On an approximation of a J-Bessel integral and its applications
(with an appendix by J.S. Friedman∗).

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December 9, 2020

Abstract
The convergence of $\int_0^\infty tJ_0(at) \prod_{m=0}^{n} J_0(r_m t)dt$ has been established for $n \geq 3$, where $J_0$ is the order zero Bessel function of the first kind, and $a$ and $r_m$ are certain positive real numbers. We investigate the excluded case of $n = 2$, showing that the integral converges for all but finitely many arguments of the integrand and characterize its asymptotic behavior at the points of divergence. As an application we establish a formulation for the Mahler measure of a complex linear form in three variables, which in turn results in certain combinatorial identities, and verify our theoretical investigations by numerical calculations presented in the Appendix.

1 Introduction
Our investigation of the stated integral arose primarily out of considering a problem in Analytic Number Theory, namely that of establishing a particular formulation for the logarithmic Mahler measure of a linear form in three variables. Bessel integrals however are of even broader interest; they appear prominently as probability density functions ([Bor+12]), which in turn exhibit connections to quantum field theory ([ZHO19]). In the next section we point out their emergence in the former context and collect some useful results.

1.1 Random Walks
Suppose a man wanders into the complex plane, finds himself at the origin and determines to go on a ramble. He walks from his starting point for some distance $r_m$ at angle $\theta_m$, both chosen at whim, and does this $n$-times successively. Curious observers wish to know the probability his distance from the origin at the conclusion of the $n$ stretches is between $r$ and $r + \delta r$, for some pre-determined $r$, $\delta > 0$. This is the well-known problem of the random walk in the plane ([Wat44]). A closer examination of this problem in fact surfaces the integral which is the main object of this paper’s investigation. In the following we summarize the relevant discussion of [CJS20] from pages 15-17, which may be referenced for further detail.

Let $S$ be the subset in the affine chart $Z_0 \neq 0$ of $\mathbb{C}P^n$ consisting of $n$-tuples of affine coordinates $(z_1, ..., z_n)$ satisfying

$$(z_1, ..., z_n) = (e^{i\theta_1}, ..., e^{i\theta_n}) \text{ with } (\theta_1, ..., \theta_n) \in [0, 2\pi]^n.$$  

We equip $\mathbb{C}P^n$ with the Fubini-Study metric $\mu = \mu_{FS}$ associated to a particular Kähler form $\omega$, denoting the Fubini-Study distance between two points $z, w \in \mathbb{C}P^n$ as $d_{FS}(z, w)$, and furthermore endow $S$ with the

∗The views expressed in this article are the author’s own and not those of the U.S. Merchant Marine Academy, the Maritime Administration, the Department of Transportation, or the United States government.
†The authors acknowledge the support of NSF grant DMS-1820731
measure
\[ \mu_S(z) = \frac{1}{(2\pi)^n} \frac{dz_1}{z_1} \cdots \frac{dz_n}{z_n} = \frac{1}{(2\pi)^n} d\theta_1 \cdots d\theta_n. \]

For \( n \geq 2 \) let \( x \in [0, 1] \) be a function of \( z, D \in \mathbb{C}P^n \)
\[ x = x(z, D) := (\cos(d_{FS}(z, D)))^2 = \cos^2 r, \]
h(\( x \)) be \( L^1([0, 1]) \). Consider the integral
\[ I(D; h) = \int_S h(x(z, D)))\mu_S(z). \]

For \( W_m \in \mathbb{C} \) and \( z_m \in S \), write \( W_m = r_m e^{x_m} \) and \( Z_m = e^{t_m} \) for each integer \( m \) from 0 to \( n \). With this, set
\[ \mathcal{X} := \sum_{m=0}^{n} Z_m W_m = \sum_{m=0}^{n} r_m e^{i(\theta_m - \varphi_m)}. \]

For \( n = 2 \) we may view \( \mathcal{X} \) as the endpoint of an 3--step random walk in two dimensions, where step number \( m \) is of length \( r_m \), and the walk occurs in the direction with angle \( (\theta_m - \varphi_m) \in [-\pi, \pi] \). The directions are viewed as independent and identically distributed random variables, and the probability distribution of each is uniform on the interval \( [-\pi, \pi] \). Let
\[ d(D) = |W_0| + \cdots + |W_n| \]
be the \( \ell^1 \) norm of \( D \). Let \( \mathcal{Y} \) be the random variable which is the distance of \( \mathcal{X} \) to the origin. It can be shown ([Wat44], page 420) that for any \( u \in [0, d(D)] \) the cumulative distribution \( F_D(u) \) of \( \mathcal{Y} \) is given by
\[ \text{Prob}(\mathcal{Y} \leq u) = F_D(u) = u \int_0^{\infty} J_1(ut) \prod_{m=0}^{2} J_0(r_m t) dt. \]

\( F_D(u) = 0 \) for \( u < 0 \) and \( F_D(u) = 1 \) for \( u > d(D) \), where \( J_0 \) and \( J_1 \) are the classical \( J \)-Bessel functions of order zero and one respectively. The probability density function \( f_D(u) \) of \( \mathcal{Y} \) is obtained by differentiating \( F_D(u) \) with respect to \( u \). One may employ formula 8.472.1 of [GR07] to obtain that for \( u \in [0, d(D)] \) the function \( f_D(u) \) is given by
\[ f_D(u) = \int_0^{\infty} ut J_0(ut) \prod_{m=0}^{2} J_0(r_m t) dt; \]

note that \( f_D(u) \) vanishes outside \([0, d(D)]\) and define \( c(D) := \sqrt{(n + 1)(|W_0|^2 + \cdots + |W_n|^2)}. \)

Consideration of the above, further manipulation of the integral \( I(D; h) \) and letting \( v = u/c(D) \) yields the change of variables formula
\[ \int_S h(x(z, D)))\mu_S(z) = c(D)^2 \int_0^{d(D)/c(D)} h(v^2) \left( \int_0^{\infty} vt J_0(c(D)ut) \prod_{m=0}^{2} J_0(r_m t) dt \right) dv. \]
where the inner integral is precisely the object of our examination.

