Generalized Sums over Histories for Quantum Gravity

II. Simplicial Conifolds

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ABSTRACT: This paper examines the issues involved with concretely implementing a sum over conifolds in the formulation of Euclidean sums over histories for gravity. The first step in precisely formulating any sum over topological spaces is that one must have an algorithmically implementable method of generating a list of all spaces in the set to be summed over. This requirement causes well known problems in the formulation of sums over manifolds in four or more dimensions; there is no algorithmic method of determining whether or not a topological space is an n-manifold in five or more dimensions and the issue of whether or not such an algorithm exists is open in four. However, as this paper shows, conifolds are algorithmically decidable in four dimensions. Thus the set of 4-conifolds provides a starting point for a concrete implementation of Euclidean sums over histories in four dimensions. Explicit algorithms for summing over various sets of 4-conifolds are presented in the context of Regge calculus.
INTRODUCTION

The sum over histories approach to formulating quantum amplitudes provides a convenient and powerful tool for the study of many aspects of quantum field theories. This approach is especially useful in the study of topology and topology change in Euclidean gravitational amplitudes; Euclidean gravitational histories consist both of a topological space and a metric and thus a sum over histories formulation incorporates contributions from histories of different topology in a very natural manner. For example, quantities such as the transition amplitude between a set of manifolds $\Sigma^{n-1}$ with metrics $h$,

$$G[\Sigma^{n-1}, h] = \sum_{(K^n, g)} \exp(-I[g])$$

$$I[g] = -\frac{1}{16\pi G} \int_{K^n} (R - 2\Lambda) d\mu(g) - \frac{1}{8\pi G} \int_{\Sigma^{n-1}} K d\mu(h)$$

(1)

where $d\mu$ denotes the covariant volume element with respect to the indicated metric are formed by taking the sum over an appropriate set of compact physically distinct $K^n$ and $g$ weighted by the Euclidean Einstein action. Quantities formulated in terms of sums over closed connected topological spaces $K^n$ such as

$$< A > = \frac{\sum_{(K^n, g)} A(g) \exp\left(-I[g]\right)}{\sum_{(K^n, g)} \exp\left(-I[g]\right)}$$

(2)

correspond to the expectation values of geometrical objects $A$. Thus expressions such as (1) and (2) manifestly incorporate contributions from histories of different topology. They are especially useful in the qualitative analysis of topology and topology change because even without a precise implementation of the sum, they can be evaluated in semiclassical approximation. These qualitative studies have led to many very interesting results.\(^1\)

However, the study of topology change and other consequences of topology in semiclassical approximation is limited as semiclassical calculations consider only the contribution from
a certain restricted set of spaces $K^n$; those that are classical Euclidean instantons. Therefore, though a useful qualitative guide, semiclassical evaluations do not yield a method of studying topology in depth as they do not include contributions from all topologies.

In order to go beyond such semiclassical evaluations, it is necessary to replace the heuristic expression (1) with a more concrete definition that explicitly implements the sum over histories with different topology. Of course the goal of finding a complete, well defined sum over histories for Euclidean gravity is out of reach due to well known problems such as the unboundedness of the Euclidean Einstein action and perturbative nonrenormalizability. However, these problems are directly related to properties of the quantum mechanics of the metric alone. Moreover, the topological aspects of forming a sum over histories can be isolated from those involving the metric. Indeed it is easy to observe that the sum over histories for a quantum amplitude such as (2) can be written in the form of an explicit sum over spaces $K^n$ and a functional integral over metrics $g$ on each topological space:

$$<A> = \frac{\sum_{K^n} \int Dg A(g) \exp(-I[g])}{\sum_{K^n} \int Dg \exp(-I[g])}. \tag{3}$$

Thus the topological aspects of formulating a sum over histories can be studied independently of those involving the metric. Moreover, these topological aspects are of particular interest as they are not linked to the dynamics of the metric itself. Therefore the issues that arise in defining a sum over topological spaces such as manifolds or conifolds should be relevant in any theory involving such a sum, not just Einstein gravity.

In attempting to concretely formulate any sum over topologies such as that in (3), two important and intimately related issues must be addressed: what kinds of histories should be included in the sum and whether or not a sum over these spaces can be explicitly carried out. The first issue was addressed in part I (Ref.[2]). The second issue will be the topic of this paper.
As discussed in part I, it is well known that the space of histories for a theory is larger than the set of the classical histories of the theory. For example, the space of histories for a field theory includes nondifferentiable field configurations as well as smooth ones. Similarly, one anticipates that the space of histories for Euclidean gravity also includes some sort of histories that are less regular than classical histories. Now, a classical history in Euclidean gravity consists of both a differentiable metric $g$ and smooth manifold $M^n$. It follows that a less regular history for gravity can be less regular in two different ways; it can consist of a less regular metric $g$, or a less regular topological space $K^n$. Additionally, these two ways can be separated; nondifferentiable metrics can be defined on smooth manifolds and conversely, regular metrics can be defined on smooth topological spaces that are not manifolds. Therefore, it is natural to consider whether or not more general topological spaces than manifolds should be included in a sum over topologies. Moreover, as the topological generalizations are distinct from the metric generalizations, this issue can be studied in the context of classical paths. As discussed in detail in part I, semiclassical results indicate that allowing only manifolds in the sum over histories is too restrictive; limits of sequences of smooth manifolds lead to non-manifold stationary points of the Euclidean action. Given that such spaces occur in the semiclassical limit, it is logical that they should be included in the space of histories for Euclidean functional integrals for gravity. These non-manifold spaces are elements of a more general set of topological spaces called conifolds:

**Definition (1.1).** A $n$-dimensional conifold $X^n$, $n \geq 2$, is a metrizable space such that given any $x_0 \in X^n$ there is an open neighborhood $N_{x_0}$ and some closed connected $(n-1)$-manifold $\Sigma_{x_0}^{n-1}$ such that $N_{x_0}$ is homeomorphic to the interior of a cone over $\Sigma_{x_0}^{n-1}$ with $x_0$ mapped to the apex of the cone.

Thus semiclassical results indicate that the topology of histories in the space of histories for expressions such as (3) should be generalized to include smooth conifolds.
The second issue, how to explicitly carry out a sum over a set of topological spaces such as manifolds or conifolds, will be discussed in this paper. Recall that the standard rule for formulating any sum over histories is that only physically distinct histories should be included in the sum. In Euclidean gravity, two histories \((K^n, g)\) and \((K'^n, g')\) are physically distinct if they have metrics which are not diffeomorphic; however, they are also physically distinct if their underlying topological spaces \(K^n\) and \(K'^n\) are not diffeomorphic. Thus, in formal terms, a sum over physically distinct histories for Euclidean gravity should consist of a sum over smooth topological spaces that are not diffeomorphic to each other. In four or more dimensions, it turns out that topological spaces that are homeomorphic are not necessarily diffeomorphic as they can admit different smooth structures. Thus a sum over physically distinct topological spaces \(K^n\) in (3) must include a sum over inequivalent smooth structures as well as a sum over inequivalent (that is nonhomeomorphic) topological spaces.

Now, in order to make an expression such as (3) concrete, one must replace the heuristic summation sign with a concrete method of taking such a sum. However, even though the number of closed topological spaces is countable, and even though the number of distinct smooth structures on these spaces is countable, it turns out that such a concrete method does not exist in all dimensions or for all sets of topological spaces. For example, consider the formulation of a sum over n-manifolds. One imagines explicitly implementing this sum by first making a list of all physically distinct n-manifolds, that is a list of all manifolds that are not diffeomorphic to each other. One then simply takes the sum in (3) to be over all distinct n-manifolds in the list. However, though such a technique sounds reasonable, it turns out that it cannot be carried out for general dimension n. There is no way to explicitly list all physically distinct n-manifolds for \(n \geq 4\) because whether or not two manifolds are diffeomorphic cannot be determined by a finite procedure; n-manifolds for \(n \geq 4\) are not classifiable. Additionally, there is no known algorithm for classifying 3-
manifolds. Even worse, in five or more dimensions, one can prove that there is no finite algorithm for determining whether or not a given topological space satisfies the definition of a manifold and there is no known finite algorithm for doing so in four dimensions. Moreover, without such a finite algorithm, even the first step in concretely implementing an expression of the form (3) cannot be carried out. Thus, as it stands, expressions such as (3) are not well defined for a sum over smooth n-manifolds in arbitrary dimension.

It turns out that the ability to carry out an explicit formulation of a sum over topological spaces depends intimately on two things: the set of topological spaces at hand and the criteria by which they are to be classified as distinct. Different sets of topological spaces other than the set of manifolds are explicitly known to be algorithmically decidable in four or more dimensions. Additionally there are algorithmically decidable sets that include all n-manifolds as a subset. Thus by choosing a different set of topological spaces, it is possible to explicitly construct a set of spaces that includes all classical histories.

Given a constructible set of more general topological spaces, the next task is to find a set of unique representatives of physically distinct topological spaces. As in the case of n-manifolds, the criteria for doing so is diffeomorphism invariance. However, it turns out that the problems with classifying n-manifolds for \( n \geq 4 \) extends to any set of more general topological spaces that includes all n-manifolds. Thus allowing more general sets of topological spaces only addresses the first factor involved in the explicit construction of sums over physically distinct histories, not the second. Therefore, in order to explicitly formulate a sum over distinct topological spaces, the issue of when two topological spaces are to be identified as distinct must be reexamined.

These observations about algorithmic decidability were used as motivation by Hartle for studying “unruly topologies” in 2-dimensional simplicial gravity.\(^3\) Hartle argued that sums over 2-pseudomanifolds would produce the same qualitative results in the classical limit as sums over 2-manifolds in expressions such as (3). However, little concrete work has
been done in higher dimensions in either formulating sums over more general topological spaces or exploring their consequences; there are many algorithmically decidable spaces so without further information it is difficult to select a viable candidate. However, part I of this paper provides precisely the further information needed to select such a candidate: a physically motivated set of spaces, conifolds. Furthermore, it turns out that conifolds can be shown to be algorithmically decidable in four or fewer dimensions. Thus, the set of smooth conifolds provides a starting point for an explicit implementation of the sum over topologies. Given this starting point, different criteria for classifying conifolds can be formulated and their consequences studied in terms of explicitly constructible amplitudes.

This paper will give a comprehensive discussion of algorithmic decidability and classifiability of both manifolds and conifolds in general dimension and will provide explicit implementations of sums over conifolds in four dimensions. In order to discuss the problems with implementing the sum over topological spaces, it is necessary to have a finite representation of the topological space. A well known method of doing so is to use simplicial complexes. The topology of the simplicial complex is completely carried by the simple set of rules used to assemble it from a countable set of elements, the simplices. Section 2 will discuss simplicial complexes and then present the definitions of combinatorial manifolds and conifolds, the simplicial analogs of continuum manifolds and conifolds. Section 3 will describe precisely how these simplicial analogs are related to their smooth counterparts. It turns out that in less than seven dimensions, the set of smooth manifolds uniquely corresponds to that of combinatorial manifolds and likewise for the conifold case. Therefore combinatorial manifolds and conifolds are the desired finite representations of smooth manifolds and conifolds in less than seven dimensions. Additionally, a sum over combinatorial spaces automatically incorporates a sum over smooth structures in less than seven dimensions. Section 4 will begin by discussing the algorithmic decidability and classifiability of manifolds. Certain important points about these results that are usually brushed
over in discussions of sums over manifolds in Euclidean gravity will be emphasized; these points are especially relevant and a misunderstanding of them can lead to erroneous claims. After this discussion of the manifold case, the results on the algorithmic decidability and classifiability of conifolds will be derived. Section 5 will discuss the consequences of the results of section 4 on the definition of Euclidean functional integrals for gravity. It will illustrate these consequences in terms of Regge calculus and provide explicit algorithms for finding different sets of distinct 4-conifolds for implementing these sums. However, it is important to note that, although Regge calculus is especially useful for studying effects of topology and topology change numerically, the results on the explicit formulation of sums over topology apply generally due to the results of section 3.

2. COMBINATORIAL MANIFOLDS
AND COMBINATORIAL CONIFOLDS

It is useful to begin by summarizing certain definitions and theorems on simplicial complexes; even though simplicial complexes are well known and often used in topology, there are many instances of different authors using the same terminology to refer to slightly different objects. Thus it is best to explicitly present the definitions used for the reader’s understanding. After this summary, the definition of a combinatorial manifold and that of a combinatorial conifold are provided. Finally, certain aspects of these definitions particularly relevant to their use in this paper are discussed.

Simplicial complexes are a subset of the set of polyhedra discussed in section 4 of part I. Thus it is useful to recall the definition of a polyhedra. First note that two subspaces $X$ and $Y$ of $\mathbb{R}^n$ are said to be in general position if for any distinct points $x_1, x_2 \in X$ and $y_1, y_2 \in Y$, the line segments connecting $x_1$ to $y_1$ and $x_2$ to $y_2$ do not intersect. With this terminology,
Definition (2.1). Let two subspaces $X$ and $Y$ of $\mathbb{R}^n$ be in general position. Their PL join is the union of all line segments joining points of $X$ to points of $Y$. The join will be denoted $XY$.

A PL cone is defined to be PL join of space $X$ with a point $a$. It will be denoted by $aX$.

Then, given the definition of a PL cone,

Definition (2.2). A polyhedron $P$ is a subset of $\mathbb{R}^n$ such that each point $p$ has a cone neighborhood $N = aL \subseteq P$ where $L$ is a compact topological space. $L$ is called the link of the neighborhood $N$.

The term polyhedra will be used in the strict sense of Def.(2.2) in this paper in contrast to its usage in part I. One can show that as topological spaces, the PL join when it exists is homeomorphic to the topological join $X \ast Y$. This means that as topological spaces, there is no difference between the PL join and the topological join. The difference is that the PL join has more structure and can only be defined when the two spaces can be positioned in this nice way. In order to see this, note that the natural equivalence of a polyhedra is given by piecewise linear maps, referred to as PL maps.

Definition (2.3). A map $f : P \rightarrow Q$ between two polyhedra is piecewise linear if each point $a$ in $P$ has a cone neighborhood $N = aL$ in $P$ such that $f(\lambda a + \mu x) = \lambda f(a) + \mu f(x)$ where $x$ is in $L$ and $\lambda, \mu \geq 0$, $\lambda + \mu = 1$.

It is clear that a PL map from a join to itself cannot be defined for a join that does not satisfy the conditions of Def.(2.1). Thus the definition of a PL map characterizes the extra structure inherent in the definition of a PL join. Finally, of particular interest are PL homeomorphisms, that is PL maps that are continuous and have a continuous PL inverse. Two polyhedra are said to be PL equivalent if there is a PL homeomorphism between them.
Given the previous definitions, the first step in defining a simplicial complex is to define a simplex.\textsuperscript{6}

**Definition (2.4).** Let $v_1, v_2 \ldots, v_{n+1}$ be affinely independent points\textsuperscript{7} in $\mathbb{R}^{n+1}$. An $n$-simplex $\sigma^n$ is the convex hull of these points:

$$\sigma^n = \{x \mid x = \sum_{i=1}^{n+1} \lambda_i v_i; \lambda_i \geq 0; \sum_{i=1}^{n+1} \lambda_i = 1\}.$$ 

A 0-simplex is a point, a 1-simplex is a line segment or edge, a 2-simplex is a triangle, a 3-simplex is a tetrahedra. Higher dimensional simplices are generalizations of tetrahedra to higher dimensions. A simplex spanned by a subset of the vertices is called a face. Thus the vertices are all faces of a n-simplex; they are 0-simplices. Similarly, 1-simplices or edges formed from any two distinct vertices are faces. Note that by Def.(2.4), a simplex is uniquely determined by its vertices. This property is very important for both using and understanding simplicial complexes:

**Definition (2.5).** A simplicial complex $K$ is a topological space $|K|$ and a collection of simplices $K$ such that

i) $|K|$ is a closed subset of some finite dimensional Euclidean space

ii) if $F$ is a face of a simplex $K$, then $F$ is also contained in $K$

iii) if $B, C$ are simplices in $K$, then $B \cap C$ is a face of both $B$ and $C$.

The topological space $|K|$ is the union of all simplices in $K$.

A particularly simple example of a simplicial complex is provided by the square in Figure 1a). It consists of four 0-simplices or vertices, five 1-simplices and two 2-simplices. The rules for constructing the square are given by enumerating which vertices and edges are contained in which triangle. This set of rules encodes the topology of the square; these rules are conveniently given the drawing itself, i.e. the two triangles have one edge in common. An alternate method of giving these rules is to provide a list of which vertices are...
contained in each simplex of the complex as indicated in Figure 1a). As higher dimensional simplices are uniquely specified by their vertices, note that it is not really necessary to label them in any way other than by the subset of the vertices they contain.

Observe that simplicial complexes form a subset of the set of polyhedra of Def.(2.2). However, some spaces that are homeomorphic to polyhedra are not simplicial complexes because they do not satisfy all the conditions of Def.(2.5). For example, a nice one dimensional polyhedron is a circle represented by two line segments connected at each endpoint in Figure 1c). This space is homeomorphic to a polyhedron as it is a closed subset of $\mathbb{R}^2$; however it is not a simplicial complex. Both line segments have the same endpoints $a, b$ and thus are not uniquely determined by these endpoints. This example may seem obvious; however, it is easy to forget this important property of simplicial complexes when dealing with higher dimensional spaces. Finally, another simplicial description of the same topological space as Figure 1a) is given in Figure 1b) and some other examples of simplicial complexes are given in Figures 2-6.

The dimension of a simplicial complex is the largest dimension of any simplex contained in the complex. Condition $i)$ implies that the simplicial complexes of Def.(2.5) are finite dimensional. One can define infinite dimensional simplicial complexes by changing this condition. However, this paper is concerned with finite dimensional spaces, so such a change is unnecessary. Similarly, abstract simplicial complexes can be defined by using the last two conditions in Def.(2.5) and replacing the first condition with some topology for the space $|K|$ other than that induced by Euclidean space (See Appendix A). However, there is generally no advantage to do this for finite dimensional complexes because, if the topology used to define the complex is reasonable, then the simplicial complex can be realized as a subset of Euclidean space.

Thus, a simplicial complex describes both the building blocks of the space $|K|$ and gives the rules for how these building blocks are connected together. Consequently the
simplicial complex completely describes the topology of the space $|K|$. Finally, as each simplex in $K$ is uniquely determined by its vertices, the simplicial complex itself is uniquely determined the vertices and the rules for which simplices they are contained in. It is clear that this property is especially valuable for computational purposes.

Simplicial complexes can describe topological spaces containing subspaces of different dimension, compact and noncompact spaces, and spaces with boundary. Thus, there are simplicial analogs of various standard definitions in topology: A *pure* simplicial complex is one in which every lower dimensional simplex is contained in at least one $n$-simplex where $n$ is the dimension of the simplicial complex. A *compact* simplicial complex is one which contains a finite number of simplices. A *connected* simplicial complex is one in which any two vertices are connected by a sequence of edges. One can easily verify that an equivalent definition of a connected complex is that the underlying space $|K|$ of the complex is a connected topological space. The properties of these definitions are illustrated by the examples of Figure 2: All of the simplicial complexes drawn are compact. All of the simplicial complexes are pure except Figure 2a); the flagpole is not contained in any triangle. All of the simplicial complexes are connected except Figure 2b). It is easy to see that these properties can all be tested for simplicial complexes presented in the form of a list as well these pictorial representations.

