NON-AUTONOMOUS 2D NEWTON-BOUSSINESQ EQUATION
WITH OSCILLATING EXTERNAL FORCES
AND ITS UNIFORM ATTRACTOR

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ABSTRACT. We consider a non-autonomous two-dimensional Newton-Boussinesq equation with singularly oscillating external forces depending on a small parameter $\varepsilon$. We prove the existence of the uniform attractor $A^\varepsilon$ when the Prandtl number $Pr > 1$. Furthermore, under suitable translation-compactness and divergence type condition assumptions on the external forces, we obtain the uniform (with respect to $\varepsilon$) boundedness of the related uniform attractors $A^\varepsilon$ as well as the convergence of the attractor $A^\varepsilon$ to the attractor $A^0$ as $\varepsilon \to 0^+$.

1. Introduction. In this paper, we consider the following 2D Newton-Boussinesq equation

\[
\begin{aligned}
\partial_t \xi &+ u \partial_x \xi + v \partial_y \xi = \Delta \xi - \frac{Ra}{Pr} \partial_x \theta + f_0(x, y, t) + \varepsilon^{-p} f_1\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}, t\right), \\
\Delta \Psi &= \xi, u = \Psi_y, v = -\Psi_x, \\
\partial_t \theta &+ u \partial_x \theta + v \partial_y \theta = \frac{1}{Pr} \Delta \theta + g_0(x, y, t) + \varepsilon^{-p} g_1\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}, t\right),
\end{aligned}
\]

where $\vec{U} = (u, v)$ is the velocity vector of the fluid, $\theta$ is the flow temperature, $\Psi$ is the flow function, $\xi$ is the vortex. $(x, y) \in \Omega, \Omega \subseteq \mathbb{R}^2$ is an open bounded domain with sufficiently smooth boundary $\partial \Omega$. The positive constants $Pr$ and $Ra$ are the Prandtl number and the Rayleigh number, respectively. The Prandtl number is a dimensionless scalar in fluid mechanics. It reflects the mutual influence of the energy and momentum transfer processes in the fluid, and plays an important role in thermal calculations. In fluid mechanics, the Rayleigh number of a fluid is a

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dimensionless number related to buoyancy-driven convection. When the Rayleigh number of a certain fluid is lower than the critical value, the main form of heat transfer is heat conduction; when the Rayleigh number exceeds the critical value, the main form of heat transfer is convection.

Notice that system (1) can be rewritten as follows: for every \((x, y) \in \Omega\) and \(t > 0\),
\[
\begin{align*}
\frac{\partial \xi}{\partial t} - \Delta \xi + J(\Psi, \xi) + \frac{R_p}{\mathcal{R}} \frac{\partial \theta}{\partial t} &= f_0(x, y, t) + \varepsilon^{-\rho} f_1(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}, t), \\
\frac{\partial \theta}{\partial t} - \frac{1}{\mathcal{R}} \Delta \theta + J(\Psi, \theta) &= g_0(x, y, t) + \varepsilon^{-\rho} g_1(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}, t),
\end{align*}
\]
with the boundary conditions
\[\xi|_{\partial \Omega} = 0, \quad \theta|_{\partial \Omega} = 0, \quad \Psi|_{\partial \Omega} = 0,\]
and the initial conditions
\[\xi(x, y, \tau) = \xi_\tau(x, y), \quad \theta(x, y, \tau) = \theta_\tau(x, y),\]
where the function \(J\) is given by
\[J(a, b) = a \varepsilon b_2 - a_2 b_\varepsilon.\]

The functions
\[f_\varepsilon(x, y, t) = \begin{cases} f_0(x, y, t) + \varepsilon^{-\rho} f_1(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}, t), & 0 < \varepsilon \leq 1, \\
f_0(x, y, t), & \varepsilon = 0, \end{cases}\]
\[g_\varepsilon(x, y, t) = \begin{cases} g_0(x, y, t) + \varepsilon^{-\rho} g_1(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}, t), & 0 < \varepsilon \leq 1, \\
g_0(x, y, t), & \varepsilon = 0, \end{cases}\]
represent the external force terms.

It is easy to verify that the function \(J\) in (5) satisfies:
\[
\int\limits_\Omega J(u, v) \, dx \, dy = 0, \quad \text{for all } u \in H^1(\Omega), v \in H^2(\Omega) \cap H^1_0(\Omega),
\]
\[
\| J(u, v) \| \leq C \| u \|_{H^2} \| v \|_{H^2}, \quad \text{for all } u \in H^2(\Omega), v \in H^2(\Omega),
\]
\[
\| J(u, v) \| \leq C \| u \|_{H^2} \| \nabla v \|, \quad \text{for all } u \in H^3(\Omega), v \in H^1(\Omega).
\]

Throughout this paper, we frequently use the Poincaré inequality
\[\| u \| \leq \lambda \| \nabla u \|, \forall u \in H^1_0(\Omega),\]
where \(\lambda\) is a positive constant.

Natural convection is becoming an interesting and important topic because it is usually encountered in various natural and industrial processes, such as drying, combustion of gas mixtures, chemical reaction, thermal storage system and so on. Bénard convection is one typical natural convection driven by the buoyancy force in a fluid which is simultaneously heated and cooled from below and above, respectively. The Newton-Boussinesq equation describes the Bénard flow. There are some works concerning problem (1). For example, Guo studied the existence and uniqueness of weak solutions of two-dimensional Newton-Boussinesq equation by spectral method and Galerkin method in [15] and [14], respectively. In [13], Fucci et al. proved the existence of a global attractor in \(L^2(\Omega) \times L^2(\Omega)\) for the Newton-Boussinesq equation defined in a two-dimensional channel by the uniform estimates on the tails of solutions. In [12], Fang et al. considered a class of periodic initial value problems for two-dimensional Newton-Boussinesq equation. Using iterative method, they obtained the local existence of solution, and proved the global existence of solution by the method of a priori estimates. In [23], Song
et al. investigated a class of non-autonomous Newton-Boussinesq equation in two-dimensional bounded domains. They proved the existence of pullback attractors in $L^2(\Omega) \times L^2(\Omega)$ and $H^1(\Omega) \times H^1(\Omega)$. In [24], Song et al. studied the existence of $H^1_0(\Omega) \times H^1_0(\Omega)$ and $H^2(\Omega) \times H^2(\Omega)$-global attractors in two-dimensional bounded domains. Meanwhile, they obtained the existence of $H^1_0(\Omega) \times H^1_0(\Omega)$-global attractors in two-dimensional channels. In [20], Qiu et al. considered the two-dimensional Newton-Boussinesq equation with the incompressibility condition. They obtained a regularity criterion for the Newton-Boussinesq equation by virtue of the commutator estimate. In [18], Ma et al. investigated the global well-posedness for the 3D Newton-Boussinesq equations in a large class of non-decaying vorticity. With the help of the Fourier analysis and the coupling structure, they established the global-in-time estimate of vorticity in non-Lipschitz vector field. In [35], Wang et al. studied asymptotic autonomy of the kernel sections for Newton-Boussinesq equations on unbounded zonary domains. They showed that the forward compactness of the kernel sections for the process is a necessary and sufficient condition such that the kernel sections are attracted by the global attractor for the semigroup. And they obtained nonempty, uniformly bounded and forward compact kernel sections for the non-autonomous equation defined on an unbounded zonary domain and perturbed by longtime convergent forces.

Attractor is an important concept in the study of dynamical systems. There are many works concerning this subject, see, e.g., [2, 16, 17, 21, 22, 32, 34, 36, 37]. Stability of attractors for a dynamical system with some oscillating external forces is also important in natural phenomenon. Indeed, this issue has been considered by some mathematicians and engineers, see [1, 3, 4, 5, 7, 8, 9, 10, 11, 19, 27, 28, 29, 30, 31, 33]. However, as far as we know, there is no paper dealing with the non-autonomous Newton-Boussinesq equation with rapidly oscillating terms.

This paper follows the key ideas of the paper [8] devoted to the non-autonomous 2D Navier-Stokes system with singularly oscillating forces. In [8], Chepyzhov and Vishik proposed the divergence conditions assumption for the first time. Later, Tachim applied this method in [27, 28, 30, 31] to discuss the uniform attractors of the solutions of several types of partial differential equations with singular oscillating external force terms. Motivated by the idea of [8], we will investigate the asymptotic behavior of the non-autonomous Newton-Boussinesq equation depending on the small parameter $\varepsilon$, which reflects the rate of oscillations in the term $\varepsilon^{-\rho}f_1(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}, t)$ and $\varepsilon^{-\rho}g_1(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}, t)$ with amplitude of order $\varepsilon^{-\rho}$. Both terms $f_0(x, y, t), g_0(x, y, t), f_1(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}, t), g_1(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}, t)$ are supposed to be translation bounded in the space $L^2_{\text{loc}}(\mathbb{R}; L^2(\Omega))$. We aim to prove the stability of the uniform attractors $A^\varepsilon$ associated to problem (2)-(4) as $\varepsilon \to 0^+$ in $L^2(\Omega) \times L^2(\Omega)$. The main purpose of this paper is to show:

1. the uniform (with respect to $\varepsilon$) boundedness of the family $A^\varepsilon$ in $L^2(\Omega) \times L^2(\Omega)$:
   $$\sup_{\varepsilon \in [0,1]} \|A^\varepsilon\|_{L^2(\Omega) \times L^2(\Omega)} < +\infty;$$

2. the convergence of $A^\varepsilon$ to $A^0$ as $\varepsilon \to 0^+$ in the standard Hausdorff semidistance in $L^2(\Omega) \times L^2(\Omega)$:
   $$\lim_{\varepsilon \to 0^+} \text{dist}_{L^2(\Omega) \times L^2(\Omega)}(A^\varepsilon, A^0) = 0.$$

The outline of this paper is as follows: In the following section, we introduce some preliminary knowledge that will be used in this paper. In section 3, we derive some a priori estimates and prove the existence of uniform attractors in $L^2(\Omega) \times L^2(\Omega)$. 

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In Section 4, we study the boundedness of the uniform attractors when the external forces satisfy a divergence type condition. In Section 5, we investigate the deviation of a solution of the singular system from the solution of the “limiting” system. In Section 6, we investigate the structure of the uniform attractor of the Newton-Boussinesq equation. The last section is devoted to prove the main result of this article, which is the convergence of the uniform attractors of the oscillating Newton-Boussinesq equation to the attractor of the “limiting” system.

2. Preliminaries. In this section, we introduce some notations and preliminaries, which will be used throughout this paper.

First, we recall some basic concepts and results of uniform attractor theory which can be found in [25, 26].

The Hausdorff semidistance in X from one set \( B_1 \) to another set \( B_2 \) is defined as
\[
\text{dist}_X(B_1, B_2) = \sup_{b_1 \in B_1} \inf_{b_2 \in B_2} \| b_1 - b_2 \|_X.
\]

\( L^p(\Omega)(1 \leq p \leq +\infty) \) is the generic Lebesgue space and \( H^s(\Omega) \) is the usual Sobolev space. We denote by \( \| \cdot \|, (\cdot, \cdot) \) the norm and scalar product in \( L^2(\Omega) \), respectively.

We also denote by \( C \) a generic constant, which is different from line to line or even in a same line.

Definition 2.1. Let \( E \) be a Banach space and \( \Sigma \) be a symbol space. A two-parameter family of mappings \( \{ U_\sigma(t, \tau) : \sigma \in \Sigma \} \), is said to be a process in \( E \) with symbol space \( \Sigma \) if
\[
\begin{align*}
U_\sigma(t, \tau) &= U_\sigma(s, \tau), \quad \forall t \geq s \geq \tau, \tau \in \mathbb{R}, \\
U_\sigma(\tau, \tau) &= I, \quad \tau \in \mathbb{R},
\end{align*}
\]
where \( I \) is the identity operator.

Let \( \{ T(r) | r \geq 0 \} \) be the translation semigroup on \( \Sigma \), we say that a family of processes \( \{ U_\sigma(t, \tau) : \sigma \in \Sigma \} \) satisfies the translation identity if
\[
T(r)\Sigma = \Sigma,
\]
and
\[
U_\sigma(t + r, \tau + r) = U_{T(r)\sigma}(t, \tau), \forall \sigma \in \Sigma, t \geq \tau, \tau \in \mathbb{R}, r \geq 0.
\]

By \( B(E) \) we denote the collection of the bounded subsets of \( E \).

Definition 2.2. (1) We say that the family of processes \( \{ U_\sigma(t, \tau) : \sigma \in \Sigma \} \), is uniformly (with respect to (w.r.t.) \( \sigma \in \Sigma \)) bounded if for any \( B \in B(E) \) the set
\[
\bigcup_{\sigma \in \Sigma} \bigcup_{\tau \in \mathbb{R}} U_\sigma(t, \tau)B \in B(E).
\]

(2) A bounded set \( B_0 \in B(E) \) is said to be a bounded uniformly (w.r.t. \( \sigma \in \Sigma \)) absorbing set for \( \{ U_\sigma(t, \tau) : \sigma \in \Sigma \} \) if for any \( \tau \in \mathbb{R} \) and \( B \in B(E) \), there exists a time \( T_0 = T_0(B, \tau) \geq \tau \) such that
\[
\bigcup_{\sigma \in \Sigma} U_\sigma(t, \tau)B \subseteq B_0, \forall t \geq T_0.
\]

Definition 2.3. A subset \( A \subseteq E \) is said to be uniformly (w.r.t. \( \sigma \in \Sigma \)) attracting for the processes \( \{ U_\sigma(t, \tau) : \sigma \in \Sigma \} \) if for any fixed \( \tau \in \mathbb{R} \) and any \( B \in B(E) \),
\[
\lim_{t \to +\infty} \left( \sup_{\sigma \in \Sigma} \text{dist}_E(U_\sigma(t, \tau)B, A) \right) = 0.
\]
Definition 2.4. We say that a closed set $A_{\Sigma} \subset E$ is the uniform (w.r.t. $\sigma \in \Sigma$) attractor of the family processes $\{U_\sigma(t, \tau) : \sigma \in \Sigma\}$, if it is
(1) uniformly (w.r.t. $\sigma \in \Sigma$) attracting;
(2) it is contained in any closed uniformly (w.r.t. $\sigma \in \Sigma$) attracting set $A'$ of the family of processes $\{U_\sigma(t, \tau) : \sigma \in \Sigma\}$.

Definition 2.5. A process $\{U_\sigma(t, \tau)\}$ in a metric space $E$ is said to be uniformly (w.r.t. $\sigma \in \Sigma$) asymptotically compact if, for each fixed $\tau \in \mathbb{R}$, each sequence $\{t_n\} \subset [\tau, +\infty)$ with $t_n \to \infty$ (as $n \to \infty$), and each bounded sequence $\{u_n\} \subset E$, $\{\sigma_n\} \subset \Sigma$ the sequence $\{U_{\sigma_n}(t_n, \tau)u_n\}$ has a convergent subsequence in $E$.

