Analytic Bethe Ansatz and $T$-system in $C_2^{(1)}$ vertex models

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ABSTRACT

Eigenvalues of the commuting family of transfer matrices are expected to obey the $T$-system, a set of functional relation, proposed recently. Here we obtain the solution to the $T$-system for $C_2^{(1)}$ vertex models. They are compatible with the analytic Bethe ansatz and Yang-Baxterize the classical characters.

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Solvable lattice models in two-dimensions possess a commuting family of the row-to-row transfer matrices [1]. Recently, a set of functional relations (FRs), the $T$-system, are proposed among them [2] for a wide class of models associated with any classical simple Lie algebra or its quantum affine analogue [3,4]. In the QISM terminology [5], the $T$-system relates the transfer matrices with various fusion types in the auxiliary space but acting on a common quantum space. It generalizes earlier FRs [6-9] and enables the calculation of various physical quantities [10]. The structure that underlies the $T$-system is an (short) exact sequence of the finite dimensional modules of the above mentioned algebras [2]. As discussed therein, there is an intriguing connection between the $T$-system, the thermodynamic Bethe ansatz (TBA) and dilogarithm identities, indicating some deep interplay among these subjects.

In this Letter we report the solution to the $C_2(1)$ $T$-system that is compatible with the analytic Bethe ansatz [11,12] and Yang-Baxterizes the classical characters. To explain the problem, let $W_m^a$ ($a = 1, 2, m \in \mathbb{Z}_{\geq 1}$) be the irreducible finite dimensional representation (IFDR) of the quantum affine algebra $U_q(C_2(1))$ ($q$: generic) as sketched in section 3.2 of [2]. As the $C_2$-module, it decomposes as

$$W_m^{(1)} \simeq V_{m\omega_1} \oplus V_{(m-2)\omega_1} \oplus \cdots \oplus \left\{ V_{\omega_m} \begin{cases} m \text{ even} \\ V_{\omega_1} \end{cases} \right., \quad (1)$$

$$W_m^{(2)} \simeq V_{m\omega_2},$$

where $\omega_1, \omega_2$ are the fundamental weights and $V_{\omega}$ denotes the IFDR of $C_2$ with highest weight $\omega$. Thus $\text{dim} W_m^{(1)} = (m+2)(m+4)(m^2 + 6m + 6)/48$ for $m$ even, $= (m + 1)(m + 3)^2(m + 5)/48$ for $m$ odd and $\text{dim} W_m^{(2)} = (m + 1)(m + 2)(2m + 3)/6$. For $W, W' \in \{ W_m^a \mid a = 1, 2, m \in \mathbb{Z}_{\geq 1} \}$, there exists the quantum $R$-matrix $R_{W,W'}(u)$ acting on $W \otimes W'$ and satisfying the Yang-Baxter equation

$$R_{W,W'}(u)R_{W,W''}(u + v)R_{W'',W'}(v) = R_{W'',W'}(v)R_{W,W''}(u + v)R_{W,W'}(u) \quad (2)$$

with $u, v \in \mathbb{C}$ being the spectral parameters. For $W = W' = W_1^{(1)}$, the $R$-matrix has been explicitly written down in [13,14], from which all the other $R_{W,W'}$ may be constructed via the fusion procedure [15]. $W_m^{(a)}$ is an analogue of the $m$-fold symmetric tensor representation of $W_1^{(a)}$. The transfer matrix with auxiliary space $W_m^{(a)}$ is then defined by

$$T_m^{(a)}(u) = \text{Tr}_{W_m^{(a)}} \left( R_{W_m^{(a)}}, W_m^{(p)}(u-w_1) \cdots R_{W_m^{(a)}, W_m^{(s)}}(u-w_N) \right) \quad (3)$$

up to an overall scalar multiple. Here $N \in 2\mathbb{Z}$ denotes the system size, $w_1, \ldots, w_N$ are complex parameters representing the inhomogeneity, $p = 1, 2$ and $s \in \mathbb{Z}_{\geq 1}$. We say that (3) is the row-to-row transfer matrix with fusion type $W_m^{(a)}$ acting on the quantum space $(W_s^{(p)})^\otimes N$. We shall reserve the letters $p$ and $s$ for this meaning throughout. Thanks to the Yang-Baxter equation (2), the transfer matrices (3) form a commuting family

$$[T_m^{(a)}(u), T_m^{(a')} (u')] = 0. \quad (4)$$
We shall write the eigenvalues of $T_m^{(a)}(u)$ as $\Lambda_m^{(a)}(u)$. Our goal is to find an explicit formula for them.

