A GENERIC CLASSIFICATION OF EXCEPTIONAL
ORTHOGONAL $X_1$-POLYNOMIALS BASED ON PEARSON
DISTRIBUTIONS FAMILY

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Abstract. The so-called exceptional orthogonal $X_1$-polynomials arise as eigen-
functions of a Sturm-Liouville problem. In this paper, a generic classification
of these polynomials is presented based on Pearson distributions family. Then,
six special differential equations of the aforesaid classification are introduced
and their polynomial solutions are studied in detail.

1. INTRODUCTION

Classical orthogonal polynomials are known to play a fundamental role in the
construction of bound-state solutions to exactly solvable potentials in quantum
mechanics [24]. They are eigenfunctions of some Sturm-Liouville problems and
form complete sets with respect to some positive-definite measures [12, 14, 20].

Consider the second order differential equation

$$
\frac{d}{dx} \left( k(x) \frac{dy}{dx} \right) - \left( \lambda \rho(x) + q(x) \right) y = 0,
$$

on an open interval, say $(a, b)$, with the boundary conditions

$$
\alpha_1 y(a) + \beta_1 y'(a) = 0,
\alpha_2 y(b) + \beta_2 y'(b) = 0,
$$

in which $\alpha_1, \alpha_2$ and $\beta_1, \beta_2$ are given constants and the functions $k(x) > 0$, $k'(x)$, $q(x)$
and $\rho(x) > 0$ in (1.1) are assumed to be continuous for $x \in [a, b]$. The boundary
value problem (1.1)-(1.2) is called a regular Sturm-Liouville problem and if one of
the points $a$ and $b$ is singular (i.e. $k(a) = 0$ or $k(b) = 0$), it is called a singular
Sturm-Liouville problem [3]. Sturm-Liouville problems appear in various banches
of engineering, physics and biology. Recently in [18], some generalized Sturm-
Liouville problems in three different continuous, discrete and q-discrete spaces have
been introduced and classified.

Let $y_n(x)$ and $y_m(x)$ be two solutions of equation (1.1). Following Sturm-
Liouville theory [3, 21], these functions are orthogonal with respect to the positive

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weight function \( \rho(x) \) on \((a, b)\) under the given conditions \([1.2]\), i.e.

\[
\int_{a}^{b} \rho(x)y_n(x)y_m(x) \, dx = \left( \int_{a}^{b} \rho(x)y_n^2(x) \, dx \right) \delta_{n,m},
\]

where

\[
\delta_{n,m} = \begin{cases} 
0 & (n \neq m), \\
1 & (n = m). 
\end{cases}
\]

Many special functions in theoretical and mathematical physics are solutions of a regular or singular Sturm-Liouville problem satisfying the orthogonality condition \([1.3]\), see e.g. \([3, 5, 19, 27]\).

There are totally six sequences of real polynomials \([15, 17]\) that are orthogonal with respect to the Pearson distributions family

\[
W_{d^*, e^* a, b, c}(x) = \exp \left( \int_{a}^{b} d^* x + e^* ax^2 + bx + c \, dx \right) \quad (a, b, c, d^*, e^* \in \mathbb{R}).
\]

Three of them (i.e. Jacobi, Laguerre and Hermite polynomials \([20]\)) are infinitely orthogonal with respect to three special cases of the positive function \([1.4]\) (i.e. Beta, Gamma and Normal distributions \([13]\)) and three other ones are finitely orthogonal with respect to F-Fisher, Inverse Gamma and Generalized T-Student distributions \([16, 13]\), limited to some parametric constraints. Table 1 shows the main properties of these six sequences.

**Table 1. Characteristics of six sequences of orthogonal polynomials**

| Polynomial notation | Distribution | Weight function | Kind, Interval and Parameters constraint |
|---------------------|--------------|-----------------|-----------------------------------------|
| \(P_n^{(\alpha,\beta)}(x)\) | Beta | \(W_{-\alpha - \beta, -\alpha + \beta}(-1, 0, 1) = (1 - x)^\alpha(1 + x)^\beta\) | Infinite \([-1, 1]\), \(\forall n, \alpha > -1, \beta > -1\) |
| \(L_n^{(\alpha)}(x)\) | Gamma | \(W_{-1, \alpha}(-1, 0, 1) = x^\alpha \exp(-x)\) | Infinite \([0, \infty)\), \(\forall n, \alpha > -1\) |
| \(H_n(x)\) | Normal | \(W_{-2, 0}(-2, 0, 1) = \exp(-x^2)\) | Infinite \((-\infty, \infty)\) |
| \(M_n^{(p,q)}(x)\) | Fisher F | \(W_{-p, q}(-p, q 1, 1, 0) = x^p(1 + x)^{-q}\) | Finite \([0, \infty)\), max \(n < (p - 1)/2\), \(q > -1\) |
| \(N_n^{(p)}(x)\) | Inverse Gamma | \(W_{-p, 1}(-p, 1, 0) = x^{-p} \exp(-1/x)\) | Finite \([0, \infty)\), max \(n < (p - 1)/2\) |
| \(J_n^{(p,q)}(x)\) | Generalized T | \(W_{-2p, q}(-2p, q 1, 1, 0) = (1 + x^2)^{-p} \exp(q \arctan x)\) | Finite \((-\infty, \infty)\), max \(n < p - 1/2\) |
It has been proved by S. Bochner [4] that if an infinite sequence of polynomials \( \{P_n\}_{n=0}^{\infty} \) satisfies a second-order eigenvalue equation of the form
\[
\sigma(x)P''_n(x) + \tau(x)P'_n(x) + r(x)P_n(x) = \lambda_n P_n(x) \quad n = 0, 1, 2, \ldots,
\]
then \( \sigma(x) \), \( \tau(x) \) and \( r(x) \) must be polynomials of degree 2, 1 and 0, respectively. Moreover, if the sequence \( \{P_n\}_{n=0}^{\infty} \) is orthogonal, then it has to be one of the classical Jacobi, Laguerre or Hermite polynomials which satisfy a second order differential equation of the form [1, 2, 4]
\[
(1.5) \quad \sigma(x)y''_n(x) + \tau(x)y'_n(x) - \lambda_n y_n(x) = 0,
\]
where
\[
\sigma(x) = ax^2 + bx + c \quad \text{and} \quad \tau(x) = dx + e,
\]
and
\[
\lambda_n = n(d + (n - 1)a),
\]
is the eigenvalue depending on \( n = 0, 1, 2, \ldots \). However, there are three other sequences of hypergeometric polynomials that are solutions of the equation \( (1.5) \) but finitely orthogonal [16, 17].

It is usually supposed in the literature that the orthogonal polynomial systems start with a polynomial of degree 0. Nevertheless, from Sturm-Liouville theory point of view, this restriction is not necessary [9].

Recently in [9, 10], two new families of exceptional orthogonal polynomials, i.e. Jacobi \( X_1 \)-polynomials \( \hat{P}^{(\alpha,\beta)}_n(x) \) and Laguerre \( X_1 \)-polynomials \( \hat{L}^{(\alpha)}_n(x) \) have been introduced as solutions of a second-order eigenvalue equation of the form
\[
(1.6) \quad \left( k_2(x - b)^2 + k_1(x - b) + k_0 \right)y''_n(x) + \frac{ax - ab - 1}{x - b} \left( k_1(x - b) + 2k_0 \right)y'_n(x) - \left( \frac{a}{x - b} \left( k_1(x - b) + 2k_0 \right) + \lambda_n \right)y_n(x) = 0,
\]
for \( n \geq 1 \), where
\[
\lambda_n = (n - 1)(nk_2 - ak_1),
\]
and \( k_0 \neq 0, k_1, k_2 \) are real constants. In this sense, the authors in [10] have proved a converse statement similar to Bochner’s theorem for the classical orthogonal polynomials: if a self-adjoint second order operator has a polynomial eigenfunctions \( \{P_1(x)\}_{n=1}^{\infty} \), then it must be either the \( X_1 \)-Jacobi or the \( X_1 \)-Laguerre Sturm-Liouville problem. Moreover, the functions
\[
(1.7) \quad \tilde{W}_{\alpha,\beta}(x) = \left( x - \frac{\beta + \alpha}{\beta - \alpha} \right)^{-2} (1 - x)^{\alpha}(1 + x)^{\beta} \quad \text{for} \quad x \in (-1, 1),
\]
with the restrictions \( \alpha, \beta > -1, \alpha \neq \beta, \text{sgn} \alpha = \text{sgn} \beta \), and
\[
(1.8) \quad \tilde{W}_{\alpha}(x) = (x + \alpha)^{-2}x^{\alpha}e^{-x} \quad \text{for} \quad x \in (0, \infty),
\]
with the restriction \( \alpha > 0 \), are the weight functions corresponding to \( \hat{P}^{(\alpha,\beta)}_n(x) \) and \( \hat{L}^{(\alpha)}_n(x) \), respectively.

Exceptional orthogonal polynomials have been recently of great interest due to their important applications in exactly solvable potentials and supersymmetry [24], Dirac operators minimally coupled to external fields [11] and entropy measures in quantum information theory [6]. Moreover, the relationship between exceptional orthogonal polynomials and Darboux transformations has been observed, giving
rise to new families of $X_2$ polynomials of codimension two [24 25]. See also [22 23] for higher-order codimensional families.

