Interactive Configuration by Regular String Constraints

Esben Rune Hansen\textsuperscript{1} and Henrik Reif Andersen\textsuperscript{1}

\textsuperscript{1} IT University of Copenhagen,
Rued Langgaards Vej 7, DK-2300 Copenhagen S, Denmark
{esben,hra}@itu.dk

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Abstract A product configurator which is complete, backtrack free and able to compute the valid domains at any state of the configuration can be constructed by building a Binary Decision Diagram (BDD). Despite the fact that the size of the BDD is exponential in the number of variables in the worst case, BDDs have proved to work very well in practice. Current BDD-based techniques can only handle interactive configuration with small finite domains. In this paper we extend the approach to handle string variables constrained by regular expressions. The user is allowed to change the strings by adding letters at the end of the string. We show how to make a data structure that can perform fast valid domain computations given some assignment on the set of string variables.

We first show how to do this by using one large DFA. Since this approach is too space consuming to be of practical use, we construct a data structure that simulates the large DFA and in most practical cases are much more space efficient. As an example a configuration problem on $n$ string variables with only one solution in which each string variable is assigned to a value of length of $k$ the former structure will use $\Omega(k^n)$ space whereas the latter only need $O(kn)$. We also show how this framework easily can be combined with the recent BDD techniques to allow both boolean, integer and string variables in the configuration problem.

1 Introduction

Interactive configuration is a special Constraint Satisfaction Problem (CSP), where a user is assisted in configuration by interacting with a configurator – a computer program. In configuration the user repeatedly chooses an unassigned variable and assigns it a value until all variables are assigned. The task of the configurator is to state the valid choices for each of the unassigned variable during the configuration. The set of valid choices for an unassigned variable $x$ is denoted the valid domain of $x$ [HSLJ+04], [SLHJ+04].

As an example consider the problem of assigning values to the variables $x_1$, $x_2$ and $x_3$ where $x_1 \in \{1, \ldots, 5\}$ and $x_2, x_3 \in \{1, \ldots, 10\}$ with the requirement that $x_1 = 1 \lor x_1 = 2 \lor x_2 = 2$ and $x_2 = x_3$. Initially the user can choose to assign...
a value from \{1, \ldots, 5\} to \(x_1\) or assign a value from \{1, \ldots, 10\} to \(x_2\) or \(x_3\). Suppose now the user assigns 3 to \(x_3\). In this case the valid domain of \(x_2\) is \{3\} and the valid domain of \(x_1\) is \{1, 2\}. We obtain the requirement \(x_2 = 3\) by \(x_2 = x_3\) and \(x_3 = 3\). Further we obtain \(x_1 \in \{1, 2\}\) by \(x_1 = 1 \lor x_1 = 2 \lor x_2 = 2\) and \(x_2 = 3\).

The valid domain of each unassigned variable has to be updated every time a value is assigned to a variable as the assignment might make other assignments invalid as in the example above. The user interaction with the configurator has to be real-time which in practice means that the configurator has to update the valid domains within 250 milliseconds [Ras00]. Calculating the valid domains is NP-hard since it can be used to solve 3SAT. However by making an off-line construction of a Binary Decision Diagram that represents the constraints we are able to keep the computation time polynomial in the size of the BDD. The BDD constructed can be exponentially large, but in practice BDDs have proved themselves to be far from exponential in size for many configuration problems.

As BDDs use binary variables to represent the domains of the variables we normally assume small finite domains. In this paper we will consider the case of variables that take strings as their values, hence their domain might not be finite. Therefore the standard BDD approach will not be able to handle the problem.

As an example suppose that a user has to fill in a form were there is a lot of constraints on the data. Consider a CSP with the variables \texttt{phone}, \texttt{country}, \texttt{zip} and \texttt{district} along with the following constraints:

I. The prefix of \texttt{phone} is “+45” \iff \texttt{country} = “Denmark”

II. \texttt{country} = “Denmark” \implies \texttt{zip} has four digits

III. \texttt{zip} = “2300” \land \texttt{country} = “Denmark” \iff \texttt{district} = “Copenhagen S”

Suppose in the CSP above that the user entered \texttt{district} = “Copenhagen S”. This restricts the valid domain of \texttt{zip} to the singleton set \{“2300”\} and the valid domain of \texttt{country} to \{“Denmark”\} by (III). The valid domain of \texttt{phone} is decreased to the set of strings which has “+45” as a prefix by (I).

Suppose instead that the user have entered \texttt{phone} = “+45 23493844”. This decreases the valid domain of \texttt{country} to \{“Denmark”\} by (I), and the valid domain of \texttt{zip} to strings consisting of 4 digits. Actually this restriction will be performed as soon as the user have entered “+45”, since every completion of \texttt{phone} achieved by appending additional letters at the end of \texttt{phone} still will have “+45” as a prefix.

2 Related Work

It has recently been proposed to introduce global constraints that require that the variables in the CSP considered in some order has to belong to a regular language, supposing that the domain of each variable is contained in the alphabet of the regular language [Pes04]. This approach has this year (in 2006) been extended to global constraints where the variables of the CSP have to belong to a specified context-free grammar [QW06][Sel06]. Both results give algorithms for ensuring generalized arc consistency which corresponds to valid domains in the case of interactive configuration.
Consider a CSP stated as $C = (X, \Sigma, F)$. By $X = \{x_1, x_2, \ldots, x_n\}$ we denote the variables of the problem. By $\Sigma$ we denote an alphabet. By $F = \{f_1, \ldots, f_o\}$ we denote formulas written using the following syntax

$$f ::= f \lor f | \neg f | \text{match}(x, \alpha),$$

where $\alpha$ is a regular expression over $\Sigma$. The expression $\text{match}(x, \alpha)$ is true if and only if $x \in L(\alpha)$, where $L(\alpha)$ is the language defined by the regular expression $\alpha$. We use $f \land g, f \Rightarrow g$ and $f \Leftrightarrow g$ as shortcuts for $\neg(\neg f \lor \neg g), \neg f \lor g$ and $(f \Rightarrow g) \land (g \Rightarrow f)$ respectively.

Regular expression are written on the syntax:

$$\alpha ::= \alpha\alpha | \alpha|\alpha^*$$

listed in increasing order of strength of binding. The expression $\alpha^*$ is zero or more repetitions of $\alpha$. The expression $\alpha\alpha$ is the concatenation of two regular expressions. The expression $\alpha_1|\alpha_2$ means that either $\alpha_1$ or $\alpha_2$. For instance $L(a|c|(abc^*)d) = L((a|c)(ab(c^*)d)) = \{\text{“ad”}, \text{“cd”}, \text{“abcd”}, \text{“abcccd”}, \ldots\}$. We further use “.” as a shortcut for any letter in $\Sigma$ — i.e. “$w_1|w_2|\ldots|w_\Sigma$” where $\{w_k | 1 \leq k \leq \Sigma\} = \Sigma$.

In the example where user had to fill in some data the restriction (I) from last section would be stated as:

$$\text{match}(\text{phone, “+45.*”}) \iff \text{match}(\text{country, “Denmark”})$$

where phone and country are two variables in $X$.

We denote by $\rho = \{(x_1, w_1), \ldots, (x_n, w_n)\}$ a complete assignment of the values $w_1, \ldots, w_n \in \Sigma^*$ to the variables $x_1, \ldots, x_n$ that is all the variables in $X$. We define $\Sigma^*$ in the usual way as $\epsilon \cup \Sigma \cup \Sigma^2 \cup \cdots$. The set of solutions to $C$ is the set of assignments to $X$ that satisfy all formulas in $F$, stated formally:

$$\text{sol}(C) = \{\rho | \rho \models f \text{ for all } f \in F\}$$

**Definition 1 (Valid Domains).** The valid domain of $x_i \in X$ relative to an assignment $\rho$, denoted $V^\rho_{x_i}$, is the set of values $w \in \Sigma^*$ for which appending $w$ to the current assignment to $x_i$ can be extended to a solution to $C$ by appending an appropriate string to values to the assignment to the remaining variables $X \setminus \{x_i\}$. Stated formally:

$$V^\rho_{x_i} = \{w \in \Sigma^* \mid \exists \rho' : \rho'(x_i) = w \land \rho \rho' \in \text{sol}(C)\}$$

where $\rho$ and $\rho'$ are assignments to $X$ and the concatenation $\rho \rho'$ is defined by $\rho \rho' = \{(x_1, \rho(x_1)\rho'(x_1)), \ldots, (x_n, \rho(x_n)\rho'(x_n))\}$
The following theorem will be proved in the next section:

**Theorem 1.** For any \( x \in X \) and any assignment \( \rho \) to \( X \) it holds that \( V^\rho_x \) is a regular language.

