PARTICLE RELABELLING SYMMETRIES AND NOETHER’S THEOREM FOR VERTICAL SLICE MODELS

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Abstract. We consider the variational formulation for vertical slice models introduced in Cotter and Holm (Proc Roy Soc, 2013). These models have a Kelvin circulation theorem that holds on all materially-transported closed loops, not just those loops on isosurfaces of potential temperature. Potential vorticity conservation can be derived directly from this circulation theorem. In this paper, we show that this property is due to these models having a relabelling symmetry for every single diffeomorphism of the vertical slice that preserves the density, not just those diffeomorphisms that preserve the potential temperature. This is developed using the methodology of Cotter and Holm (Foundations of Computational Mathematics, 2012).

1. Introduction. Vertical slice models are models of 3D fluids that assume that all fields are independent of \( y \), except for the (potential) temperature, which is assumed to take the form \( \theta(x, y, z, t) = \theta_S(x, z, t) + sy \), where \( s \) is a time-independent coefficient, this allows for a model of North-South temperature gradient \( s \). These models provide a simplification of the equations that allows to study geophysical phenomena such as frontogenesis in idealised geometries. They also provide a useful testbed for numerical algorithms for atmospheric dynamical cores, since they only require computation with 2D data, and so can be run quickly on a desktop computer.

A hierarchy of vertical slice models have been used to study the formation and evolution of fronts. The vertical slice non-hydrostatic incompressible Boussinesq equations, the hydrostatic Boussinesq equations and the corresponding semi-geostrophic equations all exhibit frontogenesis, whilst representing solutions of the full 3D equations. As summarised in [4], the semi-geostrophic equations have an optimal transport interpretation. The optimal transport formulation proves that geostrophic and hydrostatic balance can be achieved while respecting Lagrangian

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conservation properties. The optimal transport formulation has also been used to
develop numerical algorithms for the equations in Lagrangian form, that form
fronts even at low resolution. These solutions provide a useful comparison for stan-
dard Eulerian numerical algorithms for the non-hydrostatic incompressible Boussi-
nesq equations, by considering their solutions in the semi-geostrophic limit, as de-
scribed in [9] who found that the limiting Eulerian solutions deviated from the
semi-geostrophic solution, suggesting that fronts need some extra parameterisation
in Eulerian models. The limiting Eulerian numerical solutions satisfied geostrophic
and hydrostatic balance to the expected extent, but the Lagrangian conservation
properties were systematically violated. This shows that the computed solution was
fundamentally diffusive, which is not physically realistic on this scale. Thus, there
is a strong incentive to devise computable formulations which retain Lagrangian
conservation properties, as stated in [5].

It would be even more useful to operational weather forecasting to be able to
make these comparisons with compressible models. [6] attempted to do this, but
encountered the problem that it is not possible to make a compressible vertical slice
model whose solutions represent solutions of the full 3D compressible equations. A
vertical slice model was proposed that can be obtained as a small modification of
a 3D compressible model, but since the model did not conserve energy, it was not
possible to make meaningful comparisons on a long timescale. Then, [2], developed
a variational framework for deriving fluid models in the vertical slice framework that
conserves energy and potential vorticity. As well as recovering the incompressible
Euler-Boussinesq equations, [2] provided a number of new models, including an
α-
regularised vertical slice model, and a compressible vertical slice model, which fixed
the lack of conservation in the compressible model of [6].

In [2], it was noted that the slice Euler-Poincaré equations with advected density
D and potential temperature θS have a Kelvin circulation theorem
\[
\frac{d}{dt} \oint_{c(uS)} \left( s \left( \frac{1}{D} \frac{\delta l}{\delta uS} \right) - \left( \frac{1}{D} \frac{\delta l}{\delta uT} \right) \nabla \thetaS \right) \cdot dx = 0,
\]
where c(uS) is a loop advected by the in-slice velocity uS, uT is the transverse ve-
clocity (the component orthogonal to the slice), the potential temperature is written
as
\[
\theta(x, y, z, t) = \thetaS(x, z, t) + sy,
\]
where s is the constant temperature gradient in the y-direction, and l[uS, uT, θS, D] is
the reduced Lagrangian. For the case of the incompressible Euler-Boussinesq slice
model, this becomes
\[
\frac{d}{dt} \oint_{C(t)} (suS - (uT + fx)\nabla \thetaS) \cdot dx = 0.
\]
This is curious because the circulation theorem for EP equations on the diffeomor-
phism group (instead of the semidirect product used in the slice models) has a
baroclinic term that only vanishes when the curve is on a θ-isosurface. In the slice
model case, this term is replaced by an additional term in the circulation, leading to
a conserved circulation on any loop, θ-isosurface or not. Since the contour used to
define the circulation theorem is a material contour, it cannot cross fronts in the
semi-geostrophic limits. In the semi-geostrophic solutions, fronts are singularities in
the optimally transporting map where the boundary of the domain has been mapped
into the interior. Thus solutions which are discontinuous in physical space are still
compatible with the circulation theorem, as long as this restriction is made. These formulations will preserve Lagrangian conservation properties. However, computing the solutions remains challenging. Care with use of potential vorticity conservation near the semi-geostrophic limit for standard models is therefore also required.

