ON THE ENERGY MOMENTUM TENSOR
OF THE M-THEORY FIVEBRANE

Oliver Bärwald, Neil D. Lambert and Peter C. West
oliver, lambert, pwest@mth.kcl.ac.uk

Department of Mathematics, King’s College London
Strand, London WC2R 2LS, Great Britain

Abstract: We construct the energy momentum tensor for the bosonic fields of the
covariant formulation of the M-theory fivebrane within that formalism. We then
obtain the energy for various solitonic solutions of the fivebrane equations of motion.

1 Introduction

There exist two basic extended solitonic objects in eleven dimensional M-theory, a membrane and
a fivebrane. The membrane’s dynamics are described by eight scalar fields and their spinorial
superpartners and can be derived from the usual super-p-brane action [1]. On the other hand the
dynamics of the fivebrane are much more complicated since they include a self-dual three form
tensor field. The equations of motion for this system have only been obtained relatively recently
and provide us with an interesting physical system which has received substantial attention.

There are essentially two formalisms for the dynamics of the M-fivebrane. The first is based
on the manifestly covariant superembedding formalism [2, 3] which we will study here. There is a
second approach based on the non-covariant description of [4, 5]. This latter approach was then
later expanded into a covariant form [6] by the introduction of an auxiliary scalar. In this case the
equations of motion can be obtained from an action and hence many physical quantities such as
the energy density can be readily obtained. However, there are serious objections to the use of an
action for the M-fivebrane [7, 8] due to subtleties associated with the self-duality constraint.

We are therefore interested here in extending the covariant superembedding approach which
does not invoke the use of an action. To date a draw back has been that various physical quantities
such as the energy have not been identified within this formalism. Thus in this paper we aim to
derive directly from the covariant equations of motion the full non-linear energy momentum tensor
for bosonic fields of the M-theory fivebrane.

In addition the covariant equations of motion can lead to lengthy calculations and one expects
that a knowledge of the energy, and more generally the energy momentum tensor, will offer physical
insights into the structure of this highly nonlinear theory. This should also be helpful when finding
and studying new solutions. Finally, since in the action framework the energy is known it might
also lead to a better understanding of the relation between the two approaches.

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This paper is organised as follows: we start by giving a brief introduction into the covariant formulation of the M-fivebrane. We then construct a two parameter family of all symmetric second rank tensors which are covariantly conserved when the fields obey their equations of motion. After fixing the free parameters using the six dimensional supersymmetry we are naturally led to a unique form for the energy momentum tensor. As expected the resulting energy density is positive definite.

Following this we will evaluate the tensor for a number of solitonic solutions to the M-fivebrane equations of motion which involve the self-dual three form \([9, 10]\). We reproduce the results for the energy in all cases where it was previously obtained using the Hamiltonian formalism and also obtain it for one additional case; two intersecting self-dual strings.

2 The M-Theory Fivebrane

Let us consider an M-fivebrane in the \(x^0, x^1, x^2, \ldots, x^5\) plane. The field content consists of five scalars \(X^{a'}, a' = 6, 7, 8, 9, 10\) and a 16 component spinor \(\Theta\) corresponding to the breaking of translation and half the spacetime supersymmetry respectively. However it also contains an antisymmetric second rank tensor gauge field \(B\) whose field strength obeys a self-duality condition.

The classical equations of motion of the fivebrane in the absence of fermions and background fields are \([3]\)

\[
G^{mn} \nabla_m X^{a'} = 0, \tag{1}
\]

and

\[
G^{mn} \nabla_m H_{npq} = 0. \tag{2}
\]

We use \(m, n, p, \ldots = 0, 1, \ldots, 5\) and \(a, b, c, \ldots = 0, 1, \ldots, 5\) for world and tangent indices respectively. The symbols that occur in the equations of motion are defined as follows: the usual induced metric for a \(p\)-brane is given, in static gauge and flat background superspace, by

\[
g_{mn} \equiv \eta_{mn} + \partial_m X^{a'} \partial_n X^{a'} \delta_{a'a'}. \tag{3}
\]

The covariant derivative in the equations of motion is defined using the Levi-Civita connection associated with the metric \(g_{mn}\).

