On Non-Positively Curved Homogeneous Finsler Metrics

B. Najafi and A. Tayebi

April 7, 2021

Abstract

In this paper, we prove two rigidity results for non-positively curved homogeneous Finsler metrics. Our first main result yields an extension of Hu-Deng’s well-known result proven for the Randers metrics. Indeed, we prove that every connected homogeneous Finsler space with non-positive flag curvature and isotropic S-curvature is Riemannian or locally Minkowskian. We extend the Szabó’s rigidity theorem for Berwald surfaces and show that homogeneous isotropic Berwald metrics with non-positive flag curvature are Riemannian or locally Minkowskian. We prove that a homogeneous $(\alpha, \beta)$-metrics has isotropic mean Berwald curvature if and only if it has vanishing mean Berwald curvature generalizing result previously only known in the case of Randers metrics. Our second main result is to show that every homogeneous $(\alpha, \beta)$-metric with non-positive flag curvature and almost isotropic S-curvature is Riemannian or locally Minkowskian.

Keywords: Homogeneous Finsler metric, flag curvature, isotropic Berwald metric, $(\alpha, \beta)$-metric, E-curvature, S-curvature.

1 Introduction

In Riemannian geometry, there is only one curvature derived from the Levi-Civita connection, namely the Riemannian curvature. It defines the notion of sectional curvature, which is a way to describe the curvature of Riemannian manifolds whose dimensions are greater than two. Finslerian geometry is the most natural generalization of Riemannian geometry. In the Finslerian setting, there are three curvatures derived from a Finsler connection, which are functions not merely of position but also direction. The notion of flag curvature obtained from the hh-curvature of the Finslerian connection is a natural extension of the sectional curvature in Riemannian geometry, which tells us how curved is the Finsler manifold at a point. Alternatively, flag curvatures can be treated as Jacobi endomorphisms.

The world of Riemannian manifolds is single-colored, while the world of Finsler manifolds is like a crayon box with unlimited colors. This beauty stems from the nature of the Finsler metrics and the other two non-Riemannian curvatures of their connections. Indeed, besides the Riemannian curvature and its family, there are many interesting and essential non-Riemannian curvatures that vanish for Riemannian metrics. These non-Riemannian quantities describe the “color” and its rate of change over the manifold, such as the Cartan torsion $C$, the Berwald curvature $B$, the mean Berwald curvature $E$, the Landsberg curvature $L$, the mean Landsberg curvature $J$, and the S-curvature $S$.

\footnote{2000 Mathematics subject Classification: 53B40,53C60.}
One of the central problems in Finsler geometry is studying and classifying Finsler metrics of non-positive flag curvature. There are some elegant global rigidity results for the Finsler metrics with $K \leq 0$. The most well-known result regarding this case is the contribution of Akbar-Zadeh. He proved that every closed Finsler manifold with negative constant flag curvature $K < 0$ must be Riemannian, and every closed Finsler manifold with $K = 0$ must be locally Minkowskian [2]. In [21], Mo-Shen showed that every compact Finsler manifold of dimension $\geq 3$ and negative scalar curvature is a Randers metric. The problem of Finsler metrics with non-positive flag curvature in Finsler geometry is well-studied. However, up to now, very little attention has been paid to the subject of homogeneous Finsler metrics. A Finsler manifold $(M, F)$ is said to be homogeneous if its group of isometries acts transitively on $M$. In [15], Hu-Deng initiated the study of homogeneous Finsler manifolds of positive flag curvature. They obtained a classification of homogeneous Randers metrics with isotropic S-curvature and positive flag curvature. In [9], Deng-Hu proved that a homogeneous Finsler manifold with non-positive flag curvature and negative Ricci scalar is a simply connected manifold. In [10], Deng-Hu classified homogeneous Finsler manifolds of positive flag curvature. In [14], Heintze classified the class of homogeneous Riemannian manifolds with negative sectional curvature. Thus, it is natural to consider the class of non-positively curved homogeneous Finsler metrics. The first step to this problem was taken by Hu-Deng in [15], where they showed that every connected homogeneous Randers metric with almost isotropic S-curvature and negative Ricci scalar reduces to a Riemannian metric. It turns out that every connected homogeneous Randers metric with almost isotropic S-curvature and negative flag curvature is Riemannian. In this paper, we extend this result for general homogeneous Finsler metrics. More precisely, we prove the following.

**Theorem 1.1.** Every connected $n$-dimensional homogeneous Finsler space $(M, F)$ with isotropic S-curvature is Riemannian or locally Minkowskian, provided that $F$ is of non-positive flag curvature for $n = 2$ and is of negative flag curvature for $n > 2$.

It is natural to consider the non-positive curved homogeneous Finsler manifolds without the restriction imposed on their S-curvature. However, since an explicit formula of the S-curvature of the homogeneous Finsler manifolds has not been obtained, this problem is still open. An interesting question is whether Theorem 1.1 holds for homogeneous Finsler manifold of positive flag curvature. For now, only this can be said that every connected homogeneous Landsberg metric with positive flag curvature and isotropic S-curvature is Riemannian.

In [11], Deng-Hu proved that a homogeneous Randers metric of Berwald type whose flag curvature is non-zero everywhere must be Riemannian. Every Finsler metric of isotropic S-curvature has isotropic mean Berwald curvature or equivalently almost isotropic S-curvature. Then by considering Theorem 1.1 and Deng-Hu’s theorem, it is interesting to consider homogeneous Finsler manifolds with non-positive flag curvature and almost isotropic S-curvature. However, it is challenging to investigate this question for general Finsler metrics. The first step to solve this problem is to consider it for a class of Finsler metrics that are tangible and computable. Restricting our attention to the class of $(\alpha, \beta)$-metrics, we prove the following.

**Theorem 1.2.** Every homogeneous $(\alpha, \beta)$-metric with non-positive flag curvature, and almost isotropic S-curvature is Riemannian or locally Minkowskian.

Every Berwald metric satisfies $S = 0$. Thus, Theorem 1.2 is a natural extension of the Deng-Hu’s result that proved only for Randers metrics. In [30], Xu-Deng gave a complete clas-
sification of positively curved homogeneous \((\alpha, \beta)\)-metrics with vanishing S-curvature. Therefore, Theorem 1.2 can be also considered as the complement of the Deng-Hu’s works.

