NOTE ON THE GENERALIZATION OF THE HIGHER ORDER
\textit{q}-\textit{GENOCCHI NUMBERS AND \textit{q}-EULER NUMBERS}

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Abstract Cangul-Ozden-Simsek\cite{1} constructed the \textit{q}-Genocchi numbers of high order using a fermionic \textit{p}-adic integral on \(\mathbb{Z}_p\), and gave Witt’s formula and the interpolation functions of these numbers. In this paper, we present the generalization of the higher order \textit{q}-Euler numbers and \textit{q}-Genocchi numbers of Cangul-Ozden-Simsek. We define \textit{q}-extensions of \(w\)-Euler numbers and polynomials, and \(w\)-Genocchi numbers and polynomials of high order using the multivariate fermionic \textit{p}-adic integral on \(\mathbb{Z}_p\). We have the interpolation functions of these numbers and polynomials. We obtain the distribution relations for \textit{q}-extensions of \(w\)-Euler and \(w\)-Genocchi polynomials. We also have the interesting relation for \textit{q}-extensions of these polynomials. We define \((h, q)\)-extensions of \(w\)-Euler and \(w\)-Genocchi polynomials of high order. We have the interpolation functions for \((h, q)\)-extensions of these polynomials. Moreover, we obtain some meaningful results of \((h, q)\)-extensions of \(w\)-Euler and \(w\)-Genocchi polynomials.

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1. Introduction, Definitions and Notations

Many authors have been studied on the multiple Genocchi and Euler numbers, and multiple zeta functions (cf. [1-2], [4-6], [9-10], [14], [17], [19], [22], [24]). In [10], Kim, the first author of this paper, presented a systematic study of some families of multiple \textit{q}-Euler numbers and polynomials. By using the \textit{q}-Volkenborn integration on \(\mathbb{Z}_p\), Kim constructed the \textit{p}-adic \textit{q}-Euler numbers and polynomials of higher order, and gave the generating function of these numbers and the Euler \(q\)-\(\zeta\)-function. In [14], Kim studied some families of multiple \textit{q}-Genocchi and \textit{q}-Euler numbers by using the multivariate \textit{p}-adic \textit{q}-Volkenborn integral on \(\mathbb{Z}_p\), and gave interesting identities related to these numbers.

Recently, Cangul-Ozden-Simsek\cite{1} constructed the \textit{q}-Genocchi numbers of high order by using a fermionic \textit{p}-adic integral on \(\mathbb{Z}_p\), and gave Witt’s formula and the interpolation functions of these numbers. In [17], Kim gave another constructions of the \textit{q}-Euler and \textit{q}-Genocchi numbers, which were different from those of Cangul-Ozden-Simsek. Kim obtained the interesting relationship between the \textit{q}-\(w\)-Euler numbers and \textit{q}-\(w\)-Genocchi numbers, and gave the interpolation functions of these numbers. In this paper, we will present the generalization of the higher order \textit{q}-Euler numbers and \textit{q}-Genocchi numbers of Cangul-Ozden-Simsek approaching as Kim did in [17].

Throughout this paper, let \(p\) be a fixed odd number and the symbols \(\mathbb{Z}_p, \mathbb{Q}_p, \mathbb{C}\) and \(\mathbb{C}_p\) denote the ring of \(p\)-adic rational integers, the field of \(p\)-adic rational numbers, the complex number field and the completion of algebraic closure of \(\mathbb{Q}_p\), respectively.
From the definition of the $q$-normalized exponential valuation of $C$, respectively. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let $v_p$ be the normalized exponential valuation of $\mathbb{C}_p$, with $|p|_p = p^{-v_p(p)} = \frac{1}{p}$.

The symbol $q$ can be treated as a complex number, $q \in \mathbb{C}$, or as a $p$-adic number, $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, then we always assume that $|q| < 1$. If $q \in \mathbb{C}_p$, then we usually assume that $|1 - q|_p < 1$.

Now we will recall some $q$-notations. The $q$-basic natural numbers are defined by $[n]_q = \frac{1 - q^n}{1 - q}$, $[n]_q! = \frac{1 - q^n}{1 - q}$, and the $q$-factorial by $[n]_q! = [n]_q[n - 1]_q \cdots [2]_q[1]_q$. In this paper, we use the notation $[x]_q = \frac{1 - q^x}{1 - q}$. Hence $\lim_{q \to 1} [x]_q = x$ for any $x$ with $|x|_p \leq 1$ in the $p$-adic case (see [1-25]).

