INFINITELY MANY SOLUTIONS FOR GENERALIZED QUASILINEAR SCHRÖDINGER EQUATIONS WITH SIGN-CHANGING POTENTIAL

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Abstract. We investigate a class of generalized quasilinear Schrödinger equations

$$-\text{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u = f(x,u) \quad \text{in} \quad \mathbb{R}^N,$$

where $g(u) : \mathbb{R} \to \mathbb{R}^+$ is a nondecreasing function with respect to $|u|$, the potential $V(x)$ and the primitive of the nonlinearity $f(x,u)$ are allowed to be sign-changing. Under some suitable assumptions, we obtain the existence of infinitely many nontrivial solutions. The proof is based on a change of variable as well as symmetric Mountain Pass Theorem.

1. Introduction and main results. We are concerned with the following generalized quasilinear Schrödinger equations:

$$-\text{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u = f(x,u) \quad \text{in} \quad \mathbb{R}^N, \quad (1.1)$$

where $N \geq 1$ and $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$. Moreover, we assume the function $g(u)$ and the potential $V(x)$ satisfy the following assumptions:

$(g)$ $g(u) : \mathbb{R} \to \mathbb{R}^+$ is an even function with $g'(u) \geq 0$ for all $u \geq 0$, $ug'(u) < g(u)$ for all $u \in \mathbb{R}$ and $g''(u) \geq 0$ is strict on a subset of positive measure in $\mathbb{R}$;

$(V_1)$ $V \in C(\mathbb{R}^N, \mathbb{R})$ and $\inf_{x \in \mathbb{R}^N} V(x) > -\infty$;

$(V_2)$ for each $M > 0$, there exists a constant $r > 0$ such that

$$\lim_{|y| \to +\infty} \text{meas}(\{x \in \mathbb{R}^N : |x - y| \leq r, V(x) \leq M\}) < +\infty = 0,$$

where $\text{meas}(\cdot)$ denotes the Lebesgue measure in $\mathbb{R}^N$.

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The quasilinear Schrödinger equations, of the form similar to problem (1.1), arise in various branches of mathematical physics, and they are related to the existence of standing wave solutions for quasilinear Schrödinger equations

\[ i_z = -\Delta z + W(x)z - k(x, z) - \Delta (l(|z|^2))l'(|z|^2)z, \quad x \in \mathbb{R}^N, \quad (1.2) \]

where \( z : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C} \), \( W : \mathbb{R}^N \to \mathbb{R} \) is a given potential, \( l \) and \( k(x, z) \) are real functions. We would like to mention that quasilinear equations of the form (1.2) have been derived as models of several physical phenomena corresponding to various types on nonlinear terms \( l \). For instance, in the case of \( l(s) = s \), problem (1.2) was used as a model of the time evolution of the condensate wave function in superfluid film, and was called the superfluid film equation in fluid mechanics by Kurihura (see [10]); the case \( l(s) = (1 + s)^{1/2} \) models the self-channeling of a high-power ultrashort laser in matter, see [2, 3, 4, 19]. Equation (1.2) also appears in plasma physics and fluid mechanics [10, 11, 14, 17], in mechanics [9] and in condensed matter theory [15].

Putting \( z(t, x) = \exp(-iEt)u(x) \) in (1.2), where \( E \in \mathbb{R} \) and \( u(x) > 0 \) is a real function, we obtain a corresponding of elliptic type

\[ -\Delta u + V(x)u - \Delta (l(|u|^2))l'(|u|^2)u = k(x, u), \quad x \in \mathbb{R}^N. \quad (1.3) \]

If we let

\[ g^2(u) = 1 + \frac{[l(|u|^2)]^2}{2}, \]

then (1.3) turns into (1.1) (see [8, 20]). And if we set \( g^2(u) = 1 + 2u^2 \), i.e., \( l(s) = s \), we get the superfluid film equation in plasma physics:

\[ -\Delta u + V(x)u - \Delta (u^2)u = k(x, u), \quad x \in \mathbb{R}^N. \quad (1.4) \]

In the past, the research on the existence of standing wave solutions for equation (1.3) mainly concentrates upon some given special function \( l(s) \), a natural question is whether there is a unified method to study (1.3) with general functions \( l(s) \)?

Recently, Shen and Wang [20] have given an affirmative answer and obtained the existence of positive solutions for (1.1) when \( f(x, u) \) is superlinear and subcritical. We must point out that the authors introduced a new change of variable in this work which provides us a useful method in studying problem (1.1). Later, the results was extended by Deng et al. in [7, 8], and they established the existence of positive solutions when \( f(x, u) \) is critical. In [6], Deng et al. constructed the existence of infinitely many sign-changing solutions for problem (1.1) under some assumptions on \( g, f \) and \( V \). And we have also studied the existence and multiplicity of solutions for problem (1.1) by using this change of variable in [21, 22, 23].

However, in the aforementioned papers [6, 7, 8, 20, 21, 22, 23], the authors all assumed the potential \( V(x) \) is non-negative. As far as we know there are no papers dealing with the case which the problem (1.1) has a sign-changing potential. We also note that in most of the above papers, the condition of the type of Ambrosetti-Rabinowitz (or shortly, (AR)-condition), that is
Theorem 1.1. Suppose that the assumptions 
\( \rho > 0 \) (weak) convergence is denoted by \( \rightarrow N \) which tends to zero when 
\( f(x,u) = \int_0^u f(x,s)ds \) and \( G(u) = \int_0^u g(s)ds \), is necessary. However, there are many functions which do not satisfy the condition (f).

Motivated by all facts mentioned above, it is very natural for us to pose some questions as follows:

(Q1) Can one establish the suitable variational framework for problem (1.1) with the sign-changing potential, precisely, the potential \( V(x) \) satisfies the assumptions (V1) and (V2).

(Q2) If the condition (f) is replaced by a weaker condition or some other suitable conditions in problem (1.1), will the problem admit one or multiple nontrivial solutions?