The goal of this paper is to construct a data structure that based on a CSP \( C = (X, \Sigma, F) \) support three operations:

- **BUILD(\( C \))** that constructs the data structure from \( C \),
- **APPEND(\( x_i, w \))** that updates \( \rho \) by setting \( \rho(x_i) \) to \( \rho(x_i)w \) and makes the data structure conform to the new \( \rho \), and
- **VALIDDOMAIN(\( x_i \))** that returns a regular expression that corresponds to the valid domain of \( x_i \) on \( \rho \) that is a regular expression \( \alpha \) for which \( L(\alpha) = V^\rho_x \).

As the two latter algorithms has to be used during user interaction the goal is to make these two operations run as fast as possible without using too much space.

One might consider a fourth operation **COMPLETE(\( x_i \))** that indicates that there will be no more updates to the value of some string variable which will imply an additional reduction of the valid domains. In the context of form validation this corresponds to the event that the user hits the return key or leaves the current input field. In Section 10.3 we show that this extra functionality easily can be supported by the three operations already mentioned.

In order to check whether \( w \in L(\alpha) \) we use a deterministic finite automaton (DFA). We denote DFAs deciding the regular expressions that occurs in \( F \) by the name *match-DFAs*.

### 4 A Solution based on a single DFA

In this section we will prove that \( V^\rho_x \) is a regular language. However we want to do more than that. We will present a construction of a DFA that for any \( x_i \in X \) and any assignment \( \rho \) to \( X \), can be turned into a DFA deciding \( V^\rho_x \). This proves that \( V^\rho_x \) is a regular language but the data structure that will be presented in this section uses too much space to be of any practical use. However it gives us a good starting point for making a smaller efficient data structure supporting the operations **BUILD**, **APPEND** and **VALIDDOMAIN** mentioned in the last section.

The DFA we want to construct is denoted \( M_C \), and is the DFA deciding a language we denote \( L_C \). We will now spend some time on defining the language \( L_C \). The basic property of \( L_C \) is that:

\[ w \in L_C \iff \rho_w \in \text{sol}(C) \quad \text{(1)} \]

where \( w \) is a word that induces the assignment \( \rho_w \), where the meaning of induces will be defined in (3).

Intuitively we make the alphabet of \( L_C \), denoted \( \Sigma_C \), consist of all possible **APPEND**-operations. More formally stated \( \Sigma_C \subseteq (\Sigma \cup \{\epsilon\})^n \) where each letter in \( \Sigma_C \) only contain one element different from \( \epsilon \) that is:

\[ \Sigma_C \overset{def}{=} \bigcup_{1 \leq i \leq n} \bigcup_{w \in \Sigma} \{ (\epsilon, \ldots, \epsilon, w, \epsilon, \ldots, \epsilon) \}_{i-1}^{n-i} \quad \text{(2)} \]
Every word \( w \) in \( L_C \) is a concatenation of letters from \( \Sigma_C \) that is \( L_C \subseteq \Sigma_C^n \). We say that:

\[
w = w_1 \cdots w_k \quad \text{induces} \quad \rho_w = \{(x_1, w_{1,1} \cdots w_{1,k}), \ldots, (x_n, w_{1,n} \cdots w_{k,n})\} \tag{3}
\]

where \( w_{1,i} \) denotes the \( i \)th element in the letter \( w_1 \in \Sigma_C \) and \( 1 \leq l \leq k \) and \( 1 \leq i \leq n \) and \( \rho_w \) is an assignment to \( C \). Note that for any \( w = w_1 \cdots w_k \) every word \( w' \) that consist of the exactly the letters \( w_1, \ldots, w_k \) permuted in a way that maintains the ordering of \( w_1, \ldots, w_k \) for every \( 1 \leq i \leq n \) we have \( \rho_w' = \rho_w \).

Hence every assignment \( \rho_w \) corresponds to exactly the \( w \) words. For convenience we will in the following, when we use \( w \) and \( \rho_w \) in the same calculations, implicitly assume that \( w \) induces \( \rho_w \) as defined in (3).

**Example 1.** Consider the CSP where \( X = \{x_1, x_2, x_3\} \) and \( \Sigma = \{a,b\} \). In this case

\[
\Sigma_C = \{(a,\epsilon,\epsilon),(b,\epsilon,\epsilon),(a,\epsilon,\epsilon),(b,\epsilon,\epsilon),(\epsilon,\epsilon,\epsilon),(\epsilon,a,\epsilon),(\epsilon,\epsilon,b),(\epsilon,\epsilon,a)\}
\]

and for instance does the word \( w = (a,\epsilon,\epsilon)(\epsilon,\epsilon,a)(b,\epsilon,\epsilon)(a,\epsilon,\epsilon) \) induce the assignment \( \rho_w = \{(x_1,aba),(x_2,\epsilon),(x_3,a)\}, \) and so does for instance \( w' = (a,\epsilon,\epsilon)(b,\epsilon,\epsilon)(a,\epsilon,\epsilon)(\epsilon,\epsilon,a) \) and \( w'' = (a,\epsilon,\epsilon)(b,\epsilon,\epsilon)(\epsilon,\epsilon,a)(a,\epsilon,\epsilon) \). In the case of \( w, (1) \) becomes:

\[
(b,\epsilon,\epsilon)(\epsilon,\epsilon,b)(b,\epsilon,\epsilon)(a,\epsilon,\epsilon) \in L_C \iff \{(x_1,aba),(x_2,\epsilon),(x_3,a)\} \in \text{sol}(C)
\]

Note however that for instance \( w''' = (b,\epsilon,\epsilon)\epsilon,\epsilon,a)(a,\epsilon,\epsilon) \) does not induce \( \rho_w, \) since \( \rho_w''' = \{(x_1,baa),(x_2,\epsilon),(x_3,a)\}. \)

Hence if we can make a DFA that decides \( L_C \) this DFA can be used to decide for any assignment \( \rho \) whether \( \rho \in \text{sol}(C) \). In the following we will construct such a DFA and we will show how we based on this construction for any \( V^\rho_{x_i} \) can make a DFA that decides the language \( V^\rho_{x_i} \) thereby showing that \( V^\rho_{x_i} \) is a regular language. Before we begin the construction we formally define a DFA:

**Definition 2 (DFA).** A deterministic finite automaton DFA = \((Q, \Sigma, \delta, s, A)\), has a finite set of states \( Q \), a transition function \( \delta : Q \times \Sigma \rightarrow Q \), where \( \Sigma \) is some alphabet, a starting state \( s \in Q \) and a set of accepting states \( A \subseteq Q \). We use \( \delta(s,w) \) as a shorthand for \( (\cdots \delta(\cdots \delta(\delta(q,w_1),w_2),\ldots,w_l) \), where \( (w_1,\ldots,w_l) \) are the letters of the \( w \in \Sigma^* \). If \( q = s \) we write \( \delta(q,w) \) as \( \delta(w) \).

**Definition 3 (Reachability in a DFA).** In a DFA \( M = (Q, \Sigma, \delta, s, A) \) a state \( q \) is reachable from a state \( p \) by the string \( w \in \Sigma^* \) if and only if \( \delta(p,w) = q \). In particular any state is reachable from itself by the empty string. The state \( q \) is reachable from \( p \) if and only if \( q \) is reachable from \( p \) by some string. We say that a state is reachable in \( M \) if it is reachable from the source.

In the rest of this paper we will use the notation \( p \sim q \) to denote that \( q \) is reachable from \( p \). We will also assume that \( M_\gamma = (Q_\gamma, \Sigma_\gamma, \delta_\gamma, s_\gamma, A_\gamma) \) for any subscript \( \gamma \).

In the rest of this section we will do the following. First we construct the DFA \( M_C \) based on the match-DFAs of \( F \). We then reduce the DFA \( M_C \), by replacing the alphabet and defining \( A_C \). We thereby obtain that \( M_C \) decides the language \( L_C \) where \((q_1,\ldots,q_m) \in Q_C \) and \((w_1,\ldots,w_n) \in \Sigma_C \). Finally we show
how we can turn $M_C$ into an automaton deciding $V_x^n$ by changing the source and the alphabet in $M_C$.

