ON WELL-POSEDNESS OF PARABOLIC EQUATIONS OF NAVIER-STOKES TYPE WITH $BMO^{-1}$ DATA

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Abstract. We develop a strategy making extensive use of tent spaces to study parabolic equations with quadratic nonlinearities as for the Navier-Stokes system. We begin with a new proof of the well-known result of Koch and Tataru on the well-posedness of Navier-Stokes equations in $\mathbb{R}^n$ with small initial data in $BMO^{-1}(\mathbb{R}^n)$. We then study another model where neither pointwise kernel bounds nor self-adjointness are available. In this case, $BMO^{-1}(\mathbb{R}^n)$ has to be replaced by an adapted space.

1. Introduction

In [32], it was shown that the incompressible Navier-Stokes equations in $\mathbb{R}^n$ are well-posed for small initial data in $BMO^{-1}(\mathbb{R}^n)$. The result was a breakthrough, and it is believed to be best possible, in the sense that $BMO^{-1}(\mathbb{R}^n)$ is the largest possible space with the scaling of $L^n(\mathbb{R}^n)$ where the incompressible Navier-Stokes equations are proved to be well-posed. Ill-posedness is shown in the largest possible space $B_{\infty,\infty}^{-1}(\mathbb{R}^n)$ in [15], and in a space between $BMO^{-1}(\mathbb{R}^n)$ and $B_{\infty,\infty}^{-1}(\mathbb{R}^n)$ in [42]. See also some counter-examples of this type in [13].

The proof in [32] reduces to establishing the boundedness of a bilinear operator. This proof has two main ingredients: bounds coming from the representation of the Laplacian (such as the estimates for the Oseen kernel) and, in the crucial step, self-adjointness of the Laplacian to obtain an energy estimate using a clever integration by parts. Our new proof is rather based on operator theoretical arguments with emphasis on use of tent spaces, maximal regularity operators and Hardy spaces. In particular, we do not make use of self-adjointness of the Laplacian: we obtain the energy estimate by using Hardy space estimates for the main term and cruder estimates for a remainder term. Although more involved for the Navier-Stokes system as compared to the original proof, our argument is flexible enough to adapt to other models. We illustrate this at the end of the article by treating a more complicated model with rougher operators.

That our techniques have generalisations to rougher operators is thanks to recent works on maximal regularity in tent spaces (cf. [9] and [7]) and on Hardy spaces associated with (bi-)sectorial operators (cf. [5], [8], [25], [26] and followers). Using those results, it is possible to adapt our new proof to operators whose gradient of the semigroup (or the semigroup itself, although we do not do it here) only satisfies bounds of non-pointwise type. This could open up the way to possible generalisations for Navier-Stokes equations on rougher domains and in other type of geometry (cf. [40], [38], [36], [37] for Lipschitz domains in Riemannian manifolds, and [14] on the Heisenberg group), geometric flows (cf. [30]), or other semilinear parabolic equations of a similar structure, but for rougher domains or operators (cf. [33] for dissipative quasi-geostrophic equations, and [22], [23] for abstract formulations of parabolic equations with quadratic nonlinearity). Let us
also mention the survey article [31], which considers parabolic equations with a similar structure. The solution spaces considered have some similarities with the ones we consider in Section 5. The approach in [31] seems more suitable for applications on uniform manifolds, but restricted to operators with pointwise bounds, whereas one of the key aspects of this article is to show that our methods can be adapted to operators that satisfy non-pointwise bounds.

Potential applications may also be stochastic Navier-Stokes equations (cf. e.g. [35] and the references therein). The maximal regularity operators on tent spaces we are relying on in our proof, have proven useful already for other stochastic differential equations (cf. [10]).

2. The new proof of Koch-Tataru’s result

Consider the incompressible Navier-Stokes equations

\[
\begin{aligned}
\text{(NSE)} \\
\begin{cases}
  u_t + (u \cdot \nabla)u - \Delta u + \nabla p &= 0, \\
  \text{div } u &= 0, \\
  u(0, .) &= u_0,
\end{cases}
\end{aligned}
\]

where \(u(t, x)\) is the velocity and \(p(t, x)\) the pressure with \((t, x) \in \mathbb{R}_+^{n+1} = (0, \infty) \times \mathbb{R}^n\). As usual, the pressure term can be eliminated by applying the Leray projection \(P\). It is known from [21] that the differential Navier-Stokes equations are equivalent to their integrated counterpart

\[
\begin{aligned}
\begin{cases}
  u(t, .) &= e^{t\Delta}u_0 - \int_0^t e^{(t-s)\Delta}P \text{div}(u(s, .) \otimes u(s, .)) \, ds, \\
  \text{div } u_0 &= 0,
\end{cases}
\end{aligned}
\]

under an assumption of uniform local square integrability of \(u\). (In fact, under such a control on \(u\), most possible formulations of the Navier-Stokes equations are equivalent, as shown by the nice note of Dubois [18].) Using the Picard contraction principle, matters reduce to showing that the bilinear operator \(B\), defined by

\[
B(u, v)(t, .) := \int_0^t e^{(t-s)\Delta}P \text{div}(u \otimes v(s, .)) \, ds,
\]

is bounded on an appropriately defined admissible path space to which the free evolution \(e^{t\Delta}u_0\) belongs. This is what we reprove with an argument based on boundedness of singular integrals like operators on parabolically scaled tent spaces.

For a ball \(B := B(x, r) \subseteq \mathbb{R}^n\), denote \(\lambda B = \lambda B(x, r) = B(x, \lambda r)\), and \(S_0(B) = B, S_j(B) = 2^j B \setminus 2^{j-1} B\) for \(j \geq 1\). We use the following tent spaces on \(\mathbb{R}_+^{n+1}\).

**Definition 2.1.** The tent space \(T^{1, 2}(\mathbb{R}_+^{n+1})\) is defined as the space of all measurable functions \(F\) in \(\mathbb{R}_+^{n+1}\) such that

\[
\|F\|_{T^{1, 2}(\mathbb{R}_+^{n+1})} = \left( \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}_+^{n+1}} t^{-n/2} 1_{B(x, \sqrt{t})}(y) |F(t, y)|^2 \, dydt \right)^{1/2} \, dx \right) < \infty.
\]

The tent spaces \(T^{\infty, 1}(\mathbb{R}_+^{n+1})\) and \(T^{\infty, 2}(\mathbb{R}_+^{n+1})\) are defined as the spaces of all measurable functions \(F\) in \(\mathbb{R}_+^{n+1}\) such that

\[
\|F\|_{T^{\infty, p}(\mathbb{R}_+^{n+1})} = \sup_{x \in \mathbb{R}^n} \sup_{t > 0} \left( \int_0^t \int_{B(x, \sqrt{t})} |F(s, y)|^p \, dyds \right)^{1/p} < \infty,
\]
for \( p \in \{1, 2\} \), respectively. The tent space \( T^{1,\infty}(\mathbb{R}^{n+1}_+) \) is defined as the space of all continuous functions \( F : \mathbb{R}^{n+1}_+ \to \mathbb{C} \) such that the parabolic non-tangential limit \( \lim_{(t,y) \to x} F(t,y) \) exists for a.e. \( x \in \mathbb{R}^n \) and
\[
\|F\|_{T^{1,\infty}(\mathbb{R}^{n+1}_+)} = \|N(F)\|_{L^1(\mathbb{R}^n)} < \infty,
\]
where \( N \), defined by \( N(F)(x) := \sup_{(t,y) : x \in \mathcal{B}(y, \sqrt{t})} |F(t,y)| \), denotes the non-tangential maximal function.

The tent spaces were introduced in [16], but in elliptic scaling. It is easy to check that
\[
F \in T^{1,2}(\mathbb{R}^{n+1}_+) \iff G \in T_{\text{ell}}^{1,2}(\mathbb{R}^{n+1}_+), \quad \text{where } G(t,.) := tF(t^2,.)
\]
and \( T_{\text{ell}}^{1,2}(\mathbb{R}^{n+1}_+) \) denotes the tent space in elliptic scaling denoted by \( T_2^{1} \) in [16]. The same correspondence holds true for \( T^{\infty,2}(\mathbb{R}^{n+1}_+) \). For \( T^{1,\infty}(\mathbb{R}^{n+1}) \), the correspondence is \( G(t,.) := F(t^2,.) \), and for \( T^{\infty,1}(\mathbb{R}^{n+1}_+) \), \( G(t,.) := t^2F(t^2,.) \).

One has the duality \( (T^{1,2}(\mathbb{R}^{n+1}_+))' = T^{\infty,2}(\mathbb{R}^{n+1}_+) \) and \( (T^{1,\infty}(\mathbb{R}^{n+1}_+))' = T^{\infty,1}(\mathbb{R}^{n+1}_+) \) with duality form \( \int_{\mathbb{R}^{n+1}} f(t,y)g(t,y) \, dydt \).

We recall the definition of the admissible path space for \( (\text{NSE}) \) in [32] (with the notation as in [33]).

**Definition 2.2.** Let \( T \in (0,\infty] \). Define
\[
\mathcal{E}_T := \{ u \text{ measurable in } (0,T) \times \mathbb{R}^n : \|u\|_{\mathcal{E}_T} < \infty \},
\]
with
\[
\|u\|_{\mathcal{E}_T} := \left\| t^{1/2} u \right\|_{L^\infty((0,T) \times \mathbb{R}^n)} + \sup_{x \in \mathbb{R}^n} \sup_{0 < t < T} \left( t^{-n/2} \int_0^t \int_{B(x,\sqrt{t})} |u(s,y)|^2 \, dyds \right)^{1/2}.
\]

**Remark 2.3.** (i) Observe that for \( T = \infty \), one has
\[
\|u\|_{\mathcal{E}_\infty} = \left\| t^{1/2} u \right\|_{L^\infty(\mathbb{R}^{n+1}_+)} + \|u\|_{T^{\infty,2}(\mathbb{R}^{n+1}_+)}. \tag{2.2}
\]
(ii) The corresponding adapted value space \( \mathcal{E}_T \) is defined as the space of \( u_0 \in \mathcal{S}'(\mathbb{R}^n) \) with \( (e^{i(t\Delta)u_0})_{0 < t < T} \in \mathcal{E}_T \). For \( T = \infty \), observe that the first part of the norm in (2.2) corresponds to the adapted value space \( \dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^n) \) and the second part to \( BMO^{-1}(\mathbb{R}^n) \). Since \( BMO^{-1}(\mathbb{R}^n) \to \dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^n) \), one has \( E_\infty = BMO^{-1}(\mathbb{R}^n) \).

**Theorem 2.4.** Let \( T \in (0,\infty] \). The bilinear operator \( B \) defined in (2.1) is continuous from \((\mathcal{E}_T)^n \times (\mathcal{E}_T)^n\) to \((\mathcal{E}_T)^n\).

**Proof.** We restrict ourselves to the case \( T = \infty \). The same argument works otherwise.

**Step 1** (From linear to bilinear). In a first step, one reduces the bilinear estimate to a linear estimate. We use the following fact, which is a simple consequence of Hölder’s inequality:
\[
\|u \otimes v\|_{(\mathcal{E}_\infty)^n} = \|\alpha\|_{(\mathcal{E}_\infty)^n}, \quad \alpha := u \otimes v \Rightarrow \quad \alpha \in T^{\infty,1}(\mathbb{R}^{n+1}_+; \mathbb{C}^n \otimes \mathbb{C}^n),
\]
\[
s^{1/2} \alpha(s,.) \in T^{\infty,2}(\mathbb{R}^{n+1}_+; \mathbb{C}^n \otimes \mathbb{C}^n),
\]
\[
\alpha(s,.) \in L^\infty(\mathbb{R}^{n+1}_+; \mathbb{C}^n \otimes \mathbb{C}^n).
\]

It thus suffices to show that for the linear operator \( A \), defined by
\[
A(\alpha)(t,.) = \int_0^t e^{(t-s)\Delta} \text{div} \alpha(s,.) \, ds,
\]
\[
\]
there exists a constant $C > 0$ such that for all $\alpha$ satisfying the conditions in (2.3),

\begin{align}
(2.5) \quad \|t^{1/2}A(\alpha)\|_{L^\infty(\mathbb{R}^{n+1}_+;C^n)} &\leq C \|\alpha\|_{T^{\infty,1}(\mathbb{R}^{n+1}_+;C^n)} + C \|s\alpha(s, .)\|_{L^\infty(\mathbb{R}^{n+1}_+;C^n)}, \\
(2.6) \quad \|A(\alpha)\|_{T^{\infty,2}(\mathbb{R}^{n+1}_+;C^n)} &\leq C \|\alpha\|_{T^{\infty,1}(\mathbb{R}^{n+1}_+;C^n)} + C \|s^{1/2}\alpha(s, .)\|_{T^{\infty,2}(\mathbb{R}^{n+1}_+;C^n)}.
\end{align}

**Step 2 (L^\infty estimate).**

The proof of (2.5) is the one found in [32]. Notice that the argument only uses the polynomial bounds on the Oseen kernel $k_t(x)$ of $e^{t\Delta \mathbb{P}}$ (See e.g. [33] Chapter 11) for $|\beta| = 1$,

\begin{equation}
(2.7) \quad \left|\partial_{\beta}k_t(x)\right| \leq Ct^{-n/2}(1 + t^{-1/2}|x|)^{-n-|\beta|} \quad \forall \beta \in \mathbb{N}^n, \forall x \in \mathbb{R}^n, \forall t > 0
\end{equation}

and no other special properties on the corresponding operator $e^{t\Delta \mathbb{P}}$. We shall see later that such assumptions can be weakened.