Definition 2.6. The set
$$\omega_{\tau, \Sigma}(B) = \bigcap_{t \geq \tau} \bigcup_{\sigma \in \Sigma} \bigcup_{s \geq t} U_\sigma(s, \tau)B,$$
where $B$ is a bounded subset of $E$ is said to be the uniform (w.r.t. $\sigma \in \Sigma$) $\omega$-limit set of $B$.

Theorem 2.7. Let $E$ be a complete metric space, $\{U_\sigma(t, \tau)\}, \sigma \in \Sigma$ be a family of processes on $E$ satisfying the translation identity (12)-(13). Then $\{U_\sigma(t, \tau)\}, \sigma \in \Sigma$ has a compactly uniform attractor (w.r.t. $\sigma \in \Sigma$) $A_{\Sigma}$ in $E$ and satisfies
$$A_{\Sigma} = \omega_{0, \Sigma}(B_0) = \omega_{\tau, \Sigma}(B_0) = \bigcup_{\sigma \in \Sigma} \omega_{\tau, \Sigma}(B), \forall \tau \in \mathbb{R},$$
if and only if $\{U_\sigma(t, \tau)\}, \sigma \in \Sigma$
(1) has a bounded uniformly (w.r.t. $\sigma \in \Sigma$) absorbing set $B_0$;
(2) is uniformly (w.r.t. $\sigma \in \Sigma$) asymptotically compact.

In the non-autonomous system, we usual choose $\Sigma = \mathcal{H}(\sigma_0)$ as symbol space of the system,
$$\mathcal{H}(\sigma_0) = [(\sigma_0(\cdot + h)|h \in \mathbb{R})]_X$$
for every fixed $\sigma_0 \in X$, where $[(\cdot)]$ denotes the closure of a set in a topological space $X$. If $\mathcal{H}(\sigma_0)$ is compact in $X$ then we call $\sigma_0 \in X$ is translation compact (tr.c.). The translation semigroup $\{T(r)|r \geq 0\}$ satisfies (12), (13), that is, $T(r)\mathcal{H}(\sigma_0) = \mathcal{H}(\sigma_0)$, $U_\sigma(t + r, \tau + r) = U_{T(r)\sigma}(t, \tau), \forall \sigma \in \mathcal{H}(\sigma_0), t \geq \tau, \tau \in \mathbb{R}, r \geq 0$.

In order to clarify the assumptions on the external forces $f^e, g^e$, we introduce the following notations. Given a Banach space $X$, we denote by $L^2_{\text{loc}}(\mathbb{R}; X)$ the metrizable space of function $\varphi(s), s \in \mathbb{R}$ with value in $X$ that are locally 2-power integrable in the Bochner sense. It is equipped with the local 2-power mean convergence topology. We will also denote by $L^2_{\text{loc}}(\mathbb{R}; X)$ the subspace of $L^2_{\text{loc}}(\mathbb{R}; X)$ of translation bounded functions; i.e., for $\varphi(s) \in L^2_{\text{loc}}(\mathbb{R}; X)$, we have
$$\| \varphi \|^2_{L^2_{\text{loc}}(\mathbb{R}; X)} = \sup_{t \in \mathbb{R}} \int_t^{t+1} \| \varphi(s) \|^2_X ds < \infty.$$

Hereafter, we assume that
$$f_0 \in L^2_{\text{loc}}(\mathbb{R}; L^2(\Omega)), g_0 \in L^2_{\text{loc}}(\mathbb{R}; L^2(\Omega)).$$
To describe the function $f_1, g_1$, we use the space $Z = L^2_{\text{loc}}(\mathbb{R}^2; \mathbb{R})$. By definition, a function $\psi \in Z$ if
$$\| \psi \|^2_{Z} = \| \psi \|^2_{L^2_{\text{loc}}(\mathbb{R}^2; \mathbb{R})} = \sup_{(z_1, z_2) \in \mathbb{R}^2} \int_{z_2}^{z_2+1} \int_{z_1}^{z_1+1} |\psi(\zeta_1, \zeta_2)|^2 d\zeta_1 d\zeta_2 < \infty.$$
(14)
We now assume that the function $f_1(\cdot,t) \in Z, g_1(\cdot,t) \in Z$ for almost every $t \in \mathbb{R}$, and has finite norms in the space $L^2_{\text{loc}}(\mathbb{R}; Z)$, that is
\[
\| f_1 \|_{L^2_{\text{loc}}(\mathbb{R}; Z)}^2 = \sup_{t \in \mathbb{R}} \int_t^{t+1} \| f_1(\cdot,s) \|_Z^2 \, ds < \infty, \tag{15}
\]
\[
\| g_1 \|_{L^2_{\text{loc}}(\mathbb{R}; Z)}^2 = \sup_{t \in \mathbb{R}} \int_t^{t+1} \| g_1(\cdot,s) \|_Z^2 \, ds < \infty. \tag{16}
\]

Now, let us recall the known Gagliardo-Nirenberg inequality as follows.

**Lemma 2.8.** Let $\Omega = \mathbb{R}^n$ or $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial \Omega$, and $u \in L^q(\Omega), D^mu \in L^r(\Omega), 1 \leq q, r \leq \infty$. Then, there exists a constant $c$, such that,
\[
\| D^j u \|_{L^p(\Omega)} \leq c \| D^m u \|_{L^p(\Omega)}^a \| u \|_{L^p(\Omega)}^{1-a},
\]
where
\[
\frac{1}{p} = \frac{j}{n} + a\left(\frac{1}{r} - \frac{m}{n}\right) + (1-a)\frac{1}{q}, 1 \leq p \leq \infty, 0 \leq j \leq m, \frac{j}{m} \leq a \leq 1,
\]
c depends only on $(n, m, j, a, q, r)$.

3. Some a priori estimates and the uniform attractor. In this section, we first derive some a priori estimates on the solutions to (2)-(4). We then use these estimates to construct the bounded uniformly (with respect to $\tau \in \mathbb{R}$) absorbing sets in $L^2(\Omega) \times L^2(\Omega)$ and $H^1_0(\Omega) \times H^1_0(\Omega)$. We will need the following lemma, whose proof is given in [8].

**Lemma 3.1.** ([8]) Let a real function $z(t), t \geq 0$ be uniformly continuous and satisfy the inequality
\[
\frac{dz(t)}{dt} + \gamma z(t) \leq f(t), \forall t \geq 0,
\]
where $\gamma > 0, f(t) \geq 0$ for all $t \geq 0$ and $f \in L^1_{\text{loc}}(\mathbb{R}^+)$. Suppose also that
\[
\int_t^{t+1} f(s)ds \leq M, \forall t \geq 0.
\]
Then,
\[
z(t) \leq z(0)e^{-\gamma t} + M(1 + \gamma^{-1}), \forall t \geq 0.
\]

We first estimate the boundedness of the solution $(\xi(t), \theta(t))$ in $L^2(\Omega) \times L^2(\Omega)$.

**Proposition 1.** Given $f^\tau, g^\tau \in L^2_0(\mathbb{R}; L^2(\Omega)), \xi_\tau, \theta_\tau \in L^2(\Omega)$ and $P_r > 1$. Then the problem (2)-(4) has a unique weak solution $(\xi(t), \theta(t))$ satisfying
\[
\xi \in C^0([\tau, \infty), L^2(\Omega)) \cap L^2(\tau, T; H^1_0(\Omega)), \theta \in C^0([\tau, \infty), L^2(\Omega)) \cap L^2(\tau, T; H^1_0(\Omega)).
\]

And the following estimates hold:
\[
\| \xi(t) \|^2 \leq \| \xi(\tau) \|^2 e^{-\frac{2\lambda^2}{P_r}(t-\tau)} + \frac{4P_r^2\lambda^2}{P_r(P_r - 1)} e^{-\frac{2P_r^2\lambda^2}{P_r}(t-\tau)} \| \theta(\tau) \|^2
\]
\[
+ 2\lambda^2(1 + 2\lambda^2) \| f^\tau \|^2_{L^2(\mathbb{R}; L^2(\Omega))} + \frac{4P_r^2\lambda^2}{P_r}(1 + 2P_r\lambda^2) \| g^\tau \|^2_{L^2(\mathbb{R}; L^2(\Omega))}. \tag{17}
\]
\[
\| \theta(t) \|^2 \leq \| \theta(\tau) \|^2 e^{-\frac{2P_r^2\lambda^2}{P_r}(t-\tau)} + 2P_r\lambda^2(1 + 2P_r\lambda^2) \| g^\tau \|^2_{L^2(\mathbb{R}; L^2(\Omega))}. \tag{18}
\]
We notice that the following inequalities hold by a standard method as in [13, 15]. To derive (17)-(20), we proceed as follows.

The existence and uniqueness of solutions for problem (2)-(4) can be proved by taking the inner product of (2)

\[ \| \nabla \xi(t) \| ^2 + \frac{1}{2} \int _{\tau} ^{t} \| \nabla \xi(s) \| ^2 ds + \frac{1}{2 \lambda ^2} \int _{\tau} ^{t} \| \xi(s) \| ^2 ds \leq \| \xi(\tau) \| ^2 + 4 \frac{R _{2} ^{2} \lambda ^{2}}{\nu} (\| \theta(\tau) \| ^2 + 2 \nu \lambda ^{2} (t - \tau + 1) \| g ^{\varepsilon} \| _{L _{2} ^{0}} ^{2}) + 2 \lambda ^{2} (t - \tau + 1) \| f ^{\varepsilon} \| _{L _{2} ^{0}} ^{2} . \]

The inequality, it follows from (23) that

\[ \frac{1}{2} \frac{d}{dt} \| \theta \| ^{2} + \frac{1}{P_r} \| \nabla \theta \| ^{2} = (g ^{\varepsilon}, \theta) \leq \| g ^{\varepsilon} \| \| \theta \| \leq \frac{1}{4 P_r \lambda ^{2}} \| \theta \| ^{2} + P_r \lambda ^{2} \| g ^{\varepsilon} \| ^{2} . \]

Using (11), we have

\[ \frac{d}{dt} \| \theta \| ^{2} + \frac{1}{P_r} \| \nabla \theta \| ^{2} + \frac{1}{2 P_r \lambda ^{2}} \| \theta \| ^{2} \leq 2 P_r \lambda ^{2} \| g ^{\varepsilon} \| ^{2} . \]

Applying Lemma 3.1 with \( z(t) = \| \theta(t) \| ^{2} \), \( f(t) = 2 \nu \lambda ^{2} \| g ^{\varepsilon}(t) \| ^{2} \), \( \gamma = \frac{1}{2 P_r \lambda ^{2}} \), we obtain

\[ \| \theta(t) \| ^{2} \leq \| \theta(\tau) \| ^{2} e ^{-\frac{1}{2 P_r \lambda ^{2}} \int _{\tau} ^{t} ds} + 2 P_r \lambda ^{2} (1 + 2 \nu \lambda ^{2}) \| g ^{\varepsilon} \| _{L _{2} ^{0}(R; L ^{2}(\Omega))} ^{2} . \]

Taking the inner product of (2)_1 with \( \xi \) in \( L ^{2}(\Omega) \) and using (8) we obtain

\[ \frac{1}{2} \frac{d}{dt} \| \xi \| ^{2} + \| \nabla \xi \| ^{2} + \frac{R _{a}}{P_r} \int _{\Omega} \theta \xi dx dy = (f ^{\varepsilon}, \xi) . \]

We notice that the following inequalities hold

\[ \frac{R _{a}}{P_r} \int _{\Omega} \theta x \xi dx dy = \left[ \frac{R _{a}}{P_r} \right] \int _{\Omega} \xi dx dy \leq \frac{R _{a}}{P_r} \| \nabla \xi \| \| \theta \| \leq \frac{1}{4} \| \nabla \xi \| ^{2} + \frac{R _{a} ^{2}}{P _{r} ^{2}} \| \theta \| ^{2} , \]

and

\[ |(f ^{\varepsilon}, \xi)| \leq \| f ^{\varepsilon} \| \| \xi \| \leq \lambda \| f ^{\varepsilon} \| \| \nabla \xi \| \leq \frac{1}{4} \| \nabla \xi \| ^{2} + \lambda ^{2} \| f ^{\varepsilon} \| ^{2} . \]

By (24), (25), using the Poincaré inequality, it follows from (23) that

\[ \frac{d}{dt} \| \xi \| ^{2} + \frac{1}{2} \| \nabla \xi \| ^{2} + \frac{1}{2 \lambda ^{2}} \| \xi \| ^{2} \leq \frac{2 R _{a} ^{2}}{P _{r} ^{2}} \| \theta \| ^{2} + 2 \lambda ^{2} \| f ^{\varepsilon} \| ^{2} . \]

