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Strict Intuitionistic Fuzzy Distance/Similarity Measures Based on Jensen-Shannon Divergence

Xinxing Wu, Zhiyi Zhu, Guanrong Chen, Tao Wang, Peide Liu

Abstract—Being a pair of dual concepts, the normalized distance and similarity measures are very important tools for decision-making and pattern recognition under intuitionistic fuzzy sets framework. To be more effective for decision-making and pattern recognition applications, a good normalized distance measure should ensure that its dual similarity measure satisfies the axiomatic definition. In this paper, we first construct some examples to illustrate that the dual similarity measures of two nonlinear distance measures introduced in [A distance measure for intuitionistic fuzzy sets and its application to pattern classification problems, IEEE Trans. Syst., Man, Cybern., Syst., vol. 51, no. 6, pp. 3980–3992, 2021] and [Intuitionistic fuzzy sets: spherical representation and distances, Int. J. Intell. Syst., vol. 24, no. 4, pp. 399–420, 2009] do not meet the axiomatic definition of intuitionistic fuzzy similarity measure. We show that (1) they cannot effect Petroleum Liguish some intuitionistic fuzzy values (IFVs) with obvious size relationship; (2) except for the endpoints, there exist infinitely many pairs of IFVs, where the maximum distance 1 can be achieved under these two distances; leading to counter-intuitive results. To overcome these drawbacks, we introduce the concepts of strict intuitionistic fuzzy distance measure (SIFDiSM) and strict intuitionistic fuzzy similarity measure (SIFSiSM), and propose an improved intuitionistic fuzzy distance measure based on Jensen-Shannon divergence. We prove that (1) it is a SIFDiSM; (2) its dual similarity measure is a SIFSiSM; (3) its induced entropy is an intuitionistic fuzzy entropy. Comparative analysis and numerical examples demonstrate that our proposed distance measure is completely superior to the existing ones.

Index Terms—Distance measure, similarity measure, strict similarity measure, intuitionistic fuzzy set, Jensen-Shannon divergence.

I. INTRODUCTION

To deal with the ubiquitous uncertainty and fuzziness in real life, Zadeh (1965) [1] presented the fuzzy set (FS) theory by applying membership degrees to measure the importance of a fuzzy element. Zadeh’s FS generalized the concept of crisp sets, which is described by a characteristic function that can take any value in the interval [0, 1]. However, due to the limitation of a membership function that only indicates two (supporting and opposing) states of fuzziness, the fuzzy sets cannot express the neutral state of “this and also that”. As a remedy to this, Atanassov [2] generalized Zadeh’s fuzzy set by introducing the concept of intuitionistic fuzzy sets (IFSs) (see also [3]), which was characterized by a membership function and a non-membership function simultaneously, satisfying the condition that the sum of the membership degree and the non-membership degree at every point is less than or equal to 1. Moreover, in dealing with many practical problems, it is not appropriate for experts to precisely express their decisions with crisp numbers due to the complexity and uncertainty of available information. Based on this observation, Atanassov and Gargov [4] further extended IFSs to interval-valued intuitionistic fuzzy sets (IVIFSs) (see also [5]), replacing the membership degree and the non-membership degree by some closed intervals in [0, 1].

To assess the differences of IFSs, the normalized IF distance measure (IFDiSM) and the IF similarity measure (IFSiSM), being a pair of dual concepts (see [6]), are important tools for decision-making ([7], [8], [9], [10]), pattern recognition ([11], [12], [13], [14], [15], [16], [17], [18]), medical diagnosis ([19], [20], [10]), clustering analysis ([21], [20]), image processing ([22], [10]), and data mining under intuitionistic fuzzy sets framework. Motivated by the similarity measure for Zadeh’s fuzzy sets, Li and Cheng [11] introduced the concept of similarity measure (SimM) for IFSs, which was improved by Mitchell [23], and applied to pattern recognition problems. Recently, Szmidt [24] presented an overview on IFDiSMs and IFSiSMs. The research on IFDiSMs and IFSiSMs focuses on two aspects, one is the two-dimensional (2-D) representation of IFSs, which only considers the membership and nonmembership degrees, and the other is the three-dimensional (3-D) representation of IFSs, which simultaneously considers the membership, nonmembership, and indeterminacy degrees. However, noticeably, because the indeterminacy degree is uniquely determined by the membership and nonmembership degrees, the space of all IFVs is essentially a 2-D topological structure. This indicates a serious problem that many existing distance and similarity measures via 3-D representation of IFSs, including Euclidean similarity measure in [24] and Minkowski similarity measure in ([7], [25]), do not meet the axiomatic definition of IFSiSMs (see [26, Section 3]).

For 2-D IFDiSMs and IFSiSMs, Szmidt and Kacprzyk [27] presented the normalized Hamming distance and normalized Euclidean distance for IFSs. Grzegorzewski [28] proposed an
IFDisM by using the Hausdorff metric for closed intervals (also see [29]). Wang and Xin [30] introduced the axiomatic definition of distance measure as a dual concept of the similarity measure for IFSs and constructed some new 2-D IFDisMs by combining the 2-D Hamming IFDisM [27] and the 2-D Hausdorff IFDisM [28]. Further, based on IFDisMs introduced by Wang and Xin [30], Xu and Chen [31] presented some new IFSimMs.

For 3-D IFDisMs and IFSimMs, Xu and Chen [31] suggested some IFSimMs based on the idea of TOPSIS of Hwang and Yoon [32]. Wu et al. [26] pointed out that the IFSimM in [31, Eq. 14] based on the normalized Euclidean distance does not satisfy the axiomatic definition of IFSimMs. Motivated by the divergence measure in information theory, Chen et al. [15] introduced a novel IFSimM using centroid points of transformed right-angled TrFNs. Yang and Chichlala [33] constructed a nonlinear spherical distance measure for IFSSs by using the ‘arccos’ function. Hung and Yang [34] constructed a $J_{r}$-divergence ($\gamma > 0$) for IFSs and proved that it satisfies the axiomatic definition of IFDisM of Wang and Xin [30] for $\gamma \in [1, 2]$. Joshi and Kumar [35] obtained a dissimilarity Jensen-Shannon divergence measure for IFSs. In fact, this dissimilarity measure is equivalent to the $J_{1}$-divergence in [34]. Recently, Xiao [18] proposed a new IFDisM $d_\chi$ based on the Jensen-Shannon divergence and showed that this distance measure is superior to those in [27, 28, 30, 36, 37, 38, 39]. By direct calculation, it is easy to see that $\ln 2 \cdot d_\chi = \sqrt{\frac{J_{1}}{2}},$ i.e., Xiao’s distance measure $d_\chi$ is a special case of Hung and Yang’s $J_{r}$-divergence (see Section III). For more results on the IFDisMs and IFSimMs, see [24], [25], [40]. It should be noted that because the axiomatic definition (S4) of IFSimMs requires the admissibility with Atanassov’s partial order ‘⊂’, and that Atanassov’s partial order only indicates the size relationship for membership degrees and non-membership degrees, implying that many existing 3-D IFSimMs may violate the axiomatic definition (S4) of IFSimMs.

We will provide Examples 1–3 below to show that Xiao’s distance measure [18], Hung and Yang’s $J_{1}$-divergence [34], and Joshi and Kumar’s dissimilarity Jensen-Shannon divergence measure in [35] have the follows three drawbacks: (1) its dual similarity measure does not satisfy the axiomatic definition (S4) of Definition 2; (2) it cannot effectively distinguish some IFVs/IFSs with obvious size relationship; (3) Except for the endpoints (1, 0) and (0, 1), there exist infinitely many pairs of IFVs, where the maximum distance 1 can be achieved under these distances. Meanwhile, observing from Examples 4 and 10 below, Yang and Chichlala’s spherical distance measure in [33] has the same drawbacks. To distinguish IFVs more effectively and overcome the above three drawbacks for IFDisMs/IFSimMs, we introduce the concepts of strict intuitionistic fuzzy similarity measure (SIFSimM) and strict intuitionistic fuzzy distance measure (SIFDisM) and construct a novel IFDisM based on Jensen-Shannon divergence measure. We demonstrate that its dual similarity measure is a SIFSimM and its induced entropy measure meets the axiomatic definition of IF entropy. Moreover, we present some comparative analysis to illustrate that our proposed distance measure is completely superior to the existing IFDisMs; in particular, it is much better than Xiao’s distance measure in [18], Hung and Yang’s $J_{r}$-divergence in [34], Joshi and Kumar’s dissimilarity divergence in [35], and Yang and Chichlala’s spherical distance in [33]. Finally, to demonstrate the effectiveness of our proposed IFSimM, we apply it to a practical pattern recognition problem.

II. Preliminaries

A. Intuitionistic fuzzy set (IFS)

Definition 1 ([3, Definition 1.1]): Let $X$ be the universe of discourse (UOD). An intuitionistic fuzzy set (IFS) $I$ in $X$ is defined as

$$I = \left\{ \frac{\langle \mu_{I}(x), \nu_{I}(x) \rangle}{x} \mid x \in X \right\},$$

(1)

where the functions $\mu_{I} : X \rightarrow [0, 1]$ and $\nu_{I} : X \rightarrow [0, 1]$ define the degree of membership and the degree of non-membership of the element $x$ in the set $I$, respectively, and for every $x \in X$, $\mu_{I}(x) + \nu_{I}(x) \leq 1$.

