Linear equations for the number of intervals which are isomorphic with Boolean lattices and the Dehn–Sommerville equations

Gábor Hegedűs
Johann Radon Institute for Computational and Applied Mathematics

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Abstract

Let \( P \) be a finite poset. Let \( L := J(P) \) denote the lattice of order ideals of \( P \). Let \( b_i(L) \) denote the number of Boolean intervals of \( L \) of rank \( i \).

We construct a simple graph \( G(P) \) from our poset \( P \). Denote by \( f_i(P) \) the number of the cliques \( K_{i+1} \), contained in the graph \( G(P) \).

Our main results are some linear equations connecting the numbers \( f_i(P) \) and \( b_i(L) \).

We reprove the Dehn–Sommerville equations for simplicial polytopes.

In our proof we use free resolutions and the theory of Stanley–Reisner rings.

1 Introduction

In my article I use the following proof technique:

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Let $R$ denote the polynomial ring $\mathbb{Q}[x_1, \ldots, x_n]$. Let $I$ be a monomial ideal. If we know a graded free resolution of the quotient module $M := R/I$:

$$0 \to F_n \to \ldots \to F_1 \to M \to 0$$

then we can compute the Hilbert function (or the Hilbert polynomial) of $M = R/I$.

On the other hand, we can compute the Hilbert function of the module $M = R/I$ by counting standard monomials.

If we compare these computations we get a new equation, and we can prove some linear equations for the combinatorial invariants of the monomial ideal $I$.

We give here two applications in lattice theory and in the theory of polytopes.

Throughout this paper we use the following notations.

Let $(P, \preceq)$ be a fixed poset. We say that $J \subseteq P$ is an order ideal of $P$, if $a \in J$ and $b \preceq_P a$, then $b \in J$. Let $L := J(P)$ denote the distributive lattice of order ideals of $P$.

We say that a lattice $B$ is a Boolean lattice, if $B$ is distributive, $B$ has 0 and 1 and each $a \in B$ has a complement $a' \in B$.

Let $M$ denote an arbitrary finite distributive lattice. Let $l, k \in M$ with $l \leq k$. Then the set

$$[l, k] := \{m \in M : l \leq m \leq k\} \subseteq M$$

is called an interval in $M$. Let $b_i(M)$ denote the number of intervals of $M$, which are isomorphic to the Boolean lattice of rank $i$.

In our main result we describe some linear equations for the numbers $b_i(L)$.

We can state our results in a more compact form if we associate the following graphs $G(P)$ to the poset $P$.

Let $P = \{q_1, \ldots, q_p\}$ be a finite poset, $|P| = p$. We define a simple graph $G(P)$ as follows: let the vertex set of $G(P)$ be the disjoint union $\{x_1, \ldots, x_p\} \cup \{y_1, \ldots, y_p\}$. Define the edge set of $G(P)$ as

$$\{(x_i, x_j) : 1 \leq i < j \leq p\} \cup \{(y_i, y_j) : 1 \leq i < j \leq p\} \cup \{(x_i, y_j) : q_i \not\preceq q_j\}.$$

(1)

Let $j$ be a nonnegative integer. Denote by $f_j(P)$ the number of the cliques $K_{j+1}$, contained in the graph $G(P)$. 

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In particular, \( f_0(P) = 2p \) and \( f_1(P) = |E(G(P))| \). Let \( f_{-1}(P) = 1 \).

Our main results are the following formulas, which connect the numbers \( f_i(P) \) to the numbers \( b_i(J(P)) \).

**Theorem 1.1** Let \( P \) be a fixed poset, \( p := |P| \). Let \( L := J(P) \) denote the distributive lattice of order ideals of \( P \). Let \( k \) be the Sperner number of the poset \( P \), i.e., the maximum of the cardinalities of antichains of \( P \). Then

\[
f_{2p-i-1}(P) = \sum_{m=0}^{k} (-1)^m b_m(L) \binom{p-m}{2p-i}
\]

(2)

for each \( p \leq i \leq 2p \).

For example, let \( P \) be an antichain with \( |P| = p \). Clearly the Sperner number of this poset \( P \) is \( p \). It can be shown easily that \( L := J(P) \) is isomorphic to the Boolean lattice of rank \( p \). The vertex set of the graph \( G(P) \) is the disjoint union \( \{x_1, \ldots, x_p\} \cup \{y_1, \ldots, y_p\} \), the edge set of \( G(P) \)

\[
\{\{x_i, x_j\} : 1 \leq i < j \leq p\} \cup \{\{y_i, y_j\} : 1 \leq i < j \leq p\} \cup \{\{x_i, y_j\} : i \neq j, 1 \leq i, j \leq p\}.
\]

(3)

It can be shown easily that \( b_i(L) = \binom{p}{i}2^{p-i} \) for each \( 0 \leq i \leq p \) and \( f_i(P) = \binom{p}{i+1}2^{i+1} \) for each \(-1 \leq i \leq p-1\). Hence equation (2) becomes

\[
\left( \frac{p}{2p-i} \right)2^{2p-i} = \sum_{m=0}^{p} (-1)^m \binom{p}{m} \binom{p-m}{2p-i}2^{p-m}
\]

for each \( p \leq i \leq 2p \).