It is also interesting to define the simplicial analog of continuum manifolds and conifolds with boundary. It is clear that the notion of boundary in the sense used in the continuum theory can only be made meaningful for a subset of pure complexes. It turns out that there is an additional requirement needed to specify this subset:

**Definition (2.6).** A nonbranching simplicial complex is a simplicial complex of dimension $n$ in which every $(n-1)$-simplex is contained in at most two $n$-simplices.

For example, Figure 2c) is a branching simplicial complex as it has an edge that is contained
in four triangles. Then

**Definition (2.7).** The **boundary of a pure nonbranching simplicial complex** is the set of 
\((n-1)\)-simplices that are faces of only one \(n\)-simplex.

The boundary of fig.(1a) is composed of the four edges \(\alpha, \beta, \gamma, \delta\) that are in only one triangle. Figures 2b) and 2d) have no boundary by the above definition. Note especially 
that the common vertex of the two tetrahedra of Figure 2d) is not a boundary as a boundary 
is a \((n-1)\) dimensional simplicial complex by definition. This is completely in keeping with 
continuum definition of boundary.\(^8\)

Def.(2.7) implies that a pure nonbranching complex without boundary is one in which 
every \((n-1)\)-simplex is contained in exactly two \(n\)-simplices. Therefore, again there is an 
easily implementable test for boundary in terms of any concrete description of the complex. 
Finally, it can be proven that the boundary of a pure nonbranching simplicial complex 
is a topological invariant of the underlying space; it is the nonbranching condition that 
provides this necessary and desirable property. Therefore, Def.(2.7) is the desired analog 
of the continuum definition of boundary.

From the above examples, it is apparent that there are many simplicial complexes which 
describe the same space. First note that given any simplicial complex \(K\), a **subdivision** 
of \(K\) is a simplicial complex \(L\) such that \(|K| = |L|\) and any simplex in \(L\) is contained in 
a simplex in \(K\). Then two simplicial complexes \(K\) and \(K'\) are **combinatorially equivalent** 
if both can be subdivided so that the simplices of the resulting subdivisions \(L\) and \(L'\) 
can be put into one to one correspondence by relabeling the vertices. For example, the 
complexes in Figures 1a) and 1b) are combinatorially equivalent; in fact Figure 1b) is 
itself a subdivision of Figure 1a) and thus the combinatorial equivalence follows trivially. 
Another example is given by the simplicial complexes in Figure 3; here both must be 
subdivided in order to make the one to one correspondence of the vertices.
The combinatorial equivalence of complexes is related to homeomorphisms of the simplicial complexes which preserve their linear structure; since simplicial complexes are polyhedra, PL maps are well defined on them and such homeomorphisms are PL homeomorphisms. In fact, one can easily show that if two simplicial complexes are PL homeomorphic, then they are combinatorially equivalent. One can also define a more restricted PL map between simplicial complexes called a simplicial map: A simplicial map \( f : K \to K' \) is a PL map that maps simplices of \( K \) to simplices of \( K' \) and is linear on each simplex. Simplicial maps are more restricted in that they are determined entirely by their behavior on the vertices of the complex. Consequently, simplicial homeomorphisms are simply permutations or relabeling of the vertices of a simplicial complex. Thus one can restate the definition of combinatorial equivalence of two complexes in the following way; \( K \) and \( K' \) are combinatorially equivalent if their subdivisions \( L' \) and \( L' \) are related by a simplicial homeomorphism.

At this point it is possible begin defining special sets of simplicial complexes. Historically, a subset of pure nonbranching simplicial complexes has been studied because their homology has many properties similar to that of manifolds. They are given the special name of pseudomanifolds:

\textbf{Definition (2.8).} A pseudomanifold \( P^n \) is a pure nonbranching simplicial complex such that i) any two \( n \)-simplices can be connected by a sequence of \( n \)-simplices, each intersecting along some \( (n-1) \)-simplex.

The reason for the further requirement on pure nonbranching simplicial complexes is so that the \( n \)th homology group of a pseudomanifold, \( H_n(P^n) \), has a single generator. For closed pseudomanifolds, this condition implies that the \( \mathbb{Z}_2 \) homology always satisfies \( H_n(P^n; \mathbb{Z}_2) = \mathbb{Z}_2 \). This is equivalent to saying that all closed pseudomanifolds are \( \mathbb{Z}_2 \) orientable. In fact the homology groups yield an equivalent description of closed pseudoman-
ifolds, namely, a pure nonbranching closed simplicial complex is a closed pseudomanifold if and only if $H_n(P^n; \mathbb{Z}_2) = \mathbb{Z}_2$. If the simplicial complex is orientable this is equivalent to $H_n(P^n) = \mathbb{Z}$. A disadvantage of condition i) is that the boundary of a pseudomanifold is not necessarily itself a pseudomanifold; the boundary can fail to be connected and hence fail to satisfy Def.(2.8). The homology properties of pseudomanifolds that follow from Def.(2.8) mean that these spaces are very similar to manifolds; indeed both connected manifolds and conifolds are subsets of the set of pseudomanifolds. Figures 1a), 3a) and 3b) are all examples of pseudomanifolds as they are homeomorphic to manifolds. Figure 4b) is an example of a pseudomanifold that is not homeomorphic to a manifold. Finally, note that Figure 2d) is pure and nonbranching but is not a pseudomanifold as it does not satisfy condition i).

In order to study subsets of pseudomanifolds, three more definitions are required. These definitions are used to characterize the local topology of simplicial complexes and thus provide the means for defining simplicial equivalents of smooth manifolds and conifolds.

**Definition (2.9).** The combinatorial star $St(v)$ of a vertex $v$ is the complex consisting of all simplices that contain $v$.

**Definition (2.10).** The combinatorial link $L(v)$ of a simplex is the subset of the star of $v$ consisting of all simplices in the star that do not intersect $v$ itself.

For example, the star of vertex $f$ in Figure 3a) consists of the four triangles $fab$, $fbc$, $fcd$, $fda$ that contain $f$ and their constituent edges and vertices. The link of this vertex is the set of vertices and edges $a$, $b$, $c$, $d$, $ab$, $bc$, $cd$, $da$ that form a square. Thus the link of $f$ is homeomorphic to a circle. Secondly, consider the vertex $a$ of the pinched torus in Figure 4b). Its star consists of the six triangles $acd$, $acb$, $abc$, $aeg$, $agf$, $aef$ and their faces. Its link consists of the two disconnected subsets $b$, $c$, $d$, $cd$, $bc$, $bd$ and $e$, $f$, $g$, $ef$, $fg$, $eg$ that are homeomorphic to two disjoint circles. Thus, the difference in the topology of the
neighborhoods of these two vertices is carried by the difference in their links. In general it will be seen that the topology of a neighborhood of a vertex will be closely related to the properties of its link.

Using Def.(2.1), it is easily verified that the underlying space of the star \( |St(v)| \) is the PL cone over \( |L(v)| \) with apex \( v \). The relations between the combinatorial links, stars, and PL cones helps to define a simplicial cone of a simplicial complex in the following way:

**Definition (2.11).** The simplicial cone of a simplicial complex \( K \) with apex \( v \), \( C(K) \), is the simplicial complex consisting of the simplicial cones of each \( \sigma \in K \) with apex \( v \); that is it consists of all simplices containing the vertex \( v \) and subsets of the vertices of \( K \) corresponding to the simplices of \( K \).

The simplicial cone of Figure 1a) is a complex consisting of two tetrahedra and all of their faces as illustrated in Figure 5a). Similarly, the simplicial suspension of a simplicial complex, \( S(K) \), can be defined by gluing together two simplicial cones \( C(K) \) along their common boundary; an example of a suspension is illustrated in Figure 5b). Observe that the PL cone of a \( k \)-simplex with apex \( v \) is \((k+1)\)-simplex; for this special case the correspondence of simplicial cones to PL cones is obvious. In general, the underlying space of the simplicial cone \( C(K) \) is the PL cone over the space \( |K| \), that is \( C(|K|) \); in other words \( |C(K)| = C(|K|) \). Similarly, the underlying space of the simplicial suspension satisfies \( |S(K)| = S(|K|) \). Thus a simplicial cone is a simplicial representation of the space corresponding to a PL cone.

Given these definitions, the simplicial counterparts of various continuum sets of topological spaces can now be defined. For ease of presentation, the definitions given below will be for spaces without boundary. However, note that their generalizations to the case with boundary are obvious. First it is very useful to give the simplicial counterpart of the continuum definition of a homology \( n \)-manifold which is the following:
Definition (2.12). A metric space $Q^n$ is a homology $n$-manifold if and only if each point $x_0 \in Q^n$ has a neighborhood $N_{x_0}$ which is homeomorphic to a topological cone over a compact space $L_{x_0}$ which has the same integer homology as $S^{n-1}$.

Observe that although all manifolds are also homology manifolds, homology manifolds are not necessarily manifolds; in fact they are not necessarily conifolds as the compact space $L_{x_0}$ need not be a closed (n-1)-manifold! However, there is a close correspondence between the homology properties of a manifold and those of a homology manifold. Because of this close correspondence, it is common to generalize the definition of a homology sphere to include not only manifolds but homology manifolds with the same integer homology as a sphere. This convention will be used in Thm.(2.15).

The simplicial counterpart of Def.(2.12) is a subset of pseudomanifolds:

Definition (2.13). A combinatorial homology $n$-manifold is a $n$-pseudomanifold for which the link of every vertex has the same integer homology as an $(n-1)$-sphere.

An example of a combinatorial homology $n$-manifold is given later in this section. Although not proven here, one should note that any simplicial complex which is homeomorphic to a homology manifold as given by Def.(2.12) satisfies Def.(2.13).

At this point $n$-manifolds can now be defined:

Definition (2.14). A combinatorial $n$-manifold is a $n$-pseudomanifold for which the link of every vertex is a combinatorial $(n-1)$-sphere.

The main reason for defining combinatorial manifolds in the above way is that it is a homogeneous definition; namely, no vertex of the simplicial complex has preferred treatment and the links of the vertices are all homeomorphic. This is similar to the idea of a topological manifold where each point has a neighborhood homeomorphic to a ball. As a necessary
and sufficient condition for the star of a vertex to be a combinatorial n-ball is for the link to be a combinatorial (n-1)-sphere, the definition of a combinatorial n-manifold can be phrased in terms of the links. It is easy to find examples of combinatorial manifolds. For example figure 3a) is a combinatorial manifold. The link of every vertex is topologically a circle. However, Figure 4b) is not a combinatorial manifold as the link of vertex a is topologically two disjoint circles.

It follows immediately from Def.(2.14) that combinatorial manifolds are homeomorphic to topological manifolds. However, it is important to note that there are simplicial complexes that are not combinatorial manifolds that are homeomorphic to topological manifolds as well. One can readily construct such a simplicial complex which is a topological manifold but is not a combinatorial manifold. Consider space \( SO(3)/I \) in which all points of the group \( SO(3) \) which differ by an element of the icosahedral group \( I \) are identified. (Recall that the icosahedral group is the group of symmetries of a icosahedron.) The resulting space \(|\Sigma|\) is a closed smooth 3-manifold because \( SO(3) \) is a Lie group and \( I \), being a finite subgroup of \( SO(3) \), must act freely. Furthermore, \(|\Sigma|\) is homeomorphic to a combinatorial manifold \( \Sigma \) and has the same integer homology as a 3-sphere. The PL suspension of \(|\Sigma|\), \( S(|\Sigma|) \) is not homeomorphic to a manifold because \(|\Sigma|\) is not simply connected; it follows from the construction of \(|\Sigma|\) that the fundamental group is the binary icosahedral group. This implies that the simplicial suspension \( S(\Sigma) \) is not a combinatorial manifold; in fact it is not topologically a manifold, but rather is a homology manifold. However, the double suspension \( S^2(|\Sigma|) \), that is the PL suspension of \( S(\Sigma) \), is homeomorphic to a 5-sphere. This follows from the double suspension theorem\textsuperscript{10} or by explicit construction. However, even though the space \( S^2(|\Sigma|) \) is homeomorphic to a 5-sphere, the simplicial complex \( S^2(\Sigma) \) is not a combinatorial manifold. In order to see this, note that the apex of either of the two cones in the second suspension has a link \( S(\Sigma) \) which is not a 4-sphere or even a 4-manifold. Thus \( S^2(\Sigma) \) does not satisfy Def.(2.14). Therefore the
The definition of combinatorial manifold carries more structure than simply the topology.

The above example raises the issue of how general will a simplicial complex be if it is homeomorphic to a n-manifold. As just demonstrated, the links of vertices of such a simplicial complex can be nonmanifolds. However, it turns out that the links cannot be arbitrary; one can verify that the link of every vertex of a simplicial complex homeomorphic to a n-manifold has the same integer homology as a sphere even though the link is not necessarily a manifold. The following theorem gives some necessary conditions on the simplicial complex.

**Theorem (2.15).** Given a connected simplicial complex $K^n$ such that $|K^n|$ is homeomorphic to a closed n-manifold, then $K^n$ is a combinatorial homology n-manifold. Furthermore, if $n \geq 3$, then the link of each vertex is also simply connected.

First, the simplicial complex $K^n$ must be pure; otherwise there would be points in $K^n$ which have neighborhoods with dimension less than $n$. Second, it must be nonbranching because neighborhoods of points on (n-1)-simplices at which three or more n-simplices meet are not homeomorphic to the interior of a $n$-ball. Hence, $K^n$ is pure and nonbranching. Since $|K^n|$ is a manifold and all closed manifolds are $\mathbb{Z}_2$ orientable, it follows that $H_n(|K^n|; \mathbb{Z}_2) = \mathbb{Z}_2$. Therefore, $K^n$ is a closed pseudomanifold.

Next, let $v_0$ be any vertex in $K^n$ and let $U = |K^n| - |St(v_0)|$; then the following relative homology groups satisfy

$$H_*(|K^n|, |K^n| - \{v_0\}) = H_*(|K^n| - U, (|K^n| - \{v_0\}) - U)$$

by the excision property.\(^{11}\) Observe that $|K^n| - U = |St(v_0)|$ and that $(|K^n| - \{v_0\}) - U = |St(v_0)| - \{v_0\}$ which is homotopic to $|L(v_0)|$. Thus it follows from (4) that

$$H_*(|St(v_0)|, |L(v_0)|) = H_*(|K^n|, |K^n| - \{v_0\}).$$
However, $|K^n|$ is a manifold so

$$H_*(|K^n|, |K^n| - v_0) = H_*(B^n, S^{n-1}).$$

(6)

Since $B^n$ is contractible, the exact sequence for relative homology groups implies that $H_k(B^n, S^{n-1}) = H_{k-1}(S^{n-1})$. Hence, $H_k(|St(v_0)|, |L(v_0)|) = H_{k-1}(S^{n-1})$. Using the exact sequence again with the contractibility of $|St(v_0)|$, it follows that $H_k(|St(v_0)|, |L(v_0)|) = H_{k-1}(|L(v_0)|)$. Hence, $H_*(|L(v_0)|) = H_*(S^{n-1})$. Therefore, $K^n$ is a combinatorial homology manifold.

Finally, let $v_0$ be any vertex in $K^n$ where $n \geq 3$, then $C(|L(v_0)|)$ is a manifold by assumption. Observe that if $M^n$ is any manifold with $n \geq 3$, then $\pi_1(M^n - \{p\}) = \pi_1(M^n)$ for point $p \in M^n$. This is due to the fact that any curve can be moved around an isolated point in three or more dimensions without intersecting the point. Hence, $\pi_1(C(|L(v_0)| - \{a\}) = \pi_1(C(|L(v_0)|)) = 1$ where $a$ is the apex of the cone. Furthermore, $\pi_1(C(|L(v_0)| - \{a\}) = \pi_1(I \times |L(v_0)|) = \pi_1(|L(v_0)|)$. Therefore, $\pi_1(|L(v_0)|) = 1$. Q.E.D.

Finally, the simplicial counterparts of topological conifolds as defined in Def.(1.1) can be presented. By analogy with Def.(2.14) for a combinatorial manifold, combinatorial conifolds are defined as follows:

**Definition (2.16).** A combinatorial $n$-conifold is a $n$-pseudomanifold for which the link of every vertex is a closed connected combinatorial $(n-1)$-manifold.

Clearly, all of the singular points of a combinatorial conifold are a subset of the set of vertices of the simplicial complex. Hence, a combinatorial conifold is a manifold everywhere except possibly at a countable set of vertices. This parallels the definition of a topological $n$-conifold, for which the neighborhoods of all but a countable set of points are homeomorphic to $n$-balls. The requirement that the links of vertices be combinatorial $(n-1)$-manifolds is the natural extension of the requirement that the links be combinatorial $(n-1)$-spheres.
in the definition of combinatorial manifold. The class of combinatorial n-conifolds quite clearly includes all combinatorial n-manifolds by definition; again as in the continuum case, the class of combinatorial n-conifolds differs from that of n-manifolds only for \( n \geq 3 \). Thus Def.(2.16) is the logical analog of Def.(2.14). Figure 6b) provides an example of a combinatorial conifold that is not a combinatorial manifold.

As for combinatorial manifolds, a connected combinatorial conifold is a pseudomanifold by definition. However, again the converse is not true. The simplicial suspension of any n-pseudomanifold will be a \((n+1)\)-pseudomanifold; for example the simplicial suspension of the pinched torus in Figure 5b) is a 3-pseudomanifold. However, in general such suspensions will not be combinatorial conifolds as the links of vertices need not be connected manifolds. Indeed, the link of the apex of the suspension of Figure 4b) is clearly not a manifold. Furthermore, all simplicial complexes that are topologically n-conifolds are not necessarily combinatorial n-conifolds. This follows immediately from the fact that the set of combinatorial n-conifolds includes all combinatorial n-manifolds. Again the double suspension \( S^2(\Sigma) \) provides an easily understood example; recall that the links are not combinatorial 4-manifolds. Therefore, \( S^2(\Sigma) \) fails to be a combinatorial 5-conifolds as well as a combinatorial 5-manifold. It is easy to persuade oneself that a similar construction can be done to form more general examples of simplicial complexes that are topologically conifolds but fail to be combinatorial conifolds. Thus, combinatorial conifolds, like combinatorial manifolds, are simplicial counterparts of topological conifolds with additional nice structure. The precise nature of this nice structure will be discussed in the next section.

