Points in algebraic geometry

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Abstract

We give scheme-theoretic descriptions of the category of fibre functors on the categories of sheaves associated to the Zariski, Nisnevich, étale, rh, cdh, ldh, eh, qfh, and h topologies on the category of separated schemes of finite type over a separated noetherian base. Combined with a theorem of Deligne on the existence of enough points, this provides an algebraic description of a conservative family of fibre functors on these categories of sheaves. As an example of an application we show direct image along a closed immersion is exact for all these topologies except qfh. The methods are transportable to other categories of sheaves as well.

Introduction

Stalks play an important rôle in the theory of sheaves on a topological space, principally via the fact that a morphism of sheaves is an isomorphism if and only it induces an isomorphism on every stalk. In general topos theory, this is no longer true (see [SGA72a, IV.7.4] for an example of a topos with no fibre functors). However there is an abstract topos theoretic theorem of Deligne which says that under some finiteness hypotheses which are almost always satisfied in algebraic geometry, one can indeed detect isomorphisms using fibre functors (Theorem 1.5).

For the étale topology (on the category of finite type étale morphisms over a noetherian scheme $X$) one has an extremely useful algebraic description of the fibre functors as a certain class of morphisms $\text{Spec}(R) \to X$ where $R$ is a strictly henselian local ring. One can define a ring $R$ to be a strictly henselian local ring if every étale morphism of finite type $U \to \text{Spec}(R)$ admits a section. Let $S$ be a separated noetherian scheme, and let $\tau$ be a topology on the category $\text{Sch}/S$ of finite type separated $S$-schemes.

Definition 0.1. Let us say that an $S$-scheme $P \to S$ (not necessarily subject to any finiteness conditions) is $(\text{Sch}/S, \tau)$-local if for every every $\tau$-cover $\{U_i \to X\}_{i \in I}$ in $\text{Sch}/S$ the canonical morphism

$$\Pi_{i \in I} \text{hom}_S(P, U_i) \to \text{hom}_S(P, X)$$

is surjective.

Our first goal is to observe that for many nice topologies $\tau$ on $\text{Sch}/S$, there is a canonical equivalence between the category of $(\text{Sch}/S, \tau)$-local affine$^1$ $S$-schemes, and the category of fibre functors...
functors on \(\text{Shv}_\tau(\text{Sch}/S)\) (Theorem 2.3). For this, we pass through a third category: the category of \(\tau\)-local pro-objects (Definition 1.2) in \(\text{Sch}/S\) (one should think of such pro-objects as the system of neighbourhoods of the “point” in question). One has the following equivalences of categories

\[
\begin{align*}
\left\{ \text{fibre functors on } \text{Shv}_\tau(\mathcal{C}) \right\}^{\text{op}} & \cong \left\{ \tau\text{-local pro-objects in } \mathcal{C} \right\} \\
& \cong \left\{ \text{(Sch}/S, \tau)\text{-local affine } S\text{-schemes} \right\}
\end{align*}
\]

(1)

The first equivalence is an old topos theoretic result valid for any category \(\mathcal{C}\) admitting finite limits and equipped with a topology \(\tau\). The second equivalence is a standard application of the limit arguments for schemes in [Gro66, Section 8] applied to \(\mathcal{C} = \text{Sch}/S\), and is valid for any topology \(\tau\) finer than the affine topology (by affine topology we mean the topology generated by jointly surjective families of affine morphisms).\(^2\) The combination of this equivalence with Deligne’s theorem leads to statements such as the following:

**Theorem 0.2** (cf. Theorem 2.3). Suppose that \(S\) is a separated noetherian scheme, \(\tau\) a topology on \(\text{Sch}/S\) finer than the affine topology, for which every covering family is refinable by one indexed by a finite set. Then a morphism \(f : F \to G\) in \(\text{Shv}_\tau(\text{Sch}/S)\) is an isomorphism if and only if \(f(P)\) is an isomorphism\(^3\) for every \((\text{Sch}/S, \tau)\)-local scheme \(P\).

A second goal is to give an algebraic description of \((\text{Sch}/S, \tau)\)-local schemes for various topologies arising in algebraic geometry.

**Theorem** (2.6). Suppose that \(S\) is a separated noetherian scheme. An affine \(S\)-scheme is \((\text{Sch}/S, \tau)\)-local if and only if it is \((\ast)\) where \(\tau\) and \((\ast)\) are as in Table 1.

The descriptions for Zariski, Nisnevich, étale, rh, and h are already in the literature (we give references in the main text). The cases \(\tau = \text{cdh}, \text{ltdh}, \) and \(\text{eh}\) are immediate corollaries of these. The description for \(\tau = \text{finite}\) appears to be new; the qfh case is an immediate consequence of this case. We remark that the cases qfh, rh, cdh, eh, and h of this theorem were observed by Gabber at Oberwolfach in August 2002 and circulated in an email to the conference attendees, but never appeared in print.

The reader will notice that the fppf-topology is missing from Table 1. We make some remarks about this in Section 3.

In Section 4 as an example of an application of Theorem 2.3 we show that the direct image between abelian sheaves along a closed immersion is exact for a number of topologies.

My original motivation for thinking about Theorem 2.3 was the hope of finding a shortcut to the main result of [Kel12, Chapter 3]. In the end, due to non-noetherian rings being so much more complicated than noetherian ones, all I got was alternative proofs of some statements without any noticeable reduction in length.

