KdV-type equations linked via Bäcklund transformations: remarks and perspectives

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Abstract

Third order nonlinear evolution equations, that is the Korteweg-deVries (KdV), modified Korteweg-deVries (mKdV) equation and other ones are considered: they all are connected via Bäcklund transformations. These links can be depicted in a wide Bäcklund Chart which further extends the previous one constructed in [22]. In particular, the Bäcklund transformation which links the mKdV equation to the KdV singularity manifold equation is reconsidered and the nonlinear equation for the KdV eigenfunction is shown to be linked to all the equations in the previously constructed Bäcklund Chart. That is, such a Bäcklund Chart is expanded to encompass the nonlinear equation for the KdV eigenfunctions [30], which finds its origin in the early days of the study of Inverse scattering Transform method, when the Lax pair for the KdV equation was constructed. The nonlinear equation for the KdV eigenfunctions is proved to enjoy a nontrivial invariance property. Furthermore, the hereditary recursion operator it admits [30] is recovered via a different method. Then, the results are extended to the whole hierarchy of nonlinear evolution equations it generates. Notably, the established links allow to show that also the nonlinear equation for the KdV eigenfunction is connected to the Dym equation since both such equations appear in the same Bäcklund chart.

Keywords: Nonlinear Evolution Equations; Bäcklund Transformations; Recursion Operators; Korteweg deVries-type equations; Invariances; Cole-Hopf Transformations.

AMS Classification: 58G37; 35Q53; 58F07

1 Introduction

The relevance of Bäcklund transformations in soliton theory is well established, see [3] [46] [43] [47] [27] where a wide variety of applications of Bäcklund
and Darboux Transformations and their connections with partial differential equations admitting soliton solutions is given. Here, the concern is on Bäcklund transformations as a tool to investigate structural properties of nonlinear evolution equations. Specifically, the Bäcklund chart in [22] is reconsidered to show that it can be further extended to incorporate also the nonlinear equation for the KdV eigenfunction. The latter, named hereafter KdV eigenfunction equation for short, is studied in [30] where, among many other ones, it is proved to be integrable via the inverse spectral transform (IST) method. Indeed, this equation was firstly derived in a founding article of the IST method [38] and also [53], later further investigated in [30, 34] wherein a wide variety of nonlinear evolution equations is studied. Nevertheless, the KdV eigenfunction equation does not appear in subsequent classification studies of integrable nonlinear evolution equations, such as [2, 54, 35, 36] until very recently when, in [1], linearizable nonlinear evolution equations are classified. The KdV eigenfunction equation is a third order nonlinear equation of KdV-type since it is connected via Bäcklund transformations with the Korteweg deVries (KdV), the modified Korteweg deVries (mKdV), the Korteweg deVries interacting soliton (int.sol.KdV) [20] and the Korteweg deVries singularity manifold (KdV sing.) equations. The KdV eigenfunction equation is, then, proved to enjoy an invariance property. In addition, since it is connected via Bäcklund transformations to the other KdV-type equations, according to [18, 19], its hereditary recursion operator [30] can be recovered. The heritariness of all the recursion operators admitted by the equations in the Bäcklund chart allow to extend all the links to the whole corresponding hierarchies; hence, previous results [22] are generalised. Generalisations to non-Abelian KdV-type equations and hierarchies are comprised in [7, 8, 11, 13], wherein the links among them are depicted in a noncommutative Bäcklund chart analogous of that one in [22].

The material is organized as follows. The opening Section 2 is devoted to recall the definition of Bäcklund transformation adopted throughout this work together with its consequences which are most relevant to the present investigation. In the following Section 3 the nonlinear equation for the KdV eigenfunction is obtained. Specifically, it is shown to be linked, via Bäcklund transformations with the mKdV and the KdV singularity manifold equations. Notably, both the equations introduced in [13] represent non-Abelian counterparts of this equation when commutativity is assumed. In Section 4 the KdV eigenfunction equation is proved to enjoy a non trivial invariance property. The subsequent Section 5 concerns the Bäcklund chart in [22], which is further extended to include also the KdV eigenfunction equation. In Section 6 via the links in the Bäcklund chart, the hereditary recursion operator, firstly obtained in [30], admitted by the KdV eigenfunction equation is recovered in explicit form. Thus, the generated hierarchy follows. As a

\[\text{UrKdV or Schwarz-KdV}\]
consequence \cite{18}, the whole hierarchy of nonlinear evolution equations turns out to be connected to all the hierarchies in the Bäcklund chart. Concluding remark as well as how the present work is related to previous ones and, in particular, with the research program devoted to the study of non-Abelian nonlinear evolution equations \cite{7,8,11,13}, are comprised in the closing Section 7.

