Local convexity of the TAP free energy and AMP convergence for
\(Z_2\)-synchronization

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Abstract

We study mean-field variational Bayesian inference using the TAP approach, for \(Z_2\)-synchronization as a prototypical example of a high-dimensional Bayesian model. We show that for any signal strength \(\lambda > 1\) (the weak-recovery threshold), there exists a unique local minimizer of the TAP free energy functional near the mean of the Bayes posterior law. Furthermore, the TAP free energy in a local neighborhood of this minimizer is strongly convex. Consequently, a natural-gradient/mirror-descent algorithm achieves linear convergence to this minimizer from a local initialization, which may be obtained by a constant number of iterations of Approximate Message Passing (AMP). This provides a rigorous foundation for variational inference in high dimensions via minimization of the TAP free energy.

We also analyze the finite-sample convergence of AMP, showing that AMP is asymptotically stable at the TAP minimizer for any \(\lambda > 1\), and is linearly convergent to this minimizer from a spectral initialization for sufficiently large \(\lambda\). Such a guarantee is stronger than results obtainable by state evolution analyses, which only describe a fixed number of AMP iterations in the infinite-sample limit.

Our proofs combine the Kac-Rice formula and Sudakov-Fernique Gaussian comparison inequality to analyze the complexity of critical points that satisfy strong convexity and stability conditions within their local neighborhoods.

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1 Introduction

Variational inference is an increasingly popular method for performing approximate Bayesian inference, and is widely used in applications ranging from document classification to population genetics [BNJ03, LPJK07, CS12, RSP14]. For large-scale problems, variational methods provide an appealing alternative to Markov Chain Monte Carlo procedures, particularly in settings where MCMC may be computationally prohibitive to apply. We refer readers to the classical expositions [JGJS99, WJ08] and the recent review [BKM17] for an introduction.

In “mean-field” models where the posterior distribution $p(x|Y)$ of parameters $x$ given data $Y$ may be close to being a product measure, a common approach to variational inference is to approximate $p(x|Y)$ by a product law. The most widely used such approximation minimizes the KL-divergence to $p(x|Y)$ over the class $Q$ of product measures,

$$
\hat{q}(x) = \arg\min_{q \in Q} D_{KL}(q(x) \| p(x|Y)). \tag{1.1}
$$

When $x \in \mathbb{R}^n$ is high-dimensional, a problematic phenomenon may occur in which this distribution $\hat{q}(x)$ provides inconsistent approximations to the posterior marginals and posterior means, even in models where all low-dimensional marginals of $p(x|Y)$ have approximately independent coordinates. Such a phenomenon was first investigated by Thouless, Anderson, and Palmer for the Sherrington-Kirkpatrick (SK) model of spin glasses, where a simple method of addressing this inaccuracy—now often called the “TAP correction”—was
also proposed [TAP77]. Manifestations of this phenomenon and analogues of the TAP free energy for several high-dimensional statistical models have been studied in [KMTZ14, RFSK16, GJM19, FMM21, QS22], and we provide further discussion in Section 1.3.

The TAP approach to variational inference constructs a free energy functional $F_{\text{TAP}}$ by adding a correction term to the KL-divergence objective (1.1). This TAP correction accounts for dependences between pairs of coordinates of $x$ in their posterior law, which are individually weak but may have a non-negligible aggregate effect in high dimensions. Variational inference is performed by minimizing $F_{\text{TAP}}$, or by solving the TAP stationary equations

$$0 = \nabla F_{\text{TAP}}.$$  

(1.2)

Since the pioneering work of [Kab03, DMM09, Bol14], both the theory and implementation of TAP-variational inference have been closely connected to Approximate Message Passing (AMP) algorithms, which provide specific iterative procedures for solving (1.2). TAP-variational inference has been successfully applied via AMP to a variety of high-dimensional statistical problems. We highlight in particular the line of work [RF12, DM14, MR15, LKZ15, BDM16*, LM19, MV21] on low-rank matrix estimation, of which the $\mathbb{Z}_2$-synchronization problem is a specific example.

The goal of our current paper is to address several foundational questions regarding TAP-variational inference that, despite the above successes, remain poorly understood. First, the convergence of AMP is usually known only in a weak sense, guaranteeing $\|\sqrt{n} \cdot \nabla F_{\text{TAP}}\|^2 < \varepsilon$ in the limit $n \to \infty$ for a constant number of AMP iterations $k \equiv k(\varepsilon)$ independent of $n$. Such a guarantee is too weak to ensure, for example, even the high-probability existence of a critical point of $F_{\text{TAP}}$ to which the AMP iterates converge. It does not establish whether the minimizer of $F_{\text{TAP}}$ is close to the true Bayes posterior mean, and indeed, these properties remain conjectural in most models to which the AMP/TAP approach has been applied. Second, regularity properties of the landscape of $F_{\text{TAP}}$ are largely unknown, making it unclear whether optimization algorithms other than AMP can successfully implement the TAP-variational inference paradigm.

In this paper, we clarify these properties of $F_{\text{TAP}}$ and the convergence of AMP and other descent algorithms for the specific model of $\mathbb{Z}_2$-synchronization. We build upon previous results and techniques of [FMM21], which studied this model in a regime of large signal-to-noise. Our main results will show that for any signal strength above the weak-recovery threshold, there exists a unique local minimizer $m_*$ of $F_{\text{TAP}}$ near the Bayes posterior mean, and $F_{\text{TAP}}$ is strongly convex in a local neighborhood (of non-trivial size) around $m_*$. Consequently, a generic natural-gradient-descent (NGD) algorithm exhibits linear convergence to $m_*$, from a local initialization, which may be obtained by a finite number of iterations of AMP. We also show that the Jacobian of the AMP map is stable at $m_*$, so that AMP initialized in a (potentially very) small neighborhood of $m_*$ will also converge for fixed $n$ as the number of iterations $t \to \infty$. In the large signal-to-noise regime of [FMM21], we show that both NGD and AMP exhibit linear convergence to $m_*$ from a spectral initialization.

Formalizing these properties of $F_{\text{TAP}}$ and the convergence of generic optimization algorithms has several appeals over the existing theory around AMP. First, it clarifies a concrete objective function for high-dimensional variational inference, which can serve a number of practical purposes such as assessing algorithm convergence. Second, the convergence and state evolution of AMP are tied to probabilistic aspects of the model, whereas NGD is always a strict descent algorithm (for small enough step size, even in misspecified models) and may provide a more flexible and robust approach for optimization in practice. Finally, understanding the landscape of $F_{\text{TAP}}$ may be useful in other contexts. For example, following the initial posting of our work, [AMS22, Cel22] have used the local strong convexity of $F_{\text{TAP}}$ in the related SK model to argue that its stationary point is Lipschitz in the external field. This is a central technical ingredient in these works to show the correctness of an algorithmic stochastic localization procedure for sampling from the SK measure.

We review relevant background on the $\mathbb{Z}_2$-synchronization model in Section 1.1, and we describe our results in more detail in Section 1.2.
1.1 $\mathbb{Z}_2$-synchronization and the TAP free energy

In $\mathbb{Z}_2$-synchronization, we wish to estimate an unknown binary vector $x \in \{-1,+1\}^n$ having the entry-wise symmetric Bernoulli prior $x_i \overset{iid}{\sim} \text{Unif}\{-1,+1\}$. For a signal-to-noise parameter $\lambda > 0$, we observe

$$Y = \frac{\lambda}{n}xx^\top + W,$$  \hspace{1cm} (1.3)

Thus $W$ is symmetric Gaussian noise, having entries $(w_{ij} : i = 1, \ldots, n) \overset{iid}{\sim} \mathcal{N}(0,2/n)$ independent of $(w_{ij} : 1 \leq i < j \leq n) \overset{iid}{\sim} \mathcal{N}(0,1/n)$. Equivalently, $W = (Z + Z^\top)/\sqrt{2n}$ where $(z_{ij} : i, j = 1, \ldots, n) \overset{iid}{\sim} \mathcal{N}(0,1)$.

The parameter $x$ is identifiable only up to ± sign, and the posterior law $p(x|Y)$ has the corresponding sign symmetry $p(x|Y) = p(-x|Y)$. Thus we will consider estimation of the sign-invariant rank-one matrix $X = xx^\top$. The Bayes posterior-mean estimate of this matrix is

$$\hat{X}_{\text{Bayes}} = \mathbb{E}[xx^\top | Y].$$  \hspace{1cm} (1.4)

The asymptotic squared-error Bayes risk of this estimator was characterized by Deshpande, Abbe, and Montanari in [DAM16]:

$$\lim_{n \to \infty} \frac{1}{n^2} \mathbb{E}[\|\hat{X}_{\text{Bayes}} - xx^\top\|^2] = \begin{cases} 1 - q_*(\lambda)^2 & \text{if } \lambda > 1 \\ 1 & \text{if } \lambda \leq 1, \end{cases}$$

(1.5)

where $q_*(\lambda) > 0$ is the solution to a fixed-point equation (4.1). Thus for $\lambda < 1$, no non-trivial estimation is possible in the large-$n$ limit, as the optimal Bayes risk coincides with that of the trivial estimator $\hat{X} = 0$. In contrast, for $\lambda > 1$, the Bayes estimator achieves positive entry-wise correlation with $xx^\top$.

[DAM16] studied also an AMP algorithm for approximately computing $\hat{X}_{\text{Bayes}}$. Starting from initializations $h^0, m^{-1} \in \mathbb{R}^n$, this algorithm takes the form

$$m^k = \tanh(h^k)$$

$$h^{k+1} = \lambda Ym^k - \lambda^2[1 - Q(m^k)]m^{k-1}$$

(AMP)

where $Q(m) = \|m\|^2/2n$. The analyses of [DAM16] imply that for any $\lambda > 1$ and $\varepsilon > 0$, starting from an informative initialization $h^0$, there exists an iterate $k \equiv k(\varepsilon) \in \mathbb{N}$ of AMP for which $\|m^k(m^k)^\top - \hat{X}_{\text{Bayes}}\|^2/n^2 < \varepsilon$, with high probability for all large $n$. More recent results of [MV21] imply that such a guarantee holds also for AMP with a spectral initialization.

The TAP free energy in this $\mathbb{Z}_2$-synchronization model is defined for $m \in (-1,1)^n$ by

$$F_{\text{TAP}}(m) = -\frac{\lambda}{2n}(m, Ym) - \frac{1}{n} \sum_{i=1}^n h(m_i) - \frac{\lambda^2}{4}[1 - Q(m)]^2$$

(TAP)

where $Q(m) = \|m\|^2/2n$ as above, and $h(m)$ is the binary entropy function

$$h(m) = -\frac{1 + m}{2} \log \frac{1 + m}{2} - \frac{1 - m}{2} \log \frac{1 - m}{2}.$$  \hspace{1cm} (1.6)

This function $F_{\text{TAP}}$ has the sign symmetry $F_{\text{TAP}}(m) = F_{\text{TAP}}(-m)$, corresponding to the above sign symmetry of the posterior law. The first two terms of (TAP) coincide\footnote{Up to an additive constant, and a replacement of $\mathbb{E}_{x \sim q}(x|yx)$ by $\langle m, Ym \rangle$ which incurs negligible error} with the KL-divergence $D_{\text{KL}}(q(x)||p(x|Y))$ for a product measure $q(x)$ on $\{-1, +1\}^n$, upon parameterizing $q$ by its mean $m = \mathbb{E}_{x \sim q}[x] \in (-1,1)^n$. The third term of (TAP) is the TAP correction. Applying $h'(m) = -\arctanh(m)$, the stationary condition $0 = \nabla F_{\text{TAP}}(m)$ may be rearranged as the TAP mean-field equations

$$m = \tanh \left( \lambda Ym - \lambda^2[1 - Q(m)]m \right),$$

and the AMP algorithm (AMP) is an iterative scheme for computing a fixed point of these equations.
In [FMM21], an upper bound for the expected number of critical points of $F_{\text{TAP}}$ in sub-regions of the domain $(-1,1)^n$ was derived for any $\lambda > 0$. Using this result, for $\lambda > \lambda_0$ a large enough absolute constant, it was shown that the global minimizer $m_\ast$ of $F_{\text{TAP}}$ satisfies $E[\|m_\ast m_\ast^\top - \hat{X}_{\text{Bayes}}\|_F^2]/n^2 \to 0$, and that this holds more generally for any critical point $m$ of $F_{\text{TAP}}$ in the domain

$$S = \{m \in (-1,1)^n : F_{\text{TAP}}(m) < -\lambda^2/3\}.$$ 

As a consequence, it was also shown that $E[\|m_\ast m_\ast^\top - \hat{X}_{\text{Bayes}}\|_F^2]/n^2$ must be bounded away from 0 for the minimizer $m_\ast$ of $F_{\text{TAP}}$ in the domain $(-1,1)^n$ parametrized similarly by $m$. We note that the landscape guarantees in [FMM21] do not extend to the entire weak-recovery regime $\lambda > 1$. The analyses for large $\lambda > \lambda_0$ also fall short of showing uniqueness (up to sign) of the TAP critical point $m_\ast$ in $S$, and of establishing polynomial-time convergence of AMP or other optimization algorithms for computing $m_\ast$.

### 1.2 Contributions

Our current work establishes the following properties of $F_{\text{TAP}}(m)$ and of descent algorithms for minimizing this objective function.

1. **Existence of Bayes-optimal TAP local minimizer.** For any $\lambda > 1$, we show there exists a local minimizer $m_\ast$ of $F_{\text{TAP}}$ such that $\|m_\ast m_\ast^\top - \hat{X}_{\text{Bayes}}\|_F^2/n^2 \to 0$ in probability. This strengthens the guarantee of [FMM21] that was shown for large $\lambda > \lambda_0$. Subject to the validity of a numerical conjecture about a deterministic low-dimensional variational problem (see Remark 4.5), our results imply that this is also the global minimizer of $F_{\text{TAP}}$ for any $\lambda > 1$.

2. **Local strong convexity of the TAP free energy.** For any $\lambda > 1$, we show that $F_{\text{TAP}}$ is strongly convex in a $\sqrt{\varepsilon n}$-neighborhood of this local minimizer $m_\ast$. Hence this local minimizer is the unique critical point satisfying $\|m, m_\ast^\top - \hat{X}_{\text{Bayes}}\|_F^2/n^2 < \varepsilon$, for some constant $\varepsilon$.

3. **Local convergence of natural gradient descent.** We introduce a natural gradient descent (NGD) algorithm for minimizing $F_{\text{TAP}}$, which is equivalently a mirror descent procedure that adapts to the curvature of $F_{\text{TAP}}$ near the boundaries of $(-1,1)^n$. For any $\lambda > 1$, we prove that NGD achieves linear convergence to $m_\ast$ from an initialization within this $\sqrt{\varepsilon n}$-neighborhood. This initialization may be obtained by first performing a fixed number of iterations of AMP, thus yielding a polynomial-time algorithm for computing $m_\ast$.

4. **Stability of AMP.** For any $\lambda > 1$, we show that the AMP map is stable at $m_\ast$, in the sense of having a Jacobian with spectral radius strictly less than 1. Thus, AMP initialized in a sufficiently small neighborhood of $m_\ast$ will also linearly converge to $m_\ast$.

5. **Finite-$n$ convergence of AMP and NGD.** Finally, for $\lambda > \lambda_0$ a large enough absolute constant, our results combine with those of [FMM21] to show that $m_\ast$ is the global minimizer and unique critical point (up to sign) of $F_{\text{TAP}}$ in the domain $\{m : F_{\text{TAP}}(m) < -\lambda^2/3\}$. In this signal-to-noise regime, we prove that both AMP and NGD alone exhibit linear convergence to $m_\ast$ from a spectral initialization.

We emphasize that this convergence of AMP is established in the sense $\lim_{k \to \infty} m^k = m_\ast$ for fixed dimension $n$, which is stronger than the guarantee $\limsup_{n \to \infty} \|m^k - m_\ast\|_F^2/n < \varepsilon$ for fixed $k \equiv k(\lambda, \varepsilon)$ that is obtainable by standard analyses of the AMP state evolution.

The main challenge in understanding the landscape of $F_{\text{TAP}}$ locally near $m_\ast$ is that—for any constant signal strength $\lambda$—this point $m_\ast$ does not converge to the true signal vector $x \in \{-1,+1\}^n$ as $n \to \infty$, but rather remains random in $(-1,1)^n$. Thus it is not enough to study the landscape of $F_{\text{TAP}}$ in a vanishing neighborhood of $x$ using, for example, the uniform convergence arguments [SQW18, MBM18]. The above results instead pertain to the geometry of $F_{\text{TAP}}$ in a random region of the cube $(-1,1)^n$.

We will prove these results using a combination of the Kac-Rice formula and Gaussian comparison inequalities. We provide a detailed overview of this proof in Section 4. The Kac-Rice formula has been successfully applied to study the complexity of critical points for various non-convex function landscapes. However, to our knowledge, our argument for using Kac-Rice to study also the local geometry around a
particular critical point is novel. We believe that this technique may be of independent interest for some recent analyses of related disordered systems [Bol18, DS19, FW21], where conditioning on a sequence of AMP iterates was used as a surrogate for conditioning on an actual TAP critical point.

1.3 Further related literature

1.3.1 Variational inference

The terminology “variational inference” encompasses a large family of methods for approximate Bayesian inference [Blei12, Pea82, Min01, YFW03], based upon approximating a variational representation to the evidence or marginal log-likelihood of the observed data. Variational inference has been incorporated into many software packages including Pyro [BCJ19], Infer.NET [MWG+14], and Edward [TKD+16].

There has been renewed interest in theoretical analyses of variational inference in recent years, focusing on a number of common desiderata: [HOW11, HPWW11, BCCZ13, WB19, GK21] study properties of consistency and asymptotic normality for estimates of low-dimensional parameters in latent variable models (i.e. of the prior “hyperparameters” in Bayesian contexts), using variational approximations for the marginal log-likelihood. In particular, [BCCZ13, GK21] establish such guarantees for the mean-field variational approximation in stochastic block models (SBMs), which are closely related to the $Z_2$-synchronization model of our work. [MSWY18, PPB20, ZZ20] study the optimization landscape and convergence properties of iterative coordinate ascent (CAVI) and block coordinate ascent (BCAVI) algorithms, with [ZZ20] showing that BCAVI achieves an optimal exponentially-vanishing rate of estimation error for the latent community membership vector in SBMs with asymptotically diverging signal strength. [ZG20, AR20, CA19, YPB20, RS21] study rates of posterior contraction for both variational Bayes and $\alpha$-fractional variational Bayes methods, establishing conditions under which the variational posteriors may enjoy the same optimal rates of contraction in a frequentist Bernstein-von-Mises sense as the true Bayes posteriors. In particular, [AR20, CA19, YPB20] discuss applications of these results to low-rank matrix estimation problems, including matrix completion, probabilistic PCA, and topic models.

In our work, we study the $Z_2$-synchronization model with bounded signal strength, which is in a different asymptotic regime from the above posterior contraction results for SBMs and low-rank matrix estimation. Fixing the true parameter $x$ as the all-1’s vector, the Bayes estimate for $x$ in our setting has a marginal distribution of coordinates that converges to a non-degenerate limit law, and an asymptotically non-vanishing per-coordinate Bayes risk.

Our focus on such a setting is motivated in part by our belief that in many applications, Bayesian approaches to inference may be favored because the data is in a regime of limited signal-to-noise that is far from theoretical regimes of posterior contraction. Instead, information in the hypothesized prior is important in informing inference, and the desideratum is then to obtain an accurate estimate of the posterior distribution under this prior. Our results are oriented towards this goal, showing (in a simple but illustrative model) that minimizing the TAP free energy yields a variational approximation which consistently estimates the posterior marginals, even when the posterior distribution itself does not concentrate strongly around the true parameter.

1.3.2 TAP free energy and the naive mean-field approximation

Thouless, Anderson, and Palmer introduced in [TAP77] the TAP equations (and the associated TAP free energy) as a system of asymptotically exact mean-field equations in the SK model. For spin glasses, the validity of the TAP equations and their relation to the Gibbs measure have been extensively studied—see for example [Ple82, DGY83, BMY84, CGPM03] in the physics literature, and [Tal10, Cha10, Bol14, AJ19, CP18, CPS18, BK19, Sub21] for rigorous mathematical results. Direct optimization of an analogous TAP free energy (a.k.a. approximate Bethe free energy) was proposed for Bayesian linear and generalized linear models in [KMTZ14, RFSK16], which recognized that its critical points are in exact correspondence with fixed points of AMP. $Z_2$-synchronization corresponds to the SK model with an added ferromagnetic bias, and the form of the TAP free energy that we study is identical to the (high-temperature) TAP free energy in the SK model with this added ferromagnetic component.

We emphasize that both the TAP approach and the “naive” mean-field approach of (1.1) have received significant attention in the theoretical literature. A line of work [CD16, BM17, Eld18, JKM18, Yan20, 6
Aug20] on the theory of non-linear large deviations establishes that the naive mean-field approximation to the free energy (i.e. the marginal log-likelihood in Bayesian models) is asymptotically accurate to leading order, without the need for a TAP correction, under a condition that the log-density has a “low-complexity gradient”. In Ising models with couplings matrix $Y \in \mathbb{R}^{n \times n}$ having $O(1)$ operator norm, such a condition holds when $Y$ is nearly low-rank in the sense $\|Y\|_F^2 = o(n)$ [BM17]. It does not hold for $\mathbb{Z}_2$-synchronization with any fixed signal strength $\lambda$, where [GJM19, FMM21] contrasted variational inference based on the TAP and naive mean-field approximations. In particular, [GJM19] showed that for $\lambda \in (1/2, 1)$, naive mean-field variational Bayes may yield a “falsely informative” variational posterior, and [FMM21] showed that critical points of the naive mean-field free energy cannot correspond to consistent approximations of the posterior mean for any sufficiently large but fixed value of $\lambda$.

1.3.3 Spiked matrix models and $\mathbb{Z}_2$-synchronization

Spiked matrix models have been a mainstay in the statistical literature since their introduction by [Joh01]. $\mathbb{Z}_2$-synchronization is a specific example of the spiked model with Bernoulli prior, and also of more general synchronization problems over compact groups [Sin11, BCLS20]. The Bayes risks in $\mathbb{Z}_2$-synchronization and other spiked matrix models were studied in [DAM16, BDM+16, KXXZ16, LM19]. For $\mathbb{Z}_2$-synchronization, non-trivial signal estimation above the weak-recovery threshold $\lambda = 1$ can also be achieved by spectral methods [BAP05, Péc06] and semi-definite programming [MS16, JMRT16], although such methods do not achieve the asymptotically optimal Bayes risk (1.5).

$\mathbb{Z}_2$-synchronization has been studied in part as a simpler analogue of the symmetric two-component SBM that replaces the noise $A - \mathbb{E}[A]$ of the adjacency matrix $\mathbb{A}$ by Gaussian noise, and it is possible to make formal connections between estimation in these models via universality arguments [DAM16, MS16]. We believe that certain aspects of our analyses and results may also be extendable to the SBM via universality arguments developed for AMP in [BLM15, CL21, WZF22, DSL22] and for minimizers of optimization objective functions with random data in [MN17, HL20, MS22, HS22], and this would be interesting to explore in future work.

1.3.4 AMP algorithms

AMP algorithms were proposed and studied in [Kab03, DMM09] for Bayesian linear regression and compressed sensing. They may be derived by approximating belief propagation on dense graphical models, see e.g. [DMM10, Mon12]. Various generalizations of AMP have been developed, including the Generalized AMP algorithm of [Ran11] and the Vector AMP algorithm of [RSF19], and we refer to [FVR+22] for a recent review. The state evolution formalism of AMP was introduced in [DMM09] and rigorously established in [Bol14, BM11]. This has since been generalized in [JM13, BMN20, MV21]. A finite-$n$ analysis of AMP was performed in [RV18], which extended the validity of the state evolution to $o(\log n / \log \log n)$ iterations. Following the initial posting of our work, [LW22] established a different finite-$n$ guarantee for AMP via a novel decomposition of the AMP iterates, which applies for $o(n / (\log^2 n))$ iterations in the $\mathbb{Z}_2$-synchronization problem with signal strength $\lambda \in (1, 1.2)$.

1.3.5 Gaussian comparison inequalities

The proofs of our main results rely heavily on Slepian’s comparison inequality [Sle62] and its later development by Sudakov-Fernique [Sud71, Sud79, Fer75], to reduce the study of $\mathcal{F}_{\text{TAP}}$ to a simpler Gaussian process. This approach is related to a recent line of work that generalizes Gordon’s inequality [Gor85, Kah86] to a Convex Gaussian Minimax Theorem (CGMT) [Sto13, OTH13, TOH15, MM21, CMW20].

1.3.6 Kac-Rice formula and complexity analysis

Physics calculations of the complexity of critical points in spin glass models using the Kac-Rice formalism can be found in [BM80, CGPM03, CLPR03, Fyo04, CLR05]. This method was made rigorous for spherical spin glasses in [AAČ13, AA13, Sub17], and a more recent line of work [AMMN19, MAB20, FMM21, BKMN21, BKMN22, ABM21] has used this approach to analyze non-convex function landscapes in other high-dimensional probabilistic and statistical models.
2 Main results

2.1 Local analysis of the TAP free energy

Our first result shows the existence and uniqueness of a local minimizer of the TAP free energy $F_{\text{TAP}}$ near the Bayes estimator (c.f. Eq. (1.4)), for any signal strength $\lambda > 1$. We also establish strong convexity of $F_{\text{TAP}}$ in a $\sqrt{\varepsilon n}$-neighborhood around this minimizer, as well as the stability of the AMP map

$$T_{\text{AMP}}(m, m_-) = \left( \tanh \left( \lambda Ym - \lambda^2 [1 - Q(m)] m_- \right), m \right)$$

(2.1)

at this local minimizer. This is the map for which the AMP iterations ($\text{AMP}$) may be expressed as $(m^{k+1}, m^k) = T_{\text{AMP}}(m^k, m^{k-1})$.

**Theorem 2.1** (Local convexity and AMP stability). Fix any $\lambda > 1$. There exist $\lambda$-dependent constants $\varepsilon, t > 0$ and $r \in (0, 1)$ such that for any fixed $\iota > 0$, with probability approaching 1 as $n \to \infty$, the following all occur.

(a) (Bayes-optimal TAP local minimizer) Let $\hat{X}_{\text{Bayes}} = \mathbb{E}[xx^\top | Y]$. There exists a critical point and local minimizer $m_*$ of $F_{\text{TAP}}(m)$ such that

$$\frac{1}{n^2} \| m_* m_*^\top - \hat{X}_{\text{Bayes}} \|_F^2 < \iota.$$  

(2.2)

For sufficiently small $\iota > 0$ (which is $\lambda$-dependent and $n$-independent), this is the unique critical point satisfying (2.2) up to $\pm$ sign.

(b) (Local strong convexity of TAP free energy) Let $\lambda_{\text{min}}(\cdot)$ denote the smallest eigenvalue. For this local minimizer $m_*$, we have

$$\lambda_{\text{min}} \left( n \cdot \nabla^2 F_{\text{TAP}}(m) \right) > t > 0$$

for all $m \in (-1, 1)^n \cap B_{\sqrt{\varepsilon n}}(m_*)$.

In particular, $F_{\text{TAP}}$ is strongly convex over $(-1, 1)^n \cap B_{\sqrt{\varepsilon n}}(m_*)$.

(c) (Local stability of AMP) Let $dT_{\text{AMP}} \in \mathbb{R}^{2n \times 2n}$ be the Jacobian of the AMP map (2.1), and let $\rho(\cdot)$ denote the spectral radius. For this local minimizer $m_*$, we have

$$\rho \left( dT_{\text{AMP}}(m_*, m_*) \right) < r < 1.$$  

Combining with the global landscape analysis of [FMM21], this implies the following immediate corollary for large enough signal strength $\lambda$.

**Corollary 2.2** (Global landscape for large $\lambda$). For an absolute constant $\lambda_0 > 0$, suppose $\lambda > \lambda_0$. Then with probability approaching 1 as $n \to \infty$, the local minimizers $\pm m_*$ guaranteed by Theorem 2.1 are the global minimizers of $F_{\text{TAP}}$. Furthermore, they are the only critical points of $F_{\text{TAP}}$ in the domain

$$S = \left\{ m \in (-1, 1)^n : F_{\text{TAP}}(m) < -\lambda^2/3 \right\}.$$  

A proof sketch of Theorem 2.1 can be found in Section 4, and its detailed proof can be found in Appendix B. The proof of Corollary 2.2 can be found in Appendix C.1.

2.2 Convergence of algorithms

We study convergence of the AMP algorithm ($\text{AMP}$), with the spectral initialization

$$h^0 = \text{principal eigenvector of } Y \text{ with } \| h^0 \|_2 = \sqrt{n \lambda^2 (\lambda^2 - 1)}, \quad m^{-1} = \lambda h^0.$$  

(SI)

We choose this scaling for $h^0$ as in [MV21, Section 2.4] to simplify the AMP state evolution.
We introduce also the following more “generic” first-order natural gradient descent (NGD) algorithm, with a step size parameter $\eta > 0$:

\[
\begin{align*}
    m^k &= \tanh(h^k) \\
    h^{k+1} &= h^k - \eta m \cdot \nabla F_{TAP}(m^k) \\
    &= (1 - \eta)h^k + \eta \left( \lambda Y m^k - \lambda^2 [1 - Q(m^k)]m^k \right). 
\end{align*}
\]

We call this algorithm “natural gradient descent” because we may apply $(d/dh) \tanh(h) = 1 - \tanh(h)^2$ to write the $m$-gradient $\nabla F_{TAP}(m^k)$ equivalently as a preconditioned $h$-gradient,

\[
\nabla F_{TAP}(m^k) = I(m^k)^{-1} \cdot \nabla h F_{TAP}(\tanh(h^k)), \quad I(m) = \text{diag} \left( \frac{1}{1 - m^2} \right),
\]

where $I(m)$ is proportional to the Fisher information matrix in a model of $n$ independent Bernoulli $\{-1, +1\}$ variables with mean $m \in \mathbb{R}^n$. This identifies (NGD) as a natural gradient method [Ama98]. We note that setting the step size $\eta = 1$ yields an algorithm similar to (AMP), but with $m^{k-1}$ replaced by $m^k$. For simplicity, we will consider the same spectral initialization $h^0$ for this algorithm as for AMP in (S1), although here this specific choice of initialization is less important.

Alternatively, the iterations (NGD) may be understood as a mirror-descent/Bregman-gradient method in the $m$-parameterization [NY83, BT03]. Recalling the binary entropy function $h$ from (1.6), we define

\[
L = \frac{1}{\eta}, \quad H(m) = \frac{1}{n} \sum_{i=1}^n h(m_i), \\
D_{-H}(m, m') = -H(m) + H(m') + \langle \nabla H(m'), m - m' \rangle
\]

where $L$ is the inverse step size, $-H(m)$ is a separable convex prox function, and $D_{-H}(m, m')$ is its associated Bregman divergence. Then it may be checked that (NGD) takes the equivalent mirror-descent form

\[
m^{k+1} = \text{arg min}_{m \in \{-1, +1\}^n} F_{TAP}(m^k) + \langle \nabla F_{TAP}(m^k), m - m^k \rangle + L \cdot D_{-H}(m, m^k). 
\]

One motivation for studying this algorithm, rather than ordinary gradient descent in the $m$-parameterization, is that the Hessian $\nabla^2 F_{TAP}(m)$ is not uniformly bounded over $(-1, 1)^n$, and instead diverges as $m$ approaches the boundaries of the cube. The form (2.4) naturally adapts to this non-uniform curvature of $F_{TAP}$, allowing for a convergence analysis using techniques of [BBT17, LFN18] for minimizing functions that are not strongly smooth in the Euclidean metric.

