Galerkin Format and Analysis of Mathematical Model of Groundwater Pollution

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Abstract. As an important resource for people to survive, water plays an important role in human life and industrial and agricultural production. However, with the development of industry, groundwater pollution and surface decline are becoming more and more serious. As one of the important pollution sources of water pollution, nitrogen-containing pollutants play a very important role in the strategic reserve of water resources. It can be seen that scientific management and evaluation of the quality of water resources are of great significance for the rational development and utilization of water resources. Based on the mathematical model of groundwater pollution, four full discrete Galerkin finite element schemes are constructed. In this paper, the convergence and stability of these four schemes are discussed in due course. In order to estimate the maximum error of the scheme, it can not only solve the seepage problem with finite elements, but also understand the transportation law of matter in porous media to a certain extent.

1. Introduction
Groundwater pollutants are mainly composed of three parts: ammonia nitrogen, nitrate nitrogen and nitrite nitrogen pollutants. With the rapid development of modern cities, the degree of groundwater pollution of Triaz is becoming more and more serious, which will seriously affect the reserve and use of groundwater resources in China. Normally, the nitrogen in groundwater comes from the organic matter produced by the decomposition of animals, plants and bacteria, and the content is relatively small. However, in recent years, with the excessive exploitation of human industry and the random discharge of waste, the content of nitrogen in groundwater has seriously exceeded the standard, and the overall imbalance of the system has caused serious nitrogen pollution. In addition, groundwater pollution also involves other complex and changeable pollutants. Based on the mixed boundary condition, this paper constructs four Galerkin finite element schemes, and finally gives numerical examples to provide a reference path for groundwater pollution control[1].

2. Issues raised
Because all kinds of sewage discharged in human life and pesticides, fertilizers and other pollutants in the field of industry and agriculture will gradually penetrate into the ground and become pollutants in groundwater. When these solutes are dissolved in groundwater, they can move in convection with groundwater and diffuse continuously due to the diffusion of water molecular concentration. In other words, the groundwater pollution model is composed of solute transport and water flow. Therefore, researchers should solve the problem of solute transport and water flow at the same time. This paper assuming that \( \Omega \subset \mathbb{R}^n \) is a bounded region satisfying the condition[2], and \( \Gamma \) is smooth in segments,
when the bounded area satisfying the condition is smooth one by one, the solution of the water pollution model is shown in figure 1

\[
\begin{align*}
\frac{\partial H}{\partial t} - 5 \cdot (K(H)5H) &= f(t), (x, t) \in Q_T \quad (1) \\
\frac{\partial s}{\partial t} - 5 \cdot (D(x)5s) + 5 \cdot (us) &= g(t), (x, t) \in Q_T \\
H(x, 0) &= H_0(x), \quad s(x, 0) = s_0(x) \quad x \in K \quad (3) \\
H(x, t)|_{r_1} &= 0, \quad s(x, t)|_{r_1} = 0 \quad t \in [0, T] \quad (4) \\
K(H)5H \cdot n|_{r_2} &= H_1(x, t), D(x)5s \cdot n|_{r_2} = s_1(x, t), \quad t \in [0, T] \quad (5)
\end{align*}
\]

Figure 1. Determination of Water Pollution Model.

In this model, “n” means the unit outward normal vector of \( \Gamma \), \( \Gamma_1 \cup \Gamma_2 = \Gamma \), \( \text{meas} \ (\Gamma_1) > 0 \), \( \text{meas} \ (\Gamma_1 \cap \Gamma_2 ) = 0 \). Equation (1) is the flow equation of groundwater, (2) is the solute transport equation, \( H(x, t) \) is the groundwater head function, and \( S(x, t) \) is the concentration function of pollution mass. Moreover, \( f(t) \) and \( g(t) \) both represent the source flow rate [3], \( u \) is the velocity vector of groundwater pollution, \( f(x) \) refers to the temperature of groundwater, \( _w(x) \) is the degree of water supply, and \( (x) \) is the mass diffusion coefficient matrix. The formula is as follows: \( D(x) = H(x) \cdot I \), where \( H(x) \) represents the porosity of the medium, the diffusion coefficient, and the unit matrix [4].