2 Some background definitions

This section is devoted to provide some background definitions which are of use throughout the whole article. Since many definitions are not unique in the literature, here, those ones here adopted are provided.

First of all, the notion of Bäcklund Transformation, according to Fokas and Fuchssteiner \cite{18} is recalled (see also the book by Rogers and Shadwick \cite{46}). Consider non linear evolution equations of the type

$$u_t = K(u)$$  \hspace{1cm} (2.1)

where the unknown function \(u\) depends on the independent variables \(x, t\) and, for fixed \(t\), \(u(x, t) \in M\), a manifold modeled on a linear topological space so that the generic fiber \(T_u M\), at \(u \in M\), can be identified with \(M\) itself\footnote{It is generally assumed that \(M\) is the space of functions \(u(x, t)\) which, for each fixed \(t\), belong to the Schwartz space \(S\) of rapidly decreasing functions on \(\mathbb{R}^n\), i.e. \(S(\mathbb{R}^n) := \{f \in C^\infty(\mathbb{R}^n) : \|f\|_{\alpha, \beta} < \infty, \forall \alpha, \beta \in \mathbb{N}_0^n \}\), where \(\|f\|_{\alpha, \beta} := \sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta f(x)|\), and \(D^\beta := \partial^\beta / \partial x^\beta\); throughout this article \(n = 1\).} and \(K : M \rightarrow TM\), is an appropriate \(C^\infty\)-vector field on a manifold \(M\) to its tangent manifold \(TM\). Let, now

$$v_t = G(v)$$  \hspace{1cm} (2.2)

denote another nonlinear evolution equation. If it assumed \(u(x, t) \in M_1\) and \(v(x, t) \in M_2\) where \(M_1, M_2\) represent manifolds modeled on a linear topological space so that, \(K : M_1 \rightarrow TM_1\) and \(G : M_2 \rightarrow TM_2\) represent appropriate \(C^\infty\)-vector fields on the manifolds \(M_i, i = 1, 2\), then

\[
\begin{align*}
    u_t &= K(u), \quad K : M_1 \rightarrow TM_1, \quad u : (x, t) \in \mathbb{R} \times \mathbb{R} \rightarrow u(x, t) \in M_1 \quad (2.3) \\
    v_t &= G(v), \quad G : M_2 \rightarrow TM_2, \quad v : (x, t) \in \mathbb{R} \times \mathbb{R} \rightarrow v(x, t) \in M_2. \quad (2.4)
\end{align*}
\]

Here, according to the usual choice when soliton solutions are considered, it is further assumed \(M := M_1 \equiv M_2\). Then, \cite{18} a Bäcklund transformation can defined as follows.

**Definition** Given two evolution equations, \(u_t = K(u)\) and \(v_t = G(v)\), then \(B(u, v) = 0\) represents a Bäcklund transformation between them whenever given two solutions of such equations, say, respectively, \(u(x, t)\) and \(v(x, t)\) such that

$$B(u(x, t), v(x, t))|_{t=0} = 0$$  \hspace{1cm} (2.5)
it follows that,
\[ B(u(x, t), v(x, t))|_{t=\bar{t}} = 0, \quad \forall \bar{t} > 0, \quad \forall x \in \mathbb{R}. \tag{2.6} \]

Hence, solutions admitted by the two equations are connected via the Bäcklund transformation which establishes a correspondence between them: it can graphically represented as
\[ u_t = K(u) B_{\bar{t}} v_t = G(v). \tag{2.7} \]

In addition, if, the nonlinear evolution equation (2.1) admits a hereditary recursion operator \([18, 41]\), denoted as \(\Phi(u)\), it can be written as
\[ u_t = \Phi(u) u_x \quad \text{where} \quad K(u) = \Phi(u) u_x. \tag{2.8} \]

