Canards and black swans in a model of a 3-D autocatalator

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Abstract.

The mathematical model of a 3-D autocatalator is studied using the geometric theory of singular perturbations, namely, the black swan and canard techniques.

Critical regimes are modeled by canards (one-dimensional stable-unstable slow integral manifolds). The meaning of criticality here is as follows. The critical regime corresponds to a chemical reaction which separates the domain of self-accelerating reactions from the domain of slow reactions.

A two-dimensional stable-unstable slow integral manifold (black swan) consisting entirely of canards, which simulate the critical phenomena for different initial data of the dynamical system, is constructed. It is shown that this procedure leads to the phenomenon of auto-oscillations in the chemical system.

The geometric approach combined with asymptotic and numerical methods permits us to explain the strong parametric sensitivity and to obtain asymptotic representations of the critical behavior of the chemical system.

1. Introduction

Recall that a canard is a trajectory of a singularly perturbed system of differential equations if it, at first, follows a stable slow integral manifold, and then an unstable one [1]. In both cases the distances travelled are more than infinitesimally small. The slow integral manifold is defined as an invariant surface of slow motions [21].

A canard trajectory may be considered as the result of gluing stable (attractive) and unstable (repulsive) slow integral manifolds at one point of the breakdown surface, due to the availability of an additional scalar parameter in the differential system. If we take an additional function of a vector variable parameterizing the breakdown surface, we can glue the stable (attractive) and unstable (repulsive) slow integral manifolds at all points of the breakdown curve at the same time. As a result we obtain a continuous stable/unstable (attractive/repulsive) integral surface or black swan [20]. Such surfaces are considered as a multidimensional analogue of the notion of a canard. It is possible to consider the gluing function as a special kind of partial feedback control. This guarantees the safety of chemical regimes, even with perturbations, during a chemical process.

The main object of our consideration is the following singularly perturbed system

\[ \dot{x} = f(x, y, z, \varepsilon), \]
\[ \dot{y} = g(x, y, z, \varepsilon), \]
\[ \varepsilon \dot{z} = p(x, y, z, \alpha, \varepsilon), \]

where \( \varepsilon \) is a small positive parameter, \( \alpha \) is a scalar parameter, \( x \) and \( z \) are scalar variables, \( y \) is a vector of dimension \( n \), and the dot refers to differentiation with respect to time \( t \). We detach the variable \( x \) for the following reason: it will be used as a new independent variable when the original variable \( t \) is excluded. For nonautonomous systems the variable \( t \) plays the role of the variable \( x \) and \( f \equiv 1 \) in this case.

Recall that the slow surface \( S \) of the system (1) is the surface described by the equation

\[ p(x, y, z, \alpha, 0) = 0. \]

Let \( z = \phi(x, y, \alpha) \) be an isolated solution of equation (2). We call the subset \( S^s \) (\( S^u \)) of \( S \), defined by

\[ \frac{\partial p}{\partial z}(x, y, \phi(x, y), \alpha, 0) < 0, \ (>) 0, \]

the stable (unstable) leaf (or part) of \( S \).

The subset of \( S \) defined by

\[ \frac{\partial p}{\partial z}(x, y, \phi(x, y), \alpha, 0) = 0 \]

is called the breakdown surface. Its dimension is equal to \( \dim y \).

In an \( \varepsilon \)-neighborhood of \( S^s \) (\( S^u \)) there exists a stable (unstable) slow integral manifold. This means that the slow surface is an approximation to the slow integral manifold (for \( \varepsilon = 0 \)) [21].

The availability of the additional scalar parameter \( \alpha \) provides the possibility of gluing the stable and unstable integral manifolds at one point of the breakdown surface. The canard trajectory passes through this point.