In this paper, we show that this Kelvin circulation theorem occurs because of relabelling symmetries that occur in the variational description of [2], by using the variational tools for applying Noether’s Theorem to the Euler-Poincaré theory discussed in [1]. The rest of this paper is structured as follows. In Section 2, we review the variational formulation of vertical slice models. In Section 3, we show how the conserved potential vorticity arises from relabelling symmetries that are specific to the slice geometry. Finally in Section 4 we provide a summary and outlook.

2. Variational formulations of slice models. In this section, we briefly review the variational formulation of slice models. In a vertical slice model, we assume that the velocity field is independent of \( y \), but still has a component in the \( y \)-direction. This means that the Lagrangian flow map can be written

\[
\Phi(X, Y, Z, t) = (x(X, Z, t), y(X, Z, t) + Y, z(X, Z, t)),
\]

where \( X, Y, Z \) are Lagrangian labels, \( (x, y, z) \) are particle locations and \( t \) is time.

Equivalently, the Lagrangian flow map can be represented by a diffeomorphism\(^1\) of the vertical slice \( x - z \) plane combined with an \( (x, z) \)-dependent displacement in the \( y \)-direction, which we write as an element \((\phi, f)\) of the semidirect product \( \text{Diff}(\Omega) \circledS \mathcal{F}(\Omega) \), where \( \circledS \) denotes the semidirect product, and \( \mathcal{F}(\Omega) \) denotes an appropriate space of smooth functions on \( \Omega \) that specify the displacement of Lagrangian particles in the \( y \)-direction at each point in \( \Omega \). Then, if we write \( X = (x, z) \), the transformation takes the form

\[
(x, y) = \Phi(X, Y) = (\phi(X), Y + f(X)).
\]

In this representation, composition of two slice flow maps \((\phi_1, f_1)\) and \((\phi_2, f_2)\) is obtained from the semi-direct product formula \([7]\),

\[
(\phi_1, f_1) \cdot (\phi_2, f_2) = (\phi_1 \circ \phi_2, f_1 \circ \phi_2 + f_2).
\]

We see that the vertical slice diffeomorphisms \( \phi_1 \) and \( \phi_2 \) compose in the normal way, whilst the combined vertical deflection is obtained by moving \( f_1 \) with \( \phi_2 \) before adding \( f_2 \).

We represent Eulerian velocity fields by splitting into two components \((u_S, u_T)\) where \( u_S \) is the “slice” component in the \( x-z \) plane, and \( u_T \) is the “transverse” component in the \( y \) direction. \((u_S, u_T)\) is considered as an element in the semidirect product Lie algebra \( \mathcal{X}(\Omega) \circledS \mathcal{F}(\Omega) \) where \( \mathcal{X}(\Omega) \) denotes the vector fields on \( \Omega \), representing the two components of the velocity \( u_S \in \mathcal{X}(\Omega) \) and \( u_T \in \mathcal{F}(\Omega) \). This Lie algebra has a Lie bracket, which we shall make use of below, given by

\[
[(u_S, u_T), (w_S, w_T)] = ([u_S, w_S], u_S \cdot \nabla w_T - w_S \cdot \nabla u_T),
\]

\(^1\)The assumption that the invertible map \( \Phi \) is smooth is necessary to compute the potential vorticity equation. However, the optimal transport formulation relaxes the smoothness requirements, enabling weaker solutions such as the semigeostrophic frontal solutions which are weak Lagrangian solutions. The Lagrangian formulation is likely to have some sort of global existence, based on conservation properties, but the Eulerian formulation may only work in smooth cases, which may not be generic for weather phenomena such as fronts.
where \([u_S, w_S] = u_S \cdot \nabla w_S - w_S \cdot \nabla u_S\) is the Lie bracket for the time-dependent vector fields \((u_S, w_S) \in \mathfrak{X}(\Omega)\), and \(\nabla\) denotes the gradient in the \(x-z\) plane.

A time-dependent Lagrangian flow map must satisfy the equation
\[
\frac{\partial}{\partial t} (\phi, f) = (u_S, u_T)(\phi, f) = (u_S \circ \phi, u_T \circ \phi),
\]
for some slice vector field \((u_S, u_T) \in \mathfrak{X}(\Omega) \circ \mathcal{F}(\Omega)\). Similarly, if within Hamilton’s principle we consider a one-parameter family of perturbations \((\phi_\epsilon, f_\epsilon)\) (parameterised by \(\epsilon\)) to \((\phi, f)\), then
\[
\delta(\phi, f) = \lim_{\epsilon \to 0} \frac{(\phi_\epsilon, f_\epsilon) - (\phi, f)}{\epsilon} = (w_S, w_T)(\phi, f) = (w_S \circ \phi, w_T \circ \phi),
\]
for some time-dependent slice vector field \((u_S, u_T)\) that generates the infinitesimal perturbations at \(\epsilon = 0\). Taking \(\epsilon\) and time-derivatives of these two expressions and comparing leads to
\[
\delta(u_S, u_T) = \frac{\partial}{\partial t} (w_S, w_T) + [(u_S, u_T), (w_S, w_T)],
\]
which gives us a formula telling us how perturbations in \((\phi, f)\) lead to perturbations in \((u_S, u_T)\) that we can use in Hamilton’s principle.

