ON UNIQUENESS PROPERTIES OF SOLUTIONS OF THE GENERALIZED FOURTH-ORDER SCHRÖDINGER EQUATIONS

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Abstract. In this paper, we study uniqueness properties of solutions to the generalized fourth-order Schrödinger equations in any dimension \( d \) of the following forms,

\[
i\partial_t u + \sum_{j=1}^{d} \partial_{x_j}^4 u = V(t,x)u, \quad \text{and} \quad i\partial_t u + \sum_{j=1}^{d} \partial_{x_j}^4 u + F(u,\overline{u}) = 0.
\]

We show that a linear solution \( u \) with fast enough decay in certain Sobolev spaces at two different times has to be trivial. Consequently, if the difference between two nonlinear solutions \( u_1 \) and \( u_2 \) decays sufficiently fast at two different times, it implies that \( u_1 \equiv u_2 \).

1. Introduction

In this work, we consider both the linear generalized fourth-order Schrödinger equations of the following form

\[
\begin{align*}
\left\{ \begin{array}{ll}
i\partial_t u + \Delta^2 u = V(t,x)u, & (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\
u(0,x) = u_0(x),
\end{array} \right.
\end{align*}
\]

and nonlinear ones of the type

\[
\begin{align*}
\left\{ \begin{array}{ll}
i\partial_t u + \Delta^2 u + F(u,\overline{u}) = 0, & (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\
u(0,x) = u_0(x),
\end{array} \right.
\end{align*}
\]

where \( \Delta^2 := \sum_{j=1}^{d} \partial_{x_j}^4 \). Note that \( \Delta^2 \) is a fourth-order differential operator that removes the mixed derivative terms \( \partial_{x_k x_l x_j x_j} \) (\( k \neq j \)) from the regular bi-Laplacian operator \( \Delta^2 \). We will use a slight abuse of language – we will refer to \( \Delta^2 \) as a ‘separable’ fourth-order Schrödinger operator in the rest of this work.

We establish two main results. First, we give sufficient conditions on the decay of the solution \( u \) at two different times \( t = 0 \) and \( t = 1 \) which guarantee that \( u \equiv 0 \) is the unique solution of (1.1). Second, we deduce sufficient conditions on the decay of the difference of two solutions of (1.2) at two times \( t = 0 \) and \( t = 1 \) so that the two solutions are in fact equal. It is worth noting that in order to deduce a nonlinear unique continuation result from a linear one, the potential \( V \) in (1.1) must (a) allow complex values, and (b) be time-dependent (this is because when considering the difference between two nonlinear solutions \( (u \) and \( v) \), the difference in their nonlinearities \( F(u,\overline{u}) - F(v,\overline{v}) \) becomes complex-valued and time-dependent).

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1.1. Motivation.

1.1.1. Unique continuation questions. The study of unique continuation for partial differential equations (PDE) has been historically an active area of research. This line of research addresses the question of under what conditions two solutions of a PDE must coincide. In the context of dispersive PDEs, there are lots of works along this research line, see [20, 21, 22, 31, 32] and reference therein. These uniqueness results typically assume that two solutions coincide in a large subdomain of $\mathbb{R}^d$ at two different times, then they conclude that they are identical on $\mathbb{R}^d$.

In Kenig-Ponce-Vega [29] and Escauriaza-Kenig-Ponce-Vega [10], the authors were motivated by Hardy’s uncertainty principle [17] and initiated a different way to answer the unique continuation question for free Schrödinger equations, that is, they consider the solutions of linear Schrödinger equation of the following form,

$$i\partial_t u + \Delta u = V(t, x)u, \quad (t, x) \in [0, T] \times \mathbb{R}^d,$$

that do not agree on a certain subset but have comparable (fast spatial Gaussian) decays at certain times.

Roughly speaking, [10, 29] showed that if a solution has fast decay at two different times, the solution has to be trivial. Accordingly, for the nonlinear equation, they deduced the uniqueness of the solution from information on the decay of the difference of two possible solutions at two different times. Later, in a series of papers [12, 13, 14, 28], the authors obtained the sharpest fast decay requirement. In particular, [14] obtained that for classical Schrödinger equations if the potential $V$ satisfies certain boundedness properties (we neglect the assumption on $V$ here, but $V$ is allowed to be complex-valued and time-dependent, hence nonlinear uniqueness results available), then for a solution $u$ with fast enough decay, one has that the solution must be trivial.

To best connect Hardy’s uncertainty principle to this form of results, let us first recall the fast decay type result in [14].

**Theorem A** (Theorem 1 in [14]). Assume that $u \in C([0, T], L^2(\mathbb{R}^d))$ verifies (1.3). $A, B > 0$, $AB > 1/16$ both $\left\|e^{A|x|^2}u(0, x)\right\|_{L^2(\mathbb{R}^d)}$ and $\left\|e^{B|x|^2}u(1, x)\right\|_{L^2(\mathbb{R}^d)}$ are finite, and the potential $V$ satisfies certain boundedness properties. Then $u \equiv 0$. Moreover, if $AB = \frac{1}{16}$, $u$ is a constant multiple of $e^{-B+i/4T|x|^2}$.

Recall also the following Hardy’s uncertainty principle.

**Theorem B** (Hardy’s uncertainty principle in [17]). For any function $f : \mathbb{R} \rightarrow \mathbb{C}$, if the function $f$ itself and its Fourier transform $\hat{f}$ satisfy

$$f(x) = \mathcal{O}(e^{-Ax^2}) \quad \text{and} \quad \hat{f}(\xi) = \mathcal{O}(e^{-B\xi^2})$$

with $A, B > 0$ and $AB > \frac{1}{16}$, then $f \equiv 0$. Moreover, if $A = B = \frac{1}{4}$, then $f(x) = Ce^{-\frac{1}{4}x^2}$.

The potential $V$ here can be thought as a perturbation of the free Schrödinger equation, with which the uniqueness of solutions for nonlinear equations will be obtained as a direct consequence of Theorem A (by considering the equation satisfied by the difference of two nonlinear solutions). To see the heart of the problem, the discussion below safely ignores the potential, $V$.

Now one can see how Theorem B motivates Theorem A by writing the free Schrödinger flow (taking $V = 0$) with initial data $f$ in the following form using Fourier transforms

$$u(t, x) := e^{it\Delta}f = (4\pi it)^{-d/2} \int_{\mathbb{R}^d} e^{i|x-y|^2/4it}f(y) \, dy = (2\pi it)^{-d/2} e^{i|x|^2/4t} e^{i|y|^2/4} f \left(\frac{x}{2t}\right),$$

(1.4)

Roughly speaking, the solution of the free Schrödinger equation at time $t$ is a rescaled multiple of the Fourier transform of the initial data. One would connect the decay requirement at time $t = 1$ in Theorem A to the decay requirement on the Fourier transform of $f$ in Theorem B by simply evaluating (1.4) at, for example, time $t = 1$.

Let us point out that the fast enough decay measured in such Gaussian weight fashion is sharp since in the same work [14], the authors provided an example for the threshold case ($T\alpha\beta = 1/4$) which is a nonzero smooth solution with corresponding decay.
It is then natural to ask whether one can give quantitative unique continuation properties from two times for more general dispersive equations with a similar flavor. The answer is Yes. In [11], the $k$-generalized KdV equations

$$
\partial_t u + \partial_x^k u + u^k \partial_x u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}, \quad k \in \mathbb{N}_+,
$$

were considered and the authors obtained that the difference of two distinct solutions $u_1 - u_2$ cannot decay faster than $e^{-\lambda x^{3/2}}$ at two different times for $\lambda > 0$. It is worth pointing out that the decay rate $e^{-cx^{3/2}}$ corresponds to the behavior of the fundamental solution for the linear problem, which appears as a scaled Airy function. Moreover, [24] showed that such decay assumption is optimal by finding an example solution persisting the spatial decay that initial data enjoys in a long time. There are also works on higher order KdV type equation [7, 23].

We would like to mention that unique continuation results have been established for various dispersive models including Schrödinger with gradient terms [9], discrete Schrödinger equations [4, 25], variable coefficient Schrödinger equations [15] and Schrödinger equations with magnetic potential in [2, 3, 6]. For further details and additional relevant references, we refer the interested reader to the aforementioned works.

As for higher order Schrödinger equations, a recent work by Huang-Huang-Zheng [19] obtained a unique continuation result of such fast decay type in one spatial dimension for the model of the following form

$$
i \partial_t u - (-\Delta)^m u = V(x)u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}, \quad m = 2, 3, 4, \ldots.
$$

The result reads that a non-trivial cannot decay faster than $e^{-\lambda|x|^{2m/(2m-1)}}$ at two different times.

In the general case, in order to realize the correct decay rate, one recalls the following result by Hörmander [18], which is a variation of Hardy’s uncertainty principle with conjugate convex weights.

**Theorem C** (Corollary in [18]). If $\varphi$ and $\psi$ are conjugate convex functions, for example, $\varphi = |x|^p/p$ and $\psi = |x|^q/q$, with $\frac{1}{p} + \frac{1}{q} = 1$. Then $f \equiv 0$ if

$$
\int_{\mathbb{R}} |f(x)| e^{\varphi(x)} dx < \infty, \quad \text{and} \quad \int_{\mathbb{R}} |f(\xi)| e^{\psi(\xi)} d\xi < \infty.
$$

The work [19] is considered sharp in terms of the decay exponent, since the exponent $\frac{2m}{2m-1}$ in [19] agrees with Theorem C (recall the kernel of associated fundamental solutions is of the form $e^{O(|x|^{2m/(2m-1)})}$).

1.1.2. **Fourth-order Schrödinger equations.** Now let us come back to the topic of the current work. Fourth-order Schrödinger equations with bi-Laplacian were introduced by Karpman [27] and Karpman-Shagalov [26] to investigate the stabilizing role of higher-order dispersive effects for the soliton instabilities. The following work by Fibich-Ilan-Papanicolaou [16] studies the self-focusing and singularity formation of such fourth-order Schrödinger equations from the mathematical viewpoint.

Analogues of unique continuation questions remain widely open for many high-dimensional dispersive equations, in particular, of the higher-order of Schrödinger equation type. As we pointed out [19] is the only work obtaining a unique continuation result of the fast decay for higher order Schrödinger equations in one spatial dimension. However, extending their argument to higher-dimensional analogues poses challenges. The authors themselves commented that the main obstacle lies in obtaining a suitable higher-dimensional Carleman estimate. This difficulty arises due to a potential phase degeneracy problem in the restriction estimate employed in their proof. It is worth noting that the result obtained by [19] pertains to linear unique continuation, but is not strong enough to deduce a nonlinear version. This limitation arises from the requirement that the potential function $V$ must be real, bounded and not time-dependent in their analysis, where the nonlinear version requires $V$ to be at least complex-valued and time-dependent.

In this work, our goal is to extend a fast decay type of unique continuation property (initiated in [10, 29]) to ‘separable’ fourth-order Schrödinger equations (both linear and nonlinear), especially in higher dimensions. To the best of authors’ knowledge, we believe that the current paper is the first result towards obtaining the unique continuation property of higher-order Schrödinger equations in higher dimensions. It is worth mentioning that this type of higher degree generalizations of the Schrödinger equation is not uncommon, see for example [1] for the same generalization in the context of the study of pointwise convergence of Schrödinger operators.
1.2. Main results and their sharpness. Now let us present the main results.

**Theorem 1.1** (Linear unique continuation). Let $d \geq 1$. Assume that $u \in C^1([0, 1] : H^k(\mathbb{R}^d))$ solves \eqref{EE} with 
$V(t, x), \nabla_x V(t, x), \nabla_x^2 V(t, x), \nabla_x^3 V(t, x) \in L^\infty([0, 1] \times \mathbb{R}^d)$. If there exists $\lambda > 0$ and $\alpha > \frac{4}{9}$ such that 
\begin{align}
  u(0, x), u(1, x) \in H^3(e^{\lambda|x|^{\alpha}} dx), \tag{1.5}
\end{align}
and 
\begin{align}
  \lim_{r \to \infty} \int_0^1 \sup_{|x| > r} |V(t, x)| dt = 0, \tag{1.6}
\end{align}
then $u(t, x) \equiv 0$.

**Theorem 1.2** (Nonlinear unique continuation). Let $d \geq 1$. Assume that $u_1, u_2 \in C^1([0, 1] : H^k(\mathbb{R}^d))$, with 
k $\in \mathbb{Z}^+, k > \max\{\frac{d}{2}, 6\}$ are strong solutions of \eqref{EE} on $[0, 1] \times \mathbb{R}^d$ with 
$F : \mathbb{C}^2 \to \mathbb{C}, F \in C^k$ and $F(0) = \partial_u F(0) = \partial_\nu F(0) = 0$. If there exists $\lambda > 0$ and $\alpha > \frac{4}{9}$ such that 
\begin{align}
  u_1(0, x) - u_2(0, x), \quad u_1(1, x) - u_2(1, x) \in H^3(e^{\lambda|x|^{\alpha}} dx),
\end{align}
then $u_1 \equiv u_2$.

**Remark 1.3** (Decay notation). Note that we say that $f \in L^2(e^{\lambda|x|^{\alpha}} dx)$ if 
\begin{align}
  \int_{\mathbb{R}^d} |f(x)|^2 e^{\lambda|x|^{\alpha}} dx < \infty,
\end{align}
and that $f \in H^3(e^{\lambda|x|^{\alpha}} dx)$ if $f, \partial_x f, \partial_{x_j} f, \partial_{x_j x_k} f, \partial_{x_j x_k x_p} f \in L^2(e^{\lambda|x|^{\alpha}} dx)$ for all $j, k, p = 1, \cdots d$.

**Remark 1.4** (Sharpness of the result and discussion on assumptions). With the main results stated, let us make a few comments on the order of exponential weight, $e^{\lambda|x|^{\alpha}}$, $\alpha > \frac{4}{9}$.

1. As one can see from \cite{10}, such super-Gaussian weight in the measurement of the decay of data is closely related to the quadratic weight in Hardy’s uncertainty principle. For the general case, as we recalled in Theorem C, the analogue of Gaussian weight in our case would be expected to be the conjugate convex weights, $e^{O(|x|^2)}$. This implies that our decay power $\alpha > \frac{4}{9}$ is almost sharp, even for the case of complex valued and time-dependent potential $V(t, x)$.

2. The decay rate which is described by $\lambda|x|^{\alpha}$, $\alpha > \frac{4}{9}$ in \eqref{EE} can be made better by replacing the exponential weight in \eqref{EE} by $\lambda|x|^{\frac{4}{9}}$, where $\lambda > \lambda_0$, for some $\lambda_0 > 0$ well chosen. The choice of such $\lambda_0$ can be made using the same argument done in the proof of Theorem 1.1 in \cite{19}.

3. The $H^3$ regularity requirement for both solutions and the potential is not necessary. We included it in the statement of Theorem 1.1 simply because in the proof of it, we need to differentiate the equation when deriving an exponential decay estimate for solutions with derivatives. In fact, by following the strategy in \cite{13} and introducing an artificial diffusion into the equation, we should be able to get rid of the regularity assumption. That is, we consider the modified equation (to fix the idea, we consider the $V = 0$ case)
\begin{align}
  \partial_t u = (A + iB)\Delta^2 u
\end{align}
where $A > 0$. An inherent decay given by the artificial diffusion $A\Delta^2$ allows one to do integration by parts freely and prove the solution up to certain derivatives preserves the same decay properties (via a logarithmic convexity) as the initial and terminal data without requiring extra regularity of the solution at all (since no differentiation of the equation is needed). Hence as a byproduct, we could even remove the $H^3$ regularity requirement on the solution. Then by taking the parameter $A \to 0$, a limiting argument gives the unique continuation properties that we desire. We do not plan to introduce any artificial diffusion in our proof, but instead make use of frequency cut-off operators to allow complex $V$, such as integration by parts, needed to obtain our results.

4. We note that \eqref{1.6} is the same assumption made on $V$ in \cite{10}, and it will be verified when used in the nonlinear result.

5. The following example shows that our theorem is essentially sharp for $V(x)$ real-valued and constant in time. Indeed, we show that for any $V \in L^\infty(\mathbb{R}^d)$, there exists non-zero $u(t, x)$ such that 
\begin{align}
  \|e^{\alpha|x|^{4/3}} u(0, x)\|_{L^2(\mathbb{R}^d)} \quad \text{and} \quad \|e^{\beta|x|^{4/3}} u(1, x)\|_{L^2(\mathbb{R}^d)}
\end{align}
are both finite for some $\alpha, \beta > 0$. 

Indeed, let \( f(x) = e^{-2|x|^{4/3}}. \) Now, let
\[
 u(t, x) = e^{-(\epsilon + it)(\Delta^2 + V)} f,
\]
which solves
\[
 i\partial u = \Delta^2 u + V(x)u.
\]

Next, we state a version of Lemma 2.1 from [19] which holds for our modified operator \( \Delta^2 + V(x) \) by virtue of the higher order heat kernel estimate that is Theorem 1 in [8].

**Lemma 1.5.** Suppose \( A, B \in \mathbb{R} \) and \( V(x) \in L^{\infty}(\mathbb{R}^d) \) is real-valued. Then there exists \( N_1, N_2 > 0 \) independent of \( A, B, V \) and \( \Theta_{A,B}(\gamma) > 0 \) such that
\[
 \left\| e^{\Theta_{A,B}(\gamma)|x|^{4/3}} e^{-(A+iB)(\Delta^2 + V)} f \right\|_{L^2} \leq N_1 e^{\omega_0 A}\|V\|_{L^{\infty}} (1 + B^2/A^2)^{n/2} \left\| e^{\gamma |x|^{4/3}} f \right\|_{L^2}.
\]

Letting \( A = \epsilon, B = t, \gamma = 1, \) we have
\[
 \left\| e^{\Theta_{\epsilon,t}(1)|x|^{4/3}} u(t, x) \right\|_{L^2} \lesssim_d (1 + t^2/\epsilon^2)^{d/2} \left\| e|x|^{4/3} f \right\|_{L^2}.
\]
Hence, we see that both \( \left\| e^{\epsilon|x|^{4/3}} u(0, x) \right\|_{L^2(\mathbb{R}^d)} \) and \( \left\| e^{|x|^{4/3}} u(1, x) \right\|_{L^2(\mathbb{R}^d)} \) are finite and \( u(t, x) \) is non-trivial, demonstrating the sharpness of our result.

1.3. **Outline of and challenges in the proof.** As we mentioned, in a series of works [10, 12, 13, 14, 28, 29], the authors set out a systematic procedure to tackle the unique continuation problems with a fast decay flavor. This general method is based on a contradiction argument. We will outline the major step of the method below while listing the main ingredients needed in the contradiction arguments. Additionally, we will highlight the new ingredients we introduce to adapt to our specific fourth-order Schrödinger case (both linear and nonlinear).

1. **Persistence of fast decay.** The solution to (1.1) is initially assumed to have fast decay only at times \( t = 0, 1. \) By examining the evolution of the weighted norm \( \| e^{\lambda|x|/d} u(t, x) \|_{L^2_d} \), one obtains an exponential decay estimate of the following form, for \( \lambda > 0, \)
\[
 \| e^{\lambda|x|/d} u(t, x) \|_{L^2_d} \leq C \| e^{\lambda|x|} u(0, x) \|_{L^2_d} + C \| e^{\lambda|x|} u(1, x) \|_{L^2_d},
\]
via an energy estimate\(^1\). This allows passing the fast decay property to any intermediate times \((0 < t < 1)\) using a technique based on the earlier result [29]. This result is sufficient for our purposes, though it is slightly weaker than a similar logarithmic convexity result pertaining to solutions \( u(t, x) \) of (1.3),
\[
 \| e^{\lambda|x|^2/2} u(t, x) \|_{L^2_d} \leq C \| e^{\lambda|x|^2} u(0, x) \|_{L^2_d}^{1-t} \| e^{\lambda|x|^2} u(1, x) \|_{L^2_d}^t,
\]
for \( \lambda > 0. \)

2. **Carleman estimates with well-chosen weights.** As we mentioned earlier, the Carleman type of inequalities was introduced into the consideration of uniqueness principles, and is now widely used to tackle unique continuation problems. Here is an example of Carleman type estimates that were used in [10]:
\[
 C(\alpha) \left\| e^{\alpha \phi(t,x)} g \right\|_{L^2_{t,x}^s} \leq \left\| e^{\alpha \phi(t,x)} (i\partial_t + \Delta) g \right\|_{L^2_{t,x}^s}.
\]
It is worth emphasizing that the major challenge in obtaining such type of inequality falls on hunting a suitable (carefully designed) weight function\(^2\) \( \phi \) that allows one to get the estimate. As expected, one might utilize very different weights when considering different models. In [13], the weight function is quadratic in space with a proper translation in the first spatial variable. We use a different but still quadratic weight to prove our Carleman estimate.