The Bäcklund transformation allows to construct the operator \(\Psi(v)\)
\[ \Psi(v) = \Pi \Phi(u) \Pi^{-1} \tag{2.9} \]

where
\[ \Pi := -B_v^{-1} B_u, \quad \Pi : TM_1 \to TM_2, \tag{2.10} \]
while \(B_u \) and \(B_v\) denote the Frechet derivatives of the Bäcklund transformation \(B(u, v)\). Then, \([18]\), the operator \(\Psi(v)\) represents the hereditary recursion operator admitted by the equation \(v_t = G(v)\) which, thus, can be written under the form
\[ G(v) = \Psi(v) v_x. \]

That is, according to \([18]\), given the Bäcklund transformation \(B(u, v)\), and the hereditary recursion operator \(\Phi(u)\) admitted by equation (2.1), then, also equation (2.2) admits a hereditary recursion operator: it is obtained on use of the operator \(\Pi\), (2.10), via the trasformation formula (2.9).

On subsequent applications of the admitted recursion operators are, respectively, the two hierarchies
\[ u_t = [\Phi(u)]^n u_x \quad \text{and} \quad v_t = [\Psi(v)]^n v_x, \quad n \in \mathbb{N} \tag{2.11} \]
of evolution equations can be constructed \([19]\); their base members equations, which correspond to \(n = 1\), coincide with equations (2.1) and (2.2). Fixed any \(n_0 \in \mathbb{N}\), the two equations \(u_t = [\Phi(u)]^{n_0} u_x\) and \(v_t = [\Psi(v)]^{n_0}\) are connected, via the same Bäcklund Transformation which connects the two base members equations. This extension to the whole hierarchies is graphically represented by the following Bäcklund chart
\[ u = [\Phi(u)]^n u_x B_{\bar{t}} v_t = [\Psi(v)]^n v_x. \tag{2.12} \]

which emphasizes that the link between the two equations (2.1) and (2.2) is extended to corresponding members of the two hierarchies generated, respectively, by the recursion operators \(\Phi\) and \(\Psi\).
3 A third order KdV-type equation

In this Section the Bäcklund chart in [22] is further extended to incorporate the KdV eigenfunction equation:

\[ w_t = w_{xxx} - 3 \frac{w_x w_{xx}}{w} . \] (3.1)

This nonlinear evolution equation was introduced in [38], one of the IST founding articles. Later, it was investigated in [53] and, subsequently, in [30] where its integrability via the inverse spectral transform (IST) method of soliton eigenfunction equations is proved. Among the many equations studied in the extended article [30] also the KdV eigenfunction equation is included: ist IST integrability is proved and also, via Lax pair representation, its recursion operator is provided. In Section 6 the explicit form of the recursion operator admitted by (3.1) is constructed via a different approach. Indeed, to obtain such a recursion operator, we apply the connections, via Bäcklund transformations, of equation (3.1) with other KdV-type equations according to the result presented in Section 5 and here. Notably, equation (3.1) appears also in recent works [53, 1]. The latter finds this equation in classifying linearizable evolution equations.

In this Section, equation 3.1 is shown to be linked with the mKdV and the KdV singularity manifold equations. The following proposition can be proved.

Proposition 1

Equation (3.1) is linked to the mKdV equation

\[ v_t = v_{xxx} - 6v^2 v_x \] (3.2)

via the Cole-Hopf [16, 28] transformation

\[ \text{CH} : \quad vw - w_x = 0 . \] (3.3)

Proof

On substitution, in (3.2), of \( v \) in terms of \( w \) according to the latter gives:

\[ \left( \frac{w_x}{w} \right)_t = \left( \frac{w_x}{w} \right)_{xxx} - 6 \left( \frac{w_x}{w} \right)^2 \left( \frac{w_x}{w} \right)_x \]

which, since the assumed regularity of \( v \) and \( w \) implies Schwartz theorem on order of partial derivation holds, delivers

\[ \left( \frac{w_t}{w} \right)_x = \left[ \frac{w_{xxx}}{w} - 3 \frac{w_x w_{xx}}{w^2} + 2 \left( \frac{w_x}{w} \right)^3 - 2 \left( \frac{w_x}{w} \right)^3 \right]_x \]

hence, after simplification, equation (3.1) follows. \( \Box \)

\(^3\text{see also [6]}\)
Proposition 2

Equation (3.1) is linked to the KdV sing manifold equation

\[ \varphi_t = \varphi_x \{ \varphi; x \}, \quad \{ \varphi; x \} = \left( \frac{\varphi_{xx}}{\varphi_x} \right)_x - \frac{1}{2} \left( \frac{\varphi_{xx}}{\varphi_x} \right)^2 \]  