Let IFS(X) denote the set of all IFSs in X. For $I \in$ IFS(X), the indeterminacy degree $\pi_{I}(x)$ of an element $x$ belonging to $I$ is defined by $\pi_{I}(x) = 1 - \mu_{I}(x) - \nu_{I}(x)$.

Clearly, each subset $A$ of $X$ can be expressed as an IFS, $A = \left\{ \frac{\langle \mu_{A}(x), \nu_{A}(x) \rangle}{x \in X} \right\}$, which is also called a crisp set.

In [41], [40], the pair $\langle \mu_{I}(x), \nu_{I}(x) \rangle$ is called an intuitionistic fuzzy value (IFV) or an intuitionistic fuzzy number (IFN). For convenience, use $\alpha = (\mu_{\alpha}, \nu_{\alpha})$ to represent an IFV $\alpha$, which satisfies $\mu_{\alpha} \in [0, 1], \nu_{\alpha} \in [0, 1], \text{ and } 0 \leq \mu_{\alpha} + \nu_{\alpha} \leq 1$. Let $\Theta$ denote the set of all IFVs, i.e., $\Theta = \left\{ \langle \mu, \nu \rangle : \mu, \nu \in [0, 1], \mu + \nu \leq 1 \right\}$. For $\alpha = (\mu_{\alpha}, \nu_{\alpha}) \in \Theta$, the complement $\alpha^{C}$ of $\alpha$ is $\alpha^{C} = (\nu_{\alpha}, \mu_{\alpha})$.

Atanassov’s order ‘⊂’ [3], defined by the property that $\alpha \subset$ $\beta$ if and only if $\alpha \cap \beta = \alpha$ is a partial order on $\Theta$. Clearly, $\alpha \subset \beta$ if and only if $\mu_{\alpha} \leq \mu_{\beta}$ and $\nu_{\alpha} \geq \nu_{\beta}$. The order ‘$\not\subset$’ on $\Theta$ is defined by the property $\alpha$ $\not\subset \beta$ if and only if $\alpha \subset \beta$ and $\alpha \neq \beta$.

B. Similarity/Distance measures for IFSs

Definition 2 ([40], [25]): A mapping $S : \Theta \times \Theta \rightarrow [0, 1]$ is called an intuitionistic fuzzy similarity measure (IFSimM) on $\Theta$ if it satisfies the following conditions: for any $\alpha_{1}, \alpha_{2}, \alpha_{3} \in \Theta$,

(S1) $0 \leq S(\alpha_{1}, \alpha_{2}) \leq 1$.

(S2) $S(\alpha_{1}, \alpha_{2}) = 1$ if and only if $\alpha_{1} = \alpha_{2}$.

(S3) $S(\alpha_{1}, \alpha_{2}) = S(\alpha_{2}, \alpha_{1})$.

(S4) If $\alpha_{1} \subset \alpha_{2} \subset \alpha_{3}$, then $S(\alpha_{1}, \alpha_{3}) \leq S(\alpha_{1}, \alpha_{2})$ and $S(\alpha_{1}, \alpha_{3}) \leq S(\alpha_{2}, \alpha_{3})$.

Definition 3 ([40], [25]): Let $X$ be a UOD. A mapping $S : \text{IFS}(X) \times \text{IFS}(X) \rightarrow [0, 1]$ is called an intuitionistic fuzzy similarity measure (IFSimM) on IFS(X) if it satisfies the following conditions: for any $I_{1}, I_{2}, I_{3} \in \text{IFS}(X)$,

(S1) $0 \leq S(I_{1}, I_{2}) \leq 1$.

(S2) $S(I_{1}, I_{2}) = 1$ if and only if $I_{1} = I_{2}$.

(S3) $S(I_{1}, I_{2}) = S(I_{2}, I_{1})$.

(S4) If $I_{1} \subset I_{2} \subset I_{3}$, then $S(I_{1}, I_{3}) \leq S(I_{1}, I_{2})$ and $S(I_{1}, I_{3}) \leq S(I_{2}, I_{3})$. 
To effectively compare and distinguish IFVs and IFSs, we introduce the concept of strict intuitionistic fuzzy similarity/distance measure as follows.

**Definition 4:** A mapping $S : \Theta \times \Theta \rightarrow [0,1]$ is called a strict intuitionistic fuzzy similarity measure (SIFSimM) on $\Theta$ if, for any $\alpha_1, \alpha_2, \alpha_3 \in \Theta$, it satisfies (S1)–(S3) in Definition 3 and (S4') and (S5) described by

(S4') (Strict distinctiveness) If $\alpha_1 \not\subseteq \alpha_2 \not\subseteq \alpha_3$, then $S(\alpha_1, \alpha_3) < S(\alpha_1, \alpha_2)$ and $S(\alpha_1, \alpha_3) < S(\alpha_2, \alpha_3)$.

(S5) (Extreme dissimilarity on endpoints) $S(\alpha_1, \alpha_2) = 0$ if and only if $(\alpha_1 = (0,1)$ and $\alpha_2 = (1,0))$ or $(\alpha_1 = (1,0)$ and $\alpha_2 = (0,1))$.

Property (S4') indicates that the similarity measure $S$ can strictly distinguish every pair of different IFVs under Atanassov-order “$\subseteq$”. Property (S5) indicates that it is extremely unsimilar (similarity measure is zero) for a pair of IFVs depending only on two endpoints.

**Definition 5:** Let $X$ be a UOD. A mapping $S : IFS(X) \times IFS(X) \rightarrow [0,1]$ is called a strict intuitionistic fuzzy similarity measure (SIFSimM) on IFS(X) if, for any $I_1$, $I_2$, $I_3 \in IFS(X)$, it satisfies (S1)–(S3) in Definition 3 and (S4') and (S5) described by

(S4') If $I_1 \not\subseteq I_2 \not\subseteq I_3$, then $S(I_1, I_3) < S(I_1, I_2)$ and $S(I_1, I_3) < S(I_2, I_3)$.

(S5) $S(I_1, I_2) = 0$ if and only if, for any $x \in X$, $(I_1(x) = (0,1)$ and $I_2(x) = (1,0))$ or $(I_1(x) = (1,0)$ and $I_2(x) = (0,1))$.

**Remark 1:** Property (S5) can be equivalently expressed as that $S(I_1, I_2) = 0$ if and only if $I_1$ is a crisp set and $I_2 = I_2^c$.

Dually, a mapping $d : IFS(X) \times IFS(X) \rightarrow [0,1]$ is called a strict intuitionistic fuzzy distance measure (SIFDisM) on IFS(X) if it satisfies the following conditions:

(D1) The mapping $d$ is a distance measure on IFS(X) (see [24, Definition 3.1]);

(D2) The mapping $S(\alpha, \beta) = 1 - d(\alpha, \beta)$ is a SIFSimM on IFS(X).

### C. Entropy measure for IFSs

Entropy is an important information measure. Szmidt and Kacprzyk [42] gave the axiomatic definition of entropy measure for IFSs as follows:

**Definition 6 ([42]):** A mapping $E : \Theta \rightarrow [0,1]$ is called an intuitionistic fuzzy entropy measure (IFEM) on $\Theta$ if it satisfies the following conditions: for any $\alpha, \beta \in \Theta$,

(E1) $E(\alpha) = 0$ if and only if $\alpha = (1,0)$ or $\alpha = (0,1)$,

(E2) $E(\alpha) = 1$ if and only if $\mu_\alpha = \nu_\alpha$,

(E3) $E(\alpha) = E(\alpha^c)$,

(E4) $E(\alpha) \leq E(\beta)$ whenever it holds either $\mu_\alpha \leq \mu_\beta \leq \nu_\beta \leq \nu_\alpha$ or $\mu_\alpha \geq \mu_\beta \geq \nu_\beta \geq \nu_\alpha$.

**Definition 7 ([42]):** Let $X$ be a UOD. A mapping $E : IFS(X) \rightarrow [0,1]$ is called an intuitionistic fuzzy entropy measure (IFEM) on IFS(X) if it satisfies the following conditions: for any $I_1, I_2 \in IFS(X)$,

(E1) $E(I_1) = 0$ if and only if $I_1$ is a crisp set.

(E2) $E(I_1) = 1$ if and only if, for any $x \in X$, $\mu_{I_1}(x) = \nu_{I_1}(x)$.

(E3) $E(I_1) = E(I_1^c)$.

(E4) $E(I_1) \leq E(I_2)$ if, for any $x \in X$, it holds either $\mu_{I_1}(x) \leq \mu_{I_2}(x) \leq \nu_{I_2}(x) \leq \nu_{I_1}(x)$ or $\mu_{I_1}(x) \geq \mu_{I_2}(x) \geq \nu_{I_2}(x) \geq \nu_{I_1}(x)$.

