Soliton Solutions for the Super mKdV and sinh-Gordon Hierarchy

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Abstract
The dressing and vertex operator formalism is employed to study the soliton solutions of the \( N = 1 \) super mKdV and sinh-Gordon models. Explicit two and four vertex solutions are constructed. The relation between the soliton solutions of both models is verified.

1 Introduction

A systematic construction of supersymmetric integrable hierarchies within the algebraic formalism was proposed in [1]. An interesting feature of such approach is that it allows time evolutions according to both, positive and negative grades. In particular the first negative grade time evolution is always associated to the relativistic integrable model (with appropriated choice of space-time coordinates). Specific examples were given in connection with the \( sl(2,1) \) super Lie algebra yielding the super mKdV and sinh-Gordon models. The supersymmetric sinh-Gordon was proposed in [2] by introducing a pair of Grassmann coordinates whilst the super mKdV, involving a single Grassmann coordinate, was proposed later in [3]. By extending Hirota’s method for superfields, solutions for the super mKdV was obtained [4]. The relation between the bosonic mKdV and sinh-Gordon models was observed by a number of authors [5], [6]. This fact was explained and further extended to other integrable models in [7] (e.g. Lund-Regge and Non linear Schroedinger) and in [8] the relation between soliton solutions of both models were verified explicitly. This fact was explored in [1] to show that the super mKdV and sinh-Gordon models share the same algebraic structure and henceforth belong to the same integrable hierarchy. This is verified explicitly here by comparing the soliton solutions of both models.

In this paper we employ the algebraic formalism to study the \( N = 1 \) super mKdV and sinh-Gordon models by decomposing the affine \( \hat{sl}(2,1) \) into half-integer graded subspaces following ref. [1]. In order to obtain a general non-trivial soliton solution from the dressing formalism and vertex operators we employ a slightly different gradation and loop automorphism from those of ref. [1].

This paper is organized as follows. In Sect. 2 we discuss the decomposition of the affine \( \hat{sl}(2,1) \) super algebra into half integer graded subspaces and construct the mKdV and sinh-Gordon models. In Sect. 3 we follow refs. [9], [6] to derive the tau functions from the dressing formalism and construct the vertex operators leading to the soliton solutions for those integrable models. In the appendices we discuss the relevant subalgebra of the affine \( sl(2,1) \) and the general four vertex solution.
2 The super mKdV and sinh-Gordon Hierarchy

In this section we employ the algebraic formalism to construct an integrable hierarchy containing the mKdV and sinh-Gordon supersymmetric models. Consider the $sl(2,1)$ super Lie algebra with generators

$$h_1 = \alpha_1 \cdot H, \quad h_2 = \alpha_2 \cdot H, \quad E_{\pm \alpha_1}, \quad E_{\pm \alpha_2}, \quad E_{\pm (\alpha_1 + \alpha_2)} \quad (2.1)$$

where $\alpha_1$ and $\alpha_2, \alpha_1 + \alpha_2$ are bosonic and fermionic roots respectively. The integrable hierarchy is defined by choosing the grading operator

$$Q = 2d + \frac{1}{2} h_1 \quad (2.2)$$

where $d$ satisfy $[d, T^{(m)}_a] = m T^{(m)}_a$, $T^{(m)}_a$ denote both $E^{(m)}_\alpha$ or $H^{(m)}_i$. The hierarchy is further specified by the constant grade one element, $E = E^{(1)}$ where,

$$E^{(2n+1)} = h_1^{(n+1/2)} + 2 h_2^{(n+1/2)} - E_{\alpha_1} - E_{-\alpha_1} \quad (2.3)$$

A key ingredient to construct the desired integrable models is the judicious choice of a subalgebra $\hat{G}$ of the affine $sl(2,1)$. This is discussed in the appendix A. The grading operator $Q$ and $E$ decomposes the associated affine super Kac-Moody algebra $\hat{G} = \hat{G}_{l} = \mathcal{K} \oplus \mathcal{M}$ where $l$ is the degree of the subspace $\mathcal{G}_{l}$ and $\mathcal{K} = \{ x \in \hat{G}, [x, \mathcal{K}] = 0 \}$ denote the kernel of $E$ and $\mathcal{M}$ its complement, i.e.,

$$\mathcal{K}_{Bose} = \{ K^{(2n+1)}_1, K^{(2n+1)}_2 \}, \quad \mathcal{K}_{Fermi} = \{ F^{(2n+3/2)}_1, F^{(2n+1/2)}_2 \}, \quad \mathcal{M}_{Bose} = \{ M^{(2n+1)}_1, M^{(2n+1)}_2 \}, \quad \mathcal{M}_{Fermi} = \{ G^{(2n+1/2)}_1, G^{(2n+3/2)}_2 \} \quad (2.4)$$

where the generators $K_i, M_i, F_i$ and $G_i$ are constructed in terms of generators of $\hat{sl}(2,1)$ in app. A. Define the Lax operator $L = \partial_x + E + A_0 + A_{1/2} = \partial_x + \mathcal{A}_x$ where $\mathcal{A}_x = A_0 + A_{1/2} \in \mathcal{M}$ mod $\hat{c}$, i.e.,

$$A_0 = u M_2^{(0)} + \eta \hat{c}, \quad A_{1/2} = \bar{\psi} G_1^{(1/2)} \quad (2.5)$$

