On the relation between quantum mechanical probabilities and event frequencies

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Abstract
The probability ‘measure’ for measurements at two consecutive moments of time is non-additive. These probabilities, on the other hand, may be determined by the limit of relative frequency of measured events, which are by nature additive. We demonstrate that there are only two ways to resolve this problem. The first solution places emphasis on the precise use of the concept of conditional probability for successive measurements. The physically correct conditional probabilities define additive probabilities for two-time measurements. These probabilities depend explicitly on the resolution of the physical device and do not, therefore, correspond to a function of the associated projection operators. It follows that quantum theory distinguishes between physical events and propositions about events, the latter are not represented by projection operators and that the outcomes of two-time experiments cannot be described by quantum logic.

The alternative explanation is rather radical: it is conceivable that the relative frequencies for two-time measurements do not converge, unless a particular consistency condition is satisfied. If this is true, a strong revision of the quantum mechanical formalism may prove necessary. We stress that it is possible to perform experiments that will distinguish the two alternatives.

1 Introduction
Quantum mechanics is a probabilistic theory. It provides a set of rules that allows us to associate probabilities to specific physical events. There is little doubt that these rules have been proved remarkably successful in the description of any physical phenomenon that we have been able to study experimentally, but gravitational ones.

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The reasoning in terms of probability in physical theories, however, is not entirely unproblematic. Most discussions about the interpretation of quantum theory – and the associated "paradoxes" – are related to the appropriate use of quantum probabilities. Some of the issues raised are not specifically quantum – they refer to the physical applicability of the general concepts of probability theory and date at least back to Boltzmann.

One may ask, for instance, whether the probabilities are subjective or objective – namely whether they refer to our knowledge about a physical system or to the physical system itself. In the latter case, one may further ask whether probabilities refer to an individual system – denoting perhaps its propensity to manifest one behavior or another - or to statistical ensembles. One may also question whether there exists a sample space for quantum phenomena, or all predictions have to make reference to a concrete measurement set-up.

There exists a common denominator in all interpretations of probability, either classical or quantum. We may not agree whether probabilities refer to the properties of the things themselves or not, but we do accept that probabilities refer to the statistics of measurement outcomes. Probabilities may or may not be physically meaningful a priori (before an experiment), but they can definitely be determined a posteriori, namely after a large number of experimental runs.

The probability of an event is defined as the limit of the relative frequency of this event as the number of trials goes to infinity. It may be argued that this is not the only way we employ probability in physics – after all statistical arguments enter into the design and preparation of any experiment. Still, whenever we want to compare the theoretical probabilities with concrete empirical data, we invariably employ the relation of probability to event frequencies.

In this paper we analyse the basic properties of quantum mechanical probability for two-time measurements in two consecutive moments of time (two-time measurements). The key point of our argumentation is the empirical determination of probabilities as limits of relative frequencies. We employ this relation without committing to a frequency interpretation of probability – we need not assume that probability, as a concept, is defined as a limit of relative frequencies. Neither do we commit to a specific interpretation of quantum theory. We only assume that the outcomes of measurements (that have actually been performed) are described by the probabilities obtained from the rules of quantum theory. Quantum mechanical probabilities may refer to other aspects of physical reality, but we need not make such an assumption. This thesis can hardly be rejected by any interpretation of quantum theory.

In a two-time measurement one determines specific properties of a physical system at two successive moments of time. The measurement outcomes may be sampled in the same manner they are sampled in the single-time measurements. Probabilities are then still determined by the limits of relative frequencies. We may still employ the rules of quantum theory to associate a probability to each possible measurement outcome. The problem is that the quantum mechanical probability ‘measure’ for two-time histories does not satisfy the additivity property of probabilities. On the other hand, relative frequencies are always additive, since they are constructed by counting specific and indivisible physical
events.

We next proceed to resolve this conflict. A physical theory must explain the observed phenomena – namely the frequencies of measured events. If these frequencies define probabilities, we have to accept that the conventional rule for probabilities of two-time measurements fails. We show that the derivation of this rule employs the concept of conditional probability in a rather ambiguous way. We address this problem and define thereby additive probabilities for two-time measurements. But the new probability assignment depends explicitly on the resolution of the physical device. The probabilities assigned to a specific sample set of measurements depend therefore on the physical characteristics of the apparatus and they are not a function of the associated projection operators. Projection operators cannot represent events universally. It follows that the YES-NO experiments cannot reconstruct all probabilistic aspects of a physical system, and for this reason the outcomes of two-time experiments cannot be represented by any form of quantum logic.

The alternative is rather radical, but cannot be a priori rejected. It is conceivable that the relative frequencies for the two-time measurements do not converge (see for a relevant interpretation of quantum probability). In that case, probabilities can only be defined for two-time events that satisfy a consistency condition – the same condition that appears in the consistent histories interpretation of quantum theory. The failure of the frequencies to converge is eventually due to the interference between the two alternatives. This alternative explanation implies, of course, that one would need an all-new reformulation of quantum theory.