After this brief overview we begin the actual construction. We start by constructing $M_C$. This construction can be divided into three steps:

1. For every match-expression $\text{match}(x_i, \alpha)$ that occurs in $F$ we construct a match-DFA that decides the regular language $L(\alpha)$. We denote these match-DFA $M_1, \ldots, M_m$, where $M_j$ is the match-DFA deciding the regular expression in the $j$th match-expression in $F$, assuming some order on the match-expressions in $F$. We define the mapping $I : \{1, \ldots, m\} \rightarrow \{1, \ldots, n\}$ such that $x_{I_j}$ is the variable that occurs in the $j$th match-expression.

2. For every state $q$ in the DFA $M_1, \ldots, M_m$ we add a self-looping transition on the empty string $\epsilon \notin \Sigma$ i.e. the transition $\delta(q, \epsilon) = q$. This results in DFA as the ones shown in Figure 1.

3. We construct a DFA $M_C = (Q_C, \Sigma_C, \delta_C, s_C, A_C)$ defined by:

   \[
   Q_C = Q_1 \times \cdots \times Q_m
   \]

   \[
   s_C = (s_1, \ldots, s_m)
   \]

   \[
   \delta_C : \delta_C((q_1, \ldots, q_m), (w_1, \ldots, w_n)) = (\delta_1(q_1, w_{I_1}), \ldots, \delta_m(q_m, w_{I_m})) \quad \text{where} \quad (q_1, \ldots, q_m) \in Q_C \text{ and } (w_1, \ldots, w_n) \in \Sigma_C
   \]

   \[
   A_C = \{(q_1, \ldots, q_m) \in Q_C \mid \{y_1, (q_1 \in A_1)\}, \ldots, (y_m, (q_m \in A_m))\} \models f[\text{match}(x_{I_j}, \alpha_j) \leftarrow y_j] \quad \text{for all } f \in F \text{ where we by } f[\text{match}(x_{I_j}, \alpha_j) \leftarrow y_j] \text{ mean the formula } f \text{ where every match-expression on the form } \text{match}(x_{I_j}, \alpha_j) \text{ is replaced by the boolean variable } y_j.
   \]

   The definition of $Q_C$ and $s_C$ should be straightforward. The definitions of $\delta_C$ and $A_C$ need some explanation.

In order to explain the definition of $\delta_C$ we break it down to four steps:

1. Since every state in $M_C$ is a vector of $m$ states a straightforward definition of $\delta_C$ would be on the alphabet $\Sigma^m$ on vectors on $m$ letter. Making every transition correspond to taking exactly one step in each of the $m$ underlying DFAs.

2. For our use we need to ensure that we follow transitions on the same letter in every set of DFAs that evaluates the same variable. This is ensured by using the mapping $I : \{1, \ldots, m\} \rightarrow \{1, \ldots, n\}$ defined earlier in the Section. The mapping $I$ is used to map every vector of letters in $(w_1, \ldots, w_n) \in \Sigma^m$ to a vector $(w_{I_1}, \ldots, w_{I_m}) \in \Sigma^m$ where $w_{I_i} = w_{I_j}$ if the two match-DFA $M_i$ and $M_j$ evaluates the same variable.

3. By extending the alphabet $\Sigma^m$ to $(\Sigma \cup \{\epsilon\})^m$ we make it possible to make movements that corresponds to appending a letter to the value of a subset of the variables.

4. Finally we replace the alphabet $(\Sigma \cup \{\epsilon\})^m$ by $\Sigma_C$ as defined in (2) – that is we remove all letters from the alphabet that does not correspond to appending a letter to the value of exactly one variable.
The above four steps are described in terms of the alphabet and not in terms of transitions. However, by exchanging letters above we implicitly mean that the definitions of the transitions are exchanged as well. If we for instance exchanged a word \( w_1 \in \Sigma_1 \) by \( w_2 \in \Sigma_2 \) the transition \( \delta(p, w_1) = q \) would be exchanged by the transition \( \delta(p, w_2) = q \).

**Example 2.** Consider the example of a DFA \( M_C \) in Figure 2 based on the two match-DFAs from Figure 1. In figure 2 we have indicated all transitions corresponding to taking a single move in both of the two match-DFAs by arrows.

For all DFAs in this paper accepting states are indicated by double circles and the source is assumed to be the leftmost state in the graph. Further each state are labeled with the regular expression corresponding to the state, that is every state \( q \) is labeled with the regular expression \( \alpha \) for which \( w \in L(\alpha) \iff \hat{\delta}(w) = q \).

When the alphabet is of the DFA is a subset of \( \Sigma_2 \) we label the states by two regular expressions \( \alpha \) and \( \beta \) such that \( w_1 \in L(\alpha) \land w_2 \in L(\beta) \iff \hat{\delta}(w_1w_2) = q \).

To check for some \( q \in Q_C \) whether there exists a \( w \in \Sigma_0^* \) for which \( \hat{\delta}(w) = q \) is a simple task but checking whether \( \rho_w \in \text{sol}(\mathcal{C}) \) holds for every \( w \) for which \( \hat{\delta}(w) = q \) requires some explanation.

Every \( j \)th match-expression in \( \mathcal{F} \) evaluated by the match-DFA \( M_j \) is a term that either is true or false depending on whether the current state in the \( M_j \)
Figure 2: The DFA $M_C$ built based on the DFAs corresponding to $L("ab")$ and $L("ac")$ showed in Figure 1

is accepting or not. Every state $q$ in $M_C$ corresponds to the combination of states $(q_1, \ldots, q_m) = q$ in the match-DFAs $M_1, \ldots, M_m$. Because of this we might intuitively consider every match-expression as a boolean variable. Let us denote the boolean variables $y_1, \ldots, y_m$, and let $y_j$ correspond to the $j$th match-expression in $F$ for $1 \leq j \leq m$. Observe that every state $q = (q_1, \ldots, q_m) \in Q_C$ can be conceived as a complete assignment of boolean values to such $y_1, \ldots, y_m$ by acceptance/rejection of $q_1, \ldots, q_m$ by $M_1, \ldots, M_m$ respectively. If this complete assignment satisfies every formula $f \in \mathcal{F}$, then we have for every $\hat{\delta}(w) = q$ that $\rho_w \in \text{sol}(\mathcal{C})$, otherwise we have for every $\hat{\delta}(w) = q$ that $\rho_w \notin \text{sol}(\mathcal{C})$.

We restate this in formal terms. We first define the assignment

$$\tau_q = \{(y_1, (q_1 \in A_1)), \ldots, (y_m, (q_m \in A_m))\}$$

to the boolean variables $y_1, \ldots, y_m$. We let $\alpha_j$ be the regular-expression in the $j$th match-expression and obtain

$$\rho_w \in \text{sol}(\mathcal{C}) \iff \tau_{\hat{\delta}(w)} \models f[\text{match}(x_{I_j}, \alpha_j) \leftarrow y_j] \text{ for all } f \in \mathcal{F}$$

where we by $f[\text{match}(x_{I_j}, \alpha_j) \leftarrow y_j]$ we mean the formula $f$ where every match-expression on the form $\text{match}(x_{I_j}, \alpha_j)$ is replaced by the boolean variable $y_j$.

Using equation (5) this can be rewritten as:

$$A_C = \{q \in Q_C \mid \tau_q \models f[\text{match}(x_{I_j}, \alpha_j) \leftarrow y_j] \text{ for all } f \in \mathcal{F}\}$$

Checking for some $q$ whether $\tau_q \models f[\text{match}(x_{I_j}, \alpha_j) \leftarrow y_j]$ can be done by simply plugging in some values in the boolean formula $f$ and checking whether this makes $f$ true or false.
Having explained $\delta_C$ and $A_C$ we now consider how to turn $M_C$ into a DFA that decides $V^p_x$. We start by stating the definition of valid domains (Definition 1) in terms of the language $L_C$ as:

$$V^p_x = \{ w \in \Sigma^* | \exists w_C \in L_C : \rho_{w_C}(x_i) = \rho(x_i)w \}$$

If we want to change $M_C$ such that it decides $V^p_x$ we have to do two things:

1. Set the source in $M_C$ to $\hat{\delta}_C(w)$, where $w \in \Sigma_C$ is the word corresponding to $\rho$

2. Project the alphabet on $x_i$—that is, replace every letter $w = (w_1, \ldots, w_n) \in \Sigma_C$ by $w_i \in \Sigma \cup \{\epsilon\}$

Note that the second step turn all transitions on $w$ for which $w_i = \epsilon$ into $\epsilon$-transitions, hence we have made a non-deterministic automaton on the alphabet $\Sigma$, deciding $V^p_x$. Using basic automata theory we obtain a corresponding DFA and the corresponding regular expression.