In this paper, we consider six sequences of $X_1$ orthogonal polynomials as special solutions of a generic Sturm-Liouville equation of the form

\begin{equation}
(1.9) \quad (x - r) \left( a_2 x^2 + a_1 x + a_0 \right) y''_n(x) + \left( b_2 x^2 + b_1 x + b_0 \right) y'_n(x) - \left( \lambda_n(x - r) + c_0 \right) y_n(x) = 0, \quad n \geq 1,
\end{equation}

where $r$ is a real parameter such that $a_2 r^2 + a_1 r + a_0 \neq 0$ and the roots of $b_2 x^2 + b_1 x + b_0$ are supposed to be real.

Both infinite and finite types of exceptional orthogonal $X_1$-polynomials can be extracted from the above equation (1.9). Although three infinite polynomial sequences have been investigated in [8] for only some particular parameters, the finite cases of exceptional $X_1$-polynomials orthogonal with respect to three particular weight functions on infinite intervals are introduced in this paper for the first time. A fundamental point is that the weight functions corresponding to these six sequences are exactly a multiplication of Pearson distributions family introduced in table 1.

2. A Review on Classical Orthogonal Polynomials

It is shown in [15] that the monic polynomial solution of equation (1.5) can be represented as

\begin{equation}
(2.1) \quad y_n(x) = P_n \left( \frac{d, e}{a, b, c} \right) x^n = \sum_{k=0}^{n} \binom{n}{k} G_k^{(n)}(a, b, c, d, e) x^k,
\end{equation}

where

\begin{equation}
G_k^{(n)} = \left( \frac{2a}{b + \sqrt{b^2 - 4ac}} \right)^{k-n} {}_2F_1 \left( \frac{k - n, 2a - bd - \Delta}{2a(\sqrt{b^2 - 4ac}) + 1 - d + 2n} \right),
\end{equation}

and

\begin{equation}
{}_2F_1 \left( \frac{a, b}{c} \right) x^k = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{x^k}{k!},
\end{equation}

is the Gauss hypergeometric function [26] for $(a)_k = (a+1) \ldots (a+k-1)$.

The general formula (2.1) is a suitable tool to compute the coefficients of $x^k$ for any fixed degree $k$ and arbitrary $a$. For example, to obtain the coefficient $x^{n-1}$, it is enough to calculate the term

\begin{equation}
G_{n-1}(a, b, c, d, e) = \left( \frac{2a}{b + \Delta} \right)^{n-1} {}_2F_1 \left( -1, \frac{2a - bd - (d + (2n - 2)a) \Delta}{2a \Delta} \right)
\end{equation}

\begin{equation}
= \left( \frac{b + \Delta}{2a} \right) \left( 1 + \frac{2ac - bd - (d + (2n - 2)a) \Delta}{2a \Delta} \right) \left( 2a \Delta b + \Delta d + (2n - 2)a \right)^{-1} \frac{a}{d + (2n - 2)a},
\end{equation}

in which $\Delta = \sqrt{b^2 - 4ac}$. Note in (2.2) that all parameters are free and can adopt any value including zero because neither both values $a$ and $d$ nor both values $b$ and $c$ are supposed to be real.
and $e$ can vanish together in (2.3). After simplifying $G_k^{(n)}(a, b, c, d, e)$ for $k = n - 1, n - 2, \ldots$ we eventually obtain

$$
\bar{P}_n \begin{pmatrix} d & e \\ a & b & c \end{pmatrix} | x \rangle = x^n + \left( \frac{n}{2} \right) \frac{e + (n-1)b}{d + (2n-2)a} x^{n-1} + \cdots
$$

where

$$
\bar{P}_0 \begin{pmatrix} d & e \\ a & b & c \end{pmatrix} | x \rangle = 1,
$$

$$
\bar{P}_1 \begin{pmatrix} d & e \\ a & b & c \end{pmatrix} | x \rangle = x + \frac{e}{d},
$$

$$
\bar{P}_2 \begin{pmatrix} d & e \\ a & b & c \end{pmatrix} | x \rangle = x^2 + \frac{2e + b}{d + 2a} x + \frac{c(d + 2a) + e(e + b)}{(d + 2a)(d + a)},
$$

$$
\bar{P}_3 \begin{pmatrix} d & e \\ a & b & c \end{pmatrix} | x \rangle = x^3 + 3\frac{e + 2b}{d + 4a} x^2 + 3\frac{c(d + 4a)(e + 2b)}{(d + 4a)(d + 3a)} x + \frac{2c(d + 3a)(e + 2b) + cc(d + 4a) + e(e + 2b)}{(d + 4a)(d + 3a)(d + 2a)}.
$$

Moreover, by referring to the Nikiforov and Uvarov approach [21] and considering equation (1.5) as a self-adjoint form, the Rodrigues representation of the monic polynomials is derived as

$$
(2.3) \quad \bar{P}_n \begin{pmatrix} d & e \\ a & b & c \end{pmatrix} | x \rangle = \frac{1}{\left( \prod_{k=1}^{n} d + (n + k - 2)a \right)} W \left( \frac{d & e \\ a & b & c \end{pmatrix} | x \rangle \right)
$$

$$
\times d^n \left( \left( ax^2 + bx + c \right) W \left( \frac{d & e \\ a & b & c \end{pmatrix} | x \rangle \right) \right),
$$

where

$$
(2.4) \quad W \left( \frac{d & e \\ a & b & c \end{pmatrix} | x \rangle \right) = \exp \left( \int \frac{(d - 2a)x + e - b}{ax^2 + bx + c} dx \right).
$$

By using the formulæ (2.1) or (2.3) we can also obtain a generic three term recurrence equation for the polynomials as follows

$$
\bar{P}_{n+1}(x) = \left( x + \frac{2n(n+1)ab + (d - 2a)(e + 2nb)}{(d + 2na)(d + 2n - 2a)} \right) \bar{P}_n(x)
$$

$$
+ n(d + (n-2)a) x + \frac{c(d + 2n - 2a)^2 - nb^2(d + (n-2)a) + (e - b)(a(e + b) - bd)}{(d + 2n - 3a)(d + 2n - 2a)^2(d + 2n - 1a)} \bar{P}_{n-1}(x),
$$

$$
\frac{c(d + 2n - 2a)^2 - nb^2(d + (n-2)a) + (e - b)(a(e + b) - bd)}{(d + 2n - 3a)(d + 2n - 2a)^2(d + 2n - 1a)}\bar{P}_{n-1}(x),
$$
in which \( \bar{P}_n(x) \) denotes the same as monic polynomials of (2.1) with the initial values
\[
\bar{P}_0(x) = 1 \quad \text{and} \quad \bar{P}_1(x) = x + \frac{e}{d}.
\]

Finally, the norm square value of the monic polynomials (2.1) can be simplified and computed as follows:

Let \([L, U]\) be a predetermined orthogonality interval which consists of the zeros of \( \sigma(x) = ax^2 + bx + c \) or \( \pm \infty \). By noting the Rodrigues representation (2.3) we have

\[
(2.5) \quad \left\| \bar{P}_n \right\|^2 = \int_L^U \bar{P}_n^2 \left( \begin{array}{ccc} d & e & x \\ a & b & c \end{array} \right) W \left( \begin{array}{ccc} d & e & x \\ a & b & c \end{array} \right) dx = \frac{1}{\prod_{k=1}^{n} d + (n + k - 2)a} \times \int_L^U \bar{P}_n \left( \begin{array}{ccc} d & e & x \\ a & b & c \end{array} \right) \frac{d^n}{dx^n} \left( (ax^2 + bx + c)^n W \left( \begin{array}{ccc} d & e & x \\ a & b & c \end{array} \right) \right) dx.
\]

So, integrating by parts from the right hand side of (2.5) yields

\[
(2.5) \quad \left\| \bar{P}_n \right\|^2 = \frac{n! (-1)^n}{\prod_{k=1}^{n} d + (n + k - 2)a} \int_L^U (ax^2 + bx + c)^n \left( \exp \int (d - 2a)x + e - b ax^2 + bx + c \right) dx.
\]

Although the Jacobi polynomials
\[
\bar{P}^{(\alpha,\beta)}_n(x) = \bar{P}_n \left( \begin{array}{ccc} -\alpha - \beta - 2, & \beta - \alpha \\ -1, & 0, & 1 \end{array} \right),
\]

Laguerre polynomials
\[
\bar{L}^{(\alpha)}_n(x) = \bar{P}_n \left( \begin{array}{ccc} -1, & \alpha + 1 \\ 0, & 1, & 0 \end{array} \right),
\]

and Hermit polynomials
\[
\bar{H}_n(x) = \bar{P}_n \left( \begin{array}{ccc} -2, & 0 \\ 0, & 0, & 1 \end{array} \right),
\]

are three polynomial solutions of equation (1.5), there are three other sequences of hypergeometric polynomials that are finitely orthogonal with respect to the generalized T, inverse Gamma and F distributions [16, 17] and are solutions of equation (1.6).