We define the world surface sechsbein associated with the above metric in the usual way via \(e_m^a \eta_{ab} e_n^b \equiv g_{mn}\). There is another inverse metric \(G^{mn}\) which occurs in the equations of motion, mediating the coupling between the scalars and the gauge field and the self-coupling of the latter. It is related to the usual induced metric given above by the equation

\[
G^{mn} \equiv (e^{-1})_m^c \eta^{ca} m_a^d m_d^b (e^{-1})_b^n, \tag{4}
\]

where the matrix \(m\) is given by

\[
m_a^b \equiv \delta_a^b - 2h_{acd} h^{bcd}. \tag{5}
\]

The field \(h_{abc}\) is a three-form which is self-dual

\[
h_{abc} = \frac{1}{3!} \varepsilon_{abcde} h^{def}, \tag{6}
\]

with \(\varepsilon^{012345} = 1\) and \(\eta_{ab} = \text{diag}(-1, +1, \ldots, +1)\), but it is not the curl of a three-form gauge field. It is related to the field \(H_{mnp} \equiv 3\partial_m B_{np}\) which appears in the equations of motion and is the curl of the two-form gauge field \(B_{np}\), but \(H_{mnp}\) is not self-dual in the linear sense. The relationship between the two fields is given by

\[
H_{mnp} = e_m^a e_n^b e_p^c (m^{-1})_c^d h_{abd}. \tag{7}
\]

Clearly, the self-duality of \(h_{abc}\) transforms into a condition for \(H_{mnp}\) and vice versa for the Bianchi identity \(dH = 0\).
3 Constructing the Tensor

3.1 The Three-Form Case

With all scalar fields set to zero the fivebrane dynamics reduce to a system involving a self-interacting three-form in 6-dimensional flat Minkowski-spacetime. For convenience we shall work in the tangent frame in this section. In this case the self-dual three-form tensor field $h_{abc}$ equation of motion becomes

$$m^{ab}\partial_a h_{bcd} = 0,$$

where we write $m_{ab}$ as

$$m_{ab} = \eta_{ab} - 2k_{ab},$$

defining a new matrix $k_{ab}$ by

$$k_{ab} \equiv h_{a}^{\;\;cd}h_{bcd}.$$

(10)

We shall need some consequences of the self-duality of $h_{abc}$, namely

$$k_{aa} = 0,$$

(11)

and

$$k^{ab}k_{bc} = \frac{1}{6}\delta^a_c k^2,$$

(12)

where

$$k^2 \equiv k_{ab}k_{ab}.$$

(13)

We now want to find the energy momentum tensor associated with this system. To be precise we want to construct a second rank symmetric tensor $T_{ab}$ obeying the conservation equation

$$\partial^a T_{ab} = 0.$$

(14)

Observe that the equation of motion (8) has a symmetry, we can send $h_{abc}$ to $-h_{abc}$ and the equation remains unchanged. Demanding that the energy momentum tensor respects this symmetry implies that $h_{abc}$ can appear only quadratically i.e. in the form of $k^2$ or $k_{ab}$.

The most general ansatz compatible with this restriction is

$$T_{ab} = f_1(k^2)\eta_{ab} + f_2(k^2)k_{ab},$$

(15)

where $f_1$ and $f_2$ are two arbitrary functions. From (8) we can obtain an equation of motion for $k_{ab}$, namely

$$m^{ab}\partial_a k_{bc} = \partial^a k_{ac} - 2k_{ab}\partial_a k_{bc} = 0.$$

(16)

We can iterate this equation as follows

$$\partial^a k_{ab} = 2k^{ac}\partial_a k_{cb},$$

$$= 2\partial_a(k^{ac}k_{cb}) - 2k_{cb}\partial_a k^{ac},$$

$$= \frac{1}{3}\partial_b k^2 - 4k_{cb}k^{ad}\partial_a k_{df},$$

$$= \frac{1}{3}\partial_b k^2 + 4k_{cb}k^{ad}\partial_a k_{df} - 4k_{cd}\partial_a k^{ad}k_{dc},$$

$$= \frac{1}{3}\partial_b k^2 + 4k^2\partial^a k_{ab} - \frac{4}{3}k_{ab}\partial^a k^2,$$

$$= \frac{1}{3}m_{ab}\partial^a k^2 + \frac{4}{3}k^2\partial^a k_{ab}. $$

(17)
Rewriting the last line gives
\[ \partial^a k_{ab} = \frac{1}{3} m_{ab} \partial^a k^2 - \frac{1}{3} \frac{k}{k^2}. \] \hspace{1cm} (18)

