The congruence of Wolstenholme and generalized binomial coefficients related to Lucas sequences

CHRISTIAN BALLOT
Département de Mathématiques et Mécanique
Université de Caen
F14032 Caen Cedex, France
christian.ballot@unicaen.fr

Abstract

Wolstenholme’s congruence says that \( \binom{2p-1}{p-1} \equiv 1 \pmod{p^3} \) for all primes \( p \geq 5 \). Kimball and Webb established an analogue of the congruence of Wolstenholme using Fibonomial coefficients. This note answers the question: ‘Is there a common generalization to the Wolstenholme and the Kimball and Webb congruences?’ Tinted by a positive answer, valid for all fundamental Lucas sequences, we go up the ladder. We give a broad generalization of several congruences such as Ljunggren et al’s\( \binom{kp}{\ell p} \equiv \binom{k}{\ell} \pmod{p^3} \), \( p \geq 5 \), or McIntosh’s: \( \binom{2p-1}{p-1} \equiv 1 - p^2 \sum_{0 < t < p} \frac{1}{t^2} \pmod{p^5} \), \( p \geq 7 \), replacing ordinary binomials by generalized binomial coefficients \( \binom{\cdot}{\cdot}_U \), where \( U = U(P,Q) \) is an arbitrary fundamental Lucas sequence. That is, a sequence which satisfies \( U_0 = 0, U_1 = 1 \) and \( U_{t+2} = P U_{t+1} - Q U_t \), for all \( t \geq 0 \).

1 Introduction

In 1862 Joseph Wolstenholme \cite{28} established a now well-known congruence for binomial coefficients, namely
Theorem 1. Let \( p \) be a prime number \( \geq 5 \). Then

\[
\binom{2p-1}{p-1} \equiv 1 \pmod{p^3}.
\] (1)

Charles Babbage [1], in 1819, had actually shown that congruence (1) held modulo \( p^2 \) for all primes \( p \) greater than 2. There is a survey paper [18] on the numerous generalizations of Theorem 1 discovered in the last 150 years. This survey also contains many other related results.

We focus first our attention on the slightly more general congruence

\[
\binom{(k+1)p-1}{p-1} \equiv 1 \pmod{p^3},
\] (2)

which holds for all primes \( p \geq 5 \) and all nonnegative integers \( k \). According to the survey [18], congruence (2) was proved in 1900 by Glaisher ([9] p. 21, [10] p. 33).

Lemma 3 of the paper [14], which we rewrite as a theorem below, is an analogue of (2).

Theorem 2. Let \( p \) be a prime at least 7 whose rank of appearance \( \rho \) in the Fibonacci sequence is equal to \( p - \epsilon_p \), where \( \epsilon_p \) is \( \pm 1 \). Then for all integers \( k \geq 0 \)

\[
\binom{(k+1)p-1}{p-1}_F \equiv \epsilon_p^k \pmod{p^3},
\] (3)

where the symbol \( \binom{*}{*}_F \) stands for the Fibonomial coefficient.

If \( A = (a_n)_{n \geq 0} \) is a sequence of complex numbers where \( a_0 = 0 \) and all \( a_n \neq 0 \) for \( n > 0 \), then one defines, for \( m \) and \( n \) nonnegative integers, the generalized binomial coefficient

\[
\binom{m}{n}_A = \begin{cases} 
\frac{a_m a_{m-1} \ldots a_{m-n+1}}{a_n a_{n-1} \ldots a_1}, & \text{if } m \geq n \geq 1; \\
1, & \text{if } n = 0; \\
0, & \text{otherwise}.
\end{cases}
\] (4)

The well-written paper [11] contains a number of early references about these coefficients and investigated several of their general properties. We point out another early reference [26], not often quoted, in which Ward gives two equivalent criteria that imply the integrality of the generalized coefficients.
One of them is that \( A \) be a strong divisibility sequence, i.e., one for which \( \gcd(m,n) = \gcd(a_m,a_n) \) for all \( m > n > 0 \); the other criterion is expressed in terms of ranks of appearance of prime powers in \( A \). The equivalence of these two criteria was essentially rediscovered in [16]. When \( A \) is the Fibonacci sequence these binomial coefficients are called Fibonomials and many papers have studied their properties. Some papers have considered the generalized binomial coefficients when \( A \) is a fundamental Lucas sequence, that is, a sequence \( U = U(P, Q) \) satisfying

\[
U_0 = 0, \quad U_1 = 1 \quad \text{and} \quad U_{n+2} = PU_{n+1} - QU_n, \quad \text{for all} \quad n \geq 0,
\]

where \((P, Q)\) is a pair of integers, \( Q \) nonzero. We will refer to these generalized binomials as Lucanomial coefficients in the sequel. Ordinary binomials are Lucanomial coefficients with parameters \((P, Q) = (2, 1)\), whereas the Fibonomials correspond to \((P, Q) = (1, -1)\).

Therefore it makes sense to look for a simple congruence for the general Lucanomial

\[
\left( \frac{(k + 1)\rho - 1}{\rho - 1} \right)_U \pmod{p^3}, \tag{6}
\]

valid for an arbitrary Lucas sequence \( U \), that would encompass both the congruence (2) and Theorem 2.

Here \( \rho \) represents the rank of appearance of the prime \( p \) in \( U \), that is, the least positive integer \( t \) such that \( p \mid U_t \). It is known to exist for all primes \( p \) not dividing \( Q \) and to divide \( p - \epsilon_p \), where \( \epsilon_p \) is the Legendre character \((D \mid p) \) and \( D \) is \( P^2 - 4Q \). It is necessary to require, as in Theorem 2, that the rank \( \rho \) be maximal, i.e., be equal to \( p - \epsilon_p \). Note that the rank of any prime \( p \) is maximal and equal to \( p \) for \( U_n = n \) \((D = 0, \epsilon_p = 0)\). However, the case \( \epsilon_p = 0 \) only occurs for \( p = 5 \) for the Fibonacci sequence \( F = U(1, -1) \), a case that Theorem 2 does not address. A calculation for \( p = 5 \) yields

\[
\left( \frac{2\rho - 1}{\rho - 1} \right)_F = \left( \frac{9}{4} \right)_F \equiv 1 \pmod{125}. \tag{7}
\]

This residue of 1 is at least conform to what one gets in (2), but does not match the expression \( \epsilon_p^k \) of Theorem 2 which would yield 0.

Thus, one needs to generalize the results of the paper [14] from Fibonomial coefficients to Lucanomial coefficients and include the case \( \epsilon_p = 0 \) in the analysis. However, some of the results leading to Theorem 2 in [14] seem, at
first sight, to depend on idiosyncracies of the Fibonacci sequence. Thus, a few numerical calculations helped us believe in the existence of a generalization and were useful in guiding us to it.

**Theorem 3.** Let $U = U(P, Q)$ be a fundamental Lucas sequence with parameters $P$ and $Q$. Let $p \geq 5$, $p \nmid Q$, be a prime whose rank of appearance $\rho$ in $U$ is equal to $p - \epsilon_p$, where $\epsilon_p$ is the Legendre character $(D \mid p)$, $D = P^2 - 4Q$. Then for all integers $k \geq 0$

$$\binom{(k+1)\rho - 1}{\rho - 1}_U \equiv (-1)^k p Q^{k\rho(\rho-1)/2} \pmod{p^3},$$

where the symbol $\binom{*}{U}$ stands for the Lucanomial coefficient.

**Remark 4.** Theorem 3 implies that for all $k \geq 0$

$$\binom{(k+1)\rho - 1}{\rho - 1}_U \equiv \left(\frac{2\rho - 1}{\rho - 1}\right)^k \pmod{p^3}.$$

**Remark 5.** Congruence (2), Theorem 2 and, as readily checked, congruence (7) are implied by Theorem 3. Indeed, the sequence $a_n = n$ is $U_n(2, 1)$, for which $Q = 1$ and $\epsilon_p = 0$ for all primes. To see that Theorem 2 is a corollary of Theorem 3 it suffices to check that

$$\epsilon_p = -(-1)^{\rho(\rho-1)/2},$$

for every odd prime $p \geq 5$ of maximal rank in the Fibonacci sequence $U(1, -1)$. All primes of rank $p \pm 1$ in the Fibonacci sequence must be congruent to 3 (mod 4), since by Euler’s criterion for Lucas sequences (19) we need to have $(-1 \mid p) = -1$. If $\epsilon_p = 1$, that is, if $\rho = p - 1$, then $\rho(\rho - 1) \equiv 2 \pmod{4}$ so that $-(-1)^{\rho(\rho-1)/2} = +1 = \epsilon_p$. If $\epsilon_p = -1$, that is, $\rho = p + 1$, then $\rho(\rho - 1) \equiv 0 \pmod{4}$ so $-(-1)^{\rho(\rho-1)/2} = -1 = \epsilon_p$.

