Determinant Expressions for Hyperelliptic Functions in Genus Three

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1. Introduction

Let $\sigma(u)$ and $\varphi(u)$ be the usual functions in the theory of elliptic functions. The following two formulae were found in the nineteenth-century. First one is

$$(-1)^{n(n-1)/2}1!2! \cdots n! \frac{\sigma(u_0 + u_1 + \cdots + u_n) \prod_{i<j} \sigma(u_i - u_j)}{\sigma(u_0)^{n+1} \sigma(u_1)^{n+1} \cdots \sigma(u_n)^{n+1}}$$

$$= \begin{vmatrix}
1 & \varphi(u_0) & \varphi'(u_0) & \varphi''(u_0) & \cdots & \varphi^{(n-1)}(u_0) \\
1 & \varphi(u_1) & \varphi'(u_1) & \varphi''(u_1) & \cdots & \varphi^{(n-1)}(u_1) \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \varphi(u_n) & \varphi'(u_n) & \varphi''(u_n) & \cdots & \varphi^{(n-1)}(u_n)
\end{vmatrix}. \quad (1.1)$$

This formula appeared in the paper of Frobenius and Stickelberger [7]. Second one is

$$(-1)^{n(n-1)/2}(1!2! \cdots (n-1)!)^2 \frac{\sigma(nu)}{\sigma(u)^n} = \begin{vmatrix}
\varphi' & \varphi'' & \cdots & \varphi^{(n-1)} \\
\varphi'' & \varphi''' & \cdots & \varphi^{(n)} \\
\vdots & \vdots & \ddots & \vdots \\
\varphi^{(n-1)} & \varphi^{(n)} & \cdots & \varphi^{(2n-3)}
\end{vmatrix}(u). \quad (1.2)$$

Although this formula can be obtained by a limiting process from (1.1), it was found before [7] by the paper of Kiepert [9].

If we set $y(u) = \frac{1}{2} \varphi'(u)$ and $x(u) = \varphi(u)$, then we have an equation $y(u)^2 = x(u)^3 + \cdots$, that is a defining equation of the elliptic curve to which the functions $\varphi(u)$ and $\sigma(u)$ are attached. Here the complex number $u$ and the coordinate $(x(u), y(u))$ correspond by the equality

$$u = \int_{\infty}^{(x(u), y(u))} \frac{dx}{2y}.$$ 

Then (1.1) and (1.2) is easily rewritten as

$$\begin{vmatrix}
1 & x(u_0) & y(u_0) & x^2(u_0) & yx(u_0) & x^3(u_0) & \cdots \\
1 & x(u_1) & y(u_1) & x^2(u_1) & yx(u_1) & x^3(u_1) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
1 & x(u_n) & y(u_n) & x^2(u_n) & yx(u_n) & x^3(u_n) & \cdots
\end{vmatrix}. \quad (1.3)$$

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and
\[
1!2! \cdots (n-1)! \frac{\sigma(nu)}{\sigma(u)^n} = \left| \begin{array}{cccccc}
x' & y' & (x^2)' & (yx)' & (x^3)' & \cdots \\
x'' & y'' & (x^2)'' & (yx)'' & (x^3)'' & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
x^{(n-1)} & y^{(n-1)} & (x^2)^{(n-1)} & (yx)^{(n-1)} & (x^3)^{(n-1)} & \cdots \\
\end{array} \right| (u),
\]
respectively.

The author recently gave a generalization of the formulae (1.3) and (1.4) to the case of genus two in [13]. Our aim is to give a quite natural generalization of (1.3) and (1.4) and the results in [13] to the case of genus three (see Theorem 3.2 and Theorem 4.2). Our generalization of the function in the left hand side of (1.4) is along a line which appeared for a curve of genus two in the paper [8] of D. Grant. Although Fay’s famous formula, that is (44) in p.33 of [6], possibly relates with our generalizations, no connections are known.

Now we prepare the minimal fundamentals to explain our results. Let \( f(x) \) be a monic polynomial of \( x \) of degree 7. Assume that \( f(x) = 0 \) has no multiple roots. Let \( C \) be the hyperelliptic curve defined by \( y^2 = f(x) \). Then \( C \) is of genus 3 and it is ramified at infinity. We denote by \( \infty \) the unique point at infinity. Let \( C^3 \) be the coordinate space of all values of the integrals, with their initial points \( \infty \), of the first kind with respect to the basis \( dx/2y, xdx/2y, x^2dx/2y \) of the differentials of first kind. Let \( \Lambda \subset C^3 \) be the lattice of their periods. So \( C^3/\Lambda \) is the Jacobian variety of \( C \). We have an embedding \( \iota : C \hookrightarrow C^3/\Lambda \) defined by \( P \mapsto (\int_{\infty}^{P} \frac{dx}{2y}, \int_{\infty}^{P} \frac{xdx}{2y}, \int_{\infty}^{P} \frac{x^2dx}{2y}) \). Therefore \( \iota(\infty) = (0,0,0) \in C^3/\Lambda \). We also have a canonical map \( \kappa : C^3 \twoheadrightarrow C^3/\Lambda \). An algebraic function on \( C \), that we call a hyperelliptic function in this article, is regarded as a function on a universal covering \( \kappa^{-1}\iota(C) (\subset C^3) \) of \( C \). If \( u = (u^{(1)}, u^{(2)}, u^{(3)}) \) is in \( \kappa^{-1}\iota(C) \), we denote by \( (x(u), y(u)) \) the coordinate of the corresponding point on \( C \) by
\[
\begin{align*}
u^{(1)} &= \int_{\infty}^{(x(u), y(u))} \frac{dx}{2y}, & u^{(2)} &= \int_{\infty}^{(x(u), y(u))} \frac{xdx}{2y}, & u^{(3)} &= \int_{\infty}^{(x(u), y(u))} \frac{x^2dx}{2y},
\end{align*}
\]
with appropriate choice of a path of the integrals. Needless to say, we have \( (x(0,0,0), y(0,0,0)) = \infty \).

