Decidability problems in automaton semigroups

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Abstract We consider decidability problems in self-similar semigroups, and in particular in semigroups of automatic transformations of $X^*$. We describe algorithms answering the word problem, and bound its complexity under some additional assumptions. We give a partial algorithm that decides in a group generated by an automaton, given $x, y$, whether an Engel identity ($[\ldots[[x,y],y],\ldots,y] = 1$ for a long enough commutator sequence) is satisfied. This algorithm succeeds, importantly, in proving that Grigorchuk’s 2-group is not Engel. We consider next the problem of recognizing Engel elements, namely elements $y$ such that the map $x \mapsto [x,y]$ attracts to $1$. Although this problem seems intractable in general, we prove that it is decidable for Grigorchuk’s group: Engel elements are precisely those of order at most 2. We include, in the text, a large number of open problems. Our computations were implemented using the package Fiat within the computer algebra system GAP.

1 Introduction

Automata are infinity — come down to computer scientists’ level

Louis-Ferdinand Céline, Journey to the End of the Night

The theory of groups, and even more so of semigroups, is fundamentally example-driven: on the one hand, these algebraic objects encode the symmetry, regularity, and operations present in any kind of structure under consideration,
and are thus given to us for study; on the other hand, there is such a diversity of groups and semigroups that the most one can hope for, in their theory, is a description of the phenomena that may occur.

Groups and semigroups are fundamentally (semi)groups of self-maps of a set. To construct infinite (semi)groups, we should therefore give oneself an infinite set and a collection of self-maps, or possibly a generating set of self-maps. Automata are of fundamental use, in the guise of transducers, in giving finite, recursive descriptions of infinite self-maps; see [17].

Now, even if the description of the (semi)group’s generators are completely explicit, this does not mean that the (semi)group is well understood. For example, one may want to know, given two words \(u, v\) representing products of generators, whether they are equal in the semigroup; semiconjugate \((\exists w : uw = vw)\), conjugate \((\exists \text{ invertible } w : uw = vw)\), etc. These decision problems are, usually, undecidable, and the question is which extra conditions on the semigroup’s generators guarantee the problem’s decidability.

In this text, I will survey a general construction of self-similar semigroups, highlight important decision problems, and describe stronger and stronger restrictions on the self-similarity structure in parallel with solutions to decision problems. There is a wealth of unsolved problems in this area, and I hope that the panorama provided by this text will have some value in highlighting interesting, yet-unexplored areas of mathematics and theoretical computer science.

The new results included in this text are in particular a partial algorithm that answers, in self-similar groups, whether the Engel property holds (for all \(x, y \in G\), some long-enough iterated commutator \([\cdots [x, y], \ldots, y]\) is trivial). Remarkably, this partial algorithm, once implemented, proved that the first Grigorchuk group is not Engel.

2 Self-similar semigroups

A self-similar semigroup is a semigroup acting in a self-similar fashion on a self-similar set.

We are thus given a set \(\Omega\) and a finite family of self-maps \(X = \{x : \Omega \to \Omega\}\). The set \(\Omega\) is self-similar in the sense that subsets \(x(\Omega) \subseteq \Omega\) are defined and identified (via \(x\)) with \(\Omega\), for a collection of \(x \in X\). Such systems are often called iterated function systems. The fundamental example is \(\Omega = X^\mathbb{N}\) the set of infinite words over an alphabet \(X\), and in that case \(x \in X\) acts on an infinite word by pre-catenation: \(x(x_1x_2\ldots) = xx_1x_2\ldots\); another example is \(\Omega = X^*\), the tree of finite words, with again the same identification of \(X\) with self-maps of \(\Omega\) by pre-catenation.

**Definition 1** ([32]) Let \(G\) be a semigroup acting on the right on a set \(\Omega\). The action is called self-similar if for every \(x \in X, g \in G\) there exist \(h \in G, y \in X\) such that

\[
x(\omega)^g = y(\omega^h) \quad \text{for all } \omega \in \Omega.
\]
In other words, the action of \( g \in G \) on \( \Omega \) is as follows: it carries the subset \( x(\Omega) \) to \( y(\Omega) \), and along the way transforms \( \Omega \) by \( h \).

If we were to write the function application \( x(\omega) \) on the right, as \( (\omega)x \), then we could rephrase (1) as “\( xg = hy \)” qua composition of self-maps of \( \Omega \).

The elements \( h \in G, y \in X \) are not necessarily unique in Definition 1. Let us assume that some choices are made for them; then they may be encoded in a map \( \Phi: X \times G \to G \times X \), given by \((x, g) \mapsto (h, y)\) when (1) holds. This map satisfies some axioms following from the fact that \( G \) is a semigroup acting on \( \Omega \). We summarize them in the following

**Definition 2** A self-similarity structure for a semigroup \( G \) is the data of a set \( X \) and a map \( \Phi: X \hat{\times} G \to G \hat{\times} X \) satisfying

\[
\Phi(x, 1) = (1, x), \quad \Phi(x, g) = (h, y) \land \Phi(y, g') = (h', z) \Rightarrow \Phi(x, gg') = (hh', z).
\]

From a self-similarity structure, one can reconstruct a self-similar action on \( \Omega = X^\ast \) by defining recursively (for \( \varepsilon \) the empty word in \( X^\ast \))

\[
\varepsilon^g = \varepsilon, \quad (xw)^g = y(u^h) \quad \text{whenever} \quad \Phi(x, g) = (h, y).
\]

The action on \( X^\ast \) extends uniquely by continuity to an action on infinite words \( X^\infty \).

A self-similar semigroup may be defined by specifying a generating set \( Q \), an alphabet \( X \), and a map \( \Phi: X \times Q \to Q^\ast \times X \). The semigroup defined is then the semigroup of self-maps of \( X^\ast \) given by (2). We write that semigroup as \( G(\Phi) \).

It is quite convenient to describe a self-similar semigroup \( G = \langle Q \rangle \) by writing its self-similarity structure on the perimeters of squares: there is a square for each \( x \in X, g \in Q \); the left, bottom, right, top labels are respectively \( x, g, y, h \) when \( \Phi(x, g) = (h, y) \). In this manner, to compute the action of \( g = q_1 \ldots q_n \) on a word \( x_1 \ldots x_m \), one writes \( x_1 \ldots x_m \) on the left and \( q_1 \ldots q_n \) on the bottom of an \( m \times n \) rectangle, and one fills in the rectangle’s squares one at a time. The right label will then be \( y_1 \ldots y_m \), the image of \( x_1 \ldots x_m \) under \( g \). Here are two examples of self-similar semigroups given in this manner:
Example 1 (Grigorchuk's group) The generating set is $Q = \{a, b, c, d, 1\}$ and the alphabet is $X = \{1, 2\}$. The self-similarity structure is given by

$$
\begin{align*}
1 & \xrightarrow[1]{\Phi} 2 \\
1 & \xrightarrow[a]{\Phi} 1 \\
2 & \xrightarrow[b]{\Phi} 1 \\
2 & \xrightarrow[c]{\Phi} 1 \\
1 & \xrightarrow[d]{\Phi} 1 \\
1 & \xrightarrow[1]{\Phi} 1
\end{align*}
$$

The associated automaton is depicted in Figure 1, left. The Grigorchuk group will be denoted by $G_0$ throughout this text.

