Equations of Electromagnetic Self-Consistency in a Plasma

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Abstract The set of equations governing a system consisting of an electromagnetic field plus charges in it is obtained by varying the appropriate action. It is not assumed that the currents are given, which in fact leads to the Maxwell equations governing the fields. Nor is it assumed that the fields are given, which in fact would lead to the determination of the motions of the charges (the currents) through the Lorentz force. On the contrary, currents and fields are left free to interplay, and they can be found simultaneously from the equations obtained.

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1 Introduction

It is evident that petroleum is diminished day by day, at the same time polluting the atmosphere more and more. It is sure that life could not exist on Earth after perhaps a few decades, when the petroleum deposits would finish, if not earlier because of the pollution. This is the case because of the exclusive dependence of the industry, and consequently the whole of human life, on this resource of energy. Thus, in the near future the Earth could not be dwelled at all, unless an alternative solution would be found concerning the energy problem.

In this respect the problem of controlled thermo-nuclear fusion is more than imperative to be solved in our days. In fact, the solution to this problem would be the ideal one for the energy problem, mainly for two reasons. First, there are almost inexhaustible resources, namely the oceans. Because from sea water one could obtain huge amounts of hydrogen by electrolysis, which hydrogen could then be subject to fusion and thus enormous amounts of energy could be obtained. Second, this kind of energy would be a pure one, without pollution to the atmosphere, or the surface of the Earth. This is the case because the nuclear reactions involved do not leave radioactive remainders.

The problem of controlled thermo-nuclear fusion is not a new one. People have been working (both theoretically and experimentally) on this problem for some decades. In order for fusion to take place extremely high temperatures are required. At these temperatures all of hydrogen is ionized. Thus we have to deal with a plasma. On the other hand no material bottle can suffer these temperatures without being melted. So we have to store the plasma using electromagnetic techniques, specifically in a magnetic bottle. It is therefore clear that the controlled thermo-nuclear fusion can be studied theoretically using classical electrodynamics.[1]

Now one could ask why the controlled thermo-nuclear fusion has not been achieved as yet. The answer is that a lot of instabilities are developed in the plasma, so fusion cannot last more than a small fraction of a second. Thus, the big problem is the one of plasma stability. In my opinion, the instabilities being developed are due to inadequate theoretical treatment of plasma stability. To be clear, the basic equations used are the Maxwell equations and the expression for the Lorentz force. But neither of them is self-consistent. Thus the Maxwell equations govern the fields when the currents are given. But they do not take into account the influence of the fields on the currents themselves. On the other hand, the Lorentz force can govern the currents, given that the fields are known. But the currents have an influence on the fields themselves, which is left out of the problem.

Thus, it is evident that the correct theoretical treatment must be based on self-consistent equations, which therefore would be gladly welcomed. The purpose of the present paper is to obtain these equations. And, when we speak about self-consistent equations, we mean a set of equations containing both the fields and currents, from the solution of which we could obtain both of them (fields and currents) simultaneously.

The situation which has to be expected is analogous to the Einstein equations in gravitation, which are self-consistent with the meaning that by solving them we can determine both the gravitational field, and the distribution and motion of the matter producing it, without considering either of the above two factors as given. To this direction we will now focus on obtaining the equivalent of the Einstein equations in gravitation in the case (and in the framework) of Classical Electrodynamics. It has to be noted that the equations we will obtain are not at all restricted to the special case of a plasma, but they hold quite general for any system consisting of an electromagnetic field plus charges located in it.
2 Derivation of Equations

(i) Generalities

The exposition below is in the style of Landau and Lifshitz.\[5\] The reader can consult this book for reference on the basic electromagnetic theory (see also Ref. \[3\]).

We will use for the plasma (a fully ionized gas) the continuum approach, that is the fluid approximation. We will assume that it is about a two-component plasma consisting of negative charges (electrons) and positive charges (ions) of one type only (e.g. protons). This assumption is being done without loss of generality and merely for simplicity. It is easy to generalize the results for the case of more than one types ions. In the following the indices (+) or (−) in a quantity refer accordingly to ions or electrons respectively. Quantity without any index means either of them.

In describing the fields, we will use electromagnetic potentials \( A' \) rather than the fields themselves. This fourvector is given by

\[
A^i = (\phi, \vec{A}) ,
\]

where \( \phi \) is the scalar potential and \( \vec{A} \) the vector potential.