From our standing point of view, the following three features stand out on the formulae (1.3) and (1.4). Firstly, the sequence of functions of \( u \) whose values at \( u = u_j \) are displayed in the \( (j+1) \)-th row of the determinant of (1.3) is a sequence of the monomials of \( x(u) \) and \( y(u) \) displayed according to the order of their poles at \( u = 0 \). Secondly, while the right hand sides of (1.3) and (1.4) are polynomials of \( x(u) \) and \( y(u) \), where \( u = u_0 \) for (1.4), the left hand sides are expressed in terms of theta functions, whose domain is properly the universal covering space \( C \) (of the Jacobian variety) of the elliptic curve. Thirdly, the expression of the left hand side
of (1.4) states the function of the two sides themselves of (1.4) is characterized as a hyperelliptic function such that its zeroes are exactly the points different from ∞ whose n-plication is just on the standard theta divisor in the Jacobian of the curve, and such that its pole is only at ∞. In the case of the elliptic curve above, the standard theta divisor is just the point at infinity.

Surprisingly enough, these three features just invent good generalizations for hyperelliptic curves. More concretely, our generalization of (1.4) is obtained by replacing the sequence of the right hand side by the sequence

\[1, x(u), x^2(u), x^3(u), y(u), x^4(u), yx(u), \cdots,\]

where \(u = (u^{(1)}, u^{(2)}, u^{(3)})\) is on \(\kappa^{-1}(C)\), of the monomials of \(x(u)\) and \(y(u)\) displayed according to the order of their poles at \(u = (0, 0, 0)\) with replacing the derivatives with respect to \(u \in C\) by those with respect to \(u^{(1)}\) along \(\kappa^{-1}(C)\); and the left hand side of (1.4) by

\[1!2! \cdots (n-1)! \sigma(nu)/\sigma_2(u)^{n^2},\]

where \(\sigma(u) = \sigma(u^{(1)}, u^{(2)}, u^{(3)})\) is a well-tuned Riemann theta series and \(\sigma_2(u) = (\partial \sigma/\partial u^{(2)})(u)\). Therefore, the hyperelliptic function that is the right hand side of the generalization of (1.4) is naturally extended to a function on \(C^3\) via theta functions. Although the extended function on \(C^3\) is no longer a function on the Jacobian, it is expressed simply in terms of theta functions and is treated really similar way to the elliptic functions. The most difficult problem is to find the left hand side of the expected generalization of (1.3). The answer is remarkably pretty and is

\[
\frac{\sigma(u_0 + u_1 + \cdots + u_n) \prod_{i < j} \sigma_3(u_i - u_j)}{\sigma_2(u_0)^{n+1} \sigma_2(u_1)^{n+1} \cdots \sigma_2(u_n)^{n+1}},
\]

where \(u_j = (u_j^{(1)}, u_j^{(2)}, u_j^{(3)})\) are variables on \(\kappa^{-1}(C)\) and \(\sigma_2(u) = (\partial \sigma/\partial u^{(3)})(u)\). If we once find this, we can prove the formula, roughly speaking, by comparing the divisors of the two sides. As same as the formula (1.4) is obtained by a limiting process from (1.3), our generalization of (1.4) is obtained by similar limiting process from the generalization of (1.3).

Although this paper is almost based on [13], several critical facts are appeared in comparison with [13]. Sections 3 and 4 are devoted to generalize (1.3) and (1.4), respectively. We recall in Section 2 the necessary facts for Sections 3 and 4.

The author started this work by suggestion of S. Matsutani concerning the paper [13]. After having worked out this paper, the author tried to generalize the formula (1.3) further to the case of genus larger than three and did not succeed. The author hopes that publication of this paper would contribute to generalize our formula of type (1.3) or (1.1) to cases of genus larger than three in the line of our investigation. Matsutani also pointed out that (1.4) can be generalized to all of hyperelliptic curves. The reader who is interested in the generalization of (1.4) should be consult with the paper [10].
Cantor [5] gave another determinant expression of the function that is characterized in the third feature above for any hyperelliptic curve. The expression of Cantor should be seen as a generalization of a formula due to Brioschi (see [4], p.770, ℓ.3). The Appendix of [10] written by Matsutani reveals the connection of our formula, that is Theorem 4.2 below, and the determinant expression of [5]. So we have three different proofs for the generalization of (1.4) in the case of genus three or below.

There are also various generalizations of (1.1) (or (1.3)) in the case of genus two different from our line. If the reader is interested in them, he should be refered to Introduction of [13].

We use the following notations throughout the rest of the paper. We denote, as usual, by $\mathbb{Z}$ and $\mathbb{C}$ the ring of rational integers and the field of complex numbers, respectively. In an expression of the Laurent expansion of a function, the symbol $(d^o(z_1, z_2, \cdots, z_m) \geq n)$ stands for the terms of total degree at least $n$ with respect to the given variables $z_1, z_2, \cdots, z_m$. When the variable or the least total degree are clear from the context, we simply denote them by $(d^o \geq n)$ or the dots “…”.

For cross references in this paper, we indicate a formula as (1.2), and each of Lemmas, Propositions, Theorems and Remarks also as 1.2.
2. The Sigma Function in Genus Three

In this Section we summarize the fundamental facts used in Sections 3 and 4. Detailed treatment of these facts are given in [1], [2] and [3] (see also Section 1 of [12]).

Let

\[ f(x) = \lambda_0 + \lambda_1 x + \lambda_2 x^2 + \lambda_3 x^3 + \lambda_4 x^4 + \lambda_5 x^5 + \lambda_6 x^6 + \lambda_7 x^7, \]

where \( \lambda_1, \ldots, \lambda_7 \) are fixed complex numbers. Assume that the roots of \( f(x) = 0 \) are different from each other. Let \( C \) be a smooth projective model of the hyperelliptic curve defined by \( y^2 = f(x) \). Then the genus of \( C \) is \( g \). We denote by \( \infty \) the unique point at infinity. In this paper we suppose that \( \lambda_7 = 1 \). The set of forms

\[ \omega^{(1)} = \frac{dx}{2y}, \quad \omega^{(2)} = \frac{x dx}{2y}, \quad \omega^{(3)} = \frac{x^2 dx}{2y} \]

is a basis of the space of differential forms of first kind. We fix generators \( \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \) and \( \beta_3 \) of the fundamental group of \( C \) such that their intersections are \( \alpha_i \cdot \alpha_j = \beta_i \cdot \beta_j = 0, \quad \alpha_i \cdot \beta_j = \delta_{ij} \) for \( i, j = 1, 2, 3 \). If we set