It should be noted that in the early papers devoted to canards in the case \( \dim y = 0 \), the existence of a unique canard corresponding to a unique value of the parameter \( \alpha = \alpha^* \) was stated (more precisely, the “canard” value of the parameter \( \alpha^* \) exists on an interval of order \( O(e^{-1/\varepsilon}) \)). This property is known as the short life of canards [1, 3, 7, 24]. But, in the case when \( \dim y = 1 \), another picture is beginning to emerge. It was shown that a one-parameter family of canards exists [22]. In different situations canards for systems with a vector slow variable were studied in [2, 12, 16] and others. If we take the parameter \( \alpha \) as a function of \( y \), we can glue the stable and unstable integral manifolds along all points of the breakdown curve at the same time. As a result we obtain the continuous stable/unstable integral surface or black swan. Such surfaces may be considered as a multidimensional analogue of a canard.

Note that the notion of a black swan can be used in the case \( \dim z = m > 1 \), see [17]. The discussion of this notion with a number of simple examples can be found in [18, 19].

Canards and black swans have interesting applications in the modelling of critical phenomena in various problems, including the problems of chemical kinetics. We shall use canards as separating solutions. This means that canards simulate the critical regimes separating the basic types of chemical regimes. This approach was proposed for the first time in [10] and was then applied in [9, 22, 23] and other papers. Later this approach was extended to black swans, see, for example, [16, 18]. It should be noted that the first use of canards in applied problems for models different from the van der Pol equation, namely, for mechanical and chemical systems, is in [4, 6, 10]. In last few years there has been a considerable number of papers devoted to canards in applied problems (see, for example, [5, 8, 13] and references therein).

To explain the situation in the model of a 3-D autocatalator we give the following examples.

**Example 1 (dim y = 0).**
Consider the van der Pol equation in the form of a plane system with an additional parameter $\alpha$:

$$\dot{x} = z - \alpha, \quad \varepsilon \dot{z} = -1/3z^3 + z - x.$$

The jump points $(-2/3, -1)$ and $(2/3, 1)$ divide the slow curve $x = -1/3z^3 + z$ into stable ($z < -1$ and $z > 1$) and the unstable ($-1 < z < 1$) parts, and we can observe the canard phenomenon near the first jump point with $\alpha = -1 + \varepsilon/8 + \varepsilon^2 \ldots$ or near the second one with $\alpha = 1 - \varepsilon/8 + \varepsilon^2 \ldots$, see Figure 1.

**Example 2 (dim $y = 1$).**

Consider now the 3-D modification of system (3):

$$\dot{x} = z - \alpha, \quad \dot{y} = -0.3y, \quad \varepsilon \dot{z} = -1/3z^3 + z - x.$$

The system (4) has a black swan. This black swan is a cylindrical attractive surface, see Figure 2. All trajectories on this surface are canards, but only one of them is a cycle on the plane $y = 0$. We can observe a similar situation in a 3-D autocatalator model.

![Figure 1. The slow curve (the dashed line) and the canard (the solid line) of the system (3) near jump point $(-2/3, -1)$.

Figure 2. The trajectory of the system (4) with $\varepsilon = 0.1$ and initial point $x = 0.02, y = 0.1, z = -0.2$.](image1)

2. **3-D autocatalator**

The application of black swans consisting entirely of canards to the modelling of critical phenomena permits us to take into account small perturbations in the chemical systems. Moreover we can use black swans to model of critical phenomena in chemical problems without fixed initial conditions.
As an illustration let us consider a model of the three–dimensional autocatalator [14, 15]:

\[
\begin{align*}
\frac{dx}{d\tau} &= \mu \left( \frac{5}{2} + y \right) - xz^2 - x, \\
\frac{dy}{d\tau} &= z - y, \\
\frac{dz}{d\tau} &= xz^2 + x - z,
\end{align*}
\]

(5)

where

\[
x \geq 0, \quad y \geq 0, \quad z \geq 0, \quad 0 \leq \mu < 1.
\]

(6)

The system (5) simulates a sort of the Belousov–Zhabotinsky reaction. The variables \(x, y\) and \(z\) represent dimensionless concentrations of three chemical reagents, \(\varepsilon\) is a small positive parameter, and \(\mu\) is a bifurcation parameter. Note that this and similar models have been studied in [6, 11] and elsewhere.

