Towards loop quantum supergravity (LQSG): II. $p$-form sector

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Abstract
In our companion paper, we focused on the quantization of the Rarita–Schwinger sector of supergravity theories in various dimensions by using an extension of loop quantum gravity to all spacetime dimensions. In this paper, we extend this analysis by considering the quantization of additional bosonic fields necessary to obtain a complete SUSY multiplet next to graviton and gravitino in various dimensions. As a generic example, we study concretely the quantization of the 3-index photon of minimal 11$d$ SUGRA, but our methods easily extend to more general $p$-form fields. Due to the presence of a Chern–Simons term for the 3-index photon, which is due to local SUSY, the theory is self-interacting and its quantization far from straightforward. Nevertheless, we show that a reduced phase space quantization with respect to the 3-index photon Gauß constraint is possible. Specifically, the Weyl algebra of observables, which deviates from the usual CCR Weyl algebras by an interesting twist contribution proportional to the level of the Chern–Simons theory, admits a background independent state of the Narnhofer–Thirring type.

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1. Introduction
In our companion papers [1–5], we studied the canonical formulation of general relativity (gravitons) coupled to standard matter in terms of connection variables for a compact gauge group without second class constraints in order that loop quantum gravity (LQG) quantization methods, so far formulated only in three and four spacetime dimensions [6, 7], apply.

The actual motivation for doing this comes from supergravity and string theory [8, 9]. String theory is considered as a candidate for a UV completion of general relativity, which in its present formulation requires extra dimensions and supersymmetry. Supergravity is considered as the low energy effective field theory limit of string theory. One may therefore call string
theory a top–bottom approach. In this series of papers we take first steps towards a bottom–top approach in that we try to canonically quantize the supergravity theories by LQG methods. While string theory in its present form needs a background dependent and perturbative quantum formulation, the LQG quantum formulation is by design background independent and non-perturbative. On the other hand, quantum string theory is much richer above the low energy field theory limit, containing an infinite tower of higher excitation modes of the string, which come into play only when approaching the Planck scale and which are necessary in order to find a theory which is finite at least order by order in perturbation theory.

The quantization of supergravity is therefore the ideal arena in which to compare these two complementary approaches to quantum gravity, which was not possible so far. At least at low energies, that is, in the semiclassical limit, the two theories should agree with each other, as otherwise they would quantize two different classical theories. Evidently, this opens the very exciting possibility of cross fertilization between the two approaches, which we are going to address in future publications.

The new field content of supergravity theories as compared to standard matter Lagrangians are (1) Majorana (or Majorana–Weyl) spinor fields of spin 1/2, 3/2 including the Rarita–Schwinger field (gravitino) and (2) additional bosonic fields that appear in order to obtain a complete supersymmetry multiplet in the dimension and the amount \( N \) of supersymmetry charges under consideration. The treatment of the Rarita–Schwinger sector and its embedding in the framework of [1–5] was accomplished in [10]. In this paper, we complete the quantization of the extra matter content of many supergravity theories by considering the quantization of the additional bosonic fields, in particular, \( p \)-form fields. Specifically, for reasons of concreteness, we quantize the 3-index photon of 11\( d \) supergravity but it will transpire that the methods employed generalize to arbitrary \( p \).

What makes the quantization possible is that the Gauß constraints of the 3-index photon form an Abelian ideal in the constraint algebra. If this ideal (or subalgebra) would be non-Abelian, then our methods would be insufficient and we most probably would have to use methods from higher gauge theory [11–15] such as \( p \)-groups, \( p \)-holonomies etc, a subject which at the moment is not yet sufficiently developed from the mathematical perspective (see [16] for the state of the art of the subject). Despite the Abelian character of this additional Gauß constraint, the quantization of the theory is not straightforward and cannot be performed in complete analogy to the treatment of the Abelian Gauß constraint of standard 1-form matter [17]. This is due to a Chern–Simons term in the supergravity action, whose presence is dictated by supersymmetry and which makes the theory in fact self-interacting, that is, the Hamiltonian is a fourth order polynomial in the 3-connection and its conjugate momentum just like in Yang–Mills theory. In particular, while one can define a holonomy flux algebra as for Abelian Maxwell-theory, the Ashtekar–Isham–Lewandowski representation [18, 19] is inadequate because the Abelian gauge group does not preserve the holonomy flux algebra.

A solution to the problem lies in performing a reduced phase space quantization in terms of a twisted holonomy flux algebra, which is in fact Gauß invariant. We were not able to find a background independent representation of the corresponding Heisenberg algebra, which also differs by a twist from the usual one, however, one succeeds when formulating the quantum theory in terms of the corresponding Weyl elements. The resulting Weyl algebra is not of standard form and to the best of our knowledge it has not been quantized before. We show that it admits a state of the Narnhofer–Thirring type [20] whence the Hilbert space representation follows by the GNS construction. The Hamiltonian (constraint) can be straightforwardly expressed in terms of the Weyl elements, in fact it is quadratic in terms of the classical observables, that is, the generators of the Heisenberg algebra.
This paper’s architecture is as follows. In section 2, we sketch the Hamiltonian analysis of the 3-index photon in a self-contained fashion for the benefit of the reader and in order to settle our notation. We also describe in detail why one cannot straightforwardly apply methods from LQG as mentioned above. In section 3, we display the reduced phase space quantization solution in terms of the twisted holonomy flux algebra. Finally, in section 4, we summarize and conclude.

