Entanglement negativity as a universal non-Markovianity witness

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In order to engineer an open quantum system and its evolution, it is essential to identify and control the memory effects. These are formally attributed to the non-Markovianity of dynamics that manifests itself by the evolution being indivisible in time, a property which can be witnessed by a non-monotonic behavior of contractive functions or correlation measures. We show that by monitoring directly the entanglement behavior of a system in a tripartite setting it is possible to witness all invertible non-Markovian dynamics, as well as all (also non-invertible) qubit evolutions. This is achieved by using negativity, a computable measure of entanglement, which in the usual bipartite setting is not a universal non-Markovianity witness. We emphasize further the importance of multipartite states by showing that non-Markovianity cannot be faithfully witnessed by any contractive function of single qubits. We support our statements by an explicit example of eternally non-Markovian qubit dynamics, for which negativity can witness non-Markovianity at arbitrary time scales.

Introduction. Describing effective dynamics of any realistic quantum system that interacts with its environment inevitably requires the theory of open quantum systems [1, 2]. In recent years, a growing interest has been devoted to the determination of dynamical properties that can be pinpointed when studying solely the system evolution, in particular, distinguishing memory-less - Markovian - dynamics from ones that exhibit memory effects. Although it is hard to unambiguously define the concept of memory or, more precisely, non-Markovianity at the level of the quantum evolutions [3–5] (see [6–9] for detailed reviews on the topic), one may formulate a natural generalisation of the Chapman-Kolmogorov equation, which assures the time-divisibility of stochastic maps in case of classical Markovian processes [10]. In particular, focusing on the family of quantum operations, i.e., completely positive (CP) trace-preserving (TP) maps Λ that represent the system evolution from the initial time $t = 0$ to each $t > 0$, one may verify their CP-divisibility [11, 12], inspecting whether at any intermediate time $0 \leq s \leq t$ each of them could be decomposed (concatenated) as

$$\Lambda_t = V_{t,s} \circ \Lambda_s$$  \hspace{1cm} (1)

with a valid dynamical (CPTP) map $V_{t,s}$.

The above criterion, however, can be weakened in order to construct witnesses of non-Markovianity that despite not always being able to certify the non-CP character of $V_{t,s}$ can have a clear operational meaning. The most commonly used notion indicating non-Markovianity is the temporal behaviour of distinguishability, as measured by the trace distance $\|\rho - \sigma\|_1 / 2$ with the trace norm $\|M\|_1 = \text{Tr} \sqrt{M^\dagger M}$, between a pair of evolving quantum states $\rho$ and $\sigma$. Its increase for a given time $t$ assures information backflow from the environment to the system at that time instance [3]. Moreover, if the corresponding dynamical map $\Lambda_t$ is invertible [13, 14], image non-increasing [15], or acts on a single qubit [16], by allowing for an ancilla of system dimension $d$ one may formulate a statement based on distinguishability:

$$\frac{d}{dt} \| \Lambda_t \otimes \mathbb{1}_d | p_1 \rho_1 - p_2 \rho_2 \|_1 \leq 0,$$  \hspace{1cm} (2)

which is, in fact, equivalent to the condition (1) if valid for all bipartite system-ancilla initial states $\rho_1, \rho_2$ and probabilities $p_1, p_2$.

Nonetheless, an important question has been left open whether non-Markovianity can be faithfully verified by considering solely the evolution of correlations, in particular, dynamics of the entanglement between the system and some ancillae [12]. This would allow to certify non-Markovianity at any time $t > 0$ by preparing the system and ancillae in a suitable single correlated state, in order to observe an increase of some entanglement measure [17, 18] at $t > 0$, without need to consider ensembles of initial states and distinguishability tasks [19]. Previous results suggest that traditional correlation quantifiers, such as entanglement measures [20] and mutual information [21] fail to witness all non-Markovian evolutions, while a recently proposed correlation measure [20] can witness "almost all" of them. Surprisingly, in this Letter we show that negativity, a well known computable quantifier of bipartite entanglement [22, 23], can witness all non-Markovian qubit dynamics $\Lambda_t$ and all invertible evolutions of arbitrary dimension.

In the following, we first discuss possibilities and limitations to witness non-Markovianity in single-qubit systems, proving that a certain class of non-Markovian evolutions cannot be witnessed with contractive functions. We then show that their non-Markovian character can neither be witnessed by studying entanglement negativity in a bipartite system-ancilla setting. In stark contrast, we then prove that negativity can still be used to witness all non-Markovian single-qubit evolutions when multipartite states are employed. Our results extend to systems of arbitrary dimension when considering invertible dynamics. We illustrate our statements, applying them to eternally non-Markovian qubit evolutions [24].
**Witnessing non-Markovianity with contractive functions.** A general witness of non-Markovianity can be built from any contractive function $f(\rho, \sigma)$ of two quantum states $\rho$ and $\sigma$, where contractivity means that

$$f(\Lambda(\rho), \Lambda(\sigma)) \leq f(\rho, \sigma)$$

for any quantum operation $\Lambda$. Important examples for contractive functions are the trace distance $||\rho - \sigma||_1/2$, infidelity $1 - F(\rho, \sigma)$ with fidelity $F(\rho, \sigma) = ||\sqrt{\rho} \sqrt{\sigma}||_1$, and the quantum relative Rényi entropy $S(\rho|\sigma) = \text{Tr}[\rho \log_2 \rho] - \text{Tr}[\rho \log_2 \sigma]$. Recently, a family of contractive functions, named quantum relative Rényi entropy, has been introduced as $[25]

$$D_\alpha^q(\rho|\sigma) = \frac{1}{\alpha - 1} \log_2 \left[ \alpha \frac{1}{\pi^2} \rho_0 \frac{1}{\pi^2} \sigma_0 \right],$$

with $\alpha \geq 1/2$. In the limit $\alpha \rightarrow 1$ the function $D_\alpha^q(\rho|\sigma)$ coincides with the relative entropy $S(\rho|\sigma)$, and for $\alpha = 1/2$ we obtain $D_{1/2}^q(\rho|\sigma) = -2 \log_2 F(\rho, \sigma)$.

