Phase-matched four wave mixing and quantum beam splitting of matter waves in a periodic potential

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We show that the dispersion properties imposed by an external periodic potential ensure both energy and quasi-momentum conservation such that correlated pairs of atoms can be generated by four wave mixing from a Bose-Einstein condensate moving in an optical lattice potential. In our numerical solution of the Gross-Pitaevskii equation, a condensate with initial quasi-momentum $k_0$ is transferred almost completely ($>95\%$) into a pair of correlated atomic components with quasi-momenta $k_1$ and $k_2$, if the system is seeded with a smaller number of atoms with the appropriate quasi-momentum $k_1$.

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Bose-Einstein condensates in optical lattices provide flexible systems for studying the behavior of coherent matter in periodic potentials. Considerable attention has been and continues to be subject to theoretical and experimental investigations [2, 3, 4].

The process we wish to consider is a four wave mixing (FWM) process, which transfers pairs of atoms coherently from an initial momentum state $k_0$ to new states with momenta $k_1$ and $k_2$. We consider a Bose-Einstein condensate in a quasi-1D geometry and we will consider only the longitudinal dynamics of the condensate. This geometry is relevant, e.g., for atomic wave guides and atom interferometers based on atom chips [5].

In Refs. [6, 7], it was shown that nonlinear interaction originating from the s-wave scattering between atoms leads to depletion of the condensate and emission of pairs of atoms at other momenta when a continuous matter wave beam passes through a finite region with enhanced interactions. For a larger condensate, however, the process will not be effective unless it conserves both energy and momentum, i.e., the waves must be phase-matched over the extent of the sample. We shall show how the characteristic energy band structure in a one-dimensional optical lattice can be used to conserve both energy and quasi-momentum conservation, i.e., phase-matching of the FWM process.

Our tailoring of the dispersion properties of matter waves by an external potential is inspired by approaches to non-linear optics, which employ various means to ensure phase-matching, e.g., of the FWM process [8, 9, 10]. A similar phase-matched FWM process has been used to explain giant amplification from semiconductor microcavities, where the polariton dispersion properties can be controlled by the strong photon-exciton coupling [11]. We also note that a recent analysis [12] of the break-up of a bright matter wave soliton was analyzed in terms of dispersion and phase-matching. Phase-matched FWM has been realized in collisions of two condensates in two dimensions [13, 14, 15], but in the present paper we show that the process can take place with atomic motion along a single direction, for example inside an atomic waveguide.

The basic idea of our proposal is illustrated in Fig. 1. In a periodic potential $V(z)$, the energy spectrum constitutes a band structure, and the figure shows the lowest energy band for the corresponding linear Schrödinger equation. When two atoms with momentum $k_0$ collide and leave with momenta $k_1$ and $k_2$ momentum conservation requires

$$2k_0 = k_1 + k_2 \text{ modulo } Q,$$

where $Q$ is a reciprocal lattice vector. In the periodic potential the energy does not vary quadratically with the wave number, and as indicated by example in Fig. 1 it is possible to identify sets of wave numbers with conservation of the total energy

$$2\varepsilon_0 = \varepsilon_1 + \varepsilon_2.$$  

To investigate the effectiveness of this degenerate FWM process, we have performed a mean-field analysis of the dynamics of the condensate based on the one-dimensional Gross-Pitaevskii equation

$$i\hbar \frac{d\Psi}{dt} = \left(-\frac{\hbar^2}{2m} \frac{d^2}{dz^2} + V(z) + \gamma |\Psi|^2 \right)\Psi,$$ 

where the periodic potential $V(z)$ is given by

$$V(z) = -\beta E_R \cos(2k_L z),$$

where $E_R \equiv \hbar^2 k_L^2 / 2m$ is the recoil energy. The periodic potential can be generated with a standing wave of a laser with wavelength $\lambda = 2\pi / k_L$. The factor