2. Classical Hamiltonian analysis of the 3-index-photon action

The Hamiltonian analysis of the full 11d SUGRA Lagrangian has been performed in [21]. We will review the analysis of the contribution of the 3-index-photon 3-form \( F_{\mu_1\cdots\mu_4} \) to the 11d SUGRA Lagrangian with Chern–Simons term. This part of the Lagrangian is given up to a numerical constant by

\[
\mathcal{L}_C = -\frac{1}{2} |g|^{1/2} F_{\mu_1\cdots\mu_4} F^{\mu_1\cdots\mu_4} - \alpha |g|^{1/2} F_{\mu_1\cdots\mu_4} J^{\mu_1\cdots\mu_4}
\]

\[
- \frac{c}{2} |g|^{1/2} F_{\mu_1\cdots\mu_4} J_{\mu_5\cdots\mu_9} \epsilon^{\mu_1\cdots\mu_4\nu_1\cdots\nu_9} g^{\nu_1\nu_2 \cdots \nu_9}. \tag{2.1}
\]

Here, \( F = da \). \( F_{\mu_1\cdots\mu_4} = \partial_\mu A_{\nu_1\cdots\nu_4} \) is the curvature of the 3-index-photon and indices are moved with the spacetime metric \( g_{\mu\nu} \). Furthermore, \( J \) is a totally skew tensor current bilinear in the graviton field not containing derivatives, whose explicit form does not need to concern us here, except that it does not depend on any other fields. Finally, \( c, \alpha \) are positive numerical constants whose value is fixed by the requirement of local supersymmetry [22]. The number \( c \) could be called the level of the Chern–Simons theory in analogy to \( a = D + 1 = 3 \).

We proceed to the 10 + 1 split of this Lagrangian in a coordinate system with coordinates \( t, x^\alpha; \alpha = 1, \ldots, 10 \) adapted to a foliation of the spacetime manifold. The result of a tedious calculation is given by

\[
F_{\mu_1\cdots\mu_4} F^{\mu_1\cdots\mu_4} = 4 F_{\alpha_1\cdots\alpha_4} F^{\alpha_1\cdots\alpha_4} + F_{\alpha_1\cdots\alpha_3} F^{2\alpha_1\cdots\alpha_3},
\]

\[
G_{\alpha_1\cdots\alpha_3, \beta_1\cdots\beta_3} = g^{\alpha_1\beta_1} g^{\alpha_2\beta_2} g^{\alpha_3\beta_3} + 3 g^{\alpha_1\beta_1} g^{\alpha_2\beta_2} g^{\alpha_3\beta_3} - g^{\alpha_1\beta_1} g^{\alpha_2\beta_2} g^{\alpha_3\beta_3},
\]

\[
M_{\alpha_1\cdots\alpha_3, \beta_1\cdots\beta_4} = g^{\alpha_1\beta_1} g^{\alpha_2\beta_2} g^{\alpha_3\beta_3} g^{\alpha_4\beta_4} - 3 g^{\alpha_1\beta_1} g^{\alpha_2\beta_2} g^{\alpha_3\beta_3} g^{\alpha_4\beta_4},
\]

\[
F_{\alpha_1\cdots\alpha_4} F_{\beta_1\cdots\beta_4} = V_1 \epsilon_{\alpha_1\cdots\alpha_4, \beta_1\cdots\beta_4} - 4 \epsilon_{\alpha_1\cdots\alpha_4, \beta_1\cdots\beta_4} F_{\alpha_1\cdots\alpha_4} F_{\beta_1\cdots\beta_4},
\]

\[
V_1 = g^{\alpha_1\beta_1} \cdots g^{\alpha_4\beta_4} F_{\alpha_1\cdots\alpha_4} F_{\beta_1\cdots\beta_4},
\]

\[
F_{\mu_1\cdots\mu_4} F_{\nu_1\cdots\nu_4} A_{\rho_1\cdots\rho_4} \epsilon^{\mu_1\cdots\mu_4, \nu_1\cdots\nu_4, \rho_1\cdots\rho_4} = \epsilon_{\alpha_1\cdots\alpha_4, \beta_1\cdots\beta_4} F_{\alpha_1\cdots\alpha_4} F_{\beta_1\cdots\beta_4} A_{\epsilon_1\cdots\epsilon_4},
\]

\[
J^{\mu_1\cdots\mu_4} = 4 j^{\mu_1\cdots\mu_4} F_{\mu_1\cdots\mu_4} + V_2,
\]

\[
V_2 = j^{\mu_1\cdots\mu_4} F_{\mu_1\cdots\mu_4},
\]

(2.2)

where we used \( \epsilon^{\alpha_1\cdots\alpha_4} = \epsilon_{\alpha_1\cdots\alpha_4} \) and defined \( j^{\mu_1\cdots\mu_4} := j^{\mu_1\cdots\mu_4} \). The potential terms \( V_1, V_2 \) only depend on the spatial components of the curvature and do not contain time derivatives.