3. **Lower bounds for the solution.** Having derived a Carleman estimate, with proper localization of the solution, we are able to obtain an absolute lower bound away from \( t = 0, 1 \) for non-trivial solutions (supported on an annulus domain as a consequence). Let us remark that the lower bound depends greatly on the weight function chosen in the Carleman inequality. In fact, in order to reach a contradiction in the next step, such a lower bound has to match the fast decay rate of the solution.

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\(^1\)The weight \( e^{\lambda|x|} \) can be upgraded to a weight \( e^{\lambda|x|/d} \) with a subordination-type inequality

\(^2\)In the context of Carleman estimates, the weight function usually is the function appearing in the exponential. When we say a ‘linear weight function’ or a ‘quadratic weight function’, we mean \( \phi \) is linear or quadratic in spatial variables.
A contradiction argument. At this point, there are two different rates discussed: (a) the fast decay rate of non-zero solutions at intermediate time inherit the fast decay at $t = 0, 1$ (due to the log-convexity); (b) the asymptotic lower bounds established via Carleman estimates. The difference between these two rates forces such solutions with assumed fast decay to be trivial, and completes the proof of unique continuation for the linear equation. To handle the nonlinear equation, we consider the difference of two solutions and the resulting nonlinearity as a potential, after which we can repeat the steps for the linear unique continuation result as the hypotheses of Theorem 1.2 guarantee that the new potential in the non-linear setting satisfies the necessary conditions in Theorem 1.1.

The main challenges in our paper lie in the proof of the exponential decay estimate for $L^2$ norms of solutions to an inhomogeneous version of (1.1) as well as that of our Carleman inequality.

For the persistence of fast decay part, if one employs the strategy of the logarithmic convexity type (as [19] did in their work), tools such as introducing artificial dissipation into the equation, utilizing parabolic estimates, and subordination-type inequalities are commonly involved. However, this route only would yield decay estimates for real-valued and time-independent potentials. While these estimates are sufficient to establish a linear unique continuation result, it is not enough to deduce a nonlinear unique continuation result. To this end, we decided to approach the problem slightly differently. Inspired by [29], we in fact are able to obtain an $L^2$-based exponential decay for solutions to (1.1) with respect to a measure of the form $e^{\beta|x|}$, $\beta > 0$, then extend it via a subordination type inequality (Corollary 2.2 in [10]) to a super-linear exponential measure of the form $e^{\lambda|x|^\alpha}$, $\lambda > 0, \alpha > 1$, from which we are able to prove persistence of fast decay at two times. More precisely, to obtain Lemma 3.1, our fundamental decay estimate, that we later modify suitably, we must first cut off a weight function multiplying a solution to (3.2) to be able to rigorously apply various technologies such as integration by parts to the resulting $L^2$ norm of such weighted solutions. The difficult part lies in controlling the growth of $L^2$ norms cutoff and projected in frequency space of the weighted solutions to (3.2) since there is a large number of terms that each require qualitatively different techniques to suitably bound.

To obtain a Carleman type inequality for the operator $i\partial_t + \Delta^2$, we expect to see many more terms in the computation of commutators (arising from splitting the conjugate operator

$$ \hat{T}f(x) := e^{\phi(x)}(\partial_x)_4 \left[e^{\phi(x)}f(x)\right] $$

into a symmetric and an anti-symmetric components) compared to the case for the operator $i\partial_t + \Delta$ (since the number of derivatives is twice as high as that of the classical Schrödinger case). Among these commutators, most terms are computed manually, but in a couple of cases during the computation, we use a computational software to simplify extremely lengthy expressions. This is in fact the major reason why we consider the operator $\Delta^2$. After such simplifications, we need to manipulate certain $L^2$ inner-products containing many terms in such a way that they can be lower-bounded in a positive fashion (for more details see the proof of Lemma 5.1). We use this lower bound to derive a new Carleman inequality, which we would later use in our lower bound proof.

1.4. Organization of the paper. In Section 2, we discuss some notations and define some cutoff functions that will be used in the rest of the paper; in Section 3, we present an exponential decay estimate for solutions to (1.1) with fast decay and we upgrade it to a super-linear exponential decay estimate in Section 4; in Section 5, we derive a Carleman inequality for the ‘separable’ fourth-order Schrödinger operator; in Section 6, we prove a lower bound for the fast decay solutions; in Section 7, we prove the linear and nonlinear unique continuation results, by combining the lower bound and the exponential decay proved in previous sections.

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2. Preliminaries

In this section, we list some notations and define some cutoff functions that will be used in the rest of the paper.
2.1. **Notations.** We use the usual notation that $A \lesssim B$ or $B \gtrsim A$ to denote an estimate of the form $A \leq C B$, for some constant $0 < C < \infty$ depending only on the a priori fixed constants of the problem.

We define the Fourier transform on $\mathbb{R}^d$ by

$$\hat{f}(\xi) := \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-ix \cdot \xi} f(x) \, dx,$$

and Fourier inversion

$$f(x) := \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{ix \cdot \xi} \hat{f}(\xi) \, d\xi.$$

For a time interval $I$, we have the following spacetime norms $L^\varphi_t L^q_x (I \times \mathbb{R}^d)$

$$\|u\|_{L^\varphi_t L^q_x (I \times \mathbb{R}^d)} := \left( \int_I \left( \int_{\mathbb{R}^d} |u(t,x)|^q \, dx \right)^{\frac{\varphi}{q}} \, dt \right)^{\frac{1}{\varphi}}.$$

2.2. **Cutoff functions and chain rule.** We will frequently apply cutoff functions to the solution $u$ in later sections, hence we provide a general formula for the chain rule and product rule calculation here.

If $u$ solves (1.1), we then can find the equation for the modified $u$. That is, for a smooth function $\sigma(t,x)$, then the modified solution $\sigma(t,x)u(t,x)$ satisfies

$$(i\partial_t + \Delta^2)[\sigma(t,x)u(t,x)] = i\partial_t(\sigma u) + \Delta^2(\sigma u)$$

$$= i(\partial_t \sigma)u + i\sigma(\partial_t u) + \sum_{j=1}^d \sigma \partial_{x_j}^4 u + 4(\partial_{x_j} \sigma)\partial_{x_j}^3 u + 6(\partial_{x_j}^2 \sigma)\partial_{x_j}^2 u + 4(\partial_{x_j}^3 \sigma)\partial_{x_j} u + (\partial_{x_j}^4 \sigma)u$$

$$= i(\partial_t \sigma)u + \sigma(i\partial_t + \Delta^2)u + \sum_{j=1}^d 4(\partial_{x_j} \sigma)\partial_{x_j}^3 u + 6(\partial_{x_j}^2 \sigma)\partial_{x_j}^2 u + 4(\partial_{x_j}^3 \sigma)\partial_{x_j} u + (\partial_{x_j}^4 \sigma)u. \quad (2.1)$$

When $\sigma = \sigma(x)$, we write

$$(i\partial_t + \Delta^2)[\sigma(x)u(t,x)]$$

$$= \sigma(i\partial_t + \Delta^2)u + \sum_{j=1}^d 4(\partial_{x_j} \sigma)\partial_{x_j}^3 u + 6(\partial_{x_j}^2 \sigma)\partial_{x_j}^2 u + 4(\partial_{x_j}^3 \sigma)\partial_{x_j} u + (\partial_{x_j}^4 \sigma)u. \quad (2.1)$$

3. **Linear Exponential Decay Estimate**

In this section, we prove an $L^2$-based decay estimate for solutions to an inhomogeneous version of (1.1) with an exponentially weighted measure in one spatial direction.

**Lemma 3.1.** There exists $\varepsilon_0 > 0$ such that if $V : [0,1] \times \mathbb{R}^d \to \mathbb{C}$ satisfies

$$\|V\|_{L^1_t L^\infty_x} \leq \varepsilon_0, \quad (3.1)$$

and $u \in C([0,1] : L^2_x(\mathbb{R}^d))$ is a solution of the following perturbed equation

$$\begin{cases}
(i\partial_t + \Delta^2)u = Vu + H, & (t, x) \in [0,1] \times \mathbb{R}^d \\
u(0, x) = u_0(x)
\end{cases} \quad (3.2)$$

with $H \in L^1_t([0,1] : L^2(\mathbb{R}^d))$ and for some $\beta \in \mathbb{R}$,

$$u_0, u_1 \in L^2(e^{2\beta x_1} \, dx), \quad H \in L^1_t([0,1] : L^2(e^{2\beta x_1} \, dx)).$$

Then

$$\sup_{t \in [0,1]} \|u(t)\|_{L^2(e^{2\beta x_1} \, dx)}^2 \leq C(\|u_0\|_{L^2(e^{2\beta x_1} \, dx)}^2 + \|u_1\|_{L^2(e^{2\beta x_1} \, dx)}^2 + \|H\|_{L^1_t L^2(e^{2\beta x_1} \, dx)}^2). \quad (3.3)$$

Note the constant $C$ is independent on $\beta$. 

Remark 3.2 (A formal proof). The proof of this lemma is very computational and involves introducing cutoff functions and handling error term produced by such truncation. Before starting the proof, let us present the main idea of the calculation.

Let us forget the perturbation $H$ and potential $V$ for a moment. By considering a change of variables: $v = e^{\beta x_1} u$, we reduce (3.3) into the following inequality

$$\sup_{t \in [0,1]} \|v(t)\|_{L^2}^2 \leq C(\|v(0)\|_{L^2}^2 + \|v(1)\|_{L^2}^2).$$

Under such change of variables, we have

$$\partial_t u = e^{-\beta x_1} \partial_t v,$$
$$\partial_{x_1}^4 v = e^{-\beta x_1} \partial_{x_1}^4 v - 4\beta e^{-\beta x_1} \partial_{x_1}^3 v + 6\beta^2 e^{-\beta x_1} \partial_{x_1}^2 v - 4\beta^3 e^{-\beta x_1} \partial_{x_1} v + \beta^4 e^{-\beta x_1} v,$$
$$\partial_{x_j}^4 v = e^{-\beta x_1} \partial_{x_j}^4 v, \quad j = 2, \ldots, d.$$

which gives the differential equation that $v$ solves

$$(i\partial_t + \Delta^2)v = -4\beta \partial_{x_1}^4 v + 6\beta^2 \partial_{x_1}^2 v - 4\beta^3 \partial_{x_1} v + \beta^4 v.$$

Taking Fourier transforms on both sides, we obtain a separable differential equation for $\hat{v}$

$$i\partial_t \hat{v} + \sum_{j=1}^d (i\xi_j)^4 \hat{v} = -4\beta (i\xi_1)^3 \hat{v} + 6\beta^2 (i\xi_1)^2 \hat{v} - 4\beta^3 (i\xi_1) \hat{v} + \beta^4 \hat{v}$$

which implies

$$\partial_t \hat{v} = i\hat{v}(\sum_{j=1}^d \xi_j^4 + 6\beta^2 \xi_1^2 - \beta^4) + \hat{v}(4\beta^3 - 4\beta^3 \xi_1).$$

Then we obtain $\hat{v}$ of the following form

$$\hat{v}(t) = Ce^{i((\sum_{j=1}^d \xi_j^4 + 6\beta^2 \xi_1^2 - \beta^4) t + 4i(\beta^3 - \beta^3 \xi_1) t)}$$

where $C$ is some initial data. Then compute the $L^2$ norm of $v$ by Plancherel theorem

$$\|v(t)\|_{L^2} = \|\hat{v}(t)\|_{L^2} = \left\|Ce^{i((\sum_{j=1}^d \xi_j^4 + 6\beta^2 \xi_1^2 - \beta^4) t + 4i(\beta^3 - \beta^3 \xi_1) t)} \right\|_{L^2}.$$ (3.4)

Now we see that when $\beta^3 \xi_1^3 - \beta^3 \xi_1 > 0$, $\left\|Ce^{i((\sum_{j=1}^d \xi_j^4 + 6\beta^2 \xi_1^2 - \beta^4) t + 4i(\beta^3 - \beta^3 \xi_1) t)} \right\|_{L^2}$ increases, hence

$$\left\|P_{\beta^3 \xi_1^3 - \beta^3 \xi_1 > 0} v(t) \right\|_{L^2} \leq \left\|v(0)\right\|_{L^2},$$

and when $\beta^3 \xi_1^3 - \beta^3 \xi_1 < 0$, $\left\|Ce^{i((\sum_{j=1}^d \xi_j^4 + 6\beta^2 \xi_1^2 - \beta^4) t + 4i(\beta^3 - \beta^3 \xi_1) t)} \right\|_{L^2}$ decreases, hence

$$\left\|P_{\beta^3 \xi_1^3 - \beta^3 \xi_1 < 0} v(t) \right\|_{L^2} \leq \left\|v(0)\right\|_{L^2}.$$

Combining these two inequalities, we arrive at our conclusion that for $t \in [0,1]$

$$\left\|v(t)\right\|_{L^2}^2 = \left\|P_{\beta^3 \xi_1^3 - \beta^3 \xi_1 > 0} v(t) \right\|_{L^2}^2 + \left\|P_{\beta^3 \xi_1^3 - \beta^3 \xi_1 < 0} v(t) \right\|_{L^2}^2 \leq \left\|v(0)\right\|_{L^2}^2 + \left\|v(1)\right\|_{L^2}^2.$$

This computation is considered formal since initially we did not know the $L^2$-finiteness of the new variable $v = e^{\beta x_1} u$. However, the real proof utilizes the same idea in this remark. To make sense of such a change of variables and ensure its finiteness, we need to introduce several cutoff functions and carefully handle the resulting error terms through Calderón’s first commutator estimates.

In the rest of this section, we prove Lemma 3.1 by employing the strategy outlined in this formal proof which includes the careful treatment of error terms.

Now we are ready to start the proof.
3.1 Then putting the derivatives above, we get the following equation

\[ \text{Note here } v \text{ is unknown. To address this concern, we introduce cutoff functions as a means of handling this issue. This proof is based on an energy estimate.} \]

- Define \( \varphi_n \in C^\infty(\mathbb{R}) \), \( 0 \leq \varphi_n \leq 1 \) such that
  \[
  \varphi_n(s) = \begin{cases} 
  1, & s \leq n, \\
  0, & s > 10n,
  \end{cases}
  \]

  and
  \[
  |\varphi_n^{(k)}(s)| \leq \frac{c_k}{n^k}.
  \]

- Based on \( \varphi_n \), we define \( \theta_n \in C^\infty(\mathbb{R}) \),
  \[
  \theta_n(s) := \int_0^s \varphi_n^2(\ell) \, d\ell,
  \]
  which satisfies
  \[
  \theta_n(s) = \begin{cases} 
  \beta s, & s \leq n, \\
  c_n \beta, & s > 10n.
  \end{cases}
  \]

  and
  \[
  \theta'_n(s) = \beta \varphi_n^2(s) \leq \beta, \quad |\theta_n^{(k)}(s)| \leq \frac{ck\beta}{n^k-1}.
  \]

- Finally, we obtain the important modification of the weight \( e^{\beta x_1} \), which is given by
  \[
  \Phi_n(s) = e^{\theta_n(s)}
  \]
  and satisfies
  \[
  \Phi_n(s) \leq e^{\beta s} \quad \text{and} \quad \lim_{n \to \infty} \Phi_n(s) = e^{\beta s}.
  \]

Recalling the change of variables that we did in Remark 3.2, we write

\[
 v_n(t, x) = \Phi_n(x_1)u(t, x) = e^{\theta_n(x_1)}u.
\]

Note here \( v_n \) is almost the \( v = e^{\beta x_1}u \) in the change of variables that we did in Remark 3.2.

Then we want to find a differential equation that \( v_n \) satisfies. First, we compute

\[
i \Phi_n \partial_t u = i \partial_t v_n, \quad \Phi_n \partial_x u = -\theta'_n v_n + \partial_x v_n, \quad \Phi_n \partial_{x_1} u = (-\theta'_n)^2 v_n + (-\theta''_n)v_n + 2(-\theta'_n)\partial_x v_n + \partial^2 v_n;
\]

\[
\Phi_n \partial_{x_1}^2 u = (-\theta'_n)^3 v_n + 3(-\theta'_n)(-\theta''_n)v_n + (-\theta'''_n)v_n + 3(-\theta'_n)^2 \partial_x v_n + 3(-\theta'_n)\partial_{x_1} v_n + 3(-\theta'_n)\partial_{x_1}^2 v_n + \partial^3 v_n;
\]

\[
\Phi_n \partial_{x_1}^4 u = (-\theta'_n)^4 v_n + 6(-\theta'_n)^2(-\theta''_n)v_n + 4(-\theta'_n)(-\theta''_n)v_n + 3(-\theta''_n)2 \partial_x v_n + 3(-\theta''_n) \partial_{x_1} v_n + (-\theta''_n)\partial_{x_1}^2 v_n + (-\theta'''_n)v_n + 3(-\theta''_n)\partial_{x_1}^2 v_n + \partial^3 v_n;
\]

\[
\Phi_n \partial_{x_1}^4 u = \Phi_n \partial_{x_1}^4(e^{-\theta_n} v_n) = \partial_{x_1}^4 v_n, \quad j = 2, \ldots, d.
\]

Then putting the derivatives above, we get the following equation

\[
\Phi_n H + \Phi_n Vu = \Phi_n(i \partial_t u + \Delta^2 u) = i \partial_t v_n + \Delta^2 v_n
\]

\[
+ [(-\theta'_n)^4 v_n + 6(-\theta'_n)^2(-\theta''_n)v_n + 4(-\theta'_n)(-\theta''_n)v_n + 3(-\theta''_n)2 \partial_x v_n + (\theta'''_n)v_n + 3(-\theta''_n)\partial_{x_1}^2 v_n + \partial^3 v_n].
\]
Using (1.1), we write
\[ i\partial_t v_n + \Delta^2 v_n = -v_n(-\theta_n')^4 + 6(-\theta_n')^2(-\theta_n'') + 4(-\theta_n')(-\theta_n''') + 3(-\theta_n'')^2 + (-\theta_n''') \]
\[ - \partial_x v_n[4(-\theta_n')^3 + 12(-\theta_n')(-\theta_n'') + 4(-\theta_n''')] 
\[ - \partial_x^2 v_n[6(-\theta_n')^2 + 6(-\theta_n'')] 
\[ - \partial_x^3 v_n[4(-\theta_n')] + \Phi_n H + V v_n, \]
where
\[ \theta_n' = \beta \varphi_n, \]
\[ \theta_n'' = 2\beta \varphi_n \varphi_n', \]
\[ \theta_n''' = 2\beta \left[ (\varphi_n')^2 + \varphi_n \varphi_n'' \right], \]
\[ \theta_n'''' = 2\beta \left[ 3\varphi_n' \varphi_n'' + \varphi_n \varphi_n''' \right]. \]

Step 2: Second change of variables. We wish to compute \( \partial_t \| v_n \|_{L^2}^2 \), however, there is a constant multiply of \( \| v_n \|_{L^2}^2 \) on the right-hand side of (3.5) (to be more precise, it is the first term \( -v_n(-\theta_n')^4 \)), which will not be made small when doing estimates at the very end (recall \( \varphi_n \sim 1 \) when \( s \leq n \)). Hence we remove this term by another change of variables (this only removes the non-vanishing term and will not change the \( L^2 \) norm at all) before computing \( \partial_t \| v_n \|_{L^2}^2 \) via
\[ w_n = e^{-i(-\theta_n')^4 t} v_n =: e^{\mu} v_n. \]

Similarly, we need to find a differential equation that \( w_n \) satisfies.
\[ e^{\mu} \partial_t v_n = i(-\theta_n')^4 w_n + \partial_t w_n, \]
\[ e^{\mu} \partial_x w_n = e^{\mu} \partial_x (e^{-\mu} w_n), \]
\[ \partial_x^2 w_n = 4\partial_x^3 w_n \mu' + \partial_x^2 w_n[6(\mu')^2 - 6\mu''] + \partial_x w_n[12\mu' \mu'' - 4(\mu')^3 - 4\mu'''] \]
\[ + w_n[-6(\mu')^2 \mu'' + (\mu')^4 + 4\mu''' \mu' + 3(\mu'')^2 - \mu'''], \]
\[ e^{\mu} \partial_x^3 w_n = e^{\mu} \partial_x^4 e^{-\mu} w_n = \partial_x^4 w_n, \quad j = 2, \ldots, d. \]
where
\[ \mu = -i(-\theta_n')^4 t, \]
\[ \mu' = -it[4(-\theta_n')^3(-\theta_n''')], \]
\[ \mu'' = -it[12(-\theta_n')^2(-\theta_n'')^2 + 4(-\theta_n')^3(-\theta_n''')], \]
\[ \mu''' = -it[24(-\theta_n')(-\theta_n'')^3 + 36(-\theta_n')^2(-\theta_n'')(-\theta_n''') + 4(-\theta_n')^3(-\theta_n''')], \]
\[ \mu'''' = -it[24(-\theta_n')^4 + 144(-\theta_n')(-\theta_n'')^2(-\theta_n'')^2 + 36(-\theta_n')^2(-\theta_n'')^2] \]
\[ + 48(-\theta_n')^2(-\theta_n'')(\theta_n''') - 4(-\theta_n')^3(-\theta_n'''). \]

Putting the derivatives together, we obtain
\[ e^{\mu} (i\partial_t + \Delta^2) w_n = \mu + \partial_t w_n + \Delta^2 w_n \]
\[ - 4\partial_x^3 w_n \mu' + \partial_x^2 w_n[6(\mu')^2 - 6\mu''] + \partial_x w_n[12\mu' \mu'' - 4(\mu')^3 - 4\mu'''] \]
\[ + w_n[-6(\mu')^2 \mu'' + (\mu')^4 + 4\mu''' \mu' + 3(\mu'')^2 - \mu'']. \]

Substituting the RHS of (3.5) into the LHS of (3.6), we then write
\[ e^{\mu} (-v_n[-(-\theta_n')^4 + 6(-\theta_n')^2(-\theta_n'') + 4(-\theta_n')(-\theta_n''') + 3(-\theta_n'')^2 + (-\theta_n''')]) \]
\[ - \partial_x v_n[4(-\theta_n')^3 + 12(-\theta_n')(-\theta_n'') + 4(-\theta_n''')] 
\[ - \partial_x^2 v_n[6(-\theta_n')^2 + 6(-\theta_n'')] 
\[ - \partial_x^3 v_n[4(-\theta_n')] + \Phi_n H + V v_n \}
\[ = -(-\theta_n')^4 w_n + i\partial_t w_n + \Delta^2 w_n \]
\[ - 4\partial_x^3 w_n \mu' + \partial_x^2 w_n[6(\mu')^2 - 6\mu''] + \partial_x w_n[12\mu' \mu'' - 4(\mu')^3 - 4\mu'''] \]

\[ + w_n[-6(\mu')^2 \mu'' + (\mu')^4 + 4\mu''' \mu' + 3(\mu'')^2 - \mu'']. \]
+ w_n[-6(\mu''')^2 \mu'' + (\mu')^4 + 4\mu''' \mu' + 3(\mu'')^2 - \mu'''].

