Cohomological invariants of algebraic curves, part 2

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Abstract

The continuous cohomological invariants of the stacks of hyperelliptic curves of even genus and genus 3 are computed. The techniques used are based on the equivariant version of Rost’s Chow groups with coefficients and the description of these stacks in Arsie and Vistoli’s paper [AV04].

1 introduction

In what follows we fix a base field $k_0$ and a prime number $p$. We will always assume that the characteristic of $k_0$ is different from $p$ and that $k_0$ contains a non-trivial $p$-th root of 1. If $X$ is a $k_0$-scheme we will denote by $H^*_{et}(X)$ the étale cohomology ring of $X$ with coefficients in $\mathbb{F}_p$. If $R$ is a $k_0$-algebra, we set $H^*_{et}(R) = H(Spec(R))$.

In Part 1 we defined a continuous cohomological invariant of an algebraic stack $\mathcal{M}$ to be a natural transformation between the functor of points of $\mathcal{M}$ and étale cohomology, seen as functors from the category of extensions of a given base field to set, satisfying a certain natural conditions for maps from a henselian DVR to $\mathcal{M}$. The continuous cohomological invariants of $\mathcal{M}$ form a graded ring $\text{CInv}^*(\mathcal{M})$, the assignment $X \mapsto \text{CInv}^*(X)$ is functorial and moreover defines a sheaf on an appropriate site of algebraic stacks, the smooth-Nisnevich site [Pir14, 2.11].

For smooth stacks, continuous cohomological invariants are naturally isomorphic to the smooth-Nisnevich sheafification of étale cohomology [Pir14, 2.17]. On algebraic spaces, this is isomorphic to Rost’s zero-codimensional Chow group with coefficients [Ros96, 6.5], where the coefficients are taken in étale cohomology.

This makes Rost’s Chow groups with coefficients the instrument of choice for computing continuous cohomological invariants, especially if we are considering a quotient stack $[X/G]$, as in this case we have at our disposal an equivariant Chow ring with coefficients $A^*_G(X)$ and there is an isomorphism $A^*_G(X) = \text{CInv}^*([X/G])$.

In this paper we use the theory we set up in Part 1 to compute the continuous cohomological invariants of the stacks of Hyperelliptic curves of all even genus, and of genus three. The main result is the following:

Theorem 1. Suppose our base field $k_0$ is algebraically closed, of characteristic different from 2, 3.
Suppose \( g \) is even. For \( p = 2 \), the cohomological invariants of \( \mathcal{H}_g \) are generated as a graded \( \mathbb{F}_2 \)-module by 1 and invariants \( x_1, \ldots, x_{g+2} \), where the degree of \( x_i \) is \( i \).

If \( p \neq 2 \), then the cohomological invariants of \( \mathcal{H}_g \) are nontrivial if and only if \( 2g + 1 \) is divisible by \( p \). In this case they are generated by a single nonzero invariant of degree one.

Suppose \( g = 3 \). For \( p = 2 \) the cohomological invariants of \( \mathcal{H}_3 \) are freely generated as a \( \mathbb{F}_2 \)-module by 1 and elements \( x_1, x_2, w_2, x_3, x_4, x_5 \), where the degree of \( x_i \) is \( i \) and \( w_2 \) is the second Stiefel-Whitney class coming from the cohomological invariants of \( \text{PGL}_2 \).

If \( p \neq 2 \), then the cohomological invariants of \( \mathcal{H}_3 \) are trivial for \( p \neq 7 \) and generated by a single nonzero invariant of degree one for \( p = 7 \).

Our method is based on the presentation by Vistoli and Arsie [AV04, 4.7] of the stacks of hyperelliptic curves as the quotient of an open subset of an affine space by a group \( G \), which is equal to \( GL_2 \) for even genera and \( PGL_2 \times G_m \) for odd genera. We use a technique similar to the stratification method introduced by Vezzosi in [Vez00] and used by various authors afterwards (Gu08, VM06).

The idea is, given a quotient stack \([X/G]\), to construct a stratification \( X = X_0 \supset X_1 \supset \ldots \supset X_n = \emptyset \) of \( X \) such that the geometry of \( X_i \setminus X_{i+1} \) is simple enough that we can compute inductively the invariants for \( X_i \) using the result for \( X_{i+1} \) and the open-closed exact sequence [Ros96, p. 356].

1.1 Description of content

In the second section we describe the presentation of the stacks \( \mathcal{H}_g \) as quotients \([U_g/G]\). If we see \( \mathbb{A}^{2g+3} \) as the space of binary forms of degree \( 2g+2 \), the scheme \( U_g \subset \mathbb{A}^{2g+3} \) is the open subscheme of nonzero forms with distinct roots. We show that cohomological invariant can be computed on the projectivized space \( \mathcal{Z}_g = X_g/G_m \), where \( G_m \) acts by multiplication, and we introduce a stratification \( \mathcal{P}_{2g+2} \supset \Delta_{1,2g+2} \supset \ldots \supset \Delta_{g+1,2g+2} \) which will be the base of our computation. We can see \( \Delta_{1,2g+2} \) as the closed subscheme of homogeneous binary forms divisible by the square of a form of degree \( i \), and we have \( \mathcal{Z}_g = \mathcal{P}_{2g+2} \setminus \Delta_{1,2g+2} \).

The third section is dedicated to computing the continuous cohomological invariants of \( \mathcal{M}_2 \) over an algebraically closed field. The proof is kept as elementary as possible to give the reader a gradual introduction to the techniques used, and using the fact that we are working over an algebraically closed field everything is worked out with considerations on the equivariant Chow groups mod \( p \). The argument is based on the fact that the equivariant Chow groups with coefficients of \( \Delta_{1,2g+2} \) are isomorphic to those of \( P^{n-2i} \setminus \Delta_{1,2g+2} \times P^i \), giving rise to an inductive reasoning.

In the fourth section we compute the invariants for \( \mathcal{H}_g \) for all even \( g \). The techniques used for \( \mathcal{M}_2 \) are improved and generalized as more specific arguments involving the functoriality of the open-closed exact sequence and the projection formula are introduced.

The fifth section is dedicated to extending the results to fields that are not algebraically closed. The extension turns out to be immediate when the prime \( p \) is different from 2, and rather troublesome for \( p = 2 \). The main difficulty lies
in understanding if the pushforward through the closed immersion $\Delta_{1,6} \to P^6$ induces the zero map on $A^0_{GL_2}$. To do so, we construct an element in degree 0 which belongs to the annihilator of the image of $A^0_{GL_2}(\Delta_{1,6})$ but does not belong to the annihilator of any nonzero element of $A^0_{GL_2}(P^6)$. This has to be done at cycle level, and the construction relies heavily on the explicit description of the first Chern class of a line bundle given in Part 1.

In the sixth section we compute the equivariant Chow groups with coefficients of the classifying spaces of $\mu_2, O_2, O_3$ and $SO_3$. This is needed as $H_3$ is described as a quotient by an action of $PGL_2 \times G_m$, and the equivariant Chow ring $A^*_G(X(\text{spec}(k_0)))$ is isomorphic to $A^*_PGL_2(\text{spec}(k_0))$. We follow step by step the stratification method used in [VM06], with some minor changes. The computations are done both with coefficients in étale cohomology and in Milnor’s $K$-theory, as the proofs can be adapted easily.

In the seventh section we compute the continuous cohomological invariants of $H$. The fact that $PGL_2$ is not special creates some additional complications, as the equivariant ring $A^*_PGL_2(\text{spec}(k_0))$ has several nonzero elements in positive degree even for an algebraically closed $k_0$. The main difficulty is again proving that the map $A^*_PGL_2(\Delta_{1,n}) \to A^*_PGL_2(P^n)$ is zero. Luckily the richer structure of $A^*_PGL_2(\text{spec}(k_0))$ allows us to inductively construct an element $f_n$ in the annihilator of the image of $A^*_PGL_2(\Delta_{1,n})$ which for $n \leq 8$ does not belong to the annihilator of any nonzero element in $A^*_PGL_2(P^n)$, allowing us to conclude.

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2 Preliminaries

In this section we state some general considerations that will be needed for all the computations in the paper. As we will always be working with the notion of continuous cohomological invariants defined in Part 1, throughout the paper we will often call them cohomological invariants or just invariants for brevity.

We begin by recalling the presentations of the stacks we will work with, all due to Vistoli and Arsie [AV04].

**Theorem 2.1.** Let $g$ be an even positive integer. Consider the affine space $\mathbb{A}^{2g+3}$, seen as the space of all binary forms $\phi(x) = \phi(x_0, x_1)$ of degree $2g + 2$.

Denote by $X$ the open subset consisting of nonzero forms with distinct roots. Consider the action of $GL_2$ on $X_g$ defined by $A(\phi(x)) = \det(A)^g \phi(A^{-1}x)$.

Denote by $H_g$ the stack of smooth hyperelliptic curves of genus $g$. In particular, as any smooth curve of genus 2 is hyperelliptic, $H_2 = \mathcal{H}_2$. Then we have

$$H_g \simeq [X_g/GL_2]$$

And the canonical representation of $GL_2$ yields the Hodge bundle of $H_g$.

Let $g$ be an odd positive integer. Consider $\mathbb{A}^{2g+3}$ as the space of all binary forms of degree $2g + 2$. Denote by $X_g$ the open subset consisting of nonzero forms with distinct roots, and let $PGL_2 \times G_m$ act on it by $([A], \alpha)(f)(x) = \det(A)^g \alpha^{-2} f(A^{-1}(x))$. 

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Then for the stack $\mathcal{H}_g$ of smooth hyperelliptic curves of genus $g$ we have

$$\mathcal{H}_g = [X_g/(\text{PGL}_2 \times G_m)]$$

Proof. This is corollary 4.7 of [AV01]. When $g = 2$, the presentation of $\mathcal{H}_2$ was originally shown by Vistoli in [Vis96] 3.1.

In both cases, the quotient of $X_g$ by the usual action of $G_m$ defined by $(x_1, \ldots, x_7, t) \rightarrow (tx_1, \ldots, tx_7)$, which we will name $Z_g$, is naturally an open subset of the $\text{GL}_2$ (resp. $\text{PGL}_2 \times G_m$)-scheme $P(\mathbb{A}^{2g+3})$, namely the complement of the discriminant locus.

Let $G$ be either $\text{GL}_2$ or $\text{PGL}_2 \times G_m$. We will first construct the invariants of the quotient stack $[Z/G]$, then use the principal $G_m$-bundle $[X/G] \rightarrow [Z/G]$ to compute the invariants of $\mathcal{H}_g$ for $g$ even and $g = 3$.

Lemma 2.2. Let $G$ be either $\text{GL}_2$ or $\text{PGL}_2 \times G_m$. If $p$ differs from 2, the principal $G_m$-bundle $[X/G] \rightarrow [Z/G]$ induces an isomorphism on cohomological invariants. If $p$ is equal to 2, it induces an injective map.

Proof. The statement for $p = 2$ can be immediately deduced by the fact that a $G_m$-torsor is a smooth-Nisnevich cover.

For $p \neq 2$ we first reduce to the case where $G$ is special, so that $X \rightarrow [X/G]$ is a smooth-Nisnevich cover. We already have that $\text{GL}_2$ is special, and for $\text{PGL}_2 \times G_m$ we do the following. Consider a representation $V$ of $\text{PGL}_2 \times G_m$ satisfying the usual condition that the action is free on an open subset $U$ whose complement has codimension at least 2. Then the quotient $X' = (X \times U)/\text{PGL}_2$ is an algebraic space and its cohomological invariants are equal to these of $[X/\text{PGL}_2]$. It has two different actions by $G_m$ given by the two actions on $X$, the first one defined by $(x, t) \rightarrow xt^{-2}$ and the second one defined by multiplication.

We need to compare the cohomological invariants of $[X'/G_m]$ and those of $[X'/G_m \times G_m]$ when $p \neq 2$.

