Captions of figures

Fig. 1: The schematic diagram of the production and the consequent decay of the complex kaon system. The double lines represent the propagator matrix.

Fig. 2: The s-channel Kaon exchange contributions to the resonant transition amplitude.
The Quantum Field Theory of the Kaon Oscillations

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ABSTRACT

We consider the covariant formulation of the kaon mixing in the context of the propagator method. It results important to check the possibility of a sizable effect in the vacuum regeneration of kaons. We discuss all those terms which may give relevant contributions to modify the exponential decay law of the Wigner–Weisskopf narrow width approximation. Moreover, we examine the characteristic structure of the complex singularities of the matrix propagator and we provide a generalized form of the Bell-Steinberger unitarity sum rule.

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I. Introduction

The origin of $CP$ violation is still not explained, although the Standard Model of the electroweak interactions can accommodate a complex violating phase in the quark mixing matrix [1]. However, the eventuality that $CP$ violation originates from some effects at a much higher energy scale is not excluded. Indeed, the high sensitivity of the $K^0 - \bar{K}^0$, makes it a testing ground of even more speculative proposals like $CPT$-violation [2] and deviations from the conventional Quantum Mechanics [3]. In view of the planned high precision experiments in kaon physics, it is then important to reconsider the fundamental aspects of the space-time evolution of the $K^0 - \bar{K}^0$ system also beyond the generalized single pole or Lee-Oehme-Yang (LOY) approximation [4].

Physically, it is worthwhile to examine the possibility of a sizable effect in the vacuum regeneration of kaons [5]. We cannot neglect another complaint of the LOY formalism which seems to exclude neutral mesons systems from the realm of its sensible applications. In fact, its assumption that the mixing of elementary particles is independent of the momentum, appears too drastic. The inclusion of the $q^2$ dependence, usually thought to be small, could have unexpectedly large effects mainly in the resonant $CP$-violating processes [6]. The usual method of the LOY approximation neglects these effects. Another severe limitation in the application of this approximation consists in the fact that it rests completely inappropriate to implement the notion of the rest frame for an oscillating unstable composite system. In this case, there is a subtle point worth noting here, referring to the phase ambiguities that arise in non relativistic theories when the superposition of states with different mass and momentum are described in a Galilean invariant form [6]. Indeed, an explicit relativistic description of the neutral meson systems seems to be required. It is important to stress that if we consider the problem in the formalism of the group representations, we may find difficulties to describe unstable particles on the same footing of the stable ones because non-unitary representations of the Poincaré group are requested by a fundamental complex mass [6].
In this paper, we consider the covariant formulation of the kaon mixing in the context of the propagator method. Similar works were done along this line [7], but here we do not use a specific dynamical model and general \( q^2 \)-dependent relations are considered. In fact, the structure of the propagator is considered only from the general assumptions of Lorentz invariance and causality. Furthermore, we discuss all those terms which may give significant contributions and we do not restrict to the pole approximation which gives the dominant exponential law in the time evolution. Moreover, we consider the analytical continuation of the Fourier transformed propagator in the second Riemann sheet and we analyze the characteristic structure of the complex singularities. Apart the previous theoretical warnings, it will be clear that there are some advantages to work with the propagator method rather than in the LOY approximation. Indeed, it can be easily seen that the main results are equivalent.

II. The Shortcomings of the Pole Approximation in the Kaon System

The phenomenological description of the evolution of a \( K^0 \) meson consists in the application of a generalized Wigner–Weisskopf narrow width approximation in the \( K^0, \bar{K}^0 \) subspace. Usually, one restricts to a single pole or LOY approximation which is supplemented by the unitarity sum rule of Bell and Steinberger [8], connecting the two dimensional subspace with the space of all final states for the decaying system. In this method, the neutral \( K \)-meson system is described by a scattering theory with the Hamiltonian \( H = H_0 + H_{\text{int}} \), where \( H_0 \) contains the strong interactions under which the \( K^0 \) and \( \bar{K}^0 \) mesons appear as stable particles, and \( H_{\text{int}} \) induces their decay into the continuous spectrum of \( H_0 \). Let \( \mathcal{H} = \mathcal{H}_K \oplus \mathcal{H}_F \) be the Hilbert space of the neutral \( K \) mesons together all their decay products and \( \mathcal{P} \) the projection into the two-dimensional subspace \( \mathcal{H}_K \) spanned by \( K^0, \bar{K}^0 \) (or by \( K_{1,2} = \frac{1}{\sqrt{2}}(K^0 \pm \bar{K}^0) \)) and \( \mathcal{Q} = \mathcal{I} - \mathcal{P} \) the projection on the continuous final states part \( \mathcal{H}_F \) of the spectrum \( H_0 \). Then \( H_{\text{int}} \) induces the decay of the \( K \)-mesons only if \( \mathcal{Q}H_{\text{int}}\mathcal{P} \neq 0 \), so that the evolution of the whole system is given by the dynamical semigroup relation:

\[
U(t) = \exp[-iHt] .
\]

