Explicit averages of square-free supported functions: beyond the convolution method

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Abstract

We give a general statement of the convolution method so that one can provide explicit asymptotic estimations for all averages of square-free supported arithmetic functions that have a sufficiently regular order on the prime numbers and observe how the nature of this method gives in general error term estimations of order $X^{-\delta}$, where $\delta$ belongs to an open real positive set $I$. In order to have a better error estimation, a natural question is whether or not we can achieve an error term of critical order $X^{-\delta_0}$, where $\delta_0$, the critical exponent, is the right hand endpoint of $I$. We reply positively to that question by presenting a new method that improves qualitatively almost all instances of the convolution method; now, the asymptotic estimation of averages of sufficiently regular square-free supported arithmetic functions can be given with its critical exponent and a reasonable explicit error constant. We illustrate this new method by analyzing a particular average related to the work of Ramaré–Akhilesh (2017), which leads to notable improvements when imposing non-trivial coprimality conditions.

1 Introduction

The convolution method was first investigated by Ramar in 1995, particularly in [15, Lemma 3.2], where it was given in a somewhat hidden version with respect to the one we present in this article. It is a technique that relies upon a convolution identity and helps obtaining explicit estimations of averages of arithmetic functions, under some conditions. It is particularly meaningful when these arithmetic functions are supported on the square-free numbers, having a sufficiently regular behavior on all large prime numbers.

While the convolution method provides the main term of a asymptotic expansion for the average of an arithmetic function with ease, it is at the remainder term where it shows its true potential, as it succeeds in giving a good enough estimation, explicit, for the error term: if the average is performed for the range $(0, X]$, where $X > 0$, then the convolution methods gives error term explicit estimations of magnitude $X^{-\delta}$ when $\delta$ belongs to a maximal real open and positive interval $I$.

Nevertheless, the nature of the convolution method does not allow one to obtain an error term estimation of magnitude $X^{-\delta_0}$ where $\delta_0$ is the right endpoint of $I$. Since it
is usually a subject of interest in the explicit theory of numbers to improve error term magnitudes of expressions of interest, it is thus natural to ask whether or not one can provide, necessarily by a different method, an error term of critical order \( \delta_0 \) so that the overall estimation is qualitatively improved.

We first present in §3 a general form of the convolution method involving sufficiently regular square-free supported functions, as shown in Theorem 3.2.1. As it relies upon some complex analytic facts, this method is related to a typical complex analytic approach for estimating the asymptotic expansion for the average of an arithmetic function by means of residue theory.

Our main result, presented in §4, differs from complex analysis. In §4.3, we see how the use of some very particular estimations given in §4.2 constitute the main ingredient to obtain reasonable explicit estimations of critical exponent in almost all cases where the convolution method may be applied. Indeed, since our technique also relies upon the convergence of infinite products, some extra conditions on the regularity of the arithmetic function that is being averaged are needed, as condition (A) in Theorem 4.3.1 tells, and therefore there is a small range of functions that are not considered in our improvements, namely when the values of \( \alpha \) and \( \beta \) defined in Theorem 4.3.1 have a difference of absolute value smaller than \( \frac{1}{2} \). However, as most of the applications we mention throughout this article do not involve that missing case, we then claim that every one of these ones can be improved up to its critical exponent.

Previous work towards the obtention of error terms of critical exponent can be found in \[18\] and \[19\], where a completely different approach is used, using some results known as the covering remainder lemma and the unbalanced Dirichlet hyperbola formula as well as strong explicit bounds on some summatory functions involving the Möbius functions that, unlike our case of study, do oscillate. Furthermore, it is important to point out that whereas a similar path to the mentioned ones could have been followed, these results consider specific properties of the functions that are being averaged and they are thus not easy to generalize to a broader class of functions. This is the reason why \[18\] Thm. 1.2 improves on the classic convolution method result presented in Corollary 3.2.2 (a) but still requires the convolution method to estimate related averages of less simple arithmetic functions; for example, with the result we present in Theorem 4.3.1 one can now immediately derive stronger estimations for \[18\] Lemmas 7.1, 7.2, 7.6, 7.7, 7.8, 7.9 that may lead to further improvements on the cited article of Ramar–Akhilesh. In that aspect, our result might help as a reference for further improvements on all the places where the convolution method has been employed.

As an application of our main result, we deduce how the improvement on the convolution method produces better savings on the error term constant of \( \sum_{\ell \leq X} \frac{\mu^2(\ell)}{\varphi(\ell)} \), \( X > 0, q \in \mathbb{Z}_{>0} \) than the one in \[18\] Thm. 1.1, when non-trivial coprimality conditions are introduced. This situation is examined in §4.4 and we have for instance the following result.

**Lemma.** Let \( X > 0 \), then

\[
\sum_{\ell \leq X} \frac{\mu^2(\ell)}{\varphi(\ell)} = \frac{\varphi(q)}{q} (\log (X) + a_2) + O^* \left( \frac{2.169 \sqrt{X}}{X} \right),
\]
2 Details and basic definitions

The $O$ and $O^*$ notation. We write $f(x) = O(g(x))$ as $x \to a$ ($a = \pm \infty$ is allowed), for a real valued function $g$ such that $g > 0$ in a neighborhood of $a$, and for a real or complex valued function $f$, we mean that there is an independent constant $C$, such that $|f(x)| \leq Cg(x)$ in that neighborhood. We write $f(x) = O^*(h(x))$, as $x \to a$ to indicate that $|f(x)| \leq h(x)$ in a neighborhood of $a$. Therefore, as $x \to a$, $f(x) = O(g(x))$ if and only if $f(x) = O^*(Cg(x))$ for some constant $C > 0$. In absence of definition of $a$, then $a$ corresponds to $\infty$, where a neighbourhood of $\infty$ corresponds to the set of $x$ bigger that any predetermined constant.

The Euler $\varphi$ and Kappa $\kappa$ functions. Let $s$ be any complex number. We define $\varphi_s : \mathbb{Z}_{>0} \to \mathbb{C}$ as $q \mapsto q^s \prod_{p|q} \left(1 - \frac{1}{p^s}\right)$ and $\kappa_s : \mathbb{Z}_{>0} \to \mathbb{C}$ as $q \mapsto q^s \prod_{p|q} \left(1 + \frac{1}{p^s}\right)$.

Computational details. Every constant in this text has been estimated using interval arithmetic. Early numerical analysis was carried out using the ARB implementation, under the SageMath commands RBF and RIF, implemented in Python. We decided, however, to use Platt’s implementation in C++, used for example in [13], as it provides results with double precision, when compared to ARB, and at higher performance and faster speed.

Throughout the calculations that we present in this article, we have set a precision order equal to $6 \times 10^9$ and run a .cpp script compiled with C++. We have also written a .ipynd script (compiled by SageMath) to verify some of our results.

3 A general version of the convolution method

In the convolution method, it is crucial to preserve regularity conditions, that is, conditions that do not impose specific ranges other than the variable itself being a positive integer, under, perhaps, some coprimality restrictions.

To put an example, when one carries out a summation on a variable $e \in \mathbb{Z}_{>0}$ such that $e \leq X$ for certain real number $X > 0$ and a positive integer $d$, it is often implicitly assumed that $\frac{X}{d} \geq 1$, so that the set $\{e \in \mathbb{Z}_{>0}, e \leq \frac{X}{d}\}$ is not empty. If $d$ is itself a variable, that means that we have the range condition $\{d \leq X\}$ on the variable $d$. Hence, if we are able to estimate asymptotically a summation on the variable $e \in \mathbb{Z}_{>0}$ such that $e \leq \frac{X}{d}$, regardless of whether or not an empty condition sum is performed, that is an empty sum, then the range condition on the variable $d$ will be absent.

3.1 Regularity conditions: estimating empty summations

The following lemma estimates asymptotically some sums even when they have an empty condition.
Lemma 3.1.1. Let $X > 0$ and $\alpha > 0$. If $0 < \delta \leq 1$, we have

$$
\sum_{n \leq X} \frac{1}{n^\alpha} = \log(X) + \gamma + O^\ast \left( \frac{\Delta^1_f}{X^\delta} \right); \tag{3.1.1}
$$

if $\max\{0, \alpha - 1\} < \delta \leq \alpha$ and $\alpha \neq 1$, we have

$$
\sum_{n \leq X} \frac{1}{n^\alpha} = \zeta(\alpha) - \frac{1}{(\alpha - 1)X^{\alpha - 1}} + O^\ast \left( \frac{\Delta^1_f}{X^\delta} \right), \tag{3.1.2}
$$

where

$$
\Delta^1_f = \max \left\{ \gamma, \frac{1}{\delta e^{x+1}} \right\},
$$

and for $\alpha \neq 1$,

$$
\Delta^n_f = \begin{cases} 
\max \left\{ \frac{1}{\delta}, \left( \frac{\left( \frac{\delta - \alpha + 1}{\zeta(\alpha)} \right)^{\delta - \alpha + 1}}{\delta e^{x+1}} \right), \zeta(\alpha) - \frac{1}{\alpha-1} \right\}, & \text{if } \alpha - 1 < \delta < \alpha, \\
\frac{1}{\delta}, \zeta(\alpha) - \frac{1}{\alpha-1}, & \text{if } \delta = \alpha.
\end{cases}
$$

Proof. By [18] Lemma 2.1 and [3] Lemma 2.9, for $X > 0$ we have

$$
\sum_{n \leq X} \frac{1}{n^\alpha} = \log(X) + \gamma + O^\ast \left( \frac{\gamma}{X^\delta} \right), \tag{3.1.3}
$$

$$
\sum_{n \leq X} \frac{1}{n^\alpha} = \zeta(\alpha) - \frac{1}{(\alpha - 1)X^{\alpha - 1}} + O^\ast \left( \frac{1}{2X^\alpha} \right), \tag{3.1.4}
$$

respectively. Thus, if $X \geq 1$, the result holds trivially as $\delta' \mapsto X^{\delta'}$ is increasing and $\delta < \alpha$. Otherwise, when $0 < X < 1$ the above summations are empty; write $X = \frac{1}{n}$ with $Y > 1$ and observe first that the function $f: Y \geq 1 \mapsto \frac{\log(Y)^{\gamma}}{Y^{\delta}}$ has a single critical point at $y_0 = e^{\frac{1}{\delta}+\gamma} > 1$ taking the value $f(y_0) = \frac{1}{\delta e^{\frac{1}{\delta}+\gamma}} > 0$. As $f(1) = -\gamma$ and $\lim_{Y \to \infty} f(Y) = 0$, $f$ is increasing in $[1, y_0]$ and decreasing in $[y_0, \infty)$, and hence $\sup_{\{Y > 1\}} |f(Y)| = \max \left\{ \gamma, \frac{1}{\delta e^{x+1}} \right\}$.