**Step 3 (T^{\infty,2} estimate - New decomposition).**

We split $A$ into three parts:

\[
A(\alpha)(t, .) = \int_0^t e^{(t-s)\Delta \mathbb{P}} \text{div} \alpha(s, .) \, ds = \int_0^t e^{(t-s)\Delta} (s\Delta)^{-1}(I - e^{2s\Delta})s^{1/2}\mathbb{P} \text{div} s^{1/2}\alpha(s, .) \, ds
\]

\[
+ \int_0^\infty e^{(t+s)\Delta \mathbb{P}} \text{div} \alpha(s, .) \, ds - \int_t^\infty e^{(t+s)\Delta \mathbb{P}}s^{-1/2} \text{div} s^{1/2}\alpha(s, .) \, ds =: A_1(\alpha)(t, .) + A_2(\alpha)(t, .) + A_3(\alpha)(t, .).
\]

**Step 3(i) (Maximal regularity operator).** To treat $A_1$, we use the fact that the maximal regularity operator

\[
M^+: T^{\infty,2}(\mathbb{R}^{n+1}_+) \to T^{\infty,2}(\mathbb{R}^{n+1}_+),
\]

\[
(M^+F)(t, .) := \int_0^t e^{(t-s)\Delta} F(s, .) \, ds,
\]

is bounded. The result for $T^{2,2}(\mathbb{R}^{n+1}_+) = L^2(\mathbb{R}^{n+1}_+)$ was established by de Simon in [39]. The extension to $T^{\infty,2}(\mathbb{R}^{n+1}_+)$ was implicit in [32], but not formulated this way. It is an application of [9] Theorem 3.2, taking $\beta = 0$, $m = 2$ and $L = -\Delta$, noting that the Gaussian bounds for the kernel of $t\Delta e^{t\Delta}$ yield the needed decay. This extends to $\mathbb{C}^n$-valued functions $F$ straightforwardly.

Next, for $s > 0$, define $T_s := (s\Delta)^{-1}(I - e^{2s\Delta})s^{1/2}\mathbb{P} \text{div}$.) Observe that $T_s$ is bounded uniformly from $L^2(\mathbb{R}^n; \mathbb{C}^n \otimes \mathbb{C}^n)$ to $L^2(\mathbb{R}^n; \mathbb{C}^n)$, and that standard Fourier computations show that $T_s$ is a convolution operator with kernel $k_s$ satisfying a pointwise estimate of order $n + 1$ at $\infty$, more precisely,

\begin{equation}
(2.9) \quad |k_s(x)| \leq C s^{-n/2}(s^{-1/2}|x|)^{-n-1} \quad \forall x \in \mathbb{R}^n, \forall s > 0, |x| \geq s^{1/2}.
\end{equation}
We show in Lemma 3.1 below, stated under weaker assumptions in form of $L^2$-$L^\infty$ off-diagonal estimates, that the operator $T$, defined by

$$T : T^\infty,2(\mathbb{R}^{n+1}_+; C^n \otimes C^n) \to T^\infty,2(\mathbb{R}^{n+1}_+; C^n),$$

is bounded. With the definitions in (2.8) and (2.10), we then have $A_1(\alpha) = \mathcal{M}^+ T(s^{1/2} \alpha(s, \cdot))$ and the boundedness of these operators imply

$$\|A_1(\alpha)\|_{T^\infty,2} = \|\mathcal{M}^+ T(s^{1/2} \alpha(s, \cdot))\|_{T^\infty,2} \lesssim \|T(s^{1/2} \alpha(s, \cdot))\|_{T^\infty,2} \lesssim \|s^{1/2} \alpha(s, \cdot)\|_{T^\infty,2}.$$

**Step 3(ii)** (Hardy space estimates). This is the main new part of the proof. We use in the following that the Leray projection $P$ commutes with the Laplacian and the above bounds on the Oseen kernel to show that

$$A_2 : T^\infty,1(\mathbb{R}^{n+1}_+; C^n \otimes C^n) \to T^\infty,2(\mathbb{R}^{n+1}_+; C^n),$$

$$\langle A_2 F \rangle(t, \cdot) := \int_0^\infty e^{(t+s)\Delta} P \text{div} F(s, \cdot) \, ds,$$

is bounded. We work via duality, and it is easy to show that

$$A_2^* : T^{1,\infty}(\mathbb{R}^{n+1}_+; C^n) \to T^{1,\infty}(\mathbb{R}^{n+1}_+; C^n \otimes C^n),$$

$$\langle A_2^* G \rangle(s, \cdot) = e^{s\Delta} \int_0^\infty \nabla P e^{t\Delta} G(t, \cdot) \, dt,$$

is bounded. To see this, we factor $A_2^*$ through the Hardy space $H^1(\mathbb{R}^n; C^n \otimes C^n)$. We know from classical Hardy space theory, that $H^1(\mathbb{R}^n)$ can either be defined via non-tangential maximal functions or via square functions (here in parabolic scaling instead of the more commonly used elliptic scaling). First, the operator

$$\mathcal{S} : T^{1,2}(\mathbb{R}^{n+1}_+; C^n) \to H^1(\mathbb{R}^n; C^n \otimes C^n),$$

$$\mathcal{S} G(\cdot) = \int_0^\infty \nabla P e^{t\Delta} G(t, \cdot) \, dt,$$

is bounded. This uses the polynomial decay of order $n + 1$ at $\infty$ of the kernel of $\nabla P e^{t\Delta}$ in (2.7) (some weaker decay of non-pointwise type would suffice for this, in fact). The precise calculations are given in [20] (cf. also [16]). Second, again by [20], we have for $h \in H^1(\mathbb{R}^n)$ that $(s, x) \mapsto e^{s\Delta} h(x) \in T^{1,\infty}$ and $\|N(e^{s\Delta} h)\|_{L^1(\mathbb{R}^n)} \lesssim \|h\|_{H^1(\mathbb{R}^n)}$. The same holds componentwise for $C^n \otimes C^n$-valued functions. A combination of both estimates gives the expected result for $A_2^*$.

**Step 3(iii)** (Remainder term). The considered integral in $A_3$ is not singular in $s$ and is an error term. It suffices to show that

$$\mathcal{R} : T^{\infty,2}(\mathbb{R}^{n+1}_+; C^n \otimes C^n) \to T^{\infty,2}(\mathbb{R}^{n+1}_+; C^n),$$

$$(\mathcal{R} F)(t, \cdot) := \int_t^\infty e^{(t+s)\Delta} P s^{-1/2} \text{div} F(s, \cdot) \, ds$$

is bounded as $A_3(\alpha) = \mathcal{R}(s^{1/2} \alpha(s, \cdot))$. This can be seen as a special case of [7, Theorem 4.1 (2)]. As parts of this proof refer to earlier arguments, we give a self-contained proof for $\mathcal{R}$ in Lemma 3.3 below.
3. Technical results

**Lemma 3.1.** Let \((T_t)_{t>0}\) be a measurable family of uniformly bounded operators in \(L^2(\mathbb{R}^n)\), which satisfy \(L^2-L^\infty\) off-diagonal estimates of the form

\[
(3.1) \quad \| \mathds{1}_E T_s \mathds{1}_E \|_{L^2(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)} \leq C s^{-\frac{n}{4}} \left( s^{-1/2} \text{dist}(E, \hat{E}) \right)^{-\frac{3}{2}}
\]

for all \(s > 0\) and Borel sets \(E, \hat{E} \subseteq \mathbb{R}^n\) with \(\text{dist}(E, \hat{E}) \geq s^{1/2}\). Then the operator \(T\), defined by

\[
T : T^{\infty,2}(\mathbb{R}^n_{+}) \to T^{\infty,2}(\mathbb{R}^n_{+}),
\]

\[
(TF)(s, \cdot) := T_s(F(s, \cdot)),
\]

is bounded.

**Remark 3.2.** This statement obviously extends to vector-valued functions. A straightforward calculation shows that the kernel estimates in \((2.9)\) imply the \(L^2-L^\infty\) off-diagonal estimates in \((3.1)\).

**Proof.** The proof is a slight modification of \([27, \text{Theorem 5.2}]\). Let \(F \in T^{\infty,2}(\mathbb{R}^n_{+})\) and fix \((t, x) \in \mathbb{R}^n_{+}\). Define \(F_0 := \mathds{1}_{B(x,2\sqrt{t})} F\) and \(F_j := \mathds{1}_{B(x,2j+1,\sqrt{t})} \setminus B(x,2j,\sqrt{t}) F\) for \(j \geq 1\). On the one hand, the uniform boundedness of \(T_s\) in \(L^2(\mathbb{R}^n)\) yields

\[
\| T_s F_0(s, \cdot) \|_{L^2(B(x,\sqrt{t}))} \lesssim \| F(s, \cdot) \|_{L^2(B(x,2\sqrt{t}))}.
\]

On the other hand, Hölder’s inequality and \((3.1)\) yield for \(s < t\) and \(j \geq 1\),

\[
\| T_s F_j(s, \cdot) \|_{L^2(B(x,\sqrt{t}))} \lesssim t^{\frac{n}{4}} \| T_s F_j(s, \cdot) \|_{L^\infty(B(x,\sqrt{t}))} \lesssim t^{-j} \left( \frac{\sqrt{s}}{2j} \right)^{j+1} \| F(s, \cdot) \|_{L^2(B(x,2j+1,\sqrt{t}))} \lesssim 2^{-j(\frac{n}{4}+1)} \| F(s, \cdot) \|_{L^2(B(x,2j+1,\sqrt{t}))}.
\]

Thus,

\[
\left( t^{-n/2} \int_0^t \| T_s F(s, \cdot) \|^2_{L^2(B(x,\sqrt{t}))} \, ds \right)^{1/2} \lesssim \sum_{j \geq 0} 2^{-j(\frac{n}{4}+1)} 2^{j/2} \left( \frac{2j}{\sqrt{t}} \right)^{j-n} \int_0^t \| F(s, \cdot) \|^2_{L^2(B(x,2j+1,\sqrt{t}))} \, ds \right)^{1/2} \lesssim \| F \|_{T^{\infty,2}(\mathbb{R}^n_{+})}.
\]

\(\square\)

**Lemma 3.3.** The operator \(\mathcal{R}\) defined in \((2.13)\) is bounded.

**Proof.** We write

\[
(\mathcal{R}F)(t, \cdot) = \int_t^\infty K(t, s) F(s, \cdot) \, ds,
\]

with \(K(t, s) := e^{(t+s)\Delta} \mathbb{P} s^{-1/2} \text{div}\) for \(s, t > 0\). We first show the boundedness of \(\mathcal{R}\) on \(L^2(\mathbb{R}^n_{+})\) and the proof gives a meaning to this integral. This follows from the easy bound \(\| K(t, s) \|_{L^2 \to L^2} \leq C s^{-1/2}(t + s)^{-1/2}\). Indeed, pick some \(\beta \in (-\frac{n}{2}, 0)\), set \(p(t) := t^{\beta}\) and observe that \(k(t, s) := \mathds{1}_{(t,\infty)}(s) \| K(t, s) \|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)}\) satisfies

\[
\int_0^\infty k(t, s) p(t) \, dt \lesssim \int_0^t s^{-1/2} t^{-1/2} t^{\beta} \, dt \lesssim s^\beta = p(s), \forall s > 0,
\]

\[
\int_0^\infty k(t, s) p(s) \, ds \lesssim \int_t^\infty s^{-1/2} s^{-1/2} s^\beta \, ds \lesssim t^{\beta} = p(t), \forall t > 0.
\]
This allows to apply Schur’s lemma and the $L^2$ boundedness is proved.

Next, we show that $\mathcal{R}$ extends to a bounded operator on $\mathcal{T}^{\infty,2}(\mathbb{R}^{n+1}_+)$. Note that for all $s, t > 0$, the operator $K(t,s)$ is an integral operator of convolution with $k_{t,s}$, which satisfies
\[
|k_{t,s}(x)| \leq C s^{-1/2} (t + s)^{-1/2} (t + s)^{-n/2} \left(1 + (t + s)^{-1/2} |x|\right)^{n-1} \quad \forall x \in \mathbb{R}^n, \forall s, t > 0.
\]
These estimates imply $L^2-L^\infty$ off-diagonal estimates of the form
\[
\left\| \mathbb{1}_E K(t,s) \mathbb{1}_E \right\|_{L^2\to L^\infty} \leq C s^{-1/2} (t + s)^{-1/2} (t + s)^{-n/2} \left(1 + (t + s)^{-1/2} \text{dist}(E, \tilde{E})\right)^{-n/2 - 1}
\]
for all Borel sets $E, \tilde{E} \subseteq \mathbb{R}^n$ and $s, t > 0$. Let $F \in \mathcal{T}^{\infty,2}(\mathbb{R}^{n+1}_+)$ and fix $(r, x_0) \in \mathbb{R}^{n+1}_+$. Define $B_j := (0, 2^j r) \times B(x_0, \sqrt{2^j} r)$ for $j \geq 0$ and $C_j := B_j \setminus B_{j-1}$ for $j \geq 1$. Then set $F_0 := \mathbb{1}_{B_0} F$ and $F_j := \mathbb{1}_{C_j} F$ for $j \geq 1$. Using Minkowski’s inequality, we have
\[
\left( r^{-n/2} \int_0^r \left\| (\mathcal{R} F)(t, \cdot) \right\|_{L^2(B(x_0, \sqrt{r}))}^2 \right)^{1/2} \leq \sum_{j \geq 0} \left( r^{-n/2} \int_0^r \left\| (\mathcal{R} F_j)(t, \cdot) \right\|_{L^2(B(x_0, \sqrt{r}))}^2 \right)^{1/2} =: \sum_{j \geq 0} I_j.
\]
For $j \leq 2$, the boundedness of $\mathcal{R}$ on $L^2(\mathbb{R}^{n+1}_+)$ yields the desired estimate $\|I_j\| \lesssim \|F\|_{T^{\infty,2} \mathcal{R}^{n+1}_+}$.

