Characterization of two parameter matrix-valued BMO by commutator with the Hilbert transform

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Abstract

In this paper we prove that the space of two parameter, matrix-valued BMO functions can be characterized by considering iterated commutators with the Hilbert transform. Specifically, we prove that

\[ \|B\|_{BMO} \lesssim \|[M_B, H_1], H_2\|_{L^2(\mathbb{R}^2; \mathbb{C}^d) \rightarrow L^2(\mathbb{R}^2; \mathbb{C}^d)} \lesssim \|B\|_{BMO}. \]

The upper estimate relies on Petermichl’s representation of the Hilbert transform as an average of dyadic shifts, and the boundedness of certain paraproduct operators, while the lower bound follows Ferguson and Lacey’s proof for the scalar case.

1 Introduction

It is well known, by the work of R. Coifman, R. Rochberg, and G. Weiss [3], that the space of functions of bounded mean oscillation (BMO) can be characterized by commutators with the Hilbert transform (and in general, with the Riesz transforms). Given \( b \in BMO \), let \( M_b \) represent the multiplication operator \( M_b(f) = bf \), if \( H \) represents the Hilbert transform, defined as

\[ Hf(x) = \text{p.v.} \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(y)}{x - y} \, dy, \]

then we have

\[ \|b\|_{BMO} \lesssim \|[M_b, H]\|_{L^2 \rightarrow L^2} \lesssim \|b\|_{BMO}. \]

The study of the norm of the commutator has several implications in the characterization of Hankel operators, the problem of factorization and weak factorization of function spaces and the div-curl problem. Several extensions and generalizations have been made in different settings. In the two parameter version of this result, the upper bound was shown by S. Ferguson and C. Sadosky in [6], while the lower bound was proved by S. Ferguson and M. Lacey in [5]. The formulation in this case is the following: If \( H_i \) represents the Hilbert transform in the \( i \)-th variable, then

\[ \|b\|_{BMO} \lesssim \|[M_b, H_1], H_2\|_{L^2 \rightarrow L^2} \lesssim \|b\|_{BMO}. \]

Here, we are considering the product BMO of S.Y. Chang and R. Fefferman [2]. These results were later extended to the multi-parameter case by M. Lacey and E. Terwilleger [8].
The idea of the present work, is to obtain the same characterization in the two parameter case, for a matrix-valued BMO function. In the one parameter setting, we have the desired characterization due to S. Petermichl [12], and also F. Nazarov, G. Pisier, S. Treil and A. Volberg [10].

Consider the collection \( \mathcal{D} \) of dyadic intervals, that is \( \mathcal{D} := \{ [k2^{-j}, (k+1)2^{-j}) : j, k \in \mathbb{Z} \} \), and the collection of “shifted” dyadic intervals \( \mathcal{D}^{\alpha, r} = \{ \alpha + r[k2^j, (k+1)2^j) : k, j \in \mathbb{Z} \} \), for \( \alpha, r \in \mathbb{R} \).

Define the dyadic Haar function as \( h_I := \frac{1}{\sqrt{|I|}} (1_{I_L} - 1_{I_R}) \), where \( I_L \) and \( I_R \) represent the left and right half of the interval \( I \), respectively. Denote also \( h_{1}^{\mathcal{D}} = \frac{1}{\sqrt{|I|}} \) (non-cancelative Haar function). The family \( \{ h_I : I \in \mathcal{D} \} \) (or \( I \in \mathcal{D}^{\alpha, r} \)), is an orthonormal basis for \( L^2(\mathbb{R}; \mathbb{C}^d) \); here, for two Banach spaces \( X \) and \( Y \), we use the notation \( L^p(X; Y) \) to denote the set \( \{ f : X \to Y : \int_X \| f \|_Y^p < \infty \} \).

Define the dyadic Haar shift by \( \text{III}^{\alpha, r}(h_I) = \frac{1}{\sqrt{2}} (h_{I_L} - h_{I_R}) \), and extend to a general function \( f \) by

\[
\text{III}^{\alpha, r}(f) = \sum_{I \in \mathcal{D}} \hat{f}(I) \text{III}^{\alpha, r}(h_I) = \sum_{I \in \mathcal{D}} \hat{f}(I) \frac{1}{\sqrt{2}} (h_{I_L} - h_{I_R}).
\]

Note that \( \text{III}^{\alpha, r} \) is bounded from \( L^2(\mathbb{R}; \mathbb{C}^d) \) to \( L^2(\mathbb{R}; \mathbb{C}^d) \), with operator norm \( 1 \). As proven by Petermichl in [12], the kernel for the Hilbert transform can be written as an average of dyadic shifts, in particular

\[
K(t, x) = \lim_{L \to \infty} \frac{1}{2 \log L} \int_{L/2}^L \lim_{R \to \infty} \frac{1}{2R} \int_{-R}^R K^{\alpha, r}(t, x) \, dr.
\]

Where \( K^{\alpha, r}(t, x) = \sum_{I \in \mathcal{D}^{\alpha, r}} h_I(t) \text{III}^{\alpha, r}(h_I(x)) \). Therefore, it is enough to prove the upper bound for the commutator with the shift \( [M_B, \text{III}] \) (the estimates don’t depend on \( \alpha \) or \( r \)).

Let \( B \) be a function with values in the space of \( d \times d \) matrices. We consider the commutator \( [M_B, H] \) acting on a vector-valued function \( f \) by

\[
[M_B, H]f = BH(f) - H(Bf).
\]

The result obtained by Petermichl is based on a decomposition in paraproducts, and uses the estimates obtained by Katz [7], and Nazarov, Treil and Volberg [11] independently. We have

\[
\| [M_B, H] \|_{L^2(\mathbb{R}^2; \mathbb{C}^d) \to L^2(\mathbb{R}^2; \mathbb{C}^d)} \lesssim \log(1 + d) \| B \|.
\]

Motivated by this result, we wish to find a generalization in a two parameter setting, with the corresponding definition of the product BMO space (analogous to the one given by Chang and Fefferman in [2]). The main result of the paper can be stated as follows.

