Dynamics of ‘small-world’ network phononic lattices: spectral gaps and diffusive transport

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Abstract

We investigate a family of elastic lattices with disorder introduced through non-local network connections. Inspired by the Watts-Strogatz "small-world" model, a single parameter determining the probability of local connections being re-wired allows for transitions between regular to completely disordered lattices. These connections are added as non-local springs to otherwise regular lattices of different topologies, such as one-dimensional (1D) lattices and two-dimensional (2D) square, triangular and hexagonal lattices. We illustrate that spectral band gaps emerge in 1D lattices for increasing degrees of disorder, being persistent across multiple lattice realizations. These gaps progressively disappear for 2D lattices of increasing connectivity, from the hexagonal (3 connections per node) to the triangular (6 connections per node) case. We also investigate their transient behavior in terms of transport properties, revealing transitions from ballistic to super-diffusive or diffusive transport that occur for increasing degrees of disorder, and for a high enough ratio between the strength of the network connections and that of the underlying lattice coupling. Such unconventional transport properties robustly manifest in all the considered lattice topologies, both 1D and 2D. These numerical investigations unveil the potential for the disorder in the form of non-local connections to enable additional functionalities of metamaterials, such as disorder-induced spectral gaps, and impact mitigation from diffusive-type transport that does not occur in regular periodic materials.

I. INTRODUCTION

Complex networks are used to describe a wide variety of systems in nature and society. In particular, the ‘small-world’ network model proposed by Watts and Strogatz [1] has emerged as a popular model that allows for exploring the space between regular and random networks. In such model, the vertices of a network are initially connected to the neighbors forming a regular network, and then re-wired with a probability $p$ to another random node. Therefore, they simultaneously exhibit a high degree of clustering characteristic of regular networks, and short vertex-to-vertex distances characteristic of random networks. Apart from the general characterization of its properties, [2–6] such a celebrated model has been applied to

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a variety of scenarios such as in the dynamics of epidemic spreading, [7, 8] in the modeling of brain networks, [9, 10] social networks, [11] and transportation networks. [12, 13] The physics of condensed matter systems based on small-world networks has also been explored, such as Ising models, [14] and localization [15, 16] and transport [17–19] in quantum lattices.

In this paper, we investigate the dynamics of phononic lattices characterized by non-local connections inspired by the small-world network model. In the context of phononics and metamaterials, [20] there has been considerable interest in the role of disorder, for instance in rainbow-based materials for bandgap widening and energy trapping [21–27], in disorder induced energy localization [28], topological phase transitions [29] and signal processing [30], in the role of disorder in phonon and thermal transport [31, 32], and the exploration of quasi-periodic lattices. [33–41] So far, most of these works consider disorder introduced to local parameters or couplings; the idea of introducing disorder through non-local connections according to a network model has not yet been explored. Our model consists on regular monoatomic spring-mass lattices that define underlying lattices of different topologies, such as 1D lattices and 2D hexagonal, square and triangular lattices. Additional network connections are then added based on the small-world model. We first investigate the emergence of spectral gaps as a function of disorder in 1D lattices, which seem to be preserved across multiple disorder realizations and averaging. These gaps are shown to progressively disappear in 2D lattices for increasing degrees of connectivity, from the hexagonal (3 connections per node) to the triangular (6 connections per node) case. We then explore the transient behavior and transport properties of both 1D and 2D lattices, where we identify transitions from ballistic to super-diffusive and diffusive transport, partly similar to what experienced by quantum and photonic lattices in the presence of on-site disorder [42, 43]. These investigations identify a new route for introducing disorder to metamaterials in the form of non-local connections, with potential to induce disorder-resilient spectral gaps and for applications in impact mitigation relying on diffusive energy transport, as opposed to the ballistic spreading typically observed in periodic materials.

This paper is organized as follows: following this introduction, section II describes the modeling of the small world phononic lattices and the associated equations of motion. Next, section III describes the spectral properties of the lattices and the emergence of band gaps through the non-local disordered links. Section IV then describes the transport properties of the network lattices and characterizes the transition from ballistic to diffusive transport.
Finally, section V summarizes the main findings of this work and outlines future research directions.