For $j \geq 3$, split $C_j = (0, 2^j r) \times (B(x_0, \sqrt{2^j} r) \setminus B(x_0, \sqrt{2^j} r)) \cup (2^j r, 2^{j+1} r) \times B(x_0, \sqrt{2^j} r) =: C_j^{(0)} \cup C_j^{(1)}$. Define $F_j^{(0)} := \mathbb{1}_{C_j^{(0)}} F$ and $F_j^{(1)} := \mathbb{1}_{C_j^{(1)}} F$, and $I_j^{(0)}, I_j^{(1)}$ correspondingly. For $I_j^{(0)}$, we split the integral in $s$ and use Hölder’s inequality to obtain
\[
I_j^{(0)} \lesssim \sum_{k \geq 0} \left( r^{-n/2} \int_0^r \int_0^{2^k} \left(2^k t\right)^{1/2} \left\| K(t,s) F_j^{(0)}(s, \cdot) \right\|_{L^2(B(x_0, \sqrt{r}))}^2 ds \right)^{1/2}.
\]
Now observe that for $j \geq 3, k \geq 0, t \in (0, r)$ and $s \in (2^k t, 2^{k+1} t)$, Hölder’s inequality and (3.3) yield for any $\delta \in (0, 1]$
\[
\|K(t,s) F_j^{(0)}(s, \cdot)\|_{L^2(B(x_0, \sqrt{r}))} \lesssim r^{n/4} \|K(t,s) F_j^{(0)}(s, \cdot)\|_{L^{\infty}(B(x_0, \sqrt{r}))} \lesssim r^{n/4} s^{-1/2} (t + s)^{-1/2} (t + s)^{-n/4} \left(1 + \sqrt{2^j - 1} r - \sqrt{r} (t + s)^{1/2} \right)^{-n/2 - \delta} \|F_j(s, \cdot)\|_{L^2} \lesssim (2^j)^{-n/2 - \delta/2} \|F_j(s, \cdot)\|_{L^2}.
\]
Inserting this into (3.4), interchanging the order of integration and choosing $\delta < 1$ finally gives
\[
\sum_{j \geq 1} I_j^{(0)} \lesssim \sum_{k \geq 0} 2^{-j \delta/2 - k (1/2 - \delta/2)} \left(2^j r\right)^{-n/2} \int_0^{2^j r} \left\| F_j(s, \cdot) \right\|_{L^2}^2 ds \lesssim \|F\|_{T^{\infty,2} \mathcal{R}^{n+1}_+}.
\]
For $I_j^{(1)}$, we can only use $L^2-L^\infty$ boundedness for $K(t,s)$ instead of off-diagonal estimates. For $s \in (2^j r, 2^{j+1} r)$ and $t \in (0, r)$, one obtains
\[
\|K(t,s) F_j^{(1)}(s, \cdot)\|_{L^2(B(x_0, \sqrt{r}))} \lesssim s^{n/4} \|K(t,s) F_j^{(1)}(s, \cdot)\|_{L^{\infty}(B(x_0, \sqrt{r}))} \lesssim s^{n/4} s^{-1/2} (t + s)^{-1/2} (t + s)^{-n/4} \|F_j(s, \cdot)\|_{L^2} \lesssim 2^{-j n/4} \|F_j(s, \cdot)\|_{L^2}.
\]
Plugging this into $I_j^{(1)}$ then gives
\[
I_j^{(1)} \lesssim \left( r^{-n/2} \int_0^r \int_{2j-1}^{2j} (2^j r)^2 |K(t, s)| F_j^{(1)}(s, .) \|_{L^2(B(x_0, \sqrt{r}))} ds dt \right)^{1/2}
\]

\[
\lesssim (2^j r)^{-1/2} \left( 2^j r)^{-n/2} \int_0^{2j} \| F_j(s, .) \|_{L^2} ds \right)^{1/2} \lesssim 2^{-j/2} \| F \|_{T^\infty, 2(R_+^{n+1})}.
\]
Summing over $j$ gives the assertion. 

\[4. \text{ Comments}\]

Let us temporarily denote by $T_{1/2}^{\infty, 2}(R_+^{n+1})$ the weighted tent space defined by $F \in T_{1/2}^{\infty, 2}(R_+^{n+1})$ if and only if $s^{1/2} F(s, .) \in T^{\infty, 2}(R_+^{n+1})$. Respectively for $T_{1/2}^{2, 2}(R_+^{n+1})$.

The first comment is that the $T_{1/2}^{\infty, 2}$ estimate for $\alpha$ is not used in [32].

The second comment is that our proof is non local in time. By this, we mean that we need to know $\alpha = u \otimes v$ on the full time interval $[0, T]$ to get estimates for $B(u, v)$ at all smaller times $t$. In contrast, the proof in [32] is local in time: bounds for $u, v$ on the time interval $[0, t]$ suffice to get bounds at time $t$ for $B(u, v)$.

The third comment is on the optimality of the estimate in [26], which could be related to the second comment. We have seen in Section 2 that both $A_1$ and $A_3$ are bounded operators from $T_{1/2}^{\infty, 2}$ to $T^{\infty, 2}$. It is thus a natural question whether the same holds for $A_2$ as it would eliminate the $T_{1/2}^{\infty, 1}$ term in the right hand side of [26]. We show that this is not the case. It is therefore necessary to use a different argument for $A_2$, as is done in Step 3(ii) above. In [32], this operator does not arise.

**Proposition 4.1.** The operator $A_2$ is neither bounded as an operator from $T_{1/2}^{2, 2}(R_+^{n+1}; \mathbb{C}^n \otimes \mathbb{C}^n)$ to $T^{2, 2}(R_+^{n+1}; \mathbb{C}^n)$, nor from $T_{1/2}^{\infty, 2}(R_+^{n+1}; \mathbb{C}^n \otimes \mathbb{C}^n)$ to $T^{\infty, 2}(R_+^{n+1}; \mathbb{C}^n)$.

We adapt the argument of [1, Theorem 1.5].

**Proof.** We first show the result for $T^{2, 2}$. We work with the dual operator $A_2^*$ defined in [2.12] and show that

\[G \mapsto s^{-1/2}(A_2^* G)(s, .) = s^{-1/2} e^{s \Delta} \int_0^{\infty} \nabla \mathbb{P} e^{t \Delta} G(t, .) dt\]

is not bounded from $L^2(R_+^{n+1}; \mathbb{C}^n)$ to $T^{2, 2}(R_+^{n+1}; \mathbb{C}^n)$.

There exists $u \in L^2(R^n; \mathbb{C}^n)$ with $\nabla (-\Delta)^{-1/2} (-\Delta)^{-1/2} \mathbb{P}(e^{\Delta} - e^{2\Delta}) u \neq 0$ in $L^2(R^n; \mathbb{C}^n \otimes \mathbb{C}^n)$. Define $G(t, .) = u$ for $t \in (1, 2)$, and $G(t, .) = 0$ otherwise. Clearly $G \in L^2(R_+^{n+1}; \mathbb{C}^n)$. Then, for $s < 1$,

\[s^{-1/2}(A_2^* G)(s, .) = e^{s \Delta} \nabla (-\Delta)^{-1/2} \int_0^2 (-\Delta)^{1/2} \mathbb{P}(e^{\Delta} - e^{2\Delta}) u dt\]

\[= e^{s \Delta} \nabla (-\Delta)^{-1/2} \int_0^2 (-\Delta)^{1/2} \mathbb{P}(e^{\Delta} - e^{2\Delta}) u dt\]

(4.1)

and

\[\left\| s^{-1/2}(A_2^* G)(s, .) \right\|_{L^2(R_+^{n+1})}^2 \geq \int_0^1 \left\| e^{s \Delta} \nabla (-\Delta)^{-1/2} \int_0^2 (-\Delta)^{1/2} \mathbb{P}(e^{\Delta} - e^{2\Delta}) u \right\|_{L^2}^2 \frac{ds}{s} = \infty,
\]
as $e^{s\Delta} \to I$ for $s \to 0$.

For the result on $T^{∞,2}$, we argue similarly. There is some ball $B = B(x, 1)$ in $\mathbb{R}^n$ such that $\nabla (-\Delta)^{-1/2}(-\Delta)^{-1/2} P(e^{\Delta} - e^{2\Delta}) u \neq 0$ in $L^2(B; \mathbb{C}^n \otimes \mathbb{C}^n)$. Let $G$ be defined as above. Then $G \in T^{∞,2}(\mathbb{R}^{n+1}; \mathbb{C}^n)$, since the Carleson norm of $G$ can be restricted to balls of radius larger than 1 by definition of $G$ and

$$\|G\|^2_{T^{∞,2}} = \sup_{x_0 \in \mathbb{R}^n} \sup_{r > 1} \int_0^r \int_{B(x_0, \sqrt{r})} |G(t, x)|^2 \, dx \, dt \leq \int_1^2 \int_{\mathbb{R}^n} |u(x)|^2 \, dx \, dt = \|u\|^2_2.$$ 

Now, using again (4.1), we get as above

$$\left| \frac{1}{s} \left( \int_0^s e^{\Delta} \nabla (-\Delta)^{-1/2}(-\Delta)^{-1/2} P(e^{\Delta} - e^{2\Delta}) u \right) \right|^2_{L^2(B)} \, ds = \infty.$$

\[ \square \]

5. A MODEL CASE

We illustrate that we do not use self-adjointness and pointwise bounds by considering a model case. See also [29] for other models of similar type.

Let $A \in L^∞(\mathbb{R}^n; \mathcal{L}(\mathbb{R}^n))$ with $\text{Re}(A(x)) \geq \kappa I > 0$ for a.e. $x \in \mathbb{R}^n$. Let $L = -\text{div}(A \nabla)$. Consider the equation

$$(5.1) \quad \left\{ \begin{array}{l}
  u_t(t, x) + Lu(t, x) - \text{div} f(u^2(t, x)) = 0, \\
  u(0, .) = u_0,
\end{array} \right.$$ 

where we assume that $f : \mathbb{R} \to \mathbb{R}^n$ is globally Lipschitz continuous, and satisfies

$$|f(x)| \leq C|x|, \quad x \in \mathbb{R}.$$ 

As before, we study mild solutions, i.e., we consider solutions $u : \mathbb{R}^{n+1}_+ \to \mathbb{R}$ of the integral equation

$$(5.2) \quad u(t, .) = e^{-tL}u_0 - \int_0^t e^{-(t-s)L} \text{div} f(u^2(s, .)) \, ds.$$ 

Here too, we put appropriate assumptions on $u_0$ so as to construct mild solutions with Carleson type control. Again, using the Picard contraction principle, matters reduce to showing that the operator $B$, defined by

$$(5.3) \quad B(u)(t, .) := \int_0^t e^{-(t-s)L} \text{div} f(u^2(s, .)) \, ds,$$ 

is bounded on an appropriately defined admissible path space to which the free evolution $e^{-tL}u_0$ belongs.

For this model case, we work with a slightly different path space than previously. We use the notation $\hat{f}$ to denote averages.

**Definition 5.1.** For $(t, x) \in \mathbb{R}^{n+1}_+$, define the (parabolic) Whitney box of standard size as

$$W(t, x) := (t, 2t) \times B(x, \sqrt{t}).$$ 

For $1 \leq q, r \leq \infty$, $F$ measurable in $\mathbb{R}^{n+1}_+$ and $(t, x) \in \mathbb{R}^{n+1}_+$, the Whitney average of $F$ is defined as

$$\left( W_{q, r} F \right)(t, x) = \left( \int_t^{2t} \left( \int_{B(x, \sqrt{t})} |F(s, y)|^q \, dy \right)^{r/q} \, ds \right)^{1/r}.$$
with the usual essential supremum modification when \( q = \infty \) or/and \( r = \infty \). For \( q = r \), we write \( W_q F = W_{q,q} F \), that is
\[
(W_q F)(t,x) := \| W(t,x) \|_{L^q(W(t,x))}^{-1/q} \| F \|_{L^q(W(t,x))}
\]
or the essential supremum on \( W(t,x) \) for \( q = \infty \). The tent spaces \( T^{\infty,1,q,r}(\mathbb{R}^{n+1}_+) \) and \( T^{\infty,2,q,r}(\mathbb{R}^{n+1}_+) \) are defined as the spaces of all measurable functions \( F \) in \( \mathbb{R}^{n+1}_+ \) such that
\[
\| F \|_{T^{\infty,p,q,r}(\mathbb{R}^{n+1}_+)} = \| W_{q,r} F \|_{T^{\infty,p}(\mathbb{R}^{n+1}_+)} < \infty,
\]
for \( p \in \{1,2\} \), respectively.

The tent space \( T^{1,\infty,2}(\mathbb{R}^{n+1}_+) \) is defined as the space of all measurable functions \( F \) in \( \mathbb{R}^{n+1}_+ \) such that
\[
\| F \|_{T^{1,\infty,2}(\mathbb{R}^{n+1}_+)} = \| N(W_2 F) \|_{L^1(\mathbb{R}^n)} < \infty.
\]

For \( p \in [1,\infty) \) and \( F \) measurable in \( \mathbb{R}^{n+1}_+ \), set
\[
N_p F(t,x) = |B(x,\sqrt{t})|^{-1/p} \| F(t,\cdot) \|_{L^p(B(x,\sqrt{t}))}, \quad (t,x) \in \mathbb{R}^{n+1}_+.
\]

This is well-defined almost everywhere.

We quote \textsuperscript{28} Theorem 3.1, Theorem 3.2\textsuperscript{[1]} which gives a Carleson duality result for tent spaces with Whitney averages.

**Proposition 5.2.** There exists \( C > 0 \) such that for functions \( F,G \) measurable in \( \mathbb{R}^{n+1}_+ \),
\[
\| FG \|_{L^1(\mathbb{R}^{n+1}_+)} \leq C \| F \|_{T^{1,\infty,2}(\mathbb{R}^{n+1}_+)} \| G \|_{T^{\infty,1,2}(\mathbb{R}^{n+1}_+)}.\]

Moreover, \( (T^{1,\infty,2}(\mathbb{R}^{n+1}_+), T^{\infty,1,2}(\mathbb{R}^{n+1}_+)) \) form a dual pair with respect to the duality \( (F,G) \rightarrow \int_{\mathbb{R}^{n+1}} FG \, dx \, dt \) in the sense that for all \( F \in T^{1,\infty,2}(\mathbb{R}^{n+1}_+) \),
\[
\| F \|_{T^{1,\infty,2}(\mathbb{R}^{n+1}_+)} \sim \sup_{\| G \|_{T^{\infty,1,2}(\mathbb{R}^{n+1}_+)} = 1} |(F,G)|,
\]
and for all \( G \in T^{\infty,1,2}(\mathbb{R}^{n+1}_+) \),
\[
\| G \|_{T^{\infty,1,2}(\mathbb{R}^{n+1}_+)} \sim \sup_{\| F \|_{T^{1,\infty,2}(\mathbb{R}^{n+1}_+)} = 1} |(F,G)|.
\]

Let us define a path space for the model equation (5.1), which we again denote by \( \mathcal{E}_T \). Let \( p \in [1,\infty) \).