Combining the local strong convexity of Theorem 2.1, the state evolution of spectrally-initialized AMP, and this type of convergence analysis for NGD, we deduce the following result, whose proof can be found in Section 5.1 and Appendix C.

**Theorem 2.3** (Computation of Bayes-optimal TAP minimizer). Fix any $\lambda > 1$. There exist $\lambda$-dependent constants $C, \mu, \eta_0 > 0$ and $T \geq 1$ such that with probability approaching 1 as $n \to \infty$, the following occurs.

- Fix any step size $\eta \in (0, \eta_0)$, let $m^T \in (-1, 1)^n$ be the $T$th iteration of (AMP) from the spectral initialization (S1), and let $m^{T+k} \in (-1, 1)^n$ be obtained by $k$ iterations of (NGD) with step size $\eta$ from the initialization $m^T$. Let $m_*$ be the Bayes-optimal local minimizer of $F_{TAP}$ in Theorem 2.1. Then for some choice of sign $\pm$ and every $k \geq 1$,

\[
F_{TAP}(m^{T+k}) - F_{TAP}(\pm m_*) < C(1 - \mu \eta)^k, \\
\|m^{T+k} - (\pm m_*)\|_2 < C(1 - \mu \eta)^k \sqrt{n}. 
\]

In particular, $\lim_{k \to \infty} m^{T+k} \in \{+m_*, -m_*\}$.

This theorem implies that for any fixed value of $\lambda > 1$, the Bayes-optimal local minimizer $m_*$ of $F_{TAP}$ guaranteed by Theorem 2.1 may be computed in time that is polynomial in the problem size $n$ (in the usual sense of linear convergence). Let us remark that the convergence analysis of NGD in this result is purely
geometric, relying only on the smoothness and local convexity properties of $F_{\text{TAP}}$. We hence expect that a similar convergence analysis may be performed for momentum-accelerated or stochastic variants of NGD, such as those developed recently in [HRX21, GP22, DEH21].

For sufficiently large signal strength $\lambda$, where the more global landscape of $F_{\text{TAP}}$ is clarified by Corollary 2.2, our next result Theorem 2.4 verifies that the hybrid AMP/NGD approach in Theorem 2.3 is not needed, and that either algorithm alone can achieve linear convergence to the global TAP minimizer $m_*$ from a spectral initialization. The proof of Theorem 2.4 can be found in Section 5.2 and 5.3, and Appendix C.

**Theorem 2.4 (Convergence of AMP and NGD for large $\lambda$).** For an absolute constant $\lambda_0 > 0$, suppose $\lambda > \lambda_0$ and let $m_*$ be the global minimizer of $F_{\text{TAP}}$ in Corollary 2.2. Then there exist $\lambda$-dependent constants $C, \mu, \eta_0 > 0$ and $\alpha \in (0, 1)$ such that with probability approaching 1 as $n \to \infty$, the following all occur.

(a) (Convergence of AMP) Let $m^k$ be the $k^{th}$ iterate of AMP from the spectral initialization (SI). For some choice of sign $\pm$ and every $k \geq 1$,

$$F_{\text{TAP}}(m^k) - F_{\text{TAP}}(\pm m_*) < C\alpha^k, \quad \|m^k - (\pm m_*)\|_2 < C\alpha^k \sqrt{n}.$$ 

(b) (Convergence of NGD) Fix any step size $\eta \in (0, \eta_0)$, and let $m^k$ be the $k^{th}$ iterate of NGD from the spectral initialization (SI) with step size $\eta$. For some choice of sign $\pm$ and every $k \geq 1$,

$$F_{\text{TAP}}(m^k) - F_{\text{TAP}}(\pm m_*) < C(1 - \mu)\eta^k, \quad \|m^k - (\pm m_*)\|_2 < C(1 - \mu)\eta^k \sqrt{n}.$$ 

In particular, for both algorithms, $\lim_{k \to \infty} m^k \in \{+m_*, -m_*\}$.

**Remark 2.5.** We believe that the requirement $\lambda > \lambda_0$ sufficiently large in Theorem 2.4 is artificial, and that this result also holds for all $\lambda > 1$. This is supported by numerical simulations in Section 3 below. Let us clarify that such a guarantee for AMP does not follow from its state evolution combined with its local stability shown in Theorem 2.1(c): The state evolution ensures convergence to a $\sqrt{n}$-neighborhood of $m_*$, for any fixed $\varepsilon > 0$, in a finite number of AMP iterations. However, the local stability in Theorem 2.1(c) does not quantify the size of the neighborhood of $m_*$ in which AMP is then guaranteed to converge to $m_*$.

**Remark 2.6.** Part of our analysis of Theorem 2.4(a) still uses the state evolution for AMP with spectral initialization developed in [MV21]. This result would hold equally if AMP is initialized with a vector $m_1$ that is independent of the noise matrix $W$ and has non-vanishing correlation with $m_*$, by the validity of the AMP state evolution also in this setting. For a random initialization that is uncorrelated with $m_*$, we note that an analysis of AMP seems challenging even in this setting of large but fixed $\lambda > \lambda_0$, as the algorithm would still require $O(\log(n))$ iterations to achieve a non-negligible correlation with $m_*$, and existing finite-$n$ analyses of AMP [RV16, LW22] do not seem to immediately apply to describe this early phase of optimization. In Theorem 2.4(b), the spectral initialization is used to ensure that NGD is initialized in a basin of attraction of $m_*$, and analyses of the global landscape of $F_{\text{TAP}}$ in [FMM21] are also insufficient to show that this basin of attraction includes random initializations.

3 Numerical simulations

3.1 Convergence of algorithms

We perform numerical simulations to confirm the global convergence of AMP and NGD for all $\lambda > 1$, and to compare their convergence rates. We initialize both AMP and NGD using the spectral initialization (SI).

In Figure 1(a), we plot the residual squared error $\min\{\|m^k - m_*\|_2^2/n, \|m^k + m_*\|_2^2/n\}$, where $m^k$ is the $k^{th}$ iterate of AMP or NGD with different step sizes, and $m_* = \arg\min_m F_{\text{TAP}}(m)$. (We first compute $m_*$ up to high numerical accuracy using AMP.) For each algorithm, we simulated 10 random instances of $Y \in \mathbb{R}^{n \times n}$ according to the $Z_2$-synchronization model (1.3), with $n = 500$ and $\lambda = 1.5$. Figure 1(a) shows that AMP and NGD with step sizes 0.1 and 0.5 all consistently achieve convergence to $m_*$, where AMP has the fastest rate of convergence.

In Figure 1(b), we report the success probability of NGD for achieving convergence to $m_*$, for various step sizes $\eta$ (horizontal axis) and signal-to-noise ratios $\lambda > 1$ (vertical axis). The success probability is
Gaussian ensembles. In Figure 1, we verify this numerically for three examples of symmetric non-Gaussian noise matrices of the TAP free energy landscape to be robust under sufficiently light-tailed distributions of noise entries. Although we analyze AMP and NGD for Gaussian noise, we expect the properties of these estimators and the properties of the TAP minimizers and of the AMP and NGD iterates are indeed robust to these distributions, for varying signal-to-noise ratios $\lambda$ and step sizes $\eta$. In both panels, $n = 500$.

defined as the fraction of the 10 random instances of $Y$ for which NGD achieved residual squared error $10^{-4}$ within $k = 12000$ iterations. Figure 1(b) suggests that NGD with step size $\eta < 0.4$ converges for any $\lambda > 1$, and illustrates that as $\lambda$ increases, NGD allows for a larger step size in achieving this convergence.

3.2 Universality with respect to the noise distribution

Although we analyze AMP and NGD for Gaussian noise, we expect the properties of these estimators and of the TAP free energy landscape to be robust under sufficiently light-tailed distributions of noise entries. Here, we verify this numerically for three examples of symmetric non-Gaussian noise matrices $W$:

- Rademacher: $(w_{ij} : 1 \leq i \leq j \leq n) \overset{iid}{\sim} \text{Unif}\{-1/\sqrt{n}, 1/\sqrt{n}\}$.
- Double-exponential (Laplace): $W = (G + G^T)/\sqrt{2n}$, where $(G_{ij} : 1 \leq i,j \leq n) \overset{iid}{\sim} (1/\sqrt{2}) \exp\{-\sqrt{2} \cdot |x|\}$.
- Student’s t: $W = (G + G^T)/\sqrt{2n}$, where $(G_{ij} : 1 \leq i,j \leq n) \overset{iid}{\sim} t(\nu)/\sqrt{\nu/(\nu-2)}$ and the degrees-of-freedom is $\nu = 4$.

In all three examples, all entries $w_{ij}$ have mean 0, and all off-diagonal entries $w_{ij}$ have variance $1/n$.

In Figure 2(a), we report the estimation mean squared error (MSE) $\min\{\|m^k - m^*\|^2/n, \|m^k + m^*\|^2/n\}$ versus $\lambda$, where $m^* = \arg\min_m F_{\text{TAP}}(m)$ is computed from AMP up to high accuracy as before, and the noise matrix $W$ is generated from either the assumed Gaussian (GOE) model or from the above three non-Gaussian ensembles. In Figure 2(b), we report the residual squared error $\min\{\|m^k - m^*\|^2/n, \|m^k + m^*\|^2/n\}$ versus the number of algorithm iterations $k$, for the same four noise ensembles. These figures show that properties of the TAP minimizers and of the AMP and NGD iterates are indeed robust to these distributions of the noise entries, even for some heavy-tailed distributions.

We also tested Student’s t-distribution with degrees-of-freedom $\nu = 3$, and observed that when $\lambda \in (1, 2)$ and $n = 500$, AMP oscillates between two points rather than converging to a fixed point. Instead, the NGD algorithm with a sufficiently small step size continues to converge to the global minimizer.

3.3 Comparing TAP and mean-field variational Bayes

We compare the TAP approach to naive mean-field variational Bayes (mean-field VB), under both a correctly specified noise model and a misspecified model that lies outside of the preceding universality class.
In this section, we describe the main ideas and steps in the proof of Theorem 2.1.

We will prove that each statement of the theorem holds with probability approaching 1 conditional on the signal vector $\mathbf{x} \in \{-1, +1\}^n$. By symmetry, this conditional probability is the same for any given vector $\mathbf{x} \in \{-1, +1\}^n$, so we may assume without loss of generality

$$\mathbf{x} = \mathbf{1} = (1, 1, \ldots, 1).$$
Figure 3: Comparison of TAP with mean-field VB. The plot shows mean squared errors of the TAP and VB minimizers in both a correctly specified and a misspecified model, for signal-to-noise ratio $\lambda \in [1,2]$ and $n = 500$. The mean curve is averaged over 10 independent instances, and the error bars report $1/\sqrt{10}$ times the standard deviation across instances.

Conditional on $x$, the only remaining randomness is in the noise matrix $W \sim \text{GOE}(n)$, and $F_{\text{TAP}}(m)$ is a Gaussian process indexed by $m \in (-1,1)^n$.

The proof combines information derived from the Kac-Rice formula for the expected number of critical points of Gaussian processes, the Sudakov-Fernique Gaussian comparison inequality, and the AMP state evolution. It is helpful to summarize the type of information each of these tools will provide:

**Kac-Rice formula.** We use the Kac-Rice formula to upper bound the expected number of critical points of the TAP free energy in certain regions of the domain $(-1,1)^n$, or for which the TAP Hessian or AMP Jacobian violate the stated properties of Theorem 2.1. In particular, by establishing upper bounds that are vanishing as $n \to \infty$, we prove the non-existence of such critical points with high probability.

**Sudakov-Fernique inequality.** We use the Sudakov-Fernique inequality to lower bound the infima of Gaussian processes defined by $W \sim \text{GOE}(n)$ with the infima of Gaussian processes defined by a standard Gaussian vector $g \in \mathbb{R}^n$. We then analyze the latter to obtain variational lower bounds for large $n$. There are three Gaussian processes to which we apply this technique:

- The TAP free energy itself, to obtain lower bounds on its minimum value over regions of $(-1,1)^n$.
- A Gaussian process whose infimum gives the minimum eigenvalue of the TAP Hessian over subsets of $(-1,1)^n$, to show local strong convexity of the TAP free energy.
- A Gaussian process whose infimum is related to the spectral radius of the AMP Jacobian, to show local stability of the AMP map.

**AMP state evolution.** We use the AMP state evolution to evaluate the TAP free energy at the iterates of AMP, giving upper bounds for the TAP free energy value near the Bayes estimator.

The information provided by each of these three tools is distinct, and the proof of Theorem 2.1 combines the information we can extract from each.

We outline the four main steps of the proof in Section 4.1. These steps are discussed in Sections 4.2 through 4.5, and the technical arguments that execute each step are deferred to Appendix B.

### 4.1 Proof outline

For small parameters $\delta, \eta > 0$, we define two deterministic subsets $B_\delta, D_\eta \subset (-1,1)^n$ based on the empirical distribution of coordinates of $m \in (-1,1)^n$. These subsets will contain the desired TAP local minimizer $m_*$ with high probability (conditional on $x = 1$).
For $\lambda > 1$, let $q_\star = q_\star(\lambda)$ be the unique solution in $(0, 1)$ (cf. Proposition A.2) to the fixed-point equation
\begin{equation}
q_\star = \mathbb{E}_{G \sim \mathcal{N}(0, 1)} \left[ \tanh(\lambda^2 q_\star + \lambda \sqrt{q_\star} G)^2 \right].
\end{equation}
Define
\begin{align}
h_\star &= \mathbb{E}_{G \sim \mathcal{N}(0, 1)} [\log 2 \cosh(\lambda^2 q_\star + \lambda \sqrt{q_\star} G)] - \lambda^2 q_\star, \\
e_\star &= -\frac{\lambda^2}{4} (1 - 2 q_\star - q_\star^2) - \mathbb{E}_{G \sim \mathcal{N}(0, 1)} [\log 2 \cosh(\lambda^2 q_\star + \lambda \sqrt{q_\star} G)].
\end{align}
For any point $m \in (-1, 1)^n$, denote
\begin{align}
Q(m) &= \frac{1}{n} \|m\|^2_2, \\
M(m) &= \frac{1}{n} m^T 1, \\
H(m) &= \frac{1}{n} \sum_{i=1}^n h(m_i)
\end{align}
where $h(\cdot)$ is the binary entropy function from (1.6). We define the first subset $\mathcal{B}_\delta$ as
\begin{equation}
\mathcal{B}_\delta = \left\{ m \in (-1, 1)^n : |Q(m) - q_\star|, |M(m) - q_\star|, |H(m) - h_\star| < \delta \right\}.
\end{equation}
Let
$\mu_\star = \text{distribution of } \tanh(\lambda^2 q_\star + \lambda \sqrt{q_\star} G) \text{ when } G \sim \mathcal{N}(0, 1),$
\begin{equation}
\hat{\mu}_m = \frac{1}{n} \sum_{i=1}^n \delta_{m_i}.
\end{equation}
Denote by $W(\mu, \mu')$ the Wasserstein-2 distance between arctanh$\mu$ and arctanh$\mu'$, where arctanh$\mu$ is shorthand for the law of arctanh$\mu$ when $\mu \sim \mu$. That is, we have
\begin{equation}
W(\mu, \mu') \equiv W_2(\text{arctanh} \mu, \text{arctanh} \mu')
= \left( \inf_{\text{couplings } \nu \text{ of } (\mu, \mu')} \int (\text{arctanh} m - \text{arctanh} m')^2 d\nu(m, m') \right)^{1/2}.
\end{equation}
We review properties of this distance in Appendix A.3. We define the second subset $\mathcal{D}_\eta$ as
\begin{equation}
\mathcal{D}_\eta = \left\{ m \in (-1, 1)^n : W(\hat{\mu}_m, \mu_\star) < \eta \right\}.
\end{equation}
The proof of Theorem 2.1 then consists of four steps (all conditional on $x = 1$):
1. For sufficiently small $\delta > 0$, we use the Sudakov-Fernique inequality to lower bound the value of $\mathcal{F}_{\text{TAP}}$ on $\mathcal{B}_\delta \setminus \mathcal{B}_{\delta/2}$. Comparing with the value of $\mathcal{F}_{\text{TAP}}$ achieved by an iterate $m^k \in \mathcal{B}_{\delta/2}$ of AMP, we show that $\mathcal{F}_{\text{TAP}}$ must have a local minimizer $m_\star$ in $\mathcal{B}_\delta$, and $\mathcal{F}_{\text{TAP}}(m_\star) \approx e_\star$.
2. For any fixed $\eta > 0$, we use a Kac-Rice upper bound to show that with high probability, any such local minimizer $m_\star$ cannot belong to $\mathcal{B}_\delta \setminus \mathcal{D}_\eta$. Thus it must belong to $\mathcal{B}_\delta \cap \mathcal{D}_\eta$.
3. For $\varepsilon, \delta > 0$ sufficiently small, we apply a second Kac-Rice upper bound to show that for all critical points $m_\star \in \mathcal{D}_\eta$, $\lambda_{\min}(n \cdot \nabla^2 \mathcal{F}_{\text{TAP}}) \geq t$ everywhere in a $\sqrt{\varepsilon n}$-ball around $m_\star$. We analyze the Kac-Rice bound by representing $\lambda_{\min}(n \cdot \nabla^2 \mathcal{F}_{\text{TAP}})$ over this ball as the infimum of a Gaussian process, and lower bounding its value by a second application of the Sudakov-Fernique inequality. This implies that $\mathcal{F}_{\text{TAP}}$ is strongly convex near any local minimizer $m_\star \in \mathcal{B}_\delta \cap \mathcal{D}_\eta$ of Steps 1 and 2. This convexity then ensures that there exists a unique such local minimizer satisfying (2.2), establishing Theorem 2.1(a–b).
4. To show Theorem 2.1(c), we relate each (possibly complex) eigenvalue $\mu$ of $dT_{\text{AMP}}(\mathbf{m}_*, \mathbf{m}_*)$ to a zero eigenvalue of a corresponding “Bethe Hessian” of $F_{\text{TAP}}$ [SKZ14]. We extend the Kac-Rice/Sudakov-Fernique argument of Step 3 from $\nabla^2 F_{\text{TAP}}$ to this Bethe Hessian, and show that it is positive definite whenever $|\mu|$ exceeds some constant $r(\lambda) \in (0, 1)$. Thus all eigenvalues of $dT_{\text{AMP}}$ satisfy $|\mu| \leq r(\lambda)$.

The next four sections describe these steps in greater detail.

4.2 Sudakov-Fernique lower bound for the TAP free energy

We record here the following application of the Slepian/Sudakov-Fernique comparison inequality for Gaussian processes.

**Lemma 4.1.** Let $\mathcal{X}$ be a separable metric space, and let $f : \mathcal{X} \to \mathbb{R}$ and $v : \mathcal{X} \to \mathbb{R}^n$ be bounded measurable functions on $\mathcal{X}$. Let $W \sim \text{GOE}(n)$ and $g \sim \mathcal{N}(0, I_n)$. Then

$$E \left[ \sup_{x \in \mathcal{X}} v(x)^\top W v(x) + f(x) \right] \leq E \left[ \sup_{x \in \mathcal{X}} \frac{2}{\sqrt{n}} \|v(x)\|_2 \langle g, v(x) \rangle + f(x) \right].$$

Note that (conditional on $x = 1$) $-F_{\text{TAP}}(\mathbf{m})$ is a Gaussian process of this form, where $\mathcal{X} = (-1, 1)^n$ and $v(\mathbf{m}) = \sqrt{n/2n} \mathbf{m}$. Then applying this comparison lemma and an analysis of the comparison process, we obtain the following lower bound for $F_{\text{TAP}}(\mathbf{m})$ in terms of a low-dimensional, deterministic variational formula.

**Lemma 4.2.** Fix any $\lambda > 1$, and suppose $x = 1$. Fix any $\varepsilon > 0$ and two compact sets $K \subseteq [0, 1]^2 \times [0, \log 2]$ and $K' \subseteq \mathbb{R}^3$. Then for some $(\lambda, K', \varepsilon)$-dependent constant $c > 0$ and all large $n$, with probability at least $1 - e^{-cn},$

$$\inf_{\mathbf{m} \in (-1, 1)^n: (Q(\mathbf{m}), M(\mathbf{m}), H(\mathbf{m})) \in K} F_{\text{TAP}}(\mathbf{m}) > \inf_{(q, \varphi, h) \in K} \sup_{(\gamma, \tau, \nu) \in K'} E_\lambda(q, \varphi, h; \gamma, \tau, \nu) - \varepsilon \quad (4.9)$$

where

$$E_\lambda(q, \varphi, h; \gamma, \tau, \nu) = -\frac{\lambda^2}{2} \varphi^2 - \frac{\lambda^2}{4}(1 - q)^2 - h + \frac{q^2}{2} + \varphi \tau + \nu h$$

$$- \mathbb{E}_{G \sim \mathcal{N}(0, 1)} \left\{ \sup_{m \in (-1, 1)} \left[ \frac{\lambda m^2}{2} + \frac{\gamma m^2}{2} + \tau m + \nu h(m) \right] \right\} \quad (4.10)$$

**Lemma 4.2** makes precise the statement that

$$\widetilde{E}_\lambda(q, \varphi, h) = \sup_{(\gamma, \tau, \nu) \in K'} E_\lambda(q, \varphi, h; \gamma, \tau, \nu) \quad (4.11)$$

is a lower bound for $F_{\text{TAP}}(\mathbf{m})$ when $Q(\mathbf{m}) \approx q$, $M(\mathbf{m}) \approx \varphi$, and $H(\mathbf{m}) \approx h$. We may show that $\widetilde{E}_\lambda(q, \varphi, h)$ has a local minimizer at $(q, \varphi, h) = (q_*, \varphi_*, h_*)$ and is strongly convex around this minimizer, and hence give a more explicit lower bound for $F_{\text{TAP}}(\mathbf{m})$ when $\mathbf{m} \in B_{\delta}$ for sufficiently small $\delta > 0$.

**Lemma 4.3.** Fix any $\lambda > 1$, and let $E_\lambda(q, \varphi, h; \gamma, \tau, \nu)$ be as defined in Lemma 4.2. Then

$$\sup_{(\gamma, \tau, \nu) \in \mathbb{R}^3} E_\lambda(q_*, \varphi_*, h_*; \gamma, \tau, \nu) = E_\lambda(q_*, \varphi_*, h_*; 0, \lambda^2 q_*, 1) = e_* \quad (4.12)$$

Fix any subset $K' \subseteq \mathbb{R}^3$ containing $(0, \lambda^2 q_*, 1)$ in its interior, and define $\widetilde{E}_\lambda$ by (4.11). Then for some $\lambda, K'$-dependent constants $\delta, c > 0$ and all $(q, \varphi, h)$ satisfying $|q - q_*|$, $|\varphi - \varphi_*|$, $|h - h_*| \leq \delta$,

$$\widetilde{E}_\lambda(q, \varphi, h) \geq e_* + c(q - q_*)^2 + c(\varphi - \varphi_*)^2 + c(h - h_*)^2. \quad (4.13)$$

Lemmas 4.2 and 4.3 together imply that the energy value $F_{\text{TAP}}(\mathbf{m})$ is bounded away from $e_*$ on the domain $\mathbf{m} \in B_{\delta} \setminus B_{\delta/2}$. The AMP state evolution may be applied to show that AMP iterates eventually enter $B_{\delta/2}$, and achieve a TAP free energy value arbitrarily close to $e_*$ (cf. Lemma A.7). Combined, these yield the following corollary.
Corollary 4.4. Fix any $\lambda > 1$ and $\delta > 0$, and suppose $x = 1$. Then with probability approaching 1 as $n \to \infty$, there exists a critical point and local minimizer $m_*$ of $\mathcal{F}_{TAP}$ belonging to $\mathcal{B}_\delta$ and satisfying $|\mathcal{F}_{TAP}(m_*) - e_*| < \delta$.

The detailed proofs of this section are contained in Appendix B.1.

Remark 4.5. We conjecture, based on numerical evidence, that $(q_*, q_*, h_*)$ is in fact the global minimizer of $\bar{E}_\lambda(q, \varphi, h)$ for all $\lambda > 1$: We may first restrict $E_\lambda$ to $\nu = 1$ and $\tau = \lambda^2 \varphi$, to obtain the further lower bound

$$\bar{E}_\lambda(q, \varphi, h) \geq \bar{E}_\lambda(q, \varphi) = \sup_{\gamma} E_\lambda(q, \varphi; \gamma) \quad (4.14)$$

where

$$E_\lambda(q, \varphi; \gamma) = \frac{\lambda^2}{2} \varphi^2 - \frac{\lambda^2}{4}(1 - q)^2 + \frac{q_\gamma^2}{2} - \mathbb{E}_{G \sim \mathcal{N}(0,1)} \left[ \sup_{m \in (-1,1)} \lambda \sqrt{q} \cdot G m + \frac{\gamma m^2}{2} + \lambda^2 \varphi m + h(m) \right].$$

Numerical evaluations of this function $\bar{E}_\lambda(q, \varphi)$ over the relevant domain $q \in (0, 1)$ and $|\varphi| < \sqrt{q}$ are presented in Figure 4.2. For all tested values of $\lambda > 1$, these evaluations support the claim that $\bar{E}_\lambda(q, \varphi)$ has the unique global minimizer $(q_*, \varphi_*)$ (c.f. Eq. (4.14)). This claim then implies that $(q_*, q_*, h_*)$ is also the unique global minimizer of $\bar{E}_\lambda(q, \varphi, h)$, by the global convexity of $h \mapsto \bar{E}_\lambda(q_*, \varphi_*, h)$ and its strong convexity near its minimizer $h_*$. Subject to the validity of this numerical conjecture, Lemma 4.2 may be used to show that $\mathcal{F}_{TAP}(m)$ is also bounded away from $e_*$ for all $m \in (-1,1)^n \setminus \mathcal{B}_\delta$. Our subsequent arguments will then imply that for any $\lambda > 1$, with probability approaching 1, the (unique) local minimizer $m_*$ described by Corollary 4.4 and Theorem 2.1 is in fact the global minimizer of $\mathcal{F}_{TAP}$. (All theoretical results stated in this work will be established using only that $m_*$ is a local minimizer of $\mathcal{F}_{TAP}$, and they will not require the validity of this conjecture.)

4.3 Kac-Rice localization of critical points

We now use a Kac-Rice upper bound to show that the critical point(s) $m_*$ described by Corollary 4.4 must belong to the more restrictive set $\mathcal{B}_\delta \cap \mathcal{D}_g$ (c.f. Eq. (4.4) and (4.8)).

Define functions $g$ and $H$, which are the gradient and Hessian of the renormalized TAP free energy

$$g(m) = n \cdot \nabla \mathcal{F}_{TAP}(m) = -\lambda Y m + \text{arctanh}(m) + \lambda^2 [1 - Q(m)] m, \quad (4.15)$$

$$H(m) = n \cdot \nabla^2 \mathcal{F}_{TAP}(m) = -\lambda Y + \text{diag} \left( \frac{1}{1 - m^2} \right) + \lambda^2 [1 - Q(m)] I - \frac{2 \lambda^2}{n} m m^T. \quad (4.16)$$

We apply the following Kac-Rice upper bound from [FMM21].
Lemma 4.6. Fix any $\lambda > 0$, suppose $x = 1$, and let $T \subseteq (-1, 1)^n \setminus \{0\}$ be any (deterministic) Borel-measurable set. Then

$$\mathbb{E}\left[\left|\{m \in T : g(m) = 0\}\right|\right] \leq \int_T \mathbb{E}\left[|\det H(m)| \right| g(m) = 0 \right] p_{g(m)}(0) \, dm$$

where $p_{g(m)}(0)$ is the Lebesgue-density of the distribution of $g(m)$ at $g(m) = 0$.

Applying this bound, we eliminate the possibility that the critical point(s) described by Corollary 4.4 belong to $B_\delta \setminus D_\eta$, as stated in the following lemma. Thus they belong to $B_\delta \cap D_\eta$ as desired.

Lemma 4.7. Fix any $\lambda > 1$ and $\eta > 0$, and suppose $x = 1$. Then for some $(\lambda, \eta)$-dependent constants $c, \delta > 0$ and all large $n$,

$$\mathbb{P}\left[\text{there exists } m \in B_\delta : g(m) = 0, |F_{\text{TAP}}(m) - e_*| < \delta, m \notin D_\eta\right] < e^{-cn}.$$ 

Let us make two high-level clarifications regarding the proof: First, to show Lemma 4.7, we wish to apply Lemma 4.6 with $T$ being the set

$$\left\{m \in B_\delta \setminus D_\eta : |F_{\text{TAP}}(m) - e_*| < \delta\right\}.$$ 

We cannot do so directly, because $F_{\text{TAP}}(m)$ is random, and hence this is not a deterministic subset of $(-1, 1)^n$. However, restricted to points $m$ where $g(m) = 0$, the identity $0 = m^\top g(m)$ allows us to re-express $m^\top Y m$ and $F_{\text{TAP}}(m)$ as deterministic functions of $m$. Lemma 4.7 is then obtained by replacing $|F_{\text{TAP}}(m) - e_*| < \delta$ with an equivalent deterministic condition to define $T$.

Second, we remark that the Sudakov-Fernique argument of the preceding section cannot be used here to similarly localize $m_*$ to $D_\eta$, by bounding the TAP free energy value outside $B_\delta \setminus D_\eta$. This is because there exists $m \in (-1, 1)^n$ with one coordinate very close to $\pm 1$, so that $W(\mu_m, \mu_*)$ is arbitrarily large (c.f. Eq. (4.7)) and $m \notin D_\eta$, but $F_{\text{TAP}}(m)$ is arbitrarily close to $e_*$ in value. Thus, we use this separate Sudakov-Fernique argument, and the condition $m \in B_\delta$ as an input to the Kac-Rice analysis, to establish the localization to $D_\eta$ in Lemma 4.7.

The detailed proofs of this section are contained in Appendix B.2.

4.4 Sudakov-Fernique lower bound for local strong convexity

We now show that the TAP free energy is strongly convex in a local neighborhood of any critical point $m_* \in D_\eta$. For a parameter $\varepsilon > 0$, define

$$\ell_\varepsilon^+(m, W) = \inf \left\{\lambda_{\min}(H(u)) : u \in (-1, 1)^n \cap B_{\sqrt{m}(m)}\right\}.$$ (4.17)

The dependence of $\ell_\varepsilon^+$ on $W$ is via $H(u)$. We will make this dependence implicit in what follows, and write simply $\ell_\varepsilon^+(m) = \ell_\varepsilon^+(m, W)$. If $\ell_\varepsilon^+(m_*) > t > 0$, then the TAP free energy is strongly convex on a $\sqrt{m}$-ball around $m_*$, as desired. We use a Kac-Rice upper bound to show, with high probability, no critical points of $F_{\text{TAP}}$ belong to the set

$$\left\{m \in D_\eta : \ell_\varepsilon^+(m) < t\right\}$$

for some sufficiently small constant $t > 0$.

The condition $\ell_\varepsilon^+(m) < t$ is again random, so this is not a deterministic subset of $(-1, 1)^n$. We address this using the following extension of the Kac-Rice upper bound in Lemma 4.6.

