TILING SYSTEMS AND HOMOLOGY OF LATTICES IN TREE PRODUCTS

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Abstract. Let $\Gamma$ be a torsion free cocompact lattice in $\text{Aut}(T_1) \times \text{Aut}(T_2)$, where $T_1, T_2$ are trees whose vertices all have degree at least three. The group $H_2(\Gamma, \mathbb{Z})$ is determined explicitly in terms of an associated 2-dimensional tiling system. It follows that under appropriate conditions the crossed product $C^*$-algebra $\mathcal{A}$ associated with the action of $\Gamma$ on the boundary of $T_1 \times T_2$ satisfies $\text{rank } K_0(\mathcal{A}) = 2 \cdot \text{rank } H_2(\Gamma, \mathbb{Z})$.

1. Introduction

This article is motivated by the problem of calculating the K-theory of certain crossed product $C^*$-algebras $\mathcal{A}(\Gamma, \partial \Delta)$, where $\Gamma$ is a higher rank lattice acting on an affine building $\Delta$ with boundary $\partial \Delta$. Here we examine the case where $\Delta$ is a product of trees. We determine the K-theory rationally, thereby proving some conjectures in [KR].

Let $T_1$ and $T_2$ be locally finite trees whose vertices all have degree at least three. Consider the direct product $\Delta = T_1 \times T_2$ as a two dimensional cell complex. Let $\Gamma$ be a discrete subgroup of $\text{Aut}(T_1) \times \text{Aut}(T_2)$ which acts freely and cocompactly on $\Delta$. Associated with the action $(\Gamma, \Delta)$ is a tiling system whose set of tiles is the set $\mathfrak{R}$ of “directed” 2-cells of $\Gamma \backslash \Delta$. There are vertical and horizontal adjacency rules $tHs$ and $tVs$ between tiles $t, s \in \mathfrak{R}$ illustrated below. Precise definitions will be given in Section 2.

\[
\begin{array}{c}
| & | \\
\text{t} & \text{s} \\
| & | \\
\text{s} & \text{t}
\end{array}
\]

There are homomorphisms $T_1, T_2 : \mathbb{Z} \mathfrak{R} \to \mathbb{Z} \mathfrak{R}$ defined by

$T_1 t = \sum_{tHs} s, \quad T_2 t = \sum_{tVs} s$.

Consider the homomorphism $\mathbb{Z} \mathfrak{R} \to \mathbb{Z} \mathfrak{R} \oplus \mathbb{Z} \mathfrak{R}$ given by

\[
\begin{pmatrix}
T_1 - I \\
T_2 - I
\end{pmatrix} : t \mapsto (T_1 t - t) \oplus (T_2 t - t).
\]

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The main result of this article is the following Theorem, which is formulated more precisely in Theorem 1.1.

**Theorem 1.1.** There is an isomorphism

\[ H_2(\Gamma, \mathbb{Z}) \cong \ker \left( \frac{T_1 - I}{T_2 - I} \right). \]

The proof of (1) is elementary, but care is needed because the right hand side is defined in terms of “directed” 2-cells rather than geometric 2-cells. A square complex \( X \) is VH-T if every vertex link is a complete bipartite graph and if there is a partition of the set of edges into vertical and horizontal, which agrees with the bipartition of the graph on every link [BM]. The universal covering space \( \Delta \) of a VH-T complex \( X \) is a product of trees \( T_1 \times T_2 \) and the fundamental group \( \Gamma \) of \( X \) is a subgroup of \( \text{Aut}(T_1) \times \text{Aut}(T_2) \) which acts freely and cocompactly on \( T_1 \times T_2 \). Conversely, every finite VH-T complex arises in this way from a free cocompact action of a group \( \Gamma \) on a product of trees. Recall that a discrete group which acts freely on a CAT(0) space is necessarily torsion free.

The group \( \Gamma \) acts on the (maximal) boundary \( \partial \Delta \) of \( \Delta \), which is the set of chambers of the spherical building at infinity, endowed with an appropriate topology [KR]. This boundary may be identified with a direct product of Gromov boundaries \( \partial T_1 \times \partial T_2 \). The boundary action \((\Gamma, \partial \Delta)\) gives rise to a crossed product \( C^\ast\)-algebra \( \mathcal{A}(\Gamma, \partial \Delta) = C_\mathbb{C}(\partial \Delta) \rtimes \Gamma \) as described in [KR].