**Theorem 1.1** Let \( B \) be a \( d \times d \) matrix-valued BMO function on \( \mathbb{R}^2 \). If \( M_B \) denotes the operator “multiplication by \( B \)”, and \( H_i \) represents the Hilbert transform in the \( i \)-th parameter, for \( i = 1, 2 \), then the norm of the iterated commutator \( [[M_B, H_1], H_2] \) satisfies

\[
d^{-2} \| B \|_{BMO} \lesssim \| [M_B, H_1], H_2 \|_{L^2(\mathbb{R}^2; \mathbb{C}^d) \to L^2(\mathbb{R}^2; \mathbb{C}^d)} \lesssim d \| B \|_{BMO}.
\]

The paper is organized as follows. Section 2, contains the proof of the upper bound for the norm of the commutator, using a decomposition in paraproducts. Section 3 contains the proof of the lower bound, that is
a reproduction of the proof for the scalar case by S. Ferguson and M. Lacey in [5]. Throughout the paper, we use the notation \( A \lesssim B \) to indicate that there is a positive constant \( C \), such that \( A \leq CB \).

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## 2 Upper bound

Consider \( \mathcal{R} = \mathcal{D} \times \mathcal{D} \), the class of rectangles consisting on products of dyadic intervals. Given a subset \( E \) of \( \mathbb{R}^2 \), denote by \( \mathcal{R}(E) \) the family of dyadic rectangles contained in \( E \).

Consider the wavelet \( w_I \) constructed by Meyer in [9], and the two-parameters wavelet \( v_{R}(x, y) = w_{I}(x)w_{J}(y) \) for \( R = I \times J \), with all its properties listed in [5]. We start by giving the definitions of product \( BMO \) and product dyadic \( BMO \).

**Definition 2.1 (BMO)** A function \( B \) is in \( BMO(\mathbb{R} \times \mathbb{R}) \) if and only if there are constants \( C_1 \) and \( C_2 \) such that, for any open set \( U \subseteq \mathbb{R}^2 \) we have

\[
\left( \frac{1}{|U|} \sum_{R \in \mathcal{R}(U)} \langle B, v_R \rangle \langle B, v_R \rangle^* \right)^{1/2} \leq C_1 I_d \quad \text{and} \quad \left( \frac{1}{|U|} \sum_{R \in \mathcal{R}(U)} \langle B, v_R \rangle^* \langle B, v_R \rangle \right)^{1/2} \leq C_2 I_d.
\]

The inequalities are considered in the sense of operators, \( I_d \) is the identity \( d \times d \) matrix. The \( BMO \)-norm is defined as the smallest constant, denoted by \( \| B \|_{BMO} \), for which the two inequalities are satisfied simultaneously. If we take the supremum only over rectangles \( U \), we obtain the rectangular \( BMO \)-norm, denoted by \( \| B \|_{BMO_{rec}} \).

If \( h_I \) represents the Haar function associated to a dyadic interval \( I \), define

\[ h_R(x, y) = h_I(x)h_J(y), \quad \text{for} \quad R = I \times J. \]

That is \( h_R = h_I \otimes h_J \). The family \( \{ h_R \}_{R \in \mathcal{R}} \) is an orthonormal basis for \( L^2(\mathbb{R}^2, \mathbb{C}^d) \). We have the following definition of dyadic \( BMO \). Note that it is the same definition, but considering the Haar wavelet instead of the Meyer wavelet.

**Definition 2.2 (Dyadic BMO)** A matrix-valued function \( B \) is in \( BMO_d(\mathbb{R} \times \mathbb{R}) \) (dyadic \( BMO \)) if and only if, there are constants \( C_1 \) and \( C_2 \) such that for any open subset \( U \) of the plane, we have

\[
\left( \frac{1}{|U|} \sum_{R \in \mathcal{R}(U)} \langle B, h_R \rangle \langle B, h_R \rangle^* \right)^{1/2} \leq C_1 I_d \quad \text{and} \quad \left( \frac{1}{|U|} \sum_{R \in \mathcal{R}(U)} \langle B, h_R \rangle^* \langle B, h_R \rangle \right)^{1/2} \leq C_2 I_d.
\]

Where the inequality is in the sense of operators. And the corresponding norm \( \| B \|_{BMO_d} \) is, again, the best constant for the two inequalities.

It is known that \( \| B \|_{BMO_d} \leq \| B \|_{BMO} \); this fact can be found in [14]. In that paper, the proof of the inequality is given in the multiparameter setting, for Hilbert space-valued functions, by means of the dual inequality
\[ \|f\|_{H^1} \leq \|B\|_{H^1} \] (Estimate 2.3 in [14]). Using this fact, for the proof of the upper bound, it’s enough to consider the dyadic version of BMO for the computations. For the rest of this section, we use \(\hat{B}(R)\) to denote the Haar coefficient of the function \(B\), associated to the function \(h_R\), that is

\[ \hat{f}(R) = \langle f, h_R \rangle = \int_{\mathbb{R}^2} f(x, y) h_R(x, y) \, dx \, dy. \]

Since \(\hat{B}(R)\hat{B}(R)^*\) is a positive semi-definite matrix, we have

\[
\sqrt{\frac{1}{|U|} \sum_{R \subseteq U} \|\hat{B}(R)\|^2} \leq \sqrt{\text{Tr} \left( \frac{1}{|U|} \sum_{R \subseteq U} \hat{B}(R)\hat{B}(R)^* \right)} \leq \sqrt{\frac{1}{|U|} \sum_{R \subseteq U} \hat{B}(R)\hat{B}(R)^*}. 
\]

So, if

\[
\sqrt{\frac{1}{|U|} \sum_{R \subseteq U} \hat{B}(R)\hat{B}(R)^*} \leq CI_d, \quad \text{or} \quad \sqrt{\frac{1}{|U|} \sum_{R \subseteq U} \hat{B}(R)^*\hat{B}(R)} \leq CI_d,
\]

taking the trace on both sides, we get

\[
\sqrt{\frac{1}{|U|} \sum_{R \subseteq U} \|\hat{B}(R)\|^2} \leq Cd. \tag{1}
\]

Some of the following computations rely on the decomposition used by L. Dalenc and Y. Ou in [4] for the scalar case. If \(f = \sum_{I \in D} \hat{f}(I) h_I\), the dyadic shift operator \(\text{III}(f) = \sum_{I \in D} \hat{f}(I) \frac{1}{\sqrt{2}}(h_{I^-} - h_{I^+})\) corresponds to the operator \(S^{1,0}\) described by Dalenc and Ou, given by

\[
S^{1,0}f = \sum_{K \in D} \sum_{I \subseteq K} \sum_{J \subseteq K} a_{JKI}(f, h_I) h_J, \quad \text{where} \quad a_{JKI} = \begin{cases} \frac{1}{\sqrt{2}}, & \text{if } J = K, \\ \frac{1}{\sqrt{2}}, & \text{if } J = K. \end{cases}
\]

Here, the symbol \(\sum_{I \subseteq J}^{(k)}\) represents summing over those dyadic intervals \(I\) such that \(I \subseteq J\) and \(|I| = 2^{-k}|J|\). Let \(\tilde{I}\) represent the parent of the dyadic interval \(I\), that is, the unique dyadic interval containing \(I\) with \(|\tilde{I}| = 2|I|\), then, the shift can also be expressed in a simpler way by

\[
\text{III}(f) = \sum_{I \in D} a_I \hat{f}(I) h_I, \tag{2}
\]

where \(a_I = \frac{1}{\sqrt{2}}\) if \(I = \tilde{I}_-\), and \(-\frac{1}{\sqrt{2}}\) if \(I = \tilde{I}_+\).

