Global Seiberg–Witten Maps for $U(n)$-Bundles on Tori and $T$-duality

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Abstract. Seiberg–Witten maps are a well-established method to locally construct noncommutative gauge theories starting from commutative gauge theories. We revisit and classify the ambiguities and the freedom in the definition. Geometrically, Seiberg–Witten maps provide a quantization of bundles with connections. We study the case of $U(n)$-vector bundles on two-dimensional tori, prove the existence of globally defined Seiberg–Witten maps (induced from the plane to the torus) and show their compatibility with Morita equivalence.

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1. Introduction

The geometry of noncommutative tori is one of the most studied and inspiring examples in the mathematics and physics literature. Yang–Mills theory on noncommutative tori was defined in [1,2]; it was shown that noncommutative tori emerge as backgrounds for compactifications of M-theory. The study of Yang–Mills and Born–Infeld theories on noncommutative tori has proven very fruitful: On the one hand, it allows to realize M-theory and string theory duality transformations within the low energy physics of noncommutative (Super) Yang–Mills theories [2]. On the other hand, it provides exact low-energy D-branes effective actions (in a given $\alpha' \to 0$ sector of string theory where closed strings decouple). This is a general feature of low-energy effective actions of open strings in the presence of a constant background flux ($B$-field) and has led to the Seiberg–Witten map between commutative and noncommutative gauge fields [3]. This is a field transformation that allows to rewrite a gauge theory on commutative space as a gauge theory on noncommutative space. Particularly relevant are then action functionals, like the Yang–Mills one, that are invariant in form under this map (background independent).

Most of the literature on Seiberg–Witten map does not consider global geometric aspects and focusses on the local properties. Furthermore, this map has mainly been studied in the context of formal deformation quantization. An interesting result there concerns abelian gauge theories, in that case the Seiberg–Witten map has been generalized to nonconstant background $B$-fields using Kontsevich formality theorem [4] and shown to quantize line bundles on a Poisson manifold to quantum line bundles [5].

In this paper, we study global aspects of the Seiberg–Witten map for non-abelian $U(n)$-gauge fields on noncommutative tori and show that it quantizes vector bundles on tori with connections to vector bundles on noncommutative tori with noncommutative connections, and these are the nonformal (Hilbert) modules over noncommutative tori studied in the mathematics and physics literature [2,6–9]. In this global nonformal context, we also revisit the relation between Seiberg–Witten maps and Morita equivalence ($T$-duality) transformations showing their compatibility.

Since the torus is a quotient of the plane, gauge and matter fields associated with a bundle on the torus are just gauge and matter fields (of trivial bundles) on the plane that satisfy twisted periodicity conditions, determined by $U(n)$-valued functions $\Omega_\alpha$ on the plane. While these are thought as “transition functions” of the torus, from the plane perspective they are endomorphisms of the trivial rank $n$ bundle on the plane, or equivalently, sections of the trivial rank $n \times n$ bundle, transforming in the adjoint representation. The basic idea is then that the Seiberg–Witten map quantization of these sections defines the quantum bundle on the noncommutative torus via noncommutative twisted periodicity condition. In this way, the Seiberg–Witten map on the plane induces a quantization of bundles on tori. Consistently, we show that the Seiberg–Witten map for matter fields on the plane induces a Seiberg–Witten map for matter fields on the torus.
A key point in order to obtain as explicit solutions the quantum sections $\phi^\theta$ studied via different methods in [7] is to exploit the freedom in the Seiberg–Witten differential equation defining the Seiberg–Witten map. This has led us to an exhaustive study of the ambiguities in the definition of the Seiberg–Witten map extending the previous results in [10,11].

The paper is organized as follows. We begin by reviewing the Seiberg–Witten map on $\mathbb{R}^d$, including the original Seiberg–Witten differential equations and their recursive solutions as formal power series in the noncommutativity parameter $\theta$. We present a simple way to classify the ambiguities of the Seiberg–Witten map by classifying all terms allowed in the Seiberg–Witten equations, including those breaking covariance under constant $GL(d, \mathbb{R})$ rotations on $\mathbb{R}^d$, that is anyhow broken on the $d$-dimensional torus.

In chapter three, we present quantized $U(n)$-bundles on tori following [7] and establish the relation to the projective modules description of the more mathematical literature, where $\theta$ is a real number. We also describe endomorphisms of these bundles and connections. All these fields are seen as functions on the plane satisfying twisted periodicity conditions.

In the following chapter four, we prove the main result, the compatibility of the Seiberg–Witten map with the twisted periodicity conditions defining bundles with connections on the torus, and show that the quantized sections are obtained via the Seiberg–Witten map. This establishes a global and converging Seiberg–Witten map for $U(n)$-bundles on the torus. In particular, different ordering prescriptions for the quantization of the algebra of sections in the adjoint correspond to different choices of Seiberg–Witten map. In chapter five, we review Morita equivalence and its $T$-duality transformations and show its compatibility with the Seiberg–Witten map.

### 2. Seiberg–Witten Map on $\mathbb{R}^d$

In this section, we recall the Seiberg–Witten map between the noncommutative gauge fields $\hat{A}$ and gauge parameters $\hat{\epsilon}$ on the noncommutative $n$-dimensional plane $\mathbb{R}^d_\theta$ and the ordinary ones $A$ and $\epsilon$ on $\mathbb{R}^d$. The $n$-dimensional plane $\mathbb{R}^d_\theta$ is described by the noncommutative algebra of formal power series in $\theta$ with coefficients in complex-valued smooth functions on $\mathbb{R}^d$. Noncommutativity is given by the Moyal–Weyl star product, with the following conventions:

$$(f \star g)(\sigma) = \exp \left( i\pi \theta^{\mu\nu} \frac{\partial}{\partial \sigma^\mu} \frac{\partial}{\partial \rho^\nu} \right) f(\sigma)g(\rho)|_{\sigma=\rho}, \quad (2.1)$$

so that $[\sigma^\mu, \sigma^\nu]_\star = 2\pi i \theta^{\mu\nu}$ (with $\theta^{\mu\nu} = -\theta^{\nu\mu}$). We adopt conventions typically used in the literature on tori, which differs from conventions used for the noncommutative plane by a factor of $2\pi$ (i.e., $\vartheta^{\mu\nu} = 2\pi \theta^{\mu\nu}$ with $\vartheta^{\mu\nu}$ the noncommutativity parameter of [3]). The Seiberg–Witten map relates the noncommutative gauge potential $\hat{A}$ and the noncommutative gauge parameters $\hat{\epsilon}$ to the ordinary $A$ and $\epsilon$ so as to satisfy:

$$\hat{A}(A + \delta_\epsilon A) = \hat{A}(A) + \delta_\hat{\epsilon} \hat{A}(A) \quad (2.2)$$
with
\[
\delta_\epsilon A_\mu = \partial_\mu \epsilon - i A_\mu \epsilon + i \epsilon A_\mu, \tag{2.3}
\]
\[
\hat{\delta}_\overline{\epsilon} \hat{A}_\mu = \partial_\mu \hat{\epsilon} - i \hat{A}_\mu \star \hat{\epsilon} + i \hat{\epsilon} \star \hat{A}_\mu. \tag{2.4}
\]

Condition (2.2) states that the dependence of the noncommutative gauge field on the ordinary one is such that ordinary gauge variations of \( A \) inside \( \hat{A}(A) \) produce the noncommutative gauge variation of \( \hat{A} \). In a gauge theory, physical quantities are gauge invariant: They do not depend on the gauge potential but on the gauge equivalence class of the potential given. The Seiberg–Witten map relates the noncommutative gauge fields to the commutative ones by requiring noncommutative fields to have the same gauge equivalence classes as the commutative ones. Equation (2.2) can be solved order by order in \( \theta \) yielding \( \hat{A} \) and \( \hat{\epsilon} \) as power series in \( \theta \):
\[
\hat{A}(A) = A + A^1(A) + A^2(A) + \cdots + A^n(A) + \cdots \tag{2.5}
\]
\[
\hat{\epsilon}(\epsilon, A) = \epsilon + \epsilon^1(\epsilon, A) + \epsilon^2(\epsilon, A) + \cdots + \epsilon^n(\epsilon, A) + \cdots \tag{2.6}
\]
where \( A^n(A) \) and \( \epsilon^n(\epsilon, A) \) are of order \( n \) in \( \theta \). Note that \( \hat{\epsilon} \) depends on the ordinary \( \epsilon \) and also on \( A \).

The Seiberg–Witten condition (2.2) holds for any value of the noncommutativity parameter \( \theta \). If we consider it at \( \theta' \) and at \( \theta \), we easily obtain that gauge equivalence classes of the \( \theta' \)-noncommutative theory have to correspond to gauge equivalent classes of the \( \theta \)-noncommutative theory, i.e., we generalize (2.2) to
\[
\hat{A}'(A + \hat{\delta}_\epsilon \hat{A}) = \hat{A}'(\hat{A}) - \hat{\delta}'_{\overline{\epsilon}} \hat{A}'(\hat{A}), \tag{2.7}
\]
where we denoted by \( \star' \), \( \hat{A}' \), \( \hat{\epsilon}' \), \( \hat{\delta}'_{\overline{\epsilon}} \) the star product, the gauge potential, the gauge parameter and the gauge variation; \( \hat{\delta}'_{\overline{\epsilon}} \hat{A}'_\kappa = \partial_\kappa \hat{\epsilon}' - i \hat{A}'_\kappa \star' \hat{\epsilon}' + i \hat{\epsilon}' \star' \hat{A}'_\kappa \) at noncommutativity parameter \( \theta' \). By considering \( \theta \) and \( \theta' \) infinitesimally close, so that \( \theta' = \theta + \delta \theta \) and \( \hat{A} = \hat{A} + \delta \theta \mu\nu \overline{\partial} A'_{\mu\nu} \) (we use the convention \( \overline{\partial} \mu\nu \overline{\partial} \rho\sigma \)) independent from \( \overline{\partial} \mu\nu \overline{\partial} \), and hence, we sum over all \( \mu, \nu \) indices), in [3, §3.1] it is shown that if \( \hat{A} \) and \( \hat{\epsilon} \) solve the differential equations,
\[
\frac{\partial}{\partial \theta \mu\nu} \hat{A}_\kappa = -\frac{\pi}{4} \left( \{ \hat{A}_\mu, \partial_\nu \hat{A}_\kappa + \hat{F}_{\nu\kappa} \}_s + \{ \hat{A}_\nu, \partial_\mu \hat{A}_\kappa + \hat{F}_{\mu\kappa} \}_s \right) \tag{2.8}
\]
\[
\frac{\partial}{\partial \theta \mu\nu} \hat{\epsilon} = -\frac{\pi}{4} \left( \{ \hat{A}_\mu, \partial_\nu \hat{\epsilon} \}_s + \{ \hat{A}_\nu, \partial_\mu \hat{\epsilon} \}_s \right) \tag{2.9}
\]
with the definitions
\[
\hat{F}_{\mu\nu} := \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - i \hat{A}_\mu \star \hat{A}_\nu + i \hat{A}_\nu \star \hat{A}_\mu \tag{2.10}
\]
\[
\{ f, g \}_s := f \star g + g \star f \tag{2.11}
\]
then \( \hat{A}'(\hat{A}) \) and \( \hat{\epsilon}'(\hat{\epsilon}, \hat{A}) \) satisfy also the Seiberg–Witten condition (2.7) for arbitrary values of \( \theta' \) and \( \theta \), and in particular, therefore, solve the Seiberg–Witten condition (2.2).
The differential Eqs. (2.8) and (2.9) admit solutions in terms of formal power series in $\theta$, and they are given recursively by [12]

$$A^{n+1}_\mu = -\frac{\pi}{2(n+1)} \theta^{\rho\sigma} \{ \hat{A}_\rho, \partial_\sigma \hat{A}_\mu + \hat{F}_{\rho\mu} \}^n, \quad (2.12)$$

$$\varepsilon^{n+1} = -\frac{\pi}{2(n+1)} \theta^{\rho\sigma} \{ \hat{A}_\rho, \partial_\sigma \varepsilon \}^n, \quad (2.13)$$

where $\{ \hat{f}, \hat{g} \}^n_*$ is the $n$-th order term in $\{ \hat{f}, \hat{g} \}_*$, so that, for example,

$$\{ \hat{A}_\rho, \partial_\sigma \varepsilon \}^n_* \equiv \sum_{r+s+t=n} (A^r_\rho \star^s \partial_\sigma \varepsilon^t \star^s A^t_\mu), \quad (2.14)$$

here $\star^s$ indicates the $s$th order term in the star product expansion.1

Similar considerations hold for matter fields $\phi$ transforming in the fundamental or in the adjoint representation of the gauge group. The Seiberg–Witten condition reads [14]

$$\hat{\phi}(A + \delta \varepsilon A, \phi + \delta \varepsilon \phi) = \hat{\phi}(A, \phi) + \delta \varepsilon \hat{\phi}(A, \phi), \quad (2.15)$$

or more generally,

$$\hat{\phi}^\prime(\hat{A} + \delta \varepsilon \hat{A}, \hat{\phi} + \delta \varepsilon \hat{\phi}) = \hat{\phi}^\prime(\hat{A}, \hat{\phi}) + \delta \varepsilon \hat{\phi}^\prime(\hat{A}, \hat{\phi}), \quad (2.16)$$

and it is satisfied if the matter fields solve the differential equation

$$\delta \theta^{\mu\nu} \frac{\partial \hat{\phi}}{\partial \theta^{\mu\nu}} = -\frac{\pi}{2} \delta \theta^{\mu\nu} \hat{A}_\mu \star (\partial_\nu \hat{\phi} + D_\nu \hat{\phi}) \quad \text{fundamental rep., i.e.,} \quad \delta \varepsilon \hat{\phi} = i \varepsilon \hat{\phi}, \quad (2.17)$$

$$\delta \theta^{\mu\nu} \frac{\partial \hat{\Psi}}{\partial \theta^{\mu\nu}} = -\frac{\pi}{2} \delta \theta^{\mu\nu} \{ \hat{A}_\mu, (\partial_\nu \hat{\Psi} + D_\nu \hat{\Psi}) \}_* \quad \text{adjoint rep., i.e.,} \quad \delta \varepsilon \hat{\Psi} = i \varepsilon \hat{\Psi} - i \hat{\Psi} \star \varepsilon. \quad (2.18)$$

The explicit solutions order by order in $\theta$ are

$$\phi^{n+1} = -\frac{\pi}{2(n+1)} \theta^{\mu\nu} \left( \hat{A}_\mu \star (\partial_\nu \hat{\phi} + D_\nu \hat{\phi}) \right)^n \quad \text{(fundamental)} \quad (2.19)$$

$$\Psi^{n+1} = -\frac{\pi}{2(n+1)} \theta^{\mu\nu} \left( \hat{A}_\mu, \partial_\nu \hat{\Psi} + D_\nu \hat{\Psi} \right)^n_* \quad \text{(adjoint)} \quad (2.20)$$

where

$$D_\nu \hat{\phi} = \partial_\nu \hat{\phi} - i \hat{A}_\nu \star \hat{\phi}, \quad D_\nu \hat{\Psi} = \partial_\nu \hat{\Psi} - i \hat{A}_\nu \star \hat{\Psi} + i \hat{\Psi} \star \hat{A}_\nu \quad (2.21)$$

are the covariant derivative in the fundamental and in the adjoint.

