Self-consistency and Symmetry in 
d-dimensions

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Abstract

Bethe approximation is shown to violate Bravais lattices translational invariance. A new scheme is then presented which goes over the one-site Weiss model yet preserving initial lattice symmetry. A mapping to a one-dimensional finite closed chain in an external field is obtained. Lattice topology determines the chain size. Using recent results in percolation, lattice connectivity between chains is argued to be \( q^{(d-1)-2} \) where \( q \) is the coordination number and \( d \) is the space dimension. A new self-consistent mean-field equation of state is derived. Critical temperatures are thus calculated for a large variety of lattices and dimensions. Results are within a few percent of exact estimates. Moreover onset of phase transitions is found to occur in the range \( (d - 1)q > 2 \). For the Ising hypercube it yields the Golden number limit \( d > \frac{1 + \sqrt{5}}{2} \).
1 Introduction

The Weiss theory of ferromagnetism, also known as the mean-field theory, has been proposed almost a century ago [1]. Its major achievement is to provide a simple and universal model for collective phenomena and phase transitions. In most cases it yields phase diagrams which are qualitatively correct. However, quantitative results are very poor. In particular, all aspects of the critical behavior are grossly misrepresented.

Thirty years latter, Bethe extended the Weiss one-spin treatment to a cluster of \((q+1)\) fluctuating spins where \(q\) is the coordination number [2]. For the one-dimensional Ising model, this scheme reproduces the exact results. At higher dimensions, though some non-universal critical parameters are better represented, the critical behavior stays classical.

However it was recently pointed that within a Bethe scheme different site magnetizations are produced respectively in the \((q+1)\) cluster and its surrounding [3]. Therefore the Bethe scheme violates Bravais lattices translational invariance. This symmetry violation which has been overlooked for decades, is indeed consistent with a well known result. Bethe equations of state correspond to the exact solution of a model on the deep interior of a Cayley tree (which by definition, has no translational symmetry and exhibits singularities at its surface [4]).

In this paper we present a new self-consistent approach which embodies Bethe ideas, including fluctuations of a spin cluster, but preserves the lattice translational symmetry.

The outline of this paper is as follows. A generalized lattice version of the usual mean field treatment of the Ising ferromagnet is presented in Section 2. The underlying use of symmetry is thus explicited. Along the same lines, the Bethe scheme is revisited in Section 3. It shows the initial lattice symmetry is always broken. Section 4 is devoted to the presentation of a new self-consistent treatment which does preserves the symmetry. The lattice is divided into finite closed loops of fluctuating spins and mean-field spins respectively. The problem is thus mapped onto a one-dimensional closed fluctuating chain in an external field which allows an exact solution. The field is \(h \equiv \delta Jm\) where \(J\) is the coupling exchange, \(m\) is the mean-field average spin magnetisation and \(\delta\) is the chain connectivity. Using very recent results on percolation [5] it is argued that \(\delta = \frac{q(d-1)-2}{d}\) where \(d\) is the space dimension. Critical temperatures are calculated in Section 5 for a large variety of lattices.
at several dimensions. Discrepancies with available exact estimates are within few percent. In the last Section phase transitions are found to occur only in the range \((d-1)q > 2\) for \(d\)-dimensional Bravais lattices with coordination \(q\). In the hypercubic ferromagnetic Ising case it gives the Golden number limit \(d > \frac{1+\sqrt{5}}{2}\).

2 The mean field scheme

Mean-field theory is basically a one-site approach. However we present here a new generalized lattice scheme which enhances its interplay with symmetry. We illustrate it in the case of a ferromagnetic Ising system on a lattice with \(q\) nearest neighbors. The Hamiltonian is,

\[
H = -J \sum_{(i,j)} S_i S_j ,
\]

where \(S_i = \pm 1\) is an Ising variable, and interactions \(J\) are restricted to nearest neighbors (nn). A mean-field treatment of Eq. (1) processes indeed in two steps.