3. TRIANGULATION OF MANIFOLDS AND CONIFOLDS

Obviously, combinatorial manifolds and conifolds as defined in Def.(2.14) and Def.(2.16) are all topological manifolds and conifolds. However, as seen in the last section, not all
simplicial complexes that are topological manifolds and conifolds are actually combinatorial manifolds and conifolds. Finally, a priori, it is not clear what the relationship is between these combinatorial spaces and the corresponding smooth versions. Therefore, an explicit characterization of the connection of smooth manifolds and conifolds to their combinatorial counterparts is desirable and necessary. Indeed one anticipates that a close connection between smooth and combinatorial spaces exists; very much like smooth spaces, combinatorial spaces have a nice structure that will enable integration and differentiation to be well defined as necessary for physical applications. It turns out that all smooth manifolds and conifolds have combinatorial counterparts and in less than seven dimensions, any combinatorial manifold or conifold corresponds to a unique smooth manifold or conifold respectively.

In order to discuss the connection of smooth spaces to combinatorial spaces, the first concept needed is that of a triangulation. Given any topological space $P$, a triangulation consists of a simplicial complex $K$ and a homeomorphism $t : |K| \rightarrow P$. One can show that all polyhedra as defined in Def.(2.2) admit a triangulation. Moreover, any topological space that admits a triangulation is homeomorphic to a polyhedron. Therefore spaces that admit a triangulation are nice in the sense that they have the same properties as polyhedra. A combinatorial triangulation of a manifold $M^n$ consists of a combinatorial manifold $K^n$ and a homeomorphism $t : |K^n| \rightarrow M^n$. Similarly, a combinatorial triangulation of a conifold $X^n$ consists of a combinatorial conifold $K^n$ and a homeomorphism $t : |K^n| \rightarrow X^n$.

It is important to note that not all triangulations of manifolds are combinatorial triangulations as was illustrated with the $S^2(\Sigma)$ example of the previous section. Such non-combinatorial triangulations of manifolds are referred to as weak triangulations. However, note that a direct consequence of Thm.(2.15) and the observed properties of manifolds and pseudomanifolds is that all weak triangulations of closed n-manifolds for $n \leq 3$ are in fact combinatorial triangulations. In one dimension, this result follows by construction. In two
and three dimensions, recall that by Thm.(2.15), the links of a weak triangulation of a
n-manifold must be homology spheres. However, in dimensions one and two, a homology
sphere is a combinatorial sphere; this observation is trivial in one dimension and in two
dimensions follows immediately by recognizing that the only 2-pseudomanifold with the
same integer homology as a 2-sphere is the 2-sphere itself. Therefore, in one, two and
three dimensions, all weak triangulations are in fact combinatorial triangulations. The
example of the 5-sphere given in section 2 already shows this result fails in five dimensions
and it is unsolved for 4-manifolds.

The first task is to show that every smooth manifold and conifold admit combinatorial
triangulations. Intuitively, one should expect this to be true because these spaces admit
a smooth atlas by definition and it seems that by judicious choice, the smooth charts can
be taken to correspond to the simplices of a combinatorial manifold.

In order to prove that all smooth n-manifolds have triangulations, assume that $M^n$ is
a closed smooth n-manifold. By a standard embedding theorem it can be smoothly em-
bedded in $\mathbb{R}^{2n+1}$ as a closed subset. Note that $M^n$ inherits a metric and curvature from
the embedding. Next observe that $\mathbb{R}^{2n+1}$ has a nice family of triangulations consisting of
combinatorial manifolds $K$ and a simplicial homeomorphism; i.e. this family of triangu-
lations consists of tessellating $\mathbb{R}^{2n+1}$ with simplices. Pick one of these triangulations of
$\mathbb{R}^{2n+1}$ such that the size of each simplex in the image of $K_0$, $\text{Im}(K_0)$, is small compared
to the curvature of $M^n$; in other words, choose the simplices of the triangulation to be
small enough that the manifold appears to be approximately flat inside the simplices. Ob-
serve that $M^n$ intersects $\text{Im}(K_0)$ and the triangulation can be chosen such that $M^n$ is in
general position with respect to the n-simplices of $\text{Im}(K_0)$. This is easy to see as if $M^n$
does not intersect $\text{Im}(K_0)$ in general position for the initial choice of triangulation, the
vertices of $\text{Im}(K_0)$ can be moved so that it does. But moving the vertices corresponds to a
simplicial homeomorphism, i.e. to another nice triangulation, therefore one can choose this
triangulation for $\mathbb{R}^{2n+1}$ from the start. Since it is in general position with respect to $M^n$, no interiors of simplices in $Im(K_0)$ with dimension less than $(n+1)$ intersect $M^n$. This result combined with the fact that the size of the simplices of $Im(K_0)$ is small compared to the curvature of $M^n$ means that each $(n+1)$-simplex which has non-empty intersection with $M^n$ intersects it in a unique point which is in the interior of the $(n+1)$-simplex. Furthermore, if the simplices are chosen small enough, then the intersection of $M^n$ with each $(2n+1)$-simplex is a convex n-ball. The collection of all of these n-balls will yield curved simplices which correspond to the image of a simplicial complex $K^n$ homeomorphic to $M^n$. Thus any closed smooth n-manifold has a combinatorial triangulation. Observe that the homeomorphism is actually a smooth map on the interior of every simplex by construction; therefore, this triangulation of the n-manifold is particularly well behaved.

In order to prove that smooth manifolds that are not compact have triangulations, note that the compactness of $M^n$ is used in picking the triangulation of $\mathbb{R}^{2n+1}$ to be small relative to the curvature of the manifold. If the manifold is not compact, the curvature may become larger as one moves toward infinity. However, note that the triangulation can be chosen such that the images of the simplices are shrinking as a function of distance. Then the rest of the above argument for closed n-manifolds follows through. A similar technique also applies in the case of a smooth manifold with boundary. Thus, in all cases the above method produces a combinatorial triangulation of the smooth manifold.

Given the result that all smooth n-manifolds have combinatorial triangulations, it immediately follows that all smooth n-conifolds $X^n$ have them too. Recall that the singular set $S$ of a n-conifold is the set of points whose neighborhoods are not homeomorphic to the interior of a cone over a (n-1)-sphere. Delete conical neighborhoods of the singular set $S$ and then triangulate the resulting smooth manifold with boundary, $X^n - N(S)$. Finally, take a simplicial cone as given in Def.(2.11) of each boundary (n-1)-manifold to yield a combinatorial triangulation of the smooth n-conifold $X^n$. Therefore, any smooth manifold
or conifold has a combinatorial triangulation; that is every smooth manifold or conifold has a combinatorial counterpart.

The next step is to find the conditions for which the converse of the above statement is true; that is under what conditions does a combinatorial manifold or conifold have a smooth counterpart. In order to do so, it is useful to introduce a more general set of manifolds than smooth manifolds, namely PL manifolds.

**Definition (3.1).** A topological manifold $M^n$ is a PL manifold if and only if there is an atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \Lambda}$ such that the mapping

$$
\varphi_\beta \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \rightarrow \varphi_\beta(U_\alpha \cap U_\beta)
$$

is a PL mapping between subsets of $\mathbb{R}^n_+$.

Note that there is no ambient embedding space needed in the definition of a PL manifold, in contrast to the definition of a combinatorial manifold. Additionally,

**Theorem (3.2).** Combinatorial manifolds are equivalent to PL manifolds.

Let $K^n$ be a combinatorial n-manifold; then $|K^n|$ is a PL manifold with $K^n$ as its triangulation. This can be demonstrated by defining a collection of neighborhoods $U_v = St(v) - L(v)$ for each vertex $v$ in $K^n$. These charts cover $|K^n|$ and since each $U_v$ is an open subset of the polyhedron $|K^n|$, it follows that each $U_v$ is itself a polyhedron. Furthermore, each $U_v$ is PL homeomorphic to the interior of a standard n-simplex $\sigma^n$ as $K^n$ is a combinatorial n-manifold. Choose a set of PL homeomorphisms $\varphi_v : U_v \rightarrow |\sigma^n|$ to be such maps. Next the intersections $U_v \cap U_w$ are also polyhedra which are PL homeomorphic to an open set contained in the interior of $|\sigma^n|$. Furthermore, the overlap maps of the intersection are PL homeomorphisms as given in Def.(3.1). Therefore $|K^n|$ is a PL manifold with triangulation $K^n$ by construction.
Conversely, given any PL manifold $M^n$, it has a combinatorial triangulation $K^n$. The following lemma, proven in Appendix A, is needed in order to prove this:

**Lemma (3.3).** Let $P_1$ and $P_2$ be polyhedra, $S_1 \subseteq P_1$ and $S_2 \subseteq P_2$ be subpolyhedra and $\psi : S_1 \to S_2$ be a PL homeomorphism. Then $P = P_1 \cup \psi P_2$ is a polyhedron where $P_1 \cup \psi P_2$ is the disjoint union of the two polyhedra with points $x_1 \in S_1$ and $x_2 \in S_2$ identified if and only if $\psi(x_1) = x_2$.

Denote the atlas of $M^n$ by $\{U_\alpha, \varphi_\alpha\}_{\alpha \in \Lambda}$. Each set $\varphi_\alpha(U_\alpha) \subseteq \mathbb{R}^n$ is a polyhedron because it is an open subset of $\mathbb{R}^n$. Next $\varphi_\alpha(U_\alpha \cap U_\beta) \subseteq \varphi_\alpha(U_\alpha)$ and $\varphi_\beta(U_\alpha \cap U_\beta) \subseteq \varphi_\beta(U_\beta)$ are open subsets and therefore subpolyhedra. Moreover, each map $\psi_{\alpha\beta} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ where $\psi_{\alpha\beta} = \varphi_\beta \varphi_\alpha^{-1}$ is a PL homeomorphism. The above lemma implies that $\varphi_\alpha(U_\alpha) \cup \psi_{\alpha\beta} \varphi_\beta(U_\beta)$ is a polyhedron. Thus for any $U_\alpha$ and $U_\beta$, the above construction produces another polyhedron that is the disjoint union of the two spaces. Next note that this construction can be repeated between the neighborhood $U_\beta$ and another neighborhood $U_\gamma$ to form a new polyhedron $\varphi_\beta(U_\beta) \cup \psi_{\beta\gamma} \varphi_\gamma(U_\gamma)$. Furthermore, this polyhedron can be joined to the first to form another polyhedron by noting that the polyhedron $\varphi_\beta(U_\beta)$ is common to both. Finally, by successively adding the remaining neighborhoods $U_\alpha$ in this fashion, it follows that the resulting space $P$ is a polyhedron homeomorphic to $M^n$. Furthermore all polyhedra have a triangulation so $P = |K^n|$ where $K^n$ is a simplicial complex and hence $M^n = |K^n|$. Finally the star of every vertex of $K^n$ is PL homeomorphic to a n-ball because each point in $M^n$ has a neighborhood PL homeomorphic to a subset of $\mathbb{R}^n$. Therefore, $K^n$ is a combinatorial n-manifold and thus $M^n$ has a combinatorial triangulation. Moreover, this constructive proof shows that the map $t : |K^n| \to M^n$ is PL in the sense that $\varphi_\alpha t$ is a PL homeomorphism for all charts in the atlas. Q.E.D.

Thus PL manifolds can be thought of as coordinate independent representations of combinatorial manifolds. Now, the result that every smooth n-manifold has a combinatori-
rial triangulation immediately implies that all smooth n-manifolds have a PL counterpart in any dimension. Moreover, the correspondence of combinatorial manifolds to smooth manifolds can be addressed using the known results from topology for the correspondence of PL manifolds to smooth manifolds.\(^\text{14}\)

In order to understand the results on this correspondence, recall that the structure group of the tangent bundle of a smooth manifold is \(GL(n, \mathbb{R})\). This follows from the fact that the maps on the overlaps of the charts of the smooth manifold are diffeomorphisms on \(\mathbb{R}^n\) and diffeomorphisms act on vectors in \(\mathbb{R}^n\) by general linear transformations. In fact, without loss of generality, by appropriately choosing a Riemannian metric so that orthonormal vectors are defined, one can assume that the structure group is \(O(n)\). The structure group of the tangent bundle over a PL manifold is not so simple; it is the group of PL homeomorphisms, \(PL(n)\). Observe that the structure groups on the tangent bundles carry the information about the structure (PL or smooth) of the base manifold. Therefore the question of whether or not a given PL manifold admits a smoothing is equivalent to the question of whether or not one can change the structure group on the given PL tangent bundle from \(PL(n)\) to \(O(n)\).

Whether or not there is an obstruction to placing an \(O(n)\) structure group on a given PL tangent bundle is determined by the cohomology of the PL manifold.\(^\text{14}\)

**Theorem (3.4).** Let \(M^n\) be a PL manifold and \(\Gamma_k\) be the group of diffeomorphisms of the k-sphere, \(f : S^k \to S^k\), modulo those which extend to diffeomorphisms of the \((k+1)\)-ball, \(f : B^{k+1} \to B^{k+1}\). Then \(M^n\) is smoothable if and only if the obstructions \(c_k(M^n) \in H^{k+1}(M^n; \Gamma_k)\) vanish for \(0 \leq k \leq n - 1\).

Note that \(\Gamma_k\) is the same as the set of inequivalent smooth structures on a PL k-sphere. If the obstructions vanish, then there is at least one smoothing; that is one can find a smooth atlas on \(M^n\) that is diffeomorphic its PL atlas when restricted to the interior of
each neighborhood. If they do not vanish, then there is no smoothing of the given PL manifold. The number of smoothings of a PL manifold that is smoothable is determined by the following\textsuperscript{14}

**Theorem (3.5).** A PL homeomorphism $f : M^n \to N^n$ is equivalent to a smooth map if the obstructions of Thm.(3.4) vanish and $c_k(f) \in H^{k+1}(M^n; \Gamma_{k+1})$ vanish for $0 \leq k \leq n$.

These cohomology conditions are necessary and sufficient in any dimension although note that the results on smoothing 2-manifolds and 3-manifolds can be proven independently.

Using these results, it can be proven that every PL manifold in less than seven dimensions has a unique smoothing; this follows from the fact that $\Gamma_k = 0$ for $k \leq 6$. In seven dimensions, all PL manifolds have smoothings, but one can show that there are PL manifolds that do not have a unique smoothing. In eight or more dimensions, one can show that there are both PL manifolds that do not correspond to smooth manifolds and PL manifolds that do not have a unique smoothing. Therefore, PL manifolds are a more general set of topological manifolds than smooth manifolds; however in less than seven dimensions, PL manifolds have a unique correspondence to smooth manifolds. Thus in less than seven dimensions, combinatorial manifolds uniquely correspond to smooth manifolds. This connection between smooth manifolds and PL manifolds is the reason that combinatorial triangulations are preferred over weak triangulations for the purposes of this paper.\textsuperscript{15}

Similarly, the conditions under which a combinatorial conifold has a smooth counterpart are best discussed in terms of PL conifolds. PL conifolds are defined by requiring that the topological n-conifold of Def.(1.1) admit a PL atlas;

**Definition (3.6).** A PL atlas on a n-conifold $X^n$ is a collection $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \Lambda}$ of open sets and homeomorphisms indexed by a set $\Lambda$ satisfying the following:
i) The sets $U_\alpha$ cover $X^n$.

ii) $X^n - S = \bigcup_{\alpha \in \Lambda_0} U_\alpha$ for some subset $\Lambda_0 \subset \Lambda$.

iii) For $\alpha \in \Lambda_0$, $\varphi_\alpha$ is a PL homeomorphism of $U_\alpha$ to an open set in $\mathbb{R}^n_+$.

iv) For each $\alpha \in \Lambda - \Lambda_0$, $U_\alpha$ is a conical neighborhood of a singular point and $\varphi_\alpha$ is a PL homeomorphism onto the interior of a cone.

Again, it can be proven that PL conifolds are equivalent to combinatorial conifolds. This result can be seen easily from the corresponding result for manifolds, Thm.(3.2): Let $K^n$ be a combinatorial n-conifold; then it follows immediately by the same arguments as given in Thm.(3.2) that $|K^n|$ is a PL conifold with $K^n$ as its triangulation. Next let $X^n$ be a PL conifold. Excise neighborhoods of all the singular points, $N(S)$, of $X^n$; the result is a PL manifold with boundary. Next observe that by Thm.(3.2), the PL manifold is equivalent to a combinatorial manifold. Additionally the neighborhood of each singular point is a PL cone over a (n-1)-manifold and is easily seen to be PL homeomorphic to a simplicial cone over the manifold. Thus $X^n = |K^n|$ where $K^n$ is a simplicial complex. Finally, as each piece in the construction of the simplicial n-conifold $K^n$ is combinatorial, the conifold itself is combinatorial. Thus PL conifolds can be thought of as coordinate independent representations of combinatorial conifolds.

As in the manifold case, PL conifolds are closely related to smooth conifolds; again it follows immediately that every smooth n-conifold has a PL counterpart in any dimension. Furthermore, the results for smoothing PL manifolds can be extended to prove similar results for PL conifolds although the details will not be presented here. In particular, it can be proven that any PL conifold of dimension less than seven has a unique smoothing. Thus in fewer than seven dimensions, smooth manifolds and conifolds have a unique correspondence to combinatorial or equivalently PL manifolds and conifolds. Moreover, it is precisely this connection that allows for a concrete discussion of the issues involved
in summing over physically distinct manifolds and conifolds to be formulated in terms of combinatorial manifolds and conifolds.

The final issue is the relation of topological n-manifolds and n-conifolds to smooth n-manifolds and n-conifolds. Although only smooth spaces are relevant to physics, it is useful to understand their relationship with topological spaces as it characterizes the additional structure that smooth spaces carry. Two smooth structures are said to be equivalent if they are diffeomorphic to each other. Similarly, two PL structures are equivalent if they are PL homeomorphic to each other. Clearly, by the previous discussion, the number of inequivalent PL structures on a given topological manifold or conifold determines the number of inequivalent smooth structures and there is a unique correspondence between the numbers in less then seven dimensions. Thus results on PL structures and smooth structures are interchangable in less than seven dimensions.

It turns out that whether or not all smooth structures on a manifold are equivalent depends on dimension. In dimension three or less, every topological manifold admits a smooth structure. Furthermore, one can prove that this smooth structure is unique. Therefore, in dimension three or less, there is no difference between topological, PL and smooth manifolds. In more than five dimensions, the number of inequivalent PL structures can be characterized in a manner similar to that used in the smoothing of PL manifolds. Whether or not a given topological manifold \( M^n \) admits a PL structure is determined by whether or not the structure group of the topological manifold, \( Top(n) \), can be replaced by \( PL(n) \). The obstruction to placing a PL structure on a given topological manifold \( M^n \) is determined by the following:

**Theorem (3.7).** Let \( M^n \) be a topological manifold with \( n \geq 5 \). Then \( M^n \) has a smooth structure if and only if the invariant \( ks(M^n) \in H^4(M^n, \partial M^n; \mathbb{Z}_2) \) satisfies \( ks(M^n) = 0 \). Furthermore, given a continuous homeomorphism \( h : M^n \to N^n \) between PL manifolds, it
is equivalent to a PL homeomorphism if and only if the invariant $k_s(h) \in H^3(M^n, \partial M^n; \mathbb{Z}_2)$ satisfies $k_s(h) = 0$.