It is important to emphasise that the two possible resolutions of our problem may be empirically distinguished. It is possible – in theory and we believe in practice too – to design experiments that will determine whether the relative frequencies of two-time events converge or not. Either way, such experiments would shed much light in many counter-intuitive aspects of quantum probability.

2 Probabilities for two-time measurements

First we describe the relation of probabilities to event frequencies. We assume an ensemble of a large number of identically prepared systems. In each system we measure some physical properties say \( A \), which take value in a set \( \Omega \). We then perform the measurements one by one – thus constructing a sequence \( A_N \) of points of \( \Omega \), where \( N \) is an integer that labels the experiments. We next sample the measurement outcomes into subsets \( U \) of \( \Omega \).

\[1\]

In standard probability theory such sets are denoted as events, because they represent possible outcomes of the measurement process. Not any subset of \( \Omega \) can play that role – unless \( \Omega \) is a denumerable set. In the standard theory \( U \) are usually taken to be Borel sets, namely sets that can be obtained from denumerable unions and intersections of the open subsets of \( \Omega \). The description of probabilities in terms of frequencies, however, is related directly to sampling of measurement outcomes. If \( \Omega \) is a continuous set, Borel subsets consisting of discrete points are irrelevant to what we may actually measure and should be, therefore, excluded.
We next define \( n(U, N) \) as the number of times that the result of the measurement is found in \( U \) in the first \( N \) experiments. It is evident that \( n(U, N) \) satisfies the following properties

\[
\begin{align*}
n(U, N) & \geq n(U, M), \text{ if } N > M \quad (2.1) \\
n(U \cup V, N) & = n(U, N) + n(V, N), \text{ if } U \cap V = \emptyset \quad (2.2) \\
n(\Omega, N) & = N \quad (2.3) \\
n(\emptyset, N) & = 0 \quad (2.4)
\end{align*}
\]

One then may define the a-posteriori probability that an event in \( U \) has been realised as

\[
p(U) = \lim_{N \to \infty} \frac{n(U, N)}{N}, \quad (2.5)
\]

provided the limit exists. These probabilities satisfy all axioms of ordinary probability.

We then consider a measurement at two successive moment of time. At time \( t = 0 \) a particle described the density matrix \( \hat{\rho}_0 \) is emitted from a source. At time \( t_1 \) it passes through a device, which measures its position. The device may simply consist of a thin strip of a medium, which registers a track of the particles that penetrate it. The particle then passes at time \( t_2 > t_1 \) from an identical measurement device, which we placed immediately behind the first. It leaves a second track there.

The experiment described above may be repeated \( N \) times, each time recording the mark left by the particle on the measurement devices. To study the statistics of the measurements, we split the possible values of particle position at each moment of time into a set of \( n \) exclusive alternatives. Each alternative labelled by the index \( i \) corresponds to a subset \( U_i \) of the real line, such that \( \bigcup_i U_i = \mathbb{R} \) and \( U_i \cap U_j = \emptyset \), for \( i \neq j \).

The sample space \( \Omega = \mathbb{R}^2 \) for the two-times measurements is partitioned into the \( n^2 \) sets \( U_{ij} = U_i \times U_j \), which are labelled by the ordered pair of integers \((i, j)\). From the results of the measurements we may immediately read the numbers \( n(U_{ij}, N) \). According to the relation between relative frequencies and probabilities these numbers should satisfy

\[
\frac{n(U_{ij}, N)}{N} \to p(U_i, t_1; U_j, t_2), \quad (2.6)
\]

as \( N \to \infty \). Here \( p(U_i, t_1; U_j, t_2) \) refers to the probability first \( i \) and then \( j \) are realised.

The rules of quantum theory allow us to express this probability in terms of the projection operators \( \hat{P}_i \) that correspond to the interval \( U_i \) of the particle's position.

\[
p(U_i, t_1; U_j, t_2) = \text{Tr}(\hat{Q}_j \hat{P}_i \rho(t_1) \hat{P}_i), \quad (2.7)
\]
where we denoted for simplicity $Q_j = e^{i\hat{H}(t_2-t_1)}P_j e^{-i\hat{H}(t_2-t_1)}$. $\hat{H}$ the Hamiltonian of the particle and $\rho(t_1)$ the initial density matrix evolved until time $t_1$.

