The Cardinal Complexity of Comparison-based Online Algorithms

Nick Gravin∗ Enze Sun† Zhihao Gavin Tang∗

Abstract
We consider ordinal online problems, i.e., those tasks that only depend on the pairwise comparisons between elements in the input. E.g., the secretary problem and the game of googol. The natural approach to these tasks is to use ordinal online algorithms that at each step only consider relative ranking among the arrived elements, without looking at the numerical values of the input. We formally study the question of how cardinal algorithms (that can use numerical values of the input) can improve upon ordinal algorithms.

We give a universal construction of the input distribution for any ordinal online problem, such that the advantage of the cardinal algorithms over the ordinal algorithms is at most $1 + \varepsilon$ for arbitrary small $\varepsilon > 0$. However, the value range of the input elements in this construction is huge: $O\left(\frac{n^{n-1}}{\varepsilon^n}\right)$ for an input sequence of length $n$. Surprisingly, we also identify a natural family of hardcore problems that achieve a matching advantage of $1 + \Omega\left(\frac{1}{\log(c)\log N}\right)$, where $\log^{(c)} N = \log \log \ldots \log N$ with $c$ iterative logs and $c$ is an arbitrary constant $c \leq n - 2$.

We also consider a simpler variant of the hardcore problem, which we call maximum guessing and is closely related to the game of googol. We provide a much more efficient construction with cardinal complexity $O\left(\frac{1}{\varepsilon}\right)$ for this easier task. Finally, we study the dependency on $n$ of the hardcore problem. We provide an efficient construction of size $O(n)$, if we allow cardinal algorithms to have constant factor advantage against ordinal algorithms.

1 Introduction

Sorting is one of the most well known problems in computer science with an endless list of existing algorithms such as bubble-sort, heapsort, quicksort, etc. The vast majority of these algorithms are ordinal algorithms, i.e., they only do pair-wise comparisons between the input elements. It is also well known that any such algorithm has a fundamental limitation: on average, it must perform at least $\Omega(n \log n)$ comparisons to produce the correct output. On the other hand, there are a few algorithms such as pigeonhole, counting, and radix sorts that utilize the cardinal information about the input. I.e., these algorithms are not comparison based and thus are not limited by the $\Omega(n \log n)$ barrier. Some of them have faster than $O(n \log n)$ running time for the practical task of sorting integers in the range from 0 to $K$, see, e.g., $O(n \sqrt{\log \log K})$ randomized algorithm of [12], or deterministic $O(n \log \log K)$ algorithm of [11] in the word RAM model of computations.

The story of the sorting algorithms illustrates how cardinal information may be advantageous in performing ordinal tasks, i.e., problems whose outputs only depend on the pair-wise comparisons between the elements of the input. In this paper, we study what advantage one can get by using the cardinal information in ordinal tasks, but instead of computational problems (which can be tricky to formalize due to the differences between many models of computations) we consider online problems.

∗ITCS, Shanghai University of Finance and Economics. Email: {nikolai,tang,zhihao}@mail.shufe.edu.cn
†Shanghai Jiaotong University. Email: sun.en.ze@sjtu.edu.cn
with the focus on the information theoretic guarantees. One such online problem is the celebrated secretary problem\footnote{Also known under the names of marriage problem, the sultan’s dowry problem, the fussy suitor problem, the best choice problem, and the googol game.} from the optimal stopping theory. According to \cite{Bertsekas1985}, the secretary problem was first published by Martin Gardner \cite{Gardner1960} in the form of the game of googol:

**Game of Googol.** Ask someone to take as many slips of paper as he pleases, and on each slip write a different positive number. The numbers may range from small fractions of 1 to a number the size of a googol (10\(^{100}\)) or even larger. These slips are turned face down and shuffled over the top of a table. One at a time you turn the slips face up. The aim is to stop turning when you come to the number that you guess to be the largest of the series. You cannot go back and pick a previously turned slip. If you turn over all the slips, then of course you must pick the last one turned.

A similar version of this game is the secretary problem in which one observes a sequence of candidates arriving in a random order and wants to hire the best secretary. The difference with the game of googol is that the online algorithm does not see the numerical values of the candidates, and can only do pairwise comparisons between them. I.e., one can only use ordinal online algorithm for the secretary problem, while in the game of googol, the algorithm can use cardinal (numerical) values. It is well known that the optimal competitive ratio achieved for the ordinal variant is \(1/\varepsilon\), see, e.g., \cite{Bertsekas1985}. The original paper \cite{Gardner1960} also solved the ordinal variant and made an appealing but informal argument that for the large enough numbers (say between 1 and 10\(^{100}\)) the cardinal and ordinal problems are the same. It was later observed that the difference between winning probabilities in the two variants is at most \(\varepsilon\) for arbitrary small \(\varepsilon > 0\) when the game is played over sufficiently large integers, and it completely vanishes when the values can be real numbers \cite{Cheraghchi2012}. We note however that using arbitrary real numbers, or numbers as large as the 10\(^{100}\) is out of practical range in almost any imaginable scenario, e.g., if anyone was to assign numerical scores to the candidates she would most likely use integer scores less than 100 and possibly even smaller than 10. Thus it is natural to ask what range of numerical values is necessary or sufficient to have the ordinal and cardinal problems be within an \(\varepsilon\) distance from each other.

For the special case of \(n = 2\) the game of googol has advantage of \(\varepsilon = \Theta(1/N)\) when the numbers are integers in \([N]\) (see, e.g., the numbers game in \cite{Knuth1997}, section 19.3.3, p. 822). We would also like to consider more general ordinal problems than secretary problem. As we will see later, the following ordinal task which we call a die guessing game turns out to be quite interesting.

**Die Guessing Game.** Consider a fair die with \(n\) faces, e.g., the standard die with \(n = 6\). Imagine two players playing the following game. The first player secretly writes \(n\) distinct integers from \(\{1, 2, \ldots, N\}\) on each face and then roll the die. The second player sees all faces but one, which is at the bottom. The second player wins if he guesses correctly the rank of the hidden number compared to all visible ones. Without seeing the numbers, by guessing any rank between 1 and \(n\), the second player wins with probability \(\frac{1}{n}\). It seems intuitive that one should not get much advantage when seeing the numbers on the faces, and indeed as we will formally show later for arbitrary ordinal problems this intuition is correct.

We would like to analyze how much better as a function of \(N\) the second player can do by seeing the actual numbers on the \(n - 1\) faces. More generally, the same question applies to any ordinal task and that is the question we address in our paper.
1.1 Our Contributions

We only consider online problems with random arrival orders in this paper, which in certain sense can be viewed as an average-case analysis of online algorithms. The reason for this is that the worst-case adversary is just too powerful. E.g., observe that the range of $N = 2^n$ values is already sufficient to make cardinal algorithms no better than ordinal algorithms for any ordinal worst-case online problem. In many cases the bound of $N = 2^n$ can be brought down to a small polynomial in $n$ (like $n^2$) at no or very small advantage to the cardinal algorithms over the ordinal ones. The question of cardinal vs. ordinal online algorithms however is much more interesting when there is certain randomness to the input, specifically the arrival order, that cannot be controlled by the adversary.

Universal Cardinal Complexity. First, we study the question if it is always possible to mitigate the advantage of the cardinal algorithms over ordinal algorithms for any given ordinal task. We answer positively to this question provided that there are sufficiently many different values for each number in the input, i.e., Martin Gardner’s intuition is essentially correct for arbitrary ordinal tasks, if we allow the integers to be sufficiently large. Further, we initiate the study of cardinal complexity of online ordinal problems, i.e., the minimum number of different values required so that the advantage of cardinal algorithms over ordinal algorithms is no more than $1 + \varepsilon$. See Section 2 for the formal definitions.

Our main theorem is a universal upper bound for the cardinal complexity. Namely, for any online ordinal problem, any input size $n$, distribution of arrival orders $\pi$, and any $\varepsilon > 0$ we give an explicit construction of a distribution $\mathcal{F}$ of numerical inputs $S$ such that any cardinal online algorithm may not have its expected reward higher than the expected reward of the best ordinal algorithm times $(1 + \varepsilon)$ (see Theorem 3.1 for the formal statement). Specifically, the cardinal complexity is of order $O\left(\frac{n^3 n!}{\varepsilon}\right) \uparrow\uparrow (n - 1)$ for a given $\varepsilon$, where $\uparrow\uparrow$ is the Knuth’s up-arrow notation for the iterated exponentiation, e.g., $4 \uparrow\uparrow 3 = 4^4 = 4^{256}$.

Tight Lower Bound: Die Guessing. Second, our bound $O\left(\frac{n^3 n!}{\varepsilon}\right) \uparrow\uparrow (n - 1)$ may be a really huge number, which makes it not practical in almost any imaginable application. E.g., consider the game of googol with $N = 2^{2^{1/2}}$ for $n = 4$, $\varepsilon = \frac{1}{100}$, then just writing such a number would require $2^{100}$ bits. Perhaps our most surprising result in the paper is that the bound is almost tight regarding the dependency on $\varepsilon$, i.e., the tower of $(n - 1)$ exponents is necessary. Specifically, we construct a cardinal algorithm with $\frac{1}{n} \left(1 + \Omega\left(\frac{1}{\log^{(c)} N}\right)\right)$ probability of guessing correctly for the die guessing game described in the introduction, where $\log^{(c)}(x) = \underbrace{\log \log \ldots \log}_{c \text{ times}} x$ and $c \leq n - 2$ is any constant.

This suggests that a simple looking die guessing game is the hardcore task among the cardinal vs. ordinal problems. Another important implication of this result is that, in general, using the Martin Gardner’s intuition might be infeasible in practice. Indeed, the cardinal values with only a doubly exponential dependency on $1/\varepsilon$ may easily get to the order of $2^{2^{100}}$, which are too large to be compared with each other or even stored on a computer. On the positive side, our result suggests that in some cases one can use cardinal information to improve upon performance of the ordinal algorithm if the numerical values are not very big.

\footnote{We simply use binary encoding for $n$ numbers to implement any ordinal ranking: let the first number be $s_1 = 10\ldots0$; the second number be $s_2 = 010\ldots0$ if we want $s_1 > s_2$, or $s_2 = 110\ldots0$ if we want $s_1 < s_2$; in general, we pick $s_i = x_1 \ldots x_{i-1}10\ldots0$ where $x_1 \ldots x_{i-1}0\ldots0 = s_j$ is the binary code of the largest previous number $s_j < s_i$.}
Easier Tasks: Maximum Guessing. Our previous results do not say anything about the specific task of the googol game, i.e., the task of identifying the maximum in a random sequence. We believe that the task in the googol game is easier than the die guessing game and thus it likely admits a more efficient cardinal-to-ordinal reduction than is necessary for the die guessing game. We obtain much more efficient construction of cardinal complexity $N = O(\frac{1}{\epsilon}n - 1)$ for a natural variant of the die guessing game related to the game of googol. This game, which we call maximum guessing, has the same setup as the die guessing game, but with a different objective to guess correctly whether the hidden face is the maximum among $n$ numbers written on the faces, or if it is not. Our construction of the distribution uses stationary distribution for a certain Markov chain over all permutations $\pi \in \text{Sym}(n)$ and also shares some ideas with the universal construction.

We conjecture that our construction would also work for the game of googol, although an intuitive and straightforward reduction argument fails (we discuss the implications and potential difficulties in using the construction for maximum guessing to the game of googol). We believe that the maximum guessing problem is an interesting setting in its own right and understanding its cardinal complexity makes a solid step towards characterizing the cardinal complexity of the game of googol.