In the present paper, we restrict our attention to the existence of infinitely many solutions for problem (1.1) and try to seek definite answers to questions (Q1) and (Q2).

Before stating our main results, we need to introduce some notations.

**Notation.** Throughout this paper, we denote \( 2^* = \infty \) if \( N = 1, 2 \) and \( 2^* = \frac{2N}{N-2} \) if \( N \geq 3 \); \( C \) and \( C_i \) will denote various positive constants; the strong (respectively weak) convergence is denoted by \( \rightharpoonup \) (respectively \( \rightarrow \)); \( o(1) \) denotes any quantity which tends to zero when \( n \to \infty \); \( B_\rho(0) \) denotes a ball centered at the origin with radius \( \rho > 0 \).

Now we make the following assumptions:

\( (f_1) \) \( f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R}) \), and there exist constants \( c_1, c_2 > 0 \) such that for all \( u \in \mathbb{R} \),
\[
|f(x,u)| \leq c_1 g(u)|G(u)| + c_2 g(u)|G(u)|^{p-1},
\]
where \( 2 < p < 2^* \);

\( (f_2) \) \( \lim_{|u| \to \infty} F(x,u) = \infty \) uniformly in \( x \) and there exists \( r_0 > 0 \) such that
\( F(x,u) \geq 0 \) for all \( x \in \mathbb{R}^N \) and \( |u| \geq r_0 \);

\( (f_3) \) \( \bar{F}(x,u) = \frac{f(x,u)G(u)}{2g(u)} - F(x,u) \geq 0 \), and there exist \( c_3 > 0 \) and \( \sigma > \max\{1, \frac{N}{2} \} \) such that
\[
|F(x,u)|^\sigma \leq c_3 |u|^{2\sigma} \bar{F}(x,u) \text{ for all } (x,u) \in \mathbb{R}^N \times \mathbb{R} \text{ with } |u| \text{ large enough};
\]

\( (f_4) \) \( f(x,u) = -f(x,-u) \) for all \( (x,u) \in \mathbb{R}^N \times \mathbb{R} \).

We summarize our main results as follows:

**Theorem 1.1.** Suppose that the assumptions \( (g), (V_1)-(V_2), (f_1)-(f_4) \) are satisfied, then problem (1.1) possesses infinitely many solutions \( \{u_n\} \) such that \( \|u_n\| \to \infty \) and \( I(u_n) \to \infty \) (I will be defined later).

**Remark 1.** From (V1) and (f2), we easily see that \( V(x) \) and \( F(x,u) \) are allowed to be sign-changing. Moreover, the usual “superlinear condition” at the origin \( f(x,u) = o(u) \) uniformly in \( x \) as \( u \to 0 \), which is all assumed in the aforementioned works for problem (1.1), is not needed in our result.

**Remark 2.** Note that \( g \) is an even function with \( g'(u) \geq 0 \) for all \( u \geq 0 \), and \( ug'(u) < g(u) \) for all \( u \in \mathbb{R} \), it is not difficult to deduce that
\[
|g(u)| \leq c_4 |u| + c_5 \text{ and } g'(u) \leq c_6,
\]
with some constants \( c_4, c_5, c_6 > 0 \).
Before proceeding to the proof of the main results, we give an example to illustrate the above assumptions.

**Example 1.** Let \( F(x,u) = \frac{1}{2} |G(u)|^2 \ln((|G(u)|^{1/2} + 2) \), by a simple computation, it is easy to check that the function satisfies the assumptions \((f_1)-(f_4)\), but does not satisfy condition \((f)\).

The rest of the paper is organized as follows. After presenting some preliminary results in section 2, we give the proof of our main result in section 3.

2. **Variational setting and preliminaries.** Before establishing the variational framework for problem (1.1), we first notice a fact: from \((V_1)\), we easily see that there exists a constant \(V_0 > 0\) such that

\[
\overline{\tilde{V}}(x) := V(x) + V_0 > 0 \quad \text{for all} \quad x \in \mathbb{R}^N.
\]

Let \( \overline{\tilde{f}}(x,u) := f(x,u) + V_0 u \) and consider the following new equation

\[
-\text{div}(g^2(u) \nabla u) + g(u) g'(u) |\nabla u|^2 + \overline{\tilde{V}}(x) u = \overline{\tilde{f}}(x,u) \quad \text{in} \quad \mathbb{R}^N. \tag{2.1}
\]

We note that problem (2.1) is equivalent to the problem (1.1) and the hypotheses \((V_2)\) and \((f_1)-(f_4)\) still hold for \(\overline{\tilde{V}}\) and \(\overline{\tilde{f}}\) if those hold for \(V\) and \(f\).

In what follows, we just need to study the equivalent problem (2.1). Throughout this section, we assume that \(g(0) = 1\). Besides, we make the following assumption instead of \((V_1)\):

\[
(V_1) \quad V \in C(\mathbb{R}^N, \mathbb{R}) \quad \text{and} \quad \inf_{x \in \mathbb{R}^N} V(x) > 0.
\]

We next give the following notations. As usual, for \(1 \leq s < +\infty\), we let

\[
\|u\|_s = \left( \int_{\mathbb{R}^N} |u(x)|^s \, dx \right)^{\frac{1}{s}}, \quad u \in L^s(\mathbb{R}^N).
\]

Let

\[
H^1(\mathbb{R}^N) = \{ u \in L^2(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N) \}
\]

with the inner product

\[
(u,v)_{H^1} = \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + uv) \, dx
\]

and the norm

\[
\|u\|_{H^1} = (u,u)^{\frac{1}{2}}_{H^1}.
\]

Here we consider the following function space

\[
E := \{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x) u^2 \, dx < +\infty \},
\]

which is a Hilbert space endowed with the inner product

\[
(u,v) = \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + V(x)uv) \, dx
\]

and the norm

\[
\|u\| = (u,u)^{\frac{1}{2}}.
\]

It is well known that under the assumption \((\overline{V}_1)\), the embedding \(E \hookrightarrow L^s(\mathbb{R}^N)\) is continuous for \(s \in [2,2^*)\), and \(E \hookrightarrow L_{loc}^s(\mathbb{R}^N)\) is compact for \(s \in [2,2^*)\), i.e., there exist constants \(\gamma_s > 0\) such that

\[
\|u\|_s \leq \gamma_s \|u\|, \quad \forall u \in E, \quad s \in [2,2^*]. \tag{2.2}
\]

Furthermore, we have the following compactness lemma due to [1].
Lemma 2.1. Under the assumptions \((\tilde{V}_1)\) and \((V_2)\), the embedding from \(E\) into \(L^s(\mathbb{R}^N)\) is compact for each \(2 \leq s < 2^*\).

