THE OBSTACLE AND DIRICHLET PROBLEMS
ASSOCIATED WITH \( p \)-HARMONIC FUNCTIONS
IN UNBOUNDED SETS IN \( \mathbb{R}^n \) AND METRIC SPACES

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Abstract. We study the obstacle problem for unbounded sets in a proper metric measure space supporting a \((p, p)\)-Poincaré inequality. We prove that there exists a unique solution. We also prove that if the measure is doubling and the obstacle is continuous, then the solution is continuous, and moreover \( p \)-harmonic in the set where it does not touch the obstacle. This includes, as a special case, the solution of the Dirichlet problem for \( p \)-harmonic functions with Sobolev type boundary data.

1. Introduction

The classical Dirichlet problem is the problem of finding a harmonic function, that is, a solution of the Laplace equation that takes prescribed boundary values. According to Dirichlet’s principle, this is equivalent to minimizing the Dirichlet energy integral,

\[
\int_{\Omega} |\nabla u|^2 \, dx,
\]

among all functions \( u \), in the domain \( \Omega \), that have the required boundary values and continuous partial derivatives up to the second order.

A more general (nonlinear) Dirichlet problem considers the \( p \)-Laplace equation,

\[
\Delta_p u := \text{div}(|\nabla u|^{p-2} \nabla u) = 0, \quad 1 < p < \infty
\]

(which reduces to the Laplace equation when \( p = 2 \)). Solving this problem is equivalent to the variational problem of minimizing the \( p \)-energy integral,

\[
\int_{\Omega} |\nabla u|^p \, dx,
\]

among all admissible functions \( u \), and a minimizer/solution that is continuous is said to be \( p \)-harmonic.

The nonlinear potential theory of \( p \)-harmonic functions has been studied since the 1960s. Initially for \( \mathbb{R}^n \), and later generalized to weighted \( \mathbb{R}^n \), Riemannian manifolds, and other settings. The interested reader may consult the monograph Heinonen–Kilpeläinen–Martio [20] for a thorough treatment in weighted \( \mathbb{R}^n \).

It is not clear how to employ partial differential equations in a general metric measure space. However, by using the notion of minimal \( p \)-weak upper gradients, as substitutes for the modulus of the usual gradients, the variational approach...
becomes available. This has led to the more recent development of nonlinear potential theory on complete metric spaces equipped with a doubling measure supporting a p-Poincaré inequality.

In this paper, instead of just studying the Dirichlet problem for p-harmonic functions, we study the associated obstacle problem with a given obstacle and given boundary values. We minimize the p-energy integral among admissible functions lying above the obstacle ψ. This problem reduces to the Dirichlet problem when ψ ≡ −∞. The obstacle problem has been studied for bounded sets in (weighted) \( \mathbb{R}^n \) (see, e.g., Heinonen–Kilpeläinen–Martio [20] and the references therein) and later also in metric spaces (see, e.g., Björn–Björn [3], [4], Björn–Björn–Mäkeläinen–Parviainen [6], Björn–Björn–Shanmugalingam [8], Eleuteri–Farnana–Kansanen–Korte [12], Farnana [13], [14], [15], [16], Kinnunen–Martio [25], Kinnunen–Shanmugalingam [26], and Shanmugalingam [31]).

Suppose that Ω is a nonempty (possibly unbounded) open subset of a proper metric measure space that supports a \((p, p)\)-Poincaré inequality. Furthermore, suppose that the capacity of the complement of Ω is nonzero (this is needed for the boundary data to make sense). Let ψ be an extended real-valued function and let \( f \) be a function in \( D^p(\Omega) \) (see Section 2 for definitions). In this setting, we prove Theorem 3.4, which asserts that there exists a unique (up to sets of capacity zero) solution of the \( K_{\psi, f}(\Omega) \)-obstacle problem whenever the space of admissible functions is nonempty.

Moreover, by adding the assumption of the measure being doubling, we obtain Theorem 4.4, which, as a special case, implies that there is a unique solution of the Dirichlet problem for p-harmonic functions with boundary values in \( D^p(\Omega) \) taken in Sobolev sense (i.e., that the \( K_{\psi, f}(\Omega) \)-obstacle problem has a unique continuous solution whenever \( \psi \equiv -\infty \)).

To the best of the author’s knowledge, these results are new also for \( \mathbb{R}^n \).

2. Notation and preliminaries

We assume throughout the paper that \((X, \mathcal{M}, \mu, d)\) is a metric measure space (which we will refer to as \( X \)) equipped with a metric \( d \) and a measure \( \mu \) such that

\[
0 < \mu(B) < \infty
\]

for all balls \( B \) in \( X \) (we make the convention that balls are nonempty and open). The \( \sigma \)-algebra \( \mathcal{M} \) on which \( \mu \) is defined is the completion of the Borel \( \sigma \)-algebra.

We start with the assumption that \( 1 \leq p < \infty \). However, in the next section (and for the rest of the paper), we will assume that \( 1 < p < \infty \).

The measure \( \mu \) is said to be doubling if there exists a constant \( C_\mu \geq 1 \) such that

\[
0 < \mu(2B) \leq C_\mu \mu(B) < \infty
\]

for all balls \( B \) in \( X \). We use the notation that if \( B \) is a ball with radius \( r \), then the ball with radius \( \lambda r \) that is concentric with \( B \) is denoted by \( \lambda B \).

The characteristic function \( \chi_E \) of a set \( E \) is defined by \( \chi_E(x) = 1 \) if \( x \in E \) and \( \chi_E(x) = 0 \) if \( x \notin E \). The set \( E \) is compactly contained in \( A \) if \( \overline{E} \) (the closure of \( E \)) is a compact subset of \( A \). We denote this by \( E \subseteq \overline{A} \). The extended real number system is denoted by \( \overline{\mathbb{R}} := [-\infty, \infty] \). Recall that \( f_+ = \max\{f, 0\} \) and \( f_- = \max\{-f, 0\} \), and hence that \( f = f_+ - f_- \) and \( |f| = f_+ + f_- \).

By a curve in \( X \), we mean a rectifiable nonconstant continuous mapping \( \gamma \) from a compact interval into \( X \). Since our curves have finite length, they may be
parametrized by arc length, and we will always assume that this has been done. We will abuse notation and denote both the mapping and the image by \( \gamma \).

Unless otherwise stated, the letter \( C \) will be used to denote various positive constants whose exact values are unimportant and may vary with each usage.

We follow Heinonen–Koskela [21], [22] in introducing upper gradients. (Heinonen and Koskela, however, called them very weak gradients.)

**Definition 2.1.** A Borel function \( g : X \to [0, \infty] \) is said to be an *upper gradient* of a function \( f : X \to \mathbb{R} \) whenever

\[
|f(x) - f(y)| \leq \int_\gamma g \, ds
\]  

holds for all pairs of points \( x, y \in X \) and every curve \( \gamma \) in \( X \) joining \( x \) and \( y \). We make the convention that the left-hand side is infinite when at least one of the terms is.

Recall that a Borel function \( g : X \to Y \) is a function such that the inverse image \( g^{-1}(G) = \{ x \in X : g(x) \in G \} \) is a Borel set for every open subset \( G \) of \( Y \).

Observe that upper gradients are not unique (if we add a nonnegative Borel function to an upper gradient of \( f \), then we obtain a new upper gradient of \( f \)) and that \( g \equiv \infty \) is an upper gradient of all functions. Note also that if \( g \) and \( \tilde{g} \) are upper gradients of \( u \) and \( \tilde{u} \), respectively, then \( g - \tilde{g} \) is not in general an upper gradient of \( u - \tilde{u} \). However, upper gradients are subadditive, that is, if \( g \) and \( \tilde{g} \) are upper gradients of \( u \) and \( \tilde{u} \), respectively, and \( \alpha \in \mathbb{R} \), then \( |\alpha|g \) and \( g + \tilde{g} \) are upper gradients of \( \alpha u \) and \( u + \tilde{u} \), respectively.

A drawback of upper gradients is that they are not preserved by \( L^p \)-convergence. Fortunately, it is possible to overcome this problem by relaxing the conditions. Therefore, we define the \( p \)-modulus of a curve family, and then follow Koskela–MacManus [27] in introducing \( p \)-weak upper gradients.

**Definition 2.2.** Let \( \Gamma \) be a family of curves in \( X \). The *\( p \)-modulus* of \( \Gamma \) is

\[
\text{Mod}_p(\Gamma) := \inf_{\rho} \int_X \rho^p \, d\mu,
\]

where the infimum is taken over all nonnegative Borel functions \( \rho \) such that

\[
\int_\gamma \rho \, ds \geq 1 \quad \text{for all curves } \gamma \in \Gamma.
\]

Whenever a property holds for all curves except for a curve family of zero \( p \)-modulus, it is said to hold for *\( p \)-almost every* (\( p \)-a.e.) curve.

The \( p \)-modulus (as the module of order \( p \) of a system of measures) was defined and studied by Fuglede [17]. Heinonen–Koskela [22] defined the \( p \)-modulus of a curve family in a metric measure space and observed that the corresponding results by Fuglede carried over directly.