Section 2 of the paper is devoted to some relevant additional remarks on Lucas sequences, some useful lemmas and to a proof of Theorem 3.

For all primes $p \geq 5$ and all nonnegative integers $k$ and $\ell$, we have the congruence

$$\binom{kp}{\ell p} \equiv \binom{k}{\ell} \pmod{p^3}.$$

This congruence supersedes congruence (2) and was first proved in a collective paper [7] which appeared in 1952. It was reproved by Bailey some 30 years
later in the paper [2], where the case \((k, \ell) = (2, 1)\), which is equivalent to Wolstenholme’s congruence (1), is proved first before an induction on \(k\) yielded congruence (2) and another proof by induction gave (9). Interestingly another simple argument, combinatorial, reduces the proof of (9) to that of the case \((k, \ell) = (2, 1)\) in the book [21] (see solution of exercise 1.14 p. 165).

Similarly in [14], Theorem 2 is used by the authors to produce an analogue of (9) for the Fibonacci sequence \(U = F\). That is, in our notation, for primes \(p \geq 7\) of rank \(\rho = p - \epsilon\), where \(\epsilon = \pm 1\), their result ([14], p. 296) states that

\[
\left(\frac{kp}{\ell p}\right)_{F'} \equiv \epsilon_p (k - \ell)_{F'} \left(\frac{k}{\ell}\right)_{F'} \pmod{p^3},
\]

where \(F'_t = F_{pt}\) for all \(t \geq 0\), \(k, \ell\) are integers satisfying \(k \geq \ell \geq 1\). Section 3 states and proves a congruence, Theorem 10, for Lucasnomials \(\left(\frac{kp}{\ell p}\right)_U \pmod{p^3}\) that subsumes the congruences (9) and (10). Here again the proof of this more general result is easily derived from Theorem 3. We raise in passing the question of the existence of a combinatorial argument that would reduce Theorem 10 to the case \((k, \ell) = (2, 1)\). Note that Lucasnomial coefficients were given a combinatorial interpretation in [6]. Also a \(q\)-analogue of (9) that uses \(q\)-binomial coefficients was established in the paper [22].

In a fourth section, we selected three congruences for binomials \(\binom{2p-1}{p-1} \pmod{p^5}\), namely (25), (26) and (27), and establish for each a generalization to Lucasnomial coefficients \(\binom{2p-1}{p-1}_U \pmod{p^5}\) for primes \(p \geq 7\) of maximal rank \(\rho\) in \(U\). Not to lengthen an already long introduction we only state the example of congruence (27), i.e.,

\[
\binom{2p - 1}{p - 1} \equiv 1 - p^2 \sum_{0 < t < p} \frac{1}{t^2} \pmod{p^5},
\]

which generalizes into

\[
\binom{2\rho - 1}{\rho - 1}_U \equiv (-1)^{\rho/p} Q^{\rho(p-1)/2}_p \left[1 - 4 \frac{U^2}{p^2} \sum_{0 < t < \rho} \frac{Q^t}{U^t} \right] \pmod{p^5},
\]

where \(U(P, Q)\) is a fundamental Lucas sequence and \(V(P, Q)\) is its companion sequence.

Note that the condition that \(p\) be of maximal rank in \(U\) may be viewed as a quadratic analogue of Artin’s conjecture which gives a positive density (equal to a positive rational number times Artin’s constant) for the set
of primes $p$ for which a given $a$ is a primitive root (mod $p$), when $a$ is a non-square integer and $|a| \geq 2$. Hooley [12] proved Artin’s conjecture conditionally to some generalized Riemann hypotheses. So did Roskam ([23], [24]) for the set of primes $p$ for which a fundamental unit of a quadratic field has maximal order modulo $(p)$. Thus, given $U(P, Q)$, $Q$ not a square, our theorems presumably should also concern sets of primes of positive densities.

In recent years congruences for ordinary binomials $\binom{2p-1}{p-1}$ (mod $p'$) have been established for larger and larger values of $l$ (see [18], p. 4-6). No doubt there must be higher corresponding congruences for Lucanomials. In fact, we end the paper with such a congruence modulo $p^6$. Generalizations of (25) are stated in Theorems 15 and 22, those of (26) and the above congruence (27) appear in Theorems 16 and 19 respectively. We added an appendix as a short fifth section where the integrality of all Lucanomial coefficients $\binom{m}{n}_U$ is asserted for all $U$ Lucas sequences.

Familiarity with Lucas sequences is assumed throughout the paper, but the reader may want to consult the introduction of [5] and the references it mentions. Chapter 4 of the book [27] is a useful introduction to these sequences.

Lucanomial coefficients have already been the object of generalizations of classical arithmetic properties of ordinary binomial coefficients. Kummer’s theorem giving the exact power of a prime $p$ in the binomial coefficient $\binom{m+n}{n}$ as the number of carries in the addition of $m$ and $n$ in radix $p$ was generalized to all strong divisibility sequences of positive integers [16]. That includes, in particular, all Lucas sequences $U(P, Q)$ with positive terms when $P$ and $Q$ are coprime.

Also a generalization of the celebrated theorem of Lucas:

$$\binom{mp+r}{np+s} \equiv \binom{m}{n}\binom{r}{s} \pmod{p},$$

where $r$ and $s$ are nonnegative integers less than the prime $p$, was achieved in terms of Lucanomials $\binom{mp+r}{np+s}_U$, under the hypothesis that $U(P, Q)$ is a Lucas sequence with gcd($P, Q$) = 1, $P \neq 0$ and $P^2Q \neq 1$ (see [13]).

In fact both the theorems of Kummer and of Lucas had been generalized in an earlier paper [8] but with respect to $q$-binomial coefficients.
2 Preliminaries and a proof of Theorem 3

Lucas theory is often developed with the two hypotheses that \( U(P, Q) \) is nondegenerate and \( \gcd(P, Q) = 1 \). The Lucas sequence \( U(P, Q) \) is called degenerate whenever the ratio of the zeros \( \alpha \) and \( \beta \) of \( x^2 - Px + Q \) is a root of unity. We do not make any of these assumptions here. If \( U \) is degenerate then we must have \( U_2 U_3 U_4 U_6 = 0 \). Indeed, if \( \alpha \neq \beta \) then \( U_t = \alpha^t - \beta^t \) and the ratio \( \alpha/\beta \), lying in the quadratic field \( \mathbb{Q}(\sqrt{D}) \), must be a second, third, fourth or sixth root of unity. Thus, some terms of the sequence \( U \) will be 0, but rather than discard those Lucas sequences from our analysis, we make a small amendment to the definition (4) to ensure that the corresponding Lucas polynomials \( (m_n)_U \) are well defined as rational numbers. Although the hypotheses of Theorems 3, 10 or of the theorems of Section 4 if applied to a prime \( p \geq 11 \) prevent the corresponding Lucas polynomials from having zero terms, this is not necessarily the case if \( p = 5 \) or \( p = 7 \). With \( \gcd(P, Q) > 1 \), the Lucas sequence \( A = U(P, Q) \) is no longer a strong divisibility sequence. Nevertheless, \( A \), or \( \lambda A \), \( \lambda \) an integer, satisfies some ‘convexity’ property. Namely for all prime powers \( p^a \) (\( a \geq 1 \)), \( p^a \nmid 2Q \), and for all \( x \geq 2 \), we have

\[
\# \{ t \in [x], p^a \mid A_t \} \geq \# \{ t \in [y], p^a \mid A_t \} + \# \{ t \in [x-y], p^a \mid A_t \},
\]

(11)

for all \( y \in [x-1] \). Here, if \( z \) is an integer \( \geq 1 \), \( [z] \) denotes the set of natural numbers \( 1, 2, \ldots, z \). This property holds because for such prime powers \( p^a \), we have \( p^a \mid U_t \) iff \( \rho(p^a) \mid t \), where \( \rho(p^a) \) is the rank of appearance of \( p^a \) in \( U \), and because \( [x+y] \geq [x] + [y] \) for all real numbers \( x \) and \( y \).