II. SMALL-WORLD PHONONIC LATTICES: DESCRIPTION AND EQUATIONS

We consider 1D (Fig. 1) and 2D (Figs. 2,3,4) lattices of equal masses $m$ (in red) connected to its neighbors and to the grounds by springs of stiffness $k_0$, represented by black lines, forming periodic baseline lattices with unitary lattice constants (or spacing). Additionally, the masses are connected through a network of springs, represented in blue, which is inspired by the small-world model [1], but implemented here over straight line (Fig. 1), hexagonal (Fig. 2), square (Fig. 3) and triangular (Fig. 4) lattice topologies, instead of the conventional ring-type topology. The chosen underlying lattice topologies are respectively characterized by 2, 3, 4 and 6 nearest-neighbor connections (with the exception of the boundary nodes). The network springs initially connect each mass to its nearest neighbors, replicating the same nearest-neighbor couplings of the underlying lattices, and are then re-wired with probability $p$ (varying from 0 to 1) to another random node. Examples are illustrated in Fig. 1(b) for a 1D lattice with 20 masses; $p = 0$ recovers a regular lattice with additional nearest-neighbor connections only, while increasing values of $p$ generate lattices with higher degrees of disorder emanating from the non-local connections. Similarly, examples of 2D lattices are illustrated in Figs. 2,3,4 for hexagonal, square and triangular lattices, and for increasing values of $p$. The presence of an underlying lattice is necessary to guarantee that all the masses are connected, since that property is not guaranteed by the network (blue) links alone for arbitrary values of $p$. [1] The spring constants of the network springs are defined to be inversely proportional to the distance, i.e $k_{n,s} = \alpha k_0 / d_{n,s}$, where $d_{n,s}$ denotes the distance separating masses indexed by $n$ and $s$, while $\alpha$ is a parameter that indicates the strength of the network connections compared to the underlying lattice (note that for nearest neighbor connections the spring constant is simply $k_{n,s} = \alpha k_0$). This choice represents a more physical coupling behavior, for example the equivalent stiffness of elastic rods is known to be inversely proportional to its length, and is more realistic than considering the coupling strength to be independent of the distance.

The equation of motion for a mass of index $n$ in the absence of external forces is expressed
as:

\[ m \ddot{u}_n + k_0 u_n + \sum_r k_0 (u_n - u_r) + \alpha \sum_s \frac{k_0}{d_{n,s}} (u_n - u_s) = 0, \quad (1) \]

where \( r \) runs over the nearest neighbors, and \( s \) runs over the network links that connect to that mass, while \( u_n \) represents the displacement of the \( n_{th} \) mass. For a finite lattice of \( N \) masses, the equations of motion can be assembled in matrix form:

\[ \mathbf{M} \ddot{\mathbf{u}} + \mathbf{K} \mathbf{u} = 0, \quad (2) \]

where \( \mathbf{u} = [u_1, u_2, \ldots, u_N]^T \), and \( \mathbf{M}, \mathbf{K} \) respectively denote the mass and stiffness matrices. The numerical results presented in this paper rely on standard procedures such as numerical solution of the eigenvalue problem \( \mathbf{K} \mathbf{u} = \omega^2 \mathbf{M} \mathbf{u} \) for the natural frequencies and mode shapes of the lattice, and numerical integration of the equations of motion for evaluating the transient behavior under a set of initial conditions.

### III. SPECTRAL PROPERTIES AND BAND GAP EMERGENCE

We investigate the spectral properties of the network lattices and the emergence of band gaps. Due to the absence of periodicity, we rely on eigenfrequencies and mode shapes.

![Schematics of 1D 'small-world' network phononic lattices. Each mass (red) is connected to the neighbors and to the ground by a spring of constant \( k_0 \) (black lines) (a). Additional network connections are represented by blue links, which are initially nearest neighbor links that are re-wired with probability \( p \) to another random node. Examples with \( p = 0, 0.3, 0.8 \) (b).](image-url)
computations performed on representative finite lattices. Figure 5(a) illustrates an example for a finite lattice with $N = 50$ masses and $\alpha = 5$, where eigenfrequencies computed for

FIG. 2. Examples of 2D hexagonal lattices with small-world network connections for increasing values of $p$. The top and bottom rows respectively display the perspective and top views, with network links represented by blue lines.

FIG. 3. Examples of 2D square lattices with small-world network connections for increasing values of $p$. The top and bottom rows respectively display the perspective and top views, with network links represented by blue lines.
two lattices with $p = 0$ and $p = 0.6$ under free-free boundary conditions are compared. Throughout this paper, $\Omega = \omega / \omega_0$ is a normalized frequency, where $\omega_0 = \sqrt{k_0/m}$. The color of the dots represents the Inverse Participation Ratio (IPR) of the corresponding modes,

FIG. 4. Examples of 2D triangular lattices with small-world network connections for increasing values of $p$. The top and bottom rows respectively display the perspective and top views, with network links represented by blue lines.