1There is a simple proof of (2.12), (2.13) [13]: multiplying the differential equations by $\theta^{\mu\nu}$ and analyzing them order by order yields

$$\theta^{\mu\nu} \frac{\partial}{\partial \theta^{\mu\nu}} A^{n+1}_\rho = (n+1) A^{n+1}_\rho = -\frac{\pi}{2} \theta^{\mu\nu} \{ \hat{A}_\mu, \partial_\nu \hat{A}_\rho + \hat{F}_{\nu\rho} \}^n_*, \quad (2.19)$$

$$\theta^{\mu\nu} \frac{\partial}{\partial \theta^{\mu\nu}} \varepsilon^{n+1} = (n+1) \varepsilon^{n+1} = -\frac{\pi}{2} \theta^{\mu\nu} \{ \hat{A}_\mu, \partial_\nu \varepsilon \}^n_*, \quad (2.20)$$

since $A^{n+1}_\rho$ and $\varepsilon^{n+1}$ are homogeneous functions of $\theta$ of order $n+1$.1
2.1. Ambiguities in the Seiberg–Witten Map

The solution to the Seiberg–Witten conditions (2.2), (2.15) is not unique. For example, if \( \hat{A}_\mu \) is a solution, any noncommutative gauge transformation of \( \hat{A}_\mu \) gives another solution. Another source of ambiguities is that of field redefinitions of the gauge potential (e.g., if \( \hat{A}_\mu \) is a solution, then so is \( \hat{A}_\mu + \theta^{\rho\sigma} \theta^{\lambda\eta} \hat{F}_{\rho\lambda} \phi D_{\sigma} \hat{F}_{\eta\mu} \)). We present a novel study of these ambiguities by considering the freedom in modifying the differential Eqs. (2.8), (2.9) and (2.17) leading to the Seiberg–Witten conditions.

We generalize the Seiberg–Witten equations allowing for three extra terms \( \hat{D}_{\mu\nu\rho}(\hat{A}), \hat{E}_{\mu\nu}(\hat{A}, \hat{\varepsilon}) \) and \( \hat{C}_{\mu\nu}(\hat{A}, \hat{\phi}) \) (for the fundamental or adjoint representation) that are a priori arbitrary functions of their arguments and derivatives thereof that are (formal) power series in \( \theta \) and that are antisymmetric in the \( \mu, \nu \) indices. We hence consider the equations

\[
\delta^\theta \hat{A}_\kappa = \frac{\delta \theta^{\mu\nu}}{\partial \theta^{\mu\nu}} \frac{\partial \hat{A}_\kappa}{\partial \theta^{\mu\nu}} = \frac{\pi}{2} \delta \theta^{\mu\nu} \left( \hat{A}_\mu \ast (\partial_\nu \hat{A}_\kappa + \hat{F}_{\nu\kappa}) \right) + (\partial_\nu \hat{A}_\kappa + \hat{F}_{\nu\kappa}) \ast \hat{A}_\mu + \hat{D}_{\mu\nu\kappa}(\hat{A}), \tag{2.22} \right.
\]

\[
\delta^\theta \hat{\varepsilon} = \frac{\delta \theta^{\mu\nu}}{\partial \theta^{\mu\nu}} \frac{\partial \hat{\varepsilon}}{\partial \theta^{\mu\nu}} = \frac{\pi}{2} \delta \theta^{\mu\nu} \left( \hat{A}_\mu \ast \hat{\varepsilon} \ast \hat{A}_\nu + \hat{A}_\nu \ast \partial_\mu \hat{\varepsilon} + \hat{E}_{\mu\nu}(\hat{A}, \hat{\varepsilon}) \right), \tag{2.23} \right.
\]

\[
\delta^\theta \hat{\phi} = \frac{\delta \theta^{\mu\nu}}{\partial \theta^{\mu\nu}} \frac{\partial \hat{\phi}}{\partial \theta^{\mu\nu}} = -\frac{\pi}{2} \delta \theta^{\mu\nu} \left( \hat{A}_\mu \ast \partial_\nu \hat{\phi} + \hat{A}_\nu \ast \hat{D}_{\nu\mu}(\hat{A}) \right), \tag{2.24} \right.
\]

\[
\delta^\theta \hat{\Psi} = \frac{\delta \theta^{\mu\nu}}{\partial \theta^{\mu\nu}} \frac{\partial \hat{\Psi}}{\partial \theta^{\mu\nu}} = -\frac{\pi}{2} \delta \theta^{\mu\nu} \left( \hat{A}_\mu \ast (\partial_\nu \hat{\Psi} + \hat{D}_{\nu\mu}(\hat{A})) \right) + (\partial_\nu \hat{\Psi} + \hat{D}_{\nu\mu}(\hat{A})) \ast \hat{A}_\mu + \hat{C}_{\mu\nu}(\hat{A}, \hat{\Psi}) \right), \tag{2.25} \right.
\]

and observe that \( \hat{E}_{\mu\nu} \) must be linear in \( \hat{\varepsilon} \) since all terms in (2.23) but \( \hat{E}_{\mu\nu} \) are linear in \( \hat{\varepsilon} \), similarly \( \hat{C}_{\mu\nu} \) must be linear in \( \hat{\phi} \) because of the linearity in \( \hat{\phi} \) of all other terms in (2.24), and similarly for \( \hat{C}_{\mu\nu}(\hat{A}, \hat{\Psi}) \) in (2.25). We further constrain the dependence of \( \hat{D}_{\mu\nu\kappa}, \hat{E}_{\mu\nu}, \) and \( \hat{C}_{\mu\nu} \) on \( \theta, \hat{A}, \hat{\varepsilon} \) and their derivatives by requiring the Seiberg–Witten conditions (2.7), (2.16). Let us first consider condition (2.7), we use (2.22) in \( \hat{A}'(\hat{A} + \hat{\delta}_\varepsilon \hat{A}) = \hat{A} + \delta\varepsilon \hat{A} + \delta^\theta \hat{A} (\hat{A} + \delta\varepsilon \hat{A}) \) as well as in \( \hat{A}'(\hat{A}) = \hat{A} + \delta^\theta \hat{A}, \) and then, we use \( f \ast h = f \ast h + i \pi \delta \theta^{\mu\nu} \partial_\mu f \ast \partial_\nu h + O(\theta^2) \) and (2.23) in \( \delta^\theta \hat{A}'_\kappa = \partial_\kappa \hat{\varepsilon} - i \hat{\varepsilon}'_\kappa \ast \hat{\varepsilon}' + i \hat{\varepsilon}' \ast \hat{\varepsilon}'_\kappa \) and finally recall that for \( \hat{D}_{\mu\nu\kappa} = \hat{E}_{\mu\nu} \) to be zero the equation is satisfied, we thus obtain the condition

\[
\hat{D}_{\mu\nu\kappa}(\hat{A} + \delta\varepsilon \hat{A}) - \hat{D}_{\mu\nu\kappa}(\hat{A}) - i [\hat{\varepsilon}, \hat{D}_{\mu\nu\kappa}(\hat{A})]_* = -D_{\kappa} \hat{E}_{\mu\nu}(\varepsilon, \hat{A}). \tag{2.26} \right.
\]

Similarly, the Seiberg–Witten condition (2.16) for matter fields in the fundamental and in the adjoint constraints \( \hat{C}_{\mu\nu} \) to satisfy
\[ \mathcal{C}_{\mu\nu}(\hat{A} + \delta_{\varepsilon} \hat{A}, \hat{\phi} + \delta_{\varepsilon} \hat{\phi}) - \mathcal{C}_{\mu\nu}(\hat{A}, \hat{\phi}) - i\varepsilon \star \mathcal{C}_{\mu\nu}(\hat{A}, \hat{\phi}) = -i\hat{E}_{\mu\nu}(\hat{A}, \varepsilon) \star \hat{\phi}, \]
\[ \mathcal{C}_{\mu\nu}(\hat{A} + \delta_{\varepsilon} \hat{A}, \hat{\Psi} + \delta_{\varepsilon} \hat{\Psi}) - \mathcal{C}_{\mu\nu}(\hat{A}, \hat{\Psi}) - i[\varepsilon, \mathcal{C}_{\mu\nu}(\hat{A}, \hat{\Psi})]_\ast = -i[\hat{E}_{\mu\nu}(\hat{A}, \varepsilon), \hat{\Psi}]_\ast, \]  

where \([f, g]_\ast = f \star g - g \star f\) is the star commutator.

An immediate comment is that any \(\hat{D}_{\mu\nu\kappa}\) and \(\hat{C}_{\mu\nu}\) covariant under gauge transformations solve (2.26), (2.27) with \(\hat{E}_{\mu\nu} = 0\).

In summary, we have shown that the most general solution \(\hat{A}(A), \hat{\varepsilon}(A, \varepsilon), \hat{\phi}(A, \phi), \hat{\Psi}(A, \Psi)\) of the Seiberg–Witten conditions (2.2), (2.15) is given by the differential equations (2.22)–(2.25) where \(\hat{D}, \hat{E}, \hat{C}\) are constrained by (2.26) and (2.27). Further constraints on the \(\hat{D}, \hat{E}, \hat{C}\) terms are obtained by requiring that Seiberg–Witten map respects hermiticity and charge conjugation in the sense that the hermiticity and charge conjugation properties of the commutative fields imply those of the noncommutative fields [13,15].

It is useful to compare these results with the previous ones in the literature. To this aim, we constrain \(\hat{D}, \hat{E}, \hat{C}\) to be \(\ast\)-polynomials in the variables \(\hat{A}_\mu, D_\mu, \hat{\varepsilon}, \hat{\phi}, \hat{\Psi}, \theta^{\mu\nu}, \partial_\mu\) that is in the variables \(\hat{A}_\mu, D_\mu, \hat{\varepsilon}, \hat{\phi}, \hat{\Psi}, \theta^{\mu\nu}\), indeed the partial derivatives \(\partial_\mu\) can be always expressed in terms of the covariant derivatives and the gauge potentials. We then define the operator \(\mathcal{L}_{\varepsilon}\) to satisfy the Leibniz rule and to be the adjoint action of \(\hat{\varepsilon}\) on fields in the adjoint, and the fundamental action otherwise: \(\mathcal{L}_{\varepsilon} (A_\mu) = [\hat{\varepsilon}, A_\mu]_\ast, \mathcal{L}_{\varepsilon}(D_\mu) = [\hat{\varepsilon}, D_\mu]_\ast, \mathcal{L}_{\varepsilon}(\hat{\varepsilon}) = 0, \mathcal{L}_{\varepsilon}(\hat{\phi}) = \hat{\varepsilon} \star \hat{\phi}, \mathcal{L}_{\varepsilon}(\hat{\Psi}) = [\hat{\varepsilon}, \hat{\Psi}]_\ast\). Next, as in [10], we introduce the operator

\[ \delta'_{\varepsilon} = \partial_\mu \varepsilon \frac{\partial}{\partial A_\mu} \]  

that acts only on the gauge potential \(A_\mu\) and does not act on the covariant derivatives: \(\delta'_{\varepsilon}(D_\mu) = 0\). This operator just substitutes \(A_\mu\) with \(\partial_\mu \varepsilon\) and satisfies the Leibniz rule. We can consider (2.26)–(2.27) as a combinatorial problem in the words (symbols) \(A_\mu, D_\mu, \hat{\phi}, \hat{\Psi}, \hat{\varepsilon}, \theta^{\mu\nu}\). Since \(\delta'_{\varepsilon} = \delta_{\varepsilon} - i\mathcal{L}_{\varepsilon}\) (as is easily seen on the generators \(A_\mu, D_\mu, \hat{\phi}, \hat{\varepsilon}, \theta^{\mu\nu}\) and then recalling the Leibniz rule), we can rewrite (2.26) and (2.27) as

\[ \delta'_{\varepsilon} \hat{D}_{\mu\nu\kappa}(\hat{A}) = -D_\kappa \hat{E}_{\mu\nu}(\hat{A}, \varepsilon), \]
\[ \delta'_{\varepsilon} \hat{C}_{\mu\nu}(\hat{A}, \hat{\phi}) = -i\hat{E}_{\mu\nu}(\hat{A}, \varepsilon) \star \hat{\phi}, \delta'_{\varepsilon} \hat{C}_{\mu\nu}(\hat{A}, \hat{\Psi}) = -i[\hat{E}_{\mu\nu}(\hat{A}, \varepsilon), \hat{\Psi}]_\ast. \]

If we constrain \(\hat{D}, \hat{E}, \hat{C}\) to be words (\(\ast\)-polynomials) only of \(A_\mu, D_\mu, \hat{\phi}, \hat{\Psi}, \hat{\varepsilon}\) and not of the other letters \(\theta^{\mu\nu}\), we then have a finite number of letters that can match the dimensionality of \(\hat{D}, \hat{E}, \hat{C}\) and the linearity constraints on \(\hat{\varepsilon}, \hat{\phi}\) and \(\hat{\Psi}\) discussed immediately after (2.25). In \(\mathbb{R}^d\), the star product \(f \star g\) is invariant under constant \(GL(d, \mathbb{R})\) coordinate transformations; if we do not fix the gauge potential \(A_\mu\), it is natural to implement this symmetry also in the Seiberg–Witten map, i.e., to ask all expressions to be tensorial under constant coordinate transformations. Since \(\theta^{\mu\nu}, A_\mu, \partial_\mu\) and \(D_\mu\) transform covariantly,
this is achieved, as usual, by contracting indices tensorially and by matching
the index structure of $\tilde{D}, \tilde{E}, \tilde{C}$.

We have written the most general linear combination of letters for $\tilde{D}$,
for $\tilde{E}$ and $\tilde{C}$, substituted them in (2.29) and (2.30) and shown (using also
Bianchi identities that are indeed combinatorial identities) that the most gen-
eral $GL(d,\mathbb{R})$-covariant solution with no explicit dependence on $\theta^{\mu\nu}$ is

$$
\tilde{D}_{\mu\nu\kappa} = \alpha D_\kappa \tilde{F}_{\mu\nu} + \beta D_\kappa [\tilde{A}_\mu, \tilde{A}_\nu]_*, \quad \tilde{E}_{\mu\nu} = 2\beta [\partial_\mu \tilde{\varepsilon}, \tilde{A}_\nu]_* , \quad (2.31)
$$

$$
\tilde{C}_{\mu\nu} = -2i\beta [\tilde{A}_\mu, \tilde{A}_\nu]_* \ast \hat{\phi} + \gamma \tilde{F}_{\mu\nu} \ast \hat{\phi} \quad \text{(fundamental)} \quad (2.32)
$$

$$
\tilde{C}^\prime_{\mu\nu} = -2i\beta [\tilde{A}_\mu, \tilde{A}_\nu]_* \ast \tilde{\Psi} + \gamma^\prime \tilde{F}_{\mu\nu} \ast \tilde{\Psi} + \tilde{\gamma} \tilde{\Psi} \ast \tilde{F}_{\mu\nu} \quad \text{(adjoint)} \quad (2.33)
$$

with $\alpha, \beta, \gamma, \gamma'$ and $\tilde{\gamma}$ arbitrary constants (terms like $D_\mu D_\nu \tilde{\phi}$ are proportional
to $F_{\mu\nu}$ due to antisymmetry in $\mu, \nu$). This shows that the results in [10,11]
are the most general solutions covariant under constant $GL(d,\mathbb{R})$ coordinate
transformations and that do not depend explicitly on $\theta^{\mu\nu}$.

If we relax the covariance under constant $GL(d,\mathbb{R})$ coordinate transforma-
tions, we obtain further solutions. Let $\tilde{D}^{\text{gauge-cov}, \text{GL-cov}}_{\mu\nu\kappa}$ and $\tilde{C}^{\text{gauge-cov}, \text{GL-cov}}_{\mu\nu}$ denote gauge covariant terms that are not $GL(d,\mathbb{R})$-covariant (with slight
abuse of notation we use the same notation for $\tilde{C}_{\mu\nu}$-terms in the fundamental
and in the adjoint). As noted after (2.27), these terms automatically satisfy the
Seiberg–Witten conditions (2.26) and (2.27) with $\tilde{E}_{\mu\nu} = 0$. Then, the terms

$$
\tilde{D}_{\mu\nu\kappa} = \alpha D_\kappa \tilde{F}_{\mu\nu} + \beta D_\kappa [\tilde{A}_\mu, \tilde{A}_\nu]_* + \tilde{D}^{\text{gauge-cov}, \text{GL-cov}}_{\mu\nu\kappa} , \quad \tilde{E}_{\mu\nu} = 2\beta [\partial_\mu \tilde{\varepsilon}, \tilde{A}_\nu]_* , \quad (2.34)
$$

$$
\tilde{C}_{\mu\nu} = -2i\beta [\tilde{A}_\mu, \tilde{A}_\nu]_* \ast \tilde{\phi} + \gamma \tilde{F}_{\mu\nu} \ast \tilde{\phi} + \tilde{C}^{\text{gauge-cov}, \text{GL-cov}}_{\mu\nu} \quad \text{(fundamental)} \quad (2.35)
$$

$$
\tilde{C}^\prime_{\mu\nu} = -2i\beta [\tilde{A}_\mu, \tilde{A}_\nu]_* \ast \tilde{\Psi} + \gamma^\prime \tilde{F}_{\mu\nu} \ast \tilde{\Psi} + \tilde{\gamma} \tilde{\Psi} \ast \tilde{F}_{\mu\nu} + \tilde{C}^{\text{gauge-cov}, \text{GL-cov}}_{\mu\nu} \quad \text{(adjoint)} \quad (2.36)
$$

give well-defined Seiberg–Witten equations (2.22)–(2.25) (i.e., satisfying
Seiberg–Witten conditions (2.7), (2.16)). In particular, we can consider

$$
\tilde{C}_{\mu\nu} = -2i\beta [\tilde{A}_\mu, \tilde{A}_\nu]_* \ast \tilde{\phi} + \gamma \tilde{F}_{\mu\nu} \ast \tilde{\phi} + \rho D_\mu D_\nu \tilde{\phi} \quad \text{for} \mu < \nu , \quad (2.37)
$$

(and similarly for the adjoint case). Here, $\rho$ is a constant, and $GL(d,\mathbb{R})$-
covariance is broken because $\tilde{C}_{\mu\nu}$ has not the term $\rho D_\nu D_\mu \tilde{\phi}$. Another example
is provided by $\tilde{C}^{\text{gauge-cov}, \text{GL-cov}}_{\mu\nu}$ terms like the linear combination $\eta D_\mu D_\nu \tilde{\phi} + 
\omega D_\nu D_\nu \tilde{\phi}$ that even more clearly breaks $GL(d,\mathbb{R})$ coordinate transformations.

