Re(A₀), Re(A₂) and RG evolution for N_f = 3*

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We present results of Re(A₀) and Re(A₂) calculated using HYP staggered fermions on the lattice of 16³ x 64 at β = 6.0. These results are obtained using leading order chiral perturbation in quenched QCD. Buras’s original RG evolution matrix develops a removable singularity for N_f = 3. This subtlety is resolved by finding a finite solution to RG equation and the results are presented.

In the standard model, we can express Re(A_I) in terms of the matrix elements of the effective weak Hamiltonian between hadronic states as follows:

\[
Re(A_I) = \frac{G_F}{\sqrt{2}} V_{ud} V_{us} \left[ \sum_{i=1,2} z_i(\mu) \langle Q_i(\mu) \rangle_I \right. + Re(\tau) \sum_{i=3}^{10} y_i(\mu) \langle Q_i(\mu) \rangle_I \right]
\]

(1)

where V_{ij} is an element of CKM matrix. Here, z_i(\mu) and y_i(\mu) are the Wilson coefficients which are obtained analytically. The hadronic matrix elements \langle Q_i(\mu) \rangle_I are defined as

\[
\langle Q_i(\mu) \rangle_I = \langle \pi\pi | Q_i(\mu) | K \rangle
\]

(2)

Since these matrix elements are in a highly non-perturbative regime of QCD, it is necessary to calculate them using non-perturbative tools such as a numerical method based on lattice gauge theory. In fact, we calculate \langle \pi | Q_i | K \rangle and \langle 0 | Q_i | K \rangle on the lattice using HYP staggered fermions and we construct \langle \pi\pi | Q_i | K \rangle out of them using chiral perturbation in its leading order [1]. The Re(\tau) in Eq. (1) is defined as

\[
Re(\tau) = -Re(\lambda_2/\lambda_u) = 0.002
\]

\[
\lambda_f = V_{f3} V_{f4}
\]

In Eq. (1), Q_i (i = 3, 4, \cdots, 10) represents the QCD and electroweak penguin operators. Since z_i, y_i \approx 1, the contribution from the penguin operators to Re(A_I) is suppressed by Re(\tau). Therefore, in practice, only the current-current operators \langle Q_i \rangle (i = 1, 2) can contribute dominantly to Re(A_I) whereas the contribution from penguin operators is negligible. In the case of the direct CP violation, only penguin operators can contribute to \epsilon'/\epsilon [2] and current-current operators do not play any role in it. In this regards, we can say that \epsilon'/\epsilon and Re(A_I) are completely different physics.

In Fig. 1, we show individual channels contributing to Re(A₀) with the total sum in the last column. The dominant contribution comes from \langle Q_i \rangle (i = 1, 2) as expected. Each channel is obtained by fitting the data to a function: \( f_1(m_K) = c_0 + c_1 m_K^2 \) and extrapolating to the chiral limit. We use the matching formula given in [3,4] to convert lattice results into continuum values in the NDR scheme at the scale of \mu = 1/a. By assuming that these values are close enough to those for N_f = 3, we run them from \mu = 1/a down to \mu = m_e, using the RG evolution equation for N_f = 3.

There are two alternative methods to transcribe the Q_6 and Q_5 operators to the lattice: the standard (STD) method and the Golterman & Pallante (GP) method [5]. In the case of \epsilon'/\epsilon, the dominant contribution to the \Delta I = 1/2 amplitude comes from \langle Q_6 \rangle and so there is a big difference in \epsilon'/\epsilon between the STD and GP methods [2]. In the case of Re(A₀), however, the contribution from \langle Q_6 \rangle is extremely suppressed by Re(\tau) and so both methods do not make much difference to Re(A₀).
The Hamiltonian can be written as follows:

$$\mathcal{H} = \frac{G_F}{\sqrt{2}} \sum_{i=1}^{10} C_i(\mu)Q_i(\mu)$$  \hspace{2cm} (3)$$

In Fig. 1 we show the same kind of plot as in Fig. 1 except for the scale at which the Wilson coefficient and the matrix elements are combined. We set the scale to $\mu = 1/\alpha$ in Fig. 2 and to $\mu = m_c$ (charm quark mass) in Fig. 1. The contribution from each individual channel fluctuates depending on this matching scale but the total sum should be invariant.

In Fig. 1, we show the same kind of plot as in Fig. 2 except for the fitting function. Each channel is obtained by fitting the data to a function: $f_2(m_K) = c_0 + c_1 m_K^2 + c_2 m_K^4$ and extrapolating it to the chiral limit. Comparing Fig. 1 and 3, we observe that the dependence of the chiral extrapolation on the fitting function is large. These results are consistent with the experimental value of $Re(A_0) = 33.3 \times 10^{-8}$ GeV within the systematic and statistical uncertainty.

In Fig. 3, we show the same kind of plot as in Fig. 3 except for the fitting function. Each channel is obtained by fitting the data to a linear function $f_1(m_K)$ and extrapolating it to the chiral limit. The result of Fig. 3 is slightly lower than the experimental value of $Re(A_2) = 1.5 \times 10^{-8}$ GeV.