The cohomological invariants of $[X'/G_m]$ are equal to the invariants of $X'$ satisfying the sheaf condition for the map $X' \rightarrow [X'/G_m]$, and the invariants of $[X'/G_m \times G_m]$ are in the same way equal to the invariants of $X'$ satisfying the glueing conditions for $X' \rightarrow [X'/G_m \times G_m]$. We want to show that there are no elements in $\text{ChInv}^*(X')$ that satisfies the glueing conditions for $X' \rightarrow [X'/G_m]$ and does not satisfy the conditions for $X' \rightarrow [X'/G_m \times G_m]$.

Denote by $m_1 : X' \times G_m \rightarrow X'$ the multiplication relative to the first action, by $m_2 : X' \times G_m \rightarrow X'$ the multiplication relative to the second action and by $M : X' \times G_m \times G_m \rightarrow X'$ the combined multiplication map. Note that the first multiplication factors through the second as $X' \times G_m \xrightarrow{(x^{-2}, \text{Id})} X' \times G_m \xrightarrow{m_2} X'$

Where $x^{-2}$ is the map $G_m \rightarrow G : m$ defined by $\lambda \rightarrow \lambda^{-2}$.

Consider now a cohomological invariant $\alpha \in \text{ChInv}(X')$ satisfying the glueing conditions for $m_1$, that is, $\text{Pr}_{1*}(\alpha) = m_1^*(\alpha)$. This happens if and only if we have $((m_1^* - \text{Pr}_{1*})(\alpha))(\xi) = 0$, where $\xi$ is the generic point of $X \times G_m$. The extension $\xi \rightarrow \xi'$ of generic points induced by $(x^{-2}, \text{Id})$ is a map of degree 2, which implies that it is injective on étale cohomology for $p \neq 2$. As $m_1^* - \text{Pr}_{1*} = (x^{-2}, \text{Id})^* \circ (m_2^* - \text{Pr}_{1*})$ we conclude that $(m_2^* - \text{Pr}_{1*})(\alpha)$ must be zero itself,
that a polynomial with \( r \) double roots must be divisible by the square of a polynomial of degree \( r \).

We can repeat the same reasoning almost word by word for the case of \( GL_2 \) by using the inclusion \( G_m \rightarrow GL_2 \) to compare the different glueing conditions.

We generalize the family of equivariant schemes in Theorem 2.1 this way: let \( F \) be the dual of the standard representation of \( GL_2 \). We can see \( F \) as the space of all binary forms \( \phi = \phi(x_0, x_1) \) of degree 1. It has the natural action of \( GL_2 \) defined by \( A(\phi)(x) = \phi(A^{-1}(x)) \). We denote by \( E_i \) the \( i \)-th symmetric power \( \text{Sym}^i(F) \). We can see \( E_i \) as the space of all binary forms of degree \( i \), and the action of \( GL_2 \) induced by the action on \( F \) is again \( A(\phi)(x) = \phi(A^{-1}(x)) \). If \( i \) is even we can consider the additional action of \( GL_2 \) given by \( A(\phi)(x) = \det(A)^{i/2} \phi(A^{-1}(x)) \), and the action of \( PGL_2 \) given by \( [A] (\phi)(x) = \det(A)^{i/2} f(A^{-1}(x)) \).

We denote \( \Delta_{r,i} \) the closed subspace of \( E_i \) composed of forms \( \phi \) such that there exists a form \( f \) of degree \( r \) dividing \( \phi \). With this notation the scheme \( X_g \) in theorem 2.1 is equal to \( E_{2g+2} \setminus \Delta_{1,2g+2} \).

We denote \( \Delta_{r,i} \) the closed locus of the projectivized \( P(E_i) \) composed of forms \( \phi \) such that there exists a form \( f \) of degree \( r \) dividing \( \phi \). With this notation we have \( Z_g = P(E_{2g+2}) \setminus \Delta_{1,2g+2} \).

Thanks to the open-closed exact sequence on Chow groups with coefficients, understanding the cohomological invariants of \( [P^i \setminus \Delta_{1,i}] / G \) can be reduced to understanding the invariants of \( [P^i / G] \), which are understood thanks to the projective bundle formula, the top Chow group with coefficients \( A^0_G(\Delta_{1,i}) \) (which is not equal to the cohomological invariants of \( [\Delta_{1,r} / G] \), as \( \Delta_{1,i} \) is not smooth) and the pushforward map \( A^0_G(\Delta_{1,i}) \rightarrow A^2_G(P^i) \).

The computation of \( A^0_G(\Delta_{1,i}) \) will be based on the following two propositions.

**Proposition 2.3.** Let \( \pi_{r,i} : P(E_{i-2r}) \times P(E_r) \rightarrow \Delta_{r,i} \) be the map induced by \((f,g) \rightarrow fg^2 \). The equivariant morphism \( \pi_{r,i} \) restricts to a universal homeomorphism on \( \Delta_{r,i} \setminus \Delta_{r,i+1} \). Moreover, if \( \text{char}(k_0) > 2r \) or \( \text{char}(k) = 0 \) then any \( k \)-valued point of \( \Delta_{r,i} \setminus \Delta_{r,i+1} \) can be lifted to a \( k \)-valued point of \( P(E_{i-2r}) \times P(E_r) \).

**Proof.** See [Vis96] 3.2. The reasoning holds in general as long as we can say that a polynomial with \( r \) double roots must be divisible by the square of a polynomial of degree \( r \). This is clearly true for \( \text{char}(k) = 0 \), but in positive characteristic it holds only as long as \( 2r < \text{char}(k) \), as we can find irreducible polynomials of degree \( \text{char}(k) \) with only one distinct root. It is however always true that the map \( \pi_{r,i} \) is a bijection when restricted to \( \Delta_{r,i} \setminus \Delta_{r,i+1} \). Being proper and bijective, it is a universal homeomorphism.

**Proposition 2.4.** The pushforward of a (equivariant) universal homeomorphism induces an isomorphism on (equivariant) Chow groups with coefficients in \( H^* \).

**Proof.** Note first that the non-equivariant statement implies the equivariant one, as if \( X, Y \) are \( G \)-schemes on which \( G \) acts freely then an equivariant universal homeomorphism between them induces a universal homeomorphism on quotients.
Let \( f : X \to Y \) be a universal homeomorphism. Given a point \( y \in Y \), its fibre \( x \) is a point of \( X \) and the map \( f_x : x \to y \) is a purely inseparable field extension. The pullback \((f_x)^* : H^*(y) \to H^*(x)\) is an isomorphism, and the projection formula yields \((f_x)_*((f_x)^*\alpha) = [k(x) : k(y)]\alpha\). As the characteristic of \( k(x) \) is different from \( p \), the degree \([k(x) : k(y)]\) is invertible modulo \( p \) and the corestriction map is an isomorphism. This implies that \( f_* \) induces an isomorphism on cycle level.

In the next sections we will exploit the stratification \( \Delta_{1,i} = \Delta_{1,i} \setminus \Delta_{2,i} \coprod \Delta_{2,i} \setminus \Delta_{3,i} \coprod \cdots \coprod \Delta_{[i/2],i} \) and the isomorphism

\[
A^0_G(\Delta_{r,i}) \simeq A^0_G((P^{i-2r} \setminus \Delta_{1,i-2r}) \times P^r)
\]

to inductively compute \( A^0_G(\Delta_{1,i}) \) and \( A^0_G(P^i \setminus \Delta_{1,i}) \).

### 3 The invariants of \( \mathcal{M}_2 \)

In this section we will compute the cohomological invariants of \( \mathcal{M}_2 \), assuming we are working over an algebraically closed field. Thanks to \( GL_2 \) being a special group, we will be able to do our computation by just looking at the Chow groups mod \( p \). The result is as follows:

**Theorem 3.1.** Suppose the base field \( k_0 \) is algebraically closed, of characteristic different from 2, 3. Then the cohomological invariants of \( \mathcal{M}_2 \) are trivial if \( p \notin \{2,5\} \), and:

- If \( p = 2 \), \( \text{Inv}(\mathcal{M}_2) \) is generated as a graded \( \mathbb{F}_2 \)-module by 1 and nonzero elements \( x_1, x_2, x_3, x_4 \) respectively of degree 1, 2, 3, 4.
- If \( p = 5 \), \( \text{Inv}(\mathcal{M}_2) \) is generated as an \( \mathbb{F}_5 \)-module by 1 and a nonzero element \( x_1 \) of degree 1.

A few last considerations on equivariant Chow rings are needed.

**Lemma 3.2.** Let \( F \) be a vector bundle of rank 2 on a smooth variety \( S \), \( P = P(F) \) the projective bundle of lines in \( F \), \( \Delta \) the image of the diagonal embedding \( \delta : P \to P \times_S P \). Let \( x_1, x_2 \) in \( \Lambda^*(P \times_S P) \) be the two pullbacks of the first Chern class of \( O_P(1) \), \( c_2 \in \Lambda^*(P \times_S P) \) the pullback of the first Chern class of \( F \). Then the class of \( \Delta \) is \( x_1 + x_2 + c_1 \).

**Proof.** This is \([\text{Vis96}, 3.8]\). \( \square \)

Using the previous lemma we are able to compute the classes of \( \Delta_{1,i} \) in \( \text{CH}^1_{GL_2}(P^i) \). Recall that the \( GL_2 \)-equivariant Chow ring of \( P^i \) is generated by the Chern classes \( \lambda_1, \lambda_2 \) of the Hodge bundle and the first Chern class of \( O_P(1) \), which we will call \( t_i \), and the only relation is a polynomial \( f_i(t_i, \lambda_1, \lambda_2) \) of degree \( i + 1 \) \([\text{EC95}, 3.2, \text{prop}6]\) and the formula for projective bundles.)
Proposition 3.3. The class of $\Delta_{1,2}$ in $\text{CH}^1_{\text{GL}_2}(P^2)$ is $2t_2 - 2\lambda_1$. The class of $\Delta_{1,4}$ in $\text{CH}^1_{\text{GL}_2}(P^4)$ is $6t_4 - 12\lambda_1$, and the class of $\Delta_{1,6}$ in $\text{CH}^1_{\text{GL}_2}(P^6)$ is $10t_6 - 30\lambda_1$.

Proof. Consider the following commutative diagram:

$$
\begin{array}{ccc}
(P^1)^{(i-2)} \times P^1 & \xrightarrow{i} & (P^1)^i \\
\downarrow{\rho_1} & & \downarrow{\rho_2} \\
(P^{i-2}) \times P^1 & \xrightarrow{\pi_{1,i}} & P^i \\
\end{array}
$$

The map $\rho_1$ is defined by $(f_1, \ldots, f_{i-2}, g) \to (f_1 \ldots f_{i-2}, g)$, the map $\rho_2$ is defined by $(f_1, \ldots, f_i) \to (f_1 \ldots f_i)$, and the map $i$ is defined by $(f_1, \ldots, f_{i-2}, g, g) \to (f_1, \ldots, f_{i-2}, g, g)$. All the maps in the diagram are $\text{GL}_2$-equivariant, $i$ is a closed immersion, $\rho_1, \rho_2$ are finite of degree respectively $(i-2)!$ and $i!$.

Now, the class of $\Delta_{1,i}$ is the image of $1$ through $(\pi_{1,i})_*$. Following the left side of the diagram we obtain $[\Delta_{1,i}] = \frac{1}{(i-2)!}(\pi_{1,i} \circ \rho_1)^*(1)$. Consider now the right side of the diagram. The equivariant chow ring of $(P^1)^i$ is generated by all the different pullbacks of $t_1$, which we will call $x_1, \ldots, x_i$, plus $\lambda_1$ and $\lambda_2$. It is easy to check that the pullback of $t_i$ is $x_1 + \ldots + x_i$, which by the projection formula, and by symmetry, implies that $\rho_*(x_j) = (i-1)!t_i$.

Using lemma 3.2 we see that $i_*(1) = x_1 + x_{i-1} + \lambda_1$. Its image is $2(i-1)!t_i + i!\lambda_1$. By comparing the two formulas we obtain the statement of the proposition.

We are ready to prove theorem 3.1. The proof will require a few steps.