Hardcore setting: Dependency on $n$. The previous results mostly refer to the regime of a constant $n$ and $\epsilon \to 0$. However, in many problems a typical scenarios are the ones where $n$ is large and maybe it is fine to have a constant factor gap between cardinal and ordinal algorithms. To this end, we also study the regime when $\epsilon$ is a fixed constant and $n \to \infty$. We provide preliminary results for the $n$-faces die guessing game. We give an efficient construction with $N = O(n)$ in which the best cardinal algorithm has only a $2 + \delta$ advantage over the ordinal algorithms when $\epsilon = 1 + \delta$.

1.2 Related Work

Our paper is mostly motivated by the extensive study of secretary problem and its variants. Below we survey the most relevant ones to our paper, that study ordinal algorithms.

Correa et al. [4] studied the two-sided game of googol with numbers written on both sides of each slip. The goal is to select the maximum hidden number with the largest probability. They designed an algorithm that succeeds with probability 0.453. Buchbinder, Jain, and Singh [1] proposed the $(J, K)$-secretary problem, in which an algorithm is allowed to select at most $J$ items and the goal is to maximize the number of items selected among the best $K$ items. They proposed an optimal LP-based algorithm that only does pairwise comparisons between the items. Observe that the reward function in this setting is ordinal. Soto, Turkieltaub, and Verdugo [15] studied the ordinal matroid secretary problem and introduced a stronger notion of ordinal-competitiveness. However, their competitive ratio cannot be modeled as an ordinal reward function in our setting.

Another line of work studies secretary problems by restricting the algorithms to only use ordinal information, while the objective function is cardinal. For such settings, it is widely believed that a gap between the best cardinal algorithm and the best ordinal algorithm exists. A concrete example is the $(2, 2)$-secretary with a cardinal objective (i.e. the sum of the weights of the selected items), Chan, Chen, and Jiang [2] proved that the best ordinal algorithm is 0.488 while a cardinal algorithm can be 0.492-competitive. Hoefer and Kodric [13] studied combinatorial secretary problems with ordinal information, and designed efficient algorithms for a large family of constraints. They also showed that the reduction by Feldman and Zenklusen [7] from submodular to linear matroid secretary problem can be implemented in the ordinal model.

We are aware of two related prior works that implicitly analyze the advantage of ordinal algorithms over cardinal algorithms to obtain results in their cardinal models. First, Correa et al. [3]
consider the setting of unknown i.i.d. prophet inequality, proving among other results that no online algorithm has competitive ratio better than $\frac{1}{e}$ Note that the $\frac{1}{e}$ ratio can be achieved by the standard ordinal algorithm for the classic secretary problem despite the fact that the objective is cardinal. Second, Erza et al. [6] study the secretary matching setting. They introduce an ordinal version of the problem to establish a tight lower bound of $\frac{3}{4}$. Their ordinal version is a multi-choice secretary setting with the objective to select the maximum element.

Both papers among other things (i) analyse settings with the goal of selecting the maximum element; (ii) apply a nontrivial Ramsey theory argument to reduce what we call “cardinal” algorithms (i.e., algorithms that observe numerical values) to what we call “ordinal” algorithms (i.e., algorithms that only use relative ranking of the elements). In fact, Erza et al. [6] explicitly do a two step reduction from their original setting with cardinal objective: first to the “Hybrid setting” which is exactly captured by our notion of an ordinal objective; then to the “Ordinal setting” where not only the objective but also the algorithm are ordinal. The latter step of their reduction is much more difficult than the former one and was inspired by the Ramsey theory argument from Correa et al. [3]. Our universal construction can be used as an alternative proof for the reduction from the hybrid to the ordinal setting. Interestingly, given the connection between i.i.d. prophet inequality and the secretary settings, the approach of Correa et al. [3] can be almost verbatim applied to the game of googol and the size of their construction is similar to our universal bound in Section 3.

2 Preliminaries

2.1 Cardinal Complexity

We formalize the family of online ordinal tasks and state the question we study in this paper.

Online Ordinal Tasks. Adversary chooses an ordered set $S = (s_1, s_2, \ldots, s_n) \subseteq [N]$ of $n$ distinct numbers, where $s_1 < s_2 < \ldots < s_n$. We use $S_i$ to denote the ordered numbers of $(s_i)_{i=1}^I$ for arbitrary $I \subseteq [n]$ and $S_i$ to denote the set by deleting $s_i$ from $S$, i.e. $S_i \overset{\text{def}}{=} (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$. Nature (not adversary) samples a random order $\pi \in \text{Sym}(n)$ from a priori known distribution $D$. We use $\pi(k)$ to denote the $k$-th element in the order $\pi$, and use $\pi[k]$ to denote the first $k$ elements in $\pi$. At each step $1 \leq k \leq n$, the number $s_{\pi(k)}$ is revealed to the algorithm, and the algorithm selects an action $a_k$ from the action set $A_k$. Thus at step $k$, the algorithm observes the previous actions $a_{[k-1]} = (a_1, \ldots, a_{k-1})$, and the first $k$ numbers $S_{\pi[k]} = (s_{\pi(1)}, \ldots, s_{\pi(k)})$. It makes a decision $a_k = \text{ALG}_k(a_{[k-1]}, S_{\pi[k]})$. The final output of the algorithm after step $n$ is $a = (a_1, \ldots, a_n)$. There could be some constraints on the feasible action space of the algorithm: $A \subseteq A_1 \times A_2 \times \ldots \times A_n$.

We think of the algorithm as $\text{ALG} : [N]^n \times \text{Sym}(n) \rightarrow A$, i.e., $\text{ALG}(S, \pi)$.

We study ordinal reward functions $R : A \times \text{Sym}(n) \rightarrow \mathbb{R}_+$ and assume that if $a \notin A$, then $R(a, \pi) = 0$. We refer to such a setting as ordinal tasks since the reward function does not depend on the actual values of the elements of $S$. Then the performance of each algorithm is $E_{\pi \sim D}[R(a(S, \pi), \pi)]$.

We illustrate the above concepts by examples.

\footnote{They only give existential result and understandably did not explicitly calculate its size.}
**Game of Googol.** The ordered set $S$ corresponds to the numbers written on the cards, that are distinct integers from 1 to $N = 10^{100}$. The distribution $D$ is a uniform random order. The $n$ numbers are shuffled according to $\pi \sim D$. At each step $k$, the algorithm observes the $k$-th number and gets two options, i.e., $A_k = \{\text{accept}, \text{reject}\}$. The feasible action space allows at most one accept throughout the $n$ steps. Then the reward function $R$ equals 1 when the algorithm accepts the largest number and equals 0 otherwise.

**Die Guessing.** The ordered set $S$ corresponds to the numbers written on each side of the die, that are $n$ distinct integers between 1 and $N$. The distribution $D$ is a uniform random order. The $n$ numbers are shuffled according to $\pi \sim D$. The action space $A_k$ is empty for $k \leq n - 2$ and $k = n$. At step $n - 1$, the algorithm observes the first $n - 1$ numbers and makes a guess from $A_{n-1} = \{1, 2, \ldots, n\}$. Then the reward function $R$ equals 1 when the algorithm guesses correctly $\pi(n)$ at step $n - 1$ and equals 0 otherwise.

**Ordinal (Comparison-based) Algorithms.** We study a subfamily of the online algorithms that only use pairwise comparisons to determine which actions to take at every step. Formally, let $\sigma_k = \sigma(\pi[k])$ be the ranking of the first $k$ elements of $\pi$ (i.e., $\sigma_k \in \text{Sym}(k)$ is a permutation of $k$ elements). An ordinal algorithm takes action $a_k = \text{ALG}(a[k-1], \sigma_k)$ at step $k$, where $\sigma_k = \sigma(\pi[k])$ only depends on the ordinal comparisons of the elements in $S_{\pi[k]}$. I.e., the algorithm $\text{ALG} : \text{Sym}(n) \to A$ only depends on the relative order of the arrived elements. We use $\text{Ord}$ to denote the family of all ordinal algorithms and use $\text{Card}$ to denote the family of all algorithms.

**Cardinal Complexity.** Intuitively, for an arbitrary $n$-round online ordinal task with reward function $R : A \times \text{Sym}(n) \to \mathbb{R}_+$, distribution $D$ of orders $\pi$, and any $\varepsilon > 0$, there exists a sufficiently large integer $N \in \mathbb{N}$ and a distribution $\mathcal{F}$ of ordered $n$-element sets $S \subseteq [N]$ such that

$$\max_{\text{ALG} \in \text{Card}} \max_{\pi \sim D} \mathbb{E}_{S \sim \mathcal{F}}[R(\text{ALG}(S, \pi))] \leq (1 + \varepsilon) \cdot \max_{\text{ALG} \in \text{Ord}} \max_{\pi \sim D} \mathbb{E}_\pi[R(\text{ALG}(\pi), \pi)].$$

I.e., for a sufficiently large size of the universe $[N]$ the online algorithm for any ordinal task $R(a, \pi)$ does not have much advantage over the ordinal online algorithms. We confirm this intuition and study how large the size $N$ of the universe need to be. For a given online ordinal task and a parameter $\varepsilon > 0$, we define its cardinal complexity to be the minimum $N$ required so that such a distribution $\mathcal{F}$ exists.\(^4\)

### 2.2 Total Variation Distance

Throughout the paper, we shall study discrete random objects, including integers and ordered sets of integers. Consider two random objects $X, Y$ sampled from probability mass functions $p_X, p_Y$ over a discrete domain $\mathcal{T}$. The total variation distance between random variables $X, Y$ is defined as the following.

$$d_{\text{TV}}(X, Y) \overset{\text{def}}{=} d_{\text{TV}}(p_X, p_Y) \overset{\text{def}}{=} \frac{1}{2} \sum_{t \in \mathcal{T}} |p_X(t) - p_Y(t)|$$

\(^4\)Our cardinal complexity measure is similar to the support size of $\mathcal{F}$, but it is more convenient to use since it directly refers to the range of possible values $N$ instead of $\binom{N}{n}$ possible subsets of $[N]$ of size $n$. 


The following three lemmas summarize certain standard and useful properties of the TV-distance, which we state here for the ease of reference without proofs.

**Lemma 2.1 (Triangle Inequality)** Let \( X, Y, Z \) be random objects over a discrete domain \( T \), then \( d_{TV}(X, Z) \leq d_{TV}(X, Y) + d_{TV}(Y, Z) \).

**Lemma 2.2 (Mixture)** Let \( \{X^\lambda\}, \{Y^\lambda\} \) be a set of random objects parameterized by \( \lambda \in \Lambda \). If \( X, Y \) are random objects generated by two steps: 1) sample \( \lambda \) with respect to the same distribution and 2) sample from \( X^\lambda, Y^\lambda \), then \( d_{TV}(X, Y) \leq \max_{\lambda \in \Lambda} d_{TV}(X^\lambda, Y^\lambda) \).

**Lemma 2.3 (Mapping)** Let \( X, Y \) be random objects over a discrete domain \( T \) and \( f \) be an arbitrary (random) mapping from \( T \rightarrow U \). Then \( d_{TV}(f(X), f(Y)) \leq d_{TV}(X, Y) \).

We also have the following bound on the total variation distance of uniform distributions.

**Lemma 2.4 (Uniform Distributions)** Suppose \( x_1 \sim \text{Uni}[\alpha_1, \beta_1] \) and \( x_2 \sim \text{Uni}[\alpha_2, \beta_2] \) with positive integers \( 0 \leq \alpha_2 \leq \beta_2 \leq \beta_1 - \alpha_1 \), then \( d_{TV}(x_1, x_2) \leq \frac{\beta_2}{\beta_1 - \alpha_1 + 1} \).

