On the motion of an Earth satellite after fixing the magnitude of its acceleration as a problem with nonholonomic third-order constraint

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Abstract. The motion of an artificial Earth satellite with constant absolute value of the acceleration is considered. This requirement is equivalent to imposing a second-order nonlinear nonholonomic constraint or a third-order linear nonholonomic constraint. Two theories of motion of nonholonomic systems with high-order constraints are used for solving this problem. According to the first theory, a consistent system of differential equations is constructed with respect to the generalized coordinates and the Lagrange multipliers; the second theory is based on the application of the generalized Gauss principle. The results are different, although the constraints are satisfied in both theories. It turns out that infinitely many solutions can be built, but using these theories one can find two specific solutions. The question of the difference of these two solutions from the set of all other possible solutions is raised. We also simplify the previously obtained differential equations. The transition to dimensionless variables is made. Three parameters of motion prior to imposition of the constraint are single out, which control the motion after the application of the constraint. The solutions obtained from these theories of motion of nonholonomic systems are compared.

1. Introduction
The motion of nonholonomic systems with high-order constraints has been investigated for many years at the Department of Theoretical and Applied Mechanics of the Faculty of Mathematics and Mechanics of St. Petersburg University. The results of this extended work are reflected in the monographs [1, 2, 3]. In these studies, much attention was paid to the problem of motion of an Earth artificial satellite after fixing the magnitude of its acceleration at a given time, which is equivalent to the imposition of a second-order nonlinear nonholonomic constraint or a third-order linear nonholonomic constraint. This problem was formulated as one of the first examples of motion of real mechanical systems under superposition of a high-order constraint (see [4, 5]).

Two different theories of motion of nonholonomic systems with high-order constraints are used for solving this problem. According to the first theory, three differential equations are composed with respect to the polar coordinates $r, \varphi$ of the satellite and the generalized control force $\Lambda$. As a result, the satellite moves between two concentric circles. The second theory is based on the generalized Gauss principle [6]. It is interesting to note that, in this setting, the spacecraft becomes a space body and asymptotically tends to a uniformly accelerated motion along a straight line. For detailed numerical calculations of the motion of the satellite with fixed
magnitude of its acceleration in the perigee and apogee of the orbit, see [7, 8]. The present
paper continues the study of such motions.

2. Statement of the problem
Consider the motion of an artificial Earth satellite. Let the origin of the coordinate system be
at the Earth center. The satellite position will be measured in the polar coordinates \( r, \varphi \). In
this case, the satellite moves in an ellipse.

We will use the following notation:

- \( G \) is the gravitational constant;
- \( R_E \) is the Earth radius;
- \( M \) is the Earth mass;
- \( m \) is the satellite mass;
- \( e \) is the orbital eccentricity;
- \( p \) is the latus rectum.

2.1. Satellite motion equations
The Newton second law for the satellite is as follows:

\[
m \ddot{\vec{w}} = \vec{F} + \vec{R}, \quad \vec{R} \text{ is the constraint force},
\]

in this case, the covariant components of the vectors \( \vec{w}, \vec{F} \) are calculated by the formulas (the
standard basis of the polar coordinate system is denoted by \( \vec{e}_1, \vec{e}_2 \)):

\[
\begin{align*}
 w_1 &= \dot{r} - r \dot{\varphi}^2, \quad & F_1 &= -GMm/r^2, \\
 w_2 &= r (r \ddot{\varphi} + 2 \dot{r} \dot{\varphi}), \quad & F_2 &= 0.
\end{align*}
\]

The covariant component \( R_1, R_2 \) are zero prior to imposition of the constraints.

