Self-similar solution for power-law liquid flow down an inclined plane

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Abstract. Within the Stokes film approximation, unsteady plane-parallel spreading of a thin layer of a heavy non-Newtonian fluid along an inclined solid surface is studied. The forced spreading regime induced by mass supply is considered. An evolution equation for the film thickness is derived. The group classification problem is solved. Self-similar solution is constructed for the power time dependence on mass supply.

Keywords: non-Newtonian fluid, Stokes film approximation, evolution equation, symmetry operator, self-similar solution

1. Introduction
In different applications it is necessary to consider motion of non-Newtonian fluids. Such fluids often occur in nature and are used in industry. Mudflow, volcanic lava, oil, paint and lacquer, polymer melts and solutions, granulated material are the examples of non-Newtonian liquids. Nonlinear dependence of shear rate at each point of liquid volume on shear stress at the same point characterizes liquids with complex rheology properties. Fundamental characteristics of liquefied and solid-like objects, methods of experimental determination of rheology properties and detailed description of hydraulic flows are given in [1]. Determination of the principles of the non-Newtonian laminar flow is presented in [2]. The reference of rheological equations for different materials is stated in [3].

Slow spreading of a thin incompressible layer of a heavy non-Newtonian fluid along an inclined solid surface is studied at a given localized mass supply. The surface tension effect is neglected. A rheological equation is formulated in invariant terms [3]. An evolution equation for the film thickness is derived. The group classification problem is solved. Self-similar solution form is found in case when the law of mass supply in the film is a power function of time.

2. Rheological model of flow
A rheological equation is formulated in invariant terms [3] for two-dimension flow

\[ \tau_{ij} = 2KI_2^{n-1}e_{ij}, \]

where \( \tau_{ij} \) are the components of the viscous stresses tensor, \( K \) and \( n \) are empirical constants, \( e_{ij} \) are the components of the deformation rate tensor and \( I_2 \) is the second invariant of the deformation rate tensor

\[ I_2 = e^{ij}e_{ij} = u_x^2 + w_z^2 + \frac{1}{2}(w_x + u_z)^2. \]
3. The equations of motion and boundary conditions

The two-dimension slow source flow of a thin layer of a heavy non-Newtonian fluid is considered. The law of mass supply in the film is a power function of time. The flow is on an inclined solid surface. The angle of inclination of the layer equals $\alpha$. The flow is described by equations of mass and momentum conservation. The system of equations is closed by rheological formula. We align coordinates so that the plane occupies $z = 0$, and $x$ points downslope, origin of the coordinate system is related to the source. The flow is considered slow and laminar, all its characteristics depend on the spatial coordinates $x$ and $z$ and time $t$. The film thickness does not depend on coordinate $z$. The equation of motion in the projections on the coordinate axes has, respectively, the form

$$
\rho \frac{du}{dt} = \rho g \sin \alpha + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xz}}{\partial z},
$$
$$
\rho \frac{dw}{dt} = -\rho g \cos \alpha + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{zz}}{\partial z},
$$

where $\tau_{xx}$, $\tau_{xz}$ and $\tau_{zz}$ are the components of the viscous stresses tensor. Let us write the conservation law

$$
\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0.
$$

The boundary conditions at free surface $z = h$ are

$$
p = \tau_{xz} = 0, \quad \tau_{zz} = 0.
$$

No-slip condition is suggested at the solid wall $z = 0$

$$
u = 0, \quad w = 0.
$$

Let us write the equations of mass and momentum conservation in a dimensionless form. To remove dimensions we use $L$ for the distance downslope from the source. Then the conservation law (2) gives

$$(x, z) = L(x^*, \gamma z^*),$$
$$(u, w) = U(u^*, \gamma w^*),$$

$$t = \frac{L}{U} t^*, \quad p = \rho g \sin \alpha \gamma L p^*,$$

$$\tau_{ij} = \rho \nu \frac{U}{\gamma L} \tau_{ij}^*.$$

Dimensionless values are marked with an asterisk. The thin boundary-layer approximation is invoked by neglecting terms of quadratic order in the cross-stream aspect ratio ($\gamma^2 \ll 1$), so the pressure field is hydrostatic. The characteristic flow speed $U$ and an effective kinematic viscosity $\nu$ have the following form

$$U = \frac{\gamma^3 L^2 g \cos \alpha}{\nu},$$
$$\nu = \frac{K}{\rho} \left( \frac{U}{\gamma L} \right)^{n-1}.$$

Then the the asterisk decoration is lowered for convenience.