If \( p \) is prime then \( \text{PGL}_2(\mathbb{Q}_p) \) acts on its Bruhat-Tits tree \( T_{p+1} \), which is a homogeneous tree of degree \( p + 1 \). If \( p, \ell \) are prime then the group \( \text{PGL}_2(\mathbb{Q}_p) \times \text{PGL}_2(\mathbb{Q}_\ell) \) acts on the \( \Delta = T_{p+1} \times T_{\ell+1} \). Let \( \Gamma \) be a torsion free irreducible lattice in \( \text{PGL}_2(\mathbb{Q}_p) \times \text{PGL}_2(\mathbb{Q}_\ell) \). Then \( \mathcal{A}(\Gamma, \partial \Delta) \) is a higher rank Cuntz-Krieger algebra and fits into the general theory developed in [RS1, RS2]. In particular, it is classified up to isomorphism by its K-theory. It is a consequence of Theorem 1.1 (see Section 5) that

\[ \text{rank } K_0(\mathcal{A}(\Gamma, \partial \Delta)) = 2 \cdot \text{rank } H_2(\Gamma, \mathbb{Z}). \]

This proves a conjecture in [KR]. The normal subgroup theorem [Mar, IV, Theorem (4.9)] implies that \( H_1(\Gamma, \mathbb{Z}) \) is a finite group. Equation (2) can therefore be expressed as

\[ \chi(\Gamma) = 1 + \frac{1}{2} \text{rank } K_0(\mathcal{A}(\Gamma, \partial \Delta)). \]

One easily calculates that \( \chi(\Gamma) = \frac{(p-1)(\ell-1)}{4} |X^0| \), where \( |X^0| \) is the number of vertices of \( X \). Therefore the rank of \( K_0(\mathcal{A}(\Gamma, \partial \Delta)) \) can be expressed explicitly in terms of \( p, \ell \) and \( |X^0| \). Examples are constructed in [M3, Section 3], where \( p, \ell \equiv 1 \pmod{4} \) are two distinct primes.
2. Products of trees and their automorphisms.

If $T$ is a tree, there is a type map $\tau$ defined on the vertex set of $T$, taking values in $\mathbb{Z}/2\mathbb{Z}$. Two vertices have the same type if and only if the distance between them is even. Any automorphism $g$ of $T$ preserves distances between vertices, and so there exists $i \in \mathbb{Z}/2\mathbb{Z}$ (depending on $g$) such that $\tau(gv) = \tau(v) + i$, for every vertex $v$.

Suppose that $\Delta$ is the 2 dimensional cell complex associated with a product $T_1 \times T_2$ of trees. Let $\Delta^k$ denote the set of $k$-cells in $\Delta$ for $k = 0, 1, 2$. The 0-cells are vertices and the 2-cells are geometric squares. Denote by $u = (u_1, u_2)$ a generic vertex of $\Delta$. There is a type map $\tau$ on $\Delta^0$ defined by

$$\tau(u) = (\tau(u_1), \tau(u_2)) \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$  

Any 2-cell $\delta \in \Delta^2$ has one vertex of each type. For every $g \in \text{Aut}T_1 \times \text{Aut}T_2$ there exists $(k, l) \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ such that, for each vertex $u$,

$$\tau(gu) = (\tau(u_1) + k, \tau(u_2) + l).$$  

Let $\Gamma < \text{Aut}T_1 \times \text{Aut}T_2$ be a torsion free discrete group acting cocompactly on $\Delta$. Then $X = \Gamma \backslash \Delta$ is a finite cell complex with universal covering $\Delta$. Let $X^k$ denote the set of $k$-cells of $X$ for $k = 0, 1, 2$.

The first step is to formalize the notion of a directed square in $X$. We modify the terminology of [BM, Section 1], in order to fit with [RS1, RS2, KR]. Let $\sigma$ be a model typed square with vertices $00, 01, 10, 11$, as illustrated in Figure 2. Assume that the vertex $ij$ of $\sigma$ has type $\tau(ij) = (i, j) \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

![Figure 2. The model square $\sigma$.](image_url)

The vertical and horizontal reflections $v, h$ of $\sigma$ are the involutions satisfying $v(00) = 01, v(10) = 11, h(00) = 10, h(01) = 11$. An isometry $r : \sigma \rightarrow \Delta$ is said to be type rotating if there exists $(k, l) \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ such that, for each vertex $ij$ of $\sigma$

$$\tau(r(ij)) = (i + k, j + l).$$  

Let $R$ denote the set of type rotating isometries $r : \sigma \rightarrow \Delta$. If $g \in \text{Aut}T_1 \times \text{Aut}T_2$ and $r \in R$ then it follows from (3) that $g \circ r \in R$. If $\delta^2 \in \Delta^2$ then for each $(k, l) \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ there is a unique $r \in R$ such that $r(\sigma) = \delta^2$ and $r(00)$ has type $(k, l)$. Therefore each geometric square $\delta^2 \in \Delta^2$ is the image of each of the four elements of $\{r \in R; r(\sigma) = \delta^2\}$ under the map $r \mapsto r(\sigma)$. The next lemma records this observation.
Lemma 2.1. The map \( r \mapsto r(\sigma) \) from \( R \) to \( \Delta^2 \) is 4-to-1.