The \( p \)-modulus has the following properties (as observed in [22]): \( \text{Mod}_p(\emptyset) = 0 \), \( \text{Mod}_p(\Gamma_1) \leq \text{Mod}_p(\Gamma_2) \) whenever \( \Gamma_1 \subset \Gamma_2 \), and \( \text{Mod}_p(\bigcup_{j=1}^\infty \Gamma_j) \leq \sum_{j=1}^\infty \text{Mod}_p(\Gamma_j) \). If \( \Gamma_0 \) and \( \Gamma \) are two curve families such that every curve \( \gamma \in \Gamma \) has a subcurve \( \gamma_0 \in \Gamma_0 \), then \( \text{Mod}_p(\Gamma) \leq \text{Mod}_p(\Gamma_0) \). For proofs of these properties and all other results in this section, we refer to Björn–Björn [4]. (Some of the references that we mention below may not provide a proof in the generality considered here, but such proofs are given in [4].)
Definition 2.3. A measurable function \( g : X \to [0, \infty] \) is said to be a \( p \)-weak upper gradient of a function \( f : X \to \mathbb{R} \) if (2.1) holds for all pairs of points \( x, y \in X \) and \( p \)-a.e. curve \( \gamma \) in \( X \) joining \( x \) and \( y \).

Note that a \( p \)-weak upper gradient, as opposed to an upper gradient, is not required to be a Borel function. It is convenient to demand upper gradients to be Borel functions, since then the concept of upper gradients becomes independent of the measure, and all considered curve integrals will be defined. The situation is a bit different for \( p \)-weak upper gradients, as the curve integrals need only be defined for \( p \)-a.e. curve, and therefore, it is in fact enough to require that \( p \)-weak upper gradients are measurable functions. There is no disadvantage in assuming only measurability, since the concept of \( p \)-weak upper gradients would depend on the measure anyway (as the \( p \)-modulus depends on the measure). The advantage is that some results become more appealing (see, e.g., Björn–Björn [4]).

Since the \( p \)-modulus is subadditive, it follows that \( p \)-weak upper gradients share the subadditivity property with upper gradients.

Definition 2.4. The Dirichlet space on \( X \), denoted by \( D^p(X) \), is the space of all extended real-valued functions on \( X \) that are everywhere defined, measurable, and have upper gradients in \( L^p(X) \).

If \( E \) is a measurable set, then we can consider \( E \) to be a metric space in its own right (with the restriction of \( d \) and \( \mu \) to \( E \)). Thus the Dirichlet space \( D^p(E) \) is also given by Definition 2.4. Note, however, that the collection of upper gradients with respect to \( E \) can differ from those with respect to \( X \) (unless \( E \) is open).

The local Dirichlet space is defined analogously to the local space \( L^p_{\text{loc}}(X) \). Thus we say that a function \( f \) on \( X \) belongs to \( D^p_{\text{loc}}(X) \) if for every \( x \in X \) there is a ball \( B \) such that \( x \in B \) and \( f \in D^p(B) \).

Lemma 2.4 in Koskela–MacManus [27] asserts that if \( g \) is a \( p \)-weak upper gradient of a function \( f \), then for all \( q \) such that \( 1 \leq q \leq p \), there is a decreasing sequence \( \{g_j\}_{j=1}^\infty \) of upper gradients of \( f \) such that \( \|g_j - g\|_{L^q(X)} \to 0 \) as \( j \to \infty \).

This implies that a measurable function belongs to \( D^p(X) \) whenever it (merely) has a \( p \)-weak upper gradient in \( L^p(X) \).

If \( u \) belongs to \( D^p(X) \), then \( u \) has a minimal \( p \)-weak upper gradient \( g_u \in L^p(X) \).

It is minimal in the sense that \( g_u \leq g \) a.e. for all \( p \)-weak upper gradients \( g \) of \( u \).

This was proved for \( p > 1 \) by Shanmugalingam [31] and \( p \geq 1 \) by Hajłasz [18]. Minimal \( p \)-weak upper gradients \( g_u \) are true substitutes for \( |\nabla u| \) in metric spaces.

One of the important properties of minimal \( p \)-weak gradients is that they are local in the sense that if two functions \( u, v \in D^p(X) \) coincide on a set \( E \), then \( g_u = g_v \) a.e. on \( E \). Moreover, if \( U = \{ x \in X : u(x) > v(x) \} \), then \( g_u \chi_U + g_v \chi_{X \setminus U} \) is a minimal \( p \)-weak upper gradient of \( \max\{u, v\} \), and \( g_v \chi_U + g_u \chi_{X \setminus U} \) is a minimal \( p \)-weak upper gradients of \( \min\{u, v\} \). These results are from Björn–Björn [2].

It is well-known that the restriction of a minimal \( p \)-weak upper gradient to an open subset remains minimal with respect to that subset. As a consequence, the results above about minimal \( p \)-weak upper gradients extend to functions in \( D^p_{\text{loc}}(X) \) having minimal \( p \)-weak upper gradients in \( L^p_{\text{loc}}(X) \).

With the help of \( p \)-weak upper gradients, it is possible to define a type of Sobolev space on the metric space \( X \). This was done by Shanmugalingam [30]. We will, however, use a slightly different (semi)norm. The reason for this is that when we define the capacity in Definition 2.6, it will be subadditive.
Definition 2.5. The Newtonian space on \( X \) is
\[
N^{1,p}(X) := \{ u \in D^p_{\text{loc}}(X) : \| u \|_{N^{1,p}(X)} < \infty \},
\]
where \( \| \cdot \|_{N^{1,p}(X)} \) is the seminorm defined by
\[
\| u \|_{N^{1,p}(X)} = \left( \int_X |u|^p \, d\mu + \int_X g^p_u \, d\mu \right)^{1/p}.
\]

We emphasize the fact that our Newtonian functions are defined everywhere, and not just up to equivalence classes of functions that agree almost everywhere. This is essential for the notion of upper gradients to make sense.

The associated normed space defined by \( \tilde{N}^{1,p}(X) = N^{1,p}(X)/\sim \), where \( u \sim v \) if and only if \( \| u - v \|_{N^{1,p}(X)} = 0 \), is a Banach space (see Shanmugalingam [30]).

Note that some authors denote the space of the everywhere defined functions by \( N^{1,p}(X) \), and then define the Newtonian space, which they denote by \( N^{1,p}(X) \), to be the corresponding space of equivalence classes.

The local space \( N^{1,p}_{\text{loc}}(X) \) and the space \( N^{1,p}(E) \) when \( E \) is a measurable set are defined analogously to the Dirichlet spaces.

Recall that a metric space is said to be proper if all bounded closed subsets are compact. In particular, this is true if it is complete and the measure is doubling.

Various definitions of capacities for sets can be found in the literature (see, e.g., Kinnunen–Martio [24] and Shanmugalingam [30]). We will use the following definition.

Definition 2.6. The (Sobolev) capacity of a subset \( E \) of \( X \) is
\[
C_p(E) := \inf_u \| u \|^p_{N^{1,p}(X)},
\]
where the infimum is taken over all \( u \in N^{1,p}(X) \) such that \( u \geq 1 \) on \( E \).

Whenever a property holds for all points except for points in a set of capacity zero, it is said to hold quasieverywhere (q.e.). Note that we follow the custom of refraining from making the dependence on \( p \) explicit here.

Trivially, we have \( C_p(\emptyset) = 0 \), and \( C_p(E_1) \leq C_p(E_2) \) whenever \( E_1 \subset E_2 \). Furthermore, the proof in Kinnunen–Martio [24] for capacities of Hajlasz–Sobolev spaces on metric spaces can easily be modified to show that \( C_p \) is countably subadditive, that is, \( C_p \left( \bigcup_{j=1}^\infty E_j \right) \leq \sum_{j=1}^\infty C_p(E_j) \). Thus \( C_p \) is an outer measure. Note that \( C_p \) is finer than \( \mu \) in the sense that the capacity of a set may be positive even when the measure of the same set equals zero.

Shanmugalingam [30] showed that if two Newtonian functions are equal almost everywhere, then they are in fact equal quasieverywhere. This result extends to functions in \( D^p_{\text{loc}}(X) \).

When \( E \) is a subset of \( X \), we let \( \Gamma_E \) denote the family of all curves in \( X \) that intersect \( E \). Lemma 3.6 in Shanmugalingam [30] asserts that \( \text{Mod}_p(\Gamma_E) = 0 \) whenever \( C_p(E) = 0 \). This implies that two functions have the same set of \( p \)-weak upper gradients whenever they are equal quasieverywhere.

In order to be able to compare boundary values of Dirichlet functions (and Newtonian functions), we introduce the following spaces.
Proposition 2.9. Let $H$ be an open subset of $X$. Then $D^p_0(H) = D^p_0(\Omega \setminus \overline{\Omega})$. 

Definition 2.7. The Dirichlet space with zero boundary values in $A \setminus E$, for subsets $E$ and $A$ of $X$, where $A$ is measurable, is

$$D^p_0(E; A) := \{ f | f \in D^p(A) \text{ and } f = 0 \text{ in } A \setminus E \}.$$ 

The Newtonian space with zero boundary values in $A \setminus E$, denoted by $N^1_0(E; A)$, is defined analogously.