### III. The drawbacks of Xiao’s distance measure $d_\chi$

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite UOD and $I_1 = \{\alpha^{(1)}_j \mid 1 \leq j \leq n, \alpha^{(1)}_j \in \Theta\}$ and $I_2 = \{\alpha^{(2)}_j \mid 1 \leq j \leq n, \alpha^{(2)}_j \in \Theta\}$ be two IFSs on $X$. Based on Jensen-Shannon divergence, Xiao [18] introduced a new distance measure $d_\chi$ for IFSs as follows:

$$d_\chi(I_1, I_2) = \frac{1}{n} \sum_{j=1}^{n} \left[ \left( \mu_{I_1}(x_j) \cdot \log_2 \frac{2\mu_{I_1}(x_j)}{\mu_{I_1}(x_j) + \mu_{I_2}(x_j)} + \mu_{I_2}(x_j) \cdot \log_2 \frac{2\mu_{I_2}(x_j)}{\mu_{I_1}(x_j) + \mu_{I_2}(x_j)} \right) + \nu_{I_1}(x_j) \cdot \log_2 \frac{2\nu_{I_1}(x_j)}{\nu_{I_1}(x_j) + \nu_{I_2}(x_j)} + \nu_{I_2}(x_j) \cdot \log_2 \frac{2\nu_{I_2}(x_j)}{\nu_{I_1}(x_j) + \nu_{I_2}(x_j)} \right]^{0.5}$$

where $\alpha^{(1)}_j = \langle \mu_{I_1}(x_j), \nu_{I_1}(x_j) \rangle$ and $\alpha^{(2)}_j = \langle \mu_{I_2}(x_j), \nu_{I_2}(x_j) \rangle$. Its dual similarity measure $S_\chi$ is defined by $S_\chi(I_1, I_2) = 1 - d_\chi(I_1, I_2)$.

The following examples show that Xiao’s distance measure defined by Eq. (2) has some drawbacks: (1) its dual similarity measure $S_\chi$ does not satisfy the axiomatic definition (S4) of Definition 2; (2) it cannot effectively distinguish some IFVs/IFSs with obvious size relationship; (3) Except for the endpoints $(1,0)$ and $(0,1)$, there exist infinitely many pairs such that the maximum distance 1 can be achieved under this distance.

**Example 1:** Let the UOD $X = \{x\}$ and choose $I_1 = \{\alpha^{(1)} \mid x \in (0.33,0.36)\}$, $I_2 = \{\alpha^{(2)} \mid x \in (0.5,1)\}$, and $I_3 = \{\alpha^{(3)} \mid x \in (0.973,0.99)\}$ \in $IFS(X)$. Clearly, $I_1 \subset I_2 \subset I_3$. Meanwhile, by Eq. (2) and direct calculation, we have $S_\chi(I_1, I_2) = 1 - d_\chi(I_1, I_2) = 0.973$ and $S_\chi(I_1, I_3) = 1 - d_\chi(I_1, I_3) = 0.9741713$, and thus $S_\chi(I_1, I_2) < S_\chi(I_1, I_3)$. This contradicts the axiomatic definition (S4) of Definition 2 because $I_1 \subset I_2 \subset I_3$.

Furthermore, take $I_1^{(\lambda)} = \{\alpha^{(1)} \mid x \in (0.33,0.36)\}$ and $I_2^{(\lambda)} = \{\alpha^{(2)} \mid x \in (0.33,0.36)\}$ \in $IFS(X)$ in Example 1. By varying the parameter $\lambda$ from 0 to 0.36, Fig. 1 visualizes the changing trend of distances between $I_1^{(\lambda)}$ and $I_1$, and between $I_1^{(\lambda)}$ and $I_1$, based on the distance measure $d_\chi$. Observing from Fig. 1, it is clear that there exists $\lambda^* \in (0,0.36)$ (the intersection of the two curves in Fig. 1) such that, for any $0 < \lambda < \lambda^*$, it holds that $d_\chi(I_1^{(\lambda)}, I_2^{(\lambda)}) > d_\chi(I_1, I_2^{(\lambda)})$. This further illustrates the unreasonable of the distance measure $d_\chi$ since $I_1 \subset I_2^{(\lambda)} \subset I_3^{(\lambda)}$. 
Hung and Yang [29] introduced the $J_\gamma$-divergence to measure the difference between two IFVs as follows: for $\alpha, \beta \in \Theta$,

- If $\gamma \in (0, 1) \cup (1, +\infty)$, $J_\gamma(\alpha, \beta) = \frac{1}{\gamma-1} \left( \left( \frac{\mu_\alpha + \mu_\beta}{2} \right)^\gamma - \frac{1}{2} (\mu_\alpha^\gamma + \mu_\beta^\gamma) \right) - \frac{1}{2} (\nu_\alpha^\gamma + \nu_\beta^\gamma) + \left( \frac{\pi_\alpha + \pi_\beta}{2} \right)^\gamma - \frac{1}{2} (\pi_\alpha^\gamma + \pi_\beta^\gamma) \right)$;

- If $\gamma = 1$, $J_1(\alpha, \beta) = \frac{1}{\lambda-1} \left( (\mu_\alpha + \mu_\beta) \ln \left( \frac{\mu_\alpha + \mu_\beta}{2} \right) - \mu_\alpha \cdot \ln \mu_\alpha - \mu_\beta \cdot \ln \mu_\beta + (\nu_\alpha + \nu_\beta) \ln \left( \frac{\nu_\alpha + \nu_\beta}{2} \right) - \nu_\alpha \cdot \ln \nu_\alpha - \nu_\beta \cdot \ln \nu_\beta \right) \ln (\pi_\alpha + \pi_\beta) - \pi_\alpha \cdot \ln \pi_\alpha - \pi_\beta \cdot \ln \pi_\beta$.

Remark 2: Joshi and Kumar [35] introduced an IF dissimilarity divergence $DJS$, which is equivalent to the $J_1$-divergence of Hung and Yang [29]. By direct calculation, it can be verified that $\sqrt{DJS(\alpha, \beta)} = \sqrt{J_1(\alpha, \beta)} = \ln 2 \cdot d_\chi(\alpha, \beta)$, and thus Example 1 also indicates that [29, Theorem 1 (D3)] and [35, Subsection 3.3] do not hold.

The following example illustrates the unreasonableess of Xiao’s distance measure $d_\chi$ from another perspective.

**Example 2:** Let the UOD $X = \{x\}$ and choose $I_1 = \{\frac{1+\lambda}{2}, \frac{1-\lambda}{2}\}$ and $I_2^{(\lambda)} = \{\frac{\lambda(0,0.00901)}{x}\} \in IFS(X)$. By varying the parameter $\lambda$ from $\frac{1}{3}$ to $1$, Fig. 2 visualizes the changing trend of distances between $I_2^{(\lambda)}$ and $I_1$ by using Xiao’s distance measure $d_\chi$. Observing from Fig. 2, it is clear that, for any $\lambda_1, \lambda_2 \in \left(\frac{1}{3}, 0.5\right)$ with $\lambda_1 < \lambda_2$, it holds that $d_\chi(I_1, I_2^{(\lambda_1)}) > d_\chi(I_1, I_2^{(\lambda_2)})$, which contradicts the fact that $I_1 \subset I_2^{(\lambda_1)} \subset I_2^{(\lambda_2)}$. This also illustrates the unreasonableess of the distance measure $d_\chi$.

By direct calculation, it follows from Eq. (2) that

$$d_\chi(I_1, I_2^{(\lambda)}) = \frac{1}{\gamma} \left[ \log_2 \left( \frac{2}{1 + \lambda} + \lambda \log_2 \left( \frac{2\lambda}{1 + \lambda} + (1 - \lambda) \log_2 \frac{2(1 - \lambda)}{1 - \lambda} \right) \right) \right]$$

and

$$d_\chi(I_1, I_3^{(\lambda)}) = \frac{1}{\gamma} \left[ \log_2 \left( \frac{2}{1 + \lambda} + \lambda \log_2 \left( \frac{2\lambda}{1 + \lambda} + (1 - \lambda) \log_2 \frac{2(1 - \lambda)}{1 - \lambda} \right) \right) \right]$$

(i) For $\lambda \neq 1$, it is clear that $I_2^{(\lambda)} \nsubsetneq I_3^{(\lambda)}$. It follows from Eq. (3) that Xiao’s distance measure $d_\chi$ cannot effectively distinguish $I_2^{(\lambda)}$ and $I_3^{(\lambda)}$ for $I_1$, outputing an unreasonable result with $I_1 \subset I_2^{(\lambda)} \nsubsetneq I_3^{(\lambda)}$.

(ii) For $0 < \lambda_1 < \lambda_2 < 1$, it is clear that $I_2^{(\lambda_1)} \nsubsetneq I_2^{(\lambda_2)}$. It follows from Eq. (4) that Xiao’s distance measure $d_\chi$ cannot effectively distinguish $I_2^{(\lambda_1)}$ and $I_2^{(\lambda_2)}$ with $\lambda_1 \neq \lambda_2$ from $I_1$, also outputing an unreasonable result with $I_2^{(\lambda_1)} \nsubsetneq I_2^{(\lambda_2)} \subset I_1$. Moreover, it follows from Eq. (4) that there exist infinite IFSs $I_2^{(\lambda)} (\lambda \in [0, 1])$ such that the distance from $I_1$ is equal to the maximum value of 1.

**IV. THE DRAWBACKS OF YANG AND CHICLANA’S SPHERICAL DISTANCE $d_{yc}$**

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite UOD and $I_1 = \left\{ \frac{\alpha^{(1)}}{x_j} \mid 1 \leq j \leq n, \alpha^{(1)} \in \Theta \right\}$ and $I_2 = \left\{ \frac{\alpha^{(2)}}{x_j} \mid 1 \leq j \leq n, \alpha^{(2)} \in \Theta \right\}$ be two IFSs on $X$. Based on the $\arccos$ function, Yang and Chiclana [33] introduced a spherical distance $d_{yc}$ for IFSs as follows:

$$d_{yc}(I_1, I_2) = \frac{2}{n\pi} \sum_{j=1}^{n} \arccos \left( \sqrt{\mu_{I_1}(x_j) \mu_{I_2}(x_j)} + \sqrt{\nu_{I_1}(x_j) \nu_{I_2}(x_j)} \right)$$

for $\alpha^{(1)} = (\mu_{I_1}(x_j), \nu_{I_1}(x_j))$ and $\alpha^{(2)} = (\mu_{I_2}(x_j), \nu_{I_2}(x_j))$. Its dual similarity measure $S_{yc}$ is defined by $S_{yc}(I_1, I_2) = 1 - d_{yc}(I_1, I_2)$.

