Parity and Fermions in Front-Form: An Unexpected Result

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Abstract

We show that under the operation of parity the front-form $(1/2, 0)$ and $(0, 1/2)$ Weyl spinors (massive or massless) do not get interchanged. This has the important consequence that if a front-form theory containing $(1/2, 0) \oplus (0, 1/2)$ representation space has to be parity covariant then one must study the evolution of a physical system not only along $x^+$ but also along the $x^-$ direction. As a result of our analysis, we find an indication that there may be no halving of the degrees of freedom in the front form of field theories.

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1. Introduction

During a recent series of investigations on the \((j, 0) \oplus (0, j)\) representation space, we have come across a problem in the front form of field theory with regard to the operation of parity (\(\mathcal{P}\)). It is the purpose of this paper to bring attention to this problem and present its solution. It is not the first time that the problem of parity in the front form has been addressed and yet, as will be seen, an \textit{ab initio} analysis is needed. One of the earliest considerations of this problem appears in the 1971 thesis of Soper \cite{14}. More recently, Jacob \cite{15} considered the question of parity while considering the quantization of the scalar field, and he arrived at similar conclusions to those presented in this paper in a different context. We look at the transformation properties of the \((1/2, 0)\) and \((0, 1/2)\) Weyl spinors in the front form and find that demand for \(\mathcal{P}\)-covariance necessitates considering front-form evolution not only along \(x^+\) (or \(x^-\)) but simultaneously along \(x^+\) and \(x^-\). Our considerations are valid for massive as well as massless particles, and are based upon space-time symmetries that unambiguously establish the conclusions, thus disproving a widely-favored conjecture regarding massive particles. Consensus, however, already exists on McCartor’s results \cite{16} for massless particles where, while investigating the quantization of massless fields in the front form, he concluded that a spin-1/2 system must be specified on both \(x^+\) and \(x^-\) surfaces. Our analysis can be easily extended for all \((j, 0)\) and \((0, j)\) Weyl spinors and hence for all generalised Dirac-like \((j, 0) \oplus (0, j)\) fields. However, for conceptual clarity and general familiarity, we choose

\footnote{For instance, in Ref. \cite{1} it was found that the \((1, 0) \oplus (0, 1)\) representation space supports a Bargmann-Wightman-Wigner-type quantum field theory in which a boson and its antiparticle have \textit{opposite} relative intrinsic parity. In Ref. \cite{4}, we applied the approach used previously for the instant-form formalism \cite{3} to the front-form case and obtained the \((j, 0) \oplus (0, j)\) spinors and generalized Melosh transformations for any spin. The front-form formalism was seen to be endowed with several advantages. The work of Ref. \cite{4}, apart from the indicated generalization, reproduced some of the well-known results of Melosh \cite{11}, Lepage and Brodsky \cite{12}, and Dziembowski \cite{13} for spin-1/2. Due to certain magic of Wigner’s time-reversal operator, in Ref. \cite{1} we were able to present a Majorana-like construct in the \((j, 0) \oplus (0, j)\) representation space.}
spin-1/2 as an example case.

The present communication involves purely kinematical considerations. However, kinematical considerations cannot be considered as devoid of dynamical consequences. In general, if a theory is not $\mathcal{P}$-covariant kinematically then dynamics cannot restore $\mathcal{P}$-covariance. On the other hand, suppose that, we introduce an interaction via the principle of local gauge invariance (without the $\mathbb{1} \pm \gamma^5$ type projectors in the $(1/2, 0) \oplus (0, 1/2)$ representation space) in a theory that is $\mathcal{P}$-covariant at the kinematical level; then the resulting dynamical theory is guaranteed to be $\mathcal{P}$-covariant as long as this covariance is not violated by the imposed boundary conditions [17]. It is in this context that the kinematical considerations that follow are presented.

The remarks that we present are seemingly trivial, but in view of their suggested relevance, we take the liberty of presenting them in this brief essay.

