CONSECUTIVE PIATETSKI-SHAPIRO PRIMES BASED ON THE HARDY-LITTLEWOOD CONJECTURE

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Abstract. The Piatetski-Shapiro sequences are of the form $N^{(c)} := ([n^c])_{n=1}^\infty$ with $c > 1, c \notin \mathbb{N}$. In this paper, we study the distribution of pairs $(p, p^\#)$ of consecutive primes such that $p \in N^{(c_1)}$ and $p^\# \in N^{(c_2)}$ for $c_1, c_2 \in (1, 2)$ and give a conjecture with the prime counting functions of the pairs $(p, p^\#)$. We give a heuristic argument to support this prediction based on a model by Lemke Oliver and Soundararajan which relies on a strong form of the Hardy-Littlewood conjecture. Moreover, we prove a proposition related to the average of singular series with a weight of a complex exponential function.

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1. Introduction

The Piatetski-Shapiro sequences are sequences of the form

$$N^{(c)} := ([n^c])_{n=1}^\infty \quad (c > 1, c \notin \mathbb{N}).$$

Piatetski-Shapiro [6] proved that if $c \in (1, \frac{12}{11})$ the counting function

$$\pi^{(c)}(x) := |\{\text{prime } p \leq x : p \in N^{(c)}\}|$$

satisfies the asymptotic relation

$$\pi^{(c)}(x) \sim \frac{x^{1/c}}{\log x} \quad \text{as } x \to \infty.$$  

The admissible range for $c$ of the above formula has been extended many times and is currently known to hold for all $c \in (1, \frac{243}{235})$ thanks to Rivat and Sargos [7]. Rivat and Wu [8] also showed that there are infinitely many Piatetski-Shapiro primes for $c \in (1, \frac{243}{235})$. We refer the readers to see [3] for more details of the improvements of $c$. The asymptotic relation is expected to hold for all values of $c \in (1, 2)$. The estimation of Piatetski-Shapiro primes is an approximation of the well-known conjecture that there exist infinitely many primes of the form $n^2 + 1$.

For a better understanding of the distribution of primes, it is natural to study consecutive primes, for example the twin prime conjecture. In this article, to understand the distribution of Piatetski-Shapiro primes, we are interested in the counting function of consecutive primes in Piatetski-Shapiro sequences. Fix real numbers $c_1, c_2 \in (1, 2)$. For every prime $p$, let $p^\#$ denote the next larger prime. We define the counting function

$$\pi(x; N^{(c_1)}, N^{(c_2)}) := |\{p \leq x : p \in N^{(c_1)} \text{ and } p^\# \in N^{(c_2)}\}|.$$  

Our idea is inspired by a breakthrough in 2016 by Lemke Oliver and Soundararajan [4]. Let $p_n$ be the sequence of primes in ascending order. Let $q \geq 3$ and $a := (a_1, \cdots, a_r)$ with $(a_i, q) = 1$ for all $1 \leq i \leq r$. Applying a model based on a modified version of the Hardy-Littlewood
conjecture, Lemke Oliver and Soundararajan [4] investigated the biases in the occurrence of the pattern $a$ in strings of $r$ consecutive primes reduced modulo $q$. In fact, they analyzed the counting function.

$$\pi(x; q, a) := |\{p_n \leq x : p_{n+i-1} \equiv a_i \mod q \text{ for } 1 \leq i \leq r\}|.$$  

The method has been applied to analyze other consecutive sequences. David, Devin, Nam and Schlitt [2] applied Lemke Oliver and Soundararajan’s method to study consecutive sums of two squares. Let $$\mathbb{E} := \{a^2 + b^2 : a, b \in \mathbb{Z}\} := \{E_n : n \in \mathbb{N}\}.$$ By the Hardy-Littlewood conjectures in arithmetic progressions for sum of two squares, they gave a heuristic argument of a conjecture of the counting function

$$|\{E_n \leq x : E_n \equiv a \pmod{q}, E_{n+1} \equiv b \pmod{q}\}|.$$  

For any given real numbers $\alpha > 0$ and $\beta \geq 0$, the associated (generalized) Beatty sequence is defined by $$\mathcal{B}_{\alpha,\beta} := (\lfloor \alpha m + \beta \rfloor)_{m \in \mathbb{N}},$$ which is also called the generalized arithmetic progression. Banks and Guo [1] gave a conjecture of the estimation of the counting function

$$\pi(x; \mathcal{B}_{\alpha,\beta}, \mathcal{B}_{\alpha,\beta}^c) := |\{p \leq x : p \in \mathcal{B}_{\alpha,\beta} \text{ and } p^\sharp \in \mathcal{B}_{\alpha,\beta}^c\}|$$ by a heuristic argument based on the method of Lemke Oliver and Soundararajan. In this article, we apply a similar model to give a heuristic argument of the following conjecture.

**Conjecture 1.1.** For any fixed positive number $\varepsilon > 0$, the counting function (1.1) satisfies that

$$\pi(x; \mathcal{N}(c_1), \mathcal{N}(c_2)) = \frac{x^{1/c_1+1/c_2-1}}{c_1 c_2 \log x} + O\left(\frac{x^{1/c_1+1/c_2-1}}{\log x}^{3/2-\varepsilon}\right),$$  

where the implied constant depends only on $c_1$, $c_2$ and $\varepsilon$.

In what follows we give a short survey of the breakthrough of Lemke Oliver and Soundararajan’s biases. We will end the introduction by the main proposition and key improvement to this topic.

### 1.1. The Hardy-Littlewood conjecture

Let $\mathcal{H}$ be a finite subset of $\mathbb{Z}$, and let $1_\mathcal{P}$ denote the indicator function of the primes. A strong form of the Hardy-Littlewood conjecture for $\mathcal{H}$ asserts that the estimate

$$\sum_{n \leq x} \prod_{h \in \mathcal{H}} 1_\mathcal{P}(n + h) = \mathcal{G}(\mathcal{H}) \int_2^x \frac{du}{(\log u)^{\#\mathcal{H}}} + O(x^{1/2+\varepsilon})$$  

holds for every fixed $\varepsilon > 0$, where $\mathcal{G}(\mathcal{H})$ is the singular series given by

$$\mathcal{G}(\mathcal{H}) := \prod_p \left(1 - \frac{|(\mathcal{H} \mod p)|}{p}\right) \left(1 - \frac{1}{p}\right)^{-\#\mathcal{H}}.$$  

For their work on primes in short intervals, Montgomery and Soundararajan [5] have introduced the modified singular series

$$\mathcal{G}_0(\mathcal{H}) := \sum_{T \subseteq \mathcal{H}} (-1)^{|\mathcal{H}\setminus T|} \mathcal{G}(T),$$
for which one has the relation
\[ \mathcal{G}(\mathcal{H}) = \sum_{T \subseteq \mathcal{H}} \mathcal{G}_0(T). \]

Note that \( \mathcal{G}(\emptyset) = \mathcal{G}_0(\emptyset) = 1 \). The Hardy-Littlewood conjecture (1.2) can be reformulated in terms of the modified singular series as follows:
\[
\sum_{n \leq x} \prod_{h \in \mathcal{H}} \left( 1 - \frac{1}{\log n} \right) = \mathcal{G}_0(\mathcal{H}) \int_{2}^{x} \frac{dy}{(\log y)^{|\mathcal{H}|}} + O(x^{1/2+\epsilon}). \quad (1.3)
\]