The following examples show that Yang and Chiclana’s spherical distance has the same drawbacks as Xiao’s distance measure.

**Example 4:** Let $X = \{x\}$ and choose $I_1 = \left\{ \frac{(0.5,0.5)}{x} \right\}$, $I_2 = \left\{ \frac{(0.6,0.3)}{x} \right\}$, and $I_3 = \left\{ \frac{(0.7,0.3)}{x} \right\} \in IFS(X)$. Clearly, $I_1 \subset I_2 \subset I_3$. Meanwhile, by Eq. (5) and direct calculation, we have $S_{yc}(I_1, I_2) = 1 - d_{yc}(I_1, I_2) = 1 - \frac{2}{\pi} \arccos(\sqrt{0.3 + \sqrt{0.15}}) < 1 - \frac{2}{\pi} \arccos(\sqrt{0.15} + \sqrt{0.15}) = 1 - d_{yc}(I_1, I_3) = S_{yc}(I_1, I_3)$. This contradicts the axiomatic definition (S4) of Definition 2 because $I_1 \subset I_2 \subset I_3$.
Example 5: Assume that IFSs $I_1$, $I_2$, and $I_3$ on UOD $X = \{x\}$ are given by $I_1 = \{\frac{(1,0)}{x}, \frac{(0,1)}{x}\}$, $I_2 = \{\frac{\lambda}{x}, \frac{(0,1)}{x}\}$, and $I_3 = \{\frac{\lambda}{x}, \frac{(0,1)}{x}\}$, where $0 \leq \lambda \leq 1$. By direct calculation, we have

$$d_{vc}(1, I_2(\lambda)) = d_{vc}(1, I_3(\lambda)) = \frac{2}{\pi} \arccos(\sqrt{\lambda}),$$

and

$$d_{vc}(1, I_2(\lambda)) = 1.$$

(i) For $\lambda \neq 1$, it is clear that $I_2(\lambda) \neq I_3(\lambda)$. Form follows from Eq. (6) that Yang and Chiclana’s spherical distance $d_{vc}$ cannot effectively distinguish $I_2(\lambda)$ and $I_3(\lambda)$ from $I_1$, outputting a unreasonable result with $I_1 \supseteq I_2(\lambda) \supseteq I_3(\lambda)$.

(ii) For $\lambda \neq 0$, it is clear that $I_2(\lambda) \neq I_2(\lambda)$, it follows from Eq. (7) that Yang and Chiclana’s spherical distance $d_{vc}$ cannot effectively distinguish $I_2(\lambda)$ and $I_2(\lambda)$ with $\lambda \neq 0$ from $I_1$, also outputting a unreasonable result with $I_2(\lambda) \supseteq I_2(\lambda) \supseteq I_1$. Meanwhile, it follows from Eq. (7) that there exist infinite IFSs $I_2(\lambda)$ ($\lambda \in [0, 1]$), where the distance from $I_1$ is equal to the maximum value 1.

V. A NOVEL SIFDisM/SIFSimM BASED ON JENSEN-SHANNON DIVERGENCE

In this section, we propose a new strict distance measure and a new strict similarity measure for IFVs and IFSs based on Jensen-Shannon divergence, which can overcome the drawbacks of Xiao’s distance measure and Hung and Yang’s $J_{\gamma}$-divergence discussed in the previous section.

A. A new distance/similarity measure on IFVs

Clearly, an IFV $\alpha = (\mu, \nu)$ can be equivalently expressed as an interval $[\nu, 1 - \mu]$, and thus we can use $-(\nu \ln \nu + (1 - \mu) \ln(1 - \mu))$ to express the Shannon entropy $H(\alpha)$ of $\alpha$. For $\alpha = (\mu_\alpha, \nu_\alpha)$, $\beta = (\mu_\beta, \nu_\beta) \in \Theta$, by applying Jensen-Shannon divergence, we define the Jensen-Shannon IF divergence measure $JS_{\beta_{\alpha}}(\alpha, \beta)$ between $\alpha$ and $\beta$ as follows:

$$JS_{\beta_{\alpha}}(\alpha, \beta) = H(\alpha) = \frac{1}{2} \cdot (1 - \mu_\alpha) \cdot \ln \frac{(1 - \mu_\alpha)}{(1 - \mu_\alpha) + (1 - \mu_\beta)} + (1 - \mu_\beta) \cdot \ln \frac{(1 - \mu_\beta)}{(1 - \mu_\alpha) + (1 - \mu_\beta)} + \nu_\alpha \cdot \ln \frac{2 \nu_\alpha}{\nu_\alpha + \nu_\beta} + \nu_\beta \cdot \ln \frac{2 \nu_\beta}{\nu_\alpha + \nu_\beta},$$

where $0 \leq \ln 0 = 0 \cdot \ln 0 = 0$ and $\frac{\alpha + \beta}{2} = \frac{(\mu_\alpha + \mu_\beta, \nu_\alpha + \nu_\beta)}{2}$.

Let $R^+ = \{x \in R \mid x \geq 0\}$. To obtain a new metric for probability distributions, Endres and Schindelin [43] introduced the following function $L$ and presented some basic properties.

Definition 8 ([43, Definition 1]): Define the function $L(p, q) : R^+ \times R^+ \rightarrow R^+$ by

$$L(p, q) = p \cdot \log_2 \frac{2p}{p + q} + q \cdot \log_2 \frac{2q}{p + q}.$$

Remark 3: Endres and Schindelin [43] proved that the function $L(\mu, \nu)$ is well defined, i.e., $L(p, q) \geq 0$ holds for all $p, q \in R^+$.

Lemma 1 ([43, Lemma 1]): Let $p, q, r \in R^+$. Then,

$$\sqrt{L(p, q)} \leq \sqrt{L(p, r)} + \sqrt{L(r, q)}.$$

For simplicity of presentation, denote $Z(\alpha, \beta) = L(1 - \mu_\alpha, 1 - \mu_\beta) + L(\nu_\alpha, \nu_\beta)$.

Define a function $\zeta(x) = x \cdot \log_2 (2x) + (1 - x) \cdot \log_2 (2(1 - x))$ ($x \in (0, 1)$). By direct derivation, we have

$$\zeta'(x) = \log_2 \frac{x}{1 - x} = \begin{cases} < 0, & 0 < x < 0.5, \\ > 0, & 0.5 < x < 1, \end{cases}$$

implying that $\zeta(\mu)$ is strictly decreasing on $(0, 0.5)$ and strictly increasing on $(0.5, 1)$. Thus, for any $x \in (0, 1)$,

$$0 = \zeta(0.5) \leq \zeta(x) \leq \max \{\lim_{x \to 0^+} \zeta(x), \lim_{x \to 1^-} \zeta(x)\} = 1.$$  

Meanwhile, by direct calculation, we get

$$Z(\alpha, \beta) = (1 - \mu_\alpha) \cdot \log_2 \frac{2(1 - \mu_\alpha)}{(1 - \mu_\alpha) + (1 - \mu_\beta)} + (1 - \mu_\beta) \cdot \log_2 \frac{2(1 - \mu_\beta)}{(1 - \mu_\alpha) + (1 - \mu_\beta)} + \nu_\alpha \cdot \log_2 \frac{2\nu_\alpha}{\nu_\alpha + \nu_\beta} + \nu_\beta \cdot \log_2 \frac{2\nu_\beta}{\nu_\alpha + \nu_\beta}.$$

This, together with Eq. (9), implies that $Z(\alpha, \beta) \geq 0$, and thus $JS_{\beta_{\alpha}}(\alpha, \beta) \geq \ln \frac{2}{Z(\alpha, \beta)} \geq 0$ by applying Eq. (8).

Applying the square root of $JS_{\beta_{\alpha}}$, we define the normalized Jensen-Shannon IF divergence measure $JS_{\beta_{\alpha}}(\alpha, \beta)$ as follows:

$$JS_{\beta_{\alpha}}(\alpha, \beta) = \sqrt{Z(\alpha, \beta)}.$$

Clearly, $JS_{\beta_{\alpha}}(\alpha, \beta) = \sqrt{Z(\mu_\alpha, 1 - \mu_\beta) + L(\nu_\alpha, \nu_\beta)}$

where $\frac{2(1 - \mu_\alpha)}{(1 - \mu_\alpha) + (1 - \mu_\beta)} + (1 - \mu_\beta) \cdot \log_2 \frac{2(1 - \mu_\beta)}{(1 - \mu_\alpha) + (1 - \mu_\beta)} + \nu_\alpha \cdot \log_2 \frac{2\nu_\alpha}{\nu_\alpha + \nu_\beta} + \nu_\beta \cdot \log_2 \frac{2\nu_\beta}{\nu_\alpha + \nu_\beta}$.

Property 1: $JS_{\beta_{\alpha}}(\alpha, \beta) \leq JS_{\beta_{\alpha}}(\beta, \alpha)$.

Proof: It follows directly from Eq. (11).