**Definition 5.3.** Let \( T \in (0,\infty] \). Define
\[
\mathcal{E}_T := \{ u \text{ measurable in } (0,T) \times \mathbb{R}^n : \| u \|_{\mathcal{E}_T} < \infty \},
\]
with
\[
\| u \|_{\mathcal{E}_T} := \left( \int_{(0,T) \times \mathbb{R}^n} |N_{2p}(s^{1/2} u(s,\cdot))|^2 \, dx \, dt \right)^{1/2} + \sup_{t \in (0,T)} \left( \int_0^t \int_{B(x,\sqrt{t})} (W_{2p} u)(s,y)^2 \, dy \, ds \right)^{1/2}.
\]

We obtain the following well-posedness result. As before, we restrict ourselves to the case \( T = \infty \).

**Theorem 5.4.** Suppose \( p \in [2,\infty) \) with \( 2p > n \). There exists \( \varepsilon > 0 \) such that for all \( u_0 \in BMO^{-1}_L(\mathbb{R}^n) \) with \( \| u_0 \|_{BMO^{-1}_L} < \varepsilon \), the equation (5.1) has a unique global mild solution \( u \) in a ball of \( \mathcal{E}_\infty \).
We will define $\text{BMO}_L^{-1}(\mathbb{R}^n)$ later. Compared with the path space for Navier-Stokes, we have a weaker requirement on the $L^\infty$ term (control of local $L^{2p}$ norms with finite $p$ instead of $p = \infty$) and stronger requirement in the Carleson control ($L^{2p}$ integrability with $2p > n$ whereas $2p = 2$ works for Navier-Stokes).

Our proof relies on the following estimates on $(e^{-tL})_{t > 0}$. The same estimates hold true for $L$ replaced by $L^*$.

**Lemma 5.5.** (i) Denote by $w_t(x, y)$ the kernel of $e^{-tL}$. It is a Hölder continuous function and there exist constants $C, c > 0$ such that for all $t > 0, x, y \in \mathbb{R}^n$, 

$$|w_t(x, y)| \leq Ct^{-\frac{n}{2}} \exp(-ct^{-1}|x - y|^2).$$

(ii) There exists $\varepsilon > 0$ such that $\nabla e^{-tL}$ is bounded from $L^1(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ if $1 \leq q < 2 + \varepsilon$. Moreover, one has $L^1-L^q$ off-diagonal estimates for $\nabla e^{-tL}$ of the form

$$\|\mathbb{1}_{E}\sqrt{tL}e^{-tL}e_{E}\|_{L^1(\mathbb{R}^n)\rightarrow L^q(\mathbb{R}^n)} \leq Ct^{-\frac{n}{4}}t^{-\frac{n}{2}(1-\frac{1}{q})}\exp(-ct^{-1}\text{dist}(E, \hat{E})^2)$$

for all Borel sets $E, \hat{E} \subseteq \mathbb{R}^n$ and $t > 0$.

Proof. For (i), see [1] Theorem 3.23. For (ii), see [2] Proposition 1.24. \[\square\]

**Remark 5.6.** The absence of pointwise bounds for $\nabla e^{-tL}$ is responsible for not taking $2p = 1$ in the Carleson control and not taking $2p = \infty$ in the $L^\infty$ term. One can also weaken the estimate of Lemma 5.4. The pointwise bounds of the kernel of $e^{-tL}$ to $L^r-L^s$ off-diagonal estimates with $r > n$ and by [3] the ones on $\nabla e^{-tL}$ become from $L^r$ to $L^s$. It implies that for dimensions $n = 1, 2, 3, 4$, one could take $L$ to have complex coefficients or even an elliptic system if one wishes (see [3]). The proof of this possible generalisation is a little more involved and we do not include details.

5.1. The free evolution. We need to make sense to the free evolution term $e^{-tL}u_0$. Recall that in the case of the Navier-Stokes systems (with the Laplacian in the background), the adapted value space consists of divergence free elements $u_0$ in $\text{BMO}^{-1}(\mathbb{R}^n; \mathbb{C}^n)$ and is characterized by $e^{i\Delta}u_0$ in the path space. We consider a similar procedure, but here we have to work with a space adapted to the operator $L$.

We define the space $\text{BMO}_L^{-1}(\mathbb{R}^n)$ as the dual space of $H^{1,1}_L(\mathbb{R}^n)$ introduced in [26] Section 8.4]. The latter is the completion of the homogeneous Sobolev space $\dot{W}^{1,2}(\mathbb{R}^n)$ for the norm 

$$\|(t, x) \mapsto tL^*e^{-tL}t^{-1/2}L^{1/2}h(x)|_{T^{1,2}}\|,$$

that is

$$\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^{n+1}} t^{-n/2}1_{B(x, \sqrt{t})}(y)\left|tL^*e^{-tL}t^{-1/2}L^{1/2}h(y)\right|^2 dy dt\right)^{1/2} dx < \infty.$$

Note that $L^{1/2}h \in L^2(\mathbb{R}^n)$ is equivalent to $h \in \dot{W}^{1,2}(\mathbb{R}^n)$ by [3]. Also it is shown that the completion can be realized as the Triebel-Lizorkin space $F^{1,2}_1(\mathbb{R}^n) \cong \dot{H}^{1,1}(\mathbb{R}^n)$ ([26] Proposition 8.43) together with Lemma 5.5 above which shows $p_{-}(L) = 1$ in the notation of [26]). We choose this realization. In particular, since the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ is dense in $F^{1,2}_1(\mathbb{R}^n)$ ([11] Theorem 2.3.3]), it makes $\text{BMO}_L^{-1}(\mathbb{R}^n)$ a space of tempered distributions.

**Lemma 5.7.** Assume $u_0$ is a tempered distribution. Then $u_0 \in \text{BMO}_L^{-1}(\mathbb{R}^n)$ if and only if there exists $G \in T^{\infty,2}(\mathbb{R}^{n+1})$ such that

$$(u_0, h) = \int_{\mathbb{R}^{n+1}} G(s, y)sL^*e^{-sL}t^{-1/2}L^{1/2}h(y) dyds \quad \forall h \in \mathcal{S}(\mathbb{R}^n),$$

where $G$ is the distributional kernel of $-L$. The expression

$$(w_t, u_0) = \int_{\mathbb{R}^{n+1}} G(s, y)sL^*e^{-sL}t^{-1/2}L^{1/2}h(y) dyds$$

is defined using the distributional properties of $\nabla e^{-tL}$.
the integral converging absolutely, and
\[ \|u_0\|_{BMO_{L^1}^{-1}} \sim \inf \{ \|G\|_{T^{\infty,2}}; (5.7) \text{ holds} \}. \]
In that case, the integral exists for all \( h \in \dot{W}^{1,2}(\mathbb{R}^n) \cap \dot{H}^{1,1}(\mathbb{R}^n) \) and gives \( \langle u_0, h \rangle \).

Proof. This is a straightforward consequence of the definition. \( \square \)

Thus, we may introduce the map \( S : T^{\infty,2} \to BMO_{L^1}^{-1}, G \mapsto \int_0^\infty s^{-1/2}L^{1/2} sLe^{-sL} G(s, .) \, ds \)
defined by (5.7), which is bounded and onto.

**Lemma 5.8.** The map \( V : T^{\infty,2} \to T^{\infty,2}, G \mapsto H \) with
\[ H(t, .) = \int_0^\infty s^{-1/2}L^{1/2} sLe^{-(t+s)L} G(s, .) \, ds \]
is bounded.

Proof. The proof is analogous to that of Lemma 3.3. One proves the \( T^{2,2} \) boundedness first using the Schur test. Next, one has \( L^2-L^\infty \) estimates like (5.3) with extra multiplicative factor \( \frac{s}{s+t} \)
for the operator-valued kernel \( K(t, s) = s^{-1/2}L^{1/2} sLe^{-(s+t)L} \) compared to the one in Lemma 4.9
(this is needed to allow integration on the full interval \((0, \infty))\). This suffices to run the same argument as for \( R \). \( \square \)

**Corollary 5.9.** Let \( u_0 \in BMO_{L^1}^{-1}(\mathbb{R}^n) \).

1. For each \( t > 0 \), \( e^{-tL}u_0 \in BMO_{L^1}^{-1}(\mathbb{R}^n) \) with \( \langle e^{-tL}u_0, h \rangle = \langle u_0, e^{-tL^*}h \rangle \) say for each \( h \in S(\mathbb{R}^n) \), \( \|e^{-tL}u_0\|_{BMO_{L^1}^{-1}} \leq C\|u_0\|_{BMO_{L^1}^{-1}} \) uniformly and we have the semigroup property
\( e^{-(s+t)L}u_0 = e^{-sL}(e^{-tL}u_0) \) for any \( s, t > 0 \).
2. \( t \mapsto e^{-tL}u_0 \) belongs to \( C^\infty(0, \infty; BMO_{L^1}^{-1}(\mathbb{R}^n)) \) and is a strong solution in \((0, \infty)\) of \( \partial_t u + Lu = 0 \).
3. \( e^{-\varepsilon L}u_0 \to u_0 \) weak-* \( \varepsilon \to 0 \).
4. Moreover, \( u(t, x) := e^{-tL}u_0(x) \in T^{\infty,2} \) and \( \|u\|_{T^{\infty,2}} \lesssim \|u_0\|_{BMO_{L^1}^{-1}} \).

Proof. By construction of \( BMO_{L^1}^{-1}, L \) has \( H^\infty \)-functional calculus on \( BMO_{L^1}^{-1} \) and in particular
we obtain item (1). Item (2) is then an easy consequence of semigroup theory in Banach spaces.
Item (3) is proved by duality provided one can show strong convergence \( e^{-\varepsilon L^*}h \to h \) as \( \varepsilon \to 0 \)
in \( H_{L^1}^{1,1}(\mathbb{R}^n) \). By density and the uniform boundedness of the semigroup in \( H_{L^1}^{1,1}(\mathbb{R}^n) \), it suffices to assume \( h \in W^{1,2}(\mathbb{R}^n) \) for which the right hand side of (5.6) is finite. But the theory of (26) allows to change \( tL^* e^{-tL^*} \) by \( (tL^*)^k e^{-tL^*} \) for any integer \( k \geq 1 \) and to have an equivalent norm
for the pre-complete space (see in particular Corollary 4.17 there). Now, one can follow the proof of [11, Proposition 4.5] given in a different but similar context to show the strong convergence.
We skip details.
To prove item (4), pick \( G \) such that \( u_0 = SG \). It remains to see that \( V G(t, x) = (e^{-tL}u_0)(x) \) for example in the distributions in \( \mathbb{R}^{n+1}_+ \) since we can see both functions as distributions. Pick a test
We continue with the following auxiliary lemma.

**Remark 5.12.** It is easy to show using the integral representation that

\[ \int (G, \varphi \otimes h) = \iint G(s, y) s L^* e^{-s L^*} y \int_{T^1/2} (e^{-t L^*} h)(y) \varphi(t) ds dy dt \]

\[ = \int (u_0, e^{-t L^*} h) \varphi(t) dt \]

\[ = \int (e^{-t L} u_0, h) \varphi(t) dt \]

\[ = \langle u, \varphi \otimes h \rangle. \]

Each line can be appropriately justified and we leave details to the reader. \[ \square \]

Remark that \(e^{-t L} u_0(x)\) is not defined by integration against the kernel \(w_t(x, y)\) in Lemma 5.9. Nevertheless, one has the following properties.

**Lemma 5.10.** Let \(u_0 \in BMO_L^{-1}(R^n)\). Then \(t \mapsto e^{-t L} u_0 \in C^\infty(0, \infty; L^1_{loc}(R^n))\) and

\[ e^{-(t+s)L} u_0(x) = \int_{R^n} w_s(x, y) e^{-t L} u_0(y) dy \]

for almost every \(t, s > 0\) and \(x \in R^n\). As a consequence, \((t, x) \mapsto e^{-t L} u_0(x)\) is (almost everywhere equal to) a locally bounded and Hölder continuous function (to which it is now identified).

**Proof.** Using the same analysis, one can replace \(e^{-t L}\) by \((tL)^m e^{-t L} = (-1)^m t^m \partial_t^m e^{-t L}\) for each positive integer \(m\) and obtain that \(t^m \partial_t^m e^{-t L} u_0(x)\) exists for all \(m\) in \(T^\infty_1\), hence in \(L^2_{loc}(R^{n+1})\).

Thus, we may see \(t \mapsto e^{-t L} u_0 \in C^\infty(0, \infty; L^2_{loc}(R^n))\).

To show the integral representation for \(e^{-(s+t)L} u_0(x)\), we use for any \(h \in S(R^n)\),

\[ \langle e^{-(s+t)L} u_0, h \rangle = \langle e^{-t L} u_0, e^{-s L^*} h \rangle \]

and then use the integral representation for \(e^{-s L^*} h\) with the adjoint of kernel of \(w_s(x, y)\). Next, we use for any \(a > 0\) and \(z \in R^n\),

\[ \int_0^{2a} \int |(e^{-s L} u_0)(y)| dydt \lesssim \sqrt{a} \| u_0 \|_{BMO_L^{-1}} \]

and the estimates of Lemma 5.9 together with the decay of \(h\) to show that for any \(a, s > 0\)

\[ \int_0^{2a} \int_{R^n} \int_{R^n} |h(x) w_s(x, y) (e^{-t L} u_0)(y)| dy dx dt < \infty \]

hence the integral \(\int_{R^n} w_s(x, y) e^{-t L} u_0(y) dy\) exists for all \(s\) and almost every \(t, x\) and by Fubini’s theorem, for almost every \(t > 0\),

\[ \langle e^{-t L} u_0, e^{-s L^*} h \rangle = \int_{R^n} \int_{R^n} w_s(x, y) e^{-t L} u_0(y) dy \overline{h(x)} dx. \]

The conclusion follows. \[ \square \]

**Remark 5.11.** It can be shown that for \(u_0 \in BMO_L^{-1}(R^n)\), \((t, x) \mapsto e^{-t L} u_0(x)\) is also a weak solution of the parabolic equation \(\partial_t u - \text{div} A \nabla u = 0\) (with \(u\) and \(\nabla u\) in \(L^2_{loc}\) in space-time).