The first finite sequence of classical orthogonal polynomials as

\[
(2.6) \quad \bar{M}^{(p,q)}_n(x) = \bar{P}_n \left( \begin{array}{ccc} 2 - p, & 1 + q \\ 1, & 1, & 0 \end{array} \right),
\]

satisfies the differential equation

\[
(2.7) \quad (x^2 + x)y''_n(x) + ((2 - p)x + q + 1)y'_n(x) - n(n + 1 - p)y_n(x) = 0,
\]

and is finitely orthogonal with respect to the weight function

\[
W_1(x; p, q) = x^q (1 + x)^{-(p+q)},
\]

on \([0, \infty)\) if and only if [17]

\[
p > 2\{\max n\} + 1 \quad \text{and} \quad q > -1.
\]
The second finite sequence defined as
\[
\tilde{N}_n^{(p)}(x) = P_n \left( \begin{array}{ccc}
2 - p, & 1, & 0 \\
0, & 0, & 1
\end{array} \right) x^n,
\]
satisfies the differential equation
\[
x^2 y_n''(x) + ((2 - p)x + 1)y_n'(x) - n(n + 1 - p)y_n(x) = 0,
\]
and is finitely orthogonal with respect to the weight function \(W_2(x; p) = x^{-p}e^{-\frac{1}{x}}\),
on \((0, \infty)\) for \(n = 0, 1, 2, \ldots, N < \frac{p - 1}{2}\).

Finally, the third finite sequence, which is finitely orthogonal with respect to the

generalized T-student distribution weight function
\[
W_3(x; p, q) = (1 + x^2)^{-p} \exp(q \arctan x),
\]
is defined on \((-\infty, \infty)\) as
\[
\tilde{m}_n^{(p, q)}(x) = P_n \left( \begin{array}{ccc}
2 - 2p, & q, & 1 \\
0, & 1, & 0
\end{array} \right) x^n,
\]
satisfying the equation
\[
(1 + x^2) y_n''(x) + (2(1 - p)x + q) y_n'(x) - n(n + 1 - 2p) y_n(x) = 0,
\]
and the orthogonality property holds if
\[
n = 0, 1, 2, \ldots, N < p - \frac{1}{2} \text{ and } q \in \mathbb{R}.
\]

3. A Generic Classification of Exceptional \(X_1\)-Orthogonal Polynomials

Since for any arbitrary real parameters \(A, B\) and \(C\) the relation
\[
Ax^2 + Bx + C = A(x - r)^2 + (2Ar + B)(x - r) + Ar^2 + Br + C,
\]
always holds true, another form of equation (3.3) is as
\[
(x - r) \left( a_2(x - r)^2 + (2a_2r + a_1)(x - r) + a_2r^2 + a_1r + a_0 \right) y_n''(x)
+ \left( b_2(x - r)^2 + (2b_2r + b_1)(x - r) + b_2r^2 + b_1r + b_0 \right) y_n'(x)
- \left( \lambda_n(x - r) + c_0^* \right) y_n(x) = 0, \quad n \geq 1.
\]

The eigenvalue \(\lambda_n\) in (3.1) is to be determined such that for every \(n \geq 1\), the
solution \(y_n\) is a polynomial of degree \(n\). For this purpose, we first consider a
subspace of the whole space of polynomials of degree at most \(n\) as
\[
\Pi_{n, r, \nu} = \text{span} \left\{ (x - r - \nu), (x - r)^2, \ldots, (x - r)^n \right\},
\]
in which \(\nu\) is a real constant. Then substituting \(y_1(x) = x - r - \nu\) and \(y_n(x) = (x - r)^n\) for \(n \geq 2\) into (3.1) respectively yield
\[
(b_2 - \lambda_1)(x - r)^2 + (2b_2r + b_1 - c_0^* + \nu \lambda_1)(x - r) + b_2r^2 + b_1r + b_0 + \nu c_0^* = 0,
\]
and
\[ n(n-1)\left(a_2(x-r)^2 + (2a_2r + a_1)(x-r) + a_2r^2 + a_1r + a_0\right)(x-r)^{n-1} + n\left(b_2(x-r)^2 + (2b_2r + b_1)(x-r) + b_2r^2 + b_1r + b_0\right)(x-r)^{n-1} - \left(\lambda_n(x-r) + c_0^*\right)(x-r)^n = 0 \quad n \geq 2. \]

Therefore
\[ \lambda_n = n((n-1)a_2 + b_2) \quad \text{for} \quad n \geq 1, \]
and
(3.3) \[ \begin{cases} 2b_2r + b_1 - c_0^* + \nu b_2 = 0, \\ b_2r^2 + b_1r + b_0 + \nu c_0^* = 0. \end{cases} \]

By solving the system (3.3) we get
\[ \nu = \frac{-(2b_2r + b_1) \pm \sqrt{b_1^2 - 4b_0b_2}}{2b_2} = \begin{cases} r_1 - r, \\ r_2 - r, \end{cases} \]
where \( r_1, r_2 \) are roots of \( b_2x^2 + b_1x + b_0 \), and

\[ c_0^* = \frac{2b_2(b_2r + b_1 + b_0)}{2b_2r + b_1 \pm \sqrt{b_1^2 - 4b_0b_2}} = \begin{cases} b_2(r - r_2), \\ b_2(r - r_1). \end{cases} \]

**Corollary 3.1.** If we take \( b_2x^2 + b_1x + b_0 = b_2(x-r_1)(x-r_2) \) and
\[ \Pi_{n,r,\nu} = \text{span}\{e_k(x)\}_{k=1}^n, \]
then
(i) \( e_1(x) = x - r_1 \) and \( \{e_k(x)\}_{k=2}^\infty = \{(x-r)^k\}_{k=2}^\infty \) lead to \( c_0^* = b_2(r - r_2). \)

(ii) \( e_1(x) = x - r_2 \) and \( \{e_k(x)\}_{k=2}^\infty = \{(x-r)^k\}_{k=2}^\infty \) lead to \( c_0^* = b_2(r - r_1). \)

We will use this important corollary in the next sections.

We are now in a good position to prove that the polynomial solutions of equation (3.1) in \( \Pi_{n,r,\nu} \) are orthogonal on an interval, say \([a, b]\), with respect to a weight function in the form
(3.4) \[ \rho(x) = (x-r)\omega(x), \]
where \( \omega(x) \) satisfies the differential equation
(3.5) \[ \frac{\omega'(x)}{\omega(x)} = \frac{(b_2 - 3a_2)x^2 + (b_1 - 2a_1 + 2a_2r)x + b_0 - a_0 + a_1r}{(a_2x^2 + a_1x + a_0)}(x-r). \]

To prove this claim, we first consider the self-adjoint form of equation (3.1) as
(3.6) \[ \left(\omega(x)(x-r)(a_2x^2 + a_1x + a_0)y_n'\right)' = \omega(x)\left(\lambda_n(x-r) + c_0^*\right)y_n(x), \]
and for the index \( m \) as
(3.7) \[ \left(\omega(x)(x-r)(a_2x^2 + a_1x + a_0)y_m'\right)' = \omega(x)\left(\lambda_m(x-r) + c_0^*\right)y_m(x). \]
Multiplying by \( y_m(x) \) and \( y_n(x) \) in relations (3.3) and (3.4) respectively, subtracting them and then integrating from both sides we get

\[
(3.8) \quad \left[ \omega(x)(x-r)\left(a_2x^2 + a_1x + a_0\right) \left(y'_n(x)y_m(x) - y'_m(x)y_n(x)\right) \right]_a^b = (\lambda_n - \lambda_m) \int_a^b (x-r)\omega(x)y_n(x)y_m(x) \, dx.
\]

Now if the following relations
\[
\omega(a)(a-r)(a_2a^2 + a_1a + a_0) = 0,
\omega(b)(b-r)(a_2b^2 + a_1b + a_0) = 0,
\]
hold, the left hand side of (3.8) is equal to zero and therefore
\[
\int_a^b (x-r)\omega(x)y_n(x)y_m(x) \, dx = 0 \quad m \neq n,
\]
which shows the orthogonality of polynomial sequence \( \{y_n(x)\}_{n=1}^\infty \) with respect to the weight function \( \rho(x) = (x-r)\omega(x) \). On the other hand, according to (3.5), \( \rho(x) \) should have “seven” free parameters (regardless \( r \), one parameter more than the number of parameters in Pearson distribution) because the explicit solution of equation (3.5) is as

\[
(3.9) \quad \omega(x) = \exp \left( \int \frac{(b_2 - 3a_2)x^2 + (b_1 - 2a_1 + 2a_2r)x + b_0 - a_0 + a_1r}{(a_2x^2 + a_1x + a_0)(x-r)} \, dx \right).
\]

