Undergraduate students’ conceptual understanding on rational inequalities

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Abstract. Rational inequalities are in part of algebraic inequality problems. Ability to solve the problems related to this type of inequality can be brought by the decent understanding about algebraic logic and manipulation. This study aims to investigate undergraduate students’ conceptual understanding on dealing with rational inequality tasks. In this analysis, we considered indicators of mathematical conceptual understanding in order to develop an instrument including five tasks. The tasks were administered to ten first-year undergraduate students in mathematics education. Then, we analyzed students’ solution to indicate their difficulties and misconceptions. We underlined two conclusions. First, the participating students must reconsider their conceptual understanding related to rational inequality topic, especially on translating graphical situations into the algebraic situations and vice-versa. Second, the lack of students’ abilities related to mathematical conceptual understanding indicators can be viewed as the sources of revealed students’ difficulties and misconceptions. We also summarize the potential difficulties and misconceptions categorized by their sources.

1. Introduction
Inequality is an important but difficult topic in mathematics[1, 2, 3, 4]. The concept of inequality used for other topics or works like proofing, optimizing, and an investigation of function[5]. Unfortunately, a lot of previous study revealed that students often have the difficulties and misconceptions when dealing with mathematics inequality problems. Some of those, there were students’ difficulties to use logical connections (‘and’ and ‘or’) on finding solution set of inequality and students’ tendencies to use equations solving steps for solving inequalities[2,6]. For examples related to logical connection, when solving \((x + 1)(x - 1) > 0\), student wrote \(x + 1 > 0\), \(x - 1 > 0\) and finally he didn’t find the solution. Related to tendency to use equation solving steps, for example, solving \(\frac{x-1}{x+1} > 1\) with multiplying both side with \(x + 1\) without consider the case of negative or positive of \(x + 1\).

In the case of quadratic and rational inequalities, previous researchers found three other common students’ errors beside the errors related to logical connection. There were misconception about ‘quadratic expressions always positive (ignoring zero)’, multiplying both sides with not-necessarily positive expression, and ignoring the other case[6], [7]. For example, students answer that set of whole real number is the solution set for \((x - 1)^2 > 0\). Their reason said because quadratic expressions is always positive. But, literally we have that \(x = 1\) is not satisfy that inequality. Related to multiplying both sides with not-necessarily positive expression, for example, multiplying both sides of \(\frac{x-1}{x+1} > 1\) with \(x + 1\), instead of multiplying both these sides with \((x + 1)^2\). Mean while, related to...
ignoring the other case, for example, students consider \( a > 0 \) and \( b > 0 \) is the only condition that causes \( \frac{a}{b} > 0 \). In other words, ignoring the other condition, of course \( a < 0 \) and \( b < 0 \).

The problems include misconceptions when dealing with inequalities also happened on undergraduate students in mathematics education. The evidences was revealed by Naseer [8], which found the undergraduate students misconceptions on some algebraic concept, including inequalities. More focused on inequalities topic, there was a study on 328 pre-service mathematics teacher students about their work on equations and inequalities [9]. This previous study revealed that on eight paired equation-inequality tasks, more students made incorrect answer on solving an inequality, instead of solving its corresponding equation, except on the last paired tasks. Figure 1 gives us information about their work on solving eight paired equations and inequalities.

![Figure 1](image)

In the discussion part of that mentioned study, the researchers put forward the comments about students’ solution of each paired tasks. Related to second, fourth, and seventh paired tasks, the researchers revealed there was a tendency of the students to merely carry out an inequality as if it was an equation. Especially, the faulty use of null factors on seventh task. Where in the solution, student replaced the equals sign for ‘greater than’ sign [9].

Based on those explained studies, we construe that the inequalities which include the multiplication or division of two algebraic expression may build a challenge for students’ conceptual understanding about basic algebraic concepts. Ability to solve that challenge can be provided by the decent understanding about algebraic logic and manipulation. That ability is necessary as a success key in learning college algebra, pre-calculus, and teaching mathematics [9].

Although the mentioned studies have been revealed the difficulties and misconceptions about students dealing with inequalities, those studies not yet linked the misconceptions or difficulties with aspect of mathematics proficiency. Also, the studies still focused on general algebraic inequalities. Taking those into account, we carried out a qualitative study on describing mathematics education undergraduate students’ conceptual understanding on solving rational inequalities, a type of algebraic inequalities which, based on previous explained studies, students made several misconceptions. We construct the indicators related to conceptual understanding on solving rational inequalities. Then, we developed the tasks based on these indicators. The term conceptual understanding which involved in this study is based on theory proposed by Kilpatrick, Swafford, and Findell [10].