Noting that any contractive function is monotonically decreasing with $t$ for any Markovian evolution, an increase of $f$ for some $t > 0$ serves as a witness of non-Markovianity. It is now reasonable to ask whether any non-Markovian evolution can be witnessed by some suitably chosen contractive function. As we show in the Theorem 1 below, the answer to this question is negative for single-qubit systems. An important type of evolutions in this context is given by Eq. (1), where $V_{t,s}$ admits the decomposition

$$V_{t,s}[\rho] = pE_1[\rho] + (1 - p)E_2[\rho^T]$$

with probabilities $p$ and CPTP-maps $E_1$ and $E_2$ which can further depend on $t$ and $s$ with $s \leq t$. Note that $V_{t,s}$ is a positive map (P-map) which is not necessarily CP. Evolutions admitting decompositions with $V_{t,s}$ being P are generally called $P$-divisible, an example is presented below in Eq. (22). We are now ready to present the first main result of this Letter.

**Theorem 1.** For any non-Markovian evolution $\Lambda_t = V_{t,s} \circ \Lambda_s$ with $V_{t,s}$ fulfilling Eq. (5) it holds that:

$$\frac{d}{dt} f(\Lambda_t[\rho], \Lambda_t[\sigma]) \leq 0$$

for any contractive function $f(\rho, \sigma)$ and any single-qubit states $\rho$ and $\sigma$.

**Proof.** First, we will show that for any two single-qubit states $\rho$ and $\sigma$ there exists a CPTP map $\Phi_{t,s}$ (that may in general depend on both $\rho$ and $\sigma$) such that

$$V_{t,s}[\rho] = \Phi_{t,s}[\rho], \quad V_{t,s}[\sigma] = \Phi_{t,s}[\sigma].$$

This statement can be proven by considering the Bloch vectors $r$ and $s$ of the states $\rho$ and $\sigma$. The Bloch vector $\tilde{r}$ of the transposed state $\rho^T$ is related to $r = (r_x, r_y, r_z)$ via a reflection on the x-z plane, i.e., $\tilde{r} = (r_x, -r_y, r_z)$, and similar for $\sigma$. In particular, this means that transposition preserves the lengths of the two Bloch vectors and the angle between them. This implies that for any two states $\rho$ and $\sigma$ there exists a unitary rotation $U$ such that

$$\rho^T = U\rho U^t, \quad \sigma^T = U\sigma U^t.$$  (8)

The CPTP map $\Phi_{t,s}$ fulfilling Eqs. (7) is thus given as

$$\Phi_{t,s}[\rho] = pE_1[\rho] + (1 - p)E_2[U\rho U^t],$$

where the unitary $U$ is chosen such that Eqs. (8) hold. Note that – in general – the unitary $U$ depends on the two states $\rho$ and $\sigma$.

Combining the above arguments, we obtain the following for any contractive function $f$ and any two single-qubit states $\rho$ and $\sigma$:

$$f(\Lambda_t[\rho], \Lambda_t[\sigma]) = f(V_{t,s} \circ \Lambda_s[\rho], V_{t,s} \circ \Lambda_s[\sigma])
= f(\Phi_{t,s}[\rho], \Phi_{t,s}[\Lambda_s[\sigma]])
\leq f(\Lambda_s[\rho], \Lambda_s[\sigma])$$

which proves that any contractive function is monotonically decreasing with $t$. □

While Theorem 1 applies only to single-qubit systems, this constraint can be lifted if one considers only specific functions, namely the trace distance, the relative entropy, and the quantum relative Rényi entropy $D_\alpha^q(\rho|\sigma)$ for $\alpha > 1$. Noting that these functions are contractive under positive maps $[26]$, it follows that they are monotonic under non-Markovian evolutions which are P-divisible.

A question which is left open in Theorem 1 is whether it is still possible to detect non-Markovianity via the behavior of a contractive function $f$. Even if $f$ is monotonically decreasing with $t$, its overall behavior might depend on whether the evolution is Markovian or not. We answer this question in Appendix A, showing that the monotonic behavior of any contractive function can be reproduced by Markovian dynamics.

**Witnessing non-Markovianity with entanglement.** The above results tell us that to witness all non-Markovian evolutions, our input state must be of higher dimension, possibly a compound state of the system extended by ancillae, i.e., we need to consider the evolution $\Lambda^A_t \otimes I^B$ acting on a bipartite state $\rho = \rho^{AB}$. The behavior of any entanglement measure $E^{A|B}$ of the final state

$$\sigma_t = \Lambda^A_t \otimes I^B[\rho]$$

then serves as a witness of non-Markovianity, as for any Markovian evolution the entanglement must monotonically decrease. However, this approach is not suitable to create a universal witness of non-Markovianity, as for any evolution $\Lambda_t$ which consists of an entanglement...
with maximally entangled states

Proof. For any single-qubit non-Markovian evolution, the state \( \sigma_i \) will have zero entanglement for all \( t \geq t' \) [20].