2. Instant-Form Parity Transformation for Weyl and Dirac Spinors

Within the above framework, to define the problem, we recall that in the instant form $(1/2, 0)$ and $(0, 1/2)$ Weyl spinors (for $m \neq 0$ as well as $m = 0$) transform as [13,18]

\begin{align*}
(1/2, 0) : \quad \phi_R(p^\mu) &= \exp \left[ + \varphi \cdot \frac{\sigma}{2} \right] \phi_R(\hat{p}^\mu) , \\
(0, 1/2) : \quad \phi_L(p^\mu) &= \exp \left[ - \varphi \cdot \frac{\sigma}{2} \right] \phi_L(\hat{p}^\mu) .
\end{align*}

(1)

In Eqs. (1), $p^\mu$ represents the four-momentum of the particle and $\hat{p}^\mu$ corresponds to the particle at rest. The boost parameter $\varphi$ that appears in Eq. (1) is defined as

\begin{align*}
\cosh(\varphi) &= \gamma = \frac{1}{\sqrt{1 - v^2}} = \frac{E}{m} , \\
\sinh(\varphi) &= v \gamma = \frac{|p|}{m} , \quad \hat{\varphi} = \frac{p}{|p|} , \quad \varphi = |\varphi| ,
\end{align*}

(2)

with $p$ the three-momentum of the particle. It is immediately obvious from Eqs. (1) and (2) that under the operation parity, $\mathcal{P}$, $(1/2, 0)$ and $(0, 1/2)$ representation spaces get interchanged,

\begin{align*}
\mathcal{P} : \quad (1/2, 0) &\leftrightarrow (0, 1/2) .
\end{align*}

(3)

As is well known, it is because of the result (3) that any parity-covariant description involving a spin-1/2 system must include both the $(1/2, 0)$ and $(0, 1/2)$ Weyl spinors. One of the easiest, and
most familiar, way to incorporate the result (3) into a theory that includes spin-1/2 fermions is to introduce the \((1/2, 0) \oplus (0, 1/2)\) Dirac spinor, which in the chiral representation (argument \(p^\mu\) of spinors in the chiral representation are enclosed in curly brackets) reads

\[
\psi\{p^\mu\} = \begin{pmatrix}
\phi_R(p^\mu) \\
\phi_L(p^\mu)
\end{pmatrix}.
\]

These results are well known and can indeed be found in any modern textbook on quantum field theory [18–20]. For the sake of later reference, let’s note that the familiar [21] canonical representation is defined as (argument \(p^\mu\) of spinors in the canonical representation are enclosed in square brackets)

\[
\psi[p^\mu] = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & 1 \\
1 & -1
\end{pmatrix} \psi\{p^\mu\},
\]

where \(1\) is a \(2 \times 2\) identity matrix. Under the operation of parity operator \(S(\mathcal{P}) = \gamma^0\) in the \((1/2, 0) \oplus (0, 1/2)\) representation space, the particle and antiparticle spinors, in the usual notation of Refs. [3,21], transform as

\[
u_\sigma[p'^\mu] = + \gamma^0 \nu_\sigma[p^\mu],
\]

\[
u_\sigma[p'^\mu] = - \gamma^0 \nu_\sigma[p^\mu].
\]

The \(p'^\mu\) is the parity-transformed \(p^\mu\).

### 3. Front-Form Parity Transformation for Weyl and Dirac Spinors

The front-form counterpart of the simple and important instant-form relation (3), and other equations such as (3), is a little subtler. To see this, note that the counterpart of transformation properties of the Weyl spinors in the front-form of evolution associated with \(x^+ = x^0 + x^3\) reads (as was recently shown in Ref. [3])

\[
(1/2, 0)^{[x^+]} : \phi_R^{[x^+]}(p^\mu) = \exp \left[ + \beta \cdot \frac{\sigma}{2} \right] \phi_R^{[p^\mu]}(\tilde{p}^\mu), \]

\[
(0, 1/2)^{[x^+]} : \phi_L^{[x^+]}(p^\mu) = \exp \left[ - \beta^* \cdot \frac{\sigma}{2} \right] \phi_L^{[p^\mu]}(\tilde{p}^\mu).
\]
The superscript \([x^+]\) in the above equations serves the purpose of reminding that these relations hold true for the evolution along \(x^+\). The \(\sigma\) are the standard Pauli matrices. The boost parameter \(\beta\) that appears in Eq. (7) is defined \([4]\) as:

\[
\beta = \eta (\alpha v^r, -i \alpha v^r, 1) ,
\]

where \(\alpha = [1 - \exp(-\eta)]^{-1}\), \(v^r = v_x + iv_y\) (and \(v^\ell = v_x - iv_y\)). In terms of the front-form variable \(p^+ \equiv E + p_x\), one can show that

\[
\cosh(\eta/2) = \Omega \left(p^+ + m\right) , \quad \sinh(\eta/2) = \Omega \left(p^+ - m\right) ,
\]

with \(\Omega = [1/(2m)] \sqrt{m/p^+}\). Under the operation of parity, \(\mathcal{P}\), an inspection of Eqs. (7), indicates that \((1/2, 0)\) and \((0, 1/2)\) representation spaces do not get interchanged;

\[
\mathcal{P} : \quad (1/2, 0)^{[x^+]} \not\leftrightarrow (0, 1/2)^{[x^+]} .
\]

In fact we ourselves had failed to notice this in Ref. [4]. (See the paragraph after Eq. (15) of Ref. [4]. However, the results obtained in that article remain unaffected.)