Hence shuffling the terms in the equation above, we get the following

\[
(i \partial_t + \Delta^2)w_n = -\{-4\partial_x^2 w_n \mu' + \partial_x^2 w_n (6(\mu')^2 - 6\mu'') + \partial_x w_n [12\mu'' - 4(\mu')^3 - 4\mu'''

+ w_n[-6(\mu')^2 \mu'' + (\mu')^4 + 4\mu''' \mu' + 3(\mu'')^2 - \mu''']

- e^\nu v_n [6(-\theta_n')^2(-\theta''_n) + 4(-\theta'_n)(-\theta'''_n) + 3(-\theta''_n)^2 + (-\theta''')_n]

- e^\nu \partial_x v_n [4(-\theta'_n)^3 + 12(-\theta'_n)(-\theta''_n) + 4(-\theta'''_n)]

- e^\nu \partial_x^2 v_n [6(-\theta''_n)^2 + 6(-\theta''')_n]

- e^\nu \partial_x^3 v_n [4(-\theta''_n)] + e^\nu [\Phi_n H + V v_n].
\]

(3.7)

Noticing that the \(v_n\) terms in (3.7) can be rewritten as

\[
e^\nu \partial_x v_n = -\mu' w_n + \partial_x w_n,

e^\nu \partial_x^2 v_n = (-\mu')^2 w_n + (-\mu'' \partial_x w_n + 2(-\mu') \partial_x w_n + \partial_x^2 w_n,

e^\nu \partial_x^3 v_n = (-\mu')^3 w_n + 3(-\mu') (-\mu'' w_n + (-\mu''') w_n + 3(-\mu')^2 \partial_x w_n + 3(-\mu')^3 \partial_x w_n + 3(-\mu')^3 \partial_x w_n + 3(-\mu') \partial_x w_n + \partial_x^3 w_n,
\]

then we have the following equivalent form for the last four lines involving \(v_n\) in (3.7)

Last four lines in (3.7) = \[w_n \{-6(-\theta'_n)^2(-\theta''_n) - 6(-\theta'_n)(-\theta'''_n) - 3(-\theta''_n)^2 - (-\theta''')_n\} - 4(-\theta''_n)^3 + 12(-\theta'_n)(-\theta''_n) + 2(-\mu') [6(-\theta''_n)^2 - 6(-\theta''')_n] + [3(-\mu')^3 + \partial_x w_n [6(-\theta''_n)^2 - 6(-\theta''')_n] + e^\nu \Phi_n H + V w_n.
\]

Now we finally find a differential equation that \(w_n\) satisfies, and (3.7) becomes

\[
(i \partial_t + \Delta^2)w_n = w_n[g_0(x_1)] + \partial_x w_n [a_1^2(x_1) + q_1(x_1)] + \partial_x^2 w_n [-a_2^2(x_1) + q_2(x_1)]

+ \partial_x^3 w_n [a_3^2(x_1) + itb(x_1)] + e^\nu \Phi_n(x_1) H + V w_n,
\]

(3.8)

where

\[
q_0(x_1) = [-6(\mu')^2 \mu'' + (\mu')^4 + 4\mu''' \mu' + 3(\mu'')^2 - \mu'''] - 6(-\theta''_n)^2(-\theta''')_n

- 4(-\theta'_n)(-\theta''')_n - 3(-\theta''_n)^2 - (-\theta''')_n \cdot [-4(-\theta'_n)^3 - 12(-\theta'_n)(-\theta''_n) - 4(-\theta'''_n)]

+ [(-\mu')^2 + (-\mu'')] [6(-\theta''_n)^2 - 6(-\theta''')_n] + [(\mu')^3 + 3(-\mu'')(\mu''') + (-\mu''')_n] \cdot [-4(-\theta'_n)],

a_1^2(x_1) = -4(-\theta''_n)^3,

q_1(x_1) = [-12\mu' \mu'' - 4(\mu')^3 - 4\mu''']

- 12(-\theta'_n)(-\theta''_n) - 4(-\theta''_n)^2 + [3(-\mu'')(\mu''') + (-\mu''')_n] \cdot [-4(-\theta'_n)],

-a_2^2(x_1) = -6(-\theta''_n)^2 \cdot [-4(-\theta'_n)],

q_2(x_1) = [-6(\mu')^2 - 6\mu''] + [6(-\theta''_n)^2 - 6(-\theta''')_n] \cdot [-4(-\theta''_n)],

a_3^2(x_1) = -4(-\theta''_n)^3,

i tb(x_1) = 4\mu'.
\]

Recall

\[
\mu = -i(-\theta'_n)^4 t,
\mu' = -it[4(-\theta'_n)^3(-\theta''_n)],
\mu'' = -it[12(-\theta'_n)^2(-\theta''_n)^2 + 4(-\theta''_n)^3(-\theta''')_n],
\]

\[
\mu''' = -it[6(-\theta''_n)^2(-\theta''')_n + 3(-\theta''_n)^3(-\theta''''_n)].
\]
We observe the following decay properties from the coefficients in (3.8), that is, for \( k \in \mathbb{N} \):

\[
\| \partial_{x_i}^k q_j \|_{L^\infty} \leq \frac{c}{n^{k+1}}, \quad j = 0, 1, 2, \\
\| \partial_{x_i}^k a_j^2 \|_{L^\infty} \leq \frac{c}{n^k}, \quad j = 1, 2, 3, \\
\| \partial_{x_i}^k b \|_{L^\infty} \leq \frac{c}{n^{k+1}}.
\]

We remark here that due to the second change of variables \( v_n \rightarrow w_n \), we successfully removed a constant multiply of \( v_n \) and only left with a decaying coefficient \( q_0(x_1) \) times \( w_n \).

**Step 3: An energy estimate on \( w_n \).** In the rest of the proof, we work on estimating the \( L^2 \) norm of \( w_n \). Starting by introducing a couple of Fourier multipliers.

- Define

\[
\chi_+(\xi) = \begin{cases} 
1 & \text{if } \xi_1 \in (-\beta, 0) \cup (\beta, \infty), \\
0 & \text{if } \xi_1 \in (-\infty, -\beta] \cup [0, \beta], 
\end{cases}
\]

and

\[
\chi_-(\xi) = \begin{cases} 
1 & \text{if } \xi_1 \in (-\infty, -\beta] \cup [0, \beta], \\
0 & \text{if } \xi_1 \in (-\beta, 0) \cup (\beta, \infty). 
\end{cases}
\]

- We also define \( \eta \in C_0^\infty(\mathbb{R}^d) \) with \( 0 \leq \eta(x) \leq 1 \) and

\[
\eta(x) = \begin{cases} 
1 & \text{if } |x| \leq \frac{1}{2}, \\
0 & \text{if } |x| \geq 1. 
\end{cases}
\]

- Then we define two projections, for \( \varepsilon \in (0, 1] \)

\[
\tilde{P}_\varepsilon f(\xi) := \eta_\varepsilon(\xi) \hat{f}(\xi) = \eta(\varepsilon \xi) \hat{f}(\xi), \\
\tilde{P}_\pm f(\xi) := \chi_\pm(\xi) \hat{f}(\xi).
\]

We remark that (1) the projections \( P_\pm \) are defined based on the formal calculation in Remark 3.2, which allow the dominant term in (3.4) in the \( L^2 \) norm of \( v_n \) (or \( w_n \)) to have a sign; (2) the projection \( P_\varepsilon \) permits the freedom to do any integration by parts in the frequency space.

We now want to derive equations for \( P_\varepsilon P_\pm w_n \) by applying the projection to each term in (3.8):

\[
i\partial_t P_\varepsilon P_+ w_n + \Delta^2 P_\varepsilon P_+ w_n = P_\varepsilon P_+ w_n[q_0(x_1)] + P_\varepsilon P_+ \partial_{x_i} w_n[a_1^2(x_1) + q_1(x_1)] \\
+ P_\varepsilon P_+ \partial_{x_i}^2 w_n[-a_2^2(x_1) + q_2(x_1)] + P_\varepsilon P_+ \partial_{x_i}^3 w_n[a_3^2(x_1) + itb(x_1)] + P_\varepsilon P_+ \varepsilon^2 \Phi_n(x_1)H + P_\varepsilon P_+ V w_n,
\]

and

\[
-i\partial_t \tilde{P}_\varepsilon P_+ w_n + \Delta^2 \tilde{P}_\varepsilon P_+ w_n = \tilde{P}_\varepsilon P_+ w_n[q_0(x_1)] + \tilde{P}_\varepsilon P_+ \partial_{x_i} w_n[a_1^2(x_1) + q_1(x_1)] \\
+ \tilde{P}_\varepsilon P_+ \partial_{x_i}^2 w_n[-a_2^2(x_1) + q_2(x_1)] + \tilde{P}_\varepsilon P_+ \partial_{x_i}^3 w_n[a_3^2(x_1) + itb(x_1)] + \tilde{P}_\varepsilon P_+ \varepsilon^2 \Phi_n(x_1)H + \tilde{P}_\varepsilon P_+ V w_n.
\]
Multiplying (3.10) and (3.11) by $P_zP_+w_n$ and $-P_zP_+w_n$, respectively, and adding the result, we obtain
\[ i\partial_t |P_zP_+w_n|^2 + \Delta^2 P_zP_+w_n \cdot P_zP_+w_n - \Delta^2 P_zP_+w_n \cdot P_zP_+w_n \]
\[ = P_zP_+w_n[q_0(x_1)] \cdot \overline{P_zP_+w_n} - P_zP_+w_n[q_0(x_1)] \cdot P_zP_+w_n \]
\[ + P_zP_+\partial_{x_1}w_n[a_1^2(x_1) + q_1(x_1)] \cdot \overline{P_zP_+w_n} - P_zP_+\partial_{x_1}w_n[a_1^2(x_1) + q_1(x_1)] \cdot P_zP_+w_n \]
\[ + P_zP_+\partial_{x_2}w_n[-a_2^2(x_1) + q_2(x_1)] \cdot \overline{P_zP_+w_n} - P_zP_+\partial_{x_2}w_n[-a_2^2(x_1) + q_2(x_1)] \cdot P_zP_+w_n \]
\[ + P_zP_+\partial_{x_3}^2w_n[a_3^2(x_1) + itb(x_1)] \cdot \overline{P_zP_+w_n} - P_zP_+\partial_{x_3}^2w_n[a_3^2(x_1) + itb(x_1)] \cdot P_zP_+w_n \]
\[ + P_zP_+e^a\Phi_n(x_1)H \cdot \overline{P_zP_+w_n} - P_zP_+e^a\Phi_n(x_1)H \cdot P_zP_+w_n \]
\[ + P_zP_+Vw_n \cdot P_zP_+w_n - P_zP_+Vw_n \cdot P_zP_+w_n, \]
and taking the imaginary part in the equation above yields
\[ \partial_t |P_zP_+w_n|^2 + 2\text{Im}(\Delta^2 P_zP_+w_n \cdot \overline{P_zP_+w_n}) \]
\[ = 2\text{Im}(P_zP_+w_n[q_0(x_1)] \cdot \overline{P_zP_+w_n}) \]
\[ + 2\text{Im}(P_zP_+\partial_{x_1}w_n[a_1^2(x_1) + q_1(x_1)] \cdot \overline{P_zP_+w_n}) \]
\[ + 2\text{Im}(P_zP_+\partial_{x_2}w_n[-a_2^2(x_1) + q_2(x_1)] \cdot \overline{P_zP_+w_n}) \]
\[ + 2\text{Im}(P_zP_+\partial_{x_3}^2w_n[a_3^2(x_1) + itb(x_1)] \cdot \overline{P_zP_+w_n}) \]
\[ + 2\text{Re}(P_zP_+\partial_{x_3}^2w_n[tb(x_1)] \cdot \overline{P_zP_+w_n}) \]
\[ + 2\text{Im}(P_zP_+e^a\Phi_n(x_1)H \cdot \overline{P_zP_+w_n}) \]
\[ + 2\text{Im}(P_zP_+Vw_n \cdot \overline{P_zP_+w_n}). \] (3.12)

Now we will integrate (3.12) and estimate each term in this integration.

**Easy terms.** Since for all $n \in \mathbb{N}$, $w_n \in L^2_\infty(\mathbb{R}^d)$, $e^a\Phi_n(x_1)H \in L^2_\infty(\mathbb{R}^d)$, we have
\[ \text{Im} \int_{\mathbb{R}^d} \Delta^2 P_zP_+w_n \cdot \overline{P_zP_+w_n} \, dx = 0. \]

Also we have for terms (3.18) and (3.19)
\[ |\text{Im} \int_{\mathbb{R}^d} P_zP_+e^a\Phi_n(x_1)H \cdot \overline{P_zP_+w_n} \, dx| \leq c \| e^a\Phi_n(x_1)H \|_{L^2_\infty} \| P_zw_n \|_{L^2_\infty}, \]
\[ |\text{Im} \int_{\mathbb{R}^d} P_zP_+Vw_n \cdot \overline{P_zP_+w_n} \, dx| \leq \| V \|_{L^\infty_\infty} \| P_zw_n \|_{L^2_\infty}^2, \]
and term (3.13) by (3.9)
\[ |\text{Im} \int_{\mathbb{R}^d} P_zP_+w_n[q_0(x_1)] \cdot \overline{P_zP_+w_n} \, dx| \leq c \| q_0 \|_{L^\infty_\infty} \| P_zw_n \|_{L^2_\infty}^2 \leq \frac{c}{n} \| P_zw_n \|_{L^2_\infty}^2. \]

**Preparation.** To deal with other terms (3.14) - (3.17), we recall Calderón first commutator estimates in [5, 30] which were also used in [29]
\[ \| [P_z; a]\partial_{x_1}f \|_{L^2} \leq c \| \partial_{x_1}a \|_{L^\infty} \| f \|_{L^2}, \]
\[ \| \partial_{x_1}[P_z; a]f \|_{L^2} \leq c \| \partial_{x_1}a \|_{L^\infty} \| f \|_{L^2}, \]
\[ \| [P_z; a]\partial_{x_1}f \|_{L^2} \leq c \| \partial_{x_1}a \|_{L^\infty} \| f \|_{L^2}, \]
\[ \| \partial_{x_1}[P_z; a]f \|_{L^2} \leq c \| \partial_{x_1}a \|_{L^\infty} \| f \|_{L^2}. \] (3.20)

We also recall Claim 1 and Claim 2 in [29] here.

**Claim 3.3** (Claim 1 and Claim 2 in [29]). Using Calderón first commutator estimates, we have

1. For $a^2(x_1) \geq 0$
\[ \text{Im} \int_{\mathbb{R}^d} P_zP_+(a^2(x_1)\partial_{x_1}w_n) \cdot \overline{P_zP_+w_n} \, dx = \text{Im} \int_{\mathbb{R}^d} \partial_{x_1}P_zP_+(a(x_1)w_n) \cdot \overline{P_zP_+(a(x_1)w_n)} \, dx + O\left(\frac{\| P_zw_n \|_{L^2_\infty}^2}{n}\right). \]
(2) For \( b(x_1) \) pure imagery

\[
\text{Im} \int_{\mathbb{R}^d} P_{x} P_+ (b(x_1) \partial_{x_1} w_n) \cdot \overline{P_{x} P_+ w_n} \, dx = O(\|P_{x} w_n\|_{L^2}^2). 
\]

**Term (3.14).** For the contribution from \( a_1^2(x_1) \) in term (3.14), Item (1) in Claim 3.3 gives

\[
\text{Im} \int_{\mathbb{R}^d} P_{x} P_+ (a_1^2(x_1) \partial_{x_1} w_n) \cdot \overline{P_{x} P_+ w_n} \, dx = \text{Im} \int_{\mathbb{R}^d} \partial_{x_1} P_{x} P_+ (a_1(x_1) w_n) \cdot \overline{P_{x} P_+ (a_1(x_1) w_n)} \, dx + O(\|P_{x} w_n\|_{L^2}^2). 
\]

Then using Parseval’s identity, we write

\[
\text{Im} \int_{\mathbb{R}^d} \partial_{x_1} P_{x} P_+ (a_1(x_1) w_n) \cdot \overline{P_{x} P_+ (a_1(x_1) w_n)} \, dx = \text{Im} \int_{\mathbb{R}^d} (i \xi_1) P_{x} P_+ (a_1(x_1) w_n) \cdot \overline{P_{x} P_+ (a_1(x_1) w_n)} \, d\xi 
= \text{Re} \int_{\mathbb{R}^d} \xi_1 |P_{x} P_+ (a_1(x_1) w_n)|^2 \, d\xi. 
\tag{3.21}
\]

Now let us turn to the contribution from \( q_1(x_1) \) to (3.14). Since \( \|q_1(x)\|_{L^\infty} \leq \frac{c}{n} \), hence for \( n \) large enough,

\[ a_1^2(x_1) + \text{Re} q_1(x_1) = \tilde{a}_1^2(x_1) \geq 0. \]

Then item (2) in Claim 3.3 yields

\[
\text{Im} \int_{\mathbb{R}^d} P_{x} P_+ (\text{Im} q_1(x_1) \partial_{x_1} w_n) \cdot \overline{P_{x} P_+ w_n} \, dx = O(\|P_{x} w_n\|_{L^2}^2). 
\]

Combining (3.21) with

\[
\int (3.14) = \text{Re} \int_{\mathbb{R}^d} \xi_1 |P_{x} P_+ (a_1(x_1) w_n)|^2 \, d\xi + O(\|P_{x} w_n\|_{L^2}^2). 
\]

**Term (3.15).** Let us then take (3.15) and start with the contribution of \( a_2^2 \) term. First using the product rule, we write

\[
a_2^2 \partial_{x_1} w_n = a_2 \partial_{x_1}^2 (a_2 w_n) - a_2 (\partial_{x_1}^2 a_2) w_n - 2a_2 (\partial_{x_1} a_2) (\partial_{x_1} w_n). 
\tag{3.22}
\]

Notice that using (3.9)

\[
|\text{Im} \int_{\mathbb{R}^d} P_{x} P_+ a_2(\partial_{x_1}^2 a_2) w_n \cdot \overline{P_{x} P_+ w_n} \, dx| = O(\|P_{x} w_n\|_{L^2}^2). 
\]

Since by (3.9)

\[
\|a_2(\partial_{x_1} a_2)\|_{L^\infty} \leq \frac{c}{n},
\]

hence when \( n \) large enough it can be similarly absorbed by (3.21) without changing the sign of \( a_1^2 \) (just replace \( a_1^2 \) by a slightly different \( \tilde{a}_1^2 \)).