Suppose, however, that we want to consider the probability that the particle first crossed through either $U_1$ or $U_2$ and then through $U_j$. The projection operator corresponding to $U_1 \cup U_2$ is $\hat{P}_1 + \hat{P}_2$, hence the corresponding probability is

$$p(U_1 \cup U_2, t_1; U_j, t_2) = \text{Tr}(\hat{Q}_j (\hat{P}_1 + \hat{P}_2) \hat{\rho}(t_1)(\hat{P}_1 + \hat{P}_2))$$

$$= p(U_1, t_1; U_j, t_2) + p(U_2, t_1; U_j, t_2) + 2\text{Re} d(U_1, U_2, t_1; U_j, t_2), \quad (2.8)$$

where

$$d(U_1, U_2, t_1; U_j, t_2) = \text{Tr}(\hat{Q}_j \hat{P}_1 \hat{\rho}(t_1) \hat{P}_2) \quad (2.9)$$

is known as the *decoherence functional* in the consistent histories approach. It provides a measure of the interference between the events 1 and 2.

On the other hand the elementary properties of the frequencies $n(U_{ij}, N)$ state that $n([U_1 \cup U_2] \times U_j, N) = n(U_1 \times U_j, N) + n(U_2 \times U_j, N)$, so that in the limit $N \to \infty$

$$p(U_1 \cup U_2, t_1; U_j, t_2) = p(U_1, t_1; U_j, t_2) + p(U_2, t_1; U_j, t_2) \quad (2.10)$$

In other words, the quantum mechanical probabilities are not additive (unless the consistency condition $\text{Red}(U_1, U_2, t_1; U_j, t_2) = 0$ is satisfied), while the measured frequencies of events are additive. We shall see that there exist only two possible resolutions to the problem. The first one is close to conventional wisdom about quantum theory, but has, nonetheless, disturbing implications. The other is more radical, but cannot be discounted a priori.

### 3 Probabilities are contextual

#### 3.1 The correct use of conditional probability

Our first alternative involves the assumption that the sequences (2.6) converge, while the second that they do not converge. In the former case the physically relevant probabilities are defined by the limit of the relative frequencies. These probabilities are a datum of experiment, and as such they should be explained by the physical theory. If the theory fails in that regard, then there must be a mistake somewhere in the analysis. It follows that if the probabilities can be defined, the derivation of equation (2.7) should be reexamined.

We start from a density matrix $\hat{\rho}$ at $t = 0$, which is evolved unitarily until time $t_1$, when the particle enters the measuring device. If we register the particle in the interval labelled by $i$, the outcoming density matrix will equal

$$\frac{\hat{P}_i \hat{\rho}(t_1) \hat{P}_i}{\text{Tr}(\hat{\rho}(t_1) \hat{P}_i)} \quad (3.1)$$
We need make no commitments about the interpretation of the measurement process. It is irrelevant whether the measuring device is classical like in Copenhagen quantum theory, or quantum mechanical and a physical process of wave packet reduction has taken place. And it makes little difference whether the density matrix refers to an individual system, or a statistical ensemble, because at the end of the day our results will be interpreted by statistical processing of the measurement outcomes. What is important is that the density matrix (3.1) allows us to compute the conditional probabilities that the event $j$ takes place at $t_2$ provided the event $i$ took place at $t_1$

$$\frac{\text{Tr} \left( \hat{P}_i \hat{\rho}(t_1) \hat{P}_i \hat{Q}_j \right)}{\text{Tr} (\hat{\rho}(t_1) \hat{P}_i)}.$$

(3.2)

from which the classical definition of conditional probability leads us to expression (2.7) for the probability that first the event $i$ takes place at $t_1$ and then the event $j$ takes place at time $t_2$.

The problem lies in equation (3.1). If, instead of sampling the measurement outcomes in the set, say $U_1$, we sampled it into $U_1 \cup U_2$, we would have employed the projector $\hat{P}_1 + \hat{P}_2$ and the out-coming density matrix would read

$$\frac{(\hat{P}_1 + \hat{P}_2) \hat{\rho}(t_1)(\hat{P}_1 + \hat{P}_2)}{\text{Tr}(\hat{\rho}(t_1)(\hat{P}_1 + \hat{P}_2))}.$$

(3.3)

We would then obtain the result (2.8), which is inconsistent with the probabilities defined through relative frequency.

However, there is no a priori reason to use equation (3.1) for the out-coming density matrix. The action of the projection $\hat{P}_i$ depends on our choice of sampling of measurement outcomes and not on the measurement outcome itself. What has actually taken place is that the particle left a mark on a specific point, and we then choose to place that point into one or the other set. If we had a measurement at a single moment of time, this would not have been a problem, because the density matrix (3.1) does not appear in any physical predictions for single-time measurements. In a single-time measurement the only physically relevant quantities are the probabilities $\text{Tr}(\hat{P}_i \hat{\rho}(t_1))$, which are additive and for this reason they do not depend on our choice of sampling.