(3.4)

via the Bäcklund transformation

\[ B : \quad w^2 - \varphi_x = 0. \]  

(3.5)

Proof

The Bäcklund transformation (3.5) implies:

\[
\begin{align*}
2ww_t &= \varphi_x, \\
2\frac{w_{xx}}{w} &= \frac{\varphi_{xxx}}{\varphi_x} - \frac{1}{2} \frac{\varphi^2_{xx}}{\varphi_x^2}, \\
2\frac{w_{xxxx}}{w} &= \frac{\varphi_{xxxx}}{\varphi_x} - \frac{3}{2} \frac{\varphi_{xx}\varphi_{xxx}}{\varphi_x^2} + \frac{3}{4} \frac{\varphi^3_{xx}}{\varphi_x^3}
\end{align*}
\]

Substitution of the latter in (3.1) gives:

\[ \varphi_t = \varphi_{xxxx} - 3 \frac{\varphi_{xx}\varphi_{xxx}}{\varphi_x} + \frac{3}{2} \frac{\varphi^3_{xx}}{\varphi_x^2} \]

since

\[ \{ \varphi; x \}_x = \varphi_{xxxx} - 3 \frac{\varphi_{xx}\varphi_{xxx}}{\varphi_x} + \frac{3}{2} \frac{\varphi^3_{xx}}{\varphi_x^2} \]

on integration with respect to \( x \), the KdV sing manifold equation (3.4) follows and the proof is complete. \( \Box \)

Remark

Both the two new non-Abelian nonlinear evolution equations

\[
\begin{align*}
W_t &= W_{xxx} - 3W_{xx} W^{-1}W_x, & Z_t &= Z_{xxx} - 3Z_x Z^{-1}Z_x
\end{align*}
\]

(3.6)

introduced in \cite{13}, reduce to equation (3.1) when the assumption of a non-Abelian unknown is removed.

4 A non trivial invariance

Some properties the introduced equation (3.1) enjoys are studied in this section and in the following ones. In particular, this section is devoted to prove an invariance property it enjoys.

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4 Capital case is used to emphasize that the unknown functions \( Z \) and \( W \) are non-Abelian ones, in \cite{13} operators on a Banach space.
First of all, equation (3.1) is scaling invariant since substitution of $\alpha w$, $\forall \alpha \in \mathbb{C}$, to $w$ leaves it unchanged. In addition, the following proposition holds.

**Proposition 2**

The nonlinear evolution equation (3.1) is invariant under the transformation

$$I : \hat{w}^2 = \frac{ad - bc}{(cD^{-1}(w^2) + d)^2} w^2, \quad a, b, c, d \in \mathbb{C} \text{ s.t. } ad - bc \neq 0,$$

(4.1)

where

$$D^{-1} := \int_{-\infty}^{x} d\xi$$

is well defined since so called soliton solutions are looked for in the Schwartz space $S(\mathbb{R}^n)$.

**Proof**

The KdV singularity manifold equation (3.4) is invariant under the Möbius group of transformations

$$M : \hat{\varphi} = \frac{a\varphi + b}{c\varphi + d}, \quad a, b, c, d \in \mathbb{C} \text{ such that } ad - bc \neq 0.$$  

(4.2)

Recalling that, according to proposition 2, the KdV singularity manifold equation (3.4) and equation (3.1) are connected to each other via the Bäcklund transformation $B$ (3.5), the result is readily obtained. Indeed, the following Bäcklund chart

$$M : \hat{\varphi} = \frac{a\varphi + b}{c\varphi + d} \quad \begin{cases} w_t = w_{xxx} - \frac{3w_x w_{xx}}{w} & w^2 - \varphi_x = 0 \quad \varphi_t = \varphi_x\{\varphi; x\} \\ \hat{\varphi}_t = \hat{\varphi}_x\{\hat{\varphi}; x\} \end{cases} \quad \uparrow I \quad \uparrow M$$

$$\forall a, b, c, d \in \mathbb{C} \mid ad - bc \neq 0$$

shows that the invariance $I$ is obtained via composition of the Möbius transformation $M$ with the two Bäcklund transformations