**Proof:** We calculate the total variation distance directly. For every \( x \in [\alpha_1 + \beta_2, \beta_1] \), we have that

\[
\Pr [x_1 + x_2 = x] = \sum_{i=\alpha_2}^{\beta_2} \Pr [x_1 = x - i, x_2 = i] = \sum_{i=\alpha_2}^{\beta_2} \frac{1}{\beta_1 - \alpha_1 + 1} \cdot \frac{1}{\beta_2 - \alpha_2 + 1} = \frac{1}{\beta_1 - \alpha_1 + 1}.
\]

Thus,

\[
d_{TV}(x_1, x_2) = \frac{1}{2} \sum_{x=\alpha_1}^{\alpha_1 + \beta_2 - 1} |\Pr [x_1 = x] - \Pr [x_1 + x_2 = x]| \\
= \sum_{x=\alpha_1}^{\alpha_1 + \beta_2 - 1} |\Pr [x_1 = x] - \Pr [x_1 + x_2 = x]| \leq \sum_{x=\alpha_1}^{\alpha_1 + \beta_2 - 1} \frac{1}{\beta_1 - \alpha_1 + 1} = \frac{\beta_2}{\beta_1 - \alpha_1 + 1}.
\]

\[\square\]

### 3 Universal Cardinal Complexity

Our universal cardinal complexity bound is inspired by the hardcore die guessing problem, from which we abstract the following desired property of the random set \( S \).

**Lemma 3.1** For any \( \varepsilon > 0 \) and \( N = O\left(\frac{1}{\varepsilon}\right) \uparrow \uparrow (n-1) \), there exists a distribution \( \mathcal{F}(\varepsilon) \) over ordered \( n \)-element sets \( S \subseteq [N] \) such that

\[
d_{TV}(S_i, S_j) \leq \varepsilon, \ \forall i, j \in [n]
\]

This is the main technical lemma in this section and the proof is deferred to the end of the section. We proceed by discussing its implications. As an immediate corollary, we give an upper bound on the cardinal complexity of the die guessing game, when the set \( S \) is drawn according to the above lemma.

**Corollary 3.1** The cardinal complexity of the die guessing game is at most \( O\left(\frac{n}{\varepsilon}\right) \uparrow \uparrow (n-1) \).
Proof: We use the distribution $F\left(\frac{\varepsilon}{2n}\right)$ from Lemma 3.1. Recall that any ordinal algorithm wins with probability $\frac{1}{n}$. It suffices to verify that no algorithm can guess correctly with probability larger than $1 + \frac{\varepsilon}{n}$. Since there is effectively only one action of the game, let $\text{ALG} : \binom{[N]}{n-1} \to [i]$ be an arbitrary guessing algorithm. We use $f(S)$ to denote the probability mass function of $F$. Then, the winning probability of $\text{ALG}$ is the following.

$$E[R(\text{ALG})] = \sum_{S \in \binom{[N]}{n}} f(S) \cdot \sum_{i \in [n]} \frac{1}{n} \cdot 1[\text{ALG}(S,i) = i]$$

(each $s_i$ is deleted w.p. $\frac{1}{n}$)

$$= \frac{1}{n} \sum_{S \in \binom{[N]}{n-1}} \sum_{i \in [n]} 1[\text{ALG}(\tilde{S}) = i] \cdot \sum_{S \in \binom{[N]}{n}} \left( f(S) \cdot 1[S,i = \tilde{S}] \right)$$

($\tilde{S}$ is the observed set)

$$= \frac{1}{n} \sum_{S \in \binom{[N]}{n-1}} \sum_{i \in [n]} 1[\text{ALG}(\tilde{S}) = i] \cdot \Pr[S,i = \tilde{S}]$$

$$\leq \frac{1}{n} \sum_{S \in \binom{[N]}{n-1}} \max_{i \in [n]} \Pr[S,i = \tilde{S}]$$

$$\leq \frac{1}{n} \sum_{S \in \binom{[N]}{n-1}} \left( \Pr[S,1 = \tilde{S}] + \sum_{i \neq 1} \Pr[S,i = \tilde{S}] - \Pr[S,1 = \tilde{S}] \right)$$

$$= \frac{1}{n} \left( 1 + 2 \cdot \sum_{i \neq 2} d_{TV}(S,i, S,1) \right) \leq \frac{1 + \varepsilon}{n}.$$

Furthermore, building on the distribution from Lemma 3.1 our next Lemma 3.2 gives another distribution that satisfies stronger conditions which we will use to obtain our universal cardinal complexity bound.

Lemma 3.2 For any $\varepsilon > 0$ and $N = O\left(\frac{n^2}{\varepsilon}\right) \uparrow \uparrow (n-1)$, there exists a distribution $F(\varepsilon)$ over ordered $n$-element sets $S \subseteq [N]$ such that

$$d_{TV}(S_I, S_J) \leq \varepsilon, \quad \forall I, J \subseteq [n], |I| = |J|. \tag{3.1}$$

Proof: We use the distribution $F\left(\frac{\varepsilon}{2n}\right)$ constructed in Lemma 3.1 which has cardinal complexity of $N = O\left(\frac{n^2}{\varepsilon}\right) \uparrow \uparrow (n-1)$. For a given pair of index sets $I$ and $J$, we iteratively construct a sequence of index sets $\{I_s\}, \{J_t\}$ in the following way:
Let $I_0 = I$ and $J_0 = J$ and $s = t = 0$.

We continue the construction of the sequence until $I_s = I_t$. For each intermediate step, we write the elements in $I_s, J_t$ in ascending order:

\[ I_s = (i_1 < i_2 < \ldots < i_k), \quad J_t = (j_1 < j_2 < \ldots < j_k). \]

- Let $i_r \neq j_r$ be the first different element. We have $i_\ell = j_\ell$ for $\ell \in [r - 1]$.
- If $i_r > j_r$, let $I_{s+1} = \{i_1, i_2, \ldots, i_{r-1}, i_r - 1, i_{r+1}, \ldots, i_k\}$ and increase $s$ to $s+1$.
- Else, let $J_{t+1} = \{j_1, j_2, \ldots, j_{r-1}, j_r - 1, j_{r+1}, \ldots, j_k\}$ and increase $t$ to $t+1$.

It is easy to see that the earth mover’s distance between $I_s = \{i_1, i_2, \ldots, i_k\}$ and $J_t = \{j_1, j_2, \ldots, j_k\}$, i.e., the value of \(\sum_{\ell \in [k]} |i_\ell - j_\ell|\) decreases by 1 after each iteration, the above procedure ends after at most $n^2$ steps, since $\sum_{\ell \in [k]} |i_\ell - j_\ell| \leq kn \leq n^2$. Let there be $m_1$ different sets in $\{I_s\}$ and $m_2$ sets in $\{J_t\}$. We have $m_1 + m_2 \leq n^2$.

Each pair of $I_s$ and $I_{s+1}$ differs only by a single element: $i_r \in I_s, i_r - 1 \notin I_s$ and $i_r \notin I_{s+1}, i_r - 1 \in I_{s+1}$. Thus, we can express both $S_{I_s}, S_{I_{s+1}}$ as the same (projection) function applied to $S_{(i_r-1)}$, or $S_{i_r}$, which deletes a subset of coordinates in either $S_{(i_r-1)}$, or $S_{i_r}$ with ranks $[n]\setminus \{i_1, \ldots, i_{r-1}, i_r - 1, i_r, i_{r+1}, \ldots, i_k\}$. By Lemma 2.3, $d_{TV}(S_{I_s}, S_{I_{s+1}}) \leq d_{TV}(S_{(i_r-1)}, S_{i_r}) \leq \frac{\varepsilon}{n^2}$, due to the property from Lemma 3.1. Similarly, we also have $d_{TV}(S_{J_t}, S_{J_{t+1}}) \leq \frac{\varepsilon}{n^2}$. Therefore, by triangle inequality for TV-distance

\[
d_{TV}(S_I, S_J) \leq \sum_{s=0}^{m_1-1} d_{TV}(S_{I_s}, S_{I_{s+1}}) + d_{TV}(S_{I_{m_1}}, S_{J_{m_2}}) + \sum_{t=0}^{m_2-1} d_{TV}(S_{J_t}, S_{J_{t+1}}) \leq \frac{\varepsilon}{n^2} \cdot n^2 \leq \varepsilon.
\]

Based on the above construction, we establish a universal upper bound of cardinal complexity for all ordinal tasks.

**Theorem 3.1** For any online problem with $n$ rounds, ordinal reward function $R : A \times \text{Sym}(n) \to \mathbb{R}_+$, distribution over arrival orders $\pi \sim D, \pi \in \text{Sym}(n)$, its cardinal complexity is at most $N = O\left(\frac{n^3}{\varepsilon} \right) \uparrow \uparrow (n - 1)$. That is, there exists a distribution $F$ over subsets of $[N]$, such that the advantage of the cardinal algorithms over ordinal algorithms is at most $1 + \varepsilon$, i.e.,

\[
\max_{\text{Card} S \sim F, \pi} \mathbb{E} [R(\text{ALG}(S, \pi), \pi)] \leq (1 + O(\varepsilon)) \cdot \max_{\text{Ord} \pi} \mathbb{E} [R(\text{ALG}(\pi), \pi)].
\]

**Proof:** Let $F(\frac{\varepsilon}{n^2})$ be the distribution from Lemma 3.2. Consequently, the largest number used is $N = O\left(\frac{n^3}{\varepsilon} \right) \uparrow \uparrow (n - 1)$.

Next, we verify the stated inequality. Given an arbitrary cardinal algorithm ALG*. Consider the following ordinal algorithm that mimics the behavior of ALG*:

- At step $k$, we have the ranking of the first $k$ elements $\sigma_k$ and the actions $\tilde{a}_{[k-1]} = (\tilde{a}_1, \ldots, \tilde{a}_{k-1})$ that we have made in the previous rounds.
- Sample $\tilde{S} \sim F$ and select an arbitrary $\tilde{\pi} \in \text{Sym}(n)$ that is consistent with $\sigma_k$.
- Take action $\tilde{a}_k \sim \text{ALG}^*(\tilde{a}_{[k-1]}, \tilde{S}[k])$. 


Lemma 3.3 For all $\pi \in \text{Sym}(n)$, $d_{TV}(a, \bar{a}) \leq \frac{\varepsilon}{n}$, where $a$ are the actions taken by $\text{ALG}^*$ when the instance is drawn from $\pi \sim \mathcal{D}$ and $S \sim \mathcal{F}$; $\bar{a}$ is the actions taken by the ordinal algorithm defined above.

Proof: We prove the following stronger statement. For all $\sigma_k \in \text{Sym}(k)$, we have $d_{TV}(a_{k}, \bar{a}_{k}) \leq k \cdot \frac{\varepsilon}{n \cdot n!}$, where $a_{k}$ are the actions taken by $\text{ALG}^*$ in the first $k$ steps when the instance is drawn from $\pi \sim \mathcal{D}_{\sigma_k}$ and $S \sim \mathcal{F}$; $\bar{a}_{k}$ are the actions taken by the ordinal algorithm defined above.

