Explicit Multimonopole Solutions
in SU(N) Gauge Theory

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ABSTRACT

We construct multimonopole solutions containing \(N - 1\) distinct fundamental monopoles in SU(\(N\)) gauge theory. When the gauge symmetry is spontaneously broken to \(U(1)^{N-1}\), the monopoles are all massive, and we show that the fields can be written in terms of elementary function for all values of the monopole positions and phases. In the limit of unbroken \(U(1) \times SU(N - 2) \times U(1)\) symmetry, the configuration can be viewed as containing a pair of massive monopoles, each carrying both \(U(1)\) and \(SU(N - 2)\) magnetic charges, together with \(N - 3\) massless monopoles that condense into a cloud of non-Abelian fields. We obtain explicit expressions for the fields in the latter case and use these to analyze the properties of the non-Abelian cloud.

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1 Introduction

The massive magnetic monopoles of spontaneously broken gauge theories have long been the objects of considerable study. Although these monopoles arise as spatially extended solutions of the classical field equations, they correspond to single-particle states of the full quantum theory. Their dynamics, at least at low energies, can be described by a small number of degrees of freedom, just like that of the elementary particles of the theory [1]. Indeed, these magnetically charged states can be regarded as the counterparts, related by an exact duality symmetry in certain supersymmetric theories, of the massive electrically charged states built from the elementary excitations of the theory [2].

Recent studies of low-energy monopole dynamics have shown that when the unbroken gauge group has a non-Abelian component there are degrees of freedom that can be attributed to the presence of massless non-Abelian monopoles; these can be seen as the dual counterparts of the massless elementary gauge bosons [3]. In contrast with the massive monopoles, these massless monopoles cannot be exhibited as isolated classical solutions, but can be studied classically only as part of multimonopole configurations. In the simplest example, an $SO(5)$ solution [4] with one massive and one massless monopole, the massless monopole is manifested as a spherically symmetric “cloud” of non-Abelian fields surrounding the massive monopole. In this paper we examine a somewhat more complex class of configurations containing two massive and $N - 3$ massless monopoles in the Bogomolny-Prasad-Sommerfield (BPS) [5] limit of an $SU(N)$ gauge theory. By obtaining explicit analytic expressions for the gauge and Higgs fields we can see clearly how the massless monopoles condense into a non-Abelian cloud and can verify that the properties of this cloud inferred from the form of the moduli space metric are indeed present.

The origin of these massless monopoles can be understood by recalling that the magnetically charged BPS solutions for an arbitrary gauge group $G$ can all be analyzed in terms of fundamental monopoles of various types. The simplest case is when the adjoint Higgs field breaks a group of rank $r$ maximally, to $U(1)^r$. There are then $r$ quantized topological charges, one for each of the unbroken $U(1)$ factors. Corresponding to each of these is a fundamental monopole solution carrying a single unit of topological charge. Each of these is described by four collective coordinates, three specifying its position and one corresponding to an overall $U(1)$ phase. These solutions can be realized explicitly by embedding the unit $SU(2)$ monopole using a preferred set of simple roots. All higher charged solutions may be regarded as multimonopole solutions containing appropriate numbers of the various fundamental monopoles. Not only does one find that the energy of the solution is the sum of the masses of the component monopoles, but an index theory analysis shows
that the number of zero modes, and hence of collective coordinates, is precisely four times the number of component monopoles.

This can be illustrated in a particularly simple fashion when the gauge group is $SU(N)$. By means of a gauge transformation, the asymptotic value of the Higgs field in any fixed direction can always be brought into the form

$$\Phi = \text{diag} \left( t_N, t_{N-1}, \ldots, t_1 \right)$$ (1.1)

with $t_1 \leq t_2 \leq \ldots \leq t_N$. We may write the asymptotic magnetic field in the same direction as

$$F_{ij} = \epsilon_{ijk} Q_M / r^3,$$

with

$$Q_M = 4\pi e \text{diag} \left( n_{N+1}, n_N - n_{N-2}, \ldots, n_2 - n_1, -n_1 \right).$$ (1.2)

The generalized topological quantization condition implies that the $n_j$ must be integers.

If the eigenvalues of the asymptotic Higgs field are all distinct, then the unbroken gauge group is $U(1)^r$ and the $n_j$ are the topological charges. Apart from a constant Higgs field contribution, the $k$th fundamental monopole, with $n_j = \delta_{jk}$, is obtained by embedding the $SU(2)$ monopole solution (rescaled appropriately so as to give the correct Higgs expectation value) in the $2 \times 2$ block at the intersections of the $(N-k)$th and $(N+1-k)$th rows and columns. The resulting $SU(N)$ monopole then has a mass

$$M_k = \frac{2\pi(t_{k+1} - t_k)}{e}$$ (1.3)

where $e$ is the gauge coupling. Although there are other possible $SU(2)$ embeddings, both the mass formulas and the zero mode counting indicate that these are merely special cases of multimonopole solutions.

Varying the asymptotic Higgs field so that two or more of its eigenvalues are equal enlarges the unbroken symmetry group so that some of the $U(1)$ factors are replaced by a non-Abelian group $K$. The magnetic charge must still be of the form of Eq. (1.2), but the $n_j$ that correspond to roots of $K$ are no longer topologically conserved charges; in fact, they are not even gauge-invariant. According to Eq. (1.3), the corresponding monopoles should become massless in this limit. From the point of view of electric-magnetic duality this seems quite reasonable, since one would expect the massless gauge bosons carrying electric-type charges in the subgroup $K$ to have massless counterparts carrying magnetic charges. On the other hand, one would not expect to find zero energy solitons. Indeed, the classical one-monopole solution tends toward the vacuum solution as the limit of unbroken symmetry is approached. However, examination of the moduli space Lagrangian that describes the low-energy dynamics of a collection of BPS monopoles suggests that
the degrees of freedom corresponding to these monopoles can survive even in the massless limit. Specifically, if a number of massless monopoles are combined with one or more massive monopoles to give an \( n \)-monopole solution whose total magnetic charge is invariant under \( K \), the dimension of the moduli space, and hence the number of collective coordinates, remains \( 4n \) even in the limit of non-Abelian unbroken symmetry \( [9] \). Furthermore, examination of specific examples suggests that the moduli space metric, and hence the Lagrangian, behaves smoothly in this limit.

As noted above, the simplest examples, containing one massive and one massless monopole, arise in the context of an \( SO(5) \) gauge theory spontaneously broken to \( SU(2) \times U(1) \) \( [4] \). These contain a massive monopole core surrounded by a spherically symmetric “non-Abelian cloud”, of arbitrary radius, that can be viewed as the remnant of the massless monopole. Within the cloud there is a Coulomb magnetic field corresponding to a magnetic charge with components lying both in the unbroken \( U(1) \) and in the unbroken \( SU(2) \), while outside the cloud only the \( U(1) \) Coulomb field is present. The solution is described by eight collective coordinates. Four of these are readily identified as the position and \( U(1) \) phase of the massive monopole. The other four coordinates describe the cloud, with three determining its overall \( SU(2) \) orientation and one specifying its radius.