Lemma 4.8. Fix any $\lambda > 0$, and suppose $x = 1$. Let $\text{Sym}_n$ be the space of real symmetric $n \times n$ matrices, $T \subseteq (-1, 1)^n \setminus \{0\}$ any (deterministic) Borel-measurable set, and $\ell : T \times \text{Sym}_n \to \mathbb{R}$ any Borel-measurable function. Let $c > 0$ and $t \in \mathbb{R}$ be any (possibly $n$-dependent) values, and let $U \sim \text{Unif}([-c, c])$ be a uniform random variable independent of $W$. Define

$$C = \left\{m \in T : g(m) = 0 \text{ and } \ell(m, W) + U < t\right\}.$$
Then
\[ E[|C|] \leq \int_T E\left[ |\det \mathbf{H}(\mathbf{m})| \cdot 1\{\ell(\mathbf{m}, \mathbf{W}) + U < t\} \mid \mathbf{g}(\mathbf{m}) = 0\right] p_{\mathbf{g}(\mathbf{m})}(0) \, d\mathbf{m}, \tag{4.18} \]
where \( p_{\mathbf{g}(\mathbf{m})}(0) \) is the Lebesgue-density of the distribution of \( \mathbf{g}(\mathbf{m}) \) at \( \mathbf{g}(\mathbf{m}) = 0 \), and the expectations are over both \( U \) and \( \mathbf{W} \).

(Introducing this auxiliary variable \( U \) alleviates the need to check a technical condition that \( \ell(\mathbf{m}, \mathbf{W}) = t \) and \( \mathbf{g}(\mathbf{m}) = 0 \) do not simultaneously occur at any \( \mathbf{m} \in T \), when applying the Kac-Rice lemma.)

In [FMM21], an upper bound on the determinant \( |\det \mathbf{H}(\mathbf{m})| \) was established via a spectral analysis of \( \mathbf{H}(\mathbf{m}) \), which shows \( E[|\det \mathbf{H}(\mathbf{m})|^2 \mid \mathbf{g}(\mathbf{m}) = 0] \leq e^{c(\eta)n} \) for \( \mathbf{m} \in D_\eta \) and a constant \( c(\eta) \to 0 \) as \( \eta \to 0 \). Thus, to show that (4.18) is vanishing, we complement this by showing an exponentially small upper bound for the probability \( \mathbb{P}[\ell_+^+(\mathbf{m}) + U < t \mid \mathbf{g}(\mathbf{m}) = 0] \). We do this again using the Sudakov-Fernique inequality of Lemma 4.1, to obtain the variational lower bound on the conditional mean \( E[\ell_+^+(\mathbf{m}) \mid \mathbf{g}(\mathbf{m}) = 0] \) stated in part (a) of the following lemma. This bound is shown to be positive in part (b).

**Lemma 4.9.** Suppose \( \lambda > 1 \) and \( \mathbf{x} = 1 \). Define
\[ H_\lambda^+(p,u;\alpha,\kappa,\gamma) = -\left[ 2\lambda^2 p^2 + \lambda^2 u^2 - 2\lambda^2 (1-q_*) p^2/q_* - \alpha u - \kappa p \right] + \lambda^2(1-q_*) + \gamma \]
\[ -E_{m \sim \nu_*}\left[ \left( 4\lambda^2 p^2 - q_*/q_* \right) + (2z(m)p/q_* + \alpha + \kappa m)^2 \right]/\left( \frac{4}{1-m^2} - 4\gamma \right) \]  
where \( z(m) = \text{arctanh} \, m - \lambda^2q_* + \lambda^2(1-q_*)m \).

(a) Fix any \( t > 0 \) and compact domain \( K' \subset \mathbb{R}^2 \times (-\infty,1) \). For some \( (\lambda,K',t) \)-dependent constants \( \varepsilon,\eta > 0 \), and all large \( n \),
\[ \inf_{\mathbf{m} \in D_\eta} E\left[ \ell_+^+(\mathbf{m}) \mid \mathbf{g}(\mathbf{m}) = 0\right] \geq \inf_{u \in [-1,1]} \sup_{\nu \in [-\sqrt{T},\sqrt{T}]} H_\lambda^+(p,u;\alpha,\kappa,\gamma) - t \]

(b) Suppose \( K' \) contains \( (0,0,0) \) in its interior. Then there is a \( (\lambda,K',t) \)-dependent constant \( t_0 > 0 \) for which
\[ \inf_{u \in [-1,1]} \sup_{\nu \in [-\sqrt{T},\sqrt{T}]} H_\lambda^+(p,u;\alpha,\kappa,\gamma) > t_0 > 0. \]

The desired upper bound for \( \mathbb{P}[\ell_+^+(\mathbf{m}) + U < t \mid \mathbf{g}(\mathbf{m}) = 0] \) then follows by concentration of \( \ell_+^+(\mathbf{m}) \) around its mean. Applying this to (4.18) yields the following corollary on local strong convexity.

**Corollary 4.10.** Fix any \( \lambda > 1 \), and suppose \( \mathbf{x} = 1 \). Then there exist \( \lambda \)-dependent constants \( \varepsilon,\eta,t,c > 0 \) such that, for all large \( n \),
\[ \mathbb{P}\left[ \text{there exist } \mathbf{m} \in D_\eta \text{ and } \mathbf{u} \in (-1,1)^n \cap B_{\sqrt{m}}(\mathbf{m}) : \mathbf{g}(\mathbf{m}) = 0 \text{ and } \lambda_{\text{min}}(\mathbf{H}(\mathbf{u})) < t \right] < e^{-cn}. \tag{4.20} \]

Finally, this convexity implies Theorem 2.1(a–b) by the following argument: Letting \( \mathbf{m}^k \) be a sufficiently large iterate of AMP, we may pick a local minimizer \( \mathbf{m}_* \) in Corollary 4.4 such that there is a strict descent path from \( \mathbf{m}^k \) to \( \mathbf{m}_* \). Strong convexity of \( \mathcal{F}_{\text{TAP}} \) around \( \mathbf{m}_* \) and an upper bound on \( \mathcal{F}_{\text{TAP}}(\mathbf{m}^k) - \mathcal{F}_{\text{TAP}}(\mathbf{m}_*) \) then imply an upper bound on the Euclidean distance \( \|\mathbf{m}_* - \mathbf{m}^k\|_2 \). Then this point \( \mathbf{m}_* \) must satisfy (2.2) by the Bayes-optimality of the AMP iterate \( \mathbf{m}^k \). Furthermore, the local convexity of \( \mathcal{F}_{\text{TAP}} \) implies that such a point \( \mathbf{m}_* \) is unique. We provide the details of this argument in Appendix B.3.

### 4.5 Local stability of AMP

We now describe the proof of Theorem 2.1(c). Let us write the input and output of \( T_{\text{AMP}} \) in (2.1) as
\[ (\mathbf{m}_+, \mathbf{m}) = T_{\text{AMP}}(\mathbf{m}_-, \mathbf{m}_-). \]
Differentiating by the chain rule, the Jacobian of $T_{\text{AMP}}$ may be expressed as

$$
dT_{\text{AMP}}(m, m_-) = \left( \text{diag}(1 - m_+^2) \cdot \lambda Y + 2\lambda^2 m_- m^T/n \right) I - \text{diag}(1 - m_+^2) \cdot \lambda^2[1 - Q(m)] 0 \right).
$$

(4.21)

At any point $m_* \in (-1, 1)^n$ where $g(m_*) = 0$, we have $T_{\text{AMP}}(m_*, m_*) = (m_*, m_*)$. Thus $dT_{\text{AMP}}(m_*, m_*) = B(m_*)$ for the matrix

$$
B(m) = \left( \text{diag}(1 - m^2) \cdot \lambda Y + 2\lambda^2 mm^T/n \right) I - \text{diag}(1 - m^2) \cdot \lambda^2[1 - Q(m)] 0 \right).
$$

In Appendix B.4, we first verify the simple algebraic identity that the eigenvalues $\mu \in \mathbb{C}$ of this matrix $B(m)$, for any $m \in (-1, 1)^n$, are exactly those values $\mu \in \mathbb{C}$ for which the “Bethe Hessian” matrix

$$
\mu \left( -\lambda Y - \frac{2\lambda^2}{n} mm^T \right) + \lambda^2[1 - Q(m)] I + \mu^2 \text{diag} \left( \frac{1}{1 - m^2} \right)
$$

is singular. Applying this relation, we then show the following deterministic lemma relating the spectral radius of $B(m)$ to the smallest eigenvalue of the above matrix for real arguments $\mu = \pm r$.

**Lemma 4.11.** Fix any $\lambda > 1$. There exist $\lambda$-dependent constants $\delta > 0$ and $r_0 \in (0, 1)$ such that for any $r \in (r_0, 1)$ and $m \in (-1, 1)^n$ with $|Q(m) - q_*| < \delta$, if we have

$$
\lambda_{\text{min}} \left[ \pm r \left( -\lambda Y - \frac{2\lambda^2}{n} mm^T \right) + \lambda^2[1 - Q(m)] I + r^2 \text{diag} \left( \frac{1}{1 - m^2} \right) \right] > 0
$$

(4.23)

for both choices of sign $\pm$, then $\rho(B(m)) < r < 1$.

To prove Theorem 2.1(c), by a simple continuity argument, it will suffice to consider exactly $r = 1$ in (4.23) and to show that (4.23) holds with high probability at $m = m_*$ for both choices of sign $\pm$. For $r = 1$ and sign $+$, the matrix in (4.23) is precisely the Hessian $H(m)$, whose smallest eigenvalue at $m = m_*$ was bounded in the preceding section. The case of sign $-$ is a minor extension of these arguments: Define

$$
H^-(m) = \left( \lambda Y + \frac{2\lambda^2}{n} mm^T \right) + \text{diag} \left( \frac{1}{1 - m^2} \right) + \lambda^2[1 - Q(m)] I,
$$

$$
\ell^-_m(u) = \inf \left\{ \lambda_{\text{min}}(H^-(u)) : u \in (-1, 1)^n \cap B_{\sqrt{\epsilon}}(m) \right\}.
$$

We show the following lemma using the Sudakov-Fernique inequality, analogously to Lemma 4.9.

**Lemma 4.12.** Suppose $\lambda > 1$ and $x = 1$. Define

$$
H^-_\lambda(p, u; \alpha, \kappa, \gamma) = \left[ 2\lambda^2 p^2 + \lambda^2 u^2 - 2\lambda^2(1 - q_*) p^2/q_* - \alpha u - \kappa p \right] + \lambda^2(1 - q_*) + \gamma
$$

$$
- E_{m \sim \mu_\lambda} \left[ 4\lambda^2(1 - p^2/q_*) \left( 2z(m)p/q_* + \alpha + \kappa m \right)^2 \right] \left( \frac{4}{1 - m^2} - 4\gamma \right)
$$

where $z(m) = \text{arctanh} m - \lambda^2 q_* + \lambda^2(1 - q_*) m$. Then the statements of Lemma 4.9 hold also with $\ell^+_m(u)$ and $H^+_\lambda$ replaced by $\ell^-_m(u)$ and $H^-_\lambda$.

Now applying this result in the Kac-Rice upper bound of Lemma 4.8 for $\ell(m, W) = \ell^-_m(u)$, we obtain that (4.23) also holds with high probability for $r = 1$ and sign $-$, implying Theorem 2.1(c).

The detailed proofs of this section are contained in Appendix B.4.

5 Convergence of optimization algorithms

In this section, we describe the main ideas in the proofs of Theorems 2.3 and 2.4. It again suffices to show that the results hold with high probability conditional on $x = 1$. The detailed proofs of this section are contained in Appendix C.
5.1 Convergence of NGD with local initialization

Theorem 2.3 is a consequence of the local strong convexity of $F_{\text{TAP}}$ established in Theorem 2.1(b) and the following local convergence result for the natural gradient algorithm (NGD).

**Lemma 5.1.** Fix any $\lambda > 1$, $t > 0$, and $\varepsilon \in (0, 1)$. Consider the event where $m_\ast$ in Theorem 2.1(a) exists and is unique up to sign, and $\|W\|_{\text{op}} < 3$ and $\lambda_{\min}(n \cdot \nabla^2 F_{\text{TAP}}(m)) > t$ for every $m \in (-1,1)^n \cap B_{\sqrt{\varepsilon n}}(m_\ast)$. Consider any initialization $m^0 = \tanh(h^0)$ such that

$$F_{\text{TAP}}(m^0) < F_{\text{TAP}}(m_\ast) + t\varepsilon/8, \quad \|m^0 - m_\ast\|_2 < \sqrt{\varepsilon n}. \quad (5.1)$$

There exist $(\lambda, t, \varepsilon)$-dependent constants $C, \mu, \eta_0 > 0$ such that if (NGD) with any step size $\eta \in (0, \eta_0)$ is initialized at $m^0$, then on this event, for every $k \geq 1$ we have

$$F_{\text{TAP}}(m^k) < F_{\text{TAP}}(m_\ast) + C \left( 1 + \frac{\|\arctanh(m^0)\|_2}{\sqrt{n}} \right) (1 - \mu\eta)^k, \quad (5.2)$$

$$\|m^k - m_\ast\|_2 < C\sqrt{n} \left( 1 + \frac{\|\arctanh(m^0)\|_2}{\sqrt{n}} \right) (1 - \mu\eta)^k, \quad (5.3)$$

The proof of this lemma applies the mirror-descent form of NGD given in (2.4), together with an observation that on the above event, $F_{\text{TAP}}$ is strongly smooth and strongly convex over $(-1,1)^n \cap B_{\sqrt{\varepsilon n}}(m_\ast)$ relative to the prox function $-H(m)$, in the sense of [BBT17, LFN18]

$$\mu \cdot \nabla^2 (-H(m)) \preceq \nabla^2 F_{\text{TAP}}(m) \preceq L \cdot \nabla^2 (-H(m))$$

for some constants $L, \mu > 0$. We may then adapt a convergence analysis of [LFN18] to show that, for the above initialization, NGD with sufficiently small step size $\eta > 0$ must remain in this strongly convex neighborhood and exhibit the above linear convergence to $m_\ast$. For any $\lambda > 1$, the event in Lemma 5.1 holds with high probability by Theorem 2.1. The required initial condition (5.1) is also with high probability achieved by a sufficiently large iteration of AMP, as may be deduced from the AMP state evolution. Combined, this yields Theorem 2.3. The detailed proofs of Lemma 5.1 and Theorem 2.3 are contained in Appendix C.2.

5.2 Convergence of NGD from spectral initialization

For large $\lambda$, to show the result of Theorem 2.4(b) that NGD alone converges to $\pm m_\ast$ from a spectral initialization, recall the domain

$$S = \left\{ m \in (-1,1)^n : F_{\text{TAP}}(m) < -\lambda^2/3 \right\}$$

as defined in Corollary 2.2. For a parameter $q \in (0, 1)$, define the deterministic subset

$$M_q = \left\{ m \in (-1,1)^n : M(m) > q \right\},$$

where recall $M(m) = m^\top 1/n$. We first establish the following more quantitative characterization of the landscape of $F_{\text{TAP}}$.

**Lemma 5.2.** Fix any integer $a \geq 5$, and set $q = 1 - \lambda^{-a}$. Suppose $x = 1$. For a constant $\lambda_0(a) > 0$, if $\lambda > \lambda_0(a)$, then there are $(a, \lambda)$-dependent constants $C, c, t > 0$ such that with probability at least $1 - Ce^{-cn}$,

(a) Every point $m \in S \setminus M_q$ satisfies

$$\|\sqrt{n} \cdot \nabla F_{\text{TAP}}(m)\|_2^2 > t.$$

(b) Every point $m \in S \cap M_q$ satisfies

$$n \cdot \nabla^2 F_{\text{TAP}}(m) > \frac{1}{2} \text{diag} \left( \frac{1}{1 - |m|^2} \right) \succeq \frac{1}{2} I.$$

Part (b) of this lemma is sufficient to imply Theorem 2.4(b) on the convergence of NGD: The initialization $m^0 = \tanh(h^0)$ defined by (SI) will belong to the region $S \cap M_q$ with high probability, so Lemma 5.1 may again be used to show linear convergence to $\pm m_\ast$. The detailed proof of Lemma 5.2 is contained in Appendix C.1 and the detailed proof of Theorem 2.4(b) is contained in Appendix C.2.
5.3 Convergence of AMP from spectral initialization

For Theorem 2.4(a) on the convergence of AMP, we directly prove contractivity of the map \( T_{\text{AMP}} \) defined in (2.1) locally near \( m_* \), in a parameterization by coordinates \( p \) that lie “between” \( h \) and \( m = \tanh(h) \): Define two strictly increasing functions \( \Gamma, \Lambda : \mathbb{R} \rightarrow \mathbb{R} \) as

\[
\Gamma(h) = \int_0^h \sqrt{1 - \tanh(s)^2} \, ds, \quad \Lambda(p) = \tanh(\Gamma^{-1}(p)),
\]

and consider \( p = \Gamma(h) \). Then \( m = \tanh(h) = \Lambda(p) \). We write as shorthand

\[
\frac{dm}{dh} = d_h \tanh(h), \quad \frac{dp}{dh} = d_h \Gamma(h), \quad \frac{dm}{dp} = d_p \Lambda(p)
\]

where these are vectors in \( \mathbb{R}^n \), and the derivatives are applied entry-wise. These definitions of \( \Gamma \) and \( \Lambda \) are designed so as to factor the identity \( 1 - m^2 = dm/dh \) into the pair of identities

\[
\sqrt{1 - m^2} = \frac{dm}{dp} \frac{dp}{dh}.
\]

(This reparameterization by \( p \) may seem mysterious, and is carefully chosen to precondition the Jacobian of the AMP map and enable an operator norm bound for this Jacobian. We provide a heuristic motivation for this reparameterization in Remark 5.3 in Appendix C.3.)

The range of \( p = \Gamma(h) \) is the cube \( \Omega(p) = (-\pi/2, \pi/2)^n \). We denote the AMP map (2.1) in the \( p \)-parameterization as \( T_{\text{AMP}}^{(p)} : \Omega(p) \times \Omega(p) \rightarrow \Omega(p) \times \Omega(p) \), defined by

\[
T_{\text{AMP}}^{(p)}(p, p-) = (\Lambda \otimes \Lambda)^{-1} \circ T_{\text{AMP}}((\Lambda \otimes \Lambda)(p, p_-)).
\]

Thus, reparameterizing by \( p^k = \Gamma(h^k) \), the AMP iterations (AMP) take the form \( (p^{k+1}, p^k) = T_{\text{AMP}}^{(p)}(p^k, p^{k-1}) \).

**Lemma 5.3.** Consider the metric \( \| (p, p') \|_\lambda = \| p \|_2 + \lambda^{-1/5} \| p' \|_2 \). Fix \( q = 1 - \lambda^{-5} \) and \( x = 1 \). For an absolute constant \( \lambda_0 > 0 \), suppose \( \lambda > \lambda_0 \). Then with probability at least \( 1 - Ce^{-cn} \) for \( \lambda \)-dependent constants \( C, c > 0 \), the following holds: If there exists a critical point \( m_* \in M_q \) of \( \mathcal{F}_{\text{TAP}} \), then for \( p_* = \Lambda^{-1}(m_*) \), any \( p, p_- \in B_{\lambda^{-1/5}}(p_*) \cap \Omega(p) \), and \( (p_+, p) = T_{\text{AMP}}^{(p)}(p, p_-) \), we have \( p_+ \in B_{\lambda^{-1/5}}(p_*) \cap \Omega(p) \) and

\[
\|(p_+, p) - (p_+, p_*)\|_\lambda \leq 2\lambda^{-1/5}\| (p, p_-) - (p_*, p_*) \|_\lambda.
\]  

The AMP state evolution guarantees that with probability approaching 1 as \( n \rightarrow \infty \), \( p^{k-1}, p^k \in B_{\lambda^{-1/5}}(p_*) \) for a sufficiently large iteration \( k \). Then the contractivity guaranteed in Lemma 5.3 implies Theorem 2.4(a). The detailed proofs of Lemma 5.3 and Theorem 2.4(a) are contained in Appendix C.3.

6 Discussion

In this paper, we showed the local strong convexity of the TAP free energy for \( Z_2 \)-synchronization around its Bayes-optimal local minimizer, and studied the finite-\( n \) convergence of optimization algorithms for computing this minimizer. Numerical simulations confirm that the TAP free energy can be efficiently optimized, and that properties of its minimizer are robust to model misspecification. Our results provide theoretical justification for using the TAP free energy to perform variational inference in this model.

In terms of proof techniques, our work introduced a method of using the Kac-Rice formula to study the local geometry of a non-convex function around its critical points. Some intermediate results in the proof, for example the convergence of the empirical distribution of coordinates of the TAP minimizer, are of independent interest. We note that an analogous TAP free energy function may be defined in broader contexts, such as for spiked matrix models with more general priors or for linear and generalized linear models, and some of our techniques may be useful also for analyzing the local geometries of these TAP free energy functions around their informative fixed points. However, the Rademacher \( \{+1, -1\} \) prior in \( Z_2 \)-synchronization does have several conveniences, including a fixed second moment, an explicit form for
both its entropy and its posterior mean function, and a unique fixed-point for the equation (4.1) that defines $q_\star$. Analyses of models having priors that lack these properties would have additional technical hurdles, and we leave the exploration of such extensions to future work.

Finally, we proved the finite-$n$ convergence of a well-studied AMP algorithm for this problem, which is not implied by analysis of the AMP state evolution alone. Our proof of this result required sufficiently large $\lambda$, but we conjecture that the result holds for any $\lambda > 1$. This conjecture is supported by our numerical simulations and also by the stability of the AMP map around its fixed point, which indeed holds for any $\lambda > 1$. We leave this conjecture as an open question, and hope that the techniques developed in this paper can perhaps inspire a proof.

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A Preliminaries

A.1 Uniform continuity of $\mathcal{F}_\text{TAP}$

Proposition A.1. For any $m, m' \in (-1, 1)^n$,

$$|Q(m) - Q(m')| \leq \sqrt{2} \cdot \frac{\|m - m'\|_2}{\sqrt{n}}, \quad |H(m) - H(m')| \leq (\log 2 + 1) \left(\frac{\|m - m'\|_2}{n}\right)^{1/4},$$

$$|\mathcal{F}_\text{TAP}(m) - \mathcal{F}_\text{TAP}(m')| \leq \left(\frac{\lambda}{2}\|Y\|_{\text{op}} + \frac{\lambda^2 \sqrt{2}}{2}\right) \cdot \frac{\|m - m'\|_2}{\sqrt{n}} + (\log 2 + 1) \left(\frac{\|m - m'\|_2}{n}\right)^{1/4}.$$  

Proof. First, by Cauchy-Schwarz,

$$|Q(m) - Q(m')| = \frac{|(m - m', m + m')|}{n} \leq \sqrt{2} \cdot \frac{\|m - m'\|_2}{\sqrt{n}}.$$  

Next, by Markov’s inequality, for any $t > 0$,

$$\frac{1}{n} \sum_{i=1}^{n} 1\{|m_i - m'_i| \geq t\} \leq \frac{1}{t^2} \cdot \frac{\|m - m'\|_2}{n}.$$  

Concavity of $h(m)$ implies $|h(m) - h(m')| \leq h(|m - m'| - 1) - h(-1) = h(|m - m'| - 1)$. Applying the bounds $h(m) \leq (m + 1)^{2/3}$ and $h(m) \in [0, \log 2]$ for $m \in [-1, 1]$,

$$|H(m) - H(m')| \leq \frac{1}{t^2} \cdot \frac{\|m - m'\|_2}{n} \cdot \log 2 + \left(1 - \frac{1}{t^2} \cdot \frac{\|m - m'\|_2}{n}\right) \cdot t^{2/3}.$$  

Then choosing $t = (\|m - m'\|_2^3/n)^{3/8}$ yields

$$|H(m) - H(m')| \leq (\log 2 + 1) \left(\frac{\|m - m'\|_2}{n}\right)^{1/4}.$$  

Finally, observe that

$$\frac{m^T Y m - m'^T Y m'}{n} \leq \frac{m^T Y (m - m')}{n} + \frac{m'^T Y (m - m')}{n},$$

$$\leq 2\|Y\|_{\text{op}} \cdot \frac{\|m - m'\|_2}{\sqrt{n}},$$

$$\left|(1 - Q(m))^2 - (1 - Q(m'))^2\right| \leq 2|Q(m) - Q(m')| \leq 2\sqrt{2} \cdot \frac{\|m - m'\|_2}{\sqrt{n}}.$$  

Combining these bounds and applying to (TAP) yields the stated bound for $\mathcal{F}_\text{TAP}$.  

A.2 Properties of $q_*$ and $\mu_*$

For $\lambda > 1$, recall $q_*, h_*, e_*$ from (4.1–4.3) and the distribution $\mu_*$ from (4.5), and define in addition

$$b_* = E_{G \sim \mathcal{N}(0, 1)}\left(\tanh(\lambda^2 q_* + \lambda \sqrt{q_*} G)^4\right) \quad (A.1)$$

Proposition A.2. For any $\lambda > 1$, there is a unique solution $q_* \in (0, 1)$ to (4.1). This solution $q_* = q_*(\lambda)$ is strictly increasing in $\lambda > 1$, and satisfies $\lim_{\lambda \to 1} q_*(\lambda) = 0$ and $\lim_{\lambda \to 1} q_*(\lambda)/(\lambda - 1) = 2$. Furthermore, we have

$$q_* = E_{m \sim \mu_*}[m^2] = E_{m \sim \mu_*}[m], \quad b_* = E_{m \sim \mu_*}[m^4] = E_{m \sim \mu_*}[m^3],$$

$$\lambda^2 q_* = E_{m \sim \mu_*}[m \arctanh m], \quad h_* = E_{m \sim \mu_*}[h(m)].$$

Finally, we have $q_*(\lambda) > 1 - 1/\lambda^2$.  

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Proof. [DAM16, Appendix B.2] shows that the function $f(\gamma) = \mathbb{E}[\tanh(\gamma + \sqrt{G})^2]$ is strictly increasing and strictly concave over $\gamma \in [0, \infty)$. By simple calculus, we have $f(0) = 0$, $f'(0) = 1$, and $f''(0) = -2$. Hence, for any $\lambda > 1$, there is a unique solution $\gamma_\lambda \in (0, \lambda^2)$ to $f(\gamma_\lambda) = \gamma_\lambda / \lambda^2$, and we identify $q_* = \gamma_\lambda / \lambda^2$. Furthermore, the same argument shows that $q_*(\lambda)$ is strictly increasing in $\lambda$, and $\lim_{\lambda \to 1} q_*(\lambda) = 0$. Finally, simple calculus shows that $\lim_{\lambda \to 1} q_*(\lambda) / (\lambda - 1) = 2$.

The identities $q_* = \mathbb{E}[m^2]$ and $b_* = \mathbb{E}[m^4]$ follow by definition. [DAM16, Appendix B.2] also shows $\mathbb{E}[\tanh(\gamma + \sqrt{G})^{2k}] = \mathbb{E}[\tanh(\gamma + \sqrt{G})^{2k-1}]$ for any integer $k \geq 1$, so $q_* = \mathbb{E}[m]$ and $b_* = \mathbb{E}[m^3]$. The identity $\lambda^2 q_* = \mathbb{E}[m \arctanh m]$ follows from combining $q_* = \mathbb{E}[m]$ and Gaussian integration by parts,

$$
\mathbb{E}[\lambda \sqrt{q_*G} \cdot \tanh(\lambda^2 q_* + \lambda \sqrt{q_*G})] = \lambda^2 q_* \mathbb{E}[1 - \tanh(\lambda^2 q_* + \lambda \sqrt{q_*G})^2] = \lambda^2 q_*(1 - q_*).$
$$

The identity $h_* = \mathbb{E}[h(m)]$ follows from this, $h(\tanh(x)) = \log 2 \cosh(x) - x \tanh(x)$, and the definition of $h_*$. To show $q_* > 1 - 1/\lambda^2$, note that for an observation $Z \sim \mathcal{N}(\lambda^2 q_* X, \lambda^2 q_*)$ with prior $X \sim \text{Unif}\{-1, +1\}$, $\tanh(Z) = \mathbb{E}[X|Z]$ is the posterior mean estimate of $X$. Applying $q_* = \mathbb{E}[m^2] = \mathbb{E}[m]$, its Bayes risk is $\mathbb{E}[(\mathbb{E}[X|Z] - X)^2] = q_* - 2q_* + 1 = 1 - q_*$, which may be compared to $\mathbb{E}[(1 + \lambda^2 q_* - 2Z - X)^2] = (1 + \lambda^2 q_*)^{-1}$ for the linear estimator $(1 + \lambda^2 q_* - 1)Z$. Since $q_* > 0$, this linear estimator is not almost surely equal to the Bayes estimator, so $q_* = 1 - (1 + \lambda^2 q_*)^{-1}$ strictly. Rearranging yields $q_* > 1 - 1/\lambda^2$. 

Proposition A.3. For an absolute constant $\lambda_0 > 0$ and all $\lambda > \lambda_0$, we have $q_* > 1 - e^{-\lambda^2/8}$.

Proof. From the identity $q_* = \mathbb{E}_{m \sim \mu_*}[m]$, monotonicity of $\tanh$, and the bound $\tanh(x) \geq -1$, we have

$$
q_* = \mathbb{E}_{G \sim \mathcal{N}(0,1)}[\tanh(\lambda^2 q_* + \lambda \sqrt{q_*G})]
\geq \tanh(\lambda^2 q_*/4) \cdot \mathbb{P}[G \geq -3\lambda \sqrt{q_*/4}] - \mathbb{P}[G < -3\lambda \sqrt{q_*/4}].
$$

Since $q_*(\lambda) \to 1$ as $\lambda \to \infty$, for sufficiently large $\lambda$ we have $q_* > 1/2$. Then, applying $\mathbb{P}[G < -t] < e^{-t^2/2}$ and $\tanh(t) > 1 - 2e^{-2t}$ for $t > 0$,

$$
q_* > \tanh(\lambda^2/8)(1 - e^{-9\lambda^2/64}) - e^{-9\lambda^2/64} > 1 - e^{-\lambda^2/8}.
$$

A.3 Properties of Wasserstein-2 distance

Let $W_2(\mu, \mu')$ denote the Wasserstein-2 distance between distributions $\mu$ and $\mu'$ on $\mathbb{R}$, i.e.

$$
W_2(\mu, \mu') = \left( \inf_{\text{couplings } \nu \text{ of } (\mu, \mu')} \int (x - x')^2 d\nu(x, x') \right)^{1/2}.
$$

Note that if $\mu, \mu'$ are the empirical distributions of coordinates of $x, x' \in \mathbb{R}^n$, then this implies

$$
W_2(\mu, \mu')^2 \leq \frac{1}{n} \|x - x'\|_2^2.
$$

This distance induces the weak convergence $W_2(\mu_n, \mu) \to 0$ if and only if $\mathbb{E}_{x \sim \mu_n}[U(x)] \to \mathbb{E}_{x \sim \mu}[U(x)]$ for all continuous functions $U : \mathbb{R} \to \mathbb{R}$ satisfying $\sup_{x \in \mathbb{R}} |U(x)|/(1 + x^2) < \infty$, see [Vil08, Definition 6.7].