The slow surface (see Figure 3) of the system (5) is described by the equation

\[F(x, y, z) = xz^2 + x - z = 0.\]

The breakdown surface, which is described by

\[F = 0, \quad \frac{\partial F}{\partial z} = 2xz - 1 = 0,\]

consists of two straight lines, but only one point

\[x = 0.5, \quad z = 1\]

(7)

has physical meaning. The breakdown surface divides the slow surface into three leaves \(S_1^u\) (\(z > 1\)), \(S_2^u\) (\(z < 1\)), \(S^a\) (\(|z| < 1\), see Figure 3), which are zeroth order approximations for the corresponding slow integral manifolds \(S_1^{u,\varepsilon}\), \(S_2^{u,\varepsilon}\), and \(S^a_{\varepsilon}\). Manifolds \(S_1^{u,\varepsilon}\) and \(S_2^{u,\varepsilon}\) are unstable and \(S^a_{\varepsilon}\) is stable. Note, that the part of \(S^a_{\varepsilon}\) with \(0 \leq x < 0.5\) and \(S_1^{u,\varepsilon}\) are situated in the domain of interest to us (6).
3. Location of equilibrium

The system (5) has an equilibrium at \( P \) given by

\[
\left( \frac{10\mu(1 - \mu)}{29\mu^2 - 8\mu + 4}, \frac{5\mu}{2(1 - \mu)}, \frac{5\mu}{2(1 - \mu)} \right),
\]

and with \( \mu = 0 \) this equilibrium is a stable node at the origin.

To analyze the type of equilibrium at \( P \) when \( 0 < \mu < 1 \), we introduce new variables

\[
\bar{x} = x - B, \quad \bar{y} = y - A, \quad \bar{z} = z - A,
\]

where

\[
A = \frac{5\mu}{2(1 - \mu)}, \quad B = \frac{10\mu(1 - \mu)}{29\mu^2 - 8\mu + 4}.
\]

The corresponding Jacobian matrix is

\[
\begin{pmatrix}
-D & \mu & -2AB \\
0 & -1 & 1 \\
\varepsilon^{-1}D & 0 & -\varepsilon^{-1}C
\end{pmatrix},
\]

where \( C = 1 - 2AB \) and \( D = 1 + A^2 \).

Depending on the value of the parameter \( \mu \), the equilibrium point \( P \) lies either on the stable leaf of the slow surface, or on the unstable one. \( P \) belongs to a stable leaf of the slow surface \((0 \leq z < 1)\) and in the domain of interest (6), if the following conditions are satisfied:

\[
\left\{ \begin{array}{l}
\mu/(1 - \mu) \geq 0, \\
5\mu/(2(1 - \mu)) < 1.
\end{array} \right.
\]

Hence, with \( 0 \leq \mu < 2/7 \) the equilibrium \( P \) lies on the stable leaf of the slow surface \( S^s \); with \( \mu = 2/7 \) it lies on the breakdown surface; with \( 2/7 < \mu < 1 \) it lies on the unstable leaf \( S^u \).

The characteristic polynomial

\[
\varepsilon \lambda^3 + \lambda^2(1 - 2AB + \varepsilon(2 + A^2)) + \lambda(A^2 - 2AB + 2 + \varepsilon(1 + A^2)) + (1 + A^2)(1 - \mu) = 0
\]

is stable (i.e. all its roots have negative real parts) if and only if the conditions

\[
\left\{ \begin{array}{l}
D(1 - \mu) > 0, \\
C + \varepsilon(1 + D) > 0, \\
C + D + \varepsilon D > 0, \\
(C + \varepsilon(1 + D))(C + D + \varepsilon D) - \varepsilon D(1 - \mu) > 0
\end{array} \right.
\]

are satisfied. From this system it follows that \( P \) is a stable equilibrium for \( 0 < \mu < 2/7 \).

