On the baryon acoustic oscillation amplitude as a probe of radiation density

Will Sutherland* and Lukasz Mularczyk

School of Physics and Astronomy, Queen Mary University of London, Mile End Road, London E1 4NS, UK

Accepted 2013 December 13. Received 2013 December 12; in original form 2013 July 4

ABSTRACT

The baryon acoustic oscillation (BAO) feature in the distribution of galaxies has been widely studied as an excellent standard ruler for probing cosmic distances and expansion history, and hence dark energy. In contrast, the amplitude of the BAO feature has received relatively little study, mainly due to limited signal-to-noise, and complications due to galaxy biasing, effects of non-linear clustering and dependence on several cosmological parameters. As expected, the amplitude of the BAO feature is sensitive to the cosmic baryon fraction: for standard radiation content, the cosmic microwave background (CMB) acoustic peaks constrain this precisely and the BAO amplitude is largely a redundant cross-check. However, the CMB mainly constrains the redshift of matter-radiation equality, \( z_{\text{eq}} \), and the baryon/photon ratio: if a non-standard radiation density (\( N_{\text{eff}} \)) is allowed, increasing \( N_{\text{eff}} \) while matching the CMB peaks leads to a reduced baryon fraction and a lower relative BAO amplitude. We construct an observable for the relative area of the BAO feature from the galaxy correlation function (equation 8); from linear-theory models, we find that this is mainly sensitive to \( N_{\text{eff}} \) and quite insensitive to other cosmological parameters. More detailed work from \( N \)-body simulations will be needed to constrain the effects of non-linearity and scale-dependent galaxy bias on this observable.

Key words: cosmic background radiation – cosmological parameters – cosmology: theory – large-scale structure of Universe.

1 INTRODUCTION

The detection of baryon acoustic oscillations (BAOs) in the large-scale distribution of galaxies in both the SDSS (Eisenstein et al. 2005) and 2dFGRS (Cole et al. 2005) redshift surveys was a major milestone for cosmology, strongly supporting the standard paradigm for structure formation based on gravitational instability including cold (or warm) dark matter. Recently, there have been several new independent measurements of the BAO feature in galaxy redshift surveys, e.g. from SDSS-DR8 (Percival et al. 2010), WiggleZ (Blake et al. 2011), 6dFGRS (Beutler et al. 2011), an angular measurement from SDSS-DR9 (Seo et al. 2012) and a first measurement from BOSS (Anderson et al. 2012), which are all consistent with the concordance \( \Lambda \)CDM model at the few-per cent level.

The BAO feature in galaxy clustering (Peebles & Yu 1970; Bond & Efstathiou 1984; Eisenstein & Hu 1998; Meiksin, White & Peacock 1999) has a very similar origin to the acoustic peaks in the cosmic microwave background (CMB) temperature power spectrum. Most recent attention has focused on the length-scale of the BAO feature, used as a standard ruler to measure cosmic distances in units of the sound horizon \( r_s(z_{\text{d}}) \) at the baryon drag epoch.

Many theoretical and computational studies (Seo et al. 2008, 2010) have concluded that the comoving length-scale of the BAO feature evolves by \( \sim 0.5 \) per cent between the CMB era and the recent past \( z \sim 0.3 \) due to the non-linear growth of structure, but this shift can be corrected down to the 0.1 per cent level using reconstruction methods (Eisenstein et al. 2007; Padmanabhan et al. 2012). Therefore, the BAO feature is probably the best-understood standard ruler in the moderate-redshift Universe, and when combined with CMB observations it offers great power for probing the cosmic expansion history and therefore the properties of dark energy (Weinberg et al. 2013). These BAO distance measurements are complementary to those from Type-Ia supernovae, have potentially smaller systematic errors and can offer direct information on the time-variable \( H(z) \) without differentiation. On the downside, cosmic variance sets a floor on the BAO precision in the low-redshift Universe, \( \sim 1 \) per cent at \( z \sim 0.25 \) and worsening below this (Seo & Eisenstein 2007).

However, in this paper we look at a different property, specifically the overall amplitude of the BAO feature, rather than the length-scale. As expected, the amplitude is mainly sensitive to the cosmic baryon fraction (relative to total matter), \( f_b \). Until now, the BAO amplitude has received much less attention than the length-scale, for two main reasons: first, recent CMB results from WMAP (Hinshaw et al. 2012), SPT (Keisler et al. 2011; Story et al. 2013) and ACT (Sievers et al. 2013) measure the baryon fraction to around 4 per cent...
relative precision (given standard assumptions), while the strongest detection of the BAO peak (Anderson et al. 2012) gives $\sim 6\sigma$ significance or $16\%$ per cent error in amplitude. Secondly, complications due to galaxy bias, the non-linear growth of structure, redshift-space distortions and the uncertain global shape of the power spectrum make it challenging to extract the baryon fraction from the BAO feature, even with very large future redshift surveys.

However, we note that parameter estimates from the CMB are subject to a significant degeneracy between $f_b$ and the total radiation density in the CMB era, usually parametrized by an effective number of neutrino species $N_{\text{eff}}$. Recent reviews of $N_{\text{eff}}$ are given by e.g. Riemer-Sorensen, Parkinson & Davis (2013a) and Abazajian et al. (2012).

In contrast, the BAO feature is sensitive to the baryon fraction rather directly; therefore, combining CMB measurements (primarily sensitive to the physical baryon density $o_b$ and the redshift of matter-radiation equality $z_{eq}$) with a BAO measurement sensitive to $f_b$, may provide an interesting probe of the radiation density, $N_{\text{eff}}$. This may be less precise than other methods, but is largely complementary.

The plan of the paper is as follows: in Section 2, we review the main effects of varying cosmological parameters, including $f_b$ and radiation density, on CMB and BAO observations. In Section 3, we present numerical predictions of the BAO feature for a set of models (selected to give a good match to WMAP) with varying matter density and $N_{\text{eff}}$, and derive a statistic based on the galaxy correlation function, which is sensitive to $f_b$ and $N_{\text{eff}}$, but cancels galaxy bias and dark energy to leading order. We summarize our conclusions in Section 4. Most of this work was completed before the Planck release in 2013 March, so we mainly use WMAP-9 fit parameters (Hinshaw et al. 2012) as our baseline. The adjustments post-Planck are moderate, and we discuss the implications of recent Planck results (Planck Collaboration, Ade et al. 2013) in Section 3.3.

Throughout the paper, we use the standard notation that $\Omega_i$ is the present-day density of species $i$ relative to the critical density; and the physical density $o_i = \Omega_i h^2$, with $h = H_0/(100\text{ km s}^{-1}\text{ Mpc}^{-1})$.