We observe that formally problem (1.1) is the Euler-Lagrange equation associated with the natural energy functional \(I : E \to \mathbb{R}\) given by
\[
I(u) = \frac{1}{2} \int_{\mathbb{R}^N} g^2(u)|\nabla u|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x)u^2 \, dx - \int_{\mathbb{R}^N} F(x, u) \, dx. \tag{2.3}
\]

It is well known that \(I\) may be not well defined in general in \(E\). To overcome this difficulty, we make a change of variables constructed by Shen and Wang in [20], as
\[
v = G(u) = \int_0^u g(t) \, dt.
\]

Let us first collect some properties of the change of variables \(G : \mathbb{R} \to \mathbb{R}\), which will be used frequently in the sequel of the paper.

Lemma 2.2. The function \(G(t)\) and \(G^{-1}(s)\) enjoy the following properties:

1. \(G\) is odd, \(C^2\) and invertible;
2. for all \(t \geq 0, s \geq 0\), \(G(t) \leq g(t)t, G^{-1}(s) \leq s\);
3. for all \(s \geq 0\), \(\frac{G^{-1}(s)}{s}\) is nonincreasing and
   \[
   \lim_{|s| \to 0} \frac{G^{-1}(s)}{s} = 1, \quad \lim_{s \to +\infty} \frac{G^{-1}(s)}{s} = \begin{cases} 
   \frac{1}{g'(\infty)} & \text{if } g \text{ is bounded,} \\
   o(1) & \text{if } g \text{ is unbounded};
   \end{cases}
   \]
4. if \(g\) is unbounded,
   \[
   \lim_{s \to +\infty} \frac{|G^{-1}(s)|^2}{s} = \frac{2}{g'(\infty)};
   \]
5. \(0 \leq \frac{1}{g(t)}g'(t) \leq 1\), for all \(t \in \mathbb{R}\);
6. there exists a positive constant \(C\) such that
   \[
   |G^{-1}(s)| \geq \begin{cases} 
   C|s|, & |s| \leq 1, \\
   C|s|^{1/2}, & |s| \geq 1;
   \end{cases}
   \]
7. for each \(\alpha > 0\), there exists a positive constant \(C(\alpha)\) such that
   \[
   |G^{-1}(\alpha s)|^2 \leq C(\alpha)|G^{-1}(s)|^2.
   \]

Proof. By the definition of \(g\), properties (1), (3) and (4) are obvious. Since \(g(t) > 0\) is nondecreasing on \([0, +\infty)\), then \(g(0)t \leq G(t) \leq g(t)t\) for all \(t \geq 0\), which implies (2). By direct calculation, we get (5). (6) and (7) are consequences of (3) and (4). \(\square\)

Remark 3. It follows from \((f_2)\) and \((f_3)\) that
\[
\tilde{F}(x, u) \geq \frac{1}{c_3} \left( \frac{|F(x, u)|}{|u|^2} \right)^\sigma \geq \frac{1}{c_3} \left( \frac{|F(x, u)|}{|G(u)|^2} \right)^\sigma \to \infty
\]
uniformly in \(x\) as \(|u| \to \infty\).

Therefore, after the change of variables, we obtain the following functional
\[
J(v) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x)|G^{-1}(v)|^2 \, dx - \int_{\mathbb{R}^N} F(x, G^{-1}(v)) \, dx. \tag{2.4}
\]
Note that the critical points of $J$ are the weak solutions of the following equation (see [20]):

$$-\triangle v + V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))} - \frac{f(x, G^{-1}(v))}{g(G^{-1}(v))} = 0. \quad (2.5)$$

It is clear that if $v$ is a critical point of $J$, $u = G^{-1}(v)$ is a critical point of $I$, i.e. $u = G^{-1}(v)$ is a solution of (1.1).

**Lemma 2.3.** $J$ is well defined in $E$ and $J \in C^1(E, \mathbb{R})$. Moreover,

$$\langle J'(v), \psi \rangle = \int_{\mathbb{R}^N} \left[ \nabla v \nabla \psi + V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))} \psi - \frac{f(x, G^{-1}(v))}{g(G^{-1}(v))} \psi \right] dx, \forall \psi \in E.$$

**Proof.** First we prove that $J$ is well defined in $E$. Since $g$ is a nondecreasing positive function, we get

$$|G^{-1}(v)| \leq |v|.$$

We obtain that for each $v \in E$,

$$\int_{\mathbb{R}^N} V(x)|G^{-1}(v)|^2 dx \leq \int_{\mathbb{R}^N} V(x)v^2 dx.$$

It follows from the condition $(f_1)$ that

$$\int_{\mathbb{R}^N} F(x, G^{-1}(v)) dx \leq \int_{\mathbb{R}^N} \left( \frac{c_1}{2} |v|^2 + \frac{c_2}{p} |v|^p \right) dx, \text{ for all } v \in E.$$

Thus, $J$ is well defined in $E$.

