Bell test in a classical pilot-wave system

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Since its discovery in 2005, the hydrodynamic pilot-wave system has provided a concrete macroscopic realization of wave-particle duality and concomitant classical analogs of many quantum effects. The question naturally arises as to whether this hydrodynamic pilot-wave system might provide a platform for violating Bell’s inequality, and so yield a classical analog of quantum entanglement. We here present the results of a static Bell test performed with a numerical model of the hydrodynamic pilot-wave system, specifically a coupled bipartite tunneling system. We demonstrate that, under certain conditions, the Bell inequality is violated owing to the wave-mediated coupling between the two subsystems. Our system represents a new platform for exploring whether Bell’s Theorem, typically taken to be a no-go theorem for all local hidden variable theories, need be respected by the class of hidden variable theories based on non-Markovian pilot-wave dynamics.

In 2005, Yves Couder and Emmanuel Fort [1, 2] discovered that a millimetric droplet may self-propel along the surface of a vibrating fluid bath through a resonant interaction with its own wave field. The resulting ‘walker’ consists of a droplet dressed in a quasi-monochromatic wave field, and represents a concrete, macroscopic example of wave-particle duality [3]. Remarkably, this hydrodynamic pilot-wave system exhibits many features previously thought to be exclusive to the microscopic, quantum realm [4, 5]. Notable examples include single-particle diffraction and interference [2, 6, 7], quantized orbits [3, 8], unpredictable tunneling [9], Friedel oscillations [10], spin lattices [11], and quantum-like statistics and statistical projection effects in corrals [12, 13]. In all instances, the emergent quantum behavior may be rationalized in terms of the droplet’s non-Markovian pilot-wave dynamics [5]. Specifically, the instantaneous wave force imparted to the drop during impact depends on the droplet’s history. Thus, the drop navigates a potential landscape of its own making [5], and the hydrodynamic pilot-wave system is said to be endowed with ‘memory’ [14].

In several settings, features emerge that might be misinterpreted as being non-local if the influence of the pilot-wave field were not duly considered [5]. For example, long-range lift forces are generated when a walking droplet interacts with a submerged pillar [15] or well [16], and could be misconstrued as indicating action at a distance if the influence of the pilot-wave field were not adequately resolved. Long-range correlations between distant walkers may be established through the influence of the intervening wave field [10, 17]. Recently, Papatryfonos et al. [18] established a walking-droplet analog of superradiance, an effect originally attributed to quantum interference of two or more entangled atoms [19–21], but subsequently rationalised in terms of classical electromagnetic wave interference [22]. The totality of the quantum-like features evident in the hydrodynamic pilot-wave system naturally raises the question as to whether this system might provide a platform for demonstrating a classical analog of entanglement [23, 24]. Specifically, might the walking-droplet system permit violations of Bell’s inequality?

Bell’s Theorem was derived by John Bell in 1964 [25] with a view to informing the Bohr-Einstein debate concerning the completeness of quantum theory [26]. Hidden
variables are those variables that would be required for a complete description of quantum dynamics, including the position and momentum of a microscopic particle. Bell tests inform what class of hidden variable theories are viable candidates for a causally complete quantum theory. A Bell test can be performed on any probabilistic system consisting of two subsystems (A and B) on which one measures a dichotomic property X (with stochastic outcomes of +1 or -1) that depends on some ‘analyzer setting’ (α or β). The measurement $X_A$ made at the left measurement system depends on the analyzer setting α which may take values α or $\alpha'$; likewise, the measurement ($X_B$) made at the right measurement system depends on β which may take values $\beta$ or $\beta'$. In the derivation of the Bell inequality (Eq. 1), it is assumed that the two subsystems undergo only local interactions; specifically, $X_A$ depends on α and not $\beta$; likewise, $X_B$ depends on $\beta$ and not α. This assumption is referred to as ‘Bell locality’. Another assumption made is that the hidden variables that prescribe X are independent of α and $\beta$, a condition referred to as ‘measurement independence’ or equivalently ‘freedom-of-choice’. Bell’s theorem implies that for any classical system for which Bell locality and freedom-of-choice hold, the quantity $S(\alpha = a, \beta = b, \alpha = a', \beta = b') = M(a, b) + M(a', b) - M(a, b') - M(a', b')$ must satisfy the inequality

$$|S(a, b, a', b')| \leq 2$$  \(1\)  

for any choice of measurement settings (a, a', b, b'). Here, $M(\alpha, \beta)$ is the average product, $M(\alpha, \beta) = \sum_{X_A, X_B} X_A X_B P(X_A, X_B|\alpha, \beta)$, where $P(X_A, X_B|\alpha, \beta)$ is the joint probability of measurements ($X_A, X_B$) when the left and right analyzers are set to (α, β). We note that Eq. 1 is cast in the form of the CHSH inequality.