We let $D^p_0(E)$ and $N^1_0(E)$ denote $D^p_0(E; X)$ and $N^1_0(E; X)$, respectively.

The assumption “$f = 0$ in $A \setminus E$” can in fact be replaced by “$f = 0$ q.e. in $A \setminus E$” without changing the obtained spaces.

It is easy to verify that the function spaces that we have introduced are vector spaces and lattices. This means that if $u, v \in D^p(X)$ and $a, b \in \mathbb{R}$, then we have $au + bv, \max\{u, v\}, \min\{u, v\} \in D^p(X)$, and furthermore, as a direct consequence, we also have $u_+, u_-, |u| \in D^p(X)$.

The following lemma is useful for asserting that certain functions belong to a Dirichlet space with zero boundary values.

Lemma 2.8. Suppose that $E$ is a measurable subset of $X$ and that $u \in D^p(E)$. If there exist two functions $u_1$ and $u_2$ in $D^p_0(E)$ such that $u_1 \leq u \leq u_2$ q.e. in $E$, then $u \in D^p_0(E)$.

This was proved for Newtonian functions in open sets in Björn–Björn [3], and with trivial modifications, it provides a proof for our version of the lemma. For the reader’s convenience, we give the proof here.

Proof. Let $v_1$ and $v_2$ be functions in $D^p(X)$ such that $v_1|_E = u_1$, $v_2|_E = u_2$, and $v_1 = v_2 = 0$ outside $E$, and let $g_1 \in L^p(X)$ and $g_2 \in L^p(X)$ be upper gradients of $v_1$ and $v_2$, respectively. Let $g \in L^p(E)$ be an upper gradient of $u$ and define

$$v = \begin{cases} u & \text{in } E, \\ 0 & \text{in } X \setminus E \end{cases} \quad \text{and} \quad \tilde{g} = \begin{cases} g_1 + g_2 + g & \text{in } E, \\ g_1 + g_2 & \text{in } X \setminus E. \end{cases}$$

To complete the proof, it suffices to show that $\tilde{g} \in L^p(X)$ is a $p$-weak upper gradient of $v$.

Let $E'$ be a subset of $E$ with $C_p(E') = 0$ and such that $u_1 \leq u \leq u_2$ in $E \setminus E'$. Let $\gamma$ be an arbitrary curve in $X \setminus E'$ with endpoints $x$ and $y$. Then $\text{Mod}_p(\Gamma_{E'}) = 0$, so the following argument asserts that $\tilde{g}$ is a $p$-weak upper gradient of $v$.

If $\gamma \subset E \setminus E'$, then

$$|v(x) - v(y)| = |u(x) - u(y)| \leq \int_\gamma g \, ds \leq \int_\gamma \tilde{g} \, ds.$$ 

On the other hand, if $x, y \in X \setminus E$, then

$$|v(x) - v(y)| = 0 \leq \int_\gamma \tilde{g} \, ds.$$ 

Hence, by splitting $\gamma$ into two parts, and possibly reversing the direction, we may assume that $x \in E \setminus E'$ and $y \in X \setminus E$. Then it follows that

$$|v(x) - v(y)| = |u(x)| \leq |v_1(x) + v_2(x)| = |v_1(x) - v_1(y)| + |v_2(x) - v_2(y)| \leq \int_\gamma g_1 \, ds + \int_\gamma g_2 \, ds \leq \int_\gamma \tilde{g} \, ds. \qed$$

Proposition 2.9. Let $\Omega$ be an open subset of $X$. Then $D^p_0(\Omega) = D^p_0(\Omega \setminus \overline{\Omega})$. 

The proof is very similar to the proof of Lemma 2.8 (see, e.g., Proposition 2.39 in Björn–Björn [4] for a corresponding proof for Newtonian functions).

The next two results from Björn–Björn–Parviainen [7] (Lemma 3.2 and Corollary 3.3), following from Mazur’s lemma (see, e.g., Theorem 3.12 in Rudin [29]), will play a major role in the existence proof for the obstacle problem.

**Lemma 2.10.** Assume that $1 < p < \infty$. Assume further that $g_j$ is a $p$-weak upper gradient of $u_j$, $j = 1, 2, \ldots$, and that $\{u_j\}_{j=1}^\infty$ and $\{g_j\}_{j=1}^\infty$ are bounded in $L^p(X)$. Then there exist functions $u$ and $g$, both in $L^p(X)$, convex combinations $v_j = \sum_{i=1}^{N_j} a_{j,i} u_i$ with $p$-weak upper gradients $\tilde{g}_j = \sum_{i=1}^{N_j} a_{j,i} \tilde{g}_i$, $j = 1, 2, \ldots$, and a subsequence $\{u_{j_k}\}_{k=1}^\infty$, such that

(a) both $u_{j_k} \to u$ and $g_{j_k} \to g$ weakly in $L^p(X)$ as $k \to \infty$;

(b) both $v_j \to u$ and $\tilde{g}_j \to g$ in $L^p(X)$ as $j \to \infty$;

(c) $v_j \to u$ q.e. as $j \to \infty$;

(d) $g$ is a $p$-weak upper gradient of $u$.

Recall that $\alpha_1 v_1 + \cdots + \alpha_n v_n$ is said to be a convex combination of $v_1, \ldots, v_n$ whenever $\alpha_k \geq 0$ for all $k = 1, \ldots, n$ and $\alpha_1 + \cdots + \alpha_n = 1$.

**Corollary 2.11.** Assume that $1 < p < \infty$. Assume also that $\{u_j\}_{j=1}^\infty$ is bounded in $N^{1,p}(X)$ and that $u_j \to u$ q.e. on $X$ as $j \to \infty$. Then $u \in N^{1,p}(X)$ and

$$\int_X g^p \, d\mu \leq \liminf_{j \to \infty} \int_X g_j^p \, d\mu.$$ 

In general, the upper gradients of a function give no control over the function. This is obviously so when there are no curves. Requiring a Poincaré inequality to hold is one possibility of gaining such a control by making sure that there are enough curves connecting any two points.

**Definition 2.12.** Let $q \geq 1$. We say that $X$ supports a $(q,p)$-Poincaré inequality (or that $X$ is a $(q,p)$-Poincaré space) if there exist constants $C_{P1} > 0$ and $\lambda \geq 1$ (dilation constant) such that for all balls $B$ in $X$, all integrable functions $u$ on $X$, and all upper gradients $g$ of $u$, it is true that

$$\left( \int_B |u - u_B|^q \, d\mu \right)^{1/q} \leq C_{P1} \text{diam}(B) \left( \int_B g^p \, d\mu \right)^{1/p},$$

where

$$u_B := \int_B u \, d\mu := \frac{1}{\mu(B)} \int_B u \, d\mu.$$ 

For short, we say $p$-Poincaré inequality instead of $(1,p)$-Poincaré inequality, and if $X$ supports a $p$-Poincaré inequality, we say that $X$ is a $p$-Poincaré space.

By using Hölder’s inequality, one can show that if $X$ supports a $(q,p)$-Poincaré inequality, then $X$ supports a $(\tilde{q},\tilde{p})$-Poincaré inequality for all $\tilde{q} \leq q$ and $\tilde{p} \geq p$.

From the next section on, we will assume $X$ to support a $(p,p)$-Poincaré inequality. Then we have the following useful assertion that implies that a function can be controlled by its minimal $p$-weak upper gradient. This was proved for Euclidean spaces by Maz’ya (see, e.g., [28]), and later J. Björn [9] observed that the proof goes through also for metric spaces. The following version is from Björn–Björn [4] (Theorem 5.53).
Theorem 2.13 (Maz’ya’s inequality.). Suppose that $X$ supports a $(p,p)$-Poincaré inequality. Then there exists a constant $C_{MI} > 0$ such that if $B$ is a ball in $X$, $u \in N^{1,p}_{\text{loc}}(X)$, and $S = \{x \in X : u(x) = 0\}$, then
\[
\int_{2B} |u|^p \, d\mu \leq \frac{C_{MI} (\text{diam}(B))^p + 1) \mu(2B)}{C_p(B \cap S)} \int_{2\lambda B} g_\mu^p \, d\mu,
\]
where $\lambda$ is the dilation constant in the $(p,p)$-Poincaré inequality.

The following result from Björn–Björn [4] (Proposition 4.14) is also a useful consequence of the $(p,p)$-Poincaré inequality.

Proposition 2.14. Suppose that $X$ supports a $(p,p)$-Poincaré inequality. Let $\Omega$ be an open subset of $X$. Then $D^p_{\text{loc}}(\Omega) = N^1_{p,\text{loc}}(\Omega)$.

3. The obstacle problem

In this section, we assume that $1 < p < \infty$, that $X$ is proper and supports a $(p,p)$-Poincaré inequality with dilation constant $\lambda$, and that $\Omega$ is a nonempty open subset of $X$ such that $C_p(X \setminus \Omega) > 0$.

