MAGNETO-ELASTICITY ON THE DISK

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ABSTRACT. A model problem of magneto-elastic body is considered. Specifically, the case of a two dimensional circular disk is studied. The functional which represents the magneto-elastic energy is introduced. Then, the minimisation problem, referring to the simplified two-dimensional model under investigation, is analysed. The existence of a minimiser is proved and its dependence on the eigenvalues of the problem is investigated. A bifurcation result is obtained corresponding to special values of the parameters.

1. Introduction

The interest in magneto-elastic materials finds its motivation in the growing variety of new materials among which magneto-rheological elastomers or magneto-sensitive polymeric composites (Hossain \textit{et al.} 2015a,b) may be mentioned. A special issue of the journal \textit{Materials} devoted to \textit{Magnetoelastic Materials} is currently being published (Szewczyk 2020). Many applications of magneto-elastic materials, covering a wide area of interest from technological to biomedical devices (see, e.g., Ren \textit{et al.} 2019), can be listed. In particular, also two dimensional problems are subject of applicative investigations (Hadda and Tilioua 2012). The model we consider is a two dimensional simplified one, however, we believe that, it might open the way to further applications, possibly, via perturbative methods (Bernard \textit{et al.} 2019).

We study the functional energy of a magneto-elastic material, that is a material which is capable of deformation and magnetisation. The magnetisation is a phenomenon that does not appear at a macroscopic level, it is characterised by the magnetisation vector whose magnitude is independent of the position while its direction which can vary from one point to another. In this context, the magnetisation vector \( \mathbf{m} \) is a map from \( \Omega \) (a bounded open set of \( \mathbb{R}^2 \)) to \( S^2 \) (the unit sphere of \( \mathbb{R}^3 \)). In particular, here we assume \( \Omega \) is the unit disk of \( \mathbb{R}^2 \). The magnetisation distribution is well described by a free energy functional which we assume composed of three terms, namely the exchange energy \( E_{\text{ex}} \), the elastic energy \( E_{\text{el}} \) and the elastic-magnetic energy \( E_{\text{em}} \). In Section 2 we detail the three energetic terms and, after some simplifications, derive the proposed functional for describing some phenomena.
Assuming the hypothesis of radially symmetric maps, i.e.,
\[ \mathbf{m} = (\cos \theta \sin h(r), \sin \theta \sin h(r), \cos h(r)) , \]
we get to the analysis of a one-dimensional energy functional that can be expressed in terms of the only scalar function \( h \). The effect of the elastic deformation reveals through a positive parameter \( \mu \) which characterizes the connection between the magnetic and elastic processes. In Section 3 the minimisation of the energy functional, namely
\[ E(h) = \pi \int_0^1 \left[ h_r^2 + \left( \frac{\sin h}{r} \right)^2 - \frac{\mu}{2} (\sin 2h)^2 \right] r \, dr , \]
is the aim of our paper. In particular, we prove that there exists a critical value \( \mu^0 \) such that for \( \mu \leq \mu^0 \) the functional energy is not negative and there is only a global minimiser that is the trivial solution \( h \equiv 0 \); for \( \mu > \mu^0 \) other nontrivial minimisers appear, moreover the energy takes negative values. The local bifurcation analysis is carried out. More precisely we prove that at the point \( \mu^0 \), two branches of minimisers, with small norm, bifurcate from the trivial stable solution. This local analysis does not exclude the existence of other solutions of the minimisation problem even for \( \mu = 0 \) (see also the results by Brezis and Coron 1983, concerning the solutions of harmonic maps from the unit disk in \( \mathbb{R}^2 \) to the sphere \( S^2 \)).

For the modelling of magneto-elastic interactions see also Brown (1966), He (1999), Bertsch et al. (2001), Valente and Vergara Caffarelli (2007), Cerimele et al. (2008), and Chipot et al. (2008, 2009). Magneto-viscoelastic problems were studied by Carillo et al. (2011, 2012, 2017). Moreover we recall that the phenomenon of bifurcation of minimising harmonic maps has been studied by Bethuel et al. (1992) in a different physical context.

2. The model

We start with the general three-dimensional theory. We assume \( \Omega \subset \mathbb{R}^3 \) is the volume of the magneto-elastic material and \( \partial \Omega \) its boundary. Let \( x_i, i = 1,2,3 \), be the position of a point \( x \) of \( \Omega \) and denote by
\[ u_i = u_i(x), \quad i = 1,2,3 \]
the components of the displacement vector \( \mathbf{u} \) and by
\[ \varepsilon_{kl}(\mathbf{u}) = \frac{1}{2} (u_{k,l} + u_{l,k}), \quad k,l = 1,2,3 \]
the deformation tensor where, as a common praxis, \( u_{k,l} \) stands for \( \frac{\partial u_k}{\partial x_l} \). Moreover we denote by
\[ m_j = m_j(x), \quad j = 1,2,3 \]
the components of the magnetisation vector \( \mathbf{m} \) that we assume of unit modulus, i.e., \( |\mathbf{m}| = 1 \). In the sequel, where not specified, the Latin indices vary in the set \{1,2,3\} and the summation over repeated indices is assumed. We first define the exchange energy which arises from exchange neighbourhood interactions as
\[ E_{ex}(\mathbf{m}) = \frac{1}{2} \int_{\Omega} a_{ij} m_{k,i} m_{k,j} d\Omega \] (1)
where \(a_{ijkl} = a_1 \delta_{ijkl} + a_2 \delta_{ij} \delta_{kl}\) with \(a_1, a_2 \geq 0\) and \(\delta_{ijkl} = \delta_{ik} \delta_{jl}\) is the fourth-order identity tensor. This integral represents the interface energy between magnetised domains with different orientations. For most magnetic materials \(\text{div} \mathbf{m} = \delta_{ij} m_{i,j} = 0\), so hereafter we assume \(a_1 = a > 0\) and \(a_2 = 0\) (see Landau and Lifshitz 1935). The magneto-elastic energy is due to the coupling between the magnetic moments and the elastic lattice. For cubic crystals it is assumed to be