The electromagnetic field tensor \( F_{ik} \) is then given by

\[
F_{ik} = A_{k,i} - A_{i,k} ,
\]

where the comma means partial differentiation. By the way, with this definition, the first pair of Maxwell equations (the one without sources), reading

\[
F_{ik,i} + F_{ki,i} + F_{ii,k} = 0 ,
\]

is merely an identity.

We will also use the current \( j_{i}^{(±)} \) for the description of the sources. This is given by

\[
j_{i}^{(±)} = \rho(±) \frac{dx^i}{dt} ,
\]

or

\[
\vec{j}^{(±)} = (c \rho(±), \vec{v}(±)) ,
\]

where by definition

\[
\vec{j}_{i}^{(±)} = \rho_{(±)} \bar{v}_{i}^{(±)}
\]

with \( \rho(±) \) being the charge density and \( \bar{v}_{i}^{(±)} \) the velocity of the elementary space volume which contains the elementary charge under consideration (and which gives the density \( \rho_{(±)} \)).

We have for the space-time element \( ds \)

\[
\frac{ds}{dt} = c ,
\]

where by definition

\[
\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} .
\]

But we may write \( \gamma \) as

\[
\gamma = \frac{1}{\sqrt{1 - (\rho \vec{v})^2/(\rho c)^2}} ,
\]

which gives

\[
\gamma = \frac{\rho c}{\sqrt{\vec{j}^2_{i} j_{i}}} ,
\]

so that, from Eq. (7),

\[
\frac{ds}{dt} = \frac{\sqrt{j_{i}^2 j_{i}}}{\rho} .
\]

The state of a fluid system of charged particles with (rest) masses \( m_± \) and charges \( ±e \), located in an electromagnetic field, is obviously completely described by \( A', j^±_i, \vec{j}^±_i \). We assume that the particles do not interact with each other directly, but only through the field. This is justified because we consider a gaseous plasma of low density, so that we ignore collisions, i.e., we deal with a collisionless plasma. The action \( S \) for this system is then given as a sum

\[
S = S_m + S_{mf} + S_f ,
\]

where \( S_m \) is the mechanical action, given by

\[
S_m = - \sum_{±} \int m_± c ds ,
\]

the part \( S_{mf} \) is due to the interaction of the particles with the field,

\[
S_{mf} = - \sum_{±} \left( \frac{e^2}{c} \right) A_k dx^k ,
\]

and \( S_f \) describes the field,

\[
S_f = - \frac{1}{16\pi c} \int F_{ik} F^{ik} d\Omega ,
\]

where \( d\Omega \) is the element of four-volume.

(ii) The variation of the material part of the action

Now, since we are working in the fluid approximation, we have to consider a continuous distribution of the material. Then equation (13) becomes

\[
S_m = - c \sum_{±} \int \mu_± dV dV ,
\]

or

\[
S_m = - c \sum_{±} \int \mu_± \frac{ds}{dt} dV dt ,
\]

where \( \mu_± \) is the (rest) mass density for each of the two species, and \( dV \) is the spatial volume element.

Substituting Eq. (11) into Eq. (17), we find

\[
S_m = - \sum_{±} \int \frac{\mu_±}{\rho_±} \sqrt{j_{i}^{(±)} j_{i}^{(±)}} dV dt ,
\]

or

\[
S_m = - \sum_{±} \frac{1}{\lambda_±} \int \sqrt{j_{i}^{(±)} j_{i}^{(±)}} d\Omega ,
\]

where we have set

\[
\lambda_± = \frac{\rho_±}{\mu_±}
\]

for the charge per unit (rest) mass. This is a scalar, because if we write it as \( \lambda_± = (\rho_± dV)/(\mu_± dV) \), both the
numerator and denominator become scalars. Evidently then
\[ \lambda_{\pm} = \pm \frac{e}{m_{\pm}}, \quad (20') \]
i.e., \( \lambda_{\pm} \) equals the ratio of the charge over mass for the particles of each of the two species, which is known.

