The symmetries of the Fokker - Planck equation in one dimension

Igor A. Tanski
tanski@protek.ru
ZA OC VP rotek

ABSTRACT

We calculate all point symmetries of the Fokker - Planck equation in one-dimensional Euclidean space. General expression of symmetry group action on arbitrary solution of Fokker - Planck equation is presented. We propose new notation for the group-theoretic analysis of PDE. The Lie prolongation formula is derived as an example of the new notation.

1. The symmetries of the Fokker - Planck equation in one dimension

The object of our considerations is a special case of Fokker - Planck equation, which describes evolution of 1D continuum of non-interacting particles imbedded in a dense medium without outer forces. The interaction between particles and medium causes combined diffusion in physical space and velocities space. The only force, which acts on particles, is damping force proportional to velocity.

The 3D variant of this equation was investigated in our work [1]. In this work fundamental solution of 3D equation was obtained by means of Fourier transform.

In present work we choose another object of research. Our goal is to find all symmetries of the Fokker - Planck equation. For this purpose we use the classic techniques of group-theoretic analysis of PDE, developed in fundamental works [2-5].

The Fokker - Planck equation in one dimension is

$$\frac{\partial n}{\partial t} + v \frac{\partial n}{\partial x} - av \frac{\partial n}{\partial v} - an - k \frac{\partial^2 n}{\partial v^2} = 0. \quad (1)$$

where

- $n = n(t, x, v)$ - density;
- $t$ - time variable;
- $x$ - space coordinate;
- $v$ - velocity;
- $a$ - coefficient of damping;
- $k$ - coefficient of diffusion.

The list of symmetries of the Fokker - Planck equation in one dimension follows. The calculations of symmetries are rather awkward. They are carried out to APPENDIX 2.

Instead of classic "$\xi - \phi$" notation we use another ("$\delta$") notation. This notation is presented in APPENDIX 1. The Lie prolongation formula is derived as an example of the new notation.
The symmetries of the Fokker-Planck equation in one dimension are:

\[ v_1 = A \frac{\partial}{\partial n}. \]

where \( A \) is arbitrary solution of the (1) equation.

Scaling of density

\[ v_2 = n \frac{\partial}{\partial n}. \]

The reason of symmetries (2-3) is linearity of PDE (1).

Time shift

\[ v_3 = \frac{\partial}{\partial t}. \]

Space translation

\[ v_4 = \frac{\partial}{\partial x}. \]

Extended Galilean transformation, which besides time and space coordinates affects the density

\[ v_5 = \frac{\partial}{\partial v} + \frac{\partial}{\partial x} - \frac{\alpha n}{2k} (ax + v) \frac{\partial}{\partial n}. \]

Negative exponent transformation - it affects time and space coordinates, contains time-dependent common multiplier (negative exponent). It does not affect density. This transformation is evident from [1].

\[ v_6 = e^{-\alpha t} \left( -a \frac{\partial}{\partial v} + \frac{\partial}{\partial x} \right) \]

Positive exponent transformation - it affects time, space and density, contains time-dependent common multiplier (positive exponent).

\[ v_7 = e^{\alpha t} \left( a \frac{\partial}{\partial v} + \frac{\partial}{\partial x} - \frac{\alpha^2}{k} n v \frac{\partial}{\partial n} \right) \]

One-parameter groups, generated by vector fields \( v_1 - v_7 \), are enumerated in the following list. The list contains images of the point \((n, t, x, v)\) by transformation \( \exp(\epsilon v_i) \)

\[ G_1: \quad (n + \epsilon A, t, x, v); \]
\[ G_2: \quad (\epsilon^\alpha n, t, x, v); \]
\[ G_3: \quad (n, t + \epsilon, x, v); \]
\[ G_4: \quad (n, t, x + \epsilon, v); \]
\[ G_5: \quad (\exp \left[ - \frac{a}{2k} (\epsilon (ax + v) + \frac{1}{2} \epsilon^2 (at + 1)) \right] n, t, x + \epsilon t, v + \epsilon); \]
\[ G_6: \quad (n, t, x + \epsilon e^{-\alpha t}, v - \epsilon a e^{-\alpha t}); \]
\[ G_7: \quad (n \exp \left[ - \frac{a^2}{k} e^{\alpha t} \left( \epsilon v + \frac{1}{2} \epsilon^2 a e^{\alpha t} \right) \right], t, x + \epsilon e^{\alpha t}, v + \epsilon a e^{\alpha t}). \]
Relatively nontrivial integration for $G_5$ and $G_7$ is carried out to APPENDIX 3 and APPENDIX 4.

The fact, that $G_i$ are symmetries of PDE (1) means, that if $f(t, x, v)$ is arbitrary solution of (1), the functions

\[
\begin{align*}
  u^{(1)} & : f(t, x, v) + \varepsilon A(t, x, v); \\
  u^{(2)} & : e^\varepsilon f(t, x, v); \\
  u^{(3)} & : f(t - \varepsilon, x, v); \\
  u^{(4)} & : f(t, x - \varepsilon, v); \\
  u^{(5)} & : \exp \left[ -\frac{a}{2k} \left( \varepsilon(ax + v) - \frac{1}{2} \varepsilon^2(at + 1) \right) \right] f(t, x - \varepsilon t, v - \varepsilon); \\
  u^{(6)} & : f(t, x - \varepsilon e^{-at}, v + \varepsilon a e^{-at}); \\
  u^{(7)} & : \exp \left[ -\frac{a^2}{k} e^{\varepsilon t} \left( \varepsilon v - \frac{1}{2} \varepsilon^2 a e^{at} \right) \right] f(t, x - \varepsilon e^{at}, v - \varepsilon ae^{at}).
\end{align*}
\]

where $\varepsilon$ - arbitrary real number , also are solutions of (1). Here $A$ is another arbitrary solution of (1).