2.2. Transition to the dimensionless variables
Let us change to the dimensionless variables by the formulas:

\[
t = T \tau, \quad r = R_E \rho, \quad T^2 = R_E^3 / (GM).
\]

Then the vector equation of motion (1) can be rewritten as

\[
\begin{align*}
 \ddot{\vec{w}}_1 &= \vec{F}_1 + \vec{R}_1, \\
 \ddot{\vec{w}}_2 &= \vec{F}_2 + \vec{R}_2,
\end{align*}
\]

where we set

\[
\begin{align*}
 \ddot{w}_1 &= T^2 w_1 / R_E, \quad \vec{F}_1 = R_E^2 F_1 / (GMm), \quad \vec{R}_1 = T^2 R_1 / (mR_E), \\
 \ddot{w}_2 &= T^2 w_2 / R_E^2, \quad \vec{F}_2 = 0, \quad \vec{R}_2 = T^2 R_2 / (mR_E^2).
\end{align*}
\]

Prior to imposition of the constraints (\( R_1 = R_2 = 0 \)), the system of equations (2) has the form

\[
\begin{align*}
 \dot{\rho} - \rho \dot{\varphi}^2 &= -1/\rho^2, \\
 \rho (\rho \ddot{\varphi} + 2 \dot{\rho} \dot{\varphi}) &= 0.
\end{align*}
\]
According to (3), the point will move in an ellipse \( \rho = p/(1 + e \cos \varphi) \).

Suppose that the constraint is imposed at \( t_0 = 0 \), when the satellite has the coordinates \( \rho_0, \varphi_0 \) related by \( \rho_0 = p/(1 + e \cos \varphi_0) \). Then the initial data for further motion can be written as

\[
\begin{align*}
\varphi(0) &= \varphi_0, \\
\dot{\varphi}(0) &= \sqrt{p/\rho_0^2}, \\
\ddot{\varphi}(0) &= -2e \sin \varphi_0/\rho_0^3, \\
\rho(0) &= \rho_0, \\
\dot{\rho}(0) &= e \sin \varphi_0/\sqrt{p}, \\
\ddot{\rho}(0) &= e \cos \varphi_0/\rho_0^2.
\end{align*}
\]

The quantities \( \varphi_0, p, e \) are parameters, which control the further motion.

2.3. Nonholonomic constraint

Assume that the satellite acceleration changes with the motion of the satellite. We impose the constraint \( f_2 = 0 \), which fixes the absolute value of the acceleration vector

\[
f_2 = \ddot{\rho} - \rho \dot{\varphi}^2 = (\ddot{\rho} - \rho \dot{\varphi}^2)^2 + (\rho \ddot{\varphi} + 2 \dot{\rho} \dot{\varphi})^2 - w_0^2 = 0,
\]

and use the initial conditions (4) to find \( w_0 \)

\[
w_0^2 = (-1/\rho_0^3)^2 \Rightarrow w_0 = -1/\rho_0^2.
\]

Here it is taken into account that \( \tilde{w}_1(0) = w_0, \tilde{w}_2(0) = 0 \).

Note that the constraint (5) is nonlinear; however, the theories of motion of nonholonomic systems are applied to linear relations with respect to higher derivatives. To derive a linear constraint, we differentiate the constraint (5) with respect to time

\[
f_3 = \dot{f}_2 = ( \ddot{\rho} - \rho \dot{\varphi}^2 ) ( \dddot{\rho} - 2 \dot{\varphi} \ddot{\varphi} + 2 \dot{\rho} \dot{\varphi} + 2 \dot{\rho} \dot{\varphi} ) + ( \rho \dddot{\varphi} + 3 \dot{\rho} \dot{\varphi} + 2 \ddot{\rho} \dot{\varphi} + 3 \rho \dddot{\varphi} + 2 \ddot{\rho} \dot{\varphi} ) = 0.
\]

Now the constraint (6) (unlike (5)) is linear with respect to the higher derivatives of the generalized coordinates \( \rho, \varphi \).

3. Application of the first theory

In the first theory, the vector differential equation (1) is written with respect to the generalized coordinates and the generalized control forces. It is assumed that the reaction vector of the constraint \( f_n = 0 \), which is linear with respect to higher derivatives of the generalized coordinates, has the form

\[
\tilde{R} = \Lambda \nabla^{(n)} f_n,
\]

where the partial derivatives of \( f_n \) with respect to \( \tilde{q}^{(n)} \) are used in the construction of the vector \( \nabla^{(n)} f_n \).