For the first term on the right hand side of (3.22), using (3.20) and integration by parts, we write

\[
- \text{Im} \int_{\mathbb{R}^d} P_{x} P_+ a_2(x_1) \partial_{x_1}^2 w_n \cdot \overline{P_{x} P_+ w_n} \, dx 
= - \text{Im} \int_{\mathbb{R}^d} P_{x} P_+ a_2(x_1) \partial_{x_1} (a_2(x_1) w_n) \cdot \overline{P_{x} P_+ w_n} \, dx + O(\|P_{x} w_n\|_{L^2}^2)
= - \text{Im} \int_{\mathbb{R}^d} a_2(x_1) \partial_{x_1} P_{x} P_+ \partial_{x_1} (a_2(x_1) w_n) \cdot \overline{P_{x} P_+ w_n} \, dx + O(\|P_{x} w_n\|_{L^2}^2)
= \text{Im} \int_{\mathbb{R}^d} P_{x} P_+ \partial_{x_1} (a_2(x_1) w_n) \cdot \overline{P_{x} P_+ w_n} \, dx + O(\|P_{x} w_n\|_{L^2}^2). 
\tag{3.23}
\]

Using (3.20) again, the second factor inside the integral in (3.23) can be written as

\[
\partial_{x_1} a_2(x_1) P_{x} P_+ w_n = \partial_{x_1} a_2(x_1) P_{x} P_+ w_n = \partial_{x_1} P_{x} a_2(x_1) P_{x} w_n + O(\|P_{x} w_n\|_{L^2}^2). 
\]
\[
\partial_{x_1} P_\varepsilon P_\varepsilon a_2(x_1) w_n + O\left(\frac{\|P_\varepsilon P_\varepsilon w_n\|_{L^2}^2}{n}\right) = P_\varepsilon P_\varepsilon \partial_{x_1}(a_2(x_1) w_n) + O\left(\frac{\|P_\varepsilon w_n\|_{L^2}^2}{n}\right).
\]

Here the big O notation means that \(\partial_{x_1} a_2(x_1) P_\varepsilon P_\varepsilon w_n - \partial_{x_1} P_\varepsilon a_2(x_1) P_\varepsilon w_n\) as an operator acting on \(w_n\) is bounded in \(L^2\) with norm \(O\left(\frac{1}{n}\right)\).

Therefore

\[
(3.23) = \text{Im} \int_{\mathbb{R}^d} P_\varepsilon P_\varepsilon \partial_{x_1}(a_2(x_1) w_n) \cdot \overline{P_\varepsilon P_\varepsilon \partial_{x_1}(a_2(x_1) w_n)} \, dx + O\left(\frac{\|P_\varepsilon w_n\|_{L^2}^2}{n}\right)
\]

\[
= \text{Im} \int_{\mathbb{R}^d} |P_\varepsilon P_\varepsilon \partial_{x_1}(a_2(x_1) w_n)|^2 \, dx + O\left(\frac{\|P_\varepsilon w_n\|_{L^2}^2}{n}\right) = O\left(\frac{\|P_\varepsilon w_n\|_{L^2}^2}{n}\right) + O\left(\frac{\|P_\varepsilon P_\varepsilon \partial_{x_1} w_n\|_{L^2}^2}{n}\right).
\]

Using the definition \(P_\varepsilon\) and the support of its multiplier \(|\eta| \leq \frac{1}{\varepsilon}\), we have

\[
\|P_\varepsilon P_\varepsilon \partial_{x_1} w_n\|_{L^2} = \|\eta \varepsilon \chi + \xi_1 \overline{w_n}\|_{L^2} \leq \frac{C}{\varepsilon} \|P_\varepsilon P_\varepsilon w_n\|_{L^2} \leq \frac{C}{\varepsilon} \|P_\varepsilon w_n\|_{L^2}.
\]

As a consequence, we also have

\[
\|P_\varepsilon P_\varepsilon \partial_{x_1}^2 w_n\|_{L^2} \leq \frac{C}{\varepsilon^2} \|P_\varepsilon P_\varepsilon w_n\|_{L^2} \leq \frac{C}{\varepsilon^2} \|P_\varepsilon w_n\|_{L^2}.
\]

For the contribution of \(q_2\) to (3.15), using (3.20) and integration by parts, we have

\[
- \text{Im} \int_{\mathbb{R}^d} P_\varepsilon P_\varepsilon q_2(x_1) \partial_{x_1}^2 w_n \cdot \overline{P_\varepsilon P_\varepsilon w_n} \, dx
\]

\[
= - \text{Im} \int_{\mathbb{R}^d} q_2(x_1) P_\varepsilon P_\varepsilon \partial_{x_1}^2 w_n \cdot \overline{P_\varepsilon P_\varepsilon w_n} \, dx + O\left(\frac{\|P_\varepsilon P_\varepsilon \partial_{x_1} w_n\|_{L^2}^2}{n}\right)
\]

\[
= - \text{Im} \int_{\mathbb{R}^d} q_2(x_1) \partial_{x_1} P_\varepsilon P_\varepsilon \partial_{x_1} w_n \cdot \overline{P_\varepsilon P_\varepsilon w_n} \, dx + O\left(\frac{\|P_\varepsilon w_n\|_{L^2}^2}{\varepsilon^2 n}\right)
\]

\[
= \text{Im} \int_{\mathbb{R}^d} P_\varepsilon P_\varepsilon \partial_{x_1} P_\varepsilon \varepsilon P_\varepsilon \partial_{x_1} w_n \cdot \partial_{x_1}(q_2(x_1) \overline{P_\varepsilon P_\varepsilon w_n}) \, dx + O\left(\frac{\|P_\varepsilon w_n\|_{L^2}^2}{\varepsilon^2 n}\right)
\]

\[
= - \text{Im} \int_{\mathbb{R}^d} P_\varepsilon P_\varepsilon \partial_{x_1} w_n \cdot \partial_{x_1} q_2(x_1) |P_\varepsilon P_\varepsilon P_\varepsilon |^2 P_\varepsilon w_n \, dx - 2 \text{Im} \int_{\mathbb{R}^d} P_\varepsilon P_\varepsilon \partial_{x_1} w_n \cdot (\partial_{x_1} q_2(x_1)) \overline{P_\varepsilon P_\varepsilon \partial_{x_1} w_n} \, dx
\]

\[
- \text{Im} \int_{\mathbb{R}^d} P_\varepsilon P_\varepsilon \partial_{x_1} w_n \cdot q_2(x_1) |P_\varepsilon P_\varepsilon \partial_{x_1} w_n|^2 \, dx + O\left(\frac{\|P_\varepsilon w_n\|_{L^2}^2}{\varepsilon^2 n}\right)
\]

where the first term in above is of the size

\[
\text{Im} \int_{\mathbb{R}^d} P_\varepsilon P_\varepsilon \partial_{x_1} w_n \cdot (\partial_{x_1} q_2(x_1)) |P_\varepsilon P_\varepsilon \partial_{x_1} w_n| \, dx = O\left(\frac{\|P_\varepsilon w_n\|_{L^2}^2}{n}\right)
\]

and the second term in above

\[
\text{Im} \int_{\mathbb{R}^d} P_\varepsilon P_\varepsilon \partial_{x_1} w_n \cdot (\partial_{x_1} q_2(x_1)) |P_\varepsilon P_\varepsilon \partial_{x_1} w_n| \, dx
\]

can be absorbed by (3.21) when \(n\) is large enough.

Now, for the third term in above, notice that \(\|q_2\|_{L^\infty} \leq \frac{C}{n}\), and we have

\[
|\text{Im} \int_{\mathbb{R}^d} P_\varepsilon P_\varepsilon \partial_{x_1} w_n \cdot q_2(x_1) |P_\varepsilon P_\varepsilon \partial_{x_1} w_n| \, dx| \leq \|q_2\|_{L^\infty} \|P_\varepsilon P_\varepsilon \partial_{x_1} w_n\|_{L^2}^2 \leq \frac{C}{\varepsilon^2 n} \|P_\varepsilon w_n\|_{L^2}^2.
\]

We will choose \(\varepsilon\) (depending on \(n\)) later.

Hence

\[
\int (3.15) = O\left(\frac{\|P_\varepsilon w_n\|_{L^2}^2}{n}\right) + O\left(\frac{\|P_\varepsilon w_n\|_{L^2}^2}{\varepsilon^2 n}\right)
\]
Similarly, we have
\[
O\left(\frac{\|P_{x}\partial_{x_{1}}^{3}w_{n}\|_{L_{2}^{2}}^{2}}{n}\right) = O\left(\frac{\|P_{x}w_{n}\|_{L_{2}^{2}}^{2}}{\varepsilon^{2}n}\right)
\]

**Term (3.16).** Take \(a_{2}^{2}(x_{1})\partial_{x_{1}}^{3}w_{n}\) in (3.16), and write
\[
a_{2}^{2}(x_{1})\partial_{x_{1}}^{3}w_{n} = a_{3}\partial_{x_{1}}^{3}(a_{3}(x_{1})w_{n}) - a_{3}(\partial_{x_{1}}^{3}a_{3}(x_{1}))w_{n}
\]
\[
- 3a_{3}(x_{1})((\partial_{x_{1}}^{2}a_{3}(x_{1}))(\partial_{x_{1}}w_{n}) - 3a_{3}(x_{1})(\partial_{x_{1}}a_{3}(x_{1}))((\partial_{x_{1}}^{2}w_{n})).
\]
(3.25)

Then we bound the contribution of the second term in (3.25) by
\[
|\text{Im} \int_{\mathbb{R}^{d}} P_{x}P_{x_{1}}[a_{3}(\partial_{x_{1}}^{3}a_{3}(x_{1}))w_{n}] \cdot \overline{P_{x}P_{x_{1}}w_{n}} dx| = O\left(\frac{\|P_{x}w_{n}\|_{L_{2}^{2}}^{2}}{n}\right)
\]
and have (3.21) absorb the contribution of the third term in (3.25)
\[
\text{Im} \int_{\mathbb{R}^{d}} P_{x}P_{x_{1}}[a_{3}(x_{1})(\partial_{x_{1}}^{2}a_{3}(x_{1}))(\partial_{x_{1}}w_{n})] \cdot \overline{P_{x}P_{x_{1}}w_{n}} dx.
\]

Using the same calculation in (3.15) and (3.24), we have the bound for the contribution of the fourth term in (3.25)
\[
|\text{Im} \int_{\mathbb{R}^{d}} P_{x}P_{x_{1}}[a_{3}(x_{1})(\partial_{x_{1}}a_{3}(x_{1}))(\partial_{x_{1}}^{2}w_{n})] \cdot \overline{P_{x}P_{x_{1}}w_{n}} dx| = O\left(\frac{\|P_{x}w_{n}\|_{L_{2}^{2}}^{2}}{n}\right) + O\left(\frac{\|P_{x}w_{n}\|_{L_{2}^{2}}^{2}}{\varepsilon^{2}n}\right)
\]

Now we only need to control the contribution of the first term in (3.25), that is,
\[
\text{Im} \int_{\mathbb{R}^{d}} P_{x}P_{x_{1}}a_{3}^{2}(x_{1})\partial_{x_{1}}^{3}w_{n} \cdot P_{x}P_{x_{1}}w_{n} dx
\]
\[
= \text{Im} \int_{\mathbb{R}^{d}} P_{x}P_{x_{1}}[a_{3}(x_{1})(\partial_{x_{1}}^{2}a_{3}(x_{1}))w_{n}] \cdot P_{x}P_{x_{1}}w_{n} dx + O\left(\frac{\|P_{x}w_{n}\|_{L_{2}^{2}}^{2}}{n}\right) + O\left(\frac{\|P_{x}w_{n}\|_{L_{2}^{2}}^{2}}{\varepsilon^{2}n}\right)
\]
\[
= \text{Im} \int_{\mathbb{R}^{d}} a_{3}(x_{1})P_{x}P_{x_{1}}\partial_{x_{1}}^{2}(a_{3}(x_{1}))w_{n} \cdot P_{x}P_{x_{1}}w_{n} dx + O\left(\frac{\|P_{x}w_{n}\|_{L_{2}^{2}}^{2}}{n}\right) + O\left(\frac{\|P_{x}w_{n}\|_{L_{2}^{2}}^{2}}{\varepsilon^{2}n}\right) + O\left(\frac{\|P_{x}\partial_{x_{1}}^{2}w_{n}\|_{L_{2}^{2}}^{2}}{n}\right)
\]
\[
= - \text{Im} \int_{\mathbb{R}^{d}} P_{x}P_{x_{1}}\partial_{x_{1}}^{2}(a_{3}(x_{1}))w_{n} \cdot \partial_{x_{1},a_{3}(x_{1})}P_{x}P_{x_{1}}w_{n} dx + O\left(\frac{\|P_{x}w_{n}\|_{L_{2}^{2}}^{2}}{n}\right) + O\left(\frac{\|P_{x}w_{n}\|_{L_{2}^{2}}^{2}}{\varepsilon^{2}n}\right) + O\left(\frac{\|P_{x}\partial_{x_{1}}^{2}w_{n}\|_{L_{2}^{2}}^{2}}{n}\right)
\]
\[
= - \text{Im} \int_{\mathbb{R}^{d}} P_{x}P_{x_{1}}\partial_{x_{1}}^{2}(a_{3}(x_{1}))w_{n} \cdot \partial_{x_{1},a_{3}(x_{1})}P_{x}P_{x_{1}}w_{n} dx + O\left(\frac{\|P_{x}w_{n}\|_{L_{2}^{2}}^{2}}{n}\right) + O\left(\frac{\|P_{x}w_{n}\|_{L_{2}^{2}}^{2}}{\varepsilon^{2}n}\right) + O\left(\frac{\|P_{x}\partial_{x_{1}}^{2}w_{n}\|_{L_{2}^{2}}^{2}}{n}\right)
\]
\[
= - \text{Im} \int_{\mathbb{R}^{d}} P_{x}P_{x_{1}}\partial_{x_{1}}^{2}(a_{3}(x_{1}))w_{n} \cdot \partial_{x_{1},a_{3}(x_{1})}P_{x_{1}}P_{x_{1}}(a_{3}(x_{1}))w_{n} dx + O\left(\frac{\|P_{x}w_{n}\|_{L_{2}^{2}}^{2}}{n}\right) + O\left(\frac{\|P_{x}w_{n}\|_{L_{2}^{2}}^{2}}{\varepsilon^{2}n}\right) + O\left(\frac{\|P_{x}\partial_{x_{1}}^{2}w_{n}\|_{L_{2}^{2}}^{2}}{n}\right)
\]
\[
= \text{Im} \int_{\mathbb{R}^{d}} P_{x}P_{x_{1}}(a_{3}(x_{1})w_{n}) \cdot \partial_{x_{1}}^{3}P_{x_{1}}P_{x_{1}}(a_{3}(x_{1})w_{n}) dx + O\left(\frac{\|P_{x}w_{n}\|_{L_{2}^{2}}^{2}}{n}\right) + O\left(\frac{\|P_{x}w_{n}\|_{L_{2}^{2}}^{2}}{\varepsilon^{2}n}\right) + O\left(\frac{\|P_{x}\partial_{x_{1}}^{2}w_{n}\|_{L_{2}^{2}}^{2}}{n}\right)
\]
\[
= \text{Im} \int_{\mathbb{R}^{d}} P_{x}P_{x_{1}}(a_{3}(x_{1})w_{n}) \cdot (i\xi_{1})^{3}P_{x_{1}}P_{x_{1}}(a_{3}(x_{1})w_{n}) d\xi + O\left(\frac{\|P_{x}w_{n}\|_{L_{2}^{2}}^{2}}{n}\right) + O\left(\frac{\|P_{x}w_{n}\|_{L_{2}^{2}}^{2}}{\varepsilon^{2}n}\right) + O\left(\frac{\|P_{x}\partial_{x_{1}}^{2}w_{n}\|_{L_{2}^{2}}^{2}}{n}\right)
\]
\[
= - \text{Re} \int_{\mathbb{R}^{d}} \xi_{1}^{3}\|P_{x}P_{x_{1}}(a_{3}(x_{1})w_{n})\|_{L_{2}^{2}}^{2} d\xi + O\left(\frac{\|P_{x}w_{n}\|_{L_{2}^{2}}^{2}}{n}\right) + O\left(\frac{\|P_{x}w_{n}\|_{L_{2}^{2}}^{2}}{\varepsilon^{2}n}\right) + O\left(\frac{\|P_{x}\partial_{x_{1}}^{2}w_{n}\|_{L_{2}^{2}}^{2}}{n}\right).
\]

**Term (3.17).** Similarly, we compute
\[
\int_{\mathbb{R}^{d}} P_{x}P_{x_{1}}b(x_{1})\partial_{x_{1}}^{3}w_{n} \cdot \overline{P_{x}P_{x_{1}}w_{n}} dx
\]
\[
= \int_{\mathbb{R}^{d}} b(x_{1})P_{x}P_{x_{1}}\partial_{x_{1}}^{3}w_{n} \cdot \overline{P_{x}P_{x_{1}}w_{n}} dx + O\left(\frac{\|P_{x}\partial_{x_{1}}^{2}w_{n}\|_{L_{2}^{2}}^{2}}{n}\right)
\]
\[ \begin{align*}
&= \int_{\mathbb{R}^d} b(x_1) \partial^3_{x_1} P_{\varepsilon} P_{\varepsilon}^+ w_n \cdot P_{\varepsilon} P_{\varepsilon}^+ w_n \, dx + \mathcal{O}\left( \frac{\| P_{\varepsilon} w_n \|_{L^2}^2}{\varepsilon^4 n} \right) \\
&= - \int_{\mathbb{R}^d} \partial^2_{x_1} P_{\varepsilon} P_{\varepsilon}^+ w_n \cdot \partial_x b(x_1) P_{\varepsilon} P_{\varepsilon}^+ w_n \, dx + \mathcal{O}\left( \frac{\| P_{\varepsilon} w_n \|_{L^2}^2}{\varepsilon^4 n} \right) \\
&= \int_{\mathbb{R}^d} P_{\varepsilon} P_{\varepsilon}^+ w_n \cdot \partial^3_{x_1} P_{\varepsilon} P_{\varepsilon}^+ (b(x_1) w_n) \, dx + \mathcal{O}\left( \frac{\| P_{\varepsilon} w_n \|_{L^2}^2}{\varepsilon^4 n} \right) \\
&= - \int_{\mathbb{R}^d} P_{\varepsilon} P_{\varepsilon}^+ w_n \cdot \partial^3_{x_1} P_{\varepsilon} P_{\varepsilon}^+ (b(x_1) w_n) \, dx + \mathcal{O}\left( \frac{\| P_{\varepsilon} w_n \|_{L^2}^2}{\varepsilon^4 n} \right).
\end{align*} \]

Then

\[ \text{Re} \int_{\mathbb{R}^d} P_{\varepsilon} P_{\varepsilon}^+ b(x_1) \partial^3_{x_1} w_n \cdot P_{\varepsilon} P_{\varepsilon}^+ w_n \, dx = \mathcal{O}\left( \frac{\| P_{\varepsilon} w_n \|_{L^2}^2}{n} \right) + \mathcal{O}\left( \frac{\| P_{\varepsilon} w_n \|_{L^2}^2}{\varepsilon^4 n} \right) \]

Finally, (3.12) becomes

\[ \partial_t |P_{\varepsilon} P_{\varepsilon}^+ w_n|^2 = \text{Re} \int_{\mathbb{R}^d} \xi_1 |P_{\varepsilon} P_{\varepsilon}^+ (\overline{a_1(x_1) w_n})|^2 \, d\xi - \text{Re} \int_{\mathbb{R}^d} \xi_1^3 |P_{\varepsilon} P_{\varepsilon}^+ (a_3(x_1) w_n)|^2 \, d\xi + c \| V \|_{L^\infty} \| P_{\varepsilon} w_n \|_{L^2}^2 + c \| e^{i \xi \cdot \Phi_n (x_1)} H \|_{L^2} \| P_{\varepsilon} w_n \|_{L^2} \\
+ \mathcal{O}\left( \frac{\| P_{\varepsilon} w_n \|_{L^2}^2}{\varepsilon^4 n} \right) + \mathcal{O}\left( \frac{\| P_{\varepsilon} w_n \|_{L^2}^2}{\varepsilon^4 n} \right) \]

where \( a_1^2(x_1) = 4(\beta \varphi_1^2) \), \( a_1^2(x_1) = 4(\beta \varphi_1^2) \), and \( a_3^2(x_1) = 4 \beta \varphi_n^3 \).

