SPECTRAL CHARACTERIZATION OF THE QUADRATIC VARIATION OF MIXED BROWNIAN-FRACTIONAL BROWNIAN MOTION

EHSAN AZMOODEH AND ESKO VALKEILA

Abstract. Dzhaparidze and Spreij [5] showed that the quadratic variation of a semimartingale can be approximated using a randomized periodogram. We show that the same approximation is valid for a special class of continuous stochastic processes. This class contains both semimartingales and non-semimartingales. The motivation comes partially from the recent work by Bender et al. [2], where it is shown that the quadratic variation of the log-returns determines the hedging strategy.

Keywords: fractional Brownian motion, quadratic variation, randomized periodogram

2010 AMS subject classification: 60G15, 62M15

1. Introduction

Spectral characterization of the bracket. It is well-known that for a semimartingale $X$, the bracket $[X, X]$ can be identified with

$$[X, X]_t = \mathbb{P}^{-}\lim_{|\pi| \to 0} \sum_{t_k \in \pi} (X_{t_k} - X_{t_{k-1}})^2,$$

where $\pi = \{t_k : 0 = t_0 < t_1 < \cdots < t_n = t\}$ is a partition of the interval $[0, t]$, $|\pi| = \max \{t_k - t_{k-1} : t_k \in \pi\}$, and $\mathbb{P}^{-}\lim$ means convergence in probability. Statistically speaking, the sums of squared increments (realized quadratic variation) is a consistent estimator for the bracket as the volume of observations tends to infinity. Barndorff-Nielsen and Shephard [1] studied precision of the realized quadratic variation estimator for a special class of continuous semimartingales. They showed that sometimes the realized quadratic variation estimator can be rather noisy estimator. So one should seek for new estimators of the quadratic variation.

Dzhaparidze and Spreij [5] suggested another characterization of the bracket $[X, X]$. Let $\mathcal{F}^X\tau$ be the filtration of $X$ and $\tau$ be a finite stopping time. For...
\( \lambda \in \mathbb{R} \), define the \textit{periodogram} \( I_\tau(X; \lambda) \) of \( X \) at \( \tau \) by

\[
I_\tau(X; \lambda) := \left| \int_0^\tau e^{i\lambda s} \, dX_s \right|^2 = 2 \Re \int_0^\tau \int_0^t e^{i\lambda(t-s)} \, dX_s \, dX_t + [X, X]_\tau \quad \text{(by Ito formula)}.
\]

Given \( L > 0 \) and \( \xi \) be a symmetric random variable with a density \( g_\xi \), real characteristic function \( \varphi_\xi \), and independent of the filtration \( \mathcal{F}^X \). Define the randomized periodogram by

\[
E_\xi I_\tau(X; L\xi) = \int \mathbb{R} I_\tau(X; Lx) g_\xi(x) \, dx.
\]

If the characteristic function \( \varphi_\xi \) is of bounded variation, then Dzhaparidze and Spreij have shown that we have the following characterization of the bracket as \( L \to \infty \)

\[
E_\xi I_\tau(X; L\xi) \overset{P}{\to} [X, X]_\tau.
\]

**Robust Black & Scholes pricing by hedging.** Next we give another motivation for our estimation problem. Bender et al. \[2\] consider a class of pricing models, where the continuous stock price \( S \) has the following quadratic variation as a functional of the observed path \( S \):

\[
d[S, S]_t = \sigma^2(S_t) \, dt.
\]

Here \( \sigma : \mathbb{R} \to \mathbb{R} \) is a continuously differentiable function of linear growth. A typical example of this kind of stock price models is the classical Black & Scholes model with constant volatility \( \sigma \), where the stock price \( \tilde{S} \) is given by

\[
\tilde{S}_t = s_0 e^{\sigma W_t - \frac{1}{2} \sigma^2 t},
\]

where \( W \) is a standard Brownian motion. We have

\[
d[\tilde{S}, \tilde{S}]_t = \sigma^2 \tilde{S}_t^2 \, dt,
\]

and the bracket \([\tilde{S}, \tilde{S}]\) has the form of (1.3). On the other hand, let \( X \) be a continuous process with quadratic variation \([X, X]_t = \sigma^2 t\); take for example \( X_t = \sigma W_t + \eta B^H_t \), \( B^H \) is a fractional Brownian motion with Hurst parameter \( H \geq \frac{1}{2} \), independent of \( W \), and \( \eta \) is a constant. Then, for the process \( S_t = s_0 e^{X_t - \frac{1}{2} \sigma^2 t} \), we have that

\[
d[S, S]_t = \sigma^2 S_t^2 \, dt,
\]

and again the bracket has the functional form of (1.3). Here we have examples, where the quadratic variation of the driving process \( X \) determines the structure of the quadratic variation of the stock price. Moreover, if this is the case, then Bender et al. have shown that within a fixed model class, determined by the relation (1.3), the hedging of options has the same functional form as in the classical Black & Scholes pricing model. The options, which can be hedged, includes European options, path dependent options like look-back options, and Asian options.
The results. We show that the result of Dzhaparidze and Spreij holds for the mixed Brownian-fractional Brownian motion. Let \((\Omega, \mathcal{F}, P)\) be a complete probability space and fix \(T > 0\). Throughout the paper, we assume that \(W = \{W_t\}_{t \in [0,T]}\) is a standard Brownian motion and \(B^H = \{B^H_t\}_{t \in [0,T]}\) is a fractional Brownian motion with Hurst parameter \(H \in \left(\frac{1}{2}, 1\right)\), independent of the Brownian motion \(W\). Define the mixed Brownian-fractional Brownian motion \(X_t\) by

\[ X_t = W_t + B^H_t \quad t \in [0, T]. \]

