1. Introduction

In [1], Bloch constructs symbols in $K_2(E)$ for a CM elliptic curve $E$ defined over $\mathbb{Q}$, corresponding to divisors supported on torsion points of the curve. This construction, and the special properties of such curves, allowed him to prove the Beilinson conjecture for such curves. In [2], Deninger extends Bloch’s results, for certain elliptic curves ‘of Shimura type’ or ‘type (S)’. For simplicity assume $E$ has complex multiplication by the ring of integers $\mathcal{O}_K$ of the complex quadratic field $K$, and $E$ is defined over an extension $F$ of $K$. Shimura showed [15, Theorem 7.44] that the following conditions are equivalent, and we will take either of them to mean that $E$ is of type (S).

**Theorem.** $F(E_{\text{tors}})$ is contained in $K^{ab}$ if and only if $F$ is abelian over $K$ and the corresponding Hecke character $\psi$ on the ideals of $F$ factors through the norm map from $F$ to $K$.

This condition is closely related to one considered by Gross in [3]. He calls a CM curve defined over a Galois extension $F$ of $\mathbb{Q}$ a ‘$\mathbb{Q}$-curve’ if it is isogenous over $F$ to all its Galois conjugates. Similarly one defines a $K$-curve via isogeny with all Gal($F/K$) conjugates. If $E$ is type (S) then it is a $K$-curve, since then the Hecke character $\psi$ is clearly Galois invariant and this is an isogeny invariant [3, Proposition (9.1.3)]. Conversely suppose $F$ is abelian over $K$ and $E$ is a $K$-curve defined over $F$. If $\mathcal{O}_K^\times = \{\pm 1\}$ and the 2-Sylow subgroup of Gal($F/K$) is cyclic, then $E$ is type (S) [10, Proposition 2].

The key fact for results about special values of $L$-functions is that if $E$ is of type (S), then $L(s, E)$ factors as a product of $L$ functions of Hecke characters of $K$. In [12], results on the conjecture of Birch and Swinnerton-Dyer were obtained this way.

To explain the Beilinson conjectures in the context of a curve $E$ defined over a number field $F$ in a simple format, we tensor $K_2(E_F)$ with $\mathbb{Q}$ to get a vector space $\mathbb{Q}K_2(E_F)$. Let $\xi_i$ be a basis, and let $\Phi_j$ denote the embeddings of $F$ into $\mathbb{C}$. There are regulator maps for each embedding, described in [2] below (along with some facts about...
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\( K\)-theory). Roughly speaking, the Beilinson conjectures consists of two parts.

1. ‘Dimension conjecture’:
   \[ \dim \mathbb{Q}K_2(E_F) = [F : \mathbb{Q}] \]

2. ‘\( L\)-value conjecture’:
   \[ L(0, E)^{([F : \mathbb{Q}])} \approx_{\mathbb{Q}^\times} \det[\text{reg}(\xi_i)_{\Phi_j}] \]

Note that for a CM curve, the functional equation gives the order of the zero at \( s = 0 \) as \([F : \mathbb{Q}]\). What is known by the work of Bloch and Deninger is that there are at least \([F : \mathbb{Q}]\) linearly independent symbols \( \xi_i \) which make the \( L\)-value conjecture hold.

In §3 below we prove a negative result: if the curve \( E \) is not of Shimura type, then the Bloch construction gives only symbols with regulator equal 0. The problem is caused by the Galois action on the torsion points.

In §4 we try to use the Galois action on \( K_2(E_F) \) to an advantage, by developing an elliptic curve analog of the Stark conjecture. No claim is made that this conjecture is not implied by more general motivic conjectures already in the literature. The relevant \( L\)-functions are Artin-Hecke \( L\)-functions. Although we would like to work more generally, we restrict attention to CM curves for three reasons. First, for elliptic curves over number fields the continuation of the \( L\)-function to \( s = 0 \) is still only conjectural. More significant is that CM curves have at worst additive bad reduction at any prime. The \( K\)-theory of \( E \) becomes more complicated if there is split multiplicative bad reduction. Finally, a useful realization of the regulator map on \( K_2(E) \) requires choosing a basis of the lattice in \( \mathbb{C} \) corresponding to \( E \). For curves which are not defined over \( \mathbb{R} \) there is no canonical way to do this. For CM curves we show in §2 that it is possible to make a choice so that our determinant is well defined up to an element of \( \mathbb{Q}^\times \).

In §5 assuming the Dimension conjecture (1) and the \( L\)-value conjecture (2) above, we prove the analog Stark’s result [16] for rational characters.

In §6 assuming the Dimension conjecture and that \( E \) is type (S), we prove the elliptic curve analog of the Stark conjecture for an abelian extension of the complex quadratic field \( K \). In particular, taking a trivial character of the Galois group, we have re-derived the result of Deninger in [4] that (2) above holds. This is not an independent proof; the results rely on the same facts about curves of type (S) from [4] that [2] uses. However, it is a very classical proof. The two main ideas are
an extension of the Frobenius determinant theorem, and a distribution relation for values of Kronecker-Eisenstein series on isogenous curves.

In §7 we prove a result for a general CM elliptic curve $E$ (i.e.; not necessarily type (S)) defined over an abelian extension $F$ of $K$: There exists an extension $M$ of $F$, and a character $\chi$ of $\text{Gal}(M/F)$, such that the elliptic Stark conjecture is true for $E$ and $\chi$.

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2. Analytic prerequisites

2.1. The Bloch-Wigner Dilogarithm. The regulator on $K_2$ of curves over number fields is a map into cohomology. Here, however, we will follow the philosophy of [9] where one finds the advice “In general, the more concrete one is able to make the [Borel] regulator map, the more explicit the information one is able to extract from it.” So we will use Bloch’s original, function theoretic approach to the regulator as in [1].

Recall that the classical Euler dilogarithm is defined by

$$Li_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2} \quad |z| < 1$$

$$= \int_0^z -\frac{\log(1-t)}{t} dt \quad z \in \mathbb{C}\setminus[1, \infty)$$

after analytic continuation. The Bloch-Wigner dilogarithm

$$D(z) = \text{Im}(Li_2(z)) + \log |z| \text{arg}(1-z)$$

is well defined independent of path used to continue $Li_2$ and arg. For a torus $\mathbb{C}/\Lambda$, $\Lambda = [\omega_1, \omega_2]$ corresponding to a point $\tau$ in $\mathcal{H}$, we have the $q$-symmetrized, or elliptic, dilogarithm

$$D_q(z) = \sum_{n \in \mathbb{Z}} D(zq^n) \quad q = \exp(2\pi i\tau) \quad z \in \mathbb{C}^\times/q^\mathbb{Z} \cong \mathbb{C}/\Lambda.$$  

We define another real valued function $J(z) = \log |z| \log |1-z|$, and

$$J_q(z) = \sum_{n=0}^{\infty} J(zq^n) - \sum_{n=1}^{\infty} J(z^{-1}q^n) + \frac{2}{3} \log |q|^2 B_3(\log(|z|/|q|)),$$

where $B_3(t)$ is the third Bernoulli polynomial $B_3(t) = t^3 - 3/2t^2 + 1/2t$.

Together these functions make the regulator function:

$$R_q(z) = D_q(z) - iJ_q(z).$$  

(This normalization differs by $-i$ from the one usually taken.)
Recall that Weil [13] defines the Kronecker-Eisenstein-Lerch series for \( x, x_0 \) points in \( \mathbb{C}/\Lambda \), and \( \text{Re}(s) > a/2 + 1 \) as

\[
K_a(x, x_0, s, \Lambda) = \sum \langle x_0, \omega \rangle (\overline{x + w}^a |x + w|^{-2s}).
\]

Here

\[
\langle x_0, \omega \rangle = \exp((x_0 \overline{w} - x_0 \overline{w})/A(\Lambda)),
\]

where \( A(\Lambda) = \omega_1 \overline{\omega_1} \text{Im}(\tau)/\pi \), so that \( \omega_1 \omega_2 - \omega_1 \omega_2 = -2\pi i \delta A(\Lambda) \) with \( A(\Lambda) > 0 \) and \( \delta = \pm 1 \) chosen so that \( \omega_2/\omega_1 = \delta \tau \) with \( \tau \) in \( \mathcal{H} \). In this way \( A(\Lambda) \) is independent of choice of generators for the lattice, while \( A([1, \tau]) = \text{Im}(\tau)/\pi \). And * means sum over \( w \in \Lambda, w \neq x \) if \( x \in \Lambda \). We have the functional equation

\[
(1) \quad \Gamma(s) K_a(x, x_0, s, \Lambda) = A(\Lambda)^{a+1-2s} \Gamma(a+1-s) K_a(x_0, x, a+1-s) \langle x_0, x \rangle
\]

For the special case \( a = 1, x = 0, s = 2 \), we will use the more concise notation

\[
K_{2,1}(u, \Lambda) = K_1(0, u, 2, \Lambda) = \sum \langle u, \omega \rangle \text{Im}(\tau)/\pi|w|^4.
\]

Observe that the behavior under homothety is simple:

\[
A^2(c\Lambda) K_{2,1}(cu, c\Lambda) = \tau A^2(\Lambda) K_{2,1}(u, \Lambda).
\]

Bloch showed in [1] that for \( \Lambda = [1, \tau] \)

\[
(2) \quad R_q(\exp(2\pi i u)) = \pi A^2([1, \tau]) K_{2,1}(u, [1, \tau]).
\]

It is worth observing how this function depends on the choice of the lattice basis. Suppose we have a basis \( \omega_1, \omega_2 \) and \( \tau = \omega_2/\omega_1 \). Let

\[
\bar{\omega}_1 = d\omega_1 + c\omega_2
\]

\[
\bar{\omega}_2 = b\omega_1 + a\omega_2
\]

and \( \bar{\tau} = (a\tau + b)/(c\tau + d) \). Then

\[
(3) \quad \bar{\omega}_1 A^2([1, \tau]) K_{2,1}(u/\omega_1, [1, \tau]) = A^2([\omega_1, \omega_2]) K_{2,1}(u, [\omega_1, \omega_2]) = A^2([\bar{\omega}_1, \bar{\omega}_2]) K_{2,1}(u, [\bar{\omega}_1, \bar{\omega}_2]) = \bar{\omega}_1 A^2([1, \tau]) K_{2,1}(u/\omega_1, [1, \bar{\tau}]).
\]

This is a problem in general, as there is no canonical choice for the lattice basis.
2.2. Curves over number fields. Now let $E$ be an elliptic curve defined over a number field $F$, with complex multiplication by a complex quadratic field $K$. Let $M$ be an extension of $F$, and $P$ a point in $E(M)$. To each embedding $\Phi : M \hookrightarrow \mathbb{C}$ we get an elliptic curve $E_{\Phi}$ over $\mathbb{C}$ corresponding to a lattice $\Lambda$. If $\Phi$ restricts to a real embedding of $F$, the real period gives a canonical choice for $\omega_1$. Extend to a lattice basis with any complex $\omega_2$ such that $\tau = \omega_2/\omega_1$ is in the upper half plane, let $q = \exp(2\pi i \tau)$ and $u$ in $\mathbb{C}/\Lambda$ corresponding to $\Phi(P)$. Then

$$R_q(\exp(2\pi i u/\omega_1)) = \pi A^2([1, \tau])K_{2,1}(u/\omega_1, [1, \tau])$$

is well defined. On the other hand, if $\Phi$ restricts to a complex embedding of $F$, we choose any basis $\omega_1, \omega_2$ for $\Lambda$, and define $\tau$ and $q$ as before. If $\Phi'$ is the embedding which differs from $\Phi$ by complex conjugation, we can certainly choose basis $\omega'_1 = \overline{\omega_1}, \omega'_2 = -\overline{\omega_2}$ for $\Lambda' = \overline{\Lambda}$, so $\tau' = -\overline{\tau}$ and $q' = \overline{q}$.

