Integrable quantum many-body scars in a non-integrable large-N spin chain

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Exact quantum many-body scar states in non-integrable models that have been found in recent years all share the common property of being lowly entangled, therefore representable by matrix product states. Here, we present a novel type of partly integrable one-dimensional multicomponent model that can be viewed as a minimal generalization to the spin-\(\frac{1}{2}\) XXZ model with reduced frustration. In certain subspaces of the Hilbert space, exact eigenstates are Bethe Ansatz integrable, embedding the spectra of spin-\(\frac{1}{2}\) XXZ Hamiltonians. For the rest of Hilbert space, it’s non-integrable in the sense that its scattering matrices violate Yang-Baxter’s equation. This model in turn can be viewed as a low-energy effective Hamiltonian for the “coprime quantum spin ladder” model \cite{1}. We show the entanglement entropy of its ground state scales linearly with the size of the subchain with sufficiently large number of local degrees of freedom, and argue that the scar states have entanglement entropy scalings similar to those established for Bethe Ansatz integrable spin-\(\frac{1}{2}\) chains.

I. INTRODUCTION

Recent progress in cold atom experiments have made possible the simulation of a larger variety of models of strongly correlated systems\cite{2}. In particular, alkine-earth atoms such as \(^{137}\)Yb and \(^{87}\)Sr in optical atoms can be used to realize the Mott insulating phase of Fermi-Hubbard model, which can be described by the SU(N) Heisenberg Hamiltonian \cite{3,4,5,6,7}. This multi-component generalization of the spin-\(\frac{1}{2}\) Heisenberg model was solved based on Yang’s gernalization to the Bethe Ansatz \cite{8,9} method by Sutherland \cite{10}. Using the same nested Bethe Ansatz method, Babelon, de Vega, and Viallet constructed an integrable \(\mathbb{Z}_{n+1} \times \mathbb{Z}_{n+1}\) symmetric generalization of the XXZ model, from the solution of Yang-Baxter’s equations \cite{11}. However, the anisotropy in this model is artificially designed so that the model is exactly solvable, and not realistic to be realized in experiment. In this paper, we propose a minimal generalization from the spin-\(\frac{1}{2}\) XXZ Hamiltonian, written in terms of permutation operators, whose SU(N) symmetry is explicitly broken to the symmetric group \(\mathfrak{S}_N\) by a term composed of Cartan operators. As was adopted in Sutherland’s original formulation of the isotropic model \cite{10}, such a multicomponent spin model can be equivalently formulated in the language of hard-core bosonic particles moving in a one-dimensional space with density-density interactions. This was in part motivated by the recently proposed models bearing a number theoretic analogy \cite{1,12}, whose low energy effective theory becomes the current model in certain limits of its parameters. The questions raised in those papers, such as the scaling of the energy gap from ground state, are partially answered in this paper.

While this model is not integrable, a subset of its eigenstates with energies spanning the entire spectrum are nonetheless integrable. Such models fall in the category of quasi-exactly solvable models, which has been studied since the time of quantum mechanics \cite{13}. In many-body systems, there is a class of models called frustration-free, whose zero energy ground state can be explicitly written as the common lowest energy eigenstate of each local operator in the Hamiltonian. Examples include the Majumdar-Ghosh model \cite{14}, AKLT model \cite{15}, Motzkin and Fredkin spin chains \cite{16,17,18,19,20}, and in two dimensions Kitaev’s toric code model \cite{21}, to name a few. Many of these models boast soluble eigenstates beyond ground state as well. For instance, stabilizer code Hamiltonians consist of mutually commuting operators, so the entire spectrum is soluble. Despite not satisfying that condition, Majumdar-Ghosh \cite{22}, and spin-1 AKLT model \cite{23} are also long been known as having exact excited states. Recently they have generated stronger interests due to the experimental observation of anomalous dynamics in Rydberg atom experiments \cite{24} that intrapolates between ergodicity breaking and thermalization. Such dynamics was numerically attributed to a large overlap of the initial state with a tower of ETH-violating eigenstates with equally spaced energies in the so-called PXP Hamiltonian \cite{25}, used to model the Rydberg blockade experiment \cite{26}. The name “quantum many-body scar (QMBS)” , which originally refers to such a paradigm of ergodicity breaking, later by a slight abuse of terminology predominantly refers to analytically solvable towers of equally spaced ETH-violating eigenstates, as more and more such examples are found \cite{27,28,29,30,31,32,33,34,35,36}. More recently, these individual examples are to some extent unified in the framwork of spectrum generating algebra \cite{37}, which explains the equal-distant spacing of the energy spectrum of these scar states.
However, the relation between rational value of energies and the existence of scar states is yet to be established. The tremendous success in searching these scar states is largely thanks to the approach of numerically examining the entanglement spectrum for lowly entangled states\cite{29}, such that they can be exactly represented with matrix product states \cite{33} by reverse engineering. Therefore, an example of quantum many-body scar state with area-law violating entanglement is so far absent, and in general, whether the sparsity of entanglement spectrum is a necessary condition for the existence of exact eigenstates remains to be understood.

The plethora of the class of frustration-free model and its success in providing insights in understanding various aspects of many-body systems and quantum computing through rigorous mathematical theorems \cite{39–44}, naturally poses the question of "to what extent can a frustration-free Hamiltonian intrapolate between specifying only a ground state, and a stabilizer code that gives the entire solvable spectrum". The examples of exact QMBS in AKLT models partially answered this question, in that the additional eigenstates are of momenta $\pi$, allowing the diffractive scattering to be cancelled out in a momentum eigenstate\cite{29}. In this work, we try to extend this idea to non-diffractive scattering with other momenta using Bethe Ansatz, in a non-integrable multicomponent model with a "weaker" frustration. Such a strategy of easing frustration by enlarging local degrees of freedom has been employed by models with $SU(N)$-singlet simplex solid ground states\cite{45}, in the form of geometric frustration, which also has one-dimensional cousin models with longer range interaction, such as $SU(3)$ spin chains with trimer and valance bond solid (VBS) ground states\cite{46}, and $SU(N)$ VBS\cite{47}. Here, instead of designing a Hamiltonian as sum of projectors pinning down a desired ground state, we work with a minimal multicomponent generalization to the spin-$\frac{1}{2}$ XXZ spin chain with 2-local interaction, which by violation of Yang-Baxter’s equation (YBE) is non-integrable, and show that the reduced frustration with larger local Hilbert space allows integrable subspaces solvable by Bethe Ansatz, with countinuous spectra, and area-law violating entanglement.

Entanglement entropy and its scaling has been a major tool in probing many-body systems with strong correlations, see \cite{48} for a review. In one-dimension, Hastings have given a rigorous proof that the ground state entanglement entropy scaling with system size is bounded by a constant for gapped systems\cite{49}. Meanwhile, people have been searching for area-law violating grounds states in gapless systems with logarithmic\cite{16, 19}, power law\cite{17}, and even linear scaling\cite{18, 20, 21, 50, 51}. In the study of quantum many-body scars, entanglement spectrum and the violation of volume law scaling of entanglement entropy has been used to show the exact excited states violate strong eigenstate thermalization hypothesis\cite{29}. In this paper, we show the entanglement entropy of the ground state of our translationally invariant model with only local interactions, when the dimension of local Hilbert space is large enough, scales linearly with system size. The excited states being embeded Bethe Ansatz states is expected to have similar area-law breaking scalings of entanglement entropy as in the spin-$\frac{1}{2}$ XXZ model\cite{52, 53}, depending on the anisotropy.

The paper is organized as follows. In Sec. II we introduce the model and discuss its symmetries, which we will take advantage of later in its diagonalization. In Sec. III, we first construct a basis to diagonalize the integrable subspaces of the Hilbert space in III A, first with an elementary approach for the two-body example, then with the general coordinate Bethe Ansatz, establishing the embedding of the spectrum of spin-$\frac{1}{2}$ Heisenberg in our model. In III B, we show the sector with multiple identical particles is non-integrable unless the anisotropy is turned off, and outline the method of diagonalization for the isotropic case. In section IV, we analytically calculate the entanglement entropy of the ground state in the case of having enough number of species of particles, and show that in the thermaldynamic limit, it scales linearly with system size. In Sec. V, we give a summary and point out a few future direction worth exploring.