An obvious step toward gaining further understanding of these massless monopoles and their associated non-Abelian clouds would be to investigate solutions containing larger numbers of monopoles. Solutions corresponding to one massless and two identical massive monopoles have been studied in \( SU(3) \) broken to \( SU(2) \times U(1) \) \( [10, 11] \) and in \( Sp(4) = SO(5) \), also broken to \( SU(2) \times U(1) \) \( [12] \). (In the latter case the unbroken \( SU(2) \times U(1) \) is a different subgroup than that considered in Ref. \( [4] \).) In both cases the moduli space metric was found explicitly, but analytic expressions for the gauge and Higgs fields could only be found for special configurations.

The complexity of these solutions is perhaps not surprising if one recalls the rather nontrivial form of the Atiyah-Hitchin metric for the moduli space of two identical \( SU(2) \) monopoles \( [13] \). By contrast, the moduli space metric for two \( [14] \), or even an arbitrary number \( [15] \), of distinct fundamental monopoles is relatively simple, and so one might expect the corresponding solutions for the fields to be more tractable. As we will see, this is indeed the case. We consider here \( SU(N) \) solutions comprising \( N - 1 \) distinct fundamental monopoles. For both the case of maximal symmetry breaking, where all of the monopoles are massive, and the case where the unbroken group is \( U(1) \times SU(N - 2) \times U(1) \), where all but two of the monopoles become massless, the gauge

\[^{1}\text{One encounters a number of pathologies when dealing with configurations whose total magnetic charge has a non-Abelian component} \]
and Higgs fields can be expressed in terms of elementary functions for all values of the collective coordinates.

We use Nahm’s method to construct these solutions \[16\]. In Sec. II we review the details of this construction for BPS monopoles in an $SU(N)$ theory. The implementation of the construction for configurations containing many distinct fundamental monopoles is described in Sec. III. In Sec. IV we consider the case where the unbroken group is $U(1) \times SU(N-2) \times U(1)$ and obtain explicit expressions for the fields. These expressions simplify considerably in the regions outside the cores of the massive monopoles. We discuss these asymptotic forms in Sec. V. Section VI contains some concluding remarks. There is an Appendix containing details of some of the calculations.

2 The Nahm construction

The fundamental elements in Nahm’s construction \[16\] of the BPS monopole solutions are a triplet of matrices $T_a(t)$, the Nahm data, that satisfy a set of nonlinear ordinary differential equations. These $T_a$ then define a linear differential equation for a second set of matrices, $v(t, r)$, from which the fields $\Phi(r)$ and $A(r)$ can be constructed. In this section we review the details of this construction for the case of an $SU(N)$ theory \[17\] with the asymptotic Higgs field and magnetic charge given by Eqs. (1.1) and (1.2), respectively. Here, and for the remainder of the paper, we set the gauge coupling equal to unity.

The matrices $T_a(t)$ are defined for $t_1 < t < t_N$. The $t_j$ divide this range into $N-1$ intervals. On the $j$th interval, $t_j < t < t_{j+1}$, we define $k(t) = n_j$ and require that the $T_a$ have dimension $k(t) \times k(t)$. In addition, whenever two adjacent intervals have the same value for $k(t)$, there are three matrices $\alpha_j$, of dimension $k(t_j) \times k(t_j)$, defined at the interval boundary $t_j$. These matrices satisfy the Nahm equation

$$
\frac{dT_a}{dt} = \frac{i}{2} \epsilon_{abc}[T_b, T_c] + \sum_j (\alpha_j)_{a\delta}(t - t_j),
$$

where the sum in the last term (and similar sums in later equations) should be understood to run only over those values of $j$ such that $n_j = n_{j-1}$. (The $T_a$ are singular at $t_{j+1}$ if $|n_{j+1} - n_j| \geq 2$. Because we will not be considering such situations here, we will not describe the requirements obeyed by these singularities.)

Having found the Nahm data, the next step is to find a $2k(t) \times N$ matrix function $v(t, r)$ and

\footnote{We will in general follow the notation of Ref. \[18\].}
$N$-component row vectors $S_j(r)$ obeying the differential equation

$$0 = \left[ -\frac{d}{dt} + (T_a + r_a) \otimes \sigma_a \right] v + \sum_j a_j^\dagger S_j \delta(t - t_j) \tag{2.2}$$

and the normalization condition

$$I = \int dt v^\dagger v + \sum_j S_{j}^\dagger S_{j}. \tag{2.3}$$

Here $a_j$ is a $2k(t_j)$-component row vector obeying

$$a_j^\dagger a_j = \alpha_j \cdot \sigma - i(\alpha_j)_0 I \tag{2.4}$$

with $(\alpha_j)_0$ chosen so that the above matrix has rank 1.

Finally, the spacetime fields are given by

$$\Phi = \int dt t v^\dagger v + \sum_j t_j S_{j}^\dagger S_{j} \tag{2.5}$$

$$A = -i \int dt v^\dagger \nabla v - i \sum_j S_{j}^\dagger \nabla S_{j}$$

$$= -\frac{i}{2} \int dt \left[ v^\dagger \nabla v - \nabla v^\dagger \right] - \frac{i}{2} \sum_j \left[ S_{j}^\dagger \nabla S_{j} - \nabla S_{j}^\dagger S_{j} \right] \tag{2.6}$$

where the second equality in Eq. (2.6) is obtained with the aid of the normalization condition Eq. (2.3). These satisfy the self-dual BPS equations

$$B = D\Phi \tag{2.7}$$

where

$$B_a = \frac{1}{2} \epsilon_{abc} F_{bc} = \frac{1}{2} \epsilon_{abc} (\partial_b A_c - \partial_c A_b + i[A_b, A_c]) \tag{2.8}$$

$$D_a \Phi = \partial_a \Phi + i[A_a, \Phi]. \tag{2.9}$$

Equations (2.2) and (2.3) do not completely determine $v$ and the $S_j$. Given any solution of these equations, a second solution can be obtained by multiplication on the right by an $N \times N$ unitary matrix function of $r$; this corresponds to an ordinary gauge transformation. In addition, there is also some freedom to multiply $v$ and the $S_j$ on the left, with corresponding transformations on the Nahm data. Such transformations have no effect on the spacetime fields, but can be used to simplify the intermediate calculations, as we will see in Sec. 4.
3 Construction of $(1, 1, \ldots, 1)$ monopole solutions