Proof of Theorem 3.1. During this proof, we will often use the fact that by [Pit13 4.3, 4.17], if our base field is algebraically closed then the rings $A^*_{\text{GL}_2}(P(E_i))$ consist only of geometric elements, i.e. the map

$$
\text{CH}^*_{\text{GL}_2}(P(E_i))/(p) \to A^*_{\text{GL}_2}(P(E_i))
$$

is an isomorphism.

1. If $p = 2$ then $A^0_{\text{GL}_2}(\Delta_{1,4}) = \langle 1, z_1 \rangle$, where the degree of $z_1$ is 1, otherwise $A^0_{\text{GL}_2}(\Delta_{1,4}) = \langle 1 \rangle$.

We begin by considering the exact sequence

$$
0 \to A^0_{\text{GL}_2}(P^2) \to A^0(P^2 \setminus \Delta_{1,2}) \xrightarrow{i_*} A^0(\Delta_{1,2}) \xrightarrow{q} A^1(P^2)
$$

As $i_*(1) = 2t_2 - 2\lambda_1$, $i_*$ is zero for $p = 2$ and injective otherwise. In the first case, we have an element $z_1 \in A^0(P^2 \setminus \Delta_{1,2})$ of degree one, corresponding to an equation for $\Delta_{1,2}$, seen as an element of $k(P^2)^*/k(P^2)^{\cdot 2}$. In the second case, it is trivial.

Now, if $p \neq 2$, then $A^0((P^2 \setminus \Delta_{1,2}) \times P^1) \simeq A^0(\Delta_{1,4} \setminus \Delta_{2,4})$ is trivial, implying the same for $\Delta_{1,4}$.

If $p = 2$, we first consider the following commutative diagram with exact rows:
As before, we consider the commutative diagram:

\[
\begin{array}{c}
\text{CH}_2^{GL_2}(\Delta_{1,2} \times P^1) 
\xrightarrow{(\pi_1)\ast} \text{CH}_2^{GL_2}(\Delta_{2,4}) \\
\xrightarrow{i_\ast} \text{CH}_2^{GL_2}(P^2 \times P^1) 
\xrightarrow{\pi_\ast} \text{CH}_2^{GL_2}(\Delta_{1,4}) \\
\xrightarrow{j_\ast} \text{CH}_2^{GL_2}((P^2 \setminus \Delta_{1,2}) \times P^1) 
\xrightarrow{(\pi_2)\ast} \text{CH}_2^{GL_2}(\Delta_{1,4} \setminus \Delta_{2,4})
\end{array}
\]

The first horizontal map is multiplication by two, and the third one is an isomorphism. \((i_\ast)\) and \((i_2)_\ast\) are injective because an irreducible effective divisor of a projective scheme cannot be numerically zero. We need to understand whether the class of \(\Delta_{2,4}\) is divisible by two or not in the Picard group of \(\Delta_{1,4}\).

By the diagram we obtain the relation \(2[\Delta_{2,4}] = 2\pi_\ast(t_2) + 2\pi_\ast\lambda_1\). Suppose now that \([\Delta_{2,4}] = 2\alpha\) for some \(\alpha \in \text{CH}_2^{GL_2}(\Delta_{1,4})\). Then \(2\alpha\) must belong to the kernel of \(j_\ast\), but the only elements in the image of \(\pi_\ast\) whose double belongs to the kernel are the multiples of \(\pi_\ast(t_2) + 2\pi_\ast\lambda_1\). This implies that \(2\alpha\) is equal to \(2k[\Delta_{2,4}]\) for some \(k\), and thus \((2k - 1)[\Delta_{2,4}] = 0\), contradicting the injectivity of \((i_2)_\ast\).

Consider now the exact sequence:

\(0 \rightarrow A^0_{GL_2}(\Delta_{1,4}) \rightarrow A^0_{GL_2}(\Delta_{1,4} \setminus \Delta_{2,4}) \xrightarrow{\partial} A^0_{GL_2}(\Delta_{2,4}) \xrightarrow{i_\ast} A^1_{GL_2}(\Delta_{1,4})\)

By proposition \(\text{2.3}\) \(\Delta_{2,4}\) is universally homeomorphic to \(P^2\), so that \(A^0(\Delta_{2,4})\) is concentrated in degree zero. Then the map \(i_\ast\) must be injective, and \(\partial\) must be zero.

2. We have \(A^0_{GL_2}(\Delta_{2,6}) = \langle 1 \rangle\).

As \(\Delta_{2,6} \setminus \Delta_{3,6}\) is universally homeomorphic to \((P^2 \setminus \Delta_{1,2}) \times P^2\), for \(p \neq 2\) the result is immediate. If \(p = 2\) we consider the exact sequence:

\(0 \rightarrow A^0_{GL_2}(\Delta_{2,6}) \rightarrow A^0_{GL_2}(\Delta_{2,6} \setminus \Delta_{3,6}) \xrightarrow{\partial} A^0_{GL_2}(\Delta_{3,6}) \xrightarrow{i_\ast} A^1_{GL_2}(\Delta_{2,6})\)

We know that \(A^0_{GL_2}(\Delta_{2,6} \setminus \Delta_{3,6}) = \langle 1, x \rangle\) and \(x\) has degree one. In order to prove that \(x\) does not come from an element of \(A^0_{GL_2}(\Delta_{2,6})\), we just have to prove that \(i_\ast\) is not injective (recall that \(\partial\) lowers degree by one.)

As before, we consider the commutative diagram:

\[
\begin{array}{c}
\text{CH}_3^{GL_2}(\Delta_{1,2} \times P^2) 
\xrightarrow{(\pi_1)_\ast} \text{CH}_3^{GL_2}(P^2 \times P^2) \\
\xrightarrow{i_\ast} \text{CH}_3^{GL_2}(\Delta_{3,6}) 
\xrightarrow{\pi_\ast} \text{CH}_3^{GL_2}(\Delta_{2,6})
\end{array}
\]

The map \((\pi_1)_\ast\) is multiplication by three, and the image of 1 through \(i_\ast\) is \(2(t_2 + \lambda_1)\), where \(t_2\) is the pullback of \(t_2 \in \text{CH}_2^{GL_2}(P^2)\) through the first projection. Then we have \(3[\Delta_{3,6}] = 2(t_2 + \lambda_1)\), implying \([\Delta_{3,6}] = 2(t_2 + \lambda_1 - [\Delta_{3,6}])\).
3. $A_{GL_2}^0(\Delta_{1,6})$ is equal to $\langle 1, y_1, y_2 \rangle$, with deg$(x_1) = 1$, deg$(x_2) = 2$ if $p = 2$. Otherwise, $A_{GL_2}^0(\Delta_{1,6})$ is trivial.

Recall that $\Delta_{1,6} \setminus \Delta_{2,6}$ is universally homeomorphic to $(P^4 \setminus \Delta_{1,4}) \times P^1$, and $A_{GL_2}^0((P^4 \setminus \Delta_{1,4}) \times P^1) = A_{GL_2}^0(P^4 \setminus \Delta_{1,4})$. We can compute the latter by using the usual exact sequence:

$$0 \to A_{GL_2}^0(P^4) \to A_{GL_2}^0(P^4 \setminus \Delta_{1,4}) \xrightarrow{\partial} A_{GL_2}^0(\Delta_{1,4}) \xrightarrow{i_*} A_{GL_2}^1(P^4)$$

The image of 1 through $i_*$ is $6t_4 - 12\lambda_1$. For $p = 2$, $i_* = 0$ as $A^1(P^4)$ is concentrated in degree zero and the class of $\Delta_{1,4}$ is 2-divisible. This implies that $A_{GL_2}^0((P^4 \setminus \Delta_{1,4})$ contains both an inverse image of $x_1$ through $\partial$, which we will name $y_2$ as well, and a new element of degree 1, $y_1$, corresponding to an equation for $\Delta_{1,4}$.

For $p = 3$ again $i_*$ is zero, so that the element $\beta$ in degree one appears. For $p \neq 2, 3$ the map $i_*$ is injective, implying that $\partial = 0$, so $A_{GL_2}^0(P^4 \setminus \Delta_{1,4})$ is trivial. This proves the statement for $p \neq 2, 3$, as an open immersion induces an injective map on $A_{GL_2}^0(-)$.

We now consider one more exact sequence:

$$0 \to A_{GL_2}^0(\Delta_{1,6}) \to A_{GL_2}^0(\Delta_{1,6} \setminus \Delta_{2,6}) \xrightarrow{\partial} A_{GL_2}^0(\Delta_{2,6}) \xrightarrow{i_*} A_{GL_2}^1(\Delta_{1,6})$$

We want to prove that $i_*$ is injective for $p = 2$ and zero for $p = 3$. It suffices to prove that the class of $\Delta_{2,6}$ is divisible by three and not divisible by two in $CH_{GL}^4(\Delta_{1,6})$. We proceed as before:

$$\begin{align*}
CH_{GL}^4(\Delta_{1,4} \times P^1) & \xrightarrow{(\pi_1)_*} CH_{GL}^4(\Delta_{2,6}) \\
\downarrow i_* & \downarrow (\iota_2)_* \\
CH_{GL}^4(P^4 \times P^1) & \xrightarrow{\pi_*} CH_{GL}^4(\Delta_{1,6}) \\
\downarrow j_* & \downarrow j'_* \\
CH_{GL}^4(\Delta_{1,4} \setminus P^1) & \xrightarrow{(\pi_2)_*} CH_{GL}^4(\Delta_{1,6} \setminus \Delta_{2,6})
\end{align*}$$

the map $(\pi_1)_*$ is just multiplication by two. This implies that $2[\Delta_{2,6}] = 6\pi_*(t_3 - 2\lambda_1)$ by the same reasoning as above, the only elements in the image of $\pi_*$ whose double belongs to the kernel of $j_*$ are multiples of $3\pi_*(t_3 - 2\lambda_1)$, and we easily get to the same contradiction as before. The class of $\Delta_{2,6}$ is divisible by three as $3(-2\pi_*(t_4 - 2\lambda_1) + [\Delta_{2,6}]) = [\Delta_{2,6}]$.

4. If $p = 2$, then $A_{GL_2}^0(P^6 \setminus \Delta_{1,6})$ is equal to $\langle 1, x_1, x_2, x_3 \rangle$, where the degrees of $x_1, x_2, x_3$ are respectively 1, 2, 3. If $p = 5$ then $A_{GL_2}^0(P^6 \setminus \Delta_{1,6}) = \langle 1, x_1 \rangle$ where $x_1$ is the class of an equation for $\Delta_{1,6}$. Otherwise, $A_{GL_2}^0(P^6 \setminus \Delta_{1,6})$ is trivial.

This is instantly obtained by looking at the exact sequence:

$$0 \to A_{GL_2}^0(P^6) \to A_{GL_2}^0(P^6 \setminus \Delta_{1,6}) \xrightarrow{\partial} A_{GL_2}^0(\Delta_{1,6}) \xrightarrow{i_*} A_{GL_2}^1(P^6)$$
6. Consider the exact sequence:

\[ 0 \to A^1_{GL_2}(P^2) \to A^1_{GL_2}(P^2 \setminus \Delta_{1,2}) \to A^1_{GL_2}(\Delta_{1,2}) \]

We need to understand what \( t_2z_1 \) is. By the compatibility of Chern classes and boundary maps [Pir14, 4.2], we know that \( \partial(t_2z_1) = \partial(c_1(\mathcal{O}_{P^2(-1)})(z_1)) = c_1(i^*\mathcal{O}_{P^2(-1)})(\partial(z_1)) = c_1(i^*\mathcal{O}_{P^2(-1)})(1) \). As the pullback of \( \mathcal{O}_{P^2(-1)} \) through \( P^1 \times_{\Delta_{1,2}} P^2 \to P^2 \) is equal to \( \mathcal{O}_{P^2(-1)} \), we see that \( \partial(t_2z_1) = 0 \). Then \( t_2z_1 \) must be the image of some \( \gamma \in A^1_{GL_2}(P^2) \), but there are no element of positive degree in \( A^*_{GL_2}(P^2) \), so \( t_2z_1 = 0 \).