It is known (see [3]) the process \(X\) is a \((\mathcal{F}^X, \mathcal{I}^X, P)\) semimartingale, if \(H \in \left(\frac{3}{4}, 1\right)\), and for \(H \in \left(\frac{1}{2}, \frac{3}{4}\right)\), \(X\) is not a semimartingale with respect to its own filtration \(\mathcal{F}^X\). Moreover in both cases we have

\[
[X, X]_t = [X]_t = \lim_{\pi \to 0} \sum_{t_i \leq t} (X_{t_i} - X_{t_{i-1}})^2 = t.
\]

If the partitions in (1.4) are nested, then the convergence can be strengthened to almost sure convergence. Hereafter, we always assume that the sequence of partitions are nested. For \(\lambda \in \mathbb{R}\), define the complex-valued stochastic process \(Y\) by

\[ Y_t = \int_0^t e^{i\lambda s} dX_s \quad t \in [0, T], \]

where the stochastic integral is understood in a path-wise way, and it is defined by integration by parts formula (see [12]):

\[
\int_0^t e^{i\lambda s} dX_s = e^{i\lambda t} X_t - i\lambda \int_0^t X_s e^{i\lambda s} ds.
\]

Therefore, \(Y = \{Y_t\}_{t \in [0,T]}\) is a process with continuous sample paths. Moreover, it is straightforward to check that for \(t \in [0, T]\), we have that

\[
[Y, Y]_t = [Y]_t = \lim_{|\pi| \to 0} \sum_{t_i \leq t} (Y_{t_i} - Y_{t_{i-1}})(Y_{t_i} - Y_{t_{i-1}}) = [\text{Re } Y]_t + [\text{Im } Y]_t = [X]_t = t,
\]

where \(\overline{Y}_t\) is complex conjugate of \(Y_t\) ([12, p.84]). Given \(\lambda \in \mathbb{R}\), define the periodogram of \(X\) at \(T\) as (1.1), i.e.

\[
I_T(X; \lambda) = \left| \int_0^T e^{i\lambda t} dX_t \right|^2 = \left| e^{i\lambda T} X_T - i\lambda \int_0^T X_t e^{i\lambda t} dt \right|^2 = X_T^2 + X_T \int_0^T i\lambda (e^{i\lambda(T-t)} - e^{-i\lambda(T-t)}) X_t dt + \lambda^2 \int_0^T e^{i\lambda t} X_t dt \right|^2.
\]
the filtration $\mathcal{F}^X$. Define for any positive real number $L$ the randomized periodogram by
\begin{equation}
\mathbb{E}_\xi I_T(X; L\xi) := \int_{\mathbb{R}} I_T(X; Lx)g_\xi(x)dx.
\end{equation}

Our main result is the following.

**Theorem 1.1.** Assume that $X$ is a mixed Brownian-fractional Brownian motion, $\mathbb{E}_\xi I_T(X; L\xi)$ is the randomized periodogram given by (1.5) and $\mathbb{E}_\xi \xi^2 < \infty$.

Then as $L \to \infty$ we have
\begin{equation}
\mathbb{E}_\xi I_T(X; L\xi) \xrightarrow{\mathbb{P}} [X, X]_T.
\end{equation}

**Remark 1.1.** To compare our result to the results of Dzhaparidze and Spreij:

- They take any finite stopping time $\tau$, whereas we must assume a constant stopping time $T$.
- They assume that the characteristic function $\varphi_\xi$ is of bounded variation, whereas instead we assume that $\xi$ is a square integrable random variable.

Note that under our assumption of deterministic stopping time $\tau = T$, when $X$ is a Gaussian martingale, they can drop the condition of the bounded variation on $[0, \infty)$ of the characteristic function $\varphi_\xi$ of the random variable $\xi$ (see [5, Remark, p.170]).

Next we give some auxiliary material and then finish the proof.

## 2. Auxiliary results

### 2.1. Path-wise Ito formula

Föllmer [6] obtained a path-wise calculus for continuous functions with finite quadratic variation. The next theorem is essentially due to Föllmer. For a nice exposition, and its use in finance, see Sondermann [14].

**Theorem 2.1.** [14] Let $X : [0, T] \to \mathbb{R}$ be a continuous process with continuous quadratic variation $[X, X]_t$ and $F \in C^2(\mathbb{R})$. Then for any $t \in [0, T]$, the limit of the Riemann-Stieltjes sums
\begin{equation}
\lim_{|\pi| \to 0} \sum_{t_{i-1} \leq t} F_x(X_{t_{i-1}})(X_{t_i} - X_{t_{i-1}}) := \int_0^t F_x(X_s)dX_s,
\end{equation}
exists almost surely. Moreover, we have
\begin{equation}
F(X_t) = F(X_0) + \int_0^t F'_x(X_s)dX_s + \frac{1}{2} \int_0^t F''_x(X_s)d[X, X]_s.
\end{equation}

**Lemma 2.1.** For the mixed Brownian-fractional Brownian motion $X$ we have
\begin{equation}
I_T(X; \lambda) = [X]_T + 2 \operatorname{Re} \int_0^T \int_0^t e^{i\lambda(t-s)} dX_s dX_t
\end{equation}
where the iterated stochastic integral in the right hand side is understood in path-wise way, i.e. as the limit of the Riemann-Stieltjes sums.
Proof: We apply Ito type formula (2.1) to the real part $\text{Re} \, Y$ and the imaginary part $\text{Im} \, Y$ of the process $Y = \{Y_t\}_{t \in [0,T]}$ with the function $F(x) = x^2$. We obtain

$$F(\text{Re} \, Y_T) = \int_0^T \left( 2 \int_0^t \text{Re} \, e^{i\lambda s}dX_s \right) \text{Re} \, e^{i\lambda t}dX_t + |\text{Re} \, Y|_T.$$ 