Since \( a_1^2(x_1) \) and \( a_1^2(x_1) \) are close enough, we consider the error term of the following form

\[ \text{Re} \int_{\mathbb{R}^d} \xi_1 |P_{\varepsilon} P_{\varepsilon}^+ (\overline{a_1(x_1) w_n})|^2 \, d\xi - \text{Re} \int_{\mathbb{R}^d} \xi_1^3 |P_{\varepsilon} P_{\varepsilon}^+ (a_3(x_1) w_n)|^2 \, d\xi \]

\[ = 4 \text{Re} \int_{\mathbb{R}^d} \xi_1 \beta^3 |P_{\varepsilon} P_{\varepsilon}^+ (\varphi_n^2 w_n)|^2 - \xi_1^3 \beta |P_{\varepsilon} P_{\varepsilon}^+ (\varphi_n w_n)|^2 \, d\xi \]

\[ = 4 \text{Re} \int_{\mathbb{R}^d} \xi_1 \beta^3 |P_{\varepsilon} P_{\varepsilon}^+ (\varphi_n^2 w_n)|^2 - \beta^3 |P_{\varepsilon} P_{\varepsilon}^+ (\varphi_n w_n)|^2 \, d\xi + \text{Re} \int_{\mathbb{R}^d} (\xi_1 \beta^3 - \xi_1^3 \beta) |P_{\varepsilon} P_{\varepsilon}^+ (\varphi_n w_n)|^2 \, d\xi \]

The second term above is negative under the definition of the projection \( P_{\varepsilon}^+ \).

Next, we focus on the error term

\[ |\text{Re} \int_{\mathbb{R}^d} \xi_1 \beta^3 (|P_{\varepsilon} P_{\varepsilon}^+ (\varphi_n^3 w_n)|^2 - |P_{\varepsilon} P_{\varepsilon}^+ (\varphi_n w_n)|^2) \, d\xi| \]

\[ \leq |\text{Re} \int_{\mathbb{R}^d} \xi_1 \beta^3 (|P_{\varepsilon} P_{\varepsilon}^+ (\varphi_n^3 w_n)|^2 - |P_{\varepsilon} P_{\varepsilon}^+ (\varphi_n w_n)|^2) \, d\xi| + |\text{Re} \int_{\mathbb{R}^d} \xi_1 \beta^3 (|P_{\varepsilon} P_{\varepsilon}^+ (\varphi_n w_n)|^2 - |P_{\varepsilon} P_{\varepsilon}^+ (\varphi_n w_n)|^2) \, d\xi| \]

where \( P_{\varepsilon} P_{\varepsilon}^+ (\varphi_n^3 w_n) \rightarrow P_{\varepsilon} P_{\varepsilon} w \), with \( w = e^{-i \beta x_1} e^{i \beta x_1} u \) in \( L^2 \), hence \( |\xi_1| |\gamma P_{\varepsilon} P_{\varepsilon}^+ w_n \rightarrow |\xi_1| |\gamma P_{\varepsilon} P_{\varepsilon}^+ w_n | \) in \( L^2 \). Then

\[ \int_{\mathbb{R}^d} \xi_1 |P_{\varepsilon} P_{\varepsilon}^+ (\varphi_n^3 w_n)|^2 = \int_{\mathbb{R}^d} \xi_1 |P_{\varepsilon} P_{\varepsilon}^+ w|^2 + o(1) \]

\[ \int_{\mathbb{R}^d} \xi_1 |P_{\varepsilon} P_{\varepsilon}^+ (\varphi_n w_n)|^2 = \int_{\mathbb{R}^d} \xi_1 |P_{\varepsilon} P_{\varepsilon}^+ w|^2 + o(1) \]

and

\[ \text{Re} \int_{\mathbb{R}^d} \xi_1 \beta^3 (|P_{\varepsilon} P_{\varepsilon}^+ (\varphi_n w_n)|^2 - |P_{\varepsilon} P_{\varepsilon}^+ (\varphi_n w_n)|^2) \, d\xi = o(1) \]
Therefore
\[ \text{Re} \int_{\mathbb{R}^d} \xi_1 |P_x P_+ (a_1(x) w_n)|^2 \, dx - \text{Re} \int_{\mathbb{R}^d} \xi_1^3 |P_x P_+ (a_3(x) w_n)|^2 \, dx = o(1) \]

Combining all the computation above, we have
\[
\frac{\partial}{\partial t} \int_{\mathbb{R}^d} |P_x P_+ w_n|^2 \, dx \leq c \|e^{\mu} \Phi_n(x_1) H\|_{L_2^2} \|P_x w_n\|_{L_2^2} + c \|V\|_{L_\infty^2} \|P_x w_n\|_{L_2^2}^2
\]
\[
+ O \left( \frac{\|P_x w_n\|_{L_2^2}^2}{\varepsilon n} \right) + O \left( \frac{\|P_x w_n\|_{L_2^2}^2}{\varepsilon^2 n} \right) + O \left( \frac{\|P_x w_n\|_{L_2^2}^2}{\varepsilon^4 n} \right) + o(1)
\]
\[
\leq c \|e^{\mu} \Phi_n(x_1) H\|_{L_2^2} \|P_x w_n\|_{L_2^2} + c \|V\|_{L_\infty^2} \|P_x w_n\|_{L_2^2}^2 + O \left( \frac{\|P_x w_n\|_{L_2^2}^2}{\varepsilon^4 n} \right) + o(1) \quad (3.26)
\]

Arguing similarly for \( P_- \), we obtain
\[
\frac{\partial}{\partial t} \int_{\mathbb{R}^d} |P_x P_- w_n|^2 \, dx \geq -c \|e^{\mu} \Phi_n(x_1) H\|_{L_2^2} \|P_x w_n\|_{L_2^2} - c \|V\|_{L_\infty^2} \|P_x w_n\|_{L_2^2}^2
\]
\[
+ O \left( \frac{\|P_x w_n\|_{L_2^2}^2}{\varepsilon n} \right) + O \left( \frac{\|P_x w_n\|_{L_2^2}^2}{\varepsilon^2 n} \right) + O \left( \frac{\|P_x w_n\|_{L_2^2}^2}{\varepsilon^4 n} \right) + o(1)
\]
\[
\geq -c \|e^{\mu} \Phi_n(x_1) H\|_{L_2^2} \|P_x w_n\|_{L_2^2} - c \|V\|_{L_\infty^2} \|P_x w_n\|_{L_2^2}^2 + O \left( \frac{\|P_x w_n\|_{L_2^2}^2}{\varepsilon^4 n} \right) + o(1) \quad (3.27)
\]

**Step 4: Estimating \( L^2 \) norm of \( w_n \).** By the definition of the supremum, there exists \( t_n \in [0, 1] \) such that
\[
\|P_x w_n(t_n, \cdot)\|_{L_2^2}^2 \geq \frac{1}{2} \sup_{t \in [0, 1]} \|P_x w_n(t, \cdot)\|_{L_2^2}^2 \quad (3.28)
\]

when we choose \( \varepsilon = \frac{n}{\sqrt{t_n}} \).

Now the fundamental theorem of calculus in time \( t \) applying on (3.26) and (3.27) yields
\[
\|P_x P_+ w_n(1, \cdot)\|_{L_2^2}^2 - \|P_x P_+ w_n(t_n, \cdot)\|_{L_2^2}^2
\]
\[
\geq - \int_{t_n}^{t_n} \|V\|_{L_\infty^2} \|P_x w_n\|_{L_2^2} + \int_{t_n}^{1} \|e^{\mu} \Phi_n H\|_{L_2^2} \|P_x w_n\|_{L_2^2} \, dt + O \left( \frac{\|P_x w_n\|_{L_2^2}^2}{\varepsilon^4 n} \right) + o(1)
\]
\[
\|P_x P_- w_n(t_n, \cdot)\|_{L_2^2}^2 - \|P_x P_- w_n(0, \cdot)\|_{L_2^2}^2
\]
\[
\leq \int_{0}^{t_n} \|V\|_{L_\infty^2} \|P_x w_n\|_{L_2^2} + \int_{0}^{t_n} \|e^{\mu} \Phi_n H\|_{L_2^2} \|P_x w_n\|_{L_2^2} \, dt + O \left( \frac{\|P_x w_n\|_{L_2^2}^2}{\varepsilon^4 n} \right) + o(1). \quad (3.29)
\]

Then
\[
\|P_x w_n(t_n, \cdot)\|_{L_2^2}^2 = \|P_x P_+ w_n(t_n, \cdot)\|_{L_2^2}^2 + \|P_x P_- w_n(t_n, \cdot)\|_{L_2^2}^2
\]
\[
\leq \|P_x P_+ w_n(1, \cdot)\|_{L_2^2}^2 + \int_{t_n}^{1} \|V\|_{L_\infty^2} \|P_x w_n\|_{L_2^2}^2 \, dt + \int_{t_n}^{1} \|e^{\mu} \Phi_n H\|_{L_2^2} \|P_x w_n\|_{L_2^2} \, dt
\]
\[
+ \|P_x P_- w_n(0, \cdot)\|_{L_2^2}^2 + \int_{0}^{t_n} \|V\|_{L_\infty^2} \|P_x w_n\|_{L_2^2}^2 \, dt + \int_{0}^{t_n} \|e^{\mu} \Phi_n H\|_{L_2^2} \|P_x w_n\|_{L_2^2} \, dt
\]
\[
+ O \left( \frac{\|P_x w_n\|_{L_2^2}^2}{\varepsilon^4 n} \right) + o(1)
\]
\[
\leq \|P_x w_n(1, \cdot)\|_{L_2^2}^2 + \|P_x w_n(0, \cdot)\|_{L_2^2}^2 + \int_{0}^{1} \|V\|_{L_\infty^2} \|P_x w_n\|_{L_2^2}^2 \, dt + \int_{0}^{1} \|e^{\mu} \Phi_n H\|_{L_2^2} \|P_x w_n\|_{L_2^2} \, dt
\]
\[
+ \frac{c}{\varepsilon^4 n} \|P_x w_n\|_{L_2^2}^2 + o(1). \quad (3.29)
\]
Since we chose \( \varepsilon = n^{-\frac{1}{4n}} \), for \( n \) large enough, we will have
\[
1 - \frac{c}{\varepsilon^4} = 1 - \frac{c}{n^{\frac{1}{8}}} > \frac{1}{2}.
\]

Then let the left-hand side of (3.29) absorb the \( \frac{c}{\varepsilon^4} \|P_n w_n\|_{L^2}^2 \) term, and we write
\[
\|P_n w_n(t_n, \cdot)\|_{L^2}^2 \leq 2 \|P_n w_n(1, \cdot)\|_{L^2}^2 + 2 \|P_n w_n(0, \cdot)\|_{L^2}^2 + 2 \int_0^1 \|V\|_{L^\infty} \|P_n w_n\|_{L^2} dt + 2 \int_0^1 \|e^\mu \Phi_n H\|_{L^2} \|P_n w_n\|_{L^2} dt
\]
\[
\leq \|P_n w_n(t_n, \cdot)\|_{L^2}^2 \leq 2 \|P_n w_n(1, \cdot)\|_{L^2}^2 + 2 \|P_n w_n(0, \cdot)\|_{L^2}^2
\]
\[
+ 2 \sup_{t \in [0, 1]} \|P_n w_n\|_{L^2}^2 \int_0^1 \|V\|_{L^\infty} dt + \frac{1}{100} \sup_{t \in [0, 1]} \|P_n w_n\|_{L^2}^2 + 100(\int_0^1 \|e^\mu \Phi_n H\|_{L^2} dt)^2.
\]

By choosing \( \varepsilon_0 \) in (3.1) such that
\[
\int_0^1 \|V\|_{L^\infty} dt < \frac{1}{100}
\]
we have
\[
\|P_n w_n(t_n, \cdot)\|_{L^2}^2 \leq 2 \|P_n w_n(1, \cdot)\|_{L^2}^2 + 2 \|P_n w_n(0, \cdot)\|_{L^2}^2
\]
\[
+ \frac{1}{50} \sup_{t \in [0, 1]} \|P_n w_n\|_{L^2}^2 + \frac{1}{100} \sup_{t \in [0, 1]} \|P_n w_n\|_{L^2}^2 + 100(\int_0^1 \|e^\mu \Phi_n H\|_{L^2} dt)^2
\]
\[
\leq 2 \|P_n w_n(1, \cdot)\|_{L^2}^2 + 2 \|P_n w_n(0, \cdot)\|_{L^2}^2 + \frac{1}{10} \sup_{t \in [0, 1]} \|P_n w_n\|_{L^2}^2 + 100(\int_0^1 \|e^\mu \Phi_n H\|_{L^2} dt)^2.
\]

Using (3.28), we write
\[
\frac{1}{2} \sup_{t \in [0, 1]} \|P_n w_n(t, \cdot)\|_{L^2}^2 \leq \|P_n w_n(t_n, \cdot)\|_{L^2}^2
\]
\[
\leq 2 \|P_n w_n(1, \cdot)\|_{L^2}^2 + 2 \|P_n w_n(0, \cdot)\|_{L^2}^2 + \frac{3}{100} \sup_{t \in [0, 1]} \|P_n w_n\|_{L^2}^2 + 100(\int_0^1 \|e^\mu \Phi_n H\|_{L^2} dt)^2
\]
then
\[
\sup_{t \in [0, 1]} \|P_n w_n(t, \cdot)\|_{L^2}^2 \leq 10 \|P_n w_n(1, \cdot)\|_{L^2}^2 + 10 \|P_n w_n(0, \cdot)\|_{L^2}^2 + 1000(\int_0^1 \|e^\mu \Phi_n H\|_{L^2} dt)^2.
\]

Notice that we chose \( \varepsilon = n^{-\frac{1}{4n}} \) and now we only have one limit in \( n \) to take.

Letting \( n \to \infty \), we obtain the desired inequality,
\[
\sup_{t \in [0, 1]} \|u(t)\|_{L^2(x^{2a_1} dx)}^2 \lesssim \|u_1\|_{L^2(x^{2a_1} dx)}^2 + \|u_0\|_{L^2(x^{2a_1} dx)}^2 + \|H\|_{L^1(x^{2a_1} dx)}^2 L^2(x^{2a_1} dx).
\]

Now we complete the proof of Lemma 3.1. \( \square \)

4. **Upgraded Exponential Decay Estimate**

In this section, we prove an interior estimate for the rapidly decaying solutions, and upgrade it to a super-linear exponential decrease estimate.
4.1. Linear exponential decay estimate.

Lemma 4.1 (Linear exponential decay estimate in all directions). If in addition to the hypothesis in Lemma 3.1 one has that for some $\beta > 0$

$$u_0, u_1 \in L^2_x(e^{2\beta|x|} \, dx)$$

and $H \in L^1_t([0,1] : L^2_x(e^{2\beta|x|} \, dx))$, then

$$\sup_{t\in[0,1]} \|u(t)\|^2_{L^2_x(e^{2\beta|x|}/d) \, dx} \leq C(d)(\|u_0\|^2_{L^2_x(e^{2\beta|x|} \, dx)} + \|u_1\|^2_{L^2_x(e^{2\beta|x|} \, dx)} + \|H\|^2_{L^2_tL^2_x(e^{2\beta|x|} \, dx)})$$

with $C(d)$ independent of $\beta > 0$.

Proof of Lemma 4.1. By Lemma 3.1, we have that for any $\beta > 0$,

$$\sup_{t\in[0,1]} \|u(t)\|^2_{L^2_x(e^{2\beta|x|} \, dx)} \leq C(\|u_0\|^2_{L^2_x(e^{2\beta|x|} \, dx)} + \|u_1\|^2_{L^2_x(e^{2\beta|x|} \, dx)} + \|H\|^2_{L^2_tL^2_x(e^{2\beta|x|} \, dx)}) := \Phi,$$

for any $j = 1, \ldots, d$. Hence, for any $t \in [0,1]$,

$$\|u(t)\|^2_{L^2_x(e^{2\beta|x|}/d) \, dx} = \int_{\mathbb{R}^d} |u(t,x)|^2 e^{2\beta|x|}/d \, dx \leq \int_{\mathbb{R}^d} |u(x)|^2 e^{2\beta \sum_i |x_i|/d} \, dx$$

$$= \int_{\mathbb{R}^d} \prod_j \left( |u|^{2\beta} e^{2\beta|x_j|/d} \right) \, dx \leq \prod_j \left( \int_{\mathbb{R}^d} |u(x)|^2 e^{2\beta|x_j|} \, dx \right)^{\frac{1}{\beta}}$$

$$\leq \sum_j \int_{\mathbb{R}^d} |u(x)|^2 e^{2\beta|x_j|} \, dx = \sum_j \|u\|^2_{L^2_x(e^{2\beta|x_j|} \, dx)} \leq C(d) \Phi$$

$$= C(d)(\|u_0\|^2_{L^2_x(e^{2\beta|x|} \, dx)} + \|u_1\|^2_{L^2_x(e^{2\beta|x|} \, dx)} + \|H\|^2_{L^2_tL^2_x(e^{2\beta|x|} \, dx)}).$$

This completes the proof of Lemma 4.1. \qed

4.2. Super-linear exponential decrease.

Lemma 4.2 (Super-linear exponential decay estimate). In addition to the hypotheses of Lemma 3.1, we assume that for some $\lambda > 0$, and $\alpha > 1$, $u_0, u_1 \in L^2(e^{\lambda|x|^\alpha} \, dx)$. Additionally, let $u \in C^1([0,1] : H^1(\mathbb{R}^d))$, then, there exists $c_\alpha > 0$ such that

$$\sup_{t\in[0,1]} \int_{|x| \geq c_\alpha} |u(t,x)|^2 e^{\lambda|x|^\alpha/(10d)^\alpha} \, dx \lesssim \|u_0\|^2_{L^2_x(e^{\lambda|x|\alpha} \, dx)} + \|u_1\|^2_{L^2_x(e^{\lambda|x|\alpha} \, dx)}$$

$$+ \int_0^1 \int_{\mathbb{R}^d} |H(t,x)|^2 e^{\lambda|x|^\alpha} \, dx \, dt + \sum_{j=1}^d \sum_{l=0}^{\min(d,3)} \int_0^1 \int_{\mathbb{R}^d} |\partial_{x_j} u(t,x)|^{2\alpha} \, dx \, dt.$$

We note that the factor 10 on the left-hand side of the inequality is not essential, and it suffices for it to be slightly greater than 2.

Proof of Lemma 4.2. Let $\eta(x) \in C^\infty$ be non-decreasing, radial and such that

$$\eta(x) = \begin{cases} 0, & \text{if } |x| \leq 1, \\ 1, & \text{if } |x| \geq 2. \end{cases}$$

We also define $\eta_R(x) = \eta(x/R)$.