The $q$-shift factorial is given by

$$(a : q)_0 = 1, \quad (a : q)_k = (1 - a)(1 - aq) \cdots (1 - aq^{k-1}).$$

We note that $\lim_{q \to 1} (a : q)_k = (1 - a)^k$. It is known that

$$(a : q)_\infty = (1 - a)(1 - aq)(1 - aq^2) \cdots = \prod_{i=1}^{\infty} (1 - aq^{i-1}), \quad \text{(see [8]).}$$

From the definition of the $q$-shift factorial, we note that

$$(a : q)_k = \frac{(a : q)_\infty}{(aq^k : q)_\infty}.$$
Hence it follows that
\[
\frac{1}{(z : q)^\alpha} = \sum_{n=0}^{\infty} \binom{n + \alpha - 1}{n} z^n q^n,
\]
which converges to \( \frac{1}{(1 - z)^\alpha} = \sum_{n=0}^{\infty} \binom{n + \alpha - 1}{n} z^n \) as \( q \to 1 \).

We say that \( f \) is a uniformly differentiable function at a point \( a \in \mathbb{Z}_p \), and write \( f \in UD(\mathbb{Z}_p) \), the set of uniformly differentiable function, if the difference quotients \( F_g(x, y) = \frac{f(x) - f(y)}{x - y} \) have a limit \( l = f'(a) \) as \( (x, y) \to (a, a) \). For \( f \in UD(\mathbb{Z}_p) \), the \( q \)-deformed bosonic \( p \)-adic integral is defined as
\[
I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \to \infty} \sum_{x=0}^{p^N-1} f(x) \frac{q^x}{[p^N]_q},
\]
and the \( q \)-deformed fermionic \( p \)-adic integral is defined by
\[
I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{N \to \infty} \sum_{x=0}^{p^N-1} f(x) \frac{(-q)^x}{[p^N]_{-q}}.
\]
The fermionic \( p \)-adic integral on \( \mathbb{Z}_p \) is defined as
\[
I_{-1}(f) = \lim_{q \to 1} I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x).
\]
It follows that \( I_{-1}(f_1) = -I_{-1}(f) + 2f(0) \), where \( f_1(x) = f(x + 1) \). For details, see [4-17].

The classical Euler polynomials \( E_n(x) \) are defined as
\[
\frac{2}{e^t + 1} e^{xt} = \sum_{x=0}^{\infty} E_n(x) \frac{t^n}{n!},
\]
and the Euler numbers \( E_n \) are defined as \( E_n = E_n(0) \), (see [1-25]). The Genocchi numbers are defined as
\[
\frac{2t}{e^t + 1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!} \quad \text{for} \quad |t| < \pi,
\]
and the Genocchi polynomials \( G_n(x) \) are defined as
\[
\frac{2t}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!}, \quad \text{(see [12], [14], [21])}.
\]

It is known that the \( w \)-Euler polynomials \( E_{n,w}(x) \) are defined as
\[
\frac{2}{we^t + 1} e^{xt} = \sum_{x=0}^{\infty} E_{n,w}(x) \frac{t^n}{n!},
\]
and \( E_{n,w} = E_{n,w}(0) \) are called the \( w \)-Euler numbers. The \( w \)-Genocchi polynomials \( G_{n,w}(x) \) are defined as
\[
\frac{2t}{we^t + 1} e^{xt} = \sum_{x=0}^{\infty} G_{n,w}(x) \frac{t^n}{n!},
\]
and \( G_{n,w} = G_{n,w}(0) \) are called the \( w \)-Genocchi numbers, (see [1]).
The $w$-Euler polynomials $E_{n,w}^{(r)}(x)$ of order $r$ are defined as
\[
\left( \frac{2}{w+1} \right)^r e^{xt} = \sum_{n=0}^{\infty} E_{n,w}^{(r)}(x) \frac{t^n}{n!},
\]
(see [1]), and $E_{n,w}^{(r)} = E_{n,w}^{(r)}(0)$ are called the $w$-Euler numbers of order $r$. The $w$-Genocchi polynomials $G_{n,w}^{(r)}(x)$ of order $r$ are defined as
\[
\frac{2te^{xt}}{w+1} = \sum_{n=0}^{\infty} G_{n,w}^{(r)}(x) \frac{t^n}{n!},
\]
(see [1]), and $G_{n,w}^{(r)} = G_{n,w}^{(r)}(0)$ are called the $w$-Euler numbers of order $r$. When $r = 1$ and $w = 1$, $E_{n,w}^{(r)}(x)$ and $E_{n,w}^{(r)}$ are the ordinary Euler polynomials and numbers, and $G_{n,w}^{(r)}(x)$ and $G_{n,w}^{(r)}$ are the ordinary Genocchi polynomials and numbers, respectively.