2.1 Breaking the lattice symmetry

Two interpenetrated lattices \(A\) and \(B\) are created within a staggered symmetry. Thermal fluctuations are then ignored on the B-sublattice. Spins \(S^B_j\) are substituted to their common thermal average \(< S^B_j >\) denoted \(m \equiv < S^B_j >\). At this stage spins \(S^A_i\) are no longer equivalent to their \(q\) nn \(S^B_j\). The translational lattice symmetry has been broken. Eq. (1) turns to,

\[
H_{MF} = -J \sum_{i \in A} S^A_i \sum_{j \in B, j=1}^q < S^B_j > ,
\]

which is a decoupled one-site Hamiltonian. The partition function can now be calculated integrating over \(S^A_i = \pm 1\). Magnetization at site \(i \in A\) is then,

\[
<S^A_i> = \tanh\{Km\} ,
\]

where \(K \equiv \beta J\), \(\beta \equiv \frac{1}{k_B T}\), \(k_B\) is the Boltzmann constant and \(T\) the temperature.
2.2 Restoring the symmetry

At this stage symmetry has been artificially broken through the introduction of two interpenetrated sublattices $A$ and $B$. The second step in the mean-field scheme is then to restore the initial lattice symmetry. It is done requiring the equality $< S^A_i > = m$ which in turn produces the well known self-consistent mean-field equation,

$$ m = \tanh\{K q m\} . $$

(4)

Although fluctuations have been included only at one sublattice, the system is again homogeneous with the same magnetization $m$ at every lattice site. The critical temperature $T_c$ is given by,

$$ K_{cW} = \frac{1}{q} . $$

(5)

Numerical estimates from Eq. (5) are rather poor. For instance, it is $K_{cW} = 0.25$ for the square lattice $(q = 4)$ instead of the exact value $0.4407$ [6]. Moreover $K_{cW} = 0.50$ at $d = 1$ $(q = 2)$ whereas it should be $T_c = 0$.

3 The Bethe approximation revisited

We now present an unusual viewpoint about the Bethe scheme to enlighten its symmetry content. First a finite cluster is considered. Second the Bethe approximation is implemented starting from a lattice. On this basis it then is shown translational invariance is always lost in the process.

3.1 A cluster in a surface field

We consider a finite cluster of one spin $S_i$ coupled ferromagnetically to $q$ nn spins $S_j$ with an exchange $J$. The associated partition function yields $< S_i > = < S_j > = 0$ as expected for a finite spin system.

An external field $h$ is then applied but only on the surface shell of the cluster, i.e., only on the $q$ spins $S_j$ and not on the central spin $S_i$. The Hamiltonian is,

$$ H_i = -J S_i \sum_{j=1}^{q} S_j - h \sum_{j=1}^{q} S_j . $$

(6)
A non-zero magnetization is induced in the cluster from $h \neq 0$ which is a symmetry breaking field. However since the field is not applied to the center spin, the cluster is no longer homogeneous, i.e., $< S_i > \neq < S_j >$ with both thermal averages being a function of $h$. At order one in $\alpha \equiv \beta h$ we have,

$$< S_i > = q \tanh(K) \alpha ,$$  
and

$$< S_j > = \{1 + (q - 1) \tanh^2(K)\} \alpha .$$

We can then ask which value of the surface field $h$, if any, makes the cluster homogeneous with a non-zero magnetization $< S_i > = < S_j > \neq 0$. It leads to the self-consistent equation in $\alpha$,

$$\alpha = (q - 1) \ln \frac{\cosh(\alpha + K)}{\cosh(\alpha - K)} .$$

Expanding the right-hand side of Eq. (9) around $\alpha = 0$ two solutions are obtained. First,

$$h = k_B T \{3(q - 1)(K - K_c)\}^{1/2} ,$$
for $K \geq K^B_c$ where $(\tanh K_c = \frac{1}{q-1})$,

$$K^B_c = \frac{1}{2} \ln \frac{q}{q - 2} .$$

Second we have $h = 0$ for $K \leq K^B_c$.

We thus have found that for each value of temperature $K > K^B_c$, there exists a well defined non-zero value of $h$ which makes the cluster homogeneous, i.e., $< S_i > = < S_j > \neq 0$. However for $K \leq K^B_c$ such a field does not exist, i.e., only $h = 0$ is compatible with an homogeneous cluster. In this case $< S_i > = < S_j > = 0$ due to the cluster finiteness. Otherwise $< S_i > \neq < S_j >$.

### 3.2 The Bethe implementation

The Bethe model starts from a lattice in which a $(q + 1)$ cluster similar to the one treated above, is singled out. However the difference from above comes from the definition of the field $h$. Above, $h$ is an external magnetic field. Here $h$ is argued to represent an effective field which accounts for
the coupling between the cluster surface and the remainder of the lattice. Calculations are nevertheless exactly identical to those done in Subsection (3.1) leading to Eqs. (7) and (8).

It is then usually stated that common magnetization in the cluster, i.e.,
\[ m \equiv \langle S_i \rangle = \langle S_j \rangle \]
is the magnetization in the lattice bulk. Eq. (7) gives,
\[ m = \frac{q}{q - 1} \alpha, \]
where \( K \) is taken at \( K_c \).