Kinnunen–Martio [25] defined an obstacle problem for Newtonian functions in open sets in a complete $p$-Poincaré space with a doubling measure. They proved that there exists a unique solution whenever the set is bounded and such that the complement has nonzero measure and the set of feasible solutions is nonempty (Theorem 3.2 in [25]). Shanmugalingam [30] had earlier solved the Dirichlet problem (i.e., the obstacle problem with obstacle $\psi \equiv -\infty$).

Roughly, Kinnunen and Martio defined their obstacle as follows.

Definition 3.1. Suppose that $V$ is a nonempty bounded open subset of $X$ with $C_p(X \setminus V) > 0$. Let $\psi : V \to \mathbb{R}$ and let $f \in N^{1,p}(V)$. Define
\[
\mathcal{K}_{\psi,f}^n(V) = \{v \in N^{1,p}(V) : v - f \in N^1_{\text{loc}}(V) \text{ and } v \geq \psi \text{ q.e. in } V\}.
\]
Then $u$ is said to be a solution of the $\mathcal{K}_{\psi,f}^n(V)$-obstacle problem if $u \in \mathcal{K}_{\psi,f}^n(V)$ and
\[
\int_V g_\mu^p \, d\mu \leq \int_V g_v^p \, d\mu \text{ for all } v \in \mathcal{K}_{\psi,f}^n(V).
\]

They required that $\mu(X \setminus V) > 0$ and merely that $v \geq \psi$ a.e. instead of q.e. This does not matter if the obstacle $\psi$ is in $D^p_{\text{loc}}(V)$, since then $v \geq \psi$ a.e. implies that $v \geq \psi$ q.e. This follows from Corollary 3.3 in Shanmugalingam [30]; see also Corollary 1.60 in Björn–Björn [4]. However, the distinction may be important. For example, if $K$ is a compact subset of $V$ such that $C_p(K) \geq \mu(K) = 0$, then the solution of the $\mathcal{K}_{\chi_K,0}^n(V)$-obstacle problem takes the value $1$ on $K$, whereas the solution of the corresponding obstacle problem defined by Kinnunen–Martio [25] is the trivial solution (because their candidate solutions do not “see” this obstacle). Moreover, it is possible to have no solution of the $\mathcal{K}_{\psi,f}^n(V)$-obstacle problem when there is a solution of the corresponding obstacle problem defined by [25] (see, e.g., the discussion following Definition 3.1 in Farnana [13]).

See also Farnana [13], [14], [15], [16] for the double obstacle problem, and Björn–Björn [5] for obstacle problems on nonopen sets.

Now we define our obstacle problem (without the boundedness requirement).
**Definition 3.2.** Suppose that $V$ is a nonempty (possibly unbounded) open subset of $X$ such that $C_p(X \setminus V) > 0$. Let $\psi : V \to \mathbb{R}$ and let $f \in D^p(V)$. Define

$$K_{\psi,f}(V) = \{ v \in D^p(V) : v - f \in D^p_0(V) \text{ and } v \geq \psi \text{ q.e. in } V \}.$$ 

We say that $u$ is a solution of the $K_{\psi,f}(V)$-obstacle problem (with obstacle $\psi$ and boundary values $f$) if $u \in K_{\psi,f}(V)$ and

$$\int_V g_v^p \, d\mu \leq \int_V g_w^p \, d\mu \text{ for all } v \in K_{\psi,f}(V).$$

When $V = \Omega$, we denote $K_{\psi,f}(\Omega)$ by $K_{\psi,f}$ for short.

Observe that we only define the obstacle problem for $V$ with $C_p(X \setminus V) > 0$. This is because the condition $u - f \in D^p_0(V)$ becomes empty when $C_p(X \setminus V) = 0$, since then we have $D^p_0(V) = D^p(V)$.

Note also that we solve the obstacle problem for boundary data $f \in D^p(V)$. Since such a function is not defined on $\partial V$, we do not really have boundary values, and hence the definition should be understood in a weak Sobolev sense.

**Remark 3.3.** If $V$ is bounded, then Proposition 2.7 in Björn–Björn [5] asserts that $D^p_0(V) = N^1_p(V)$, and hence we have $K_{\psi,f}(V) = K^n_{\psi,f}(V)$. Thus Definition 3.2 is a generalization of Definition 3.1 to Dirichlet functions and to unbounded sets.

The main result in this paper shows that the $K_{\psi,f}$-obstacle problem has a unique solution under the natural condition of $K_{\psi,f}$ being nonempty.

**Theorem 3.4.** Let $\psi : \Omega \to \mathbb{R}$ and let $f \in D^p(\Omega)$. Then there exists a unique (up to sets of capacity zero) solution of the $K_{\psi,f}$-obstacle problem whenever $K_{\psi,f}$ is nonempty.

The assumption that $X$ is proper is needed only in the end of the existence part of the proof.

In the uniqueness part of the proof, we use the fact that $L^p(\Omega)$ is strictly convex. Clarkson [11] introduced the notions of strict convexity and uniform convexity (the latter being a stronger condition), and proved that all $L^p$-spaces, $1 < p < \infty$, are uniformly convex. A Banach space $Y$ (with norm $\| \cdot \|$) is strictly convex if $x = cy$ for some constant $c > 0$ whenever $x$ and $y$ are nonzero and $\|x + y\| = \|x\| + \|y\|$. In particular, $x = y$ whenever $\|x\| = \|y\| = \\frac{1}{2}(x + y)$.

The idea used in the uniqueness part of the proof comes from Cheeger [10].

**Proof.** (Existence.) We start by choosing a ball $B \subset X$ such that $C_p(B \setminus \Omega) > 0$ and $B \cap \Omega$ is nonempty. Clearly, we have $B \subset 2B \subset 3B \subset \ldots \subset X = \bigcup_{t=1}^{\infty} tB$.

Let

$$I = \inf_{v} \int_\Omega g_v^p \, d\mu,$$

with the infimum taken over all $v \in K_{\psi,f}$. Then $0 \leq I < \infty$ as $K_{\psi,f}$ is nonempty. Let $\{u_j\}_{j=1}^{\infty} \subset K_{\psi,f}$ be a minimizing sequence such that

$$I_j := \int_\Omega g_{u_j}^p \, d\mu \to I \text{ as } j \to \infty.$$

Let $w_j \in D^p(X)$ be such that $w_j = u_j - f$ in $\Omega$ and $w_j = 0$ outside $\Omega$, $j = 1, 2, \ldots$. We claim that both $\{w_j\}_{j=1}^{\infty}$ and $\{g_{w_j}\}_{j=1}^{\infty}$ are bounded in $L^p(tB)$ for all $t \geq 1$.

To show that, we first observe that $g_{w_j} \leq (g_{u_j} + g_f) \chi_\Omega$ a.e., and hence

$$\|g_{w_j}\|_{LP(X)} \leq \|g_{u_j}\|_{LP(\Omega)} + \|g_f\|_{LP(\Omega)} \leq \|g_{u_j}\|_{LP(\Omega)} + \|g_f\|_{LP(\Omega)} =: C' < \infty.$$
Let $t \geq 1$ be arbitrary and let $S = \bigcap_{j=1}^{\infty} \{ x \in X : w_j(x) = 0 \}$. Then
\[ C_p(tB \cap S) \geq C_p(tB \setminus \Omega) \geq C_p(B \setminus \Omega) > 0. \]

Maz'ya’s inequality (Theorem 2.13) asserts the existence of a constant $C_{tB} > 0$ such that
\[ \int_{2tB} |w_j|^p \, d\mu \leq C_{tB}^p \int_{2tB} |g_{w_j}|^p \, d\mu. \]
This implies that we also have
\[ \|w_j\|_{L^p(tB)} \leq C_{tB} \|g_{w_j}\|_{L^p(X)} \leq C_{tB} C' =: C_{tB} < \infty, \quad (3.1) \]
and the claim follows.

Consider the ball $B$. Lemma 2.10 asserts that we can find a function $\varphi_1 \in L^p(B)$ and convex combinations
\[ \varphi_{1,j} = \sum_{k=j}^{N_{1,j}} a_{1,j,k} w_k \quad \text{in } D^p(X), \quad j = 1, 2, \ldots, \quad (3.2) \]
such that $\varphi_{1,j} \to \varphi_1$ q.e. in $B$ as $j \to \infty$. Because $\varphi_{1,j} = 0$ outside $\Omega$, we must have $\varphi_1 = 0$ q.e. in $B \setminus \Omega$, and hence we may choose $\varphi_1$ so that $\varphi_1 = 0$ in $B \setminus \Omega$.