\(^2\) The second is valid in many other situations which the reader can work out according to their needs. For example, the étale, qfh, Nisnevich, Zariski, etc topologies admit various “small” sites, and when \(\mathcal{C}\) is taken to be such a small site instead of \(\text{Sch}/S\), we obtain the full subcategory of the category of \((\mathcal{C}, \tau)\)-local schemes consisting of those which are obtainable as inverse limits of pro-objects of \(\mathcal{C}\).

\(^3\) By abuse of notation, by \(f(P)\) we mean \(\lim_{\rightarrow f(P|_{\text{Sch}/S})} f(X)\), where \((P|_{\text{Sch}/S})\) is the category of factorisations \(P \to X \to S\) with \(X \in \text{Sch}/S\).
τ

| \( \tau \) | (*) |
|-----------------|-----|
| Zariski         | local |
| Nisnevich       | local henselian |
| étale           | local strictly henselian |
| closed          | integral |
| finite          | integral and a.i.c. (Definition 3.1) |
| qfh             | local, integral, and a.i.c. (Definition 3.1) |
| rh              | a valuation ring |
| cdh             | a henselian valuation ring |
| ldh             | a henselian valuation ring whose fraction field has no finite extensions of degree prime to \( l \) (cf. [Dat12, Definition 6.10]). |
| eh              | a strictly henselian valuation ring |
| h               | an a.i.c. valuation ring (Definition 3.1) |

Table 1: \((\text{Sch}/S, \tau)\)-local schemes. The words “Spec of” have been omitted from the last five.

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I am indebted to Ofer Gabber for helping me realise that the equivalences (1) convert Deligne’s Theorem 1.5 into Theorem 0.2 so there is no need to try and prove the latter directly. I am also grateful to Aise Johan de Jong for pointing out mistakes in my attempted description of \((\text{Sch}/S, \text{fppf})\)-local schemes, and explaining the need to use almost cocontinuous morphisms of sites (Definition 4.1). I also thank the mathematicians who encouraged me to put this material in print, despite the fact that much of it is already known to some.

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1 Points of locally coherent topos

In this section we recall Deligne’s theorem (Theorem 1.5) on the existence of a conservative family of fibre functors. We also recall the equivalence between the category of fibre functors, and a subcategory of pro-objects of the underlying site (Proposition 1.4). The material in this section is well-known to topos theorists.

Recall that a fibre functor of a category \( S \) is a functor \( S \to \text{Set} \) towards the category of sets which preserves finite limits and small colimits. By category of fibre functors of \( S \), we mean the full subcategory of the category of functors from \( S \) to \( \text{Set} \) whose objects are fibre functors. We will write \( \text{Fib}(S) \) for the category of fibre functors [SGA72a, Definition IV.6.2]. If \( S \) is a category of sheaves, the category of points of \( S \) is by definition \( \text{Fib}(S)^{\text{op}} \) [SGA72a, Définition IV.6.1].

Recall as well, that a pro-object of a category \( C \) is a (covariant) functor \( P_{\bullet} : \Lambda \to C \) from a cofiltered \(^4\) category. The pro-objects of a category \( C \) are the objects of a category \( \text{Pro}(C) \)

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\(^4\)By cofiltered we mean a category \( \Lambda \) that is non-empty, for every pair of objects \( i, j \) there exists an object \( k \)
where one defines
\[
\text{hom}_{\text{Pro}(\mathcal{C})} \left( (\Lambda \xrightarrow{P} \mathcal{C}), (\Lambda' \xrightarrow{P'} \mathcal{C}) \right) = \lim_{\lambda' \in \Lambda'} \lim_{\lambda \in \Lambda} \text{hom}_\mathcal{C}(P_\lambda, P'_{\lambda'}). 
\]
There is an obvious fully faithful functor $\mathcal{C} \to \text{Pro}(\mathcal{C})$ which sends an object $X \in \mathcal{C}$ to the constant pro-object $X : \ast \to \mathcal{C}$ with value $X$, where $\ast$ is the category with one morphism. Sometimes the category $\mathcal{C}$ is considered as a subcategory of $\text{Pro}(\mathcal{C})$ in this way.

Suppose that $\mathcal{C}$ is equipped with a topology $\tau$. Following Suslin and Voevodsky (and contrary to Artin and therefore Milne), we use the terms topology and covering as in [SGA72a, Definition II.1.1] and [SGA72a, Definition II.1.2] respectively. In particular, we have the following two properties.

**Lemma 1.1.**

1. If a family of morphisms admits a refinement by a $\tau$-covering, then that family itself is also a covering [SGA72a, Proposition II.1.4].

2. Suppose $\mathcal{C}$ is essentially small and admits a conservative family of fibre functors $\{\phi_j\}_{j \in J}$. Then a set of morphisms $\{p_i : U_i \to X\}_{i \in I}$ is a $\tau$-covering family if and only if for each $j \in J$ the family $\{\phi_j(p_i)\}_{i \in I}$ is a jointly surjective family of morphisms of sets [SGA72a, Theorem II.4.4].

**Definition 1.2.** Say that a pro-object $\Lambda \xrightarrow{P} \mathcal{C}$ is $\tau$-local if for every $X \in \mathcal{C}$ and every $\tau$-covering family $\{U_i \to X\}_{i \in I}$ the morphism
\[
\Pi_{i \in I} \text{hom}_{\text{Pro}(\mathcal{C})}(P_\lambda, U_i) \to \text{hom}_{\text{Pro}(\mathcal{C})}(P_\lambda, X)
\]
is surjective. Write $\text{Pro}_\tau(\mathcal{C})$ for the full subcategory of $\tau$-local pro-objects in $\text{Pro}(\mathcal{C})$.