Plugging this into our ansatz we get
\[ \partial^a T_{ab} = \partial_b f_1(k^2) + k_{ab} \partial^a f_2(k^2) + f_2(k^2) \partial^a k_{ab}, \]
\[ = f_1' \partial_b k^2 + k_{ab} f_2' \partial^a k^2 + f_2 \frac{m_{ab} \partial^a k^2}{3 - 2k^2}, \] \hspace{1cm} (19)
\[ = \left( f_1' + f_2 \frac{1}{3 - 2k^2} \right) \partial_b k^2 + \left( f_2' - \frac{2f_2}{3 - 2k^2} \right) k_{ab} \partial^a k^2. \]

Demanding conservation implies that the expressions in the two brackets should vanish. This gives two ordinary differential equations for \( f_1 \) and \( f_2 \). The general solutions are
\[ f_2 = \frac{\alpha}{3 - 2k^2}, \quad \text{and} \quad f_1 = -\frac{1}{2} \frac{\alpha}{3 - 2k^2} + \beta, \] \hspace{1cm} (20)
with two constants \( \alpha \) and \( \beta \). Hence the most general conserved symmetric tensor built out of \( k_{ab} \) has the form
\[ T_{ab} = \beta \eta_{ab} - \frac{1}{2} \frac{\alpha}{3 - 2k^2} \eta_{ab} + \frac{\alpha}{3 - 2k^2} m_{ab} = \beta \eta_{ab} - \frac{1}{2} \frac{\alpha}{3 - 2k^2} m_{ab}. \] \hspace{1cm} (21)

### 3.2 The Covariant Generalisation

There is an obvious generalisation of this tensor to the case of active scalars, namely by interpreting the flat-space coordinates of the previous section as coordinates of the tangent-frame. However it is not obvious that this doesn’t spoil our earlier reasoning. The setup remains the same apart from the equation of motion (8) which becomes
\[ m_{ab} \nabla_a h_{bcd} = 0. \] \hspace{1cm} (22)

The main difference between working in flat space and working in the moving frame is the fact that the covariant derivatives do not commute. But since all our calculations only involve a single derivative everything goes through as before and we find that
\[ T_{ab} = \beta \eta_{ab} - \frac{1}{2} \frac{\alpha}{3 - 2k^2} m_{ab}, \] \hspace{1cm} (23)

is covariantly conserved
\[ \nabla^a T_{ab} = 0. \] \hspace{1cm} (24)

We can now find the canonical tensor simply by switching to the coordinate frame using the sechsbeins. We will furthermore include a conventional factor of \( \sqrt{-g} \) into our tensor to make sure that
\[ E_{\text{tot}} = - \int d^5 x T^{00}, \] \hspace{1cm} (25)

is the invariant total energy. This allows us to interpret the fivebrane-model alternatively as a nonlinear theory in six-dimensional flat space where \( d^5 x \) rather than \( \sqrt{-g} d^6 x \) is the natural measure.

Rescaling \( \alpha \) for later convenience we get
\[ T^{mn} = \beta \sqrt{-g} g^{mn} + \alpha \sqrt{-g} Q^{-1} m^{mn}, \] \hspace{1cm} (26)
where we introduced \( Q \equiv 1 - \frac{2}{3} k^2 \).
3.3 Fixing the Parameters

The alert reader might be surprised about the appearance of two free parameters in the conserved tensor. This is however to be expected since the energy contains two arbitrary parameters, namely the scale and the origin. Indeed the second constant $\beta$ multiplies the metric $g_{mn}$ which is always covariantly constant and therefore by our construction $\beta$ is entirely arbitrary.

On the other hand in a supersymmetric theory there is an alternative way of computing energy and momentum; they appear on the right hand side of the anti-commutator of two supersymmetry transformations. This relation will allow us to fix one of the parameters.