For the purpose, we postulate the (unrestricted) $T$-system [2]:

$$T_{2m}^{(1)}(u - \frac{1}{2})T_{2m}^{(1)}(u + \frac{1}{2}) = T_{2m+1}^{(1)}(u)T_{2m-1}^{(1)}(u)$$

$$+ g_m^{(1)}(u)T_m^{(2)}(u - \frac{1}{2})T_m^{(2)}(u + \frac{1}{2}), \quad (5a)$$

$$T_{2m+1}^{(1)}(u - \frac{1}{2})T_{2m+1}^{(1)}(u + \frac{1}{2}) = T_{2m+2}^{(1)}(u)T_{2m}^{(1)}(u)$$

$$+ g_{2m+1}^{(1)}(u)T_{m+1}^{(2)}(u)T_{m+1}^{(2)}(u), \quad (5b)$$

$$T_{m}^{(2)}(u - 1)T_{m}^{(2)}(u + 1) = T_{m+1}^{(2)}(u)T_{m-1}^{(2)}(u) + g_m^{(2)}(u)T_{2m}^{(1)}(u). \quad (5c)$$

Here $g_m^{(a)}(u)$ is a scalar function that depends on $W_s^{(p)}$ and overall normalization of the transfer matrices. Due to (4) the eigenvalues $\Lambda_m^{(a)}(u)$ also obey the same system as (5), which can be solved successively yielding an expression of the $\Lambda_m^{(a)}(u)$ in terms of $\Lambda_{1}^{(1)}(u + \text{shift})$ and $\Lambda_{1}^{(2)}(u + \text{shift})$. Thus the first step to achieve the goal is to find the formula for the eigenvalues $\Lambda_{1}^{(1)}(u)$ and $\Lambda_{1}^{(2)}(u)$. This we do by the analytic Bethe ansatz. The method consists of assuming the so-called “dressed vacuum form” for the eigenvalues and determining the unknown parts thereby introduced from some functional properties and asymptotic behaviors. See [12,16] for the detail. To present the results for our problem, we prepare a few notations. Let $\alpha_1, \alpha_2$ be the simple roots of $C_2$. We take $\alpha_2$ to be a long root and normalize it as $\langle \alpha_2|\alpha_2 \rangle = 2$ via the bilinear form (|). Then one has $\langle \alpha_a|\omega_b \rangle = \delta_{ab}/t_a$, where $t_1 = 2$, $t_2 = 1$. We set

$$\phi(u) = \prod_{j=1}^{N}[u - w_j], \quad [u] = q^u - q^{-u},$$

$$\phi_m^{(a)}(u) = \phi(u + \frac{m-1}{t_a})\phi(u + \frac{m-3}{t_a})\cdots\phi(u - \frac{m-1}{t_a}) \quad a = 1, 2, m \in \mathbb{Z}_{\geq 1}, \quad (6)$$

$$Q_a(u) = \prod_{j=1}^{N_a}[u - iu_j^{(a)}] \quad a = 1, 2.$$

Here $N_1$, $N_2$ are non-negative integers such that $\omega^{(p)} \overset{\text{def}}{=} N s \omega_p - N_1 \alpha_1 - N_2 \alpha_2$ is a non-negative weight. The numbers $\{u_j^{(a)} \mid a = 1, 2, 1 \leq j \leq N_a\}$ are the solutions to the Bethe ansatz equation [16]

$$- \frac{\phi(iu_k^{(a)} + \frac{s_w}{t_{p_a}}\delta_{p_a})}{\phi(iu_k^{(a)} - \frac{s_w}{t_{p_a}}\delta_{p_a})} = \prod_{b=1}^{2}Q_b(iu_k^{(a)} + (\alpha_a|\alpha_b)) / Q_b(iu_k^{(a)} - (\alpha_a|\alpha_b)), \quad a = 1, 2, 1 \leq k \leq N_a. \quad (7)$$

Under these definitions, the result of the analytic Bethe ansatz reads as follows.