It is well-known, however, that the projection of this time evolution into a subspace
cannot, in general, have the semigroup property, if $H$ is not Hermitian. Therefore, the total evolution of Eq. (1) does not conserve the subspace $\mathcal{H}_K = \mathcal{P}\mathcal{H}$ of the Hilbert space $\mathcal{H}$. In fact, the evolution of the $K$-meson system is governed by

$$U'(t) = \mathcal{P}U(t)\mathcal{P}$$  \hspace{1cm} (2)$$

where $\mathcal{P}$ does not commute with $H$ and then $U'(t)$ does not satisfy the semigroup law

$$U'(t_1)U'(t_2) \neq U'(t_1 + t_2).$$  \hspace{1cm} (3)$$

On the other hand, the Wigner–Weisskopf (WW) approximation with constant (not Hermitian) Hamiltonian assures the validity of the semigroup property. On the other side, the assumption of constant decay rates as they arise in the unitary sum rule is also not justifiable if the time-reversal $T$ invariance is not a symmetry of the underlying Hamiltonian [9]. In fact, in this case, the condition of reciprocity is not properly satisfied. Relaxing the WW pole approximation, we shall now obtain an approximate form for $U'(t)$ which enable us to derive the unitary sum rule in a generalized form.

In general, $U(t)$ can be represented as the Laplace transform of the resolvent $G(s) = [s\mathcal{I} - H]^{-1}$

$$U(t) = \frac{1}{2\pi i} \oint G(s) \exp(-ist) \, ds$$  \hspace{1cm} (4)$$

where the integration path is around the spectrum of $H$. Hence $U'(t)$ can be expressed in terms of the analytic properties of the reduced resolvent (the propagator in the $K$ meson space)

$$G'(s) = \mathcal{P}G(s)\mathcal{P}$$  \hspace{1cm} (5)$$

in the form

$$U'(t) = \frac{1}{2\pi i} \oint G'(s) \exp(-ist) \, ds$$  \hspace{1cm} (6)$$

where $G$ and $G'$ are $2 \times 2$ matrices. The resolvent $G(s)$ of the total Hamiltonian $H$ can be assumed analytic on the entire complex plane except for the spectrum of $H$. Since the discontinuity of $G(s)$ is unbounded across the cut along the spectrum of $H$, the cut forms a natural boundary and the resolvent cannot be analytically continued across. For a suitable $H_{int}$, however, the discontinuity of the reduced resolvent $G'(s)$ can be regular on some open set belonging to the spectrum of $H$ and there may then
exist an analytic continuation of \( G'(s) \). In a model in which \( QH_{\text{int}}Q = 0 \) (no final state interactions), and for sufficiently weak coupling, we may assume that the degenerate eigenvalues in the discrete spectrum of \( H_0 \) appear as two poles in the second sheet of the reduced total resolvent \( G' \) which are not-degenerate due to the different phase space strength of coupling to the decay channels. We further assume that the rank of \( G'(s) \) remains two in its domain of regularity, and hence \( G'(s) \) admits an inverse. The inverse of \( G'(s) \equiv \Delta'(s) \) restricted to the subspace \( \mathcal{P} \mathcal{H} = \mathcal{H}_K \) is \( (\Delta'(s))^{-1} \) so that

\[
G'(s)\Delta'^{-1}(s) = \Delta'^{-1}(s)G'(s) = \mathcal{P}.
\] (7)

In order to give our notation some physical content, we may consider the specific problem of the kaon mixing, although it should be clear that the method is general. In absence of weak interactions, \( K^0 \) and \( \bar{K}^0 \) are eigenstates of the strong interactions and form a degenerate particle–antiparticle pair in flavour state, with a common mass \( m_\circ \) (whatever we assume \( CPT \) invariance). When higher order weak interactions are introduced, transitions are induced between \( K^0 \) and \( \bar{K}^0 \). Thus, mixing prohibits the \( K^0 \), \( \bar{K}^0 \) scalar mesons from propagating independently of each other. Including the effects of the weak interactions, the full dressed matrix propagator is given by

\[
\Delta'(q^2) = [q^2I - \Lambda(q^2)]^{-1},
\] (8)

where the effective \( \Lambda \) matrix consists in the following sum of the bare square-mass matrix and the proper self-energy \( \Pi \) contributions

\[
\Lambda(q^2) = [m_\circ^2I + \Pi(q^2)],
\] (9)

in a representation referring to the set of the factorized constituents \( K^0, \bar{K}^0 \). As a consequence of \( CPT \) invariance, the diagonal matrix elements are equal, whereas the off-diagonal elements \( \Delta'_{ij} \) are equal only in the case the interactions are all \( CP \) invariant. Anyway, any invariance of an underlying theory will reflect itself in an invariance of the propagator and then also of the square mass matrix \( \Lambda \). Dropping the superscript prime, the time evolution of the flavour states

\[
\begin{pmatrix}
|K^0(t)\rangle \\
|\bar{K}^0(t)\rangle
\end{pmatrix} = \mathcal{U}(t) \begin{pmatrix}
|K^0\rangle \\
|\bar{K}^0\rangle
\end{pmatrix},
\] (10)
and similarly for the mass right–eigenstates

\[
\begin{pmatrix}
|K_S(t)\rangle \\
|K_L(t)\rangle
\end{pmatrix} = \mathcal{V}(t) \begin{pmatrix}
|K_S\rangle \\
|K_L\rangle
\end{pmatrix}
\] (11)

are governed by the evolution matrices $\mathcal{U}$ and $\mathcal{V}$ which are related by the following similarity transformation

\[
\mathcal{U} = \mathcal{R}^{-1}\mathcal{V}\mathcal{R} \quad .
\] (12)