We will be concerned in this paper with solutions consisting of $N-1$ distinct fundamental monopoles. The $n_j$ are then all equal to unity and $k(t) = 1$ for the entire range of $t$. Since $k(t)$ is unchanged at each of the intermediate $t_j$, there is an $\alpha_j$ and an $S_j$ for each value of $j$ from 2 through $N-1$. The commutator term vanishes, and so Eq. (2.2) is easily solved to give the piecewise constant solution

$$T(t) = -x_j, \quad t_j < t < t_{j+1}. \quad (3.1)$$

The $x_a$ have a natural interpretation as the positions of the individual monopoles. The $a_j$ of Eq. (2.4) are simply two-component row vectors, which we take to be

$$a_j = \sqrt{2|\alpha_j - x_{j-1}|} \begin{pmatrix} \cos(\theta/2)e^{i\phi/2}, \sin(\theta/2)e^{-i\phi/2} \end{pmatrix} \quad (3.2)$$

where $\theta$ and $\phi$ specify the direction of the vector $\alpha_j = x_{j-1} - x_j$.

The next step is to find a $2 \times N$ matrix $v(t)$ and a set of $N$-component row vectors $S_k$ ($k = 2, 3, \ldots, N-1$) that satisfy Eq. (2.2). To this end, we first define for each interval $t_k \leq t \leq t_{k+1}$ a function $f_k(t)$, with

$$f_1(t) = e^{(t-t_2)(r-x_1)\cdot\sigma},$$

$$f_k(t) = e^{(t-t_k)(r-x_k)\cdot\sigma} f_{k-1}(t_k), \quad k > 1. \quad (3.3)$$

These have been defined so that their values at the endpoints of the intervals satisfy

$$f_k(t_k) = f_{k-1}(t_k) \equiv g_k, \quad k = 2, 3, \ldots, N-1. \quad (3.4)$$

An arbitrary solution of Eq. (2.2) can then be written in the form

$$v(t) = f_k(t)\eta_k, \quad t_k < t < t_{k+1}, \quad (3.5)$$

with discontinuities at the intermediate $t_k$ obeying

$$\eta_k = \eta_{k-1} + g_k^{-1}a_k^\dagger S_k. \quad (3.6)$$

The normalization condition, Eq. (2.3), becomes

$$I = \sum_{j=2}^{N-1} S_j^\dagger S_j + \sum_{k=1}^{N-1} \eta_k^\dagger N_k \eta_k \quad (3.7)$$

where

$$N_k = \int_{t_k}^{t_{k+1}} dt f_k^\dagger(t)f_k(t). \quad (3.8)$$
We will find it convenient to distinguish between the first two and the last \((N - 2)\) columns of \(v\) and the \(S_k\), labeling the former by Roman superscripts from the beginning of the alphabet and the latter by Greek superscripts that run from 3 to \(N\). We choose the \(v^a\) to be continuous, so that

\[ S_k^a = 0, \quad a = 1, 2. \]  

(3.9)

A properly normalized solution for the \(v^a\) is then obtained by taking

\[ \eta_k^a = N^{-1/2} \theta^a, \quad a = 1, 2, \]

(3.10)

where

\[ N = \sum_{k=1}^{N-1} N_k, \]

(3.11)

and the \(\theta^a\) are the two-component objects \(\theta^1 = (1, 0)^t\) and \(\theta^2 = (0, 1)^t\).

Orthogonality of each of the last \(N - 2\) columns of \(v\) with the first two implies that

\[ 0 = \sum_{k=1}^{N-1} N_k \eta_k^\mu, \quad \mu = 3, 4, \ldots N. \]

(3.12)

Together with the discontinuity Eq. (3.6), this gives

\[ \eta_j^\mu = N^{-1} \sum_{k=1}^{N-1} \sum_{l=2}^{N-1} c_{jkl} N_k g_l^{-1} a_i^\dagger S_l^\mu \]  

(3.13)

where

\[ c_{jkl} = \begin{cases} 1, & j \geq l > k, \\ -1, & j < l \leq k, \\ 0, & \text{otherwise}. \end{cases} \]

(3.14)

Substituting these solutions for the \(\eta_j^\mu\) into the orthogonality condition, Eq. (3.7), gives

\[ \delta^{\mu\nu} = \sum_{i,j=2}^{N-1} S_i^\mu [\delta_{ij} + a_i \nu M_{ij} a_j^\dagger] S_j^\nu \equiv \sum_{i,j=2}^{N-1} S_i^\mu K_{ij} S_j^\nu \]  

(3.15)

where

\[ M_{ij} = \sum_{k,l,m=1}^{N-1} c_{kli} c_{kmj} g_i^{-1} N_j^\dagger N^{-1} N_k N^{-1} N_m g_j^{-1}. \]

(3.16)

Hence,

\[ S_j^\nu = \left( K^{-1/2} \right)_{j\mu} U_{\mu\nu} \]  

(3.17)

where \(U\) is any \((N - 2) \times (N - 2)\) unitary matrix\footnote{In applying this equation, one must be careful to take into account our convention that the upper index on \(S\) runs from 3 to \(N\), while the lower runs from 2 to \(N - 1\).}. The freedom to choose \(U\) corresponds to a \(U(N - 2)\) subgroup of the \(SU(N)\) gauge symmetry; the remaining gauge symmetry has already been fixed by our choices for the first two columns of \(v\).
Substituting this expression for the $S_j^\nu$ into Eq. (3.13) gives the $\eta^\mu_j$ and thus, through Eq. (3.5), determines $v(t)$. It is then a straightforward, although tedious, matter to substitute these results into Eqs. (2.5) and (2.6) and thus obtain the fields $A(r)$ and $\Phi(r)$. We will not carry this out explicitly for the case of maximal symmetry breaking. However, we note that it is clear from the above equations that the result can be expressed in terms of elementary functions.

4 Solutions for $SU(N) \rightarrow U(1) \times SU(N - 2) \times U(1)$

Our main interest is in the case where the middle $N - 2$ eigenvalues of the asymptotic Higgs field are all equal, so that the unbroken gauge group is $U(1) \times SU(N - 2) \times U(1)$. If we adjust the Higgs field in this fashion, then, as was argued in Ref. [3], the $(N - 1)$-monopole solutions of the previous section can be viewed as being composed of two massive and $N - 3$ massless monopoles, with the latter condensing into a non-Abelian “cloud”. The massive monopoles are located at $x_1$ and $x_{N-1}$; without any loss of generality we may take these to lie on the $z$-axis, with $z_{N-1} = z_1 + R \geq z_1$. The locations of the massless monopoles are less well-defined. Extrapolating from the maximally broken case, one would take these to be the points $x_2, x_3, \ldots, x_{N-2}$. However, as we will now show, many different choices for these points yield the same solution.