Based on that theory, this term refers to a mathematical proficiency strand that focused on comprehension of mathematical concepts, operations, and its relations. Student with decent conceptual understanding could have ability to: (1) comprehend a mathematical concept and its relation to other concepts; (2) recognize what mathematical concept that useful for solve a problem; and (3) represent a mathematical situation into the other mathematical situation, according to the specific purpose [10]. For additional information, this topic also included in Indonesian mathematics curriculum in first year senior high school [11].
2. The Method
This study aims to investigate how undergraduate students’ conceptual understanding about rational inequalities determines the existences of difficulties and misconceptions when dealing with rational inequality tasks. In order to do so, we considered the indicators which adapted from indicators of conceptual understanding[10](see Table 1), to develop five rational inequality tasks. Next, we asked ten of mathematics education undergraduate students to solve the tasks in forty five minutes. Finally, we analyzed the students’ responses on all five tasks. Table 2 shows our developed tasks and related indicator of each task, also our explanation including description and expected strategies for each one. The expected strategies based on our consideration to the indicators.

### Table 1. The indicators and adapted indicators

| No. | Indicators of conceptual understandings | Adapted indicators for rational inequalities topics | Codes |
|-----|----------------------------------------|---------------------------------------------------|-------|
| 1.  | comprehend a mathematical concept and its relation to other concepts | figure out the concept of rational inequality in relation with algebraic manipulations, mathematical operations, and sets | F1    |
| 2.  | recognize what mathematical concept which useful for solve a problem | recognize and use the certain useful concepts in order to solving a rational inequality | R2    |
| 3.  | represent a mathematical situation into the other mathematical situation, according to the specific purpose | represent a mathematical situation into the other mathematical situation related to solving rational inequalities | R3    |

### Table 2. The tasks

| No. | Tasks | Related indicators | Explanations |
|-----|-------|--------------------|--------------|
| 1.  | Find the solution set of \( \frac{2x - 1}{x + 1} \geq 1 \). | F1 | Students are expected to perform an appropriate algebraic manipulation or case partition. They are also expected to find the correct solution set by working correctly with set union and intersection. |
| 2.  | Are the following inequalities: \( \frac{x - 1}{x + 2} > 0 \) and \( x^2 + x - 2 > 0 \) having same solution set? Explain. | F1 | We challenged the students to be aware to the possible conditions which cause a division and a multiplication between two algebraic expressions provide a positive or negative value. Students may not need to solve each inequality. |
| 3.  | Find the solution set of \( \frac{x - 1}{x^2 + x + 1} \leq 0 \). | R2 | Students may recognize the rule of discriminant or the form of \( p^2 + k \). (\( k > 0 \)), and use either to claim a positive definite of the denominator. |
Table 2. The Tasks (Continued)

| No. | Tasks | Related indicators | Explanations |
|-----|-------|-------------------|--------------|
| 4.1 | Given a linear function $f$ and a quadratic function $h$. The graphs of $f$ and $h$ are shown below. | | |
|     | ![Graph](image1) | | |
|     | Based on above figure, find the solution set of $f(x) / h(x) \leq 0$. | | Students are challenged to translate the graphical situation into the algebraic situation. They may exploit the intervals whenever each graphic lie above or below $X$-axis in order to determine the solution set of the inequalities. On the last task, students may exploit the grid. |
| 4.2 | Given a quadratic function $f$ and a linear function $h$. The graphs of $f$ and $h$ are shown below. | | |
|     | ![Graph](image2) | | |
|     | Based on above figure, find the solution set of $f(x) / h(x) \geq 1$. | | |

Note: the participants worked with the tasks in Bahasa Indonesia.

3. Results and Discussion

Table 3 showed the summary of the students’ solutions for each given task. Correct answer means the steps and final answer, e.g. solution set, which are appropriate with a task.
Table 3. Students’ solution on rational inequality tasks

| Tasks no. | Related indicators | Number of correct answers |
|-----------|--------------------|--------------------------|
| 1.        | F1                 | 4                        |
| 2.        | F1                 | 4                        |
| 3.        | R2                 | 1                        |
| 4.1       | R3                 | 0                        |
| 4.2       | R3                 | 1                        |

For the first task, all the correct answers provided by doing the algebraic manipulation steps started from subtracting both sides by $-1$. The errors made are dominantly about the multiplication both side with $x+1$ without case partitioning (see figure 2(a)), i.e. whenever $x+1$ positive or negative. This errors similar with the findings on previous studies [6, 7], related to multiplying both sides with not-necessarily positive expression and ‘misconception related to students’ tendency to solving inequality using steps to solving equation (i.e. $\frac{2x-1}{x+1} = 1$).

![Figure 2](image1.png)

Figure 2. Incorrect answers for first task

![Figure 3](image2.png)

Figure 3. Simplify an inequality correctly but cannot state the possible condition of numerator and denominator
We also found a student which use an appropriate idea to multiplying both side with \((x + 1)^2\), but in last two steps, he/she didn’t reconsidered \(x + 1\) as the denominator of the initial expression of inequality. This type of solution just made by a student that we said as ‘almost correct’ answer (see Figure 2(b)).