**Remark 1.3.** Rather than the surjectivity condition of Definition 0.1 one is tempted to use a condition like “every $\tau$-cover of $P$ admits a section”. We have avoided this because some topologies (such as the cdh and h for example) have multiple equivalent definitions, which are no longer equivalent when working with non-noetherian schemes (cf. GL01, Example 4.5). As the schemes $P$ are often non-noetherian, we have chosen this statement to avoid the choice of non-noetherian versions of these topologies.

**Proposition 1.4** ([Joh77, Proposition 7.13]). Suppose that $\mathcal{C}$ is a category that admits finite limits and is equipped with a topology $\tau$. Then the functor $\text{Pro}(\mathcal{C}) \to \text{Fib}(\text{PreShv}(\mathcal{C}))$ which sends a pro-object $\Lambda \xrightarrow{P} \mathcal{C}$ to the functor $F \mapsto \lim_{\lambda \in \Lambda} F(P_\lambda)$ induces an equivalence of categories.

\[
\text{Pro}_\tau(\mathcal{C}) \to \text{Fib}(\text{Shv}(\mathcal{C}))^{\text{op}}
\]

(2)

and morphisms $k \to i, k \to j$, and for every pair of parallel morphisms $i \Rightarrow j$ there exists a morphism $k \to i$ such that the two compositions are equal [SGA72a, Definition 1.2.7].
An inverse $\text{Fib}(\text{Shv}(\mathcal{C}))^{op} \to \text{Pro}_\tau(\mathcal{C})$ is given as follows. For any functor $\text{Shv}_\tau(\mathcal{C}) \xrightarrow{\phi} \text{Set}$ let $(\star \downarrow \phi)$ be the category whose objects are pairs $(X, s)$ with $X \in \mathcal{C}$, $s \in \phi(X)$ (where we identify $X$ with the $\tau$-sheafification of the presheaf it represents). The morphisms $(X, s) \to (Y, t)$ are those morphisms $f : X \to Y$ such that $\phi(f)(s) = t$. One can check that when $\phi$ is a fibre functor, $(\star \downarrow \phi)$ is cofiltered, and therefore the canonical projection $(\star \downarrow \phi) \to \mathcal{C}$ is a pro-object, and is in fact the $\tau$-local pro-object corresponding to $\phi$. Informative examples include the case where $(\mathcal{C}, \tau)$ is the site associated to a classical topological space, or the small étale (or Zariski) site of a noetherian scheme.

**Theorem 1.5** (Deligne [SGA72b, Proposition VI.9.0] or [Joh77, Theorem 7.44, Corollary 7.17]). Suppose that $\mathcal{C}$ is a category in which fibre products are representable, and is equipped with a topology $\tau$ such that every covering family of every object admits a finite subfamily which is still a covering family.

Then a morphism $f$ in $\text{Shv}_\tau(\mathcal{C})$ is an isomorphism if and only if $\phi(f)$ is an isomorphism for every fibre functor $\phi \in \text{Fib}(\text{Shv}_\tau(\mathcal{C}))$.

In fact, there exists a (proper) set of fibre functors which is still a conservative family.

**Remark 1.6.** The statement in [SGA72b] is for a locally coherent topos. A topos is locally coherent [SGA72b, Definition VI.2.3] if and only if it is equivalent to a category of sheaves on a site such that every object is quasi-compact, and all fibre products are representable. By definition [SGA72b, Definition VI.1.1], an object is quasi-compact if and only if every covering family admits a finite refinement which is still a covering family. What [SGA72b] calls a locally coherent topos, Johnstone calls a coherent topos [Joh77, Theorem 7.35].

The existence of a proper set of of fibre functors which is still a conservative family is [Joh77, Corollary 7.17] which is stated for a Grothendieck topos, i.e., one which is equivalent to the category of sheaves on a site [Joh77, Definition 0.41].

## 2 Points of algebro-geometric categories of sheaves

In this section we recall various topologies on the category of schemes of finite type over a separated noetherian base scheme, and give a geometric description the fibre functors for some of these sites.

Let $\text{Aff}/S$ denote the category of affine $S$-schemes of finite type. That is, the category of $S$-schemes of finite type whose structural morphism is an affine morphism. The material in [Gro66, Section 8] allows us to replace pro-objects by honest schemes. The following is a direct consequence of the definitions and [Gro66, Proposition 8.13.5].

**Proposition 2.1.** Suppose that $S$ is a separated noetherian scheme and $\alpha$ a topology on $\text{Aff}/S$. The functor which sends a pro-object $\Lambda \xrightarrow{P} \text{Aff}/S$ to $\varprojlim_{\lambda \in \Lambda} P_\lambda$ induces an equivalence of categories between $\text{Pro}_\alpha(\text{Aff}/S)$ (Definition 1.2) and the category of $(\text{Aff}/S, \alpha)$-local affine $S$-schemes (Definition [L.7]).

An explicit inverse is given as follows. For an $S$-scheme $P \to S$, define $(P/\text{Sch}/S)$ to be the category whose objects are factorisations $P \to X \to S$ with $X \in \text{Aff}/S$ and morphisms
are commutative diagrams $P \xleftarrow{Y} X \xrightarrow{r} S$. Since this category is co-filtered, projecting $P \to X \to S$ towards $X \to S$ gives a pro-object.

**Remark 2.2.** Note that the adjective “affine” is necessary if we want an equivalence of categories. For example, since $\text{hom}_S(\text{Spec}(f_*O_P), X) = \text{hom}_S(P, X)$ for any $S$-scheme $P$ and any affine $S$-scheme $X$, the $S$-schemes $\text{Spec}(f_*O_P)$ and $P$ determine the same pro-object of Aff/$S$.