If we write \(B = \sum_{I \in D} \hat{B}(I) h_I\), and \(f = \sum_{J \in D} \hat{f}(J) h_J\), then we can write

\[
Bf = \sum_{I} \sum_{J} \hat{B}(I) h_I \hat{f}(J) h_J.
\]

Therefore the commutator

\[
[M_B, \text{III}](f) = M_B \text{III}(f) - \text{III}(M_B f) = B \text{III}(f) - \text{III}(Bf),
\]

can be written as

\[
[M_B, \text{III}](f) = \sum_{I, J} \hat{B}(I) \hat{f}(J) h_I \text{III}(h_J) - \sum_{I, J} \hat{B}(I) \hat{f}(J) \text{III}(h_I h_J) = \sum_{I, J} \hat{B}(I) \hat{f}(J) [M_{h_I, \text{III}}](h_J).
\]
Note that the terms are non-zero, only when $I \cap J \neq \emptyset$, also, if $J \subsetneq I$, we have that $h_I$ is constant in $I \cap J$, therefore, for every $x \in I \cap J$, we have

$$[M_{h_I}, \mathcal{I}_I](h_J) = h_I(x)\mathcal{I}_I(h_J(x)) - \mathcal{I}_I(h_I(x)h_J(x)) = h_I(x)\mathcal{I}_I(h_J(x)) - h_I(x)\mathcal{I}_I(h_J(x)) = 0.$$ 

Then, the only non-trivial terms are those for which $I \subset J$.

We consider the two parameter commutator $[[M_B, H_1], H_2]$ acting on a vector-valued function $f$ by

$$[[M_B, H_1], H_2]f = BH_1(H_2(f)) - H_1(B(H_2(f))) - H_2(BH_1(f)) + H_2(H_1(Bf)).$$

Where $H_1$ and $H_2$ represent the Hilbert transform, on the first and second variable respectively. That is,

$$H_1f(x, y) = p.v. \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(z, y)}{x-z} \, dz, \quad H_2f(x, y) = p.v. \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(x, z)}{y-z} \, dz.$$

The main result that we want to prove in this section is the following

**Theorem 2.1** Let $B$ be a matrix-valued $\text{BMO}_d(\mathbb{R}^2)$ function and $f$ in $L^2(\mathbb{R}^2; \mathbb{C}^d)$, then

$$\|[[M_B, H_1], H_2]\|_{L^2(\mathbb{R}^2; \mathbb{C}^d) \rightarrow L^2(\mathbb{R}^2; \mathbb{C}^d)} \lesssim \|B\|_{\text{BMO}_d}.$$

**Proof:** Let $\mathcal{I}_1$ and $\mathcal{I}_2$ represent the dyadic shift operator in the first and second variable respectively, that is, $\mathcal{I}_1(h_R) = \mathcal{I}_1(h_I) \otimes h_J$, and $\mathcal{I}_2(h_R) = h_I \otimes \mathcal{I}_1(h_J)$, for $R = I \times J$, and extending to a function $f$ by

$$\mathcal{I}_j(f) = \sum_{R \in \mathcal{D}} \hat{f}(R) \mathcal{I}_j(h_R), \quad j = 1, 2.$$

Or in the notation of (2),

$$\mathcal{I}_1(f) = \sum_{I, J \in \mathcal{D}} a_{I,J} \hat{f}(I \times J) h_I \otimes h_J, \quad \mathcal{I}_2(f) = \sum_{I, J \in \mathcal{D}} a_{I,J} \hat{f}(I \times J) h_I \otimes h_J.

Again, due to the representation of $H$ as an average of shifts, it is enough to prove the result for the commutator $[[M_B, \mathcal{I}_1], \mathcal{I}_2]$. By an iteration of the computation for the one parameter case, using the Haar expansion of the functions $B$ and $f$ and taking their formal product, we obtain

$$[M_B, \mathcal{I}_1](f) = \sum_{R,S \in \mathcal{D}} \hat{B}(R) \hat{f}(S)(h_R \mathcal{I}_1(h_S) - \mathcal{I}_1(h_R h_S)) = \sum_{R,S \in \mathcal{D}} \hat{B}(R) \hat{f}(S)[M_{h_R}, \mathcal{I}_1](h_S)$$

$$= \sum_{I, J \in \mathcal{D}} \sum_{K, L \in \mathcal{D}} \hat{B}(I \times J) \hat{f}(K \times L)(h_I \mathcal{I}_1 h_K \otimes h_j h_L - \mathcal{I}_1(h_I h_K) \otimes h_j h_L).$$

Repeating the same computations, in the two-parameters case we get

$$[[M_B, \mathcal{I}_1], \mathcal{I}_2](f) = \sum_{I, J \in \mathcal{D}} \sum_{K, L \in \mathcal{D}} \hat{B}(I \times J) \hat{f}(K \times L) h_I \mathcal{I}_1 h_K \otimes h_j \mathcal{I}_2 h_L$$

$$- \sum_{I, J \in \mathcal{D}} \sum_{K, L \in \mathcal{D}} \hat{B}(I \times J) \hat{f}(K \times L) \mathcal{I}_1(h_I h_K) \otimes h_j \mathcal{I}_2 h_L$$

$$- \sum_{I, J \in \mathcal{D}} \sum_{K, L \in \mathcal{D}} \hat{B}(I \times J) \hat{f}(K \times L) h_I \mathcal{I}_1 h_K \otimes \mathcal{I}_2(h_I h_L)$$

$$+ \sum_{I, J \in \mathcal{D}} \sum_{K, L \in \mathcal{D}} \hat{B}(I \times J) \hat{f}(K \times L) \mathcal{I}_1(h_I h_K) \otimes \mathcal{I}_2(h_I h_L).$$
Proof of proposition:

We have that for

\[
= T_1f - T_2f - T_3f + T_4f
\]

\[
= \sum_{I,J\in \mathcal{D}} \sum_{K,L\in \mathcal{D}} \hat{B}(I \times J) \hat{f}(K \times L) [M_{h_I}, \mathbb{I}_I](h_K) \otimes [M_{h_J}, \mathbb{I}_J](h_L).
\]