To begin, note that of the $N - 1$ intervals into which the range of $t$ was divided, only the leftmost ($t_1 \leq t \leq t_2$) and rightmost ($t_2 = t_{N-1} \leq t \leq t_N$) now have nonzero width; we shall use subscripts $L$ and $R$ to label quantities related to these two intervals. Hence, of the $N - 1$ integrals defined by Eq. (3.8), only $N_1 \equiv N_L$ and $N_{N-1} \equiv N_R$ are nonzero. Also, the $g_k = f_k(t_k)$ are all equal to unity. As a result, Eq. (3.15) simplifies to

$$\delta^{\mu\nu} = \sum_{i,j} S_i^{\mu\dagger} \left[ \delta_{ij} + a_i^\dagger M a_j^\dagger \right] S_j^\nu \quad (4.1)$$

where

$$M = \left( N_L^{-1} + N_R^{-1} \right)^{-1} \quad (4.2)$$

is Hermitian. Once a set of $S_j^\nu$ satisfying Eq. (4.1) has been found, the $\eta_k$, and hence $v(t)$, can be found from Eq. (3.13), which now reduces to

$$\eta_1^\mu = -N_L^{-1} M \sum_j a_j^\dagger S_j^\mu$$

$$\eta_{N-1}^\mu = N_R^{-1} M \sum_j a_j^\dagger S_j^\mu \quad (4.3)$$

With maximally broken symmetry, the monopole positions enter both through the functions $f_k(t)$ and through the various $a_j$. In the present case, where the middle intervals have zero width,
the \( x_k \) associated with the massless monopoles enter only through the \( a_j \), which, as can be seen from Eqs. (4.1) and (4.3), appear only in the combination \( \sum_j a_j^\dagger S_j^\mu \). With this in mind, consider two sets of monopole positions \( x_k \) and \( \tilde{x}_k \) with identical locations for the massive monopoles, but with the massless monopoles constrained only by the requirement that \( \tilde{a}_j = W_{jk} a_k \), with \( W \) some \((N - 2) \times (N - 2)\) unitary matrix. If \( S_j^\mu \) gives a solution of Eq. (4.1) for the former set of positions, \( \tilde{S}_j^\mu = W_{jk} S_k^\mu \) is a solution for the transformed set. (Note that this transformation has no effect on either \( v \) or on the fields themselves.)

The possibility of performing such transformations implies that the positions of the massless monopoles are not all physically meaningful quantities. In fact, these yield only a single physical parameter, which can be identified by noting that these transformations leave invariant the quantity

\[
\sum_j a_j^\dagger a_j = \sum_j [a_j \cdot \sigma - i\alpha_j 0] = (x_1 - x_{N-1}) \cdot \sigma + \sum_{j=2}^{N-1} |x_j - x_{j-1}|. \tag{4.4}
\]

The first term on the right hand side is fixed by the positions of the two massive monopoles. The second term is just the sum of the distances between successive monopoles. It is precisely this sum that was identified in Ref. [3], from the properties of the moduli space, as the unique gauge-invariant quantity characterizing the non-Abelian cloud. It will be convenient to express it in terms of the “cloud parameter” \( b \geq 0 \), defined by the equation

\[
2b + R = \sum_{j=2}^{N-1} |x_j - x_{j-1}|. \tag{4.5}
\]

It is not hard to show that any two sets of massless monopole positions corresponding to the same value of \( b \) can be transformed into one another. We may therefore define a canonical set of positions by placing one of the massless monopoles on the \( z \)-axis at \( z_2 = z_1 - b \) and the remaining \( N - 3 \) massless monopoles on top of the massive monopole at \( z_{N-1} \). The only nonvanishing \( a_j \) are \( a_2 \) and \( a_3 \), which can be combined as the rows of a real \( 2 \times 2 \) matrix

\[
A = \left( \begin{array}{cc} \sqrt{2b} & 0 \\ 0 & \sqrt{2(b+R)} \end{array} \right). \tag{4.6}
\]

With this canonical choice of monopole positions, the solution for the \( S_k \) given in Eq. (3.17) takes on a particularly simple form. If these \( N - 2 \) row vectors are combined to form an \((N - 2) \times N\) matrix \( S \), the solution corresponding to the choice \( U = I \) can be written as

\[
S = \left( \begin{array}{cc} 0 & S \\ 0 & I_{N-4} \end{array} \right), \tag{4.7}
\]

where the columns (rows) have been grouped in blocks of 2, 2, and \( N - 4 \) (2 and \( N - 4 \)) and the \( 2 \times 2 \) matrix

\[
S = S^\dagger = (I + A A)^{-1/2}. \tag{4.8}
\]
Equations (3.5) and (4.3) then give $v(t)$, which in a similar block notation takes the form

$$
v(t) = \begin{cases} 
  f_1(t) \left( N^{-1/2}, -N_L^{-1}(ML)^{1/2}, 0 \right), & t_1 \leq t \leq t_2, \\
  f_{N-1}(t) \left( N^{-1/2}, N_R^{-1}(ML)^{1/2}, 0 \right), & t_2 \leq t \leq t_N.
\end{cases} \tag{4.9}
$$

Here $N = N_L + N_R$ and we have defined

$$L = MA^2(I + AMA)^{-1}. \tag{4.10}$$

Note that, as we will see explicitly below, $A$, $M$, and $L$ are all diagonal and hence commute.

These results can now be combined to yield the gauge and Higgs fields. To express these, it is convenient to introduce the integrals

$$H_L = \int_{t_1}^{t_2} dt [f_1(t), \nabla f_1(t)]$$
$$K_L = \int_{t_1}^{t_2} dt t f_1(t)^2 - t_2 N_L \tag{4.11}$$

and the similarly defined quantities $K_R$ and $H_R$ involving $f_{N-1}(t)$. The fields can then be written as

$$A = \begin{pmatrix} N^{-1/2} & 0 & 0 \\ 0 & (LM)^{1/2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a^{(1)} & a^{(3)} \\ a^{(3)*} & a^{(2)} \end{pmatrix} \begin{pmatrix} N^{-1/2} & 0 & 0 \\ 0 & (ML)^{1/2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.12}$$

$$\Phi = t_2 I + \begin{pmatrix} N^{-1/2} & 0 & 0 \\ 0 & (LM)^{1/2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \phi^{(1)} & \phi^{(3)} \\ \phi^{(3)*} & \phi^{(2)} \end{pmatrix} \begin{pmatrix} N^{-1/2} & 0 & 0 \\ 0 & (ML)^{1/2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.13}$$

where the $a^{(a)}$ and $\phi^{(a)}$ are the $2 \times 2$ matrices

$$a^{(1)} = -\frac{i}{2} (H_L + H_R) + \frac{i}{2} \left[ N^{1/2}, \nabla N^{1/2} \right]$$
$$\phi^{(1)} = (K_L + K_R)$$
$$a^{(2)} = -\frac{i}{2} (N_L^{-1} H_L N_L^{-1} + N_R^{-1} H_R N_R^{-1})$$
$$\phi^{(2)} = N_L^{-1} K_L N_L^{-1} + N_R^{-1} K_R N_R^{-1}$$
$$a^{(3)} = \frac{i}{2} \left[ (\nabla N_R - H_R)N_R^{-1} - (\nabla N_L - H_L)N_L^{-1} \right]$$
$$\phi^{(3)} = K_R N_R^{-1} - K_L N_L^{-1}. \tag{4.14}$$