1.2. A modified Hardy-Littlewood conjecture with congruence conditions. To investigate the distribution of primes in arithmetic progressions, we introduce a modification of the Hardy-Littlewood conjecture with congruence conditions \( (\mod q) \) from Lemke Oliver and Soundararajan’s model [4]. For any integer \( q \geq 1 \) and a finite subset \( \mathcal{H} \subset \mathbb{Z} \), define the singular series away from \( q \) by
\[ \mathcal{G}_q(\mathcal{H}) := \prod_{p \mid q} \left( 1 - \frac{|(\mathcal{H} \mod p)|}{p} \right) \left( 1 - \frac{1}{p} \right)^{|\mathcal{H}|}. \]

We require that \( a \pmod{q} \) is such that \( (h + a, q) = 1 \) for all \( h \in \mathcal{H} \), then it asserts that
\[
\sum_{n \leq x} \prod_{h \in \mathcal{H}} 1_{\mathbb{P}}(n + h) \sim \mathcal{G}_q(\mathcal{H}) \left( \frac{q}{\varphi(q)} \right)^{|\mathcal{H}|} \frac{1}{q} \int_{2}^{x} \frac{dy}{(\log y)^{|\mathcal{H}|}}. \quad (1.4)
\]

Now similar to \( \mathcal{G}_0 \), define
\[ \mathcal{G}_{q,0}(\mathcal{H}) := \sum_{T \subset \mathcal{H}} (-1)^{|\mathcal{H} \setminus T|} \mathcal{G}_q(T), \]
which gives that
\[ \mathcal{G}_q(\mathcal{H}) = \sum_{T \subset \mathcal{H}} \mathcal{G}_{q,0}(T). \]

Conditioning \( (h + a, q) = 1 \) for all \( h \in \mathcal{H} \), we expect that
\[
\sum_{n \leq x} \prod_{h \in \mathcal{H}} \left( 1_{\mathbb{P}}(n + h) - \frac{q}{\varphi(q) \log n} \right) \sim \mathcal{G}_{q,0}(\mathcal{H}) \left( \frac{q}{\varphi(q)} \right)^{|\mathcal{H}|} \frac{1}{q} \int_{2}^{x} \frac{dy}{(\log y)^{|\mathcal{H}|}}. \quad (1.4)
\]

The term \( q/(\varphi(q) \log n) \) is expected to be the probability that \( n + h \) is prime, with the fact that \( (n + h, q) = 1 \).

1.3. Lemke Oliver and Soundararajan’s method. Based on a model by assuming a modified version of the Hardy-Littlewood conjecture (1.4), Lemke Oliver and Soundararajan [4] conjectured that
\[ \pi(x; q, a) = \frac{\text{li}(x)}{\varphi(q)^r} \left( 1 + c_1(q; a) \frac{\log \log x}{\log x} + c_2(q; a) \frac{1}{\log x} + O((\log x)^{-7/4}) \right), \]

where \( c_1(q; a) \) and \( c_2(q; a) \) are explicit constants.

They rewrited the sum of the characteristic function
\[
\sum_{n \leq x} \frac{1}{\mathbb{P}}(n) 1_{\mathbb{P}}(n + h) \prod_{0 < t < h \atop (t + a, q) = 1} (1 - 1_{\mathbb{P}}(n + t))
\]
into a sum related to singular series and achieved that
\[ \pi(x; q, (a, b)) \sim \frac{1}{q} \int_2^x \alpha(y)^r \left( \frac{q}{\varphi(q) \log y} \right)^2 D(a, b; y) dy \]
where \( D(a, b; y) \) is a sum depending on the average of singular series. The key point to analyze \( D(a, b; y) \) is a detailed estimation of
\[ \sum_{h > 0} \mathcal{G}_{q,0}(\{0, h\}) e^{-h/H}, \]
which was calculated as the main proposition in \[4\].

1.4. Key proposition of this article. To complete the heuristic of Conjecture 1.1, the key point is to analyze the average of singular series with a weight of exponential functions which differs from the case in \[4\]. Let \( \nu(u) = 1 - 1/\log u \). By Lemke Oliver and Soundararajan’s idea \[4\], one can estimate the following expression
\[ \sum_{h \equiv r \pmod{q}} (\log h)^{\nu(u)} e^{f(h, u)}. \]
Since the counting function of the Piatetksi-Shapiro sequence requires us to express the fractional part of a function into a sum of exponential sums, we need to estimate
\[ \sum_{h \equiv r \pmod{q}} (\log h)^{\nu(u)} e^{f(h, u)}, \quad (1.5) \]
where the function \( f(h, u) \) is “smooth”. The case when the function \( f(h, u) \) is linear was estimated in \[1\], but the method has to be revised to adapt the smooth case (1.5). One can compare the following proposition to Lemma 2.4 in \[1\]. A detailed proof is in Section 4.

**Proposition 1.2.** Fix \( \theta \in [0, 1] \) and \( \vartheta = 0 \) or 1. Let \( \gamma_1, \gamma_2 \in (0, 1) \) be two real numbers. For all \( j, k \in \mathbb{R} \) and \( u \geq 3 \), let \( c(j, k, u, h) \) be a complex number with \( |c(j, k, u, h)| = 1 \) and \( c(j, k, u, h) = 1 \) if \( j = k = 0 \). We define
\[ R_{\theta, \vartheta, j, k}(u) := \sum_{\substack{h \geq 1, 2 \not| k}} h^\theta (\log h)^{\vartheta} e^{f(h, u)} c(j, k, u, h), \]
\[ S_{j,k}(u) := \sum_{\substack{h \geq 1, 2 \not| k}} \mathcal{G}_0(\{0, h\}) e^{f(h, u)} c(j, k, u, h). \]

When \( j = k = 0 \) we have the estimates
\[ R_{\theta, 0, 0, 0}(u) = \frac{1}{2} \Gamma(1 + \theta)(\log u)^{1+\theta} + O(1), \]
\[ R_{\theta, 1, 0, 0}(u) = \frac{1}{2} \Gamma(2 + \theta)(\log u)^{1+\theta} + O(1), \]
\[ S_{0,0}(u) = \frac{1}{2} \log u - \frac{1}{2} \log \log u + O(1). \]

On the other hand, if \( k \) is such that \( |k| \geq (\log u)^{-1} \), then
\[ \max \{|R_{\theta, \vartheta, j, k}(u)|, |S_{j,k}(u)|\} \ll |k|^{-4}. \]
If \( j, k \) are such that \( |jk| \geq (\log u)^{-1} \), then
\[ \max \{|R_{\theta, \vartheta, j, k}(u)|, |S_{j,k}(u)|\} \ll |jk|^{-4}. \quad (1.6) \]
2. Preliminaries

2.1. Notation. We denote by $|t|$ and $\{t\}$ the greatest integer $\leq t$ and the fractional part of $t$, respectively. We also write $e(t) := e^{2\pi i t}$ for all $t \in \mathbb{R}$, as usual. We make considerable use of the sawtooth function defined by

$$\psi(t) := t - |t| - \frac{1}{2} - \{t\} - \frac{1}{2} \quad (t \in \mathbb{R}).$$

Let $\mathbb{P}$ denote the set of primes in $\mathbb{N}$. In what follows, the letter $p$ always denotes a prime number, and $p^\sharp$ is used to denote the smallest prime greater than $p$. In other words, $p$ and $p^\sharp$ are consecutive primes with $p^\sharp > p$. We also put

$$\delta_p := p^\sharp - p \quad (p \in \mathbb{P}).$$

Let $\gamma := e^{-1}, \gamma_1 := c_1^{-1}$ and $\gamma_2 := c_2^{-1}$. Throughout the paper, $\varepsilon$ is always a sufficiently small positive number.