It has been well established that bipartite quantum systems can violate inequality for a judicious choice of (a, a', b, b'), with a maximum value $S = 2\sqrt{2}$ corresponding to the Tsirelson bound. Bell tests were first performed with static analyzer settings, so could not strictly rule out signalling between the two subsystems. Specifically, the left measurement $X_A$ could in principle be influenced by the right analyzer setting $\beta$ (or, similarly, $X_B$ by $\alpha$) through long-range interactions between the two subsystems. Such static Bell tests have also been performed with pairs of massive entangled particles. Technological advances in quantum optics and electronics have enabled this ‘locality loophole’ to be closed via dynamic Bell tests, in which the detector settings α and $\beta$ are altered just prior to measurement, so that the two measurement events are space-like separated. Other loopholes, including the ‘detection loophole’ (that posits that the detection efficiency depends on (α, $\beta$)) and the ‘freedom-of-choice’ loophole, have been addressed through a series of Bell tests of increasing sophistication and precision. Nevertheless, there remains a debate as to whether the freedom-of-choice loophole can be closed in systems with a background medium, wherein the hidden field variables may be influenced by the analyzer settings. With a view to informing this debate, we here devise and execute a static Bell test on the pilot-wave hydrodynamic system, as a first step towards a dynamic test.

We consider a pair of walking droplets in the bipartite tunneling system introduced by Papatryfonos et al. in their demonstration of a hydrodynamic analog of quantum superradiance (See Figure 1a). The two subsystems, labelled A and B, correspond to single, wave-generating particles confined to a pair of identical cavities separated by a barrier across which the particles may tunnel. Each particle generates waves and moves in response to them.
according to the mathematical model of walking droplets detailed in the Methods section. In each subsystem, the preferred cavity corresponds to the ground state $|\rightarrow \rangle$ and the other to the excited state $|+ \rangle$. The ground state may be either the inner or outer cavity, depending on the length of the outer cavities [15]. For the specific geometry considered here, the inner cavity of each subsystem is the ground state $|\rightarrow \rangle$; the outer cavity the excited state, $|+ \rangle$.

The two subsystems are separated by a coupling cavity of variable length $L_c$, and by barriers that are sufficiently high as to preclude the particles from tunneling into the coupling cavity. Waves are transmitted across the central cavity, and so provide the coupling between subsystems $A$ and $B$. The efficiency of this coupling is prescribed by the geometry of the central cavity: by increasing its depth $d_c$, the coupling may be increased, allowing the coupling cavity to serve as a nearly resonant transmission line [14]. Transitions between ground and excited states in the subsystems correspond to individual tunneling events, the rate of which depends on the depths of the left and right barriers, and the length of the coupling cavity, $L_c$. As illustrated in Fig. 1a and 1c, we identify the state $\langle + | \text{or} |\rightarrow \rangle$ of each subsystem with the dichotomous property $X$ in the optical Bell test, and the depths of the left and right barriers with the analyzer settings (respectively, $\alpha$ and $\beta$). The four possible combinations of $(X_A, X_B)$ are shown in Fig. 1c. We proceed by employing the numerical simulation method developed by Nachbin [16, 41] to identify conditions under which Bell’s inequality is violated.

