Examples of $S$–expansions of Lie Algebras

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Abstract. In this note, we give examples of $S$–expansions of Lie algebras of finite and infinite dimension. For the finite dimensional case, we expand all real three-dimensional Lie algebras. In the case of infinite dimension, we perform contractions obtaining new Lie algebras of infinite dimension.

1. Introduction

$S$–expansions were introduced in the field of Physics in [1], and have been used in different contexts. This method uses abelian semigroups, which play an important role in this construction, since different semigroups generate different $S$–expanded algebras. In this work, we apply the $S$–expansions method an real three-dimensional Lie algebras given in [2].

The theory of infinite dimensional Lie algebras is a very active field of research. Krichever-Novikov algebras $KN$ are of interest because they arise from geometric objects and are generalizations of the Virasoro algebra. In this note we carry out the $S$–expansions method to an example of $KN$ algebras. Furthermore, we will perform contractions to $S$–expanded algebras, comparing them with the $S$–expansions of the contractions.

2. Preliminaries

Consider a Lie algebra $G$ of dimension $n$, over the field $\mathbb{R}$ or $\mathbb{C}$. Let $\{e_i\}_{1 \leq i \leq n}$ be a basis for $G$ with Lie brackets $[e_i, e_j] = C_{ij}^k e_k$, where the coefficients $C_{ij}^k$ are the structure constants. We recall the classification of real three dimensional Lie algebras give in [2];
Table 1. Three-dimensional real Lie Algebra

| Lie algebra | Lie product |
|-------------|-------------|
| $G^1$       | 0           |
| $G^2$       | $[e_1, e_3] = e_1$ |
| $G^3$       | $[e_1, e_3] = e_2$ |
| $G^4$       | $[e_1, e_3] = e_1$, $[e_2, e_3] = e_1 + e_2$ |
| $G^5$       | $[e_1, e_3] = e_1$, $[e_2, e_3] = e_2$ |
| $G^6$       | $[e_1, e_3] = e_1$, $[e_2, e_3] = -e_2$ |
| $G^7$       | $[e_1, e_3] = e_1$, $[e_2, e_3] = ae_2$, $0 < |a| < 1$ |
| $G^8$       | $[e_1, e_3] = -e_2$, $[e_2, e_3] = e_1$ |
| $G^9$       | $[e_1, e_3] = be_1 - e_2$, $[e_2, e_3] = e_1 + be_2$, $b > 0$ |
| $G^{10}$    | $[e_1, e_3] = e_1$, $[e_2, e_3] = -e_2$, $[e_1, e_2] = 2e_3$ |
| $G^{11}$    | $[e_1, e_2] = e_3$, $[e_2, e_3] = e_1$, $[e_3, e_1] = e_2$ |

2.1. The $S-$expansions procedure

In general, this process consists of through an Abelian semigroup $S$ and Lie algebra $G$, a new Lie algebra is generated [1]. In this work, we will consider $S-$expansions of Lie algebra that admit the following subspace decomposition $G = G_0 \oplus G_1$, with a $\mathbb{Z}_2$-graded structure,

$$[G_0, G_0] \subset G_0, [G_0, G_1] \subset G_1, [G_1, G_1] \subset G_0.$$  (2.1)

We will consider expansions with the finite semigroup $S = \{\lambda_i\}$, provided with a closed associative and commutative product. In particular we consider the semigroup $S_N = \{\lambda_0, ..., \lambda_{N+1}\}$, with product $\lambda_i \lambda_j = \lambda_{i+j}$, for $i + j \leq N$ and $\lambda_i \lambda_j = \lambda_{N+1}$ for $i + j > N$, and $\lambda_{N+1}$ acts like the element zero [4]. The semigroup is considered together with the following resonant decomposition

$$S^0_N = \{\lambda_{2k}\} \cup \{\lambda_{N+1}\}, S^1_N = \{\lambda_{2k+1}\} \cup \{\lambda_{N+1}\},$$

which is compatible with (2.1) in the sense that

$$S^0_N \cdot S^0_N \subset S^0_N, S^0_N \cdot S^1_N \subset S^1_N, S^1_N \cdot S^1_N \subset S^1_N.$$
and therefore resonant with the choice of \( G_0, G_1 \). Thus, a reduced resonant expanded algebra can be defined as the

\[ G_{SN} = \langle \{ S_N \otimes G_0 \} \rangle \cup \langle \{ S_N \otimes G_1 \} \rangle, \]

where the reduction condition in the algebra is implemented by the constraints

\[ \lambda_{N+1} \otimes e_i = 0, \]

for all \( i \).