Case $p = 1$;
\[ \Lambda_1^{(1)}(u) = \phi_s^{(1)}(u+3)\phi_s^{(1)}(u+1) \frac{Q_1(u-\frac{3}{2})}{Q_1(u+\frac{1}{2})} + \phi_s^{(1)}(u+2)\phi_s^{(1)}(u) \frac{Q_1(u+\frac{3}{2})}{Q_1(u+\frac{3}{2})} + \phi_s^{(1)}(u+3)\phi_s^{(1)}(u) \left( \frac{Q_1(u+\frac{3}{2})Q_2(u-\frac{1}{2})}{Q_1(u+\frac{1}{2})Q_2(u+\frac{3}{2})} + \frac{Q_1(u+\frac{3}{2})Q_2(u+\frac{3}{2})}{Q_1(u+\frac{3}{2})Q_2(u+\frac{3}{2})} \right), \] (8a)

\[ \Lambda_1^{(2)}(u) = \phi_s^{(1)}(u+\frac{5}{2}) \left( \frac{Q_2(u-1)}{Q_2(u+1)} + \frac{Q_1(u)Q_2(u+3)}{Q_1(u+2)Q_2(u+1)} \right) + \phi_s^{(1)}(u+\frac{11}{2}) \left( \frac{Q_2(u+4)}{Q_2(u+2)} + \frac{Q_1(u+3)Q_2(u)}{Q_1(u+1)Q_2(u+2)} \right), \] (8b)

Case \( p = 2; \)

\[ \Lambda_1^{(1)}(u) = \phi_s^{(2)}(u+\frac{5}{2})\phi_s^{(2)}(u+\frac{3}{2}) \left( \frac{Q_1(u-\frac{1}{2})}{Q_1(u+\frac{1}{2})} + \frac{Q_1(u+\frac{3}{2})Q_2(u-\frac{1}{2})}{Q_1(u+\frac{1}{2})Q_2(u+\frac{3}{2})} \right) + \phi_s^{(2)}(u+\frac{7}{2}) \left( \frac{Q_1(u+\frac{3}{2})Q_2(u+\frac{1}{2})}{Q_1(u+\frac{1}{2})Q_2(u+\frac{3}{2})} + \frac{Q_1(u+\frac{3}{2})Q_2(u+\frac{3}{2})}{Q_1(u+\frac{3}{2})Q_2(u+\frac{3}{2})} \right), \] (8c)

\[ \Lambda_1^{(2)}(u) = \phi_s^{(2)}(u+3)\phi_s^{(2)}(u+2) \frac{Q_2(u-1)}{Q_2(u+1)} + \phi_s^{(2)}(u+1)\phi_s^{(2)}(u) \frac{Q_2(u+4)}{Q_2(u+2)} + \phi_s^{(2)}(u+3)\phi_s^{(2)}(u) \left( \frac{Q_1(u)Q_2(u+3)}{Q_1(u+2)Q_2(u+1)} + \frac{Q_1(u)Q_1(u+3)}{Q_1(u+1)Q_1(u+2)} \right) \] (8d)