Here the key point is that the function (3.9) is exactly a multiplication of Pearson distribution given in (1.4), because if the integrand function of (3.9) is written as a sum of two fractions with linear and quadratic denominators in the form

\[
\frac{(b_2 - 3a_2)x^2 + (b_1 - 2a_1 + 2a_2)x + b_0 - a_0 + a_1r}{(a_2x^2 + a_1x + a_0)(x-r)} = \frac{b_2x^2 + b_1r + b_0}{a_2x^2 + a_1x + a_0} - \frac{1}{x-r} + \frac{(b_2 - a_2(2 + \frac{b_2x^2 + b_1r + b_0}{a_2x^2 + a_1x + a_0}))x + b_1 + b_2r - \left(\frac{b_2x^2 + b_1r + b_0}{a_2x^2 + a_1x + a_0}\right)(a_1 + a_2r) - a_1}{a_2x^2 + a_1x + a_0},
\]
then we obtain

\[
(3.10) \quad \omega(x) = (x-r)^\frac{b_2x^2 + b_1r + b_0}{a_2x^2 + a_1x + a_0} - 1 \times \exp \left( \int \frac{(b_2 - a_2(2 + \frac{b_2x^2 + b_1r + b_0}{a_2x^2 + a_1x + a_0}))x + b_1 + b_2r - \left(\frac{b_2x^2 + b_1r + b_0}{a_2x^2 + a_1x + a_0}\right)(a_1 + a_2r) - a_1}{a_2x^2 + a_1x + a_0} \, dx \right) = (x-r)^\frac{b_2x^2 + b_1r + b_0}{a_2x^2 + a_1x + a_0} - 1 \times \left( b_2 - a_2(2 + \frac{b_2x^2 + b_1r + b_0}{a_2x^2 + a_1x + a_0}), b_1 + b_2r - \left(\frac{b_2x^2 + b_1r + b_0}{a_2x^2 + a_1x + a_0}\right)(a_1 + a_2r) - a_1 \right) \times W \left( b_2 - a_2(2 + \frac{b_2x^2 + b_1r + b_0}{a_2x^2 + a_1x + a_0}), b_1 + b_2r - \left(\frac{b_2x^2 + b_1r + b_0}{a_2x^2 + a_1x + a_0}\right)(a_1 + a_2r) - a_1 \bigg| x \right),
\]
and accordingly, 
\begin{equation}
\rho(x) = (x-r)^{b_2 + b_1 r + b_0} \frac{b_2 - a_2(2 + \frac{b_2 x^2 + b_1 x + b_0}{a_2 x^2 + a_1 x + a_0})}{b_1 + b_2 r - \frac{b_2 x^2 + b_1 x + b_0}{a_2 x^2 + a_1 x + a_0}} W \left( \frac{a_2}{a_1}, x \right).
\end{equation}

**Corollary 3.2.** The eigenfunctions of the equation 
\begin{equation}
(x-r)(a_2 x^2 + a_1 x + a_0) y''_n(x) + (b_2 x^2 + b_1 x + b_0) y'_n(x) - \left(n(b_2 + (n-1)a_2)(x-r) + c_0^*\right)y_n(x) = 0 \quad n \geq 1,
\end{equation}
where \((-1)^\theta = 1\), are exceptional $X_1$-polynomials orthogonal with respect to the weight function \([3.11]\).

Let us make a contract here that the polynomial solution of equation \([3.12]\) is indicated as 
\begin{equation}
y_n(x) = Q_{n,r} \left( \frac{b_2, b_1, b_0}{a_2, a_1, a_0} \right) x.
\end{equation}

By referring to the Pearson distributions family \([1.4]\), we can now follow an inverse process and suppose that a simplified case of the weight function \([3.11]\) is given as 
\begin{equation}
\rho(x) = (x-r)^\theta W \left( \frac{d^*, e^*}{a, b, c} \right),
\end{equation}
in which \((-1)^\theta = 1\). Then, by noting the equation \([3.12]\) the unknown polynomials $p_2(x)$ and $q_2(x)$ of degree 2 in the differential equation 
\begin{equation}
(x-r)p_2(x)y''_n(x) + q_2(x)y'_n(x) - (\lambda_n(x-r) + c_0^*)y_n = 0,
\end{equation}
can be directly derived by computing the logarithmic derivative of the function 
\begin{equation}
\rho(x) = (x-r)^{\theta-1}W \left( \frac{d^*, e^*}{a, b, c} \right) = (x-r)^{\theta-1}W(x),
\end{equation}
as 
\begin{equation}
\frac{((x-r)^{\theta-1}W(x))'}{(x-r)^{\theta-1}W(x)} = \frac{\theta - 1}{x-r} + \frac{W'(x)}{W(x)} = \frac{\theta - 1}{x-r} + \frac{d^* x + e^*}{a x^2 + b x + c} = \frac{(d^* + (\theta - 1)a)x^2 + (e^* - rd^* + (\theta - 1)b)x - re^* + (\theta - 1)c}{(x-r)(a x^2 + b x + c)},
\end{equation}
and then equating the result with 
\begin{equation}
\frac{q_2(x) - ((x-r)p_2(x))'}{(x-r)p_2(x)},
\end{equation}
so that we finally obtain 
\begin{equation}
p_2(x) = ax^2 + bx + c,
\end{equation}
and 
\begin{equation}
q_2(x) = (d^* + (\theta + 2)a)x^2 + (e^* - r(d^* + 2a) + (\theta + 1)b)x + \theta c - r(e^* + b),
\end{equation}
provided that the roots of \( q_2 \) are real.

Relations (3.16) and (3.17) show that the polynomial solution of equation (3.15) can be represented in terms of the symbol (3.13) as

\[
y_n(x) = Q_{n,r}(d^* + (\theta + 2)a, e^* - r(d^* + 2a) + (\theta + 1)b, \theta c - r(e^* + b)) = x^N_n, b, c
\]

with the eigenvalue

\[
\lambda_n = n((n + 1 + \theta)a + d^*) \quad n \geq 1.
\]

Also, according to the Corollary 3.1, \( c_0^* \) in (3.16) directly depends on the roots of \( q_2(x) \) in (3.17) and is therefore computed as

\[
c_0^* = 2\theta(ar^2 + br + c)(d^* + (\theta + 2)a) \times
\]

\[
\left( e^* + rd^* + (\theta + 1)(2ra + b) \mp \left( (e^* - r(d^* + 2a) + (\theta + 1)b)^2 - 4(d^* + (\theta + 2)a)(\theta c - r(e^* + b)) \right)^{\frac{1}{2}} \right)^{-1}.
\]

As we observed, \( \rho(x) \) was indeed the product of \((x - r)^\theta\) for

\[
(3.18) \quad \theta = \frac{b_2r^2 + b_1r + b_0}{a_2r^2 + a_1r + a_0},
\]

and a special case of Pearson distributions family. This means that we can classify the exceptional \( X_1 \)-orthogonal polynomials into six main sequences.

**Corollary 3.3.** By referring to table 4 and relation (3.14), there are totally six sequences of \( X_1 \)-polynomials as follows:

1. **Infinite \( X_1 \)-Jacobi polynomials orthogonal with respect to the weight function**
   \[
   \rho_1(x) = (x - r)^\theta(1 - x)^\alpha(1 + x)^\beta, \quad (-1 \leq x \leq 1).
   \]

2. **Infinite \( X_1 \)-Laguerre polynomials orthogonal with respect to the weight function**
   \[
   \rho_2(x) = (x - r)^\theta x^\alpha \exp(-x), \quad (0 \leq x < \infty).
   \]

3. **Infinite \( X_1 \)-Hermite polynomials orthogonal with respect to the weight function**
   \[
   \rho_3(x) = (x - r)^\theta \exp(-x^2), \quad (-\infty < x < \infty).
   \]

4. **Finite \( X_1 \)-polynomials orthogonal with respect to the weight function**
   \[
   \rho_4(x) = (x - r)^\theta x^\alpha(x + 1)^{-(p+q)}, \quad (0 \leq x < \infty).
   \]

5. **Finite \( X_1 \)-polynomials orthogonal with respect to the weight function**
   \[
   \rho_5(x) = (x - r)^\theta x^{-p} \exp\left(-\frac{1}{x}\right), \quad (0 \leq x < \infty).
   \]

6. **Finite \( X_1 \)-polynomials orthogonal with respect to the weight function**
   \[
   \rho_6(x) = (x - r)^\theta (1 + x^2)^{-\alpha} \exp(q \arctan(x)), \quad (-\infty < x < \infty).
   \]

In all six above mentioned cases \( r \in \mathbb{R} \) and \( \theta \) is a real parameter such that \((-1)^\theta = 1\).

Note that for \( \theta = -2 \), the weight functions (studied in [9, 10] and represented in (1.7) and (1.8)) are retrieved for exceptional \( X_1 \)-Jacobi and \( X_1 \)-Laguerre polynomials when \( r = \frac{2+\alpha}{\beta-\alpha} \) and \( r = -\alpha \), respectively.
4. On the Series Solutions of Equation (3.1)

Let us reconsider equation (3.1) in the form

\[ (4.1)\]

\[
y_n''(x) + \frac{b_2 x^2 + b_1 x + b_0}{(x-r)(a_2 x^2 + a_1 x + a_0)} y_n'(x) - \frac{\lambda_n(x-r) + c_0^*}{(x-r)(a_2 x^2 + a_1 x + a_0)} y_n(x) = 0, \quad n \geq 1.
\]

Since \[ \lim_{x \to r} (x-r) \frac{b_2 x^2 + b_1 x + b_0}{(x-r)(a_2 x^2 + a_1 x + a_0)} = \frac{b_2 r^2 + b_1 r + b_0}{a_2 r^2 + a_1 r + a_0}, \]
and

\[
\lim_{x \to r} (x-r)^2 \frac{\lambda_n(x-r) + c_0^*}{(x-r)(a_2 x^2 + a_1 x + a_0)} = 0,
\]
the indicial equation corresponding to (4.1) is as

\[
t^2 + \left( \frac{b_2 r^2 + b_1 r + b_0}{a_2 r^2 + a_1 r + a_0} - 1 \right) t = 0.
\]

By using the Frobenius method, one can obtain series solutions of equation (3.1) when

\[
t_1 = 1 - \frac{b_2 r^2 + b_1 r + b_0}{a_2 r^2 + a_1 r + a_0} = 1 - \theta,
\]
for different values of \( \theta \).