If either \(I \cap K = \emptyset\), \(J \cap L = \emptyset\), \(K \subseteq I\) or \(L \subseteq J\), then we have that \([M_{h_I}, \mathbb{I}_I](h_K) \otimes [M_{h_J}, \mathbb{I}_J](h_L) = 0\); therefore, the terms are non-trivial only when \(I \subseteq K\) and \(J \subseteq L\). We have four different cases, that can be analyzed independently for each term in the sum. The computations for the four terms are similar, only the complete details for the term \(T_2\) will be provided. Let \(\tilde{T}_j\) represent \(T_j\) restricted to the case \(I \subseteq K\) and \(J \subseteq L\), then we have

\[
\tilde{T}_2f = \mathbb{M}_1 \left( \sum_{K} \sum_{L} \sum_{I,J} \sum_{K \subseteq I \cap L \subseteq J} \hat{B}(I \times J) \hat{f}(K \times L) h_I h_K \otimes h_J h_L \right).
\]

To analyze each of the four cases, we need the following proposition.

Proposition 2.1 Consider the following paraproducts

\[
(i) \ P^1_B(f) = \sum_{I,J\in \mathcal{D}} \pm \hat{B}(I \times \hat{J}) \langle f, h_I \otimes h_J \rangle h_I^1 \otimes h_J |I|^{-1/2} |\hat{J}|^{-1/2}.
\]

\[
(ii) \ P^2_B(f) = \sum_{I,J\in \mathcal{D}} \pm \hat{B}(I \times \hat{J}) \langle f, h_I^1 \otimes h_J \rangle h_I \otimes h_J |I|^{-1/2} |\hat{J}|^{-1/2}.
\]

\[
(iii) \ P^3_B(f) = \sum_{I,J\in \mathcal{D}} \hat{B}(I \times J) \langle f, h_I \otimes h_J^1 \rangle h_I \otimes h_J |I|^{-1/2} |\hat{J}|^{-1/2}.
\]

\[
(iv) \ P^4_B(f) = \sum_{I,J\in \mathcal{D}} \hat{B}(I \times J) \langle f, h_I \otimes h_J \rangle h_I^1 \otimes h_J |I|^{-1/2} |\hat{J}|^{-1/2}.
\]

\[
(v) \ P^5_B(f) = \sum_{I,J\in \mathcal{D}} \hat{B}(I \times J) \langle f, h_I \otimes h_J \rangle h_I \otimes h_J |I|^{-1/2} |\hat{J}|^{-1/2}.
\]

We have that for \(i = 1, 2, 3, 4\), \(\|P^i_B(f)\|_{L^2(\mathbb{R}^2; \mathbb{C}^4)} \lesssim \|B\|_{BMO} \|f\|_{L^2(\mathbb{R}^2; \mathbb{C}^4)}\).

Proof of proposition: In the following computations, for simplification we will write \(L^2(Y) = L^2(\mathbb{R}^2; Y)\), since all the functions that we are considering are defined on \(\mathbb{R}^2\).

(i) We make use of a well known result:

Theorem 2.2 (Carleson Embedding Theorem) Let \(\{a_R\}_{R \in \mathbb{R}}\) be a sequence of nonnegative numbers, indexed by the grid of dyadic rectangles. Then the following are equivalent:

\[
(i) \ \sum_{R \in \mathbb{R}} a_R \langle f \rangle_R^2 \leq C_1 \|f\|_{L^2}^2, \ \text{for all } f \in L^2.
\]

\[
(ii) \ \frac{1}{|U|} \sum_{R \in \mathbb{R}(U)} a_R \leq C_2.
\]

Moreover, \(C_1 \simeq C_2\).

We have the following basic estimates

\[
| \langle P^1_B f, g \rangle_{L^2} | = \left| \int_{\mathbb{R}^2} \langle P^1_B f, g \rangle_{\mathbb{C}^4} \, dx \, dy \right|
\]

\[
= \left| \int_{\mathbb{R}^2} \left( \sum_I \sum_J \pm \hat{B}(I \times \hat{J}) \hat{f}(I \times J) \mathbb{1}_I |I|^{-1} \otimes h_J |\hat{J}|^{-1/2}, g \right)_{\mathbb{C}^4} \, dx \, dy \right|
\]

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Here, we used the fact that since $B \in BMO_d$, then by (1), the second condition in Theorem 2.2 is satisfied with $a_R = \| \hat{B}(R) \|^2_{S_2^4}$. Note, that we have a linear dependence on the dimension of the matrix, due to the use of the trace. Note also that the same computations allow us to replace each individual $I$ and $J$ for a parent or “great parent” of $I$ and $J$, the implied constant will depend also on the “generation” in this case; we will use $P_B^1$ to denote any of these kind of paraproducts.

(ii) A direct computation shows that $(P_B^2)^*$ is of the type $P_B^1$, therefore, by the symmetry of the definition of $BMO_d$-norm, the boundedness for $P_B^2$ follows from that of $P_B^1$.

(iii) Denote by $S_2^d$ the space of $d \times d$ complex matrices, equipped with the norm derived from the inner product $\langle A, B \rangle_{\text{Tr}} = \text{tr}(AB^*)$, that is $\| A \|^2_{S_2^d} = \text{tr}(AA^*)$. To estimate the $L^2$-norm of this operator, we perform the following computation

$$\langle P_B^3(f), g \rangle = \int_{\mathbb{R}^2} \sum_{I,J} \left\langle \hat{B}(I \times J) \left\langle f, h_I^1 \otimes h_J^1 \right\rangle h_I \otimes h_J |I|^{-1/2} |J|^{-1/2}, g \right\rangle_{C^d} \, dx \, dy$$

$$= \sum_{I,J} \int_{\mathbb{R}^2} \langle \hat{B}(I \times J) \langle f, h_I^1 \otimes h_J^1 \rangle, g h_I \otimes h_J |I|^{-1/2} |J|^{-1/2} \rangle_{C^d} \, dx \, dy$$

$$= \sum_{I,J} \langle \hat{B}(I \times J) \langle f, h_I^1 \otimes h_J^1 \rangle, \langle g, h_I \otimes h_J \rangle |I|^{-1/2} |J|^{-1/2} \rangle_{C^d}$$

$$= \sum_{I,J} \langle \hat{B}(I \times J), \langle f, h_I \otimes h_J \rangle \langle f, h_I^1 \otimes h_J^1 \rangle^* |I|^{-1/2} |J|^{-1/2} \rangle_{\text{Tr}}$$
Therefore, by duality, it is enough to prove that

$$\|\Pi_1(f, g)\|_{H^1} \lesssim \|S(\Pi_1(f, g))\|_{L^1} \lesssim \|f\|_{L^2}\|g\|_{L^2}. $$