We employ the convention such that the eigenvalue of \( \hat{R}_{W_{1}^{(1)},W_{1}^{(1)}}(u) \) on the highest component \( V_{2\Lambda_1} \) is \( [u+3][u+1] \) and let the overall normalization of \( \Lambda_1^{(a)}(u) \) as specified by (8). (The common factor \( \phi_s^{(2)}(u+\frac{3}{2}) \) in (8c) has been attached so as to simplify the forthcoming formula (12).) The \( \Lambda_1^{(a)}(u) \) consists of \( \dim W_1^{(a)} = 4,5 \) \((a = 1,2)\) terms and its pole-free conditions are given by (7) in accordance with the analytic Bethe ansatz. The formulas (8) coincide with those in [16,17] for some special cases. In particular, ratio of \( Q_a \)'s in \( \Lambda_1^{(1)}(u) \) are just those appearing in [16] for the \( C_2^{(1)} \) vertex model with \( W_{s}^{(p)} = W_{1}^{(1)} \) (upon some convention adjustment). Namely, the \( Q_a \)-part is determined only from the auxiliary space choice, while the quantum space dependence enters \( \phi_s^{(p)} \)-part. This is also the case in the formula eq.(3.17) of [8] for the \( sl(n) \) case. Similary, \( Q_a \)-part in \( \Lambda_1^{(2)}(u) \) are those appearing in the \( B_2^{(1)} \) case of [16] due to the equivalence \( C_2 \simeq B_2 \).

To proceed to \( \Lambda_m^{(a)}(u) \) with higher \( m \), we introduce a few more notations.

\[ G(u) = \begin{cases} \phi_s^{(1)}(u)G(u) & \text{for } a = 1, \\
G(u) & \text{for } a = 2, \end{cases} \quad H_a(u) = \begin{cases} H(u) & \text{for } a = 1, \\
\phi_s^{(2)}(u)H(u) & \text{for } a = 2, \end{cases} \]

\[ G(u) = \frac{Q_2(u+\frac{1}{2})Q_2(u-\frac{1}{2})}{Q_1(u+\frac{1}{2})Q_1(u-\frac{1}{2})}, \quad H(u) = \frac{Q_1(u)}{Q_2(u+1)Q_2(u-1)}. \] (9)
We consider the $T$-system (5) for $\Lambda^{(a)}_m(u)$ with the initial condition for $m = 1$ as (8) and

$$\Lambda_0^{(1)}(u) = \phi_s^{(1)}(u + 5/2)\phi_s^{(1)}(u + 1/2), \quad \Lambda_0^{(2)}(u) = \phi_s^{(1)}(u + 3/2) \quad \text{for} \quad p = 1,$$

$$\Lambda_0^{(1)}(u) = \Lambda_0^{(2)}(u) = \phi_s^{(2)}(u + 1)\phi_s^{(2)}(u + 2) \quad \text{for} \quad p = 2. \quad (10)$$

Then our main result is

**Theorem.** The functions

$$\Lambda^{(1)}_m(u) = Q_1(u - \frac{m}{2})Q_1(u + \frac{m}{2} + 3) \sum_{0 \leq i \leq j \leq m} \sum_{l = \left[\frac{i+1}{2}\right]}^{\left[\frac{j}{2}\right]} \sum_{k = \left[\frac{l+1}{2}\right]} G_p(u + \frac{m + 5}{2} - i) G_p(u + \frac{m + 1}{2} - j) H_p(u + \frac{m}{2} - 2l + 2) H_p(u + \frac{m}{2} - 2k + 1),$$

$$\Lambda^{(2)}_m(u) = Q_2(u - m)Q_2(u + m + 3) \sum_{j=0}^{\left[\frac{m}{2}\right]} \sum_{l=0}^{\left[\frac{m}{2}\right]} \sum_{k=\left[\frac{l+1}{2}\right]} G_p(u + m + \frac{3}{2} - j) H_p(u + m + 2l + 2) H_p(u + m - 2k + 1) \quad (11)$$

are the solutions to the $T$-system (5) with the initial condition (8), (10) and $g^{(a)}_m(u)$ given by

$$g^{(a)}_m(u) = \begin{cases} \phi_s^{(p)}(u + \frac{m}{t_p} + 3)\phi_s^{(p)}(u - \frac{m}{t_p}) & \text{if } a = p \\ 1 & \text{otherwise} \end{cases} \quad (12)$$