FIG. 5. Eigenfrequencies (a) and selected mode shapes (b) of finite lattice with $N = 50$ masses and free-free boundary conditions for $p = 0$ and $p = 0.6$. The frequencies in (a) are color-coded according to their IPR, signaling whether the modes are localized or not.
defined as:  
\[ IPR = \frac{\sum_{n} |u_n|^4}{(\sum_{n} |u_n|^2)^2}, \]
where \( u_n \) is the \( n_{th} \) component of the eigenvector. The IPR varies from 0 to 1 and signals whether a mode is localized or not when its value is high or low, respectively. A few modes are marked in Fig. 5(a) and have their mode shapes displayed in Fig. 5(b) for reference. When \( p = 0 \), the frequencies of the lattice define a continuous band with only non-localized modes, as expected of a periodic monoatomic lattice [20]. However, the lattice with \( p = 0.6 \) supports a series of localized modes, with two examples displayed in Fig. 5(b). While localized modes are expected to appear due to the presence of disorder, we also note that a few frequency bands are not populated by any modes and may potentially define band gaps. Naturally, these results correspond to a single realization of the lattice, and are not sufficient to draw conclusions about the behavior expected in general of any realization, as they can be different from each other due to the randomness of the non-local connections.

To obtain further insight into the spectral properties of the network lattices, we compute the modes of a large finite lattice comprising \( N = 500 \) masses for \( p \) varying from 0 to 1, for multiple realizations and for multiple values of \( \alpha \), with results summarized in Fig. 6(a). Each column corresponds to one \( \alpha \) value ranging from 0.5 to 5: the top row displays the frequencies as a function of \( p \) for a single lattice realization, while the bottom row superimposes the frequencies of 100 different realizations, with the color representing the IPR in both cases. The results confirm the existence of frequency bands where no modes exist, which emerge for increasing values of \( p \), and are more pronounced for larger values of \( \alpha \). The presence of these gaps is further confirmed by harmonic response computations conducted on a lattice with \( N = 200 \) masses, excited at the center mass, where the response is averaged across 100 different lattice realizations. The results are displayed in Fig. 6(b) for the different \( \alpha \) values, where the colormap represents the transmission averaged along the entire lattice as a function of frequency \( \Omega \) and \( p \), in log scale. In particular, we note that for \( \alpha = 0.5 \) and \( \alpha = 1 \) only small gaps appear for a single realization, which are mostly filled by localized modes (with high IPR) when multiple realizations are considered. For \( \alpha = 3 \) and \( \alpha = 5 \), some gaps are formed and persist without any existing modes even for multiple realizations, signaling a higher degree of robustness. The transmission results in Fig. 6(b) confirm the existence of the spectral gaps of both types: in addition to the robust gaps which are not
filled by any modes, the ones which are populated by localized modes still define regions of low transmission, since such localized modes do not transmit energy across the lattice. We also note the presence of flat transmission bands that are pronounced for $\alpha = 3$ (around $\Omega = 2.5$) and $\alpha = 5$ (around $\Omega = 2.8$), which separate the gap with high density of localized modes from the gap not populated by any modes. These results illustrate that the small-world network of springs of sufficient strength ($\alpha > 1$) produces well defined spectral gaps that emerge due to the disordered network connections, and persist across multiple lattice realizations.

We also investigate the spectral properties of 2D network lattices. Results for $\alpha = 5$ are summarized in Fig. 7, which displays the frequencies of $61 \times 61$ lattices as a function of $p$, color-coded by the IPR, and considering 100 different realizations of hexagonal (a), square (b) and triangular (c) lattices. The bottom panels (d,e,f) display the transmission as a function of $p$ and $\Omega$, calculated by exciting $41 \times 41$ lattices at the center mass and averaging the response across the lattices over 100 realizations. While no "robust" gaps (not populated by any modes) are found, there is evidence of regions populated only by localized modes emerging with increasing $p$. In the hexagonal case, a couple of frequency gaps filled with localized modes are observed to emerge in Fig. 7, which are confirmed by low transmission regions in (d). The size of these gaps are diminished in the square lattice case (b,e), and almost disappear in the triangular lattice case (c,f). These results illustrate how the lattice topology plays an important role in the emergence of bandgaps, and suggest that the progressive increase in connectivity from the 1D lattice (2 connections per node) to the triangular lattice (6 connections per node) causes a decrease in band gap occurrence. We also note that low frequency gaps seem to emerge as a function of $p$ as the first mode of the lattice is separated by a gap from the collective of the other modes of the lattice, whose size increases with connectivity (from hexagonal to triangular lattices). Although the spectral properties of the network lattices are largely influenced by the topology, in the next section we illustrate that the transport properties are qualitatively similar and that diffusive transport can be observed in all cases.
IV. TRANSIENT BEHAVIOR: FROM BALLISTIC TO DIFFUSIVE TRANSPORT