It is a remarkable example of a group: among its properties, it is an infinite, finitely generated torsion group, namely, the group is infinite, but every element generates a finite subgroup. It is also a group of intermediate word-growth, namely, the number $v(n)$ of group elements that are products of at most $n$ generators is a function growing asymptotically as

$$
\exp(n^{0.51}) < v(n) < \exp(n^{0.76}).
$$

(the exact growth asymptotics are not known; see [4, 5, 11].)

Example 2 (A two-state automaton of intermediate growth) The generating set is $\{q, r\}$, and the alphabet is $X = \{1, 2\}$. The self-similarity structure is given by

$$
\begin{align*}
1 & \xrightarrow[r]{\Phi} 2 \\
1 & \xrightarrow[s]{\Phi} 2 \\
2 & \xrightarrow[r]{\Phi} 2 \\
1 & \xrightarrow[r]{\Phi} 1 \\
2 & \xrightarrow[s]{\Phi} 1
\end{align*}
$$

The associated automaton is depicted in Figure 1, right. The semigroup $G_1 = \langle r, s \rangle$ is infinite, and the growth of $G_1$ is better understood than that of $G_0$; letting $v(n)$ denote the number of elements of $G_1$ that are products of at most $n$ generators $r, s$, we have

$$
v(n) \approx 2^{5/2} 3^{\pi/2} \pi^{-2} n^{1/4} \exp(\sqrt{n}/6).
$$

Automata can be naturally composed, in two manners; see [16]. Let us consider a single automaton $\Phi$, with stateset $Q$ and alphabet $X$. Then, for every $m, n \in \mathbb{N}$, there is an automaton $\Phi_{m,n}$ with stateset $Q^n$ and alphabet $X^m$, described by squares as follows. For all words $g \in Q^n$ and $u \in X^m$, one writes $g, u$ respectively at the left and bottom of an $m \times n$ rectangle, and fills it by the $1 \times 1$ squares of the automaton $\Phi$.

The meaning of these automata is the following. In $\Phi_{m,1}$, the automaton has stateset $Q$ and alphabet $X^m$: its arrows are length-$m$ directed paths in the automaton $\Phi$. In formula, this automaton $\Phi_{m,1}$ is defined by

$$
\Phi_{m,1}(x_1 \ldots x_m, s_0) = (s_m, y_1 \ldots y_m) \text{ if } \Phi(x_i, s_{i-1}) = (s_i, y_i) \text{ for all } i = 1, \ldots, m.
$$
The automaton $\Phi_{m,1}$ expresses the action of $Q$ on words of length (a multiple of) $m$. Similarly, the automaton $\Phi_{1,n}$ has stateset $Q^n$ and alphabet $X$, and is given by

$$\Phi_{1,n}(x_0, s_1 \ldots s_n) = (t_1 \ldots t_n, x_n)$$

if $\Phi(x_{i-1}, s_i) = (t_i, x_i)$ for all $i = 1, \ldots, n$.

It expresses the action on $X^*$ of words of length $n$ in $Q$. These products may naturally be combined so as to give an automaton $\Phi_{m,n}$ with stateset $Q^n$ and alphabet $X^m$.

We shall abuse notation and write $\Phi_{p,u,g}q$ instead of $\Phi_{m,n}p,u,g,q$, since it is always clear from the arguments $u, g$ what the values of $m, n$ are.

### 3 Decision problems

The study of decision problems is commonly attributed to Dehn [13], though its origins can be traced to Hilbert’s work. Let $G$ be a finitely generated semi-group, and consider a finite generating set $Q$. There is therefore an evaluation map $Q^* \to G$, written $w \mapsto w$. Consider the following questions:

**Word problem (WP):** Given $u, v \in Q^*$, does one have $u = v$?

**Division problem:** Given $u, v \in Q^*$, is $u$ a left divisor of $v$? I.e. does one have $uG \ni v$? Is it a right divisor?

**Order problem (OP):** Given $u \in Q^*$, is $\langle u \rangle$ finite? If so, what is its structure, i.e. what are the minimal $m < n$ with $u^m = u^n$?

**Inverse problem:** Given $u \in Q^*$, is $u$ invertible?

**Conjugacy problem:** Given $u, v \in Q^*$, are they semiconjugate, i.e. are there $g \in G$ with $g^u = v$? Are they conjugate, i.e. is there an invertible $g \in G$ with $g^u = v$?

**Membership problem (MP):** Given $u, v_1, \ldots, v_n \in Q^*$, does one have $v \in \langle v_1, \ldots, v_n \rangle$?

**Structure problem:** Given $u_1, \ldots, u_n \in Q^*$, is the semigroup $\langle u_1, \ldots, u_n \rangle$ free? Is it finite?

**Engel problem:** Given $u, v \in Q^*$, and assuming $\pi, \pi$ are invertible, are they an Engel pair, i.e. does there exist $n \in \mathbb{N}$ such that the $n$-fold iterated commutator satisfies $[\ldots [\pi, \pi], \ldots, \pi] = 1$?

**Ad-nilpotence problem:** Given $v \in Q^*$ and assuming $\pi$ is invertible, is it ad-nilpotent, i.e. is $(g, \pi) \text{ an Engel pair for all invertible } g \in G$?

**Orbit problem (OP):** Assume that a countable set $\Omega$ is given via a computable bijection with (say) $\mathbb{N}$, and that $G$ acts on the right on $\Omega$. Given $\omega_1, \omega_2 \in \Omega$, does there exist $g \in G$ with $\omega_1^g = \omega_2$? If so, which one?

In all cases, what is required is an algorithm that answers the question. Equivalently, the inputs (a word, a finite list of words, . . .) may be encoded into $\mathbb{N}$ by a computable bijection. Let $\mathcal{P} \subseteq \mathbb{N}$ denote the set of inputs for which the answer to the question is “yes”. One then asks whether $\mathcal{P}$ is recursive, namely whether there exists an algorithmic enumeration of $\mathcal{P}$ and of $\mathbb{N}\setminus\mathcal{P}$. 
There are $2^\aleph_0$ finitely generated semigroups up to isomorphism, and only $\aleph_0$ algorithms, so for “most” finitely semigroups all the above decision problems have a negative solution.

Note also that each of these decision problems are stated for a fixed semigroup $G$. One may also ask directly some questions on $G$:

**Semigroup structure:** Is $G$ trivial? finite? commutative? free?

**Invertibility:** Is the subgroup $G^*$ of invertibles finite? If so, what is it?

**Group structure:** Assume $G$ is a group. Is $G$ nilpotent? free? Engel, i.e. is every invertible element ad-nilpotent?