2 Preliminaries

Let \((M, F)\) be an \(n\)-dimensional Finsler manifold, and \(TM\) be its tangent space. We denote the slit tangent space of \(M\) by \(TM_0\), i.e., \(T_xM_0 = T_xM - \{0\}\) at every \(x \in M\). The fundamental tensor \(g_y : T_xM \times T_xM \to \mathbb{R}\) of \(F\) is defined by following

\[
g_y(u, v) := \frac{1}{2} \frac{\partial^2}{\partial s \partial t} \left[ F^2(y + su + tv) \right]|_{s, t = 0}, \quad u, v \in T_xM.
\]

Let \(x \in M\) and \(F_x := F|_{T_xM}\). To measure the non-Euclidean feature of \(F_x\), define \(\mathbf{C}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}\) by

\[
\mathbf{C}_y(u, v, w) := \frac{1}{2} \frac{d}{dt} \left[ g_{yw}(u, v) \right]|_{t = 0}, \quad u, v, w \in T_xM.
\]

The family \(\mathbf{C} := \{\mathbf{C}_y\}_{y \in TM_0}\) is called the Cartan torsion. By definition, \(\mathbf{C}_y\) is a symmetric trilinear form on \(T_xM\). It is well known that \(\mathbf{C} = 0\) if and only if \(F\) is Riemannian.

Let \((M, F)\) be a Finsler manifold. For \(y \in T_xM_0\), define \(\mathbf{I}_y : T_xM \to \mathbb{R}\) by

\[
\mathbf{I}_y(u) = \sum_{i=1}^{n} g^{ij}(y) \mathbf{C}_y(u, \partial_i, \partial_j),
\]

where \(\{\partial_i\}\) is a basis for \(T_xM\) at \(x \in M\). The family \(\mathbf{I} := \{\mathbf{I}_y\}_{y \in TM_0}\) is called the mean Cartan torsion. By definition, \(\mathbf{I}_y(u) := I_i(y) u^i\), where \(I_i := g^{jk} C_{ijk}\). By Deicke’s theorem, every positive-definite Finsler metric \(F\) is Riemannian if and only if \(\mathbf{I} = 0\).

Given a Finsler manifold \((M, F)\), then a global vector field \(\mathbf{G}\) is induced by \(F\) on \(TM_0\), and in a standard coordinate \((x^i, y^i)\) for \(TM_0\) is given by \(\mathbf{G} = y^i \partial/\partial x^i - 2G^i(x, y) \partial/\partial y^i\), where \(G^i = G^i(x, y)\) are scalar functions on \(TM_0\) given by

\[
G^i := \frac{1}{4} g^{ij} \left\{ \frac{\partial^2 [F^2]}{\partial x^k \partial y^j} y^k - \frac{\partial [F^2]}{\partial x^i} \right\}, \quad y \in T_xM. \tag{2.1}
\]

The vector field \(\mathbf{G}\) is called the spray associated with \((M, F)\).

A natural volume form \(dV_F = \sigma_F(x) dx^1 \cdots dx^n\) of a Finsler metric \(F\) on an \(n\)-dimensional manifold \(M\) is defined by

\[
\sigma_F(x) := \frac{\text{Vol}(\mathbb{B}^n)}{\text{Vol}\{ (y^i) \in \mathbb{R}^n \mid F(y^i \partial/\partial x^i) < 1 \}}, \tag{2.2}
\]

where \(\mathbb{B}^n = \{y \in \mathbb{R}^n \mid |y| < 1\}\). The S-curvature is defined by

\[
\mathbf{S}(y) := \frac{\partial G^i}{\partial y^i}(x, y) - y^i \frac{\partial}{\partial x^i} \left[ \ln \sigma_F(x) \right],
\]

where \(y = y^i \partial/\partial x^i|_x \in T_xM\). We say \(F\) has almost isotropic S-curvature if

\[
\mathbf{S} = (n + 1)cF + \eta
\]

where \(c = c(x)\) is a scalar function and \(\eta = \eta_i(x) y^i\) is a closed 1-form on \(M\). If \(\eta = 0\), then \(F\) has isotropic S-curvature.
For \( y \in T_xM_0 \), define \( B_y : T_xM \times T_xM \times T_xM \to T_xM \) by \( B_y(u, v, w) := B^i_{jkl}(y)u^jv^kw^l \partial_{x^i}|_x \) where
\[
B^i_{jkl}(y) := \frac{\partial^i G^j_{y}^{y}}{\partial y^j \partial y^k \partial y^l}.
\]
The quantity \( B \) is called the Berwald curvature of the Finsler metric \( F \). We call a Finsler metric \( F \) a Berwald metric, if \( B = 0 \). Moreover, \( F \) is called of isotropic Berwald curvature if
\[
B_y(u, v, w) = cF^{-1}\left\{ h(u, v)\left(w - g_y(w, \ell)\ell\right) + h(v, w)\left(u - g_y(u, \ell)\ell\right) + h(w, u)\left(v - g_y(v, \ell)\ell\right) + 2FC_y(u, v, w)\ell \right\}. \tag{2.3}
\]
where \( c = c(x) \) is a scalar function on \( M \), and \( h = g_{ij}dx^i \otimes dx^j \) is the angular metric.

Define the mean of Berwald curvature by \( E_y : T_xM \times T_xM \to \mathbb{R} \), where
\[
E_y(u, v) := \frac{1}{2} \sum_{i=1}^n g^{ij}(y)g_y \left(B_y(u, v, e_i), e_j\right). \tag{2.4}
\]
The family \( E = \{E_y\}_{y \in TM \setminus \{0\}} \) is called the mean Berwald curvature or E-curvature. In a local coordinates, \( E_y(u, v) := E_{ij}(y)u^iv^j \), where
\[
E_{ij} := \frac{1}{2} B^m_{nij}.
\]
By definition, \( E_y(u, v) \) is symmetric in \( u \) and \( v \) and we have \( E_y(y, v) = 0 \). \( E \) is called the mean Berwald curvature. \( F \) is called a weakly Berwald metric if \( E = 0 \). \( F \) is called of isotropic E-curvature, if
\[
E = \frac{n+1}{2} cF^{-1}h,
\]
where \( c = c(x) \) is a scalar function on \( M \).

For \( y \in T_xM \), define the Landsberg curvature \( L_y : T_xM \times T_xM \times T_xM \to \mathbb{R} \) by
\[
L_y(u, v, w) := -\frac{1}{2} g_y \left(B_y(u, v, w), y\right).
\]
\( F \) is called a Landsberg metric if \( L_y = 0 \). By definition, every Berwald metric is a Landsberg metric.