Next we verify that $J \in C^1(E, \mathbb{R})$. Note that for any $v, \psi \in E$ fixed, using the mean value theorem gives

$$\frac{1}{2} \int_{\mathbb{R}^N} V(x)(|G^{-1}(v + t\psi)|^2 - |G^{-1}(v)|^2) dx = \int_{\mathbb{R}^N} V(x) \frac{G^{-1}(\zeta)}{g(G^{-1}(\zeta))} \psi dx,$$

where

$$\min\{v(x), v(x) + t\psi(x)\} \leq \zeta(x) \leq \max\{v(x), v(x) + t\psi(x)\}.$$

Since $|t| \leq 1$, we get

$$\left| V(x) \frac{G^{-1}(\zeta)}{g(G^{-1}(\zeta))} \psi \right| \leq CV(x)|G^{-1}(\zeta)|^2 \leq CV(x)|\zeta(x)|^2 \leq CV(x)(|v(x)|^2 + |\psi(x)|^2) \in L^1(\mathbb{R}^N).$$

It is easily seen that

$$V(x) \frac{G^{-1}(\zeta)}{g(G^{-1}(\zeta))} \psi \rightarrow V(x) \frac{G^{-1}(v)}{g(G^{-1}(v))} \psi, \text{ a.e. in } \mathbb{R}^N \text{ as } t \rightarrow 0.$$

Then by the Lebesgue dominated convergence theorem, we conclude that

$$\lim_{t \rightarrow 0} \frac{1}{2} \int_{\mathbb{R}^N} V(x)(|G^{-1}(v + t\psi)|^2 - |G^{-1}(v)|^2) dx = \int_{\mathbb{R}^N} V(x) \frac{G^{-1}(v)}{g(G^{-1}(v))} \psi dx.$$

Similarly, using the properties of $G^{-1}(v)$, the assumption $(f_1)$ and the Lebesgue dominated convergence theorem lead to

$$\lim_{t \rightarrow 0} \int_{\mathbb{R}^N} \frac{F(x, G^{-1}(v + t\psi)) - F(x, G^{-1}(v))}{t} dx = \int_{\mathbb{R}^N} \frac{f(x, G^{-1}(v))}{g(G^{-1}(v))} \psi dx.$$
These indicate that \( J \in C^1(E, \mathbb{R}) \) and that for any \( \psi \in E \),
\[
\langle J'(v), \psi \rangle = \int_{\mathbb{R}^N} \left[ \nabla v \nabla \psi + V(x) \frac{G^{-1}(v)}{g(G^{-1}(v))} \psi - \frac{f(x, G^{-1}(v))}{g(G^{-1}(v))} \psi \right] \, dx.
\]
The proof is complete.

Recall that a sequence \( \{v_n\} \subset E \) is said to be a \((C)_c\)-sequence if \( J(v_n) \to c \) and \((1 + \|v_n\|)J'(v_n) \to 0\), \( J \) is said to satisfy the \((C)_c\)-condition if any \((C)_c\)-sequence has a convergent subsequence.

**Lemma 2.4.** Assume that \((g), (\tilde{V}_1), (V_2)\) and \((f_1)-(f_3)\) hold, then any \((C)_c\)-sequence for the functional \( J \) is bounded in \( E \).

**Proof.** Let \( \{v_n\} \subset E \) be a \((C)\)-sequence for \( J \) at level \( c \in \mathbb{R} \), that is,
\[
J(v_n) \to c \quad \text{and} \quad (1 + \|v_n\|)J'(v_n) \to 0.
\]
(2.6)

Therefore, there is a constant \( C > 0 \) such that
\[
J(v_n) - \frac{1}{2} \langle J'(v_n), v_n \rangle \leq C.
\]
(2.7)

We first prove that there exists \( C_1 > 0 \) such that
\[
\int_{\mathbb{R}^N} (|\nabla v_n|^2 + V(x)|G^{-1}(v_n)|^2) \, dx \leq C_1.
\]
If it is not true, we suppose that
\[
\|v_n\|_0^2 := \int_{\mathbb{R}^N} (|\nabla v_n|^2 + V(x)|G^{-1}(v_n)|^2) \, dx \to \infty.
\]
Setting \( w_n = G^{-1}(v_n)/\|v_n\|_0 \), then
\[
\|w_n\| = \frac{1}{\|v_n\|_0} \int_{\mathbb{R}^N} \left( \frac{|\nabla v_n|^2}{|g(G^{-1}(v_n))|^2} + V(x)|G^{-1}(v_n)|^2 \right) \, dx \leq 1.
\]

Passing to a subsequence, we may assume that \( w_n \to w \) in \( E \), \( w_n \to w \) in \( L^s(\mathbb{R}^N) \) for \( 2 \leq s < 2^* \), and \( w_n \to w \) a.e. on \( \mathbb{R}^N \).

It follows from (2.4) and (2.6) that
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} \frac{|f(x, G^{-1}(v_n))|}{\|v_n\|_0^2} \, dx = \frac{1}{2}.
\]
(2.8)

On the other hand, from (2.6) we obtain that
\[
C \geq J(v_n) - \frac{1}{2} \langle J'(v_n), v_n \rangle
\]
\[
= \frac{1}{2} \int_{\mathbb{R}^N} V(x)|G^{-1}(v_n)|^2 \, dx - \frac{1}{2} \int_{\mathbb{R}^N} V(x) \frac{G^{-1}(v_n)}{g(G^{-1}(v_n))} v_n \, dx
\]
\[
+ \int_{\mathbb{R}^N} \left( \frac{f(x, G^{-1}(v_n))}{2g(G^{-1}(v_n))} v_n - F(x, G^{-1}(v_n)) \right) \, dx
\]
\[
\geq \int_{\mathbb{R}^N} \tilde{F}(x, G^{-1}(v_n)) \, dx.
\]
(2.9)