When considering Seiberg–Witten maps on the $d$-dimensional torus, $GL(d,\mathbb{R})$-
covariance is anyhow broken, and $\tilde{C}_{\mu\nu}$ terms like those listed can be considered,
and, as we will see, are important in order to explicitly solve the Seiberg-Witten
differential equations.

We further remark that it is very natural to allow for polynomials that
depend explicitly also on $\theta$ (e.g., the field redefinition $\tilde{A}_\mu \rightarrow \tilde{A}_\mu + \theta^{\rho \sigma} \theta^{\lambda \eta} \tilde{F}_{\rho \lambda} \ast$
$D_{\sigma}\hat{F}_{\mu\nu}$ is of this kind). In this case, (2.31) is not the most general solution. As an example, consider
\[ \hat{D}_{\mu\nu\kappa} = \gamma(\theta^{\rho\sigma}\hat{F}_{\rho\sigma})^{\mu}D_{\kappa}\hat{F}_{\mu\nu} \] (with $\gamma$ arbitrary constant, $p$ integer) and $\hat{E} = 0$.

3. Noncommutative Tori

Noncommutative tori are among the most studied objects in noncommutative geometry. In physics, they serve as key examples to study $T$-duality. We review the classification of bundles (finitely generated projective modules) with connections on noncommutative tori.

3.1. $U(n)$-Vector Bundles on Commutative Tori

We consider hermitian $n$-dimensional vector bundles over the two-dimensional torus $T$, i.e., rank $n$ complex vector bundles canonically associated (via the fundamental representation) with a $U(n)$-principal bundle. For short, we will call these bundles $U(n)$-vector bundles or simply $U(n)$-bundles.

While a usual description of bundles is via local sections defined on opens and transition functions on overlaps, since the torus $T$ is given by the quotient $\mathbb{R}^2/(2\pi\mathbb{Z})^2$, bundles on $T$ are most easily described by sections of the trivial vector bundle on the plane $\mathbb{R}^2 \times \mathbb{C}^n \to \mathbb{R}^2$ that obey twisted periodicity conditions, often called boundary conditions. These conditions can be seen as arising from the transition functions of the bundle on $T$ in the limiting case of open sets that overlap only on the boundary of the fundamental domain determining the torus as a quotient of $\mathbb{R}^2$. Let us describe the smooth sections $\mathcal{E}_{n,m}$ of a $U(n)$-vector bundle with topological charge $m$ [16] (we follow [7,17], see also [18]). Define the fundamental domain in $\mathbb{R}^2$ to be the square of length $2\pi$, so that $T = \mathbb{R}^2/(2\pi\mathbb{Z})^2$ and functions on $T$ are $2\pi$-periodic functions on $\mathbb{R}^2$, and define the $U(n)$-matrix-valued functions (transition functions)

\[ \Omega_1(\sigma^2) = e^{im\sigma^2/n}U, \quad \Omega_2(\sigma^1) = V, \] (3.1)

where $U$ and $V$ are the clock and shift $U(n)$-matrices with entries $U_{kl} = e^{2\pi i k m/n} \delta_{kl}, V_{kl} = \delta_{(k+1)l}$ for $k < n$, $V_{n,l} = \delta_{1l}$. Let $\phi$ be an $n$-dimensional vector of complex-valued functions on $\mathbb{R}^2$ (a section of the trivial bundle $\mathbb{R}^2 \times \mathbb{C}^n \to \mathbb{R}^2$), and then, the twisted periodicity conditions defining the sections $\phi \in \mathcal{E}_{n,m}$ is the system of $2n$ equations in $\mathbb{R}^2$:

\[ \phi(\sigma^1 + 2\pi, \sigma^2) = \Omega_1(\sigma^2)\phi(\sigma^1, \sigma^2), \]
\[ \phi(\sigma^1, \sigma^2 + 2\pi) = \Omega_2(\sigma^1)\phi(\sigma^1, \sigma^2), \] (3.2)

where the $U(n)$-matrix-valued functions $\Omega_\alpha$ satisfy the cocycle condition

\[ \Omega_1(\sigma^2 + 2\pi)\Omega_2(\sigma^1) = \Omega_2(\sigma^1 + 2\pi)\Omega_1(\sigma^2). \] (3.3)

We remark that (3.2) and (3.3) are equations for functions $(\phi_k, \Omega_{1,kl}, \Omega_{2,kl})$ on $\mathbb{R}^2$. More geometrically, $\phi$ are sections of $\mathbb{R}^2 \times \mathbb{C}^n \to \mathbb{R}^2$ (the trivial $U(n)$-vector bundle on $\mathbb{R}^2$), and $\Omega_\alpha : \mathbb{R}^2 \to U(n) \subset M_{n \times n}(\mathbb{C})$, $\alpha = 1,2$, are endomorphisms of this vector bundle that transform sections to sections...
\( \phi \mapsto \Omega_\alpha \phi \). (They are sections of the endomorphism bundle \( \text{End}(E) \).) Denoting by \( E = C^\infty(\mathbb{R}^2) \oplus \mathbb{C}^n \), the module of sections of \( \mathbb{R}^2 \times \mathbb{C}^n \to \mathbb{R}^2 \) we write \( \Omega_\alpha \in \text{End}(E) = C^\infty(\mathbb{R}^2) \oplus (n \times n) \) where in the last equality we used that endomorphisms of a trivial (smooth) \( n \)-dimensional vector bundle are just (smooth) maps from the base space to linear maps on the fiber. Thus \( \Omega_\alpha \) are sections of the complexified adjoint bundle, for short, sections in the adjoint.

An explicit description of the sections \( \phi \in \mathcal{E}_{n,m} \) solving (3.2) was provided in [19] and requires defining:

\[
A := \frac{m}{n} \left( \frac{\sigma^2}{2\pi} + k + ns \right) + j, \quad B := i\sigma^1, \tag{3.4}
\]

an arbitrary section \( \phi = (\phi_k)_{k=1,\ldots,n} \in \mathcal{E}_{n,m} \) is then given by

\[
\phi_k(\sigma^1, \sigma^2) = \sum_{s \in \mathbb{Z}, j = 1}^{m} e^{AB} \tilde{\phi}_j \left( \frac{n}{m} A \right), \tag{3.5}
\]

where \( m \) is the topological charge, \( k = 1, \ldots n \) and \( \tilde{\phi}_j \) are \( m \) arbitrary complex-valued Schwartz functions on \( \mathbb{R} \). More elegantly, they define a Schwartz function \( \tilde{\phi} : \mathbb{R} \times \mathbb{Z}_m \to \mathbb{C} \).

In an analogous manner, covariant derivatives \( D_\mu \) on the module of sections \( \mathcal{E}_{n,m} \) are described by covariant derivatives of the trivial \( U(n) \)-bundle on \( \mathbb{R}^2 \) satisfying the appropriate periodicity:

\[
D_\mu |_{(\sigma^1 + 2\pi, \sigma^2)} = \Omega_1(\sigma^2) D_\mu |_{(\sigma^1, \sigma^2)} \Omega_1^{-1}(\sigma^2),
\]

\[
D_\mu |_{(\sigma^1, \sigma^2 + 2\pi)} = \Omega_2(\sigma^1) D_\mu |_{(\sigma^1, \sigma^2)} \Omega_2^{-1}(\sigma^2). \tag{3.6}
\]

A particular solution to these conditions, suitable for later generalizations to the noncommutative torus, is the one with only a single nonvanishing component of the gauge potential, proportional to the unit \( n \times n \)-matrix:

\[
D_1 = \partial_1 - iA_1 = \partial_1, \quad D_2 = \partial_2 - iA_2 = \partial_2 - \frac{im}{n} \sigma^1 1. \tag{3.7}
\]

The field strength is given by \( F = F_{12} = i[D_1, D_2] = \frac{1}{2\pi} \frac{m}{n} 1 \) and indeed the topological charge of \( \mathcal{E}_{n,m} \) is \( m : \frac{1}{2\pi} \int \text{tr}(F)d\sigma^1 d\sigma^2 = m \).

The set of sections \( \mathcal{E}_{n,m} \) is a module over the algebra \( C^\infty(T) \) of smooth functions on \( T \), and the action \( \phi \mapsto \phi f \) (it is customary to multiply functions from the right rather than the left) is simply the product of the vector \( \phi \) with the periodic function \( f \). It is immediate to check that \( \phi f \) satisfies (3.2) if so does \( \phi \).

Moreover, \( \mathcal{E}_{n,m} \) is a module with respect to the algebra of endomorphisms of \( \mathcal{E}_{n,m} \) itself. By definition, an endomorphism is a fiberwise linear map on the vector bundle that acts as the identity on the base space (the torus), and hence, it is a linear map on sections that is the identity on functions on the torus: \( \phi f \mapsto \Psi(\phi f) = \Psi(\phi) f \). Therefore, the \( \text{End}(\mathcal{E}_{n,m}) \)- and \( C^\infty(T) \)-actions

\[\text{Since the bundle is a positive definite hermitian complex vector bundle and the algebra of continuous functions } C(T) \text{ is a } C^*\text{-algebra, we also have that } \mathcal{E}_{n,m} \text{ is a Hilbert module over } C(T).\]
commute, and $\mathcal{E}_{n,m}$ is a bimodule with respect to the algebras $\text{End}(\mathcal{E}_{n,m})$ and $C^\infty(T)$, and we write this as

$$\mathcal{E}_{n,m} \in \text{End}(\mathcal{E}_{n,m})\mathcal{M}_{C^\infty(T)}. \quad (3.8)$$

We conclude with an explicit description of the algebra of endomorphisms of $\mathcal{E}_{n,m}$. First of all, the algebra of endomorphisms of $E$, the module of sections of $\mathbb{R}^2 \times \mathbb{C}^n \to \mathbb{R}^2$, as already observed, is given by all $n \times n$ matrix-valued functions on $\mathbb{R}^2$, $\text{End}(E) = \{ \Psi : \mathbb{R}^2 \to M_{n \times n}(\mathbb{C}) \}$; the action on sections is simply the matrix transformation $\phi \mapsto \Psi \phi$. The algebra $\text{End}(\mathcal{E}_{n,m})$ is the subalgebra of $\text{End}(E)$ that preserves the twisted periodicity conditions (3.2): If $\phi$ satisfies (3.2), then so does $\Psi \phi$. That is, endomorphisms of $\mathcal{E}_{n,m}$ are endomorphisms of $E$ that satisfy the twisted boundary conditions in the adjoint representation:

$$\Psi(\sigma^1 + 2\pi, \sigma^2) = \Omega_1(\sigma^2)\Psi(\sigma^1, \sigma^2)\Omega_1^{-1}(\sigma^2),$$

$$\Psi(\sigma^1, \sigma^2 + 2\pi) = \Omega_2(\sigma^1)\Psi(\sigma^1, \sigma^2)\Omega_2^{-1}(\sigma^2). \quad (3.9)$$

We see that they are the sections of the adjoint $U(n)$-vector bundle on the torus (i.e., the complex vector bundle canonically associated via the adjoint representation, rather than the fundamental, to the $U(n)$-principal bundle).

For $m$ and $n$ coprime, the algebra $\text{End}(\mathcal{E}_{n,m})$ is generated by the $U(n)$-valued functions on $\mathbb{R}^2$ (cf. [7,8]):

$$Z_1 = e^{i\sigma^1/n}V^b, \quad Z_2 = e^{i\sigma^2/n}U^{-b}, \quad (3.10)$$

where $b \in \mathbb{Z}$ is such that $an - bm = 1$ with $a \in \mathbb{Z}$. (If $a'$ and $b'$ are another couple satisfying $a'n - b'm = 1$, the algebra is the same since $m, n$ coprime implies $(a - a') = ms, (b - b') = ns, s \in \mathbb{Z}$ and we have $U^m = V^n = 1$.) It is easy to see that $Z_1Z_2 = e^{2\pi ib/n}Z_2Z_1$, henceforth $\text{End}(\mathcal{E}_{n,m}) = T_{b/n}$ the algebra of the noncommutative torus with rational noncommutativity parameter $b/n$.

### 3.2. $U(n)$-Vector Bundles on Noncommutative Tori

The description of bundles on the torus via modules on the algebra $C^\infty(T)$ of smooth functions on the torus, and the description of these modules via vector-valued functions $\phi$ on $\mathbb{R}^2$ (sections of the trivial $U(n)$-vector bundle on $\mathbb{R}^2$) satisfying twisted periodicity conditions determined by the two matrix-valued functions $\Omega_\alpha : \mathbb{R}^2 \to U(n)$ (bundle endomorphisms) is particularly well suited to noncommutative generalizations.

Consider as in Sect. 2 the noncommutative Moyal–Weyl algebra $\mathbb{R}^2_\theta$, with $\theta = \theta^{12} = -\theta^{21}$,

$$(f \ast g)(\sigma^1, \sigma^2) = fg(\sigma^1, \sigma^2) + i\pi \theta(\partial_{\sigma^1}f \partial_{\sigma^2}g - \partial_{\sigma^2}f \partial_{\sigma^1}g) + \mathcal{O}(\theta^2). \quad (11.11)$$

The $\ast$-product between $2\pi$-periodic functions on the plane is again a $2\pi$-periodic function, and therefore, $2\pi$-periodic functions form a subalgebra of $\mathbb{R}^2_\theta$, this is the noncommutative torus $T_{(-\theta)}$. Explicitly, $T_{(-\theta)}$, for $\theta \in \mathbb{R}$, is defined as the algebra over $\mathbb{C}$ generated by two invertible elements $U_1, U_2$ that satisfy the relations

$$U_1U_2 = e^{2\pi i(-\theta)}U_2U_1 \quad (3.12)$$
with involution given by $U_1^* = U_1^{-1}$, $U_2^* = U_2^{-1}$. The smooth noncommutative torus $T_{(-\theta)}$ is $T_{(-\theta)} = \{ \sum_{p,q\in \mathbb{Z}} a_{p,q} U_1^p U_2^q : a_{p,q} \in \mathbb{C} \}$, where $a : \mathbb{Z}^2 \to \mathbb{C}$, $(p,q) \mapsto a_{p,q}$ are Schwarz functions on $\mathbb{Z}^2$. We can also consider $\theta$ as a formal parameter (so that $U_1$ and $U_2$ generate the algebra over the ring of formal power series $\mathbb{C}[[\theta]]$). Since in $\mathbb{R}_\theta^2$

\[
e^{-i\sigma_1} \ast e^{-i\sigma_2} = e^{2\pi i(-\theta)} e^{i\sigma_2} \ast e^{i\sigma_1},
\]

(3.13)

setting $U_1 = e^{i\sigma_1}$, $U_2 = e^{i\sigma_2}$ realizes this algebra as a subalgebra of $\mathbb{R}_\theta^2$. Notice that restricting to periodic functions allows to specialize the formal parameter $\theta$ to a real number.