If \( \theta \not\in \mathbb{Z} \), the two basic solutions of (4.1) are respectively in the forms

\[
y_{n,1}(x) = \sum_{k=0}^{\infty} C_k (x-r)^k, \quad C_0 \neq 0,
\]
and

\[
y_{n,2}(x) = (x-r)^{1-\theta} \sum_{k=0}^{\infty} d_k (x-r)^k, \quad d_0 \neq 0.
\]

If \( \theta \in \mathbb{Z} \), three cases can occur for the basis solutions as follows:

\[
\theta = 1 \quad \Rightarrow \quad \begin{cases} 
y_{n,1}(x) = \sum_{k=0}^{\infty} C_k (x-r)^k, \quad C_0 \neq 0, \\
y_{n,2}(x) = y_{n,1}(x) \ln |x-r| + \sum_{k=1}^{\infty} d_k (x-r)^k,
\end{cases}
\]
and

\[
\theta < 1 \quad \Rightarrow \quad \begin{cases} 
y_{n,1}(x) = (x-r)^{1-\theta} \sum_{k=0}^{\infty} C_k (x-r)^k, \quad C_0 \neq 0, \\
y_{n,2}(x) = wy_{n,1}(x) \ln |x-r| + \sum_{k=0}^{\infty} d_k (x-r)^k, \quad d_0 \neq 0, \quad w \in \mathbb{R},
\end{cases}
\]
and finally

\[
\theta > 1 \quad \Rightarrow \quad \begin{cases} 
y_{n,1}(x) = \sum_{k=0}^{\infty} C_k (x-r)^k, \quad C_0 \neq 0, \\
y_{n,2}(x) = wy_{n,1}(x) \ln |x-r| + (x-r)^{1-\theta} \sum_{k=0}^{\infty} d_k (x-r)^k, \quad d_0 \neq 0, \quad w \in \mathbb{R}.
\end{cases}
\]
Let us assume that
\begin{equation}
(4.2) \quad y_n(x) = \sum_{k=0}^{\infty} C_k(x - r)^{k-\theta+1},
\end{equation}
for \( \theta \in \mathbb{Z} \) and \( \theta < 1 \). Since
\begin{equation}
y_n'(x) = \sum_{k=0}^{\infty} (k - \theta + 1)C_k(x - r)^{k-\theta},
\end{equation}
and
\begin{equation}
y_n''(x) = \sum_{k=0}^{\infty} (k - \theta + 1)(k - \theta)C_k(x - r)^{k-\theta-1},
\end{equation}
substituting the above results in equation (3.1) eventually leads to
\begin{equation}
(4.3) \quad C_{k-1} \left(a_2(k-\theta)(k-\theta-1) + b_2(k-\theta) - \lambda_n\right)
+ C_k \left((2a_2r + a_1)(k-\theta + 1)(k-\theta) + (2b_2r + b_1)(k-\theta + 1) - c_0^*\right)
+ C_{k+1} \left((a_2r^2 + a_1r + a_0)(k-\theta + 2)(k-\theta + 1) + (b_2r^2 + b_1r + b_0)(k-\theta + 2)\right) = 0.
\end{equation}

In a similar way, for \( \theta \in \mathbb{Z} \) and \( \theta \geq 1 \), or \( \theta \notin \mathbb{Z} \) the assumption
\begin{equation}
y_n(x) = \sum_{k=0}^{\infty} C_k(x - r)^k,
\end{equation}
leads to the same as recurrence relation (4.3) for \( \theta = 1 \).

4.1. Some polynomials solutions of Equation (3.1)

According to corollary (3.1) since the coefficients of the polynomial \( B(x) = b_2x^2 + b_1x + b_0 \) in the main differential equation of (3.1) or (3.12) have a significant role in determining the parameter \( c_0^* \) in relations (3.3), in this section we investigate six special cases of \( B(x) \) based on its roots and the real value \( r \) leading to particular cases of equation (3.12).

First, suppose that \( b_2 \neq 0 \) and \( r \) is a root of \( B(x) \). So,
\begin{equation}
b_2r^2 + b_1r + b_0 = 0,
\end{equation}
and relations (3.3) are reduced to
\begin{equation}
(4.4) \quad \begin{cases}
2b_2r + b_1 - c_0^* + \nu b_2 = 0, \\
\nu c_0^* = 0.
\end{cases}
\end{equation}
The relation \( \nu c_0^* = 0 \) in (4.4) gives three different cases as follows:

- **Case 1.** \( \nu = 0 \) and \( c_0^* = 2b_2r + b_1 = B'(r) \neq 0 \),
- **Case 2.** \( c_0^* = 0 \) and \( \nu = \frac{-2b_2r + b_1}{b_2} = -\frac{B'(r)}{b_2} \neq 0 \),
- **Case 3.** \( c_0^* = 0 \) and \( \nu = 0 \), leading to \( B'(r) = 0 \) which means that \( r \) is a multiple root of \( B(x) \).
Second, suppose that \( b_2 = 0 \) and \( b_1 \neq 0 \). So, relations \((4.3)\) are reduced to

\[
\begin{align*}
  c_0^* &= b_1, \\
  \nu c_0^* &= -(b_1 r + b_0).
\end{align*}
\]

Now, if \( r \) is a root of \( B(x) \), we have \( \nu c_0^* = 0 \) leading to

- **Case 4.** \( c_0^* = b_1 \neq 0 \) and \( \nu = 0 \).

otherwise we get

- **Case 5.** \( c_0^* = b_1 \neq 0 \) and \( \nu = -\frac{b_1 r + b_0}{c_0^*} = -\frac{B(r)}{b_1} \neq 0 \).

Finally, suppose that \( b_2 = b_1 = 0 \) so that relations \((4.3)\) are reduced to

\[
\begin{align*}
  c_0^* &= 0, \\
  b_0 + \nu c_0^* &= 0,
\end{align*}
\]

which yield \( b_0 = 0 \) leading to \( B(x) \equiv 0 \). Therefore we get

- **Case 6.** \( c_0^* = 0 \) and \( \nu \) is arbitrary.

Under the conditions stated in Case 1, the differential equation \((5.12)\) reads as

\[
\begin{align*}
  (a_2 x^2 + a_1 x + a_0) y''_n(x) + (b_2 x + b_2 r + b_1) y'_n(x) - \left(n (b_2 + (n-1)a_2) + \frac{2b_2 r + b_1}{x - r}\right) y_n(x) &= 0,
\end{align*}
\]

for \( n \geq 1 \), whose solutions belong to the polynomial space

\[
\Pi_{n,r,0} = \text{span}\left\{(x - r), (x - r)^2, \ldots, (x - r)^n\right\}.
\]

Without loss of generality, for \( r = 0 \) the above equation takes the form

\[
\begin{align*}
  (a_2 x^2 + a_1 x + a_0) y''_n(x) + (b_2 x + b_1) y'_n(x) - \left(n (b_2 + (n-1)a_2) + \frac{b_1}{x}\right) y_n(x) &= 0,
\end{align*}
\]

for \( a_0 = 0 \) reads as

\[
\begin{align*}
  x^3 (a_2 x + a_1) y''_n(x) + x(b_2 x + b_1) y'_n(x) - \left(n (b_2 + (n-1)a_2) x + b_1\right) y_n(x) &= 0.
\end{align*}
\]

Note in this case that we have \( \theta = 0 \) and for \( \lambda_n = n (b_2 + (n-1)a_2) \), the coefficients \( C_k \) in \((4.3)\) are recursively given by

\[
C_k = \frac{\lambda_n - \lambda_k}{k (b_1 + (k + 1) a_1)} C_{k-1} \quad k = 1, 2, \ldots.
\]

Setting \( C_0 = 1 \), it can be easily observed that

\[
C_k = \frac{1}{k!} \frac{\prod_{j=0}^{k-1} \lambda_n - \lambda_{j+1}}{\prod_{j=0}^{k-1} b_1 + (j + 2) a_1} \quad \text{for} \quad k = 0, 1, \ldots, n - 1.
\]
and \( C_k = 0 \) for \( k \geq n \). So, by assuming
\[
p_{n-1}(x) = \sum_{k=0}^{n-1} C_k x^k,
\]
and noting (4.12) we obtain \( y_n(x) = x p_{n-1}(x) \) as a polynomial solution of equation (4.8). In this sense, replacing \( y' = p_{n-1} + x p'_{n-1} \) and \( y'' = 2p'_{n-1} + x p''_{n-1} \) in (4.8) yields the differential equation
\[
(a_2 x^2 + a_1 x) p''_{n-1} + ((2a_2 + b_2)x + 2a_1 + b_1) p'_{n-1} - (n-1)(na_2 + b_1)p_{n-1} = 0,
\]
which has a polynomial solution of type (2.1) as
\[
p_{n-1}(x) = P_{n-1} \left( \begin{array}{ccc} \frac{b_2 + 2a_2}{a_2} & \frac{b_1 + 2a_1}{a_1} & 0 \end{array} \right) x,
\]
orthogonal with respect to a weight function of type (1.4) as
\[
W \left( \begin{array}{ccc} \frac{b_2}{a_2} & \frac{b_1}{a_1} & 0 \end{array} \right) x = x^{1 + \frac{b_1}{a_1}} \left( a_2 x + a_1 \right)^{\frac{b_2}{a_2} - \frac{b_1}{a_1} - 1}.
\]