## 2. BAOs, Radiation Density $N_{\text{eff}}$ and the Cosmic Baryon Fraction

### 2.1 Overview of BAOs

The BAO feature appears as a single hump in the galaxy correlation function $\xi(r)$, or equivalently a series of decaying wiggles in the power spectrum (see Eisenstein, Seo & White (2007) for a clear explanation in real space, and Bassett & Hlozek (2010) and Weinberg et al. (2013) for reviews]. The length-scale of the hump is very close to the comoving sound horizon $r_s(z_d)$ at the baryon drag epoch $z_d \approx 1020$; this $z_d$ is commonly defined by the fitting formula equation 4 of Eisenstein & Hu (1998). (This formula is defined for standard $N_{\text{eff}}$; however, the dependence of $z_d$ on $N_{\text{eff}}$ is weak, so the error from adopting the fitting formula is small.) This comoving length is predicted precisely mainly from CMB constraints, and several very large redshift surveys are ongoing or planned to exploit this as a standard ruler to measure cosmic distances at $0.2 \lesssim z \lesssim 2.5$ and thus probe dark energy.

Standard cosmological models contain a density of collisionless dark matter $\sim 5$ times larger than the baryon density. This explains naturally why the acoustic peaks in the CMB power spectrum have large relative amplitude, while the BAO feature is relatively weak in the late-time galaxy correlation function. Qualitatively, this occurs because the acoustic peaks at last scattering appeared only in the power spectrum of baryons, not dark matter: the peaks are prominent in the CMB because almost all CMB photons last scattered off a free electron. After decoupling, the distribution of baryons and dark matter became averaged together by the gravitational growth of the structure over the next few $e$-foldings between the CMB era and redshift $z \sim 20$ (Eisenstein et al. 2007), well before the formation of large galaxies. The dark matter dominates in this averaging, so the BAO signal in the galaxy correlation function becomes diluted by a factor $\sim f_b$. In the following, we define the baryon fraction as

$$f_b \equiv \frac{o_b}{o_\gamma + o_b},$$

so that the denominator includes CDM and baryons, but excludes massive neutrinos.

Thus, the BAO peak amplitude provides a potential measure of the baryon fraction; this has been used as a simple and compelling argument against Modified Newtonian Dynamics (MOND)-type modified gravity theories without non-baryonic dark matter (Dodelson 2011).

We can make this more quantitative and use the BAO feature to directly estimate the cosmic baryon fraction. However, there are several reasons why this has received little attention to date:

(i) Recent observations of the CMB power spectrum (Hinshaw et al. 2012) measure the physical baryon density $o_b$ to $\pm 2\%$ per cent precision, and (assuming standard $N_{\text{eff}}$) measure the physical matter density $o_m$ to $3\%$ per cent; the errors on these are weakly correlated, so this gives an estimate of the cosmic baryon fraction $f_b \leq 5\%$ per cent relative precision. (Recent results from the Planck mission have improved this to the $\sim 2\%$ per cent level, see Section 3.3).

(ii) The BAO feature is affected by several effects: it is blurred by the non-linear growth of structure, and amplified by galaxy bias, which is challenging to measure and may also be scale dependent.

(iii) The overall large-scale shape of the galaxy power spectrum also depends on other cosmological parameters, and this will also have some influence on the BAO peak shape.

Thus, at first sight it appears that BAOs cannot compete in precision with the CMB as a probe of $f_b$. This is partly true, but with an important caveat: the CMB-based estimates of $f_b$ are significantly degenerate with the total radiation density in the CMB era.

### 2.2 Radiation density

At this point, we define our notation on densities: as usual, we define the parameter $N_{\text{eff}}$ such that the radiation density at matter-radiation equality is

$$\rho_{\text{rad}} = \rho_\gamma (1 + 0.2271 N_{\text{eff}}),$$

where $\rho_{\text{rad}}$ and $\rho_\gamma$ are densities of (total) radiation and photons, respectively. Here, $N_{\text{eff}}$ is an ‘effective’ number of neutrinos, but in fact it is not specific to neutrinos and counts any species (except photons) which were relativistic until around matter-radiation equality. Assuming the standard population of only three very light neutrinos with the oscillation parameters given by solar, atmospheric and beam-based neutrino experiments, the value of $N_{\text{eff}}$ can be accurately predicted as $N_{\text{eff}} = 3.046$ (Mangano et al. 2005); here, the additional 0.046 arises from a small residual coupling of neutrinos to baryons and photons at the epoch of electron/positron annihilation.

It is also convenient to define the scaled radiation density $X_{\text{rad}}$ by

$$X_{\text{rad}} \equiv \frac{\rho_{\text{rad}}}{1.6918 \rho_\gamma} = 1 + 0.134 (N_{\text{eff}} - 3.046),$$

This may be less precise than other methods, but is largely complementary.
where the factor of 1.6918 is the parentheses in equation (2) for 
$N_{\rm eff} = 3.046$; therefore, $X_{\text{rad}} = 1.00$ for standard radiation content, and for example $X_{\text{rad}} = 1.134$ for $N_{\text{eff}} = 4.046$, i.e. the case of additional ‘dark radiation’ with energy density equal to one standard neutrino flavour. Here, $N_{\text{eff}}$ and $X_{\text{rad}}$ are equivalent, but the latter is convenient later since several parameters of interest scale almost as half-integer powers of $X_{\text{rad}}$.

There are now several known routes to probe $N_{\text{eff}}$ from observations: historically, it was first constrained by big bang nucleosynthesis (Steigman, Schramm & Gunn 1977; Mangano & Serpico 2011). However, $^4\text{He}$ is the nuclide with the main sensitivity to $N_{\text{eff}}$, and observational measurements of the primordial $^4\text{He}$ abundance, $Y_p$, appear to be limited by systematic errors; over the past 25 years the estimates of $Y_p$ have shifted significantly upwards, but the realistic error bars have not much improved. Unless a new better method of measuring $Y_p$ can be found, we cannot expect dramatic progress from the Helium route. Recently, a constraint on $N_{\text{eff}}$ has been derived from deuterium abundance (Pettini & Cooke 2012), but this currently relies on only a single object, and also uses the baryon density derived from the CMB.

Secondly, the CMB damping tail at high multipoles $\ell \gtrsim 1400$ is sensitive to $N_{\text{eff}}$ (Jungman et al. 1996; Bashinsky & Seljak 2003; Hou et al. 2013); and several recent measurements (Hou et al. 2012; Riemer-Sorensen et al. 2013a; Ade et al. 2013) give tantalizing but not decisive hints for a value higher than the standard 3.046. However, the CMB damping tail method is significantly degenerate with other possible new parameters, including running of the spectral index $n_s$ and non-standard helium abundance $Y_p$ (Hou et al. 2013; Joudaki 2013), so other complementary probes of $N_{\text{eff}}$ are desirable.