Let $v_{1,j} = f + \varphi_1|_{\Omega}$. Then
\[ v_{1,j} = f + \sum_{k=j}^{N_{1,j}} a_{1,j,k} w_k|_{\Omega} = \sum_{k=j}^{N_{1,j}} a_{1,j,k}(f + w_k|_{\Omega}) = \sum_{k=j}^{N_{1,j}} a_{1,j,k} u_k \geq \psi \quad \text{q.e. in } \Omega. \]
We also have
\[ g_{v_{1,j}} \leq \sum_{k=j}^{N_{1,j}} a_{1,j,k} g_{w_k} \quad \text{a.e. in } \Omega \quad \text{and} \quad g_{\varphi_{1,j}} \leq \sum_{k=j}^{N_{1,j}} a_{1,j,k} g_{w_k} \quad \text{a.e.} \]

A sequence of convex combinations of functions taken from a bounded sequence must also be bounded, and therefore we can apply Lemma 2.10 repeatedly here. Hence, for every $n = 2, 3, 4, \ldots$, we can find a function $\varphi_n \in L^p(nB)$ such that $\varphi_n = 0$ in $nB \setminus \Omega$ and convex combinations
\[ \varphi_{n,j} = \sum_{k=j}^{N_{n,j}} a_{n,j,k} \varphi_{n-1,k} \quad \text{in } D^p(X), \quad j = 1, 2, \ldots, \quad (3.3) \]
such that $\varphi_{n,j} \to \varphi_n$ q.e. in $nB$ as $j \to \infty$. Let $v_{n,j} = f + \varphi_{n,j}|_{\Omega}$. Then
\[ v_{n,j} = \sum_{k=j}^{N_{n,j}} a_{n,j,k}(f + \varphi_{n-1,k}|_{\Omega}) = \sum_{k=j}^{N_{n,j}} a_{n,j,k} v_{n-1,k} \geq \psi \quad \text{q.e. in } \Omega, \]
and also
\[ g_{v_{n,j}} \leq \sum_{k=j}^{N_{n,j}} a_{n,j,k} g_{v_{n-1,k}} \quad \text{a.e. in } \Omega \quad \text{and} \quad g_{\varphi_{n,j}} \leq \sum_{k=j}^{N_{n,j}} a_{n,j,k} g_{\varphi_{n-1,k}} \quad \text{a.e.} \]
Let $u = f + \varphi|_{\Omega}$, where $\varphi$ is the function on $X$ defined by
\[ \varphi(x) = \sum_{n=1}^{\infty} \varphi_n(x) \chi_{nB \setminus (n-1)B}(x), \quad x \in X. \]
We shall now show that $u$ is indeed a solution of the $K_{\psi,f}$-obstacle problem. To do that, we first establish that $u \in K_{\psi,f}$, and then show that $u$ is a minimizer.
Because $\varphi = u - f$ in $\Omega$ and $\varphi = 0$ outside $\Omega$, it suffices to show that $\varphi \in D^p(X)$ in order to establish that $u - f \in D^p_0(\Omega)$ and $u \in D^p(\Omega)$.

Consider the diagonal sequences $\{v_{n,n}\}_{n=1}^\infty$ and $\{\varphi_{n,n}\}_{n=1}^\infty$. Observe that the latter is bounded in $L^p(tB)$ for $t \geq 1$, since $\|\varphi_{n,j}\|_{L^p(tB)} \leq C_{tB}$ for all $n$ and $j$, by (3.1), (3.2), and (3.3).

We claim that $\varphi_{n,n} \to \varphi$ q.e. as $n \to \infty$. To prove that, we start by fixing an integer $n \geq 1$ and consider $nB$. Then

$$|\varphi_{n+1} - \varphi_n| \leq |\varphi_{n+1} - \varphi_{n+1,j}| + |\varphi_{n+1,j} - \varphi_n|$$

$$\leq |\varphi_{n+1} - \varphi_{n+1,j}| + \sum_{k=j}^{N_{n+1,j}} a_{n+1,j,k} |\varphi_{n,k} - \varphi_n| \to 0$$

q.e. in $\Omega$ as $j \to \infty$. Thus $\varphi_{n+1} = \varphi_n$ q.e. in $nB$ for $n = 1, 2, \ldots$.

By definition, we have $\varphi = \varphi_1$ in $B$. Now assume that $\varphi = \varphi_n$ q.e. in $nB$ for some positive integer $n$. By definition also, we have $\varphi = \varphi_{n+1}$ in $(n+1)B \setminus nB$, and because $\varphi_{n+1} = \varphi_n$ q.e. in $nB$, it follows that $\varphi = \varphi_{n+1}$ q.e. in $(n+1)B$.

Hence, by induction, we have $\varphi = \varphi_n$ q.e. in $nB$ for $n = 1, 2, \ldots$.

For $n = 1, 2, \ldots$, let $E_n$ be the subset of $nB$ where $\varphi_{n,j} \to \varphi_n = \varphi$ as $j \to \infty$ and let $E = \bigcup_{n=1}^\infty (nB \setminus E_n)$. Then we have $C_p(E) \leq \sum_{n=1}^\infty C_p(nB \setminus E_n) = 0$.

Let $x \in X \setminus E$. Clearly, $x \in mB$ and $\varphi(x) = \varphi_m(x)$ for some positive integer $m$. Given $\varepsilon > 0$, choose a $J$ such that $j \geq J$ implies that

$$|\varphi_{m,j}(x) - \varphi_m(x)| < \varepsilon.$$

Assume that for some $n \geq m$, we have $|\varphi_{n,j}(x) - \varphi_m(x)| < \varepsilon$ for $j \geq J$. Then

$$|\varphi_{n+1,j}(x) - \varphi_m(x)| \leq \sum_{k=j}^{N_{n+1,j}} a_{n+1,j,k} |\varphi_{n,k}(x) - \varphi_m(x)| < \varepsilon$$

for $j \geq J$. By induction, it follows that $|\varphi_{n,j}(x) - \varphi_m(x)| < \varepsilon$ for $n \geq m$ and $j \geq J$, and hence, for $n \geq \max\{m, J\}$, we have

$$|\varphi_{n,n}(x) - \varphi(x)| = |\varphi_{n,n}(x) - \varphi_m(x)| < \varepsilon.$$

We conclude that $\varphi_{n,n} \to \varphi$ q.e., and also that $v_{n,n} \to u$ q.e. in $\Omega$, as $n \to \infty$.

By using Jensen’s inequality, we can see that

$$\int g^p_{v_{n,j}} \, d\mu \leq \int \left( \sum_{k=j}^{N_{n,j}} a_{1,j,k} g_{u_k} \right)^p \, d\mu \leq \sum_{k=j}^{N_{n,j}} a_{1,j,k} \int g^p_{u_k} \, d\mu \leq \int g^p_{u_j} \, d\mu$$

and

$$\int g^p_{\varphi_{n,j}} \, d\mu \leq \int \left( \sum_{k=j}^{N_{n,j}} a_{1,j,k} g_{u_k} \right)^p \, d\mu \leq \sum_{k=j}^{N_{n,j}} a_{1,j,k} \int (g_{u_k} + g_{f})^p \, d\mu$$

$$\leq 2^p \sum_{k=j}^{N_{n,j}} a_{1,j,k} \int (g^p_{u_k} + g^p_{f}) \, d\mu \leq 2^p \int (g^p_{u_j} + g^p_{f}) \, d\mu.$$

Assume that for some positive integer $n$, it is true that

$$\int g^p_{v_{n,j}} \, d\mu \leq \int g^p_{u_j} \, d\mu \quad \text{and} \quad \int g^p_{\varphi_{n,j}} \, d\mu \leq 2^p \int (g^p_{u_j} + g^p_{f}) \, d\mu.$$
Then
\[ \int_{\Omega} g_{n+1,j}^p \, d\mu \leq \int_{\Omega} \left( \sum_{k=j}^{N_{n+1,j}} a_{n+1,j,k} g_{v_{n,k}} \right)^p \, d\mu \leq \sum_{k=j}^{N_{n+1,j}} a_{n+1,j,k} \int_{\Omega} g_{v_{n,k}}^p \, d\mu \]
and
\[ \int_{\Omega} g_{v_{n+1,j}}^p \, d\mu \leq \int_{\Omega} \left( \sum_{k=j}^{N_{n+1,j}} a_{n+1,j,k} g_{v_{n,k}} \right)^p \, d\mu \leq \sum_{k=j}^{N_{n+1,j}} a_{n+1,j,k} \int_{\Omega} g_{v_{n,k}}^p \, d\mu \]
\[ \leq 2p \sum_{k=j}^{N_{n+1,j}} a_{n+1,j,k} \int_{\Omega} (g_f^p + g_{g_{u_k}}^p) \, d\mu \leq 2p \int_{\Omega} (g_f^p + g_{g_{u_j}}^p) \, d\mu. \]

By induction, and letting \( j = n \), it follows that
\[ \int_{\Omega} g_{v_{n,n}}^p \, d\mu \leq \int_{\Omega} g_{v_{n,n}}^p \, d\mu \quad \text{and} \quad \int_{\Omega} g_{v_{n,n}}^p \, d\mu \leq 2p \int_{\Omega} (g_f^p + g_{g_{u_j}}^p) \, d\mu, \quad n = 1, 2, \ldots. \]

Fix an integer \( m \geq 1 \). Since \( \{\varphi_{n,n}\}_{n=1}^\infty \) and \( \{g_{\varphi_{n,n}}\}_{n=1}^\infty \) are bounded in \( L^p(mB) \) and \( \varphi_{n,n} \to \varphi \) q.e. in \( mB \) as \( n \to \infty \), Corollary 2.11 asserts that \( \varphi \in N^{1,p}(mB) \). This implies that \( \varphi \in D^p_{loc}(X) \). Note that \( g_{\varphi} \) and \( g_{\varphi_{n,n}} \) are minimal \( p \)-weak upper gradients of \( \varphi \) and \( \varphi_{n,n} \), respectively, with respect to \( mB \). Hence, by Corollary 2.11 again, it follows that
\[ \int_{mB} g_{\varphi}^p \, d\mu \leq \liminf_{n \to \infty} \int_{mB} g_{\varphi_{n,n}}^p \, d\mu \leq \liminf_{n \to \infty} \int_{\Omega} g_{\varphi_{n,n}}^p \, d\mu \]
\[ \leq 2p \liminf_{n \to \infty} \int_{\Omega} (g_f^p + g_{g_{u_j}}^p) \, d\mu = 2p \int_{\Omega} g_f^p \, d\mu + 2pI. \]

Letting \( m \to \infty \) yields
\[ \int_{\Omega} g_{\varphi}^p \, d\mu = \lim_{m \to \infty} \int_{mB} g_{\varphi}^p \, d\mu \leq 2p \int_{\Omega} g_f^p \, d\mu + 2pI < \infty, \]
and hence \( \varphi \in D^p(X) \). We conclude that \( u - f \in D^p_0(\Omega) \) and \( u \in D^p(\Omega) \).