Applying the Gronwall inequality to (26), we have

\[ \| \xi(t) \| ^{2} \leq \| \xi(\tau) \| ^{2} e ^{-\frac{1}{2 \lambda ^{2}} \int _{\tau} ^{t} ds} + \int _{\tau} ^{t} \left[ \frac{2 R _{a} ^{2}}{P _{r} ^{2}} \| \theta(s) \| ^{2} + 2 \lambda ^{2} \| f ^{\varepsilon}(s) \| ^{2} \right] e ^{-\frac{1}{2 \lambda ^{2}} \int _{s} ^{t} ds} ds \]

\[ = \| \xi(\tau) \| ^{2} e ^{-\frac{1}{2 \lambda ^{2}} \int _{\tau} ^{t} ds} + 2 \lambda ^{2} \int _{\tau} ^{t} e ^{-\frac{1}{2 \lambda ^{2}} \int _{s} ^{t} ds} \| f ^{\varepsilon}(s) \| ^{2} ds \]

\[ + \frac{2 R _{a} ^{2}}{P _{r} ^{2}} \int _{\tau} ^{t} e ^{-\frac{1}{2 \lambda ^{2}} \int _{s} ^{t} ds} \| \theta(s) \| ^{2} ds . \]
Noticing
\[
2\lambda^2 \int_\tau^t e^{-\frac{1}{2}\lambda\tau (t-s)} \| \frac{d}{dt} f^\varepsilon (s) \|^2 \, ds \\
\leq 2\lambda^2 \left[ \int_{t-1}^t e^{-\frac{1}{2}\lambda\tau (t-s)} \| f^\varepsilon (s) \|^2 \, ds + \int_{t-2}^{t-1} e^{-\frac{1}{2}\lambda\tau (t-s)} \| f^\varepsilon (s) \|^2 \, ds + \ldots \right] \\
\leq 2\lambda^2 \left[ \int_{t-1}^t \| f^\varepsilon (s) \|^2 \, ds + e^{-\frac{1}{2}\lambda\tau} \int_{t-2}^{t-1} \| f^\varepsilon (s) \|^2 \, ds + \ldots \right] \\
\leq 2\lambda^2 \left[ 1 + e^{-\frac{1}{2}\lambda\tau} + e^{-\frac{1}{2}\lambda\tau} + \ldots \right] \| f^\varepsilon \|^2_{L^2_t} \\
\leq 2\lambda^2 (1 - e^{-\frac{1}{2}\lambda\tau})^{-1} \| f^\varepsilon \|^2_{L^2_t} \\
\leq 2\lambda^2 (1 + 2\lambda^2) \| f^\varepsilon \|^2_{L^2_t},
\]
and
\[
\frac{2R^2}{P_t^2} \int_\tau^t e^{-\frac{1}{2}\lambda\tau (t-s)} \| \theta (s) \|^2 \, ds \\
\leq \frac{2R^2}{P_t^2} \int_\tau^t e^{-\frac{1}{2}\lambda\tau (t-s)} \left[ \| \theta (\tau) \|^2 + 2P_r \lambda^2 (1 + 2P_r \lambda^2) \| g^\varepsilon \|^2_{L^2_t} \right] ds \\
\leq \frac{2R^2}{P_t^2} \| \theta (\tau) \|^2 \int_\tau^t e^{-\frac{1}{2}\lambda\tau (t-s)} e^{-\frac{1}{2}\lambda\tau (s-\tau)} ds \\
+ \frac{4R^2\lambda^2}{P_t} (1 + 2P_r \lambda^2) \int_\tau^t e^{-\frac{1}{2}\lambda\tau (t-s)} \| g^\varepsilon \|^2_{L^2_t} ds \\
\leq \frac{4R^2\lambda^2}{P_t (P_t - 1)} \left[ e^{-\frac{1}{2}\lambda\tau (t-\tau)} - e^{-\frac{1}{2}\lambda\tau (t-s)} \right] \| \theta (\tau) \|^2 \\
+ \frac{4R^2\lambda^2}{P_t} (1 + 2P_r \lambda^2)(1 + 2\lambda^2) \| g^\varepsilon \|^2_{L^2_t} \\
\leq \frac{4R^2\lambda^2}{P_t (P_t - 1)} e^{-\frac{1}{2}\lambda\tau (t-\tau)} \| \theta (\tau) \|^2 + \frac{4R^2\lambda^2}{P_t} (1 + 2P_r \lambda^2)(1 + 2\lambda^2) \| g^\varepsilon \|^2_{L^2_t}.
\]

it follows from (27) that
\[
\| \xi (t) \|^2 \leq \| \xi (\tau) \|^2 e^{-\frac{1}{2}\lambda\tau (t-\tau)} + \frac{4R^2\lambda^2}{P_t (P_t - 1)} e^{-\frac{1}{2}\lambda\tau (t-\tau)} \| \theta (\tau) \|^2 \\
+ 2\lambda^2 (1 + 2\lambda^2) \| f^\varepsilon \|^2_{L^2_t} + \frac{4R^2\lambda^2}{P_t} (1 + 2P_r \lambda^2)(1 + 2\lambda^2) \| g^\varepsilon \|^2_{L^2_t}. \quad (30)
\]

Integrating inequality (22) on \([\tau, t]\) we have
\[
\| \theta (t) \|^2 + \frac{1}{P_t} \int_\tau^t \| \nabla \theta (s) \|^2 \, ds + \frac{1}{2P_r \lambda^2} \int_\tau^t \| \theta (s) \|^2 \, ds \\
\leq \| \theta (\tau) \|^2 + 2P_r \lambda^2 \int_\tau^t \| g^\varepsilon (s) \|^2 \, ds \\
\leq \| \theta (\tau) \|^2 + 2P_r \lambda^2 (t - \tau + 1) \| g^\varepsilon \|^2_{L^2_t}.
\]
Integrating inequality (26) on $[\tau, t]$, and by (19) we get
\[
\| \xi(t) \|^2 + \frac{1}{2} \int_\tau^t \| \nabla \xi(s) \|^2 \, ds + \frac{1}{2\lambda^2} \int_\tau^t \| \xi(s) \|^2 \, ds \\
\leq \| \xi(\tau) \|^2 + \frac{2R^2}{P_r} \int_\tau^t \| \theta(s) \|^2 \, ds + \frac{4R^2\lambda^2}{P_r} \int_\tau^t \| f^e(s) \|^2 \, ds \\
\leq \| \xi(\tau) \|^2 + \frac{2R^2}{P_r} \| \theta(\tau) \|^2 + \frac{4R^2\lambda^2}{P_r} \| g^e \|_{L^2_b}^2 \\
+ 2\lambda^2(t - \tau + 1) \| f^e \|_{L^2_b}^2.
\]
Proposition 1 is proved.

Corollary 1. If the functions $f_0, g_0 \in L^2_b(\mathbb{R}; L^2(\Omega))$ and $f_1, g_1 \in L^2_b(\mathbb{R}; Z)$, where $Z = L^2_b(\mathbb{R}^2; \mathbb{R})$, then the functions $f^e, g^e$ belongs to $L^2_b(\mathbb{R}; L^2(\Omega))$ and
\[
\| f^e \|_{L^2_b(\mathbb{R}; L^2(\Omega))} \leq C_0 \| f_0 \|_{L^2_b(\mathbb{R}; L^2(\Omega))} + C_1 \| f_1 \|_{L^2_b(\mathbb{R}; Z)} + C_2 \| g_0 \|_{L^2_b(\mathbb{R}; L^2(\Omega))} + C_3 \| g_1 \|_{L^2_b(\mathbb{R}; Z)}.
\]
where the constant $C$ is independent of $\varepsilon$.

Proof. It is similar to the proof of Corollary 2.5 in [31].

Now from (17) and (18) we derive
\[
\| \xi(t) \|^2 \leq \| \xi(\tau) \|^2 e^{-\frac{1}{2\lambda^2}(t-\tau)} + \frac{4R^2\lambda^2}{P_r} e^{-\frac{1}{2\lambda^2}(t-\tau)} \| \theta(\tau) \|^2 + C_0^2 + C_1^2 \varepsilon^{-2p},
\]
where the constant $C_0$ depends on $\lambda, P_r, R_a$ and the norms $\| f_0 \|_{L^2_b(\mathbb{R}; L^2(\Omega))}$ and $\| g_0 \|_{L^2_b(\mathbb{R}; L^2(\Omega))}$, $C_1$ depends on $\lambda, P_r, R_a$ and the norms $\| f_1 \|_{L^2_b(\mathbb{R}; Z)}$ and $\| g_1 \|_{L^2_b(\mathbb{R}; Z)}$. And
\[
\| \theta(t) \|^2 \leq \| \theta(\tau) \|^2 e^{-\frac{1}{2\lambda^2}(t-\tau)} + C_2^2 + C_3^2 \varepsilon^{-2p},
\]
where the constant $C_2$ depends on $P_r, \lambda$ and the norm $\| g_0 \|_{L^2_b(\mathbb{R}; L^2(\Omega))}$, $C_3$ depends on $P_r, \lambda$ and the norm $\| g_1 \|_{L^2_b(\mathbb{R}; Z)}$.

Now we consider the process
\[
U(\xi(t), \theta(t)), t \geq \tau, \tau \in \mathbb{R},
\]
corresponding to the problem (2)-(4). More precisely, the mapping $U(\xi', \theta')(t, \tau) : L^2(\Omega) \times L^2(\Omega) \rightarrow L^2(\Omega) \times L^2(\Omega)$ is defined by: $U(\xi', \theta')(t, \tau)(\xi, \theta) = (\xi(t), \theta(t))$ for all $(\xi, \theta) \in L^2(\Omega) \times L^2(\Omega), t \geq \tau, \tau \in \mathbb{R}$, where $(\xi(t), \theta(t))$ is the solution to the system (2)-(4).

It follows from (33) and (34) that for every $\varepsilon, 0 < \varepsilon \leq 1$, the process $\{U(\xi', \theta')(t, \tau)\}$ has the uniform (with respect to $\tau \in \mathbb{R}$) absorbing set
\[
B_{0,\varepsilon} = \{ (\xi, \theta) \in L^2(\Omega) \times L^2(\Omega), \| \xi \|^2 \leq 2(C_0^2 + C_1^2 \varepsilon^{-2p}), \| \theta \|^2 \leq 2(C_2^2 + C_3^2 \varepsilon^{-2p}) \},
\]
and the set $B_{0,\varepsilon}$ is bounded in $L^2(\Omega) \times L^2(\Omega)$ for $\varepsilon > 0$ fixed. Therefore, for any bounded (in $L^2(\Omega) \times L^2(\Omega)$) set $B$, there exists a time $t(B)$ such that $U(\xi', \theta')(t, \tau) B \subset B_{0,\varepsilon}$ for all $t \geq t(B)$ and $\tau \in \mathbb{R}$.

Now let us estimate the boundedness of the solution $(\xi(t), \theta(t))$ in $H^1_0(\Omega) \times H^1_0(\Omega)$. 
Proposition 2. The assumptions are the same as in Proposition 1. Then the solution \((\xi(t), \theta(t))\) satisfying
\[
(t - \tau)(\| \nabla \xi \|^2 + \| \nabla \theta \|^2)^\frac{1}{2} \leq C(t - \tau, \| \xi(\tau) \|, \| \theta(\tau) \|, \| f^\xi \|_{L^2}^2, \| g^\xi \|_{L^2}^2),
\]
where \(C(R_1, R_2, R_3, R_4, R_5)\) is a positive increasing function of \(R_1, R_2, R_3, R_4\) and \(R_5\).

Proof. Taking the inner product of (2) with \(-(t - \tau)\Delta \theta\) in \(L^2(\Omega)\), we have
\[
\frac{1}{2} (t - \tau) \frac{\partial}{\partial t} \| \nabla \theta \|^2 + \frac{1}{P_r} (t - \tau) \| \Delta \theta \|^2
= (t - \tau) \int_\Omega J(\Psi, \theta) \Delta \theta \, dx dy - (t - \tau) \int_\Omega g^\xi \Delta \theta \, dx dy. \tag{37}
\]
Note that the first term on the right-hand side of (37) is given by
\[
(t - \tau) \int_\Omega J(\Psi, \theta) \Delta \theta \, dx dy = (t - \tau) \int_\Omega \Psi_y \theta_x \Delta \theta \, dx dy - (t - \tau) \int_\Omega \Psi_x \theta_y \Delta \theta \, dx dy. \tag{38}
\]
We now estimate the first term on the right-hand side of (38). Using the Hölder inequality, the Gagliardo-Nirenberg inequality and the Poincaré inequality, we have
\[
(t - \tau) \int_\Omega \Psi_y \theta_x \Delta \theta \, dx dy \leq (t - \tau) \| \Psi_y \|_4 \| \theta_x \|_4 \| \Delta \theta \| \tag{39}
\leq C(t - \tau) \| \Psi_y \|_4^\frac{1}{2} \| \Psi_y \|_H^\frac{1}{2} \| \theta_x \|_2 \| \theta_x \|_H^\frac{1}{2} \| \Delta \theta \| \tag{40}
\leq C(t - \tau) \| \Psi \|_H \| \nabla \theta \|_2 \| \Delta \theta \|_2 \leq \frac{1}{4P_r} (t - \tau) \| \Delta \theta \|^2 + C(t - \tau) \| \xi \|_4 \| \nabla \theta \|_2^2.
\]
Similarly for the second term on the right-hand side of (38) we have
\[
(t - \tau) \int_\Omega \Psi_x \theta_y \Delta \theta \, dx dy \leq \frac{1}{4P_r} (t - \tau) \| \Delta \theta \|^2 + C(t - \tau) \| \xi \|_4 \| \nabla \theta \|_2^2. \tag{41}
\]
Note that
\[
(t - \tau) \int_\Omega g^\xi \Delta \theta \, dx dy \leq (t - \tau) \| g^\xi \| \| \Delta \theta \| \leq \frac{1}{4P_r} (t - \tau) \| \Delta \theta \|^2 + P_r (t - \tau) \| g^\xi \|^2.
\]
Combining (38), (39), (40), (41) with (37), and multiplying the resulting inequality by 2, we get
\[
\frac{d}{dt} \left((t - \tau) \| \nabla \theta \|^2\right) + \frac{1}{2P_r} (t - \tau) \| \Delta \theta \|^2 \leq \| \nabla \theta \|^2 + C(t - \tau) \| \xi \|_4 \| \nabla \theta \|_2^2
+ 2P_r (t - \tau) \| g^\xi \|^2. \tag{42}
\]
Taking the inner product of (2) with \(-(t - \tau)\Delta \xi\) in \(L^2(\Omega)\), we have
\[
\frac{1}{2} (t - \tau) \frac{\partial}{\partial t} \| \nabla \xi \|^2 + (t - \tau) \| \Delta \xi \|^2 = (t - \tau) \int_\Omega J(\Psi, \xi) \Delta \xi \, dx dy
+ \frac{R_a}{P_r} (t - \tau) \int_\Omega \theta_x \Delta \xi \, dx dy - (t - \tau) \int_\Omega f^\xi \Delta \xi \, dx dy. \tag{43}
\]
We notice that the first term on the right-hand side of (43) can be written as
\[
(t - \tau) \int_\Omega J(\Psi, \xi) \Delta \xi \, dx dy = (t - \tau) \int_\Omega \Psi_y \xi_x \Delta \xi \, dx dy - (t - \tau) \int_\Omega \Psi_x \xi_y \Delta \xi \, dx dy. \tag{44}
\]
Similar with (39) and (40), we have
\[
(t - \tau) \int_\Omega \Psi_y \xi_x \Delta \xi dx dy \leq \frac{1}{8} (t - \tau) \| \Delta \xi \|^2 + C(t - \tau) \| \xi \|^4 \| \nabla \xi \|^2, \tag{45}
\]
and
\[
(t - \tau) \int_\Omega \Psi_x \xi_y \Delta \xi dx dy \leq \frac{1}{8} (t - \tau) \| \Delta \xi \|^2 + C(t - \tau) \| \xi \|^4 \| \nabla \xi \|^2. \tag{46}
\]
Note that the last two terms on the right-hand side of (43) are bounded by
\[
\left| \frac{R_n}{P_r} (t - \tau) \int \theta_x \Delta \xi dx dy \right| + \left| (t - \tau) \int \theta_x \Delta \xi dx dy \right|
\leq \frac{1}{4} (t - \tau) \| \Delta \xi \|^2 + \frac{2R_n^2}{P_r^2} (t - \tau) \| \nabla \theta \|^2 + 2(t - \tau) \| f^\varepsilon \|^2. \tag{47}
\]
Combining (44), (45), (46), (47) with (43), and multiplying the resulting inequality by 2, we have
\[
\frac{d}{dt} (t - \tau) \| \nabla \xi \|^2 + (t - \tau) \| \Delta \xi \|^2
\leq \| \nabla \xi \|^2 + C(t - \tau) \| \xi \|^4 \| \nabla \xi \|^2 + \frac{4R_n^2}{P_r^2} (t - \tau) \| \nabla \theta \|^2
+ 4(t - \tau) \| f^\varepsilon \|^2. \tag{48}
\]
Adding (42) to (48), we get
\[
\frac{d}{dt} (t - \tau)(\| \nabla \xi \|^2 + \| \nabla \theta \|^2) \leq (t - \tau)\left( \| \nabla \xi \|^2 + \| \nabla \theta \|^2 \right)(C \| \xi \|^4 + \frac{4R_n^2}{P_r^2}) + \| \nabla \theta \|^2 + \| \nabla \xi \|^2 + 2P_r(t - \tau) \| g^\varepsilon \|^2 + 4(t - \tau) \| f^\varepsilon \|^2. \tag{49}
\]
Let
\[
\psi(t) = (t - \tau)(\| \nabla \xi \|^2 + \| \nabla \theta \|^2), k(t) = C \| \xi \|^4 + \frac{4R_n^2}{P_r^2},
\phi(t) = \| \nabla \theta \|^2 + \| \nabla \xi \|^2 + 2P_r(t - \tau) \| g^\varepsilon \|^2 + 4(t - \tau) \| f^\varepsilon \|^2. \tag{50}
\]
Then from (49)-(50) we have
\[
\frac{d\psi}{dt} \leq k\psi + \phi, \tag{51}
\]
which yields (using the Gronwall lemma) for all \( t \geq \tau \),
\[
(t - \tau)(\| \nabla \xi \|^2 + \| \nabla \theta \|^2) \leq \int_\tau^t \phi(s) \exp \left( \int_s^t k(\mu)d\mu \right) ds
\leq \left( \int_\tau^t \phi(s) ds \right) \exp \left( \int_\tau^t k(\mu) d\mu \right). \tag{52}
\]
Now let us estimate \( \int_\tau^t \phi(s) ds \) and \( \int_\tau^t k(\mu) d\mu \). From (19)-(20), we note that
\[
\int_\tau^t \phi(s) ds = \int_\tau^t (\| \nabla \Theta(s) \|^2 + \| \nabla \xi(s) \|^2 + 2P_r(s - \tau) \| g^\varepsilon(s) \|^2
+ 4(s - \tau) \| f^\varepsilon(s) \|^2) ds
\leq C(t - \tau, \| \xi(\tau) \|, \| \Theta(\tau) \|, \| f^\varepsilon \|_{L^2}, \| g^\varepsilon \|_{L^2}). \tag{53}
\]
and
\[ \int_{s}^{t} k(\mu) \, d\mu = \int_{s}^{t} (C \| \xi(s) \|^{4} + \frac{4R_{\mu}^{2}}{P_{\mu}^{2}}) \, ds \]
\begin{align*}
& \leq C \left( \sup_{\tau \in [s, t]} \| \xi(s) \|^{2} \right)^{2} (t - \tau) + \frac{4R_{\mu}^{2}}{P_{\mu}^{2}} (t - \tau) \\
& \leq C (t - \tau, \| \xi(\tau) \|, \| \theta(\tau) \|, \| f^{\varepsilon} \|_{L_{\mu}^{2}}, \| g^{\varepsilon} \|_{L_{\mu}^{2}}),
\end{align*}
where \( C(R_{1}, R_{2}, R_{3}, R_{4}, R_{5}) \) is a positive increasing function of \( R_{1}, R_{2}, R_{3}, R_{4} \) and \( R_{5} \). It follows from (52)-(54) that
\[ (t - \tau)(\| \nabla \xi \|^{2} + \| \nabla \theta \|^{2}) \leq C(t - \tau, \| \xi(\tau) \|, \| \theta(\tau) \|, \| f^{\varepsilon} \|_{L_{\mu}^{2}}, \| g^{\varepsilon} \|_{L_{\mu}^{2}}). \]
Thus (36) is proved. \( \square \)

