Universal observable detecting all two-qubit entanglement and determinant based separability tests

Remigiusz Augusiak, Maciej Demianowicz, and Paweł Horodecki
Faculty of Applied Physics and Mathematics, Gdańsk University of Technology, 80–952 Gdańsk, Poland

We construct a single observable measurement of which mean value on four copies of an unknown two-qubit state is sufficient for unambiguous decision whether the state is separable or entangled. In other words, there exists a universal collective entanglement witness detecting all two-qubit entanglement. The test is directly linked to a function which characterizes to some extent the entanglement quantitatively. This function is an entanglement monotone under so-called local pure operations and classical communication (pLOCC) which preserve local dimensions. Moreover it provides tight upper and lower bounds for negativity and concurrence. Elementary quantum computing device estimating unknown two-qubit entanglement is designed.

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Introduction .- One of the main challenges of both theoretical and experimental Quantum Information Theory is a determination of entanglement properties of a given state. There is an extensive literature covering the problem of deciding entanglement of a state [1, 2, 3, 4, 5, 6]. As one knows from the seminal paper of Peres and Wooters [5] collective measurement on several copies of a system in a given quantum state may provide better results than measurements performed on each copy separately. This fact was reflected in the method of entanglement detection with collective measurements. The method initiated for pure states [6] then developed for mixed states with help of quantum networks [7, 8]. The concept of collective entanglement witnesses [9] has found its first experimental demonstration in coalescence-anti coalescence coincidence experiment [10]. In particular, somewhat surprisingly, it was shown how to estimate and/or even measure amount of entanglement (concurrence) without prior state reconstruction [11, 12, 13]. Recently the method got the new twist thanks to application of such collective measurements [14, 15, 16] that are directly related to quantum concurrence (see [17]) including photon polarization-momentum experimental demonstration for pure states in distant laboratories paradigm [18]. The idea of collective entanglement witnesses was also implemented in continuous variables setup [21].

We show that a single observable if measured on four copies of a unknown two-qubit state is sufficient for discrimination between entanglement and separability of it. Moreover it can serve for limited quantitative purposes. To this aim we explore the two-qubit separability test (equivalent to the PPT one [2, 25]) stating that a state is separable iff the determinant of its partially transposed density matrix is nonnegative [26, 27]. The result, known for a few years, was barely mentioned in the literature in that form (see e.g. [28, 29]) and up to our knowledge this is the first time an operative physical meaning is assigned to it. Namely we introduce a state function, straightforwardly connected to the test, which is a monotone under pLOCC with fixed dimensions (see [29, 31]) and only single collective observable is enough to measure it experimentally, and provides tight upper and lower bounds for the two-qubit negativity and concurrence.

Further we discuss how the result allows to build a small quantum device implementing a kind of elementary algorithm, namely, detecting entanglement in an unknown two-qubit state. Our method has a significant advantage over prior methods [12, 13] as we require only one collective measurement. In comparison to the result of Ref. [21], where a single observable provides a concurrence lower bound which sometimes is not conclusive, we achieved sharp test which is to some extent quantitative.

We also discuss higher dimensional and multiparty generalizations. In particular, we find that reduction criteria [32, 33] on composite systems with the map applied to the second subsystem is equivalent to a single determinant condition and as such can be checked via measurement of a single observable.

The criterion.- Here we discuss the necessary and sufficient condition for two-qubit separability in terms of a determinant of a partially transposed density matrix. The observation follows from the facts from papers of Sanpera et al. [26] and Verstraete et al. [27]. Here we prove more general statement about the reduction criterion, exploiting its equivalence to PPT test on two qubits. Let us consider the reduction map defined as $\Lambda_r(A) = \text{Tr}(A)\mathbb{I}_d - A$ on any $d \times d$ matrix $A$ with $\mathbb{I}_d$ standing for an identity acting on $\mathbb{C}^d$. The following proposition holds.

\textbf{Proposition 1.-} For any $2 \otimes d$ state $\varrho$ the reduction criterion with respect to the system $B$ is satisfied iff

$$\det([I \otimes \Lambda_r](\varrho)) \geq 0.$$  \hspace{1cm} (1)

In particular any two-qubit state is separable iff

$$\det\varrho^F \geq 0.$$  \hspace{1cm} (2)

