Positronium Hyperfine Splitting in Non-commutative Space at the Order $\alpha^6$

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Abstract

We obtain positronium Hyperfine Splitting owing to the non-commutativity of space and show that, in the leading order, it is proportional to $\theta \alpha^6$ where, $\theta$ is the parameter of non-commutativity. It is also shown that spatial non-commutativity splits the spacing between $n = 2$ triplet excited levels $E(2^3S_1) \rightarrow E(2^3P_2)$ which provides an experimental test on the non-commutativity of space.
1 Introduction

The question of measuring the spatial non-commutativity effects, in physical processes, is under intensive interest. Non-commutative QED (NCQED) seems to be a straightforward method to examine such effects. For this purpose, one needs a precise experimental data such as positronium hyperfine splitting (HFS) among the other processes. The basic difference between NCQED and QED is the existence of new interactions (3-photon and 4-photon vertices) which complicate the calculations in NCQED. Although the Feynman rules of this theory are given in [1, 2], to apply these rules to bound state, one needs special treatments like Bethe-Salpeter (BS) approach [3] or non-relativistic QED (NRQED) [4]. In our preceding letter [5], using BS equation, we have shown that up to the order $\alpha^4$ no spin-dependent correction owing to the spatial non-commutativity appears in the positronium spectrum. Therefore one should calculate the higher order corrections. In this letter we calculate the corrections to the positronium by using NRQED method. In section 2, we introduce NRQED-vertices in the NC-space. Consequently, in section 3, we use the modified NRQED to determine HFS at the lowest order. In this section, we show that our calculations at the leading order lead to the corrections at the order of $\theta \alpha^6$, where $\theta$ is the parameter of non-commutativity. At the end, we summarize our results.

2 NRQED in non-commutative space

NRQED is an effective field theory which simplifies the bound state calculation. To apply this technique in non-commutative space one should modify the NRQED vertices by performing $\frac{p}{m_e}$ expansion on NCQED scattering amplitude. In doing so, one obtains an effective theory of non-relativistic particles which permits the direct application of well tested techniques based on Schrödinger’s equation. Now, comparing NCQED scattering amplitudes with NRQED can completely determine the matching coefficients. Some of the vertices with their appropriate matching coefficients, are shown in Fig. 1. They contribute to the tree level matching to get the leading order bound state energy shift. One should note that these coefficients apart from a phase factor are very similar to the standard NRQED [4, 5]. This similarity is owing to the fact that the scattering amplitude of $e^+e^-$ in NCQED is independent of the parameter of non-commutativity of space [6]. The other vertices which are not shown in Fig. 1 and have not counterpart in the standard NRQED, due to the existence of the three and four photon vertices have contributions to higher order corrections to energy shift. Now, by using the first graph of Fig. 1 and expanding the vertices up to order $\theta$, one can easily verify the results of refs.[3, 9] at the order $\theta \alpha^4$ as

$$\Delta E = \left< \frac{\alpha \Theta \mathbf{L}}{r^3} \right> = \theta \alpha^4 \frac{P_{nl}}{l(l+1)(l+\frac{1}{2})(l+1)},$$

(1)

Such an energy shift is spin-independent and therefore has not any contribution to HFS.
3 Positronium HFS at the leading order

By using the modified NRQED we can determine the diagrams which contribute to the lowest order of HFS (Fig. 2). We can now calculate each diagram separately as follows:

\[ \Delta E_a = \int \frac{d^3p d^3p'}{(2\pi)^6} \psi^*(p') \Gamma_a(p, p') \psi(p), \]  

with

\[ \Gamma_a(p, p') = \left[ \frac{-ie(p' - p) \times \sigma}{2m_e} e^{i p / p'} \right] \left[ \delta_{ij} - \frac{(p - p')_i(p - p')_j}{(p - p')^2} \right] \left[ \frac{1}{2m_e} e^{i p / p'} \right], \]  

where \( p \wedge p' = \frac{1}{2} \theta_{\mu\nu} p_\mu p'_\nu \) and \( \theta_{\mu\nu} \), the parameter of the non-commutativity is given as

\[ \theta_{\mu\nu} = i [x_\mu, x_\nu]. \]  