\[
\omega' = \begin{bmatrix}
\int_{\alpha_1} \omega^{(1)} & \int_{\alpha_2} \omega^{(1)} & \int_{\alpha_3} \omega^{(1)} \\
\int_{\alpha_1} \omega^{(2)} & \int_{\alpha_2} \omega^{(2)} & \int_{\alpha_3} \omega^{(2)} \\
\int_{\alpha_1} \omega^{(3)} & \int_{\alpha_2} \omega^{(3)} & \int_{\alpha_3} \omega^{(3)}
\end{bmatrix},
\omega'' = \begin{bmatrix}
\int_{\beta_1} \omega^{(1)} & \int_{\beta_2} \omega^{(1)} & \int_{\beta_3} \omega^{(1)} \\
\int_{\beta_1} \omega^{(2)} & \int_{\beta_2} \omega^{(2)} & \int_{\beta_3} \omega^{(2)} \\
\int_{\beta_1} \omega^{(3)} & \int_{\beta_2} \omega^{(3)} & \int_{\beta_3} \omega^{(3)}
\end{bmatrix}
\]

the lattice of periods of our Abelian functions appearing below is given by

\[ \Lambda = \omega' \begin{bmatrix} Z \\ Z \\ Z \end{bmatrix} + \omega'' \begin{bmatrix} Z \\ Z \\ Z \end{bmatrix} \subset \mathbb{C}^3. \]

Let \( J \) be the Jacobian variety of the curve \( C \). We identify \( J \) with the Picard group \( \text{Pic}^0(C) \) of linear equivalence classes of the divisors of degree 0 of \( C \). Let \( \text{Sym}^3(C) \) be the symmetric product of three copies of \( C \). Then we have a birational map

\[ \text{Sym}^3(C) \to \text{Pic}^0(C) = J \]

\[ (P_1, P_2, P_3) \mapsto \text{the class of } P_1 + P_2 + P_3 - 3 \cdot \infty. \]

We may also identify (the \( \mathbb{C} \)-rational points of) \( J \) with \( \mathbb{C}^3/\Lambda \). We denote by \( \kappa \) the canonical map \( \mathbb{C}^3 \to \mathbb{C}^3/\Lambda \) and by \( \iota \) the embedding of \( C \) into \( J \) given by mapping \( P \) to the class of \( P - \infty \). The image of the triples of the form \((P_1, P_2, \infty)\), by the birational map above, is a theta divisor of \( J \), and is denoted by \( \Theta \). The image \( \iota(C) \) is obviously contained in \( \Theta \). We denote by \( O \) the origin of \( J \). Obviously \( \Lambda = \kappa^{-1}(O) = \kappa^{-1}(\iota(\infty)) \).
Lemma 2.1. As a subvariety of $J$, the divisor $\Theta$ is singular only at the origin of $J$.

A proof of this fact is seen, for instance, in Lemma 1.7.2(2) of [12].

Let

\[
\eta^{(1)} = \frac{\lambda_3 x + 2\lambda_4 x^2 + 3\lambda_5 x^3 + 4\lambda_6 x^4 + 5\lambda_7 x^5}{2y} dx,
\]

\[
\eta^{(2)} = \frac{\lambda_5 x^2 + 2\lambda_6 x^3 + 3\lambda_7 x^4}{2y} dx,
\]

\[
\eta^{(3)} = \frac{x^3 dx}{2y}.
\]

Then $\eta^{(1)}$, $\eta^{(2)}$, and $\eta^{(3)}$ are differential forms of the second kind without poles except at $\infty$ (see [1, p.195, Ex.i] or [2, p.314]). We also introduce matrices

\[
\eta' = \begin{bmatrix}
\int_{\alpha_1} \eta^{(1)} & \int_{\alpha_2} \eta^{(1)} & \int_{\alpha_3} \eta^{(1)} \\
\int_{\alpha_1} \eta^{(2)} & \int_{\alpha_2} \eta^{(2)} & \int_{\alpha_3} \eta^{(2)} \\
\int_{\alpha_1} \eta^{(3)} & \int_{\alpha_2} \eta^{(3)} & \int_{\alpha_3} \eta^{(3)}
\end{bmatrix},
\eta'' = \begin{bmatrix}
\int_{\beta_1} \eta^{(1)} & \int_{\beta_2} \eta^{(1)} & \int_{\beta_3} \eta^{(1)} \\
\int_{\beta_1} \eta^{(2)} & \int_{\beta_2} \eta^{(2)} & \int_{\beta_3} \eta^{(2)} \\
\int_{\beta_1} \eta^{(3)} & \int_{\beta_2} \eta^{(3)} & \int_{\beta_3} \eta^{(3)}
\end{bmatrix}.
\]

The modulus of $C$ is $Z := \omega^{-1} \omega''$. If we set

\[
\delta'' = \begin{bmatrix} 1/2 & 1/2 & 1/2 
\end{bmatrix}, \quad \delta' = \begin{bmatrix} 0 & 1/2 & 1 
\end{bmatrix},
\]

then the sigma function attached to $C$ is defined, as in [3], by

\[
\sigma(u) = c \exp\left(-\frac{1}{2} u\eta' \omega^{-1} u\right) \cdot \sum_{n \in \mathbb{Z}^2} \exp\left[2\pi i \frac{1}{2} \left( t(n + \delta'') Z(n + \delta'') + t(n + \delta')(\omega^{-1} t u + \delta') \right) \right]
\]

with a constant $c$. This constant $c$ is fixed by the following lemma.

Lemma 2.2. The Taylor expansion of $\sigma(u)$ at $u = (0, 0, 0)$ is, up to a multiplicative constant, of the form

\[
\sigma(u) = u^{(1)} u^{(3)} - u^{(2)}^2 - \frac{\lambda_0}{3} u^{(1)}^3 u^{(2)} - \frac{\lambda_1}{3} u^{(1)}^2 u^{(2)}^2 - \frac{\lambda_2}{3} u^{(1)} u^{(2)}^3 - \frac{\lambda_3}{3} u^{(1)}^4 u^{(2)} - \frac{2\lambda_2}{3} u^{(1)}^3 u^{(3)} - \frac{\lambda_5}{3} u^{(2)} u^{(3)} - \frac{\lambda_6}{2} u^{(2)}^2 u^{(3)}^2 + \frac{\lambda_6}{6} u^{(1)} u^{(3)}^3 - \frac{\lambda_7}{3} u^{(2)}^3 (d'' \geq 6), \quad (\lambda_7 = 0),
\]

with the coefficient of the term $u^{(3)}^6$ being $\frac{\lambda_7}{18}$.

