Clifford algebras, noncommutative tori and universal quantum computers

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Abstract
Recently author suggested [23] an application of Clifford algebras for construction of a “compiler” for universal binary quantum computer together with later development [24] of the similar idea for a non-binary base. The non-binary case is related with application of some extension of idea of Clifford algebras. It is noncommutative torus defined by polynomial algebraic relations of order \( l \). For \( l = 2 \) it coincides with definition of Clifford algebra. Here is presented the joint consideration and comparison of both cases together with some discussion on possible physical consequences.

1 Introduction
Application of Clifford algebras [1, 2] in theory of quantum computers [9, 10] already was discussed in relation with product operator formalism [6, 7] used for description of NMR based quantum computers [6, 8]. In the present paper is discussed application of Clifford algebras in more abstract models of quantum computations for constructions of universal sets of quantum gates. It is related already with early works about quantum computers, as “universal simulator” of behavior of quantum systems [11, 12, 13].

Due to such approach it is reasonable to consider some sets of elements that may be used for construction or approximation of any unitary transformation, like AND, NOT gates for usual Boolean function used in classical computations.

In most common model of quantum computation each elementary system is described by finite-dimensional Hilbert space and quantum gates act on some composite system with Hilbert space described as tensor product.

In simplest case the elementary system is described by two-dimensional Hilbert space and called qubit. More general case with higher-dimensional systems [12] is used less often, but also is considered in the paper. For quantum

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computation with \( n \) qubits total Hilbert space has dimension \( 2^n \) and most general quantum evolution may be described by \( 2^n \times 2^n \) unitary matrix.

In such a case the problem of universality is existence of some set of unitary matrices \( U_\kappa \) with possibility of presentation of any unitary matrix \( U \in U(2^n) \) as product

\[
U = U_\kappa_1 U_\kappa_2 \cdots U_\kappa_p
\]

or approximation \( U \approx U_{k_1} U_{k_2} \cdots U_{k_p} \) with necessary or arbitrary precision, if the set is finite \( k = 1, \ldots, K \).

It should be mentioned also that the Hilbert space \( \mathcal{H} \) has special structure due to representation as tensor power of \( n \) two-dimensional spaces \( \mathcal{H}_2 \)

\[
\mathcal{H} = \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_2. \tag{2}
\]

Due to such structure any unitary transformation of some subsystem with \( m < n \) qubits, \( U_S \in U(2^m) \) can be enclosed in the \( U(2^n) \) as transformation that acts only on \( m \) terms in the Hilbert space decomposition as tensor product with \( n \) terms Eq. (2). Such transformations are called \( m \)-qubit gates.

The consideration has straightforward generalization on systems with dimension of Hilbert space \( \mathcal{H}_l, l > 2 \), with \( l^n \)-dimensional space of composition

\[
\mathcal{H} = \mathcal{H}_l \otimes \cdots \otimes \mathcal{H}_l, \tag{3}
\]

and unitary group \( U(l^n) \). Non-binary quantum \( m \)-gates correspond to elements of \( U(l^n), m \leq n \).

An important result of quantum theory of computation is existence of universal set with two-qubit quantum gates \([14, 15, 16]\).

An application of Clifford algebras to quantum circuits with qubit is justified even from formal point of view, for example universal complex Clifford algebra \( \mathfrak{C}(2n, \mathbb{C}) \) is isomorphic with \( \mathbb{C}(2^n \times 2^n) \), algebra of all \( 2^n \times 2^n \) complex matrices \([1]\), but any quantum gate may be represented as unitary \( 2^n \times 2^n \) matrix and considered as element of the Clifford algebra \( \mathfrak{C}(2n, \mathbb{C}) \).

Less formal relation of Clifford algebras with quantum computation may be also due to physical properties of system used for implementation of quantum computer. For example property of spin group is important in realization of quantum computers on NMR \([7, 8]\) systems. Secondary quantization of fermions using Clifford systems \([3, 4]\) is also may be applied to quantum computation \([21, 22]\).