The convention we adopt for the generalized binomials \( (m_n)_A \) of definition (4) is that if there are zero terms in the product \( \prod_{i=1}^{n} \frac{m_{a_{i+1}}}{a_{i}} \) then a 0 in the numerator and a 0 in the denominator cancel out as a 1. (12)

With convention (12), property (11) satisfied by \( A = \lambda U \), for all Lucas sequences \( U \), guarantees that the generalized binomial \( (m_n)_A \) is a well defined rational number. Indeed this property implies that the number of 0 terms in the numerator of \( \prod_{i=1}^{n} \frac{m_{a_{i+1}}}{a_{i}} \) is at least that of its denominator. It also implies that \( (m_n)_A \) \( m \) and \( n \) nonnegative integers, is well defined \( p \)-adically for all primes \( p \nmid 2Q \). In fact we can show it is always a rational integer.  

\[\text{1See our short Appendix}\]
To each fundamental Lucas sequence \( U(P, Q) \) we associate a companion Lucas sequence \( V = V(P, Q) \) which obeys recursion (15), but has initial values \( V_0 = 2 \) and \( V_1 = P \). The following identities are all classical ones and are all valid no matter what the value of \( \gcd(P, Q) \) is. We will use them throughout the paper.

\[
\begin{align*}
2U_{s+t} &= U_s V_t + U_t V_s, & (13) \\
2V_{s+t} &= V_s V_t + DU_s U_t, & (14) \\
V_t^2 - DU_t^2 &= 4Q^t, & (15) \\
U_{2t} &= U_t V_t, & (16) \\
V_{2t} &= V_t^2 - 2Q^t, & (17) \\
2Q^t U_{s-t} &= U_s V_t - U_t V_s. & (18)
\end{align*}
\]

We referred to Euler’s criterion for Lucas sequences in our introduction. The criterion states that

\[
p \mid U_{(p - \epsilon_p)/2} \text{ iff } Q \text{ is a square modulo } p,
\]

where \( U(P, Q) \) is a fundamental Lucas sequence and \( p \) is a prime that does not divide \( 2DQ \) (see [27], pp. 84–85).

Note that our theorems and the lemmas of Section 4 all deal with primes \( p \geq 5 \) of maximal rank. In their statements, we sometimes omit to mention the condition \( p \nmid Q \), because that condition is necessary. Indeed, if \( p \mid Q \), then, by (5), \( U_t \equiv P^{t-1} \pmod{p} \). Thus, \( p \) has no rank, because if \( p \) divided \( P \), then \( \rho(p) \) would be equal to 2, as \( U_2 = P \), a contradiction.

Given a prime \( p \) of rank \( \rho \) and a nonnegative integer \( \nu \), we write

\[
\Sigma_\nu := \sum_{0 < t < \rho} \frac{V_t^\nu}{U_t^\nu} \text{ and } \Sigma_{1,1} := \sum_{0 < s < t < \rho} \frac{V_s V_t}{U_s U_t}.
\]

The proof of Theorem 3 we are about to write uses a few lemmas which we state first.

**Lemma 6.** Let \( (U, V) \) be a pair of Lucas sequences with parameters \( P \) and \( Q \). Let \( \nu \) be a nonnegative integer. If \( p \nmid Q \) is a prime at least \( \nu + 3 \) of
maximal rank $\rho$, i.e., of rank $p - \epsilon_p$, where $\epsilon_p = 0$ or $\pm 1$, then

$$\Sigma_\nu \equiv \begin{cases} 0 \pmod{p^2}, & \text{if } \nu \text{ is odd;} \\ 0 \pmod{p}, & \text{if } \epsilon_p = -1 \text{ or } 0; \\ -2D^{\nu/2} \pmod{p}, & \text{if } \nu \text{ is even and } \epsilon_p = 1. \end{cases} \quad (21)$$

Moreover, if $p$ is an odd prime not dividing $Q$ of rank $\rho$, then

$$\Sigma_\nu \equiv 0 \pmod{p}, \quad \text{when } \nu \text{ is odd.} \quad (22)$$

Proof. The case $\nu$ odd of (21) is Theorem 3 of [3]. (The case $\nu = 1$ first appeared, nearly complete, as the main theorem of the paper [15], but also (nearly) as a corollary of the main theorem of [20], and as a particular case of Theorem 4.1 of [4], or of Theorems 3 and 12 of [5].)

The case $\nu$ even can be treated with the very same arguments used in the last part of the proof of Theorem 4, p. 5, of [3]. (The basic facts, noted first in [15], are that, by (18), all $V_t/U_t$ are distinct (mod $p$) for $t \in (0, \rho)$ and no $V_t/U_t$ is $\pm \sqrt{D}$ (mod $p$) by (19); also $p \mid \sum_{t=1}^{p} t^e$ if $p - 1 \nmid e$). The condition $p \geq \nu + 3$ is a sufficient condition which guarantees that $p - 1 \nmid \nu$ for $\nu \geq 2$ even.

The additional congruence (22) for $\nu$ odd, but without the restrictions that $\rho$ be maximal and $p \geq \nu + 3$, is a consequence of the congruence (mod $p^2$) on the sixth line of the proof of Theorem 4 of [3].

Lemma 7. Let $U = U(P, Q)$ be a fundamental Lucas sequence. If $p \nmid 6Q$ is a prime of maximal rank $\rho$ in $U$, then

$$\Sigma_{1,1} \equiv \begin{cases} 0 \pmod{p}, & \text{if } \epsilon_p = 0 \text{ or } -1; \\ D \pmod{p}, & \text{if } \epsilon_p = 1. \end{cases}$$

Proof. We have $\Sigma_1^2 = \Sigma_2 + 2\Sigma_{1,1}$ so that $\Sigma_{1,1} \equiv -\frac{1}{2} \Sigma_2 \pmod{p}$, since, by Lemma 6 $p^4$ divides $\Sigma_1^2$ and $\Sigma_2$ is either 0 or $-2D \pmod{p}$.

Lemma 8. Let $U = U(P, Q)$ be a fundamental Lucas sequence. If $p \nmid Q$ is an odd prime of even rank $\rho$ in $U$ and $k \geq 1$ is an odd integer, then

$$\frac{V_{k\rho}}{2} \equiv -Q^{k\rho/2} \pmod{p^2}.$$
Proof. Since \( p \) divides \( U_{k\rho} \), but not \( U_{k\rho/2} \), we find by (16) that \( p \) divides \( V_{k\rho/2} \). Therefore, from (17) with \( t = k\rho/2 \), we deduce that \( V_{k\rho} \equiv -2Q^{k\rho/2} \pmod{p^2} \).

\[ \square \]

\textbf{Lemma 9.} Let \( V = V(P, Q) \) be a companion Lucas sequence. Let \( m \) be an integer \( \geq 2 \). Suppose \( V_t \equiv \pm 2Q^t/2 \pmod{m} \). Then

\[ V_{2t} \equiv 2Q^t \pmod{m}. \]

Proof. We have \( V_{2t} = V_t^2 - 2Q^t \equiv 2Q^t \pmod{m} \).

We are now ready for a proof of Theorem 3.