Lemma 2.2 is proved in Proposition 2.1.1(3) of [12] by the same argument of [12], p.96. We fix the constant $c$ in (2.1) such that the expansion is exactly of the form in 2.2.
Lemma 2.3. Let \( \ell \) be an element of \( \Lambda \). The function \( u \mapsto \sigma(u) \) on \( \mathbb{C}^3 \) satisfies the translational formula

\[
\sigma(u + \ell) = \chi(\ell)\sigma(u) \exp L(u + \ell, \ell),
\]

where \( \chi(\ell) = \pm 1 \) is independent of \( u \), \( L(u, v) \) is a form which is bilinear over the real field and \( \mathbb{C} \)-linear with respect to the first variable \( u \), and \( L(\ell_1, \ell_2) \) is \( 2\pi\sqrt{-1} \) times an integer if \( \ell_1 \) and \( \ell_2 \) are in \( \Lambda \).

The detail of 2.3 is given in [12], p.286 and Lemma 3.1.2 of [12].

**Lemma 2.4.** (1) The function \( \sigma(u) \) on \( \mathbb{C}^3 \) vanishes if and only if \( u \in \kappa^{-1}(\Theta) \).

(2) Suppose that \( v_1, v_2, v_3 \) are three points of \( \kappa^{-1}(C) \). The function \( u \mapsto \sigma(u - v_1 - v_2 - v_3) \) is identically zero if and only if \( v_1 + v_2 + v_3 \) is contained in \( \kappa^{-1}(C) \). If the function is not identically zero, it vanishes only at \( u = v_j \) modulo \( \Lambda \) for \( j = 1, 2, 3 \) of order 1 or of multiple order according as coincidence of some of the three points.

(3) Let \( v \) be a fixed point of \( \kappa^{-1}(C) \). There exist two points \( v_1 \) and \( v_2 \) of \( \kappa^{-1}(C) \) such that the function \( u \mapsto \sigma(u - v - v_1 - v_2) \) on \( \kappa^{-1}(C) \) is not identically zero and vanishes at \( u = v \) modulo \( \Lambda \) of order 1.

**Proof.** The assertion 2.4(1) and (2) is proved in [1], pp.252-258, for instance. The assertion (3) obviously follows from (2).

We introduce the functions

\[
\varphi_{jk}(u) = -\frac{\partial^2}{\partial u^{(j)} \partial u^{(k)}} \log \sigma(u), \quad \varphi_{jk\ldots r}(u) = \frac{\partial}{\partial u^{(j)}} \varphi_{k\ldots r}(u)
\]

which are defined by Baker. Lemma 2.3 shows that these functions are periodic with respect to the lattice \( \Lambda \). By 2.4(1) we know that the functions \( \varphi_{jk}(u) \) and \( \varphi_{jk\ell}(u) \) have its poles along \( \Theta \). We also use the notation

\[
\sigma_j(u) = \frac{\partial}{\partial u^{(j)}} \sigma(u), \quad \sigma_{jk\ldots r}(u) = \frac{\partial}{\partial u^{(j)}} \sigma_{k\ldots r}(u).
\]

Let \( (u^{(1)}, u^{(2)}, u^{(3)}) \) be an arbitrary point in \( \mathbb{C}^3 \). Then we can find a set of three points \( (x_1, y_1), (x_2, y_2), \) and \( (x_3, y_3) \) on \( C \) such that

\[
\begin{align*}
  u^{(1)} &= \int_{\infty}^{(x_1, y_1)} \omega^{(1)} + \int_{\infty}^{(x_2, y_2)} \omega^{(1)} + \int_{\infty}^{(x_3, y_3)} \omega^{(1)}, \\
  u^{(2)} &= \int_{\infty}^{(x_1, y_1)} \omega^{(2)} + \int_{\infty}^{(x_2, y_2)} \omega^{(2)} + \int_{\infty}^{(x_3, y_3)} \omega^{(2)}, \\
  u^{(3)} &= \int_{\infty}^{(x_1, y_1)} \omega^{(3)} + \int_{\infty}^{(x_2, y_2)} \omega^{(3)} + \int_{\infty}^{(x_3, y_3)} \omega^{(3)}
\end{align*}
\]

(2.2)

with certain choices of the three paths in the integrals. If \( (u^{(1)}, u^{(2)}, u^{(3)}) \) does not belongs to \( \kappa^{-1}(\Theta) \), the set of the three points is uniquely determined. In this situation, one can show the following.
LEMMA 2.5. With the notation above, we have

\[ \wp_{13}(u) = x_1 x_2 x_3, \quad \wp_{23}(u) = -x_1 x_2 - x_1 x_3 - x_3 x_1, \quad \wp_{33}(u) = x_1 + x_2 + x_3. \]

For a proof of this, see [2], p.377. This fact is entirely depends on the choice of forms \( \omega^{(j)} \)'s and \( \eta^{(j)} \)'s.

LEMMA 2.6. If \( u = (u^{(1)}, u^{(2)}, u^{(3)}) \) is on \( \kappa^{-1}_\iota(C) \), then we have

\[ u^{(1)} = \frac{1}{5} u^{(3)^5} + (d^9(u^{(3)}) \geq 6), \quad u^{(2)} = \frac{1}{3} u^{(3)^3} + (d^9(u^{(3)}) \geq 4). \]

This is mentioned in [12], Lemma 2.3.2(2). If \( u \) is a point on \( \kappa^{-1}_\iota(C) \) the \( x \)- and \( y \)-coordinates of \( \iota^{-1}_\kappa(u) \) will be denoted by \( x(u) \) and \( y(u) \), respectively. As is shown in Lemma 2.3.1 of [12], for instance, we see the following.

LEMMA 2.7. If \( u \in \kappa^{-1}_\iota(C) \) then

\[ x(u) = \frac{1}{u^{(3)^2}} + (d^9 \geq -1), \quad y(u) = -\frac{1}{u^{(3)^5}} + (d^9 \geq -4). \]

LEMMA 2.8. (1) Let \( u \) be an arbitrary point on \( \kappa^{-1}_\iota(C) \). Then \( \sigma_2(u) \) is 0 if and only if \( u \) belongs to \( \kappa^{-1}(O) \).