Even if the evolution is not entanglement breaking, we can show that certain entanglement quantifiers fail to detect non-Markovianity. In the following, we quantify the amount of entanglement via negativity [22, 23]

\[
E_{AB}(\rho) = \frac{\|\rho_T\|_1 - 1}{2},
\]

where \( T_B \) denotes the partial transpose with respect to the subsystem \( B \). As is shown in Appendix B, negativity is monotonic under local positive maps of the form (5), i.e.,

\[
P^A \otimes \mathbb{1}^B[\rho] = pE^A_1 \otimes \mathbb{1}^B[\rho] + (1 - p)E^A_2 \otimes \mathbb{1}^B[\rho^T],
\]

for any bipartite state \( \rho = \rho^{AB} \) and probability \( p \) [27].

This implies that negativity is monotonically decreasing for any local evolution \( \Lambda^A_t = V_{t,s}^A \otimes \Lambda^A_t \) with \( V_{t,s} \) being of the form (5). An example for a non-Markovian evolution admitting this form will be given in Eq. (22). As we further show in Appendix A, negativity cannot be used to witness non-Markovianity if \( E_{AB}(\Lambda^A_t \otimes \mathbb{1}^B[\rho]) \) is monotonically decreasing with \( t \), as a decreasing behavior can always be reproduced by Markovian dynamics.

From this, we conclude that negativity \( E_{AB} \) fails to witness some non-Markovian evolutions on subsystem \( A \) even if they are not entanglement breaking [28].

In the light of these results, it is tempting to conclude that negativity is not suitable for construction of a universal non-Markovianity witness. Quite surprisingly, the situation changes completely by adding an extra particle \( C \), and considering the negativity \( E^{ABC} \) of the state

\[
\tau^{ABC}_t = \Lambda^A_t \otimes \mathbb{1}^{BC}[\rho^{ABC}],
\]

where \( \rho^{ABC} \) is a suitably chosen initial state. The following theorem shows, that in a tripartite setting negativity is a universal non-Markovianity witness for all single-qubit evolutions.

**Theorem 2.** For any single-qubit non-Markovian evolution \( \Lambda_t \) there exists a quantum state \( \rho^{ABC} \) such that

\[
\frac{d}{dt} E^{ABC}(\Lambda^A_t \otimes \mathbb{1}^{BC}[\rho^{ABC}]) > 0
\]

for some \( t > 0 \).

Proof. We introduce the following state

\[
\rho^{ABC} = p_1\rho^{AB_1}_{1}\otimes|\Psi^+\rangle\langle\Psi^+|^{B_2C} + p_2\rho^{AB_2}_{2}\otimes|\Psi^-\rangle\langle\Psi^-|^{B_2C}
\]

with maximally entangled states \( |\Psi^\pm\rangle = (|01\rangle \pm |10\rangle)/\sqrt{2} \) and suitably chosen states \( \rho_i \) and probabilities \( p_i \), which will be specified in more detail below. If now an evolution \( \Lambda^A_t \) acts on the state \( \rho^{ABC} \), the time-evolved state takes the form

\[
\tau^{ABC}_t = p_1\Lambda^A_t[\rho^{AB_1}][\rho^{AB_1}_{1}\otimes|\Psi^+\rangle\langle\Psi^+|^{B_2C} + p_2\Lambda^A_t[\rho^{AB_2}_{2}\otimes|\Psi^-\rangle\langle\Psi^-|^{B_2C}.
\]

To evaluate the negativity in \( AB|C \) cut we notice that the partial transposition with respect to \( C \) is given by

\[
\tau^{AC}_{t_i} = \frac{1}{2} \Lambda^A_t[p_1\rho^{AB_1}_{1} + p_2\rho^{AB_2}_{2}] \otimes (|01\rangle\langle01|^{B_2C} + |10\rangle\langle10|^{B_2C})
\]

\[
+ \frac{1}{2} \Lambda^A_t[p_1\rho^{AB_1}_{1} - p_2\rho^{AB_2}_{2}] \otimes (|\Phi^\pm\rangle\langle\Phi^\pm|^{B_2C} - |\Phi^\mp\rangle\langle\Phi^\mp|^{B_2C})
\]

with \( |\Phi^\pm\rangle = (|00\rangle \pm |11\rangle)/\sqrt{2} \). Since the states \( |\Phi^\pm\rangle \) are orthogonal to \( |01\rangle \) and \( |10\rangle \), the trace norm of \( \tau^{AC}_{t_i} \) can be evaluated as

\[
\|\tau^{AC}_{t_i}\|_1 = 1 + \|\Lambda^A_t[p_1\rho^{AB_1}_{1} - p_2\rho^{AB_2}_{2}]\|_1,
\]

where we used the fact that \( \mu := p_1\Lambda^A_t[\rho^{AB_1}_{1} - p_2\rho^{AB_2}_{2}] \) is a valid quantum state, and thus \( \|\mu\|_1 = 1 \). The negativity of \( \tau^{ABC}_t \) is thus given as

\[
E^{ABC}(\tau^{ABC}_t) = \frac{1}{2} \|\Lambda^A_t[p_1\rho^{AB_1}_{1} - p_2\rho^{AB_2}_{2}]\|_1.
\]

To complete the proof of the theorem, recall that for any non-Markovian single-qubit evolution \( \Lambda^A_t \) there exists states \( \rho^{AB_i}_i \) and probabilities \( p_i \) such that Eq. (2) is violated [16].