Proof. We have

\[
\binom{(k+1)\rho-1}{\rho-1} = \frac{\prod_{t=1}^{\rho-1} U_{k\rho+t}}{\prod_{t=1}^{\rho-1} U_t}.
\]

By the addition formula (13), we find that

\[
2^{\rho-1} \prod_{t=1}^{\rho-1} U_{k\rho+t} = \prod_{t=1}^{\rho-1} (V_{k\rho} U_t + U_{k\rho} V_t)
\]

\[
\equiv (V_{k\rho}^{\rho-1} + V_{k\rho}^{\rho-2} U_{k\rho} \Sigma_1 + V_{k\rho}^{\rho-3} U_{k\rho}^2 \Sigma_{1,1}) \times \prod_{t=1}^{\rho-1} U_t
\]

\[
\equiv (V_{k\rho}^{\rho-1} + V_{k\rho}^{\rho-2} U_{k\rho}^2 \Sigma_{1,1}) \times \prod_{t=1}^{\rho-1} U_t \pmod{p^3},
\]

since \( p \) divides \( U_{k\rho} \) and, by Lemma 7, \( \Sigma_1 \) is \( 0 \pmod{p^2} \).

We first examine the cases \( \rho \) is \( p+1 \) and \( \rho \) is \( p \). In those cases \( U_{k\rho}^2 \Sigma_{1,1} \) is \( 0 \pmod{p^3} \) by Lemma 7. Hence,

\[
\binom{(k+1)\rho-1}{\rho-1} \equiv \left(\frac{V_{k\rho}}{2}\right)^{\rho-1} \pmod{p^3}.
\]

If \( \rho \) is \( p \), then, by (15) and the fact that \( p^3 \mid DU_{k\rho}^2 \), we see that \( V_{k\rho}^2 \equiv 4Q^{k\rho} \pmod{p^3} \). Therefore,

\[
\binom{(k+1)\rho-1}{\rho-1} \equiv (Q^{k\rho})^{(\rho-1)/2} \pmod{p^3},
\]

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yielding the result in that case. If \( \rho = p + 1 \) and \( k \) is odd, then by Lemma 8 there is an integer \( \lambda \) such that \( \frac{V_{kp}}{2} = -Q^{k\rho/2} + \lambda p^2 \). Raising members of the previous equation to the \( p \)th power gives \( (V_{kp}/2)^p \equiv -Q^{k\rho/2} \pmod{p^3} \). But \(-1 = (-1)^{-k}\) so the theorem follows in that case.

If \( \rho = p + 1 \) and \( k = 2^a\ell \), where \( \ell \) is odd and \( a \geq 1 \), then, by Lemma 8 we have \( V_{\ell p} \equiv -2Q^{\ell\rho/2} \pmod{p^2} \). Applying \( a \) times Lemma 9 we see that \( V_{kp} \equiv 2Q^{k\rho/2} \pmod{p^2} \). As we did in the case \( k \) odd, we raise both sides of the congruence to the \( p \)th power to obtain \( (V_{kp}/2)^p \equiv Q^{k\rho(p-1)/2} = (-1)^{k\rho}Q^{k\rho(p-1)/2} \pmod{p^3} \) and the theorem follows.

Suppose now \( \epsilon_p = 1 \), that is, \( \rho = p - 1 \). By Lemma 10 \( \Sigma_{1,1} \equiv D \pmod{p} \) so that \( U_{kp}^2 \Sigma_{1,1} \equiv DU_{kp}^2 \pmod{p^3} \). But, by \([15]\), \( DU_{kp}^2 = V_{kp}^2 - 4Q^{kp} \). Therefore, we have

\[
2^{p-1}\left(\frac{(k+1)\rho - 1}{\rho - 1}\right)_U \equiv 2V_{kp}^{p-1} - 4Q^{kp}V_{kp}^{p-3} \pmod{p^3}.
\]

This gives

\[
\left(\frac{(k+1)\rho - 1}{\rho - 1}\right)_U \equiv \left(\frac{V_{kp}}{2}\right)^p \left[2\left(\frac{2}{V_{kp}}\right)^2 - Q^{kp}\left(\frac{2}{V_{kp}}\right)^4\right] \pmod{p^3}. \tag{23}
\]

By Lemma 8 we have \( V_{kp}/2 \equiv -Q^{kp/2} \pmod{p^2} \) in case \( k \) is odd. Using Lemma 9 as for the case \( \rho = p + 1 \), we get that \( V_{kp}/2 \equiv Q^{kp/2} \pmod{p^2} \) if \( k \) is even. Thus, generally, \( V_{kp}/2 \equiv (-1)^k Q^{kp/2} \pmod{p^2} \). Raising the previous congruence to the \( p \)th power yields \( (V_{kp}/2)^p \equiv (-1)^k Q^{kpp/2} \pmod{p^3} \), while inverting it yields the existence of an integer \( \mu \) such that \( 2/V_{kp} \equiv (-1)^k Q^{-kp/2} + \mu p^2 \pmod{p^3} \). Thus, with \( \alpha_{p,k} := \) the bracket factor of the righthand side of \([23]\), we find that modulo \( p^3 \)

\[
\alpha_{p,k} \equiv 2((-1)^k Q^{-kp/2} + \mu p^2)^2 - Q^{kp}((-1)^k Q^{-kp/2} + \mu p^2)^4 \\
\equiv (2Q^{-kp} + (-1)^k 4Q^{-kp/2} \mu p^2) - Q^{kp}(Q^{-2kp} + (-1)^k 4Q^{-3kp/2} \mu p^2) \\
= Q^{-kp}.
\]

Thus, we end up with

\[
\left(\frac{(k+1)\rho - 1}{\rho - 1}\right)_U \equiv (-1)^k Q^{kpp/2}Q^{-kp} = (-1)^{k\epsilon_p} Q^{k\rho(p-2)/2} \pmod{p^3},
\]

which yields the theorem. \( \square \)
The above proof is the first that came to us. It proceeds case by case according to whether the value of the rank of \( p \) is \( p + 1 \), \( p \) or \( p - 1 \) and, thus, appears somewhat miraculous. Although we initially wrote case by case proofs for the higher congruences of Section 4, we ended up finding a global and more natural approach at least for Theorems 16 and 19.

3 Lucanomials \( \binom{k\rho}{\ell\rho} U \pmod{p^3} \)

Here is our common generalization of the Ljunggren et al. congruence (9) and Kimball and Webb’s theorem (10).

Theorem 10. Let \( U, V \) be a pair of Lucas sequences with parameters \( P \) and \( Q \). Let \( p \geq 5 \), \( p \nmid Q \), be a prime whose rank of appearance \( \rho \) is equal to \( p - \epsilon_p \), \( \epsilon_p \) being 0 or \( \pm 1 \). Then, for all nonnegative integers \( k \) and \( \ell \), we have

\[
\binom{k\rho}{\ell\rho} U \equiv \binom{k}{\ell} U' \cdot (\prod_{i=k-\ell}^{k-1} U_i) \cdot \left( \prod_{t=1}^{\ell-1} U_t \right) \cdot \left( \prod_{t=1}^{\rho-1} U_t \right) \pmod{p^3},
\]

where \( U' \) is the sequence \( U_\rho \times U(V_\rho, Q^\rho) \).

Proof. We only need a proof in case \( k > \ell \geq 1 \). With convention \( \text{(12)} \) we may write

\[
\binom{k\rho}{\ell\rho} U = \frac{U_{kp} U_{kp-1} \cdots U_{(k-\ell)\rho+1}}{U_{\ell\rho} U_{\ell\rho-1} \cdots U_1} \cdot \prod_{i=k-\ell}^{k-1} U_i \cdot \prod_{t=1}^{\ell-1} U_t \cdot \prod_{t=1}^{\rho-1} U_t
\]

\[
= \binom{k}{\ell} U' \cdot \prod_{i=k-\ell}^{k-1} \binom{(i+1)\rho - 1}{\rho - 1} U_i \cdot \left( \prod_{t=1}^{\ell-1} U_t \right) \cdot \left( \prod_{t=1}^{\rho-1} U_t \right) \pmod{p^3},
\]

yielding, by Theorem 3, the theorem. \( \square \)
Remark 11. If, in Theorem 10, \( U_\rho \neq 0 \) then we might as well set \( U' \) equal to \( U(V_\rho, Q_\rho) \).

Remark 12. If \( U = U(2, 1) \), then \( U_t = t \) and \( U'_t = pt \), or \( U'_t = t \) by the above remark. Thus the theorem implies that

\[
\binom{kp}{\ell p} \equiv \binom{k}{\ell} U' \pmod{p^3},
\]

which is the classical congruence (9) of Ljunggren et alii. For \( U = U(1, -1) \) and \( \epsilon_p = \pm 1 \) we saw in Remark 3 that \( \epsilon_p = -(-1)^{\rho(p-1)/2} = -Q^{\rho(p-1)/2} \) so that Theorem 10 implies (10).