Applying this change of variables formula to the integral \( \int_S (1 - x(z, D))^\mu \, \mu_s(z) \) one has that

\[
\int_S (1 - x(z, D))^\mu \, \mu_s(z) = c(D)^2 \int_0^{\frac{2\pi}{\mu}} (1 - v^2)^\mu v \left( \int_0^{\infty} t J_0(c(D)vt) \prod_{m=0}^2 J_0(r_m t) dt \right) dv. 
\] (1)

an identity which will later prove useful.

1.2 Logarithmic Mahler measure

The Mahler measure \( M(P) \) of a \( (n+1) \)-variable complex polynomial \( P \) is defined by

\[
M(P) = \exp \left( \frac{1}{(2\pi)^{n+1}} \int_0^{2\pi} \cdots \int_0^{2\pi} \log \left| P(e^{i\theta_0}, e^{i\theta_1}, \ldots, e^{i\theta_n}) \right| d\theta_0 d\theta_1 \cdots d\theta_n \right).
\]

The logarithmic Mahler measure is defined as \( m(P) := \log M(P) \). Let

\[
P_D(Z_0, Z_1, \ldots, Z_n) := W_0 Z_0 + W_1 Z_1 + \cdots + W_n Z_n
\]

be a linear form in \( n+1 \) complex variables, \( D := (W_0, \ldots, W_n) \) is its tuple of coefficients and \( c(D) \) be as above.

1.3 Our main results

The main results of our paper are the following.

**Theorem 1.** Let \( I := \int_0^{\infty} t J_0(at) \prod_{m=0}^2 J_0(r_m t) dt, S := \{ r_0 \pm r_1 \pm r_2 \} \), and \( b \in S \). Then

(a) For any \( v > 0, a := c(D)v \) with \( a \notin S \), the integral \( I \) is finite, and

(b) For \( b \in S, I = O \left( \sum_{a \in S} \log |a - b| \right) \); for \( 0 < a < b \) with \( a \to b^- \) and for \( 0 < b < a \) with \( b \to a^- \).

**Corollary 1.1.** With notation as in 1.2, let

\[
a(n, k, D) = \sum_{l_0 + \cdots + l_n = k, l_m \geq 0} \left( \begin{array}{c} k \\ l_0, l_1, \ldots, l_n \end{array} \right)^2 |W_0|^{2k} \cdots |W_n|^{2k}
\]

where \( \left( \begin{array}{c} k \\ l_0, l_1, \ldots, l_n \end{array} \right) \) is the multinomial coefficient. Then for \( n = 2 \), the logarithmic Mahler measure \( m(P_D) \) of the linear polynomial \( P_D \) is given by

\[
m(P_D) = \log c(D) - \frac{1}{2} \sum_{j=1}^\infty \sum_{k=0}^j \binom{j}{k} \frac{(-1)^k a(n, k, D)}{c(D)^{2k}}
\]

**Corollary 1.2.** Set \( H_0 := 0 \) and for any integer \( l \geq 1 \), let \( H_l = 1 + \frac{1}{2} + \cdots + \frac{1}{l} \), and for any integer \( l \geq 0 \) define
\[ S_D(l) := \sum_{j=0}^{\infty} \frac{2j + l}{j(j + l)} \sum_{k=0}^{j} \binom{j + l + k - 1}{k} (-1)^k a(n, k, D) c(D)^{2k}. \]

(i) For \( n = 2 \) and all \( l \geq 0 \) with \( D = r(1, 1, \ldots, 1) \) for some \( r \neq 0 \), we have that
\[
m(P_D) = \log c(D) - \frac{1}{2} H_l - \frac{1}{2} S_D(l). \tag{3}
\]

(ii) Additionally, if \( l \in \{0, 1\} \), then (2) holds for any \( D \).

Note that the paper [CJS20] treats the case of \( n \geq 3 \) and [Bor+12] addresses the case of \( D = r(1, 1, \ldots, 1) \) for \( r \neq 0 \). The case when \( n = 1 \) is met in standard complex analysis texts using Jensen’s formula (see [Lan05], p. 345).

### 1.4 Applications

One key result shedding light on our findings is Theorem 1 of [Smy81]: If an \( n+1 \) variable complex polynomial \( P(Z_0, \ldots, Z_n) \in \mathbb{C}[Z_0, \ldots, Z_n] \) splits in \( N \) linear factors as \( \prod_{j=1}^{N} (C_j + W_0 Z_0 + W_1 Z_1 + \ldots + W_n Z_n) \) where \( C_j, W_{ij} \in \mathbb{C} \) and for each \( j \) there is a coefficient \( W_{kj} \) satisfying
\[
|W_{kj}| \geq \sum_{i \neq k} |W_{ij}|
\]

Then
\[
M(P) = \prod_{j=1}^{N} |W_{kj}|.
\]

We are primarily concerned with the case \( N = 1 \) and \( n = 2 \), i.e. a linear polynomial in 3 variables, in which case we ignore the subscript \( j \) and note that the logarithmic Mahler measure will then be given by \( m(P) = \log |W_k| \). Employing this result together with the computational tool provided by Joshua Friedman yields the following observations.