To collect statistics in a manner comparable to the optical Bell tests with entangled photons, we proceed as follows. Each run begins by placing the two particles at random positions within their own subsystem. Their trajectories are then calculated for 2000 Faraday periods. At that time, measurements are made in each subsystem. Specifically, we note the cavity in which the particle is located, and assign $X$ the value of $+1$ if the particle is in the outer cavity or $-1$ if it is in the inner cavity. The methodology for collecting the data is detailed in the Methods section. As in our recent study of hydrodynamic superradiance [15], the barrier depths $(\alpha, \beta)$ prescribe the local tunneling probability, and may also influence that of the distant partner drop. We proceed by demonstrating that for judicious choice of pairs of measurement settings $(\alpha, \beta)$, Bell’s inequality may be violated. Our main result is shown in Fig. 2, indicating a narrow parameter range in which the CHSH inequality is violated. In the remainder of this section we will analyze in detail how this violation comes about.

While the Bell inequality can be violated in our system with four different values of the measurement settings $(a, a', b, b')$, our exploration of the $(a, a', b, b')$ parameter-space indicates that a local maximum of $S$ arises when $b = a$ and $b' = a'$, the symmetric case in which one may write $S(a, b, a', b') = S(a, a, a', a') = M(a, a) - M(a', a') + 2M(a, a')$. We deduced a maximum violation of $S_{\max} = 2.49 \pm 0.04$ when $a = b = a' = 0.099$ cm and $a' = b' = a^* = 0.1033$ cm. In Fig. 2a, we plot $S$ as a function of $a'$ for fixed $a = a^*$, with the dashed line showing the limit $S = 2$ above which the CHSH inequality (Eq. 1) is violated. While the inequality is violated only for a narrow range of parameters settings, in this parameter regime, the violation is clear, and the statistical confidence of the violation is above 20 standard deviations. This behavior is reminiscent of the quantum case, where, without guidance from the theory, it is relatively difficult to find analyzer settings that allow for violation of the CHSH inequality, but for these particular settings, the inequality is violated substantially. Figure 2b and 2c shows a typical example of the convergence of the ‘running average’ with the number of runs which determines the relative error of our statistics. This approach indicates when our statistics have converged for each $M(\alpha, \beta)$ calculation, specifically when the relative error has fallen below the prescribed tolerance.

We proceed by detailing the manner in which Bell’s inequality is violated in this classical pilot-wave system. The maximum $S$ value occurs for moderate barrier depths, for which the droplets may become strongly correlated through the background wave field. In Fig. 3a, we show typical trajectories for the three combinations of measurement settings $(\alpha, \beta) \in \{(a^*, a^*), (a^*, a^*), (a'^*, a'^*)\}$ that maximize $S$. For $(a, a') = (a^*, a^*)$, $S$ is maximized because $M(a^*, a^*) - M(a'^*, a'^*)$ is large (see Fig. 3a top and lower panels), while $M(a^*, a^*)$ is relatively small (Fig. 3a middle panel). Fig. 3b corresponds to a shallow barrier, $a' = 0.0937$ cm (the left-most value in Fig. 2a) and Fig. 3c to a relatively deep barrier, $a' = 0.11$ cm (the right-most value in Fig. 2a). Figures 3b and c correspond to minima of $S$ occurring when the $a'$ barrier is either too shallow (Fig. 3b) or too deep (Fig. 3c).

The degree of synchronization in the droplet tunneling depends on the extent to which the droplets are affected by the barrier depth in the distant station. When the barrier depth in one station is too small, the local particle is prevented from tunneling, regardless of the barrier depth in the other. The synchronization of states is thus reduced substantially. Conversely, when the barrier depth is too large, the particle generally tunnels across it, unaffected by the distant particle. Thus, the synchronization again remains relatively low. For intermediate barrier depths, each particle tunnels with a moderate probability that is strongly affected by the behavior of its distant partner. (We note that when only one drop is present, the variation of the measurement setting in the other subsystem does not influence its behavior. See Supplementary Information, Figure S1.) As in our previous study of superradiance [15], the wave-mediated correlation creates a collective behavior of the droplet pairs. In particular, when one of the droplets tunnels to its excited state, the probability of the second droplet doing likewise.
in the establishment of statistical indistinguishability of partite hydrodynamic pilot-wave systems, most notably dynamics and other stochastic bipartite classical systems. Experimental execution of Bell tests in pilot-wave hydrodynamical systems may serve as a proxy for analyzer settings. In our system, the barrier depths play the role of polarizer angles in the photonic Bell tests. We expect this concept to be used for performing dynamic Bell tests or for building analogs to qubits and quantum computing and therefore, such classical states cannot be used for performing dynamic Bell tests or for building analogs to qubits and quantum computing. Conversely, our bipartite system involves spatially separated subsystems, and so introduces the possibility of exploring non-classical computing, and performing more sophisticated Bell tests.