We say that a polytope \( Q \) is *simplicial*, if all proper faces of \( Q \) are simplices.

Let \( Q \) be a \( d \)-dimensional simplicial polytope. We define the \( f \)-vector of \( Q \) as follows:

\[
f(Q) := (f_{-1}(Q), f_0(Q), \ldots, f_{d-1}(Q)) \in \mathbb{N}^{d+1},
\]

where \( f_i(Q) \) is the number of \( i \)-dimensional faces of \( Q \) and \( f_{-1}(Q) = 1 \). The \( h \)-vector of \( Q \):

\[
h(Q) := (h_0(Q), \ldots, h_d(Q)) \in \mathbb{N}^{d+1},
\]
where
\[ h_k(Q) := \sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{d-k} f_{i-1}(Q) \]
for each \(0 \leq k \leq d\).

In particular, \(h_0(Q) = 1\), \(h_1(Q) = f_0(Q) - d\), and
\[ h_d(Q) = f_{d-1}(Q) - f_{d-2}(Q) + \ldots + (-1)^{d-1} f_0(Q) + (-1)^d. \]

The following equations describe the complete set of linear equations for the coordinates of the \(f\)-vector of a simplicial polytope.

**Theorem 1.2 (Dehn–Sommerville equations)** Let \(f(Q) = (f_{-1}(Q), f_0(Q), \ldots, f_{d-1}(Q))\) be the \(f\)-vector of a \(d\)-dimensional simplicial polytope. Then
\[ f_{k-1}(Q) = \sum_{i=k}^{d} (-1)^{d-i} \binom{i}{k} f_{i-1}(Q). \]  
(4)

for each \(0 \leq k \leq d\).

We can write in the following short form these equations:
\[ h_k(Q) = h_{d-k}(Q) \]
for each \(0 \leq k \leq d\).

An important special case is the following **Euler–Poincaré formula**:
\[ f_0(Q) - f_1(Q) + \ldots + (-1)^d f_{d-1}(Q) = 1 - (-1)^d. \]  
(5)

The history of the Dehn–Sommerville equations starts with M. Dehn, who proved the case \(d = 5\) in [8]. Later Sommerville in [18] proved the general case for simplicial polytopes. V. Klee in [15] gave an elementary proof on the level of simplicial semi–Eulerian complexes. This class includes all the triangulated manifold (without boundary).

Here we reprove these equations in the special case of simplicial polytopes.

The outline of the present paper is the following.

First in Chapter 2 we collected the preliminary definitions and results about simplicial complexes, Stanley–Reisner rings, graphs, free resolutions and the Hibi ideal of the poset \(P\). In Chapter 3 we provide a short proof for our Theorem 1.1. In Chapter 4 we reprove the Dehn–Sommerville equations (4) and give an application using a formula of Peskin and Szpiro [17]. In our proof we use the homological algebra of free resolutions and the theory of Stanley–Reisner rings. Our results are based on the results of H. Hibi and J. Herzog in [11].
2 Preliminaries

2.1 Simplicial complexes

We say that $\Delta \subseteq 2^{[n]}$ is a simplicial complex on the vertex set $[n] = \{1, 2, \ldots, n\}$, if $\Delta$ is a set of subsets of $[n]$ such that $\Delta$ is a down–set, that is, $G \in \Delta$ and $F \subseteq G$ implies that $F \in \Delta$, and $\{i\} \in \Delta$ for all $i$.

The elements of $\Delta$ are called faces and the dimension of a face is one less than its cardinality. An $r$-face is an abbreviation for an $r$-dimensional face. The dimension of $\Delta$ is the dimension of a maximal face. We use the notation $\dim(\Delta)$ for the dimension of $\Delta$.

Let $f_i(\Delta)$ denote the number of $i$–faces of $\Delta$. If $\dim(\Delta) = d - 1$, then the $(d + 1)$–tuple $(f_{-1}(\Delta), \ldots, f_{d-1}(\Delta))$ is called the $f$-vector of $\Delta$, where $f_i(\Delta)$ denotes the number of $i$–dimensional faces of $\Delta$.