Thirdly, combining CMB data with a direct local measurement of $H_0$ can also probe $N_{\text{eff}}$; however, using CMB+$H_0$ alone is critically dependent on other assumptions such as $w = -1$ and flatness. An improvement on the $H_0$ method is given by Sutherland (2012), who showed that a theory-free measurement of $r_\ell$ can be obtained by combining a low-$z$ BAO redshift survey and a suitable absolute distance measurement to a matched redshift (specifically $4\pi/3$, where $z^2$ is the characteristic redshift of the BAO survey). This almost cancels the distance effects from dark energy and curvature; comparing such a direct $r_\ell$ measurement to CMB data (which mainly constrains $r_\ell \sqrt{X_{\text{rad}}}$ rather than $r_\ell$ alone) therefore probes $N_{\text{eff}}$. The above-mentioned method is less theory dependent than the CMB damping tail, but requires a challenging measurement of a distance to $z \sim 0.3$ to $\sim 2\%$ absolute accuracy.

We will demonstrate below that the amplitude of the BAO feature provides a fourth possible probe of $N_{\text{eff}}$: this is currently much less precise than the known methods above, but involves different assumptions and systematics; with future massive redshift surveys expected in the next decade, it may provide a useful complement to the better-known methods above.

### 2.3 Cosmological parameter set

The present-day photon density $\omega_\gamma = \Omega_\gamma h^2$ is very well constrained by the observed CMB temperature (Fixsen 2009) and spectrum to be $\omega_\gamma \simeq 1/40440$; we define $\omega_{\text{b,eff}} \equiv \omega_\gamma + \omega_\text{b}$ to be the physical matter density today, specifically CDM and/or WDM plus baryons, excluding neutrinos. Defining $z_{\text{eq}}$ as the redshift of matter-radiation equality, and simply assuming that the photon density scales with redshift as $\propto (1+z)^4$, and matter conservation so CDM and baryon densities scale $\propto (1+z)$ (i.e. no decaying dark matter, dark energy to dark matter transitions or other exotic effects) leads to the following identities:

$$\omega_{\text{b,eff}} = \frac{23904}{(1+z_{\text{eq}}) X_{\text{rad}}}$$

$$h = \sqrt{\frac{(1+z_{\text{eq}}) X_{\text{rad}}}{23904 \Omega_{\text{b,eff}}}}$$

where the first three and $X_{\text{rad}}$ are defined above, as usual $A$ is the scalar perturbation amplitude (which cancels in the following), $n_s$ is the scalar spectral index and $\tau$ is the optical depth to last scattering. Then, $h$, $\omega_{\text{b,eff}}$ and $f_\text{b}$ are derived parameters from equations (4)–(6). We may add optional parameters, curvature $\Omega_k$, dark energy equation of state $w$ and present-day neutrino density $\Omega_\nu$, defaulting to 0, $-1$, $\Omega_\nu = 0.0013$, respectively (for minimal neutrino mass). Then, $\Omega_{\text{b,eff}} \equiv 1 - \Omega_k$, and the dark energy density $\Omega_{\text{DE}}$ is another derived parameter, via $\Omega_{\text{DE}} = 1 - \Omega_{\text{b,eff}} - \Omega_\nu - \Omega_k$.

This parameter set (7) including $z_{\text{eq}}$ and $\Omega_{\text{b,eff}}$ in the basic six looks unconventional compared to the more common choice including $\omega_{\text{b,eff}}$ and $\Omega_{\text{DE}}$ as two of the basic parameters; but for variable $N_{\text{eff}}$, our set links more naturally to observational constraints as we will see below; see also Appendix A, and the discussion in section 4.2 of Sutherland (2012). To summarize the latter, dimensionless observables, such as the CMB acoustic wavenumber $\ell_*,1$ and distance ratios from BAO and SNe provide good constraints on dimensionless parameters including $z_{\text{eq}}$ and $\Omega_{\text{b,eff}}$, but there remains one overall scale degeneracy between $N_{\text{eff}}$ and dimensionful parameters such as $H_0$, $\Omega_\text{b,eff}, P_\text{b,eff}$. (Parameters such as $h$, $\omega_{\text{b,eff}}$ are only pseudo-dimensionless since they are relative to an arbitrary choice of $100 \text{ km s}^{-1} \text{ Mpc}^{-1}$, and these are affected by this degeneracy.)

It has been shown by several authors (Hu & Sugiyama 1996; Jungman et al. 1996; Bashinsky & Seljak 2003; Jimenez et al. 2004; Komatsu et al. 2011) that the heights of the first few acoustic peaks in the CMB primarily constrain the redshift of matter-radiation equality $z_{\text{eq}}$, not the physical matter density $\omega_{\text{b,eff}}$. These latter two parameters are equivalent if we force $N_{\text{eff}} \simeq 3.04$, but if we allow $N_{\text{eff}}$ to be free they are no longer equivalent, and then $z_{\text{eq}}$ is constrained much better than $\omega_{\text{b,eff}}$ by CMB data (see e.g. Komatsu et al. 2011).

$^1$ Here, following the WMAP convention, $\ell_* = \pi/\theta_\text{eq}$, with acoustic angle $\theta_\text{eq} = r_\ell(z_{\text{eq}})/(1+z_{\text{eq}})D_A(z_{\text{eq}})$ and $z_{\text{eq}}$ is the redshift of decoupling.

$^2$ Strictly, it is the ratio $(1+z_{\text{eq}})/(1+z_*)$ which is important in the CMB, where $z_*$ is the decoupling redshift; however, in practice the relative uncertainty in $z_*$ is much smaller than in $z_{\text{eq}}$, so we ignore this for simplicity.
The WMAP data also constrain the baryon density $\omega_b$ accurately. The effect of baryons on the CMB derives mainly from the baryon/photon ratio; given the photon density measured very accurately by COBE (Fixsen 2009), the $\omega_b$ estimate from WMAP is only weakly dependent on $N_{\text{eff}}$ or $X_{\text{rad}}$.

Measuring both $\omega_b$ and $z_{eq}$ immediately gives us the product $f_b X_{\text{rad}}$ from equation (6); so, the key point from the above is that the first few CMB acoustic peaks provide an accurate constraint on the product $f_b X_{\text{rad}}$, but $f_b$ and $X_{\text{rad}}$ are significantly degenerate. Therefore, adding a non-CMB observable which is sensitive to $f_b$ can provide another probe of $N_{\text{eff}}$.

