A note on the Prodi-Serrin conditions for the regularity of a weak solution to the Navier-Stokes equations

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Abstract - The paper is concerned with the regularity of weak solutions to the Navier-Stokes equations. The aim is to investigate on a relaxed Prodi-Serrin condition in order to obtain regularity for \( t > 0 \). The most interesting aspect of the result is that no compatibility condition is required to the initial data \( v_0 \in J^2(\Omega) \).

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1 Introduction

We consider the 3-d Navier-Stokes initial boundary value problem:

\[
\begin{align*}
\frac{\partial v}{\partial t} + v \cdot \nabla v + \nabla \pi v &= \Delta v, \quad \nabla \cdot v = 0, \quad \text{in } (0, T) \times \Omega, \\
v &= 0 \text{ on } (0, T) \times \partial \Omega, \\
v_0 = v(0) &= v_0 \text{ on } \{0\} \times \Omega.
\end{align*}
\] (1)

In system (1) \( v \) is the kinetic field, \( \pi v \) is the pressure field. We set \( b_k := \frac{\partial b}{\partial x_k} \) and \( b \cdot \nabla d := b_k \frac{\partial d}{\partial x_k} \). In order to highlight the main ideas we assume: \( \Omega \subseteq \mathbb{R}^3 \) smooth bounded or exterior domain, zero body force and homogeneous boundary data.

The symbol \( \mathcal{C}_0(\Omega) \) stands for the subset of \( C_0^\infty(\Omega) \) whose elements are divergence free. We set \( J^2(\Omega):= \text{completion of } \mathcal{C}_0(\Omega) \) with respect to the \( L^2 \)-norm, and \( J^{1,2}(\Omega):= \text{completion of } \mathcal{C}_0(\Omega) \) with respect to the \( W^{1,2}(\Omega) \)-norm. We set \( (u, g)_D := \int_D u \cdot g \, dx \), and in the case of \( D \equiv \Omega \) we drop the subscript \( D \).

Following Prodi [24], we set

**Definition 1** Assuming \( v_0 \in J^2(\Omega) \), a field \( v : (0, \infty) \times \Omega \to \mathbb{R}^3 \) is said a weak solution to problem (1) if

i) for all \( T > 0 \), \( v \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; J^{1,2}(\Omega)) \),

ii) \( \lim_{t \to 0} \| v(t) - v_0 \|_2 = 0 \),

iii) for all \( t, s \in (0, T) \) the field \( v \) satisfies the equation:

\[
\int_s^t \left[ (v, \varphi_t) - (\nabla v, \nabla \varphi) + (v \cdot \nabla \varphi, v) \right] \, d\tau + (v(s), \varphi(s)) = (v(t), \varphi(t)),
\]

for all \( \varphi \in H'(\Omega_T) \).

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where the test functions set is defined as
\[ \mathcal{W}(\Omega_T) := \{ \varphi \in C(0, T; J^1(\Omega)), \ \text{with} \ \varphi_t \in L^2(0, T; L^2(\Omega)), \] and \( \varphi = 0 \) in neighborhood of \( T \}(PUNTO). \]

The following existence result holds:

**Theorem 1** For any \( v_0 \in J^2(\Omega) \) there exists a weak solution to problem (1) such that
\[
\|v(t)\|_2^2 + 2 \int_0^t \|\nabla v(\tau)\|_2^2 d\tau \leq \|v_0\|_2^2, \text{ for all } t > 0, \] (2)

and \((v(t), \psi) \in C([0, T])\) for all \( \psi \in J^2(\Omega)\).

The above existence result is due to Hopf in [15]. Inequality (2) is called energy inequality in weak form. It is different from the following one
\[
\|v(t)\|_2^2 + 2 \int_s^t \|\nabla v(\tau)\|_2^2 d\tau \leq \|v(s)\|_2^2, \text{ for all } t > s, \text{ a.e. in } s > 0 \text{ and for } s = 0, \] (3)
called energy inequality in strong form, and from the following one, which is a localized form of (2) and (3), that we state for the Cauchy problem
\[
\int_{\mathbb{R}^3} |v(t)|^2 \phi(t) dx + 2 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |\nabla v(\tau)|^2 \phi dxd\tau \leq \int_{\mathbb{R}^3} |v(s)|^2 \phi(s) dx \] + \int_{s}^{t} \int_{\mathbb{R}^3} |v|^2 (\phi_\tau + \Delta \phi) dx d\tau + \int_{s}^{t} \int_{\mathbb{R}^3} (|v|^2 + 2\pi_v) v \cdot \nabla \phi dxd\tau, \] (4)
for all \( t \geq s \), for \( s = 0 \) and a.e. in \( s > 0 \), and for all nonnegative \( \phi \in C_0^\infty(\mathbb{R} \times \mathbb{R}^3)\).

Inequality (3) is due to Leray in [20] for the Cauchy problem. Subsequently, in the case of an IBVP, we have the energy inequality in strong form for solutions in \( \Omega \) bounded considering again a Hopf weak solution but constructed by means of the Heywood’s device [14]. For exterior domains and more in general for unbounded domains there are several contributions, see for example [23, 9]. Inequality (4) is due to Caffarelli, Khon and Nirenberg in [2]. To date, in unbounded three dimensional domains the energy inequality in strong form and in localized form is not proved for the solutions furnished by Hopf’s technique, [15]. Moreover, regardless of the kind of weak solution, it is not known if the energy inequality in strong form holds for \( \Omega \subset \mathbb{R}^n \), \( n > 4 \) and \( \Omega \) unbounded domain. However the result claimed in Theorem 1, or its variant in [20] and in [2], is the unique existence result at disposal for arbitrary data.

It is known that the regularity of a weak solution to problem (1) is an open question (see e.g. [20, 19, 2]). In the interval between the two essays [20] and [2], Prodi and Serrin, in the papers [24] and [26, 27], respectively, introduce the idea of searching for sufficient conditions in order to obtain the energy equality, the uniqueness and the regularity of a weak solution. This approach translates into extra assumptions that are apt to obtain the
well posedness of the problem. A well known and classical result concerns the regularity and the uniqueness:

if \( v \) is a weak solution and \( v \in L^\rho(0,T;L^\sigma(\Omega)) \), then \( v \) is smooth and unique,

provided that \( \frac{3}{\sigma} + \frac{2}{\rho} = 1 \) with \( \sigma \in (3,\infty) \), \( \rho \in [2,\infty) \). A proof of this result is given by Giga in [13] both in the cases of the Cauchy problem and IBVP with \( \Omega \) bounded. In arbitrary domains the problem is considered by Galdi and Maremonti in [12]. The limit case \( L^\infty(0,T;L^3(\Omega)) \) is not considered and is studied by Escurazia, Seregin and Sverák in [7]. In [17], Kim and Kozono establish interior regularity considering Lorentz spaces in place of Lebesgue spaces. In this connection a further contribution is given by Bosia, Pata and Robinson in [1]. There exists a wide literature on sufficient conditions for the regularity of a weak solution. A possible key tool to obtain these results is the one based on the mild solutions to the integral equation associated to problem (1). They are established on the wake of the ones due to Kato in [16] and to Giga in [13]. In this connection see the paper [8], where Farwig provides an interesting review of the state of the art on the problematic and on the techniques. Different assumptions, closely connected with the one by Prodi and by Serrin, are considered in a series of papers. An interesting update on the topic is given in the recent paper by Tran and Yu [30].