For a good review of the classical presentation of the Bethe model, see the textbook of Pathria [7]. Major improvement over Weiss model is twofold. Values of critical temperatures are slightly better. It is \( K_B^{c} = 0.35 \) instead of 0.25 for the square case. At \( d = 1 \) results are exact with in particular \( K_c^{B} = 0. \)

### 3.3 The Bethe violation of translational invariance

Nevertheless it has been proven recently [3] that a Bethe implementation on a lattice breaks translational invariance. To enlight this statement we present here a generalized lattice view similar to the one developped above for the mean-field model.

To implement a Bethe scheme on a lattice 3 distinct interpenetrated sub-lattices must be introduced artificially. We start from one fluctuating spin labelled \( i (S_i) \) to which we refer as an A-spin. The set of its nn spins are labelled by index \( j (S_j^i) \) and called B-spins. Last, nn spins of \( j \) spins, except the \( i \) spin, are labelled by the index \( k (S_k^{i,j}) \) and called C-spins. The A-spin plus these B and C shells constitute a supercell for the lattice. This supercell is then used to pave the space and reproduce the whole lattice. Eq. (1) writes,
\[ H = -J \sum_{i \in A} \sum_{j \in B} S_i^j \sum_{j \in B, j=1}^{q} S_j^i + \sum_{j \in B, j=1}^{q} S_j^i \left( \sum_{k \in C} S_k^{i,j} \right). \]
The prime over last sum means the summation over \( k \) \( j \)-nn spins does not includes the \( i \)-spin. The scheme is easily visualized in the square case. For lattices with closed loops nn like triangular, Eq. (13) has to be modified to account for interactions between nn \( S_j^i \).
3.3.1 Breaking the lattice symmetry

It is worth noticing Eq. (13) is identical to Eq. (1). At this stage no approximation was done. Only a formal relabelling has been used.

Now we perform a mean-field transformation of the Hamiltonian. More remote C-sublattice spins $S_{k,j}^i$ are substituted to their thermal average $m_C \equiv < S_{k,j}^i >$. In so doing the lattice symmetry has been artificially broken. However it allows a calculation of the partition function. Eq. (13) reduces to $H = \sum_{i \in A} H_i$ with,

$$H_i = -JS_i \sum_{j=1}^q S_{j}^i - h \sum_{j=1}^q S_{j}^i,$$  (14)

where $h \equiv J \sum_{k=1}^{q-1} < S_{k,j}^i >$ and $k$ runs for all (q-1) nn of respectively each $j$ spin, spin $i$ being excluded. Hence

$$h = J(q-1)m_C,$$  (15)

for the effective field. From Eq. (14) we calculate thermal averages $m_A \equiv < S_i >$ and $m_B \equiv < S_j^i >$ as a function of $m_C$. We end up with three unknown magnetizations which are respectively $m_A$, $m_B$ and $m_C$.

3.3.2 The symmetry breaking is irreversible

At this stage, as in the mean-field scheme, the artificially broken symmetry has to be restored to recover the initial physical problem we started from. It is of importance to stress even a drastic approximation should not change the symmetry of the problem. It means here the 3 different magnetizations have to be identical.

First the symmetry within the fluctuating cluster A-B can be restored putting $m_A = m_B$. This equality yields a self-consistent equation in $m_C$ identical to Eq. (9),

$$m_C = \frac{1}{K} \ln \frac{\cosh(K[(q-1)m_C + 1])}{\cosh(K[(q-1)m_C - 1])},$$  (16)

where $\alpha = K(q-1)m_C$ has been used.

Next step is to extend the cluster homogeneity to the C-sublattice. Denoting $m \equiv < S_i > = < S_{j}^i >$ we have to fulfill the equality $m_C = m$. However
Eq. (7) shows that \( m = q \tanh(KK(q - 1)m_C \) which in turn obviously demonstrates \( m \neq m_C \), as long as Eq. (16) holds.

Alternatively we could have achieved \( m_A = m_C \). Then it is at the expense of getting another self-consistent equation in \( m_C \),

\[
m_C = q(q - 1)K \tanh(Km_C) ,
\]

which is nevertheless not compatible with Eq. (16). In this case \( m_A \neq m_B \).

In both cases, translational invariance is lost. There exist one super-sublattice A and B (A and C) embedded in the C-sublattice (B-sublattice), both having different magnetizations. We thus have clearly demonstrated that the Bethe scheme produces always an irreversible lattice symmetry breaking.