In the next section, we show that the relative amplitude of the BAO peak in galaxy clustering may provide such a test: it depends mainly on $f_b$, with weak sensitivity to other parameters. Thus, comparing a BAO-based estimate of $f_b$ to a CMB-based measurement (approximately $f_b X_{\text{rad}}$) can provide a new probe of the radiation density which is largely independent of existing methods. A strong point of this method is that the CMB can measure $z_{eq}$ using only the first three acoustic peaks; for models near concordance parameter values, the ratio of the third to first peak height is especially sensitive to $z_{eq}$ (Hu et al. 2001; Page et al. 2003), and the third peak at $\ell \approx 800$ is only weakly affected by the Silk damping which dominates at $\ell \gtrsim 1500$. Thus, while we need CMB data at $\ell \lesssim 1000$, this method is not strongly dependent on the CMB damping tail and other possible early-time nuisance parameters.

3 ESTIMATING BARYON FRACTION FROM THE BAO PEAK

3.1 The BAO equivalent width

We noted above that the CMB power spectrum from WMAP constrains $f_b X_{\text{rad}}$ to $\sim 4$ per cent, and this improves to $\sim 2$ per cent with Planck data; therefore, an estimate of $f_b$ derived from the BAO amplitude can translate into a probe of $X_{\text{rad}}$ or equivalently $N_{\text{eff}}$.

However, deriving $f_b$ from the BAO feature is affected by several complications listed below: first there is galaxy bias, which may be scale dependent. Here, we choose to work with the correlation function rather than the power spectrum, since the former makes the BAO feature a single hump which simplifies the analysis. In the linear-bias approximation, the galaxy and matter correlation functions are related by $\xi_{gg}(r) \simeq b^2 \xi_{mm}(r)$; therefore, a suitable ratio of correlation functions near the BAO bump versus outside the bump can cancel galaxy bias if it is scale independent. This is believed to be a good approximation at linear scales $k < 0.1 \text{ h Mpc}^{-1}$ (Angulo et al. 2008; Baugh 2013), but a better understanding of galaxy formation may be required to clarify this.

Secondly, the BAO bump is blurred by the non-linear growth of structure, mainly due to peculiar motions (Eisenstein et al. 2007); this both lowers its height and broadens its shape, and causes a small shift in the central position. However, it is shown by Orban & Weinberg (2011) that non-linear growth almost conserves the total area of the bump; thus, measuring the bump area rather than its height is relatively insensitive to the non-linear growth of structure.

Thirdly, the global shape of $\xi(r)$ with $r$ in $h^{-1}$ Mpc units, depends on other quantities including $\Omega_m$ and the dark energy equation of state, which are not tightly constrained by the CMB. However, we show later that if we define $\mu = r/r_o$, to be the ratio of comoving separation $r$ to the sound horizon scale, then the broad-band shape of $\xi(u)$ on intermediate scales depends mostly on $z_{eq}$; nearly all shape dependence on other parameters is collapsed into $z_{eq}$, which is already well determined by the CMB.

Therefore, we define the following observable $W_b$ from the measured galaxy correlation function $\xi_{gg}$, as a measure of the BAO ‘equivalent width’: we define

$$W_b = \frac{\int_{u_{min}}^{u_{max}} u \left[ \xi_{gg}(u) - \xi_{bg}(u) \right] du}{\int_{u_{min}}^{u_{max}} u^2 \xi_{gg}(u) du},$$

(8)

where $\xi_{gg}(u)$ is the observed galaxy correlation function in units of $u = r/r_o$, $r_o$ is the bump scale (here the value of $r$ at the peak in $r^2 \xi_{gg}(r)$), and $\xi_{bg}(u)$ is a smooth ‘no bump’ curve, here a polynomial fitted to the regions of $\xi_{gg}(u)$ just outside the BAO bump. Then, $u_1, \ldots, u_4$ are arbitrary dimensionless limits of integration, where $u_1, u_2$ span almost the full area of the bump; while $u_3, u_4$ are intermediate scales non-overlapping with the bump, used for normalization. There is a compromise here, since we want to avoid the non-linear regime $u_1 \gtrsim 0.15$, while at $u_4 \gtrsim 0.6$ the measurement noise in $u^2 \xi(u)$ generally increases, and becomes more sensitive to systematic errors. In the following, we choose $u_1 = 1/3$, $u_2 = 2/3$ as simple values which give a well-measured signal in the linear regime, and are not too far below the bump scale to minimize the possible effects of scale-dependent bias.

In the above definition, a constant bias in $\xi_{gg}$ cancels out as long as it is scale independent on large scales $r \gtrsim 30 h^{-1}$ Mpc, while a multiplicative stretch of cosmic distance scales (e.g. from varying dark energy) also cancels in $W_b$, since we are measuring at fixed fractions of the comoving scale $r_o$ which is fitted from the data. The ratio between $r_o$ and the horizon size at matter-radiation equality is determined almost entirely by $z_{eq}$, so we expect $W_b$ defined as above to be mainly sensitive to the baryon fraction $f_b$ as desired.

To verify this and test parameter dependences, we next evaluate $W_b$ from the linear matter power spectrum for some example theoretical models generated by CAMB.3

3.2 Dependence of $W_b$ on $N_{\text{eff}}$ and $z_{eq}$

Here, we evaluate $W_b$ (defined above) for a set of six representative models which are all consistent with CMB data up to 2012. All models are flat $\Lambda$CDM ($\Omega_{\text{tot}} = 1, \omega_b = -1$), and have $n_s$ fixed to 0.96 and $\omega_b = 0.0226$ in accordance with WMAP. We vary $z_{eq}$ and $N_{\text{eff}}$, and also adjust $\Omega_b$, to preserve the CMB acoustic scale $\ell_c$.

For our ‘base’ model (hereafter C3), we set $N_{\text{eff}} = 3.046, z_{eq} = 3264, \Omega_{b,0} = 0.279$; therefore, $\omega_b, 0.1366$ and $h = 0.700$.

For model C4, we add a fourth light neutrino species, but retain identical values of $z_{eq}$ and $\Omega_{b,0}$; therefore, C4 has $\omega_b$ and $h$ increased by 13.4 and 6.5 per cent, respectively, relative to C3. (Here, $\omega_b$ is held at 0.0226 for both models, so model C4 has dark matter density $\omega_b$ increased by slightly more than 13.4 per cent, while $f_b$ is reduced by a factor of 0.882.)