In [2], a new highlight on the Prodi-Serrin ideas is given by Caffarelli, Khon and Nirenberg. They detect that a solution \( v \) and the condition \( v \in L^\rho(s,t;L^\sigma(D)) \) \((\frac{3}{\sigma} + \frac{2}{\rho} = 1)\) suitably satisfy a scaling invariance property. That is if \( v \) is a solution then for all \( \lambda > 0 \) we get that \( v_\lambda(t,x) := \lambda v(\lambda^2 t, \lambda x) \) is still a solution, and \( \|v_\lambda\|_{L^\rho(s,t;L^\sigma(D))} \equiv \|v\|_{L^\rho(s,t;L^\sigma(D))} \) independently of the domain \((s,t) \times D\). It is just the case to recall that the kind of scaling invariant norm is connected with dimensional balance of equation (1). In [2] by means of Proposition 1 and Proposition 2 the authors realize local (that is on a space-time parabolic neighborhood of points) estimates in norms which are scaling invariant. Then they obtain new sufficient conditions for the regularity proving partial regularity for a suitable weak solution with an initial data in \( J^2(\Omega) \). In this connection, starting from an idea already contained in [2], in the recent paper [6] Crispo and Maremonti detect that the bound of the metrics employed in [2] for a weak solution is ensured by some weighted norms, whose advantage is that they hold for all \( t > 0 \) provided that the initial data satisfies suitable assumptions.

Hence the possibility of realizing the regularity seems connected with the existence of scaling invariant norms of a weak solution \( v \) on \((0,T) \times \Omega\).

The above considerations on the scaling invariant metrics have generated, explicitly or tacitly, a way of thinking for which in order to obtain regularity for a weak solution also the initial data \( v_0 \) has to belong to some function space which is scaling invariant. As a matter of course this leads to consider initial data in more regular spaces than \( J^2(\Omega) \). Conversely, it is natural to inquire about the compatibility between an initial data \( a \ priori \) in \( J^2(\Omega) \) and the regularity of solutions for \( t > 0 \). In other words one questions if it is well posed the following

**Definition 2** We say that a weak solution \( u \) is a regular solution to problem (1) if for all \( \varepsilon > 0 \) and \( T > \varepsilon \) we have \( u \in L^\infty((\varepsilon,T) \times \Omega) \).

An analogous question can be posed on the extra assumption related to a weak solution
\( v \) of Theorem 1 in order to obtain the energy equality:

\[
\|v(t)\|_2^2 + 2 \int_s^t \|\nabla v(\tau)\|_2^2 d\tau = \|v(s)\|_2^2, \quad \text{for all } t > s \geq 0. \tag{5}
\]

In [24], under the extra assumption \( v \in L^4(0,T;L^4(\Omega)) \) Prodi proves the energy equality. More recently, in [3] and [4] the assumptions are different. In [3] Cheskidov, Friedlander and Shvydkoy assume \( \int_0^T \|A^{\frac{1}{2}} u(t)\|_3^3 dt \), where \( A \) is the Stokes operator. In [4] Cheskidov, Costantin, Friedlander and Shvydkoy consider the assumption \( v \in L^3(0,T;B_{1,3}^{\frac{3}{2}}(\mathbb{R}^3)) \), where \( B_{1,3}^{\frac{3}{2}} \) is the Besov space. Finally in [8] Farwig assumes \( \int_0^T \|A^{\frac{1}{2}} v(t)\|_3^3 dt \). These assumptions have the same scaling, in particular we get \( v \in L^3(0,T;L^9(\Omega)) \) which furnishes \( \frac{3}{2} + \frac{2}{3} = \frac{4}{3} > \frac{3}{4} + \frac{1}{2} \), the last one being the Prodi condition. In any case these conditions are not compatible on \((0,T)\) with an initial data in \( J^2(\Omega) \).

The main goal of this note is to prove the following results related to the regularity and to the energy equality, and their compatibility with the initial data in \( J^2(\Omega) \).

**Theorem 2** Assume that the weak solution \( v \) of Theorem 1 satisfies the condition:

for all \( \varepsilon > 0 \) \( v \in L^p(\varepsilon, T; L^q(\Omega)) \) with \( \frac{3}{q} + \frac{2}{p} = 1 \), \( p < \infty \), \( \tag{6} \)

then \( v \) is a regular solution, and

\[
(t - \varepsilon)\|v_0\|_2 \leq B(\|v_0\|_2, \varepsilon, t), \quad \text{for all } t > \varepsilon, \tag{7}
\]

where \( B(\|v_0\|_2, \varepsilon, t) := c \|v_0\|_2 \exp[t - \varepsilon + c \int_\varepsilon^t \|v(\tau)\|_2^2 d\tau] \), and \( c \) is a constant independent of \( v \) and \( \varepsilon \).

For the energy equality we do not consider the conditions furnished in [3, 4, 8]. We limit ourselves to prove

**Theorem 3** Assume that the weak solution \( v \) of Theorem 1 satisfies the condition:

for all \( \varepsilon > 0 \) \( v \in L^p(\varepsilon, T; L^q(\Omega)) \), \( \tag{8} \)

then for \( v \) the energy equality (5) holds. In particular we get that \( v \in C([0,T); J^2(\Omega)) \).

Assumptions (6) and (8) are a weak form of Prodi-Serrin conditions that yield the analogous results.

By interpolation of the spaces \( L^\infty(0,T;L^2(\Omega)) \) and \( L^\infty((\varepsilon, T) \times \Omega) \) all regular solutions belong to \( L^p(\varepsilon, T; L^q(\Omega)) \) with \( \frac{3}{2} + \frac{2}{p} = 1 \), therefore the set of regular solutions is characterized by means of the extra condition (6).

Via property (7) a regular solution is a classical solution for \( t > 0 \) (see e.g. [26] and also [25]).

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1 For more general domains see also [10].