In order to build geophysical models, we also need to consider advected tracers and densities. In this modelling framework, we assume that densities are independent of \(y\), so that the continuity equation becomes
\[
\frac{\partial}{\partial t} D + \nabla \cdot (u_S D) = 0,
\]
which we write in geometric notation as
\[
\frac{\partial}{\partial t} D d S + \mathcal{L}_{u_S} D d S = 0,
\]
where \(d S\) is the area form (so that \(D d S \in \Lambda^2(\Omega)\), the space of 2-forms on \(\Omega\)) and \(\mathcal{L}_{u_S}\) is the Lie derivative with respect to \(u_S\), given via Cartan’s Magic Formula as
\[
\mathcal{L}_{u_S} D d S = d((u_S \lrcorner D) d S),
\]
recalling that \(d(D d S) = 0\). Under an infinitesimal perturbation to \((\phi, f)\) the density changes according to
\[
\delta D d S + \mathcal{L}_{w_S} D d S = 0.
\]

We assume that tracers can be written as sum of a \(y\)-independent component \(\theta_S\) plus a time-independent component \(s y\) with linear dependence on \(y\) (geophysically this allows for North-South temperature gradients that are necessary for the baroclinic instability that leads to frontogenesis). This means that the flow map \((\phi, f)\) transports initial conditions \((\theta_{S,0}, s_0)\) according to
\[
\theta_S = \phi_\ast (\theta_{S,0} - s_0 f), \quad s = s_0.
\]
Time-differentiation then leads to the transport equation
\[
\frac{\partial}{\partial t} \theta_S + u_S \cdot \nabla \theta_S + s w_T = 0, \quad \frac{\partial s}{\partial t} = 0,
\]
which we write in geometric notation as
\[
\frac{\partial}{\partial t} (\theta_S, s) + \mathcal{L}_{(u_S, w_T)} (\theta_S, s) = 0,
\]
where
\[ L_{(u_S, w_T)}(\theta_S, s) = (u_S \triangledown d \theta_S + s w_T, 0), \]
and where we have considered \((\theta_S, s) \in F(\Omega) \times \mathbb{R})\). Similarly, infinitesimal perturbations to \((\phi, f)\) lead to infinitesimal perturbations to \((\theta_S, s)\) given by
\[ \delta(\theta_S, s) + L_{(w_S, w_T)}(\theta_S, s) = 0. \]

Given a Lagrangian \(l[(u_S, u_T), D, (\theta_S, s)]\), we define
\[ \delta l[(u_S, u_T), D, (\theta_S, s); (\delta u_S, \delta u_T), \delta D, (\delta \theta_S, \delta s)] = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} [l[(u_S, u_T) + \epsilon(\delta u_S, \delta u_T), D + \epsilon \delta D, (\theta_S, s) + \epsilon(\delta \theta_S, \delta s)] - l[(u_S, u_T), D, (\theta_S, s)]] , \]
\[ = \int_{\Omega} \frac{\delta l}{\delta u_S} \cdot \delta u_S + \frac{\delta l}{\delta u_T} \cdot \delta u_T + \frac{\delta l}{\delta D} \delta D + \frac{\delta l}{\delta \theta_S} \delta \theta_S + \frac{\delta l}{\delta s} \delta s \, dS \, dD, \]
for all \((\delta u_S, \delta u_T) \in \mathcal{X}(\Omega) \otimes \mathcal{F}(\Omega), \delta D \in \Lambda^2(\Omega), (\delta \theta_S, \delta s) \in F(\Omega) \times \mathbb{R}\). Geometrically, we interpret \(\frac{\delta l}{\delta u_S} \cdot d \times d S\) as being a 1-form-density in \(\Lambda^1(\Omega) \otimes \Lambda^2(\Omega)\), the dual space of vector fields \(\mathcal{X}(\Omega)\). \(\frac{\delta l}{\delta \theta_S} \delta \theta_S + \frac{\delta l}{\delta s} \delta s \) are interpreted as being densities in \(\Lambda^2(\Omega)\), whilst \(\frac{\delta l}{\delta u_T} \delta u_T\) is interpreted as being a function in \(F(\Omega)\), and \(\frac{\delta l}{\delta D} \delta D\) is being a function in \(\Lambda^2(\Omega)\).