The transformation between the physical and flavour bases are given by

\[
\begin{pmatrix}
|K_S\rangle \\
|K_L\rangle
\end{pmatrix} = \mathcal{R} \begin{pmatrix}
|K_0\rangle \\
|K_0\rangle
\end{pmatrix}
\] (13)

where $\mathcal{R}$ is usually parameterized according to the following relations

\[
\mathcal{R} = \frac{1}{\sqrt{2(1+|\epsilon|^2)}} \begin{pmatrix}
(1+\epsilon) & -(1-\epsilon) \\
(1+\epsilon) & (1-\epsilon)
\end{pmatrix} = \frac{1}{\sqrt{1+|\eta|^2}} \begin{pmatrix}
1 & \eta \\
1 & -\eta
\end{pmatrix} = \begin{pmatrix}
p & -q \\
p & q
\end{pmatrix}
\] (14)

where the normalization factor $(|p|^2 + |q|^2)^{-\frac{1}{2}}$ is assumed here unity. We remember that the phases of $p, q$ may be altered by redefining the phases of the $K$ states, so that both $p$ and $q$ are not measurable quantities, whereas the overlap between $K_S$ and $K_L$

\[
\langle K_S|K_L \rangle = \frac{2\text{Re } \epsilon}{1+|\epsilon|^2} = \frac{1 - |\eta|^2}{1 + |\eta|^2}
\] , (15)

is independent of any phase convention at the same strength of the magnitude of the variable

\[
\eta = -\frac{q}{p} = -\frac{1 - \epsilon}{1 + \epsilon}
\] (16)

Assuming $\Delta S = \Delta Q$ rule conserved, this last quantity is directly connected to the amount of the kaon semileptonic charge rate

\[
A_{SL} = \frac{\Gamma(K_L \to \ell^+\nu X) - \Gamma(K_L \to \ell^-\nu X)}{\Gamma(K_L \to \ell^+\nu X) + \Gamma(K_L \to \ell^-\nu X)} = \frac{1 - |\eta|^2}{1 + |\eta|^2}.
\] (17)

Nevertheless, in order to describe the kaon system in terms of uncoupled channels, we need to diagonalize $\Delta'$. The fact that $\Lambda$ is, in general, momentum dependent does not introduce any additional complications, in practice, since $s = q^2$ is always fixed by the on-shell condition of the initial particles. Anyway, the resulting eigen-physical fields are those with a definite propagation behaviour.
Neglecting the prime superscript, the regularized inverse propagator can be rewritten as
\[
\Delta^{-1}_{ij} = [sI - \Lambda] = R^{-1}_{\alpha\alpha} \Delta^{-1}_{\alpha\beta} R_{\beta j}
\] (18)
where
\[
\Delta^{-1}_{\alpha\beta}(s) = [s\delta_{\alpha\beta} - N_{\alpha\beta}]
\] (19)
and
\[
N_{\alpha\beta} = [RAR^{-1}]_{\alpha\beta} .
\] (20)

Here, according to a standard prescription [6], the latin indices denote $K^0$, $\bar{K}^0$ and the greek letters denote $K_S$, $K_L$, respectively. Although, in general, $\Pi$ (and hence $\Lambda$), is momentum dependent, in principle the matrix propagator can be brought into a diagonal form
\[
\Delta_{\alpha\beta}(q^2) = \begin{pmatrix} \Delta_S(q^2) & 0 \\ 0 & \Delta_L(q^2) \end{pmatrix}
\] (21)
through a $q^2$-dependent transformation. It is worth noting that $\Lambda(q^2)$ shares all the properties of the effective Hamiltonian in the description of the kaon system. In particular, $CPT$ invariance requires that $\Lambda_{11} = \Lambda_{22}$ and $CP$ invariance prescribes the equality of the off-diagonal elements $\Lambda_{12} = \Lambda_{21}$. Thus, the $K^0 - \bar{K}^0$ mixing gives rise to $CP$ violation through the effective mass-squared matrix $\Lambda(q^2)$. The basic parameter which characterizes the indirect $CP$ violation induced by the mixing in the kaon system is then given by
\[
\eta = \frac{-q}{p} = \frac{-(1 - \epsilon)}{(1 + \epsilon)} = \sqrt{\frac{\Lambda_{21}}{\Lambda_{12}}}
\] (22)
which is a rephasing invariant quantity and hence physically meaningful.