Based on Proposition 2, we have the following result.

**Proposition 3.** Suppose \( P_{\mu} > 1 \). For any \( \varepsilon > 0 \), the process \( \{U_{(f^{\varepsilon}, g^{\varepsilon})}(t, \tau)\} \) associated to (2)-(4) is uniformly compact in \( L^{2}(\Omega) \times L^{2}(\Omega) \) and it has a uniform absorbing set \( B_{1, \varepsilon} \) (bounded in \( H_{0}^{1}(\Omega) \times H_{0}^{1}(\Omega) \)) defined by
\[ B_{1, \varepsilon} = \bigcup_{\tau \in \mathbb{R}} U_{(f^{\varepsilon}, g^{\varepsilon})}(\tau + 1, \tau) B_{0, \varepsilon}, \]
where \( B_{0, \varepsilon} \) is given by (35). Moreover, there exists a uniform attractor \( A^{\varepsilon} \), which satisfies
\[ A^{\varepsilon} \subset B_{0, \varepsilon} \cap B_{1, \varepsilon}. \]

**Proof.** From (36) and (55), it is clear that \( B_{1, \varepsilon} \) is bounded in \( H_{0}^{1}(\Omega) \times H_{0}^{1}(\Omega) \), and hence compact in \( L^{2}(\Omega) \times L^{2}(\Omega) \). And it is also clear that \( B_{1, \varepsilon} \) is uniform (with respect to \( \tau \in \mathbb{R} \)) absorbing set for the process \( \{U_{(f^{\varepsilon}, g^{\varepsilon})}(t, \tau)\} \). The rest of the Proposition follows the general theory on global attractors. [6, 32]. \( \square \)

We now consider the “limiting” system
\[ \begin{cases}
\frac{\partial \xi^{0}}{\partial \tau} - \Delta \xi^{0} + J(\Psi^{0}, \xi^{0}) + \frac{R_{\mu}}{P_{\mu}} \frac{\partial \theta^{0}}{\partial \tau} = f_{0}(x, y, t), \\
\Delta \Psi^{0} = \xi^{0}, \\
\frac{\partial \theta^{0}}{\partial \tau} - \frac{1}{P_{\mu}} \Delta \theta^{0} + J(\Psi^{0}, \theta^{0}) = g_{0}(x, y, t),
\end{cases} \]
where \( f_{0}, g_{0} \in L_{\mu}^{2}(\mathbb{R}; L^{2}(\Omega)) \).

We can check that
\[ \| \theta^{0}(t) \|^{2} \leq \| \theta^{0}(\tau) \|^{2} e^{-\frac{1}{2P_{\mu}}(t-\tau)} + 2P_{\mu} \lambda^{2}(1 + 2P_{\mu} \lambda^{2}) \| g_{0} \|_{L_{2}^{2}(\mathbb{R}; L^{2}(\Omega))}^{2}, \]
\[ \| \xi^{0}(t) \|^{2} \leq \| \xi^{0}(\tau) \|^{2} e^{-\frac{1}{2P_{\mu}}(t-\tau)} + \frac{4R_{\mu}^{2} \lambda^{2}}{P_{\mu}(P_{\mu} - 1)} \| \theta^{0}(\tau) \|^{2} e^{-\frac{1}{2P_{\mu}}(t-\tau)} + 2\lambda^{2}(1 + 2\lambda^{2}) \| f_{0} \|_{L_{2}^{2}}^{2} + \frac{4R_{\mu}^{2} \lambda^{2}}{P_{\mu}}(1 + 2P_{\mu} \lambda^{2})(1 + 2\lambda^{2}) \| g_{0} \|_{L_{2}^{2}}^{2}, \]
and
\[ \| \theta^{0}(t) \|^{2} + \frac{1}{P_{\mu}} \int_{\tau}^{t} \| \nabla \theta^{0}(s) \|^{2} \, ds \leq \| \theta^{0}(\tau) \|^{2} + 2P_{\mu} \lambda^{2}(t - \tau + 1) \| g_{0} \|_{L_{2}^{2}}^{2}. \]
\[ \| \xi^0(t) \|^2 + \frac{1}{2} \int_0^t \| \nabla \xi^0(s) \|^2 \, ds \leq \| \xi^0(\tau) \|^2 + \frac{4R_2^2 \lambda^2}{P_r} (\| \theta^0(\tau) \|^2 + 2P_\lambda \lambda^2 (t - \tau + 1) \| g_0 \|_{L^2_\lambda}^2) + 2\lambda^2 (t - \tau + 1) \| f_0 \|_{L^2_\lambda}^2. \]  

(60)

It follows that

\[ \| \theta^0(t) \|^2 \leq \| \theta^0(\tau) \|^2 e^{-\frac{2\lambda}{P_r} (t - \tau)} + C_4^2, \]

(61)

\[ \| \xi^0(t) \|^2 \leq \| \xi^0(\tau) \|^2 e^{-\frac{4\lambda R_2^2}{P_r (P_r - 1)} (t - \tau)} + 2 \| \theta^0(\tau) \|^2 e^{-\frac{2\lambda}{P_r} (t - \tau)} + C_5^2, \]

(62)

where \( C_4 \) depends on \( P_r, \lambda \) and \( \| g_0 \|_{L^2_\lambda}^2 \), \( C_5 \) depends on \( P_r, R_\lambda, \| f_0 \|_{L^2_\lambda} \) and \( \| g_0 \|_{L^2_\lambda} \). This proves that the process \( \{ U(t_{f_0, g_0})(t, \tau) \} \) has the uniform (with respect to \( \tau \in \mathbb{R} \)) absorbing set

\[ B_{0,0} = \{ (\xi^0, \theta^0) \in L^2(\Omega) \times L^2(\Omega), \| \xi^0 \|^2 \leq 2C_0^2, \| \theta^0 \|^2 \leq 2C_4^2 \}, \]

(63)

and the set \( B_{0,0} \) is bounded in \( L^2(\Omega) \times L^2(\Omega) \).

Similar to the proof of Proposition 2, we can get

\[ (t - \tau) (\| \nabla \xi^0 \|^2 + \| \nabla \theta^0 \|^2) \leq C(t - \tau) \| \xi^0(\tau) \|, \| \theta^0(\tau) \|, \| f_0 \|_{L^2_\lambda}, \| g_0 \|_{L^2_\lambda}). \]

(63)

Choosing \( \tau = t - 1 \), we have

\[ \| \nabla \xi^0 \|^2 + \| \nabla \theta^0 \|^2 \leq C(\| \xi^0(t - 1) \|, \| \theta^0(t - 1) \|, \| f_0 \|_{L^2_\lambda}, \| g_0 \|_{L^2_\lambda}). \]

According to (63), when \( t \) is large enough, we have

\[ \| \nabla \xi^0 \|^2 + \| \nabla \theta^0 \|^2 \leq C(C_5, C_4, \| f_0 \|_{L^2_\lambda}, \| g_0 \|_{L^2_\lambda}), \]

where \( C(R_1, R_2, R_3, R_4) \) is a positive increasing function of \( R_1, R_2, R_3 \) and \( R_4 \). Thus, there exists an absorbing set \( B_{1,0} \) in \( H_1^0(\Omega) \times H_1^0(\Omega) \), i.e.,

\[ B_{1,0} = \{ (\xi^0, \theta^0) \in H_1^0(\Omega) \times H_1^0(\Omega), \| \nabla \xi^0 \|^2 \leq 2C_0^2, \| \nabla \theta^0 \|^2 \leq 2C_5^2 \}, \]

(64)

where \( C_6, C_7 \) depend on \( C_4, C_5, \| f_0 \|_{L^2_\lambda} \) and \( \| g_0 \|_{L^2_\lambda} \). Therefore the process \( \{ U(t_{f_0, g_0})(t, \tau) \} \) is uniform compact. In particular, the process has a compact uniform attractor \( A^0 \) such that

\[ A^0 \subset B_{0,0} \cap B_{1,0}. \]

(65)

4. Divergence type condition and boundedness of \( A^\varepsilon \). Let \( f^\varepsilon = f_0(x, y, t) + \varepsilon^{-\rho} f_1(\xi^0, \theta^0, t), g^\varepsilon = g_0(x, y, t) + \varepsilon^{-\rho} g_1(\xi^0, \theta^0, t) \), where \( f_0, f_1, g_0 \) and \( g_1 \) satisfy

\[ \| f_0(\cdot) \|_{L^2_\xi(\mathbb{R}; L^2(\Omega))} < \infty, \| g_0(\cdot) \|_{L^2_\xi(\mathbb{R}; L^2(\Omega))} < \infty, \]

\[ \| f_1(\cdot) \|_{L^2_\xi(\mathbb{R}; Z)} < \infty, \| g_1(\cdot) \|_{L^2_\xi(\mathbb{R}; Z)} < \infty, \]

where \( Z = L^2_\lambda(\mathbb{R}^2; \mathbb{R}) \). We assume that \( f_1(x, y, t) \) and \( g_1(x, y, t) \) satisfy the following condition referred to hereafter (see [8]) as the divergence type condition.

There exist functions \( F_i(x, y, t), G_i(x, y, t) \in L^2(\mathbb{R}; Z), i = 1, 2 \) such that

\[ \frac{\partial F_1}{\partial x}(x, y, t) + \frac{\partial F_2}{\partial y}(x, y, t) + \frac{\partial G_1}{\partial x}(x, y, t) + \frac{\partial G_2}{\partial y}(x, y, t) \in L^2_\xi(\mathbb{R}; Z), \]

\[ f_1(x, y, t) = \frac{\partial F_1}{\partial x}(x, y, t) + \frac{\partial F_2}{\partial y}(x, y, t), g_1(x, y, t) = \frac{\partial G_1}{\partial x}(x, y, t) + \frac{\partial G_2}{\partial y}(x, y, t), \]

\[ \forall (x, y) \in \mathbb{R}^2, t \in \mathbb{R}. \]
Theorem 4.1. We assume that $f_1, g_1$ satisfy (66). Then for every $\rho$, $0 \leq \rho \leq 1$, the uniform attractors $A^\varepsilon$ are uniformly (with respect to $\varepsilon \in [0, 1]$) bounded in $L^2(\Omega) \times L^2(\Omega)$, that is

$$\| A^\varepsilon \|_{L^2(\Omega) \times L^2(\Omega)} \leq C, \forall \varepsilon \in (0, 1),$$

where $C$ is independent of $\varepsilon$.