In many papers about quantum computation is used presentation of quantum gates in product operator formalism with Pauli matrices similar with one of constructions of Clifford algebras \([1]\). The relation is directly mentioned and used by some authors \([6, 7, 21]\). It should be mentioned that in \([21, 23]\) and in present paper are used complex Clifford algebras instead of real \([6, 8]\).
2 Universal sets of quantum gates

In present paper is discussed application of Clifford algebras to construction of universal set of quantum gates. It has some difference with example discussed above, because instead of representation of unitary group of quantum gates here is used representation of Lie algebra of this group. It was suggested in [23] for quantum gates with qubits, and later was found that for non-binary case very similar construction can be used with some analogue of Clifford algebra, noncommutative torus [24]. Here both cases are considered jointly, to show similarity of constructions with noncommutative tori and Clifford algebras together with some specific differences.

Algebraic tools discussed here produce directs algorithms for decomposition of some unitary matrix on set of quantum gates [23, 24] (sometimes such algorithms are called “quantum compilers” [19]). It also can be convenient as a background for design of quantum processors [25, 26].

2.1 Lie algebras

Here is discussed infinitesimal approach to construction of universal set of quantum gates using Lie algebra \( \mathfrak{u}(N) \). Here \( N = 2^n \) for quantum computation with \( n \) qubits. It was already discussed in many works [14, 16, 17] and described here only briefly. In the approach is considered Lie algebra \( \mathfrak{u}(N) \) of Lie group of unitary matrices and finite set of elements \( A_k \) of this algebra represented as some anti-Hermitian matrices \( A_k = -A_k^\dagger \). Quantum gates are generated as

\[
U_k^\tau = \exp(A_k \tau)
\]

with some real parameter \( \tau \). Now instead of universal set of elements of Lie group \( U(N) \) used in Eq. (1) it is possible to work with elements \( A_k \) of Lie algebra \( \mathfrak{u}(N) \).

If the elements \( A_k \) generate full Lie algebra \( \mathfrak{u}(N) \) by commutators, then it is possible to use set of gates described by Eq. (4) as universal set of gates [14, 19, 17].

The passage from Lie algebra to Lie group of gates due to Eq. (4) is not discussed with more detail, see [13, 14, 16, 17, 23, 24]. In physical applications \( A_k = iH_k \) correspond to Hamiltonians and real parameter \( \tau \), may be infinitesimal [14], irrational [16] or it is possible to consider Eq. (4) as one-parametric family, where \( \tau \) is time [17], but the main theme of this work is construction of the set of elements \( A_k \) itself using Clifford algebras and noncommutative tori.

2.2 Clifford algebras

Any associative algebra is Lie algebra in respect to commutator \([a, b] = ab - ba\). It was already mentioned that universal Clifford algebra \( \mathfrak{C}(2n, \mathbb{C}) \) is isomorphic with associative matrix algebra \( \mathbb{C}(2^n \times 2^n) \). Lie algebra \( \mathfrak{u}(2^n) \) can be represented
as algebra of anti-Hermitian $2^n \times 2^n$ matrices in respect to commutator and so can be considered as subalgebra of $\mathfrak{cl}(2n, \mathbb{C})$.

Let us denote Pauli matrices $\sigma_x, \sigma_y, \sigma_z \in \mathbb{C}(2 \times 2) \cong \mathfrak{cl}(2, \mathbb{C})$. Then anti-Hermitian matrices

\[
\Gamma_{2k} = i \mathbb{I} \otimes \cdots \otimes \mathbb{I} \otimes \sigma_x \otimes \sigma_z \otimes \cdots \otimes \sigma_z,
\]

\[
\Gamma_{2k+1} = i \mathbb{I} \otimes \cdots \otimes \mathbb{I} \otimes \sigma_y \otimes \sigma_z \otimes \cdots \otimes \sigma_z,
\]

where $k = 0, \ldots, n - 1$ are generators of $\mathfrak{cl}(2n, \mathbb{C})$ [1, 3, 4, 23]. The generators meet to usual rule:

\[
\{\Gamma_k, \Gamma_j\} \equiv \Gamma_k \Gamma_j + \Gamma_j \Gamma_k = -2\delta_{kj}
\]

where unit of algebra as multiplier is omitted for simplicity in right part of this equation.

Sums of all possible products of the elements Eq. (5) with complex coefficients generate full algebra $\mathfrak{cl}(2n, \mathbb{C})$, but for applications with Lie algebras and quantum gates we may use only commutators with real coefficients. It can be simply shown [23] that elements Eq. (5) generate only $(2n^2 + n)$-dimensional subalgebra in $4^n$-dimensional Lie algebra $\mathfrak{u}(2^n)$ and the algebra is isomorphic with Lie algebra $\mathfrak{so}(2n + 1)$ and the subgroup of $U(2^n)$ generated by the elements Eq. (5) is isomorphic with $(2n^2 + n)$-dimensional group $\text{Spin}(2n + 1)$. So the gates are not universal.