For example, let $Q$ be a simplicial polytope. The boundary complex $\Delta(Q)$ is formed by the set of vertices of all proper faces of $Q$.

A flag complex is a simplicial complex with the property that every minimal nonface has precisely two elements.

Let $\mathcal{F} \subseteq 2^{[n]}$ be an arbitrary set system. Define the complement of $\mathcal{F}$ as

$$\mathcal{F}' := 2^{[n]} \setminus \mathcal{F}.$$ 

Consider the following set system

$$\co(\mathcal{F}) := \{[n] \setminus F : F \in \mathcal{F}\}.$$ 

We denote by $\mathcal{F}^*$ the Alexander dual of $\mathcal{F}$

$$\mathcal{F}^* := \co(\mathcal{F}') = (\co(\mathcal{F}))' \subseteq 2^{[n]}.$$ 

We collect here some definition from the theory of Stanley–Reisner rings.

Let $\mathbb{Q}$ denote the rational field. Let $R$ stand for the polynomial ring $\mathbb{Q}[x_1, \ldots, x_n]$. We denote by $\mathbb{Q}[x_1, \ldots, x_n]_{\leq s}$ the vector space of all polynomials over $\mathbb{Q}$ with degree at most $s$.

Let $\Delta$ be an arbitrary simplicial complex. We associate the Stanley–Reisner ideal $I(\Delta)$ to the simplicial complex $\Delta$:

$$I(\Delta) := \langle x_F : F \notin \Delta \rangle \subseteq R.$$ 

Clearly $I(\Delta)$ is a monomial ideal.
The Stanley-Reisner ring of a simplicial complex $\Delta$ is the quotient ring

$$Q[\Delta] := R/I(\Delta).$$

Let $I$ be an arbitrary ideal of $R = Q[x_1, \ldots, x_n]$. The Hilbert function of the algebra $R/I$ is the sequence $h_{R/I}(0), h_{R/I}(1), \ldots$. Here $h_{R/I}(m)$ is the dimension over $Q$ of the factor-space $Q[x_1, \ldots, x_n]_{\leq m}/(I \cap Q[x_1, \ldots, x_n]_{\leq m})$ (see [4, Section 9.3]).

On the other hand, if we know the $f$-vector of $\Delta$, then we can compute easily the Hilbert function $h_{Q[\Delta]}(t)$ of the Stanley-Reisner ring $Q[\Delta]$.

**Lemma 2.1** (Stanley, see Theorem 5.1.7 in [3]) The Hilbert function of the Stanley-Reisner ring $Q[\Delta]$ of a $(d-1)$-dimensional simplicial complex $\Delta$ is

$$h_{Q[\Delta]}(t) = \sum_{j=0}^{d-1} f_j(\Delta) \left( \binom{t-1}{j} \right).$$

(6)

Let $\Delta^*$ denote the Alexander dual of the simplicial complex $\Delta$. We can easily compute $f^*(\Delta)$, the $f$-vector of $\Delta^*$:

**Lemma 2.2** Let $f(\Delta) = (f_{-1}(\Delta), \ldots, f_{d-1}(\Delta))$ be the $f$-vector of a $(d-1)$-dimensional simplicial complex $\Delta$. Then the $f$-vector of the simplicial complex $\Delta^*$ is:

$$f^*(\Delta) = f((\Delta)^*) = \left[ f_{-1}^*, \left( \binom{n}{1} - f_0 \right), \ldots, \left( \binom{n}{n-d-1} - f_{d-1} \right), \left( \binom{n}{n-d} - f_{d-1} \right), \ldots, \left( \binom{n}{2} - f_1 \right) \right].$$

(7)

**Corollary 2.3** Let $f(\Delta) = (f_{-1}(\Delta), \ldots, f_{d-1}(\Delta))$ be the $f$-vector of a $(d-1)$-dimensional simplicial complex $\Delta$. Then the Hilbert function $h_M(t)$ of the quotient ring $M = Q[\Delta^*] = R/I(\Delta^*)$ is

$$h_M(t) = h_{Q[\Delta^*]}(t) = \sum_{i=0}^{n-d-1} \binom{n}{i} \left( \binom{t}{i} \right) + \sum_{j=2}^{d} \left( \binom{n}{j} - f_{j-1} \right) \left( \binom{t}{n-j} \right).$$

(8)

**Proof.** If we substitute the $f$-vector $f^*(\Delta)$ of the Alexander dual $\Delta^*$ from Lemma 2.2 to the equation (6), we get the result. \qed
2.2 Graph theory

Let $G$ be a finite graph on the vertex set $[n] = \{1, 2, \ldots, n\}$ with no loops and no multiple edges. We will assume in the following that $G$ possesses no isolated vertex. Let $R = \mathbb{Q}[x_1, \ldots, x_n]$ denote the polynomial ring in $n$ variables over the field $\mathbb{Q}$.