The most general form of the (2,0) supersymmetry algebra in six dimensions is

$$\{Q^i_\alpha, Q^j_\beta\} = \eta^{ij} \gamma^m_{\alpha\beta} P_m + \gamma^m_{\alpha\beta} Z^m_{ij} + \gamma^m_{\alpha\beta} Z^m_{ij},$$

where $\eta^{ij}$ is the Spin(5) invariant tensor, $P_m$ is the momentum and $Z^m_{ij}$ and $Z^m_{ij}$ are central charges. The spinor indices $\alpha, \beta, \ldots$ run from 1 to 4 as do the internal Spin(5) indices $i, j, \ldots$. The $\gamma$-matrices should not be confused with ordinary $\Gamma$-matrices satisfying $\{\Gamma^m, \Gamma^n\} = 2 \eta^{mn}$. They arise as building blocks of the eleven dimensional $\Gamma$-matrices via

$$\Gamma^m = \begin{pmatrix} 0 & \gamma^m \\ \tilde{\gamma}^m & 0 \end{pmatrix}.$$  

The basic relations are

$$\{\gamma^m, \gamma^n\} \equiv \gamma^m \tilde{\gamma}^n + \gamma^n \tilde{\gamma}^m = 2 \eta^{mn},$$

with $\tilde{\gamma}^m = \gamma^m$ for $m \neq 0$ and $-\tilde{\gamma}^0 = \gamma^0 = 1$. The antisymmetric product is defined as

$$\gamma^{m_1 m_2 m_3 \ldots} \equiv \gamma^{[m_1 \tilde{\gamma}^{m_2} \gamma^{m_3} \ldots},$$

and one also has the following duality relation

$$\gamma^{m_1 m_2 \ldots m_n} = \frac{1}{(6-n)!} (-1)^{\frac{n(n+1)}{2}} \varepsilon^{m_1 m_2 \ldots m_n m_{n+1} \ldots m_6} \gamma_m^{m_{n+1} \ldots m_6}.$$  

We need the local version of equation (27). Recall that to every symmetry of a physical system one can construct an associated conserved current $j_\mu$ which upon integration gives rise to a time independent charge

$$Q = \int d^{d-1}x j_0,$$

which acts as a generator of the symmetry on the fields of the model via

$$\delta \phi = [Q, \phi].$$

The expression on the right hand side of the last equation is understood to be evaluated in terms of the Poisson brackets of the fundamental fields.

Hence the local version of equation (27) is given by

$$\int d^d x \{Q^i_\alpha, j^j_{\alpha\beta}\} = \int d^d x \eta^{ij} \gamma^m_{\alpha\beta} T_{0m} + \text{central charge terms},$$

where $j^j_{m\beta}$ is the supercurrent and $T_{mn}$ the energy momentum tensor. We can rewrite this as

$$\int d^d x \delta^i_{\alpha\beta} j^j_{0\beta} = \int d^d x \eta^{ij} \gamma^m_{\alpha\beta} T_{0m} + \text{central charge terms},$$

from which we learn that the energy momentum tensor can be obtained as the supervariation of the supercurrent.

To make our calculation simpler we shall consider the linearised supersymmetry and also ignore all interactions. This will allow us to use the free-field Poisson brackets and yet will still determine the constant $\beta$. 
Since we are only interested in the bosonic part of the energy we can focus on the part of the supercurrent which is of the form

\[ j = \Theta B + \text{terms cubic in } \Theta, \]

where \( B \) is constructed out of bosonic fields only. The full supervariation of the supercurrent is then given by

\[ \delta j = B \delta \Theta, \]

plus terms which vanish if we set the Fermions to zero. To determine \( B \) consider the supervariation of \( \Theta \), the Dirac conjugate of \( \Theta \). We have

\[ \delta \Theta = \{ Q, \Theta \} = \int d^5x \{ j, \Theta \} = \int d^5x \{ \Theta B, \Theta \} = \int d^5x B \{ \Theta, \Theta \} = B, \]

where we used the free Fermion Poisson bracket \( \{ \Theta(x), \tilde{\Theta}(x') \} = \delta(x-x') \) and ignored terms which vanish if we set the Fermions to zero. Hence we can read off all information from the supervariation of \( \Theta \). We have, at linearised level,

\[ 2 \overline{\Theta} = 1 \delta_{ij} h_{m_1 m_2 m_3} \gamma^m \gamma^{m_1} \gamma^{m_2} \gamma^{m_3}. \]

For the supervariation of the supercurrent we find from our earlier reasoning

\[ \delta j = \frac{1}{4} \delta_{i j} h_{m_1 m_2 m_3} \gamma^m \gamma^{m_1} \gamma^{m_2} \gamma^{m_3}. \]