The LOY approximation is equivalent to assume the following weak coupling
\[
\Pi_{ij}(q^2) \simeq \Pi_{ij}(m_0^2) .
\] (23)

In this case,
\[
\Lambda = [m_0^2 I + \Pi(m_0^2)]
\] (24)
can be diagonalized by a complex matrix $R$:
\[
[RAR^{-1}]_{\beta\alpha} = \lambda^2_{\alpha} \delta_{\beta\alpha}
\] (25)
where, in brief we obtain

\[ \lambda^2_S = \frac{1}{2} [(\Lambda_{11} + \Lambda_{22}) - Q] \]
\[ \lambda^2_L = \frac{1}{2} [(\Lambda_{11} + \Lambda_{22}) + Q] \]  

with

\[ Q = \sqrt{(\Lambda_{11} - \Lambda_{22})^2 + 4\Lambda_{12}\Lambda_{21}}. \]

The physical fields \( K_L \) and \( K_S \) corresponding to the eigenvalues

\[ \lambda^2_{S,L} = \left( m_{S,L} - i\gamma_{S,L} \right)^2 \simeq m^2_{S,L} - im_{S,L}\gamma_{S,L} \]

are combinations of the \( K^0 \) and \( \bar{K}^0 \) for which only the diagonal elements of the propagator matrix contain poles in the \( K_L, K_S \) basis according to the following expression

\[ \Delta'_{\alpha\beta}(q^2) = \begin{pmatrix} \frac{1}{q^2 - \lambda^2_S} & 0 \\ 0 & \frac{1}{q^2 - \lambda^2_L} \end{pmatrix}. \]

In this approximation, extracting the two non degenerate poles \( \lambda^2_{\alpha} \), we obtain the following form for the time evolution matrix

\[ U(t) = \begin{pmatrix} g_1(t) & \eta g_2(t) \\ \eta g_2(t) & g_1(t) \end{pmatrix} \]

where \( g_{1,2}(t) = \frac{1}{2}(V_{SS} \pm V_{LL}) = \frac{1}{2}(e^{-i\lambda_ST} \pm e^{-i\lambda_LT}) \).

Nevertheless, since the LOY method is an approximate theory, it is not surprising that it cannot satisfy exactly the unitarity requirement [9], which is essential for the basic interpretation of any theory. The assumption of constant decay rates, as they arise by the unitarity sum rules [8] connecting the kaon system with the space of all the decaying final states, then, cannot be justified in an exact sense. In fact, for instance, the modulus of the ratio of the off-diagonal elements

\[ r(t) = \frac{U_{12}(t)}{U_{21}(t)} = \left( \frac{q}{p} \right)^2 \]

differs from unity and could vary with time [5] if the time-reversal \( T \)-invariance is not a symmetry of the underlying Hamiltonian. In fact, in this case, the condition of reciprocity is not properly satisfied [9]. Indeed, the inclusion of off-diagonal terms \( V_{SL} = -V_{LS} \) in the evolution matrix \( V \), induces a modification both of the time evolution
matrix and also of the previous ratio. Assuming a global $CPT$-invariant propagation, in fact, it becomes

$$r(t) = \left(\frac{q}{p}\right)^2 \left(\frac{1 - A}{1 + A}\right) .$$

(32)

On more general assumptions like causality and analyticity, the time dependence of the vacuum regeneration term $A = (V_{LS} - V_{SL})/(V_{SS} - V_{LL})$ can be obtained to study the $s = q^2$ dependence of the proper self energy contributions $\Pi$. In particular, the very short time behaviour cannot be derived taking solely into account the mentioned poles.

The non exponential contributions can be estimated with somewhat greater generality to introduce some analytical techniques used in the recent literature on decay problems [10]. In fact, there is a resorted interest about the validity of the exponential decay law. In general, the time evolution of a metastable quantum state is roughly described by three distinct trends. At very short time, the decay rate were noted to be characterized by a Gaussian behaviour. The exponential decay is expected within a limited time interval. An inverse power law will remain at long times in dependence of the structure of the initial state. Technically, we can say that at short times with respect to the inverse width there is no exponential evolution because other poles, lying further from the real axis, may become important. On the other side, for very long times, the cut contribution exceeds the exponential decrement so that we get a residual inverse power law. We shall not be concerned with the power law here, but we confine to study the breakdown of the exponential decay law due to the occurrence of bound states (poles on the real energy axis). Therefore, we neglect the effects of the interactions of the (in)elastic rescattering of the final decay modes (branch points on the real axis). However, the matrix elements of $\Pi(s)$ have a cut along the positive real axis whose threshold is $m_{th}^2$ and are expected to have no poles there (Herglotz property). If the interaction is small, $\Delta(s)$ has two poles and we obtain the exponential law for the evolution of the kaon complex. But since the pole residues are nonorthogonal matrices rather than numbers, the dynamical semigroup law Eq. (3) of the time evolution is violated, unless there is some symmetry which forces the Hamiltonian to be Hermitian. Nevertheless, new poles emerging from the analytical continuation become very important when they are close enough to the axis. The dynamics for an eventual complex pole $s_p$ of the matrix propagator in the second Riemann sheet is however regulated by the following relation which locates the
position of the poles
\[
\det \left[ s_p \delta_{\alpha\beta} - \Lambda_{\alpha\beta} \right] = 0,
\]
where \( \Pi \) and hence \( \Lambda \) are evaluated in the second Riemann sheet. Of course, these eventual poles become more important as they are closer to the real axis. The factorization of the \( s_p \) pole in the transition propagator, yields
\[
\Delta'^{-1}(s) = (s - s_p) \mathcal{Z}^{-1}.
\]