Secondly, by [12] Cor. 1.14], we have that $\zeta(\alpha) > \frac{1}{\alpha-1}$ and $\zeta(\alpha)/(\alpha - 1) > 0$ for all $\alpha \geq 0$ and $\alpha \neq 1$, therefore the function $g: Y > 0 \mapsto \frac{1}{\delta} \left( \zeta(\alpha) - \frac{1}{\alpha-1} \right)$ has a critical point $y_0$ satisfying $y_0^{\alpha - 1} = \left( \frac{\zeta(\alpha) - \frac{1}{\alpha-1}}{\delta} \right) > 0$, since $\delta > \alpha - 1$ and $\delta > 0$. In this case, we have that $\lim_{Y \to \infty} g(Y) = 0$ and thus $|g|$ is decreasing in $[y_0, \infty)$. We conclude then that

$$
\max_{[y_0, \infty)} |g(Y)| = |g(y_0)|, \text{ where}
$$

$$
|g(y_0)| = \left( \frac{1}{\delta} \left( \frac{\delta - \alpha + 1}{\zeta(\alpha)} \right)^{\delta - \alpha + 1} \right) \zeta(\alpha) - \frac{1}{\alpha-1} \right\}
$$

If $y_0 \leq 1$, then $|g(1)| = g(1) \leq |g(y_0)|$ and $\sup_{\{Y > 1\}} |g(Y)| = g(1)$; otherwise, if $y_0 > 1$, as $g$ is also monotonic between 1 and $y_0$, we derive that $\sup_{\{Y > 1\}} |g(Y)| = \max\{g(1), |g(y_0)|\}$, which gives us the desired result. \[\blacksquare\]
It is important to point out that in case that \( \alpha > 1 \), it would have been possible to give an error term expression even if \( \delta = \alpha - 1 > 0 \). Moreover, if \( \delta < \alpha - 1 \) then \(|g| \) would have been unbounded in \([1, \infty)\).

On the other hand, as pointed out at the beginning of §3, it is essential to have an estimation of the above summations when they have actually an empty condition, that is when \( X \in (0, 1) \). Indeed, this will provide regularity for some sum conditions during the proof of Theorem 3.2.1 that otherwise would impose some variables to be at least 1 and some sums to be non-empty. It should be expected, though, that the fact of imposing regularity conditions, or rather asking for estimations of sums up to the variable \( X \) with \( X > 0 \), will worsen a bit the constants on the involved error terms; for instance, when \( \alpha = 1 \) and when we are restricted to the range \( X \geq 1 \), the value of \( \gamma = 0.57721 \ldots \) given in (3.1.3) can be improved to \( 2(\log(2) + \gamma - 1) = 0.54072 \ldots \) (refer to [18, Lemma 2.1] ). Moreover, concerning only to upper bounds, one can obtain even better, as \( \sum_{\alpha \leq X} \frac{1}{n} - \log(X) - \gamma \leq \frac{1}{2} = 0.5 \) for \( X \geq 1 \) (see [5, Lemma 2.8] for example).

### 3.2 The convolution method

The following theorem will help us to state Corollary 3.2.2. Although inspired by [15, Lemma 3.2], it is presented in a much general framework, in an attempt to understand and deduce with ease the order of averages of sufficiently regular square-free supported arithmetic functions. By sufficiently regular, we mean an arithmetic function having a specific constant dominant term on all sufficiently large prime numbers. As it turns out, it is precisely the regularity of an arithmetic function that helps to derive the asymptotic expansion of its average under the method of convolution.

**Theorem 3.2.1.** Let \( X > 0 \) be a real number and \( q \) a positive integer. Consider a multiplicative function \( f : \mathbb{Z}^+ \to \mathbb{C} \) such that

(i) \( f(p) = \frac{1}{p^\beta} + O \left( \frac{1}{p} \right) \), for every sufficiently large prime number \( p \) coprime to \( q \),

(ii) \( f(p) \neq -1 \) for any prime number

where \( \beta > \alpha > \frac{1}{2} \). Then for any real number \( \delta > 0 \) such that \( \max\{0, \alpha - 1\} < \delta < \alpha - \frac{1}{2} \) we have the estimation

\[
\sum_{\ell \leq X \atop (\ell,q)=1} \mu^2(\ell)f(\ell) = F^q_{\alpha}(X) + O^{\star} \left( \Delta_{\alpha}^{\delta} \frac{K_{\alpha-\delta}(q)}{q^{\alpha-\delta}} \times \frac{H^q_f(-\delta)}{X^\delta} \right),
\]

where

\[
F^q_{\alpha}(X) = \frac{H^q_f(0) \zeta(\alpha) \varphi_{\alpha}(q)}{q^\alpha} - \frac{H^q_f(1-\alpha) \varphi(q)}{(\alpha - 1)q} \times \frac{1}{X^{\alpha - 1}}, \quad \text{if } \alpha \neq 1 \left( \text{and } \alpha > \frac{1}{2} \right),
\]

\[
F^q_1(X) = \frac{H^q_f(0) \varphi(q)}{q} \left( \log(X) + T^q_f + \gamma + \sum_{p \mid q} \frac{\log(p)}{p - 1} \right),
\]

\[
T_j^\delta = \sum_{p \neq q} \frac{\log(p)(1 - (p - 2)f(p))}{(f(p) + 1)(p - 1)},
\]

\[
\Delta_j^\delta \text{ being defined as in Lemma 3.2.1, and where } H_j^\theta := \{s \in \mathbb{C}, \Re(s) > \frac{1}{2} - \alpha \} \rightarrow \mathbb{C} \text{ is an analytic function satisfying}
\]

\[
H_j^\theta(s) = \prod_{p \neq q} \left(1 - \frac{1 - f(p)p^\alpha}{p^s + \alpha} - \frac{f(p)}{p^{2s} + \alpha}\right) = \sum_{d = 1} h_j^\theta(d) \frac{\mu(d)}{d^{s+\alpha}},
\]

\[
H_j^\theta(s) = \prod_{p \neq q} \left(1 + \frac{|1 - f(p)p^\alpha|}{p^{s} + \alpha} + \frac{|f(p)|}{p^{2s} + \alpha}\right) = \sum_{d = 1} h_j^\theta(d) \frac{\mu(d)}{d^{s + \alpha}}.
\]

Proof. By (i), the Dirichlet series \(D_j^\theta\) associated with \(\ell \mapsto \mu^2(\ell)f(\ell)\mathbb{1}_q(\ell)\), where \(\mathbb{1}_q\) is defined as the multiplicative function \(\ell \mapsto 1\) for every prime \(\ell\), converges absolutely for any \(s \in \mathbb{C}\) such that \(\Re(s) > 1 - \alpha\). Thus, in the set \(\{s \in \mathbb{C}, \Re(s) > 1 - \alpha\}\), the equality

\[
D_j^\theta(s) = \sum_{(\ell,q)=1} \frac{\mu^2(\ell)f(\ell)}{\ell^s} = \prod_{p \neq q} \left(1 + \frac{f(p)}{p^s}\right)
\]

holds and \(s \mapsto \zeta(s + \alpha)\) can be expressed by an Euler product. For any \(s\) such that \(\Re(s) > 1 - \alpha\), we have then

\[
\frac{D_j^\theta(s)}{\zeta(s + \alpha)} = \prod_{p \neq q} \left(1 + \frac{f(p)}{p^s}\right) \left(1 - \frac{1}{p^s + \alpha}\right) \prod_{p \neq q} \left(1 - \frac{1}{p^s + \alpha}\right)
\]

\[
= \frac{\zeta(s + \alpha)}{q^{s + \alpha}} \prod_{p \neq q} \left(1 - \frac{1 - f(p)p^\alpha}{p^s + \alpha} - \frac{f(p)}{p^{2s} + \alpha}\right) = \frac{\zeta(s + \alpha)}{q^{s + \alpha}} \times H_j^\theta(s).
\]

Also by (i), we have that \(\frac{1 - f(p)p^\alpha}{p^{s} + \alpha} = O\left(\frac{1}{p^{s(1/2 + \alpha)}}\right)\) and \(\frac{f(p)}{p^{s} + \alpha} = O\left(\frac{1}{p^{s(1/2 + \alpha)}}\right)\). Since \(\beta > \alpha > \frac{1}{2}\), we have that \(\max\{1 - \alpha - \beta, \frac{1}{2} - \alpha\} = \frac{1}{2} - \alpha\) and hence \(H\) can be extended analytically from \(\{s \in \mathbb{C}, \Re(s) > 1 - \alpha\}\) onto \(\{s \in \mathbb{C}, \Re(s) > \frac{1}{2} - \alpha\}\). Further, as \(0 > \frac{1}{2} - \alpha\), \(H_j^\theta(0)\) exists and is non-zero, since each factor defining it can be factorized as \((1 + f(p))(1 - \frac{1}{p^{\alpha}})\), \(\alpha\) is different from 0, and by (ii), \(f(p) \neq -1\) for every prime \(p\).