Where $S$ represents the square function. We have

\[
\begin{align*}
[S(\Pi_1(f, g))]^2 &= \sum_{l,j} \left\| \left( g, h_I \otimes h_J \right) \left( f, h_I^1 \otimes h_J^1 \right)^* |I|^{-1/2}|J|^{-1/2} \right\|_{S^2}^2 \frac{1_{I \times J}(y)}{|I|} \\
&\leq \sum_{l,j} \left\| \left( g, h_I \otimes h_J \right) \left( f, h_I^1 \otimes h_J^1 \right)^* |I|^{-1/2}|J|^{-1/2} \right\|_{C^2}^2 \frac{1_{I \times J}(x, y)}{|I|} \\
&\leq \sup_{(x,y) \in I \times J} \left\| \left( f, h_I^1 \otimes h_J^1 |I|^{-1/2}|J|^{-1/2} \right) \right\|_{C^2}^2 \sum_{l,j} \left\| \left( g, h_I \otimes h_J \right) \right\|_{C^2}^2 \frac{1_{I \times J}(x, y)}{|I|} \\
&\leq \left[ M(\|f\|_{C^2})^2 \right] [S(g)]^2.
\end{align*}
\]

Here, $M$ represents the Hardy-Littlewood maximal function. Using the $L^2$-boundedness of the maximal and square functions, we conclude

$$\|\Pi_1(f, g)\|_{H^1} \lesssim \|S(\Pi_1(f, g))\|_{L^1} \lesssim \|M(\|f\|_{C^2})S(g)\|_{L^1} \lesssim \|f\|_{L^2}\|g\|_{L^2}. $$

(iv) The idea is the same as in the previous case

\[
\begin{align*}
\langle P^2_{B}(f, g) \rangle &= \int_{\mathbb{R}^2} \left\langle \sum_{l,j} \hat{B}(I \times J) \left( f, h_I \otimes h_J \right) h_I^1 \otimes h_J |I|^{-1/2}|J|^{-1/2}, g \right\rangle_{C^4} \, dx \, dy \\
&= \sum_{l,j} \int_{\mathbb{R}^2} \left\langle \hat{B}(I \times J) \left( f, h_I \otimes h_J^1 \right), \left( g, h_I \otimes h_J \right) |I|^{-1/2}|J|^{-1/2} \right\rangle_{C^4} \, dx \, dy \\
&= \sum_{l,j} \left\langle \hat{B}(I \times J), \left( g, h_I \otimes h_J \right) \left( f, h_I \otimes h_J^1 \right)^* |I|^{-1/2}|J|^{-1/2} \right\rangle_{C^4} \\
&= \sum_{l,j} \left\langle \hat{B}(I \times J), \left( g, h_I \otimes h_J \right) \left( f, h_I \otimes h_J^1 \right)^* |I|^{-1/2}|J|^{-1/2} \right\rangle_{C^4} \\
&= \sum_{l,j} \left\langle \hat{B}(I \times J), \left( g, h_I \otimes h_J \right) \left( f, h_I \otimes h_J^1 \right)^* |I|^{-1/2}|J|^{-1/2} \right\rangle_{C^4} \\
&= \int_{\mathbb{R}^2} \left\langle B_{h_I} \otimes h_J, \left( g, h_I \otimes h_J \right) \left( f, h_I \otimes h_J^1 \right)^* |I|^{-1/2}|J|^{-1/2} \right\rangle_{C^4} \, dx \, dy \\
&= \int_{\mathbb{R}^2} \left\langle B_{h_I} \otimes h_J, \left( g, h_I \otimes h_J \right) \left( f, h_I \otimes h_J^1 \right)^* |I|^{-1/2}|J|^{-1/2} \right\rangle_{C^4} \, dx \, dy.
\end{align*}
\]
\[ = \left\langle B, \sum_{I,J} (g, h_I^1 \otimes h_J^1) \langle f, h_I^1 \otimes h_J^1 \rangle^* h_I \otimes h_J |I|^{-1/2} |J|^{-1/2} \right\rangle = \langle B, \Pi_2(f, g) \rangle. \]

Therefore, by duality, it is enough to prove that
\[ \|\Pi_2(f, g)\|_{H^1} \lesssim \|f\|_{L^2} \|g\|_{L^2}. \]

For this, we proceed again to find a pointwise estimate for the square function of this expression
\[ [S(\Pi_2(f, g))]^2 = \sum_{I,J} \left\| \left( \langle g, h_I^1 \otimes h_J^1 \rangle \langle f, h_I^1 \otimes h_J^1 \rangle^* \right) \right\|_{L^2(I \times J)}^2 \frac{1}{|I \times J|} \]
\[ \leq \sum_{I,J} \left( \| \langle g, h_I^1 \rangle \|_{L^2(I)} \| \langle f, h_I^1 \rangle \|_{L^2(I)} \right)^2 \frac{1}{|I|} \leq \sum_{I,J} \left( \| \langle g, h_I^1 \rangle \|_{C^2(I)} \right)^2 \frac{1}{|I|} \]
\[ \leq \left( \sum_I (\mathcal{M}_2 \| \langle f, h_I \rangle \|_{C^2(I)})^2 \frac{1}{|I|} \right) \left( \sum_J (\mathcal{M}_1 \| \langle g, h_J \rangle \|_{C^2(J)})^2 \frac{1}{|J|} \right). \]

Where \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) represent the maximal function in the first and second variable, respectively. These last two factors are symmetric to each other, so it is enough to prove the \( L^2 \)-boundedness for the operator
\[ \tilde{S}f(x, y) = \left( \sum_I (\mathcal{M}_2 \| \langle f, h_I \rangle \|_{C^2(I)})^2 \frac{1}{|I|} \right)^{1/2}. \]

This can be done by consecutive applications of Cauchy-Schwarz and Fefferman-Stein inequality. More details can be found in [4].

