Revisiting quantum relativistic effects from phase transition by the catastrophe theory

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Abstract – In this paper we start from the Schrödinger equation to revisit some classical quantum mechanics from the perspective of phase transition process. Here the relativistic effect of particles moving at high speed can be regarded as the phase transition process when the velocity variable increases. Considering that the catastrophe theory could describe qualitatively any phase transition process, we adopt the simplest folding catastrophe type as the potential function in the Schrödinger equation to obtain a revised Schrödinger relativistic equation through the dimensionless analysis first, and then further to derive out the steady-state Klein-Gordon equation and Dirac relativistic equation gradually. These results reveal that the quantum relativistic effect could be considered as the phase transition process, which could be described by adopting the catastrophe models as the potential function in the classical Schrödinger equation.

Introduction. – Due to the fundamentality and complexity of quantum mechanics, its explanation and exploration have always been one of the hotspots in theoretical physics. Jammer has provided a comprehensive classic treatise on fundamental problems and various explanations in quantum mechanics [1].

Dirac proposed that “the present form of quantum mechanics should not be regarded as the final form”, it is just “the best theory one can give so far”, perhaps in the future “there will be an improved quantum mechanics that brings it back to determinism”, but this has to give up some basic ideas that are now considered acceptable [2].

Classical physics from Newton to General Relativity is essentially the theory of various kinds of smooth behavior. However, there are many kinds of jump phenomena in which sudden changes are caused also by smooth alterations. The sudden changes involved were christened by Thom catastrophes, and the techniques involved to cover a broad range of such phenomena in a coherent manner have become known as catastrophe theory [3]. The catastrophe theory is a highly generalized mathematical theory that summarizes the rules of non-equilibrium phase transition by several catastrophe models, which can explain the phenomenon of gradual quantitative change to sudden qualitative change [4]. In one of our previous papers, the general non-equilibrium phase transition process of fluid has been investigated quantitatively by using the catastrophe theory [5]. More recently, further we have derived out a novel kind of partial differential equations to link the quantum dynamics and the classical wave motions by adopting the catastrophe models as the potential function in the Schrödinger equation and through the dimensionless analysis [6]. In this paper we will continue to revisit some classical quantum mechanics from the perspective of phase transition processes by using the simplest catastrophe model and starting from the classical Schrödinger equation.

A revised Schrödinger relativistic equation by the catastrophe theory. – First we start from the time-dependent Schrödinger equation that can be expressed as

\[ i\hbar \frac{\partial \varphi (r,t)}{\partial t} = \left( -\frac{\hbar^2}{2m} \nabla^2 + V (r) \right) \varphi (r,t), \quad (1) \]
where $\nabla$ is Hamilton’s operator, $\varphi$ the wave function, $\hbar$ Planck’s constant, $m$ the rest mass of the particle, and $V(r)$ is the three-dimensional potential function.

To derive out the Dirac relativistic equation from the Schrödinger equation, the relativistic effect of electrons moving at high speed can be regarded as the phase transition process when the velocity variable increases. Therefore the potential function $V(r)$ in eq. (1) could adopt one of the 7 types of elementary catastrophe models [3,6].

For simplicity and without losing generality, by adopting the simplest folding catastrophe type $V(x) = x^n + nx$ (where $x$ denotes the variable and $n$ is the control parameter), the potential function in eq. (1) could be expressed as $V(r) = \frac{n^2}{2m}[x^n + n(r)x]$, and the variable $x$ might be expanded to the form of power exponent product with respect to the related parameters as

$$x \sim m^\alpha h^{\alpha_1} c^{\alpha_2} \omega^{\alpha_3},$$  \hspace{1cm} (2)

where $c$ is the speed of light in vacuum, $\omega$ is the circular frequency, and all the indices $\alpha_1$, $\alpha_2$, $\alpha_3$ and $\alpha_4$ are constants to be determined by the non-dimensional analysis.

In addition, the parameter $n(r)$ might be taken as $n \sim r^\alpha_0$ ($\alpha_0$ is also a constant).

By introducing the three basic dimensions $T(s)$, $L(m)$, and $M$(kg), it is noted that the dimension of $m$ is $M$, the dimensions of $h$, $\omega$ and $c$ are $ML^2T^{-1}$, $T^{-1}$, $LT^{-1}$, respectively. It is noted that the dimension of $x$ is the same as $L^{-2/3}$. Thus the relationships among the power exponents by the dimensionless analysis are listed in table 1.

| $m(\alpha_1)$ | $h(\alpha_2)$ | $c(\alpha_3)$ | $\omega(\alpha_4)$ | $x$ |
|----------------|----------------|----------------|-------------------|-----|
| $L$            | 0              | 2              | 1                 | $-2/3$ |
| $T$            | 0              | $-1$           | $-1$              | 0   |
| $M$            | 1              | 1              | 0                 | 0   |