\[
E_{\text{em}}(\mathbf{m}, \mathbf{u}) = \frac{1}{2} \int_{\Omega} \lambda_{ijkl} m_i m_j \varepsilon_{kl}(\mathbf{u}) d\Omega
\]  

(2)

where \(\mathbb{L} = \{\lambda_{klmn}\}\) denotes the magneto-elasticity tensor whose entries \(\lambda_1, \lambda_2, \lambda_3 \geq 0\), and \(\lambda_{ijkl} = \lambda_1 \delta_{ijkl} + \lambda_2 \delta_{ij} \delta_{kl} + \lambda_3 (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})\) with \(\delta_{ijkl} = 1\) if \(i = j = k = l\) and \(\delta_{ijkl} = 0\) otherwise. Moreover we introduce the elastic energy

\[
E_{\text{el}}(\mathbf{u}) = \frac{1}{2} \int_{\Omega} \sigma_{ijkl} \varepsilon_{ij}(\mathbf{u}) \varepsilon_{kl}(\mathbf{u}) d\Omega
\]  

(3)

where \(\mathbb{E} = \{\varepsilon_{lm}\}\) indicates the strain tensor \(\sigma_{ijkl}\) satisfying the following symmetry property

\[
\sigma_{ijkl} = \sigma_{kl ij} = \sigma_{jilk}
\]

and moreover the inequality

\[
\sigma_{ijkl} \varepsilon_{ij} \varepsilon_{kl} \geq \beta \varepsilon_{ij} \varepsilon_{ij}
\]

holds for some \(\beta > 0\). In the isotropic case

\[
\sigma_{ijkl} = \tau_1 \delta_{ijkl} + \tau_2 \delta_{ij} \delta_{kl}, \quad \tau_1, \tau_2 \geq 0.
\]

The resulting energy functional \(E\) is given by

\[
E(\mathbf{m}, \mathbf{u}) = E(m, u) = E_{\text{ex}}(m) + E_{\text{em}}(m, u) + E_{\text{ve}}(u),
\]  

(4)

which, after some manipulations (Bertsch et al. 2001; Carillo et al. 2012), under the assumption the material is isotropic, reads

\[
E(\mathbf{m}, \mathbf{u}) = \frac{1}{2} \int_{\Omega} a |\nabla \mathbf{m}|^2 d\Omega + \frac{1}{2} \int_{\Omega} \left[ \tau_1 |\nabla \mathbf{u}|^2 + \tau_2 (\text{div} \mathbf{u})^2 \right] d\Omega + \frac{1}{2} \int_{\Omega} \left[ \lambda_1 \delta_{ijkl} u_{ij} m_k m_l + \lambda_2 |\mathbf{m}|^2 \text{div} \mathbf{u} + 2 \lambda_3 (\nabla u_i \cdot \mathbf{m}) m_i \right] d\Omega .
\]  

(5)

2.1. A simplified 2D model. To get the proposed model we make some approximations. First of all we assume \(\Omega \subset \mathbb{R}^2\) and neglect the components in plane of the displacement vector \(\mathbf{u}\), i.e., we assume \(\mathbf{u} = (0, 0, w)\), which implies \(\text{div} \mathbf{u} = 0\) since \(w\) depends only on the plane coordinates. Let \(\lambda_3 = \lambda\) be a positive constant, setting\(^1\) \(\tau_1 = 1\) and \(a = 1\), the functional \(E\) reduces to

\[
E(\mathbf{m}, w) = \frac{1}{2} \int_{\Omega} \left( |\nabla \mathbf{m}|^2 + 2 \lambda m_3 (m_\alpha w, \alpha) + |\nabla w|^2 \right) d\Omega
\]  

(6)

where the Greek indices vary in the set \(\{1, 2\}\).

