An Application of Interval Valued Fuzzy Matrix in Modeling Clinical Waste Incineration Process

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Abstract. This paper exploits an interval valued fuzzy matrix in representing combustion process of clinical waste incineration system which previously modelled by using Fuzzy Autocatalytic Set (FACS). We defined a membership degree of interval-valued fuzzy edge connectivity using concept of infimum and supremum and proposed a mapping from FACS of clinical waste incineration process to an interval valued fuzzy matrix of FACS. Basic characteristics of the matrix are observed and some proven theorems are derived.

1. Introduction

Most of physical quantities involved in our real life situations are not necessarily perfect, deterministic and crisp in value due to the existence of uncertainties element in the data collection. Due to this, Interval Valued Fuzzy matrix (IVFM) is developed to deals with the vagueness and ambiguities present in most of our daily life problems [1]. The parameterization tool of IVFM enhances the flexibility of its applications. The concept of IVFM as a simplification of fuzzy matrix was established and presented in [2] by broadening the max-min operations on fuzzy algebra.

On the other hand, Fuzzy Autocatalytic Set (FACS) is developed through combination of both theory of fuzzy graph and Autocatalytic Set. It is described as a subgraph in which each node has minimum one incoming link with membership value $\mu(e_i) \in (0,1], \forall e_i \in E$ [3, 4]. First application of the concept is in modelling a process in the clinical waste incinerator [3] and later, FACS had also been used to explain the process of combustion in circulating fluidized bed boiler [5, 6, 7] and pressurized water reactor [8]. The models of those applications are in a form of directed graph in which the nodes represent significant variables namely chemical compounds involved in the process and the edges connecting the nodes representing the strength of connection between the variables (refer Figure 1). In the case of incineration process, fuzzy graph type-3 which has identified nodes and edges, but unidentified edge connectivity was chosen to best describe the nature of the process. The construction of FACS of fuzzy graph type-3 involving fuzzy head $h(e_i)$, fuzzy tail $t(e_i)$ and fuzzy edge connectivity of the edge $e_i$. The fuzzy tail $t(e_i) = 1, \forall e_i \in E$, is due to the fact that each variable is taken as a whole before it changed to other variables. The fuzzy head, $h: E \rightarrow [0,1]$ is based on the reaction taken place and the strength of relation to the other variables in the system. Following from this definition of fuzzy head and tail leads to the next definition on fuzzy edge connectivity and adjacency matrix of FACS as follows:
**Definition 1:** Fuzzy edge connectivity $C$ for $e_i \in E$ denoted as $C(e_i)$ is a tuple of $(t(e_i), h(e_i))$. Let $C$ be the set of all ordered pairs of fuzzy edge connectivity defined by $C = \{ (t(e_i), h(e_i)) : e_i \in E \}$ and let $\mu(e_i)$ denoted the membership value for fuzzy edge connectivity for each edge $i$. Then, the membership value for fuzzy edge connectivity is defined as $\mu(e_i) = \min \{ t(e_i), h(e_i) \}$. [3]

**Definition 2:** Let $C_F$ be the adjacency matrix of FACS, then the matrix is defined as

$$C_F = \begin{cases} 0 & \text{for } i = j \text{ and } e_i \neq 0 \\ \mu(e_i) & i \neq j \end{cases}$$

The concept is then applied to model the clinical waste incineration process [3] and the model is represented as follows:

![Fig.1. FACS of fuzzy graph type-3 for clinical waste incineration process](image)