We systematically replaced "old coordinates" by their expressions through "new coordinates". Note, that due to these replacements terms with $\varepsilon^2$ in $u^{(5)}$ and $u^{(7)}$ change their signs.

We have trivial solution $n = e^{at}$ at our disposal. If we act on this solution by transformations (10), we obtain 2 new solutions:

\[
\begin{align*}
  n & = \exp \left[ at - \frac{a}{2k} \left( \varepsilon(ax + v) - \frac{1}{2} \varepsilon^2(at + 1) \right) \right]; \\
  n & = \exp \left[ at - \frac{a^2}{k} e^{\varepsilon t} \left( \varepsilon v - \frac{1}{2} \varepsilon^2 a e^{at} \right) \right].
\end{align*}
\]

General expression is

\[
\begin{align*}
  u & = e^{\varepsilon^2} \exp \left[ -\frac{a}{2k} \left( \varepsilon_3(ax + v) - \frac{1}{2} \varepsilon_3^2(at + 1) \right) \right] \exp \left[ -\frac{a^2}{k} e^{\varepsilon t} \left( \varepsilon_7(v - \varepsilon_5) - \frac{1}{2} \varepsilon_7^2 a e^{at} \right) \right] \times \\
  & \times f(t - \varepsilon_3, x - \varepsilon_4 - \varepsilon_5 t - \varepsilon_6 e^{-at} - \varepsilon_7 e^{at}, v - \varepsilon_5 + \varepsilon_6 a e^{-at} - \varepsilon_7 a e^{at}) + \varepsilon_1 A(t, x, v).
\end{align*}
\]
DISCUSSION
Looking at the list of all point symmetries of the Fokker-Planck equation in one-dimensional Euclidean space, we see, that there is no simple way to get, for example, fundamental solution of PDE, using these symmetries. We have not at our disposal such an instrument, as scaling of independent variables $x, u, v$.

Indirect way of use of Galilean transformation (6) was demonstrated in [1]. The transformation was used for generalization of solution, which was obtained in the form of exponent of quadratic form of space coordinates and velocities with time dependent coefficients.

There is need of further investigations of Fokker-Planck equation and its set of symmetries, which may lead to another physically interesting results.

ACKNOWLEDGMENTS
We wish to thank Jos A. M. Vermaseren from NIKHEF (the Dutch Institute for Nuclear and High-Energy Physics), for he made his symbolic computations program FORM release 3.1 available for download for non-commercial purposes (see [5]). This wonderful program makes difficult task of symmetries search more accessible.

REFERENCES
[1] Igor A. Tanski. Fundamental solution of Fokker-Planck equation. arXiv:nlin.SI/0407007 v1 4 Jul 2004
[2] L. P. Eisenhart, Continuous Groups of Transformations, Princeton University Press, Princeton, 1933.
[3] L. V. Ovsyannikov, Group analysis of differential equations, Moscow, Nauka, 1978.
[4] Peter J. Olver, Applications of Lie groups to differential equations. Springer-Verlag, New York, 1986.
[5] N. H. Ibragimov, CRC Handbook of Lie Group Analysis of Differential Equations, V. 3, CRC Press, New York, 1996.
[6] Michael M. Tung. FORM Matters: Fast Symbolic Computation under UNIX. arXiv:cs.SC/0409048 v1 27 Sep 2004
APPENDIX 1

We propose following notation.

\( u^\alpha \) - the set of dependent variables;
\( x^i \) - the set of independent variables;
\( \frac{\partial u^\alpha}{\partial x^i} = u^\alpha_{,i} \) - first derivatives;
\( \frac{\partial^2 u^\alpha}{\partial x^i \partial x^j} = u^\alpha_{,ij} \) - second derivatives.

The space of dependent and independent variables undergo small point transformation. The small displacements are:
\( \delta u^\alpha, \delta x^i \) - variations of independent and dependent variables. We call them "the components of infinitesimal transformation vector field" also.

Variations \( \delta u^\alpha, \delta x^i \) are functions of independent and dependent variables. Therefore derivatives of variations on dependent and independent variables are:
\( \frac{\partial \delta u^\alpha}{\partial x^i} = (\delta u^\alpha)^{,i}, \frac{\partial \delta u^\alpha}{\partial u^\beta} = (\delta u^\alpha)^{,\beta} \) - derivatives of variations of dependent variables;
\( \frac{\partial \delta x^i}{\partial x^j} = (\delta x^i)^{,j}, \frac{\partial \delta x^i}{\partial u^\beta} = (\delta x^i)^{,\beta} \) - derivatives of variations of independent variables.

In the same manner we write expressions for derivatives of the second and higher orders.