Using these results, it can be proven that there are topological $n$-manifolds that do not admit a PL structure in five or more dimensions. In addition, it follows from the above theorem that there are a finite number of inequivalent PL structures on all $n$-manifolds in five or more dimensions if the cohomology is finitely generated. In particular, all compact $n$-manifolds in five or more dimensions have finitely generated cohomology and thus admit a finite number of smooth structures.

Thm.(3.7) breaks down in four dimensions. It turns out that the vanishing of $k_s(M^n)$ is only a necessary condition in four dimensions; it is not sufficient. This is the reason that many issues concerning smooth structures on 4-manifolds remain open. However, there are several important results on 4-manifolds that have recently been proved. First, as shown by Freedman,\textsuperscript{17} there are topological 4-manifolds that admit no PL structure. Even worse, one can show that there are topological 4-manifolds that do not even admit a weak triangulation from results of Donaldson.\textsuperscript{18} Therefore, there are topological 4-manifolds that cannot be realized in terms of a simplicial complex. A discussion of such a topological 4-manifold, $|E8|$, is provided in Appendix B. In addition, it is well known that some 4-manifolds admit more than one smooth structure; in fact, there are an infinite number of inequivalent smooth structures on certain 4-manifolds. For compact 4-manifolds, it can be shown that the number is countably infinite; for example, the 4-manifold $CP^2\#9(-CP^2)$, a connected sum of complex projective space with nine copies of itself with the opposite orientation, has a countably infinite number.\textsuperscript{14} Even more interesting is the result that the number of different smooth structures is uncountable for certain open manifolds. In particular, $\mathbb{R}^4$ and $\mathbb{R} \times S^3$ both have an uncountable number of distinct smooth structures.\textsuperscript{17} This is shockingly different than the case for $\mathbb{R}^n$ in any other dimension; by Thm.(3.7), all other $\mathbb{R}^n$ have a unique PL structure as their cohomology vanishes. Moreover they have a
unique smooth structure by Thm.(3.4) and Thm.(3.5) for the same reason. It is clear that
the issue of smooth structures on manifolds in four dimensions is much more complicated
than in any other dimension.

Similar results on smooth structures apply to n-conifolds; these results are summarized
below as the details are not directly relevant to this paper.\(^{19}\) All conifolds are manifolds in
dimensions one and two and therefore have a unique smooth structure in these dimensions.
The result that 3-manifolds have a unique smooth structure implies that 3-conifolds do
as well. By removing conical neighborhoods around singular points of the conifold, one
obtains a manifold with boundary. This manifold has a unique smooth structure; by
gluing back the conical neighborhood of the singular points, one produces a unique smooth
structure on the conifold. Consequently, there is no difference between topological, smooth
and combinatorial conifolds in dimension three or less. In four or more dimensions, n-
conifolds may admit more than one inequivalent smooth structure or may admit no smooth
structure. In addition to the obvious examples of n-conifolds that are topological manifolds
that do not admit a PL structure, one can show that the suspension of the 4-manifold \(||E_8|||
will be a topological conifold which does not have a PL structure. Therefore, there are
topological conifolds besides those that are also topological manifolds that do not admit
PL structures.

The unique correspondence of smooth manifolds and conifolds to combinatorial man-
ifolds and conifolds in less than seven dimensions implies that the information about the
smooth structure of the space is carried by its combinatorial triangulation. Indeed, equiv-
ance of smooth structures can also be reexpressed in terms of the properties of the
combinatorial spaces; combinatorially equivalent n-manifolds have both the same topology
and equivalent smooth structures. Similarly, two combinatorial n-conifolds that are com-
binatorially equivalent have both the same topology and equivalent smooth structures.
Thus it follows from the results on smooth structures that all combinatorial manifolds
that are topologically equivalent are actually combinatorially equivalent in three or fewer dimensions. In four dimensions, it follows that there are an infinite number of combinatorially inequivalent triangulations of some closed 4-manifolds such as $CP^2 \# 9(-CP^2)$ as combinatorially inequivalent triangulations correspond to distinct smooth structures. Similarly, the result that there are an uncountable number of inequivalent smooth structures on $\mathbb{R}^4$ implies that there are an uncountable number of combinatorially inequivalent triangulations of it. In dimensions five and six, there are a finite number of combinatorially inequivalent triangulations of some n-manifolds as there are a finite number of distinct smooth structures. In dimension seven or greater, as combinatorial manifolds no longer necessarily correspond to smooth manifolds, a combinatorial manifold no longer necessarily specifies a unique smooth structure on the manifold. Finally, similar results for combinatorial n-conifolds in less than seven dimensions follow from the corresponding results on smooth n-conifolds.

These first two sections provide the tools to study the topological issues involved formulating quantum amplitudes involving sums over smooth spaces such as (3). Since the simplicial complexes are discrete descriptions of smooth manifolds and conifolds, the question of the algorithmic decidability of manifolds and conifolds can be phrased using them. Furthermore, the unique correspondence of smooth manifolds and conifolds and their combinatorial counterparts in less than seven dimensions provides the means of taking into account inequivalent smooth structures in a sum over topological spaces.

4. CLASSIFIABILITY AND DECIDABILITY OF MANIFOLDS AND CONIFOLDS

The canonical rule in the formulation of quantum amplitudes via a sum over histories is that only physically distinct histories are included in the sum. As discussed in section 2 of
part I (Ref.[2]), a history in Euclidean gravity consists of both a topological space such as a manifold or a conifold and a metric; thus physically distinct histories must consist not only of physically distinct metrics but also physically distinct topological spaces. Therefore, one needs to have a method for determining whether or not two topological spaces are physically distinct. Having an abstract definition of the set of topological spaces is not actually enough; the existence of such an abstract definition does not ensure that the set can be constructed. The issue of whether or not there is a finite procedure for determining if a given topological space actually satisfies the abstract definition of the set is called algorithmic decidability. Secondly, an abstract definition of the set does not ensure that one can find a unique collection of representatives. The issue of whether or not there is a finite procedure for determining if a given member of the set is distinct from another member of the set is called classifiability. Therefore the vital question at hand is whether manifolds and/or conifolds are algorithmically decidable and classifiable.

As mentioned in part I, any finite set has an algorithmic description. Thus the issue of whether or not a set is algorithmically decidable is only of interest in the case of infinite sets. The sets of compact connected combinatorial n-manifolds and compact combinatorial n-conifolds are countably infinite: A compact connected manifold contains a finite number of n-simplices. Since one can obtain at most a countable number of different spaces by gluing together a finite number of n-simplices, it follows there are at most a countable number of compact connected combinatorial n-manifolds. (One should observe that the compactness of the manifolds is why there are only a countable number of manifolds. In fact, in three or more dimensions there are an uncountable number of n-manifolds which are not compact.) Similarly the set of compact connected combinatorial n-conifolds is countably infinite. Therefore, finite algorithms can be expressed in terms of this finite set of elements in a given dimension n. Thus there is the possibility that there may exist finite algorithms for the algorithmic decidability and classifiability of compact connected
n-manifolds and n-conifolds. It turns out that the existence of such algorithms depends on dimension.

4.1 Algorithmic Decidability and Classifiability

It is useful to discuss certain examples of algorithmic decidability and classifiability before discussing the particular case of n-manifolds and n-conifolds. First, note that there are examples of infinite sets where no algorithms exist. One can make the simple observation that if a set is uncountable, then there is no way that one can describe it algorithmically because there is no way to place the members of the set in one to one correspondence with the integers to facilitate the development of any finite procedure. Thus simple examples of sets that do not have algorithmic descriptions are the set of all subsets of the integers and the set of real numbers. Therefore the questions of algorithmic decidability and classifiability are nontrivial only in the case of countably infinite sets. It turns out there are examples with and without finite algorithms for this case.

An example of an infinite set which can be described algorithmically is the set of prime numbers. Given any natural number \( N \) one can write an algorithm which will decide whether or not it is prime, namely, test whether or not any integer between 1 and \( N \) divides \( N \): Start with the number 2. If 2 divides \( N \), then \( N \) is not prime and the test stops. However, if 2 does not divide \( N \), then the same test is repeated for each consecutive integer until either the test stops or the integer equals \( N \). If \( N \) is reached before the test stops then \( N \) is prime. This is quite clearly a finite algorithm for deciding whether or not an integer is prime.

An important example of an infinite algorithmically decidable set is the set of all finitely presented groups. Such finitely presented groups are very important for topology as they correspond to the fundamental groups of manifolds. A finitely presented group is
a group with a finite number of generators and relations. A generator is represented by \(a_i\) and has an inverse \(a_i^{-1}\). Elements of the group correspond to finite products of these generators, e.g. \(a_1, a_2a_1^{-1}a_2, a_3^2a_2a_1^{-1}\), and are called words. The identity element 1 of the group is the empty word, that is the word containing no generators. A relation \(r_i = 1\) sets particular words of the group equal to the identity element. Two words \(w_1\) and \(w_2\) are said to be equivalent if one can be transformed into the other by a finite sequence of insertions or deletions of the relations or of the trivial relations \(a_i a_i^{-1} = a_i^{-1} a_i = 1\). The presentation of a finitely presented group is given by \(<a_1, a_2, \ldots, a_n; r_1, r_2, \ldots, r_n>\) which denotes the set of all equivalence classes of finite words in the generators \(a_i\) with respect to the relations \(r_i\). For example one presentation of the group \(Z\) is \(<a; ->\) consists of one generator \(a\) and no relations. Similarly, one presentation of \(Z_2 = <a; a^2>\) consists of one generator and one relation and one presentation of the permutation group on three objects \(P = <a_1, a_2; a_1^3, a_2^2, a_1 a_2 a_1^{-2} a_2^{-1}>\) consists of two generators and three relations. It is clear from its definition that the set of all finitely presented groups is algorithmically decidable.

Although it is easy to algorithmically describe what a finitely presented group is, note that distinct sets of generators and relations may actually be different presentations of the same group. For example, the presentations \(Z = <a_1, a_2; a_2>, Z_2 = <a; a^2, a^4>\) and \(P = <a_1, a_2; a_1^3, a_2^2, a_1 a_2 a_1 a_2^{-1}>\) are all different presentations of the groups \(Z, Z_2\) and \(P\). Note that both the number of generators and number of relations can differ in different presentations of the same group. Therefore the set of all finitely presented groups contains more than one representative of the same group.

Clearly it would be useful to eliminate this redundancy and find the set of unique representatives of all finitely presented groups. Such a set is a classification of the set of all finitely presented groups. However, there is no finite algorithm for finding unique representatives of such groups. As finite presentations of the same group can have different
numbers of generators and relations as well as different relations between the generators, there is no way to determine whether or not a given finite presentation is equivalent to a fiducial presentation by comparing generators and relations alone. One instead has to determine whether or not the given finite presentation generates the same group as the fiducial one by a finite algorithm. In order to have such an algorithm, one must have a finite algorithm for determining whether or not an arbitrary word \( w \) is equivalent to 1. This problem is called the word problem for finitely presented groups and one can prove that there is no solution to it.\(^{21}\) An important point is that one can show that there is no solution to the word problem for particular presentations of groups, not just for the set of all finite presentations of groups. There are several known examples of such groups;\(^ {22,23,24} \) one such finite presentation of a group is given in Appendix C. The fact that there are explicit finite presentations that can be proven to be unsolvable emphasizes the point that the unsolvability of the word problem is not a property of the set but rather of a particular finite presentation of a group. Thus the issues raised by the unsolvability of the word problem inevitably arise whenever these particular presentations appear, not just with the set of all finite presentations of groups in its entirety.

Finally, another related problem which has no algorithmic solution is the isomorphism problem for finitely presented groups, i.e. there is no finite algorithm to prove whether or not two arbitrary presentations are isomorphic groups.\(^ {25} \) Intuitively, this is related to the word problem because in order to prove that two presentations generate isomorphic groups, one must first identify which elements are trivial in each presentation. In fact, one can prove that the two problems are completely equivalent.\(^ {26} \) The unsolvability of the word problem and the isomorphism problem will be seen to be directly relevant to the algorithmic decidability and classifiability of manifolds.

These examples illustrate the basic procedure for determining whether or not a countably infinite set is algorithmically decidable or classifiable. That is, one begins with an
algorithmically decidable set that includes all members of the set of interest; i.e. the set of all positive integers or the set of all finitely generated groups. Then one asks whether or not there is an algorithm to select out the set of elements satisfying the more restrictive definition; i.e. the set of all primes or the set of all unique representatives of groups. Finally the important part of the procedure is not the initial algorithmically decidable set (so long as it contains all members of the subset of interest) but whether or not there is an algorithm for selecting out the elements that satisfy the more restrictive definition. Thus the starting point of the discussion of algorithmic decidability of n-manifolds and n-conifolds is to find an appropriate algorithmically decidable set that includes these spaces.

4.2 Algorithmic Decidability of Manifolds and Conifolds

For simplicity it is useful to restrict the discussion of algorithmic decidability to closed connected manifolds and conifolds. The results are easily extended to the case of compact connected spaces with boundary by doubling the complex over at the boundary and applying the closed results. The set of complexes $K$ containing a finite number of simplices is clearly an algorithmically decidable set as one simply checks that the list of simplices in $K$ and the set of rules for constructing the topological space $|K|$ satisfy Def.(2.5). The algorithm must be finite as the number of simplices is finite. However, for reasons of efficiency it is better to use a smaller algorithmically decidable set as the starting point, the set of all closed n-pseudomanifolds. It is easy to see that this set is algorithmically decidable: First observe that all complexes in this set contain a finite number of simplices as they are closed and connected. Therefore, begin with the decidable set of complexes containing a finite number of simplices. Next, given a complex $K$ in this set, find the maximum dimension $n$ of any simplex in $K$ and then check that all simplices of dimension less than $n$ are contained in an $n$-simplex to verify that the complex is pure. As the the number of
simplices to be checked is finite, the algorithm to do this check is finite. Second, verify that the simplicial complex is nonbranching, that is each (n-1)-simplex is contained in precisely two n-simplices. This can be readily done in terms of the rules used for constructing the complex and again is a finite procedure. Finally, verify condition i) of Def.(2.8). Again this procedure can be readily done and is finite:

**Procedure (4.1).** Observe that condition i) is transitive; if simplex A is connected to simplex B by an appropriate sequence of n-simplices and if B is connected to C by another appropriate sequence, then A is connected to C by the sequence consisting of the concatenation of these two sequences. Next simply begin with an arbitrary n-simplex A and construct the set $A^1$ consisting of A and all n-simplices that adjoin A by a (n-1)-simplex. Then construct the set $A^2$ consisting of $A^1$ and all simplices that adjoin $A^1$ by a (n-1)-simplex. Repeat this procedure $N$ times where $N$ is the number of n-simplices in the complex to construct $A^N$. Either $A^N$ contains all n-simplices in the complex or it does not; simply counting the number of n-simplices in $A^N$ will determine this.

Therefore this finite procedure determines whether or not the requirement is satisfied. These three finite algorithms determine whether or not $K$ is a closed connected pseudo-manifold and therefore the set of closed connected n-pseudomanifolds is algorithmically decidable.

In one dimension, all closed combinatorial 1-conifolds are 1-manifolds and all closed connected 1-manifolds are 1-pseudomanifolds by definition. Therefore, immediately from the above result, closed connected 1-manifolds are algorithmically decidable. Alternately, this can be seen explicitly; let $P^1$ be a closed 1-pseudomanifold. Observe that there are a minimum of three 1-simplices in $P^1$ because the vertices of a simplicial complex completely determine the complex. Next pick any vertex $v$ in $P^1$. As $P^1$ is a closed pseudomanifold, there are exactly two 1-simplices which meet at $v$. Thus $St(v)$ is a line segment. Therefore
the neighborhood of every vertex manifestly satisfies Def.(2.14). There are precisely two 1-manifolds; the circle $S^1$ which is a closed manifold and the real line which is not a closed manifold. As the $P^1$ is closed, it must be equivalent to $S^1$. Thus, given a space that is a closed connected 1-pseudomanifold, it is a closed connected 1-manifold and no additional algorithm is needed to differentiate it.

In two or more dimensions, closed n-pseudomanifolds are more general than closed connected combinatorial n-manifolds and n-conifolds. However, these spaces are subsets of the set of closed n-pseudomanifolds. Thus a first step to an algorithmic description of these combinatorial spaces is to have a description of how they differ from pseudomanifolds. A useful tool is the following result.

**Theorem (4.2).** Let $v$ be any vertex of a closed n-pseudomanifold $P^n$. Then the link $L(v)$ is a pure nonbranching (n-1)-simplicial complex without boundary.

As $X^n$ is pure, the star $St(v)$ of each vertex $v$ is a pure simplicial complex of dimension n. Each n-simplex in $St(v)$ is uniquely specified by (n+1) vertices, one of which is $v$ and thus each n-simplex in $St(v)$ has precisely one (n-1)-simplex $\sigma_{L_{n-1}}$ that does not contain $v$. It also follows that this (n-1)-simplex is uniquely specified by n vertices. Additionally, all lower dimensional simplices in the n-simplex that do not contain $v$ are subsets of set of these n vertices uniquely specifying $\sigma_{L_{n-1}}$. Consequently, each n-simplex in $St(v)$ contributes to $L(v)$ the complex consisting of this one (n-1)-simplex $\sigma_{L_{n-1}}$ and its lower dimensional faces. (For example, consider the star of the vertex $i$ in Figure 6b) of the suspension of $RP^2$. Tetrahedra $icbg$ contributes triangle $cgb$ and its faces to $L(i).$) Therefore, $L(v)$ is a pure (n-1)-complex.

In order to prove that $L(v)$ is nonbranching, it must be shown that each (n-2)-simplex in $L(v)$ is contained in precisely two (n-1)-simplices. Note that each (n-2)-simplex that is in $L(v)$ is also a face of a (n-1)-simplex containing $v$. As $X^n$ is nonbranching, each (n-1)-
simplex in \( St(v) \) that contains the vertex \( v \) is in exactly two \( n \)-simplices. (For example, edge \( bg \) is in triangle \( bgi \) in Figure 6b) and triangle \( bgi \) is in tetrahedra \( bgci \) and \( bgfi \).

It follows that the two \((n-1)\)-simplices in \( L(v) \) that come from these adjoining \( n \)-simplices share a common \((n-2)\)-simplex in \( L(v) \). (For example, edge \( bg \) is in triangles \( cbg \) and \( fbg \).)

Thus each \((n-2)\)-simplex is in at least two \((n-1)\)-simplices in \( L(v) \). Next, if some \((n-2)\)-simplex in \( L(v) \) were to be contained in more than two \((n-1)\)-simplices in \( L(v) \), it would have to be a \((n-2)\)-simplex in more than two \( n \)-simplices in \( St(v) \). However, this would imply that \( X^n \) is branching; the \((n-1)\) vertices of the \((n-2)\)-simplex and \( v \) uniquely specify a \((n-1)\)-simplex containing \( v \) and thus the \((n-1)\)-simplex containing \( v \) would be a face of more than two \( n \)-simplices. Consequently each \((n-2)\)-simplex is contained in precisely two \((n-1)\)-simplices in \( L(v) \). Therefore \( L(v) \) is a pure, nonbranching simplicial complex. QED.