Using

\[
F_{\alpha_1\cdots\alpha_4} = \frac{1}{2} \epsilon_{\alpha_1\cdots\alpha_4, \beta_1\cdots\beta_4} J^{\beta_1\cdots\beta_4} - 3 \epsilon_{\alpha_1\cdots\alpha_4, \beta_1\cdots\beta_4} A_{\beta_1\cdots\beta_4}, \tag{2.3}
\]

we may perform the Legendre transform. The momentum conjugate to \( A \) reads

\[
\pi^{\alpha_1\cdots\alpha_4} = \frac{\partial \mathcal{L}}{\partial \dot{A}_{\alpha_1\cdots\alpha_4}} = -|g|^{1/2} \left[ G^{\alpha_1\cdots\alpha_4, \beta_1\cdots\beta_4} F_{\beta_1\cdots\beta_4} - M^{\alpha_1\cdots\alpha_4, \beta_1\cdots\beta_4} F_{\beta_1\cdots\beta_4} + \alpha J^{\alpha_1\cdots\alpha_4} \right] - c \epsilon_{\alpha_1\cdots\alpha_4, \beta_1\cdots\beta_4} A_{\beta_1\cdots\beta_4} \epsilon^{\alpha_1\cdots\alpha_4}, \tag{2.4}
\]
We may solve (2.4) for $F_{\alpha_1\alpha_2\alpha_3}$

$$F_{\alpha_1\alpha_2\alpha_3} = - \frac{1}{8} |g|^{1/2} G_{\alpha_1\alpha_2\alpha_3} \sum_{b_1} \left( \pi^{b_1} b_1 + B^{b_1} b_1 + \alpha |g|^{1/2} \right),$$

$$G_{\alpha_1\alpha_2\alpha_3} = \frac{1}{2} \epsilon^{\alpha_1\alpha_2\alpha_3} \epsilon_{\alpha_1\alpha_2} \epsilon_{\alpha_1\alpha_3} q^{\alpha_1} q^{\alpha_2} q^{\alpha_3}$$

where

$$G_{\alpha_1\alpha_2\alpha_3} = - \frac{1}{8} \sum_{b_1} \left( \pi^{b_1} b_1 + B^{b_1} b_1 + \alpha |g|^{1/2} \right) G_{b_1 b_1 b_1},$$

(2.5)

defines the inverse of $G$.

Inverting (2.3) for $\Lambda$ and using (2.4) and (2.2) we obtain for the Hamiltonian after a longer calculation

$$H = \int d^{10} x \left[ \pi^{\alpha_1\ldots\alpha_5} - \pi^{\alpha_1\ldots\alpha_5} \right] - L$$

$$= - \int d^{10} x \left[ 3 A_{\alpha_1\alpha_2} G_{\alpha_3}^{\alpha_4\alpha_5} + 2 |g|^{-1/2} G_{\alpha_1\alpha_2\alpha_3} b_1 b_2 \left( \pi + B + \alpha j \right) \right]^{\alpha_4\alpha_5} \pi^{\alpha_1\ldots\alpha_5} + |g|^{-1/2} [V_1 / 2 + \alpha V_2]],$$

$$G_{\alpha_3} := - \delta_{\alpha_3} \pi^{\alpha_1\ldots\alpha_5} - \frac{1}{2} \epsilon^{\alpha_1\alpha_2} \epsilon_{\alpha_1\alpha_2} q^{\alpha_1} q^{\alpha_2} q^{\alpha_3},$$

(2.7)

where an integration by parts has been performed in order to isolate the Lagrange multiplier $A_{\alpha_1\alpha_2}$. Using the ADM frame metric components

$$g^{\alpha} = -1 / N^2, \quad g^a = \alpha / N^2, \quad g^{ab} = q^{ab} - N^a N^b / N^2;$$

$$g^{\alpha} = -N^2, \quad g^a = q^{ab} N^b, \quad g^{ab} = q^{ab},$$

(2.8)

with $q^{ab}$ the induced metric on the spatial slices and lapse respectively shift functions $N, N^a$ we can easily decompose the piece of $H$ independent of the 3-index Gauß constraint $G_{\alpha_3}^{\alpha_4\alpha_5}$ into the contributions $N^a H_{\alpha_3} + N H_{\alpha_4}$ to the spatial diffeomorphism constraint and Hamiltonian constraint, however, we will not need this at this point.

We will drop the subscript $C$ in what follows, since in this paper we are only interested in the $p$-form sector. We smear the Gauß constraint with a 2-form $\Lambda$, that is

$$G[\Lambda] := \int d^{10} x \Lambda_{ab} G^{ab}$$

(2.9)

and study the gauge transformation behavior of the canonical pair $(A_{abc}, \pi^{abc})$ with non-vanishing Poisson brackets

$$\{ \pi^{\alpha_1\ldots\alpha_5}, \Lambda_{b_1 b_2 b_3} \} = \delta^{(10)} (x, y) \delta^{\alpha_1}_{b_1} \delta^{\alpha_2}_{b_2} \delta^{\alpha_3}_{b_3}.$$  

(10)

We find

$$\{ G[\Lambda], A_{\alpha_1\ldots\alpha_5} \} = - \delta_{\alpha_1} \Lambda_{\alpha_2\alpha_3},$$

$$\{ G[\Lambda], \pi^{\alpha_1\ldots\alpha_5} \} = c \epsilon^{\alpha_1\ldots\alpha_5} \epsilon_{\alpha_1\alpha_2} \epsilon_{\alpha_1\alpha_3} \delta_{\alpha_1} \Lambda_{\alpha_2\alpha_3} F_{\alpha_4 \alpha_5}.$$  

(2.11)

These equations can be written more compactly in differential form language, in terms of which they are easier to memorize. Introducing the dual 7-pseudo-form

$$(* \pi)_{\alpha_1 \ldots \alpha_7} := \frac{1}{12} \epsilon^{\alpha_1 \ldots \alpha_7} \epsilon_{\alpha_1 \ldots \alpha_5} \pi^{\alpha_5 \alpha_7},$$

(12)

we may write (2.11) as

$$\delta_\Lambda A = - d \Lambda, \quad \delta_\Lambda \pi = c (d \Lambda) \wedge F,$$

(13)

3 Notice that $\pi^{abc}$ is a tensor density of type $T_0^3$ and density weight one.
Since the right hand side of (2.13) is closed, in fact exact, it would seem that the observables of the theory can be coordinatized by integrals of $A$ and $\ast \pi$ respectively over closed 3-submanifolds or 7-submanifolds respectively.