For simplicity we take only one of the scalars fields to be active, namely \( X \equiv X^5' \). Using the identities

\[ h_{m_1 m_2 m_3} h_{m_4 m_5 m_6} \gamma^m \gamma^{m_1} \gamma^{m_2} \gamma^{m_3} = 2(3!)^2 \gamma_m \gamma_k, \]

and

\[ \partial_m X \partial_n X \gamma^m \gamma^0 \gamma^m = 2 \gamma_m (\frac{1}{2} \eta^{0m} \partial^p X \partial_p X - \partial^m X \partial^0 X), \]

and neglecting the last term which is a central charge contribution we find

\[ \delta j = \frac{1}{2} \overline{\epsilon}_1 \gamma_{m_1} \epsilon_2 \left\{ \frac{1}{2} \eta^{m_1 0} \partial^p X \partial_p X - \partial^m X \partial^0 X - 4k^{n_0} \right\}. \]

Expanding (22) up to terms quadratic in fields gives

\[ T^{mn} = (\alpha + \beta) (\eta^{mn} + \frac{1}{2} \eta^{m 0} \partial^p X \partial_p X - \partial^m X \partial^0 X) - 2ak^{mn}. \]

The two expressions look similar except for the first term in (26) which does not appear in (43). This is to be expected, however, since a configuration with all fields set to zero corresponds to the vacuum and hence must have zero energy in a supersymmetric theory. From the 11-dimensional
viewpoint the same configuration is a flat, static brane which has the constant energy density normalised to one. Comparing the other coefficients gives the relation $\beta = -\frac{1}{2} \alpha$ and we find that

$$T^{mn} = \alpha \frac{1}{2} \sqrt{-g} ((2 - Q) g^{mn} - 4k^{mn}), \quad (45)$$

is the unique conserved, symmetric rank two tensor that is compatible with supersymmetry, justifying the name energy momentum tensor. Following our earlier reasoning we can normalise the tensor by demanding that it reduces to $\eta^{mn}$ if all fields are set to zero. This gives $\alpha = 2$, a choice we adopt from now on. For a static configuration we find the following simple formula for the energy density

$$E = \frac{\sqrt{-g}}{Q} (2 - Q + 4k^{00}). \quad (46)$$

Recall that the field $h_{mnp}$ is not closed and the physics is most naturally described by $H_{mnp}$. We would therefore like to find $T^{mn}$ in terms of $H_{mnp}$. To this end we note two identities which can be readily derived [10]

$$Q = \frac{3}{H^2} \left( 1 - \sqrt{1 + \frac{2}{3} H^2} \right), \quad (47)$$

$$h_{mnp} = Q H_{mnp}^{(+)} \quad (48)$$

where $H_{mnp}^{(+)}$ is the self-dual part of $H_{mnp}$. Therefore one finds

$$k^{mn} = Q^2 H_{mnp}^{(+)} H^{(+)n}_{pq} \quad (49)$$

Finally we note that we can rewrite this tensor in a much simpler form in terms of the natural metric $G^{mn}$ occurring in the superembedding formalism. To this end recall the definition of the inverse metric [4],

$$G^{mn} = (1 + \frac{2}{3} k^2) g^{mn} - 4k^{mn} = (2 - Q) g^{mn} - 4k^{mn}. \quad (50)$$

Using this we find that

$$T^{mn} = \sqrt{-g} Q^{-1} G^{mn}. \quad (51)$$

We can replace the determinant as well. Using

$$\sqrt{-G} = Q^{-3} \sqrt{-g} \quad (52)$$

we find a third expression for the tensor, namely

$$T^{mn} = \sqrt{-G} Q^2 G^{mn}. \quad (53)$$

From this final expression it is obvious that the energy, given by $E = -T^{00}$, is always positive definite. Note that $\sqrt{-G}$ and $Q^2$ are always positive and $G^{00}$ is the time-time-component of the metric which occurs naturally in the embedding formalism. Using our conventions this implies that $G^{00}$ is negative definite and hence the energy is positive definite. We note that this agrees with the energy momentum tensor obtained using the action formulation [12].