Let us derive the Lie prolongation formula for variations of the first order derivatives.

Due to point transformation of space of independent and dependent variables first derivatives undergo some transformation. To find this transformation, we write relation between differentials of independent and dependent variables

\( du^\alpha = \frac{\partial u^\alpha}{\partial x^i} dx^i. \) (A1-1)

The same relation take place for variables after the transformation. Therefore we can take variation of (A1-1) and get

\( \delta(du^\alpha) = \delta\left(\frac{\partial u^\alpha}{\partial x^i}\right) dx^i + \frac{\partial u^\alpha}{\partial x^i} \delta(dx^i). \) (A1-2)

or

\( \delta\left(\frac{\partial u^\alpha}{\partial x^i}\right) dx^i = \delta(du^\alpha) - \frac{\partial u^\alpha}{\partial x^k} \delta(dx^k). \) (A1-3)

Variations of differentials are equal to
\( \delta(du^\alpha) = D_i(\delta u^\alpha) dx^i; \)
\( \delta(dx^k) = D_j(\delta x^j) dx^j; \)

where symbol \( D_i \) denotes the full partial derivative.

\( D_i = \frac{\partial}{\partial x^i} + \frac{\partial u^\alpha}{\partial x^j} \frac{\partial}{\partial u^\alpha} + \cdots. \) (A1-6)

After substitution of (A1-4) and (A1-5) to (A1-3) we get

\( \delta\left(\frac{\partial u^\alpha}{\partial x^i}\right) = D_i(\delta u^\alpha) - \frac{\partial u^\alpha}{\partial x^k} D_j(\delta x^j). \) (A1-7)
This is the famous Lie formula for the first derivatives (see ref. [1]). It is named "prolongation formula", because it prolongs the action of infinitesimal transformation to first derivatives.

We shall use this formula in expanded form

\[ \delta \frac{\partial u^\alpha}{\partial x^m} = \frac{\partial (\delta u^\alpha)}{\partial x^k} + \frac{\partial (\delta u^\alpha)}{\partial u^\beta} u^{\beta, \nu} - u^{\alpha, \nu} \left( \frac{\partial (\delta x^m)}{\partial x^k} + \frac{\partial (\delta x^m)}{\partial u^\beta} u^{\beta, \nu} \right) \]  

(A1-8)

In the same way we get Lie prolongation formula for variations of the second order derivatives. We begin with

\[ \delta \left( \frac{\partial u^\alpha}{\partial x^m} \right) = D_m(\delta(u^\alpha)) - \frac{\partial(u^\alpha)}{\partial x^p} D_m(\delta x^p). \]  

(A1-9)

which is (A1-7) equation for \( u^{\alpha, \nu} \).

or

\[ \delta(u^\alpha) = D_m(\delta(u^\alpha)) - u^{\alpha, \nu} D_m(\delta x^p), \]  

(A1-10)

where \( \delta(u^\alpha) \) we take from (7).

We expand expression for \( D_m(\delta(u^\alpha)) \) and obtain

\[ \delta(u^\alpha) = \frac{\partial(\delta(u^\alpha))}{\partial(x^m)} u^\beta, \nu p + \frac{\partial(\delta(u^\alpha))}{\partial(x^m)} D_m(\delta x^\gamma, j) + \frac{\partial(\delta(u^\alpha))}{\partial(x^m)} D_m(\delta x^\gamma, \beta) + \]  

\[ + \frac{\partial(\delta(u^\alpha))}{\partial(x^m)} D_m(\delta x^\gamma, j) + \frac{\partial(\delta(u^\alpha))}{\partial(x^m)} D_m(\delta x^\gamma, \beta) - u^{\alpha, \nu} D_m(\delta x^p). \]  

(A1-11)

From the (A1-7) we get

\[ \frac{\partial(\delta(u^\alpha))}{\partial(x^m)} \]  

(A1-12)

and it follows

\[ \frac{\partial(\delta(u^\alpha))}{\partial(x^m)} u^\beta, \nu p = \frac{\partial(\delta(u^\alpha))}{\partial(x^m)} u^{\beta, \nu p} - u^{\alpha, \nu} \left( \frac{\partial(\delta x^m)}{\partial(x^k)} + \frac{\partial(\delta x^m)}{\partial u^\gamma} u^{\gamma, \nu k} \right) - u^{\alpha, \nu} u^{\gamma, \nu} \frac{\partial(\delta x^m)}{\partial u^\gamma}. \]  

(A1-13)

In the similar way we have

\[ \frac{\partial(\delta(u^\alpha))}{\partial(x^m)} = -u^{\alpha, \nu} \delta_{\beta \gamma}; \Rightarrow \frac{\partial(\delta(u^\alpha))}{\partial(x^m)} D_m(\delta x^\gamma, j) = -u^{\alpha, \nu} D_m(\delta x^\gamma, \beta). \]  

(A1-14)

\[ \frac{\partial(\delta(u^\alpha))}{\partial(x^m)} = -u^{\alpha, \nu} \delta_{\beta \gamma}; \Rightarrow \frac{\partial(\delta(u^\alpha))}{\partial(x^m)} D_m(\delta x^\gamma, j) = -u^{\alpha, \nu} u^{\beta, \nu} \frac{\partial(\delta x^\gamma)}{\partial u^\gamma} D_m(\delta x^\gamma, \beta). \]  