Let \( (M, F) \) be a Finsler manifold. For \( y \in T_xM_0 \), define \( J_y : T_xM \to \mathbb{R} \) by
\[
J_y(u) = \sum_{i=1}^n g^{ij}(y)L_y(u, \partial_i, \partial_j).
\]
The quantity \( J \) is called the mean Landsberg curvature or J-curvature of Finsler metric \( F \). A Finsler metric \( F \) is called a weakly Landsberg metric if \( J_y = 0 \). By definition, every Landsberg metric is a weakly Landsberg metric. Mean Landsberg curvature can also be defined as following
\[
J_i := y^m \frac{\partial I_i}{\partial x^m} - I_m \frac{\partial G^m}{\partial y^i} - 2G^m \frac{\partial I_i}{\partial y^m}.
\]
By definition, we get

\[ J_y(u) := \frac{d}{dt} \left[ I_{\dot{\sigma}(t)}(U(t)) \right]_{t=0}, \]

where \( y \in T_x M, \sigma = \sigma(t) \) is the geodesic with \( \sigma(0) = x, \dot{\sigma}(0) = y \), and \( U(t) \) is a linearly parallel vector field along \( \sigma \) with \( U(0) = u \). The mean Landsberg curvature \( J_y \) is the rate of change of \( I_y \) along geodesics for any \( y \in T_x M_0 \).

For an arbitrary non-zero vector \( y \in T_x M_0 \), the Riemann curvature is a linear transformation \( R_y : T_x M \rightarrow T_x M \) with homogeneity \( R_{\lambda y} = \lambda^2 R_y, \forall \lambda > 0 \), which is defined by

\[ R_i^k(y) := \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j} \frac{\partial^2 G^j}{\partial x^i \partial x^k} - \frac{\partial G^i}{\partial y^j} \frac{\partial G^j}{\partial y^k}. \] (2.5)

The family \( R := \{ R_y \}_{y \in T M_0} \) is called the Riemann curvature of the Finsler manifold \((M, F)\).

For a flag \( P := \text{span}\{y, u\} \subset T_x M \) with flagpole \( y \), the flag curvature \( K = K(x, y, P) \) is defined by

\[ K(x, y, P) := \frac{g_y(u, R_y(u))}{g_y(y, y) g_y(u, u) - g_y(y, u)^2}. \] (2.6)

The flag curvature \( K(x, y, P) \) is a function of tangent planes \( P = \text{span}\{y, v\} \subset T_x M \). This quantity tells us how curved space is at a point. A Finsler metric \( F \) is of scalar flag curvature if \( K = K(x, y, P) \) is independent of flags \( P \) containing \( y \in T_x M_0 \).

### 3 Proof of Theorem 1.1

In this section, we are going to prove Theorem 1.1. For this aim, we need the following.

**Lemma 3.1.** Let \( \Psi : \mathbb{R} \rightarrow \mathbb{R} \) be a bounded smooth function. If \( \Psi'' \) is a non-negative function, then \( \Psi \) is a constant function.

**Proof.** Suppose that \( \Psi \) is not a constant function. Then, there exists an interval \((a, b)\) such that \( \Psi(a) \neq \Psi(b) \). Without loss of generality, we suppose that \( \Psi(a) < \Psi(b) \). By the mean value theorem, there exists \( t_0 \in (a, b) \) such that

\[ \Psi'(t_0) = \frac{\Psi(b) - \Psi(a)}{b - a} > 0. \]

Since \( \Psi'' \) is a non-negative function, for every \( t > t_0 \) we have \( \Psi'(t) \geq \Psi'(t_0) > 0 \). For any natural number \( n \), there exists \( t_n > t_0 \) such that

\[ \frac{\Psi(t_0 + n) - \Psi(t_0)}{n} = \Psi'(t_n) \geq \Psi'(t_0), \]

which implies that

\[ \Psi(t_0 + n) \geq \Psi(t_0) + n \Psi'(t_0), \quad \forall n \in \mathbb{N}. \]

This yields that \( \Psi \) is not bounded, which is a contradiction. This completes the proof. \( \square \)
In [18], Latifi-Razavi proved that every homogeneous Finsler manifold is forward geodesically complete. In [26], Tayebi-Najafi improved their result and proved the following.

**Lemma 3.2.** ([27]) Every homogeneous Finsler manifold is complete.

By definition, every two points of a homogeneous Finsler manifold $(M, F)$ map to each other under an isometry. This causes the norm of an invariant tensor under the isometries of a homogeneous Finsler manifold is a constant function on $M$, and consequently, it has a bounded norm. Then, we conclude the following.

**Lemma 3.3.** ([26]) Let $(M, F)$ be a homogeneous Finsler manifold. Then, every invariant tensor under the isometries of $F$ has a bounded norm with respect to $F$.

Here, we prove a result that is the crucial lemma throughout the paper.

**Lemma 3.4.** Let $(M, F)$ be a homogeneous Finsler manifold of isotropic $S$-curvature. Suppose that $F$ has non-positive flag curvature. Then, $F$ is a weakly Landsberg metric.

**Proof.** Combining formula (5) in [21] and formula (21) in [6], one can obtain

$$J_{k|m}y^m + I_{m}R^m_k = S_{k|m}y^m - S_k,$$

which is equivalent to

$$I_{[p|q]}y^p y^q + R^m_m = g^{ik}\left\{S_{k|m}y^m - S_k\right\}.$$  \hspace{1cm} (3.2)

It is proved that every homogeneous Finsler metric of isotropic $S$-curvature $S = (n + 1)cF$ has vanishing $S$-curvature $S = 0$ (Corollary 4.3. in [16]). Thus, (3.2) and vanishing of the $S$-curvature imply that $I_{[p|q]}y^p y^q + R^m_m = 0$ or equivalently

$$I_{[p|q]}y^p y^q = -R^m_m I^m.$$  \hspace{1cm} (3.3)

Suppose that $F$ is a non-Riemannian metric. Then by famous Deicke’s theorem $||I||_{(x,y)} \neq 0$ at some non-zero tangent vectors $y \in T_x M$. Suppose $y \in T_x M_0$ is such a vector and let $\sigma = \sigma(t)$ be the unit geodesic of $F$ with $\sigma(0) = x$ and $\sigma'(0) = y$. Let us define

$$\Psi(t) := ||I||_{(\sigma(t), \sigma'(t))}$$

and suppose that $(a, b)$ is the maximal interval on which $\Psi(t)$ is positive.