**Example 3.** Consider the example $C = (X, \Sigma, F)$ where $X = \{x_1, x_2\}$, $F = f_1, f_2$ where $f_1 = \text{match}(x_1, \text{“ab”}) \lor \text{match}(x_2, \text{“abc”})$ and $f_2 = \text{match}(x_2, \text{“abd”})$. We construct the match-DFA $M_1, M_2$ and $M_3$ on the regular languages $L(\text{“ab”})$, $L(\text{“abc”})$ and $L(\text{“abd”})$ respectively. To each state in $M_1, M_2$ and $M_3$ we add $\epsilon$-transitions that are self-loops. The resulting DFAs are shown in Figure 3.

We now begin the construction of the DFA $M_C$. Since $Q_C = Q_1 \times Q_2 \times Q_3$ this DFA will have $|Q_C| \cdot |Q_1| \cdot |Q_2| = 3 \cdot 4 \cdot 3 = 36$ states. However not all the states are reachable by $\delta_C$ since $\delta_C$ is only defined on $\Sigma_C = \bigcup_{w \in \Sigma} \{(w, \epsilon) \cup (\epsilon, w)\}$.

The remaining 14 states and the transitions in $M_C$ are shown in Figure 4.

We can check for each state in $M_C$ whether it is accepting by checking if its corresponding states in the match DFAs $M_1, \ldots, M_m$ yield an assignment to the match-expressions by acceptance/rejection that make $F$ true. In this example only the state labeled “(ab, abd*)” is accepting.

Suppose now we want to calculate $V^p_{x_2}$ where $\rho = \{(x_1, \text{“a”})(x_2, \text{“ab”})\}$. We first set $s_C = \hat{\delta}_C((\text{“a”}, \epsilon)(\text{“a”})(\epsilon, \text{“b”}))$ and then replace every letter $w = (w_1, w_2) \in \Sigma_C$ by $w_2 \in \Sigma \cup \{\epsilon\}$ that is every $(\epsilon, w_2)$ by $w_2$ and every $(w_1, \epsilon)$ by $\epsilon$. The resulting non-deterministic automaton and its corresponding DFA is shown in Figure 5. In this example we get $V^p_{x_2}\{(x_1, \text{“a”}),(x_2, \text{“ab”})\} = \text{“d*”}$.

Figure 3: The upper two DFAs stems from $\text{match}(x_1, \text{“ab”})$ and $\text{match}(x_2, \text{“abc”})$ respectively. The lower DFA stems from $\text{match}(x_2, \text{“abd*”})$. 

Diagram 3
Figure 4: A DFA $M_C$ built on the formula $f_1 = \text{match}(x_2, \text{“abc”}) \lor \text{match}(x_1, \text{“ab”}) \land \text{match}(x_2, \text{“abd*”})$. The transition-labels 1:$w$ and 2:$w$ where $w \in \Sigma$ corresponds to the assignments $\rho = \{(x_1, w)(x_2, \epsilon)\}$ and $\{(x_1, \epsilon), (x_2, w)\}$ respectively. For simplicity the states corresponding to rejection of any of the match-expressions are not included. A DFA with all states are shown in Figure 7 in the Appendix.

Figure 5: To the left: The non-deterministic automaton deciding valid domains $V_{x_2}^\rho$ where $\rho = \{(x_1, \text{“a”}), (x_2, \text{“ab”})\}$ derived from the DFA $M_C$ in Figure 4. To the right: the corresponding DFA.

The size of the valid domains DFA Though both updating $x_i$ and computing valid domains will be fast using this solution, the size of the DFA is too large for the solution to be of any use for larger problems. As an example a problem on $n$ variables containing a single solution $\{(x_1, w_1), \ldots, (x_n, w_n)\}$ where $|w_i| = k$ for all $1 \leq i \leq n$ the $M_C$ will contain $\Omega(k^n)$ states. The construction that we will achieve at the end of this paper will contain $O(kn)$ states.

4.1 Simulating the valid domains DFA

In order to make a less space consuming construction we separate the valid domains DFA into smaller DFAs, that is instead of joining all the match-DFAs into the DFA $M_C$ we only join match-DFAs on the same variable. The drawback of this approach is that we cannot encode the boolean logic of $\mathcal{F}$ into the DFAs on the variables as each of these DFAs only constitutes a partial solution to $\mathcal{F}$. We therefore build a BDD on the boolean logic of $\mathcal{F}$. In this BDD every match-expression is considered as a boolean variable. Given any combination of states in the DFAs on the values the we can compute the value of $A_C$ on
the fly by restricting the BDD to the acceptance and rejections of the various values. In this way we are able to simulate the DFA $M_C$ by a much smaller data-structure. This structure will perform well in terms of updating values and deciding $L_C$ and reporting $V_{x_i}$. Performing this construction is the main of this paper.

In Section 5 we describe how to encode a set of the DFAs on the same variable into a Multi-DFA that can simulate many DFAs simultaneously on the same string. In Section 6 we encode into every state $q$ in the Multi-DFA which combinations of acceptance/rejection by the simulated match-DFAs that can be reached by following transitions corresponding to some word from $q$ in the Multi-DFA. In Section 7 we construct the BDD taking care of the boolean logic in $F$ as the constraint problem $D$, where every match-expression is replaced by a boolean variable. In Section 8 we present the algorithms $\textsc{Build}(C)$, $\textsc{Append}(x_i, w)$ and $\textsc{ValidDomain}(x_i)$. Finally in Section 10 we consider various extensions to the data structure.

5 DFAs and Multi-DFAs

By the construction of the DFA $M_C$ in the previous section we have ensured two properties:

1. All small DFAs on the same variable are synchronized
2. All states that cannot be a valid solution are removed

In order to reduce the space consumption of the DFAs we will present solution that only join match-DFAs on the same variable. By doing this we ensure (1). In the last section we could ensure (2) simply by minimizing the DFA. We do not have this option if we separate DFAs on different variables since the DFAs will not have the logic of $F$ encoded in their structure. This problem will be addressed in Section 7.

Since DFAs on a single variable often will be the combination of more than one match- DFA and since the value of one variable is not enough to determine whether or not $F$ is satisfied, we cannot use acceptance and rejection in the same way as in Section 7. We therefore replace the notion of accepting states by an bit-vector denoted acceptance value assigned to each state containing true or false for each of the match-DFAs accepting or rejecting for each in the current state. This is the idea behind the following generalization of the definition of a DFA.

Definition 4. A multi-DFA (MDFA) $(Q, \Sigma, \delta, s, a)$ of acceptance size $k$, has a finite set of states $Q$, a transition function $\delta : Q \times \Sigma \to Q$, where $\Sigma$ is some alphabet, a starting state $s$ and an acceptance value $a(q) \in \mathbb{B}^k$ for every $q \in Q$. The acceptance value of a word $w$ is defined as $a(\delta(w)) \in \mathbb{B}^k$.

Note that the definition above assigns exactly one acceptance value to every finite string in $\Sigma^*$. Note further that an MDFA with acceptance size 1 is a standard DFA with the set of accepting states $\{ q \mid a(q) = \text{true} \}$.

As we in the rest of this paper only use the alphabet $\Sigma$ given by the CSP $C = (X, \Sigma, F)$ we will from now on not state the alphabet $\Sigma$ explicitly in our definitions of DFAs and MDFAs. In other words we use $(Q, \delta, s, A)$ and $(Q, \delta, s, a)$
as a shortcuts for \((Q, \Sigma, \delta, s, A)\) and \((Q, \Sigma, \delta, s, a)\) for DFAs and MDFAs respectively, where \(\Sigma\) is the alphabet given by \(\mathcal{C}\).

**Construction of an MDFA** We might build an MDFA by slightly modifying the construction of the DFA \(M_C\). However this might make the intermediate structure very large. Instead we use a simple approach making a simultaneous DFS in the DFAs that has to be joined as described in the next two bits of pseudocode. We let \(\mu, Q, \delta, s, a, k\) and \(Q_i, \delta_i, s_i, a_i\), for \(1 \leq i \leq k\) be globals.