### 3.3.3 The Bethe scheme is forbidden by symmetry

Overlooked for several decades, this symmetry breaking could indeed be found readily from the Bethe scheme cluster character. Cluster center has all its nn spins which are fluctuating while surface cluster spins have \((q - 1)\) mean-field nn spins and one fluctuating spin. Simultaneously mean-field spins have all their nn which are fluctuating spins, making their environment identical to the cluster center. On this basis the Bethe requirement \( < S_i > = < S_j > \) is not compatible with the equality \( m_C = < S_j > \) which should also hold to ensure translational invariance. The Bethe topology is therefore forbidden by symmetry [3]. It is not the case for one-site Weiss theory.

Last but not least, it is worth noticing it is this very symmetry problem which makes the Bethe model exact on the Caley tree lattice. This lattice does not exhibit translational invariance by construction. However the Cayley tree (also called Bethe lattice in the tree’s deep interior) is pathological with singularities at its surface [4]. Spins on the surface have a non-negligible contribution to the partition function in the thermodynamic limit. This is not the case for a regular lattice.

### 4 A preserving symmetry scheme

Once the Bethe model is found forbidden by symmetry, the underlying question is to find out if a self-consistent treatment which includes more than one
site is indeed possible. We show below that such a model does exist.

4.1 A 2-spins model

Ogushi [8] considered a two nn fluctuating spin cluster. For one pair \((1, 2)\) of fluctuating spins the equality \(<S_1>=<S_2>\) holds by construction. It allows then to fulfill \(<S_1>=m\) as required by lattice symmetry. Critical temperatures are given by,

\[
(q - 1)K^O\{1 + \tanh(K^O)\} = 1. \tag{18}
\]

Improvement over the Weiss model is only marginal with, for instance, \(K^O_c = 0.26\) for the square lattice. Moreover a transition is still wrongly predicted at \(d = 1\).

4.2 A \(N\)-spin model

To go over a two-spin cluster while preserving initial lattice symmetry, requires to start with a topology of only two spin species A and B. Therefore fluctuating spins \(S_i\) make one species A and mean-field spins characterize the other species B. A two-sublattice coverage has to be built up under the constraint of identical A-spins and B-spins with respect to their respective topology (nn-surroundings).

4.2.1 Breaking the symmetry

Above Bethe approximation analysis emphasisizes the role of the cluster center in the breaking of symmetry. Therefore to avoid such a symmetry breaking no fluctuating center should exist within the fluctuating cluster. Such a constraint can be achieved via compact closed linear loops within the lattice topology. For instance 4-spin squares and 3-spin triangles for respectively square and triangular lattices.

Each one of these plaquettes is then set respectively as A-species and B-species with a staggered-like covering pattern. A-plaquettes (B-plaquettes) have thus all their nn plaquettes as B-plaquettes (A-plaquettes). For a given plaquette, each spin has two nn spins of the same species within the plaquette itself and \((q - 2)\) nn spins of the other species belonging to nn plaquettes.
At this stage the problem has been turned into a series of fluctuating one-dimensional finite closed chains in an external field $h$. The number of spins $N$ is determined by the lattice topology. It is $N = 4$, $N = 3$, $N = 3$ and $N = 6$ for respectively square, triangular, Kagomé and Honeycomb lattices. Interactions with nn mean-field spin plaquettes produce the field $h$. We have $h = \delta J m$ where $\delta$ accounts for connectivity to B-sublattices.

Above transformation allows then an exact solution since the problem of a nn Ising ferromagnet on a closed chain in an external field can be solved exactly [7]. In particular, the chain site magnetization is,

$$< S_i > = \beta \exp 2K \left\{ \frac{1 - \tanh(K)^N}{1 + \tanh(K)^N} \right\} h ,$$

at order one in $h$. Here $i \in A$-plaquettes.

### 4.2.2 Restoring the lattice symmetry

Then putting $< S_i > = m$ restores initial lattice symmetry. Here symmetry restoring is possible since only two sublattices were involved. It was not the case for the three sublattice Bethe scheme. The self-consistent equation of state is,

$$m = \delta K \exp 2K \left\{ \frac{1 - \tanh(K)^N}{1 + \tanh(K)^N} \right\} m + ... ,$$

at order one in $m$ using $h = \delta J m$. To solve Eq. (20) we need to determine the value of $\delta$.