To proceed further we need explicit expressions for the various integrals that we have defined. These are most easily expressed in terms of the vectors

$$y_L = r - x_1$$
$$y_R = r - x_{N-1} \tag{4.15}$$
and the quantities

\[ s_L = (t_2 - t_1)|y_L| = (t_2 - t_1)y_L \]
\[ s_R = (t_N - t_2)|y_R| = (t_N - t_2)y_R . \]  

(4.16)

Straightforward integration yields

\[ N_L = \frac{1}{2} y_L^2 \sinh s_L e^{-s_L y_L} \sigma \]
\[ N_R = \frac{1}{2} y_R^2 \sinh s_R e^{s_R y_R} \sigma \]
\[ K_L = \frac{1}{2} (t_2 - t_1) e^{-2s_L y_L} \sigma - N_L \]
\[ K_R = \frac{1}{2} (t_N - t_2) e^{2s_R y_R} \sigma - N_R \]
\[ H_L = -i y_L^2 (\hat{y}_L \times \sigma) (\sinh s_L \cosh s_L - s_L) \]
\[ H_R = -i y_R^2 (\hat{y}_R \times \sigma) (\sinh s_R \cosh s_R - s_R) . \]  

(4.17)

Substituting these expressions into Eq. (4.14) yields the remarkably simple formulas

\[ a^{(2)} = -\frac{1}{2} (\mathbf{V} \times \sigma) \]
\[ \phi^{(2)} = -\frac{1}{2} (\mathbf{V} \cdot \sigma) \]
\[ a^{(3)} = i \frac{1}{2} F \sigma \]
\[ \phi^{(3)} = i \frac{1}{2} F \]  

(4.18)

where

\[ \mathbf{V} = \hat{y}_L \left[ \coth s_L - \frac{s_L}{\sinh^2 s_L} \right] + \hat{y}_R \left[ \coth s_R - \frac{s_R}{\sinh^2 s_R} \right] \]  

(4.19)

and

\[ F = \frac{\hat{y}_L \cdot \sigma}{y_L} - \frac{\hat{y}_R \cdot \sigma}{y_R} + (t_2 - t_1) (1 - \hat{y}_L \cdot \sigma \coth s_L) + (t_N - t_2) (1 + \hat{y}_R \cdot \sigma \coth s_R) . \]  

(4.20)

Combining the expressions for \( N_L \) and \( N_R \) with Eq. (4.2), and using the fact that

\[ (y_L - y_R) \cdot \sigma = R \sigma_3 \]

(4.21)

we obtain

\[ M = (y_L \coth s_L + y_R \coth s_R + R \sigma_3)^{-1} . \]  

(4.22)

Combining this with Eqs. (4.6) and (4.10) yields

\[ L = \frac{2b + R - R \sigma_3}{y_L \coth s_L + y_R \coth s_R + R + 2b} . \]  

(4.23)

Note that the cloud parameter \( b \) enters the solution only through the matrix \( L \).
5 Asymptotic behavior

In this section we will examine in some detail the solutions that we have found. From the form of Eqs. (4.12) and (4.13), it is clear that for any $N \geq 5$ all $SU(N)$ solutions are essentially embeddings of solutions with one massless and two massive monopoles in $SU(4)$ broken to $U(1) \times SU(2) \times U(1)$. Therefore, without any loss of generality we can simplify our notation by specializing to the case $N = 4$. Each adjoint representation elementary multiplet of the theory can then be decomposed into five massless fields (an $SU(2)$ triplet and two singlets) together with a pair of massive doublets and a pair of massive singlets. Because of the “twisting” of the topologically nontrivial Higgs field, this decomposition will not in general have a simple correspondence with the matrix components of the fields. However, one might hope that matters would simplify in the region outside the cores of the massive monopoles (i.e., the region where $s_L$ and $s_R$ are both much greater than unity), where the massive fields would be expected to be exponentially small.

The first hint of this simplification comes from noting that $N_L$ and $N_R$ each have one exponentially large eigenvalue. Hence, the eigenvalues of $N = N_L + N_R$ are in general both exponentially large, implying that those of $N^{-1/2}$ are both exponentially small. The only exception to this occurs when the large-eigenvalue eigenvectors of $N_L$ and $N_R$ are almost parallel to each other, which happens only near the line joining the centers of the two massive monopoles. If we exclude the region close to this intermonopole axis, no elements of the matrices $F$, $M$, and $L$ are ever exponentially large. We then immediately see from Eq. (4.14) that the terms containing $a^{(3)}$ and $\phi^{(3)}$ are exponentially small.

If we therefore restrict ourselves to the region where $s_L, s_R \gg 1$ (so that we are outside the massive cores) and

$$\frac{y_L + y_R - R}{R} \gg e^{-2s_L} + e^{-2s_R}$$

(to avoid the intermonopole axis), we may approximate the fields by the block diagonal form

$$A = \begin{pmatrix} A^{(1)} & 0 \\ 0 & A^{(2)} \end{pmatrix}$$

$$\Phi = \begin{pmatrix} \Phi^{(1)} & 0 \\ 0 & \Phi^{(2)} \end{pmatrix}.$$  

With the fields written in this form, their group theoretic interpretation is fairly clear. The traceless parts of the nonvanishing blocks correspond to two commuting $SU(2)$ subgroups, one of which (the lower right, as we shall see) must be the unbroken $SU(2)$. The massless $U(1)$ fields are then

\footnote{For the special case $R = 0$ and $t_1 = -t_N$, these actually reduce to embeddings of the $SO(5)$ solution of Ref. 12.}
contained in the traceless part of the other block and in the two traces. Examining Eqs. (4.12) and (4.13) and recalling that the dependence on the cloud parameter is only through \( L \), we see that \( A^{(2)} \) and \( \Phi^{(2)} \) may be \( b \)-dependent, but \( A^{(1)} \) and \( \Phi^{(1)} \) are not. (Because the fields are traceless, the \( b \)-dependence must be entirely in the traceless parts of \( A^{(2)} \) and \( \Phi^{(2)} \).)