To investigate the equilibrium on the unstable leaf of the slow surface \( S^u \) consider an auxiliary system (the vector form of the corresponding linearized system):

\[
\left\{ \begin{array}{l}
\dot{a} = W_1 a + W_2 b, \\
\varepsilon \dot{b} = W_3 a + W_4 b.
\end{array} \right.
\]

Here,

\[
W_1 = \begin{pmatrix} -D & \mu \\ 0 & -1 \end{pmatrix}, \quad W_2 = \begin{pmatrix} -2AB \\ 1 \end{pmatrix}, \quad a = \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix},
\quad W_3 = \begin{pmatrix} -\varepsilon^{-1}D & 0 \end{pmatrix}, \quad W_4 = \begin{pmatrix} \varepsilon^{-1}C & 0 \end{pmatrix}, \quad b = \bar{z},
\]

and the matrix $W_4$ defines the kind of stability on the slow surface.

On the slow surface we have

$$W_3a + W_4b = 0$$

or

$$b = -W_4^{-1}W_3a,$$

and then

$$\dot{a} = \left(W_1 - W_2W_4^{-1}W_3\right)a.$$

The Jacobian matrix of the vector field associated with system (5)

$$\left(W_1 - W_2W_4^{-1}W_3\right) = \begin{pmatrix} -D/C & \mu \\ D/C & -1 \end{pmatrix}$$

has the characteristic polynomial

$$\lambda^2 + \lambda(1 + D/C) + D(1 - \mu)/C = 0.$$ 

This polynomial is stable if and only if

$$\begin{cases} 
1 + D/C > 0, \\
D(1 - \mu)/C > 0.
\end{cases}$$

Hence, $P$ is a stable equilibrium (node) on the stable leaf of slow surface (with $0 \leq \mu < 2/7$); $P$ is an unstable equilibrium (saddle) on the unstable leaf $S_u^1$ (with $\mu > 2/7$).

The slow surface is an approximation to a slow integral manifold (for $\varepsilon = 0$), hence the previous analysis implies that it is possible to determine the basic types of chemical regimes and corresponding values of the control parameter $\mu$.

With $0 \leq \mu < 2/7$ a trajectory of the system (5), starting from an initial point in the basin of attraction of the stable slow integral manifold $S_s^1$, after a short time follows $S_s^1$ and tends to the stable equilibrium $P$ as $\tau \to \infty$. This behavior corresponds to the slow chemical regime, see Figures 4, 5.
Figure 6. Projections of the slow surface (the dashed line) and the trajectory (the solid line) of the system (5) on the plane $xOz$ for $\mu = 0.4$, $\varepsilon = 0.01$ and initial point $(0.1, 1, 2)$. $P$ lies on the unstable part of the slow surface.

With $\mu > 2/7$ a trajectory of (5) will follow the $S^u_\varepsilon$ to the breakdown line. After this time, $z(\tau)$ will increase rapidly. This behavior characterizes the explosive regime, see Figures 6, 7.

Due to the continuous dependence of the right-hand side of (5) on the parameter $\mu$ we can say that there is a critical trajectory in the neighborhood of $\mu = 2/7$, which separates the regions of the chemical reactions shown above.

The availability of the additional scalar parameter $\mu$ provides the possibility of gluing $S^u_\varepsilon$ and $S^s_{1,\varepsilon}$ at a point of the breakdown line (7). The canard trajectory passes through this point.

The canard plays the role of a separating solution, and is taken as a mathematical object to model the critical trajectory corresponding to the critical value $\mu = \mu^* = 2/7 + O(\varepsilon)$, $(\varepsilon \to 0)$. This means that canard simulates a chemical reaction separating the domain of self-accelerating reactions ($\mu > \mu^*$) and the domain of slow reactions ($\mu < \mu^*$).