\(^1\)No need to prescribe nor \(\lambda_2\) nor \(\tau_2 \geq 0\) since they both appear only as factors of \(\text{div} \mathbf{u}\); also \(\lambda_1\) can be left arbitrary; indeed, \(\delta_{ijkl} u_{ij} = 0\) since \(u_{ij} \neq 0\) only if \(j = 3\) and \(i = 1, 2\) but \(\delta_{ij} = 0\) when \(j \neq i\).
Setting $\Omega \equiv D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ and assuming radial symmetry, further to $w = w(r)$, we can express the components of the vector $\mathbf{m}$ in terms of $r$, that is

$$\mathbf{m} = \left( \frac{x}{r} \sin h(r), \frac{y}{r} \sin h(r), \cos h(r) \right), \quad r = \sqrt{x^2 + y^2},$$

where $h : (0, 1) \subset \mathbb{R} \to \mathbb{R}$ is an unknown regular function. Using the fact that $\partial_r r = \frac{x}{r}$ and $\partial_r r = \frac{y}{r}$ we deduce by the chain rule, where $h_r := \frac{dh}{dr}$ denotes the derivatives of the $h$ with respect to the variable $r$:

$$\partial_r \mathbf{m} = \left( \frac{x}{r} \left( \frac{\sin h}{r} \right)_r + \frac{x^2}{r^2} \left( \frac{\sin h}{r} \right)_r, \frac{y}{r} \left( \frac{\sin h}{r} \right)_r, \frac{x}{r} \left( \cos h \right)_r \right),$$

$$\partial_r \mathbf{m} = \left( \frac{xy}{r} \left( \frac{\sin h}{r} \right)_r, \frac{x}{r} \left( \frac{\sin h}{r} \right)_r, \frac{y}{r} \left( \cos h \right)_r \right).$$

Thus we get

$$|\nabla \mathbf{m}|^2 = \left[ \frac{\sin h}{r} + \frac{x^2}{r} \left( \frac{\sin h}{r} \right)_r \right]^2 + \left[ \frac{\sin h}{r} + \frac{y^2}{r} \left( \frac{\sin h}{r} \right)_r \right]^2 + 2 \left[ \frac{xy}{r} \left( \frac{\sin h}{r} \right)_r \right]^2 + \left[ \left( \cos h \right)_r \right]^2 \bigg[\cos h \right]_r \bigg]_r \bigg]_r + \left[ \left( \cos h \right)_r \right]_r$$

$$= 2 \left( \frac{\sin h}{r} \right)^2 + \frac{x^2 + 2xy^2 + y^4}{r^2} \left[ \frac{\sin h}{r} \right]^2 + 2 \left( \frac{xy}{r} \right)^2 + \left[ \left( \cos h \right)_r \right]_r$$

Thus we get

$$|\nabla \mathbf{m}|^2 = 2 \left( \frac{\sin h}{r} \right)^2 + \left[ \frac{\sin h}{r} + r \left( \frac{\sin h}{r} \right)_r \right]^2 + h_r^2 (\sin h)^2$$

So the energy (6), when we recall the assumed radial symmetry implies also $w = w(r)$, adopting the notation $w_r := \frac{dw}{dr}$, becomes

$$E(h, w) = \pi \int_0^1 \left[ h_r^2 + \left( \frac{\sin h}{r} \right)_r^2 + \lambda \sin 2hw_r + w_r^2 \right] \, dr$$

and from that we deduce the governing equations

$$\begin{cases}
  h_{rr} + \frac{h_r}{r} - \frac{\sin 2h}{2r^2} - \lambda \cos 2hw_r = 0 \\
  w_{rr} + \frac{w_r}{r} + \frac{\lambda}{2} \left[ \left( \sin 2h \right)_r + \frac{\sin 2h}{r} \right] = 0
\end{cases}$$

(7)

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We prescribe the following boundary conditions
\[ w_r(0) = 0, w(1) = 0, \] 
where the first condition is motivated by the symmetry assumptions, while the second one corresponds to prescribe the boundary of \( \Omega \) is fixed, and
\[ h_r(1) = 0. \] 
Solving the second equation of (7) which can be written
\[ (r w_r)_r + \frac{\lambda}{2} (r \sin 2h)_r = 0 \Leftrightarrow w_r = -\frac{\lambda}{2} \sin 2h, \]
where the double implication is guaranteed when we set \( h(0) = 0 \), then letting \( \mu = \lambda^2 / 2 \) we get the equation
\[ h_{rr} + \frac{h_r}{r} - \frac{\sin 2h}{2r^2} + \mu \sin 2h \cos 2h = 0 \] 
and the energy \( E \) becomes
\[ E(h) = \pi \int_0^1 \left[ h_r^2 + \left( \frac{\sin h}{r} \right)^2 - \frac{\mu}{2} (\sin 2h)^2 \right] r \, dr. \] 

The variational analysis of the functional \( E(h) \) is the objective of the following section.

3. The minimisation problem

Lemma 3.1. Let us define
\[ V = \{ v \mid v_r, \frac{v}{r} \in L^2(0, 1; r \, dr) \}. \]

\( V \) is a Hilbert space equipped with the norm
\[ \| v \|^2 = \int_0^1 (v_r^2 + \frac{v^2}{r^2}) \, r \, dr. \]

Proof. Let \( v_n \) be a Cauchy sequence in \( V \), \( \{(v_n)_r\}, \{\frac{v_n}{r}\} \) are Cauchy sequences in
\[ L^2(r \, dr) = L^2(0, 1; r \, dr) \]
and there exist \( h, g \) such that
\[ \{(v_n)_r\}, \{\frac{v_n}{r}\} \rightarrow g, h \text{ in } L^2(r \, dr) \] 
Set \( \tilde{h} = hr \). Since
\[ \int_0^1 \left( \frac{v_n}{r} - h \right)^2 \, r \, dr \rightarrow 0 \] 
one has
\[ v_n \rightarrow rh \text{ in } L^2(0, 1; \frac{dr}{r}). \] 
but also in \( \mathcal{D}'(0, 1) \) so that
\[ (v_n)_r \rightarrow \tilde{h}_r \text{ in } \mathcal{D}'. \]
We deduce from (15) that $\tilde{h}_r = g$ and thus $\tilde{h} \in V$ and since
\[(v_n)_r, \frac{v_n}{r} \to \tilde{h}_r, \frac{\tilde{h}}{r} \text{ in } L^2(rdr) \tag{18}\]
one has $v_n \to \tilde{h} \in V$. This completes the proof of Lemma 3.1. \hfill \Box

