Can we predict the failure point of a loaded composite material?

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Abstract

As a model of composite material, the fiber bundle model has been chosen - where a bundle of fibers is subjected to external load and fibers have distributed thresholds. For different loading conditions, such a system shows few precursors which indicate that the complete failure is imminent. When external load is increased quasi-statically - bursts (number of failing fibers) of different sizes are produced. The burst statistics shows a robust crossover behavior near the failure point, around which the average burst size seems to diverge. If the load is increased by discrete steps, susceptibility and relaxation time diverge as failure point is approached. When the bundle is overloaded (external load is more than critical load) the rate of breaking shows a minimum at half way to the collapse point. The pattern and statistics of energy emission bursts show characteristic difference for below-critical and over-critical load levels.

Keywords: fiber bundle model, burst distribution, susceptibility, relaxation time, breaking rate, energy burst

1. Introduction

Prediction of the failure point is a major challenge in various scenarios of fracture and breakdown - ranging from fracturing in nano-materials to onset of earthquakes. A material body can tolerate certain level of load or force on it and beyond that level it collapses. If the load is increased continuously - when does the collapse point come ? Is there any precursor which signals that the complete failure is imminent ?

Fiber bundle model (FBM) has been proved to capture the essentials of failures in composite materials. FBM contains a large number of fibers with statistically distributed strength thresholds. It has simple geometry and clear-cut rules for how stress caused by a failed fiber is redistributed on intact fibers. Most importantly, this model can be solved analytically to an extent (For reviews, see [7]) that is not possible for any other model of material-failure. The statistical properties of FBM is well studied [8, 9, 12], and the failure dynamics at a constant load has been formulated [11] through recursion relations which in turn explore the phase transition and associated critical behavior in this model.

In this article we discuss how we can predict the failure point of a loaded FBM from the available precursors. The term precursor usually means some prior indications of an upcoming incident and in current context such incident is the complete failure (collapse) of a fiber bundle under external load. We consider equal-load-sharing model, in which the load previously carried by a failed fiber is shared equally by all the remaining intact fibers [7]. The bundle consisting of $N$ elastic fibers, clamped at both ends (Fig. 1). All the fibers obey Hooke’s law with force constant set to unity for simplicity. Each fiber $i$ is associated with a breakdown threshold $x_i$ for its elongation. When the length exceeds $x_i$ the fiber breaks immediately, and does not contribute to the strength of the bundle thereafter. The individual thresholds $x_i$ are assumed to be independent random variables.

Figure 1: The fiber bundle model of composite materials.
with the same cumulative distribution function \( P(x) \) and a corresponding density function \( p(x) \):

\[
\text{Prob}(x_i < x) = P(x) = \int_0^x p(y) \, dy. \tag{1}
\]

We analyse three different loading cases: quasi-static loading, load increment by equal steps and overloaded situation. For prediction purpose it is important that precursors can be seen in a single system. Therefore, throughout this article we will present and discuss results that can be seen in a single bundle containing large number of fibers. For simplicity, we consider the uniform distribution of fiber thresholds: \( P(x) = x \) for \( 0 \leq x \leq 1 \).

### 2. Quasi-static loading

The quasi-static loading is a strain controlled method, where at each step the whole bundle is stretched till the weakest fiber (among the intact ones) fails. At an elongation \( x \) per surviving fiber the total force on the bundle is \( x \) times the number of intact fibers. The expected or average force at this stage is therefore

\[
F(x) = N \times (1 - P(x)). \tag{2}
\]

The maximum \( F_c \) of \( F(x) \) corresponds to the value \( x_c \) for which \( dF/dx \) vanishes. Thus

\[
1 - P(x_c) - x_c p(x_c) = 0; \tag{3}
\]

where \( x_c \) is the critical elongation value above which the bundle collapses.

#### 2.1. Burst or avalanche of failing fibers

When a fiber fails, the stress on the intact fibers increases. This may in turn trigger further fiber failures, which can produce bursts (avalanches) that either lead to a stable situation or to breakdown of the whole bundle. A burst is usually defined as the amount or number \((\Delta)\) of simultaneous fiber failure during loading. It was shown in \([9]\) that the average number of burst events of size \( \Delta \) follows a power law of the form

\[
D(\Delta)/N = C \Delta^{-\xi} \tag{4}
\]

in the limit \( N \to \infty \). Here, \( \xi = \frac{5}{2} \) is the universal burst exponent and \( C \) is a constant. The value of \( \xi \) is, under mild assumptions, independent of the threshold distribution \( P(x) \); the probability density needs to have a quadratic maximum somewhere within the range of threshold values.