II. THE HAMILTONIAN AND ITS SYMMETRY

We consider a minimal generalization to the spin-1/2 anti-ferromagnetic Heisenberg model with an anisotropy parameter $\Delta$. The local Hilbert space $C^s$ is spanned by the $s$ components of the SU(N) group, or $s$ species of hard-core bosons. The model is defined on a one-dimensional lattice of length $L$, with periodic boundary condition $C_{L+1} \equiv C_1$. Its Hamiltonian takes the form

$$H = \sum_{i=1}^{L} [P_{i,i+1} + 2\Delta C_{i,i+1}]$$ (1)
where transposition operator $P_{i,i+1}$ and the diagonal anisotropy term $C_{i,i+1}$ are defined as

$$P_{i,i+1} = \sum_{a,b=1}^{s} e^{(ab)}_i \otimes e^{(ba)}_{i+1},$$  \hspace{1cm} (2)

$$C_{i,i+1} = \sum_{a=1}^{s} e^{(aa)}_i \otimes e^{(aa)}_{i+1},$$  \hspace{1cm} (3)

where $(e^{(ab)})_{cd} = \delta^a_c \delta^b_d$ is the standard basis of $s \times s$ matrices, so that $P_{i,i+1}(v_i \otimes v_{i+1}) = v_{i+1} \otimes v_i$ for any $v_i, v_{i+1}$. The transposition term is SU(N) invariant, as it can be written as traces over product of SU(N) generators. The diagonal term however, can be written only in terms of the generators of the Cartan subalgebra of SU(N), and therefore breaks the symmetry down to the symmetric group $S_N$.

Apart from the multicomponent chain interpretation, one can also look at this model as a hard-core boson with a color degree of freedom propagating in a repulsive density-density interaction. From now on, we will interchange freely between these two languages for the convenience of presentation. However, while one can map each boson with different color to a prime number, and consider the diagonal term the “coprime interaction”, unlike the model discussed in [54], we do not have a quasi-2d lattice, and each boson doesn’t have a composite structure consisting of more elementary degrees of freedom. Moreover, there is no onsite dynamical terms in the Hamiltonian that allows bosons of one species to transmute into others, or raising or lowering spins. Nonetheless, the coprime spin ladder model does reproduce this Hamiltonian when reduced to a certain sector of particle content $\{c_1^{n_1}, c_2^{n_2}, \ldots, c_s^{n_s}\}$, where $c_A$ denotes the particle of the $A$’th color, and $n_A$ denotes the number of times it appear along the chain. In each sector, the restricted Hamiltonian will be equivalent to [10]

$$H(\{c_1^{n_1}, c_2^{n_2}, \ldots, c_s^{n_s}\}) = \sum_{i=1}^{L} \left[ (1 + 2\Delta) \sum_{A} N_{i,i+1}^{(AA)} + \sum_{A<B} P_{i,i+1}^{(AB)} \right],$$  \hspace{1cm} (4)

where $N^{(AA)}$ counts the number of neighboring pairs of species $A$, and $P^{(AB)}$ transposes only neighboring pairs of species $A$ and $B$.

This restricted Hamiltonian manifests permutation symmetries in addition to the translational invariance. Firstly, each total number of each species $n_A$ is conserved. Secondly, each particle being indistinguishable, this Hamiltonian is invariant under $S_{n_1} \otimes S_{n_2} \otimes \cdots \otimes S_{n_s}$, where $\sum_{A=1}^{s} n_A = L$, and $\otimes$ denote the outer product between symmetric groups [54, 60]. Notice that this is a subgroup of the original $\hat{S}_L$ symmetry. Therefore, eigenvectors of this Hamiltonian can be search within each of its irreducible representations (irrep’s). This step would already reduce the complexity of the problem significantly enough so that one can diagonalize the Hamiltonian written in terms of standard irreducible representations matrices by brute force, as was done in [57, 58]. With some more thoughts, Gaudin [59, 60] turned the Young tableauxs, equivalently described in the language of Hund’s method usually more preferable to physicist, into a linear system of constraints from Fock’s conditions, which by one remarkable algebraic identity after another, arrived at the same Bethe-Yang Hypothesis states. However, the anisotropy term in our Hamiltonian mixes irreducible representations, making this approach not applicable.

### III. Diagonalization of the Hamiltonian

The Hamiltonian [1] without the anisotropy term is diagonalized by Sutherland using nested Bethe Ansatz [10, 61]. As will be shown at the end of this section, the anisotropy breaks integrability by violating the Yang-Baxter Equation (YBE), except for certain irrep’s. The large number of local degrees of freedom, or species of particles, can be both a curse and a blessing. In this papaer, we take advantage of this property by constructing a particular basis, in which the solution of relatively low-lying eigenstates using Bethe Ansatz is drastically simplified. In this section, we first show how this trick helps solving eigenstates in the sector where only one species of particle appear multiple times, while the rest each appears only once. Then at the end of the section, we show the YBE is still violated in this basis for a generic sector where multiple species appear more than once.

#### A. One identical species case: the $\{c_1^n, c_2, \ldots, c_{L-n+1}\}$ sector

The key observation leading to this method is made by realizing that each transposition operator is minimized at eigenvalue $-1$. So the ground state is achieved by antisymmetrizing as many neighbors as possible. This can
only reconcile among different species, as antisymmetrizing a symmetric pair gives 0. Therefore, we can restrict our searching for the ground state in the subspace of the Hilbert space spanned by the basis
\[
| i_1, i_2, \ldots, i_n \rangle = \sum_{\sigma \in S_L} \text{sgn}(\sigma)(-\sum_{a=1}^{n} \sigma_a | i_1, i_2, \ldots, i_n ; \sigma (c_2 c_3 \cdots c_{L-n}) \rangle),
\]
where \( i_1 < i_2 < \cdots < i_n \) labels the location of the \( n \) identical species, here chosen to be the one corresponding to the smallest prime number, and \( \text{sgn}(\sigma) \) denotes the signature of the permutation \( \sigma \) acting on the reference state in a certain order of species with \( c_j \) denoting particle of the \( j \)-th color, before permuting the identical species to their final locations. An illustration of the \( n = 2, L = 5 \) case is given in Fig. 1. The reason for this convenient choice of basis is such that
\[
P_{j,j+1} | i_1, \ldots, i_n \rangle = \begin{cases} 
-|i_1, \ldots, i_n \rangle, & j \neq i_a, i_a - 1, \forall a \\
-i_1, \ldots, i_{a-1}, i_a + 1, i_{a+1}, \ldots, i_n \rangle, & j = i_a \neq i_{a+1} - 1 \\
|i_1, \ldots, i_n \rangle, & j = i_a = i_{a+1} - 1
\end{cases}
\]
for any \( i_a < L \). Since the transposition operators act the kinetic term of the Hamiltonian, antisymmetrizing the species is equivalent to picking a subspace where all the particles with different species except the identical one that appears multiple times to have momenta \( \pi \). Periodic boundary condition on the physical chain leads to (anti-)periodic boundary condition on our artificial basis, depending on the parity of the length of the chain.

\[
| i_1, \ldots, i_{n-1}, L + 1 \rangle \equiv (-1)^L | i_1, \ldots, i_{n-1} \rangle,
\]
where \( i_1 > 1 \). For simplicity, we restrict our discussion to the case of even length from now on. The odd length case can be treated likewise.

In this basis, we can diagonalize the Hamiltonian by expanding the eigenvector as
\[
| v \rangle = \sum_{1 \leq i_1 < \cdots < i_n \leq L} w_{i_1, \ldots, i_n} | i_1, i_2, \ldots, i_n \rangle.
\]
When keeping in mind that our basis vectors are already eigenvectors of transposition operators unless it acts on the identical species, the eigenvalue equation
\[
H|v\rangle = E_v|v\rangle.
\]
results in a difference equation version of the Helmholtz equation of weights \( w_{i_1, \ldots, i_n} \)
\[
\nabla^2 w_{i_1, \ldots, i_n} = -(E_v + L)w_{i_1, \ldots, i_n},
\]
where we used \( \nabla^2 \) to denote the discrete Laplacian operator so as to avoid confusion with the anisotropy parameter. If none of the position \( i \)'s is consecutive,
\[
\nabla^2 w_{i_1, \ldots, i_n} \equiv \sum_{a=1}^{n} (w_{i_1, \ldots, i_{a-1}, \ldots, i_n} + w_{i_1, \ldots, i_{a+1}, \ldots, i_n} - 2w_{i_1, \ldots, i_n}),
\]
where \( w_{i_1, \ldots, i_{n-1}, L+1} \equiv (-1)^{n-1} w_{i_1, \ldots, i_{n-1}} \) due to (7). When certain position \( i \)'s form consecutive strings \( l_a, l_a + 1, \ldots, r_a \), \( \nabla^2 \) becomes
\[
\nabla^2 w_{i_1, \ldots, i_{l_a}, \ldots, i_{r_a}} = \sum_{a=1}^{s_{l_a}} (w_{i_1, \ldots, i_{l_a-1}, l_a+1, \ldots, r_a, \ldots, i_{r_a}} + w_{i_1, \ldots, i_{l_a-1}, r_a+1, \ldots, i_{r_a}} \cdots - |\sum_{a=l_a}^{r_a-1} (2 + 2\Delta) + 2|w_{i_1, \ldots, i_{l_a}}|\).
\]
Special remark is due here to make the distinction with ferromagnetic spin-1/2 Heisenberg model as solved by Bethe Ansatz, there the coefficient of the third term in (12) this term would remain $-2$ regardless of how long the string of consecutive spin downs would be. This difference eventually will lead to our different solution from Bethe Ansatz equations demonstrated below.