**Remark 5.12.** It is easy to show using the integral representation that \(L^n(R^n)\) imbeds into \(BMO_L^{-1}(R^n)\) (both are spaces of tempered distributions, so they can be compared) when \(n \geq 1\).

We continue with the following auxiliary lemma.
Lemma 5.13. For $F \in T^{\infty,2}(\mathbb{R}^{n+1}_+)$ and $t > 0$, set

$$G(t, \cdot) = \frac{1}{t} \int_{\frac{t}{4}}^{t} e^{-(t-s)L} F(s, \cdot) \, ds. \tag{5.8}$$

Suppose $q, r \in [2, \infty]$. Then there exists $C > 0$, independent of $F$, such that

$$\|N_q(t^{1/2}G)\|_{L^\infty(\mathbb{R}^{n+1}_+)} \leq C \|F\|_{T^{\infty,2}}$$

and

$$\|G\|_{T^{\infty,2; q, r}} \leq C \|F\|_{T^{\infty,2}}.$$

Proof. Let $(\tau, x) \in \mathbb{R}^{n+1}_+$ and $t \in [\tau, 2\tau]$. Using Minkowski's inequality, $L^2-L^q$ off-diagonal estimates for $(e^{-tL})_{t>0}$, and Hölder's inequality gives for any $N \geq 0$

$$\left( \int_{B(x, \sqrt{t})} |G(t, y)|^q \, dy \right)^{1/q} \leq \frac{1}{t} \int_{\frac{t}{4}}^{t} \left( \int_{B(x, \sqrt{t})} |e^{-(t-s)L} F(s, \cdot)|^q \, dy \right)^{1/q} \, ds \leq \sum_{j=0}^{\infty} 2^{-j(2N-\frac{q}{2})} \left( \int_{0}^{2^j \tau} \int_{B(x, \sqrt{t})} |F(s, y)|^2 \, dy \, ds \right)^{1/2} \leq \sum_{j=0}^{\infty} 2^{-j(2N-\frac{q}{2})} \left( \int_{0}^{2^j \tau} \int_{B(x, \sqrt{t})} |F(s, y)|^2 \, dy \, ds \right)^{1/2}.$$

By taking $t = \tau$, we have an estimate of $N_q(t^{1/2}G)(\tau, x)$ and the right hand side is bounded by

$$\sum_{j=0}^{\infty} 2^{-j(2N-\frac{q}{2})} \left( \int_{0}^{2^j \tau} \int_{B(x, \sqrt{t})} |F(s, y)|^2 \, dy \, ds \right)^{1/2} \leq \|F\|_{T^{\infty,2}}$$

if $N > \frac{n}{4}$. Remark that the argument applies with $q = \infty$, taking essential supremum. Next, by estimating the $L^r$ average in time, we also have

$$\left( \int_{\tau}^{2\tau} \left( \int_{B(x, \sqrt{\tau})} |G(t, y)|^q \, dy \right)^{r/q} \, dt \right)^{1/r} \leq \sum_{j=0}^{\infty} 2^{-j(2N-\frac{q}{2})} \left( \int_{0}^{2^j \tau} \int_{B(x, \sqrt{t})} |F(s, y)|^2 \, dy \, ds \right)^{1/2}.$$

Hence, Fubini's theorem and $N > \frac{n}{2}$ finally yield

$$\|G\|_{T^{\infty,2; q, r}} = \|W_{q,r}G\|_{T^{\infty,2}} \leq \sum_{j=0}^{\infty} 2^{-j(2N-\frac{q}{2})} \sup_{(r,x) \in \mathbb{R}^{n+1}} \left( \int_{0}^{\tau} \int_{B(x, \sqrt{\tau})} \int_{0}^{\tau} \int_{B(x, \sqrt{\tau})} |F(s, y)|^2 \, dy \, ds \, dx \, d\tau \right)^{1/2} \leq \sum_{j=0}^{\infty} 2^{-j(2N-n)} \sup_{(r,x) \in \mathbb{R}^{n+1}} \left( \int_{0}^{\tau} \int_{2^j B(x, \sqrt{\tau})} |F(s, y)|^2 \, dy \, ds \right)^{1/2} \leq \|F\|_{T^{\infty,2}}.$$

Again, the argument applies for $q$ and/or $r = \infty$. \qed
Corollary 5.14. Let $p \in [1, \infty]$. There exists $C > 0$ such that for all $u_0 \in BMO^{-1}_C$,
\[
\|N_{2p}(t^{1/2}e^{-tL}u_0)\|_{\infty} + \|e^{-tL}u_0\|_{T^{\infty, 2.2p}} \leq C\|u_0\|_{BMO^{-1}_C}
\]

Proof. As $\|e^{-tL}u_0\|_{T^{\infty, 2}} \lesssim \|u_0\|_{BMO^{-1}_C}$, it suffices to apply Lemma 5.13 with $q = r = 2p$ and $F(s, \cdot) = e^{-sL}u_0$, noting that $G = F$ in (5.8). Let us this last point. We have seen that $F(t, \cdot) = e^{-(t-s)L}(F(s, \cdot))$ for almost every $0 < s < t$ and it suffices to average for $t/4 < s < t/2$, and (5.8) holds for almost every $t > 0$. This suffices to get the conclusions. \[\square\]

5.2. From linear to quadratic.

Lemma 5.15. With $\mathcal{E}_\infty$ as defined in (5.4), we have
\[
(5.9) \quad u, v \in \mathcal{E}_\infty, \quad \alpha := f(u^2) - f(v^2) \Rightarrow \begin{cases} \alpha \in T^{\infty, 1, p}(\mathbb{R}^{n+1} \setminus \mathbb{C}^n), \\ s^{1/2}\alpha(s, \cdot) \in T^{\infty, 2, p, 2p}(\mathbb{R}^{n+1} \setminus \mathbb{C}^n), \\ N_p(\alpha(s, \cdot)) \in L^\infty(\mathbb{R}^{n+1} \setminus \mathbb{C}^n). \end{cases}
\]

Proof. By the Lipschitz property of $f$, observe that $|f(u^2) - f(v^2)| \leq ab$, with $a = C|u - v|$ and $b = |u + v|$ which satisfy the same conditions as $u$ and $v$. By repeated use of Hölder’s inequality, one obtains
\[
\|\alpha\|_{T^{\infty, 1, p}} = \|C_1(W_p\alpha)\|_{\infty} \leq \|C_1(W_2p a \cdot W_2p b)\|_{\infty} \leq \|C_2(W_2p a)\|_{\infty}\|C_2(W_2p b)\|_{\infty} = \|a\|_{T^{\infty, 2, p}}\|b\|_{T^{\infty, 2, p}}.
\]
Similarly,
\[
N_p(s^{1/2}v(s, \cdot)) \leq N_{2p}(a)N_{2p}(s^{1/2}b(s, \cdot)) \leq N_{2p}(a)\|N_{2p}(s^{1/2}b(s, \cdot))\|_{\infty}
\]
hence
\[
W_{2p, 2p}(s^{1/2}\alpha(s, \cdot)) \leq W_{2p}(a)\|N_{2p}(s^{1/2}b(s, \cdot))\|_{\infty}
\]
and
\[
\|s^{1/2}\alpha(s, \cdot)\|_{T^{\infty, 2, p, 2p}} = \|C_2(W_{2p, 2p}(s^{1/2}\alpha(s, \cdot)))\|_{\infty} \leq \|a\|_{T^{\infty, 2, p}}\|N_{2p}(s^{1/2}b(s, \cdot))\|_{\infty}.
\]
Finally,
\[
\|N_p(\alpha(s, \cdot))\|_{\infty} \leq \|N_{2p}(s^{1/2}\alpha(s, \cdot))\|_{\infty}\|N_{2p}(s^{1/2}b(s, \cdot))\|_{\infty}.
\]
\[\square\]

With this lemma in hand, it suffices to study the linear operator $A$ defined by
\[
A(\alpha)(t, \cdot) = \int_0^t e^{-(t-s)L} \text{div} \alpha(s, \cdot) \, ds.
\]
Instead of (2.5) and (2.6), we are going to show that there exists $C > 0$ such that for all $\alpha$ satisfying the conditions in (5.4), and $p \in [2, \infty)$ with $2p > n$,
\[
(5.10) \quad \|N_{2p}(t^{1/2}A(\alpha))\|_{\infty} \leq C\|\alpha\|_{T^{\infty, 1, 2}} + C\|N_p(\alpha(s, \cdot))\|_{\infty},
\]
\[
(5.11) \quad \|A(\alpha)\|_{T^{\infty, 2, 2p}} \leq C\|\alpha\|_{T^{\infty, 1, p}} + C\|s^{1/2}\alpha(s, \cdot)\|_{T^{\infty, 2, p, 2p}}.
\]
5.3. The $L^\infty$ estimate. We establish (5.10) in the following lemma.

**Lemma 5.16.** Assume $n < 2p < \infty$ and $p \geq 2$. There exists $C > 0$ such that for all $\alpha \in T^{\infty, p}(\mathbb{R}^{n+1}_+, \mathbb{C}^n)$ and with $N_p(s\alpha(s, .)) \in L^\infty(\mathbb{R}^{n+1}_+, \mathbb{C}^n)$,

$$\|N_{2p}(t^{1/2}A(\alpha))\|_\infty \lesssim \|\alpha\|_{T^{\infty, 1, p}} + \|N_p(s\alpha(s, .))\|_\infty.$$

**Proof.** Fix $(t, x) \in \mathbb{R}^{n+1}_+$, and let $a \in (0, 1)$ be arbitrary (1/2 for example). We are going to estimate the quantity $t^{-n/4p}\|t^{1/2}A\alpha(t, .)\|_{L^{2p}(B(x, \sqrt{t}))}$, where we split $A$ into the two parts

$$t^{1/2}A\alpha(t, .) = t^{1/2} \int_0^a e^{-(t-s)L} \div \alpha(s, .) \, ds + t^{1/2} \int_a^t e^{-(t-s)L} \div \alpha(s, .) \, ds. \tag{5.12}$$

For the second part, $L^p$-$L^{2p}$ off-diagonal estimates for $(e^{-tL} \div)_{t>0}$ yield

$$t^{-n/4p}\|t^{1/2} \int_0^t e^{-(t-s)L} \div \alpha(s, .) \, ds\|_{L^{2p}(B(x, \sqrt{t}))} \leq \sum_{j=0}^\infty t^{-n/4p}\|t^{1/2} \int_0^t \left(\frac{t}{t-s}\right)^{\frac{n}{4p}+\frac{1}{2}} e^{-(t-s)L}\div 1_{S_j(B(x, \sqrt{t}))} \alpha(s, .) \|_{L^{2p}(B(x, \sqrt{t}))} \, ds$$

$$\lesssim t^{-n/2p}\int_0^t \left(\frac{t}{t-s}\right)^{\frac{n}{4p}+\frac{1}{2}} \|\alpha(s, .)\|_{L^p(B(x, 8\sqrt{t}))} \, ds$$

$$+ \sum_{j=3}^\infty t^{-n/2p}\int_0^t \left(\frac{t}{t-s}\right)^{\frac{n}{4p}+\frac{1}{2}} \left(\frac{t-s}{2^{2j}t}\right)^N \|\alpha(s, .)\|_{L^p(2^j B(x, \sqrt{t}))} \, ds$$

$$\lesssim \|N_p(s\alpha(s, .))\|_{L^\infty} \left(\int_0^t \left(\frac{t}{t-s}\right)^{\frac{n}{4p}+\frac{1}{2}} \, ds \right) + \sum_{j=3}^\infty 2^{-2jN}2^{j\frac{n}{p}} \int_0^t \left(\frac{t-s}{t}\right)^{N-\frac{n}{4p}-\frac{1}{2}} \, ds \right) \lesssim \|N_p(s\alpha(s, .))\|_{L^\infty},$$

where the assumption $2p > n$ is used in the last step.

Consider now the first part in (5.12). Decompose

$$t^{-n/4p}\|t^{1/2} \int_0^a e^{-(t-s)L} \div \alpha(s, .) \, ds\|_{L^{2p}(B(x, \sqrt{t}))} \leq t^{-n/4p}\|t^{1/2} \int_0^a e^{-(t-s)L} \div 1_{B(x, 8\sqrt{t})} \alpha(s, .) \, ds\|_{L^{2p}(B(x, \sqrt{t}))}$$

$$+ \sum_{j=3}^\infty t^{-n/4p}\|t^{1/2} \int_0^a e^{-(t-s)L} \div 1_{S_j(B(x, \sqrt{t}))} \alpha(s, .) \, ds\|_{L^{2p}(B(x, \sqrt{t}))}.$$
For the on-diagonal part, we write

$$
(5.13) \quad t^{-n/4p} t^{1/2} \int_0^t e^{-(t-s)L} \text{div} \mathbb{1}_{B(x,8\sqrt{t})} \alpha(s, \cdot) \, ds \|L^{2p}(B(x,\sqrt{t}))
$$

$$
= \sup_{g \in L^{2p}(B(x,\sqrt{t})) \|g\|_{2p'} = 1} t^{-n/4p} \left| \int_0^t e^{-(t-s)L} t^{1/2} \text{div} \mathbb{1}_{B(x,8\sqrt{t})} \alpha(s, \cdot) \, ds \right|,
$$

$$
= \sup_{g \in L^{2p}(B(x,\sqrt{t})) \|g\|_{2p'} = 1} t^{-n/4p} \left| \int_0^t \langle \alpha(s, \cdot), \beta_0(s, \cdot) \rangle \, ds \right|,
$$

with

$$
\beta_0(s, y) = \mathbb{1}_{(0,at) \times B(x,8\sqrt{t})}(s, y) t^{1/2} \nabla e^{-(t-s)L}^* g(y).
$$

Since $\alpha \in T^{\infty,1,p}(\mathbb{R}_+^{n+1}; \mathbb{C}^n)$ by assumption and $p \geq 2$, we have $\alpha \in T^{\infty,1,2}(\mathbb{R}_+^{n+1}; \mathbb{C}^n)$. By Proposition 5.2, it suffices to show that $N(W_2\beta_0) \in L^1(\mathbb{R}^n)$ with $\|N(W_2\beta_0)\|_1 \lesssim t^{n/4p}$. To do so, split $\beta_0 = \beta_0^0 + \beta_0^1$ with

$$
\beta_0^0(s, y) = \mathbb{1}_{(0,at) \times B(x,8\sqrt{t})}(s, y) t^{1/2} \nabla e^{-tL^*} g(y) =: \mathbb{1}_{(0,at)}(s) h(y),
$$

$$
\beta_0^1(s, y) = \mathbb{1}_{(0,at) \times B(x,8\sqrt{t})}(s, y) t^{1/2} \nabla (e^{-(t-s)L^*} - e^{-tL^*}) g(y).
$$