Observe now that the formal equality \(D_j^\theta(s) = H_j^\theta(s) \times \prod_{p \neq q} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \ldots\right)\) hides the convolution product

\[
\ell^\alpha \mu^2(\ell)f(\ell) \mathbb{1}_{(\ell,q)=1}(\ell) = (h_j^\theta \ast \mathbb{1}_q)(\ell) = \sum_{d | \ell} h_j^\theta(d) \mathbb{1}_q\left(\frac{\ell}{d}\right),
\]

where \(h\) is a multiplicative function defined on the prime numbers as

\[
h_j^\theta(p) = (f(p)p^\alpha - 1) \times \mathbb{1}_q(p),
\]

\[
h_j^\theta(p^2) = -f(p)p^\alpha \times \mathbb{1}_q(p),
\]
\[ h^\alpha_j(p^k) = 0, \quad k > 2. \]

Therefore, from (3.2.2), we conclude that
\[
\sum_{\ell \leq X} \mu^2(\ell)f(\ell) = \sum_{\ell \leq X} \left( \frac{h_j^\alpha(1)}{\ell^\alpha} \right) = \sum_d \frac{h_j^\alpha(d)}{d^\alpha} \sum_{e \leq d} \frac{1}{e^\alpha}
\]
\[
= \sum_d \frac{h_j^\alpha(d)}{d^\alpha} \sum_{e \leq d} \frac{1}{e^\alpha} \sum_{d'|e,d'|q} \mu(d') = \sum_d \frac{h_j^\alpha(d)}{d^\alpha} \sum_{d'|q} \mu(d') \sum_{e \leq d} \frac{1}{e^\alpha} \quad (3.2.4)
\]

where there is no upper bound conditions on the variables \( d \) and \( d' \) present in the outer sums above, their being encoded by the innermost sum of (3.2.4), which, in order to continue our analysis, we must estimate regardless of whether or not it is empty: Lemma 3.1.1 allow us to handle this situation.

Hence, if \( \alpha \neq 1 \), as \( \max\{0, \alpha - 1\} < \delta < \alpha - \frac{1}{2} < \alpha \), we derive that the second sum in (3.2.4) can be expressed as
\[
\sum_{d'|q} \frac{\mu(d')}{d'^\alpha} \sum_{e \leq \frac{d'}{d''}} \frac{1}{e^\alpha} = \sum_{d'|q} \frac{\mu(d')}{d'^\alpha} \left( \zeta(\alpha) - \frac{(dd')^{\alpha-1}}{(\alpha-1)X^{\alpha-1}} + O^* \left( \Delta_\alpha^\delta (dd')^\delta \right) \right)
\]
\[
= \frac{\zeta(\alpha)\varphi_\alpha(q)}{q^\alpha} - \frac{\varphi(q)}{(\alpha-1)q} \times \frac{d^{\alpha-1}}{X^{\alpha-1}} + O^* \left( \Delta_\alpha^\delta \kappa_{\alpha-\delta}(q) \times \frac{d^\delta}{X^\delta} \right). \quad (3.2.5)
\]

Similarly, if \( \alpha = 1 \), as \( 0 < \delta < \frac{1}{2} \), we derive that
\[
\sum_{d'|q} \frac{\mu(d')}{d'^\alpha} \sum_{e \leq \frac{d'}{d''}} \frac{1}{e^\alpha} = \sum_{d'|q} \frac{\mu(d')}{d'^\alpha} \left( \log \left( \frac{X}{dd'} \right) + \gamma + O^* \left( \Delta_1^\delta (dd')^\delta \right) \right)
\]
\[
= \frac{\varphi_\alpha(q)}{q^\alpha} \left( \log \left( \frac{X}{d} \right) + \gamma \right) - \sum_{d'|q} \frac{\mu(d') \log(d')}{d'^\alpha} + O^* \left( \Delta_1^\delta \kappa_{\alpha-\delta}(q) \times \frac{d^\delta}{X^\delta} \right)
\]
\[
= \frac{\varphi_\alpha(q)}{q^\alpha} \left( \log \left( \frac{X}{d} \right) + \gamma \right) + \sum_{p|q} \frac{\log(p)}{p^{\alpha-1}} + O^* \left( \Delta_1^\delta \kappa_{\alpha-\delta}(q) \times \frac{d^\delta}{X^\delta} \right), \quad (3.2.6)
\]

where we have used that
\[
- \sum_{d'|q} \frac{\mu(d') \log(d')}{d'^\alpha} = \left( \sum_{d'|q} \frac{\mu(d')}{d'^{\alpha+\alpha}} \right)' = \left( \frac{\varphi_\alpha(q)}{q^{\alpha+\alpha}} \right)' = \frac{\varphi_\alpha(q)}{q^\alpha} \sum_{p|q} \frac{\log(p)}{p^{\alpha-1}} \quad (3.2.7)
\]

Therefore, from (3.2.4), the sum \( \sum_{\ell \leq X} \mu^2(\ell)f(\ell) \) can be estimated either as
\[
\sum_d \frac{h_j^\alpha(d)}{d^\alpha} \left( \zeta(\alpha)\varphi_\alpha(q) \right) = \frac{\varphi(q)}{(\alpha-1)q} \times \frac{d^{\alpha-1}}{X^{\alpha-1}} + O^* \left( \Delta_\alpha^\delta \kappa_{\alpha-\delta}(q) \times \frac{d^\delta}{X^\delta} \right)
\]

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\[
\begin{align*}
H'_j(0) &= H'_j(0) - \frac{\zeta(\alpha) \varphi_{\alpha}(q)}{q^\alpha} \left( \varphi(q) \right) \left( 1 - \frac{\varphi(q)}{\alpha - 1} q^\alpha \right) \left( 1 - \frac{H'_j(1 - \alpha)}{X^{\alpha - 1}} \right) + O^* \left( \frac{\Delta_\alpha^d \kappa_{\alpha - \delta}(q)}{q^{\alpha - \delta}} \times \frac{\mathcal{P}_{\alpha}(-\delta)}{X^\delta} \right), \tag{3.2.8}
\end{align*}
\]

if \( \alpha \neq 1 \) and \( \delta \in (\max\{0, \alpha - 1\}, \alpha - \frac{1}{2}) \), by using Corollary 3.2.2, or as

\[
\begin{align*}
\sum_d h'_j(d) \left( \frac{\varphi_{\alpha}(q)}{q^\alpha} \left( \log \left( \frac{X}{d} \right) + \gamma + \sum_{p^s q} \frac{\log(p)}{p^{s/\alpha} - 1} \right) + O^* \left( \frac{\Delta_\alpha^d \kappa_{\alpha - \delta}(q)}{q^{\alpha - \delta}} \times d^\delta \right) \right) \\
= H'_j(0) \frac{\varphi_{\alpha}(q)}{q^\alpha} \left( \log(X) + H'_j(0) \frac{H'_j(0)}{H'_j(0)} + \gamma + \sum_{p^s q} \frac{\log(p)}{p^{s/\alpha} - 1} \right) + O^* \left( \frac{\Delta_\alpha^d \kappa_{\alpha - \delta}(q)}{q^{\alpha - \delta}} \times \frac{\mathcal{P}_{\alpha}(-\delta)}{X^\delta} \right), \tag{3.2.9}
\end{align*}
\]

if \( \alpha = 1 \) and \( \delta \in (0, \alpha - \frac{1}{2}) \), by using Corollary 3.2.3. Finally, observe that \( H'_j(1 - \alpha) \) and \( \mathcal{P}_{\alpha}(-\delta) \) are well-defined, as \( \min\{1 - \alpha, -\delta\} > \frac{1}{2} - \alpha \), that \( H'_j(0) \neq 0 \) and that \( -\sum_d h'_j(d) \log(d) = H'_j(0) \). The result is thus obtained by noticing that

\[
\begin{align*}
\frac{H'_j(0)}{H'_j(0)} &= \left( \prod_{p^s q} \left( 1 - \frac{1 - f(p)p^\alpha}{p^{s/\alpha} - 1} \right) \right) \sum_{p^s q} \frac{\log(p)(1 - f(p)p^\alpha + 2f(p))}{(f(p) + 1)(p^{s/\alpha} - 1)}. \tag{3.2.10}
\end{align*}
\]

\textbf{Corollary 3.2.2.} Let \( X > 0 \) and \( q \in \mathbb{Z}_{>0} \). The following estimations hold

\[
\begin{align*}
(a) \quad & \sum_{\ell \leq X} \frac{\mu^2(\ell)}{\varphi(\ell)^{\alpha}} = \frac{\varphi(q)}{q} \left( \log(X) + a_q \right) + O^* \left( \frac{7.36 \times \mathcal{A}_q}{X^\pi} \right), \tag{3.2.10} \\
(b) \quad & \sum_{\ell \leq X} \frac{\mu^2(\ell)}{\ell^\alpha} = \frac{6}{\pi^2} \frac{q}{\kappa(q)} \left( \log(X) + b_q \right) + O^* \left( \frac{2.554 \times \mathcal{B}_q}{X^\pi} \right), \tag{3.2.11}
\end{align*}
\]

where

\[
\mathcal{A}_q = \prod_{p^s q} \left( 1 - \frac{1 - f(p)p^\alpha - 2}{(p - 1)p^{s/\alpha} + p^{s/\alpha} + 1} \right), \quad \mathcal{B}_q = \prod_{p^s q} \left( 1 + \frac{p^{s/\alpha} - 1}{p^{s/\alpha} + 1} \right),
\]

and

\[
\begin{align*}
a_q &= \sum_{p} \frac{\log(p)}{p(p - 1)} + \gamma + \sum_{p^s q} \frac{\log(p)}{p}, \quad \sum_{p} \frac{\log(p)}{p(p - 1)} + \gamma = 1.33258228 \ldots, \\
b_q &= \sum_{p} \frac{2\log(p)}{p^{s/\alpha} - 1} + \gamma + \sum_{p^s q} \frac{\log(p)}{p + 1}, \quad \sum_{p} \frac{2\log(p)}{p^{s/\alpha} - 1} + \gamma = 1.71713766 \ldots.
\end{align*}
\]
Proof. For the case (a) (respectively (b)), apply Theorem 3.2.1 with \( f(p) = \frac{1}{\varphi(p)} = \frac{1}{p-1} \) (respectively \( f(p) = \frac{1}{p} \)), \( \alpha = 1 \) and \( 0 \leq \delta = \frac{1}{2} < \frac{1}{2} \).