To gain some physical understanding of this result we note that under the operation of parity, \(\mathcal{P}\), unlike the instant-form direction of evolution \(t = x^0\), the front-form direction of evolution that we picked above, \(x^+\), gets interchanged with \(x^-\); and we should therefore obtain counterparts of \((7)\) and \((10)\) for the evolution along the parity-transformed \(x^+\), that is \(x^-\), and see how the Weyl spinors transform. Algebraically this exercise is not trivial, but it parallels our previous analysis of Ref. [4]. Here we just quote the result of our calculations. We find that the \((1/2, 0)\) and \((0, 1/2)\) Weyl spinors in the front form of evolution associated with \(x^-\) direction transform as follows

\[
(1/2, 0)^{[x^-]} : \quad \phi^{[x^-]}_K(p^\mu) = \exp \left[+\beta^* \cdot \frac{\sigma}{2} \right] \phi^{[x^-]}_K(p^\mu) ;
\]

\[
(0, 1/2)^{[x^-]} : \quad \phi^{[x^-]}_L(p^\mu) = \exp \left[-\beta \cdot \frac{\sigma}{2} \right] \phi^{[x^-]}_L(p^\mu) ;
\]

and under the operation of parity, \(\mathcal{P}\), \((1/2, 0)\) and \((0, 1/2)\) representation spaces do not get interchanged \([4]\).

\[\text{2The superscript } [x^-] \text{ in the above equations serves the purpose of reminding that these relations hold true for the evolution along } x^- \text{.}\]
\[ \mathcal{P} : \ (1/2, 0)^{[x^-]} \not\leftrightarrow \ (0, 1/2)^{[x^-]} . \]  

Comparison of transformation properties (8) and (11) yields the front-form counterpart of the instant-form relation (3),

\[ \mathcal{P} : \begin{cases} 
(1/2, 0)^{[x^+]} \leftrightarrow (0, 1/2)^{[x^-]} \\
(0, 1/2)^{[x^+]} \leftrightarrow (1/2, 0)^{[x^-]} 
\end{cases} . \]  

(13)  

Thus, under the operation of parity, the representation space \((1/2, 0)^{[x^+]} \oplus (0, 1/2)^{[x^+]}\) maps one-to-one onto \((1/2, 0)^{[x^-]} \oplus (0, 1/2)^{[x^-]}\). To be more explicit, one may carry out an exercise similar to the one presented in our recent work [4] and obtain the \(u_\mu^{[x^-]}[p'^\mu]\) and \(u_\mu^{[x^-]}[p'^\mu]\) spinors in the \((1/2, 0)^{[x^-]} \oplus (0, 1/2)^{[x^-]}\) representation space. To obtain \(u_\mu^{[x^-]}[p'^\mu]\) and \(u_\mu^{[x^-]}[p'^\mu]\) the reader may find it convenient to first rewrite (14) as

\[ (1/2, 0)^{[x^-]} : \ \phi^R_{[x^-]}(p'^\mu) = \exp \left[ - \beta^* \cdot \frac{\sigma}{2} \right] \phi^R_{[x^-]}(\hat{p}'^\mu) , \]

\[ (0, 1/2)^{[x^-]} : \ \phi^L_{[x^-]}(p'^\mu) = \exp \left[ + \beta \cdot \frac{\sigma}{2} \right] \phi^L_{[x^-]}(\hat{p}'^\mu) , \]  