Consider now the elements \( t_4\alpha, t_4\beta \) in \( A^1_{GL_2}(P^4 \setminus \Delta_{1,4}) \). We will use the exact sequence

\[ 0 \to A^1_{GL_2}(P^4 \setminus \Delta_{2,4}) \to A^1_{GL_2}(P^4 \setminus \Delta_{1,4}) \to A^1_{GL_2}(\Delta_{1,4} \setminus \Delta_{2,4}) \]

Again by [Pir14, 4.2] we see that \( \partial(t_4\gamma_1) \) is equal to \( c_1(i^*\mathcal{O}_{P^4(-1)})(\partial(y_1)) = c_1(i^*\mathcal{O}_{P^4(-1)})(1) \). We can now apply the projection formula to the map \( \pi_{1,4} : (P^2 \setminus \Delta_{1,2}) \times P^1 \to \Delta_{1,4} \). As the pullback of \( \mathcal{O}_{P^2(-1)} \) to \( (P^2 \setminus \Delta_{1,2}) \times P^1 \) is equal to \( pr_1^*\mathcal{O}_{P^2(-1)} \otimes pr_2^*\mathcal{O}_{P^2(-1)} \), we have \( c_1(i^*\mathcal{O}_{P^2(-1)})(1) = (\pi_{1,4}) \neq 0 \), implying that \( t_4\gamma_1 \) must be nonzero.

On the other hand, applying the same reasoning to \( t_4y_2 \) we obtain \( \partial(t_4y_2) = (\pi_{1,4})_t2z_1 = 0 \). Then \( t_4\alpha \) must belong to the image of \( A^1_{GL_2}(P^4 \setminus \Delta_{2,4}) \), but looking at the exact sequence \( \Delta_{2,4} \to P^4 \) we see that \( A^1_{GL_2}(P^4 \setminus \Delta_{2,4}) \) contains elements of degree at most one, so \( t_4\beta \) must be zero.

6. Let \( X \) be the open subscheme of \( \mathbb{A}^7 \) consisting of nondegenerate forms of degree six. The graded \( \mathbb{P}_2 \)-module \( A^0_{GL_2}(X) \) is generated by 1 and nonzero elements \( x_1, x_2, x_3, x_4 \) of respective degrees 1, 2, 3, 4.

Let \( E \xrightarrow{r} \mathbb{P}^6 \setminus \Delta_{1,6} \) be the total space of the \( GL_2 \)-equivariant line bundle \( \mathcal{O}^{-1} \otimes \mathcal{O}^2 \) on \( \mathbb{P}^6 \setminus \Delta_{1,6} \). Let \( s : \mathbb{P}^6 \setminus \Delta_{1,6} \to E \) be the zero section. We consider the exact sequence:

\[ 0 \to A^0_{GL_2}(E) \xrightarrow{r} A^0_{GL_2}(X) \to A^0_{GL_2}(\mathbb{P}^6 \setminus \Delta_{1,6}) \xrightarrow{s} A^1_{GL_2}(E) \]

If we identify the equivariant Chow groups with coefficients of \( E \) with those of \( \mathbb{P}^6 \setminus \Delta_{1,6} \) using the isomorphism \( r \) defined by Rost in [Ros96, section 9] we obtain the following exact sequence:
Proof. trivial.

Proposition 4.1. hyperelliptic curves of even genus. Again we suppose that our base field $k$ and $(\alpha, \beta)$ are when using the open-closed exact sequence, and in fact in most computations it we get the element $\{\alpha\}$.

In analogy with the case of $P^4 \setminus \Delta_{1,4}$, we consider the exact sequence

$$0 \to A^0_{\text{GL}_2}(P^6 \setminus \Delta_{1,6}) \xrightarrow{J^s \circ \phi} A^0_{\text{GL}_2}(X) \xrightarrow{\partial} A^0_{\text{GL}_2}(P^6 \setminus \Delta_{1,6}) \xrightarrow{\text{res}_s} A^1_{\text{GL}_2}(P^6 \setminus \Delta_{1,6})$$

The second map is equal to the pullback from $A^0_{\text{GL}_2}(P^6 \setminus \Delta_{1,6})$ to $A^0_{\text{GL}_2}(X)$, and the fourth map is equal to $c_1(\mathcal{O}(-1) \otimes \mathcal{O}_2)$ which is then equal to multiplication by $t_6$ as we are working modulo two. So the question boils down to whether $t_6 x_1, t_6 x_2$ and $t_6 x_3$ are zero or not in $A^1_{\text{GL}_2}(P^6 \setminus \Delta_{1,6})$.

In analogy with the case of $P^4 \setminus \Delta_{1,4}$, we consider the exact sequence

$$0 \to A^1_{\text{GL}_2}(P^6 \setminus \Delta_{2,6}) \to A^1_{\text{GL}_2}(P^6 \setminus \Delta_{1,6}) \xrightarrow{\partial} A^1_{\text{GL}_2}(\Delta_{1,6} \setminus \Delta_{2,6})$$

Following the same reasoning as before we see that the boundaries of $t_6 x_1$ and $t_6 x_2$ cannot be zero as they are respectively equal to $(\pi_1,\alpha), t_4$ and $(\pi_1,\alpha) t_4 y_1$. The differential of $t_6 \alpha$ on the other hand is equal to $(\pi_1,\alpha) t_4 y_2 = 0$, and $A^1_{\text{GL}_2}(P^6 \setminus \Delta_{2,6})$ contains elements of degree at most one because $A^0_{\text{GL}_2}(\Delta_{2,6}) = \langle 1 \rangle$, implying that it must be zero.

\[ \square \]

Remark 3.4. In this case we can understand almost completely the multiplicative structure of $\text{Cl}(\mathcal{M}_2)$. We have $\phi^2 = \phi \phi = \phi \beta = \phi \gamma = 0$ as there are no elements of degree higher than $\phi$, and simarily $\alpha^2 = \alpha \beta = \alpha \gamma = 0$ as these elements are pullbacks from $\text{Cl}(\mathcal{P}^6 \setminus \Delta_{1,6}/\text{GL}_2))$ and we can apply the same reasoning. The squares $\gamma^2, \beta^2$ are both zero, as the second is of degree 4 and there are no elements of degree 4 in $\text{Cl}(\mathcal{P}^6 \setminus \Delta_{1,6}/\text{GL}_2))$, and the first is represented by an element $\gamma \in H^2(k(P^6)) = k(P^6) \otimes (k(P^6))^2$ and squaring it we get the element $\{1\} \gamma = H^2(k(P^6))$ which is zero as $k$ contains a square root of $-1$. The product $\gamma \beta$ may be either equal to zero or to $\alpha$.

In general we have no instruments to understand the multiplicative structure of $\text{Cl}(\mathcal{M}_2^g)$. The reason is that it is difficult to keep track of what our elements are when using the open-closed exact sequence, and in fact in most computations on classical cohomological invariants the multiplicative structure stems from an explicit a priori description of the invariants.

4 The invariants of $\mathcal{M}_g, g$ even

In this section we will extend the result we obtained for $\mathcal{M}_2$ to all the stacks of hyperelliptic curves of even genus. Again we suppose that our base field $k_0$ is algebraically closed.

Proposition 4.1. Let $p \neq 2$. The class of $\Delta_{1,2i}$ is divisible by $p$ in $\text{CH}^1_{\text{GL}_2}(P^{2i})$ then $A^0_{\text{GL}_2}(P^{2i} \setminus \Delta_{1,2i})$ is generated by $\langle 1, \alpha \rangle$, where $\alpha \neq 0$ is the invariant in degree 1 corresponding to an equation for $\Delta_{1,2i}$. Otherwise $A^0_{\text{GL}_2}(P^{2i} \setminus \Delta_{1,2i})$ is trivial.

Proof. We will proceed by induction on $i$, the base step being the groups we computed in the proof of theorem 2.1. Due to the open-close exact sequence and proposition 2.2, the statement is equivalent to proving that for all $i$ the group $A^0_{\text{GL}_2}(\Delta_{1,2i})$ is concentrated in degree 0.

Consider the exact sequence:
Due to the universal homeomorphism $A_{GL_2}^0 (P^{2i-2} \setminus \Delta_{1,2i-2}) \times P^1 \to \Delta_{1,2i-2} \setminus \Delta_{2,2i}$, the group $A_{GL_2}^0 (\Delta_{1,2i-2} \setminus \Delta_{2,2i})$ is isomorphic to $A_{GL_2}^0 (P^{2i-2} \setminus \Delta_{1,2i-2})$, so by the inductive hypothesis $A_{GL_2}^0 (\Delta_{1,2i-2} \setminus \Delta_{2,2i})$ has at most a single nontrivial element $\alpha$ in degree one. If the group is trivial, we are done. Otherwise, we must show that the map $\iota_*$ vanishes in degree zero. We know that the class of $\Delta_{1,2i-2}$ is divisible by $p$ in $A_{GL_2}^1 (P^{2i-2})$. Consider the diagram:

\[
\begin{array}{c}
\vdots \\
CH_{2i-2}^{GL_2}(\Delta_{1,2i-2} \times P^1) \\
\downarrow \iota_* \\
CH_{2i-2}^{GL_2}(P^{2i-2} \times P^1) \\
\downarrow j^* \\
CH_{2i-2}^{GL_2}(P^{2i-2} \setminus \Delta_{1,2i-2} \times P^1) \\
\downarrow j^*_2 \\
CH_{2i-2}^{GL_2}
\end{array}
\]

As before, $\pi_1$ is multiplication by two, and we obtain that two times the class of $\Delta_{2,2i}$ is equal to the image of the class of $\Delta_{1,2i-2} \times P^1$. As the latter is by hypothesis divisible by $p$, there is an element $\psi$ of $CH_{2i-2}^{GL_2}(\Delta_{1,2i})$ such that $p\psi = 2 [\Delta_{2,2i}]$. As $p$ is odd, $\frac{p-1}{2} p\psi = p - 1 [\Delta_{2,2i}]$ so that $p ([\Delta_{2,2i}] - \frac{p-1}{2} \psi) = [\Delta_{2,2i}]$. As $[\Delta_{2,2i}]$ is divisible by $p$, the map $\iota_*$ must vanish in degree zero. \[\square\]

**Remark 4.2.** The result for $p \neq 2$ does not really require $k_0$ to be algebraically closed; it works in the same way by considering instead of a single element the $H^*(k_0)$-module it generates, and using the fact that all the maps we are considering are maps of $H^*(k_0)$-modules. This shows that the cohomological invariants of $[P^{2i-2} \setminus \Delta_{1,2i}/GL_2]$ are generated at most by a single element of degree 1 for $p \neq 2$.

**Proposition 4.3.** Let $p = 2$. If $r$ is odd, the inclusion map $\Delta_{r,2i} \setminus \Delta_{r+1,2i} \xrightarrow{j} \Delta_{r,2i}$ induces an isomorphism on $A_{GL_2}^0$. If $r$ is even, $A_{GL_2}^0 (\Delta_{r,2i})$ is trivial.

**Proof.** As $A_{GL_2}^0 (\Delta_{r,2i})$ is isomorphic to $A_{GL_2}^0 (\Delta_{r,2i} \setminus \Delta_{r+2,2i})$ (because $\Delta_{r+2,2i}$ has codimension two in $\Delta_{r,2i}$) we can compute it using the following exact sequence:

\[
0 \to A_{GL_2}^0 (\Delta_{r,2i} \setminus \Delta_{r+2,2i}) \to A_{GL_2}^0 (\Delta_{r,2i}) \xrightarrow{\partial} A_{GL_2}^0 (\Delta_{r+1,2i} \setminus \Delta_{r+2,2i})
\]

We want to prove that the kernel of $\partial$ is contained in degree zero when $r$ is even, and that $\partial$ is zero when $r$ is odd. The map $(P^{2i-2r} \setminus \Delta_{2,2r}) \times P^r \xrightarrow{i_2} \Delta_{r,2i} \setminus \Delta_{r+2,2i}$ yields the following commutative diagram with exact rows:

\[
\begin{array}{c}
\vdots \\
A_{GL_2}^0 (\Delta_{r,2i} \setminus \Delta_{r+2,2i}) \\
\downarrow \iota_* \\
A_{GL_2}^0 (\Delta_{r,2i} \setminus \Delta_{r+1,2i}) \\
\downarrow j^* \\
A_{GL_2}^0 (\Delta_{r,2i}) \\
\downarrow j^*_2 \\
A_{GL_2}^0
\end{array}
\]
The second horizontal map is an isomorphism because $\pi_\ast$ is a universal homeomorphism when restricted to $\Delta_{r,2i} \setminus \Delta_{r+1,2i}$. The kernel of $\partial_1$ is contained in degree one because $A^0_{GL_2}((\Delta_{r,2i} \setminus \Delta_{r+1,2i}) \times P^r)$ is trivial. We claim that when $r$ is even the third horizontal map is an isomorphism, implying that the kernel of $\partial$ must be contained in degree zero too, and when $r$ is odd the third horizontal map is zero, so that $\partial$ must be zero too.