#### 1.4.1 Application 1

Let \( D = (W_0, W_1, W_2) \), \( |W| = \max\{|W_0|, |W_1|, |W_2|\} \) and suppose \( |W| \) is greater than or equal to the sum of the smaller two elements of \( \{|W_0|, |W_1|, |W_2|\} \). Then the conditions of [Smy81] hold and we have the the formula
\[
S_D(1) = \log \left( \frac{3(|W_0|^2 + |W_1|^2 + |W_2|^2)}{|W|^2} \right) - 1. \tag{4}
\]

**Example 1.1.** If \( D = (1, 2, 1) \), then \( S_D(1) = \log \left( \frac{9}{4} \right) - 1 \).

\( P_D \) satisfies the assumptions of [Smy81] Theorem 1. Thus \( m(P_D) = \log |2| \). Corollary 1.1 then implies that \( S_D(1) = \log \left( \frac{9}{4} \right) - 1 \approx 0.5040 \). Indeed the explicit calculation (given in the Appendix) yields \( S_D(1) = 0.5040 \).

**Example 1.2.** If \( D = (4, 1, 1) \), then \( S_D(1) = \log \left( \frac{27}{5} \right) - 1 \).
Example 1.3. \( \log \) and therefore continuous. Poisson’s formal expansion of \( J \) implies \( S_D(1) = \log \left( \frac{2\pi}{8} \right) - 1 \approx 0.21639 \) and explicit calculation yields \( S_D(1) \approx 0.2164 \).

1.4.2 Application 2

On the other hand one may obtain an expression for \( \log(\alpha) \) by setting \( \alpha = \left( 3 \left( |W_0|^2 + |W_1|^2 + |W_2|^2 \right) \right) / |W|^2 \).

Note that \( \left( |W_0|^2 + |W_1|^2 + |W_2|^2 \right) / |W|^2 \in (1, 2) \) and a suitable choice of \( D \) will yield any \( \alpha \in (3, 6) \). In other words, whenever \( D = (W_0, W_1, W_2) \) satisfies Smyth’s condition we have

\[
\log \left( \frac{\alpha}{\epsilon} \right) = S_{(W_0, W_1, W_2)}(1).
\]

Example 1.3. \( \log \left( \frac{a}{\pi} \right) = S_{(m, m, 2m)}(1) \), for any integer \( m \in \mathbb{Z}_{>0} \).

Explicit computation yields \( S_{(100, 100, 200)}(1) \approx 0.50398 \) so that \( S_{(1, 2, 1)}(1) = S_{(100, 100, 200)}(1) \) by Example 1.1.

Example 1.4. \( \log \left( \frac{\alpha}{\pi} \right) = S_{(2, \sqrt{5}, 6)}(1) \).

We have \( \log \left( \frac{3}{\pi} \right) \approx 0.3863 \) and explicit computation yields \( S_{(2, \sqrt{5}, 6)} \approx 0.38629 \).

Example 1.5. \( \log \left( \frac{\alpha}{\pi} \right) = S_{\left( 5 - \frac{\sqrt{100/3}}{2}, 3 + \frac{\sqrt{100/3}}{10} \right)}(1) \).

We have \( \log \left( \frac{5}{\pi} \right) \approx 0.6094 \) and explicit computation yields \( S_{\left( 5 - \frac{\sqrt{100/3}}{2}, 3 + \frac{\sqrt{100/3}}{10} \right)} \approx 0.60932 \).

1.5 Organization of the paper

In Section 2 we include relevant facts from the literature. In Section 3 we examine the integral in question, with the goal of expressing it in a manner suitable to our purpose, and in Section 4 we utilize the findings of Section 3 to establish our main results. Finally in Section 5 we present numerical evaluations which confirm our theoretical findings.

1.6 Acknowledgements

We express our heartfelt gratitude to Professors Lejla Smajlović and Gautam Chinta, whose constructive feedback proofreading our work was invaluable. We are especially indebted to Professor Jay Jorgenson, whose patience, expertise, enthusiasm and encouragement enabled this project’s completion.

2 Background

2.1 J-Bessel functions

Recall that \( J_0(t) \) is defined as the solution to a particular differential equation\( [\text{Har09}] \), hence differentiable and therefore continuous. Poisson’s formal expansion of \( J_0(t) \) \( [\text{Wat44}], \text{p.194} \) for large arguments (i.e., \( |t| \geq 45, \text{[Har09]} \)) is given by

\[
J_0(t) = \sqrt{\frac{2}{\pi t}} \left[ \cos \left( t - \frac{\pi}{4} \right) P_0(t) + \sin \left( t - \frac{\pi}{4} \right) Q_0(t) \right]
\] (5)

We use this expansion for \( t \geq 1 \), without loss of generality. Stieltjes discovered very useful estimates for the series \( P_0(t) \) and \( Q_0(t) \), for a finite number of terms as follows \( [\text{Wat44}], \text{p.208} \).
\[ P_0(t) = 1 - \frac{1^2 \cdot 3^2}{2!(8t)^2} + \cdots + (-1)^{p-1} \frac{1^2 \cdot 3^2 \cdot 5^2 \cdots (4p-5)^2}{(2p-2)!(8t)^{2p-2}} + (-1)^p \theta_1 \frac{1^2 \cdot 3^2 \cdot 5^2 \cdots (4p-1)^2}{(2p)!(8t)^{2p}} \]  

\[ Q_0(t) = -\frac{1^2}{18t} + \frac{1^2 \cdot 3^2 \cdot 5^2 \cdot 7^2}{3!(8t)^3} - \cdots + (-1)^p \frac{1^2 \cdot 3^2 \cdot 5^2 \cdots (4p-3)^2}{(2p-1)!(8t)^{2p-1}} + (-1)^{p+1} \theta_2 \frac{1^2 \cdot 3^2 \cdot 5^2 \cdots (4p+1)^2}{(2p+1)!(8t)^{2p+1}} \]

where \(0 < \theta_1, \theta_2 < 1\), and \(p\) is any non-negative integer. In our calculations, we will utilize the expressions for \(p = 1\), namely

\[ P_0(t) = 1 - \theta_1 \frac{9}{128t^2} \quad \text{and} \quad Q_0(t) = -\frac{1}{8t} + \theta_2 \frac{225}{3072} \cdot \frac{1}{t^3} \]