Now, since $h$ is constant with respect to $s$, one has for every $x_0 \in \mathbb{R}^n$,

$$
N(W_2\beta_0^0)(x_0) = \sup_{|x_0 - z| < \sqrt{\sigma}} \left( \frac{\sigma - \frac{2}{p' - 1}}{\frac{2}{p' - 1} - 1} \right)^{1/2} \left( \int_{W(\sigma,z)} |\mathbb{1}_{(0,at)}(s) h(y)|^2 \, dy ds \right)^{1/2},
$$

$$
\lesssim (\mathcal{M} h^2)^{1/2}(x_0) =: \mathcal{M}_2 h(x_0),
$$

where $\mathcal{M}$ denotes the uncentred Hardy-Littlewood maximal operator. Moreover, note that $\text{supp} \beta_0^0 \subseteq B(x,8\sqrt{t}) \times (0,at)$ implies $\text{supp} N(W_2\beta_0^0) \subseteq B(x,c\sqrt{t})$ for some constant $c > 0$, $i = 0, 1$, independent of $x$ and $t$. Choose $q \in (2, 2 + \varepsilon)$, where $\varepsilon > 0$ is given in Lemma 5.5 (ii). Using H"older's inequality, the support property of $N(W_2\beta_0^0)$, boundedness of $\mathcal{M}_2$ in $L^q(\mathbb{R}^n)$ and $L^{2p'}-L^q$ boundedness of $(t^{1/2} \nabla e^{-tL^*})_{t>0}$, one obtains

$$
\|N(W_2\beta_0^0)\|_1 \lesssim |B(x,c\sqrt{t})|^{1/q} \|\mathcal{M}_2 h\|_q \lesssim t^{n/2q'} \|h\|_q \lesssim t^{n/2q'} t^{-\frac{2}{(2q')-\varepsilon}} \|g\|_{2p'} = t^{n/4p}.
$$

This gives the desired estimate for $\beta_0^0$. To handle $\beta_0^1$, we first observe that a simple geometric argument shows that for every $x_0 \in \mathbb{R}^n$, there exists a parabolic cone $\tilde{\Gamma}(x_0)$, with aperture independent of $x_0$, such that

$$
(\sigma, z) \in \Gamma(x_0) \Rightarrow W(\sigma, z) \subset \tilde{\Gamma}(x_0),
$$

Therefore,

$$
N(W_2\beta_0^1)(x_0)^2 \lesssim \int_{\tilde{\Gamma}(x_0)} |\beta_0^1(s, y)|^2 \, dy ds \frac{1}{s^{n/2+1}}.
$$
and, using Fubini in the second step,
\[
\int_{B(x, c\sqrt{t})} N(W_2\beta^1_0)(x_0) \, dx_0 \lesssim t^{n/4} \left( \int_{B(x, c\sqrt{t})} N(W_2\beta^1_0)(x_0)^2 \, dx_0 \right)^{1/2} \lesssim t^{n/4} \left( \int_{\mathbb{R}^n} \int_0^{at} |t^{1/2} \nabla (e^{-(t-s)L^*} - e^{-tL^*})g(y)|^2 \, dy ds \right)^{1/2}.
\] (5.14)

Now write
\[
t^{1/2} \nabla (e^{-(t-s)L^*} - e^{-tL^*})g = t^{1/2} e^{-\frac{1}{2}L^*} \int_{t/2-s}^t L^* e^{-rL^*} g \, dr.
\]

Since \((t^{1/2} \nabla e^{-tL^*})_{t>0}\) is bounded from \(L^{(2p)'}\) to \(L^2\), and \((e^{-tL^*})_{t>0}\) is analytic in \(L^{(2p)'}\), one has for \(s \in (0, at)\),
\[
\|t^{1/2} \nabla (e^{-(t-s)L^*} - e^{-tL^*})g\|_2 \lesssim t^{-\frac{n}{2} \left( \frac{1}{(2p)' - \frac{1}{2}} \right)} \left( \int_{t/2-s}^t L^* e^{-rL^*} g \, dr \right) \lesssim t^{-\frac{n}{2} \left( \frac{1}{(2p)' - \frac{1}{2}} \right)} \frac{1}{t} \|g\|_{(2p)'}.
\] (5.15)

Plugging this into (5.14) yields
\[
\int_{B(x, c\sqrt{t})} N(W_2\beta^1_0)(x_0) \, dx_0 \lesssim t^{n/4p} \|g\|_{(2p)'} \left( \int_0^{at} \frac{s}{t} \, ds \right)^{1/2} \lesssim t^{n/4p}.
\]

To handle the off-diagonal part, we follow the same path and replace \(\beta_0\) by
\[
\beta_j = \mathds{1}_{(0,at) \times S_j(B(x,\sqrt{t}))}(s, y) t^{1/2} \nabla e^{-tL^*} \mathds{1}_{B(x,\sqrt{t})}g(y)
\]
for \(j \geq 4\), and split \(\beta_j = \beta^0_j + \beta^1_j\) in the same way as for \(\beta_0\), with \(h\) replaced by
\[
h_j(y) = \mathds{1}_{S_j(B(x,\sqrt{t}))}(y) t^{1/2} \nabla e^{-tL^*} g(y).
\]

According to Lemma 3.5 (ii), \((t^{1/2} \nabla e^{-tL^*})_{t>0}\) satisfies \(L^{(2p)'}-L^2\) off-diagonal estimates, which yield for any \(N \geq 0\),
\[
\||h_j||_q \lesssim t^{-\frac{n}{2} \left( \frac{1}{(2p)' - \frac{1}{2}} \right)} \left( 1 + \frac{2j + t}{t} \right)^{-N} \|g\|_{(2p)'} \lesssim t^{-\frac{n}{2} \left( \frac{1}{(2p)' - \frac{1}{2}} \right)} \frac{1}{t} 2^{-jN}.
\]

Observe that similarly to above, the support property of \(\beta^1_j\) implies \(\supp N(W_2\beta^1_j) \subseteq B(x, c2\sqrt{t})\), for \(i = 0, 1\) with \(c\) independent of \(x, t, j\) and \(j\). Thus,
\[
\|N(W_2\beta^0_j)\|_1 \lesssim \|B(x, c2\sqrt{t})\|^{1/q'} \|M_2 h_j\|_q \lesssim 2^{j/q'} t^{n'/q'} \|h_j\|_q \lesssim 2^{-\frac{j}{2} \left( 2N - \frac{n}{2} \right)} t^{n'/q'} \|g\|_{(2p)'} = 2^{-j \left( 2N - \frac{n}{2} \right) / q'} \|g\|_{(2p)'}
\]

Choosing \(N > \frac{n}{2q'}\) allows to sum over \(j\) and gives the assertion for \(\beta^0_j\). Finally, for \(\beta^1_j\), one can repeat the argument for \(\beta^0_j\), and replace (5.14) by
\[
\|N(W_2\beta^1_j)\|_1 \lesssim 2^{j/q} \left( \int_{S_j(B(x,\sqrt{t}))} \int_0^{at} |t^{1/2} \nabla (e^{-(t-s)L^*} - e^{-tL^*})g(y)|^2 \, dy ds \right)^{1/2}.
\] (5.16)

Combining \(L^{(2p)'}-L^2\) off-diagonal estimates for \(t^{1/2} \nabla e^{-tL^*}\) in \(t\) with \(L^{(2p)'}\) off-diagonal estimates for \(rL^* e^{-rL^*}\) in \(r \approx t\) then refines the estimate (5.15) to
\[
\|t^{1/2} \nabla (e^{-(t-s)L^*} - e^{-tL^*})g\|_{L^2(S_j(B(x,\sqrt{t})))} \lesssim t^{-\frac{n}{2} \left( \frac{1}{(2p)' - \frac{1}{2}} \right)} \left( 1 + \frac{2j + t}{t} \right)^{-N} \frac{1}{t} \|g\|_{(2p)'}.
\]
Plugging the estimate back into (5.16) and integrating over $s$ gives
\[ \|N(W_2\beta_j^j)\|_1 \lesssim 2^{jN/4}2^{jN/4}. \]
Summing over $j$ finally gives the assertion of the lemma provided $N > n/4$. $\square$

5.4. The Carleson measure estimate. In order to show (5.11), we use a similar splitting for $A$ as in Section 2. Write
\[ A(\alpha)(t, .) = \int_0^t e^{-(t-s)L} \operatorname{div} \alpha(s, .) \, ds \]
\[ = \int_0^t e^{-(t-s)L}(sL)^{-1}(I - e^{-2sL})s^{1/2} \operatorname{div} s^{1/2} \alpha(s, .) \, ds \]
\[ + \int_0^\infty e^{-(t+s)L} \operatorname{div} \alpha(s, .) \, ds \]
\[ - \int_t^\infty e^{-(t+s)L} s^{-1/2} \operatorname{div} s^{1/2} \alpha(s, .) \, ds \]
\[ =: A_1(\alpha)(t, .) + A_2(\alpha)(t, .) + A_3(\alpha)(t, .). \]

In the following, we use without further mentioning that $L$ has a bounded $H^\infty$ functional calculus in $L^p(\mathbb{R}^n)$ for any $1 < p < \infty$ (which follows from [19] Theorem 3.1] combined with Lemma 5.3).

For the estimate on $A_1$, we apply the following two lemmata. The first one is an extension of [9] Theorem 3.2 using the structure of the maximal regularity operator.

**Lemma 5.17.** Suppose $q \in [2, \infty)$. The operator
\[ M^+ : T^{\infty, 2q}(\mathbb{R}^{n+1}_+) \to T^{\infty, 2q}(\mathbb{R}^{n+1}_+), \]
\[ (M^+F)(t, .) := \int_0^t Le^{-(t-s)L}F(s, .) \, ds, \]
is bounded.

**Proof.** According to Lemma 6.5 and [12] Lemma 1.19, $(tLe^{-(t-s)L})_{t > 0}$ satisfies Gaussian estimates, therefore in particular the weaker $L^2$ off-diagonal estimates of [9] Definition 2.3]. Hence, we can apply [9] Theorem 3.2] to obtain that
\[ M^+ : T^{\infty, 2q}(\mathbb{R}^{n+1}_+) \to T^{\infty, 2q}(\mathbb{R}^{n+1}_+) \to T^{\infty, 2}(\mathbb{R}^{n+1}_+) \]
is bounded. To show it is bounded into the smaller space $T^{\infty, 2q}(\mathbb{R}^{n+1}_+)$, we argue as follows. Set
\[ \tilde{M}^+ F(t, .) := M^+ F(t, .) - \int_0^t e^{-(t-s)L}M^+ F(s, .) \, ds. \]
According to Lemma 6.13 and (5.18), we have for the last term
\[ \|\int_0^t e^{-(t-s)L}M^+ F(s, .) \, ds\|_{T^{\infty, 2q}} \lesssim \|M^+ F\|_{T^{\infty, 2}} \lesssim \|F\|_{T^{\infty, 2q}}. \]
Thus $M^+ : T^{\infty, 2q}(\mathbb{R}^{n+1}_+) \to T^{\infty, 2q}(\mathbb{R}^{n+1}_+)$ is bounded if and only if $\tilde{M}^+ : T^{\infty, 2q}(\mathbb{R}^{n+1}_+) \to T^{\infty, 2q}(\mathbb{R}^{n+1}_+)$ is bounded. To show the latter, observe that
\[ \tilde{M}^+ F(t, .) = \int_0^t \int_s^t Le^{-(t-\sigma)L}F(\sigma, .) \, d\sigma \, ds, \]
Let \( F \) consider now the case \( j \geq N \) where \( t \). Since the last expression is independent of \( \tilde{\tau} \) implies boundedness of \( \tilde{\tau} \) therefore, for any \( \tau \geq 0 \) and \( t \in (\tau, 2\tau) \), we have the time localisation formula
\[
(5.19) \quad \tilde{M}^+ F(t, .) = \tilde{M}^+ (1_{(\tau/4, \tau)} F)(t, .),
\]
hence for fixed \( (\tau, x) \),
\[
W_q(\tilde{M}^+ F)(\tau, x) = W_q(\tilde{M}^+ (1_{(\tau, 2\tau)} F))(\tau, x).
\]
Let \( F \in T^{\infty, 2, q}(\mathbb{R}^n_+) \). Fix \( (r, x_0) \in \mathbb{R}^n_+ \), and set \( B := B(x_0, \sqrt{\tau}) \). By Minkowski’s inequality,
\[
\left( r^{-n/2} \int_0^r \int_B (W_q(\tilde{M}^+ F)(\tau, x))^2 \, dx \, d\tau \right)^{1/2} \leq \sum_{j=0}^\infty \left( r^{-n/2} \int_0^r \int_B (W_q(\tilde{M}^+ 1_{S_j(B(x_0, \sqrt{\tau}))} F)(\tau, x))^2 \, dx \, d\tau \right)^{1/2} =: \sum_{j=0}^\infty I_j.
\]
Consider first the case \( j \leq 3 \). According to [17, Theorem 1.2], combined with Lemma 5.5, \( L \) has \( L^q \)-maximal regularity on \( L^q(\mathbb{R}^n) \), that is, \( \tilde{M}^+ \) is bounded on \( L^q((0, \infty); L^q(\mathbb{R}^n)) \), which also implies boundedness of \( \tilde{M}^+ \) on \( L^q((0, \infty); L^q(\mathbb{R}^n)) \). Using this bound and (5.19), one obtains
\[
W_q(\tilde{M}^+ (1_{B(x, 8\sqrt{\tau})} F))(\tau, x) \lesssim \left( \tau^{-n/2 - 1} \int_{\tau/4}^{2\tau} \int_{B(x, 8\sqrt{\tau})} |F(s, y)|^q \, ds \, dy \right)^{1/q} = C \tilde{W}_q(F)(\tau, x),
\]
where \( \tilde{W}_q \) denotes the average over the rescaled Whitney box \( (\frac{\tau}{4}, 2\tau) \times B(x, 8\sqrt{\tau}) \). By covering this Whitney box by many Whitney boxes of standard size, one obtains
\[
\left( r^{-n/2} \int_0^r \int_B (W_q(\tilde{M}^+ 1_{B(x, 8\sqrt{\tau})} F)(\tau, x))^2 \, dx \, d\tau \right)^{1/2} \lesssim \| F \|_{T^{\infty, 2, q}}.
\]
Consider now the case \( j \geq 4 \). Denote \( F_j(s, y) := F(s, y) 1_{S_j(B(x_0, \sqrt{\tau}))}(y) 1_{(0, 2\tau)}(s) \). Using Minkowski’s inequality and \( L^q \) off-diagonal estimates for the semigroup, which are a consequence of the kernel estimates stated in Lemma 5.3 (i), one obtains for fixed \( (\tau, x) \in (0, r) \times B \), \( t \in (\tau, 2\tau) \) and any \( N \geq 1 \),
\[
\| \tilde{M}^+ F_j(t, .) \|_{L^q(B(x, \sqrt{\tau}))} \leq \int_{\tau}^{\frac{\tau}{4}} \int_{\tau}^t (t - \sigma)^{-1} \| (t - \sigma) Le^{-(t-\sigma)L} F_j(\sigma, .) \|_{L^q(B(x, \sqrt{\tau}))} \, d\sigma \, ds \leq \int_{\tau}^{\frac{\tau}{4}} (t - \sigma)^{-1} \left( \frac{t - \sigma}{2j T} \right)^N \| F_j(\sigma, .) \|_{L^q(B(x, \sqrt{\tau}))} \, d\sigma \lesssim 2^{-2j N} \tau^{-1} \int_{\tau}^{\frac{\tau}{4}} \| F_j(\sigma, .) \|_{L^q(B(x, \sqrt{\tau}))} \, d\sigma.
\]
Since the last expression is independent of \( t \) and by definition of \( F_j \), we therefore have
\[
\left( \int_{W(\tau, x)} |\tilde{M}^+ F_j(t, y)|^q \, dy \, dt \right)^{1/q} \lesssim 2^{-2j N} \left( \int_{\tau}^{\frac{\tau}{4}} \int_{\tau}^{2\tau} |F(\tau, y)|^q \, dy \, d\sigma \right)^{1/q}.
\]
By change of angle in tent spaces [3, Theorem 1.1], choosing \( N \) large enough and summing over \( j \), one obtains the assertion. \( \square \)
Hence we have shown (5.20) with \( \tilde{\rho} \leq q^* \) (with \( q^* = \frac{nq}{n-q} \) if \( q < n \) and \( q^* = \infty \) otherwise) and \( r \in [2, \infty) \). Then the operator
\[
\mathcal{T} : T^{\infty,2q,r}(\mathbb{R}^{n+1}_+; \mathbb{C}^n) \to T^{\infty,2\tilde{\rho},r}(\mathbb{R}^{n+1}_+),
\]
\[
(\mathcal{T} F)(s, \cdot) := T_s(F(s, \cdot)),
\]
is bounded.