Let \( \mathcal{R} = \Gamma \backslash R \) and call \( \mathcal{R} \) the set of directed squares of \( X = \Gamma \backslash \Delta \). There is a commutative diagram

\[
\begin{array}{ccc}
R & \xrightarrow{r \mapsto r(\sigma)} & \Delta^2 \\
\downarrow & & \downarrow \\
\mathcal{R} & \xrightarrow{\eta} & X^2
\end{array}
\]

where the vertical arrows represent quotient maps and \( \eta \) is defined by \( \eta(\Gamma r) = \Gamma r(\sigma) \). The next result makes precise the fact that each geometric square in \( X^2 \) corresponds to exactly four directed squares.

Lemma 2.2. The map \( \eta : \mathcal{R} \to X^2 \) is surjective and 4-to-1.

Proof. Fix \( \delta^2 \in R \). By Lemma 2.1 the set \( \{ r \in R : r(\sigma) = \delta^2 \} = \{ r_1, r_2, r_3, r_4 \} \) contains precisely 4 elements. Since \( \Gamma \) acts freely on \( \Delta \), the set \( \{ \Gamma r_1, \Gamma r_2, \Gamma r_3, \Gamma r_4 \} \subset \mathcal{R} \) also contains precisely four elements, each of which maps to \( \Gamma \delta^2 \) under \( \eta \). Now suppose that \( \eta(\Gamma r) = \Gamma \delta^2 \) for some \( r \in R \). Then \( \gamma r(\sigma) = \delta^2 \) for some \( \gamma \in \Gamma \). Thus \( \gamma r \in \{ r_1, r_2, r_3, r_4 \} \) and \( \Gamma r \in \{ \Gamma r_1, \Gamma r_2, \Gamma r_3, \Gamma r_4 \} \). This proves that \( \eta \) is 4-to-1. \( \square \)

The vertical and horizontal reflections \( v, h \) of the model square \( \sigma \) act on \( \mathcal{R} \) and generate a group \( \Sigma = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \) of symmetries of \( \mathcal{R} \). The \( \Sigma \)-orbit of each \( r \in \mathcal{R} \) contains four elements. Choose once and for all a subset \( \mathcal{R}^+ \subset \mathcal{R} \) containing precisely one element from each \( \Sigma \)-orbit. The map \( \eta \) restricts to a 1-1 correspondence between \( \mathcal{R}^+ \) and the set of geometric squares \( X^2 \). For each \( \phi \in \Sigma - \{ 1 \} \), let \( \mathcal{R}^\phi \) denote the image of \( \mathcal{R}^+ \) under \( \phi \). Then \( \mathcal{R} \) may be expressed as a disjoint union

\[ \mathcal{R} = \mathcal{R}^+ \cup \mathcal{R}^v \cup \mathcal{R}^h \cup \mathcal{R}^{vh}. \]

Now we formalize the notion of horizontal and vertical directed edges in \( X \). Consider the two directed edges \([00, 10], [00, 01]\) of the model square \( \sigma \).

![Directed edges of the model square \( \sigma \).](image)

Let \( A \) be the set of type rotating isometries \( r : [00, 10] \to \Delta \), and let \( B \) be the set of type rotating isometries \( r : [00, 01] \to \Delta \). There is a natural 2-to-1 mapping \( r \mapsto \text{range} \ r \), from \( A \cup B \) onto \( \Delta^1 \). Let \( \mathfrak{A} = \Gamma \backslash A \) and \( \mathfrak{B} = \Gamma \backslash B \). Call \( \mathfrak{A}, \mathfrak{B} \) the sets of horizontal and vertical directed edges of \( X = \Gamma \backslash \Delta \). Let \( \mathcal{E} = \mathfrak{A} \cup \mathfrak{B} \), the set of all directed edges of \( X \).
If \( a = \Gamma r \in \mathfrak{A} \), let \( o(a) = \Gamma r(00) \in X^0 \) and \( t(a) = \Gamma r(10) \in X^0 \), the origin and terminus of the directed edge \( a \). Similarly, if \( b = \Gamma r \in \mathfrak{B} \), let \( o(b) = \Gamma r(00) \in X^0 \) and \( t(b) = \Gamma r(01) \in X^0 \). Note that it is possible that \( o(e) = t(e) \).

A straightforward analogue of Lemma 2.2 shows that each geometric edge in \( X^1 \) is the image of each of two directed edges. The horizontal and vertical reflections on \( \sigma \) induce an inversion on \( E \), denoted by \( e \mapsto e \), with the property that \( e = e \) and \( o(e) = t(e) \). The pair \((E, X^0)\) is thus a graph in the sense of \( \text{[Se]} \). Choose once and for all an orientation of this graph: that is a subset \( E^+ \) of \( E \), with \( E = E^+ \cup E^- \). Write \( \mathfrak{A}^+ = \mathfrak{A} \cap E^+ \) and \( \mathfrak{B}^+ = \mathfrak{B} \cap E^+ \). The images of \( \mathfrak{A} \) [respectively \( \mathfrak{B} \)] in \( X^1 \) are the edges the horizontal [vertical] 1-skeleton \( X^1_h [X^1_v] \).