Note that the above theorem holds for any qubit evolution \( \Lambda_t \), including also dynamics which are not invertible. The theorem extends to arbitrary dimension \( d \) if the evolution \( \Lambda_t \) is invertible [13, 14], or even image non-increasing [15], noting that for invertible and image non-increasing non-Markovian evolutions there exist bipartite states \( \rho_i \) and probabilities \( p_i \) violating Eq. (2).

**Applications.** We apply the results presented above to qubit eternally non-Markovian (ENM) dynamics [24], an evolution exhibiting non-Markovianity at any \( t > 0 \), even at arbitrarily small and large timescales. Such a model falls into well-studied categories of random-unitary [29] and phase-covariant [30] qubit commutative evolutions. Yet, it constitutes an important example with its non-Markovian features being hard to witness [31, 32].

In general, a random-unitary qubit dynamics is described by a time-dependent master equation:

\[
\frac{d\rho(t)}{dt} = \mathcal{L}_i[\rho(t)] = \sum_{i=1}^{3} \gamma_i(t)\left[\sigma_i\rho(t)\sigma_i - \rho(t)\right],
\]
where the mixing probabilities \( p_i(t) \), and their time-dependence, can be explicitly expressed as a function of \( \gamma_i(t) \) [29]. Crucially, for any such evolution the CP-divisibility condition (1) is equivalent to the statement that for all \( t > 0 \) all the decay rates are non-negative, \( \gamma_i(t) \geq 0 \), while the P-divisibility criterion corresponds to a weaker requirement that at all times \( t > 0 \) each pair \((i \neq j)\) of decay parameters satisfies \( \gamma_i(t) + \gamma_j(t) \geq 0 \) [29].

The ENM model introduced in Ref. [24] corresponds then to the choice:

\[
\gamma_1 = \gamma_2 = \frac{\alpha c}{2}, \quad \gamma_3(t) = -\frac{\alpha c}{2} \tanh(ct) \tag{23}
\]

with \( \alpha \geq 1 \) and \( c > 0 \). Note that this yields a stringent form of the dynamical generator \( \mathcal{L}_t \) in Eq. (21) that is not generally rescalable, i.e., by rescaling \( \mathcal{L}_t \rightarrow \beta \mathcal{L}_t \) with any \( 0 < \beta < 1/\alpha \) the map \( \Lambda_t \) in (22) ceases to be CP [33, 34].

Crucially, ENM dynamics exhibits non-Markovianity at all times, as \( \gamma_3(t) < 0 \) for all \( t > 0 \). In contrast, it is always P-divisible due to \( \gamma_i + \gamma_3(t) = \alpha \xi(1 - \tanh(ct)) \geq 0 \) for \( \ell \in \{1, 2\} \) and any \( t \geq 0 \). Still, the resulting CP-map (22) is invertible, i.e., for every \( t \geq 0 \) one can find a linear map \( \Lambda_t^{-1} \) such that \( \Lambda_t^{-1} \circ \Lambda_t = \mathbb{1} \). As a result, one can unambiguously define \( V_{ts} = \Lambda_t \circ \Lambda_s^{-1} \) in (1) and explicitly compute its Choi-Jamiołkowski (CJ) matrix, \( \Omega_{V_{ts}} := 2V_{ts} \otimes \mathbb{1} \langle \Phi^+\rangle\langle \Phi^+\rangle \), associated with it:

\[
\Omega_{V_{ts}} = \frac{1}{2} \begin{pmatrix} 1 + \lambda_1^{2t_s} & 0 & 0 & 2\Gamma_1^{t_s} \\ 0 & 1 - \lambda_1^{2t_s} & 0 & 0 \\ 0 & 0 & 1 - \lambda_1^{2t_s} & 0 \\ 2\Gamma_1^{t_s} & 0 & 0 & 1 + \lambda_1^{2t_s} \end{pmatrix} \tag{24}
\]

where \( \lambda_1 = e^{-ct} \) and \( \Gamma_1^{t_s} = \lambda_1^{t_s} \cosh(c t_s) \text{sech}(c t_s) \). It may be explicitly verified that \( \Omega_{V_{ts}} \) is non-positive for any \( 0 < s < t \), confirming the “eternal non-Markovianity” of dynamics, unless \( s = 0 \) for which \( \Omega_{V_{ts}} = \Omega_{\Lambda_t} \geq 0 \) assures the physicality of the overall evolution.