(2) The Taylor expansion of the function \( \sigma_2(u) \) on \( \kappa^{-1}_\iota(C) \) at \( u = (0,0,0) \) is of the form

\[ \sigma_2(u) = -u^{(3)^3} + (d^9(u^{(3)}) \geq 5). \]

Proof. For (1), assume that \( u \in \kappa^{-1}_\iota(C) \) and \( u \not\in \kappa^{-1}(O) \). Then we have

\[ \frac{\sigma_1(u)}{\sigma_2(u)} = \frac{\wp_{13}(u)}{\wp_{23}(u)} = \frac{x_1 x_2 x_3}{-x_1 x_2 - x_2 x_3 - x_3 x_1} \bigg|_{x_1 = x_2 = x_3 = \infty} = -x(u), \quad \frac{\sigma_3(u)}{\sigma_2(u)} = \frac{\wp_{33}(u)}{\wp_{23}(u)} = 0. \]

by using 2.4(1) and 2.5. Hence it must be \( \sigma_3(u) = 0 \) by the second formula. If \( \sigma_2(u) = 0 \) then the first formula yields \( \sigma_1(u) = 0 \). This contradicts to 2.1, 2.4(1) and (2). So it must be \( \sigma_2(u) \neq 0 \). The assertion (2) follows from 2.2 and 2.6. \( \square \)

LEMMA 2.9. Let \( u \) be a point on \( \kappa^{-1}(\Theta) \). The function \( \sigma_3(u) \) vanishes if and only if \( u \in \kappa^{-1}_\iota(C) \).

Proof. We have already proved in the proof of 2.8 that if \( u \in \kappa^{-1}_\iota(C) \) then \( \sigma_3(u) = 0 \). So we prove the converse. Assume that \( u \in \kappa^{-1}(\Theta), \ u \not\in \kappa^{-1}_\iota(C), \) and \( u \) corresponds to the pair of points \( (x_1,y_1) \) and \( (x_2,y_2) \). Then we have

\[ \frac{\sigma_1(u)}{\sigma_3(u)} = \frac{\wp_{13}(u)}{\wp_{33}(u)} = -x_1 x_2, \quad \frac{\sigma_2(u)}{\sigma_3(u)} = \frac{\wp_{23}(u)}{\wp_{33}(u)} = -x_1 - x_2. \]

by using 2.4(1) and 2.5. If \( \sigma_3(u) = 0 \), then the second formula says that \( \sigma_2(u) = 0 \), and the first one says that \( \sigma_1(u) = 0 \). This contradicts to 2.1 by 2.4(1) and (2). So it must be \( \sigma_3(u) \neq 0 \). \( \square \)
Lemma 2.10. Let \( v \) be a fixed point in \( \kappa^{-1}(C) \) different from any point of \( \kappa^{-1}(O) \). Then the function
\[
u \mapsto \sigma_3(u - v)
\]
vanishes of order 2 at \( u = (0, 0, 0) \). Precisely, one has
\[
\sigma_3(u - v) = \sigma_2(v)u^{(3)2} + (d^0(u^{(3)}) \geq 3).
\]

Proof. Since \( u - v \) is on \( \Theta \), we have \( \sigma(u - v) = 0 \). If we write \( u \) as \((x_1, y_1)\) and \( v \) as \((x_2, y_2)\), 2.4(1), 2.5 and 2.7 imply that
\[
\sigma_3(u - v) = \sigma_2(u - v)(x_1 + x_2 + x_3) + (d^0(u^{(3)}) \geq 3).
\]

There exist two points \( v_1 \) and \( v_2 \) in \( \kappa^{-1}(C) \) such that the function \( u \mapsto u^{(j)} - v^{(j)} \) on \( \kappa^{-1}(C) \) is not identically zero and vanishes at \( u = v \) of order 1 by 2.4(3). Let \( m \) be the vanishing order of the function \( u \mapsto u^{(j)} - v^{(j)} \). Then the
vanishing orders of $u \mapsto u^{(j)} - v^{(j)}$ ($j = 1, 2$) are equal to or larger than $m$ by (2.3). Furthermore the expansion

$$\sigma(u - v - v_1 - v_2) = \sigma_1(-v_1 - v_2)(u^{(1)} - v^{(1)}) + \sigma_2(-v_1 - v_2)(u^{(2)} - v^{(2)}) + \sigma_3(-v_1 - v_2)(u^{(3)} - v^{(3)}) + (d^6(u^{(1)} - v^{(1)}), u^{(2)} - v^{(2)}, u^{(3)} - v^{(3)} \geq 2)$$

shows that the vanishing order of $u \mapsto \sigma(u - v - v_1 - v_2)$ is higher than or equal to $m$. Hence $m$ must be 1. On the other hand, 2.2 and (2.3) imply that

$$\sigma_3(u - v) = (u^{(1)} - v^{(1)}) + (d^6(u^{(1)} - v^{(1)}) \geq 2).$$

Thus the statement follows. \qed

**Lemma 2.12.** If $u$ is a point of $\kappa^{-1}_{\iota}(C)$, then

$$\frac{\sigma_3(2u)}{\sigma_2(u)^4} = -2y(u).$$

**Proof.** We first prove that the left hand side is a function on $\iota(C)$. Since $[2]^*\Theta$, the pull-back by duplication in $J$ of $\Theta$, is linearly equivalent to $4\Theta$ as is shown by Corollary 3 of [11], p.59, and an equality $[-1]^*\Theta = \Theta$, the function $\sigma_3(2u)/\sigma(u)^4$ is a function on $J$. For $u \notin \kappa^{-1}_{\iota}(C)$, after multiplying

$$\frac{\wp_{333}(2u)}{\wp_{33}(2u)\wp_{22}(u)^2} = \frac{-2\sigma_3^3 + 3\sigma_3^2 - \sigma_3\sigma_3^2}{\sigma_3^2 - \sigma_3\sigma}(2u) \cdot \frac{\sigma_2^2}{\sigma_2^2 - \sigma_2\sigma}(u)$$