\textbf{Proof.} The necessity of the condition is obvious. Let us prove sufficiency. To this aim we may assume that
our 2 $\otimes$ d state $\varrho$ has nonsingular reduced density matrix $\varrho_A = Tr_B \varrho$, as otherwise it would be a product state. Applying a local filter $\mathcal{F}_A = (\varrho_A^{1/2})^{1/2}$ and utilizing previous observation, one obtains $det\{I \otimes \Lambda_r\}(\varrho) = [det(\varrho_A)/2]^2 det\{I \otimes \Lambda_r\}((\varrho))$, where the state $\tilde{\varrho}$ is a result of local filtering. Now there is an immediate observation that for any positive $\Lambda$ positivity of $\{I \otimes \Lambda\}(\varrho)$ is equivalent to the positivity of the new state being the result of local filtering on system $A$ with arbitrary nonsingular filter $\mathcal{F}_A$. Since we deal with the nonsingular $\mathcal{F}_A$, the original state $\varrho$ violates the reduction criterion iff the state $\tilde{\varrho}$ does. Suppose this is the case. Since the first subsystem of the latter is in a maximally mixed state, i.e., $\tilde{\varrho}_A = (1/2)I_2$ one easily infers (cf. [12]) that in order to violate the criterion $\tilde{\varrho}$ must have one eigenvalue that is greater than one-half. Then the operator $\{I \otimes \Lambda_r\}((\varrho)) = (1/2)I_2 - \varrho$ clearly has the spectrum with all nonzero values in which only one is negative. This finally gives $det\{I \otimes \Lambda_r\}(\varrho) < 0$ which (as we already mentioned) is equivalent to $det\{I \otimes \Lambda_r\}(\varrho) < 0$. Thus violation of reduction criterion by 2 $\otimes$ d state on the second subsystem is equivalent to violation of (1).

To prove the second part, we only need to observe that $det \varrho^F = det\{I_2 \otimes \sigma_F \varrho^F I_2 \otimes \sigma_F\} = det\{I \otimes \Lambda_r\}(\varrho)$ and recall that reduction criterion is equivalent to PPT test on two-qubit states. This concludes the proof.

Quantifying entanglement.- A question important from an experimental point of view is whether a function of determinant of a partially transposed density matrix can serve for quantitative purposes. We obtain partial positive answer.

First we introduce the function defined on $d \otimes d$ states

$$\pi_d(\varrho) = \begin{cases} 0, & \text{det } \varrho^F \geq 0; \\ d^{-2} \sqrt{|det \varrho^F|}, & \text{det } \varrho^F < 0. \end{cases}$$

(3)

Let us observe that $\pi_d(\ket{\psi}) = d \sqrt{\text{det } A^{\psi} |2/d for any pure state } \ket{\psi} = \sum_{i,j} A^{\psi}_{ij} |i\rangle\langle j|$. This leads to the fact that $\pi_d(\ket{\psi}) = G_d(\ket{\psi})$, where $G_d$ is called G-concurrence and is defined as $\sqrt{\text{det } G}$, the minimum of Schmidt numbers (see [30,31]). The latter is known to be a monotone under LOCC not changing dimensions of the state, and as such is considered as an entanglement measure [29,30,31]. Below we prove that $\pi_d$ satisfies monotonicity property under some restricted class of LOCC (invariance under local unitary operations is obvious due to properties of determinant), namely the ones for which local operations are pure in a sense they consist only of single Kraus operators. We call them pure LOCC (pLOCC). To this aim let us assume that $\varrho$ is entangled. Then we have the following.

**Proposition 2.** For any pLOCC not changing the dimension of a state, which transform initial state $\varrho$ to $\varrho^{(i)}$ with probability $p_i$, the following holds

$$\sum_i p_i \pi_d(\varrho^{(i)}) \leq \pi_d(\varrho).$$

(4)

**Proof.** Reasoning from [34] (the measure is symmetric under the change of particles) allows to restrict ourselves to single measurement on Bob’s side. These are described by the family of completely positive operators $\mathcal{M}_i$ with single Kraus decomposition (we consider only pLOCC) i.e. their action is as follows $\mathcal{M}_i(\varrho) = I_d \otimes M_i \varrho I_d \otimes M_i^\dagger$. We take square $M_i$ (the family of the states $\sum_i M_i^\dagger M_i \leq I_d$) to fulfill the requirement of not changing the dimension. Since $[\mathcal{M}_i(\varrho)]^{1/2} = \mathcal{M}_i(\varrho^{1/2})$, we have

$$\sum_i p_i \pi_d(\varrho^{(i)}) = d \sum_i p_i \sqrt{\text{det}(1/p_i) (\mathbb{1}_d \otimes M_i \varrho I_d \otimes M_i^\dagger)^\Gamma}$$

$$= \sum_i \sqrt{\text{det}(1/p_i) (\mathbb{1}_d \otimes M_i \varrho I_d \otimes M_i^\dagger) \pi_d(\varrho)} \leq d \sqrt{\text{det } M_i \varrho I_d \otimes M_i^\dagger \pi_d(\varrho)}$$

where the last inequality follows from Minkowski determinant theorem. Now taking into account normalization condition for $M_i$ we conclude that the last term is less or equal to $\pi_d$ finishing the proof.

