TRANSFORMATIONS OF NEVANLINNA OPERATOR-FUNCTIONS
AND THEIR FIXED POINTS

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To Eduard R. Tsekanovskii on the occasion of his 80-th birthday

ABSTRACT. We give a new characterization of the class \( N_{\mathbb{H}}^0[-1,1] \) of the operator-valued Hilbert space \( \mathbb{H} \) Nevanlinna functions that admit representations as compressed resolvents (\( m \)-functions) of selfadjoint contractions. We consider the automorphism \( \Gamma : M(\lambda) \to M_\Gamma(\lambda) := ((\lambda^2 - 1)M(\lambda))^{-1} \) of the class \( N_{\mathbb{H}}^0[-1,1] \) and construct a realization of \( M_\Gamma(\lambda) \) as a compressed resolvent. The unique fixed point of \( \Gamma \) is the \( m \)-function of the block-operator Jacobi matrix related to the Chebyshev polynomials of the first kind. We study a transformation \( \hat{\Gamma} : M(\lambda) \mapsto M_{\hat{\Gamma}}(\lambda) := -(M(\lambda) + \lambda I_{\mathbb{H}})^{-1} \) that maps the set of all Nevanlinna operator-valued functions into its subset. The unique fixed point \( M_0 \) of \( \hat{\Gamma} \) admits a realization as the compressed resolvent of the "free" discrete Schrödinger operator \( \hat{J}_0 \) in the Hilbert space \( H_0 = \ell^2(N_0) \otimes \mathbb{H} \). We prove that \( M_0 \) is the uniform limit on compact sets of the open upper/lower half-plane in the operator norm topology of the iterations \( \{M_{n+1}(\lambda) = -(M_n(\lambda) + \lambda I_{\mathbb{H}})^{-1}\} \) of \( \hat{\Gamma} \). We show that the pair \( \{H_0, \hat{J}_0\} \) is the inductive limit of the sequence of realizations \( \{\hat{J}_n, A_n\} \) of \( \{M_n\} \). In the scalar case \( (\mathbb{H} = \mathbb{C}) \), applying the algorithm of I.S. Kac, a realization of iterates \( \{M_n\} \) as \( m \)-functions of canonical (Hamiltonian) systems is constructed.

1. Introduction and preliminaries

Notations. We use the symbols \( \text{dom} T, \text{ran} T, \text{ker} T \) for the domain, the range, and the null-subspace of a linear operator \( T \). The closures of \( \text{dom} T, \text{ran} T \) are denoted by \( \overline{\text{dom} T}, \overline{\text{ran} T} \), respectively. The identity operator in a Hilbert space \( \mathcal{H} \) is denoted by \( I \) and sometimes by \( I_\mathcal{H} \). If \( \mathcal{L} \) is a subspace, i.e., a closed linear subset of \( \mathcal{H} \), the orthogonal projection in \( \mathcal{H} \) onto \( \mathcal{L} \) is denoted by \( P_\mathcal{L} \). The notation \( T\big|_\mathcal{L} \) means the restriction of a linear operator \( T \) on the set \( \mathcal{L} \subset \text{dom} T \). The resolvent set of \( T \) is denoted by \( \rho(T) \). The linear space of bounded operators acting between Hilbert spaces \( \mathcal{H} \) and \( \mathfrak{K} \) is denoted by \( \mathcal{B}(\mathcal{H}, \mathfrak{K}) \) and the Banach algebra \( \mathcal{B}(\mathcal{H}) \) by \( \mathcal{B}(\mathcal{H}) \). Throughout this paper we consider separable Hilbert spaces over the field \( \mathbb{C} \) of complex numbers. \( \mathbb{C}_+ / \mathbb{C}_- \) denotes the open upper/lower half-plane of \( \mathbb{C} \), \( \mathbb{R}_+ := [0, +\infty) \), \( \mathbb{N} \) is the set of natural numbers, \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \).

Definition 1.1. A \( \mathcal{B}(\mathfrak{M}) \)-valued function \( M \) is called a Nevanlinna function (\( R \)-function \cite{15,20}, Herglotz function \cite{12}, Herglotz-Nevanlinna function \cite{1,3}) if it is holomorphic outside the real axis, symmetric \( M(\lambda)^* = M(\lambda) \), and satisfies the inequality \( \text{Im} \lambda \text{Im} M(\lambda) \geq 0 \) for all \( \lambda \in \mathbb{C} \setminus \mathbb{R} \).

This class is often denoted by \( \mathcal{R}[\mathfrak{M}] \). A more general is the notion of Nevanlinna family, cf. \cite{9}.

\[ \begin{align*}
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\end{align*} \]
Definition 1.2. A family of linear relations $\mathcal{M}(\lambda)$, $\lambda \in \mathbb{C}\setminus\mathbb{R}$, in a Hilbert space $\mathfrak{M}$ is called a Nevanlinna family if:

1. $\mathcal{M}(\lambda)$ is maximal dissipative for every $\lambda \in \mathbb{C}_+$ (resp. accumulative for every $\lambda \in \mathbb{C}_-$);
2. $\mathcal{M}(\lambda)^* = \mathcal{M}(\lambda)$, $\lambda \in \mathbb{C}\setminus\mathbb{R}$;
3. for some, and hence for all, $\mu \in \mathbb{C}_+(\mathbb{C}_-)$ the operator family $(\mathcal{M}(\lambda) + \mu I_{\mathfrak{M}})^{-1}(\in B(\mathfrak{M}))$ is holomorphic on $\mathbb{C}_+(\mathbb{C}_-)$. 

The class of all Nevanlinna families in a Hilbert space $\mathfrak{M}$ is denoted by $\tilde{R}(\mathfrak{M})$. Each Nevanlinna family $\mathcal{M} \in \tilde{R}(\mathfrak{M})$ admits the following decomposition to the operator part $M_s(\lambda)$, $\lambda \in \mathbb{C}\setminus\mathbb{R}$, and constant multi-valued part $M_\infty$:

$$\mathcal{M}(\lambda) = M_s(\lambda) \oplus M_\infty, \quad M_\infty = \{0\} \times \text{mul} \mathcal{M}(\lambda).$$

Here $M_s(\lambda)$ is a Nevanlinna family of densely defined operators in $\mathfrak{M} \oplus \text{mul} \mathcal{M}(\lambda)$.

A Nevanlinna $B(\mathfrak{M})$-valued function admits the integral representation, see [15], [20],

$$(1.1) \quad M(\lambda) = A + B\lambda + \int_\mathbb{R} \left( \frac{1}{t - \lambda} - \frac{t}{t^2 + 1} \right) d\Sigma(t), \quad \int_\mathbb{R} \frac{d\Sigma(t)}{t^2 + 1} \in B(\mathfrak{M}),$$

where $A = A^* \in B(\mathfrak{M})$, $0 \leq B = B^* \in B(\mathfrak{M})$, the $B(\mathfrak{M})$-valued function $\Sigma(\cdot)$ is nondecreasing and $\Sigma(t) = \Sigma(t - 0)$. The integral is uniformly convergent in the strong topology; cf. [8], [15]. The following condition is equivalent to the definition of a $B(\mathfrak{M})$-valued Nevanlinna function $M(\lambda)$ holomorphic on $\mathbb{C}\setminus\mathbb{R}$: the function of two variables

$$K(\lambda, \mu) = \frac{M(\lambda) - M(\mu)^*}{\lambda - \mu}$$

is a nonnegative kernel, i.e., $\sum_{k,l=1}^n (K(\lambda_k, \lambda_l) f_i, f_k) \geq 0$ for an arbitrary set of points 

$\{\lambda_1, \lambda_2, \ldots, \lambda_n\} \subset \mathbb{C}_+/\mathbb{C}_-$ and an arbitrary set of vectors $\{f_1, f_2, \ldots, f_n\} \subset \mathfrak{M}$.

It follows from (1.1) that

$$B = s - \lim_{y \uparrow \infty} \frac{M(iy)}{y} = s - \lim_{y \uparrow \infty} \frac{\text{Im} M(iy)}{y},$$

$$\text{Im} M(iy) = B y + \int_\mathbb{R} \frac{y}{t^2 + y^2} d\Sigma(t),$$

and this implies that $\lim_{y \to \infty} y \text{Im} M(iy)$ exists in the strong resolvent sense as a selfadjoint relation; see e.g. [5]. This limit is a bounded selfadjoint operator if and only if $B = 0$ and $\int_\mathbb{R} d\Sigma(t) \in B(\mathfrak{M})$, in which case $s - \lim_{y \to \infty} y \text{Im} M(iy) = \int_\mathbb{R} d\Sigma(t)$. In this case one can rewrite the integral representation (1.1) in the form

$$(1.2) \quad M(\lambda) = E + \int_\mathbb{R} \frac{1}{t - \lambda} d\Sigma(t), \quad \int_\mathbb{R} d\Sigma(t) \in B(\mathfrak{M}),$$

and $E = \lim_{y \to \infty} M(iy)$ in $B(\mathfrak{M})$.

The class of $B(\mathfrak{M})$-valued Nevanlinna functions $M$ with the integral representation (1.2) with $E = 0$ is denoted by $\mathcal{R}_0[\mathfrak{M}]$. In this paper we will consider the following subclasses of the class $\mathcal{R}_0[\mathfrak{M}]$.

Definition 1.3. A function $N$ from the class $\mathcal{R}_0[\mathfrak{M}]$ is said to belong to the class

1. $\mathcal{N}[\mathfrak{M}]$ if $s - \lim_{y \to \infty} iy N(iy) = -I_{\mathfrak{M}}$,
2. $\mathcal{N}_{\text{gr}}[\mathfrak{M}]$ if $N \in \mathcal{N}[\mathfrak{M}]$ and $\bar{N}$ is holomorphic at infinity,
(3) \( N_{0}^{0}[-1, 1] \) if \( N \in N_{0}^{0} \) and is holomorphic outside the interval \([-1, 1] \).

Thus, we have inclusions
\[
N_{0}^{0}[-1, 1] \subset N_{0}^{0} \subset N[\mathcal{M}] \subset R_{0}[\mathcal{M}] \subset R[\mathcal{M}] \subset \widetilde{R}(\mathcal{M}).
\]

A selfadjoint operator \( T \) in the Hilbert space \( \mathcal{H} \) is called \( \mathcal{M} \)-simple, where \( \mathcal{M} \) is a subspace of \( \mathcal{H} \), if \( \overline{\text{span}} \{ T - \lambda I \}^{-1} \mathcal{M}, \lambda \in \mathbb{C} + \mathbb{C} - \} = \mathcal{H} \). If \( T \) is bounded then the latter condition is equivalent to \( \overline{\text{span}} \{ T^n \mathcal{M}, n \in \mathbb{N}_0 \} = \mathcal{H} \).

The next theorem follows from \([8, \text{Theorem 4.8}]\) and the Naimark’s dilation theorem \([8, \text{Theorem 1, Appendix I}]\), see \([2]\) and \([3]\) for the case \( M \in N_{0}^{0} \).