1. Pedagogical two-body isotropic example

The following two subsections serve as a very elementary introduction to readers who aren’t currently familiar with the tricks of the trade of Bethe Ansatz. More experienced readers should jump directly to our main result in III A 3. The case of $n = 2$ as we will see shortly corresponds to a 2-quasiparticle excitation from the background of momentum-$\pi$ species. Considering this case is particularly illuminating as it allows an exact diagonalization by solving a second-order linear homogeneous recurrence relation. To see this, we perform the transformation

$$w_{1,i_2} \rightarrow a_r^{(s)} = w_{r,r+s},$$

where $r$ labels the site of the first occurrence of the identical species, while $s$ denotes the distance between the two of them. The eigenvector is then expressed as

$$|v\rangle = \sum_{r=1}^{L} \sum_{s=1}^{L-r} a_r^{(s)} |r, r+s\rangle.$$

And the anti-periodic boundary condition is given by

$$a_{r}^{(L-r+1)} = -a_1^{(r-1)}.$$

We first exploit the translational invariance of the Hamiltonian by diagonalizing in the basis formed by eigenvectors of the translation operator

$$T|v\rangle = t^{-1}|v\rangle.$$

According to the anti-periodic boundary condition (7), we have

$$T^L|1,2\rangle = T^2|L-1,L\rangle = -T|1,L\rangle = |1,2\rangle.$$

So $T^N = 1$ still applies, giving

$$t = e^{2\pi \theta}, \quad \theta = \frac{i\pi k}{L}, \quad k = 1, \ldots, L.$$

Therefore, (16) implies $a_r^{s+1} = ta_r^s$, and up to a normalization constant, we can choose $a_1^r = t^r$ as the initial condition for recurrence equations of $a_r^s$ with respect to index $s$.

The recurrence relation (10) now becomes

$$a_r^{(2)} + a_{r-1}^{(2)} = ea_r^{(1)}, \quad a_r^{(s-1)} + a_{r-1}^{(s-1)} + a_r^{(s+1)} + a_{r-1}^{(s+1)} = ea_r^{(s)}, \quad s > 1,$$

where $e = -(E_v + N - 4)$. Using translation invariance, the first equation gives the second initial condition necessary to solve the second-order recurrence relation $a_r^2 = \frac{e^{2^2 + 1}}{1 + t}$, while the second equation becomes

$$a_r^{(s+1)} - \frac{et}{1 + t} a_r^{(s)} + ta_r^{(s-1)} = 0.$$

The solution of its characteristic equation depends on its discriminant $D = \frac{e^2 + 1}{1 + t} - 4t$.

a. $D = 0$: Identical root: In this scenario, the general solution is of the form $a_r^s = (c_0 + c_1 s)t^s$, which cannot satisfy the anti-periodic boundary condition (15) and the initial conditions simultaneously. We conclude that the discriminate must be nonzero.
b. $D \neq 0$: Different root: The characteristic equation has two roots in this case
\[
\lambda_{\pm} = \left( \frac{e}{4 \cos \theta} \pm i \sqrt{1 - \left( \frac{e}{4 \cos \theta} \right)^2} \right) e^{i\theta},
\]
\[
= e^{i(\theta \pm \alpha)},
\]
where $\alpha = \arccos\left( \frac{e}{4 \cos \theta} \right)$. Here, we have assumed $|e| \leq 4 \cos \theta$, which is necessary for the anti-periodic boundary to hold. Together with the initial conditions given by $a^{(s)}_r$ and $a^{(s)}_r$, (22) gives
\[
a^{(s)}_r = t^{r+(s-1)/2} \sin \frac{s\alpha}{\sin \alpha},
\]
Plugging this into the anti-periodic boundary condition (15), we get
\[
\alpha = \theta + \theta', \quad \theta' = \frac{2\pi k'}{L}, \quad k' = 1, 2, \ldots, L.
\]
We can parametrize the energy eigenvalues with $\theta_1 \equiv \theta - \alpha = -\theta'$ and $\theta_2 \equiv \theta + \alpha = 2\theta + \theta'$, and the eigenvectors with $\mu_{1,2} \equiv e^{i\theta_{1,2}}$ as
\[
E(\theta_1, \theta_2) = -L + 4 - 2\cos \theta_1 - 2\cos \theta_2
\]
\[
|\theta_1, \theta_2\rangle = \frac{1}{N} \sum_{1 \leq i,j \leq L} (\mu_1^\dagger \mu_2^\dagger - \mu_1^\dagger \mu_2^\dagger) |i,j\rangle,
\]
where $\theta_{1,2} = i2\pi k_{1,2}/N$, with $k_{1,2} = 1, 2, \ldots, N$. Since its coefficients $a_{i,j} = -a_{j,i}$, we can alternatively express it as
\[
|\theta_1, \theta_2\rangle = \frac{1}{N} \sum_{1 \leq i,j \leq L} \mu_1^\dagger \mu_2^\dagger |i,j\rangle,
\]
with $|i,i\rangle \equiv 0$. Notice the $\theta_1 = \theta_2$ solutions give vanishing eigenvectors, as is expected when one attempts to antisymmetrize a symmetrized pair of identical species. In other words, when the anisotropy or density-density interaction is absent, these identical particles of same species behave as free fermions. The fact that our original bosonic degrees of freedom has ended up behaving as fermions is exactly due to the unitary transformation pointed out by Sutherland [10] of multiplying all wavefunctions by completely antisymmetric one in all objects, which flips the sign of the Hamiltonian and the exchange statistics of the particles at the same time. The ground state energy in this sector is therefore $-L + 2 - 2\cos \frac{2\pi}{L}$. In the thermodynamic limit, it gives a vanishing gap that scales with $\frac{1}{L^2}$ counting from the universal ground state of $L$ different species.

The partial spectrum and corresponding eigenstates given above agree with the results from nested Bethe Ansatz by Sutherland [10], which starts from a reference state of identical species and consider the different ones as quasiparticles moving around. To map the eigenstates solved from our approach of treating the different species as background and identical ones as quasiparticles, to the framework of Bethe Ansatz, the different species are understood to have $\pi$ momenta. Our solution corresponds to two particles having momenta $\theta_1$ and $\theta_2$, with a scattering phase of $\pi$, which says the two identical species albeit impossible to be antisymmetrized in real space, is antisymmetrized in the momentum space.

2. Two-body problem with anisotropy/interaction

Since the interaction term acts on neighboring sites, it would only change the $e$ in (19) to $e + 2\Delta$ in determining the initial condition $a^2_0$, leaving the recurrence equation unchanged. Their solution is given by
\[
a^{(s)}_r(\Delta) = \frac{t^{r+(s-1)/2}}{i \sin \alpha} \left[ \left( \frac{e + 2\Delta}{2 \cos \theta} - e^{-i\alpha} \right) e^{i(s-1)\alpha} + \left( e^{i\alpha} - \frac{e + 2\Delta}{2 \cos \theta} \right) e^{-i(s-1)\alpha} \right]
\]
\[
= t^{r+(s-1)/2} \left( \frac{\sin(s\alpha)}{\sin \alpha} + \frac{\Delta}{\cos \theta} \frac{\sin(s-1)\alpha}{\sin \alpha} \right),
\]
which reproduces (24) when $\Delta$ is taken to be 0. Quantization of the energy is once again given by the solution of the antiperiodic boundary condition, now written as
\[
e^{iL(\alpha - \theta)} = \frac{\cos \theta + \Delta e^{i\alpha}}{\cos \theta + \Delta e^{-i\alpha}}
\]
Combining terms from permutations differing by a transposition \((34)\) still holds in this case, we can subtract this equation from it to get

\[
\sigma \sum_{a=1}^{\infty} \frac{\Delta \sin \alpha_f}{2 \cos \theta + \Delta \cos \alpha_f}, \quad J = 1, 2, \ldots, L. \tag{30}
\]

Parametrizing as before, \(\theta_1 = \theta - \alpha, \theta_2 = \theta + \alpha\), and \(\mu_{1,2} = e^{i\theta_{1,2}}\), we still have

\[
E(\theta_1, \theta_2) = -L + 4 - 2\cos \theta_1 - 2\cos \theta_2 \tag{31}
\]

\[
|\theta_1, \theta_2| = \frac{1}{N} \sum_{1 \leq i < j \leq L} (\mu_1 \mu_2 - \mu_{12} (\mu_1^2 - \mu_2^2)) |i, j\rangle, \tag{32}
\]

where \(S_{12} = \mu_1 - \mu_2, \) since \((\mu_1 \mu_2)^L = 1\), and normalization constant \(N^2 = L^2 - L e^{iL \theta} \sin(L - 1)\theta / \sin \alpha\).