**Lemma 3.2.**

\[V \subset \{ v \in C([0,1]) \mid v(0) = 0 \}\]

**Proof.** For $x, y \in (0,1]$ one has
\[
|vx(x) - vy(y)| = \left| \int_x^y (rv) dr \right| = \left| \int_x^y r(v_r + \frac{v}{r}) dr \right| \leq \int_x^y r(|v_r| + |\frac{v}{r}|) dr. \tag{19}\]

Using the Cauchy-Young inequality $a \leq \frac{1}{2}a^2 + \frac{1}{2}$ one gets
\[
|vx(x) - vy(y)| \leq \int_x^y \left\{ \frac{r}{2} \left( v_r^2 + \frac{v^2}{r^2} \right) + r \right\} dr \to 0 \text{ when } y \to x. \tag{20}\]

It follows that $v$ is continuous at any point where $r \neq 0$ on $[0,1]$. Now, one has also
\[
v(x)^2 - v(y)^2 = \int_x^y \frac{d}{dr} v(r)^2 dr = \int_x^y 2v_r v dr = \int_x^y 2\sqrt{r} v_r \frac{v}{\sqrt{r}} dr \tag{21}\]
\[
\leq \int_x^y \left\{ r(v_r^2 + \frac{v^2}{r^2}) \right\} dr \leq \epsilon
\]
for $x, y$ small enough (we used again the Cauchy-Young inequality). Thus, when $x \to 0$, $v(x)^2$ is a Cauchy sequence and there exist $l \geq 0$ such that
\[
\lim_{x \to 0} v(x)^2 = l. \tag{22}\]

If $l > 0$ one has for $\epsilon$ small enough
\[
\|v\|^2 \geq \int_0^1 \frac{v^2}{r} dr \geq \int_{\epsilon^2}^{\epsilon} \left( \frac{l}{2} \right)^2 \frac{dr}{r} = \frac{l^2}{4} (\ln \epsilon - 2 \ln \epsilon) = -\frac{l^2}{4} \ln \epsilon \tag{23}\]
and a contradiction when $\epsilon \to 0$. Thus, $l = 0$ and this completes the proof of the Lemma 3.2. \hfill \Box

**Remark** Since $V \subset H^1(\epsilon, 1)$, it follows that $V \subset C^{1/2}(\epsilon, 1)$ for every $\epsilon$.

One sets
\[
E(h) = \pi \int_0^1 \left\{ h^2 + \left( \frac{\sin h}{r} \right)^2 - \frac{\mu}{2} (\sin 2h)^2 \right\} r dr. \tag{24}\]

One would like to show that $E(h)$ possesses a minimiser on $V$ for any $\mu$. 

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Lemma 3.3. The energy $E(h)$ is bounded from below on $V$ and one can find a minimising sequence $v_n$ such that
\[
0 \leq v_n \leq \frac{\pi}{2}.
\] (25)

Proof. One has clearly for every $h \in V$
\[
E(h) \geq -\pi |\mu| \int_0^1 r dr = -\pi |\mu| \frac{4}{4}.
\] (26)

Thus
\[
I = \inf_{h \in V} E(h)
\]
exists. Let us denote by $v_n$ a sequence such that
\[
E(v_n) \to I.
\]
If $v_n \in V$, then also $|v_n| \in V$ and one has
\[
E(v_n) = E(|v_n|)
\]
so, without loss of generality, we assume $v_n \geq 0$.

Then on $v_n > \frac{\pi}{2}$, we replace $v_n$ by $-v_n + \pi$ (cfr. Fig. 1). It is clear that
\[
\bar{v}_n = v_n X_{\{v_n \leq \frac{\pi}{2}\}} + (-v_n + \pi) X_{\{v_n > \frac{\pi}{2}\}}
\]
(27)
satisfies $\bar{v}_n \in V$ and
\[
E(\bar{v}_n) = E(v_n).
\]
This completes the proof of the Lemma 3.3. □

Remark 3.4. It could be that $-v_n + \pi$ achieves negative values, but clearly, after a finite number of operations like the one we just did we get a $v_n$ satisfying (25).