Since we took care of including all cases of Lucas sequences in our theorems, we provide an example of an application of Theorem 10 to a degenerate Lucas sequence.

Example 13. Consider \( U(2, 2) \). Its first terms are

\[
0, 1, 2, 2, 0, -4, -8, -8, 0, 16, 32, 32, 0, \ldots
\]

So Theorem 10 applies to \( p = 5 \) since its rank is maximal and equal to 4. Choose, say \( k = 3 \) and \( \ell = 2 \). By our extended definition of (4), we have \( \binom{3}{2}_{U'} = 1 \) and \( (-1)^{\ell(k-\ell)} \epsilon_p Q^{\ell(k-\ell)\rho(p-1)/2} = 2^{12} \). Computing \( \binom{12}{8}_U \) we may verify the congruence modulo 125, which in that case is an equality, since

\[
\binom{12}{8}_U = \frac{U_{11} \cdot U_{10} \cdot U_9}{U_3 \cdot U_2 \cdot U_1} = \frac{16 \cdot 32 \cdot 32}{2 \cdot 2 \cdot 1} = 2^{12}.
\]

4 Lucanomials \( \binom{2p-1}{p-1}_U \pmod{p^5} \)

The congruence of Wolstenholme has been studied to prime powers higher than the third. In particular, we have, for all primes \( p \geq 7 \),

\[
\binom{2p-1}{p-1} \equiv 1 + p \sum_{0 < t < p} \frac{1}{t} + p^2 \sum_{0 < s < t < p} \frac{1}{st} \pmod{p^5}
\]

(25)

\[
\equiv 1 + 2p \sum_{0 < t < p} \frac{1}{t} \pmod{p^5}
\]

(26)

\[
\equiv 1 - p^2 \sum_{0 < t < p} \frac{1}{t^2} \pmod{p^5}.
\]

(27)
We will find congruences for the Lucanomial coefficients \((2p^{-1})_U\), valid for a general fundamental Lucas sequence \(U\), modulo the fifth power of a prime of maximal rank \(\rho\), which generalize the three congruences above. Expanding the binomial \((2p^{-1})_U\), as was done more generally for Lucanomials in the proof of Theorem 3, one falls naturally on the congruence (25). This expansion appears, for instance, in the proof of Proposition 1 in \([19]\). Congruence (26) is a special case of Theorem 3 of the paper \([29]\) and was known to hold for primes \(p \geq 5\) modulo \(p^4\) much earlier, while congruence (27) appears in \([17]\), p. 385.

To complete the notation introduced in (20) we define the symbols \(\Sigma_{1,\nu}\) (\(\nu = 2\) or 3), \(\Sigma_{1,1,1}\), \(\Sigma_{1,1,2}\) and \(\Sigma_{1,1,1,1}\), respectively, as the sums

\[
\sum_{s,t} \frac{V_s V'_t}{U_s U'_t}, \sum_{s < t} \frac{V_s V'_t}{U_s U'_t}, \sum_{r < s < t} \frac{V_r V_s V'_t}{U_r U_s U'_t}, \sum_{t \in (0, \rho)} \frac{V_q V_r V_s V'_t}{U_q U_r U_s U'_t},
\]

where in each sum \(q, r, s\) and \(t\) are distinct integers in the interval \((0, \rho)\) and \(\rho\) is the rank of a prime \(p\).

Lemma 14. We have for all primes \(p \geq 7\) of maximal ranks

\[
\Sigma_{1,1,1} \equiv 0 \pmod{p^2} \quad \text{and} \quad \Sigma_{1,1,1,1} \equiv 0 \pmod{p}, \quad \text{if} \ \epsilon_p = 0 \text{ or} \ -1;
\]

\[
D^2 \pmod{p}, \quad \text{if} \ \epsilon_p = 1.
\]

Proof. We have the linear system

\[
\Sigma_1^3 - \Sigma_3 = 3\Sigma_{1,2} + 6\Sigma_{1,1,1}, \quad \Sigma_1 \cdot \Sigma_{1,1} = \Sigma_{1,2} + 3\Sigma_{1,1,1}.
\]

Because \(p^2\) divides both \(\Sigma_1\) and \(\Sigma_3\), \(\Sigma_1^3 - \Sigma_3\) and \(\Sigma_1 \cdot \Sigma_{1,1}\) are each \(0 \pmod{p^2}\). Since the determinant of the system is prime to \(p\), \(\Sigma_{1,2}\) and \(\Sigma_{1,1,1}\) are both \(0 \pmod{p^2}\).

From Lemma 6 with \(p > 5\), which yields the values of \(\Sigma_2\) and \(\Sigma_4 \pmod{p}\), we deduce that

\[
\Sigma_{2,2} = \frac{1}{2} [\Sigma_2^2 - \Sigma_4] \equiv \begin{cases}
0 \pmod{p}, & \text{if } \epsilon_p = 0 \text{ or } -1; \\
3D^2 \pmod{p}, & \text{if } \epsilon_p = 1.
\end{cases}
\]

Now \(\Sigma_{1,3} = \Sigma_1 \cdot \Sigma_3 - \Sigma_4 \implies \Sigma_{1,3} \equiv -\Sigma_4 \pmod{p}\). Moreover, \(2\Sigma_{1,1,2} + 2\Sigma_{2,2} + \Sigma_{1,3} = \Sigma_{1,2} \cdot \Sigma_1 \equiv 0 \pmod{p}\).
Thus, $\Sigma_{1,1,2}$ is $0 \pmod{p}$, if $\epsilon_p$ is 0 or $-1$, and $\Sigma_{1,1,2}$ is $-4D^2 \pmod{p}$, if $\epsilon_p$ is 1.

Therefore, as $6\Sigma_{1,1,1,1} = \Sigma_{1,1} - \Sigma_{2,2} - 2\Sigma_{1,1,2}$, we obtain, using Lemma 7, the desired congruences for $\Sigma_{1,1,1,1}$. \hfill $\blacksquare$

Our first theorem is a generalization of congruence (25).

**Theorem 15.** Let $(U, V)$ be a pair of Lucas sequence with parameters $P$ and $Q$. Let $p$ be a prime at least 7 of maximal rank $\rho$ equal to $p - \epsilon_p$. Then

$$
\left(\frac{2\rho - 1}{\rho - 1}\right)_U \equiv \left(\frac{V_\rho}{2}\right)^{\rho - 1} \left[ 1 + \frac{U_\rho}{V_\rho} \sum_{0 < t < \rho} \frac{V_t}{U_t} + \sum_{0 < s < t < \rho} \frac{V_s V_t}{U_s U_t} + R \right] \pmod{p^5},
$$

where $R = \frac{\epsilon_p (1 + \epsilon_p) D^2 U_\rho^4}{2 V_\rho^4}$ =

$$\begin{cases} 
0, & \text{if } \epsilon_p = 0 \text{ or } -1; \\
D^2 U_\rho^4 / V_\rho^4, & \text{if } \epsilon_p = 1.
\end{cases}$$

**Proof.** Expanding the product $2^{\rho - 1} \prod_{t=1}^{\rho - 1} U_{\rho + t} = \prod_{t=1}^{\rho - 1} (V_\rho U_t + U_\rho V_t)$ as we did early in the proof of Theorem 3 but up to the fourth power of $U_\rho$, yields that $2^{\rho - 1} \left(\frac{2\rho - 1}{\rho - 1}\right)_U$ is congruent to

$$V_\rho^{\rho - 1} + V_\rho^{\rho - 2} U_\rho \Sigma_1 + V_\rho^{\rho - 3} U_\rho^2 \Sigma_{1,1} + V_\rho^{\rho - 4} U_\rho^3 \Sigma_{1,1,1} + V_\rho^{\rho - 5} U_\rho^4 \Sigma_{1,1,1,1} \pmod{p^5}.$$

Applying the congruences obtained in Lemma 14 to the last two terms of the above sum yields the theorem. \hfill $\blacksquare$

We now prove a congruence formula that generalizes (26), but also generalizes Theorem 3 when $k = 1$. The method of proof brings out the factor $(-1)^{\epsilon_p} Q^{\rho(\rho - 1)/2}$ naturally. It is particularly appealing because it only contains two terms, no more than (26), and is valid regardless of the values of the maximal rank $\rho$.