Let $\psi$ be the map from $(P^{2i-2r-2} \setminus \Delta_{2i-2r-2}) \times P^r \times P^1$ to $(P^{2i-2r-2} \setminus \Delta_{2i-2r-2}) \times P^{r+1}$ sending $(f, g, h)$ to $(f, gh)$. We have a commutative diagram:

\[
\begin{array}{c}
(P^{2i-2r-2} \setminus \Delta_{2i-2r-2}) \times P^r \\
\downarrow \psi \\
(P^{2i-2r-2} \setminus \Delta_{2i-2r-2}) \times P^{r+1}
\end{array}
\begin{array}{c}
\pi_1 (\Delta_{2i-2r} \setminus \Delta_{2i-2r}) \times P^r \\
\pi_2 \Delta_{r+1,2i} \setminus \Delta_{r+2,2i}
\end{array}
\]

Where $\pi_1$ and $\pi_2$ are defined respectively by $(f, g, h) \rightarrow (fg^2, h)$ and $(f, g) \rightarrow (fg^2)$. As $\pi_1$ and $\pi_2$ are universal homeomorphisms, if we prove that $\psi_\ast$ is an isomorphism then $\pi_\ast$ will be an isomorphism too, and if $\psi_\ast$ is zero then $\pi_\ast$ will be zero too. Consider this last diagram:

\[
\begin{array}{c}
(P^{2i-2r-2} \setminus \Delta_{2i-2r-2}) \times P^r \\
\downarrow \psi \\
(P^{2i-2r-2} \setminus \Delta_{2i-2r-2}) \times P^{r+1}
\end{array}
\begin{array}{c}
p_1 (P^{2i-2r-2} \setminus \Delta_{2i-2r-2}) \\
p_2 \Delta_{r+1,2i} \setminus \Delta_{r+2,2i}
\end{array}
\]

The pullbacks of $p_1$ and $p_2$ are both isomorphism, implying that the pullback of $\psi$ is an isomorphism. By the projection formula, $\psi_\ast(\psi^\ast \alpha) = \deg(\psi) \alpha$. Then as the degree of $\psi$ is $r+1$, $\psi_\ast$ is an isomorphism if $r$ is even and zero if $r$ is odd.

\[\square\]

**Remark 4.4.** Again this result does not require any hypothesis on $k_0$. The result can be also proven by direct computation on the Chow groups mod 2, with the slight additional complication that the map $CH^{GL_2}_{2i}(P^{2i-2r-2}) \xrightarrow{\psi^\ast} CH^{GL_2}_{2i}(\Delta_{r,2i} \setminus \Delta_{r+1,2i})$ is no longer surjective, as some points will only lift up to a purely inseparable extension.

Interestingly, the proposition above also gives us information on the Chow groups of $\Delta_{r,2i}$, as it implies that for the class of $\Delta_{r+1,2i}$ is 2-divisible in $CH^{GL_2}_{2i}(\Delta_{r,2i})$ if and only if $r$ is even.

From the next corollary on we will rely heavily on the algebraic closure of $k_0$, which is necessary to prove that the image of $i_\ast : A^0_{GL_2}(\Delta_{1,2i}) \rightarrow A^0_{GL_2}(P^{2i})$. 

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is zero. In the next two sections we will explore some ideas to get around this obstacle.

**Corollary 4.5.** If \( p = 2 \), then \( A^0_{GL_2}(P^{2i} \setminus \Delta_{1,2i}) = \langle 1, x_1, \ldots, x_i \rangle \), where the degree of \( x_i \) is \( i \), and all the \( x_i \) are nonzero.

**Proof.** We obtain the corollary immediately by induction using the last proposition and the open-close exact sequence.

**Theorem 4.6.** Suppose our base field \( k_0 \) is algebraically closed, of characteristic different from 2, 3, and let \( g \) be even. For \( p = 2 \), the cohomological invariants of \( \mathcal{H}_g \) are generated as a graded \( \mathbb{F}_2 \)-module by nonzero invariants \( x_1, \ldots, x_{g+2} \), where the degree of \( x_i \) is \( i \). If \( p \neq 2 \), then the cohomological invariants of \( \mathcal{H}_g \) are nontrivial if and only if \( 2g + 1 \) is divisible by \( p \). In this case they are generated by a single nonzero invariant in degree one.

**Proof.** In the case \( p \neq 2 \) this is a direct consequence of lemma 2.2 and proposition 3.1. The explicit result for \( p \neq 2 \) can be obtained by looking at whether the class of \( \Delta_{1,2g+2} \) is divisible by \( p \) in the equivariant Picard group of \( P^{2g+2} \), which can be easily done using proposition 3.3. If \( p = 2 \) we start from corollary 4.5. As we did at the end of the proof of theorem 3.1, we have to consider the exact sequence

\[
0 \to A^0_{GL_2}(P^{4g+2} \setminus \Delta_{1,4g+2}) \overset{\varphi}{\to} A^0_{GL_2}(A^{4g+3} \setminus \Delta) \overset{\partial}{\to} A^0_{GL_2}(P^{4g+2} \setminus \Delta_{1,4g+2}) \to 0
\]

So the question again boils down to understanding whether the products \( t_{4g+2}x_j \) are zero or not. Suppose by induction that for \( 2i < 4g + 2 \) we know that \( t_{2i}x_j \in A^0_{GL_2}(P^{2i} \setminus \Delta_{1,2i}) \) is equal to zero if and only if \( j = i \). We already know that for \( i \leq 3 \), giving us the base for the induction. Again, we can use the properties of Chern classes to see that, if we consider the exact sequence

\[
0 \to A^1_{GL_2}(P^{4g+2} \setminus \Delta_{2,4g+2}) \to A^1_{GL_2}(P^{4g+2} \setminus \Delta_{1,4g+2}) \overset{\partial}{\to} A^1_{GL_2}(\Delta_{1,4g+2} \setminus \Delta_{2,4g+2})
\]

Then the image of \( t_{4g+2}x_j \) through \( \partial \) is equal to \( (\pi_{1,4g+2})_t t_{4g}x_{j-1} \), where we consider \( x_0 = 1 \). This immediately implies the thesis for \( j < 2g + 1 \). As \( \varphi(t_{4g+2}x_{2g+1}) = 0 \), it must come from an element of \( A^1_{GL_2}(P^{4g+2} \setminus \Delta_{2,4g+2}) \). The elements of \( A^1_{GL_2}(P^{4g+2} \setminus \Delta_{2,4g+2}) \) can have degrees only up to one plus the maximum degree of an element of \( A^0_{GL_2}(\Delta_{2,4g+2}) \). Then by proposition 3.1, their degree is equal or lesser than one, and \( t_{4g+2}x_{2g+1} \) must be zero, concluding the proof.

## 5 The non algebraically closed case

In this section we obtain a partial result on the cohomological invariants of \( \mathcal{H}_g \) for a general base field. This should give an idea of the inherent problems that arise when we have nontrivial elements of positive degree in our base rings. Note that this will happen even for an algebraically closed field if we are considering quotients by groups that are not special, making the development of techniques and ideas to treat these type of problems crucial for the future development of the theory.
Theorem 5.1. Suppose that the characteristic of \( k_0 \) is different from 2, 3.

Let \( p \) be a prime different from 2. Then the cohomological invariants of \( \mathcal{M}_g \) are nontrivial if and only if 4q + 1 is divisible by \( p \). In this case they are freely generated by 1 and a single nonzero invariant in degree one.

Let \( p \) be equal to 2. Then the cohomological invariants of \( \mathcal{M}_2 \) are isomorphic as a \( H^* (\text{Spec}(k)) \)-module to a direct sum \( M \oplus K \) where \( M \) is freely generated by 1 and elements \( x_1, x_2, x_3 \) of respective degrees 1, 2, 3 and \( K \) is a submodule of \( H^* (\text{Spec}(k)) \) [4].

The first statement is a direct consequence of remark (4.2) and proposition (2.2). The case \( p = 2 \) will require some work, and in the rest of the section we always work in this case. We begin by simplifying the last step:

Lemma 5.2. The map \( A^0_{GL_2}(\Delta_{1,6}) \to A^1_{GL_2}(P^6) \) is zero if and only if the map \( A^0(\Delta_{1,6}) \to A^1(\Delta_{1,6}) \) is zero.

Proof. One arrow is trivial: the equivariant groups for \( P^6 \) map surjectively on the non-equivariant groups and the assignment is functorial, so if the equivariant map is trivial the same must be true for the non-equivariant map.

We now remove \( \Delta_{2,6} \) from both sides, so that we are reduced to considering the map \((P^4, \Delta_{1,4}) \times P^1 \to P^6 \setminus \Delta_{2,6} \). All elements in \( A^0_{GL_2}(P^4 \setminus \Delta_{1,4}) \times P^1 \) are pullbacks through the first projection. An element \( \alpha \in A^0_{GL_2}(P^4 \setminus \Delta_{1,4}) \) satisfies the equation \((t^5 + t^3\lambda_1^2)\alpha = 0\), where \( t \) is the first Chern class of \( \mathcal{O}_{P^1}(-1) \) and \( \lambda_1 \) is the first Chern class of the determinant line bundle. The pullback \( \alpha \in A^0((P^4 \setminus \Delta_{1,4}) \times P^1) \) must then satisfy the same equation.

Note now that modulo two the pullback of \( \mathcal{O}_{P^4}(-1) \) is equal to \( \mathcal{O}_{P^4}(-1) \). As \( \lambda_1 \) is also the pullback of the corresponding equivariant line bundle on \( P^6 \), by the projection formula we see that the image of \( \alpha \) must satisfy the same equation. As \( i_*(\alpha) \) is an element of \( A^1_{GL_2}(P^6) \) we can write \( i_*(\alpha) = \lambda_1 \cdot a + t \cdot b \) with \( a, b \in H^* (\text{spec}(k)) \). Then we have \((t^5 + t^3\lambda_1^2)(\lambda_1 \cdot a + t \cdot b) = 0 \in A^0_{GL_2}(P^6 \setminus \Delta_{2,6}) \).

Suppose that we know the result in the non-equivariant case, that is, we know that \( b = 0 \). We want to show that \((t^5 + t^3\lambda_1^2)\lambda_1 \cdot a \) belongs to the image of \( A^0_{GL_2}(\Delta_{2,6}) \) if and only if \( a = 0 \). Recall that \( \Delta_{2,6} \) can be seen as the disjoint union of \((P^2 \setminus \Delta_{1,2}) \times P^2 \) and \( \Delta_{3,6} \). We can divide the elements in \( A^0_{GL_2}(\Delta_{2,6}) \) in three categories: those that come from \( P^2 \times P^2 \), those that come from \( \Delta_{3,6} \) (which is universally homeomorphic to \( P^3 \)) and the elements of \( A^0_{GL_2}(P^2 \setminus \Delta_{1,2} \times P^2) \) that are ramified on \( \Delta_{1,2} \times P^2 \) but unramified on \( \Delta_{3,6} \).