Lemma 2: Fix an IFV $\alpha = (\mu_\alpha, \nu_\alpha) \in \Theta$. Then, for any $\beta_1, \beta_2 \in \Theta$ with $\alpha \subseteq \beta_1 \subseteq \beta_2$, we have $Z(\alpha, \beta_1) = Z(\beta_1, \alpha) \leq Z(\alpha, \beta_2) = Z(\beta_2, \alpha)$.

Because we define $0 \cdot \log_0 0 = 0$, one has $\lim_{x \to 0^+} \zeta(x) = 1 = \zeta(0)$ and $\lim_{x \to 1^-} \zeta(x) = 1 = \zeta(1)$. 

\[\]
Proof: For any \( \beta = (\mu, \nu) \in \Theta \), it is clear that

\[
\mathcal{Z}(\alpha, \beta) = \left[ (1 - \mu_\alpha) \cdot \log_2 \frac{2(1 - \mu_\alpha)}{(1 - \mu_\alpha) + (1 - \mu)} + (1 - \mu) \cdot \log_2 \frac{2(1 - \mu)}{(1 - \mu_\alpha) + (1 - \mu)} + \nu_\alpha \cdot \log_2 \frac{2\nu_\alpha}{\nu_\alpha + \nu} \right].
\]

By Eq. (12) and direct calculation, we have that, for \( \mu \geq \mu_\alpha \) and \( \nu \leq \nu_\alpha \),

\[
\frac{\partial \mathcal{Z}}{\partial \mu} = \frac{1 - \mu_\alpha}{\ln 2} \cdot \left( \frac{1}{(1 - \mu_\alpha) + (1 - \mu)} - \log_2 \frac{1}{(1 - \mu_\alpha) + (1 - \mu)} - \frac{1 - \mu_\alpha}{\ln 2} \cdot \frac{1}{(1 - \mu_\alpha) + (1 - \mu)} \right) \geq 0,
\]

and

\[
\frac{\partial \mathcal{Z}}{\partial \nu} = -\frac{\nu_\alpha}{(\nu_\alpha + \nu) \cdot \ln 2} + \log_2 \frac{2\nu}{\nu_\alpha + \nu} + \frac{\nu_\alpha}{(\nu_\alpha + \nu) \cdot \ln 2} \leq 0.
\]

Let \( \beta_1 = (\mu_1, \nu_1) \) and \( \beta_2 = (\mu_2, \nu_2) \). From \( \alpha \subseteq \beta_1 \subseteq \beta_2 \), it follows that \( \mu \leq \mu_\alpha \leq \mu_\beta \) and \( \nu \geq \nu_\alpha \geq \nu_\beta \). Together with Eqs. (13) and (14), we get \( \mathcal{Z}(\alpha, \beta_1) \leq \mathcal{Z}(\alpha, \beta_2) \leq \mathcal{Z}(\beta_1, \beta_2) \).

**Lemma 3**: Fix an IFV \( \alpha = (\mu_\alpha, \nu_\alpha) \in \Theta \). Then, for any \( \beta_1, \beta_2 \in \Theta \) with \( \alpha \subseteq \beta_1 \subseteq \beta_2 \), we have \( \mathcal{Z}(\alpha, \beta_1) = \mathcal{Z}(\beta_1, \alpha) < \mathcal{Z}(\beta_2, \alpha) \) and \( \mathcal{J}_{SP}^\alpha(\alpha, \beta_1) < \mathcal{J}_{SP}^\alpha(\beta_2, \alpha) \).

**Proof**: For any \( \beta = (\mu, \nu) \in \Theta \) with \( \alpha \not\subseteq \beta \), by Eqs. (13) and (14), we have

1. If \( \mu \geq \mu_\alpha \) and \( \nu \leq \nu_\alpha \), then
   \[
   \frac{\partial \mathcal{Z}}{\partial \mu} = -\log_2 \frac{2(1 - \mu)}{(1 + (1 - \mu_\alpha)} > 0.
   \]
2. If \( \mu \geq \mu_\alpha \) and \( \nu \leq \nu_\alpha \), then
   \[
   \frac{\partial \mathcal{Z}}{\partial \nu} = \log_2 \frac{2\nu}{\nu_\alpha + \nu} < 0.
   \]

For \( \beta_1 = (\mu_1, \nu_1) \) and \( \beta_2 = (\mu_2, \nu_2) \) with \( \alpha \not\subseteq \beta_1 \subseteq \beta_2 \), consider the following cases:

1. If \( \mu_\alpha < \mu_\beta \leq \mu_\beta \) and \( \nu_\alpha \geq \nu_\beta \), then, by Eqs. (14) and (15), we have
   \[
   \mathcal{Z}(\alpha, \beta_1) = \mathcal{Z}(\alpha, \beta_2) < \mathcal{Z}(\alpha, \beta_2), \quad \text{(by Eq. (15))}
   \]
   \[
   \mathcal{Z}(\alpha, \beta_2) = \mathcal{Z}(\beta_2, \alpha) \quad \text{(by Eq. (14))},
   \]
   \[
   \mathcal{Z}(\alpha, \beta_1) = \mathcal{Z}(\beta_2, \alpha) \quad \text{(by Eq. (14))}.
   \]

2. If \( \mu_\alpha \leq \mu_\beta \leq \mu_\beta \) and \( \nu_\alpha > \nu_\beta \), then, by Eqs. (13) and (16), we have
   \[
   \mathcal{Z}(\alpha, \beta_1) = \mathcal{Z}(\alpha, \beta_2) \quad \text{(by Eq. (16))},
   \]
   \[
   \mathcal{Z}(\alpha, \beta_2) = \mathcal{Z}(\alpha, \beta_2) \quad \text{(by Eq. (16))}.
   \]

Summing up the above shows that \( 0 \leq \mathcal{Z}(\alpha, \beta) \leq 2 \).
Property 3: \( JS_{IP}(\alpha, \beta) = 1 \) if and only if \((\alpha = (0, 1) \text{ and } \beta = (1, 0)) \) or \((\alpha = (1, 0) \text{ and } \beta = (0, 1)) \).

Proof: Sufficiency. By direct calculation and Eq. (11), it follows that \( JS_{IP}(1, 0) = JS_{IP}(0, 1) = 1 \).

Necessity. Fix \( \beta = (\mu_\beta, \nu_\beta) \in \Theta\{(0, 1), (1, 0)\} \). For any \( \alpha \in \Theta\{(0, 1), (1, 0)\} \), according to the proof of Property 2, consider the following four cases:

1. If \( \mu_\alpha \geq \mu_\beta \) and \( \nu_\alpha \leq \nu_\beta \), by \( \nu_\beta < 1 \), we have \( Z(\alpha, \beta) \leq (1 - \mu_\beta) + \nu_\beta < (1 - \mu_\beta) + 1 \leq 2 \), i.e., \( Z(\alpha, \beta) \leq 2 \).

2. If \( \mu_\alpha \leq \mu_\beta \) and \( \nu_\alpha \geq \nu_\beta \), by \( \mu_\beta < 1 \), we have \( Z(\alpha, \beta) \leq 1 + \log_2 \frac{\nu_\alpha}{1 - \nu_\alpha} < 2 \).

3. If \( \mu_\alpha \leq \mu_\beta \) and \( \nu_\alpha \leq \nu_\beta \), by \( \nu_\beta < 1 \), we have \( Z(\alpha, \beta) \leq 1 + \log_2 \frac{\nu_\alpha}{1 - \nu_\alpha} \).

4. If \( \mu_\alpha \geq \mu_\beta \) and \( \nu_\alpha \geq \nu_\beta \), by \( \nu_\beta < 1 \), we have \( Z(\alpha, \beta) \leq 1 + \log_2 \frac{\nu_\alpha}{1 - \nu_\alpha} \).

This, together with Eq. (17), implies that \( JS_{IP}(\alpha, \beta) = 1 \).

Property 4: \( JS_{IP}(\alpha, \beta) = 0 \) if and only if \( \alpha = \beta \).

Proof: Sufficiency. By Eq. (11), it is clear that \( JS_{IP}(\alpha, \beta) = 0 \) if \( \alpha = \beta \).

Necessity. Assume that \( JS_{IP}(\alpha, \beta) = 0 \). It will be shown that \( \alpha = \beta \).

Suppose on the contrary that \( \alpha \neq \beta \). Without loss of generality, assume \( 0 \leq \nu_\alpha < \nu_\beta \). From Eq. (10), it follows that \( Z(\alpha, \beta) \geq (\nu_\alpha + \nu_\beta) \cdot \zeta(\frac{\nu_\alpha}{\nu_\alpha + \nu_\beta}) \cdot (1 - \mu_\beta) \cdot \zeta(\frac{\nu_\alpha}{\nu_\alpha + \nu_\beta}) \) This, together with the fact that \( \zeta(\cdot) \) is strictly decreasing on \([0, 0.5] \) and \( \frac{\nu_\alpha}{\nu_\alpha + \nu_\beta} \in \left(0, \frac{1}{2}\right) \), implies that \( Z(\alpha, \beta) \geq (\nu_\alpha + \nu_\beta) \cdot \zeta(\frac{\nu_\alpha}{\nu_\alpha + \nu_\beta}) > 0 \).

Therefore, \( 0 = JS_{IP}(\alpha, \beta) = \sqrt{\frac{Z(\alpha, \beta)}{2}} > 0 \), which is a contradiction.

Property 5: Let \( \alpha, \beta, \gamma \in \Theta \).

1. If \( \alpha \subset \beta \subset \gamma \), then \( JS_{IP}(\alpha, \beta) \leq JS_{IP}(\alpha, \gamma) \) and \( JS_{IP}(\beta, \gamma) \leq JS_{IP}(\alpha, \gamma) \).