By [Sze39] Theorem 7.31.2, \(J_0\) is bounded. In particular we have

\[ |J_0(c(D)v)| \leq \sqrt{\frac{2}{\pi c(D)v}} \quad \text{for all } t \geq 1. \]

### 2.2 Integral Evaluations involving \(J\)-Bessel functions

A compendium of various integral evaluations is given in [GR07]. 6.699-1 and 6.699-2 (p.731) are two which are especially useful. Letting \(a := c(D)v\), we employ the case where \(\lambda = -\frac{1}{2}\) and \(\nu = 0\) in which case we have

\[ \text{For } 0 < b < a, \int_0^\infty t^{-\frac{1}{2}} J_0(at) \sin(bt) dt = 2^{\frac{1}{2}} a^{-\frac{3}{2}} b \frac{\Gamma \left( \frac{1}{4} \right)}{\Gamma \left( \frac{3}{4} \right)} F \left( \frac{3}{4}, \frac{3}{4}; \frac{1}{2}; \left( \frac{b}{a} \right)^2 \right), \]

\[ \text{For } 0 < a < b, \int_0^\infty t^{-\frac{1}{2}} J_0(at) \sin(bt) dt = b^{\frac{1}{2}} \frac{\Gamma \left( \frac{1}{2} \right)}{\Gamma (1)} \sin \left( \frac{\pi}{4} \right) F \left( \frac{3}{4}, \frac{1}{4}; 1; \left( \frac{a}{b} \right)^2 \right), \]

\[ \text{For } 0 < b < a, \int_0^\infty t^{-\frac{1}{2}} J_0(at) \cos(bt) dt = 2^{\frac{1}{2}} a^{-\frac{3}{2}} b \frac{\Gamma \left( \frac{1}{4} \right)}{\Gamma \left( \frac{3}{4} \right)} F \left( \frac{1}{4}, \frac{1}{4}; \frac{1}{2}; \left( \frac{b}{a} \right)^2 \right), \]

\[ \text{For } 0 < a < b, \int_0^\infty t^{-\frac{1}{2}} J_0(at) \cos(bt) dt = b^{\frac{1}{2}} \frac{\Gamma \left( \frac{1}{2} \right)}{\Gamma (1)} \cos \left( \frac{\pi}{4} \right) F \left( \frac{3}{4}, \frac{3}{4}; 1; \left( \frac{a}{b} \right)^2 \right). \]

where \(\Gamma\) denotes the Gamma function and \(F\) denotes the Gaussian hypergeometric series (function). Note the given arguments of the respective functions yield that the above evaluations are indeed finite.

### 2.3 The Ramanujan asymptotic formula for the Gaussian hypergeometric series function

A result due to Ramanujan provides an invaluable aid in describing the asymptotic behavior of the above integrals as \(a \to b^-\) and as \(b \to a^-\), in the situations where \(0 < a < b\) and \(0 < b < a\) respectively (that is, as \(v \to \frac{b}{|c(D)|}\)). Given arguments \(\alpha, \beta, \gamma\) and \(z\), of the Gaussian hypergeometric series \(F\), \(B(\alpha, \beta)\) denotes the Euler Beta function,
\[ B(\alpha, \beta) := \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}. \]

Define \( R := R(\alpha, \beta) = -\psi(\alpha) - \psi(\beta) - 2\gamma_{EM}, \psi(\alpha) = \frac{\Gamma'(\alpha)}{\Gamma(\alpha)} (\psi(\beta) \text{ is defined analogously}) \)

where \( \gamma_{EM} \) denotes the Euler-Mascheroni constant given by

\[ \gamma_{EM} = \lim_{n \to \infty} \left( \sum_{k=1}^{n} k^{-1} - \log n \right) = .57721566 \ldots \]

The above arguments \( \alpha, \beta \) and \( \gamma \) of \( F \) satisfy \( \alpha + \beta = \gamma \), so that as \( a \to b \), the argument \( 0 < z < 1 \) of \( F \) in the above evaluations approaches 1 and one has the Ramanujan asymptotic formula (BP98, p.96)

\[ F(\alpha, \beta; \alpha + \beta; z) = \frac{1}{B(\alpha, \beta)} \left[ R - \log(1-z) + O((1-z)\log(1-z)) \right] \tag{14} \]

## 3 Preliminaries

We now set ourselves to the task of investigating the integral \( \int_{0}^{\infty} tJ_{0}(at) \prod_{m=0}^{2} J_{0}(r_{m}t)dt \). We’ll first require a bound on the auxiliary function \( Q_{0}(t) \), which in turn will yield information regarding a possible bound on the given integral.

**Lemma 3.1.** For all \( t \geq 1 \), we have \( |Q_{0}(t)| \leq C_{2}/t \) where \( C_{2} = \frac{669}{3072} \).

**Proof.**

By \( (8) \), \( |Q_{0}(t)| \leq \frac{1}{8t} + \frac{\tilde{C}}{t^{3}} \leq \frac{1}{8t} + \frac{\tilde{C}}{t} \) for all \( t \geq 1 \), where \( \tilde{C} = \frac{225}{3072} \). Hence \( |Q_{0}(t)| \leq \frac{C_{2}}{t} \), as claimed. \( \Box \)

Utilizing \( (8) \) and Lemma 3.1 we now obtain a partial bound for the integral in question.