Taking a horizontal derivation of $\Psi(t)^2 = I_i(\sigma(t), \sigma'(t))I_i(\sigma(t), \sigma'(t))$ along geodesics and using the Chaushi-Schwarz inequality, we get

$$\Psi(t)\Psi'(t) = I_i(\sigma(t), \sigma'(t))J_i(\sigma(t), \sigma'(t)) \leq ||I||_{(\sigma(t), \sigma'(t))} ||J||_{(\sigma(t), \sigma'(t))} = \Psi(t) ||J||_{(\sigma(t), \sigma'(t))}$$  \hspace{1cm} (3.4)

which implies

$$\Psi'(t) \leq ||J||_{(\sigma(t), \sigma'(t))}.$$  \hspace{1cm} (3.5)

By assumption, $K \leq 0$ and (3.4), we have

$$[\Psi^2]''(t) \overset{(3.3)}{=} 2\left[ -R^m_m(\sigma(t), \sigma'(t))I^m(\sigma(t), \sigma'(t))I_i(\sigma(t), \sigma'(t)) + ||J||_{(\sigma(t), \sigma'(t))}^2 \right] \geq 2||J||_{(\sigma(t), \sigma'(t))}^2.$$  \hspace{1cm} (3.6)

We obtain that $\Psi''(t) \geq 0$. Using Lemma 3.1, we conclude that $\Psi$ is a constant function and then $\Psi'(t) \equiv 0$. By (3.6) and $K \leq 0$, we get $||J||_{(\sigma(t), \sigma'(t))} = 0$. Due to the arbitrariness of the non-zero vector $y \in T_x M$, it follows that $F$ is a weakly Landsberg metric. \qed
Proof of Theorem 1.1: We first deal with Finsler surfaces. The special and useful Berwald frame was introduced and developed by Berwald [4]. Let \((M, F)\) be a two-dimensional Finsler manifold. One can define a local field of orthonormal frame \((\ell^i, m^i)\) called the Berwald frame, where \(\ell^i = y^i/F\), \(m^i\) is the unit vector with \(\ell_i m^i = 0\), \(\ell_i = g_{ij} \ell^j\) and \(g_{ij}\) is defined by \(g_{ij} = \ell_i \ell_j + m_i m_j\). The Berwald curvature of Finsler surfaces is given by

\[
B^i_{jkl} = F^{-1}(-2I_1 \ell^i + I_2 m^i)m_j m_k m_l, \tag{3.7}
\]

where \(I = I(x, y)\) is 0-homogeneous function called the main scalar of Finsler metric and \(I_2 = I_2 + I_{1/2}\) (see page 689 in [1]). In [28], it is proved that the Berwald curvature of a Finsler surface can be written as follows

\[
B^i_{jkl} = \mu C_{jkl} \ell^i + \lambda (h_{ij} h_{kl} + h_{ik} h_{jl} + h_{il} h_{jk}), \tag{3.8}
\]

where \(\mu := -2I_1/I\) and \(\lambda := I_2/(3F)\) are homogeneous functions on \(TM\) of degrees 0 and -1 with respect to \(y\), respectively. Taking a trace of (3.8) yields

\[
E_{jk} = \frac{3}{2} \lambda h_{jk}. \tag{3.9}
\]

Contracting (3.8) with \(y_i\) implies that

\[
L_{jk} + \frac{\mu}{2} FC_{jkl} = 0. \tag{3.10}
\]

Multiplying (3.8) with \(g^{jk}\) yields

\[
J_l + \frac{\mu}{2} FI_l = 0. \tag{3.11}
\]

By Lemma 3.4, \(F\) satisfies \(J = 0\). By putting it in (3.11), we conclude that \(F\) is Riemannian or \(\mu = 0\). On the other hand, by assumption, we have \(S = 0\) and then \(E = 0\). Substituting it in (3.9) gives us \(\lambda = 0\). Plugging \(\mu = \lambda = 0\) in (3.8) implies that \(F\) is a Berwald metric. In [24], Szabó proved that every connected Berwald surface is Riemannian or locally Minkowskian. This completes the proof for 2-dimensional Finsler spaces.

Now, let us consider a Finsler manifold \((M, F)\) with dimension greater than two. Suppose that \(K < 0\). In this case, let us consider the scalar function \(f : TM \to \mathbb{R}\) defined by \(f(x, y) := F^2(x, y) g_{y^t} (I_y, I_y)\). The scalar function \(f\) is a homogeneous function of degree zero on \(TM_0\).

It is known that the Lie derivative of \(F\) along the spray of \(F\) vanishes, i.e., \(\mathcal{L}_G(F) = 0\). Therefore, we have

\[
\mathcal{L}_G(f) = f_y^s F^2 I_{i|s} y^s + F^2 I_{i|y} y^s I_s = 2F^2 J_i I_i = 0, \tag{3.12}
\]

where we have used \(J = 0\). The relation (3.12) means that \(f\) is constant along geodesics of \(F\). Using a Ricci identity given in [21], we get

\[
f_y^p R^p_i + f_i|p|y^p y^p = 0. \tag{3.13}
\]

Let \(\phi\) be a local isometry of \(F\). It is easy to see that \(f\) is invariant under \(\phi\), i.e., in a standard local coordinates, we have

\[
f(x^i, y^i) = f(\phi(x), y^t \frac{\partial \phi^t}{\partial x^i}). \tag{3.14}
\]
Let us put
\[ \bar{x}^i = \phi^i(x), \quad \bar{y}^i = y^j \frac{\partial \phi^i}{\partial x^j}. \]
Thus
\[ f(x^i, y^i) = f(\bar{x}^i, \bar{y}^i). \tag{3.15} \]
Let us define \( \phi^i_j := \frac{\partial \phi^i}{\partial x^j}. \) Since \( \phi \) is an isometry, the matrix \( (\phi^i_j) \) is invertible. Put \( (\psi^i_j) := (\phi^i_j)^{-1}. \)