In Appendix C, we explicitly show that the CJ-matrix (24) admits a convex decomposition:

\[
\Omega_{V_{ts}} = p_1 P_{\Phi_1} + p_2 P_{\Phi_2} + (1 - p_1 - p_2) P_T^{t_s} \tag{25}
\]

with probabilities \( p_1 = \frac{1}{2} (\lambda_1^{2t_s} + \Gamma_1^{t_s}) \) and \( p_2 = \frac{1}{2} (1 - \Gamma_1^{t_s}) \), and \( P_{\Phi_1} = 2|\psi\rangle\langle \psi| \). Hence, it follows (see Appendix D for a general discussion) that the decomposition (25) of the CJ-matrix assures the map \( V_{ts} \) for the ENM dynamics to admit a decomposition (5). As a direct consequence, Theorem 1 applies to the ENM dynamics, implying that no contractive function \( f(\rho, \sigma) \) evaluated on single-qubit states \( \rho \) and \( \sigma \) will be able to witness non-Markovianity of

the ENM model. Moreover, as Eq. (5) naturally generalizes to Eq. (13), it becomes evident that negativity cannot be used in the usual bipartite setting \( E^{ABC}(A_i^A \otimes 1^B | \rho) \) to witness the non-Markovianity of the ENM evolution.

However, we explicitly demonstrate that, in accordance with the Theorem 2, negativity in the tripartite setting, \( E^{ABC} \), can indeed be used to faithfully witness the non-Markovianity of the ENM evolution for any \( t > 0 \). In order to choose the initial state \( \rho^{ABC} \) – in particular, its constituents \( \rho_{\ell}^{AB_i} \) \( (\ell = 1, 2) \) such that \( E^{ABC} \) increases at a given \( t > 0 \) – we follow the constructive method of Bylicka et al. [14]. We choose \( \rho_{\ell}^{AB_i} \in \mathcal{B}(C_2 \otimes C_3) \) and mixing probabilities \( \rho_{\ell} \), see App. E, such that the trace norm in Eq. (20) is assured to increase at \( t \) [14]. In Fig. 1, we plot the dynamical behaviour of \( E^{ABC} \) for the ENM model (23) with \( \alpha = 2 \) and \( c = 1/2 \). The initial state \( \rho^{ABC} \) has been set as in Eq. (16) with probabilities \( p_i \) and states \( \rho_i^{AB_i} \) chosen according to the constructive method of Bylicka et al. [14], leading to violation of Eq. (2) for specific time \( t > 0 \). The plot shows detection of non-Markovianity for \( t = 1 \) \((t = 0.01 \text{ within the inset})\), marked with a red dashed vertical line.

**Figure 1.** Negativity \( E^{ABC} \) as a function of time \( t \) for the eternal non-Markovian (ENM) qubit dynamics (23) with \( \alpha = 2 \) and \( c = 1/2 \). The initial state \( \rho^{ABC} \) has been set as in Eq. (16) with probabilities \( p_i \) and states \( \rho_i^{AB_i} \) chosen according to the constructive method of Bylicka et al. [14], leading to violation of Eq. (2) for specific time \( t > 0 \). The plot shows detection of non-Markovianity for \( t = 1 \) \((t = 0.01 \text{ within the inset})\), marked with a red dashed vertical line.

**Conclusions.** In this Letter we discuss possibilities and limitations to detect non-Markovianity in qubit systems and beyond. It is shown that a very general class of quantities based on contractive functions fails to detect non-Markovianity of all qubit evolutions. This includes widely studied quantifiers such as trace distance, fidelity, and quantum relative entropy. It is shown that all of them fail to witness non-Markovianity in a certain class of evolutions, which includes eternal non-Markovian dynamics exhibiting non-Markovianity at all times \( t > 0 \).

If entangled systems are employed to witness non-Markovianity, we show that the situation strongly depends on the number of particles used. Surprisingly, for three particles \( A, B, \) and \( C \) it is possible to witness all non-Markovian evolutions on the qubit system \( A \) by...
considering entanglement in the cut $AB|C$. We show this explicitly for entanglement negativity, a computable measure of entanglement, which is non-monotonic for any non-Markovian qubit dynamics and a suitable chosen initial state. For invertible and image non-increasing dynamics our results apply to systems of any dimension. As an example, we show numerical results for the eternal non-Markovianity model, where the non-monotonic behavior of negativity can be observed at arbitrary small times.

Our results demonstrate that well-established entanglement quantifiers are suitable as non-Markovianity witnesses for very general classes of evolutions. An important question left open in this work is whether entanglement measures can universally witness non-Markovianity of all evolutions, including non-invertible dynamics beyond qubits.

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[1] H.-P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems* (Oxford University Press, Oxford, 2002).

[2] D. A. Lidar, arXiv:1902.00967 (2019).

[3] H.-P. Breuer, E.-M. Laine, and J. Piilo, Phys. Rev. Lett. 103, 210401 (2009).

[4] D. Chruściński and S. Maniscalco, Phys. Rev. Lett. 112, 120404 (2014).

[5] F. A. Pollock, C. Rodríguez-Rosario, T. Frauenheim, M. Paternostro, and K. Modi, Phys. Rev. Lett. 120, 040405 (2018).

[6] H.-P. Breuer, E.-M. Laine, J. Piilo, and B. Vacchini, Rev. Mod. Phys. 88, 021002 (2016).

[7] L. Li, M. J. Hall, and H. M. Wiseman, Phys. Rep. 759, 1 (2018).

[8] A. Rivas, S. F. Huelga, and M. B. Plenio, Rep. Prog. Phys. 77, 094001 (2014).

[9] I. de Vega and D. Alonso, Rev. Mod. Phys. 89, 015001 (2017).

[10] B. Vacchini, A. Smirne, E.-M. Laine, J. Piilo, and H.-P. Breuer, New J. Phys. 13, 093004 (2011).

[11] M. M. Wolf and J. I. Cirac, Commun. Math. Phys. 279, 147 (2008).