to the function $\sigma(2u)/\sigma(u)^4$, bringing $u$ close to any point of $\kappa^{-1}_{\iota}(C)$, we obtain the left hand side of the desired formula. Here we have used the fact that $u \mapsto \sigma_3(2u)$ does not vanish, which follows from 2.9. Thus the the function $\sigma_3(2u)/\sigma_2(u)^4$ is a function on $\iota(C)$, that is

$$\frac{\sigma_3(2(u + \ell))}{\sigma_2(u + \ell)^4} = \frac{\sigma_3(2u)}{\sigma_2(u)^4}$$

for $u \in \kappa^{-1}_{\iota}(C)$. Lemma 2.8(1) states this function has its only pole at $u = (0,0,0)$ modulo $\Lambda$. Lemma 2.2 and 2.8(2) give that its Laurent expansion at $u = (0,0,0)$ is

$$2 \left( \frac{1}{5}u^{(3)5} \right) - \lambda_7 \cdot 2 \left( \frac{1}{3}u^{(3)3} \right) (2u^{(3)})^2 + \frac{6\lambda_7}{45} (2u^{(3)})^5 + \cdots = \frac{2}{u^{(3)7}} + \cdots.$$ 

Here we have used the assumption $\lambda_7 = 1$. Hence the function must be $-2y(u)$ by 2.7. \qed
Definition-Proposition 2.13. Let $n$ be a positive integer. If $u \in \kappa^{-1} \iota(C)$, then

$$
\psi_n(u) := \frac{\sigma(nu)}{\sigma_2(u)^n}
$$

is periodic with respect to $\Lambda$. In other words it is a function on $\iota(C)$.

This is proved by a similar argument of 2.12. For details, see Proposition 3.2.2 in [12], p.396. By 2.8(2) the function $\psi_n(u)$ has its only pole at $u = (0, 0, 0)$ modulo $\Lambda$. Hence it is a polynomial of $x(u)$ and $y(u)$.

3. A Generalization of the Formula of Frobenius and Stickelberger

The following formula is a natural generalization of the corresponding formula for Weierstrass’ functions $\sigma(u)$ and $\wp(u)$, that is (1.3) for $n = 1$.

Proposition 3.1. If $u$ and $v$ are two points in $\kappa^{-1} \iota(C)$, then

$$
\frac{\sigma_3(u+v)\sigma_3(u-v)}{\sigma_2(u)^2\sigma_2(v)^2} = \begin{vmatrix} 1 & x(u) \\ 1 & x(v) \end{vmatrix}.
$$

Proof. If we regard $u$ to be a variable on $C^3$, the function

$$
u \mapsto \frac{\sigma(u+v)\sigma(u-v)}{\sigma(u)^2\sigma(v)^2}
$$

is periodic with respect to the lattice $\Lambda$. Indeed, the theorem of square ([11], Coroll. 4 in p.59) yields the linear equivalence of $T_v^*\Theta + T_v^*\Theta$ and $2\Theta$, where $T_v^*$ denotes the pull-back of the translation by $v$. After multiplying

$$
\frac{1}{2} \frac{\wp_{33}(u+v)\wp_{33}(u-v)}{\wp_{22}(u)\wp_{22}(v)}
$$

to the function above, bringing $u$ and $v$ close to points on $\kappa^{-1} \iota(C)$, we have the left hand side of the claimed formula because of $\sigma(u \pm v) = \sigma(u) = \sigma(v) = 0$ by 2.4(1) (or (2)). So the left hand side as a function of $u$ is periodic with respect to $\Lambda$. Now we compare divisors modulo $\Lambda$ of the two sides. The left hand side has its only pole at $u = (0, 0, 0)$ modulo $\Lambda$ by 2.8(1). The two zeroes modulo $\Lambda$ of the two sides are coincide by 2.9 (or 2.11). Lemmas 2.8(2) and 2.10 gives its Laurent expansion at $u = (0, 0, 0)$ as follows

$$
-\frac{\sigma_2(v)(u^{(3)}2 + \cdots )\sigma_2(v)(u^{(3)}2 + \cdots )}{(-u^{(3)}2 + \cdots )^2\sigma_2(v)^2} = -\frac{1}{u^{(3)}2 + \cdots }.
$$

The leading term of this coincides with that of the right hand side by 2.7. Hence the desired formula holds for all $v$. □

Our generalization of the formula (1.3) in Introduction is the following.
THEOREM 3.2. Let $n \geq 2$ be an integer. Assume that $u_0$, $u_1$, $\ldots$, $u_n$ belong to $\kappa^{-1} \iota(C)$. Then

$$\frac{\sigma(u_0 + u_1 + \ldots + u_n) \prod_{i<j} \sigma_3(u_i - u_j)}{\sigma_2(u_0)^{n+1} \sigma_2(u_1)^{n+1} \cdots \sigma_2(u_n)^{n+1}}$$

is equal to

\[
\begin{array}{cccccccccc}
1 & x(u_0) & x^2(u_0) & x^3(u_0) & y(u_0) & x^4(u_0) & y(x(u_0)) & \cdots & yx^{(n-5)/2}(u_0) & x^{(n+3)/2}(u_0) \\
1 & x(u_1) & x^2(u_1) & x^3(u_1) & y(u_1) & x^4(u_1) & y(x(u_1)) & \cdots & yx^{(n-5)/2}(u_1) & x^{(n+3)/2}(u_1) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
1 & x(u_n) & x^2(u_n) & x^3(u_n) & y(u_n) & x^4(u_n) & y(x(u_n)) & \cdots & yx^{(n-5)/2}(u_n) & x^{(n+3)/2}(u_n)
\end{array}
\]

or

\[
\begin{array}{cccccccccc}
1 & x(u_0) & x^2(u_0) & x^3(u_0) & y(u_0) & x^4(u_0) & y(x(u_0)) & \cdots & yx^{(n+2)/2}(u_0) & yx^{(n-4)/2}(u_0) \\
1 & x(u_1) & x^2(u_1) & x^3(u_1)^3 & y(u_1) & x^4(u_1) & y(x(u_1)) & \cdots & yx^{(n+2)/2}(u_1) & yx^{(n-4)/2}(u_1) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
1 & x(u_n) & x^2(u_n) & x^3(u_n) & y(u_n) & x^4(u_n) & y(x(u_n)) & \cdots & yx^{(n+2)/2}(u_n) & yx^{(n-4)/2}(u_n)
\end{array}
\]

according as $n$ is odd or even. Here both of the matrices are of size $(n+1) \times (n+1)$.