These questions make a lot of sense for a human, especially if the semigroup is given implicitly as in Example 1 or Example 2. They make no sense as decision problems: either they hold or they don’t, but they have an unequivocal answer for every given $G$.

These last questions become much more interesting if one is given, rather than a semigroup $G$, a countably infinite family of semigroups. They could be given by semigroup presentations

$$G = \langle s_1, \ldots, s_n \mid r_1 = r'_1, \ldots, r_m = r'_m \rangle;$$

or by “self-similar presentations”

$$G = G(\Phi) \quad (3)$$

meaning, in the sense of the previous section, the semigroup acting faithfully on $X^*$ with the action given via (2) by $\Phi: X \times Q \to Q^* \times X$.

## 4 Some negative results

One seldom solves these decision problems directly. Rather, one uses Turing reduction: a problem $\mathcal{A}$ Turing-reduces to a problem $\mathcal{B}$ if there exists an algorithm answering $\mathcal{A}$ given an oracle for $\mathcal{B}$. In this manner, if $\mathcal{A}$ is unsolvable then so is $\mathcal{B}$.

It is a well-known fact that there are finitely-presented semigroups [29], [35], and even finitely-presented groups [34], with unsolvable word problem. Indeed Turing machines may be encoded in semigroup presentations, in such a manner that a word is trivial if and only if the corresponding Turing machine computation halts.

Let $Q = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_m \rangle$ be a finitely presented group with unsolvable word problem. Mihailova considers in [31]

$$M = \langle (x_1, x_1), \ldots, (x_n, x_n), (1, r_1), \ldots, (1, r_m) \rangle \subset F_n \times F_n.$$ 

Then $(1, u) \in M$ holds if and only if $u = 1$ holds in $Q$; so the membership problem is unsolvable in $F_2 \times F_2$. (Note however that the membership problem is solvable in free groups).
Fig. 1 The automata generating the Grigorchuk group (left, Example 1) and the Sushchansky semigroup (right, Example 2)

Now it is well-known that $F_n$ embeds in $GL_2(\mathbb{Z})$, so $F_n \times F_n$ embeds in $GL_4(\mathbb{Z})$. Clearly, if $G$ acts on the right on $\Omega$, and $H \leq G$ is a subgroup with $G, H$ finitely generated, and $\omega \in \Omega$ has trivial stabilizer in $G$, then the membership problem for $H$ in $G$ Turing-reduces to the orbit problem for $G$ acting on $\Omega$: given $H = \langle v_1, \ldots, v_n \rangle$ and $u \in G$, one has $u \in H$ if and only if $\omega, \omega^u$ are in the same $H$-orbit.

In particular, if $K$ is a finitely generated group and $H$ acts on $K$, then the orbit problem of $H$ on $K$ Turing-reduces to the conjugacy problem in $K \rtimes H$. It follows that there exist finitely generated subgroups of $GL_4(\mathbb{Z})$ with unsolvable membership and conjugacy problems.

5 Automaton semigroups

Consider a semigroup $G$ given by a self-similar presentation: there are finite sets $X, Q$ and a map $\Phi: X \times Q \to Q \times X$, defining a faithful action of $G = G(\Phi)$ on $X^\ast$ by $\Phi_Q$.

Question 1 Is the word problem in $G$ decidable?

I suspect the answer is “no”, but I don’t know. Let us put restrictions on $\Phi$ to make the problem more tractable.

Definition 3 A Mealy automaton is a map $\Phi: X \times Q \to Q \times X$.

We display the automaton as a graph with stateset $Q$ and, for every $x \in X, q \in Q$ with $\Phi(x, q) = (t, y)$, an edge starting in $q$, ending in $t$, labeled ‘$(x, y)$’ and called a transition. The letters $x$ and $y$ are respectively called the input and output labels. Since the two formalisms are obviously equivalent, we call automaton either the map $\Phi$ or its representation as a graph. The examples 1 and 2 above are Mealy automata, depicted respectively left and right in Figure 1.

In this graph interpretation, the action of $\langle Q \rangle$ on $X^\ast$ and on $X^Q$ are directly visible: given $q \in Q$ and a word $x_1x_2 \ldots$, find a path in the graph
starting at \( q \) and having input labels \( x_1, x_2, \ldots \). Let the output label on this path be \( y_1, y_2, \ldots \); then the result of the action is

\[
(x_1 x_2 \ldots)^q = y_1 y_2 \ldots.
\]

We call automaton semigroup a self-similar semigroup presented by a Mealy automaton as in (3), and we write the semigroup \( G = G(\Phi) \).

Let \( \Phi \) be an automaton with stateset \( Q \) and alphabet \( X \). For \( x \in X \) and \( q \in Q \), we denote by \( q \Phi^x \) the endpoint of the transition in \( \Phi \) starting at \( q \) and with input label \( x \), and we denote by \( \pi(q) \) the transformation of \( X \) induced by the edges starting at \( q \). Thus every transition in the Mealy automaton gives rise to

\[
q \xrightarrow{(x, x \pi(q))} q \Phi^x.
\]

**Proposition 1** The word problem in an automaton group is solvable in linear space (and therefore in exponential time).

*Proof* Let \( \Phi \) be a Mealy automaton, and let \( u, v \in Q^n \) be given words. By adding an identity state to \( Q \), we may suppose \( |u| = |v| = n \). Consider the graph with vertex set \( Q^n \) and with an edge from \( q_1 \ldots q_n \) to \( t_1 \ldots t_n \) labeled \( '(x_0, x_n)' \) whenever there are edges from \( q_i \) to \( t_i \) labeled \( '(x_{i-1}, x_i)' \) in \( \Phi \) for all \( i = 1, \ldots, n \). Then \( \overline{u} = \overline{v} \) if and only if the following holds in this graph: the vertices \( u, v \) may be identified, and outgoing edges with matching input may be identified, repeatedly, never causing an identification of vertices \( y, z \) with different \( \pi(y) \neq \pi(z) \).

Since the graph is finite and every identification reduces its size, this proves that the word problem is decidable. By carefully arranging the order in which the graph is explored, this may be done in \( O(n) \) space.

**Question 2** Is there an automaton for which the lower bound on the solution of the word problem is linear in space?

For every \( n \in \mathbb{N} \), one may embed the matrix semigroup \( M_n(\mathbb{Z}) \) into an automaton semigroup. More precisely, consider the semigroup \( G \) of affine transformations \( v \mapsto Av + w \), for all \( A \in M_n(\mathbb{Z}) \) and \( w \in \mathbb{Z}^n \). Consider the alphabet \( X = (\mathbb{Z}/2) \), and identify \( X^n \) with \( n \)-tuples of 2-adics \( (\mathbb{Z}/2)^n \). Let \( G \) act on \( (\mathbb{Z}/2)^n \) by extending the natural action on \( \mathbb{Z} \) by continuity. It is easy to see that this makes \( G \) a self-similar semigroup, and furthermore every element of \( G \) is contained in an automaton subsemigroup of \( G \).