**Proof.** We first obtain \( L^q - L^\tilde{\rho} \) off diagonal estimates for \( (T_s)_{s > 0} \)
\[
(5.20) \quad \|1_E T_s 1_E\|_{L^q \to L^q} \leq C s^{-n/2(1/q-1/\tilde{\rho})} \exp(-cs^{-1} \text{dist}(E, \tilde{E})^2)
\]
for all Borel sets \( E, \tilde{E} \) and all \( s > 0 \).

Assuming first \( q < n \), we show \( \|T_s\|_{L^q \to L^q^*} \leq s^{1/2} \). The solution of the Kato square root problem \[6.24\] in \( L^2 \) with its extension to \( L^p \) spaces (see [12, Theorem 4.1] or [2]) implies that \( \nabla (L^*)^{-1/2} \) is bounded in \( L^q \) as \( 1 < q < 2 \), therefore
\[
L^{-1/2} \text{div} : L^q(\mathbb{R}^n; \mathbb{C}^n) \to L^q(\mathbb{R}^n)
\]
is bounded. Moreover, \( L^{-1/2} : L^q(\mathbb{R}^n) \to L^{q^*}(\mathbb{R}^n) \) is bounded, see [2, Proposition 5.3]. Combining this with the fact that the semigroup \( (e^{-tL})_{t > 0} \) is bounded on \( L^q^* \) gives the claim. This in particular yields \[5.20\] with \( \tilde{\rho} = q^* \) and when \( \text{dist}(E, \tilde{E}) \leq cs^{1/2} \). For \( \text{dist}(E, \tilde{E}) \geq s^{1/2} \), we can obtain the stronger \( L^2-L^\infty \) off-diagonal estimates from Lemma \[5.5\] by writing
\[
(5.21) \quad T_s = -s^{-1/2} \int_0^{2s} e^{-uL} \text{div} \ du,
\]
which then gives
\[
\|1_E T_s 1_E\|_{L^q \to L^q^*} \leq s^{-1/2} \int_0^{2s} \|1_E e^{-uL} \text{div} 1_E\|_{L^q \to L^q^*} du
\]
\[
\leq s^{-1/2} \int_0^{2s} u^{-1} \exp(-cu^{-1} \text{dist}(E, \tilde{E})^2) du
\]
\[
\leq s^{-1/2} \exp(-c's^{-1} \text{dist}(E, \tilde{E})^2).
\]
Hence we have shown \[5.20\] with \( \tilde{\rho} = q^* \) and this implies \[5.20\] for any \( q \leq \tilde{\rho} \leq q^* \). When \( q \geq n \), the first part of the argument does not work but one can instead use \[5.21\] and still obtain an integrable factor when \( \text{dist}(E, \tilde{E}) \leq cs^{1/2} \) when plugging in the \( L^q \) to \( L^{q^*} \) norm. Details are left to the reader.

Now \[5.20\] implies for \( (\tau, x) \in \mathbb{R}^{n+1}_+ \) and \( s \in (\tau, 2\tau) \)
\[
\left( \int_{B(x, \sqrt{\tau})} |T_s F(s, \cdot)(y)|^q \, dy \right)^{1/q} \leq \sum_{j=0}^\infty \left( \int_{B(x, \sqrt{\tau})} |T_{2^j} S_j(B(x, \sqrt{\tau})) F(s, \cdot)(y)|^q \, dy \right)^{1/q} \nabla \left( \int_{2^j B(x, \sqrt{\tau})} |F(s, y)|^q \, dy \right)^{1/q}
\]
\[
\leq \sum_{j=0}^\infty \left( \frac{s}{2^{2j}} \right)^N \left( \int_{2^j B(x, \sqrt{\tau})} |F(s, y)|^q \, dy \right)^{1/q}
\]
\[
\leq \sum_{j=0}^\infty 2^{-j(2N-n/q)} \left( \int_{2^j B(x, \sqrt{\tau})} |F(s, y)|^q \, dy \right)^{1/q}.
\]
Choosing $N$ large enough and using change of angle in tent spaces \[3, \text{Theorem 1.1}\], we therefore have
\[
\| TF \|_{T^{\infty,2,q,r}} \leq \| W_{q^*,r}(TF) \|_{T^{\infty,2}}
\]
\[
\lesssim \sum_{j=0}^{\infty} 2^{-j(2N - \frac{n}{q} - \frac{q}{r})} \sup_{(r,x_0) \in \mathbb{R}^{n+1}_+} \left( \int_0^r \int_{2^{j+1}B(x_0,\sqrt{r})} \left( \int_{\mathbb{R}^n} \left| F(s,y) \right|^q \, dy \right)^{r/q} \, ds \right)^{2/r} \, dxd\tau
\]
\[
\lesssim \| F \|_{T^{\infty,2,q,r}}.
\]
□

We use the theory of Hardy spaces associated with operators for the estimate of $A_2$.

**Lemma 5.19.** The operator
\[
A_2^*: T^{1,2}(\mathbb{R}^{n+1}) 	o T^{1,\infty,2}(\mathbb{R}^{n+1}; C^n),
\]
\[
(A_2^* G)(s, \cdot) = \nabla e^{-sL^*} \int_0^\infty e^{-tL^*} G(t, \cdot) \, dt,
\]
is bounded.

Note that we cannot commute $\nabla$ and the semigroup as in Navier-Stokes. Well, in fact, one can if we imbed the scalar operator in a vector operator as in the proof below.

**Proof.** We outline how to obtain the result from \[11, \text{Theorem 9.1}\]. A direct proof is possible but is long and tedious. Recall that $L = -\text{div}(A\nabla)$ with $A \in L^\infty(\mathbb{R}^n; \mathcal{L}(\mathbb{R}^n))$, $\text{Re}(A(x)) \geq \kappa I > 0$ for a.e. $x \in \mathbb{R}^n$. Associated with $L^*$ are the operators
\[
D := \begin{bmatrix} 0 & \text{div} \\ -\nabla & 0 \end{bmatrix}, \quad B^* := \begin{bmatrix} 1 & 0 \\ 0 & A^* \end{bmatrix},
\]
and
\[
DB^* := \begin{bmatrix} 0 & \text{div} A^* \\ -\nabla & 0 \end{bmatrix}, \quad DB^* DB^* := \begin{bmatrix} -\text{div} A^* \nabla & 0 \\ 0 & -\nabla \text{div} A^* \end{bmatrix},
\]
the latter acting as bisectorial and sectorial operators in $L^2(\mathbb{R}^n; C^{1+n})$, respectively. Following \[11\], for a vector $v = \begin{bmatrix} v_\perp \\ v_\parallel \end{bmatrix} \in C^{1+n}$, we call $v_\parallel \in C^n$ the tangential part of $v$. Observe that
\[
-\nabla e^{-sL^*} \int_0^\infty e^{-tL^*} G(t, \cdot) \, dt
\]
is the tangential part of
\[
DB^* e^{-sDB^* DB^*} \int_0^\infty \begin{bmatrix} e^{-tL^*} G(t, \cdot) \\ 0 \end{bmatrix} \, dt,
\]
hence the tangential part of $e^{-sDB^* DB^*} h$, with
\[
h = \begin{bmatrix} 0 \\ -\int_0^\infty \nabla e^{-tL^*} G(t, \cdot) \, dt \end{bmatrix}.
\]
An equivalent formulation of the Kato square root estimate for $L^*$ \[6, 24\] is the square function estimate
\[
\iint_{\mathbb{R}^{n+1}_+} |(e^{-tL^*} \text{div} F)(x)|^2 \, dxdt \lesssim \| F \|_2^2
\]
for all $F \in L^2(\mathbb{R}^n; \mathbb{C}^n)$, hence $(t, x) \mapsto (e^{-tL_t} \text{ div } F)(x)$ is bounded from $L^2$ to $T^{2,2}$ and by duality this defines the bounded map

$$S : T^{2,2}(\mathbb{R}^{n+1}_+) \to L^2(\mathbb{R}^n; \mathbb{C}^n),$$

$$SG = \int_0^{\infty} \nabla e^{-tL^*} G(t, .) \, dt.$$

By application of Lemma 5.5 (ii) for $\nabla e^{-tL^*}$, one can show (e.g., by adapting the proof of Theorem 6), using $L^1-L^2$ off-diagonal estimates instead of kernel estimates) that this operator maps $T^{1,2}(\mathbb{R}^{n+1}_+)$ to $H^1(\mathbb{R}^n; \mathbb{C}^n)$. Thus, $G \in T^{1,2}(\mathbb{R}^{n+1}_+)$ implies $h \in H^1_D(\mathbb{R}^n; \mathbb{C}^{n+1})$ where this space is a closed subspace of $H^1$ defined, for example in [11]. As $B^*$ has real coefficients, [11] Corollary 13.3] shows that $H^1_D(\mathbb{R}^n; \mathbb{C}^{n+1}) = H^1_D(\mathbb{R}^n; \mathbb{C}^{n+1})$ (the former space also being defined in [11]). Therefore, [11] Theorem 9.1] is applicable. Combined with [11] Remark 9.8, it yields for every $h \in H^1_D(\mathbb{R}^n; \mathbb{C}^{n+1})$,

$$\|\tilde{N}(e^{-sDB^*DB^*}h)\|_1 \lesssim \|h\|_{H^1}.$$ 

where $\tilde{N}$ is the variant of the non-tangential maximal function $N$ used in [11] and $\tilde{N}F$ and $NF$ have (by a purely geometrical argument) equivalent $L^1$ norms. This gives the assertion. □

Finally, to handle $A_3$, we show

**Lemma 5.20.** The sublinear operator $\tilde{R} : F \mapsto \tilde{F}$ with

$$\tilde{F}(t, .) = \int_0^\infty |e^{-(t+s)L} - 1/2 \text{ div } F(s, .)| \, ds$$

is bounded from $T^{\infty,2}(\mathbb{R}^{n+1}_+; \mathbb{C}^n)$ to $T^{\infty,2}(\mathbb{R}^{n+1}_+)$.

**Proof.** Write

$$\tilde{F}(t, .) = \int_0^\infty |K(t, s)F(s, .)| \, ds,$$

with $K(t, s) := e^{-(t+s)L} - 1/2 \text{ div } F(s, .)$ for $s, t > 0$. As a consequence of uniform boundedness of $(e^{-(t+s)L} - 1/2 \text{ div } F)_{t>0}$ in $L^2$, one has

$$\|K(t, s)\|_{L^2 \to L^2} = s^{-1/2}(t+s)^{-1/2} \|e^{-(t+s)L} - 1/2 \text{ div } F\|_{L^2 \to L^2} \lesssim s^{-1/2}(t+s)^{-1/2}.$$

This allows to apply Schur’s lemma as for Lemma 3.3 and implies boundedness of $\tilde{R}$ from $L^2(\mathbb{R}^{n+1}_+; \mathbb{C}^n)$ to $L^2(\mathbb{R}^{n+1}_+)$.

For the extension to $T^{\infty,2}$, observe that Lemma 5.5 yields $L^2-L^\infty$ off-diagonal estimates of the form 3.3. The second part of the proof of Lemma 3.3 directly carries over to the present situation, and yields boundedness of $\tilde{R}$ on $T^{\infty,2}$.