4. Canard in a 3–D autocatalator

We can find the canard solution, and corresponding critical value of the parameter $\mu = \mu^*$, by including the following asymptotic expansions

$$z = z(x, \varepsilon) = \varphi_0(x) + \varepsilon \varphi_1(x) + \varepsilon^2 \varphi_2(x) + \ldots,$$

$$y = y(x, \varepsilon) = \psi_0(x) + \varepsilon \psi_1(x) + \varepsilon^2 \psi_2(x) + \ldots,$$

$$\mu^* = \mu(\varepsilon) = \mu_0 + \varepsilon \mu_1 + \varepsilon^2 \mu_2 + \ldots.\quad (8)$$

From (5) and (8) we have

$$\varepsilon z'(x, \varepsilon) \left[ \mu \left( 5/2 + y(x, \varepsilon) \right) - x z^2(x, \varepsilon) - x \right] = x z^2(x, \varepsilon) + x - z(x, \varepsilon),$$

$$y'(x, \varepsilon) \left[ \mu \left( 5/2 + y(x, \varepsilon) \right) - x z^2(x, \varepsilon) - x \right] = z(x, \varepsilon) - y(x, \varepsilon)$$

or, in more detailed form,

$$\varepsilon \left[ \varphi'_{0} + \varepsilon \varphi'_{1} + \varepsilon^2 \varphi'_{2} + \ldots \right] \left[ \left( \mu_0 + \varepsilon \mu_1 + \varepsilon^2 \mu_2 + \ldots \right) \left( 5/2 + \psi_0 + \varepsilon \psi_1 + \varepsilon^2 \psi_2 + \ldots \right) \right]$$
\[-x \left( \varphi_0 + \varepsilon \varphi_1 + \varepsilon^2 \varphi_2 + \ldots \right)^2 - x \right] = x \left( \varphi_0 + \varepsilon \varphi_1 + \varepsilon^2 \varphi_2 + \ldots \right)^2
\]
\[+ x - \left( \varphi_0 + \varepsilon \varphi_1 + \varepsilon^2 \varphi_2 + \ldots \right), \quad (9)\]
\[
\left[ \psi'_0 + \varepsilon \psi'_1 + \varepsilon^2 \psi'_2 + \ldots \right] \left[ \left( \mu_0 + \varepsilon \mu_1 + \varepsilon^2 \mu_2 + \ldots \right) \left( 5/2 + \psi_0 + \varepsilon \psi_1 + \varepsilon^2 \psi_2 + \ldots \right)
- x \left( \varphi_0 + \varepsilon \varphi_1 + \varepsilon^2 \varphi_2 + \ldots \right)^2 - x \right] = \left( \varphi_0 + \varepsilon \varphi_1 + \varepsilon^2 \varphi_2 + \ldots \right) - \left( \psi_0 + \varepsilon \psi_1 + \varepsilon^2 \psi_2 + \ldots \right). \quad (10)\]

Setting \( \varepsilon = 0 \) in (9), (10) we obtain
\[
x \varphi_0^2 - \varphi_0 + x = 0,
\]
\[
\psi'_0 \left[ \mu_0 (5/2 + \psi_0) - x \varphi_0^2 - x \right] = \varphi_0 - \psi_0,
\]
or
\[
x \varphi_0^2 - \varphi_0 + x = 0,
\]
\[
\psi'_0 \left[ \mu_0 (5/2 + \psi_0) - \varphi_0 \right] = \varphi_0 - \psi_0. \quad (11)\]

Then equation (11) defines the function \( \varphi_0 = \varphi_0(x) \), and equation (12) defines the function \( \psi_0 = \psi_0(x) \).

Next, we equate terms in \( \varepsilon^1 \) in (9), (10):
\[
\varphi'_0 \left[ \mu_0 (5/2 + \psi_0) - x \varphi_0^2 - x \right] = \varphi_1 (2x \varphi_0 - 1),
\]
\[
\psi'_1 \left[ \mu_0 (5/2 + \psi_0) - x \varphi_0^2 - x \right] + \psi'_0 \left[ \mu_1 (5/2 + \psi_0) + \mu_0 \psi_1 - 2x \varphi_0 \varphi_1 \right] = \varphi_1 - \psi_1
\]
or, taking into account (11),
\[
\varphi'_0 \left[ \mu_0 (5/2 + \psi_0) - \varphi_0 \right] = \varphi_1 (2x \varphi_0 - 1), \quad (13)
\]
\[
\psi'_1 \left[ \mu_0 (5/2 + \psi_0) - \varphi_0 \right] + \psi'_0 \left[ \mu_1 (5/2 + \psi_0) + \mu_0 \psi_1 - 2x \varphi_0 \varphi_1 \right] = \varphi_1 - \psi_1. \quad (14)
\]