**Theorem 1.4.**

1) If \( M \in N[\mathcal{M}] \), then there exist a Hilbert space \( \mathcal{H} \) containing \( \mathcal{M} \) as a subspace and a selfadjoint operator \( T \) in \( \mathcal{H} \) such that \( T \) is \( \mathcal{M} \)-simple and
\[
(1.3) \quad M(\lambda) = P_{\mathcal{M}}(T - \lambda I)^{-1}\mathcal{M}.
\]

for \( \lambda \) in the domain of \( M \). If \( M \in N_{0}^{0} \), then \( T \) is bounded and if \( M \in N_{0}^{0}[-1, 1] \), then \( T \) is a selfadjoint contraction.

2) If \( T_1 \) and \( T_2 \) are selfadjoint operators in the Hilbert spaces \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \), respectively, \( \mathcal{M} \) is a subspace in \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \), \( T_1 \) and \( T_2 \) are \( \mathcal{M} \)-simple, and
\[
(1.4) \quad M(\lambda) = P_{\mathcal{M}}(T_1 - \lambda I_{\mathcal{H}_1})^{-1}\mathcal{M} = P_{\mathcal{M}}(T_2 - \lambda I_{\mathcal{H}_2})^{-1}\mathcal{M}, \lambda \in \mathbb{C} \setminus \mathbb{R},
\]
then there exists a unitary operator \( U \) mapping \( \mathcal{H}_1 \) onto \( \mathcal{H}_2 \) such that
\[
U|\mathcal{M} = I_{\mathcal{M}} \quad \text{and} \quad UT_1 = T_2 U.
\]

The right hand side in \((1.3)\) is often called compressed resolvent/\( \mathcal{M} \)-resolvent/the Weyl function/m-function, \([6, 11]\). A representation \( M \in N_{0}^{0} \) in the form \((1.3)\) will be called a realization of \( M \).

We show in Section \( \text{2.6} \) that \( M(\lambda) \in N_{0}^{0}[-1, 1] \iff (\lambda^2 - 1)^{-1}M(\lambda)^{-1} \in N_{0}^{0}[-1, 1] \). It follows that the transformation
\[
(1.4) \quad N_{0}^{0}[-1, 1] \ni M(\lambda) \overset{\Gamma}{\mapsto} M_{\Gamma}(\lambda) := \frac{M(\lambda)^{-1}}{\lambda^2 - 1} \in N_{0}^{0}[-1, 1]
\]
maps the class \( N_{0}^{0}[-1, 1] \) onto itself and \( \Gamma^{-1} = \Gamma \). In Theorem \( \text{2.6} \) we construct a realization of \((\lambda^2 - 1)^{-1}M(\lambda)^{-1}\) as a compressed resolvent by means of the contraction \( T \) that realizes \( M \). The mapping \( \Gamma \) has the unique fixed point \( M_0(\lambda) = -\frac{I_{\mathcal{M}}}{\sqrt{\lambda^2 - 1}} \) that is compressed resolvent \( P_{\mathcal{M}_0}(J_0 - \lambda I)^{-1}\mathcal{M}_0 \) of the block-operator Jacobi matrix

\[
J_0 = \begin{bmatrix}
0 & \frac{1}{\sqrt{2}} I_{\mathcal{M}} & 0 & 0 & 0 & \cdots \\
\frac{1}{\sqrt{2}} I_{\mathcal{M}} & 0 & \frac{1}{2} I_{\mathcal{M}} & 0 & 0 & \cdots \\
0 & \frac{1}{2} I_{\mathcal{M}} & 0 & \frac{1}{2} I_{\mathcal{M}} & 0 & \cdots \\
0 & 0 & \frac{1}{2} I_{\mathcal{M}} & 0 & \frac{1}{2} I_{\mathcal{M}} & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots
\end{bmatrix},
\]

acting in the Hilbert space \( \ell^2(N_0) \otimes \mathcal{M} \), and \( \mathcal{M}_0 = \mathcal{M} \oplus \{0\} \oplus \cdots \), see Proposition \( \text{2.7} \).
A selfadjoint linear relation $\tilde{A}$ in the orthogonal sum $\mathcal{M} \oplus \mathcal{K}$ is called \textit{minimal with respect to} $\mathcal{M}$ (see [9] page 5366) if

$$\mathcal{M} \oplus \mathcal{K} = \text{span} \left\{ \mathcal{M} + (\tilde{A} - \lambda I)^{-1} \mathcal{M} : \lambda \in \rho(\tilde{A}) \right\}.$$ 

One of the statements obtained in [9] in the context of the Weyl family of a boundary relation is the following:

\textbf{Theorem 1.5.} \textit{Let $\mathcal{M}$ be a Nevanlinna family in the Hilbert space $\mathcal{M}$. Then there exists unique up to unitary equivalence a selfadjoint linear relation $\tilde{A}$ in the Hilbert space $\mathcal{M} \oplus \mathcal{K}$ such that $\tilde{A}$ is minimal with respect to $\mathcal{M}$ and the equality}

\begin{equation}
\mathcal{M}(\lambda) = -\left( P_{\mathcal{M}} \left( \tilde{A} - \lambda I \right)^{-1} | \mathcal{M} \right)^{-1} - \lambda I_{\mathcal{M}}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R} \tag{1.6}
\end{equation}

\textit{holds.}

The equivalent form of (1.6) is

$$P_{\mathcal{M}}(\tilde{A} - \lambda I)^{-1} | \mathcal{M} = -(\mathcal{M}(\lambda) + \lambda I_{\mathcal{M}})^{-1}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$ 

The compressed resolvent $P_{\mathcal{M}}(\tilde{A} - \lambda I)^{-1} | \mathcal{M}$ belongs to the class $\mathcal{R}_0[\mathcal{M}]$ and even to its more narrow subclass, see Corollary 2.4.

In Section 3 we consider the following mapping defined on the whole class $\tilde{R}(\mathcal{M})$ of Nevanlinna families:

\begin{equation}
\mathcal{M}(\lambda) \mapsto \tilde{\mathcal{M}}_{\lambda} := -(\mathcal{M}(\lambda) + \lambda I_{\mathcal{M}})^{-1}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{1.7}
\end{equation}

We prove (Theorem 3.1) that the mapping $\tilde{\mathcal{M}}$ and each its degree $\tilde{\mathcal{M}}^k$ has the unique fixed point

$$\mathcal{M}_0(\lambda) = \frac{-\lambda + \sqrt{\lambda^2 - 4}}{2} I_{\mathcal{M}}$$

and the sequence of iterations

$$\mathcal{M}_1(\lambda) = -(\mathcal{M}(\lambda) + \lambda I_{\mathcal{M}})^{-1}, \quad \mathcal{M}_{n+1}(\lambda) = -(\mathcal{M}_n(\lambda) + \lambda I_{\mathcal{M}})^{-1}, \quad n \in \mathbb{N},$$

starting with an arbitrary Nevanlinna family $\mathcal{M}$, converges to $\mathcal{M}_0$ in the operator norm topology uniformly on compact sets lying in the open left/right half-plane of the complex plane. The function $\mathcal{M}_0(\lambda)$ can be realized by the free discrete Schrödinger operator given by the block-operator Jacobi matrix

\begin{equation}
\tilde{J}_0 = \begin{bmatrix}
0 & I_{\mathcal{M}} & 0 & 0 & 0 & \cdots \\
I_{\mathcal{M}} & 0 & I_{\mathcal{M}} & 0 & 0 & \cdots \\
0 & I_{\mathcal{M}} & 0 & I_{\mathcal{M}} & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\end{equation}

acting in the Hilbert space $\ell^2(\mathbb{N}) \otimes \mathcal{M}$. Besides we construct a sequence $\{\tilde{J}_n, \tilde{A}_n\}$ of realizations of functions $\mathcal{M}_n$, $\mathcal{M}_n(\lambda) = P_{\mathcal{M}}(\tilde{A}_{n-1} - \lambda I)^{-1} | \mathcal{M}$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$) and show that the Hilbert space $\ell^2(\mathbb{N}) \otimes \mathcal{M}$ and the block-operator Jacobi matrix $\tilde{J}_0$ are the inductive limits of $\{\tilde{J}_n\}$ and $\{\tilde{A}_n\}$, respectively. Observe that when $\mathcal{M} = \mathbb{C}$, the Jacobi matrices $J_0$ and $\frac{1}{2} \tilde{J}_0$ are connected with Chebyshev polynomials of the first and second kinds, respectively [6].
Let $\mathcal{H}(t) = \begin{bmatrix} h_{11}(t) & h_{12}(t) \\ h_{21}(t) & h_{22}(t) \end{bmatrix}$ be symmetric and nonnegative $2 \times 2$ matrix-function with scalar real-valued entries on $\mathbb{R}_+$. Assume that $\mathcal{H}(t)$ is locally integrable on $\mathbb{R}_+$ and is trace-normed, i.e., $\text{tr} \mathcal{H}(t) = 1$ a.e. on $\mathbb{R}_+$. Let $\mathcal{J} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. The system of differential equations

$$\mathcal{J} \frac{d\bar{x}}{dt} = \lambda \mathcal{H}(t)\bar{x}(t), \quad \bar{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad t \in \mathbb{R}_+, \quad \lambda \in \mathbb{C},$$

is called the canonical system with the Hamiltonian $\mathcal{H}$ or the Hamiltonian system.