Proof. For $\theta$, we have

$$\frac{1}{2} \frac{d}{dt} \| \theta \|^2 + \frac{1}{2} \| \nabla \theta \|^2 = (g_0, \theta) + \varepsilon^{-\rho}(g_1 \varepsilon, \theta, t, \theta).$$

Noting

$$| (g_0, \theta) | \leq \lambda \| g_0 \| \| \nabla \theta \| \leq \frac{1}{4P_r} \| \nabla \theta \|^2 + P_r \lambda^2 \| g_0 \|^2,$$

and using the boundary condition on $\theta$, we have

$$\varepsilon^{-\rho}(g_1 \varepsilon, \theta, t, \theta) = \varepsilon^{-\rho} \left( \frac{\partial G_1}{\partial x}(x, y, t) + \frac{\partial G_2}{\partial y}(x, y, t), \theta \right)$$

$$= \varepsilon^{-\rho} \left( \frac{\partial}{\partial x} G_1(x, y, t) + \frac{\partial}{\partial y} G_2(x, y, t), \theta \right)$$

$$= -\varepsilon^{-\rho} (G_1(x, y, t), \frac{\partial \theta}{\partial y}) + \varepsilon^{-\rho} (G_2(x, y, t), \frac{\partial \theta}{\partial y})$$

$$\leq \varepsilon^{-\rho} || G_1(x, y, t) || || \nabla \theta || + P_r \varepsilon^{2(1-\rho)} || G_1(x, y, t) ||^2 + P_r \varepsilon^{2(1-\rho)} || G_2(x, y, t) ||^2.$$ 

Combining (68)–(70), we have

$$\frac{d}{dt} \| \theta \|^2 + \frac{1}{2} \frac{d}{dt} \| \nabla \theta \|^2 \leq 2P_r \lambda^2 \| g_0 \|^2$$

$$+ 2P_r \varepsilon^{2(1-\rho)} (|| G_1(x, y, t) ||^2 + || G_2(x, y, t) ||^2).$$

By assumptions, we have

$$\int_t^{t+1} \| g_0(s) \|^2 \, ds \leq \| g_0 \|^2 \| G_i \|_{L^2(\Omega)}^2 \equiv M_0, \forall t \in \mathbb{R},$$

$$\int_t^{t+1} \int_{\Omega} | G_i(x, y, s) |^2 \, dx \, dy \, ds \leq C \| G_i(\cdot) \|^2 \| L^2(\Omega) \equiv M_i, i = 1, 2, \forall t \in \mathbb{R},$$

where $C > 0$ is independent of $\varepsilon$.

It follows from Lemma 3.1 that

$$\| \theta(t) \| \leq \varepsilon^{-\rho} \left( 1 + 2P_r \lambda^2 \right) (1 + 2P_r \lambda^2).$$

For $\xi$, we have

$$\frac{1}{2} \frac{d}{dt} \| \xi \|^2 + \| \nabla \xi \|^2 + \frac{R_0}{P_r} \int_{\Omega} \theta \xi \, dx \, dy = (f_0, \xi) + \varepsilon^{-\rho}(f_1 \varepsilon, \xi, t, \xi).$$

Note that

$$\left| \frac{R_0}{P_r} \int_{\Omega} \theta \xi \, dx \, dy \right| \leq \frac{R_0}{P_r} \int_{\Omega} \xi \theta \, dx \, dy \leq \frac{R_0}{P_r} \| \nabla \xi \| \| \theta \| \leq \frac{1}{4} \| \nabla \xi \|^2 + \frac{R^2_0}{P^2_r} \| \theta \|^2.$$ 

(76)
and
\[ (f_0, \xi) \leq \| f_0 \| \| \xi \| \leq \lambda \| f_0 \| \| \nabla \xi \| \leq \frac{1}{4} \| \nabla \xi \|^2 + \lambda^2 \| f_0 \|^2. \] (77)

Similar with (70), we have
\[ \varepsilon^{-\rho}(f_1(x, y, t), \xi) \leq \frac{1}{4} \| \nabla \xi \|^2 + 2\varepsilon^{2(1-\rho)}(\| F_1(x, y, t) \|^2 + \| F_2(x, y, t) \|^2). \]

Combining (75)-(78) we have
\[ \frac{d}{dt} \| \xi \|^2 + \frac{1}{2\lambda^2} \| \xi \|^2 \leq \frac{2R^2_2}{P_r} \| \theta \|^2 + 2\lambda^2 \| f_0 \|^2 + 4\varepsilon^{2(1-\rho)}(\| F_1(x, y, t) \|^2 + \| F_2(x, y, t) \|^2). \] (79)

By assumptions, we have
\[ \int_{t_0}^{t+1} \| f_0(s) \|^2 ds \leq \| f_0 \|^2_{L^2([\mathbb{R};L^2(\Omega)])} = N_0, \forall t \in \mathbb{R}, \] (80)
\[ \int_{t_0}^{t+1} \int_\Omega |F_i(x, y, s)|^2 \, dx \, dy \, ds \leq C \| F_i(\cdot) \|^2_{L^2(\mathbb{R};\mathbb{R})} = N_i, \forall t \in \mathbb{R}, \] (81)
where \( C > 0 \) is independent of \( \varepsilon \).

Using (74), we have
\[ \int_{t_0}^{t+1} \left\{ \frac{2R^2_2}{P_r} \| \theta(s) \|^2 + 2\lambda^2 \| f_0(s) \|^2 + 4\varepsilon^{2(1-\rho)}(\| F_1(x, y, s) \|^2 + \| F_2(x, y, s) \|^2) \right\} ds \]
\[ \leq \frac{2R^2_2}{P_r} \int_{t_0}^{t+1} \{ \| \theta(t) \|^2 e^{-\frac{1}{2\lambda^2}(t-s)} + [2P_r\lambda^2 M_0 + 2P_r\varepsilon^{2(1-\rho)}(M_1 + M_2)](1 + 2P_r\lambda^2) \} ds + 2\lambda^2 N_0 + 4\varepsilon^{2(1-\rho)}(N_1 + N_2) \] (82)
\[ \leq \frac{4R^2_2\lambda^2}{P_r} \| \theta(t) \|^2 e^{-\frac{1}{2\lambda^2}(t-s)} + \frac{2R^2_2}{P_r} [2P_r\lambda^2 M_0 + 2P_r\varepsilon^{2(1-\rho)}(M_1 + M_2)] \cdot (1 + 2P_r\lambda^2) + 2\lambda^2 N_0 + 4\varepsilon^{2(1-\rho)}(N_1 + N_2). \]

By (82), it follows from Lemma 3.1 that
\[ \| \xi(t) \|^2 \leq \| \xi(\tau) \|^2 e^{-\frac{1}{2\lambda^2}(t-s)} + (1 + 2\lambda^2) \frac{4R^2_2\lambda^2}{P_r} \| \theta(\tau) \|^2 e^{-\frac{1}{2\lambda^2}(t-s)} \]
\[ + (1 + 2\lambda^2) \frac{2R^2_2}{P_r} [2P_r\lambda^2 M_0 + 2P_r\varepsilon^{2(1-\rho)}(M_1 + M_2)](1 + 2P_r\lambda^2) \] (83)
\[ + 2(1 + 2\lambda^2) \lambda^2 N_0 + 4(1 + 2\lambda^2)\varepsilon^{2(1-\rho)}(N_1 + N_2). \]

Since \( 0 \leq \rho \leq 1 \) and \( 0 < \varepsilon \leq 1 \), it follows that the processes \( \{U(f^*, g^*)(t, \tau)\} \) has a uniform absorbing set
\[ \tilde{B} = \{ (\xi, \theta) \in L^2(\Omega) \times L^2(\Omega), \| \xi \|^2 \leq C_8, \| \theta \|^2 \leq C_9 \}, \] (84)
where
\[ C_8 = 2 \left\{ (1 + 2\lambda^2) \frac{2R^2_0}{P_r} [2P_r\lambda^2 M_0 + 2P_r(M_1 + M_2)](1 + 2P_r\lambda^2) + 2(1 + 2\lambda^2)\lambda^2 N_0 + 4(1 + 2\lambda^2)(N_1 + N_2) \right\}. \]
Thus,

$$
\| A^x \|_{L^2(\Omega) \times L^2(\Omega)} \leq C, \forall 0 < \varepsilon \leq 1,
$$

where $C$ is a positive constant which depends on $C_8$ and $C_9$. Theorem 4.1 is proved.

5. Estimate for the deviation of solutions. We consider the problem (2)-(4). We assume that $f_0, g_0 \in L^2_0(\mathbb{R}; L^2(\Omega))$ and $f_1, g_1 \in L^2_0(\mathbb{R}; Z)$. Moreover, we assume that $f_1$ and $g_1$ satisfy the divergence condition (66).

Along with (2), we consider the corresponding “limiting” equation (56). We associate with (2) and (56) the same initial data at $t = \tau$

$$
(\xi |_{t=\tau} = \xi_\tau, \xi_0 |_{t=\tau} = \xi_\tau, \theta |_{t=\tau} = \theta_\tau, \theta^0 |_{t=\tau} = \theta_\tau),
$$

where $(\xi_\tau, \theta_\tau) \in \bar{B}$ and $\bar{B}$ is the absorbing set defined in (84). Note that $\bar{B}$ is independent of $\rho, 0 \leq \rho \leq 1$ and $\varepsilon, 0 < \varepsilon \leq 1$.

**Theorem 5.1.** Suppose that $f_1$ and $g_1$ satisfy (66). Then for every $(\xi_\tau, \theta_\tau) \in \bar{B}$, the difference $w = \theta - \theta^0, v = \xi - \xi^0$ of the solutions to (2) and (56) with the same initial data $(\xi_\tau, \theta_\tau) \in \bar{B}$ satisfies the following inequality:

$$
\| v(t) \|^2 + \| w(t) \|^2 \leq C \varepsilon^{2(1-\rho)} \varepsilon^{2r(t-\tau)}, \forall t \geq \tau,
$$

where $C$ and $r$ are independent of $\varepsilon, (\xi_\tau, \theta_\tau) \in \bar{B}$ and $0 \leq \rho \leq 1$.

**Proof.** Let $w = \theta - \theta^0, v = \xi - \xi^0$, where $(\xi, \theta)$ and $(\xi^0, \theta^0)$ are the solutions to (2) and (56), respectively, with the same initial data taken in $\bar{B}$. Then $w, v$ satisfy (with $\tau = 0$ for simplicity)

$$
\frac{\partial v}{\partial t} - \Delta v + J(\Psi, \xi) - J(\Psi^0, \xi^0) + \frac{R_a}{P_r} \frac{\partial w}{\partial x} = \varepsilon^{-\rho} f_1(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}, t),
$$

$$
\frac{\partial w}{\partial t} - \frac{1}{P_r} \Delta w + J(\Psi, \theta) - J(\Psi^0, \theta^0) = \varepsilon^{-\rho} g_1(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}, t),
$$

$$
w |_{t=0} = 0, v |_{t=0} = 0.
$$

Taking the inner product of (88) with $v$ in $L^2(\Omega)$, we have

$$
\frac{1}{2} \frac{d}{dt} \| v \|^2 + \| \nabla v \|^2 = - \int_{\Omega} [J(\Psi, \xi) - J(\Psi^0, \xi^0)] v dx dy - \frac{R_a}{P_r} \int_{\Omega} w_x v dx dy + \varepsilon^{-\rho} (f_1, v).
$$

And because $J(\Psi, \xi) - J(\Psi^0, \xi^0) = J(\Psi, v) + J(\Psi - \Psi^0, \xi^0), so$

$$
- \int_{\Omega} [J(\Psi, \xi) - J(\Psi^0, \xi^0)] v dx dy = - \int_{\Omega} J(\Psi, v) v dx dy - \int_{\Omega} [J(\Psi - \Psi^0, \xi^0)] v dx dy.
$$

Similar with (78), we have

$$
\varepsilon^{-\rho} (f_1(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}, t), v) \leq \frac{1}{4} \| \nabla v \|^2 + 2\varepsilon^{2(1-\rho)} (\| F_1(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}, t) \|^2 + \| F_2(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}, t) \|^2).
$$

(93)
Combining (94) with (97) we have
\[
\frac{1}{2} \frac{d}{dt} \| v \|^2 + \| \nabla v \|^2 = -\int_{\Omega} [J(\Psi, v) + J(\Psi - \Psi^0, \xi^0)] v dx dy - \frac{R_a}{P_r} \int_{\Omega} w x v dx dy \\
+ \varepsilon^{-\rho}(f_1, v)
\leq \| J(\Psi - \Psi^0, \xi^0) \| \| v \| + \frac{R_a}{P_r} \| \nabla w \| \| v \| + \frac{1}{4} \| \nabla v \|^2 \\
+ 2\varepsilon^{2(1-\rho)}(\| \nabla f_1(x, \xi, t) \|^2 + \| \nabla f_2(x, \xi, t) \|^2)
\leq C \| \nabla v \| \| \nabla \xi^0 \| \| v \| + \frac{R_a}{P_r} \| \nabla w \| \| v \| + \frac{1}{4} \| \nabla v \|^2 \\
+ 2\varepsilon^{2(1-\rho)}(\| \nabla f_1(x, \xi, t) \|^2 + \| \nabla f_2(x, \xi, t) \|^2)
\]
\[
\leq \frac{3}{4} \| \nabla v \|^2 + C \| \nabla \xi^0 \| \| v \|^2 \frac{1}{2P_r} \| \nabla w \|^2 + \frac{R_a^2}{2P_r} \| v \|^2 \\
+ 2\varepsilon^{2(1-\rho)}(\| \nabla f_1(x, \xi, t) \|^2 + \| \nabla f_2(x, \xi, t) \|^2).
\]

Taking the inner product of (89) with \( w \) in \( L^2(\Omega) \), we have
\[
\frac{1}{2} \frac{d}{dt} \| w \|^2 + \frac{1}{P_r} \| \nabla w \|^2 = -\int_{\Omega} [J(\Psi, w) + J(\Psi - \Psi^0, \theta^0)] w dx dy + \varepsilon^{-\rho}(g_1, w).
\]
\[
(95)
\]

Similar with (70), we have
\[
\varepsilon^{-\rho}(g_1(x, \xi, t), w) \leq \frac{1}{2P_r} \| \nabla w \|^2 + P_r \varepsilon^{2(1-\rho)}(\| G_1(x, \xi, t) \|^2 + \| G_2(x, \xi, t) \|^2).
\]
\[
(96)
\]

Using (8), (10), from (95) and (96) we get
\[
\frac{1}{2} \frac{d}{dt} \| w \|^2 + \frac{1}{P_r} \| \nabla w \|^2 \leq \| J(\Psi - \Psi^0, \theta^0) \| \| w \| + \frac{1}{2P_r} \| \nabla w \|^2 \\
+ P_r \varepsilon^{2(1-\rho)}(\| G_1(x, \xi, t) \|^2 + \| G_2(x, \xi, t) \|^2)
\leq C \| \nabla v \| \| \nabla \theta^0 \| \| w \| + \frac{1}{P_r} \| \nabla w \|^2 \\
+ P_r \varepsilon^{2(1-\rho)}(\| G_1(x, \xi, t) \|^2 + \| G_2(x, \xi, t) \|^2)
\leq \frac{1}{2P_r} \| \nabla w \|^2 + \frac{1}{4} \| \nabla v \|^2 + C \| \nabla \theta^0 \| \| w \|^2 \\
+ P_r \varepsilon^{2(1-\rho)}(\| G_1(x, \xi, t) \|^2 + \| G_2(x, \xi, t) \|^2).
\]
\[
(97)
\]