Figure 1. Graphical representation.
Lemma 3.5. There exists a minimiser \( \tilde{h} \) of \( E \) in \( V \) satisfying

\[
0 \leq \tilde{h} \leq \frac{\pi}{2}.
\]

Proof. We consider the sequence \( \{v_n\} \) constructed in Lemma 3.3. We claim that \( \{v_n\} \) is bounded in \( V \) independently of \( n \). Indeed, one has, since for some constant \( \lambda > 0 \) one has

\[
\left( \frac{\sin x}{x} \right)^2 \geq \lambda, \quad \forall x \in [0, \frac{\pi}{2}],
\]

\[
\int_0^1 r \left\{ (v_n)_r^2 + \frac{v_n^2}{r^2} \right\} dr \leq \int_0^1 r \left\{ (v_n)_r^2 + \frac{1}{\lambda} \left( \frac{\sin v_n}{r} \right)^2 \right\} dr
\]

\[
\leq \left( 1 \vee \frac{1}{\lambda} \right) \int_0^1 r \left\{ (v_n)_r^2 + \left( \frac{\sin v_n}{r} \right)^2 \right\} dr \leq C
\]

where \( C \) is a constant independent of \( n \) and \( \vee \) denotes the maximum of two numbers. Recall that since \( v_n \) is a minimising sequence one has, for \( n \) large enough,

\[
E(v_n) \leq E(0) = 0
\]

i.e., see the definition of \( E \)

\[
\pi \int_0^1 \left\{ (v_n)_r^2 + \left( \frac{\sin v_n}{r} \right)^2 \right\} r dr \leq \pi \frac{|\mu|}{2} \int_0^1 \sin^2(2v_n) r dr \leq \pi \frac{|\mu|}{4}.
\]

Since \( \{(v_n)_r\}, \{\frac{v_n}{r}\} \) are bounded in \( L^2(r dr) \) one finds a subsequence, still labelled by \( n \), such that

\[
\frac{v_n}{r} \rightarrow h, \quad (v_n)_r \rightarrow g \text{ in } L^2(r dr).
\]

Set \( h = hr \). The first weak convergence above reads

\[
\int_0^1 \frac{v_n}{r} \Psi r dr \rightarrow \int_0^1 h \Psi r dr, \quad \forall \Psi \in L^2(r dr).
\]

In particular, taking \( \Psi \in \mathcal{D}(0,1) \) one see that

\[
v_n \rightarrow h = hr \text{ in } \mathcal{D}'(0,1)
\]

and thus, by the continuity of the derivative in \( \mathcal{D}' \)

\[
(v_n)_r \rightarrow h_r = g \text{ in } \mathcal{D}'(0,1).
\]

Thus, we have \( \tilde{h} \in V \). For any \( k \geq 2 \) one has also, thank to (29), that \( v_n \) is bounded in \( H^1\left( \frac{1}{k}, 1 \right) \). Thus, by induction, one can find a subsequence \( \{n_k\} \) extracted from \( \{n_{k-1}\} \) such that

\[
v_{n_k} \rightarrow \tilde{h} \text{ in } L^2\left( \frac{1}{k}, 1 \right) \text{ and a.e.}
\]

Then clearly

\[
v_{n_k} \rightarrow \tilde{h} \text{ a.e. on } (0,1).
\]
By the dominated Lebesgue theorem one has then that
\begin{align*}
r \sin 2v_n & \to r \sin 2\tilde{h} \quad \text{in } L^2(0, 1), \\
\sin v_n & \to \sin \tilde{h} \quad \text{a.e. on } (0, 1).
\end{align*}

Then, since \( x \mapsto x^2 \) is convex by the Fatou lemma one has
\begin{align*}
I = \lim_{\lambda \to 0^+} E(v_n) &= \pi \lim_{\lambda \to 0^+} \int_0^1 \left\{ (v_n)^2 r + \left( \frac{\sin v_n}{r} \right)^2 \right\} r \, dr - \pi \int_0^1 \frac{\mu}{2} (\sin(2v_n))^2 r \, dr \\
&\geq \pi \lim_{\lambda \to 0^+} \int_0^1 (v_n)^2 r \, dr + \pi \lim_{\lambda \to 0^+} \int_0^1 \left( \frac{\sin v_n}{r} \right)^2 r \, dr - \frac{\pi \mu}{2} \int_0^1 (\sin(2\tilde{h}))^2 r \, dr \\
&\geq \int_0^1 (\tilde{h}_r)^2 r \, dr + \pi \int_0^1 \lim_{\lambda \to 0^+} \left( \frac{\sin v_n}{r} \right)^2 r \, dr - \frac{\pi \mu}{2} \int_0^1 (\sin(2\tilde{h}))^2 r \, dr = \\
&= E(\tilde{h}) = I . \tag{31}
\end{align*}

This shows that \( \tilde{h} \) is the minimiser that we are looking for.

\textbf{Lemma 3.6.} \textit{The Euler equation of the minimising problem is given by}
\begin{equation}
\begin{cases}
- h_{rr} - \frac{h_r}{r} + \frac{\sin 2h}{r^2} = \mu \sin 2h \cos 2h \quad \text{in } (0, 1) \\
h(0) = h_r(1) = 0
\end{cases} \tag{32}
\end{equation}

\textit{Proof.} If \( h \) is a minimiser of \( E \) on \( V \) one has
\begin{equation}
\frac{d}{d\lambda} E(h + \lambda v)|_{\lambda = 0} = 0 , \quad \forall v \in V
\end{equation}

Since
\begin{equation}
E(h + \lambda v) = \pi \int_0^1 \left\{ (h + \lambda v)^2 r + \frac{\sin(h + \lambda v)^2}{r^2} - \frac{\mu}{2} \sin(2(h + \lambda v))^2 \right\} r \, dr . \tag{33}
\end{equation}