The $m$-function $m_{\mathcal{H}}$ of the canonical system (1.9) can be defined as follows:

$$m_{\mathcal{H}}(\lambda) = \frac{x_2(0, \lambda)}{x_1(0, \lambda)}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

where $\bar{x}(t, \lambda)$ is the solution of (1.9), satisfying

$$x_1(0, \lambda) \neq 0 \quad \text{and} \quad \int_{\mathbb{R}_+} \bar{x}(t, \lambda)^* \mathcal{H}(t)\bar{x}(t, \lambda) dt < \infty.$$

The $m$-function of a canonical system is a Nevanlinna function. As has been proved by L. de Branges [7], see also [22], for each Nevanlinna function $m$ there exists a unique trace-normed canonical system such that its $m$-function $m_{\mathcal{H}}$ coincides with $m$. In the last Section 4, applying the algorithm suggested by I.S. Kac in [14], we construct a sequence of Hamiltonians $\{\mathcal{H}_n\}$ such that the $m$-functions of the corresponding canonical systems coincides with the sequence of the iterates $\{m_n\}$ of the mapping $\hat{\Gamma}$

$$m_1(\lambda) = -\frac{1}{m(\lambda) + \lambda}, \ldots, m_{n+1}(\lambda) = -\frac{1}{m_n(\lambda) + \lambda}, \ldots, \lambda \in \mathbb{C} \setminus \mathbb{R},$$

where $m(\lambda)$ is a non-rational Nevanlinna function form the class $\mathcal{N}_0^0$. This sequence $\{m_n\}$ converges locally uniformly on $\mathbb{C}_+ / \mathbb{C}_-$ to the function $m_0(\lambda) = \frac{-\lambda + \sqrt{\lambda^2 - 4}}{2}$ that is the $m$-function of the canonical system with the Hamiltonian

$$\mathcal{H}_0(t) = \begin{bmatrix} \cos^2(j + 1)\frac{\pi}{2} & 0 \\ 0 & \sin^2(j + 1)\frac{\pi}{2} \end{bmatrix}, \quad t \in [j, j + 1) \forall j \in \mathbb{N}_0.$$

For the constructed Hamiltonian $\mathcal{H}_n$ the property $\mathcal{H}_n| [0, n+1) = \mathcal{H}_0| [0, n+1)$ is valid for each $n \in \mathbb{N}$. Moreover, our construction shows that for the Hamiltonian $\mathcal{H}$ such that the $m$-function $m_{\mathcal{H}}$ of the corresponding canonical system belongs to the class $\mathcal{N}_0^0$, the Hamiltonian $\mathcal{H}_{\hat{\Gamma}}$ of the canonical system having $\hat{\Gamma}(m)$ as its $m$-function, is of the form

$$\mathcal{H}_{\hat{\Gamma}}(t) = \begin{cases} 
\mathcal{H}_0(t), & t \in [0, 2) \\
\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \mathcal{H}(t-1), & t \in [2, +\infty) 
\end{cases}.$$
2. Characterizations of subclasses

2.1. The subclass $R_0[\mathcal{M}]$. The next proposition is well known, cf.\cite{Arlinski}.  

**Proposition 2.1.** Let $M(\lambda)$ be a $B(\mathcal{M})$-valued Nevanlinna function. Then the following statements are equivalent:

(i) $M \in R_0[\mathcal{M}]$;
(ii) the function $y \| M(iy) \|$ is bounded on $[1, \infty)$,
(iii) there exists a strong limit $s - \lim_{y \to +\infty} iy M(iy) = -C$, where $C$ is a bounded selfadjoint nonnegative operator in $\mathcal{M}$;
(iv) $M$ admits a representation

\begin{equation}
M(\lambda) = K^* (T - \lambda I)^{-1} K, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},
\end{equation}

where $T$ is a selfadjoint operator in a Hilbert space $\mathcal{K}$ and $K \in B(\mathcal{M}, \mathcal{K})$; here $\mathcal{K}$, $T$, and $K$ can be selected such that $T$ is $\text{ran} \ K$-simple, i.e.,

$$\text{span} \{(T - \lambda)^{-1} \text{ran} K : \lambda \in \mathbb{C} \setminus \mathbb{R}\} = \mathcal{K}.$$  

**Proof.** Our proof is based on the Schur-Frobenius formula for the resolvent $(\tilde{A} - \lambda I)^{-1}$

\begin{equation}
(\tilde{A} - \lambda I)^{-1} = \begin{bmatrix}
-V(\lambda)^{-1} & V(\lambda)^{-1} K^* (T - \lambda I)^{-1} \\
(T - \lambda I)^{-1} K V(\lambda)^{-1} & (T - \lambda I)^{-1} (I_K - K V(\lambda)^{-1} K^* (T - \lambda I)^{-1})
\end{bmatrix},
\end{equation}

$$V(\lambda) := \lambda I_{\mathcal{M}} - D + K^* (T - \lambda I)^{-1} K, \quad \lambda \in \rho(T) \cap \rho(\tilde{A}).$$

Actually, \eqref{2.2} implies the equivalences

$$\text{span} \left\{ \mathcal{M} + (\tilde{A} - \lambda I)^{-1} \mathcal{M} : \lambda \in \mathbb{C} \setminus \mathbb{R} \right\} = \mathcal{M} \oplus \mathcal{K},$$

$$\iff \mathcal{K} \bigcap_{\lambda \in \mathbb{C} \setminus \mathbb{R}} \ker \left( P_{\mathcal{M}} (\tilde{A} - \lambda I)^{-1} \right) = \{0\} \iff \bigcap_{\lambda \in \mathbb{C} \setminus \mathbb{R}} \ker \left( K^* (T - \lambda I)^{-1} \right) = \{0\}$$

$$\iff \text{span} \left\{ (T - \lambda)^{-1} \text{ran} K : \lambda \in \mathbb{C} \setminus \mathbb{R} \right\} = \mathcal{K}. \quad \square$$

In the sequel we will use the following consequence of \eqref{2.2}:

\begin{equation}
P_{\mathcal{M}} (\tilde{A} - \lambda I)^{-1} \upharpoonright \mathcal{M} = - (D - K^* (T - \lambda I_{\mathcal{M}})^{-1} K + \lambda I_{\mathcal{M}})^{-1}, \quad \lambda \in \rho(T) \cap \rho(\tilde{A}).
\end{equation}

**Proposition 2.3.** cf.\cite{Arlinski} the proof of Theorem 3.9. For a $B(\mathcal{M})$-valued Nevanlinna function $M$ the following statements are equivalent:

(i) the limit value $C := -s - \lim_{y \to +\infty} iy M(iy)$ satisfies $0 \leq C \leq I_{\mathcal{M}}$;
(ii) $M$ admits a representation
\[ M(\lambda) = P_{\mathfrak{M}}(\tilde{A} - \lambda I)^{-1} | \mathfrak{M}, \; \lambda \in \mathbb{C} \setminus \mathbb{R}, \]
where $\tilde{A}$ is a selfadjoint linear relation in a Hilbert space $\tilde{\mathfrak{H}} \supseteq \mathfrak{M}$ and $P_{\mathfrak{M}}$ is the orthogonal projection from $\tilde{\mathfrak{H}}$ onto $\mathfrak{M}$;
(iii) $M$ admits a representation (2.1) with a contraction $K \in \mathcal{B}(\mathfrak{M}, \mathfrak{F})$;
(iv) the following inequality holds
\[ \frac{\mathrm{Im} M(\lambda)}{\mathrm{Im} \lambda} - M(\lambda)M(\lambda)^* \geq 0, \; \lambda \in \mathbb{C} \setminus \mathbb{R}. \]

In (ii) $\mathfrak{F}$ and $\tilde{A}$ can be selected such that $\tilde{A}$ is minimal w.r.t. $\mathfrak{M}$. Moreover, $\tilde{A}$ in (2.4) can be taken to be a selfadjoint operator if and only if $C = I_{\mathfrak{M}}$. The operator $K$ in (iii) is an isometry if and only if $C = I_{\mathfrak{M}}$.

**Proof.** The equivalence (i)$\iff$ (iii) follows from Proposition 2.1.

(i)$\impl (iv)
Since (2.1) holds, we get $C = K^*K$ and the inequality $0 \leq C \leq I_{\mathfrak{M}}$ implies $||K|| \leq 1$ and, therefore, holds the inequality.
\[ \frac{\mathrm{Im} M(\lambda)}{\mathrm{Im} \lambda} - M(\lambda)M(\lambda)^* \geq 0, \; \lambda \in \mathbb{C} \setminus \mathbb{R}. \]

(iv)$\impl (ii)
Consider $-M(\lambda)^{-1}$. Then
\[ \frac{\mathrm{Im} (-M(\lambda)^{-1}h - \lambda h, h)}{\mathrm{Im} \lambda} = \frac{\mathrm{Im} (-M(\lambda)^{-1}h, h)}{\mathrm{Im} \lambda} - ||h||^2 \geq 0, \; h \in \mathfrak{M}. \]
Hence $\mathcal{M}(\lambda) := -M(\lambda)^{-1} - \lambda I_{\mathfrak{M}}$ is a Nevanlinna family. Due to Theorem 1.5 and (1.6) we have
\[ -(\mathcal{M}(\lambda) + \lambda I_{\mathfrak{M}})^{-1} = P_{\mathfrak{M}}(\tilde{A} - \lambda I_{\tilde{\mathfrak{H}}})^{-1} | \mathfrak{M}, \; \lambda \in \mathbb{C} \setminus \mathbb{R}, \]
where $\tilde{A}$ is a selfadjoint linear relation in some Hilbert space $\tilde{\mathfrak{H}} = \mathfrak{M} \oplus K$.

(ii)$\impl (i)$
Let $\tilde{A}_0$ be the operator part of $\tilde{A}$ acting in a subspace $\mathfrak{F}_0$ of $\mathfrak{F}$. Decompose $\tilde{A}$ as $H = \text{Gr} \tilde{A}_0 \oplus \{0, \mathfrak{F} \ominus \mathfrak{F}_0\}$. Then
\[ P_{\mathfrak{M}}(\tilde{A} - \lambda I)^{-1} | \mathfrak{M} = P_{\mathfrak{M}}(\tilde{A}_0 - \lambda I)^{-1}P_{\mathfrak{F}_0} | \mathfrak{M} = P_{\mathfrak{M}}P_{\mathfrak{F}_0}(\tilde{A}_0 - \lambda I)^{-1}P_{\mathfrak{F}_0} | \mathfrak{M}. \]
Set $K = P_{\mathfrak{F}_0} | \mathfrak{M} : \mathfrak{M} \to \mathfrak{F}_0$. Then $K^* = P_{\mathfrak{M}}P_{\mathfrak{F}_0}$. $||K|| \leq 1$,
\[ M(\lambda) = K^*(\tilde{A}_0 - \lambda I)^{-1}K, \; \lambda \in \mathbb{C} \setminus \mathbb{R}, \]
and
\[ s - \lim_{x \to +\infty} iyM(iy) = -K^*K, \; C = K^*K \in [0, I_{\mathfrak{M}}]. \]

(iii)$\impl (ii)
Since $||K|| \leq 1$, $\mathcal{M}(\lambda) = -M^{-1}(\lambda) - \lambda I_{\mathfrak{M}}$ is a Nevanlinna family. By Theorem 1.5 there is a Hilbert space $\mathfrak{K}$ and a selfadjoint linear relation $\tilde{A}$ in $\mathfrak{M} \oplus \mathfrak{K}$ minimal w.r.t. $\mathfrak{M}$ such that $\mathcal{M}(\lambda) = -\left(P_{\mathfrak{M}}(\tilde{A} - \lambda I)^{-1} | \mathfrak{M}\right)^{-1} - \lambda I_{\mathfrak{M}}, \; \lambda \in \mathbb{C} \setminus \mathbb{R}. \]

$\square$
Corollary 2.4. There is a one-to-one correspondence between all Nevanlinna families $\mathcal{M}$ in $\mathbb{M}$ and all $\mathcal{B}(\mathbb{M})$-valued Nevanlinna functions $M$ satisfying the condition (ii) in Proposition 2.1 with $C \in [0, I_{\mathbb{M}}]$. This correspondence is given by the relations

$$M(\lambda) = -(\mathcal{M}(\lambda) + \lambda I_{\mathbb{M}})^{-1}, \quad \mathcal{M}(\lambda) = -M(\lambda)^{-1} - \lambda I_{\mathbb{M}}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$ 

Remark 2.5. For the case $\mathbb{M} = \mathbb{C}$ the statement of Corollary 2.4 can be found in [6, Chapter VII, §1, Lemma 1.7].