The $G(A)$ generate an Abelian ideal in the constraint algebra since
\[\{G[A_1], G[A_2]\} = 0, \quad \{G[A], H(\varepsilon)\} = 0,\]
where $H(\varepsilon)$ is the integrand of $H$ in (2.7) and since the only $\pi$ or $A$ dependent contributions to the Hamiltonian and spatial diffeomorphism constraints are contained in $H(\varepsilon)$.

We see that due to the non-vanishing Chern–Simons constant $c$, the transformation behaviour of $\ast \pi$ differs from the spatial transformation behaviour with respect to the higher dimensional analogue of the usual Maxwell type of Gauß law, which would be just the divergence term $\partial_{\mu} \pi^\mu$. In particular, $\pi^{abc}$ itself is not gauge invariant. This ‘twisted’ Gauß constraint (2.7) can be written in the form
\[G^{a_1 a_2} := \partial_{\theta_i} \left[ \pi^{a_1 a_2} - \frac{1}{2} \epsilon^{a_1 a_3 a_4 b_1 b_2 c_1 c_2} \partial_{b_1} \partial_{b_2} \epsilon_{c_1 c_2} A_{b_3 b_4} F_{c_3 c_4} \right] =: \partial_{\theta_i} \pi^{a_1 a_2},\] which suggests to introduce a new momentum $\pi'$. Unfortunately, this does not work because $\ast (\pi' - \pi) = A \wedge F$ does not have a generating functional $K$ with $\delta K/\delta A = A \wedge F$, since the only possible candidate $K = \int A \wedge A \wedge F \equiv 0$ identically vanishes in the dimensions considered here. Since this is not the case, the Poisson brackets of $\pi'$ with itself do not vanish and neither is $\pi'$ gauge invariant as we will see below, so that there is no advantage of working with $\pi'$ as compared to $\pi$.

The presence of the twist term in the Gauß constraint leads to the following difficulty when trying to quantize the theory on the usual LQG type kinematical Hilbert space.

Such a Hilbert space would roughly be generated by a holonomy flux algebra constructed from holonomies
\[A(e) = \exp \left( i \int_e A \right), \quad \pi(S) = \int_S \ast \pi,\]
where $e$ and $S$ are oriented three-dimensional and seven-dimensional submanifolds respectively, which we call ‘edges’ and surfaces in what follows. One could then study the GNS Hilbert space representation generated by the LQG type of positive linear functional
\[\omega(f(\pi(S_1) \cdots \pi(S_n))) = 0, \quad \omega(f) = \mu[f],\]
where $\mu$ is an LQG type measure on a space of generalized connections $\mathcal{A}$. One can define it abstractly by requiring that the charge network functions
\[T_{\gamma, n} = \prod_{e \in \gamma} A(e)^{n_e}, \quad n_e \in \mathbb{Z}\]
form an orthonormal basis in the corresponding $\mathcal{H} = L_2(\mathcal{A}, \mu)$, see [7] for details. Here, a graph $\gamma$ is a collection of edges which are disjoint up to intersections in ‘vertices’, which are oriented 2-manifolds. The possible intersection structure of these cobordisms should be tamed by requiring that all submanifolds are semi-analytic.

Up to here everything is in full analogy with LQG. The problem is now to isolate the Gauß invariant subspace of the Hilbert space. While the connection transforms as in a theory with untwisted Gauß constraint, it appears that we can solve it by requiring that charges add up to zero at vertices. However, this does not work because while such a vector is annihilated by the divergence term in $G^{a b}$, it is not by the second term $\propto A \wedge F$. Even more disastrous, the term $A \wedge F$ does not exist in this representation which is strongly discontinuous in the holonomies so that operators $A, F$ do not exist. Finally, although $\pi$ is not Gauß invariant, it leaves this would be gauge invariant subspace invariant, which reveals that this subspace is not the kernel of the twisted Gauß constraint.
We therefore must be more sophisticated. Since the $A$ dependent terms in $G$ cannot be quantized on the kinematical Hilbert space, we must exponentiate it.