**Theorem:** Let the Lie algebra \( G \) and the semigroup \( S_N \) then \( G_{SN} = S_N \times G \) is a Lie algebra.

3. \( S_N \)-expansions of three-dimensional Lie Algebras

We will write an element \( E_{ij} \in S_N \times G \) as \( E_{ij} = \lambda_i \otimes e_j \). Let \( G \) be a three dimensional Lie algebra together with the \( \mathbb{Z}_2 \)-graduation;

\[ G^0 = \langle \{ e_3 \} \rangle \]
\[ G^1 = \langle \{ e_1, e_2 \} \rangle \]

We will illustrate a case in detail, the other cases are deduced analogously. For example, let us take the algebra \( G^{11} \) from the table and perform the process of \( S_N \) expansion. If \( N = 2n + 1 \), then an algebra is generated

\[ G^{11}_{SN} = \langle \{ E_{(1+2l)1}, E_{(1+2l)2}, E_{(2m)3} \} \rangle \]

where \( 0 \leq l \leq \left[ \frac{N-1}{2} \right], 0 \leq m \leq \left[ \frac{N}{2} \right] \). The brackets are given by

\[ [E_{(2m)3}, E_{(1+2l)1}] = \lambda_{2m} \otimes e_3, \lambda_{1+2l} \otimes e_1 ] \]
\[ = \lambda_{1+2l+2m} \otimes [e_3, e_1] \]
\[ = \lambda_{1+2l+2m} \otimes e_2 \]
\[ = E_{(1+2l+2m)2}, \]
\[ [E_{(1+2l)2}, E_{(2m)3}] = \lambda_{1+2l} \otimes e_2, \lambda_{2m} \otimes e_3 ] \]
\[ = \lambda_{1+2l+2m} \otimes [e_2, e_3] \]
\[ = \lambda_{1+2l+2m} \otimes e_1 \]
\[ = E_{(1+2l+2m)1}, \]

for \( 0 \leq l \leq \left[ \frac{N-1}{2} \right], 0 \leq m \leq \left[ \frac{N}{2} \right] \) and \( 1 + 2l + 2n \leq N + 1 \).
\[ [E_{(1+2\tilde{l})1}, E_{(1+2\tilde{m})2}] = [\lambda_{1+2\tilde{l}} \otimes e_1, \lambda_{1+2\tilde{m}} \otimes e_2] = \lambda_{2+2\tilde{l}+2\tilde{m}} \otimes [e_1, e_2] = \lambda_{2+2\tilde{l}+2\tilde{m}} \otimes e_3 = E_{(2+2\tilde{l}+2\tilde{m})3}, \]

where \( 0 \leq \tilde{l}, \tilde{m} < \left\lfloor \frac{N-1}{2} \right\rfloor \).

For \( N = 2n + 2 \), the generated algebra can be written as

\[ G_{S_N}^{11} = \langle \{ E_{(1+2\tilde{l})1}, E_{(1+2\tilde{l})2}, E_{(2\tilde{m})3} \} \rangle \]

where \( 0 \leq l \leq \left\lfloor \frac{N-1}{2} \right\rfloor \), \( 0 \leq m \leq \left\lfloor \frac{N}{2} \right\rfloor + 1 \). The brackets are given by (3.1)-(3.8), and

\[ [E_{(1+2\tilde{l})1}, E_{(1+2\tilde{m})2}] = [\lambda_{1+2\tilde{l}} \otimes e_1, \lambda_{1+2\tilde{m}} \otimes e_2] = \lambda_{2+2\tilde{l}+2\tilde{m}} \otimes [e_1, e_2] = \lambda_{2+2\tilde{l}+2\tilde{m}} \otimes e_3 = E_{(2+2\tilde{l}+2\tilde{m})3}, \]

where \( 0 \leq \tilde{l}, \tilde{m} \leq \left\lfloor \frac{N-1}{2} \right\rfloor \).

The algebras generated by \( S_N \)-expansion process are given by:

(i) If \( N = 2n + 1 \), algebras of dimension \( 3(n + 1) \) are generated
for $0 \leq l \leq \left[\frac{N-1}{2}\right]$, $0 \leq m \leq \left[\frac{N}{4}\right]$ and $0 \leq \tilde{l}, \tilde{m} < \left[\frac{N-1}{2}\right]$.

(ii) If $N = 2n + 2$, algebras of dimension $3(n + 1) + 1$ are generated. These algebras, except for the case $G^{10}$ and $G^{11}$, are central extensions of the algebras generated for the case $N = 2n + 1$.