Proceeding with Hamilton’s principle \(\delta S = 0\), where
\[ \delta S = \int_{t_0}^{t_1} l[(u_S, u_T), D, (\theta_S, s)] \, dt, \]
considering variations
\[ \delta (u_S, u_T) = \frac{\partial}{\partial t} (w_S, w_T) + [(u_S, u_T), (w_S, w_T)], \]
\[ \delta D \, dS = -L_{w_S} D \, dS, \]
\[ \delta (\theta_S, s) = -L_{w_S} (\theta_S, s), \]
for perturbation-generating velocity fields \((w_S, w_T)\) that vanish at \(t = t_0\) and \(t = t_1\), we obtain
\[ 0 = \delta S \]
\[ = \int_{t_0}^{t_1} \int_{\Omega} \frac{\delta l}{\delta u_S} \cdot \delta u_S + \frac{\delta l}{\delta u_T} \cdot \delta u_T + \frac{\delta l}{\delta D} \delta D + \frac{\delta l}{\delta \theta_S} \delta \theta_S + \frac{\delta l}{\delta s} \delta s \, dS \, dD, \]
\[ = \int_{t_0}^{t_1} \int_{\Omega} \frac{\delta l}{\delta u_S} \cdot \left( \frac{\partial}{\partial t} w_S + [u_S, w_S] \right) \, dS + \frac{\delta l}{\delta u_T} \left( \frac{\partial}{\partial t} w_T + L_{u_S} w_T - L_{w_S} w_T \right) \, dS \]
\[ - \frac{\delta l}{\delta D} L_{w_S} (D \, dS) - L_{(w_S, w_T)}(\theta_S) \frac{\delta l}{\delta \theta_S} \, dS \, dD, \]
\[ = \int_{t_0}^{t_1} \int_{\Omega} \omega_s \cdot \left( \frac{\partial}{\partial t} \omega_S + \frac{\delta l}{\delta u_S} \cdot d \times d S + \frac{\delta l}{\delta u_T} \cdot d \times d S \right) \]
\[ + \frac{\delta l}{\delta \theta_S} d \theta_S \otimes d S - D \, d \frac{\delta l}{\delta D} \otimes d S \]
\[ + \int_{t_0}^{t_1} \omega_T \left( \frac{\partial}{\partial t} \omega_S + \frac{\delta l}{\delta u_S} \cdot d \times d S + \frac{\delta l}{\delta \theta_S} \cdot d S \right) \, dt, \]
where we have integrated by parts in time and space, and have used the identity
\[ \int_{\Omega} m \cdot [u, v] \otimes dS = - \int_{\Omega} v \cdot \mathcal{L}_{u_S} m \cdot x \otimes S, \]
for all \( u, v \in \mathcal{X}(\Omega) \), \( m \cdot dx \otimes dS \in \Lambda^1(\Omega) \otimes \Lambda^2(\Omega) \), and where the Lie derivative of a one-form density is defined as

\[
\mathcal{L}_{us}(m \cdot dx \otimes dS) = (\mathcal{L}_{us} m \cdot dx) \otimes dS + m \cdot dx \otimes dS.
\]  

(27)

Since \((w_S, w_T)\) are arbitrary, save for the endpoint conditions, we obtain (for sufficiently smooth solutions),

\[
\frac{\partial}{\partial t} + \mathcal{L}_{us} \frac{\delta l}{\delta u_S} \cdot dx \otimes dS + \frac{\delta l}{\delta u_T} \delta u_T \otimes dS + \frac{\delta l}{\delta \theta_S} \delta \theta_S \otimes dS - D \delta \theta_S \delta dS = D \delta dS,
\]

(28)

in which case

\[
\frac{\partial}{\partial t} + \mathcal{L}_{us} \frac{\delta l}{\delta u_S} \cdot dx \otimes dS = \left( \frac{\partial}{\partial t} + \mathcal{L}_{us} \frac{1}{\delta u_S} \cdot dx \right) \otimes dS - D \delta dS,
\]

(29)

after making use of the continuity equation for \( D dS \). Similarly, we have

\[
\frac{\partial}{\partial t} + \mathcal{L}_{us} \frac{1}{\delta u_T} \delta u_T \otimes dS = \left( \frac{\partial}{\partial t} + \mathcal{L}_{us} \frac{1}{\delta u_T} \right) D dS.
\]

(30)

Hence we obtain

\[
\frac{\partial}{\partial t} + \mathcal{L}_{us} \frac{1}{\delta u_S} \cdot dx + \frac{\delta l}{\delta u_T} \delta u_T + \frac{\delta l}{\delta \theta_S} \delta \theta_S - D \delta dS = 0,
\]

(31)

after making use of the continuity equation for \( D dS \). Similarly, we have

\[
\frac{\partial}{\partial t} + \mathcal{L}_{us} \frac{1}{\delta u_T} \delta u_T + \frac{\delta l}{\delta \theta_S} = 0.
\]

(32)

Translating back into vector calculus notation, we get

\[
\frac{\partial}{\partial t} + u_S \cdot \nabla + (\nabla u_S)^T \cdot \frac{1}{\delta u_S} \delta u_S + \frac{\delta l}{\delta u_T} \delta u_T + \frac{\delta l}{\delta \theta_S} \delta \theta_S - \nabla \cdot \delta \delta \delta = 0,
\]

(33)

\[
\frac{\partial}{\partial t} + u_S \cdot \nabla \left( \frac{1}{\delta u_T} \delta u_T + \frac{1}{\delta \theta_S} \right) = 0.
\]

(34)