In Section 4, we denoted $W(\mu, \mu') = W_2(\arctanh \mu, \arctanh \mu')$. Then, defining the function class

$$
\mathcal{Q} = \left\{ \text{continuous functions } U : (-1,1) \to \mathbb{R} \text{ s.t. } \sup_{m \in (-1,1)} \frac{|U(m)|}{1 + (\arctanh m)^2} < \infty \right\}
$$

this distance $W(\mu, \mu')$ induces the weak convergence

$$
\lim_{n \to \infty} W(\mu_n, \mu) = 0 \text{ if and only if } \lim_{n \to \infty} \mathbb{E}_{m \sim \mu_n}[U(m)] = \mathbb{E}_{m \sim \mu}[U(m)] \text{ for all } U \in \mathcal{Q}.
$$

Proposition A.4. Let $X_1, \ldots, X_n \sim \mathcal{N}(0,1)$ and let $\hat{\mu}$ be the empirical distribution of $X_1, \ldots, X_n$. There is a universal constant $C > 0$ such that for any $t > 0$,

$$
\mathbb{P}[W_2(\hat{\mu}, \mathcal{N}(0,1)) \geq t + Cn^{-1/2}] \leq e^{-nt^2/2}.
$$
For (a), let 

\[ W_2(\mu, \mathcal{N}(0, 1)) \leq Cn^{-1/2} \]

for a universal constant \( C > 0 \). By the Wasserstein-2 triangle inequality, if \( \mu, \mu' \) are the empirical distributions of \( x, x' \in \mathbb{R}^n \), then

\[
|W_2(\mu, \mathcal{N}(0, 1)) - W_2(\mu', \mathcal{N}(0, 1))| \leq W_2(\mu, \mu') \leq \left( n^{-1} \|x - x'\|_2^2 \right)^{1/2}.
\]

So \( W_2(\mu, \mathcal{N}(0, 1)) \) is \( n^{-1/2} \)-Lipschitz in \((X_1, \ldots, X_n)\), and the result follows by concentration of Gaussian measure.

\[ \square \]

**Proposition A.5.** Let \( \mu, \mu' \) be two probability distributions on \( \mathbb{R} \), let \( X \sim \mu \) and \( X' \sim \mu' \), and denote \( \|\mu\|_{L_2} = (E X^2)^{1/2} \) and \( \|\mu'\|_{L_2} = (E X'^2)^{1/2} \).

(a) For any \( \alpha \in (0, 1) \), suppose \( q_\alpha, q'_\alpha \) satisfy \( P(|X| \geq q_\alpha) = P(|X'| \geq q'_\alpha) = \alpha \). Then

\[
E[X^2 1_{|X| \geq q_\alpha}] - E[X'^2 1_{|X'| \geq q'_\alpha}] \leq W_2(\mu, \mu') \cdot (\|\mu\|_{L_2} + \|\mu'\|_{L_2}).
\]

(b) Let \( f \) be any function such that \( |f(x) - f(x')| \leq L(1 + |x| + |x'|)|x - x'| \) for all \( x, x' \in \mathbb{R} \) and some constant \( L > 0 \). Then

\[
E[f(X)] - E[f(X')] \leq L \cdot W_2(\mu, \mu') \cdot (1 + \|\mu\|_{L_2} + \|\mu'\|_{L_2}).
\]

**Proof.** For (a), let \( W_2(-|X|, -|X'|) \) denote the Wasserstein-2 distance between the laws of \(-|X|\) and \(-|X'|\). Any coupling of \((X, X')\) defines also a coupling of \((-|X|, -|X'|)\), so \( W_2(-|X|, -|X'|)^2 \leq E((-|X| + |X'|)^2) \leq E((X - X')^2) \). Taking the infimum over all couplings \((X, X')\) gives

\[ W_2(-|X|, -|X'|) \leq W_2(\mu, \mu'). \]

Now consider the quantile function \( G(u) \) of \(-|X|\), satisfying \( G(u) \leq -x \) if and only if \( u \leq P[-|X| \leq -x] = P[|X| \geq x] \). Let \( G'(u) \) be the quantile function of \(-|X'|\), let \( U \sim \text{Unif}(0, 1) \), and consider the coupling of \((-|X|, -|X'|)\) given by \(-|X| = G(U)\) and \(-|X'| = G'(U)\). This is the optimal coupling that yields

\[ W_2(-|X|, -|X'|) = E[(-|X| + |X'|)^2]^{1/2} = E[(G(U) - G'(U))^2]^{1/2}. \]

We have \( |X| \geq q_\alpha \) if and only if \( G(U) \leq -q_\alpha \) if and only if \( U \leq \alpha \). Hence

\[
E[X^2 1_{|X| \geq q_\alpha}] - E[X'^2 1_{|X'| \geq q'_\alpha}] = E[(G(U)^2 - G'(U)^2)1_{U \leq \alpha}],
\]

so by the Cauchy-Schwarz and Minkowski inequalities,

\[
\leq E[(G(U)^2 - G'(U)^2)] \leq E[(G(U) - G'(U))^2]^{1/2} \leq W_2(\mu, \mu') \cdot (\|\mu\|_{L_2} + \|\mu'\|_{L_2}).
\]

For (b), consider any coupling of \((X, X')\). Applying again Cauchy-Schwarz and Minkowski,

\[
|E[f(X)] - E[f(X')]| \leq E[|f(X) - f(X')|] \leq L \cdot E[(1 + |X| + |X'|)^2]^{1/2} \cdot E[(X - X')^2]^{1/2} \leq L \cdot (1 + \|\mu\|_{L_2} + \|\mu'\|_{L_2}) \cdot E[(X - X')^2]^{1/2}.
\]

The left side does not depend on the coupling, so taking the infimum over couplings yields (b).

\[ \square \]
A.4 Sudakov-Fernique bound

Proof of Lemma 4.1. Denote

\[ G(x) = v(x)^T W v(x) + f(x), \quad g(x) = \frac{2}{\sqrt{n}} \|v(x)\|_2 \langle g, v(x) \rangle + f(x). \]

For all \( x \in \mathcal{X} \), we have \( \mathbb{E}[G(x)] = \mathbb{E}[g(x)] = f(x) \). Furthermore, for any \( v, v' \in \mathbb{R}^n \), we have

\[
\mathbb{E} \left[ \langle v, Wv \rangle \cdot \langle v', Wv' \rangle \right] = \sum_{i=1}^{n} \mathbb{E}[W_{ii}^2 v_i^2] + 4 \sum_{i<j} \mathbb{E}[W_{ij}^2 v_i v_j] = \frac{2}{n} \langle v, v' \rangle^2.
\]

Then

\[
\mathbb{E} \left[ (\langle v, Wv \rangle - \langle v', Wv' \rangle)^2 \right] = \frac{2}{n} \left( \|v\|_2^4 + \|v'\|_2^4 - 2\langle v, v' \rangle^2 \right)
\leq \frac{2}{n} \left( \|v\|_2^4 + \|v'\|_2^4 + 2\|v\|_2^2 \|v'\|_2^2 - 4\|v\|_2 \|v'\|_2 \langle v, v' \rangle \right)
\leq \frac{2}{n} \left( 2\|v\|_2^4 + 2\|v'\|_2^4 - 4\|v\|_2 \|v'\|_2 \langle v, v' \rangle \right)
\leq \mathbb{E} \left[ \left( \frac{2}{\sqrt{n}} \|v\|_2 (g, v) - \frac{2}{\sqrt{n}} \|v'\|_2 (g, v') \right)^2 \right].
\]

So \( \mathbb{E}[(G(x) - G(x'))^2] \leq \mathbb{E}[(g(x) - g(x'))^2] \), and the result follows from the Sudakov-Fernique inequality, see e.g. [AT09, Theorem 2.2.3].

A.5 Kac-Rice upper bounds

We prove the Kac-Rice upper bounds of Lemmas 4.6 and 4.8 by small extensions of arguments in [FMM21].

Proof of Lemma 4.6. This follows from taking \( \delta \to 0 \) on both sides of [FMM21, Lemma A.1], using the monotone convergence theorem.

Lemma A.6. In the setting of Lemma 4.8, suppose further that \( T \subset (-1,1)^n \setminus \{0\} \) is compact, and its boundary \( \partial T \) has zero Lebesgue measure in \( \mathbb{R}^n \). Then with probability 1 over \( W \) and \( U \), there are no points \( m \in T \) satisfying \( g(m) = 0 \) together with any of the following three conditions:

- \( \det H(m) = 0 \), or
- \( m \in \partial T \), or
- \( \ell(m, W) + U = t \).

On this event of probability 1,

\[
|C| = \lim_{r \to 0} \int_T 1\{\ell(m) + U < t\} \cdot \frac{1}{\text{Vol}(B_r(0))} 1\{g(m) \in B_r(0)\} : |\det H(m)| \, dm.
\]  

(3)

Proof. It is shown in [FMM21, Lemma A.3] that on an event \( \mathcal{E} \) of probability 1 defined by \( W \), no point \( m \in T \) satisfies both \( g(m) = 0 \) and either \( \det H(m) = 0 \) or \( m \in \partial T \). Letting \( \bar{T} \) be the interior of \( T \), it remains to check that with probability 1, also no point \( m \in \bar{T} \) satisfies both \( g(m) = 0 \) and \( \ell(m, W) + U = t \).

Conditioning on \( W \) and on this event \( \mathcal{E} \), the TAP free energy \( F_{\text{TAP}}(m) \) is a Morse function over \( \bar{T} \). Thus the critical points \( m \in \bar{T} \) where \( g(m) = 0 \) are isolated, and there are at most countably many such points. So

\[
\{t - \ell(m, W) : m \in \bar{T}, g(m) = 0\}
\]

is a countable set of values. Since \( U \) is independent of \( W \), this implies that \( U \) does not belong to this set with probability 1 conditional on \( W \) and \( \mathcal{E} \). Hence also unconditionally with probability 1 over \( W \) and \( U \), no point \( m \in \bar{T} \) satisfies both \( g(m) = 0 \) and \( \ell(m, W) + U = t \), as desired.

The statement (3) then follows from [AT09, Theorem 11.2.3].

}
Proof of Lemma 4.8. We write as shorthand $\ell(m)$ for $\ell(m, W)$. Suppose first that $T$ is compact with boundary $\partial T$ having Lebesgue measure 0. Then applying (A.3), Fatou’s lemma, and Fubini’s theorem,

$$
E[|C|] 
\leq \liminf_{r \to 0} \int_T \mathbb{E} \left[ 1\{\ell(m) + U < t\} \cdot \frac{1}{\text{Vol}(B_r(0))} \cdot \mathbb{I}\{g(m) \in B_r(0)\} \cdot |\det H(m)| \right] \, dm
$$

$$
= \liminf_{r \to 0} \int_T \left( \frac{1}{\text{Vol}(B_r(0))} \int_{B_r(0)} \mathbb{E} \left[ 1\{\ell(m) + U < t\} \cdot |\det H(m)| \cdot g(m) = h \right] p_{g(m)}(h) \, dh \right) \, dm
$$

where $p_{g(m)}(h)$ is the Lesbesgue-density of $g(m)$ at $g(m) = h$. It may be checked from the forms (4.15) and (4.16) for $g$ and $H$ that for any fixed $r_0 > 0$, both $p_{g(m)}(h)$ and $\mathbb{E}[|\det H(m)| \cdot g(m) = h]$ are continuous functions of $(m, h) \in T \times B_{r_0}(0)$ (where $B_{r_0}(0)$ is the closure of $B_{r_0}(0)$). Then applying

$$
\mathbb{E} \left[ 1\{\ell(m) + U < t\} \cdot |\det H(m)| \right] g(m) = h p_{g(m)}(h) \leq \mathbb{E} \left[ |\det H(m)| \cdot g(m) = h \right] p_{g(m)}(h),
$$

this continuity, and the compactness of $T \times B_{r_0}(0)$, the left side is bounded over $(m, h) \in T \times B_{r_0}(0)$. Thus the above integrand

$$
\frac{1}{\text{Vol}(B_r(0))} \int_{B_r(0)} \mathbb{E} \left[ 1\{\ell(m) + U < t\} \cdot |\det H(m)| \cdot g(m) = h \right] p_{g(m)}(h) \, dh
$$

is also bounded over $m \in T$, and furthermore by continuity in $h$, its limit as $r \to 0$ is

$$
\mathbb{E} \left[ 1\{\ell(m) + U < t\} \cdot |\det H(m)| \right] g(m) = 0 p_{g(m)}(0).
$$

Then applying the bounded convergence theorem, we obtain (4.18).

This establishes the result for all compact $T \in (-1, 1)^n \setminus \{0\}$ whose boundary has zero Lebesgue measure, and in particular for all hyper-rectangles $T$ in this domain. The result for all Borel-measurable $T$ then follows from the same argument of outer measure as in the conclusion of the proof of [FMM21, Lemma A.1].

A.6 AMP state evolution

The following lemma collects some implications of the state evolution for AMP starting from a spectral initialization, as characterized in [MV21].

Lemma A.7. Suppose $\lambda > 1$. Let $\{h^k, m^k\}_{k \geq 0}$ be the iterates of the AMP algorithm (AMP) with the initializations of (SI), where we take the sign $\langle x, h^0 \rangle \geq 0$. Set $\gamma_0 = \lambda^2 - 1$ and define recursively $\gamma_{k+1} = \lambda^2 \mathbb{E}_{G \sim \mathcal{N}(0, 1)}[\tanh(\gamma_k + \sqrt{\gamma_k} G)^2]$.

(a) For any fixed $k \geq 0$ and any function $\psi : \mathbb{R}^2 \to \mathbb{R}$ satisfying $|\psi(x, y) - \psi(x', y')| \leq C(1 + \|x, y\|_2 + \|x', y'\|_2)\|x, y - (x', y')\|_2$ for a constant $C > 0$, almost surely

$$
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^n \psi(x_i, h_i^k) = \mathbb{E}[\psi(X, \gamma_k X + \sqrt{\gamma_k} G)] \text{ where } X \sim \text{Unif}\{-1, +1\} \perp G \sim \mathcal{N}(0, 1).
$$

(b) We have $\lim_{k \to \infty} \gamma_k = \lambda^2 q_s$, and almost surely

$$
\lim_{k \to \infty} \frac{\langle x, m^k \rangle}{n}, \frac{Q(m^k), H(m^k), \frac{m^k}{n}, \frac{h^k}{n}}{\frac{n}{2}} = (q_s, q_s, h_s, \lambda^2 q_s, \mathbb{E}_{m \sim \mu} \langle \text{arctanh}(m) \rangle).
$$

(c) For any fixed $\varepsilon > 0$,

$$
\lim_{k \to \infty} \lim_{n \to \infty} \mathbb{P} \left[ \frac{1}{n} \|X_{\text{Bayes}} - m^k(m^k)^\top\|^2_F > \varepsilon \right] = 0.
$$
(d) For any fixed \( \varepsilon > 0 \),
\[
\lim_{k \to \infty} \lim_{n \to \infty} \mathbb{P}\left[ |\mathcal{F}_{\text{TAP}}(m^k) - e_*| > \varepsilon \right] = 0.
\]

Proof. Part (a) follows from [MV21, Theorem 2], specializing to the prior distribution \( X \sim \text{Unif}\{-1,+1\} \) and the optimal nonlinearity \( f_k(h) = \lambda \tanh(h) \) in each iteration. (The required initialization \( m^{-1} = \lambda h^0 \) was not specified in [MV21]; this condition may be derived from the observation that the principal eigenvector \( h \) of \( Y \) is a fixed point of the linear AMP iterations
\[
m^k = \alpha h^k, \quad h^{k+1} = Y m^k - \alpha m^{k-1} = \alpha Y h^k - \alpha^2 h^{k-1}
\]
when \( 1 = \alpha \lambda_{\max}(Y) - \alpha^2 \). In the limit \( n \to \infty \), we have \( \lambda_{\max}(Y) \to \lambda + 1/\lambda \), so this is satisfied by \( \alpha = \lambda \).

For part (b), recall from the proof of Proposition A.2 that \( f(\gamma) = \mathbb{E}\{\tanh(\gamma + \sqrt{n}G)^2\} \) is increasing and concave over \( \gamma \in [0,\infty) \), with \( f(0) = 0 \) and \( f'(0) = 1 \). Then the iterations \( \gamma_{k+1} = \lambda^2 f(\gamma_k) \) must converge to the unique fixed point \( \gamma_* = \lambda^2 q_* \) from any positive initialization \( \gamma_0 > 0 \). Applying part (a) with
\[
\psi(x,y) \in \{x \tanh(y), \tanh(y)^2, h(\tanh(y)), y \tanh(y), y^2\}
\]
and using Proposition A.2 to evaluate the Gaussian expectations in the limit \( k \to \infty \), part (b) follows.

For part (c), we have by part (b) almost surely
\[
\lim_{k \to \infty} \lim_{n \to \infty} \frac{1}{n^2} \left\| xx^\top - m^k(m^k)^\top \right\|_F^2 = \lim_{k \to \infty} \lim_{n \to \infty} \left( 1 - 2 \left( \frac{\langle x, m^k \rangle}{n} \right)^2 + Q(m^k)^2 \right) = 1 - q_*^2.
\]
This coincides with the Bayes risk \( \lim_{n \to \infty} \mathbb{E}\left[ \|\hat{X}_{\text{Bayes}} - XX^\top\|_F^2 \right]/n^2 = 1 - q_*^2 \), see e.g. [LM19, Section 2.3]. Since \( \hat{X}_{\text{Bayes}} = \mathbb{E}[xx^\top \mid Y] \) and \( m^k(m^k)^\top \) is a function of \( Y \), we have the Pythagorean relation
\[
\mathbb{E}\left[ \| xx^\top - m^k(m^k)^\top \|_F^2 \right] = \mathbb{E}\left[ \| \hat{X}_{\text{Bayes}} - m^k(m^k)^\top \|_F^2 \right] + \mathbb{E}\left[ \| \hat{X}_{\text{Bayes}} - xx^\top \|_F^2 \right],
\]
hence \( \lim_{k \to \infty} \lim_{n \to \infty} \mathbb{E}\left[ \| \hat{X}_{\text{Bayes}} - m^k(m^k)^\top \|_F^2 \right]/n^2 = 0 \). Part (c) then follows by Markov’s inequality.

Finally, for part (d), let us use the notational shorthand \( X \to c \) to mean the convergence in probability \( \lim_{k \to \infty} \lim_{n \to \infty} \mathbb{P}\{ |X - c| > \varepsilon \} = 0 \) for any fixed \( \varepsilon > 0 \). Observe that (c) implies
\[
\frac{1}{n^2} \left\| m^k(m^k)^\top - m^{k+1}(m^{k+1})^\top \right\|_F^2 \to 0.
\]

Writing
\[
\frac{1}{n^2} \left\| m^k(m^k)^\top - m^{k+1}(m^{k+1})^\top \right\|_F^2 = Q(m^k)^2 + Q(m^{k+1})^2 - 2 \left( \frac{\langle m^k, m^{k+1} \rangle}{n} \right)^2,
\]
this and the statements \( Q(m^k), Q(m^{k+1}) \to q_* \) from (b) imply \( \langle m^k, m^{k+1} \rangle/n \to q_* \). Applying again \( Q(m^k), Q(m^{k+1}) \to q_* \) and Cauchy-Schwarz, this in turn implies \( \| m^k \pm m^{k+1} \|_F^2/n \to 0 \) for some choice of sign \( \pm \). Part (b) shows \( \langle x, m^k \rangle/n \to q_* \) and \( \langle x, m^{k+1} \rangle/n \to q_* \) for the same positive constant \( q_* \), so \( \| m^k - m^{k+1} \|_F^2/n \to 0 \) must hold with the sign \( - \). Then by part (b) and Cauchy-Schwarz, we have also
\[
\langle m^k, m^{k+1} \rangle/n \to q_*, \quad \langle m^k, h^{k+1} \rangle/n \to \lambda^2 q_*.
\]

Writing \( \lambda Y m^k = h^{k+1} + \lambda^2 (1 - Q(m)) m^{k-1} \), this yields
\[
\mathcal{F}_{\text{TAP}}(m^k) = -\frac{1}{2n} \left[ \langle m^k, h^{k+1} \rangle + \lambda^2 (1 - Q(m)) \langle m^k, m^{k-1} \rangle \right] - h(m^k) - \frac{\lambda^2}{4} \left( 1 - Q(m) \right)^2
\]
\[
\to -\frac{1}{2} \left( \lambda^2 q_* + \lambda^2 (1 - q_*) q_* \right) - h_* - \frac{\lambda^2}{4} (1 - q_*^2) = e_*.
\]

We remark that this proves Lemma A.7 marginally over \( x \sim \text{Unif}\{-1,+1\}^n \). The claims of parts (b–d) then hold also conditional on any fixed \( x \in \{-1,+1\}^n \), and in particular conditional on \( x = 1 \), by sign symmetry.
B Proofs for the local analysis of TAP and AMP

B.1 Proofs for Section 4.2

Proof of Lemma 4.2. Fixing $x = 1$, we have

$$-\mathcal{F}_{\text{TAP}}(m) = \frac{\lambda}{2n} \langle m, Wm \rangle + \frac{\lambda^2}{2} M(m)^2 + \frac{\lambda^2}{4} \left(1 - Q(m)\right)^2 + H(m).$$

Let $g \sim \mathcal{N}(0, I_n)$, and define an auxiliary Gaussian process $G(m)$ to be

$$G(m) = \frac{\lambda}{n^{3/2}} \|m\|_2 \langle g, m \rangle + \frac{\lambda^2}{2} M(m)^2 + \frac{\lambda^2}{4} \left(1 - Q(m)\right)^2 + H(m).$$

Then by Lemma 4.1, for any domain $\Omega \subset (-1, 1)^n$ we have

$$\mathbb{E} \left[ \sup_{m \in \Omega} -\mathcal{F}_{\text{TAP}}(m) \right] \leq \mathbb{E} \left[ \sup_{m \in \Omega} G(m) \right]. \quad (B.1)$$

Introduce

$$\Gamma(g; p, \gamma, \tau, \nu) = \sup_{m \in (-1, 1)^n} \lambda p \cdot gm + \frac{\gamma m^2}{2} + \tau m + \nu h(m).$$

Then

$$\sup_{m \in (-1, 1)^n; \langle Q(m), M(m), H(m)\rangle \in K} G(m)$$

$$\leq \sup_{(q, \varphi, h) \in K} \sup_{m \in (-1, 1)^n; \langle Q(m), M(m), H(m)\rangle = (q, \varphi, h)} \frac{\lambda}{n} \langle g, m \rangle \sqrt{q} + \frac{\lambda^2}{2} \varphi^2 + \frac{\lambda^2}{4} (1 - q)^2 + h$$

$$+ \tau \left( M(m) - \varphi \right) + \frac{\gamma}{2} \left( Q(m) - q \right) + \nu \left( H(m) - h \right)$$

$$= \sup_{(q, \varphi, h) \in K} \inf_{(\gamma, \tau, \nu) \in K'} \left[ \frac{\lambda^2}{2} \varphi^2 + \frac{\lambda^2}{4} (1 - q)^2 + h - \frac{\tau^2}{2} - \varphi \tau - h \nu + \frac{1}{n} \sum_{i=1}^{n} \Gamma(g_i; \sqrt{q}, \gamma, \tau, \nu) \right]$$

$$\leq \sup_{(q, \varphi, h) \in K} \inf_{(\gamma, \tau, \nu) \in K'} \left[ \frac{\lambda^2}{2} \varphi^2 + \frac{\lambda^2}{4} (1 - q)^2 + h - \frac{\tau^2}{2} - \varphi \tau - h \nu \right] 
+ \sup_{(q, \varphi, h) \in K} \left[ \frac{1}{n} \sum_{i=1}^{n} \Gamma(g_i; \sqrt{q}, \gamma, \tau, \nu) - \mathbb{E}_{G \sim \mathcal{N}(0, 1)}[\Gamma(G; \sqrt{q}, \gamma, \tau, \nu)] \right].$$

Let $\omega = (p, \gamma, \tau, \nu)$, let $K_{\omega} = [0, 1] \times K'$, and let $C, c > 0$ be $(\lambda, K', \varepsilon)$-dependent constants changing from instance to instance. For any fixed $\omega \in K_{\omega}$, the function $\Gamma(g; \omega)$ is $\lambda p$-Lipschitz in $g$. Then the Gaussian concentration and Hoeffding inequalities show $\|\Gamma(G; \omega) - \mathbb{E}[\Gamma(G; \omega)]\|_\psi_2 \leq C$ and hence

$$\mathbb{E} \left[ \left| \frac{1}{n} \sum_{i=1}^{n} \Gamma(g_i; \omega) - \mathbb{E}[\Gamma(G; \omega)] \right| \right] \leq C / \sqrt{n}. \quad (B.3)$$

To obtain uniform control over $K_{\omega}$, let $\mathcal{N}$ be a $n^{-1/6}$-net of $K_{\omega}$ of cardinality $|\mathcal{N}| \leq C n^{1/4}$. Fixing $g$ and applying $h(m) \in [0, \log 2]$, observe that $\Gamma(g; \omega)$ is $1$-Lipschitz in $\gamma$, 1-Lipschitz in $\tau$, $(\log 2)$-Lipschitz in $\nu$, and $\lambda|g|$-Lipschitz in $g$. Then $n^{-1} \sum_{i=1}^{n} \Gamma(g_i; \omega)$ is $C(1 + n^{-1} \sum_{i=1}^{n} |g_i|)$-Lipschitz in $\omega$. Applying (B.3) for
\( \omega \in \mathcal{N} \), we then obtain for all large \( n \) that

\[
\mathbb{E} \left[ \sup_{\omega \in \mathcal{N}} \left| \frac{1}{n} \sum_{i=1}^{n} \Gamma(g_i; \omega) - \mathbb{E}[\Gamma(G; \omega)] \right| \right] \\
\leq \mathbb{E} \left[ \sup_{\omega \in \mathcal{N}} \left| \frac{1}{n} \sum_{i=1}^{n} \Gamma(g_i; \omega) - \mathbb{E}[\Gamma(G; \omega)] \right| \right] + Cn^{-1/16} \cdot \mathbb{E} \left[ 1 + \frac{1}{n} \sum_{i=1}^{n} |g_i| \right] \\
\leq |\mathcal{N}| \cdot \frac{C}{\sqrt{n}} + Cn^{-1/16} \cdot \mathbb{E} \left[ 1 + \frac{1}{n} \sum_{i=1}^{n} |g_i| \right] < \epsilon/2. \tag{B.4}
\]

Combining (B.1), (B.2), and (B.4) and negating the sign, we arrive at

\[
\mathbb{E} \left[ \inf_{m \in (-1,1)^n: (Q(m),M(m),H(m)) \in K} \mathcal{F}_{\text{TAP}}(m) \right] \geq \inf_{(q,\varphi,h) \in K} \sup_{(\gamma,\tau,\nu) \in K'} E_\lambda(q, \varphi, h; \gamma, \tau, \nu) - \epsilon/2
\]

for all large \( n \). Finally, writing \( W = (Z + Z^T)/\sqrt{2n} \) where \( Z \) has i.i.d. \( \mathcal{N}(0,1) \) entries, observe that \( Z \mapsto (m, Wm) \) is \( \sqrt{2/n \|m\|_2^2} \)-Lipschitz with respect to the Frobenius norm of \( Z \), and \( \|m\|_2^2 \leq n \). Then \( \inf_{m \in (-1,1)^n: (Q(m),M(m),H(m)) \in K} \mathcal{F}_{\text{TAP}}(m) \) is \( C/\sqrt{n} \)-Lipschitz in \( Z \). Applying Gaussian concentration of measure,

\[
\mathbb{P} \left[ \inf_{m \in (-1,1)^n: (Q(m),M(m),H(m)) \in K} \mathcal{F}_{\text{TAP}}(m) \\
\leq \mathbb{E} \left( \inf_{m \in (-1,1)^n: (Q(m),M(m),H(m)) \in K} \mathcal{F}_{\text{TAP}}(m) \right) - \epsilon/2 \right] < e^{-cn}.
\]

Combining the above two displays shows (4.9).

**Proof of Lemma 4.3.** Observe that \( h'(m) = -\text{arctanh} \ m \) and \( h''(m) = 1/(1-m^2) \leq -1 \) for all \( m \in (-1,1) \).

Thus, for \( \nu > \max(\gamma,0) \) and any realization of \( G \), the function \( m \mapsto \lambda\sqrt{q}. \ Gm + \gamma m^2/2 + \tau m + \nu h(m) \) is strictly concave over \( m \in (-1,1) \), with derivative diverging to \( \mp \infty \) as \( m \to \pm 1 \). Then its supremum is achieved at the unique value \( m = m(G; q, \gamma, \tau, \nu) \in (-1,1) \) satisfying the stationary condition

\[
0 = \lambda\sqrt{q} G + \tau + \gamma m - \nu \text{arctanh}(m). \tag{B.5}
\]

The above strict concavity implies that the derivative in \( m \) of the right side of (B.5) is non-zero, so by the implicit function theorem, \( m(G; q, \gamma, \tau, \nu) \) is analytic over \( \{(q, \gamma, \tau, \nu) \in \mathbb{R}^5 : q > 0, \nu > \max(\gamma,0)\} \).

Denote

\[
F(G; q, \gamma, \tau, \nu) = \lambda\sqrt{q}. \ Gm(G; q, \gamma, \tau, \nu) + \frac{\gamma m(G; q, \gamma, \tau, \nu)^2}{2} + \tau m(G; q, \gamma, \tau, \nu) + \nu h(m(G; q, \gamma, \tau, \nu)).
\]

Applying that (B.5) holds at \( m = m(G; q, \gamma, \tau, \nu) \) to cancel the terms involving the derivatives of \( F \) in \( m \), we obtain

\[
\partial_q F = \frac{\lambda Gm}{2\sqrt{q}}, \quad \partial_\gamma F = \frac{m^2}{2}, \quad \partial_\nu F = m, \quad \partial_\nu F = h(m). \tag{B.6}
\]

Differentiating (B.5) implicitly, we have also

\[
\nabla_{q,\gamma,\tau,\nu} m = \left( \frac{\nu}{1-m^2} - \gamma \right)^{-1} \left( \frac{\lambda G}{2\sqrt{q}}, m, 1, -\text{arctanh} \ m \right).
\]
Then, differentiating (B.6) a second time and applying these forms,

\[
\nabla_{q, \gamma, \tau, \nu} \partial_q F = -\frac{\lambda G m}{4 q^3} + \frac{\lambda G}{2 \sqrt{q}} \left( \frac{\nu}{1 - m^2} - \gamma \right)^{-1} \left( \frac{\lambda G}{2 \sqrt{q}}, m, 1, -\text{arctanh } m \right),
\]

\[
\nabla_{q, \gamma, \tau, \nu} \partial_\gamma F = m \left( \frac{\nu}{1 - m^2} - \gamma \right)^{-1} \left( \frac{\lambda G}{2 \sqrt{q}}, m, 1, -\text{arctanh } m \right),
\]

\[
\nabla_{q, \gamma, \tau, \nu} \partial_\tau F = \left( \frac{\nu}{1 - m^2} - \gamma \right)^{-1} \left( \frac{\lambda G}{2 \sqrt{q}}, m, 1, -\text{arctanh } m \right),
\]

\[
\nabla_{q, \gamma, \tau, \nu} \partial_\nu F = -\text{arctanh}(m) \left( \frac{\nu}{1 - m^2} - \gamma \right)^{-1} \left( \frac{\lambda G}{2 \sqrt{q}}, m, 1, -\text{arctanh } m \right).
\]

Applying these expressions and the identity \( \text{arctanh } m = (\lambda \sqrt{q} G + \tau + \gamma m)/\nu \) from (B.5), we may check that over a local neighborhood of any \((q, \gamma, \tau, \nu)\) satisfying \( q > 0 \) and \( \nu > \max(\gamma, 0) \), we have the bounds \(|\partial F|, |\partial^2 F| \leq C(G^2 + 1)\) for all first and second partial derivatives of \( F \) and some constant \( C > 0 \) depending on \( \lambda \) and this neighborhood. Thus, the dominated convergence theorem may be applied to differentiate \( E_{G \sim \mathcal{N}(0,1)}[F(G; q, \gamma, \tau, \nu)] \) twice under the integral. This implies also that \( E_\lambda(q, \varphi, h; \gamma, \tau, \nu) \) as defined in (4.10) is twice continuously-differentiable in all arguments.