Lemma 2.3. There is a well defined injective map

\[
 t \mapsto (a(t), b(t)) : \mathfrak{R} \rightarrow \mathfrak{A} \times \mathfrak{B}
\]

which is surjective if \( X \) has one vertex.

![Figure 4. Directed edges in X.](image)

Proof. The map \( r \mapsto (r|_{[00,10]}, r|_{[00,01]}) : R \rightarrow A \times B \) is injective because each geometric square of \( \Delta \) is uniquely determined by any two edges containing a common vertex.

If \( t = \Gamma r \in \mathfrak{R} \) then define

\[
 a(t) = \Gamma r|_{[00,10]}, \quad b(t) = \Gamma r|_{[00,01]}
\]

Using the fact that \( \Gamma \) acts freely on \( \Delta \) it is easy to see that the map \( t \mapsto (a(t), b(t)) \) is injective.

If \( X \) has one vertex, then any two elements \( a \in \mathfrak{A}, b \in \mathfrak{B} \) are represented by type rotating isometries \( r_1 : [00,10] \rightarrow \Delta, r_2 : [00,01] \rightarrow \Delta \) with \( r_1(00) = r_2(00) \). The isometries \( r_1, r_2 \) are restrictions of an isometry \( r \in R \), which defines an element \( t = \Gamma r \in \mathfrak{R} \) with \( a = a(t) \) and \( b = b(t) \).

If \( t = \Gamma r \in \mathfrak{R} \), define directed edges \( a'(t) \in \mathfrak{A}, b'(t) \in \mathfrak{B} \) opposite to \( a(t), b(t) \), as follows.

\[
 a'(t) = \Gamma (r \circ v|_{[00,10]}),
 b'(t) = \Gamma (r \circ h|_{[00,01]}).
\]

In other words

\[
 a'(t) = a(t^v); \quad b'(t) = b(t^h).
\]
3. Some related graphs

Associated to the VH-T complex $X$ are two graphs whose vertices are directed edges of $X$. Denote by $G_v(\mathfrak{A})$ the graph whose vertex set is $\mathfrak{A}$ and whose edge set is $\mathfrak{R}$, with origin and terminus maps defined by $t \mapsto a(t)$ and $t \mapsto a'(t)$ respectively. Similarly $G_h(\mathfrak{B})$ is the graph whose vertex set is $\mathfrak{B}$ and whose edge set is $\mathfrak{R}$, with the origin and terminus maps defined by $t \mapsto b(t)$ and $t \mapsto b'(t)$.

Now define two directed graphs whose vertices are elements of $\mathfrak{R}$. The “horizontal” graph $G_h(\mathfrak{R})$ has vertex set $\mathfrak{R}$. A directed edge $[t, s]$ is defined as follows. Consider the model rectangle $H$ made up of two adjacent squares with vertices $\{(i, j) \in \mathbb{Z}^2 : i = 0, 1, 2, j = 0, 1\}$ where the vertex $(i, j)$ has type $(i + 2\mathbb{Z}, j + 2\mathbb{Z})$. The model square $\sigma$ of Figure 2 is considered as the left hand square of $H$.

An isometry $r : H \to \Delta$ is said to be type rotating if there exists $(k, l) \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ such that, for each vertex $(i, j)$ of $H$, $\tau(r((i, j))) = (i + k, j + l)$. A directed edge of $G_h(\mathfrak{R})$ is $\Gamma r$ where $r : H \to \Delta$ be a type rotating isometry. The origin of $\Gamma r$ is $t = \Gamma r_1$, where $r_1 = r|_\sigma$ and the terminus of $\Gamma r$ is $s = \Gamma r_2$, where $r_2 : \sigma \to \Delta$ is defined by $r_2(i, j) = r(i + 1, j)$. There is a similar definition for the “vertical” graph $G_v(\mathfrak{R})$ with vertex set $\mathfrak{R}$. Edges $[t, s]$ of $G_h(\mathfrak{R})$ and $G_v(\mathfrak{R})$ are illustrated in Figure 8 by the ranges of representative isometries.
An edge of $\mathcal{G}_h(\mathcal{R})$  
An edge of $\mathcal{G}_v(\mathcal{R})$

**Figure 8.**

Since $\Gamma$ acts freely on $\Delta$, it is easy to see that the existence of a directed edge $[t, s]$ of $\mathcal{G}_h(\mathcal{R})$ with origin $t \in \mathcal{R}$ and terminus $s \in \mathcal{R}$ is equivalent to

$$b(s) = b'(t), \quad s \neq t^h.$$  \hspace{1cm} (5)

Similarly the existence of a directed edge $[t, s]$ of $\mathcal{G}_v(\mathcal{R})$, with origin $t \in \mathcal{R}$ and terminus $s \in \mathcal{R}$ is equivalent to

$$a(s) = a'(t), \quad s \neq t^v.$$  \hspace{1cm} (6)

The next Lemma will be used later. Recall that a lattice $\Gamma$ in $\text{PGL}_2(\mathbb{Q}_p) \times \text{PGL}_2(\mathbb{Q}_\ell)$ is automatically cocompact \cite[IX Proposition 3.7]{Mar}}.