This means that automaton semigroups are at least as powerful as linear semigroups, and also shows that there exist automaton semigroups with unsolvable conjugacy problem.

**Question 3** Do there exist automaton groups with unsolvable order problem?

It is known [18] that there exist automaton semigroups with unsolvable order problem. This is proven by Turing-reducing the order problem to a tiling problem.
**Definition 4** Let $\Phi : X \times Q \rightarrow Q \times X$ be a Mealy automaton. It is called *bounded* if there is a constant $C$ such that, for all $n \in \mathbb{N}$, there are at most $C$ elements in $\Phi(X^n \times Q) \setminus \{1\} \times X^n$.

In terms of graphs describing the automaton, that condition says that, apart from the identity state and its self-loops, there are no paths in the automaton that follow more than one loop.

Bondarenko, Sidki and Zapata prove in [10] that, if $G$ is an automaton group generated by a bounded automaton, then the order problem is solvable in $G$.

**Definition 5** Let $\Phi : X \times Q \rightarrow Q \times X$ be a Mealy automaton. It is called *nuclear* if, for every $g \in Q^*$, there exists $n \in \mathbb{N}$ such that for all $u \in X^n$ we have $g^@u \in Q$ as elements of $G(\Phi)$.

In other words, for every $g \in G(\Phi)$, its action on all remote-enough subtrees $uX^*$ may be described by elements of $Q$. An automaton semigroup is called *contracting* if it may be presented by a nuclear automaton.

For instance, the Grigorchuk group $G_0$ is contracting, and the automaton $\Phi$ presenting it is nuclear. On the other hand, the semigroup $G_1$ is not contracting: any nuclear automaton presenting it must contain the infinitely many distinct states $(rs)^n$ for all $n \in \mathbb{N}$.

If an automaton $\Phi$ is nuclear, then this may be verified in finite time: it suffices to check the condition of the definition for every $g \in Q^2$.

**Question 4** Let $\Phi$ be an automaton. Is it decidable if $\Phi$ is nuclear? Or if the automaton $\Phi_{1,n}$ (on the stateset $Q^n$) is nuclear?

**Proposition 2** Let $\Phi$ be a nuclear automaton. Then the word problem in $G(\Phi)$ is solvable in polynomial time.

**Proof** Denote by $|\cdot|$ the word metric on $G(\Phi)$. It follows from the definition that there is a constant $n$ such that $|g^@v| \leq \frac{1}{4}(|g| + 1)$ for all $g \in G(\Phi), v \in X^n$. The preprocessing step is to compute which elements of $Q$ are equal in $G(\Phi)$, and to determine for each word $q_1q_2$ of length 2 and each $v \in X^n$ an element $q \in Q$ such that $q_1q_2^@v = q$.

Then, given $g, h \in Q^*$, one computes $\#X^n$ words of length $\frac{1}{4}(|g| + 1), \frac{1}{4}(|h| + 1)$ representing the action on subtrees $uX^*$ for all $u \in X^n$; and compares them recursively. The complexity is polynomial of degree $n \log_2(\#X)$.

**Question 5** Let $G(\Phi)$ be a contracting automaton semigroup. Is its torsion problem decidable?

Let $G(\Phi)$ be a contracting automaton group. Is its conjugacy problem decidable?

We remarked in the introduction that the Grigorchuk group is an infinite torsion group.

**Question 6** Is there an algorithm that, given an nuclear automaton $\Phi$, decides whether $G(\Phi)$ is infinite? Whether it is torsion?
5.1 More constructions

Let $G$ be a self-similar semigroup. We generalize the notation $q@x$ to arbitrary semigroup elements and words: consider a word $v \in X^*$ and an element $g \in G$; denote by $v^g$ the image of $v$ under $g$. There is then a unique element of $G$, written $g@v$, with the property

$$(v w)^g = (v^g)(w)^g @ v$$

for all $w \in X^*$.

We call by extension this element $g@v$ the state of $g$ at $v$; it is the state, in the Mealy automaton defining $g$, that is reached from $g$ by following the path $v$ as input; thus in the Grigorchuk automaton $b@1 = a$ and $b@222 = b$ and $(bc)@2 = cd$. There is a reverse construction: by $v^g$ we denote the transformation of $X^*$ (which need not belong to $G$) defined by

$$(v w)^v g = v w^g,$$

$w v^g = w$ if $w$ does not start with $v$.

Given a word $w = w_1 \ldots w_n \in X^*$ and a Mealy automaton $\Phi$ of which $g$ is a state, it is easy to construct a Mealy automaton of which $w * g$ is a state: add a path of length $n$ to $\Phi$, with input and output $(w_1, w_1), \ldots, (w_n, w_n)$ along the path, and ending at $g$. Complete the automaton with transitions to the identity element. Then the first vertex of the path defines the transformation $w * g$. For example, here is $12 * d$ in the Grigorchuk automaton:

Note the simple identities $(g@v_1)@v_2 = g@(v_1 v_2)$, $(v_1 v_2) * g = v_1 * (v_2 * g)$, and $(v * g)@v = g$. Recall that we write conjugation in $G$ as $g^h = h^{-1} g h$. For any $h \in G$ we have

$$(v * g)^h = v^h * (g^h @ v).$$

An automaton semigroup is called regular weakly branched if there exists a non-trivial subsemigroup $K$ of $G$ such that for every $v \in X^*$ the semigroup $v * K$ is contained in $K$, and therefore also in $G$. Abért proved in [1] that regular weakly branched groups satisfy no law.
5.2 Grigorchuk’s example

The first Grigorchuk group $G_0$, defined in Example 1, is an automaton group which appears prominently in group theory, for example as a finitely generated infinite torsion group [21] and as a group of intermediate word growth [22]. This section is not an introduction to Grigorchuk’s first group, but rather a brief description of it with all information vital for the calculation in §8. For more details, see e.g. [7].

Fix the alphabet $X = \{1, 2\}$. The first Grigorchuk group $G_0$ is a permutation group of the set of words $X^*$, generated by the four non-trivial states $a, b, c, d$ of the automaton given in Example 1. Alternatively, the transformations $a, b, c, d$ may be defined recursively as follows:

\begin{align*}
(1x_2 \ldots x_n)^a &= 2x_2 \ldots x_n, & (2x_2 \ldots x_n)^a &= 1x_2 \ldots x_n, \\
(1x_2 \ldots x_n)^b &= 1(2x_2 \ldots x_n)^a, & (2x_2 \ldots x_n)^b &= 2(1x_2 \ldots x_n)^c, \\
(1x_2 \ldots x_n)^c &= 1(2x_2 \ldots x_n)^a, & (2x_2 \ldots x_n)^c &= 2(2x_2 \ldots x_n)^d, \\
(1x_2 \ldots x_n)^d &= 1x_2 \ldots x_n, & (2x_2 \ldots x_n)^d &= 2(1x_2 \ldots x_n)^b
\end{align*}

which directly follow from $d@1 = 1$, $d@2 = b$, etc.