The overall shape of the correlation function also depends significantly on $z_{eq}$: to explore this dependence, we choose two models (labelled L3, L4) with $z_{eq}$ fixed to 5 per cent lower than C3, and, respectively, $N_{\text{eff}} = 3.04$ and 4.04; likewise another two models (H3, H4) with $z_{eq}$ fixed 5 per cent higher than C3. For these models, $\Omega_{b,0}$ is adjusted in order to preserve the CMB acoustic scale $\ell_c \equiv \pi/\theta_c$. The resulting parameter values are shown in Table 1. Since these models are all flat, they do not quite follow the CMB geometrical

3 We used the 2011 January release of CAMB, by A. Lewis and A. Challinor, available from http://camb.info/
Table 1. Cosmological parameters for the six example models discussed in the text. All models have $\Omega_{eq} = 1$, $w = -1$ and $\omega_b = 0.0226$. Model C3 (bold) is our baseline, while model C4 has $N_{\text{eff}} = 4.04$ but unchanged $z_{eq}$ and $\Omega_{cb}$. Models labelled L and H have $z_{eq}$ forced, respectively, 5 per cent lower/higher than C, then $\Omega_{cb}$ adjusted to preserve the CMB acoustic scale. Values of $\omega_b$, $f_b$ and $H_0$ are derived from the first three. The last column gives the value of $W_b$ as defined in equation (8), calculated by integrating the linear-theory matter correlation function.

| Model | $z_{eq}$ | $\Omega_{cb}$ | $N_{\text{eff}}$ | $\omega_b$ | $f_b$ | $H_0$ (km s$^{-1}$ Mpc$^{-1}$) | $W_b$ |
|-------|----------|---------------|-----------------|------------|-------|-------------------------------|-------|
| L3    | 3101     | 0.247         | 3.04            | 0.1298     | 0.174 | 72.5                          | 0.612 |
| L4    | 3101     | 0.247         | 4.04            | 0.1471     | 0.154 | 77.1                          | 0.561 |
| C3    | 3264     | 0.279         | 3.04            | 0.1366     | 0.165 | 70.0                          | 0.614 |
| C4    | 3264     | 0.279         | 4.04            | 0.1549     | 0.146 | 74.5                          | 0.561 |
| H3    | 3428     | 0.315         | 3.04            | 0.1434     | 0.158 | 67.5                          | 0.608 |
| H4    | 3428     | 0.315         | 4.04            | 0.1626     | 0.139 | 71.9                          | 0.560 |

Figure 1. This figure shows the CMB temperature power spectra for the six example models from Table 1; all are normalized to match model C3 at $\ell = 100$. The horizontal axis is linear in $\sqrt{\ell}$ for improved resolution at low $\ell$. Models with $N_{\text{eff}} = 3.04$ are solid lines; models with $N_{\text{eff}} = 4.04$ are dashed lines. The values of $z_{eq}$ are labelled.

degeneracy, but they do follow the related degeneracy of constant $\ell_*$ or horizon angle as outlined in Percival et al. (2002).

We used CAMB to evaluate the CMB temperature power spectra for the above six models; these are shown in Fig. 1, normalized to match model C3 at $\ell = 100$. Clearly, the CMB spectra are very similar for all our models, since the acoustic scales are matched by construction, and the variations in $z_{eq}$ are only ±5 per cent. Minor differences are apparent, notably around the third peak (which is positively correlated with $z_{eq}$), while the effects of $N_{\text{eff}}$ appear mainly in the damping tail and are small at $\ell < 1000$. We repeat here that $\omega_b = 0.0226$ and $n_s = 0.96$ have been held fixed in all models for simplicity, in order to highlight the effects of $z_{eq}$ and $N_{\text{eff}}$. Clearly, allowing $n_s$ and $\omega_b$ to float to fit CMB data would result in model spectra that are even more similar, especially if a running spectral index is also allowed.

We took the linear-theory matter power spectra for the above six models generated by CAMB, and then Fourier transformed them to obtain the real-space matter correlation functions; these are shown in Fig. 2, with the $x$-axis in units of $h^{-1}$ Mpc corresponding to the observable from a low-$z$ redshift survey.

For the matter correlation functions in Fig. 2, the differences between models are much more obvious than in the CMB: the position of the BAO bump (in $h^{-1}$ Mpc units) is insensitive to $N_{\text{eff}}$ for fixed $z_{eq}$, $\Omega_{cb}$, but it does shift with $z_{eq}$. In fact, as explained in Appendix A, the BAO bump location is more sensitive to $\Omega_{cb}$ than $z_{eq}$, but changing $z_{eq}$ required us to adjust $\Omega_{cb}$ to conserve the CMB
acoustic scale, and it is actually the change in $\Omega_b$, which dominates the shift of the bump location. The other notable feature in Fig. 2 is that all the $N_{\text{eff}} = 4.04$ models have a slightly reduced BAO peak amplitude, as qualitatively expected given their smaller $f_b$.

To highlight the effects of varying $N_{\text{eff}}$, in Fig. 3 we plot the matter correlation functions as a function of $u = r/r_b$, so that the BAO bump appears at $u = 1$. This figure shows clearly that the bump amplitude is mainly sensitive to $N_{\text{eff}}$, while the broad-band shape (the ratio of power at $u \sim 0.2$ to that at $u \gtrsim 0.6$) is governed mainly by $z_{\text{eq}}$. This is understandable since the broad-band shape is determined by the scale of the turnover in the matter power spectrum, which is directly proportional to the particle horizon size at $z_{\text{eq}}$. This scale in observable $h^{-1}$ Mpc units depends on several cosmological parameters. However, as noted in e.g. equation B2 of Sutherland (2012), the ratio of the BAO sound horizon $r_s(z_d)$ to the particle horizon $r_H$ at $z_{\text{eq}}$ (both in comoving units) has a simpler dependence: this ratio is well approximated by simply

$$\frac{r_s(z_d)}{r_H(z_{\text{eq}})} \simeq \frac{1 + z_{\text{eq}}}{3201} 1.275^{0.75} ;$$

(9)

since the sound speed $c_s(z)$ is well constrained by the WMAP baryon density. The dependence on other parameters such as $\Omega_b$, $h$, $X_{\text{rad}}$ is almost entirely compressed into $z_{\text{eq}}$, and the ratio is completely independent of late-time parameters such as $w$, $\Omega_\Lambda$. Thus, the changes in $\xi(u)$ in Fig. 3 are largely driven by the differing $z_{\text{eq}}$ and $f_b$ between the six models, and adding optional parameters, such as $\Omega_b$, $w$ will have minimal effect.

Here, it is also noteworthy that the zero-crossing in $\xi(u)$ occurs close to $u \sim 1.2$ for all the models; this offers an interesting possible consistency test for the $\Lambda$CDM framework which is largely insensitive to galaxy bias. However, this is observationally challenging to measure since the zero-crossing is much more sensitive than the BAO peak position and amplitude to broad-band systematic errors in the observed $\xi(u)$.