The Lagrangian for the incompressible Euler-Boussinesq slice equations is

\[
l = \int_{\Omega} \frac{D}{2} \left( |u_S|^2 + u_T^2 \right) + Df u_T x + \frac{g}{\theta_0} \delta \left( z - \frac{H}{2} \right) \delta \theta_S + p(1 - D) \delta dS,
\]

(35)

where \( f \) is the Coriolis parameter, \( g \) is the acceleration due to gravity, \( \theta_0 \) is a reference potential temperature, \( H \) is the height of the vertical slice, and \( p \) is a Lagrange multiplier introduced to enforce that the density stays constant. In this case we have

\[
\frac{1}{\delta u_S} \delta \delta \delta = u_S, \quad \frac{1}{\delta u_T} \delta \delta \delta = u_T + f x,
\]

\[
\frac{\delta l}{\delta \delta \delta} = \frac{1}{2} \left( |u_S|^2 + u_T^2 \right) + f u_T x - p + g \theta_0 \delta \left( z - \frac{H}{2} \right),
\]

(36)

\[
\frac{1}{\delta \theta_S} = \frac{g}{\theta_0} \left( z - \frac{H}{2} \right).
\]

(37)
Substituting these formula and rearranging leads us to the Euler-Boussinesq vertical slice equations

\[ \begin{align*}
\partial_t u_S + u_S \cdot \nabla u_S - f u_T \hat{x} &= -\nabla p + \frac{g}{\theta_0} \theta_S \hat{z}, \\
\partial_t u_T + u_S \cdot \nabla u_T + f u_S \cdot \hat{x} &= -\frac{g}{\theta_0} \left( z - \frac{H}{2} \right) s, \\
\nabla \cdot u_S &= 0, \\
\partial_t \theta_S + u_S \cdot \nabla \theta_S + u_T s &= 0,
\end{align*} \]

(39)

where \( \hat{x} \) and \( \hat{z} \) are the unit normals in the \( x \)- and \( z \)-directions, respectively. In addition to providing a variational derivation of the incompressible Euler-Boussinesq slice model, [2] also provided Lagrangians that lead to an alpha-regularised Euler-Boussinesq slice model, and a compressible Euler slice model. Since these models have a variational derivation, they all have a conserved energy; they also have a conserved potential vorticity as we shall now discuss.

Returning to (33-34), [2] made the following direct calculation to derive Kelvin’s circulation theorem and thus conservation of potential vorticity. Using the fact that the exterior derivative \( d \) commutes with \( \partial/\partial t \) and \( L_{u_S} \), we deduce that

\[ \left( \frac{\partial}{\partial t} + L_{u_S} \right) d \theta_S = -s d u_T. \]

(40)

Combining with (34), we obtain that

\[ \left( \frac{\partial}{\partial t} + L_{u_S} \right) \frac{1}{D} \frac{\delta l}{\delta u_T} d \theta_S = \frac{1}{D} \frac{\delta l}{\delta u_T} \left( \frac{\partial}{\partial t} + L_{u_S} \right) d \theta_S + \frac{1}{D} \frac{\delta l}{\delta \theta_S} \frac{1}{D} \frac{\delta l}{\delta u_T} d \theta_S, \]

\[ = -s \left( \frac{1}{D} \frac{\delta l}{\delta u_T} d u_T + \frac{1}{D} \frac{\delta l}{\delta \theta_S} d \theta_S \right), \]

\[ = \left( \frac{\partial}{\partial t} + L_{u_S} \right) \frac{1}{D} \frac{\delta l}{\delta u_S} \cdot d x - d \frac{\delta l}{\delta D}, \]

(41)

where we used (33) in the last equality. Hence, we obtain

\[ \left( \frac{\partial}{\partial t} + L_{u_S} \right) \left( s \frac{1}{D} \frac{\delta l}{\delta u_S} \cdot d x - \frac{1}{D} \frac{\delta l}{\delta \theta_S} d \theta_S \right) = d \frac{\delta l}{\delta D}. \]

(42)

Integrating this around a closed curve \( C(t) \) that is moving with velocity \( u_S \), we obtain a Kelvin circulation theorem

\[ \frac{d}{dt} \oint_{C(t)} \left( s \frac{1}{D} \frac{\delta l}{\delta u_S} \cdot d x - \frac{1}{D} \frac{\delta l}{\delta \theta_S} d \theta_S \right) = d \frac{\delta l}{\delta D}. \]

(43)

In the case of the incompressible Euler Boussinesq slice model, the circulation theorem reads

\[ \frac{d}{dt} \oint_{C(t)} (s u_S - (u_T + f x) \nabla \theta_S) \cdot d x = 0. \]

(44)

Applying \( d \) to Equation (42), we obtain

\[ \left( \frac{\partial}{\partial t} + L_{u_S} \right) d \left( s \frac{1}{D} \frac{\delta l}{\delta u_T} - \frac{1}{D} \frac{\delta l}{\delta \theta_S} d \theta_S \right) = 0. \]

(45)

Finally combining with the continuity equation we obtain conservation of potential vorticity,

\[ (\partial_t + L_{u_S}) q = 0, \]

(46)
In the case of the incompressible Euler-Boussinesq slice model this becomes
\[ q = s \nabla^\perp \cdot (u_S - (u_T + f x) \nabla \theta_S) . \]  

(48)

The circulation theorem is curious, because usually in the presence of advected temperatures, we obtain a baroclinic source term, so that circulation is only preserved on isosurfaces of \( \theta_S \). In the slice model case, this baroclinic term can be replaced by the Lie derivative of an additional quantity, so that we get a conservation law for any circulation loop. The new contribution of this paper is to show that these extra conservation laws arise from new relabelling symmetries that exist in the slice model framework.