3. Many-body problem

When there are more than two identical particles in one species, the recursive equations as in the previous section will involve multiple indices. Therefore, it is necessary to construct the Bethe Ansatz wave function

\[
w_{i_1, \ldots, i_n} = \sum_{\sigma \in \mathcal{S}_n} A_{\sigma} \prod_{a=1}^{n} \mu_{\sigma a}^{i_a}, \tag{33}
\]

where \(\mathcal{S}_n\) denotes symmetric group of order \(n\), and \(\mu_a = e^{i\theta_a}\). When \(i_{a+1} > i_a + 1\) for all \(a\), \((10)\) takes the form

\[
\sum_{\sigma \in \mathcal{S}_n} A_{\sigma} \prod_{a=1}^{n} \frac{\mu_{a}^{-1} + \mu_{\sigma a} - 2}{\mu_{\sigma a}} = -(E_v + L) \sum_{\rho \in S_n} A_{\rho} \prod_{a=1}^{n} \mu_{\rho a}^{i_a}. \tag{34}
\]

The common factor on the l.h.s. can be taken out of the sum over \(\sigma\), which gives \(E_v = -L + \sum_{a=1}^{n} (2 - 2\cos \theta_a)\). When \(i_{a+1} = i_a + 1\) for some \(a_0\), but \(i_{a+1} > i_a + 1\) for \(a \neq a_0\), \((10)\) becomes

\[
\sum_{\sigma \in \mathcal{S}_n} A_{\sigma} \left(\mu_{\sigma a_0}^{-1} + \mu_{\sigma (a_0+1)} - 2 - 2\Delta + \sum_{a \neq a_0} (2\cos \theta_{a \sigma a_0} - 2)\right) \prod_{a=1}^{n} \mu_{\sigma a}^{i_a}
\]

\[
= -(E_v + L) \sum_{\rho \in S_n} A_{\rho} \prod_{a=1}^{n} \mu_{\rho a}^{i_a}. \tag{35}
\]

Noticing that \((34)\) still holds in this case, we can subtract this equation from it to get

\[
\sum_{\sigma \in \mathcal{S}_n} A_{\sigma} (\mu_{\sigma a_0} + \mu_{\sigma (a_0+1)} - 2\Delta) \prod_{a=1}^{n} \mu_{\sigma a}^{i_a} = 0, \tag{36}
\]

or using explicitly \(i_{a_0+1} = i_{a_0} + 1\),

\[
\sum_{\sigma \in \mathcal{S}_n} A_{\sigma} \left(\mu_{\sigma a_0} \mu_{\sigma (a_0+1)} + 1 + 2\Delta \mu_{\sigma (a_0+1)}\right) \prod_{a=1}^{n} \mu_{\sigma a}^{i_a} = 0. \tag{37}
\]

Combining terms from permutations differing by a transposition \((a_0 a_0 + 1)\), one can see that for this equation to be satisfied for any coordinates \(i_a\’s\) and momenta \(k_a\’s\), it requires

\[
\frac{A_{\sigma'}(\Delta)}{A_{\sigma}(\Delta)} = -\frac{\mu_{\sigma a_0} \mu_{\sigma (a_0+1)} + 1 + 2\Delta \mu_{\sigma (a_0+1)}}{\mu_{\sigma a_0} \mu_{\sigma (a_0+1)} + 1 + 2\Delta \mu_{\sigma a_0}}, \tag{38}
\]

where \(\sigma' = \tau_{a_0} \sigma\), with \(\tau_{a_0}\) transposing \(a_0\) and \(a_0 + 1\). In appendix \(A\) we introduce different parametrization of the momenta for different magnitudes of the anisotropy, in which the scattering phase depends only on the difference of the rapidities of the two scattering particles, so as to facilitate a unified treatment of nested and algebraic Bethe
Ansatz in the next section. Quantization condition comes from the periodic boundary condition \([7]\). Decomposing

\[ \prod_{a=1}^{n} \frac{\Delta \cos \frac{1}{2}(\theta_{\sigma n} - \theta_{\sigma a}) + \cos \frac{1}{2}(\theta_{\sigma n} + \theta_{\sigma a}) + i\Delta \sin \frac{1}{2}(\theta_{\sigma n} - \theta_{\sigma a})}{\Delta \cos \frac{1}{2}(\theta_{\sigma n} - \theta_{\sigma a}) + \cos \frac{1}{2}(\theta_{\sigma n} + \theta_{\sigma a}) - i\Delta \sin \frac{1}{2}(\theta_{\sigma n} - \theta_{\sigma a})} = e^{i\theta_{\sigma n}L}, \]  

(39)

where we have absorbed the minus signs into the scattering phases. Since \([12]\) holds for any \(\sigma \in S_n\), we have the Bethe Ansatz equations

\[ L\theta_{\sigma} + 2\pi J_{\sigma} = 2 \sum_{b=1}^{n} \arctan \frac{\Delta \sin \frac{1}{2}(\theta_{\sigma b} - \theta_{\sigma a})}{\Delta \cos \frac{1}{2}(\theta_{\sigma b} - \theta_{\sigma a}) + \cos \frac{1}{2}(\theta_{\sigma b} + \theta_{\sigma a})}, \]

(40)

where \(J_{\sigma} = 1, ..., L\), for any \(a = 1, ..., n\). Taking the sum on both sides over \(a\), we get \(\sum_{a=1}^{n} \theta_{\sigma a} = 2\pi k/L\), with \(k = 1, ..., L\).

Once the momenta are solved from the Bethe Ansatz equations, \((38)\) can be used to deduce the relative weights of the permutations among momenta, which gives the eigenstates in this sector. If the interaction term is turned off, \((38)\) becomes

\[ A_{\sigma} = -A_{\sigma'}, \]

(41)

Therefore \(A_{\sigma} = (-1)^{\sigma}\). Boundary condition now becomes

\[ \sum_{\sigma} A_{\sigma} L_{\sigma}^{n+1} \prod_{a=1}^{n-1} \mu_{\sigma a}^{i_a} = (-1)^{n-1} \sum_{\rho} A_{\rho} \mu_{\rho a}^{i_a} \prod_{a=2}^{n} \mu_{\rho a}^{i_a}, \]

(42)

for any \(i_1, ..., i_n\). Comparing the terms on both sides differing by a global translation \(\tau = (1 2 \cdots n)\), we have

\[ (-1)^{n-1} = \frac{A_{\sigma^{-1}}}{A_{\sigma}} = (-1)^{n-1} \mu_{\sigma n}, \quad \forall \sigma \in S_n. \]

(43)

Hence, \(\theta_{\sigma} = 2\pi k_{\sigma}/N\), and the ground state energy in the sector with \(n\) particles belonging to the identical species is \(E = -L + 2 \sum_{k=-m}^{m} \cos \frac{2\pi k}{L}\) for \(n = 2m+1\), or \(E = -L + 2 \sum_{k=-m}^{m} \cos \frac{2\pi k}{N}\) for \(n = 2m\), since if any two of the momenta coincide, the anti-symmetrized coefficient \((33)\) would vanish. We can use the Lagrange’s trigonometric identity to estimate their gap from the ground state in the thermodynamic limit \(n \to \infty\), \(E - E_0 \to 2L(\frac{m}{N} - \frac{1}{2} \sin \frac{\pi m}{N})\).