Using the computations in [Vis96] we see that the first two images form the ideal \((t^6 + t^5\lambda_1 + t^4\lambda_1^2 + t^3\lambda_1^3)\). For the latter, the computation reduces to finding out the kernel of the pushforward \( A^*_{GL_2}(P^1 \times P^2) \to A^* (P^6) \). Using the fact that the map is finite of degree 3 one sees that if we write \( t = c_1(\mathcal{O}_{P^1}(-1)), s = c_1(\mathcal{O}_{P^2}(-1)) \) the kernel is generated as a \( A^*_{GL_2}(\text{Spec}(k)) \)-module by 1, \( s, st \). Then any element in codimension 4 belonging to the kernel of our pushforward can be written down as a sum \( \lambda_1 a_1 + \lambda_2 a_2 \), and the same must hold for any element in \( A^0_{GL_2}(\Delta_{2,6}) \) belonging to the third category. By the projection formula we can conclude that the image of \( A^0_{GL_2}(\Delta_{2,6}) \) must be contained in the ideal \((t^6 + t^5\lambda_1 + t^4\lambda_1^2 + t^3\lambda_1^3, \lambda_2)\), which does not contain \((t^5 + t^3\lambda_1^2)\lambda_1 \cdot a \) unless \( a = 0 \). \( \square \)

Of course the same trick will not work on the non-equivariant case, as the relation would be \( t^6 \cdot a = 0 \) and \( \Delta_{2,6} \) contains rational points. We will have to
dirty our hands and work at cycle level. Recall that the first Chern class of a line bundle $L$ can be defined on cycles up to choosing a coordination for $L$.

**Lemma 5.3.** Let $X \times \mathbb{A}^1$ be a trivial vector bundle with zero section $\sigma$ and let $\tau$ be any coordination. Then $c_{1,\tau}(X \times \mathbb{A}^1) = r_\tau \circ \sigma_* : C^*(X) \to C^*(X)$ has the property that for every $\alpha$ there is $\beta$ such that $r_\tau \circ \sigma_*(\alpha) = d(\beta)$.

**Proof.** We will proceed by induction on the length $n$ of our coordination $\tau = (X_0 = X, X_1, \ldots, X_n = \emptyset)$. Recall that if we have constructed the map $r_{\tau_1}$ where $\tau_1$ is the coordination restricted to $X_1$ we obtain the map $r_\tau$ by the formula:

$$r_\tau = \begin{pmatrix} r_{\text{triv}} \\ r_{\tau_1} \circ \partial_{X_1 \times \mathbb{A}^1} \circ H_{\text{triv}} \circ r_{\tau_1} \end{pmatrix}$$

We first prove by induction that $r_\tau(\{t\} \pi^*(\alpha))$ is zero for all $\alpha$, where $\mathbb{A}^1 = \text{Spec}(k[t])$. This element is well defined as $\pi^*(\alpha)$ does not have any component lying on the zero section. When the coordination is trivial $r_\tau(\{t\} \pi^*(\alpha))$ is zero because $\{t\} \{t^{-1}\} = -\{t\} \{t^{-1}\} = 0$. In general, we have $H_{\text{triv}}(\{t\} \pi^*(\alpha)) = 0$ by direct computation in the same way that $H_{\text{triv}} \circ \pi^* = 0$, so the formula above allows us to conclude.

Consider now an element $\sigma_*(\alpha)$. If the coordination is trivial the result is trivially true. Consider now a general coordination $\tau$, and suppose the result holds for $\tau_1$. By direct computation we see that $H_{\text{triv}}(\sigma_*(\alpha)) = \{t\} \pi^*(\alpha)$. We separate the boundary map $\partial_{X_1 \times \mathbb{A}^1}$ in two components:

$$\partial_{X_1 \times \mathbb{A}^1} = \partial_{X_1 \times (\mathbb{A}^1 \setminus \{0\})} + \sigma_1^* \circ \partial_{X_1 \times \mathbb{A}^1}$$

When computing the first component on the right we can consider $t$ as an invertible element so that $\{t\}$ and $\partial_{X_1 \times (\mathbb{A}^1 \setminus \{0\})}$ anti-commute and we obtain an element in the form $\{t\} \pi_{X_1}^*(\beta)$ for some $\beta$, so that when we apply $r_{\tau}$ we get zero. The second component is contained in the zero section of $X_1$ so we can apply the inductive hypothesis. \qed

**Lemma 5.4.** Let $E \to X$ be a line bundle that is isomorphic to $L \otimes W^\mathbb{G}_p$ for some line bundles $L, W$. Let $\tau_L, \tau_E$ be coordinations respectively for $L$ and $W$, and consider the coordination for $\tau_L \cup \tau_W$ for $E$. Then for all $\alpha$ there is a $\beta$ such that $c_{1,\tau_L \cup \tau_W}(E)(\alpha) = c_{1,\tau}(\alpha) + d(\beta)$.

**Proof.** It can be seen directly as in [Pir14][4.8] that given a compatible choice of a trivialization and coordination for $E$ the Chern class $c_{1,\tau}(E)$ is the sum of $c_{1,\tau}(X \times \mathbb{A}^1)$ and a function that is linear in the coordinate change elements $\alpha_{i,j} \in \mathcal{O}^*(U_i \times X U_j)$. In the above situation the elements $\alpha_{i,j} \in \mathcal{O}^*(U_i \times X U_j)$ satisfy $\alpha_{i,j} = \alpha_{i,j,L} \cdot \alpha_{i,j,W}$, allowing us to conclude. \qed

**Proposition 5.5.** The map $A^0(\Delta_{1,6}) \to A^1(P^6)$ is zero.
Proof. Given an element \( \alpha \) in \( A^0(\Delta_{1,6}) \to A^1(P^6) \) we know that it must come from \( A^0((P^1_4,\Delta_{1,4}) \times P^1) \), which in turn comes from \( \beta \in A^0(P^1_4,\Delta_{1,4}) \). Consider a cycle \( z \in C^0(P^1) \) mapping to \( \alpha \). Then if we consider \( L = \mathcal{O}_{P^1}(-1) \), with the standard coordination \( \gamma \) given by the hyperplane at infinity, we have \( c_{1,\gamma}(L)^\delta(z) = 0 \). Then the pullback \( L' \) of \( L \) to \( P^4 \times P^1 \) must satisfy \( c_{1,\gamma}(L')^\delta(\beta) = 0 \).

Consider now \( E = \mathcal{O}_{P^1}(-1) \) with the standard coordination \( \gamma \) given again by the hyperplane at infinity. The pullback \( E' \) of \( E \) is \( \mathcal{O}_{P^1}(-1) \otimes \mathcal{O}_{P^1}(-1)^2 \), so we can see by the last lemma that \( c_{1,\gamma}(E')^\delta(\beta) = \delta(\zeta) \) for some \( \zeta \) in \( C^4(P^1 \times P^1) \). The projection formula (on cycles) now tells us that the pushforward of \( \beta \), which is an unramified element, must satisfy \( c_{1}(E)^\delta(i_*(\beta)) = 0 \) which implies \( i_*(\beta) = 0 \).

Proposition 5.6. The pullback \( A^0_{GL_2}(\Delta_{1,2i}) \to A^0_{GL_2}(\Delta_{1,2i} \setminus \Delta_{2,2i}) \) is an isomorphism.

Proof. This is remark (4.4). \( \square \)

Proof of Theorem 5.7. 1. The maps
\[
A^0_{GL_2}(\Delta_{1,2}) \to A^1_{GL_2}(P^2), \quad A^0_{GL_2}(\Delta_{1,4}) \to A^1_{GL_2}(P^4)
\]
are both zero. The first statement is due to the projection formula. To check the second, note that by the previous points we have that \( A^0_{GL_2}(\Delta_{1,4}) = \langle 1, \alpha \rangle \) as an \( H^*(\text{Spec}(k)) \)-module, where \( \alpha \) is an element of degree 1. Moreover, as before we see that if we call \( c_1 \) the pullback of first Chern class of \( \mathcal{O}_{P^1}(-1) \) we have \( \partial(c_1 \cdot \alpha) = 0 \), and consequently by the projection formula \( c_1(\mathcal{O}_{P^1}(-1)(i_*(\alpha))) = 0 \), which by the structure of the Chow groups with coefficient of a projective bundle implies \( i_* \alpha = 0 \).

2. The points above and the preliminary results we have proven in the rest of this section easily imply that \( A^0_{GL_2}(P^6 \setminus \Delta_{1,6}) \) is freely generated as an \( H^*(\text{Spec}(k)) \)-module by 1 and elements \( x_1, x_2, x_3 \) of degree respectively 1, 2, 3. All that is left to understand is the kernel of \( c_1(\mathcal{O}_{P^1}(-1)) : A^0_{GL_2}(P^6 \setminus \Delta_{1,6}) \to A^0_{GL_2}(P^6 \setminus \Delta_{1,6}) \). We can proceed as in the previous sections to prove by induction that the map is injective on the submodule generated by 1, \( x_1, x_2 \). Unfortunately the reasoning we used before to prove that \( \gamma \) must belong to the kernel of \( c_1(\mathcal{O}_{P^1}(-1)) \) no longer works, as it relied heavily on the algebraic closure of \( k \), so we have to add the unspecified module \( K \) to our final result. \( \square \)

6 Some equivariant Chow groups with coefficients

In this section we compute some equivariant Chow groups with coefficients leading to \( A^*_{SO_3}(\text{spec}(k_0)) \), which we will use to compute the cohomological invariants of \( \mathscr{H}_3 \) using the isomorphism \( SO_3 \simeq PGL_2 \).

The computation has some interest by itself, and it does not require much effort to extend it to Chow groups with coefficients in Milnor’s \( K \)-theory. We begin by computing the \( \mu_p \)-equivariant Chow ring with coefficients of a point.
Proposition 6.1. Let \( k \) be a field and \( p \) be a prime different from the characteristic of \( k \).

- If \( M \) is Milnor’s \( K \)-theory, then \( A^*\mu_p(Spec(k)) \) is equal to \( M(Spec(k)) \cdot [\xi]/p\xi \). Here \( \xi \) is the first Chern class of the standard one-dimensional representation of \( \mu_n \).

- If \( M \) is Galois cohomology with coefficients in \( \mathbb{F}_q, q \neq p \), then \( A^*\mu_p(Spec(k)) \) is equal to \( M(Spec(k)) \).

- If \( M \) is Galois cohomology with coefficients in \( \mathbb{F}_p \), then \( A^*\mu_p(Spec(k)) \) is equal to \( M(Spec(k))[t, \xi] \). Here \( t \) is an element in codimension 0 and degree one, corresponding to a generator for the cohomological invariants of \( \mu_p \).

Proof. We consider the action of \( \mu_p \) on \( G_m \) induced by the inclusion. This action extends linearly to \( \mathbb{A}^1_k \). Then there is a long exact sequence:

\[
0 \to A^0_{\mu_p}(\mathbb{A}^1_k) \to A^0_{\mu_p}(G_m) \xrightarrow{\partial} A^0_{\mu_p}(Spec(k)) \xrightarrow{\xi} A^1_{\mu_p}(\mathbb{A}^1_k) \to \ldots
\]

Using the retraction \( r \) described in [Ros96] section 9 we identify \( A^*_{\mu_p}(\mathbb{A}^1_k) \) with \( A^*_{\mu_p}(Spec(k)) \) and consequently the inclusion pushforward with the first Chern class for the equivariant vector bundle \( \mathbb{A}^1_k \to Spec(k) \) [[Pir14, 4.1]. Note now that \( [G_m/\mu_p] \simeq G_m \), so that \( A^*_{\mu_p}(G_m) \xrightarrow{\partial} M(Spec(k)) \oplus t \cdot M(Spec(k)) \), where \( t \) is an element in codimension zero and degree one. The differential of this element at the origin is equal to \( p \). The computation immediately follows.

The reasoning works the same for an algebraic space being acted on trivially by \( \mu_p \).

Lemma 6.2. Let \( X \) be an algebraic space over a field \( k \), and let \( \mu_p \) act trivially on it. Then \( A^*_{\mu_p}(X) = A^*(X) \otimes M(Spec(k)) A^*_{\mu_p}(Spec(k)) \).