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**Figure 2:** The burst distributions: all bursts (squares) and bursts within an interval \( 0.9 x_c \) and \( x_c \) (circles). The figure is based on a single bundle containing \( N = 10^7 \) fibers with uniformly distributed fiber thresholds within \( 0 \) and 1.

#### 2.2. The crossover behavior

When all the bursts are recorded for the entire failure process, the burst distribution follows the asymptotic power law \( D(\Delta) \propto \Delta^{-5/2} \). If we just sample bursts that occur near the breakdown point, a different power law is seen—-which can be explained analytically \([9, 10]\). Fig. 2 compares the complete burst distribution with what we get when we sample merely bursts in the threshold interval \((0.9 x_c, x_c)\).

This observation may be of practical importance, as it gives a criterion for the imminence of complete failure \([9]\). The bursts or avalanches can be recorded from outside -without disturbing the ongoing failure process. Therefore, any signature in burst statistics that can warn of imminent system failure would be very useful in the sense of wide scope of applicability. It is enticing to note the recent observation \([14]\) of a crossover behavior in the magnitude distribution of earthquakes before the largest earthquake appears. A similar crossover behavior is also seen \([10, 12]\) in the burst distribution and energy distribution of the fuse model which is a standard model for studying fracture and breakdown phenomena in disordered systems. Most important is that this crossover signal does not hinge on observing rare events and is seen in a single system (see Fig. 2). Therefore, such crossover signature has a strong potential to be used as a detection tool.

#### 2.3. Variation of average burst size

We have seen that if external load is increased quasi-statically on a bundle of large number of fibers, bursts of different sizes occur during the whole breaking process till complete failure. One can ask - what is the average burst size at a particular elongation \((x)\) value? The average burst size is indeed a very relevant quantity that can
that survive after step applied stress and of stress redistribution steps. We call by a recursion relation: Let the failure dynamics of the bundle can be represented a bundle by equal amount at each loading step. Here, 3. Load increment by equal steps

**Figure 3:** Average burst size $\Delta_{av}$ vs. elongation $x$ for the same fiber bundle as in Fig. 2

**Figure 4:** Inverse of average burst size is plotted against $x$ for the same data set as in Fig. 3. A straight line can be fitted near $x_c$ from which one can predict the failure point.

be measured easily during the failure process. It seems (Fig. 3) that average burst size( $\Delta_{av}$) around some elongation value $x$ goes as

$$\Delta_{av}(x) \approx (x_c - x)^{0.9}. \quad (5)$$

This means if we plot $\Delta_{av}^{1/0.9}$ vs. $x$, we should get a straight line which touches the $X$ axis at $x = x_c$. Even in a single system we can see this signature (Fig. 4).

### 3. Load increment by equal steps

In the force-controlled case, load can be increased on a bundle by equal amount at each loading step. Here, the failure dynamics of the bundle can be represented by a recursion relation: Let $N_t$ be the number of fibers that survive after step $t$, where $t$ indicates the number of stress redistribution steps. We call $\sigma = F/N$, the applied stress and $U_t = N_t/N$, the surviving fraction of total fibers. Then the effective stress after $t$ step becomes $x_t = \frac{F}{N_t}$ and after $t + 1$ steps the surviving fraction of total fibers will be $U_{t+1} = 1 - P(x_t)$. Therefore we can construct the following recursion relation \[11\]

$$U_{t+1} = 1 - P(\sigma/U_t); U_0 = 1. \quad (6)$$

At equilibrium $U_{t+1} = U_t \equiv U^*$. For uniform fiber strength distribution, the cumulative distribution becomes $P(\sigma/U_t) = \sigma/U$. Therefore the recursion relation becomes

$$U_{t+1} = 1 - \frac{\sigma}{U_t} \quad (7)$$

At the fixed point the above relation takes a quadratic form $U^{2} - U^{*} + \sigma = 0$, with the solutions

$$U^{*}(\sigma) = \frac{1}{2} \pm (\sigma_c - \sigma)^{1/2}; \sigma_c = \frac{1}{4}. \quad (8)$$

One can define the breakdown susceptibility $\chi$, as the change of $U^{*}(\sigma)$ due to an infinitesimal increment of the applied stress $\sigma$:

$$\chi = \left| \frac{dU^{*}(\sigma)}{d\sigma} \right| = \frac{1}{2}(\sigma_c - \sigma)^{-\beta}; \beta = \frac{1}{2}. \quad (9)$$