We prove the statement by induction on $k$. The base of induction trivially holds for $k = 0$.

$$d_{TV}(a_{k}, \bar{a}_{k}) = d_{TV}((a_{k-1}, a_{k}), (\bar{a}_{k-1}, \bar{a}_{k})) \leq d_{TV}((\bar{a}_{k-1}, a_{k}(\bar{a}_{k-1}, S_{\pi[k]})), (\bar{a}_{k-1}, \bar{a}_{k}))$$

$$+ d_{TV}((a_{k-1}, a_{k}(a_{k-1}, S_{\pi[k]})), (\bar{a}_{k-1}, a_{k}(\bar{a}_{k-1}, S_{\pi[k]})))$$

$$\leq \max_{\pi[k-1]} d_{TV}(a_{k}(b_{k-1}, S_{\pi[k]}), \bar{a}_{k}(b_{k-1}, \sigma_k)) + d_{TV}(a_{k-1}, \bar{a}_{k-1})$$

$$\leq \max_{b_{k-1}} d_{TV}(a_{k}(b_{k-1}, S_{\pi[k]}), a_{k}(b_{k-1}, \bar{a}_{k}(b_{k-1}, S_{\pi[k]}))) + (k - 1) \cdot \frac{\varepsilon}{n \cdot n!}$$

$$\leq d_{TV}(S_{\pi[k]}, \bar{S}_{\pi[k]}) + (k - 1) \cdot \frac{\varepsilon}{n \cdot n!} \leq k \cdot \frac{\varepsilon}{n \cdot n!},$$

where the first inequality follows from the triangle inequality of Lemma 2.1 in the second inequality we use that $d_{TV}((x, f(x)), (x, g(x))) \leq \max_{\pi[k-1]} d_{TV}(f(x), g(x))$ and that $d_{TV}((x, f(x)), (y, f(y))) = d_{TV}(x, y)$ for any randomized mappings $f(\cdot), g(\cdot)$; in the third inequality we used induction hypothesis for $k - 1$ and the definition of $\bar{a}_{k}$; in the forth inequality we apply Lemma 2.3 to the mapping $a_{k}(b_{k-1}, \cdot)$ with inputs $S_{\pi[k]}$ and $\bar{S}_{\pi[k]}$; in the last inequality we used Lemma 3.2.

Corollary 3.2 Disadvantage of the best $\text{ALG} \in \text{Ord}$ compared to the best $\text{ALG} \in \text{Card}$ is bounded as $\max_{\text{Ord}} \mathbb{E}_\pi[R(\text{ALG}(\pi), \pi)] \geq \max_{\text{Card}} \mathbb{E}_\pi[R(\text{ALG}(S, \pi), \pi)] - \frac{\varepsilon}{n!} \cdot \mathbb{E}_\pi[\max_{\pi} R(a, \pi)].$

Proof: Let $\text{ALG}^* \in \text{Card}$ be the optimal cardinal algorithm that results in a distribution of actions $a(\pi)$ for each $\pi \in \text{Sym}(n)$ and random $S \sim \mathcal{F}$. We consider the above ordinal algorithm that results in the action sequence $\bar{a}(\pi)$ for each $\pi \in \text{Sym}(n)$. Then $\max_{\text{Ord}, \pi} \mathbb{E}_\pi[R(\text{ALG}(\pi), \pi)] \geq \mathbb{E}_\pi[R(\bar{a}(\pi), \pi)]$ and $\max_{\text{Card}, \pi, S} \mathbb{E}_\pi[R(\text{ALG}(S, \pi), \pi)] = \mathbb{E}_\pi[R(a(\pi), \pi)]$. Lemma 3.2 says that $d_{TV}(a(\pi), \bar{a}(\pi)) \leq O(\varepsilon)$, for any fixed $\pi \in \text{Sym}(n)$ which gives us for any fixed $\pi \in \text{Sym}(n)$

$$\mathbb{E}_{a, a}[R(a(\pi), \pi) - R(\bar{a}(\pi), \pi)] \leq d_{TV}(a(\pi), \bar{a}(\pi)) \cdot \max_{b} [R(b, \pi) - 0] \leq \frac{\varepsilon}{n!} \cdot \max_{b} R(b, \pi)$$

We conclude the proof by taking expectation over $\pi \sim \mathcal{D}$.

Next, we have a trivial ordinal algorithm that guesses the arrival order to be

$$\pi^* = \arg \max_{\pi} \left( \Pr_{\mathcal{D}}[\pi] \cdot \max_{a} R(a, \pi) \right),$$

and then chooses the corresponding optimal actions at each step. This algorithm achieves at least $\frac{1}{n!}$ fraction of the offline optimum $\mathbb{E}_\pi[\max_{a} R(a, \pi)]$, since there are at most $n!$ possible orders and we chose the one with maximal expected contribution. This means that

$$\max_{\text{ALG} \in \text{Ord}} \mathbb{E}_\pi[R(\text{ALG}(\pi), \pi)] \geq \frac{1}{n!} \cdot \mathbb{E}_\pi \left[ \max_{a} R(a, \pi) \right].$$

(3.2)

Finally, we combine Corollary 3.2 with equation (3.2) to conclude the proof of the theorem.
3.1 Proof of Lemma 3.1

Now, we give an explicit construction of the distribution that satisfies the stated property. As a warm up we first describe how to construct such distribution $\mathcal{F}$ for $n = 2, 3$.

**Warm up for $n = 2$.** For $N = \Theta\left(\frac{1}{\varepsilon}\right)$, consider a uniform distribution over consecutive numbers $(i, i + 1)$ for all $1 \leq i \leq N - 1$. Then $S_1$ is a uniform distribution over $\{1, 2, \ldots, N - 1\}$ and $S_2$ is a uniform distribution over $\{2, \ldots, N - 1, N\}$. Thus, $d_{\text{TV}}(S_1, S_2) = \frac{1}{N-1} \leq \varepsilon$, for $N = O\left(\frac{1}{\varepsilon}\right)$.

**Warm up for $n = 3$.** For $N = \left(\frac{1}{\varepsilon}\right)^\frac{1}{3}$, consider a uniform distribution over $(i, i + 2^\ell, i + 2^\ell+1)$ for all $\ell \leq \frac{1}{\varepsilon}$ and all $i$’s as long as $i + 2^{\ell+1} \leq N$. For an observed set $(i, i + 2^\ell)$, unless $\ell \in \{1, \frac{1}{\varepsilon}\}$ or $i \leq 2^\ell$, or $i + 2^{\ell+1} > N$, it is equally likely that the observed set was obtained after deleting $i - 2^\ell$, or $i + 2^\ell-1$, or $i + 2^{\ell+1}$. Therefore, to calculate the total variation distance, it suffices to count the number of the problematic boundary cases, that is roughly $\frac{1}{\varepsilon} = \varepsilon$ portion of the possibilities.

**Inductive Construction.** The general construction proceeds by induction on $n$. For each $n \geq 2$ we construct a distribution $\mathcal{F}_n$ with cardinal complexity $N = O\left(\frac{1}{\varepsilon}\right) \uparrow \uparrow \left( n - 1 \right)$. The base of inductive construction is specified above for $n = 2$. For the inductive step, we assume that there is a distribution of $T = (t_1 < t_2 < \ldots < t_{n-1}) \sim \mathcal{F}_{n-1}(\varepsilon)$ with desired properties and with cardinal complexity $O\left(\frac{1}{\varepsilon}\right) \uparrow \uparrow \left( n - 2 \right)$ for each $\varepsilon > 0$. I.e., the maximum possible value of $t_i$ is $O\left(\frac{1}{\varepsilon}\right) \uparrow \uparrow \left( n - 2 \right)$. Then we construct $S = (s_1 < s_2 < \ldots < s_n) \sim \mathcal{F}_n$ as follows

1. Consider equivalent representation of $S$ as $(s_1, d_1, \ldots, d_{n-1})$, where $d_i = s_{i+1} - s_i$ for $i \in [n]$.

2. Let $s_1 \sim \text{Uni}\left[ 1, O\left(\frac{1}{\varepsilon}\right) \uparrow \uparrow \left( n - 1 \right) \right]$, and $d_i \sim \text{Uni}[C^n, 2 \cdot C^n]$ for $C = \frac{6}{\varepsilon}$ independently from each other, where $(t_i)_{i=1}^{n-1}$ are defined by $(t_1 < t_2 < \ldots < t_{n-1}) = T \sim \mathcal{F}_{n-1}\left(\frac{\varepsilon}{3}\right)$.

We first calculate the cardinal complexity of our distribution $\mathcal{F}_n$:

$$N = \max s_n = \max \left( s_1 + \sum_{i=1}^{n-1} \max d_i \right) \leq O\left(\frac{1}{\varepsilon}\right) \uparrow \uparrow \left( n - 1 \right) + \sum_{i=1}^{n-1} 2 \cdot \left(\frac{6}{\varepsilon}\right)^{\max t_i}$$

$$\leq O\left(\frac{1}{\varepsilon}\right) \uparrow \uparrow \left( n - 1 \right) + O\left(\frac{1}{\varepsilon}\right)^{\max t_{n-1} + 1} \leq O\left(\frac{1}{\varepsilon}\right) \uparrow \uparrow \left( n - 1 \right),$$

where the last inequality uses the induction hypothesis that the largest possible value of $t_{n-1}$ is $O\left(\frac{1}{\varepsilon}\right) \uparrow \uparrow \left( n - 2 \right)$.

Next, we verify the stated total variation bound of the lemma. Consider $S_i$ in the alternative representation for each $i \in [n]$: $S_i = (s_1, d_1, \ldots, d_{i-2}, d_{i-1} + d_i, d_{i+1}, \ldots, d_{n-1})$.

$$S_1 = (s_1 + d_1, d_2, \ldots, d_{n-1}).$$

We define auxiliary random ordered sets $U_i \overset{\text{def}}{=} (s_1, d_1, \ldots, d_{i-1}, d_{i+1}, \ldots, d_{n-1})$ for $i \in [n - 1]$ in the same alternative representation as $S_i$ and independent distributions of $s_1$, and all $d$’s. Notice

\footnote{For brevity and transparency of presentation we omit precise estimates of the boundary cases.}
that
\[ d_{TV}(S_i, U_{i-1}) = d_{TV}(d_i, d_i + d_{i-1}) \quad \text{for } i \geq 2, \quad \text{and} \quad d_{TV}(S_1, U_1) = d_{TV}(s_1, s_1 + d_1). \]

Moreover, we have by Lemma 2.21: (i) \( d_{TV}(d_i, d_i + d_{i-1}) \leq \frac{2C_i^{i-1}}{C_i^i} \leq \frac{2}{\epsilon} = \frac{\delta}{d} \) for arbitrary fixed \( t_i \leq t_i - 1 \); (ii) \( d_{TV}(s_1, s_1 + d_1) \leq \frac{2C_i^{i-1}}{O((\log(n-1)))} < \frac{\delta}{d} \) for arbitrary fixed \( t_i \leq O\left(\frac{1}{\epsilon}\right) \uparrow (n-2) \). We can also apply Lemma 2.3 to the random mapping from \( T_i, T_j \) to \( U_i, U_j \) and get
\[ d_{TV}(U_i, U_j) \leq d_{TV}(T_i, T_j). \]

Finally, we are ready to conclude the proof of the lemma. We consider two cases. First, we assume that \( i, j \geq 2 \) in the lemma’s statement. Then
\[
\begin{align*}
    d_{TV}(S_i, S_j) &\leq d_{TV}(S_i, U_{i-1}) + d_{TV}(S_j, U_{j-1}) + d_{TV}(U_{i-1}, U_{j-1}) = d_{TV}(d_i, d_i + d_{i-1}) \\
    &\quad + d_{TV}(d_j, d_j + d_{j-1}) + d_{TV}(U_{i-1}, U_{j-1}) \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + d_{TV}(T_{(i-1)}, T_{(j-1)}) \leq \epsilon.
\end{align*}
\]
Second, we assume that \( j = 1, i \geq 2 \). Then, similar to the previous case we have
\[
\begin{align*}
    d_{TV}(S_1, S_i) &\leq d_{TV}(S_1, U_1) + d_{TV}(S_i, U_{i-1}) + d_{TV}(U_1, U_{i-1}) = d_{TV}(s_1, s_2) \\
    &\quad + d_{TV}(d_i, d_i + d_{i-1}) + d_{TV}(U_1, U_{i-1}) \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + d_{TV}(T_1, T_{(i-1)}) \leq \epsilon.
\end{align*}
\]
This concludes the proof of Lemma 3.1.