4 Applications

To make contact with previous work on the energy of fivebrane configurations we now evaluate our tensor for some of the known solitonic solutions to the fivebrane equations of motion. For most cases expressions for the energy are also known [9, 10] from the noncovariant Hamiltonian formalism [1, 2, 3]. Here we are able to reproduce these known results and also to determine the energy for the intersecting self-dual string solution of [10, 14].
4.1 The Self-Dual String

We are looking for a string-soliton whose world sheet lies in the \((x^0, x^5)\)-plane and hence take all fields to be independent of these two coordinates. We shall denote indices ranging from 0 to 5 from indices ranging from 1 to 4 by putting a hat on the former. We use the following ansatz:

\[
\begin{align*}
X^6' &\equiv \phi, \\
h_{05a} &\equiv v_a, \\
h_{abc} &\equiv \epsilon_{abcd}v^d,
\end{align*}
\]

the last equation being a consequence of the self-duality of \(h\). All other scalars and all other components of \(h^{\hat{a}\hat{b}\hat{c}}\) are set to zero. To evaluate the energy-momentum tensor we have to calculate the \(m\)-matrix. In the tangent frame we get:

\[
m_{\hat{a}\hat{b}} = \begin{pmatrix} 1 + 4v^2 & 0 & 0 \\ 0 & (1 - 4v^2)\delta_{ab} + 8v_a v_b & 0 \\ 0 & 0 & 1 + 4v^2 \end{pmatrix}.
\]

The usual induced metric reduces in static gauge to

\[
g_{\hat{m}\hat{n}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \delta_{mn} + \partial_m \phi \partial_n \phi & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]

We shall also need the inverse metric and the associated sechsbein, they take the form of

\[
g^{\hat{m}\hat{n}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \delta_{mn} + \partial_m \phi \partial_n \phi & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]

and

\[
e_{m}^{\hat{a}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \delta^n_{m} + c\partial_m \phi \partial^n \phi & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]

where \(-g = -\det g = 1 + |\partial \phi|^2\) and \(c = \frac{-1 + \sqrt{-g}}{\sqrt{-g}}\).

We are only interested in the energy of this configuration. We find

\[
-E = T^{00} = \sqrt{-g}g^{00}e_0^0 e^0_0 (2m_0 Q^{-1} - \delta^0_0) = -2\sqrt{-g} (1 + 4v^2)Q^{-1} + \sqrt{-g}.
\]

Using \(Q = 1 - \frac{2}{3}k^2 = 1 - 16v^4\) this reduces to

\[
E = \sqrt{-g} \frac{2}{1 - 4v^2} - \sqrt{-g}.
\]

Demanding that the solution preserves half the supersymmetries leads to the Bogomol’nyi condition

\[
v_a = \frac{1}{2} \frac{\partial_a \phi}{\sqrt{1 + |\partial \phi|^2}}.
\]

A bit of algebra gives \(1 - 4v^2 = \frac{2}{1 + \sqrt{-g}}\) and hence finally

\[
E = -g = 1 + |\partial \phi|^2,
\]

which agrees with the result obtained using the non-covariant formalism.
4.2 Neutral Strings: The Instanton

This solution is obtained by setting all scalars to zero, and taking the remaining fields to be independent of the \(x^0\) and \(x^5\) directions. Since with all scalars inactive the induced metric is the flat metric we shall work in the tangent frame in this section. Furthermore we take the three-form to be

\[
h_{0ab} = \pm h_{5ab} = F_{ab}, \tag{63}\]

with all other components set to zero. Depending on the sign the two-form \(F_{ab}\) is taken to be either self-dual or anti-self-dual. We get the following expression for \(k^A_b\),

\[
k^A_b = \begin{pmatrix} -F^2 & 0 & F^2 \\ 0 & 0 & 0 \\ -F^2 & 0 & F^2 \end{pmatrix}, \tag{64}\]

where \(F^2 = F^{ab}F_{ab}\) and the scalar \(k^2\) vanishes. Focusing on the energy we have

\[
E = -T^{00} = 1 + 4F^2. \tag{65}\]

This also agrees, up to a rescaling of \(F \to \frac{1}{4}F\), with the result obtained from the non-covariant formalism \([10]\).

4.3 Combining Neutral and Self-Dual Strings

If we superpose the solutions of the two previous subsections we get the following expression \([10]\) for \(k^A_b\),

\[
k^A_b = \begin{pmatrix} -F^2 - 2v^2 & \sqrt{2v_c}F^{bc} & F^2 \\ 4\sqrt{2v_c}F_{ac} & 2v^2\delta^b_c - 4v_av^b & -4\sqrt{2v_c}F_{ac} \\ -F^2 & \sqrt{2v_c}F^{bc} & F^2 - 2v^2 \end{pmatrix}. \tag{66}\]

Using the Bogomol’nyi condition \([11]\) gives after a lengthy calculation the following expression for the energy

\[
E = 1 + |\partial\phi|^2 + (1 + \sqrt{1 + |\partial\phi|^2})^2F^2. \tag{67}\]