**Lemma 3.1.** If $p, \ell$ are prime and $\Gamma$ is a torsion free irreducible lattice in $\text{PGL}_2(\mathbb{Q}_p) \times \text{PGL}_2(\mathbb{Q}_\ell)$ acting on the corresponding product of trees, then the directed graphs $\mathcal{G}_h(\mathcal{R})$, $\mathcal{G}_v(\mathcal{R})$ are connected.

**Proof.** This follows from \cite[Proposition 2.15]{M3}, using the topological transitivity of an associated shift system. The proof uses the Howe-Moore theorem for $p$-adic semisimple groups and is explained in \cite[Lemma 2]{M2}.

\hspace{1cm} \square

4. **Tilings and $H_2(\Gamma, \mathbb{Z})$**

Throughout this section, $\mathcal{T}_1$ and $\mathcal{T}_2$ are locally finite trees whose vertices all have degree at least three. The group $\Gamma$ acts freely and cocompactly on the 2 dimensional cell complex $\Delta = \mathcal{T}_1 \times \mathcal{T}_2$ and we continue to use the notation introduced in the preceding sections.

For $t, s \in \mathcal{R}$ write $tHs$ [respectively $tVs$] to mean that there is a “horizontal” [respectively “vertical”] directed edge $[t, s]$ in $\mathcal{G}_h(\mathcal{R})$ [respectively $\mathcal{G}_v(\mathcal{R})$]. Define homomorphisms $T_1, T_2: \mathbb{Z}\mathcal{R} \rightarrow \mathbb{Z}\mathcal{R}$ by

$$T_1 t = \sum_{tHs} s, \quad T_2 t = \sum_{tVs} s.$$
It follows from (5), (6) that

\[ T_1 t = \left( \sum_{b(s) = b'(t)} s \right) - t^h, \]

\[ T_2 t = \left( \sum_{a(s) = a'(t)} s \right) - t^v. \]

Consider the homomorphism

\[
\begin{pmatrix} T_1 - I \\ T_2 - I \end{pmatrix} : \mathbb{Z}\mathcal{R} \to \mathbb{Z}\mathcal{R} \oplus \mathbb{Z}\mathcal{R},
\]

\[ t \mapsto (T_1 t - t) \oplus (T_2 t - t). \]

Define \( \varepsilon : \mathbb{Z}\mathcal{E} \to \mathbb{Z}\mathcal{E}^+ \) by

\[
\varepsilon(x) = \begin{cases} 
  x & \text{if } x \in \mathcal{E}^+, \\
  -\overline{x} & \text{if } x \in \mathcal{E}^-. 
\end{cases}
\]

The boundary map \( \partial : \mathbb{Z}\mathcal{R}^+ \to \mathbb{Z}\mathcal{E}^+ \) is defined by

\[ \partial t = \varepsilon(a(t) + b'(t) - a'(t) - b(t)) \]

and since \( X \) is 2-dimensional, \( H_2(\Gamma, \mathbb{Z}) = \ker \partial \). Define a homomorphism

\[ \varphi_2 : \mathbb{Z}\mathcal{R}^+ \to \mathbb{Z}\mathcal{R} \]

by

\[ \varphi_2 t = t - t^v - t^h + t^{vh}. \]

The rest of this section is devoted to proving the following result, which is a more precise version of Theorem 1.1.

**Theorem 4.1.** The homomorphism \( \varphi_2 \) restricts to an isomorphism from \( H_2(\Gamma, \mathbb{Z}) \) onto \( \ker \left( \frac{T_1 - I}{T_2 - I} \right) \).

Define a homomorphism \( \varphi_1 : \mathbb{Z}\mathcal{E} \to \mathbb{Z}\mathcal{R} \oplus \mathbb{Z}\mathcal{R} \) by

\[
\varphi_1(a) = 0 \oplus \left( \sum_{a(s) = \overline{a}} s - \sum_{a(s) = a} s \right), \quad \text{if } a \in \mathcal{A},
\]

\[
\varphi_1(b) = \left( \sum_{b(s) = b} s - \sum_{b(s) = \overline{b}} s \right) \oplus 0, \quad \text{if } b \in \mathcal{B}.
\]

Note that if \( x \in \mathcal{E} \) then \( \varphi_1(\overline{x}) = -\varphi_1(x) \) and so \( \varphi_1(\varepsilon(x)) = \varphi_1(x) \).
Lemma 4.2. The homomorphisms $\varphi_1, \varphi_2$ are injective and the following diagram commutes:

$$
\begin{array}{ccc}
\mathbb{Z}E^+ & \xleftarrow{\partial} & \mathbb{Z}R^+ \\
\varphi_1 \downarrow & & \downarrow \varphi_2 \\
\mathbb{Z}R \oplus \mathbb{Z}R & \xleftarrow{(T_1-I) + (T_2-I)} & \mathbb{Z}R
\end{array}
$$