One gets \( \forall v \)
\begin{equation}
\int_0^1 \left\{ 2h_r v_r + 2 \frac{\sin h \cos h}{r^2} v - 2\mu \sin(2h) \cos(2h) v \right\} r \, dr = 0 \\
\iff \int_0^1 \left\{ h_r v_r + \frac{\sin(2h)}{2r^2} v - \mu \sin(2h) \cos(2h) v \right\} r \, dr = 0 , \quad \forall v \in V . \tag{34}
\end{equation}
Thus, in the distributional sense

\[-(rh_r) + \frac{\sin(2h)}{2r} - \mu r \sin(2h) \cos(2h) = 0\]

\[\implies -r h_r - h_r + \frac{\sin(2h)}{2r} - \mu r \sin(2h) \cos(2h) = 0.
\]

Dividing by \( r \) we get the first equation of (32). Integrating by parts in (34) and using (35) we get

\[\int_0^1 (rh_v)_r - (rh_r)_r v + \frac{\sin 2hv}{2r} - \mu \sin(2h) \cos 2hv = 0, \quad \forall v \in V\]

i.e.,

\[\int_0^1 (rh_v)_r = 0, \quad \forall v \in V\]

which gives

\[h_r(1) = 0.
\]

(in a weak sense) \( h(0) = 0 \) follows from \( h \in V \). This completes the proof of the Lemma. \( \square \)

**Lemma 3.7.** If \( h \neq 0 \) is a nonnegative minimiser of \( E \) on \( V \) then \( h > 0 \) on \((0, 1)\).

**Proof.** Indeed, if \( h \) vanishes at \( r_0 \in (0, 1) \) then, since \( h \) is smooth and \( r_0 \) is a minimum for \( h \), one would have

\[h(r_0) = h_r(r_0) = 0\]

then from the theory of o.d.e’s (see [1]), \( h \equiv 0 \). \( \square \)

**Lemma 3.8.** If \( h \) is a positive minimiser of \( E \) then \( 0 < h \leq \frac{\pi}{2} \).

**Proof.** If not then \( h \) constructed as in the figure before (Fig.1) is a minimiser but it has a jump in the derivative unless this one is 0. But then \( h = \frac{\pi}{2} \) is solution of the o.d.e. on \( h > \frac{\pi}{2} \) and a contradiction follows. Note that the solution of the elliptic equation (14) is smooth on \((0, 1)\). \( \square \)

**Lemma 3.9.** A minimiser cannot vanish on \((0, 1)\) unless it vanishes identically.

**Proof.** If \( h \) is a minimiser, \( |h| \) is also a minimiser. But, then, \( |h| \) would have a jump discontinuity in its derivative unless when it vanishes so does \( h_r \). This implies (theory of o.d.e’s), \( h = 0 \). \( \square \)

**Lemma 3.10.** If \( h \in V \) then \( \sin(kh) \in V \), \( \forall k \in \mathbb{R} \).

**Proof.** One has

\[\sin(kh)_r = kh_r \cos(kh), \quad |\sin(kh)| \leq |kh|.
\]
Therefore one has
\[
\|\sin(kh)\|^2 = \int_0^1 \left\{ \sin(kh)^2 + \left( \frac{\sin(kh)}{r} \right)^2 \right\} r \, dr \\
\leq \int_0^1 \left\{ k^2 \cos(kh)^2 h^2 + \frac{k^2 h^2}{r^2} \right\} r \, dr \leq k^2 \|h\|^2
\]  
(37)

It is easy to check that \( h \equiv 0 \) solves (9), (10) and hence it is a stationary point of the functional (11).

Let \( \gamma_0 \) be the first eigenvalue of the problem
\[
\begin{cases}
-\phi_{rr} - \frac{\phi_r}{r} + \frac{\phi}{r^2} = \gamma \phi \\
\phi(0) = 0, \quad \phi_r(1) = 0.
\end{cases}
\]  
(38)

**Lemma 3.11.**
\[ \gamma_0 > 1. \]

**Proof.** Suppose not, i.e., \( \gamma_0 \leq 1 \). Let \( \phi \) be the corresponding positive (or nonnegative) eigenfunction. One has
\[
-\phi_{rr} - \frac{\phi_r}{r} = \phi(\gamma_0 - \frac{1}{r^2}) \leq 0 \quad \text{since } r \in (0, 1)
\]
(39)

\( (r \phi_r)_r \geq 0 \implies r \phi_r \nearrow \implies r \phi_r \leq 0 \quad \text{since } \phi_r(1) = 0. \)

Thus, the maximum of \( \phi \) is achieved at 0 but, since \( \phi(0) = 0 \), we get a contradiction i.e., \( \phi \equiv 0 \). \( \square \)

We have the following bifurcation lemma.

**Lemma 3.12.** If \( \mu \leq \gamma_0/2 \) we have \( E(h) \geq 0 \) and the global minimum is attained only for \( h \equiv 0 \). For \( \mu > \gamma_0/2 \) the global minimum is negative.