The transient behavior of the small-world lattices is investigated next. In the absence of periodicity and without the possibility of dispersion analyses, we rely on an approach commonly used to characterize the transport in disorder materials that considers the dynamic evolution of the lattice motion resulting from an initial perturbation. The evolution is quantified through the Mean Square Displacement (MSD) defined as

$$MSD(t) = \left\langle \sum_{n} (d_{n,n_0})^2 |u_n(t)|^2 \right\rangle \approx t^\gamma,$$

(4)

FIG. 6. Spectral properties of 1D small-world phononic lattices. Eigenfrequencies of finite lattice with $N = 500$ masses computed as a function of $p$ for multiple $\alpha$ values and 1 (top) or 100 (bottom) lattice realizations, color coded by the IPR (a). Transmission of finite lattice with $N = 200$ masses when excited at the center as a function of $p$, averaged along the lattice and across 100 realizations, for multiple $\alpha$ values (b).
where $\langle \rangle$ denotes averaging across multiple realizations, $n_0$ is the site where the initial perturbation is applied, and $d_{n,n_0}$ is the distance between sites $n$ and $n_0$. The MSD is usually found to scale as $t^\gamma$, where the exponent $\gamma$ quantifies the speed of the spreading and is used to classify the type of transport that occurs along the medium. In quantum and photonic lattices, which follow similar equations of motion, periodic lattices exhibit ballistic transport characterized by $\gamma = 2$ [43, 44]. In the presence of disorder, $\gamma$ decreases and quantifies the slower spread when compared to regular periodic materials. Depending on the amount of disorder, the lattice may exhibit super-diffusive ($\gamma = 1.5$) or diffusive ($\gamma = 1$) transport, or absence of transport (i.e. Anderson Localization) [45] when $\gamma \approx 0$ [42, 43]. In addition, this approach has been recently applied to other types of aperiodic systems, for example in fractal lattices where $\gamma$ is found to be related to the fractal dimension of the lattice, [46] and also for the characterization of wave packet spreading in disordered non-linear architected materials [47].

Herein, we utilize the MSD to characterize the transport in the small-world phononic lattices through a similar procedure. Starting from the 1D case, we consider a large lattice

![FIG. 7. Spectral properties of 2D network lattices. Top panels (a,b,c) display the eigenfrequencies of 61 × 61 hexagonal, square and triangular lattices, considering 100 realizations, and color-coded according to the IPR. The bottom panels (d,e,f) display the transmission averaged over 100 realizations for 41 × 41 hexagonal, square and triangular lattices.](image)
with $N = 1000$ masses and apply a perturbation to the $n_0 = 500$ site. The perturbation is applied in the form of initial conditions $u_{n_0}(0) = 0, \dot{u}_{n_0}(0) = 1$, which is equivalent to an impulse excitation $f(t) = \delta(t)$ applied to that mass [48]. Such excitation includes the entire spectrum of the lattice since the Fourier Transform of the delta function is unitary, and is similar to the excitation applied to photonic lattices at $z = 0$, where $z$ is the propagation dimension [43]. We observe the spreading for a series of 1D lattices of different $\alpha$ values and its variation with $p$, where the MSD is computed with averaging over 200 realizations, with results summarized in Fig. 8. The simulation time window is adjusted for each $\alpha$ value to stop right before reflections occur in the $p = 0$ case, which would affect the MSD computations. For each $\alpha, p$ pair, $\gamma$ is extracted from the fitting of the tail of the corresponding $MSD(t)$ curve in log scale. Figure 8(a) displays the resulting variation of $\gamma$ with $p$, for multiple $\alpha$ values ranging from 0.1 to 7.5, while a few examples marked in (a) are illustrated in (b,c) (the dashed red lines indicate the numerical fitting of the tail of the MSD curves). We first note that for $p = 0$, ballistic spreading characterized by $\gamma = 2$ occurs regardless of the $\alpha$ value, as expected from periodic lattices. As $p$ increases, $\gamma$ decreases along different transitions that are intensified for higher values of $\alpha$. Two transitions are illustrated for $\alpha = 0.5$ and $\alpha = 2.5$, where selected examples marked in Fig. 8(a) have their MSD fitting illustrated in Figs. 8(b,c), and the corresponding displacement field (averaged across the 200 realizations) displayed in Fig. 8(d,e). For $\alpha = 0.5$, a transition to super-diffusive behavior is illustrated ($\gamma \approx 1.5$), while $\alpha = 2.5$ causes a transition to diffusive behavior ($\gamma$ near 1). In addition to the characterization by the extracted $\gamma$ values, the averaged displacement fields confirm the reduction in spread associated with the super-diffusive and diffusive behaviors.