Proof. Let $t \in \mathcal{R}$. Then

\begin{align*}
(T_1 - I)t &= \left( \sum_{b(s)=b'(t)} s \right) - t^h - t, \\
(T_1 - I)t^v &= \left( \sum_{b(s)=\overline{b}(t)} s \right) - t^{vh} - t^v, \\
(T_1 - I)t^h &= \left( \sum_{b(s)=b(t)} s \right) - t - t^h, \\
(T_1 - I)t^{vh} &= \left( \sum_{b(s)=\overline{b}(t)} s \right) - t^v - t^{vh}.
\end{align*}

Therefore

\begin{align*}
(T_1 - I) \circ \varphi_2(t) &= (T_1 - I)(t - t^v - t^h + t^{vh}) \\
&= \left( \sum_{b(s)=b'(t)} s - \sum_{b(s)=\overline{b}(t)} s \right) - \left( \sum_{b(s)=b(t)} s - \sum_{b(s)=\overline{b}(t)} s \right).
\end{align*}

By definition of $\varphi_1$, this implies that

$$
\varphi_1(b'(t) - b(t)) = (T_1 - I)\varphi_2(t) \oplus 0.
$$

Similarly

$$
\varphi_1(a(t) - a'(t)) = 0 \oplus (T_2 - I)\varphi_2(t).
$$

Therefore

$$
\left( \frac{T_1-I}{T_2-I} \right) \circ \varphi_2(t) = \varphi_1(b'(t) - b(t) + a(t) - a'(t)) \\
= \varphi_1 \circ \varepsilon(b'(t) - b(t) + a(t) - a'(t)) \\
= \varphi_1 \circ \partial(t).
$$

This shows that (7) commutes.

It is obvious that $\varphi_2$ is injective. To verify that $\varphi_1$ is injective, define $\psi : \mathbb{Z}R \oplus \mathbb{Z}R \to \mathbb{Z}E^+$ by $\psi(s, t) = \varepsilon(b(s) - a(t))$. Then $\psi \circ \varphi_1(x)$ is a nonzero multiple of $x$, for all $x \in \mathcal{E}$. It follows that $\psi \circ \varphi_1 : \mathbb{Z}E^+ \to \mathbb{Z}E^+$ is injective and therefore so is $\varphi_1$. \qed
Lemma 4.3. The homomorphism $\varphi_2$ restricts to an isomorphism from $H_2(\Gamma, \mathbb{Z})$ onto $\varphi_2(\mathcal{R}^+) \cap \ker \left( \frac{T_1}{T_2} - I \right)$.

Proof. Let $\varphi_2(\beta) \in \ker \left( \frac{T_1}{T_2} - I \right)$, where $\beta \in \mathcal{R}^+$. It follows from (7) that $\varphi_1 \circ \partial(\beta) = 0$. But $\varphi_1$ is injective, so $\partial \beta = 0$ i.e. $\beta \in H_2(\Gamma, \mathbb{Z})$.

Conversely, if $\beta \in H_2(\Gamma, \mathbb{Z})$ then $\left( \frac{T_1}{T_2} - I \right) \circ \varphi_2(\beta) = 0$ by (7), so $\varphi_2(\beta) \in \ker \left( \frac{T_1}{T_2} - I \right)$. Since $\varphi_2$ is injective, the conclusion follows.  

The next result, combined with Lemma 4.3, completes the proof of Theorem 4.1.

Lemma 4.4. There is an inclusion $\ker \left( \frac{T_1}{T_2} - I \right) \subset \varphi_2(\mathcal{R}^+)$. 

Proof. Let $\alpha = \sum_{t \in \mathcal{R}} \lambda(t) t \in \ker \left( \frac{T_1}{T_2} - I \right)$. We show that $\alpha \in \varphi_2(\mathcal{R}^+)$. If $s \in \mathcal{R}$ then the coefficient of $s$ in the sum representing $(T_1 - I)\alpha$ is

$$
\left( \sum_{t \in \mathcal{R}, t \neq s^h} \lambda(t) - \lambda(s) = \left( \sum_{t \in \mathcal{R}} \lambda(t) \right) - \lambda(s) - \lambda(s^h).
$$

This coefficient is zero, since $\alpha \in \ker(T_1 - I)$. Therefore

$$
\lambda(s) + \lambda(s^h) = \sum_{t \in \mathcal{R}} \lambda(t)\quad\text{when } t \neq b(s).
$$