2. If \( \alpha \not\subset \beta \not\subset \gamma \), then \( JS_{IP}(\alpha, \beta) < JS_{IP}(\alpha, \gamma) \) and \( JS_{IP}(\beta, \gamma) < JS_{IP}(\alpha, \gamma) \).

Proof: (1) Fix the IFV \( \alpha \). It follows directly from Eq. (12) that \( Z(\alpha, \beta) \leq Z(\alpha, \gamma) \). Thus, \( JS_{IP}(\alpha, \beta) = \sqrt{\frac{Z(\alpha, \beta)}{2}} \leq \sqrt{\frac{Z(\alpha, \gamma)}{2}} = JS_{IP}(\alpha, \gamma) \). Similarly, we can prove \( JS_{IP}(\beta, \gamma) \leq JS_{IP}(\alpha, \gamma) \) by Lemma 4).

(2) Similarly, we can prove \( JS_{IP}(\alpha, \beta) < JS_{IP}(\alpha, \gamma) \) and \( JS_{IP}(\beta, \gamma) < JS_{IP}(\alpha, \gamma) \) by Lemmas 3 and 5.

Property 6 (Triangle inequality): Let \( \alpha, \beta, \gamma \in \Theta \). Then, \( J S_{IP}(\alpha, \gamma) \geq JS_{IP}(\alpha, \beta) + JS_{IP}(\beta, \gamma) \).

Proof: For convenience, denote \( \sqrt{L(1 - \mu_\alpha, 1 - \mu_\beta)} = \xi_1 \), \( \sqrt{L(\nu_\alpha, \nu_\beta)} = \eta_1 \), \( \sqrt{L(1 - \mu_\beta, 1 - \mu_\gamma)} = \xi_2 \), \( \sqrt{L(\nu_\beta, \nu_\gamma)} = \eta_2 \), \( \sqrt{L(1 - \mu_\alpha, 1 - \mu_\gamma)} = \xi_3 \), and \( \sqrt{L(\nu_\alpha, \nu_\gamma)} = \eta_3 \). By Lemma 1, we have

\[
\xi_1 \leq \xi_3 + \xi_2, \quad \eta_1 \leq \eta_3 + \eta_2.
\]

Meanwhile, from \( (\xi_1^2 + \eta_1^2) \cdot (\xi_2^2 + \eta_2^2) - (\xi_1 \xi_2 + \eta_1 \eta_2)^2 \geq 0 \), it follows that \( (\sqrt{\xi_1^2 + \eta_1^2} + \sqrt{\xi_2^2 + \eta_2^2})^2 = \xi_1^2 + \eta_1^2 + \xi_2^2 + \eta_2^2 + 2\sqrt{\xi_1^2 + \eta_1^2} \cdot \sqrt{\xi_2^2 + \eta_2^2} \geq \xi_1^2 + \eta_1^2 + \xi_2^2 + \eta_2^2 + 2(\xi_1 \xi_2 + \eta_1 \eta_2) = (\xi_1 + \xi_2)^2 + \eta_1^2 + \eta_2^2.\]

This, together with Eq. (17), implies that \( JS_{IP}(\alpha, \gamma) = \sqrt{\frac{1}{2} (\xi_1 + \xi_2)^2 + (\eta_1 + \eta_2)^2} \). Summing Properties 1–6, we have the following results.

**Theorem 1:** (1) The divergence measure \( JS_{IP} \) is a SIFDisM on \( \Theta \).

2. The function \( S_{IP}(\alpha, \beta) = 1 - JS_{IP}(\alpha, \beta) \) is a SIFSimM on \( \Theta \).

**Theorem 2:** The mapping \( E \) defined by

\[
E : \Theta \rightarrow [0, 1],
\]

\[
\alpha \rightarrow 1 - JS_{IP}(\alpha, \alpha^\beta),
\]

is an IFEM on \( \Theta \).

Proof: (E1), (E2), and (E3) follow directly from Properties 3 and 4, and Eq. (18), respectively.

(E4) For \( \alpha, \beta \in \Theta \), consider the following two cases:

E4-1 If \( \mu_\alpha \leq \mu_\beta \leq \nu_\beta \leq \nu_\alpha \), then \( \alpha \subset \beta \subset \beta^\alpha \subset \alpha^\beta \). This, together with Property 5, implies that \( E(\alpha) = 1 - JS_{IP}(\alpha, \alpha^\beta) \leq 1 - JS_{IP}(\alpha, \beta) \leq 1 - JS_{IP}(\beta, \beta^\beta) = E(\beta) \).

E4-2 If \( \mu_\alpha \geq \mu_\beta \geq \nu_\beta \geq \nu_\alpha \), then \( \alpha \subset \beta \subset \beta^\alpha \subset \alpha^\beta \). This, together with Property 5, implies that \( E(\alpha) = 1 - JS_{IP}(\alpha, \alpha^\beta) \leq 1 - JS_{IP}(\alpha, \beta) \leq 1 - JS_{IP}(\beta, \beta^\beta) = E(\beta) \).

**Fig. 3** shows the graph of IFEM in Theorem 2.

\[
\text{Fig. 3. The graph of IFEM } E \text{ in Theorem 2.}
\]
Applying the the normalized Jensen-Shannon IF divergence measure $\mathcal{JS}_{1P}$ on IFVs, we define a new normalized distance measure $d_{wu}(I_1, I_2)$ as follows:

\[
d_{wu}(I_1, I_2) = \sum_{j=1}^{n} \omega_j \cdot \mathcal{JS}_{1P}(\alpha_j^{(1)}, \alpha_j^{(2)})
\]

\[
= \sum_{j=1}^{n} \omega_j \left[ \frac{1}{2} \left( \log_2 \frac{2(1 - \mu_1(x_j))}{1 - \mu_1(x_j) + 1 - \mu_2(x_j)} + \log_2 \frac{2(1 - \mu_2(x_j))}{1 - \mu_1(x_j) + 1 - \mu_2(x_j)} + \log_2 \frac{2\nu_1(x_j)}{\nu_1(x_j) + \nu_2(x_j)} \right) \right] 0.5.
\]

where $\alpha_j^{(1)} = (\mu_1(x_j), \nu_1(x_j))$, $\alpha_j^{(2)} = (\mu_2(x_j), \nu_2(x_j))$, and $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weight vector of $x_j (j = 1, 2, \ldots, n)$ with $\omega_j \in [0, 1]$ and $\sum_{j=1}^{n} \omega_j = 1$. Clearly,

\[
d_{wu}(I_1, I_2) = \sum_{j=1}^{n} \omega_j \cdot \sqrt{Z(\alpha_j^{(1)}, \alpha_j^{(2)}) / 2}.
\]

By applying Properties 1–6, one can easily verify that the function $d_{wu}$ has the following properties.

Property 7: $d_{wu}(I_1, I_2) = d_{wu}(I_2, I_1)$.

Property 8: $0 \leq d_{wu}(I_1, I_2) \leq 1$.

Property 9: $d_{wu}(I_1, I_2) = 1$ if and only if, for any $1 \leq j \leq n$, $(\alpha_j^{(1)} = (0, 1))$ or $(\alpha_j^{(2)} = (1, 0))$ and $(\alpha_j^{(1)} = (0, 1))$ or $(\alpha_j^{(2)} = (1, 0))$.

Property 10: $d_{wu}(I_1, I_2) = 0$ if and only if $I_1 = I_2$.

Property 11: Let $X = \{x_1, x_2, \ldots, x_3\}$ and $I_1, I_2, I_3 \in IFS(X)$.

1. If $I_1 \subseteq I_2 \subseteq I_3$, then $d_{wu}(I_1, I_2) \leq d_{wu}(I_1, I_3)$ and $d_{wu}(I_2, I_3) \leq d_{wu}(I_1, I_3)$.

2. (1) If $I_1 \subseteq I_2 \subseteq I_3$, then $d_{wu}(I_1, I_2) < d_{wu}(I_1, I_3)$ and $d_{wu}(I_2, I_3) < d_{wu}(I_1, I_3)$.

Summing Properties 7–11, similarly to the proof of Theorem 2, we have the following results.

Theorem 3: (1) The distance measure $d_{wu}$ defined by Eq. (19) is a SIFDisM on IFS(X).

(2) The function $S_{wu}(I_1, I_2) = 1 - d_{wu}(I_1, I_2)$ is a SIFSImM on IFS(X).

Theorem 4: The mapping $E$ defined by

\[
E : \Theta \rightarrow [0, 1], \quad \alpha \longmapsto 1 - d_{wu}(\alpha, \alpha^c),
\]

is an IFEM on IFS(X).

VI. COMPARATIVE ANALYSIS

Xiao [18] showed that the distance measure $d_\chi$ is better than other existing distance measures proposed in [27, 28, 30, 36, 37, 38, 39] by some numerical comparisons. However, for the nonlinear distance measure $d_{vc}$ introduced by Yang and Chiclana [33], only one figure (see [18, Fig. 4]) was used to show that the curve of $d_\chi$ is sharper than that of $d_{vc}$ for some special cases. This does not convincingly explain the superiority of the distance measure $d_\chi$. This section demonstrates that our proposed distance measure $d_{wu}$ is completely superior to Xiao’s distance measure $d_\chi$ and Yang and Chiclana’s distance measure $d_{vc}$. Because of the duality of distance and similarity measures, for brevity we only compare and analyze distance measures.