The dimensionless second invariant of the deformation rate tensor and the dimensionless components of the deformation rate tensor become

$$I_2 = u_x^2 + w_z^2 + \frac{1}{2} \left( \gamma w_x + \frac{u_z}{\gamma} \right)^2,$$
$$\epsilon_{ij} = \frac{1}{2} \left( \begin{array}{c}
2u_x \\
2w_z \\
u + \gamma^2 w_x \\
u + \gamma^2 w_z
\end{array} \right).$$

(5)
When neglecting small terms in (5), we obtain

\[ I_2 = \frac{1}{2} u_z^2, \]

\[ e_{ij} = \frac{1}{2} \begin{pmatrix} 2u_x & u_z \\ u_z & 2w_z \end{pmatrix}. \]

Let us write equations of motion (1) in dimensionless form

\[ \frac{U^2}{L} \left( u_t + uu_x + wu_z \right) = -\frac{\nu U}{\gamma^2 L^2} p_x + \frac{\nu U}{\gamma^2 L^2} \gamma + \frac{\nu U}{\gamma^2 L^2} (\gamma^2 \tau_{xx,x} + \tau_{xz,z}), \]

\[ \frac{\gamma U^2}{L} \left( w_t + uu_x + wu_z \right) = -\frac{\nu U}{\gamma^4 L^2} p_z - \frac{\nu U}{\gamma^4 L^2} + \frac{\nu U}{\gamma L^2} (\tau_{xz,x} + \tau_{zz,z}). \]

Let \( S = \tan \alpha / \gamma \) be a slope parameter that characterizes the angle of inclination of the underlying surface and the Reynolds number \( Re = \delta^2 \gamma^2 \nu U / L \). Then the equations of motion (6) become

\[ \gamma^2 Re (u_t + uu_x + wu_z) = -p_x + S + \gamma^4 \tau_{xx,x} + \tau_{xz,z}, \]

\[ \gamma^4 Re (w_t + uu_x + wu_z) = -p_z - 1 - \gamma^2 (\tau_{xz,x} + \tau_{zz,z}) \]

4. Equation for the film thickness

The flow is considered slow so that \( \gamma^2 Re \ll 1 \). We obtain the following system of equations by neglecting small terms in (7)

\[ p_x = S + \tau_{xx,z}, \]

\[ p_z = -1. \]

We find the solution for the system of equations (8) taking into account boundary conditions (3) and (4) that are written in dimensionless form. Hence it follows that

\[ \tau_{xz} = (S - h_x)(h - z). \]

On the other hand

\[ \tau_{xz} = \frac{u_z^n}{B}, \]

where \( B = 2^{\frac{n+1}{2}} \). As a result we have a formula for the \( x \)-component of the velocity

\[ u = \frac{n}{n + 1} B^{\frac{n}{2}} (S - h_x)^{\frac{1}{2}} (h - z)^{\frac{1}{2} + 1}. \]

We use continuity equation (2) in dimensionless form to find \( w = w(x, y, z) \). Thus

\[ w|_{z=h} = \frac{B^{\frac{1}{2}}}{(n+1)(2n+1)} (S - h_x)^{\frac{1}{2} - 1} h_x^{\frac{1}{2} + 1} ((n+1)hh_{xx} - (2n+1)(S - h_x)h_x). \]

The kinematic condition \( w = h_t + uh_x \) is valid at \( z = h = h(t, x) \)

\[ w|_{z=h} = h_t + \frac{nB^{\frac{1}{2}}}{(n+1)} (S - h_x)^{\frac{1}{2}} h_x^{\frac{1}{2} + 1}. \]
We set equal the right parts of expressions (9) and (10) and obtain the equation for the film thickness

\[(2n + 1)h_t + B^{\frac{1}{n}}(S - h_x)^{\frac{1}{n}}h^{\frac{1}{n} + 1}((2n + 1)(S - h_x)h_x - hh_{xx}) = 0.\]  