Let
\[
h(r) := \inf \{ \tilde{F}(x, G^{-1}(v_n)) | x \in \mathbb{R}^N \text{ with } |G^{-1}(v_n)| \geq r \}
\]
for \( r \geq 0 \). It easily follows from Remark 3 that \( h(r) \to \infty \) as \( r \to \infty \). For \( 0 \leq a < b \), set
\[
\Omega_n(a, b) = \{ x \in \mathbb{R}^N : a \leq |G^{-1}(v_n)| < b \}.\]
Hence, it follows from (2.9) that

\[
C \geq \int_{\Omega_n(0,r)} F(x, G^{-1}(v_n)) \, dx + \int_{\Omega_n(r, +\infty)} F(x, G^{-1}(v_n)) \, dx \\
\geq \int_{\Omega_n(0,r)} F(x, G^{-1}(v_n)) \, dx + h(r) \text{meas}(\Omega_n(r, +\infty)),
\]

which implies that \(\text{meas}(\Omega_n(r, +\infty)) \to 0\) uniformly in \(n\) as \(r \to \infty\). Thus, for any \(2 \leq s < 2^*\), by Hölder inequality and Sobolev embedding theorem, we deduce that

\[
\int_{\Omega_n(r, +\infty)} |w_n|^s \, dx \leq \left( \int_{\Omega_n(r, +\infty)} |w_n|^{2^*} \, dx \right)^{\frac{s}{2^*}} (\text{meas}(\Omega_n(r, +\infty)))^{\frac{2^*-s}{2^*}}
\]

\[
= \frac{1}{\|v_n\|^s_0} \left( \int_{\Omega_n(r, +\infty)} |G^{-1}(v_n)|^{2^*} \, dx \right)^{\frac{s}{2^*}} (\text{meas}(\Omega_n(r, +\infty)))^{\frac{2^*-s}{2^*}}
\]

\[
\leq \frac{C_2}{\|v_n\|^s_0} \left( \int_{\Omega_n(r, +\infty)} |\nabla G^{-1}(v_n)|^2 \, dx \right)^{\frac{s}{2}} (\text{meas}(\Omega_n(r, +\infty)))^{\frac{2^*-2}{2^*}}
\]

\[
\leq \frac{C_2}{\|v_n\|^s_0} \left( \int_{\Omega_n(r, +\infty)} |\nabla v_n|^2 \, dx \right)^{\frac{s}{2}} (\text{meas}(\Omega_n(r, +\infty)))^{\frac{2^*-2}{2^*}}
\]

\[
\leq C_2 (\text{meas}(\Omega_n(r, +\infty)))^{\frac{s}{2^*}} \to 0
\]

(2.10)

uniformly in \(n\) as \(r \to \infty\).

If \(w = 0\), then \(w_n \to 0\) in \(L^s(\mathbb{R}^N)\) for \(2 \leq s < 2^*\). For any \(0 < \varepsilon < \frac{1}{8}\), there exist large \(r_1, N_0\) such that

\[
\int_{\Omega_n(0, r_1)} \frac{|F(x, G^{-1}(v_n))|}{|G^{-1}(v_n)|^2} |w_n|^2 \, dx \leq C_3 \int_{\Omega_n(0, r_1)} \frac{|G^{-1}(v_n)|^2 + |G^{-1}(v_n)|^p}{|G^{-1}(v_n)|^2} |w_n|^2 \, dx
\]

\[
\leq C_3 (1 + r_1^{p-2}) \int_{\Omega_n(0, r_1)} |w_n|^2 \, dx
\]

\[
\leq C_3 (1 + r_1^{p-2}) \int_{\mathbb{R}^N} |w_n|^2 \, dx < \varepsilon
\]

(2.11)

for all \(n > N_0\). Set \(\sigma' = \frac{\sigma}{\sigma - 1}\). Since \(\sigma > \max\{1, \frac{N}{2}\}\), one sees that \(2\sigma' \in (2, 2^*)\).

Hence, it follows from (f3) and (2.10) that

\[
\int_{\Omega_n(r_1, +\infty)} \frac{|F(x, G^{-1}(v_n))|}{|G^{-1}(v_n)|^2} |w_n|^2 \, dx
\]

\[
\leq \left( \int_{\Omega_n(r_1, +\infty)} \left( \frac{|F(x, G^{-1}(v_n))|}{|G^{-1}(v_n)|^2} \right)^{\frac{\sigma}{1}} \right)^{1/\sigma} \left( \int_{\Omega_n(r_1, +\infty)} |w_n|^{2\sigma'} \, dx \right)^{1/\sigma'}
\]

\[
\leq C_4 \left( \int_{\Omega_n(r_1, +\infty)} |w_n|^{2\sigma'} \, dx \right)^{1/\sigma'} < \varepsilon
\]

(2.12)
for all $n$. Combining (2.11) with (2.12), we have
\[
\int_{\mathbb{R}^N} \frac{|F(x, G^{-1}(v_n))|}{\|v_n\|_0^2} \, dx = \int_{\Omega_n(0, r_1)} \frac{|F(x, G^{-1}(v_n))|}{|G^{-1}(v_n)|^2} |w_n|^2 \, dx \\
+ \int_{\Omega_n(r_1, +\infty)} \frac{|F(x, G^{-1}(v_n))|}{|G^{-1}(v_n)|^2} |w_n|^2 \, dx < 2\varepsilon < \frac{1}{4}
\]
for all $n > N_0$, which is a contradiction with (2.8).

If $w \neq 0$, then $\text{meas}(\Omega) > 0$, where $\Omega := \{x \in \mathbb{R}^N : w \neq 0\}$. For $x \in \Omega$, we have $|G^{-1}(v_n)| \to \infty$ as $n \to \infty$. Hence $\Omega \subset \Omega_n(r_0, \infty)$ for large $n$, where $r_0$ is given in (f2). By (f2), we obtain
\[
F(x, G^{-1}(v_n)) \to 0 \quad \text{as} \quad n \to \infty.
\]
Hence, using Fatou’s lemma, we have
\[
\int_{\Omega} \frac{F(x, G^{-1}(v_n))}{|G^{-1}(v_n)|^2} \, dx \to \infty \quad \text{as} \quad n \to \infty. \tag{2.13}
\]