This theorem will be used as a tool in finding algorithms for deciding whether or not a closed \( n \)-pseudomanifold is a closed connected combinatorial \( n \)-manifold or \( n \)-conifold in low dimensions. The case of closed connected \( n \)-manifolds will be discussed first. In order to decide whether or not a given \( n \)-pseudomanifold is actually a closed connected combinatorial \( n \)-manifold by Def.(2.14), one needs a finite algorithm for deciding whether or not the link of every vertex of the \( n \)-pseudomanifold is actually a combinatorial \((n-1)\)-sphere. Whether or not this can be done is dependent on the dimension. It turns out that closed connected combinatorial \( n \)-manifolds are algorithmically decidable in two and three dimensions, and can be proven to be not decidable in five or more dimensions. Whether or not 4-manifolds are decidable is an open question.

In order to show that 2-manifolds are algorithmically decidable, consider how they differ from 2-pseudomanifolds. Let \( P^2 \) be a closed connected 2-pseudomanifold. First note that the link of every vertex of \( P^2 \) is the disjoint union of circles. In order to prove this, let \( v \) be any vertex of \( P^2 \). The link \( L(v) \) is a pure nonbranching 1-dimensional simplicial complex without boundary by Thm.(4.2). Note that \( L(v) \) may have several disconnected
components. However, each component must be a 1-pseudomanifold because condition i) in Def.(2.8) is equivalent to being connected in one dimension. Therefore, \( L(v) \) is the disjoint union of closed 1-pseudomanifolds, or equivalently the disjoint union of circles. Thus if \( L(v) \) is connected for every vertex \( v \) of \( P^2 \), then \( P^2 \) is actually a 2-manifold. Furthermore, the character of the links implies any closed 2-pseudomanifold that is not a 2-manifold is obtained by identifying vertices of a closed connected combinatorial 2-manifold.

Given these properties, it is easy to find an algorithm to determine whether or not a given 2-pseudomanifold \( P^2 \) is actually a 2-manifold. First find all the links of the 2-pseudomanifold. Then check that every link is connected; this can be done using the procedure (4.1) in this one dimensional case. These two steps are manifestly both finite procedures; thus they constitute a finite algorithm for deciding 2-manifolds. Therefore closed connected 2-manifolds are decidable.

In order to show that closed connected 3-manifolds are algorithmically decidable, again consider how they differ from closed 3-pseudomanifolds. Let \( P^3 \) be a closed connected 3-pseudomanifold and \( v \) be any vertex in \( P^3 \). The link \( L(v) \) is a pure nonbranching 2-complex. Therefore, in order to test whether or not \( P^3 \) is a 3-manifold one needs an algorithm to test for whether or not a pure nonbranching 2-complex is a 2-sphere. This can be done in two finite steps: First check whether or not \( L(v) \) is a closed 2-pseudomanifold. This can be done by applying the finite algorithm for deciding 2-pseudomanifolds discussed previously. Next, if \( L(v) \) is found to be a 2-pseudomanifold, whether or not it is a 2-sphere can be determined by computing its Euler characteristic. The Euler characteristic of any pure n-complex is given by the alternating sum

\[
\chi(K^n) = \sum_{i=0}^{n} (-1)^i \alpha_i
\]

where \( \alpha_i \) is the number of i-simplices in the complex. The Euler characteristic of a 2-sphere is two and it is a well known fact that the Euler characteristic of any other closed
2-manifold is less than two. In addition it is easy to prove that the Euler characteristic of any closed 2-pseudomanifold is less than or equal to two by using the previously mentioned fact that all 2-pseudomanifolds are obtained from 2-manifolds by the identification of vertices. Each identification on a 2-manifold lowers the Euler characteristic of the resulting 2-pseudomanifold by one; therefore the Euler characteristic of a 2-pseudomanifold is two if and only if it is a 2-sphere. Thus there is a finite algorithm for determining whether or not \( L(v) \) is a 2-sphere.

Given these properties, the algorithm for determining whether or not \( P^3 \) is a 3-manifold follows directly. First find all the links of \( P^3 \). Next check that every link is a 2-sphere by the algorithm described above. If all links are found to be 2-spheres by this procedure then \( P^3 \) is a 3-manifold. This algorithm is manifestly finite and thus closed connected 3-manifolds are decidable.

However in dimensions higher than three, one runs into difficulty. Recall that the link of a vertex being a combinatorial \((n-1)\)-sphere is equivalent to its star being a combinatorial n-ball. Moreover, one can show that there is no algorithm for recognizing a n-ball for \( n \geq 6 \).\(^5\) This proof has become part of mathematics folklore; it is originally due to S. Novikov but has never been published. Furthermore, using modern results one can extend this proof to demonstrate that there is no algorithm for recognizing a 5-ball, as will be outlined below.\(^{17}\) It is a topological, not explicitly simplicial proof but obviously applies to the simplicial case as well. The proof utilizes the h-cobordism theorem:\(^5,17\) Given any compact \((n+1)\)-manifold \( Y^{n+1} \) with \( n \geq 4 \) and two boundary components \( M_1^n \) and \( M_2^n \) such that \( \pi_1(M_1^n) = \pi_1(M_2^n) = \pi_1(Y^{n+1}) = 1 \) and \( H_*(Y^{n+1}, M_1^n) = 0 \), then \( Y^{n+1} \) is homeomorphic to \( M_1^n \times I \). Next, note that one can construct a special type of closed \( n \)-manifold \( M^n \) with \( n \geq 5 \) that has any finitely presented group as its fundamental group. This part of the construction is given in the next subsection as it is also directly relevant to the classifiability of \( n \)-manifolds. Next, let \( W^n \) be \( M^n \) minus the interiors of two
disjoint n-balls; thus $W^n$ has boundary consisting of two disjoint n-spheres and has the
same fundamental group as $M^n$. Furthermore, one can show that there is an infinite set
of groups so that $\pi_1(W^n) = 1$ implies that $H_*(W^n, S^{n-1}) = 0$. Thus $W^n$ satisfies the
conditions of the h-cobordism theorem if $\pi_1(W^n) = 1$. In particular, the above manifold
$W^n$ for $n \geq 5$ is a product $S^{n-1} \times I$ if and only if its fundamental group is trivial. Therefore,
by capping off one of the boundary spheres of $W^n$ to form the manifold $B^n$, it follows that
$B^n$ is a n-ball if and only if its fundamental group is trivial. However, one can prove that
there are groups for which the word problem is unsolvable in the set of fundamental groups
satisfying the conditions of the h-cobordism theorem. Thus there is no algorithm to show
that every word in $\pi_1(B^n)$ is trivial for all possible groups. The conclusion that there is no
algorithm follows directly; $B^n$ is homeomorphic to an n-ball if and only if its fundamental
group is trivial but no algorithm exists for determining this fact. Therefore, there is no
algorithm to decide if a given space is a n-manifold for $n \geq 5$ because there is no way to
decide if the neighborhoods of the space are equivalent to n-balls.

The algorithmic decidability of 4-manifolds is an open problem. An algorithm for rec-
ognizing a combinatorial 4-manifold requires an algorithm for recognizing a combinatorial
3-sphere. There are two sufficient conditions for the existence of such an algorithm: a so-
lution to the Poincaré conjecture, and a solution to the word problem for the fundamental
groups of 3-manifolds, both open problems in topology.\textsuperscript{20} The Poincaré conjecture is that
any closed simply connected 3-manifold is actually a 3-sphere. If the Poincaré conjecture
is true, then it provides a starting point for developing an algorithm for recognizing a
3-sphere. The next step is to find a computable method of determining whether or not the
fundamental group of a 3-manifold is trivial. Whether or not such an algorithm exists is
also an open issue; one can prove that the set of all fundamental groups of 3-manifolds is
only a subset of the set of all finitely presented groups. Moreover, it is currently unknown
as to whether or not the word problem is solvable for this subset of finitely presented
groups. Therefore, if the Poincaré conjecture holds and the word problem for the fundamental groups of 3-manifolds has an algorithmic solution, then there will exist a finite algorithm for recognizing a combinatorial 3-sphere. If the Poincaré conjecture is not true, it may still be possible that a finite algorithm exists for recognizing a 3-sphere. For example, if the word problem for the fundamental groups of 3-manifolds is solvable and if a known set of counterexamples to the Poincaré conjecture that are all characterized by a finite algorithm is found, then a finite algorithm for recognizing the 3-sphere could also be constructed by combining these results. Alternatively, there may be some method of recognizing a 3-sphere that does not rely on either the Poincaré conjecture or the word problem for the fundamental groups of 3-manifolds. In any case, it does not seem likely that the algorithm for recognizing a combinatorial 3-sphere will be simple if any does exist; after all, mathematicians have tried to find one for nearly 100 years. Without such an algorithm, it is not possible to decide whether or not a given 4-pseudomanifold $P^4$ is a 4-manifold. Thus there is currently no known finite algorithm for algorithmically deciding 4-manifolds.

The algorithmic decidability of n-conifolds is studied in exactly the same manner as that for n-manifolds. By Def.(2.16), conifolds in one and two dimensions are manifolds; thus it follows from the previous results that closed connected conifolds are algorithmically decidable in one and two dimensions. In three or more dimensions, conifolds are more general topological spaces than manifolds and therefore the algorithmic decidability of conifolds in these dimensions is a separate issue. In the case of n-conifolds, one needs a finite algorithm for determining whether or not the link of every vertex is a combinatorial (n-1)-manifold. Again, as in the manifold case, whether or not this can be done is dependent on dimension. It turns out that closed connected n-conifolds are algorithmically decidable not only in three but also in four dimensions. Additionally it can be proven that closed connected n-conifolds are not decidable in six or more dimensions and the question
remains open in five.

In order to show that closed connected 3-conifolds are decidable, again begin by considering how they differ from closed 3-pseudomanifolds. Let \( v \) be a vertex of a closed 3-pseudomanifold \( P^3 \). By Thm.(4.2), the link \( L(v) \) is a pure nonbranching simplicial 2-complex without boundary. Therefore, in order to test whether or not \( P^3 \) is a 3-conifold, one needs an algorithm to test for whether or not a pure nonbranching 2-complex is a 2-manifold. This can be done in two finite steps: First check whether or not \( L(v) \) is a closed 2-pseudomanifold. This can be done by applying the finite algorithm for deciding 2-pseudomanifolds discussed previously. Next, if \( L(v) \) is indeed a 2-pseudomanifold, apply the finite algorithm for deciding whether or not a 2-pseudomanifold is a closed 2-manifold. Therefore, given the algorithm for determining whether or not \( L(v) \) is a closed 2-manifold, the algorithm for determining whether or not \( P^3 \) is a 3-conifold is to repeat this procedure for each vertex of \( P^3 \). If each link \( L(v) \) is a closed 2-manifold, then \( P^3 \) is a closed combinatorial 3-conifold. Thus 3-conifolds are algorithmically decidable.

Given that all 3-manifolds are 3-conifolds, it is useful to have an algorithm for determining whether or not a given closed 3-conifold \( X^3 \) is actually a combinatorial 3-manifold. First, note that all triangulations of 3-manifolds are combinatorial triangulations as observed in section 3. Next, recall from Thm.(5.2) of part I that a 3-conifold is a 3-manifold if and only if its Euler characteristic is zero. Thus calculate the Euler characteristic of \( X^3 \) using (7); this procedure is obviously finite. It follows that if \( \chi(X^3) = 0 \), then \( X^3 \) is a combinatorial 3-manifold. Thus the test for whether or not a given combinatorial 3-conifold is actually a combinatorial 3-manifold is very simple.

In order to show that closed connected 4-conifolds are decidable, again consider how they differ from 4-pseudomanifolds. Let \( P^4 \) be a closed 4-pseudomanifold. The link \( L(v) \) of each vertex \( v \) is a pure nonbranching simplicial 3-complex without boundary and thus, in order to proceed, an algorithm is needed to determine whether or not a pure nonbranching

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3-complex is a closed connected 3-manifold. Such an algorithm is the following: First test whether or not $L(v)$ is a 3-pseudomanifold by applying the finite algorithm for deciding 3-pseudomanifolds. If $L(v)$ is indeed a 3-pseudomanifold, then test whether or not the Euler characteristic of $L(v)$ is zero; if the Euler characteristic is nonvanishing then $L(v)$ cannot be a 3-manifold. If the Euler characteristic indeed vanishes, test that $L(v)$ is actually a closed connected combinatorial 3-manifold by applying either the finite algorithm for deciding 3-manifolds or the finite algorithm for deciding 3-conifolds. This is a finite algorithm for deciding when $L(v)$ satisfies Def. (2.14). From it one deduces that the finite algorithm for deciding when $P^4$ is combinatorial 4-conifold is to repeat this algorithm for every vertex $v$ in $P^4$. If each link is found to be a closed connected 3-manifold then $P^4$ is a combinatorial 4-conifold; otherwise it is not. Thus, 4-conifolds are algorithmically decidable.

In five dimensions, whether or not the class of 5-conifolds is algorithmically decidable is open; it depends on whether or not 4-manifolds are algorithmically decidable. Obviously, if closed connected combinatorial 4-manifolds are eventually found to be algorithmically decidable, then 5-conifolds will also be algorithmically decidable and conversely. Finally, it is easy to prove that n-conifolds are not decidable in six or more dimensions by observing that n-manifolds are not decidable in five or more dimensions. Therefore there is no algorithm for recognizing when the link of a vertex in a n-pseudomanifold is actually a combinatorial (n-1)-manifold for $n \geq 6$. Thus combinatorial n-conifolds in six or more dimensions are not algorithmically decidable.

Finally, it should be emphasized that there is no method of constructing a set of topological spaces that are not proven to be algorithmically decidable contrary to a certain suggestion in the literature by Hartle. Hartle asserts that a set of all 4-manifolds can be constructed in terms of combinations of a set of known combinatorial manifolds. Such an assertion may seem reasonable given certain theorems in surgery of n-manifolds as described below. Given the set of all closed n-manifolds, one can define an equivalence
relation by defining two to be equivalent if and only if they are cobordant. Applying this equivalence to the set of all closed n-manifolds yields a finite set consisting of the equivalence classes in dimension n. One can now pick a known explicit manifold to represent each equivalence class. These manifolds can be taken to be the finite set of building blocks for generating all n-manifolds. The surgical result used to do this is that two n-manifolds are cobordant if and only if they differ by a finite number of handle surgeries. Namely, embed $S^p \times B^{q+1}$ with $p + q + 1 = n$ in a n-manifold $M^n$, remove its interior yielding a compact n-manifold with boundary $S^p \times S^q$, then glue $B^{p+1} \times S^q$ in along the boundary $S^p \times S^q$ to yield a closed n-manifold. In two dimensions, this procedure is simply that of adding handles to $S^2$ and $\mathbb{RP}^2$ to generate all 2-manifolds. Thus this is just the natural higher dimensional generalization. However note that surgery techniques can be used to construct all n-manifolds and independently, it is known that n-manifolds are not algorithmically decidable in five or more dimensions as outlined above for the particular case of the n-ball. The problem is that in order to algorithmically implement such surgery techniques, one must prove that they can be encoded in an explicit finite algorithm. This clearly can be done in two and three dimensions. However, it turns out that the different embeddings of the handles are not nice in higher dimensions and it is this step that is not algorithmically describable. Thus Hartle’s scheme for building the set of all 4-manifolds cannot be proven to actually generate this set without an algorithm for recognizing the 3-sphere. Therefore the issue of the algorithmic decidability of 4-manifolds cannot be avoided.

4.3 Classifiability of Manifolds and Conifolds

In order to discuss the classifiability of manifolds and conifolds, it is first necessary to state the criteria by which two combinatorial manifolds or conifolds will be judged to be physically distinct. As just seen in the algorithmic decidability subsection, the criteria
used for defining a set is strongly coupled to the existence of an algorithm for implementing it. The standard criteria in the mathematics literature for classifying n-manifolds or n-conifolds in the continuum is equivalence under homeomorphisms; two topological spaces are said to be equivalent if they are homeomorphic to each other. However, in more than four dimensions, two spaces can be homeomorphic but not diffeomorphic as they can have distinct smooth structures. Moreover, the physically natural invariance applied to histories appearing in expressions (1) or (3) is diffeomorphism invariance. Thus the desired criteria for physics is to classify two smooth topological spaces as physically equivalent if they are both homeomorphic to each other and have equivalent smooth structures. In less than seven dimensions, this criteria is equivalent to the combinatorial equivalence of the corresponding combinatorial counterparts of the smooth topological spaces. Thus the natural criteria for the classification of combinatorial n-manifolds and n-conifolds is combinatorial equivalence. Note that by the results of section 3, a sum over combinatorially inequivalent topological spaces incorporates a sum over smooth structures. Thus combinatorial equivalence takes care of both the issue of equivalence under homeomorphisms and equivalence of smooth structures in a very natural fashion.

Again the classification of n-manifolds and n-conifolds will be discussed for the case of closed connected combinatorial manifolds and conifolds; it is easy to prove that the results can be directly extended to compact spaces. As all conifolds are manifolds in dimensions one and two, it follows immediately that they can be classified if 1-manifolds and 2-manifolds can be classified. Closed connected 1-manifolds are obviously classifiable; the only closed connected 1-manifold is a circle. Therefore the set of distinct 1-manifolds has only one element. Closed connected 2-manifolds are classified by the orientability or nonorientability of the manifold and its Euler characteristic. Given a closed combinatorial 2-manifold $M^2$, its orientability or nonorientability can be determined by computing its second homology; $H_2(M^2) = \mathbb{Z}$ if it is orientable and $H_2(M^2) = 0$ if it is not. This
computation clearly takes a finite number of steps as the number of simplices in the closed combinatorial 2-manifold is finite. The Euler characteristic (7) of the 2-manifold is also clearly calculated in a finite number of steps. Thus these two procedures form a finite algorithm and a set of distinct 2-manifolds can be generated using it. As the Euler characteristic is unbounded below, this set contains a countably infinite number of representatives.

Whether or not combinatorial 3-manifolds or 3-conifolds are classifiable is an open problem. Clearly, a necessary step in solving this problem is an algorithmic method of recognizing combinatorial 3-spheres. As discussed extensively in the algorithmic decidability subsection, sufficient conditions for solving this problem are an algorithmic solution to the Poincaré conjecture and to the word problem for fundamental groups of 3-manifolds. Given an affirmative solution to the problem of algorithmically recognizing a 3-sphere, it may be possible to provide a finite algorithm for classifying closed connected 3-manifolds if other open conjectures for 3-manifolds can also be solved. However, the issue is clearly unresolved at the present time. Similarly it turns out that the classifiability of 3-conifolds is also an open problem. It follows from the observation that all finite presentations of groups do not appear as the fundamental groups of 3-conifolds that there is no easy proof that 3-conifolds are not classifiable. Then the observation that the problem of classifying 3-manifolds is open immediately implies that the issue is also open for 3-conifolds as 3-manifolds are a proper subset of 3-conifolds. Thus there is no known method of generating a sets of either combinatorially inequivalent 3-manifolds or 3-conifolds.