For the variation of \( S_m \) upon variation of \( j^i \), we have
\[ \delta \sqrt{j_n j^i} = \frac{\delta (j_i j^i)}{2 \sqrt{j_n j^i}} = \frac{j_i \delta j^i}{\sqrt{j_n j^i}}, \quad (21) \]
so that the relation
\[ \delta S_m = - \sum_{+,-} \frac{1}{\lambda_{\pm}} \int \frac{j_i \delta j^i}{\sqrt{j_n j^i}} d\Omega, \quad (22) \]
becomes
\[ \delta S_m = - \sum_{+,-} \frac{1}{\lambda_{\pm}} \int \frac{j_i \delta j^i}{\sqrt{j_n j^i}} d\Omega. \quad (23) \]

But we actually have to vary not the currents \( j^i \) but the coordinates \( x^k \). To do this we proceed as follows. On the one hand
\[ j_i = u_i \sqrt{j_n j^i} \quad (23.1) \]
where \( u_i \) is the four-velocity, obeying the restriction
\[ \sqrt{u_i u^i} = 1. \quad (23.2) \]
On the other hand we may write
\[ j^i = cn u^i, \quad (23.3) \]
where \( n \) is the proper charge density (see Ref. [7]), which is a scalar (while \( \rho \) is not). We note that we can use for the determination of the state of the plasma fluid \( n \) and any three components of the four-velocity (for example the space components \( u^\alpha \) with \( \alpha = 1, 2, 3 \), of \( u^i \)) (because of the restriction (23.2)), instead of the four components of the current \( j^i \) themselves. It also has to be noted that the plasma considered is at very high temperature, so that its components do not interchange charges by collisions, since the plasma will be constantly fully ionized. In such a situation charge will be conserved for each of the species, so that we may assume that the continuity equation holds for all species separately, and thus we have (in four-dimensional notation)
\[ \frac{\partial j^i}{\partial x^i} = 0 \quad (23.4) \]
with \( j^i \) given by Eq. (23.3).[7]

Now we can write for the action
\[ S_m = - \sum m_e \frac{d x_i}{d s} \frac{d x^i}{d s}, \quad (23.5) \]
which results, changing to integration, in
\[ S_m = - \int \rho \left( \frac{d x^i}{d t} \right) u_i d\Omega, \quad (23.6) \]
which gives finally
\[ S_m = - \frac{1}{\lambda} \int j^i u_i d\Omega. \quad (23.7) \]
Upon variation, this gives
\[ \delta S_m = - \frac{1}{\lambda} \int u_i \delta j^i d\Omega - \frac{1}{\lambda} \int j^i \delta u_i d\Omega. \quad (23.8) \]
But the second term gives zero, because of the restriction (23.2) (if we take in mind Eq. (23.3)). Thus we are left with
\[ \delta S_m = - \frac{1}{\lambda} \int u_i \delta j^i d\Omega. \quad (23.9) \]