With some trick explained in appendix \([3]\), one can even show that the normalization constant to be \(N = L^{n/2}\). The eigenstates are then expressed as

\[ |\theta_1, ..., \theta_n\rangle = L^{-n/2} \sum_{1 \leq i_1 < \cdots < i_n \leq L} \sum_{\sigma \in S_n} (-1)^{\sigma} e^{i \sum_{a=1}^{n} \theta_{\sigma a} i_a} |i_1, ..., i_n\rangle. \]

(44)

The attentive reader might recognize now, that choosing \(n\) momenta among their \(L\) possible values allowed by quantization condition, the dimension of this eigenvector subspace \((L_n)\), is precisely the dimension of the irreducible representation corresponding to the one-column Young tableaux with \(n\) rows, whereas the absolute ground state of the full Hilbert space expressed by Slater determinant correspond to the 1 dimensional irreducible representation of the \(L\) row, \(1\) column tableaux.

B. Multiple identical bosons with different colors case: the \(\{c_1^{n_1}, \cdots, c_m^{n_m}, c_{m+1}, \cdots, c_{L-n+m}\}\) sector

The spectrum of a generic sector, where multiple colors appear multiple times in the particle content, is solved by recourse to the Bethe-Yang hypothesis \([3]\), used in Yang’s solution to the one-dimensional Fermi problem with repulsive \(\delta\) interaction, (which was originally independently solved by Gaudin, using a less physical and more algebraically involved approach \([59]\)) based on the pioneering work of McGuire \([62, 63]\), Lieb and Flicker \([64]\), and subsequently further generalized by Sutherland \([8, 10]\).

Yang’s approach differs from Bethe Ansatz used in the previous section in that it’s not assuming any constraint on the permutation symmetry of the wave function, which is indeed the case when multiple species are involved. Therefore, if we still label the sites of all identical species with \(i\)’s, \([6]\) no longer holds. Consequently, the basis
$|i_1, \ldots, i_n\rangle$ no longer spans the whole Hilbert space when restricted to $i_1 < \cdots < i_n$. Instead, we should write the eigenvector of the Hamiltonian as

$$|v\rangle = \sum_{Q \in \mathfrak{S}_n} \sum_{1 \leq x_{Q1} < \cdots < x_{Qn} \leq L} \psi_Q(x)|x\rangle,$$

where $x$ is an $n$-component array of the combinations of coordinate and spin/color $\{q_i; c_i\}$, and $x_i < x_j$ if $q_i < q_j$, or $c_i < c_j$ if $q_i = q_j$. The simultaneous swapping of a pair both of them gives a minus sign due to the fermionic nature of the basis,

$$|x\tau\rangle = \text{sgn}(\tau)|x\rangle,$$

where $x\tau = (x_{\tau(1)}, \ldots, x_{\tau(n)}) \equiv (\{q_{\tau(1)}; c_{\tau(1)}\}, \ldots, \{q_{\tau(n)}; c_{\tau(n)}\})$. Therefore, only the anti-symmetric part of the wave function multiplying an antisymmetric basis is meaningful

$$\psi_{\tau^{-1}Q}(x\tau) = \text{sgn}(\tau)\psi_Q(x).$$

While we are discussing permutation symmetry, it’s worth mentioning theorems on the ordering of energy levels of the isotropic Hamiltonian. According to Lieb and Mattias’ theorem on the ordering of energy levels of anti-ferromagnets, the lowest energy eigenstate is determined by the so-called “pouring principle” [58]. More recently, this result is generalized to a lattice model with higher spin with both open and periodic boundary conditions [57], which says the lowest energy eigenstate corresponds to the Young Tableaux that gives the highest weight state of the $sl(n)$ algebra. Their proof is given much in the same spirit as that of Lieb and Matthias’, namely, showing that the representation corresponding to the Young Tableaux of highest weight state is required by Perron-Frobenius theorem to have non-negative components and therefore overlaps with the ground state, at the same time, the Hamiltonian doesn’t mix representations, so one concludes that it has to be the only representation corresponding to the ground state. Irreducible representations of outer product can be constructed according to a simple rule from the irreducible representations of each group in the outer product, which in our case, has to be the single row Young tableaux $[n_1, \ldots, n_m]$, as one cannot antisymmetrize among an identical species. Then the Young Tableaux in the outcome of the outer product that every other one can “pour into”, alias corresponding to the highest weight state, is $[n_1, \ldots, n_m]$, where each row is filled with the same species, assuming $n_1 > n_2 > \cdots > n_m$. The above analysis relies purely on the permutation symmetry of the Hamiltonian. As we will see shortly, Yang’s approach utilizes both permutation symmetry and the translational invariance, and is more convenient in practice. It starts with a first layer of Bethe Ansatz treating all particles as if they have the same color, while keeping in mind that their colors could be different later by keeping track of the swapping of color indices

$$\psi_Q(x) = \sum_{P \in \mathfrak{S}_n} A_{Q,P} \prod_{i=1}^n \pi_{P_i}^{q_{Q_i}},$$

for each sector $1 \leq x_{Q1} < x_{Q2} < \cdots < x_{Qn} \leq L$ in the coordinate space labelled by a permutation $Q \in \mathfrak{S}_n$, which are a priori independent. Each $Q$ corresponds to a particular sector of the whole coordinate space, with the $Q = \text{id}$ one called the fundamental sector. $A_{Q,P}$ is an $n! \times n!$ matrix, whose columns are denoted by $\xi_P$. To relate the wave functions defined in these separate sectors, we use the boundary conditions between two sectors $Q = Q_{\tau_a}$ that differ by a transposition between indices $a$ and $a + 1$. In appendix [C] we give the detailed derivation of Yang’s scattering matrix $Y$ from comparing the eigenvalue equations of nonadjacent and adjacent cases, which requires

$$\xi_{P\sigma_{ij}} = Y_{ij}^{\sigma} \xi_P,$$

for $P(a) = i, P(a + 1) = j$, where the $Y$ matrix as defined in Appendix [C]. These $n!(n-1)$ equations are consistent for the isotropic Hamiltonian only. For completeness, we carry out its diagonalization following Sutherland [10].

The above Yang’s $Y$ matrices are given in the reflection(-diagonal) representation of the asymptotic wave function, according to Sutherland [65], which is convenient for relating the a priori independent wave functions defined in separate sectors, as the coordinate of each particle remains invariant after scattering in this representation. However, in order to utilize the periodic boundary condition as a quantization condition for the momenta, we switch to the transmission(-diagonal) representation, where the diagonal terms of scattering $S$ matrices are the transmission amplitudes. In the transmission representation, momentum is fixed to a particle after scattering. Picking a particular permutation of momenta $P_0 = Q$, such that $i = a, j = a + 1$, we have

$$S_{ij} = \pi(\tau_{ij}) Y_{ij}^{\sigma} = T_{ij} Z + R_{ij} \pi(\tau_{ij}),$$

for $P(a) = i, P(a + 1) = j$.
where the transmission and reflection coefficients are defined as in (C11), (C10). In Appendix C we show that in the presence of anisotropy in the Hamiltonian, the Yang-Baxter’s equation, which is necessary for the following procedures to work, is only satisfied for subsectors of the Hilbert space corresponding to two types of Young Diagram, namely those of the shape illustrated in Fig. 2(c) and (d). The former corresponds to the solution of spin-$\frac{1}{2}$ XXZ model, while the latter corresponds to our scarred sectors.

The periodic boundary condition in the multiple identical species case relates wave functions in different sectors of the coordinate space

$$\psi_Q(x_1, \ldots, \{q_j = 1; \ c_j\}, \ldots, x_n) = \psi_{\tilde{Q}}(x_1, \ldots, \{q_j = L + 1; \ c_j\}, \ldots, x_n),$$

where $\tilde{\tau}$ denotes the translation $(2 \ 3 \ \cdots \ n \ 1)$. Upon applying the ansatz wave function, this implies

$$A_{Q, P} = A_{\tilde{Q}, \tilde{P} + P} \mu_L^P,$$

and consequently

$$\pi(\tilde{\tau}) \xi_P = \mu_L^P \xi_{\tilde{P} + P},$$

since the left regular representation acts as $\pi(\tilde{\tau}) A_{Q, P} = A_{\tilde{Q} - 1, P}$. Taking $P^0 = \tau_{12} \cdots \tau_{(j-1)j}$, $\varphi = \xi_{P^0}$, and using the braiding property $S_{k\bar{j}} \pi(\tau_{ik}) = \pi(\tau_{ik}) S_{ij}$, the periodic boundary condition then leads to

$$\mu_j^L \varphi = S_{(j+1)j}^L S_{(j+2)j}^L \cdots S_{nj}^L S_{1j}^L S_{2j} \cdots S_{(j-1)j}^L \varphi,$$