**Lemma 1.** Suppose the embeddings $\Phi, \Phi'$ differ by complex conjugation. Then

(4) $$R_q(\exp(2\pi i u/\omega_1)) = \overline{R_{q'}(\exp(2\pi i u'/\omega'_1))}$$

**Proof.** We have

$$\text{Im}(\tau)^2K_{2,1}(u/\omega_1, \Lambda) = \text{Im}(\tau')^2K_{2,1}(u'/\omega'_1, \overline{\Lambda})$$
$$= \text{Im}(\tau')^2K_{2,1}(u'/\omega'_1, \Lambda')$$

So (4) follows from (3). \qed

**Remark.** Of course this still depends on the choice of the basis. If we change $\tau$ to $(a\tau + b)/(c\tau + d)$, then (3) implies that $R_q$ changes by $c\tau + d$.

To motivate what follows, we will summarize some relevant facts from $K$-theory. For a commutative ring $R$, recall that $K_0(R)$ is just the Grothendieck group, with generators $[M]$ for each projective $R$-module $M$, and relations $[M] + [M'] = [M \oplus M']$. In particular for a field $k$, $K_0(k) = \mathbb{Z}$. We will say no more about $K_1$ than the fact that for fields, $K_1(k) = k^\times$. For $K_2(k)$, the Matsumoto relations give that

$$K_2(k) = k^\times \otimes k^\times / \{f \otimes 1 - f \mid f \neq 0, 1\}.$$ 

The class of $f \otimes g$, denoted $\{f, g\}$, is called a symbol.
For a curve \( E \) over a field \( F \), we have the function field \( F(E) \), and a map

\[
K_1(F(E)) \to \bigsqcup_{P \in E(F)} K_0(F) = \text{Div}(E)
\]

\[
f \mapsto \sum \text{ord}_P f(P)
\]

The kernel \( F^\times = K_1(F) \) of this map is relatively uninteresting; the cokernel \( \text{Pic}(E) \) is important.

For functions \( f, g \) in \( F(E)^\times \), and fixed \( P \in E \), the tame symbol at \( P \) is defined by

\[
T_P(f, g) = (-1)^{\text{ord}_P g} \frac{f^{\text{ord}_P g}}{g^{\text{ord}_P f}}(P),
\]

and is trivial on tensors \( f \otimes 1 - f \), thus is a function on symbols \( \{f, g\} \).

In analogy to the divisor map above we have

\[
K_2(F(E)) \xrightarrow{\prod T_P} \bigsqcup_{P \in E(F)} K_1(F)
\]

Here the cokernel is mysterious. The kernel is, modulo torsion, our object of study \( K_2(E) \).

When \( F \) is a number field we get a group \( K_2(E_\Phi) \) for each embedding \( \Phi \) of \( F \) into \( \mathbb{C} \). Associated to a symbol \( \{f, g\} \) in \( K_2(E_\Phi) \) we have the divisors \( \text{div}(f), \text{div}(g) \) of the elliptic functions \( f, g \). Let

\[
\sum_{Q, Q'} \text{ord}_Q(f) \text{ord}_{Q'}(g)(Q - Q') = \text{say} \sum_P a_P(P)
\]

be their convolution \( \text{div}(f) \ast \text{div}(g) \). Let \( u_P \) be the point on \( \mathbb{C}/\Lambda \) corresponding to \( P \) on \( E \). The regulator associated to the symbol \( \{f, g\} \) and the embedding \( \Phi \) is defined to be

\[
\text{reg}(\{f, g\})_\Phi = \sum_P a_P R_q(\exp(2\pi i u_P/\omega_1))/\pi
\]

\[
= \sum_P a_P A^2([1, \tau]) K_{2,1}(u_P/\omega_1, [1, \tau]),
\]

where the dependence of each of the parameters \( q, \tau, \omega_1 \), and \( u_P \) on the embedding \( \Phi \) is suppressed. One can show [1] that this is a Steinberg function, i.e. trivial on the relations that define \( K_2 \).

Now suppose that the number field \( F \) has degree \( n \) over \( \mathbb{Q} \), and the extension \( M \) is Galois over \( F \) with Galois group \( G \). Let \( \Sigma = \text{Hom}(M, \mathbb{C}) \), and let \( M_\mathbb{C} \) be the complex vector space with basis \( \Sigma \). A typical element in \( M_\mathbb{C} \) is written \( \sum_\Phi z_\Phi \Phi \). Complex conjugation acts on both \( \mathbb{C} \) and \( \Sigma \), and we define Minkowski space \( M_\mathbb{R} \) to be the points
such that $z_{\Phi} = \overline{z_\Phi}$. This Euclidean space is canonically isomorphic to $\mathbb{R}^\Sigma$ [8, chapter I, §5].

We define a regulator map

$$
\lambda : K_2(E_M) \to M_{\mathbb{R}}
$$

$$
\xi \mapsto \sum \operatorname{reg}(\xi)_\Phi \Phi.
$$

The relation (4) of Lemma 1 show that $\lambda$ actually maps to $M_{\mathbb{R}}$, not just $M_{\mathbb{C}}$. The action of $G$ on $\Sigma$ on the left is the opposite action on the field: for $\gamma$ an element of $G$, $\gamma^{-1} \cdot \Phi(x) = \Phi(\gamma \cdot x)$, which we are writing $\Phi(x^\gamma)$. So the $\Phi$ coefficient of $\gamma \cdot \lambda(\xi)$, which is $\operatorname{reg}(\xi)_{\gamma^{-1} \cdot \Phi}$, is equal to $\operatorname{reg}(\xi^\gamma)_\Phi$, the $\Phi$ coefficient of $\lambda(\xi^\gamma)$. Thus the map $\lambda$ is a $G$ module homomorphism.

**Remark.** Of course, this $\lambda$ still depends on the choice of lattice basis at each embedding. Suppose as before that $\Phi, \Phi'$ are related by complex conjugation and we have chosen $\tau$ and $\tau' = -\overline{\tau}$. If we have a vector of symbols $\overrightarrow{\xi}$, then by the remark after Lemma 1, we compute that changing $\tau$ to $(a\tau + b)/(c\tau + d)$ changes the vectors

$$
\begin{align*}
\operatorname{reg}(\overrightarrow{\xi})_\Phi & \quad \text{into} \quad (c\overline{\tau} + d)\operatorname{reg}(\overrightarrow{\xi})_\Phi \\
\operatorname{reg}(\overrightarrow{\xi})_{\Phi'} & \quad \text{into} \quad (c\tau + d)\operatorname{reg}(\overrightarrow{\xi})_{\Phi'}.
\end{align*}
$$

This changes the determinant of any matrix in which these vectors appear, by $|c\tau + d|^2$. Since our curve has complex multiplication, $\tau$ is in $K$ and this factor is in $\mathbb{Q}^\times$. Thus modulo $\mathbb{Q}^\times$, our determinants will be independent of choice of lattice basis.

### 2.3. Isogenies between curves.

Suppose we have elliptic curves $E$ and $E'$ defined over a number field $F$, and an $F$-isogeny

$$
\phi : E \to E'.
$$

We identify the isogeny with a scalar $\phi \in \mathbb{C}^\times$ such that for the corresponding lattices, $\phi \Lambda \subset \Lambda'$ and

$$
\phi : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda'
$$

$$
z \mapsto \phi z.
$$

The isogeny $\phi$ gives a contravariant map on the function fields of the two curves

$$
\phi^* f = f \circ \phi.
$$

This respects the Matsumoto relations so we get a map on $K_2$ of the function fields. It is easy to show the tame symbol satisfies

$$
T_\Phi(\phi^* f, \phi^* g) = T_{\Phi(r)}(f, g)
$$
so we get a map
\[ \phi^* : K_2(E') \to K_2(E) \]
\[ \{f, g\} \mapsto \{\phi^* f, \phi^* g\}. \]

**Theorem 2.** We have the distribution relation
\[ \phi \cdot K_{2,1}(\phi(x), \Lambda') = \deg \phi \cdot \sum_{t \in \ker \phi} K_{2,1}(x - t, \Lambda). \]

As a consequence, the regulator map \( \lambda \) of §2.2 satisfies
\[ \lambda(\phi^* \{f, g\}) = \phi \cdot \lambda(\{f, g\}), \]
i.e. we have a commutative diagram

\[ \begin{array}{ccc}
K_2(E') & \xrightarrow{\phi^*} & K_2(E) \\
\downarrow \lambda & & \downarrow \lambda \\
\mathbb{F}_\mathbb{R} & \xrightarrow{\phi} & \mathbb{F}_\mathbb{R}
\end{array} \]

**Proof** We will first need a distribution relation for the isogeny given by multiplication by \( d \) on \( \mathbb{C}/\Lambda \) (found in [7, Lemme 2.4.2]). Fix \( x \) in \( \mathbb{C}/\Lambda \) and \( x_0 \in d^{-1}\Lambda \), then
\[ d^{2+a-2s} K_a(x_0, dx, s, \Lambda) = \sum_{t \in d^{-1}\Lambda/\Lambda} (-dx_0, x + t)K_a(0, x + t, s, \Lambda). \]

This is easy to prove as the left side is just
\[ d^2 \sum_{\omega_0} \langle \omega_0, dx \rangle \frac{(d\omega_0 + dx_0)^a}{|d\omega_0 + dx_0|^{2s}} \]
while the right is just
\[ \sum_{\omega \neq -dx_0} \sum_t \langle \omega, x + t \rangle \frac{(\omega + dx_0)^a}{|\omega + dx_0|^{2s}}. \]

The result follows from the orthogonality relation
\[ \sum_{t \in d^{-1}\Lambda/\Lambda} \langle \omega, t \rangle = \begin{cases} d^2, & \text{in case } \omega = d\omega_0 \\ 0, & \text{otherwise}. \end{cases} \]

Now suppose \( \Lambda, \Lambda' \), and \( \phi \) are as above, so \( \phi \Lambda \subset \Lambda' \). By the fundamental theorem of elementary divisors (see, for example, [13, Lemma 3.11]), we can choose bases \( \{\omega_1, \omega_2\} \) of \( \Lambda \) and \( \{\omega'_1, \omega'_2\} \) of \( \Lambda' \) so that
\[ \phi \omega_1 = d_1 \omega'_1, \quad \phi \omega_2 = d_2 \omega'_2 \]
for some integers \( d_1, d_2 \), so
\[ [\Lambda' : \phi \Lambda] = d_1 d_2 = \deg \phi. \]
Note that $|\phi|^2 A(\Lambda) = d_1 d_2 A(\Lambda')$, a fact we will use to compare the pairing $\langle \cdot, \cdot \rangle'$ corresponding to $\Lambda'$ with $\langle \cdot, \cdot \rangle$. We compute

$$K_a(0, \phi x, s, \Lambda') = \sum_{\omega' \neq 0 \in \Lambda'} \langle \omega', \phi x \rangle' \frac{\overline{\omega'}}{|\omega'|^{2s}}$$

$$= \frac{\phi^a}{|\phi|^{2s}} \sum_{\omega \neq 0 \in \phi^{-1} \Lambda'} \langle \omega, dx \rangle \frac{\overline{\omega}}{|\omega|^{2s}}$$

$$= \frac{\phi^a}{|\phi|^{2s}} \sum_{\tau \in \ker \phi} \langle \tau, dx \rangle K_a(\tau, dx, s, \Lambda)$$

$$= \frac{\phi^a}{|\phi|^{2s}} \deg \phi^{2s-a-2} \sum_{\tau \in \ker \phi} \langle [\deg \phi] \tau, t \rangle K_a(0, x-t, s, \Lambda)$$

by the distribution relation for the isogeny $[\deg \phi]$ above. We use the orthogonality relation, for each $t \in \ker [\deg \phi]$

$$\sum_{\tau \in \ker \phi} \langle [\deg \phi] \tau, t \rangle = \begin{cases} \deg \phi, & \text{if } t \in \ker \phi \\ 0, & \text{if } t \in \ker [\deg \phi] \backslash \ker \phi \end{cases}$$

to deduce

$$K_a(0, \phi x, s, \Lambda') = \frac{\phi^a}{|\phi|^{2s}} \deg \phi^{2s-a-1} \sum_{t \in \ker \phi} K_a(0, x-t, s, \Lambda).$$

The distribution relation follows from taking $s = 2$, $a = 1$, and $\phi \bar{\phi} = \deg \phi$.