$$\gamma = gN / A_L$$

describes the strength of the nonlinearity, where $N$ is the total number of atoms in the condensate, $A_L$ is the area
of the transverse ground state, and \( g \) relates to the s-wave scattering length \( a_s \) and the mass \( m \) of the atoms as \( g = 4\pi\hbar^2 a_s/m \). In our calculation presented below we assume \( \beta = 1/2 \) and \( \gamma = 40.8E_R/k_L \), corresponding to \( N = 100000 = 87Rb \) atoms confined to a transverse area of \( A_\perp = 42 \mu m^2 \) and a grid of 512 periods of the potential. The effective area \( A_\perp = \frac{2\pi\hbar}{mc^2} \sqrt{1 + 2a_s N/L} \) results from a gaussian variational ansatz \([16, 17]\), to the radial distribution in a harmonic potential with \( \omega_\perp = 2\pi \times 44 \) Hz and a constant longitudinal density \( N/L \).

An eigenstate \( \Psi_0 \) of the condensate with quasi-momentum \( k_0 \) is found using the method of steepest descent in imaginary time, assuming a solution according to Bloch’s theorem on a single period of the lattice potential. Subsequently, a seed at variable \( k_1 \) is applied giving the following initial wave function

\[
\Psi_{\text{init}}(z) = \frac{1}{\sqrt{1 + \alpha^2}} \left[ 1 + \alpha e^{i(k_1 - k_0)z} \right] \Psi_0(z),
\]

where \( \alpha = 0.1 \) has been used in our calculations with the Gross-Pitaevskii equation. This wave function does not fulfill Bloch’s theorem, and we hence restrict ourselves to values of \( k_0 \) and \( k_1 \) with interference patterns which are periodic on an extended grid of 512 periods of the potential. To test the importance of phase-matching in the FWM process we propagate the wave function \( \Psi_{\text{init}} \) on this grid, with different values for the seeded momentum component \( k_1 \). As a function of time, we can observe the evolution of the Gross-Pitaevskii wave function and build-up of amplitude at different momenta. We are particularly interested in the quasi-momentum regions around \( k_0, k_1 \) and \( k_0 = 2k_0 - k_1 \). Let \( \psi(k, t) \) denote the Fourier transform of the time-dependent Gross-Pitaevskii wave-function \( \Psi(z, t) \). The distribution in momentum space folded into the single Brillouin zone from \( k = 0 \) to

\[
Q = 2k_L
\]

is given by the following expression

\[
P_k(t) = \sum_n \int_{-\Delta k/2}^{\Delta k/2} \left| \psi(k + nQ, t) \right|^2 dk,
\]

where we have introduced a sampling over a narrow momentum window with \( \Delta k = k_L/32 \).

Fig. 2 (a) shows the part of the condensate, \( P_{k_0}(t) \), remaining at the initial quasi-momentum \( k_0 \) when the condensate is seeded with different values of \( k_1 \). The most important features in the figure occur when the condensate is seeded with \( k_1 = 1.055k_L \) and \( k_1 = 0.289k_L \). The original condensate fraction at \( k_0 \) is almost completely depleted, and strong growth of the population of the seeded momentum state is shown in \( P_{k_1}(t) \) in part b, accompanied by simultaneous growth in the phase-matched component \( k_2 = 2k_0 - k_1 \), shown as \( P_{k_2}(t) \) in part c of the figure. Comparing the set \((k_0, k_1, k_2) = (0.672k_L, 1.055k_L, 0.289k_L)\) with the set of phase-matched wave vectors in Fig. 1 we find extremely good agreement and we conclude that the structure at \( k_1 = 1.055k_L \) is indeed due to the phase-matched FWM process.