We can associate a useful ideal $I(G)$ to the graph $G$. The edge ideal of $G$ is the ideal $I(G)$ of $R$ generated by the squarefree quadratic monomials $x_ix_j$ such that $\{i, j\}$ is an edge of $G$.

A finite graph $G$ is bipartite if there is a partition $[n] = T \cup T'$ such that each edge of $G$ is of the form $\{j, k\}$, where $j \in T$ and $k \in T'$. It is a well-known fact from graph theory that a finite graph $G$ is bipartite if and only if $G$ possesses no cycle of odd length.

The complementary graph of $G = (V, E)$ is the graph $\overline{G}$ with the vertices of $V$ and edges all the couples $\{v_i, v_j\}$ such that $i \neq j$ and $\{v_i, v_j\} \notin E$.

A clique of a graph $G$ is a complete subgraph of $G$. We can associate to a graph $G$ the clique complex $\Delta(G)$: this is the collection of all the cliques of the graph $G$, which forms a simplicial complex.

The following Lemma is an easy consequence of the definitions.

**Lemma 2.4** Let $G$ be a simple graph. Then

$$I(\overline{G}) = I(\Delta(G)).$$

(9)

2.3 Free resolutions

We introduce some terminology for describing free resolutions.

Let $\mathbb{Q}$ denote the rational field. Let $R$ be the graded ring $\mathbb{Q}[x_1, \ldots, x_n]$. The vector space $R_s = \mathbb{Q}[x_0, \ldots, x_n]_s$ consists of the homogeneous polynomials of total degree $s$, together with 0.

Recall that $M$ over $R$ is a graded module with a family of subgroups $\{M_t : t \in \mathbb{Z}\}$ of the additive group, where $M_t$ are the homogeneous elements of degree $t$, if we can write $M$ in the form

$$M = \bigoplus_{t \in \mathbb{Z}} M_t$$

and

$$R_s M_t \subseteq M_{s+t}$$
for all $s \geq 0$ and $t \in \mathbb{Z}$. If $M$ is finitely generated, then it can be shown easily that $M_t$ are finite dimensional vector spaces over $\mathbb{Q}$.

Let $M$ be a graded $R$-module and let $d \in \mathbb{Z}$ be an arbitrary integer. We can define

$$M(d) := \bigoplus_{t \in \mathbb{Z}} M(d)_t,$$

where $M(d)_t := M_{d+t}$. Then $M(d)$ is again a graded $R$-module.

Consider the graded free modules of the form $R(d_1) \oplus \ldots \oplus R(d_n)$ for any integers $d_1, \ldots, d_n$. We say that these free modules are the twisted graded free modules.

Let $M$ be a graded $R$–module. A graded resolution of $M$ is a resolution of the form

$$0 \rightarrow F_n \rightarrow \ldots \rightarrow F_1 \rightarrow M \rightarrow 0,$$

where each $F_l$ is a twisted graded free module and each homomorphism $\phi_l : F_l \rightarrow F_{l-1}$ is a graded homomorphism such that $\phi(F_l)_t \subseteq (F_{l-1})_t$ for all $t \in \mathbb{Z}$.

It is a well–known fact from the theory of free resolutions that every finitely generated $R$–module has a finite graded resolution of length at most $n$ (see [6, Chapter 6, Theorem 3.8]).

We say that the resolution

$$0 \rightarrow F_n \rightarrow \ldots \rightarrow F_1 \rightarrow M \rightarrow 0$$

is minimal iff $\phi_l : F_l \rightarrow F_{l-1}$ takes the standard basis of $F_l$ to a minimal generating set of $\text{im}(\phi_l)$ for each $l \geq 1$.

Let $M$ be a finitely generated graded $R$–module. Then we define the Hilbert function $H_M(t)$ by

$$H_M(t) := \dim_{\mathbb{Q}} M_t.$$

Now we specialize this definition for the case of the homogeneous ideals.

Let $I \trianglelefteq R$ be a homogeneous ideal of $R$. Then the quotient ring $R/I$ has a natural graded module structure, set $(R/I)_t := R_t/I_t$, where $I_t := I \cap R_t$. Thus it comes out from the definitions that if $M := R/I$ is the quotient graded $R$-module, then $H_M(t) = h_{R/I}(t)$ for each $t \geq 0$.