Let $u_R(t,x) = \eta_R(x)u(t,x)$, where $\eta_R(x)$ defined above. Then using (2.1), we have

$$(i \partial_t + \Delta^2) u_R = Vu_R + \bar{H}_R,$$

where

$$\bar{H}_R = \eta_R H - 4 \sum_{j=1}^d (\partial_{x_j} \eta_R) \partial_{x_j}^2 u - 6 \sum_{j=1}^d (\partial_{x_j}^2 \eta_R) \partial_{x_j}^2 u - 4 \sum_{j=1}^d (\partial_{x_j}^3 \eta_R) \partial_{x_j} u - \sum_{j=1}^d (\partial_{x_j}^4 \eta_R) u.$$


We now use Lemma 3.1 to conclude that
\[ \|u_R(t)\|^2_{L^2(e^{2\beta|x|}/dx)} \lesssim \|u_R(0)\|^2_{L^2(e^{2\beta|x|}/dx)} + \|u_R(1)\|^2_{L^2(e^{2\beta|x|}/dx)} + \|\tilde{H}_R\|^2_{L^1_t L^2_x(e^{2\beta|x|}/dx)}. \] (4.1)

Hence, using the definition of \( \eta_R \) and (4.1), we have
\[
\int_{|x|>2R} |u(t,x)|^{2e^{2\beta|x|}/d} \, dx \leq \int_{R^d} |u_R(t,x)|^{2e^{2\beta|x|}/d} \, dx
\]
\[
\lesssim \sum_{k=0}^d \int_{|x|>R} |u_{k,R}|^{2e^{2\beta|x|}} \, dx + \int_0^1 \int_{|x|>R} |H(t,x)|^{2e^{2\beta|x|}} \, dx \, dt
\]
\[
+ \sum_{j=1}^d \int_{R<x|<2R} e^{2\beta|x|}(|u|^2 R^{-8} + |\partial_x u|^2 R^{-6} + |\partial^2_x u|^2 R^{-4} + |\partial^3_x u|^2 R^{-2}) \, dx
\]
\[
\lesssim \sum_{k=0}^d \int_{|x|>R} |u_{k,R}|^{2e^{2\beta|x|}} \, dx + \int_0^1 \int_{|x|>R} |H(t,x)|^{2e^{2\beta|x|}} \, dx \, dt
\]
\[
+ \sum_{j=1}^d \int_{R<x|<2R} (|u|^2 R^{-8} + |\partial_x u|^2 R^{-6} + |\partial^2_x u|^2 R^{-4} + |\partial^3_x u|^2 R^{-2}) \, dx
\]
where \( u_{0,R} = u_R(0,0) \), and \( u_{1,R} = u_R(1,0) \). By multiplying the inequality above by \( e^{-4\beta R} \), we write
\[ A := e^{-4\beta R} \int_{|x|>2R} |u(t,x)|^{2e^{2\beta|x|}/d} \, dx \] (4.2)
\[
\lesssim e^{-4\beta R} \sum_{k=0}^d \int_{|x|>R} |u_{k,R}|^{2e^{2\beta|x|}} \, dx + e^{-4\beta R} \int_0^1 \int_{|x|>R} |H(t,x)|^{2e^{2\beta|x|}} \, dx \, dt
\]
\[
+ \sum_{j=1}^d \int_{R<x|<2R} (|u|^2 R^{-8} + |\partial_x u|^2 R^{-6} + |\partial^2_x u|^2 R^{-4} + |\partial^3_x u|^2 R^{-2}) \, dx
\]
\[ =: D_1 + D_2 + D_3. \] (4.3)

We fix \( 4\beta R = 2bR^\alpha \). Integrating the inequality (4.2) in \( R \) in the interval \([0, \infty)\), and consider the resulting terms separately. Using Fubini’s theorem, the \( D_1 \) term can be written as
\[
\int_0^\infty e^{-2bR^\alpha} \sum_{k=0}^d \int_{|x|>R} |u_j(x)|^{2e^{2\beta|x|}} \, dx \, dR = \sum_{j=0}^d \int_{|x|>R} \left( \int_1^\infty e^{-2bR^\alpha+2bR^\alpha-1} \, dR \right) |u_j(x)|^{2} \, dx,
\]
where \( r = |x| \).

To deduce an upper bound for this expression we see that \( \varphi(R) = bR^\alpha-1(r - 2R) \) has its maximum at
\[ R_M = \frac{(\alpha - 1)r}{2\alpha} < \frac{r}{2}, \]
hence
\[
\int_1^r e^{-2bR^\alpha+bR^\alpha-1} \, dR \leq re^{\varphi(R_M)} = re^{b(\alpha-1)^{\alpha-1}r/(2^{\alpha-1} \alpha)} = re^{b_\alpha r^\alpha} = |x|e^{b_\alpha |x|^\alpha},
\]
that is
\[ b_\alpha = b(\alpha - 1)^{\alpha-1}/(2^{\alpha-1} \alpha). \]

This estimate yields the bound
\[
\sum_{j=0}^d \int_{R|x|}^1 |u_j(x)|^{2e^{2\beta|x|}} \, dx.
\]
A similar argument provides the following upper bound for the term coming from \( D_2 \) in (4.3)
\[
\int_0^1 \int_{R^d} |H(t,x)|^{2e^{2\beta|x|}} \, dx \, dt.
\]
Next, we shall deduce a lower bound for the term arising from $A$. Using again Fubini’s theorem this can be written as
\[
\int_1^\infty e^{-2bR^a} \int_{|x| > 2R} |u(t,x)|^2 e^{2\beta|x|/d} \, dx \, dR = 2 \int_2^\infty \int_{S^{d-1}} \left( \int_1^2 e^{-2bR^a+bR^a-r/d} \, dR \right) |u(t,x)|^2 r^{d-1} \, dSdr.
\]
Since $\eta(R) = -2bR^a + bR^a-1r/d = bR^a-1(r/d - 2R)$ has its maximum at
\[
\tilde{R}_M = \frac{(\alpha - 1)r}{2ad} < \frac{r}{2d} \leq \frac{r}{2},
\]
we take $R_0 = \frac{(\alpha - 1)r}{10ad}$, $r > c_a$ and $c_a > \frac{10ad}{\alpha - 1} > 2$ to bound from below the integral as
\[
\int_1^2 e^{-2bR^a+bR^a-1r/d} \, dR \geq \int_{R_0}^{\tilde{R}_M} e^{bR^a-1(r/d-2R)} \, dR \geq e^{bR_0^{-1}(r/d-2R_0)}(\tilde{R}_M - R_0) \geq e^{bR_0^{-1}(r/d-2\tilde{R}_M)}(\tilde{R}_M - R_0) \geq \frac{2}{5} \frac{\alpha - 1}{d} e^{b(\alpha - 1)r^\alpha/10^{\alpha-1}\alpha^\alpha d^\alpha} = \frac{2}{5} \frac{\alpha - 1}{d} e^{b(\alpha - 1)r^\alpha/10^{\alpha-1}\alpha^\alpha d^\alpha}.
\]
This last expression is thus a lower bound for the exponential part of the integrand on the left hand side of (4.2).

Fixing $b$ such that $b_a + \varepsilon = b(\alpha - 1)^{\alpha-1}/(2^{\alpha-1}\alpha^\alpha) + \varepsilon = \lambda$ with $\varepsilon > 0$ small enough.

To bound the $D_3$ term, we compute for $R \gg 1$
\[
\sum_{j=1}^d \int_0^1 \int_{R < |x| < 2R} \left( |u|^2 R^{-8} + |\partial_{x_j} u|^2 R^{-6} + |\partial_{x_j}^2 u|^2 R^{-4} + |\partial_{x_j}^3 u|^2 R^{-2} \right) \, dx \, dt \, dR
\leq \sum_{j=1}^d \int_0^1 \int_{0 < |x| < 2R} \left( |u|^2 R^{-2} + |\partial_{x_j} u|^2 R^{-2} + |\partial_{x_j}^2 u|^2 R^{-2} + |\partial_{x_j}^3 u|^2 R^{-2} \right) \, dx \, dt \, dR
= \sum_{j=1}^d \int_0^1 \int_{|x| \geq 1} \left( |u|^2 + |\partial_{x_j} u|^2 + |\partial_{x_j}^2 u|^2 + |\partial_{x_j}^3 u|^2 \right) \left( \int_{|x|}^{2R} \frac{dR}{R^2} \right) \, dx \, dt
\leq \sum_{j=1}^d \int_0^1 \int_{|x| \geq 1} \left( |u|^2 + |\partial_{x_j} u|^2 + |\partial_{x_j}^2 u|^2 + |\partial_{x_j}^3 u|^2 \right) \, dx \, dt.
\]
Hence, we obtain that
\[
\sup_{t \in [0,1]} \int_{|x| \geq c_a} |u(t,x)|^2 e^{\lambda|x|/\alpha d^\alpha} \, dx \lesssim \|u_0\|^2_{L_2(e^{\alpha|x|/\alpha d^\alpha} \, dx)} + \|u_1\|^2_{L_2(e^{\alpha|x|/\alpha d^\alpha} \, dx)} + \int_0^1 \int_{\mathbb{R}^d} |H(t,x)|^2 e^{\alpha|x|^\alpha} \, dx \, dt + \sum_{j=1}^3 \sum_{l=0}^{3} \int_0^1 \int_{\mathbb{R}^d} |\partial_{x_j}^l u(t,x)|^2 \, dx \, dt.
\]
This completes the proof of Lemma 4.2. \hfill \Box

**Remark 4.3.** The results in this section extend to equations of the form
\[
i\partial_t u + \Delta u = V_1 u + V_2 \pi + H
\]
with the potentials $V_j$, where $j = 1, 2$ satisfying the assumption (3.1).

4.3. **Logarithmic convexity generalized from [19].** A key ingredient in [19] is the following logarithmic convexity in any dimensions.

**Lemma 4.4** (Proposition 1.3 in [19]). Suppose $V \in L^\infty(\mathbb{R}^d)$ is real-valued, and $u \in C(\mathbb{R} ; L^2(\mathbb{R}^d))$ solves
\[
i\partial_t u - (-\Delta)^m u = V(x) u.
\]
If there exists $\gamma > 0$ such that
\[
e^{\gamma|x|^\alpha \alpha d^\alpha} u(0,x), \quad e^{\gamma|x|^\alpha \alpha d^\alpha} u(1,x) \in L_2^\alpha(\mathbb{R}^d),
\]
then for \( t \in [0, 1] \), we have
\[
\left\| e^{\gamma |x| \frac{2m}{2m+n}} u(t,x) \right\|_{L^2_x} \leq C e^{(1-t)\|V\|_{L^\infty}} \left\| e^{\gamma |x| \frac{2m}{2m+n}} u(0,x) \right\|_{L^2_x}^{1-t} \left\| e^{\gamma |x| \frac{2m}{2m+n}} u(1,x) \right\|_t.
\]

**Remark 4.5** (Comparison between Lemma 4.4 (which is Proposition 1.3 in [19]) and Lemma 3.1). In [19], the authors established a logarithmic convexity for higher order Schrödinger operators in any dimensions, considering real, bounded, and time-independent potentials. Their proof relied on two key ingredients: (i) an estimate for higher-order heat kernels, which can be extended to our ‘separable’ fourth-order Schrödinger operator (ii) a formal commutator estimate, as presented in Lemma 2 of [13], which remain valid in our case as well.

(2) To obtain a nonlinear unique continuation result, it becomes necessary to allow the potential \( V \) take complex values and be time-dependent, which is missing in Lemma 4.4. In our Lemma 3.1, the energy estimate method, inspired by [29], allows complex-valued and time-dependent potentials, thereby enabling us to obtain the unique continuation for nonlinear equations.

(3) In Lemma 3.1, the \( H^3 \) regularity of the solution is initially required due to the need for taking multiple derivatives. However, Lemma 4.4 only requires \( L^2 \) regularity. This is done by introducing an artificial diffusion term into the differential equation, as was first proposed in [13]. We believe that this \( H^3 \) regularity can be relaxed to \( L^2 \) with artificial diffusion.

5. A Carleman Inequality

In this section, we prove a Carleman estimate with a quadratic exponential weight for the ‘separable’ equation (1.1), which will be used in Section 7.

**Lemma 5.1.** Assume that \( R > 0 \) and \( \varphi : [0,1] \to \mathbb{R} \) is a smooth function. Let \( u(t,x) \in C_c^\infty(\mathbb{R} \times \mathbb{R}^d) \) with support contained in the set
\[
\{(t,x) \in [0,1] \times \mathbb{R}^d : |\frac{x_1}{R} + \varphi(t)| \geq 1\}.
\]
Then there exists \( c = c(d, \|\varphi\|_{L^\infty}, \|\varphi''\|_{L^\infty}) \) such that the inequality
\[
\left\| e^{\alpha(t) + \varphi(t)(t)} + \frac{1}{2} \sum_{j=2}^d \left( \frac{x_j}{R} \right)^2 (i \partial_t + \Delta^2) \right\|_{L^2_{t,x}}^2 \geq \frac{c \alpha^7}{R^6} \left\| e^{\alpha(t) + \varphi(t) + 1/2} \sum_{j=2}^d \left( \frac{x_j}{R} \right)^2 u \right\|_{L^2_{t,x}}^2
\]
holds when \( \alpha \geq cR^{\frac{2}{3}} \).

**Proof of Lemma 5.1.** Let \( f = e^{\Phi^2} u \), where
\[
\Phi^2(t,x) := \left( \frac{x_1}{R} + \varphi(t) \right)^2 + \sum_{j=2}^d \left( \frac{x_j}{R} \right)^2 = \psi^2 + \sum_{j=2}^d \left( \frac{x_j}{R} \right)^2,
\]
and
\[
\psi = \frac{x_1}{R} + \varphi(t).
\]
Under this change of variables, we reduce to proving
\[
\left\| e^{\alpha \Phi^2} (i \partial_t + \Delta^2) e^{-\alpha \Phi^2} f \right\|_{L^2_{t,x}}^2 \geq \frac{c \alpha^7}{R^6} \|f\|_{L^2_{t,x}}^2.
\]
We first write
\[
e^{\alpha \Phi^2} (i \partial_t + \Delta^2) e^{-\alpha \Phi^2} f =: Sf + Af
\]
where \( S \) and \( A \) are respectively symmetric and anti-symmetric operators (with respect to the \( L^2 \) norm). A direct computation gives that
• For $j = 1$

\[
e^{\alpha \phi^2} \partial_{x_1}^4 (e^{\alpha \phi^2} f) = \partial_{x_1}^4 f + \partial_{x_1}^3 f \left[ \frac{-8 \alpha \psi}{R} \right] + \partial_{x_1}^2 f \left[ \frac{24 \alpha^2 \psi^2}{R^2} - \frac{12 \alpha}{R^2} \right] + \partial_{x_1} f \left[ -\frac{32 \alpha^3 \psi^3}{R^3} + \frac{48 \alpha^2 \psi}{R^3} \right] + \left[ 16 \alpha^4 \psi^4 - \frac{48 \alpha^3 \psi^2}{R^3} + \frac{12 \alpha^2}{R^2} \right].
\]

• For $j = 2, \ldots, d$

\[
e^{\alpha \phi^2} \partial_{x_j}^4 (e^{\alpha \phi^2} f) = \partial_{x_j}^4 f + \partial_{x_j}^3 f \left[ \frac{8 \alpha \psi}{R^2} \right] + \partial_{x_j}^2 f \left[ \frac{24 \alpha^2 \psi^2}{R^4} - \frac{12 \alpha}{R^4} \right] + \partial_{x_j} f \left[ -\frac{32 \alpha^3 \psi^3}{R^5} + \frac{12 \alpha}{R^5} \right] + \left[ \frac{48 \alpha^2 \psi}{R^3} \right] + \left[ \frac{16 \alpha^4 \psi^4}{R^5} - \frac{48 \alpha^3 \psi^2}{R^5} + \frac{12 \alpha^2}{R^4} \right].
\]

Then adding these two cases yields

\[
e^{\alpha \phi^2} \Delta^2 (e^{\alpha \phi^2} f) = \Delta^2 f + \sum_{j=2}^d \partial_{x_j}^3 f \left[ \frac{8 \alpha \psi}{R^2} \right] + \sum_{j=2}^d \partial_{x_j}^2 f \left[ \frac{24 \alpha^2 \psi^2}{R^4} - \frac{12 \alpha}{R^4} \right] + \sum_{j=2}^d \partial_{x_j} f \left[ -\frac{32 \alpha^3 \psi^3}{R^5} + \frac{48 \alpha^2 \psi}{R^5} \right] + \sum_{j=2}^d f \left[ \frac{16 \alpha^4 \psi^4}{R^5} - \frac{48 \alpha^3 \psi^2}{R^5} + \frac{12 \alpha^2}{R^4} \right].
\]

and

\[
e^{\alpha \phi^2} i \partial_t (e^{\alpha \phi^2} f) = e^{\alpha \phi^2} i \left( e^{-\alpha \phi^2} \partial_t f - e^{-\alpha \phi^2} \alpha \left( \frac{2x_1}{R} \varphi' + 2 \psi \varphi' \right) f \right) = i \partial_t f - i \alpha \left( \frac{2x_1}{R} \varphi' + 2 \psi \varphi' \right) f. \tag{5.2}
\]

We recognize the symmetric and the anti-symmetric parts of the operator in (5.1) and (5.2). The symmetric operator $S$ is given by

\[
Sf = i \partial_t f + \Delta^2 f
\]

\[
+ \sum_{j=2}^d \partial_{x_j}^3 f \left[ \frac{8 \alpha \psi}{R^2} \right] + \sum_{j=2}^d \partial_{x_j}^2 f \left[ \frac{24 \alpha^2 \psi^2}{R^4} - \frac{12 \alpha}{R^4} \right] + \sum_{j=2}^d \partial_{x_j} f \left[ -\frac{32 \alpha^3 \psi^3}{R^5} + \frac{48 \alpha^2 \psi}{R^5} \right] + \sum_{j=2}^d f \left[ \frac{16 \alpha^4 \psi^4}{R^5} - \frac{48 \alpha^3 \psi^2}{R^5} + \frac{12 \alpha^2}{R^4} \right].
\]

We decompose $S$ into

\[
Sf = S_t f + \sum_{j=2}^d S_{x_j} f,
\]

where

\[
S_t f = i \partial_t f,
\]

\[
S_{x_j} f = \partial_{x_j}^4 f + \partial_{x_j}^2 f \left[ \frac{24 \alpha^2 \psi^2}{R^2} \right] + \partial_{x_j} f \left[ \frac{48 \alpha^2 \psi}{R^3} \right] + f \left[ \frac{16 \alpha^4 \psi^4}{R^4} + \frac{12 \alpha^2}{R^4} \right].
\]
\[ \mathcal{S}_x f = \partial_x^2 f + \partial_x^4 f \left[ \frac{24\alpha^2 x^2}{R^3} + \partial_x f \left[ \frac{48\alpha x}{R^4} \right] + f \left[ \frac{16\alpha^4 x^4}{R^8} + \frac{12\alpha^2}{R^4} \right] \right], \quad j = 2, \ldots, d. \]

Then the anti-symmetric operator \( A \) is given by
\[
Af = f \left[ -2i\alpha \left( \frac{\partial_x}{R} \varphi + \varphi' \right) \right] + \partial_x^3 f \left[ -\frac{8\alpha \varphi}{R} \right] + \sum_{j=2}^d \partial_x^j f \left[ -\frac{8\alpha x}{R^2} \right] + \partial_x^2 f \left[ -\frac{12\alpha}{R^3} \right] + \sum_{j=2}^d \partial_x^j f \left[ -\frac{12\alpha}{R^3} \right] + \partial_x f \left[ -\frac{32\alpha^3 \varphi^3}{R^6} \right] + \sum_{j=2}^d \partial_x^j f \left[ -\frac{32\alpha^3 \varphi^3}{R^6} \right] + f \left[ -\frac{48\alpha^3 \varphi^2}{R^4} \right] + \sum_{j=2}^d \left( -\frac{48\alpha^3 x^2}{R^6} \right). 
\]

We again decompose \( A \) into
\[
Af = A_t f + \mathcal{A}_x f + \sum_{j=2}^d \mathcal{A}_j f,
\]
where
\[
A_t f = f \left[ -2i\alpha \left( \frac{\partial_x}{R} \varphi + \varphi' \right) \right],
\]
\[
\mathcal{A}_x f = \partial_x^4 f \left[ -\frac{8\alpha \varphi}{R} \right] + \partial_x^3 f \left[ -\frac{12\alpha}{R^3} \right] + \partial_x^2 f \left[ -\frac{32\alpha^3 \varphi^3}{R^6} \right] + f \left[ -\frac{8\alpha \varphi}{R} \right] + f \left[ -\frac{48\alpha^3 \varphi^2}{R^4} \right] + \sum_{j=2}^d \partial_x^j f \left[ -\frac{8\alpha x}{R^2} \right] + \partial_x^3 f \left[ -\frac{12\alpha}{R^3} \right] + \partial_x^2 f \left[ -\frac{32\alpha^3 x^3}{R^6} \right] + f \left[ -\frac{48\alpha^3 x^2}{R^6} \right], \quad j = 2, \ldots, d.
\]