**Lemma 3.2.** Suppose \( t \geq 1 \), and \( v > 0 \). Then

\[ \int_{1}^{\infty} tJ_{0}(cDvt) \prod_{m=0}^{2} J_{0}(r_{m}t)dt = \int_{1}^{\infty} tJ_{0}(cDvt) \prod_{m=0}^{2} \sqrt{\frac{2}{\pi r_{m}t}} \cos(r_{m}t - \pi/4)dt + \int_{1}^{\infty} B(t)dt, \]

where \( B(t) \) is a function satisfying \( |B(t)| \leq \left( \prod_{m=0}^{2} \sqrt{\frac{2}{\pi r_{m}}} \right) \tilde{C}t^{-\frac{3}{2}} \) and \( \tilde{C} \) is a constant given by \( \tilde{C} = M(C_{4} + C_{5}) \) such that \( M \) is the least upper bound of \( |J_{0}(cDvt)| \) and \( C_{4} \) and \( C_{5} \) are explicit real-valued constants. In particular, we have that \( \int_{1}^{\infty} B(t)dt < \infty. \)
Proof. By (5) and (8), we have
\[
\prod_{m=0}^{2} J_0(r_m t) = \prod_{m=0}^{2} \left( \frac{2}{\pi r_m t} \cos(r_m t - \pi/4)P_0(r_m t) + \sin(r_m t - \pi/4)Q_0(r_m t) \right),
\]
\[
= \prod_{m=0}^{2} \left( \frac{2}{\pi r_m t} \right) \left( \prod_{m=0}^{2} (\cos(r_m t - \pi/4)P_0(r_m t) + \sin(r_m t - \pi/4)Q_0(r_m t) \right). \quad (*)
\]

Expanding the terms of the inner product in (*) we obtain an expression consisting of a sum of 8 terms, call this \(T_2\), each of which is the product of some combination of cosines and sines, along with their respective auxiliary functions \(P_0(t)\) and \(Q_0(t)\). The first of these 8 terms, utilizing (8) yields another sum of 8 terms, denote this as \(T_1\), each of which is the product of only cosine terms with their respective auxiliary functions.

Now define
\[
B_1(t) \text{ as } \left( \prod_{m=0}^{2} \frac{2}{\pi r_m t} \right) \cdot T_1,
\]
\[
B_2(t) \text{ as } \left( \prod_{m=0}^{2} \frac{2}{\pi r_m t} \right) \cdot T_2 \text{ and }
\]
\[
B(t) \text{ as } tJ_0(c(D)vt) \cdot (B_1(t) + B_2(t)).
\]

By construction we then get
\[
\prod_{m=0}^{2} J_0(r_m t) = \left( \prod_{m=0}^{2} \frac{2}{\pi r_m t} \right) \left( \prod_{m=0}^{2} \cos \left( r_m t - \frac{\pi}{4} \right) \right) + (B_1(t) + B_2(t)).
\]

Thus by the definition of \(B(t)\),
\[
\int_{1}^{\infty} tJ_0(c(D)vt) \prod_{m=0}^{2} J_0(r_m t) dt = \int_{1}^{\infty} tJ_0(c(D)vt) \prod_{m=0}^{2} \frac{2}{\pi r_m t} \cos(r_m t - \pi/4) dt + \int_{1}^{\infty} B(t).
\]

This proves the first part of our claim. Applying standard inequalities and calculation yields
\[
|B_1(t)| \leq \prod_{m=0}^{2} \frac{2}{\pi r_m} C_4 t^{-\frac{3}{2}} \text{ and } |B_2(t)| \leq \prod_{m=0}^{2} \frac{2}{\pi r_m} C_5 t^{-\frac{3}{2}},
\]
where \(C_4 = \frac{611289}{2097152} \approx .2914\) and \(C_5 = \frac{954044363}{1073741874} \approx .8885\). Hence
\[
|B(t)| = t |J_0(c(D)vt)| |B_1(t) + B_2(t)|,
\]
\[
\leq Mt \left( \prod_{m=0}^{2} \frac{2}{\pi r_m} (C_4 + C_5) t^{-\frac{3}{2}} \right),
\]
\[
= \left( \prod_{m=0}^{2} \frac{2}{\pi r_m} \right) C t^{-\frac{3}{2}}.
\]

so that \(|B(t)| \leq \left( \prod_{m=0}^{2} \frac{2}{\pi r_m} \right) C t^{-\frac{3}{2}}\) and \(\int_{1}^{\infty} B(t) dt < \infty\) then follows. \(\square\)
Lemma 3.3. With notation as above we have

\[
\int_1^\infty t J_0(c(D)vt) \sum_{m=0}^2 \sqrt{\frac{2}{\pi r_m t}} \cos(r_m t - \pi/4) dt + \int_1^\infty B(t) dt = 4 \sum_{i=1}^4 \left( \alpha_i \int_1^\infty t^{-\frac{1}{2}} J_0(c(D)vt) \cos(a_i t) dt + \beta_i \int_1^\infty t^{-\frac{1}{2}} J_0(c(D)vt) \sin(a_i t) dt \right)
\]

where

\[
\begin{align*}
\alpha_1 &= \beta_4 = -\frac{1}{2} \sum_{m=0}^2 \sqrt{\frac{1}{\pi r_m}}, \\
\alpha_2 &= \alpha_3 = \alpha_4 = \beta_1 = \beta_2 = \beta_3 = \frac{1}{2} \sum_{m=0}^2 \sqrt{\frac{1}{\pi r_m}}, \\
\end{align*}
\]

\[\begin{align*}
a_1 &= r_0 + r_1 + r_2, & a_2 &= r_0 + r_1 - r_2, & a_3 &= r_0 - r_1 + r_2, & a_4 &= r_0 - r_1 - r_2.
\end{align*}\]