In the following Theorem we connect the computation of the Hilbert function $H_M(t)$ to the computation of the dimensions of the free graded modules in a graded resolution of $M$. 8
Theorem 2.5 ([6, Chapter 6, Proposition 4.7]) Let $M$ be a graded $R$-module with the graded free resolution

$$0 \rightarrow F_n \rightarrow \ldots \rightarrow F_1 \rightarrow M \rightarrow 0. \quad (12)$$

If each $F_j$ is the twisted free graded module $F_j = \bigoplus_{i=1}^{\beta_j} R(d_{i,j})$, then

$$H_M(t) = \sum_{j=1}^{k} (-1)^{\beta_j} \sum_{i=1}^{\beta_j} \binom{n + d_{i,j} + t}{n}. \quad (13)$$

The numbers $\beta_j$ are the Betti numbers of the module $M$.

Let $I \subseteq R$ be an arbitrary graded ideal with graded minimal free resolution

$$0 \rightarrow \bigoplus_{j=1}^{\beta_1} R(-a_{sj}) \rightarrow \ldots \rightarrow \bigoplus_{j=1}^{\beta_s} R(-a_{1j}) \rightarrow R/I \rightarrow 0$$

Suppose that $\text{height}(I) = h$.

Denote by $e(R/I)$ the Hilbert–Samuel multiplicity of the ring $R/I$. Then by a formula of Peskine and Szpiro \[17\]

$$e(R/I) = \frac{(-1)^i}{h!} \sum_{j=1}^{s} (-1)^i \sum_{j=1}^{\beta_j} (a_{ij})^h. \quad (14)$$

2.4 Hibi ideals of a poset $P$

We give here a short summary about the results of H. Hibi and J. Herzog (see \[11\]).

Let $P$ be a finite poset, $|P| = p$. Let $\mathbb{Q}$ denote the rational field. Consider

$$S := \mathbb{Q}[\{x_p, y_p\}_{p \in P}],$$

the polynomial ring in $2p$ variables.

Let $K \subseteq P$ be an arbitrary order ideal of $P$. We associate with $K$ the square–free monomial

$$u_K := \prod_{p \in K} x_p \prod_{p \in P \setminus K} y_p \in S.$$
In particular, \( u_P := \prod_{p \in P} x_p \) and \( u_\emptyset = \prod_{p \in P} y_p \).

H. Hibi and J. Herzog defined in [11] the Hibi ideal

\[ H(P) := \langle u_K : K \in J(P) \rangle \subseteq S \]

of \( P \), which is generated by all \( u_K \).

They described the following beautiful graded free resolution of \( H(P) \) (see Theorem 2.1 of [11]).

**Theorem 2.6** Let \( P \) be an arbitrary poset with \( |P| = p \) and denote by \( L := J(P) \) the distributive lattice of order ideals of \( P \).

Let \( S := \mathbb{Q}[\{x_p, y_p\}_{p \in P}] \) denote the polynomial ring in \( 2p \) variables. Let \( H(P) \subseteq S \) denote the Hibi ideal of \( P \). Then \( H(P) \) has the following \( F_P \) graded minimal free \( S \)-resolution:

\[
F_P : 0 \rightarrow S(-p - k)^{b_k(L)} \rightarrow S(-p - k + 1)^{b_{k-1}(L)} \rightarrow \ldots \\
\rightarrow S(-p - 1)^{b_1(L)} \rightarrow S(-p)^{b_0(L)} \rightarrow H(P) \rightarrow 0,
\]

where \( k \) is the Sperner number of \( P \), i.e., the maximum of the cardinalities of antichains of \( P \).

Let \( \Gamma_P \) denote the simplicial complex attached to the squarefree monomial ideal \( H(P) \), that is, \( H(P) = I(\Gamma_P) \). H. Hibi and J. Herzog described also the Stanley–Reisner ideal of the \( (\Gamma_P)^* \) Alexander dual of \( \Gamma_P \) (see Lemma 3.1 of [11]).

**Lemma 2.7** The Stanley–Reisner ideal of the Alexander dual \( (\Gamma_P)^* \) is generated by those squarefree monomials \( x_i y_j \) such that \( p_i \leq p_j \) in \( P \).

# 3 Proof of Theorem 1.1

We follow the following strategy in our proof.