In [10] (see also [4]) it is established that an $\mathcal{B}(\mathbb{M})$-valued function $M(\lambda)$, $\lambda \in \mathcal{D} \subset \mathbb{C}_+ / \mathbb{C}_-$ admits the representation (2.4) iff the kernel

$$K(\lambda, \mu) = \frac{M(\lambda) - M(\mu)^*}{\lambda - \bar{\mu}} - M(\mu)^*M(\lambda)$$

is nonnegative on $\mathcal{D}$.

2.2. The subclass $N^0_{\mathbb{M}}[-1, 1]$. Notice, that if $M \in N^0_{\mathbb{M}}[-1, 1]$, then

$$\left\{ \begin{array}{ll} (M(x)g, g) > 0 & \forall g \in \mathbb{M} \setminus \{0\}, \ x < -1, \\ (M(x)g, g) < 0 & \forall g \in \mathbb{M} \setminus \{0\}, \ x > 1. \end{array} \right.$$ 

Therefore, see [16, Appendix]

$$(1 + \lambda)M(\lambda), \ (1 - \lambda)M(\lambda) \in \mathcal{R}[\mathbb{M}].$$

Theorem 2.6. 1) A $\mathcal{B}(\mathbb{M})$-valued Nevanlinna function $M$ belongs to $N^0_{\mathbb{M}}[-1, 1]$ if and only if the function

$$L(\lambda, \xi) = \frac{(1 - \lambda^2)M(\lambda) - (1 - \bar{\xi}^2)M(\xi)^* - (\lambda - \bar{\xi})I_{\mathbb{M}}}{\lambda - \xi},$$

with $\lambda, \xi \in \mathbb{C} \setminus [-1, 1]$, $\lambda \neq \bar{\xi}$ is a nonnegative kernel.

2) If $M \in N^0_{\mathbb{M}}[-1, 1]$, then the function

$$\frac{M(\lambda)^{-1}}{\lambda^2 - 1}, \ \lambda \in \mathbb{C} \setminus [-1, 1]$$

belongs to $N^0_{\mathbb{M}}[-1, 1]$ as well.

3) If a selfadjoint contraction $T$ in the Hilbert space $\mathcal{H}$, containing $\mathbb{M}$ as a subspace, realizes $M$, i.e., $M(\lambda) = P_{\mathbb{M}}(T - \lambda I)^{-1} \uparrow \mathbb{M}$, for all $\lambda \in \mathbb{C} \setminus [-1, 1]$, then

$$\frac{M(\lambda)^{-1}}{\lambda^2 - 1} = P_{\mathbb{M}}(T - \lambda I)^{-1} \uparrow \mathbb{M}, \ \lambda \in \mathbb{C} \setminus [-1, 1],$$

where a selfadjoint contraction $\mathcal{T}$ is given by

$$T := \begin{bmatrix} -P_{\mathbb{M}}T \uparrow \mathbb{M} & P_{\mathbb{M}}D_T \\ D_T \uparrow \mathbb{M} & T \end{bmatrix}, \quad \mathbb{M} \oplus \mathbb{M} \rightarrow \mathbb{D}_T \oplus \mathbb{D}_T,$$

and $D_T := (I - T^2)^{1/2}$, $\mathbb{D}_T := \overline{\text{ran}} D_T$. Moreover, if $T$ is $\mathbb{M}$-simple, then $\mathcal{T}$ is $\mathbb{M}$-simple as well and the operator $T \uparrow \mathbb{D}_T$ is unitarily equivalent to the operator $P_{\mathbb{M}}T \uparrow \mathbb{M}$.
Proof. The statement in 1) follows from [2, Theorem 6.1]. Observe that if $M(\lambda) = P_{\mathfrak{m}}(T - \lambda I)^{-1} | \mathfrak{m}$ $\forall \lambda \in \mathbb{C} \setminus [-1, 1]$, where $T$ is a selfadjoint contraction, then

\begin{equation}
L(\lambda, \xi) = \frac{(1 - \lambda^2)M(\lambda) - (1 - \xi^2)M(\xi)^* - (\lambda - \xi)I_{\mathfrak{m}}}{\lambda - \xi}
\end{equation}

\begin{equation}
= P_{\mathfrak{m}}(T - \lambda I)^{-1}(I - T^2)(T - \bar{\xi}I)^{-1} | \mathfrak{m}, \lambda, \xi \in \mathbb{C} \setminus [-1, 1], \lambda \neq \bar{\xi}.
\end{equation}

2) Let $\lambda \in \mathbb{C} \setminus [-1, 1]$, then

$$|(T - \lambda I)h, h)| \geq d(\lambda)||h||^2 \forall h \in \mathfrak{h},$$

where $d(\lambda) = \text{dist}(\lambda, [-1, 1])$. Set $h = (T - \lambda I)^{-1}f, f \in \mathfrak{m}$. Then

$$||M(\lambda)f|| ||f|| \geq |(f, M(\lambda)f)| = |(f, (T - \lambda I)^{-1}f)| = |(h, (T - \lambda I)h)| \geq d(\lambda)||h||^2 \geq c(\lambda)||f||^2, c(\lambda) > 0.$$

Hence, $||M(\lambda)f|| \geq c(\lambda)||f||$ and since $M(\lambda) = M(\lambda)^*$, we get $||M(\lambda)^*f|| \geq c(\lambda)||f||$. It follows that $M(\lambda)^{-1} \in \mathcal{B}(\mathfrak{m})$ for all $\lambda \in \mathbb{C} \setminus [-1, 1]$.

Set

$$L(\lambda) := (1 - \lambda^2)M(\lambda) - \lambda I_{\mathfrak{m}}, \lambda \in \mathbb{C} \setminus [-1, 1].$$

Then from (2.6) we get

$$L(\lambda) - L(\lambda)^* = (1 - \lambda^2)M(\lambda) - (1 - \bar{\lambda}^2)(M(\lambda)^* - (\lambda - \bar{\lambda})I_{\mathfrak{m}}$$

$$= (\lambda - \bar{\lambda})P_{\mathfrak{m}}(T - \lambda I)^{-1}(I - T^2)(T - \bar{\lambda}I)^{-1} | \mathfrak{m}.$$}

It follows that $L(\lambda)$ and the functions

$$(1 - \lambda^2)M(\lambda) = L(\lambda) + \lambda I_{\mathfrak{m}}, \lambda \in \mathbb{C} \setminus [-1, 1]$$

and

$$-(1 - \lambda^2)M(\lambda)^{-1} = \frac{M(\lambda)^{-1}}{\lambda^2 - 1}, \lambda \in \mathbb{C} \setminus [-1, 1]$$

are Nevanlinna functions. Then from the equality $M(\lambda) = -\lambda^{-1} + o(\lambda^{-1}), \lambda \to \infty$, we get that also

$$\frac{M(\lambda)^{-1}}{\lambda^2 - 1} = -\lambda^{-1} + o(\lambda^{-1}), \lambda \to \infty,$$

i.e.,

$$\frac{M(\lambda)^{-1}}{\lambda^2 - 1} \in \mathbb{N}_{\mathfrak{m}}[-1, 1].$$

3) Observe that the subspace $\mathfrak{D}_T$ is contained in the Hilbert space $\mathfrak{h}$. Let $H := \mathfrak{m} \oplus \mathfrak{D}_T$ and let $T$ be given by (2.5). Since $T$ is a selfadjoint contraction in $\mathfrak{h}$, we get for an arbitrary $\varphi \in \mathfrak{m}$ and $f \in \mathfrak{D}_T$ the equalities

$$\left(\begin{bmatrix} \varphi \\ f \end{bmatrix}, \begin{bmatrix} \varphi \\ f \end{bmatrix}^{+} \right) \pm \left(\begin{bmatrix} \varphi \\ f \end{bmatrix}, T \begin{bmatrix} \varphi \\ f \end{bmatrix} \right) = ||(I \mp T)^{1/2}\varphi \pm (I \mp T)^{1/2}f||^2.$$

Therefore $T$ is a selfadjoint contraction in the Hilbert space $H$. 

Applying \[2.3] we obtain
\[
P\mathfrak{M}(T - \lambda I)^{-1}|\mathfrak{M} = - (\lambda I + P_{\mathfrak{M}} T | \mathfrak{M} + P_{\mathfrak{M}} D_T (T - \lambda I)^{-1} D_T | \mathfrak{M})^{-1}
\]
\[
= - (\lambda I + P_{\mathfrak{M}} (T(T - \lambda I) + I - T^2) (T - \lambda I)^{-1} | \mathfrak{M})^{-1}
\]
\[
= - (\lambda I + P_{\mathfrak{M}} (I - \lambda T)(T - \lambda I)^{-1} | \mathfrak{M})^{-1} = - ((1 - \lambda^2) P_{\mathfrak{M}} (T - \lambda I)^{-1} | \mathfrak{M})^{-1}
\]
\[
= \frac{M^{-1}(\lambda)}{\lambda^2 - 1}, \quad \lambda \in \mathbb{C} \setminus [-1, 1].
\]
Suppose that \(T\) is \(\mathfrak{M}\)-simple, i.e.,
\[
\overline{\text{span}} \{T^n \mathfrak{M}, \ n \in \mathbb{N}_0\} = \mathfrak{M} \oplus \mathcal{K} \iff \bigcap_{n=0}^{\infty} \ker (P_{\mathfrak{M}} T^n) = \{0\}.
\]
Hence, since
\[
\mathcal{D}_T \oplus \{\overline{\text{span}} \{T^n D_T \mathfrak{M}, \ n \in \mathbb{N}_0\}\} = \bigcap_{n=0}^{\infty} \ker (P_{\mathfrak{M}} T^n D_T),
\]
we get \(\overline{\text{span}} \{T^n D_T \mathfrak{M}, \ n \in \mathbb{N}_0\} = \mathcal{D}_T\). This means that the operator \(T\) is \(\mathfrak{M}\)-simple.