The positive hierarchy is given in terms of the zero curvature condition

$$[\partial_x + E + A_0 + A_{1/2}, \partial_n + D^{(n)} + D^{(n-1/2)} + \cdots D^{(0)}] = 0 \quad (2.6)$$

where $D^{(i)} \in \mathcal{G}_i$ and can be solved recursively decomposing eqn. (2.6) grade by grade (see for instance ref. ([1])). The solution is local and the image part of the zero and one-half grade components of (2.6) yields the time evolution for the fields defined in (2.5). For $n = 3$ we find the equations of motion for the $N = 1$ super mKdV, i.e.,

$$4 \partial_{\tilde{t}_3} \bar{\psi} = \partial^2_{\tilde{t}_3} \bar{\psi} - 3 u \partial_x (u \bar{\psi}), \quad 4 \partial_{\tilde{t}_3} u = \partial^3_{\tilde{t}_3} u - 6 u^2 \partial_x u + 3 \bar{\psi} \partial_x (u \partial_x \bar{\psi}) \quad (2.7)$$

Observe that the equation of motion for the field $\eta$ is not fixed due to ambiguity in determining $D^{(0)} \rightarrow D^{(0)} + \rho \hat{c}$. In general, for non relativistic theories, it is necessary a more
restrictive structure given by the dressing transformations explained in the next section. Other integrable equations are obtained for different values of \( n \) in similar manner. The negative hierarchy is obtained by

\[
[\partial_x + E + A_0 + A_{1/2}, \partial_{t_{-m}} + D^{(-m)} + D^{(-m+1/2)} + \cdots D^{(-1)} + D^{(-1/2)}] = 0 \tag{2.8}
\]

Both, positive and negative hierarchies (2.6) were shown to be derived from the Riemann-Hilbert problem for the homogeneous gradation [11]. The solution for the negative hierarchy (2.8) is, in general non-local, however, the simplest member of the negative hierarchy for \( m = 1 \) in (2.8) has a closed local solution in terms of the zero grade group element \( B \in G_0 \),

\[
A_0 = -\partial_x BB^{-1}, \quad A_{1/2} = \bar{\psi}G_1^{(1/2)}, \quad D^{(-1/2)} = \psi BG_2^{(-1/2)}B^{-1}, \quad D^{(-1)} = BE^{(-1)}B^{-1} \tag{2.9}
\]

where \( E^{(-1)} \) is given by (2.3) for \( n = -1 \). According to \( Q \) in (2.2) the zero grade subalgebra is generated by \( G_0 = \{ M_2^{(0)}, \hat{c} \} \), i.e.

\[
B = \exp (\phi M_2^{(0)} + \nu \hat{c}) \tag{2.10}
\]

The time evolution for \( t_{-1} \) is obtained from (2.8) and coincides with the Leznov-Saveliev’s equation [12] when we identify \((x, t_{-1})\) with the light cone coordinates,

\[
\begin{align*}
\partial_{t_{-1}} \partial_x BB^{-1} &= -[E, BE^{(-1)}B^{-1}] - [\bar{\psi}G_1^{(1/2)}, \psi BG_2^{(-1/2)}B^{-1}], \\
\partial_{t_{-1}} \bar{\psi}G_1^{(1/2)} &= [E, \psi BG_2^{(-1/2)}B^{-1}] \tag{2.11}
\end{align*}
\]

leading in components to the \( N = 1 \) super sinh-Gordon equations of motion,

\[
\begin{align*}
\partial_{t_{-1}} \partial_x \phi &= 2 \sinh 2\phi + 2 \bar{\psi} \psi \sinh \phi, \\
\partial_{t_{-1}} \partial_x \nu &= \bar{\psi} \psi \cosh \phi + (1 - e^{2\phi}), \\
\partial_{t_{-1}} \bar{\psi} &= 2 \psi \cosh \phi, \\
\partial_x \psi &= 2 \bar{\psi} \cosh \phi \tag{2.12}
\end{align*}
\]

The above equations are invariant under supersymmetry transformation

\[
\bar{\psi}' = \bar{\psi} + \epsilon \partial_x \phi, \quad \phi' = \phi + \epsilon \bar{\psi} \tag{2.13}
\]

The zero group element \( B \) defined in (2.9) when parametrized as in (2.10) establishes a correspondence between relativistic (sinh-Gordon) and non relativistic (mKdV) field variables, i.e.

\[
u = -\partial_x \phi, \quad \eta = -\partial_x \nu \tag{2.14}
\]

3 Dressing and Soliton Solutions

We now construct the soliton solution for both, mKdV and sinh-Gordon models from the dressing transformation generated by \( \Theta_\pm \) which relates two solutions of the equations of
motion written in the zero curvature representation. In particular, it relates the vacuum and the 1-soliton solutions by a gauge transformation,

\[ \mathcal{A}_\mu = \Theta_\pm \mathcal{A}^\text{vac}_\mu \Theta_\pm^{-1} - (\partial_\mu \Theta_\pm) \Theta_\pm^{-1} \]  