**Corollary 4.1.** The polynomial solution of the differential equation (4.7) is given by
\[
y_n(x) = (x-r) P_{n-1} \left( \begin{array}{ccc} \frac{b_2 + 2a_2}{a_2} & \frac{b_1 + 2a_1}{a_1} & 0 \end{array} \right) \left( x-r \right).
\]

This means that
\[
Q_{n,r} \left( \begin{array}{ccc} \frac{b_2}{a_2} & \frac{b_1}{a_1} & 0 \end{array} \right) x = (x-r) P_{n-1} \left( \begin{array}{ccc} \frac{b_2 + 2a_2}{a_2} & \frac{b_1 + 2a_1}{a_1} & 0 \end{array} \right) \left( x-r \right).
\]

Note that for \( b_2 = 0 \) in the above corollary the Case 4 is retrieved.

For cases 2, 3 and 6, where \( \theta \) is equal to zero, the differential equation (3.12) respectively reads as
\[
(a_2 x^2 + a_1 x + a_0) y''_n(x) + (b_2 x + b_2 r + b_1) y'_n(x) - n(b_2 + (n-1)a_2) y_n(x) = 0 \quad n \geq 1,
\]
\[
(a_2 x^2 + a_1 x + a_0) y''_n(x) + b_2(x-r) y'_n(x) - n(b_2 + (n-1)a_2) y_n(x) = 0 \quad n \geq 1,
\]
and
\[
(a_2 x^2 + a_1 x + a_0) y''_n(x) - n(n-1)a_2 y_n(x) = 0 \quad n \geq 1,
\]
which are all special cases of equation (1.5) and their polynomial solutions belong to the spaces
\[
\Pi_{n,r,0} \text{ and } \Pi_{n,r,\nu} \text{ where } \nu \text{ is an arbitrary value.}
\]

**Corollary 4.2.** The monic polynomial solutions of equations (4.9) and (4.11) can be respectively denoted by
\[
Q_{n,r} \left( \begin{array}{ccc} \frac{b_2}{a_2} & \frac{b_1}{a_1} & 0 \end{array} \right) x = P_n \left( \begin{array}{ccc} b_2 & b_2 r + b_1 & a_0 \end{array} \right) \left( x-r \right),
\]
\[
\tilde{Q}_{n,r} \left( \begin{array}{ccc} \frac{b_2}{a_2} & \frac{b_1}{a_1} & 0 \end{array} \right) x = \tilde{P}_n \left( \begin{array}{ccc} b_2 & -b_2 r & a_0 \end{array} \right) \left( x-r \right),
\]
and
\[ \bar{Q}_{n,r} \left( \begin{array}{ccc} 0, & 0, & 0 \\ a_2, & a_1, & a_0 \end{array} \right) = \bar{P}_{n} \left( \begin{array}{ccc} 0, & 0 \\ a_2, & a_1, & a_0 \end{array} \right) (x - r). \]

Finally, let us consider Case 5 where the differential equation (3.12) is reduced to
\[ (a_2 x^2 + a_1 x + a_0) y''(x) + \left( b_1 + \frac{b_0 r + b_0}{x - r} \right) y'(x) - \left( n(n - 1)a_2 + \frac{b_1}{x} \right) y_n(x) = 0, \]
with the polynomial space
\[ \Pi_{n,r,\nu} = \text{span}\left\{ (x + \frac{b_0}{b_1}) (x - r)^2, \ldots, (x - r)^n \right\}. \]

Again, without loss of generality, for \( r = 0 \) the differential equation (4.12) takes the form
\[ (a_2 x^2 + a_1 x + a_0) y''(x) + \left( b_1 + \frac{b_0}{x} \right) y'(x) - \left( n(n - 1)a_2 + \frac{b_1}{x} \right) y_n(x) = 0. \]
In this case, for \( b_2 = 0, c_0' = b_1 \) and \( r = 0 \), the relation (4.13) can be written as
\[ C_{k-1} a_2 ((k - \sigma)(k - \sigma - 1) - n(n - 1)) \]
\[ + C_k (k - \sigma)(a_1(k - \sigma + 1) + b_1) \]
\[ + C_{k+1} (k - \sigma + 2)(a_0(k - \sigma + 1) + b_0) = 0, \]
where
\[ \sigma = \begin{cases} \theta & \text{if } \theta < 1, \theta \in \mathbb{Z} \\ 1 & \text{if } \theta \geq 1, \theta \in \mathbb{Z} \text{ or } \theta \notin \mathbb{Z.} \end{cases} \]

Clearly, it is not possible generally to obtain \( C_k \) explicitly. However, in particular cases, if \( a_2 = 0 \) then \( y(x) = 1 + \frac{b_1}{b_0} x \) and if \( a_1 = b_1 = 0 \), it is in contradiction with hypothesis in Case 5. Moreover, \( a_0 = b_0 = 0 \) lead to the special case 4 for \( \sigma = 0 \).

Let us consider (4.14) for \( a_0 = b_0 = 0 \) when \( \sigma = 1 \), which occurs if \( \theta \in \mathbb{N} \) or \( \theta \notin \mathbb{Z} \), as
\[ C_{k-1} a_2 ((k - 1)(k - 2) - n(n - 1)) + C_k (k - 1)(a_1 k + b_1) = 0. \]

So, the coefficients \( C_k \) in \( y_n(x) = \sum_{k=1}^{\infty} C_k x^k \) are recursively given by
\[ C_k = \frac{n(n - 1) - (k - 1)(k - 2)}{(k - 1)(a_1 k + b_1)} a_2 C_{k-1} \quad \text{for} \quad k = 2, 3, \ldots. \]

Setting \( C_1 = 1 \), it can be easily observed that
\[ C_k = \frac{a_2^{k-1}}{(k - 1)!} \prod_{j=0}^{k-2} \frac{n(n - 1) - j(j + 1)}{\prod_{j=0}^{k-2} b_1 + (j + 2)a_1} \quad \text{for} \quad k = 1, 2, 3, \ldots, n, \]
and \( C_k = 0 \) for \( k > n \).
5. Six Classes of Exceptional X₁-Orthogonal Polynomials

By noting the corollary [3.3] in this section we consider six special cases of the main equation [3.12] and study their properties.

5.1. On the differential equation of exceptional X₁-Jacobi Polynomials

As a generalization of Jacobi differential equation for \( \theta = 0 \), consider the equation

\[
(x-\rho)(1-x^2)y''_n(x) + \left( - (\alpha + \beta + \theta + 2)x^2 + (\beta - \alpha + r(\alpha + \beta + 2))x - r(\beta - \alpha) \right)y'_n(x) + \left( n(n + \alpha + \beta + \theta + 1)(x - \rho) - c_0^{(P)} \right)y_n(x) = 0 \quad n \geq 1,
\]

where \( r, \theta, \alpha, \beta \) are real parameters such that \( \alpha, \beta > -1 \), \((-1)^\theta = 1 \) and

\[
c_0^{(P)} = 2\theta(1 - r^2)(\alpha + \beta + \theta + 2) \left( (\alpha + \beta + 2\theta + 2)r + \alpha - \beta \right) \pm \sqrt{((\alpha + \beta + 2)r - \beta - \alpha)^2 + 4(\alpha + \beta + \theta + 2)(\theta - r(\beta - \alpha))} \right)^{-1}.
\]

The polynomial solution of equation (5.1), i.e.