This result uses the unphysical field \(F_{ab}\) which is not the curl of a one-form gauge field. Recall that in general the self-dual three-form \(h_{abc}\) is not equal to the physical three-form \(H_{abc}\). The relation between \(F_{ab}\) and the physical field \(K_{ab} = H_{0ab}\) is given by

\[
K_{ab} = (1 - 4v^2)^{-1}F_{ab} = \frac{1}{2}(1 + \sqrt{1 + |\partial\phi|^2})F_{ab}. \tag{68}\]

Hence in terms of \(K\) the energy is given by

\[
E = 1 + |\partial\phi|^2 + 4K^2, \tag{69}\]

which up to a rescaling \(K \to \frac{1}{4}K\) agrees with the expression for the energy obtained using the Hamiltonian formalism \([10]\).

4.4 Intersecting Self-Dual Strings: The Monopole

As a final example we now consider two intersecting self-dual strings \([10, 14]\). This soliton can be related to monopole configurations in \(N = 2\) supersymmetric Yang-Mills theories \([14]\) and the evaluation of the energy is therefore of some interest.

We have two active scalars, \(X^6\) and \(X^7\), depending on the 4 spacetime coordinates \(x^0, \ldots, x^3\) and two additional coordinates \(x^4\) and \(x^5\). It will be useful to introduce complex coordinates for these

\[
z \equiv x^4 + ix^5, \tag{70}\]

\[
E = 1 + |\partial\phi|^2 + 4K^2, \tag{69}\]

which up to a rescaling \(K \to \frac{1}{4}K\) agrees with the expression for the energy obtained using the Hamiltonian formalism \([10]\).
and also to combine the two real scalar fields into a single complex scalar field

\[ s = X^0 + iX^7. \]  

(71)

We denote the associated derivatives by

\[ \partial \equiv \partial_4 + i\partial_5. \]  

(72)

In this section we use indices \( a, b, c, \ldots = 0, 1, 2, 3 \) in the tangent-frame, where we will perform all calculations, indices \( m, n, p, \ldots = 0, 1, 2, 3 \) in the world-frame and put hats on these to denote indices taking the full range from 0 to 5. In the tangent frame we shall also use \( i, j, k, \ldots = 1, 2, 3 \) to denote purely spatial indices.

In the complex coordinate system the flat metric and its inverse are given by

\[ \eta_{\hat{a}\hat{b}} = \begin{pmatrix} \eta_{ab} & 0 \\ 0 & \eta \end{pmatrix}, \quad \text{and} \quad \eta^{\hat{a}\hat{b}} = \begin{pmatrix} \eta^{ab} & 0 \\ 0 & \eta \end{pmatrix}, \]  

(73)

where \( \eta_{ab} \) and \( \eta^{ab} \) denote the usual four-dimensional flat Minkowski-metric and its inverse. In the world-frame we find

\[ g_{\tilde{m}\tilde{n}} = \eta_{\tilde{m}\tilde{n}} + \frac{1}{2} \partial_{\tilde{m}} s \partial_{\tilde{n}} \tilde{s} + \frac{1}{2} \partial_{\tilde{n}} s \partial_{\tilde{m}} \tilde{s}. \]  

(74)

We also need the determinant which is given by

\[ -g = - \det g = (1 + |\partial s|^2 - |\bar{\partial} s|^2)^2 + 4|\bar{\partial} s|^2 \\
+ |\bar{\partial} s|^2 (1 + |\partial s|^2 - |\bar{\partial} s|^2) + (\partial_i s) (\bar{\partial} s) + (\partial_i \bar{s})^2 \partial s \partial \bar{s}. \]  

(75)

We will only consider static solutions, i.e. have set \( \partial_0 \equiv 0 \) and also have expanded all expressions up to second order in the spatial derivatives for simplicity. The process of solving the fivebrane equations starts with making an ansatz for the six-dimensional three-form. We decompose \( h \) into four-dimensional two-forms and vectors as follows

\[ h_{ab\hat{z}} = \kappa \mathcal{F}_{ab}, \quad h_{ab\hat{z}} = \bar{\kappa} \mathcal{F}_{ab}, \quad h_{a\hat{z}z} = i\nu_a. \]  

(76)

Self-duality of \( h \) implies that \( h_{abc} = 2\epsilon_{abcd} e^d \) and \( \mathcal{F}_{ab} = \frac{1}{6} \epsilon_{abcd} \mathcal{F}^{cd}. \)