The right hand side of equation (8) depends only on $b(s)$, so for any $b \in \mathcal{B}$ we define

$$
\mu(b) = \sum_{t \in \mathcal{R}} \lambda(t)\quad\text{when } t \neq b(s).
$$

Thus (8) may be rewritten as

$$
\lambda(s) + \lambda(s^h) = \mu(b(s)).
$$

It follows from (8) and (9) that

$$
\mu(b(s)) = \mu(b(s^h)) = \mu(b'(s)).
$$

![Figure 9. $\mu(b(s)) = \mu(b'(s))$]

Fix an element $b_0 \in \mathcal{B}$, and let $C$ be the connected component of the graph $\mathcal{G}_h(\mathcal{B})$ containing $b_0$. Then $C$ is a connected graph with vertex set $C^0 \subset \mathcal{B}$ and edge set $C^1 \subset \mathcal{R}$. The graph $C$ has a natural orientation $C^+ = C^1 \cap (\mathcal{R}^+ \cup \mathcal{R}^v)$ and it is clear that $C^1 = C^+ \cup \{t^h : t \in C^+\}$. Each
vertex of $C$ has degree at least three, since the same is true of the tree $T_1$. Therefore the number of vertices of $C$ is less than the number of geometric edges i.e. $|C^0| < |C^+|$. 

If $b \in C^0$ then there is a path in $C^0$ from $b_0$ to $b$. It follows by induction from (10) that $\mu(b_0) = \mu(b)$. Thus

$$\mu(b_0) = \sum_{t \in R} \lambda(t) = \sum_{t \in C_1} \lambda(t).$$

Therefore

$$|C^0| \mu(b_0) = \sum_{b \in C^0} \sum_{t \in C_1} \lambda(t) = \sum_{t \in C_1} \lambda(t)$$

$$= \sum_{t \in C^+} (\lambda(t) + \lambda(t^h)) = \sum_{t \in C^+} \mu(b(t))$$

$$= \sum_{t \in C^+} \mu(b_0) = |C^+| \mu(b_0).$$

Since $|C^0| < |C^+|$, it follows that $\mu(b_0) = 0$ for all $b_0 \in \mathfrak{B}$. In other words, by (9),

(11) 

$$\lambda(s) = -\lambda(s^h)$$

for all $s \in \mathfrak{R}$. A similar argument, using $\alpha \in \ker(T_2 - I)$ and interchanging the roles of horizontal and vertical reflections, shows that

(12) 

$$\lambda(s) = -\lambda(s^v)$$

for all $s \in \mathfrak{R}$. Combining (11) and (12) gives

(13) 

$$\lambda(s) = \lambda(s^{vh})$$

for all $s \in \mathfrak{R}$. Finally,

$$\alpha = \sum_{t \in \mathfrak{R}^+} \left( \lambda(s)s + \lambda(s^v)s^v + \lambda(s^h)s^h + \lambda(s^{vh})s^{vh} \right)$$

$$= \sum_{t \in \mathfrak{R}^+} \lambda(s) \left( s - s^v - s^h + s^{vh} \right)$$

$$= \sum_{t \in \mathfrak{R}^+} \lambda(s) \varphi_2(s) \in \varphi_2(\mathbb{Z}\mathfrak{R}^+).$$

5. K-theory of the boundary $C^*$-algebra

The (maximal) boundary $\partial \Delta$ of $\Delta$ is defined in [KR]. It is homeomorphic to $\partial T_1 \times \partial T_2$, where $\partial T_j$ is the totally disconnected space of ends of the tree $T_j$. The group $\Gamma$ acts on $\partial \Delta$ and hence on $C_\mathfrak{C}(\partial \Delta)$ via $g \mapsto \alpha_g$, where $\alpha_g f(\omega) = f(g^{-1} \omega)$, for $f \in C_\mathfrak{C}(\partial \Delta)$, $g \in \Gamma$. The full crossed product $C^*$-algebra $\mathcal{A}(\Gamma, \partial \Delta) = C_\mathfrak{C}(\partial \Delta) \rtimes \Gamma$ is the completion
of the algebraic crossed product in an appropriate norm. We present examples where the rank of the analytic $K$-group $K_0(\mathcal{A}(\Gamma, \partial \Delta))$ is determined by Theorem 4.1.

5.1. One vertex complexes. The case where the quotient VH-T complex $X$ has one vertex was studied in [KR]. The group $\Gamma$ acts freely and transitively on the vertices of $\Delta$ and $\mathcal{A}(\Gamma, \partial \Delta)$ is isomorphic to a rank-2 Cuntz-Krieger algebra, as described in [RS1, RS2]. The proof of this fact given in [KR, Theorem 5.1]. It follows from [RS1] that $\mathcal{A}(\Gamma, \partial \Delta)$ is classified by its K-theory. By the proofs of [RS2, Proposition 4.13] and [KR, Lemma 4.3, Theorem 5.3], we have

$$K_0(\mathcal{A}(\Gamma, \partial \Delta)) = K_1(\mathcal{A}(\Gamma, \partial \Delta))$$

and

$$\text{rank}(K_0(\mathcal{A}(\Gamma, \partial \Delta))) = 2 \cdot \dim \ker \left( \frac{I_1}{I_2 - I} \right).$$

Together with Theorem 4.1, this proves

(14) $$\text{rank } K_0(\mathcal{A}(\Gamma, \partial \Delta)) = 2 \cdot \text{rank } H_2(\Gamma, \mathbb{Z}).$$

This verifies a conjecture in [KR].