In Section 2, we define $q$-extensions of $w$-Euler numbers and polynomials of order $r$ and $w$-Genocchi numbers and polynomials of order $r$, respectively, using the multivariate fermionic $p$-adic integral on $\mathbb{Z}_p$. We obtain the interpolation functions of these numbers and polynomials. We have the distribution relations for $q$-extensions of $w$-Euler polynomials and those of $w$-Genocchi polynomials. We obtain the interesting relation for $q$-extensions of these polynomials. We also define $(h, q)$-extensions of $w$-Euler and $w$-Genocchi polynomials of order $r$. We have the interpolation functions for $(h, q)$-extensions of these polynomials. Moreover, we obtain some meaningful results of $(h, q)$-extensions of $w$-Euler and $w$-Genocchi polynomials when $h = r - 1$.

2. ON THE EXTENSION OF THE HIGHER ORDER $q$-GENOCCHI NUMBERS AND $q$-EULER NUMBERS OF CANGUL-OZDEN-SIMSEK

In this section, we assume that $w \in \mathbb{C}_p$ with $|1 - w|_p < 1$ and $q \in \mathbb{C}_p$ with $|1 - q|_p < 1$. Recently, Cangul-Ozden-Simsek[1] constructed $w$-Genocchi numbers of order $r$, $G_{n,w}^{(r)}$, as follows.

\[
\begin{align*}
& t^r \int_{\mathbb{Z}_p} w^{x_1 + \cdots + x_r} e^{t(x_1 + \cdots + x_r)} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \\
& = 2^r \left( \frac{t}{w+1} \right)^r = \sum_{n=0}^{\infty} G_{n,w}^{(r)} \frac{t^n}{n!},
\end{align*}
\]
where $\int_{\mathbb{Z}_p} = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}$ ($r$-times) and $r \in \mathbb{N}$. They also consider the $q$-extension of $G_{n,w}^{(r)}$ as follows.

\[
\begin{align*}
& t^r \int_{\mathbb{Z}_p} \left( \sum_{i=1}^{\infty} \frac{(h-i+1)x_i}{e} \right)^r e^{t(x_1 + \cdots + x_r)} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \\
& = \left( \frac{2^rt^r}{(q^h+1) \cdots (q^h-r+1e+1)} \right) = \sum_{n=0}^{\infty} G_{n,q}^{(h,r)} \frac{t^n}{n!}.
\end{align*}
\]
From (2), they obtained the following interesting formula:

$$G_{n+r,q}^{(r-1,r)} = 2^r r! \left( \binom{n+r}{r} \sum_{v=0}^{\infty} \binom{r+v-1}{v}_q (-1)^v v^n. \right)$$

(3)

It seems to be interested to find the numbers corresponding to

$$2^r r! \left( \binom{n+r}{r} \sum_{v=0}^{\infty} \binom{r+v-1}{v}_q (-1)^v[v]^n. \right)$$

In the viewpoint of the $q$-extension of (1) using the multivariate $p$-adic integral on $\mathbb{Z}_p$, we define the $q$-analogue of $w$-Euler numbers of order $r$, $E_{n,w,q}^{(r)}$, as follows.

$$E_{n,w,q}^{(r)} = \int_{\mathbb{Z}_p} w^{x_1 + \cdots + x_r} [x_1 + \cdots + x_r]_q^n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r).$$

(4)

From (4), we note that

$$E_{n,w,q}^{(r)} = \frac{2^r}{(1-q^n)} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \left( \frac{1}{1+qw} \right)^r$$

$$= \frac{2^r}{(1-q^n)} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \sum_{m=0}^{\infty} \binom{m+r-1}{m} (-1)^m q^m w^m$$

$$= 2^r \sum_{m=0}^{\infty} \binom{m+r-1}{m} (-1)^m w^m [m]^n_q.$$

Therefore, we obtain the following theorem.