We prove this Theorem by induction on $n$. First of all we prove the cases of $n = 2$ and $n = 3$. We quote these cases as Lemmas 3.3 and 3.4 below.

LEMMA 3.3. Assume that $u$, $u_1$ and $u_2$ are belong to $\kappa^{-1} \iota(C)$. Then

$$\frac{\sigma(u + u_1 + u_2) \sigma_3(u - u_1) \sigma_3(u - u_2) \sigma_3(u_1 - u_2)}{\sigma_2(u)^3 \sigma_2(u_1)^3 \sigma_2(u_2)^3} = \begin{vmatrix}
1 & x(u) & x^2(u) \\
1 & x(u_1) & x^2(u_1) \\
1 & x(u_2) & x^2(u_2)
\end{vmatrix}.$$

Proof. We suppose that $u$, $u_1$, $u_2$ are any points not on $\kappa^{-1} \iota(C)$. Since the sum of pull-backs of translations $T^*_{u_1 + u_2} \Theta + T^*_{u_1} \Theta + T^*_{u_2} \Theta$ is linearly equivalent to $3 \Theta$ by the theorem of square ([11], Coroll. 4 in p.59), the function

$$\frac{\sigma(u + u_1 + u_2) \sigma(u - u_1) \sigma(u - u_2) \sigma(u_1 - u_2)}{\sigma(u)^3 \sigma(u_1)^3 \sigma(u_2)^3}$$

of $u$ is periodic with respect to the lattice $\Lambda$. As in the proof of 3.1, after multiplying

$$\frac{\varphi_{333}^{333}(u - u_1) \varphi_{333}^{333}(u - u_2) \varphi_{333}^{333}(u_1 - u_2)}{\varphi_{222}^{222}(u) \varphi_{222}^{222}(u_1) \varphi_{222}^{222}(u_2)}$$

to the function above, by bringing $u$, $u_1$, and $u_2$ close to points on $\kappa^{-1} \iota(C)$, we have the left hand side of the claimed formula. Here we have used the fact that $\sigma(u - u_1)$, $\sigma(u - u_2)$, and $\sigma(u_1 - u_2)$ vanish for $u$, $u_1$, and $u_2$ on $\kappa^{-1} \iota(C)$ by Lemma 2.4(2). So the left hand side as a function of $u$ on $\kappa^{-1} \iota(C)$ is periodic with respect to $\Lambda$. Now we regard the both sides to be functions of $u$ on $\kappa^{-1} \iota(C)$. We see the left hand side has its only pole at $u = (0, 0, 0)$ modulo $\Lambda$ by 2.8(1), and has its
Laurent expansion of the left hand side is\(\sigma\) of the left hand side are at \(u\) by 2.8(2) and 2.10, the order of the pole is \(4\times 3.2\). These two are known to be equal by 3.3 and desired formula is proved. It is probably to demonstrate only the case \(n\) from 3.4.

The right hand side is by 2.10. Such coefficient for the right hand side is \(\sigma\) and is contributed only by the functions \(\sigma\) and 2.10 as follows:

\[
\frac{\sigma_3(u_1 + u_2)\sigma_2(u_1)\sigma_2(u_2)\sigma_3(u_1 - u_2)}{\sigma_2(u_1)^3\sigma_2(u_2)^3} \left(\frac{1}{u^{(3)}} + \cdots\right).
\]

The right hand side is

\[
\begin{vmatrix}
1 & x(u_1) & x^2(u_1) & x^3(u_1) \\
1 & x(u_2) & x^2(u_2) & x^3(u_2) \\
1 & x(u_3) & x^2(u_3) & x^3(u_3)
\end{vmatrix}
\]

Hence the leading terms of these expansions coincide by 3.1, and the sides must be equal.

**Lemma 3.4.** Assume that \(u, u_1, u_2\) and \(u_3\) belong to \(\kappa^{-1}(C)\). Then

\[
\frac{\sigma(u + u_1 + u_2 + u_3)\sigma_3(u - u_1)\sigma_3(u - u_2)\sigma_3(u - u_3)\sigma_3(u_1 - u_2)\sigma_3(u_1 - u_3)\sigma_3(u_2 - u_3)}{\sigma_2(u)^3\sigma_2(u_1)^3\sigma_2(u_2)^3\sigma_2(u_3)^3}
= \begin{vmatrix}
1 & x(u) & x^2(u) & x^3(u) \\
1 & x(u_1) & x^2(u_1) & x^3(u_1) \\
1 & x(u_2) & x^2(u_2) & x^3(u_2) \\
1 & x(u_3) & x^2(u_3) & x^3(u_3)
\end{vmatrix}
\]

**Proof.** We know the left hand side of the claimed formula is, as a function of \(u\), a periodic function with respect to \(\Lambda\). Its pole is only at \(u = (0, 0, 0)\) modulo \(\Lambda\) and is contributed only by the functions \(\sigma_2(u)^4, \sigma_3(u - u_1), \sigma_3(u - u_2), \sigma_3(u - u_3)\). By 2.8(2) and 2.10, the order of the pole is \(4 \times 3 - 3 \times 2\), that is 6. The zeroes of the left hand side are at \(u = -u_1, -u_2, \) and \(u_3\) modulo \(\Lambda\) which are coming from \(\sigma(u + u_1 + u_2 + u_3)\); and at \(u = u_1, u_2, u_3\) which are coming from \(\sigma(u - u_1), \sigma(u - u_2), \sigma(u - u_3)\), respectively. These 6 zeroes are of order 1 by 2.11. Thus we see that the divisors of two sides coincide. The coefficient of leading term of the Laurent expansion of the left hand side is

\[
\frac{\sigma(u_1 + u_2 + u_3)\sigma_2(u_1)\sigma_2(u_2)\sigma_2(u_3)\prod_{i<j} \sigma_3(u_i - u_j)}{\sigma_2(u_1)^4\sigma_2(u_2)^4\sigma_2(u_3)^4}
\]

by 2.10. Such coefficient for the right hand side is

\[
\begin{vmatrix}
1 & x(u_1) & x^2(u_1) \\
1 & x(u_2) & x^2(u_2) \\
1 & x(u_3) & x^2(u_3)
\end{vmatrix}
\]