Fixing any \((q, \varphi, h)\), observe that \((\gamma, \tau, \nu) \mapsto E_\lambda(q, \varphi, h; \gamma, \tau, \nu)\) is concave on all of \( \mathbb{R}^3 \), by its definition. At \((q, \varphi, h) = (q_*, q_*, h_*)\) and \((\gamma, \tau, \nu) = (0, \lambda^2 q_*, 1)\), the supremum over \( m \) in

\[
\sup_{m \in (-1, 1)} \lambda \sqrt{q} \cdot Gm + \frac{\gamma m^2}{2} + \tau m + \nu h(m) = \sup_{m \in (-1, 1)} \lambda \sqrt{q} \cdot Gm + \lambda^2 q_* m + h(m)
\]

is achieved at \( m = \text{tanh}(\lambda^2 q_* + \lambda \sqrt{q} G) \), which has the distribution \( \mu_* \). Then

\[
E_\lambda(q_*, q_*, h_*; 0, \lambda^2 q_*, 1) = \epsilon_*
\]

by Proposition A.2. Specializing (B.6) to the point \((q_*, q_*, h_*; 0, \lambda^2 q_*, 1)\), we have

\[
\nabla_{\gamma, \tau, \nu} E_\lambda(q_*, q_*, h_*; 0, \lambda^2 q_*, 1) = \left( q_* / 2 - \mathbb{E}_{m \sim \mu_*} [m^2 / 2], q_* - \mathbb{E}_{m \sim \mu_*} [m], h_* - \mathbb{E}_{m \sim \mu_*} [h(m)] \right) = 0.
\]

So \((\gamma, \tau, \nu) = (0, \lambda^2 q_*, 1)\) is a critical point of \((\gamma, \tau, \nu) \mapsto E_\lambda(q_*, q_*, h_*; \gamma, \tau, \nu)\). Then by concavity of this function, \((\gamma, \tau, \nu) = (0, \lambda^2 q_*, 1)\) is a global maximizer, and we obtain (4.12).

To show the lower bound (4.13), we first check strict concavity around \((\gamma, \tau, \nu) = (0, \lambda^2 q_*, 1)\): Specializing (B.7) to the point \((q, \varphi, h; \gamma, \tau, \nu) = (q_*, q_*, h_*; 0, \lambda^2 q_*, 1)\), we have

\[
\nabla^2_{\gamma, \tau, \nu} E_\lambda(q_*, q_*, h_*; 0, \lambda^2 q_*, 1) = -\mathbb{E}_{m \sim \mu_*} \begin{pmatrix} (1 - m^2) & m & m \\ -m & 1 & 1 \\ -m & -1 & \text{arctanh } m \end{pmatrix} \begin{pmatrix} m \\ -m \\ -\text{arctanh } m \end{pmatrix}^T.
\]

The distribution of \( m \sim \mu_* \) is supported on the full interval \((-1, 1)\), and the curve \( \{(m, 1 - \text{arctanh } m) : m \in (-1, 1)\} \) is not contained in any 2-dimensional subspace. Thus this Hessian is strictly negative definite.

Applying again the implicit function theorem, this implies that the maximizer of \((\gamma, \tau, \nu) \mapsto (q_*, \varphi, h; \gamma, \tau, \nu)\) is implicitly defined as a twice continuously-differentiable function of \((q_*, \varphi, h)\), in a neighborhood of \((q_*, q_*, h_*)\). Fixing any subset \( K' \subseteq \mathbb{R}^3 \) containing \((0, \lambda^2 q_*, 1)\) in its interior, this maximizer must then belong to the interior of \( K' \) for all

\[
(q, \varphi, h) \in K = \left\{ (q, \varphi, h) : |q - q_*|, |\varphi - q_*|, |h - h_*| \leq \delta \right\}
\]

and some sufficiently small \( \delta \). Defining

\[
\bar{E}_\lambda(q, \varphi, h) = \sup_{(\gamma, \tau, \nu) \in K'} E_\lambda(q, \varphi, h; \gamma, \tau, \nu),
\]

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this function $\tilde{E}_λ(q, ϕ, h)$ is then twice continuously-differentiable on $K$. We proceed to show two properties of $\tilde{E}_λ$: For sufficiently small constants $c, δ > 0$ and any $(q, ϕ, h) ∈ K$,

\begin{align}
\tilde{E}_λ(q, ϕ, h) &≥ e_∗ + c(q - q_∗)^2 + c(ϕ - q_∗)^2, \quad (B.10) \\
\tilde{E}_λ(q_∗, q_*, h) &≥ e_∗ + c(h - h_*)^2. \quad (B.11)
\end{align}

To show (B.10), note that restricting to $ν = 1$ removes the dependence of $E$ on $h$. We denote this restriction as $E_λ(q, ϕ; γ, τ)$. Restricting the supremum in (B.9) to $ν = 1$ yields the lower bound

$$
\tilde{E}_λ(q, ϕ, h) ≥ \tilde{E}_λ(q, ϕ) = \sup_{(γ, τ) ∈ K'} E_λ(q, ϕ; γ, τ).
$$

We compute the Hessian of $E_λ(q, ϕ; γ, τ)$ at $(q_∗, q_∗; 0, λ^2 q_*)$: Differentiating (B.5) implicitly now in $q$ and $G$ at $(q, ϕ, h; γ, τ, ν) = (q_∗, q_∗, h_∗; 0, λ^2 q_∗, 1)$,

$$
\partial_Г m = λ\sqrt{q_*}(1 - m^2), \quad \partial_q m = \frac{λ}{2\sqrt{q_*}} G(1 - m^2). \quad (B.12)
$$

Recalling that $m = \tanh(λ^2 q_* + λ\sqrt{q_*} G)$ at this point, we write $E$ for the expectation over $G ∼ N(0, 1)$. Then, applying $b_3 = E[m^3] = E[m^4]$ from Proposition A.2 and $E[Гm] = E[Гm] = λ\sqrt{q_*}(1 - q_*)$ by Gaussian integration by parts,

$$
\nabla^2 E_λ(q, ϕ; γ, τ) \big|_{(q, ϕ; γ, τ) = (q_∗, q_∗; 0, λ^2 q_∗)} = \text{diag}(\frac{1}{2}, 1, 1, 1) \cdot \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \cdot \text{diag}(\frac{1}{2}, 1, 1, 1)
$$

where

\begin{align}
A_{11} &= \begin{pmatrix}
-2λ^2 - \text{E} \left[ -\frac{λ^2}{q_*^2} m + \frac{2λ^2}{q_*} \partial_q m \right] & 0 \\
0 & -λ^2
\end{pmatrix} \\
&= \begin{pmatrix}
-2λ^2 + \frac{λ^2(1-q_*)}{q_*} - \frac{λ^2}{q_*} \text{E}[G^2(1-m^2)] & 0 \\
0 & -λ^2
\end{pmatrix} \\
A_{12} &= A_{21}^T = \begin{pmatrix}
1 - 2\text{E}[m_0m] & -2\text{E}[\partial_q m] \\
0 & 1
\end{pmatrix} \begin{pmatrix}
1 - \frac{λ}{\sqrt{q_*}} \text{E}[Гm(1-m^2)] & \frac{λ}{\sqrt{q_*}} \text{E}[Гm^2] \\
0 & 1
\end{pmatrix} \\
A_{22} &= \begin{pmatrix}
-\text{E}[m_0m] & -\text{E}[\partial_q m] \\
-\text{E}[m_0m] & -\text{E}[\partial_q m]
\end{pmatrix} = \begin{pmatrix}
q_* - b_3 & q_* - b_3 \\
q_* - b_3 & 1 - q_*
\end{pmatrix}. \quad (B.13)
\end{align}

We may simplify the above expressions for $A_{11}$ and $A_{12}$ further using the integration by parts identities

\begin{align}
\text{E}[Гm^2] &= \text{E}[2m_0Гm] = 2λ\sqrt{q_*}(q_* - b_3) \\
\text{E}[Гm(1-m^2)] &= \text{E}[(1 - 3m^2)Гm] = λ\sqrt{q_*}(1 - 4q_* + 3b_3) \\
\text{E}[Г^2(1-m^2)] &= \text{E}[(1-m^2) - 2ГmГm] \\
&= 1 - q_* - 2λ\sqrt{q_*} \text{E}[Гm(1-m^2)] = 1 - q_* - 2λ^2 q_*(1 - 4q_* + 3b_3),
\end{align}

yielding

\begin{align}
A_{11} &= \begin{pmatrix}
-2λ^2 + 2λ^4(1 - 4q_* + 3b_3) & 0 \\
0 & -λ^2
\end{pmatrix}, \quad A_{12} = \begin{pmatrix}
1 - λ^2(1 - 4q_* + 3b_3) & 2λ^2(q_* - b_3) \\
0 & 1
\end{pmatrix}.
\end{align}

Here $A_{22}$ is the upper-left $2 \times 2$ submatrix of (B.8), which we have argued satisfies $A_{22} ≪ 0$. Computing explicitly its inverse, the Hessian of $\tilde{E}_λ(q, ϕ)$ at $(q_∗, q_*)$ is then given by

$$
\nabla^2 \tilde{E}_λ(q, ϕ) \big|_{(q, ϕ) = (q_∗, q_*)} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A_{11} - A_{12} A_{22}^{-1} A_{21} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} c_1 & -c_2 \\ -c_2 & c_2 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}
$$
where, after some algebraic simplification,
\[
c_1 = \frac{(1 - q_*) - 2\lambda^2(1 - 2q_* + b_*)^2 + \lambda^4(1 - 2q_* + b_*)^3}{(1 - 2q_* + b_*)(q_* - b_*)} - \lambda^4(1 - 2q_* + b_*) ,
\]
\[
c_2 = \frac{1}{1 - 2q_* + b_*} - \lambda^2.
\]
From Proposition A.2, we have the inequalities
\[
q_* - b_* = \mathbb{E}[m^2(1 - m^2)] > 0 ,
\]
\[
1 - 2q_* + b_* = \mathbb{E}((1 - m^2)^2) > 0 ,
\]
\[
1 - 2q_* + b_* < 1 - q_* < \frac{1}{\lambda^2}
\]
so \(c_2 > 0\). We may compute the Schur-complement
\[
c_1 - (c_2)\c_2^{-1} - c_2 = c_1 - c_2
\]
where the last inequality applies \(q_* - b_* > 0\) and \(1 - \lambda^2(1 - 2q_* + b_*) > 0\). The statements \(c_2 > 0\) and \(c_1 - (c_2)\c_2^{-1} - c_2 > 0\) together imply \(A_{11} - A_{12}A_{22}^{-1}A_{21} > 0\), so \(\tilde{E}_\lambda(q, \varphi)\) is strongly convex at \((q, \varphi) = (q_*, q_*)\). Then for small enough \(\delta, c > 0\), we obtain \(\tilde{E}_\lambda(q, \varphi) \geq e_* + c(q - q_*)^2 + c(\varphi - q_*)^2\) for all \((q, \varphi, h) \in K\), and hence (B.10) holds.

To show (B.11), let us restrict \(E\) to \((q, \varphi; \gamma, \tau) = (q_*, q_*)\) and denote this restriction as \(E_\lambda(h; \nu)\). Restricting the supremum in (B.9) to \((\gamma, \tau) = (0, \lambda^2 q_*)\) yields
\[
\tilde{E}_\lambda(q_*, q_*, h) \geq \hat{E}_\lambda(h) = \sup_{\nu : (0, \lambda^2 q_*, \nu) \in \tilde{K}} E_\lambda(h; \nu).
\]
The Hessian of \(E_\lambda(h; \nu)\) is
\[
\nabla^2 E_\lambda(h; \nu) \bigg|_{(h; \nu) = (h_*, \tau)} = \begin{pmatrix} 0 & 1 \\ 1 & \mathbb{E}[(\text{arctanh } m) \partial_{\nu} m] \end{pmatrix}.
\]
The lower-right entry is the (3, 3) entry of (B.8), which we have argued is negative. Then
\[
\nabla^2 \tilde{E}_\lambda(h) \bigg|_{h = h_*} = -1 \cdot \mathbb{E}[(\text{arctanh } m) \partial_{\nu} m]^{-1} \cdot 1 > 0,
\]
so \(\tilde{E}_\lambda(h)\) is strongly convex at \(h = h_*\). Then \(\tilde{E}_\lambda(h) \geq e_* + c(h - h_*)^2\) for small enough \(c, \delta > 0\), implying (B.11).

Finally, let \(C_3 > 0\) be an upper bound for \(\|\nabla^2 \tilde{E}_\lambda(q, \varphi, h)\|_{\text{op}}\) over \((q, \varphi, h) \in K\). Since \(\tilde{E}_\lambda(q, \varphi, h)\) is twice continuously-differentiable on \(K\) and \(\nabla^2 \tilde{E}_\lambda(q_*, q_*, h_*) = 0\), (B.11) implies
\[
\tilde{E}_\lambda(q, \varphi, h) \geq \tilde{E}_\lambda(q_*, q_*, h) - |\tilde{E}_\lambda(q_*, \varphi, h) - \tilde{E}_\lambda(q_*, q_*, h)|
\]
\[
\geq e_* + c(h - h_*)^2 - \sup_{(q, \varphi, h) \in K} \|\nabla \tilde{E}_\lambda(q, \varphi, h)\| : \left[|q - q_*| + |\varphi - q_*| + |h - h_*| \right] \cdot \left[|q - q_*| + |\varphi - q_*| + |h - h_*| \right].
\]
If \(|q - q_*| + |\varphi - q_*| < c/(4C_3) \cdot |h - h_*|\), we apply this bound to obtain \(\tilde{E}_\lambda(q, \varphi, h) \geq e_* + (c/2)(h - h_*)^2\). Otherwise, we apply (B.10), and combining these cases yields (4.13).

Proof of Corollary 4.4. Throughout the proof, for any set \(K \subseteq \mathbb{R}^d\), we denote \(\overline{K}\) as the closure of \(K\). Fix any compact set \(K' \subset \mathbb{R}^3\) containing \((0, \lambda^2 q_*, 1)\) in its interior. Define
\[
K_\delta = \{(q, \varphi, h) : |q - q_*|, |\varphi - q_*|, |h - h_*| < \delta\}.
\]
For $\delta > 0$ sufficiently small, we apply Lemma 4.2 once with $K = K_\delta$ and once with $K = K_\delta \setminus K_{\delta/2}$. Then Lemmas 4.2 and 4.3 combine to show, almost surely for all large $n$ and a $\lambda$-dependent constant $c_0 > 0$,

\begin{align}
\inf_{m \in B_{\delta}} \mathcal{F}_{\text{TAP}}(m) &> e_* - \delta, \\
\inf_{m \in B_{\delta} \setminus B_{\delta/2}} \mathcal{F}_{\text{TAP}}(m) &> e_* + c_0\delta^2.
\end{align}

Lemma A.7(b) and (d) imply that for some sufficiently large $k$, with probability approaching 1 as $n \to \infty$, the AMP iterate $m^k$ satisfies $m^k \in B_{\delta/2}$ and $\mathcal{F}_{\text{TAP}}(m^k) < e_* + (c_0/2)\delta^2$. Together with (B.16), this implies that there must exist a critical point and local minimizer $m_*$ of $\mathcal{F}_{\text{TAP}}$ in $B_{\delta/2}$, satisfying $\mathcal{F}_{\text{TAP}}(m_*) < e_* + (c_0/2)\delta^2 < e_* + \delta$. Applying (B.15), we get $|\mathcal{F}_{\text{TAP}}(m_*) - e_*| < \delta$. □

### B.2 Proofs for Section 4.3

For a parameter $\varepsilon > 0$, define the truncation of the cube

$$
\Sigma_{\varepsilon} = \left[ -1 + e^{-\alpha\varepsilon}, 1 - e^{-\alpha\varepsilon} \right]^n \setminus B_{\sqrt{n\varepsilon}}(0).
$$

We obtain from Lemma 4.6 and arguments similar to [FMM21] the following complexity upper bound.

**Lemma B.1.** Fix any $\lambda, \varepsilon > 0$ and suppose $x = 1$. Let $T \subset (-1,1)^n$ be any deterministic Borel-measurable set, and let

$$
\mathcal{C} = \left\{ m \in T : g(m) = 0 \right\}.
$$

Then for a $(\lambda, \varepsilon)$-dependent constant $C > 0$ and all large $n$,

$$
E[|\mathcal{C} \cap \Sigma_{\varepsilon}|] \leq \sqrt{\frac{n}{2\pi\lambda^2}} \int_{T \times \mathbb{R}} \exp \left( n \cdot J(m, y) - \frac{ny^2}{2\lambda^2} + Cn^{0.9} \right) \, dm \, dy
$$

where

$$
J(m, y) = \frac{\lambda^2(1 - Q(m))^2}{2} - \frac{1}{2} \log \left( 2\pi\lambda^2Q(m) \right)
+ \frac{1}{n} \sum_{i=1}^{n} \left( \log \frac{1}{1 - m_i} - \frac{\arctanh m_i - \lambda^2 M(m) + \lambda^2(1 - Q(m))m_i - ym_i)^2}{2\lambda^2 Q(m)} \right).
$$

**Proof.** Lemma 4.6 shows

$$
E[|\mathcal{C} \cap \Sigma_{\varepsilon}|] \leq \int_{T \cap \Sigma_{\varepsilon}} E\left[ |\det H(m)| \bigg| g(m) = 0 \right] p_{g(m)}(0) \, dm.
$$

The result then follows from applying [FMM21, Proposition 3.2 and Lemma 3.3] under the identification $\beta = \lambda$, and noting that the integrand is non-negative so we may upper-bound the integral over $T \cap \Sigma_{\varepsilon}$ by that over $T$. □

**Proof of Lemma 4.7.** We introduce $A(m) = n^{-1} \sum_{i=1}^{n} m_i \arctanh m_i$. If $g(m) = 0$, then

$$
0 = \frac{1}{n} m^T g(m) = -\frac{\lambda}{n} m^T Y m + A(m) + \lambda^2 [1 - Q(m)] Q(m)
= 2\mathcal{F}_{\text{TAP}}(m) + 2H(m) + \frac{\lambda^2}{2} [1 - Q(m)]^2 + A(m) + \lambda^2 [1 - Q(m)] Q(m).
$$

Consider any $m \in B_{\delta}$ where $|\mathcal{F}_{\text{TAP}}(m) - e_*| < \delta$ and $g(m) = 0$. Evaluating $2e_* + 2h_* + (\lambda^2/2)(1 - q_*)^2 + \lambda^2(1 - q_*)q_* = -\lambda^2 q_*$, we obtain for any such $m$ that

$$
|A(m) - \lambda^2 q_*| < (3\lambda^2 + 4)\delta.
$$
Thus, it suffices to show \( P[|C| \geq 1] < e^{-cn} \) where we define

\[
    C = \left\{ m \in (-1,1)^n : g(m) = 0, |A(m) - \lambda^2q_\ast| < (3\lambda^2 + 4)\delta, m \in B_\delta, m \not\in D_\eta \right\}.
\]

We have replaced the random condition \( |F_{\text{TAP}}(m) - e_\ast| < \delta \) with a deterministic condition involving \( A(m) \).

Fix \( \varepsilon = q_\ast/2 \). Applying Lemma B.1,

\[
P[|C \cap \Sigma_\varepsilon| \geq 1] \leq \mathbb{E}[|C \cap \Sigma_\varepsilon|] \leq \sqrt{n \over 2\pi \lambda^2} \int_{(-1,1)^n \times \mathbb{R}} \exp \left( n \cdot J(m, y) - n y^2 \over 2\lambda^2 \right) + C_{n,0.9} \cdot 1_{|A(m) - \lambda^2q_\ast| < (3\lambda^2 + 4)\delta}. 
\]

We expand the square in the last term of \( J(m, y) \), to write

\[
    {1 \over n} \sum_{i=1}^n \left( \arctanh m_i - \lambda^2 M(m) + \lambda^2 (1 - Q(m)) m_i - y m_i \right)^2
\]

\[
= {1 \over n} \sum_{i=1}^n \left( \arctanh m_i - \lambda^2 M(m) \right)^2 - {1 \over n} \sum_{i=1}^n \left( [y - \lambda^2 (1 - Q(m))] + [y - \lambda^2 (1 - Q(m))] \right)^2
\]

Then, for any \( m \in B_\delta \) satisfying \( |A(m) - \lambda^2q_\ast| < (3\lambda^2 + 4)\delta \), we have

\[
    J(m, y) - y^2 \over 2\lambda^2
\]

\[
= \lambda^2 (1 - q_\ast)^2 - {1 \over 2} \log(2\pi \lambda^2 q_\ast) + (1 - q_\ast) \cdot \left( y - \lambda^2 (1 - q_\ast) \right) - \left( y - \lambda^2 (1 - q_\ast) \right)^2 + {y^2 \over 2\lambda^2}
\]

and applying this above, we obtain

\[
P[|C \cap \Sigma_\varepsilon| \geq 1] \leq {e^{C_{n,0.9}} \over \sqrt{2}} \int_{(-1,1)^n \times \mathbb{R}} \exp \left( nr_\lambda(\delta) \left( 1 + {1 \over n} \sum_{i=1}^n \left( \arctanh m_i \right)^2 + |y| \right) \right) \cdot 1_{m \not\in D_\eta} \cdot \mathbb{E} \left[ \prod_{i=1}^n {1 \over \sqrt{2\pi \lambda^2 q_\ast}} \cdot \left( 1 - m_i^2 \right) \exp \left( - \left( \arctanh m_i - \lambda^2 q_\ast \right) \right) \right] dm_i
\]

Let us change variables to \( x_i = \arctanh m_i \), and write this as an expectation over independent random variables \( Y \sim \mathcal{N}(\lambda^2 (1 - q_\ast), {\lambda^2 \over 2m_i}) \) and \( X_i \sim \mathcal{N}(\lambda^2 q_\ast, \lambda^2 q_\ast) \). Then the set \( D_\eta \) is defined by the condition \( W_2(\hat{\mu}_X, \mathcal{N}(\lambda^2 q_\ast, \lambda^2 q_\ast)) < \eta, \) where \( \hat{\mu}_X \) is the empirical distribution of \( x_1, \ldots, x_n \) and \( W_2 \) is the Wasserstein-2 distance. Thus, applying this representation and Cauchy-Schwarz,

\[
P[|C \cap \Sigma_\varepsilon| \geq 1]
\]

\[
\leq {e^{C_{n,0.9}} \over \sqrt{2}} \cdot \mathbb{E} \left[ \mathbb{1}\left\{ W_2(\hat{\mu}_X, \mathcal{N}(\lambda^2 q_\ast, \lambda^2 q_\ast)) \geq \eta \right\} \cdot \exp \left( nr_\lambda(\delta) \left( 1 + {1 \over n} \sum_{i=1}^n X_i^2 + |Y| \right) \right) \right]
\]

\[
\leq {e^{C_{n,0.9}} \over \sqrt{2}} \cdot \mathbb{P} \left\{ W_2(\hat{\mu}_X, \mathcal{N}(\lambda^2 q_\ast, \lambda^2 q_\ast)) \geq \eta \right\} ^{1/2} \mathbb{E} \left[ \exp \left( 2nr_\lambda(\delta) \left( 1 + {1 \over n} \sum_{i=1}^n X_i^2 + |Y| \right) \right) \right]^{1/2} .
\]
By Proposition A.4, for a constant $c_0 > 0$ depending only on $(\lambda, \eta)$ and for all large $n$,

$$
P \left[ W_2(\hat{\mu}_X, \mathcal{N}(\lambda^2 q_*, \lambda^2 q_*)) \geq \eta \right] < e^{-c_0 n}.
$$

(B.17)

Applying the moment generating functions of the Gaussian and non-central chi-squared distributions, together with $\log(1 + t) \leq t$ for any $t < 1/2$ we have

$$
\mathbb{E}[e^{-ntY}] \leq \mathbb{E}[e^{-nY}] + \mathbb{E}[e^{-ntY}] \leq 2e^{n\lambda^2(1-q_*) + n^2\lambda^2/4},
$$

and

$$
\mathbb{E}[e^{\sum_{i=1}^{n} X_i^2}] \leq \exp \left( \frac{n\lambda^2 q_*^2}{1 - 2t} \right) \cdot (1 - 2t)^{n/2} \leq \exp \left( \frac{n\lambda^2 q_*^2}{1 - 2t} - nt \right).
$$

Now choosing $\delta = \delta(\lambda, \eta)$ sufficiently small so that $r_\lambda(\delta)$ is sufficiently small, we may guarantee

$$
\mathbb{E} \left[ \left( 2nr_\lambda(\delta) \left( 1 + \frac{1}{n} \sum_{i=1}^{n} X_i^2 + |Y| \right) \right)^{1/2} \right] \leq e^{c_0 n/8}
$$

where $c_0$ is the constant in (B.17). Thus $\mathbb{P}[|C \cap \Sigma_\varepsilon| \geq 1] < e^{-c_0 n/8}$.

Finally, recalling $\varepsilon = q_*/2$, the set $B_\varepsilon$ does not intersect the ball $B_{\sqrt{c\varepsilon}}(0)$ for any $\delta < q_*/2$. Thus

$$
|C \cap \Sigma_\varepsilon| \leq \left\{ m \in (-1, 1)^n : g(m) = 0 \text{ and } \|m\|_\infty \in (1 - e^{-n^{0.6}}, 1) \right\}.
$$

On the event of probability $1 - e^{-cn}$ where $\|W\|_{op} < 3$ and hence $\|Y\|_{op} < \lambda + 3$, we have $\|\lambda Y m - \lambda^2[1 - Q(m)]m\|_2 < (2\lambda^2 + 3\lambda)\sqrt{n}$ for all $m \in (-1, 1)^n$. When $\|m\|_\infty \in (1 - e^{-n^{0.6}}, 1)$, we also have $\|\text{arctanh}(m)\|_2 \geq n^{0.6}/2$. Applying this to (4.15), we must have $g(m) \neq 0$. Thus also $\mathbb{P}[|C| \geq 1] < e^{-cn}$, so $\mathbb{P}[|C| \geq 1] \leq e^{-cn}$ for all large $n$, as desired.

**B.3 Proofs for Section 4.4**

Let us write

$$
H^\chi(m) = \chi \left( -\lambda Y - \frac{2\lambda^2}{n} mm^T \right) + \text{diag} \left( \frac{1}{1 - m^2} \right) + \lambda^2[1 - Q(m)]I, \quad \chi \in \{ +, - \}
$$

(B.18)

and

$$
\ell_\varepsilon^\chi(m) = \text{inf} \left\{ \lambda_{\text{min}}(H^\chi(u)) : u \in (-1, 1)^n \cap B_{\sqrt{\varepsilon}}(m) \right\}.
$$

These coincide with the definitions of $\ell_+^\varepsilon$ and $\ell_-^\varepsilon$ in Sections 4.4 and 4.5.

**Lemma B.2.** Suppose $x = 1$. For $u, m \in (-1, 1)^n$, let $P_m^\perp$ be the orthogonal projection onto the $(n - 1)$-dimensional subspace orthogonal to $m$, and let

$$
z(m) = \text{arctanh}(m) - \lambda^2 M(m)1 + \lambda^2[1 - Q(m)]m,
$$

(B.19)

$$
M^\chi(m, u) = \chi \left( \frac{1}{\|m\|_2^2} (mz(m)^T + z(m)m^T) - \frac{m, z(m)}{\|m\|_2^4} \cdot mm^T \right.
$$

$$
+ \frac{\lambda^2}{n} 11^T + \frac{2\lambda^2}{n} uu^T \left) - \text{diag} \left( \frac{1}{1 - u^2} \right) - \lambda^2[1 - Q(u)]I. \right)
$$

(B.20)

Then for any $m \in (-1, 1)^n$ and either $\chi \in \{ +, - \}$, conditional on the event $g(m) = 0$,

$$
\left\{ - H^\chi(u) : u \in (-1, 1)^n \right\}_{g(m) = 0} \overset{d}{=} \left\{ \lambda P_m^\perp \widetilde{W} P_m^\perp + M^\chi(m, u) : u \in (-1, 1)^n \right\}
$$

(B.21)

where $\widetilde{W} \sim \text{GOE}(n)$ is an independent copy of $W$, and this holds as an equality in law of two Gaussian processes indexed by $u \in (-1, 1)^n$. 

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Proof. Writing $Y = (\lambda/n)11^T + W$, $g(m) = 0$ is equivalent to $Wm = \lambda^{-1}z(m)$. Hence conditioned on this event, the law of $W$ is (see e.g. [FMM21, Lemma 4.1])

$$W|_{g(m)=0} \overset{d}{=} \frac{1}{\lambda \|m\|_2^2}(mz(m)^T + z(m)m^T) - \frac{\langle m, z(m) \rangle}{\lambda \|m\|_2^2} \cdot mm^T.$$

The result follows from substituting this into the expression for $H^X(u)$ in (B.18).

Proof of Lemma 4.9(a) and Lemma 4.12. Recall that by Lemma B.2, we have

$$\left\{ -H^X(u) : u \in (-1,1)^n \right\} \overset{d}{=} \left\{ \lambda P_m W^P_m + M^X(m,u) : u \in (-1,1)^n \right\}.$$

Then

$$-\ell^X(m) \overset{d}{=} \sup_{u \in (-1,1)^n \cap B_{\sqrt{n}}(m)} \sup_{v \in \mathbb{R}^n : \|v\|_2 = 1} \lambda \langle P_m^v v, W^P_m v \rangle + \langle v, M^X(m,u)v \rangle.$$

We introduce $g \sim \mathcal{N}(0,1) \in \mathbb{R}^n$ and define the auxiliary Gaussian process

$$G(m,u,v) = 2\lambda \sqrt{\frac{1}{n}} \langle P_m^v v, g \rangle \|P_m^v v\|_2 + \langle v, M^X(m,u)v \rangle.$$

Then by Lemma 4.1, we obtain for any fixed $m \in D_\eta$ that

$$E \left[ -\ell^X(m) \mid g(m) = 0 \right] \leq E \left[ \sup_{u \in (-1,1)^n \cap B_{\sqrt{n}}(m)} \sup_{v \in \mathbb{R}^n : \|v\|_2 = 1} G(m,u,v) \right].$$

(B.22)

We now analyze $G(m,u,v)$. Throughout the proof, we write $r(n, \varepsilon, \eta)$ for any error term depending only on $n, \varepsilon, \eta, \lambda, K'$ and satisfying

$$\lim_{\varepsilon, \eta \to 0} \lim_{n \to \infty} r(n, \varepsilon, \eta) = 0.$$

We write $C, c > 0$ for constants depending only on $(\lambda, K')$ and changing from instance to instance. Suprema over $(u,v)$ are implicitly over $u \in (-1,1)^n \cap B_{\sqrt{n}}(m)$ and $v \in \mathbb{R}^n$ with $\|v\|_2 = 1$, unless otherwise stated.