for $j = 1, \ldots, n$. Unlike the single identical species case, this is no longer a scalar equation, but an eigenvalue problem that requires diagonalization of the r.h.s. To proceed from here, we have to specify an irreducible representation of the permutation group $\mathfrak{S}_n$ corresponding to sector $\{c_1^1, \ldots, c_m^m, c_{m+1}, \ldots, c_{L-n+m}\}$ of the Hilbert space. To find the lowest energy of this sector, let’s take the irrep $R = [m n^m, (m-1)n^{m-1}, \ldots, 2n^2-n^2, 1^{n-n}]$. Sutherland’s approach of diagonalization of this irrep corresponding to the multi-row Young tableaux using Bethe-Yang Hypothesis is to first treat all the other $m-1$ species as the same, except the first one, then all the other $m-2$ species as the same, except the second one, and so on, so forth. This way, at each stage of the nesting, we are dealing with an irrep corresponding to a 2-row Young tableaux, which can be solved by recourse to the algebraic Bethe Ansatz as used in diagonalizing the spin-half Heisenberg model. Notice, however, at each stage, while the irrep of the spin part of the wave function forms a 2-row Young Tableaux, the spatial wave function must form the irrep corresponding to different Young Tableaux. (c) Young diagram corresponding to spin-$\frac{1}{2}$ XXZ model. (d) Young diagram corresponding to the scarred sectors of our model.
terms of the the scattering matrix in the Heisenberg spin-$\frac{1}{2}$ problem of a length $n$ chain. So we can readily apply the results from algebraic Bethe Ansatz at each stage to write $\varphi$ as

$$\varphi = \sum_{\sigma \in \mathcal{S}_n} A_\sigma F(A_{\sigma 1}, y_1) F(A_{\sigma 2}, y_2) \cdots F(A_{\sigma (n-1)}, y_{n-1}),$$

(57)

where $y_1 < y_2 < \cdots < y_{n-1}$ are the coordinates of the $n_1$ “down spins”, and

$$F(A, y) = \prod_{j=1}^{y-1} \frac{\lambda_j - \Lambda + i/2}{\lambda_j + i/2}$$

(58)

are defined in terms of the set of unequal numbers to be solved from the set of coupled algebraic equations

$$\mu_j^L = -\prod_{k=1}^{n} \frac{\lambda_j - \lambda_k - i}{\lambda_j - \lambda_k + i} \prod_{\alpha=1}^{n-n_1} \frac{\lambda_j - \Lambda_\alpha^{(1)} + i/2}{\lambda_j - \Lambda_\alpha^{(1)} - i/2}, \quad j = 1, \ldots, n,$$

(59)

$$\prod_{j=1}^{n-n_1} \frac{\Lambda_{\alpha}^{(1)} - \lambda_j - i/2}{\Lambda_{\alpha}^{(1)} - \lambda_j + i/2} = -\prod_{\beta=1}^{n-n_1} \frac{\Lambda_{\beta}^{(1)} - \Lambda_{\alpha}^{(1)} - i}{\Lambda_{\beta}^{(1)} - \Lambda_{\alpha}^{(1)} + i} \prod_{\gamma=1}^{n-n_1-n_2} \frac{\Lambda_\alpha^{(1)} - \Lambda_\gamma^{(2)} + i/2}{\Lambda_\alpha^{(1)} - \Lambda_\gamma^{(2)} - i/2}, \quad \alpha = 1, \ldots, n - n_1,$$

(60)

$$\prod_{\delta=1}^{n_{m-1}+n_m} \frac{\Lambda_\zeta^{(m)} - \Lambda_\delta^{(m-1)} - i/2}{\Lambda_\zeta^{(m)} - \Lambda_\delta^{(m-1)} + i/2} = -\prod_{\epsilon=1}^{n_m} \frac{\Lambda_\zeta^{(m)} - \Lambda_\epsilon^{(m)} - i}{\Lambda_\zeta^{(m)} - \Lambda_\epsilon^{(m)} + i}, \quad \zeta = 1, \ldots, n_m.$$  

(61)

This set of equations are historically called Lieb-Wu equations [69], the details of their derivation of these equations can be found in standard texts such as [70]. Now if there’s a cut-off on the number of components, the ground state of our Hamiltonian will no longer be a superposition of fully antisymmetrized spin configurations. In stead, from the reasoning above, it will be in the sector where each color appear the same number of times along the chain, say the number of colors $s$ divides $L$, and $l = L/s$. In this sector, the Young tableaux corresponding to the lowest energy eigenstate in this sector will be $[l^s]$. The Lieb-Wu equations for this irrep become

$$L \theta(\lambda_j) = 2\pi J_j^{(0)} + \sum_{k=1}^{L} \theta(\frac{1}{2}(\lambda_k - \lambda_j)) + \sum_{\alpha=1}^{(s-1)l} \theta(\lambda_j - \Lambda_\alpha^{(1)}), \quad j = 1, \ldots, L,$$

(62)

$$\sum_{j=1}^{L} \theta(\lambda_j - \Lambda_\alpha^{(1)}) = 2\pi J_\alpha^{(1)} + \sum_{\beta=1}^{(s-1)l} \theta(\frac{1}{2}(\Lambda_\beta^{(1)} - \Lambda_\alpha^{(1)})) + \sum_{\gamma=1}^{(s-2)l} \theta(\Lambda_\alpha^{(1)} - \Lambda_\gamma^{(2)}), \quad \alpha = 1, \ldots, (s-1)l,$$

(63)

$$\vdots$$

$$\sum_{\delta=1}^{2l} \theta(\Lambda_\delta^{(s-1)} - \Lambda_\zeta^{(s)}) = 2\pi J_\zeta^{(s)} + \sum_{\epsilon=1}^{l} \theta(\frac{1}{2}(\Lambda_\epsilon^{(s)} - \Lambda_\zeta^{(s)})), \quad \zeta = 1, \ldots, l,$$

(64)

where $\theta(\lambda) = 2\cot^{-1}(2\lambda)$, and the quantum numbers that labels the eigenstates, $J$’s, are half-integers.

### IV. ENTANGLEMENT ENTROPY OF EXACT EIGENSTATES

As a highlight peculiarity of our model, the integrable scar states actually does not satisfy the area law of entanglement entropy, as opposed to all known examples of such states in non-integrable systems so far (see for instance Ref. [29]). In this section, we first show that the ground state has exactly linear scaling of entanglement entropy in the thermodynamic limit. Then we argue that the entanglement entropy of other exact scar states are lower-bounded by the corresponding Bethe Ansatz states in spin-$\frac{1}{2}$ XXZ chain, which already violate area law [52] [53].
A. Ground state entanglement entropy for \( s = L \)

In the case of \( s = L \), the ground state can be expressed as

\[
|GS\rangle = \frac{1}{\sqrt{L!}} \sum_{\mathbf{P} \in S_L} \text{sgn}(\mathbf{P}) |c_{P_1}, c_{P_2}, \ldots, c_{P_L}\rangle,
\]

where \(|...,c_i,...\rangle\) denotes the configuration with the \( i \)-th site having particle of color \( c_i \).

This ground state is highly entangled as the color at each site depends on those at all the others. This can be seen by calculation the entanglement entropy between two halves of the system. Let \( L = 2l \), then the Schmidt decomposition of the ground state is

\[
|GS\rangle = \sum_{\mathbf{P}_0 \in \mathbb{S}_{L/2}} \text{sgn}(\mathbf{P}_0) \frac{l!}{\sqrt{(2l)!}} |P_01, P_02, \ldots, P_0l\rangle \otimes |P_0(l+1), P_0(l+2), \ldots, P_0L\rangle,
\]

where \(|i_1, i_2, \ldots, i_l\rangle = \frac{1}{\sqrt{l!}} \sum_{\mathbf{P} \in \mathbb{S}_l} \text{sgn}(\mathbf{P}) |c_{P_1}, s_{P_2}, \ldots, s_{P_l}\rangle\). The entanglement entropy between two subsystems is then

\[
S = - \left( \frac{2l}{l} \right) \left( \frac{l!}{(2l)!} \right)^2 \log \left( \frac{l!^2}{(2l)!} \right).
\]

In the thermodynamic limit \( l \to \infty \), this is approximated by the stderr formula as

\[
S \simeq 2l \log(2l) - 2l \log l - l \log 2.
\]

B. Entanglement entropy for \( s < L \)

When the number of particle species is not enough to support a totally antisymmetrized state as the ground state, we nonetheless have exactly excited state solved from Bethe Ansatz in our fully antisymmetrized basis constructed in certain subspaces of Hilbert space. These fully antisymmetrized basis states serve as the counterpart of pseudovacuum state in the Bethe Ansatz solution of the corresponding spin-\( \frac{1}{2} \) XXZ model. However, unlike the pseudovacuum there, which doesn’t contribute to entanglement between subsystems, since it’s a uniform product state, here they could contribute to entanglement if separated in two subsystems, just because the basis is a superposition of all the permutations among them. For this reason, we believe that the entanglement entropy of these scar states in our model is lower-bounded by those of the corresponding Bethe Ansatz states in spin-\( \frac{1}{2} \) model, which are know to be area law breaking [52, 53].