Consider the Hamiltonian flow of $G[A]$
\[ \exp(iG[A], \cdot)A = A - d\Lambda, \quad \exp(iG[A], \cdot) \star \pi = \star \pi + c(d\Lambda) \wedge F, \]
which is a Poisson automorphism $\alpha_A$ (canonical transformation) and one would like to secure that an implementation of the corresponding automorphism group $\alpha_A \circ \alpha_{A'} = \alpha_{A+\Lambda'}$ by unitary operators $U(\Lambda)$ exists. The $U(\Lambda)$ would correspond to the desired exponentiation of the Gauß constraint. One way of securing this is by looking for an invariant state $\omega = \omega \circ \alpha_A$ on the holonomy-flux algebra (see [23]) for the details for this construction). This would then open the possibility that the Gauß constraint can be solved by group averaging methods. The first problem is that the automorphisms do not preserve the holonomy flux algebra because there appears an $F$ on the right hand side of (2.19) which should appear exponentiated in order that the algebra closes. This forces us to pass to exponentiated fluxes, that is, to the corresponding Weyl algebra defined by exponentials of $\pi, A$. This algebra is now preserved by the automorphisms, as one can see by an appeal to the Baker–Campbell–Hausdorff formula. However, we now see that the state (2.17) is not invariant, because
\[ \omega(e^{i\pi(S)}) = 1, \quad \omega(\alpha_A(e^{i\pi(S)})) = \omega(e^{i\pi(S)} - c(F \wedge F)) = 0 \]
for suitable choices of $\Lambda$. In the GNS Hilbert space we would like to have unitary operators $U(\Lambda)$ such that for any element $W$ in the Weyl algebra we have $U(\Lambda)\pi(W)U(\Lambda)^* = \pi(\alpha_A(W))$. Then (2.20) is compatible with unitarity only if the LQG vacuum $\Omega$ is not invariant under $U(\Lambda)$. Now the operator $U(\Lambda)$ should correspond to $\exp(iG[A])$ and using a calculation similar to (2.14) and the BCH formula one shows that on the LQG vacuum $\Omega = 1$ it reduces formally to $U(\Lambda)\Omega = \exp(i\epsilon/2 \int \Lambda \wedge F)\Omega$ which is ill defined as it stands. We must therefore define $U(\Lambda)\Omega$ to be some state in the GNS Hilbert space which has a component orthogonal to the vacuum and such that the representation property $U(\Lambda)U(\Lambda') = U(\Lambda + \Lambda'), \quad U(\Lambda)^* = U(-\Lambda)$ (possibly up to a projective twist) holds. We did not succeed to find a solution to this problem indicating that a unitary implementation of the Gauss constraint is impossible in the LQG representation and even if it were possible, the strategy outlined in the next section is certainly more natural. We also remark that solving the constraint by group averaging methods becomes non-trivial if not impossible in case of the non-existence of $U(\Lambda)$. Even if we could somehow construct the Gauß invariant Hilbert space, the observables $A(e), \exp(i\pi(S))$ with $\delta e = \delta S = 0$, which leave the physical Hilbert space invariant, are insufficient to approximate (for small $e, S$) the $\pi$ dependent terms appearing in the Hamiltonian (2.7), as one can check explicitly.

3. Reduced phase space quantization

In the previous section, we established that a quantization in strict analogy to the procedure followed in LQG does not work. While a rigorous kinematical Hilbert space can be constructed, the Dirac operator constraint method of looking for the kernel of the Gauß constraint is problematic. As an alternative, a reduced phase space quantization suggests itself. This has a chance to work due to the observation (2.14) which demonstrates that $H(x)$ only depends on observables. Indeed, $H(x)$ depends, except for $G^{ab}$ which is a trivial observable since it is constrained to vanish, only on the combination $\pi + B + \alpha_j$. Obviously $j$ trivially Poisson commutes with $G$. Unpacking $B$ from (2.5), we see that $\pi + B$ is a linear combination (with only metric dependent coefficients) of $F$ and
\[ p_{abc} := \pi_{abc} + c\epsilon^{abc}d_1d_2d_3d_4e_1 \cdots e_7 F_{d_1 \cdots d_4} A_{e_1 \cdots e_7} \iff \star P = \star \pi + c A \wedge F, \]

which suggests that \( [G(\Lambda), P^{\mu\nu\lambda}(x)] = 0 \) because \( F \) is already invariant. This indeed can be verified using (2.13)

\[
\delta_\Lambda \ast P = \delta_\Lambda \ast \pi + c \delta_\Lambda A \wedge F = 0.
\] (3.2)

Our classical observables therefore are coordinatized by the 4-form and 7-form \( F = dA \) and \( \ast P = \ast \pi + c A \wedge F \) respectively. Since \( F \) is exact, it is determined entirely by a 3-form modulo an exact form, which in turn is parametrized by a 2-form. This 2-form worth of gauge freedom matches the number of Gauss constraints which can be read as a condition on \( \pi \). Thus, on the constraint surface, the number of degrees of freedom contained in \( F \) and \( P \) match.

We compute the observable algebra. Let \( f \) be a 3-form and \( h \) a 6-form with dual \( \ast h \) (a totally skew 4-times contravariant tensor pseudodensity) and smear the observables with these

\[
P[f] := \int d^{10}x f_{\mu_1 \ldots \mu_4} P^{\mu_1 \ldots \mu_4} = \int f \wedge \ast P, \quad F[h] := \int d^{10}x (\ast h)^{\mu_1 \ldots \mu_4} F_{\mu_1 \ldots \mu_4} = \int h \wedge F.
\] (3.3)

Then, we find after a short computation

\[
[F[h], F[h']] = 0, \quad [P[f], F[h]] = \int h \wedge df, \quad [P[f], P[f']] = -3c \, F[f \wedge f'].
\] (3.4)

Thus, the observable algebra closes but \( P \) is not conjugate to \( F \).

The form of the observable algebra (3.4) reveals the following. Typically, background independent representations tend to be discontinuous in at least one of the configuration or the momentum variable. For instance, in LQG electric fluxes exist in non-exponentiated form, but connections do not. Let us assume that we find such a representation in which \( F[h] \) does not exist so that we have to consider instead its exponential (Weyl element). Then (3.4) tells us that in such a representation automatically also \( P[f] \) cannot be defined, because if it could, then its commutator would exist, which however is proportional to some \( F \) which is a contradiction. Hence, either both \( F, P \) exist or only both of their corresponding Weyl elements.