Figure 1 shows six variables that play vital roles in the process which are waste ($v_1$), fuel ($v_2$), Oxygen ($v_3$), Carbon Dioxide ($v_4$), Carbon Monoxide ($v_5$) and other gases including water ($v_6$). The nodes in the graph correspond to the variables and a directed link from node $i$ to node $j$ indicates that variable $i$ catalyzes the production of variable $j$. The five different colors signify different range of membership value for the fuzzy edge connectivity which was abstracted from the chemical reaction relations between the variables and subsequently constitute to the element of the adjacency matrix $C_F$ of the graph as follows:

$$C_F = \begin{bmatrix} 0 & 0 & 0.06529 & 0 & 0 & 0.13401 \\ 0.00001 & 0 & 0 & 0 & 0 & 0 \\ 0.15615 & 0 & 0 & 0 & 0 & 0 \\ 0.51632 & 0.68004 & 0.63563 & 0 & 0.99999 & 0 \\ 0.00001 & 0.00001 & 0.00002 & 0 & 0 & 0 \\ 0.32752 & 0.31995 & 0.29906 & 0.00001 & 0 & 0 \end{bmatrix}$$

From the matrix, the higher the value of connectivity $C_F$, the thicker the link between vertices $i$ and $j$ in the graph. Further investigation on the evolution of the variables for longer period is discussed in [3] using Perron-Frobenius eigenvector and in [5] using left Perron vector of transition matrix in which both methods give information on by-product of the incineration process which are $CO_2$ and $H_2O$ with other pollutants and waste and $H_2O$ with other pollutants respectively. However, since no physical quantity can be measured with perfect certainty and the measurement of certain quantity is always differs over time, therefore further investigation of other representation of FACS is explored. This paper reports the investigation of the concept of interval value fuzzy matrix (IVFM) in representing FACS.

2. **Interval Value Fuzzy Matrix**

Fuzzy matrix is a matrix with elements having values in the closed interval $[0, 1]$. A number of researches had been done on fuzzy matrices and its variant [9, 10, 11]. According to Anita and Madhumangal [12] the membership value is well known to be dependable to mentality of the expert or evaluator and its habits. Due to that reason, sometimes it happens that the membership value cannot be measured as a point, but as an interval. Furthermore, sometimes it becomes impossible to measured due to the rapid variation of the characteristics of
the system whose membership value are to be determined. Recently, the development of the concept of Interval-Valued Fuzzy Matrix as a simplification of fuzzy matrix was established by Shyam and Pal [2] involving the extension of max-min operation on fuzzy algebra, $F = [0,1]$ for element $a$ and $b$ in $F$, and $a \cdot b = \min\{a, b\}$. The following definitions related to Interval-Valued Fuzzy Matrix which are exploited in the investigation are listed as follow:

**Definition 3:** An interval-valued fuzzy matrix of order $m \times n$ is defined as $A = \{a_{ij}\}_{m \times n}$, where $a_{ij} = [a_{ij}^-, a_{ij}^+]$ is the $ij$th element of $A$, which represents the membership value. All the element of an interval-valued fuzzy matrix are in a form of interval and they are members of $D$ where $D$ denote the set of all subintervals of the interval $[0, 1]$. [13]

**Definition 4:** If all the elements are zero then an interval-valued fuzzy matrix is said to be a null interval-valued fuzzy matrix. [12]

**Definition 5:** Let $A$ be a square interval-valued fuzzy matrix of order $n$. If all the diagonal entries of $A$ are $[1, 0]$ and all other entries are $[0, 0]$ then $A$ is an identity interval-valued fuzzy matrix. [12]

**Definition 6:** An $n \times n$ interval-valued fuzzy matrix $A$ is said to be,

- i) Reflexive iff $A \geq I_n$.
- ii) Symmetric iff $A = A'$.
- iii) Transitive iff $A^2 \leq A$.
- iv) Indempotent iff $A^2 \leq A$. [12]

**Definition 7:** Let $(a_{ij})$ be a square interval-valued fuzzy matrix or order $n$ where $a = [a_{ij}^-, a_{ij}^+]$. If there exist an integer $m$ such that $A^{m+1} = A^m$ holds, then the power of an interval-valued fuzzy matrix is said to be converge. Generally, an interval-valued fuzzy matrix is said to converge when its powers converge. [12]