Proof. An application of elementary trigonometric identities gives us

\[
\prod_{m=0}^2 \cos(r_m t - \pi/4) = 4 \sum_{m=1}^4 \left( \hat{\alpha}_i \cos(a_i t) + \hat{\beta}_i \sin(a_i t) \right),
\]

where \(\hat{\alpha}_1 = \hat{\beta}_4 = -\frac{\sqrt{2}}{8}, \hat{\alpha}_2 = \hat{\alpha}_3 = \hat{\alpha}_4 = \hat{\beta}_1 = \hat{\beta}_2 = \hat{\beta}_3 = \frac{\sqrt{2}}{8}\) and \(a_i\) is as in the claim. Hence

\[
\prod_{m=0}^2 \sqrt{\frac{2}{\pi r_m t}} \cos(r_m t - \pi/4) = t^{-\frac{1}{2}} \sum_{m=1}^4 \left( \alpha_i \cos(a_i t) + \beta_i \sin(a_i t) \right),
\]

where \(\alpha_i\) and \(\beta_i\) are as in the statement of the claim, and the result follows.

We’re now in a position to show the above integral is finite. Note we may suppose \(b > 0\) without loss of generality, for \(b \in \{a_i : i = 1, \ldots, 4\}\).

Lemma 3.4. Assume \(v > 0, b > 0\) for \(b \in \{a_i : i = 1, \ldots, 4\}\). Let \(a := c(D)v\), then

\[
I_1(v; b, c(D)) := \int_1^\infty t^{-\frac{1}{2}} J_0(at) \cos(bt) dt \quad \text{and} \quad I_2(v; b, c(D)) := \int_1^\infty t^{-\frac{1}{2}} J_0(at) \sin(bt) dt.
\]

(i) For \(j \in \{1, 2\}\) and any \(v \neq \frac{b}{c(D)}\), \(i.e., a \neq b\) the integrals which define \(I_j(v; b, c(D))\) converge.

(ii) Furthermore, for \(0 < a < b\) with \(a \rightarrow b^-\) and \(0 < b < a\) with \(b \rightarrow a^-\), we have that \(I_j = O(\log |a - b|)\).

Proof. (i) By (10) through (13), the integrals \(\int_0^\infty t^{-\frac{1}{2}} J_0(at) \cos(bt) dt\) and \(\int_0^\infty t^{-\frac{1}{2}} J_0(at) \sin(bt) dt\) are finite, proving (i).

(ii) We prove the claim for \(j = 1\). The case of \(j = 2\) is completely analogous.

Set \(D_1 := \int_0^1 t^{-\frac{1}{2}} J_0(at) \cos(bt) dt\), so that by the additivity of the integral

\[
I_1(v, c(D), b) = \int_0^\infty t^{-\frac{1}{2}} J_0(at) \cos(bt) dt - D_1.
\]
Suppose $0 < b < a$. Then we have
\[
I_1(v, c(D), b) = 2^{-\frac{1}{2}}(a)^{-\frac{1}{2}}\Gamma\left(\frac{1}{2}\right) \frac{1}{\Gamma\left(\frac{3}{4}\right)} F \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}; \left(\frac{b}{a}\right)^2\right) - D_1, \text{ by } [12],
\]
and for $b \to a^-$, we get
\[
= 2^{-\frac{1}{2}}(a)^{-\frac{1}{2}}\Gamma\left(\frac{1}{2}\right) \frac{1}{\Gamma\left(\frac{3}{4}\right) B\left(\frac{1}{4}, \frac{1}{4}\right)} \left[ R - \log \left(1 - \left(\frac{b}{a}\right)^2\right) + O \left(\left(1 - \left(\frac{b}{a}\right)^2\right) \log \left(1 - \left(\frac{b}{a}\right)^2\right)\right)\right] - D_1, \text{ by } [14],
\]
\[
= O\left(\log \left(1 - \left(\frac{b}{a}\right)^2\right)\right) = O\left(\log |a - b|\right).
\]
On the other hand, if $0 < a < b$, then [13] yields
\[
I_1(v, c(D), b) = b^{-\frac{1}{2}}\frac{\cos\left(\frac{\pi v}{4}\right)}{\Gamma\left(\frac{1}{2}\right)} F \left(\frac{1}{4}, \frac{3}{4}, 1; \left(\frac{a}{b}\right)^2\right) - D_1, \text{ and for } a \to b^-
\]
\[
= b^{-\frac{1}{2}}\frac{\cos\left(\frac{\pi v}{4}\right)}{\Gamma\left(\frac{1}{2}\right)} \left[ R - \log \left(1 - \left(\frac{a}{b}\right)^2\right) + O \left(\left(1 - \left(\frac{a}{b}\right)^2\right) \log \left(1 - \left(\frac{a}{b}\right)^2\right)\right)\right] - D_1, \text{ by } [14],
\]
\[
= O(\log \left(1 - \left(\frac{a}{b}\right)^2\right)) = O\left(\log |a - b|\right).
\]
so that in both cases we have $I_1(v, c(D), b) = O\left(\log |a - b|\right)$, as claimed. 