Put
\[ g_{ij} := \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j}, \quad \bar{g}_{ij} := \frac{1}{2} \frac{\partial^2 F^2}{\partial \bar{y}^i \partial \bar{y}^j}. \]
Thus
\[ g_{ij} = \bar{g}_{rs} \phi^r_i \phi^s_j. \tag{3.16} \]
Equivalently
\[ \bar{g}_{ij} = g_{rs} \psi^r_i \psi^s_j. \tag{3.17} \]
It follows that
\[ \bar{g}_{ij} = g_{pq} \phi^p_i \phi^q_j. \tag{3.18} \]
By definition, we have \( \bar{y}^i = y^l \phi^i_l. \) Thus
\[ \bar{y}_i := \frac{\partial F}{\partial \bar{y}^i} = y_j \psi^i_j. \tag{3.19} \]
The following holds
\[ \frac{\partial f}{\partial y^i} = \phi^i_j \frac{\partial f}{\partial y^j}. \tag{3.20} \]
This means that the tensor field with components \( \partial f / \partial y^i \) is invariant under isometries of \( F. \) Thus, it has bounded norm with respect to \( F. \) This means that the scalar function
\[ \tilde{f} := F^2 g_{ij} \frac{\partial f}{\partial y^i} \frac{\partial f}{\partial y^j} \]
is invariant under the isometries of \( F. \) The scalar function \( \tilde{f} \) is a homogeneous function of degree zero on \( TM_0. \) Let \( \sigma : \mathbb{R} \to M \) be an arbitrary unit speed geodesic of \( F. \) To avoid clutter, we use the abbreviation \( \tilde{f}(t) := \tilde{f}(\sigma(t), \dot{\sigma}(t)). \) Our argument shows that \( \tilde{f}(t) : \mathbb{R} \to \mathbb{R} \) is a bounded smooth function. It is easy to see that
\[ \tilde{f}''(t) = 2 f_{i|pl} \hat{\sigma}^p \hat{\sigma}^q g^{ij} + 2 f_{i|lp} f_{j|q} \hat{\sigma}^p \hat{\sigma}^q g^{ij}. \tag{3.21} \]
Plugging (3.13) into (3.20), we get
\[ \tilde{f}''(t) = -2 R^k_{ij} f_{i|j} g^{jk} + 2 f_{i|lp} f_{j|q} \hat{\sigma}^p \hat{\sigma}^q g^{ij}. \tag{3.22} \]
Since \( F \) has non-positive flag curvature, then we have \( \tilde{f}''(t) \geq 0. \) It follows from Lemma 3.1 that \( \tilde{f} \) is a constant function. Thus, \( \tilde{f} \) is zero function, and consequently, \( \tilde{f}'' = 0. \) It follows form (3.21)
\[ R^i_{jk} f_{i|j} g^{jk} = f_{i|lp} f_{j|q} \hat{\sigma}^p \hat{\sigma}^q g^{ij} = 0. \tag{3.23} \]
Negatively curved condition and the arbitrariness of the geodesic $\sigma$ imply that $f_i = 0$. It means that $f$ is a function of position. From (3.12), we get
\[ \frac{\partial f}{\partial x^i} y^i = 0. \]
Thus, $\partial f/\partial x^i = 0$ and as a result $f$ is a constant. We recall that $I_i = \partial\tau/\partial y^i$. For a fixed point $x_0 \in M$, the distorsion attains its extremum on indicatrix of $F$ at $x_0$. At this point $f$ vanishes, and constancy of $f$ implies that $f = 0$. The proof follows from Deicke’s theorem.

**Remark 3.1.** With the notation of Theorem 1.1, suppose that $F$ has vanishing flag curvature. According to [2], any positively complete Finsler metric with zero flag curvature must be locally Minkowskian if the first and second Cartan torsions are bounded. For the homogeneous Finsler metrics, the first and second Cartan torsions are bounded. Then in this case, $F$ reduces to a locally Minkowskian metric.

In [15], Hu-Deng proved that every homogeneous Randers metric of isotropic S-curvature and negative flag curvature is Riemmanian. Then Theorem 1.1 is an extension of their result.

In [27], the authors proved that every homogeneous isotropic Berwald metric on a manifold $M$ of dimension $n \geq 3$ is either a Berwald metric or a Randers metric of Berwald type. Here, we prove the following.

**Corollary 3.1.** Every connected homogeneous isotropic Berwald manifold with non-positive flag curvature is Riemannian or locally Minkowskian.

**Proof.** In [29], it is proved that isotropic Berwald metric has isotropic S-curvature. By Theorem 1.1, we get the proof.

Also, according to the Szabó’s theorem, every Berwald manifold is Riemannian (if $K \neq 0$) or locally Minkowsian (if $K = 0$). Thus the Corollary 3.1 is an extension of his result for homogeneous isotropic Berwald manifolds.

**Corollary 3.2.** Every connected homogeneous Einstein manifold with non-positive flag curvature is Riemannian or locally Minkowskian.

**Proof.** In [3], Bao-Robles proved that every Einstein Finsler metric has constant S-curvature. By Theorem 1.1, we get the proof.

Finsler metrics of sectional flag curvature are those Finsler metrics whose flag curvatures $K(x, y, P)$ only depend on sections $P$. Trivial examples are Riemannian metrics and isotropically or constantly curved Finsler metrics. In [17], Huang-Shen proved that there is no non-trivial Finsler metric of sectional flag curvature.

**Corollary 3.3.** Every homogeneous Finsler metric of non-positive sectional flag curvature and isotropic S-curvature is Riemannian or locally Minkowskian.

**Proof.** Every homogeneous Finsler metric of non-positive sectional flag curvature is of non-positive scalar flag curvature [17]. By Theorem 1.1, we get the proof.
4 Proof of Theorem 1.2

A Killing frame for an $n$-dimensional Finsler manifold $(M,F)$ is a set of local vector fields $\{X_i\}_{i=1}^n$, defined on an open subset $U \subseteq M$ around a given point, such that: (1) The set of tangent vectors $\{X_i(x)\}$ gives a basis for every tangent space $T_x(M)$, at any point $x \in U$; (2) In $U$, each $X_i$ satisfies $X_i(F) = 0$.

Though Killing frames are rare in the general study of Finsler geometry, they can be easily found for a homogeneous Finsler space at any given point [16]. Let the homogeneous Finsler space $(M,F)$ be presented as $M = G/H$, where $H$ is the isotropy subgroup for the given $x$. The tangent space $T_xM$ can be identified as the quotient $\mathfrak{m} = \mathfrak{g}/\mathfrak{h}$, where $\mathfrak{g}$ and $\mathfrak{h}$ are the Lie algebras of $G$ and $H$, respectively. Take any basis $\{v_1, \ldots, v_n\}$ of $\mathfrak{m}$, with the pre-images $\{\hat{v}_1, \ldots, \hat{v}_n\}$ in $\mathfrak{g}$. Then the Killing vector fields $\{\hat{X}_1, \ldots, \hat{X}_n\}$ on $M$ corresponding to $\hat{v}_i$s defines a Killing frame around $x$. The choice of $\hat{v}_i$s or $X_i$s identifies the quotient space $\mathfrak{m}$ with a subspace of $\mathfrak{g}$. Thus, we can write the decomposition of linear space

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m}. \quad (4.1)$$

With respect to the decomposition $(4.1)$ there is a projection map $\text{pr} : \mathfrak{g} \rightarrow \mathfrak{m}$. Note that for the bracket operation $[\cdot, \cdot]$ on $\mathfrak{g}$ we have $[\cdot, \cdot]_\mathfrak{m} = \text{pr}[\cdot, \cdot]$.