Writing $P_m = I - P_m^\perp$ for the projection onto the span of $m$, observe that

$$E \left[ \sup_{\|v\|_2 = 1} 2\lambda \sqrt{\frac{1}{n}} \langle P_m^v v, g \rangle \|P_m^v v\|_2 \right] \leq 2\lambda \sqrt{\frac{2}{n}}.$$

Then

$$E \left[ \sup_{u,v} G(m,u,v) \right] \leq E \left[ \sup_{u,v} 2\lambda \sqrt{\frac{1}{n}} \langle v, g \rangle \|P_m^v v\|_2 + \langle v, M^X(m,u)v \rangle \right] + r(n, \varepsilon, \eta).$$

In the expression $\langle v, M^X(m,u)v \rangle$, applying $\|u - m\|_2^2 \leq \varepsilon n$, we may bound

$$\frac{2\lambda^2}{n} \langle u, v \rangle^2 - \langle m, v \rangle^2 + \lambda^2 \left| Q(m) - Q(u) \right| \leq \frac{3\lambda^2}{n} \|m - u\|_2 \|m + u\|_2 \leq 6\lambda^2 \varepsilon^{1/2}$$

to replace $Q(u)$ and $\langle u, v \rangle^2$ by $Q(m)$ and $\langle m, v \rangle^2$, up to $r(n, \varepsilon, \eta)$ error. We may then apply, for $m \in D_\eta,$

$$\left\{ -H^X(u) : u \in (-1,1)^n \right\} \overset{d}{=} \left\{ \lambda P_m W^P_m + M^X(m,u) : u \in (-1,1)^n \right\}.$$

$$\left\{ -H^X(u) : u \in (-1,1)^n \right\} \overset{d}{=} \left\{ \lambda P_m W^P_m + M^X(m,u) : u \in (-1,1)^n \right\}.$$
where \( z(m) \) in this last equality denotes the entry-wise application of \( z(m) = \arctanh m - \lambda^2 q_* + \lambda^2 (1 - q_*) m \) to \( m \), and differs from \( z(m) \) defined in (B.19) in the replacement of \( M(m) \) and \( Q(m) \) by \( q_* \). This yields

\[
\mathbb{E} \left[ \sup_{u,v} G(m, u, v) \right] \leq \mathbb{E} \left[ \sup_{u,v} G_*(m, u, v) \right] + r(n, \varepsilon, \eta)
\]

where we define

\[
G_*(m, u, v) = 2\lambda \sqrt{\frac{1}{n}} \langle v, g \rangle \|P_m v\|_2^2 + \chi \left( \frac{2 \langle z(m), v \rangle - \langle m, v \rangle^2}{\|m\|_2^2} - 2 \lambda^2 (1 - q_*) \|m\|_2^2 \right) + \lambda^2 \left( v, 1 \right)^2 + 2 \lambda^2 q_* \left( \|m\|_2^2 \right) - \sum_{i=1}^n \frac{v_i^2}{1 - u_i^2} - \lambda^2 (1 - q_*).
\]

Next, we analyze \( G_*(m, u, v) \). We introduce

\[
p = \sqrt{q_*} \cdot \frac{\langle m, v \rangle}{\|m\|_2} \in \left[ -\sqrt{q_*}, \sqrt{q_*} \right], \quad u = \frac{\langle v, 1 \rangle}{\sqrt{n}} \in [-1, 1], \quad K = \left\{ (p, u) : |p| \leq \sqrt{q_*}, |u| \leq 1 \right\}
\]

to bound the supremum over \( v \) for fixed \( u \) as

\[
\sup_{v \in \mathbb{R}^n : \|v\|_2 = 1} G_*(m, u, v) \leq \sup_{(p, u) \in K} \inf_{v \in \mathbb{R}^n : \|v\|_2 = 1} \sup_{\|m\|_2 = p/\sqrt{q_*}} \left\{ 2\lambda \sqrt{\frac{1}{n}} \langle v, g \rangle \left( 1 - \frac{p^2}{q_*} \right)^{1/2} + \chi \left( \frac{2 \langle z(m), v \rangle - \langle m, v \rangle^2}{\|m\|_2^2} - 2 \lambda^2 p^2 \frac{1 - q_*}{q_*} + \lambda^2 u^2 + 2 \lambda^2 p^2 \right) - \sum_{i=1}^n \frac{v_i^2}{1 - u_i^2} - \lambda^2 (1 - q_*) + \gamma \left( \|v\|_2^2 - 1 \right) + \chi \alpha \left( \frac{\langle v, 1 \rangle}{\sqrt{n}} - u \right) + \chi \kappa \left( \frac{\langle m, v \rangle}{\sqrt{n}} - p \|m\|_2 \right) \right\}.
\]

Collecting the terms above that depend on \( v \), substituting \( x_i \) for \( u_i \sqrt{n} \), and defining

\[
\Theta_\lambda(g, m, u) = \sup_{x \in \mathbb{R}} \left\{ 2\lambda x g \left( 1 - \frac{p^2}{q_*} \right)^{1/2} - \frac{x^2}{1 - u^2} + \gamma x^2 + \chi x \left( 2z(m) \cdot \frac{p}{q_*} + \alpha + \kappa \|m\|_2 \right) \right\},
\]

this yields

\[
\sup_{v \in \mathbb{R}^n : \|v\|_2 = 1} G_*(m, u, v) \leq \sup_{(p, u) \in K} \inf_{(\alpha, \kappa, \gamma) \in K'} \left\{ \chi \left[ 2\lambda^2 p^2 + \lambda^2 u^2 - 2 \lambda^2 \frac{1 - q_*}{q_*} \right] - \alpha u - \kappa \|m\|_2 \right\} - \lambda^2 (1 - q_*) + \gamma \frac{1}{n} \sum_{i=1}^n \Theta_\lambda(g_i, m_i, u_i). \tag{B.23}
\]

Since \( \gamma < 1 \) and \( 1/(1 - u^2) \geq 1 \), the maximum over \( x \) in the definition of \( \Theta_\lambda(g, m, u) \) is achieved at

\[
x = \left( 2\lambda g(1 - p^2/q_*)^{1/2} + \chi \left[ 2z(m) \cdot \frac{p}{q_*} + \alpha + \kappa \|m\|_2 \right] \right) / \left( 2 \frac{1}{1 - u^2} - 2 \gamma \right),
\]

yielding the explicit form

\[
\Theta_\lambda(g, m, u) = \left( 2\lambda g(1 - p^2/q_*)^{1/2} + \chi \left[ 2z(m) \cdot \frac{p}{q_*} + \alpha + \kappa \|m\|_2 \right] \right)^2 / \left( 4 \frac{1}{1 - u^2} - 4 \gamma \right). \tag{B.24}
\]
Since $\gamma$ is less than and bounded away from 1 on the compact domain $K'$, 

$$\left| \frac{1}{1-u^2} - \frac{1}{1-m^2} \right| = \left| \frac{1-u^2}{1-\gamma(1-u^2)} - \frac{1-m^2}{1-\gamma(1-m^2)} \right| \leq C|u - m|$$

for a constant $C > 0$ depending on $K'$. Then, applying this to (B.24) and recalling the definition of $z(m)$,

$$\left| \frac{1}{n} \sum_{i=1}^{n} \Theta_\lambda(g_i, m_i, u_i) - \Theta_\lambda(g_i, m_i, m_i) \right| \leq \frac{1}{n} \sum_{i=1}^{n} C \left( 1 + g_i^2 + (\arctanh m_i)^2 \right) |u_i - m_i|$$

doing for a different constant $C > 0$ depending also on $\lambda$. Now applying this to (B.23), and taking also the supremum over $u$ and the expectation over $g$, we arrive at

$$\mathbb{E} \left[ \sup_{u, v} G_*(m, u, v) \right] \leq \mathbb{E} \left[ \sup_{(p, n) \in K(\alpha, \kappa, \gamma)} -H_n^\lambda(g, m; p, u, \alpha, \kappa, \gamma) \right] + R_n(m)$$

where

$$H_n^\lambda(g, m; p, u, \alpha, \kappa, \gamma) = -\lambda \left[ 2\lambda^2 p^2 + \lambda^2 u^2 - 2\lambda^2 (1-q_\lambda)p^2/q_\lambda - au - \kappa p \right] + \lambda^2 (1-q_\lambda) + \gamma - \frac{1}{n} \sum_{i=1}^{n} \Theta_\lambda(g_i, m_i, m_i),$$

and

$$|R_n(m)| \leq C \left( \frac{\|m\|_2}{\sqrt{m_q}} - 1 + \mathbb{E} \left[ \sup_{u} \frac{1}{n} \sum_{i=1}^{n} \left( 1 + g_i^2 + (\arctanh m_i)^2 \right) |u_i - m_i| \right] \right).$$

To bound this remainder $R_n(m)$, note that $m \in D_\eta$ implies for the first term

$$\frac{\|m\|_2}{\sqrt{m_q}} = 1 + r(n, \varepsilon, \eta).$$

For the second term, applying $n^{-1} \|u - m\|_2^2 \leq \varepsilon$ and Markov's inequality,

$$|u_i - m_i| \leq \varepsilon^{1/4} + 21 |u_i - m_i| \geq \varepsilon^{1/4}, \quad \frac{1}{n} \sum_{i=1}^{n} 1_{|u_i - m_i| \geq \varepsilon^{1/4}} \leq \varepsilon^{1/2}.$$ 

Set $a_i = \arctanh m_i$, define $|g_i|_{(1)} \geq \ldots \geq |g_i|_{(n)}$ as the values $|g_1|, \ldots, |g_n|$ sorted in decreasing order, and define similarly $|a_i|_{(1)} \geq \ldots \geq |a_i|_{(n)}$. Then

$$\mathbb{E} \left[ \sup_u \frac{1}{n} \sum_{i=1}^{n} \left( 1 + g_i^2 + a_i^2 \right) |u_i - m_i| \right] \leq \frac{1}{n} \sum_{i=1}^{n} \left( 2 + a_i^2 \right) \varepsilon^{1/4} + \frac{C}{n} \sum_{i=1}^{\lfloor \varepsilon^{1/2} \rfloor} \left( 1 + \mathbb{E}[|g_i|_{(i)}^2] + |a_i^2|_{(i)} \right).$$

The Wasserstein-2 distance between the empirical distribution of $(a_1, \ldots, a_n)$ and the law $X_\star \sim \mathcal{N}(\lambda^2 q_\star, \lambda^2 q_\star)$ is at most $\eta$, by the condition $m \in D_\eta$. Then, letting $q_\varepsilon$ be the $1 - \varepsilon^{1/2}$ quantile of the distribution of $|X_\star|$ and applying Proposition A.5(a),

$$\frac{1}{n} \sum_{i=1}^{\lfloor \varepsilon^{1/2} \rfloor} |a_i|_{(i)}^2 \leq \mathbb{E}[X_\star^2 1_{|X_\star| \geq q_\varepsilon}] + C\eta \leq r(n, \varepsilon, \eta).$$

Similarly, letting $q_\varepsilon$ be the $1 - \varepsilon^{1/2}$ quantile of $|G|$ for $G \sim \mathcal{N}(0, 1)$ and applying Proposition A.4,

$$\mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{\lfloor \varepsilon^{1/2} \rfloor} |g_i|_{(i)}^2 \right] \leq \mathbb{E}[G^2 1_{|G| \geq q_\varepsilon}] + r(n, \varepsilon, \eta) \leq r(n, \varepsilon, \eta).$$
Combining these observations yields $|R_n(m)| \leq r(n, \varepsilon, \eta)$, and thus
\[
E \left[ \sup_{u, v} G(m, u, v) \right] \leq E \left[ \sup_{(p, u) \in K} \min_{(\alpha, \kappa, \gamma) \in K'} -H_n^\lambda(g, m; p, u, \alpha, \kappa, \gamma) \right] + r(n, \varepsilon, \eta). \tag{B.26}
\]

Finally, comparing $H_n^\lambda$ in (4.19) with $H_n^\lambda$ in (B.25), observe that
\[
\begin{align*}
H_n^\lambda(p, u; \alpha, \kappa, \gamma) - H_n^\lambda(g, m; p, u, \alpha, \kappa, \gamma) \\
= \frac{1}{n} \sum_{i=1}^{n} \Theta_\lambda(g, m, i, m_i) - E_{G \sim N(0, 1), \mu \sim \mu_*} \left[ \Theta_\lambda(G, m, m) \right],
\end{align*}
\tag{B.27}
\]
where $\Theta_\lambda$ is defined in (B.24). Let us now make the dependence of $\Theta_\lambda$ on $p, \alpha, \kappa, \gamma$ explicit, and write
\[
\Theta_\lambda(g, m, m) = \Xi(g, m; 2\lambda(1 - p^2/q), 2p/q, \alpha, \kappa, \gamma)
\]
where
\[
\Xi(g, m; p, p, \alpha, \kappa, \gamma) = (p_1g + \chi p_2z(m) + \chi \alpha + \chi \kappa m)^2 \left( \frac{4}{1 - m^2} - 4\right).
\]
Bounding (B.27) is similar to the argument in Lemma 4.2: Set $\omega = (p_1, p_2, \alpha, \kappa, \gamma)$ and
\[
K_\omega = \{ \omega : p_1 \in [0, 2\lambda], p_2 \in [-2/\sqrt{n}, 2/\sqrt{n}], (\alpha, \kappa, \gamma) \in K' \}.
\]
For any fixed $m \in (-1, 1)$ and $\omega \in K_\omega$, let $G \sim N(0, 1)$, and note that $\Xi(G, m; \omega)$ is the square of a Gaussian variable with mean bounded by $C(1 + |z(m)|)$ and variance bounded by $C$. Then $E(\Xi(G, m; \omega)) - E[\Xi(G, m; \omega)] = (C(1 + \omega^2), C)$-sub-Gamma, see e.g. [MM21, Definition G.2 and Proposition G.5]. So $n^{-1} \sum_{i=1}^{n} \Xi(g, m_i; \omega) - E[\Xi(G, m_i; \omega)] = (C(n + \|z(m)\|^2)/n, C/n)$-sub-Gamma for any fixed $m \in D_n$. Applying $\|z(m)\|^2/n \leq C$, we obtain that this is $(C/n, C/n)$-sub-Gamma, so Bernstein’s inequality yields
\[
E \left[ \frac{1}{n} \sum_{i=1}^{n} \Xi(g, m_i; \omega) - E[\Xi(G, m_i; \omega)] \right] \leq C/\sqrt{n}. \tag{B.28}
\]
To obtain uniform control over $K_\omega$, let $N$ be a $n^{-1/20}$-net of $K_\omega$ of cardinality $|N| \leq Cn^{1/4}$. Observe that $n^{-1} \sum_{i=1}^{n} \Xi(g, m_i; \omega) = C(1 + \|g\|^2/n + \|z(m)\|^2/n)$-Lipschitz in $\omega \in K_\omega$. Then, applying (B.28) over $N$, we get
\[
E \left[ \sup_{\omega \in K_\omega} \left| \frac{1}{n} \sum_{i=1}^{n} \Xi(g, m_i; \omega) - E[\Xi(G, m_i; \omega)] \right| \right] \leq |N| \cdot C/\sqrt{n} + n^{-1/20} \cdot E \left[ C \left( 1 + \frac{\|g\|^2}{n} + \frac{\|z(m)\|^2}{n} \right) \right] \leq r(n, \varepsilon, \eta).
\]
Now observe that the function $m \mapsto E_{G \sim N(0, 1)}[\Xi(G, m; \omega)]$ satisfies
\[
\left| E_{G \sim N(0, 1)}[\Xi(G, m; \omega)] - E_{G \sim N(0, 1)}[\Xi(G, m'; \omega)] \right| \leq C(1 + |\text{arctanh } m| + |\text{arctanh } m'|)|\text{arctanh } m - \text{arctanh } m'|
\]
for all $m, m' \in (0, 1)$, uniformly over $\omega \in K_\omega$ and for a constant $C > 0$ depending only on $(\lambda, K')$. Then, applying Proposition A.4(b) and $m \in D_n$, we have
\[
\sup_{\omega \in K_\omega} \left| \frac{1}{n} \sum_{i=1}^{n} E[\Xi(G, m_i; \omega)] - E_{G \sim N(0, 1), \mu \sim \mu_*}[\Xi(G, m; \omega)] \right| \leq r(n, \varepsilon, \eta).
\]
Combining these two bounds, we obtain
\[
E \left[ \sup_{(p, u) \in K} \min_{(\alpha, \kappa, \gamma) \in K'} -H_n^\lambda(g, m, i, m_i) - E_{G \sim N(0, 1), \mu \sim \mu_*} \left[ \Theta_\lambda(G, m, m) \right] \right] \leq r(n, \varepsilon, \eta).
\]

Applying this back to (B.26) and (B.27), we get
\[ E \left[ \sup_{u \in D_\eta} G(m, u, v) \right] \leq E \left[ \inf_{(p, u) \in K(\alpha, \kappa, \gamma)} -H_\lambda(p, u; \alpha, \kappa, \gamma) \right] + r(n, \varepsilon, \eta). \]

This holds for all \( m \in D_\eta \), where \( r(n, \varepsilon, \eta) \) is independent of the specific point \( m \in D_\eta \). Then, applying this to (B.22), negating the sign, and choosing \( \varepsilon, \eta > 0 \) sufficiently small depending on \( t \) yields the result. \( \square \)

**Proof of Lemma 4.9(b).** We write as shorthand \( H_\lambda = H_\lambda^+ \) and \( E \) for \( E_{m \sim \mu} \). Recall
\[
H_\lambda(p, u; \alpha, \kappa, \gamma) = -2\lambda^2 p^2 - \lambda^2 u^2 + 2\lambda(1 - q_*) p^2 / q_* + \alpha u + \kappa p + \gamma + \lambda^2 (1 - q_*)
\]
\[
- E \left[ \left( 4\lambda^2 (1 - p^2 / q_*) + (2z(m)p/q_* + \alpha + \kappa m)^2 \right) \left( \frac{4}{1 - m^2} - 4\gamma \right) \right].
\]

Note that the dependence of \( H_\lambda \) on the first four variables \( (p, u; \alpha, \kappa) \) is homogeneous of degree 2. Collecting these quadratic terms yields
\[
H_\lambda(p, u; \alpha, \kappa, \gamma) = (p, u, \alpha, \kappa)^T A^{(\gamma)}(p, u, \alpha, \kappa) + \gamma + \lambda^2 (1 - q_*) - \lambda^2 E \left[ \left( \frac{1}{1 - m^2} - \gamma \right)^{-1} \right]
\]
where
\[
A^{(\gamma)} = \begin{pmatrix}
A_{11}^{(\gamma)} & A_{12}^{(\gamma)} \\
A_{21}^{(\gamma)} & A_{22}^{(\gamma)}
\end{pmatrix}
\]
and
\[
A_{11}^{(\gamma)} = \begin{pmatrix}
-2\lambda^2 + 2\lambda^2 \frac{1 - q_*}{q_*} + \lambda^2 E \left[ \frac{1}{1 - m^2} - \gamma \right]^{-1} - \frac{1}{q_*} E \left[ z(m) \frac{1}{1 - m^2} - \gamma \right]^{-1} & 0 \\
0 & -\lambda^2
\end{pmatrix}
\]
\[
A_{12}^{(\gamma)} = A_{21}^{(\gamma)^T} = \begin{pmatrix}
\frac{1}{2} - \frac{1}{q_*} E \left[ mz(m) \frac{1}{1 - m^2} - \gamma \right]^{-1} - \frac{1}{q_*} E \left[ z(m) \frac{1}{1 - m^2} - \gamma \right]^{-1} & 0
\end{pmatrix}
\]
\[
A_{22}^{(\gamma)} = -\frac{1}{4} \begin{pmatrix}
E \left[ m^2 \left( \frac{1}{1 - m^2} - \gamma \right) \right] & E \left[ m \left( \frac{1}{1 - m^2} - \gamma \right) \right] \\
E \left[ m \left( \frac{1}{1 - m^2} - \gamma \right) \right] & E \left[ \left( \frac{1}{1 - m^2} - \gamma \right) \right]^{-1}
\end{pmatrix}
\]

We first specialize this matrix \( A^{(\gamma)} \) to \( \gamma = 0 \), and apply the representation \( z(m) = \lambda \sqrt{q_*} G + \lambda^2 (1 - q_*) m \) where \( m = \tanh(\lambda q_* + \lambda \sqrt{q_*} G) \) and \( G \sim N(0, 1) \). Recalling \( q_* = E[m] = E[m^2] \) and \( b_* = E[m^3] = E[m^4] \) from Proposition A.2, this yields
\[
A_{11}^{(0)} = \begin{pmatrix}
-2\lambda^2 + 3\lambda^2 \frac{1 - q_*}{q_*} - \lambda^2 \frac{1 - q_*}{q_*^2} (q_* - b_*) - \frac{2\lambda^2 (1 - q_*)}{q_*} E[Gm(1 - m^2)] - \frac{\lambda^2}{q_*} E[G^2(1 - m^2)] & 0 \\
0 & -\lambda^2
\end{pmatrix}
\]
\[
A_{12}^{(0)} = \begin{pmatrix}
\frac{1}{2} - \frac{\lambda^2 (1 - q_*)}{2q_*} (q_* - b_*) - \frac{\lambda^2}{2q_*} E[Gm(1 - m^2)] & \frac{\lambda^2 (1 - q_*)}{2q_*} (q_* - b_*) + \frac{\lambda^2}{2q_*} E[Gm^2] \\
0 & \frac{1}{2} (q_* - b_*) (1 - q_*)
\end{pmatrix}
\]
\[
A_{22}^{(0)} = \begin{pmatrix}
\frac{1}{4} (q_* - b_*) (1 - q_*) & \frac{1}{4} (q_* - b_*) (1 - q_*)
\end{pmatrix}
\]

Now, recalling the matrices \( A_{11}, A_{12}, A_{21}, A_{22} \) defined in (B.13) from the proof of Lemma 4.3, we may check that
\[
A_{11}^{(0)} = 0
\]
\[
A_{12}^{(0)} = \frac{1}{2} \left( A_{12} + \begin{pmatrix}
\frac{\lambda^2 (1 - q_*)}{q_*} & 0 \\
0 & 0
\end{pmatrix} A_{22}
\right)
\]
\[
A_{11}^{(0)} = A_{11} + A_{12} \begin{pmatrix}
\frac{\lambda^2 (1 - q_*)}{q_*} & 0 \\
0 & 0
\end{pmatrix} A_{21} + \begin{pmatrix}
\frac{\lambda^2 (1 - q_*)}{q_*} & 0 \\
0 & 0
\end{pmatrix} A_{22} \begin{pmatrix}
\frac{\lambda^2 (1 - q_*)}{q_*} & 0 \\
0 & 0
\end{pmatrix}
\]
It was verified in the proof of Lemma 4.3 that $A_{22} < 0$ and $A_{11} - A_{12}A_{22}^{-1}A_{21} > 0$ strictly. Then also $A_{22}^{(0)} < 0$ and

$$A_{11}^{(0)} - A_{12}^{(0)}(A_{22}^{(0)})^{-1}A_{21}^{(0)} = A_{11} - A_{12}A_{22}^{-1}A_{21} > 0$$

strictly. Then by continuity in $\gamma$, also $A_{11}^{(\gamma)} < 0$ and $A_{11}^{(\gamma)} - A_{12}^{(\gamma)}(A_{22}^{(\gamma)})^{-1}A_{21}^{(\gamma)} > 0$ strictly for all $|\gamma|$ sufficiently small.

Applying this to the form (B.29) for $H_\lambda$, the condition $A_{22}^{(\gamma)} < 0$ shows that for fixed $p, u, \gamma$, the quadratic function $H_\lambda$ is strictly concave in $(\alpha, \kappa)$. At $(p, u) = (0, 0)$, $H_\lambda$ is maximized at $(\alpha, \kappa) = (0, 0)$. Here, $(0, 0, \gamma)$ belongs to the interior of the given domain $K'$ for $|\gamma|$ sufficiently small. Thus, for some $(\lambda, K')$-dependent constant $c > 0$ and any $|\gamma| < c$, the function

$$H_\lambda(p, u; \gamma) = \sup_{(\alpha, \kappa) : (\alpha, \kappa, \gamma) \in K'} H_\lambda(p, u; \alpha, \kappa, \gamma)$$

is quadratic in $(p, u)$ in a neighborhood of $(p, u) = (0, 0)$. The condition $A_{11}^{(\gamma)} - A_{12}^{(\gamma)}(A_{22}^{(\gamma)})^{-1}A_{21}^{(\gamma)} > 0$ shows that $H_\lambda(p, u; \gamma)$ is strictly convex in $(p, u)$ near $(0, 0)$, and is minimized at $(p, u) = (0, 0)$. We thus obtain for any fixed $\gamma$ with $|\gamma| < c$ that

$$\inf_{(p, u) \in \mathbb{R}^2} \sup_{(\alpha, \kappa, \gamma) \in K'} H_\lambda(p, u; \alpha, \kappa, \gamma) = H_\lambda(0, 0; 0, 0, 0, 0, 0)$$

for $|\gamma| = 1 - q_* - \lambda^2(1 - q_*) < 1$, so

$$\partial_\gamma H_\lambda(0, 0; 0, 0, 0, 0)\big|_{\gamma = 0} > 0.$$ 

Then there exists $\gamma_* > 0$ sufficiently small such that, bounding the supremum over $\gamma$ by the value at $\gamma = \gamma_*$,

$$\inf_{(p, u) \in \mathbb{R}^2} \sup_{(\alpha, \kappa, \gamma) \in K'} H_\lambda(p, u; \alpha, \kappa, \gamma) \geq H_\lambda(0, 0; 0, 0, 0, 0, 0) > 0.$$ 

Identifying $t_0 = H_\lambda(0, 0; 0, 0, 0, 0, 0)\big|_{\gamma = 0}$ concludes the proof. □

We combine these results with the following minor extension of the analysis of $\det H(m)$ from [FMM21], to show Corollary 4.10.

**Lemma B.3.** Fix any $\lambda > 1$, and suppose $x = 1$. For any $c > 0$, there exists $\eta > 0$ depending on $(\lambda, c)$ such that for all large $n$,

$$\int_{\mathcal{D}_\eta} \mathbb{E} \left[ |\det H(m)|^2 \big| g(m) = 0 \right]^{1/2} \rho_{g(m)}(0) \, dm \leq e^{cn}.$$ 

**Proof.** Define

$$A(m) = \frac{1}{n} m^T \text{arctanh } m, \quad E(m) = -H(m) - \frac{1}{2} A(m) - \frac{\lambda^2}{4} [1 - Q(m)^2].$$

The functions $m \mapsto m^2, m, \text{arctanh } m, h(m)$ all belong to the class (A.2). Thus, by Proposition A.2, the convergence $W(p, m, \mu_*) \to 0$ implies $(Q(m), M(m), A(m), E(m)) \to (q_*, q_*, \lambda^2 q_*, e_*)$. So for any $\delta > 0$, there is a constant $\eta = \eta(\delta) > 0$ for which

$$\mathcal{D}_\eta \subseteq \mathcal{E}_\delta = \left\{ m \in (-1, 1)^n : |Q(m) - q_*|, |M(m) - q_*|, |A(m) - \lambda^2 q_*|, |E(m) - e_*| < \delta \right\}.$$
Let
\[ L(m) = \frac{\lambda^2[1 - Q(m)]^2}{2} + \frac{1}{n} \sum_{i=1}^{n} \log \left( \frac{1}{1 - m_i^2} \right). \]

Then a small modification of the proof of [FMM21, Proposition 3.2] shows that for any fixed \( m \in \mathcal{E}_\delta \),
\[ \mathbb{E} \left[ |\det H(m)|^2 \mid g(m) = 0 \right]^{1/2} \leq \exp(n \cdot L(m) + Cn^{0.9}) \quad \text{(B.31)} \]
for all large \( n \) and some constant \( C > 0 \). Indeed, defining \( l(x) = \log |x - i \cdot n^{-0.11}| = \text{Re} \log(x - i \cdot n^{-0.11}) \) and applying this spectrally to the matrix \( H(m) \) by the functional calculus, we have
\[ \mathbb{P} \left[ \text{Tr} \, l(H(m)) \geq c_n + nt \mid g(m) = 0 \right] \leq 2e^{-cn^{1.78}t^2}, \quad c_n = n \cdot L(m) + Cn^{0.89} \]
as established in the proof of [FMM21, Proposition 3.2]. Then, applying
\[ \mathbb{E}[e^X] \leq e^t + E[(e^X - e^t)1\{X \geq t\}] = e^t + \int_t^{\infty} e^s \cdot \mathbb{P}[X > s] ds \]
we obtain
\[ \mathbb{E} \left[ |\det H(m)|^2 \mid g(m) = 0 \right] \leq \mathbb{E} \left[ e^{2\text{Tr} \, l(H(m))} \mid g(m) = 0 \right] \]
\[ \leq e^{2c_n + n^{0.89}} + \int_{2c_n + n^{0.89}}^{\infty} e^s \cdot \mathbb{P} \left[ 2 \text{Tr} \, l(H(m)) > s \mid g(m) = 0 \right] ds \]
\[ \leq e^{2c_n + n^{0.89}} + \int_{2c_n + n^{0.89}}^{\infty} e^s \cdot 2e^{-cn^{0.22}(s/2-c_n)^2} ds \]
\[ = e^{2c_n + n^{0.89}} + \int_{n^{0.89}/2}^{\infty} 4e^{2(t+c_n)-cn^{0.22}t^2} dt \leq e^{2c_n+Cn^{0.89}} \]
for all large \( n \), which implies (B.31). Then substituting (B.31) in place of the bound for \( \mathbb{E}[|\det H(m)| \mid g(m) = 0] \) in the proof of [FMM21, Theorem 1.1], we obtain
\[ \limsup_{n \to \infty} \frac{1}{n} \log \int_{\mathcal{E}_\delta} \mathbb{E} \left[ |\det H(m)|^2 \mid g(m) = 0 \right]^{1/2} p_g(m)(0) \, dm \]
\[ \leq \sup_{(q,\varphi,a,e) \in \mathbb{R}^+ \cup \{q=0\} : |q-\varphi,|,|a-\lambda^2q,|,|e-e_*| < \delta} S_*(q,\varphi,a,e) \]
where the complexity functional \( S_*(q,\varphi,a,e) \) is as defined in [FMM21, Eq. (1.10)]. This function \( S_* \) is continuous by definition, and [FMM21, Proposition 5.2] shows (for any \( \lambda > 0 \)) that \( S_*(q_*,\varphi_*,\lambda^2q_*,e_*) = 0 \). Thus, for any \( c > 0 \), there is \( \delta > 0 \) small enough so that this supremum is less than \( c \), and the result follows. \( \square \)