Notice that we have absorbed all renormalization effects into the matrix \( \mathcal{Z} \) which represents the residue of the full propagator at the \( s_p \) pole. Extracting the leading term of the Laurent expansion about \( s_p \), in terms of renormalized quantities we obtain
\[
\mathcal{Z}^{-1} = [I - \Pi'(s_p)] .
\]

However, it is possible to determine an exact expression for the residue of the full propagator in terms of the projection operators
\[
P_{S,L} = |K_{S,L}(s)\rangle\langle K'_{S,L}(s)| .
\]

In order to obtain an explicit form for \( \mathcal{Z}_{\alpha,\beta} \), we can separate the reduced resolvent \( \Delta' \) into two parts
\[
\Delta'(s) = \frac{P_S(s)}{s - \lambda^2_S(s)} + \frac{P_L(s)}{s - \lambda^2_L(s)}
\]
where \( \lambda^2_{S,L}(s) \) denote the two eigenvalues of \( \Lambda \) and \( P_{S,L} \) satisfy the following properties
\[
\begin{cases}
P_S(s)P_L(s) = P_L(s)P_S(s) = 0 \\
(P_{S,L})^2 = P_{S,L}
\end{cases}
\]

Hence \( \Delta' \) has two simple poles corresponding to the values \( s_S \) and \( s_L \), obtained by the equations
\[
s_S - \lambda^2_S(s_S) = 0, \quad s_L - \lambda^2_L(s_L) = 0.
\]

Therefore for any given \( s \), \( \Lambda(s) \) may be brought to a diagonal form by a complex transformation whose columns are determined by these eigenvalues. In the case that \( \Lambda \) may be treated as a constant, independent of \( s \), over the entire range of interest, this
transformation is independent of $s$. A direct calculation of the residue of the propagator at the $s_S$ yields (an analogous expression is given for $Z(s_L)$ replacing the subscript $L$ with $S$)

$$Z(s_S) = \frac{\mathcal{P}_S(s_S)}{1 - \frac{d\lambda_\pm(s)}{ds}\big|_{s=s_S}}.$$ (40)

Of course, we could obtain a similar result from the viewpoint of dispersion relations, using the separation of the one-particle states from the $q^2$-dependent terms and indeed without talking about fields at all [10].

### III. Unitarity Sum Rules

The question of the correct treatment of the complex kaon faced with the problem that we must consider an Hilbert space $\mathcal{H} = \mathcal{H}_K \oplus \mathcal{H}_F$, composed by the two dimensional neutral K–meson space $\mathcal{H}_K$ and the space $\mathcal{H}_F$ of all their decay final states. The evolution of the entire system should be defined by a semi–group of unitary operators in $\mathcal{H}$ with a total self–adjoint Hamiltonian. Therefore, once the decay channels being included into the base, the resulting total Hamiltonian will respect a semi–group evolution. If we suppose that the set of final states $\{F\}$ forms an orthonormal basis in $\mathcal{H}_F$, the phenomenological description of the $K^0\overline{K}^0$ meson decays [11] consists in the application of the LOY approximation supplemented by the unitarity sum rule of Bell and Steinberger [8] connecting the two dimensional kaon subspace with the space of all final states for the decaying system. In order to show the generalization of the Bell and Steinberger’s relation, it is necessary to be specific about their assumptions.

The time evolution of the entire state vector $|\Psi(t)\rangle$, composed of the neutral K-meson system with its decay channels $F$, is governed by the total Hamiltonian and it satisfies a completeness relation

$$\sum_i |\langle K_i|\Psi(t)\rangle|^2 + \sum_F |\langle F|\Psi(t)\rangle|^2 = ||\Psi||^2.$$ (41)

Introducing the reduced evolution $\mathcal{U}$ in $\mathcal{H}_K$ in contour representation (we omit the prime index for brevity)

$$\mathcal{U}_{ij}(t) = \frac{1}{2\pi i} \int_{\text{Spec}(H)} ds e^{-ist} \Delta_{ij}'(s).$$ (42)
where the integration extends over the whole spectrum of the Hamiltonian, at any given time, initial kaon pure states \((U_{ij}(0) = \delta_{ij})\) decay into an \(F\) channel mode with an amplitude probability

\[
A_{iF}(t) = \langle F|K_i(t)\rangle = \langle F|U_{ij}(t)|K_j\rangle
\]  

being

\[
A_D = (A_{iF}) = \begin{pmatrix} \langle F|K^0(0)\rangle \\ \langle F|K^0(0)\rangle \end{pmatrix} = \begin{pmatrix} A(K^0 \rightarrow F) \\ A(K^0 \rightarrow F) \end{pmatrix} .
\]