In the two-time measurement, however, the probabilities turn out to be non-additive, and this should urge some caution on the use of conditional probability. If we sample events into larger sets than the ones being manifested in the experiments, then we employ less information than what we have actually obtained. Our use of conditional probabilities will be, therefore, improper. (See the discussion in [9] about the way conditional probabilities may lead to erroneous predictions, if we fail to make use of all available information.)

The physically correct procedure would be to incorporate in our probabilities all information that has been obtained from the measurements. In other words we must construct the out-coming density matrix not on the basis of our arbitrary choice of sampling events, but on what we have actually observed.
We should not use an arbitrary set of projectors, but only the finest possible projectors compatible with the resolution of the apparatus. If $\delta$ is the sharpest resolution of the measuring device (say the width of the dots indicating the particle’s position) the relevant projectors are $\hat{P}_{x}^{\delta}$, which project onto the interval $[x - \frac{\delta}{2}, x + \frac{\delta}{2}]$. Using these projectors we construct the probabilities
\begin{equation}
 p_{\delta}(x_1, t_1; x_2, t_2) = \text{Tr} \left( e^{i\hat{H}(t_2-t_1)} \hat{P}_{x_2}^{\delta} e^{-i\hat{H}(t_2-t_1)} \hat{P}_{x_1}^{\delta} \hat{\rho}(t_1) \hat{P}_{x_1}^{\delta} \right),
\end{equation}
that a dot will be found centered at the point $x_1$ in the first measurement and then a dot centered at the point $x_2$ in the second measurement.

We may then construct the probabilities for a particle to be found within a subset $U_i$ of $\mathbb{R}$ at time $t_1$ and then within a subset $U_j$ at time $t_2$. For this purpose, we split each set $U_i$ into mutually exclusive cells $u_{\alpha i}$ of size $\delta$, such that
\begin{equation}
 \cup_{\alpha} u_{\alpha i} = U_i,
\end{equation}
\begin{equation}
 u_{\alpha i} \cap u_{\beta i} = \emptyset, \alpha \neq \beta. \tag{3. 6}
\end{equation}
If we denote select points $x_{\alpha i} \in u_{\alpha i}$, for all $i$ ($x_{\alpha i}$ may be the midpoint of $u_{\alpha i}$), we may construct the probability
\begin{equation}
 p_{\delta}(U_i, t_1; U_j, t_2) = \sum_{\alpha} \sum_{\beta} p_{\delta}(x_{\alpha i}, t_1; x_{\beta j}, t_2) \tag{3. 7}
\end{equation}
In the limit that the typical size of the sets $U_j$ is much larger than $\delta$, we may approximate the summation by an integral,
\begin{equation}
 p_{\delta}(U_i, t_1|U_j, t_2) = \frac{1}{\delta^2} \int_{U_i} dx_1 \int_{U_j} dx_2 p_{\delta}(x_1, t_1; x_2, t_2), \tag{3. 8}
\end{equation}
In other words, the objects $\frac{1}{\delta^2} p_{\delta}(U_i, t_1|U_j, t_2)$ play the role of probability densities. The probabilities (3.8) are compatible with the relative frequencies, because they do satisfy the additivity criterion. Note, however, that they depend strongly on the resolution $\delta$ of the measuring device.\(^2\)

There exists a systematic error in the definition of the probabilities (3.8), which is due to the approximation of the sums (3.7) by integrals. This is related to the fact that the dots have a finite size and hence cannot be definitely ascertained whether they lie in a sample set $U_i$ or its neighboring one. For sufficiently large sets $U_j$, characterised by a typical size $L$, the ambiguity may be approximated by a Gaussian distribution and is of the order $e^{-L^2/\delta^2}$.

\(^2\)If the two -time probabilities did not depend on $\delta$, it would have been possible to describe the quantum mechanical system in terms of a stochastic process that reproduces all $n$-point functions of the quantum mechanical description without making any reference to the measurement apparatus. The generic dependence of $n$-time probabilities on the measuring device renders this impossible, in agreement with many constraints placed by Bell’s theorem.
The important feature of the probabilities (3.8) is that they are not functions of the projection operators \( P_{U_i} \). They depend on the resolution \( \delta \) of the measuring device and the way the sample sets \( U_i \) are partitioned into subsets of size \( \delta \). For this reason, different measuring devices lead to different values of the probabilities (3.8). We may consider, for instance two different measuring devices, one with resolution \( \delta \) and one with resolution \( 2\delta \). The probabilities corresponding to the former will be constructed from the minimal projectors \( \hat{P}_\delta^x \), while the latter from the projectors \( \hat{P}_{2\delta}^x \). Given that \( \hat{P}_{2\delta}^x = \hat{P}_\delta^x - \frac{\delta^2}{4} + \hat{P}_\delta^x + \frac{\delta^2}{4} \), the difference between the probabilities \( p_\delta(U_{i_1}, t_1 | U_{j_1}, t_2) \) and \( p_{2\delta}(U_{i_1}, t_1 | U_{j_1}, t_2) \) equals