The infinite products that participate in the main and error terms as well as the infinite summation that participates in the main term can be estimated by using a rigorous implementation of interval arithmetic, and some techniques for accelerating convergence.

Remarks. Condition \( \alpha > \frac{1}{2} \) in Theorem 3.2.1 is necessary. Nonetheless, we can derive an analogous result for any multiplicative arithmetic function \( f \) satisfying the conditions of Theorem 3.2.1 with \( \alpha \leq \frac{1}{2} \), by means of a summation by parts. In this instance, there will not be any secondary term appearing and the error term magnitude will be \( O(X^{1-\alpha-\delta}) \) for any \( \delta \in (0, \frac{1}{2}) \).

Furthermore, it is enough to state condition (i) of Theorem 3.2.1 implicitly, unlike for condition (A) in Theorem 4.3.1, meaning that it is sufficient to know that there exist \( \alpha \) such that \( \lim_{p \to \infty} \frac{f(p) - \frac{1}{p}}{H_q f \left( -\delta \right)} = 0 \).

Upon having Theorem 3.2.1 at our disposal, the asymptotic estimation of averages \( \sum_{\ell \leq X} \mu^2(\ell) f(\ell) \) satisfying conditions of that theorem becomes an automatized, but not uninteresting task, that involves each time a choice of parameters: a value for \( \delta \) and a precision value in order to obtain a rigorous estimation of some infinite products.

In general, we have freedom to choose the error term parameter \( \delta \) described in §3 but some of them are not optimal. For instance, if \( \alpha = 1 \), then in terms of Theorem 3.2.1 and Lemma 3.1.1, \( \Delta^1 \to \infty \) as \( \delta \to 0^+ \). Since \( \prod_f^\delta (-\delta) \) converges, that makes the expression \( \Delta^1 \prod_f^\delta (-\delta) \) tending to \( \infty \) as well, thus not providing a numerical acceptable value. On the other hand, when \( \delta \to \frac{1}{2}^- \), the infinite product given by \( \prod_f^\delta (-\delta) \) tends to \( \infty \), whereas \( \Delta^1 \to \Delta^1 \), thus bounded, so that one also derives that the expression \( \Delta^1 \prod_f^\delta (-\delta) \) becomes too big to be practical. The search looks for a value of \( \delta \) not too close to the boundaries of \((0, \frac{1}{2})\); and in almost all cases it seems acceptable to set \( \delta = \frac{1}{3} \); indeed, with respect to a higher choice, we have a larger term but quantitatively more acceptable, as the constant that accompanies it is rather small in comparison to the corresponding one for smaller magnitudes.

A natural question is then whether or not we can derive an estimation, mandatorily with a different method, of exponent \( \delta = \frac{1}{2} \), if \( \alpha = 1 \) or \( \delta = \alpha - \frac{1}{2} \) if \( \alpha > \frac{1}{2} \), \( \alpha \neq 1 \), or, by summation by parts (see Theorem 4.3.2), of exponent \( \frac{1}{2} - \alpha \), if \( \alpha \leq \frac{1}{2} \). The answer to that question is given in [13]; it is positive and it constitutes our main result; we provide in addition explicit estimations for those critical exponents.

Out of the results above, the sum 3.2.10 is classical and it has been thoroughly studied by Ramaré and Akhilesh in [18], by Ramar in [19, Thm. 3.1], [15, Lemma 3.4] and given in our simpler form by Helfgott in [10, §6.1.1].
4 Improvements on the convolution method

During the proof of Theorem 3.2.1 it was crucial to have an empty sum estimation for the inner sum given in (3.2.4) so that, thanks to the regularity on the variable $d$ we find convergent main and error term coefficients, as shown in (3.2.8) and (3.2.9).

This general idea misses the fact that the function $h_q^f$ defined in (3.2.3) vanishes on all non cube-free numbers, and that the particular function $h_q^f : p, (p, q) = 1 \mapsto \frac{1}{p^\alpha}$, with $\alpha > \frac{1}{2}$, satisfies $h_q^f(p) = 0$. Moreover, the fact that that particular function is meaningful only on the square of the prime numbers, will allow us to achieve the critical exponent $\delta = \frac{1}{2}$, if $\alpha = 1$ or $\delta = \alpha - \frac{1}{2}$, if $\alpha \neq 1$ and $\alpha > \frac{1}{2}$.

4.1 Background

In order to continue our analysis, we recall some bounds that will be useful throughout this section.

\[
\sup_{\{X \geq 1573\}} \frac{1}{\sqrt{X}} \left| \sum_{\ell \leq X, (\ell, 2) = 1} \mu^2(\ell) - \frac{4}{\pi^2} X \right| \leq \frac{9}{70} \quad \text{[10, Lemma 5.2]}, \tag{4.1.1}
\]

\[
\sup_{\{X \geq 1\}} \sqrt{X} \left\{ \sum_{\ell \leq X} \frac{\mu^2(\ell)}{\ell} - \frac{6}{\pi^2} (\log(X) + b_1) \right\} \leq 0.43 \quad \text{[19, Cor. 1.2]}, \tag{4.1.2}
\]

where $b_1$ is defined as in Lemma 4.2.1.

4.2 A particular case

Let us see how we can improve the estimation (b) given in Corollary 3.2.2.

\textbf{Lemma 4.2.1.} Let $X > 0$. Then

\[
\sum_{\ell \leq X} \frac{\mu^2(\ell)}{\ell} = \frac{6}{\pi^2} (\log(X) + b_1) + O^* \left( \frac{1.044}{\sqrt{X}} \right), \tag{4.2.1}
\]

\[
\sum_{\ell \leq X, (\ell, 2) = 1} \frac{\mu^2(\ell)}{\ell} = \frac{4}{\pi^2} (\log(X) + b_2) + O^* \left( \frac{\sqrt{2}}{\varphi(2)} \frac{0.232}{\sqrt{X}} \right), \tag{4.2.2}
\]

where $b_1 = \gamma + \sum_p \frac{2 \log(p)}{p^2 - 1} = 1.71713766 \ldots$, $b_2 = b_1 + \frac{\log(2)}{3} = 1.94818672 \ldots$

\textbf{Proof.} Equation (3.2.11) gives the main term of (4.2.2) and from that, we can conclude by summation by parts that for all $X \geq 1$, $\sum_{\ell \leq X, (\ell, 2) = 1} \frac{\mu^2(\ell)}{\ell}$ equals

\[
\frac{4}{\pi^2} (\log(X) + b_2) + \left( \sum_{\ell \leq X, (\ell, 2) = 1} \mu^2(\ell) - \frac{4}{\pi^2} X \right) \frac{1}{X} - \int_X^{\infty} \left( \sum_{\ell \leq t, (\ell, 2) = 1} \mu^2(\ell) - \frac{4}{\pi^2} t \right) \frac{dt}{t^2},
\]