Similarly, we have

$$F(\text{Im} \, Y_T) = \int_0^T \left( 2 \int_0^t \text{Re} \, -ie^{i\lambda s}dX_s \right) \text{Re} \, -ie^{i\lambda t}dX_t + |\text{Im} \, Y|_T.$$ 

Summing the left and right hand sides of the identities, we get

$$\left| \int_0^T e^{i\lambda t}dX_t \right|^2 = \left( \int_0^T \text{Re} \, e^{i\lambda t}dX_t \right)^2 + \left( \int_0^T \text{Re} \, -ie^{i\lambda t}dX_t \right)^2 + 2 \int_0^T \int_0^t \text{Re} \, e^{i\lambda s} \text{Re} \, e^{i\lambda t}dX_s dX_t$$

$$+ 2 \int_0^T \int_0^t -\text{Re} \, ie^{i\lambda s} - \text{Re} \, ie^{i\lambda t}dX_s dX_t$$

$$= |X|_T + 2 \int_0^T \int_0^t \text{Re} \, e^{i\lambda s} \text{Re} \, e^{i\lambda t}dX_s dX_t$$

$$+ 2 \int_0^T \int_0^t -\text{Re} \, ie^{i\lambda s} - \text{Re} \, ie^{i\lambda t}dX_s dX_t$$

Note that $\text{Re} \, e^{i\lambda(t-s)} = \text{Re} \, e^{i\lambda s} \text{Re} \, e^{i\lambda t} + \text{Re} \, -ie^{i\lambda s} \text{Re} \, -ie^{i\lambda t}.$

2.2. Path-wise stochastic integration in fractional Besov-type spaces.

A stochastic process $X$ is a semimartingale if and only if one has a version of the Lebesgue dominated convergence theorem (see [12]). Fractional Brownian motion is not a semimartingale, and hence the stochastic integral with respect to fractional Brownian motion $B^H$ must be defined. Using the smoothness of the sample paths of the fractional Brownian motion $B^H$, when $H \in \left(\frac{1}{2}, 1\right)$, one can define the so-called generalized Lebesgue-Stieltjes integral. For more information, see [10], [16] and [9].

Definition 2.1. [10] Fix $0 < \alpha < \frac{1}{2}$.

(i) For $f : [0, T] \to \mathbb{R}$, define

$$\|f\|_{\alpha, 1} := \int_0^T \frac{|f(t)|}{t^\alpha}dt + \int_0^T \int_0^t \frac{|f(t) - f(s)|}{(t-s)^{\alpha+1}}dsdt,$$

and

$$W_{\alpha, 1}[0, T] = \{f : [0, T] \to \mathbb{R} : \|f\|_{\alpha, 1} < \infty\}.$$

(ii) Also, for $f : [0, T] \to \mathbb{R}$, define

$$\|f\|_{1-\alpha, \infty, T} := \sup_{0<s<t<T} \left( \frac{|f(t) - f(s)|}{(t-s)^{1-\alpha}} + \int_s^t \frac{|f(y) - f(s)|}{(y-s)^{2-\alpha}}dy \right),$$

and

$$W_{T}^{1-\alpha, \infty}[0, T] := \{f : [0, T] \to \mathbb{R} : \|f\|_{1-\alpha, \infty, T} < \infty\}.$$
Denote by $C^\lambda[0, T]$ the space of $\lambda$-Hölder continuous functions on the interval $[0, T]$. Then $\forall \varepsilon > 0$, we have the inclusions

\[
C^{1-\alpha+\varepsilon}[0, T] \subseteq W^{1-\alpha, \infty}_T[0, T] \subseteq C^{1-\alpha}[0, T]
\]

\[
C^{\alpha+[\varepsilon]}[0, T] \subseteq W^{\alpha-1, 1}_T[0, T].
\]

Recall that almost surely sample paths of $B^H$ for any $0 < \gamma < H$, belong to $C\gamma[0, T]$. This follows from the Kolmogorov continuity theorem. Hence the sample paths of $B^H$ belong to $W^{\alpha, \infty}_T[0, T]$ for any $0 < \alpha < H$.

In the following $D^\alpha f_r$ (resp. $D^\alpha_0 f_r$) stand for right-sided (resp. left-sided) fractional derivatives (\cite{13}). For $g \in W^{1-\alpha, \infty}_T[0, T]$, define

\[
\Lambda_\alpha(g) := \frac{1}{\Gamma(1-\alpha)} \sup_{0<s<t<T} |(D^1_{t-} g_t)(s)| \leq \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)} \|g\|_{1-\alpha, \infty, T}.
\]

**Definition 2.2.** \cite{10} Fix $0 < \alpha < \frac{1}{2}$. Let $f \in W^{\alpha, 1}_0[0, T]$ and $g \in W^{1-\alpha, \infty}_T[0, T]$. Then the Lebesgue integral

\[
\int_0^T D^\alpha_0 f_0^+(t) D^{1-\alpha}_{t-} g_{T-}(t) dt
\]

exists, and we can define the generalized Lebesgue-Stieltjes integral by

\[
\int_0^T f_t dg_t := \int_0^T D^\alpha_0 f_0^+(t) D^{1-\alpha}_{t-} g_{T-}(t) dt,
\]

where $f_0^+(t) = f(t) - f(0^+)$ and $g_{T-}(t) = g(T^-) - g(t)$.