Because of the factors of \( N^{-1/2} \), the analysis required to obtain the asymptotic form for the upper left block is in general somewhat tedious. However, the calculation simplifies considerably if \( e^{2s_L}/y_L \) is either much less than or much greater than \( e^{2s_R}/y_R \) (which is the case in almost of space.) If we define the unit vector \( \hat{n} \) to be equal to \( \hat{y}_R \) (\( \hat{y}_L \)) in the former (latter) region, then, up to exponentially small corrections, the gauge and Higgs fields are

\[
A^{(1)} = \frac{(y_L + y_R)}{2((y_L + y_R)^2 - R^2)} \left\{ (\hat{y}_L \times \hat{y}_R) + [\hat{n} \times (\hat{y}_L + \hat{y}_R)] \hat{n} \cdot \sigma \right\} + \frac{i}{4} [\nabla \cdot \sigma, \hat{n} \cdot \sigma] \quad (5.4)
\]

and

\[
\Phi^{(1)} = \left( t_4 - \frac{1}{2y_R} \right) \left( \frac{1 + \hat{n} \cdot \sigma}{2} \right) + \left( t_1 + \frac{1}{2y_L} \right) \left( \frac{1 - \hat{n} \cdot \sigma}{2} \right) \quad (5.5)
\]

while the asymptotic field strength is

\[
B^{(1)} = \frac{\hat{y}_R}{2y_R} \left( \frac{1 + \hat{n} \cdot \sigma}{2} \right) - \frac{\hat{y}_L}{2y_L} \left( \frac{1 - \hat{n} \cdot \sigma}{2} \right). \quad (5.6)
\]

(In fact, it is not hard to show that \( \Phi^{(1)} \) and \( B^{(1)} \) must be of this form (although with a more complicated expression for \( \hat{n} \)) at all points outside the massive cores.)

Although the calculation of the fields in the lower right corner is in principle straightforward, somewhat lengthy manipulations needed to put the result in a simple form. We leave the details of these to the Appendix, and state the results here. From Eqs. (4.12) and (4.18) we obtain

\[
A^{(2)} = -\frac{1}{2}(LM)^{1/2}V \times \sigma (ML)^{1/2} \quad (5.7)
\]

where we can now use the asymptotic forms

\[
V = \hat{y}_L + \hat{y}_R \quad (5.8)
\]

\[
M = (y_L + y_R + R\sigma_3)^{-1} \quad (5.9)
\]

and

\[
L = \frac{2b + R - R\sigma_3}{y_L + y_R + R + 2b} \equiv h(2b + R - R\sigma_3). \quad (5.10)
\]

For the Higgs field we obtain the particularly simple form

\[
\Phi^{(2)} = t_2 I + L^{1/2} \left[ \frac{1}{2y_R} \left( \frac{1 - \hat{q} \cdot \sigma}{2} \right) - \frac{1}{2y_L} \left( \frac{1 + \hat{q} \cdot \sigma}{2} \right) \right] L^{1/2} \quad (5.11)
\]
where \( \hat{q} \) is a unit vector with components
\[
\hat{q}_a = \begin{cases} 
\frac{(y_L)_a + (y_R)_a}{\sqrt{(y_L + y_R)^2 - R^2}}, & a = 1, 2, \\
y_R - y_L & a = 3.
\end{cases}
\] (5.12)

(Note that at large distances \( \hat{q} \), like \( \hat{n} \), approaches the radial unit vector.) The field strength is
\[
B^{(2)} = L^{1/2} \left\{ \left[ \hat{y}_L + h \frac{(\hat{y}_L + \hat{y}_R)}{y_L} \right] \left( 1 + \hat{q} \cdot \sigma \right) - \left[ \hat{y}_R + h \frac{(\hat{y}_L + \hat{y}_R)}{y_R} \right] \left( 1 - \hat{q} \cdot \sigma \right) \right\} L^{1/2}
\] (5.13)

where \( h \) is defined by Eq. (5.10) and
\[
f_a = \begin{cases} 
\sqrt{(y_L + y_R)^2 - R^2} \sigma_a, & a = 1, 2, \\
(y_L + y_R) \sigma_3 - R & a = 3.
\end{cases}
\] (5.14)

It is instructive to consider several limiting cases. First, suppose that \( b = 0 \). From Eq. (4.23) we see that \( L \) is then proportional to \( 1 - \sigma_3 \) so that, except for a constant term in \( \Phi \), the third rows and columns of all fields vanish. The solutions are then essentially embeddings of \( SU(3) \to U(1) \times U(1) \) solutions [19]. Since the unbroken subgroup of \( SU(3) \) is Abelian, it should be possible to choose a gauge in which the asymptotic fields are simply superpositions of single monopole fields. Indeed, these can be written in the form
\[
\Phi = U_{1}^{-1}(r) \begin{pmatrix} t_4 - \frac{1}{2y_R} & 0 & 0 & 0 \\ 0 & t_2 - \frac{1}{2y_L} + \frac{1}{2y_R} & 0 & 0 \\ 0 & 0 & t_2 & 0 \\ 0 & 0 & 0 & t_1 + \frac{1}{2y_L} \end{pmatrix} U_{1}(r) \] (5.15)

\[
B = U_{1}^{-1}(r) \begin{pmatrix} \frac{\hat{y}_R}{2y_R} & 0 & 0 & 0 \\ 0 & \frac{\hat{y}_L}{2y_L} - \frac{\hat{y}_R}{2y_R} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{\hat{q}_a}{2y_L} \end{pmatrix} U_{1}(r). \] (5.16)

[Here, in order to make the \( U(1) \times SU(2) \times U(1) \) structure of the theory clearer, we have reordered the rows and columns to correspond with the order in Eqs. (1.1) and (1.2). With this reordering, the unbroken \( SU(2) \) is contained in the middle \( 2 \times 2 \) block, and a fundamental monopole solution with a single nonzero \( n_j \) corresponds to an embedding in a pair of adjacent rows and columns.] Viewed as an \( SU(3) \) solution, this corresponds to a configuration containing one each of the two massive fundamental monopoles of the theory, located at points \( x_1 \) and \( x_3 \). Viewed as an \( SU(4) \)
solution, it can be interpreted as containing a massive fundamental monopole with \( n_j = \delta_{j1} \) at \( x_1 \) and a superposition of a massive monopole with \( n_j = \delta_{j3} \) and a massless monopole\(^3\) with \( n_j = \delta_{j2} \) at \( x_3 \). Even though the underlying \( SU(3) \) solution has purely Abelian long-range fields, the long-range part of the \( SU(4) \) solution is non-Abelian in the sense that the unbroken \( SU(2) \) subgroup acts nontrivially on \( A^{(2)} \) and \( \Phi^{(2)} \). The crucial point, however, is that the \( SU(2) \) orientations of the Coulomb field centered at \( x_1 \) and the Coulomb field centered at \( x_3 \) are aligned, so that the net \( SU(2) \) component is a purely dipole field that falls as \( R/y^3 \) for \( y \equiv (y_L + y_R)/2 \gg R \).