Substituting \( j^i \) from Eq. (23.3) into Eq. (23.9), if we also take in mind Eqs. (23.2) and (23.3) again, we have
\[ \delta S_m = - \frac{1}{\lambda} \int \frac{\partial n}{\partial x^k} \delta x^k d\Omega - \frac{1}{\lambda} \int j_i \frac{d \delta x^i}{d s} d\Omega. \quad (23.10) \]
Integrating the second term by parts, we obtain
\[ \delta S_m = - \frac{1}{\lambda} \int \frac{\partial n}{\partial x^k} \delta x^k d\Omega - \frac{1}{\lambda} \int \frac{d j_i}{d s} \delta x^i d\Omega. \quad (23.11) \]
Now the integrand in the second term can be written as
\[ \frac{d}{d s} (j_i \delta x^i) = u_k \frac{\partial}{\partial x^k} (j_i \delta x^i), \quad (23.12) \]
or, because of the continuity equation (23.4),
\[ \frac{d}{d s} (j_i \delta x^i) = u_k \frac{\partial}{\partial x^k} (j_i \delta x^i) - (u_i \delta x^i) \frac{\partial j^k}{\partial x^k}, \quad (23.13) \]
or,
\[ \frac{d}{d s} (j_i \delta x^i) = u_k \frac{\partial}{\partial x^k} (nu_i \delta x^i) - (u_i \delta x^i) \frac{\partial u^k}{\partial x^k}, \quad (23.14) \]
or, expanding and simplifying,
\[ \frac{d}{d s} (j_i \delta x^i) = cn u^k \frac{\partial}{\partial x^k} (u_i \delta x^i) - (u_i \delta x^i) cu \frac{\partial u^k}{\partial x^k}, \quad (23.15) \]
where
\[ cn u^k \frac{\partial}{\partial x^k} (u_i \delta x^i) = cn u^k \frac{\partial u_i}{\partial x^k} \delta x^i + cn u^k u_i \frac{\partial}{\partial x^k} (\delta x^i). \quad (23.15') \]
Now,
\[ \frac{\partial}{\partial x^k} (nu_i \delta x^i) = \frac{\partial n}{\partial x^k} u_i \delta x^i + cn \frac{\partial u_i}{\partial x^k} \delta x^i \]
\[ + cn u^k \frac{\partial u_i}{\partial x^k} \delta x^i + cn u^k u_i \frac{\partial}{\partial x^k} (\delta x^i). \quad (23.16) \]
Thus, by means of Eq. (23.16), we obtain from Eq. (23.15) (taking also in mind Eq. (23.15'))
\[ \frac{d}{d s} (j_i \delta x^i) = \frac{\partial}{\partial x^k} (nu_i \delta x^i) - cn \frac{\partial n}{\partial x^k} u_i \delta x^i \]
\[ - cn \frac{\partial u_k}{\partial x^k} u_i \delta x^i - (u_i \delta x^i) \frac{\partial u^k}{\partial x^k}. \quad (23.17) \]
But because of the continuity equation (23.4)
\[ \frac{\partial n}{\partial x^k} u_k + cn \frac{\partial u_k}{\partial x^k} = 0, \quad (23.18) \]
so that equation (23.17) becomes
\[
\frac{d}{ds}(j_i\delta x^i) = c\frac{\partial}{\partial x^k}(nu^k u_i\delta x^i) - (u_i\delta x^i)cn\frac{\partial u^k}{\partial x^k}.
\]  
(23.19)

Thus, from Eq. (23.11), we get, due to Eq. (23.19),
\[
\delta S_m = -\frac{1}{\lambda} \int c\frac{\partial n}{\partial x^k}\delta x^k d\Omega - \frac{1}{\lambda} \int c\frac{\partial}{\partial x^k}(nu^k u_i\delta x^i) d\Omega + \frac{1}{\lambda} \int (u_i\delta x^i)cn\frac{\partial u^k}{\partial x^k} d\Omega.
\]  
(23.20)

or
\[
\delta S_m = -\frac{1}{\lambda} \int c\frac{\partial n}{\partial x^k}\delta x^k d\Omega - \frac{1}{\lambda} \int \frac{\partial}{\partial x^k}(cn u^k u_i \delta x^i) d\Omega + \frac{1}{\lambda} \int \frac{\partial}{\partial x^k}(ju^k)\delta x^i d\Omega.
\]  
(23.21)

But, because of the continuity equation (23.4), if we also use Eq. (23.3), we take for the last term in Eq. (23.21)
\[
\frac{\partial}{\partial x^k}(ju^k) = cnu^k\frac{\partial}{\partial x^k},
\]  
(23.22)

so that finally equation (23.21) gives
\[
\delta S_m = -\frac{1}{\lambda} \int c\frac{\partial n}{\partial x^k}\delta x^k d\Omega - \frac{1}{\lambda} \int \frac{\partial}{\partial x^k}(nu^k u_i \delta x^i) d\Omega + \frac{1}{\lambda} \int cnu^k\frac{\partial}{\partial x^k} \delta x^k d\Omega.
\]  
(23.23)

But concerning the second integral, because the integrand is a 4-divergence, we can transform it into a (hyper-)surface integral over a hyper-surface enclosing our system by just applying the Gauss theorem. Thus we have
\[
\int \frac{\partial}{\partial x^k}(nu^k u_i \delta x^i) d\Omega = \int \nu u^k u_i \delta x^i dS_k.
\]  
(23.24)

At the limits of integration, that is on the hyper-surface, the variation of the coordinates is assumed to be zero according to the principle of least action.[6] Thus this integral vanishes.