(3.15)

where

\[ \Theta_- = e^{m(-1/2) + m(-1)} \quad \Theta_+ = B e^{v(1/2) + v(1)} \]  

(3.16)

where \( m(-i) \in G_{-i} \) and \( v(i) \in G_i \). The zero curvature representation implies for pure gauge solutions:

\[ \mathcal{A}_\mu^\text{vac} = -\partial_\mu T_0 T_0^{-1}, \quad \mathcal{A}_\mu = -\partial_\mu TT^{-1} \]  

(3.17)

which leads to the following relation

\[ \Theta_+^{-1} \Theta_- = T_0 g T_0^{-1}, \]  

(3.18)

where \( g \in \hat{G} \) is an arbitrary constant element of the corresponding affine group. Suppose \( T_0 \) represents the vacuum solution,

\[ T_0 = \exp(-t_n E^{(n)}) \exp(-x E^{(1)}), \]  

(3.19)

i.e.,

\[ \mathcal{A}_{t_n}^\text{vac} = E^{(n)}, \quad \mathcal{A}_x^\text{vac} = E^{(1)} \]  

(3.20)

As consequence of (3.15) with (3.20) and (3.16) we can determine \( \Theta_\pm \). Consider for instance eqn. (3.15) for \( \mathcal{A}_x \) and \( \Theta_- \). Its zero and half grade components determine the \( \mathcal{M} \) components of \( m(-1/2) \) and \( m(-1) \) through

\[ A_{1/2} = [m(-1/2), E], \quad A_0 = [m(-1), E] + \frac{1}{2} [m(-1/2), A_{1/2}] \]  

(3.21)

The same equation (3.15), for grades \(-1/2\) and \(-1\) yields respectively

\[ \partial_x m(-1/2) = [m(-3/2), E] + \frac{1}{2} [m(-1/2), [m(-1), E]] \]

\[ + \frac{1}{2} [m(-1), A_{1/2}] + \frac{1}{3!} [m(-1/2), [m(-1/2), A_{1/2}]] \]

\[ \partial_x m(-1) = -\frac{1}{2} m(-1/2) \partial_x m(-1/2) + \frac{1}{2} \partial_x m(-1/2) m(-1/2) + [m(-2), E] \]

\[ + \frac{1}{2} [m(-1/2), [m(-3/2), E]] + \frac{1}{2} [m(-1), [m(-1), E]] + \frac{1}{2} [m(-3/2), A_{1/2}] \]

\[ + \frac{1}{3!} [m(-1/2), [m(-1/2), [m(-1), E]]] + \frac{1}{3!} [m(-1/2), [m(-1), A_{1/2}]] \]

\[ + \frac{1}{3!} [m(-1), [m(-1/2), A_{1/2}]] + \frac{1}{4!} [m(-1/2), [m(-1/2), [m(-1/2), A_{1/2}]]] \]  

(3.22)

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and determines the kernel, \( K \), components of \( m(-1/2) \) and \( m(-1) \) (which, in principle is non local) together with the image, \( \mathcal{M} \), components of \( m(-3/2) \) and \( m(-2) \). From \( A_0 \) and \( A_{1/2} \) given by (2.5) we find for the super mKdV

\[
m(-1/2) = \alpha_1 G_2^{(-1/2)} + \alpha_2 F_1^{(-1/2)},
\]

\[
m(-1) = \beta_1 M_1^{(-1)} + \beta_2 K_1^{(-1)} + \beta_3 K_2^{(-1)}
\]

(3.23)

where

\[
\alpha_1 = -\frac{1}{2} \bar{\psi}, \quad \alpha_2 = -\frac{1}{2} \chi, \quad \beta_1 = \frac{1}{2} (u - \frac{1}{2} \bar{\psi} \chi)
\]

\[
\beta_2 = \frac{1}{4} \int (\bar{\psi} \partial_x \bar{\psi} - \chi \partial_x \chi) dx - \frac{1}{2} \int u^2 dx \quad \beta_3 = \frac{1}{4} \int (\bar{\psi} \partial_x \bar{\psi} + \chi \partial_x \chi) dx,
\]

(3.24)

and \( \partial_x \chi = u \bar{\psi} \). It also leads to the eqn. for \( \eta \),

\[
2 \partial_x \eta = u^2 - \partial_x u - \bar{\psi} \partial_x \bar{\psi}.
\]

(3.25)

The full dressing transformation \( \Theta_{\pm} \) is then determined by considering higher grade terms of (3.15) with (3.20) and (3.16).