Given a topology $\tau$ on Sch/$S$ we will call the induced topology on Aff/$S$ the affine $\tau$-topology or $\tau^{\text{aff}}$-topology. That is, $\tau^{\text{aff}}$ is the finest topology on Aff/$S$ such that the image in Sch/$S$ of any $\tau^{\text{aff}}$-covering family is a $\tau$-covering family [SGA72a, Definitions III.1.1, III.3.1].

**Theorem 2.3.** Suppose that $S$ is a separated noetherian scheme and $\tau$ a topology on Sch/$S$ coarser than the affine topology. The functor which sends a (Sch/$S$, $\tau$)-local $S$-scheme $P \to S$ to the functor $\phi_P : F \mapsto \lim_{\to (P \to X \to S)} F(X)$ (where the colimit is indexed by factorisations with $X \in \text{Sch}/S$) induces an equivalence of categories:

\[
\left\{\begin{array}{c}
\text{(Sch}/S, \tau)\text{-local} \\
\text{affine } S\text{-schemes}
\end{array}\right\} \cong \text{Fib}\left(\text{Shv}_\tau(\text{Sch}/S)\right). \tag{3}
\]

If moreover, every covering family is refinable by one indexed by a finite set, then the family $\{\phi_P\}$ indexed by (Sch/$S$, $\tau$)-local affine $S$-schemes is a conservative family of fibre functors.

**Remark 2.4.** It seems to be a non-trivial problem in general to show for a given topology $\tau$ on Sch/$S$ that every covering family is refinable by one indexed by a finite set. For example, the Riemann-Zariski space [GL01, Section 3] is used in [GL01] for this purpose for the h and rh topologies.

**Proof.** If $\tau$ is a topology on Sch/$S$ coarser than the affine topology, then the canonical functor $\text{Shv}_\tau(\text{Sch}/S) \to \text{Shv}_{\tau^{\text{aff}}}(\text{Aff}/S)$ is an equivalence. Hence there is an equivalence of categories of fibre functors

$$\text{Fib}(\text{Shv}_\tau(\text{Sch}/S)) \cong \text{Fib}(\text{Shv}_{\tau^{\text{aff}}}(\text{Aff}/S)).$$

Now we have the equivalences

$$\text{Fib}(\text{Shv}_{\tau^{\text{aff}}}(\text{Aff}/S))^{op} \cong \text{Pro}_{\tau^{\text{aff}}}(\text{Aff}/S)$$

and

$$\text{Pro}_{\tau^{\text{aff}}}(\text{Aff}/S) \cong \left\{\begin{array}{c}
\text{(Aff}/S, \tau)\text{-local} \\
\text{affine } S\text{-schemes}
\end{array}\right\}$$

of Proposition [1.4] and Proposition [2.1] and finally, we use again the fact that $\tau$ is coarser that the affine topology to obtain the equivalence

$$\left\{\begin{array}{c}
\text{(Aff}/S, \tau)\text{-local} \\
\text{affine } S\text{-schemes}
\end{array}\right\} \cong \left\{\begin{array}{c}
\text{(Sch}/S, \tau)\text{-local} \\
\text{affine } S\text{-schemes}
\end{array}\right\}.$$

The statement about a conservative family follows from Theorem [1.5] of Deligne. 

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5i.e., such that every covering family is refinable by a covering family $\{U_i \to X\}_{i \in I}$ with each $U_i$ in Aff/$S$. 

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Let us now consider specific topologies.

**Definition 2.5.** Let $S$ be a separated noetherian scheme. We consider the following topologies on $\text{Sch}/S$. If $\sigma$ and $\rho$ are two topologies then we denote by $\langle \sigma, \rho \rangle$ the coarsest topology which is finer than both $\sigma$ and $\rho$.

1. The **Zariski topology (or Zar)** is generated by families
   \[ \{ f_i : U_i \to X \}_{i \in I} \] (4)
   which are jointly surjective (i.e., $\bigsqcup U_i \to X$ is surjective on the underlying topological spaces) and such that each $U_i \to X$ is an open immersion.

2. The **Nisnevich topology (or Nis)** is generated by completely decomposed families (4) with each $f_i$ an étale morphism. By *completely decomposed* we mean that for each $x \in X$ there is an $i$ and a $u \in U_i$ such that $f_i(u) = x$ and $[k(u_i) : k(x)] = 1$ ([Nis89]).

3. The **étale topology (or Et)** is generated by families (4) which are jointly surjective and such that each $f_i$ is étale.

4. The **closed topology (or cl)** is generated by families (4) which are jointly surjective and such that each $f_i$ is a closed immersion.

5. The **finite-flat-surjective-prime-to-$l$ topology (or fps$l'$)** is generated by families $\{ Y \to X \}$ containing a single finite flat surjective morphism of constant degree prime to $l$, an prime integer.

6. The **finite-flat topology (or fps)** is generated by families $\{ Y \to X \}$ containing a single finite flat surjective morphism.

7. The **envelope topology (or cdp)** is generated by completely decomposed families $\{ Y \to X \}$ containing a single proper morphism ([Ful98, Definition 18.3]).

8. The **finite topology (or f)** is generated by families (4) which are jointly surjective and such that each $f_i$ is a finite morphism.