$B : \quad w^2 - \varphi_x = 0 \quad \text{and} \quad \hat{B} : \quad \hat{w}^2 - \hat{\varphi}_x = 0.$  

$\square$

5 Extended Bäcklund chart

In this section, the equation (3.1) is inserted in the Bäcklund chart in [22] which, then, is further extended. Indeed, combination of the two transformations $CH$ (3.3) and $B$ in the previous section gives

$$v - \frac{1}{2} \frac{\varphi_{xx}}{\varphi_x} = 0$$  

(5.1)

---

5 see footnote on page 4.
which coincides with the link in [22] between mKdV and KdV sing. equations. The links given up to this stage are summarised in the following Bäcklund chart:

\[
\begin{align*}
\text{KdV}(u) & \quad \text{mKdV}(v) & \quad \text{new eq}(w) & \quad \text{KdV sing.}(\varphi) & \quad \text{int. sol KdV}(s) & \quad \text{Dym}(\rho)
\end{align*}
\]

where all the third order nonlinear evolution equations are, respectively

\[
\begin{align*}
u_t &= u_{xxx} + 6uu_x, & \quad \text{(KdV)}, \\
v_t &= v_{xxx} - 6v^2v_x, & \quad \text{(mKdV)}, \\
\varphi_t &= \varphi_x \{\varphi; x\}, & \quad \text{where } \{\varphi; x\} := \left(\frac{\varphi_{xx}}{\varphi_x}\right)_x - \frac{1}{2} \left(\frac{\varphi_{xx}}{\varphi_x}\right)^2, & \quad \text{(KdV sing.)}, \\
s^2s_t &= s^2s_{xxx} - 3ss_x s_{xx} + \frac{3}{2} s_x^3, & \quad \text{(int. sol KdV)}, \\
\rho_t &= \rho^3 \rho_{\xi\xi}, & \quad \text{(Dym)}.
\end{align*}
\]

The Bäcklund transformations linking these equations them, are, following their order in the Bäcklund chart:

\[
\begin{align*}
\text{(a)} & \quad u + v_x + v^2 = 0, & \quad \text{(b)} & \quad v - \frac{w_x}{w} = 0, & \quad \text{(5.2)} \\
\text{(c)} & \quad w^2 - \varphi_x = 0, & \quad \text{(d)} & \quad s - \varphi_x = 0, & \quad \text{(5.3)}
\end{align*}
\]

and

\[
\begin{align*}
\text{(e)} & \quad \bar{x} := D^{-1}s(x), \quad \rho(\bar{x}) := s(x), & \quad \text{where } D^{-1} := \int_{-\infty}^{x} d\xi, & \quad \text{(5.4)}
\end{align*}
\]

so that \(\bar{x} = \bar{x}(s, x)\) and, hence, \(\rho(\bar{x}) := \rho(\bar{x}(s, x))\). The transformation (e) denotes the reciprocal transformation, as it is termed to stress it interchanges the role of the dependent and independent variables. It represents a Bäcklund transformation between the extended manifold consisting of both the dependent and the independent variables; namely, the manifold given by pairs formed by the dependent and the independent variables. Then, the transformation (e) defines, at least locally [5, 18], a Bäcklund transformation:

\[
T_{(s, x)} \rightarrow T_{(\rho, \bar{x})}
\]

between the two respective tangent spaces. Therefore, it is possible to transfer the infinitesimal structure using the transformation formulae [22, 18].

---

6Combination of (1.12) with (1.19), respectively, Cole-Hopf and introduction of a bona fide potential when connecting the interacting soliton KdV with the KdV sing. equation [22] produce the transformation [5, 11].

7see, for instance, [46] for a general introduction and application of reciprocal transformations. Details on the transformation (e) are given in [5, 22].
Now, taking into account the invariance under the Möbius group of transformations enjoyed by the singularity manifold equation, the Bäcklund chart can be duplicated to obtain

\[
\begin{array}{cccccc}
\text{KdV}(u) & \text{mKdV}(v) & \text{new eq}(w) & \text{KdV sing.}(\varphi) & \text{int. sol KdV}(s) & \text{Dym}(\rho) \\
AB_1 \downarrow & AB_2 \downarrow & AB_3 \downarrow & M \downarrow & AB_4 \downarrow & AB_5 \downarrow \\
\text{KdV}(u) & \text{mKdV}(v) & \text{new eq}(w) & \text{KdV sing.}(\varphi) & \text{int. sol KdV}(s) & \text{Dym}(\rho)
\end{array}
\]