From eqn. (3.18) the following \( \tau \)-functions are found,

\[
\tau_0 = e^\nu = <\lambda_0 | T_0 g T_0^{-1} | \lambda_0 >,
\]

\[
\tau_1 = e^{\phi + \nu} = <\lambda_1 | T_0 g T_0^{-1} | \lambda_1 >,
\]

\[
\tau_2 = \frac{1}{2} (\bar{\psi} - \chi) e^\nu = <\lambda_0 | G_1^{(1/2)} T_0 g T_0^{-1} | \lambda_0 >,
\]

\[
\tau_3 = \frac{1}{2} (\bar{\psi} + \chi) e^{\phi + \nu} = <\lambda_1 | G_1^{(1/2)} T_0 g T_0^{-1} | \lambda_1 >
\]

(3.26)

where \( \lambda_i, \ i = 0, 1 \) denote the first two fundamental weights of \( \hat{sl}(2,1) \) satisfying

\[
\hat{c} | \lambda_i > = | \lambda_i >, \quad M_2^{(0)} | \lambda_i > = \delta_{i,1} | \lambda_i >
\]

(3.27)

and are annihilated by the positive grade generators. The soliton solution is therefore given in terms of representations of the \( sl(2,1) \) affine Lie super algebra,

\[
\phi = ln \left( \frac{\tau_1}{\tau_0} \right), \quad \bar{\psi} = \frac{\tau_3}{\tau_1} + \frac{\tau_2}{\tau_0},
\]

(3.28)

For the relativistic sinh-Gordon \( t_n = t_{-1} \) whilst for the non relativistic mKdV model \( t_n = t_3 \), \( u = -\partial_x \phi \) and \( \eta = -\partial_x \nu \).

The soliton solutions are classified in terms of the constant element \( g \) in (3.18) which is constructed in terms of eigenvectors of \( E^{(n)} \), i.e.,

\[
[E^{(2n+1)}, F_{\pm}(\gamma)] = \pm 2 \gamma^{2n+1} F_{\pm}(\gamma)
\]

(3.29)

where

\[
F_-(\gamma) = \sum_{n \in \mathbb{Z}} M_1^{(2n+1)} \gamma^{-2n-1} + (M_2^{(2n)} - \frac{1}{2} \hat{c} \delta_{n,0}) \gamma^{-2n},
\]

\[
F_+(\gamma) = \sum_{n \in \mathbb{Z}} G_1^{(2n+1/2)} \gamma^{-2n} + G_2^{(2n+3/2)} \gamma^{-2n-1}
\]

(3.30)
3.1 Two Vertex Solution

Consider as an illustration the case where
\[ g = e^{b_1 F_-(\gamma_1)} e^{c_1 F_+(\gamma_3)} \]
where \( b_1 \) and \( c_1 \) are bosonic and fermionic coefficients respectively. By virtue of (3.29) the explicit space-time dependence on the r.h.s. of (3.26) is
\[ T_0 g T_0^{-1} = e^{b_1 \rho_1^+(\gamma_1) F_-(\gamma_1)} e^{c_1 \rho_3^- (\gamma_3) F_+(\gamma_3)} \]
where
\[ \rho_i^\pm = e^{\pm (2 \gamma_i x + 2 \gamma_i^2 t_{2n+1})} \]
The \( \tau \) functions (3.26) become
\[ \tau_0 = e^{\nu_0} = 1 - \frac{1}{2} b_1 \rho_1^+ + b_1 c_1 \rho_1^+ \rho_3^- < \lambda_0 | F_- (\gamma_1) F_+ (\gamma_3) | \lambda_0 >, \]
\[ \tau_1 = e^{\phi + \nu} = 1 + \frac{1}{2} b_1 \rho_1^+ + b_1 c_1 \rho_1^+ \rho_3^- < \lambda_1 | F_- (\gamma_1) F_+ (\gamma_3) | \lambda_1 >, \]
\[ \tau_2 = \frac{1}{2} (\bar{\psi} - \chi) e^{\nu} = c_1 \rho_3^- \gamma_3 + b_1 c_1 \rho_1^+ \rho_3^- < \lambda_0 | G_1^{(1/2)} F_- (\gamma_1) F_+ (\gamma_3) | \lambda_0 >, \]
\[ \tau_3 = \frac{1}{2} (\bar{\psi} + \chi) e^{\phi + \nu} = c_1 \rho_3^- \gamma_3 + b_1 c_1 \rho_1^+ \rho_3^- < \lambda_1 | G_1^{(1/2)} F_- (\gamma_1) F_+ (\gamma_3) | \lambda_1 >, \]
where the matrix elements can be evaluated from the representation theory of the affine \( \hat{sl}(2,1) \) super algebra (3.26) yielding
\[ < \lambda_i | F_- (\gamma_1) F_+ (\gamma_3) | \lambda_i > = 0 \]
\[ < \lambda_i | G_1^{(1/2)} F_- (\gamma_1) F_+ (\gamma_3) | \lambda_i > = \frac{\gamma_3 (\gamma_1 + \gamma_3)}{2 (\gamma_1 - \gamma_3)} (1 - 2 \delta_{i,1}), \quad i = 0, 1 \]
Since, in our formulation, the super mKdV belongs to the same hierarchy as the super sinh-Gordon, its solution is determined using the same vertex functions (3.30) and substituting \( \gamma_i^{-1} t_{-1} \) by \( \gamma_i^3 t_3 \) together with the change of variables \( u = -\partial_x \phi, \eta = -\partial_x \nu \). The explicit space-time dependence according to the sinh-Gordon or mKdV are given respectively by,
\[ \rho_i^{S-G} = e^{\pm (2 \gamma_i x + 2 \gamma_i^2 t_{-1})} \quad \rho_i^{mKdV} = e^{\pm (2 \gamma_i x + 2 \gamma_i^3 t_3)} \]
The corresponding two-vertex solution for the super mKdV is then given by
\[ u = -\partial_x \phi = -b_1 \gamma_1 \rho_1^+ \left( \frac{1}{1 + \frac{1}{2} b_1 \rho_1^+} + \frac{1}{1 - \frac{1}{2} b_1 \rho_1^+} \right), \]
\[ \bar{\psi} = c_1 \rho_3^- \gamma_3 - b_1 c_1 \rho_1^+ \rho_3^- \sigma_{1,3} \frac{1 + \frac{1}{2} b_1 \rho_1^+}{1 - \frac{1}{2} b_1 \rho_1^+} + c_1 \rho_3^- \gamma_3 + b_1 c_1 \rho_1^+ \rho_3^- \sigma_{1,3} \frac{1 - \frac{1}{2} b_1 \rho_1^+}{1 - \frac{1}{2} b_1 \rho_1^+}, \]
\[ \eta = -\partial_x \nu = \frac{b_1 \gamma_1 \rho_1^+}{1 - \frac{1}{2} b_1 \rho_1^+}, \quad \sigma_{1,3}(\gamma_1, \gamma_3) = \frac{\gamma_3 (\gamma_1 + \gamma_3)}{2 (\gamma_1 - \gamma_3)} \]
For the particular case where \( \gamma_1 = -\gamma_3 = k, b_1 = -2 \) and \( c_1 = -\frac{\gamma}{k} \) our solution for \( u \) and \( \bar{\psi} \) (3.37) coincide (after scaling \( t_3 \)) with the one obtained in [4] by extending the bilinear approach to the supersymmetric mKdV equation.
3.2 Four Vertex Solution