(A1-15)

\[ \frac{\partial(\delta(u^\alpha))}{\partial(x^m)} = \delta_{\alpha \beta} \delta_{\gamma j}; \Rightarrow \frac{\partial(\delta(u^\alpha))}{\partial(x^m)} D_m(\delta u^\beta, j) = D_m(\delta u^\beta). \]  

(A1-16)

\[ \frac{\partial(\delta(u^\alpha))}{\partial(x^m)} = u^{\gamma, \nu} \delta_{\alpha \beta}; \Rightarrow \frac{\partial(\delta(u^\alpha))}{\partial(x^m)} D_m(\delta u^\beta, j) = u^{\gamma, \nu} D_m(\delta u^\beta). \]  

(A1-17)

We collect all these results and obtain Lie prolongation formula for variations of the second order derivatives:
\[
\delta(u^\alpha,_{\delta x}m) = \frac{\partial u^\alpha}{\partial x^n} u^\beta,_{\delta x}m - u^\alpha,_{\delta x}m (\frac{\partial \delta x^\beta}{\partial x^n} + \frac{\partial \delta x^\beta}{\partial u^\gamma} u^\gamma) -
\]

\[
- u^\beta,_{\delta x}m u^\alpha,_{\delta x} \frac{\partial \delta x^\beta}{\partial u^\gamma} - u^\alpha,_{\delta x} D_m((\delta x^\beta),_{\delta x}) - u^\alpha,_{\delta x} \frac{\partial \delta x^\beta}{\partial u^\gamma} D_m((\delta x^\beta),_{\delta x}) +
\]

\[
+D_m((\delta u^\alpha),_{\delta x}) + u^\alpha,_{\delta x} D_m((\delta u^\alpha),_{\gamma}) - u^\alpha,_{\delta x} D_m(\delta x^\beta).
\]

We immediately use (A1-8), (A1-18) for our task. In our case the only dependent variable is \( u \), independent variables are \( t, x, v \). Expanding (A1-8), we have

\[
\delta \frac{\partial n}{\partial x} = \frac{\partial}{\partial n} (\delta n) + \frac{\partial}{\partial x} (\delta n) - \frac{\partial}{\partial x} (\delta x) - \frac{\partial}{\partial n} (\delta x) -
\]

\[
\delta \frac{\partial n}{\partial v} = \frac{\partial}{\partial n} (\delta v) + \frac{\partial}{\partial x} (\delta v) - \frac{\partial}{\partial x} (\delta x).
\]

Expanding (A1-18), we have

\[
\delta \frac{\partial^2 n}{\partial v^2} = \frac{\partial}{\partial n} (\delta n) \frac{\partial^2 n}{\partial v^2} - \frac{\partial^2 n}{\partial v^2} \frac{\partial}{\partial x} (\delta x) - \frac{\partial^2 n}{\partial v^2} \frac{\partial}{\partial n} (\delta x) -
\]

\[
- \frac{\partial}{\partial n} (\delta t) + \frac{\partial}{\partial x} (\delta t) - \frac{\partial}{\partial n} (\delta x) + \frac{\partial}{\partial x} (\delta x) - \frac{\partial}{\partial n} (\delta v) + \frac{\partial}{\partial x} (\delta v)
\]

These expressions are typical. It is obvious, that to manipulate such expressions by bare hands is very hard task. We use simple JavaScript program to generate these expressions.
APPENDIX 2

The infinitesimal invariance criteria for equation (1) is

\[ \delta \left( \frac{\partial n}{\partial t} \right) + \delta v \frac{\partial n}{\partial x} + v \delta \left( \frac{\partial n}{\partial x} \right) - a \delta v \frac{\partial n}{\partial v} - av \delta \left( \frac{\partial n}{\partial v} \right) - a \delta n - k \delta \left( \frac{\partial^2 n}{\partial v^2} \right) = 0; \]  

(A2-1)

We eliminate \( \frac{\partial n}{\partial t} \) from (A1) using original equation (1):

\[ \frac{\partial n}{\partial t} = -v \frac{\partial n}{\partial x} + av \frac{\partial n}{\partial v} + an + k \frac{\partial^2 n}{\partial v^2}. \]  

(A2-2)

Substitute (A1-19 - A1-22) to (A2-1), collect terms by different combinations of \( n \) derivatives and equate these terms to zero. We obtain following equations:

\[ \frac{\partial n}{\partial x} \frac{\partial n}{\partial v^2} \]

\[-kv \frac{\partial^2}{\partial n^2} \delta t + k \frac{\partial^2}{\partial n^2} \delta x = 0 \]  

(A2-3)

\[ \frac{\partial n}{\partial x} \frac{\partial n}{\partial v} \]

\[-kv \frac{\partial^2}{\partial n \partial v} \delta t - kv \frac{\partial^2}{\partial n \partial v} \delta t + 2k \frac{\partial^2}{\partial n \partial v} \delta x = 0 \]  