2 Concerning the energy equality, also in connection with the one in local form (4), an interesting analysis can be found in [22].
We remark that, setting $L^\sigma := \text{completion of } C_0$ in the Lorentz space $L(\sigma, \infty)$, we can consider the assumption $v \in L^p(\varepsilon, T; L^\sigma(\Omega))$ which is a weaker spatial assumption, close to the one employed in [17]. It is important to point out that, by interpolation, for $\sigma \in [3, 6]$ the assumption $v \in L^\sigma$ is automatically satisfied.

We stress that for a weak solution of Theorem 1 condition (6) furnishes the regularity in the sense of Definition 2, but we are not able to prove that assumption (6) also implies uniqueness. This makes the difference with the Prodi-Serrin condition which ensures both the properties.

*Mutatis mutandis* the notion of regular solution can be also given for solutions $u$ to the Stokes problem (that is (1) dropping the convective term). In this case, if $u_0 \in L^p(\Omega)$, then it is possible to give a behavior in $t = 0$ of the $L^q$-norm of the solutions, $q \geq p$. Actually, we get $\lim_{t \to 0} \| t^{\frac{2}{q} - \frac{2}{p}} u(t) \|_q = 0$ and $\| u(t) \|_q \leq c \| u_0 \|_p t^{-\frac{2}{q} + \frac{2}{p} - \frac{1}{2}}$, with $c$ independent of $u_0$. The case of data in $L(p, \infty)(\Omega)$ is different. There the above $L^p - L^p$ properties are admissible for initial data $u_0 \in L(p, \infty)$ (see e.g. [21]). In the two dimensional case the definition of regular solution is well posed (see Ladyzhenskaya [19]). In this case we also get behavior in $t = 0$ of the solution for $q \geq p = 2$. This is consequence of the estimate $\| \nabla v(t) \|_2 \leq \| v_0 \|_2 \exp(c \| v_0 \|_2) t^{-\frac{1}{2}}$, the energy equality and the Gagliardo-Nirenberg inequality. In [18], Kozono-Yamazaki furnish a generalization of Ladyzhenskaya’s result. They assume $v_0 \in L(2, \infty)$ and divergence free, but in this case no result about the behavior of the solution $v$ in a neighborhood of $t = 0$ is known. In particular the estimate for $q = p = 2$ does not hold. We stress that in all the listed cases of regular solutions $u$ the uniqueness holds.

The plain of the note is the following. In sect. 2 we prove Theorem 3. This is done by means of the properties of solutions to a suitable linearized Navier-Stokes problem. In particular we deduce the energy equality. Out of respect for G. Prodi, our proof is developed following in the first step the argument lines given in [24]. Sect. 3 is devoted to some auxiliary results. Finally, in sect. 4 we give the proof of Theorem 2. In doing this we partially follow the argument lines by Galdi in [11], that follows in turn the ones by Galdi and Maremonti in [12].

## 2 An improvement of the Prodi result: the energy equality.

Fundamental for our aims is the study of the linearized Navier-Stokes problem:

\[
\begin{align*}
    w_t + a_r \cdot \nabla w + \nabla \pi_w &= \Delta w, \quad \nabla \cdot w = 0, \quad \text{in } (0, T) \times \Omega, \\
    w &= 0 \quad \text{on } (0, T) \times \partial \Omega, \quad w = w_0 \quad \text{on } \{0\} \times \Omega, \\
\end{align*}
\]

(9)

where for the coefficient $a_r$ the subscript $r$ ranges between 1 or 2. Respectively, the coefficients enjoy the following integrability properties:

\[
\begin{align*}
    a_1 &\in L^2(0, T; J^{1,2}(\Omega)) \cap L^4(\varepsilon, T; L^4(\Omega)), \\
    a_2 &\in L^2(0, T; J^{1,2}(\Omega)) \cap L^6(\varepsilon, T; L^6(\Omega)), \\
\end{align*}
\]

(10)

Roughly speaking, assuming the coefficients only in $L^2(0, T; J^{1,2}(\Omega))$ the weak solutions to the linearized problem (9) reveal the same difficulties in order to prove either an
energy equality relation and uniqueness of the solutions. We are interested in both the questions, because we reduce the proof of Theorem 2 to the study of a suitable linearized problem. For these aims it is also crucial to consider the mollified system

\[
\begin{align*}
  w_t + J_n [a] \cdot \nabla w + \nabla \pi_w &= \Delta w, \quad \nabla \cdot w = 0, \quad \text{in } (0, T) \times \Omega, \\
  w &= 0 \text{ on } (0, T) \times \partial \Omega, \quad w = w_0 \text{ on } \{0\} \times \Omega,
\end{align*}
\]

(11)

where, for all \( n \in \mathbb{N} \), \( J_n [\cdot] \) is the time-space Friderichs mollifier, and \( a \in L^2(0, T; J^{1,2}(\Omega)) \), and it is extended to 0 for \( t < 0 \).

The following result holds (cf. Solonnikov [29]):

**Theorem 4** For all \( \phi_0 \in J^{1,2}(\Omega) \) there exists a unique solution \((\phi, \pi_\phi)\) to problem (11) such that \( \phi \in C(0, T; J^{1,2}(\Omega)) \) \( \cap \) \( L^2(0, T; J^{2,2}(\Omega)) \) and \( \phi_t, \nabla \pi_\phi \in L^2(0, T; L^2(\Omega)) \). Moreover, uniformly with respect to \( n \in \mathbb{N} \), the energy equality holds:

\[
\| \phi(t) \|^2 + \int_s^t \| \nabla \phi(\tau) \|^2 d\tau = \| \phi(s) \|^2, \quad \text{for all } t > s \geq 0.
\]

(12)

In order to work with the weak solutions to problem (1), it is better to study a weak formulation of problem (9).

**Definition 3** Assuming \( w_\circ \in J^2(\Omega) \), a field \( w : (0, \infty) \times \Omega \to \mathbb{R}^3 \) is said a weak solution to problem (9) if

i) for all \( T > 0 \), \( w \in L^\infty(0, T; L^2(\Omega)) \) \( \cap \) \( L^2(0, T; J^{1,2}(\Omega)) \),

ii) \( \lim_{t \to 0} \| w(t) - w_\circ \|_2 = 0 \),

iii) for all \( t, s \in (0, T) \) the field \( w \) satisfies the equation:

\[
\int_s^t \left[ (w, \varphi_\tau) - (\nabla w, \nabla \varphi) + (a \cdot \nabla \varphi, w) \right] d\tau + (w(s), \varphi(s)) = (w(t), \varphi(t)),
\]

for all \( \varphi \in \mathcal{W}(\Omega_T) \).