We now explicit display the general 4-vertex solution where

$$g = e^{b_1 F_-(\gamma_1)} e^{b_2 F_-(\gamma_2)} e^{c_1 F_+(\gamma_3)} e^{c_2 F_+(\gamma_4)}$$

(3.38)

where $b_i$ and $c_i$, $i = 1, 2$ are bosonic and fermionic coefficients respectively and

$$T_0 g T_0^{-1} = e^{b_1 \rho_+^+(\gamma_1) F_-(\gamma_1)} e^{b_2 \rho_+^+(\gamma_2) F_-(\gamma_2)} e^{c_1 \rho_-^-(\gamma_3) F_+(\gamma_3)} e^{c_2 \rho_-^-(\gamma_4) F_+(\gamma_4)}$$

(3.39)

The $\tau$ functions in (3.26) become

$$\tau_0 = e^\nu = 1 - \frac{1}{2} b_1 \rho_1^+ - \frac{1}{2} b_2 \rho_2^+ + b_1 b_2 \rho_1^+ \rho_2^+ \alpha_{1,2}$$

$$+ c_1 c_2 \rho_3^- \rho_4^- (\beta_{3,4} - b_1 \rho_1^+ \delta_{1,3,4} - b_2 \rho_2^+ \delta_{2,3,4} + b_1 b_2 \rho_1^+ \rho_2^+ \theta_{1,2,3,4}),$$

$$\tau_1 = e^{\phi^+\nu} = 1 + \frac{1}{2} b_1 \rho_1^+ + \frac{1}{2} b_2 \rho_2^+ + b_1 b_2 \rho_1^+ \rho_2^+ \alpha_{1,2}$$

$$+ c_1 c_2 \rho_3^- \rho_4^- (\beta_{3,4} + b_1 \rho_1^+ \delta_{1,3,4} + b_2 \rho_2^+ \delta_{2,3,4} + b_1 b_2 \rho_1^+ \rho_2^+ \theta_{1,2,3,4}),$$

$$\tau_2 = \frac{1}{2} (\bar{\psi} - \chi) e^\nu = c_1 \rho_3^- (\gamma_3 + b_1 \rho_1^+ \sigma_{1,3} + b_2 \rho_2^+ \sigma_{2,3} + b_1 b_2 \rho_1^+ \rho_2^+ \lambda_{1,2,3})$$

$$+ c_2 \rho_4^- (\gamma_4 + b_1 \rho_1^+ \sigma_{1,4} + b_2 \rho_2^+ \sigma_{2,4} + b_1 b_2 \rho_1^+ \rho_2^+ \lambda_{1,2,4}),$$

$$\tau_3 = \frac{1}{2} (\bar{\psi} + \chi) e^{\phi^+\nu} = c_1 \rho_3^- (\gamma_3 - b_1 \rho_1^+ \sigma_{1,3} - b_2 \rho_2^+ \sigma_{2,3} + b_1 b_2 \rho_1^+ \rho_2^+ \lambda_{1,2,3})$$

$$+ c_2 \rho_4^- (\gamma_4 - b_1 \rho_1^+ \sigma_{1,4} - b_2 \rho_2^+ \sigma_{2,4} + b_1 b_2 \rho_1^+ \rho_2^+ \lambda_{1,2,4}).$$