Now we will compute the commutator \([\mathcal{S}, A]\) using term by term. First notice that \([\mathcal{S}_x, A_t] = 0 \text{ when } i \neq j, \]
and \([\mathcal{S}_x, \mathcal{A}_x] = [\mathcal{S}_x, \mathcal{A}_t] = 0, \text{ for } i \neq 1. \) This observation implies that we only need to compute the following five cases:

1. \([\mathcal{S}_t, A_t]\)
2. \([\mathcal{S}_t, \mathcal{A}_x]\)
3. \([\mathcal{S}_x, A_t]\)
4. \([\mathcal{S}_x, \mathcal{A}_x]\)
5. \([\mathcal{S}_x, \mathcal{A}_t]\)

For Case (1) in the list, we have
\[
[\mathcal{S}_t, A_t] f = \left[ i \partial_t, -2i\alpha \left( \frac{\partial_x}{R} \varphi + \varphi' \right) \right] f = 2\alpha \left[ \partial_t, \left( \frac{\partial_x}{R} \varphi + \varphi' \right) \right] f
= 2\alpha \left( \frac{\partial_x}{R} \varphi' + \varphi'' \right) f - \frac{48\alpha^3 \varphi^2}{R^4} f
= 2\alpha \left( \frac{\partial_x}{R} \varphi'' + \varphi'^2 + \varphi'' \right) f.
\]

Then consider Case (2), we compute
\[
\left[ i \partial_t, \frac{8\alpha \varphi}{R} \partial_x^3 \right] f = i \partial_t \left( \frac{8\alpha \varphi}{R} \partial_x^3 f \right) - \frac{8\alpha \varphi}{R} \partial_x^3 (\partial_t f) = i \frac{8\alpha \varphi'}{R} \partial_x^3 f,
\]
\[
\left[ i \partial_t, \frac{12\alpha}{R^2} \partial_x^2 \right] f = 0,
\]
\[
\left[ i\partial_t, \frac{32\alpha^3\psi^3}{R^3} \partial_{x_1} \right] f = i\partial_t \left( \frac{32\alpha^3\psi^3}{R^3} \partial_{x_1} f \right) - i\frac{32\alpha^3\psi^3}{R^3} \partial_{x_1} (\partial_t f) = i\frac{96\alpha^3\psi^2\varphi'}{R^4} \partial_{x_1} f,
\]
\[
\left[ i\partial_t, \frac{48\alpha^2\psi^2}{R^4} \right] f = i\partial_t \left( \frac{48\alpha^2\psi^2}{R^4} f \right) - i\frac{48\alpha^2\psi^2}{R^4} (\partial_t f) = i\frac{96\alpha^3\psi\varphi'}{R^4} f.
\]
Then adding them together, we have
\[
[S_1, A_{x_1}] f = -i\frac{8\alpha\varphi'}{R} \partial_{x_1}^3 f - i\frac{96\alpha^3\psi^2\varphi'}{R^3} \partial_{x_1} f - i\frac{96\alpha^3\psi\varphi'}{R^4} f.
\]
For Case 3, we compute first
\[
\left[ \partial_{x_1}^2, 2\alpha \left( \frac{x_1}{R} \varphi' + \varphi \varphi' \right) \right] f = 2\alpha \partial_{x_1}^4 \left( \left( \frac{x_1}{R} \varphi' + \varphi \varphi' \right) f \right) - 2\alpha \left( \frac{x_1}{R} \varphi' + \varphi \varphi' \right) \partial_{x_1}^4 f = 2\alpha \frac{4}{R} \varphi' \partial_{x_1}^3 f,
\]
\[
\left[ 24\alpha^2\psi^2 \frac{R^2}{R^2} \partial_{x_1}^2, 2\alpha \left( \frac{x_1}{R} \varphi' + \varphi \varphi' \right) \right] f = 2\alpha \frac{24\alpha^2\psi^2}{R^2} \partial_{x_1}^2 \left( \left( \frac{x_1}{R} \varphi' + \varphi \varphi' \right) f \right) - 2\alpha \left( \frac{x_1}{R} \varphi' + \varphi \varphi' \right) \frac{24\alpha^2\psi^2}{R^2} \partial_{x_1}^2 f
\]
\[
= 2\alpha \frac{24\alpha^2\psi^2}{R^2} \frac{2}{R} \varphi' \partial_{x_1} f,
\]
\[
\left[ 48\alpha^2\psi \frac{R^3}{R^3} \partial_{x_1}, 2\alpha \left( \frac{x_1}{R} \varphi' + \varphi \varphi' \right) \right] f = 2\alpha \frac{48\alpha^2\psi}{R^3} \partial_{x_1} \left( \left( \frac{x_1}{R} \varphi' + \varphi \varphi' \right) f \right) - 2\alpha \left( \frac{x_1}{R} \varphi' + \varphi \varphi' \right) \frac{48\alpha^2\psi}{R^3} \partial_{x_1} f
\]
\[
= 2\alpha \frac{48\alpha^2\psi}{R^3} \frac{1}{R} \varphi' f,
\]
\[
\left[ \left( \frac{16\alpha^4\psi^4}{R^4} + 12\alpha^2 \frac{R^2}{R^2} \right), 2\alpha \left( \frac{x_1}{R} \varphi' + \varphi \varphi' \right) \right] f = 0.
\]
Then summing up all the terms above, we have
\[
[S_{x_1}, A_t] f = -2\alpha \frac{4}{R} \varphi' \partial_{x_1}^3 f - 2\alpha \frac{48\alpha^2\psi^2}{R^3} \varphi' \partial_{x_1} f - 2\alpha \frac{48\alpha^2\psi}{R^4} \varphi' f.
\]
Next for Cases (4) and (5) in the list, using Mathematica we get
\[
[S_{x_1}, A_{x_1}] f = f \left[ -\frac{1536\alpha^5\psi^2}{R^8} + \frac{2048\alpha^7\psi^6}{R^8} \right] - \frac{6144\alpha^5\psi^3 \partial_{x_1} f}{R^7} + \frac{384\alpha^3 \partial_{x_1}^2 f}{R^6} - \frac{1536\alpha^5\psi^4 \partial_{x_1}^2 f}{R^6} - \frac{1536\alpha^5 x^2_{x_1} \partial_{x_1} f}{R^6} + \frac{384\alpha^3 x^2_{x_1} \partial_{x_1}^2 f}{R^6} - \frac{32\alpha \partial_{x_1}^3 f}{R^2}.
\]
and
\[
[S_{x_1}, A_{x_1}] f = f \left[ -\frac{1536\alpha^5 x^2_{x_1}}{R^{10}} + \frac{2048\alpha^7 x^6_{x_1}}{R^{10}} \right] - \frac{6144\alpha^5 x^3_{x_1} \partial_{x_1} f}{R^{10}} + \frac{384\alpha^3 x^3_{x_1} \partial_{x_1}^2 f}{R^{10}} + \frac{384\alpha^3 x^2_{x_1} \partial_{x_1}^2 f}{R^{10}} - \frac{1536\alpha^5 x^4_{x_1} \partial_{x_1}^2 f}{R^{10}} + \frac{384\alpha^3 x^2_{x_1} \partial_{x_1}^4 f}{R^{10}} - \frac{32\alpha \partial_{x_1}^3 f}{R^2}.
\]
Combining (5.3) and (5.4), we obtain a part of the commutator [S, A]
\[
[S_{x_1}, A_{x_1}] f + \sum_{j=2}^{d} [S_{x_1}, A_{x_1}] = f \left[ \frac{1536\alpha^5\psi^2}{R^8} + \frac{2048\alpha^7\psi^6}{R^8} + \sum_{j=2}^{d} \left( \frac{-1536\alpha^5 x^2_{x_1}}{R^{10}} + \frac{2048\alpha^7 x^6_{x_1}}{R^{10}} \right) \right]
\]
\[
- \frac{6144\alpha^5\psi^3 \partial_{x_1} f}{R^7} + \sum_{j=2}^{d} \left( \frac{-6144\alpha^5 x^3_{x_1} \partial_{x_1} f}{R^{10}} \right)
\]
\[
+ \frac{384\alpha^3 \partial_{x_1}^2 f}{R^6} - \frac{1536\alpha^5\psi^4 \partial_{x_1}^2 f}{R^6} + \sum_{j=2}^{d} \left( \frac{384\alpha^3 x^2_{x_1} \partial_{x_1}^2 f}{R^6} - \frac{1536\alpha^5 x^4_{x_1} \partial_{x_1}^2 f}{R^{10}} \right).
\]
Next we compute the inner product \( \langle f, [S, A] f \rangle \) and our aim is to find an lower bound for it.

\[
\langle f, [S, A] f \rangle_{L^2_{\mathbb{R}^3} \times L^2_{\mathbb{R}^3}} = \int f \left[ 2 \alpha \left( \frac{\phi'}{R} \phi'' + \phi'^2 + \phi \phi'' \right) - i \frac{16 \alpha \phi'}{R} \phi' \frac{\partial^3 \phi}{\partial x^3} f - i \frac{192 \alpha^3 \phi^2 \phi'}{R^3} \frac{\partial \phi}{\partial x_1} f - i \frac{192 \alpha^3 \phi \phi'}{R^4} f \right] \ dx \ dt \tag{5.5}
\]

\[
+ \int |f|^2 \left[ - \frac{1536 \alpha^5 \phi^2}{R^8} + \frac{2048 \alpha^7 \phi^6}{R^8} + \sum_{j=2}^{d} \left( i \frac{1536 \alpha^5 \phi^2}{R^{10}} + \frac{2048 \alpha^7 \phi^6}{R^{14}} \right) \right] \ dx \ dt \tag{5.6}
\]

\[
+ \int f \partial_{x_1} f \left[ - \frac{6144 \alpha^5 \phi^3}{R^7} \right] + \sum_{j=2}^{d} \tilde{f} \partial_{x_j} \left[ - \frac{6144 \alpha^5 \phi^3}{R^{10}} \right] \ dx \ dt \tag{5.7}
\]

\[
+ \int \tilde{f} \partial_{x_1}^2 f \left[ \frac{384 \alpha^3 \phi^2}{R^6} - \frac{1536 \alpha^5 \phi^4}{R^6} \right] + \sum_{j=2}^{d} \tilde{f} \partial_{x_j}^2 \left[ \frac{384 \alpha^3 \phi^2}{R^6} - \frac{1536 \alpha^5 \phi^4}{R^{10}} \right] \ dx \ dt \tag{5.8}
\]

\[
+ \int \tilde{f} \partial_{x_1}^3 f \left[ \frac{1536 \phi}{R^5} \right] + \sum_{j=2}^{d} \tilde{f} \partial_{x_j}^3 \left[ \frac{1536 \alpha^3 \phi_x}{R^6} \right] \ dx \ dt \tag{5.9}
\]

\[
+ \int \tilde{f} \partial_{x_1}^4 f \left[ \frac{384 \alpha^3 \phi^2}{R^4} \right] + \sum_{j=2}^{d} \tilde{f} \partial_{x_j}^4 \left[ \frac{384 \alpha^3 \phi^2}{R^6} \right] \ dx \ dt \tag{5.10}
\]
To this end, we need to perform a few integration by parts term by term. 

**Term (5.5).** First take the last term in (5.5). An integration by parts yields

\[
\iint -i \frac{192\alpha^3\psi^2\varphi'}{R^3} \bar{f} \partial_x f \, dx \, dt = \iint i \frac{192\alpha^3\psi^2\varphi'}{R^3} \partial_x f \, dx \, dt + \iint i \frac{192\alpha^3\psi^2\varphi'}{R^4} |f|^2 \, dx \, dt,
\]

which implies

\[
\iint i \frac{192\alpha^3\psi^2\varphi'}{R^4} |f|^2 \, dx \, dt = -\iint i \frac{192\alpha^3\psi^2\varphi'}{R^3} \text{Re}(\bar{f} \partial_x f) \, dx \, dt.
\]

Hence the last two terms in (5.5) is given by

\[
\iint -i \frac{192\alpha^3\psi^2\varphi'}{R^3} \bar{f} \partial_x f - i \frac{192\alpha^3\psi^2\varphi'}{R^4} |f|^2 \, dx \, dt
= \iint -i \frac{192\alpha^3\psi^2\varphi'}{R^3} \bar{f} \partial_x f \, dx \, dt + \iint i \frac{192\alpha^3\psi^2\varphi'}{R^3} \text{Re}(\bar{f} \partial_x f) \, dx \, dt
= \iint \frac{192\alpha^3\psi^2\varphi'}{R^3} \text{Im}(\bar{f} \partial_x f) \, dx \, dt.
\]

The second term in (5.5) can be written as

\[
\iint -i \frac{16\alpha^2}{R} \bar{f} \partial_x^3 f \, dx \, dt = \iint i \frac{16\alpha^2}{R} \bar{f} \partial_x^2 f \, dx \, dt = \iint -i \frac{16\alpha^2}{R} \bar{f} \partial_x^2 f \, dx \, dt
\]

which implies

\[
\iint -i \frac{16\alpha^2}{R} \bar{f} \partial_x^3 f \, dx \, dt = \iint - \frac{16\alpha^2}{R} \text{Im}(\partial_x \bar{f} \partial_x^2 f) \, dx \, dt.
\]

Therefore,

\[
(5.5) = \iint |f|^2 \left[ 2\alpha \left( \left( \frac{x_1}{R} + \varphi \right) \varphi'' + \varphi^2 \right) \right] - \frac{16\alpha^2}{R} \text{Im}(\partial_x \bar{f} \partial_x^2 f) + \frac{192\alpha^3\psi^2\varphi'}{R^3} \text{Im}(\bar{f} \partial_x f) \, dx \, dt.
\]

**Term (5.6).** Rewriting it in the following form

\[
(5.6) = \iint |f|^2 \left[ - \frac{1536\alpha^5\psi^2}{R^8} + \frac{2048\alpha^7\psi^6}{R^8} + \frac{d}{j=2} \left( - \frac{1536\alpha^5 x_j^2}{R^{10}} + \frac{2048\alpha^7 x_j^6}{R^{14}} \right) \right] \, dx \, dt
= \iint |f|^2 \left[ - \frac{1536\alpha^5\psi^2}{R^8} + \frac{2048\alpha^7\psi^6}{R^8} \left( \psi^6 + \frac{d}{j=2} \left( \frac{x_j}{R} \right)^6 \right) \right] \, dx \, dt.
\]

**Terms (5.7) and (5.8).** Again integrating by parts yields

\[
(5.8) = \iint \bar{f} \partial_x^2 f \left[ \frac{384\alpha^3}{R^6} - \frac{1536\alpha^5\psi^4}{R^6} \right] + \sum_{j=2}^d \bar{f} \partial_x^2 f \left[ \frac{384\alpha^3}{R^6} - \frac{1536\alpha^5 x_j^4}{R^{10}} \right] \, dx \, dt
= \iint |\partial_x f|^2 \left[ \frac{384\alpha^3}{R^6} + \frac{1536\alpha^5\psi^4}{R^6} \right] + \sum_{j=2}^d |\partial_x f|^2 \left[ \frac{384\alpha^3}{R^6} + \frac{1536\alpha^5 x_j^4}{R^{10}} \right] \, dx \, dt
+ \iint \bar{f} \partial_x f \left[ \frac{6144\alpha^5\psi^3}{R^3} \right] + \sum_{j=2}^d \bar{f} \partial_x f \left[ \frac{6144\alpha^5 x_j^3}{R^{10}} \right] \, dx \, dt.
\]
Noticing that the last two terms in (5.12) is the opposite of (5.7), hence

\[
(5.7) + (5.8) = \int \int |\partial_x f|^2 \left[ \frac{-384\alpha^3}{R^6} + \frac{1536\alpha^5\psi^4}{R^6} \right] + \sum_{j=2}^{d} |\partial_x f|^2 \left[ \frac{-384\alpha^3}{R^6} + \frac{1536\alpha^5 x_j^4}{R^{10}} \right] \, dx \, dt.
\]

Terms (5.9) and (5.10). Performing integration by parts again, we write

\[
\text{Term 1 in (5.10)} = \int \int \bar{f} \partial_x^4 f \frac{384\alpha^3 \psi^2}{R^4} \, dx \, dt
\]

\[
= \int \int \left( -\partial_x \bar{f} \partial_x^3 f \frac{384\alpha^3 \psi^2}{R^4} - \bar{f} \partial_x^3 f \frac{768\alpha^3 \psi}{R^5} \right) \, dx \, dt
\]

\[
= \int \int |\partial_x f|^2 \left( \frac{384\alpha^3 \psi^2}{R^4} + \partial_x f \bar{f} \partial_x^2 f \frac{768\alpha^3 \psi}{R^5} - \bar{f} \partial_x^3 f \frac{768\alpha^3 \psi}{R^5} \right) \, dx \, dt
\]

\[
= \int \int |\partial_x f|^2 \left( \frac{384\alpha^3 \psi^2}{R^4} - \bar{f} \partial_x^3 f \frac{768\alpha^3 \psi}{R^5} - \bar{f} \partial_x^2 f \frac{768\alpha^3 \psi}{R^5} - \bar{f} \partial_x f \frac{768\alpha^3 \psi}{R^5} \right) \, dx \, dt,
\]

and

\[
\text{Term 2 in (5.10)} = \int \int \bar{f} \partial_x^2 f \frac{384\alpha^3 x_j^2}{R^6} \, dx \, dt
\]

\[
= \int \int \left( -\partial_x \bar{f} \partial_x f \frac{384\alpha^3 x_j^2}{R^6} - \bar{f} \partial_x^2 f \frac{768\alpha^3 x_j^2}{R^7} \right) \, dx \, dt
\]

\[
= \int \int |\partial_x f|^2 \left( \frac{384\alpha^3 x_j^2}{R^6} + \partial_x f \bar{f} \partial_x f \frac{768\alpha^3 x_j^2}{R^7} - \bar{f} \partial_x^2 f \frac{768\alpha^3 x_j^2}{R^7} \right) \, dx \, dt
\]

\[
= \int \int |\partial_x f|^2 \left( \frac{384\alpha^3 x_j^2}{R^6} - \bar{f} \partial_x^2 f \frac{768\alpha^3 x_j^2}{R^7} - \bar{f} \partial_x f \frac{768\alpha^3 x_j^2}{R^7} - \bar{f} \partial_x f \frac{768\alpha^3 x_j^2}{R^7} \right) \, dx \, dt
\]

Noticing that the second terms in (5.13) and (5.14) show up in (5.9) with the opposite sign, we have

\[
(5.9) + (5.10) = \int \int |\partial_x^2 f|^2 \left( \frac{384\alpha^3 \psi^2}{R^4} + |\partial_x f|^2 \frac{768\alpha^3 x_j^2}{R^6} + \sum_{j=2}^{d} |\partial_x^2 f|^2 \frac{384\alpha^3 x_j^2}{R^6} \right) + \sum_{j=2}^{d} |\partial_x f|^2 \frac{768\alpha^3}{R^6} \, dx \, dt.
\]

Term (5.11). Finally, we write

\[
(5.11) = \int \int \bar{f} \partial_x^6 f \left[ \frac{32\alpha}{R^2} \right] + \sum_{j=2}^{d} |\partial_x^5 f|^2 \left[ \frac{32\alpha}{R^2} \right] \, dx \, dt = \int \int |\partial_x^3 f|^2 \left[ \frac{32\alpha}{R^2} \right] + \sum_{j=2}^{d} |\partial_x^2 f|^2 \frac{32\alpha}{R^2} \, dx \, dt.
\]