For an arbitrary set $S$, we use $1_S$ to denote its indicator function:

$$1_S(n) := \begin{cases} 
1 & \text{if } n \in S, \\
0 & \text{if } n \notin S,
\end{cases}$$

and let $1_{\mathcal{A}}(n) := 1_{\mathcal{A}(\mathcal{A})}(n)$. Throughout the paper, implied constants in symbols $O$, $\ll$ and $\gg$ may depend (where obvious) on the parameters $\gamma_1, \gamma_2, \varepsilon$ but are absolute otherwise. For given functions $F$ and $G$, the notations $F \ll G$, $G \gg F$ and $F = O(G)$ are all equivalent to the statement that the inequality $|F| \leq c|G|$ holds with some constant $c > 0$.

2.2. Technical lemmas. We start by a simple average estimation of singular series.

Lemma 2.1. Let $h$ be a positive integer. We have

$$\sum_{1 \leq t \leq h-1} \mathcal{S}_0(\{0, t\}) \ll h^{1/2 + \varepsilon},$$

$$\sum_{1 \leq t \leq h-1} \mathcal{S}_0(\{t, h\}) \ll h^{1/2 + \varepsilon},$$

$$\sum_{1 \leq t_1 < t_2 \leq h-1} \mathcal{S}_0(\{t_1, t_2\}) = -\frac{1}{2}h \log h + \frac{1}{2}Ah + O(h^{1/2 + \varepsilon}),$$

where $A := 2 - C_0 - \log 2\pi$ and $C_0$ denotes the Euler-Mascheroni constant.

Proof. See [1, Lemma 2.2].

Recall that

$$\nu(u) = 1 - \frac{1}{\log u} \quad (u > 1).$$

Note that $\nu(u)$ is the same as $\alpha(u)$ in the notation of [4] and $\nu(u) \asymp 1$ if $u \geq 3$.

Lemma 2.2. Let $c > 0$ be a constant, and suppose that $f$ is a function such that $|f(h)| \leq h^c$ for all $h \geq 1$. Then, uniformly for $3 \leq u \leq x$ and a real function $g(u, h) \in \mathbb{R}$ we have

$$\sum_{h \leq (\log x)^3} f(h)\nu(u)^h e(g(u, h)) = \sum_{h \geq 1} f(h)\nu(u)^h e(g(u, h)) + O_c(x^{-1}).$$
Proof. Let $H := -(\log \nu(n))^{-1}$. We write $\nu(u)^h = e^{-h/H}$. By $H \lesssim \log u$ for $u \geq 3$, $h > (\log x)^3$, we conclude that $h/H \gtrsim h^{2/3}$ with $u \leq x$. Hence
\[
\left| \sum_{h > (\log x)^3} f(h)\nu(u)^h e(g(u, h)) \right| \leq \sum_{h > (\log x)^3} h^c e^{-h^{2/3}} \lesssim x^{-1} \sum_{h > (\log x)^3} h^c e^{h^{1/3} - h^{2/3}} \ll_c x^{-1}.
\]

\[\square\]

**Lemma 2.3.** Assuming the Hardy-Littlewood conjecture (1.3), let $h \lesssim \log^3 x$ and write
\[
f_h(n) := \mathbf{1}_P(n) \mathbf{1}_P(n + h) \prod_{0 < t < h} \left(1 - \mathbf{1}_P(n + t)\right) = \begin{cases} 1 & \text{if } n = p \in P \text{ and } \delta_p = h, \\ 0 & \text{otherwise.} \end{cases}
\]

Let
\[
S_h(x) := \sum_{n \leq x} f_h(n).
\]

For every integer $L \geq 0$ we denote
\[
D_{h,L}(u) := \sum_{A \subseteq \{0, h\}} \sum_{T \subseteq [1, h-1]} (-1)^{|T|} \mathbf{S}_0(A \cup T)(\nu(u) \log u)^{-|T|} \nu(u)^h.
\]

Then
\[
S_h(x) = \sum_{L=0}^{h+1} \int_0^x \nu(u)^{-1} (\log u)^{-2} D_{h,L}(u) \, du + O(x^{1/2+\varepsilon}).
\]

Proof. First, write $\widetilde{\mathbf{1}}_P(n) := \mathbf{1}_P(n) - 1/\log n$, and put
\[
\widetilde{f}_h(n) := \left(\widetilde{\mathbf{1}}_P(n) + \frac{1}{\log n}\right) \left(\widetilde{\mathbf{1}}_P(n + h) + \frac{1}{\log n}\right) \prod_{0 < t < h} \left(1 - \frac{1}{\log n} - \widetilde{\mathbf{1}}_P(n + t)\right).
\]

Recalling that $h \lesssim (\log x)^3$, we have the uniform estimate
\[
f_h(n) = \widetilde{f}_h(n) \left(1 + O\left(\frac{(\log x)^6}{x^{1/2}}\right)\right)
\]
for all $n > x^{1/2}$. Since $f_h(n)$ and $\widetilde{f}_h(n)$ are bounded, it follows that
\[
S_h(x) = \sum_{n \leq x} \widetilde{f}_h(n) + O(x^{1/2+\varepsilon}).
\]

It follows that, up to an error term of size $O(x^{1/2+\varepsilon})$, the quantity $S_h(x)$ equals
\[
\sum_{A \subseteq \{0, h\}} \sum_{T \subseteq [1, h-1]} (-1)^{|T|} \sum_{n \leq x} \left(\frac{1}{\log n}\right)^{2-|A|} \left(1 - \frac{1}{\log n}\right)^{h - 1 - |T|} \prod_{t \in A \cup T} \widetilde{\mathbf{1}}_P(n + t)
\]
(compare to [4, Equations (2.5) and (2.6)].)
By the modified Hardy-Littlewood conjecture (1.3) the estimate
\[
\sum_{n \leq x} (\log n)^{-c} \prod_{t \in \mathcal{H}} \tilde{I}_p(n + t) = \int_{3}^{x} (\log u)^{-c} \left( \sum_{n \leq u} \prod_{t \in \mathcal{H}} \tilde{I}_p(n + t) \right) \, du = \mathcal{G}_0(\mathcal{H}) \int_{3}^{x} (\log u)^{-c - |\mathcal{H}|} \, du + O(x^{1/2 + \varepsilon})
\]
holds uniformly for any constant \( c > 0 \); consequently, up to an error term of size \( O(x^{1/2 + \varepsilon}) \) the quantity \( S_h(x) \) is equal to
\[
\sum_{A \subseteq \{0, h\}} \sum_{\mathcal{T} \subseteq [1, h-1]} (-1)^{|\mathcal{T}|} \mathcal{G}_0(A \cup \mathcal{T}) \int_{3}^{x} (\log u)^{-2 - |\mathcal{T}|} \nu(u)^{h - 1 - |\mathcal{T}|} \, du.
\]
By the definition (2.2), we have
\[
S_h(x) = \sum_{L = 0}^{h+1} \int_{3}^{x} \nu(u)^{-1} (\log u)^{-2} D_{h,L}(u) \, du + O(x^{1/2 + \varepsilon}).
\]

We need the following well known approximation of Vaaler.

**Lemma 2.4.** For any \( H \geq 1 \) there are numbers \( a_h, b_h \) such that
\[
\left| \psi(t) - \sum_{0 < |h| \leq H} a_h e(h t) \right| \leq \sum_{|h| \leq H} b_h e(h t), \quad a_h \ll \frac{1}{|h|}, \quad b_h \ll \frac{1}{H}.
\]

*Proof.* See [9].