**Proof.** The first equation of (38) can also be written after a multiplication by \( r \) as
\[
-(r \phi_r)_r + \frac{\phi}{r} = \gamma r \phi.
\]
Multiplying by \( \phi \) and integrating over \( (0, 1) \) we derive by definition of \( \gamma_0 \) that
\[
\int_0^1 \left( \phi_r^2 + \frac{\phi^2}{r^2} \right) r \, dr \geq \gamma_0 \int_0^1 \phi^2 r \, dr \quad \forall \phi \quad \text{with } \phi(0) = 0, \quad \phi_r(1) = 0.
\]  
(40)

We divide the proof in two parts:

(i) \( \mu \leq \gamma_0/2 \)
In this case we have (using (40) with \( \phi = \sin h \))

\[
E(h) = \pi \int_0^1 \left[ (\cosh^2 h r^2 + \left( \frac{\sinh h}{r} \right)^2 - 2\mu (\sinh h)^2 \right] r dr
\]

\[
\geq \int_0^1 \left[ (\gamma_0 (\sin h)^2 - 2\mu (\sin h)^2 (\cosh h)^2 + (1 - (\cos h)^2) h_r^2 \right] r dr
\]

\[
= \int_0^1 \left[ (\gamma_0 - 2\mu) (\sin h)^2 + (1 - (\cos h)^2)(2\mu (\sin h)^2 + h_r^2) \right] r dr
\]

\[
\geq 0 = E(0)
\]

the equality taking place only for \( h = 0 \).

(ii) \( \mu > \gamma_0/2 \)

Let us denote by \( \phi_0 \) the first positive normalised eigenfunction to (38). One has, for \( \varepsilon > 0 \): \[
E(\varepsilon \phi_0) \leq \pi \int_0^1 \left[ (\varepsilon \phi_0)^2 + \left( \frac{\varepsilon \phi_0}{r} \right)^2 - \frac{\mu}{2} (\sin(2\varepsilon \phi_0))^2 \right] r dr
\]

\[
= \pi \int_0^1 \left[ (\gamma_0 (\varepsilon \phi_0)^2 - \frac{\mu}{2} (\sin(2\varepsilon \phi_0))^2 \right] r dr.
\]

Using with \( x = 2\varepsilon \phi_0 \) the formula

\[
\sin x = x - \int_0^1 (1 - \cos(tx)) x dt
\]

\( E(\varepsilon \phi_0) \) can be written as

\[
E(\varepsilon \phi_0) = \pi \int_0^1 \left[ (\varepsilon \phi_0)^2 \left\{ \gamma_0 - 2\mu (1 - \int_0^1 (1 - \cos(2\varepsilon \phi_0)) dt \right\)^2 \right] r dr
\]

\[
< 0 = E(0)
\]

for \( \varepsilon \) small, since

\[
\int_0^1 (1 - \cos(2\varepsilon \phi_0)) dt \to 0
\]

when \( \varepsilon \to 0 \). \( \square \)

**Alternative proof of (ii)**

Suppose \( h \neq 0 \) is a minimiser of \( E \) one has

\[
E(h) < E(0)
\]

(41)
i.e.,
\[
\int_0^1 \left\{ h_r^2 + \left( \frac{\sin h}{r} \right)^2 \right\} r dr < \frac{\mu}{2} \int_0^1 (\sin 2h)^2 r dr = 2\mu \int_0^1 \sin^2 h \cos^2 h r dr
\]

\[\Rightarrow \int_0^1 \left\{ \cos^2 h^2 r^2 + \left( \frac{\sin h}{r} \right)^2 \right\} r dr < \gamma_0 \int_0^1 \sin^2 h r dr \tag{42}\]

\[\Rightarrow \gamma_0 > \frac{\int_0^1 \left\{ \sin h^2 r^2 + \left( \frac{\sin h}{r} \right)^2 \right\} r dr}{\int_0^1 \sin^2 h r dr} \]

and a contradiction since \(\sin h \in V\) with the definition of \(\gamma_0\).

Consider the problem

\[
\begin{cases}
- h_{rr} - \frac{h_r}{r} + \frac{\sin 2h}{r^2} = \mu \sin 2h \cos 2h & \text{on } (0, 1) \\
h(0) = h_r(1) = 0
\end{cases}
\tag{43}\]

**Lemma 3.13.** If \(\mu \leq \gamma_0/2\) the only solution of (43) such that \(h \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right]\) is \(h \equiv 0\).

**Proof.** Recall that for any \(\phi \in V\) one has by definition of \(\gamma_0\)

\[
\gamma_0 \int_0^1 \phi^2 r dr \leq \int_0^1 \left( \phi_r^2 + \frac{\phi^2}{r^2} \right) r dr . \tag{44}\]

Let us write the equation (43) as

\[
(rh_r)_r + \frac{\sin 2h}{2r} = \mu \sin 2h \cos 2h r . \tag{45}\]

Multiply both sides by \(\sin 2h\) and integrate on \((0, 1)\). It comes

\[
\int_0^1 \left( h_r (\sin 2h)_r + \frac{(\sin 2h)^2}{2r^2} \right) r dr = \mu \int_0^1 (\sin 2h)^2 \cos 2h r dr . \tag{46}\]

One has

\[
(\sin 2h)_r = 2 \cos 2h h_r \iff h_r = \frac{(\sin 2h)_r}{2 \cos 2h} . \tag{47}\]

Thus, the equation above becomes

\[
\int_0^1 \left( (\sin 2h)_r^2 \frac{1}{\cos 2h} + \frac{(\sin 2h)^2}{r^2} \right) r dr = 2\mu \int_0^1 (\sin 2h)^2 \cos 2hr dr . \tag{48}\]
Suppose that \( h \) is such that \( h \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \) then since \(-1 \leq \cos 2h \leq 1\) one gets
\[
\int_{0}^{1} r \left\{ (\sin 2h)^2 + \frac{(\sin 2h)^2}{r^2} \right\} dr < \gamma_0 \int_{0}^{1} (\sin 2h)^2 rdr
\]
\[\text{(49)}\]
i.e, \( \sin 2h \in V \) and satisfies an inequality contradicting (44), except if \( h \equiv 0 \).