5.2. Irreducible lattices in $\text{PGL}_2(\mathbb{Q}_p) \times \text{PGL}_2(\mathbb{Q}_\ell)$. If $p, \ell$ are prime then the group $\text{PGL}_2(\mathbb{Q}_p) \times \text{PGL}_2(\mathbb{Q}_\ell)$ acts on the $\Delta = \mathbb{T}_{p+1} \times \mathbb{T}_{\ell+1}$ and on its boundary $\partial \Delta$, which can be identified with a direct product of projective lines $\mathbb{P}_1(\mathbb{Q}_p) \times \mathbb{P}_1(\mathbb{Q}_\ell)$. Let $\Gamma$ be a torsion free irreducible lattice in $\text{PGL}_2(\mathbb{Q}_p) \times \text{PGL}_2(\mathbb{Q}_\ell)$. Then $\Gamma$ acts freely on $\Delta$ and $\mathcal{A}(\Gamma, \partial \Delta)$ is a rank-2 Cuntz-Krieger algebra in the sense of [RS1]. The irreducibility condition (H2) of [RS1] follows from Lemma 3.1. The proofs of the remaining conditions of [RS1] are exactly the same as in [KR, Lemma 4.1]. It follows that (14) is also true in this case. Since $\Gamma$ is irreducible, the normal subgroup theorem [Mar, IV, Theorem (4.9)] implies that $H_1(\Gamma, \mathbb{Z}) = \Gamma/[\Gamma, \Gamma]$ is finite. Equation (14) can therefore be written

(15) $$\chi(\Gamma) = 1 + \frac{1}{2} \text{rank } K_0(\mathcal{A}(\Gamma, \partial \Delta)).$$

On the other hand, one easily calculates $\chi(\Gamma) = \frac{(p-1)(\ell-1)}{4} |X^0|$, where $|X^0|$ is the number of vertices of $X$. Therefore the rank of $K_0(\mathcal{A}(\Gamma, \partial \Delta))$ can be expressed explicitly in terms of $p, \ell$ and $|X^0|$.

Explicit examples are studied in [M3, Section 3]. If $p, l \equiv 1 \pmod{4}$ are two distinct primes, Mozes constructs an irreducible lattice $\Gamma_{p,l}$ in $\text{PGL}_2(\mathbb{Q}_p) \times \text{PGL}_2(\mathbb{Q}_\ell)$ which acts freely and transitively on the vertex set of $\Delta$. Here is how $\Gamma_{p,l}$ is constructed. Let $\mathbb{H}(\mathbb{Z}) = \{a = a_0 + a_1 i + a_2 j + a_3 k; a_j \in \mathbb{Z}\}$, the ring of integer quaternions, let $i_p$ be a square root of $-1$ in $\mathbb{Q}_p$ and define

$$\psi : \mathbb{H}(\mathbb{Z}) \to \text{PGL}_2(\mathbb{Q}_p) \times \text{PGL}_2(\mathbb{Q}_\ell)$$
by
\[
\psi(a) = \left( \begin{array}{cc}
a_0 + a_1i & a_2 + a_3i_p \\
a_2 + a_3i_p & a_0 - a_1i_p
\end{array} \right) = \left( \begin{array}{cc}
a_0 + a_1i_\ell & a_2 + a_3i_\ell \\
a_2 + a_3i_\ell & a_0 - a_1i_\ell
\end{array} \right).
\]

Let \( \tilde{\Gamma}_{p,\ell} = \{ a = a_0 + a_1i + a_2j + a_3k \in \mathbb{H}(\mathbb{Z}); a_0 \equiv 1 \pmod{2}, a_j \equiv 0 \pmod{2}, j = 1, 2, 3, |a|^2 = p^{r_1} \ell \} \). Then \( \Gamma_{p,\ell} = \psi(\tilde{\Gamma}_{p,\ell}) \). The fact that \( \Gamma_{p,\ell} \) is irreducible follows easily from [RR, Corollary 2.3], where it is observed that the only nontrivial direct product subgroup of \( \Gamma_{p,\ell} \) is \( \mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^2 \).

Since \(|X^0| = 1\), it follows from [LR] that the rank of \( K_0(A(\Gamma_{p,\ell}, \partial \Delta)) \) is \( \frac{(p-1)(\ell-1)}{2} - 2 \). This proves an experimental observation of [KR, Example 6.2]. The construction of Mozes has been generalized in [Rat, Chapter 3] to all pairs \((p,l)\) of distinct odd primes and the same conclusion applies.

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