\[
\epsilon_\delta(U_{i_1}, t_1; U_{j_1}, t_2) = \text{Re} \int_{U_{i_1}} dx_1 \int_{U_{j_1}} dx_2 d_\delta(x_1 + \delta/2, x_1 - \delta/2, t_1 : x_2, t_2), \quad (3.9)
\]

in terms of the interference term

\[
d_\delta(x_1 + \delta/2, x_1 - \delta/2, t_1 : x_2, t_2) = \text{Tr} \left( e^{iH(t_2-t_1)} \hat{P}_{2\delta}^x e^{-iH(t_2-t_1)} \hat{P}_\delta^x_{x_1 + \delta/2} \hat{\rho}(t_1) \hat{P}_\delta^x_{x_1 - \delta/2} \right). \quad (3.10)
\]

The probabilities for the same events depend on the resolution, unless the interference term vanishes for all \( \delta \) or if it is number of the order of \( e^{-L^2/\delta^2} \). Only then would the probabilities have a functional dependence on the projectors \( P_{U_i} \). This is the case, for instance, when the final projector is equal to the unity, in which case we recover the single-time results.

### 3.2 An explicit example

It is instructive to compute the probabilities and the interference term in a concrete physical system. We assume a non-relativistic free particle in one dimension, with Hamiltonian \( H = \frac{\hat{p}^2}{2m} \), where \( m \) is the particle’s mass and \( \hat{p} \) its momentum. It is convenient to employ a Gaussian initial state

\[
\psi(x) = \frac{1}{(\pi\sigma^2)^{1/4}} e^{-\frac{x^2}{2\sigma^2}} e^{ipx}. \quad (3.11)
\]

The parameter \( \sigma \) is the position uncertainty, \( p \) its mean momentum, and we assumed without loss of generality that it is centered around \( x = 0 \).

Our device measures position. We shall employ smeared Gaussians instead of sharp projection operators \(^3\).

\[
\langle x | \hat{P}_{\delta}^x_{x_0} | y \rangle = e^{-\frac{1}{2\sigma^2}(x-x_0)^2} \delta(x, y) \quad (3.12)
\]

We are interested in the case that \( \delta << |x_0| << \sigma \), namely that the device may distinguish between different readings that lie close to the center of the initial state. In that case

\[
\hat{P}_{\delta}^x_{x_0} \psi(x) = \frac{1}{(\pi\sigma^2)^{1/4}} e^{-\frac{(x-x_0)^2}{2\sigma^2}} e^{ipx}. \quad (3.13)
\]

\(^3\)The smeared Gaussians project, in effect, into a fuzzy set, which is quite appropriate for realistic position measurements.
The above expression is, in fact, a reasonable approximation for any wave function with spread \( \sigma \), whose structure does not vary much in the scale of \( \delta \).

Assuming that the second measurement takes place at time \( t \), we find

\[
p_\delta(x, 0; x', t) = \frac{\delta}{\sigma} \frac{r}{1 + r} e^{-\frac{r}{1 + r} (x' - x - \frac{p_m}{t})^2} \tag{3.14}
\]

\[
d_\delta(x + \delta/2, x - \delta/2, 0: x' t) = p_\delta(x, 0; x', t)e^{-\frac{1}{2} \left( \frac{x' - x - \frac{p_m}{t}}{\delta} \right)^2} \tag{3.15}
\]

where

\[
a = 1 + \frac{2r}{(1 + r)^2}; \tag{3.16}
\]
\[
b = 2r \frac{(1 - r)^2}{1 + r^2}; \tag{3.17}
\]
\[
c = \frac{\frac{r(1 + 2r - 3r^2 + 2r^3)}{2(1 + r^2)^2}}{1 + r^2}; \tag{3.18}
\]
\[
r = \frac{m\delta^2}{t}. \tag{3.19}
\]

The parameter \( r \) is the time-of-flight phase space uncertainty, namely the uncertainty \( m\delta/t \) in a time-of-flight determination of the momentum times the resolution \( \delta \). From the equations above, we see that the interference term is of the same order as the probability irrespective of the value of \( r \).