**Theorem 16.** Let $(U, V)$ be a pair of Lucas sequence with parameters $P$ and $Q$. Let $p$ be a prime at least 7 of maximal rank $\rho$ equal to $p - \epsilon_p$. Then

$$
\left(\frac{2\rho - 1}{\rho - 1}\right)_U \equiv (-1)^{\epsilon_p} Q^{\rho(\rho - 1)/2} \left[ 1 + 2 \frac{U_\rho}{V_\rho} \sum_{0 < t < \rho} \frac{V_t}{U_t} \right] \pmod{p^5}.
$$
Proof. All unmarked sums and products are for \( t \) running from 1 to \( \rho - 1 \). Note that \( \prod U_t = \prod U_{\rho-t} \). Thus by (18) we may write

\[
2^{\rho-1}Q^\sum_t \prod U_t = \prod 2Q^t U_{\rho-t} = \prod (U_{\rho} V_t - V_{\rho} U_t) = (-V_{\rho})^{\rho-1} \prod \left(1 - \frac{U_{\rho} V_t}{V_{\rho} U_t}\right) \prod U_t.
\]

Therefore

\[
(-1)^{\rho-1}Q^{\rho(\rho-1)/2} = \left(\frac{V_{\rho}}{2}\right)^{\rho-1} \prod \left(1 - \frac{U_{\rho} V_t}{V_{\rho} U_t}\right),
\]

so that

\[
(-1)^{\rho-1}Q^{\rho(\rho-1)/2} \equiv \left(\frac{V_{\rho}}{2}\right)^{\rho-1} \text{ (mod } p^5\text{)}.
\] (28)

Note that from (28) we recover the congruence

\[
(-1)^{\rho-1}Q^{\rho(\rho-1)/2} \equiv \left(\frac{V_{\rho}}{2}\right)^{\rho-1} \text{ (mod } p^2\text{)}.
\] (29)

Subtracting the expansion in (28) from that of \( (2^{\rho-1})^U \) obtained in the proof of Theorem 15, we find that

\[
\binom{2\rho-1}{\rho-1} U - (-1)^{\rho-1}Q^{\rho(\rho-1)/2} \equiv \left(\frac{V_{\rho}}{2}\right)^{\rho-1} \left(2 \frac{U_{\rho} V_t \Sigma_1}{V^2_{\rho}} + \frac{U_{\rho} T V_t}{V^3_{\rho}} \Sigma_{1,1,1,1}\right)
\]

\[
\equiv 2 \left(\frac{V_{\rho}}{2}\right)^{\rho-1} \frac{U_{\rho} V_t \Sigma_1}{V^2_{\rho}} \text{ (mod } p^5\text{)},
\]

since \( \Sigma_{1,1,1} \) is 0 (mod \( p^2 \)) by Lemma 14. In the above congruence as \( \frac{U_{\rho} V_t \Sigma_1}{V^2_{\rho}} \) is 0 (mod \( p^3 \)) we may, by (29), replace \( \left(\frac{V_{\rho}}{2}\right)^{\rho-1} \) by \( (-1)^{\rho-1}Q^{\rho(\rho-1)/2} \) and deduce our theorem. \)

Lemma 17. Suppose \( \nu \) is a nonnegative integer. Let \( p \geq \nu + 5 \) be a prime of maximal rank, say \( \rho \). Then

\[
\sum_{0 < t < \rho} \frac{4Q^t V_t^\nu}{U^2_t U_t^{\nu}} = \Sigma_{\nu+2} - D \Sigma_{\nu} \equiv \begin{cases} 0 & \text{ (mod } p^2\text{), if } \nu \text{ is odd;} \\ 0 & \text{ (mod } p\text{), if } \nu \text{ is even.} \end{cases}
\]
Proof. We have
\[ \sum_{0 < t < \rho} 4Q^t V_\nu U_t \equiv \sum_{0 < t < \rho} \frac{(V_t^2 - DU_t^2) V_\nu}{U_t^4} \equiv \Sigma_{\nu+2} - D\Sigma_\nu. \]

If \( \nu \) is odd, then, \( p \geq \nu + 5 \) implies, by Lemma 6, that both \( \Sigma_\nu \) and \( \Sigma_{\nu+2} \) are 0 (mod \( p^2 \)). If \( \nu \) is even, then both \( \Sigma_{\nu+2} \) and \( D\Sigma_\nu \) are 0 (mod \( p \)), when \( \rho \) is \( p \) or \( p + 1 \), by Lemma 6. If \( \rho \) is \( p - 1 \), then by the same lemma \( \Sigma_{\nu+2} - D\Sigma_\nu \equiv -2D^{\nu+2} - D(-2D^{\nu/2}) \equiv 0 \) (mod \( p \)).

\[ \text{Lemma 18. We have for all primes } p \geq 7 \text{ of maximal rank } \rho \]
\[ -2\Sigma_1 \equiv \frac{U_\rho}{V_\rho} \sum_{0 < t < \rho} \frac{4Q^t}{U_t^2} \quad (\text{mod } p^4). \]

Proof. All sums are over an index \( t \) running from 1 to \( \rho - 1 \).
\[ -2\Sigma_1 = - \sum \frac{(V_t + V_{\rho-t})}{U_t U_{\rho-t}} = -2U_\rho \sum \frac{1}{U_t U_{\rho-t}}, \quad \text{by (13)}, \]
\[ = -2U_\rho \sum \frac{2Q^t}{U_t (U_t V_t - U_{\rho-t} V_{\rho-t})}, \quad \text{using (13)}, \]
\[ = \frac{2U_\rho}{V_\rho} \sum \frac{2Q^t}{U_t^2 [1 - \frac{V_t}{U_t} V_{\rho-t}]} \]
\[ \equiv \frac{U_\rho}{V_\rho} \sum \frac{4Q^t}{U_t^2} \left[ 1 + \frac{V_t U_{\rho-t}}{U_{\rho-t} V_{\rho-t}} + \frac{V_t^2 U_{\rho-t}^2}{U_{\rho-t} V_{\rho-t}^2} \right] \quad (\text{mod } p^4), \]

because, by Lemma 17, \( U_{\nu+1}^2 \sum \frac{4Q^t V_\nu}{U_t^2 V_t} \equiv 0 \) (mod \( p^4 \)), for \( \nu = 1 \) and \( \nu = 2 \), if \( p \geq 7 \). \( \square \)

From Theorem 16, it is not difficult to reach a third theorem that generalizes (27).

\[ \text{Theorem 19. Let } (U, V) \text{ be a pair of Lucas sequence with parameters } P \text{ and } Q. \text{ Let } p \text{ be a prime at least } 7 \text{ of maximal rank } \rho \text{ equal to } p - \epsilon_\rho. \text{ Then} \]
\[ \binom{2\rho - 1}{\rho - 1} \equiv (-1)^{\rho} Q^{\frac{p(\rho - 1)}{2}} \left[ 1 - 4 \frac{U_{\rho}}{V_\rho} \sum_{0 < t < \rho} \frac{Q^t}{U_t^2} \right] \quad (\text{mod } p^5). \]

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Proof. In the congruence for the Lucanomial \( \binom{2^{p-1}}{p-1} \) of Theorem 16 we may replace \( \frac{U^2}{V^2} \sum_1 \) by \( -\frac{U^2}{V^2} \sum_1 \frac{Q'}{U'_t} \) since by Lemma 18 the two expressions are congruent modulo \( p^5 \). \qed

Remark 20. In stating Theorem 19 we chose the expression \( -4\frac{U^2}{V^2} \sum Q' \) rather than \( -\frac{U^2}{V^2} \sum \frac{Q'}{U'_t} \) because it contains only one term; that term is 0 \((\text{mod } p^3)\) and it reduces to \( -p^2 \sum \frac{1}{t^2} \) for \( U = U(2,1) \).