It is remarkable that most properties of $G_0$ derive from a careful study of the automaton (or equivalently this action), usually using inductive arguments.

For example,

**Proposition 3** ([21]) The group $G_0$ is infinite, and all its elements have order a power of 2.

The self-similar nature of $G_0$ is made apparent in the following manner:

**Proposition 4** ([6 §4]) Define $x = [a, b]$ and $K = \langle x, x^a \rangle$. Then $K$ is a normal subgroup of $G_0$ of index 16, and $\psi(K)$ contains $K \times K$.

In other words, for every $g \in K$ and every $v \in X^*$ the element $v * g$ belongs to $G_0$.

6 Engel Identities

In this section, we restrict ourselves to invertible Mealy automata and self-similar groups.

A law in a group $G$ is a word $w = w(x_1, x_2, \ldots, x_n)$ such that $w(g_1, \ldots, g_n) = 1$, the identity element, for all $g_1, \ldots, g_n \in G$; for example, commutative groups satisfy the law $[x_1, x_2] = x_1^{-1}x_2^{-1}x_1x_2$. A variety of groups is a maximal class of groups satisfying a given law; e.g. the variety of commutative groups (satisfying $[x_1, x_2]$) or of groups of exponent $p$ (satisfying $x_1^p$); see [33, 36].

Consider now a sequence $\mathcal{W} = (w_0, w_1, \ldots)$ of words in $n$ letters. Say that $(g_1, \ldots, g_n)$ almost satisfies $\mathcal{W}$ if $w_i(g_1, \ldots, g_n) = 1$ for all $i$ large enough, and say that $G$ almost satisfies $\mathcal{W}$ if all $n$-tuples from $G$ almost satisfy $\mathcal{W}$.
For example, $G$ almost satisfies $(x_1, \ldots, x_i^1, \ldots)$ if and only if $G$ is a torsion group.

The problem of deciding algorithmically whether a group belongs to a given variety has received much attention (see e.g. [25] and references therein); we consider here the harder problems of determining whether a group (respectively a tuple) almost satisfies a given sequence. This has, up to now, been investigated mainly for the torsion sequence above [19].

The Engel law is

$$E_c = E_c(x, y) = [x, y, \ldots, y] = [[[x, y], y], \ldots, y]$$

with $c$ copies of ‘$y$’; so $E_0(x, y) = x$, $E_1(x, y) = [x, y]$ and $E_c(x, y) = [E_{c-1}(x, y), y]$.

See below for a motivation. Let us call a group (respectively a pair of elements) Engel if it almost satisfies $\delta = (E_0, E_1, \ldots)$. Furthermore, let us call $h \in G$ an Engel element if $(g, h)$ is Engel for all $g \in G$.

A concrete consequence of our investigations is:

**Theorem 1** The first Grigorchuk group $G_0$ is not Engel. Furthermore, an element $h \in G_0$ is Engel if and only if $h^2 = 1$.

We prove a similar statement for another prominent example of automaton group, the Gupta-Sidki group, see Theorem 2.

Theorem 1 follows from a partial algorithm, giving a criterion for an element $y$ to be Engel. This algorithm proves, in fact, that the element $ad$ in the Grigorchuk group is not Engel. Our aim is to solve the following decision problems in an automaton group $G$:

- **Engel($g, h$):** Given $g, h \in G$, does there exist $c \in \mathbb{N}$ with $E_c(g, h) = 1$?
- **Engel($h$):** Given $h \in G$, does Engel($g, h$) hold for all $g \in G$?

The algorithm is described in [7]. As a consequence,

**Corollary 1** Let $G$ be an automaton group acting on the set of binary sequences $\{1, 2\}^*$, that is contracting with contraction coefficient $\eta < 1$. Then, for torsion elements $h$ of order $2^e$ with $2^{2e} \eta < 1$, the property Engel($h$) is decidable.

The Engel property attracted attention for its relation to nilpotency: indeed a nilpotent group of class $c$ satisfies $E_c$, and conversely among compact [30] and solvable [23] groups, if a group satisfies $E_c$ for some $c$ then it is locally nilpotent. Conjecturally, there are non-locally nilpotent groups satisfying $E_c$ for some $c$, but this is still unknown. It is also an example of iterated identity, see [3, 14]. In particular, the main result of [3] implies easily that the Engel property is decidable in algebraic groups.

It is comparatively easy to prove that the first Grigorchuk group $G_0$ satisfies no law [1, 28]; this result holds for a large class of automaton groups. In fact, if a group satisfies a law, then so does its profinite completion. In the class mentioned above, the profinite completion contains abstract free subgroups, precluding the existence of a law. No such arguments would help for the Engel
property: the restricted product of all finite nilpotent groups is Engel, but the unrestricted product again contains free subgroups. This is one of the difficulties in dealing with iterated identities rather than identities.

If $\mathfrak{A}$ is a nil algebra (namely, for every $a \in \mathfrak{A}$ there exists $n \in \mathbb{N}$ with $a^n = 0$) then the set of elements of the form $\{1 + a : a \in \mathfrak{A}\}$ forms a group $1 + \mathfrak{A}$ under the law $(1 + a)(1 + b) = 1 + (a + b + ab)$. If $\mathfrak{A}$ is defined over a field of characteristic $p$, then $1 + \mathfrak{A}$ is a torsion group since $(1 + a)^p = 1$ if $a^p = 0$. Golod constructed in [20] non-nilpotent nil algebras $\mathfrak{A}$ all of whose 2-generated subalgebras are nilpotent (namely, $\mathfrak{A}^n = 0$ for some $n \in \mathbb{N}$); given such an $\mathfrak{A}$, the group $1 + \mathfrak{A}$ is Engel but not locally nilpotent.

Golod introduced these algebras as means of obtaining infinite, finitely generated, residually finite (every non-trivial element in the group has a non-trivial image in some finite quotient), torsion groups. Golod's construction is highly non-explicit, in contrast with Grigorchuk's group for which much can be derived from the automaton's properties.

It is therefore of high interest to find explicit examples of Engel groups that are not locally nilpotent, and the methods and algorithms presented here are a step in this direction.