Xiao [18] used some examples to illustrate the superiority of the distance measure $d_\chi$. The following shows that our proposed distance measure has the same superiority for the same numerical examples.

Example 6 ([18, Examples 3]): Assume that the IFSs $I_1$ and $I_2$ on UOD $X = \{x\}$ are given by $I_1 = \{(\mu, \nu)\}$ and $I_2 = \{(\mu, \nu)\}$.

Fig. 4 shows the changing trend of distance $d_{wu}(I_1, I_2)$ with varying parameters $\mu$ and $\nu$ satisfying $(\mu, \nu) \in \Theta$. Observing from Fig. 4, it can be seen that our proposed distance measure $d_{wu}$ has a form similar to the distance measure $d_\chi$ shown in [18, Fig. 2] with $\mathcal{JS}_{1P}(0, 1, \langle 1, 0 \rangle) = \mathcal{JS}_{1P}(0, 1, \langle 1, 0 \rangle) = 0$.

**Example 7** ([18, Examples 5 and 6]): Assume that the IFSs $A_i$ and $B_i$ on the UOD $X = \{x_1, x_2\}$ in Case $i (i = 1, 2, 3, 4, 5)$ are given as shown in Table I.

| Case | A_i | B_i |
|------|-----|-----|
| 1    | (0.30, 0.20, 0.40, 0.30) | (0.15, 0.25, 0.25, 0.35) |
| 2    | (0.16, 0.26, 0.26, 0.36) | (0.16, 0.26, 0.26, 0.36) |
| 3    | (0.50, 0.40, 0.40, 0.30) | (0.50, 0.40, 0.40, 0.30) |
| 4    | (0.15, 0.25, 0.25, 0.35) | (0.15, 0.25, 0.25, 0.35) |
| 5    | (0.55, 0.45, 0.55, 0.45) | (0.45, 0.55, 0.45, 0.55) |

The comparative results produced by Xiao’s distance measure $d_\chi$ and our distance measure $d_{wu}$ are displayed in Table II, which indicates that both distance measures can effectively distinguish $A_i$ and $B_i$ in Cases 1–5.
A. Comparative analysis between Xiao’s distance measure \( d_X \) and our proposed distance \( d_{\text{wu}} \)

**Example 8 (Continuation of Example 3):** Let IFSs \( I_1 \), \( I_1' \), \( I_2(\lambda) \), and \( I_3(\lambda) \) on UOD \( X = \{ x \} \) be given as in Example 3. By direct calculation, it follows from Eq. (2) that

\[
\begin{align*}
\chi(I_1, I_3(\lambda)) &= \sqrt{\frac{1}{2} \log_2 \left( \frac{2\lambda}{\lambda} + (1 - \lambda) \log_2 \frac{2(1 - \lambda)}{2 - \lambda} \right)}, \\
\chi(I_1', I_2(\lambda)) &= \sqrt{\frac{1}{2} \log_2 \left( \frac{2\lambda}{\lambda} + (1 - \lambda) \log_2 \frac{2(1 - \lambda)}{2 - \lambda} \right)}.
\end{align*}
\]

By applying our distance measure \( d_{\text{wu}} \) defined in Eq. (19), we obtain

\[
\begin{align*}
d_{\text{wu}}(I_1, I_2(\lambda)) &= \sqrt{\frac{1}{2} \log_2 \left( \frac{2\lambda}{\lambda} + (1 - \lambda) \log_2 \frac{2(1 - \lambda)}{2 - \lambda} \right)}, \\
&= \sqrt{\frac{1}{2} \log_2 \left( \frac{2\lambda}{\lambda} + (1 - \lambda) \log_2 \frac{2(1 - \lambda)}{2 - \lambda} \right)}.
\end{align*}
\]

and

\[
\begin{align*}
d_{\text{wu}}(I_1', I_2(\lambda)) &= \sqrt{\frac{1}{2} \log_2 \left( \frac{2\lambda}{\lambda} + (1 - \lambda) \log_2 \frac{2(1 - \lambda)}{2 - \lambda} \right)}, \\
&= \sqrt{\frac{1}{2} \log_2 \left( \frac{2\lambda}{\lambda} + (1 - \lambda) \log_2 \frac{2(1 - \lambda)}{2 - \lambda} \right)}.
\end{align*}
\]

Directly observing the values of \( I_1 \), \( I_1' \), \( I_2(\lambda) \), and \( I_3(\lambda) \), we find that \( I_1' \subset I_3(\lambda) \subset I_2(\lambda) \subset I_1 \), and thus \( I_2(\lambda) \) is more similar to \( I_1 \) than \( I_3(\lambda) \), and \( I_3(\lambda) \) is more similar to \( I_1' \) than \( I_2(\lambda) \), which are consistent with our computed results, since \( S_{\text{wu}}(I_1, I_3(\lambda)) = 1 - d_{\text{wu}}(I_1, I_3(\lambda)) = 1 - \sqrt{\frac{1}{2} \log_2 \left( \frac{2\lambda}{\lambda} + (1 - \lambda) \log_2 \frac{2(1 - \lambda)}{2 - \lambda} \right)} = 1 \), \( d_{\text{wu}}(I_1, I_2(\lambda)) = S_{\text{wu}}(I_1, I_2(\lambda)) \), and \( d_{\text{wu}}(I_1', I_2(\lambda)) = 1 - \sqrt{\frac{1}{2} \log_2 \left( \frac{2\lambda}{\lambda} + (1 - \lambda) \log_2 \frac{2(1 - \lambda)}{2 - \lambda} \right)} = 1 - d_{\text{wu}}(I_1', I_2(\lambda)). \)

By varying the parameter \( \lambda \) from 0 to 1, the results in Figs. 5 and 6 visualize the changing trend of distances between \( I_2(\lambda) \) and \( I_1(I_1') \) and between \( I_2(\lambda) \) and \( I_1(I_1') \) discussed in Example 8 by using Xiao’s distance measure \( d_X \) and our proposed distance measure \( d_{\text{wu}} \). The simulation results are consistent with our calculation. This example indicates that our proposed distance is far superior to Xiao’s distance measure \( d_X \) in [18].

**Fig. 5.** The distances between \( I_2(\lambda) \) and \( I_1 \), and between \( I_3(\lambda) \) and \( I_1 \) in Example 8

**Fig. 6.** The distances between \( I_2(\lambda) \) and \( I_1' \), and between \( I_3(\lambda) \) and \( I_1' \) in Example 8

**Example 9:** Figs. 7 and 8 visualize the changing trend of distances between \( I_2 = \left\{ \left( \frac{2}{x} \right) \right\} \) and \( I_1 = \left\{ \left( \frac{1}{x} \right) \right\} \), and between \( I_2 = \left\{ \left( \frac{2}{x} \right) \right\} \) and \( I_1' = \left\{ \left( \frac{1}{x} \right) \right\} \) with varying the parameters \( \mu \) and \( \nu \) satisfying \( (\mu, \nu) \in \Theta \), by using Xiao’s distance measure \( d_X \) and our proposed distance measure \( d_{\text{wu}} \), respectively.

From Fig. 7 (e) and (f), we can observe that (1) the distance \( d_X(I_1, I_2) \) between \( I_1 = \left\{ \left( \frac{1}{x} \right) \right\} \) and \( I_2 = \left\{ \left( \frac{2}{x} \right) \right\} \) remains unchanged when the membership degree \( \mu \) is fixed; (2) the distance \( d_X(I_1', I_2) \) between \( I_1' = \left\{ \left( \frac{1}{x} \right) \right\} \) and \( I_2 = \left\{ \left( \frac{2}{x} \right) \right\} \) remains unchanged when the non-membership degree \( \nu \) is fixed. These are consistent with the following calculation results:

\[
\begin{align*}
d_X(I_1, I_2) &= \sqrt{\frac{1}{2} \log_2 \left( \frac{2\mu}{\lambda + \mu} + \mu \cdot \log_2 \frac{2\mu}{\lambda + \mu} + (1 - \mu) \right),} \\
&= \sqrt{\frac{1}{2} \log_2 \left( \frac{2\mu}{\lambda + \mu} + \mu \cdot \log_2 \frac{2\mu}{\lambda + \mu} + (1 - \mu) \right)}.
\end{align*}
\]

and

\[
\begin{align*}
d_X(I_1', I_2) &= \sqrt{\frac{1}{2} \log_2 \left( \frac{2\nu}{\lambda + \nu} + \nu \cdot \log_2 \frac{2\nu}{\lambda + \nu} + (1 - \nu) \right),} \\
&= \sqrt{\frac{1}{2} \log_2 \left( \frac{2\nu}{\lambda + \nu} + \nu \cdot \log_2 \frac{2\nu}{\lambda + \nu} + (1 - \nu) \right)}.
\end{align*}
\]

From Figs. 8 (c) and (d), we can observe that (1) the distance \( d_{\text{wu}}(I_1, I_2) \) between \( I_1 = \left\{ \left( \frac{1}{x} \right) \right\} \) and \( I_2 = \left\{ \left( \frac{2}{x} \right) \right\} \) increases strictly with the increase of \( \nu \in [0, 1 - \mu] \) when the
membership degree $\mu$ is fixed; (2) the distance $d_{\text{wu}}(I_1', I_2)$ between $I_1' = \left\{ \frac{(0.1)}{x} \right\}$ and $I_2 = \left\{ \frac{(\mu, \nu)}{x} \right\}$ increases strictly with the increase of $\mu \in [0, 1 - \nu]$ when the membership degree $\nu$ is fixed. These are reasonable and consistent with our results in Lemmas 2–5.