[12] A. Rivas, S. F. Huelga, and M. B. Plenio, Phys. Rev. Lett. 105, 050403 (2010).

[13] D. Chruściński, A. Kossakowski, and Á. Rivas, Phys. Rev. A 83, 052128 (2011).

[14] B. Bylicka, M. Johansson, and A. Acín, Phys. Rev. Lett. 118, 120501 (2017).

[15] D. Chruściński, Á. Rivas, and E. Størmer, Phys. Rev. Lett. 121, 080407 (2018).

[16] S. Chakraborty and D. Chruściński, arXiv:1901.03476 (2019).

[17] V. Vedral, M. B. Plenio, M. A. Rippin, and P. L. Knight, Phys. Rev. Lett. 78, 2275 (1997).
SUPPLEMENTAL MATERIAL

Appendix A: Monotonically decreasing functions and entanglement measures cannot witness non-Markovianity

Here we will show that a contractive function $f$ cannot witness non-Markovianity of $\Lambda_t$ if $f(\Lambda_t[\rho],\Lambda_t[\sigma])$ is monotonically decreasing. We will show this for the case of discrete time steps $t_i$ with $t_0 = 0$. Then, there exists a CP-divisible family of maps $W_{i,t}$ such that

$$f(\mu_i,\tau_i) = f(\Lambda_t[\rho],\Lambda_t[\sigma])$$ (A1)

is true for all $i$, where the states $\mu_i$ and $\tau_i$ are defined recursively via

$$\mu_{i+1} = W_{i+1,i}[\mu_i], \quad \tau_{i+1} = W_{i+1,i}[\tau_i],$$ (A2)

and $\mu_0 = \rho$, $\tau_0 = \sigma$. CP-divisible family $W_{i,t}$ which achieves this is given by

$$W_{i+1,i}[\rho] = \alpha_{i+1}^{i+1-i} \rho + (1 - \alpha_{i+1}^{i+1-i}) \frac{1}{d}$$, (A3)

where the parameters $0 \leq \alpha_i \leq 1$ are chosen such that Eq. (A1) is fulfilled. By continuity, such values for $\alpha_i$ always exist, as $f(W_{i+1,i}[\rho],W_{i+1,i}[\sigma])$ monotonically decreases with decreasing $\alpha_i$, achieving minimal value for $\alpha_i = 0$.

By similar arguments it follows that the behavior of any entanglement measure $E^{ABC}$ cannot witness non-Markovianity of $\Lambda_t$ if $E^{ABC}(\Lambda_t^A \otimes 1^{BC}[\rho^{ABC}])$ is monotonically decreasing with $t$. For this, we recursively define tripartite states

$$\rho_{i+1}^{ABC} = W_{i+1,i}^{A} \otimes 1^{BC}[\rho_{i}^{ABC}]$$

with $\rho_{0}^{ABC} = \rho^{ABC}$ and $W_{i,t}$ is a local CP-divisible family defined in Eq. (A3). Here, the parameters $\alpha_i$ are chosen such that

$$E^{ABC}(\rho_{i}^{ABC}) = E^{ABC}(\Lambda_i^A \otimes 1^{BC}[\rho_{i}^{ABC}])$$ (A4)

Again, such values of $\alpha_i$ always exist by continuity, as $E^{ABC}(W_{i+1,i}^{A} \otimes 1^{BC}[\rho])$ monotonically decreases with decreasing $\alpha_i$, achieving minimal value for $\alpha_i = 0$.

Appendix B: Negativity and local positive maps

Here we will show that negativity is monotone under local positive maps of the form

$$P^A \otimes 1^B[\rho] = pE^A \otimes 1^B[\rho] + (1-p)E^B \otimes 1^B[\rho^T]$$, (B1)

for any CPTP maps $E_i$, bipartite state $\rho = \rho^{AB}$, and probability $p$. Noting that $P^A$ commutes with partial-transposition $T_B$, we obtain

$$E^{AB}(P^A \otimes 1^B[\rho]) = \frac{1}{2} \left( \left\| P^A \otimes 1^B[\rho^T] \right\|_1 - 1 \right)$$ (B2)

$$\leq \frac{1}{2} \left( p \left\| E^A \otimes 1^B[\rho^T] \right\|_1 + (1-p) \left\| E^B \otimes 1^B[\rho] \right\|_1 - 1 \right)$$

$$= \frac{p}{2} \left( \left\| E^A \otimes 1^B[\rho] \right\|_1 - 1 \right)$$

$$= pE^{AB}(E^A \otimes 1^B[\rho]) \leq E^{AB}(\rho),$$

where we used convexity of the trace norm and its monotonicity under CPTP maps, and the fact that $\| E^A \otimes 1^B[\rho] \|_1 = 1$.