The twisted boundary conditions (3.2) and the cocycle conditions are equations for the functions $(\phi_k, \Omega_{1kl}, \Omega_{2kl})$ on $\mathbb{R}^2$, and deforming the commutative product in the $\ast$-product we obtain the noncommutative deformation of these conditions

\[
\phi^\theta (\sigma^1 + 2\pi, \sigma^2) = \Omega_1(\sigma^2) \ast \phi^\theta (\sigma^1, \sigma^2),
\]

\[
\phi^\theta (\sigma^1, \sigma^2 + 2\pi) = \Omega_2(\sigma^1) \ast \phi^\theta (\sigma^1, \sigma^2),
\]

(3.14)

\[
\Omega_1(\sigma^2 + 2\pi) \ast \Omega_2(\sigma^1) = \Omega_2(\sigma^1 + 2\pi) \ast \Omega_1(\sigma^2).
\]

(3.15)

The solutions (3.1) of the commutative cocycle conditions are also solutions of the $\ast$-cocycle condition. The solutions $\phi^\theta$ to (3.14) are immediately seen to be a module with respect to the noncommutative torus subalgebra $T_\theta \subset \mathbb{R}_\theta^2$; if and only if $f$ is a periodic function we have that a $\phi^\theta$ satisfying (3.14) implies that $\phi^\theta \ast f$ satisfies (3.14). The solutions $\phi^\theta$ therefore span the module $\mathcal{E}^\theta_{n,m}$ of sections of a rank $n$ complex vector bundle on the noncommutative torus.

The explicit solution of (3.14) requires the use of a normal ordered function $E(A, B)$ defined by

\[
E(A, B) := \frac{1}{1 - [A, B]} \sum_{l=0}^{\infty} \frac{1}{l!} A^l \ast B^l,
\]

(3.16)

where the definition of $A$ and $B$ in terms of the coordinates $\sigma^1, \sigma^2$ and the integers $n$ and $m$ is given in (3.4). For later work with the function $E(A, B)$, we collect some of its properties, whose proof follows for the relation $[A, B]_\ast = \frac{m}{n} \theta$. (The last two are easily derived from the corresponding differential equation in $\lambda$.)

**Lemma 1.** The function $E(A, B)$ satisfies

\[
A \ast E(A, B) = \frac{1}{1 - c} E(A, B) \ast A,
\]

(3.17)

\[
B \ast E(A, B) = E(A, B)(1 - c) \ast B,
\]

(3.18)

\[
E(-B, A)E(A, B) = 1,
\]

(3.19)

\[
E(A + \lambda, B) = E(A, B) \ast e^{\lambda B},
\]

(3.20)

\[
E(A, B + \lambda) = e^{\lambda A} \ast E(A, B),
\]

(3.21)

where we have set $c := [A, B]_\ast = \frac{m}{n} \theta$, and $\lambda \in \mathbb{C}$. 

We can now recall the solution $\phi^\theta = (\phi_k^\theta)_{k=1,\ldots,n}$ presented in [7], see [8] for a derivation, to the twisted boundary conditions (3.14)

$$
\phi_k^\theta(\sigma^1, \sigma^2) = \sum_{s \in \mathbb{Z}} \sum_{j=1}^{m} E\left(\frac{m}{n} \left( \frac{\sigma^2}{2\pi} + k + ns \right) + j, i\sigma^1 \right) \star \tilde{\phi}_j\left( \frac{\sigma^2}{2\pi} + k + ns + \frac{n}{m}j \right),
$$

that for short we rewrite as

$$
\phi_k^\theta(\sigma^1, \sigma^2) = \sum_{s \in \mathbb{Z}} \sum_{j=1}^{m} E(A, B) \star \tilde{\phi}_j(\frac{n}{m} A),
$$

where $A$ and $B$ are defined in (3.4) and $k = 1, \ldots,n$ and $\tilde{\phi}_j : \mathbb{R} \to \mathbb{C}, j = 1, \ldots,m$ are arbitrary Schwartz functions on $\mathbb{R}$, denoted by $\tilde{\phi}_j$ in [7]. The module of sections $\mathcal{E}_{n,m}^\theta$ can be directly described in terms of these functions $\tilde{\phi}_j$, that is, in terms of Schwartz functions $\tilde{\phi} : \mathbb{R} \times \mathbb{Z}_m \to \mathbb{C}$, thus recovering the more mathematical presentation used in [2] (see also [9,20]) of the module $\mathcal{E}_{n,m}^\theta \in \mathcal{M}_{T_{\{-\theta\}}}$. We denote by $\triangleleft$, the action of the torus algebra on the sections $\tilde{\phi}$. The action of the generators $U_1 = e^{i\sigma^1}, U_2 = e^{i\sigma^2}$ is induced by that on $\phi^\theta$ by defining $\tilde{\phi} \triangleleft U_\mu (\mu = 1, 2)$ such that

$$
(\phi_k^\theta \star U_\mu)(\sigma^1, \sigma^2) = \sum_{s \in \mathbb{Z}} \sum_{j=1}^{m} E(A, B) \star (\tilde{\phi} \triangleleft U_\mu)_j(\frac{n}{m} A). \tag{3.24}
$$

This gives (use $U_1^{-1} \star \tilde{\phi}_j(\frac{n}{m} A) \star U_1 = \tilde{\phi}_j(\frac{n}{m} A + \theta)$ and (3.20))

$$
(\tilde{\phi} \triangleleft U_1)_j(x) = \tilde{\phi}_{j-1}(x - \frac{n}{m} + \theta),
$$

$$
(\tilde{\phi} \triangleleft U_2)_j(x) = \tilde{\phi}_j(x) e^{2\pi i(x-\frac{jn}{m})}. \tag{3.25}
$$

Thus, we have a module isomorphism between the module of sections $\phi_k^\theta(\sigma^1, \sigma^2)$ (satisfying the boundary conditions) and that of sections $\tilde{\phi}_j(x)$; we identify these two modules over the noncommutative torus and use the same notation $\mathcal{E}_{n,m}^\theta$. Equations (3.25) provide a more explicit definition of the module of sections $\mathcal{E}_{n,m}^\theta$ on the noncommutative torus $T_{\{-\theta\}}$ because the twisted periodicity conditions (3.14) have been solved, it is, however, less geometric than the implicit one with the constrained sections $\phi^\theta$. The geometric description holds for $n \in \mathbb{N} - \{0\}$ and $m \in \mathbb{Z} - \{0\}$; then, we have the modules $\mathcal{E}_{n,0}^\theta$ that are defined to be the direct sums $T_{\{-\theta\}}^{\oplus n}$ of $n$ copies of the trivial module $T_{\{-\theta\}}$, i.e., the modules of sections of the trivial $U(n)$-bundles on the noncommutative torus. The algebraic definition (3.25) allows to consider also the case $-n \in \mathbb{N} - \{0\}$, but there is no new module since $\mathcal{E}_{n,m}^\theta = \mathcal{E}_{-n,-m}^\theta$. Finally, (3.25) defines also the modules $\mathcal{E}_{0,m}^\theta$ that, however, coincide with the modules $\mathcal{E}_{m,m}^{\theta+1}$. In particular, $\mathcal{E}_{0,1}^{\theta} = \mathcal{E}_{1,1}^{\theta+1}$ is the module of sections of the $U(1)$-bundle over $T_{\{-\theta-1\}} = T_{\{-\theta\}}$ with charge $m = 1$. It will play a key role in Sect. 5.
Similarly to the classical case, covariant derivatives $D_\mu$ on the module of sections $E_{n,m}^\theta$ are described by covariant derivatives of the trivial $U(n)$-bundle on $\mathbb{R}_\theta^2$ satisfying the appropriate periodicity:

$$D_\mu|_{(\sigma^1+2\pi,\sigma^2)} = \Omega_1(\sigma^2) \star D_\mu|_{(\sigma^1,\sigma^2)} \Omega_1^\dagger(\sigma^2),$$

$$D_\mu|_{(\sigma^1,\sigma^2+2\pi)} = \Omega_2(\sigma^1) \star D_\mu|_{(\sigma^1,\sigma^2)} \Omega_2^\dagger(\sigma^2).$$

(3.26)

A particular solution to these conditions, which in the commutative limit reduces to the previous solution, is

$$D_1 = \partial_{\sigma^1} - iA_1^\theta = \partial_{\sigma^1}, \quad D_2 = \partial_{\sigma^2} - iA_2^\theta = \partial_{\sigma^2} - \frac{i}{2\pi} \frac{m\sigma^1}{n - m\theta} \mathbf{1}. \quad (3.27)$$

Notice that the derivations $\partial_{\sigma^\nu}$ on $T_{(-\theta)}$ can be defined intrinsically by $\partial_{\sigma^\nu}U_\nu = i\partial_\mu U_\nu$ and therefore independently from the realization $U_\mu = e^{i\sigma^\mu}$ with $[\sigma^\mu, \sigma^\nu]_\ast = 2\pi i\theta^{\mu\nu}$.

The curvature corresponding to the connection $A^\theta$ is $F^\theta = i[D_1, D_2]$, a constant proportional to the unit matrix. One can check that, taking into account the appropriate normalization [2] of the integral on the noncommutative torus, the topological charge is indeed $\frac{1}{2\pi} \int tr(F^\theta) = m$. Since the covariant derivatives satisfy the Heisenberg algebra the modules $E_{n,m}^\theta$ with connection $(A_1^\theta, A_2^\theta)$ as in (3.27) are called Heisenberg modules.

The algebra of endomorphisms of $E^\theta := (\mathbb{R}_\theta^2)^{\otimes n}$ is $\text{End}(E^\theta) = (\mathbb{R}_\theta^2)^{\otimes (n \times n)}$, that is, that of all $n \times n$ matrix-valued noncommutative functions on $\mathbb{R}_\theta^2$. The action on sections is the matrix transformation $\phi^\theta \mapsto \Psi^\theta \ast \phi^\theta$. Of course, since this action is via $\ast$-multiplication form the left, it commutes with the action from the right of $\mathbb{R}_\theta^2$, we therefore have the bimodule $E^\theta \in \text{End}(E^\theta) \mathcal{M}_{\mathbb{R}_\theta^2}$. The algebra of endomorphisms $\text{End}(E_{n,m}^\theta)$ is the subalgebra of $\text{End}(E^\theta)$ that preserves the twisted boundary conditions (3.2): If $\phi^\theta$ satisfies (3.2), then so does $\Psi^\theta \ast \phi^\theta$. That is, endomorphisms of $E_{n,m}^\theta$ are endomorphisms of $E^\theta$ that satisfy the twisted boundary conditions in the adjoint representation

$$\Psi^\theta(\sigma^1 + 2\pi, \sigma^2) = \Omega_1(\sigma^2) \star \Psi^\theta(\sigma^1, \sigma^2) \star \Omega_1^{-1}(\sigma^2),$$

$$\Psi^\theta(\sigma^1, \sigma^2 + 2\pi) = \Omega_2(\sigma^1) \star \Psi^\theta(\sigma^1, \sigma^2) \star \Omega_2^{-1}(\sigma^2).$$

(3.28)

The algebra $\text{End}(E_{n,m}^\theta)$ is generated by $\ast$-multiplication with the $U(n)$-valued functions (see [8] for a proof)

$$Z_1^\theta = e^{i\sigma^1 \theta} V^b, \quad Z_2^\theta = e^{i\sigma^2 \theta} U^{-b}. \quad (3.29)$$

Since $Z_1^\theta \ast Z_2^\theta = e^{2\pi i \theta} Z_2^\theta \ast Z_1^\theta$, with $\tilde{\theta} = \frac{a(-\theta) + b}{m(-\theta) + n}$, and $an - bm = 1$, $a, b \in \mathbb{Z}$, we see that with this choice of generators the endomorphisms algebra is the torus algebra $T_{\tilde{\theta}}$. Thus, we have the bimodule $E_{n,m}^\theta \in T_{\tilde{\theta}} \mathcal{M}_{T_{(-\theta)}}$.

The connection (3.7) and the endomorphisms can be described directly on the solutions $\phi^\theta$ rather than on the more geometric sections $\phi^\theta$. Proceeding as before, by star multiplying from the left with the $U(n)$-valued functions
By definition, a complete (or gauge) Morita equivalence bimodule is a module with a (right module) constant curvature connection $D_\mu$ proportional to the identity and such that the induced derivations $\hat{\delta}_\mu$ on $\text{End}(\mathcal{E}_{n,m}^\theta) \simeq T_\theta$ are an invertible linear combination of the canonical derivations $\partial_{\sigma^\mu}$ on $T_\theta$, see [9]. We have seen that the Heisenberg bimodules $\mathcal{E}_{n,m}^\theta$ are complete Morita equivalence bimodules.
Notice that $D^L_\mu := (n - m\theta) D_\mu$ satisfies the left Leibniz rule
\[ D^L_\mu (Z_\nu \triangleright \tilde{\phi}) = \partial_\nu (Z_\nu \triangleright \tilde{\phi}) + Z_\nu \triangleright D^L_\mu \tilde{\phi} = i\delta_{\mu\nu} Z_\nu \triangleright \tilde{\phi} + Z_\nu \triangleright D^L_\mu \tilde{\phi}; \quad (3.37) \]

hence, $D^L_\mu$ is a left connection on the left $T\tilde{\theta}$-module $E^\theta_{n,m}$. Another way of characterizing complete (or gauge) Morita equivalence bimodules $E \in T\tilde{\theta} \mathcal{M}_{T(-\theta)}$ is then by requiring the right $T(-\theta)$-module constant curvature connection $D_\mu$ to be also, up to an invertible linear transformation, a left $T\tilde{\theta}$-module connection.

In this section, we have described noncommutative vector bundles using $\star$-products. The advantage of this deformation quantization approach is that we have a manifest dependence on the deformation parameter $\theta$. This naturally leads to generalize the Seiberg–Witten map of Sect. 2 by establishing a Seiberg–Witten map between classical sections in $E_{n,m}$ and quantum sections in $E^\theta_{n,m}$. This Seiberg–Witten map is compatible with the bimodule structure of the Heisenberg modules
\[ E^\theta_{n,m} \in T\tilde{\theta} \mathcal{M}_{T(-\theta)}, \quad \tilde{\theta} = \frac{a(-\theta) + b}{m(-\theta) + n}, \quad (3.38) \]

with connection $A^\theta_\mu$ and constant curvature $F^\theta = \frac{1}{2\pi} \frac{m}{n-m\theta} 1$.

4. The Seiberg–Witten Map on Tori

In the preceding section, we have described $U(n)$-vector bundles with connections and topological charge $m$ on tori and on noncommutative tori. On the other hand, in the first section we have recalled that the Seiberg–Witten map relates commutative to noncommutative gauge theories. Here, we first see that it is a quantization of $U(n)$-bundles with connections on $\mathbb{R}^2$ to $U(n)$-bundles with connections on noncommutative $\mathbb{R}^2$. Then, we construct an induced Seiberg–Witten map that quantizes $U(n)$-bundles with connections on $T$ to $U(n)$-bundles with connections on $T\tilde{\theta}$. While the treatment in Sect. 2 and 4.1 is local (because it is performed on a single open chart of $\mathbb{R}^2$), in Sect. 4.2 we achieve a global description of the Seiberg–Witten map on tori. In Sect. 4.3, we then compare the general construction we perform with the description in Sect. 3.2 of bundles on noncommutative tori in terms of the module of noncommutative sections $E^\theta_{n,m}$. We find full agreement. On the one hand, this frames the solution found in $E^\theta_{n,m}$ in the general Seiberg–Witten map deformation scheme. On the other hand, it provides an explicit solution to the Seiberg–Witten equations and an example of a formal deformation quantization that is actually nonformal, since $\theta$ can be specialized to a real number and power series in $\theta$ can be summed and analytically continued to well-defined complex-valued functions.