The above results also show the unreasonableness of Xiao’s distance measure $d_{\tilde{\chi}}$ and indicate the superiority of our proposed distance measure.

**Example 10 (Continuation of Example 5):** Let IFSs $I_1, I_2^{(\lambda)}$, and $I_3^{(\lambda)}$ on UOD $X = \{x\}$ be given as in Example 5. Together with Eqs. (6), (23), and (25), by varying the parameter $\lambda$ from 0 to 1, Fig. 9 visually shows the changing trend of distances between $I_2^{(\lambda)}$ and $I_1$, and between $I_3^{(\lambda)}$ and $I_1$ in Example 10, by using the distance measure $d_{\text{vc}}$ and our proposed distance measure $d_{\text{wu}}$. The simulation results are consistent with our calculation. This example demonstrates that our proposed distance is far superior to Yang and Chiclana’s spherical distance in [33].

**Example 11:** Fig. 10 visualizes the changing trend of distances between $I_2 = \left\{ \frac{(\mu, \nu)}{x} \right\}$ and $I_1 = \left\{ \frac{(1, 0)}{x} \right\}$, and between $I_2 = \left\{ \frac{(\mu, \nu)}{x} \right\}$ and $I_1' = \left\{ \frac{(0.1)}{x} \right\}$ with varying the parameters $\mu$ and $\nu$ satisfying $(\mu, \nu) \theta \Theta$, by using Yang and Chiclana’s spherical distance $d_{\text{vc}}$.

From Figs. 10 (c) and (d), we can observe that (1) the distance $d_{\text{vc}}(I_1, I_2)$ between $I_1 = \left\{ \frac{(1, 0)}{x} \right\}$ and $I_2 = \left\{ \frac{(\mu, \nu)}{x} \right\}$ remains unchanged when the membership degree $\mu$ is fixed; (2) the distance $d_{\text{vc}}(I_1', I_2)$ between $I_1' = \left\{ \frac{(0.1)}{x} \right\}$ and $I_2 = \left\{ \frac{(\mu, \nu)}{x} \right\}$ remains unchanged when the non-membership degree $\nu$ is fixed. These are consistent with the following calculation results:

$$d_{\text{vc}}(I_1, I_2) = \frac{2}{\pi} \arccos \sqrt{\mu},$$

and

$$d_{\text{vc}}(I_1', I_2) = \frac{2}{\pi} \arccos \sqrt{\nu}.$$

Contrary to Fig. 8, the above results also show the unreasonableness of Yang and Chiclana’s spherical distance $d_{\text{vc}}$ and indicate the superiority of our distance measure.
IV. APPLICATION TO PATTERN RECOGNITION

In practical applications, in order to better distinguish highly similar but inconsistent IFSs, we introduce the following parametric distance and similarity measures for IFSs.

Let \( X = \{ x_1, x_2, \ldots, x_n \} \) be a finite UOD and \( I_1 = \{ \alpha_j^{(1)} = \frac{\nu_j(x)}{\mu_j(x)} | 1 \leq j \leq n, \alpha_j^{(1)} \in \Theta \} \) and \( I_2 = \{ \alpha_j^{(2)} = \frac{\nu_j(x)}{\mu_j(x)} | 1 \leq j \leq n, \alpha_j^{(2)} \in \Theta \} \) be two IFSs on \( X \). For \( \lambda > 0 \), define

\[
\begin{align*}
    d^{(\lambda)}_{\text{wa}}(I_1, I_2) &= \sum_{j=1}^{n} \omega_j \left[ 1 \right. \\
    &\quad + \frac{1}{2} \left( (1 - (\mu_1(x_j))^\lambda) \cdot \log_2 \left( \frac{1}{1 - (\mu_1(x_j))^\lambda} + (1 - (\mu_2(x_j))^\lambda) \right) \\
    &\quad + (1 - (\mu_2(x_j))^\lambda) \cdot \log_2 \left( \frac{1}{1 - (\mu_2(x_j))^\lambda} + (1 - (\mu_1(x_j))^\lambda) \right) \\
    &\quad + (\nu_1(x_j))^\lambda \cdot \log_2 \left( \frac{2}{(\nu_1(x_j))^\lambda + (\nu_2(x_j))^\lambda} \right) \\
    &\quad + (\nu_2(x_j))^\lambda \cdot \log_2 \left( \frac{2}{(\nu_1(x_j))^\lambda + (\nu_2(x_j))^\lambda} \right) \left. \right) \right]^{0.5},
\end{align*}
\]

and

\[
S^{(\lambda)}_{\text{wa}}(I_1, I_2) = 1 - d^{(\lambda)}_{\text{wa}}(I_1, I_2),
\]

where \( \alpha_j^{(1)} = \langle \mu_1(x_j), \nu_1(x_j) \rangle \), \( \alpha_j^{(2)} = \langle \mu_2(x_j), \nu_2(x_j) \rangle \), and \( \omega = (\omega_1, \omega_2, \ldots, \omega_n) \) is the weight vector of \( x_j \) \((j = 1, 2, \ldots, n)\) with \( \omega_j \in (0, 1) \) and \( \sum_{j=1}^{n} \omega_j = 1 \).

Similarly to the discussions in Section V, we have the following result.

Theorem 5: For \( \lambda > 0 \),

(1) the function \( d^{(\lambda)}_{\text{wa}} \) defined by Eq. (27) is a SIFDisM on IFS\((X)\);

(2) the function \( S^{(\lambda)}_{\text{wa}} \) defined by Eq. (28) is a SIFSimM on IFS\((X)\).

Now, we utilize a practical example to illustrate the effectiveness of our proposed distance measure.

Example 12 ([18, Application 2], [19, Example 4.3]): Consider a pattern classification problem with three classes and three attributes \( A = \{ x_1, x_2, x_3 \} \), described by three patterns \( P = \{ P_1, P_2, P_3 \} \) and a test sample \( S_1 \) expressed by the IFSs listed in Table III.

| Attribute | \( \mu_1 \) | \( \nu_1 \) | \( \mu_2 \) | \( \nu_2 \) | \( \mu_3 \) | \( \nu_3 \) |
|----------|--------|--------|--------|--------|--------|--------|
| Pattern   | \( P_1 \) | \( P_2 \) | \( P_3 \) |
| \( x_1 \) | 0.15   | 0.25   | 0.35   | 0.35   | 0.45   |
| \( x_2 \) | 0.05   | 0.15   | 0.25   | 0.25   | 0.35   |
| \( x_3 \) | 0.16   | 0.26   | 0.36   | 0.36   | 0.46   |

| Sample    | \( S_1 \) |
|-----------|-----------|
| \( x_1 \) | 0.30      |
| \( x_2 \) | 0.20      |
| \( x_3 \) | 0.40      |

\( \times \) denotes that it cannot be determined.

The details for distance measures in Table IV can be found in [18, Section III].

If we take the weight vector \( \omega = \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right) \), then by the principle of maximum degree of similarity measures, the pattern classification results obtained by using different distance measures are listed in Table IV and Fig. 11. Observing from Table IV and shown in Fig. 11, one can see that the test sample \( S_1 \) is classified to the pattern \( P_3 \) by our proposed distance measure, which is consistent with the results obtained by the distance measures \( d^{E}_{\text{sm}} \), \( d^{R}_{\text{sm}} \), \( d^{\text{sk}} \), \( d^{\text{h}} \), \( d^{\text{L}} \), \( d^{\text{sm}} \), \( d^{\text{H}} \), and \( d^{\text{M}} \); however, the methods based on the distance measures \( d^{\text{h}} \), \( d^{E}_{\text{sk}} \), \( d^{\text{R}} \), \( d^{\text{H}} \), \( d^{\text{M}} \), \( d^{\text{L}} \), \( d^{\text{sm}} \), \( d^{\text{H}} \), \( d^{\text{M}} \), and \( d^{\text{W}} \), cannot determine to which pattern the test sample \( S_1 \) belongs.

VIII. CONCLUSION

This paper is devoted to the construction of SIFDisM and SIFSimM, which can effectively measure the differences between IFSs. First, we show some examples to demonstrate that Xiao’s distance measure in [18] and Yang and Chichala’s spherical distance in [33] have some shortcomings, which may cause counter-intuitive results. To overcome these shortcomings, we present the concepts of strict intuitionistic fuzzy...
distance measure (SIFDisM) and strict intuitionistic fuzzy similarity measure (SIFSimM), and propose a novel IFSDisM based on Jensen-Shannon divergence. Moreover, we prove that the dual similarity measure of our proposed distance measure is an SIFSimM and its induced entropy measure is an IF entropy measure. Meanwhile, we perform some comparative analysis to illustrate that our proposed distance measure is completely superior to the existing IFSDisMs; in particular, it is much better than Xiao’s distance measure in [18], Hung and Yang’s \( J_\alpha \)-divergence in [34], Joshi and Kumar’s dissimilarity divergence in [35], and Yang and Chiclana’s spherical distance in [33]; consequently, it is better than distance measures in [27], [28], [30], [36], [37], [38], [39]. Finally, to illustrate the availability of our proposed IFSM, we apply it to a practical pattern recognition problem. In the future, we will apply our methods to establish new distance/similarity measures for Pythagorean fuzzy sets, q-rung orthopair fuzzy sets, spherical fuzzy sets, picture fuzzy sets, and some other interval-valued fuzzy sets.

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