For the commutative diagram, notice that if for each $Q$ we fix a $P$ in $\phi^{-1}(Q)$,

$$\text{div}(\phi^* f) = \sum_Q \sum_{T \in \ker \phi} \text{ord}_Q(f)(P - T),$$

and similarly with $\text{div}(\phi^* g)$. Thus

$$\text{div}(\phi^* f) * \text{div}(\phi^* g) = \deg (\phi) \cdot \phi^*(\text{div}(f) * \text{div}(g)),$$

and the result follows from the distribution relation.

\[\square\]

**Remark.** Suppose the curve $E$ has complex multiplication by $\mathcal{O}_K$. The theorem says the image of $K_2(E)$ in $F_\mathbb{R}$ is an $\mathcal{O}_K$ module. If the ‘Dimension conjecture’ \([\Pi]\) is true, this is a lattice in $F_\mathbb{R}$ with complex multiplication.
3. Torsion divisorial support

As mentioned in the introduction, a weak form of the Beilinson conjectures is known for CM elliptic curves of type (S). What can be done more generally? Throughout this section we assume that $E$ is defined over an abelian extension $F$ of $K$, since the most interesting case is the smallest extension of $K$ over which $E$ is defined, the Hilbert class field $F = K(j(E))$.

Recall that $F(E_{\text{tors}})$ is always an abelian extension of $F$, and that $K_{\text{ab}}$ is equal to $K(j(E), x(E_{\text{tors}}))$, where $x$ is the $x$-function for $E$. Thus if $F$ is abelian over $K$, so is $F(x(E_{\text{tors}}))$.

Remark. If the class number of $K$ is one, $K(j(E)) = K$. Then $F$ abelian over $K$ implies

$$F \subseteq K(x(E_{\text{tors}})) \subseteq K(E_{\text{tors}})$$

so

$$F(E_{\text{tors}}) \subseteq K(E_{\text{tors}}) \subseteq K_{\text{ab}}$$

and so $E$ is of type (S). We assume for the rest of this section that the class number of $K$ is greater than one.

We would like to know when $\text{reg}(\xi)_\Phi$ is equal to zero. In the paper [13], Schappacher considers this question for curves over $\mathbb{Q}$. Concerning the $d$ torsion, he remarks in (5.6) that since the function $x \to K_1(x', x, s)$ is odd, if a divisor $a$ is fixed under $a \to -a$, then $K_1(0, a, s) = 0$.

Similar considerations appear in the thesis of Ross [11], from which we will borrow the idea of ‘torsion divisorial support’: Let $L$ be any extension of $F$, and let $i : F \hookrightarrow L$ denote inclusion. This gives rise to two maps in $K$ theory,

$$i^* : K_2(E_F) \to K_2(E_L),$$

induced by base extension, and

$$i_* : K_2(E_L) \to K_2(E_F),$$

induced by restriction of scalars. The map $i_\ast \circ i^*$ acts by multiplication by $[L : F]$, and $i^* \circ i_\ast$ is the norm map

$$i^* \circ i_* : \{f, g\} \to \prod_{\sigma \in \text{Gal}(L/F)} \{f, g\}_\sigma.$$

If we fix an embedding $\Phi : L \hookrightarrow \mathbb{C}$, and let $\{f, g\} \in K_2(E_L)$, we see

$$\text{reg}(i_*\{f, g\})_\Phi = \text{reg}(i^* \circ i_*\{f, g\})_{\Phi|_F} = \sum_{\sigma \in \text{Gal}(L/F)} \text{reg}(\{f, g\}_\sigma)_\Phi.$$
One sees immediately that the divisor corresponding to \( i_* \{ f, g \} \) is invariant under \( \text{Gal}(L/F) \). Ross then makes a definition similar to

**Definition 3.** Let \( \mathcal{N} \) an ideal of \( \mathcal{O}_K \). Then a symbol in \( K_2(E_F) \) is said to have \( \mathcal{N} \) torsion divisorial support if it is of the form \( i_* \prod_i \{ f_i, g_i \} \) with all \( f_i, g_i \) defined over \( F(E_{\mathcal{N}}) \), and such that the divisors \( \text{div}(f_i), \text{div}(g_i) \) are supported on \( E_{\mathcal{N}} \).

If there is a \( \sigma \) in \( \text{Gal}(F(E_{\mathcal{N}})/F) \) such that \( P^\sigma = -P \) for all \( \mathcal{N} \) torsion points \( P \), then we see the regulator is zero on any element \( \xi \) with \( \mathcal{N} \) torsion divisorial support. In this context the following lemma will be useful.

**Lemma 4.** For an ideal \( \mathcal{N} \) of \( \mathcal{O}_K \) not dividing \( 2\mathcal{O}_K \), the following are equivalent:

1. There exists an \( \mathcal{N} \) torsion point \( P \) of \( E \) such that \( \forall \sigma \in \text{Gal}(F(E_{\mathcal{N}})/F), \quad P^\sigma \neq -P \).
2. For all ideals \( \mathcal{A} \) in \( \mathcal{O}_F \), \( \psi(\mathcal{A}) \neq -1 \mod \mathcal{N} \), where \( \psi \) is the Hecke character of \( E \)
3. \( F(E_{\mathcal{N}}) = F(x(E_{\mathcal{N}})) \).

**Proof.** If \( \sigma \) is the Artin symbol of an ideal \( \mathcal{A} \), then \( P^\sigma = \psi(\mathcal{A})P \). Thus 1 and 2 are equivalent. Since \( E \) is defined over \( F \), it is isomorphic over \( F \) to an equation of the form \( y^2 = x^3 + Ax + B \). For a point \( P = (x, y) \), we have \( P^\sigma = -P \) if and only if \( x^\sigma = x \) and \( y^\sigma = -y \). If \( F(E_{\mathcal{N}}) = F(x(E_{\mathcal{N}})) \) then clearly such a Galois action can not happen.

Conversely suppose \( P \) is \( \mathcal{N} \) torsion such that for all \( \sigma \), \( P^\sigma \neq -P \), with \( \mathcal{N} \) minimal for \( P \). Let \( \sigma \in \text{Gal}(F(E_{\mathcal{N}})/F(x(E_{\mathcal{N}}))) \). So for all \( \tilde{P} \) in \( E_{\mathcal{N}} \), \( x(\tilde{P})^\sigma = x(\tilde{P}) \). Choose a prime \( \mathcal{Q} \) of \( \mathcal{O}_F \) so that \( \sigma \) is the Artin symbol for \( \mathcal{Q} \); then \( \tilde{P}^\sigma = \psi(\mathcal{Q})\tilde{P} \). From the Weierstrass equation, if \( y(P)^\sigma \neq -y(P) \), it must equal \( y(P) \). So \( \psi(\mathcal{Q})P = P \), thus \( \psi(\mathcal{Q}) \equiv 1 \) modulo \( \mathcal{N} \). Then for all \( \tilde{P} \), \( \psi(\mathcal{Q})\tilde{P} = \tilde{P} \) and \( \tilde{P}^\sigma = \tilde{P} \), so \( \sigma \) is trivial. This shows 1 \( \iff \) 2. \( \square \)

**Definition 5.** \( E \) is of type \((R)\) if there exists an ideal \( \mathcal{N} \) of \( \mathcal{O}_K \) not dividing \( 2\mathcal{O}_K \) such that any of the equivalent conditions above hold.

**Remark.** This is a necessary condition for the regulator of a symbol with \( \mathcal{N} \) torsion divisorial support to be nonzero.

**Lemma 6.** If \( E \) is of type \((S)\), then it is of type \((R)\)
Proof. This is Lemma 4.7 of [4], where they show that (S) implies that for any \( \mathcal{N} \) divisible by both the conductor of \( E \) and the conductor of \( F \) over \( K \), \( F(E_\mathcal{N}) = F(x(E_\mathcal{N})) \), and is in fact the ray class field of \( K \) modulo \( \mathcal{N} \).

**Theorem 7.** \( E \) is of type (R) if and only if it is of type (S).

**Proof.** Recall we are assuming the class number of \( K \) is greater than 1, and thus \( \mathcal{O}_K^\times = \{ \pm 1 \} \). By Corollaire 2 we know there exists an elliptic curve \( E' \) defined over \( F \) which is of type (S). The proof constructs a Hecke character \( \psi' \) which has the relevant property. By Theorem 9.1.3 of [3], we may assume that \( E' \) and \( E \) have the same \( j \) invariant. Thus \( E' \) is a model of \( E \) and so \( \psi = \chi \psi' \) for some quadratic Dirichlet character \( \chi \) associated to an extension \( M/F \). We will show that that \( M \) is abelian over \( K \).

Let \( \mathcal{N} \) be an ideal of \( \mathcal{O}_K \) such that the Hecke character \( \psi \) of \( E \) is never \(-1 \) modulo \( \mathcal{N} \). Since \( E' \) is of type (S) it is of type (R) by Lemma 6, and we may assume there exists an ideal \( \mathcal{N}' \) divisible by \( \mathcal{N} \) such that \( F(x(E'_\mathcal{N}')) = F((E'_\mathcal{N}'')) \). Let \( \mathcal{Q} \) a prime ideal of \( \mathcal{O}_F \) which splits completely in \( F(x(E'_\mathcal{N}')) \). We will show that \( \mathcal{Q} \) splits in \( M \). Since \( \mathcal{Q} \) splits completely, the corresponding Frobenius automorphism \( \sigma \) is trivial, so \( P^\sigma = P \) for all \( \mathcal{N} \) torsion \( P \) on \( E' \), thus \( \psi'(\mathcal{Q}) \equiv 1 \mod \mathcal{N}' \). This means

\[
\psi(\mathcal{Q}) = \chi(\mathcal{Q})\psi'(\mathcal{Q}) \equiv 1 \mod \mathcal{N}'
\]

because \( \chi(\mathcal{Q}) = \pm 1 \). Thus \( \psi(\mathcal{Q}) \equiv 1 \mod \mathcal{N} \) as \( \mathcal{N} \) divides \( \mathcal{N}' \). By hypothesis on \( \mathcal{N} \), we must have \( \psi(\mathcal{Q}) \equiv 1 \mod \mathcal{N} \), and therefore \( \chi(\mathcal{Q}) = 1 \). Thus \( \mathcal{Q} \) splits in \( M \). This then is enough to say that

\[
M \subset F(x(E'_\mathcal{N}')),
\]

a ray class field of \( K \). Thus \( M \) is abelian over \( K \).

By Lemme 1 of [10], we see that \( E \) is of type (S). The point is that \( E' \) is of type (S) so \( F(E'_\text{tors}) \) is contained in \( K^{ab} \). With \( M \) also contained in \( K^{ab} \), we get \( F(E'_\text{tors}) \) is contained in \( K^{ab} \).

**Remark.** If the curve \( E \) is not of Shimura type, then any symbol with torsion divisorial support has regulator equal 0.