### RecConstructMDFA\((q_1, \ldots, q_k)\)
1. if \(\mu(q_1, \ldots, q_k)\) is defined
2. then return \(\mu(q_1, \ldots, q_k)\)
3. create a new state \(q \notin Q\)
4. \(Q \leftarrow Q \cup \{q\}\)
5. \(\mu(q_1, \ldots, q_k) \leftarrow q\)
6. \(a(q) \leftarrow ((q_1 \in A_1), \ldots, (q_k \in A_k))\)
7. for each \(w \in \Sigma\)
8. do \(\delta(q, w) \leftarrow \text{RecConstructMDFA}(\delta_1(q_1, w), \ldots, \delta_k(q_k, w))\)
9. return \(q\)

### ConstructMDFA\((DFA_1, \ldots, DFA_k)\)
1. \(Q \leftarrow \delta \leftarrow a \leftarrow \mu \leftarrow \emptyset\)
2. \(s \leftarrow \text{RecConstructMDFA}(s_1, \ldots, s_k)\)
3. return \((Q, \delta, s, a)\)

The function \(\mu\) is used to ensure a new state in the MDFA corresponding to a position \((q_1, \ldots, q_k)\) in the DFAs is created only once. We only create new states (by proceeding to Line 3) if \(\mu(q_1, \ldots, q_k)\) is undefined, which is the case if and only if \((q_1, \ldots, q_k)\) has not been visited before. Otherwise we return the previously created state that is assigned to \(\mu(q_1, \ldots, q_k)\) to the caller in Line 2. In Line 6 we by “\(q_j \in A_j\)” mean true if \(q_j \in A_j\) and false otherwise.

For instance the requirements \(\text{match}(x_1, "abc")\) and \(\text{match}(x_1, "abd\,*")\) on \(x_1\) will result in the MDFA drawn in figure 6.

![Figure 6](image_url)

**Acceptance values**

1. \((false, false)\)
2. \((false, false)\)
3. \((false, true)\)
4. \((true, false)\)
5. \((false, true)\)

Note that the state \((true, true)\) corresponding to \(\text{match}(x_2, "abc") \land \text{match}(x_2, "abd\,*")\) is not contained in the MDFA due to the fact that \(L("abc") \cap L("abd\,*") = \emptyset\).

Note also that this construction could be easily adapted to construct \(M_C\) if use the alphabet \(\Sigma_C\) and following the transition in the DFAs.

We want to make sure that the construction of the MDFA is minimal in the number of states it is contained. In order to prove this we need to define
what means to have a minimal number of states. This can be done by a natural generalizing the definition of a minimized DFA to a minimized MDFA.

**Definition 5.** A MDFA is minimized if all states in the MDFA are reachable from s and no pair of states in the MDFA are equivalent. For any pair of nodes \( p, q \in Q : p \) and \( q \) are equivalent by definition if and only if for all words \( w \in \Sigma^* : a(\hat{\delta}(p, w)) = a(\hat{\delta}(q, w)) \).

**Lemma 1.** If the DFAs given as input to ConstructMDFA are minimized then the constructed MDFA will be minimized.

*Proof.* We first note that all states in \( Q \) are reachable. This is due to the fact that every state created except \( s \) will be a result of a recursive call made at line 7. Hence every created state in the MDFA will be assigned to a \( \delta(q, w) \) for state \( q \) reachable by \( s \) and some \( w \in \Sigma \).

We then prove that no pair of states in the constructed MDFA is equivalent if every DFA \( DFA_1, \ldots, DFA_k \) is minimal. Consider any pair of distinct nodes \( p, q \in Q \). Suppose \( \mu(p_1, \ldots, p_k) = p \) and \( \mu(q_1, \ldots, q_k) = q \). Since \( p \neq q \) we know by the initial check on line 1-2 that \( (p_1, \ldots, p_k) \neq (q_1, \ldots, q_k) \). Hence for some \( 1 \leq i \leq k \) we have \( p_i, q_i \in Q_i \) for which \( p_i \neq q_i \). Since \( DFA_i \) is minimized we know that \( p_i \) is not equivalent to \( q_i \) which implies that there exists an \( w \in \Sigma^* \) for which \( a(\hat{\delta}(p_i, w)) \neq a(\hat{\delta}(q_i, w)) \). This implies that \( a(\hat{\delta}(\mu(p_1, \ldots, p_k), w)) \neq a(\hat{\delta}(\mu(q_1, \ldots, q_k), w)) \) which by is the same as \( a(\hat{\delta}(p, w)) \neq a(\hat{\delta}(q, w)) \). Hence \( p \) and \( q \) are not equivalent.

### 6 Reachable acceptance values

As we noticed earlier then main problem we face by not joining all match-expression into one big DFA is that we lack the logic. We will present a notion we call Reachable acceptance values. The reachable acceptance values of a state \( p \) in an MDFA is the set containing exactly the acceptance values of every state \( q \) that can be reached from the state \( p \) by following zero or more transitions from \( p \). Formally:

\[
R(p) = \{a(q) \mid p \sim q\}, \text{ where } p, q \in Q
\]

**Example 4.** The states in the MDFA on Figure 6 has that following reachable acceptance values:

- \( R(1) = R(2) = R(3) = \{(true, false), (false, true), (false, false)\} \)
- \( R(4) = \{(true, false), (false, false)\} \) and
- \( R(5) = \{(false, true), (false, false)\} \)

The goal in this section is to compute and store the set of reachable acceptance values for each of the states in an MDFA. When this set is stored we can at any state of the MDFA know in advance which acceptance values that we might end up in. Hence we can use this to constrain the logical structure, by only allowing values that can be reached from the current state. The exact meaning of “constraining the logical structure” will be clear in Section 7.

Having defined the set of reachable acceptance values we now consider how to compute the set for every state in an MDFA in an efficient way.
6.1 Computing the reachable acceptance values

We start by pointing out two obvious facts about the reachable acceptance values $R$ for the nodes in an MDF $A$

**Fact 1:** If a state $p$ has transitions to exactly the states $\{q_1, \ldots, q_l\}$ then $R(p) = a(p) \cup R(q_1) \cup \ldots \cup R(q_l)$

**Fact 2:** If two states $p, q$ belong to the same strongly connected component we have $R(p) = R(q)$.

A strongly connected component in an MDF $(Q, \delta, s, a)$ is defined as a set of states $C \subseteq Q$ for which it for any $p \in C$ holds that $p \sim q$ and $q \sim p$ if and only if $q \in C$. Calculating the strongly connected components in an MDF $A$ easily be done in linear time in the size of the MDF [CLRS01].

```plaintext
COMPUTEreachableAcceptanceStates(M)
1 Let $C'$ be the set of strongly connected components in $Q$
2 for each $C_1, C_2 \in C'$
3    do if $\delta(p, w) = q$ for some $p \in C_1, q \in C_2, w \in \Sigma$
4        then $\Gamma(C_1) \leftarrow \Gamma(C_1) \cup C_2$
5    for each $C \in C'$
6        do $R'(C) \leftarrow \bigcup_{q \in C} \{a(q)\}$ \quad \triangleright Ensure Fact 1
7    for each $C_1 \in C'$ in reverse topological order
8        do $R'(C_1) = R'(C_1) \cup \bigcup_{C_2 \in \Gamma(C_1)} R'(C_2)$ \quad \triangleright Ensure Fact 2
9    for each $C \in C'$
10       do for each $q \in C$
11          do $R(q) \leftarrow R'(C)$
12 return $R$
```

We assume that $M = (Q, \delta, s, a)$ is an MDF and that initially $R = R' = C' = \Gamma = \emptyset$. In Line 2-4 we construct the neighbor function $\Gamma(C)$ mapping any strongly connected component into the set of “children” of the strongly connected component. In Line 5-6 every $R'(C)$ is assigned to the set of acceptance values of the states contained in $C$. In Line 7-8 for every connected component $C_1$, the set $R(C_1)$ is assigned to the union of all $R'(C_2)$s for which $C_1 \sim C_2$ in $C'$. Note that the topological order in $C'$ is well defined since $C'$ is a DAG [CLRS01]. Finally in Line 9-11 the reachable acceptance states of the strongly connected components are assigned to the reachable acceptance states of the states in $Q$

7 The boolean logic of $\mathcal{F}$

We now return to the problem of representing the boolean logic of $\mathcal{F}$. In Section 4 the boolean logic was contained in the DFA, in the way that every $C$ in the DFA $M_C$ constructed in Section 4 was encoded by whether a state was an accepting state or not.

Since we have divided the match-expressions in $\mathcal{F}$ into MDFAs on each of the variables $x \in \mathcal{X}$ no MDFA is can in it self decide whether $\mathcal{F}$ is satisfied or not. This is why the MDFAs are neither accepting or rejecting. However if we pick
a state from each of the MDFAs this set of states is a complete assignment to the variables in \( \mathcal{X} \). Such a set is an accepting state if and only if evaluating the match-expression by the rejection/acceptance of the match-DFAs used during the construction of the MDFAs, on the states corresponding to the states in the MDFAs, makes \( \mathcal{F} \) true – exactly as in Section 4. We denote such a set an accepting set. Furthermore every state in an MDFA is valid if it occurs in some accepting set. If it occurs in no accepting set it is invalid. We observe that every accepting set of states correspond to an accepting state in \( M_C \).