### 4 Perturbed Rank Guessing

In this section, we establish a lower bound on the cardinal complexity of the die guessing game, by designing an algorithm that efficiently utilizes the cardinal information in the game. For technical reasons, we study a slightly more general version of the die guessing game, which we call *perturbed rank guessing*. The new game gives more power to the adversary, which we will need in our inductive proof later.

**Perturbed Rank Guessing.** Given \( n, N, p = (p_1, \ldots, p_n) \in \Delta_n \), the adversary (first player) chooses a set \( S \subseteq [N] \) of \( n \) distinct integers \( s_1 < s_2 < \cdots < s_n \), with a technical condition that \( s_i - s_{i-1} \geq 20 \) for all \( i \geq 2 \). Then \( S_i \) is generated by deleting a single random number from \( S \), where each \( S_i \) is deleted with probability \( p_i \). Upon seeing \( S_i \), the adversary can modify every number of \( S_i \) by \( \pm 1 \) or 0 and show modified numbers \( \tilde{S} \) to the algorithm (second player). Finally, the algorithm guesses the index \( i \in [n] \) of the deleted number \( s_i \in S \). If the algorithm guesses correctly, the reward is \( \frac{1}{n} \). Otherwise, the reward is 0.

**Remark 4.1** The perturbed ranking guessing game does not belong to the family of online ordinal tasks we defined in Section 2, as the adversary has an extra power to perturb each number before it is observed by the algorithm. On the other hand, this game is harder for the algorithm than the die guessing game, in the sense that if we have a cardinal algorithm \( ALG \) for the perturbed rank guessing game, we can apply it to the die guessing game and achieve the same expected reward. Indeed, we first set all probabilities \( p_i = \frac{1}{n} \) for every \( i \); and in order to meet the technical condition that gaps between consecutive numbers are at least 20, we multiply each observed number by 20 before we call \( ALG \) as a black box. Effectively, we translate the instance from set \( S = (s_1, \ldots, s_n) \) to \( S' = (20s_1, \ldots, 20s_n) \).
Our main result is a randomized algorithm with the following performance guarantee for the perturbed rank guessing game. We remark that we do not try to optimize the dependency on \( n \).

The most important regime for us is when \( n \) is a constant and \( N \to \infty \).

**Theorem 4.1** There exists an algorithm for the perturbed rank guessing game with expected reward
\[
1 + \frac{1}{(6n)^{n^2}} \cdot \Omega \left( \frac{1}{\log^{(n-2)} N} \right),
\]
where \( \log^{(n)}(x) \defeq \log \log \ldots \log x \).

As a corollary, by applying the algorithm to the die guessing game as explained above, we establish a lower bound on the cardinal complexity.

**Corollary 4.1** The cardinal complexity of the die guessing game is at least \( \Omega \left( \frac{2}{(6n)^{n^2}} \right) \).

The next corollary shows that the dependency on \( \varepsilon \) of the \( n \)-face die guessing game is a tower of exponents for any \( n \) (\( n \) is not necessarily a constant) of arbitrary constant height \( c \leq n - 2 \).

**Corollary 4.2** For any constant \( c \leq n \) the cardinal complexity of the \( n \)-faces die guessing game is at least \( \Omega \left( \frac{2^c}{c^{c-2} \log n} \right) \).

**Proof:** We reduce the \( n \)-face die guessing game to the \( c \)-face perturbed rank guessing game (the reduction does not use any perturbations, only the non-uniform probabilities \( p = (p_1, \ldots, p_c) \in \Delta_c \) for the hidden face). The algorithm for the \( n \)-face die guessing game works as follows: consider the largest \( c - 1 \) numbers among \( n - 1 \) visible faces and try to guess the rank of the hidden number relative to them using the algorithm for \( c \)-face perturbed rank guessing game with probabilities \( p = (\frac{n-c-1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}) \); if our guess in the \( c \)-face game is that the hidden number is the smallest number, then we pick our answer uniformly at random among the smallest \( n - c + 1 \) numbers in the \( n \)-face game; otherwise we simply report the same rank as in the \( c \)-face game. This algorithm guesses correctly with probability \( \frac{1}{n} \left( 1 + \Omega \left( \frac{1}{\log^{(c-2)} N} \right) \right) \) and concludes the proof of the corollary. We omit a straightforward calculation of the performance guarantee.

Before we delve into technical details of Theorem 4.1 proof, we give a high level overview of our approach in the next subsection.

### 4.1 Proof Sketch

Consider the alternative representation of set \( S = (s_1, d_1, \ldots, d_{n-1}) \), where \( d_i = s_{i+1} - s_i \). After the deletion of a number, our algorithm observes \( \tilde{S} = (\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_{n-1}) \). Observe that the \( n - 1 \) numbers partition \([N]\) into \( n \) intervals \( I_1 = [1, \tilde{s}_1], I_2 = (\tilde{s}_1, \tilde{s}_2), \ldots, I_n = (\tilde{s}_{n-1}, N] \). It is equivalent between guessing the index \( j \) of the deleted number and guessing which interval \( I_j \) the deleted number belongs to. We shall describe our algorithm below as guessing the interval, which is more intuitive. And for now, say we are playing the original die guessing game.

Our first step is to show that a hard instance must be like \( d_1 \ll d_2 \ll \ldots \ll d_{n-1} \) (or \( d_1 \gg d_2 \gg \ldots \gg d_{n-1} \)). We introduce two subroutines, **Mono-Gaps** (refer to Lemma 1.1) and **Exp-Gaps** (refer to Lemma 4.2) that achieve a constant advantage (that only depends on \( n \) but does not depend on \( N \)) over ordinal algorithms, unless the instance has this specific shape, and perform not worse than any ordinal algorithm for this case.
Our second step focuses on instances with \( d_1 \ll d_2 \ll \ldots \ll d_{n-1} \). We use \( g_i = \bar{s}_{i+1} - \bar{s}_i, i \in [n-2] \) to denote the gaps observed by our algorithm. Our recursive algorithm only looks at those gaps and views them as a random (perturbed) subset of \( \{d_i\}_{i \in [n-1]} \) with \( n-2 \) numbers. E.g., when \( s_i \) is deleted, the gaps we observe are

\[
(d_1, \ldots, d_{i-2}, d_{i-1} + d_i, d_{i+1}, \ldots, d_{n-1}) \approx (d_1, \ldots, d_{i-2}, d_i, d_{i+1}, \ldots, d_{n-1}),
\]

since \( d_{i-1} \ll d_i \). We formalize this idea by taking the logarithm of \( g_i \)'s. Then we can treat \( \{\lfloor \log_2 g_i \rfloor\}_{i \in [n-2]} \) as a random subset of \( \{\lfloor \log_2 d_i \rfloor\}_{i \in [n-1]} \), within a tiny error of at most 1:

\[
\lfloor \log_2 d_1 \rfloor, \ldots, \lfloor \log_2 d_{i-2} \rfloor, \lfloor \log_2 (d_{i-1} + d_i) \rfloor, \lfloor \log_2 d_{i+1} \rfloor, \ldots, \lfloor \log_2 d_{n-1} \rfloor
\]

That is the reason why we introduced perturbation to the setting. Moreover, notice that the \( n \) possible deletions of \( \{s_i\}_{i \in [n]} \) result in only \( n-1 \) possible gap vectors. Indeed, the two cases when \( s_1 \) or \( s_2 \) is deleted lead to (almost) the same set of observed gaps:

\[
\lfloor \log_2 (d_1 + d_2) \rfloor, \lfloor \log_2 d_3 \rfloor, \ldots, \lfloor \log_2 d_{n-1} \rfloor = \lfloor \log_2 d_2 \rfloor + 0/1, \lfloor \log_2 d_3 \rfloor, \ldots, \lfloor \log_2 d_{n-1} \rfloor
\]

In particular, a uniform deletion of the \( n \) numbers from \( S \) leads to a non-uniform deletion of the \( n-1 \) gaps with probabilities \( \{\frac{2}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\} \). This is why we consider non-uniform deletion of the numbers in the perturbed version of the die guessing game. Those changes to the setting do not affect too much the analysis for the first step, but allow us to strengthen our induction hypothesis in the second step. Finally, notice that our recursive step reduces the cardinal complexity \( N \) by applying a logarithmic function after each step, from which we derive the stated algorithm’s performance with iterative logarithms.

### 4.2 Formal Proof of Theorem 4.1

We will use notation \( \bar{S} \) for the set of ordered values that cardinal algorithm observes before making its guess.

**Ordinal Algorithm.** An ordinal algorithm cannot see the numbers of \( \bar{S} \), i.e. its decision does not depend on the input. On the other hand, notice that the gaps between consecutive numbers are at least 20. The perturbation would not change the relative order of the numbers. Hence, the guess of \( i \) wins with probability \( p_i \) with the expected reward of \( 1 = p_1 \cdot \frac{1}{p_1} \), for all \( i \in [n] \). Therefore, any ordinal algorithm has expected reward of 1.

**Cardinal Algorithms.** We prove that an algorithm can achieve a noticeably better reward than 1 using \( \bar{S} \). Within this section, we are interested in the regime when \( N \to \infty \) and \( n \) is a small constant. As a warm up we first describe the algorithms for \( n = 2, 3 \).

**Warm-up for \( n = 2 \).** Consider the following algorithm: it sees \( \bar{S} = \{\bar{s}\} \) and guesses 1 with probability \( \frac{\bar{s}}{N} \) and 2 otherwise. Let \( S = (s_1, s_2) \) be the ordered set chosen by the adversary. With probability \( p_1 \), we see \( \bar{s} \in \{s_2 \pm 1, s_2\} \) and win with probability \( \frac{\bar{s}}{N} \geq \frac{s_2 - 1}{N} \) and get reward \( \frac{1}{p_1} \). With probability \( p_2 \), we see \( \bar{s} \in \{s_1 \pm 1, s_1\} \) and win with probability at least \( 1 - \frac{s_1 + 1}{N} \) and get reward \( \frac{1}{p_2} \). Therefore, the expected reward \( \text{ALG} \) of the algorithm is at least

\[
\text{ALG} \geq p_1 \cdot \frac{s_2 - 1}{N} \cdot \frac{1}{p_1} + p_2 \cdot \left( 1 - \frac{s_1 + 1}{N} \right) \cdot \frac{1}{p_2} = 1 + \frac{s_2 - s_1 - 2}{N} \geq 1 + \frac{18}{N}.
\]
Warm-up for $n = 3$. Consider the following algorithm: it sees $\tilde{S} = (s_1, s_2)$ and guesses 2 with probability $\frac{\log_2(s_2 - s_1)}{\log_2 N}$ and guesses 1, 3 uniformly at random otherwise. Let $S = \{s_1, s_2, s_3\}$ be the set chosen by the adversary.