(3.40)

where the coefficients are given by

$$\alpha_{1,2} = \frac{1}{4} \frac{(\gamma_1 - \gamma_2)^2}{(\gamma_1 + \gamma_2)^2},$$

$$\beta_{3,4} = \frac{\gamma_3 \gamma_4}{(\gamma_3 + \gamma_4)^2},$$

$$\delta_{j,3,4} = \frac{\gamma_3 \gamma_4}{2 (\gamma_3 + \gamma_4)^2} \frac{(\gamma_3 - \gamma_4)(\gamma_j + \gamma_3)(\gamma_j + \gamma_4)}{(\gamma_j - \gamma_3)(\gamma_j - \gamma_4)} (j = 1, 2),$$

$$\sigma_{j,k} = \frac{\gamma_k (\gamma_j + \gamma_k)}{2 (\gamma_j - \gamma_k)} (j = 1, 2) (k = 3, 4),$$

$$\lambda_{1,2,j} = \frac{\gamma_j (\gamma_1 - \gamma_2)^2}{4 (\gamma_1 + \gamma_2)^2} \frac{(\gamma_1 + \gamma_j)(\gamma_2 + \gamma_j)}{(\gamma_1 - \gamma_j)(\gamma_2 - \gamma_j)} (j = 3, 4),$$

\footnote{we have used the Mathematica program of ref. [10]}
\[ \theta_{1,2,3,4} = \frac{\gamma_3 \gamma_4 (\gamma_1 - \gamma_2)^2 (\gamma_1 + \gamma_3) (\gamma_2 + \gamma_3) (\gamma_3 - \gamma_4) (\gamma_1 + \gamma_4) (\gamma_2 + \gamma_4)}{4 (\gamma_1 + \gamma_2)^2 (\gamma_1 - \gamma_3) (\gamma_2 - \gamma_3) (\gamma_3 + \gamma_4)^2 (\gamma_1 - \gamma_4) (\gamma_2 - \gamma_4)}. \]  

(3.41)

The solution for the super sinh-Gordon is then given as

\[ \phi = \ln \left( \frac{1 + \frac{1}{2} b_1 \rho_1^+ + \frac{1}{2} b_2 \rho_2^+ + b_1 b_2 \rho_1^+ \rho_2^+ \alpha_{1,2}}{1 - \frac{1}{2} b_1 \rho_1^- - \frac{1}{2} b_2 \rho_2^- + b_1 b_2 \rho_1^- \rho_2^- \alpha_{1,2}} \right) + c_1 c_2 \rho_3 \rho_1^- \left( \beta_{3,4} + b_1 \rho_1^+ \delta_{1,3,4} + b_2 \rho_2^+ \delta_{2,3,4} + b_1 b_2 \rho_1^+ \rho_2^+ \theta_{1,2,3,4} \right) \]

\[ + \ c_1 c_2 \rho_3 \rho_1^+ \left( \beta_{3,4} + b_1 \rho_1^- \delta_{1,3,4} - b_2 \rho_2^- \delta_{2,3,4} + b_1 b_2 \rho_1^- \rho_2^- \theta_{1,2,3,4} \right). \]  

(3.42)

\[ \bar{\psi} = c_1 \rho_3 \left( \gamma_3 - b_1 \rho_1^+ \sigma_{1,3} - b_2 \rho_2^+ \sigma_{2,3} + b_1 b_2 \rho_1^+ \rho_2^+ \lambda_{1,2,3} \right) \]

\[ + \ c_2 \rho_4 \left( \gamma_4 - b_1 \rho_1^+ \sigma_{1,4} - b_2 \rho_2^+ \sigma_{2,4} + b_1 b_2 \rho_1^+ \rho_2^+ \lambda_{1,2,4} \right) \]

\[ + \ c_1 \rho_3 \left( \gamma_3 + b_1 \rho_1^+ \sigma_{1,3} + b_2 \rho_2^+ \sigma_{2,3} + b_1 b_2 \rho_1^+ \rho_2^+ \lambda_{1,2,3} \right) \]

\[ + \ c_2 \rho_4 \left( \gamma_4 + b_1 \rho_1^+ \sigma_{1,4} + b_2 \rho_2^+ \sigma_{2,4} + b_1 b_2 \rho_1^+ \rho_2^+ \lambda_{1,2,4} \right). \]  

(3.43)

\[ \nu = \ln \left( 1 - \frac{1}{2} b_1 \rho_1^+ - \frac{1}{2} b_2 \rho_2^+ + b_1 b_2 \rho_1^+ \rho_2^+ \alpha_{1,2} \right) \]

\[ + \ c_1 c_2 \rho_3 \rho_1^- \left( \beta_{3,4} - b_1 \rho_1^+ \delta_{1,3,4} - b_2 \rho_2^+ \delta_{2,3,4} + b_1 b_2 \rho_1^+ \rho_2^+ \theta_{1,2,3,4} \right). \]  

(3.44)

where \( \rho_i^\pm = \rho_i^{\pm, S-G} \).