Let \( A_n = \{ x \in \Omega : v_{n,n}(x) < \psi(x) \} \) for \( n = 1, 2, \ldots \), and let \( A = \bigcup_{n=1}^\infty A_n \). Then, since \( v_{n,n} \to u \) q.e. in \( \Omega \) as \( n \to \infty \), it follows that \( u \geq \psi \) q.e. in \( \Omega \setminus A \).

Because \( v_{n,n} \geq \psi \) q.e. in \( \Omega \), we have \( C_p(A_n) = 0 \), and hence \( C_p(A) = 0 \) by the subadditivity of the capacity. Thus \( u \geq \psi \) q.e. in \( \Omega \), and we conclude that \( u \in K_{\psi,f} \).

Proposition 2.14 asserts that \( f \in N^{1,p}_{W} (\Omega) \), and hence \( f \in L^p(\Omega') \) for all open \( \Omega' \subset \Omega \). Let
\[ \Omega_t = \left\{ x \in tB \cap \Omega : \inf_{y \in \partial \Omega} d(x,y) > \delta/t \right\}, \quad 1 \leq t < \infty, \]
where \( \delta > 0 \) is chosen small enough so that \( \Omega_t \) is nonempty. Then we have \( \Omega_t \subset \Omega_{t+1} \subset \cdots \subset \Omega = \bigcup_{t=1}^\infty \Omega_t \). Moreover, \( \{v_{n,n}\}_{n=1}^\infty \) is bounded in \( L^p(\Omega_t) \), since
\[ \|v_{n,n}\|_{L^p(\Omega_t)} \leq \|\varphi_{n,n}\|_{L^p(\Omega_t)} + \|f\|_{L^p(\Omega_t)} \leq C_{t,B} + \|f\|_{L^p(\Omega_t)} < \infty. \]

Fix an integer \( m \geq 1 \). Since \( \{v_{n,n}\}_{n=1}^\infty \) and \( \{g_{\varphi_{n,n}}\}_{n=1}^\infty \) are bounded in \( L^p(\Omega_m) \), \( v_{n,n} \to u \) q.e. in \( \Omega_m \) as \( n \to \infty \), and \( g_u \) and \( g_{\varphi_{n,n}} \) are minimal \( p \)-weak upper
The gradients of $u$ and $v_{n,n}$, respectively, with respect to $\Omega_m$, by Corollary 2.11, it follows that
\[
\int_{\Omega_m} g_{u}^n \,d\mu \leq \lim_{n \to \infty} \inf \int_{\Omega_m} g_{v_{n,n}}^n \,d\mu \leq \lim_{n \to \infty} \inf \int_{\Omega} g_{v_{n,n}}^n \,d\mu \leq \lim_{n \to \infty} \inf \int_{\Omega} g_{u}^n \,d\mu = I.
\]
Letting $m \to \infty$ completes the existence part of the proof by showing that
\[
I \leq \int_{\Omega} g_{u}^n \,d\mu = \lim_{m \to \infty} \int_{\Omega_m} g_{u}^n \,d\mu \leq I.
\]
(Uniqueness.) Suppose that $u'$ and $u''$ are solutions to the $K_{\psi,f}$-obstacle problem. We begin this part by showing that $g_{u'} = g_{u''}$ a.e. in $\Omega$.

Clearly, $\frac{1}{2}(u' + u'') \in K_{\psi,f}$, and hence
\[
\|g_{u'}\|_{L^p(\Omega)} \leq \|g_{\frac{1}{2}(u' + u'')}\|_{L^p(\Omega)} \leq \|\frac{1}{2}(g_{u'} + g_{u''})\|_{L^p(\Omega)}
\]
\[
\leq \frac{1}{2}\|g_{u'}\|_{L^p(\Omega)} + \frac{1}{2}\|g_{u''}\|_{L^p(\Omega)} = \|g_{u'}\|_{L^p(\Omega)} = \|g_{u'}\|_{L^p(\Omega)}.
\]
Thus $g_{u'} = g_{u''}$ a.e. in $\Omega$ by the strict convexity of $L^p(\Omega)$.

Now we show that $g_{u'} - g_{u''} = 0$ a.e. in $\Omega$. Fix a real number $c$ and let
\[
u = \max\{u', \min\{u'', c\}\}.
\]
The following shows that $u \in K_{\psi,f}$. Clearly, $u \in D^p(\Omega)$. Furthermore, we have $u \geq u' \geq \psi$ q.e. in $\Omega$, and $u - f \in D^p_0(\Omega)$ by Lemma 2.8, since
\[
-u - f \leq \max\{u', u''\} - f = \max\{u' - f, u'' - f\} \in D^p_0(\Omega)
\]
and $u - f \geq u' - f \in D^p_0(\Omega)$.

Let $U_c = \{x \in \Omega : u'(x) < c < u''(x)\}$. Then we have $g_u = 0$ a.e. in $U_c$, since $U_c \subset \{x \in \Omega : u(x) = c\}$. The minimizing property of $g_u$ then implies that
\[
\int_{\Omega} g_{u'}^n \,d\mu = \int_{\Omega \setminus U_c} g_{u'}^n \,d\mu = \int_{\Omega \setminus U_c} g_{u''}^n \,d\mu,
\]
so $g_u = g_{u'} = g_{u''}$ a.e. in $\Omega \setminus U_c$. Hence $g_{u'} = g_{u''} = 0$ a.e. in $U_c$ for all $c \in \mathbb{R}$, and because
\[
\{x \in \Omega : u'(x) < u''(x)\} \subset \bigcup_{c \in \mathbb{Q}} U_c,
\]
we have $g_{u'} = g_{u''} = 0$ a.e. in $\{x \in \Omega : u'(x) < u''(x)\}$. Analogously, the same is true for $\{x \in \Omega : u'(x) > u''(x)\}$, and hence
\[
g_{u'} - g_{u''} = (g_{u'} + g_{u''})\chi_{\{x \in \Omega : u'(x) \neq u''(x)\}} = 0 \text{ a.e. in } \Omega.
\]
Since $u' - u'' = u' - f = (u'' - f) \in D^p_0(\Omega)$, there exists $w \in D^p(X)$ such that $w = u' - u''$ in $\Omega$ and $w = 0$ outside $\Omega$. We have $g_w = g_{u'' - u''} \chi_{\Omega} = 0$ a.e.

Let $\tilde{S} = \{x \in X : w(x) = 0\}$ and let $t \geq 1$ be arbitrary. Then
\[
C_{\rho}((tB \cap \tilde{S}) \geq C_{\rho}(tB \cap \Omega) \geq C_{\rho}(B \setminus \Omega) > 0.
\]
Max'ya's inequality (Theorem 2.13) applies to $w$, and hence there exists a constant $\tilde{C}_{tB} > 0$ such that
\[
\int_{2tB} |u' - u''|^p \,d\mu \leq \int_{2tB} |w|^p \,d\mu \leq \tilde{C}_{tB} \int_{2mtB} g_w^p \,d\mu = 0.
\]
This implies that $u' = u''$ q.e. in $tB \cap \Omega$.

Let $V_m = \{x \in mB \cap \Omega : u'(x) \neq u''(x)\}$, $m = 1, 2, ..., \infty$, and let $V = \bigcup_{m=1}^{\infty} V_m$. Then $u' = u''$ in $\Omega \setminus V$. Since $C_{\rho}(V_m) = 0$ for all $m$, the subadditivity of the capacity implies that $C_{\rho}(V) = 0$, hence $u' = u''$ q.e. in $\Omega$. We conclude that the solution of the $K_{\psi,f}$-obstacle problem is unique (up to sets of capacity zero). □
If \( v = u \) q.e. in \( \Omega \) and \( u \) is a solution of the \( K_{\psi,f} \)-obstacle problem, then so is \( v \). Indeed, \( v = u \) q.e. implies that \( g_u \) is a \( p \)-weak upper gradient of \( v \). Thus \( v \in D^p(\Omega) \) and \( \int_{\Omega} g_v^p \, d\mu \leq \int_{\Omega} g_u^p \, d\mu \). Clearly, we have \( v \geq \psi \) q.e., and since Lemma 2.8 asserts that \( v - f \in D^p_0(\Omega) \), it follows that \( v \in K_{\psi,f} \).