With probability $p_1$, we see $\tilde{s}_1 \in \{s_2 + 1, s_2\}, \tilde{s}_2 \in \{s_3 + 1, s_3\}$ and win with probability $\frac{1}{2} \left( 1 - \frac{\log_2(s_2 - s_1)}{\log_2 N} \right) \geq \frac{1}{2} \left( 1 - \frac{\log_2(s_3 - s_2 + 2)}{\log_2 N} \right)$. Similarly, with probability $p_3$ when the largest number is deleted, we win with probability at least $\frac{1}{2} \left( 1 - \frac{\log_2(s_2 - s_1 + 2)}{\log_2 N} \right)$.

With probability $p_2$, we see $\tilde{s}_1 \in \{s_1 + 1, s_1\}, \tilde{s}_2 \in \{s_3 + 1, s_3\}$ and win with probability $\frac{\log_2(s_2 - s_1)}{\log_2 N} \geq \frac{\log_2(s_3 - s_1 - 2)}{\log_2 N}$.

Therefore, the expected reward $\text{ALG}$ of the algorithm is at least

$$\text{ALG} \geq p_1 \frac{1}{2} \left( 1 - \frac{\log_2(s_3 - s_2 + 2)}{\log_2 N} \right) + p_2 \frac{1}{2} \left( 1 - \frac{\log_2(s_3 - s_1 + 2)}{\log_2 N} \right) + p_3 \frac{1}{2} \left( 1 - \frac{\log_2(s_2 - s_1 + 2)}{\log_2 N} \right) \geq 1 + \Omega \left( \frac{1}{\log N} \right),$$

where we use Jensen’s inequality $\log a + \log b \leq 2 \log \left( \frac{a+b}{2} \right)$ for the concave $\log(x)$ function in the second inequality, and in the last inequality, we know that $s_3 - s_1 \geq 40$ according to our technical assumption.

**General Guessing Algorithm.** We prove the theorem by induction and construct the algorithm recursively for each $n \geq 4$ using the algorithm for $n - 1$. The input to our algorithm is an increasing sequence $\tilde{S} = (\tilde{s}_i)_{i \in [n-1]}$. We observe the gaps $g_i \eqdef \tilde{s}_{i+1} - \tilde{s}_i$ for all $i \in [n-2]$.

We first introduce a strategy called **Mono-Gaps**, that has expected reward significantly higher than 1 when the sequence of gaps $\mathbf{d} \eqdef (d_i = s_{i+1} - s_i)_{i \in [n-1]}$ is not monotone, and does as good as random guessing for any instance. Recall a technical assumption that every $d_i \geq 20$.

| Algorithm 1: Mono-Gaps$(n, S)$ |
|---------------------------------|
| Select a pair of two adjacent gaps $(g_i, g_{i+1})$, with $i \sim \text{Uni}[n-3]$ |
| if $g_i + 4 < g_{i+1}$ then |
| | With probability $\frac{2}{3}$, return $i + 2$ |
| | With probability $\frac{1}{3}$, return $i$ |
| end |
| if $g_i > g_{i+1} + 4$ then |
| | With probability $\frac{2}{3}$, return $i + 1$ |
| | With probability $\frac{1}{3}$, return $i + 3$ |
| end |
| return $j \sim \text{Uni}\{i, i + 3\}$ |

**Lemma 4.1** For any $S$, the expected reward $\text{MG}$ of **Mono-Gaps** satisfies the following:

1. $\text{MG} \geq 1$;
2. If $\mathbf{d}$ is not monotone, then $\text{MG} \geq 1 + \frac{1}{3(n-3)}$;
3. If $\mathbf{d}$ is increasing (or decreasing) and there exists $i$ with $d_i + d_{i+1} > d_{i+2} + 8$ (or $d_i + 8 < d_{i+1} + d_{i+2}$), then $\text{MG} \geq 1 + \frac{1}{3(n-3)}$.
Intuition behind Mono-Gaps algorithm. It is useful to think about the random selection of the pair \((g_i, g_{i+1})\) as first guessing the deleted element to be among \(\{i, i+1, i+2, i+3\}\). If this guess is correct, our decision only depends on \((g_i, g_{i+1})\), i.e., we reduce the problem to the case \(n = 4\) for \(S = (s_t, s_{t+1}, s_{t+2}, s_{t+3})\) and if the corresponding part \((d_i, d_{i+1}, d_{i+2})\) of \(d\) is not monotone we get a certain advantage over the random guessing strategy. For \(n = 4\) there are three following cases

1. \(d_1 \leq d_2 \geq d_3\). In this case, when nature deletes \(i = 2\), we observe \((g_1 = d_1 + d_2 > g_2 = d_3)\) and when nature deletes \(i = 3\), we observe \((g_1 = d_1 < g_2 = d_2 + d_3)\). By guessing the deleted element \(s_i\) to be inside of the larger gap \((g_1, g_2)\) we identify correctly the case \(i = 2\) and \(i = 3\) with probability \(\frac{2}{3}\). This gives higher expected reward than 1.

2. \(d_1 \geq d_2 \leq d_3\). In this case, the natural strategy of guessing-inside-the-larger-gap \((g_1, g_2)\) does not give us any advantage over the random guessing (but, it does not give us any disadvantage over random guessing). On the other hand, the strategy of guessing \(i = 1\) when \(g_1 < g_2\) and \(i = 4\) when \(g_1 > g_2\) is correct when \(i = 1\) or \(i = 4\). This allows us to improve upon random guessing strategy when we mix the guess-inside-the-large-gap and the guess-outside-in-the-direction-of-smaller-gap strategies.

3. In the case \(d_1 < d_2 < d_3\) or \(d_1 > d_2 > d_3\), either of the strategies gives expected reward of 1.

We note that the actual algorithm and the formal proof of Lemma [4, 1] are more complicated than the above intuition, as the adversary can perturb a little the observed set \(\bar{S}\) and since the reduction to the case \(n = 4\) is only an informal statement.

Proof: With probability \(p_i\), \(s_i\) is deleted and we observe \(\bar{S}\), where \(\bar{s}_j \in \{s_j \pm 1, s_j\}\) for each \(j \leq i - 1\), and \(\bar{s}_j \in \{s_{j+1} \pm 1, s_{j+1}\}\) for each \(j \geq i\). We consider the sequence of distances \(d = (d_i = s_{i+1} - s_i)_{i \in [n-1]}\) for the original instance \(S\).

We have \(|g_j - d_j| \leq 2\) for each \(j \neq i - 1\) and \(|g_{i-1} - d_{i-1} - d_i| \leq 2\). We first calculate the expected reward of algorithm [1] if \(s_i\) was deleted from \(S\). There are at most four possibilities for the random pair \((g_j, g_{j+1})\) that can lead to the correct guessing of \(i\). Namely, \(j \in \{i - 3, i - 2, i - 1, i\}\):

(a) Algorithm selects the pair \((g_{i-3}, g_{i-2})\). If \(g_{i-3} > g_{i-2} + 4\), then our expected reward is \(\frac{1}{7}\). If \(g_{i-2} - 4 \leq g_{i-3} \leq g_{i-2} + 4\), then our expected reward is \(\frac{1}{7}\). Thus when \(d_{i-3} \geq d_{i-2}\), we have \(g_{i-3} \geq g_{i-2} - 4\) and algorithm’s expected reward is at least \(\frac{1}{7}\).

(b) Algorithm selects the pair \((g_{i-2}, g_{i-1})\). If \(g_{i-2} < g_{i-1} - 4\) (happens when \(d_{i-2} < d_{i-1} + d_i - 8\)), then our expected reward is \(\frac{2}{7}\).

(c) Algorithm selects the pair \((g_{i-1}, g_i)\). If \(g_{i-1} > g_i + 4\) (happens when \(d_{i-1} + d_i - 8 > d_{i+1}\)), then our expected reward is \(\frac{2}{7}\).

(d) Algorithm selects the pair \((g_i, g_{i+1})\). If \(g_i + 4 < g_{i+1}\), then our expected reward is \(\frac{1}{3}\). If \(g_i - 4 \leq g_{i+1} \leq g_i + 4\), then our expected reward is \(\frac{1}{2}\). Thus when \(d_{i+2} \geq d_{i+1}\), we have \(g_{i+2} \geq g_{i+1} - 4\) and algorithm’s expected reward is at least \(\frac{1}{3}\).
Therefore, the expected reward $\text{MG}$ of the algorithm for any adversarial choice of $\tilde{S}$ is at least

$$
\mathbb{E}\left[\text{MG}(\tilde{S}) \cdot 1\left[s_i \text{ is deleted}\right]\right] \geq \frac{1}{n} \left[3 \leq i \leq n-1\right] \cdot \left(\frac{2}{3} \cdot 1\left[d_{i-1} + d_i > d_{i-2} + 8\right]\right)
$$

$$
+ \frac{1}{n} \left[2 \leq i \leq n-2\right] \cdot \left(\frac{2}{3} \cdot 1\left[d_{i-1} + d_i > d_{i+1} + 8\right]\right) + \frac{1}{n} \left[4 \leq i \leq n\right] \cdot \left(\frac{1}{3} \cdot 1\left[d_{i-3} \geq d_{i-2}\right]\right)
$$

$$
+ \frac{1}{n} \left[1 \leq i \leq n-3\right] \cdot \left(\frac{1}{3} \cdot 1\left[d_{i+2} \geq d_{i+1}\right]\right),
$$

(4.1)

where the randomness in expectation is over the randomness of $\text{Mono-Gaps}$. Hence,

$$
\text{MG}(S) = \sum_{i \in [n]} \mathbb{E}\left[\text{MG}(\tilde{S}) \cdot 1\left[s_i \text{ is deleted}\right]\right]
$$

$$
\geq \frac{2}{3(n-3)} \sum_{i=2}^{n-2} \left(1\left[d_i + d_{i+1} > d_{i-1} + 8\right] + 1\left[d_{i-1} + d_i > d_{i+1} + 8\right]\right)
$$

$$
+ \frac{1}{3(n-3)} \cdot \left(1\left[d_1 \geq d_2\right] + 1\left[d_{n-1} \geq d_{n-2}\right] + n - 4\right),
$$

(4.2)

where the inequality follows from (4.1) and the fact that $1\left[d_i \geq d_{i+1}\right] + 1\left[d_i \leq d_{i+1}\right] \geq 1$. Note that

$$
1\left[d_i + d_{i+1} > d_{i-1} + 8\right] + 1\left[d_{i-1} + d_i > d_{i+1} + 8\right] \geq 1 \text{ for every } i \in [n-1].
$$

(4.3)

Indeed, if both of the indicators are 0, we would have $0 < d_i - 8 \leq d_{i-1} - d_i \leq 8 - d_i < 0$ (recall that $d_i \geq 12$), a contradiction.

Now, if we estimate every $1\left[d_i + d_{i+1} > d_{i-1} + 8\right] + 1\left[d_{i-1} + d_i > d_{i+1} + 8\right]$ term by 1 and ignore the terms $1\left[d_1 \geq d_2\right], 1\left[d_{n-1} \geq d_{n-2}\right]$, then the right hand side of (4.2) is at least

$$
\text{MG}(S) \geq \frac{2(n-3) + (n-4)}{3(n-3)} = 1 - \frac{1}{3(n-3)}.
$$

As it turns out, we can slightly improve this bound. First, observe that if there is an index $i$ such that both indicators $1\left[d_i + d_{i+1} > d_{i-1} + 8\right] = 1\left[d_{i-1} + d_i > d_{i+1} + 8\right] = 1$, then $\text{MG}(S) \geq 1 - \frac{1}{3(n-3)} + \frac{2}{3(n-3)} + \frac{1}{3(n-3)} = 1 + \frac{1}{3(n-3)}$. The latter immediately implies all three statements of the lemma. Therefore, it suffices to consider the case when all inequalities (4.3) are tight.