(11)

The statement of the problem for finding the surface shape \(h(t, x)\) is completed by the formulation of an integral condition following from the mass supply law and the requirement of zero film thickness at the leading front \(x_f\) of the wetted area

\[\int_0^{x_f} h(x, t)dx = Q(t), \quad h(t, x_f) = 0.\]  

(12)

5. Group classification

Symmetry operators of equation (11) are found as

\[X = \xi^1 \frac{\partial}{\partial x} + \xi^2 \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial h},\]

where \(\xi^1 = \xi^1(t, x, h), \xi^2 = \xi^2(t, x, h)\) and \(\eta = \eta(t, x, h)\). Applying the criterion of invariance [4] we obtain the system of the determining equations. The analysis of the system solution shows that the kernel of the symmetry operators consists of the symmetry operator

\[X_1 = x \frac{\partial}{\partial x} - t \frac{\partial}{\partial t} + h \frac{\partial}{\partial h}.\]  

(13)

6. Self-similar solution

The transformation group corresponding to symmetry operator (13) is the group of inhomogeneous dilations. The group invariants have the following form

\[I_1 = xt^n, \quad I_2 = ht^n.\]

The solution of equation (11) is found in the form \(h = t^{-n}F(\eta), \) where \(\eta = Cxt^n, C = const.\) As a result we have an ordinary differential equation for the determining of \(F(\eta)\)

\[(2n + 1)n(\eta F' - F) + (2n + 1)CB^{\frac{1}{n}}(S - CF')^{\frac{1}{n}}F'F^{\frac{1}{n} + 1} - C^2B^{\frac{1}{n}}(S - CF')^{\frac{1}{n} - 1}F''F^{\frac{1}{n} + 2} = 0.\]  

(14)

Volume of the liquid bounded by the film surface shape according to condition (12) is calculated as

\[Q(t) = \frac{1}{C} t^{-2n} \int_0^{\eta_f} F(\eta)d\eta.\]

The value of \(\eta_f\) is a constant as \(F(\eta_f) = 0.\) Thus, the mass supply law in the film is a power function of time and it is written as

\[Q(t) = At^{-2n}, \quad A = \frac{1}{C} \int_0^{\eta_f} F(\eta)d\eta = const > 0.\]

The value of the constant \(C\) is chosen so that the self-similar coordinate of the leading edge \(\eta_f\) is equal to unity. Next the constant \(C\) is defined by formula

\[C = A^{-1} \int_0^1 F(\eta) d\eta.\]  

(15)
The solution of the equation (14) is defined by boundary condition $F(1) = 0$ and formula (15). Asymptotic of the solution of equation (14) in the neighborhood of point $\eta = 1$ is found as

$$F(\eta) = F_0(1 - \eta)^p + \cdots .$$

After the substitution of the sought solution asymptotic expansion in equation (14) we obtain

$$p = \frac{1}{n + 2} ,$$

$$F_0^{1 + \frac{2}{p}} C^{1 + \frac{1}{p}} B^p = -(2n + 1) .$$

Further case of $0 < p < 1$ is considered. So

$$-1 < n < -\frac{1}{2} .$$

And as a result we have

$$F(\eta) \simeq \left( -\frac{2n + 1}{C^{1 + \frac{1}{p}} B^p} \right)^{-\frac{p}{n+2}} (1 - \eta)^{\frac{1}{n+2}} . \quad (16)$$

Equation (14) was solved numerically using the asymptotic (16) and condition (15).

Figure 1 shows the results of numerical solution of equation (14) at $A = 0.0085$, $n = -0.6$ and $-0.7$. It has the second order and it is solved numerically using Runge–Kutta method.

**Figure 1.** Function $F(\eta)$

7. Conclusion

The evolution equation for the film thickness is derived in the work. The group classification problem is solved. Self-similar solution form is found in case when the law of mass supply in the film is a power function. The results of numerical calculations are presented.

Acknowledgments

This work was supported by RFBR grant 18-01-00890.

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