3. Relabelling symmetries and Noether’s theorem for slice models. In this section we describe the relabelling symmetry for slice models and compute the corresponding conserved quantities via Noether’s theorem. Relabelling symmetry in fluid dynamics is the statement that the reference configuration for the Lagrangian flow map is arbitrary. In the slice framework, this means that we can arbitrarily select an alternative reference configuration, which can be transformed back to the original reference configuration by the slice map represented by the relabelling transformation \((\psi, g) \in \text{Diff}(\Omega) \circ \mathcal{F}(\Omega)\). After this change of base coordinates, the Lagrangian flow map is transformed according to
\[
(\phi, f) \mapsto (\phi \circ \psi, f \circ \psi + g).
\]

(49)

Relabelling symmetries are such transformations that leave the initial data \( \theta_{S,0}, D_0, s_0 \) all invariant. In the group variable notation of previous sections, relabelling symmetries form a group
\[
\mathcal{G}_{D_0,\theta_{S,0},s_0} = \{ (\psi, g) \in \text{Diff}(\Omega) \circ \mathcal{F}(\Omega) \mid \psi^* (D_0 \, dS) = D_0 \, dS \text{ and } g = (\theta_{S,0} - \psi^* \theta_{S,0}) / s_0 \}.
\]

To compute infinitesimal relabelling transformations, we consider a 1-parameter family of relabellings \((\psi_\epsilon, g_\epsilon)\) for \( \epsilon > 0 \), with \((\psi_0, g_0) = (\text{Id}, 0)\), so that
\[
(\psi_\epsilon, g_\epsilon) = (\text{Id}, 0) + \epsilon (v_S, v_T) + \mathcal{O}(\epsilon).
\]

(51)

Then,
\[
\delta(\phi, f) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left( ((\phi \circ \psi_\epsilon, f \circ \psi_\epsilon + g_\epsilon) - (\phi, f)) \right) = (\nabla \phi \cdot v_S, \nabla f \cdot v_S + v_T).
\]

(52)

Following (8), we define \((w_S, w_T)\) to be the unique element of \(X(\Omega) \circ \mathcal{F}(\Omega)\) such that
\[
w_S \circ \phi = \nabla \phi \cdot v_S := \delta \phi, \quad w_T \circ \phi = \nabla f \cdot v_S + v_T := \delta f.
\]

(53)

Time differentiation gives
\[
\frac{\partial}{\partial t} (w_S \circ \phi) = \frac{\partial w_S}{\partial t} \circ \phi + u_S \circ \phi \cdot (\nabla w_S) \circ \phi, \quad \frac{\partial}{\partial t} (w_T \circ \phi) = \frac{\partial w_T}{\partial t} \circ \phi + u_S \circ \phi \cdot (\nabla w_T) \circ \phi.
\]

(54)
On the other hand,

$$\frac{\partial}{\partial t} (\nabla \phi \cdot v_S) = \nabla (u_S \circ \phi) \cdot v_S,$$

$$= (\nabla u_S) \circ \phi \cdot (\nabla \phi \cdot v_S),$$

$$= (\nabla u_S) \circ \phi \cdot w_S \circ \phi,$$

$$= (\nabla u_T) \circ \phi \cdot (\nabla \phi \cdot v_S),$$

$$= (\nabla u_T) \circ \phi \cdot w_S \circ \phi.$$  \hspace{1cm} (55)

Equating and composing with $\phi^{-1}$, we obtain

$$\frac{\partial}{\partial t} (w_S, w_T) + [(u_S, u_T), (w_S, w_T)] = 0.$$  \hspace{1cm} (57)

This means that

$$\delta u_S = \dot{w}_S + [u_S, w_S] = 0,$$  \hspace{1cm} (58)

$$\delta u_T = \dot{w}_T + u_S \cdot \nabla w_T - w_S \cdot \nabla u_T = 0,$$  \hspace{1cm} (59)

i.e. $u_S$ and $u_T$ are left invariant under relabelling transformations.

To leave $\theta_S$ invariant, the infinitesimal symmetries $(w_S, w_T)$ also need to satisfy

$$\delta \theta_S = -L_{(w_S, w_T)}(\theta_S, s) = -w_S \cdot \nabla \theta_S - s w_T = 0.$$  \hspace{1cm} (60)

In the $s = 0$ case, this restricts $w_S$ to being tangential to the contours of $\theta$, hence the baroclinic torque term in the vorticity equation. However, if $s \neq 0$, we can take any $w_S$ and then pick

$$w_T = - \frac{1}{s} w_S \cdot \nabla \theta_S,$$  \hspace{1cm} (61)

i.e. any change in $\theta_S$ caused by advection with $w_S$ can be corrected by a source term from $w_T$. For this to work, we need equation (61) to be compatible with (59). This is verified by the following proposition.