Appendix C: Eternally non-Markovian qubit dynamics

For the general solution to the master equation (21) describing random unitary dynamics we refer the reader to Ref. [29]. Still, for the choice of decay parameters (23) corresponding to the ENM model, the mixing probabilities $p_i(t)$ in the dynamical (Pauli) map $\Lambda_t$ defined in (22) read:

$$p_0(t) = \frac{1}{4} \left[ 1 + e^{-2\alpha t} \left( 1 + 2e^{\alpha t} \cosh(c t) \right) \right]$$, (C1a)

$$p_1(t) = p_2(t) = \frac{1}{4} \left( 1 - e^{-2\alpha t} \right)$$, (C1b)

$$p_3(t) = \frac{1}{4} \left[ 1 + e^{-2\alpha t} \left( 1 - 2e^{\alpha t} \cosh(c t) \right) \right]$$, (C1c)

where with a superscript we have specially stated the dependence on the parameter $\alpha \geq 1$. We have kept the $\alpha$-dependence explicit, so that we can conveniently express the inverse map of $\Lambda_i$ (i.e., $\Lambda_i^{-1}$ s.t. $\Lambda_i^{-1} \circ \Lambda_i = 1$), which also takes the Pauli form (22), as $\Lambda_i^{-1}[\rho] = \sum_{\mu} p_{\mu}^{(n)}(t) \alpha_{\mu}^{i} \rho_{\mu}^{i}$ by simply changing the sign of $\alpha$.

As a result, the CJ matrix of the map $V_{i,t} = \Lambda_t \circ \Lambda_i^{-1}$ stated in Eq. (24) can be directly computed as

$$\Omega_{V_{ij}} = \left( \Lambda_t \circ \Lambda_i^{-1} \right) \otimes 1 \{ P_{\Phi_i} \}$$ (C2)

$$= \sum_{\mu} \left( \sum_{\nu} p_{\mu}^{(n)}(t) p_{\nu}^{(-n)}(s) \alpha_{\mu}^{i} \alpha_{\nu}^{j} P_{\Phi_i} \alpha_{\nu}^{j} \right)$$

$$+ \sum_{i \neq j} \left( \sum_{\nu} p_{\mu}^{(n)}(t) p_{\nu}^{(-n)}(s) \alpha_{\mu}^{i} \alpha_{\nu}^{j} P_{\Phi_i} \alpha_{\nu}^{j} \right)$$

$$+ \sum_{j} \left( \sum_{\nu} p_{\mu}^{(n)}(t) p_{\nu}^{(-n)}(s) + p_{\mu}^{(-n)}(t) p_{\nu}^{(n)}(s) \right) \alpha_{\mu}^{i} \Phi_i \alpha_{\nu}^{j},$$

where $P_{\Phi} = 2|\psi\rangle\langle\psi|$. Using, the properties of Pauli operators and Bell states (e.g., $\sigma_1^{A} |\Phi_+\rangle = |\Psi_+\rangle$), as well as $P_{\Phi_i} = 1_4 - P_{\Phi_i}$, one arrives at the decomposition (25):

$$\Omega_{V_{ij}} = p_1 P_{\Phi_i} + p_2 P_{\Phi} + (1 - p_1 - p_2) P_{\Psi_i}^{T_A}$$, (C3)
with
\[ p_1 = \frac{1}{2} e^{-c(t-s)^a} \left[ e^{-c(t-s)^a} + (\cosh(cs) \, \text{sech}(ct))^{-a} \right], \quad (C4a) \]
\[ p_2 = \frac{1}{2} \left( 1 - e^{-c(t-s)^a} (\cosh(cs) \, \text{sech}(ct))^{-a} \right). \quad (C4b) \]
In order to prove that \(|p_1, p_2, 1-p_1-p_2\) constitutes a valid probability distribution, it is enough to demonstrate that \(p_1, p_2 \geq 0\) and \(p_1 + p_2 \leq 1\). Being a sum of positive quantities, clearly \(p_1 > 0\). Since \(s \leq t\) and \(c, a\) are positive,
\[ 0 \leq e^{-2c(t-s)^a} \leq 1, \quad (C5) \]
showing that \(p_1 + p_2 \leq 1\). Thus, it remains only to show that \(p_2 \geq 0\), which is equivalent to
\[ \left[ e^{c(t-s)^a} \cosh(cs) \, \text{sech}(ct) \right]^{-a} \leq 1 \quad (C6) \]
\[ \Leftrightarrow e^{c(t-s)^a} \cosh(cs) \, \text{sech}(ct) \geq 1 \]
\[ \Leftrightarrow 1 + e^{2cs} \frac{1}{1 + e^{-2ct}} \geq 1 \]
\[ \Leftrightarrow s \leq t, \]
which is true.

Appendix D: Choi-Jamiolkowski matrix decomposition for the P-maps of interest

The action of any linear TP-map \(\Lambda\) on \(\rho \in \mathcal{B}(\mathcal{H}_d)\) can be generally expressed as
\[ \Lambda[\rho] = \text{Tr}_B \left[ \Omega_{\Lambda}(\mathds{1}_d \otimes \rho^T) \right], \quad (D1) \]
where \(\Omega_{\Lambda} := \Lambda \otimes \mathds{1}_d [d|\Phi^+\rangle\langle\Phi^+|]\) is the “effective” CJ-matrix satisfying \(\text{Tr}_A \Omega_{\Lambda} = \mathds{1}_d\), yet not being necessarily positive semi-definite.

Now, let us show that if \(\Omega_{\Lambda}\) admits a convex decomposition:
\[ \Omega_{\Lambda} = p \, \Omega_{\ell_1} + (1-p) \, \Omega_{\ell_2}^T \quad (D2) \]
with \(0 \leq p \leq 1\), \(\Omega_{\ell_1} \geq 0\) and \(\text{Tr}_A \Omega_{\ell_1} = \mathds{1}_d\) for both \(\ell = 1, 2\), then the linear map \(\Lambda\) can always be decomposed as stated in the main text in Eq. (5).