**Remark 2.1.** \cite{9}, \cite{16} The definition of the generalized Lebesgue-Stieltjes integral does not depend on the choice of $\alpha$.

**Remark 2.2.** \cite{16} If $f$ and $g$ are Hölder continuous of orders $\alpha$ and $\beta$ with $\alpha + \beta > 1$, then the generalized Lebesgue-Stieltjes integral exists and coincides with the Riemann-Stieltjes integral. This fact is based on the integration theory developed by Young \cite{15}.

Since fractional Brownian motion is not a semimartingale, the next theorem and corollary can be used instead of the Lebesgue dominated convergence theorem for fractional Brownian motion.

**Theorem 2.2.** \cite{10} Let $g \in W^{1-\alpha, \infty}_T[0, T]$ and $f \in W^{\alpha, 1}_0[0, T]$. Then we have the estimate

\[
\left| \int_0^T f_t dg_t \right| \leq \Lambda_\alpha(g) C_{\alpha, T} \|f\|_{\alpha, 1}
\]

for some constant $C = C_{\alpha, T}$.

**Corollary 2.1.** Assume $f, f^n \in W^{\alpha, 1}_0[0, T]$, and $\|f^n - f\|_{\alpha, 1} \to 0$ as $n \to \infty$ for some $\alpha \in (1 - H, \frac{1}{2})$. Then as $n \to \infty$

\[
\int_0^T f^n_t dB^n_t \to \int_0^T f_t dB^H_t, \quad a.s.
\]

Next we use this machinery to prove a stochastic Fubini type result that is cornerstone of the proof of our main result.
Proposition 2.1. Assume that $X = W + B^H$ is a mixed Brownian-fractional Brownian motion, and $\xi$ is a square integrable random variable with a density $g_\xi$, and independent of the filtration $\mathcal{F}^X$. Then the iterated integrals

\begin{align*}
I_1 &= \int_{\mathbb{R}} \left( \int_0^T \phi_W(t,x) dB^H_t \right) g_\xi(x) dx \\
I_2 &= \int_0^T \left( \int_{\mathbb{R}} \phi_W(t,x) g_\xi(x) dx \right) dB^H_t
\end{align*}

exist, and moreover $I_1 = I_2$ almost surely, where $\phi_W(t,x) = \int_0^t e^{ix(t-s)} dW_s$ and the stochastic integrals in $I_1$ and $I_2$ are understood in path-wise way as the limit of Riemann-Stieltjes sums.

Proof: We split the proof into four steps.

Step 1: The existence of $I_1$. Using integration by parts formula and simple manipulations, we see that the sample paths of the complex-valued stochastic process $\{\phi_W(t,x)\}_{t \in [0,T]}$ parametrized by $x \in \mathbb{R}$, are Hölder continuous of any order less than half almost surely. Moreover, for any $\alpha \in (0, \frac{1}{2})$, $x \in \mathbb{R}$, and $s, t \in [0,T]$, we have

\begin{equation}
|\phi_W(t,x) - \phi_W(s,x)| \leq C(\omega,T)(1 + |x| + |x|^2)|t - s|^\alpha \quad \text{and} \quad |\phi_W(t,x)| \leq C(\omega,T)(1 + T|x|),
\end{equation}

where $C(\omega,T)$ is an almost surely finite and positive random variable that may be different from line to line. Hence the interior stochastic integral in $I_1$ can be defined as limit of Riemann-Stieltjes sums almost surely.

Step 2: The existence of $I_2$. Using (2.3), we see that for any $\alpha \in (0, \frac{1}{2})$,

\begin{equation*}
\left| \int_{\mathbb{R}} \phi_W(t,x) g_\xi(x) dx - \int_{\mathbb{R}} \phi_W(s,x) g_\xi(x) dx \right| \leq C(\omega,T)(1 + 2\mathbb{E}|\xi|)^2|t - s|^\alpha.
\end{equation*}

Therefore, the stochastic integral $I_2$ can be defined as limit of the Riemann-Stieltjes sums almost surely.

Step 3. Define for any $N \in \mathbb{N}$,

\begin{align*}
I_1^N &= \int_{-N}^N \left( \int_0^T \phi_W(t,x) dB^H_t \right) g_\xi(x) dx \\
I_2^N &= \int_0^T \left( \int_{-N}^N \phi_W(t,x) g_\xi(x) dx \right) dB^H_t
\end{align*}

Clearly $I_1^N$ converges to $I_1$ almost surely as $N$ tends to infinity. We aim to show that $I_1^N = I_2^N$ almost surely. By definition of the Riemann integral,
there exists a sequence of partitions \( \{\pi_n^N\}_{n=1}^\infty \) of the interval \([-N, N]\), such that \(|\pi_n^N| \to 0\) as \(n \to \infty\) and

\[
I_1^N = \lim_{n \to \infty} \sum_{x_i^n \in \pi_n^N} \left( \int_0^T \phi_W(t, x_{i-1}^n)dB_t^H \right) g_\xi(x_{i-1}^n) \Delta x_i^n,
\]

and

\[
\lim_{n \to \infty} \sum_{x_i^n \in \pi_n^N} \phi_W(t, x_{i-1}^n)g_\xi(x_{i-1}^n) \Delta x_i^n = \int_{-N}^{N} \phi_W(t, x)g_\xi(x)dx
\]

hold. Assume that \(\pi_n^N = \{-N = x_0^n < x_1^n < \cdots < x_{k_n}^n = N\}\). For each \(x_{i-1}^n \in \pi_n^N, 1 \leq i \leq k_n + 1\), we can find a sequence \(\{\pi_m^T,x_i^{n-1}\}\) of partitions of the interval \([0, T]\), such that \(|\pi_m^T,x_i^{n-1}| \to 0\) as \(m \to \infty\) and

\[
\lim_{m \to \infty} \sum_{t_j^m \in \pi_m^T,x_i^{n-1}} \phi_W(t_j^m, x_{i-1}^n) \Delta B_{t_j^m}^H = \int_0^T \phi_W(t, x_{i-1}^n)dB_t^H.
\]