(A2-4)

\[ \frac{\partial n}{\partial x} \]

\[-av^2 \frac{\partial}{\partial v} \delta t + av \frac{\partial}{\partial n} \delta t + av \frac{\partial}{\partial v} \delta x - an \frac{\partial}{\partial n} \delta x - kv \frac{\partial^2}{\partial v^2} \delta t + \]

\[ +k \frac{\partial^2}{\partial v^2} \delta x + v^2 \frac{\partial}{\partial x} \delta t - v \frac{\partial}{\partial x} \delta x + v \frac{\partial}{\partial t} \delta x + \delta t - \frac{\partial}{\partial t} \delta x = 0 \]  

(A2-5)

\[ \frac{\partial n}{\partial v^3} \]

\[ akv \frac{\partial^2}{\partial n^2} \delta t + k \frac{\partial^2}{\partial n^2} \delta v = 0 \]  

(A2-6)

\[ \frac{\partial n}{\partial v^2} \frac{\partial^2 n}{\partial v^2} \]

\[ k^2 \frac{\partial^2}{\partial n^2} \delta t = 0 \]  

(A2-7)

\[ \frac{\partial n}{\partial v^2} \]

\[ 2akv \frac{\partial^2}{\partial n \partial v} \delta t + akn \frac{\partial^2}{\partial n^2} \delta t - k \frac{\partial^2}{\partial n^2} \delta n + 2k \frac{\partial^2}{\partial n \partial v} \delta v = 0 \]  

(A2-8)

\[ \frac{\partial n}{\partial v} \frac{\partial^2 n}{\partial v^2} \]

\[ 2k^2 \frac{\partial^2}{\partial n \partial v} \delta t + 2k \frac{\partial}{\partial n} \delta v = 0 \]  

(A2-9)

\[ \frac{\partial n}{\partial v} \frac{\partial^2 n}{\partial v \partial x} \]
\[ 2k \frac{\partial}{\partial n} \delta x = 0 \quad \text{(A2-10)} \]

\[ \frac{\partial n}{\partial v} \frac{\partial^2 n}{\partial t \partial v} \]

\[ 2k \frac{\partial}{\partial n} \delta t = 0 \quad \text{(A2-11)} \]

\[ \frac{\partial n}{\partial v} \]

\[ a^2 v^2 \frac{\partial}{\partial v} \delta t - a^2 v n \frac{\partial}{\partial n} \delta t + akv \frac{\partial^2}{\partial v^2} \delta t + 2akn \frac{\partial^2}{\partial n \partial v} \delta t - av^2 \frac{\partial}{\partial x} \delta t + av \frac{\partial}{\partial v} \delta v - \]

\[ -av \frac{\partial}{\partial t} \delta t - an \frac{\partial}{\partial n} \delta v - a \delta v - 2k \frac{\partial^2}{\partial n \partial v} \delta n + k \frac{\partial^2}{\partial v^2} \delta v - v \frac{\partial}{\partial x} \delta v - \frac{\partial}{\partial t} \delta v = 0 \quad \text{(A2-12)} \]

\[ \frac{\partial^2 n}{\partial v^2} \]

\[ akv \frac{\partial}{\partial v} \delta t - akn \frac{\partial}{\partial n} \delta t + k^2 \frac{\partial^2}{\partial v^2} \delta t - kv \frac{\partial}{\partial x} \delta t + 2k \frac{\partial}{\partial v} \delta v - k \frac{\partial}{\partial t} \delta t = 0 \quad \text{(A2-13)} \]

\[ \frac{\partial^2 n}{\partial v \partial x} \]

\[ 2k \frac{\partial}{\partial v} \delta x = 0 \quad \text{(A2-14)} \]

\[ \frac{\partial^2 n}{\partial t \partial v} \]

\[ 2k \frac{\partial}{\partial v} \delta t = 0 \quad \text{(A2-15)} \]

\[ 1 \]

\[ a^2 v n \frac{\partial}{\partial v} \delta t - a^2 n^2 \frac{\partial}{\partial n} \delta t + akn \frac{\partial^2}{\partial v^2} \delta t - avn \frac{\partial}{\partial x} \delta t - av \frac{\partial}{\partial v} \delta n + \]

\[ +an \frac{\partial}{\partial n} \delta n - an \frac{\partial}{\partial t} \delta t - a \delta n - k \frac{\partial^2}{\partial v^2} \delta n + v \frac{\partial}{\partial x} \delta n + \frac{\partial}{\partial t} \delta n = 0. \quad \text{(A2-16)} \]

From (A2-10), (A2-11), (A2-14), (A2-15) we see, that \( \delta x = \delta x(x,t); \delta t = \delta t(x,t) \). Using this, we simplify the rest of equations (A2-3 - A2-16).