Lemma 21. We have for all primes \( p \geq 7 \) of maximal rank \( \rho \)
\[
\frac{U^2}{V^2} \sum_1 \equiv -\frac{1}{2} \frac{U^2}{V^2} \sum_{1<\rho} 4Q' \quad (\text{mod } p^5).
\]

Proof. By Lemma 18 we see that
\[
\frac{U^2}{V^2} \sum_1 \equiv -\frac{1}{2} \frac{U^2}{V^2} \sum_{0<\rho} 4Q' \quad (\text{mod } p^5).
\]

By Lemma 17
\[
\sum_{0<\rho} 4Q' \equiv \Sigma_2 - D(\rho - 1).
\]
Thus, as \( \Sigma_2 = \Sigma_2^2 - 2\Sigma_{1,1} \equiv -2\Sigma_{1,1} \quad (\text{mod } p^4) \), the lemma follows. \qed

By using Lemma 21 and Theorem 16 we obtain another generalization of 25 slightly different from that given in Theorem 15 which we now state.

Theorem 22. Let \((U, V)\) be a pair of Lucas sequences with parameters \( P \) and \( Q \). Let \( p \) be a prime at least 7 of maximal rank \( \rho \) equal to \( p - \epsilon \). Then \( \binom{2^{p-1}}{p-1} \) is congruent to
\[
(-1)^\epsilon pQ^{\rho(p-1)/2} \left[ 1 + \frac{U^2}{V^2} \sum_{0<\rho} V_t U_t + \frac{U^2}{V^2} \sum_{0<\rho} V_t U_t - \frac{1}{2} D \frac{U^2}{V^2} (\rho - 1) \right] \quad (\text{mod } p^5).
\]

We end the paper with a congruence for \( \binom{2^{p-1}}{p-1} \) modulo \( p^6 \). It generalizes Theorem 2.4 of 25 which says that
\[
\left( \binom{2p-1}{p-1} \right) \equiv 1 + 2p \sum_{0<\rho} \frac{1}{t} + \frac{2p^3}{3} \sum_{0<\rho} \frac{1}{t^3} \quad (\text{mod } p^6),
\]
for all primes \( p \geq 7 \), and also generalizes our Theorem 16.
Theorem 23. Let \((U, V)\) be a pair of Lucas sequences with parameters \(P\) and \(Q\). Let \(p\) be a prime at least 7 of maximal rank \(\rho\). Then
\[
\binom{2\rho - 1}{\rho - 1}_U \equiv (-1)^{\rho-1}Q^{\rho(\rho-1)/2} \left[ 1 + \frac{2}{\sqrt{\rho}} \sum_{0 < t < p} V_t \frac{U_t}{U_t} + \frac{2}{3} \frac{U^3_{\rho}}{V^3_{\rho}} \sum_{0 < t < p} \frac{V^3_t}{U^3_t} \right] \pmod{p^6}.
\]

Proof. We proceed as in Lemma 14 to show that \(\Sigma_{1,1,1,1,1} \equiv 0 \pmod{p}\) (in fact 0 modulo \(p^2\)). First we extend the definitions made before Lemma 14 to define analogously the sums \(\Sigma_{1,4}, \Sigma_{1,1,3}, \Sigma_{2,3}, \Sigma_{1,2,2}\) and \(\Sigma_{1,1,1,2}\). The expressions \(\Sigma_1 \cdot \Sigma_4 - \Sigma_5, \Sigma_3 \cdot \Sigma_1, \Sigma_1 \cdot \Sigma_3\) and \(\Sigma_1 \cdot \Sigma_{2,2}\) are all 0 (mod \(p^2\)), so we deduce, successively, that the sums \(\Sigma_{1,4}, \Sigma_{1,1,3}, \Sigma_{2,3}\) and \(\Sigma_{1,2,2}\) are each 0 (mod \(p^2\)). Therefore, modulo \(p^2\), the two expressions \(\Sigma_1 \cdot \Sigma_{1,1,1,1}\) and \(\Sigma_5 - \Sigma_5\) are linear combinations of \(\Sigma_{1,1,1,1,1}\) and \(\Sigma_{1,1,1,2}\). Because these two expressions are each 0 (mod \(p^2\)) we deduce that \(\Sigma_{1,1,1,1,1} \equiv 0 \pmod{p^2}\).

Since \(\Sigma_{1,1,1,1,1}\) is 0 (mod \(p\)), both the congruence for \(\binom{2\rho - 1}{\rho - 1}_U\), derived from the proof of Theorem 15 and congruence (28) remain valid when we raise the modulus from \(p^5\) to \(p^6\). Hence,
\[
\binom{2\rho - 1}{\rho - 1}_U \equiv (-1)^{\rho-1}Q^{\rho(\rho-1)/2} \equiv \left( \frac{V^2_{\rho}}{2} \right)^{\rho-1} \left( 2 \frac{U^3_{\rho}}{V^3_{\rho}} \Sigma_{1,1,1,1}^U \right) \pmod{p^6}.
\]
Suppose first that \(\epsilon_p = -1\) or \(\epsilon_p = 0\). Then, as \(\Sigma_{1,1} \equiv 0 \pmod{p}\), we find that (29) is valid modulo \(p^3\). Thus, we may replace \((V^2_{\rho}/2)^{\rho-1}\) in (30) by \((-1)^{\rho-1}Q^{\rho(\rho-1)/2}\) and obtain that
\[
\binom{2\rho - 1}{\rho - 1}_U \equiv (-1)^{\rho-1}Q^{\rho(\rho-1)/2} \left( 1 + 2 \frac{U^3_{\rho}}{V^3_{\rho}} \Sigma_{1,1,1,1} \right) \pmod{p^6}.
\]
Looking at the linear system at the start of the proof of Lemma 14 modulo \(p^3\) we find the system of congruences
\[
3\Sigma_{1,2} + 6\Sigma_{1,1,1} \equiv -\Sigma_3, \\
\Sigma_{1,2} + 3\Sigma_{1,1,1} \equiv 0.
\]
Solving for \(\Sigma_{1,1,1}\), we see that \(\Sigma_{1,1,1} \equiv \frac{\Sigma_3}{3} \pmod{p^3}\), which inserted in congruence (31) yields the theorem.

Suppose now \(\epsilon_p = 1\) so that congruence (29), when the modulus is increased to \(p^3\), becomes
\[
(-1)^{\rho-1}Q^{\rho(\rho-1)/2} \equiv \left( \frac{V^2_{\rho}}{2} \right)^{\rho-1} (1 + DU^2_{\rho}/V^2_{\rho}) \pmod{p^3}.
\]
Thus we may replace \((V_\rho/2)^{\rho-1}\) in (30) by \((-1)^{\rho-1}Q^{(\rho-1)/2}(1-DU_\rho^2/V_\rho^2)\), multiply out the resulting expression and remove the term in \(U_\rho^5\Sigma_{1,1,1}\) which is 0 \((\text{mod } p^7)\) to find that

\[
\left(\frac{2\rho - 1}{\rho - 1}\right)_{U} \equiv (-1)^{\rho-1}Q^{(\rho-1)/2}\left(1 + \frac{2U_\rho}{V_\rho^2}\Sigma_1 + \frac{2U_\rho^3}{V_\rho^3}(\Sigma_{1,1,1} - D\Sigma_1)\right) \quad \text{(mod } p^6)\).
\]

Because \(\Sigma_1\) is 0 \((\text{mod } p^2)\) and \(\Sigma_{1,1} \equiv D \pmod{p}\), the linear system of Lemma 14 taken modulo \(p^3\) is

\[
3\Sigma_{1,2} + 6\Sigma_{1,1,1} \equiv -\Sigma_3,
\]

\[
\Sigma_{1,2} + 3\Sigma_{1,1,1} \equiv D\Sigma_1.
\]

Solving for \(\Sigma_{1,1,1}\) yields \(\Sigma_{1,1,1} \equiv D\Sigma_1 + \Sigma_3/3\) and the theorem holds.

5 Appendix on the integrality of Lucanomials

The question of the integrality of Lucanomials has appeared in various places, but we want to formally prove that with convention (12) they are integral in full generality.