V. SUMMARY AND OUTLOOKS

In this paper, we explore the effect of easing frustration by enlarging local degrees of freedom, using a \( \mathbb{S}_N \)-invariant antiferromagnetic multicomponent Heisenberg model, which at the same time describes hard-core bosons in one-dimension with density-density interaction, the latter bearing a number-theoretic resemblance to the coprime spin [71]. The physics of this model depends heavily on the number of components of the spins or colors of bosons. When their number is sufficiently large compared to the size of the system, the Hamiltonian is frustration free, and its ground state is analytically solvable, with entanglement entropy shown to scale linearly with the system size. When the dimensionality of local Hilbert space is smaller, the Hamiltonian become frustrated, but integrable eigenstates are still given by a Bethe Ansatz solution on top of an antisymmetrized basis, despite the model being non-integrable by violation of YBE. These states do not fall into the category of exact quantum many-body scar states, as they are unconventional in the following three ways. First, unlike many known examples, such as AKLT and Majumdar-Ghosh models which start from a frustration-free ground state to build momentum \( \pi \) single modes [29], here we were able to solve excited states despite the ground state being frustrated and not solvable when the number of species is finite. Second, given these states are solved with Bethe Ansatz, their corresponding energy are not integer or rational valued, nor are they equal distant in the spectrum. This contradicts the unified framework of spectrum generating algebra that describes known examples of exact scars so far [38]. Third and probably the most significant, they are not the lowly entangled states that can be exactly expressed in terms of matrix product states [33, 72]. In fact, their entropy
doesn’t even obey area law. These novelties challenging the present theory of quantum many-body scars naturally pose the question of whether the existence of exact eigenstates in non-integrable systems could perhaps be much more abundant and the comes in more variety than we have searched. One promising follow-up in this direction is to see if the tower of QMBSs in $SU(N)$ VBS and impure solid models allow for more possibilities than $\pi$-momenta.

We emphasize that violation of YBE only implies the absence of non-diffractive scattering. When Bethe Ansatz fails, Sommerfeld diffraction ansatz can still apply if internal consistency depending on solution of certain Riemann-Hilbert problem is met. Indeed, McGuire and Hurst has developed algebraic formulation for solving the three-body problem with different scattering between particles of different species [73, 74]. Yet so far, this approach has only been applied to three- or four-body problems [75], except for scattering with bound states [76]. It is worth exploring how this approach can be generalized to the N-body problem in our case, given our scattering matrices only differ between the scattering of like-particles and of unlike-paritcles. Another direction to pursue in terms of diagonalization without going into the most general Sommerfeld diffraction ansatz is to calculate the diffractive scattering amplitude for the $\Delta \to 0$ case, when the Yang-Baxter equation is weakly violated [77].

Our model also provides a natural playground for exploring the dependency of ground state entanglement with the number of local degrees of freedom. We see our ground state entanglement changing from volume law when the local Hilbert space is sufficiently large, to logarithmic scaling when the dimensionality is smaller. Characterising where and how smoothly such a transition happens can deepen our understanding of the role local degrees of freedom and frustration plays on entanglement.

**Appendix A: Parametrization of Scattering matrices**

The reparametrization of momenta into rapidities for different anisotropy parameters is due to Orbach [78].

a. $\Delta > 1$: Using parametrization $\Delta \equiv \cosh \eta$, and

\[
\mu = -\frac{\sin(\lambda + i\eta/2)}{\sin(\lambda - i\eta/2)},
\]

becomes

\[
\frac{A_{\sigma'}(\eta)}{A_{\sigma}(\eta)} = \frac{\sin(\lambda\sigma(a_0+1) - \lambda\sigma_0 + i\eta)}{\sin(\lambda\sigma(a_0+1) - \lambda\sigma_0 - i\eta)},
\]

(A2)

b. $\Delta = 1$: In this case, we parametrize

\[
\mu = \lambda + i/2
\]

(A3)

and (38) takes the form

\[
\frac{A_{\sigma'}(\eta)}{A_{\sigma}(\eta)} = \frac{\lambda\sigma(a_0+1) - \lambda\sigma_0 + i}{\lambda\sigma(a_0+1) - \lambda\sigma_0 - i}
\]

(A4)

c. $0 \leq \Delta < 1$: Using parametrization $\Delta \equiv \cos \eta$, and

\[
\mu = -\frac{\sinh(\lambda + i\eta/2)}{\sinh(\lambda - i\eta/2)},
\]

(A5)

then becomes

\[
\frac{A_{\sigma'}(\eta)}{A_{\sigma}(\eta)} = \frac{\sinh(\lambda\sigma(a_0+1) - \lambda\sigma_0 + i\eta)}{\sinh(\lambda\sigma(a_0+1) - \lambda\sigma_0 - i\eta)}
\]

(A6)

Summarizing, we have

\[
\varphi^\Delta(\lambda^\Delta \pm i\eta) = \begin{cases} 
\sin(\lambda^\Delta \pm i\arccosh\Delta), & \Delta > 1, \\
\lambda^\Delta \pm i, & \Delta = 1 \\
\sinh(\lambda^\Delta \pm i\arccos\Delta), & 0 \leq \Delta < 1,
\end{cases}
\]

(A7)

and

\[
\mu = \begin{cases} 
-\frac{\sin(\lambda^\Delta + i\eta/2)}{\sin(\lambda^\Delta - i\eta/2)}, & \Delta > 1, \\
\lambda^\Delta + i/2, & \Delta = 1 \\
-\frac{\sinh(\lambda^\Delta + i\eta/2)}{\sinh(\lambda^\Delta - i\eta/2)}, & 0 \leq \Delta < 1.
\end{cases}
\]

(A8)
Appendix B: Normalization of eigenstates

In the absence of the interaction term, the eigenvalues of translation operator are roots of unity. Therefore it admits a cute calculation of the normalization constant, as we demonstrate below.

\[
\mathcal{N}^2 = \sum_{1 \leq i_1 < \ldots < i_n \leq N} |w_{i_1 \ldots i_n}|^2 \\
= \sum_{1 \leq i_1 < \ldots < i_n \leq L} \sum_{\sigma, \rho \in S_n} (-1)^{\sigma + \rho} \prod_{a=1}^n (\mu_\sigma a \mu_\rho a^*)^{i_a} \\
= \sum_{1 \leq i_1 < \ldots < i_n \leq L} \sum_{\sigma, \tau \in S_n} (-1)^\tau \prod_{a=1}^n (\mu_\sigma a \mu_\tau a^*)^{i_a} \\
= \sum_{1 \leq i_1 < \ldots < i_n \leq L} \sum_{\sigma, \tau \in S_n} (-1)^\tau \prod_{a=1}^n (\mu_\sigma a \mu_\tau a^*)^{i_a} \\
= \sum_{1 \leq i_1 < \ldots < i_n \leq L} \sum_{\tau \in S_n} (-1)^\tau \prod_{a=1}^n (\delta_{a, \tau \sigma} L) \\
- \sum_{\exists b, c, \tau \in S_n} (-1)^\tau \prod_{a=1}^n (\mu_\sigma a \mu_\tau a^*)^{i_a} \sum_{\tau \in \{id, (b, c)\}} (-1)^\tau (\mu_\sigma b \mu_\tau c \mu_\tau c^*)^{i_b} \\
= \sum_{\tau \in S_n} (-1)^\tau \prod_{a=1}^n \delta_{\tau, id} L^n \\
= L^n.
\]

As a special case, when \( n = L \), this norm can be easily calculated by representing the wave function as Slater determinant, and evaluating the product of determinants with the determinant of the product of the two matrices.