In the remainder of this text, we concentrate on the Engel property, which is equivalent to nilpotency for finite groups. In particular, if an automaton group $G$ is to have a chance of being Engel, then its image under the map $\pi: G \to \text{Sym}(X)$ should be a nilpotent subgroup of $\text{Sym}(X)$. Since finite nilpotent groups are direct products of their $p$-Sylow subgroups, we may reduce to the case in which the image of $G$ in $\text{Sym}(X)$ is a $p$-group. A further reduction lets us assume that the image of $G$ is an abelian subgroup of $\text{Sym}(X)$ of prime order. We therefore make the following

**Standing assumption 1** The alphabet is $X = \{1, \ldots, p\}$ and automaton groups $G(\Phi)$ are generated by automata $\Phi: X \times Q \to Q \times X$ such that for every $q \in Q$ the corresponding map $\pi(q) \in \text{Sym}(X)$ describing the action of $q$ on $X$ takes values in the cyclic subgroup $\mathbb{Z}/p$ of $\text{Sym}(X)$ generated by the cycle $(1, 2, \ldots, p)$.

We make a further reduction in that we only consider the Engel property for elements of finite order. This is not a very strong restriction: given $h$ of infinite order, one can usually find an element $g \in G$ such that the conjugates $\{g^n : n \in \mathbb{Z}\}$ are independent, and it then follows that $h$ is not Engel. We content ourselves with an example:

**Example 3 (The Brunner-Sidki-Vieira group [12])** The generating set is $Q = \{\tau^{\pm 1}, \mu^{\pm 1}, 1\}$ and the alphabet is $X = \{1, 2\}$. The self-similarity structure is given by

$$
\begin{array}{c c}
1 & 2 \\
\Phi & \tau
\end{array}
\quad
\begin{array}{c c}
1 & 2 \\
\Phi & \tau
\end{array}
\quad
\begin{array}{c c}
1 & 2 \\
\Phi & \mu
\end{array}
\quad
\begin{array}{c c}
1 & 2 \\
\Phi & \mu^{-1}
\end{array}
$$
Let $G_2$ denote the group generated by $Q$. The elements $\tau, \mu$ have infinite order, and in fact act transitively on $X^n$ for all $n$.

Let us show that $(\mu, \tau)$ is not an Engel pair, namely $E_c(\mu, \tau) \neq 1$ for all $c \in \mathbb{N}$. We rely on the calculations in [3], which compute the lower 2-central series of $G_2$, namely the series of subgroups $\delta_1 = G_2$ and $\delta_{n+1} = [\delta_n, G_2]\{g^2 : g \in \delta_n\}$ for all $n \geq 1$. In that article, a basis of the $\mathbb{F}_2$-vector space $\delta_n/\delta_{n+1}$ is given for all $n$, and in particular one of the basis vectors of $\delta_n/\delta_{n+1}$ is $E_{n-1}(\mu, \tau)$.

7 A semi-algorithm for deciding the Engel property

We start by describing a semi-algorithm to check the Engel property. It will sometimes not return any answer, but when it returns an answer then that answer is guaranteed correct. It is guaranteed to terminate as long as the contraction property of the automaton group $G$ is strong enough.

Algorithm 1 Let $G$ be a contracting automaton group with alphabet $X = \{1, \ldots, p\}$ for prime $p$, with the contraction property $\|g \circ j\| \leq \eta\|g\| + C$.

For $n \in p\mathbb{N}$ and $R \in \mathbb{F}$ consider the following finite graph $\Gamma_{n,R}$. Its vertex set is $B(R)^n \cup \{\text{fail}\}$, where $B(R)$ denotes the set of elements of $G$ of length at most $R$. Its edge set is defined as follows: consider a vertex $(g_1, \ldots, g_n)$ in $\Gamma_{n,R}$, and compute

$$(h_1, \ldots, h_n) = (g_1^{-1} g_2, \ldots, g_n^{-1} g_1).$$

If $h_i$ fixes $X$ for all $i$, i.e. all $h_i$ have trivial image in $\text{Sym}(X)$, then for all $j \in \{1, \ldots, p\}$ there is an edge from $(g_1, \ldots, g_n)$ to $(h_1 \circ j, \ldots, h_n \circ j)$, or to fail if $(h_1 \circ j, \ldots, h_n \circ j) \notin B(R)^n$. If some $h_i$ does not fix $X$, then there is an edge from $(g_1, \ldots, g_n)$ to $(h_1, \ldots, h_n)$, or to fail if $(h_1, \ldots, h_n) \notin B(R)^n$.

Given $g, h \in G$ with $h^n = 1$: Set $t_0 = (g, g^h, g^{h^2}, \ldots, g^{h^{n-1}})$. If there exists $R \in \mathbb{N}$ such that no path in $\Gamma_{n,R}$ starting at $t_0$ reaches fail, then Engel$(g, h)$ holds if and only if the only cycle in $\Gamma_{n,R}$ reachable from $t_0$ passes through $(1, \ldots, 1)$.

If the contraction coefficient satisfies $2^n \eta < 1$, then it is sufficient to consider $R = (\|g\| + \|h\|)^n C/(1 - 2^n \eta)$.

Given $n \in \mathbb{N}$: The Engel property holds for all elements of exponent $n$ if and only if, for all $R \in \mathbb{N}$, the only cycle in $\Gamma_{n,R}$ passes through $(1, \ldots, 1)$.

If the contraction coefficient satisfies $2^n \eta < 1$, then it is sufficient to consider $R = 2^n C/(1 - 2^n \eta)$.

Given $G$ weakly branched and $n \in \mathbb{N}$: If for some $R \in \mathbb{N}$ there exists a cycle in $\Gamma_{n,R}$ that passes through an element of $K^n \setminus \{1\}$, then no element of $G$ whose order is a multiple of $n$ is Engel.

If the contraction coefficient satisfies $2^n \eta < 1$, then it is sufficient to consider $R = 2^n C/(1 - 2^n \eta)$.

We consider the graphs $\Gamma_{n,R}$ as subgraphs of a graph $\Gamma_{n,\infty}$ with vertex set $G^n$ and same edge definition as the $\Gamma_{n,R}$.
We note first that, if $G$ satisfies the contraction condition $2^n \eta < 1$, then all cycles of $\Gamma_{n,\infty}$ lie in fact in $\Gamma_{n,2^n C/(1-2^n \eta)}$. Indeed, consider a cycle passing through $(g_1, \ldots, g_n)$ with $\max_i |g_i| = R$. Then the cycle continues with $(g_1^{(1)}, \ldots, g_n^{(1)}), (g_1^{(2)}, \ldots, g_n^{(2)}), \ldots$ etc. with $|g_i^{(k)}| \leq 2^k R$; and then for some $k \leq n$ we have that all $g_i^{(k)}$ fix $X$; namely, they have a trivial image in $\text{Sym}(X)$, and the map $g \mapsto g@j$ is an injective homomorphism on them. Indeed, let

$$\pi_1, \ldots, \pi_n, \pi_1^{(1)}, \ldots, \pi_n^{(1)} \in \mathbb{Z}_p \subset \text{Sym}(X)$$

be the images of $g_1, \ldots, g_n, g_1^{(1)}, \ldots, g_n^{(1)}$ respectively, and denote by $S$: $\mathbb{Z}_p^n \rightarrow \mathbb{Z}_p^n$ the cyclic permutation operator. Then $(\pi_1^{(n)}, \ldots, \pi_n^{(n)}) = (S - 1)^n(\pi_1, \ldots, \pi_n)$, and $(S - 1)^n = \sum_i S^i(n) = 0$ since $p/n$ and $S^n = \mathbb{1}$. Thus there is an edge from $(g_1^{(k)}, \ldots, g_n^{(k)})$ to $(g_1^{(k+1)}@j, \ldots, g_n^{(k+1)}@j)$ with $|g_i^{(k+1)}@j| \leq \eta|g_i^{(k)}| + C \leq \eta 2^n R + C$. Therefore, if $R > 2^n C/(1 - 2^n \eta)$ then $2^n \eta R + C < R$, and no cycle can return to $(g_1, \ldots, g_n)$.