**Lemma 2.5.** Let \( g(n) \) be bounded, \( |a_j| \ll 1/|j| \). Then
\[
\sum_{3 < n \leq x} \sum_{1 \leq |j| \leq J} g(n) a_j \left( e(j(n + h + 1)^\gamma) - e(j(n + h)^\gamma) \right) \ll x^{\gamma - 1} \max_{N \leq x} \sum_{1 \leq j \leq J} \left| \sum_{3 < n \leq N} g(n) e(j(n + h)^\gamma) \right|.
\]

*Proof.* Let 
\[
\phi_{j,h}(t) := e\left( j((t + h + 1)^\gamma - (t + h)^\gamma) \right) - 1.
\]
Then
\[
\phi_{j,h}(t) \ll |j|^t \gamma - 1 \quad \text{and} \quad \frac{\partial \phi_{j,h}(t)}{\partial t} \ll |j|^t \gamma - 2,
\]
so we have
\[
\sum_{3 < n \leq x} \sum_{1 \leq |j| \leq J} g(n) a_j \left( e(j(n + h + 1)^\gamma) - e(j(n + h)^\gamma) \right) \ll \sum_{1 \leq j \leq J} |j|^{-1} \max_{3 < n \leq x} \left| \sum_{3 < n \leq x} g(n) \phi_{j,h}(n) e(j(n + h)^\gamma) \right|
\]
\[
\ll \sum_{1 \leq j \leq J} |j|^{-1} \phi_{j,h}(x) \max_{3 < n \leq x} \left| \sum_{3 < n \leq x} g(n) e(j(n + h)^\gamma) \right|
\]
\[
\ll \sum_{1 \leq j \leq J} |j|^{-1} \phi_{j,h}(x) \max_{3 < n \leq x} \left| \sum_{3 < n \leq x} g(n) e(j(n + h)^\gamma) \right|
\]}
\[ + \int_{\sqrt[3]{x}}^{x} \sum_{1 \leq j \leq J} j^{-1} \left| \frac{\partial \phi_{j,h}(u)}{\partial u} \sum_{3 < n \leq u} g(n) e(j(n+h)^{\gamma}) \right| du \]

\[ \leq x^{\gamma-1} \max_{\mathbb{N} \leq x} \left| \sum_{3 < n \leq N} g(n) e(j(n+h)^{\gamma}) \right|. \]

Finally, we use the following well-known lemma, which provides a characterization of the numbers that occur in the Piatetski-Shapiro sequence \( \mathcal{N}(c) \).

**Lemma 2.6.** A natural number \( m \) has the form \( \lfloor n^c \rfloor \) if and only if \( 1^{(c)}(m) = 1 \), where \( 1^{(c)}(m) := \lfloor -m^\gamma \rfloor - \lfloor -(m+1)^\gamma \rfloor \). Moreover,

\[ 1^{(c)}(m) = \gamma m^{\gamma-1} + \psi(-(m+1)^\gamma) - \psi(-m^\gamma) + O(m^{\gamma-2}). \]

### 3. Heristic of Conjecture 1.1

For every even integer \( h \geq 2 \) we denote

\[ \pi_h(x; \mathcal{N}(c_1), \mathcal{N}(c_2)) := \left| \{ p \leq x : p \in \mathcal{N}(c_1), \ p^\gamma \in \mathcal{N}(c_2) \ \text{and} \ \delta_p = h \} \right| \]

\[ = \sum_{n \leq x} 1^{(c_1)}(n) 1^{(c_2)}(n+h) f_h(n), \]

where

\[ f_h(n) := 1_p(n) 1_p(n+h) \prod_{0 < t < h} (1 - 1_p(n+t)) \]

\[ = \begin{cases} 1 & \text{if } n = p \in \mathbb{P} \text{ and } \delta_p = h, \\ 0 & \text{otherwise.} \end{cases} \]

Clearly,

\[ \pi(x; \mathcal{N}(c_1), \mathcal{N}(c_2)) = \sum_{h \leq (\log x)^3} \pi_h(x; \mathcal{N}(c_1), \mathcal{N}(c_2)) + O \left( \frac{x}{(\log x)^3} \right). \]

Fixing an even integer \( h \in [1, (\log x)^3] \) for the moment, our initial goal is to express \( \pi_h(x; \mathcal{N}(c_1), \mathcal{N}(c_2)) \) in terms of the function \( S_h(x) = \sum_{n \leq x} f_h(n) \)

introduced by Lemke Oliver and Soundararajan [4, Equation (2.5)]. Recall that \( \gamma_1 := c_1^{-1} \in (0, 1) \) and \( \gamma_2 := c_2^{-1} \in (0, 1) \). By Lemma 2.6 and Lemma 2.4, taking \( J := x^{1-\gamma_1+\varepsilon} \) with \( a_j \ll 1/|j| \) and \( b_j \ll 1/J \) we write

\[ 1^{(c_1)}(n) = \gamma_1 n^{\gamma_1-1} + (\psi(-(n+1)^{\gamma_1}) - \psi(-n^{\gamma_1})) + O(n^{\gamma_1-2}) \]

\[ = \gamma_1 n^{\gamma_1-1} + \sum_{1 \leq |j| \leq J} a_j (e(j(n+1)^{\gamma_1}) - e(jn^{\gamma_1})) \]

\[ + O \left( \sum_{0 \leq |j| < J} b_j (e(jn^{\gamma_1}) + e(j(n+1)^{\gamma_1})) \right) + O(n^{\gamma_1-2}). \]
Similarly, taking that $K := x^{1-\gamma+\varepsilon}$ with $a_k \ll 1/|k|$ and $b_k \ll 1/K$ we have

$$1^{(c_2)}(n+h) = \gamma_2(n+h)^{2-1} + (\psi(-(n+h+1)^{2}) - \psi(-(n+h)^{2})) + O(n^{2-2})$$

$$= \gamma_2(n+h)^{2-1} + \sum_{1 \leq |k| \leq K} a_k \left( e(k(n+h+1)^{2}) - e(k(n+h)^{2}) \right)$$

$$+ O \left( \sum_{0 \leq |k| \leq K} b_k \left( e(k(n+h)^{2}) + e(k(n+h+1)^{2}) \right) \right) + O(n^{2-2}).$$

Hence we derive the estimate

$$\pi_h(x; N^{(c_1)}, N^{(c_2)}) = \sum_{h \leq \log x^3} \left( S_1 + S_2 + S_3 + S_4 ight)$$

$$+ O \left( S_5 + S_6 + S_7 + S_8 + S_9 + S_{10} + S_{11} + S_{12} \right) + O \left( \frac{x}{\log^3 x} \right),$$

where

$$S_1 := \sum_{n \leq x} \gamma_1 \gamma_2 n^{\gamma_1-1}(n+h)^{\gamma_2-1} f_h(n);$$

$$S_2 := \sum_{n \leq x} \gamma_1 n^{\gamma_1-1} f_h(n) \sum_{1 \leq |k| \leq K} a_k \left( e(k(n+h+1)^{2}) - e(k(n+h)^{2}) \right);$$

$$S_3 := \sum_{n \leq x} \gamma_2(n+h)^{\gamma_2-1} \sum_{1 \leq |j| \leq J} f_h(n) a_j \left( e(j(n+1)^{\gamma_1}) - e(jn^{\gamma_1}) \right);$$

$$S_4 := \sum_{n \leq x} f_h(n) \left( \sum_{1 \leq |j| \leq J} a_j \left( e(j(n+1)^{\gamma_1}) - e(jn^{\gamma_1}) \right) \right)$$

$$\cdot \left( \sum_{1 \leq |k| \leq K} a_k \left( e(k(n+h+1)^{2}) - e(k(n+h)^{2}) \right) \right);$$