3.1. Derivation of the differential equations

With the use of (7) the reaction of the constraint (6) can be written as

\[
\tilde{R} = \Lambda \tilde{\nabla}^{mn} f_3 = \Lambda \left( \frac{\partial f_3}{\partial \tilde{q}} e^1 + \frac{\partial f_3}{\partial \tilde{\varphi}} e^2 \right),
\]

and hence the covariant components \( \tilde{R}_1, \tilde{R}_2 \) can be found by the formulas

\[
\tilde{R}_1 = \Lambda ( \ddot{\rho} - \rho \dot{\varphi}^2 ) / m, \quad \tilde{R}_2 = \Lambda \rho ( \dot{\rho} \dot{\varphi} + 2 \dot{\rho} \dot{\varphi} ) / m.
\]
Substituting (8) in equations (2), we find that
\[
\begin{align*}
\ddot{\rho} - \rho \dot{\phi}^2 &= -m / (\rho^2 (m - \Lambda)), \\
\rho (\rho \ddot{\phi} + 2 \dot{\rho} \dot{\phi}) &= 0.
\end{align*}
\]

(9)

Next, to find \( \Lambda \) we use the constraint (5). We have
\[\Lambda = m \pm m / (w_0 \rho^2).\]

Assuming that \( \Lambda(0) = 0 \), we rewrite system (9) in the form
\[
\begin{align*}
\ddot{\rho} - \rho \dot{\phi}^2 &= w_0, \\
\rho (\rho \ddot{\phi} + 2 \dot{\rho} \dot{\phi}) &= 0,
\end{align*}
\]

that is, \( \ddot{w}_1 = w_0 \), \( \ddot{w}_2 = 0 \). The resulting system (10) can be written as
\[
\begin{align*}
\ddot{\rho} &= \rho / \rho^3 + w_0, \\
\dot{\phi} &= \sqrt{p / \rho^2}
\end{align*}
\]

(11)

and then proceed with numerical integration with the initial conditions (4).

3.2. Analysis of the results and graphs

According to the first theory, the satellite moves between two concentric circles. We choose parameters \( \varphi_0, p, e \) so that the motion pattern will be clearly visible on the graphs. For example,
\[
\begin{align*}
\varphi_0 &= 5\pi / 8, \\
p &= 1.95, \\
e &= 0.5.
\end{align*}
\]

(12)

With these parameters, the dimensionless altitude of the satellite above the Earth in the perigee and apogee is 0.3 and 2.9, respectively. We integrate system (11) up to \( \tau = 70 \).

![Image](image_url)

**Figure 1.**
The trajectory of an Earth satellite according to the first theory.

Figure 1 shows: the trajectories before and after imposition of the constraint (5) was imposed; the Earth, as well as the circles \( \rho = \rho_{\text{min}} \) and \( \rho = \rho_{\text{max}} \), between which the satellite moves after the constraint was imposed. The acceleration components do not change during the motion.

In figure 2, we show the modulus of the vector \( \vec{R} \); note that \( |\vec{R}| = \rho e_1 \cdot \vec{R} = \vec{R}_1 \), and \( \rho e_2 \cdot \vec{R} = 0 \).
4. Application of the second theory

The second theory is based on the application of the generalized Gauss principle. For the constraint $f_n = 0$, the generalized Gauss principle reads as

$$\delta^{(n)} \left( \frac{d^{n-2} \vec{R}}{dt^{n-2}} \right)^2 = (n-2) \frac{\vec{R}}{\delta^{(n)}} \cdot \delta^{(n)} \vec{R} = 0,$$  \hspace{1cm} (13)

where $\delta^{(n)}$ indicates that only the derivatives of the generalized coordinates $\dot{q}$ vary.