Now, let us address a subtle issue on the RG (renormalization group) equation for the Wilson coefficient $C_i$ is

$$\left[ \mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} \right] \tilde{C} = \gamma^T(g, \alpha) \tilde{C}$$  \hspace{2cm} (4)$$

where $\beta(g)$ is the QCD beta function:

$$\beta(g) = -\beta_0 \frac{g^3}{16\pi^2} - \beta_1 \frac{g^5}{(16\pi^2)^2} - \beta_2 \frac{g^7}{(16\pi^2)^3} - \cdots$$  \hspace{2cm} (5)$$

and $\gamma(g, \alpha)$ is the anomalous dimension matrix:

$$\gamma(g, \alpha) = \gamma_s(g^2) + \frac{\alpha}{4\pi} \Gamma(g^2) + \cdots$$  \hspace{2cm} (6)$$

$$\gamma_s(g^2) = \gamma_s^{(0)} + \frac{\alpha}{4\pi} \gamma_s^{(1)} + \frac{\alpha^2}{(4\pi)^2} + \cdots$$  \hspace{2cm} (7)$$

A solution to the RG equation can be expressed in terms of the evolution matrix.

$$\tilde{C}(\mu) = U(\mu, \mu', \alpha) \tilde{C}(\mu')$$  \hspace{2cm} (8)$$

Here the RG evolution matrix is, in general,

$$U(m_1, m_2, \alpha) = T_g \exp \left( \int_{g(m_2)}^{g(m_1)} \frac{d\gamma}{\gamma} \frac{T(g', \alpha)}{\beta(g')} \right)$$  \hspace{2cm} (9)$$

In the perturbative expansion, we can express the RG evolution matrix as follows:

$$U(m_1, m_2, \alpha) = U(m_1, m_2) + \frac{\alpha}{4\pi} R(m_1, m_2)$$  \hspace{2cm} (10)$$

where $U(m_1, m_2)$ represents the pure QCD evolution and $R(m_1, m_2)$ describes the additional evolution in the presence of the electromagnetic interaction. The pure QCD evolution matrix which
Buras, et al. provided originally \[57\], is

\[
U(m_1, m_2) = \left(1 + \frac{\alpha_s(m_1)}{4\pi} J \right) U^{(0)}(m_1, m_2)
\cdot \left(1 - \frac{\alpha_s(m_2)}{4\pi} J \right)
\]

(11)

where \( U^{(0)}(m_1, m_2) \) denotes the evolution in the leading logarithmic approximation and \( J \) corresponds to the next-to-leading correction.

The \( J \) matrix can be expressed as follows:

\[
J = V S V^{-1}, \quad G = V^{-1} \Gamma_{s(1)} V
\]

(12)

\[
S_{ij} = \delta_{ij} \gamma_i^{(0)} - \frac{\beta_i^2}{2} \frac{G_{ij}}{2\beta_0 + \gamma_i^{(0)} - \gamma_j^{(0)}}
\]

(13)

where \( V \) is a matrix which diagonalize the \( \gamma_s^{(0)T} \) matrix and \( \gamma_i^{(0)} \) represents corresponding diagonal elements. Here, note that the \( J \) or \( S \) matrix is singular if \( 2\beta_0 + \gamma_i^{(0)} - \gamma_j^{(0)} = 0 \). In fact, for \( N_f = 3, 2\beta_0 + \gamma_i^{(0)} - \gamma_j^{(0)} = 0 \) when \( i = 8, j = 7 \). However, since \( U(m_1, m_2) \) is not singular, this singularity must be removable at the next-to-leading order. After we make a correct combination at the next leading order and remove the singularity, we can make the \( U(m_1, m_2) \) finite.

\[
U(m_1, m_2) = U_0(m_1, m_2) + \frac{1}{4\pi} V A(m_1, m_2) V^{-1}
\]

(14)

where, if \( a_j \neq a_i + 1 \) (note that \( a_i = \gamma_i^{(0)}/(2\beta_0) \)),

\[
A(m_1, m_2) = \delta_{ij} \left[ \alpha_s(m_1) \left(\frac{\alpha_s(m_2)}{\alpha_s(m_1)} \right)^{a_j} \right]
\]

(15)

and if \( a_j = a_i + 1 \),

\[
A(m_1, m_2) = \frac{G_{ij}}{2\beta_0} \alpha_s(m_2) \left(\frac{\alpha_s(m_2)}{\alpha_s(m_1)} \right)^{a_j} \cdot \ln \left(\frac{\alpha_s(m_2)}{\alpha_s(m_1)} \right)
\]

(16)

Here, note that the \( A \) matrix is finite while \( J \) matrix is singular.

In the case of the \( R(m_1, m_2) \), the same kind of removable singularity makes the RG evolution much more complicated, which we will present in Ref. [8].

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