$$S_5 := \sum_{n \leq x} \gamma_1 n^{\gamma_1-1} f_h(n) \sum_{0 \leq |k| \leq K} b_k \left( e(k(n+h)^{2}) + e(k(n+h+1)^{2}) \right);$$

$$S_6 := \sum_{n \leq x} \gamma_2(n+h)^{\gamma_2-1} f_h(n) \sum_{0 \leq |j| \leq J} b_j \left( e(jn^{\gamma_1}) + e(j(n+1)^{\gamma_1}) \right);$$

$$S_7 := \sum_{n \leq x} f_h(n) \left( \sum_{1 \leq |j| \leq J} a_j \left( e(j(n+1)^{\gamma_1}) - e(jn^{\gamma_1}) \right) \right)$$

$$\cdot \left( \sum_{0 \leq |k| \leq K} b_k \left( e(k(n+h)^{2}) + e(k(n+h+1)^{2}) \right) \right);$$

$$S_8 := \sum_{n \leq x} f_h(n) \left( \sum_{0 \leq |j| \leq J} b_j \left( e(-jn^{\gamma_1}) + e(-j(n+1)^{\gamma_1}) \right) \right).$$
\[
\sum_{1 \leq |k| \leq K} a_k \left( e(k(n + h + 1)^{\gamma_2}) - e(k(n + h + 1)^{\gamma_1}) \right);
\]

\[ S_9 := \sum_{n \leq x} f_h(n) \left( \sum_{0 \leq |j| \leq J} b_j \left( e(j n^{\gamma_1}) + e(j(n + 1)^{\gamma_1}) \right) \right) \cdot \left( \sum_{0 \leq |k| \leq K} b_k \left( e(k n^{\gamma_2}) + e(k(n + h + 1)^{\gamma_2}) \right) \right); \]

\[ S_{10} := \sum_{n \leq x} 1^{(c_1)}(n) n^{\gamma_2 - 2}; \]

\[ S_{11} := \sum_{n \leq x} 1^{(c_2)}(n + h) n^{\gamma_1 - 2}; \]

\[ S_{12} := \sum_{n \leq x} n^{\gamma_1 + \gamma_2 - 4}. \]

It is easy to see that the contribution from \( S_{10}, S_{11} \) and \( S_{12} \) are negligible. We work on the other sums separately.

3.1. **Estimation of \( S_1 \).** We write \( S_1 = S_{11} + O(S_{12}) \), where

\[ S_{11} := \gamma_1 \gamma_2 \sum_{n \leq x} n^{\gamma_1 + \gamma_2 - 2} f_h(n) \]

and

\[ S_{12} := \sum_{n \leq x} h n^{\gamma_1 + \gamma_2 - 3} f_h(n). \]

We consider \( S_{11} \). By the definition (2.1) we have

\[ S_{11} = \gamma_1 \gamma_2 \int_{J_3}^{x} u^{\gamma_1 + \gamma_2 - 2} d (S_h(u)) + O(\log x). \]

Then by Lemma 2.3, it follows

\[ \sum_{h \leq (\log x)^3} S_{11} = \gamma_1 \gamma_2 \sum_{h \leq (\log x)^3} \sum_{L=0}^{h+1} \int_{J_3}^{x} \frac{u^{\gamma_1 + \gamma_2 - 2}}{\nu(u) \log^2 u} D_{h,L}(u) du + O(\log x). \]

By a similar argument in [1, P. 170], we conclude that the contribution of the terms with \( L \geq 3 \) is negligible. Taking in account Lemma 2.2, we have

\[ \sum_{h \leq (\log x)^3} S_{11} = \gamma_1 \gamma_2 \sum_{l=1}^{5} \int_{J_3}^{x} \frac{u^{\gamma_1 + \gamma_2 - 2}}{\nu(u) \log^2 u} F_{1,l}(u) du + O(\log x), \quad (3.1) \]

where

\[ F_{1,1}(u) := \sum_{h \geq 1} \nu(u)^h; \]
\[ \mathcal{F}_{1,2}(u) := \sum_{h \geq 1 \atop 2 \mid h} \mathcal{G}_0 \{ \{0, h\} \} \nu(u)^h; \]

\[ \mathcal{F}_{1,3}(u) := \frac{(-1)}{\nu(u) \log u} \sum_{h \geq 1 \atop 2 \mid h} \sum_{1 \leq t < h-1} \mathcal{G}_0 \{ \{0, t\} \} \nu(u)^h; \]

\[ \mathcal{F}_{1,4}(u) := \frac{(-1)}{\nu(u) \log u} \sum_{h \geq 1 \atop 2 \mid h} \sum_{1 \leq t < h-1} \mathcal{G}_0 \{ \{t, h\} \} \nu(u)^h; \]

\[ \mathcal{F}_{1,5}(u) := \frac{1}{(\nu(u) \log u)^2} \sum_{h \geq 1 \atop 2 \mid h} \sum_{1 \leq t_1 < t_2 < h-1} \mathcal{G}_0 \{ \{t_1, t_2\} \} \nu(u)^h. \]

By Lemma 1.2 we have

\[ \mathcal{F}_{1,1}(u) = R_{0,0,0,0}(u) = \frac{1}{2} \log u + O(1) \quad (3.2) \]

and

\[ \mathcal{F}_{1,2}(u) = S_{0,0}(u) = \frac{1}{2} \log u - \frac{1}{2} \log \log u + O(1). \quad (3.3) \]

Then combining (3.2) and (3.3), we have the main term

\[ \sum_{l=1}^{2} \int_{1}^{x} \frac{u^{\gamma_1 + \gamma_2 - 2}}{\nu(u) \log^2 u} \mathcal{F}_{1,l}(u) \, du = \int_{1}^{x} \frac{u^{\gamma_1 + \gamma_2 - 2}}{\nu(u) \log^2 u} \left( \log u - \frac{1}{2} \log \log u + O(1) \right) \, du \]

\[ = \frac{x^{\gamma_1 + \gamma_2 - 1}}{\log x} + O \left( \frac{x^{\gamma_1 + \gamma_2 - 1} \log \log x}{\log^2 x} \right). \]

By Lemma 2.1 and Lemma 1.2, we have

\[ \mathcal{F}_{1,3}(u) \ll \frac{1}{\log u} \sum_{h \geq 1 \atop 2 \mid h} h^{1/2+\varepsilon} \nu(u)^h \ll (\log u)^{1/2+\varepsilon} \quad (3.4) \]

and

\[ \mathcal{F}_{1,4}(u) \ll (\log u)^{1/2+\varepsilon}, \]

hence for \( l = 3, 4 \) we get that

\[ \int_{1}^{x} \frac{u^{\gamma_1 + \gamma_2 - 2}}{\nu(u) \log^2 u} \mathcal{F}_{1,l}(u) \, du \ll \frac{x^{\gamma_1 + \gamma_2 - 1}}{(\log x)^{3/2-\varepsilon}}. \]

By Lemma 2.1, we have

\[ \mathcal{F}_{1,5}(u) = \frac{1}{(\nu(u) \log u)^2} \sum_{h \geq 1 \atop 2 \mid h} \left( -\frac{1}{2} h \log h + \frac{1}{2} Ah + O(h^{1/2+\varepsilon}) \right) \nu(u)^h \]

\[ = \frac{1}{(\nu(u) \log u)^2} \left( -\frac{1}{2} R_{1,1;0,0}(u) + \frac{1}{2} AR_{1,0;0,0}(u) + O(R_{1/2+\varepsilon/2,0,0,0}(u)) \right) \]

\[ \ll 1, \]

then

\[ \int_{1}^{x} \frac{u^{\gamma_1 + \gamma_2 - 2}}{\nu(u) \log^2 u} \mathcal{F}_{1,5}(u) \, du \ll \int_{1}^{x} \frac{u^{\gamma_1 + \gamma_2 - 2}}{\nu(u) \log^2 u} \, du \ll \frac{x^{\gamma_1 + \gamma_2 - 1}}{\log^2 x}. \]
Therefore, we conclude that
\[
\sum_{h \leq (\log x)^3 \atop 2 \mid h} S_{11} = \gamma_1 \gamma_2 x^{\gamma_1 + \gamma_2 - 1} \log x + O\left(\frac{x^{\gamma_1 + \gamma_2 - 1}}{\log x} \right).
\]