Second, we observe that if $d_{i-1} \leq d_i \leq d_{i+1}$ for any $2 \leq i \leq n-1$, then $1\left[d_i + d_{i+1} > d_{i-1} + 8\right] = 1\left[d_{i-1} + d_i > d_{i+1} + 8\right] = 1$, which we assumed to be impossible. I.e., the sequence $d$ does not have any internal $(1 < i < n-1)$ local maximums. This means that the sequence $d$ is either

1. strictly increasing, then $1\left[d_{n-1} \geq d_{n-2}\right] = 1$ and $\text{MG}(S) \geq 1$;
2. or strictly decreasing, then $1\left[d_1 \leq d_2\right] = 1$ and $\text{MG}(S) \geq 1$;
3. or strictly decreasing and then strictly increasing, then $1\left[d_{n-1} \geq d_{n-2}\right] = 1\left[d_1 \leq d_2\right] = 1$ and $\text{MG}(S) \geq 1 + \frac{1}{3(n-3)}$. This implies all three statements of the Lemma.

To conclude the proof, we note that $\text{MG}(S) \geq 1$, which implies the first statement of the Lemma. Moreover, we have $\text{MG}(S) \geq 1 + \frac{1}{3(n-3)}$, unless $d$ is a strictly monotone sequence, which implies the second statement of the Lemma. Finally, if $d_i + d_{i+1} > d_{i+2} + 8$ (or $d_i + 8 < d_{i+1} + d_{i+2}$) and $d$ is strictly increasing (decreasing) sequence, i.e., $d_i < d_{i+1} < d_{i+2}$ (or $d_i > d_{i+1} > d_{i+2}$), then
\[ d_i + d_{i+1} > d_{i-1} + 8 \] implies \( d_i + d_{i+1} > d_{i-1} + 8 \) and \( \text{MG}(S) \geq 1 + \frac{1}{d(n-3)} \), which concludes the proof of the third part of Lemma 4.1.

Now, if we use Mono-Gaps strategy with probability \( 1 - O \left( \frac{1}{n} \right) \), we can ensure that sequence \( d \) has a nice structure, i.e., \( d \) is monotone and does not satisfy the Fibonacci-like inequality in the third point of Lemma 4.1. Indeed, if the expected reward of Mono-Gaps is at least \( 1 + \Omega \left( \frac{1}{n} \right) \), we already have the expected reward to be higher than that of the random guessing regardless of what other strategy we use with probability \( O \left( \frac{1}{n} \right) \). Thus, if \( d \) is monotone and does not satisfy the Fibonacci-like inequality, then we use Mono-Gaps strategy with probability \( 1 + \Omega \left( \frac{1}{n} \right) \) and use the other strategy with probability \( O \left( \frac{1}{n} \right) \) to amplify the Fibonacci-like guarantee from Lemma 4.1.

In the following, we want to amplify the Fibonacci-like guarantee from Lemma 4.1 to much stronger condition that \( d_{i+1} \geq C \cdot d_i \) for a large constant \( C \), every \( i \in [n-1] \) (analogously \( d_i \geq C \cdot d_{i+1} \) for the decreasing \( d \)). To do this, we introduce our next strategy Exp-Gaps\((n, \tilde{S})\). This strategy has an additional parameter \( \ell \in [6] \) which we call a level of Exp-Gaps. We are going to run Exp-Gaps at every level \( \ell \), with diminishing in \( \ell \) probability.

**Algorithm 2: Exp-Gaps\((\ell, n, \tilde{S})\)**

Set level constants at \( L_1 = 2, L_2 = 4, L_3 = 16, L_4 = 225, L_5 = 42374, L_6 = 2^{21} \).

Let \( I = \{1, 2, n\} \) (or \( I = \{1, n-1, n\} \) when \( g \) is decreasing).

If \( g \) is increasing (or decreasing) then

- \( I \leftarrow I \cup \{i \mid n-1 \geq i \geq 3, \ g_{i-1} \geq L_\ell \cdot g_{i-2} + 2 \cdot L_\ell + 2\} \)
- \( I \leftarrow I \cup \{i \mid n-2 \geq i \geq 2, \ g_{i-1} \geq L_\ell \cdot g_i + 2 \cdot L_\ell + 2\} \) when \( g \) is decreasing.

Return \( i \sim \text{Uni}(I) \).

We say that an increasing (decreasing) \( d \) satisfies level-\( \ell \) condition for \( \ell \in [6] \) if and only if \( d_{i+1} \geq L_\ell \cdot d_i \) (\( d_i \geq L_\ell \cdot d_{i+1} \)) for every \( i \in [n-2] \). We also introduce the level-0 condition which just refers to the Fibonacci-like condition from Lemma 4.1.

\[ \forall i \in [n-3] \quad d_i + d_{i+1} \leq d_{i+2} + 8 \quad \text{(or } d_i + 8 \geq d_{i+1} + d_{i+2} \text{ for decreasing } d) \]

**Lemma 4.2** If an increasing sequence \( d \) satisfies level-(\( \ell - 1 \)) condition, then the expected reward \( \text{EG}(\ell) \) of Exp-Gaps\((\ell, n, \tilde{S})\) is at least

1. \( \text{EG}(\ell) \geq 1; \)

2. If \( d \) violates level-\( \ell \) condition, then \( \text{EG}(\ell) \geq 1 + \frac{n-3}{n(n-1)} \).

**Proof:** We first observe the following useful property of the set \( I \) in the Exp-Gaps strategy.

**Claim 4.1** \( \forall i \in [n], \text{ when } s_i \text{ is deleted from } S, \text{ then } i \in I(\tilde{S}) \) for any adversarial choice of \( \tilde{S} \).

**Proof:** The case when \( i \in \{1, 2, n\} \) is trivial according to our algorithm. We consider the case when \( s_i \) is deleted from \( S \) for a given \( i \notin \{1, 2, n\} \). Then \( g_{i-2} = \tilde{s}_{i-1} - \tilde{s}_{i-2} \) and \( g_{i-1} = \tilde{s}_{i+1} - \tilde{s}_{i-1} \) satisfy \( |g_{i-2} - d_{i-2}| \leq 2 \) and \( |g_{i-1} - d_{i-1} - d_i| \leq 2 \). We consider two cases.

1. When \( \ell = 1 \). The level-0 condition implies \( d_{i-1} + d_{i-2} \leq d_i + 8 \). Then we have,

\[ g_{i-1} \geq d_{i-1} + d_i - 2 \geq 2d_{i-1} + d_{i-2} - 10 \geq 2d_{i-2} + 10 \geq 2g_{i-2} + 6, \]

where the third inequality follows from the fact that \( d_{i-1} \geq d_{i-2} \geq 20 \).
2. When \( \ell > 1 \). The level-\((\ell - 1)\) condition implies \( L_{\ell-1}^2 \cdot d_{i-2} \leq L_{\ell-1} \cdot d_{i-1} \leq d_i \). Hence

\[
g_{i-1} \geq d_{i-1} + d_i - 2 \geq (L_{\ell-1}^2 + L_{\ell-1}) \cdot d_{i-2} - 2 \geq L_\ell \cdot (d_{i-2} + 2) + 2L_\ell + 2 \geq L_\ell \cdot g_{i-2} + 2L_\ell + 2.
\]

Here, the third inequality holds since \( L_\ell \leq \frac{(L_{\ell-1}^2 + L_{\ell-1})^{20-4}}{d_{i-2}+4} \leq \frac{(L_{\ell-1}^2 + L_{\ell-1})^{d_{i-2}+4}}{d_{i-2}+4} \), and according to the choice of the constants \( L = \langle 2, 4, 16, 225, 42374, 2^{21} \rangle \).

In both cases, the algorithm adds \( i \) to \( I(\tilde{S}) \), since \( g_{i-1} \geq L_\ell \cdot g_{i-2} + 2L_\ell + 2 \).

Now, we prove the first statement of the lemma. Note that when \( s_i \) is deleted from \( S \) for any \( i \in [n] \), then \( i \in I(\tilde{S}) \) and the expected reward of \( \text{Exp-Gaps}(\ell, n, \tilde{S}) \) is at least \( \frac{1}{|I(\tilde{S})|} \geq \frac{1}{n} \).

\[
\text{EG}(S) = \sum_{i \in [n]} \mathbb{E} \left[ \text{EG}(\tilde{S}) \cdot 1 \left[ s_i \text{ is deleted} \right] \right] \geq \sum_{i \in [n]} \frac{1}{n} = 1.
\]

Next, if \( d \) violates level-\( \ell \) condition then there are cases when \( I(\tilde{S}) \) has less than \( n \) elements.

**Claim 4.2** If \( L_\ell \cdot d_{j-1} > d_j \) for \( n > j > 1 \), then

1. for every deletion of \( i \geq j + 2 \) and every \( \tilde{S} \) we have \( j + 1 \notin I(\tilde{S}) \);

2. for every deletion of \( i \leq j - 2 \) and every \( \tilde{S} \) we have \( j \notin I(\tilde{S}) \);

**Proof:** We only prove the first statement, as the second statement only differs by a shift of indexes. We have \( g_j \leq d_j + 2 < L_\ell \cdot d_{j-1} + 2 \leq L_\ell \cdot (g_{j-1} + 2) + 2 = L_\ell \cdot g_{j-1} + 2L_\ell + 2 \), since \( g_j \leq d_j + 2 \) and \( d_{j-1} \leq g_{j-1} + 2 \). I.e., \( j + 1 \notin I(\tilde{S}) \) when \( i \geq j + 2 \) was deleted. \( \Box \)

Now, suppose there exists a \( 2 \leq j \leq n - 1 \) with \( L_\ell \cdot d_{j-1} > d_j \). Then for every \( i \in P \overset{\text{def}}{=} [n] \setminus \{j - 1, j, j + 1\} \), when \( s_i \) is deleted from \( S \), the corresponding \( I(\tilde{S}) \) has size at most \( n - 1 \), which results in an improved performance of our algorithm. Namely,

\[
\text{EG}(S) = \sum_{i \in [n]} \mathbb{E} \left[ \text{EG}(\tilde{S}) \cdot 1 \left[ s_i \text{ is deleted} \right] \right] \geq \sum_{i \in P} \frac{1}{n - 1} + \sum_{i \notin P} \frac{1}{n} \geq 1 + \frac{n - 3}{n^2 - n}.
\]
Finally, we present our recursive guessing algorithm.

**Algorithm 3: Guess** \((n, \tilde{S})\)

```plaintext
if \(n = 3\) then
    Run the algorithm in the warm-up.
else
    With probability \(1 - \frac{1}{6n}\), return Mono-Gaps\((n, \tilde{S})\);
    For each \(\ell \in [6]\), with probability \(\frac{1}{(6n)^\ell} - \frac{1}{(6n)^{\ell+1}}\), return Exp-Gaps\((\ell, k, \tilde{S})\);
    With remaining probability \(\frac{1}{(6n)^7}\), let \(\tilde{T} = \{\tilde{t}_i \triangleq \lceil \log_2 g_i \rceil \}_{i \in [n-2]}\).
    if \((t_i)_{i \in [n-2]}\) is increasing (or decreasing) then
        return \(\text{Guess}(n-1, \tilde{T}) + 1\) (or \(\text{Guess}(n-1, \text{Reverse}(\tilde{T}))\))
    else
        return \(i \sim \text{Uni}\{1, 2, \ldots, n\}\)
end
```

*The \text{Reverse} function reverses the descending vector \(\tilde{S}\) to ascending.*

**Lemma 4.3** If an increasing \(d\) sequence violates level-6 condition, the expected reward \(\text{Guess}\) of our algorithm is at least \(1 + \Omega\left(\frac{1}{n^7}\right)\).