Consider now an element $h \in G$ with $h^n = \mathbb{1}$. For all $g \in G$, there is an edge in $\Gamma_{n,\infty}$ from $(g, g^h, \ldots, g^{h^{n-1}})$ to $(g, h@v, [g, h]@v, [g, h]^h@v)$ for some word $v \in \{x\} \sqcup X$, and therefore for all $c \in \mathbb{N}$ there exists $d \leq c$ such that, for all $v \in X^d$, there is a length-$c$ path from $(g, g^h, \ldots, g^{h^{n-1}})$ to $(E_c(g, h)@v, \ldots, E_c(g, h)^{h^{n-1}}@v)$ in $\Gamma_{n,\infty}$.

We are ready to prove the first assertion: if $\text{Engel}(g, h)$, then $E_c(g, h) = \mathbb{1}$ for some $c$ large enough, so all paths of length $c$ starting at $(g, g^h, \ldots, g^{h^{n-1}})$ end at $(\mathbb{1}, \ldots, \mathbb{1})$. On the other hand, if $\text{Engel}(g, h)$ does not hold, then all long enough paths starting at $(g, g^h, \ldots, g^{h^{n-1}})$ end at vertices in the finite graph $\Gamma_{n,2^n C/(1-2^n \eta)}$ so must eventually reach cycles; and one of these cycles is not $(\{\mathbb{1}, \ldots, \mathbb{1}\})$ since $E_c(g, h) \neq \mathbb{1}$ for all $c$.

The second assertion immediately follows: if there exists $g \in G$ such that $\text{Engel}(g, h)$ does not hold, then again a non-trivial cycle is reached starting from $(g, g^h, \ldots, g^{h^{n-1}})$, and independently of $g, h$ this cycle belongs to the graph $\Gamma_{n,2^n C/(1-2^n \eta)}$.

For the third assertion, let $\tilde{k} = (k_1, \ldots, k_n) \in K^n \setminus \mathbb{1}^n$ be a vertex of a cycle in $\Gamma_{n,2^n C/(1-2^n \eta)}$. Consider an element $h \in G$ of order $s_n$ for some $s \in \mathbb{N}$. By the condition that $#X = p$ is prime and the image of $G$ in $\text{Sym}(X)$ is a cyclic group, $s_n$ is a power of $p$, so there exists an orbit $\{v_1, \ldots, v_{s_n}\}$ of $h$, so labeled that $v_i^h = v_{i-1}$, indices being read modulo $s_n$. For $i = 1, \ldots, s_n$ define

$$h_i = (h@v_1)^{-1} \cdots (h@v_i)^{-1},$$

noting $h_i(h@v_i) = h_{i-1}$ for all $i = 1, \ldots, s_n$ since $h^{s_n} = \mathbb{1}$. Denote by $\langle i \rangle$ the unique element of $\{1, \ldots, n\}$ congruent to $i$ modulo $n$, and consider the element

$$g = \prod_{i=1}^{s_n} (v_i \ast k_{\langle i \rangle}^{h_i}),$$

which belongs to $G$ since $G$ is weakly branched. Let $(k_1^{(1)}, \ldots, k_n^{(1)})$ be the next vertex on the cycle of $\tilde{k}$. We then have, using (2),

$$[g, h] = g^{-1} g^h = \prod_{i=1}^{s_n} (v_i \ast k_{\langle i \rangle}^{-h_i}) \prod_{i=1}^{s_n} (v_{i-1} \ast k_{\langle i \rangle}^{h_i} h@v_i) = \prod_{i=1}^{s_n} (v_i \ast (k_{\langle i \rangle}^{(1)})^h).$$
and more generally $E_c(g,h)$ and some of its states are read off the cycle of $k$. Since this cycle goes through non-trivial group elements, $E_c(g,h)$ has a non-trivial state for all $c$, so is non-trivial for all $c$, and $Engel(g,h)$ does not hold.

8 Proof of Theorem 1

The Grigorchuk group $G_0$ is contracting, with contraction coefficient $\eta = 1/2$. Therefore, the conditions of validity of Algorithm 1 are not satisfied by the Grigorchuk group, so that it is not guaranteed that the algorithm will succeed, on a given element $h \in G_0$, to prove that $h$ is not Engel. However, nothing forbids us from running the algorithm with the hope that it nevertheless terminates. It seems experimentally that the algorithm always succeeds on elements of order 4, and the argument proving the third claim of Algorithm 1 (repeated here for convenience) suffices to complete the proof of Theorem 1.

Below is a self-contained proof of Theorem 1, extracting the relevant properties of the previous section, and describing the computer calculations as they were keyed in.

Consider first $h \in G_0$ with $h^2 = 1$. It follows from Proposition 3 that $h$ is Engel: given $g \in G_0$, we have $E_{1+k}(g,h) = [g,h]^{-2k}$ so $E_{1+k}(g,h) = 1$ for $k$ larger than the order of $[g,h]$.

For the other case, we start by a side calculation. In the Grigorchuk group $G_0$, define $x = [a,b]$ and $K = \langle x \rangle^{G_0}$ as in Proposition 4, consider the quadruple

$$A_0 = (A_{0,1}, A_{0,2}, A_{0,3}, A_{0,4}) = \left( x^{-2}a^{-2}, x^{-2}a^{-2}x^{-2}a, x^{-2}a^{-2}x^{-2}, x^{-2}a^{-2}x^{-2} \right)$$

of elements of $K$, and for all $n \geq 0$ define

$$A_{n+1} = (A_{n,1}^{-1}A_{n,2}, A_{n,2}^{-1}A_{n,3}, A_{n,3}^{-1}A_{n,4}, A_{n,4}^{-1}A_{n,1}).$$

**Lemma 1** For all $i = 1, \ldots, 4$, the element $A_{9,i}$ fixes 111112, is non-trivial, and satisfies $A_{9,i} \circ 111112 = A_{9,i}$.