where the vertical lines represent auto-Bäcklund transformations which are all induced by the combination of the Bäcklund transformations linking the other equations with invariance enjoyed by the KdV singularity equation. In detail, starting from the left hand side, the invariance $AB_1$ and $AB_2$ are the well known KdV and mKdV [37], auto-Bäcklund transformations, given in [3, 22], $AB_3 \equiv I$ is the invariance admitted by the novel nonlinear evolution equation (3.1) proved in Proposition 2. The last two, $AB_4$, and $AB_5$, respectively, are auto-Bäcklund transformations, [22], admitted by the int. sol. KdV and Dym equations. Notably, the connection between KdV and Dym equation [45, 22] finds applications in the construction of solutions admitted by the Dym equation [24, 26]. The extension to 2 + 1 dimensional equations is given in [42, 39].

6 Admitted recursion operator & hierarchy

This section is devoted to the construction of the recursion operator admitted by the equation (3.1). Specifically, the Cole-Hopf link (3.5) between equation (3.1) and the mKdV equation allows to prove, according to [19, 18], it admits a recursion operator.

Proposition 3
The nonlinear evolution equation (3.1) admis the recursion operator

\[
\Psi(w) = \frac{1}{2w} D w^2 \left[ D^2 + 2U + D^{-1} 2UD \right] \frac{1}{w^2} D^{-1} 2w ,
\]  

where

\[
U := \frac{w_{xx}}{w} - 2 \frac{w_x^2}{w^2} .
\]

Proof
Consider the Bäcklund transformation (3.5), then, the transformation operator $\Pi$, recalling (2.10), reads

\[
\Pi := -B^{-1}_w B_\varphi
\]

where

\[
B_w[q] = \frac{\partial}{\partial \varepsilon} B(w + \varepsilon q, \varphi) \bigg|_{\varepsilon=0} = \frac{\partial}{\partial \varepsilon} \left[ (w + \varepsilon q)^2 - \varphi_x \right] \bigg|_{\varepsilon=0} = 2wq
\]
and,

$$B_{\varphi}[q] = \frac{\partial}{\partial \varepsilon} B(w, \varphi + \varepsilon q) \bigg|_{\varepsilon=0} = \frac{\partial}{\partial \varepsilon} \left[ w^2 + (\varphi + \varepsilon q)_{x} \right] \bigg|_{\varepsilon=0} = q_x \tag{6.5}$$

hence

$$B_w = 2w \quad , \quad B_{\varphi} = D \quad \Rightarrow \quad \Pi = -B_w^{-1} B_{\varphi} = \frac{1}{2w} D \ . \tag{6.6}$$

Now, substitution of the latter in (2.9) gives

$$\Psi(w) = B_w^{-1} B_{\varphi} \Phi(\varphi) B^{-1}_{\varphi} B_w \bigg|_{w^2 - \varphi_x = 0} \ , \tag{6.7}$$

where $\Phi(\varphi)$, according to formulae (1.23)-(1.24) in [22], is

$$\Phi(\varphi) = \varphi_x \left[ D^2 + \{ \varphi; x \} + D^{-1} \{ \varphi; x \} \right] \frac{1}{\varphi_x} D^{-1} \ . \tag{6.8}$$

Then, substitution of the latter and of the transformation operator $\Pi$ in (6.7) gives the operator

$$\Psi(w) = \frac{1}{2w} D \varphi_x \left[ D^2 + \{ \varphi; x \} + D^{-1} \{ \varphi; x \} \right] D^{-1} 2w \bigg|_{\varphi_x = w^2} \ . \tag{6.9}$$

recalling the definition (3.4) of the Schwarzian derivative $\{ \varphi; x \}$, on substitution of $\varphi_x = w^2$, it follows

$$\{ \varphi; x \} \big|_{\varphi_x = w^2} = 2 \left( \frac{w_{xx}}{w} - 2 \frac{w_x^2}{w^2} \right) , \tag{6.10}$$

the latter, on use of $U$, introduced in (6.2) to simplify the notation, gives

$$\{ \varphi; x \} \big|_{\varphi_x = w^2} = 2U$$

and, hence,

$$\Psi(w) = \frac{1}{2w} D w^2 \left[ D^2 + 2U + D^{-1} 2UD \right] \frac{1}{w^2} D^{-1} 2w \ ,$$

which coincides with (6.1) and completes the proof. \qed

In addition, the following proposition holds. \textbf{Proposition 5}

The recursion operator (6.1) admitted by the nonlinear evolution equation (3.1) is hereditary.