On the other hand, there is another sequence \(\{\hat{\pi}_m^T\}\) of partitions of the interval \([0, T]\) such that

\[
\lim_{m \to \infty} \sum_{t_j^m \in \hat{\pi}_m^T} \left( \int_{-N}^{N} \phi_W(t_j^m, x)g_\xi(x)dx \right) \Delta B_{t_j^m}^H = \int_0^T \left( \int_{-N}^{N} \phi_W(t, x)g_\xi(x)dx \right) dB_t^H.
\]

Let

\[
\pi_m^T,x_i = \bigcup_{i=1}^{k_n+1} \pi_m^T,x_i^{n-1} \quad \text{and} \quad \pi_m^T = \hat{\pi}_m^T \cup \pi_m^T,x_i.
\]

Therefore, for any \(n \in \mathbb{N}\), the partition \(\pi_m^T\) of the interval \([0, T]\) contains all points of the partitions \(\hat{\pi}_m^T\) and \(\pi_m^T,x_i\), and denote the points of \(\pi_m^T\) by \(t_k^m, k = 0, \cdots, l_m\). Then for any \(x_{i-1}^n \in \pi_n^N\), we can write

\[
\lim_{m \to \infty} \sum_{t_j^m \in \pi_m^T,x_i^{n-1}} \phi_W(t_j^m, x_{i-1}^n) \Delta B_{t_j^m}^H = \int_0^T \phi_W(t, x_{i-1}^n)dB_t^H.
\]

Now for \(n, m \in \mathbb{N}\), we can have the estimate

\[
|I_1^N - I_2^N| \leq |I_1^N - \Delta_{n,m}| + |I_2^N - \Delta_{n,m}| := A_{n,m} + B_{n,m},
\]

where

\[
\Delta_{n,m} := \sum_{x_i^n \in \pi_n^N} \sum_{t_j^m \in \pi_m^T,x_i^{n-1}} \phi_W(t_j^m, x_{i-1}^n) \Delta B_{t_j^m}^H g_\xi(x_{i-1}^n) \Delta x_i^n.
\]
For the first term \( A_{n,m} \), we have

\[
|A_{n,m}| \leq |I_1^n - \sum_{x^n_i \in \pi^n_N} \left( \int_0^T \phi_W(t, x^n_{i-1})dB_t^H \right) g_\xi(x^n_{i-1})\Delta x^n_i | + \sum_{x^n_i \in \pi^n_N} \left| \int_0^T \phi_W(t, x^n_{i-1})dB_t^H \right| - \sum_{t^n_m \in \pi^n_T} \phi_W(t^n_{m-1}, x^n_{i-1})\Delta B^H_{t^n_j} |g_\xi(x^n_{i-1})\Delta x^n_i |.
\]

For fix \( n \) and \( x^n_{i-1} \), when \( m \) tends to infinity, we have

\[
\left| \int_0^T \phi_W(t, x^n_{i-1})dB_t^H - \sum_{t^n_m \in \pi^n_T} \phi_W(t^n_{m-1}, x^n_{i-1})\Delta B^H_{t^n_j} \right| \to 0.
\]

Therefore, \( \lim_{n \to \infty} \lim_{m \to \infty} A_{n,m} = 0 \). Similarly, for the second term \( B_{n,m} \)

\[
|B_{n,m}| \leq |I_2^n - \sum_{t^n_j \in \pi^n_T} \left( \int_{-N}^N \phi_W(t^n_{j-1}, x)g_\xi(x)dx \right) \Delta B^H_{t^n_j} | + \Delta_{n,m} - \sum_{t^n_j \in \pi^n_T} \left( \int_{-N}^N \phi_W(t^n_{j-1}, x)g_\xi(x)dx \right) \Delta B^H_{t^n_j} |
\]

So, it is enough to show that the second term in the right hand side converges to 0 as \( n, m \) tend to infinity. Note that the second term can be written as

\[
\left| \Delta_{n,m} - \sum_{t^n_j \in \pi^n_T} \left( \int_{-N}^N \phi_W(t^n_{j-1}, x)g_\xi(x)dx \right) \Delta B^H_{t^n_j} \right| = \left| \sum_{i=0}^{k_n} \int_{x^n_{i-1}}^{x^n_i} f_m(x, x^n_{i-1})dx \right|
\]

where

\[
f_m(x, x^n_{i-1}) = \sum_{t^n_j \in \pi^n_T} \left( \phi_W(t^n_{j-1}, x^n_{i-1})g_\xi(x^n_{i-1}) - \phi_W(t^n_{j-1}, x)g_\xi(x) \right) \Delta B^H_{t^n_j}.
\]

So, when \( m \) tends to infinity, we have that

\[
f_m(x, x^n_{i-1}) \rightarrow \int_0^T \left( \phi_W(t, x^n_{i-1})g_\xi(x^n_{i-1}) - \phi_W(t, x)g_\xi(x) \right)dB_t^H.
\]