(v) The computations are symmetric to those for (iv), exchanging the roles of \( I \) and \( J \).

\[ \blacksquare \]

We proceed now to prove the upper bound for the four different cases. In each of them, the idea is to reduce the term to an expression of the form \( \Pi_3 \circ P_B \circ \Pi_2 \), therefore, by Proposition 2.1 and the boundedness of the shifts, we get the desired result. The estimates for the rest of the terms are similar, since they are reduced to find an upper bound for the norm of the four variants of paraproduct studied above. More specifically, they correspond to expressions of the form \( \Pi_3(P_B(\Pi_2(f))) \), \( \Pi_3(\Pi_3(P_B(f))) \) and \( \Pi_3(\Pi_3(P_B(f))) \), \( \Pi_3(\Pi_3(P_B(f))) \), or duals of operators of the form \( \Pi_3(P_B, (\Pi_3(f))) \), \( \Pi_3(\Pi_3(P_B, f)) \), \( \Pi_3(\Pi_3(P_B, f)) \) and \( \Pi_3(\Pi_3(P_B, f)) \).

Case I = K, J = L. In this case, using the definition of the shift, we have
\[ \Pi_3 \left( \sum_I \sum_J \hat{B}(I \times J) \hat{f}(I \times J) h_I^2 h_J \right) = \Pi_3 \left( \sum_I \sum_J \hat{B}(I \times J) \hat{f}(I \times J) h_I^2 \otimes h_J \right). \]

Notice that \( \Pi_2 \langle f, h_I \rangle = \sum_L a_L \hat{f}(I \times L) h_L \), and so, \( \langle \Pi_3 \langle f, h_I \rangle, h_J \rangle = a_J \hat{f}(I \times J) \). Then, the previous expression is equal to
\[ \Pi_3 \left( \sum_I \sum_J \hat{B}(I \times J) \langle \Pi_2 \langle f, h_I \rangle, h_J \rangle h_I^2 \otimes h_J \right) \]
\[ = \Pi_3 \left( \sum_I \sum_J \pm \hat{B}(I \times J) \langle \Pi_2 \langle f, h_I \rangle, h_J \rangle H^{-1} \otimes h_J |J|^{-1/2} \right) \]
\[
= \mathbb{I}_1 \left( \sum_I \sum_J \pm \hat{B}(I \times J) \langle \mathbb{I}_2 f, h_I \otimes h_J \rangle h_I^1 \otimes h_J |I|^{-1/2} |J|^{-1/2} \right).
\]

\(\mathbb{I}_1(P_B^1(\mathbb{I}_2 f)).\)

**Case I \( \subseteq K, J \subseteq L\).** Here we have

\[
\mathbb{I}_1 \left( \sum_K \sum_{I \subseteq K} \sum_L \sum_{J \subseteq L} \hat{B}(I \times J) \hat{f}(K \times L) h_I h_K \otimes h_J \mathbb{I}_2 h_L \right)
= \mathbb{I}_1 \left( \sum_K \sum_{I \subseteq K} \sum_J \hat{B}(I \times J) h_I h_K \otimes \left( \sum_{L \supseteq J} \langle \langle f, h_K \rangle, h_L \rangle \mathbb{I}_2 h_L \mathbb{I}_J \right) h_J \right).
\]

By using the definition of the shift, and the known average identity \(\langle g, h_J^1 \rangle |J|^{-1/2} = \sum_{I \supseteq J} \hat{f}(I) h_I \mathbb{I}_J\), we have

\[
\sum_{L \supseteq J} \langle \langle f, h_K \rangle, h_L \rangle \mathbb{I}_2 h_L \mathbb{I}_J = \mathbb{I}_2 \left( \sum_{L \supseteq J} \langle \langle f, h_K \rangle, h_L \rangle h_L \mathbb{I}_J \right) \mathbb{I}_J = \sum_{L \supseteq J} a_L \langle \langle f, h_K \rangle, h_L \rangle h_L \mathbb{I}_J
= \langle \mathbb{I}_2 \langle f, h_K \rangle, h_J^1 \rangle |J|^{-1/2} + \langle \mathbb{I}_2 \langle f, h_K \rangle, h_J \rangle h_J.
\]

This divides the original sum into two sums \(S_1 + S_2\).

\[
S_1 = \mathbb{I}_1 \left( \sum_K \sum_{I \subseteq K} \sum_J \hat{B}(I \times J) \langle \mathbb{I}_2 \langle f, h_K \rangle, h_J^1 \rangle h_I h_K \otimes h_J |I|^{-1/2} \right)
= \mathbb{I}_1 \left( \sum_I \sum_J \hat{B}(I \times J) \left( \sum_{K \supseteq I} \langle \langle \mathbb{I}_2 f, h_J^1 \rangle, h_K \rangle h_K \mathbb{I}_I \right) h_I \otimes h_J |I|^{-1/2} \right)
= \mathbb{I}_1 \left( \sum_I \sum_J \hat{B}(I \times J) \langle \langle \mathbb{I}_2 f, h_J^1 \rangle, h_J \rangle h_I \otimes h_J |I|^{-1/2} |J|^{-1/2} \right)
= \mathbb{I}_1 \left( \sum_I \sum_J \hat{B}(I \times J) \langle \mathbb{I}_2 f, h_I^1 \otimes h_J^1 \rangle h_I \otimes h_J |I|^{-1/2} |J|^{-1/2} \right).
\]

Which has the form \(\mathbb{I}_1(P_B^1(\mathbb{I}_2 f)).\) And with similar computations, we get

\[
S_2 = \mathbb{I}_1 \left( \sum_I \sum_J \hat{B}(I \times J) \langle \mathbb{I}_2 f, h_I^1 \otimes h_J \rangle h_I \otimes h_J |I|^{-1/2} |J|^{-1/2} \right) = \mathbb{I}_1(P_B^1(\mathbb{I}_2 f)).
\]

**Case I = K, J \subset L.** In this case we get

\[
\mathbb{I}_1 \left( \sum_I \sum_L \sum_{J \subseteq L} \hat{B}(I \times J) \hat{f}(I \times L) h_I^2 \otimes h_J \mathbb{I}_2 h_L \right)
= \mathbb{I}_1 \left( \sum_I \sum_J \hat{B}(I \times J) h_I^2 \otimes \left( \sum_{L \supseteq J} \langle \langle f, h_I \rangle, h_L \rangle \mathbb{I}_2 h_L \mathbb{I}_J \right) h_J \right).
\]
2.1 Remark: Logarithmic estimate

\[
= \Pi_1 \left( \sum_I \sum_J \hat{B}(I \times J) h_I^2 \otimes \left( \langle \Pi_2(f, h_K), h_J \rangle \right) \right)
= S_1 + S_2.
\]