The remaining structures in Fig. 2 are less prominent but for instance the structure in Fig. 2(b) after 25 msec at \( k_1' = 0.492k_L \) can be identified as a double FWM process with the following steps: \( 2k_0 \rightarrow k_1' + k_2' \) and \( k_0 + k_2' \rightarrow k_1' + k_3' \), where \( k_3' = k_2' + (k_0 - k_1') \). These steps do not

![FIG. 1: Band structure for atomic motion in the periodic potential Eq. 4. Quasi-momentum conservation and energy conservation is fulfilled in the crosses where two atoms with momentum \( k_0 \) collide and separate at momenta \( k_1 \) and \( k_2 \) illustrated in the figure.](image)

![FIG. 2: Population of different momentum components (a): \( P_{k_0}(t) \), (b): \( P_{k_1}(t) \) and (c): \( P_{k_2}(t) \) as a function of time and as function of the seeding wave vector \( k_1 \). The calculations are performed with a potential modulation \( \beta = 1/2 \), \( N = 100000 \) atoms, and an initial wave vector of \( k_0 = 0.672k_L \). When \((k_0, k_1, k_2)\) fulfill the phase-matching conditions in Eq. 1 and 2, which is the case for \( k_1 = 0.289k_L, 1.055k_L \), the condensate originally having wave vector \( k_0 \) is efficiently transferred into a set of atomic clouds with wave vectors \( k_1 \) and \( k_2 \).](image)
and k phase-matched sets of wave vectors (energy and quasi-momentum conservation as illustrated occurring sets of wave vectors we performed a simple procedure for quasi-momentum states of atoms outside the original condensate. When the process is phase-matched, however, the calculations show an extremely high conversion efficiency (up to 95%). Furthermore the populations \( P_k(t) \), displayed more clearly in Fig. 4(a)-(c) for the set of phase-matched wave vectors \((k_0, k_1, k_2) = (0.672k_L, 1.047k_L, 0.297k_L)\) from a Gross-Pitaevskii simulation with half the amount of atoms as compared with Fig. 4(a) increasing \( k_1 \) (and decreasing \( k_2 \)) lowers the energy of both states, whereby energy conservation restricts the coupling to a narrow part of the momentum continuum, in which case the dynamics passes to Rabi oscillatory dynamics. The expected frequency of the Rabi-oscillations is proportional to \( N \). Fig. 4(d)-(f) illustrate \( P_{k_0}(t), P_{k_1}(t), P_{k_2}(t) \) for the phase-matched \((k_0, k_1, k_2) \) due to the effect of excitations, and we solve the following equation:

\[
\left(-\frac{\hbar^2}{2m} \frac{d^2}{dz^2} + V(z) + 2\gamma|\Psi(z)|^2 \right) u(z) = \varepsilon u(z), \tag{8}
\]

sometimes referred to as the Popov approximation to the coupled Bogoliubov-de Gennes equations for the problem. The mean-field interaction term \( 2\gamma|\Psi(z)|^2 \) depends on \( k_0 \) and amounts to less than a five percent correction to \( V(z) \). Hence, we identify \( \Psi_0(z) \) and we solve Eq. 8 for each quasi-momentum \( k_0 \) and derive band structures, from which we extract the phase-matched pair \( k_1, k_2 \) of final momenta. Sets of \((k_0, k_1, k_2)\) found with this method are shown by the dashed curve in Fig. 3. As expected the interactions only slightly change the phase-matching condition.

Our original expectations were that the phase-matched, degenerate FWM could be achieved in a perturbative regime with only few atoms expelled from the original condensate. For completeness we present also a calculation including sometimes referred to as the Popov approximation to the coupled Bogoliubov-de Gennes equations for the problem. The mean-field interaction term \( 2\gamma|\Psi(z)|^2 \) depends on \( k_0 \) and amounts to less than a five percent correction to \( V(z) \). Hence, we identify \( \Psi_0(z) \) and we solve Eq. 8 for each quasi-momentum \( k_0 \) and derive band structures, from which we extract the phase-matched pair \( k_1, k_2 \) of final momenta. Sets of \((k_0, k_1, k_2)\) found with this method are shown by the dashed curve in Fig. 3. As expected the interactions only slightly change the phase-matching condition.

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be spontaneously scattered, and that our seeded process
will be dominant.

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