4.1. Derivation of the differential equations

Using (2), we rewrite the generalized Gauss principle (13) as

$$\left( \dot{\vec{w}} - \dot{\vec{F}} \right) \cdot \delta''' a = 0.$$  \hspace{1cm} (14)

We set $\vec{a} = \dot{\vec{w}}$, $\vec{b} = \dot{\vec{F}}$ and rewrite the inner product in (14) as

$$(a_1 - b_1) \delta''' a^1 + (a_2 - b_2) \delta''' a^2 = 0.$$  \hspace{1cm} (15)

The covariant components of the vectors $\vec{a}, \vec{b}$ can be calculated by the formulas [1, 2]:

$$a_i = \dot{w}_i - \Gamma^k_{ij} \dot{w}_k \dot{q}^j, \hspace{1cm} b_i = \dot{F}_i - \Gamma^k_{ij} \dot{F}_k \dot{q}^j, \hspace{1cm} i, j, k = 1, 2,$$

where $\Gamma^k_{ij}$ are the Christoffel symbols of the second kind; the nonzero polar coordinates are as follows:

$$\Gamma^2_{12} = \Gamma^2_{21} = 1/\rho, \hspace{2cm} \Gamma^1_{22} = -\rho.$$

Now the components of the vectors $\vec{a}, \vec{b}$ are expressed as

$$a_1 = \ddot{\rho} - 3\dot{\rho} \ddot{\phi} - 3\rho \dot{\phi}^2, \hspace{1cm} a^1 = a_1, \hspace{1cm} b_1 = 2\dot{\rho}/\rho^3,$$

$$a_2 = \rho^2 \ddot{\phi} + 3 \rho \dot{\rho} \ddot{\phi} + 3 \rho \dot{\phi}^2 - \rho^2 \dot{\phi}^3, \hspace{1cm} a^2 = a_2/\rho^2, \hspace{1cm} b_2 = -\ddot{\phi}/\rho.$$

The variations of the contravariant components of $\vec{a}$ are related to the derivatives of the generalized coordinates as follows:

$$\delta''' a^1 = \delta''' \ddot{\rho}, \hspace{2cm} \delta''' a^2 = \delta''' \ddot{\phi}.$$
Consider the system of equations (15) and (6). Substituting the resulting expressions for $\vec{a}, \vec{b}$, we get

$$\begin{cases}
(\dddot{\rho} - 3\dot{\rho}\dddot{\varphi} - 3\dot{\rho}\dot{\varphi}^2 - 2\dot{\rho}/\rho^3) \ddot{\varphi} + (\rho^2 \dot{\varphi} + 3\rho\dot{\varphi} + 3\rho\ddot{\varphi} - \rho^2\varphi^3 + \dddot{\varphi}/\rho) \delta'''' \varphi = 0, \\
(\dddot{\rho} - \rho\dot{\varphi}^2) \ddot{\varphi} + \rho (\ddot{\varphi} + 2\dot{\varphi}) \delta'''' \varphi = 0.
\end{cases}$$

(16)

The variations in (16) are independent, and so

$$\rho (\rho \ddot{\varphi} + 2\dot{\varphi}) (\dddot{\rho} - 3\dot{\rho}\dddot{\varphi} - 3\dot{\rho}\dot{\varphi}^2 - 2\dot{\rho}/\rho^3)$$

$$- (\ddot{\rho} - \rho\dot{\varphi}^2) (\rho^2 \dddot{\varphi} + 3\rho\dot{\varphi} + 3\rho\ddot{\varphi} - \rho^2\varphi^3 + \dddot{\varphi}/\rho) = 0.$$

(17)

Considering (17) and (6) as a system of algebraic equations with respect to the higher derivatives and taking into account the constraint (5), we get the system

$$\begin{cases}
\ddot{\rho} = 3\dot{\varphi} (\rho \ddot{\varphi} + \dot{\varphi}) + \frac{1}{w_0^2} \left( \frac{\ddot{\varphi}}{\rho^2} + 2\frac{\dot{\varphi}^2}{\rho} - 2\frac{\dot{\varphi}^3}{\rho^3} + 8\frac{\dot{\varphi}^4}{\rho^4} + 8\frac{\ddot{\varphi}}{\rho^2} + 2\frac{\ddot{\varphi}}{\rho^3} - \varphi^3 \ddot{\varphi} \right), \\
\dddot{\varphi} = -3\dddot{\rho} + \dddot{\varphi} + \frac{\dddot{\varphi}}{\rho^3} + \frac{1}{w_0^3} \left( 2\frac{\dddot{\varphi}}{\rho^2} - \frac{\dddot{\varphi}}{\rho} + 4\frac{\dot{\varphi}^3}{\rho^3} + 2\frac{\ddot{\varphi}}{\rho^2} - 2\frac{\ddot{\varphi}}{\rho^3} - 4\frac{\dddot{\varphi}}{\rho^4} \right).
\end{cases}$$

(18)

Now the solution of the problem can be found via numerical integration of system (18) with the initial data (4).