The solitary solutions for the super mKdV are obtained by replacing \( \rho_i^\pm = \rho_i^{\pm, mKdV} \) in (3.36) and writing \( u = -\partial_x \phi \) and \( \eta = -\partial_t \nu \). We have verified that our solution for the four-vertex super mKdV agrees with the one found in ref. [4] when \( \gamma_1 = -\gamma_3 = k_1, \gamma_2 = -\gamma_4 = k_2, b_1 = b_2 = -2 \) and \( c_i = -\frac{\delta}{k_i}, \ i = 1, 2 \) (after scaling \( t_3 \)).

It becomes clear that the solitary solutions are classified in terms of the number and in terms of the type of vertices employed in constructing \( g \). Other integrable equations within the same hierarchy and associated to higher grade time evolution, \( t_{2n+1} \), can be constructed from the zero curvature condition (2.6) by replacing \( E^{(1)} \) by \( E^{(2n+1)} \) (given in (2.3)). They all share the same solitary solutions for fields \( u(x, t_{2n+1}) = -\partial_x \phi, \ \eta(x, t_{2n+1}) = -\partial_t \nu, \ \bar{\psi}(x, t_{2n+1}) \) with \( \rho_i^\pm \) given by \( \rho_i^\pm = e^{\pm 2\gamma_i x + 2\gamma_i^2 t_{2n+1}} \).
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4 Appendix A

We now discuss how to implement the relevant subalgebra of the affine $\hat{sl}(2, 1)$ Kac-Moody algebra in order to construct the integrable hierarchy we are interested in. Consider the $sl(2, 1)$ Lie algebra with generators

$$\{ h_1 = \frac{2\alpha_1 \cdot H}{\alpha_1^2}, \lambda_2 \cdot H, E_{\pm\alpha_1}, E_{\pm\alpha_2}, E_{\pm(\alpha_1 + \alpha_2)} \}$$

(4.45)

where $\alpha_1, \alpha_2$ are the bosonic and fermionic simple roots respectively and $\lambda_2$ is the second fundamental weight. The affine $\hat{sl}(2, 1)$ structure is implemented by extending each generator $T_a \in sl(2, 1)$ to $T_a^{(q)}$ where $[d, T_a^{(q)}] = qT_a^{(q)}$. The relevant subalgebra of the affine $sl(2, 1)$ is constructed as follows. The grade one constant element $E^{(1)}$ be given in (2.3) decomposes the affine algebra into

$$\mathcal{K}_{\text{Bose}} = \{ K_1^{(2n+1)} = -(E_{\alpha_1}^{(n)} + E_{-\alpha_1}^{(n+1)}), \quad K_2^{(2n+1)} = \lambda_2 \cdot H^{(n+1/2)} \}$$

$$\mathcal{M}_{\text{Bose}} = \{ M_1^{(2n+1)} = -E_{\alpha_1}^{(n)} + E_{-\alpha_1}^{(n+1)}, \quad M_2^{(2n)} = h_1^{(n)} \}$$

(4.46)

and the fermionic sector

$$\mathcal{K}_{\text{Fermi}} = F_1^{(2n+3/2)} = (E_{\alpha_1 + \alpha_2}^{(n+1/2)} - E_{\alpha_2}^{(n+1/2)}) + (E_{-\alpha_1 - \alpha_2}^{(n+1/2)} - E_{-\alpha_2}^{(n+1/2)})$$

$$F_2^{(2n+1/2)} = -E_{\alpha_1 + \alpha_2}^{(n/2)} + E_{\alpha_2}^{(n/2)} + (E_{-\alpha_1 - \alpha_2}^{(n+1/2)} - E_{-\alpha_2}^{(n+1/2)})$$

$$\mathcal{M}_{\text{Fermi}} = G_1^{(2n+1/2)} = -E_{\alpha_1 + \alpha_2}^{(n+1/2)} + (E_{\alpha_1}^{(n+1/2)} + E_{\alpha_2}^{(n+1/2)}) + (E_{-\alpha_1 - \alpha_2}^{(n+1/2)} - E_{-\alpha_2}^{(n+1/2)})$$

(4.47)