10
so that, by (4.1.1), we conclude that
\[
\sum_{\ell \leq X} \frac{\mu^2(\ell)}{\ell} = \frac{4}{\pi^2} (\log(X) + b_2) + O^* \left( \frac{27}{70} \frac{1}{\sqrt{X}} \right), \quad \text{if } X \geq 1573,
\]
where \(27/70 = 0.385\ldots\). We further verify by interval arithmetic that
\[
\sup_{\{1 \leq \ell \leq [1573]\}} \sqrt{X} \left| \sum_{\ell \leq X} \frac{\mu^2(\ell)}{\ell} - \frac{4}{\pi^2} (\log(X) + b_2) \right| \leq 0.407
\]
the above upper bound being almost achieved when \(X \to 3^-\). Hence, by using (4.1.2), when \(v \in \{1, 2\}\), we have the bounds
\[
\sup_{\{X \geq 1\}} \sqrt{X} \left| \sum_{\ell \leq X} \frac{\mu^2(\ell)}{\ell} - \frac{v}{\kappa(\ell)^2} \frac{6}{\pi^2} (\log(X) + b_v) \right| \leq \begin{cases} 0.43, & \text{if } v = 1, \\ 0.407, & \text{if } v = 2, \end{cases} \quad (4.2.3)
\]
In order to conclude the result, it is sufficient to obtain bounds for (4.2.3) when \(X \in (0, 1)\), in which case the above summation vanishes. By defining \(Y = \frac{1}{\pi} > 1\) and \(t_v : Y \mapsto \frac{6v(\log(Y) + b_v)}{\kappa(v)\pi^2\sqrt{Y}}\), we need to find \(\sup_{Y > 1} |t_v(Y)|\). By calculus, the function \(t_v\) has a critical point at \(y_0 = e^{2v+b_v}\), with value \(t_v(y_0) = \frac{12v}{\kappa(v)\pi^2e^{1+2v}}\), and it is monotonic in \([1, y_0]\) and in \([y_0, \infty)\). As \(\lim_{Y \to \infty} t_v(Y) = 0\) and \(t_v(y_0) > 0\), we conclude that \(t_v\) is decreasing in \([y_0, \infty)\). Similarly, as \(t_v(1) = -\frac{6v\phi_v}{\kappa(v)e^{2}} < 0\), \(t_v\) is increasing in \([1, y_0]\). Therefore
\[
\sup_{\{0 < X < 1\}} \sqrt{X} \left| \sum_{\ell \leq X} \frac{\mu^2(\ell)}{\ell} - \frac{v}{\kappa(\ell)^2} \frac{6}{\pi^2} (\log(X) + b_v) \right| = \max\{|t_v(1)|, |t_v(y_0)|\} = \frac{6v\phi_v}{\kappa(v)e^{2}}
\]
\[
= \begin{cases} 1.044, & \text{if } v = 1, \\ 0.79, & \text{if } v = 2. \end{cases} \quad (4.2.4)
\]
Finally, whenever either \(v = 1\) or \(v = 2\), the constant in the error term is obtained by taking the maximum between the bounds (4.2.3) and (4.2.4) and then multiplying it by \(\varphi_\alpha^{(v)}(\varphi_i^v)\).

**Lemma 4.2.2.** Let \(X > 0\) and \(\alpha > \frac{1}{2}\). If \(\alpha \neq 1\), then
\[
\sum_{\ell \leq X} \frac{\mu^2(\ell)}{\ell^\alpha} = \frac{\zeta(\alpha)}{\zeta(2\alpha)} - \frac{6}{(\alpha - 1)\pi^2} \frac{1}{X^{\alpha - 1}} + O^* \left( \frac{E^{(1)}_\alpha}{X^{\alpha - \frac{1}{2}}} \right), \quad (4.2.5)
\]
\[
\sum_{\ell \leq X} \frac{\mu^2(\ell)}{(\ell^2)^\alpha} = \frac{2^\alpha}{(2^\alpha + 1)} \frac{\zeta(\alpha)}{(\alpha - 1)\pi^2} - \frac{4}{(\alpha - 1)\pi^2} \frac{1}{X^{\alpha - 1}} + O^* \left( \frac{\sqrt{2} E^{(2)}_\alpha}{\varphi_\alpha^{(2)} X^{\alpha - \frac{1}{2}}} \right), \quad (4.2.6)
\]

\[11\]
where, for \( v \in \{1, 2\} \), we have

\[
E_{\alpha}^{(v)} = \max \left \{ D_v \left ( 1 + \frac{|\alpha - 1|}{\alpha - \frac{1}{2}} \right ), \frac{\varphi_{\ell,v}^{(v)}}{v} \left | \kappa_\alpha(v) \zeta(2\alpha) - \frac{v}{\kappa(v)} \frac{6}{(\alpha - 1)^2} \right | \right \}
\]

\[
\varphi_{\ell,v}(v) = \frac{\alpha - 1}{v^{\frac{1}{2}} (\alpha - \frac{1}{2}) \left ( \frac{3\kappa_\alpha(v)\zeta(2\alpha)}{(\alpha - \frac{1}{2}) v^{\alpha - 1} \kappa(v)^2 \zeta(\alpha)(\alpha - 1)} \right )^{\frac{1}{4}}}
\]

and

\[D_1 = 0.43, \quad D_2 = 0.12.\]

If \( X \geq 1 \), we can replace \( E_{\alpha}^{(v)} \) by \( D_v \left ( 1 + \frac{|\alpha - 1|}{\alpha - \frac{1}{2}} \right ) \).

Proof. If \( X \geq 1 \), by summation by parts, we can write \( \sum_{\ell \leq X} \frac{\mu^2(\ell)}{\ell} \) as

\[
\left ( \sum_{\ell \leq X} \frac{\mu^2(\ell)}{\ell} \right ) \frac{v}{\kappa(v)} \frac{6}{(\alpha - 1)^2} \left ( \log(X) + b_v \right ) \frac{1}{X^{\alpha - 1}} - \frac{v}{\kappa(v)} \frac{6}{(\alpha - 1)^2} \frac{1}{X^{\alpha - 1}} + (\alpha - 1) \frac{1}{\alpha} \int_1^X \left ( \sum_{\ell \leq t} \frac{\mu^2(\ell)}{\ell} - \frac{v}{\kappa(v)} \frac{6}{(\alpha - 1)^2} (\log(t) + b_v) \right ) \frac{dt}{t^{\alpha}} + \frac{v}{\kappa(v)} \frac{6(b_\alpha(\alpha - 1) + 1)}{\pi^2(\alpha - 1)} = \frac{v^\alpha}{\kappa_\alpha(v) \zeta(2\alpha)}. \]

Further, by equation (4.2.8), we conclude that \( \sum_{\ell \leq X} \frac{\mu^2(\ell)}{\ell} \) is equal to

\[
\frac{v^\alpha}{\kappa_\alpha(v) \zeta(2\alpha)} - \frac{v}{\kappa(v)} \frac{6}{(\alpha - 1)^2} X^{\alpha - 1} + O^* \left ( \sqrt{\varphi_{\ell,v}^{(v)}} \left ( \frac{1}{X^{\alpha - \frac{1}{2}}} + \frac{|\alpha - 1|}{\alpha - \frac{1}{2}} \right \int_X^\infty \frac{dt}{t^{\alpha + \frac{1}{2}}} \right ) \right )
\]

\[
= \frac{v^\alpha}{\kappa_\alpha(v) \zeta(2\alpha)} - \frac{v}{\kappa(v)} \frac{6}{(\alpha - 1)^2} X^{\alpha - 1} + O^* \left ( \sqrt{\varphi_{\ell,v}^{(v)}} \left ( 1 + \frac{|\alpha - 1|}{\alpha - \frac{1}{2}} \right ) \frac{1}{X^{\alpha - \frac{1}{2}}} \right ), \tag{4.2.8}
\]

for all \( X \geq 1 \), where \( D_1 = 0.43 \) and \( \frac{\varphi_{\ell,v}^{(v)}}{\sqrt{2}} \leq 0.407 \leq D_2 = 0.12. \)

Suppose now that \( X \in (0, 1) \). Define \( g : X > 0 \rightarrow \frac{v^{\alpha - 1} \kappa_\alpha(v) \zeta(\alpha)(\alpha - 1)^2}{6\kappa_\alpha(v) \zeta(2\alpha)} X^{-\frac{1}{2}} - \sqrt{X} \).

We have by (12) Cor. 1.14 that \( 1 < \zeta(\alpha)(\alpha - 1) < \alpha \). If \( \alpha > 1 \), we derive that \( \frac{\zeta(\alpha)(\alpha - 1)}{\zeta(2\alpha)} > \frac{1}{\alpha} \). As \( \frac{v^{\alpha - 1} \kappa_\alpha(v)}{\kappa_\alpha(v)} = \frac{1 + \frac{1}{\alpha}}{1 + \frac{1}{\alpha}} > 1 \) we conclude that \( g(1) > 0 \) and \( g \) has a critical point \( x_0 \).
satisfying \( 0 < x_0^{\alpha - 1} = \frac{3\kappa_\alpha(v)(2\alpha)}{(\alpha - \frac{1}{2})\nu^{\alpha-1}\kappa_\nu\pi^2|\zeta(\alpha)(\alpha - 1)|} < 1 \), with value \( g(x_0) = \frac{1 - \alpha}{\alpha - \frac{3}{2}} \sqrt{x_0} < 0 \).

As \( g(0) = 0 \), we conclude that if \( \alpha > 1 \), then \( \sup_{0 < x < 1} g(X) = \max\{g(1), g(x_0)\} \).

On the other hand, if \( \frac{1}{2} < \alpha < 1 \), then \( 2\alpha - 1 < 1, \zeta(\alpha)(\alpha - 1) < \alpha < 1 \), and \( \frac{\nu^{\alpha-1}\kappa(v)}{\kappa_\alpha(v)} < 1 \), so that \( g(1) < 0 \). Moreover, the critical point \( x_0 \) of \( g \) satisfies \( x_0^{1-\alpha} < 1 \), so that \( x_0 < 1 \), and \( g(x_0) > 0 \). Therefore, if \( \frac{1}{2} < \alpha < 1 \), then \( \sup_{0 < x < 1} |g(X)| = \max\{|g(1)|, |g(x_0)|\} \).