**Proposition 1.** Let $w_S$ be a vector field with arbitrary initial condition, and let $w_T$ be a function satisfying (61) initially. Let $(w_S, w_T)$ satisfy the particle relabelling symmetry conditions (58-59), and let $\theta$ evolve according to

$$\frac{\partial \theta_S}{\partial t} = -L_{(u_S, u_T)}(\theta_S, s) = -(u_S \cdot \nabla \theta_S + su_T).$$  \hspace{1cm} (62)

Then $\theta_S$, $w_T$ satisfy (61) for all time.

**Proof.** First note that $\mathcal{L}_{(u_S, u_T)}(\theta_S, s)$ is a Lie algebra action of $\mathcal{X}(\Omega) \mathbb{S} \mathcal{F}(\Omega)$ on $\mathcal{F}(\Omega) \times \mathbb{R}$, i.e.

$$\mathcal{L}_{(u_S, u_T)}(\mathcal{L}_{(u_S, u_T)}(\theta_S, s) - \mathcal{L}_{(w_S, w_T)}(\mathcal{L}_{(u_S, u_T)}(\theta_S, s) = \mathcal{L}_{[(u_S, u_T), (w_S, w_T)]},$$  \hspace{1cm} (63)

where $[(u_S, u_T), (w_S, w_T)]$ is the bracket on $\mathcal{X}(\Omega) \mathbb{S} \mathcal{F}(\Omega)$ defined in [2], given by

$$[(u_S, u_T), (w_S, w_T)] = ([u_S, w_S], u_S \cdot \nabla w_T - w_S \cdot \nabla u_T).$$
Then we have
\[
\frac{\partial}{\partial t} L_{(w_S, w_T)}(\theta_S, s) \\
= L_{\frac{\partial}{\partial t}(w_S, w_T)}(\theta_S, s) + L_{(w_S, w_T)}(\theta_S, 0), \\
= L_{\frac{\partial}{\partial t}(w_S, w_T)}(\theta_S, s) + L_{(u_S, u_T)}(\theta_S, s), \\
= L_{\frac{\partial}{\partial t}(w_S, w_T)}(\theta_S, s) - L_{(w_S, w_T)}(\theta_S, 0), \\
= L_{\frac{\partial}{\partial t}(w_S, w_T)}(\theta_S, s) - L_{[(w_S, w_T), (w_S, w_T)]}(\theta_S, s) - L_{(u_S, u_T)} L_{(w_S, w_T)}(\theta_S, 0),
\]
\[
= L \frac{\partial}{\partial t}(w_S, w_T) + [(u_S, u_T), (w_S, w_T)](\theta_S, s) = 0,
\]
as required.

**Proposition 2.** Let \(u_S, u_T, \theta_S, D\) solve the equations (28-29). Then the potential vorticity \(q\) (weakly) satisfies the Lagrangian conservation law
\[
\frac{\partial q}{\partial t} + u_S \cdot \nabla q = 0,
\]
as a consequence of Noether’s theorem.

**Proof.** Given initial condition \(D_0\) for density, we pick arbitrary \(\psi_0 \in \Lambda^0(\Omega)\), compactly supported in the interior of \(\Omega\). Then we choose \(\psi\) as the solution of
\[
\left(\frac{\partial}{\partial t} + L_{u_S}\right) \psi = 0,
\]
with initial condition \(\psi_0\). We then define \(w_s\) via
\[
w_s \cdot D \, dS = d\psi.
\]
Then
\[
\left(\frac{\partial}{\partial t} + L_{u_S}\right) w_s \cdot D \, dS = \left(\frac{\partial}{\partial t} + L_{u_S}\right) d\psi = 0,
\]
and we deduce from the chain rule and Equation (14) that \(w_s\) satisfies Equation (58). Further, Equation (67) implies that
\[
\delta(D \, dS) = L_{w_S}(D \, dS) = d(w_S \cdot D \, dS) = 0,
\]
for all times, i.e. \(w_s\) is a relabelling symmetry for \(D\). We then choose
\[
w_T = -\frac{1}{s} w_S \cdot \nabla \theta_S,
\]
which defines a relabelling symmetry for \(\theta_S\) by Lemma 1.