9. The **proper topology (or prop)** is generated by families $\{ Y \to X \}$ containing a single proper morphism.

10. The **rh topology** is $\text{rh} = \langle \text{Zar}, \text{cdp} \rangle$ ([GL01, Definition 1.2]).

11. The **cdh topology** is $\text{cdh} = \langle \text{Nis}, \text{rh} \rangle$ ([SV00, Definition 5.7]).

12. The **ldh topology** is $\text{ldh} = \langle \text{cdh}, \text{fps}$-$l' \rangle$. ([Kel12, Definition 3.2.1(2)]).

13. The **eh topology** is $\text{eh} = \langle \text{Et}, \text{rh} \rangle$ ([Gei06, Definition 2.1]).

14. The **fppf topology** is $\text{fppf} = \langle \text{Nis}, \text{fps} \rangle$ ([SGA70, Section IV.6.3], cf. [Gro67, Corollaire 17.16.12, Théorème 18.5.11(c)]).
15. The qfh topology is \( \text{qfh} = (\text{Et}, f) \) (cf. [Voe96, Definition 3.1.2, Lemma 3.4.2]).

16. The h topology is \( h = (\text{Zar}, \text{prop}) \) (cf. [Voe96, Definition 3.1.2], [GL01, Definition 1.1, Theorem 4.1]).

The following diagram indicates some relationships. Not all relationships are shown.

\[
\begin{align*}
\text{cl} & \quad \text{fps} & \quad f & \quad \text{prop} \\
\text{Zar} & \quad \text{Nis} & \quad \text{Et} & \quad \text{fppf} & \quad \text{qfh} \\
\text{cdp} & \quad \text{rh} & \quad \text{cdh} & \quad \text{eh} & \quad h
\end{align*}
\]

Amongst the geometric descriptions of \((\text{Sch}/S, \tau)-\text{local}\) affine \(S\)-schemes which we give, the only one which is not either already in the literature, or follows from the others is \(\tau = f\). This we postpone to the next section. We do not discuss further the proper or envelope topologies as the induced topology on \(\text{Aff}/S\) is difficult to describe; they were included to make the diagram more conceptually complete.

Our convention for the term valuation ring is a ring \(A\) which is an integral domain (i.e., \(ab = 0 \implies a = 0\) or \(b = 0\)) and such that for every \(a \in \text{Frac}(A)\), either \(a \in A\) or \(a^{-1} \in A\). We allow the totally ordered set of prime ideals of \(A\) to have any order type.

**Theorem 2.6.** Suppose that \(S\) is a separated noetherian scheme. An affine \(S\)-scheme is \((\text{Sch}/S, \tau)\)-local if and only if it is \((*)\) where \(\tau\) and \((*)\) are as in Table 7.

**Remark 2.7.** The cases \text{qfh}, \text{rh}, \text{cdh}, \text{eh}, and \(h\) in the above statement were observed by Gabber at Oberwolfach in August 2002.

**Proof.** The case \(\tau = h\) (resp. \(\text{rh}\)) is [GL01, Proposition 2.2] (resp. [GL01, Proposition 2.1]). The proof of [GL01, Proposition 2.1] does the cases \(\tau = \text{Zar}\) and \(\tau = \text{cl}\). The cases \(\tau = \text{Nis}\) and \(\tau = \text{Et}\) are classical.

The case \(\tau = f\) is Lemma 3.2 treated in the next section.

Finally, one notices that for any two topologies \(\sigma, \rho\), an \(S\)-scheme is \((\text{Sch}/S, (\sigma, \tau))\)-local if and only if it is both \((\text{Sch}/S, \sigma)\)-local and \((\text{Sch}/S, \tau)\)-local. So the cases \(\tau = \text{cdh}, \text{ldh}, \text{eh}, \text{and qfh}\) follow from the others. □

### 3 The finite, qfh, and fppf-topologies

This sections contains the commutative algebra required to characterise \((\text{Sch}/S,f)\)-local (and consequently \((\text{Sch}/S,\text{qfh})\)-local) schemes. We also make some basic comments on the fppf-case.
**Definition 3.1.** An integral ring $A$ is said to be **absolutely integrally closed** or a.i.c. if it is normal with algebraically closed fraction field. An integral scheme $X$ is said to be **absolutely integrally closed** (or a.i.c.) if all its local rings are a.i.c., or equivalently, if $A$ is a.i.c. for every open affine $\text{Spec}(A) \subset X$ ([Dat12, Lemma 5.8]).

For various properties about absolutely integrally closed rings and schemes see [Dat12, Sections 3, 4, and 5]. For example, suppose that $A$ is an a.i.c. integral domain. Let $B \subset A$ be a subring which is integrally closed in $A$, let $p \subseteq A$ be a prime ideal, and let $S \subseteq A$ be a multiplicatively closed subset not containing zero. Then $B, A/p$ and $S^{-1}A$ are all a.i.c. integral domains [Dat12, Lemmas 3.5, 4.1, and 5.4]. Furthermore, an integral domain $A$ is a.i.c. if and only if $A_p$ (resp. $A_m$) is a.i.c. for all prime ideals $p$ (resp. maximal ideals $m$) of $A$ [Dat12, Lemma 5.8].

**Lemma 3.2.** Let $S$ be a separated noetherian scheme and suppose that $P = \varprojlim P_\lambda$ is a projective limit of schemes $P_\lambda$ in $\text{Aff}/S$. Then the following are equivalent.

1. $P$ is $(\text{Aff}/S, f)$-local.

2. Every open subscheme of $P$ is $(\text{Aff}/S, f)$-local.

3. $P$ is an a.i.c. integral affine $S$-scheme.