All in all, we derive

\[
\sup_{0 < x < 1} X^{\alpha - \frac{1}{2}} \left| \sum_{\ell \leq X \atop (\ell, v) = 1} \frac{\mu^2(\ell)}{\ell^\alpha} - \frac{\nu^\alpha}{\kappa_\alpha(v) \zeta(2\alpha)} + \frac{v}{\kappa(v)} \frac{6}{(\alpha - 1)\pi^2} X^{\alpha - 1} \right| = \frac{v}{\kappa(v)} \frac{6}{(\alpha - 1)\pi^2} \max\{|g(1)|, |g(x_0)|\},
\]

where,

\[
\frac{\varphi_\pm(v)}{\sqrt{v}} \frac{6}{\kappa(v) |\alpha - 1|\pi^2} |g(1)| = \frac{\varphi_\pm(v)}{\sqrt{v}} \frac{v^\alpha}{\kappa_\alpha(v) \zeta(2\alpha)} - \frac{v}{\kappa(v)} \frac{6}{(\alpha - 1)\pi^2} \quad (4.2.9)
\]

\[
\frac{\varphi_\pm(v)}{\sqrt{v}} \frac{6}{\kappa(v) |\alpha - 1|\pi^2} |g(x_0)| = \frac{\varphi_\pm(v)}{\sqrt{v}} \frac{v^\alpha}{\alpha - \frac{1}{2}} \left( \frac{3\kappa_\alpha(v)\zeta(2\alpha)}{(\alpha - \frac{1}{2}) \nu^{\alpha-1}\kappa_\nu\pi^2|\zeta(\alpha)(\alpha - 1)|} \right)^{\frac{1}{2}} \quad (4.2.10)
\]

The result is obtained by defining \( E_\alpha^{(v)} \), \( v \in \{1, 2\} \) as the maximum between \( D_v \left( 1 + \frac{|\alpha - 1|}{\alpha - \frac{1}{2}} \right) \) and the bounds presented in (4.2.9) and (4.2.10).

Define the constants given in Lemma 4.2.1 as \( C_1 = 1.044 \) and \( C_2 = 0.232 \). We can also derive the following.

**Lemma 4.2.3.** Let \( X > 0, \alpha > \frac{1}{2} \) and \( q \in \mathbb{Z}_{>0} \). Then

\[
\sum_{\ell \leq X \atop (\ell, q) = 1} \frac{\mu^2(\ell)}{\ell^\alpha} = \frac{q}{\kappa(q) \pi^2} (\log(X) + b_q) + O^* \left( \frac{\sqrt{q}}{\varphi_\pm(q)} \frac{C_1 \prod_{2|q} \frac{C_1}{\sqrt{X}}}{\sqrt{X}} \right),
\]

where \( b_q \) is defined in Lemma 3.2.3 and

\[
\sum_{\ell \leq X \atop (\ell, q) = 1} \frac{\mu^2(\ell)}{\ell^\alpha} = \frac{q^\alpha}{\kappa_\alpha(q) \zeta(2\alpha)} - \frac{q}{\kappa(q) (\alpha - 1)\pi^2} X^{\alpha - 1} + O^* \left( \frac{\sqrt{q}}{\varphi_\pm(q)} \frac{E_\alpha^{(1)} \prod_{2|q} \frac{E_\alpha^{(2)}}{\sqrt{X}}}{X^{\alpha - \frac{1}{2}}} \right),
\]

if \( \alpha \neq 1 \), where \( E_\alpha^{(v)} \), \( v \in \{1, 2\} \), is defined as in Lemma 4.2.2.
Proof. Proceed as in [12, Lemma 2.17]. Define $D_r = \{ p \text{ prime}, p|r \} \subset \mathbb{Z}_{\geq 0}$. Consider $v \in \{1,2\}$ and write $q = v^k r, k \in \mathbb{Z}_{>0}$, with $(v, r) = 1$ (where, if $v = 1$, then $k = 0$). Then for all $s \in \mathbb{C}$ such that $\Re(s) > 1 - \alpha$, we have the identity

$$
\sum_{(\ell, q)=1} \frac{\mu^2(\ell)}{\ell^{s+\alpha}} = \prod_{p|r} \left(1 + \frac{1}{p^{s+\alpha}}\right)^{-1} \times \sum_{(\ell, v)=1} \frac{\mu^2(\ell)}{\ell^{s+\alpha}} = \sum_{d \in D_r} \lambda(d) \int_{e^\alpha}^{\infty} \sum_{(e,v)=1} \frac{\mu^2(e)}{e^{s+\alpha}},
$$

where $\lambda$ corresponds to the Liouville function: the completely multiplicative function taking the value $-1$ at every prime number. Hence

$$
\sum_{\ell \leq X} \frac{\mu^2(\ell)}{\ell^s} = \sum_{d \in D_r} \frac{\lambda(d)}{d^s} \sum_{e \leq X \atop (e,v)=1} \frac{\mu^2(e)}{e^{\alpha}},
$$

which, as in Lemma 3.1.11 does not require the condition $\{d \leq X\}$. We are considering thus an infinite range of values of $d$ for the above outer sum, which can be estimated as long as the inner sum is expressed asymptotically with an error term valid even when it has an empty condition plus the fact that the series of error terms for this expression, formed by the outer sum, converges.

If $\alpha = 1$, by using Lemma [1.2.1] in [1.2.11], we derive the same main term as the one given in Corollary 3.2.2 (b), but a better error term magnitude, since $\sum_{\ell \leq X} \frac{\mu^2(\ell)}{\ell^s}$ can be written as

$$
\sum_{d \in D_r} \frac{\lambda(d)}{d^s} \left( \frac{v}{\pi^2 \kappa(v)} \left( \log \left( \frac{X}{d} \right) + b_v \right) + O^* \left( \frac{\sqrt{v}}{\varphi(v)} \frac{C_v \sqrt{d}}{\sqrt{X}} \right) \right)
$$

$$
= \frac{vr}{\kappa(v)} \frac{6}{\pi^2} \left( \log(X) + b_v \right) - \frac{v}{\kappa(v)} \frac{6}{\pi^2} \sum_{d \in D_r} \frac{\lambda(d) \log(d)}{d} + O^* \left( \frac{\sqrt{v}}{\varphi(v)} \frac{C_v \sqrt{d}}{\sqrt{X}} \right)
$$

$$
= \frac{q}{\kappa(q)} \frac{6}{\pi^2} \left( \log(X) + b_q \right) + O^* \left( \frac{\sqrt{v}}{\sqrt[4]{(q)}} \frac{C_v \prod_{p|q} \frac{C_q}{p}}{\sqrt{X}} \right),
$$

where we have used that

$$
\sum_{d \in D_r} \frac{-\lambda(d) \log(d)}{d} = \frac{r}{\kappa(r)} \left( \sum_{d \in D_r} \frac{\lambda(d)}{d^s} \right)_{s=1}^{r} \times \left( \sum_{d \in D_r} \frac{\lambda(d)}{d^s} \right)_{s=1}^{r} = \frac{r}{\kappa(r)} \sum_{p|r} \frac{\log(p)}{p+1},
$$

and that $\frac{v}{\kappa(v)} \frac{\sqrt{v}}{\varphi(v)} = \frac{\sqrt{v}}{\sqrt[4]{(q)}} \sum_{p|v} \frac{\log(p)}{p+1} + \sum_{p|r} \frac{\log(p)}{p+1} = \sum_{p|q} \frac{\log(p)}{p+1}.$
Finally, if \( \alpha \neq 1 \), then by using Lemma 4.2.2 and by noticing that \( \frac{(\alpha r)^{\alpha}}{\kappa(r)} \) we derive that \( \sum_{\ell \leq X} \frac{\mu^2(\ell)}{\ell^2} \) can be expressed as

\[
\sum_{d \in D_d} \frac{\lambda(d)}{d^a} \left( \frac{v^\alpha}{\kappa(a)} \zeta(\alpha) - v \frac{6}{\kappa(v) (\alpha - 1) \pi^2 X^{\alpha - 1}} + O^* \left( \frac{1}{X^{1 - \frac{1}{\alpha}}} \right) \right) \]

which, again, has the expected main term according to Theorem 3.2.1 but an error term of lower magnitude.

Let us recall that the requirement of the empty sum estimation, as in Lemma 3.1.1, worsens a bit the error term constants with respect to the ones under condition \( X \geq 1 \), say, as shown in Lemmas 4.2.1 and 4.2.2, but we gain regularity in our expressions in the variable \( d \). It is precisely that regularity that allows us to derive the coprimality restrictions products in a simpler manner: for example, we derive immediately that \( \sum_{d \leq X} \frac{\lambda(d)}{d} = \frac{\lambda(\alpha)}{\pi(r)} \) whereas condition \( \frac{\Lambda}{\alpha} \geq 1 \) would have imposed us to analyze \( \sum_{d \leq X} \frac{\lambda(d)}{d} \), or, rather, \( \sum_{d \geq X} \frac{\lambda(d)}{d} \). This last observation is key for the work carried out in [16] and [19].

**Corollary 4.2.4.** Let \( X > 0 \). Then

\[
\sum_{\ell \leq X} \frac{\mu^2(\ell)}{\ell^2} = \frac{q^a}{\kappa(a)} \zeta(\alpha) - \frac{q}{\kappa(a)} \frac{1}{\pi^2 X^{\alpha - 1}} + O^* \left( \frac{1}{X^{1 - \frac{1}{\alpha}}} \right) \]

\[
= \frac{q}{\kappa(a)} \frac{1}{\pi^2 X^{\alpha - 1}} + O^* \left( \frac{1}{X^{1 - \frac{1}{\alpha}}} \right) \]

**Proof.** By applying Lemma 4.2.3 with \( \alpha = 2 \), we have

\[
\sum_{\ell \leq X} \frac{\mu^2(\ell)}{\ell^2} = \frac{q^2}{\kappa(a)} \zeta(\alpha) - \frac{q}{\kappa(a)} \frac{1}{\pi^2 X^{\alpha - 1}} + O^* \left( \frac{1}{X^{1 - \frac{1}{\alpha}}} \right)
\]

where, for \( v \in \{1, 2\} \), we have

\[
E_2(v) = \max \left\{ \frac{5}{3} \frac{\varphi_2(v)}{v^2} \left| \zeta(\alpha) - \frac{v^2}{\kappa(a)} \right| \frac{6}{\kappa(a)} \frac{1}{\pi^2 X^{\alpha - 1}} \right. \right. \]

\[
\left. \left. \varphi_2(v) \frac{2}{v} \left( \frac{2k_2(v) \zeta(4)}{v \kappa(v) \pi^2 \zeta(2)} \right)^2 \right\}\right) \]

\[
\leq \begin{cases} 0.912, & \text{if } v = 1, \\ 0.238, & \text{if } v = 2, \end{cases}
\]
We obtain the result by observing that

$$\sum_{\ell > X \atop (\ell, q) = 1} \frac{\mu^2(\ell)}{\ell^2} = \frac{q^2}{\varphi_2(q)} \frac{\zeta(2)}{\zeta(4)} = \sum_{\ell \leq X \atop (\ell, q) = 1} \frac{\mu^2(\ell)}{\ell^2}.$$  

■

4.3 Achieving the critical exponent

We present here a new method to achieve the critical exponent of estimation of averages of the form studied in Theorem 3.2.1.