Proof of Corollary 4.10. For \( t > 0 \) to be determined, let \( U \sim \text{Unif}([-t,t]) \) be independent of \( W \). Applying Lemma 4.8 with \( T = \mathcal{D}_q \) and \( \ell(m,W) = \ell^+_x(m) \), we have
\[ \mathbb{P} \left[ \text{there exist } m \in \mathcal{D}_q \text{ and } u \in (-1,1)^n \cap B_{\sqrt{3n}}(m) : g(m) = 0 \text{ and } \lambda_{\text{min}}(H(u)) < t \right] \]
\[ = \mathbb{P} \left[ \text{there exists } m \in \mathcal{D}_q : g(m) = 0 \text{ and } \ell^+(m) < t \right] \]
\[ \leq \mathbb{P} \left[ \text{there exists } m \in \mathcal{D}_q : g(m) = 0 \text{ and } \ell^+(m) + U < 2t \right] \]
\[ \leq \int_{\mathcal{D}_q} \mathbb{E} \left[ |\det H(m)| \cdot 1\{\ell^+(m) + U < 2t\} \mid g(m) = 0 \right] p_g(m)(0) \, dm \]
\[ \leq \int_{\mathcal{D}_q} \mathbb{E} \left[ |\det H(m)| \cdot 1\{\ell^+(m) < 3t\} \mid g(m) = 0 \right] p_g(m)(0) \, dm. \]
Then by Cauchy-Schwarz,

\[ \mathbb{P} \left[ \text{there exist } m \in \mathcal{D}_\eta \text{ and } u \in (-1, 1)^n \cap \mathcal{B}_{\sqrt{2n}}(m) : g(m) = 0 \text{ and } \lambda_{\min}(H(u)) = \frac{1}{t} \right] \leq \int_{\mathcal{D}_\eta} \mathbb{E} \left[ \left| \det H(m) \right|^2 \left| g(m) = 0 \right|^2 \right]^{1/2} p_g(m)(0) d\mathbf{m} \cdot \sup_{m \in \mathcal{D}_\eta} \mathbb{P} \left[ \ell^+_s(m) < 3t \left| g(m) = 0 \right| \right]^{1/2}. \quad (B.32) \]

Let us express the conditional law of \{ -H(u) : u \in (-1, 1)^n \} |_{g(m)=0} as the right side of (B.21), and write \( W = (Z + Z^T)/\sqrt{2n} \) where \( Z \in \mathbb{R}^{n \times n} \) has i.i.d. \( \mathcal{N}(0, 1) \) entries. Then

\[ Z \mapsto \inf \left\{ v^T H(u) v : u \in (-1, 1)^n \cap \mathcal{B}_{\sqrt{2n}}(m), v \in \mathbb{R}^n, \|v\|_2 = 1 \right\} \]

is \( \lambda \sqrt{2/n} \)-Lipschitz with respect to the Frobenius norm of \( Z \). Hence by the Gaussian concentration inequality, for any \( s > 0 \),

\[ \mathbb{P} \left[ \ell^+_s(m) < \mathbb{E}[\ell^+_s(m) | g(m) = 0] - s \left| g(m) = 0 \right| \right] \leq e^{-s^2 n/4\lambda^2}. \]

Recalling the constant \( t_0 > 0 \) from Lemma 4.9(b), let us set \( 3t = s = t_0/3 \). Then by Lemma 4.9(a–b), for some sufficiently small \( \lambda \)-dependent constants \( \varepsilon, \eta, c_1 > 0 \), all \( m \in \mathcal{D}_\eta \), and all large \( n \),

\[ \mathbb{P} \left[ \ell^+_s(m) < 3t \left| g(m) = 0 \right| \right] \leq e^{-c_1 n}. \]

Now applying Lemma B.3, for this constant \( c_1 \), some sufficiently small \( \eta > 0 \), and all large \( n \),

\[ \int_{\mathcal{D}_\eta} \mathbb{E} \left[ \left| \det H(m) \right|^2 \left| g(m) = 0 \right|^2 \right]^{1/2} p_g(m)(0) d\mathbf{m} \leq e^{c_1 n/4 \lambda^2}. \]

Applying these two bounds to (B.32) shows (4.20). \( \square \)

**Proof of Theorem 2.1(a–b).** Fix \( \eta, \varepsilon, t > 0 \) small enough as described in Corollary 4.10. Fix any \( \lambda > 0 \). We choose \( \delta = \delta(\lambda, \eta, \varepsilon, t, i) \) small enough, to be determined, so that the conclusions of Corollary 4.4 and Lemma 4.7 hold for this \( \delta \). Then with probability approaching 1, Corollary 4.4 establishes the existence of a local minimizer \( m_* \) of \( \mathcal{F}_{\text{TAP}} \) belonging to \( \mathcal{B}_\delta \), such that \( \mathcal{F}_{\text{TAP}}(m_*) - e_* < \delta \). Lemma 4.7 ensures that this local minimizer \( m_* \) belongs also to \( \mathcal{D}_\eta \), and Corollary 4.10 then ensures that for all \( u \in \mathcal{B}_{\sqrt{2n}}(m_*) \), we have \( \lambda_{\min}(\nabla^2 \mathcal{F}_{\text{TAP}}(u)) > t/n \) (where we recall that \( H \) is the Hessian of \( \mathcal{F}_{\text{TAP}} \) rescaled by \( n \)).

Let us now show that with probability approaching 1, we may pick such a local minimizer \( m_* \) to satisfy the Bayes optimality condition (2.2). Lemma A.7(b–d) shows that for sufficiently large \( k \), with probability approaching 1, the AMP iterate \( m^k \) satisfies \( m^k \in \mathcal{B}_{\delta/2}, \mathcal{F}_{\text{TAP}}(m^k) < e_* + \delta \wedge (c_0 \delta^2) \) with \( c_0 \) as in the proof of Corollary 4.4, and also

\[ \frac{1}{n^2} \| \hat{X}_{\text{Bayes}} - m^k(m^k)^\top \|^2 \leq \delta. \]

Now we claim that we may pick the local minimizer \( m_* \) in the proof of Corollary 4.4 so that there is a path \( \Gamma \) connecting \( m^k \) to \( m_* \) for which \( \mathcal{F}_{\text{TAP}}(m) \leq \mathcal{F}_{\text{TAP}}(m^k) < e_* + \delta \) for all \( m \in \Gamma \). To see this, consider running gradient flow \( \frac{d}{dt} m = -\nabla \mathcal{F}_{\text{TAP}}(m) \) initialized at \( m^k \). Because (1) \( \mathcal{F}_{\text{TAP}}(m) \) must be non-increasing along this trajectory, (2) by Eqs. (B.15) and (B.16) \( \mathcal{F}_{\text{TAP}}(m) \) is larger on \( \mathcal{B}_\delta \setminus \mathcal{B}_{\delta/2} \) than it is at \( m^k \), and (3) the gradient flow would need to, by the continuity of \( \mathcal{F}_{\text{TAP}} \), pass through \( \mathcal{B}_{\delta/2} \) were it ever to leave \( \mathcal{B}_{\delta/2} \), we can conclude that the gradient flow stays in \( \mathcal{B}_{\delta/2} \) for all time. Moreover, by the Lojasiewicz Theorem [Loj83], the gradient flow converges to a critical point \( m_\star \) of \( \mathcal{F}_{\text{TAP}} \). By Corollary 4.4, this critical point must be a local minimizer, and the gradient flow guarantees the the property \( \mathcal{F}_{\text{TAP}}(m) \leq \mathcal{F}_{\text{TAP}}(m^k) < e_* + \delta \) on the full path.

Now, choosing \( \delta < \varepsilon t/6 \), if \( m^k \notin \mathcal{B}_{\sqrt{6\delta n/t}}(m_*) \), then at the point \( m \in \Gamma \) where \( \Gamma \) crosses the boundary of \( \mathcal{B}_{\sqrt{6\delta n/t}}(m_*) \), the strong convexity of \( \mathcal{F}_{\text{TAP}} \) on \( \mathcal{B}_{\sqrt{2n}}(m_*) \) implies that

\[ \mathcal{F}_{\text{TAP}}(m) - \mathcal{F}_{\text{TAP}}(m_*) \geq \frac{t}{2n} \cdot (\sqrt{6\delta n/t})^2 = 3\delta. \]
But this is a contradiction because \( e_* - \delta < F_{\text{TAP}}(m_*) \leq F_{\text{TAP}}(m) < e_* + \delta \). Thus, for this choice of \( m_* \), we must have \( \| m_* - m^K \|_2 \leq \sqrt{6\delta/n} / t \). Then with probability approaching 1,

\[
\frac{1}{n^2} \| \hat{X}_{\text{Bayes}} - m_* m_*^T \|_F^2 
\leq \frac{2}{n^2} \| \hat{X}_{\text{Bayes}} - m^K (m^K)^T \|_F^2 + \frac{2}{n^2} \| m^K (m^K)^T - m_* m_*^T \|_F^2 
< 2\delta + 24\delta / t
\]

for all large \( n \). Choosing \( \delta \) small enough so that \( 2\delta + 24\delta / 4 < \iota \), we obtain that \( m_* \) satisfies (2.2).

Finally, for sufficiently small \( \iota > 0 \), uniqueness of the critical point \( m_* \) up to sign follows from the strong convexity of \( F_{\text{TAP}} \) near \( m_* \): If \( m, m' \) both satisfy \( n^{-2} \| \hat{X}_{\text{Bayes}} - m m^T \|_F^2 < \iota \), then

\[
\min \left( \frac{1}{n} \| m + m' \|_2, \frac{1}{n} \| m - m' \|_2 \right)^2 
\leq \frac{1}{n^2} \| m + m' \|_2^2 \| m - m' \|_2^2 
= \frac{1}{n^2} \left( \| m \|_2^4 + \| m' \|_2^4 + 2 \| m \|_2^2 \| m' \|_2^2 - 4(m^T m')^2 \right) 
\leq \frac{2}{n^2} \left( \| m \|_2^4 + \| m' \|_2^4 - 2(m^T m')^2 \right) 
\leq \frac{2}{n^2} \| m m^T - m'(m')^T \|_F^2 < 8\iota.
\]

Thus either \( m'_* \in B_{(8\iota)^{1/4} \sqrt{m}(m_*)} \) or \( m'_* \in B_{(8\iota)^{1/4} \sqrt{m}(-m_*)} \). When \( (8\iota)^{1/4} < \sqrt{n} \), the strong convexity of \( F_{\text{TAP}} \) on \( B_{\sqrt{m}(m_*)} \) ensures in the first case that \( m'_* = m_* \). By sign symmetry, \( F_{\text{TAP}} \) is also strongly convex on \( B_{\sqrt{m}(-m_*)} \) with a local minimizer at \( -m_* \), so \( m'_* = -m_* \) in the latter case.

**B.4 Proofs for Section 4.5**

*Proof of Lemma 4.11.* Define

\[
X(m) = \text{diag}(1 - m^2)^{1/2} \cdot (\lambda Y + 2\lambda^2 m m^T / n) \cdot \text{diag}(1 - m^2)^{1/2}, \\
D(m) = \lambda^2 [1 - Q(m)] \cdot \text{diag}(1 - m^2).
\]

Then for any \( \mu \in \mathbb{C} \), note that

\[
\det(B(m) - \mu I) = \det \left[ \begin{pmatrix} \text{diag}(1 - m^2)^{1/2} \cdot X(m) \cdot \text{diag}(1 - m^2)^{-1/2} - \mu I & -D(m) \\ I & -\mu I \end{pmatrix} \right] 
= \det \left[ \mu^2 I - \mu \text{diag}(1 - m^2)^{1/2} \cdot X(m) \cdot \text{diag}(1 - m^2)^{-1/2} + D(m) \right] 
= \det \left[ \mu^2 I - \mu X(m) + D(m) \right].
\]

Then \( \mu \) is an eigenvalue of \( B(m) \) if and only if \( \mu^2 I - \mu X(m) + D(m) \) is singular. (Note that this is equivalent to singularity of the matrix (4.22) as discussed in the main text.)

For any \( \xi \in \mathbb{C} \), define the matrix

\[
K(\xi, \mu, m) = \xi \mu^2 I - \xi \mu X(m) + \xi D(m).
\]

Then for any \( v \in \mathbb{C}^n \), we have

\[
v^* K(\xi, \mu, m) v = v^* [R K(\xi, \mu, m)] v + i \cdot v^* [\Im K(\xi, \mu, m)] v,
\]

where \( R \) and \( \Im \) denote the entry-wise real and imaginary parts. Since \( I \), \( X(m) \), and \( D(m) \) are all real and symmetric, we have that \( R K(\xi, \mu, m) \) and \( \Im K(\xi, \mu, m) \) are also real and symmetric, so \( v^* [R K(\xi, \mu, m)] v \) and \( v^* [\Im K(\xi, \mu, m)] v \) are both real. If there exists some \( \xi \in \mathbb{C} \) for which \( R K(\xi, \mu, m) \) is non-singular, then this implies the real part of \( v^* K(\xi, \mu, m) v \) is non-zero for any non-zero vector \( v \in \mathbb{C}^n \), so \( \mu \) is not an eigenvalue of \( B(m) \).

By Proposition A.2, \( \lambda^2 (1 - q_* + \delta) < 1 \) for some \( \delta > 0 \). Let us take \( r_0 = \sqrt{\lambda^2 (1 - q_* + \delta)} \), and suppose that (4.23) holds for \( r \in (r_0, 1) \). We show that for any \( \mu \in \mathbb{C} \) with \( |\mu| \geq r \), we may pick \( \xi \) to ensure that \( R K(\xi, \mu, m) > 0 \), and hence \( R K(\xi, \mu, m) \) is non-singular. There are four cases.
Case 1 (positive imaginary part) Suppose \( \mu = \rho e^{i\varphi} \) with \( \rho \geq r \) and \( \varphi \in (0, \pi) \). Take \( \xi = -i e^{i\varphi} \). Then
\[
\Re K(-i e^{-i\varphi}, \rho e^{i\varphi}, m) = \rho^2 \cos(3\pi/2 + \varphi)I + \cos(3\pi/2 - \varphi)D(m)
\]
\[
= \cos(3\pi/2 + \varphi)(\rho^2 I - \lambda^2 [1 - Q(m)] \text{diag}(1 - m^2))
\]
Applying \( \text{diag}(1 - m^2) \leq I \), \( \lambda^2[1 - Q(m)] < \lambda^2(1 - q_\star + \delta) < r_0^2 < \rho^2 \), and \( \cos(3\pi/2 + \varphi) > 0 \), this ensures \( \Re K(-i e^{-i\varphi}, \rho e^{i\varphi}, m) > 0 \).

Case 2 (negative imaginary part) Suppose \( \mu = \rho e^{i\varphi} \) with \( \rho \geq r \) and \( \varphi \in (\pi, 2\pi) \). Take \( \xi = i e^{i\varphi} \). Then
\[
\Re K(i e^{-i\varphi}, \rho e^{i\varphi}, m) = \cos(\pi/2 + \varphi)(\rho^2 I - \lambda^2 [1 - Q(m)] \text{diag}(1 - m^2)).
\]
By the same argument as in case 1, \( \Re K(i e^{-i\varphi}, \rho e^{i\varphi}, m) > 0 \).

Case 3 (real and positive) Suppose \( \mu = \rho \geq r \). Take \( \xi = 1 \), and note that
\[
\text{diag}(1 - m^2)^{-1/2} \Re K(1, \rho, m) \text{diag}(1 - m^2)^{-1/2}
\]
\[
= \rho \left[-(\lambda Y + 2\lambda^2 mm^T/n) + \rho^{-1}\lambda^2[1 - Q(m)]I + \rho \text{diag}(1 - m^2) \right].
\]
When \( \rho = r \), this is positive-definite by assumption. For any \( a,b > 0 \), the function \( \rho \rightarrow a/\rho + b/\rho \) is increasing for \( \rho \geq \sqrt{a/b} \). Then, applying \( \lambda^2[1 - Q(m)] < \lambda^2(1 - q_\star + \delta) < r_0^2 \) and \( \text{diag}(1/(1 - m^2)) \geq I \), the matrix inside the parenthesis is increasing in \( \rho \) in the positive-definite ordering, for \( \rho \geq r \). Hence this matrix is positive definite for all \( \rho \geq r \), implying also \( \Re K(1, \rho, m) > 0 \).

Case 4 (real and negative) Suppose \( \mu = -\rho \leq -r \). Take \( \xi = 1 \), and note that
\[
\text{diag}(1 - m^2)^{-1/2} \Re K(1, -\rho, m) \text{diag}(1 - m^2)^{-1/2}
\]
\[
= \rho \left[(\lambda Y + 2\lambda^2 mm^T/n) + \rho^{-1}\lambda^2[1 - Q(m)]I + \rho \text{diag}(1 - m^2) \right].
\]
When \( \rho = r \), this is again positive definite by assumption, so \( \Re K(1, -\rho, m) > 0 \) for all \( \rho \geq r \) by the same argument as in case 3.

Combining these cases, \( B(m) \) does not have any eigenvalue \( \mu \in \mathbb{C} \) with \( |\mu| \geq r \), as desired.

Proof of Lemma 4.12(b). The argument is similar to that of Lemma 4.9(b): Let us write as shorthand \( H_\lambda = H_\lambda^* \) and \( \mathbb{E} \) for \( \mathbb{E}_{m \sim \mu_*} \) . Then
\[
H_\lambda(p, u; \alpha, \kappa, \gamma)
\]
\[
= 2\lambda^2 p^2 + \lambda^2 u^2 - 2\lambda^2(1 - q_\star)p^2/q_\star - \alpha u - \kappa p + \gamma + \lambda^2(1 - q_\star)
\]
\[
= -E\left[\left(4\lambda^2(1 - p^2)/q_\star + (2z(m)p/q_\star + \alpha + \kappa m)^2\right)/\left(1 - m^2\right) - 4\gamma\right]
\]
\[
= (p, u, \alpha)^T \begin{bmatrix}
A_{11}(-\gamma) & A_{12}(-\gamma) \\
A_{21}(-\gamma) & A_{22}(-\gamma)
\end{bmatrix} (p, u, \alpha) + \gamma + \lambda^2(1 - q_\star) - \lambda^2 E\left[\left(1 - m^2\right) - \gamma\right]^{-1}
\]
where, specializing to \( \gamma = 0 \), we have
\[
A_{11}(-\gamma) = \begin{bmatrix}
2\lambda^2 - \frac{\lambda^2(1 - q_\star)}{q_\star} & \frac{\lambda^2(1 - q_\star)}{q_\star} \\
\frac{\lambda^2(1 - q_\star)}{q_\star} & 0
\end{bmatrix}
\]
\[
A_{12}(-\gamma) = \begin{bmatrix}
\frac{\lambda^2(1 - q_\star)}{2q_\star} & 0 \\
0 & \frac{\lambda^2(1 - q_\star)}{2q_\star}
\end{bmatrix}
\]
\[
A_{22}(-\gamma) = -\frac{1}{4} \begin{bmatrix}
q_\star - b_\star & q_\star - b_\star \\
q_\star - b_\star & 1 - q_\star
\end{bmatrix}
\]

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Recalling $A_{11}, A_{12}, A_{21}, A_{22}$ from (B.13), we then may check that

$$A_{22}^{(-,0)} = \frac{1}{4} A_{22}$$
$$A_{12}^{(-,0)} = \frac{1}{2} \left( A_{12} + \begin{pmatrix} \lambda^2 (1-q_* - \frac{\kappa}{q_*} & 0 \\ 0 & 0 \end{pmatrix} A_{22} - 2I \right)$$
$$A_{11}^{(-,0)} = A_{11} + A_{12} \left( \begin{pmatrix} \lambda^2 (1-q_* - \frac{\kappa}{q_*} & 0 \\ 0 & 0 \end{pmatrix} A_{22} - 4 \right) A_{21} + \begin{pmatrix} \lambda^2 (1-q_* - \frac{\kappa}{q_*} & 0 \\ 0 & 2\lambda^2 \end{pmatrix}.$$ 

Thus $A_{22}^{(-,0)} < 0$, and

$$A_{11}^{(-,0)} - A_{12}^{(-,0)} (A_{22}^{(-,0)})^{-1} A_{21}^{(-,0)} = \left( A_{11} - A_{12} A_{22}^{-1} A_{21} \right) + 2A_{12} A_{22}^{-1} + 2A_{21} A_{22}^{-1} - 4A_{22}^{-1} + \begin{pmatrix} 4\lambda^2 & 0 \\ 0 & 2\lambda^2 \end{pmatrix}$$

Computing explicitly $A_{22}^{-1}$, we obtain after some simplification

$$A_{11}^{(-,0)} - A_{12}^{(-,0)} (A_{22}^{(-,0)})^{-1} A_{21}^{(-,0)} = \begin{pmatrix} c_1 & -c_2 \\ -c_2 & c_2 \end{pmatrix}$$

where now

$$c_1 = \frac{(1-\alpha_*) + 2\lambda^2 (1-2\alpha_* + b_*) + \lambda^4 (1-2\alpha_* + b_*)^3}{(1-2\alpha_* + b_*)(q_* - b_*)} - \lambda^4 (1-2\alpha_* + b_*)$$
$$c_2 = \frac{1}{1-2\alpha_* + b_*} + \lambda^2$$

Applying again (B.14), together with

$$1 - \lambda^2 (q_* - b_* + \lambda^2 (1-2\alpha_* + b_*) > 1 - \lambda^2 (1-\alpha_*) > 0,$$

we get $c_2 > 0$ and

$$c_1 - (-c_2) c_2^{-1} (-c_2) = c_1 - c_2$$
$$= -\lambda^2 \left(1 + \lambda^2 (1-2\alpha_* + b_*) \right) + \frac{1}{q_* - b_*} \left(1 + \lambda^2 (1-2\alpha_* + b_*) \right)^2$$
$$= \frac{1}{q_* - b_*} \left(1 + \lambda^2 (1-2\alpha_* + b_*) \right) \left(1 - \lambda^2 (q_* - b_*) + 1 + \lambda^2 (1-2\alpha_* + b_*) \right) > 0.$$ 

Thus $A_{11}^{(-,0)} - A_{12}^{(-,0)} (A_{22}^{(-,0)})^{-1} A_{21}^{(-,0)} > 0$. This implies, as in the proof of Lemma 4.9(b), that for all $|\gamma| < c$ small enough,

$$\inf_{(p,u) \in B_{\gamma,m}^{\infty}} \sup_{(\alpha,\kappa) \in K} H_\lambda(p,u;\alpha,\kappa,\gamma) = H_\lambda(0,0;0,0,\gamma) = \gamma + \lambda^2 (1-\alpha_*) - \lambda^2 [\frac{1}{1-m^2} - \gamma]^{-1}.$$ 

The conclusion then follows as in the proof of Lemma 4.9(b).

**Corollary B.4.** Fix any $\lambda > 1$, and suppose $x = 1$. Then there exist $\lambda$-dependent constants $\epsilon, \eta, t, c > 0$ such that, for all large $n$,

$$\mathbb{P} \left[ \text{there exist } m \in D_{\eta} \text{ and } u \in (-1,1)^n \cap B_{\sqrt{m}}(m) : g(m) = 0 \text{ and } \lambda_{\min}(H^- (u)) < t \right] < e^{-cn}.\]
Proof. The proof is identical to that of Corollary 4.10, applying Lemma 4.12 in place of Lemma 4.9. □

Proof of Theorem 2.1(c). Fix η, δ > 0 small enough to satisfy Lemma 4.11 and Corollaries 4.10 and B.4. Let \( m_* \) be the local minimizer identified in Theorem 2.1(a), which belongs to \( B_0 \cap D_\eta \). Corollaries 4.10 and B.4 imply that for some \( t > 0 \), with probability approaching 1, both \( H^+(m_*) > tI \) and \( H^-(m_*) > tI \). Consider now the matrices

\[
\pm r^{-1}(\lambda Y + 2\lambda^2 m_* m_*^T/n) - r^{-2}\lambda^2[1 - Q(m_*)]I - \operatorname{diag}\left(\frac{1}{1-m_*^2}\right),
\]

where \( r = 1 \) corresponds to \( H^+(m_*) \). We have \( \|\lambda^2[1 - Q(m_*)]I\|_{op} \leq \lambda^2 \), and also \( \|\lambda Y + 2\lambda^2 m_* m_*^T/n\|_{op} < 3\lambda + 3\lambda^2 \) on the event of probability \( 1 - e^{-cn} \) where \( \|W\|_{op} < 3 \). Thus, on this event and for some constant \( r(t, \lambda) \in (0, 1) \), the above matrices must also be positive definite by continuity for all \( r \in (r(t, \lambda), 1) \). Multiplying by \( r^2 \) and recalling \( r_0 \) from Lemma 4.11, this ensures that for some \( r \in (0, 1) \) with \( r > \max(r(t, \lambda), r_0) \), almost surely for all large \( n \), the matrices (4.23) are positive definite at \( m_* \). Then by Lemma 4.11,

\[
\rho(d_{\text{AMP}}(m_*, m_*)) = \rho(B(m_*)) < r < 1.
\]

□

C Proofs for algorithm convergence

C.1 Analysis of TAP landscape

We prove Corollary 2.2 and Lemma 5.2 on properties of the global landscape of \( F_{\text{TAP}} \) when \( \lambda > \lambda_0 \).

Proof of Corollary 2.2. Fix any \( \iota > 0 \). By [FMM21, Theorem 1.2], with probability approaching 1, there exists a critical point \( m \in S \) of \( F_{\text{TAP}} \), and furthermore all critical points \( m \in S \) satisfy

\[
\frac{1}{n^2}\|mm^T - \hat{X}_{\text{Bayes}}\|_F^2 < \iota.
\]

For small enough \( \iota > 0 \), Theorem 2.1(a) then implies that the critical point \( m \in S \) is unique up to sign, so it must be the global minimizer of \( F_{\text{TAP}} \) by the definition of the set \( S \). □

We next show the following strengthened version of [FMM21, Lemma C.3].

Lemma C.1. Fix any integer \( a \geq 0 \), and set \( q = 1 - \lambda^{-a} \) and \( t = \lambda^{2-a} \). Suppose \( x = 1 \). For a constant \( \lambda_0(a) > 0 \), if \( \lambda > \lambda_0(a) \), then for some \( (a, \lambda) \)-dependent constants \( C, c > 0 \), with probability at least \( 1 - Ce^{-cn} \),

\[
\left\{ m \in (-1, 1)^n : M(m) + Q(m) > 1.01 \text{ and } n \cdot \|\nabla F_{\text{TAP}}(m)\|_2 < t \right\} \subseteq M_q. \tag{C.1}
\]

Proof. We induct on \( a \). The base case \( a = 0 \) is trivial, because \( M(m) + Q(m) > 1.01 \) implies \( M(m) > 0 \).

Suppose by induction that the statement holds for an integer \( a \geq 0 \). Let \( g(m) = n \cdot \nabla F_{\text{TAP}}(m) \) be the normalized gradient from (4.15), and denote its coordinates as \( g_i(m) \). Fix \( \delta = 0.001 \), and consider

\[
S_1 = \left\{ i \in \{1, \ldots, n\} : |g_i(m)| < \delta \lambda^2 \right\}.
\]

If \( m \) satisfies \( n \cdot \|\nabla F_{\text{TAP}}(m)\|_2^2 = n^{-1} \sum_i g_i(m)^2 < \lambda^{2-a} \), then by Markov’s inequality,

\[
|S_1| > n \left( 1 - \frac{1}{\delta^2 \lambda^{2+a}} \right).
\]

Next, write \( W \sim \text{GOE}(n) \) as \( W = (Z + Z^T)/\sqrt{2n} \), where \( Z_{ij} \overset{iid}{\sim} \mathcal{N}(0, 1) \). Let \( z_i \) be the \( i \)th column of \( Z \). Note that if \( m \in M_{1-\lambda^{-a}} \), then

\[
\|m - 1\|_2/\sqrt{n} \leq \sqrt{2 - 2M(m)} \leq \sqrt{2 \lambda^{-a}/2}.
\]
Then applying [FMM21, Lemma C.2] with $\delta \lambda^{1+a}/2$ in place of $\lambda$, there are absolute constants $C_0, C, c > 0$ (independent of $\lambda$) such that
\[
\mathbb{P}\left[ \sup_{m \in \mathcal{M}_{1-\lambda^{-a}}} \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}\{|(z_i, m - 1)| \geq \delta \lambda \sqrt{n}/2\} \geq \frac{C_0}{\lambda^{2+a}} \right] \leq C e^{-cn}. \tag{C.2}
\]

By the Chernoff bound, for $\lambda$ large enough such that $\mathbb{P}_{Z \sim \mathcal{N}(0, 1)}[|Z| \geq \delta \sqrt{2}] < C_0/(2\lambda^{2+a})$, we also have
\[
\mathbb{P}\left[ \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}\{|(z_i, 1)| \geq \delta \lambda \sqrt{n}/2\} \geq \frac{C_0}{\lambda^{2+a}} \right] \leq C e^{-cn}. \tag{C.3}
\]

Then, applying these bounds also for the rows of $Z^\top$ and defining
\[
\mathcal{S}_2(m) = \left\{ i \in \{1, \ldots, n\} : ||W(m - 1)||_i \leq \delta \lambda \text{ and } ||W1||_i \leq \delta \lambda \right\},
\]
we obtain
\[
\mathbb{P}\left[ \inf_{m \in \mathcal{M}_{1-\lambda^{-a}}} |\mathcal{S}_2(m)| \geq n(1 - 4C_0/\lambda^{2+a}) \right] \geq 1 - 4Ce^{-cn}. \tag{C.4}
\]

Consider now the event where (C.4) holds and where (C.1) holds for $a$. This has probability at least $1 - C' e^{-cn}$ for some $(a, \lambda)$-dependent constants $C', c' > 0$, by the induction hypothesis. On this event, for any $m \in (-1, 1)^n$ belonging to the left side of (C.1), the above shows $|\mathcal{S}_1 \cap \mathcal{S}_2(m)| \geq n(1 - C_1/\lambda^{2+a})$ where $C_1 = 4C_0 + 1/\delta^2$. For this $m$ and any index $i \in \mathcal{S}_1 \cap \mathcal{S}_2(m)$, writing
\[
g_i(m) = -\lambda^2 M(m) - \lambda \cdot [W1]_i - \lambda \cdot [W(m - 1)]_i + \text{arctanh}(m_i) + \lambda^2 [1 - Q(m)] m_i,
\]
we have
\[
m_i > \tanh(\lambda^2 M(m) + \lambda \cdot [W1]_i + \lambda \cdot [W(m - 1)]_i - \lambda^2 [1 - Q(m)] m_i - \delta \lambda^2)
\geq \tanh((M(m) + Q(m) - 1)\lambda^2 - 3\delta \lambda^2) \geq \tanh(7\delta \lambda^2),
\]
where the last step uses $M(m) + Q(m) > 1.01$ and $\delta = 0.001$. For the coordinates $i \notin (\mathcal{S}_1 \cap \mathcal{S}_2(m))$, we may apply the trivial bound $m_i \geq -1$. Then on the above event, for all sufficiently large $\lambda$,
\[
M(m) = \langle m, 1 \rangle/n > (1 - C_1/\lambda^{2+a}) \cdot \tanh(7\delta \lambda^2) - C_1/\lambda^{2+a} > 1 - 3C_1/\lambda^{2+a} > 1 - 1/\lambda^{1+a}.
\]
So the left side of (C.1) is contained in $\mathcal{M}_{1-\lambda^{-a+1}}$. This completes the induction.

\[\square\]

**Lemma C.2.** For $t \in [0, 1]^n$ and $\varepsilon > 0$, denote by $\mathcal{S}(t, \varepsilon)$ the subset of indices $i \in \{1, \ldots, n\}$ for which $t_i \geq \varepsilon$. Then there exist universal constants $C, C', c > 0$ such that for $W \sim \text{GOE}(n)$ and any $\varepsilon > 0$ and $0 < s < 1$,
\[
\mathbb{P}\left[ \sup_{t_1, t_2 \in [0, 1]^n: \max(||S(t_1, \varepsilon)||, ||S(t_2, \varepsilon)||) \leq ns} \|\text{diag}(t_1)W\text{diag}(t_2)\|_{\text{op}} \geq C' \cdot (\varepsilon + \sqrt{s \log(\varepsilon/s)}) \right] \leq C e^{-cn}. \tag{C.5}
\]

**Proof of Lemma C.2.** Denote $P_S : \mathbb{R}^n \to \mathbb{R}^n$ to be the projection operator onto the subspace associated to $S \subseteq \{1, \ldots, n\}$, and let $P_S^\perp = I - P_S$. Then
\[
\|\text{diag}(t_1)W\text{diag}(t_2)\|_{\text{op}} \leq \|\text{diag}(t_1)P_{S(t_1, \varepsilon)}WP_{S(t_2, \varepsilon)}\text{diag}(t_2)\|_{\text{op}}
+ \|\text{diag}(t_1)P_{S(t_1, \varepsilon)}WP_{S(t_2, \varepsilon)}^\perp\text{diag}(t_2)\|_{\text{op}}
+ \|\text{diag}(t_1)P_{S(t_1, \varepsilon)}^\perpWP_{S(t_2, \varepsilon)}\text{diag}(t_2)\|_{\text{op}}
+ \|\text{diag}(t_1)P_{S(t_1, \varepsilon)}^\perpWP_{S(t_2, \varepsilon)}^\perp\text{diag}(t_2)\|_{\text{op}}
\leq \|P_{S(t_1, \varepsilon)}WP_{S(t_2, \varepsilon)}\|_{\text{op}} + 3\varepsilon \|W\|_{\text{op}}.
\]

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We have $\|W\|_\text{op} < 3$ with probability $1 - Ce^{-cn}$. Thus it suffices to show that

$$P \left[ \sup_{|S_1|,|S_2| \leq n,s} \|W_{S_1,S_2}\|_\text{op} \geq C\sqrt{s\log(e/s)} \right] \leq e^{-csn}, \tag{C.6}$$

where $W_{S_1,S_2}$ is the submatrix of rows in $S_1$ and columns in $S_2$. Let $B_s = \{ v \in \mathbb{R}^n : \|v\|_2 \leq 1, \|v\|_0 \leq ns \}$, and note that

$$\sup_{|S_1|,|S_2| \leq n,s} \|W_{S_1,S_2}\|_\text{op} = \sup_{v,v' \in B_s} v^T W v'. $$

For any support set $S$ of size $\lfloor ns \rfloor$, the 1/4-covering number of $\{ v \in \mathbb{R}^n : \|v\|_2 \leq 1, \text{supp}(v) \subseteq S \}$ is no larger than $(1 + 1/8)^{ns}/(1/8)^{ns} = 9^{ns}$. Then applying $\lfloor ns \rfloor \leq (e/s)^{ns}$, the 1/4-covering number of $B_s$ is no larger than $(9e/s)^{ns}$. Letting $\mathcal{N}$ be a 1/4-cover of $B_s$,

$$\sup_{v,v' \in \mathcal{N}} v^T W v' \leq \sup_{v,v' \in \mathcal{N}} \sup_{\delta v, \delta v' \in B_s/4} (v + \delta v)^T W (v + \delta v') \leq \sup_{v,v' \in \mathcal{N}} v^T W v' + \frac{9}{16} \sup_{v,v' \in B_s} v^T W v', $$

hence

$$\sup_{v,v' \in \mathcal{N}} v^T W v' \leq 16 \sup_{v,v' \in \mathcal{N}} v^T W v'. $$

For each $v, v' \in \mathcal{N}$, a Gaussian tail bound yields $P(v^T W v' \geq t) \leq e^{-nt^2/4}$. The result then follows by choosing $t = \sqrt{8s \log(9e/s)}$ and taking a union bound over $\mathcal{N}$.