First we compute the Hilbert function \( h_M(t) \) of the quotient module \( M := S/H(P) \) from the graded free resolution of \( H(P) \). Then we compute this Hilbert function \( h_M(t) \) from the theory of Stanley–Reisner rings. These computations yield to a new equation and if we compare the coefficients of \( \binom{t}{i} \) on both side, then the desired equation (2) follows.
Let $M := S/H(P)$ denote the quotient module of the Hibi ideal $H(P)$. From Theorem 2.5 and Theorem 2.6 we conclude that the Hilbert function $h_M(t)$ of $M$ is
\[
h_M(t) = \binom{t + 2p}{2p} + \sum_{i=0}^{k} (-1)^{i+1} b_i(L) \binom{t + 2p - (p + i)}{2p} = \binom{t + 2p}{2p} + \sum_{i=0}^{k} (-1)^{i+1} b_i(L) \binom{t + p - i}{2p}.
\]
(15)

Lemma 3.1 Let $P = \{q_1, \ldots, q_p\}$ be a finite poset with $|P| = p$. Define the graph $G(P)$ as follows: let the vertex set of $G(P)$ be the disjoint union $\{x_1, \ldots, x_p\} \cup \{y_1, \ldots, y_p\}$. Define the edge set of $G(P)$ as
\[
\{\{x_i, x_j\} : 1 \leq i < j \leq p\} \cup \{\{y_i, y_j\} : 1 \leq i < j \leq p\} \cup \{\{x_i, y_j\} : q_i \not\leq q_j\}.
\]
(16)
Denote by $\Gamma_P$ the simplicial complex attached to the squarefree monomial ideal $H(P)$, that is, $H(P) = I(\Gamma_P)$. Then
\[
(\Gamma_P)^* = \Delta(G(P)).
\]

**Proof.** We write $G_2(P)$ for the bipartite graph on the vertex set $\{x_1, \ldots, x_p\} \cup \{y_1, \ldots, y_p\}$ whose edges are those $\{x_i, y_j\}$ such that $p_i \leq p_j$ in $P$. It follows from Lemma 2.7 that $I((\Gamma_P)^*) = I(G_2(P))$, that is, the Stanley–Reisner ideal of $(\Gamma_P)^*$ is the edge ideal of $G_2(P)$. Since $G(P) = G_2(P)$ by definition, hence
\[
I((\Gamma_P)^*) = I(G_2(P)) = I(G(P)) = I(\Delta(G(P))),
\]
where we applied Lemma 2.4 in the last equality. This means that $(\Gamma_P)^*$ is the clique complex of the graph $G(P)$. 

It follows from Lemma 3.1 that $f(P) := (f_{-1}(P), \ldots, f_{d-1}(P))$ is the $f$–vector of $(\Gamma_P)^*$. Let $\Delta$ stand for $(\Gamma_P)^*$, the Alexander dual of $\Gamma_P$. Then $\Delta^* = ((\Gamma_P)^*)^* = \Gamma_P$ and $M = S/H(P) = S/I(\Gamma_P) = \mathbb{Q}[\Gamma_P] = \mathbb{Q}[\Delta^*]$.

Let $d := \dim((\Gamma_P)^*) + 1$. Clearly $n = f_0((\Gamma_P)^*) = 2p$. It is easy to verify that $d = p$. Namely let $K$ denote one of the maximal $d$–clique in the graph $G(P)$. Then $d \geq p$ follows from the definition of $G(P)$. On the other hand,
if \( V(K) \cap X = \{x_i, \ldots, x_r \} \), then \( \{y_i, \ldots, y_r \} \cap (V(K) \cap Y) = \emptyset \), because 
\( (x_i, y_i) \notin E(G(P)) \) for each \( 1 \leq i \leq p \), hence

\[
d = |V(K)| = |V(K) \cap X| + |V(K) \cap Y| \leq p.
\]

We can apply Corollary 2.3 for \( \Delta^* \):

\[
h_M(t) = h_{Q[\Delta^*]}(t) = \sum_{i=0}^{n-d-1} \binom{n}{i} \binom{t}{i} + \sum_{j=2}^{d} \left( \binom{n}{j} - f_{j-1}(P) \right) \binom{t}{n-j},
\]

(17)

Hence the equations (15) and (17) imply that

\[
\left( \frac{t + 2p}{2p} \right) + \sum_{m=0}^{k} (-1)^{m+1} b_m(L) \left( \frac{t + p - m}{2p} \right) =
\]

\[
= \sum_{i=0}^{n-d-1} \binom{n}{i} \binom{t}{i} + \sum_{j=2}^{d} \left( \binom{n}{j} - f_{j-1}(P) \right) \binom{t}{n-j}.
\]

Since \( d = p \) and \( n = 2p \), we get that

\[
\left( \frac{t + 2p}{2p} \right) + \sum_{m=0}^{k} (-1)^{m+1} b_m(L) \left( \frac{t + p - m}{2p} \right) =
\]

\[
= \sum_{i=0}^{p-1} \binom{2p}{i} \binom{t}{i} + \sum_{j=2}^{p} \left( \binom{2p}{j} - f_{j-1}(P) \right) \binom{t}{2p-j}.
\]