\[ v^2 \frac{\partial}{\partial x} \delta t - v \frac{\partial}{\partial x} \delta x + v \frac{\partial}{\partial t} \delta t + \delta v - \frac{\partial}{\partial t} \delta x = 0; \quad \text{(A2-17)} \]

\[ k \frac{\partial^2}{\partial n^2} \delta v = 0; \quad \text{(A2-18)} \]

\[ -k \frac{\partial^2}{\partial n^2} \delta n + 2k \frac{\partial^2}{\partial n \partial v} \delta v = 0; \quad \text{(A2-19)} \]

\[ 2k \frac{\partial}{\partial n} \delta v = 0; \quad \text{(A2-20)} \]

\[ -av^2 \frac{\partial}{\partial x} \delta t + av \frac{\partial}{\partial v} \delta v - av \frac{\partial}{\partial t} \delta t - an \frac{\partial}{\partial n} \delta v = \quad \text{(A2-21)} \]
We solve (A2-30) and find

\[-a\delta v - 2k \frac{\partial^2}{\partial n \partial v} \delta n + k \frac{\partial^2}{\partial v^2} \delta v - v \frac{\partial}{\partial t} \delta v - \frac{\partial}{\partial t} \delta v = 0;\]

\[-kv \frac{\partial}{\partial x} \delta t + 2k \frac{\partial}{\partial v} \delta v - k \frac{\partial}{\partial t} \delta t = 0;\]  \hspace{1cm} (A2-22)

\[-avn \frac{\partial}{\partial x} \delta t - av \frac{\partial}{\partial v} \delta n + an \frac{\partial}{\partial n} \delta n - av \frac{\partial}{\partial t} \delta t - a\delta n - k \frac{\partial^2}{\partial v^2} \delta n + v \frac{\partial}{\partial x} \delta n + \frac{\partial}{\partial t} \delta n = 0.\]  \hspace{1cm} (A2-23)

From (A2-20) we see, that \(\delta v = \delta v(x, v, t)\). Using this, we simplify the rest of equations (A2-17 - A2-23).

\[v^2 \frac{\partial}{\partial x} \delta t - v \frac{\partial}{\partial x} \delta x + v \frac{\partial}{\partial t} \delta t + \delta v - \frac{\partial}{\partial t} \delta v = 0;\]  \hspace{1cm} (A2-24)

\[-k \frac{\partial^2}{\partial n \partial v} \delta n = 0;\]  \hspace{1cm} (A2-25)

\[-av^2 \frac{\partial}{\partial x} \delta t + av \frac{\partial}{\partial v} \delta v - av \frac{\partial}{\partial t} \delta t - a\delta n - k \frac{\partial^2}{\partial v^2} \delta n = 0.\]  \hspace{1cm} (A2-26)

\[-k \frac{\partial^2}{\partial n \partial v} \delta n + k \frac{\partial^2}{\partial v^2} \delta v - v \frac{\partial}{\partial x} \delta v - \frac{\partial}{\partial t} \delta v = 0;\]

\[-kv \frac{\partial}{\partial x} \delta t + 2k \frac{\partial}{\partial v} \delta v - k \frac{\partial}{\partial t} \delta t = 0;\]  \hspace{1cm} (A2-27)

\[-avn \frac{\partial}{\partial x} \delta t - av \frac{\partial}{\partial v} \delta n + an \frac{\partial}{\partial n} \delta n - an \frac{\partial}{\partial t} \delta t - a\delta n - k \frac{\partial^2}{\partial v^2} \delta n + v \frac{\partial}{\partial x} \delta n + \frac{\partial}{\partial t} \delta n = 0.\]  \hspace{1cm} (A2-28)

From (A2-25) we conclude, that

\[\delta n = A + nB;\]  \hspace{1cm} (A2-29)

where \(A = A(x, v, t), B = B(x, v, t)\). Using this, we simplify the rest of equations (A2-24 - A2-28).

\[v^2 \frac{\partial}{\partial x} \delta t - v \frac{\partial}{\partial x} \delta x + v \frac{\partial}{\partial t} \delta t + \delta v = 0;\]  \hspace{1cm} (A2-30)

\[-av^2 \frac{\partial}{\partial x} \delta t + av \frac{\partial}{\partial v} \delta v - av \frac{\partial}{\partial t} \delta t - a\delta n + k \frac{\partial^2}{\partial v^2} \delta v - 2k \frac{\partial B}{\partial v} - v \frac{\partial}{\partial x} \delta v - \frac{\partial}{\partial t} \delta v = 0;\]  \hspace{1cm} (A2-31)

\[-kv \frac{\partial}{\partial x} \delta t + 2k \frac{\partial}{\partial v} \delta v - k \frac{\partial}{\partial t} \delta t = 0;\]  \hspace{1cm} (A2-32)

\[-av \frac{\partial}{\partial x} \delta t - av \frac{\partial B}{\partial v} - a \frac{\partial}{\partial t} \delta t - k \frac{\partial^2 B}{\partial v^2} + v \frac{\partial B}{\partial x} + \frac{\partial B}{\partial t} = 0;\]  \hspace{1cm} (A2-33)

\[-av \frac{\partial}{\partial v} \delta t - av \frac{\partial A}{\partial v} - a \frac{\partial}{\partial t} \delta t - k \frac{\partial^2 A}{\partial v^2} + v \frac{\partial A}{\partial x} + \frac{\partial A}{\partial t} = 0.\]  \hspace{1cm} (A2-34)

Equation (A2-34) is simply Fokker - Planck equation for \(A\).