In order to represent the boolean logic in \( \mathcal{F} \) we define a CSP \( \mathcal{D} = (\mathcal{Y}, \mathcal{B}, \mathcal{G}) \) based on \( \mathcal{C} \). The construction of the problem has many similarities with the calculation of the set of accepting states in \( M_C \) in Section 4. The variables in \( \mathcal{Y} \) are the same as the \( y \)-variables in Section 4 and all the constraints \( \{ f[\text{match}(x_{I_j}, \alpha_j) \leftarrow y_j] \mid f \in \mathcal{F} \} \) are constraints in \( \mathcal{G} \). However we need some extra constraints in \( \mathcal{G} \) and another way to index the \( y \)-variables in \( \mathcal{Y} \) in this section, but basically this section is just an extension of the techniques used in Section 4. We will now present the notation that will be used in this section, that will help us describe the implementation of the three operations BUILD, APPEND and VALIDDOMAIN in the next section.

Let \( \mathcal{D} = (\mathcal{Y}, \mathcal{B}, \mathcal{G}) \) be a CSP, where \( \mathcal{Y} = \{ y_1, \ldots, y_m \} \) is a set of boolean variables and \( \mathcal{G} \) is a set of boolean constraints on the values that can be assigned \( \mathcal{Y} \). Let \( \phi = \{(y_1, b_1), \ldots, (y_m, b_m)\} \), where \( y_1, \ldots, y_m \in \mathcal{Y} \) and \( b_1, \ldots, b_m \in \mathcal{B} \) denote a complete assignment of the variables in \( \mathcal{Y} \) to boolean values, or in short: an assignment to \( \mathcal{Y} \). We define the solution to \( \mathcal{D} \) by:

\[
\text{sol}(\mathcal{D}) = \{ \phi \mid \phi \models \mathcal{G} \}
\]

where \( \phi \) is an assignment to \( \mathcal{Y} \). Further we let the formulas \( \{ f[\text{match}(x_{I_j}, \alpha_j) \leftarrow y_j] \mid f \in \mathcal{F} \} \) be a part of \( \mathcal{G} \).

For the use of this section we will define \( y_j^i \) as the \( y \)-variable in \( \mathcal{Y} \) replacing the \( j \)th of the match-expressions on the variable \( x_i \), for \( 1 \leq i \leq n \) and \( 1 \leq j \leq k_i \) where \( k_i \) is the number of match-expressions on \( x_i \) that occurs in \( \mathcal{F} \). Using this notation we can restate \( \mathcal{Y} \) as

\[
\mathcal{Y} = \{ y_1^1, \ldots, y_{k_1}^1, y_1^2, \ldots, y_{k_2}^2, \ldots, y_1^n, \ldots, y_{k_n}^n \} \tag{7}
\]

Using the shortcuts \( y^i = (y^i_1, \ldots, y^i_{k_i}) \) and \( b^i = (b^i_1, \ldots, b^i_{k_i}) \) for every \( 1 \leq i \leq n \) where \( b^i_1, \ldots, b^i_{k_i} \in \mathcal{B} \) we define:

\[
\phi(y^i) = (y^i_1, \ldots, y^i_{k_i})
\]

and

\[
y^i = b^i := \bigwedge_{1 \leq j \leq k_i} y^i_j = b^i_j
\]

We further define the shortcut:

\[
y^i \in B^i \iff \bigvee_{b^i \in B^i} y^i = b^i
\]

where \( B^i \in \mathcal{B}^{k_i} \). We further denote the \( j \)th element in the acceptance value of a state \( q_i \) in the MDFA on \( x_i \) by \( a^i_j(q_i) \) and the entire acceptance value of \( q_i \) as
By this we know that when

We want to ensure that

where ρ is the assignment that induces φρ. The rightward implication of (8) can be satisfied by ensuring

by including it in G, which is quite similar to what we did in Section 4.

The leftward implication in (8) was ensured in Section 4 by the definition of AC and the fact that only the accepting states that were reachable from the source of MC were the states q ∈ AC where q = (q₁, . . . , qn) = (δ₁(w₁), . . . , δₘ(wₘₙ)) for some w₁, . . . , wn ∈ Σ* where δı denotes transitions in the match-DFAs. In this section we need to ensure the leftward implication by adding the constraint:

\[
y^i ∈ R^i(s_i) \text{ for all } 1 ≤ i ≤ n
\]  

(9)
to G. From this we get that if

\[
G = \{ f[\text{match}(x, α^i_j) ← y^j] \mid f ∈ F \} ∪ \bigwedge_{1≤i≤n} y^i ∈ R^i(s_i)
\]

then (8) holds.

We define the valid domains of y^i by

\[
V^0_{y^i} = \{ b^i ∈ B^{k_i} \mid ∃ φ ∈ sol(D) : φ(y^i) = b^i \}
\]

Note that this definition is different from the definition of V^0_{x_i}, but is quite similar to the standard definition of valid domains as e.g. in [TH06]. This version however, is specialized for valid domains on the empty assignment and is a projection of the valid solution onto a vector of variables from Y.

Recall the shortcut ρρ′ defined by ρρ′ = \{(x₁, ρ(x₁)p⁺(x₁)), . . . , (xn, ρ(xn)p⁺(xₙ))\} used in the definition of V^0_{x_i} in Definition 1. Using this shortcut and that

\[
ρ ∈ sol(C) ⇐⇒ φρ ∈ sol(D) \text{ where } φρ = \{(y¹, a¹(δ₁(ρ(x₁))))), . . . , (yn, aⁿ(δₙ(ρ(xₙ))))\}
\]

we get:

\[
V^0_{x_i} = \{ w ∈ Σ⁺ \mid ∃ p⁺ : ρp⁺ ∈ sol(C) ∧ p⁺(x_i) = w \}
\]

\[
= \{ w ∈ Σ⁺ \mid ∃ p⁺ : ρp⁺ ∈ sol(C) ∧ p⁺(x_i) = ρ(x_i)w \}
\]

\[
= \{ w ∈ Σ⁺ \mid ∃ p⁺ : φp⁺ ∈ sol(D) ∧ p⁺(x_i) = ρ(x_i)w \}
\]

\[
= \{ w ∈ Σ⁺ \mid ∃ p⁺ : φp⁺ ∈ sol(D) ∧ ρp⁺(y^i) = a¹(δ₁(ρ(x_i))w) \}
\]

\[
= \{ w ∈ Σ⁺ \mid ∃ φ ∈ sol(D) ∧ φ(y^i) = a¹(δ₁(ρ(x_i))w) \}
\]

\[
= \{ w ∈ Σ⁺ \mid a¹(δ₁(ρ(x_i))w) ∈ V^0_{y^i} \}
\]

By this we know that when ρ ∈ sol(C) ⇐⇒ φρ ∈ sol(D) is ensured we can
compute $V^{\rho}_x$ using only the MDFA $M_i$ and $V^{\rho}_y$. To obtain a DFA $(Q, \Sigma, \delta, s, A)$ deciding $V^{\rho}_x$ based on the MDFA $M_i = (Q_i, \delta^i, s_i, a^i)$ on the variable $x_i$ we can do the following:

- set $Q = Q_i$ and $\delta = \delta_i$
- set $A = \{a^i(q_i) | a^i(q_i) \in V^{\rho}_y\}$
- set $s = \delta_i(\rho(x_i))$

Note that this is very close to what was done in Section 4. The main difference is that instead of making states accepting/rejecting at the preprocessing we construct $A$ during the valid domain computation by using $V^{\rho}_x = \{w | a^i(\delta_i(\rho(x_i))w) \in V^{\rho}_y\}$. Further we have no need to change the alphabet which is needed in Section 4.