It is shown that \( \theta_{0i} \neq 0 \) leads to some problems with the unitarity of field theory and the concept of causality \([10, 11]\), therefore in our calculations we consider \( \theta_{0i} = 0. \)
After some algebra Eq.(6) yields

\[ \Delta E_a = \frac{i e^2}{2 m_e^2} \int \frac{d^3 p d^3 p'}{(2\pi)^6} \psi^*(p') \sigma_1 \cdot \sigma_1 \times p' \cdot e^{i \theta_i p_i p'_i} \psi(p) \]
\[ = \frac{e^2}{8 \pi m_e^2} \int d^3 r \left[ \psi^* (r + i \theta \cdot \nabla) \frac{r \times p}{r^3} \sigma_1 \right] \psi(r) \]
\[ = \frac{\alpha}{2 m_e^2} \left( S_1 \cdot L \right) \frac{3 \alpha}{m_e^2} \int d^3 r (\Theta \cdot L) \psi^* \psi + O(\alpha^7), \]  
(5)

where \((\theta, \nabla)_i = \theta_{ij} \partial_j\) and \(\Theta = (\theta_{23}, \theta_{31}, \theta_{12})\). In the third equality we used

\[ \psi^* (r + i \theta \cdot \nabla) = \psi^* (r) + i (\nabla \psi^* (r) \cdot \theta \cdot \nabla) + O(\theta^2). \]  
(6)

One should note that the first term in Eq.(6) is the usual term in NRQED which is of the order \(\alpha^4\). But the second term which is appeared in Eq.(6), owing to the spatial non-commutativity is of the order \(\theta \alpha^6\). Nonexistence of the terms at the order of \(\alpha^4\) which carry \(\theta\)-dependence is a remarkable result which happens in all diagrams of HFS. Indeed this fact is due to appearance of \(\psi^* (r + i \theta \cdot \nabla)\) instead of \(\psi^* (r)\) in all energy-correction expressions. Therefore, to obtain the energy corrections for HFS at the order \(\alpha^6\) one should once calculate the corrections up to the lowest order of NRQED (i.e. Fig. 2). In the other words, the \(\alpha^6\)-corrections in NRQED calculations of commutative space lead to the higher order of \(\alpha\) in non-commutative space.

Now we work out the Figs. 2(b-f) as follows:

\[ \Delta E_b = \Delta E_a (S_1 \rightarrow S_2) \]
\[ \Delta E_c + \Delta E_d = \frac{1}{2} (\Delta E_a + \Delta E_b) \]  
(7)

\[ \Delta E_e = \int \frac{d^3 p d^3 p'}{(2\pi)^6} \psi^*(p') \Gamma_e (p, p') \psi(p), \]  
(8)

with

\[ \Gamma_e (p, p') = \left[ \begin{array}{c} -ie(p' - p) \times \sigma_1 e^{ip \cdot \cdot \cdot p'} \\ \frac{-1}{2m_e} \end{array} \right] \]
\[ = \left[ \begin{array}{c} \delta_{ij} - \frac{(p - p')_i (p - p')_j}{(p - p')^2} \\ \frac{-ie(p - p') \times \sigma_2}{2m_e} e^{ip \cdot \cdot \cdot p'} \end{array} \right], \]  
(9)

which results in

\[ \Delta E_e = \frac{e^2}{4 m_e^2} \int d^3 r \psi^* (r) \psi^* (r + i \theta \cdot \nabla) \left[ -\sigma_1 . \sigma_2 \nabla^2 + (\sigma_1 . \nabla)(\sigma_2 . \nabla) \right] \frac{1}{4 \pi r} \]
\[ = (\ldots) + \frac{3e^2}{16 \pi m_e^2} \int d^3 r \hat{\Gamma}_e \psi^* (r), \]  
(10)

where \((\ldots)\) means the usual part of the energy shift and

\[ \hat{\Gamma}_e = \left[ \begin{array}{c} \frac{\sigma_1 \cdot \sigma_2}{r^5} \Theta \cdot L - \frac{\sigma_2 \cdot \cdot \cdot r}{r^9} \sigma_1 \cdot (\Theta \times \cdot \cdot \cdot p) - \frac{\sigma_1 \cdot \cdot \cdot r}{r^9} \sigma_1 \cdot (\Theta \times \cdot \cdot \cdot p) - \frac{5}{r^7} (\sigma_1 \cdot r)(\sigma_2 \cdot r) \Theta \cdot L \end{array} \right], \]  
(11)
where $\hat{p} = -i \nabla$. The final diagram (Fig. 2f) has not any contribution at the order of our interest. For $S = 1$ one can easily find
\[
\Delta E_a^{\text{NC}} + \Delta E_b^{\text{NC}} = \frac{-3e^2}{4\pi m_e^2} \int d^3r \left[ \frac{\Theta \cdot L}{r^5} \psi^*(\mathbf{r}) \right] \ell \psi(\mathbf{r})
\]
\[
\Delta E_a^{\text{NC}} + \Delta E_b^{\text{NC}} = \frac{1}{2}(\Delta E_c^{\text{NC}} + \Delta E_d^{\text{NC}})
\]
\[
\Delta E_c^{\text{NC}} = \frac{3e^2}{16\pi m_e^2} \int d^3r \psi(\mathbf{r}) \Gamma \psi^*(\mathbf{r})
\]
where
\[
\Gamma = \frac{\Theta \cdot L}{r^5} - 2 \left\{ \frac{z(\Theta \times \hat{p})_3}{r^5} \mathbf{r} \cdot (\Theta \times \hat{p}) - 2z(\Theta \times \hat{p})_3 \right\} - \frac{5}{r^7} \left\{ \frac{z^2}{r^2} - \frac{2z^2}{2} \right\} \Theta \cdot L.
\]