**Proof.** $(1 \Rightarrow 2)$. Suppose that $P$ is $(\text{Aff}/S, f)$-local, $U \xrightarrow{i} P$ is an open immersion. It suffices to show that every finite surjective morphism $f : V \to U$ admits a section. There exists a $\lambda$, a finite surjective morphism $f_\lambda : V_\lambda \to U_\lambda$ and an open immersion $i_\lambda : U_\lambda \to P_\lambda$ such that $i = P_\lambda \times_{P_\lambda} i_\lambda$ and $f = P \times_{P_\lambda} f_\lambda$ [Gro66, Théorème 8.10.5]. Using Nagata’s compactification theorem followed by the Stein factorisation, we can find a commutative diagram

```
\begin{align*}
  V_\lambda & \longrightarrow Y_\lambda & \longrightarrow & \text{Spec}(p_* O_{Y_\lambda}) \\
  f_\lambda \downarrow & & \uparrow p & \downarrow g_\lambda \\
  U_\lambda & \downarrow i_\lambda & \longrightarrow & P_\lambda
\end{align*}
```

where $V_\lambda \to Y_\lambda$ is a dense open immersion, $p : Y_\lambda \to P_\lambda$ is proper, $g_\lambda$ is finite surjective, and $f_\lambda = U_\lambda \times_{P_\lambda} g_\lambda$. As $P$ is $(\text{Aff}/S, f)$-local, the morphism $P \to P_\lambda$ factors through $g_\lambda$ and so there is a $\mu \geq \lambda$ for which $P_\mu \times_{P_\lambda} g_\lambda$ admits a section. Since $f = P \times_{P_\lambda} f_\lambda = P \times_{P_\lambda} U_\lambda \times_{P_\lambda} g_\lambda$ this implies that $f$ admits a section.

$(2 \Rightarrow 1)$ is trivial.

$(1 \Rightarrow 3)$. The $f$-topology is finer than the closed topology so $P$ is integral. By $(1 \Rightarrow 2)$ every open affine subscheme of $P$ is $(\text{Aff}/S, f)$-local and so it suffices to consider the case when $P = \text{Spec}(A)$. Let $L/\text{Frac}(A)$ be a finite field extension (possibly trivial) and $a$ an element of the normalisation of $A$ in $L$. To show that $A$ is a.i.c., it suffices to show now that $a \in A$. If $\min_a(T) \in A[T]$ is the minimal polynomial of $a$ then $f : \text{Spec}(A[T]/\min_a(T)) \to \text{Spec}(A)$ is a finite surjective morphism and there is a finite surjective morphism $f_\lambda : Y_\lambda \to P_\lambda$ for some $\lambda$ such that $P \times_{P_\lambda} f_\lambda = f$ [Gro66, Théorème 8.10.5]. Since $P$ is $(\text{Aff}/S, f)$-local the morphism $P \to P_\lambda$ is
factors through $f_\lambda$, and consequently, $f$ has a section. But $f$ has a section, if and only if $\min_a(T)$ has a solution in $A$ if and only if $a$ is in $A$.

(3 $\Rightarrow$ 1). Suppose that $\{U_i \to X\}$ is an $f$-covering family of some $X \in \text{Aff}/S$ and $P \to X$ is an $S$-morphism. Choose a $U_i$ whose image in $X$ contains the image of the generic point of $P$. Then $P \times_X U_i \to P$ is a finite surjective morphism, and it suffices to show that it has a section. Since the fraction field of $P$ is algebraically closed, the inclusion of the generic point $\eta \to P$ factors through $P \times_X U_i \to P$. Let $V \subset P \times_X U_i$ be the closure of the image of $\eta$. Then $V \to P$ is a finite birational morphism of integral schemes with normal target, and is therefore an isomorphism.

There is little we can say about $(\text{Sch}/S, \text{fppf})$-local $S$-schemes.

**Lemma 3.3.** Suppose $S$ is a separated noetherian scheme, and $\text{Spec}(R) \to S$ is $(\text{Sch}/S, \text{fppf})$-local.

1. $R$ is strictly henselian.
2. All residue fields of $R$ are algebraically closed.
3. If $R$ is integral, then it is normal and therefore $(\text{Sch}/S, \text{qfh})$-local.
4. There exists at least one $(\text{Sch}/S, \text{fppf})$-local $\text{Spec}(R) \to S$ such that $R$ is not integral.
5. If $R$ is noetherian, then it is reduced. Consequently, condition (2) above is not sufficient for $\text{Spec}(R) \to S$ to be $(\text{Sch}/S, \text{fppf})$-local.

**Proof.**

1. This is an immediate consequence of the fppf-topology being finer than the étale topology.

2. Suppose that $p$ is a prime ideal of $R$, let $\kappa = \text{Frac}(R/p)$ and suppose that $f(T) = T^n + \sum_{i=0}^{n-1} b_i T^i \in \kappa[T]$ is a monic polynomial. We will confuse the $b_i, c_i \in R$ with their images in $R/p$. Let $c = \prod_{i=0}^{n-1} c_i$, let $a_i = c^{n-i} b_i \in R$ (for $0 \leq i \leq n-1$), and set $g(T) = T^n + \sum_{i=0}^{n-1} a_i T^i$. Now $\text{Spec}(R[T]/(g(T))) \to \text{Spec}(R)$ is a finite flat surjective morphism. The polynomial $g$ therefore has some solution $d \in R$ since $\text{Spec}(R)$ is $(\text{Sch}/S, \text{fppf})$-local. Notice that $c$ is not zero in $R/p$ since the $c_i$ are not zero in $R/p$, so $\frac{d}{c}$ is a well-defined element of $\kappa = \text{Frac}(R/p)$. Now in $\kappa$ we have $f(\frac{d}{c}) = \frac{1}{c^n} g(d) = 0$.