In [24], for all weak solutions to problem (1) Prodi proves the energy equality, that is

\[
\| v(t) \|^2 + 2 \int_0^t \| \nabla v(\tau) \|^2 d\tau = \| v_\circ \|^2, \quad \text{a.e. in } t > 0,
\]

provided that they enjoy the extra condition\(^3\) \( v \in L^4(0, T; L^4(\Omega)) \). Actually, Prodi’s result contains inside a uniqueness theorem for weak solutions to problem (9). The following lemma and related corollary give an improvement of the quoted results because the assumption is relaxed as follows: \( L^4(\varepsilon, T; L^4(\Omega)) \) for all \( \varepsilon > 0 \).

\(^3\)Actually, as was recognized and remarked subsequently in the literature, by means of the extra assumption \( v \in L^4(0, T; L^4(\Omega)) \), the result of the energy equality has \( n \)-dimensional validity, \( n \geq 2 \).
Lemma 1 Assume that $w$ is a weak solution to problem (9) with coefficient $a = a_1$. Assume that, for all $\varepsilon > 0$, $w \in L^4(\varepsilon, T; L^4(\Omega))$, $(w(t), \psi) \in C((0, T))$ for all $\psi \in J^2(\Omega)$, and the energy inequality (in weak form) holds:

$$\|w(t)\|_2^2 + 2 \int_0^t \|\nabla w(\tau)\|^2_2 d\tau \leq \|w_0\|_2^2, \text{ for all } t > 0.$$  \hspace{1cm} (13)

Then the energy equality holds for $w$:

$$\|w(t)\|_2^2 + 2 \int_s^t \|\nabla w(\tau)\|^2_2 d\tau = \|w(s)\|_2^2, \text{ for all } s \geq 0.$$  \hspace{1cm} (14)

Proof. The proof of property (14) is achieved in two steps. In the first step we employ the Prodi technique. We denote by $N$ the set of those $t$ such that $\|\nabla w(t)\|_2 < \infty$ and consider $s \in (0, t)$. We define

$$w^*(\tau, x) := w(\tau, x) - w(t, x) \frac{\tau - s}{t - s} \text{ for } \tau \in [s, t] \text{ and } w^*(\tau, x) = 0 \text{ for } \tau \notin [s, t].$$  \hspace{1cm} (15)

Moreover we set

$$w_\eta(\tau, x) = h(\tau, t)w(t)\frac{\tau - s}{t - s} + J_\eta[J_\eta[w^*]](\tau),$$

where the function $h(\tau, t)$ is a nonnegative smooth cutoff function such that $h(\tau, t) = 1$ for $\tau \leq t$ and $h(\tau, t) = 0$ for $\tau \geq 2t$, and $J_\eta$ is a mollifier. It is easy to check that $w_\eta \in W(\Omega_T)$. So that we use $w_\eta$ as test function in iii) of Definition 3:

$$\int_s^t \left[(w, w_\eta) - (\nabla w, \nabla w_\eta) + (a \cdot \nabla w_\eta, w)\right] d\tau + (w(s), w_\eta(s)) = (w(t), w_\eta(t)).$$  \hspace{1cm} (16)

We evaluate the terms:

$$(w(t), w_\eta(t)) - \int_s^t (w, w_\eta) d\tau =: I_1 + I_2.$$  

We get

$$I_1 = \|w(t)\|_2^2 + (w(t), J_\eta[J_\eta[w^*]](t)).$$

By virtue of the definition (15) of $w^*$, making use of an integration by parts, we get

$$I_2 = -\int_s^t (w(\tau), w(t)(t - s)^{-1} + \frac{\partial}{\partial \tau}J_\eta[J_\eta[w^*]](\tau))d\tau$$

$$= -\int_s^t (w^*(\tau) + w(t)\frac{\tau - s}{t - s}, w(t)(t - s)^{-1} + \frac{\partial}{\partial \tau}J_\eta[J_\eta[w^*]](\tau))d\tau$$

$$= -\frac{1}{2}\|w(t)\|_2^2 - \int_s^t (w^*(\tau), w(t)(t - s)^{-1} + \frac{\partial}{\partial \tau}J_\eta[J_\eta[w^*]](\tau))d\tau$$

$$-(w(t), J_\eta[J_\eta[w^*]](t)) + (t - s)^{-1} \int_s^t (w(t), J_\eta[J_\eta[w^*]](\tau))d\tau.$$
Summing we get

\[ I_1 + I_2 = \frac{1}{2} \|w(t)\|_2^2 - \int_s^t \left( w^*(\tau), \frac{\partial}{\partial \tau} J_\eta[J_\eta[w^*]](\tau) \right) d\tau + \frac{1}{t-s} \int_s^t (w(t), J_\eta[J_\eta[w^*]](\tau) - w^*(\tau)) d\tau \]

\[ = \frac{1}{2} \|w(t)\|_2^2 - H_1(\eta) + H_2(\eta). \]

Recalling the definition of \( w^* \), we obtain the identity

\[ H_1(\eta) = \int_{-\infty}^\infty \left( w(\tau), \int_{-\infty}^\infty \frac{\partial}{\partial h} J_\eta(\tau - h) \int_{-\infty}^\infty J_\eta(h - s) w^*(s) ds dh \right) d\tau \]

\[ = - \int_{-\infty}^\infty \left( \frac{\partial}{\partial h} \int_{-\infty}^\infty J_\eta(\tau - h) w^*(\tau) d\tau, \int_{-\infty}^\infty J_\eta(h - s) w^*(s) ds dh \right) \]

\[ = -\frac{1}{2} \int_{-\infty}^\infty \frac{d}{dh} \|J_\eta[w^*](h)\|_2^2 dh = 0, \text{ for all } \eta > 0. \]

Now we evaluate the limit in \( \eta \to 0 \) of \( H_2(\eta) \). Since \( J_\eta[J_\eta[w^*]] \to w^* \) in \( L^2(0, T; L^2(\Omega)) \), we have

\[ \lim_{\eta \to 0} |H_2(\eta)| \leq \lim_{\eta \to 0} \frac{1}{t-s} \int_s^t \|w(t)\|_2 \|J_\eta[J_\eta[w^*]](\tau) - w^*(\tau)\|_2 d\tau = 0. \]

For the term \((w(s), w_\eta(s))\) we get

\[ (w(s), w_\eta(s)) = (w(s), \int_{-\infty}^\infty J_\eta(s - h) \int_s^t J_\eta(h - \xi) w^*(\xi) d\xi dh) \]

\[ = \frac{1}{2} \|w(s)\|_2^2 + (w(s), \int_{-\infty}^\infty J_\eta(s - h) \int_s^t J_\eta(h - \xi)(w^*(\xi) - w(s)) d\xi dh). \]

Since \((w(s), w^*(\xi)) \in C(s, t)\), and \( \lim_{\xi \to s} (w(s), w^*(\xi) - w(s)) = 0 \), we get

\[ \lim_{\eta \to 0} (w(s), w_\eta(s)) = \frac{1}{2} \|w(s)\|_2^2. \]