It follows from (2.6) and (2.13) that
\[
0 = \lim_{n \to \infty} \frac{c + o(1)}{\|v_n\|_0^2} = \lim_{n \to \infty} \frac{J(v_n)}{\|v_n\|_0^2} \\
= \lim_{n \to \infty} \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v_n|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x)|G^{-1}(v_n)|^2 \, dx - \int_{\mathbb{R}^N} F(x, G^{-1}(v_n)) \, dx \\
= \lim_{n \to \infty} \left( \frac{1}{2} - \int_{\Omega_n(0, r_0)} \frac{|F(x, G^{-1}(v_n))|}{|G^{-1}(v_n)|^2} |w_n|^2 \, dx - \int_{\Omega_n(r_0, +\infty)} \frac{|F(x, G^{-1}(v_n))|}{|G^{-1}(v_n)|^2} |w_n|^2 \, dx \right) \\
\leq \frac{1}{2} + \limsup_{n \to \infty} \left( C_5(1 + r_0^{p-2}) \int_{\Omega_n(0, r_0)} |w_n|^2 \, dx - \int_{\Omega_n(r_0, +\infty)} \frac{|F(x, G^{-1}(v_n))|}{|G^{-1}(v_n)|^2} |w_n|^2 \, dx \right) \\
\leq C_5 - \liminf_{n \to \infty} \int_{\Omega} \frac{|F(x, G^{-1}(v_n))|}{|G^{-1}(v_n)|^2} |w_n|^2 \, dx \\
= -\infty,
\]
which is a contradiction. Thus, the inequality
\[
\int_{\mathbb{R}^N} (|\nabla v_n|^2 + V(x)|G^{-1}(v_n)|^2) \, dx \leq C_1
\]
holds true.

Next, in order to show $\{v_n\}$ is bounded in $E$, we only need to prove that there exists $C_0 > 0$ such that
\[
\|v_n\|_0^2 := \int_{\mathbb{R}^N} (|\nabla v_n|^2 + V(x)|G^{-1}(v_n)|^2) \, dx \geq C_0 \|v_n\|^2. \tag{2.14}
\]
In fact, we may assume that $v_n \neq 0$ (otherwise the conclusion is trivial). If this conclusion is not true, passing to a subsequence, we have $\frac{\|v_n\|_0^2}{\|v_n\|^2} \to 0$. Set
\[
\bar{w}_n = \frac{v_n}{\|v_n\|} \quad \text{and} \quad h_n = \frac{|G^{-1}(v_n)|^2}{\|v_n\|^2}.
\]
Then
\[ \int_{\mathbb{R}^N} (|\nabla \tilde{w}_n|^2 + V(x) h_n(x)) \, dx \to 0. \]

Hence
\[ \int_{\mathbb{R}^N} |\nabla \tilde{w}_n|^2 \, dx \to 0, \quad \int_{\mathbb{R}^N} V(x) h_n(x) \, dx \to 0, \quad \text{and} \quad \int_{\mathbb{R}^N} V(x) \tilde{w}_n^2 \, dx \to 1. \]

Similar to the idea of [27], we assert that for each \( \varepsilon > 0 \), there exists \( C_7 > 0 \) independent of \( n \) such that \( \text{meas}(\Omega_n) < \varepsilon \), where \( \Omega_n := \{ x \in \mathbb{R}^n : |v_n(x)| \geq C_7 \} \). Otherwise, there is an \( \varepsilon_0 > 0 \) and a subsequence \{\( v_{n_k} \)\} of \{\( v_n \)\} such that for any positive integer \( k \),
\[ \text{meas}(\{ x \in \mathbb{R}^N : v_{n_k}(x) \geq k \}) \geq \varepsilon_0 > 0. \]

Set \( \Omega_{n_k} := \{ x \in \mathbb{R}^N : v_{n_k}(x) \geq k \} \). By Lemma 2.2 (6) we have
\[ \|v_{n_k}\|_{0}^2 \geq \int_{\mathbb{R}^N} V(x)|G^{-1}(v_{n_k})|^2 \, dx \geq \int_{\Omega_{n_k}} V(x)|G^{-1}(v_{n_k})|^2 \, dx \geq Ck\varepsilon_0 \to \infty \]
as \( k \to \infty \), which is a contradiction. Hence the assertion is true. Notice that as \( |v_n(x)| \leq C_7 \), by (6) and (7) in Lemma 2.2, we have
\[ \frac{C}{C_7^2} v_n^2 \leq \left| G^{-1} \left( \frac{1}{C_7} v_n \right) \right|^2 \leq C_8 |G^{-1}(v_n)|^2. \]

Thus,
\[ \int_{\mathbb{R}^N \setminus \Omega_n} V(x) \tilde{w}_n^2 \, dx \leq C_9 \int_{\mathbb{R}^N \setminus \Omega_n} V(x) \frac{|G^{-1}(v_n)|^2}{\|v_n\|^2} \, dx \leq C_9 \int_{\mathbb{R}^N} V(x) h_n(x) \, dx \to 0. \tag{2.15} \]

On the other hand, by virtue of the integral absolutely continuity, there exists \( \varepsilon > 0 \) such that whenever \( \Omega' \subset \mathbb{R}^N \) and \( \text{meas}(\Omega') < \varepsilon \),
\[ \int_{\Omega'} V(x) \tilde{w}_n^2 \, dx \leq \frac{1}{2}. \tag{2.16} \]

Combining (2.15) with (2.16), we have
\[ \int_{\mathbb{R}^N} V(x) \tilde{w}_n^2 \, dx = \int_{\mathbb{R}^N \setminus \Omega_n} V(x) \tilde{w}_n^2 \, dx + \int_{\Omega_n} V(x) \tilde{w}_n^2 \, dx \leq \frac{1}{2} + o(1), \]
which implies that \( 1 \leq \frac{1}{2} \), a contradiction. This implies that (2.14) holds. Hence \{\( v_n \)\} is bounded in \( E \). The proof of this lemma is now finished. \( \square \)

**Lemma 2.5.** Assume that \((g), (\bar{V}_1), (V_2)\) and \((f_1)-(f_3)\) are satisfied, then \( J \) satisfies the \((C)\) condition.