Combining (94) with (97) we have
\[
\frac{d}{dt}(\| v \|^2 + \| w \|^2) \leq C(\| \nabla \xi^0 \|^2 + \frac{R_a}{P_r} \| v \|^2 + C \| \nabla \theta^0 \|^2 \| w \|^2 \\
+ 4\varepsilon^{2(1-\rho)}(\| \nabla f_1(x, \xi, t) \|^2 + \| \nabla f_2(x, \xi, t) \|^2)
+ 2P_r \varepsilon^{2(1-\rho)}(\| G_1(x, \xi, t) \|^2 + \| G_2(x, \xi, t) \|^2)
\]
\[
(98)
\]
\[ \leq C (\| \nabla \xi^0 \|^2 + \| \nabla \theta^0 \|^2 + \frac{R^2}{P r}) : (\| w \|^2 + \| \psi \|^2) \]
\[ + 4 \varepsilon^{2(1-\rho)} (\| F_1(x, y, t) \|^2 + \| F_2(x, y, t) \|^2) \]
\[ + 2 P r e^{2(1-\rho)} (\| G_1(x, y, t) \|^2 + \| G_2(x, y, t) \|^2). \]

Let \( \psi(t) = \| \psi(t) \|^2 + \| w(t) \|^2 \), \( k(t) = C (\| \nabla \xi^0(t) \|^2 + \| \nabla \theta^0(t) \|^2 + \frac{R^2}{P r}) \),
\( \phi(t) = 4 \varepsilon^{2(1-\rho)} (\| F_1(x, y, t) \|^2 + \| F_2(x, y, t) \|^2) + 2 P r e^{2(1-\rho)} (\| G_1(x, y, t) \|^2 + \| G_2(x, y, t) \|^2). \) Then
\[ \frac{d\psi(t)}{dt} \leq k(t) \psi(t) + \phi(t), \psi(0) = 0, \] (99)
which gives
\[ \psi(t) \leq \int_0^t \phi(s) \exp \left( \int_s^t k(\mu) d\mu \right) ds \]
\[ \leq \left( \int_0^t \phi(s) ds \right) \exp \left( \int_0^t k(\mu) d\mu \right). \] (100)

It follows from (59), (60) that
\[ \int_0^t k(s) ds = C \int_0^t (\| \nabla \xi^0(s) \|^2 + \| \nabla \theta^0(s) \|^2 + \frac{R^2}{P r}) ds \]
\[ \leq C (\| \xi(0) \|^2 + \| \theta(0) \|^2) + C(t+1) (\| f_0 \|_{L^2}^2 + \| g_0 \|_{L^2}^2) \] (101)
\[ \leq C(t+1). \]

Furthermore, we can easily deduce that
\[ \int_0^t \phi(s) ds = \int_0^t [4 \varepsilon^{2(1-\rho)} (\| F_1(x, y, s) \|^2 + \| F_2(x, y, s) \|^2) \]
\[ + 2 P r e^{2(1-\rho)} (\| G_1(x, y, s) \|^2 + \| G_2(x, y, s) \|^2)] ds \]
\[ \leq 4 \varepsilon^{2(1-\rho)} C(t+1)(N_1 + N_2) + 2 P r e^{2(1-\rho)} C(t+1)(M_1 + M_2) \]
\[ \leq C \varepsilon^{2(1-\rho)} (t+1)(N_1 + N_2 + M_1 + M_2). \] (102)

So, from (100), (101) and (102), we have
\[ \| \psi(t) \|^2 + \| w(t) \|^2 \leq C \varepsilon^{2(1-\rho)} (t+1)(N_1 + N_2 + M_1 + M_2) e^{C(t+1)} \]
\[ \leq C \varepsilon^{2(1-\rho)} e^{2rt}, \] (103)
where the constants \( C \) and \( r \) are independent of \( \varepsilon \).

6. On the structure of the uniform attractor \( A^\varepsilon \). We assume that \( f_0, g_0 \) are translation compact in the space \( L^2_{loc} (\mathbb{R}; L^2(\Omega)) \) and \( f_1, g_1 \) are translation compact in \( L^2_{loc} (\mathbb{R}; Z) \). We consider the hulls \( \mathcal{H}(f^\varepsilon), \mathcal{H}(g^\varepsilon) \) of \( f^\varepsilon \) and \( g^\varepsilon \) in the space \( L^2_{loc} (\mathbb{R}; L^2(\Omega)) \):
\[ \mathcal{H}(f^\varepsilon) = \{ f^{\varepsilon} (\cdot, t+h) | h \in \mathbb{R} \} \}_{t \in \mathbb{R}} L^2_{loc} (\mathbb{R}; L^2(\Omega)), \mathcal{H}(g^\varepsilon) = \{ g^{\varepsilon} (\cdot, t+h) | h \in \mathbb{R} \} \}_{t \in \mathbb{R}} L^2_{loc} (\mathbb{R}; L^2(\Omega)). \] (104)
Recall that $\mathcal{H}(f^\varepsilon), \mathcal{H}(g^\varepsilon)$ are compact in $L^2_{loc}(\mathbb{R}; L^2(\Omega))$ and each element $\hat{f}^\varepsilon \in \mathcal{H}(f^\varepsilon), \hat{g}^\varepsilon \in \mathcal{H}(g^\varepsilon)$ can be written as

$$
\hat{f}^\varepsilon(x, y, t) = \hat{f}_0(x, y, t) + \varepsilon^{-\alpha} f_1 \left( \frac{x}{\varepsilon}, \frac{y}{\varepsilon}, t \right), \quad \hat{g}^\varepsilon(x, y, t) = \hat{g}_0(x, y, t) + \varepsilon^{-\alpha} g_1 \left( \frac{x}{\varepsilon}, \frac{y}{\varepsilon}, t \right),
$$

with $\hat{f}_0 \in \mathcal{H}(f_0), \hat{g}_0 \in \mathcal{H}(g_0)$ and $f_1, g_1 \in \mathcal{H}(g_1)$, where $\mathcal{H}(f_0)$ and $\mathcal{H}(g_0)$ are the hulls of $f_0$ and $g_0$ in $L^2_{loc}(\mathbb{R}; L^2(\Omega)), \mathcal{H}(f_1)$ and $\mathcal{H}(g_1)$ are the hulls of $f_1$ and $g_1$ in $L^2_{loc}(\mathbb{R}; Z)$, respectively.

We also note that (see [4, 8, 31])

$$
\| \hat{f}_0 \|_{L^2_{loc}(\mathbb{R}; L^2(\Omega))} \leq \| f_0 \|_{L^2_{loc}(\mathbb{R}; L^2(\Omega))}, \quad \forall \hat{f}_0 \in \mathcal{H}(f_0),
$$

$$
\| \hat{g}_0 \|_{L^2_{loc}(\mathbb{R}; L^2(\Omega))} \leq \| g_0 \|_{L^2_{loc}(\mathbb{R}; L^2(\Omega))}, \quad \forall \hat{g}_0 \in \mathcal{H}(g_0).
$$

It follows that

$$
\| \hat{f}^\varepsilon \|_{L^2_{loc}(\mathbb{R}; L^2(\Omega))} \leq \| f_0 \|_{L^2_{loc}(\mathbb{R}; L^2(\Omega))} + C_\varepsilon \| f_1 \|_{L^2_{loc}(\mathbb{R}; Z)}, \quad \forall \hat{f}^\varepsilon \in \mathcal{H}(f^\varepsilon),
$$

$$
\| \hat{g}^\varepsilon \|_{L^2_{loc}(\mathbb{R}; L^2(\Omega))} \leq \| g_0 \|_{L^2_{loc}(\mathbb{R}; L^2(\Omega))} + C_\varepsilon \| g_1 \|_{L^2_{loc}(\mathbb{R}; Z)}, \quad \forall \hat{g}^\varepsilon \in \mathcal{H}(g^\varepsilon),
$$

where $C$ is independent of $f_0, g_0, f_1, g_1, \rho$ and $\varepsilon$.

From Section 3 we know that the system (2)-(4) generates a process $\{U_{(f^\varepsilon, g^\varepsilon)}(t, \tau)\}$ in the space $L^2(\Omega) \times L^2(\Omega)$, where every mapping $U_{(f^\varepsilon, g^\varepsilon)}(t, \tau) : L^2(\Omega) \times L^2(\Omega) \rightarrow L^2(\Omega) \times L^2(\Omega)$ is defined by

$$
U_{(f^\varepsilon, g^\varepsilon)}(t, \tau)(\xi, \theta_\tau) = (\xi(t), \theta(t)), t \geq \tau, \tau \in \mathbb{R},
$$

where $(\xi, \theta_\tau) \in L^2(\Omega) \times L^2(\Omega)$ and $(\xi(t), \theta(t))$ is the solution of (2)-(4) with the initial data $\xi|_{t=\tau} = \xi, \theta|_{t=\tau} = \theta_\tau$.

In Section 3, we proved that the process $\{U_{(f^\varepsilon, g^\varepsilon)}(t, \tau)\}$ has a uniform attractor $\mathcal{A}^\varepsilon$ such that

$$
\mathcal{A}^\varepsilon \subset B_{0,\varepsilon} \cap B_{1,\varepsilon}.
$$

To describe the structure of the attractor $\mathcal{A}^\varepsilon$, we consider the family of equations

$$
\frac{\partial \hat{\xi}}{\partial \tau} - \Delta \hat{\xi} + J(\hat{\Psi}, \hat{\xi}) + \frac{\partial}{\partial x} \frac{\partial \hat{\theta}}{\partial \tau} = \hat{f}^\varepsilon(x, y, t),
$$

$$
\Delta \hat{\Psi} = \hat{\xi},
$$

where $\hat{\xi}, \hat{\theta} \in \mathcal{H}(f^\varepsilon), \hat{\Psi} \in \mathcal{H}(g^\varepsilon)$.

For $\hat{f}^\varepsilon \in \mathcal{H}(f^\varepsilon), \hat{g}^\varepsilon \in \mathcal{H}(g^\varepsilon)$, the Eq. (111) generates a process $\{U_{(\hat{f}^\varepsilon, \hat{g}^\varepsilon)}(t, \tau)\}$ that satisfies the same properties as $\{U_{(f^\varepsilon, g^\varepsilon)}(t, \tau)\}$. From similar arguments as in Section 3, we can easily deduce that the sets $B_{0,\varepsilon}$ and $B_{1,\varepsilon}$ are absorbing sets for the family of processes $\{U_{(f^\varepsilon, g^\varepsilon)}(t, \tau)\}$ (see (106)-(108)). Moreover, the process $\{U_{(f^\varepsilon, g^\varepsilon)}(t, \tau)\}$ has a uniform attractor $\mathcal{A}_{(f^\varepsilon, g^\varepsilon)} \subset \mathcal{A}^\varepsilon$.

**Proposition 4.** Let $f_0, g_0$ be translation compact in $L^2_{loc}(\mathbb{R}; L^2(\Omega)), f_1, g_1$ be translation compact in $L^2_{loc}(\mathbb{R}; Z)$. Then for any fixed $\varepsilon, 0 < \varepsilon \leq 1$, the family of processes $\{U_{(f^\varepsilon, g^\varepsilon)}(t, \tau)\}$, $\hat{f}^\varepsilon \in \mathcal{H}(f^\varepsilon), \hat{g}^\varepsilon \in \mathcal{H}(g^\varepsilon)$ corresponding to (111) has an absorbing set $B_{1,\varepsilon}$, which is bounded in $H^1_0(\Omega) \times H^1_0(\Omega)$ and $L^2(\Omega) \times L^2(\Omega)$. Moreover, the
family \{U(f^{\ast},g^{\ast})(t,\tau)\}, \hat{f}^{\ast} \in \mathcal{H}(f^{\ast}), \hat{g}^{\ast} \in \mathcal{H}(g^{\ast})\) is \((L^2(\Omega) \times L^2(\Omega)) \times (\mathcal{H}(f^{\ast}) \times \mathcal{H}(g^{\ast}))\), \(L^2(\Omega) \times L^2(\Omega))\)-continuous. That is, if

\[
\hat{f}^{\ast} \rightarrow \hat{f}^{\ast}, \hat{g}^{\ast} \rightarrow \hat{g}^{\ast}\) in \(L^2_{\text{loc}}(\mathbb{R}; L^2(\Omega))\), \(\xi_{\tau_n} \rightarrow \xi_\tau, \theta_{\tau_n} \rightarrow \theta_\tau\) in \(L^2(\Omega)\),
\]
then

\[
U(f^{\ast},g^{\ast})(t,\tau)(\xi_{\tau_n},\theta_{\tau_n}) \rightarrow U(f^{\ast},g^{\ast})(t,\tau)(\xi_\tau,\theta_\tau)\) in \(L^2(\Omega) \times L^2(\Omega)\).
\]

**Proof.** The first part of the proposition can be proved as in Section 3. Let \(\tau_n \subset [\tau, +\infty)\) be a time sequence, \(U(f^{\ast},g^{\ast})(t,\tau)(\xi_{\tau_n},\theta_{\tau_n}) = (\hat{\xi}_n, \hat{\theta}_n)\), \(U(f^{\ast},g^{\ast})(t,\tau)(\xi_\tau,\theta_\tau) = (\hat{\xi}, \hat{\theta})\), and \((v_n, w_n) = (\xi_n - \hat{\xi}, \theta_n - \hat{\theta}) = U(f^{\ast},g^{\ast})(t,\tau)(\xi_{\tau_n},\theta_{\tau_n}) - U(f^{\ast},g^{\ast})(t,\tau)(\xi_\tau,\theta_\tau)\). Then \(v_n, w_n\) satisfy

\[
\begin{align*}
\frac{\partial v_n}{\partial t} & - \Delta v_n + J(\hat{\Psi}_n, \hat{\xi}_n) - J(\hat{\Psi}, \hat{\xi}) + \frac{R_n}{P_r} \frac{\partial w_n}{\partial x} = \hat{f}^{\ast} - \hat{f}^{\ast}, \\
\frac{\partial w_n}{\partial t} & - \frac{1}{P_r} \Delta w_n + J(\hat{\Psi}_n, \hat{\theta}_n) - J(\hat{\Psi}, \hat{\theta}) = \hat{g}^{\ast} - \hat{g}^{\ast}.
\end{align*}
\]

Taking the inner product of (114) with \(v_n\) in \(L^2(\Omega)\), using (8) and (10) we find that