Hence the limit for \( \eta \to 0 \) gives

\[ \lim_{\eta \to 0} \left[ (w(t), w_\eta(t)) - \int_s^t \left( w(w_\eta), d\tau - (w(s), J_\eta[J_\eta[w(s)]] \) \right] = \frac{1}{2} \|w(t)\|_2^2 - \frac{1}{2} \|w(s)\|_2^2. \]

We consider the weak limit of \( J_\eta[J_\eta[w^*]] \) in \( L^2(0, T; J^{1,2}(\Omega)) \). Hence recalling definition
(15) of \( w^* \), we get
\[
\lim_{\eta \to 0} \int_s^t (\nabla w, \nabla w_\eta) d\tau = \lim_{\eta \to 0} \int_s^t (\nabla w, \nabla w(t) \frac{d\tau}{\tau} + \nabla J_\eta[w^*]) d\tau = \lim_{\eta \to 0} \int_s^t (\nabla w, \nabla J_\eta[w]) d\tau = \int_s^t \|\nabla w(\tau)\|_2^2 d\tau.
\]

Finally, we evaluate the limit of the nonlinear part. In this limit we employ the assumption of \( a_1 \in L^4(\varepsilon, T; L^4(\Omega)) \). Actually, employing the fact that \( (a_1 \cdot \nabla w, w) = 0 \) almost everywhere in \( t > 0 \), recalling formula (15) and \( s - 2\eta > \frac{2}{3} \), we get
\[
\lim_{\eta \to 0} \int_s^t (a_1 \cdot \nabla w, w_\eta) d\tau = \lim_{\eta \to 0} \int_s^t (a_1 \cdot \nabla w, w_\eta - w) d\tau = \lim_{\eta \to 0} \int_s^t (a_1 \cdot \nabla w, J_\eta[\chi_{[s,t]}w] - w) d\tau,
\]
where \( \chi \) is the characteristic function of the interval \((s, t)\). Applying Hölder’s inequality to last term, we obtain
\[
| \lim_{\eta \to 0} \int_s^t (a_1 \cdot \nabla w, w_\eta) d\tau | \leq \lim_{\eta \to 0} \int_s^t \|a_1\|_4 \|\nabla w\|_2 \|J_\eta[\chi w] - w\|_4 d\tau
\leq \lim_{\eta \to 0} \left[ \int_s^t \|a_1\|_4^4 d\tau \right]^{\frac{1}{4}} \left[ \int_s^t \|\nabla w\|_2^4 d\tau \right]^{\frac{1}{4}} \left[ \int_s^t \|J_\eta[\chi w] - w\|_4^4 d\tau \right]^{\frac{1}{4}}.
\]
Hence we get
\[
\lim_{\eta \to 0} | \int_s^t (a_1 \cdot \nabla w, w_\eta) d\tau | = 0.
\]

Considering each limit for \( \eta \to 0 \) for the corresponding term of (16), we deduce
\[
\|w(t)\|_2^2 + 2 \int_s^t \|\nabla w(\tau)\|_2^2 d\tau = \|w(s)\|_2^2, \text{ for all } t \in N, \text{ and } s > 0.
\]

(17) Since (17) holds for all \( s > 0 \), by virtue of ii) of Definition 3 and the absolute continuity of the integral function, from the above equality we deduce
\[
\|w(t)\|_2^2 + 2 \int_0^t \|\nabla w(\tau)\|_2^2 d\tau = \|w_0\|_2^2, \text{ for all } t \in N.
\]

Now we prove the property for all \( t > 0 \). To this end we prove that for all \( t > 0 \)
\[
\lim_{\xi \to t^+} \|u(\xi) - u(t)\|_2 = 0 \text{ holds.}
\]
We consider problem (11) with coefficient \( J_\eta[\tilde{a}] \), with \( \tilde{a} := a_1(\xi - h, x) \) for \( h \in [t, \xi] \) otherwise \( \tilde{a} = 0 \) for \( h \notin [t, \xi] \). We have \( \tilde{a} \in L^2(0, \xi; J^{1,2}(\Omega)) \).

By virtue of Theorem 4, for all \( n \in N \), we obtain the solution \((\phi_n, \pi_{\phi_n})\). We set \( \phi_n(t, x) := \)
\( \phi_n(\xi - \tau, x) \), for all \( \tau \in [t, \xi] \). Since \( \hat{\phi}_n \) is solution backward in time, substituting \( \hat{\phi}_n \) in i) of Definition 3, and integrating by parts on \((t, \xi) \times \Omega\), we get

\[
(w(\xi), \phi_0) = (w(t), \phi_n(\xi - t)) + \int_t^\xi ((J_n[a_1](\tau) - a_1(\tau)) \cdot \nabla \phi_n, w) d\tau, \text{ for all } n \in \mathbb{N}.
\]

On the other hand, for all \( \xi > t > 0 \) and uniform in \( n \in \mathbb{N} \) estimate (12) ensures

\[
|\phi_n(\xi - t)|_2 \leq \|\phi_0\|_2 \text{ and } \int_t^\xi \|\nabla \phi_n(t - \tau)\|_2^2 d\tau \leq \|\phi_0\|_2^2.
\]

Therefore, applying Hölder’s inequality, we deduce

\[
|(w(\xi), \phi_0)| \leq \|w(t)\|_2 \|\phi_0\|_2 + \left( \int_t^\xi \|J_n[a_1](\tau) - a_1\|_4^2 d\tau \right)^{\frac{1}{2}} \left( \int_0^{\xi-t} \|\nabla \phi(h)\|_2^2 dh \right)^{\frac{1}{2}} \left( \int_t^\xi \|w(\tau)\|_4^4 d\tau \right)^{\frac{1}{4}}
\]

\[
\leq \|\phi_0\|_2 \left( \|w(t)\|_2 + \left( \int_t^\xi \|J_n[a_1](\tau) - a_1\|_4^4 d\tau \right)^{\frac{1}{4}} \left( \int_t^\xi \|w(\tau)\|_4^4 d\tau \right)^{\frac{3}{4}} \right) \text{ for all } n \in \mathbb{N}.
\]

Since \( \phi_0 \) is arbitrary in \( J^{1,2}(\Omega) \), in the limit for \( n \to \infty \) we deduce

\[
\|w(\xi)\|_2 \leq \|w(t)\|_2 \text{ for all } \xi > t > 0.
\]

This last property and the assumption of \((w(t), \psi)\) continuous function of \( t \), ensure that \( w \) is continuous in \( t \) on the right in \( L^2\)-norm, for all \( t \geq 0 \). Let \( t \in (0, T) - N \). Then for all sequence \( \{t_n\} \) which converges to \( t \) from the right hand side we have the limit property

\[
\lim_{t_n \to t} \|w(t_n)\|_2 = \|w(t)\|_2.
\]

Therefore from (18) written for the instant of the sequence \( \{t_n\} \) we deduce (14) for all \( t > 0 \) and \( s = 0 \). After that easily follows (14) for all \( t > s \geq 0 \). □

The following result immediately holds:

**Corollary 1** In the hypotheses of Lemma 1, we get:

\[
\text{for all } T > 0, \ w \in C([0, T; J^2(\Omega))
\]

and if, for some \( s \geq 0 \), \( \|w(s)\|_2 = 0 \) then \( w \) is identically null for all \( t > s \).