(iii) The interaction of the particles with the field

Also equation (14) can be written as
\[
S_{\text{int}} = -\frac{1}{c^2} \sum_{\text{+},-} \int A_i j^i_{\text{(k)}} d\Omega,
\]  
(24)

so that it gives, upon variation of both \(j^i_{\text{(k)}}\) and \(A^i\),
\[
\delta S_{\text{int}} = -\frac{1}{c^2} \sum_{\text{+},-} \int j^i_{\text{(k)}} \delta A_i d\Omega - \frac{1}{c^2} \sum_{\text{+},-} \int A_i \delta j^i_{\text{(k)}} d\Omega.
\]  
(25)

Now, for the terms with the variation in the currents (writing one term, without index, instead of two, for simplicity), we have to vary the coordinates again, so that, if we set
\[
\delta I = -\frac{1}{c^2} \int A_i \delta j^i d\Omega,
\]  
(25.1)

and if we observe that
\[
\delta j^i = c\delta (nu^i) = cu^i \delta n + cn \delta u^i,
\]  
(25.2)
or
\[
\delta j^i = \frac{\partial j^i}{\partial x^k} \delta x^k = c \frac{\partial n}{\partial x^k} u^i \delta x^k + c n \frac{\partial u^i}{\partial x^k} \delta x^k,
\]  
(25.3)

we have
\[
\delta I = -\frac{1}{c} \int A_i \frac{\partial n}{\partial x^k} u^i \delta x^k d\Omega - \frac{1}{c} \int A_i n \frac{\partial (\delta x^k)}{\partial s} d\Omega,
\]  
(25.4)
because
\[
\frac{\partial u^i}{\partial x^k} = \frac{d}{ds} \frac{\partial (\delta x^k)}{ds}.
\]  
(25.5)

Integrating the second term in Eq. (25.4) by parts
\[
\delta I = -\frac{1}{c} \int A_i u^i \frac{\partial n}{\partial x^k} \delta x^k d\Omega - \frac{1}{c} \int \frac{d}{ds}(A_i n \delta x^k) d\Omega + \frac{1}{c} \int \frac{d(A_i n)}{ds} \delta x^k d\Omega.
\]  
(25.6)

Now, in order to evaluate the second term in Eq. (25.6), we have for the integrand
\[
\frac{d}{ds}(A_i n \delta x^k) = u^k \frac{\partial}{\partial x^k}(n A_i \delta x^i),
\]  
(25.7)

so that
\[
\frac{d}{ds}(A_i n \delta x^k) = \frac{\partial}{\partial x^k}(nu^k A_i \delta x^i) - A_i \delta x^k \frac{\partial n}{\partial x^k},
\]  
(25.8)
or, because of equation (23.18), resulting from the continuity equation (23.4), we can write instead of Eq. (25.8)
\[
\frac{d}{ds}(A_i n \delta x^k) = \frac{\partial}{\partial x^k}(nu^k A_i \delta x^i) + A_i \delta x^i u^k \frac{\partial n}{\partial x^k}.
\]  
(25.9)

Adding the last term to the integrand of the first term of Eq. (25.6), we find
\[
- A_i u^i \frac{\partial n}{\partial x^k} \delta x^k - A_k u^i \frac{\partial n}{\partial x^i} \delta x^k.
\]  
(25.10)

Adding to this the integrand of the last term of Eq. (25.6), after writing it as
\[
\frac{\partial (n A_k)}{\partial x^i} u^i \delta x^k,
\]  
(25.11)

and performing the derivation, which gives
\[
A_k \frac{\partial n}{\partial x^i} u^i \delta x^k + n \frac{\partial A_k}{\partial x^i} u^i \delta x^k,
\]  
(25.12)

we find
\[
-A_k \frac{\partial n}{\partial x^i} u^i \delta x^k + n \frac{\partial A_k}{\partial x^i} u^i \delta x^k.
\]  
(25.13)

so that finally equation (25.6) becomes
\[
\delta I = -\frac{1}{c} \int \left( A_i \frac{\partial n}{\partial x^k} - n \frac{\partial A_k}{\partial x^i} \right) u^i \delta x^k d\Omega - \frac{1}{c} \int \frac{\partial}{\partial x^k}(nu^k A_i \delta x^i) d\Omega.
\]  
(25.14)

But again the last integral, because its integrand is a 4-divergence, can be transformed, according to Gauss’ theorem, into a (hyper-)surface integral over a hyper-surface enclosing our system,
\[
\int \frac{\partial}{\partial x^k}(nu^k A_i \delta x^i) d\Omega = \int nu^k A_i \delta x^i dS_k.
\]  
(25.15)
Because this hyper-surface represents the limits of integration, where the variation is taken zero according to the principle of least action,[6] our integral finally vanishes. 