Therefore, summarizing Terms (5.5) - (5.11) we conclude that

\[
\langle f, [S, A] f \rangle_{L^2_{t,x} \times L^2_{t,x}} = \int \int \left( \frac{16\alpha f'}{R} \phi'' + \frac{192\alpha^3 \psi^2 f' \phi'}{R^3} \right) - \frac{16\alpha f'}{R} \left( \frac{\alpha f'}{R} \phi'' + \frac{\alpha f'}{R} \phi'^2 \right) \right) \, dx \, dt
\]

\[
+ \int \int \left[ \frac{1536\alpha^5 \phi^2 + 2048\alpha^7 \phi^2}{R^8} \left( \psi^6 + \sum_{j=2}^{d} \left( \frac{x_j}{R} \right)^6 \right) \right] \, dx \, dt
\]

\[
+ \int \int \left| \partial_x f \right|^2 \left[ \frac{384\alpha^3}{R^6} + \frac{1536\alpha^5 \psi^4}{R^6} \right] + \sum_{j=2}^{d} |\partial_x^2 f|^2 \left[ \frac{384\alpha^3}{R^6} + \frac{1536\alpha^5 x_j^4}{R^{10}} \right] \, dx \, dt
\]

\[
+ \int \int \left| \partial_x f \right|^2 \left[ \frac{384\alpha^3 x_j^2}{R^6} + \sum_{j=2}^{d} |\partial_x^2 f|^2 \frac{384\alpha^3 x_j^2}{R^6} \right] \, dx \, dt
\]
Remark 5.2. We can only prove the estimate for the operator \(30\ L.E.E.\ and\ Y.u\)
which finishes the proof of Lemma \(\omega,\rho\)
obtain the desired inequality for the operator
with the condition
involved in the calculations, which might not be expected to sum together to produce a positive lower bound
inequality for the classical Schrödinger operator
proof is quite delicate and it relies on a series of very careful estimates that if not done correctly will give a
To estimates the mixed terms above, we employ Cauchy–Schwarz inequality to control
Now we obtain a lower bound of \(\langle f, [S, A] f \rangle\)

\[
\geq \int \int |f|^2 \left[ -\frac{1536\alpha^5}{R^6} \Phi^2 + \frac{2048\alpha^7}{R^6} \left( \psi^6 + \sum_{j=2}^d \frac{\Phi_j^6}{R} \right) + 2\alpha \left( \left( \frac{x_1}{R} + \varphi \right) \varphi'' + \varphi^2 \right) - \frac{192\alpha^3\psi^2\varphi'}{R^3} \right] \, d(xd5) \\
+ \int \int |\partial_x f|^2 \left[ \frac{384\alpha^3}{R^6} - \frac{1536\alpha^5\psi^4}{R^6} + \frac{16\alpha\varphi'}{R} - \frac{192\alpha^3\psi^2\varphi'}{\omega R} \right] + \sum_{j=2}^d |\partial_x f|^2 \left[ \frac{384\alpha^3}{R^6} + \frac{1536\alpha^5\psi^4}{R^6} \right] \, dx(\rho16) \\
+ \int \int |\partial_x^2 f|^2 \left[ \frac{384\alpha^3\psi^2}{R^6} - \frac{16\alpha\varphi'}{R} \right] + \sum_{j=2}^d |\partial_x^2 f|^2 \left[ \frac{384\alpha^3\psi^2}{R^6} \right] \, dxdt
\] (5.17)

\[
+ \int \int |\partial_x^3 f|^2 \left[ \frac{32\alpha}{R^2} + \sum_{j=2}^d |\partial_x^3 f|^2 \right] \, dxdt.
\] (5.18)

By choosing \(\rho \sim R^\gamma, \omega \sim R^\delta\) (with suitable constants and \(\alpha = c R^{2/3}\) (where \(c = c(d, \|\varphi\|_{L^\infty}, \|\varphi''\|_{L^\infty})\)), we can make the first term and last two terms (5.15) absorbed by the second term in (5.15); and hide the terms
with negative signs in (5.16) and (5.17) by the first positive terms in (5.16) and (5.17) respectively. Since the
two terms in (5.18) are both non-negative, we then finally obtain

\[
\langle f, [S, A] f \rangle_{L^2_t \times L^2_x} \geq c \left( \frac{\alpha^7}{R^6} \right) \|f\|_{L^2_t \times L^2_x}^2.
\]

Recall \(f = e^{i\alpha \Phi^2} u\) and \(\Phi^2(t, x) = \left( \frac{x_1}{R} + \varphi(t) \right)^2 + \sum_{j=2}^d \left( \frac{x_j}{R} \right)^2\), then we have

\[
\|e^{i\alpha \Phi^2}(i \partial_t + \Delta^2) u\|_{L^2_t \times L^2_x}^2 \geq \left( \int \int [S, A] f \, dxdt \right) \|f\|_{L^2_t \times L^2_x}^2 \geq c \left( \frac{\alpha^7}{R^6} \right) \|f\|_{L^2_t \times L^2_x}^2 = c \left( \frac{\alpha^7}{R^6} \right) \|e^{i\alpha \Phi^2} u\|_{L^2_t \times L^2_x}^2,
\]

which finishes the proof of Lemma 5.1. \(\square\)

**Remark 5.2.** We can only prove the estimate for the operator \(i \partial_t + \Delta^2\) because it is not obvious how to
obtain the desired inequality for the operator \(i \partial_t + \Delta^2\) containing mixed terms since the latter resist being
lower-bounded in a positive fashion. We believe that this inequality is highly delicate and nontrivial because
it is far from clear at first sight that it would follow using the same methods that proved a previous Carleman
inequality for the classical Schrödinger operator \(i \partial_t + \Delta\) as well as a result of the very large number of terms
involved in the calculations, which might not be expected to sum together to produce a positive lower bound
with the condition \(\alpha \geq c R^{2/3}\) (which gives us the sharpest possible unique continuation theorem). In fact, the
proof is quite delicate and it relies on a series of very careful estimates that if not done correctly will give a
weaker inequality that is only valid for a smaller range of \(\alpha\).

6. Lower Bound Estimates

In this section, we prove a lower bound for solutions with fast decay, which will be used in the next section
to prove the main theorem by way of a contradiction argument.
Lemma 6.1 (Lower bounds). Let $u \in C^1([0,1] : H^3(\mathbb{R}^d))$ solve (1.1) and let $B_R := \{ x \in \mathbb{R}^d : |x| \leq R \}$.

If

$$
\int_{1/2 - 1/8}^{1/2 + 1/8} \int_{B_R} |u(t,x)|^2 \, dx \, dt \geq 1,
$$

(6.1)

$$
\int_0^1 \int_{\mathbb{R}^d} \sum_{j=1}^d |u|^2 + |\partial_{x_j} u|^2 + |\partial_{x_j}^2 u|^2 + |\partial_{x_j}^3 u|^2 \, dx \, dt \leq A^2
$$

for some $A, L > 0$.

Then there exists $R_0 = R_0(d, A, L) > 0$ and $c = c(d)$ such that

$$
\gamma(R) := \left( \int_0^1 \int_{R-1 < |x| < R} \sum_{j=1}^d |u|^2 + |\partial_{x_j} u|^2 + |\partial_{x_j}^2 u|^2 + |\partial_{x_j}^3 u|^2 \, dx \, dt \right)^{\frac{2}{3}} \geq c R^\frac{2}{3} e^{-cR^\frac{1}{2}},
$$

for all $R > R_0$.

Proof of Lemma 6.1. Let us start with introducing some cutoff functions

- Let $\eta(x) \in C^\infty(\mathbb{R})$ be non-decreasing, radial and such that

  $$
  \eta(x) = \begin{cases} 0, & \text{if } |x| \leq 1, \\ 1, & \text{if } |x| \geq 2. \end{cases}
  $$

  We also define $\eta_R(x) = \eta(\frac{x}{R})$.

- Let $\theta_R(x) \in C^\infty(\mathbb{R}^d)$ be non-decreasing, radial and such that

  $$
  \theta_R(x) = \begin{cases} 1, & \text{if } |x| \leq R - 1, \\ 0, & \text{if } |x| \geq R. \end{cases}
  $$

  We also define $\varphi_R(x) = 1 - \theta_R(x)$.

- Let $\varphi \in C^\infty$ satisfy $\varphi(t) \in [0,3]$ on $[0,1]$ and

  $$
  \varphi(t) = \begin{cases} M, & \text{on } [\frac{1}{2} - \frac{1}{5}, \frac{1}{2} + \frac{1}{5}], \\ 0, & \text{on } [0, \frac{1}{4}] \cup [\frac{3}{4}, 1]. \end{cases}
  $$

We now apply Lemma 5.1 to the function

$$
\sigma(t,x) = \theta_R(x) \eta \left( \frac{x_1}{R} + \varphi(t) \right).
$$

Let $\Phi(t,x) := \left[ \frac{x_1}{R} + \varphi(t) e_1^t \right]$. We see that $f$ is compactly supported on $[0,1] \times \mathbb{R}^d$ and satisfies the hypothesis of Lemma 5.1. In fact, we notice that $f = u$ on $[\frac{1}{2} - \frac{1}{5}, \frac{1}{2} + \frac{1}{5}] \times B_{R-1}$, and $\Phi(t,x) \geq [\frac{x_1}{R} + \varphi(t)] \geq M - 1$, where $e_1$ is the unit vector $(1,0,\cdots,0)$.

Using our hypothesis (6.1) we see that

$$
\left\| e^{\alpha \Phi^2} \right\|_{L^2_t \times \mathbb{R}^d} \geq e^{(M-1)^2 \alpha} \left\| u \right\|_{L^2_t \times \left( [\frac{1}{2} - \frac{1}{5}, \frac{1}{2} + \frac{1}{5}] \times B_1 \right)} \geq e^{(M-1)^2 \alpha}.
$$

(6.2)

By the chain rule, we write

$$
(i\partial_t + \Delta^2 - V) f(t,x)
$$

$$
= \sum_{k=0}^3 \left( \sum_{j=2}^d C_k \partial_{x_j}^{3-k} \theta_R(x) + \sum_{l=1}^{3-k} D_k \partial_{x_1}^l \eta \partial_{x_1}^{3-k-l} \theta_R(x) \right) \partial_{x_j}^k u + i \theta_R \varphi' \theta_R \partial_{x_1} \eta \left( \frac{x_1}{R} + \varphi(t) \right) u.
$$

(6.3)
There are two types of terms in the decomposition of (6.3).

- \( \eta \times \partial \theta \) type: the support of such type is \([0, 1] \times B_R \setminus B_{R-1} \) and \(1 \leq |\frac{a}{R^2} + \varphi(t)| \leq M + 1\), hence we know \( \Phi^2 \in [1, (M + 1)^2 + (d - 1)]\).
- \( \partial \eta \times \theta_R \) or \( \partial \eta \times \partial \theta_R \) type: the support of this type is \([0, 1] \times B_R \) and \(1 \leq |\frac{a}{R^2} + \varphi(t)| \leq 2\), hence \( \Phi^2 \in [1, 4 + (d - 1)]\).

Combining Lemma 5.1 with the computation of \((i \partial_t + \Delta^2)f(t, x)\) in (6.3), we see that

\[
e^\frac{\alpha^2}{R^2} \left\| e^{\alpha \Phi^2} f \right\|_{L^2_t L^2_x} \leq \left\| e^{\alpha \Phi^2} (i \partial_t + \Delta^2)f \right\|_{L^2_t L^2_x} \leq \left\| e^{\alpha \Phi^2} (i \partial_t + \Delta^2 - V)f \right\|_{L^2_t L^2_x} + \left\| e^{\alpha \Phi^2} V f \right\|_{L^2_t L^2_x}
\]

\[
\leq \left\| e^{\alpha \Phi^2} \text{RHS of (6.3)} \right\|_{L^2_t L^2_x} + L \left\| e^{\alpha \Phi^2} f \right\|_{L^2_t L^2_x}
\]

\[
\leq e^{((M+1)^2 + (d-1)) \alpha} \gamma(R) + e^{(3+d)\alpha} A + e^{(3+d)\alpha} \gamma(R) + L \left\| e^{\alpha \Phi^2} f \right\|_{L^2_t L^2_x}.
\]

Choosing \( \alpha = cR^\frac{4}{7} \), we can hide the third term on the right-hand side of (6.4) when \( R \geq R_0(d, L) \). Then utilizing our lower bound for \( \left\| e^{\alpha \Phi^2} f \right\|_{L^2_t L^2_x} \) in (6.2), we deduce that

\[
e^\frac{\alpha^2}{R^2} e^{((M+1)^2 + (d-1)) \alpha} \gamma(R) + e^{(3+d)\alpha} A \]

\[
e^\frac{\alpha^2}{R^2} e^{((M+1)^2 + (d-4)-4) \alpha} \leq e^{((M+1)^2 + (d-4)-4) \alpha} \gamma(R) + A.
\]

Then by requiring \( (M-1)^2 \geq (3 + d) \) (note here the equality also works), we can hide \( A \) into left-hand side of this inequality, hence for all \( R \geq R_0(d, A, L) \),

\[
e^\frac{\alpha^2}{R^2} e^{((M-1)^2 + (d-4) \alpha} \leq e^{((M+1)^2 + (d-4)-4) \alpha} \gamma(R).
\]

Simplifying the two exponentials, we get

\[
e^\frac{\alpha^2}{R^2} \leq e^{((M+1)^2 + (d-4)-4) \alpha} \gamma(R)
\]

which implies

\[
\gamma(R) \geq e^\frac{4}{7} R^\frac{4}{7} e^{-cR^\frac{4}{7}},
\]

for all \( R \geq R_0(d, A, L) \).

Now we finish the proof of Lemma 6.1. \( \square \)

7. PROOF OF MAIN THEOREMS

In this section, we prove the main theorems by contradiction.

7.1. Proof of Theorem 1.1. If \( u \neq 0 \), we can assume that \( u \) satisfies the hypotheses of Lemma 6.1, after a translation, dilation and and multiplication by a constant. Thus, there exist constants \( R_0(B), c'(B) \) depending on \( B := \|u\|_{L^1 H^2((0,1] \times \mathbb{R}^d)} \) and a universal constant \( c \) such that

\[
\gamma(R) \geq c' R^{2/3} e^{-c R^{4/3}} \quad \text{for all } R \geq R_0.
\]

Let \( \theta_R(x) \in C^\infty \) be non-decreasing, radial and such that

\[
\theta_R(x) = \begin{cases} 1, & \text{if } |x| \leq R - 1, \\ 0, & \text{if } |x| \geq R. \end{cases}
\]

Now take \( \varphi_R(x) = 1 - \theta_R(x) \), where \( \theta_R(x) \) is as defined above. Now, (2.1) gives

\[
(i \partial_t + \Delta^2)(u \varphi_R) = V_R(u \varphi_R) - \sum_{j=1}^d 4(\partial x_j \varphi) \partial_x^2 u + 6(\partial_x^2 \varphi) \partial_x^2 u + 4(\partial x_j \varphi) \partial_x u + (\partial x_j \varphi) u := V_R(u \varphi_R) + H.
\]
where $V_R(t, x) = \varphi_{R-1}V(t, x)$. We apply Lemma 4.2 to the previous equation to find that for $\alpha > 1$
\[
\sup_{t \in [0, 1]} \int_{|x| \geq R} |u(t, x)|^2 e^{\lambda |x|^\alpha / (10d)\alpha} \, dx
\leq \sup_{t \in [0, 1]} \int_{|x| \geq c_{\alpha}} |u\varphi_R(t, x)|^2 e^{\lambda |x|^\alpha / (10d)\alpha} \, dx
\leq c \left( \|u_0\|_{L^2(\lambda |x|^{\alpha})}^2 + \|u_1\|_{L^2(\lambda |x|^{\alpha})}^2 \right) + c \int_0^1 \int_{R-1 \leq |x| \leq R} \left( |u|^2 + |\nabla u|^2 + |\nabla^2 u|^2 + |\nabla^3 u|^2 \right) e^{\lambda |x|^\alpha} \, dx \, dt
\]
\[ + c \sum_{j=1}^d \sum_{l=0}^3 \int_0^1 \int_\mathbb{R} |\partial_{x_j}^l (u\varphi)|^2 \, dx \, dt
\leq c_0 + c_0 \exp(\lambda R^\alpha).\]

In preparation for another application of Lemma 4.2, we calculate
\[
(i\partial_t + \Delta^2)\partial_{x_k}(u\varphi_R) = V_R(\partial_{x_k}(u\varphi_R)) + \partial_{x_k}V_R(u\varphi_R)
- 4(\partial_{x_j} \varphi_R)\partial_{x_j}^2 \partial_{x_k} u + 4(\partial_{x_j} \partial_{x_k} \varphi_R)\partial_{x_j}^3 u + 6(\partial_{x_j}^2 \varphi_R)\partial_{x_k}^2 \partial_{x_j} u + 6(\partial_{x_j}^2 \partial_{x_k} \varphi_R)\partial_{x_j}^2 u
+ 4(\partial_{x_j} \varphi_R)\partial_{x_j} \partial_{x_k} u + 4(\partial_{x_j} \partial_{x_k} \varphi_R)\partial_{x_j} u + (\partial_{x_j}^4 \varphi_R)\partial_{x_j} u + (\partial_{x_j}^4 \partial_{x_k} \varphi_R)u
= V_R(\partial_{x_k}(u\varphi_R)) + \tilde{H}_k.
\]

Applying Lemma 4.2 to the above equation, we find that
\[
\sup_{t \in [0, 1]} \int_{|x| \geq R} |\partial_{x_k}u(t, x)|^2 e^{\lambda |x|^\alpha / (10d)\alpha} \, dx
\leq \sup_{t \in [0, 1]} \int_{|x| \geq c_{\alpha}} |\partial_{x_k}(u\varphi_R)|^2(t, x) e^{\lambda |x|^\alpha / (10d)\alpha} \, dx
\leq c_0 + c \int_0^1 \int_{R-1 \leq |x| \leq R} \left( |u|^2 + |\nabla u|^2 + |\nabla^2 u|^2 + |\nabla^3 u|^2 \right) e^{\lambda |x|^\alpha} \, dx \, dt
\]
\[ + c \int_0^1 \int_\mathbb{R} |u\varphi_R\partial_{x_k} V_R|^2 e^{\lambda |x|^\alpha / (10d)\alpha} \, dx \, dt
\]
\[ + c \sum_{j=1}^d \sum_{l=0}^3 \int_0^1 \int_\mathbb{R} |\partial_{x_j}^l \partial_{x_k}(u\varphi)|^2 \, dx \, dt
\leq c + c \exp(\lambda R^\alpha).\]

We repeat this application of corollary up to the equations of $(i\partial_t + \Delta^2)\partial_{x_k}^2(u \varphi_R)$, $(i\partial_t + \Delta^2)\partial_{x_k}^3(u \varphi_R)$ and combine all the conclusions to see that
\[
\sup_{t \in [0, 1]} \int_{|x| \geq R} \left( |u|^2 + |\nabla u|^2 + |\nabla^2 u|^2 + |\nabla^3 u|^2 \right) e^{\lambda |x|^\alpha / (10d)\alpha} \, dx \leq c_0 + c_0 \exp(\lambda R^\alpha). \tag{7.1}
\]

Thus, for all $\mu$ such that $\mu R - 1 > R$,
\[
c_0(\mu R)^{2/3} e^{-c(\mu R)^{4/3}} e^{\lambda (\mu R - 1)^\alpha / (10d)^\alpha} \leq c_0(\mu R)^{\gamma} e^{\lambda (\mu R - 1)^\alpha / (10d)^\alpha}
\]
\[
\leq c \int_0^1 \int_{\mu R - 1 \leq |x| \leq \mu R} \sum_{l=0}^3 |\partial_{x_l}^l u(t, x)|^2 e^{\lambda (\mu R - 1)^\alpha / (10d)^\alpha} \, dx \, dt
\]
\[
\leq c \int_0^1 \int_{\mu R - 1 \leq |x| \leq \mu R} \sum_{l=0}^3 |\partial_{x_l}^l u(t, x)|^2 e^{\lambda |x|^\alpha / (10d)\alpha} \, dx \, dt
\]
\[
\leq c \int_0^1 \int_{\mu R - 1 \leq |x|} \sum_{l=0}^3 |\partial_{x_l}^l u(t, x)|^2 e^{\lambda |x|^\alpha / (10d)\alpha} \, dx \, dt.
\]
\[ \leq c \int_0^1 \int_{R \leq |x|} \sum_{l=0}^3 |\partial_x^l u(t,x)|^2 e^{\lambda|x|^\alpha/(10d)^\alpha} \, dx \, dt \]
\[ \leq c + c \exp(\lambda R^\alpha), \]
where the last inequality is just (7.1). Since \( \alpha > 4/3 \), taking large enough \( \mu \), we obtain a contradiction as the left hand side of the chain of inequalities is unbounded, while the right side is bounded. Thus, \( u \equiv 0 \) identically.

We have finished the proof of Theorem 1.1.

7.2. **Proof of Theorem 1.2.** We consider the difference of the two solutions
\[ w(t,x) = u_1(t,x) - u_2(t,x), \]
which satisfy the equation
\[ i\partial_t w + \Delta w = \left( \frac{F(u_1,\overline{u}_1) - F(u_2,\overline{u}_2)}{u_1 - u_2} \right) w. \]
We repeat the same arguments with the new potential
\[ V(t,x) = \frac{F(u_1,\overline{u}_1) - F(u_2,\overline{u}_2)}{u_1 - u_2}, \]
which satisfies the necessary hypotheses on \( V(t,x) \) given the regularity on \( F(u,\overline{u}) \). By the same contradiction argument, we obtain Theorem 1.2.

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