4.1. Seiberg–Witten Map for Bundles on $\mathbb{R}^2$

The Seiberg–Witten map presented in Sect. 2 can be seen as a quantization of $U(n)$-bundles with connections on $\mathbb{R}^2$. Let us describe a bundle with connection, together with a Poisson structure $\theta$ on $\mathbb{R}^2$ via the triple $(E \in \text{End}(E) \mathcal{M}_{C^\infty(\mathbb{R}^2)}, A_\mu, \theta)$, where $E = C^\infty(\mathbb{R}^2)^{\oplus n}$ is the module of sections
of the trivial bundle $\mathbb{R}^2 \times \mathbb{C}^n \to \mathbb{R}^2$, it is a bimodule $E \in \text{End}(E) \mathcal{M}_{C^\infty(\mathbb{R}^2)}$ and $A_\mu$ ($\mu = 1, 2$) are the components of a connection. The Seiberg–Witten map provides a quantization of this bundle to a $U(n)$-bundle with connection on the noncommutative plane $\mathbb{R}^2_\theta$:

$$
(E \in \text{End}(E) \mathcal{M}_{C^\infty(\mathbb{R}^2)}, A_\mu, \theta) \xrightarrow{\text{SW map}} (\hat{E} = E^\theta \in \text{End}(E^\theta) \mathcal{M}_{\mathbb{R}^2_\theta}, \hat{A}_\mu),
$$

where the $\hat{A}_\mu$ are determined by the recursive relation (2.12) of the Seiberg–Witten map for connections. The noncommutative connection is a connection on the module of sections $\hat{\Psi} = (\mathbb{R}^2_\theta)_{\oplus n}$ (the trivial $U(n)$-vector bundle on $\mathbb{R}^2_\theta$). To see this, we observe that sections $\psi \in E = C^\infty(\mathbb{R}^2)^{\oplus n}$ are mapped via Seiberg–Witten map to sections $\hat{\phi} \in (\mathbb{R}^2_\theta)^{\oplus n}$ (the former transform under usual gauge transformations the latter under noncommutative gauge transformations) and that the covariant derivative $D_\mu = \partial_\mu - iA_\mu : E \to E$ is mapped to the covariant derivative $\hat{\partial}_\mu - i\hat{A}_\mu : \hat{E} \to \hat{E}$ (that acts via $\star$-multiplication by $\hat{A}_\mu$).

Similarly, bundle endomorphisms $\Psi \in \text{End}(E) = C^\infty(\mathbb{R}^2)^{\oplus (n \times n)}$, i.e., sections of the trivial bundle $\mathbb{R}^2 \times \mathbb{C}^{n \times n} \to \mathbb{R}^2$ that transform in the adjoint representation are mapped via Seiberg–Witten map to bundle endomorphisms $\hat{\Psi} \in \text{End}(\hat{E}) = (\mathbb{R}^2_\theta)^{\oplus (n \times n)}$, i.e., sections transforming in the adjoint representation (with $\star$-product multiplication).

Let us explicitly compute the Seiberg–Witten map for the trivial bundle on $\mathbb{R}^2$ with connection $(A_1 = 0, A_2 = \frac{m \sigma_1}{n \pi \theta} \mathbf{1})$ as in (3.7). We see that since $A_1 = 0$, and $A_2$ depends only on the variable $\sigma^1$, then, $A_2^k = 0$, and $A_2^k$ depends only on $\sigma^1$, so that $\hat{A}_1 = 0$, and $\hat{A}_2$ depends only on $\sigma^1$. It follows that $\hat{\partial}_1 \hat{A}_2 = \hat{F}_{12}$, henceforth (2.12) simplifies to

$$
A_2^{k+1} = \frac{\pi \theta}{n + 1} \frac{\partial}{\partial \sigma^1} \sum_{p=0}^{k} A_2^p A_2^{k-p}.
$$

(4.2)

It is then easy to solve also for $\hat{A}_2$ and to obtain the noncommutative connection

$$
\hat{A}_1 = 0,
$$

$$
\hat{A}_2 = A_2 \sum_{k=0}^{\infty} \left( \frac{m \theta}{n} \right)^k = \frac{1}{2 \pi n - m \theta} \mathbf{1}.
$$

(4.3)

Notice that $(\hat{A}_1, \hat{A}_2) = (A_1^\theta, A_2^\theta)$ as defined in (3.27).

We can also compute the Seiberg–Witten map for the endomorphisms $\Omega_\alpha \in \text{End}(E) = C^\infty(\mathbb{R}^2)^{\oplus (n \times n)}$ defined in (3.1). From the recursive solution for sections in the adjoint (2.20), recalling that $\hat{A}_1 = 0$, it is easy to see that the noncommutative endomorphisms $\hat{\Omega}_\alpha \in \text{End}(\hat{E}) = (\mathbb{R}^2_\theta)^{\oplus (n \times n)}$ coincide with the commutative ones:

$$
\hat{\Omega}_1 = \Omega_1 = e^{im \sigma^2 / n} U, \quad \hat{\Omega}_2 = \Omega_2 = V.
$$

(4.4)
Notice that we obtain this same result if instead of the Seiberg-Witten map (3.1), we use the more general one (2.25) as long as \( \hat{C}_{12}(\Omega_\alpha, \hat{A}) = 0 \). Notice also that by choosing a specific expression for the gauge potential, we have fixed the gauge and therefore do not need to consider the Seiberg-Witten maps (2.13) or (2.23) quantizing local infinitesimal gauge transformations.

4.2. Seiberg–Witten Map for Bundles on \( T \)

Since \( T \simeq \mathbb{R}^2/(2\pi\mathbb{Z})^2 \), we have \( C^\infty(T) \hookrightarrow C^\infty(\mathbb{R}^2) \) as the subalgebra of \( 2\pi \)-periodic functions; moreover, sections of bundles on \( T \) can be seen as sections of bundles on \( \mathbb{R}^2 \) satisfying twisted periodicity conditions. Similarly, \( T_{(-\theta)} \hookrightarrow \mathbb{R}^2_{\theta} \), and sections of modules on \( T_{(-\theta)} \) can be seen as sections of the module \( \hat{E} = (\mathbb{R}^2_{\theta})^{\oplus n} \) on \( \mathbb{R}^2_{\theta} \) satisfying noncommutative twisted periodicity conditions.

Let \((\mathcal{E}, A_\mu)\) be the module of sections and the connection of a bundle on a torus determined by: (i) a trivial bundle \((E = C^\infty(\mathbb{R}^2) \oplus^n, A_\mu)\) on the plane and (ii) \(U(n)\)-valued functions on the plane \(\Omega_1(\sigma^2), \Omega_2(\sigma^1)\) satisfying the classical cocycle condition (3.3). Let \(\hat{A}_\mu, \hat{\Omega}_1(\sigma^2), \hat{\Omega}_2(\sigma^1)\) be the corresponding Seiberg–Witten map quantizations. Let these latter satisfy the noncommutative cocycle conditions

\[
\hat{\Omega}_1(\sigma^2 + 2\pi) \star \hat{\Omega}_2(\sigma^1) = \hat{\Omega}_2(\sigma^1 + 2\pi) \star \hat{\Omega}_1(\sigma^2),
\]

and, together with \(\hat{A}_\mu\), the analogue of the noncommutative twisted periodicity conditions (3.26). Consider next the following definition,

**Definition 1.** We denote by \((\hat{\mathcal{E}}, \hat{A}_\mu)\) the Seiberg–Witten map quantization of \((\mathcal{E}, A_\mu)\), where \(\hat{\mathcal{E}}\) is the subset of all elements in \(\hat{E} = (\mathbb{R}^2_{\theta})^{\oplus n}\) that satisfy the noncommutative twisted periodicity conditions (3.14) with \(\hat{\Omega}_1(\sigma^2), \hat{\Omega}_2(\sigma^1)\).

We also write \((\mathcal{E}, A_\mu) \xrightarrow{\text{SW induced}} (\hat{\mathcal{E}}, \hat{A}_\mu)\) because the quantization of this bundle on the torus is induced by the usual Seiberg–Witten map on the plane. It is immediate to see that \(\hat{\mathcal{E}}\) is a right \(T_{(-\theta)}\)-module.

An example is given by \((\hat{\mathcal{E}}_{n,m}, \hat{A}_\mu)\), the Seiberg–Witten quantization of \((\mathcal{E}_{n,m}, A_\mu)\), where \((A_\mu) = (A_1, A_2) = (0, \frac{m\sigma^1}{n2\pi}1)\). It is defined by the quantized connection and endomorphisms \(\hat{A}_\mu\) and \(\hat{\Omega}_\alpha\) computed in (4.3) and (4.4). They coincide with the connection of \(\mathcal{E}^\theta_{n,m}\) and the endomorphisms defining \(\mathcal{E}^\theta_{n,m}\), cf. (3.27) and (3.15). This shows \((\hat{\mathcal{E}}_{n,m}, \hat{A}_\mu) = (\mathcal{E}^\theta_{n,m}, A^\theta_{\mu})\).

We sharpen this result with the following commutative diagram:

\[
\begin{array}{ccc}
(E \in \text{End}(E)^{\mathcal{M}C^\infty(\mathbb{R}^2), A_\mu, \theta}) & \xrightarrow{\text{SW map}} & (\hat{E} = E^\theta \in \text{End}(E^\theta)^{\mathcal{M}R^2_{\theta}, \hat{A}_\mu}) \\
\text{(i)} & & \text{(i)} \\
(\mathcal{E}_{n,m} \in \text{End}(\mathcal{E}_{n,m})^{\mathcal{M}C^\infty(T), A_\mu, \theta}) & \xrightarrow{\text{SW induced}} & (\hat{\mathcal{E}}_{n,m} = \mathcal{E}^\theta_{n,m} \in \text{End}(\mathcal{E}^\theta_{n,m})^{\mathcal{M}T_{(-\theta)}, \hat{A}_\mu})
\end{array}
\]
where by $\mathcal{E}_{n,m} \in \text{End}(\mathcal{E}_{n,m}) \mathcal{M}_{C^\infty(T)} \xrightarrow{\iota} E \in \text{End}(E) \mathcal{M}_{C^\infty(\mathbb{R}^2)}$ we mean that the module of sections $\mathcal{E}_{n,m}$ is a linear subspace of $E$, and that it is a bimodule over the subalgebras $C^\infty(T) \hookrightarrow C^\infty(\mathbb{R}^2)$ and $\text{End}(\mathcal{E}_{n,m}) \hookrightarrow \text{End}(E)$, and similarly for the other map $i_\theta$.

**Theorem 3.** The induced Seiberg–Witten map on torus bundles $(\mathcal{E}_{n,m}, A_\mu)$ SW$^{\text{induced}}_{\text{SW}} \rightarrow (\hat{\mathcal{E}}_{n,m}, \hat{A}_\mu)$ satisfies the commutative diagram (4.6):

(i) Let $\phi \in E$ satisfy the twisted boundary conditions (3.2), and then, its Seiberg–Witten quantization $\hat{\phi}$ satisfies the twisted boundary conditions (3.14), hence $\hat{\phi} \in \mathcal{E}_{n,m}$.

(ii) Let $\Psi \in \text{End}(E)$ satisfy the twisted boundary conditions (3.9), then its Seiberg–Witten quantization $\hat{\Psi}$ satisfies the twisted boundary conditions (3.28), hence $\hat{\Psi} \in \text{End}(\mathcal{E}_{n,m})$.

**Proof.** We have already shown that $A_\mu \rightarrow \hat{A}_\mu = A^\theta_\mu$, $\Omega_\alpha \rightarrow \hat{\Omega}_\alpha = \Omega_\alpha$, proving that $(\mathcal{E}_{n,m}, A_\mu) \rightarrow (\hat{\mathcal{E}}_{n,m}, \hat{A}_\mu) = (\mathcal{E}_{n,m}^\theta, A^\theta_\mu)$.

(i) Let $\hat{\phi}|_{\sigma+2\pi} = (\Omega \ast \hat{\phi})|_{\sigma}$ be a shorthand notation for $\hat{\phi}(\sigma^1 + 2\pi, \sigma^2) = \Omega_1(\sigma^2) \ast \hat{\phi}(\sigma^1, \sigma^2)$ as well as for $\hat{\phi}(\sigma^1, \sigma^2 + 2\pi) = \Omega_2(\sigma^1) \ast \hat{\phi}(\sigma^1, \sigma^2)$. We show that

$$\hat{\phi}|_{\sigma+2\pi} = (\Omega \ast \hat{\phi})|_{\sigma} \Rightarrow \hat{\phi}|_{\sigma+2\pi} = (\Omega \ast \hat{\phi})|_{\sigma},$$

where we recall that $\ast$ and $\hat{\ast}$ are the star product and the Seiberg–Witten map at noncommutativity parameter $\theta'$. The proof then follows by setting $\theta = 0$ and $\theta' = \theta$ in (4.7). Recalling the uniqueness of the recursive solution (2.19) for $\hat{\phi}$, in order to prove (4.7), it is enough to prove its infinitesimal version for $\theta' = \theta + \delta \theta$, i.e.,

$$\delta^\theta \hat{\phi}|_{\sigma+2\pi} = \delta^\theta(\Omega \ast \hat{\phi})|_{\sigma}. \tag{4.8}$$

The right-hand side reads

$$\delta^\theta(\Omega \ast \hat{\phi})|_{\sigma} = (\Omega \ast \delta^\theta \hat{\phi})|_{\sigma} + i\pi \tilde{\delta} \delta^{\mu
u}(\partial_\mu \Omega \ast \partial_\nu \hat{\phi})|_{\sigma}
= -\pi \delta \theta^{21}(\Omega \ast \hat{A}_2 \ast \hat{D}_1 \hat{\phi} - i\partial_2 \Omega \ast \Omega_1 \hat{D}_1 \hat{\phi})|_{\sigma},$$

where we used that $\Omega_1$ and $\Omega_2$ are independent of $\theta = \theta^{12} = -\theta^{21}$ (cf. (4.4)), the Seiberg–Witten differential equation (2.17) for $\hat{\phi}$ with $\hat{A}_1 = 0$ (cf. (4.3)) so that $\partial_1 \hat{\phi} = \partial_1 \hat{\phi}$, and we also used that $\partial_1 \Omega_1 = \partial_1 \Omega_2 = 0$. We proceed similarly with the left-hand side and obtain

$$\delta^\theta \hat{\phi}|_{\sigma+2\pi} = -\pi \delta \theta^{21}(\hat{A}_2 \ast \hat{D}_1 \hat{\phi})|_{\sigma+2\pi} = -\pi \delta \theta^{21} \hat{A}_2|_{\sigma+2\pi} \ast \hat{D}_1 \hat{\phi}|_{\sigma+2\pi}
= -\pi \delta \theta^{21}(\Omega \ast \hat{A}_2 \ast \hat{D}_1 \hat{\phi} + i\Omega_1 \ast \partial_2 \Omega_1 \ast \Omega \ast \hat{D}_1 \hat{\phi})
= \delta^\theta(\Omega \ast \hat{\phi})|_{\sigma} \tag{4.9}$$

where in the first line we used invariance of the $\ast$-product under constant translations, while in the second line the twisted periodicity conditions satisfied

---

3With slight abuse of terminology, we call $\mathbb{C}[[\theta]]$-modules (and $\mathbb{C}[[\theta]]$-submodules) simply linear spaces (and subspaces).
by the connection: \( \hat{A}_2|_{\sigma+2\pi} = (\Omega \ast \hat{A}_2 \ast \Omega^{-1} + i \Omega \ast \partial_2 \Omega^{-1})|_{\sigma} \), cf. (3.26), and by the section \( D_1 \phi \). This proves \( i \). The proof of \( ii \) is very similar and left to the reader.