The following criterion for the existence of a unique solution is easy to prove.

**Proposition 3.5.** Suppose that \( \psi \) and \( f \) are in \( D^p(\Omega) \). Then \( K_{\psi,f} \) is nonempty if and only if \( (\psi - f)_+ \in D^p_0(\Omega) \).

**Proof.** Suppose that \( K_{\psi,f} \) is nonempty and let \( v \in K_{\psi,f} \). Since \( (v - f)_+ \in D^p_0(\Omega) \) and

\[
0 \leq (\psi - f)_+ \leq (v - f)_+ \quad \text{q.e. in } \Omega,
\]

Lemma 2.8 asserts that \( (\psi - f)_+ \in D^p_0(\Omega) \).

Conversely, suppose that \( (\psi - f)_+ \in D^p_0(\Omega) \). Let \( v = \max\{\psi, f\} \). Then we have \( v \in D^p(\Omega) \), \( v - f = (\psi - f)_+ \), and \( v \geq \psi \) in \( \Omega \). Thus \( v \in K_{\psi,f} \). \( \square \)

The following comparison principle (for the version of the obstacle problem defined in Kinnunen–Martio [25]) was obtained in Björn–Björn [3]. Their proof (with trivial modifications) is valid also for our obstacle problem.

**Lemma 3.6.** Let \( \psi_j : \Omega \to \mathbb{R} \) and \( f_j \in D^p(\Omega) \) be such that \( K_{\psi_j,f_j} \) is nonempty, and let \( u_j \) be a solution of the \( K_{\psi_j,f_j} \)-obstacle problem for \( j = 1, 2 \). If \( \psi_1 \leq \psi_2 \) q.e in \( \Omega \) and \( (f_1 - f_2)_- \in D^p_0(\Omega) \), then \( u_1 \leq u_2 \) q.e. in \( \Omega \).

**Proof.** Let \( h = u_1 - f_1 - u_2 + f_2 \). Then \( h \in D^p_0(\Omega) \) and

\[
-(f_1 - f_2)_+ - h = -\max\{-(f_2 - f_1), 0\} - \max\{-h, 0\} = \min\{f_2 - f_1, 0\} + \min\{h, 0\} \leq \min\{f_2 - f_1, h\} \leq h.
\]

Since \( -(f_1 - f_2)_+ - h \in D^p_0(\Omega) \), Lemma 2.8 asserts that \( \min\{f_2 - f_1, h\} \in D^p_0(\Omega) \).

Let \( u = \min\{u_1, u_2\} \). Then \( u \in D^p(\Omega) \), and since \( u_2 \geq \psi_2 \geq \psi_1 \) q.e. in \( \Omega \), we clearly have \( u \geq \psi_1 \) q.e. in \( \Omega \). Moreover, since \( u_1 - f_1 = u_2 - f_2 + h \), we have

\[
u - f_1 = \min\{u_1, u_2\} - f_1 = \min\{u_1 - f_1, u_2 - f_1\} = \min\{u_2 - f_2 + h, u_2 - f_1\} = u_2 - f_2 + \min\{h, f_2 - f_1\}.
\]

Hence \( u - f_1 \in D^p_0(\Omega) \), and we conclude that \( u \in K_{\psi_1,f_1} \).

Let \( v = \max\{u_1, u_2\} \). Then \( v \in D^p(\Omega) \) and \( v \geq \psi_2 \) q.e. in \( \Omega \). As

\[
\begin{align*}
v - f_2 &= \max\{u_1 - f_2, u_2 - f_2\} = \max\{u_1 - f_2, u_1 - f_1 - h\} = u_1 - f_1 + \max\{f_1 - f_2, -h\} = u_1 - f_1 - \min\{f_2 - f_1, h\},
\end{align*}
\]

we see that \( v - f_2 \in D^p_0(\Omega) \), and hence \( v \in K_{\psi_2,f_2} \).

Let \( E = \{x \in \Omega : u_2(x) \leq u_1(x)\} \). Since \( u_2 \) is a solution of the \( K_{\psi_2,f_2} \)-obstacle problem, we have

\[
\int_{\Omega} g_{u_2}^p \, d\mu \leq \int_E g_{u_2}^p \, d\mu = \int_E g_{u_1}^p \, d\mu + \int_{\Omega \setminus E} g_{u_2}^p \, d\mu,
\]

which implies that

\[
\int_E g_{u_2}^p \, d\mu \leq \int_E g_{u_1}^p \, d\mu.
\]

By using the last inequality, we see that

\[
\begin{align*}
\int_{\Omega} g_{u}^p \, d\mu &= \int_E g_{u}^p \, d\mu + \int_{\Omega \setminus E} g_{u}^p \, d\mu \leq \int_E g_{u_1}^p \, d\mu + \int_{\Omega \setminus E} g_{u}^p \, d\mu = \int_{\Omega} g_{u_1}^p \, d\mu.
\end{align*}
\]
Since \( u \in \mathcal{K}_{\psi,v} \) and \( u_1 \) is a solution of the \( \mathcal{K}_{\psi,v} \)-obstacle problem, this inequality implies that also \( u \) is a solution. Theorem 3.4 asserts that \( u_1 = u \) q.e. in \( \Omega \), and we conclude that \( u_1 \leq u_2 \) q.e. in \( \Omega \).

The following local property of solutions of the obstacle problem can be useful. In some cases it may enable the use of results from the theory for bounded sets. In this paper, we will use it in the proof of Theorems 4.4 and 4.5.

**Proposition 3.7.** Let \( \psi: \Omega \to \mathbb{R} \) and \( f \in D^p(\Omega) \) be such that \( \mathcal{K}_{\psi,f} \) is nonempty, and let \( u \) be a solution of the \( \mathcal{K}_{\psi,f} \)-obstacle problem. Suppose that \( \Omega' \) is an open subset of \( \Omega \). Then \( u \) is a solution of the \( \mathcal{K}_{\psi,u}(\Omega') \)-obstacle problem.

Moreover, if \( \Omega' \subset \Omega \), then \( u \) is a solution also of the \( \mathcal{K}_{\psi,u}(\Omega') \)-obstacle problem.

**Proof.** Let \( \Omega' \) be an open subset of \( \Omega \). Clearly, \( u \in \mathcal{K}_{\psi,u}(\Omega') \). Let \( v \in \mathcal{K}_{\psi,u}(\Omega') \) be arbitrary. To complete the first part of the proof, it is sufficient to show that
\[
\int_{\Omega'} g_u^p \, d\mu \leq \int_{\Omega'} g_v^p \, d\mu \quad (3.4)
\]

Let \( E = \Omega \setminus \Omega' \) and extend \( v \) to \( \Omega \) by letting \( v = u \) in \( E \). Since \( v - u \in D^p_0(\Omega') \), we have \( v = (v - u) + u \in D^p(\Omega) \) and \( v - f = (v - u) + (u - f) \in D^p_0(\Omega) \), and since \( v \geq \psi \) q.e. in \( \Omega' \) and \( v = u \geq \psi \) q.e. in \( E \), we conclude that \( u \in \mathcal{K}_{\psi,f} \).

Because \( u \) is a solution to the \( \mathcal{K}_{\psi,f} \)-obstacle problem, we have
\[
\int_{\Omega'} g_u^p \, d\mu + \int_E g_u^p \, d\mu = \int_\Omega g_u^p \, d\mu \leq \int_\Omega g_v^p \, d\mu = \int_{\Omega'} g_v^p \, d\mu + \int_E g_v^p \, d\mu \quad (3.5)
\]
Since \( u = v \) in \( \Omega \) implies that \( g_u = g_v \) a.e. in \( E \), we have
\[
\int_E g_u^p \, d\mu = \int_E g_v^p \, d\mu \leq \int_\Omega g_v^p \, d\mu < \infty.
\]
Subtracting the integrals over \( E \) in (3.5) yields (3.4).

For the second part, assume that \( \Omega' \subset \Omega \) and let \( v \in \mathcal{K}_{\psi,u}(\Omega') \) be arbitrary. Clearly, \( v \in \mathcal{K}_{\psi,u}(\Omega') \). The first part of the proof asserts that \( u \) is a solution of the \( \mathcal{K}_{\psi,u}(\Omega') \)-obstacle problem and hence (3.4) holds. By Proposition 2.14, we have \( u \in N^{1,p}_0(\Omega) \), and hence \( u \in N^{1,p}(\Omega') \). Thus \( u \in \mathcal{K}_{\psi,u}(\Omega') \) and the proof is complete.

There are many equivalent definitions of (super)minimizers in the literature (see Proposition 3.2 in A. Björn [1]). The first definition for metric spaces was given by Kinnunen–Martio [25]. Here we follow Björn–Björn–Mäkäläinen–Parviainen [6]. We also follow the custom of not making the dependence on \( p \) explicit in the notation.