Proof. We consider again the exact sequence:

\[
0 \to A^0_{\mu_p}(X \times \mathbb{A}^1) \xrightarrow{\partial} A^0_{\mu_p}(X \times G_m) \xrightarrow{\partial} A^0_{\mu_p}(X) \xrightarrow{\xi} A^1_{\mu_p}(X \times \mathbb{A}^1) \to \ldots
\]

As before, the quotient \( [(X \times G_m)/\mu_p] \) is isomorphic to \( X \times G_m \), so that for its Chow groups with coefficients the formula \( A^i_{\mu_p}(X \times G_m) = A^i(X) \oplus t \cdot A^i(X) \) holds.

As the first component comes from the pullback through \( X \times G_m \to X \) and this map factors through \( [(X \times \mathbb{A}^1)/\mu_p] \) we see that the first component always belongs to the image of \( J^* \), and given an element \( a \cdot t \) in the second component its image through the boundary map \( \partial \) is equal to \( p \) times \( a \). This gives us a complete understanding of the exact sequence, allowing us to conclude.

With the next proposition we compute the equivariant Chow ring \( A^*_{\mathbb{Z}_n}(Spec(k)) \) for \( n = 2, 3 \). This should serve as an example of how the Chow groups with coefficients can start behaving wildly even for well known objects, as elements of positive degree with no clear geometric or cohomological description appear almost immediately.

We will follow the method in [[VM06, 4.1]. First we need a few more lemmas, which are by themselves interesting facts about the equivariant approach.
Lemma 6.3. Let $G$ be a linear algebraic group, acting on a smooth separated algebraic space $X$, and let $H$ be a normal subgroup of $G$. Suppose the action of $H$ on $X$ is free with quotient $X/H$. Then there is a canonical isomorphism

$$A^*_G(X) \cong A^*_{G/H}(X/H).$$

Proof. The proof in [VM06] 2.1 works without any change.

Lemma 6.4. Let $H$ be a linear algebraic group with an isomorphism $\phi : H \cong A^n_k$ of varieties such that for any field extension $k' \supseteq k$ and any element $h \in H(k')$ the automorphism of $A^n_k$ corresponding through $\phi$ to the action of $h$ on $H_k$ by left multiplication is affine. Furthermore, let $G$ be a linear algebraic group acting on $H$ via group automorphisms, corresponding to linear automorphisms of $A^n_k$ under $\phi$.

If $G$ acts on a smooth separated algebraic space $X$ form the semidirect product $G \rtimes H$ and let it act on $X$ via the projection $G \rtimes H \to G$. Then the homomorphism

$$A^*_G(X) \to A^*_{G \rtimes H}(X)$$

induced by the projection $G \rtimes H \to G$ is an isomorphism.

Proof. Again the argument used in [VM06] 2.3 works for any equivariant theory defined as in [EG96].

Proposition 6.5. Let $M$ be equal to Milnor’s $K$-theory. Let $R_{n,k}$ be the tensor product of the ordinary $\mathcal{O}_n$-equivariant Chow groups of the spectrum of a field $k$ with the field’s $K$-theory, that is

$$R_{n,k} = M(\text{Spec}(k))[c_1, \ldots, c_n]/(2c_i)_{(i \text{ odd})}.$$ 

Then

$$A^n_{\mathcal{O}_n}(\text{Spec}(k)) = R_{2,k} \oplus R_{2,k}\tau_{1,1}$$

where $\tau_{1,1}$ is an element in codimension and degree one, with $2\tau_{1,1} = 0$. For $n = 3$ we have

$$A^n_{\mathcal{O}_n}(\text{Spec}(k)) = R_{3,k} \oplus R_{3,k}\tau_{1,1} \oplus R_{3,k}\tau_{1,2}$$

where $\tau_{1,1}$ and $\tau_{1,2}$ are respectively of codimension and degree (1, 1) and (1, 2), and both are of 2-torsion.

Let $M$ be equal to Galois Cohomology with coefficients in $\mathbb{F}_2$. Recall that $A^n_{\mathcal{O}_n}(\text{Spec}(k))$ is isomorphic to the ring of cohomological invariants of $O_n$, which is generated as a $M(\text{Spec}(k))$-algebra by the Steifel Whitney classes $1 = w_0, w_1, \ldots, w_n$, where $w_i$ has degree $i$. Let $c_1, \ldots, c_n$ be the Chern classes of the standard representation of $O_n$. Then for $n = 2, 3$

$$A^n_{\mathcal{O}_n}(\text{Spec}(k)) = A^n_{\mathcal{O}_n}(\text{Spec}(k))[c_1, \ldots, c_n] \oplus M(\text{Spec}(k))[c_1, \ldots, c_n] \tau_{1,1}$$

Where again $\tau_{1,1}$ is an element of codimension and degree (1, 1).

Let $M$ be equal to Galois Cohomology with coefficients in $\mathbb{F}_p$, with $p \neq 2$. Then $A^n_{\text{GL}_2}(\text{Spec}(k))$ is equal to the tensor product of $M(k)$ with the ordinary equivariant Chow ring.
The ideal (2), which means that there must be an element in codimension 2 and degree 1 mapping to 2 ∈ \mathbb{A} and can be seen for Galois cohomology as the pullback of such element is forced.

It must map to zero. This forces the differential of the element \( t \) to \( V \setminus \{ 0 \} \) and \( \{ 0 \} \) to \( V \).

Let \( N = 2 \) we know that the ring \( A^0(C) \) is equal to \( M(\text{Spec}(k)) \) and it must map to zero. This forces the differential of the element \( t \) to be equal to 1. As the map \( A^0_{\mathcal{O}_n}(V \setminus \{ 0 \}) \to A^0_{\mathcal{O}_n}(B) \) is injective, we have \( A^0_{\mathcal{O}_n}(V \setminus \{ 0 \}) = A^0_{\mathcal{O}_n}(\text{Spec}(k)) \oplus M(\text{Spec}(k))[c_1] \tau_1,1 \). We can then conclude by observing that the map \( A^1_{\mathcal{O}_n}(V) \to A^1_{\mathcal{O}_n}(V \setminus \{ 0 \}) \) is a map of rings and it is surjective in codimension 0 and in degree 0 for all codimensions; we can see that \( \tau_1,1 \) must be in the image as the second chern class \( c_2 \) is injective in degree zero.

For \( n = 3 \), it suffices to do the same calculations knowing that the map \( A^1(C) \to A^1(V \setminus \{ 0 \}) \) must again be 0 (this is obvious in the case of \( K \)-theory, and can be seen for Galois cohomology as the pullback of \( w_1 \) to \( C \) corresponds to \( w_1 \in A^0_{\mathcal{O}_3}(\text{Spec}(k)) \)). For the second long exact sequence we reason as above, except that in the case of \( K \)-theory the class \( c_3 \) has a kernel corresponding to the ideal (2), which means that there must be an element in codimension 2 and degree 1 mapping to 2 ∈ \( A^0_{\mathcal{O}_3}(\text{Spec}(k)) \). We conclude by seeing that the choice of such element is forced.

\[ A^0_{\mathcal{O}_3}(\text{Spec}(k)) [c_3, c_3] \oplus M(\text{Spec}(k)) [c_2, c_3] \tau_1,1 \]

Let \( M \) be equal to Galois cohomology with coefficients in \( \mathbb{F}_2 \). The equivariant Chow ring with coefficient \( A^0_{SO_3}(\text{Spec}(k)) \) is isomorphic to \( M(\text{Spec}(k)) \).

\[ A^0_{SO_3}(\text{Spec}(k)) \]

Let \( M \) be equal to Galois cohomology with coefficients in \( \mathbb{F}_2 \). The equivariant Chow ring with coefficient \( A^0_{SO_3}(\text{Spec}(k)) \) is isomorphic to the tensor product of \( M(k) \) with the ordinary equivariant Chow ring.

The invariants of \( \mathcal{H}_3 \)

In this section we will compute the cohomological invariants of the stack \( \mathcal{H}_3 \) of hyperelliptic curves of genus three over an algebraically closed field.

Recall that the presentation of \( \mathcal{H}_3 \) is obtained by considering \( A^{4g+1} \) as the space of all binary forms of degree 4g, removing the subset \( \Delta \) of binary forms with multiple roots and taking the \( (A^{4g+1} \setminus \Delta)/PGL_2 \times G_m \), where the action of \( PGL_2 \times G_m \) is given by \( (|A|, \alpha)(f)(x) = \text{Det}(A) \alpha^{-2} f(A^{-1}(x)) \).
There are various differences from the previous cases. First, $PGL_2$ is not special, and its Chow groups with coefficients have multiple elements in positive degree when $p = 2$.

**Proposition 7.1.** Let $p$ be equal to 2. Then $A^*_PGL_2(\text{Spec}(k))$ is freely generated as a module over $\text{CH}^*_{PGL_2}(\text{Spec}(k)) \otimes H^*(k)$ by the cohomological invariant $v_2$ and an element $\tau$ in degree and codimension 1.

If $p \neq 2$, then $A^*_PGL_2(\text{Spec}(k))$ is equal to $\text{CH}^*_{PGL_2}(\text{Spec}(k)) \otimes H^*(k)$.

**Proof.** As $PGL_2$ is isomorphic to $SO_3$, we can just apply [5.6]. □

The second difference is that the action of $PGL_2$ on $P^1$ does not come from a linear action on the space of degree one forms. This is true in general whenever we are having $PGL_2$ act on a projective space of odd dimension. The following proposition describes the ring $A^*_PGL_2(P^1)$.

**Proposition 7.2.** The kernel of the map $\pi^*: A^*_PGL_2(\text{Spec}(k)) \to A^*_PGL_2(P^1)$ is generated by $w_2, c_3, \tau$, and $A^*_PGL_2(P^1) = \text{Im}(\pi^*)\setminus t^2 + c_2$.

**Proof.** This can be proven exactly as in [FV11, 5.1]. □

We begin by proving the following lemma:

**Lemma 7.3.**

1. Suppose that the pullback
   
   $$A^0_{PGL_2}(P^{n-4} \setminus \Delta_{1,n-4}) \to A^0_{PGL_2}((P^{n-4} \setminus \Delta_{1,n-4}) \times P^1)$$
   
   is surjective. Then the pullback
   
   $$A^0(P^1) \to A^0(\Delta_{1,n} \setminus \Delta_{2,n})$$
   
   is an isomorphism.

2. Suppose that the pullback
   
   $$A^0_{PGL_2}(P^{n-6} \setminus \Delta_{1,n-6-2i}) \to A^0_{PGL_2}((P^{n-6-2i} \setminus \Delta_{1,n-6-2i}) \times P^1)$$
   
   is surjective. Then the map
   
   $$A^0_{PGL_2}(\text{Spec}(k)) \to A^0_{PGL_2}(\Delta_{2+2i,n})$$
   
   is surjective.

3. Suppose that the above holds and that the pushforward
   
   $$A^0_{PGL_2}(\Delta_{1,n}) \to A^1_{PGL_2}(P^n)$$
   
   is zero. Then the pullback
   
   $$A^0_{PGL_2}(P^n \setminus \Delta_{1,n}) \to A^0_{PGL_2}((P^n \setminus \Delta_{1,n}) \times P^1)$$
   
   is surjective and its kernel is generated by $w_2$, the second Stiefel Whitney class coming from $\text{Inv}(PGL_2)$. 21
Proof. First, note that given a \( PGL_2 \)-equivariant space \( X \), while \( X \times P^1 \to X \) is not the projectivization of an equivariant vector bundle, \( X \times P^1 \times P^1 \to X \times P^1 \) is, and so the pullback through the second map is an isomorphism in codimension zero.

Using this we see that we can apply the reasoning word by word to prove the first point.

The elements of positive degree in \( A^0_{PGL_2}((P^n \setminus \Delta_{1,n}) \times P^1) \) are completely determined by their image through the boundary map \( A^0_{PGL_2}((P^n \setminus \Delta_{1,n}) \times P^1) \xrightarrow{\partial} A^0_{PGL_2}(\Delta_{1,n} \times P^1) \).