**Theorem 1.** Let $r \in \mathbb{N}$ and $n \in \mathbb{Z}_+$. Then we have

$$E_{n,w,q}^{(r)} = 2^r \sum_{m=0}^{\infty} \binom{m+r-1}{m} (-1)^m w^m [m]^n_q.$$

(5)

Let $F^{(r)}(t,w|q) = \sum_{n=0}^{\infty} E_{n,w,q}^{(r)} \frac{t^n}{n!}$. By (4) and (5), we see that

$$F^{(r)}(t,w|q) = \int_{\mathbb{Z}_p} w^{x_1 + \cdots + x_r} e^{t[x_1 + \cdots + x_r]_q} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r)$$

$$= 2^r \sum_{m=0}^{\infty} \binom{m+r-1}{m} (-1)^m w^m e^{[m]}_q.$$

Thus we obtain the following corollary.

**Corollary 2.** Let $F^{(r)}(t,w|q) = \sum_{n=0}^{\infty} E_{n,w,q}^{(r)} \frac{t^n}{n!}$. Then we have

$$F^{(r)}(t,w|q) = 2^r \sum_{m=0}^{\infty} \binom{m+r-1}{m} (-1)^m w^m e^{[m]}_q.$$

Let us define the $q$-extension of $w$-Euler polynomials of order $r$ as follows.

$$E_{n,w,q}^{(r)}(x) = \int_{\mathbb{Z}_p} w^{x_1 + \cdots + x_r} [x+x_1 + \cdots + x_r]_q^n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r).$$

(6)
Theorem 3. Therefore, we obtain the following theorem.

By comparing the coefficients on both sides of (10), we see that

\[ E_{n,w,q}^{(r)}(x) = 2^r \sum_{m=0}^{\infty} \binom{m+r-1}{m}(-1)^m w^m [m+x]_q^n. \]

Therefore, we obtain the following theorem.

**Theorem 3.** Let \( r \in \mathbb{N} \) and \( n \in \mathbb{Z}_+ \). Then we have

\[ E_{n,w,q}^{(r)}(x) = 2^r \sum_{m=0}^{\infty} \binom{m+r-1}{m}(-1)^m w^m [m+x]_q^n. \]

Let \( F^{(r)}(t, w, x|q) = \sum_{n=0}^{\infty} E_{n,w,q}^{(r)}(x) \frac{t^n}{n!} \). By (6) and (7), we have

\[
F^{(r)}(t, w, x|q) = \int_{\mathbb{Z}_p^r} w^{x_1 + \cdots + x_r} e^{t [x_1 + \cdots + x_r]} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r)
\]

\[
= 2^r \sum_{m=0}^{\infty} \binom{m+r-1}{m}(-1)^m w^m e^{t [m+x]} q^n.
\]

Therefore we have the following corollary.

**Corollary 4.** Let \( F^{(r)}(t, w, x|q) = \sum_{n=0}^{\infty} E_{n,w,q}^{(r)}(x) \frac{t^n}{n!} \). Then we have

\[ F^{(r)}(t, w, x|q) = 2^r \sum_{m=0}^{\infty} \binom{m+r-1}{m}(-1)^m w^m e^{t [m+x]} q^n. \]

Now we define the \( q \)-extension of \( w \)-Genocchi polynomials of order \( r \), \( G_{n,w,q}^{(r)}(x) \), as follows.

\[ 2^r t^r \sum_{m=0}^{\infty} \binom{m+r-1}{m}(-1)^m w^m e^{t [m+x]} q^n = \sum_{n=0}^{\infty} G_{n,w,q}^{(r)}(x) \frac{t^n}{n!}. \]