By a similar method, we have
\[
\sum_{h \leq (\log x)^3 \atop 2 \mid h} S_{12} \ll \frac{x^{\gamma_1 + \gamma_2 - 1}}{\log x^{3/2 - \varepsilon}}.
\]

### 3.2. Estimation of $S_2$. After a partial summation, we apply Lemma 2.5 and obtain that
\[
S_2 = \gamma \sum_{3 \leq n \leq x} n^{\gamma_1 - 1} f_h(n) \sum_{1 \leq |k| \leq K} a_k (e(k(n + h)^{\gamma_2}) - e(k(n + h)^{\gamma_2}))
\]
\[
\ll x^{\gamma_2 - 1} \max_{N \leq x} \sum_{1 \leq k \leq K} \sum_{3 \leq n \leq N} n^{\gamma_1 - 1} f_h(n) e(k(n + h)^{\gamma_2}).
\]

We define a complex function $c(k, h)$ such that
\[
\left| \sum_{3 \leq n \leq N} n^{\gamma_1 - 1} f_h(n) e(k(n + h)^{\gamma_2}) \right| = c(k, h) \sum_{3 \leq n \leq N} n^{\gamma_1 - 1} f_h(n) e(k(n + h)^{\gamma_2}).
\]

Note that $|c(k, h)| = 1$ and $c(k, h) = 0$ if $k = 0$. Then by a similar argument to (3.1), we have
\[
\sum_{h \leq (\log x)^3 \atop 2 \mid h} S_2 \ll x^{\gamma_2 - 1} \max_{N \leq x} \sum_{1 \leq k \leq K} \sum_{1 \leq l \leq 5} \int_{3}^{N} \frac{u^{\gamma_1 - 1}}{\nu(u) \log^2 u} F_{2, l}(u) \, du,
\]

where
\[
F_{2,1}(u) := \sum_{h \geq 1 \atop 2 \mid h} \nu(u)^h c(k, h) e(k(u + h)^{\gamma_2});
\]
\[
F_{2,2}(u) := \sum_{h \geq 1 \atop 2 \mid h} \mathcal{G}_0(\{0, h\}) \nu(u)^h c(k, h) e(k(u + h)^{\gamma_2});
\]
\[
F_{2,3}(u) := \frac{(-1)}{\nu(u) \log u} \sum_{h \geq 1 \atop 2 \mid h} \sum_{1 \leq t \leq h - 1} \mathcal{G}_0(\{t, h\}) \nu(u)^h c(k, h) e(k(u + h)^{\gamma_2});
\]
\[
F_{2,4}(u) := \frac{(-1)}{\nu(u) \log u} \sum_{h \geq 1 \atop 2 \mid h} \sum_{1 \leq t \leq h - 1} \mathcal{G}_0(\{t, h\}) \nu(u)^h c(k, h) e(k(u + h)^{\gamma_2});
\]
\[
F_{2,5}(u) := \frac{1}{(\nu(u) \log u)^2} \sum_{h \geq 1 \atop 2 \mid h} \sum_{1 \leq t_1 < t_2 \leq h - 1} \mathcal{G}_0(\{t_1, t_2\}) \nu(u)^h c(k, h) e(k(u + h)^{\gamma_2}).
\]

By Lemma 1.2, we have
\[
\max (F_{2,1}(u), F_{2,2}(u)) \ll |k|^{-4},
\]
provided that $|k| \geq (\log u)^{-1}$, which is sufficient that $u \geq 4$. This gives that

$$x^{\gamma_2 - 1} \max_{N \leq x} \sum_{1 \leq l \leq K} \sum_{1 \leq k \leq K} \int_{\mathbb{R}} u^{\gamma_1 - 1} \frac{F_{2,l}(u)}{\nu(u) \log^2 u} du \ll x^{\gamma_2 - 1} \sum_{1 \leq k \leq K} \left(1 + x^{\gamma_1} (\log x)^{-2} k^{-4}\right) \ll x^{\gamma_1 + \gamma_2 - 1} (\log x)^{-2}.$$  

Similar to estimation of (3.4), by Lemma 2.1 and Lemma 1.2 we have

$$\max \left\{ F_{2,3}(u), F_{2,4}(u) \right\} \ll \frac{1}{\log u} \sum_{h \geq 1} h^{1/2 - \epsilon/2} \nu(u)^h c(k, h) e(k (u + h)^{\gamma_2})$$

and

$$F_{2,5}(u) = \frac{1}{(\nu(u) \log u)^2} \sum_{h \geq 1} \left(-\frac{1}{2} h \log h + \frac{1}{2} Ah + O(h^{1/2 + \epsilon})\right)$$

$$\cdot \nu(u)^h c(k, h) e(k (u + h)^{\gamma_2})$$

$$= \frac{1}{(\nu(u) \log u)^2} \left(-\frac{1}{2} R_{1,1;j,k}(u) + \frac{1}{2} AR_{1,0;j,k}(u) + O(R_{1/2 + \epsilon/2,0;j,k}(u))\right)$$

$$\ll (\log u)^{-1} k^{-4}.$$  

Combining (3.5) and (3.6), we have

$$x^{\gamma_2 - 1} \max_{N \leq x} \sum_{1 \leq l \leq K} \sum_{1 \leq k \leq K} \int_{\mathbb{R}} u^{\gamma_1 - 1} \frac{F_{2,k}(u)}{\nu(u) \log^2 u} du \ll x^{\gamma_1 + \gamma_2 - 1} (\log x)^{-2}.$$  

3.3. Estimation of $S_3$. Similar to the estimation of $S_1$, we write $S_3 = S_{31} + O(S_{32})$, where

$$S_{31} := \sum_{n \leq x} \gamma_2 n^{\gamma_2 - 1} \sum_{1 \leq j \leq J} f_h(n) a_j (e(j(n + 1)^{\gamma_1}) - e(jn^{\gamma_1}))$$

and

$$S_{32} := \sum_{n \leq x} hn^{\gamma_2 - 2} \sum_{1 \leq j \leq J} f_h(n) a_j (e(j(n + 1)^{\gamma_1}) - e(jn^{\gamma_1})).$$

We apply the partial summation as Lemma 2.5, then

$$S_{31} \ll x^{\gamma_1 - 1} \max_{N \leq x} \sum_{1 \leq j \leq J} \left| \sum_{3 < n \leq N} n^{\gamma_2 - 1} f_h(n) e(jn^{\gamma_1}) \right|.$$  

We define a complex function $c(j, h)$ such that

$$\left| \sum_{3 < n \leq N} n^{\gamma_2 - 1} f_h(n) e(jn^{\gamma_1}) \right| = c(j, h) \sum_{3 < n \leq N} n^{\gamma_2 - 1} f_h(n) e(jn^{\gamma_1}).$$