In more general terms, as it is clear from the proof, the induced Seiberg–Witten map quantizes \( U(n) \)-bundles on noncommutative tori:

\[
(\hat{\mathcal{E}}_{n,m} = \mathcal{E}_{\theta}^{\theta}_{n,m} \in \text{End}(\mathcal{E}_{\theta}^{\theta}_{n,m}) \mathcal{M}_{T(-\theta)}, \hat{A}_\mu) \xrightarrow{\text{SW}_n} (\hat{\mathcal{E}}_{n,m} = \mathcal{E}_{\theta}^{\theta}_{n,m} \in \text{End}(\mathcal{E}_{\theta}^{\theta}_{n,m}) \mathcal{M}_{T(-\theta)}, \hat{A}_\mu).
\]

(4.10)

We also notice that there are two ways to quantize the algebra of endomorphisms \( \text{End}(\mathcal{E}_{n,m}) \): On the one hand, \( \text{End}(\mathcal{E}_{n,m}) \) is just the algebra \( T_{\tilde{\theta}}|_{\theta=0} \) with \( \tilde{\theta}|_{\theta=0} = \frac{a(-\theta)+b}{m(-\theta)+n} \), and its quantization with respect to \( 0 \rightarrow \theta \) is \( \text{End}(\mathcal{E}_{n,m}) = T_{\tilde{\theta}} \) with \( \tilde{\theta} = \frac{a(-\theta)+b}{m(-\theta)+n} \). On the other hand, we can first quantize \( \mathcal{E}_{n,m} \) to \( \hat{\mathcal{E}}_{n,m} \) and then consider the algebra of endomorphisms of this latter: \( \text{End}(\hat{\mathcal{E}}_{n,m}) \). Since \( \hat{\mathcal{E}}_{n,m} = \mathcal{E}_{\theta}^{\theta}_{n,m} \) as a corollary of Theorem 3, we immediately have that these two alternative quantization routes are equivalent:

\( \text{End}(\hat{\mathcal{E}}_{n,m}) = \text{End}(\mathcal{E}_{\theta}^{\theta}_{n,m}) \).

In Sect. 2.1, we have studied the ambiguities in the Seiberg–Witten map. We generalize Theorem 3 by including these ambiguities, i.e., by including in the Seiberg–Witten differential equations the \( \hat{D}_{\mu\nu}, \hat{E}_{\mu\nu} \) and \( \hat{C}_{\mu\nu} \) terms considered in (2.34)–(2.36). In particular, we also include gauge covariant but not \( GL(d, \mathbb{R}) \)-covariant terms \( \hat{D}_{\mu\nu}^{\text{gauge-cov}}, \hat{E}_{\mu\nu}^{\text{c.f.}} \) and \( \hat{C}_{\mu\nu}^{\text{gauge-cov}}, \hat{E}_{\mu\nu}^{\text{c.f.}} \). Gauge covariance is realized by considering expressions containing only the gauge covariant fields \( \hat{F}_{\mu\nu}, \hat{\phi}, \hat{\Psi} \) and the covariant derivative \( D_\mu \); otherwise stated, by considering only sections of bundles associated with the noncommutative torus bundle, hence fields and operators satisfying twisted periodicity conditions.

**Theorem 4.** Consider the generalized Seiberg–Witten map defined by the differential Eqs. (2.22)–(2.25) with \( \hat{D}_{\mu\nu}, \hat{E}_{\mu\nu} \) and \( \hat{C}_{\mu\nu} \) as defined in (2.34)–(2.36). If the gauge covariant but not \( GL(d, \mathbb{R}) \)-covariant term \( \hat{D}_{\mu\nu}^{\text{gauge-cov}}, \hat{E}_{\mu\nu}^{\text{c.f.}} \) vanishes on \( A_\mu^\theta \) and if \( \hat{C}_{\mu\nu}(A_\mu^\theta, \Omega_\alpha) = 0 \) (\( \alpha = 1, 2 \)), the induced Seiberg–Witten map on torus bundles determined by diagram (4.6) quantizes \( (\mathcal{E}_{\theta}^{\theta}_{n,m}, A_\mu) \) to \( (\hat{\mathcal{E}}_{n,m}, \hat{A}_\mu) = (\mathcal{E}_{\theta}^{\theta}_{n,m}, A_\mu^\theta) \), and it is well defined, consistently quantizing \( \phi \in \mathcal{E}_{n,m} \) to \( \hat{\phi} \in \mathcal{E}_{\theta}^{\theta}_{n,m} \) and \( \Psi \in \text{End}(\mathcal{E}_{n,m}) \) to \( \hat{\Psi} \in \text{End}(\mathcal{E}_{\theta}^{\theta}_{n,m}) \).

**Proof.** The gauge potential \( (A_\mu^\theta) = (A_1^\theta, A_2^\theta) = (0, \frac{m\sigma_1}{n-m\sigma_1} 1) \) satisfies (2.22) because \( \hat{D}_{\mu\nu} \) vanishes on it (the first addend in (2.34) because \( F_{\mu\nu}^{\theta} \) is constant and proportional to the unit matrix, the second addend because \( A_1^\theta = 0 \)). This shows that the Seiberg–Witten quantization of \( (A_\mu) = (A_1, A_2) = (0, \frac{m\sigma_1}{n+2\pi} 1) \) according to the differential Eq. (2.22) is \( (\hat{A}_\mu) = (A_\mu^\theta) = (0, \frac{m\sigma_1}{n+2\pi} 1) \). Since \( \hat{C}_{\mu\nu}(A_\mu^\theta, \Omega_\alpha) = 0 \) we also have \( \hat{\Omega}_\alpha = \Omega_\alpha, (\mathcal{E}, A_\mu) \xrightarrow{\text{SW induced}} (\hat{\mathcal{E}}, \hat{A}_\mu) = (\mathcal{E}^\theta, A_\mu^\theta) \).

Concerning the quantization of sections, we repeat the proof of Theorem 3, we assume that \( \hat{\phi} \) satisfies twisted periodicity conditions, i.e., that it is a
section of $E_{n,m}^\theta$, and prove that $\hat{\phi}$ is a section of $E_{n,m}^{\theta', e}$, or equivalently (4.8). Now the left-hand side of this equation has the extra term $-\frac{\pi}{2} \delta \theta^{\mu\nu} \hat{\mathcal{C}}_{\mu\nu}|_{\sigma + 2\pi}$ while the right-hand side has the extra term $-\frac{\pi}{2} \delta \theta^{\mu\nu} (\Omega \ast \hat{\mathcal{C}}_{\mu\nu})|_{\sigma}$. These two extra terms are equal (i.e., $\hat{\mathcal{C}}_{\mu\nu}$ is a section of $E_{n,m}^{\theta, \theta'}$) because the gauge covariant part of $\hat{\mathcal{C}}_{\mu\nu}$ depends on $D_\mu, F$ and $\phi$ that satisfy the twisted periodicity conditions, while the only gauge noncovariant part of $\hat{\mathcal{C}}_{\mu\nu}(\hat{A}, \hat{\phi})$ is $-2i\beta[\hat{A}_\mu, \hat{A}_\nu] \ast \hat{\phi}$, cf. (2.35), that anyhow satisfies the twisted periodicity conditions being $A_1 = 0$ and $\partial_{\sigma^1, \Omega_\alpha} = 0, (\alpha = 1, 2)$. We conclude that the Seiberg–Witten quantization of $\phi \in E_{n,m}$ is a section $\hat{\phi} \in E_{n,m}^{\theta, \theta'}$. A similar argument holds for sections $\hat{\Psi}$ in the adjoint.

Theorem 4 shows that the Seiberg–Witten map quantization $(E, A_\mu) \xrightarrow{\text{SW induced}} (\hat{E}, \hat{A}_\mu) = (E^{\theta, e}, A^{\theta, e}_\mu)$, including sections in the fundamental and in the adjoint, is well defined for all the terms $\hat{D}_\mu \kappa, \hat{E}_\mu, \hat{\mathcal{C}}_{\mu\nu}$ found in (2.31)–(2.32), i.e., for all values of the parameters $\alpha, \beta, \gamma$. Furthermore, the quantization is well defined also for arbitrary terms $\hat{\mathcal{C}}^{\text{gauge-cov}, \text{GL-cov}}$ on fields in the fundamental and hence for the $\hat{\mathcal{C}}_{\mu\nu}$ terms in (2.35). Examples of nontrivial $\hat{\mathcal{C}}_{12}$ terms for fields in the adjoint that vanish on $\Omega_\alpha \in \text{End}(E) = \mathbb{C}\infty(\mathbb{R}^2)_{\oplus(n \times n)}$ are those in (2.33) with $\gamma = -\tilde{\gamma}$, examples that are not $GL(d, \mathbb{R})$-covariant are $\hat{\mathcal{C}}^{\text{gauge-cov}, \text{GL-cov}}(\hat{A}^\theta, \Omega_\alpha) = -iD_1D_2\Omega_\alpha = 0$ (indeed recall the definition of $\Omega_\alpha$ in (3.1), use that $[D_1, D_2]\Psi = -i[F, \Psi] = 0$ since $F$ is constant and proportional to the unit matrix, and that $D_1 = \partial_{\sigma^1}$).

### 4.3. Explicit Solutions: Ho’s Sections $\phi^\theta, Z^\theta_\mu$

We here show that the Seiberg–Witten quantization of the sections $\phi \in E_{n,m}$ in the fundamental, and of the sections $\Psi \in \text{End}(E_{n,m})$ in the adjoint, generated by $Z_\mu$, gives the sections $\phi^\theta \in E_{n,m}^\theta$ and $\Psi^\theta \in \text{End}(E_{n,m}^\theta)$ that are generated by $Z^\theta_\mu$ as described in Sect. 3.2. This shows that the solutions presented in [7] fit into the framework of Seiberg–Witten maps. It also provides explicit closed form solutions to the Seiberg–Witten map equations for nontrivial bundles.

We begin with the generators $Z_\mu \in \text{End}(E_{n,m})$.

**Proposition 5.** The Seiberg–Witten quantization according to Theorem 3 of the generators $Z_\mu \in \text{End}(E_{n,m})$ defined in (3.10) as $Z_1 = e^{i\pi_1^1} V^b, Z_2 = e^{i\pi_1^2} U^{-b}$ gives the generators $\hat{Z}_\mu = Z^\theta_\mu \in \text{End}(E_{n,m}^\theta)$ defined in (3.29) as $Z^\theta_1 = e^{i\pi_{-m}^1} V^b, Z^\theta_2 = e^{i\pi_{-m}^2} U^{-b}$.

**Proof.** The Seiberg–Witten equation (2.18) for sections in the adjoint representation with connection (4.3) reads

$$\frac{\partial \hat{Z}_\mu}{\partial \theta} = \pi \left\{ \hat{A}_2, \partial_{\hat{\theta}} \hat{Z}_\mu \right\}_\ast. \tag{4.11}$$

We show that $Z^\theta_1 = e^{i\pi_{-m}^1} V^b$ and $Z^\theta_2 = e^{i\pi_{-m}^2} U^{-b}$ satisfy (4.11) (and of course the initial conditions $Z^\theta_\mu|_{\theta = 0} = Z_\mu$). For $\mu = 1$, we note that due to the
dependence of $Z_1^\theta$ and $\hat{A}_2$ only on $\sigma^1$, the star product on the right-hand side of (4.11) reduces to the usual matrix product and gives (since $\hat{A}_2$ is proportional to the unit matrix)

$$2\pi \hat{A}_2 \cdot \partial_1 Z_1^\theta = \frac{im\sigma^1}{(n-m\theta)^2} e^{\frac{im^1}{n-m\theta}} V^b, \quad (4.12)$$

which coincides with the left-hand side $\partial Z_\mu^\theta / \partial \theta$. The case $\mu = 2$ is trivial due to the independence of $Z_2^\theta$ with respect to $\theta$ and its dependence solely on $\sigma^2$. \hfill \Box

In Proposition 5, we have just quantized the generators $Z_\mu$ of sections in the adjoint representation. In general, the Seiberg–Witten map quantizes an arbitrary section in the adjoint $\Psi \in \text{End}(\mathcal{E}_{n,m})$ to $\hat{\Psi} \in \text{End}(\mathcal{E}_{n,m}^\theta)$. We can use the ambiguities in the Seiberg–Witten maps to single out the one that quantizes $Z_\mu$ to $\hat{Z}_\mu = Z_\mu^\theta$ and that is compatible with a specific ordering of the generators of the algebras $\text{End}(\mathcal{E}_{n,m})$ and $\text{End}(\mathcal{E}_{n,m}^\theta)$.

For example, we study the Seiberg–Witten map that to the ordered monomial $Z_1^{p} Z_2^{q}$ associates the ordered monomial $\hat{Z}_1^{p} \hat{Z}_2^{q} = \hat{Z}_1^{p} \hat{Z}_2^{q}$ (powers and $q$-powers of $Z_1$ and $Z_2$ coincide). We compute

$$\frac{\partial}{\partial \theta} (Z_1^{p} \ast Z_2^{q}) - \pi \{ \hat{A}_2, \partial_1 (Z_1^{p} \ast Z_2^{q}) \} = \pi i \partial_1 Z_1^{p} \ast \partial_2 Z_2^{q} + \frac{\partial Z_1^{p}}{\partial \theta} \ast Z_2^{q}$$

$$\pi \left( \partial_1 Z_1^{p} \ast \partial_2 Z_2^{q} + \frac{\partial Z_1^{p}}{\partial \theta} \ast Z_2^{q} \right)$$

$$\pi \partial_1 Z_1^{p} \ast \partial_2 Z_2^{q} + \frac{\partial Z_1^{p}}{\partial \theta} \ast Z_2^{q}$$

$$\pi \left( \partial_1 Z_1^{p} \ast \partial_2 Z_2^{q} + \frac{\partial Z_1^{p}}{\partial \theta} \ast Z_2^{q} \right)$$

$$\pi \left( \partial_1 Z_1^{p} \ast \partial_2 Z_2^{q} + \frac{\partial Z_1^{p}}{\partial \theta} \ast Z_2^{q} \right)$$

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$$\pi \left( \partial_1 Z_1^{p} \ast \partial_2 Z_2^{q} + \frac{\partial Z_1^{p}}{\partial \theta} \ast Z_2^{q} \right)$$

$$\pi \left( \partial_1 Z_1^{p} \ast \partial_2 Z_2^{q} + \frac{\partial Z_1^{p}}{\partial \theta} \ast Z_2^{q} \right)$$

where in the second line we used that $Z_2^\theta = Z_2$ is independent from $\theta$ and $\sigma_1$, in the third line we used that (4.11) implies $\partial Z_1^{p} / \partial \theta = \pi \{ \hat{A}_2, \partial_1 Z_1^{p} \} \ast$ in the fourth line that $D_2 Z_2^{q}$ is independent from $\sigma_1$ and that $D_2 Z_1^{p} = 0$. We therefore see that the relevant Seiberg–Witten map is given by the Seiberg–Witten differential equation

$$\frac{\partial \hat{\Psi}}{\partial \theta} = \pi \{ \hat{A}_2, \partial_1 \hat{\Psi} \} + \pi i D_1 D_2 \hat{\Psi}, \quad (4.14)$$

comparison with (2.25) shows that it corresponds to the choice $\hat{C}_{12} = -\hat{C}_{21} = -iD_1 D_2 \hat{\Psi}$. Here, $\hat{\Psi}$ is more generally any linear combination of ordered monomials in $Z_1^\theta, Z_2^\theta$.

Remark 6. If we choose the opposite ordering, $Z_2^{q} Z_1^{p} \rightarrow \hat{Z}_2^{q} \hat{Z}_1^{p} = \hat{Z}_2^{q} \ast \hat{Z}_1^{p} = Z_2^{q} \ast Z_1^{p}$, the corresponding Seiberg–Witten differential equation is
(4.14) with $-\pi i D_1 D_2 \hat{\Psi}$ instead of $+\pi i D_1 D_2 \hat{\Psi}$. Since $Z_{11}^p \star Z_{22}^q - e^{2\pi i p q} Z_{22}^q \star Z_{11}^p = 0$, cf. text after (3.29), the ordering $Z_{11}^p \star Z_{22}^q + e^{2\pi i p q} Z_{22}^q \star Z_{11}^p$ satisfies the standard Seiberg–Witten differential equation with $\hat{C}_{12} = 0$; the corresponding map quantizes $Z_{11}^p Z_{22}^q + e^{2\pi i p q} Z_{22}^q Z_{11}^p$ to $Z_{11}^p \star Z_{22}^q + e^{2\pi i p q} Z_{22}^q \star Z_{11}^p$. These three different Seiberg–Witten maps are easily seen to coincide on the generators $Z_\mu$.

We now study the Seiberg-Witten quantization of sections in the fundamental representation. In [7], the quantum section $\phi^\theta$ in (3.22) was given. It is natural to ask how this solution fits in the framework of Seiberg–Witten quantization. To this aim, we study the $\theta$-dependence of the quantum section $\phi^\theta$ in (3.22).