Next, the transport behavior is illustrated for the 2D hexagonal, square and triangular network lattices in Figs. 9,10,11. A similar procedure is conducted on lattices of sizes $29 \times 52$, $41 \times 41$ and $49 \times 57$, respectively, where initial conditions are applied to the center mass. Multiple $p, \alpha$ values are considered and the results are averaged over 200 realizations for each case. Figures 9,10,11(a) display the variation of $\gamma$ with $p$ for multiple $\alpha$ values ranging from 0.5 to 6. Again we verify that $p = 0$ results in ballistic transport characterized by $\gamma = 2$ for any $\alpha$ value and for all the lattice topologies, as expected from periodic lattices. [44] The figures exemplify transitions for $\alpha = 2$ and $\alpha = 6$, where the MSD fittings are displayed in panels (b,c), while the corresponding averaged displacement fields are displayed in panels (d,e). The displacement plots represent the absolute value
of displacement across the lattice at 5 subsequent time instants, where visualization of the wave spread is enhanced by the displayed sectional plane view and by saturating the color axis to 10% of the maximum displacement value. Overall, the different lattice topologies exhibit qualitatively similar transport transitions, although their quantification through the $\gamma$ coefficient may differ slightly. For $\alpha = 2$, transitions to super-diffusive behavior ($\gamma \approx 1.5$) are observed, while $\alpha = 6$ produces transitions to diffusive transport ($\gamma \approx 1$). The associated decrease in the spreading can be clearly observed in the displacement plots.

These results illustrate how the disorder introduced through the network connections modify the type of transport for all the considered lattice topologies, causing a transition to super-diffusive or diffusive transport when the strength of the network connections ($\alpha$) is high enough. These transitions are reminiscent of those experienced by quantum and photonic lattices with on-site disorder [42, 43]. However, we note that for the considered range of $\alpha$ values, $\gamma$ reaches a plateau close to 1, and Anderson Localization ($\gamma \approx 0$) does not occur. Higher values $\alpha > 7.5$ are not considered herein since the network connections become much stronger and overcome the couplings of the underlying lattice, therefore connectivity if not guaranteed, [1] and the behavior may not be well captured by the considered MSD computations [17]. Such high $\alpha$ regime may be considered in future investigations.

V. CONCLUSIONS

In this paper, we investigate the dynamics of phononic lattices with "small-world" network connections. Our results illustrate the emergence of spectral gaps due to increasing degrees of disorder, which are persistent across multiple lattice realizations. The lattices of different topologies, such as 1D straight or 2D hexagonal, square and triangular lattices were also shown to feature transitions from ballistic to super-diffusive or diffusive transport. These results motivate a new route for the introduction of disorder in metamaterials through network connections, potentially leading to novel functionalities enabled by disorder such as spectral gaps and diffusive transport, which could be exploited in impact mitigation applications for example. The initial investigations presented here may be expanded in multiple directions in future studies. For example, a variety of other network modeling strategies may be considered, such as different underlying lattice topologies, specific statistical modeling of the non-local connections (instead of random re-wiring), introduction of non-linearities, and
experimental studies, to name a few.

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FIG. 8. Transport properties of 1D small world network lattices. Variation of $\gamma$ with $p$ for multiple $\alpha$ values ranging from 0.1 to 7.5 (a). Examples marked in (a) have their MSD fittings illustrated in (b,c), and the corresponding averaged displacements displayed in (d,e).
FIG. 9. Transport properties of 2D small world network hexagonal lattices. Variation of $\gamma$ with $p$ for multiple $\alpha$ values ranging from 0.1 to 6 (a). Examples marked in (a) have their MSD fittings illustrated in (b,c), and the corresponding averaged displacements displayed in (d,e).
FIG. 10. Transport properties of 2D small world network square lattices. Variation of $\gamma$ with $p$ for multiple $\alpha$ values ranging from 0.1 to 6 (a). Examples marked in (a) have their MSD fittings illustrated in (b,c), and the corresponding averaged displacements displayed in (d,e).
FIG. 11. Transport properties of 2D small world network triangular lattices. Variation of $\gamma$ with $p$ for multiple $\alpha$ values ranging from 0.1 to 6 (a). Examples marked in (a) have their MSD fittings illustrated in (b,c), and the corresponding averaged displacements displayed in (d,e).
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