The following result proves Theorem 3:

**Corollary 2** Let \( v \) be a weak solution to problem (1). Assume that for all \( \varepsilon > 0 \) we have \( v \in L^4(\varepsilon, T; L^4(\Omega)) \), then for \( v \) the energy equality holds:

\[
\|v(t)\|_2^2 + 2 \int_t^\xi \|\nabla v(\tau)\|_2^2 = \|v(s)\|_2^2, \ t > s \geq 0.
\]

**Proof.** It is enough to apply Lemma 1 considering \( v \) as a weak solution to problem (9) with \( a_1 := v \). □
3 Some auxiliary results

We recall some well known estimates.

**Lemma 2** Assume that $D^2 v \in L^p(\Omega)$ and $v \in W^{1,r}_0(\Omega)$. Then there exists a constant $c$ independent of $v$ such that

\[
\|D^2 v\|_q \leq c \|D^2 v\|_p \|\nabla v\|_2^{1-\lambda}, \quad \text{provided that } \frac{1}{q} = \lambda \left(\frac{1}{p} - \frac{1}{2}\right) + (1 - \lambda) \frac{1}{2},
\]

\[
\|v\|_\infty \leq c \|D^2 v\|_p \|\nabla v\|_2^{\frac{1-\lambda}{\lambda}}, \quad \text{provided that } \lambda = \left(\frac{1}{p} - \frac{2}{n}\right) + (1 - \lambda) \frac{1}{2}.
\]

**Proof.** For $\Omega$ exterior domain estimates (20) are particular cases of the one proved in [6]. For $\Omega$ bounded domain or $\Omega \equiv \mathbb{R}^n$, inequality (20) is a particular case of the well known Gagliardo-Nirenberg inequality.

\[\square\]

**Lemma 3** Assume that $v \in W^{2,2}(\Omega) \cap H^{1,2}(\Omega)$, then

\[
\|D^2 v\|_2 \leq c(\|P\Delta v\|_2 + \|v\|_{L^2(D)}),
\]

(21)

where $D \subseteq \Omega$ is a bounded domain with $\partial(\Omega - D) \cap \partial\Omega = \emptyset$, and the constant $c$ is independent of $v$.

**Proof.** See [15].

**Lemma 4** Let $v$ be as in Lemma 2. Assume that $u \in L^2(\Omega)$ and, for $q \in [2, 6)$, $b \in L^\frac{2q}{q-2}(\Omega)$, then we have

\[
|(b \cdot \nabla v, u)| \leq \frac{1}{3} \|P\Delta v\|_2^2 + \frac{1}{3} |u|^2 + c(\|b\|_{L^\frac{2q}{q-2}}^2 + \|b\|_{L^\frac{2q}{q-2}}^4 \|\nabla v\|_2^2).
\]

(22)

Moreover, if $w \in H^{1,2}(\Omega)$ and $q > 3$, then we have

\[
|(w \cdot \nabla w, b)| \leq c\|w\|_2^2 \|b\|_{L^{\frac{2q}{q-2}}}^2 + \frac{2}{3} \|\nabla w\|_2^2.
\]

(23)

In inequalities (22) and (23) the constant $c$ is independent of $v, u, b$ and $w$.

**Proof.** In the case of (22), applying Hölder’s inequality, we get

\[
|(b \cdot \nabla v, u)| \leq \|b\|_{L^\frac{2q}{q-2}} \|\nabla v\|_q \|u\|_2.
\]

Applying estimate (20) with $p = r = 2$, and subsequently (21) we obtain

\[
|(b \cdot \nabla v, P\Delta v)| \leq \|b\|_{L^\frac{2q}{q-2}} \|\nabla v\|_q \|u\|_2 \leq c\|b\|_{L^\frac{2q}{q-2}} \|\nabla v\|_2^{1-\lambda} \|D^2 v\|_2^\lambda \|u\|_2
\]

\[
\leq c \left[\|b\|_{L^\frac{2q}{q-2}} \|\nabla v\|_2^{1-\lambda} \|P\Delta v\|_2^\lambda \|u\|_2 + \|b\|_{L^\frac{2q}{q-2}} \|\nabla v\|_2^{1-\lambda} \|u\|_2 \|v\|_{L^2(D)}^\lambda\right],
\]

where the exponent $\lambda = \frac{3(q - 2)}{6q}$. Finally, since $D$ is bounded and on $\partial\Omega \cap \partial D$ we have $v = 0$, by means of Poincaré inequality, via the Cauchy inequality, we arrive at (22). Finally, for estimate (23), applying Hölder’s inequality, and the Gagliardo-Nirenberg inequality we easily obtain

\[
|(w \cdot \nabla w, b)| \leq c\|w\|_{L^\frac{2q}{q-2}} \|\nabla w\|_2 \|b\|_q \leq c\|w\|_2^{\frac{q-3}{q}} \|\nabla w\|_2^{\frac{q+3}{q}} \|b\|_q.
\]
Hence the Cauchy inequality leads to (23).

The following theorem concerns the existence of the so-called strong regular solutions in the special case of the \( L^2 \)-theory.

**Theorem 5** Let \( v_0 \in J^{1,2}(\Omega) \). The there exists a unique solution to (1) such that

\[
v \in C(0,T;J^{1,2}(\Omega)) \cap L^2(0,T;W^{2,2}(\Omega)), \quad \text{with } v_t, \nabla \pi \in L^2(0,T;L^2(\Omega)),
\]

and, for all \( k \in \mathbb{N}, \ell = 0, 1, 2, \varepsilon > 0 \),

\[
\nabla^\ell D_t^k v \in C(\varepsilon,T;L^2(\Omega)).
\]

**Proof.** This result is a particular case of the one proved in Theorem 3 in [14], see also Chap. V in [28]. □

## 4  Proof of Theorem 2

The proof of Theorem 2 is achieved by means of two lemmas.