Next, consider the case \( b \gg R \). Now \( L \) is approximately proportional to a unit matrix, with

\[
L = \frac{2b}{y_L + y_R + 2b} + O(R/b),
\]

while \( h \approx 1/(2b + y_L + y_R) \). Thus, in the region where \( y_L \) and \( y_R \) are much less than \( b \) the fields can be written as

\[
\Phi = U_2^{-1}(r)
\begin{pmatrix}
    t_4 - \frac{1}{2y_R} & 0 & 0 & 0 \\
    0 & t_2 + \frac{1}{2y_R} & 0 & 0 \\
    0 & 0 & t_2 - \frac{1}{2y_L} & 0 \\
    0 & 0 & 0 & t_1 + \frac{1}{2y_L}
\end{pmatrix} U_2(r) + \cdots
\]

\[
B = U_2^{-1}(r)
\begin{pmatrix}
    \frac{\hat{y}_R}{2y_R} & 0 & 0 & 0 \\
    0 & -\frac{\hat{y}_R}{2y_R} & 0 & 0 \\
    0 & 0 & \frac{\hat{y}_L}{2y_L} & 0 \\
    0 & 0 & 0 & -\frac{\hat{y}_L}{2y_L}
\end{pmatrix} U_2(r) + \cdots
\]

where the dots represent terms that are suppressed by powers of \( R/b, y_L/b, \) or \( y_R/b \). These are just the fields expected for two massive monopoles, with topological charges \( n_j = \delta_{j1} \) and \( n_j = \delta_{j3} \), centered at \( x_1 \) and \( x_3 \), respectively. In contrast with the previous case, the \( SU(2) \) components of their two magnetic charges are not aligned, and so the unbroken \( SU(2) \) contains two Coulomb fields rather than the dipole field of Eq. (5.16). These non-Abelian Coulomb fields disappear when

\(^3\)Viewing the massless monopole as being centered at \( x_3 \) is a gauge choice; moving it to \( x_1 \) simply corresponds to a re-ordering of the rows and columns.
\( y \gg b \), where we obtain

\[
\Phi = U_3^{-1}(r) \begin{pmatrix}
 t_4 - \frac{1}{2y} & 0 & 0 & 0 \\
 0 & t_2 & 0 & 0 \\
 0 & 0 & t_2 & 0 \\
 0 & 0 & 0 & t_1 + \frac{1}{2y}
\end{pmatrix} \begin{pmatrix}
 U_3(r) + O(b/y^2) \\
 U_3(r) + O(b/y^3)
\end{pmatrix}
\]  

(5.20)

\[
B = U_3^{-1}(r) \begin{pmatrix}
 \frac{\dot{y}}{2y^2} & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & -\frac{\dot{y}}{2y^2}
\end{pmatrix}
\]  

(5.21)

Thus, at distances large compared to \( b \) the magnetic fields exhibit the behavior that would be expected if a single massless monopole were added to the two massive fundamental monopoles. However, this massless monopole is not manifested as a localized structure with a well-defined center. Instead, we simply have a transition from a “cloud” region of size \( \sim b \) containing non-Abelian Coulomb magnetic fields to an outer region where these fields are cancelled.

In this discussion we have excluded the region close to the intermonopole axis. To explore the fields in this region, one must go back to the equations of the previous section. [Equations (5.8–5.10) are not valid approximations here, even outside the monopole cores.] Doing so, we find that the fields do not have the simple block diagonal structure of Eqs. (5.3) and (5.6). In addition, some components of \( \mathbf{A} \) become exponentially large as one approaches the axis, while \( \Phi \) turns out to be rapidly varying. However, these are essentially artifacts of our choice of gauge. The net magnetic charge of our solutions is a unit charge in the unbroken \( U(1) \) that is contained in \( A^{(1)} \).

The long-range twisting of the Higgs field must then be topologically equivalent to an embedding of the \( SU(2) \) hedgehog configuration in the corresponding \( SU(2) \); this can be seen in the behavior of the unit vector \( \hat{n} \) that appears in Eqs. (5.4–5.6). However, this cannot be the whole story. Because the two monopoles have different \( U(1) \) charges, there must be some additional twisting of the Higgs field near each of the monopoles. The conventions that we have adopted are such that this inevitable additional twisting is confined to the narrow region near the axis where Eq. (5.1) does not hold. In order that \( \mathbf{D} \Phi \) and \( \mathbf{B} \) not become large, these rapid variations in the direction of \( \Phi \) must be compensated by large values of \( \mathbf{A} \). That this actually happens can be verified by evaluating the field strength along the axis. One finds that, up to exponentially small corrections, the magnetic field along this axis is independent of \( b \) and can be put in the form of Eq. (5.17).
6 Concluding Remarks

In this paper we have shown how the Nahm construction can be used to obtain explicit multimonopole solutions corresponding to \(N - 1\) distinct fundamental monopoles in \(SU(N)\). In the case where the symmetry is broken maximally, to \(U(1)^{N-1}\), these solutions are described by \(3(N - 1)\) gauge-invariant parameters that specify the positions of the component monopoles; these combine with \(N - 1\) overall \(U(1)\) phases to give the full set of collective coordinates. Even though these solutions have no rotational symmetry at all, the gauge and Higgs fields can be expressed in terms of elementary (i.e., rational and hyperbolic) functions for arbitrary values of these parameters. Of particular interest is the behavior of these solutions in the limit where the unbroken group is enlarged to \(U(1) \times SU(N - 2) \times U(1)\) and \(N - 3\) of the component monopoles become massless. Examining the solutions, one sees no trace of the individual massless monopoles, but only a single “non-Abelian cloud”. Indeed, most of the massless monopole coordinates become redundant in this limit, with the solutions depending only on a sum of intermonopole distances that is conveniently described by the cloud parameter \(b\). Not only are the massless monopole positions somewhat ill-defined, but so, in a sense, is their number. As we have seen, the \(SU(N)\) solution that nominally contains \(N - 3\) massless monopoles is essentially equivalent to an embedded \(SU(4)\) solution containing a single massless monopole.

For values of \(b\) that are large compared to the distance \(R\) between the massive monopoles, the cloud is rather similar to that found in the \(SO(5)\) case. The non-Abelian Coulomb magnetic fields due to the two massive monopoles are present inside the cloud, but are screened at distances much larger than \(b\). On the other hand, when \(b \ll R\) these Coulomb fields are present outside the cloud (i.e., at points more than a distance \(b\) from either of the massive cores), but are aligned with each other in such a manner that the entire solution is simply an embedding of a purely Abelian configuration.