Non-separable states arise in multi- and bipartite systems when the state of the whole cannot be simply defined in terms of the state of its subsystems. A canonical example is the singlet state of entangled photons. In our system, non-separability manifests itself through the fact that the joint probability of the two dichotomic states $X_A$ and $X_B$, specifically $P(X_A, X_B | \alpha, \beta)$, is not equal to the product $P(X_A | \alpha)P(X_B | \beta)$. We have here demonstrated how wave-mediated coupling may lead to non-separable states with a bipartite classical system consisting of two distant particles. While classical non-separable states have been demonstrated in optical settings, these generally involved the non-separable superpositions of internal degrees of freedom of a single particle or of classical fields. Such degrees of freedom cannot be spatially separated and therefore, such classical states cannot be used for performing dynamic Bell tests or for building analogs to qubits and quantum computing. Conversely, our bipartite system involves spatially separated subsystems, and so introduces the possibility of exploring novel forms of non-classical computing, and performing more sophisticated Bell tests.

In conclusion, we have devised a platform for performing static Bell tests on a classical bipartite pilot-wave system. The maximum violation was found to be $2.49 \pm 0.04$, and arose when the system geometries were chosen such that the droplet motion was marked by strongly synchronized tunneling for one measurement setting combination, moderate and weak synchronization for the others. A key step in the process was recognizing that the system geometry may serve as a proxy for analyzer settings. In our system, the barrier depths play the role of polarizer angles in the photonic Bell tests. We expect this conceptual advance to prompt and facilitate the numerical and experimental execution of Bell tests in pilot-wave hydrodynamics and other stochastic bipartite classical systems.

Long-range correlations have been reported in other bipartite hydrodynamic pilot-wave systems, most notably in the establishment of statistical indistinguishability of

![Figure 2. Violation of Bell’s inequality.](image)

**Figure 2.** Violation of Bell’s inequality. 

- **a** Bell parameter $S(a, a', a')$ as a function of the barrier depth, $a'$, for the symmetric case of $a = b, a' = b'$. For the calculation of the corresponding correlation functions, $M(a, a)$, $M(a, a')$ and $M(a', a')$, the barrier depth $a = a^* = 0.099$ cm remains fixed. For each combination of measurement settings, runs continue until statistics converge. The maximum Bell violation appears at $a^* = 0.1033$ cm, where $S = 2.49 \pm 0.04$. 
- **b** Typical curve showing the convergence of $S_{\text{max}}$, the Bell parameter taken at the maximum point of violation ($a = b = a^* = 0.099$ cm; $a' = b' = a^{**} = 0.1033$ cm). The error bars indicate ±3 standard deviations. 
- **c** Relative error $\delta S_{\text{max}} / S_{\text{max}}$ of the estimation of the Bell parameter $S$ with the number of runs, evaluated for the maximum point of violation ($a = b = a^*; a' = b' = a^{**}$). Inset: log-log scale; the dashed line indicates a $-1/2$ slope as expected from the convergence of an ensemble average.
Figure 3. Trajectory analysis. a-c Droplet trajectories for the symmetric case \((\alpha, \beta) \in (a, a')\) with \(a = a^*\). Time evolves in the vertical direction. In (a), \(a' = a^{**} = 0.1033\) cm (the maximizing value for \(S\)); in (b), \(a' = 0.0937\) cm; in (c), \(a' = 0.11\) cm. a Trajectories corresponding to the three correlation functions \(M(a^*, a^*) = 0.94 \pm 0.01\) (upper panel), \(M(a^*, a'^*) = 0.13 \pm 0.01\) (middle panel), and \(M(a'^*, a'^*) = 0.84 \pm 0.01\) (lower panel). The tunneling events are highly correlated only in the upper and lower panels. b Trajectories corresponding to \(M(a^*, a^*)\) with \(a' = 0.0937\) cm. When the barrier depth \(a'\) is sufficiently small, the wave-mediated communication between droplets is diminished, and droplets tend to get trapped in one cavity, leading to minima of \(S\) and \(M(a^*, a') \approx 0\) when we average over the droplet’s initial conditions. c Another minimum of \(M(a^*, a')\) and \(S\) occurs when one of the barrier depths is too large, in which case one of the droplets tunnels continuously, unimpeded by the barrier, as if it were in a single cavity. Averaging over all initial conditions leads to a relatively low value of \(M(a^*, a') \approx 0\). Note that the correlation function is deduced by averaging over all initial conditions in both subsystems.