**Lemma 5** In the hypotheses of Theorem 2 for all \( \varepsilon > 0 \) we get

\[
v \in C(0,T;J^2(\Omega)) \cap C(\varepsilon,T;J^{1,2}(\Omega)) \cap L^2(\varepsilon,T;W^{2,2}(\Omega))
\]

\[
v_t, \nabla \pi_v \in L^2(\varepsilon,T;L^2(\Omega)),
\]

for arbitrary \( T > \varepsilon > 0 \). Moreover, we get

\[
\|v(t)\|_2^2 + 2 \int_s^t \|\nabla v(\tau)\|_2^2 d\tau = \|v(s)\|_2^2, \text{ for all } t > s \geq 0,
\]

and

\[
(t - \varepsilon)\|\nabla v(t)\|_2^2 + \frac{1}{2} \int_\varepsilon^t (\tau - \varepsilon)(\|P\Delta v(\tau)\|_2^2 + \|v_\tau(\tau)\|_2^2) d\tau
\]

\[
\leq \|v_0\|_2^2 \exp[t - \varepsilon + c \int_\varepsilon^t \|v(\tau)\|_2^2 d\tau] =: A^2(\|v_0\|_2, \varepsilon, t),
\]

with \( c \) independent of \( \varepsilon \) and \( t \).

**Proof.** We consider problem (11) with \( a := v \) and \( w_0 := v_0 \), where \( \{v_0^n\} \subset \mathscr{V}_0(\Omega) \) is a sequence which converges to \( v_0 \) in \( J^2(\Omega) \). By virtue of Theorem 4 we obtain a sequence \( \{w^n\} \) of solutions to problem (11). Now our task is to prove the existence of a limit \( w \) which is a regular solution to problem (9) with \( a_2 \equiv v \). We base the existence of the limit \( w \) by proving for \( w^n \) a bound with respect to the metrics

\[
C(\varepsilon,T;J^{1,2}(\Omega)) \cap L^2(\varepsilon,T;W^{2,2}(\Omega)) \text{ and } w_t, \nabla \pi_w \in L^2(\varepsilon,T;L^2(\Omega)),
\]
for arbitrary \( T > \varepsilon > 0 \) and then uniform in \( n \in \mathbb{N} \) such that \( \varepsilon - \frac{1}{n} > \frac{\varepsilon}{2} \). In particular the limit \( w \) satisfies the inequality

\[
(t - \varepsilon)\| \nabla w(t) \|_2^2 + \frac{1}{\varepsilon} \int_{\varepsilon}^{t} (\tau - \varepsilon) (| P \Delta w(\tau) |_2^2 + | w_\tau(\tau) |_2^2) d\tau
\]

\[
\leq | v_0 |_2^2 \exp[t - \varepsilon + c \int_{\varepsilon}^{t} | v(\tau) | \frac{4 \sigma}{| \nabla w_\tau(\tau) |_{2}^2} d\tau].
\]

By virtue of estimate (12), assumption for \( v \), via Lemma 1 we realize that \( w^n \in C(0, T; J^2(\Omega)) \) for all \( n \in \mathbb{N} \) with

\[
\| w^n(t) \|_2^2 + 2 \int_{s}^{t} \| \nabla w^n(\tau) \|_2^2 d\tau = \| w^n(s) \|_2^2 \leq \| v_0 \|_2^2, \text{ for all } t > s \geq 0.
\]

Now, we look for estimates for the derivatives of \( w^n(t) \). We multiply equation (11) by \( P \Delta w^n \). Integrating on \((0, T) \times \Omega \), we get

\[
\frac{1}{2} \frac{d}{dt} \| \nabla w^n(t) \|_2^2 + \| P \Delta w^n(t) \|_2^2 = (J_n[v](t) \cdot \nabla w^n(t), P \Delta w^n(t)), \text{ for all } t > 0.
\]

Analogously multiplying (11) by \( w^n_\tau \) and integrating on \((0, T) \times \Omega \), we get

\[
\frac{1}{2} \frac{d}{dt} \| \nabla w^n(t) \|_2^2 + \| w^n_\tau(t) \|_2^2 = -(J_n[v](t) \cdot \nabla w^n(t), w^n_\tau(t)), \text{ for all } t > 0.
\]

Applying estimate (22) with \( q \) such that \( \frac{2q}{q-2} = \sigma \) on the right hand side of the above relations with \( u = P \Delta w^n \) in the first relation and with \( u = w^n_\tau \) in the second relation, summing we get

\[
\frac{d}{dt} \| \nabla w^n(t) \|_2^2 + \frac{1}{2} \| P \Delta w^n(t) \|_2^2 + \frac{1}{2} \| w^n_\tau(t) \|_2^2 \leq c(\| J_n[v] \|_\sigma^2 + \| J_n[v] \|_{\sigma'}^2) \| \nabla w^n(t) \|_2^2,
\]

where, via the assumption on \( v \), we have taken \( \frac{2q}{q-2} + \frac{2}{2} = 1 \) into account. Multiplying by \( \tau - \varepsilon \) and integrating over \((\varepsilon, t)\), we deduce the boundness

\[
(t - \varepsilon)\| \nabla w^n(t) \|_2^2 + \frac{1}{\varepsilon} \int_{\varepsilon}^{t} (\tau - \varepsilon) \left[ | P \Delta w^n(\tau) |_2^2 + | w^n_\tau(\tau) |_2^2 \right] d\tau
\]

\[
\leq \exp[t - \varepsilon + c \int_{\varepsilon}^{t} | J_n[v] | \| \nabla w^n(\tau) \|_{2}^2 d\tau].
\]

By virtue of energy relation (31), and of the properties of the mollifier, estimate (32) is true for all \( \varepsilon > 0 \) and uniformly in \( n \in \mathbb{N} \):

\[
(t - \varepsilon)\| \nabla w^n(t) \|_2^2 + \frac{1}{\varepsilon} \int_{\varepsilon}^{t} (\tau - \varepsilon) \left[ | P \Delta w^n(\tau) |_2^2 + | w^n_\tau(\tau) |_2^2 \right] d\tau
\]

\[
\leq \| v_0 \|_2^2 \exp[t - \varepsilon + c \int_{\varepsilon}^{t} \| v(\tau) \| \frac{4 \sigma}{| \nabla w_\tau(\tau) |_{2}^2} d\tau].
\]
Estimates (31) and (33) allow to deduce the existence of a limit \( w \) belonging to (29). Moreover, the limit \( w \) satisfies the energy equality

\[
\|w(t)\|_2^2 + 2\int_t^\infty \|\nabla w(\tau)\|_2^2 \, d\tau = \|w(s)\|_2^2, \text{ for all } t > s \geq \varepsilon > 0
\]

and the energy inequality

\[
\|w(t)\|_2^2 + 2\int_0^t \|\nabla w(\tau)\|_2^2 \, d\tau \leq \|v_0\|_2^2, \text{ for all } t > 0.
\]