Let
\[
\mathcal{T} = \begin{bmatrix}
-P_{\mathfrak{M}} T | \mathfrak{M} & P_{\mathfrak{M}} D_T | \mathcal{D}_T \\
D_T | \mathfrak{M} & T | \mathcal{D}_T
\end{bmatrix} = \begin{bmatrix}
P_{\mathfrak{M}} T | \mathfrak{M} & P_{\mathfrak{M}} D_T \\
D_T | \mathfrak{M} & T | \mathcal{D}_T
\end{bmatrix} : \mathfrak{M} \oplus \mathcal{K} \to \mathfrak{M} \oplus \mathcal{D}_T.
\]
As has been proved above because the selfadjoint contraction \(T\) realizes the function \(Q(\lambda) := (\lambda^2 - 1)^{-1} M(\lambda)^{-1}\), i.e.,
\[
P_{\mathfrak{M}} (T - \lambda I)^{-1} | \mathfrak{M} = Q(\lambda) = \frac{M(\lambda)^{-1}}{\lambda^2 - 1}, \quad \lambda \in \mathbb{C} \setminus [-1, 1],
\]
the selfadjoint contraction \(T\) realizes the function \((\lambda^2 - 1)^{-1} Q(\lambda)^{-1} = M(\lambda)\). In addition, if \(T\) is \(\mathfrak{M}\)-simple, then \(T\) and therefore \(\mathcal{T}\) are \(\mathfrak{M}\)-simple. Since
\[
P_{\mathfrak{M}} (T - \lambda I)^{-1} | \mathfrak{M} = P_{\mathfrak{M}} (T - \lambda I)^{-1} | \mathfrak{M} = M(\lambda), \ |\lambda| > 1,
\]
the operators \(T\) and \(T\) are unitarily equivalent and, moreover, see Theorem 1.4 there exists a unitary operator \(U\) of the form
\[
U = \begin{bmatrix}
I_{\mathfrak{M}} & 0 \\
0 & U
\end{bmatrix} : \mathfrak{M} \oplus \mathcal{K} \to \mathfrak{M} \oplus \mathcal{K},
\]
where \(\mathcal{K} := \mathfrak{N} \oplus \mathfrak{M}\) and \(U\) is a unitary operator from \(\mathcal{D}_T\) onto \(\mathcal{K}\) such that
\[
T U = U T \iff \begin{bmatrix}
P_{\mathfrak{M}} T | \mathfrak{M} & P_{\mathfrak{M}} T | \mathfrak{K} \\
P_{\mathfrak{K}} T | \mathfrak{M} & P_{\mathfrak{K}} T | \mathfrak{K}
\end{bmatrix} \begin{bmatrix}
I_{\mathfrak{M}} & 0 \\
0 & U
\end{bmatrix} = \begin{bmatrix}
P_{\mathfrak{M}} T | \mathfrak{M} & P_{\mathfrak{M}} D_T | \mathcal{D}_T \\
P_{\mathfrak{K}} T | \mathfrak{M} & T | \mathcal{D}_T
\end{bmatrix} = \begin{bmatrix}
(P_{\mathfrak{M}} T | \mathfrak{K}) U = P_{\mathfrak{M}} D_T | \mathcal{D}_T \\
P_{\mathfrak{K}} T | \mathfrak{M} = U D_T | \mathfrak{M} \\
(P_{\mathfrak{K}} T | \mathfrak{K}) U = U T | \mathcal{D}_T | \mathcal{D}_T
\end{bmatrix}.
\]
In particular \(P_{\mathfrak{K}} T | \mathfrak{K}\) and \(T | \mathcal{D}_T\) are unitarily equivalent. \hfill \square
Observe that for a bounded selfadjoint $T$ the equality $M(\lambda) = P_{\mathfrak{M}}(T - \lambda I)^{-1}|\mathfrak{M}$ yields the following relation for $\lambda \in \mathbb{C}\setminus \mathbb{R}$:

\[
\frac{1 - |\lambda|^2}{\text{Im } \lambda} \text{Im } M(\lambda) - 2\text{Re } (\lambda M(\lambda)) - I_{\mathfrak{M}} = P_{\mathfrak{M}}(T - \lambda I)^{-1}(I - T^2)(T - \lambda I)^{-1}|\mathfrak{M}.
\]

Hence for $M(\lambda) \in \mathbb{N}_{2\mathfrak{M}}[-1, 1]$ we get

\[
\frac{1 - |\lambda|^2}{\text{Im } \lambda} \text{Im } M(\lambda) - 2\text{Re } (\lambda M(\lambda)) - I_{\mathfrak{M}} = \text{Im } \frac{(1 - \lambda^2)M(\lambda) - \lambda}{\text{Im } \lambda} \geq 0, \text{ Im } \lambda \neq 0.
\]

### 2.3. The fixed point of the mapping $\Gamma$

**Proposition 2.7.** Let $\mathfrak{M}$ be a Hilbert space. Then the mapping $\Gamma$ [1.4] has a unique fixed point

\[
M_0(\lambda) = -\frac{I_{\mathfrak{M}}}{\sqrt{\lambda^2 - 1}} \quad (\text{Im } \sqrt{\lambda^2 - 1} > 0 \text{ for } \text{Im } \lambda > 0).
\]

Define the weight $\rho_0(t)$ and the weighted Hilbert space $\mathfrak{H}_0$ as follows

\[
\rho_0(t) = \frac{1}{\pi \sqrt{1 - t^2}}, \quad t \in (-1, 1),
\]

\[
\mathfrak{H}_0 := L_2([-1, 1], \mathfrak{M}, \rho_0(t)) = L_2([-1, 1], \rho_0(t)) \otimes \mathfrak{M} = \left\{ f(t) : \int_{-1}^1 \frac{||f(t)||^2_{\mathfrak{M}}}{\sqrt{1 - t^2}} dt < \infty \right\}.
\]

Then $\mathfrak{H}_0$ is the Hilbert space with the inner product

\[
(f(t), g(t))_{\mathfrak{H}_0} = \frac{1}{\pi} \int_{-1}^1 (f(t), g(t))_{\mathfrak{M}} \rho_0(t) dt = \frac{1}{\pi} \int_{-1}^1 \frac{(f(t), g(t))_{\mathfrak{M}}}{\sqrt{1 - t^2}} dt.
\]

Identify $\mathfrak{M}$ with a subspace of $\mathfrak{H}_0$ of constant vector-functions $\{f(t) \equiv f, f \in \mathfrak{M}\}$. Define in $\mathfrak{H}_0$ the multiplication operator

\[
(T_0 f)(t) = tf(t), \quad f \in \mathfrak{H}_0.
\]

Then

\[
M_0(\lambda) = P_{\mathfrak{M}}(T_0 - \lambda I)^{-1}|\mathfrak{M}.
\]

Let $H_0 = \bigoplus_{j=0}^\infty \mathfrak{M} = \ell^2(\mathbb{N}_0) \otimes \mathfrak{M}$ and let $J_0$ be the operator in $H_0$ given by the block-operator Jacobi matrix of the form (1.5). Set $M_0 := \mathfrak{M} \bigoplus \{0\} \bigoplus \{0\} \bigoplus \cdots$. Then

\[
M_0(\lambda) = P_{\mathfrak{M}_0}(J_0 - \lambda I)^{-1}|\mathfrak{M}_0.
\]

**Proof.** Let $M_0(\lambda)$ be a fixed point of the mapping $\Gamma$, i.e.,

\[
M_0(\lambda) = \frac{M_0(\lambda)^{-1}}{\lambda^2 - 1} \iff M_0(\lambda) = \frac{1}{\lambda^2 - 1} I_{\mathfrak{M}}, \quad \lambda \in \mathbb{C}\setminus [-1, 1]
\]

Since $M_0(\lambda)$ is Nevanlinna function, we get (2.7).

For each $h \in \mathfrak{M}$ calculations give the equality, see [6] pages 545–546, [13],

\[-\frac{h}{\sqrt{\lambda^2 - 1}} = \frac{1}{\pi} \int_{-1}^1 \frac{h}{t - \lambda} \frac{1}{\sqrt{1 - t^2}} dt, \quad \lambda \in \mathbb{C}\setminus [-1, 1].\]
Therefore, if $T_0$ is the operator of the form (2.9), then
\[ M_0(\lambda) = P_{\mathbb{R}}(T_0 - \lambda I)^{-1} |\mathfrak{M}|, \quad \lambda \in \mathbb{C} \setminus [-1, 1]. \]
As it is well known the Chebyshev polynomials of the first kind
\[ \widehat{T}_0(t) = 1, \quad \widehat{T}_n(t) := \sqrt{2} \cos(n \arccos t), \quad n \geq 1 \]
form an orthonormal basis of the space $L_2([-1, 1], \rho_0(t))$, where $\rho_0(t)$ is given by (2.8). This polynomials satisfy the recurrence relations
\[ t\widehat{T}_0(t) = \frac{1}{\sqrt{2}} \widehat{T}_1(t), \quad t\widehat{T}_1(t) = \frac{1}{\sqrt{2}} \widehat{T}_0(t) + \frac{1}{2} \widehat{T}_2(t), \]
\[ t\widehat{T}_n(t) = \frac{1}{2} \widehat{T}_{n-1}(t) + \frac{1}{2} \widehat{T}_{n+1}(t), \quad n \geq 2. \]
Hence the matrix of the operator $\Sigma_0$ of multiplication on the independent variable in the Hilbert space $L_2(\rho(t), [-1, 1])$ w.r.t. the basis $\{\widehat{T}_n(t)\}_{n=0}^{\infty}$ (the Jacobi matrix) takes the form (1.5) when $\mathfrak{M} = \mathbb{C}$. Besides $m_0(\lambda) := ((\mathbf{J}_0 - \lambda I)^{-1} \delta_0, \delta_0) = -\frac{1}{\sqrt{\lambda^2 - 1}}$, where $\delta_0 = [1 \ 0 \ 0 \ \cdots]^T$. Since $T_0 = \Sigma_0 \otimes I_{\mathfrak{R}}$ we get that $T_0$ is unitarily equivalent to $\mathbf{J}_0 = J_0 \otimes I_{\mathfrak{R}}$ and $M_0(\lambda) = P_{\mathbb{R}}(J_0 - \lambda I)^{-1} |\mathfrak{M}|$.

Observe that $\mathfrak{M}$-valued holomorphic in $\mathbb{C} \setminus [-1, 1]$ function
\[ M_1(\lambda) := 2(-\lambda I_{\mathfrak{M}} + M_0^{-1}(\lambda)) = 2(-\lambda + \sqrt{\lambda^2 - 1})I_{\mathfrak{M}} \]
belongs to the class $\mathcal{N}_{\mathfrak{M}}^{-1}[-1, 1]$.