The denominator of equation (8) is mainly sensitive to the broad-band large-scale power at $k \lesssim 0.1 h$ Mpc$^{-1}$, which as above depends on the turnover scale in the matter power spectrum. If we measured this in a fixed range of Mpc or $h^{-1}$ Mpc, this would depend on quantities such as $\Omega_b$ and $w$, which would seriously degrade our ability to measure $f_b$; but since we chose our mid-scale power estimate as a fixed fraction of the BAO length rather than a fixed range in $h^{-1}$ Mpc, this mostly cancels the dependence on low-redshift parameters such as $\Omega_b$, $\Omega_\Lambda$ and $w$; the broad-band shape of $\xi(u)$ at $0.2 < u < 0.8$ depends almost entirely on $z_{\text{eq}}$ and $n_s$, which are already well constrained by the CMB. Therefore, we anticipate that $W_b$ should depend mainly on $f_b$ and only weakly on $z_{\text{eq}}$.

To quantify this, we evaluated the ratio $W_b$ for our six models, and the results are given in the last column of Table 1: the table shows that $W_b$ is close to 0.61 for all three $N_{\text{eff}} = 3.04$ models, and close to 0.56 for all three $N_{\text{eff}} = 4.04$ models, consistent with our expectations above. The dependence of $W_b$ on $z_{\text{eq}}$ is below 1 per cent and nearly negligible, while adding a fourth neutrino species or equivalent reduces $W_b$ by a factor close to 0.915 (i.e. 8.5 per cent suppression) in each case. This reduction is slightly less than we would expect from linear scaling $W_b \propto f_b$, since our $N_{\text{eff}} = 4.04$ models have $f_b$ reduced by a factor 1.134$^{-1} = 0.882$ relative to the corresponding $N_{\text{eff}} = 3.04$ model. The probable explanation is that baryons, in addition to causing oscillations, also affect the broad-band shape of the power spectrum (Eisenstein & Hu 1998) by suppressing power on all scales smaller than the sound horizon.
Therefore, reducing the baryon fraction slightly increases power on intermediate scales, and changes the broad-band shape of $\xi(u)$, which slightly counteracts the reduction in the bump area.

The conclusion is that if galaxy bias is scale independent on large scales and the area of the BAO peak is conserved under non-linear evolution (or can be recovered by reconstruction methods), then measurements of $W_i$ can offer a potential new probe of $N_{\text{eff}}$.

Estimates from large numerical simulations could be used to test these assumptions, and possibly attempt to correct for any resulting biases.

The largest current redshift surveys provide an ~6σ detection of the BAO peak (Anderson et al. 2012), which would translate to approximately 16 per cent uncertainty in $W_5$; this is twice as large as the 8.5 per cent shift predicted above for $N_{\text{eff}} \sim 4$, so at present the precision on $N_{\text{eff}}$ looks uncompetitive with other methods. However, future next-generation large redshift surveys can potentially offer a large improvement, and thus an interesting test of $N_{\text{eff}}$ which is complementary to the better-known methods from the CMB and nucleosynthesis.

3.3 Effect of Planck data

Most of this paper studies models with parameter choices based on the WMAP-9 cosmological parameter results (Hinshaw et al. 2012); the C3 model is near the best fit, and L and H models have $z_{\text{eq}}$ shifted by ±1.5σ in WMAP units. After this paper was nearly completed, the first Planck cosmology data release occurred in 2013 March. While there are many interesting consequences for inflation and the early Universe, for the present purposes, two results are most notable: first concerning $N_{\text{eff}}$, the evidence for $N_{\text{eff}} > 3.04$ has generally weakened (Ade et al. 2013), but the strength of this conclusion is somewhat dependent on the choice of additional data sets.

The fit $\Lambda$CDM + varying $N_{\text{eff}}$ to the data set ‘Planck + WMAP polarization + high-L + BAO’ (the right-hand column of table 10 of Ade et al. 2013) gives $N_{\text{eff}} = 3.30 \pm 0.26$, which is 1σ above the standard value and excludes $N_{\text{eff}} = 4.04$ at the 2.8σ level. However, there remains the well-publicized tension that Planck with vanilla $\Lambda$CDM (and $N_{\text{eff}} = 3.04$) prefers a value of $H_0 = 67.8 \pm 0.8 \, \text{km s}^{-1} \, \text{Mpc}^{-1}$, which is below the 2σ range given by recent local measurements (Riess et al. 2011; Freedman et al. 2012). There are many possible explanations, but this $H_0$ tension can be ameliorated by increasing $N_{\text{eff}}$: e.g. fitting Planck + $H_0$ data allowing variable $N_{\text{eff}}$ gives $H_0 = 72.1 \pm 1.9 \, \text{km s}^{-1} \, \text{Mpc}^{-1}$ and $N_{\text{eff}} = 3.62 \pm 0.25$, i.e. 2.2σ above the standard $N_{\text{eff}}$. In summary, $N_{\text{eff}} \sim 4.0$ is somewhat disfavoured by Planck, but a value of $N_{\text{eff}} \sim 3.5$ is completely allowed or perhaps even preferred by combining all current data. There are interesting possible models with extra relativistic species other than neutrinos leading to $N_{\text{eff}} \sim 3.5$ (e.g. Weinberg 2013). Secondly, concerning $z_{\text{eq}}$ and $\Omega_{\text{tot}}$, the Planck data imply values somewhat higher than WMAP; for the vanilla $\Lambda$CDM model, fits to Planck+BAO data give $z_{\text{eq}} = 3366 \pm 39$ and $\Omega_{\text{tot}} = 0.307 \pm 0.01$ (and $h = 0.678 \pm 0.008$ for standard $N_{\text{eff}}$). The Planck constraints on $z_{\text{eq}}$ are especially robust: in the many extensions of $\Lambda$CDM considered by the Planck team, the bounds $3150 < z_{\text{eq}} < 3500$ are generic, i.e. values outside this range are excluded at >2σ for all of the added-parameter models and data combinations. (Clearly, still more complicated models with even more non-vanilla parameters might widen this range; but there appears little motivation at present for adding two or more new parameters beyond the basic six.)
Comparing to our models above, the Planck central value \( \tilde{z}_{\text{eq}} \simeq 3366 \) is near the mid-point between our model pairs C and H above, but slightly closer to H. Our two L models (\( \tilde{z}_{\text{eq}} = 3101 \)) are now firmly excluded by Planck, at around the 5\( \sigma \) level for the base model or 3\( \sigma \) for extended models. Also, from the CMB then gives an interesting range \( \Omega_{b} \omega \) and central value \( z \) has narrowed the allowed range of \( z \) Planck prefers \( <3 \) or \( 4 \); this \( z \)/\( \Omega_{1} \) allowed by in equation (8) may provide an interesting data from the Legacy Archive and 3101) are now \( = \) and Planck \( N \) used as basic parameters.F 

4 CONCLUSIONS

We have shown that a measurement of the BAO peak amplitude via the observable \( W_{s} \) in equation (8) may provide an interesting measurement of the cosmic baryon fraction; this observable has been constructed so as to cancel galaxy bias, non-linearity and dark energy effects to leading order, thus being sensitive mostly to \( f_{b} \).