**Proof** This is proven purely by a computer calculation. It is performed as follows within GAP:

```gap
gap> LoadPackage("FR");
gap> AssignGeneratorVariables(GrigorchukGroup);
gap> x2 := Comm(a,b)^-2; x2ca := x2*(c*a); one := a^0;
gap> A0 := [x2^-1*x2ca,x2ca^-1*x2*x2ca^b,(x2ca^-1)^b*x2^-1,x2];
gap> v := [1,1,1,1,2];; A := A0;;
gap> for n in [1..9] do A := List([1..4],i->A[i]^-1*A[1+i mod 4]); od;
gap> ForAll([1..4],i->v^-1*A[i]=v and A[i]<one and State(A[i],v)=A0[i]);
true
```
Consider now \( h \in G_0 \) with \( h^2 \neq 1 \). Again by Proposition 3 we have \( h^2 = 1 \) for some minimal \( e \in \mathbb{N} \), which is furthermore at least 2. We keep the notation ‘\( a \equiv b \)’ for the unique number in \( \{1, \ldots, b\} \) that is congruent to \( a \) modulo \( b \).

Let \( n \) be large enough so that the action of \( h \) on \( X^n \) has an orbit \( \{v_1, v_2, \ldots, v_{2^e}\} \) of length \( 2^e \), numbered so that \( v_{i+1} = v_i \) for all \( i \), indices being read modulo \( 2^e \). For \( i = 1, \ldots, 2^e \) define

\[
h_i = (h \circ v_1)^{-1} \cdots (h \circ v_1)^{-1},
\]

noting \( h_i (h \circ v_1) = h_{i-1} \equiv h \) for all \( i = 1, \ldots, 2^e \) since \( h^{2^e} = 1 \), and consider the element

\[
g = \prod_{i=1}^{2^e} (v_i \ast A_{0,0}^{h_i}),
\]

which is well defined since \( 4 \mid 2^e \) and belongs to \( G_0 \) by Proposition 4. We then have, using (4),

\[
[g, h] = g^{-1} g^h = \prod_{i=1}^{2^e} (v_i \ast A_{0,0}^{h_i}) \prod_{i=1}^{2^e} (v_{i-1} \ast 2^{2e} \ast A_{0,0}^{h_i}),
\]

and more generally

\[
E_c (g, h) = \prod_{i=1}^{2^e} (v_i \ast A_{e,i}^{h_i}).
\]

Therefore, by Lemma 1, for every \( k \geq 0 \) we have \( E_k (g, h) \circ v_0 (11112)^k = A_{0,1} \neq 1 \), so \( E_c (g, h) \neq 1 \) for all \( c \in \mathbb{N} \) and we have proven that \( h \) is not an Engel element.

9 Other examples

Similar calculations apply to the Gupta-Sidki group \( \Gamma \) introduced in [24]. This is another example of infinite torsion group, acting on \( X^* \) for \( X = \{1, 2, 3\} \) and generated by the states of the following automaton:

![Automaton Diagram]

The transformations \( a, t \) may also be defined recursively by

\[
(1v)^a = 2v, \quad (2v)^a = 3v, \quad (3v)^a = 1v, \\
(1v)^t = 1v^a, \quad (2v)^t = 2v^{a-1}, \quad (3v)^t = 3v^t.
\]
The Gupta-Sidki group is contracting, with contraction coefficient $\eta = \frac{1}{2}$. Again, this is not sufficient to guarantee that Algorithm 1 terminates, but it nevertheless did succeed in proving

**Theorem 2** The only Engel element in the Gupta-Sidki group $\Gamma$ is the identity.

We only sketch the proof, since it follows that of Theorem 1 quite closely. Analogues of Propositions 3 and 4 hold, with $[\Gamma, \Gamma]$ in the role of $K$. An analogue of Lemma 1 holds with $A_0 = ([a^{-1}, t], [a, t]^s, [t^{-1}, a^{-1}])$ and $A_{t^s, t^a} = A_{0, t}$.

### 10 Closing remarks

An important feature of automaton groups is their amenability to computer experiments, and even as in this case of rigorous verification of mathematical assertions; see also [26], and the numerous decidability and undecidability of the finiteness property in [2, 18, 27].

The proof of Theorem 1 relies on a computer calculation. It could be checked by hand, at the cost of quite unrewarding effort. One of the purposes of this article is, precisely, to promote the use of computers in solving general questions in group theory: the calculations performed, and the computer search involved, are easy from the point of view of a computer but intractable from the point of view of a human.

The calculations were performed using the author’s group theory package Fr, specially written to manipulate automaton groups. This package integrates with the computer algebra system GAP [15], and is freely available from the GAP distribution site

http://www.gap-system.org

It would be dishonest to withhold from the reader how I arrived at the examples given for the Grigorchuk and Gupta-Sidki groups. I started with small words $g, h$ in the generators of $G_0$, respectively $\Gamma$, and computed $E_c(g, h)$ for the first few values of $c$. These elements are represented, internally to Fr, as Mealy automata. A natural measure of the complexity of a group element is the size of the minimized automaton, which serves as a canonical representation of the element.

For some choices of $g, h$ the size increases exponentially with $c$, limiting the practicality of computer experiments. For others (such as $(g, h) = ((ba)^4c, ad)$ for the Grigorchuk group), the size increases roughly linearly with $c$, making calculations possible for $c$ in the hundreds. Using these data, I guessed the period $p$ of the recursion (9 in the case of the Grigorchuk group), and searched among the states of $E_c(g, h)$ and $E_{c+p}(g, h)$ for common elements; in the example, I found such common states for $c = 23$. I then took the smallest-size quadruple of states that appeared both in $E_c(g, h)$ and $E_{c+p}(g, h)$ and belonged to $K$, and expressed the calculation taking $E_c(g, h)$ to $E_{c+p}(g, h)$ in the form of Lemma 1.
It was already shown by Bludov [9] that the wreath product $G_0^4 \rtimes D_4$ is not Engel. He gave, in this manner, an example of a torsion group in which a product of Engel elements is not Engel. Our proof is a refinement of his argument. In fact, his result may also be used to obtain another proof of the fact that $G_0$ is not Engel: the Grigorchuk contains a copy of $D_4$, say generated by $a, d$, which has an orbit of size 4, for example $\{111, 112, 211, 212\}$. The branching subgroup $K$ contains a subgroup, for example the stabilizer $K_{111}$ of 111, which maps onto $G_0$ by restriction to the subtree $111X^*$. The Grigorchuk group therefore contains the subgroup $(111 \ast K_{111}, a, d) \cong K_{111}^4 \rtimes D_4$ which maps onto the non-Engel group $G_0^4 \rtimes D_4$, so $G_0$ itself is not Engel.

A direct search for the elements $A_{0,1,\ldots,4}$ appearing in the proof of Theorem [7] would probably not be successful, and has not yielded simpler elements than those given before Lemma [1] if one restricts them to belong to $K$; one can only wonder how Bludov found the quadruple $(1, d, ca, ab)$, presumably without the help of a computer.

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