Independently of the issue of whether or not the set of closed connected combinatorial 4-manifolds is algorithmically decidable, it can be proven that they are not classifiable. This is a consequence of the following topological argument that all finitely presented groups occur as the fundamental group of some n-manifold for $n \geq 4$; although presented in general terms, it is clear that it can be implemented in terms of combinatorial triangulations of these n-manifolds and thus applies in the simplicial case. Given any finitely
presented group $G$ and some fixed $n \geq 4$, there is a closed smooth $n$-manifold with $G$ as its fundamental group. First, one can produce a $n$-manifold $M^n$ with fundamental group of $k$ generators and no relations by taking the connected sum of $k$ copies of the $n$-manifold $S^1 \times S^{n-1}$ with itself. Next, observe that if a simple closed curve $c$ is removed from any manifold of four or more dimensions, the fundamental group remains the same. Similarly, if a tubular neighborhood around that curve is removed, $\pi_1(M^n)$ also remains the same. The tubular neighborhood is $S^1 \times B^{n-1}$ and its boundary is $S^1 \times S^{n-2}$. Now, $S^1 \times S^{n-2}$ is also the boundary of $B^2 \times S^{n-2}$. Hence, one can replace the interior of the tubular neighborhood in $M^n$ with $B^2 \times S^{n-2}$ and obtain a new manifold $M^n_c$. Now, it is easy to check that the curve $c$ is now contractible to a point in $M^n_c$. Hence by taking the curve $c$ to pass through the appropriate copies of $S^1 \times S^{n-1}$ in the appropriate order, this construction gives one relation $c = 1$ among the $k$ generators. Thus $\pi_1(M^n_c)$ is a group with $k$ generators and one relation. By repeating this surgery procedure a finite number of times on other curves, any set of relations for a group with $k$ generators can be derived. Therefore, any finitely presented group is the fundamental group of some smooth closed $n$-manifold for $n \geq 4$.

Furthermore, one can prove that a subset of the set of all closed $n$-manifolds can be constructed such that their equivalence is completely determined by their fundamental groups; that is two $n$-manifolds in this subset are homeomorphic if and only if their fundamental groups are isomorphic. However, the isomorphism problem for finitely presented groups is unsolvable; immediately this implies that there is no algorithm for deciding when two $n$-manifolds in this subset are homeomorphic. Consequently, by the observation that a set is not classifiable if any subset of it is not, there can be no finite algorithm for classifying the set of all smooth closed $n$-manifolds. This result also clearly applies to the combinatorial equivalence of $n$-manifolds as well. Finally, as the set of all closed connected combinatorial $n$-conifolds contains all closed $n$-manifolds, it immediately follows that all $n$-conifolds
in dimensions four or more cannot be classified by the above arguments. Therefore, one cannot classify either closed combinatorial n-manifolds or n-conifolds of dimension \( n \geq 4 \).

It should be emphasized that such n-manifolds as described above can be explicitly constructed from known examples of finitely presented groups that have no solution to the word problem. In particular, a manifold with the fundamental group given in Appendix C can be explicitly constructed. This manifold can be combinatorially triangulated and thus it would appear in any simplicial approximation to manifolds using a sufficient number of n-simplices. Thus the fact that n-manifolds are not classifiable in four or more dimensions is not simply a difficulty in principle, but a problem that can and will actually be encountered even in the construction of a set of spaces that contain a finite number of n-simplices.

Note that the issue of classifying other topological spaces besides manifolds and conifolds can be studied by similar methods. For example, 2-pseudomanifolds are algorithmically classifiable in two dimensions. In fact 2-complexes themselves are algorithmically classifiable; however interestingly enough it is not possible to decide whether or not \( \pi_1(K) \) of a 2-complex is trivial.\(^{32}\) This is because the equivalence of 2-complexes under homeomorphisms is not in one to one correspondence with the isomorphism problem for their fundamental groups. The problem of classifying 3-pseudomanifolds is open and in four or more dimensions, n-pseudomanifolds are known to be not classifiable from the observation that n-manifolds are a subset of n-pseudomanifolds and are not classifiable.

Finally, it is very important to take care when applying results quoted in the mathematics literature on classifiability to the issue of finding a set of physically distinct spaces. As emphasized in this section, finding a set of physically distinct spaces involves having algorithms for deciding when a space is a member of the desired set and for deciding when it is distinct. However, it is common in the mathematics literature to not require that there be such algorithms for a set to be called classifiable. For example, a well known statement is that \( K(\pi, 1) \) manifolds are classifiable;\(^{33}\) there is a well known theorem that states that
two $K(\pi, 1)$ manifolds are homotopy equivalent if and only if their fundamental groups are isomorphic. However, by the isomorphism problem for groups, there is no algorithm for determining whether or not two arbitrary fundamental groups are isomorphic. Therefore, although $K(\pi, 1)$ manifolds are classifiable, there is no algorithm for doing so. Similarly, a particular example has been frequently misunderstood in physics literature is the case of simply connected spin 4-manifolds. It is a well known fact that simply connected spin 4-manifolds are classified by their Euler characteristic and signature. However, again by the word problem for groups, there is no algorithm for determining whether or not an arbitrary 4-manifold is simply connected as one can not prove that any arbitrary closed curve is contractible to a point. It follows that simply connected spin 4-manifolds cannot be algorithmically classified simply because the set of simply connected spin 4-manifolds cannot be algorithmically decided. Consequently, a sum over these spaces can not be concretely implemented contrary to what is frequently stated in the physics literature. Thus, one must be careful to check whether or not a classifiable set is algorithmically classifiable before claiming a sum can be formulated in terms of it.

5. EUCLIDEAN FUNCTIONAL INTEGRALS USING REGGE CALCULUS

The results of the last section provide the necessary background for explicitly implementing sums over topological spaces. Although the previous discussion is directly applicable to both the continuum and discrete formulations of sums over histories, it is useful to illustrate the explicit implementation of such sums in terms of simplicial complexes. The resulting sums over histories provide a discrete approximation of quantum amplitudes can be implemented numerically and thus a quantitative study of the consequences of a sum over topology can be directly carried out. Moreover, the effects of different algo-
rithms for generating the necessary lists of spaces are also readily accessible for amplitudes constructed completely in terms of simplicial complexes. Therefore, this section will first discuss Regge calculus and then discuss algorithms for summing over manifolds and conifolds with emphasis on the case of four dimensions. As in section 4, it is easiest to address the topological aspects of implementing sums over histories in terms of closed connected manifolds or conifolds such as (3). However, with more work, these results can be readily applied to sums over histories involving compact spaces such as (1) as well.

In order to translate an expression such as (3) into a concrete sum over simplicial histories it is necessary to be able to associate a metric and action with any $K^n$. To this point in the paper, no metric information has been associated with the simplicial complexes; the simplicial complexes carry only the topology and PL structure of the spaces. Thus in order to proceed, metric information must be attached. The easiest way to do so, as discussed by Regge, is to require that the metric on the interior of each n-simplex is the Euclidean metric; that is all n-simplices in the simplicial complex are flat. With this requirement, the geometry of each n-simplex is completely fixed by specifying the lengths $s_i$ of all of its edges. It follows that the geometry of the simplicial complex is also completely fixed by specifying the length of all edges $s_i$ in the complex. Therefore one anticipates that all geometrical quantities such as volume and curvature can be expressed completely in terms of the edge lengths.

Indeed this is the case. It is easy to see that volume of any pure simplicial complex can be computed by first computing the volume of each n-simplex in terms of the edge lengths and then adding the contribution of all the n-simplexes. Somewhat less obviously, curvature can also be expressed in terms of the edge lengths. As the metric on the interior of each n-simplex is flat, it is clear that the curvature of the combinatorial space is not carried on the interiors of the n-simplices. Rather, it turns out to be concentrated on the (n-2)-simplices of the simplicial complex. This is most directly apparent in two dimensions,
in which curvature is concentrated on vertices. For example, let $v$ be a vertex in some combinatorial closed 2-manifold and let $m$ denote the number of triangles in $St(v)$. Then the scalar curvature associated with this vertex is given by $\theta(v)$ where

$$\theta(v) = 2\pi - \sum_{i=1}^{m} \phi_i$$  \hspace{1cm} (8)

and $\phi_i$ is simply the angle between two unit vectors that lie in adjacent edges of the $i$th triangle as illustrated in Figure 7a). Indeed the sum of the curvature at each vertex over all vertices in the 2-manifold yields the Euler characteristic, $\sum_{v \in M^2} \theta(v) = 2\pi \chi(M^2)$ as required by the Gauss-Bonnet theorem. In general, if the index $i = 1 \ldots m$ sequentially labels adjacent n-simplices in $St(\sigma^{n-2})$, then the curvature is given by

$$\theta(\sigma^{n-2}) = 2\pi - \sum_{i=1}^{m} \phi_i$$  \hspace{1cm} (9)

where $\phi_i$ is now the dihedral angle between two unit vectors normal to $\sigma^{n-2}$ that lie in the adjacent $(n-1)$-simplices of the $i$th $n$-simplex as illustrated in the three dimensional case in Figure 7b). All dihedral angles can be computed in terms of the edge lengths by elementary trigonometry in any dimension. Therefore, the curvature can be expressed as a function of edge lengths alone. In addition, one can demonstrate that the definition of curvature given in (9) converges to the scalar curvature in the continuum in a suitably defined average in greater than two dimensions.\textsuperscript{36} Thus the Regge curvature is a suitable discrete version of curvature for use in simplicial gravity.

Note that the formulation of the Regge curvature implicitly relies on the combinatorial nature of the simplicial complex. The definition of the geometry in terms of the edge lengths relies on the fact that the n-simplices are completely defined by their vertices. Similarly the definition of curvature relies on the fact that there is a method of sensibly associating n-simplices and (n-1)-simplices to a given (n-2)-simplex. In order to do this, certain types of simplicial complexes must be excluded; examples of spaces which do not
have the necessary notions are branching simplicial complexes. In general, one requires
that the simplicial complex be pure and nonbranching such that the concept of adjacent
(n-1)-simplices is well defined. As manifolds, conifolds and even pseudomanifolds all satisfy
this condition by definition, Regge calculus can be used to compute curvatures on all of
these topological spaces.

Finally, given the above definitions, the Regge action for Einstein gravity with cosmolo-

gical constant $\Lambda$ for a closed pure nonbranching complex $K^n$ is

$$I[s_i] = -\frac{2}{16\pi G} \sum_{\sigma^{n-2} \in K^n} \theta(\sigma^{n-2}) V(\sigma^{n-2}) + \frac{2\Lambda}{16\pi G} \sum_{\sigma^n \in K^n} V(\sigma^n)$$

(10)

where the first sum is over all (n-2)-simplices in the simplicial complex and the second is
over all n-simplices in the complex. $V(\sigma^n)$ is the volume of the indicated n-simplex. The
Regge action for compact combinatorial manifolds and conifolds with boundary can also
be formulated entirely in terms of edge lengths; essentially one adds the appropriate dis-
etrizied form of the boundary term that appears in the continuum action (1). Finally, note
that discretized actions for other theories such as curvature squared theories on simplicial
complexes can also be formulated in terms of the edge lengths.

Given a method of associating a geometry and an action with any complex $K^n$ in the
sets of interest, the Regge equivalent of (3) is

$$<A> = \frac{\sum_{K^n \in \mathcal{L}} <A>_{K^n}}{\sum_{K^n \in \mathcal{L}} <1>_{K^n}}$$

$$<A>_{K^n} = \int Ds \ A(s_i) \exp(-I[s_i])$$

(11)

where the sum is over all complexes $K^n$ in an as yet unspecified list of complexes $\mathcal{L}$. The
notation $\int Ds$ denotes the integral over all edge lengths, $\int Ds = \prod_{s_i \in K^n} \int d\mu(s_i)$ where the
notation $d\mu(s_i)$ indicates the freedom available in choosing the measure on the space of
edge lengths. Note that the functional integral over metrics in (3) has been reduced to the product of integrals over edge lengths in (11) as the metric information is discrete. This product of integrals is well defined and thus the sum over all edge lengths is in principle implementable; discretizing the metric has removed the problems related to the issues of gauge fixing and nonrenormalizability associated with defining the measure on the space of metrics. (Such problems, of course, reappear when attempting to take the continuum limit of such a Regge integral.) Moreover, the technical details of implementing a sum over edge lengths are manifestly isolated in (11) from those involving the construction of the list \( L \). Thus the issues involved with algorithmically constructing such a list in different dimensions can be addressed independently.

Even though the functional integral over metrics has been explicitly implemented in terms of edge lengths, the expression (11) is still heuristic; at this point it is necessary to explicitly provide an algorithm that generates a list of suitable spaces \( K^n \) in the specified set. For example if one wished to implement (11) as a sum over physically distinct closed manifolds \( M^n \), one would need a list of combinatorially inequivalent \( M^n \). Similarly, an implementation of (11) as a sum over physically distinct conifolds necessitates a list of combinatorially inequivalent conifolds. The starting point for generating such a list is to generate an exhaustive list of all topological spaces in the specified set. The second step is to select from this exhaustive list, the set of unique representatives. Thus in order to get anywhere at all, the specified set of topological spaces must be algorithmically decidable. Next, the second step requires the space to be classifiable according to the desired criteria. Thus, the considerations of section 4 apply directly to this issue.

It is useful to begin by discussing the case of closed 2-manifolds, as it is an explicit example for which both steps can be completely carried out to form a list of physically distinct 2-manifolds. One can generate an exhaustive list of 2-manifolds by the following procedure: Generate all spaces built out of gluing together \( n \) triangles along their edges.
such that they form simplicial complexes. Then apply the algorithm for the definition of a closed 2-manifold to this set to find all 2-manifolds made of n triangles. Finally, repeat this procedure for all values of n. This procedure is computable by the last section. The result will be a list of all combinatorial 2-manifolds with a vast amount of redundancy. For example, one will have a large number of 2-manifolds that are equivalent under simplicial homeomorphisms, that is under permutations of the vertices. Additionally one will have a large number of combinatorially equivalent spaces; for example the list will include combinatorial 2-spheres composed of four triangles, those of six triangles, those of 386 triangles and so on, all of which are combinatorially equivalent. Thus it is necessary to classify by combinatorial equivalence this initial list of spaces to eliminate this redundancy. One can produce such a list of unique representatives of 2-manifolds in the following way: Pick one of the 2-manifolds in the initial set, say one with the smallest number of triangles, and compute whether or not it is orientable and compute its Euler characteristic. Record its orientability and its Euler characteristic and place it on the list of unique representatives \( L \). Next repeat this computation of orientability and Euler characteristic for each 2-manifold in the initial set of 2-manifolds. For each of these manifolds, check to see if a 2-manifold with the same Euler characteristic and orientability already appears on the list \( L \) of unique representatives; if it does, discard it and go on to the next manifold. If it does not, add it to the list. Continue through all 2-manifolds on the initial list. One will end up with a \( L \) of all physically distinct 2-manifolds; the elements in this list are uniquely specified by their Euler characteristic and orientability.

Note that the unique representatives generated by this procedure will have different numbers of triangles. It is also clear that there exist different procedures that will also select out a list of unique representatives \( L' \) that contain different combinatorial 2-manifolds as its members; that is the list \( L' \) will contain a 2-manifold with the same Euler characteristic and orientability as one in \( L \) but this 2-manifold may be composed of a different number...
of triangles or differ by a simplicial homeomorphism. However, these points do not affect
the topological results; any list of unique representatives $\mathcal{L}$ is equivalent to any other
such list $\mathcal{L}'$ by the fact that 2-manifolds are classified under combinatorial equivalence by
their orientability and Euler characteristic. Thus the sum over topologies in (11) can be
concretely implemented in terms of any such list of unique representatives.

Next consider the construction of such lists in four dimensions for manifolds and coni-
folds. The starting point is the same as that in two dimensions; an exhaustive list of the
set of topological spaces of interest. It is at this step that problems occur with formulat-
ing expressions for the set of all 4-manifolds; as discussed in section 4, there is no known
algorithm for generating an exhaustive list. Thus, there is no known starting point for the
rest of the algorithmic formulation of a list.

In comparison, closed 4-conifolds can indeed be algorithmically decided in four di-
dimensions and thus the first step can be concretely implemented. In parallel with the
2-dimensional case, one begins with $n$ 4-simplices, generates all spaces built out of all pos-
sible combinations of these 4-simplices and then applies the algorithm for the definition
of a closed 4-conifold to find all 4-conifolds made of $n$ 4-simplices. One then repeats this
procedure for all values of $n$. The resulting exhaustive list $\mathcal{L}$ contains all closed 4-conifolds
again with a large amount of redundancy. Thus the set of closed combinatorial 4-conifolds
has an immediate advantage over the set of all closed combinatorial 4-manifolds; as they
are algorithmically decidable, an exhaustive list of 4-conifolds can be explicitly constructed.

However, unlike the case in two dimensions, the second step cannot be carried out to
form a list of physically distinct 4-conifolds because of the results of section 4; neither
4-manifolds nor 4-conifolds are classifiable under combinatorial equivalence. Therefore a
sum over combinatorially inequivalent 4-conifolds cannot be implemented even though an
exhaustive list of these spaces can be constructed. Moreover, this problem with imple-
menting a sum over 4-conifolds cannot be removed by requiring them to satisfy certain
stricter criteria, such as requiring them to be simply connected as discussed in section 4. Therefore, as a list of combinatorially inequivalent 4-conifolds cannot be constructed, (11) cannot be implemented as a sum over these spaces. Additionally, the problem with constructing a list of combinatorially inequivalent 4-dimensional topological spaces cannot be solved for any algorithmically decidable set that includes all 4-manifolds.

Thus at this point one is forced to conclude that any concrete formulation of a sum over topology of form (11) in four dimensions must begin by a reexamination the criteria used in defining what is meant by a distinct topological space. It is clear that the problems that arise with the classification of 4-manifolds and 4-conifolds are closely tied with the criteria used in defining the set of interest. Thus a natural way to avoid these problems is to change the criteria for defining what a distinct topological space is, that is to use a less strict criteria than combinatorial equivalence. Given an appropriate change in criteria, it will be possible to algorithmically construct a list of 4-conifolds that are distinct according to that criteria. Of course, the question that must be addressed is whether or not such a change in criteria is reasonable. Obviously, there is no absolute answer to this question. Such a change in criteria would not result in a list of combinatorially inequivalent 4-conifolds, but a list of distinct 4-conifolds that would include more than one instance of the same combinatorial space. This overcounting of certain physically distinct 4-conifolds would lead to additional weighting factors in the Euclidean sum over histories. The effects of such overcounting on quantum amplitudes formed in terms of such sums would clearly be an issue for further study. It is certainly important that such a change in criteria lead to reasonable results. However, a test of what is reasonable can only be done by further investigations into the properties of sums over histories such as (11) constructed with lists of distinct 4-conifolds formed by using different criteria. Therefore a first step is to present candidates for such alternate criteria.