Comparing this BAO-based measurement to the measurement of (approximately) \( f_{b}\chi_{\text{rad}} \) from the CMB then gives an interesting probe of \( N_{\text{eff}} \); this is largely complementary to the better-known method based on fitting the CMB damping tail. Here, the key inputs required from the CMB are constraints on \( z_{\text{eq}} \) and \( \omega_{b} \). Assuming standard gravity and standard recombination, these two parameters are very robust against extra-parameter extensions to vanilla \( \Lambda \)CDM.

There are two main assumptions used here: first that galaxy bias is nearly scale independent on the large scales within the range 30 < \( r < 120 \) h\(^{-1} \) Mpc, and secondly, that the area (not height) of the BAO bump is conserved during the non-linear evolution of structure. Both of these assumptions are reasonably well motivated, but much more detailed numerical simulations would be needed to see how well these approximations are expected to hold in practice.

A measurement of \( W_{s} \) to useful precision will require a substantial advance on current data: the current precision on the BAO bump area is around 16 per cent, while we would need to reach around 3 per cent to get a useful distinction between \( N_{\text{eff}} = 3 \) or 4; this appears a challenging proposition. However, given that the CMB temperature measurements are now approaching the limits set by cosmic variance and foregrounds, other independent probes of \( N_{\text{eff}} \) are highly desirable, and the test here should become feasible at no extra cost from planned next-generation BAO redshift surveys.

ACKNOWLEDGEMENTS

We acknowledge the use of WMAP data from the Legacy Archive for Microwave Background Data Analysis (LAMBDA) at GSFC (lambda.gsfc.nasa.gov), supported by the NASA Office of Space Science.

REFERENCES

Abazajian K. N. et al., 2012, preprint (arXiv:1204.5379)
Ade P. A. R. et al. (Planck Collaboration XVI), 2013, A&A, preprint (arXiv:1303.5076)
Anderson L. et al., 2012, MNRAS, 427, 3435
Angulo R. E., Baugh C. M., Frenk C. S., Lacey C. G., 2008, MNRAS, 383, 755
Bashinsky S., Seljak U., 2004, Phys. Rev. D, 69, 083002
Bashsett B. A., Hlozek R., 2010, in Ruiz-Lapuente P., ed., Dark Energy. Cambridge Univ. Press, Cambridge, p. 246
Baugh C. M., 2013, Publ. Astron. Soc. Aust., 30, E30
Beutler F. et al., 2011, MNRAS, 416, 3017
Blake C. et al., 2011, MNRAS, 415, 2892
Bond J. R., Efstathiou G., 1984, ApJ, 285, L45
Cole S. et al., 2005, MNRAS, 362, 505
Dodelson S., 2011, Int. J. Mod. Phys. D, 20, 2749
Eisenstein D. J., Hu W., 1998, ApJ, 496, 605
Eisenstein D. J. et al., 2005, ApJ, 633, 560
Eisenstein D. J., Seo H., Sirko E., Spergel D. N., 2007, ApJ, 664, 675
Eisenstein D. J., Seo H., White M., 2007, ApJ, 664, 660
Fixsen D. J., 2009, ApJ, 707, 916
Freedman W. L., Madore B. F., Scowcroft V., Burns C., Monson A., Persson S. E., Seibert M., Rigby J., 2012, ApJ, 758, 24
Giassarina E., de Putter R., Ho S., Mena O., 2013, Phys. Rev. D, 88, 063515
Hinshaw G. et al., 2012, ApJS, 208, 19
Hou Z. et al., 2012, preprint (arXiv:1212.6267)
Hou Z., Keisler R., Knox L., Millea M., Reichardt C., 2013, Phys. Rev. D, 87, 083008
Hu W., Sugiyama N., 1996, ApJ, 471, 542
Hu W., Fukugita M., Zaldarriaga M., Tegmark M., 2001, ApJ, 549, 669
Jimenez R., Verde L., Peiris H., Kosowsky A., 2004, Phys. Rev. D, 70, 3005
Joudaki S., 2013, Phys. Rev. D, 87, 083523
Jungman G., Kamionkowski M., Kosowsky A., Spergel D., 1996, Phys. Rev. D, 54, 1332
Keisler R. et al., 2011, ApJ, 743, 28
Komatsu E. et al., 2011, ApJS, 192, 18
Mangano G., Serpico P. D., 2011, Phys. Lett. B, 701, 296
Mangano G., Miele G., Pastor S., Tinto P., Pisanti O., Serpico P. D., 2005, Nucl. Phys. B, 729, 221
Melissin A., White M., Peacock J. A., 1999, MNRAS, 304, 851
Orban C., Weinberg D. H., 2011, Phys. Rev. D, 84, 063501
Page L. et al., 2003, ApJS, 148, 233
Peebles P. J. E., Yu J. T., 1970, ApJ, 162, 815
Padmanabhan N., Xu X., Eisenstein D. J., Scalzo R., Cuesta A. J., Mehta K. T., Kazin E., 2012, MNRAS, 427, 2132
Percival W. J. et al., 2002, MNRAS, 337, 1068
Percival W. J. et al., 2010, MNRAS, 401, 2148
Pettini M., Cooke R., 2012, MNRAS, 425, 2477
Riemer-Soresen S., Parkinson D., Davis T. M., 2013a, Publ. Astron. Soc. Aust., 30, E029
Riemer-Soresen S., Parkinson D., Davis T. M., 2013b, preprint (arXiv:1306.4153)
Riess A. G. et al., 2011, ApJ, 730, 119
Seo H.-J., Eisenstein D. J., 2007, ApJ, 665, 14
Seo H.-J., Siegel E. R., Eisenstein D. J., White M., 2008, ApJ, 636, 13
Seo H.-J. et al., 2010, ApJ, 720, 1650
Seo H.-J. et al., 2012, ApJ, 716, 13
Sievers J. L. et al., 2013, J. Cosmol. Astropart. Phys., 10, 60
Steigman G., Schramm D. N., Gunn J. E., 1977, Phys. Lett. B, 66, 202
Story K. T. et al., 2013, ApJ, 799, 86
Sutherland W., 2012, MNRAS, 426, 1280
Weinberg D. H., Mortonson M. J., Eisenstein D. J., Hirata C., Reiss A. G., Rozo E., 2013, Phys. Rep., 530, 87
Weinberg S., 2013, Phys. Rev. Lett., 110, 241301