Squaring, we obtain the time–dependent rates

\[
\Gamma(K_i(t) \rightarrow F) = \sum_{kj} U^*_{ki} U_{kj} |A_{jF}|^2
\]

where the summation over repeated indices is tacitly assumed. Then, the rate of decay into the final state \(F\) at time \(t\) can be described in terms of the complex matrix elements \(M_{ij} = g_i g_j^*\) and of the amplitude probability at time \(t = 0\)

\[
\Gamma (K^0(t) \rightarrow F) = |A_F|^2 \left[ M_{11} + |\xi_F|^2 M_{22} - 2\text{Re}(\xi_F M_{21}) \right]
\]

\[
\Gamma (\bar{K}^0(t) \rightarrow F) = |A_F|^2 \left[ M_{22} + |\xi_F|^2 M_{11} - 2\text{Re}(\xi_F M_{12}) \right] |\eta|^{-2} .
\]

Using the explicit form of \(M_{ij}\), the following general formula for the time–dependent decay rates can be derived

\[
\Gamma (K^0(t) \rightarrow F) = |A_F|^2 e^{-\Gamma t} \frac{1}{2} \left\{ \cosh \left( \frac{\Delta \gamma}{2} t \right) + \cos(\Delta mt) + \right. \\
+ |\xi_F|^2 \left[ \cosh \left( \frac{\Delta \gamma}{2} t \right) - \cos(\Delta mt) \right] + \\
\left. + 2\text{Re} \left[ \xi_F \left[ \sinh \left( \frac{\Delta \gamma}{2} t \right) + i \sin(\Delta mt) \right] \right] \right\}
\]

for \(K^0(t)\) and

\[
\Gamma (\bar{K}^0(t) \rightarrow F) = |A_F|^2 e^{-\Gamma t} \frac{1}{2} \left\{ \cosh \left( \frac{\Delta \gamma}{2} t \right) - \cos(\Delta mt) + \right. \\
+ |\xi_F|^2 \left[ \cosh \left( \frac{\Delta \gamma}{2} t \right) + \cos(\Delta mt) \right] + \\
\left. + 2\text{Re} \left[ \xi_F \left[ \sinh \left( \frac{\Delta \gamma}{2} t \right) - i \sin(\Delta mt) \right] \right] \right\} |\eta|^{-2}
\]
for $\bar{K}^0(t)$. For convenience, in the previous formulas we have introduced the complex parameter $\xi_F = \frac{q_F}{p_F} A(\bar{K}^0 \to F)$ and the quantities

$$
\Delta m = m_S - m_L, \quad m = \frac{m_S + m_L}{2},
$$

$$
\Delta \gamma = \gamma_S - \gamma_L, \quad \Gamma = \frac{\gamma_S + \gamma_L}{2}.
$$

(49)

The time–integrated rates of interest

$$
\hat{\Gamma}(K_i \to F) = \int_0^\infty dt \Gamma(K_i(t) \to F) = \int_0^\infty dt |A_{iF}(t)|^2,
$$

(50)

which with the position $\hat{M}_{ij} = \int_0^\infty dt M_{ij}(t)$ become

$$
\hat{\Gamma}(K^0 \to F) = |A_F|^2 \left[ \hat{M}_{11} + |\xi_F|^2 \hat{M}_{22} - 2 \text{Re}(\xi_F \hat{M}_{21}) \right]
$$

$$
\hat{\Gamma}(\bar{K}^0 \to F) = |A_F|^2 \left[ \hat{M}_{22} + |\xi_F|^2 \hat{M}_{11} - 2 \text{Re}(\xi_F \hat{M}_{12}) \right] |\eta|^{-2},
$$

(51)

let us write the content of the Bell and Steinberger’s unitarity sum rule:

$$
\langle K_S | \Gamma | K_L \rangle = \left[ \left( \frac{\gamma_S + \gamma_L}{2} \right) - i(m_S - m_L) \right] \langle K_S | K_L \rangle = \sum_F \langle F | H_K | K_S \rangle^* \langle F | H_K | K_L \rangle = \sum_F \langle F \rangle (A + iB)
$$

(52)

where the last expression has been obtained by choosing the final decay modes $F$ to be CP-eigenstates and the integration with respect to the phase space must be understood.