3. Notice that if $R$ is an integral ring for which every finite flat $R$-algebra admits a retraction, then every monic in $R[T]$ splits into linear factors. Consequently, $R$ is integrally closed in its field of fractions. For the qfh-statement, since qfh $= \langle \text{Et}, f \rangle$, and Et is coarser than fppf, it suffices to show that $\text{Spec}(R) \to S$ is $(\text{Sch}/S, f)$-local. But $R$ is normal with algebraically closed fraction field, so $\text{Spec}(R)$ is a.i.c., and therefore $(\text{Sch}/S, f)$-local (Lemma 3.2).

\footnote{Strictly speaking, one should use limit arguments to lift $\text{Spec}(R[T]/(g(T))) \to \text{Spec}(R)$ to a finite flat surjective morphism $U \to X$ in $\text{Aff}/S$ equipped with an $S$-morphism $\text{Spec}(R) \to X$, and then convert a factorisation $\text{Spec}(R) \to U \to X$ into a solution $d \in R$. For example, let $A$ be the $O_S$-algebra generated by the $a_i$ and take $X = \text{Spec}(A)$.}
4. Suppose the contrary. By the previous part, this would then imply that the class of \((\text{Sch}/S, \text{fppf})\)-local affine \(S\)-schemes and the class of \((\text{Sch}/S, \text{qfh})\)-local affine \(S\)-schemes are the same. But this would then imply that the qfh and fppf topologies were equal (Lemma 1.1(2)). This is false since there are many surjective finite morphisms in \(\text{Sch}/S\) which are not refinable by flat ones.

5. Let \(I = \{ r : r^n = 0 \text{ for some } n > 0 \}\) be the nilradical. Let \(n\) be a positive integer \(n\) such that \(I^n = 0\) (existence of such an \(n\) is where we use the hypothesis that \(R\) is noetherian). For every \(r \in I\), the \(R\)-algebra \(R[T]/(T^n - r)\) is finite and flat, and therefore admits a retraction. Equivalently, there exists \(s \in R\) such that \(s^n = r\). But then \((s^n)^n = 0\) so \(s \in I\), and therefore \(s^n = 0\), and hence \(r = 0\). So \(I = \{0\}\).

Remark 3.4 ([Sta14, Tag 04C5]). Heuristically, for a ring \(A\), an ideal \(I\), and a flat, finitely presented algebra \(A/I \to B_0\), there is not necessarily any reason to suppose the existence of a flat, finitely presented \(A\)-algebra \(A \to B\) for which \(B_0 = B \otimes_A (A/I)\). So one might conjecture that Proposition 4.5 is false for the fppf-topology. If one believes such a conjecture, then one has some further information about \((\text{Sch}/X, \text{fppf})\)-local schemes.

Lemma 3.5. Suppose that there exists a closed immersion of separated noetherian schemes \(i : Z \to X\) such that the direct image \(i_* : \text{Shv}_{\text{fppf}}(\text{Sch}/Z, \text{Ab}) \to \text{Shv}_{\text{fppf}}(\text{Sch}/X, \text{Ab})\) on abelian fppf sheaves is not exact. Then there exists an \((\text{Sch}/X, \text{fppf})\)-local scheme \(P \to X\) and a closed subscheme \(Q \subset P\) such that \(Q\) is not \((\text{Sch}/X, \text{fppf})\)-local.

Proof. The converse contradicts Proposition 4.5.

4 Exactness of direct image

In this section we show that direct image along a closed immersion is exact for various topologies. Recall that a functor \(u : C \to C'\) between categories \(C, C'\) equipped with topologies \(\tau, \tau'\) respectively is continuous if for every covering family \(\{U_i \to X\}_{i \in I}\) in \(C\), the family \(\{u(U_i) \to u(X)\}_{i \in I}\) is a \(\tau'\)-covering family [SGA72a Definition III.1.1].

We would like to use the notion of a cocontinuous morphism of sites given in [SGA72a Definition III.2.1] (see also [SGA72a Definition II.1.2]). Unfortunately, not all closed immersions give rise to such a morphism (see [Sta14, Tag 00XV] for a counter-example).

Definition 4.1 ([Sta14, Tag 04CB4]). A functor \(u : C \to C'\) between categories \(C, C'\) equipped with topologies \(\tau, \tau'\) respectively is almost cocontinuous if for every \(\tau'\)-covering family \(\mathcal{U} = \{U_i \to u(X)\}_{i \in I}\) there exists a \(\tau\)-covering family \(\mathcal{V} = \{V_j \to X\}_{j \in J}\) such that either

1. the image of \(\mathcal{V}\) under \(u\) is a refinement of \(\mathcal{U}\), or

2. for each \(j\), the empty family is a covering of \(u(V_j)\) in \(C'\).

Remark 4.2. The functor \(u\) in the above definition is cocontinuous if for every \(\mathcal{U}\) there exists a \(\mathcal{V}\) such that the first condition is satisfied.
Proposition 4.3. Suppose that \( u : \mathcal{C} \to \mathcal{C}' \) is a functor between categories \( \mathcal{C}, \mathcal{C}' \) equipped with topologies \( \tau, \tau' \).