**Lemma 7.** The quantum section $\phi^\theta = (\phi^\theta_k)_{k=1,...,n}$ in (3.22) satisfies the differential equation:

$$\frac{\partial}{\partial \theta} \phi^\theta - \pi \hat{A}_2 \star \partial_1 \phi^\theta = 3\pi \hat{F} \star \phi^\theta + i\pi D_1 D_2 \phi^\theta. \quad (4.15)$$

**Proof.** For convenience, let us recall (3.22): $\phi^\theta_k(\sigma^1, \sigma^2) = \sum_{s \in \mathbb{Z}} \sum_{j=1}^{m} E(A, B) \star \tilde{\phi}_j(\frac{m}{n} A)$, where $A = \frac{m}{n} (\frac{e^{2\pi i}}{2\pi} + k + n s) + j$, $B = i\sigma^1$ and $E(A, B)$ is defined in (3.16). The derivative of $\phi^\theta = (\phi^\theta_k)_{k=1,...,n}$ with respect to $\theta$ gives

$$\frac{\partial}{\partial \theta} \phi^\theta = \sum_{s, j} \left( \left( \frac{\partial}{\partial \theta} E(A, B) \right) \star \tilde{\phi}_j(\frac{m}{n} A) + i\pi \partial_1 E(A, B) \star \partial_2 \tilde{\phi}_j(\frac{m}{n} A) \right). \quad (4.16)$$

The derivative of $E(A, B)$ with respect to $\theta$ is given by (use $\frac{p}{(p-1)!} = \frac{1}{(p-1)!} + \frac{p-1}{(p-1)!}$, and use Lemma 1 in the third line)

$$\frac{\partial}{\partial \theta} E(A, B) = \frac{\partial}{\partial \theta} \left( \frac{1}{1 - \frac{m}{n} \theta} \sum_{p=0}^{\infty} \frac{1}{p!} A^p \star B^p \right)$$

$$= \frac{m}{n - m \theta} E(A, B) + \frac{1}{2} \frac{m}{n} \left( E(A, B) + A \star E(A, B) \star B \right),$$

$$= \frac{m}{n - m \theta} E(A, B) + \frac{1}{2} \frac{m}{n} E(A, B) + \frac{1}{2} \frac{m}{n - m \theta} A \star B \star E(A, B). \quad (4.17)$$

The second term in (4.16) gives, using Lemma 2 and that $\partial_1 = D_1$,

$$i\pi \partial_1 \sum_{s, j} E(A, B) \star \partial_2 \tilde{\phi}_j(\frac{m}{n} A) = i\pi D_1 D_2 \phi^\theta. \quad (4.18)$$

Summarizing, the $\theta$-derivative of the section $\phi^\theta$ is given by

$$\frac{\partial}{\partial \theta} \phi^\theta = \left( \frac{m}{n - m \theta} + \frac{1}{2} \frac{m}{n} \right) \phi^\theta + \sum_{s, j} \frac{1}{2} \frac{m}{n - m \theta} A \star B \star E(A, B) \star \tilde{\phi}_j(\frac{m}{n} A)$$

$$+ i\pi D_1 D_2 \phi^\theta. \quad (4.19)$$
Furthermore, \( \partial_1 E(A, B) = i A \star E(A, B) \) implies
\[
\pi \dot{A}_2 \star \partial_1 \phi^\theta = \frac{1}{2} \frac{m}{n - m \theta} \sum_{s,j} B \star A \star E(A, B) \star \tilde{\phi}_j.
\] (4.20)

Hence, we arrive at
\[
\frac{\partial}{\partial \theta} \phi^\theta - \pi \dot{A}_2 \star \partial_1 \phi^\theta = \left( \frac{m}{n - m \theta} + \frac{1}{2} \frac{m}{n} \right) \phi^\theta + i \pi D_1 D_2 \phi^\theta
\]
\[
+ \sum_{s,j} \frac{1}{2} \frac{m}{n - m \theta} \left( A \star B \star A \right) \star E(A, B) \star \tilde{\phi}_j \left( \frac{n}{m} A \right)
\]
\[
= \frac{3}{2} \frac{m}{n - m \theta} \phi^\theta + i \pi D_1 D_2 \phi^\theta
\]
\[
= 3 \pi \hat{F} \star \phi^\theta + i \pi D_1 D_2 \phi^\theta
\] (4.21)

where in the last line we used \( \hat{F} = i[D_1, D_2] = \frac{1}{2 \pi} \frac{m}{n - m \theta} \mathbf{1} \).

Comparison of (4.15) with the (generalized) Seiberg–Witten differential equation for fields in the fundamental representation shows that (4.15) equals (2.24) with
\[
\hat{C}_{12}(\phi, \hat{A}) = -\hat{C}_{21}(\hat{\Phi}, \hat{A}) = -3 \hat{F} \star \phi^\theta - i D_1 D_2 \phi^\theta.
\] (4.22)

The same operator \(-3 \hat{F} - i D_1 D_2\) acting in the adjoint reads \(\hat{C}_{12}(\hat{\Psi}, \hat{A}) = -\hat{C}_{21}(\hat{\Psi}, \hat{A}) = -i D_1 D_2 \hat{\Psi}\) (use \(\hat{F}, \hat{\Psi}\), since \(\hat{F}\) is constant and proportional to the identity). In particular, for the sections \(\hat{\Omega}_\alpha = \Omega_\alpha \in \text{End}(E^\theta) = (\mathbb{R}^2_\theta)^{\oplus(n \times n)}\), we have \(\hat{C}_{12}(\hat{\Omega}_\alpha, \hat{A}) = 0\). These expressions transform covariantly under gauge transformations (they are sections of \(E^\theta = (\mathbb{R}^2_\theta)^{\oplus n}\); cf. also (2.37), and of \(\text{End}(E^\theta) = (\mathbb{R}^2_\theta)^{\oplus(n \times n)}\) and the corresponding (generalized) Seiberg–Witten map is a quantization of bundles on tori as shown in Theorem 4. We thus conclude that

**Theorem 8.** The Seiberg–Witten map with arbitrary \(\hat{D}_{\mu \nu \kappa}\) and \(\hat{E}_{\mu \nu}\) as in (2.31)–(2.33), and more generally as in Theorem 4, and with \(\hat{C}_{\mu \nu}\) given by the operator \(-3 \hat{F} - i D_1 D_2\) quantizes, following Theorem 4, \((\mathcal{E}_{n,m}, A_\mu)\) to \((\mathcal{E}_{n,m}, \hat{A}_\mu) = (\mathcal{E}_{n,m}^\theta, A_\mu^\theta)\); the sections \(\phi \in \mathcal{E}_{n,m}\) defined in (3.5) to the sections \(\hat{\phi} = \phi^\theta \in \mathcal{E}_{n,m}^\theta\) as defined in (3.22); the adjoint sections \(Z_1^p Z_2^q \in \text{End}(\mathcal{E}_{n,m})\) to \(Z_1^{p \theta} \star Z_2^{q \theta} \in \text{End}(\mathcal{E}_{n,m}^\theta)\).

We have recovered within the framework of Seiberg–Witten map Ho’s solutions \(\phi^\theta \in \mathcal{E}_{n,m}^\theta\) and \(Z_\mu^\theta \in \text{End}(\mathcal{E}_{n,m}^\theta)\) to the noncommutative periodicity conditions. It follows that this Seiberg–Witten framework, initially developed in the context of deformation quantization with \(\theta\) a formal deformation parameter, can be specialized to \(\theta \in \mathbb{R} - \{ \frac{m}{n} \}\). Indeed, the \(\star\)-product can be completed to a nonformal product à la Rieffel, and the connection \(\hat{A}_\mu\), and the sections \(\phi^\theta, Z_\mu^\theta\) are well defined for \(\theta \in \mathbb{R} - \{ \frac{m}{n} \}\).
5. Morita Equivalence, $T$-duality and Seiberg–Witten Map

Here, we briefly review how Morita equivalence implements $T$-duality of Yang–Mills theories and show the compatibility of the Seiberg–Witten maps with $T$-duality transformations.

Two (associative and unital) algebras $A$ and $\tilde{A}$ are Morita equivalent if their categories of right modules $\mathcal{M}_A$ and $\mathcal{M}_{\tilde{A}}$ are equivalent. By a theorem of Morita, two algebras $A$ and $\tilde{A}$ are Morita equivalent if and only if there exists a finitely generated and projective $\tilde{A}$-module $E \in \mathcal{M}_{\tilde{A}}$ such that every other $\tilde{A}$ module is a quotient of $E^{\oplus N}$ for some integer $N$, and such that $A \simeq \text{End}(E_{\tilde{A}})$. In this case, $E \in \mathcal{M}_A$ is called a Morita equivalence bimodule. The equivalence between the categories of representations $\mathcal{M}_A$ and $\mathcal{M}_{\tilde{A}}$ is easily constructed via $E \in \mathcal{M}_A$. The main point is that given a module $E \in \mathcal{M}_A$, the tensor product over $A$ with $E \in \mathcal{M}_{\tilde{A}}$ gives a module $\tilde{E} = E \otimes_A E \in \mathcal{M}_{\tilde{A}}$, and morphisms of $\mathcal{M}_A$ modules $\varphi : E \to F$ are mapped to morphisms of $\mathcal{M}_{\tilde{A}}$ modules $\varphi \otimes_A \text{id} : E \otimes_A E \to F \otimes_A E$. The bimodules $E_{n,m}^\theta$, $n > 0$, $m \neq 0$, $n,m$ relatively prime, in Sect. 3.2 (Heisenberg modules) are examples of Morita equivalence bimodules and prove Morita equivalence of two-dimensional tori $T_{(-\tilde{\theta})}$ and $T_{(-\theta)}$ related by a fractional $SL(2,\mathbb{Z})$ transformation: $(-\tilde{\theta}) = \frac{a(-\theta) + b}{m(-\theta) + n}$ with $(a/b, m/n) \in SL(2,\mathbb{Z})$. Since $T_{(-\theta)}$ is isomorphic to $T_\theta$, we further have that two-dimensional tori $T_{\tilde{\theta}}$ and $T_{(-\theta)}$ related by a fractional $GL(2,\mathbb{Z})$ transformation are Morita equivalent. This is also a necessary condition: The two-dimensional tori $T_{(-\tilde{\theta})}$ and $T_{(-\theta)}$ are Morita equivalent, if and only if $4(-\tilde{\theta}) = \frac{a(-\theta) + b}{m(-\theta) + n}$, with $(a/b, m/n) \in GL(2,\mathbb{Z})$, [6].

In order to study the equivalence of categories between Morita equivalent torus algebras, we consider the Heisenberg module $E_{0,1}^{1/\sigma}$, $s \in \mathbb{Z}$, that is a bimodule in $T_{(-\theta)}\mathcal{M}_{T_{(-1/\sigma)}}$, accounting for the transformation $-\theta \to \frac{1}{\sigma-\theta}$. We recall from Sect. 3.2 that $E_{0,1}^{1/\sigma} = E_{1,1}^{1/\sigma} + 1$ is the module of sections of the $U(1)$-bundle over $T_{(-1/\sigma)}$ with charge $m = 1$, algebraically it is the vector space of Schwartz functions on $\mathbb{R}$, with right $T_{(-1/\sigma)}$-action given by

$$
(\tilde{\phi} \triangleright U_1)(x) = \tilde{\phi}(x + \frac{1}{\sigma-\theta}), \quad (\tilde{\phi} \triangleleft U_2)(x) = \tilde{\phi}(x) e^{2\pi i x}.
$$

(5.1)

The algebra $T_{(-\theta)}$ of endomorphisms is generated by

$$
(Z_1 \triangleright \tilde{\phi})(x) = \tilde{\phi}(x - 1), \quad (Z_2 \triangleright \tilde{\phi})(x) = \tilde{\phi}(x) e^{2\pi i x(\theta-\sigma)} = \tilde{\phi}(x) e^{2\pi i x \theta},
$$

(5.2)

There are different notions of Morita equivalence: The one just recalled for algebras (and more generally rings), a stronger notion for $C^*$-algebras, and, in the case of (multidimensional) tori, an even stronger one called complete Morita equivalence of smooth noncommutative tori [9] (called gauge Morita equivalence in [20]). These notions for two-dimensional tori are all equivalent (the bimodules $E_{n,m}^\theta$ can be completed to full Hilbert modules providing $C^*$-algebra Morita equivalence, and they are Heisenberg modules with constant curvature connections that provide complete Morita equivalence).
while the connection is
\[ D_1 \tilde{\phi}(x) = \frac{-ix}{\theta} \tilde{\phi}_j(x), \quad D_2 \tilde{\phi}(x) = \frac{1}{2\pi} \frac{\partial}{\partial x} \tilde{\phi}(x). \] (5.3)

Since any projective module over \( T_{(-\theta)} \) is equivalent to a direct sum of Heisenberg modules \( \mathcal{E}_{n,m}^\theta \) \((n \geq 0, m \neq 0, n, m \text{ relatively prime})\) or the trivial module \( T_{(-\theta)} \), it is sufficient to describe the transformations of these modules under a Morita equivalence bimodule \( \mathcal{E}^{\frac{1}{s-\theta}} \), \( s \in \mathbb{Z} \) in order to know it on every module in \( \mathcal{M}_{T_{(-\theta)}} \).

We have an isomorphism (see, for example, the outlined explicit derivation in [9, §3])
\[ \mathcal{E}_{n,m}^\theta \otimes_{T_{(-\theta)}} \mathcal{E}^{\frac{1}{s-\theta}} \simeq \mathcal{E}^{\frac{1}{s-\theta}}_{n+m,m}, \] (5.4)
in particular, both left-hand side and right-hand side are left \( T_{\tilde{\theta}} \)-modules with \( \tilde{\theta} = \frac{a(-\theta)+b}{m(-\theta)+n} \). We see that under the Morita equivalence bimodule \( \mathcal{E}_{0,1}^{\frac{1}{s-\theta}} \), the module \( \mathcal{E}_{n,m}^{\theta} \) is mapped (up to equivalence) to the module \( \mathcal{E}_{n,m}^{-\frac{1}{s-\theta}} = \mathcal{E}^{\frac{1}{s-\theta}}_{n+m,n+n+m} \), and thus, \( \theta \mapsto \tilde{\theta} = (\frac{0}{-1} s) \theta = \frac{1}{s-\theta} \), \( (n/m) \mapsto (\frac{n}{m}) = (\frac{m}{n+n+m}) \).

Similarly, let us define the bimodule \( \mathcal{E}_{1,0}^{\theta+1} \in T_{(-\theta)} \mathcal{M}_{T_{(-\theta-1)}} \) to be \( T_{(-\theta)} \) as a left \( T_{(-\theta)} \)-module, with right \( T_{(-\theta-1)} \)-action defined by \( e \lhd U_{\mu}^{T_{(-\theta-1)}} = eU_{\mu} \), where \( e \in \mathcal{E}_{1,0}^{\theta+1}, U_{\mu}^{T_{(-\theta-1)}} \) are the generators of \( T_{(-\theta-1)} \), \( U_{\mu} \) those of \( T_{(-\theta)} \) and \( eU_{\mu} \) is the product in \( T_{(-\theta)} \). Then, it is easy to show that
\[ \mathcal{E}_{n,m}^\theta \otimes_{T_{(-\theta)}} \mathcal{E}_{1,0}^{\theta+1} \simeq \mathcal{E}_{n+m,m}^{\theta+1}. \] (5.5)

Indeed, using the nonvanishing global section \( 1 \in \mathcal{E}_{1,0}^{\theta+1} \) we write a generic section of \( \mathcal{E}_{n,m}^\theta \otimes_{T_{(-\theta)}} \mathcal{E}_{1,0}^{\theta+1} \) as \( \tilde{\phi} \otimes_{T_{(-\theta)}} 1 \). We prove the equivalence (5.5) by showing that \( \tilde{\phi} \otimes_{T_{(-\theta)}} 1 \) transforms as a section of \( \mathcal{E}_{n+m,m}^{\theta+1}, \)
\[ ((\tilde{\phi} \otimes_{T_{(-\theta)}} 1) \lhd U_1^{T_{(-\theta-1)}})_{j}(x) = (\tilde{\phi} \otimes_{T_{(-\theta)}} 1)_{j-1}(x - \frac{n+m}{m} + (\theta + 1)), \]
\[ ((\tilde{\phi} \otimes_{T_{(-\theta)}} 1) \lhd U_2^{T_{(-\theta-1)}})_{j}(x) = (\tilde{\phi} \otimes_{T_{(-\theta)}} 1)_{j}(x) e^{2\pi i(x-j(n+m)/m)}. \] (5.6)

E.g., \( (\tilde{\phi} \otimes_{T_{(-\theta)}} 1) \lhd U_1^{T_{(-\theta-1)}} = \tilde{\phi} \otimes_{T_{(-\theta)}} 1 \lhd U_1^{T_{(-\theta-1)}} = \tilde{\phi} \otimes_{T_{(-\theta)}} U_1 = \tilde{\phi} \otimes_{T_{(-\theta)}} 1, \) and \( (\tilde{\phi} \otimes_{T_{(-\theta)}} 1)_{j}(x) = \tilde{\phi}_{j-1}(x - \frac{n+m}{m} + (\theta + 1)) \). We have seen that tensoring with the Morita equivalence bimodule, \( \mathcal{E}_{1,0}^{\theta+1} \in T_{(-\theta)} \mathcal{M}_{T_{(-\theta-1)}} \) gives the map \( \mathcal{E}_{n,m}^\theta \mapsto \mathcal{E}_{n+m,m}^{\theta+1} \), and thus, \( \theta \mapsto \tilde{\theta} = (\frac{1}{1} 1) \theta = \theta + 1, \) \( (\frac{n}{m}) \mapsto (\frac{m}{n+n+m}) \).