3. The fixed point of the mapping $\widehat{\Gamma}$

Now we will study the mapping $\widehat{\Gamma}$ (1.7). Let $\mathcal{M}$ be a Nevanlinna family in the Hilbert space $\mathfrak{M}$. Then since
\[ |\text{Im} (\langle \mathcal{M}(\lambda) + \lambda I_{\mathfrak{M}}, f \rangle, f) | \geq |\text{Im} \lambda| |f|^2, \quad \text{Im} \lambda \neq 0, \quad f \in \text{dom} \mathcal{M}(\lambda), \]
the estimate
\[ ||(\mathcal{M}(\lambda) + \lambda I_{\mathfrak{M}})^{-1}|| \leq \frac{1}{|\text{Im} \lambda|}, \quad \text{Im} \lambda \neq 0. \]
holds true. It follows that $\mathcal{M}_1(\lambda) = -(\mathcal{M}(\lambda) + \lambda I_{\mathfrak{M}})^{-1}$ is $\mathcal{B}(\mathfrak{M})$-valued Nevanlinna function from the class $\mathcal{R}_0[\mathfrak{M}]$ and, moreover, $\mathcal{M}_1(\lambda) = K^*(\widehat{T} - \lambda I)^{-1}K$, $\text{Im} \lambda \neq 0$, where $\widehat{T}$ is a selfadjoint operator in a Hilbert space $\widehat{\mathcal{S}}$ and $K \in \mathcal{B}(\mathfrak{M}, \widehat{\mathcal{S}})$ is a contraction, see Corollary 2.4 and Proposition 2.1. For $\mathcal{M}_2(\lambda) = -(\mathcal{M}_1(\lambda) + \lambda I_{\mathfrak{M}})^{-1}$ one has
\[ \lim_{y \to \pm \infty} ||iy\mathcal{M}_2(iy) + I_{\mathfrak{M}}|| = 0, \]
i.e., $\mathcal{M}_2(\lambda) \in \mathcal{N}[\mathfrak{M}]$. Thus, see Corollary 2.4
\[ \text{ran} \widehat{\Gamma} = \widehat{\Gamma}(\mathcal{R}[\mathfrak{M}]) = \left\{ M(\lambda) \in \mathcal{R}_0[\mathfrak{M}] : s - \lim_{y \to +\infty} (-iyM(iy)) \in [0, I_{\mathfrak{M}}] \right\}, \quad \text{ran} \widehat{\Gamma}^k \subset \mathcal{N}[\mathfrak{M}], \quad k \geq 2. \]

**Theorem 3.1.** Let $\mathfrak{M}$ be a Hilbert space. Then
Then
\[
\hat{M}_0(\lambda) = \frac{-\lambda + \sqrt{\lambda^2 - 4}}{2} I_{2\mathbb{R}}, \quad \text{Im} \lambda \neq 0, \quad \hat{M}_0(\infty) = 0
\]

is a unique fixed point of the mapping \( \hat{\Gamma} \) (1.7);

(2) If \( \hat{\Gamma}(M) = M_0 \), then \( M(\lambda) = M_0(\lambda) \) for all \( \lambda \in \mathbb{C} \setminus \mathbb{R} \);

(3) For every sequence of iterations of the form
\[
M_1(\lambda) = -(M(\lambda) + \lambda I_{2\mathbb{R}})^{-1}, \quad M_{n+1}(\lambda) = -(M_n(\lambda) + \lambda I_{2\mathbb{R}})^{-1}, \quad n = 1, 2, \ldots,
\]
where \( M(\lambda) \) is an arbitrary Nevanlinna function, the relation
\[
\lim_{n \to \infty} ||M_n(\lambda) - M_0(\lambda)|| = 0
\]
holds uniformly on each compact subsets of the open upper/lower half-plane of the complex plane \( \mathbb{C} \);

(4) The function \( M_0(\lambda) \) is a unique fixed point for each degree of \( \hat{\Gamma} \).

Proof. (1) Since
\[
M(\lambda) = -(M(\lambda) + \lambda I_{2\mathbb{R}})^{-1} \iff M^2(\lambda) + \lambda M(\lambda) + I_{2\mathbb{R}} = 0,
\]
and \( M \) is a Nevanlinna family, we get that \( M_0 \) given by (3.2) is a unique solution.

(2) Suppose \( \hat{\Gamma}(M) = M_0 \), i.e.,
\[
-(M(\lambda) + \lambda I_{2\mathbb{R}})^{-1} = \frac{-\lambda + \sqrt{\lambda^2 - 4}}{2} I_{2\mathbb{R}}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.
\]

Then
\[
M(\lambda) = \left( -\frac{2}{-\lambda + \sqrt{\lambda^2 - 4}} - \lambda \right) I_{2\mathbb{R}} = \frac{-\lambda + \sqrt{\lambda^2 - 4}}{2} I_{2\mathbb{R}} = M_0(\lambda).
\]

(3) Let \( F \) and \( G \) be two \( B(\mathbb{R}) \)-valued Nevanlinna functions. Set
\[
\hat{F}(\lambda) = -(F(\lambda) + \lambda I_{2\mathbb{R}})^{-1}, \quad \hat{G}(\lambda) = -(G(\lambda) + \lambda I_{2\mathbb{R}})^{-1}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.
\]

Then \( \hat{F} \) and \( \hat{G} \) are \( B(\mathbb{R}) \)-valued and
\[
\hat{F}(\lambda) - \hat{G}(\lambda) = (F(\lambda) + \lambda I_{2\mathbb{R}})^{-1} (F(\lambda) - G(\lambda)) (G(\lambda) + \lambda I_{2\mathbb{R}})^{-1}.
\]

From (3.1) we get
\[
||(\hat{F}(\lambda) - \hat{G}(\lambda))|| \leq \frac{1}{|\text{Im} \lambda|^2} ||F(\lambda) - G(\lambda)||.
\]

Hence for the sequence of iterations \( \{M_n(\lambda)\} \) one has
\[
||(M_n(\lambda) - M_m(\lambda))|| \leq \frac{1}{(|\text{Im} \lambda|^2)^{m-1}} ||M_{n-m+1}(\lambda) - M_1(\lambda)||, \quad n > m.
\]

It follows that if \( |\text{Im} \lambda| > 1 \), then
\[
||(M_n(\lambda) - M_m(\lambda))|| \leq \frac{(|\text{Im} \lambda|^2)^{-m+1}}{1 - (|\text{Im} \lambda|)^{-2}} ||M_2(\lambda) - M_1(\lambda)||, \quad n > m.
\]
Therefore, the sequence of linear operators \( \{M_n(\lambda)\}_{n=1}^{\infty} \) convergence in the operator norm topology, and the limit satisfies the equality \( \hat{M}(\lambda) = -(\hat{M}(\lambda) + \lambda I)^{-1} \), i.e., is the fixed point of the mapping \( \hat{\Gamma} \). In addition due to the inequality

\[
\| (M_n(\lambda) - M_m(\lambda)) \| \leq \frac{1}{R^{m-1}} \| M_{n-m+1}(\lambda) - M_1(\lambda) \|, \quad n > m, \quad |\text{Im}\lambda| \geq R, \quad R > 1
\]

we get that the convergence is uniform on \( \lambda \) on the domain \( \{\lambda : |\text{Im}\lambda| \geq R\} \), \( R > 1 \).

Note that from

\[
\text{lim}_{n \to \infty} \| M_n(\lambda) - M_0(\lambda) \| = 0
\]

it follows that the sequence of operator-valued functions \( \{M_n(\lambda)\}_{n=1}^{\infty} \) is uniformly bounded on \( \lambda \) on each domain \( |\text{Im}\lambda| > r, \ r > 0 \). Thus, the sequence \( \{M_n\}_{n=1}^{\infty} \) is locally uniformly bounded in the upper and lower open half-planes and, in addition, \( \{M_n\}_{n=1}^{\infty} \) uniformly converges in the operator-norm topology on the domains \( \{\lambda : |\text{Im}\lambda| \geq R\} \), \( R > 1 \). By the Vitali-Porter theorem \cite{19} the relation

\[
\text{lim}_{n \to \infty} \| M_n(\lambda) - M_0(\lambda) \| = 0
\]

holds uniformly on \( \lambda \) on each compact subset of the open upper/lower half-plane of the complex plane \( \mathbb{C} \).

(4) The function \( M_0 \) is a fixed point for each degree of \( \hat{\Gamma} \). Suppose that the mapping \( \hat{\Gamma}^{l_0} \), \( l_0 \geq 2 \) has one more fixed point \( L_0(\lambda) \). Then arguing as above, we get

\[
|\| M_0(\lambda) - L_0(\lambda) \| \| \leq |\text{Im}\lambda|^{-l_0} |\| M_0(\lambda) - L_0(\lambda) \| | \forall \lambda \in \mathbb{C} \setminus \mathbb{R}.
\]

It follows that \( L_0(\lambda) \equiv M_0(\lambda) \). \( \square \)

The scalar case (\( \mathfrak{M} = \mathbb{C} \)) of the next Proposition can be found in \cite{6} pages 544–545, \cite{18}.

**Proposition 3.2.** Let \( \mathfrak{M} \) be a Hilbert space.

1. Consider the weighted Hilbert space

\[
\mathcal{L}_0 := L_2 \left( [-2, 2], \frac{1}{2\pi \sqrt{4-t^2}} \right) \otimes \mathfrak{M}
\]

and the operator

\[
(T_0 f)(t) = t f(t), \quad f(t) \in \mathcal{L}.
\]

Identify \( \mathfrak{M} \) with a subspace of \( \mathcal{L}_0 \) of constant vector-functions \( \{f(t) \equiv f, \ f \in \mathfrak{M}\} \). Then

\[
M_0(\lambda) = P_{\mathfrak{M}}(T_0 - \lambda I)^{-1} \downarrow \mathfrak{M}, \quad \lambda \in \mathbb{C} \setminus [-2, 2],
\]

where \( M_0(\lambda) \) is given by (3.2).

2. Let \( H_0 = \bigoplus_{j=0}^{\infty} \mathfrak{M} = L^2(N_0) \otimes \mathfrak{M} \) and let \( \hat{J}_0 \) be the operator in \( H_0 \) given by the block-operator Jacobi matrix of the form (1.8).

Set \( \mathfrak{M}_0 := \mathfrak{M} \bigoplus \{0\} \bigoplus \{0\} \bigoplus \cdots \). Then

\[
M_0(\lambda) = P_{\mathfrak{M}_0}(\hat{J}_0 - \lambda I)^{-1} \downarrow \mathfrak{M}_0, \quad \lambda \in \mathbb{C} \setminus [-2, 2].
\]

In the next statement we show that one can construct a sequence \( \{\hat{H}_n, \hat{A}_n\} \) of realizations for the iterates \( \{M_{n+1} = \hat{\Gamma}(M_n)\}_{n=1}^{\infty} \) that inductively converges to \( \{H_0, \hat{J}_0\} \).
Theorem 3.3. Let $\mathcal{M}(\lambda)$ be an arbitrary Nevanlinna family in $\mathcal{M}$. Define the iterations of the mapping $\Gamma \ (1.7)$:

$$
\mathcal{M}_1(\lambda) = -(\mathcal{M}(\lambda) + \lambda I_{2\mathbb{R}})^{-1}, \quad \mathcal{M}_{n+1}(\lambda) = -(\mathcal{M}_n(\lambda) + \lambda I_{2\mathbb{R}})^{-1}, \quad n = 1, 2, \ldots,
$$

$\lambda \in \mathbb{C} \setminus \mathbb{R}$.