Moreover, for each \( 1 \leq i \leq k_n \), the sequence \( f_m(x, x^n_{i-1}) \) has an integrable dominant with respect to variable \( x \). To see this, take \( \theta \in (\frac{1}{2}, H) \) and
\( \lambda \in (0, \frac{1}{2}) \) such that \( \theta + \lambda = 1 + \epsilon \). Then
\[
|f_m(x, x^n_{i-1})| \leq \sum_{t^n_{i-1} \in \pi^n_m} (\phi_W(t^n_{i-1}, x^n_{i-1})g_\xi(x^n_{i-1}) - \phi_W(t^n_{i-1}, x)g_\xi(x)) \Delta B^n_i - \int_0^T (\phi_W(t, x^n_{i-1})g_\xi(x^n_{i-1}) - \phi_W(t, x)g_\xi(x)) dB^H_t \bigg| + \int_0^T (\phi_W(t, x^n_{i-1})g_\xi(x^n_{i-1}) - \phi_W(t, x)g_\xi(x)) dB^H_t \bigg|
\]
\[
\leq C|\pi^n_m|^\epsilon \|B^H\|_{C^\alpha[0,T]} \|\phi_W(t, x^n_{i-1})g_\xi(x^n_{i-1}) - \phi_W(t, x)g_\xi(x)\|_{C^\lambda[0,T]} \leq C|\pi^n_m|^\epsilon \|B^H\|_{C^\alpha[0,T]} \|\phi_W(t, x^n_{i-1})g_\xi(x^n_{i-1}) - \phi_W(t, x)g_\xi(x)\|_{C^\lambda[0,T]} + \|\phi_W(t, x)g_\xi(x)\|_{C^\lambda[0,T]}.
\]
By the inequalities were obtained in (2.3), we see that
\[
\|\phi_W(t, x)g_\xi(x)\|_{C^\lambda[0,T]} \leq C(\omega, T)(1 + |x| + |x|^2)g_\xi(x) \in L^1[-N, N].
\]

Therefore, by the Lebesgue dominated convergence theorem, we have that as \( m \) tends to infinity
\[
\int_{x^n_{i-1}}^{x_n^i} f_m(x, x^n_{i-1}) \, dx \rightarrow \int_{x^n_{i-1}}^{x_n^i} \left( \int_0^T (\phi_W(t, x^n_{i-1})g_\xi(x^n_{i-1}) - \phi_W(t, x)g_\xi(x)) \, dB^H_t \right) \, dx.
\]

Therefore, as \( n \) tends to infinity, we have
\[
\sum_{i=0}^{k_n} \int_{x^n_{i-1}}^{x_n^i} \left( \int_0^T (\phi_W(t, x^n_{i-1})g_\xi(x^n_{i-1}) - \phi_W(t, x)g_\xi(x)) \, dB^H_t \right) \, dx \rightarrow 0.
\]

Hence, we have shown that \( \lim_{n \to \infty} \lim_{m \to \infty} B_{n,m} = 0 \).

**Remark 2.3.** The result of this step can be derived from theorem 2.6.5 of [9], with some modifications.

**Step 4:** We want to show that \( I_2^N \) converges to \( I_2 \) as \( N \) tends to infinity. Clearly, the difference is
\[
|I_2 - I_2^N| = \left| \int_0^T u^N_t \, dB^H_t \right|
\]
where
\[
u^N_t := \int_{[-N, N]}^\epsilon \phi_W(t, x)g_\xi(x) \, dx.
\]

According to Corollary 2.1, it is sufficient to show that for some \( \alpha \in (1 - H, \frac{1}{2}) \), the sequence \( u^N \in W^{\alpha,1}_{\mathbb{Q}}[0,T] \) and \( \|u^N\|_{\alpha,1} \to 0 \). Note that the sample paths of the process \( u^N \) are Hölder continuous of any order less than half almost surely. Therefore, by the Remark 2.2, the stochastic integral
appears in (2.4) coincides with the Riemann-Stieltjes integral. Now, for any \( \alpha \in (1 - H, \frac{1}{2}) \), using (2.3) and the assumption \( \mathbb{E}\xi^2 < \infty \), we have

\[
\int_0^T \frac{|u^N_t|}{t^\alpha} dt \to 0 \quad \text{as} \quad N \to \infty,
\]

by the Lebesgue dominated convergence theorem. For the second term, we take a positive real number \( \beta \in (\alpha, \frac{1}{2}) \). Then using (2.3), we have

\[
\int_0^T \int_0^t \frac{|u^N_t - u^N_s|}{(t-s)^{\alpha+1}} ds dt \leq C(\omega, T) \int_0^T \int_0^t \frac{1}{(t-s)^{\alpha+1-\beta}} ds dt \rightarrow 0 \quad \text{as} \quad N \to \infty,
\]

by the Lebesgue dominated convergence theorem, since \( \alpha + 1 - \beta < 1 \). Hence, we have shown that \( \|u^N\|_{\alpha,1} \to 0 \) as \( N \) tends to infinity.

### 3. Proof of the Main Result

Let \( \varphi_\xi \) stands for the real valued characteristic function of \( \xi \). Then the parametrized stochastic Fubini theorem for semimartingales (see [12]), Lemma (2.1) and Proposition (2.1) allow us to write the randomized periodogram of the mixed Brownian-fractional Brownian motion \( X \) as

\[
\mathbb{E}_{\xi} I_T(X; \xi) = 2 \int_0^T \int_0^t \varphi_\xi(L(t-s)) dX_s dX_t + [X]_T
\]

\[
= 2 \int_0^T \int_0^t \varphi_\xi(L(t-s)) dW_s dW_t + 2 \int_0^T \int_0^t \varphi_\xi(L(t-s)) dB_H^s dW_t
\]

\[
+ 2 \int_0^T \int_0^t \varphi_\xi(L(t-s)) dB_H^s dW_t + 2 \int_0^T \int_0^t \varphi_\xi(L(t-s)) dB_H^s dB_H^t + [X]_T
\]

\[
= 2J_1 + 2J_2 + 2J_3 + 2J_4 + [X]_T.
\]