4. $S_N$–expansions of $\mathcal{KN}$ algebras

$\mathcal{KN}$ type algebras are examples of infinite dimensional Lie algebras. The elements of the $\mathcal{KN}$ types algebras are meromorphic objects on a compact Riemann surface which are holomorphic outside a fixed set of points and the are related to the Virasoro algebra. Virasoro algebra is a much studied object, both mathematics and physics. An example of algebras of type $\mathcal{KN}$ can be obtained with the following basis [5]:

$$B_n(t) := t(t - \alpha)^{n-1}(t + \alpha)^{n-1} \frac{d}{dt},$$

$$F_n(t) := (t - \alpha)^{n-1}(z + \alpha)^{n-1} \frac{d}{dt},$$

for $\alpha \in \mathbb{C}$, which satisfy the Lie brackets
\[ [B_n, B_m] = 2(m - n)(B_{n+m-1} + \alpha^2 B_{n+m-2}) , \]
\[ [B_n, F_m] = (2(m - 1) - 1)F_{n+m-1} + 2(m - n)\alpha^2 F_{n+m-2} , \]
\[ [F_n, F_m] = 2(m - n)B_{n+m-2} . \]

Considering the semigroup $S_3$ we obtain the generators

\[ B_n^0 = \lambda_0 \otimes B_n , \]
\[ B_n^2 = \lambda_2 \otimes B_n , \]
\[ F_n^1 = \lambda_1 \otimes F_n , \]
\[ F_n^3 = \lambda_3 \otimes F_n , \]

which satisfy the Lie brackets,

\[ [B_n^0, B_m^0] = 2(m - n)(B_{n+m-1}^0 + \alpha^2 B_{n+m-2}^0) , \]
\[ [B_n^0, B_m^2] = 2(m - n)(B_{n+m-1}^2 + \alpha^2 B_{n+m-2}^2) , \]
\[ [B_n^0, F_m^1] = (2(m - 1) - 1)F_{n+m-1}^1 + 2(m - n)\alpha^2 F_{n+m-2}^1 , \]
\[ [B_n^2, F_m^1] = (2(m - 1) - 1)F_{n+m-1}^3 + 2(m - n)\alpha^2 F_{n+m-2}^3 , \]
\[ [F_n^1, F_m^1] = 2(m - n)B_{n+m-2}^2 . \]

Now we perform contractions of the $KN$ algebras and the $S_N$–expanded $KN$ algebra, obtaining six new subalgebras of infinite dimension.

### 4.1. Contractions of $S_N$–expanded $KN$ algebras

The commutation relations of a contracted Lie algebra, or contractions of a Lie algebra $\mathcal{G}$, are given by the limit [3]:

\[ [e_i, e_j]_\epsilon := \lim_{\epsilon \to \epsilon_0} U_{\epsilon}^{-1}[U_{\epsilon}(e_i), U_{\epsilon}(e_j)], \tag{4.1} \]

where $U_{\epsilon}$ is a non-singular linear transformation of $\mathcal{G}$, with $\epsilon_0$ being a singularity point of its inverse $U_{\epsilon}^{-1}$. Considering the splitting of the Lie algebra $\mathcal{G}$ into an arbitrary number of subspace $\mathcal{G} = \mathcal{G}_0 + ... + \mathcal{G}_p$ where $p \leq \dim \mathcal{G}$, and the diagonal $U_{\epsilon} = \oplus_j e^{n_j} \text{id}_{\mathcal{G}_j}$, with $\epsilon > 0, n_j \in \mathbb{R}$ and $j = 1, ..., p$ is obtained $[e_i, e_j]_\epsilon = \lim_{\epsilon \to 0} e^{n_i + n_j - nk} C_{ij}^k e_k$. Then the exponent must satisfy $n_i + n_j - nk \geq 0$ unless $C_{ij}^k = 0$. The constants of the contracted algebra are given by $(C_{\epsilon}^{ik})_{ij} = C^{ik}_{ij}$ if $n_i + n_j = n_k$, and $(C_{\epsilon})^{ik}_{ij} = 0$ if $n_i + n_j > n_k$. Two trivial contractions are: the Abelian algebra and the original Lie algebra, for which the commutation relations are unchanged.

Through contractions we can obtain new non-isomorphic Lie algebras. For this, let $U_{\epsilon}$ be such that

\[ U_{\epsilon} := e^{n_0} \text{id}_{\mathcal{G}_0} + e^{n_1} \text{id}_{\mathcal{G}_1} + e^{n_2} \text{id}_{\mathcal{G}_2} + e^{n_3} \text{id}_{\mathcal{G}_3} , \]
where $\mathcal{G}_0 = \langle \{B^0_n\} \rangle$, $\mathcal{G}_1 = \langle \{F^1_n\} \rangle$, $\mathcal{G}_2 = \langle \{B^2_n\} \rangle$ and $\mathcal{G}_3 = \langle \{F^3_n\} \rangle$ with $n \in \mathbb{Z}$. The Lie brackets are modified to