\[
y_n(x) = P^{(\alpha,\beta)}_{n,r,\theta}(x) = Q_{n,r,\theta} \left( \begin{array}{c} - (\alpha + \beta + \theta + 2), \beta - \alpha - r(\alpha + \beta + 2), \theta - r(\beta - \alpha) \\ -1, 0, 1 \end{array} \right) x,
\]

is orthogonal with respect to the weight function

\[
\rho_1(x; \alpha, \beta, \theta) = (x - \rho)^\theta(1 - x)^\alpha(1 + x)^\beta = (x - \rho)^\theta W\left( \begin{array}{c} - \alpha - \beta, \beta - \alpha \\ -1, 0, 1 \end{array} \right) x,
\]

on \([-1,1]\). Also, for \( \theta = 0 \), \( r = -1 \) or \( r = 1 \) in (5.1), \( c_0^{(P)} = 0 \) and the weight function \( \rho_1(x; \alpha, \beta, \theta) \) will be a special case of the beta distribution. Hence, the solution of equation (5.1) will be the same as classical Jacobi polynomials. In fact, in each of these circumstances equation (5.1) reads as

\[
(1 - x^2)y''_n(x) + \left( - (\alpha + \beta + 2)x^2 + (\beta - \alpha) \right)y'_n(x) + n(n + \alpha + \beta + 1)y_n(x) = 0,
\]

for \( \theta = 0 \) and

\[
(1 - x^2)y''_n(x) + \left( - (\alpha + \beta + 2)x^2 + (\beta + \theta - \alpha) \right)y'_n(x) + n(n + \alpha + \beta + \theta + 1)y_n(x) = 0,
\]

for \( r = -1 \) and

\[
(1 - x^2)y''_n(x) + \left( - (\alpha + \beta + 2)x^2 + (\beta - \theta - \alpha) \right)y'_n(x) + n(n + \alpha + \beta + \theta + 1)y_n(x) = 0,
\]

for \( r = 1 \) with the following Jacobi type polynomial solutions

\[
P^{(\alpha,\beta)}_{n,r,0}(x) = P^{(\alpha,\beta)}_{n}(x),
\]

\[
P^{(\alpha,\beta)}_{n,-1,\theta}(x) = P^{(\alpha,\beta+\theta)}_{n}(x),
\]

and

\[
P^{(\alpha,\beta)}_{n,1,\theta}(x) = P^{(\alpha+\theta,\beta)}_{n}(x).
\]
5.2. On the differential equation of exceptional $X_1$-Laguerre Polynomials

As a generalization of Laguerre differential equation for $\theta = 0$, consider the equation

\[(5.2)\]

\[x(x-r)y''_n(x) + \left(-x^2 + (\alpha + r + \theta + 1)x - r(\alpha + 1)\right)y'_n(x) + \left(n(x-r) - c_0^{(L)}\right)y_n(x) = 0\]

\[n \geq 1,\]

where $r, \theta, \alpha$ are real parameters such that $\alpha > -1$, $(-1)^\theta = 1$ and

\[c_0^{(L)} = 2r\theta \left(r - \alpha - \theta - 1 \pm \sqrt{(r + \theta)^2 + (\alpha + 1)(\alpha + 1 + 2\theta - 2r)}\right)^{-1}.\]

The polynomial solution of equation (5.2), i.e.

\[L_{n,r,\theta}^{(\alpha)}(x) = Q_{n,r} \begin{pmatrix} -1, \alpha + r + \theta + 1, -r(\alpha + 1) \\ 0, 1, 0 \end{pmatrix},\]

is orthogonal with respect to the weight function

\[\rho_2(x; \alpha, \theta) = (x-r)^\theta x^\alpha e^{-x} = (x-r)^\theta W\left(\begin{array}{c} -1, \alpha \\ 0, 1, 0 \end{array} \right),\]

on $[0, \infty)$. Also, for $\theta = 0$ or $r = 0$ in (5.2), $c_0^{(L)} = 0$ and the weight function $\rho_2(x; \alpha, \theta)$ will be a special case of Gamma distribution. Hence, the solution of equation (5.2) will be the same as classical Laguerre polynomials. In fact, in each of these circumstances equation (5.2) reads as

\[xy''_n(x) + (-x + \alpha + 1)y'_n(x) + ny_n(x) = 0,\]

for $\theta = 0$ and

\[xy''_n(x) + (-x + \alpha + \theta + 1)y'_n(x) + ny_n(x) = 0,\]

for $r = 0$ with the following Laguerre type polynomial solutions

\[L_{n,r,0}^{(\alpha)}(x) = L_{n}^{(\alpha)}(x),\]

and

\[L_{n,0,\theta}^{(\alpha)}(x) = L_{n}^{(\alpha+\theta)}(x).\]

5.3. On the differential equations of exceptional $X_1$-Hermite Polynomials

As a generalization of Hermite differential equation for $\theta = 0$, consider the equation

\[(5.3)\]

\[(x-r)y''_n(x) + (-2x^2 + 2rx + \theta)y'_n(x) + \left(2n(x-r) - \frac{2\theta}{r \pm \sqrt{r^2 + 2\theta}}\right)y_n(x) = 0\]

\[n \geq 1,\]

where $r, \theta$ are real parameters such that $(-1)^\theta = 1$.

The polynomial solution of equation (5.3), i.e.

\[H_{n,r,\theta}(x) = Q_{n,r} \begin{pmatrix} -2, 2r, \theta \\ 0, 0, 1 \end{pmatrix},\]

is orthogonal with respect to the weight function

\[\rho_3(x; \theta) = (x-r)^\theta e^{-x^2} = (x-r)^\theta W\left(\begin{array}{c} -2, 0 \\ 0, 0, 1 \end{array} \right),\]
on \((-\infty, \infty)\). Also, for \(\theta = 0\), \(\rho_3(x; \theta)\) is the same as normal distribution and the solution of equation (5.3) is the classical Hermite polynomials. In fact, in this case, equation (5.3) reads as
\[
y''_n(x) - 2xy'_n(x) + 2ny_n(x) = 0,
\]
with the usual Hermite polynomial solution
\[
H_{n,r,0}(x) = H_n(x).
\]

5.4. First Finite Sequence of Exceptional X₁-Orthogonal Polynomials

Consider the following differential equation
\[
x(x-r)(x+1)y''_n(x) + \left((\theta + 2 - p)x^2 + (q + \theta + 1 + r(p-2))x - r(q+1)\right)y'_n(x)
- \left(n(n+1+\theta-p)(x-r) + c_0^{(M)}\right)y_n(x) = 0 \quad n \geq 1,
\]
where \(r, \theta\) are real parameters such that \((-1)^\theta = 1\) and
\[
c_0^{(M)} = \frac{2\theta r(r+1)(p-\theta-2)}{rp-q-(\theta+1)(2r+1) \pm \left((q+\theta+1+r(p-2))^2 + 4r(q+1)(\theta+2-p)\right)^{1/2}}.
\]
The polynomial solution of equation (5.4), i.e.
\[
M_{n,r,\theta}^{(p,q)}(x) = Q_{n,r} \left(\begin{array}{c}
\theta + 2 - p, q + \theta + 1 + r(p-2), -r(q+1) \\
1, 1, 0
\end{array} \right),
\]
is finitely orthogonal with respect to the weight function
\[
\rho_4(x; p, q, \theta) = (x-r)^\theta x^q(x+1)^{-(p+q)} = (x-r)^\theta W \left(\begin{array}{c}
-p, q \\
1, 1, 0
\end{array} \right),
\]
on \([0, \infty)\) if and only if
\[
p > 2\{\max n\} + \theta + 1 \quad \text{and} \quad q > -1.
\]
In other words, if the self-adjoint form of equation (5.4) is written as
\[
(x-r)^\theta x^{q+1}(x+1)^{1-(p+q)}y'_n(x)'
= (x-r)^{\theta-1}x^q(x+1)^{-(p+q)}\left(n(n+1+\theta-p)(x-r) + c_0^{(M)}\right)y_n(x),
\]
and for the index \(m\) as
\[
(x-r)^\theta x^{q+1}(x+1)^{1-(p+q)}y'_m(x)'
= (x-r)^{\theta-1}x^q(x+1)^{-(p+q)}\left(m(m+1+\theta-p)(x-r) + c_0^{(M)}\right)y_m(x),
\]
then multiplying by \(y_m(x)\) and \(y_n(x)\) in relations (5.5) and (5.6) respectively and subtracting them and finally integrating from both sides gives
\[
\left[(x-r)^\theta x^{q+1}(x+1)^{1-(p+q)}(y'_n(x)y_m(x) - y'_m(x)y_n(x))\right]_0^\infty
= (n(n+1+\theta-p) - m(m+1+\theta-p)) \int_0^\infty (x-r)^\theta x^q(x+1)^{-(p+q)}y_n(x)y_m(x) \, dx.
\]
Now, since
\[
\max \deg \{ y'_n(x)y_m(x) - y'_m(x)y_n(x) \} = m + n - 1,
\]
if
\[
q > -1 \quad \text{and} \quad p > 2N + \theta + 1 \quad \text{for} \quad N = \max\{m, n\},
\]
the left hand side of (5.7) tends to zero and for \( m, n \geq 1 \) we get
\[
\int_0^\infty \frac{(x-r)^q x^t}{(x+1)^{p+q}} M^{(p,q)}_{n,r,\theta}(x) M^{(p,q)}_{m,r,\theta}(x) \, dx = 0
\]
\[
\Leftrightarrow m \neq n, \quad N = \max\{m, n\} < \frac{p-1-\theta}{2}, \quad q > -1 \quad \text{and} \quad (-1)^\theta = 1.
\]