**Theorem 4.3.1.** Let $X > 0$ be a real number and $q$ a positive integer. Consider a multiplicative function $f : \mathbb{Z}^+ \to \mathbb{C}$ such that for every prime number $p$ satisfying $(p, q) = 1$

$$(A) \ f(p) = \frac{1}{p^\alpha} + O \left( \frac{1}{p^\beta} \right), \text{ where } \beta > \alpha > \frac{1}{2} \text{ and } \beta - \alpha > \frac{1}{2}.$$  

Then

$$\sum_{\ell \leq X \atop (\ell, q) = 1} \mu^2(\ell) f(\ell) = F'_\alpha(X) + O^* \left( \frac{P_1}{\sqrt{X}} \right), \text{ or}$$

$$\sum_{\ell \leq X \atop (\ell, q) = 1} \mu^2(\ell) f(\ell) = F'_{\alpha}(X) + O^* \left( \frac{P_\alpha}{X^{\alpha - \frac{1}{2}}} \right), \text{ if } \alpha \neq 1,$$

where $F'_\alpha(X)$ is defined as in Theorem 3.2.1 and, if $2 \nmid q$,

$$w_1^q = C_2 = 0.232, \quad w_\alpha^q = E_{\alpha}^{(2)},$$

whereas, if $2 \mid q$,

$$w_1^q = \left( \frac{\sqrt{2} - 1}{\sqrt{2} - 1 + |2f(2) - 1|} \right) \left( C_1 + \frac{|2f(2) - 1|}{\varphi(2)} \frac{C_2}{E_{\alpha}^{(2)}} \right),$$

$$w_\alpha^q = \left( \frac{\sqrt{2} - 1}{\sqrt{2} - 1 + |2^\alpha f(2) - 1|} \right) \left( E_{\alpha}^{(1)} + \frac{|2^\alpha f(2) - 1|}{\varphi(2)} \frac{E_{\alpha}^{(2)}}{E_{\alpha}^{(2)}} \right),$$

$C_v$ and $E_{\alpha}^{(v)}$, $v \in \{1, 2\}$, being defined in lemmas 4.2.1 and 4.2.2 respectively, and where, for any $\alpha > \frac{1}{2}$,

$$p_\alpha(q) = \prod_{p / q} \left( 1 + \frac{1 - |f(p)p^\alpha - 1|}{\sqrt{p} - 1 + |f(p)p^\alpha - 1|} \right), \quad P_\alpha = \prod_{p} \left( 1 + \frac{|f(p)p^\alpha - 1|}{\sqrt{p} - 1} \right).$$
Proof. Consider the arithmetic function $i_f$ defined on each prime as $p \mapsto f(p)p^\alpha - 1$. Observe that

$$
\sum_{\ell \leq X} \mu^2(\ell) f(\ell) = \sum_{\ell \leq X} \frac{\mu^2(\ell)}{\ell^\alpha} \times f(\ell) \ell^\alpha = \sum_{\ell \leq X} \frac{\mu^2(\ell)}{\ell^\alpha} \times \prod_{p|\ell} (1 + i_f(p))
$$

where we have not imposed upper bound conditions on the variable $d$.

In order to continue our estimation, we must be able to estimate both innermost summations in (4.3.1), regardless of whether or not they have an empty condition, so that their remainder terms converge upon effecting their corresponding outermost summations. As $\alpha > \frac{1}{2}$, this situation can be treated with the help of Lemma 4.2.3, we distinguish two cases.

i) $2|\alpha$. Then continuing from (4.3.1), along with the ideas of the proof of Theorem 3.2.1 and Lemma 4.2.3, it is not difficult to see, as expected, that for all $\alpha > \frac{1}{2}$, the main term of $\sum_{\ell \leq X} \mu^2(\ell)f(\ell)$ is $F_\alpha^2(X)$. As for the error term, if $\alpha = 1$ it corresponds to

$$
\sum_{d \leq q} \frac{\mu^2(d)|i_f(d)|}{d} \mathcal{O}^\ast \left( \frac{\sqrt{qd}}{\varphi^2(qd)} \frac{C_2 \sqrt{d}}{\sqrt{X}} \right) = \mathcal{O}^\ast \left( \frac{\sqrt{q}}{\varphi^2(qd)} \frac{\sqrt{d} \ X^{\alpha - \frac{1}{2}}}{(\sqrt{q} \ X^{\alpha - \frac{1}{2}})^{\alpha - 1}} \sum_{p|d} \frac{E^{(2)}(i_f)}{p} \right),
$$

whereas, if $\alpha \neq 1$, it corresponds to

$$
\sum_{d \leq q} \frac{\mu^2(d)|i_f(d)|}{d^{\alpha}} \mathcal{O}^\ast \left( \frac{\sqrt{qd}}{\varphi^2(qd)} \frac{E^{(2)}_\alpha \ d^{\alpha - \frac{1}{2}}}{X^{\alpha - \frac{1}{2}}} \right) = \mathcal{O}^\ast \left( \frac{\sqrt{q}}{\varphi^2(qd)} \frac{E^{(2)}_\alpha}{p} \right),
$$

where, for any $\alpha > \frac{1}{2}$,

$$
\sum_{p|d} \frac{E^{(2)}(i_f)}{p} = \frac{\sqrt{q}}{\varphi^2(qd)} \prod_{p|d} \left( 1 + \frac{|f(p)|}{\sqrt{p} - 1} \right) \sum_{p|d} \frac{E^{(2)}(i_f)}{p} = \mathcal{P}_\alpha = p_\alpha(q) \times \mathcal{P}_\alpha.
$$

Observe that $\mathcal{P}_\alpha$ converges thanks to condition (A), as $\frac{|f(p)|}{\sqrt{p} - 1} = \frac{f(p)p^\alpha - 1}{\sqrt{p} - 1} = O \left( \frac{1}{p^{\beta - \alpha + \frac{1}{2}}} \right)$

and $\beta - \alpha + \frac{1}{2} > 1$.

ii) $2 \nmid q$. Then we can write (4.3.1) as

$$
\sum_{d \leq q} \frac{\mu^2(d)|i_f(d)|}{d^{\alpha}} \sum_{\ell \leq X} \frac{\mu^2(e)}{e^{\alpha}} + \frac{i_f(2)}{2^\alpha} \sum_{d \leq q} \frac{\mu^2(d)|i_f(d)|}{d^{\alpha}} \sum_{\ell \leq X} \frac{\mu^2(e)}{e^{\alpha}}
$$

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Again, it is not difficult to see that, for any \( \alpha > \frac{1}{2} \), the main term of \( S^\alpha_q(X) + \frac{i f(2)}{2\alpha} T^\alpha_q(X) \) is \( E^\alpha_q(X) \), defined in Theorem 3.2.1. On the other hand, the error term of \( S^\alpha_q(X) + \frac{i f(2)}{2\alpha} T^\alpha_q(X) \), it can be expressed as

\[
\sum_{d \mid (p,q)=1} \frac{\mu^2(d) |i f(d)|}{d} O^* \left( \frac{\sqrt{q d}}{\varphi_2(qd)} C_1 \sqrt{d} \right) \left( \frac{\sqrt{q d}}{\varphi_2(qd)} \frac{C_2}{\sqrt{d}} \right) \]

\[
+ \frac{|i f(2)|}{2} \sum_{d \mid (p,q)=1} \frac{\mu^2(d) |i f(d)|}{d} O^* \left( \frac{\sqrt{q d}}{\varphi_2(qd)} C_1 \sqrt{d} \right) \left( \frac{\sqrt{q d}}{\varphi_2(qd)} \frac{C_2}{\sqrt{d}} \right) \]

\[
= O^* \left( \frac{\sqrt{q}}{\varphi_2(q)} \prod_{p \mid 2q} \left( 1 + \frac{|i f(p)|}{\sqrt{p}-1} \right) \left( C_1 + \frac{|i f(2)| C_2}{\varphi_2(2)} \right) \frac{1}{\sqrt{X}} \right) \]

\[
= O^* \left( p_1(q) \left( \frac{\sqrt{q}}{\sqrt{2} - 1 + |2 f(2) - 1|} \right) \left( C_1 + \frac{|2 f(2) - 1| C_2}{\varphi_2(2)} \right) \frac{P}{\sqrt{X}} \right) ;
\]

similarly, the error term of \( S^\alpha_q(X) + \frac{i f(2)}{2\alpha} T^\alpha_q(X) \) for \( \alpha \neq 1 \) can be written as

\[
O^* \left( \frac{\sqrt{q}}{\varphi_2(q)} \prod_{p \mid 2q} \left( 1 + \frac{|i f(p)|}{\sqrt{p}-1} \right) \left( E^{(1)}_\alpha + \frac{|i f(2)| E^{(2)}_\alpha}{\varphi_2(2)} \right) \frac{1}{X^{\alpha - \frac{1}{4}}} \right) \]

\[
= O^* \left( p_\alpha(q) \left( \frac{\sqrt{q}}{\sqrt{2} - 1 + |2^\alpha f(2) - 1|} \right) \left( E^{(1)}_\alpha + \frac{|2^\alpha f(2) - 1| E^{(2)}_\alpha}{\varphi_2(2)} \right) \frac{P}{X^{\alpha - \frac{1}{4}}} \right) ;
\]

whence the result.