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} & \left\| v_n \right\|^2 + \left\| \nabla v_n \right\|^2 = - \int_{\Omega} [J(\hat{\Psi}_n, \hat{\xi}_n) - J(\hat{\Psi}, \hat{\xi})] v_n dx dy - \frac{R_n}{P_r} \int_{\Omega} w_n v_n dx dy \\
& + \int_{\Omega} (\hat{f}^{\ast} - \hat{f}^{\ast}) v_n dx dy \\
& \leq - \int_{\Omega} [J(\hat{\Psi}_n, v_n) + J(\hat{\Psi}_n - \hat{\Psi}, \hat{\xi})] v_n dx dy - \frac{R_n}{P_r} \int_{\Omega} w_n v_n dx dy \\
& + \int_{\Omega} (\hat{f}^{\ast} - \hat{f}^{\ast}) v_n dx dy \\
& \leq \left\| J(\hat{\Psi}_n - \hat{\Psi}, \hat{\xi}) \right\| \left\| v_n \right\| + \frac{R_n}{P_r} \left\| \nabla w_n \right\| \left\| v_n \right\| + \left\| \hat{f}^{\ast} - \hat{f}^{\ast} \right\| \left\| v_n \right\|
\end{align*}
\]

\[
\leq C \left\| \nabla v_n \right\| \left\| \nabla \hat{\xi} \right\| \left\| v_n \right\| + \frac{R_n}{P_r} \left\| \nabla w_n \right\| \left\| v_n \right\| + \left\| \hat{f}^{\ast} - \hat{f}^{\ast} \right\| \left\| v_n \right\|
\]

\[
\leq \frac{1}{2} \left\| \nabla v_n \right\|^2 + C \left\| \nabla \hat{\xi} \right\|^2 \left\| v_n \right\|^2 + \frac{1}{P_r} \left\| \nabla w_n \right\|^2 + \frac{R_n^2}{4P_r} \left\| v_n \right\|^2.
\]

Taking the inner product of (115) with \(w_n\) in \(L^2(\Omega)\), using (8) and (10) we have

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} & \left\| w_n \right\|^2 + \frac{1}{P_r} \left\| \nabla w_n \right\|^2 = - \int_{\Omega} [J(\hat{\Psi}_n, \hat{\theta}_n) - J(\hat{\Psi}, \hat{\theta})] w_n dx dy \\
& + \int_{\Omega} (\hat{g}^{\ast} - \hat{g}^{\ast}) w_n dx dy \\
& = - \int_{\Omega} [J(\hat{\Psi}_n, w_n) + J(\hat{\Psi}_n - \hat{\Psi}, \hat{\theta})] w_n dx dy + \int_{\Omega} (\hat{g}^{\ast} - \hat{g}^{\ast}) w_n dx dy \\
& \leq \left\| J(\hat{\Psi}_n - \hat{\Psi}, \hat{\theta}) \right\| \left\| w_n \right\| + \left\| \hat{g}^{\ast} - \hat{g}^{\ast} \right\| \left\| w_n \right\|
\end{align*}
\]

\[
\leq C \left\| \nabla v_n \right\| \left\| \nabla \hat{\theta} \right\| \left\| w_n \right\| + \left\| \hat{g}^{\ast} - \hat{g}^{\ast} \right\| \left\| w_n \right\|
\]

\[
\leq \frac{1}{2} \left\| \nabla v_n \right\|^2 + C \left\| \nabla \hat{\theta} \right\|^2 \left\| w_n \right\|^2 + \frac{1}{2} \left\| w_n \right\|^2 + \frac{1}{2} \left\| \hat{g}^{\ast} - \hat{g}^{\ast} \right\|^2.
\]
Combining (116) with (117), we have
\[
\frac{d}{dt}(\|v_n\|^2 + \|w_n\|^2) \leq C(\|\nabla \xi\|^2 + \|\nabla \hat{\theta}\|^2 + 1)(\|v_n\|^2 + \|w_n\|^2) \\
+ \|\hat{f}_n^\varepsilon - \hat{f}^\varepsilon\|^2 + \|\hat{g}_n^\varepsilon - \hat{g}^\varepsilon\|^2.
\] (118)

Applying the Gronwall’s inequality to (118) we get
\[
\|v_n(t)\|^2 + \|w_n(t)\|^2 \leq \left(\|v_n(\tau)\|^2 + \|w_n(\tau)\|^2 + \int_\tau^t (\|\hat{f}_n^\varepsilon(s) - \hat{f}^\varepsilon(s)\|^2 \\
+ \|\hat{g}_n^\varepsilon(s) - \hat{g}^\varepsilon(s)\|^2) ds\right) \exp \left\{\int_\tau^t C(\|\nabla \xi(s)\|^2 + \|\nabla \hat{\theta}(s)\|^2 + 1) ds\right\}
\] (119)

Note that (see (19) and (20))
\[
\exp \left\{\int_\tau^t C(\|\nabla \xi(s)\|^2 + \|\nabla \hat{\theta}(s)\|^2 + 1) ds\right\} \leq M(t) < +\infty, \forall t \geq \tau,
\] (120)

\[
f_n^\varepsilon \rightarrow \hat{f}^\varepsilon, g_n^\varepsilon \rightarrow \hat{g}^\varepsilon \text{ as } n \rightarrow \infty \text{ in } L^2_{\text{loc}}(\mathbb{R}; L^2(\Omega)),
\] (121)

\[
\|v_n(\tau)\|^2 = \|\xi_n(\tau) - \hat{\xi}(\tau)\|^2 = \|\xi_n - \xi\|^2 \rightarrow 0, \text{ as } n \rightarrow \infty,
\] (122)

\[
\|w_n(\tau)\|^2 = \|\hat{\theta}(\tau)\|^2 = \|\theta_n - \theta\|^2 \rightarrow 0, \text{ as } n \rightarrow \infty.
\] (123)

Therefore, it follows from (119)-(123) that
\[
\|v_n(t)\|^2 + \|w_n(t)\|^2 = \|\xi_n(t) - \hat{\xi}(t)\|^2 + \|\hat{\theta}(t)\|^2 \rightarrow 0 \text{ as } n \rightarrow \infty,
\]
and (113) is proved, i.e., the family of processes \(\{U_{(f^\varepsilon,g^\varepsilon)}(t,\tau)\}, f^\varepsilon \in \mathcal{H}(f^\varepsilon), g^\varepsilon \in \mathcal{H}(g^\varepsilon)\) is \((L^2(\Omega) \times L^2(\Omega)) \times (\mathcal{H}(f^\varepsilon) \times \mathcal{H}(g^\varepsilon))\; L^2(\Omega) \times L^2(\Omega))\)-continuous.

We denote by \(\mathcal{K}_{(f^\varepsilon,g^\varepsilon)}\), the kernel of (111) with the external force \(\hat{f}^\varepsilon \in \mathcal{H}(f^\varepsilon), \hat{g}^\varepsilon \in \mathcal{H}(g^\varepsilon)\). Let us recall that \(\mathcal{K}_{(f^\varepsilon,g^\varepsilon)}\) is the family of all complete solutions \((\xi(t),\hat{\theta}(t))\), \(t \in \mathbb{R}\) of (111), which are bounded in the norm of \(L^2(\Omega) \times L^2(\Omega)\):
\[
\|\xi(t)\|^2 + \|\hat{\theta}(t)\|^2 \leq C_{\xi,\hat{\theta}}, \forall t \in \mathbb{R}.
\] (124)

The set
\[
\mathcal{K}_{(f^\varepsilon,g^\varepsilon)}(s) = \{ (\xi(s),\hat{\theta}(s)), (\hat{\xi},\hat{\theta}) \in \mathcal{K}_{(f^\varepsilon,g^\varepsilon)} \} \subset L^2(\Omega) \times L^2(\Omega)
\] (125)
is called the kernel section at \(t = s\).

**Proposition 5.** If the functions \(f^\varepsilon, g^\varepsilon\) are translation compact in \(L^2_{\text{loc}}(\mathbb{R}; L^2(\Omega))\), then the process \(\{U_{(f^\varepsilon,g^\varepsilon)}(t,\tau)\}\) corresponding to (2)-(4) has a uniform attractor \(A_{(f^\varepsilon,g^\varepsilon)}\) and the following identity holds true
\[
A^\varepsilon \equiv A_{(f^\varepsilon,g^\varepsilon)} = \bigcup_{f^\varepsilon \in \mathcal{H}(f^\varepsilon), \hat{g}^\varepsilon \in \mathcal{H}(g^\varepsilon)} \mathcal{K}_{(f^\varepsilon,g^\varepsilon)}(0).
\] (126)

Moreover, the kernel \(\mathcal{K}_{(f^\varepsilon,g^\varepsilon)}\) is non-empty for all \(\hat{f}^\varepsilon \in \mathcal{H}(f^\varepsilon), \hat{g}^\varepsilon \in \mathcal{H}(g^\varepsilon)\).

**Proof.** It can be proved as in [6, 8] for the 2D Navier-Stokes equations. \(\square\)
Now we consider the "limiting" case (56). We suppose \( f_0, g_0 \) are translation compact in \( L^2_{\text{loc}}(\mathbb{R}; L^2(\Omega)) \). In particular, (57)-(65) hold true and the process \( \{U(t, f_0, g_0)\}(t, \tau) \) has the uniform attractor \( A^0 \).

We also consider the family of equations

\[
\begin{aligned}
\frac{\partial \xi}{\partial t} - \Delta \xi + J(\bar{\psi}, \xi) + \frac{1}{r} x \frac{\partial \theta}{\partial x} &= \hat{f}_0(x, t), \\
\Delta \bar{\psi} &= \xi,
\end{aligned}
\tag{127}
\]

where \( \hat{f}_0 \in \mathcal{H}(f_0), \hat{g}_0 \in \mathcal{H}(g_0) \).

We can easily deduce that the family of processes \( \{U(f_0, g_0)(t, \tau)\}, \hat{f}_0 \in \mathcal{H}(f_0), \hat{g}_0 \in \mathcal{H}(g_0) \) corresponding to (127) has an absorbing set \( B_{1,0} \) (bounded in \( H^1_0(\Omega) \times H^1_0(\Omega) \))

\[
\| B_{1,0} \|_{H^1_0(\Omega) \times H^1_0(\Omega)} \leq C_{10},
\tag{128}
\]

and the family \( \{U(f_0, g_0)(t, \tau)\}, \hat{f}_0 \in \mathcal{H}(f_0), \hat{g}_0 \in \mathcal{H}(g_0) \) is \((L^2(\Omega) \times L^2(\Omega)) \times (\mathcal{H}(f_0) \times \mathcal{H}(g_0)), L^2(\Omega) \times L^2(\Omega))\)-continuous. Furthermore, the attractor \( A^0 \) of (56) is given by

\[
A^0 = \bigcup_{\hat{f}_0 \in \mathcal{H}(f_0), \hat{g}_0 \in \mathcal{H}(g_0)} \hat{\mathcal{K}}(f_0, g_0)(0),
\tag{129}
\]

where \( \hat{\mathcal{K}}(f_0, g_0) \) is the kernel of (127) with the external forces \( \hat{f}_0 \in \mathcal{H}(f_0), \hat{g}_0 \in \mathcal{H}(g_0) \).

7. Convergence of the uniform attractor \( A^\varepsilon \). In this section, we consider the system (2) and (56) with \( f_0, g_0 \) translation compact in \( L^2_{\text{loc}}(\mathbb{R}; L^2(\Omega)) \) and \( f_1, g_1 \) translation compact in \( L^2_{\text{loc}}(\mathbb{R}; Z) \), respectively. We also assume that \( f_1, g_1 \) satisfies the divergence condition (66).

Let \( (\xi_\tau, \theta_\tau) \in \hat{B} \), where \( \hat{B} \) is the absorbing set given in (84). Let \( (\hat{\xi}, \hat{\theta}) = U(t, \tau)(\xi_\tau, \theta_\tau), t \geq \tau \) be the solution to (2)-(4) with \( \hat{f}^\varepsilon = \hat{f}_0 + \varepsilon^{-\rho} \hat{f}_1 \in \mathcal{H}(f^\varepsilon), \hat{g}^\varepsilon = \hat{g}_0 + \varepsilon^{-\rho} \hat{g}_1 \in \mathcal{H}(g^\varepsilon) \). Let \( (\hat{\xi}, \hat{\theta}) = U(t, \tau)(\xi_\tau, \theta_\tau), t \geq \tau \) be the solution to (56) with \( \hat{f}_0 \in \mathcal{H}(f_0), \hat{g}_0 \in \mathcal{H}(g_0) \).

**Proposition 6.** For \( f^\varepsilon(x, t) = f_0(x, t) + \varepsilon^{-\rho} f_1(\xi, \theta), t \in \mathcal{H}(f^\varepsilon), \hat{g}^\varepsilon(x, t) = \hat{g}_0(x, t) + \varepsilon^{-\rho} \hat{g}_1(\xi, \theta), t \in \mathcal{H}(g^\varepsilon) \), there exists \( \hat{f}_0 \in \mathcal{H}(f_0), \hat{g}_0 \in \mathcal{H}(g_0) \) such that the difference \( (\hat{v}(t), \hat{w}(t)) = U(t, \tau)(\xi_\tau, \theta_\tau) - U(t, \tau)(\xi_\tau, \theta_\tau) \) satisfies

\[
\| \hat{v}(t) \|^2 + \| \hat{w}(t) \|^2 \leq C \varepsilon^{2(1+\rho)} e^{2r(t-\tau)}, \forall \varepsilon, 0 < \varepsilon \leq 1,
\tag{130}
\]

where \( C \) and \( r \) are the same constants as in Theorem 5.1.