This is because one may then explicitly write:
\[ \Lambda[\rho] = \text{Tr}_B \left[ \Omega_{\Lambda}(\mathds{1} \otimes \rho^T) \right] \quad (D3) \]
\[ = \text{Tr}_B \left[ p \, \Omega_{\ell_1} (\mathds{1} \otimes \rho^T) + (1-p) \, \Omega_{\ell_2}^T (\mathds{1} \otimes \rho^T) \right] \]
\[ = p \, \text{Tr}_B \left[ \Omega_{\ell_1} (\mathds{1} \otimes \rho^T) \right] + (1-p) \, \text{Tr}_B \left[ (\mathds{1} \otimes \rho) \right] \]
\[ = p \, \text{Tr}_B \left[ \Omega_{\ell_1} (\mathds{1} \otimes \rho^T) \right] + (1-p) \, \text{Tr}_B [\rho], \]
where \(\text{Tr}_B[\rho] = \text{Tr}_B [\Omega_{\ell_1} (\mathds{1} \otimes \rho^T)]\) are the CPTP maps defined by the (positive semi-definite) CJ-matrices \(\Omega_{\ell_i}\).

Appendix E: Constructing \(\rho^{ABC}\) such that \(E^{ABC}\) is a faithful non-Markovianity witness for a given \(t > 0\)

We follow the method of Bylicka et al. [14] that describes how to construct initial states \(\rho_1, \rho_2\) such that for an invertible dynamical map \(\Lambda_t : \mathcal{B}(\mathcal{H}_d) \to \mathcal{B}(\mathcal{H}_d)\) the trace distance (2) at time \(t > 0\) is increasing iff \(\Vert t_{s+b,t} = \Lambda_{s+b} \circ \Lambda_{-1}^{-1}\) is not CP for \(\delta t \to 0\), i.e.,
\[ \frac{d}{dt} \left\| \Lambda_t \otimes \mathds{1}_3 [\rho_1 - \rho_2] \right\| > 0, \quad (E1) \]
where, thanks to considering the ancilla to be of dimension \(d + 1\), the probabilities in Eq. (2) can be assumed \(p_1 = p_2 = \frac{1}{2}\) without loss of generality.

As a consequence, once \(p_1, p_2\) are determined, by setting the initial state (16) to
\[ \rho^{ABC} = \frac{1}{2} (\rho_1 \otimes |\psi^\perp\rangle \langle\psi^\perp| + \rho_2 \otimes |\psi^-\rangle \langle\psi^-|) \quad (E2) \]
with \(\rho^{ABC} \in \mathcal{B}(\mathcal{H}_d^4 \otimes \mathcal{H}_d^4 \otimes \mathcal{H}_d^4)\), the condition (E1) assures the negativity to fulfill
\[ \frac{d}{dt} E^{ABC} (\tau^{ABC}_t) = \frac{1}{2} \frac{d}{dt} \left\| \Lambda_t \otimes \mathds{1}_3 [\rho_1 - \rho_2] \right\| > 0, \quad (E3) \]
so that \(E^{ABC}\) is, indeed, a faithful witness of non-Markovianity at \(t > 0\).

Constructing \(\rho_1, \rho_2\) for the eternally non-Markovian qubit dynamics

Here, we describe in more detail the above procedure for the case of ENM qubit dynamics, for which the behavior of \(E^{ABC}\) is depicted in Fig. 1. Note that for qubit dynamics the construction requires \(\rho_1, \rho_2 \in \mathcal{B}(\mathcal{H}_d^2 \otimes \mathcal{H}_d^2)\), i.e., to deal with qubit-qudit states.

The form of the dynamical map, \(\Lambda_t\), as well as its inverse, \(\Lambda_t^{-1}\), for the ENM model are described above in Appendix C. Following the method of [14]:

1. We choose maximally mixed state, \(\sigma = \frac{1}{2} \mathds{1}_d\), as an example of a state that lies in the image of \(\Lambda_t \otimes \mathds{1}_3\) for any \(t \geq 0\) in case of the ENM model.
2. We set \(\rho^A = |0\rangle \langle 0|\) as an exemplary state in \(\mathcal{B}(\mathcal{H}_d^2)\).
3. We compute states \(\rho'_1(\lambda) = (1 - \lambda) \sigma + \lambda |\Phi^+\rangle \langle\Phi^+|\) and \(\rho'_2(\lambda) = (1 - \lambda) \sigma + \lambda |\Phi^-\rangle \langle\Phi^-|\).
4. For a given \(t > 0\), we find maximal \(0 \leq \lambda \leq 1\) such that both \(\Lambda_t^{-1} \otimes \mathds{1}_3 [\rho'_1(\lambda)] \geq 0\) and \(\Lambda_t^{-1} \otimes \mathds{1}_3 [\rho'_2(\lambda)] \geq 0\). We obtain \(\lambda_{\max} = 1/(3e^{2\alpha ct} - 2)\).
5. We arrive at the desired initial states \(\rho_1 = \Lambda_t^{-1} \otimes \mathds{1}_3 [\rho'_1(\lambda_{\max})]\) and \(\rho_2 = \Lambda_t^{-1} \otimes \mathds{1}_3 [\rho'_2(\lambda_{\max})]\) that lead to \(\frac{d}{dt} \left\| \Lambda_t \otimes \mathds{1}_3 [\rho_1 - \rho_2] \right\| > 0\) at the desired \(t > 0\).