### 4.2.3 Rescaling the connectivity

At this stage it is worth evoking a recent work on percolation thresholds [5]. It was found that relevant connectivity variables for site and bond dilution are respectively $(d - 1)(q - 1)$ and $(d - 1)/d(q - 1)$. In other words, starting from a given site, it shows that for site percolation, the number of possible directions $(q - 1)$ has to be multiplied by $(d - 1)$. For bond percolation this effective number of sites has to be divided by dimension $d$. Using these variables, a universal power law form was found to yield all percolation thresholds for all lattices at all dimensions [5].

This percolation finding suggests to consider here a rescaled connectivity between closed loops instead of $\delta = q - 2$. Using above counting, we first start
with \( q \) instead of \((q - 1)\) since now dealing with pair exchange interactions and not percolation. Then we renormalize \( q \) by \((d - 1)\) giving \( q(d - 1) \). However the 2 neighboring sites which are treated exactly within the closed loop have to be subtracted from the effective number of sites which gives \( q(d - 1) - 2 \). Moreover, interactions being related to bonds, we divide this number by \( d \) as for bond percolation which results in,

\[
\delta = \frac{q(d - 1) - 2}{d}.
\]  

(21)

\section{5 Results}

We can now check the validity of our simple symmetry preserving model with respect to critical temperatures. From Eq. (21) we get,

\[
\delta K_e^G \exp 2K_e^G \left\{ \frac{1 - \tanh(K_e^G)^N}{1 + \tanh(K_e^G)^N} \right\} = 1.
\]  

(22)

The trivial connectivity counting \( \delta = q - 2 \) already improves Weiss model. For instance \( K_e^G = 0.29 \) in the square case and \( T_c = 0 \) at \( d = 1 \). We now proceed using Eq. (21) for connectivity.

\subsection{5.1 \( d = 2 \)}

For the square case \((q = 4, N = 4)\), \( \delta = 1 \) which gives \( K_e^G = 0.4399 \). Exact result is \( K_e^c = 0.4407 \). In the case of triangular lattice \((q = 6, N = 3)\), \( K_e^G = 0.2919 \) with \( \delta = 2 \) while the exact estimate is \( K_e^c = 0.2746 \). For Kagramé \((q = 4, N = 3)\) \( \delta = 1 \) yielding \( K_e^G = 0.4649 \) for an exact estimate of \( K_e^c = 0.4666 \). And \( K_e^G = 0.6160 \) for the honeycomb lattice \((q = 3, N = 6)\) where \( \delta = \frac{1}{2} \) for an exact estimate of \( K_e^c = 0.6585 \) (see Table I).

\subsection{5.2 \( d = 3 \)}

Going to \( d = 3 \) imposes to restrict the plaquette size to \( N = 4 \) since a one-dimensional loop cannot embody a three-dimensional topology. However there exits a \( d \)-dependence through Eq. (21). We get \( \delta = \frac{10}{3}, \delta = \frac{14}{3} \) and \( \delta = \frac{22}{3} \) for respectively cubic, fcc and bcc lattices. Corresponding critical temperatures are given by \( K_e^G = 0.2012, 0.1568, 0.1096 \) respectively for exact estimates of 0.2217, 0.1575, and 0.1021 (see Table I).
5.3 $d > 3$

Critical temperature estimates [9, 10] are available for the hypercube at $d = 5, 6, 7$. These are $K^c_e = 0.1139, 0.0923, 0.0777$ respectively. In comparison, Eq. (22) gives $0.1380, 0.0869, 0.0737$. We can also predict values for other lattices like for instance $fcc$ ones (see Table I).

5.4 $d \to \infty$

To get the $d \to \infty$ asymptotic limit of our model we take both $q \to \infty$ and $J \to 0$ under the constraint $qJ = cst$. From Eq. (21) connectivity limit is

$$\delta \rightarrow q$$

at leading order. Indeed $q$ diverges always quicker than $\frac{q}{d}$ even for $fcc$-lattices where $q = 2d(d - 1)$. In turn Eq. (22) becomes,

$$K^G_e = \frac{1}{q} \quad (24)$$

which is exactly Eq. (5). Mean-field result is thus recovered as expected in the $d \to \infty$ limit.