As mentioned before, Xu and Deng proved that a homogeneous Finsler metric has isotropic S-curvature if and only if it has vanishing S-curvature [16]. Here, we develop this fact to the E-curvature of homogeneous $(\alpha, \beta)$-metrics.

**Theorem 4.1.** Let $(G/H, F = \alpha\phi(\beta/\alpha))$ be an $n$-dimensional homogeneous $(\alpha, \beta)$-manifold, where $\alpha$ is a $G$-invariant Riemannian metric, and $\beta$ is a $G$-invariant 1-form on $G/H$. Then the following are equivalent:

(i) $F$ has isotropic mean Berwald curvature $E = (n + 1)/2cF^{-1}h$;

(ii) $F$ has vanishing E-curvature $E = 0$;

where $c = c(x)$ is a scalar function on $G/H$.

**Proof.** Suppose that $\alpha = \sqrt{a_{ij}y^i y^j}$ is a Riemannian metric and $\beta = b_i(x) y^i$ is a 1-form on $M = G/H$. Let $F := \alpha\phi(s)$, $s = \beta/\alpha$, be an $(\alpha, \beta)$-metric on the manifold $G/H$. Suppose that $H$ is the isotropy subgroup for the given $x \in M = G/H$. It suffices to prove the above equivalency at point $x$, since $(M,F)$ is a homogeneous Finsler manifold.

Here, $\phi = \phi(s)$ is a $C^\infty$ function on the interval $(-b_0, b_0)$. For $b^2 := a^{ij}b_i b_j$, let

$$\Phi := -(n\Delta + 1 + sQ)(Q - sQ') - (b^2 - s^2)(1 + sQ)Q'', \quad (4.2)$$

where

$$Q := \frac{\phi'}{\phi - s\phi'}, \quad \Delta = 1 + sQ + (b^2 - s^2)Q'.$$  \hspace{1cm} (4.3)

In [12], Deng-Wang proved that the S-curvature of a homogeneous $(\alpha, \beta)$-metric, with respect to the decomposition $(4.1)$, is given by

$$S(x,y) = \frac{1}{\alpha(x,y)} \frac{\Phi}{2\Delta^2} \left(\kappa([u, y]_m, y) + \alpha(x, y)Q([u, y]_m, u)\right), \quad \forall y \in \mathfrak{m}, \quad (4.4)$$

where $\kappa$ is a real constant and $u$ is the unique vector in $\mathfrak{m}$ corresponding to $\beta$. 

It follows from (4.4) that
\[ S(x, u) = 0 \quad \text{and} \quad S(x, -u) = 0. \]

Now, suppose that \( F \) has isotropic mean Berwald curvature
\[ E = \frac{n+1}{2F} c \cdot h, \]
where \( c = c(x) \) is a scalar function on \( G/H \). Then, we conclude that S-curvature of \( F \) is in the following form
\[ S = (n+1)cF + \eta, \tag{4.5} \]
where \( \eta \) is 1-form on \( G/H \). Putting \( y = u \) and \( y = -u \) in (4.5), respectively, we get
\[ c(x)F(x, u) + \eta(x, u) = 0, \quad c(x)F(x, -u) - \eta(x, u) = 0. \tag{4.6} \]
Therefore,
\[ c(x)(F(x, u) + F(x, -u)) = 0, \]
and consequently \( c(x) = 0 \). Putting it in (4.5) implies that \( S = \eta \) is a 1-form. Taking twice vertical derivative of \( S = \eta \) implies that the \( E \)-curvature vanishes. \( \square \)

Let \( F := \alpha \phi(s) \), where \( s = \beta/\alpha \) be an \((\alpha, \beta)\)-metric on a manifold \( M \). Let us define
\[ \Xi := \frac{(b^2Q + s)}{\Delta^2} \phi \tag{4.7} \]
In [22], the authors proved the following.

**Theorem 4.2.** ([22]) Let \( F = \alpha \phi(s), \ s = \beta/\alpha, \) be an \((\alpha, \beta)\)-metric. Suppose that \( \Xi \) is not constant. Then \( F \) is of isotropic \( S \)-curvature if and only if it is of isotropic \( E \)-curvature.

Here, we prove that \( \Xi \) is always non-constant for regular \((\alpha, \beta)\)-metrics.

**Theorem 4.3.** Let \( F = \alpha \phi(s), \ s = \beta/\alpha, \) be a regular \((\alpha, \beta)\)-metric. Then \( F \) is of isotropic \( S \)-curvature if and only if it is of isotropic \( E \)-curvature.

**Proof.** First we remark that if \( \Xi = 0 \) then we get
\[ (b^2Q + s)\Phi = 0. \]
If \( b^2Q + s = 0 \), then we have \( \phi = \sqrt{s^2 - b^2} \) which is not regular. Thus \( \Phi = 0 \). In [7], it is proved that an \((\alpha, \beta)\)-metric is a Rientannian metric if and only if \( \Phi = 0 \). Thus \( \Xi = 0 \) characterizes Riemannian metrics in the class of \((\alpha, \beta)\)-metrics.

Now, we are going to show that for regular \((\alpha, \beta)\)-metrics the quantity \( \Xi \) is not constant. On contrary, suppose that \( \Xi = c(\text{constant}) \). Then by (4.7), we have
\[ (b^2Q + s)\Phi = c\Delta^2 \]
which yields
\[ [(b^2 - s^2)\phi' + s\phi']\Phi = c(\phi - s\phi')\Delta^2. \tag{4.8} \]
By the regularity of $F = \alpha \phi(s)$, $b := ||\beta||_\alpha$ and $|s| \leq b$, we have

$$\phi(s) > 0, \quad \phi(s) - s\phi'(s) > 0, \quad \forall |s| < b_0$$

Letting $s$ approximate

$$\frac{\phi(b) + \sqrt{\phi^2(b) + 4b^2\phi'^2(b)}}{2\phi'(b)}$$

in (4.8) yields $c = 0$. In this case, $F$ is Riemannian. Thus for regular Finsler metrics, $\Xi$ is not constant.

By Theorems 4.1 and 4.3, we conclude the following.