Finally, from estimate (33) for all \( t > \varepsilon \) it follows

\[
(t - \varepsilon)\|\nabla w(t)\|_2^2 + \frac{1}{2} \int_{\varepsilon}^t (\tau - \varepsilon) \left[ \|P \Delta w(\tau)\|_2^2 + \|w_\sigma(\tau)\|_2^2 \right] \, d\tau \leq \|v_0\|_2^2 \exp\left[ t - \varepsilon + c \int_{\varepsilon}^t \|v(\tau)\|_2^2 \, d\tau \right].
\]

On the other hand, from integral equation iii) of Definition 3 we deduce that for all \( T > 0 \) \((w(t), \psi) \in C([0, T])\). Hence by Lemma 1 we have that the limit \( w \) also satisfies (14). In our hypotheses on \( v \) weak solution to problem (1), in accord with iii) of Definition 3, we can regard it as a solution to (9) with initial data \( v_0 \) and coefficient \( a_2 \equiv v \). So that for problem (9) with \( a_2 = v \) we have found two solutions corresponding to \( v_0 \): \( w \) and \( v \). By virtue of Corollary 1 they coincide. The proof of the lemma is completed.

\( \square \)

**Lemma 6** In the hypotheses of Theorem 2 for all \( \varepsilon > 0 \) we get \( \nabla^\ell D_\varepsilon^t v \in C(\varepsilon, T; L^2(\Omega)) \). In particular we get

\[
(t - \varepsilon)\|v_\ell\|_2 \leq c \exp \left[ c \int_{\varepsilon}^t \|v(\tau)\|_2^2 \, d\tau \right] A(\|v_0\|_2, \varepsilon, t) \leq B(\|v_0\|_2, \varepsilon, t), \text{ for all } t > \varepsilon. \tag{34}
\]

**Proof.** By virtue of (26), Theorem 5 and the uniqueness of strong regular solutions, we can claim that for all \( k \in \mathbb{N}, \ell = 1, 2 \) and \( \varepsilon > 0 \) there exists a \( T_{\varepsilon} \) such that \( \nabla^\ell D_\varepsilon^t v \in C(\varepsilon, T_{\varepsilon}; L^2(\Omega)) \). We can iterate the above procedure proving that \( \nabla^\ell D_\varepsilon^t v \in C(\varepsilon, T; L^2(\Omega)) \) for all \( T > 0 \). As matter of course this procedure furnishes an estimate of \( \|v_\ell(t)\|_2 \) which is not uniform. Hence in order to prove (34) we differentiate the equation (1)_1 with respect to \( t \), then, multiplying by \((t - \varepsilon)^2v_\ell\) and integrating on \( \Omega \), we get

\[
\frac{d}{dt} \left( (t - \varepsilon)^2\|v_\ell(t)\|_2^2 \right) + 2(t - \varepsilon)^2\|\nabla v_\ell\|_2^2 = -2(t - \varepsilon)^2(\nabla^\ell v_\ell \cdot \nabla v_\ell, v) + 2(t - \varepsilon)^2\|v_\ell\|_2^2.
\]

Applying estimate (23), we deduce the estimate

\[
|(\nabla^\ell v_\ell \cdot \nabla v_\ell, v)| \leq c\|v_\ell\|_2^2\|v\|_2^2 + \frac{1}{4}\|\nabla v_\ell\|_2^2.
\]

Hence applying this inequality on the right hand side of the above differential equation, and integrating on \((\varepsilon, t)\) we get

\[
(t - \varepsilon)^2\|v_\ell(t)\|_2^2 \leq c \exp \left[ c \int_{\varepsilon}^t \|v(\tau)\|_2^2 \, d\tau \right] \int_{\varepsilon}^t (\tau - \varepsilon)\|v_\ell(\tau)\|_2^2 \, d\tau, \text{ for all } t > \varepsilon.
\]

\( ^4 \) We recall that on any finite interval \((\varepsilon, T)\) the coefficient \( a_2 \) in particular enjoys the property of \( a_1 \).
Taking into account (28), we complete the proof. □

Proof of Theorem 2.

The proof of Theorem 2 is an immediate consequence of the above lemmas and of the Sobolev inequality. Actually, by virtue of the above lemmas and the local strong regularity Theorem 5, from equation (1) for all \( \varepsilon > 0 \) and \( t \in (\varepsilon, T) \) we have

\[
\| P \Delta v(t) \|_2 \leq \| P (v_t + v \cdot \nabla v) \|_2 \leq \| v_t \|_2 + \| v \|_\infty \| \nabla v \|_2 \leq \| v_t(t) \|_2 + c \| D^2 v \|_2^\frac{1}{2} \| \nabla v \|_2,
\]

where estimating \( \| v \|_\infty \) we have applied (20) of. By virtue of estimates (28) and (34) the right hand side is independent of \( T_\varepsilon \), and since \( v \in C((\varepsilon, T_\varepsilon); J^{1/2}(\Omega)) \) for all \( \varepsilon > 0 \), we can iterate the last arguments, hence for all \( \varepsilon > 0 \) and \( t > 0 \) we deduce

\[
\| P \Delta v(t) \|_2 \leq \| P (v_t + v \cdot \nabla v) \|_2 \leq \| v_t \|_2 + \| v \cdot \nabla v \|_2 \leq \| v_t(t) \|_2 + c \| D^2 v \|_2^\frac{1}{2} \| \nabla v \|_2.
\]

Hence applying Lemma 5 and the Sobolev inequality, for all \( \varepsilon > 0 \) we get

\[
\| P \Delta v \|_2 \leq \| v_t \|_2 + c \| D^2 v \|_2^\frac{1}{2} \| \nabla v \|_2^\frac{3}{2}, \quad \text{for all } t > \varepsilon,
\]

and by the Cauchy inequality we have for all \( \varepsilon > 0 \)

\[
\| D^2 v \|_2 \leq 2 \| v_t \|_2 + c \| \nabla v \|_2^\frac{3}{2} (\| \nabla v \|_2 + 1), \quad \text{for all } t > \varepsilon.
\]

Finally, via (28) and (34) for all \( \varepsilon > 0 \) we have

\[
(t - \varepsilon) \| P \Delta v \|_2 \leq c A^2 (\| v_0 \|_2, \varepsilon, t) \left[ (t - \varepsilon)^{- \frac{3}{4}} A (\| v_0 \|_2, \varepsilon, t) + 1 \right] + B (\| v_0 \|_2, \varepsilon, t), \quad \text{for all } t > \varepsilon.
\]

Now it is immediate to deduce the thesis. □

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