These non-Abelian clouds and their properties clearly warrant further study. For the configurations we have considered here, those parts of the solution associated with the cloud are particularly simple. The dependence on the cloud parameter is contained in a single matrix \(L\), while corresponding components of the Higgs and gauge fields depend on the same function of the spatial variables and differ only in their tensor structure. In addition, the actual functional forms outside the massive cores are relatively simple. This simplicity suggests that it might be feasible to determine the structure of the non-Abelian clouds associated with more complex configurations, at least in the regions outside the cores of the massive monopoles. The most compelling questions are
associated with the conjectured electric-magnetic duality. From this point of view, the massless monopoles are clearly the duals of the massless non-Abelian gauge bosons and their superpartners. How do these particles reflect the strange properties of the massless monopoles?

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Appendix

In this Appendix we outline the manipulations that lead to Eqs. (5.11) and (5.13) for $\Phi^{(2)}$ and $B^{(2)}$. Throughout, $V, M,$ and $L$ should be understood to be given by the asymptotic forms in Eqs. (5.8-5.10).

The first step is to obtain some useful identities. With the aid of the law of cosines, we obtain

$$V^2 = 2(1 + \hat{y}_L \cdot \hat{y}_R) = \frac{(y_L + y_R)^2 - R^2}{y_L y_R} \quad (A.1)$$

Because we have chosen the massive monopoles to lie along the $z$-axis, the third component of any vector can be obtained by noting that $Rw_3 = w \cdot (y_L - y_R)$. In particular, many of the subsequent results make use of the identities

$$(\hat{y}_L)_3 = \frac{(y_L + y_R)}{R} - \frac{(y_L + y_R)^2 - R^2}{2R y_L} \quad (A.2)$$

and

$$(\hat{y}_R)_3 = -\frac{(y_L + y_R)}{R} + \frac{(y_L + y_R)^2 - R^2}{2R y_R} \quad (A.3)$$

Next, we define a vector $f$ by the equation

$$M^{1/2} \sigma M^{1/2} = [(y_L + y_R)^2 - R^2]^{-1} f \quad (A.4)$$

Explicit expressions for the components of $f$ are given in Eq. (5.14). The inner products of this equation with $\hat{y}_L$ and $\hat{y}_R$ then lead, with the aid of Eqs. (A.2) and (A.3), to

$$M^{1/2} \hat{y}_L \cdot \sigma M^{1/2} = \left(\frac{\hat{q} \cdot \sigma + 1}{2y_L}\right) - M \quad (A.5)$$

and

$$M^{1/2} \hat{y}_R \cdot \sigma M^{1/2} = \left(\frac{\hat{q} \cdot \sigma - 1}{2y_R}\right) + M \quad (A.6)$$

where $\hat{q}$ is defined by Eq. (5.12). Combining these last two equations with the results of Sec. 4 gives the asymptotic expression for $\Phi^{(2)}$ shown in Eq. (5.11).
To obtain an expression for the magnetic field we need the derivatives of $L$ and $M$. Making use of the fact that these are both diagonal matrices, we find that

$$L^{-1/2} \nabla L^{1/2} = -\frac{1}{2} L \nabla L^{-1} = -\frac{1}{2} h V$$  \hspace{1cm} \text{(A.7)}$$

and

$$M^{-1/2} \nabla M^{1/2} = -\frac{1}{2} M \nabla M^{-1} = -\frac{1}{2} M V$$  \hspace{1cm} \text{(A.8)}$$

where $h = y_L + y_R + R + 2b$. In addition, we note that

$$L = 1 - hM^{-1}$$  \hspace{1cm} \text{(A.9)}$$

Using the last three identities, we can decompose the magnetic field $B^{(2)}$ as

$$B_a^{(2)} = \varepsilon_{abc} \left[ \partial_b A_c^{(2)} + i A_b^{(2)} A_c^{(2)} \right]$$

$$= L^{1/2} \left[ M^{1/2} b_a^{(A)} M^{1/2} + h M^{1/2} b_a^{(B)} M^{1/2} \right] L^{1/2}$$  \hspace{1cm} \text{(A.10)}$$

where

$$b_a^{(A)} = \varepsilon_{abc} \left[ -\frac{1}{2} \partial_b (V \times \sigma)_c - \frac{1}{2} \{ M^{-1/2} \partial_b M^{1/2}, (V \times \sigma)_c \} + i \frac{1}{4} (V \times \sigma)_b M (V \times \sigma)_c \right]$$

$$= L^{1/2} \left[ M^{1/2} b_a^{(A)} M^{1/2} + h M^{1/2} b_a^{(B)} M^{1/2} \right] L^{1/2}$$  \hspace{1cm} \text{(A.11)}$$

and

$$b_a^{(B)} = \varepsilon_{abc} \left[ \frac{1}{2} V_b (V \times \sigma)_c - i \frac{1}{4} (V \times \sigma)_b (V \times \sigma)_c \right]$$

$$= V_a V \cdot \sigma - V^2 \sigma_a$$  \hspace{1cm} \text{(A.12)}$$

The last equation, together with Eqs. (A.4-A.6), immediately gives the terms in Eq. (5.13) proportional to $h$. To obtain the $h$-independent terms, it is useful to write

$$M = \frac{y_L + y_R}{y L y_R V^2} - \frac{R \sigma_3}{(y_L + y_R)^2 - R^2} \equiv \frac{y_L + y_R}{y L y_R V^2} - g \sigma_3$$  \hspace{1cm} \text{(A.13)}$$

One then finds that

$$b_a^{(A)} = (\hat{y}_L)_a \hat{y}_L \cdot \sigma + (\hat{y}_R)_a \hat{y}_R \cdot \sigma + g(V_a \sigma_3 - \frac{1}{2} V^2 \delta_{a3})$$  \hspace{1cm} \text{(A.14)}$$

which leads to

$$M^{1/2} b_a^{(A)} M^{1/2} = \frac{(\hat{y}_L)_a}{2 y_L} \left( \frac{\hat{q} \cdot \sigma + 1}{2} \right) + \frac{(\hat{y}_R)_a}{2 y_R} \left( \frac{\hat{q} \cdot \sigma - 1}{2} \right) + \frac{1}{2} M \left[ \frac{(\hat{y}_R)_a}{y_R} - \frac{(\hat{y}_L)_a}{y_L} + g(2V_a \sigma_3 - V^2 \delta_{a3}) \right]$$  \hspace{1cm} \text{(A.15)}$$

After some algebra, one finds that the quantity in square brackets multiplying $M$ vanishes, thus leading to the final expression for $B^{(2)}$, Eq. (5.13).
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