### 4. Stark conjectures

The result of the previous section presents two alternatives. One could consider instead the construction of symbols in \( K_2(E) \) based on points of infinite order, as in [5]. Or one can try to used the Galois action to an advantage. This possibility is suggested in [3, p.187]:

“Because of the compatibility with the action of correspondences and
Tate twists, the Beilinson regulators admit a ‘motivic’ formulation. This generalizes Stark’s conjectures on the factoring of the regulator according to the Galois action, relating the eigen-pieces of unit groups to the values of Artin \(L\)-functions at \(s = 0\).

In this section we begin to work out the analog of Stark’s conjectures for \(K_2\) of an elliptic curve with complex multiplication. The approach here is as concrete and down to earth as possible.

4.1. **Notation.** For any finite group \(C\) and class functions \(\chi_1, \chi_2\) on \(C\), we let

\[
\langle \chi_1, \chi_2 \rangle_C = \frac{1}{\#C} \sum_{t \in C} \chi_1(t) \overline{\chi_2(t)}.
\]

For any field \(k\) and abelian group \(A\), we let \(kA\) denote \(k \otimes A\). Groups always act on the left, even if written \(a^\sigma\) instead of \(\sigma \cdot a\).

Suppose \(E\) is an elliptic curve over a number field \(F\) with complex multiplication by \(\mathcal{O}_K\), where \(K = \mathbb{Q}(\sqrt{D})\). There are two cases:

1. \(K \subseteq F\). There exists a Hecke character \(\psi\) for \(F\) such that

\[
L(s, E) = L(s, \psi) L(s, \overline{\psi}).
\]

2. \(K \not\subseteq F\); let \(H = F \cdot K\). There exists a Hecke character \(\psi\) for \(H\) such that

\[
L(s, E) = L(s, \psi).
\]

We take a field \(M\) Galois over \(F\) which also contains \(K\). Let \(G = \text{Gal}(M/F), N = \text{Gal}(M/H)\), and \([F : \mathbb{Q}] = n\).

Fix an embedding \(\Phi_1\) of \(M\) (and thus also \(H, K\)) into \(\mathbb{C}\), such that \(\Phi_1(j(E)) = j(\mathcal{O}_K)\). Let \(\sum_K = \text{Hom}_K(M, \mathbb{C})\), so \(\Sigma = \Sigma_K \cup \overline{\Sigma_K}\). Recall the spaces \(M_C, M_\mathbb{R}\) of \(\S 2.2\).

We need to define an appropriate \(\mathbb{Q}\) vector space \(M_\mathbb{Q}\) inside \(M_\mathbb{R}\). It seems \(M_\mathbb{Q}\) should be the field \(M\) itself, viewed as \(\sharp \Sigma_K\) copies of the field \(K\). The details of the \(\mathbb{Q}[G]\) embedding inside \(M_\mathbb{R}\) are in \(\S 4.4\) below.

The headings of the following subsections indicate which part of [17] we are imitating.

4.2. **L-Functions.** We take a finite dimensional complex vector space \(V\) and a representation

\[
\rho : G \to GL(V)
\]

with character \(\chi\). Let \(V^*\) denote the contragredient representation of \(V\). We define

\[
L(s, E \otimes \chi) = \begin{cases} 
L(s, \psi \otimes \chi) L(s, \overline{\psi} \otimes \chi), & \text{in case 1} \\
L(s, \psi \otimes \text{Res}_N \chi), & \text{in case 2}
\end{cases}
\]
in terms of (products of) Artin-Hecke $L$ functions. (Viewing the standard $L$ function of the elliptic curve as coming from a Galois representation into cohomology, this is just the $L$ function of the tensor product representation.) We see immediately this is well behaved with respect to direct sums:

\[(7) \quad L(s, E \otimes (\chi_1 \oplus \chi_2)) = L(s, E \otimes \chi_1)L(s, E \otimes \chi_2).\]

For induction, we need the following fact about Artin-Hecke $L$ functions: if $F'$ lies between $F$ and $M$, fixed by $G'$, and $\chi'$ is the character of a representation of $G'$, then

\[(8) \quad L(s, \psi \otimes \text{Ind}_{G'}^{G} (\chi')) = L(s, (\psi \circ \text{Norm}_{F'/F}) \otimes \chi').\]

One shows then that

\[(9) \quad L(s, E_{F'} \otimes \chi') = L(s, E_{F} \otimes \text{Ind}(\chi')).\]

(There are three cases to check, depending on whether both $F'$ and $F$ contain $K$, neither do, or only $F'$ does.)

**Proposition 8.** Let $n = [F : \mathbb{Q}]$. Then $L(s, E \otimes \chi)$ has a zero at $s = 0$ of order $n \cdot \dim(V)$.

**Proof.** Consider first case 2. Via Brauer induction there exist one dimensional characters $\chi_i$ on subgroups $G_i$ of $N$ and integers $n_i$ such that

\[\text{Res}_N(\chi) = \sum_i n_i \text{Ind}_{G_i}(\chi_i).\]

Thus

\[L(s, \psi \otimes \text{Res}_N(\chi)) = \prod_i L(s, \psi \otimes \text{Ind}_{G_i}(\chi_i))^{n_i}\]

\[= \prod_i L(s, (\psi \circ \text{Norm}_{M^{G_i}/H}) \otimes \chi_i)^{n_i}.\]

Each of the $L$-functions

\[L(s, (\psi \circ \text{Norm}_{M^{G_i}/H}) \otimes \chi_i)\]

has a zero at $s = 0$ of order $[M^{G_i} : K]$ so the product has a zero of order

\[\sum_i n_i [M^{G_i} : K] = \sum_i n_i [M^{G_i} : H][H : K]\]

\[= [H : K] \sum_i n_i \dim(\text{Ind}_{G_i}(\chi_i))\]

\[= [F : \mathbb{Q}] \dim(V).\]
In case 1 similarly each of $L(s, \psi \otimes \chi)$ and $L(s, \overline{\psi} \otimes \chi)$ have a zero at $s = 0$ of order $\dim(V)[F : K]$, so the product has a zero of order $n \cdot \dim(V)$. 

Remark. To get an appropriate regulator determinant, we want an automorphism of a vector space whose dimension is equal to the order of the zero.

**Proposition 9.** $n \cdot \dim(V) = \dim \text{Hom}_G(V^*, M_C)$

*Proof.* The representation of $G$ in $M_C = M_Q \otimes \mathbb{C}$ is just $n$ copies of the regular representation $\text{Ind}_e^G(1)$ of $G$, where $e$ denotes the identity element of $G$. We have

$$n \cdot \dim(V) = n \cdot \langle \text{Res}_e(\chi), 1 \rangle_e = \langle \text{Res}_e(\chi), n \cdot 1 \rangle_e = \langle \chi, n \cdot \text{Ind}_e^G(1) \rangle_G$$

by Frobenius Reciprocity. Since the character of the right regular representation takes rational integer values, it is real. So the above is equal to

$$\langle \chi \cdot n \cdot \text{Ind}_e^G(1), 1 \rangle_G = \dim (V \otimes M_C)^G = \dim \text{Hom}_G(V^*, M_C)$$

by duality. 

4.3. **Stark Regulator.**

**Proposition 10.** We suppose from now on the Dimension conjecture of §1; specifically, that for all $L, F \subseteq L \subseteq M$, we have

$$\dim \mathbb{Q}K_2(E_L) = [L : \mathbb{Q}].$$

Then $\mathbb{Q}K_2(E_M) \cong M_Q$ as $\mathbb{Q}[G]$ modules.

*Proof.* Via the corollary on p.104 of [14], we need only show that for all subgroups $C$ of $G$

$$\dim (\mathbb{Q}K_2(E_M)^C) = \dim (M_Q^C).$$

Let $L$ the fixed field of $C$. Via Galois descent for $K$ groups tensored with $\mathbb{Q}$,

$$\mathbb{Q}K_2(E_M)^C \cong \mathbb{Q}K_2(E_L).$$

By our assumption the dimension is $[L : \mathbb{Q}]$. On the other hand,

$$\text{Res}_C \text{Ind}_e^G(1) = [G : C] \text{Ind}_e^C(1)$$

by the Induction-Restriction theorem [14, p.58]. This representation contains the trivial representation $[G : C] = [L : F]$ times. In $M_Q$ we have $n = [F : \mathbb{Q}]$ copies of this representation, so

$$\dim (M_Q^C) = [L : \mathbb{Q}].$$
Remark. One would like to try to get by with the weaker assumption \( \dim QK_2(E_L) \geq [L : Q] \), since this is already known for curves of type (S), by the work of Deninger [2]. One might hope to then prove that \( M_Q \hookrightarrow QK_2(E_M) \) as \( Q[G] \) modules. But the inequality on the dimensions is not strong enough to prove this. It might, for example, be true that \( QK_2(E_L) = QK_2(E_M) \) for every \( L \), which would say \( QK_2(E_M) \) is trivial as a \( Q[G] \) module.

Now, assuming the Dimension conjecture, let \( f : M_Q \to QK_2(E_M) \) a \( Q[G] \) isomorphism. Recalling the map \( \lambda \) defined in \( \S 2.3 \), we see \( \lambda \circ f \) is a \( G \) automorphism of \( M_\mathbb{R} \), and we use the same notation when extending scalars to \( M_\mathbb{C} \). Via functoriality, this defines an automorphism \( (\lambda \circ f)_V \):

\[
\text{Hom}_G(V^*, M_\mathbb{C}) \to \text{Hom}_G(V^*, M_\mathbb{C})
\]

\[
A \mapsto \lambda \circ f \circ A
\]

We define \( R(E, \chi) \) to be the determinant of \( (\lambda \circ f)_V \). (This determinant actually depends on the choice of map \( f \), which is suppressed from the notation.) Let \( c(E, \chi) \) be the coefficient of the first term in the Taylor expansion of \( L(s, E \otimes \chi) \) at \( s = 0 \), and define

\[
A(E, \chi) = \frac{R(E, \chi)}{c(E, \chi)}.
\]

**Elliptic Stark Conjecture.** \( A(E, \chi) \) belongs to \( Q(\chi) \), and for \( \sigma \) in \( \text{Gal}(Q(\chi)/Q) \) we have

\[
A(E, \chi)^\sigma = A(E, \chi^\sigma).
\]

4.4. **Towards Stark’s version.** From our assumption on the dimensions of the \( K \)-groups in the previous subsection we deduced the existence of a \( Q[G] \) isomorphism \( f \) from \( M_Q \to QK_2(E_M) \), which implies there exists a set \( M \) of \( n \) ‘Minkowski symbols’ each of whose \( G \) conjugates generate a \(#G \) dimensional \( Q \) vector space, in which \( G \) acts by the regular representation. (These symbols collectively play a role analogous to that of the Minkowski unit in the unit group.) Conversely a set \( M \) of \( n \) symbols in distinct \( G \) orbits will let us define a \( Q[G] \) isomorphism \( f_M \). In this subsection we deduce a more explicit formula for \( R(E, \chi) \), by considering an explicit isomorphism \( f_M \).

We must first specify the \( Q[G] \) embedding of \( M \) inside \( M_\mathbb{R} \). Consider first case [4]. We choose representatives for the \( G \) orbits in \( \Sigma \) by taking
\(\Phi_1\) as in (4.1), \(\Phi_2, \ldots, \Phi_{n/2}\) any representatives of the \(G\) orbits in \(\Sigma_K\), along with their complex conjugates. Let \(m\) in \(M\) generate a normal basis of \(M\) as an \(F\) vector space, and \(\{f_i\}\) any basis of \(F\) as a \(K\) vector space. A typical element of \(M\) is then written

\[
\sum_{\sigma \in G} \sum_{i=1}^{n/2} c_i(\sigma) f_i m^\sigma
\]

with all the \(c_i(\sigma)\) in \(K\), which we identify with

\[
\sum_{\sigma \in G} \sum_{i=1}^{n/2} c_i(\sigma) \Phi_i + \sigma(\sigma) \Phi_i
\]

in \(M_R\). To get a basis over \(\mathbb{Q}\) we let

\[
f_i^\pm = \frac{(1 \pm \sqrt{D})}{2} f_i
\]

for \(i = 1, \ldots, n/2\).