By the same construction of $S_2$, we conclude that

$$\sum_{h \leq (\log x)^3} S_{31} \ll x^{\gamma_1 - 1} \max_{N \leq x} \sum_{1 \leq j \leq J} \sum_{l = 1}^{5} \int_{\mathbb{R}} u^{\gamma_1 - 1} \frac{F_{3,l}(u)}{\nu(u) \log^2 u} du.$$
where

\[ \mathcal{F}_{3,1}(u) := \sum_{h \geq 1 \atop 2 \mid h} \nu(u)^h c(j, h) e(ju^{n_{1}}); \]

\[ \mathcal{F}_{3,2}(u) := \sum_{h \geq 1 \atop 2 \mid h} \mathcal{G}_{0}(\{0, h\}) \nu(u)^h c(j, h) e(ju^{n_{1}}); \]

\[ \mathcal{F}_{3,3}(u) := \frac{(-1)}{\nu(u) \log u} \sum_{h \geq 1 \atop 2 \mid h} \mathcal{G}_{0}(\{t, h\}) \nu(u)^h c(j, h) e(ju^{n_{1}}); \]

\[ \mathcal{F}_{3,4}(u) := \frac{(-1)}{\nu(u) \log u} \sum_{h \geq 1 \atop 2 \mid h} \mathcal{G}_{0}(\{t, h\}) \nu(u)^h c(j, h) e(ju^{n_{1}}); \]

\[ \mathcal{F}_{3,5}(u) := \frac{1}{(\nu(u) \log u)^2} \sum_{h \geq 1 \atop 2 \mid h} \sum_{1 \leq t, \ell \leq h - 1} \mathcal{G}_{0}(\{t_1, t_2\}) \nu(u)^h c(j, h) e(ju^{n_{1}}). \]

Therefore, it follows that

\[ \sum_{h \in (\log x)^{3} \atop 2 \mid h} S_{31} \ll x^{n_{1} - 1} \sum_{1 \leq k \leq K} (1 + x^{n_{1}} (\log x)^{-2} k^{-4}) \ll x^{n_{1} + n_{2} - 1} (\log x)^{-2}. \]

Similarly, we know that

\[ \sum_{h \in (\log x)^{3} \atop 2 \mid h} S_{32} \ll x^{n_{1} + n_{2} - 1} (\log x)^{-2}. \]

### 3.4. Estimation of \( S_{4} \)

We apply Lemma 2.5.

\[ S_{4} \ll x^{n_{1} - 1} \max_{N \leq x} \sum_{1 \leq j, \ell \leq J} \sum_{3 \leq n \leq N} \left| \sum_{1 \leq k \leq K} a_{i} (e(k(n + h + 1)^{n_{1}}) - e(k(n + h)^{n_{1}})) \right| \]

\[ \ll x^{n_{1} + n_{2} - 2} \max_{N \leq x} \sum_{1 \leq j, \ell \leq J} \sum_{1 \leq k \leq K} \sum_{3 \leq n \leq N} f_{h}(n) e(jn^{n_{1}} + k(n + h)^{n_{1}}). \]

By the same method of \( S_{2} \), we have

\[ \sum_{h \in (\log x)^{3} \atop 2 \mid h} S_{4} \ll x^{n_{1} + n_{2} - 2} \max_{N \leq x} \sum_{1 \leq j, \ell \leq J} \sum_{1 \leq k \leq K} \left( 1 + \int_{4}^{N} \frac{1}{\nu(u) \log^{2} u} (jk)^{-4} du \right) \]

\[ + \int_{3}^{N} \frac{1}{\nu(u) \log^{2} u} \left( (\log u)^{-1} (jk)^{-4} + (\log u)^{-2} (jk)^{-4} \right) du \]

\[ \ll x^{n_{1} + n_{2} - 2} (JK + x(\log x)^{-2} + x(\log x)^{-3} + x(\log x)^{-4}) \]

\[ \ll x^{n_{1} + n_{2} - 1} \left( \frac{1}{(\log x)^{2}} \right). \]
3.5. Estimation of $S_5$. We show that

\[ \sum_{h \leq (\log x)^\frac{3}{2}} S_5 \ll x^{\gamma_1 + \gamma_2 - 1}. \]

(3.7)

The contribution from $k = 0$ of the left-hand side of (3.7) is

\[ \ll \sum_{h \leq (\log x)^\frac{3}{2}} \sum_{n \leq x} \gamma_1 n^{\gamma_1 - 1} f_h(n) K^{-1} \ll x^{\gamma_1} K^{-1} \ll x^{\gamma_1 + \gamma_2 - 1 - \varepsilon}. \]

(3.8)

By the same estimation of $S_2$, the contribution from $k \neq 0$ of the left-hand side of (3.7) is

\[ \ll K^{-1} \sum_{h \leq (\log x)^\frac{3}{2}} \sum_{n \leq x} n^{\gamma_1 - 1} f_h(n) \sum_{1 \leq k \leq K} e\left(l(n + h)^{\gamma_2}\right) \ll x^{\gamma_1 + \gamma_2 - 1 - \varepsilon}. \]

3.6. Estimation of $S_6$. We write $S_6 = S_{61} + O(S_{62})$, where

\[ S_{61} := \sum_{n \leq x} \gamma_2 n^{\gamma_2 - 1} f_h(n) \sum_{0 \leq |j| \leq J} b_j \left( e(j n^{\gamma_1}) + e(j (n + 1)^{\gamma_1}) \right) \]

and

\[ S_{62} := \sum_{n \leq x} \gamma_2 h n^{\gamma_2 - 2} f_h(n) \sum_{0 \leq |j| \leq J} b_j \left( e(j n^{\gamma_1}) + e(j (n + 1)^{\gamma_1}) \right). \]

We give a brief proof of the bound

\[ \sum_{h \leq (\log x)^\frac{3}{2}} S_{61} \ll x^{\gamma_1 + \gamma_2 - 1 - \varepsilon} \]

(3.9)

only, since the bound of

\[ \sum_{h \leq (\log x)^\frac{3}{2}} S_{62} \ll x^{\gamma_1 + \gamma_2 - 1 - \varepsilon} \]

can be derived by the same way. By a similar argument of (3.8), the contribution from $j = 0$ of the left-hand side of (3.9) is

\[ \ll \sum_{h \leq (\log x)^\frac{3}{2}} \sum_{n \leq x} n^{\gamma_2 - 1} f_h(n) J^{-1} \ll x^{\gamma_2} J^{-1} \ll x^{\gamma_1 + \gamma_2 - 1 - \varepsilon}. \]

Similar to the estimation of $S_{31}$, the contribution from $j \neq 0$ of the left-hand side (3.9) is

\[ \ll J^{-1} \sum_{h \leq (\log x)^\frac{3}{2}} \sum_{n \leq x} n^{\gamma_2 - 1} f_h(n) \sum_{1 \leq j \leq J} e(j n^{\gamma_1}) \ll x^{\gamma_1 + \gamma_2 - 1 - \varepsilon}. \]
3.7. Estimation of $S_7$ and $S_8$. We prove that
\[ \sum_{h \leq (\log x)^3} S_7 \ll x^{\gamma_1+\gamma_2-1-\varepsilon}. \quad (3.10) \]
The contribution from $k = 0$ of the left-hand side of (3.10) is
\[ = b_0 \sum_{h \leq (\log x)^3} \sum_{n \leq x} f_h(n) \left( \sum_{1 \leq j \leq J} a_j (e(j(n + 1)^{\gamma_1}) - e(jn^{\gamma_1})) \right) \]
\[ \ll J^{-1} x^{\gamma_1-1} \sum_{h \leq (\log x)^3} \max_{N \leq x} \left| \sum_{3 < n \leq N} f_h(n) e(jn^{\gamma_1}) \right| \ll x^{\gamma_1+\gamma_2-1-\varepsilon}, \]
by the same estimation of $S_{31}$. The contribution from $k \neq 0$ of the left-hand side of (3.10) is
\[ \ll \sum_{h \leq (\log x)^3} \left| \sum_{n \leq x} f_h(n) \left( \sum_{1 \leq j \leq J} a_j (e(j(n + 1)^{\gamma_1}) - e(jn^{\gamma_1})) \right) \right| \]
\[ \cdot \left( \sum_{1 \leq k \leq K} b_k e(l(n + h)^{\gamma_2}) \right) \]
\[ \ll K^{-1} x^{\gamma_1-1} \sum_{h \leq (\log x)^3} \sum_{1 \leq j \leq J} \max_{N \leq x} \left| \sum_{3 < n \leq N} f_h(n) e(jn^{\gamma_1} + l(n + h)^{\gamma_2}) \right| \]
\[ \ll x^{\gamma_1+\gamma_2-1-\varepsilon}, \]
by the same estimation of $S_4$. The estimation of $S_8$ is similar.