**Proof of Lemma 5.2.** For part (a), [FMM21, Lemma 1.1] implies that with probability $1 - Ce^{-cn}$, every point $m \in S$ satisfies $Q(m) \geq M(m)^2 \geq (1/3 - 6/\lambda - 4/\lambda^2)^{1/2}$. For $M(m) > 0$, this implies $Q(m) + M(m) > 1.01$. Then by Lemma C.1, with probability at least $1 - Ce^{-cn}$, the guarantee of (a) holds for all $m \in S \setminus M_q$ where $M(m) > 0$. The guarantee then also holds for $M(m) < 0$ by the sign symmetry $F_{\text{TAP}}(m) = F_{\text{TAP}}(-m)$.

For part (b), recall the form (4.16) for $H(m) = n \cdot \nabla^2 F_{\text{TAP}}(m)$, and denote $D(m) = \text{diag}((1 - m^2)^{1/2})$. Then

$$D(m)H(m)D(m) \geq -\frac{\lambda^2}{n} D(m) 11^\top D(m) - \lambda D(m) WW D(m) + I - \frac{2\lambda^2}{n} D(m) mm^\top D(m). \tag{C.7}$$

Observe that if $m \in M_q$, then $Q(m) \geq M(m)^2 > q^2 > 1 - 2\lambda^{-a}$. So

$$\|D(m)m\|^2_2 \leq \|D(m)1\|^2_2 = \sum_{i=1}^n (1 - m_i^2) = n(1 - Q(m)) < 2n\lambda^{-a} \leq 2n\lambda^{-5}.$$ 

Observe also that by Markov’s inequality,

$$\frac{1}{n} \sum_{i=1}^n 1\{(1 - m_i^2)^{1/2} > \lambda^{-5/4}\} \leq (1 - Q(m))\lambda^{5/2} < 2\lambda^{-a+5/2} \leq 2\lambda^{-5/2}. \tag{C.8}$$

Then applying Lemma C.2, with probability $1 - Ce^{-cn}$, $\|D(m)WD(m)\|_\text{op} \leq \lambda^5/4$ for every $m \in M_q$. Applying these bounds to (C.7), for all sufficiently large $\lambda$ and any $m \in M_q$, we have $D(m)H(m)D(m) \succ I/2$, and hence $H(m) \succ D(m)^{-2}/2 = \text{diag}(1/(1 - m^2)) / 2 \succeq I/2$. 

**C.2 Analysis of NGD**

We prove Lemma 5.1, Theorem 2.3, and Theorem 2.4(b) on the convergence of NGD.

**Proof of Lemma 5.1.** We use the mirror-descent form (2.4) for the NGD algorithm, and adapt the argument of [LFN18, Theorem 3.1].
Recall the form of $\nabla^2 \mathcal{F}_{\text{TAP}}(m)$ in (4.16), and note also that $\nabla^2(-H(m)) = n^{-1}\text{diag}(1/(1-m^2))$. When $\|W\|_{\text{op}} < 3$, we then have

$$\nabla^2 \mathcal{F}_{\text{TAP}}(m) < C \cdot \nabla^2(-H(m)), \quad \nabla^2 \mathcal{F}_{\text{TAP}}(m) > \nabla^2(-H(m)) - C' \cdot I$$

for some $\lambda$-dependent constants $C, C' > 0$. For $m \in (-1, 1)^n \cap \mathcal{B}_{\sqrt{\epsilon n}}(m_*)$, we have also $n \cdot \nabla^2 \mathcal{F}_{\text{TAP}}(m) > t$ by assumption, so taking a suitable linear combination of these two lower bounds yields

$$\nabla^2 \mathcal{F}_{\text{TAP}}(m) > \mu \cdot \nabla^2(-H(m))$$

for a $\lambda$-dependent constant $\mu > 0$. Then by [LFN18, Proposition 1.1], these imply the relative strong smoothness

$$\mathcal{F}_{\text{TAP}}(m) \leq \mathcal{F}_{\text{TAP}}(m') + \nabla \mathcal{F}_{\text{TAP}}(m')^T(m - m') + C \cdot D_H(m, m')$$

for all $m, m' \in (-1, 1)^n$ (C.9)

and the relative strong convexity

$$\mathcal{F}_{\text{TAP}}(m) \geq \mathcal{F}_{\text{TAP}}(m') + \nabla \mathcal{F}_{\text{TAP}}(m')^T(m - m') + \mu \cdot D_H(m, m')$$

for all $m, m' \in (-1, 1)^n \cap \mathcal{B}_{\sqrt{\epsilon n}}(m_*)$ (C.10).

Let us choose the inverse step-size $L = 1/\rho$ to be larger than this constant $C$ in (C.9). By [LFN18, Lemma 3.1], the minimizer $m^{k+1}$ of (2.4) satisfies the three-point inequality

$$\nabla \mathcal{F}_{\text{TAP}}(m^k)^T m^{k+1} + LD_H(m^{k+1}, m^k) + LD_H(m, m^k)$$

for any $m \in (-1, 1)^n$. Then, applying (C.9) and this inequality,

$$\mathcal{F}_{\text{TAP}}(m^{k+1})$$

$$\leq \mathcal{F}_{\text{TAP}}(m^k) + \nabla \mathcal{F}_{\text{TAP}}(m^k)^T(m^{k+1} - m^k) + LD_H(m^{k+1}, m^k)$$

$$\leq \mathcal{F}_{\text{TAP}}(m^k) + \nabla \mathcal{F}_{\text{TAP}}(m^k)^T(m - m^k) + LD_H(m, m^k) - LD_H(m, m^{k+1})$$

Taking $m = m^k$ shows in particular that $\mathcal{F}_{\text{TAP}}(m^{k+1}) \leq \mathcal{F}_{\text{TAP}}(m^k)$.

Next, we show that for sufficiently large $L$, every iterate $m^k$ satisfies

$$\|m^k - m_*\| < \sqrt{\epsilon n}, \quad \mathcal{F}_{\text{TAP}}(m^k) < \mathcal{F}_{\text{TAP}}(m_*) + t\epsilon/8.$$  (C.12)

We induct on $k$, where the base case $k = 0$ holds by assumption. Suppose that (C.12) holds for $k$. Then the above shows $\mathcal{F}_{\text{TAP}}(m^{k+1}) \leq \mathcal{F}_{\text{TAP}}(m^k) < \mathcal{F}_{\text{TAP}}(m_*) + t\epsilon/8$. Observe that for any $\rho < \epsilon$, by the strong convexity $\nabla^2 \mathcal{F}_{\text{TAP}}(m) > (t/n)I$ for $m \in (-1, 1)^n \cap \mathcal{B}_{\sqrt{\epsilon n}}(m_*)$, we have the implication

$$m \in (-1, 1)^n \cap \mathcal{B}_{\sqrt{\epsilon n}}(m_*) \text{ and } \mathcal{F}_{\text{TAP}}(m) < \mathcal{F}_{\text{TAP}}(m_*) + t\rho/2$$

$$\Rightarrow \|m - m_*\| < \sqrt{\epsilon n}.$$  (C.13)

So (C.12) in fact implies $\|m^k - m_*\|_2 < \sqrt{\epsilon n}/2$. Comparing the value of (2.4) at $m = m^k$ and at the minimizer $m = m^{k+1}$, we have

$$\nabla \mathcal{F}_{\text{TAP}}(m^k)^T (m^{k+1} - m^k) + LD_H(m^{k+1}, m^k) \leq 0.$$  

From the definition of the Bregman divergence $D_H(m, m')$ in (2.3), we have

$$\left|\nabla \mathcal{F}_{\text{TAP}}(m^k)^T (m^{k+1} - m^k) + D_H(m^{k+1}, m^k)\right|$$

$$= \left|\left(\nabla \mathcal{F}_{\text{TAP}}(m^k) + \nabla H(m^k)\right)^T (m^{k+1} - m^k) - H(m^{k+1}) + H(m^k)\right|$$

$$\leq C\left(\|m^{k+1} - m^k\|_2 + 1\right).$$

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Also, taking $L > 1$ and applying the strong convexity $\nabla^2(-H(m)) > (1/n)I$, we have

$$(L - 1)D_H(m^{k+1}, m^k) \geq \frac{L - 1}{2n} \|m^{k+1} - m^k\|^2_2.$$ 

Combining these bounds,

$$0 \geq \frac{L - 1}{2n} \|m^{k+1} - m^k\|^2_2 - \frac{C}{\sqrt{n}} \|m^{k+1} - m^k\|^2_2 - C$$

which implies

$$\|m^{k+1} - m^k\|^2_2 < \frac{C'}{\sqrt{L - 1}} \cdot \sqrt{n}.$$ 

for a $\lambda$-dependent constant $C' > 0$. Then, taking $L$ large enough so that $C' / \sqrt{L - 1} < \sqrt{\varepsilon}/2$, we obtain

$$\|m^{k+1} - m_*\|^2_2 \leq \|m^{k+1} - m^k\|^2_2 + \|m^k - m_*\|^2_2 < \sqrt{\varepsilon n},$$

completing the induction and the proof of (C.12).

The first statement of (C.12) allows us to apply the relative strong convexity (C.10) at $m^k$ and $m_*$, to obtain

$$\mathcal{F}_{\text{TAP}}(m^k) + \nabla \mathcal{F}_{\text{TAP}}(m^k)^\top (m_* - m^k) \leq \mathcal{F}_{\text{TAP}}(m_*) - \mu \cdot D_H(m_*, m^k).$$

Applying this bound to (C.11) now with $m = m_*$,

$$\mathcal{F}_{\text{TAP}}(m^{k+1}) \leq \mathcal{F}_{\text{TAP}}(m_*) + (L - \mu) D_H(m_*, m^k) - L D_H(m_*, m^{k+1}).$$

Multiplying by $(L - \mu)^{k+1}$ and summing over $k$ to telescope the sums of the last two terms,

$$\sum_{k=0}^{k-1} \left(\frac{L}{L - \mu}\right)^{j+1} \mathcal{F}_{\text{TAP}}(m^{j+1}) \leq \sum_{j=0}^{k-1} \left(\frac{L}{L - \mu}\right)^{j+1} \mathcal{F}_{\text{TAP}}(m_*) + L D_H(m_*, m^0).$$

Now applying $\mathcal{F}_{\text{TAP}}(m^{j+1}) \geq \mathcal{F}_{\text{TAP}}(m^j)$ for all $j \leq k - 1$ to the left side, we obtain

$$\mathcal{F}_{\text{TAP}}(m^k) \leq \mathcal{F}_{\text{TAP}}(m_*) + L \left(\sum_{j=0}^{k-1} \left(\frac{L}{L - \mu}\right)^{j+1}\right)^{-1} D_H(m_*, m^0).$$

From the form of $D_H(m_*, m^0)$ in (2.3), on the event $\|W\|_{op} < 3$, we have

$$D_H(m_*, m^0) < C \left(1 + \frac{\|\arctanh(m^0)\|_2}{\sqrt{n}}\right)$$

for a $\lambda$-dependent constant $C > 0$. Then the above shows

$$\mathcal{F}_{\text{TAP}}(m^k) \leq \mathcal{F}_{\text{TAP}}(m_*) + L \left(\frac{L - \mu}{L}\right)^k \cdot C \left(1 + \frac{\|\arctanh(m^0)\|_2}{\sqrt{n}}\right).$$

Identifying $(L - \mu)/L = 1 - \mu\eta$, this shows (5.2). Combining this with (C.12) and (C.13) shows (5.3).

\textbf{Proof of Theorem 2.3.} Suppose the initialization has sign $\langle x, h^0 \rangle \geq 0$. By Theorem 2.1(b), for some $\varepsilon, t > 0$, we have with probability approaching 1 that $\lambda_{\min}(n \cdot \nabla^2 \mathcal{F}_{\text{TAP}}(m)) > t$ for all $m \in (-1, 1)^n \cap B_{\sqrt{n}}(m_*)$.

By Lemma A.7, for a sufficiently large iteration $T$, the iterate $m_T$ of AMP will satisfy

$$\mathcal{F}_{\text{TAP}}(m_T) < \varepsilon_* + t\varepsilon/16, \quad \frac{1}{n^2} \|\hat{X}_{\text{Bayes}} - m_T m_T^\top\|^2_F < \varepsilon^4/8, \quad \frac{\|\arctanh(m_T)\|^2_2}{n} < C$$

for a $\lambda$-dependent constant $C > 0$. From Corollary 4.4 and Theorem 2.1(a), the local minimizer $m_*$ satisfies

$$\mathcal{F}_{\text{TAP}}(m_*) > \varepsilon_* - t\varepsilon/16, \quad \frac{1}{n^2} \|\hat{X}_{\text{Bayes}} - m_* m_*^\top\|^2_F < \varepsilon^4/8.$$
Therefore $F_{\text{TAP}}(m^T) < F_{\text{TAP}}(m_*) + t \epsilon/8$. Recalling (B.33),
\[
\frac{1}{n^2} \|m^T - m_*\|^2_2 \leq \frac{2}{n^2} \|m^T(m^T)^T - m_*m_*^T\|^2_F < \epsilon^2.
\]
This verifies that with high probability, the conditions of Lemma 5.1 hold for initializing NGD at this AMP iterate $m^T$, so the theorem follows from Lemma 5.1. If instead $\langle x, h^0 \rangle < 0$, then the theorem holds with sign $-m_*$ by sign symmetry. □

**Proof of Theorem 2.4(b).** Suppose the initialization has sign $\langle x, h^0 \rangle \geq 0$. For $m^0 = \tanh(h^0)$, applying Lemma A.7(a) with $k = 0$ and $\psi(x, y) = x \tanh(y)$, we have almost surely
\[
\lim_{n \to \infty} \langle x, m^0 \rangle/n = \mathbb{E}_{G \sim \mathcal{N}(0,1)}[\tanh(\lambda^2 - 1 + \sqrt{\lambda^2 - 1} \cdot G)].
\]
For large $\lambda$, applying monotonicity of $\tanh$ and the bounds $\mathbb{P}[G < -t] < e^{-t^2/2}$ and $\mathbb{P}[G > -\lambda/2] > 1 - 3e^{-\lambda^2/8}$, we may bound this by
\[
\mathbb{E}_{G \sim \mathcal{N}(0,1)}[\tanh(\lambda^2 - 1 + \sqrt{\lambda^2 - 1} \cdot G)] > \tanh(\lambda^2/3)\mathbb{P}[G \geq -\lambda/2] - \mathbb{P}[G < -\lambda/2] > 1 - 3e^{-\lambda^2/8}.
\]
Then for any $\epsilon > 0$, with probability approaching 1, $\|x - m_0\|^2_2/n \leq 2 - 2\langle x, m^0 \rangle/n < 6e^{-\lambda^2/8} + \epsilon$. Similarly by Corollary 4.4 and Proposition A.3, also with probability approaching 1, $\|x - m_*\|^2_2/n \leq 2 - 2q_* + \epsilon < 2e^{-\lambda^2/8} + \epsilon$. Choosing $\epsilon = e^{-\lambda^2/18}$ and combining these bounds, for all sufficiently large $\lambda$,
\[
\frac{\|m^0 - m_*\|^2_2}{n} \leq 2 \frac{\|x - m^0\|^2_2}{n} + 2 \frac{\|x - m_*\|^2_2}{n} < e^{-\lambda^2/9}.
\]
Choosing $\epsilon = e^{-\lambda^2/18}$ and combining these bounds, for all sufficiently large $\lambda$,
\[
\frac{\|m^0 - m_*\|^2_2}{n} < 2e^{-\lambda^2/36}.
\]
Then applying Proposition A.1 for sufficiently large $\lambda$, with probability approaching 1,
\[
F_{\text{TAP}}(m^0) - F_{\text{TAP}}(m_*) < 2e^{-\lambda^2/36}.
\]
On the event where (C.14) and (C.15) both hold and where the conclusion of Lemma 5.2(b) holds for $a = 5$, for sufficiently large $\lambda$, this initialization $m^0$ satisfies the conditions of Lemma 5.1 with $\epsilon = e^{-\lambda^2/18}$ and $t = 1/2$. Then Theorem 2.4(b) holds by Lemma 5.1 and the initial condition $\|\text{arctanh}(m^0)\|_2/\sqrt{n} = \|h^0\|_2/\sqrt{n} \to \sqrt{\lambda^2(\lambda^2 - 1)}$. If instead $\langle x, h^0 \rangle < 0$, then Theorem 2.4(b) holds with $-m_*$ by sign symmetry. □

### C.3 Analysis of AMP

We prove Lemma 5.3 on the contractivity of the AMP map, and Theorem 2.4(a) on the convergence of AMP. In Remark C.3, we provide more motivation for the reparameterization we have chosen. This remark can be read prior to reading the proof.

**Proof of Lemma 5.3.** Denote $m_+ = \Lambda(p_+)$, $m = \Lambda(p)$, $m_- = \Lambda(p_-)$, and $D(m) = \text{diag}((1 - m^2)^{1/2})$. Then, applying the form of $dT_{\text{AMP}}$ in (4.21) and the identity $\text{d}m/\text{d}p = (1 - m^2)^{1/2}$,
\[
dT_{\text{AMP}}(p, p_-) = \begin{pmatrix}
D(m_+) \cdot [\lambda Y + 2\lambda^2 m_- m^T]/n & D(m) \\
0 & -D(m_+) \cdot \lambda^2 [1 - Q(m)] \cdot D(m_-)
\end{pmatrix}.
\]

We first use a crude bound on the operator norms of the upper blocks: Consider the event of probability $1 - e^{-cn}$ where $\|W\|_{op} < 3$. On this event, applying $\|D(m)\|_{op} \leq 1$ and $\|m\|_2, \|1\|_2 \leq \sqrt{n}$, we have
\[
\|D(m_+) \cdot [\lambda Y + 2\lambda^2 m_- m^T]/n \cdot D(m)\|_{op} < 3\lambda + 3\lambda^2,
\]
\[
\| -D(m_+) \cdot \lambda^2 [1 - Q(m)] \cdot D(m_-)\|_{op} \leq \lambda^2.
\]
Integrating these bounds along the linear path between \((\bm{p}, \bm{p}_-)\) and the AMP fixed point \((\bm{p}_*, \bm{p}_*)\) then yields
\[
\|\bm{p}_+ - \bm{p}_*\|_2 \leq (3\lambda + 3\lambda^2)\|\bm{p} - \bm{p}_*\|_2 + \lambda^2\|\bm{p}_--\bm{p}_*\|_2 < 7\lambda^{-5}\sqrt{n}
\] (C.16)
when \((\bm{p}, \bm{p}_- \in B_{\lambda^{-1}p}(\bm{p}_*)\). Since \(\bm{m} = \Lambda(\bm{p})\) is 1-Lipschitz, we then have \(|\bm{m} - \bm{m}_*\|_2, |\bm{m}_--\bm{m}_*\|_2 < \lambda^{-7}\sqrt{n} \) and \(|\bm{m}_+ - \bm{m}_*\|_2 < 7\lambda^{-5}\sqrt{n}\). Then, applying the assumption \(M(\bm{m}_*) > 1 - \lambda^{-5}\) so that \(Q(\bm{m}_*) \geq M(\bm{m}_*)^2 > 1 - 2\lambda^{-5}\), and applying also the continuity bound for \(Q(\bm{m})\) in Proposition A.1, this shows \(Q(\bm{m}_*), Q(\bm{m}), Q(\bm{m}_*) > 1 - 12\lambda^{-5}\) for all large sufficiently \(\lambda\).

Now applying \(Q(\bm{m}) > 1 - 12\lambda^{-5}\) above, we obtain the improved bound
\[
\left\| -D(\bm{m}_+) \cdot \lambda^2(1 - Q(\bm{m})) \cdot D(\bm{m}_-) \right\|_{op} < 12\lambda^{-3}.
\]
Also \(\|D(\bm{m})\|_2^2 = \sum_i (1 - m_i^2)^2 \leq n(1 - Q(\bm{m})) < 12\lambda^{-5}n\), and the same bounds hold for \(D(\bm{m}_+)\), \(D(\bm{m}_-), D(\bm{m})\). Then
\[
\left\| D(\bm{m}_+) \cdot [\lambda Y + 2\lambda^2\bm{m}_-\bm{m}^\top/n] \cdot D(\bm{m}) \right\|_{op} \leq \lambda\left\| D(\bm{m}_+) W D(\bm{m}) \right\|_{op} + 36\lambda^{-3}.
\]
On the event of probability \(1 - C\epsilon^{-cn}\) (for \(\lambda\)-dependent \(C, c > 0\)) where Lemma C.2 holds with \(\epsilon = \lambda^{-5/4}\) and \(s = 12\lambda^{-5/2}\), applying Markov's inequality and this lemma as in (C.8), we obtain for sufficiently large \(\lambda\) that
\[
\left\| D(\bm{m}_+) \cdot [\lambda Y + 2\lambda^2\bm{m}_-\bm{m}^\top/n] \cdot D(\bm{m}) \right\|_{op} < \lambda^{-1/5}.
\]
Finally, integrating again these improved bounds along the linear path from \((\bm{p}, \bm{p}_-)\) to \((\bm{p}_*, \bm{p}_*)\), we may obtain
\[
\|\bm{p}_+ - \bm{p}_*\|_2 \leq \lambda^{-1/5}\|\bm{p} - \bm{p}_*\|_2 + 2\lambda^{-2/5}\|\bm{p}_--\bm{p}_*\|_2.
\]
In particular, for large \(\lambda\) this implies \(\|\bm{p}_+ - \bm{p}_*\|_2 \leq \max(\|\bm{p} - \bm{p}_*\|_2, \|\bm{p}_--\bm{p}_*\|_2), \) so \(\bm{p}_+ \in B_{\lambda^{-1}p}(p_*) \cap \Omega(\bm{p})\). Adding \(\lambda^{-1/5}\|\bm{p} - \bm{p}_*\|_2\) to both sides above yields (5.4).

**Remark C.3.** To motivate the reparameterization by \(\bm{p}\), consider instead an analysis of the contractivity of the AMP map without this reparameterization. Denote \((\bm{m}_+, \bm{m}_-) = T_{\text{AMP}}(\bm{m}_-, \bm{m}_-). The Jacobian of \(T_{\text{AMP}}\) is given by
\[
dT_{\text{AMP}}(\bm{m}_+, \bm{m}_-) = \begin{pmatrix}
\text{diag}(1 - m_+^2) \cdot [\lambda Y + 2\lambda^2 m_- m^\top/n] & -\text{diag}(1 - m_+^2)\lambda^2(1 - Q(\bm{m}))
\end{pmatrix}.
\] (C.17)
We may use the key fact that \(1 - m_+^2\) is close to zero in most coordinates to show that most rows of the upper two blocks are small. This is not enough, however, to establish a strong operator norm bound for these blocks, and we would like to also have that most of the columns of the upper two blocks are small. The reparameterization by \(\bm{p}\) is chosen so that these two blocks become right-multiplied also by the small factors of \(\text{diag}(1 - m^2)^{1/2}\).

In more detail, for any reparameterization \(\bm{m} = \Lambda(\bm{p})\), denoting \(\bm{p}_+ = \Lambda(m_+)\) and \(\bm{p}_- = \Lambda(m_-),\) the upper two blocks of the Jacobian in this parametrization become
\[
\frac{\partial \bm{p}_+}{\partial \text{dim}_+} \text{diag}(1 - m_+^2) \cdot [\lambda Y + 2\lambda^2 m_- m^\top/n] \frac{d\text{dim}}{dp},
- \frac{\partial \bm{p}_+}{\partial \text{dim}_+} \text{diag}(1 - m_+^2) \lambda^2(1 - Q(\bm{m})) \frac{d\text{dim}_-}{dp}.
\]
Because \(\frac{\partial \bm{p}}{\partial \text{dim}} = \left(\frac{\partial \text{dim}}{\partial \bm{p}}\right)^{-1}\), we see that a natural choice of reparameterization is to ensure \(\frac{\partial \bm{p}}{\partial \text{dim}} = 1/\sqrt{1 - \bm{m}^2}\), which holds exactly for our definition of \(\bm{p}\).

**Proof of Theorem 2.4(a).** Suppose \(\bm{x} = 1\) and \(\langle \bm{x}, \bm{h}^0 \rangle \geq 0\). By Corollary 4.4 and Proposition A.3, for all sufficiently large \(\lambda\), with probability approaching 1, \(\bm{m}_* \in M_{1-\lambda^{-5}}\). Also from the proof of Theorem 2.1(b) and the bound (B.33), for any \(\delta > 0\) sufficiently small and \(k = k(\delta)\) sufficiently large (where \(\text{"sufficiently}}
small” and “sufficiently large” depend on $\lambda$), with probability approaching 1, the AMP iterates $m^{k-1}$ and $m^k$ satisfy

$$\frac{1}{n} \|m^k - m_*\|_2 < \delta \sqrt{n}, \quad \frac{1}{n} \|m^{k-1} - m_*\|_2 < \delta \sqrt{n}.$$  

Because $\Gamma$ is continuous on the compact domain $[-1, 1]$, it has a modulus of continuity: $|\Gamma(m) - \Gamma(m')| < \epsilon(\Delta)$ whenever $|m - m'| < \Delta$, where $\epsilon(\Delta) \to 0$ as $\Delta \to 0$. Then by Markov’s inequality and the fact that $p^k, p_* \in \Omega(p) = (-\pi/2, \pi/2)^n$,

$$\frac{1}{n} \|p^k - p_*\|_2^2 < \epsilon(\sqrt{\delta})^2 + \pi^2 \cdot \frac{1}{n} \{i : |m^k_i - m_*| \geq \delta\} < \epsilon(\sqrt{\delta})^2 + \pi^2 \delta.$$  

The same applies to $p^{k-1}$. Thus, choosing $\delta$ sufficiently small, we ensure $p^k, p^{k-1} \in \Omega(p) \cap B_{\lambda-\gamma, \sqrt{n}}(p_*)$. Then on the event where also Lemma 5.3 holds, we conclude

$$\|p^{k+r} - p_*\|_2 \leq \|p^{k+r} - p^k, p^{k+r} - (p_*), p_*\|_2 \leq (2\lambda^{-1/5})^r \|p^k, p^{k-1} - (p_*), p_*\|_2 \leq (2\lambda^{-1/5})^r \lambda^{-7} \sqrt{n}. \quad (C.18)$$

Noting that $k$ is a $\lambda$-dependent constant, and choosing $\lambda$-dependent constants $C, \alpha > 0$ to also account for the first $k$ iterations, we obtain for every $r \geq 1$ that

$$\|m^r - m_*\|_2 < C\alpha^r \sqrt{n}.$$  

Then also with high probability, for modified constants $C, \alpha > 0$, we have $\mathcal{F}_{\text{TAP}}(m^k) - \mathcal{F}_{\text{TAP}}(m_*) < C\alpha^k$ by Proposition A.1. If $\langle x, h^0 \rangle < 0$, the same statements hold with $-m_*$ by sign symmetry.  

## D Numerical evaluation of eigenvalues of the linearized AMP operator

Theorem 2.1 shows that, for any $\lambda > 1$, the spectral radius of the Jacobian of the AMP map $dT_{\text{AMP}}(m_*, m_*)$ will be bounded away from 1 with high probability. Figure 5 shows a scatter plot of all eigenvalues of $dT_{\text{AMP}}(m_*, m_*)$ for a specific instance $Y$, with $n = 500$ and $\lambda = 1.5$.  

![Figure 5: The scatter plot of eigenvalues of the linearized AMP operator $dT_{\text{AMP}}(m_*, m_*)$. We choose $n = 500$ and $\lambda = 1.5$.](image-url)