Again, by the definition of the shift
\[
S_1 = \Pi_1 \left( \sum_I \sum_J \hat{B}(I \times J) h_I^2 \otimes \langle \Pi_2(f, h_I), 1_J |J|^{-1} \rangle h_J \right)
= \Pi_1 \left( \sum_I \sum_J \hat{B}(I \times J) \langle \Pi_2 f, h_I \otimes 1_J |J|^{-1} \rangle 1_J |J|^{-1} \otimes h_J \right)
= \Pi_1 \left( \sum_I \sum_J \hat{B}(I \times J) \langle \Pi_2 f, h_I \otimes h_J^1 \rangle h_I^1 \otimes h_J |J|^{-1/2} |J|^{-1/2} \right).
\]

Which has the form \( \Pi_1(P_B^4(\Pi_2 f)) \). And similarly
\[
S_2 = \Pi_1 \left( \sum_I \sum_J \hat{B}(I \times J) \langle \Pi_2 f, h_I \otimes h_J, h_I^1 \otimes h_J^3 |J|^{-1/2} |J|^{-1/2} \rangle \right) = \Pi_1((P_B^3)^*(\Pi_2 f)).
\]

Case I \( \subseteq K, J = L \). last case we have
\[
\Pi_1 \left( \sum_{I \subseteq K} \sum_I \sum_J \hat{B}(I \times J) \hat{B}(K \times J) h_I h_K \otimes h_J \Pi_2 h_J \right)
\]
\[
\Pi_1 \left( \sum_I \sum_J \hat{B}(I \times J) \left( \sum_{K \supset I} \langle (f, h_J), h_K \rangle h_K \right) h_I \otimes h_J \Pi_2 h_J \right)
\]
\[
\Pi_1 \left( \sum_I \sum_J \hat{B}(I \times J) \langle (f, h_J), h_J^1 \rangle h_I |J|^{-1/2} \otimes (h_J + h_J^1) |J|^{-1/2} \right).
\]

This is a sum of two terms of the form
\[
\Pi_1 \left( \sum_I \sum_J \pm \hat{B}(I \times J) \langle f, h_I \otimes h_J \rangle h_I \otimes h_J |J|^{-1/2} |J|^{(1)}^{-1/2} \right) = \Pi_1(P_B^2(\hat{T}_2)).
\]

This concludes the proof of the estimate for the term \( \hat{T}_2 \).

2.1 Remark: Logarithmic estimate

Note that, because of (1), the previous estimates for the upper bound depend on a dimensional constant. Using a slightly different ordering of the terms in the formal Haar expansion of the product \( Bf \), we obtain a decomposition in paraproducts of the form
\[
Bf = \sum_{R \in \mathcal{D}^2} \langle B, h_R^{(0, 0)} \rangle \langle f, h_R^{(0, 0)} \rangle h_R^{(1, 1)} + \sum_{R \in \mathcal{D}^2} \langle B, h_R^{(0, 0)} \rangle \langle f, h_R^{(0, 1)} \rangle h_R^{(1, 0)} + \sum_{R \in \mathcal{D}^2} \langle B, h_R^{(0, 1)} \rangle \langle f, h_R^{(0, 0)} \rangle h_R^{(1, 0)}
+ \sum_{R \in \mathcal{D}^2} \langle B, h_R^{(0, 1)} \rangle \langle f, h_R^{(0, 1)} \rangle h_R^{(1, 0)} + \sum_{R \in \mathcal{D}^2} \langle B, h_R^{(1, 0)} \rangle \langle f, h_R^{(0, 0)} \rangle h_R^{(0, 0)} + \sum_{R \in \mathcal{D}^2} \langle B, h_R^{(1, 0)} \rangle \langle f, h_R^{(0, 0)} \rangle h_R^{(0, 0)}
+ \sum_{R \in \mathcal{D}^2} \langle B, h_R^{(0, 1)} \rangle \langle f, h_R^{(1, 0)} \rangle h_R^{(1, 1)} + \sum_{R \in \mathcal{D}^2} \langle B, h_R^{(0, 1)} \rangle \langle f, h_R^{(1, 1)} \rangle h_R^{(1, 1)} + \sum_{R \in \mathcal{D}^2} \langle B, h_R^{(1, 1)} \rangle \langle f, h_R^{(0, 0)} \rangle h_R^{(0, 0)} + \sum_{R \in \mathcal{D}^2} \langle B, h_R^{(1, 1)} \rangle \langle f, h_R^{(1, 0)} \rangle h_R^{(0, 0)}.
\]
\[ = (T_1 + T_2 + T_3 + T_4 + T_5 + T_6 + T_7 + T_8 + T_9)(f). \]

Here, \( h^{(ε, δ)}_R = h^ε_I h^δ_J \), with \( ε, δ \in \{0, 1\} \), and \( h^0_I = h_I, h^1_I = |I|^{-1/2} \mathbb{1}_I \). Then,

\[
[[M_B, Π_1], Π_2](f) = [[T_1, Π_1], Π_2](f) + \cdots + [[T_9, Π_1], Π_2](f).
\]

Therefore, to find an upper bound for the commutator, it suffices to find upper bounds for the different paraproducts in the above expansion. By the previous section, this upper bound depends also on a dimensional constant, however, it is possible for the terms \( T_1, T_6 \), and \( T_8 \) (by duality), to find a better estimate of order \( \log^2(1 + d) \). This is possible due to a generalization of the results obtained by Pisier in [13] for the one parameter case, combined with the characterization by two index martingales given by Bernard in [1].

With the rest of the terms, it’s still not clear how to find this improved dimensional bound for the paraproduct, since we don’t have a representation in two-index martingales in these cases, or the appropriate embedding theorem.

### 3 Lower bound

The lower bound can be proved by using the result in the scalar case (proved by Ferguson and Lacey in [5]). That is, for all scalar functions \( b \) in \( BMO(\mathbb{R} \times \mathbb{R}) \), there is a constant \( C > 0 \) such that

\[
\| b \|_{BMO} \leq C \|[M_b, H_1], H_2]\|_{L^2 \to L^2}.
\]

Let us recall the definition of \( BMO \) given in the first section. A matrix-valued function \( B \) is in \( BMO(\mathbb{R} \times \mathbb{R}) \) if and only if

\[
\| B \|_{BMO} = \max \left( \sup_{U} \left[ \left| U \right|^{-1} \sum_{R \subseteq U} \hat{B}(R) \hat{B}(R)^* \right]^{1/2}, \sup_{U} \left[ \left| U \right|^{-1} \sum_{R \subseteq U} \hat{B}(R)^* \hat{B}(R) \right]^{1/2} \right) < \infty.
\]