Then we have

\[ \sum_{n=0}^{\infty} G_{n,w,q}^{(r)}(x) \frac{t^n}{n!} = t^r \int_{\mathbb{Z}_p^r} w^{x_1 + \cdots + x_r} e^{t [x_1 + \cdots + x_r]} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r)
\]

\[ = \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p^r} w^{x_1 + \cdots + x_r} [x + x_1 + \cdots + x_r]_q^n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) r! \binom{n+r}{n} \frac{t^{n+r}}{(n+r)!}.
\]

By comparing the coefficients on both sides of (10), we see that

\[ G_{0,w,q}^{(r)}(x) = G_{1,w,q}^{(r)}(x) = \cdots = G_{r-1,w,q}^{(r)}(x) = 0, \]
Theorem 6. \[ G_{n+r,w,q}^{(r)}(x) = r! \binom{n+r}{r} \int_{\mathbb{Z}_p} w^{x_1+x_2+\cdots+x_r} [x + x_1 + \cdots + x_r]^n d \mu_{-1}(x_1) \cdots d \mu_{-1}(x_r) \]

Furthermore, \[ E_{n,w,q}^{(r)}(x) = r! \binom{n+r}{r} E_{n,w,q}^{(r)}(x). \]

In the special case of \( x = 0 \), \( G_{n,w,q}^{(r)}(0) = G_{n,w,q}^{(r)} \) are called the \( q \)-extension of \( w \)-Genocchi numbers of order \( r \). By \([\text{III}]\), we have the following theorem.

Theorem 5. Let \( r \in \mathbb{N} \) and \( n \in \mathbb{Z}_+ \). Then we have

\[
\frac{G_{n+r,w,q}^{(r)}(x)}{r!(n+r)} = \int_{\mathbb{Z}_p} w^{x_1+x_2+\cdots+x_r} [x + x_1 + \cdots + x_r]^n d \mu_{-1}(x_1) \cdots d \mu_{-1}(x_r) = E_{n,w,q}^{(r)}(x),
\]

and \( G_{0,w,q}^{(r)}(x) = G_{1,w,q}^{(r)}(x) = \cdots = G_{r-1,w,q}^{(r)}(x) = 0. \)

Now we consider the distribution relation for the \( q \)-extension of \( w \)-Euler polynomials of order \( r \). For \( d \in \mathbb{N} \) with \( d \equiv 1 \pmod{2} \), by \([8]\), we see that

\[
F^{(r)}(t, w, x|q) = 2^r \sum_{m=0}^{\infty} \left( \frac{m+r-1}{m} \right) (-1)^m w^m e^{t[m+x]_q} \\
= \sum_{a_1, \ldots, a_r = 0} d=1 \left( \prod_{i=1}^{r} w^{a_i} \right) (-1)^{a_1+\cdots+a_r} 2^r \sum_{m=0}^{\infty} \left( \frac{m+r-1}{m} \right) (-1)^m w^m e^{t[d]_{q}(m+a_1+\cdots+a_r+x)_{q^d}} \\
= \sum_{a_1, \ldots, a_r = 0} d=1 \left( \prod_{i=1}^{r} w^{a_i} \right) (-1)^{a_1+\cdots+a_r} E^{(r)}([d]_{q}t, w^d, \frac{a_1+\cdots+a_r+x}{d} \mid q^d).
\]

By \([12]\), we obtain the following distribution relations for \( E_{n,w,q}^{(r)}(x) \) and \( G_{n+r,w,q}^{(r)}(x) \), respectively.

Theorem 6. Let \( r \in \mathbb{N}, n \in \mathbb{Z}_+ \) and \( d \in \mathbb{N} \) with \( d \equiv 1 \pmod{2} \). Then we have

\[
E_{n,w,q}^{(r)}(x) = [d]_{q}^{n} \sum_{a_1, \ldots, a_r = 0} d=1 \left( \prod_{i=1}^{r} w^{a_i} \right) (-1)^{a_1+\cdots+a_r} E_{n,w^d,q^d}^{(r)}(\frac{a_1+\cdots+a_r+x}{d}).
\]

Furthermore, \[
G_{n+r,w,q}^{(r)}(x) = [d]_{q}^{n} \sum_{a_1, \ldots, a_r = 0} d=1 \left( \prod_{i=1}^{r} w^{a_i} \right) (-1)^{a_1+\cdots+a_r} G_{n+r,w^d,q^d}^{(r)}(\frac{a_1+\cdots+a_r+x}{d}).
\]

For the extension of \([2]\), we consider the \((h,q)\)-extension of \( w \)-Euler polynomials of order \( r \). For \( h \in \mathbb{Z}, r \in \mathbb{N} \) and \( n \in \mathbb{Z}_+ \), let us define the \((h,q)\)-extension of \( w \)-Euler polynomial of order \( r \) as follows.
Therefore, we obtain the following theorem.