**Definition 3.8.** Let \( V \) be a nonempty open subset of \( X \). We say that a function \( u \in N^{1,p}_{\text{loc}}(V) \) is a **superminimizer** in \( V \) if
\[
\int_{\varphi \neq 0} g_u^p \, d\mu \leq \int_{\varphi \neq 0} g_{u+\varphi}^p \, d\mu \quad (3.6)
\]
holds for all nonnegative \( \varphi \in N^{1,p}_0(V) \).

Furthermore, \( u \) is said to be a **minimizer** in \( V \) if (3.6) holds for all \( \varphi \in N^{1,p}_0(V) \).

According to Proposition 3.2 in A. Björn [1], it is in fact only necessary to test (3.6) with (nonnegative and all, respectively) \( \varphi \in \text{Lip}_c(V) \).

As a direct consequence of Proposition 3.7 together with Proposition 9.25 in Björn–Björn [4], we have the following result.
Proposition 3.9. Suppose that $u$ is a solution of the $\mathcal{K}_{\psi,f}$-obstacle problem. Then $u$ is a superminimizer in $\Omega$.

4. Lsc-regularized solutions and $p$-harmonic solutions

In this section, we make the rather standard assumptions that $1 < p < \infty$, that $X$ is a complete $p$-Poincaré space, that $\mu$ is doubling, and that $\Omega$ is a nonempty open subset of $X$ such that $C_p(X \setminus \Omega) > 0$.

When $\mu$ is doubling, it is true that $X$ is proper if and only if $X$ is complete, and also that $X$ supports a $(p,p)$-Poincaré inequality if and only if $X$ supports a $p$-Poincaré inequality (the necessity follows from Hölder’s inequality, and the sufficiency was proved in Hajlasz–Koskela [19]; see also Corollary 4.24 in Björn–Björn [4]). Thus, the difference between this section and the previous is that here we make the assumption that $\mu$ is doubling.

Note that under these assumptions, Poincaré inequalities are self-improving in the sense that $X$ supports a $q$-Poincaré inequality for some $q < p$ (this was proved by Keith–Zhong [23]). Hence, in this section, we make the same assumptions as Kinnunen–Martio [25], and we can therefore use Theorems 5.1 and 5.5 in [25].

Theorem 4.1. Let $\psi : \Omega \to \mathbb{R}$ and let $f \in D^p(\Omega)$. Then there exists a unique lsc-regularized solution of the $\mathcal{K}_{\psi,f}$-obstacle problem whenever $\mathcal{K}_{\psi,f}$ is nonempty.

The lsc-regularization of a function $u$ is the (lower semicontinuous) function $u^*$ defined by

$$u^*(x) := \operatorname{ess lim inf}_{y \to x} u(y) := \lim_{r \to 0} \operatorname{ess inf}_{B(x,r)} u.$$ 

Proof. Suppose that $\mathcal{K}_{\psi,f}$ is nonempty. Theorem 3.4 asserts that there exists a solution $u$ of the $\mathcal{K}_{\psi,f}$-obstacle problem and that all solutions are equal to $u$ q.e. in $\Omega$. Proposition 3.9 asserts that $u$ is a superminimizer in $\Omega$, and hence by Theorem 5.1 in Kinnunen–Martio [25], we have $u^* = u$ q.e. in $\Omega$. Thus $u^*$ is the unique lsc-regularized solution of the $\mathcal{K}_{\psi,f}$-obstacle problem. \qed

The following comparison principle improves upon Lemma 3.6.

Lemma 4.2. Let $\psi_j : \Omega \to \mathbb{R}$ and $f_j \in D^p(\Omega)$ be such that $\mathcal{K}_{\psi_j,f_j}$ is nonempty, and let $u_j$ be the lsc-regularized solution of the $\mathcal{K}_{\psi_j,f_j}$-obstacle problem for $j = 1,2$. Then $u_1 \leq u_2$ in $\Omega$ whenever $\psi_1 \leq \psi_2$ q.e. in $\Omega$ and $(f_1 - f_2)_+ \in D^p(\Omega)$.

Proof. By Lemma 3.6, we have $u_1 \leq u_2$ q.e. in $\Omega$, and since both $u_1$ and $u_2$ are lsc-regularized, it follows that

$$u_1(x) = \operatorname{ess lim inf}_{y \to x} u_1(y) \leq \operatorname{ess lim inf}_{y \to x} u_2(y) = u_2(x) \quad \text{for all } x \in \Omega. \quad \Box$$

Definition 4.3. Let $V$ be a nonempty open subset of $X$. We say that a function $u \in N_{\text{loc}}^{1,p}(V)$ is $p$-harmonic in $V$ whenever it is a continuous minimizer in $V$.

Kinnunen and Martio proved that the solution $u$ (if it exists) of their obstacle problem for bounded sets is continuous in $\Omega$ and is a minimizer in the open set $\{ x \in \Omega : u(x) > \psi(x) \}$ whenever the obstacle $\psi$ is continuous in $\Omega$ (Theorem 5.5 in [25]). This is true also for the $\mathcal{K}_{\psi,f}^n(\Omega)$-obstacle problem (see, e.g., Theorem 8.28 in Björn–Björn [4]), and also for our obstacle problem (that allows for unbounded sets).
Theorem 4.4. Let $\psi: \Omega \to [-\infty, \infty)$ be continuous and $f \in D^p(\Omega)$ be such that $K_{\psi,f}$ is nonempty. Then the lsc-regularized solution $u$ of the $K_{\psi,f}$-obstacle problem is continuous in $\Omega$ and $p$-harmonic in the open set $A = \{ x \in \Omega : u(x) > \psi(x) \}$.

We also have the following corresponding pointwise result.

Theorem 4.5. Let $\psi: \Omega \to [-\infty, \infty)$ and $f \in D^p(\Omega)$ be such that $K_{\psi,f}$ is nonempty. Let $x \in \Omega$. Then the lsc-regularized solution $u$ of the $K_{\psi,f}$-obstacle problem is continuous at $x$ if $\psi$ is continuous at $x$.

Proof. Let $x \in \Omega$ and let $\Omega'$ be an open set such that $x \in \Omega' \subseteq \Omega$. Let $u$ be the lsc-regularized solution of the $K_{\psi,f}$-obstacle problem. Proposition 3.7 asserts that $u$ is a solution of the $K_{\psi,f}^1(\Omega')$-obstacle problem. By Theorem 8.29 in Björn–Björn [4] (which is a special case of Corollary 3.4 in Farnana [16]), it follows that $u$ is continuous at $x$.

Proof of Theorem 4.4. The first part follows directly from Theorem 4.5.

Now we prove that $u$ is a minimizer in $A$. The set $A$ is open since $\psi$ and $u$ are continuous. Choose a ball $B \subset \Omega$ and $\delta > 0$ small enough so that the sets

$$A_n := \left\{ x \in nB \cap A : \inf_{y \in \partial A} d(x,y) > \delta/n \right\}, \quad n = 1, 2, \ldots,$$

are nonempty. Then $A_1 \subset A_2 \subset \ldots \subset A = \bigcup_{n=1}^{\infty} A_n$. Fix a positive integer $n$. Since $u$ is a solution of the $K_{\psi,f}^n(A_n)$-obstacle problem, Theorem 5.5 in Kinnunen–Martio [25] asserts that $u$ is $p$-harmonic in $A_n$. From this, it follows that $u$ is $p$-harmonic in $A$ (see, e.g., Theorem 9.36 in Björn–Björn [4]).

Due to Theorem 4.4, the following definition makes sense.

Definition 4.6. The $p$-harmonic extension $H_\Omega f$ of a function $f \in D^p(\Omega)$ to $\Omega$ is the continuous solution of the $K_{-\infty,f}(\Omega)$-obstacle problem.

Then $H_\Omega f$ is the unique $p$-harmonic function in $\Omega$ such that $f - H_\Omega f \in D^p_0(\Omega)$. Note that Definition 4.6 is a generalization of Definition 8.31 in Björn–Björn [4] to Dirichlet functions and to unbounded sets (see Remark 3.3).

We conclude that we have solved the Dirichlet problem for $p$-harmonic functions in open sets with boundary values in $D^p(\Omega)$ taken in Sobolev sense, and we finish the paper by giving a short proof of the following comparison principle.

Lemma 4.7. Suppose that $f_1$ and $f_2$ are in $D^p(\Omega)$ and that $(f_1 - f_2)_+ \in D^p_0(\Omega)$. Then $H_\Omega f_1 \leq H_\Omega f_2$ in $\Omega$.

The conclusion holds also under the assumption that $f_1$ and $f_2$ belong to $D^p(\Omega)$ and that $f_1 \leq f_2$ q.e. on $\partial \Omega$.

The first part is just a special case of Lemma 4.2.

Proof. We prove the second part. Clearly, $(f_1 - f_2)_+ \in D^p(\overline{\Omega})$. Since $f_1 \leq f_2$ q.e. on $\partial \Omega$, we have $(f_1 - f_2)_+ = 0$ q.e. on $\overline{\Omega} \setminus \Omega$, and hence $(f_1 - f_2)_+ \in D^p_0(\Omega; \overline{\Omega})$. Since $D^p_0(\Omega) = D^p_0(\Omega; \overline{\Omega})$ according to Proposition 2.9, the result follows from the first part.

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