(iv) **The variation of the field part of the action**

Finally equation (15) can be written upon variation of the field (arising from variation of $A_i$) as

$$\delta S_i = -\frac{1}{8\pi c} \int F_{ik} \delta F_{ik} d\Omega,$$

which gives[5]

$$\delta S_i = -\frac{1}{4\pi c} \int \frac{\partial F_{ik}}{\partial x^k} \delta A_i d\Omega.$$  

(v) **The variation of the total action**

Thus we have for the sum

$$\delta S = \delta S_i + \delta S_{int} + \delta S_{tot},$$

if we vary independently $x^i$, $x^j$, and $A_i$ simultaneously, from Eqs. (27), (25), (25.14) and (23.23)

$$\delta S = -\frac{1}{c} \left[ \left( \frac{1}{c} \sum_{+,-} J_i^\pm \right) + \frac{1}{4\pi} \frac{\partial F_{ik}}{\partial x^k} \right] \delta A_i d\Omega$$

$$- \sum_{+,-} \left[ \frac{1}{c} \left( A_i \frac{\partial n_{i\pm}}{\partial x^k} - n_{i\pm} \frac{\partial A_i}{\partial x^k} \right) u_{i\pm}^j \right] dx^k d\Omega$$

$$\frac{c}{\lambda^2} \left( \frac{\partial n_{i\pm}}{\partial x^k} - n_{i\pm} u_{i\pm}^j \frac{\partial u_{i\pm}^j}{\partial x^k} \right) dx^k d\Omega.$$  

Concerning now the variation of the total action, we must take zero for the actual “path” of the system, according to the principle of least action.[6] But, because the variations $\delta A_i$ and $\delta x^i$, and $\delta x^i$, are arbitrary, their coefficients must be identically zero. Thus, for the coefficients of $\delta A_i$ in Eq. (29), and if we make use of Lorentz gauge

$$\frac{\partial A_k}{\partial x^k} = 0,$$

we obtain

$$\frac{c}{4\pi} \frac{\partial^2 A^i}{\partial x_i \partial x^i} = \sum_{+,-} \left( J_i^j \right),$$

where $\partial^2 / \partial x_i \partial x^i$ is the d’ Alembertian operator, defined by

$$\frac{\partial^2}{\partial x_i \partial x^i} = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2$$

with $\nabla^2$ the Laplacian operator defined by

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$  

It must be noted that equation (31) has to be taken as the second pair of Maxwell equations (the one with the sources). And for the coefficients of $\delta x^i$ in Eq. (29) we obtain

$$\frac{1}{c} \left( A_i \frac{\partial n_{i\pm}}{\partial x^k} - n_{i\pm} \frac{\partial A_i}{\partial x^k} \right) u_{i\pm}^j$$

$$+ \frac{c}{\lambda^2} \left( \frac{\partial n_{i\pm}}{\partial x^k} - n_{i\pm} u_{i\pm}^j \frac{\partial u_{i\pm}^j}{\partial x^k} \right) = 0.$$  

This equation has to be taken instead of the equation containing the Lorentz force.

3 Discussion

The set of self-consistent equations consists of Eqs. (31) (4 equations) and (34) (8 equations). To them, the following equations must be added: on the one hand equation (30) (expressing the Lorentz gauge) to Eq. (31), and on the other hand the two continuity equations (23.4) (one for + and the other for −) to the two tensor equations (34). But the two equations (23.4) are not independent, because any one of them can result from the other with the aid of the continuity equation for the total current $j_i^+(+) + j_i^-(−)$, which can be easily obtained from Eq. (31). Note that equation (31) is just the second pair of Maxwell’s equations (the one with the sources), but only in the Lorentz gauge (Eq. (30)). Thus we have totally 14 independent equations for 12 unknowns, as in the case of the Vlasov system of equations.[9] The unknowns are namely the 4 components of the four-potential $A^i$, and the $2 \times 4 = 8$ components of the currents $j_i^+(+) + j_i^-(−)$. Of course for each one of the latter (the currents), say $j^i$, we can take as unknowns the proper charge density $n$ (1 unknown) and any three of the four components of the four-velocity $u^i$ (3 unknowns).