Note that for \( \theta = 0 \), \( r = -1 \) or \( r = 0 \), \( 
\rho_4(x; p, q, \theta) \) would be a special case of F-Fisher distribution. Indeed, in each of these circumstances \( c_0^{(M)} = 0 \) and equation (5.4) reads as
\[
x(x+1)y''_n(x) + ((2-p)x + q + 1)y'_n(x) - n(n+1-p)y_n(x) = 0,
\]
for \( \theta = 0 \) and
\[
x(x+1)y''_n(x) + ((\theta + 2-p)x + q + 1)y'_n(x) - n(n+1+\theta - p)y_n(x) = 0,
\]
for \( r = -1 \) and
\[
x(x+1)y''_n(x) + ((\theta + 2-p)x + q + \theta + 1)y'_n(x) - n(n+1+\theta - p)y_n(x) = 0,
\]
for \( r = 0 \) with the following polynomial solutions
\[
M^{(p,q)}_{n,r,\theta}(x) = M^{(p,q)}_{n}(x),
\]
\[
M^{(p,q)}_{n,0,\theta}(x) = M^{(p,q)}_{n}(x),
\]
and
\[
M^{(p,q)}_{n,-1,\theta}(x) = M^{(p,q)}_{n}(x).
\]

5.5. Second Finite Sequence of Exceptional \( X_1 \)-Orthogonal Polynomials

Consider the following differential equation
\[
(x-r)x^2 y''_n(x) + \left( (\theta + 2-p)x^2 + (1+r(p-2))x - r \right) y'_n(x)
\]
\[- \left( n(n+1+\theta - p)(x-r) + c_0^{(N)}(x) \right) y_n(x) = 0 \quad \text{for} \quad n \geq 1,
\]
where \( r, \theta \) are real parameters such that \( (-1)^\theta = 1 \) and
\[
c_0^{(N)} = \frac{2\theta r^2(p-\theta-2)}{r(p-2(\theta+1)) - 1 \pm \left( (1+r(p-2))^2 + 4r(\theta+2-p) \right)^{\frac{1}{2}}}.
\]
The polynomial solution of equation (5.8), i.e.
\[
N^{(p)}_{n,r,\theta}(x) = Q_{n,r} \left( \begin{array}{c} \theta + 2-p, 1+r(p-2), -r \\ 1, 0, 0 \end{array} \right) \left( x \right),
\]
is finitely orthogonal with respect to the weight function
\[
\rho_5(x; p, \theta) = (x-r)^\theta x^{-p} e^{-\frac{1}{x}} = (x-r)^\theta W \left( \begin{array}{c} -p, 1 \\ 1, 0, 0 \end{array} \right) \left( x \right),
\]
on \([0, \infty)\) if and only if \(p > 2\{\max n\} + \theta + 1\), because if the self-adjoint form of equation (5.8) is written as

\[
(5.9) \quad \left( (x - r)^\theta x^{-p+2} e^{-\frac{x}{p} r} y_n(x) \right)' = (x - r)^\theta x^{-p+2} e^{-\frac{x}{p} r} \left( n(n + 1 + \theta - p)(x - r) + c^{(N)}_0(x) \right) y_n(x),
\]

and for the index \(m\) as

\[
(5.10) \quad \left( (x - r)^\theta x^{-p+2} e^{-\frac{x}{p} r} y_m(x) \right)' = (x - r)^\theta x^{-p+2} e^{-\frac{x}{p} r} \left( m(m + 1 + \theta - p)(x - r) + c^{(N)}_0(x) \right) y_m(x),
\]

then multiplying by \(y_m(x)\) and \(y_n(x)\) in relations (5.9) and (5.10) respectively and subtracting and finally integrating from both sides gives

\[
(5.11) \quad \left[ (x - r)^\theta x^{-p+2} e^{-\frac{x}{p} r} (y_n(x)y_m(x) - y_m(x)y_n(x)) \right]_0^\infty = (n(n + 1 + \theta - p) - m(m + 1 + \theta - p)) \int_0^\infty (x - r)^\theta x^{-p+2} y_n(x)y_m(x) dx.
\]

Now, if

\[
p > 2N + \theta + 1 \quad \text{for} \quad N = \max\{m, n\},
\]

the left hand side of (5.11) tends to zero and for \(m, n \geq 1\) we get

\[
\int_0^\infty (x - r)^\theta x^{-p+2} e^{-\frac{x}{p} r} N_{n,r,0}^{(p)}(x) N_{m,r,0}^{(p)}(x) dx = 0
\]

\[
\Leftrightarrow m \neq n, \quad N = \max\{m, n\} < \frac{p - \theta - 1}{2} \quad \text{and} \quad (-1)^\theta = 1.
\]

Note that for \(\theta = 0\) or \(r = 0\), \(\rho_5(x; p, \theta)\) would be a special case of inverse Gamma distribution. Indeed, in each of these circumstances \(c^{(N)}_0 = 0\) and equation (5.8) reads as

\[
x^2 y_n''(x) + ((2 - p)x + 1)y'_n(x) - n(n + 1 - p)y_n(x) = 0,
\]

for \(\theta = 0\) and

\[
x^2 y'_n(x) + ((\theta + 2 - p)x + 1)y_n(x) - n(n + 1 + \theta - p)y_n(x) = 0,
\]

for \(r = 0\) with the following polynomial solutions

\[
N_{n,r,0}^{(p)}(x) = N_{n,r}^{(p)}(x),
\]

and

\[
N_{n,0,0}^{(p)}(x) = \tilde{N}_{n}^{(p-\theta)}(x).
\]

5.6. Third Finite Sequence of Exceptional X₁-Orthogonal Polynomials

Consider the following differential equation

\[
(5.12) \quad (x - r)(1 + x^2)y'_n(x) + \left((\theta + 2 - 2p)x^2 + (q + 2r(p - 1))x + \theta - rq\right)y_n(x)
\]

\[
- \left(n(n + 1 + \theta - 2p)(x - r) + c^{(j)}_0\right)y_n(x) = 0 \quad n \geq 1,
\]
where $r, \theta$ are real parameters such that $(-1)^{\theta} = 1$ and

$$c_0^{(J)} = \frac{2\theta(r^2 + 1)(2p - \theta - 2)}{2r(p - \theta - 1) - q \pm \left((q + 2r(p - 1))^2 - 4(\theta + 2 - 2p)(\theta - rq)\right)^{1/2}}.$$ 

The polynomial solution of equation (5.12), i.e.

$$J_{n,r,\theta}^{(p,q)}(x) = Q_{n,r} \left( \begin{array}{c} \theta + 2 - 2p, q + 2r(p - 1), \theta - rq \\ 1, 0, 1 \end{array} \right) x,$$

is finitely orthogonal with respect to the weight function

$$\rho_0(x; p, q, \theta) = (x - r)^\theta (1 + x^2)^{-p} \exp(q \arctan x) = (x - r)^\theta W \left( \begin{array}{c} -2p, q \\ 1, 0, 1 \end{array} \right) x,$$

on $(-\infty, \infty)$ if and only if $p > \{\max n\} + \frac{\theta + 1}{2}$, because if the self-adjoint form of equation (5.12) is written as

$$(5.13) \quad \left( (x - r)^\theta (1 + x^2)^{1-p} \exp(q \arctan x) y_n'(x) \right)' = (x - r)^{\theta-1}(1 + x^2)^{-p} \exp(q \arctan x) \left( n(n + 1 + \theta - 2p)(x - r) + c_0^{(J)} \right) y_n(x),$$

and for the index $m$ as

$$(5.14) \quad \left( (x - r)^\theta (1 + x^2)^{1-p} \exp(q \arctan x) y_m'(x) \right)' = (x - r)^{\theta-1}(1 + x^2)^{-p} \exp(q \arctan x) \left( m(m + 1 + \theta - 2p)(x - r) + c_0^{(J)} \right) y_m(x),$$

then multiplying by $y_m(x)$ and $y_n(x)$ in relations (5.13) and (5.14) respectively and subtracting them and finally integrating from both sides gives

$$(5.15) \quad \left[ (x - r)^\theta (1 + x^2)^{1-p} \exp(q \arctan x) \left( y_n'(x)y_m(x) - y_m'(x)y_n(x) \right) \right]_{-\infty}^\infty = \left( n(n+1+\theta-2p)-m(m+1+\theta-2p) \right) \int_{-\infty}^\infty (x - r)^\theta (1 + x^2)^{-p} \exp(q \arctan x) J_{n,r,\theta}^{(p,q)}(x) J_{m,r,\theta}^{(p,q)}(x) \, dx.$$

Now, if

$$p > N + \frac{\theta + 1}{2} \quad \text{for} \quad N = \max\{m, n\},$$

the left hand side of (5.15) tends to zero and for $m, n \geq 1$ we have

$$\int_{-\infty}^\infty (x - r)^\theta (1 + x^2)^{-p} \exp(q \arctan x) J_{n,r,\theta}^{(p,q)}(x) J_{m,r,\theta}^{(p,q)}(x) \, dx = 0$$

$$\Leftrightarrow \quad m \neq n, \quad N = \max\{m, n\} < p - \frac{\theta + 1}{2} \quad \text{and} \quad (-1)^{\theta} = 1.$$

For $\theta = 0$, $\rho_0(x; p, q, \theta)$ is reduced to the generalized T-Student distribution. In this case $c_0^{(J)} = 0$ and equation (5.12) reads as

$$1 + x^2 y_n'(x) + (2(1-p)x + q)y_n(x) - n(n + 1 - 2p)y_n(x) = 0,$$

with the polynomial solution

$$J_{n,r,0}^{(p,q)}(x) = J_n^{(p,q)}(x).$$
A GENERIC CLASSIFICATION OF EXCEPTIONAL ORTHOGONAL X_1-POLYNOMIALS

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