We solve (A2-30) and find \(\delta v\)

\[\delta v = -v^2 \frac{\partial}{\partial x} \delta t + v \frac{\partial}{\partial x} \delta x - v \frac{\partial}{\partial t} \delta t + \frac{\partial}{\partial t} \delta x.\]  \hspace{1cm} (A2-35)

This gives for (A2-31 - A2-33)
\[-2av^2 \frac{\partial}{\partial x} \delta t - av \frac{\partial}{\partial t} \delta t - a \frac{\partial}{\partial t} \delta x - 2k \frac{\partial}{\partial x} \delta t - 2v \frac{\partial^2}{\partial t^2} \delta x = 0; \quad (A2-36)\]

\[-2k \frac{\partial B}{\partial v} + v^3 \frac{\partial^2}{\partial x^2} \delta t + 2v^2 \frac{\partial^2}{\partial t \partial x} \delta t - v^2 \frac{\partial^2}{\partial t^2} \delta x + v \frac{\partial^2}{\partial t^2} \delta t - 2v \frac{\partial^2}{\partial t^2} \delta x - \frac{\partial^2}{\partial t^2} \delta x = 0; \]

\[-5kv \frac{\partial}{\partial x} \delta t + 2k \frac{\partial}{\partial t} \delta x - 3k \frac{\partial}{\partial t} \delta t = 0; \quad (A2-37)\]

\[-av \frac{\partial}{\partial x} \delta t - av \frac{\partial B}{\partial v} - a \frac{\partial}{\partial t} \delta t - k \frac{\partial^2 B}{\partial v^2} + v \frac{\partial B}{\partial x} + \frac{\partial B}{\partial t} = 0. \quad (A2-38)\]

Now we can collect similar terms by different degrees of \(v\) in (A2-37) and so split (A2-37) into two equations:

\[\frac{\partial}{\partial x} \delta t = 0; \quad (A2-37a)\]

\[2k \frac{\partial}{\partial x} \delta x - 3k \frac{\partial}{\partial t} \delta t = 0. \quad (A2-37b)\]

From (A2-37a) we see, that \(\delta t = \delta t(t)\), which results in further simplifications

\[-av \frac{\partial}{\partial t} \delta t - a \frac{\partial}{\partial t} \delta x - 2k \frac{\partial B}{\partial v} - v^3 \frac{\partial^2}{\partial x^2} \delta x + \frac{\partial^2}{\partial t^2} \delta x = 0; \quad (A2-39)\]

\[+v \frac{\partial^2}{\partial t^2} \delta t - 2v \frac{\partial^2}{\partial t \partial x} \delta x - \frac{\partial^2}{\partial t^2} \delta x = 0; \]

\[2k \frac{\partial}{\partial x} \delta x - 3k \frac{\partial}{\partial t} \delta t = 0; \quad (A2-40)\]

\[-av \frac{\partial B}{\partial v} - a \frac{\partial}{\partial t} \delta t - k \frac{\partial^2 B}{\partial v^2} + v \frac{\partial B}{\partial x} + \frac{\partial B}{\partial t} = 0; \quad (A2-41)\]

We integrate (A2-40) and find

\[\delta x = C + \frac{3x}{2} \frac{\partial}{\partial t} \delta t; \quad (A2-42)\]

where \(C = C(t)\). Further, we substitute this expression in (A2-39), (A2-41) and obtain

\[-\frac{3ax}{2} \frac{\partial^2}{\partial t^2} \delta t - av \frac{\partial}{\partial t} \delta t - a \frac{\partial C}{\partial t} - 2k \frac{\partial B}{\partial v} - \frac{3x}{2} \frac{\partial^3}{\partial t^3} \delta t - 2v \frac{\partial^2}{\partial t^2} \delta t - \frac{\partial^2 C}{\partial t^2} = 0; \quad (A2-43)\]

\[-av \frac{\partial B}{\partial v} - a \frac{\partial}{\partial t} \delta t - k \frac{\partial^2 B}{\partial v^2} + v \frac{\partial B}{\partial x} + \frac{\partial B}{\partial t} = 0. \quad (A2-44)\]

We integrate (A2-43) and obtain following expression \((D = D(x, t)):\)

\[B = D - \frac{1}{2k} \left( \frac{3ax}{2} \frac{\partial^2}{\partial t^2} \delta t + \frac{av}{2} \frac{\partial}{\partial t} \delta t + av \frac{\partial C}{\partial t} + \frac{3ax}{2} \frac{\partial^3}{\partial t^3} \delta t + v \frac{\partial^2 B}{\partial v^2} \delta t + v \frac{\partial^2 C}{\partial t^2} \right) \quad (A2-45)\]

The substitution of (A2-45) to (A2-44), collecting and equating to zero terms by \(v\) gives

\[1/2a^2k^{-1} \frac{\partial}{\partial t} \delta t - 5/4k^{-1} \frac{\partial^3}{\partial t^3} \delta t = 0; \quad (A2-46)\]

\[3/4a^2k^{-1} x \frac{\partial^2}{\partial t^2} \delta t + 1/2a^2k^{-1} \frac{\partial C}{\partial t} - 3/4k^{-1} x \frac{\partial^4}{\partial t^4} \delta t - 1/2k^{-1} \frac{\partial^3 C}{\partial t^3} + \frac{\partial D}{\partial x} = 0; \quad (A2-47)\]
\[-1/2a \frac{\partial}{\partial t} \delta t + \frac{\partial^2}{\partial t^2} \delta t + \frac{\partial D}{\partial t} = 0. \quad (A2-48)\]