**Proof.** Let \( (\xi(t), \theta(t)) = U(t, \tau)(\xi_\tau, \theta_\tau) \) and \( (\xi^0(t), \theta^0(t)) = U(t, \tau)(\xi_\tau, \theta_\tau), \forall t \geq \tau \), be the solution to (2) and (56), respectively. Then, from (87) we have

\[
\| \xi - \xi^0 \|^2 + \| \theta - \theta^0 \|^2 = \| U(t, \tau)(\xi_\tau, \theta_\tau) - U(t, \tau)(\xi_\tau, \theta_\tau) \|^2 \\
\leq C \varepsilon^{2(1+\rho)} e^{2r(t-\tau)}, \forall (\xi_\tau, \theta_\tau) \in \hat{B}.
\tag{131}
\]

Let us check that (131) holds true for the time shifted forces

\[
f^\varepsilon_h(t) = f^\varepsilon(t+h) = f_0(t+h) + \varepsilon^{-\rho} f_1(t+h), g^\varepsilon_h(t) = g^\varepsilon(t+h) = g_0(t+h) + \varepsilon^{-\rho} g_1(t+h),
\]

\[
f_{0h}(t) = f_0(t+h), g_{0h}(t) = g_0(t+h), f_{1h}(t) = f_1(t+h), g_{1h}(t) = g_1(t+h), \forall h \in \mathbb{R}.
\]
First, it is clear that $f_{1h}(x, y, t) = f_1(x, y, t+h)$, $g_{1h}(x, y, t) = g_1(x, y, t+h)$ satisfy the divergence condition (66) with the functions

$$F_{1h}(x, y, t) = F_1(x, y, t+h), \quad G_{1h}(x, y, t) = G_1(x, y, t+h) \in L^2_0(\mathbb{R}; Z), \quad i = 1, 2.$$ 

Therefore, from (87) we have

$$\| U_{(f_{1h}, g_{1h})}(t, \tau)(\xi, \theta) - U_{(f_{0h}, g_{0h})}(t, \tau)(\xi, \theta) \|^2 \leq C e^{2(1-\rho)\epsilon^2 \tau(t-\tau)}, \quad \forall (\xi, \theta) \in \hat{B}. \quad (132)$$

Now, Let $\hat{f}^\varepsilon(x, t) = \tilde{f}_0(x, t) + \varepsilon^{-\rho} \hat{f}_1(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}, t) \in \mathcal{H}(f^\varepsilon)$, $\hat{g}^\varepsilon(x, t) = \tilde{g}_0(x, t) + \varepsilon^{-\rho} \hat{g}_1(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}, t) \in \mathcal{H}(g^\varepsilon)$. Then, there exists a sequence $\{h_i\} \subset \mathbb{R}$, such that

$$f^\varepsilon_{h_i} \to \hat{f}^\varepsilon, \quad g^\varepsilon_{h_i} \to \hat{g}^\varepsilon \quad \text{as} \quad i \to \infty \quad \text{in} \quad L^2_{\text{loc}}(\mathbb{R}; L^2(\Omega)).$$

where $f^\varepsilon_{h_i} = f^\varepsilon(t + h_i), g^\varepsilon_{h_i} = g^\varepsilon(t + h_i)$.

Since $f_0, g_0$ are translation compact in $L^2_{\text{loc}}(\mathbb{R}; L^2(\Omega))$, there exists $\tilde{f}_0 \in \mathcal{H}(f_0)$, $\tilde{g}_0 \in \mathcal{H}(g_0)$, and a subsequence of $\{f_{0h_i}\}, \{g_{0h_i}\}$ (still denoted the same) such that $f_{0h_i} \to \tilde{f}_0$, $g_{0h_i} \to \tilde{g}_0$ as $i \to \infty$ in $L^2_{\text{loc}}(\mathbb{R}; L^2(\Omega))$.

We also have

$$\| U_{(f^\varepsilon_{h_i}, g^\varepsilon_{h_i})}(t, \tau)(\xi, \theta) - U_{(f_{0h_i}, g_{0h_i})}(t, \tau)(\xi, \theta) \|^2 \leq C e^{2(1-\rho)\epsilon^2 \tau(t-\tau)}, \quad (133)$$

$\forall (\xi, \theta) \in \hat{B}, \forall i \in \mathbb{N}$. Using the $(L^2(\Omega) \times L^2(\Omega)) \times \text{continuity of the process } \{U_{(f^\varepsilon_{h_i}, g^\varepsilon_{h_i})}(t, \tau)\}$, $\hat{f}^\varepsilon \in \mathcal{H}(f^\varepsilon)$, $\hat{g}^\varepsilon \in \mathcal{H}(g^\varepsilon)$ given by Proposition 4 and passing to the limit in (133) as $i \to \infty$, we obtain that

$$\| U_{(f^\varepsilon, g^\varepsilon)}(t, \tau)(\xi, \theta) - U_{(f_0, g_0)}(t, \tau)(\xi, \theta) \|^2 \leq C e^{2(1-\rho)\epsilon^2 \tau(t-\tau)}, \quad \forall (\xi, \theta) \in \hat{B},$$

and (130) is proved.

**Theorem 7.1.** Let $\rho \in (0, 1)$ and the functions $f_0, g_0$ be translation compact in $L^2_{\text{loc}}(\mathbb{R}; L^2(\Omega))$, $f_1, g_1$ be translation compact in $L^2_{\text{loc}}(\mathbb{R}; Z)$. Then the attractors $A^\varepsilon$ of (2) converge to the attractor $A^0$ of (56) as $\varepsilon \to 0^+$ in the norm of $L^2(\Omega) \times L^2(\Omega)$, that is:

$$\text{dist}_{L^2(\Omega) \times L^2(\Omega)}(A^\varepsilon, A^0) \to 0 \quad \text{as} \quad \varepsilon \to 0^+. \quad (134)$$

**Proof.** For $\varepsilon > 0$, let $(\xi^\varepsilon, \theta^\varepsilon) \in A^\varepsilon$. Then from (126) there exists a complete bounded solution $(\xi^\varepsilon(t), \theta^\varepsilon(t))$ of (2)-(4) with some external forces

$$\hat{f}^\varepsilon(x, t) = \tilde{f}_0(x, t) + \varepsilon^{-\rho} \hat{f}_1(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}, t) \in \mathcal{H}(f^\varepsilon), \quad \hat{g}^\varepsilon(x, t) = \tilde{g}_0(x, t) + \varepsilon^{-\rho} \hat{g}_1(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}, t) \in \mathcal{H}(g^\varepsilon)$$

such that

$$\hat{\xi}^\varepsilon(0) = 0, \quad \hat{\theta}^\varepsilon(0) = 0.$$ 

For every $R \geq 0$, we have

$$\hat{\xi}^\varepsilon(-R), \quad \hat{\theta}^\varepsilon(-R) \in A^\varepsilon \subset \hat{B}.$$

We also have

$$\hat{(\xi^\varepsilon, \theta^\varepsilon)} = U_{(f^\varepsilon, g^\varepsilon)}(0, -R)(\hat{\xi}^\varepsilon(-R), \hat{\theta}^\varepsilon(-R)), \quad (135)$$

and applying Proposition 6 with $t = 0, \tau = -R$, there exists $\tilde{f}_0 \in \mathcal{H}(f_0), \tilde{g}_0 \in \mathcal{H}(g_0)$ such that

$$\| (\xi^\varepsilon, \theta^\varepsilon) - U_{(f_0, g_0)}(0, -R)(\hat{\xi}^\varepsilon(-R), \hat{\theta}^\varepsilon(-R)) \|^2 \leq C e^{2(1-\rho)\epsilon^2 R}. \quad (136)$$

On the other hand, the set $A^0$ attracts $U_{(f_0, g_0)}(t + \tau, R) \hat{B}$ in $L^2(\Omega) \times L^2(\Omega)$ as $t \to \infty$ (uniform with respect to $\tau \in \mathbb{R}$ and $\tilde{f}_0 \in \mathcal{H}(f_0), \tilde{g}_0 \in \mathcal{H}(g_0)$), then for the
particular \( f_0, \tilde{g}_0 \) and every \( \delta > 0 \), there exists some time \( T(\delta) \geq 0 \), independent of \( R \), such that

\[
\text{dist}_{L^2(\Omega) \times L^2(\Omega)}(U_{(f_0, \tilde{g}_0)}(T - R, -R)(\hat{\xi}^\varepsilon(-R), \hat{\theta}^\varepsilon(-R)), A^0) \leq \delta. 
\] (137)

Choosing \( R = T \), and collecting (136)-(137), we easily get

\[
\text{dist}_{L^2(\Omega) \times L^2(\Omega)}((\xi^\varepsilon, \theta^\varepsilon), A^0) \\
\leq ||(\xi^\varepsilon, \theta^\varepsilon) - U_{(f_0, \tilde{g}_0)}(0, -R)(\hat{\xi}^\varepsilon(-R), \hat{\theta}^\varepsilon(-R))||_{L^2(\Omega) \times L^2(\Omega)} \\
+ \text{dist}_{L^2(\Omega) \times L^2(\Omega)}(U_{(f_0, \tilde{g}_0)}(0, -R)(\hat{\xi}^\varepsilon(-R), \hat{\theta}^\varepsilon(-R)), A^0) \\
\leq C\varepsilon^{-\rho R} + \delta. 
\] (138)

Since \((\xi^\varepsilon, \theta^\varepsilon) \in A^\varepsilon \) and \( \delta > 0 \) is arbitrary, taking the limit \( \varepsilon \to 0^+ \), we can prove the theorem. \( \square \)

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**REFERENCES**

[1] C. T. Anh and N. D. Toan, *Nonclassical diffusion equations on \( \mathbb{R}^N \) with singularly oscillating external forces*, *Appl. Math. Lett.*, **38** (2014), 20–26.

[2] A. V. Babin and M. I. Vishik, *Attractors of Evolution Equations*, Studies in Mathematics and its Applications, 25, North-Holland Publishing Co., Amsterdam, 1992.

[3] V. V. Chepyzhov, M. Conti and V. Pata, *Averaging of equations of viscoelasticity with singularly oscillating external forces*, *J. Math. Pures Appl. (9)*, **108** (2017), 841–868.

[4] V. V. Chepyzhov, V. Pata and M. I. Vishik, *Averaging of 2D Navier-Stokes equations with singularly oscillating forces*, *Nonlinearity*, **22** (2009), 351–370.

[5] V. V. Chepyzhov, V. Pata and M. I. Vishik, *Averaging of nonautonomous damped wave equations with singularly oscillating external forces*, *J. Math. Pures Appl. (9)*, **90** (2008), 469–491.

[6] V. V. Chepyzhov and M. I. Vishik, *Attractors for Equations of Mathematical Physics*, American Mathematical Society Colloquium Publications, 49, American Mathematical Society, Providence, RI, 2002.

[7] V. V. Chepyzhov and M. I. Vishik, *Non-autonomous 2D Navier-Stokes system with a simple global attractor and some averaging problems*, *ESAIM Control Optim. Calc. Var.*, **8** (2002), 467–487.

[8] V. V. Chepyzhov and M. I. Vishik, *Non-autonomous 2D Navier-Stokes system with singularly oscillating external force and its global attractor*, *J. Dynam. Differential Equations*, **19** (2007), 655–684.

[9] V. V. Chepyzhov, M. I. Vishik and W. L. Wendland, *On non-autonomous sine-Gordon type equations with a simple global attractor and some averaging*, *Discrete Contin. Dyn. Syst.*, **12** (2005), 27–38.

[10] M. Efendiev and S. Zelik, *Attractors of the reaction-diffusion systems with rapidly oscillating coefficients and their homogenization*, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **19** (2002), 961–989.

[11] M. Efendiev and S. Zelik, *The regular attractor for the reaction-diffusion systems with a nonlinearity rapidly oscillating in time and its averaging*, *Adv. Differential Equations*, **8** (2003), 673–732.

[12] S.-M. Fang, L.-Y. Jin and B.-L. Guo, *Global existence of solutions to the periodic initial value problems for two-dimensional Newton-Boussinesq equations*, *Appl. Math. Mech. (English Ed.)*, **31** (2010), 405–414.

[13] G. Fucci, B. Wang and P. Singh, *Asymptotic behavior of the Newton-Boussinesq equation in a two-dimensional channel*, *Nonlinear Anal.*, **70** (2009), 2000–2013.

[14] B. Guo, *Nonlinear Galerkin methods for solving two-dimensional Newton-Boussinesq equations*, *Chinese Ann. Math. Ser. B*, **16** (1995), 379–390.
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[15] B. L. Guo, Spectral method for solving the two-dimensional Newton-Boussinesq equations, Acta Math. Appl. Sinica (English Ser.), 5 (1989), 208–218.

[16] J. K. Hale, Asymptotic behavior of dissipative systems, in Dynamics of Infinite-Dimensional Systems, NATO Adv. Sci. Inst. Ser. F Comput. Systems Sci., 37, Springer, Berlin, 1987, 123–128.

[17] Y. Hou and K. Li, The uniform attractor for the 2D non-autonomous Navier-Stokes flow in some unbounded domain, Nonlinear Anal., 58 (2004), 699–630.

[18] H. Ma and Q. Zhang, Global existence and uniqueness of Yudovich’s solutions to the 3D Newton-Boussinesq system, Acta Math. Appl. Sinica (English Ser.), 208–218.

[19] J. K. Hale, Asymptotic behavior of dissipative systems, in Dynamics of Infinite-Dimensional Systems, NATO Adv. Sci. Inst. Ser. F Comput. Systems Sci., 37, Springer, Berlin, 1987, 123–128.

[20] Y. Qin, X. Yang and X. Liu, Averaging of 3D Navier-Stokes-Voight equation with singularly oscillating forces, Nonlinear Anal., 58 (2004), 609–630.

[21] H. Ma and Q. Zhang, Global existence and uniqueness of Yudovich’s solutions to the 3D Newton-Boussinesq system, Appl. Anal., 97 (2018), 1814–1827.

[22] Y. Qin, X. Yang and X. Liu, Averaging of 3D Navier-Stokes-Voight equation with singularly oscillating forces, Nonlinear Anal. Real World Appl., 13 (2012), 893–904.

[23] H. Ma and Q. Zhang, Global existence and uniqueness of Yudovich’s solutions to the 3D Newton-Boussinesq system, Acta Math. Appl. Sinica (English Ser.), 5 (1989), 208–218.

[24] J. K. Hale, Asymptotic behavior of dissipative systems, in Dynamics of Infinite-Dimensional Systems, NATO Adv. Sci. Inst. Ser. F Comput. Systems Sci., 37, Springer, Berlin, 1987, 123–128.

[25] Y. Hou and K. Li, The uniform attractor for the 2D non-autonomous Navier-Stokes flow in some unbounded domain, Nonlinear Anal., 58 (2004), 699–630.

[26] H. Ma and Q. Zhang, Global existence and uniqueness of Yudovich’s solutions to the 3D Newton-Boussinesq system, Acta Math. Appl. Sinica (English Ser.), 5 (1989), 208–218.

[27] J. K. Hale, Asymptotic behavior of dissipative systems, in Dynamics of Infinite-Dimensional Systems, NATO Adv. Sci. Inst. Ser. F Comput. Systems Sci., 37, Springer, Berlin, 1987, 123–128.

[28] Y. Qin, X. Yang and X. Liu, Averaging of 3D Navier-Stokes-Voight equation with singularly oscillating forces, Nonlinear Anal. Real World Appl., 13 (2012), 893–904.

[29] H. Ma and Q. Zhang, Global existence and uniqueness of Yudovich’s solutions to the 3D Newton-Boussinesq system, Acta Math. Appl. Sinica (English Ser.), 5 (1989), 208–218.

[30] H. Ma and Q. Zhang, Global existence and uniqueness of Yudovich’s solutions to the 3D Newton-Boussinesq system, Acta Math. Appl. Sinica (English Ser.), 5 (1989), 208–218.

[31] H. Ma and Q. Zhang, Global existence and uniqueness of Yudovich’s solutions to the 3D Newton-Boussinesq system, Acta Math. Appl. Sinica (English Ser.), 5 (1989), 208–218.