**Proof.** Indeed, by the boundedness of \{\( v_n \)\} and the compactness of the embedding \( E \hookrightarrow L^s(\mathbb{R}^N) \) \((2 \leq s < 2^*)\), up to a subsequence, we have \( v_n \to v \) in \( E \), \( v_n \to v \) in \( L^s(\mathbb{R}^N) \) for all \( 2 \leq s < 2^* \) and \( v_n \to v \) a.e. on \( \mathbb{R}^N \). First, we claim that there exists \( C_{10} > 0 \) such that
\[ \int_{\mathbb{R}^N} \left[ |\nabla (v_n - v)|^2 + V(x) \left( \frac{G^{-1}(v_n)}{g(G^{-1}(v_n))} - \frac{G^{-1}(v)}{g(G^{-1}(v))} \right) (v_n - v) \right] \, dx \geq C_{10} \|v_n - v\|^2. \tag{2.17} \]

Indeed, we may assume \( v_n \neq v \) (otherwise the conclusion is trivial). Set
\[ \omega_n = \frac{v_n - v}{\|v_n - v\|} \quad \text{and} \quad h_n = \frac{1}{v_n - v} \left( \frac{G^{-1}(v_n)}{g(G^{-1}(v_n))} - \frac{G^{-1}(v)}{g(G^{-1}(v))} \right). \]
Argue by contradiction and assume that

\[ \int_{\mathbb{R}^N} (|\nabla \omega_n|^2 + V(x)h_n(x)\omega_n^2)dx \to 0. \]

Since

\[ \frac{d}{ds} \left( \frac{G^{-1}(s)}{g(G^{-1}(s))} \right) = \frac{g(G^{-1}(s)) - G^{-1}(s)g'(G^{-1}(s))}{g^2(G^{-1}(s))} > 0, \]

\( \frac{G^{-1}(s)}{g(G^{-1}(s))} \) is strictly increasing, and for each \( C_{11} > 0 \) there exists \( \delta_1 > 0 \) such that

\[ \frac{d}{ds} \left( \frac{G^{-1}(s)}{g(G^{-1}(s))} \right) > \delta_1, \text{ when } |s| \leq C_{11}. \]

Hence, we see that \( h_n(x) \) is positive. Hence,

\[ \int_{\mathbb{R}^N} |\nabla w_n|^2 dx \to 0, \quad \int_{\mathbb{R}^N} V(x)h_n(x)w_n^2 dx \to 0, \quad \text{and} \quad \int_{\mathbb{R}^N} V(x)w_n^3 dx \to 1. \]

By a similar argument as (2.15) and (2.16), we can conclude a contradiction.

On the other hand, we have

\[ \left| \int_{\mathbb{R}^N} \left( \frac{f(x,G^{-1}(v_n))}{g(G^{-1}(v_n))} - \frac{f(x,G^{-1}(v))}{g(G^{-1}(v))} \right) (v_n - v) dx \right| \]

\[ \leq C_{12} \int_{\mathbb{R}^N} (|v_n| + |v_n|^{p-1} + |v| + |v|^{p-1})|v_n - v| dx \]

\[ \leq C_{12} \left[ (\|v_n\|_2 + \|v\|_2)\|v_n - v\|_2 + (\|v_n\|_p^{p-1} + \|v\|_p^{p-1})\|v_n - v\|_p \right] \]

\[ \to 0. \]

By virtue of (2.17) and (2.18), we have

\[ o_n(1) = \langle J'(v_n) - J'(v), v_n - v \rangle \]

\[ = \int_{\mathbb{R}^N} \left[ |\nabla (v_n - v)|^2 + V(x) \left( \frac{G^{-1}(v_n)}{g(G^{-1}(v_n))} - \frac{G^{-1}(v)}{g(G^{-1}(v))} \right) (v_n - v) \right] dx \]

\[ - \int_{\mathbb{R}^N} \left( \frac{f(x,G^{-1}(v_n))}{g(G^{-1}(v_n))} - \frac{f(x,G^{-1}(v))}{g(G^{-1}(v))} \right) (v_n - v) dx \]

\[ \geq C_{10} \|v_n - v\|^2 + o(1), \]

which implies that \( v_n \to v \) in \( E \). This completes the proof. \( \Box \)

To complete the proof of our theorem, we state the following symmetric Mountain Pass Theorem.

**Theorem 2.6 ([18])**. Let \( X \) be an infinite dimensional Banach space, \( X = Y \oplus Z \), where \( Y \) is finite dimensional. If \( \Phi \in C^1(X, \mathbb{R}) \) satisfies \((C)_c\)-condition for all \( c > 0 \), and

(I1) \( \Phi(0) = 0, \Phi(-u) = \Phi(u) \) for all \( u \in X \);  

(I2) there exist positive constants \( p \) and \( \alpha \) such that \( \Phi |_{\partial B_p \cap Z} \geq \alpha \);  

(I3) for any finite dimensional subspace \( \bar{X} \subset X \), there is \( R = R(\bar{X}) > 0 \) such that \( \Phi(u) \leq 0 \) on \( \bar{X} \setminus B_R \);  

then \( \Phi \) possesses an unbounded sequence of critical values.
3. Proof of main results. Let \( \{e_j\} \) be a total orthonormal basis of \( E \) and define \( X_j = \mathbb{R}e_j \), then \( E = \bigoplus_{j=1}^{\infty} X_j \). Let

\[
Y_m = \bigoplus_{j=1}^{m} X_j, \quad Z_m = \bigoplus_{j=m+1}^{\infty} X_j, \quad m \in \mathbb{Z}.
\]

Then \( E = Y_m \oplus Z_m \) and \( Y_m \) is finite dimensional. Similar to Lemma 3.8 in [26], we have the following lemma.