Next, we show that as \( L \to \infty \)

\[
J_k \overset{p}{\to} 0, \quad k = 1, 2, 3, 4,
\]

using the facts that

\[
|\varphi_\xi| \leq 1 \quad \text{and} \quad \varphi_\xi(L(t-s)) \to 0 \quad \text{for} \quad s < t \quad \text{as} \quad L \to \infty.
\]

\[
J_1 \overset{p}{\to} 0;
\]

By Itô isometry, we have

\[
\mathbb{E}J_1^2 = \int_0^T \int_0^t \varphi_\xi^2(L(t-s)) ds dt \to 0 \quad \text{as} \quad L \to \infty.
\]
Since Brownian motion $W$ and fractional Brownian motion $B^H$ are independent, we can compute

$$
\mathbb{E}\left( \int_0^T \int_0^t \varphi_\xi(L(t-s))dW_s dB^H_t \right)^2
= \mathbb{E}\left( \mathbb{E}\left( \int_0^T \int_0^t \varphi_\xi(L(t-s))dW_s dB^H_t \right)^2 \mid \mathcal{F}_T \right)
= H(2H-1) \mathbb{E}\left( \int_0^T \int_0^t |u-v|^{2H-2} \int_0^u \varphi_\xi(L(u-s))dW_s \int_v^u \varphi_\xi(L(v-s))dW_s dudv \right)
= H(2H-1) \int_0^T \int_0^t |u-v|^{2H-2} \int_0^{u\wedge v} \varphi_\xi(L(u-s))\varphi_\xi(L(v-s))dsdudv
\rightarrow 0 \text{ as } L \rightarrow \infty,
$$
by the Lebesgue dominated convergence theorem.

Similar to the case $J_2$, we can compute

$$
\mathbb{E}\left( \int_0^T \int_0^t \varphi_\xi(L(t-s))dB^H_s dW_t \right)^2
= \mathbb{E}\left( \mathbb{E}\left( \int_0^T \int_0^t \varphi_\xi(L(t-s))dB^H_s dW_t \right)^2 \mid \mathcal{F}_T \right)
= H(2H-1) \int_0^T \int_0^t \varphi_\xi(L(t-u))\varphi_\xi(L(t-v))|u-v|^{2H-2}dudtdt
\rightarrow 0 \text{ as } L \rightarrow \infty.
$$

By theorem 4.1, [4] and the Lebesgue dominated convergence theorem, we have

$$
\mathbb{E}\left( \int_0^T \int_0^t \varphi_\xi(L(t-s))dB^H_s dB^H_t \right)^2
= (H(2H-1))^2 \int_0^T \int_0^T \int_0^u \varphi_\xi(L(u-s))\varphi_\xi(L(v-t))
\frac{|t-s|^{2H-2}}{|u-v|^{2H-2}}dsdtdudv
\rightarrow 0 \text{ as } L \rightarrow \infty.
$$

**4. More properties and remarks**

Assume that $X$ is a mixed Brownian-fractional Brownian motion, i.e. $X_t = W_t + B^H_t$. Let $\pi = \{t_0 = 0 < t_1 < \cdots < t_n = T\}$ be a partition of
the interval \([0, T]\). Then, we have the following properties of the realized quadratic variation estimator.

- Using the Ito type formula (2.1), we have a representation for the error term, denoted by \(e^1\), as

\[
\sum_{t_k \in \pi} (X_{t_k} - X_{t_{k-1}})^2 - [X, X]_T = 2 \sum_{t_k \in \pi} \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{t_k} dX_s dX_t.
\]

- Hence, for the error term \(e^1\) of the realized quadratic variation estimator, we obtain

\[
\mathbb{E}(e^1) = \mathbb{E} \left( 2 \sum_{t_k \in \pi} \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{t_k} dX_s dX_t \right)
\]

\[
= \mathbb{E} \left( 2 \sum_{t_k \in \pi} \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{t_k} dB_s^H dB_t^H \right) = \sum_{t_k \in \pi} (\Delta t_k)^{2H}.
\]

This implies that the realized quadratic variation is a biased estimator of the quadratic variation \([X, X]\).

- Moreover, its variance is given by

\[
\mathbb{Var}(e^1) = \mathbb{Var} \left( 2 \sum_{t_k \in \pi} \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{t_k} dX_s dX_t \right)
\]

\[
= \sum_{k=1}^{n} \mathbb{Var} \left( 2 \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{t_k} dX_s dX_t \right)
\]

\[
+ 2 \sum_{1 \leq i, j \leq n \atop i < j} \mathbb{Cov} \left( 2 \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{t_i} dX_s dX_t, 2 \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^{t_j} dX_s dX_t \right)
\]

\[
= \sum_{k=1}^{n} \left( (\Delta t_k) + (\Delta t_k)^{2H} \right)^2 + \sum_{1 \leq i, j \leq n \atop i < j} \mathbb{Cov} \left( \mathbb{E}(\Delta B_{t_i}^H)^2(\Delta B_{t_j}^H)^2 \right)^2
\]

\[
= \sum_{k=1}^{n} \left( (\Delta t_k) + (\Delta t_k)^{2H} \right)^2
\]

\[
+ \sum_{1 \leq i, j \leq n \atop i < j} \left( (t_j - t_{i-1})^{2H} + (t_{j-1} - t_i)^{2H} - (t_j - t_i)^{2H} - (t_{j-1} - t_{i-1})^{2H} \right)^2.
\]