**Corollary 4.1.** Let $(G/H, F = \alpha \phi(\beta/\alpha))$ be an $n$-dimensional homogeneous $(\alpha, \beta)$-manifold, where $\alpha$ is a $G$-invariant Riemannian metric, and $\beta$ is a $G$-invariant 1-form on $G/H$. Then the following are equivalent:

(i) $F$ has isotropic mean Berwald curvature $E = (n + 1)/2cF^{-1}h$;

(ii) $F$ has isotropic $S$-curvature $S = (n + 1)cF$;

(ii) $F$ has isotropic $S$-curvature $S = (n + 1)cF + \eta$;

where $c = c(x)$ is a scalar function and $\eta = \eta_i(x)y^i$ is a 1-form on $G/H$. In this case, $S = 0$.

**Proof of Theorem 1.2:** Let $F := \alpha \phi(s)$, where $s = \beta/\alpha$ be an $(\alpha, \beta)$-metric on a manifold $M = G/H$ introduced in Theorem 4.1. Let us define $b_{i;j}$ by $b_{i;j} := db_i - b_j \theta^j_i$, where $\theta^i := dx^i$ and $\theta^j_i := \Gamma^j_{ik}dx^k$ denote the Levi-Civita connection form of $\alpha$. Let us define

$$r_{ij} := \frac{1}{2}(b_{i;j} + b_{j;i}), \quad s_{ij} := \frac{1}{2}(b_{i;j} - b_{j;i}), \quad s_i := b^j s_{ji},$$

where $b^i := a^{im}b_m$. By definition, the 1-form $\beta$ is parallel if and only if $r_{ij} = s_{ij} = 0$.

Now, we consider two cases as follows:

**Case (i): $F$ is non-Randers type metric.** In [5], Cheng proved that every $(\alpha, \beta)$-metric $F := \alpha \phi(s)$, $s = \beta/\alpha$, of non-Randers type, i.e., $\phi \neq c_1 \sqrt{1 + c_2 s^2 + c_3 s}$ for any constants $c_1 > 0$, $c_2$ and $c_3$, has isotropic $S$-curvature $S = (n + 1)cF$ if and only if $\beta$ satisfies

$$r_{ij} = 0, \quad s_i = 0. \quad (4.9)$$

In this case, $S = 0$, regardless of the choice of a particular $\phi = \phi(s)$. In [19] Li-Shen showed that an $(\alpha, \beta)$-metric $F := \alpha \phi(s)$, $s = \beta/\alpha$, of non-Randers type on a manifold $M$ of dimension $n \geq 3$ has vanishing J-curvature $J = 0$, if and only if $\beta$ satisfies

$$r_{ij} = k(b^2 a_{ij} - b_i b_j), \quad s_{ij} = 0. \quad (4.10)$$

where $k = k(x)$ is a number depending on $x$ and $\phi = \phi(s)$ satisfies

$$\Phi = \frac{\lambda}{\sqrt{b^2 - s^2}} \Delta^\frac{3}{2},$$
where $\lambda$ is a constant. By Lemma 3.4, (4.9), and (4.10), it follows that $\beta$ is parallel. In [23], Shen proved that a regular $(\alpha, \beta)$-metric is a Berwald metric if and only if its related 1-form $\beta$ is parallel. Then $F$ is a Berwald metric. Now, we have two subcases as follows:

Case (i)a: $K = 0$. By the Numata theorem, every Berwald metric with zero flag curvature is locally Minkowskian.

Case (i)b: $K < 0$. Suppose that $K < 0$ at a point $x \in M$. From (3.3) and the assumption $J = 0$, we get $R^i_m f^m I_i = 0$ which implies $I_i(x, y) = 0$ for all $y \in T_yM_0$. By Deicke’s theorem, $F$ is Riemannian.

Case (ii): $F$ is a Randers-type metric. Now, suppose that $F$ is a Randers-type metric. It is well-known that every Randers-type metric is $C$-reducible

$$C_y(u, v, w) = \frac{1}{n+1} \left\{ I_y(u)h_y(v, w) + I_y(v)h_y(u, w) + I_y(w)h_y(u, v) \right\}.$$  (4.11)

Taking a horizontal derivation of (4.11) yields

$$L_y(u, v, w) = \frac{1}{n+1} \left\{ J_y(u)h_y(v, w) + J_y(v)h_y(u, w) + J_y(w)h_y(u, v) \right\}.$$  (4.12)

By considering Lemma 3.4, the relation (4.12) implies that $L = 0$. On the other hand, we have $S = 0$, which yields $E = 0$. In [8], Crampin showed that every Landsberg metric with vanishing mean Berwald curvature is a Berwald metric. This completes the proof.

By considering Case (i)b in Theorem 1.2, we can conclude the following rigidity result that has been proved by Deng-Hu in [11].

Corollary 4.2. ([11]) Let $(M, F)$ be a homogeneous Randers space of Berwald type. If the flag curvature of $F$ is negative everywhere, then $F$ is a Riemannian metric.

Now, we consider homogeneous isotropic Berwald metrics and prove the following.

Corollary 4.3. Every homogeneous isotropic Berwald metric on a manifold $M$ of dimension $n \geq 3$ with non-positive flag curvature is Riemannian or locally Minkowskian.

Proof. Every isotropic Berwald metric has isotropic mean Berwald curvature. In [27], the authors proved that every homogeneous isotropic Berwald metric on a manifold $M$ of dimension $n \geq 3$ is a Berwald metric or Randers metric of Berwald-type. By Numata theorem every Berwald metric of vanishing flag curvature if $K = 0$ is locally Minkowskian. Also, if $K \neq 0$, then by the same method used in Case (i)b of Theorem 1.2, $F$ is Riemannian. This completes the proof.

The Douglas metrics are an extension of Berwald metrics introduced by Douglas as a projective invariant class of Finsler metrics. A Finsler metric is called a Douglas metric if

$$G^i = \frac{1}{2} \Gamma^i_{jk}(x)y^j y^k + P(x, y) y^i,$$

where $\Gamma^i_{jk} = \Gamma^i_{jk}(x)$ are scalar functions on $M$ and $P = P(x, y)$ is a homogeneous function of degree one with respect to $y$ on $TM_0$. For homogeneous Douglas metrics, we prove the following.
Corollary 4.4. Every homogeneous Douglas $(\alpha, \beta)$-metric with non-positive flag curvature is Riemannian or locally Minkowskian.

Proof. In [20], Liu-Deng proved that every homogeneous $(\alpha, \beta)$-metric is a Douglas metric if and only if it is a Berwald metric or a Douglas metric of Randers type. By the Numata theorem for Berwald metrics and Theorem 1.2, we get the proof. 