**Definition 8:** An $n \times n$ Interval-Valued Fuzzy Matrix $A$ is said to be a constant if all its rows are equal to each other, i.e., if $a_{ik} = a_{jk}$ and $a_{ik} = a_{jk}$ for all $i, j, k$. [12]

**Definition 9:** Suppose that $A \subset R$ is a set of real numbers. If $M \in R$ is an upper bound of $A$ such that $M \leq M'$ for every upper bound $M$ of $A$, then $M$ is called the supremum of $A$, denoted as $M = \sup A$. If $m \in R$ is a lower bound of $A$ such that $m \geq m'$ for every lower bound $m'$ of $A$, then $m$ is called the infimum of $A$, denoted as $M = \inf A$. [12]

### 3. Development of Interval Valued Fuzzy Matrix of Fuzzy Autocatalytic Set

Representation of Fuzzy Autocatalytic Set concept is modified and transformed into interval-valued fuzzy matrix according to the concept of interval-valued fuzzy matrix introduced by Anita and Madhumangal [12]. Hence, some definitions are established as follow:

**Definition 10:** Interval Value of fuzzy head and tail
Let $h(e_i)$ be the interval-valued fuzzy head and $t(e_i)$ be the interval-valued fuzzy tail $\forall e_i \in E$. Then, interval-valued fuzzy head and tail of Fuzzy Autocatalytic Set is defined as $t(e_i) = [1, 1] \forall e_i \in E$ and $h(e_i) = [m - \alpha, m + \alpha]$ where the value of $m$ is the initial membership value of $e_i$ abstracted from the element of adjacency matrix of Fuzzy Autocatalytic Set $\forall e_i \in E$ and $\alpha$ is the small value represent the uncertainty of the $m$ where $|\alpha| \leq m$.

**Definition 11:** Interval value of fuzzy edge connectivity
Let Interval-valued fuzzy edge connectivity, $C_f$ for $\forall e_i \in E$ is denoted as $C_f(e_i)$ which is a tuple of $(t(e_i), h(e_i))$ where $t(e_i)$ and $h(e_i)$ are interval-valued fuzzy tail and head of FACS respectively. Thus, the set of all ordered pairs of $C_f$ is given by $C_f = \{(t(e_i), h(e_i)) : e_i \in E\}$.

Next, the membership degree of interval-valued fuzzy edge connectivity of Fuzzy Autocatalytic Set is defined.
Definition 1: Membership value of Interval value fuzzy edge connectivity

Let $\mu(e_i)$ be membership value for interval value fuzzy edge connectivity for each $e_i \in E$. Let $L_i = \min[\inf(t(e_i), h(e_i))]$ and $U_i = \min[\sup(t(e_i), h(e_i))]$ for every $e_i \in E$. Then, $\mu(e_i)$ is defined as $\mu(e_i) = [L_i, U_i]$ for $\forall e_i \in E$.

Now that FACS can be define as a subgraph where each of whose node has at least one incoming link with membership value $\mu(e_i)$ where $\mu(e_i) \subseteq [0,1]$ from a node belonging to the same graph and the interval valued fuzzy matrix of FACS is defined as in Definition 12.

Definition 12: Interval value fuzzy matrix of FACS

Let $C_{IFc}$ denoted an interval-valued fuzzy matrix of Fuzzy Autocatalytic Set. Then $C_{IFc}$ is defined as,

$$C_{IFc} = \begin{cases} [0,0] & \text{for } i = j \text{ and } e \notin E \\ \mu(e_i) & \text{for } i \neq j \text{ and } e_i \in E \end{cases}$$

where $\mu(e_i)$ is the membership degree of interval-valued for fuzzy edge connectivity of FACS and $\mu(e_i) \subseteq [0,1]$.