We will now estimate the difference \( \epsilon \) between the probabilities for two sets \( U_1 \) at time \( t = 0 \) and \( U_2 \) at time \( t \). Both sets are assumed to be of size \( L >> \delta \) so that the sampling of the data can be accurate. The set \( U_1 \) is centered around the point \( x_1 \) and the set \( U_2 \) around \( x_2 \). We will denote by \( \Delta = x_2 - x_1 - \frac{p_m}{t} \) the distance between \( X_2 \) and the evolution of \( x_1 \) according to the classical equations of motion. Using Gaussian smeared characteristic functions \( e^{-\frac{(x-x_i)^2}{2L^2}} \), to perform the integrations over the sets \( U_i \), we obtain

\[
p_\delta(U_1, 0; U_2, t) = \frac{L}{\sigma} e^{-\frac{\Delta^2}{2L^2}} \tag{3.20}
\]

\[
\epsilon_\delta(U_1, 0; U_2, t) = \frac{L}{\sigma} e^{-\frac{\Delta^2}{2L^2}} \cos \left( \frac{p_\delta + b\Delta}{\delta} \right) \tag{3.21}
\]

where \( k \) and \( k' \) denote terms of the order of unity and according to our original assumptions \( L << \sigma^4 \). Clearly \( \epsilon_\delta \) is of the same order of magnitude with \( p_\delta \), a fact that emphasises the strong dependence of the probabilities on the measuring device’s resolution.

We conclude, therefore, that the difference between the two probability assignments is of the same order of magnitude as the probabilities themselves and, consequently, the probabilities for the two-time measurements depend on the way the sample sets \( U_i \) are partitioned with respect to the resolution of the measuring device and is, therefore, not a function the sets \( U_j \).\footnote{Otherwise, the origin of the set \( U_1 \) is outside the support of the initial wave-function, and the corresponding probabilities will be close to zero.}
3.3 Consequences

Our results demonstrate that the probabilities for two-time measurements are generically not functions of the projection operators that correspond to the sample sets. This is very much unlike the single time, where the probabilities for particular events are linear functionals of the projection operators that correspond to the sample sets.

The property above of the single-time quantum probabilities has as consequence that all physical measurements may be described in terms of YES-NO experiments. A typical such experiment involves a filter –represented by a projection operator – which is set on the path of a particle beam. We may monitor, whether a particle passed through the filter or not, and it is possible after $N$ trials to determine the probability corresponding to a particle passing through that filter. If we repeat this experiment with different filters that correspond to the same physical property (the associated projectors commute), we may eventually reconstruct the probability distribution for this quantity, because the single time probabilities are additive. Hence, the probabilities $\text{Tr}(\hat{P}_U)$ determined by the filter measurement of, say, position in the set $U$ coincide with the probabilities determined by the statistical analysis of all dots that were found in the sample set $U$ in any device that records the particle positions. For this reason, the projection operator $\hat{P}_U$ represents the proposition that the particle’s position has been measured to lie in the set $U$, irrespective of the experimental procedure or the details of the measuring device. From the results of the YES-NO experiments we can unambiguously reconstruct all probabilistic information about a physical system. One is led, therefore, to the suggestion that the projection operators refer to the properties of quantum systems –as manifested in measurements– and that the structure of the lattice of projection operators represents the structure of potential quantum mechanical events. One speaks, therefore, for the quantum logic of quantum mechanical measurements –or of quantum mechanical properties, if one wishes to move beyond concrete measurement situations.

In two-time measurements the situation is different. The measured probabilities are not functions of the single-time projectors. They depend instead on the properties of the measuring device and the way each sample set is resolved into minimum resolution sets. One may still perform two-time YES-NO experiments, by directing the particles through two successive filters. The expression (2.7) may be employed to this experiment. But the filter measurements do not suffice to reconstruct the two-times probability assignment to the physical quantity they represent; the physical probabilities for the two-time experiments are given by equation (3.8) and not by equation (2.7). A filter measurement for position may be, for example, realised in terms of a wall with a slit of width $L$ in it, which represents a subset $U$ of $\mathbb{R}$. This is a very different physical system from the one we described earlier, which records any possible position with an accuracy of $\delta$. There is no a priori physical reason that the probability that the particle will cross the slit will be the same with the probability obtained from the relative frequencies of the events within $U$ in the latter measurement.
In single-time measurements they happen to be the same, but in two-time ones they are not.

In other words, the YES-NO experiments do not capture all physical information about two time measurements – the physical predictions depend strongly on the properties of the measuring devices. Hence the proposition that the particle is measured at time $t_1$ within the set $U_1$ and at time $t_2$ within the set $U_2$ is not universally represented by the ordered pair of projectors $(\hat{P}_{U_1}, \hat{P}_{U_2})$. It is only represented by these projectors when the measuring device consists of two filters – the first with a slit corresponding to $U_1$ and the second with a slit corresponding to $U_2$. There is, therefore, no universality in the representation of measurement outcomes by pairs of projection operators, with the consequence that the interpretation of two-time measurements in terms of quantum logic is not possible.