We have defined

$$
\langle \Gamma_F \rangle = \frac{1}{1 + |\eta|^2} \left[ \Gamma(K^0 \to F) + |\eta|^2 \Gamma(\bar{K}^0 \to F) \right] \simeq \frac{1}{2} \left[ \Gamma(K^0 \to F) + \Gamma(\bar{K}^0 \to F) \right].
$$

(53)

The two independent CP–violating parameters are given by

$$
A = \frac{1 - |\xi_F|^2}{1 + |\xi_F|^2},
$$

$$
B = \frac{2 \text{Im} \xi_F}{1 + |\xi_F|^2}
$$

(54)

and $\langle K_L | K_S \rangle = \frac{1 - |\eta|^2}{1 + |\eta|^2} \simeq 10^{-3}$ imposes large cancellations in the sum. The first of these parameters ($A$) vanishes in the absence of $K^0 - \bar{K}^0$ mixing. By contrast, the second parameter ($B$) yields $CP$–violation in the decay amplitude (i.e. $|\xi_F| \neq 1$).

We can reformulate this unitarity sum rule by means of the propagator method. The propagator appears in the scattering amplitude as a factor sandwiched between vertex
functions referring to the particular processes in which the unstable system is produced and detected. The vertex functions of the produced (P) and decay (D) positions are assumed to be slowly varying functions of momentum over the range corresponding to the width of the mesons. Evidently, until the dynamics of the meson system has been described in terms of initial and final asymptotic states, it results unitary and causal. Actually, the decay rates of the neutral kaons are so small to legitimate the use of $K^0$ and $\bar{K}^0$ as two asymptotic states of the scattering S-matrix. Of course, this view is not justified for a very short time interval (much shorter than the mean life of $K_S$). In fact in this time interval, decay processes cannot be described only by the pole dynamics.

Consider a process of the sort studied in CPLEAR experiments and schematically shown in Fig. 1. For definiteness, let us assume that the external lines attached to the production vertex P represent the initial I channel mode (for example $\bar{p}p$), the internal line represents the neutral kaon $K^0\bar{K}^0$ complex and the external lines attached to the decay vertex D represents the final F decay mode (for example two pions). At the CPLEAR experiment, $K^0$ and $\bar{K}^0$ are produced at point $x$ in the strong interaction of $\bar{p}p$ and subsequently decay at point $x'$ to $\pi^+\pi^-$. In pole approximation, the description of the time evolution assumes that the $q^2$ dependence of the matrix elements comes from the intermediate propagators, and none from the vertices. As a first approach, it does not seem an excessively drastic truncation to use the pole approximation and to neglect the dependence of $\Pi$ on $q^2$ in a large range. This pole approximation consists in replacing the full propagator with the pole form of Eq. (29) and the two vertices by their values at the pole. First of all, we want now to show the connection of this formalism with the usual properties of particle mixing in which we solve a Schrödinger equation for an effective Hamiltonian $H_K = M - \frac{i}{2}\Gamma$ acting on a Hilbert space in which the only states are one-particle states. It is characteristic of such approximation that the eigenvalues and the eigenstates of the effective Hamiltonian are found from the columns of the diagonalizing complex matrix $R$. The effective Hamiltonian is not Hermitean thus these eigenstates are not in general orthogonal because of the existence of common decay channels. In comparing these methods, note that in the propagator method the notation refers to the square of an effective complex mass matrix

$$\Lambda \simeq M^2 - iM\Gamma$$

(55)
whereas the effective Hamiltonian $H$ is decomposed according to the mass and decay Hermitean matrices

$$ M = \frac{1}{2} \left( H + H^\dagger \right) \quad \text{and} \quad \Gamma = i \left( H - H^\dagger \right) \quad (56) $$

In the single pole approximation, the two formalisms become equivalent if we neglect terms of order $\Gamma/4$, so that we approximate

$$ \lambda_{S,L}^2 \simeq m_{S,L}^2 - i m_{S,L} \gamma_{S,L}. \quad (57) $$

As we stressed above, within a sensible energy interval, the production $\hat{A}_P = (\hat{A}_{II})$ and decay $\hat{A}_D = (\hat{A}_{jF})$ amplitudes can be considered as momentum–independent quantities. The general s-channel scattering amplitude can be extracted from the diagrams of Fig. 2. The transition amplitude is given in a matrix notation by

$$ \tau_{FI} = \sum_{ij} \hat{A}_{F_i}[s\mathbf{I} - \Lambda]^{-1} \hat{A}_{jI} = \hat{A}_D^\dagger[s\mathbf{I} - \Lambda]^{-1} \hat{A}_P \quad (58) $$

where $\hat{A}_{D,P}$ represent the transition amplitudes in the propagator notation. After the diagonalization of $\Lambda$, in terms of the physical states the transition matrix takes the form

$$ \tau_{FI} = \sum_\alpha \left[ \tilde{A}_{F\alpha} \tilde{A}_{\alpha I} \right] \frac{\tilde{A}_{FS} \tilde{A}_{SI}}{s - \lambda_{S}^2} + \frac{\tilde{A}_{FL} \tilde{A}_{LI}}{s - \lambda_{L}^2} \quad (59) $$