The superscript NC in Eq. (12) means the non-commutative part of the energy shift and three lines in the Eq. (13) are related to $S_z = 1, 0, -1$, respectively. Meanwhile for the spin zero state ($S = 0$), all contributions to the energy shift are zero.

The average of $\Delta E_c$ over the triplet is zero, which means the spin-spin interaction part carries no correction in average and therefore the hyperfine splitting due to the non-commutativity becomes

\[
\delta E_{\text{NC}} = \frac{9e^2}{8\pi m_e^2} \int d^3r \left[ \frac{\Theta \cdot L}{r^5} \psi_{nlm}^*(\mathbf{r}) \right] \ell \psi_{nlm}(\mathbf{r}),
\]
where $\psi_{nlm}$ is the wave function of the positronium in the commutative space with the Coulomb potential and we have defined $\delta E_{\text{NC}} = \Delta E_{\text{NC}}(S = 1) - \Delta E_{\text{NC}}(S = 0)$. If the $z$-axis is chosen parallel to the vector $\Theta$, the above result simplifies into

\[
\delta E_{\text{NC}} = \frac{9e^2}{8\pi m_e^2} |\Theta| \ell m \left\langle \frac{1}{r^5} \right\rangle = |\Theta| \alpha^6 m_e \ell m f(n, l),
\]
where $\lambda_e$ is the Compton wave length of the electron and $f(n, l)$ is defined as

\[
f(n, l) = \frac{P_{n,l}^{(1)}}{l(l + \frac{3}{2})(l + 1)(l + \frac{1}{2})(l + 2)} + \frac{P_{n,l}^{(2)}}{(l - 1)(l - \frac{3}{2})(l + \frac{1}{2})(l + 1)}.
\]

One should note that the divergence of $\delta E_{\text{NC}}$ at $l = 1$ is owing to singularity of $\left\langle \frac{1}{r^5} \right\rangle$ at $r = 0$, the region where $\theta$-expansion is not well-defined. Actually, it is shown that $\theta$-expanded NCQED is not renormalizable [12].

The $\theta$ expansion imply a cut-off $\Lambda \sim \frac{1}{\sqrt{|\Theta|}}$ while the validity of NRQED requires $p \leq m_e = \frac{1}{\lambda_e}$. Since $\sqrt{|\Theta|} \leq \lambda_e$, the appropriate cut-off is $\Lambda = \frac{1}{\lambda_e}$. Therefore the energy shift for $n = 2, l = 1$ can be obtained as

\[
\delta E_{\text{NC}} = \frac{3}{512} m_e \left( \frac{|\Theta|}{\lambda_e^5} \right) [\ln 2 - \gamma - \ln \alpha] \alpha^6.
\]

The above result should be added to the values of HFS derived in NRQED at the order $\alpha^6$. The reported uncertainties on the experimental values of $E(2^3S_1) \rightarrow E(2^3P_2)$ are about 0.1 MHz [13], that give an upper bound $\frac{|\Theta|}{\lambda_e^5} \sim 10^{-1}$. Therefore determining the value of $|\Theta|$ requires more accurate experiments.
4 Summary

Using NRQED method in the non-commutative space, we have obtained that there is not any correction at the order $\alpha^4$ for the HFS of positronium, the order $\alpha^4$ corrections are spin independent. The correction to the energy shift is started at the order $\alpha^6$, Eqs. (14-15), and it depends on $\ell$ and $m$ quantum numbers. Therefore it doesn’t have any contribution to the $E(1^3S_1) \to E(1^1S_0)$ (in the spectroscopic notation $n^{2S+1}L_j$), while for $\ell \neq 0$ there is $2\ell + 1$ different shifts. Consequently, a closer look at the spacing between $n = 2$ triplet excited levels ($E(2^3S_1) \to E(2^3P_2)$), which has already been measured [14-17] can provide an experimental test on the non-commutativity of space.

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