Appendix C: Derivation of scattering matrix

In this appendix, we give the detailed derivation of Yang’s scattering \( Y \) matrix in the reflection-(diagonal) representation. Using \((45), (46)\) and \((47)\), the eigenvalue equation \( H|v\rangle = E|v\rangle \), when considering adjacent identical particles case \( Qa = Q(a + 1) - 1 \) separately, gives

\[
\sum_Q \left( \sum_{x_{Qa+1} < x_{Q(a+1)}} \psi_Q \left[ \sum_{i=1}^n \left( -[\ldots, x_i - 1, \ldots] + |\ldots, x_i + 1, \ldots| \right) - (L - 2n)[\ldots] \right] \\
+ \sum_{x_{Qa+1} = x_{Q(a+1)}} \psi_Q \left[ \sum_{i=k, k+1}^n \left( -[\ldots, x_i - 1, \ldots] + |\ldots, x_i + 1, \ldots| \right) \\
- |\ldots, x_a - 1, x_a + 1, \ldots| - [\ldots, x_a, x_a + 1, \ldots] - (L - 2n + 1)[\ldots, x_i, \ldots] \\
- |\ldots, x_a + 1, x_a, \ldots| + 2\Delta \delta^{a, a+1}[\ldots, x_a, x_a + 1, \ldots] \right] + \cdots \right)
\]

\[= \sum_Q \sum_{1 \leq x_Q < x_{Q+1} \leq L} E \psi_Q [\ldots], \]
where $\delta^{a,a+1} = 1$ if $c_Q = c_{Q(a+1)}$, and $\delta^{a,a+1} = 0$ if $c_Q \neq c_{Q(a+1)}$, and we have omitted the unchanged variables in the ket vectors, and the ellipsis in the sum on the l.h.s. denote terms with multiple adjacent identical particles. Collecting coefficients of the same basis vectors on both sides, we have

$$-\sum_{i=1}^{n}[\psi_Q(x - e_i) + \psi_Q(x + e_i)] - (L - 2n)\psi_Q(x) = E\psi_Q(x), \quad (C1)$$

for the non-adjacent case, where $e_i$ denotes the unit vector in the $i$'th coordinate component, and

$$-\sum_{i=1}^{n}[\psi_Q(x - e_i) + \psi_Q(x + e_i)] + \psi_Q(x + e_a) + \psi_Q(x - e_{a+1}) - (L - 2n + 1)\psi_Q(x) \quad (C2)$$

$$-\psi_{\tau(a+1)}Q(x\tau(a+1)) + 2\Delta \delta^{a,a+1}\psi_Q(x) = E\psi_Q(x),$$

for the adjacent case, where $\tau(a+1)$ denotes the transposition between $a$ and $a + 1$. The difference between the above two equations then gives the boundary condition

$$\psi_Q(x + e_a) + \psi_Q(x - e_{a+1}) - \psi_Q(x) - \psi_{\tau(a+1)}Q(x\tau(a+1)) + 2\Delta \delta^{a,a+1}\psi_Q(x) = 0. \quad (C3)$$

Plugging in our Bethe trial wave function for $\phi_Q$, we have

$$\sum_{P \in S_n} [(\mu_{Pa} + \mu_{P(a+1)} - 1 + \Delta p_{Qa},Q(a+1))A_{Q,P} - A_{\tau(a+1)Q,P}] \prod_{i=1}^{n} \mu_{P_i^{Q_i}} = 0. \quad (C4)$$

Denoting $Pa, P(a+1)$ with $Pa = i, P(a+1) = j$, we have

$$\sum_{P \in S_n/2_2} \left( [\mu_{ij} + 1 + (2\Delta \delta^{a,a+1} - 1)\mu_j]A_{Q,P} - \mu_j A_{\tau(a+1)Q,P} \right) + [\mu_{ij} + 1 + (2\Delta \delta^{a,a+1} - 1)\mu_j]A_{Q,P,\pi,ij} - \mu_i A_{\tau(a+1)Q,P,\pi,ij}) (\mu_{ij})_a \prod_{i \neq a+1} \mu_{P_i^{Q_i}} = 0. \quad (C5)$$

A sufficient condition for this equation to hold for any choice of $\{x_i\}$, is that the coefficients in the sum vanish term by term. Treating now $A_{Q,P}$ as components of $n!$ dimensional column vectors $\xi_P$, this becomes

$$\left( [\mu_{ij} + 1 + (2\Delta \delta^{a,a+1} - 1)\mu_j - \pi(\tau(a+1))\mu_j] \xi_P \right) + \left( [\mu_{ij} + 1 + (2\Delta \delta^{a,a+1} - 1)\mu_j - \pi(\tau(a+1))\mu_i] \xi_P,ij = 0, \quad (C6)$$

where $\pi$ is the left regular representation of $S_n$, which is associated to the scattering process of a Bethe wave function $[\pi \cdot \cdot \cdot]$. From this, we can solve Yang’s scattering $Y$ matrix, as defined in [49], to be

$$\text{Y}^{a(a+1)}(\Delta) = R^{a(a+1)}(\Delta)1 + T^{a(a+1)}(\Delta)\pi(\tau(a+1)), \quad (C7)$$

where the reflection coefficient

$$R^{ab}_{ij}(\Delta) = -\frac{(\mu_{ij} + 1)(\mu_j - \mu_i)}{(\mu_{ij} + 1 + 2\Delta \delta^{ab}\mu_i)(\mu_{ij} + 1 + 2\Delta \delta^{ab} - 1)\mu_i}, \quad (C8)$$

and transmission coefficient

$$T^{ab}_{ij}(\Delta) = -\frac{(\mu_{ij} + 1)(\mu_i - \mu_j)}{(\mu_{ij} + 1 + 2\Delta \delta^{ab}\mu_i)(\mu_{ij} + 1 + 2\Delta \delta^{ab} - 1)\mu_i}. \quad (C9)$$

Hence, for scattering between different colors, $c_a \neq c_{a+1}$, we have

$$R^{ab}_{ij}(\Delta) = \frac{-i}{\lambda_i - \lambda_j + i} \equiv R_{ij} \quad (C10)$$

$$T^{ab}_{ij}(\Delta) = \frac{\lambda_i - \lambda_j}{\lambda_i - \lambda_j + i} \equiv T_{ij}. \quad (C11)$$
where

\[ \lambda_i = \frac{i \mu_i + 1}{2 \mu_i - 1}. \]  

(C12)

Yet for scattering between the same colors, \( c_a = c_{a+1} \), we have \( \pi(\tau_{a(a+1)}) = -1 \) (due to (47)), and

\[ Y_{ij}^{a(a+1)}(\Delta) = R_{ij}^{a(a+1)}(\Delta) - T_{ij}^{a(a+1)}(\Delta) = -\frac{\mu_i \mu_j + 1 + 2 \Delta \mu_j}{\mu_i \mu_j + 1 + 2 \Delta \mu_j} \phi^\Delta(\lambda_j^\Delta - \lambda_i^\Delta + i \eta) \phi^\Delta(\lambda_j^\Delta - \lambda_i^\Delta - i \eta) = \Theta_{ij}, \]  

(C13)

with the parametrization (A8). It’s easily verified that while the unitarity relation

\[ Y_{ab}^{ij}(\Delta)Y_{ab}^{ji}(\Delta) = 1, \]  

(C14)

is satisfied for both cases. The Yang-Baxter’s equations

\[ Y_{jk}^{ab}(\Delta)Y_{ik}^{bc}(\Delta)Y_{ij}^{ab}(\Delta) = Y_{ij}^{bc}(\Delta)Y_{ik}^{ab}(\Delta)Y_{jk}^{bc}(\Delta), \]  

(C15)

reuire

\[ \Theta_{jk}R_{ik}\Theta_{ij} = R_{ij}\Theta_{ik}R_{jk} + T_{ij}R_{ik}T_{jk}, \]  

\[ R_{jk}T_{ik}\Theta_{ij} = T_{ij}\Theta_{ik}R_{jk} + R_{ij}R_{ik}T_{jk}, \]  

(C16)

(C17)

which only hold when \( \Delta = 0 \) or \( \Delta \to \infty \), see Fig.3. If \( \Delta \neq 0 \), YBE hold respectively for scattering between identical colors, and different colors, but not when they are mixed. This means the scattering among three particles and more, is only factorizable into consecutive scatterings between pairs of two particles, irrespective of their ordering, if their colors are either all the same, which leads to the solution in III.A.2 or all different, corresponding to the absolute ground state. Hence, for the generic case, where scattering between particles of both same and different colors are present, the scattering is not non-diffractive, and Yang’s nested Bethe Ansatz no longer applies. In fact, Sutherland has derived a necessary consistent condition for the scattering between different species to be non-diffractive [68], an example of such an exactly solvable anisotropic generalization of the multicomponent Heisenberg model was solved by Babelon, et. al. [11], albeit being less physically natural than our model.

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