We deal with spherically symmetric Finsler metrics. A Finsler metric $F = F(x, y)$ on a domain $\Omega \subseteq \mathbb{R}^n$ is called spherically symmetric metric if it is invariant under any rotations in $\mathbb{R}^n$ (see [25]). It is proved that a Finsler metric $F$ on a convex domain $\Omega \subseteq \mathbb{R}^n$ is spherically symmetric if and only if there exists a positive function $\phi := \phi(r, u, v)$, such that $F(x, y) = \phi(|x|, |y|, \langle x, y \rangle)$, where

$$|x| = \sqrt{\sum_{i=1}^{n} (x^i)^2}, \quad |y| = \sqrt{\sum_{i=1}^{n} (y^i)^2}, \quad \langle x, y \rangle = \sum_{i=1}^{n} x^i y^i.$$

Here, we prove the following.

Corollary 4.5. Every homogeneous spherically symmetric Finsler metric with non-positive flag curvature and isotropic Berwald curvature is Riemannian or locally Minkowskian.

Proof. In [13], it is proved that every spherically symmetric Finsler metric $F = \phi(|x|, |y|, \langle x, y \rangle)$ on the $n$-ball $\mathbb{B}^n(r)$ with isotropic Berwald curvature is a Randers metric. By Theorem 1.2 we get the proof.

References

[1] P. L. Antonelli, Handbook of Finsler Geometry, Kluwer Academic Publishers, 2005.
[2] H. Akbar-Zadeh, Sur les espaces de Finsler à courbures sectionnelles constantes, Bull. Acad. Roy. Bel. Cl, Sci, 5e Série - Tome LXXXIV (1988), 281-322.
[3] D. Bao and C. Robles, Ricci and flag curvatures in Finsler geometry, in A sampler of Riemannian-Finsler Geometry, Cambridge University Press, 2004, 197-260.
[4] L. Berwald, On Cartan and Finsler Geometries, III, Two Dimensional Finsler Spaces with Rectilinear Extremal, Ann. of Math. 42(1941), 84-122.
[5] X. Cheng, The $(\alpha, \beta)$-metrics of scalar flag curvature, Differ. Geom. Appl. 35(2014), 361-369.
[6] X. Chen(g), X. Mo and Z. Shen, On the flag curvature of Finsler metrics of scalar curvature, J. London. Math. Soc. 68(2003), 762-780.
[7] X. Cheng, H. Wang and M. Wang, $(\alpha, \beta)$-metrics with relatively isotropic mean Landsberg curvature, Publ. Math. Debrecen, 72(2008), 475-485.
[8] M. Crampin, A condition for a Landsberg space to be Berwaldian, Publ. Math. Debrecen, 8111(2018), 1-13.
[9] S. Deng and Z. Hu, *Homogeneous Finsler spaces of negative curvature*, J. Geom. Phys, **57**(2007), 657-664.

[10] S. Deng and Z. Hu, *Curvatures of homogeneous Randers spaces*, Advances in Math. **240**(2013), 194-226.

[11] S. Deng and Z. Hu, *On flag curvature of homogeneous Randers spaces*, Canad. J. Math. **65**(2013), 66-81.

[12] S. Deng and X. Wang, *The S-curvature of homogeneous $(\alpha,\beta)$-metrics*, Balkan. J. Geom. Appl. **15**(2010), 47-56.

[13] E. Guo, H. Liu and X. Mo, *On spherically symmetric Finsler metrics with isotropic Berwald curvature*, Int. J. Geom. Meth. Mod. Phys. **10**(2013), 1350054 (13 pages).

[14] E. Heintze, *On homogeneous manifolds of negative curvature*, Math. Ann. **211**(1974), 23-34.

[15] Z. Hu and S. Deng, *Homogeneous Randers spaces with isotropic S-curvature and positive flag curvature*, Math. Z. **270**(2012), 989-1009.

[16] Z. Hu and S. Deng, *Killing fields and curvatures of homogeneous Finsler manifolds*, Publ. Math. Debrecen, **94**(2019), 215-229.

[17] L. Huang and Z. Shen, *A conclusive theorem on Finsler metrics of sectional flag curvature*, arXiv:1812.09608v1.

[18] D. Latifi and A. Razavi, *On homogeneous Finsler spaces*, Rep. Math. Phys. **57**(2006), 357-366.

[19] B. Li and Z. Shen, *On a class of weakly Landsberg metrics*, Sci. China Ser. A. **50**(4) (2007), 573-589.

[20] H. Liu and S. Deng, *Homogeneous $(\alpha,\beta)$-metrics of Douglas type*, Forum Math. **27**(2015), 3149-3165.

[21] X. Mo and Z. Shen, *On negatively curved Finsler manifolds of scalar curvature*, Canad. Math. Bull. **48**(2005), 112-120.

[22] B. Najafi and A. Tayebi, *On a class of isotropic mean Berwald metrics*, Acta. Math. Acad. Paedagogicae Nyiregyhaziensis, **32**(2016), 113-123.

[23] Z. Shen, *On a class of Landsberg metrics in Finsler geometry*, Canadian. J. Math. **61**(2009), 1357-1374.

[24] Z. I. Szabó, *Positive definite Berwald spaces. Structure theorems on Berwald spaces*, Tensor (N.S.), **35**(1981), 25-39.

[25] A. Tayebi, M. Bahadori and H. Sadeghi, *On spherically symmetric Finsler metrics with some non-Riemannian curvature properties*, J. Geom. Phys. **163**(2021), 104125.

[26] A. Tayebi and B. Najafi, *A class of homogeneous Finsler metrics*, J. Geom. Phys, **140**(2019), 265-270.
[27] A. Tayebi and B. Najafi, *On homogeneous isotropic Berwald metrics*, European J Math, https://doi.org/10.1007/s40879-020-00401-4.

[28] A. Tayebi and E. Peyghan, *On Douglas surfaces*, Bull. Math. Soc. Science. Math. Roumanie, Tome 55 (103), No 3, (2012), 327-335.

[29] A. Tayebi and M. Rafie Rad, *S-curvature of isotropic Berwald metrics*, Science in China, Series A: Math. 51(2008), 2198-2204.

[30] M. Xu and S. Deng, *Homogeneous (α, β)-spaces with positive flag curvature and vanishing S-curvature*, Nonlinear Analysis, 127(2015), 45-54.

Behzad Najafi
Department of Mathematics and Computer Sciences
Amirkabir University (Tehran Polytechnic)
Hafez Ave.
Tehran. Iran
Email: behzad.najafi@aut.ac.ir

Akbar Tayebi
Department of Mathematics, Faculty of Science
University of Qom
Qom. Iran
Email: akbar.tayebi@gmail.com