Let $\mathcal{M}_1(\lambda) = K^* (\hat{T} - \lambda I)^{-1} K$, $\text{Im} \, \lambda \neq 0$ be a realization of $\mathcal{M}_1(\lambda)$, where $\hat{T}$ is a selfadjoint operator in the Hilbert space $\mathcal{H}$ and $K \in \mathcal{B}(\mathcal{M}, \mathcal{H})$ is a contraction. Further, set

$$
\mathcal{H}_1 = \mathcal{M} \oplus \mathcal{H}, \quad \mathcal{H}_2 = \mathcal{M} \oplus \mathcal{H}_1 = \mathcal{M} \oplus \mathcal{M} \oplus \mathcal{H},
$$

$$
\mathcal{H}_{n+1} = \mathcal{M} \oplus \mathcal{H}_n = \underbrace{\mathcal{M} \oplus \mathcal{M} \oplus \cdots \oplus \mathcal{M}}_{n+1} \oplus \mathcal{H}, \ldots
$$

and define the following linear operators for each $n \in \mathbb{N}$:

$$
\mathcal{M} \ni x \mapsto \mathbb{I}_{2\mathbb{R}}^{(n)} x = [x, 0, 0, \ldots, 0]^T \in \mathcal{H}_n,
$$

$$
\mathcal{H}_n \ni \begin{bmatrix} x \\ h \end{bmatrix} \mapsto P_{2\mathbb{R}}^{(0,n)} \begin{bmatrix} x \\ h \end{bmatrix} = x \in \mathcal{M} \dagger \mathcal{H}_n \forall x \in \mathcal{M}, \forall h \in \mathcal{H}_n.
$$

Define selfadjoint operators in the Hilbert spaces $\mathcal{H}_n$ for $n \in \mathbb{N}$:

$$
\hat{A}_1 = \begin{bmatrix} 0 & K^* \\ K & \hat{T} \end{bmatrix} : \mathcal{M} \oplus \mathcal{H} \rightarrow \mathcal{M} \oplus \mathcal{H}, \quad \text{dom} \, \hat{A}_1 = \mathcal{M} \oplus \text{dom} \hat{T},
$$

$$
\hat{A}_2 = \begin{bmatrix} 0 & P_{2\mathbb{R}}^{(0,1)} \\ \mathbb{I}_{2\mathbb{R}}^{(1)} & \hat{A}_1 \end{bmatrix} : \mathcal{M} \oplus \mathcal{H}_1 \rightarrow \mathcal{M} \oplus \mathcal{H}_1, \quad \text{dom} \, \hat{A}_2 = \mathcal{M} \oplus \text{dom} \hat{A}_1
$$

$$
\hat{A}_{n+1} = \begin{bmatrix} 0 & P_{2\mathbb{R}}^{(0,n)} \\ \mathbb{I}_{2\mathbb{R}}^{(n)} & \hat{A}_n \end{bmatrix} : \mathcal{M} \oplus \mathcal{H}_n \rightarrow \mathcal{M} \oplus \mathcal{H}_n, \quad \text{dom} \, \hat{A}_{n+1} = \mathcal{M} \oplus \text{dom} \hat{A}_n
$$

Then $\hat{A}_n$ is a realization of $\mathcal{M}_{n+1}$ for each $n$, i.e.,

$$
\mathcal{M}_{n+1}(\lambda) = P_{2\mathbb{R}}(\hat{A}_n - \lambda I)^{-1} |_{\mathcal{M}}, \quad n = 1, 2, \ldots, \lambda \in \mathbb{C} \setminus \mathbb{R}.
$$

If $\hat{T}$ is $\mathbb{R}$-simple, i.e., $\text{span} \{ (\hat{T} - \lambda)^{-1} \text{ran} \, K : \lambda \in \mathbb{C} \setminus \mathbb{R} \} = \mathcal{K}$, then $\hat{A}_n$ is $\mathcal{M}$-minimal for each $n \in \mathbb{N}$. Moreover, the Hilbert space $\mathcal{H}_0$ and the block-operator Jacobi matrix (1.8) are the inductive limits $\mathcal{H}_0 = \lim_{\rightarrow} \mathcal{H}_n$ and $\hat{J}_0 = \lim_{\rightarrow} \hat{A}_n$, of the chains $\{ \mathcal{H}_n \}$ and $\{ \hat{A}_n \}$, respectively.

Proof. Relations in (3.5) follow by induction from (2.3).
Note that the operator $\hat{A}_n$ can be represented by the block-operator matrix

$$
\hat{A}_n = \begin{bmatrix}
0 & I_{2\mathbb{R}} & 0 & 0 & 0 & \cdots & 0 \\
I_{2\mathbb{R}} & 0 & I_{2\mathbb{R}} & 0 & 0 & \cdots & 0 \\
0 & I_{2\mathbb{R}} & 0 & I_{2\mathbb{R}} & 0 & \cdots & 0 \\
0 & 0 & I_{2\mathbb{R}} & 0 & I_{2\mathbb{R}} & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & I_{2\mathbb{R}} & 0 & I_{2\mathbb{R}} \\
0 & 0 & \cdots & 0 & 0 & I_{2\mathbb{R}} & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & K^* \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & K^* \\
\end{bmatrix},
$$

(3.6)

Besides, if $\hat{T}$ is bounded, then all operators $\{\hat{A}_n\}_{n \geq 1}$ are bounded and each $\mathcal{M}_n(\lambda)$ belongs to the class $\mathbb{N}_{2\mathbb{R}}^0$ for $n \geq 2$.

Define the linear operators $\gamma_k^l : \mathfrak{F}_k \to \mathfrak{F}_l$, $l \geq k$, $\gamma_k^l : \mathfrak{F}_k \to \mathcal{H}_0$, $k \in \mathbb{N}$ as follows

$$
\gamma_k^l[f_1, f_2, \ldots, f_k, \varphi] = [f_1, f_2, \ldots, f_k, \underbrace{0, 0, \ldots, 0}_l, \varphi],
$$

$$
\gamma_k[f_1, f_2, \ldots, f_k, \varphi] = [f_1, f_2, \ldots, f_k, 0, 0, \ldots].
$$

Then

1. $\gamma_k^k$ is the identity on $\mathfrak{F}_k$ for each $k \in \mathbb{N}$,
2. $\gamma_k^m = \gamma_l^m \circ \gamma_k^l$ if $k \leq l \leq m$,
3. $\gamma_k^l = \gamma_l \circ \gamma_k^l$, $l \geq k$, $k \in \mathbb{N}$,
4. $\mathcal{H}_0 = \text{span} \{\gamma_k^k, k \geq 1\}$.

Note that the operators $\{\gamma_k^k\}$ are isometries and the operators $\{\gamma_k^l\}$ are partial isometries and ker $\gamma_k = \mathfrak{F}_k$ for all $k$. The family $\{\mathfrak{F}_k, \gamma_k^l, \gamma_k^k\}$ forms the inductive isometric chain [17] and the Hilbert space $\mathcal{H}_0$ is the inductive limit of the Hilbert spaces $\{\mathfrak{F}_n\}$ (3.3): $\mathcal{H}_0 = \lim_\rightarrow \mathfrak{F}_n$.

Define following [17] on $\mathcal{D}_\infty := \bigcup_{n=1}^{\infty} \gamma_n \text{dom} \hat{A}_n$ a linear operator in $\mathcal{H}_0$:

$$
\hat{A}_\infty h := \lim_{m \to \infty} \gamma_m \hat{A}_m \gamma^m h_k, \quad h = \gamma_k h_k, \quad h_k \in \mathfrak{F}_k \otimes \mathfrak{F}_k,
$$

where $\{\hat{A}_n\}$ are defined in (3.4). Due to (3.7) and (3.6) the operator $\hat{A}_\infty$ exists, densely defined and its closure is bounded selfadjoint operator in $\mathcal{H}_0$ given by the block-operator matrix $\mathfrak{J}_0$ of the form (1.8).

Note that the operator $\mathfrak{J}_0$ is called the free discrete Schrödinger operator [18]. Observe also that the function

$$
M_1(\lambda) = \frac{1}{2} \mathcal{M}_0 \left( \frac{\lambda}{2} \right) = 2(\lambda + \sqrt{\lambda^2 - 1})I_{2\mathbb{R}}, \quad \lambda \in \mathbb{C} \setminus [-1, 1],
$$

where $\mathcal{M}_0(\lambda)$ is given by (3.2), belongs to the class $\mathbb{N}_{2\mathbb{R}}^0[-1, 1]$. Besides, for all $\lambda \in \mathbb{C} \setminus [-1, 1]$ the equality $M_1(\lambda) = P_{2\mathbb{R}}(\mathcal{T}_1 - \lambda I)^{-1} | \mathfrak{M}$ holds, where $\mathcal{T}_1$ is the multiplication operator
\((\mathcal{T}_1 f)(t) = tf(t)\) in the weighted Hilbert space
\[
L_2 \left( [-1, 1], \frac{2}{\pi} \sqrt{1-t^2} \right) \otimes \mathfrak{M}.
\]
If \(\mathfrak{M} = \mathbb{C}\), then the matrix of the corresponding operator \(\mathcal{T}_1\) in the orthonormal basis of the Chebyshev polynomials of the second kind
\[
U_n(t) = \frac{\sin[(n + 1) \arccos t]}{\sqrt{1-t^2}}, \quad n = 0, 1, \ldots
\]
is of the form \(\frac{1}{2} J_0 [6]\).

4. Canonical systems and the mapping \(\hat{\Gamma}\)

Let \(m \in \mathbb{N}_0\). Then, see [6, Chapter VII, §1, Theorem 1.11], [11], [18], the function \(m\) is the compressed resolvent \((m(\lambda) = ((J - \lambda I)^{-1} \delta, \delta_0))\) of a unique finite or semi-infinite Jacobi matrix \(J = J(\{a_k\}, \{b_k\})\) with real diagonal entries \(\{a_k\}\) and positive off-diagonal entries \(\{b_k\}\) and in the semi-infinite case one has \(\{a_k\}, \{b_k\} \in \ell^\infty(\mathbb{N}_0)\). Observe that the entries of \(J\) can be found using the continued fraction (J-fraction) expansion of \(m(\lambda)\) [11], [21]
\[
m(\lambda) = \frac{-1}{\lambda - a_0 + \frac{-b_0^2}{\lambda - a_1 + \frac{-b_1^2}{\lambda - a_2 + \cdots + \frac{-b_{n-1}^2}{\lambda - a_n + \cdots}}}}.
\]
On the other hand the algorithm of I.S. Kac [14] enables to construct for given \(J(\{a_k\}, \{b_k\})\) the Hamiltonian \(H(t)\) such that the \(m\)-function of \(J(\{a_k\}, \{b_k\})\) is the \(m\)-function of the corresponding canonical system of the form (1.9).

Below we give the algorithm of Kac. Let \(J\) be a semi-infinite Jacobi matrix
\[
(4.1) \quad J = J(\{a_k\}, \{b_k\}) = \begin{bmatrix}
a_0 & b_0 & 0 & 0 & 0 & \cdots & \\
b_0 & a_1 & b_1 & 0 & 0 & \cdots & \\
0 & b_1 & a_2 & b_2 & 0 & \cdots & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \end{bmatrix}.
\]
The condition \(\{a_k\}, \{b_k\} \in \ell^\infty(\mathbb{N}_0)\) is necessary and sufficient for the boundedness of the corresponding selfadjoint operator in the Hilbert space \(\ell^2(\mathbb{N}_0)\).