a pair of distant droplets \([16]\) and in the recent analog of superradiance \([18]\). In these examples, if observers were unaware of the pilot wave field and observed only the droplets, they could only account for the observed correlations by inferring a nonlocal connection between the droplet pairs. The violation of Bell’s Inequality (Eq.1) in our static Bell test may be simply rationalized in terms of the wave-mediated coupling between the two subsystems. An assumption made in the derivation of Bell’s inequality that is not satisfied by our classical, hydrodynamic system is that of ‘Bell locality’. Specifically, the long-range influence and persistence of the pilot wave ensures that both droplets are influenced by the entire domain geometry, specifically both measurement settings.

The Bell violations reported in our static test represent a prerequisite for violations in a dynamic test. Extensions of our model system will allow for the investigation of various loopholes with more sophisticated Bell tests. For example, in order to close the ‘locality loophole’, we are currently extending our model so as to incorporate dynamic topography, thus enabling us to change \((\alpha, \beta)\) during the course of the numerical runs \([17]\). We note that, in our system, the speed of communication between the two subsystems is bounded by the fastest capillary wave speed (approximately 20 cm/s) \([14]\) rather than the speed of light; moreover, one can unequivocally eliminate communication between the two systems by the imposition of a wall between them. We shall thus explore the possibility that the violations reported here in our static Bell test might survive the isolation of the two subsystems, persist through the influence of the system memory.

**METHODS**

**Numerical method**

System parameters are chosen to correspond to a fluid bath of density 0.95 g/cm\(^3\), viscosity 16 cS and surface tension 20.9 dynes/cm vibrating vertically with amplitude \(A_0\) and frequency \(\omega_0 = 80\)Hz. The resonant bouncing of the particle at the Faraday frequency triggers a quasi-monochromatic damped wave pattern with a corresponding Faraday wavelength of \(\lambda_F = 4.75\) mm. Each of the four cavities has a fixed length of 1.2 cm, corresponding to approximately 2.5\(\lambda_F\). In all simulations, we set the coupling cavity depth to \(d_c = 6.3\lambda_F\) which ensures strong inter-cavity coupling. We thus describe our bipartite tunneling system in terms of two coupled, two-level systems, as shown schematically in Figure 1.

Nachbin et al. \([16, 41]\) formulated a theoretical model for the one-dimensional motion of walking droplets over a vibrating liquid bath with complex topography. Here we adjust this model in order to consider the cooperative tunneling of two identical particles in the geometry depicted in Figure 1. The positions, \(x_j\) \((j = 1, 2)\), of the two identical particles of mass \(m\) evolve according to Newton’s Law:

\[
\dot{x}_j + cF(t)\dot{x}_j = -F(t)\frac{\partial \eta}{\partial x}(x_j(t), t).
\]

The particle moves in response to gradients of the wave elevation \(\eta(x, t)\), which thus plays the role of a time-dependent potential. The particle motion is resisted by a drag force proportional to its speed. The drag constant \(c\)
follows from the modeling presented in Molacek & Bush [48]. The time dependence of these propulsive and drag forces is prescribed by $F(t)$, as arises in the walker system owing to the droplet’s bouncing [4 11 49]. In terms of their lateral motion, the particles are viewed as horizontal oscillators that can transition unpredictably between two neighboring cavities. The dichotomic property $X$ assessed for Bell’s inequality is assigned according to the particle location $x_j = x_j(t_m)$ ($j = A, B$) at the measurement time $t_m$. Specifically, $X = +1$ if the drop is in the outer, excited state, and $X = -1$ if it is in the inner, ground state.