We similarly construct a bimodule \( \mathcal{E}_{1,0}^{-\theta} \) that realizes the isomorphism \( T_{(-\theta)} \simeq T_{\theta} \) as a Morita equivalence. By definition, \( \mathcal{E}_{1,0}^{\theta} \in T_{(-\theta)} \mathcal{M}_{T_{(-\theta)}} \) is \( T_{(-\theta)} \) itself as a left \( T_{(-\theta)} \)-module, while the right \( T_{\theta} \)-action on \( \mathcal{E}_{1,0}^{-\theta} \) is defined by \( e \lhd U_{T_{\theta}}^{1} = eU_{1}^{-1}, e \lhd U_{T_{\theta}}^{2} = eU_{2} \), where \( e \in \mathcal{E}_{1,0}^{\theta+1}, U_{T_{\theta}}^{1} \) are the generators of
those of $T_{(-\theta)}$ and $eU^\pm_\mu$ is the product in $T_{(-\theta)}$. A generic section of $\mathcal{E}^\theta_{n,m} \otimes T_{(-\theta)}$ is $\delta \otimes 1_1$ with $\delta$ a section of $\mathcal{E}^\theta_{n,m}$. We show that

$$\mathcal{E}^\theta_{n,m} \otimes T_{(-\theta)} \mathcal{E}^\theta_{1,0} \simeq \mathcal{E}^-_{n,-m}$$

by showing that the map $\delta \otimes 1 \mapsto \iota(\delta \otimes 1)$ defined by $\iota(\delta \otimes 1)_j(x) = \tilde{\phi}|_{m-j}(x)$ (we can assume $|m| - j = 1, 2, \ldots |m|$ due to $\mathbb{Z}_{|m|}$ cyclicity) is a right $T_\theta$-module isomorphism. Proof: We have to show that $\iota((\delta \otimes 1) \preceq U^T_{\mu}) = \iota(\delta \otimes 1) \preceq U^T_{\mu}$. Indeed,

$$\iota((\delta \otimes 1) \preceq U^T_{\mu})_j(x) = \iota(\delta \otimes U^{-1}_1 \otimes 1_1)_j(x) = (\tilde{\phi} \preceq U^{-1}_1)|_{m-j}(x)$$

$$= \tilde{\phi}|_{m-j+1}(x + \frac{n}{m} - \theta)$$

$$= \iota(\delta \otimes 1)_{j-1}(x + \frac{n}{m} - \theta) = (\iota(\delta \otimes 1) \preceq U^T_{1\mu})_j(x),$$

and similarly for $U^T_{2\mu}$. We have seen that tensoring with the Morita equivalence bimodule $\mathcal{E}^-_{1,0}$ gives the transformation (up to equivalence) $\mathcal{E}^\theta_{n,m} \mapsto \mathcal{E}^\theta_{n,m} = \mathcal{E}^{\theta+1}_{n+m,m}$, and thus, $\theta \mapsto \tilde{\theta} = (0_0^0 \theta) = -\theta$, $(n_m) \mapsto (\tilde{n}_{\tilde{m}}) = (0_0^0 n_m) = (\tilde{n}_{\tilde{m}})$.

Since $SL(2, \mathbb{Z})$ is generated by $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and $GL(2, \mathbb{Z})$ by considering also $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, we see that (5.4), (5.5) and (5.7) generate the whole non-geometric $GL(2, \mathbb{Z})$ duality group that acts on the modules $\mathcal{E}^\theta_{n,m} \in \mathcal{M}_{T_{(-\theta)}}$ as

$$\theta \mapsto \tilde{\theta} = \frac{a\theta + b}{m\theta + n}, \quad (n_m) \mapsto (\tilde{n}_{\tilde{m}}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} n_m \\ \tilde{m} \end{pmatrix}, \quad (a, b, c, d) \in GL(2, \mathbb{Z}).$$

(5.9)

We now describe how the equivalence of the categories of modules $\mathcal{M}_{T_{(-\theta)}}$ and $\mathcal{M}_{T_{(-\theta)}}$ (with $\tilde{\theta}$ in the same $GL(2, \mathbb{Z})$ orbit of $\theta$) is extended to an equivalence between modules with connections, called complete (or gauge) Morita equivalence. This is due to the canonical connections of the bimodules $\mathcal{E}^{1\sigma}_{0,1}$ and $\mathcal{E}^{1\sigma}_{1,0}$ that correspondingly define these bimodules as complete (or gauge) Morita equivalence bimodules, see definition after (3.36). It is this gauge Morita equivalence that implements $T$-duality transformations between gauge theories on noncommutative bundles $\mathcal{E}^\theta_{n,m}$ on $T_{(-\theta)}$ and $\mathcal{E}^\tilde{\theta}_{\tilde{n},\tilde{m}}$ on $T_{(-\tilde{\theta})}$.

To any (right) connection $\nabla_\mu$ on $\mathcal{E}^\theta_{n,m}$, there canonically corresponds a connection on the tensor product bundle $\mathcal{E}^\theta_{n,m} \otimes T_{(-\theta)}$, $\mathcal{E}^{1\sigma}_{0,1}$ given by the sum $(\theta - s)\nabla_\mu \otimes \text{id} + \text{id} \otimes D_\mu$ where $D_\mu$ is the canonical constant curvature connection of the Heisenberg module $\mathcal{E}^{1\sigma}_{0,1}$. Explicitly, for $\delta \in \mathcal{E}^\theta_{n,m}$, $\delta' \in \mathcal{E}^{1\sigma}_{0,1}$, $((\theta - s)\nabla_\mu \otimes \text{id} + \text{id} \otimes D_\mu)(\delta \otimes T_{(-\theta)} \delta') = ((\theta - s)\nabla_\mu \otimes T_{(-\theta)} \delta') + \delta \otimes T_{(-\theta)} D_\mu \delta'$. The rescaling by $\theta - s$ is needed in order for the canonical derivations $\partial_{\sigma\mu}$ of $T_{(-\theta)}$ (entering the Leibniz rule for the connection $\nabla_\mu$) to match the derivations $\delta_\mu$ on $\text{End}(\mathcal{E}^{1\sigma}_{0,1}) \simeq T_{(-\theta)}$ induced from the connection $D_\mu$ as in (3.36). Indeed,
this matching insures that the sum \((\theta - s)\nabla_\mu \otimes \text{id} + \text{id} \otimes D_\mu\) is well defined on \(\mathcal{E}^\theta_{n,m} \otimes \mathcal{T}_{(-\theta)} \mathcal{E}^1_{0,1}\), meaning that since the tensor product is over \(T_{(-\theta)}\), the connection on \(\hat{\phi} a \otimes \mathcal{T}_{(-\theta)} \hat{\phi}'\) equals that on \(\hat{\phi} \otimes \mathcal{T}_{(-\theta)} a \hat{\phi}'\), for all \(a \in T_{(-\theta)}\).

As a special case, we can choose \(\nabla_\mu = D_\mu\), where this latter is the canonical constant curvature connection of the Heisenberg module \(\mathcal{E}^\theta_{n,m}\). Then, \(\mathcal{E}^\theta_{n,m} \otimes \mathcal{T}_{(-\theta)} \mathcal{E}^1_{0,1}\) has constant curvature connection \(D_\mu := (\theta - s)D_\mu \otimes \text{id} + \text{id} \otimes D_\mu\), indeed the curvature is easily computed to be \(\mathcal{F}_{12} = i[D_1, D_2] = \frac{1}{2\pi}(\theta - s)_{n-m} \otimes 1\). Furthermore, this curvature coincides with that of the Heisenberg module \(\mathcal{E}^\frac{1}{\pi s}_{m,-n+m}s\). This proves that the equivalence \(\mathcal{E}^\theta_{n,m} \otimes \mathcal{T}_{(-\theta)} \mathcal{E}^1_{0,1} \simeq \mathcal{E}^\frac{1}{\pi s}_{m,-n+m}s\) extends to an equivalence between Heisenberg modules, i.e., modules with constant curvature connections.

Similarly, to any connection \(\nabla_\mu\) on \(\mathcal{E}^\theta_{n,m}\) there canonically corresponds a connection on the tensor product bundle \(\mathcal{E}^\theta_{n,m} \otimes \mathcal{T}_{(-\theta)} \mathcal{E}^{\theta+1}_{1,0}\) given by the sum \(\nabla_\mu \otimes \text{id} + \text{id} \otimes \partial_\sigma\mu\) where \(\partial_\sigma\mu\) is the canonical flat connection of the trivial line bundle \(\mathcal{E}^{\theta+1}_{1,0} \simeq T_{(-\theta)}\). We see that the curvature is unchanged. It is also instructive to consider the case \(\mathcal{E}^\theta_{n,m} \otimes \mathcal{T}_{(-\theta)} \mathcal{E}^{-\theta}_{1,0}\). The canonical flat connection of the right \(T_\theta\)-module \(\mathcal{E}^{-\theta}_{1,0}\) is \(\partial_{-\sigma^1}, \partial_{\sigma^2}\), indeed these are the canonical derivations of \(T_\theta\) with generators \(U_{1}^{T_\theta} = U_1^{-1}, \ U_{2}^{T_\theta} = U_2\), if \(\partial_{\sigma^1}, \partial_{\sigma^2}\) are those of \(T_{(-\theta)}\) with generators \(U_1, U_2\). Matching the derivations \(\delta_\mu\) induced as in (3.35) by the connection \(\partial_{-\sigma^1}, \partial_{\sigma^2}\) of \(\mathcal{E}^{-\theta}_{1,0}\) with the derivations \(\partial_{\sigma^1}, \partial_{\sigma^2}\) of \(T_{(-\theta)}\) (defining the Leibniz rule of the connection \(\nabla_\mu\) of \(\mathcal{E}^\theta_{n,m}\)), we obtain the connection \(-\nabla_1 \otimes \text{id} + \text{id} \otimes \partial_{-\sigma^1}, \ \nabla_2 \otimes \text{id} + \text{id} \otimes \partial_{\sigma^2}\) on the tensor product bundle \(\mathcal{E}^\theta_{n,m} \otimes \mathcal{T}_{(-\theta)} \mathcal{E}^{-\theta}_{1,0}\). Notice that its curvature is the opposite of that of \(\mathcal{E}^\theta_{n,m}\).

As we have seen in the previous section, the Seiberg–Witten map quantizes the Heisenberg modules \(\mathcal{E}_{n,m}\). It is therefore natural to ask if Seiberg–Witten map quantization is compatible with complete Morita equivalence, i.e., if to a given Seiberg–Witten quantization there corresponds a Seiberg–Witten quantization of the \(T\)-dual modules. This is expected since the Seiberg–Witten quantization of \(\mathcal{E}_{n,m}\) is \(\check{\mathcal{E}}_{n,m} = \mathcal{E}^\theta_{n,m}\). We indeed give a positive answer in the following section.

5.1. Compatibility of Seiberg–Witten Maps with \(T\)-duality

Since \(\text{GL}(2,\mathbb{Z})\) is generated by \(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\) and \(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\), corresponding to \(\theta \mapsto -1/\theta, \ \theta \mapsto \theta + 1, \ \theta \mapsto -\theta\), it suffices to prove compatibility with these transformations. For example, the nontrivial dualities \(\theta \mapsto \frac{-1}{\theta - s}, \ s \in \mathbb{Z}\) (including that of the generator \((\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix})\)) are compatible with the Seiberg–Witten quantization maps according to the following commutative diagram.
where the vertical arrow $SW^\theta_{\theta'}$ denotes one of the Seiberg–Witten maps of Theorem 4, from $\theta$ to $\theta'$, cf. (4.10), and similarly for $SW^{\theta}_{\theta'}$, while the horizontal arrow $\otimes_{T(\theta)}(\mathcal{E}_{0,1}^{1-\vartheta}, D_\mu)$ denotes the map of Heisenberg modules with connections to Heisenberg modules with connection obtained via the complete Morita equivalence bimodule $(\mathcal{E}_{0,1}^{1-\vartheta}, D_\mu)$, and similarly for the lower horizontal arrow.

The commutativity of the diagram is due to the definition $\tilde{\theta}' = \tilde{\theta}$ in the Seiberg–Witten map $SW_{\theta'}^{\theta}$. The Seiberg–Witten map also relates the equivalence bimodules with connection $(\mathcal{E}_{0,1}^{1-\vartheta}, D_\mu)$ and $(\mathcal{E}_{0,1}^{1-\vartheta'}, D_\mu)$ used to obtain the $T$-dual modules in the right-hand side of the diagram. Just recall that $(\mathcal{E}_{0,1}^{\theta}, D_\mu) = (\mathcal{E}_{1,1}^{\theta+1}, D_\mu)$ is the module of sections of the $U(1)$-bundle over $T(\frac{1}{\vartheta}+1)$ with charge $m = 1$, cf. (5.1)–(5.3), and conclude that the corresponding Seiberg–Witten map is

\[
\begin{pmatrix}
\mathcal{E}_{0,1}^{1-\vartheta}, D_\mu
\end{pmatrix} \xrightarrow{SW_{\theta'}^{\theta+1}} \begin{pmatrix}
\mathcal{E}_{1,1}^{1-\vartheta+1}, D_\mu
\end{pmatrix} \xrightarrow{SW_{\theta'}^{\theta+1}} \begin{pmatrix}
\mathcal{E}_{0,1}^{1-\vartheta'}, D_\mu
\end{pmatrix}.
\]

(5.11)

The commutativity of the diagrams like (5.10) but for the other two $T$-duality transformations $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is straightforward, just consider the upper horizontal arrows with $(\mathcal{E}_{1,0}^{\theta+1}, D_\mu)$, respectively, $(\mathcal{E}_{1,0}^{\theta}, D_\mu)$, and similarly with $\theta \rightarrow \theta'$ for the lower horizontal arrows. Then, consider $SW_{\theta+1}^{\theta+1}$, respectively, $SW_{-\theta}^{\theta'}$ in the right-hand side of the corresponding diagrams. Correspondingly, the Seiberg–Witten map (5.11) is replaced by $SW_{\theta+1}^{\theta+1} : (\mathcal{E}_{1,0}^{\theta+1}, D_\mu) \rightarrow (\mathcal{E}_{1,0}^{\theta+1}, D_\mu)$, respectively, $SW_{-\theta}^{\theta'} : (\mathcal{E}_{1,0}^{\theta-1}, D_\mu) \rightarrow (\mathcal{E}_{1,0}^{\theta'}, D_\mu)$.

In conclusion we have that Seiberg–Witten maps are compatible with complete Morita equivalence.

Finally, we mention that the Seiberg–Witten map $SW_{\theta}^{\theta+1}$ always differs from a Morita equivalence; however, if we consider only trivial bundles ($m = 0$), then it corresponds to tensoring with $\mathcal{E}_{1,0}^{\theta+1}$, (c.f. (5.5)).
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