In the case we have more than one species of ions, it is evident that for each additional species we will have 4 additional equations of the form (34), plus 1 continuity equation (that is 5 equations), and 4 additional unknowns, the four components of the four-current due to the additional species, or $n$ and three of the $u^i$’s again. Interesting is the simple case in which there is only one type of particles, and especially electrons. We would have to deal with an electronic gas then.[2] Of course it is most likely impossible that the solution of the problem is performed analytically. But perhaps it can be possible that this can be done on a computer using numerical methods. Thus everyone interested in is welcome to take part in this endeavor.

It is straightforward, although lengthy, to find the components of the energy-momentum tensor $T^{ik}$ from its very definition.[5] Then the pressures can be found from these components[6] (and then the temperature of each species from its equation of state). And also it can be verified easily, although after lengthy calculations, that the conservation laws $T^{ik} = 0$ hold, by means of the “equations of motion” of the system found earlier, namely Eqs. (31) and (34).[5]

Up to now, the starting point for the study of plasma theory was the so-called Vlasov equations.[4,9] This set of coupled equations for the simultaneous determination of the distribution functions and the fields, plus total current, was thought of as self consistent. But it seems that
it is not so. The reason is that the force entering the equations for the determination of the distribution functions (the collisionless Boltzmann equations) is right the Lorentz force. This corresponds to an equation of motion of a charge in four-dimensional notation\(^5\)

\[
mc \frac{d\mathbf{u}^i}{ds} = -\frac{e}{c} F_{ik} u^k.
\]  

(35)

But in order to get this formula we are supposed to have varied \(A_i\) in the variation of the action as

\[
\delta A_i = \frac{\partial A_i}{\partial x^k} \delta x^k,
\]

(35')

which is indeed the case when the field (strictly speaking the four-potential) is given. But in order to get self-consistent equations we have to vary \textit{independently} the field and the trajectory (the four-potential and the coordinates in our analysis). Thus \(\delta A_i\) has to be used only in the derivation of the second pair of the Maxwell equations\(^5\) and must be absent in the derivation of the equations of motion for the charge. This remark having been taken into account amounts in an equation of motion

\[
mc \frac{d\mathbf{u}^i}{ds} = -\frac{e}{c} A^i_{\ k} u^k,
\]

(36)

rather than Eq. (35).

The Vlasov equations are also not satisfying for another reason. That is they are not relativistic. On the other hand the ones obtained above, namely Eqs. (31) and (34), are of course fully relativistic. And since we have worked in the fluid approximation, they perhaps offer the tool for the development of a detailed theory of plasma relativistic fluids, in contrast to the already existing theory of conventional relativistic fluids.\(^7\)

Another disadvantage of the Vlasov theory is the fact that it does not take into account the collisions. This is of course not the case concerning the fluid approximation we have used because of our continuum approach.

Perhaps the only advantage of the Vlasov approach is the use of distribution functions, permitting the microscopic consideration of plasma properties. But our macroscopic approach is sufficient as long as we are interested in macroscopic quantities only. Of course it might be possible to introduce relativistic distribution functions in our analysis.\(^10\) But this is not the subject of the present paper.

It remains to justify the use of the fluid approximation (strictly speaking it is about a two-fluid one, which evidently is more accurate than the one-fluid one). In fact the fluid approximation is used when the mean free path of the constituent particles is small compared with the dimensions of the system. Of course this is not the case regarding a plasma — a gas of low density. But our case must be certainly excluded, since we deal with a non-equilibrium plasma in which the electron temperature exceeds considerably the ion temperature.\(^8,9\) This is the case because of the negligible mass of the electrons, as compared with that of protons. This results in great differences in the energies, and thus the temperatures, of the two fluid components, corresponding to comparable momenta. This becomes clear from the well-known formulae

\[
E = p^2/(2m) \quad \text{and} \quad E = (1/2)kT.
\]

Of course one could then ask why we do not use the magnetohydrodynamics approach (even more crude, since it presupposes one fluid component instead of two). The answer is that again the equation of motion of MHD (the Euler equation).\(^3,8\) on the one hand, makes use of the Lorentz force, which means that the set of MHD equations\(^1,3\) is not self-consistent, and on the other hand, this equation of motion is not relativistic, as it should be, since we deal with a plasma of extremely high temperature.

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