### 5.5. Proof of Theorem 5.4

We have shown in Corollary 5.14 that for every given initial data $u_0 \in BMO^{-1}(\mathbb{R}^n)$, the free evolution $u(t, x) = e^{-tL}u_0(x)$ belongs to the path space $\mathcal{E}_\infty$ defined in 5.4. Let us assume for a moment 5.10 and 5.11. Then the theorem is a consequence of Picard’s contraction principle. The integral equation (5.2) is equivalent to

$$u(t, .) = e^{-tL}u_0 - A(f(u^2))(t, .),$$

and Lemma 5.15, 5.10, 5.11 imply

$$\|A(f(u^2)) - A(f(v^2))\|_{\mathcal{E}_\infty} \leq C\|u - v\|_{\mathcal{E}_\infty} \|u + v\|_{\mathcal{E}_\infty}.$$
The smallness condition on $u_0$ ensures that (5.2) has a global solution, unique in the open ball $B(0, \frac{1}{p})$ of $\mathcal{E}_\infty$.

We finally show (5.10) and (5.11). The $L^\infty$ estimate (5.10) is entirely contained in Lemma 5.16. As for the Carleson estimate (5.11) for $\mathcal{A}$, we argue as follows:

For $A_1$, we apply Lemma 5.18 with $q = p > n/2$ and $q = r = 2p$, in which case $q \leq q^*$.

Combining this with Lemma 5.17, we have

$$\|A_1(\alpha)\|_{T^{\infty,2},2p} = \|\mathcal{M}^+ T(s^{1/2} \alpha(s,.))\|_{T^{\infty,2},2p} \lesssim \|T(s^{1/2} \alpha(s,.))\|_{T^{\infty,2},2p} \leq \|s^{1/2} \alpha(s,.))\|_{T^{\infty,2},2p}.$$ 

Concerning $A_2$, Lemma 5.19 above establishes boundedness of the dual operator $A_2^*$, from $T^{1,2}(\mathbb{R}^n_+)$ to $T^{1,\infty,2}(\mathbb{R}^n_+; C^n)$. By duality and Proposition 5.2, respectively, we therefore obtain boundedness of the operator $A_2$ from $T^{\infty,1,2}(\mathbb{R}^n_+; C^n)$ to $T^{\infty,2}(\mathbb{R}^n_+)$.

To obtain boundedness into $T^{\infty,2,p}(\mathbb{R}^n_+)$, we apply Lemma 5.13 with $F(s) = A_2(\alpha)(s,.)$ and $q = r = 2p$. Observe that, $A_2(\alpha)(t,. ) = e^{-(t-s)L} A_2(\alpha)(\tau,. )$ for each $\tau < t$, hence one finds with the notation of Lemma 5.15, $G(t) = A_2(\alpha)(t,. )$, therefore we obtain the bootstrap estimate

$$\|A_2(\alpha)\|_{T^{\infty,2,2p}} \lesssim \|A_2(\alpha)\|_{T^{\infty,2}} \lesssim \|\alpha\|_{T^{\infty,1,2}}.$$ 

Finally, we consider $A_3$. We apply a slight variant of Lemma 5.13. Fix $t > 0$. For $s \in [t/4, t/2]$, $A_3(\alpha)(t,. ) = e^{-(t-s)L} F_t(s,. )$, with $F_t(s,. ) = \int_t^\infty e^{-(s+\sigma)L} \text{div} \alpha(s,. ) d\sigma$. Hence, $A_3(\alpha)(t,. ) = \frac{1}{t} \int_t^\infty e^{-(t-s)L} F_t(s,. ) ds$. Applying the beginning of the argument for Lemma 5.13 and observing that $|F_t(s,. )| \leq \tilde{F}(s,. ) := \int_t^\infty e^{-(s+\sigma)L} \text{div} \alpha(s,. ) d\sigma$, we obtain $\|A_3(\alpha)\|_{T^{\infty,2,2p}} \lesssim \|\tilde{F}\|_{T^{\infty,2}}$. Using Lemma 5.20 and Hölder’s inequality in the last step, we have

$$\|\tilde{F}\|_{T^{\infty,2}} = \|\tilde{R}(s^{1/2} \alpha(s,. ))\|_{T^{\infty,2}} \lesssim \|s^{1/2} \alpha(s,. )\|_{T^{\infty,2}} \lesssim \|s^{1/2} \alpha(s,. )\|_{T^{\infty,2,2p}}.$$ 

as $p \geq 2$. This finishes the proof of the theorem.

**Remark 5.21.** One wonders why we analyse $A_2$ and $A_3$ separately, while in [32], this is not needed. In Section 4, we already observed that at the level of tent spaces we used the $T^{1,2}_t$ condition on $\alpha$ and not the pointwise bounds on $\alpha$, while the latter and not the former is used in the proof of the energy estimate (15) of [32]. This proof requires an integration by parts to absorb some non absolutely convergent integrals.

Supposing we want to analyse $A_2 + A_3$ as one operator, we would have to prove a $T^{\infty,2,p}$ control for this sum. Or similarly, taking Lemma 5.15 into account, a bound in $T^{\infty,2,}$, is a Carleson measure estimate. This means that locally, we would be looking at expressions such as

$$\int_0^T \int_{B(x,\sqrt{\tau})} | \int_0^t e^{-(t-s)L} \text{div} \alpha(s,. ) ds |^2 dy dt,$$

and we would have to bound against some form of local $L^1$ estimates for $\alpha$ (and, possibly, the local averaged $L^\infty$ bounds $N_p$). Compared to [32], we face the following problems here. First, $\text{div}$ does not commute anymore with the semigroup. As explained after Lemma 5.14, we can still use a commutation property, but only by making use of the framework of first order Hodge-Dirac operators. But second, it is not clear how to compute the square and beat the lack of absolute convergence of the integral inside as in [32], as we impose no self-adjointness on $L$. The other option is to argue by duality with non-tangential maximal functions. But even in this specific situation, we do not see how to handle the terms. Thus $A_2$ contains the "singular terms" which are handled by the Hardy space technique (to absorb the non absolutely converging terms) while $A_3$ is a remainder term with no singularity and no use of Hardy spaces, thus acting on a different
tent space. It is not clear to us how to use the condition on $N_p(s\alpha)$ from the solution space in such estimates.

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References

[1] P. Auscher. *Regularity theorems and heat kernel for elliptic operators*. J. London Math. Soc. (2), 54, no. 2, 284–296, 1996.
[2] P. Auscher. On necessary and sufficient conditions for $L^p$-estimates of Riesz transforms associated to elliptic operators on $\mathbb{R}^n$ and related estimates. *Mem. Amer. Math. Soc.* 871 (2007).
[3] P. Auscher. *Change of angle in tent spaces*. C. R. Math. Acad. Sci. Paris, 349, no. 5-6, 297–301, 2011.
[4] P. Auscher, A. Axelsson. *Remarks on maximal regularity*. Parabolic problems. The Herbert Amann Festschrift. Basel: Birkhäuser. Progress in Nonlinear Differential Equations and Their Applications 80, 45-55, 2011.
[5] P. Auscher, X.T. Duong and A. McIntosh. *Boundedness of Banach space valued singular integral operators and Hardy spaces*. Unpublished manuscript, 2002.
[6] P. Auscher, S. Hofmann, M. Lacey, A. McIntosh, P. Tchamitchian. *The solution of the Kato square root problem for second order elliptic operators on $\mathbb{R}^n$*. Ann. of Math. (2), 156(2):633–654, 2002.
[7] P. Auscher, C. Kriegler, S. Monniaux and P. Portal. *Singular integral operators on tent spaces*. J. Evol. Equ., 12(4):741–765, 2012.
[8] P. Auscher, A. McIntosh and E. Russ. *Hardy spaces of differential forms on Riemannian manifolds*. J. Geom. Anal., 18(1):192–248, 2008.
[9] P. Auscher, S. Monniaux and P. Portal. *The maximal regularity operator on tent spaces*. Commun. Pure Appl. Anal., 11(6):2213–2219, 2012.
[10] P. Auscher, J. van Neerven and P. Portal. *Conical stochastic maximal $L^p$-regularity for $1 \leq p < \infty$*. Math. Ann., 359, no. 3-4:863–889, 2014.
[11] P. Auscher and S. Stahlhut. *A priori estimates for boundary value elliptic problems via first order systems*. Preprint, [arXiv:1403.5367] [math.CA], 2014.
[12] P. Auscher and P. Tchamitchian. *Square root problem for divergence operators and related topics*. Astérisque no. 249, 1998.
[13] P. Auscher and P. Tchamitchian. *Espaces critiques pour le système des équations de Navier-Stokes incompressibles*. Preprint, [arXiv:0812.1158] [math.AP], 1999.
[14] H. Bahouri and I. Gallagher. *The heat kernel and frequency localized functions on the Heisenberg group*. Progr. Nonlinear Differential Equations Appl., 78:17–35, 2009.
[15] J. Bourgain and N. Pavlović. *Ill-posedness of the Navier-Stokes equations in a critical space in 3D*. J. Funct. Anal., 255(9):2233–2247, 2008.
[16] R.R. Coifman, Y. Meyer and E.M. Stein. *Some new function spaces and their applications to harmonic analysis*. J. Funct. Anal., 62:304–335, 1985.
[17] T. Coulhon and X.T. Duong. *Maximal regularity and kernel bounds: observations on a theorem by Hieber and Prüss*. Adv. Differential Equations 5, no.1-3:343–368, 2000.
[18] S. Dubois. *What is a solution to the Navier-Stokes equations?* C. R., Math., Acad. Sci. Paris, 335(1):27–32, 2002.
[19] X.T. Duong and D.W. Robinson. *Semigroup kernels, Poisson bounds, and holomorphic functional calculus*. J. Funct. Anal., 142(1):89–128, 1996.
[20] C.L. Fefferman and E.M. Stein. *$H^p$ spaces of several variables*. Acta Math., 129:137–193, 1972.
[21] G. Furioli, P.-G. Lemarié-Rieusset and E. Terraneo. *Unicité dans $L^3(\mathbb{R}^3)$ et d’autres espaces fonctionnels limites pour Navier-Stokes*. Rev. Mat. Iberoam., 16(3):605–667, 2000.
[22] Y. Giga. *Solutions for semilinear parabolic equations in $L^p$ and regularity of weak solutions of the Navier-Stokes system*. J. Differ. Equations, 61:186–212, 1986.
[23] B.H. Haak and P.C. Kunstmann. *On Kato’s method for Navier-Stokes equations*. J. Math. Fluid Mech., 11(4):492–535, 2009.

[24] S. Hofmann, Lacey and A. McIntosh. *The solution of the Kato problem for divergence form elliptic operators with Gaussian heat kernel bounds*. Ann. of Math. (2), 156(2):623–631, 2002.

[25] S. Hofmann and S. Mayboroda. *Hardy and BMO spaces associated to divergence form elliptic operators*. Math. Ann., 344(1):37–116, 2009.

[26] S. Hofmann, S. Mayboroda and A. McIntosh. *Second order elliptic operators with complex bounded measurable coefficients in $L^p$, Sobolev and Hardy spaces*. Ann. Sci. Éc. Norm. Supér. (4), 44(5):723–800, 2011.

[27] T. Hytönen, J. van Neerven and P. Portal. *Conical square function estimates in UMD Banach spaces and applications to $H^\infty$-functional calculi*. J. Anal. Math., 106:317–351, 2008.

[28] T. Hytönen and A. Rosén. *On the Carleson duality*. Ark. Mat., 51(2):293–313, 2013.

[29] T. Iwabuchi and M. Nakamura. *Small solutions for nonlinear heat equations, the Navier-Stokes equation, and the Keller-Segel system in Besov and Triebel-Lizorkin spaces*. Adv. Differ. Equ., 18(7-8):687–736, 2013.

[30] H. Koch and T. Lamm. *Geometric flows with rough initial data*. Asian J. Math., 16(2):209–235, 2012.

[31] H. Koch and T. Lamm. *Parabolic equations with rough data*. Preprint, arXiv:1310.3658 [math.AP], 2013.

[32] H. Koch and D. Tataru. *Well-posedness for the Navier-Stokes equations*. Adv. Math., 157(1):22–35, 2001.

[33] P. G. Lemarié-Rieusset. *Recent developments in the Navier-Stokes problem*. Chapman & Hall/CRC Research Notes in Mathematics Series, 2002.

[34] P. G. Lemarié-Rieusset and F. Marchand. *Solutions auto-similaires non radiales pour l’équation quasi-géostrophique dissipative critique*. C. R., Math., Acad. Sci. Paris, 341(9):535–538, 2005.

[35] W. Liu and M. Röckner. *Local and global well-posedness of SPDE with generalized coercivity conditions*. J. Differ. Equations, 254(2):725–755, 2013.

[36] M. Mitrea and S. Monniaux. *On the analyticity of the semigroup generated by the Stokes operator with Neumann-type boundary conditions on Lipschitz subdomains of Riemannian manifolds*. Trans. Am. Math. Soc., 361(6):3125–3157, 2009.

[37] M. Mitrea and S. Monniaux. *The nonlinear Hodge-Navier-Stokes equations in Lipschitz domains*. Differ. Integral Equ., 22(3-4):339–356, 2009.

[38] M. Mitrea and M.E. Taylor. *Navier-Stokes equations on Lipschitz domains in Riemannian manifolds*. Math. Ann., 321(4):955–987, 2001.

[39] L. de Simon. *Un’applicazione della teoria degli integrali singolari allo studio delle equazioni differenziali lineare astratte del primo ordine*. Rend. Sem. Mat., Univ. Padova 205–223, 1964.

[40] M.E. Taylor. *Incompressible fluid flows on rough domains*. Progr. Nonlinear Differential Equations Appl., 42:320–334, Birkhäuser, 2000.

[41] H. Triebel. *Theory of function spaces*. Monographs in Mathematics, 78. Birkhäuser Verlag, Basel, 1983.

[42] T. Yoneda. *Ill-posedness of the 3D-Navier-Stokes equations in a generalized Besov space near BMO$^{-1}$. J. Funct. Anal., 258(10):3376–3387, 2010.

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