Unfortunately $\pi_d$ is not a general LOCC monotone. This can be shown by performing twirling on entangled Bell diagonal states, which in general increases $\pi_d$.

Let us now focus on the two–qubit states. Below we will establish a connection of $\pi_2$ with concurrence $C$ and negativity $N$. As shown in Ref. [30], the concurrence of a density matrix transformed with a filter $A \otimes B$ changes by the factor $\sqrt{\text{det } AB/\text{Tr}(AA^\dagger \otimes BB^\dagger)}$. As it turns out $\pi_2$ of the state transformed in this way changes identically. Moreover the filters are known to be sufficient for transformation of any non–singular two–qubit state to a Bell–diagonal one [37]. It is then enough to check the relation between $C$ and $\pi_2$ for these states. Taking the entangled state $\varrho$ to be the mixture of Bell states with probabilities $\{p_i\}_{i=1}^4$ we obtain $\pi_2(\varrho) = \Pi_i \sqrt{1 - 2p_i}$, which with an assumption $p_1 \geq p_1$ gives $\pi_2(\varrho) \leq 2p_1 - 1$. This however means that $\pi_2$ is bounded from below by $C$ as for Bell diagonal states it is just equal to rhs of the above. Obviously $\pi_2$ provides also an upper bound for negativity as the latter is always less or equal to $C$ [27]:

$$N(\varrho) \leq C(\varrho) \leq \pi_2(\varrho).$$

(5)

One may also provide tight lower bound on $N(\varrho)$ and $C(\varrho)$ in terms of $\pi_2(\varrho)$. To this aim notice that $\pi_2(\varrho) = 2 \sqrt{(1/2)(N(\varrho) + 2)} \lambda_1 \lambda_2 \lambda_3$, where $\lambda_i$ are the positive eigenvalues of $\varrho^F$. Their product is maximal when they are equally distributed. This observation with the aid of the fact that $\sum_{i=1}^3 \lambda_i = N/2 = 1$ lead us to

$$\pi_2(\varrho) \leq \sqrt{N(\varrho) \left(\frac{N(\varrho) + 2}{3}\right)^3} \leq \sqrt{C(\varrho) \left(\frac{C(\varrho) + 2}{3}\right)^3}.$$
It can be observed that for any $d$ the function $\pi_d$ can be measured with a single collective entanglement witness as it will be shown below, but it detects all the entanglement only in a two-qubit case.

*Universal collective entanglement witness.*- Now we address a natural question arising in the context of the results from the previous section: Is a measurement of a determinant of $\varphi^F$ possible by means of a single observable? Following Ref. [18] we define the collective witness to be a Hermitian operator $W(n)$, of which mean value on $n$-copies of $\varphi$ is nonnegative, i.e, $\langle W(n) \rangle_{\varphi^n} := \text{Tr}(W(n)\varphi^n) \geq 0$ and negative on some entangled state. Reformulating this question in terms of the above we ask if there exists such an observable that the mean value on four copies of $\varphi$ gives det $\varphi^F$. It has been shown [30] that any $m$-th degree polynomial of the elements of $\varphi$ (in particular its determinant) may be found by determining an expectation value of two observables each on $m$ copies of a state corresponding to real and imaginary part of the value of the polynomial respectively. With guarantee (a priori knowledge) that a polynomial is real valued we need only single observable (cf. [18]). In fact we deal with such a polynomial here since the determinant (2) is obviously real. It is a polynomial of the fourth degree so the necessary number of copies is four. This positively resolves the problem of the existence of a single observable $W^{(4)}_{\text{univ}}$. To find the explicit form of it we first introduce polynomials $\Pi_k(\vec{x}) = \sum_{i=1}^m x_i^k$, which for $\vec{x} = \vec{\lambda}$, a vector consisting of eigenvalues of a given matrix, are just the $k$-th moments of this matrix. We know that, for each $k$, $\Pi_k(\vec{\lambda})$ is just a mean value of single observable $O(k) = (1/2)(V(k) + V(k)^\dagger)$ on $k$ copies of $\varphi$ with permutation operators $V(k)$ defined as $V(k)|\Phi_1\ldots|\Phi_{k-1}|\Phi_k\rangle = |\Phi_1\ldots|\Phi_{k-1}\rangle|\Phi_k\rangle$ ($k = 1, \ldots, m$), with $|\Phi_i\rangle \in \mathcal{H}$.