Case 2 is more complicated as \(F\) is not a \(K\) vector space. We write \(n = r + 2s\) with \(r\) and \(s\) the number of real embeddings, resp. pairs of complex conjugate embeddings of \(F\). We choose representatives \(\Phi_1\) as above, \(\Phi_2, \ldots, \Phi_r\) in \(\Sigma_K\) so that \(\Phi_i|F\) is a real embedding for \(i \leq r\). Thus \(\Phi_i\) is equal \(\gamma_i^{-1} \cdot \Phi_i\) for some \(\gamma_i \in G\). For \(r < i \leq s\), \(\Phi_i|F\) is a complex embedding, so if we fix once and for all a \(\gamma\) in \(G \setminus N\), we can actually define \(\Phi_i + s = \gamma \cdot \Phi_i\) in this case; then \(\gamma^{-1} \cdot \Phi_i + s = \Phi_i\). Let \(m\) in \(M\) generate a normal basis of \(M\) over \(F\). Let \(\{f_i\}\) a basis of \(F\) as a \(\mathbb{Q}\) vector space. Thus a typical element of \(M\) can be written

\[
\sum_{\sigma \in G} \sum_{i=1}^n c_i(\sigma) f_i m^\sigma
\]

with the \(c_i(\sigma)\) in \(\mathbb{Q}\). For \(i \leq r\) we identify the element \(f_i m^\sigma\) in \(M\) with

\[
(1 + \sqrt{D}) \sigma \cdot \Phi_i + (1 - \sqrt{D}) \sigma \cdot \Phi_i.
\]

When \(r < i \leq s\), we identify the element \(f_i m^\sigma\) in \(M\) with

\[
\sigma \cdot \Phi_i + \sigma \cdot \Phi_i + \sqrt{D} \sigma \cdot \Phi_i + \sqrt{D} \sigma \cdot \Phi_i + \sqrt{D} \sigma \cdot \Phi_i + \sigma \cdot \Phi_i + \sigma \cdot \Phi_i + \sigma \cdot \Phi_i.
\]

and \(f_{i+j} m^\sigma\) with

\[
\sqrt{D} \sigma \cdot \Phi_i + \sqrt{D} \sigma \cdot \Phi_i + \sigma \cdot \Phi_i + \sigma \cdot \Phi_i + \sigma \cdot \Phi_i.
\]

We then extend \(\mathbb{Q}\)-linearly to get \(M_\mathbb{Q}\) inside \(M_R\).

The determinant of \((\lambda \circ f_M)_V\) in \(\text{Hom}_C(V^*, M_C)\) is equal to the determinant of \(1 \otimes \lambda \circ f_M\) in \((V \otimes M_C)^G\). We’re now ready to compute this determinant.
Consider first case I. We arbitrarily label the $n$ symbols as $\xi_{i,+}$ and $\xi_{i,-}$ for $i = 1, \ldots, n/2$. We take the isomorphism

$$f_M(f_{i}^{\pm}m^{\sigma}) = \xi_{i,\pm}^{\sigma} \quad i = 1, \ldots, n/2, \quad \sigma \in G.$$ 

For $\tau \in G$ let $\text{reg}_M(\tau)$ denote the $n \times n$ matrix of $2 \times 2$ blocks

$$\begin{bmatrix} \text{reg}(\xi_{i,+}^{\tau})_{\Phi_j} & \text{reg}(\xi_{i,+}^{\tau})_{\Phi_j} \\ \text{reg}(\xi_{i,-}^{\tau})_{\Phi_j} & \text{reg}(\xi_{i,-}^{\tau})_{\Phi_j} \end{bmatrix}$$

for $i, j = 1, \ldots, n/2$. Then

**Proposition 11.** In case I we have

$$R(E, \chi, f_M) = (-\sqrt{D})^{\dim(V)n/2} \det \left( \sum_{\tau \in G} \rho(\tau) \otimes \text{reg}_M(\tau) \right).$$

where as usual $\rho$ is the representation of $G$ with character $\chi$.

**Proof.** A typical vector in $(V \otimes M_C)^G$ looks like

$$\sum_{\sigma \in G} \sum_{i=1}^{n/2} \rho(\sigma)v_i \otimes \sigma \cdot \Phi_i + \rho(\sigma)v_i' \otimes \sigma \cdot \Phi_i'$$

with arbitrary vectors $v_i, v_i'$ in $V$. This space inherits a natural inner product from the one $\langle \cdot, \cdot \rangle$ on $V$, namely the inner product of a typical vector as above with another, formed of vectors $w_i, w_i'$ is just

$$\sum_{i=1}^{n/2} \langle v_i, w_i \rangle + \langle v_i', w_i' \rangle.$$

We let $e_p$ for $p = 1, \ldots, \dim(V)$ an orthonormal basis of $V$. We choose a basis for $(V \otimes M_C)^G$ of the form

$$v_{p,i}^{\pm} = \sum_{\sigma \in G} \rho(\sigma)e_p \otimes (f_{i}^{\pm}m^{\sigma}),$$

for $p = 1, \ldots, \dim(V)$, and $i = 1, \ldots, n/2$. We are identifying $f_{i}^{\pm}m^{\sigma}$ with its image in $M_R$ as in §[4]. This lets us compute

$$1 \otimes \lambda \circ f_M(v_{p,i}^{\pm}) = \sum_{\sigma \in G} \rho(\sigma)e_p \otimes \lambda(\xi_{i,\pm}^{\sigma})$$

$$= \sum_{\sigma \in G} \rho(\sigma)e_p \otimes \sum_{\Phi \in \Sigma} \text{reg}(\xi_{i,\pm}^{\sigma})_{\Phi} \sigma \cdot \Phi.$$
Write each \( \Phi \) as \( \tau^{-1} \cdot \Phi_k \) or \( \tau^{-1} \cdot \overline{\Phi}_k \) and change variables \( \sigma \mapsto \sigma \tau \) to get

\[
\sum_{\sigma \in G} \left\{ \sum_{\tau \in G} \text{reg}(\xi_{i,\pm}^\tau) \rho(\tau) e_p \otimes \sigma \cdot \Phi_k + \text{reg}(\xi_{i,\pm}^\tau) \rho(\tau) e_p \otimes \sigma \cdot \overline{\Phi}_k \right\}.
\]

To get matrix coefficients we compute an inner product

\[
\langle 1 \otimes \lambda \circ f_M(v_{p,i}^\pm), v_{q,j}^\pm \rangle = \left\{ \sum_{\tau \in G} \text{reg}(\xi_{i,\pm}^\tau) \rho(\tau) e_p, \frac{(1 \pm \sqrt{D})}{2} e_q \right\} + \left\{ \sum_{\tau \in G} \text{reg}(\xi_{i,\pm}^\tau) \rho(\tau) e_p, \frac{(1 \pm \sqrt{D})}{2} e_q \right\}
\]

where the choices of \( \pm \) on different sides of the inner product are of course independent. Taking all four possible choices of the \( \pm \) gives us a \( 2 \times 2 \) block:

\[
\sum_{\tau \in G} \langle \rho(\tau) e_p, e_q \rangle \begin{bmatrix}
\text{reg}(\xi_{i,+}^\tau) \rho(\tau) & \text{reg}(\xi_{i,+}^\tau) \\
\text{reg}(\xi_{i,-}^\tau) & \text{reg}(\xi_{i,-}^\tau)
\end{bmatrix} \begin{bmatrix}
\frac{1-\sqrt{D}}{2} & \frac{1+\sqrt{D}}{2} \\
\frac{1+\sqrt{D}}{2} & \frac{1-\sqrt{D}}{2}
\end{bmatrix}
\]

The determinant of the matrix with these (doubly indexed) coefficients is our regulator \( R(E, \chi, f_M) \).

Case 2 seems, at first, simpler. We have symbols \( \xi_i \) for \( i = 1, \ldots, n \) in distinct \( G \) orbits. We take the isomorphism

\[
f_M(f_i m^\sigma) = \xi_i^\sigma \quad i = 1, \ldots, n, \quad \sigma \in G.
\]

Now a typical vector in \((V \otimes M_{\mathbb{C}})^G\) looks like

\[
\sum_{\sigma \in G} \sum_{i=1}^n \rho(\sigma) v_i \otimes \sigma \cdot \Phi_i
\]

with arbitrary vectors \( v_i \) in \( V \). The inner product of a typical vector as above with another formed of vectors \( w_i \) is just

\[
\sum_{i=1}^n \langle v_i, w_i \rangle.
\]

We let \( e_p \) for \( p = 1, \ldots, \dim(V) \) an orthonormal basis of \( V \). We choose a basis for \((V \otimes M_{\mathbb{C}})^G\) of the form

\[
v_{p,i} = \sum_{\sigma \in G} \rho(\sigma) e_p \otimes (f_i m^\sigma),
\]
for $p = 1, \ldots, \dim(V)$, and $i = 1, \ldots, n$. This lets us compute

$$1 \otimes \lambda \circ f_M(v_{p,i}) = \sum_{\sigma \in G} \rho(\sigma)e_p \otimes \lambda(\xi_{i}^{\sigma})$$

$$= \sum_{\sigma \in G} \rho(\sigma)e_p \otimes \sum_{\Phi \in \Sigma} \text{reg}(\xi_{i})_{\Phi} \sigma \cdot \Phi$$

$$= \sum_{\sigma \in G} \left\{ \sum_{\tau \in G} \text{reg}(\xi_{i}^{\tau})_{\Phi_k} \rho(\tau)e_p \right\} \otimes \sigma \cdot \Phi_k$$

after writing each $\Phi$ as $\tau^{-1} \cdot \Phi_k$ and changing variables $\sigma \mapsto \sigma \tau$ just as before.

In order to compute matrix coefficients as inner products, we must rewrite the typical basis vector $v_{q,j}$ with $j \leq r$ as

$$v_{q,j} = \sum_{\sigma \in G} \rho(\sigma)e_q \otimes ((1 + \sqrt{D})\sigma \cdot \Phi_j + (1 - \sqrt{D})\sigma \cdot \overline{\Phi}_j)$$

$$= \sum_{\sigma \in G} \rho(\sigma)e_q \otimes ((1 + \sqrt{D})\sigma \cdot \Phi_j + (1 - \sqrt{D})\sigma \gamma_{j}^{-1} \cdot \Phi_j)$$

$$= \sum_{\sigma \in G} \rho(\sigma)((1 + \sqrt{D})e_q + (1 - \sqrt{D})\rho(\gamma_{j})e_q) \otimes \sigma \cdot \Phi_j$$

after using the relation $\overline{\Phi}_j = \gamma_{j}^{-1} \Phi_j$ of [1,1] above, and a change of variables. We then see that for $j \leq r$

$$\langle 1 \otimes \lambda \circ f_M(v_{p,i}), v_{q,j} \rangle = \sum_{\tau \in G} \text{reg}(\xi_{i}^{\tau})_{\Phi_j} \langle \rho(\tau)e_p, (1 + \sqrt{D})e_q + (1 - \sqrt{D})\rho(\gamma_{j})e_q \rangle$$

$$= (1 - \sqrt{D}) \sum_{\tau \in G} \text{reg}(\xi_{i}^{\tau})_{\Phi_j} \langle \rho(\tau)e_p, e_q \rangle +$$

$$+ (1 + \sqrt{D}) \sum_{\tau \in G} \text{reg}(\xi_{i}^{\tau})_{\Phi_j} \langle \rho(\tau)e_p, \rho(\gamma_{j})e_q \rangle$$

In the second sum above we use that $\rho$ acts by isometries

$$\langle \rho(\tau)e_p, \rho(\gamma_{j})e_q \rangle = \langle \rho(\gamma_{j}^{-1}\tau)e_p, e_q \rangle$$

and change the variables $\tau \mapsto \gamma_{j}\tau$. Then

$$\text{reg}(\xi_{i}^{\gamma_{j}\tau})_{\Phi_j} = \text{reg}(\xi_{i}^{\tau})_{\gamma_{j}^{-1}}_{\Phi_j} = \text{reg}(\xi_{i}^{\tau})_{\overline{\Phi}_j}$$
so finally

\[
\langle 1 \otimes \lambda \circ f_M(v_{p,i}), v_{q,j} \rangle = \sum_{\tau \in G} \left\{ (1 - \sqrt{D}) \text{reg}(\xi_i^\tau) \Phi_j + (1 + \sqrt{D}) \text{reg}(\xi_i^\tau) \overline{\Phi}_j \right\} \langle \rho(\tau)e_p, e_q \rangle
\]

for \( j \leq r \).