We did not manage to find a representation in which the Weyl elements

\[
W[h, f] := \exp \left( i(F[h] + P[f]) \right)
\] (3.5)

are strongly continuous operators in both \( f, h \). However, we did find one in which they are discontinuous in both \( h, f \). This representation was studied in the context of QED in [20] and was applied to an LQG type of quantization of the closed bosonic string in [24]. Before we define it, we must first define the Weyl algebra generated by the Weyl elements (3.5). The *-relations are obvious,

\[
W[h, f]^* = W[\ast h, \ast f].
\] (3.6)

However, the product relations are very interesting and non-trivial, because they require the generalization of the Baker–Campbell–Hausdorff formula [25–30] to higher commutators [31]. Suppose that \( X, Y \) are operators on some Hilbert space such that the triple commutators \([X, [X, Y]]\) and \([Y, [Y, X]]\) commute with both \( X \) and \( Y \). This formally applies to our case with \( X = F[h] + P[f], \ Y = F[h'] + P[f'] \), which obey the canonical commutation relations (we set \( h = 1 \) for simplicity)

\[
[X, Y] := i[X, Y] = i \left( \int (h' \wedge df - h \wedge df') \right) - 3c F[f \wedge f'].
\] (3.7)

From this follows for the triple commutators

\[
[X, [X, Y]] = -3c(i)^2 [P[f], F[f \wedge f']] = 3c \int f \wedge f' \wedge df \mathbb{1},
\]

\[
[Y, [Y, X]] = 3c(i)^2 [P[f'], F[f \wedge f']] = -3c \int f \wedge f' \wedge df' \mathbb{1},
\] (3.8)

which thus are in the centre of the algebra.
The BCH formula for the case of all triple commutators commuting with $X,Y$ reads
\[ e^X e^Y = e^{X+Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] + \frac{1}{120}[X,[X,[X,Y]]] + \cdots}, \]
which can also be proved using elementary methods. From this it is easy to derive the also useful Zassenhaus formula [31]
\[ e^{X+Y} = e^X e^Y \exp \left( \frac{1}{4} \right) [X,[X,Y]] + \left( \frac{1}{12} \right) [X,[X,[X,Y]]] + \cdots \].
Putting all these together, we obtain the Weyl relations
\[ W[h, f]W[h', f'] = W[h + h' + \frac{3c}{2} f \wedge f', f + f'] \]
\[ \times \exp \left( \frac{1}{4} \int [2(h \wedge d f' - h' \wedge d f) - c f \wedge f' \wedge d(f - f')] \right) \].
Hence also the Weyl relations get twisted as compared to the situation with $c = 0$. Notice that the first term in the phase is antisymmetric under the exchange $(h, f) \leftrightarrow (h', f')$, while the second is symmetric.

In order to obtain a representation of this $^*$-algebra $\mathfrak{A}$ generated by the Weyl elements, it is sufficient to find a positive linear functional. We consider the Narnhofer–Thirring type of functional
\[ \omega(W(h, f)) = \begin{cases} 1 & \text{if } h = f = 0 \\ 0 & \text{else} \end{cases} \]
and show that it is positive definite on $\mathfrak{A}$. Let
\[ a := \sum_{k=1}^{N} c_k W[z_k] \]
be a general element in $\mathfrak{A}$, where $N \in \mathbb{N}$, $c_k \in \mathbb{C}$ and the $z_k = (h_k, f_k)$ are arbitrary, where without loss of generality $z_k \neq z_l$ for $k \neq l$. We have
\[ \omega(a^* a) = \sum_{k,l=1}^{N} \hat{c}_k \hat{c}_l \omega(W[-z_k]W[z_l]) \]
\[ = \sum_{k,l=1}^{N} \hat{c}_k \hat{c}_l \omega(W[z_k]W[z_l]) \exp(i \alpha_{kl}), \]
\[ z_{kl} = \left( -h_k + h_l - \frac{3c}{2} f_k \wedge f_l, -f_k + f_l \right), \]
\[ \alpha_{kl} = \frac{1}{4} \int [2(-h_k \wedge d f_l + h_l \wedge d f_k) - c f_k \wedge f_l \wedge d(f_k + f_l)]. \]
For $k = l$, we have $z_{kl} = \alpha_{kl} = 0$ because $f_k, f_l$ are 3-forms. For $k \neq l$, we must have either $f_k \neq f_l$ or $h_k \neq h_l$ or both. If $f_k \neq f_l$, then obviously $z_{kl} \neq 0$. If $f_k = f_l$, then necessarily $h_k \neq h_l$ and $z_{kl} = (-h_k + h_l, 0) \neq 0$. By definition (3.12) then
\[ \omega(a^* a) = \sum_{k=1}^{N} |c_k|^2 \geq 0; \quad \omega(a^* a) = 0 \iff a = 0 \]
is positive definite. Thus, the left ideal $\mathfrak{I} = \{ a \in \mathfrak{A}; \, \omega(a^* a) = 0 \} = \{ 0 \}$ is trivial and the Hilbert space representation is given by the GNS data [23].

The cyclic vector is $\Omega = 1$, the Hilbert space $\mathcal{H}$ is the Cauchy completion of $\mathfrak{A}$ in the scalar product $(a, b) := \omega(a^* b)$ and the representation is simply $\pi(a) b := ab$ on the common dense domain $\mathcal{D} = \mathfrak{A}$.
The representation is evidently strongly discontinuous in both \( h, f \) and while cyclic, it is not irreducible. Equivalently, \( \omega \) is not a pure state [32, 33].

The question left open to answer is whether the algebra and the state \( \omega \) are still well defined when restricting the smearing functions \( (h, f) \) to the form factors of 4-surfaces and 7-surfaces respectively. The bearing of this question is that in the Hamiltonian constraint the functions \( F \) and \( *P \) appear in such a way, that in a discretization of it, which results from replacing the integral by Riemann sums in the spirit of [34], these functions are naturally smeared over 4-surfaces and 7-surfaces respectively. They could thus be approximated by Weyl elements.