**Lemma 3.1.** Under assumptions \((\tilde{V}_1)\) and \((V_2)\), for \( 2 \leq s < 2^* \),

\[
\beta_m(s) := \sup_{v \in Z_m, \|v\|_s = 1} \|v\|_s \to 0, \quad m \to \infty.
\]

Before going further, we need to show that there exists \( C_{13} > 0 \) such that

\[
\int_{\mathbb{R}^N} (|\nabla v|^2 + V(x)|G^{-1}(v)|^2) dx \geq C_{13}\|v\|^2, \quad \forall v \in S_{\rho},
\]

(3.1)

where \( S_{\rho} := \{v \in E : \|v\| = \rho\} \). Indeed, by a similar argument as (2.14), we can get this conclusion. Furthermore, by Lemma 3.1, we can choose an integer \( k \geq 1 \) such that

\[
\|v\|^2 \leq C_{13}^4\|v\|^2, \quad \|v\|^p \leq C_{13}^p\|v\|^p, \quad \forall v \in Z_k.
\]

(3.2)

**Lemma 3.2.** Suppose that \((g), (\tilde{V}_1), (V_2) \) and \((f_1)\) are satisfied, then there exist positive constants \( \rho \) and \( \alpha \) such that

\[
J(v) \geq \alpha, \quad \forall v \in Z_k : \|v\| = \rho.
\]

**Proof.** For any \( v \in Z_k \) with \( \|v\| = \rho < 1 \), since \( p \in (2, 2^*) \), by (3.1), (3.2) and Lemma 2.2 (2), we have

\[
J(v) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x)|G^{-1}(v)|^2 dx - \int_{\mathbb{R}^N} F(x, G^{-1}(v)) dx
\]

\[
\geq \frac{C_{13}}{2} \|v\|^2 - \int_{\mathbb{R}^N} \left( \frac{c_1}{2} |v|^2 + \frac{c_2}{p} |v|^p \right) dx
\]

\[
\geq \frac{C_{13}}{2} \|v\|^2 - \frac{C_{13}}{4} \|v\|^2 - \frac{C_{13}}{4} \|v\|^p
\]

\[
= \frac{C_{13}}{4} \|v\|^2 (1 - \|v\|^{p-2}) > 0.
\]

This completes the proof. \( \square \)

**Lemma 3.3.** Suppose that \((g), (\tilde{V}_1), (V_2) \) and \((f_1)-(f_2)\) are satisfied, for any finite dimensional subsequence \( \tilde{E} \in E \), there is \( R = R(\tilde{E}) > 0 \) such that

\[
J(v) \leq 0, \quad \forall v \in \tilde{E} \setminus B_R.
\]

**Proof.** For any finite dimensional subspace \( \tilde{E} \subset E \), there is a positive integral number \( m \) such that \( \tilde{E} \subset Y_m \). Suppose to the contrary that there exist \( \{v_n\} \) such that \( \|v_n\| \to \infty \), but

\[
\frac{1}{2} \int_{\mathbb{R}^N} |\nabla v_n|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x)|G^{-1}(v_n)|^2 dx > \int_{\mathbb{R}^N} F(x, G^{-1}(v_n)) dx.
\]

(3.3)

Jointly with Lemma 2.2 (2), we have

\[
\frac{\int_{\mathbb{R}^N} F(x, G^{-1}(v_n)) dx}{\|v_n\|^2} < \frac{1}{2}.
\]

(3.4)
On the other hand, set \( w_n = v_n/\|v_n\| \), then, up to a subsequence, we can assume that \( w_n \to w \) in \( E \), \( w_n \to w \) in \( L^s(\mathbb{R}^N) \) for all \( 2 \leq s < 2^* \) and \( w_n \to w \) a.e. on \( \mathbb{R}^N \).

Let \( A = \{ x \in \mathbb{R}^N : w(x) \neq 0 \} \) and \( B = \{ x \in \mathbb{R}^N : w(x) = 0 \} \).

If \( \text{meas}(A) > 0 \), then, by \((f_2)\) and Fatou’s lemma, we have

\[
\int_A \frac{F(x, G^{-1}(v_n))}{\|v_n\|^2} \, dx = \int_A \frac{F(x, G^{-1}(v_n))}{v_n^2} \|v_n\|^2 \, dx \to +\infty.
\]

By \((f_1)\) and \((f_2)\), there exists \( C_{14} > 0 \) such that

\[
F(x, t) \geq -C_{14}t^2, \quad \forall (x, t) \in \mathbb{R}^N \times \mathbb{R}.
\]

Hence

\[
\int_B \frac{F(x, G^{-1}(v_n))}{\|v_n\|^2} \, dx \geq -C_{14} \int_B \frac{|G^{-1}(v_n)|^2}{\|v_n\|^2} \, dx \geq -C_{14} \int_B w_n^2 \, dx.
\]

Since \( w_n \to w \) in \( L^2(\mathbb{R}^N) \), it is clear that

\[
\liminf_{n \to \infty} \int_B \frac{F(x, G^{-1}(v_n))}{\|v_n\|^2} \, dx \geq 0.
\]

Consequently,

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} \frac{F(x, G^{-1}(v_n))}{\|v_n\|^2} \, dx = +\infty,
\]

which contradicts with (3.4). This shows that \( \text{meas}(A) = 0 \), i.e., \( w(x) = 0 \) a.e. on \( \mathbb{R}^N \). According to the fact that all norms are equivalent on the finite dimensional space and the Sobolev embedding theorem, we have

\[
0 = \lim_{n \to \infty} \|w_n\|_s \geq C \lim_{n \to \infty} \|w_n\| = C > 0.
\]

This is a contradiction. This completes the proof. \( \square \)

**Proof of Theorem 1.1.** Set \( \Phi = J, X = E, Y = Y_m, Z = Z_m \) in Theorem 2.6.

Obviously, \( J(0) = 0 \) and \((f_4)\) implies that \( J \) is even. By Lemmas 2.5, 3.2 and 3.3, all conditions of Theorem 2.6 are satisfied. Thus, problem (2.5) possesses infinitely many nontrivial solutions sequence \( \{v_n\} \) such that \( J(v_n) \to \infty \) as \( n \to \infty \). Namely, problem (1.1) also possesses infinitely many nontrivial solutions sequence \( \{u_n\} \) such that \( I(u_n) \to \infty \) as \( n \to \infty \). \( \square \)

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