**Example 5.** Consider the CSP: $C = (X, \Sigma, F)$, where $X = \{x_1, x_2\}$, $F = \{f_1, f_2\}$, $f_1 = \text{match}(x_2, "abc") \lor \text{match}(x_1, "a")$, $f_2 = \text{match}(x_2, "abd * ")$ and $x_1 = x_2 = \epsilon$ (Assume that match-expressions are ordered in increasing order of their subscript). We define the CSP $D = (Y, \mathbb{P}, G)$. In $D$ we have $Y = \{y^1_1, y^1_1, y^2_2\}$, and disregarding the requirement (9) we have $G = \{g_1, g_2\}$ where $g_1 = y^1_1 \lor y^2_1$ and $g_2 = y^2_2$. We have the following facts:

$$
\begin{align*}
\text{sol}(D) &= \{(y^1_1, \text{false}), (y^2_1, \text{true}), (y^2_2, \text{true})\}, \\
&\quad \{(y^1_1, \text{true}), (y^2_1, \text{false}), (y^2_2, \text{true})\}, \\
&\quad \{(y^1_1, \text{true}), (y^2_1, \text{true}), (y^2_2, \text{true})\}\}
\end{align*}
$$

$$
\begin{align*}
R(s_1) &= \{(\text{true}), (\text{false})\} \\
R(s_2) &= \{(\text{false, true}), (\text{true, false})\}
\end{align*}
$$

We now impose the requirement (9), that is

$$
(y^1 \in R^1(s_1)) \cup (y^2 \in R^2(s_2))
$$

by adding it to $G$. This requirement has earlier been defined as:

$$
G \leftarrow G \cup \left( \bigvee_{b \in R(s_1)} y^1_1 = b_1 \right) \cup \left( \bigvee_{b \in R(s_2)} y^2_1 = b_1 \land y^2_2 = b_2 \right)
$$

which corresponds to the requirement:

$$
\phi(y^1_1) \in \{(\text{true}), (\text{false})\} \text{ and } \phi(y^2_1, y^2_2) \in \{(\text{false, true}), (\text{true, false})\}
$$

respectively for any $\phi \in \text{sol}(D)$. The latter constraint removes the assignments:

$$
\{(y^1_1, \text{false}), (y^2_1, \text{true}), (y^2_2, \text{true})\} \text{ and } \{(y^1_1, \text{true}), (y^2_1, \text{true}), (y^2_2, \text{true})\}
$$

from sol(D). All constraints implied by the MDFA$\text{s}$ are now contained in $D$. 

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We now have \( V_{x_i}^\emptyset = \{ w \in \Sigma^* \mid a^i(\hat{\delta}_1(w)) \in V_{y_i}^\emptyset \} \). From this we get
\[
\begin{align*}
V_{x_i}^\emptyset &= \{ w \in \Sigma^* \mid a(\hat{\delta}_1(w)) \in V_{y_i}^\emptyset \} \\
&= \{ w \in \Sigma^* \mid a(\hat{\delta}_1(w)) \in \{ \text{true} \} \} \\
&= L(\text{"a"})
\end{align*}
\]
and
\[
\begin{align*}
V_{x_i}^\emptyset &= \{ w \in \Sigma^* \mid a(\hat{\delta}_2(w)) \in V_{y_i}^\emptyset \} \\
&= \{ w \in \Sigma^* \mid a(\hat{\delta}_2(w)) \in \{ \text{false, true} \} \} \\
&= L(\text{"abd"})
\end{align*}
\]

8 The Algorithms

In this section we will present the three algorithms \textsc{Build}(\mathcal{C}), \textsc{Append}(x_i, w) and \textsc{ValidDomain}(x_i). The first algorithm \textsc{Build} constructs a data structure that is used by \textsc{Append} and \textsc{ValidDomain}. In all algorithms we assume that \( \mathcal{D}, M_1, \ldots, M_n, R_1, \ldots, R_n, a^1, \ldots, a^n, \Sigma \) and \( \rho \) are global variables. We assume that \( V_{y_i}^\emptyset \) is available. We further assume that initially \( \rho \leftarrow \{(x_1, \epsilon), \ldots, (x_n, \epsilon)\}, \mathcal{G}_2 \leftarrow \emptyset \) and \( k_1, \ldots, k_n = 0 \)

\textsc{Build}(\mathcal{C})
\[
\begin{align*}
1 & \mathcal{G}_1 \leftarrow \mathcal{F} \\
2 & \text{for } i \leftarrow 1 \text{ to } n \\
3 & \quad \text{do for } \text{each } j \text{th match expression on the variable } x_i \text{ occurring in } \mathcal{G}_1 \text{ as } \text{match}(x_i, \alpha^j_i) \\
4 & \quad \quad \text{do replace } \text{match}(x_i, \alpha^j_i) \text{ in } \mathcal{G}_1 \text{ by a new variable } y^i_j \\
5 & \quad \quad k_i \leftarrow k_i + 1 \\
6 & \quad \text{Build a DFA, } M_{i,j}' \text{ on } L(\alpha_{ij}) \\
7 & \quad y^i = (y^1_1, \ldots, y^i_{k_i}) \\
8 & \quad M_i \leftarrow \text{ConstructMDFA}(M_{i,1}', \ldots, M_{i,k_i}') \\
9 & \quad R^i \leftarrow \text{ComputeReachableAcceptanceStates}(M_i) \\
10 & \quad \mathcal{Y} = \{ y^1_1, \ldots, y^i_{k_i}, y^2_1, \ldots, y^2_{k_2}, \ldots, y^n_1, y^n_{k_n} \} \\
11 & \text{for } i \leftarrow 1 \text{ to } n \\
12 & \quad \text{do } \mathcal{G}_2 \leftarrow \mathcal{G}_2 \cup (y_i \in R(s_i)). \\
13 & \quad \mathcal{D} = (\mathcal{Y}, \mathcal{G}_1 \cup \mathcal{G}_2) \\
14 & \quad \text{if } V_{y_i}^\emptyset = \emptyset \\
15 & \quad \quad \text{then error } \text{"No feasible solutions"} \\
16 & \text{for } i \leftarrow 1 \text{ to } n \\
17 & \quad \text{do for each } q_i \in Q_i \\
18 & \quad \quad \text{do if } \{ a^i(q_i) \} \cap V_{y_i}^\emptyset = \emptyset \\
19 & \quad \quad \quad \text{then } a^i(q_i) = \emptyset \\
20 & \quad \quad R^i(q_i) \leftarrow R^i(q_i) \cap V_{y_i}^\emptyset \\
21 & \quad \text{Minimize } M_i
\end{align*}
\]

Line 1-10 constructs the first half of \( \mathcal{G} \) based on \( \mathcal{F} \). Line 11-12 constructs the second half of \( \mathcal{G} \) and Line 13 defines \( \mathcal{D} \). 14-15 check for feasible solution to \( \mathcal{C} \) the reason for using \( V_{y_i}^\emptyset \) instead of \( \text{sol}(\mathcal{D}) \) is that we have not required that \( \text{sol}(\mathcal{D}) \) is available to us. Line 16-21 tries to reduce the size of the data structure
by removing the acceptance values from $a$ and $R$ that cannot lead to a valid solution. Note that Line 18-19 might set $a(q) = \emptyset$, which is not valid according to the definition of an MDFA. However we use the value in the pseudocode to indicate that this acceptance value never can be part of a solution to $D$.

\textbf{ValidDomain}(x_i)
1. \quad \text{Let} \: A_i \leftarrow \emptyset
2. \quad \text{for each} \: q_i \in Q_i \: \text{do}
3. \quad \text{if} \: a_i(q_i) \in V^\emptyset
4. \quad \quad \text{then} \: A_i \leftarrow A_i \cup \{q_i\}
5. \quad \alpha \leftarrow \text{the regular expression corresponding to the DFA} \: (Q_i, \Sigma, \delta_i, s_i, A_i)
6. \quad \text{return} \: \alpha

This algorithm construct a DFA on the MDFA $M_i$ accepting $V^\rho_{y_i} = \{w \in \Sigma^* \mid a(\delta_i(w)) \in V^\emptyset_{y_i}\}$ and returns the regular expression corresponding to the constructed DFA. Of course we might consider other ways to indicate the valid domains than by returning a regular expression. This will be discussed in Section 10.

\textbf{Append}(x_i, w)
1. \quad \mathcal{G}' \leftarrow \mathcal{G} \cup (y^i \in R^i(\delta_i(s_i, w)))
2. \quad \text{if} \: \mathcal{G}' \models \bot
3. \quad \text{then} \: \text{error} \: \text{“invalid append”}
4. \quad \rho(x_i) \leftarrow \rho(x_i)w
5. \quad s_i \leftarrow \delta(s_i, w)
6. \quad \mathcal{G} \leftarrow \mathcal{G}'

We append the letter $w$ to $\rho(x_i)$, and add a constraint to $\mathcal{G}$ in order to remove the assignments on $Y$ that are no longer possible to attain by any $\rho$.