There are a large number of various possibilities for alternate criteria for algorithm-
mically classifying 4-conifolds. However, there is a reasonable requirement to place on such alternate criteria: The list of distinct 4-conifolds generated by such criteria should include all combinatorially inequivalent 4-conifolds, that is the set of physically distinct 4-conifolds should be a proper subset of this new list. This requirement ensures that all classical histories will be included in the explicit construction of the sum over histories. It also ensures that the contribution from all possible distinct smooth structures on a given topological 4-conifold will be included. It follows that all qualitative results obtained from a semiclassical evaluation of the sum over histories (2) in terms of Euclidean instantons will be recovered in a semiclassical evaluation of its concrete implementation (3) in terms of this alternate criteria. This requirement therefore provides a good starting point for formulating alternate criteria that lead to reasonable results.

The possibilities for such alternate criteria can be clearly illustrated in terms of algorithmic procedures on simplicial complexes; in addition, such a formulation is practical as it will directly lead to implementable sums. The first procedure is the most direct;

**Procedure (5.1).** Take the list of all distinct closed 4-conifolds to be $\mathcal{L}$, i.e. that generated by the procedure for generating an exhaustive list of all closed 4-conifolds.

Note that this method includes all combinatorially inequivalent 4-conifolds by construction. Of course this set suffers from a massive amount of redundancy; it includes both 4-conifolds that differ from each other by a simplicial homeomorphism and 4-conifolds composed of different numbers of 4-simplices that are combinatorially equivalent. However, in expressions such as (11), this overcounting of combinatorially equivalent 4-conifolds will not result in any manifest divergence as the same overcounting will occur in both numerator and denominator. Therefore the main effect of this overcounting is to induce a particular weighting of physically distinct 4-conifolds; those that are easiest to build according to the algorithm will be weighted more heavily than those that are not.
Of course Procedure (5.1) is not aesthetically very nice as it makes no attempt to eliminate even obvious redundancies. A refinement of this procedure is the following

Procedure (5.2). Beginning with the list \( \mathcal{L} \) of all 4-conifolds, derive a new list \( \mathcal{L}^1 \) in the following way: For each \( n \), define two 4-conifolds formed of \( n \) simplices to be equivalent if they are equivalent under a simplicial homeomorphism. Then \( \mathcal{L}^1 \) is the set of all 4-conifolds in \( \mathcal{L} \) under this equivalence relation.

This procedure is computable as the number of permutations of vertices is computable, though large, for each number \( n \). Note that this set will still include representatives for all physically distinct 4-conifolds as equivalence under simplicial homeomorphism is a special case of combinatorial equivalence. It also clearly considerably reduces the redundancy present in \( \mathcal{L} \). However, the list \( \mathcal{L}^1 \) will still include redundancies corresponding to 4-conifolds built of different numbers of 4-simplices that are combinatorially equivalent. A weighting of physically distinct 4-conifolds is again induced by this overcounting and it is an interesting question as to how much it differs from that of Procedure (5.1).

One can also generate procedures that partially eliminate the redundancy caused by combinatorial equivalence. Note that the problem with determining the combinatorial equivalence or inequivalence of two spaces lies in the fact that subdivisions of both spaces to arbitrarily large numbers of simplices are allowed. If only a finite number of subdivisions are allowed, the number of steps is finite. This observation can be used to form a finite procedure for weak combinatorial inequivalence:

Procedure (5.3). Begin with the list \( \mathcal{L}^1 \) generated by procedure 1. Form a sequence \( \mathcal{L}^2_n \) by the following steps: Begin with the 4-conifold in \( \mathcal{L}^1 \) containing the smallest number of 4-simplices, six, and place it in \( \mathcal{L}^2_6 \). From it form the set \( \mathcal{S}_7 \) of all subdivisions of the element in \( \mathcal{L}^2_6 \) that contain exactly seven 4-simplices. Next find the set \( \mathcal{Q}_7 \) of all elements in \( \mathcal{L}^1 \) containing seven 4-simplices that are not simplicially homeomorphic to any element
in $S_7$. Add $Q_7$ to $L_6^2$ to form $L_7^2$. In general, given $L_n^2$, form the set $S_{n+1}$ of all subdivisions of all elements in $L_n^2$ with exactly $(n+1)$ 4-simplices. Next find the set $Q_{n+1}$ all elements in $L^1$ that contain exactly $(n+1)$ 4-simplices that are not simplicially homeomorphic to any element in $S_{n+1}$. Add $Q_{n+1}$ to $L_n^2$ to form $L_{n+1}^2$. Finally repeat for all $n$ to find $L_\infty^2 = L^2$.

This procedure eliminates certain combinatorially equivalent 4-conifolds, but not all of them; for example two 4-conifolds containing different numbers of simplices may not be equivalent if only subdivisions of the one with fewer simplices into the one with more simplices are allowed, but may indeed be equivalent if one allows both to be further subdivided. Thus Procedure (5.3) again induces a weighting on physically distinct 4-conifolds.

These three procedures all provide algorithmically decidable lists of distinct 4-conifolds. Therefore one can precisely formulate (11) for 4-conifolds for any of these alternate criteria by taking $L$ to be any of the lists generated by Procedures (5.1) through (5.3). Thus (11) formulated as a sum over 4-conifolds classified with respect to alternate criteria provides a concrete computable starting point for further study of the consequences of topology and topology change.

Note that the explicit formulation of sums over topological spaces can be readily extended by application of results in section 4 to any set of algorithmically decidable spaces. For example, in two dimensions, the set of 2-pseudomanifolds itself is also algorithmically decidable and classifiable under combinatorial equivalence. Thus, a procedure analogous to that used to construct a list of physically distinct 2-manifolds can be developed that algorithmically constructs a list of physically distinct 2-pseudomanifolds. Therefore an explicit comparison of the consequences of a sum over 2-manifolds and a sum over 2-pseudomanifolds can be made. Hartle carried out a qualitative analysis of this case and concluded that a sum over 2-pseudomanifolds resulted in the same qualitative results in
the classical limit, but did not carry out a more explicit computation. It is clear that the results of such a calculation would be interesting. The case of three dimensions is particularly interesting as both 3-manifolds and 3-conifolds are algorithmically decidable; whether or not they are classifiable under combinatorial equivalence is an open issue for both. However, explicit implementations of (11) can be formed for both sets of spaces by using any of the Procedures (5.1) through (5.3) described above. Consequently, three dimensional sums over histories provide an arena in which the consequences of a sum over 3-conifolds can be tested directly against results obtained for a sum over 3-manifolds. Moreover, the issue of the effects of various alternate criteria for classifying these three dimensional spaces can also be isolated from that of their topology. In five or six dimensions, there are no algorithms for generating either n-manifolds or n-conifolds, but lists of n-pseudomanifolds can be constructed and the consequences of sums over these spaces can be studied.

Finally, it should be stressed that the topological issues illustrated in this section in terms of Regge calculus also apply to sums over histories not expressly formulated in terms of simplicial complexes. By section 3, any combinatorial n-manifold or n-conifold in less than seven dimensions uniquely corresponds to a smooth n-manifold or n-conifold respectively; the combinatorial space determines the topology and smooth charts for the definition of the smooth space. Given this smooth space, \( <A>_{K^n} \) can be computed in terms of a functional integral over the space of metrics. It is clear that changing the calculation of this expectation value in this manner in (11) in no way changes the properties of the sum over topological spaces. Thus, the conclusions about the properties of a sum over topological spaces apply in general.
In two dimensions, a concrete implementation of a sum over histories formulation of quantum amplitudes can be carried out explicitly for 2-manifolds. In four dimensions, such an implementation cannot be made for manifolds for two reasons; there is no known method of algorithmically recognizing a 4-manifold and it has been proven that 4-manifolds are not classifiable. However, 4-conifolds can be algorithmically described by a simple algorithm and by changing the criteria for distinctness, explicit algorithms for a sum over 4-conifolds can be formulated. Thus the set of 4-conifolds allows for a study of the consequences of sums over topology.

Of immediate interest is the question of whether or not enlarging the set of spaces to be summed over in an expression such as (11) changes the qualitative results in any unexpected way. Hartle studied this issue qualitatively in two dimensions and argued that 2-manifolds would dominate; however, even a qualitative assessment of the properties of a sum over histories is not as simple in four dimensions as the action (9) is no longer topological. Therefore it is necessary to estimate or evaluate the contribution from $< A >_{X^4}$ for each 4-conifold $X^4$ in order to proceed. Moreover, one needs some sort of useful and relevant quantity $A$ to compute when comparing the effects of different choices of algorithms. The choice of such a quantity is not trivial; generally speaking, one would like to compute the expectation value of a physical quantity, that is one that is diffeomorphism invariant. However, known quantities with this property are topological invariants and thus are not sensitive to the metric information. Indeed, it is difficult to formulate explicitly diffeomorphism invariant quantities that do not correspond to such invariants. Now, the expectation value of such topological invariants may indeed be of interest; however, much of the interest in explicitly computing a sum such as (11) is in precisely the consequences of topology on geometrical quantities.
These problems are not trivial to solve; however, they also do not present insurmountable obstacles. Different methods of calculating or estimating $< A >_{x^4}$ can be tried out and expectation values of different quantities $A$ can be calculated. The results of such trials will provide information not only relevant to the topological issues but to questions about the quantum mechanics of gravitational theories in general. In any case, it is clear that explicit formulations of the sum over histories as found in this paper are such invaluable in any further investigations of topology and topology change.

Acknowledgments

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APPENDIX A: PROOF OF LEMMA (3.3)

In order to prove Lemma (3.3) it is useful to have the abstract definition of a simplicial complex. This definition is equivalent to Def.(2.5) for finite dimensional complexes; however, it does not rely on the explicit embedding of the complex in Euclidean space. Its utility is that it can be applied directly in the proof of the Lemma.

Definition (A.1). An abstract simplicial complex is a topological space $|K|$ and a collection of simplices $K$ such that
The set of vertices \( K^0 \) form a countable set

II) The simplices \( K \) are a family of subsets of \( K^0 \)

iii) Each vertex is contained in only a finite number of simplices

iv) A set \( F \subseteq |K| \) is open if and only if \( F \cap |\sigma| \) is open for all simplices \( \sigma \in K \)

v) If a simplex \( \tau \subseteq \sigma \) where \( \sigma \subseteq K \) then \( \tau \subseteq K \)

vi) If \( \tau, \sigma \in K \), then \( \tau \cap \sigma \in K \).

If the maximum number of vertices contained in any simplex is less than a fixed number \((n + 1)\), then \( K \) is finite dimensional and \( n \) is its dimension.

Often in the literature, simplicial complexes which satisfy all of the above conditions are called countable locally finite simplicial complexes. Complexes which do not satisfy conditions i) and ii) are sometimes encountered; however, if a simplicial complex has finite dimension and is metrizable, then it must satisfy all of the above conditions. Thus there is no loss of generality in the finite dimensional case by imposing all of the above conditions on abstract simplicial complexes.

One can define abstract polyhedra to be the underlying topological spaces of simplicial complexes. All of the other definitions related to simplices and polyhedra used in this paper can also be expressed in abstract terms. For example, simplicial maps are defined to be continuous maps of vertices to vertices such that the simplices are also mapped to simplices. Similarly, a PL map between polyhedra \( f : |K_1| \rightarrow |K_2| \) is a continuous map such that there are subdivisions of the simplicial complexes \( K_1 \) and \( K_2 \) for which the map \( f \) is simplicial.

A characterization of the relation of Def.(A.1) to Def.(2.5) is given by the following:

**Theorem (A.2).** Any finite dimensional abstract simplicial complex \( K^n \) embeds in Euclidean space.
First some necessary background: Define \( \mathbb{R}^\infty \) to be the infinite dimensional vector space consisting of vectors of the form \((x_1, x_2, \ldots, x_k, \ldots)\) where each component is real and for any vector, all but a finite number of its components are nonzero. A topology is defined on \( \mathbb{R}^\infty \) by the componentwise convergence of sequences. Observe that an inner product can be defined by
\[
(x, y) = \sum_{i=1}^{\infty} x_i y_i
\]
where \( x, y \in \mathbb{R}^\infty \). Although not necessary for the discussion here, note that the completion of \( \mathbb{R}^\infty \) with respect to this inner product is the separable Hilbert space \( \ell^2(\mathbb{R}) \). Since \( \mathbb{R}^\infty \) is a vector space, simplices can be defined as the convex hull of affinely independent points. Thus simplicial complexes in \( \mathbb{R}^\infty \) are collections of simplices which obey the above abstract definition. One example of a simplicial complex in \( \mathbb{R}^\infty \) is the single simplex \( \sigma^\infty \) defined to be the convex hull of a set orthonormal basis vectors in \( \mathbb{R}^\infty \). However, simplicial complexes in \( \mathbb{R}^\infty \) need not be infinite dimensional. Finally, note that an equivalent definition of a simplicial map between simplicial complexes in \( \mathbb{R}^\infty \) is that it maps vertices to vertices and every simplex is mapped linearly to a simplex.

Given any abstract simplicial complex \( K \), embed its vertices in the set of vertices of \( \sigma^\infty \). This embedding of the vertices is a linearly independent set because it is a subset of the vertices of \( \sigma^\infty \). The embedding can be extended to be a simplicial map \( f : K \to \sigma^\infty \) because the vertices are embedded in linearly independent way and they completely determine all of the simplices. For example, a simplex of the form \( v_1v_2v_3\ldots v_k \) is mapped to \( f(v_1v_2v_3\ldots v_k) \) which is equal to \( f(v_1)f(v_2)f(v_3)\ldots f(v_k) \) in \( \sigma^\infty \). One can verify using the abstract definition of a simplicial complex and simplicial maps that all simplices in \( K \) are mapped to simplices in \( \sigma^\infty \). Moreover, the definition of the topology implies that \( f : |K| \to |\sigma^\infty| \) is continuous and that \( f(|K|) \) is a closed subset of \( \mathbb{R}^\infty \). Furthermore, \( f(K) \) is a subcomplex of \( \mathbb{R}^\infty \).

Finally, if \( f(K) \) is not a subset of a finite dimensional Euclidean space, then there is an
infinite sequence of simplicial complexes \( f(K) \cap \mathbb{R}^{m(k)} \) which are strictly increasing sets where \( m(k) \) is map between positive integers such that \( m(k_1) < m(k_2) \) whenever \( k_1 < k_2 \). Hence, either there are simplices of arbitrary dimension in \( K \), or there are vertices which are in an infinite number of simplices. This contradicts either the fact that \( K \) is finite dimensional or that it is locally finite. Therefore, \( f(K) \) embeds in a finite dimensional Euclidean space. Q.E.D.

Given these tools, the proof of Lemma (3.3) follows immediately. Since each \( S_i \) and \( P_i \) is a polyhedron, they all have triangulations. Furthermore, there are triangulations of \( S_1 \subseteq P_1 \) and \( S_2 \subseteq P_2 \) so that the map \( \psi \) is simplicial as it is a PL homeomorphism. Denote such triangulations by \( K(S_1) \subseteq K(P_1) \) and \( K(S_2) \subseteq K(P_2) \) and the simplicial map corresponding to \( \psi \) by \( K(\psi) : K(S_1) \rightarrow K(S_2) \) where \( \psi = |K(\psi)| \). Using the abstract definition of a simplicial complex it follows that

\[
K(P_1 \cup_{\psi} P_2) = K(P_1) \cup_{K(\psi)} K(P_2)
\]

is a simplicial complex where the simplices of \( K(S_1) \) and \( K(S_2) \) are identified via \( K(\psi) \). Finally, \( K(P_1 \cup_{\psi} P_2) \) is finite dimensional if and only if \( P_1 \) and \( P_2 \) are finite dimensional. Therefore \( K(P_1 \cup_{\psi} P_2) \) is a subset of Euclidean space and \( P_1 \cup_{\psi} P_2 = |K(P_1 \cup_{\psi} P_2)| \) is a polyhedron.

APPENDIX B: \(|E8||

It is useful to outline the proof that \(|E8|| has no smooth structure as a general, readable discussion of this result is not readily accessible. However, a complete list of references will not be provided; the interested reader should consult the work of Freedman\(^{17}\) and Donaldson\(^{18}\) for a detailed list. In order to show that the topological manifold \(|E8|| does not admit a smooth structure, the following background theorem due to Rohlin is needed.

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An orientable smooth 4-manifold admits a spin structure if and only if one can define spinors on the manifold. As PL manifolds uniquely correspond to smooth manifolds in four dimensions, a PL 4-manifold admits a spin structure if its smoothing has one. Then

**Theorem (Rohlin).** Given any closed PL 4-manifold which admits a spin structure, then the signature is a multiple of 16.

The proof of this theorem will be outlined for the case of smooth 4-manifolds; as all PL manifolds are homeomorphic to smooth manifolds in four dimensions, the results can be applied directly to PL manifolds. There are direct ways to prove the theorem without using this fact but for the purposes of the present work it is a simpler approach to outline.

Recall that the signature of a bilinear symmetric form is the number of positive eigenvalues minus the number of negative eigenvalues. The signature of any closed smooth 4-manifold $M^4$ is just the signature given by the bilinear form

$$Q(\alpha, \beta) = \int_M \alpha \wedge \beta$$

where $\alpha$ and $\beta$ are any closed 2-forms representing the second cohomology classes $H^2(M^4; \mathbb{R})$. Equivalently, this form can be constructed in terms of the homology classes of the manifold using the duality between homology and cohomology as real vector spaces. Then the signature is the oriented intersection between two 2-surfaces in general position, each corresponding to an element of the second homology class $H_2(M^4; \mathbb{R})$. Namely, the two 2-surfaces in general position will intersect in a finite number of points. At each point of intersection, a value of $+1$ or $-1$ can be assigned depending on the orientation of the intersections. The intersection form for each pair of 2-dimensional homology classes will be given by the sum of $\pm 1$ over all points of intersection and added to give the total intersection value. This equivalent symmetric bilinear form on the homology is also denoted by $Q$. 

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For smooth closed 4-manifolds with spin structure, the bilinear form $Q$ is even, i.e. $Q(\alpha, \alpha) \equiv 0 \mod 2$. Since the existence of spin structure implies that the Dirac operator is well defined, the Atiyah-Singer index theorem implies the index of the Dirac operator is an integer and equal to one eighth of the signature. In four dimensions, spinors are constructed from the usual Clifford algebra but note that this algebra has the additional structure that each nonzero element has a multiplicative inverse, namely, that it is the division algebra of quaternions. Since multiplicative inverses exist, they can be used as scalar coefficients for defining vector spaces. Since the quaternions are a 4-dimensional real vector space, it follows that quaternionic vector spaces, that is vector spaces with quaternionic coefficients, are $4k$-dimensional real or $2k$-dimensional complex vector spaces where $k$ is any positive integer. Furthermore, the spinors form such a quaternionic vector space. Also the Dirac operator commutes with multiplication by quaternionic numbers. Therefore, the kernel or cokernel of the Dirac operator is always a quaternionic vector space because multiplication of any element in the kernel of the operator by a quaternion results in another element in the kernel. The useful feature about this observation is that it means that the kernel and cokernel must be an $2k$ dimensional complex vector space or equivalently a $4k$ real vector space. Since the index of the Dirac operator is the difference of dimensions of the kernel and cokernel, it follows that it is always even in 4-dimensions. Therefore, the signature is an even multiple of 8 or equivalently a multiple of 16.