3.8. Estimation of $S_9$. We prove that
\[ \sum_{h \leq (\log x)^3} S_9 \ll x^{\gamma_1+\gamma_2-1-\varepsilon}. \quad (3.11) \]
The contribution from $j = k = 0$ of the left-hand side of (3.11) is
\[ \ll (JK)^{-1} \sum_{h \leq (\log x)^3} \sum_{n \leq x} f_h(n) \ll x^{\gamma_1+\gamma_2-1-\varepsilon}, \]
by the trivial bound. The contribution from $j = 0$ and $k \neq 0$ of the left-hand side of (3.11) is
\[ \ll (JK)^{-1} \sum_{h \leq (\log x)^3} \sum_{n \leq x} f_h(n) \sum_{1 \leq l \leq J} e(l(n + h)^{\gamma_2}) \ll x^{\gamma_1+\gamma_2-1-\varepsilon}, \]
by the same estimation of $S_2$. The contribution from $j \neq 0$ and $k = 0$ of the left-hand side of (3.11) is
\[ \ll (JK)^{-1} \sum_{h \leq (\log x)^3} \sum_{n \leq x} f_h(n) \sum_{1 \leq j \leq J} e(jn^{\gamma_2}) \ll x^{\gamma_1+\gamma_2-1-\varepsilon}, \]
by the same estimation of $S_{31}$. The contribution from $j \neq 0$ and $l \neq 0$ of the left-hand side of (3.11) is

$$\ll (JK)^{-1} \sum_{h \leq (\log x)^3 \atop 2 | h} \left| \sum_{n \leq x} f_h(n) \sum_{1 \leq j, l \leq J} e(j n^{\gamma_1} + l(n + h)^{\gamma_2}) \right| \ll x^{\gamma_1 + \gamma_2 - 1 - \varepsilon},$$

by the same estimation of $S_4$.

4. PROOF OF THE KEY PROPOSITION

The proof of Proposition 1.6 starts by a similar construction of the proof of [1, Lemma 2.4]. Note that $\nu(u) \asymp 1$ for $u \geq 3$. Let $H := -(\log \nu(n))^{-1}$, which gives that $\nu(u)^h = e^{-h/H}$. Write that

$$\nu(u)^h e(kh) = e^{-h/H_k} \quad \text{with} \quad H_k := \frac{H}{1 - 2\pi i k H}.$$ 

Since $\Re(h/H_k) = h/H > 0$ for any positive integer $h$, by the Cahen-Mellin integral it gives that

$$R_{\theta, \vartheta, j, k}(u) = \sum_{h \geq 1} h^\theta (\log h)^\vartheta f(j, k, u, h) e^{-h/H_k}$$

$$= \frac{1}{2\pi i} \int_{4 - i\infty}^{4 + i\infty} \left( \sum_{h \geq 1} \frac{h^\theta (\log h)^\vartheta}{h^s} f(j, k, u, h) \right) \Gamma(s) H_k^s ds,$$

where

$$f(j, k, u, h) := c(j, k, u, h) e(j u^{\gamma_1} + k(u + h)^{\gamma_2} - kh).$$

The case that $j = k = 0$ is the same as [1, Lemma 2.4]. When $k \neq 0$ we have

$$|R_{\theta, 0, j, k}(u)| \leq \frac{1}{2\pi} \int_{4-i\infty}^{4+i\infty} \left( \sum_{h \geq 1} \left| \frac{h^\theta (\log h)^\vartheta}{h^s} \right| \right) |\Gamma(s) H_k^s ds$$

$$\leq \frac{2^\theta}{2\pi} \int_{4-i\infty}^{4+i\infty} |2^{-4}\zeta(4 - \theta)||\Gamma(s) H_k^s| ds$$

$$\leq \frac{2^{\theta - 4}|H_k|^4}{2\pi} \int_{-\infty}^\infty |\zeta(4 - \theta)\Gamma(4 + it)| dt$$

$$\ll |H_k|^4 \left( \frac{H^2}{1 + 4\pi^2 k^2 H^2} \right)^2,$$

which gives that $R_{\theta, 0, j, k}(u) \ll k^{-4}$ if $|k| \geq (\log u)^{-1}$ since $H \asymp \log u$ for $u \geq 3$. The bound for $R_{\theta, 1, j, k}$ is proved similarly by considering $\zeta'(4 - \theta)$.

Secondly, we define

$$T_{j, k}(u) := \sum_{h \geq 1} \mathcal{S}((\{0, h\})) f(j, k, u, h) e^{-h/H_k},$$

for $j, k \in \mathbb{R}$ and $u \geq 3$. Since $\mathcal{S}_0(\{0, h\}) = \mathcal{S}(\{0, h\}) - 1$ for all integers $h$, and $\mathcal{S}(\{0, h\}) = 0$ if $h$ is odd, it follows that

$$S_{j, k}(u) = T_{j, k}(u) - R_{0, 0, j, k}(u) = T_{j, k}(u) + O(\log u).$$

Hence, to complete the proof of the lemma, it suffices to show that

$$T_{j, k}(u) \ll k^{-4} \text{ if } |k| \geq (\log u)^{-1},$$
since the case that \( j = k = 0 \) is the same as \([1, \text{Lemma 2.4}].\) As in the proof of \([4, \text{Proposition 2.1}],\) we consider the Dirichlet series

\[
F(s) := \sum_{h \geq 1} \mathbb{S}(\{0, h\}) / h^s,
\]

which can be expressed in the form

\[
F(s) = \frac{\zeta(s)\zeta(s + 1)}{\zeta(2s + 2)} \prod_p \left( 1 - \frac{1}{(p - 1)^2} + \frac{2p}{(p - 1)^2(p^{s+1} + 1)} \right),
\]

and the final product is analytic for \( \Re(s) > -1.\) Similar to the proof of the first part of the lemma, using the Cahen-Mellin integral with \( k \neq 0 \) we have that

\[
|T_{j,k}(u)| \leq \frac{|H_k|^4}{2\pi} \int_{-\infty}^{\infty} |F(4)\Gamma(4 + it)| \, dt \ll |H_k|^4 = \left( \frac{H^2}{1 + 4\pi^2 k^2 H^2} \right)^2.
\]

Hence \( T_{j,k}(u) \ll k^{-4} \) by a similar argument.

To prove (1.6), we choose

\[
\nu(u)^b e(jkh) = e^{h/H_{j,k}} \quad \text{with} \quad H_{j,k} := \frac{H}{1 - 2\pi i jk H},
\]

and

\[
f(j, k, u, h) := c(j, k, u, h)e(ju^{71} + k(u + h)^{72} - jkh).\]

Everything else follows the same.

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