Put
\[
l_{-1} = 1, \quad l_0 = 1, \quad \theta_{-1} = 0, \quad \theta_0 = \frac{\pi}{2}.
\]
Then calculate
\[
(4.3) \quad \theta_1 = \arctan a_0 + \pi, \quad l_1 = \frac{1}{l_0 b_0^2 \sin^2(\theta_1 - \theta_0)}.
\]
Find \(\theta_2\) from the system
\[
(4.4) \quad \begin{cases}
\cot(\theta_2 - \theta_1) = -a_1 l_1 - \cot(\theta_1 - \theta_0) \\
\theta_2 \in (\theta_1, \theta_1 + \pi)
\end{cases}
\]
Find successively $l_j$ and $\theta_{j+1}$, $j = 2, 3, \ldots$

\begin{equation}
  l_j = \frac{1}{l_{j-1}b_{j-1}^2 \sin^2(\theta_j - \theta_{j-1})},
\end{equation}

\begin{align*}
  \cot(\theta_{j+1} - \theta_j) &= -a_j l_j - \cot(\theta_j - \theta_{j-1}) \\
  \theta_{j+1} &\in (\theta_j, \theta_j + \pi)
\end{align*}

Define intervals $[t_j, t_{j+1})$ as follows

\begin{equation}
  t_{-1} = -1, \quad t_0 = t_{-1} + l_{-1} = 0, \quad t_1 = t_0 + l_0 = 1, \quad t_{j+1} = t_j + l_j = 1 + \sum_{k=1}^j l_k, \quad j \in \mathbb{N}.
\end{equation}

Then necessarily, we get that $\lim_{j \to \infty} t_j = +\infty$. Finally define the right continuous increasing step-function

\begin{equation}
  \theta(t) := \begin{cases} 
  \theta_0 = \frac{\pi}{2}, & t \in (t_0, t_1) = (0, 1) \\
  \theta_j, & t \in [t_j, t_{j+1}), \quad j \in \mathbb{N}
  \end{cases}
\end{equation}

and the Hamiltonian $H(t)$ on $\mathbb{R}_+$

\begin{equation}
  H(t) := \begin{bmatrix} 
  \cos \theta(t) \\
  \sin \theta(t)
  \end{bmatrix} \begin{bmatrix} 
  \cos \theta(t) & \sin \theta(t) \\
  \cos \theta(t) \sin \theta(t) & \sin^2 \theta(t)
  \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 
  1 & 0 \\
  0 & 1
  \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 
  \cos 2\theta(t) & \sin 2\theta(t) \\
  \sin 2\theta(t) & -\cos 2\theta(t)
  \end{bmatrix}.
\end{equation}

Then the Nevanlinna function $m(\lambda) = ((J - \lambda I)^{-1} \delta_0, \delta_0)$ coincides with $m$-function of the corresponding canonical system of the form (1.9). Observe that the algorithm shows that

\begin{equation}
  H(t) = \begin{bmatrix} 0 & 0 \\
  0 & 1 \end{bmatrix}, \quad t \in [0, 1).
\end{equation}

Using (4.2)–(4.8) for the Jacobi matrix $\hat{J}_0$

\begin{equation}
  \hat{J}_0 = \begin{bmatrix} 
  0 & 1 & 0 & 0 & 0 & \cdots & \\
  1 & 0 & 1 & 0 & 0 & \cdots & \\
  0 & 1 & 0 & 1 & 0 & \cdots & \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots
  \end{bmatrix},
\end{equation}

we get

\begin{align*}
  l_j^0 &= 1, \quad \theta_j^0 = (j + 1) \frac{\pi}{2}, \quad \forall j \in \mathbb{N}_0, \\
  \theta^0(t) &= (j + 1) \frac{\pi}{2}, \quad t \in [j, j + 1) \quad \forall j \in \mathbb{N}_0,
\end{align*}
Proof. Set

\[ H_0(t) = \begin{bmatrix} \cos^2(j + 1) \frac{\pi}{2} & 0 \\ 0 & \sin^2(j + 1) \frac{\pi}{2} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 - (-1)^j & 0 \\ 0 & 1 + (-1)^j \end{bmatrix}, \quad t \in [j, j + 1) \forall j \in \mathbb{N}_0. \]

Proposition 4.1. Let the scalar non-rational Nevanlinna function \( m \) belong to the class \( \mathcal{N}_C^0 \). Define the functions

\[ m_1(\lambda) = -\frac{1}{m(\lambda) + \lambda}, \ldots, m_{n+1}(\lambda) = -\frac{1}{m(\lambda) + \lambda}, \ldots, \lambda \in \mathbb{C} \setminus \mathbb{R}. \]

Let \( J \) be the Jacobi matrix with the \( m \)-function \( m \), i.e., \( m(\lambda) = ((J - \lambda I)^{-1} \delta_0, \delta_0), \forall \lambda \in \mathbb{C} \setminus \mathbb{R} \). Assume that \( H(t) \) is the Hamiltonian such that the \( m \)-function of the corresponding canonical system coincides with \( m \). Then the Hamiltonian \( H_n(t) \) of the canonical system whose \( m \)-function coincides with \( m_n \), takes the form

\[ H_n(t) = \begin{cases} H_0(t), & t \in [0, n + 1), \\ (-1)^n H(t - n) + \frac{1}{2} \begin{bmatrix} 1 - (-1)^n & 0 \\ 0 & 1 - (-1)^n \end{bmatrix}, & t \in [n + 1, \infty) \end{cases} \]

where \( \{t_j, \theta_j\}_{j \geq 1} \) are parameters of the Hamiltonian \( H(t) \).

Proof. Set

\[ J_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & \ldots \\ 1 & 0 & J & & \\ \vdots & \vdots & & & \end{bmatrix}, \ldots, J_n = \begin{bmatrix} 0 & 1 & 0 & 0 & \ldots \\ 1 & 0 & J_{n-1} & & \\ \vdots & \vdots & & & \end{bmatrix}, \ldots \]

Then (2.3) and induction yield the equalities

\[ ((J_1 - \lambda I)^{-1} \delta_0, \delta_0) = -(m(\lambda) + \lambda)^{-1} = m_1(\lambda), \ldots, \]

\[ ((J_n - \lambda I)^{-1} \delta_0, \delta_0) = -(m_{n-1}(\lambda) + \lambda)^{-1} = m_n(\lambda), \ldots, \lambda \in \mathbb{C} \setminus \mathbb{R}. \]

Let \( J = J \{a_k\}_{k=0}^\infty, \{b_k\}_{k=0}^\infty \) be of the form (4.1). Then from (4.12) it follows that for the entries of \( J_n = J_n \{a_k^{(n)}\}_{k=0}^\infty, \{b_k^{(n)}\}_{k=0}^\infty \), \( n \in \mathbb{N} \), we have the equalities

\[ a_0^{(n)} = a_1^{(n)} = \ldots = a_{n-1}^{(n)} = 0, \quad a_k^{(n)} = a_{k-n}, \quad k \geq n \]

\[ b_0^{(n)} = b_1^{(n)} = \ldots = b_{n-1}^{(n)} = 1, \quad b_k^{(n)} = b_{k-n}, \quad k \geq n. \]

In order to find an explicit form of the Hamiltonian corresponding to the Nevanlinna function \( m_n \) we apply the algorithm of Kac described by (4.2), (4.3), (4.4), (4.5), (4.6), (4.7), (4.8).
Then we obtain
\[ l_{-1}^{(n)} = l_0^{(n)} = l_1^{(n)} = \ldots = l_n^{(n)} = 1, \]
\[ \theta_{-1}^{(n)} = 0, \quad \theta_0^{(n)} = \frac{\pi}{2}, \quad \theta_1^{(n)} = \pi, \ldots, \theta_n^{(n)} = (n + 1)\frac{\pi}{2}. \]
\[ j_{n+j} = l_j, \quad \theta_{n+j} = \theta_j + (n + 2)\frac{\pi}{2}, \quad j \in \mathbb{N}. \]

Hence (4.8) and (4.10) yield (4.11).

By Theorem 3.1 the sequence \( \{m_n\} \) of Nevanlinna functions converges uniformly on each compact subset of \( \mathbb{C}_+ / \mathbb{C}_- \) to the Nevanlinna function
\[ m_0(\lambda) = -\lambda + \frac{\sqrt{\lambda^2 - 4}}{2}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \]

This function is the \( m \)-function of the Jacobi matrix \( \hat{J}_0 \) and the \( m \)-function of the canonical system with the Hamiltonian \( H_0 \). From (4.12) we see that for the sequence of selfadjoint Jacobi operators \( \{J_n\} \) in \( \ell^2(\mathbb{N}_0) \) the relations
\[ P_nJ_{n+1}P_n = P_nJ_0P_n \quad \forall n \in \mathbb{N}_0 \]
hold, where \( P_n \) is the orthogonal projection in \( \ell^2(\mathbb{N}_0) \) on the subspace
\[ E_n = \text{span} \{\delta_0, \delta_1, \ldots, \delta_{n-1}\}. \]

It follows that
\[ s - \lim_{n \to \infty} P_nJ_{n+1}P_n = \hat{J}_0. \]

For the sequence (4.11) of \( \{H_n\} \) one has
\[ (4.14) \quad H_n \mid [0, n+1) = H_0 \mid [0, n+1) \quad \forall n. \]

From (4.14) it follows that if \( \vec{f}(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} \) is a continuous function on \( \mathbb{R}_+ \) with a compact support, then there exists \( n_0 \in \mathbb{N} \) such that \( \int_0^\infty \vec{f}(t)^*H_n(t)\vec{f}(t)dt = \int_0^\infty \vec{f}(t)^*H_0(t)\vec{f}(t)dt \) for all \( n \geq n_0 \).

It is proved in [13, Proposition 5.1] that for a sequence of canonical systems with Hamiltonians \( \{H_n\} \) and \( H \) the convergence \( m_{H_n}(\lambda) \to m_H(\lambda), \ n \to \infty \) of \( m \)-functions holds locally uniformly on \( \mathbb{C}_+ / \mathbb{C}_- \) if and only if \( \int_0^\infty \vec{f}(t)^*H_n(t)\vec{f}(t)dt \to \int_0^\infty \vec{f}(t)^*H(t)\vec{f}(t)dt \) for all continuous functions \( \vec{f}(t) \) with compact support on \( \mathbb{R}_+ \).

In conclusion we note that the equalities (4.9), (4.10), and (4.11) (for \( n = 1 \)) show that for the transformation \( \hat{\Gamma} \) one has the following scheme:
\[ \mathbb{N}_0^0 \ni m \ (\text{non-rational}) \longrightarrow \mathcal{H}(t) \longrightarrow \mathcal{H}_{\hat{\Gamma}}(t) = \begin{cases} \mathcal{H}_0(t), & t \in [0, 2) \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \mathcal{H}(t-1), & t \in [2, +\infty) \end{cases} \leftarrow \hat{\Gamma}(m). \]
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