It is useful for ease of presentation to assume that the topological manifold is simply connected. This condition simplifies the definition of the signature and spin structure for nonsmooth manifolds: Given a simply connected closed smooth 4-manifold, a necessary and sufficient condition for the manifold to admit a spin structure is that $Q$ is an even form. Since $Q$ is defined for any closed 4-manifold not just smooth 4-manifolds, one can extend this theorem to provide the definition of a simply connected topological 4-manifold with spin structure: A closed simply connected topological 4-manifold with a spin structure is
one with even $Q$. In order to produce a closed simply connected topological 4-manifold which is not PL, one only need produce a spin manifold which violates the conclusion of Rohlin’s theorem. A candidate for the intersection form of such a manifold is provided by the exceptional Lie algebra $E_8$.

The exceptional Lie algebra $E_8$ is a simple 248-dimensional Lie algebra with rank 8. Recall that the rank of a Lie algebra is the dimension of maximal nilpotent subalgebra commonly called the Cartan subalgebra. For semi-simple Lie algebras the Cartan subalgebra is the maximal abelian subalgebra. Since the algebra is simple, one can use the structure constants of the algebra to construct a well defined, non-singular metric called the Killing metric. This metric is positive definite if and only if the Lie group corresponding to the particular Lie algebra is compact as in the case of $E_8$. If the Killing metric is restricted to the Cartan subalgebra, the matrix representation of this bilinear form with respect to the root vector basis is

$$
\begin{pmatrix}
2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 2 \\
\end{pmatrix}.
$$

(12)

The signature of the bilinear form defined by the above matrix is 8. The Dynkin diagram which is an equivalent graphical representation of the above matrix and of $E_8$ is given in Figure 8a).

So if there is a simply connected 4-manifold with (12) as its intersection form, then it cannot be a PL manifold. Thus it cannot have a combinatorial triangulation. Such a manifold indeed exists; it is $|E_8|$. It can be constructed by the following argument; a more detailed construction is given at the end of this appendix using the Dynkin diagram of $E_8$.

Let $P = \{(z, w, u)|z, w, u \in \mathbb{C} \text{ and } z^2 + w^3 + u^5 = 1\}$. This surface is simply connected and

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has one asymptotic region of the form $S \times \mathbb{R}_+$ where $S = SO(3)/I$ as defined in the example of the weak triangulation of a 5-sphere given below Def.(2.14). If the asymptotic region is cut off at some finite distance so that resulting 4-manifold $E$ has boundary $S$, then the bilinear form $Q$ of the compact 4-manifold $E$ is (12). Furthermore, $E$ is simply connected by construction. Since $E$ has boundary, one must cap it off to obtain the desired closed 4-manifold. Freedman proved the following important and needed result: Given any closed 3-manifold $S$ which is a homology sphere, there always exists a compact 4-manifold which is contractible to a point and has boundary $S$. Since this compact 4-manifold is contractible, gluing it onto $E$ along the common boundary $S$ will not change the second homology or the bilinear form of the resulting space. Hence, the manifold $E$ can be capped off to form a closed 4-manifold commonly called $||E8||$ which is simply connected and has the same bilinear form as $E$. One should note that using $||E8||$ to denote the 4-manifold is an abuse of notation but standard usage.

Immediately, Rohlin’s theorem implies $||E8||$ is not a PL manifold as its signature is not a multiple of 16. Hence, it does not have a combinatorial triangulation. Furthermore, using a more careful analysis it can be shown that it does not have a weak triangulation using results of Donaldson. Thus $||E8||$ is an example of a topological manifold that is not homeomorphic to a simplicial complex. Thus in general there are topological manifolds without triangulations.

Finally, plumbing techniques will be used to give a more detailed construction of the manifold $E$. The first step is to observe that the manifolds $S^p \times B^q$ and $S^q \times B^p$ have a common subspace $B^p \times B^q$ as $B^p \subset S^p$ and $B^q \subset S^q$. The plumbing of $S^p \times B^q$ onto $S^q \times B^p$ is then constructed by taking the disjoint union of $S^p \times B^q$ and $S^q \times B^p$ and identifying the common subspace $B^p \times B^q$ via the identity map. The resulting manifold $P^{p+q}$ is written formally as

$$P^{p+q} = (S^p \times B^q) \cup_{B^p \times B^q} (S^q \times B^p).$$
The plumbing of $S^1 \times B^1$ onto $S^1 \times B^1$ is illustrated in Figure 8b).

Although the above plumbing is illustrated using $S^p \times B^q$, plumbing can also be carried out to form manifolds of dimension $p+q$ using unit disk bundles. Given the tangent bundle of a 2-sphere, the unit disk bundle is the bundle of all tangent vectors on $S^2$ which have length less than or equal to one. The total space of this bundle has boundary and dimension four. This bundle is not a trivial product because there is no nowhere vanishing vector field on the 2-sphere. However, locally any bundle is a product so locally, the disk bundle can be written as a trivial bundle. Therefore, if one works on sufficiently small neighborhoods, the plumbing procedure can be applied. Indeed $E$ is constructed by plumbing together unit disk bundles in the appropriate combination.

In order to construct $E$, start with eight unit disk bundles, one associated with each circle on the Dynkin diagram of the Lie algebra $E8$ [see Figure 8a)]. Now, each time a circle on the diagram is connected with a line, plumb those two spaces. Let $E$ be the space resulting from the plumbing procedure. By construction $E$ is a compact 4-manifold with boundary. It can be given a smooth structure. It is also simply connected because each unit disk space is simply connected and the gluing is done along simply connected subspaces.

The last step is to verify that this space is $E$ by showing that the intersection form is the same as the Killing metric restricted to the Cartan subalgebra. In order to this, one must choose a set of generators of the homology of $E$. As the unit disk bundle has the same homology as the 2-sphere, each individual space has a single generator of the second homology corresponding to that of the $S^2$ before plumbing. Hence, the homology of $E$ is generated by eight generators, one for each circle in the Dynkin diagram. If two generators are not plumbed together, then their intersection form must be zero. This produces the zero entries of the matrix for $E8$. Now, given two generators for two unit disk bundles connected by the plumbing, they can only intersect once. The generators
and their intersection for the plumbing of two $S^1 \times B^1$ is illustrated in Figure 8b). It is these generators that produce the off diagonal entries. Finally, the intersection of one the generators with itself is given by pushing the generating sphere a small distance so that it intersects the original sphere and they are in general position. Basically, this intersects in two points so it can have a value of 2 or 0. However, for the two-sphere it is 2 because the intersections do not switch orientations. In general, one can repeat the same procedure using the tangent bundle to show that the intersection of any manifold with itself is the Euler characteristic which is consistent with the answer for the 2-sphere. If one is careful about signs, one can verify that the intersection form of $E$ is the desired form (12).

APPENDIX C: AN UNDECIDABLE GROUP

In Ref.[22], Boone presents a method of constructing undecidable groups from any Thue system with an unsolvable word problem. An explicit presentation of such an undecidable finitely presented group due to Boone is given below. It is based on a Thue system of Post.\textsuperscript{23} The generators of this group are

\[ s_1, s_2, s_3, s_4, q_1, q, t_1, t_2, k, x, y \]

\[ l_i, r_i \quad \text{where} \quad i = 1, 2, \ldots, 11 \]

where the notation has been chosen to simplify the presentation of the relations. The relations are most clearly presented by defining some auxiliary symbols:
\[\Sigma_1 = s_1q_1 \quad \Sigma_2 = s_1q \]
\[\Sigma_3 = s_1s_3 \quad \Sigma_4 = s_2q_1 \]
\[\Sigma_5 = s_2q \quad \Sigma_6 = s_2s_3 \]
\[\Sigma_7 = s_3 \quad \Sigma_8 = s_4s_3q_1s_1 \]
\[\Sigma_9 = s_4s_3qs_2 \quad \Sigma_{10} = s_4q_1s_3 \]
\[\Sigma_{11} = s_4qs_3 \]

and
\[\Gamma_1 = q_1s_1 \quad \Gamma_2 = qs_1 \]
\[\Gamma_3 = s_3s_1 \quad \Gamma_4 = q_1s_2 \]
\[\Gamma_5 = qs_2 \quad \Gamma_6 = s_3s_2 \]
\[\Gamma_7 = s_3s_4s_3 \quad \Gamma_8 = s_1q_1s_4 \]
\[\Gamma_9 = s_3qs_4 \quad \Gamma_{10} = s_1q_1s_3s_4 \]
\[\Gamma_{11} = s_2qs_3s_4 \]

Then the relations of the group are:
\[\Sigma_i = l_i \Gamma_i r_i \]
\[s_\beta l_i = yl_i ys_\beta \]
\[s_\beta y = yy s_\beta \]
\[t_\alpha l_i = l_i t_\alpha \]
\[t_\alpha y = yt_\alpha \]
\[sr_\beta = s_\beta xr_\alpha \]
\[xs_\beta = s_\beta x \]
\[r_k = kr_\alpha \]
\[xk = k x \]
\[kq^{-1}t^{-1}_{1}t_{2}q = q^{-1}t^{-1}_{1}t_{2}qk \]

where \(\alpha = 1, 2\) and \(\beta = 1, 2, 3, 4\). This finite presentation has 33 generators and 144 relations; however, an equivalent presentation of this group can be derived that has fewer generators and relations by standard manipulations. However, this particular presentation is convenient as it can be cleanly written down. Different undecidable groups can be derived using the Thue systems of Markov and Scott.\(^{24}\) In particular, a finitely presented
group with two generators and 32 relations can be derived from the Thue system of Scott; however, one of the relations is astronomical in length.

REFERENCES

1. See for example, J. B. Hartle and S. W. Hawking, Phys. Rev. D 28, 2960, (1983), S. Giddings and A. Strominger, Nucl. Phys. B 306, 890 (1988) and S. Coleman, Nucl. Phys. B 310, 643 (1988), J. Polchinski, Nucl. Phys. B 325, 619, (1989).

2. K. Schleich and D. M. Witt, “Generalized Sums over Histories for Quantum Gravity I: Smooth Conifolds”, UBC preprint, 1992.

3. J. B. Hartle, Class. Quantum Grav. 2, 707 (1985)

4. In particular, one should be aware that modern nomenclature differs from that of H. Seifert and W. Threlfall, A Textbook of Topology, trans. M. A. Goldman, (Academic Press, New York, 1980). (Original German edition of Lehrbuch der Topologie was published in 1934.) For example, Seifert and Threlfall’s definition of a n-manifold is actually a homology manifold in modern terms.

5. See for example, C. P. Rourke and B. J. Sanderson, Introduction to Piecewise-Linear Topology, (Springer-Verlag, New York, 1972).

6. E. H. Spanier, Algebraic Topology, (McGraw-Hill, New York, 1966).

7. A set of points \( \{v_i\} \) are affinely independent if for any fixed vector \( v_j \) in the set, the vectors \( v_i - v_j \) for \( i \neq j \) are linearly independent for all \( v_i \) in the set.

8. See for example the discussion of boundary in section 2 of Ref.[2].
9. Similarly, a compact pure nonbranching simplicial complex is a n-pseudomanifold if and only if $H_n(P^n, \partial P^n; \mathbb{Z}_2) = \mathbb{Z}_2$.

10. J. W. Cannon, *Ann. Math.* **110**, 83, (1979).

11. The excision property is the following: Let $X$ be a space with subset $A$ and $U$ be an open set of $X$ with $\bar{U} \subseteq \text{int}A$. Then $H_*(X, A) = H_*(X - U, A - U)$. See for example, Ref.[6].

12. This observation follows from the classification of 2-manifolds and is discussed in more detail in the next section.

13. S. S. Cairns, *Bull. Amer. Soc.* **41**, 549, (1935).

14. R. Kirby and L. Siebenmann, *Foundational Essays on Topological Manifolds, Smoothings, and Triangulations*, (Princeton University Press, Princeton, 1977).

15. Another reason is that theorems involving invariants such as the topological signature still hold for PL manifolds but are not true for topological manifolds admitting only a weak triangulation.

16. E. E. Moise, *Ann. Math.* **54**, 506, (1951), *Ann. Math.* **55**, 172, (1952), *Ann. Math.* **55**, 203, (1952), *Ann. Math.* **55**, 215, (1952), *Ann. Math.* **56**, 96, (1952).

17. See for example, M. Freedman and F. Quinn, *Topology of 4-Manifolds*, (Princeton University Press, Princeton, 1990) and references therein.

18. S. K. Donaldson and P. B. Kronheimer, *The Geometry of Four-Manifolds*, (Clarendon Press, Oxford, 1990).

19. K. Schleich and D. M. Witt, “Convergence of Einstein Manifolds to Einstein Conifolds”, in preparation.
20. J. Stillwell, *Classical Topology and Combinatorial Group Theory*, (Springer-Verlag, New York, 1980).

21. P. S. Novikov, *Trudy. Mat. Inst. Steklov*, 44, (1955). See also W. W. Boone, ”Certain simple, unsolvable problems of group theory”, V, VI, *Nederl. Akad. Wetensch. Ser. A* 60 22-27, 227-232, (1957).

22. W. W. Boone, *Ann. Math.* 70, 207, (1959).

23. E. L. Post, *J. Symb. Logic* 12, 1, (1947).

24. D. Scott, *J. Symb. Logic*, 21 111, (1956), A. A. Markov, *Dokl. Akad. Nauk. SSSR* 55 587, (1947) and 58, 353 (1947).

25. M. O. Rabin, *Ann. Math.* 67, 172, (1958).

26. H. Rogers Jr., *Theory of Recursive Functions and Effective Computibility*, (MIT Press, Cambridge, Massachusetts, 1987), (Original edition, McGraw-Hill, New York, 1967).

27. Strictly speaking, it may be necessary to further subdivide the simplicial complex with boundary before doubling it over so that the resulting closed space is also a simplicial complex. However, any subdivision of the initial complex is combinatorially equivalent to it by definition and results derived from the subdivided complex will obviously hold for it as well.

28. J. B. Hartle, *J. Math. Phys.* 26, 804 (1985) and *J. Math. Phys.* 27, 287 (1986). The discussion of Regge techniques in these papers is very useful and includes extensive references to the literature. However, as detailed in this section, note that a suggested scheme for constructing all 4-manifolds is flawed and that the classification of simply connected spin 4-manifolds is misunderstood.
29. See for example, W. Browder, *Surgery on Simply-Connected Manifolds*, (Springer-Verlag, New York, 1972).

30. A. A. Markov, in *Proceedings of the International Congress of Mathematicians, 1958* (Cambridge Univ. Press, London, 1960).

31. This proof breaks down in three dimensions because removing a closed curve changes the fundamental group of a 3-manifold.

32. C. D. Papakyriakopoulus, *Bull. Soc. Math. Grèce*, 22, 1 (1943)

33. $K(\pi, n)$ spaces are also known as Eilenberg-MacLane spaces. See for example Ref.[6].

34. See for example, S. W. Hawking, in *General Relativity: An Einstein Centenary Survey* edited by S. W. Hawking and W. Israel (Cambridge University Press, Cambridge, 1979).

35. T. Regge, *Nuovo Cimento* 19, 558 (1961)

36. J. Cheeger, W. Müller and R. Schrader, *Comm. Math. Phys.*, 92, 405, (1984).

37. J. B. Hartle and R. Sorkin, *Gen. Rel. Grav.*, 13, 6, (1981).

38. H. Hamber and R. Williams, *Nucl. Phys. B* 248, 145, (1984).

39. K. Schleich and D. M. Witt, “Summing over 2-manifolds and 2-pseudomanifolds in Regge Calculus”, in preparation.

40. K. Schleich and D. M. Witt, in preparation.

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FIGURE CAPTIONS

**Figure 1:**

Two representations, as an illustration and as a list of elements, of a simplicial complex homeomorphic to a disk are given in a). Another simplicial complex also homeomorphic to a disk is given in b). The diagram c) is not a simplicial complex as the two distinct edges are not uniquely specified by the vertices.

**Figure 2:**

Four examples of 2-dimensional simplicial complexes; a) is not a pure simplicial complex, b) is not a connected simplicial complex, c) is a branching simplicial complex and d) is a pure nonbranching complex, but is not a pseudomanifold.

**Figure 3:**

Two 2-complexes that are combinatorially equivalent when both are subdivided. Both complexes contain the same number of vertices, edges and faces, but note that all vertices in a) are contained in four edges; in b), vertices $a$ and $b$ are in four edges, $c$ and $d$ are in three edges, and $e$ and $f$ are in five edges. Thus both must be subdivided to show combinatorial equivalence.

**Figure 4:**

A 2-pseudomanifold corresponding to a pinched torus; it is constructed from the triangulation of the 2-sphere with two disks corresponding to triangles $bcd$ and $efg$ removed shown in a) by taking the simplicial cone over the boundary. The star of vertex $a$ consists of the eight triangles formed from vertices $abcd$ and $ae fg$ and all corresponding subsets. The link of $a$ consists of the edges defined by $abc$ and $efg$; it is homeomorphic to two disjoint circles.
Figure 5:

The simplicial cone and simplicial suspension of the complex in 1a). Note that both figures are three dimensional. The cone a) consists of the two tetrahedra $eabd$ and $ebcd$ and the corresponding faces and edges. The suspension b) consists of four tetrahedra $eabd$, $ebcd$, $fabd$, $fbcd$ and the corresponding faces and edges.

Figure 6:

A triangulation of $RP^2$ is given in a); note that the opposite edges $ab$, $bc$, and $ca$ are actually the same edge. A simplicial cone and suspension over $RP^2$ is given in b); again note that opposite faces $aic$, $cib$, $bia$, $ajc$, $cjb$ and $bja$ are actually the same. Observe that certain vertex labels and edges have been omitted in b) for clarity.

Figure 7:

Regge curvature in two and three dimensions; the total curvature is the sum of the contributions from all n-simplices that contain the (n-2)-simplex. In a), the contribution to the curvature at $v$ from $vbc$ is the angle between the two unit vectors $n_1$ and $n_2$ that lie in edges $vb$ and $vc$. In b), the contribution to the curvature at $ac$ from $adce$ is the angle between the two unit vectors $n_1$ and $n_2$ that are orthogonal to $ac$ and lie in faces $aec$ and $adc$.

Figure 8:

The Dynkin diagram of the Lie group E8 is given in a). The plumbing of $S^1 \times B^1$ onto $S^1 \times B^1$ is illustrated in b); the grey disks are identified to form the linked rings at lower right. The dotted lines are the generators of the homology and have one point of intersection.