APPENDIX A: PARAMETER DEPENDENCE OF CMB PEAKS AND BAOs

In this section, we give some accurate approximations for the dependence of CMB acoustic scale and BAO distance ratios on cosmological parameters, especially \( z_{\text{eq}} \) and \( \Omega_{b} \) used as basic parameters above. This helps us to understand the parameter choices in
Table 1, and the resulting shifts in the BAO bump position observed in Section 3.2. First, we find that a very good approximation to the CMB acoustic wavenumber \( \ell_* \) for models fairly close to standard \( \Lambda \) CDM is

\[
\ell_* \simeq 301.9 \left( \frac{1 + z_{eq}}{3201} \right)^{-0.25} \left( \frac{\Omega_{b,0}}{0.270} \right)^{0.1} \times \left( \frac{1 - f_v}{0.995} \right)^{0.4} (1 + 1.6 \Omega_{\nu}) [1 - 0.11(1 + w)],
\]

where \( f_v \equiv \Omega_{\nu}/(\Omega_{b,0} + \Omega_{\nu}) \), and this allows for small neutrino mass, weak curvature and constant \( w \neq -1 \). (This is for \( \omega_b = 0.0226 \); however, changing to the \textit{Planck} value \( \omega_b = 0.02215 \) gives only around 0.1 per cent reduction in \( \ell_* \).) This has almost negligible dependence on \( N_{\text{eff}} \), since varying \( N_{\text{eff}} \) (at fixed \( z_{eq}, \Omega_{b,0} \) as above) results in both \( r_s(z_*) \) and \( D_A(z_*) \) shrinking by a factor very close to \( X_{\text{rad}}^{-0.5} \), but these cancel almost exactly in \( \ell_* \).

Since \( \ell_* \) is measured to high precision \( \approx 0.2 \) per cent by \textit{WMAP}+\textit{ACT}+\textit{SPT}, if we vary \( z_{eq} \) (as in the \textit{L/H} models in Table 1 above), then to remain consistent with CMB data we must keep \( \ell_* \) the same as the baseline model C3; this corresponds to an increase in \( h \) by 3.6 per cent (at fixed \( N_{\text{eff}} \)). Shifts from C to H models are basically the opposite of this. We note one counter-intuitive feature: when varying parameters to conserve \( \ell_* \), it turns out that \( h \) changes in the opposite sense to \( z_{eq} \); this is distinct from the common case of fixing \( \Omega_{b,0} \) and varying \( h \), when \( 1 + z_{eq} \) varies \( \propto h^2 \).

We can also understand the resulting shifts in the BAO bump location as follows: if we copy approximation 12 from Sutherland (2012) for low-redshift BAO ratios, which is

\[
\frac{r_s}{D_A(z)} \approx 0.01868 (1 + \epsilon_v(z)) \frac{E(z)}{\sqrt{\Omega_{b,0}}} \left( \frac{1 + z_{eq}}{3201} \right)^{0.25} ;
\]

here, the LHS is a direct observable from a BAO survey at effective redshift \( z \), \( D_A(z) \) is the usual BAO dilation length (Eisenstein et al. 2005), \( E(z) \equiv H(z)/H_0 \) and \( \epsilon_v \) is a small cosmology-dependent correction term (Sutherland 2012), which is typically \( \lesssim 0.05 \). Approximation (A2) is accurate to \( \lesssim 0.7 \) per cent at \( z \lesssim 0.4 \), comparable to the cosmic variance limit, and again this is almost independent of \( N_{\text{eff}} \). At low redshift, equation (A2) is only weakly sensitive to additional non-vanilla parameters such as curvature and varying \( w \) via the \( E(2z/3) \) term; this explains why low-redshift BAO observations provide a very robust constraint on \( \Omega_{b,0} \).

In the limit \( z \to 0 \), the above simplifies to

\[
\frac{r_s}{c} H_0 \simeq 0.01868 \frac{1 + z_{eq}}{3201} \left( \frac{1 + \epsilon_v(z)}{\sqrt{\Omega_{b,0}}} \right)^{0.25}.
\]

The LHS is equivalent to a hypothetical BAO measurement at \( z = 0 \); this is not strictly observable since cosmic variance prevents us from measuring the BAO feature at \( z \lesssim 0.1 \); but it is a modest extrapolation from real low-z BAO surveys. The main point is since a galaxy redshift survey of course measures redshifts not distances, the apparent BAO bump ‘length’ presented in \( h^{-1} \) Mpc units, as in Fig. 2, is really measuring the ‘BAO velocity’ \( H_0 r_s \), in units of \( 100 \) km s\(^{-1}\). Although this quantity contains \( h \), in the case of varying \( N_{\text{eff}} \) this gets cancelled: if we vary \( N_{\text{eff}} \) while holding fixed \( z_{eq} \) (as appropriate for fitting CMB data), then \( h \) and \( r_s \) both depend on the radiation density as \( X_{\text{rad}}^{1/2} \) and \( X_{\text{rad}}^{-1/2} \), respectively; so their product is almost independent of \( X_{\text{rad}} \) and only depends on the dimensionless parameters \( z_{eq} \) and \( \Omega_{b,0} \), plus a very weak dependence on \( \omega_b \) which is negligible at the current level of accuracy. Therefore, the observed velocity scale of the BAO feature at low redshift is primarily measuring \( \Omega_{b,0} \), not \( h \), which explains why the BAO feature does not shift between the 3 and 4 neutrino model pairs in Section 3.2.

Since all of the approximations above are nearly independent of \( N_{\text{eff}} \), this was the rationale for choosing \( z_{eq} \) and \( \Omega_{b,0} \) as two of the basic parameters: observations of CMB and BAOs give us direct constraints on \( z_{eq} \) and \( \Omega_{b,0} \), nearly independently of \( N_{\text{eff}} \). These two directly give a constraint on \( h \) via \( X_{\text{rad}} \) from equation (5), but give almost no ability to measure \( h \) or \( X_{\text{rad}} \) separately; this explains why \textit{WMAP}+BAO alone currently have very weak leverage on \( N_{\text{eff}} \), unless further dimensionful data such as \( H_0 \) or \( \Omega_m \) are added.

Finally, the fact that \( \Omega_{b,0} \) appears with a \( -0.5 \) power in equation (A3) explains why the BAO feature shifts to smaller (larger) velocity scale for the models H (L) above.

This paper has been typeset from a \LaTeX\ file prepared by the author.