\section{Proof of Main Results}

\subsection{Proof of Theorem 1}

\begin{proof}
For all $t \geq 1, v > 0$ with $a := c(D)v$ and $a \neq a_i$, we have
\[
\int_1^\infty tJ_0(at) \left(\prod_{m=0}^2 J_0(r_m t)\right) dt = \int_1^\infty tJ_0(at) \left(\prod_{m=0}^2 \sqrt{\frac{2}{\pi r_m t}} \cos(r_m t - \pi/4) dt + \int_1^\infty B(t) dt\right), \text{ by Lemma 3.2,}
\]
\[
= \sum_{i=1}^4 \left(\alpha_i \int_1^\infty t^{-\frac{1}{2}} J_0(at) \cos(a_i t) dt + \beta_i \int_1^\infty t^{-\frac{1}{2}} J_0(at) \sin(a_i t) dt\right) + \int_1^\infty B(t) dt, \text{ by Lemma 3.3.}
\]
\[
\text{(**)}
\]
By Lemma 3.4 the term (**) is also finite, so $\int_1^\infty tJ_0(at) \left(\prod_{m=0}^2 J_0(r_m t)\right) dt < \infty$. By [6], the continuous function $J_0(at) \prod_{m=0}^2 J_0(r_m t)$ is bounded on the interval $[0, 1]$ so that $tJ_0(at) \prod_{m=0}^2 J_0(r_m t)$ is continuous and bounded, hence has finite integral on $[0, 1]$, and the additivity of the integral implies (a). For $a = a_i$, one applies Lemma 3.4 (ii) to Lemma 3.3 and (b) immediately follows.
\end{proof}

Armed with Theorem 1, we are now ready to establish Corollary 1.1.
4.2 Proof of Corollary 1.1

Proof. By [CJS20] Equation (44) one has the estimate

\[
2m(P_D) - 2 \log c(D) + \sum_{j=1}^{N} \frac{1}{j} \sum_{k=0}^{j} \left( \frac{j}{k} \right) \frac{(-1)^k a(n, k, D)}{c(D)^{2k}} \leq \sum_{j=N+1}^{\infty} \frac{1}{j} \int_{S} (1 - x(z, D))^j \mu_s(z). \tag{15}
\]

It suffices to derive a bound for the RHS of (15). Set \( c_b := \min \left\{ \frac{b}{c(D)}, \frac{c(D)}{b} \right\} \), so \( c_b \in (0, 1) \). By equation (1) we have

\[
\int_{S} (1 - x(z, D))^j \mu_s(z) = c(D)^2 \int_{0}^{1} \left( 1 - v^2 \right)^j v \left( \int_{0}^{\infty} tJ_0(c(D)vt) \prod_{m=0}^{2} J_0(r_m t) dt \right) dv,
\]

\[
= c(D)^2 \int_{0}^{1} \left( 1 - v^2 \right)^j v \sum_{b \in S} \log |v - c_b| \left( \frac{\int_{0}^{\infty} tJ_0(c(D)vt) \prod_{m=0}^{2} J_0(r_m t) dt}{\sum_{b \in S} \log |v - c_b|} \right) dv,
\]

\[
\leq A_D c(D)^2 \int_{0}^{1} \left( 1 - v^2 \right)^j v \log |v - c_b| dv \quad \text{by Theorem 1, for some } A_D > 0,
\]

\[
\leq A_D c(D)^2 \int_{0}^{1} \left( 1 - v^2 \right)^j v \sum_{b \in S} \log |v - c_b| dv.
\]

We show that for each \( j \) and for any \( b \in S \),

\[
\int_{0}^{1} \left( 1 - v^2 \right)^j v \log |v - c_b| dv \leq \frac{\tilde{A}}{j^{\frac{3}{2}}},
\]

for some real-valued \( \tilde{A} > 0 \), which yields the result. Note that both \( \log |v - c_b| \) and \( (1 - v^2)^j v \) are in \( L^2([0, 1]) \) and a change of variables yields that the square of the latter norm is

\[
\int_{0}^{1} (1 - v^2)^{2j} v^2 = \frac{1}{2} \int_{0}^{1} (1 - u)^{2j} u^\frac{1}{2} du.
\]

Utilizing [GR07] §3.196.3 with \( a = 0, b = 1, \mu = \frac{3}{2} \) and \( \nu = 2j + 1 \) and applying Cauchy Schwarz we obtain

\[
\int_{0}^{1} (1 - v^2)^j v \log |v - c_b| dv \leq \sqrt{\frac{\Gamma \left( \frac{3}{2} \right)}{2}} \cdot \frac{1}{j^{\frac{3}{2}}} \cdot \tilde{A}_1 = \frac{\tilde{A}}{j^{\frac{3}{2}}},
\]

where \( \tilde{A}_1 \) denotes the \( L^2 \) norm of \( \log |v - c_b| dv \) and \( \tilde{A} = \sqrt{\frac{\Gamma \left( \frac{3}{2} \right)}{2}} \cdot \tilde{A}_1 > 0 \), as claimed.

4.3 Proof of Corollary 1.2

Proof. Considering [2] for the case \( l = 1 \), [CJS20] equations (52), (53) and the fact \( \frac{d(D)}{c(D)} \leq 1 \), we see that

\[
|m(P_D) - E_2(N; n, D)| \leq \frac{C}{\sqrt{N}} c(D)^2 \int_{0}^{1} (1 - v^2)^{-\frac{3}{2}} v^\frac{1}{2} \left( \int_{0}^{\infty} tJ_0(c(D)vt) \prod_{m=0}^{2} J_0(r_m t) dt \right) dv, \tag{16}
\]
where $E_2(N; n, D)$ is the RHS of the formulation in [2], with $C$ a constant. For the case $l \geq 2$ one must assume $D \neq r(1, 1, 1)$ and [CJS20] equations (55) and (56) yield

$$\left| m(P_D) - \log c(D) + \frac{1}{2} H_l + \frac{1}{2} S_D(l) \right| \leq \frac{1}{2} \sum_{j=N+1}^{\infty} \frac{2j + 1}{j(j+1)} \int_{S} \left| P_j^{(l-1,0)}(2x(z, D) - 1) \right| \mu_S(z),$$

(17)

where

$$\int_{S} \left| P_j^{(l-1,0)}(2x(z, D) - 1) \right| \mu_S(z) \leq \frac{c(D)^2 A(D, l)}{\sqrt{2j + 1}} \int_{1}^{1} (1 - v^2)^{-\frac{1}{4}} v^{\frac{1}{4}} \left( \int_{0}^{\infty} t J_0(c(D)vt) \prod_{m=0}^{2} J_0(r_m t) dt \right) dv,$$

(18)

noting that $A(D, l)$ is a constant (as a consequence of the assumption $D \neq r(1, 1, 1)$) and $P_j^{(l-1,0)}$ is the Jacobi Polynomial (see [CJS20], page 23). In both of these cases it suffices to show that the (coincident) integrals in the RHS of (16) and (18) converge.