To study the failure dynamics around the critical point ($\sigma \to \sigma_c$), the recursion relation (Eq. 7) can be replaced by a differential equation

$$-\frac{dU}{dt} = \frac{U^2 - U + \sigma}{U}. \quad (10)$$

Close to the fixed point one gets [11]

$$U_t(\sigma) - U^{*}(\sigma) \approx \exp(-t/\tau), \quad (11)$$

where the relaxation time $\tau = \frac{1}{2} \left[ \frac{1}{2}(\sigma_c - \sigma)^{1/2} + 1 \right]$. Therefore, near the critical point $\tau$ diverges:

$$\tau \propto (\sigma_c - \sigma)^{-\beta}; \theta = \frac{1}{2}. \quad (12)$$

Since the susceptibility ($\chi$) and the relaxation time ($\tau$) follow power laws (exponent $-1/2$) with external stress and both diverge at the critical stress, therefore, if we plot $\chi^{-2}$ and $\tau^{-2}$ with external stress, we expect a linear fit near critical point and the straight lines should touch $X$ axis at the critical stress. We indeed found similar behavior (Fig. 5) in simulation experiments for a single sample.

### 4. The over-loaded situation

What happens if the initial applied load $F = N_0 \sigma$ is larger than the critical load of the bundle ? The step-wise failure process continues and the bundle collapses at some step $t_f$. If we consider the uniform distribution of fiber thresholds, and assume that the load is slightly
above the critical value: \( \sigma = \sigma_c + \epsilon \), with \( \epsilon > 0 \), we can rewrite the recursion relation (Eq. 7) as

\[
U_{t+1} = 1 - \frac{(\sigma_c + \epsilon)}{U_t}.
\]

The solution [17] of the above recursion shows that there is a relation between the minimum of the breaking rate \( R(t) = \frac{dU}{dt} \) (treating \( t \) as continuous variable) and the final step \( t_f \):

\[
t_f = 2t_0.
\]

At the minimum of the breaking rate, the bundle is just halfway to its complete collapse. Simulations on a single sample show that the breaking rate has a minimum at some value \( t_0(\epsilon) \), and that for varying \( \epsilon \) the minima all occur at a value close to \( \frac{1}{2} \) when plotted as function of the scaled variable \( t/t_f \) (Fig. 6).

5. Energy emission bursts of loaded FBM

It is well known that during fracturing process in composite materials, energy releases in the form of acoustic emissions [18, 19] and most of the cases these acoustic bursts follow power laws. Very recently the statistics of energy emission bursts has been studied in FBM [16] both analytically and through numerical simulations. As the fibers obey Hook’s law up to the failure, when a fiber fails at an elongation \( x \), elastic energy of amount \( \frac{1}{2}Kx^2 \) will be released, where \( K \) is the force constant. Therefore for a burst of size \( \Delta \) the corresponding energy release \( E_n \) can be calculated as

\[
E_n = \frac{1}{2}K \sum_{i=\text{min}}^{\Delta+\text{min}} x_i^2,
\]

where \( x_i \) is the strength of failing fiber. Now at a constant applied load, if we record \( E_n \) at each step of load redistribution, it shows different pattern (Fig. 7) depending on the stress level—an critical, over-critical or below-critical. If we record such energy emission bursts separately for below-critical and over-critical levels, the corresponding distributions exhibit convincing difference (Figs. 8, 9). For stresses below-critical level, there are many many small energy bursts which are absent in the later case. The exponent of the energy burst distributions are different for these two situations: \(-1\) and \(-1.5\) respectively.
value around half way to complete collapse. In the over-loaded case, the energy bursts also attain a minimum at half way to the collapse point. When the bundle is overloaded the rate of breakage has a minimum at half way to the collapse point and the stress value without approaching too close to the failure point. Stress value without approaching too close to the failure stress (σf) following robust power laws - from which one can predict the failure stress value without approaching too close to the failure point. The crossover signature can only warn of an imminent failure, it is possible to predict critical elongation value (x_c) in advance by measuring the average burst size. If the load is increased by equal steps, susceptibility and relaxation time seem to diverge at failure stress (σf) following robust power laws - from which one can predict the failure stress value without approaching too close to the failure point. When the bundle is overloaded the rate of breaking has a minimum at half way to the collapse point and the distributions of energy emission bursts follow different power laws for below-critical and over-critical load levels. A very recent study [20] shows that for overloaded case, the energy bursts also attain a minimum value around half way to complete collapse.

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