## 9 Implementation

In the algorithms we have supposed that we have a data structure on $D$ that supports two operations:

1. Adding constraints to $\mathcal{G}$.
2. Computing $V^\emptyset_i$ for every $1 \leq i \leq m$.

This could be done by filtering on $\mathcal{G}$ using one of the many filtering approaches (see e.g. [Dec03]). However in the setting of interactive configuration, were values are assigned one by one and valid domains and very fast valid domains computations has to be available, encoding the constraints by a BDD seems to be the obvious choice. We also choose to represent $R(q_i)$ as a BDD encoding of he constraint $y_i \in R^i(q_i)$. Hence setting $\mathcal{G} \leftarrow \mathcal{G} \cup (y_i \in R(q_i))$ can be done by setting $\text{BDD}(\mathcal{G}) \leftarrow \text{BDD}(\mathcal{G}) \wedge \text{BDD}(y_i \in R(q_i))$, where $\text{BDD}(\mathcal{H})$ is the BDD-representation of the conjunction of the set of boolean formulas in $\mathcal{H}$.

The algorithms used to minimize MDFAs in \textsc{Build} is a direct generalization of the one presented in [AHU74]. It runs in $|Q| \log |Q|$ when $Q$ are the states in the non-minimal MDFA.
The algorithm that transforms a DFA into a regular expression can be found in [HMU01]. It runs in $O(|\delta| \cdot |\alpha|)$ where $|\delta|$ is the number of transitions in the DFA and $|\alpha|$ is the number of characters in the resulting regular expression.

10 Extensions

10.1 Encompassing previous BDDs in the current context

Since $\mathcal{D}$ is encoded as a BDD we can easily provide support for boolean and integer variables allowing the same operations as usual in on-line configuration. For instance we would be able to accept constraints as $x_2 \neq 7 \land \text{match}(x_2, "7* ") \land \text{match}(x_3, "abc * ")$ on the variables $x_1, x_2, x_3$. Currently we cannot model equality of two string but it could easily be added.

One might also choose to encode the integer as a string in some cases. For instance a regular expression can be used to determine whether a integer of infinite length is a factor of 2 or a factor of 3.

10.2 k-shortest path

If we are to present the valid domain of a variable to the user, i.e. to help the user in completing a string, a regular expression might not be very intuitive – especially if the concept of regular expressions is unknown for the user. Hence one might consider other strategies.

One idea would be only to output the shortest text-completion. This can be done in $|Q| \log |Q| + |\delta|$ using Dijkstra's algorithm, where $|Q|$ and $|\delta|$ is the number of states and transitions in the MDFA respectively. We can also find the $k$ shortest paths in $O(|\delta| + |Q| \log |Q| + k)$ time [Epp94] and find the $k$ shortest simple paths in $O(k|Q||\delta| + |Q| \log |Q|))$ [Yen72].

If more than one acceptance value is valid one might consider to output the shortest path to each of the valid acceptance values one at a time.

10.3 Completing a string

We might want to support two kinds of updates:

- Appending a letter $w$ to a string $x_i \in \mathcal{X}$ as earlier described
- Completing a string $x_i \in \mathcal{X}$

To complete a variable $x_i$ is in some way to state that no more letters will be appended to $\rho(x_i)$. This could in the example of input field validation be stated by the user in hitting the return key or leaving a text field. We support this second update as the action of appending a special letter $\text{eol} \in \Sigma$ to $\rho(x_i)$, and disallowing appending letters to $\rho(x_i)$ if the last letter of $\rho(x_i)$ is $\text{eol}$.

10.4 Making savings by a simple heuristic

It might be considered to make a simple reduction. Rewritten expressions like $\text{match}(x, \alpha) \lor \text{match}(x, \beta)$ to $\text{match}(x, \alpha \cup \beta)$ and similarly $\text{match}(x, \alpha) \land \text{match}(x, \beta)$ to $\text{match}(x, \alpha \cap \beta)$. These rewritings may leads a large reduction in space as the DFA will not need to worry about 2 cases instead of 4.
10.5 Supporting initial domain of $\mathcal{X}$

In this paper we have assumed that the initial domain of any $x \in \mathcal{X}$ is $\Sigma^*$. In practice we might want to constrain the initial domain by a regular expression. For instance we might chose to constrain the zip code to only contain digits from the very start by adding $\text{match}(\text{zip}, "([0123456789]+")$ to $\mathcal{G}$ as an initial constraint.

11 Future Work

An obvious extension would be to explore whether it is possible to achieve the same functionality with languages that are more expressive than the regular languages. For instance we might investigate if we can handle context-free languages [HMU01].

Another thought that might be pursued is whether the input language used to declare the constraints of $\mathcal{F}$ is appropriate for declaring the constraints of $\mathcal{F}$. Formally it is perfect as every regular language can be expressed as a regular expression. However the length and complexity of these expressions may make it cumbersome to express even simple constraints. Consider for instance the constraint that $x$ is in the regular language of natural numbers divisible by 3. This regular language can be modeled by a DFA with 3 states and nine transitions. In our current input-language this will have to be expressed as $f = \text{match}(x, "([[0369]+*]]147258][0369]*258))0369*(147][0369]258))258])0369*(147][258][0369]*258))258])0369*(147][258][0369]*258))0369*"$). This suggest that we might consider some other ways to model the DFA constraints than the $\text{match}$-expression. The ad hoc solution to the problem stated above could be to allow expressions in the input-language on the form “$x$ modulo $y = z$” where $x, y, z \in \mathbb{Z}$. But we can easily construct similar problems that will cause other problems. Hence a challenge is to consider how the input language can be made in a way so that it is easy to express problem the numerous problems that have nice DFAs but are horrible to express as regular-expressions.

Another problem is how to make the user who in most cases will have little or no acquaintance with regular expression make constraints that can be enforced by the data structure.

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References

[AHU74] Alfred V. Aho, John E. Hopcroft, and Jeffrey D. Ullman. The design and analysis of computer algorithms. Addison-Wesley Series in Computer Science and Information Processing. Addison-Wesley, 1974.

[CLRS01] Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest, and Clifford Stein. Introduction to Algorithms. McGraw-Hill Higher Education, 2001.

[Dec03] Rina Dechter. Constraint Processing. Morgan Kaufmann Publishers, 2003.

[Epp94] David Eppstein. Finding the k shortest paths. In IEEE Symposium on Foundations of Computer Science, pages 154–165, 1994.

[HMU01] John E Hopcroft, Rajeev Motwani, and Jeffrey D Ullman. Introduction to Automata Theory, Languages and Computation. Low Price Edition. Addison Wesley Longman, Inc, Reading, Mass., USA, 2 edition, 2001.

[HSIJ+04] Tarik Hadzic, Sathiamoorthy Subbarayan, Rune Møller Jensen, Henrik Reif Andersen, Henrik Hulgaard, and Jesper Møller. Fast backtrack-free product configuration using a precompiled solution space representation. In Proceedings of the International Conference on Economic, Technical and Organizational aspects of Product Configuration Systems, pages 131–138. DTU-tryk, 2004.

[Pes04] Gilles Pesant. A regular language membership constraint for finite sequences of variables. In Proceedings of the Tenth International Conference on Principles and Practice of Constraint Programming (CP 2004), pages 482–495. Springer, 2004.

[QW06] Claude-Guy Quimper and Toby Walsh. Global grammar constraints. In Proceedings of the 12th International Conference on Principles and Practice of Constraint Programming (CP 2006), pages 751–755. Springer, 2006.

[Ras00] J. Raskin. The Humane Interface. Addison Wesley, 2000.

[Sel06] Meinolf Sellmann. The theory of grammar constraints. In Proceedings of the 12th International Conference on Principles and Practice of Constraint Programming (CP 2006), pages 530–544. Springer, 2006.

[SLJH+04] Sathiamoorthy Subbarayan, Rune Møller Jensen, Tarik Hadzic, Henrik Reif Andersen, Henrik Hulgaard, and Jesper Møller. Comparing two implementations of a complete and backtrack-free interactive configurator. In Proceedings of the CP-04 Workshop on CSP Techniques with Immediate Application, pages 97–111, 2004.

[TH06] Henrik Reif Andersen Tarik Hadzic, Rune Møller Jensen. Calculating valid domains for bdd-based interactive configuration. 2006. Available in 2006 at: http://www.itu.dk/people/tarik/cvd/cvd.pdf.
[Yen72] J. Y. Yen. Another algorithm for finding the $k$ shortest loopless network paths. Volume 10. Proc. of 41st Mtg. Operations Research Society of America, 1972.
Figure 7: A valid domains DFA built on the formulas $f_1 = \text{match}(x_2, "abc") \lor \text{match}(x_1, "a"), f_2 = \text{match}(x_2, "abds")$. Transitions 1:* an 2:* means transitions on all other letter that cannot follow any transition on the first or second variable respectively. Dashed states are states where no accepting state is reachable. If the DFA is minimized they will all be contracted to the same state.