The symbol $[x]$ in (11) denotes the greatest integer not exceeding $x$ and should not be confused with the one in (6). The function (12) satisfies $g^{(a)}_m(u - \frac{1}{t_a})g^{(a)}_m(u + \frac{1}{t_a}) = g^{(a)}_{m+1}(u)g^{(a)}_{m-1}(u)$ in accordance with eq.(3.18) of [2] (with a slight normalization change in $u$). The theorem can be proved by comparing the coefficients of $\phi_s^{(a)}$ factors on both sides of the $T$-system. In particular, the check essentially reduces to the case $p = s = 1$. A similar formula to (11) is available for $sl(n)$ case in [8].

$\Lambda^{(a)}_m(u)$ (11) Yang-Baxterizes the character of $W^{(a)}_m$ viewed as a $C_2$-module as in (1). Namely, it contains $\dim W^{(a)}_m$ terms and tends to the latter in the “braid limit”
as follows.

\[
\lim_{u \to \infty, \text{ \textstyle |q| > 1}} q^{-\psi_a} \Lambda_m^{(a)}(u) = \chi_m^{(a)}(q^{(\omega(p), \alpha_1)}, q^{(\omega(p), \alpha_2)}),
\]

\[
\psi_a = s(2Nu + 3N - 2 \sum_{j=1}^{N} a_j) \min (1, \frac{t_q}{t_p}),
\]

\[
\chi_m^{(1)}(z_1, z_2) = \sum_{0 \leq i \leq j \leq m} \sum_{l=0}^{\frac{2m-2i-2j}{2}} \sum_{k=\frac{1}{2} \left\lfloor \frac{2}{l} \right\rfloor} z_1^{2m-2i-2j} z_2^{m-2i-2j} (m-l-2k)
\]

\[
\chi_m^{(2)}(z_1, z_2) = \sum_{j=0}^{2m} \sum_{l=0}^{\frac{2m-2j}{2}} \sum_{k=\frac{1}{2} \left\lfloor \frac{2}{l} \right\rfloor} z_1^{2m-2j} z_2^{m-2j} \frac{1}{\text{ch } V_{m\omega_1} + \cdots + \text{ch } V_{m\omega_m}} \begin{cases} 
1 & m \text{ even} \\
\text{ch } V_{\omega_1} & m \text{ odd}
\end{cases},
\]

\[
\chi_m^{(1/2)} = \chi_m^{(1)} + \chi_m^{(2)},
\]

\[
\chi_m^{(1)} = \chi_m^{(1)} + \chi_m^{(2)} \chi_m^{(1)},
\]

\[
\chi_m^{(2)} = \chi_m^{(1)} \chi_m^{(2)} + \chi_m^{(1)}
\]

Here, \( \chi_m^{(a)} \) is a simple corollary of the above theorem.

In [2,10], \( \chi_m^{(a)} \) was denoted by \( Q_m^{(a)} \) and (14) was called the \( Q \)-system. As shown therein, the combinations \( y_m^{(1)}(u) = \frac{\eta^{(1)}(u)\Lambda_m^{(2)}(u)}{\Lambda_m^{(1)}(u)\Lambda_m^{(1)}(u)} \) etc from (5) yield a solution to the \( C_2^{(1)} \) \( Y \)-system [19], the TBA equation in high temperature limit:

\[
y_m^{(1)}(u) = \frac{1 + y_m^{(2)}(u)}{(1 + y_m^{(2)}(u-1))(1 + y_m^{(1)}(u))},
\]

\[
y_m^{(2)}(u) = \frac{1}{(1 + y_m^{(2)}(u-1))(1 + y_m^{(1)}(u))},
\]

\[
y_m^{(1)}(u) = \frac{1 + y_m^{(1)}(u)(1 + y_m^{(1)}(u + \frac{1}{2}))(1 + y_m^{(1)}(u - \frac{1}{2}))}{(1 + y_m^{(2)}(u-1))(1 + y_m^{(2)}(u))}.
\]

Acknowledgement

The author thanks Junji Suzuki for a useful discussion and critical reading of the manuscript.
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