(14)  

so that the arguments on the left hand side are the parity transformed \(p'^\mu\). We already have the explicit expressions for the \((1/2, 0)^{[x^+]} \oplus (0, 1/2)^{[x^+]}\) spinors, \(u_h^{[x^+]}[p'^\mu]\) and \(u_h^{[x^+]}[p'^\mu]\), from Ref. [4]. The parity operator, \(S(\mathcal{P}) = \gamma^0\), remains unaltered in going from the instant form to the front form because Melosh transformation, as was explicitly proved in [4], does not mix particle and antiparticle spinors, and the front-form \((1/2, 0) \oplus (0, 1/2)\) spinors turn out to be the superposition of the instant-form spinors with \(p'^\mu\)-dependent coefficients contained in the Melosh matrix for spin-\(1/2\). The above indicated exercise yields the front-form counterpart of the identities (8),

\[ u_\mu^{[x^-]}[p'^\mu] = + \gamma^0 u_h^{[x^+]}[p'^\mu] , \]

\[ v_\mu^{[x^-]}[p'^\mu] = - \gamma^0 v_h^{[x^+]}[p'^\mu] . \]  

(15)  

The \(h\) and \(\mu\) in the above expressions correspond to the helicity degrees of freedom associated with the front-form helicity operators (respectively associated with evolution along \(x^+\) and \(x^-\)):

\[ \mathcal{J}_3^{[x^+]} \equiv J_3 + \frac{1}{P_-} (G_1 P_2 - G_2 P_1) , \]  

\[ \mathcal{J}_3^{[x^-]} \equiv J_3 + \frac{1}{P_+} (D_1 P_2 - D_2 P_1) . \]  

(16)  

(17)
For dynamical significance and the definition of various generators involved in the above expressions, we refer the reader to Sec. II of Ref. [4]. The non-trivial space-time structure of the results we obtain, such as Eqs. (15), is intuitively satisfactory.

4. Concluding Remarks

The main result of this paper is to show that under the operation of parity the front-form \((1/2, 0)\) and \((0, 1/2)\) Weyl spinors (massive or massless) do not get interchanged. As a consequence, if we only consider \(x^+\), or equivalently \(x^-\), as the “front-form time” for the front-form evolution then such a description cannot be parity covariant; and any gauge interactions introduced for such a system would necessarily result in a parity-violating dynamical system (cf. comments made towards the end of the Introduction). Therefore, if a front-form theory containing \((1/2, 0) \oplus (0, 1/2)\) representation space has to be parity covariant then one must study the evolution of a physical system not only along \(x^+\) but also along the \(x^-\) direction.

Precisely how such a evolution is to be implemented is not yet fully clear. However, we venture a few preliminary remarks. Following the pioneering work of Chang, Root and Yan [22] we know that only half of the degrees of freedom associated with the field \(\Psi^{[x^+]}(x)\) constructed from \(u_h^{[x^+]}[p^\mu]\) and \(v_h^{[x^+]}[p^\mu]\) are dynamical. Similarly, only half of the degrees of freedom associated with the field \(\Psi^{[x^-]}(x)\), constructed from \(u_\mu^{[x^-]}[p^\nu]\) and \(v_\mu^{[x^-]}[p^\nu]\), are dynamical. Under the operation of parity, using (15), we get: \(\Psi^{[x^+]}(x) \leftrightarrow \Psi^{[x^-]}(x)\). Each of the \(\Psi(x)\) field carries two independent dynamical degrees of freedom (related via operation of parity). Since parity covariance demands that we include \(\Psi^{[x^+]}\) as well as \(\Psi^{[x^+]}\) in the theory we have four (same as in instant form) dynamical degrees of freedom.

In the context of these tentative remarks on loss of degrees of freedom we note that many physicists feel uncomfortable with the halving of degrees of freedom in the front-form of field theories. Already in 1963, a decade before front-form field theory became popular with nuclear physicists, Penrose [see Ref. [23], p. 235, footnote # 15] in a similar context had noted “This explanation leaves something to be desired, however. ... There is probably a subtler reason for this halving of
the initial data functions.” As a result of our analysis, we now find an indication that there may be no halving of the degrees of freedom in the front form of field theories.

Finally, a very different perspective on quantum field theories in the front form and our unexpected results can be gained by realising that light-like surface is a characteristic surface of the Klein-Gordon and Dirac equations. As a result, as Ligterink and Bakker [23] note “The mathematical theory of partial differential equations tells us some of the strange effects which might be expected if we use a light-like surface as boundary on which we define the initial field ... . The solutions of differential equations have a number of special properties, often independent of the fact that we have massive or massless fields... .”

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3After this work was completed, Jacob sent us a preprint [24] where he shows that the result (15), obtained by us, is essential for front-form quantization, which involves specification of a system on both $x^+ = 0$ and $x^- = 0$ surfaces.
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