Now the crucial step is to connect the determinant of a matrix with its easily measurable moments. Newton-Girard formulas [40] provide us with $\det \varphi^F = (1/24)[1 - 6\Pi_4(\vec{\lambda}) + 8\Pi_3(\vec{\lambda}) + 3\Pi_2^2(\vec{\lambda}) - 6\Pi_2(\vec{\lambda})]$. Before we proceed note that $V(k)$ can be written in a separable form as $V(k) \otimes V(k)$ where $V^k$ are permutations acting on the same subsystems of $\varphi^k$.

The approach from Ref. [14] leads to

\begin{align}
W^{(4)}_{\text{univ}} = \frac{1}{24} & - \frac{1}{8} (\tilde{V}^{(4)} \otimes \tilde{V}^{(4)} T + \tilde{V}^{(4)} T \otimes \tilde{V}^{(4)}) \\
& + \frac{1}{6} (\tilde{V}^{(3)} \otimes \tilde{V}^{(3)} T + \tilde{V}^{(3)} T \otimes \tilde{V}^{(3)}) \\
& + \frac{1}{8} V^{(2)} \otimes V^{(2)} - \frac{1}{4} I_{16} \otimes V^{(2)}
\end{align}

which mean value on four copies of $\varphi$ gives det $\varphi^F$.

*The network.*- Here we consider the problem of the designation of a network measuring $W^{(4)}_{\text{univ}}$.

The issue of avoiding unimportant data (frequency probabilities corresponding to all eigenvalues of the observable) while measuring the observable was considered in Refs. [13, 41]. The question about dimension of ancillas involved in the measurement was answered in Ref. [42] where it was shown that via unitary interaction with a single qubit and final measurement of $\sigma_z$ on it, one can get mean value of an arbitrary observable with bounded spectrum. Finally, in Ref. [39] it was shown that interaction between systems in question and the ancilla can be conducted as a controlled unitary operation. Note that the above single qubit universality in a mean value estimation is compatible with the further proof that single qubits are in a sense universal quantum interfaces [43].

The most efficient in number of systems involved network involves nine qubits interacting via unitary operation which can be constructed in a way described in [14].

We present here (Fig.3) the alternative network that requires two more ancillary qubits. However with this additional systems we achieve simplicity of the structure of the controlled unitary operations, which are just swaps. This device shows how one can easily combine mean values of many observables. We do not go into details concerning optimality of both networks in number of gates.

*Generalizing the criterion.*- Here we discuss the above approach in the context of entanglement of an arbitrary...
bipartite state $\varrho$. Let $\Lambda$ be a positive, but not completely positive, map. Following Ref. [2], $\Lambda$ constitutes a necessary separability condition for states acting on Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$. One easily reformulates this condition for separability in terms of determinant:

Fact.- If for given positive map $\Lambda$ it holds $[I \otimes \Lambda](\varrho) \geq 0$ then $\det\{[I \otimes \Lambda](\varrho)\} \geq 0$.

In a general case the converse of the Fact fails which can be shown by embedding entangled two-qubit state in a $3 \otimes 3$ space. Note that, as shown in the Proposition, converse is true for reduction applied to second subsystem of a $2 \otimes d$ system which is useful in context of entanglement distillability (see [32]).

Construction of the proper observable along the lines of Ref. [14] results in an observable which mean value on copies of the state gives the desired determinant, i.e., $\langle \hat{W}^{(n)}_{\Lambda} \rangle_{\varrho^n} = \det\{[I \otimes \Lambda](\varrho)\}$.

The idea generalizes immediately to multipartite case where maps positive on product states are involved.

Conclusions.- We have constructed single observable test that detects entanglement of an unknown two-qubit state. In addition, the function corresponding to it provides bounds for negativity and concurrence. We have also designed the quantum network that can also be interpreted as a quantum computing that solves quantitatively a problem with a quantum data structure (cf. [11]).

Some research towards higher dimensional generalizations has been initialized however the results suffer from the lack of character. Nevertheless a very natural question arises: is there any other way to generalize the main result, i.e., find single collective observable that detects entanglement of any $d \otimes d$ quantum system without ambiguity. For some $\text{SO}(3)$-invariant states Proposition 1 was shown to hold in Ref. [43] thus giving such an observable in case of these states. In general one would first need some counterpart of the analytical criterion existence of which is a long–standing open problem in quantum information theory. The first question could be whether there exists positive map which applied to one subsystem of any bipartite density matrix produces full rank matrix with odd number of negative eigenvalues so that the criterion based on the determinant would remain true.

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* Electronic address: remik@mif.pg.gda.pl

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