Similarly if \( r < j \leq r + s \), we rewrite \( v_{q,j} \) as

\[
v_{q,j} = \sum_{\sigma \in G} \rho(\sigma)(e_q - \sqrt{D}\rho(\gamma^{-1})e_q) \otimes \sigma \cdot \Phi_j + \rho(\sigma)(\sqrt{D}e_q + \rho(\gamma)e_q) \otimes \sigma \cdot \Phi_{j+s})
\]

using the relation between \( \Phi_j \) and \( \overline{\Phi}_{j+s} \) above and a change of variables. Using the same change of variables as above, we see that

\[
\langle 1 \otimes \lambda \circ f_M(v_{p,i}), v_{q,j} \rangle = \sum_{\tau \in G} \langle \rho(\tau)e_p, e_q \rangle \times \langle \text{reg}(\xi_i^\tau) \Phi_j + \text{reg}(\xi_i^\tau) \overline{\Phi}_j - \sqrt{D}\text{reg}(\xi_i^\tau) \Phi_{j+s} + \sqrt{D}\text{reg}(\xi_i^\tau) \overline{\Phi}_{j+s} \rangle.
\]

5. RATIONAL CHARACTERS

For the regulator determinant we clearly we have

\[
R(E, \chi_1 \oplus \chi_2) = R(E, \chi_1)R(E, \chi_2).
\]

Induction properties, as well, seem to follow from [17, p.29]: If \( \chi \) is the character of a representation of \( \text{Gal}(M/F') \), then

\[
R(E_{F'}, \chi) = R(E_F, \text{Ind}(\chi)).
\]

For example, with \( F' = M \) we get

\[
R(E_M) = R(E_F, \text{Ind}_e^G(1)) = \prod_{\chi \in G} R(E_F, \chi)^{\dim(\chi)},
\]

So we have factored the regulator determinant into pieces. By the usual properties of \( L \)-functions, the same holds for the \( c(E, \chi) \) and thus for the ratios \( A(E, \chi) \).

Remark. If we take the trivial representation \( 1 \) of \( G \), then comparing (3) and (10) we see we have recovered the map \( i_* : K_2(E_M) \to K_2(E_F) \), and the determinant \( R(E_F) \) is one piece of the determinant \( R(E_M) \). As in \( \S 3 \), if \( E \) is not of type \( (S) \), the regulator map is zero on symbols with torsion divisorial support. The other terms in the product are not, however, \textit{a priori} zero on symbols with torsion divisorial support. For example, take \( F = K(j(E)) \) the Hilbert class field, and \( M = F(E_Q) \), where \( \rho \mathcal{O}_K = \mathcal{Q} \mathcal{Q} \) is a split prime, and take \( \rho \) an odd character of
\[ \mathcal{G}(F) = \mathbb{F}_p^\times. \] This is a simple observation, but, as mentioned at the beginning of this section, it is the motivation for looking at this analog of Stark’s conjecture.

**Theorem 12.** Suppose \( E \) is an elliptic curve defined over \( F \), and \( M \) is a Galois extension, with \( \chi \) a character of a representation of \( \mathcal{G}(M/F) \) taking rational values. Then there exists integers \( m, n_i \) and intermediate fields \( F_i \) such that

\[
L(s, E_F \otimes \chi)^m = \prod_i L(s, E_{F_i})^{n_i}.
\]

If we assume the Dimension conjecture and the \( L \)-value conjecture of \( \S 7 \), we get that

\[
A(E, \chi)^m = \prod_i A(E_{F_i})^{n_i} \in \mathbb{Q}.
\]

**Proof.** We will treat the case when \( F \) does not contain \( K \), the other case is easier. Let \( m \) be the exponent of \( G = \mathcal{G}(M/F) \). By standard facts about representations (e.g. [14, p.103]) there are subgroups \( C_i \) of \( G \) and integers \( n_i \) such that

\[
m \cdot \chi = \sum_i n_i \text{Ind}_{C_i}^G(1), \quad m \cdot \text{Res}_N(\chi) = \sum_i n_i \text{Res}_N \text{Ind}_{C_i}^G(1).
\]

So

\[
L(s, E \otimes m \cdot \chi) = \prod_i L(s, \psi \otimes \text{Res}_N \text{Ind}_{C_i}^G(1))^{n_i}.
\]

We want to use the Induction-Restriction theorem ([14, p.58] on each term, so we need for each \( i \) a decomposition of \( G \) into double cosets

\[ G = \cup_{\gamma} N \gamma C_i. \]

Fix an element \( \delta \in G, \delta \notin N \); since \( [G : N] = 2 \), there are at most two double cosets \( N \delta C_i \) and \( N \delta C_i \). There are two cases:

1. \( C_i \notin N \); Then there is only a single double coset \( N \epsilon C_i = N \delta C_i \).

   Let \( \tilde{C}_i = C_i \cap N \), \( F_i \) the fixed field of \( C_i \) and \( F_i \) the fixed field of \( \tilde{C}_i \). The Induction-Restriction theorem says

   \[
   \text{Res}_N \text{Ind}_{\tilde{C}_i}^G(1) = \text{Ind}_{\tilde{C}_i}^N(1).
   \]

   So

   \[
   L(s, \psi \otimes \text{Res}_N \text{Ind}_{\tilde{C}_i}^G(1)) = L(s, \psi \otimes \text{Ind}_{\tilde{C}_i}^N(1))
   \]

   \[
   = L(s, \psi \circ \text{Norm}_{H_i}^{F_i})
   \]

   \[
   = L(s, E_{F_i}).
   \]
2. \( C_i < N \). The two double cosets are distinct. Let \( D_i = \delta C_i \delta^{-1} \), also a subgroup of \( N \). Let \( F_i \) the fixed field of \( C_i \), and \( L_i \) the fixed field of \( D_i \). The Induction-Restriction theorem says

\[ \text{Res}_N \text{Ind}_G (1) = \text{Ind}_N^{C_i} (1) \oplus \text{Ind}_N^{D_i} (1). \]

So

\[ L(s, \psi \otimes \text{Res}_N \text{Ind}_G C_i (1)) = L(s, \psi \otimes \text{Ind}_N^{C_i} (1)) L(s, \psi \otimes \text{Ind}_N^{D_i} (1)) \]

\[ = L(s, \psi \circ \text{Norm}^F_H) L(s, \psi \circ \text{Norm}^L_H) \]

Since \( E \) is defined over the subfield \( F \) of \( H \), and \( \delta \) generates \( \text{Gal}(H/F) \), we get from [6, Theorem 10.1.3] that \( \delta \cdot \psi = \overline{\psi} \). This gives

\[ = L(s, \psi \circ \text{Norm}^F_H) L(s, \overline{\psi} \circ \text{Norm}^F_H) \]

\[ = L(s, E_{F_i}). \]

Assuming the \( L \)-value conjecture of §5, there exist rational numbers

\[ A(E_{F_i}) = \frac{R(E_{F_i})}{c(E_{F_i})}. \]

Then the induction and direct sum properties imply the following weak form of the elliptic Stark conjecture for rational characters:

\[ A(E, \chi)^m = \prod_i A(E_{F_i})^{n_i} \in \mathbb{Q}. \]

\[ \square \]

6. ABELIAN OVER COMPLEX QUADRATIC

This section is devoted to the proof of the following

**Theorem 13.** Suppose the field \( M \) has abelian Galois group \( \Gamma \) over the complex quadratic field \( K \), and that the curve \( E \) is type (S). Then the elliptic Stark conjecture is true.

To minimize notation, we will consider the case where \( E \) is defined over \( F = K(j(E)) \). This makes \( F \) the Hilbert class field of \( K \), and \( n = [F : \mathbb{Q}] = 2h \) where \( h \) is the class number of \( \mathcal{O}_K \). For abelian Galois groups we may as well assume the character \( \chi \) satisfies \( \dim(\chi) = 1 \). By pulling the representation \( \chi \) back to a larger Galois group, we can assume \( M \) is the ray class field modulo \( \mathcal{G} \), where \( \mathcal{G} \) is principal and is divisible by the conductors of \( \chi \) and the Hecke character \( \psi \). Furthermore, since \( E \) is type (S), \( \psi = \phi \circ \text{Norm}_{F/K} \) for some Hecke character \( \phi \) of \( K \).
We begin by considering partial $L$-functions, which have only the Euler factors prime to $G$. By abuse of notation the dependence on $G$ is suppressed. Then we have

$$L(s, \overline{\psi} \otimes \chi) = L(s, \overline{\phi} \circ \text{Norm}_{F/K} \otimes \chi) =$$

$$L(s, \overline{\phi} \otimes \text{Ind}_G^F(\chi)) = \prod_{i=1}^h L(s, \overline{\phi} \otimes \chi_i) =$$

$$\prod_{i=1}^h \sum_{\gamma \in \Gamma} \chi_i(\gamma)L(s, \overline{\phi}, \gamma)$$

where $\text{Ind}_G^F(\chi)$ decomposes as $\oplus \chi_i$ and we have the partial $L$-functions associated to an Artin symbol $\gamma = [*, M/K]$ in $\Gamma$

$$L(s, \overline{\phi}, \gamma) = \sum_{\mathcal{C} \subset \mathcal{O}_K \atop [\mathcal{C} : M/K] = \gamma} \overline{\phi}(\mathcal{C})N(\mathcal{C})^{-s}.$$

This gives the $h$-th derivative at $s = 0$ as

$$(11) \quad L^{(h)}(0, \overline{\psi} \otimes \chi) = \prod_{i=1}^h \sum_{\gamma \in \Gamma} \chi_i(\gamma)L'(0, \overline{\phi}, \gamma).$$

In the next subsection we will give a generalization of what is usually called the Frobenius determinant relation (actually due to Dedekind.)