Where \( \hat{B}(R) \) represents the wavelet coefficient associated to the function \( v_R \), and the supremum is taken over open subsets of \( \mathbb{R} \times \mathbb{R} \). The lower bound estimate in the matrix-valued setting is

**Theorem 3.1 (Lower bound)** Let \( B \) be a matrix-valued function on \( \mathbb{R}^2 \), then

\[
d^{-2} \| B \|_{BMO} \lesssim \|[M_B, H_1], H_2]\|_{L^2(C^d) \to L^2(C^d)}.
\]

**Proof:** Consider the functions \( f, g \in L^2(\mathbb{C}) \). Let \( \{\tilde{e}_1, \ldots, \tilde{e}_d\} \) represent the canonical basis of \( \mathbb{R}^d \), then, for \( 1 \leq i, j \leq d \), the functions \( \tilde{f} = f\tilde{e}_i \) and \( \tilde{g} = g\tilde{e}_j \) both belong to \( L^2(\mathbb{C}^d) \). If \( B = (b_{ij}) \), an easy computation shows that

\[
\left\langle [[M_B, H_1], H_2] \tilde{f}, \tilde{g} \right\rangle_{L^2(C^d)} = \left\langle [[M_{b_{ij}}, H_1], H_2] f, g \right\rangle_{L^2(C)}
\]

Therefore, for every \( i, j \in \{1, \ldots, d\} \), we have

\[
\|[M_{b_{ij}}, H_1], H_2]\|_{L^2(C \to L^2(C)} \leq \|[M_B, H_1], H_2]\|_{L^2(C^d \to L^2(C^d)}.
\]

Let \( \{E_{ij} : 1 \leq i, j \leq d\} \) be the canonical basis for the \( d \times d \) matrices, that is, \( (E_{ij})_{kl} = \delta_{ik}\delta_{jl} \). We can write \( B = \sum_{i,j} b_{ij} E_{ij} \), and proceed to find an estimate for the \( BMO \) norm of the matrices \( B_{ij} = b_{ij} E_{ij} \).
Note that
\[ \hat{B}_{ij}(R)\hat{B}_{ij}(R)^* = \hat{B}_{ij}(R)^*\hat{B}_{ij}(R) = \hat{b}_{ij}(R)E_{ij}\hat{b}_{ij}(R)E_{ji} = |\hat{b}_{ij}(R)|^2E_{ii}. \]
Then, for any open set \( U \subseteq \mathbb{R}^2 \), we have
\[ \frac{1}{|U|} \sum_{R \subseteq U} \hat{B}_{ij}(R)\hat{B}_{ij}(R)^* \leq \frac{1}{|U|} \sum_{R \subseteq U} |\hat{b}_{ij}(R)|^2E_{ii} \leq \frac{1}{|U|} \sum_{R \subseteq U} |\hat{b}_{ij}(R)|^2I_d \leq \| \hat{b}_{ij} \|_{BMO}^2I_d. \]

Using the one parameter result, and equation 3, we get
\[ \frac{1}{|U|} \sum_{R \subseteq U} \hat{B}_{ij}(R)\hat{B}_{ij}(R)^* \lesssim \| [M_{b_{ii}}, H_1], H_2 \|_{L^2(\mathbb{C}^2 \to L^2(\mathbb{C}^2))}I_d \leq \| [M_B, H_1], H_2 \|_{L^2(\mathbb{C}^4 \to L^2(\mathbb{C}^4))}. \]

That is, \( \| \hat{B}_{ij} \|_{BMO} \lesssim \| [M_B, H_1], H_2 \|_{L^2(\mathbb{C}^4 \to L^2(\mathbb{C}^4))}. \) Therefore,
\[ \| B \|_{BMO} \leq \sum_{i,j} \| \hat{B}_{ij} \|_{BMO} \lesssim d^2\| [M_B, H_1], H_2 \|_{L^2(\mathbb{C}^4 \to L^2(\mathbb{C}^4))}. \]

Which is the desired lower bound.

References

[1] A. Bernard, Espaces \( H^1 \) de martingales à deux indices. Dualité avec les martingales de type “BMO”, Bull. Sci. Math. (2) 103 (1979), no. 3, 297–303.

[2] S.-Y. A. Chang and R. Fefferman, A continuous version of duality of \( H^1 \) with BMO on the bidisc, Ann. of Math. (2) 112 (1980), no. 1, 179–201.

[3] R. R. Coifman, R. Rochberg, and G. Weiss, Factorization theorems for Hardy spaces in several variables, Ann. of Math. (2) 103 (1976), no. 3, 611–635.

[4] L. Dalenc and Y. Ou, Upper Bound for Multi-parameter Iterated Commutators (2014). Preprint available on http://arxiv.org/abs/1401.5994.

[5] S. H. Ferguson and M. T. Lacey, A characterization of product BMO by commutators, Acta Math. 189 (2002), no. 2, 143–160.

[6] S. H. Ferguson and C. Sadosky, Characterizations of bounded mean oscillation on the polydisk in terms of Hankel operators and Carleson measures, J. Anal. Math. 81 (2000), 239–267.

[7] N. H. Katz, Matrix valued paraproducts, Proceedings of the conference dedicated to Professor Miguel de Guzmán (El Escorial, 1996), 1997, pp. 913–921.

[8] M. T. Lacey and E. Terwilleger, Hankel operators in several complex variables and product BMO, Houston J. Math. 35 (2009), no. 1, 159–183.

[9] Y. Meyer, Wavelets and operators, Cambridge Studies in Advanced Mathematics, vol. 37, Cambridge University Press, Cambridge, 1992. Translated from the 1990 French original by D. H. Salinger.

[10] F. Nazarov, G. Pisier, S. Treil, and A. Volberg, Sharp estimates in vector Carleson imbedding theorem and for vector paraproducts, J. Reine Angew. Math. 542 (2002), 147–171.

[11] F. Nazarov, S. Treil, and A. Volberg, Counterexample to the infinite-dimensional Carleson embedding theorem, C. R. Acad. Sci. Paris Sér. I Math. 325 (1997), no. 4, 383–388.

[12] S. Petermichl, Dyadic shifts and a logarithmic estimate for Hankel operators with matrix symbol, C. R. Acad. Sci. Paris Sér. I Math. 330 (2000), no. 6, 455–460 (English, with English and French summaries).

[13] G. Pisier, Notes on Banach space valued \( H^p \)-spaces, non-commutative martingale inequalities and matrix valued Harmonic Analysis, 2005. Informal Seminar Notes.

[14] S. Treil, \( H^1 \) and dyadic \( H^1 \) (2008). Available on http://arxiv.org/pdf/0809.3288.