We may make, in fact, a stronger statement: even for single-time measurements the interpretation in terms of quantum logic is not possible. The proof involves *reductio ad absurdum*. We represent the single-time lattice of projection operators on a Hilbert space $H$ as $\mathcal{L}(H)$ and assume that each measurement outcome may be uniquely represented by an element of $\mathcal{L}(H)$ – the converse need not be true. Two successive measurement outcomes should, therefore, be represented by a pair of elements of $\mathcal{L}(H)$, hence an element of $\mathcal{L}(H) \times \mathcal{L}(H)^5$. We have showed that this is not the case. Hence there exists an error in our assumptions. The statement that two measurement outcomes are represented by a pair of arguments of $\mathcal{L}(H)$ is a consequence of basic principles of logical reasoning (and a basic axiom of set theory). Unless we assume that a two-time measurement does not correspond to two single-time measurements, we are forced to conclude that the universal representation of measurement outcomes by projection operators is not valid even in single-time measurements. The reason it seems possible to do so, is because in single-time measurements the interference term $d$ of equation (2.9) always vanishes.

The conclusion above is, in a sense, complementary to many theorems about

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5The lattice of two-time measurement outcomes should contain $\mathcal{L}(H) \times \mathcal{L}(H)$ as a subset – not necessarily a sublattice. Such is the case, for instance, in Isham’s scheme [5], where the lattice of two-time measurements is represented by the lattice of projectors on $\mathcal{L}(H \otimes H)$.

6Elaborating on Bohr’s interpretation of measurement, it is possible to assume that a two-time measurement should not be analysed conceptually into two single-time ones. Any measurement refers to an irreducible physical set-up and a two-time measurement should be considered as irreducible as a single-time one. This is, however, an extreme position, which goes much further than Bohr’s explanation of the Einstein-Podolski-Rosen (EPR) argument. The present argument is not, whether there exist correlations between the results of two successive measurements, but whether we may represent – conceptually – the results of a two-time measurement as a pair of single time measurement. The corresponding thesis for the EPR experiment would be to assume that there is no way to represent separately the measurement outcomes on the two physical subsystems – to even consider physical observables that refer to each subsystem. Such a position is logically consistent, but it is so alien to the way we actually perform experiments as to be physically untenable. The results of a two-time measurement of position are always two readings of position. One simply cannot refute this fact. But even if we accepted such a destructive thesis, we would still face the fact that the outcomes of irreducible measurements (now meaning the two-time ones) cannot be universally represented by projection operators.
contextuality in quantum theory\cite{10,11}. Since a given projector may be a spectral projector of two non-commuting self-adjoint operators, one should always make a choice of the corresponding self-adjoint operator (the context) before employing that projector to represent a property of a physical system\cite{12}. Our results refer to concrete measurement situations, and for this reason their conclusions are rather stronger. Not only is it impossible to represent a measurement outcome by a projection operator, but the observed probabilities depend explicitly on specific properties of the measurement device\footnote{The same conclusion holds for approximate projectors as can be seen from the results in our model system. Fuzzy measurements cannot save us from the generic dependence of quantum mechanical probabilities on the measuring apparatus.}.

There exist many interpretation schemes that employ the lattice of projection operators to universally represent properties of physical systems, even outside the measurement context. What we have demonstrated here is that the universal representation of measurement outcomes by projection operators is untenable. It is, therefore, \textit{a priori} not contradictory to assume that projection operators refer to properties of a physical system outside the context of measurements. We will not comment in this paper, whether this thesis is tenable or not. We only remark that it is an entirely \textit{ad hoc} hypothesis in the light of our results. The quantum logic of properties of a physical systems cannot be considered as a generalisation of a quantum logic for measurements.

Our conclusions are independent of the interpretation for the quantum state (whether it is objective or subjective, whether it refers to individual systems or to ensembles). They are also independent of the interpretation of measurement theory (whether we employ subjective conditional probabilities, or we think that the reduction of the wave packet is a physical process, or there is a duality of a classical measuring device and a quantum system).

We must distinguish the two different roles of the sample sets $U$ and the corresponding projectors – a distinction that is not usually made in probability theory. A sample set $U$ may represent a physical event, if the device cannot distinguish between the elements of $U$. In that case $U$ refers to a concrete empirical fact. It may also represent a statement about the physical system, namely that an event has been found within the set $U$. The latter case, however, is not a representation of a physical fact. It is at the discretion of the experimentalist to choose the set $U$ that he will use for the sampling of its results. The physical probabilities should, therefore, be constructed with the first interpretation of the sample sets in mind. These probabilities then depend then on the construction of the physical apparatus and interaction with the measured system. They may also depend on the initial state of the measured system: ultra-fast neutrons, for instance, will leave a different trace on a recording material than slow ones. The out-coming density matrix (3.2) should, in principle, be determined by the common reasoning that is employed in the design of experiments: treating the apparatus as classical and the measured system as quantum. The results of such an analysis would not yield an expression in terms of projection operators for the minimum resolution. There exists a degree of fuzziness in any measurement, and for this reason the natural mathematical objects that
encode the effect of minimal resolution should be smeared projectors: positive operators with supremum norm less than one \[13\].