Consider now the mapping \( \Delta_{1,n} \times P^1 \to \Delta_{1,n} \). If we remove \( \Delta_{2,n} \) and its inverse image we obtain a pullback

\[
A^0_{PGL_2}((P^n-2 \setminus \Delta_{1,n-2}) \times P^1) = A^0_{PGL_2}(\Delta_{1,n} \setminus \Delta_{2,n}) \to A^0_{PGL_2}((\Delta_{1,n} \setminus \Delta_{2,n}) \times P^1) = A^1_{PGL_2}((P^n-2 \setminus \Delta_{1,n-2}) \times P^1 \times P^1) \]

Then by condition 1 we know that \( A^0_{PGL_2}(\Delta_{1,n}) \) surjects over \( A^0_{PGL_2}(\Delta_{1,n} \times P^1) \) and the fact that the map \( A^0_{PGL_2}(\Delta_{1,n}) \to A^1_{PGL_2}(P^n) \) is zero shows that for every element \( \alpha \) in the first group there is an element in \( A^1_{PGL_2}(P^n \setminus \Delta_{1,n}) \) whose boundary is exactly \( \alpha \). The compatibility of pullback and boundary maps and the surjectivity of the map above allow us to conclude. The description of the kernel stems from the fact that it must be generated by elements that are unramified on \( \Delta_{1,n} \).

The lemma almost provides an inductive step, as its conclusions provide all of its hypotheses except for the requirement that the pushforwards \( A^0_{PGL_2}(\Delta_{1,n}) \to A^1_{PGL_2}(P^n) \) are zero. The following proposition gives us some information on the annihilator of the image of these pushforwards.

We introduce some elements of \( A^*_{PGL_2}(P^n) \):

\[
f_n = \begin{cases} 
  t_i^{n+4/4} (t_i^3 + c_2 t_i + c_3)^{n/4}, & \text{if } n \text{ is divisible by } 4 \\
  t_i^{n-2/4} (t_i^3 + c_2 t_i + c_3)^{n+2/4}, & \text{if } n \text{ is not divisible by } 4 
\end{cases}
\]

We have \( A^*_{PGL_2}(P^n) = A^*_{PGL_2}(\text{Spec}(k_0))/(f_n) \) by [FV11] 6.1 and the projective bundle formula.

**Lemma 7.4.** Suppose that \( p = 2 \). Then the class of \( c_3 \) is zero in \( A^*_{PGL_2}(P^n) \) if and only if \( n \) is odd.

**Proof.** If \( n \) is even then \( P^n \) is the projectivization of a representation of \( PGL_2 \) and the projective bundle formula allows us to conclude immediately. If \( n \) is odd we just have apply the projection formula to the equivariant map \( P^1 \times P^{n-1} \to P^1 \).

**Proposition 7.5.** Let \( i \) be an even positive integer, and let \( \alpha \) be an element of \( A^0_{PGL_2}(\Delta_{1,i}) \). Then:

- If \( i \) is divisible by 4, the image of \( \alpha \) in \( A^*_{PGL_2}(P^n) \) is annihilated by \( c_3^{i/4} f_{i-4} \cdots f_{i+1} \).
- If \( i \) is not divisible by 4, the image of \( \alpha \) in \( A^*_{PGL_2}(P^n) \) is annihilated by \( c_3^{i+2/4} f_{i-4} \cdots f_{i+1} \).
Proof. Consider the map $\Delta_{1,1} \setminus \Delta_{2,1}.$. As $\Delta_{1,1} \setminus \Delta_{2,1}$ is universally homeomorphic to $P^{r-2} \setminus \Delta_{2,1}$ we know by (7.2) that the pullback of $c_3$ through $i$ must be zero. This shows that $c_3i_4\alpha = 0$. We already know that $c_3i_4\alpha$ belongs to $A_{i}^{1}(P^{n})$ it must belong to the kernel of $A_{i}^{1}(P^{n}) \rightarrow P^{n} \setminus \Delta_{2,1}$, which is the image of $A_{\Delta_{2,1}}^{1}(\Delta_{2,1})$. Let $\beta$ be a preimage of $c_3i_4\alpha$.

Consider now $\beta \in A^{2}(\Delta_{2,1})$, and let $\beta'$ be the pullback of $\beta$ to $\Delta_{1,1} \setminus \Delta_{1,2}$. We can see $\beta'$ as an element of $A_{\Delta_{1,1}}^{2}(P^{n})$, $A_{\Delta_{1,1}}^{2}((P^{r-4} \setminus \Delta_{2,1}) \times P^{2})$. We know that in this ring the equation $f_{i-4}(t_{i-4},c_{2},c_{3}) = 0$ holds and as we are working mod 2 the pullback of $t_{i} \in A_{\Delta_{1,1}}^{1}(P^{n})$ is equal to $t_{i-4} \in A_{\Delta_{1,1}}^{1}(P^{n-4})$ we see that the pullback of $f_{i-4}(t_{i},c_{2},c_{3})$ is exactly $f_{i-4}(t_{i-4},c_{2},c_{3}) = 0$, implying that $f_{i-4}(t_{i},c_{2},c_{3})i_4\beta' = 0$ in $A_{\Delta_{1,1}}^{1}(P^{n} \setminus \Delta_{2,1})$. As before, this proves that $c_3f_{i-4}i_4\alpha$ belongs to the image of $A_{\Delta_{1,1}}^{1}(\Delta_{2,1})$.

We can clearly repeat this reasoning inductively to move from $\Delta_{r,1}$ to $\Delta_{r+1,1}$, multiplying by $c_3$ and applying (7.3) if $r$ is odd, and multiplying by $f_{i-2r}$ if $r$ is even. The last thing to note is that when $r = i/2$ the process end and we obtain 0, either multiplying by $f_0 = t$ if $i$ is divisible by 4 or by $c_3$ otherwise. 

Corollary 7.6. Suppose that $p = 2$. Then the cohomological invariants of $[P^{n}/PGL_{2}]$ are freely generated as a $H^{*}(k_{0})$-module by 1 and nonzero elements $x_{1},x_{2},w_{2},x_{3},x_{4},$ where the degree of $x_{1}$ is $i$ and $w_{2}$ is the second Stiefel-Whitney class coming from the cohomological invariants of $PGL_{2}$.

If $p \neq 2$, then the cohomological invariants of $[P^{n}/PGL_{2}]$ are trivial unless $p$ divides $i-1$, in which case they are generated as a $H^{*}(k_{0})$-module by 1 and a single nonzero invariant of degree 1.

Proof. For $p = 2$, the proposition above shows that the maps $i_{4} : A_{i}^{1}(P^{n}) \rightarrow A_{i}^{1}(P^{n})$ are zero for $i \leq 8$, as the polynomial killing the image of $i_{4}$ is not divisible by $f_{i}$. Then we can apply (7.3) repeatedly to obtain the result in the same way as we did for the $g$ even case. The case $p \neq 2$ can be proven exactly as in (4.1).

The reasoning above does not work for any $i > 8$ when $p = 2$.

Theorem 7.7. Suppose that $p = 2$ and $k_{0}$ is algebraically closed. Then the cohomological invariants of $\mathscr{H}_{0}$ are freely generated as a $H^{*}(k_{0})$-module by 1 and nonzero elements $x_{1},x_{2},w_{2},x_{3},x_{4},x_{5},$ where the degree of $x_{1}$ is $i$ and $w_{2}$ is the second Stiefel-Whitney class coming from the cohomological invariants of $PGL_{2}$.

In general, for $p = 2$ the cohomological invariants of $\mathscr{H}_{0}$ are a direct sum $M \oplus K$, where $K$ is a submodule of $H^{*}(k_{0})[5]$ and $M$ is generated as a $H^{*}(k_{0})$-module by 1 and nonzero elements $x_{1},x_{2},w_{2},x_{3},x_{4},$ where the degree of $x_{1}$ is $i$ and $w_{2}$ is the second Stiefel-Whitney class coming from the cohomological invariants of $PGL_{2}$.

If $p \neq 2$ for all odd $g$ the cohomological invariants of $\mathscr{H}_{g}$ are trivial unless $p$ divides $2g+1$, in which case they are generated as a $H^{*}(k_{0})$-module by 1 and a single nonzero invariant of degree 1.

Proof. The case $p \neq 2$ is immediate from the previous corollary and lemma (2.2). For the rest of the proof we will have $p = 2$.

First, we observe that as $G_{m}$ acts trivially on $[P^{n} \setminus \Delta_{1,8}/PGL_{2}]$ the map $[P^{n} \setminus \Delta_{1,8}/PGL_{2}] \rightarrow [P^{n} \setminus \Delta_{1,8}/PGL_{2} \times G_{m}]$ induces an isomorphism on cohomological invariants.
We need to understand whether the $G_m$-torsor
\[ \mathcal{H}_3 \to \left[ P^8 \setminus \Delta_{1,8}/PGL_2 \times G_m \right] \]
generates any new cohomological invariant.

This amounts to understanding the kernel of the first Chern class of the associated line bundle on $\left[ P^8 \setminus \Delta_{1,8}/PGL_2 \times G_m \right]$, and as before for $p = 2$ this is just the first Chern class of $\mathcal{O}_1$ [FV11, 3.2].

We can follow the same reasoning we used in proving the result for $\mathcal{M}_2$. For $x_1, \ldots, x_3$ we can inductively show that they cannot be annihilated by $t_8$ as the boundary $\partial(t_8 x_i)$ is not zero. This is sufficient to prove the result for a general field.

Even if $k_0$ is algebraically closed, the matter is a bit more complicated than usual for $x_4$ as there are elements of positive degree in $A^1_{PGL_2}(\Delta_{2,i})$ coming from the Chow ring with coefficients of $\mathbb{B}(PGL_2)$. To get around this problem, we make the following consideration. There are no elements of degree 4 in $A^1_{PGL_2}(P^8)$, so $t_8 x_4$ is zero if and only if its boundary $\partial(t_8 x_4)$ is zero in $\Delta_{1,8}$. As there are no elements of degree three in $A^1_{PGL_2}(\Delta_{2,8})$ by [FV11], this is equivalent to asking that $\partial(t_8 x_4)$ is zero in $A^1_{PGL_2}([P^6 \setminus \Delta_{1,6}] \times \mathbb{P}^1)$. As the boundary of $x_4$ is the unique element of degree 3 in $A^1_{PGL_2}([P^6 \setminus \Delta_{1,6}] \times \mathbb{P}^1)$ we can proceed with the usual induction, on $(P^i \setminus \Delta_{1,i}) \times \mathbb{P}^1$. The $\mathbb{P}^1$ factor kills all elements of positive degree by [7.2], allowing us to conclude as in 3.1.

References

[AV04] Alessandro Arsie and Angelo Vistoli. Stacks of cyclic covers of projective spaces. Compositio Mathematica, 140, 647-666, 2004.

[EG96] Dan Edidin and William Graham. Equivariant intersection theory. Invent. Math, 131, 1996.

[FV11] Damiano Fulghesu and Filippo Viviani. The chow ring of the stack of cyclic covers of the projective line. Annales de l’institut Fourier, 61 no-6, 2011.

[Gui08] Pierre Guillot. Geometric methods for cohomological invariants. Documenta Mathematica, vol.12 521–545., 2008.

[Pir14] Roberto Pirisi. Cohomological invariants of algebraic curves. part 1. ArXiv preprint, 2014.

[Ros96] Markus Rost. Chow groups with coefficients. Documenta Mathematica, vol.1,319-393, 1996.

[Vez00] Gabriele Vezzosi. On the chow ring of the classifying stack of $\text{pgl}(3)$. Journal für die reine und angewandte Mathematik (Crelle), No. 523,1-54, 2000.

[Vis96] Angelo Vistoli. The chow ring of $m2$, appendix to equivariant intersection theory. Invent. Math, 131, 1996.

[VM06] Angelo Vistoli and Alberto Luis Molina. On the chow ring of classifying spaces for classical groups. Rend. Sem. Mat. Univ. Padova, v.116, 2006.