\[
[B^0_n, B^0_m] = \epsilon n_0 [2(m-n)(B^0_{n+m-1} + \alpha^2 B^0_{n+m-2})], \\
[B^0_n, B^2_m] = \epsilon n_0 [2(m-n)(B^2_{n+m-1} + \alpha^2 B^2_{n+m-2})], \\
[B^0_n, F^1_m] = \epsilon n_0 [(2(m-1) - 1)F^1_{n+m-1} + 2(m-n)\alpha^2 F^1_{n+m-2}], \\
[B^2_n, F^1_m] = \epsilon n_0 + a n_0 \cdot [(2(m-1) - 1)F^3_{n+m-1} + 2(m-n)\alpha^2 F^3_{n+m-2}], \\
[B^0_n, F^3_m] = \epsilon n_0 [(2(m-1) - 1)F^3_{n+m-1} + 2(m-n)\alpha^2 F^3_{n+m-2}], \\
[F^1_n, F^3_m] = \epsilon n_0 - a n_0 [2(m-n)B^2_{n+m-2}].
\]

Non-trivial cases in the choice of $n_0, n_1, n_2, n_3$ are given in the limit $\epsilon \to 0$:

(i) For $n_0 = 0$:

(a) $n_1 = n_2 = n_3 = 1$,

\[
[B^0_n, B^0_m] = 2(m-n)(B^0_{n+m-1} + \alpha^2 B^0_{n+m-2}), \\
[B^0_n, B^2_m] = 2(m-n)(B^2_{n+m-1} + \alpha^2 B^2_{n+m-2}), \\
[B^0_n, F^1_m] = (2(m-1) - 1)F^1_{n+m-1} + 2(m-n)\alpha^2 F^1_{n+m-2}, \\
[B^0_n, F^3_m] = (2(m-1) - 1)F^3_{n+m-1} + 2(m-n)\alpha^2 F^3_{n+m-2}.
\]

(b) $n_1 = n_2 = 1, n_3 = 2$,

\[
[B^0_n, B^0_m] = 2(m-n)(B^0_{n+m-1} + \alpha^2 B^0_{n+m-2}), \\
[B^0_n, B^2_m] = 2(m-n)(B^2_{n+m-1} + \alpha^2 B^2_{n+m-2}), \\
[B^0_n, F^1_m] = (2(m-1) - 1)F^1_{n+m-1} + 2(m-n)\alpha^2 F^1_{n+m-2}, \\
[B^0_n, F^3_m] = (2(m-1) - 1)F^3_{n+m-1} + 2(m-n)\alpha^2 F^3_{n+m-2}.
\]

(c) $n_1 = n_3 = 1, n_2 = 2$,

\[
[B^0_n, B^0_m] = 2(m-n)(B^0_{n+m-1} + \alpha^2 B^0_{n+m-2}), \\
[B^0_n, B^2_m] = 2(m-n)(B^2_{n+m-1} + \alpha^2 B^2_{n+m-2}), \\
[B^0_n, F^1_m] = (2(m-1) - 1)F^1_{n+m-1} + 2(m-n)\alpha^2 F^1_{n+m-2}, \\
[B^0_n, F^3_m] = (2(m-1) - 1)F^3_{n+m-1} + 2(m-n)\alpha^2 F^3_{n+m-2}.
\]

(ii) For $n_0 > 0$:

(a) $n_1 = 1, n_2 = 2, n_3 = 3$,

\[
[B^2_n, F^1_m] = (2(m-1) - 1)F^3_{n+m-1} + 2(m-n)\alpha^2 F^3_{n+m-2}, \\
[F^1_n, F^3_m] = 2(m-n)B^2_{n+m}.
\]
(b) \( n_1 = n_3 = 1, n_2 = 2, \)
\[
[F^1_n, F^1_m] = 2(m - n)B^2_{n+m}.
\]
(c) \( n_1 = 1, n_2 = 2, n_3 = 3, \)
\[
[B^2_n, F^1_m] = (2(m - 1) - 1)F^3_{n+m-1} + 2(m - n)\alpha^2 F^3_{n+m-2}.
\]

The cases i(b) and ii(b) correspond to the \( S_3 \)-expansions of the contractions for this examples of \( \mathcal{KN} \) algebra analogously to [3].

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5. References
[1] Izaurieta F, Rodriguez E and Salgado P 2006 J. Math. Phys. 47 123512
[2] Nesterenko M and Popovych R 2006 J. Math. Phys. 47 123515
[3] Fialowski A and Montigny M 2005 J. Phys. A: Math. Gen. 38 6335–6349
[4] Gomis J, Kleinschmidt A, Palmkvist J and Salgado P 2020 JHEP 02 009
[5] Schlichenmaier M 1993 J.Math.Phys. 34 3809-3824