Using the Vandermonde identities (see [13], 169–170)

\[
\left( \frac{t + 2p}{2p} \right) = \sum_{i=0}^{2p} \binom{2p}{i} \binom{t}{i}
\]

and

\[
\left( \frac{t + p - m}{2p} \right) = \sum_{i=0}^{2p} \binom{p-m}{2p-i} \binom{t}{i}
\]

for each \( 0 \leq m \leq k \), we get that

\[
\sum_{i=0}^{2p} \binom{2p}{i} \binom{t}{i} + \sum_{m=0}^{k} (-1)^{m+1} b_m(L) \left( \sum_{i=0}^{2p} \binom{p-m}{2p-i} \binom{t}{i} \right) =
\]
\[
\sum_{i=0}^{p-1} \binom{2p}{i} \binom{t}{i} + \sum_{j=2}^{p} \left( \binom{2p}{j} - f_{j-1}(P) \right) \binom{t}{2p-j}.
\]

After simplification we get

\[
\sum_{i=p}^{2p} \binom{2p}{i} \binom{t}{i} + \sum_{m=0}^{k} (-1)^{m+1} b_m(L) \left( \sum_{i=0}^{2p} \binom{p-m}{2p-i} \binom{t}{i} \right) =
\]

\[
= \sum_{j=2}^{p} \left( \binom{2p}{j} - f_{j-1}(P) \right) \binom{t}{2p-j}. \quad (18)
\]

Let \( p \leq i \leq 2p \) be a fixed index and compare the coefficients of \( \binom{t}{i} \) on both side of equation (18). Since \( \{ \binom{t}{i} : i \in \mathbb{N} \} \) is a basis of the vector space \( \mathbb{Q}[t] \) over \( \mathbb{Q} \), hence these coefficients are the same and equation (18) follows.

\( \square \)

4 The proof of the Dehn–Sommerville equations

Let \( Q \) be a \( d \)-dimensional simplicial polytope and let \( \Delta(Q) \) denote the boundary complex of \( Q \).

Let \( R \) stand for the polynomial ring \( \mathbb{Q}[x_1, \ldots, x_n] \). Here \( n := f_0(Q) \).

We put \( \Delta(Q)^* \) for the Alexander dual of \( \Delta(Q) \). Denote by \( M := R/I(\Delta(Q)^*) \) the Stanley–Reisner ring of \( \Delta(Q)^* \).

First we compute the Hilbert function \( h_M(t) \) of \( M \) from the following graded free resolution.

\[ \text{Theorem 4.1 (see [16], Example 4.12) The ideal } I(\Delta(Q)^*) \text{ has the following minimal graded free resolution:} \]

\[
\mathcal{F}_Q : 0 \rightarrow R(-n)^1 \rightarrow R(1-n)^{f_0(Q)} \rightarrow \ldots
\]

\[
\rightarrow R(d-n-1)^{f_{d-2}(Q)} \rightarrow R(d-n)^{f_{d-1}(Q)} \rightarrow I(\Delta(Q)^*) \rightarrow 0. \quad (19)
\]
It follows from Theorem 2.5 that the Hilbert function of $Q[\Delta(Q)^*]$ is

$$h_M(t) = h_{Q[\Delta(Q)^*]}(t) = \binom{n + t}{t} + \sum_{i=0}^{d-1} (-1)^{d-i} f_i(Q) \binom{t + n - n + i + 1}{n}$$

$$= \binom{n + t}{t} + \sum_{i=1}^{d-1} (-1)^{d-i} f_i(Q) \binom{t + i + 1}{n}.$$  \hfill (20)

Clearly $\Delta(Q) = ((\Delta(Q))^*)^*$. If we apply Corollary 2.3 for the simplicial complex $\Delta := (\Delta(Q))^*$, then we get

$$h_M(t) = h_{Q[\Delta^*]}(t) = \sum_{i=0}^{n-d-1} \binom{n-i}{t} \binom{i}{i} + \sum_{j=2}^{d} \binom{n-j}{j} - f_{j-1}(Q) \binom{t}{n-j}.$$  \hfill (21)

Hence the equations (20) and (21) imply that

$$\binom{n + t}{t} + \sum_{i=1}^{d-1} (-1)^{d-i} f_i(Q) \binom{t + i + 1}{n} =$$

$$= \sum_{i=0}^{n-d-1} \binom{n-i}{t} \binom{i}{i} + \sum_{j=2}^{d} \binom{n-j}{j} - f_{j-1}(Q) \binom{t}{n-j}.$$  