We integrate (A2-47) and obtain following expression \((E = E(t)):\)

\[D = E - \left( \frac{3a^2 x^2}{8k} \frac{\partial^2}{\partial t^2} \delta t + \frac{x a^2}{2k} \frac{\partial C}{\partial t} - \frac{3x^2}{8k} \frac{\partial^4}{\partial t^4} \delta t - \frac{x}{2k} \frac{\partial^3 C}{\partial t^3} \right). \quad (A2-49)\]

The substitution of (A2-49) to (A2-48), collecting and equating to zero terms by \(x\) gives

\[-3/8a^2 k^{-1} \frac{\partial^3}{\partial t^3} \delta t + 3/8k^{-1} \frac{\partial^5}{\partial t^5} \delta t = 0; \quad (A2-50)\]

\[-1/2a^2 k^{-1} \frac{\partial^5 C}{\partial t^5} + 1/2k^{-1} \frac{\partial^4 C}{\partial t^4} = 0; \quad (A2-51)\]

\[-1/2a \frac{\partial}{\partial t} \delta t + \frac{\partial^2}{\partial t^2} \delta t + \frac{\partial E}{\partial t} = 0; \quad (A2-52)\]

From (A2-46) and (A2-50) we conclude, that

\[\delta t = \text{const} = C_1. \quad (A2-53)\]

From (A2-51) we conclude, that

\[C = C_2 + C_3 t + C_4 e^{at} + C_5 e^{-at}. \quad (A2-54)\]

From (A2-52) and (A2-53) we see, that

\[E = C_6. \quad (A2-55)\]

We substitute expressions (A2-53 - A2-55) and obtain final expressions for variations (see (A2-35), (A2-42)):

\[\delta n = - \frac{a^2 x n}{2k} C_3 - \frac{a^2 v n}{k} C_4 e^{at} - \frac{a v n}{2k} C_3 + n C_6 + A; \quad (A2-56)\]

\[\delta x = t C_3 + C_2 + C_4 e^{at} + C_5 e^{-at}; \quad (A2-57)\]

\[\delta v = a C_4 e^{at} - a C_5 e^{-at} + C_3; \quad (A2-58)\]

\[\delta t = C_1; \quad (A2-59)\]

and these are final expressions for variations.
In this appendix we integrate equations for Galilean transformation.

The vector field of Galilean transformation is

\[ v_s = \frac{\partial}{\partial v} + \frac{\partial}{\partial x} - \frac{an}{2k} (ax + v) \frac{\partial}{\partial n}. \]  

(A3-1)

Characteristic differential equations are

\[ d\epsilon = -\frac{2k}{an} \frac{dn}{ax + v} = \frac{dt}{0} = \frac{dv}{1} = \frac{dx}{t}; \]  

(A3-2)

We get first integrals

\[ t = t_0; \quad v = \epsilon + v_0; \quad x = \epsilon t + x_0. \]  

(A3-3)

The only nontrivial equation is

\[ (ax + v) d\epsilon = -\frac{2k}{a} \frac{dn}{n}; \]  

(A3-4)

\[ (a(\epsilon t + x_0) + \epsilon + v_0) d\epsilon = -\frac{2k}{a} \frac{dn}{n}; \]  

(A3-5)

We integrate (A3-5) and get

\[ \frac{1}{2} \epsilon^2 (at + 1) + \epsilon (ax_0 + v_0) = -\frac{2k}{a} \ln\left(\frac{n}{n_0}\right); \]  

(A3-6)

Finally, we get following expression for \( n \)

\[ n = n_0 \exp\left[-\frac{a}{2k} \left(\frac{1}{2} \epsilon^2 (at + 1) + \epsilon (ax_0 + v_0)\right)\right]. \]  

(A3-7)
APPENDIX 4

In this appendix we integrate equations for positive exponent transformation.

The vector field of positive exponent transformation is

\[ v_7 = e^{at} \left( a \frac{\partial}{\partial v} + \frac{\partial}{\partial x} - \frac{a^2}{k} nv \frac{\partial}{\partial n} \right) \]  (A4-1)

Characteristic differential equations are

\[ dx = -e^{-at} \frac{k}{a^2 v} \frac{dn}{n} = \frac{dt}{0} = e^{-at} \frac{dy}{a} = e^{-at} dx; \]  (A4-2)

We get first integrals

\[ t = t_0; \quad v = v_0 + \varepsilon a e^{at}; \quad x = x_0 + \varepsilon e^{at}. \]  (A4-3)

The only nontrivial equation is

\[ -e^{at} v \, d\varepsilon = k \frac{dn}{n}; \]  (A4-4)

\[ -e^{at} (v_0 + \varepsilon a e^{at}) \, d\varepsilon = k \frac{dn}{a^2 n}; \]  (A4-5)

We integrate (A4-5) and get

\[ -e^{at} (\varepsilon v_0 + \frac{1}{2} \varepsilon^2 a \, e^{at}) = k \frac{a}{a^2} \ln\left( \frac{n}{n_0} \right); \]  (A4-6)

Finally, we get following expression for \( n \)

\[ n = n_0 \exp\left[ -\frac{a^2}{k} e^{at} \left( \varepsilon v_0 + \frac{1}{2} \varepsilon^2 a \, e^{at} \right) \right]. \]  (A4-7)