\textbf{Proof}

To prove the thesis, note that equation (3.1) is linked via Bäcklund transformations to all the nonlinear evolution equations in the Bäcklund chart; hence, since all of them admit a hereditary recursion operator, according to [18, 19], also the recursion operator (6.1) admitted by the newly obtained equation (3.1) enjoys the hereditariness property. \qed
Remark

The recursion operator (6.1) admitted by the equation (3.1) can be also obtained from the Cole-Hopf link\[vw - w_x = 0,\]with the mKdV equation, via the same method. Hence, in this case

\[
\Psi(w) = B_w^{-1}B_v\Phi_{mKdV}(v)B_v^{-1}B_w|_{vw-w_x=0},
\]

where, respectively, \(B_w\) and \(B_v\) are given by

\[
B_w[q] = \frac{\partial}{\partial \varepsilon} B(w + \varepsilon q, \varphi)\bigg|_{\varepsilon=0} = \frac{\partial}{\partial \varepsilon} \left[(w + \varepsilon q)^2 - \varphi_x\right]\bigg|_{\varepsilon=0} = 2wq = (6.12)
\]

and

\[
B_v[q] = \frac{\partial}{\partial \varepsilon} B(w, v + \varepsilon q)\bigg|_{\varepsilon=0} = \frac{\partial}{\partial \varepsilon} [w(v + \varepsilon q) - w_x]\bigg|_{\varepsilon=0} = wq = (6.13)
\]

and

\[
B_w[q] = \frac{\partial}{\partial \varepsilon} B(w + \varepsilon q, v)\bigg|_{\varepsilon=0} = \frac{\partial}{\partial \varepsilon} \left[(w + \varepsilon q)v - (w + \varepsilon q)_x\right]\bigg|_{\varepsilon=0} = vq - q_x
\]

Explicit computations allow to obtain, once again, (6.1).

Now, equation (3.1), when the hereditary recursion operator \(\Psi(w)\) is given in (6.1), can be written as

\[
w_t = \Psi(w)w_x
\]

and the corresponding hierarchy is generated

\[
w_t = [\Psi(w)]^n w_x, \ n \in \mathbb{N}.
\]

Since, as in [22], all the nonlinear evolution equations in the Bäcklund chart in Section 5 admit a hereditary recursion operator [18, 19], all the links can be extended to the corresponding whole hierarchies. Then, fixed \(n = n_0, n_0 \in \mathbb{N}\) a different Bäcklund chart is obtained which links nonlinear evolution equations of order \(2n_0 + 1\); the case \(n_0 = 1\) corresponds to the 3rd order KdV-type equations; if \(n_0 = 2, 3\), respectively, the nonlinear evolution equations in the Bäcklund chart are all of the 5th and 7th order.

The links among the corresponding members in the hierarchies can be depicted via the same Bäcklund chart in Section 5, that is, for each \(n \in \mathbb{N}\), it holds

\[
\begin{align*}
\Phi_1(w)^n w_x & \overset{(a)}{\longrightarrow} \Phi_2(v)^n v_x \overset{(b)}{\longrightarrow} \Phi_3(w)^n w_x \overset{(c)}{\longrightarrow} \Phi_4(\varphi)^n \varphi_x \overset{(d)}{\longrightarrow} \Phi_5(s)^n s_x \overset{(e)}{\longrightarrow} \Phi_6(\rho)^n \rho^{3n} \rho_{xxx}
\end{align*}
\]
where, the recursion operators\footnote{The recursion operators here listed are well known ones with the only exception of (6.1), see, for instance \cite{3}.} are, respectively