To answer this question, let \( S_4, S_7 \) be general 4 and 7 surfaces respectively. Consider the distributional forms (form factors)

\[
h_{a_1 \ldots a_4}(x) := \int_{S_4} \epsilon_{a_1 \ldots a_4 b_1 \ldots b_4} dy^{b_1} \wedge \cdots \wedge dy^{b_4} \delta(x, y),
\]

\[
f_{a_1 \ldots a_7}(x) := \int_{S_7} \epsilon_{a_1 \ldots a_7 b_1 \ldots b_7} dy^{b_1} \wedge \cdots \wedge dy^{b_7} \delta(x, y).
\]

Then

\[
F[h_{S_4}] = \int_{S_4} F, \quad P[f_{S_7}] = \int_{S_7} *P. \tag{3.17}
\]

Thus, the natural integrals of \( F, P \) over surfaces can be reexpressed in terms of distributional 6 forms and 4-forms respectively. It remains to check whether the exterior derivative and product combinations of these distributional forms appearing in the multiple Poisson brackets of (3.17) and in the Weyl relations remain meaningful. Three types of exterior derivative and product expressions appear. The first is, using formally Stokes theorem

\[
\int h_{S_4} \wedge f_{S_7} = \int_{S_4} df_{S_7} = \int_{\partial S_4} f_{S_7} = \int \frac{d\sigma^{a_1}}{\epsilon} \wedge \cdots \wedge \frac{d\sigma^{a_4}}{\epsilon} \epsilon_{a_1 \ldots a_4 b_1 \ldots b_7} \times \int_{S_7} dy^{b_1} \wedge \cdots \wedge dy^{b_7} \delta(x, y) =: \sigma(\partial S_4, S_7). \tag{3.18}
\]

The support of the integral will be on \( \partial S_4 \cap S_7 \) and we can decompose this set into components (submanifolds) which are zero-, one-, two- and three- dimensional. The number of these components will be finite if the surfaces are semi-analytic. We define the intersection number \( \sigma(\partial S_4, S_7) \) to be zero for the one-, two- and three-dimensional components and by (3.18) for the isolated intersection points, which then takes the values \( \pm 1 \). This can be justified by the same regularization as in LQG for the holonomy flux algebra [7].

The second type of integral is given by \( F[f_{S_7} \wedge f_{S_7}] \). The support of the integral will be on \( S_7 \cap S_7 \) and in \( D = 10 \) dimensions this will decompose into components that are at least four-dimensional. By the same regularization as in [7], one can remove the higher dimensional components and thus keep only the four-dimensional ones. In what follows, we thus assume that \( S_4 := S_7 \cap S_7 \) is a single four-dimensional component, otherwise the non-vanishing contributions are over a sum of those. We have

\[
F[f_{S_7} \wedge f_{S_7}] = \int_{S_7} F \wedge f_{S_7}.
\]

\[
= \int_{S_7} \frac{d\sigma^{a_1}}{\epsilon} \wedge \cdots \wedge \frac{d\sigma^{a_7}}{\epsilon} \epsilon_{a_1 \ldots a_7 b_1 \ldots b_7} \times \int_{S_7} dy^{b_1} \wedge \cdots \wedge dy^{b_7} \delta(x, y) f_{a_1 \ldots a_7}(x).
\]

By assumption, we have embeddings

\[
X_{S_7} : U \to S_7; \quad Y_{S_7} : V \to S_7; \quad Z_{S_4} : W \to S_4, \tag{3.20}
\]
with open subsets \(U, V\) of \(\mathbb{R}^7\) and an open subset \(W\) of \(\mathbb{R}^4\) respectively, whose coordinates will be denoted by \(u, v, w\) respectively. The condition \(X_{S_i}(u) = Y_{S_i}(v) = Z_{S_i}(w)\) is solved by solving \(u, v, w\), which leads to \(u = u(w), v = v(w)\). Since the integrals are reparametrization invariant, in the neighbourhood of \(S_1\) on both \(S_1\) and \(S_2\) we may use adapted coordinates so that \(u^I = u^I = v^I, I = 1, \ldots, 4\) on \(S_2\) and \(u^I, v^I, I = 5, \ldots, 7\) denote the transversal coordinates, which take the value 0 on \(S_1\). In this parametrization both \(U, V\) are of the form \(U = W \times U', V = W \times V'\) for some three-dimensional subsets \(U', V'\) of \(\mathbb{R}^3\). It follows \(Z(w) = X(w, 0) = Y(w, 0)\) in this parametrization. The \(\delta\) distribution is then supported on \(u^I = v^I, I = 1, \ldots, 4\) and \(u^I = v^I = 0, I = 5, \ldots, 7\) and we have in the neighbourhood of \(S_2\)

\[X_i^a(u) - Y_i^a(v) = -\sum_{I=1}^{4} Y_i^a(u, 0)[u^I - v^I] + \sum_{I=5}^{7} [X_i^a(u, 0)u^I - Y_i^a(u, 0)v^I]. \]  

(3.21)

We can now solve the \(\delta\) distribution in (3.19) by performing the integral over \(u^5, \ldots, u^7, v^1, \ldots, v^7\) and find with the notation \(X_i^a = \partial X_i^a(u)/\partial u^I\) and \(Y_i^a = \partial Y_i^a(v)/\partial v^I\) etc.