Let $l \geq 1$. Since $v^2 \leq v^{-\frac{1}{2}}$ on $(0, 1)$, we have

$$\int_{0}^{1} (1 - v^2)^{-\frac{1}{4}} v^{-\frac{1}{4}} \left( \int_{0}^{\infty} t J_0(c(D)vt) \prod_{m=0}^{2} J_0(r_m t) dt \right) dv,$$

$$= \int_{0}^{1} (1 - v^2)^{-\frac{1}{4}} v^{-\frac{1}{2}} \sum_{beS} \log |v - c_b| \left( \frac{\int_{0}^{\infty} t J_0(c(D)vt) \prod_{m=0}^{2} J_0(r_m t) dt}{\sum_{beS} \log |v - c_b|} \right) dv,$$

$$\leq A_D \int_{0}^{1} (1 - v^2)^{-\frac{1}{4}} v^{-\frac{1}{2}} \sum_{beS} \log |v - c_b| dv,$$

for some $A_D > 0$ by Theorem 1.

By the Cauchy-Schwarz inequality, for each $b \in S$, the integral $\int_{0}^{1} (1 - v^2)^{-\frac{1}{4}} v^{-\frac{1}{2}} \log |v - c_b| dv$ converges, yielding the claim for $l \geq 1$. The case $l = 0$ follows from the case $l = 1$ and a manipulation of the inner sum in [CJS20] equation (7).

5 Appendix: Numerical Evaluations by Joshua Friedman

5.1 Introduction

The goal of this appendix is to compute the terms $a(n, k, D)$ and $S_D(l)$ for the case of $n = 2$ using high-precision computation software. Recall that

$$a(n, k, D) = \sum_{l_0 + \cdots + l_n = k, \ell_m \geq 0} \left( \binom{k}{l_0, l_1, \ldots, l_n} \right)^2 |W_0|^{2k} \cdots |W_n|^{2k},$$

where $(l_0, l_1, \ldots, l_n) = \binom{k}{l_0, l_1, \ldots, l_n}$, and

$$S_D(l) := \sum_{j=1}^{\infty} \sum_{k=0}^{\frac{j + l - 1}{j(j+1)}} \binom{j + l - 1}{k} \binom{(-1)^k a(n, k, D)}{c(D)^{2k}}.$$

The first step towards efficient computation is to compute the multinomial in terms of a product of binomials

$$\binom{k}{l_0, l_1, \ldots, l_n} = \binom{l_0}{l_0} \binom{l_0 + l_1}{l_1} \cdots \binom{l_0 + l_1 + \cdots + l_n}{l_n}.$$
where \( l_0 + \cdots + l_n = k \).

The second step is to compute all of the \( a(n, k, D) \) terms together. That is for all values of \( k \) up to some pre-set maximum (in our code the constant \( M \)). We use a triple for loop and compute all possible sums of three indices:

```markdown
for r in 0:M
    for s in 0:M
        for t in 0:M
            k = r+s+t
```

and each time a particular \( k \)–value appears, we add it to the running sum representing \( a(n, k, D) \).

### 5.2 Technical details and results

The table below is the first four digits of output from our algorithm. It was implemented in the language Julia using the arbitrary precision data types of BigInt and BigFloat, with a precision of 512 bits and a max of \( k \leq 200 \). Each line in the table below took approximately 13 seconds on a single core of an Intel CPU (2.6 GHZ i7)

Note that we do not certify correctness of the digits below.

| \( D \) | \( l \) \( | \) \( S_D(l) \) |
|-----|----|-------|
| (1,1,−1) | 1 | 0.5511 |
| (1,1,−1) | 2 | 0.0511 |
| (1,1,−1) | 3 | −0.28 |
| (1,2,1) | 1 | 0.5040 |
| (1,2,1) | 2 | 0.0039 |
| (1,2,1) | 3 | −0.329 |
| (4,1,1) | 1 | 0.2164 |
| (4,1,1) | 2 | −0.2836 |
| (4,1,1) | 3 | −0.6169 |

### 5.3 Julia implementation of the algorithm

Note that because Julia indexes arrays starting from one rather than zero, we had to code \( a(n, k, D) \) as \( a[k + 1] \).

```markdown
#!/usr/bin/julia
const M = 200
const n = 2
const wr = BigFloat(1/2)
const ws = BigFloat(1/2)
const wt = BigFloat(1/2)
const Wr = wr^2
const Ws = ws^2
const Wt = wt^2
const C_D = (n+1)*(wr^2+ws^2+wt^2)
const l = 2
setprecision(512)

#multinomial code from https://github.com/JuliaMath/Combinatorics.jl
#We implement the multinomial as product of binomials
function multinomial(k...)
s = 0
result = 1
@inbounds for i in k
s += i
result *= binomial(s, i)
end
result
end

#main function to compute the a(n,k,D) and S_D(l) terms
function f1()
a = zeros(BigFloat,M+1)
for r in 0:M
for s in 0:M
for t in 0:M
k = r+s+t
if k <= M
a[k+1] += Wr^(r)*Ws^(s)*Wt^(t)*(multinomial(BigInt(r),BigInt(s),BigInt(t)))^2
end
end
end

#print the first 10 a(n,k,D)
print("M equals ",M, " printing first 10 ",\n"
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