6.1. Generalized Dedekind determinant. Suppose $\Gamma$ is a finite abelian group, $G < \Gamma$, and $\chi : G \to \mathbb{C}^\times$ is a character. Write

$$\pi = \text{Ind}_G^F(\chi) = \oplus \chi_i$$

for the induced representation. Fix once and for all a set $S$ of coset representatives for $G \setminus \Gamma$. Let $W$ be the vector space the induced representation acts in:

$$W = \{ F : \Gamma \to \mathbb{C} | F(\sigma x) = \chi(\sigma)F(x), \forall \sigma \in G, x \in \Gamma \},$$

where $\Gamma$ acts by multiplication: $\pi(\gamma)F(x) = F(x\gamma)$. Let

$$f : \Gamma \to \mathbb{C}$$

be any function, and define the operator on $W$

$$\pi(f) = \sum_{\gamma \in \Gamma} f(\gamma)\pi(\gamma)$$
The characters $\chi_i$ form a basis of $W$, and so do the characteristic functions of cosets $F_{G\gamma}$, where

$$F_{G\gamma}(\sigma'\gamma') := \begin{cases} \chi(\sigma') & \text{if } \gamma = \gamma' \\ 0 & \text{otherwise} \end{cases}$$

The Dedekind determinant relation computes the determinant of $\pi(f)$ with respect to these two canonical bases of $W$:

**Lemma 14.**

$$\det \pi(f) = \prod_{\gamma \in \Gamma} \sum_{\gamma \in \Gamma} f(\gamma)\chi_i(\gamma)$$

$$= \det \left[ \sum_{\tau \in G} \chi(\tau)f(\tau\gamma'/\gamma^{-1}) \right]_{\gamma,\gamma' \in S}$$

*Proof.* The functions $\chi_i$ are eigenvectors of each $\pi(\gamma)$, with eigenvalue $\chi_i(\gamma)$, thus also eigenvectors of $\pi(f)$ with eigenvalue

$$\sum_{\gamma \in \Gamma} f(\gamma)\chi_i(\gamma),$$

and so the first formula for the determinant is clear. Relative to our set of coset representatives $S$, define a function (a factor set)

$$S \times S \rightarrow G$$

$$(\gamma, \gamma') \mapsto g(\gamma, \gamma')$$

so that

$$g(\gamma, \gamma')\gamma = \gamma'\gamma'' \quad \text{where} \quad G\gamma = G\gamma'G\gamma''$$

in the quotient $G\backslash \Gamma$. An explicit computation shows that

$$\pi(f)F_{G\gamma} = \sum_{\gamma'' \in S} \sum_{\sigma \in G} \chi(\sigma g(\gamma, \gamma'))f(\sigma\gamma'')F_{G\gamma},$$

with $\gamma, \gamma', \gamma''$ related as above. A change of variables give the matrix coefficients of $\pi(f)$ as in the lemma. \qed

**Remark.** The case when $G$ is the trivial subgroup, and $\pi$ is the regular representation of $\Gamma$ is the usual Frobenius determinant relation.

Taking $L'(0, \overline{\phi}, \gamma)$ for the function $f(\gamma)$ in the Dedekind determinant, and using (11) realizes the $L$-function value as a determinant:

$$L^{(h)}(0, \overline{\psi} \otimes \chi) = \det \left[ \sum_{\tau \in G} \chi(\tau)L'(0, \overline{\phi}, \tau\gamma'\gamma^{-1}) \right]_{\gamma,\gamma'}.$$
This application of the Dedekind determinant has long been a key ingredient for special values of $L$-functions. For example, results on the conjecture of Birch and Swinnerton were obtained in \cite{4} and \cite{3}.

6.2. **Partial $L$-functions and Kronecker series.** In this subsection we present, for completeness, a calculation of the derivative at $s = 0$ of a partial $L$-function as the value at $s = 2$ of a Kronecker series.

Since $\Phi_1(j(E)) = j(O_K)$, we have, for the lattice $\Lambda$ corresponding to $E_{\Phi_1}$

$$\Lambda = \Omega O_K$$

for some $\Omega \in \mathbb{C}^\times$. Further, for $\mathcal{A}$ an ideal of $O_K$, we have $E^{[\mathcal{A}, M/K]}$ is defined over $F$, corresponding to a lattice

$$\Lambda_{\mathcal{A}} = h(\mathcal{A})\Omega\mathcal{A}^{-1}$$

for some $h(\mathcal{A})$. The isogeny between $E$ and $E^{[\mathcal{A}, M/K]}$ is just multiplication by $h(\mathcal{A})$. Our hypothesis that $E$ is type (S) implies that $E$ is isogenous over $F$ to all its Galois conjugates, which gives that $h(\mathcal{A}) \in F$ for all $\mathcal{A}$. By composing isogenies one sees that $h$ is a crossed homomorphism:

$$h(\mathcal{A}'\mathcal{A}) = h(\mathcal{A}')^{[\mathcal{A}, M/K]}h(\mathcal{A}).$$

For more details on curves of type (S) (used throughout this section) see \cite{4}.

Choose a set of representative $\mathcal{A} \in \mathcal{A}$ for the ideal class group of $K$, all prime to our fixed $\mathcal{G}$. Choose also a fixed set of representatives $\mathcal{B} \in \mathcal{B}$ so the Artin symbols $[\mathcal{B}, M/K]$ give every element of $G = \text{Gal}(M/F)$. The ideals in $\mathcal{B}$ are principal as $F$ is the Hilbert class field, and we have that $\mathcal{B} = (\phi(\mathcal{B}))$. A given element $\gamma \in \Gamma$ is then of the form $[\mathcal{A}\mathcal{B}, M/K]$ for some $\mathcal{A}$ and some $\mathcal{B}$. We get all ideals in this class by summing over $\alpha$ in $\mathcal{A}^{-1}\mathcal{G}$ since then

$$\phi(\mathcal{B}) + \alpha \equiv \phi(\mathcal{B}) \mod \mathcal{G}$$

and

$$[\mathcal{A}(\phi(\mathcal{B}) + \alpha), M/K] = [\mathcal{A}(\phi(\mathcal{B})), M/K].$$

So

$$L(s, \phi, [AB, M/K]) = \sum_{\alpha \in \mathcal{A}^{-1}\mathcal{G}} \frac{\phi(\mathcal{A}) \phi((\phi(\mathcal{B}) + \alpha))}{N(\mathcal{A})^s N(\phi(\mathcal{B}) + \alpha)^s}.$$

Now in general we have

$$\phi((\lambda)) = \phi_{\text{fin}}(\lambda)\lambda$$
where \( \phi_{\text{fin}}(\lambda) \in K \) only depends on \( \lambda \mod \mathcal{G} \). Since \( \mathcal{B} = (\phi(\mathcal{B})) \) we get that \( \phi_{\text{fin}}(\phi(\mathcal{B})) = 1 \), and
\[
\overline{\phi}(\phi(\mathcal{B} + \alpha)) = \phi_{\text{fin}}(\phi(\mathcal{B} + \alpha) \phi(\mathcal{B}) + \alpha = \phi(\mathcal{B} + \alpha).
\]
Thus
\[
L(s, \overline{\phi}, [AB, M/K]) = \frac{\overline{\phi}(A)}{N(A)^s} \sum_{\alpha \in A^{-1}G} \frac{\phi(\mathcal{B} + \alpha)}{|\phi(\mathcal{B}) + \alpha|^{2s}}.
\]

Let \( \nu \in \Omega K^\times \) so that \( (\nu/\Omega) = G^{-1} \); i.e., \( \nu \) is a \( G \) torsion point on \( C/\Lambda \). We see that
\[
\Gamma(s)L(s, \overline{\phi}, [AB, M/K]) = \frac{h(A)\nu}{|h(A)\nu|^{2s}} \sum_{\alpha \in A^{-1}G} \frac{\phi(\mathcal{B} + \alpha)}{|\phi(\mathcal{B}) + \alpha|^{2s}} = K_1(\phi(\mathcal{B})h(A)\nu, 0, s, \Lambda_A)
\]
since \( \alpha \in A^{-1}G \) exactly when \( \omega = h(A)\nu \alpha \in h(A)\Omega A^{-1} = \Lambda_A \). We combine equations (13) and (14), multiply by \( \Gamma(s) \) and use the functional equation (1) to see that
\[
\Gamma(s)L(s, \overline{\phi}, [AB, M/K]) = \frac{\overline{\phi}(A)}{h(A)\nu} |h(A)\nu|^{2s} A(\Lambda_A)^{2-2s} \Gamma(2-s) \times K_1(0, \phi(\mathcal{B})h(A)\nu, 2-s, \Lambda_A).
\]

Thus the partial \( L \)-function derivative at \( s = 0 \) is computed by the Kronecker series value at \( s = 2 \), which we are denoting \( K_{2,1} \):

**Lemma 15.**
\[
L'(0, \overline{\phi}, [AB, M/K]) = \frac{\overline{\phi}(A)}{h(A)\nu} A^2(\Lambda_A)K_{2,1}(\phi(\mathcal{B})h(A)\nu, \Lambda_A).
\]

6.3. For convenience we now number our representatives \( A_i \in A \) for the ideal class group of \( K \), and choose them so that \( A_i = \overline{A_i} \) if \( j(A_i) \) is real \((1 \leq i \leq r) \), and \( A_{i+s} = \overline{A_i} \) if \( j(A_i) \) is complex \((r < i \leq s) \).

Consider the matrix on the right hand side of equation (12). In terms of our representatives, we have
\[
\det \left[ \sum_\mathcal{B} \chi([\mathcal{B}, M/K]) L'(0, \overline{\phi}, [A_i^{-1}\mathcal{A}_j\mathcal{B}, M/K]) \right]_{i,j}
\]
To use the relation between these partial \( L \)-function derivatives and Kronecker series above, we need to change \( A_i^{-1} \) to \( A_i \). This induces a permutation of the ideal classes as well as an extra term from the representatives of the principal ideals in each row. The permutation of the rows only changes the determinant by \( \pm 1 \), but the extra principal
ideal needs to be absorbed from each row by a change of variables in
the sum, which alters the determinant. This gives the \( L \)-function value as

\[
a(\chi) \det \left[ \sum_{\mathcal{B}} \chi([\mathcal{B}, M/K]) L'(0, \tilde{\phi}, \mathcal{A}_i, \mathcal{B}, M/K) \right]_{i,j}
\]

where \( a(\chi) \) is the product of all terms introduced by these change of
variables. Note that, as the conjecture will require,

\[
a(\chi) \in \mathbb{Q}(\chi), \quad a(\chi^\sigma) = a(\chi)^\sigma \quad \forall \sigma \in \text{Gal}(\mathbb{Q}(\chi)/\mathbb{Q}).
\]

We use Lemma 15 (with \( \mathcal{A}_i, \mathcal{A}_j \) instead of \( \mathcal{A} \)) on each entry of the
matrix. Factor \( \phi(\mathcal{A}_i) \) from row \( i \) for each \( i \), and similarly \( \phi(\mathcal{A}_j) \) from
each column. Note that \( \prod_i \phi(\mathcal{A}_i)^2 \) is in \( K^\times \), since \( \prod_i \mathcal{A}_i^2 \) is principal.
In fact in the ‘other half’ of the \( L \) function (coming from \( \phi \) instead of \( \tilde{\phi} \)) we will see the complex conjugate of these terms. So modulo \( \mathbb{Q}^\times \) we
can ignore them. We have so far shown that

\[
L^{(h)}(0, \overline{\psi} \otimes \chi) \approx_{\mathbb{Q}(\chi)} \det \left[ \sum_{\mathcal{B}} \chi([\mathcal{B}, M/K]) \frac{A^2(\Lambda_{\mathcal{A}_i, \mathcal{A}_j})}{h(\mathcal{A}_i, \mathcal{A}_j)^\nu} K_{2,1}(\phi(\mathcal{B})h(\mathcal{A}_i)\nu, \Lambda_{\mathcal{A}_i, \mathcal{A}_j}) \right]_{i,j}
\]

We can now apply the distribution relation of Proposition 2 to the
isogeny

\[ h(\mathcal{A}_i)^{[A_j, F/K]} : E^{[A_j, F/K]} \to E^{[A_i, A_j, F/K]} \]

of degree \( N(\mathcal{A}_i) \). We need to use the fact that in general

\[
A(\Lambda_{\mathcal{A}_i}) = A(h(\mathcal{A}) \Omega A^{-1}) = \frac{|h(\mathcal{A})\Omega|^2 \sqrt{|D|}}{2\pi N(\mathcal{A})}
\]

where \( D \) is the discriminant of \( K \). So

\[
A^2(\Lambda_{\mathcal{A}_i, \mathcal{A}_j}) = \frac{|h(\mathcal{A}_i)^{[A_j, F/K]}|^4}{N(\mathcal{A}_i)^2} A^2(\Lambda_{\mathcal{A}_i}).
\]

In summary, we’ve got

\[
L^{(h)}(0, \overline{\psi} \otimes \chi) \approx_{\mathbb{Q}(\chi)} \det \left[ \sum_{\mathcal{B} \in \mathcal{B}} \chi([\mathcal{B}, M/K]) \frac{A^2(\Lambda_{\mathcal{A}_i})}{h(\mathcal{A}_i)^\nu} K_{2,1}(\phi(\mathcal{B})h(\mathcal{A}_i)\nu - t, \Lambda_{\mathcal{A}_i}) \right]_{i,j}
\]
By class field theory, the elements $A_j$ of the class group correspond via our fixed embedding $\Phi_1$ to the other embeddings $\Phi_j$ which have the same restriction to $K$. That is

$$C/\Lambda_{A_j} = E^{[A_j, M/K]} = E_{\Phi_j}.$$  

We use the homothety property to factor a scalar $h(A_j)\Omega$ out of the $K_{2,1}$ in each column, to convert $\Lambda_{A_j}$ to $A_j^{-1}$. If we take our representatives to be integral ideals not divisible by any rational integer but 1, it is easy to see that $A_j^{-1}$ has a lattice basis of the form $[1, \tau_j]$ as required.