where the summation over the repeated indices and integration over the phase space of the final states $F$ must be understood. The unitarized tilded amplitude $\tilde{A} = R\hat{A}$ represents the decay amplitude of a $K_\alpha$ eigenstate into a decay channel $F$. Due to the unitarity of the scattering matrix $S_{FI} = \delta_{FI} - i\tau_{FI}$ and using the optical theorem at the level of the effective anti Hermitian part of $\Lambda$ in the particular case of real values of the Mandelstam $s$ variable, we obtain

$$ R \left( \Lambda - \Lambda^\dagger \right)_{\alpha \beta} R^\dagger = -i \tilde{A}_D \tilde{A}_D^\dagger = -i \sum_F \tilde{A}_{\alpha F} \tilde{A}_{\beta F}^\ast \quad (60) $$

where

$$ \sum_F \tilde{A}_{\alpha F} \tilde{A}_{\beta F}^\ast = i(\lambda_{\alpha}^2 - \lambda_{\beta}^2) O_{\alpha \beta} \quad (61) $$
where $O_{\alpha\beta} = (RR^\dagger)_{\alpha\beta} = \langle K_\alpha | K_\beta \rangle$. Eq. (60) and Eq. (61) represent nothing but the S–matrix version of the Bell and Steinberger’s relation. The total decay width of the state $|K_\alpha\rangle$ into the decay channel $F$ is therefore

$$\sum_F |\tilde{A}_\alpha F|^2 = i(\lambda_\alpha^2 - \lambda_\alpha^{*2}) \quad (62)$$

Although, we consider only poles corresponding to the mass and the decay width of $K_S$ and $K_L$, one may consider the momentum dependence of the effective complex $\Lambda(q^2)$. In the narrow width approximation the matrix elements of $\Lambda(q^2)$ are assumed to be constant in a wide energy region. But we cannot use this approximation in all our applications since in same energy region a substructural effect could emerge. If the transition elements are meromorphic functions in the complex momentum plane having poles in the points satisfying equation $\det[sI - \Lambda] = 0$, residues in the poles are complex matrices. Let us introduce the resolvent $\Delta'(s) = [sI - \Lambda]^{-1}$ and we consider the transition matrix in the decay space

$$\tau(s) = \tilde{A}_D^\dagger \Delta'(s) \tilde{A}_P \quad (63)$$

similar to the Wigner $\Re$–matrix of the nuclear reaction theory. Actually, we neglect the non–resonant parts of the transition amplitude, that is equivalent to consider isolated narrow particles rather two overlapping resonances. Instead of working this way, we can replace back from the beginning the flavour propagator with the introduction of renormalized matrix propagator Eq. (35).

**IV. Concluding Remarks**

Usually, the Wigner–Weisskopf narrow width approximation has been applied to describe the law of the time evolution in weak nonleptonic decays of the $K^0$ and $B^0$ mesons, where some signals of $CP$–violation can be found experimentally. This technique is not rigorous, in the sense that a relativistic formulation is required to deal with the unstable kaon system. Without assuming a specific dynamical model, we have analyzed the covariant propagator method. In the pole approximation, within the spirit of the mass–mixing formalism, we take the widths to be constant, with no explicit functional $q^2$–dependence. This approach reproduces the usual exponential decay law of the
physical kaon evolution. In this paper, we have shown the connection of the propagator formalism with the usual LOY approximation, in which we solve a Schrödinger equation for an effective Hamiltonian acting on a Hilbert space in which the only states are one–particle states. In this effective approach, working with mass rather than mass–squared matrix, no momentum dependence results. Indeed, we now consider the further contributions of eventual other poles which can induce some relevant deviations. They lie on the second Riemann complex momentum sheet of the analytically continued matrix propagator closer to the cut along the real axis. We provide some phenomenological implications of these analytical complexities. Moreover, the $q^2$– dependence of the propagator matrix could be incompatible with the constraints of unitarity and causality. Independently from the specific dynamics of a given theory and from the subtleties of the higher–order terms in the matrix elements of the propagator, which are relevant only close to the $K^0\bar{K}^0$ resonance, we propose a generalization of the unitarity sum rules. All methods have advantages and disadvantages. The standard time–dependent effective Hamiltonian formalism \[11\] is expressed directly in terms of the measured quantities and is therefore more useful in a wide range of phenomenological applications. However, the propagator method arises naturally in the context of quantum field theory and hence close to the requirements of the fundamental gauge theories. Furthermore, as we have shown, this last method has the considerable advantage to provide a description of the additional complications connected with the $q^2$–dependence of the underlying interactions. In modern gauge theories, CP violating–asymmetries are generated from the interference of the tree–level amplitude with higher–order corrections to vertex, mass and wave–function. CP violating–asymmetries occur through interference between at least two possible amplitudes having different weak and strong phases. In the Standard Model a weak–phase difference is provided by the different complex phase of the CKM matrix elements of the tree and penguin diagrams, while the strong phase is given by the absorptive parts of the corresponding diagram. In this paper we analyzed carefully CP–violating effects due to intermediate oscillating mesons. These effects can be established even for vanishing strong phases difference and could be relevant in models of spontaneous CP–violation due to Higgs–bosons exchanges.
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