In addition, the functional space of groundwater pollution can be defined as \[ V = \{v|v \in H^2(K), v|_{r_1} = 0\} \]. According to the actual geological conditions, this paper makes the following assumptions:

1. In the equations of functions, the coefficients and the free terms at the left and right ends have positive upper and lower bounds, at the same time \( D(x) \) is positive.
2. The initial values \( H_0(x), s_0 \) and the function boundary value \( H_1(x, t), s_1(x, t) \) need to satisfy the following function conditions:

\[
\begin{align*}
H_0(x) &\in V \cap H^2(K), \ s_0 &\in V \cap H^2(K), \ H_1 &\in L^2(0, T; H^2(K)), \\
s_1 &\in L^2(0, T; H^2(K)).
\end{align*}
\]

When \( H_0(x), s_0(x), s_1(x, t), H_1(x, t), f(t), g(t) \) are all known functions and Meet the above two assumptions, then Function \{H(x, t), S(x, t)\} is solved as shown in Figure 2[5].

\[
\begin{align*}
H(x, 0) &= H_0, \ s(x, 0) = s_0 \\
\frac{\partial H}{\partial t} + 5 \cdot (K(H)5H) &= f(t), (x, t) \in Q_T \quad (6) \\
\frac{\partial s}{\partial t} - 5 \cdot (D(x)5s) + 5 \cdot (us) &= g(t), (x, t) \in Q_T \\
\left( \frac{\partial}{\partial t} \right)^k + a_s(s, k) + b(u, s, k) &= (g, k) + \langle H_1(x)s + s_1(x), k \rangle \quad \forall \ k \in V \quad (8) \\
H(x, 0) &= H_0, \ s(x, 0) = s_0 \\
\left| \frac{\partial}{\partial t} \right|^k + a_s(s, k) + b(u, s, k) &= (g, k) + \langle H_1(x)s + s_1(x), k \rangle \quad \forall \ k \in V \quad (7)
\end{align*}
\]

Figure 2. Function \{H(x, t), S(x, t)\} solving process.
3. Galerkin four formats

3.1. Euler -Galerkin Format

Assuming $H_h, H_h, \ldots, H_h^{-1} \in V_h$ and $s_h^0, s_h^1, \ldots, s_h^{m-1} \in W_h$ are all known functions, then the Solving equations of $\{H_h^n, s_h^n\} \in V_h \times W_h$ is shown as formula 1.

Formula 1:

\[
\frac{1}{\Delta t} \left( H_h^n - H_h^{n-1} \right), v_h \right) + a_1(H_h^n, H_h^n, V_h) + \left< H_h^n(x), v \right>_2, V_h \in V_h
\]

\[
\frac{1}{\Delta t} \left( s_h^n - s_h^{n-1}, w_h \right) + a_2(s_h^n, w_h) + b(u_h^n, s_h^n, w_h)
\]

\[
= \left< g_h^n, w_h \right> + \left< H_h^n(x), s_h^n + s_h^n(x), w_h \right>_2, \quad w_h \in W_h
\]

3.2. Crank -Nicolson -Galerkin Format

Assuming $H_h, H_h, \ldots, H_h^{-1} \in V_h$ and $s_h^0, s_h^1, \ldots, s_h^{m-1} \in W_h$ are all known functions, then the Solving equations of $\{H_h^n, s_h^n\} \in V_h \times W_h$ is shown as formula 2.

Formula 2:

\[
\frac{1}{\Delta t} \left( H_h^n - H_h^{n-1} \right), v_h \right) + a_1(H_h^n, H_h^n, V_h)
\]

\[
= \left< f_h^n, v_h \right> + \left< H_h^n(x), v_h \right>_2, \quad v_h \in V_h
\]

\[
\frac{1}{\Delta t} \left( s_h^n - s_h^{n-1}, w_h \right) + a_2(s_h^n, w_h) + b(u_h^n, s_h^n, w_h)
\]

\[
= \left< g_h^n, w_h \right> + \left< H_h^n(x), s_h^n + s_h^n(x), w_h \right>_2, \quad w_h \in W_h
\]

\[
\overline{H}_h^n = \left( H_h^n + H_h^{n-1} \right)/2, \quad \overline{s}_h^n = \left( s_h^n + s_h^{n-1} \right)/2 \quad \overline{u}_h^n = \left( u_h^n + u_h^{n-1} \right)/2
\]

3.3. Linear Crank -Nicolson -Galerkin Format

Assuming $H_h, H_h, \ldots, H_h^{-1} \in V_h$ and $s_h^0, s_h^1, \ldots, s_h^{m-1} \in W_h$ are all known functions, then the Solving equations of $\{H_h^n, s_h^n\} \in V_h \times W_h$ is shown as formula 3.

Formula 3:
4. Conclusion
As an important part of water resources, groundwater quality and stability are important guarantees for urban construction, industrial production and agricultural irrigation. Based on the Galerkin format of groundwater pollution mathematical model, this paper discusses the four formats and analyzes their stability and error, so as to provide feasible ideas for water pollution control. It plays a positive role in pollutant treatment and water resources protection. However, Galerkin format can not completely solve any problems of water pollution, and its format itself needs to be improved, so researchers and staff in related fields are actively exploring the effectiveness of Galerkin format. Strive to broaden the governance ideas, so as to protect water resources and improve underground water quality.

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