Let the index $\alpha_4 = 2/3$, due to $E = \hbar \omega$, then eq. (4a) becomes

$$V(r) = \beta \frac{E^2}{mc^2} + B \left( \frac{h^2 E}{c} \right)^{2/3} r^{-4/3}/m.$$  \hspace{1cm} (4b)

Here, we should notice that, although eq. (4b) is obtained by the folding catastrophe type, for other catastrophe types adopted as the potential function in eq. (1), the first term is the same and the second term is in the form of different power exponents of $r$ in eq. (4b), which also shows the phase transition process of the different levels and from different visions.

Substituting eq. (4b) into eq. (1) with rearrangement, we obtain the following time-dependent quantum relativistic field equation:

$$i\hbar \frac{\partial \varphi(r,t)}{\partial t} = \left[ -\frac{\hbar^2}{2m} \nabla^2 + \frac{\beta}{2mc^2} E^2 + f(r) \right] \varphi(r,t),$$  \hspace{1cm} (5)

where $f(r) = B(h^2 E/c)^{2/3}r^{-4/3}/m$ is the potential function under the action fields. Further considering the quality-energy relationship $E^2 = c^2p^2 + m^2c^4$ ($p$ is the momentum), $E^2/(mc^2) = mc^2[1 + c^4 p^2/(m^2 c^4)]$ denotes the degree of relativistic change in the phase transition process.

In the following from eq. (5) we will derive out the steady-state Klein-Gordon equation and Dirac relativistic equation gradually.

The steady-state Klein-Gordon equation and Dirac equation derived. – For the relativistic free particles, eq. (5) becomes

$$i\hbar \frac{\partial \varphi(r,t)}{\partial t} = \left[ -\frac{\hbar^2}{2m} \nabla^2 + \frac{\beta}{2mc^2} E^2 \right] \varphi(r,t).$$  \hspace{1cm} (6)

According to eq. (6), $\varphi(r,t)$ satisfies

$$2mc^2i\hbar \frac{\partial \varphi(r,t)}{\partial t} = [-\hbar^2 c^2 \nabla^2 + 2\beta E^2] \varphi(r,t).$$  \hspace{1cm} (7)

For the plane wave $\varphi(r,t) = \varphi(r) \exp(-iEt/\hbar)$, from eq. (7) we have

$$E^2 \varphi(r) = [-\hbar^2 c^2 \nabla^2 + (2\beta + 1) E^2 - 2mc^2E] \varphi(r).$$  \hspace{1cm} (8a)

Then by rearranging eq. (8a), we have

$$E^2 \varphi(r) = \{-\hbar^2 c^2 \nabla^2 + [(2\beta + 1) E^2 - 2mc^2E - m^2c^4] \} \varphi(r).$$  \hspace{1cm} (8b)

When $(2\beta + 1) E^2 - 2mc^2E - m^2c^4 = 0$, i.e.,

$$\beta = (m^2c^4 + 2mc^2E - E^2)/(2E^2)$$  \hspace{1cm} (9)

eq (8a) becomes

$$E^2 \varphi(r) = \{-\hbar^2 c^2 \nabla^2 + m^2c^4 \} \varphi(r),$$  \hspace{1cm} (10)

which is the steady-state Klein-Gordon equation.

Furthermore, let $\hat{H} = \hbar c(S_1 \frac{\partial}{\partial x} + S_2 \frac{\partial}{\partial y} + S_3 \frac{\partial}{\partial z}) + S_0 mc^2$, where $S_0$, $S_1$, $S_2$ and $S_3$ are all the operators, if the

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following conditions are satisfied:

\[ S_i S_j + S_j S_i = 2 \delta_{ij} \quad (i, j = 1, 2, 3), \]
\[ S_i S_0 + S_0 S_i = 0 \quad (i = 1, 2, 3), \]
then with \( S_0^2 = S_i^2 = S_0^2 = S_i^2 = I \) we have
\[
\hat{H}^2 = -\hbar^2 c^2 \nabla^2 + m^2 c^4.
\]

Conclusions. – The folding catastrophe model is adopted as the potential function of the Schrödinger equation, from which a revised Schrödinger relativistic equation through the dimensionless analysis is derived out. Furthermore, the steady-state Klein-Gordon equation and Dirac relativistic equation are gradually obtained. These results reveal that the quantum relativistic effect could be considered as the phase transition process, and the potential function in the classical Schrödinger equation may adopt the catastrophe models to describe the quantum relativistic effects.

Data availability statement: The data generated and/or analysed during the current study are not publicly available for legal/ethical reasons but are available from the corresponding author on reasonable request.

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