1. If \( u \) is continuous then the functor \( - \circ u \) preserves sheaves, and therefore induces a functor \( u^*_s : \text{Shv}_\tau(\mathcal{C}') \to \text{Shv}_\tau(\mathcal{C}) \) \([\text{SGA72a, Paragraph III.1.11}]\). Furthermore, \( u^*_s \) admits a left adjoint \( u^* \), and therefore \( u^*_s \) preserves all small limits. \( \text{[Sta14, Tag 04B9]} \)

2. If \( u \) is continuous and almost cocontinuous, then \( u^*_s \) preserves finite connected colimits.

**Warning 4.4.** We follow the notation \( u^*, u^*_s \) from \([\text{SGA72a}]\). Unfortunately \([\text{Sta14}]\) writes \( u^* \) for the \( u^*_s \) of \([\text{SGA72a}]\), and \( u^*_s \) for the \( u^* \) of \([\text{SGA72a}]\). So the adjunction \( (u^*, u^*_s) \) of \([\text{SGA72a}]\) is written in \([\text{Sta14}]\) as \( (u^*, u^*_s) \).

**Proposition 4.5.** Suppose that \( i : Z \to X \) is a closed immersion of separated noetherian schemes. Then the direct image

\[
i_* : \text{Shv}_\tau(\text{Sch}/Z, \text{Ab}) \to \text{Shv}_\tau(\text{Sch}/X, \text{Ab})
\]

is exact if \( \tau \) is any of Zar, Nis, ét, rh, cdh, ldh, ch, or h.

**Proof.** The functor \( Z \times_X - : \text{Sch}/X \to \text{Sch}/Z \) is a continuous morphism of sites and so \( i_* \) has a left adjoint and therefore preserves all small limits (Proposition 4.31).

If \( \tau = \text{Zar, Nis, or Et} \) we claim that the functor \( Z \times_X - : \text{Sch}/X \to \text{Sch}/Z \) is also an almost cocontinuous morphism of sites. Take \( Y \in \text{Sch}/X \) and let \( U = \{U_i \to Z \times_X Y\}_{i \in I} \) be a \( \tau \)-covering family in \( \text{Sch}/Z \). To prove the claim, we must find a \( \tau \)-covering family in \( V = \{V_j \to Y\}_{j \in J} \) that satisfies one of the two conditions in Definition 1.1.

If \( Z \times_X Y \) is the empty scheme, then \( V = \{Y \xrightarrow{id} Y\} \) satisfies the second condition. If \( Z \times_X Y \) is non-empty, we will construct a family \( V = \{p_j : V_j \to Y\}_{j \in J} \) such that for every \( \text{(Sch}/X, \tau) \)-local affine \( X \)-scheme the induced morphism \( \text{hom}_X(P,V_j) \to \text{hom}_X(P,Y) \) is surjective, and the image of \( V \) under \( Z \times_X - \) refines \( U \). Since every fibre functor of \( \text{Shv}_\tau(\text{Sch}/X) \) is induced by a \( \text{(Sch}/X, \tau) \)-local affine (Theorem 2.3) it then follows that for every fibre functor \( \phi \), the family \( \{\phi(p_j)\}_{j \in J} \) is jointly surjective and therefore \( V \) is a covering family (Lemma 1.1.2), and the second condition in Definition 1.1 is satisfied.

For each \( \text{(Sch}/X, \tau) \)-local affine \( P \to X \), the \( Z \)-scheme \( Z \times_X P \to Z \) is \( \text{(Sch}/Z, \tau) \)-local and affine (Theorem 2.6). Therefore for each \( X \)-morphism \( j : P \to Y \) there is an \( i \) such that \( Z \times_X P \to Z \times_X Y \) factors through the morphism \( U_i \to Z \times_X Y \). If one presents \( P \) as the inverse limit of a pro-object \( (P_\lambda)_{\lambda \in \Lambda} \) of \( \text{Aff}/Y \) (using our chosen morphism \( P \to Y \)), then we have \( Z \times_X P \cong Z \times_X \lim_{\lambda \in \Lambda} P_\lambda \cong \lim_{\lambda \in \Lambda} Z \times_X P_\lambda \) and the standard limit arguments \([\text{Gro66, Proposition 8.13.5}]\) provide a \( \lambda \) and a factorisation \( Z \times_X P \to Z \times_X P_\lambda \to U_i \to Z \times_X Y \). That is, we have a \( V_P \to Y \) in \( \text{Sch}/X \) (take \( V_P = P_\lambda \)) equipped with factorisations \( P \to V_P \to Y \) and \( Z \times_X (V_P) \to U_i \to Z \times_X Y \). The family \( \{V_j \to Y\}_{p_{\lambda_j Y}} \) indexed by the class \( J \) of morphisms with source a \( \text{(Sch}/X, \tau) \)-local affine and target \( Y \) satisfies our requirements.

If \( \tau = \text{rh, cdh, ldh, ch, or h} \), then the same proof works with slight modifications. We replace \( \text{Shv}_\tau(\text{Sch}/Z) \) with the equivalent category \( \text{Shv}_{\tau_{\text{red}}}(\text{Sch}_{\text{red}}/Z_{\text{red}}) \) where \( \text{Sch}_{\text{red}}/Z_{\text{red}} \) is the category of reduced separated schemes of finite type over \( Z_{\text{red}} \) equipped with the topology \( \tau_{\text{red}} \) induced
from the inclusion $\text{Sch}_{\text{red}}(Z_{\text{red}}) \subset \text{Sch}/Z$. We must also replace $Z \times_X P$ with $(Z \times_X P)_{\text{red}}$. Then the above proof works.

**Remark 4.6.** The above proof does not work with qfh because even though every integral closed subscheme of a $(\text{Sch}/S, \text{qfh})$-local scheme is $(\text{Sch}/S, \text{qfh})$-local [Dat12, Lemma 4.1], this is not necessarily true for reducible reduced closed subschemes.

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