The representation of quantum mechanical measurement outcomes by projection operators is a consequence of a principle that is often considered as a basic axiom of quantum mechanics: The possible outcomes in the measurement of a physical quantity represented by a self-adjoint operator \(\hat{A}\) lie in the spectrum of \(\hat{A}\). This postulate implies that sample sets of the measurement of \(\hat{A}\) correspond to the measurable subsets of the operator’s spectrum and, due to the spectral theorem, to the spectral projectors of \(\hat{A}\). Our results suggest strongly that this postulate may not be appropriate in quantum theory. Its abandonment would not change any physical predictions – an axiomatic framework for standard quantum mechanics is still possible in its absence. The mean values and all higher moments for measurements of any observable may still be obtained through the usual rules of quantum theory. The only difference is that the spectral resolution of an operator does not necessarily correspond to the physical resolution of the measured values of the physical quantity. We have discussed this issue in references \[14, 15\], to which we refer the reader for more details.

To summarise our results, if we accept that the relative frequencies for two-time measurements converge, we inevitably conclude that

i. The probabilities for two specific sample sets in a two-time measurement is not a function of the projection operators that, supposedly, correspond to each sample set.

ii. The YES-NO experiments do not suffice to reconstruct all physical predictions of quantum theory for two-time measurements.

iii. Projection operators cannot, in general, represent properties of a physical system. They are only relevant to the probabilities of specific YES-NO experiments.

iv. Unlike classical probability theory, quantum theory distinguishes sharply between physical events and propositions about physical events.

4 An alternative explanation

Our resolution of the ‘paradox’ of two-time measurements and the subsequent analysis was based on the assumption that the measured frequencies of events define probabilities, i.e. that the sequences (2.5) converge. We are then led to a reconsideration of the use of the conditional probability for the derivation of (2.7). The conclusion that probabilities depend rather strongly on the properties of the physical device, is rather disturbing. It implies that the results of two sets of measurements that involve an identical preparation of the physical system and very similar measuring apparatus would differ according to rather trivial
details of the apparatus’s manufacture.

There exists an alternative solution to the problem, but its implication are more disturbing rather than less. It is conceivable that the sequences do not converge to probabilities. They could perhaps exhibit an oscillating behavior as $N \to \infty$. This possibility has not, to the best of our knowledge, been refuted by any experiment that has been performed so far. In that case probabilities cannot be defined for two-time measurement. The expressions (2.7) cannot, therefore, be considered as referring to probabilities. In fact, we have no idea how to interpret them.

The hypothesis that two-times probabilities are not defined is not incompatible with the successful use of probability theory for single-time measurements. Probabilities are additive for single-time measurement – just as relative frequencies are – and there is no problem in that case to assume that the sequences (2.5) converge. The same would be true for any sufficiently coarse-grained measurements, for which the interference term vanishes. One would be, therefore, led to the interpretation of the object $|\text{Re } d(U_1, U_j; U_2, U_j)|$ as a measure of the non-convergence of the sequence of relative frequencies. Probabilities would, therefore, be definable only for specific samplings of the measurement outcomes, such that the consistency condition $\text{Re } d(U_1, U_j; U_2, U_j) = 0$ holds. This is, in fact, similar to the use of probabilities by the consistent histories approach – probabilities are defined only for sufficiently coarse-grained partitions of the two-time sample space, such that the consistency condition is satisfied.

The second alternative is perhaps too radical. It would involve a reappraisal of the use of probabilities in physical theories. We would have to extend both the theory of probability and quantum theory in a way that will deal with non-convergent sequences of relative frequencies. It is not clear, how this may be achieved, what generalisations are physically relevant, and which parts of the quantum mechanical formalism, if any, would have to be abandoned. As far as the explanation of the non-convergence of frequencies is concerned, we may only speculate. Perhaps, the set-up of two time measurements does not lead to the probabilities of quantum equilibrium – in the sense of Bohmian mechanics [16]. Or, perhaps, it is due to a physical reason unsuspected by any current interpretation or reformulation of quantum theory. In absence of conclusive empirical evidence, we find more prudent to refrain from any detailed speculation on this issue.

The hypothesis of non-converging frequencies seems much less plausible than the alternative we considered in the previous sections. However, it is a priori possible that physical phenomena cannot be entirely described in terms of probabilities. In any case, this issue can be resolved by recourse to experiment. It should not be very difficult to design and execute experiments that will measure particle positions at two moments of time. A careful statistical analysis of the measurement outcomes will then allow us to clearly distinguish whether the relative frequencies converge or not. If they do not, then it would be a strong argument in support of the incompleteness of the current formulation of quantum theory.
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