5.5 $N$-dependence

To evaluate the sensibility on the loop size, it is fruitful to expand Eq. (22) in powers of $K$. It gives

$$K^G_e(1 + 2K^G_e + \ldots + \frac{(2K^G_e)^N}{N!})(1 - (K^G_e)^N + \ldots)(1 - (K^G_e)^N + \ldots) = \frac{1}{\delta} \quad (25)$$

Since $N \geq 3$, a simple analytic expression is obtained only at order one,

$$K^G_e = \frac{1}{\delta} \quad (26)$$

At two dimensions Eq. (26) gives $K^G_e = 1, \frac{1}{2}, 1, 2$ for respectively the square, triangular, Kagomé and for honeycomb lattices. These results are rather poor and shows the importance of the finite value of $N$ which embodies part of the lattice topology.
5.6 Weiss model with δ connectivity

It is interesting to distinguish in our model the linear closed loop contribution from the use of the rescaled connectivity δ. It is readily done taking back Weiss Equation with $\frac{q(d-1)}{d}$ instead of $q$. Here we don’t have to substract the two sites of the linear loop as in Eq. (21). We thus get,

$$K^g_c = \frac{d}{q(d-1)}$$

instead of Eq. (5). Critical temperatures at $d = 2$ are, $K^g_c = 0.5000$ for the square ($q = 4$, $K^e_c = 0.4407$), $K^g_c = 0.3333$ for triangular lattice ($q = 6$, $K^e_c = 0.2746$), $K^g_c = 0.5000$ for Kagomé ($q = 4$, $K^e_c = 0.4666$) and $K^g_c = 0.6667$ for the honeycomb lattice ($q = 3$, $K^e_c = 0.6585$) (see Table I).

At $d = 3$ we get, $K^g_c = 0.2500$, 0.1875, 0.1250 respectively for exact estimates of 0.2217, 0.1575, and 0.1021 (see Table I).

6 Conclusion

We have presented a very simple self-consistent model which both preserves initial lattice symmetry and goes beyond the one-site Weiss approach. It yields values for critical temperatures within a few percent of exact results. Besides a rescaled lattice connectivity, the finite length of the loops is also taken into account. This new scheme represents a substantial improvement over existing mean-field cluster approximations. However associated critical exponents stay identical to those of the Weiss model.

Last but not least, we can determine from our model a lower critical dimension for phase transitions. It comes from the condition $h = 0$ for which we have a one-dimensional finite system. Such a system has no long range order at $T \neq 0$. Phase transitions are thus obtained only in the range $h \neq 0$ which gives,

$$q(d-1) > 2.$$  \hspace{1cm} (28)

For Ising hypercubes ($q = 2d$) it yields the Golden number limit,

$$d > \frac{1 + \sqrt{5}}{2},$$ \hspace{1cm} (29)

which both excludes the $d = 1$ case and contains $d = 2$ as it should be.
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| Dimension | Lattice     | $q$ | $\delta$ | $K_c^G$ | $K_c^{G_{c}}$ | $K_c^{g}$ |
|-----------|-------------|-----|----------|---------|--------------|---------|
| $d = 2$   | Square      | 4   | 1        | 0.4407  | 0.4399       | 0.5000  |
|           | Honeycomb   | 3   | $\frac{1}{2}$ | 0.6585  | 0.6160       | 0.6667  |
|           | Triangular  | 6   | 2        | 0.2746  | 0.2837       | 0.3333  |
| $d = 2$   | Kagome$^*$  | 4   | 1        | 0.4666  | 0.4649       | 0.5000  |
| $d = 3$   | Diamond     | 4   | 2        | 0.3698  | 0.2857       | 0.3750  |
|           | sc          | 6   | $\frac{10}{3}$ | 0.2216  | 0.2012       | 0.2500  |
|           | bcc         | 8   | $\frac{14}{3}$ | 0.1574  | 0.1568       | 0.1875  |
|           | fcc         | 12  | $\frac{22}{3}$ | 0.1021  | 0.1096       | 0.1250  |
| $d = 4$   | sc          | 8   | $\frac{22}{4}$ | 0.1497  | 0.1380       | 0.1667  |
|           | fcc         | 24  | $\frac{23}{2}$ | 0.0749  | 0.0555       |         |
| $d = 5$   | sc          | 10  | $\frac{25}{5}$ | 0.1139  | 0.1064       | 0.1250  |
|           | fcc         | 40  | $\frac{158}{5}$ | 0.0298  | 0.0312       |         |
| $d = 6$   | sc          | 12  | $\frac{29}{3}$ | 0.0923  | 0.0869       | 0.1000  |
| $d = 7$   | sc          | 14  | $\frac{25}{7}$ | 0.0777  | 0.0737       | 0.0833  |

Table 1: $K_c^G$ and $K_c^{g}$ from this work compared to “exact estimates” $K_c^e$ taken from [9, 10].