Condition \( \alpha > \frac{1}{2} \) above is necessary, as we have used Lemma 4.2.3. Nonetheless, we can derive an analogous result for any multiplicative arithmetic function \( f \) such that \( f(p) = \frac{1}{p^\alpha} + O \left( \frac{1}{p^{\beta}} \right) \) with \( \alpha \leq \frac{1}{2} \) and \( \beta > \alpha \). Indeed, we can write \( f(p) = p^{1-\alpha} f'(p) \), where \( A(t) = \sum_{\ell \leq t} \mu^2(\ell) f'(\ell) \) can be estimated by Proposition 4.3.1 with \( \alpha = 1 \), \( \beta = 1 - \alpha + \beta \). We can then estimate \( \sum_{\ell \leq X} \mu^2(\ell) f(\ell) = \sum_{(\ell,q)=1} \mu^2(\ell) f'(\ell) \ell^{1-\alpha} \) by means of a summation by parts, obtaining the following result.

**Corollary 4.3.2.** Let \( X > 0 \) be a real number and \( q \) a positive integer. Consider a multiplicative function \( f : \mathbb{Z}^+ \to \mathbb{C} \) such that for every prime number \( p \) such that \((p,q) = 1\),

\[
(A)^* \frac{f(p)}{p^\alpha} + O \left( \frac{1}{p^{\beta}} \right), \quad \text{where } \alpha \leq \frac{1}{2} \text{ and } \beta - \alpha > \frac{1}{2}.
\]
Then
\[
\sum_{\ell \leq X \atop (\ell, q) = 1} \mu^2(\ell) f(\ell) = \frac{H_{p}(0) \varphi(q)}{(1 - \alpha)q} X^{1 - \alpha} + O^* \left( p'_1(q) \times \left( 1 + \frac{2 - 2\alpha}{1 - 2\alpha} \right) w'_1 X^{\frac{\alpha}{2} - \alpha} \right),
\]
where
\[
H_{p}(0) = \prod_{p \nmid q} \left( 1 - \frac{p^{1 - \alpha} - f(p)p + f(p)}{p^{2 - \alpha}} \right),
\]
\[
p'_1(q) = \prod_{p \nmid q} \left( 1 + \frac{1 - |f(p)p^\alpha - 1|}{\sqrt{p} - 1 + |f(p)p^\alpha - 1|} \right), \quad P'_1 = \prod_p \left( 1 + \frac{|f(p)p^\alpha - 1|}{\sqrt{p} - 1} \right),
\]
and
\[
w'_1 = \begin{cases} 
C_2 = 0.232, & \text{if } 2 \mid q, \\
\left( \frac{2 - 1}{\sqrt{2 - 1} + |2^\alpha f(2) - 1|} \right) \left( C_1 + \frac{|2^\alpha f(2) - 1|}{\sqrt{2 - 1}} \right), & \text{if } 2 \nmid q.
\end{cases}
\]

Concerning the error term in Corollary 4.3.2 in some particular cases one can do much better. For instance, it is known, by [10, Lemmas 5.1-5.2] that if \( f(p) = 1 \) and \( v \in \{1, 2\} \), we have that for any \( X > 0 \) that
\[
\sum_{\ell \leq X \atop (\ell, \omega) = 1} \mu^2(\ell) \frac{6}{\pi^2} \frac{v}{\kappa(v)} X + O^*(H_v \sqrt{X}),
\]
where
\[
H_v = \begin{cases} 
\sqrt{3} \left( 1 - \frac{6}{\pi^2} \right) & \text{if } v = 1, \\
1 - \frac{6}{\pi^2} & \text{if } v = 2,
\end{cases}
\]
whereas Corollary 4.3.1 provides only an explicit error term of the form
\[
O^* \left( \frac{\sqrt{7}}{\varphi^{\frac{1}{2}}(q)} \times 3.132 \sqrt{X} \right).
\]

4.4 Consequences

Lemma 4.4.1. Let \( X > 0 \), then
\[
\sum_{\ell \leq X \atop (\ell, q) = 1} \frac{\mu^2(\ell)}{\varphi(\ell)} = \frac{\varphi(q)}{q} (\log(X) + a_q) + O^* \left( \prod_{p \nmid q} \left( 1 + \frac{p - 2}{p^2 - p - \sqrt{p} + 2} \right) \times \frac{4.4 P_{2q}}{\sqrt{X}} \right),
\]
where \( a_q \) is defined in Corollary 4.3.1.
Proof. We already know the main term of the asymptotic expression of the above sum, thanks to Corollary 3.2.2 (a); obtaining it again from Theorem 4.3.1 is an exercise. On the other hand, by Theorem 4.3.1 with \( f(p) = \frac{1}{p-1}, \alpha = 1, \beta = 2, \) its error term can be expressed as \( O^* \left( p(q) \times \frac{w^q}{\sqrt{X}} \right) \), where
\[
p(q) = \prod_{p \mid q} \left( 1 + \frac{p - 2}{p^2 - p - \sqrt{p} + 2} \right), \quad P = \prod_p \left( 1 + \frac{1}{(p - 1)(\sqrt{p} - 1)} \right) \in [9.37522, 9.3753],
\]
\[
w^q = \begin{cases} 
0.231, & \text{if } 2 \mid q, \\
\left( 1 - \frac{1}{\sqrt{2}} \right) \left( C_1 + \frac{C_2}{\sqrt{2}} \right) = 0.469 \ldots, & \text{if } 2 \nmid q
\end{cases}
\]
\[
\leq 0.47 \prod_{2 \notmid q} 0.493,
\]
where \( C_v, v \in \{1, 2\} \), are defined in Lemma 4.2.1.

When there is no coprimality conditions, we have obtained an error constant equal to 4.4, that held under the condition \( X > 0 \). Ramar and Akhilesh in [18, Thm. 1.2] have given the constant 3.95 under the condition \( X \geq 1 \), later improved by Ramar himself in [19] to 2.44 under the condition \( X > 1 \). From these last two bounds, it is not difficult to extend the range of estimation to \( X > 0 \), as we have done for example throughout Lemma 3.1.1, and these bounds continue to be better than the value 4.4.

Nonetheless, the above lemma improve considerably [15 Thm. 1.1] when coprimality conditions given by \( q \geq 2 \) are involved. For example, we have
\[
2.169 \times \frac{p(2)}{2} \leq 2.169 \leq 4.955 \leq 5.9 \times j(2),
\]
\[
4.4 \times \frac{p(3)}{3} \leq 6.186 \leq 7.221 \leq 5.9 \times j(3),
\]
\[
4.4 \times \frac{p(5)}{5} \leq 6.621 \leq 7.679 \leq 5.9 \times j(5),
\]
\[
2.169 \times \frac{p(6)}{6} \leq 3.049 \leq 6.066 \leq 5.9 \times j(6),
\]
\[
2.169 \times \frac{p(10)}{10} \leq 3.263 \leq 6.451 \leq 5.9 \times j(10),
\]
\[
2.169 \times \frac{p(14)}{14} \leq 3.166 \leq 6.424 \leq 5.9 \times j(14),
\]
where \( j \) is the error term arithmetic function defined in [15 Thm. 1.1] as \( 2 \mapsto \frac{21}{25} \) and \( p \geq 3 \mapsto 1 + \frac{p - 2}{p^2 - \sqrt{p} + 1} \). Furthermore, the estimation given in Lemma 4.4.1 is better than the one in [15 Thm. 1.1] for all \( q = p \) prime. Indeed, we observe in (4.4.2) that it is better when \( p \in \{2, 3, 5\} \); now, since
\[
\frac{p - 2}{p^2 - p - \sqrt{p} + 2} \leq \frac{1}{\sqrt{p}} \quad \text{for all } p \geq 3,
\]
\[
\frac{p - 2}{p^2 - \sqrt{p} + 1} \geq \frac{1}{2\sqrt{p}} \quad \text{for all } p \geq 5,
\]
we have, for all \( p \geq 3 \), that
\[
4.4 \times p(p) \leq 4.4 \times \left( 1 + \frac{1}{\sqrt{p}} \right) \leq 5.9 \times \left( 1 + \frac{1}{2\sqrt{p}} \right) \leq 5.9 \times j(p),
\]
where...
whence the conclusion.

As a final remark, observe that that the main contribution to the product $P$ given in Lemma 4.4.1 is precisely when $p = 2$. This is the reason why, in the present work, we have distinguished if $q$ is either odd or even. Further, as the second main contribution to the product $P$ is given by its factor at $p = 3$ (the subsequent factors when $p > 3$ being rather small, as $\sqrt{\frac{q}{p}} - 1 < 1$), the interested reader may study the behavior of the error term bounds given in Theorem 4.3.1 and therefore the error term in Lemma 4.4.1 by distinguishing whether or not $(6, q) = 1$: this procedure will require an extension of Lemma 4.2.1 to the cases $(3, q) = 1$ and, by using the inclusion-exclusion principle, to the case $(6, q) = 1$; afterwards, the analysis will continue exactly as in the current version of Theorem 4.3.1.

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