Let consider one example on how to illustrate the definitions mentioned above. Let $h(e_i)$ be the interval-valued fuzzy head and $t(e_i)$ be the interval-valued fuzzy tail for edge $e_i$. Suppose the initial membership value $m$ for edge $e_i$ is assumed as 0.06529. Based on Definition 9 and let $\alpha$ be 0.001 where $|\alpha| < m$. Therefore, the value of $t(e_i)$ and $h(e_i)$ are $[1,1]$ and $[0.06429,0.06629]$ respectively.

Then, suppose $M$ denoted the set of the greatest lower bound of interval $t(e_i)$ and $h(e_i)$ while $S$ is the set of the least upper bound of interval $t(e_i)$ and $h(e_i)$. Then, set $M = \inf[t(e_i), h(e_i)] = [1,0.06429]$ and set $S = \sup[t(e_i), h(e_i)] = [1,0.06629]$. According to the Definition 11, the membership degree of interval value fuzzy edge connectivity $\mu(e_i)$ is defined as $\mu(e_i) = [L_i, U_i]$ for $\forall e_i \in E$ where $L_i$ and $U_i$ are minimum element from set $M$ and set $S$ respectively. In this case, $L_i = \min[\inf(t(e_i), h(e_i))] = \min[1, 0.06429] = 0.06429$ and $U_i = \min[\sup(t(e_i), h(e_i))] = \min[1, 0.06629] = 0.06629$.

Hence, $\mu(e_i) = [L_i, U_i] = [0.06429,0.06629]$ which also equal to $h(e_i)$ that eventually contributes to the element of interval value fuzzy matrix of FACS.

Next, the interval-valued fuzzy matrix of Fuzzy Autocatalytic Set is implemented to represents FACS of clinical waste incineration process and the basic characteristics of the fuzzy matrix are observed.

4. Implementation and Discussion

They should also be separated from the surrounding text by one space. FACS of clinical waste incineration process as in Figure 1 which is represented by the adjacency matrix as in Eq (1) is explored. The membership values abstracted from Eq. (1) are transformed into interval-valued based on Definition 10. Here, let $\alpha = 0.00005$ be the uncertainty for the $m(e_i)$ where $|\alpha| < m(e_i)$ for each edges in graph in Figure 1. The definition of membership degree of interval-valued fuzzy edge connectivity of Fuzzy Autocatalytic Set, $\mu(e_i)$ is $\mu(e_i) = h(e_i) \forall e_i \in E$. Hence, the interval-valued fuzzy matrix of Fuzzy Autocatalytic Set for the clinical waste incineration process is presented as follows:

| Column 1 to 3 | 0.065285,0.065295 | (0.0,0) | [0.0] |
|---------------|-------------------|--------|-------|
|                | 0.065285,0.065295 | 0.000005,0.000015 | [0.0] |
| 0.065285,0.065295 | 0.000005,0.000015 | 0.000005,0.000015 | [0.0] |
| 0.065285,0.065295 | 0.000005,0.000015 | 0.000005,0.000015 | [0.0] |
| 0.065285,0.065295 | 0.000005,0.000015 | 0.000005,0.000015 | [0.0] |
All the entries are membership values which are in interval form, \( \mu(e_i) \subseteq [0, 1] \), therefore the interval-valued fuzzy matrix of Fuzzy Autocatalytic Set serves as an alternative view in describing the clinical waste incineration process. Further observations on the characteristics of the interval value fuzzy matrix of FACS are as follow:

- It is nonnegative interval-valued fuzzy matrix. The \( \alpha \) value is always less than the membership value of FACS, \( m(e_i) \) where \( |\alpha| < m(e_i) \). Hence, it is impossible for the entries of the matrix to be negative. Moreover, the intervals values in the matrix signify the connectivity between variables involved in the process are nonnegative.

- It is non-symmetrical interval-valued fuzzy matrix. The membership interval value representing the connection between \( v_i \) to \( v_j \) becomes an entry for \( a_{ji} \) in the interval-valued fuzzy matrix. If there is a link between \( v_j \) to \( v_i \), it is not necessarily providing equivalent membership interval value as link from \( v_i \) to \( v_j \). Hence, \( a_{ij} \neq a_{ji} \). Noted that every connection in the matrix is associated to the concept of autocatalytic set.