These two are known to be equal by 3.3 and desired formula is proved. \(\Box\)

**Proof of Theorem 3.2.** The best way to explain the general step of the induction is probably to demonstrate only the case \(n = 4\). The case of \(n = 4\) is claimed as
follows. Assume that \( u, u_1, u_2, u_3, \) and \( u_4 \) belong to \( \iota(C) \). Then we want to prove the equality

\[
\frac{\sigma(u + u_1 + u_2 + u_3 + u_4)\sigma_3(u - u_1)\sigma_3(u - u_2)\sigma_3(u - u_3)\sigma_3(u - u_4)\prod_{i < j} \sigma(u_i - u_j)}{\sigma_2(u)^5\sigma_2(u_1)^5\sigma_2(u_2)^5\sigma_2(u_3)^5\sigma_2(u_4)^5} = \begin{vmatrix}
1 & x(u) & x^2(u) & x^3(u) & y(u) \\
1 & x(u_1) & x^2(u_1) & x^3(u_1) & y(u_1) \\
1 & x(u_2) & x^2(u_2) & x^3(u_2) & y(u_2) \\
1 & x(u_3) & x^2(u_3) & x^3(u_3) & y(u_3) \\
1 & x(u_4) & x^2(u_4) & x^3(u_4) & y(u_4)
\end{vmatrix}.
\]

We obviously see that the left hand side of the formula above, as a function of \( u \), is periodic with respect to \( \Lambda \) by the same argument of 3.1, 3.3 and 3.4, and that it has its only pole at \( u = (0, 0, 0) \) modulo \( \Lambda \). The order of the pole is \( 5 \times 3 \) coming from \( \sigma_2(u)^5 \) minus \( 2 \times 4 \) coming from \( \sigma_3(u - u_j) \) for \( j = 1, 2, 3, 4 \); and that is equal to \( 7 \). We know, by 2.11 that there are four obvious zeroes at \( u = u_j \) modulo \( \Lambda \) of order 1 coming from \( \sigma(u - u_j) \). These are also zeroes of the right hand side. Since the right hand side is a polynomial of \( x(u) \) and \( y(u) \), it has its only pole at \( u = (0, 0, 0) \) modulo \( \Lambda \). Its order is \( 7 \) coming from the \((1,1)\)-entry \( y(u) \).

So we denote rest zeroes modulo \( \Lambda \) of the right hand side by \( \alpha, \beta, \) and \( \gamma \). Then the theorem of Abel-Jacobi implies that \( u_1 + u_2 + u_3 + u_4 + \alpha + \beta + \gamma = (0, 0, 0) \) modulo \( \Lambda \). This means \( \sigma(u + u_1 + u_2 + u_3 + u_4) \) is equal to \( \sigma(u - \alpha - \beta - \gamma) \) times a trivial theta function. Especially these two sigma functions have the same zeroes. Since the latter function has obviously zeroes at \( u = \alpha, \beta, \) and \( \gamma \) modulo \( \Lambda \) by 2.4(2), the divisors modulo \( \Lambda \) of these two sides coincide. We can show, as in the proof of 3.3 or 3.4, that the coefficients of the leading terms of the two sides in their Laurent expansions also coincide by using the formula of 3.4. The general steps in the induction is done by the same way. Thus our proof is completed. \( \square \)

4. Determinant Expression of Generalized Psi-Functions

In this section we mention a generalization of the formula of (0.2) displayed in Introduction. Our formula is natural generalization of the formula given in Section 3 of [13]. Although we can extend this generalization further to all of hyperelliptic curves as in [10], we give here the case of genus three by limiting process from 3.2.

The following formula is analogous to 3.1 in [13].

**Lemma 4.1.** Let \( j \) be 1, 2, or 3. We have

\[
\lim_{u^{(j)} \to v^{(j)}} \frac{\sigma_3(u - v)}{u^{(j)} - v^{(j)}} = \frac{1}{x^{j-1}(v)}.
\]

**Proof.** Because of 3.1 we have

\[
x(u) - x(v) = \frac{\sigma_3(u + v)}{\sigma_2(u)^2\sigma_2(v)^2} \cdot \frac{\sigma_3(u - v)}{u^{(j)} - v^{(j)}}.
\]
Now we bring $u^{(j)}$ close to $v^{(j)}$. Then the limit of the left hand side is
\[
\lim_{u^{(j)} \to v^{(j)}} \frac{x(u) - x(v)}{u^{(j)} - v^{(j)}} = \frac{dx}{du^{(j)}}(v).
\]
This is equal to $\frac{2y}{x^{j-1}}(v)$ by (2.2). The assertion follows from 2.12.

Since our proof of the following Theorem obtained by quite similar argument by using 4.1 as in the case of genus two (see [13]), we leave the proof to the reader.

Theorem 4.2. Let $n$ be an integer greater than 3. Let $j$ be any one of $\{1, 2, 3\}$. Assume that $u$ belongs to $\kappa^{-1}(C)$. Then the following formula for the function $\psi_n(u)$ of 2.13 holds:

\[
(1!2! \cdots (n-1)!)\psi_n(u) = x^{(j-1)n(n-1)/2}(u) \times \begin{vmatrix}
  x' & (x^2)' & (x^3)' & y' & (x^4)' & (yx)' & (x^5)' & \cdots \\
  x'' & (x^2)'' & (x^3)'' & y'' & (x^4)'' & (yx)'' & (x^5)'' & \cdots \\
  x''' & (x^2)''' & (x^3)''' & y''' & (x^4)''' & (yx)''' & (x^5)''' & \cdots \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
  x^{(n-1)} & (x^2)^{(n-1)} & (x^3)^{(n-1)} & y^{(n-1)} & (x^4)^{(n-1)} & (yx)^{(n-1)} & (x^5)^{(n-1)} & \cdots 
\end{vmatrix}(u).
\]

Here the size of the matrix is $n - 1$ by $n - 1$. The symbols $'$, $''$, $\cdots$, $^{(n-1)}$ denote $\frac{d}{du^{(n-1)}}(x^2), \left(\frac{d}{du^{(n-1)}}\right)^2, \cdots, \left(\frac{d}{du^{(n-1)}}\right)^{n-1}$, respectively.
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