### 4.2. Analysis of the results and graphs

According to the second theory, the satellite first moves around the Earth and then asymptotically tends to a uniformly accelerated motion along the straight line. As in the first theory, we construct the graphs for the parameters (12). System (18) is integrated up to $\tau = 17$.

![Figure 3](image-url)

Figure 3.

The trajectory of the Earth satellite (space body) according to the second theory.

In figure 3, we show the trajectories before and after the application of the constraint (5). In figure 4, as for the first theory, we construct the module of the constraint reaction vector and show its projection onto the basis vectors $\vec{e}_1, \vec{e}_2$. In figure 5, we show the first phase of satellite motion; after some time the satellite becomes a space body and moves along the straight line.
5. Comparison of the results
First, it is worth noting that the constraint must be imposed artificially, for example, by installing an engine for motion control that will create the force $\vec{R}$. We assume that initially $\vec{R}(0) = 0$. This requirement seems reasonable, since in this case a smooth change in the reaction of the constraint reaction is formed without an initial jump in its value.

Let us now look at our constraint (5). It is clear that this constraint is satisfied by a variety of functions $\rho(\tau)$, $\varphi(\tau)$. In particular, the solution could be unique if we would consider two constraints $\text{pr}_{\vec{e}_1} \vec{w} = w_01$, $\text{pr}_{\vec{e}_2} \vec{w} = w_02$ in addition to the requirement $\vec{R}(0) = 0$. This result was obtained according to the first theory, but in our case, the projections $\text{pr}_{\vec{e}_1} \vec{w}$, $\text{pr}_{\vec{e}_2} \vec{w}$ of the acceleration may vary during the motion with the imposed constraint, as was the case with the second theory.

Some additional considerations are worth making. Kinematically, the motion of a point with constant acceleration occurs either for a uniform rotation along a circle, or in the case of a rectilinear uniformly accelerated motion. The elements of the first motion were are obtained using the first theory for the satellite rotating between two concentric circles; from the second theory the satellite was found to asymptotically follow a uniformly accelerated motion along a straight line. It turns out that, in our example, two different theories of motion of nonholonomic systems with high-order constraints successfully complement each other. As already noted, there are infinitely many solutions, but these two particular solutions were found from the above theories.

An intuitively clear result was obtained from the first theory. The projections $\text{pr}_{\vec{e}_1} \vec{w}$, $\text{pr}_{\vec{e}_2} \vec{w}$ of the acceleration were found to be fixed, which gives a unique and nice solution depicted in figure 1, besides, one of the projections of the vector $\vec{R}$ is zero. Thus, the first theory gives an obviously good solution among the entire set of possible solutions to the problem.

The result obtained by the second theory also seems clear, but only from the moment the satellite starts to approach a uniformly accelerated motion along the straight line ($\tau = 17$ in figure 3). The nonholonomic constraint is actually satisfied on the entire solution, since it is derived from equations (17) and (6), of which one is the constraint. So, the second theory
provides a solution to the problem, but it is difficult to say how this solution differs from the set of all other solutions. There is no obvious transition that can be effected to change the acceleration projections so that the motion would start to follow a uniformly accelerated motion, and this causes difficulties in evaluating the resulting solution.

6. Conclusion
A simpler form of the motion equations is found. In the first theory, the variable Λ is excluded from the system and one of the equations is partially integrated. In the second theory, the nonholonomic constraint is singled out in the system. As an additional novelty, we transit to the dimensionless variables and single out three characteristic parameters controlling the trajectory of an Earth satellite after the constraint is imposed. Analysis of the possible results via two theories is carried out.

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