- All the elements in the matrix are interval values, \( a_{ij} \subseteq [0, 1] \) which denotes as the membership values between two variables, thus the matrix represent interval-valued fuzzy matrix.

- Zeros interval represent no connection between variables.

- All the diagonal entries, \( a_{ii} = [0, 0] \) since no loop is permitted in the graph.

- At least one entry of each column is not zero intervals indicates that there exist at least one out-going link for each and every node of interval-valued Fuzzy Autocatalytic Set, which imply that the graph is autocatalytic.

- It is not reflexive interval-valued fuzzy matrix since the diagonal entries of the interval-valued fuzzy matrix are always zeros interval which is less than the diagonal entries of identity interval-valued fuzzy matrix.

- It is not constant interval-valued fuzzy matrix since the rows are always not equal to each other. It is impossible due to the zero intervals in diagonal entries.

- There exist an integer \( k \) such that \( (a^k)_{ii} > 0 \), where \( a_{ij} = [a_{ij}, a_{ji}] \) for every interval-valued fuzzy matrix of Fuzzy Autocatalytic Set.

Transformation of the graph in Figure 1 to the interval value fuzzy matrix of FACS which is believed to have the above characteristics is therefore established as follows:

**Theorem 1:**
Let \( G_{kFAS}(V, E) \) be a no loop fuzzy graph which is autocatalytic; i.e interval-valued Fuzzy Autocatalytic Set defined as

\[
G_{FAS} = \begin{cases} 
[0, 0] & \text{when } i = j \text{ and } e_i \notin E \\
\mu(e_i) \subseteq [0, 1] & \text{when } i \neq j \text{ and } e_i \in E
\end{cases}
\]

for \( k = 1, 2, \ldots, n \).

Let \( G_{FAS} = \{G_{kFAS}, k = 1, 2, \ldots, n\} \) be the finite set of all fuzzy graph of FACS.

Let \( M_{wa} = \{a_i\}, a_i \subseteq [0, 1] \) with \( a_i = [0, 0] \) and define \( \theta : G_{FAS} \rightarrow M_{wa} \rightarrow \theta(G_{FAS}) = [a_i] \) where \( \theta : G_{FAS} \rightarrow M_{wa} \) is an onto and one-to-one function.
The proof of the theorem is omitted. From here, any no loop of Fuzzy Autocatalytic Set can be mapped to an interval value fuzzy matrix of FACS.

**Theorem 2:**
Any FACS of fuzzy graph type-3 \( G_{T,3}(V,E) \) can be transformed to interval-valued fuzzy matrix of Fuzzy Autocatalytic Set.

**Proof:**
Suppose \( G_{T,3}(V,E) \) is a FACS of fuzzy graph type-3 with \( n \) vertices. Then, define the \( \alpha \) to be the uncertainties of the membership values of FACS and both fuzzy head, \( h(e_i) \) and fuzzy tail, \( t(e_i) \) of FACS is transformed into interval values fuzzy set by using Definition 10. Next, the interval-valued fuzzy edge connectivity of of FACS is constructed based on the Definition 11. Then, by using Definition 12, the membership degree of interval-valued fuzzy edge connectivity of FACS \( \mu(e_i) \) for each edges in the graph are obtained where \( \mu(e_i) = h(e_i) \). Subsequently, according to the Definition 13, the interval-valued fuzzy matrix of FACS is obtained. Finally, the FACS of fuzzy graph type-3 is transformed to interval-valued fuzzy matrix of FACS.

5. **Conclusion**
The development of interval-valued fuzzy matrix of Fuzzy Autocatalytic Set is successfully presented. This concept is then implemented into the FACS of clinical waste incineration process and basic characteristics for the matrix are observed and fundamental theorems are generated.

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