For the second task, all the correct answers provided by solving each inequality. No one aware with the same condition caused positive value on multiplication and division. The incorrect answer dominated by the meaningless argument and the incorrect work in both inequalities (e.g. about multiplying both sides and wrong factorization).

From the first and second task, we found out that most students fail to figure out the concept of rational inequality in relation with mathematical operations (i.e. division and multiplication) and algebraic manipulation. This cause, mostly, an inappropriate understanding about multiplying both sides in order to simplify the inequality and also, in several, the difficulties to state the correct signs (positive or negative) for numerator and denominator that cause whenever positive or negative value, although they have simplified an inequality correctly (see Figure 3).

On the task 3, we found out just a correct answer, where student recognized the rule of discriminant on the denominator. Actually, the rule of discriminant have been recognized by 3 participants but two of them couldn’t correctly use it to make a correct answer. Students who made incorrect answer, except the two, couldn’t recognize this rule. They focused on factorization or algebraic manipulation but couldn’t find the way to simplify (see Figure 4). No participant who recognized another rule, i.e.: \(x^2 + x + 1 = x^2 + x + \frac{1}{4} + \frac{3}{4} = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4} \geq \frac{3}{4} > 0\), for all real number \(x\).

![Figure 4](image)

In this case, the lack of ability to recognize and use the certain useful concepts, may cause the impossibility to just simplify the inequality problems. In other words, students must thinking more widely to correctly solve the more types of rational inequality problems.

No one used the graphical or other representation to solving the first three tasks. Therefore, the only way to observe students’ abilities related to R3 indicator, in this case, is analyzing their answers for the last two tasks. On the fourth task, no one made a correct answer. Talking about their strategies, more students confused to do anything with the given problem. There was a student tried to exploit the conditions that cause \(\frac{f(x)}{h(x)} \leq 0\), but didn’t state the correct condition of \(x\) (see Figure 5). On the last task, we provided grid on the figure. Several students have been successfully exploited the grid to construct the equations of curve and line. But, they had errors on the further (e.g. Figure 6).
Overall, related to R3 indicator, participating students encountered the difficulties to correctly and effectively represent a graphical situation into the algebraic situations. This condition shown from the difficulties to solve task 4.1, which no grid given.

In the Table 4, we summarize the characteristics of students’ solution compared with our expected strategies based on the indicators.

**Table 4. Summary of students’ answers**

| Task no. | Related indicators | Expected strategies | #Students used expected strategies (#Correct) | #Correct answer using other strategies |
|----------|--------------------|---------------------|---------------------------------------------|---------------------------------------|
| 1.*      | F1                 | algebraic manipulation (nulling right side) | 4 (4)                                      | 0                                     |
|          |                    | case partition of $x + 1$ | 0 (0)                                      |                                       |
| 2.*      | F1                 | both $a \cdot b$ and $a/b$ are positive when $a < 0$ and $b < 0$, or $a > 0$ and $b > 0$ (without solve each equation) | 0 (0)                                      | 4                                     |
| 3.*      | R2                 | recognize the rule of discriminant | 3 (1)                                      | 0                                     |
Task no. | Related indicators | Expected strategies | #Students used expected strategies (#Correct) | #Correct answer using other strategies
---|---|---|---|---
| | recognize the form of $p^2 + k, k > 0$ | 0 (0) | |
| | | | |
| 4.1 | R3 | exploit the intervals whenever each graphic lie above or below $X$-axis | 1 (0) | 0
| 4.2 | R3 | exploit the intervals whenever each graphic lie above or below $X$-axis | 0 (0) | 0
| | exploit grid | 4 (1) | |*) No one use graphical representation to solve the tasks

4. Conclusions
From the results described, we underlined the two conclusions. First, the participating students must reconsider their conceptual understanding related to rational inequality topics, especially related to translating graphical situations into the algebraic situations, and vice-versa. Second, students’ conceptual understanding determine the existence of their difficulties and misconceptions, in the case of rational inequality. In Table 4, we summarize the potential students’ difficulties and misconceptions categorized by their sources.
Finally, we believe that the decent conceptual understanding help us to solve the problem, not only correctly, but also effectively. In further, teacher or researcher need to develop a lesson design to promote conceptual understanding in order to minimize the identified difficulties and misconceptions.

Table 5. Sources of students’ difficulties and misconceptions

| Sources | Potential difficulties | Potential misconceptions |
|---|---|---|
| the lack of understanding related to F1 indicator | difficult to perform an algebraic manipulation | multiplying both sides with denominator without case partitioning |
| | difficult to find possible conditions | |
| the lack of understanding related to R2 indicator | difficult to just simplify certain rational inequalities problems | misconception about the numerator and denominator must have null factor |
| | | |
| the lack of understanding related to R3 indicator | difficult to solve graphical problems | believes that the inequality problems must be an algebraic-presented |
| | | ignoring the use of useful graphical representation to solve an algebraic-presented rational inequality problem |

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