The particles serve as moving wave sources that establish their own time-dependent wave potential that is computed as follows. The velocity potential of the liquid bath’s reference frame, where the effective gravity is $\eta$, the wave field thus evolves according to the following:

$$\frac{\partial \eta}{\partial t} = \frac{\partial \phi}{\partial z} + 2\nu \frac{\partial^2 \eta}{\partial x^2},$$

where $\phi$ is the wave function.

$$\frac{\partial \phi}{\partial t} = -g(t)\eta + \frac{\sigma}{\rho} \frac{\partial^2 \eta}{\partial x^2} + 2\nu \frac{\partial^2 \phi}{\partial x^2} + \sum_{j=1,2} \frac{P_d(x-x_j(t))}{\rho}.$$  

The particles ($j = A, B$) generate waves on the free surface by applying local pressure terms $P_d$. The wave forcing term $P_d(x-x_j(t))$ and the coefficient $F(t)$ are activated only during a fraction of the Faraday period $T_F$, corresponding to the contact time $T_c$ in the walking-droplet system and approximated by $T_c = T_F/4$. The particle is assumed to be in resonance with the most unstable (subharmonic) Faraday mode of the bath [49], a key feature of pilot-wave hydrodynamics [4 11 50]. The numerical approach to simulating Eqs. (2)-(3) is detailed in the Supplementary Information.

**Measurement procedure and data collection**

To initialize the runs, the wave and velocity field of the bath are set to zero, and the particle positions are assigned random, uniformly distributed values. Then, the model runs for 2000 Faraday periods, a measurement is made, and all fields are reset back to zero to initialize the subsequent run. This cycle is repeated for each set of parameter settings until the relative error in the running average of $M(\alpha = a, \beta = b)$ is reduced to an acceptably small value. We set this tolerance to be 3% for parameters that violate the inequality and 7% for those that do not. While extremely accurate, this ‘discrete’ technique is computationally intensive; thus we have used it only for the most critical points of the parameter space, in which the maximal Bell violations occurred. To explore the parameter space more efficiently, we adopt an alternative, relatively expedient, ‘continuous’ approach, in which the final conditions of one run serve as the initial conditions of the next. We demonstrated the statistical equivalence of the two approaches as follows: For specific selected data points, we performed approximately 30 different runs using the two techniques, and found the results of the ‘discrete’ and ‘continuous’ runs to be in agreement to within 3%. We then executed continuous runs for 48,000 Faraday periods, during which measurements are performed frequently at uniformly distributed random times. After a sufficiently long run, the full range of initial conditions will have been effectively explored.
The consistency of the results deduced with the discrete and continuous approaches demonstrates that the system is ergodic; specifically, the long-time emergent statistics are independent of the initial conditions.

Since the inequality involves four different correlation functions (three for the symmetric case considered here), finding the combinations of measurement settings that maximized $S$ was not entirely straightforward. Figure 4 summarises the strategy we followed in seeking violations. We first investigated the evolution of a single correlation function $M(\alpha = a, \beta = a)$ as a function of $a$. This gave us a good sense of parameters that maximize the difference $\delta M(a,a') = M(a,a) - M(a',a')$ (see Figure 4a). $\delta M(a,a')$ involves two of the correlation functions of Eq. 1, in the symmetric case of interest where $a = b$ and $a' = b'$. Figure 4b shows a 2D plot of the optimisation of $\delta M$ as function of $a$ and $a'$. The black dashed lines highlight the domain in which $(\max_{a,a'}(\delta M) - \delta M)/\max_{a,a'}(\delta M) > 0.9$. The other term in the inequality, specifically $2M(\alpha = a, \beta = a')$, represents a combination of measurements from unequal

barrier depths at the two measurement stations. Figure 4c represents the dependence of $2M(\alpha = a^*, \beta = a')$ on depth $a'$ for fixed $a = a^*$, the $S$-maximizing value considered in Figures 2 and 3. Finally, Figure 4d shows the evolution of the correlation functions $(M(a,a'); M(a',a'))$ with increasing $a'$ and fixed $a = a^*$.

I. DATA AVAILABILITY

The data that support the findings of our study are available upon request.

II. CODE AVAILABILITY

The code that generated the data is available upon request.

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