The algebra is then given by the commutators

$$[K_1^{(2m+1)}, K_1^{(2n+1)}] = \hat{c}(m - n)\delta_{m+n+1,0},$$

$$[K_1^{(2m+1)}, M_1^{(2n+1)}] = -2M_2^{(m+n+1)} - \hat{c}(m + n)\delta_{m+n+1,0},$$

$$[M_2^{(2m)}, M_1^{(2n+1)}] = 2M_2^{(m+2n+1)},$$

$$[M_2^{(2m)}, M_2^{(2n)}] = \hat{c}(m - n)\delta_{m+n,0},$$

$$[M_1^{(2m+1)}, M_1^{(2n+1)}] = \hat{c}(n - m)\delta_{m+n+1,0},$$

$$[K_1^{(2m+1)}, K_2^{(2n+1)}] = 0,$$

$$[K_2^{(2m+1)}, M_1^{(2n+1)}] = 0,$$

$$[M_2^{(2m)}, K_2^{(2n+1)}] = 0,$$

$$[M_2^{(2m)}, M_1^{(2n+1)}] = 2K_1^{(2m+2n+1)}$$

$$[K_2^{(2m+1)}, K_2^{(2n+1)}] = \hat{c}(n - m)\delta_{m+n+1},$$

(4.48)

and

$$[K_1^{(2m+1)}, F_1^{(2n+3/2)}] = -F_2^{(2(m+n+1)+1/2)},$$

$$[K_1^{(2m+1)}, F_2^{(2n+1/2)}] = -F_1^{(2(m+n)+3/2)},$$

$$[K_1^{(2m+1)}, G_1^{(2n+1/2)}] = G_2^{(2(m+n)+3/2)},$$

$$[K_1^{(2m+1)}, G_2^{(2n+3/2)}] = G_1^{(2(m+n)+1/2)},$$

$$[K_1^{(2m+1)}, G_2^{(2n+3/2)}] = G_1^{(2(m+n)+1/2)},$$

$$[K_1^{(2m+1)}, G_2^{(2n+3/2)}] = G_1^{(2(m+n)+1/2)}.$$
\[
[K_2^{(2m+1)}, F_1^{(2n+3/2)}] = F_2^{(2(m+n+1)+1/2)}, \\
[K_2^{(2m+1)}, F_2^{(2n+1/2)}] = F_1^{(2(m+n)+3/2)}, \\
[M_1^{(2m+1)}, F_1^{(2n+3/2)}] = G_1^{(2(m+n)+1/2)}, \\
[M_1^{(2m+1)}, F_2^{(2n+1/2)}] = G_2^{(2(m+n)+3/2)}, \\
[M_2^{(2m)}, F_1^{(2n+3/2)}] = -G_2^{(2(m+n)+3/2)}, \\
[M_2^{(2m)}, F_2^{(2n+1/2)}] = -G_1^{(2(m+n)+1/2)}, \\
[K_2^{(2m+1)}, G_1^{(2n+1/2)}] = G_2^{(2(m+n)+3/2)}, \\
[K_2^{(2m+1)}, G_2^{(2n+3/2)}] = G_1^{(2(m+n)+1/2)}, \\
[M_1^{(2m+1)}, G_1^{(2n+1/2)}] = -F_1^{(2(m+n)+3/2)}, \\
[M_1^{(2m+1)}, G_2^{(2n+3/2)}] = -F_2^{(2(m+n)+1/2)}, \\
[M_2^{(2m)}, G_1^{(2n+1/2)}] = -F_2^{(2(m+n)+1/2)}, \\
[M_2^{(2m)}, G_2^{(2n+3/2)}] = -F_1^{(2(m+n)+3/2)},
\]

(4.49)

and the anticommutators
\[
[F_1^{(2m+3/2)}, F_1^{(2n+3/2)}]_+ = 2(K_2^{(2(m+n+1)+1)} + K_1^{(2(m+n)+1)}), \\
[F_1^{(2m+3/2)}, F_2^{(2n+1/2)}]_+ = \hat{c}(2m - 2n + 1)\delta_{m+n+1,0}, \\
[F_1^{(2m+3/2)}, G_1^{(2n+1/2)}]_+ = 2M_2^{(2(m+n+1))} + \hat{c}(2m + 2n + 1)\delta_{m+n+1,0}, \\
[F_1^{(2m+3/2)}, G_2^{(2n+3/2)}]_+ = -2M_1^{(2m+2n+3)}, \\
[F_2^{(2m+1/2)}, F_1^{(2n+3/2)}]_+ = -2(K_2^{(2(m+2n)+1)} + K_1^{(2(m+2n+1)}), \\
[F_2^{(2m+1/2)}, F_2^{(2n+1/2)}]_+ = 2M_1^{(2m+2n+1)}, \\
[F_2^{(2m+1/2)}, G_1^{(2n+1/2)}]_+ = -2M_2^{(2(m+2n)+2)} - \hat{c}(2m + 2n + 1)\delta_{m+n+1,0}, \\
[G_1^{(2m+1/2)}, G_1^{(2n+1/2)}]_+ = 2(K_2^{(2(m+2n)+1)} - K_1^{(2(m+2n+1)}), \\
[G_1^{(2m+1/2)}, G_2^{(2n+3/2)}]_+ = \hat{c}(2m - 2n - 1)\delta_{m+n+1,0}, \\
[G_2^{(2m+3/2)}, G_1^{(2n+1/2)}]_+ = -2(K_2^{(2(m+2n+3)} - K_1^{(2(m+2n+3)} \right)
\]

(4.50)

where the index \( l \) in \( K_i^{(l)}, M_i^{(l)}, F_i^{(l)} \) and \( G_i^{(l)} \) denote their grade with respect to \( Q \) given in (2.2).

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