On the breakdown surface we have \( 2x \varphi_0 - 1 = 0 \). By continuity of the function \( \varphi_1 = \varphi_1(x) \) we require the condition:
\[
\mu_0 (5/2 + \psi_0 (0.5)) - \varphi_0 (0.5) = 0.
\]

From this and (7) we obtain
\[
\mu_0 = \frac{1}{5/2 + \psi_0 (0.5)}, \quad (15)
\]
since \( \varphi_0 (0.5) = 1 \). Next, equating terms in \( \varepsilon^2 \) in (9), (10) and applying (11), we get
\[
\varphi'_0 \left[ \mu_1 (5/2 + \psi_0) + \mu_0 \psi_1 - 2x \varphi_0 \varphi_1 \right] + \varphi'_1 \left[ \mu_0 (5/2 + \psi_0) - \varphi_0 \right] = x \varphi_1^2 + (2x \varphi_0 - 1) \varphi_2, \quad (16)
\]
\[
\psi'_0 \left[ \mu_0 \psi_2 + \mu_1 \psi_1 + \mu_2 (5/2 + \psi_0) - 2x \varphi_0 \varphi_2 - \varphi_1^2 \right]
+ \psi'_1 \left[ \mu_1 (5/2 + \psi_0) + \mu_0 \psi_1 - 2x \varphi_0 \varphi_1 \right] + \psi'_2 \left[ \mu_0 (5/2 + \psi_0) - \varphi_0 \right] = \varphi_2 - \psi_2.
\]

On the breakdown line (7) the coefficient of the function \( \varphi_2 = \varphi_2(x) \) in (16) is zero. To avoid a discontinuity in this function we require, taking into account (15), the following condition
\[
\varphi'_0 (0.5) \left[ \mu_1 (5/2 + \psi_0 (0.5)) + \mu_0 \psi_1 (0.5) - \varphi_0 (0.5) \varphi_1 (0.5) \right] = 0.5 \varphi_1^2 (0.5). \quad (17)
To calculate the value $\varphi'_0(0.5)$ we differentiate (11) with respect to $x$:

$$\varphi'_0 + \varphi'_0(2x\varphi_0 - 1) + 1 = 0.$$ 

The coefficient of $\varphi'_0$ in this expression is equal to zero on the breakdown surface. Therefore, differentiating the last equation with respect to $x$, we have

$$2\varphi'_0(2\varphi_0 + x\varphi'_0) + \varphi''_0(2x\varphi_0 - 1) = 0,$$

and from this we obtain $\varphi'_0(0.5) = -4$ on the breakdown line. Substituting this value into (17) we find

$$\mu_1 = \frac{\varphi_1(0.5)[1 - \varphi_1(0.5)/8] - \mu_0\psi_1(0.5)}{5/2 + \psi_0(0.5)}. \quad (18)$$

The expressions (11)–(15), (18) define the first-order approximations to the critical value of the parameter $\mu$ and the corresponding canard of the system (5). This canard simulates the critical regime, separating slow chemical regimes from regimes with a self-acceleration.

Note, the initial data for the system (5) are not fixed. With specified initial data we can glue the stable and unstable slow integral manifold at one point on the breakdown line (7). The canard passes through this point and corresponds to the initial value problems for the nonlinear problem (12), (14). Thus, a canard is a result of gluing stable and unstable slow integral manifolds at one point of the breakdown surface.

Let $\mu = \mu(y,\varepsilon)$ be considered as a function. Then the gluing of the stable and unstable parts of slow integral manifolds can be realized at all points of the breakdown line (7) at the same time. This permits us to construct slow integral manifolds with changing stability (black swan) consisting entirely of canards. Each simulates the critical regime corresponding to the specified initial data and passes through a definite point on the breakdown line.