Using the Vandermonde identities (see 13, 169–170)

$$\binom{t + n}{n} = \sum_{i=0}^{n} \binom{n}{i} \binom{t}{i}$$

and

$$\binom{t + i + 1}{n} = \sum_{j=0}^{i+1} \binom{t}{n-j} \binom{i+1}{j}$$

for each $i \geq 0$, we get

$$\sum_{i=0}^{n} \binom{n}{i} \binom{t}{i} + \sum_{i=1}^{d-1} (-1)^{d-i} f_i(Q) \sum_{j=0}^{i+1} \binom{t}{n-j} \binom{i+1}{j} =$$

$$= \sum_{i=0}^{n-d-1} \binom{n-i}{t} \binom{i}{i} + \sum_{j=0}^{d} \left( \binom{n-j}{j} - f_{j-1}(Q) \right) \binom{t}{n-j}.$$
After simplification we conclude that

\[
\sum_{j=n-d}^{n} \binom{n}{j} \binom{t}{j} + \sum_{i=1}^{d-1} (-1)^{d-i} f_i(Q) \left( \sum_{j=0}^{i+1} \binom{t}{n-j} \binom{i+1}{j} \right) =
\]

\[
= \sum_{j=0}^{d} \left( \binom{n}{j} - f_{j-1}(Q) \right) \binom{t}{n-j},
\]

Consequently

\[
\sum_{j=n-d}^{n} \binom{n}{j} \binom{t}{j} + \sum_{j=0}^{d} \binom{t}{n-j} \left( \sum_{i=j-1}^{d-1} (-1)^{d-i} \binom{i+1}{j} f_i(Q) \right) =
\]

\[
= \sum_{j=0}^{d} \left( \binom{n}{j} - f_{j-1}(Q) \right) \binom{t}{n-j}.
\] (22)

Since

\[
\sum_{j=n-d}^{n} \binom{n}{j} \binom{t}{j} = \sum_{j=0}^{d} \binom{n}{j} \binom{t}{n-j},
\]

hence simplifying the equation (22), we get that

\[
\sum_{j=0}^{d} f_{j-1}(Q) \binom{t}{n-j} + \sum_{j=0}^{d} \binom{t}{n-j} \left( \sum_{i=j-1}^{d-1} (-1)^{d-i} \binom{i+1}{j} f_i(Q) \right) = 0.
\]

\[
\sum_{j=0}^{d} \binom{t}{n-j} \left( f_{j-1}(Q) + \sum_{i=j-1}^{d-1} (-1)^{d-i} \binom{i+1}{j} f_i(Q) \right) = 0.
\] (23)

Now we can compare the coefficients of \( \binom{t}{i} \) on both side of equation (23). We can use again the basis property of \( \{ \binom{t}{i} : i \in \mathbb{N} \} \). This implies that these coefficients are the same. Thus

\[
f_{j-1}(Q) + \sum_{i=j-1}^{d-1} (-1)^{d-i} \binom{i+1}{j} f_i(Q) = 0
\]

for each \( 0 \leq j \leq d \) and the equations (4) follow.
Finally we give an application using a formula of Peskin and Szpiro (see [17]).

Let $Q$ be a flag simplicial polytope, that is, a simplicial polytope such that the boundary complex of $Q$ is flag.

Let $\Delta(Q)$ denote this boundary complex of $Q$. Since $Q$ was a flag polytope, the Stanley–Reisner ideal attached to the Alexander dual $\Delta(Q) = ((\Delta(Q))^*)^*$ is generated by the monomials $x_p x_q$, where $\{p, q\} \notin \Delta(Q)$, hence

$$I((\Delta(Q))^*) = \bigcap_{\{p, q\} \notin \Delta(Q)} (x_p, x_q).$$

Therefore the squarefree monomial ideal $I(\Delta(Q)^*)$ is of height 2 and the multiplicity of $Q[\Delta(Q)^*]$ is given by

$$e(Q[\Delta(Q)^*]) = |\{(p, q) : \{p, q\} \notin \Delta(Q)\}|.$$

But it is easy to verify that

$$|\{(p, q) : \{p, q\} \notin \Delta(Q)\}| = \left(\binom{f_0(Q)}{2}\right) - f_1(Q).$$

**Corollary 4.2** Let $Q$ be a flag simplicial complex. Then

$$2 \cdot \left[ \binom{f_0(Q)}{2} - f_1(Q) \right] = \sum_{i=1}^{d+1} (-1)^i f_{d-i}(Q) (f_0(Q) - d + i - 1)^2.$$

**Proof.** Let $I := I((\Delta(Q))^*)$ be the Stanley–Reisner ideal attached to the Alexander dual $(\Delta(Q))^*$. Now apply the formula [14] for the free graded resolution [19].

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