**Proposition 24.** Let \(U = (U_n)\) be a Lucas sequence with parameters \(P\) and \(Q\). With the adoption of convention (12) the Lucanomial coefficients \((m_n)_U\) are rational integers for all nonnegative integers \(m\) and \(n\).

**Proof.** If all \(U_n, n > 0\), are nonzero then the frequently used induction argument (see [13], Lemma 1; or [6]) based on the general Lucas identity \(U_{n+1}U_{m-n} - QU_nU_{m-n-1} = U_m\) works fine. (The induction is on \(m\). So one proves the integrality of the Lucanomial \((m_n)_U\) for \(m > n \geq 1\) by observing that

\[
U_{n+1}\left(\begin{array}{c}
m - 1 \\
n
\end{array}\right)_U - QU_{m-n-1}\left(\begin{array}{c}
m - 1 \\
n - 1
\end{array}\right)_U =
\]

\[
\left(U_{n+1}\frac{U_{m-n}}{U_n} - QU_{m-n-1}\right)\cdot\left(\begin{array}{c}
m - 1 \\
n - 1
\end{array}\right)_U =
\]

\[
\frac{U_m}{U_n}\cdot\left(\begin{array}{c}
m - 1 \\
n - 1
\end{array}\right)_U = \left(\begin{array}{c}
m \\
n
\end{array}\right)_U,
\]

completing the induction.) If some term \(U_n, n \geq 1\), is 0 then \(U\) is degenerate and, as we saw early in Section 2, \(\rho(\infty) \in \{2, 3, 4, 6\}\), where \(\rho(\infty)\) is the least
positive integer $t$ such that $U_t = 0$. Note that we may always assume $m \geq 2n$. Thus the Lucanomial $\binom{m}{n}_U$ is the quotient of a product of $n$ consecutive $U$ terms of indices all larger than $n$ divided by $U_n U_{n-1} \cdots U_1$. If $\rho(\infty) = 2$, i.e., $U_2 = P = 0$, then $U_{2k+1} = (-1)^k Q^k$ and $U_{2k} = 0$, $(k \geq 0)$. Then $\binom{m}{n}_U$ is up to sign a positive power of $Q$. If $\rho(\infty) = 3$, then, as $U_3 = P^2 - Q$, the first few terms of $U$ are $0, 1, P, 0, -P^3, -P^4, 0, P^6, P^7, 0, \cdots$. So $|U_t| = P^{t-1}$ if $3 \nmid t$. If $\rho(\infty) = 4$, then, as $U_4 = P^3 - 2PQ$ and $P \neq 0$, $P^2 = 2Q$ and we see that $|U_t| = 2^{t/2}(P')^{t-1}$ if $4 \nmid t$, where $P = 2P'$. Omitting the 0 terms when $4 \mid t$, powers of 2 and $P'$ in $U_t$ are nondecreasing functions of $t$. A similar result holds for $\rho(\infty)$ equal to 6 when $P^2 = 3Q$ and, omitting terms divisible by 6, powers of 3 and of $P'$ in $U_t$ are nondecreasing functions of $t$, where in this case $P = 3P'$. The integrality of the Lucanomials follows readily.

References

[1] C. Babbage, Demonstration of a theorem relating to prime numbers, Edinburgh Philosophical J., 1 (1819), 46–49.

[2] D. F. Bailey, Two $p^3$ variations of Lucas’ theorem, J. Number Theory, 35 (1990), no. 2, 208–215.

[3] C. Ballot, On a congruence of Kimball and Webb involving Lucas sequences, J. Integer Seq., 17 (2014), Article 14.1.3.

[4] C. Ballot, Lucas sequences with cyclotomic root field, Dissertationes Math., 490 (2013), 92 pp.

[5] C. Ballot, A further generalization of a congruence of Wolstenholme, J. Integer Seq., 15 (2012), Article 12.8.6.

[6] A. Benjamin and S. Plott, A combinatorial approach to Fibonomial coefficients, Fibonacci Quart., 46/47 (2008/09), no. 1, 7–9.

[7] V. Brun, J. O. Stubban, J. E. Fjeldstad, R. Tambs Lyche, K. E. Aubert, W. Ljunggren, E. Jacobsthal. On the divisibility of the difference between two binomial coefficients. Den 11te Skandinaviske Matematikerkongress, Trondheim, 1949, pp. 42–54. Johan Grundt Tanums Forlag, Oslo, 1952.
[8] R. D. Fray, Congruence properties of ordinary and q-binomial coefficients. *Duke Math. J.*, 34, (1967) 467–480.

[9] J. W. L. Glaisher, Congruences relating to the sums of products of the first \( n \) numbers and to other sums of products, *Q. J. Math.* 31 (1900), 1–35.

[10] J. W. L. Glaisher, On the residues of the sums of products of the first \( p - 1 \) numbers, and their powers, to modulus \( p^2 \) or \( p^3 \), *Q. J. Math.* 31 (1900), 321–353.

[11] H. W. Gould, The bracket function and Fontené-Ward generalized binomial coefficients with application to Fibonomial coefficients, *Fibonacci Quart.*, 7.1 (1969) 23–40, 55.

[12] C. Hooley, On Artin’s conjecture, *J. Reine Angew. Math.* 225 (1967), 209–220.

[13] H. Hu and Z-W Sun, An extension of Lucas’ theorem, *Proc. Amer. Math. Soc.* 129 (2001), no. 12, 3471–3478.

[14] W. Kimball and W. Webb, A congruence for Fibonomial coefficients modulo \( p^3 \), *Fibonacci Quart.*, 33 (1995) 290–297.

[15] W. Kimball and W. Webb, Some generalizations of Wolstenholme’s theorem, *Applications of Fibonacci Numbers* 8 (Rochester, NY, 1998), Kluwer Acad. Publ., Dordrecht, (1999), 213–18.

[16] D. Knuth and H. Wilf, The power of a prime that divides a generalized binomial coefficient, *J. Reine Angew. Math.* 396 (1989), 212–219.

[17] R. J. McIntosh, On the converse of Wolstenholme’s Theorem, *Acta Arith.*, 71 (1995), 381–389.

[18] R. Meštrović, Wolstenholme’s theorem: Its generalizations and extensions in the last hundred and fifty years (1862-2012), preprint [http://arxiv.org/abs/1111.3057v2[math.NT]](http://arxiv.org/abs/1111.3057v2[math.NT]).

[19] R. Meštrović, Congruences for Wolstenholme primes, preprint [http://arxiv.org/abs/1108.4178[math.NT]](http://arxiv.org/abs/1108.4178[math.NT]).
[20] H. Pan, A generalization of Wolstenholme’s harmonic series congruence, *Rocky Mountain J. Math.*, 38 (2008), 1263–1269.

[21] R. Stanley, *Enumerative combinatorics*, Volume 1. Second edition. Cambridge Studies in Advanced Mathematics, 49. Cambridge University Press, Cambridge, (2012).

[22] A. Straub, A $q$-analogue of Ljunggren’s binomial congruence, 23rd International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2011), 897–902, *Discrete Math. Theor. Comput. Sci. Proc.*, AO, Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2011

[23] H. Roskam, A quadratic analogue of Artin’s conjecture on primitive roots. *J. Number Theory* 81 (2000), no. 1, 93–109.

[24] H. Roskam, Erratum: “A quadratic analogue of Artin’s conjecture on primitive roots” [J. Number Theory 81 (2000), no. 1, 93–109. *J. Number Theory* 85 (2000), no. 1, 108.

[25] R. Tauraso, More congruences for central binomial coefficients. *J. Number Theory*, 130 (2010), no. 12, 2639–2649.

[26] M. Ward, Note on divisibility sequences. *Bull. Amer. Math. Soc.* 42 (1936), no. 12, 843–845.

[27] H. C. Williams, *Édouard Lucas and primality testing*, Wiley, Canadian Math. Soc. Series of Monographs and Advanced Texts, (1998).

[28] J. Wolstenholme, On certain properties of prime numbers, *Q. J. Pure Appl. Math.*, 5 (1862), 35–39.

[29] J. Zhao, Bernoulli numbers, Wolstenholme’s theorem, and $p^5$ variations of Lucas’ theorem. *J. Number Theory*, 123 (2007), no. 1, 18–26.

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