From (13), we note that

(13) \[ E_{n,w,q}^{(h,r)}(x) = \int_{\mathbb{Z}_q} w^{x_1+\cdots+x_r} [x + x_1 + \cdots + x_r]^n q^{-i} \sum_{i=1} \mu_{-1}(x_1) \cdots \mu_{-1}(x_r). \]

From (13), we note that

\[ E_{n,w,q}^{(h,r)}(x) = \frac{2^r}{(1-q)^n} \sum_{l=0}^{n} \frac{\binom{n}{l}(-1)^l q^{lx}}{(1 + q^{l+h} w)(1 + q^{l+h-1} w) \cdots (1 + q^{l+r-1} w)} \]

(14) \[ = \frac{2^r}{(1-q)^n} \sum_{l=0}^{n} \frac{\binom{n}{l}(-1)^l q^{lx}}{(q^{l+h} w : q^{-1}) r} \]

(14) \[ = \frac{2^r}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l}(-1)^l q^{lx} \sum_{m=0}^{\infty} \left( \binom{m + r - 1}{m} \right) q^{-m} \sum_{q=1}^{m} (-1)^m q^{hm} w^m \]

Therefore, we obtain the following theorem.

**Theorem 7.** Let \( h \in \mathbb{Z}, r \in \mathbb{N} \) and \( n \in \mathbb{Z}_+ \). Then we have

\[ E_{n,w,q}^{(h,r)}(x) = \frac{2^r}{(1-q)^n} \sum_{l=0}^{n} \frac{\binom{n}{l}(-1)^l q^{lx}}{(q^{l+h} w : q^{-1}) r} \]

(15) \[ = \frac{2^r}{(1-q)^n} \sum_{m=0}^{\infty} \left( \binom{m + r - 1}{m} \right) q^{-m} \sum_{q=1}^{m} (-1)^m q^{hm} w^m [m + x]^n. \]

We also have the following result.

**Corollary 8.** Let \( F^{(h,r)}(t, w, x | q) = \sum_{n=0}^{\infty} E_{n,w,q}^{(h,r)}(x) t^n/n! \). Then we have

\[ F^{(h,r)}(t, w, x | q) = 2^r \sum_{m=0}^{\infty} \left( \binom{m + r - 1}{m} \right) q^{-m} \sum_{q=1}^{m} (-1)^m q^{hm} w^m e^{t[m+x]} q. \]

**Remark 1.** In the special case \( x = 0 \), \( E_{n,w,q}^{(h,r)}(0) = E_{n,w,q}^{(h,r)} \) are called the \((h, q)\)-extension of \( w\)-Euler numbers of order \( r \).

If we take \( h = r - 1 \) in (14), then we have

\[ E_{n,w,q}^{(r-1,r)}(x) = \frac{2^r}{(1-q)^n} \sum_{l=0}^{n} \frac{\binom{n}{l}(-1)^l q^{lx}}{(1 + q^{l+r-1} w)(1 + q^{l+r-2} w) \cdots (1 + q^r w)} \]

(17) \[ = \frac{2^r}{(1-q)^n} \sum_{l=0}^{n} \frac{\binom{n}{l}(-1)^l q^{lx}}{(q^{l} w : q^{-1}) r} \]

(17) \[ = \frac{2^r}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l}(-1)^l q^{lx} \sum_{m=0}^{\infty} \left( \binom{m + r - 1}{m} \right) q^{-m} \sum_{q=1}^{m} (-1)^m q^{hm} w^m \]

(17) \[ = 2^r \sum_{m=0}^{\infty} \left( \binom{m + r - 1}{m} \right) q^{-m} q^{hm} w^m [m + x]^n. \]
Then we have the following theorem.