5. Black swan construction

Let us take $\mu = \mu(y,\varepsilon)$ as control function. Then it and the black swan $x = x(y, z, \varepsilon)$ have asymptotic expansions of the form:

$$\mu = \mu_0(y) + \varepsilon\mu_1(y) + \varepsilon^2\mu_2(y) + \ldots,$$

$$x = x_0(y, z) + \varepsilon x_1(y, z) + \varepsilon^2 x_2(y, z) + \ldots.$$

Substituting these expansions into the equation

$$\frac{\partial x(y, z, \varepsilon)}{\partial z} - x(y, z, \varepsilon)z^2 + x(y, z, \varepsilon) - z + \frac{\partial x(y, z, \varepsilon)}{\partial y}(z - y) = \mu(y,\varepsilon)(5/2 + y) - x(y, z, \varepsilon)z^2 - x(y, z, \varepsilon),$$

which follows from (5), and using the slow surface equation

$$x_0z^2 + x_0 - z = 0, \quad (19)$$

we obtain

$$\left(\frac{\partial x_0}{\partial z} + \varepsilon\frac{\partial x_1}{\partial z} + \varepsilon^2\frac{\partial x_2}{\partial z} + \ldots\right)\left(x_1 + \varepsilon x_2 + \ldots\right)(1 + z^2)$$

$$+ \left(\frac{\partial x_0}{\partial y} + \varepsilon\frac{\partial x_1}{\partial y} + \varepsilon^2\frac{\partial x_2}{\partial y} + \ldots\right)(z - y) =$$
\begin{equation}
\mu_0 + \varepsilon \mu_1 + \varepsilon^2 \mu_2 + \ldots \left( 5/2 + y \right) - \left( x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \ldots \right) z^2 - \left( x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \ldots \right) .
\end{equation}

Setting \( \varepsilon = 0 \) in (20) and taking (19) into account, we get

\begin{equation}
\frac{\partial x_0}{\partial z} (1 + z^2) x_1 = \mu_0 \left( \frac{5}{2} + y \right) - z.
\end{equation}

Note, that the relationship

\[ \frac{\partial x_0}{\partial z} = 0 \]

holds on the breakdown line (7). By continuity of the function \( x_1 = x_1(z, c) \) we therefore require the following condition:

\[ \mu_0 = \frac{1}{(5/2 + y)}. \]

From this and (21) we have

\begin{equation}
x_1(y, z) = \frac{1 + z^2}{1 + z}.
\end{equation}

Equating coefficients in \( \varepsilon \) in (20), we obtain

\begin{equation}
\frac{\partial x_1}{\partial y} (z - y) + \frac{\partial x_1}{\partial z} (1 + z^2) x_1 + \frac{\partial x_0}{\partial z} (1 + z^2) x_2 = \mu_1 (5/2 + y) - x_1 (1 + z^2).
\end{equation}

To avoid a discontinuity in the function \( x_2 = x_2(y, z) \) on the breakdown line we require the following condition:

\[ \mu_1 = \frac{3}{(5/2 + y)}. \]

Applying (19), (22) and (23) yields

\[ x_2(y, z) = \frac{3(1 + z)^3 - 2 z (2 + z) (1 + z^2)^2}{(1 + z)^3 (1 - z^2)} \]

Thus, we obtain the approximate black swan

\[ x(y, z, \varepsilon) = \frac{z}{1 + z^2} + \varepsilon \frac{1 + z^2}{1 + z} + \varepsilon^2 \frac{[3(1 + z)^3 - 2z(2 + z)(1 + z^2)^2]}{(1 + z)^3(1 - z^2)} (1 + z^2) + O(\varepsilon^3), \]

and the corresponding gluing function

\[ \mu(y, \varepsilon) = \frac{\alpha(\varepsilon)}{(5/2 + y)}, \quad \alpha(\varepsilon) = 1 + 3\varepsilon + O(\varepsilon^2). \]

We can construct higher approximations to the functions \( x = x(y, z, \varepsilon) \) and \( \mu = \mu(y, \varepsilon) \) in a similar way, if necessary.