\[F[f^S] \wedge f^{S_1} = \int_U d^7u\delta^{h_{12}}\left[\sum_{h_{12}} X_i^a(u, 0)u^I - Y_i^a(u, 0)v^I\right](u)\epsilon h_{12} \int_V d^7v\delta^{h_{56}}\left[\sum_{h_{56}} X_i^a(u, 0)u^I - Y_i^a(u, 0)v^I\right](v)\]

\[\times \delta(X(u), Y(v))F_{h_{12} \rightarrow h_{56}}(X(u))\]

\[= -\int_W d^4w\epsilon^{h_{12}}\left[\sum_{h_{12}} Z_i^a(u, 0)u^I - Z_i^a(w, 0)v^I\right](w)\epsilon^{h_{56}}F_{h_{56} \rightarrow h_{12}}(Z(w))\epsilon_{h_{12} \rightarrow h_{56}}\]

\[\times \left[\text{sgn} \left(\det \left(\partial(X(u) - Y(v))\right)_{u^I = v^I = 0, I = 5, \ldots, 7}\right)\right]\]

\[= -3! 7! \hat{\sigma}(S_2, S_1)F[h^{S_1}], \]  

(3.22)

where the 10d antisymmetric symbol is in terms of the coordinates \(u^5, \ldots, u^7, v^1, \ldots, v^7\) and in the last step we noticed that the range of \(I_1, \ldots, I_4\) is restricted to \(1, \ldots, 4\). Also, we assumed that the sign function under the integral is constant and equal to \(\hat{\sigma}(S_2, S_1)\) on \(S_3\) (which defines this function), otherwise we must decompose \(S_3\) further. Under this assumption, we conclude the form factor identity

\[f_{S_1} \wedge f_{S_2} = -3! 7! \hat{\sigma}(S_2, S_1)h^{S_1 \cap S_2} \]  

(3.23)

Finally, we consider the integral of the third type, which now combining (3.18) and (3.24) is easily calculated

\[\int f_{S_1} \wedge f_{S_2} \wedge df_{S_3} = -3! 7! \hat{\sigma}(S_2, S_1)H^{S_1 \cap S_2} \wedge df_{S_3} \]

\[= -3! 7! \hat{\sigma}(S_2, S_1)\sigma(\hat{\sigma}(S_2 \cap S_3), S_1) = 0, \]  

(3.24)

because \(\hat{\sigma}(S_2 \cap S_3) \subset S_1\) for which \(\sigma\) vanishes by definition.

In order to make this restricted Weyl algebra close, we now have to decide whether the form factors should only be added with integer valued coefficients [17] or with real valued ones [35–37]. In the latter case we do not need to do anything and the restricted Weyl algebra already closes. In the former case we must replace the form factors \(f^{S_1}\) by \(\sqrt{h^{S_1}}f^{S_1}\), such that in the simplest situation we have

\[W[S_1, S_7]W[S_4, S_7] = W[S_1 + S_4 - \hat{\sigma}(S_1, S_4)S_7 \cap S_1, S_4 + S_7]\]

\[\times \exp\left(\frac{i}{2} \sigma(\hat{\sigma}(S_4, S_7) - \hat{\sigma}(S_4, S_7))\right), \]  

(3.25)

from which the general case can be easily deduced.
We conclude that the restricted Weyl algebra is well defined in either case. Thus, wherever $P$ or $F$ appear in the Hamiltonian constraint, we follow the general regularization procedure outlined in [34], which employs a combination of spatial diffeomorphism invariance and an infinite refinement limit of a Riemann sum approximation of the Hamiltonian constraint in terms of $P[S_7]$ and $F[S_4] = A[\partial S_4]$, which we approximate for instance by $\sin(P[S_7]), \sin(F[S_4])$ similar as in LQG. The details are obvious and are left to the interested reader.

4. Conclusions

Supergravity theories typically need additional bosonic fields next to the graviton, in order to obtain a SUSY multiplet (representation) containing the gravitino. In this paper, we focused on minimal 11d SUGRA for reasons of concreteness (and its relevance for lower dimensional SUGRA theories), which contains the 3-index photon in the bosonic sector. However, our analysis is easily generalized to arbitrary $p$-form fields. Without the Chern–Simons term in the action (i.e. $c = 0$) the analysis would be straightforward and in complete analogy to the background independent treatment of Maxwell theory in $D + 1 = 4$ dimensions [17]. In particular, the Hamiltonian constraint would be quadratic in the 3-form field and its conjugate momentum, which thus would reduce to a free field theory when switching off gravity. However, with the Chern–Simons term ($c \neq 0$) the Hamiltonian constraint becomes in fact quartic in the connection and thus becomes self-interacting even when switching off gravity, just like in non-Abelian Yang–Mills theories.

It is therefore the more astonishing that we can quantize the resulting $^*$-algebra of observables (with respect to the 3-index-Gauß constraint) rigorously, even though the theory is self-interacting. In fact, in terms of the observables, the Hamiltonian constraint is a quadratic polynomial, however, the price to pay is that the observable algebra is non-standard. Yet, the resulting Weyl algebra can be computed in closed form and we found at least one non-trivial and background independent representation thereof, which nicely fits into the background independent quantization of the gravitational degrees of freedom in the contribution to the Hamiltonian constraint depending on the 3-index-photon.

There are many open questions arising from the present study. One of them concerns the reducibility of the GNS representation found, which involves a mixed state. It would be nice to have control over the superselection sectors of the theory and, in particular, to analyse whether the cyclic GNS vector is not already cyclic for the Abelian subalgebra generated by the $W[h, 0]$. Next, it is worthwhile to study the question whether this algebra admits regular representations for both $P$ and $F$, because then the GNS Hilbert space would admit a measure theoretic interpretation as an $L_2$ space. Finally, it is certainly necessary to work out the cobordism theory of relevance when restricting the Weyl algebra to distributional 4-form and 7-form factors as smearing functions which is only sketched in this paper. We plan to revisit these questions in future publications.

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