**Theorem 9.** Let \( r \in \mathbb{N} \) and \( n \in \mathbb{Z}_+ \). Then we have

\[
E_{n,w,q}^{(r-1,r)}(x) = 2^r \sum_{m=0}^n \binom{n}{m} \left( \frac{q}{m+1} \right)^r q^m x^m - 2^r \sum_{m=0}^\infty \binom{m+r-1}{m} q^m (1-q)^m [m+x]^n.
\]

We also have the following corollary.

**Corollary 10.** Let \( F(t,w,x|q) = \sum_{n=0}^\infty E_{n,w,q}^{(r-1,r)}(x) t^n \). Then we have

\[
F^{(r-1,r)}(t,w,x|q) = 2^r \sum_{m=0}^\infty \binom{m+r-1}{m} q^m [m+x]^n w^m e^{t|m+x|}.
\]

From (18), we note that

\[
F^{(r-1,r)}(t,w,x|q) = 2^r \sum_{m=0}^\infty \binom{m+r-1}{m} q^m [m+x]^n w^m e^{t|m+x|}.
\]

By (19), we obtain the following distribution relation for \( F_{n,w,q}^{(r-1,r)}(x) \).

**Theorem 11.** For \( r \in \mathbb{N} \), \( n \in \mathbb{Z}_+ \), and \( d \in \mathbb{N} \) with \( d \equiv 1 \) (mod 2). Then we have

\[
E_{n,w,q}^{(r-1,r)}(x) = [d]_q^{r-1} \sum_{a_1,\ldots,a_r=0}^{d-1} \sum_{q=0}^{r-1} (-1)^{a_1+\cdots+a_r} w^{a_1+\cdots+a_r} E_{n,w,q}^{(r-1,r)}(\frac{a_1+\cdots+a_r+x}{d}).
\]

Now we define the \((h,q)\)-extension of \(w\)-Genocchi polynomials \( G_{n,w,q}^{(h,r)}(x) \) of order \( r \) as follows.

\[
G_{n,w,q}^{(h,r)}(x) = \sum_{n=0}^{\infty} G_{n,w,q}^{(h,r)}(x) \frac{t^n}{n!}.
\]
Then we have
\[ \sum_{n=0}^{\infty} G^{(h,r)}_{n,w,q}(x) \frac{t^n}{n!} \]
(21) \[ = t^r \int_{\mathbb{Z}_p^r} \sum_{i=0}^{\infty} (h-i+1)x_i w^{x_i+x_r} q^{[x+x_1+\cdots+x_r]} \mu_{-1}(x_1) \cdots \mu_{-1}(x_r) \]
\[ = \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p^r} \sum_{i=0}^{\infty} (h-i+1)x_i w^{x_i+x_r} [x+x_1+\cdots+x_r] q^n \mu_{-1}(x_1) \cdots \mu_{-1}(x_r) \]
\[ \cdot r! \binom{n+r}{r} \frac{t^{n+r}}{(n+r)!}. \]
From (13) and (21), we derive the following result.

**Theorem 12.** Let \( r \in \mathbb{N} \) and \( n \in \mathbb{Z}_+ \). Then we have
\[ G^{(h,r)}_{n+r,w,q}(x) = \frac{r!}{r!} \sum_{m=0}^{\infty} (m+r-1) m w^m [m+x] q^n \]
and \( G^{(h,r)}_{0,w,q}(x) = \cdots = G^{(h,r)}_{r-1,w,q}(x) = 0. \)

When \( h = r - 1 \) in Theorem 12, we have
\[ G^{(r-1,r)}_{n+r,w,q}(x) = \frac{r!}{r!} \sum_{m=0}^{\infty} (m+r-1) m w^m [m+x] q^n \]
\[ = 2^r \sum_{m=0}^{\infty} \binom{n}{m} (-1)^m w^m [m+x] q^n \]
\[ = 2^r \sum_{m=0}^{\infty} \binom{n}{m} (-1)^m w^m [m+x] q^n \]
\[ = E^{(r-1,r)}_{n,w,q}(x). \]

**Remark 2.** In the special case \( x = 0 \), \( G^{(h,r)}_{n,w,q}(0) = G^{(h,r)}_{n,w,q} \) are called the \((h,q)\)-extension of \(w\)-Genocchi numbers of order \(r\).

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