\[
\begin{align*}
\Phi_1(u) &\equiv \Phi_{\text{KdV}}(u) = D^2 + 2DuD^{-1} + 2u & \text{(KdV)}, \\
\Phi_2(v) &\equiv \Phi_{\text{mKdV}}(v) = D^2 - 4DvD^{-1}vD & \text{(mKdV)}, \\
\Phi_3(w) &\equiv \Psi(w) = \frac{1}{2w}Dw^2[D^2 + 2U + D^{-1}2UD] \frac{1}{w^2}D^{-2}w & \text{(new eq)}, \\
\Phi_4(\varphi) &\equiv \Phi_{\text{KdVsing}}(\varphi) = \varphi_x \left[D^2 + \{\varphi; x\} + D^{-1}\{\varphi; x\}D\right] \frac{1}{\varphi_x} & \text{(KdV sing.)}, \\
\Phi_5(s) &\equiv \Phi_{\text{KdVsol}}(s) = Ds \left[D^2 + S + D^{-1}SD\right] \frac{1}{s}D^{-1}s & \text{(int. sol KdV)}, \\
\Phi_6(\rho) &\equiv \Phi_{\text{Dym}}(\rho) = \rho^3 D^3 \rho D^{-1} \rho^{-2} & \text{(Dym)},
\end{align*}
\]

and the links among such hierarchies of nonlinear evolution equations, are indicated in (5.2).

\textbf{Remark}

The Dym hierarchy, generated on application of the hereditary recursion operator \cite{3, 22, 32, 31} to the Dym equation,

\[
\rho_t = [\Phi_{\text{Dym}}(\rho)]^n \rho^3 \rho_{xxx} , n \geq 0, \quad \text{where } \Phi_{\text{Dym}}(\rho) = \rho^3 D^3 \rho D^{-1} \rho^{-2} \quad (6.18)
\]

is connected to all the hierarchies in which appears in the Bäcklund charts in Section 5; hence, it is also related to the hierarchy of nonlinear evolution equation whose base member is (3.1).

\section{Remarks, perspectives and open problems}

This Section is devoted to collect some remarks and open problems which arise from the present results. This study is strictly connected with the research program which involves C. Schiebold, together with, lately, also M. Lo Schiavo and E. Porten, and the author concerning operator evolution equations and their properties. The results already obtained, based on the theoretical foundation in \cite{15} and references therein, concern KdV-type non-Abelian equations in \cite{7, 8, 11, 13} where, analogies as well as some notable differences which arise in the non-Abelian case are pointed out. Non-Abelian Burgers hierarchies are studied in \cite{9, 10, 12}. Comments concerning the comparison non-Abelian vs. Abelian results are comprised in \cite{7, 8, 4, 11, 12}. A remarkable property to stress in the present contest is that, also in the non-Abelian case, hereditariness is preserved under Bäcklund transformations. Hence, proved the hereditariness of one recursion operator \cite{52, 10},
the hereditariness of all the recursion operator of other non-Abelian equations linked to it follows, see [49, 50, 51, 8, 11].

Some remarks follow.

- The Bäcklund chart in Section 5 extends that one in [22, 40] finds its analogous Bäcklund chart in [5, 44], where 5th order nonlinear evolution equations, i.e. Caudrey-Dodd-Gibbon-Sawata-Kotera (CDG-SK) [14, 48] and Kaup-Kupershmidt (KK) equations, which, in turn, play the role of the KdV and mKdV equations appear. The 5th order nonlinear evolution equation analogous to the Dym eq is the Kawamoto equation [29]: it is linked via the reciprocal transformation (5.4) to the singularity manifold equation [55] related to the CDG-SK equation. However, further to the many analogies the transformations the two different Bäcklund charts are not exactly the same. Hence, the question arises whether or not there exist a 5th order analog of equation (3.1).

- All the structural properties which are preserved under Bäcklund transformations are enjoyed by all nonlinear evolution equations in the same Bäcklund chart. This is the case, in particular, of hereditariness of recursion operators. Also the Hamiltonian and/or bi-Hamiltonian structure [21, 23, 25, 33] in preserved under Bäcklund transformations. Hence, the Hamiltonian structure admitted by equation (3.1) [30] can be recovered from the presented Bäcklund chart.

- In [39] the (2 + 1) dimensional KP, mKP and Dym hierarchies are connected via Bäcklund transformations; then in [40] it is shown that, when suitable constraints are imposed, the Bäcklund chart in [22] is obtained. A further question which arises is whether the 2 + 1 KP eigenfunction equations [30] can be included in the Bäcklund chart constructed in [39].

Further questions concerning open problems and perspective investigations in the case of operator equations are referred to [11].

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