Demanding preservation of half of the supersymmetry leads to the following set of Bogomol’nyi-conditions \([10, 14]\). Given in the tangent frame they are

\[ \kappa \mathcal{F}_{0\hat{i}} = \frac{1}{8} \eta \left( \frac{1 + |\partial s|^2 - |\bar{\partial} s|^2}{X^2 - |\partial s|^2} \right) \left( X^2 \partial s + \bar{\partial} s \partial s \bar{s} \tilde{s} \right) \left( X \det e \right), \]  

\[ v_0 = + \frac{i}{16} \eta \left( \frac{1 + |\partial s|^2 - |\bar{\partial} s|^2}{X^2 - |\partial s|^2} \right) \left[ (1 + |\partial s|^2 + |\bar{\partial} s|^2) (\partial \bar{s} \partial s \partial \bar{s} \tilde{s} \bar{s}) - (\det e)^2 \right] + \frac{i}{4} \eta \left( \frac{\partial \bar{s} \partial s \partial \bar{s} \tilde{s} \bar{s}}{(X^2 - |\partial s|^2)} \right) \]  

\[ v_i = - \frac{1}{16} \eta \bar{s} \left( 1 + |\partial s|^2 - |\bar{\partial} s|^2 \right) \left( X^2 - |\partial s|^2 \right) \epsilon_{ijk} \partial \bar{s} \partial \bar{s} \partial \bar{s} \tilde{s} \bar{s} \left( X \det e \right), \]  

\[ \bar{s} = - \partial \bar{s}, \]  

(77)

where \( \eta = \pm 1 \) and the conditions for the remaining components of \( \mathcal{F} \) are obtained by using its self-duality. We have used the following convenient expressions

\[ \det e \equiv \sqrt{(1 + |\partial s|^2 - |\bar{\partial} s|^2)^2 + 4|\bar{\partial} s|^2}, \]  

(78)

\[ X^2 \equiv \frac{1}{2} (1 + |\partial s|^2 + |\bar{\partial} s|^2 + \det e). \]  

(79)

Here \( \det e \) denotes not the full determinant of the vielbein but only the part without spatial derivatives.
In the static case we have the following formula for the energy

\[ E = \sqrt{-g^2 - Q - 4k_0^0} \tag{80} \]

We only need to know \( Q \) and \( k_0^0 \) which are given by

\[ Q = 1 - 256v_0^2(v_0^2 - 2|\kappa|^2F_{0i}\bar{F}^{0i}), \tag{81} \]
\[ k_0^0 = -8v_0^2 + 8|\kappa|^2F_{0i}\bar{F}^{0i}. \tag{82} \]

Despite the complexity of the Bogomol’nyi conditions we finally get a remarkably simple answer for the energy, namely

\[ E = -\frac{g}{1 + |\partial s|^2 - |\bar{\partial} s|^2}, \tag{83} \]

where the determinant of the spacetime metric is given by (75).

If we take \( s \) to be a holomorphic function of \( z \) the energy reduces to

\[ E = 1 + |\partial s|^2 + |\bar{\partial} s|^2. \tag{84} \]

Reverting to the real scalar fields and setting \( \partial \equiv 0 \) we find

\[ E = 1 + (\partial_i X^0)^2 + (\partial_i X^7)^2, \tag{85} \]

which agrees with the energy obtained using the Hamiltonian formalism as given in [10].

5 Conclusions

In this paper we have shown that the tensor

\[ T_{mn} = \sqrt{-g} \left( (2 - Q)g^{mn} - 4Q^2H^{(+)}mpqH^{(+)}n_{pq} \right), \tag{86} \]

where \( Q = -\frac{1}{4}\sqrt{1 + \frac{2}{3}H^2} \) and \( H^{(+)} \) denotes the selfdual part of \( H \), is covariantly conserved, compatible with supersymmetry and gives expressions for the energy of solitonic configurations which agree with the Hamiltonian expressions in all known cases.

In closing we remark that although the change of variables between the covariant and action approaches is rather complicated [12] our results and those of [10] suggest that, for BPS states, we may simply identify

\[ \tilde{H}_{mn}^{\text{PST}} = \frac{1}{4}H_{mn}^{\text{covariant}}v^p, \tag{87} \]

where \( v^p \) is the unit vector in the action formulation [1]. Perhaps this observation will lead to a better understanding of the relation between the two approaches.

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