Each minimiser of \( E(h) \) solves the problem \((9), (10)\). For the solutions of this problem we can give the following existence result around the bifurcation point.

**Lemma 3.14.** There exist two positive numbers \( \rho_0 \) and \( \delta_0 \) such that, the problem \((9), (10)\) does not have non-zero solutions for \( \mu \in (\gamma_0/2 - \delta_0, \gamma_0/2] \) and \( \|h\|_0 \leq \rho_0 \). The problem has exactly two solutions \( h_1 \) and \( h_2 = -h_1 \) in the sphere \( \|h\|_0 \leq \rho_0 \) for \( \mu \in (\gamma_0/2, \gamma_0/2 + \delta_0) \).

**Proof.** The proof follows from Krasnosel’skii (1964, Theorem 6.12). Indeed the equation \((10)\) can be written in the form
\[
2\mu h = L(h,r) + C(h,r,\mu) + D(h,r,\mu)
\]
where \( L \) is the linear operator
\[
L(h,r) = -hr_r - \frac{h_r}{r} + \frac{h}{r^2}
\]
and \( C, D \) are given by
\[
C(h,r,\mu) = -\frac{2}{3} h^3 + \frac{16}{3} \mu h^3,
\]
\[
D(h,r,\mu) = - \left( \frac{2h}{2r^2} - \frac{\sin 2h}{2r^2} \right) + \frac{2}{3} h^3 + \mu (4h - \sin 4h) - \frac{16}{3} \mu h^3.
\]

It is easy to check that
\[
C(th, r, \mu) = t^3 C(h, r, \mu), \quad (-\infty < t < \infty)
\]
and
\[
\|D(h,r,\mu)\|_0 = o(\|h\|^3).
\]

Moreover we have
\[
((C(\phi^0, r, \mu), \phi^0))_0 = \int_{0}^{1} \left[ -\frac{2}{3} \frac{(\phi^0)^4}{r} + \frac{16}{3} \mu (\phi^0)^4 r \right] dr > 0, \quad \text{for } \mu \geq \frac{\gamma_0}{8}.
\]

Indeed from \(((L(h^0, r) - \gamma_0 \phi^0, (\phi^0)^3))_0 = 0\) it follows that
\[
\int_{0}^{1} \left[ -\frac{d}{dr}(\phi^0 r) (\phi^0)^3 + \frac{(\phi^0)^4}{r} - \gamma (\phi^0)^4 r \right] dr = 0
\]
\[\text{(54)}\]

The latter, on by parts integration
\[
-(\phi^0 r)(\phi^0)^3 |_{0}^{1} + \int_{0}^{1} 3(\phi^0 r)^2 (\phi^0)^2 rdr + \int_{0}^{1} \left[ \frac{(\phi^0)^4}{r} - \gamma_0 (\phi^0)^4 r \right] dr = 0
\]
that is
\[
\int_0^1 \frac{(\phi_0)^4}{r} - \gamma_0 (\phi_0)^4 r \, dr \leq 0,
\]
and the inequality (53) can be easily derived. The statements (50)- (53), together to the local Lipschitz condition on the operators \(C\) and \(D\), assure (see Krasnosel’kii 1964) the existence of exactly two branch of non-zero solutions bifurcating from the point \(\gamma_0/2\). Finally, we remark that the existence of two opposite branch follows from the odd functions in (50).

**Remark 3.15.** In order to establish the stability of the solutions to (9), (10) around the point \(\mu_0 = \gamma_0/2\), we perform a qualitative analysis of the bifurcation equation to the lowest order (see equation (55) below). From (50) setting

\[
G(h, r, \mu) = -2\mu h + L(h, r) + C(h, r, \mu) + D(h, r, \mu) = 0
\]

and

\[
2\mu = \gamma_0 + \delta, \quad |\delta| < < 1,
\]

assuming that each element \(h \in \mathcal{H}^1(0, 1)\) has the unique representation

\[
h = \beta \phi_0 + Ph, \quad (Ph, \phi_0) = 0, \quad \beta \in \mathbb{R},
\]

we have

\[
(G(h, r, \mu), \phi_0) = -\delta \beta + (C(\beta \phi_0, r, \mu), \phi_0) + \ldots
\]

Moreover from (51), (53) we can get to the simple l.o. bifurcation equation, namely

\[
-\delta \beta + \beta^3 \tilde{C} = 0, \quad \tilde{C} = (C(\phi_0, r, \mu), \phi_0) \geq 0. \tag{55}
\]

It is easy to check that:

for \(\delta \leq 0\) there is the only solution \(\beta = 0\) and this solution is stable (indeed in this case: \(-\delta + 3\beta^2 \tilde{C} \geq 0\));

for \(\delta > 0\) the trivial solution is no more stable but other two stable solutions appear, i.e., \(\beta = \pm \sqrt{\delta/\tilde{C}}\).

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