But simple commutation laws of Clifford algebra make possible to prove that it is enough to join only one element $\Gamma_u$ to $2n$ elements Eq. (5) and the new set with $2n + 1$ elements may generate full $4^n$-dimensional Lie algebra $\mathfrak{u}(2^n)$ [23]. As the extra element may be used product of any three or four matrices $\Gamma_k$, for example

\[
\Gamma_u = i \Gamma_0 \Gamma_1 \Gamma_2
\]

or

\[
\Gamma_u' = i \Gamma_0 \Gamma_1 \Gamma_2 \Gamma_3.
\]

Here $\Gamma_u$ and $\Gamma_u'$ are also anti-Hermitian and so commutators and sums with real coefficients may generate only anti-Hermitian matrices.

The set with $2n + 1$ elements Eq. (5) and Eq. (6) is universal, but elements $\Gamma_{2k}$ and $\Gamma_{2k+1}$ correspond to $(k+1)$-qubit gates. It is not very difficult to found new set there $2n + 1$ elements are only one- and two-qubit gates [23]. It is enough instead of Eq. (5) to consider $\Gamma_0$ together with $2n - 1$ elements $\Gamma_k \Gamma_{k+1}$, $k = 0, \ldots, 2n - 2$. All elements of the new set generate only one- or two-qubit gates because such products of two elements Eq. (6) have no more than two non-unit terms in tensor decomposition (other terms are $\sigma_z \sigma_z = \mathbb{I}$), but it is possible to produce all initial matrices $\Gamma_k$ using the new set [23].

On the other hand, the initial set with elements Eq. (5) may be more convenient for representation with fermionic annihilation and creation operators [1, 3, 4] used in applications for quantum circuits [21, 22, 23].
2.3 Noncommutative tori

The construction with Clifford algebra described above may not be applied directly for non-binary quantum circuits, when each system described by \( l \)-dimensional Hilbert spaces, \( l > 2 \), and operators from \( U(l^n) \) are represented in tensor product of \( n \) complex matrices \( \mathbb{C}(l \times l) \). But even in such a case the general principles are very close \([24]\).

First, let us consider instead of Pauli matrices Cayley-Weyl-Connes pair \([3, 5]\):

\[
UV = \exp(2\pi i/l)VU, \quad VV^\dagger = UU^\dagger = \mathbb{I},
\]

where \( l \times l \) complex matrices \( U \) and \( V \) may be represented as

\[
U_{kj} = \delta_{k+1(\text{mod } l), j}, \quad V_{kj} = \exp(2\pi ik/l)\delta_{kj}.
\]

Lately it is called sometime “general Pauli group” \([18]\).

Let us denote \( \zeta = \exp(2\pi i/l) \). It is possible to consider three elements

\[
\tau_x = U, \quad \tau_y = \zeta^{(l-1)/2}UV, \quad \tau_z = V,
\]

with properties \([24]\)

\[
\tau_x\tau_y = \zeta\tau_y\tau_x, \quad \tau_y\tau_z = \zeta\tau_z\tau_y, \quad \tau_x\tau_z = \zeta\tau_z\tau_x, \quad \tau_\mu^l = \mathbb{I}.
\]

It is clear, that for \( l = 2 \) Eq. (12) coincide with anti-commutation relation for Pauli matrices, but for \( l > 2 \) here is some asymmetry or order, because \( \zeta^{-1} \neq \zeta \), \( \tau_x\tau_x = \zeta^{-1}\tau_x\tau_x \neq \zeta\tau_y\tau_x \) and it is not possible to treat all three elements \( \tau_\mu \) in equal way. On the other hand, the order makes possible to introduce all necessary construction with noncommutative torus quite simply \([24]\).