Next, we follow the steps of Noether’s Theorem, considering the variations in the action \(S\) under the relabelling transformations generated by \((w_S, w_T)\) defined
above. Since these transformations leave \((u_S, u_T), D\) and \(\theta_S\) invariant, the action does not change, and we get

\[
0 = \delta S = \int_{t_1}^{t_2} \left[ \int_\Omega \frac{\delta l}{\delta u_S} \cdot w_S + \frac{\delta l}{\delta u_T} w_T \right]_{t_1}^{t_2} S, \tag{72}
\]

since \((u_S, u_T), D\) and \(\theta_S\) solve (28-29). For sufficiently smooth solutions in time, we may consider the limit \(t_1 \to t_2\), and we get

\[
0 = \frac{d}{dt} \int_{\Omega} \int_\Omega \left( \frac{\delta l}{\delta u_S} \cdot d x - \frac{1}{s} \frac{\delta l}{\delta u_T} \cdot d \theta \right) \wedge w_S \cdot d S + \frac{d}{dt} \int_{\Omega} \left( \frac{\delta l}{\delta u_S} \cdot d x - \frac{1}{s} \frac{\delta l}{\delta u_T} \cdot d \theta \right) \wedge \psi, \tag{73}
\]

\[
= \int_{\Omega} \left( \frac{\delta l}{\delta u_S} \cdot d x - \frac{1}{s} \frac{\delta l}{\delta u_T} \cdot d \theta \right) \wedge \psi + \int_{\Omega} \left( \frac{\delta l}{\delta u_S} \cdot d x - \frac{1}{s} \frac{\delta l}{\delta u_T} \cdot d \theta \right) \wedge \frac{\partial}{\partial t} \cdot d \psi,
\]

\[
= \int_{\Omega} \left( \frac{\delta l}{\delta u_S} \cdot d x - \frac{1}{s} \frac{\delta l}{\delta u_T} \cdot d \theta \right) \wedge \frac{\partial}{\partial t} \cdot d \psi - \frac{1}{D} \left( \frac{\delta l}{\delta u_S} \cdot d x - \frac{1}{s} \frac{\delta l}{\delta u_T} \cdot d \theta \right) \wedge D \cdot d \psi,
\]

\[
= - \int_{\Omega} \psi \left( \frac{\partial}{\partial t} + \mathcal{L}_{u_S} \right) \frac{1}{D} \left( \frac{\delta l}{\delta u_S} \cdot d x - \frac{1}{s} \frac{\delta l}{\delta u_T} \cdot d \theta \right), \tag{74}
\]

which holds for arbitrary \(\psi\), hence we deduce that

\[
0 = \left( \frac{\partial}{\partial t} + \mathcal{L}_{u_S} \right) (q D \cdot d S), \tag{75}
\]

hence the result (since \(D\) is positive).
Corollary 1. Let \( u_S, u_T, \theta_S, D \) solve the equations (28-29). Then, the Kelvin circulation,

\[
\int_{C(t)} \frac{\delta l}{\delta u_S} \cdot dx - \frac{1}{s} \frac{\delta l}{\delta u_T} \, d\theta,
\]

is preserved along Lagrangian closed loops \( C(t) \).

Proof. First, we note that the Lie derivative \( \mathcal{L}_{u_S} \) of a 1-form commutes with \( d \) and \( \delta = \ast d \ast \). Then if \( h \) solves

\[
\mathcal{L}_{u_S} h = 0,
\]

with harmonic initial conditions \((d h_0 = 0 \text{ and } \delta h_0 = 0)\), then \( h \) is harmonic for all times. Further, choosing \( w_{S,D} \) means that \( d(w_{S,D} D d S) = 0 \), so \( w_S \) generates a symmetry of \( D \). Choosing \( w_T \) from (71) then means that we have a relabelling symmetry. Writing

\[
v = \frac{1}{D} \left( \frac{\delta l}{\delta u_S} \cdot dx - \frac{1}{s} \frac{\delta l}{\delta u_T} \, d\theta \right),
\]

we then return to (73), leading to

\[
0 = \frac{d}{dt} \int_{\Omega} v \wedge h,
\]

\[
= \int_{\Omega} v_t \wedge h - v \wedge \mathcal{L}_u h,
\]

\[
= \int_{\Omega} \left( \frac{\partial}{\partial t} + \mathcal{L}_u \right) v \wedge h,
\]

after integration by parts. This means that \( v_t + \mathcal{L}_u v \) is orthogonal to all harmonic forms. Combining with the PV conservation law

\[
0 = \left( \frac{\partial}{\partial t} + \mathcal{L}_u \right) d v,
\]

\[
= d \left( \frac{\partial}{\partial t} + \mathcal{L}_u \right) v,
\]

we deduce from the Hodge decomposition that

\[
\left( \frac{\partial}{\partial t} + \mathcal{L}_u \right) v = d \kappa,
\]

from which we obtain the Kelvin circulation theorem after integrating around a loop.

4. Summary and outlook. In this paper, we provided a new proof that the conserved potential vorticity in vertical slice models arises through Noether’s Theorem upon consideration of the relabelling symmetries consisting of rearrangements of the vertical slice combined with transverse motion that restores the original structure of the potential temperature \( \theta_S \). The proof applies to a horizontally-periodic geometry with rigid top and bottom boundary but can be easily extended to other slice geometries.

A future direction will be to use this variational structure to build potential vorticity conserving vertical slice models, and to make further comparisons with the optimal transport formulation. It would also be interesting to use the relabelling transformations in this paper to derive balanced models using the variational asymptotics approach of [8].
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