For a given point \( y = y^* \) on the breakdown line we can find the value \( \mu^* = \mu(y^*, \varepsilon) \) from expression (24) which corresponds to the canard of the system. This trajectory lies on the black swan \( x = x(y, z, \varepsilon) \) and passes through the point \( y = y^* \) of the breakdown line. It should be noted that the choice of the gluing point \( y = y^* \) is equivalent to the choice of the starting point of the trajectory.
Figure 8. The slow surface and the canard of the system (25)–(27) with $\varepsilon = 0.01$ and initial point $(0.1, 1, 1)$.

Figure 9. $xOz$–projection of the canard of the system (25)–(27) with $\varepsilon = 0.01$ and initial point $(0.1, 1, 1)$.

Figure 10. $xOy$–projection of the canard of the system (25)–(27) with $\varepsilon = 0.01$ and initial point $(0.1, 1, 1)$.

Note that gluing the stable and unstable slow integral manifolds reduces the original system (5) to the following form

$$
\frac{dx}{d\tau} = \alpha(\varepsilon) - xz^2 - x, \tag{25}
$$

$$
\varepsilon \frac{dz}{d\tau} = xz^2 + x - z, \tag{26}
$$

$$
\frac{dy}{d\tau} = z - y. \tag{27}
$$

The system (25)–(27) has a black swan, which is a cylindrical surface, see Figure 14. All trajectories on this surface are canards (see Figures 8–13), but only one of them is a limit cycle.
Figure 11. The slow surface and the canard of the system (25)–(27) with $\varepsilon = 0.01$ and initial point $x = 0.5$, $y = 1$, $z = 1$.

Figure 12. $xOz$–projection of the canard of the system (25)–(27) with $\varepsilon = 0.01$ and initial point $x = 0.5$, $y = 1$, $z = 1$.

Figure 13. $xOy$–projection of the canard of the system (25)–(27) with $\varepsilon = 0.01$ and initial point $x = 0.5$, $y = 1$, $z = 1$.

This limit cycle is asymptotically orbitally stable. To show this, note that the equilibrium point of the system (25)–(27) has the coordinates

$$ x = 0.5 + O(\varepsilon^2), \quad y = 1 + 3\varepsilon + O(\varepsilon^2), \quad z = 1 + 3\varepsilon + O(\varepsilon^2). $$

Nonlinear system (25)–(27) has the independent of $y$ subsystem (25), (26). The corresponding Jacobian matrix

$$
\begin{pmatrix}
-2 - 6\varepsilon & -1 - 3\varepsilon \\
\varepsilon^{-1}(2 + 6\varepsilon) & -3
\end{pmatrix}
$$
Figure 14. The black swan of the system (25)–(27).

has the eigenvalues

$$
\lambda_{1,2} = 0.5 \left( 1 - 6\varepsilon \pm \sqrt{(1 - 6\varepsilon)^2 - 8\varepsilon^{-1}(1 + 3\varepsilon + O(\varepsilon^2))} \right),
$$

with negative radicand expressions ($\varepsilon \to 0$). Hence, the corresponding equilibria of the subsystem (25), (26) and the system (25)–(27) are unstable foci; the subsystem (25), (26) has a periodic solution and the system (25)–(27) has a stable limit cycle, which is a canard.

6. Conclusion
In the paper we investigated a singularly perturbed model of a sort of Belousov–Zhabotinsky reaction. The basic types of chemical reaction are defined by means of integral manifold theory. It was shown that the critical regimes, separating the domains of self–accelerating reactions and domains of slow processes, are modeled by canards. The asymptotic formulae for the calculation of the critical values of the bifurcation parameter of the system were obtained.

The procedure for constructing the two–dimensional slow integral manifold with changing stability consisting entirely of canards, each of which simulates the critical regime corresponding to specified initial data of the dynamic system, is described. The application of a black swan permits us to take into account small perturbations in the chemical systems.

The existence of an asymptotically orbitally stable canard–cycle on the black swan was shown. In [12] it was noted that in the general situation each cycle in $R^3$ is unstable, but for the same cases for the systems with nonclassical relaxation oscillations a unique stable canard–cycle exists. We observe this situation in the 3–D autocatalator.

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