Let us introduce set of \( \mathbb{C}(l^n \times l^n) \) matrices similar with construction Eq. (5) above:

\[
T_{2k} = \mathbb{I} \otimes \cdots \otimes \mathbb{I} \otimes \tau_x \otimes \tau_z \otimes \cdots \otimes \tau_z, \\
T_{2k+1} = \mathbb{I} \otimes \cdots \otimes \mathbb{I} \otimes \tau_y \otimes \tau_z \otimes \cdots \otimes \tau_z,
\]

where \( k = 0, \ldots, n-1 \). Then (see Ref. \([24]\)) \( l^{2n} \) different products of \( T_k \) \( k = 0, \ldots, 2n-1 \) generate algebra of complex matrices \( \mathbb{C}(l^n \times l^n) \) and the \( 2n \) elements \( T_k \) itself meet to special rule:

\[
T_jT_k = \zeta T_kT_j \quad (j < k), \quad (T_k)^l = \mathbb{I}.
\]

The Eq. (14) coincides with definition of Clifford algebra for \( l = 2 \), and for more general case \( l > 2 \) it is called here noncommutative torus.\(^1\)

\(^1\)The term is used for example in \([4]\) for \( n = 2 \) and irrational noncommutative torus, \( l \in \mathbb{R} \), a more general case with \( n \geq 2 \) is also widely known on some other areas of research, but here it is important to emphasize for comparison that using same number \( \zeta \) in Eq. (14) instead of matrix \( \zeta_{kj} \) is important for the application in present work.
Really matrices Eq. (13) are neither anti-Hermitian, nor Hermitian, and for \( l > 2 \) they generate complex Lie algebra \( \mathfrak{gl}(l^n, \mathbb{C}) \) without necessity of some extra elements like Eq. (16) in case of Clifford algebra. To produce generators of Lie algebra \( \mathfrak{u}(l^n) \) it is necessary to use instead of each matrix \( T_k \) Eq. (13) two anti-Hermitian matrices
\[
T_k^+ = i(T_k + T_k^\dagger), \quad T_k^- = (T_k - T_k^\dagger).
\] (15)

Here is also possible to introduce set of two-gates. It is again an analogy of case with qubits and Clifford algebras discussed above, and instead of elements Eq. (13) it is possible to use \( T_0 \) together with \( 2n-1 \) elements \( T_k^\dagger T_{k+1} \), \( k = 0, \ldots, 2n-2 \). Such products of two elements Eq. (13) have no more than two non-unit terms in decomposition (because \( T_x^\dagger T_x = 1 \)), but the new set may generate all matrices \( T_k \) and so equal with initial set. It is possible to produce two anti-Hermitian matrices for each elements of the new set using two sums like in Eq. (15), the matrices correspond to one- or two-gates (24).

### 3 Comparison and discussion

It is clear from consideration above that using Clifford algebras not only produced some alternative method for construction of universal set of quantum gates for binary quantum circuits, but also gave some guidelines for non-binary case. For example, expression for construction of noncommutative tori as tensor product of Weyl pairs Eq. (13) corresponds to Eq. (3) with generators of Clifford algebra and Pauli matrices.

The presentation of two-gates as product of two generators discussed above is also similar for both cases. It should be mentioned also that the construction has some interesting physical consequence even outside of area of quantum computation, because it is a demonstration, that evolution of set of few quantum systems(3) may be represented using only pairwise interactions (23).

On the other hand there are some differences between binary case \( l = 2 \) with qubits and Clifford algebras and non-binary case with noncommutative tori for \( l > 2 \). For example, generators of noncommutative tori are unitary like Pauli matrices, but for \( l > 2 \) they are not Hermitian and for generation of Lie algebra \( \mathfrak{u} \) it is necessary to use two sets of generators Eq. (15) instead of simple multiplication on imaginary unit in Eq. (16). But for case with Clifford algebra the Eq. (13) was also not enough, and it was necessary to use extra one element like Eq. (7).

The extra gate with product of three or four generators may be quite essential, as it already was emphasized in (23). If there is some algebra with an associative product, it is simple to introduce a structure of Lie algebra using commutator, but opposite task is not straightforward. If there is some Lie algebra with a “bracket operation \([,]\)” satisfying necessary axioms, then “restoring” of the product is related with construction of universal enveloping algebra (2).
For example under consideration the initial Lie algebra generated by $2n$ elements Eq. (5) was simply Lie algebra of rotation group of $(2n+1)$-dimensional space and expression like product with three generators Eq. (7) corresponds to consideration of complex enveloping algebra isomorphic to $\mathbb{C}(2^n \times 2^n)$ with exponentially larger dimension. If these constructions are not only abstract ones, but also correspond to some structures and symmetries of a particular physical model used for implementation of quantum computer, the extra one element like $\Gamma_u$ may have special physical entity, but it should be discussed elsewhere.

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