The divisor associated to the sum over the torsion points in the kernel of $h(A_i)$ comes from a symbol $\xi_i$ in $QK_2(E_M)$ by the theorem of Bloch \[1\]. Multiplication by $\phi(B)$ on the torsion points of one of these curves acts by Galois automorphism $[B, M/K]$. Recall these fix $F$, and since our isogenies $h(A_i)$ are defined over $F$ we can re-write the sum over $B \in \mathcal{B}$ as a sum over $\tau \in G$. Thus we see

$$L^{(h)}(0, \overline{\psi} \otimes \chi) \approx_{Q(\chi)} \det \left[ \Omega/\nu \sum_{\tau \in G} \chi(\tau) \text{reg}(\xi^{\tau}_i) \Omega_{i,j} \right]_{i,j}$$

Applying this construction to $\psi$ instead of $\overline{\psi}$, we get the same formula but with the conjugate embeddings $\overline{\Phi_j}$ instead of $\Phi_j$. However, we have so far only constructed $h$ symbols in $K_2$ which we can relate to the $L$-value. To get $2h$ symbols, we take advantage of the fact that $K_2(E_M)$ is an $O_K$ module. More specifically, the point is that we can vary the ideal $\mathcal{G}$, changing the matrix of partial $L$ functions and thus also the matrix of symbols, by a constant in $K$. For any ideal $\mathcal{P}$, one sees that for any $\gamma$ in $\Gamma$, the partial $L$-functions satisfy

$$L_\mathcal{G}(s, \overline{\phi}, \gamma) = L_{\mathcal{G}\mathcal{P}}(s, \overline{\phi}, \gamma) + \overline{\phi}(\mathcal{P}) N(\mathcal{P})^{-s} L_\mathcal{G}(s, \overline{\phi}, \gamma \cdot [\mathcal{P}, M/K]^{-1})$$

where we now, of course, need to keep track of the ideal in the notation for the partial $L$-function. Thus if $\mathcal{P}$ is principal

$$\sum_{\tau \in G} \chi(\tau)L'_{\mathcal{G}\mathcal{P}}(0, \overline{\phi}, \gamma \tau) =$$

$$(1 - \overline{\phi}(\mathcal{P})\chi([\mathcal{P}, M/K])) \times \sum_{\tau \in G} \chi(\tau)L'_{\mathcal{G}}(0, \overline{\phi}, \gamma \tau).$$

Choose two principal prime ideals $\mathcal{P}^+$ and $\mathcal{P}^-$, and let $\mathcal{G}^\pm = \mathcal{G}\mathcal{P}^\pm$. Let

$$\pi^\pm = 1 - \overline{\phi}(\mathcal{P}^\pm)\chi([\mathcal{P}^\pm, M/K]).$$

When we change $\mathcal{G}$ to $\mathcal{G}^+$ or $\mathcal{G}^-$, the matrix on the right hand side of \[12\] changes by the scalar $\pi^+$ or $\pi^-$. Following the calculation of the
previous section to the end, we see the same is true for the matrix of
regulators of symbols supported on the torsion. That is, let
\[
R = \left[ \sum_{\tau \in G} \chi(\tau) \text{reg}(\xi^\tau_i) \phi_j \right]_{i,j}
\]
the matrix corresponding to symbols on the $G$ torsion, then
\[
\pi^\pm R = \left[ \sum_{\tau \in G} \chi(\tau) \text{reg}(\xi^\tau_{i,\pm}) \phi_j \right]_{i,j}
\]
where the symbols $\xi_{i,\pm}$ come from the $G^\pm$ torsion. We have shown
above that
\[
\det(R) \det(\overline{R}) \approx_{Q(\chi)} L^{(h)}(0, \overline{\psi} \otimes \chi)L^{(h)}(0, \psi \otimes \chi) = L^{(2h)}(0, E \otimes \chi).
\]
By Proposition [1], the elliptic Stark conjecture will follow from the linear algebra

**Lemma 16.** For an $h \times h$ matrix $R$, and scalars $\pi^+, \pi^-$ in $Q(\sqrt{D})$,
\[
\det \begin{bmatrix} \pi^+ R & \pi^+ \overline{R} \\ \pi^- R & \pi^- \overline{R} \end{bmatrix} = \kappa \det(R) \det(\overline{R}), \quad \kappa \in \begin{cases} Q & \text{if } h \text{ is even} \\ Q \cdot \sqrt{D} & \text{if } h \text{ is odd} \end{cases}
\]

*Proof.* This is the fancy version of the Laplace expansion theorem, where we are expanding on the first $h$ columns. Note that in the sum over $h$ by $h$ matrices, we include row $i$ of $\pi^+ R$ if and only if we omit row $i$ of $\pi^- R$, as these are the only terms with nonzero determinant. This gives $\det(R) \det(\overline{R})$ times a sum of powers of $\pi^+$ and $\pi^-$. If $h$ is even, the Laplace theorem gives that each term occurs with sign $+1$, and it is a trace from $Q(\sqrt{D})$ to $Q$. If $h$ is odd, then group complementary terms in the sum, which necessarily occur with opposite sign. One sees that it is a ‘skew trace’, so is in $Q \cdot \sqrt{D}$. $\square$

6.4. **The simple zero.** In the Stark conjectures, the case when the $L$-function has a simple zero gets special attention. We observe here that if $[F : Q] \cdot \dim(V) = 1$, then $F = Q$. Since $E$ is defined over $F$, we must necessarily have the class number $h(K) = 1$. Since $\dim(V) = 1$, $\chi$ factors through an abelian extension $\text{Gal}(M/Q)$, and $M$ contains $K$ by hypothesis. By the remark at the beginning of §3, $E_L$ is of type $(S)$ for any intermediate field $L$ with $Q \subseteq L \subseteq M$. Assuming the Dimension conjecture, the results in the previous subsection give the elliptic Stark conjecture in this case.
7. Curves not Type (S)

The positive results so far have all been for curves of type (S), even though we introduced the elliptic Stark conjecture to study the general case. In this section we remedy this defect with the following

**Theorem 17.** If $F$ is abelian over $K$, and $E$ is any elliptic curve over $F$ with complex multiplication by $\mathcal{O}_K$, then there is a Galois extension $M$ of $F$, and a character $\chi$ of $\text{Gal}(M/F)$ such that the elliptic Stark conjecture holds for $L(s, E \otimes \chi)$.

**Remark.** This theorem does not assume the ‘Dimension conjecture’ of §1.

**Proof.** By [10] Corollaire 2 we know there exists an elliptic curve $E'$ defined over $F$ which is of type (S). By Theorem 9.1.3 of [6], we may assume that $E'$ and $E$ have the same $j$ invariant. Thus $E'$ is a model of $E$ and we can write Weierstrass equations

$$E: \quad y^2 = 4x^3 - g_2x - g_3$$

$$E': \quad y^2 = 4x^3 - d^2g_2x - d^3g_3$$

with $d$ in $F$. The curves $E$ and $E'$ become isomorphic over $M = F(\sqrt{d})$ via

$$\phi : E \rightarrow E'$$

$$(x, y) \mapsto (x' = dx, y' = d^{3/2}y)$$

We get a map on functions fields and a map on $K$-groups

$$\phi^* : \mathbb{Q}K_2(E_F') \rightarrow \mathbb{Q}K_2(E_F),$$

as in §2.

The inclusions $i : F(E) \rightarrow M(E)$ and $i' : F(E') \rightarrow M(E)$ also give maps

$$i^* : \mathbb{Q}K_2(E_F) \rightarrow \mathbb{Q}K_2(E_M)$$

$$i'^* : \mathbb{Q}K_2(E'_F) \rightarrow \mathbb{Q}K_2(E_M)$$

Note that the triangle formed by these three maps does not commute. In fact $M(E) = M(E')$ is a Galois extension of $F(x) = F(x')$, with Galois group the Klein four group. The quadratic subfields are $M(x)$, $F(E)$, and $F(E')$. The subgroups of order two which fix these fields are generated by $[-1]$, $\tau$, and $\tau'$, where

$$\tau : \begin{cases} x \rightarrow x \\ y \rightarrow y \\ \sqrt{d} \rightarrow -\sqrt{d} \end{cases}$$

$$\tau' : \begin{cases} x' \rightarrow x' \\ y' \rightarrow y' \\ \sqrt{d} \rightarrow -\sqrt{d} \end{cases}$$
Of course, $\tau' = \tau \circ [-1]$, and both $\tau$ and $\tau'$ restrict to the same nontrivial automorphism of $M$ over $F$. For $f$ in $F(E')$, we see

$$(\phi^* f)^\tau = \phi^* f, \quad f^{\tau} = f \circ [-1] = [-1]^* f.$$  

Thus for symbols $\xi$ in $\mathbb{Q}K_2(E'_F)$,

$$(i^* \phi^* \xi)^\tau = i^* \phi^* \xi, \quad (i'^* \xi)^\tau = [-1]^* i'^* \xi.$$  

Since $E'$ is type (S) there exist $n = [F : \mathbb{Q}]$ symbols $\xi_i$ such that the ‘$L$-value conjecture’ of §1 is true. In the notation of §4, this says that

$$A(E'_F) = R(E'_F)/c(E'_F)$$

is rational. Let $\chi$ be the nontrivial character of Gal($M/F$). Since

$$L(s, E'_F)L(s, E'_M) = L(s, E_M) = L(s, E_F)$$

we see that $c(E'_F) = c(E_F, \chi)$.

It remains to relate $R(E'_F)$ to $R(E_F, \chi)$. For notational convenience we suppress the inclusions $i^*$ and $i'^*$. We see that for symbols $\xi_i$ as above,

$$\left(\frac{\phi^* \xi_i + \xi_i}{2}\right)^\tau = \frac{\phi^* \xi_i - \xi_i}{2},$$

so the $n$ symbols $(\phi^* \xi_i + \xi_i)/2$ satisfy the requirement of §4 to be a set $M$ of ‘Minkowski symbols’. Fix any embedding $\Phi_j$ of $F$ into $\mathbb{C}$. In the formula for $R(E_F, \chi, f_M)$ in Proposition §1, the $i, j$ entry in the sum over the Galois group simplifies as

$$\text{reg}(\frac{\phi^* \xi_i + \xi_i}{2})_{\Phi_j} - \text{reg}(\frac{\phi^* \xi_i + \xi_i}{2})_{\Phi_j} = \text{reg}(\xi_i)_{\Phi_j}.$$  

Thus

$$R(E'_F) = R(E_F, \chi)$$

and so

$$A(E_F, \chi) = R(E_F, \chi)/c(E_F, \chi) = R(E'_F)/c(E'_F)$$

is rational.

\[\square\]

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