- For the special case of equidistant partition \(\pi_n = \{ kT/n \; ; \; k = 0, 1, \ldots, n \}\), the mean and the variance of the error term \(e^1 = e^1_{\pi_n}\) take the forms

\[
\mathbb{E}(e^1_{\pi_n}) = T^{2H} n^{1-2H},
\]

\[
\mathbb{Var}(e^1_{\pi_n}) = 2n \left( \frac{T}{n} \right)^2 + \frac{T^4}{n^4} \sum_{1 \leq i, j \leq n \atop i < j} \left( (j - i - 1)^{2H} + (j - i + 1)^{2H} - 2(j - i)^{2H} \right)^2.
\]
Therefore, we have the asymptotic behaviors
\[ \mathbb{E}(e_{1n}^1) \sim T \quad \text{as} \quad H \downarrow \frac{1}{2}, \]
\[ \mathbb{E}(e_{1n}^1) \to 0 \quad \text{as} \quad n \to \infty \quad \forall \ H > \frac{1}{2}, \]
\[ \text{Var}(e_{1n}^1) \sim 2n \left( \frac{T}{n} \right)^2 = \frac{8T^2}{n} \quad \text{as} \quad H \downarrow \frac{1}{2}. \]

Hence, we see that
\[ \lim_{n \to \infty} \lim_{H \downarrow \frac{1}{2}} \text{Var}(e_{1n}^1) = 0, \]
whereas for two independent Brownian motions \( W^1 \) and \( W^2 \), with \( Z_t = W^1_t + W^2_t \) and a simple computation we have
\[ \text{Var} \left( 2 \int_0^T \int_0^t dZ_s dZ_t \right) = \text{Var} \left( Z_T^2 - [Z, Z]_T \right) = 8T^2. \]

For randomized periodogram, we have the following properties.

- The error term, denoted by \( e^2 \), of the randomized periodogram takes a form as
  \[ \mathbb{E}_\xi I_T(X; L\xi) - [X, X]_T = 2 \int_0^T \int_0^t \varphi_{\xi}(L(t-s))dX_s dX_t. \]

- The mean of the error term \( e^2 \) can be computed as
  \[ \mathbb{E}(e^2) = \mathbb{E} \left( 2 \int_0^T \int_0^t \varphi_{\xi}(L(t-s))dX_s dX_t \right) \]
  \[ = 2H(2H - 1) \int_0^T \int_0^t \varphi_{\xi}(L(t-s))|t-s|^{2H-2}dsdt. \]

Therefore, the randomized periodogram is also a biased estimator of the quadratic variation \([X, X]\).

**Remark 4.1.** It would be interesting to know, whether the estimating based on “discretized” periodogram (or “realized periodogram”) is less noisy than the realized quadratic variation estimator.

**Remark 4.2.** It is also interesting whether one can give an unbiased estimator of the quadratic variation of mixed Brownian-fractional Brownian motion.

**References**

[1] Barndorff-Nielsen, E. O., Shephard, N., (2002). Estimating quadratic variation using realized variance. J. Appl. Econ., 17, 457-477.
[2] Bender, C., Sottinen, T., Valkeila, E., (2008). Pricing by hedging and no-arbitrage beyond semimartingales. Finance Stoch, 12, 441-468.
[3] Cheridito, P., (2001). Mixed fractional Brownian motion. Bernoulli 7, no. 6, 913-934.
[4] Dasgupta, A., Kallianpur, G., (1999). Multiple fractional integrals. Probab. Theory Related Fields, 115, 505-525
[5] Dzhaparidze. K., Spreij. P., (1994). Spectral characterization of the optional quadratic variation processes. Stoch. Procces. and Appl, 54, 165-174.
[6] Föllmer, H., (1981). Calcul d’Ito sans probabilités. Seminar on Probability, XV (Univ. Strasbourg, Strasbourg), 143-150, Lecture Notes in Math., 850, Springer.
SPECTRAL CHARACTERIZATION OF THE QUADRATIC VARIATION

[7] Janson, S., (1997). *Gaussian Hilbert Spaces*. Cambridge University Press.

[8] Memin, J., Mishura. Y., Valkeila. E., (2001). *Inequalities for the moments of Wiener integrals with respect to a fractional Brownian motion*. Statist. Probab. Lett, 51, 197-206.

[9] Mishura, Y., (2008). *Stochastic Calculus for Fractional Brownian Motion and Related Processes*. Lecture Notes in Math., Vol. 1929, Springer.

[10] Nualart, D., Răşcanu, A., (2002). *Differential equations driven by fractional Brownian motion*. Collect. Math. 53, 55-81.

[11] Nualart. D., (2005). *The Malliavin calculus and related topics*. Springer.

[12] Protter, P., (2003). *Stochastic integration and differential equations*. Springer.

[13] Samko, S. G., Kilbas, A. A., Marichev, O.I., (1993). *Fractional integrals and derivatives, Theory and applications*. Gordon and Breach Science Publishers, Yvendon.

[14] Sondermann. D., (2006). *Introduction to stochastic calculus for finance, A new didactic approach*, Lecture Note in Econ. and. Math. System, 579, Springer.

[15] Young, L. C., (1936). *An inequality of the Hölder type, connected with Stieltjes integration*. Acta Math. 67, 251-282.

[16] Zähle, M., (1998). *Integration with respect to fractal functions and stochastic calculus. I*, Probab. Theory Related Fields, 111, 333-372.

Department of Mathematics and Systems Analysis, Aalto University, P.O. Box 11100, 00076 AALTO, FINLAND
E-mail address: ehsan.azmoodeh@aalto.fi and esko.valkeila@aalto.fi