**Proof:** By Lemma 4.1 when \(d\) violates the level-0 condition, we have

\[
\text{Guess} \geq \left(1 - \frac{1}{6n}\right) \cdot \text{MG} \geq \left(1 - \frac{1}{6n}\right) \cdot \left(1 + \frac{1}{3(n-3)}\right) \geq 1 + \Omega\left(\frac{1}{n}\right).
\]

Otherwise, suppose \(d\) satisfies level-\((\ell - 1)\) condition while violates level-\(\ell\) condition. It must also satisfy level-\(j\) conditions for all \(j \leq \ell - 2\). By Lemma 4.1 and 4.2, Mono-Gaps and every Exp-Gaps with level at most \(\ell - 1\) give an expected reward of 1. Moreover, Exp-Gaps\((\ell)\) achieves an expected reward of \(1 + \frac{n-3}{n(n-1)}\). Therefore, the expected gain of our algorithm is

\[
\text{Guess} \geq \left(1 - \frac{1}{(6n)^\ell}\right) + \left(\frac{1}{(6n)^\ell} - \frac{1}{(6n)^{\ell+1}}\right) \cdot \text{EG}(\ell) \\
\geq \left(1 - \frac{1}{(6n)^\ell}\right) + \left(\frac{1}{(6n)^{\ell+1}}\right) \cdot \left(1 + \frac{n-3}{n(n-1)}\right) \geq 1 + \Omega\left(\frac{1}{n^{\ell+1}}\right).
\]

With the above lemma, when \(d\) violates the level-6 condition, the expected reward of our algorithm is \(1 + \Omega\left(\frac{1}{n^7}\right)\), which is better than the stated bound of Theorem 4.1.

In the remainder of the proof, we focus on the case when \(d\) satisfies level-6 condition. Without loss of generality, we consider the case when \(d\) is increasing and \(d_i \geq 2^{21} \cdot d_{i-1}\) for every \(i \geq 2\).

We construct an instance \(T = \{t_i \triangleq [\log_2 d_i]\}_{i \in [n-1]}\) of \((n-1)\) numbers. Note that the largest number of \(T\) is at most \(\log N\). Let each \(i\) be deleted with probability \(q_i \triangleq \begin{cases} p_1 + p_2 & i = 1 \\ p_i + p_{i+1} & i \geq 2 \end{cases} \).

First of all, we verify that the instance satisfies the technical assumption.

**Claim 4.3** For every \(2 \leq i \leq n - 1\), we have \(t_i - t_{i-1} \geq 20\).
Proof: \( t_i - t_{i-1} = \lfloor \log_2 d_i \rfloor - \lfloor \log_2 d_{i-1} \rfloor \geq \log_2 d_i - \log_2 d_{i-1} - 1 \geq \log_2 L_6 - 1 \geq 20. \)

This claim is the reason why we needed to apply multiple levels of **Exp-Gaps**.

Next, we construct a correspondence between the perturbed guessing game on \( T \) and the recursive part of the algorithm, where we guess \( \text{Guess}(n-1, T) + 1 \). Consider the following two cases:

- The case when \( i = 1, 2 \) is deleted from \( S \), which happens with probability \( p_1 + p_2 = q_1 \), corresponds to the case when 1 is deleted from \( T \). We verify that after the deletion of 1 from \( T \), all other numbers are perturbed by at most 1.
  - When \( i = 1 \), we have \( |g_i - d_{i+1}| \leq 2 \) for every \( i \in [n-2] \). Thus
    \[
    |\tilde{t}_i - t_{i+1}| = |\lfloor \log_2 g_i \rfloor - \lfloor \log_2 d_{i+1} \rfloor| \leq |\lfloor \log_2 (d_{i+1} + 2) \rfloor - \lfloor \log_2 d_{i+1} \rfloor| \\
    \leq |\lfloor \log_2 (2 \cdot d_{i+1}) \rfloor - \lfloor \log_2 d_{i+1} \rfloor| = 1.
    \]
  - When \( i = 2 \), we have \( |g_1 - d_{1} - d_2| \leq 2 \) and \( |g_i - d_{i+1}| \leq 2 \) for every \( 2 \leq i \leq n-2 \). Thus,
    \[
    |\tilde{t}_1 - t_2| = |\lfloor \log_2 g_1 \rfloor - \lfloor \log_2 d_2 \rfloor| \leq |\lfloor \log_2 (d_1 + d_2 + 2) \rfloor - \lfloor \log_2 d_2 \rfloor| \\
    \leq |\lfloor \log_2 (2 \cdot d_2) \rfloor - \lfloor \log_2 d_2 \rfloor| = 1.
    \]

The difference between \( \tilde{t}_i \) and \( t_{i+1} \) for \( i \geq 2 \) is the same as the first case.

- The case when \( i > 2 \) is deleted from \( S \), which happens with probability \( p_i = q_{i-1} \), corresponds to the case when \( i - 1 \) is deleted from \( T \). Again, we verify that after the deletion of 1 from \( T \), all other numbers are perturbed by at most 1. Observe that in this case, \( |g_j - d_j| \leq 2 \) for \( j \leq i-2 \); \( |g_{i-1} - d_{i-1} - d_i| \leq 2 \); and \( |g_j - d_{j+1}| \leq 2 \) for \( j \geq i \).
  - For \( j \leq i-2 \), \( |\tilde{t}_j - t_j| = |\lfloor \log_2 g_j \rfloor - \lfloor \log_2 d_j \rfloor| \leq |\lfloor \log_2 (d_j + 2) \rfloor - \lfloor \log_2 d_j \rfloor| \leq 1. \)
  - For \( j = i-1 \),
    \[
    |\tilde{t}_{i-1} - t_i| = |\lfloor \log_2 g_{i-1} \rfloor - \lfloor \log_2 d_{i-1} \rfloor| \leq |\lfloor \log_2 (d_{i-1} + d_i + 2) \rfloor - \lfloor \log_2 d_i \rfloor| \\
    \leq |\lfloor \log_2 (2 \cdot d_i) \rfloor - \lfloor \log_2 d_i \rfloor| \leq 1.
    \]
  - For \( j \geq i \), \( |\tilde{t}_j - t_{j+1}| = |\lfloor \log_2 g_j \rfloor - \lfloor \log_2 d_{j+1} \rfloor| \leq |\lfloor \log_2 (d_j + 2) \rfloor - \lfloor \log_2 d_{j+1} \rfloor| \leq 1.

When \( s_i \) is deleted from \( S \) for \( i = 1, 2 \), it corresponds to the same deletion of \( t_1 \) from \( T \). According to our algorithm, we will consistently guess 2 to \( S \) if the recursive algorithm makes a guess of 1 to \( T \). Though the probability of guessing correctly will be \( p_2 \leq q_1 = p_1 + p_2 \), the expected reward will be scaled up proportionally. When \( s_i \) is deleted from \( S \) for \( i > 2 \), the rewards in both games are the same.

Therefore, the expected reward of the recursive part of our algorithm equals:

\[
\text{Recursive}(S) = \text{Guess}(T) \geq 1 + \frac{1}{(6(n-1))^{\frac{n}{n-1}}} \cdot \Omega \left( \frac{1}{\log^{(n-3)}(\log N)} \right) \\
= 1 + \frac{1}{(6n)^{\frac{n}{n-1}}} \cdot \Omega \left( \frac{1}{\log^{(n-2)} N} \right),
\]

where the inequality follows from the induction hypothesis and that the largest number in \( T \) is at most \( \log N \). Finally, by Lemma 4.1 and 4.2, we have that **Mono-Gaps** and **Exp-Gaps** of all levels
have expected reward at least 1 when \( d \) satisfies level-\( \ell \) condition. With a constant probability of 
\[
\frac{1}{(6n)^7} \] 
executing the recursive step, we achieve an expected reward of
\[
1 \cdot \left( 1 - \frac{1}{(6n)^7} \right) + \left( 1 + \frac{1}{(6n)^7} \right) \cdot \Omega \left( \frac{1}{\log(n-2) \cdot N} \right) \geq 1 + \frac{1}{(6n)^7} \cdot \Omega \left( \frac{1}{\log(n-2) \cdot N} \right).
\]

5 Maximum Guessing and Game of Googol

The universal construction from Section 3 applies for arbitrary online ordinal tasks such as the Game of Googol, albeit it is prohibitively large. As we know from Section 4 this inefficiency is unavoidable in general, however, it is still possible that certain online tasks admit much smaller constructions. We consider in this section one such task: a natural modification of the Die Guessing Game of Googol, albeit it is prohibitively large. As we know from Section 4 this inefficiency is unavoidable in general, however, it is still possible that certain online tasks admit much smaller constructions. We consider in this section one such task: a natural modification of the Die Guessing Game with an immediate connection to the game of Googol. We call it Maximum Guessing Game and give a construction of \( O(\frac{1}{\varepsilon})^{n-1} \) cardinal complexity that makes the advantage of any cardinal algorithm over the ordinal algorithms to be at most \( (1 + \varepsilon) \).

Maximum Guessing Game. Given \( n, N, \) the adversary (first player) chooses a set \( S \subset [N] \) of \( n \) distinct integers \( s_1 < s_2 < \cdots < s_n \). Then \( S_i \) is generated by deleting uniformly at random a single number from \( S \). The algorithm (second player) sees \( S_i \) and guesses whether the deleted number is the maximum in \( S \) or not, i.e., guesses whether \( i = n \) (“yes”), or \( i \neq n \) (“no”). If the “yes” guess \( i = n \) is correct, the algorithm’s reward is \( n \); if the “no” guess \( i \neq n \) is correct, the reward is \( \frac{n}{n-1} \). Otherwise, if the algorithm’s “yes” or “no” guess is incorrect, then the reward is 0.

This game shares the obvious common feature with the Game of Googol, that we only need to guess whether a number is the largest or not. We start with a simple observation that the expected reward of any ordinal algorithm for the maximum guessing game is 1.

Ordinal Algorithm. An ordinal algorithm cannot see the numbers of \( S_i \), i.e., its decision does not depend on the input. The guess \( i = n \) wins with probability \( \frac{1}{n} \) with the expected reward of 1 = \( \frac{1}{n} \); the guess \( i \neq n \) wins with probability \( \frac{n-1}{n} \) with the same expected reward of 1 = \( \frac{n}{n-1} \cdot \frac{n-1}{n} \). Therefore, any ordinal algorithm has expected reward of 1.

The main technical result of this section is an explicit construction of the distribution \( F(\varepsilon) \) that satisfies (5.1) below, which quickly leads (see Theorem 5.1) to the desired result. In (5.1), the random ordered sets \( S_n = (s_1, s_2, \ldots, s_{n-1}) \) and \( S_{\text{uni}} \) are subsets of \( S = (s_1, s_2, \ldots, s_n) \sim F(\varepsilon) \): \( S_n = (s_1, s_2, \ldots, s_{n-1}) \), and \( S_{\text{uni}} \) represent a random subset \( S_i = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n) \) with \( i \) drawn uniformly from \( \{1, 2, \ldots, n-1\} \).

**Lemma 5.1** For any \( \varepsilon > 0 \) and \( N = O \left( \frac{1}{\varepsilon} \right)^{n-1} \), there exists a distribution \( F(\varepsilon) \) of ordered \( n \)-element sets \( S \subset [N] \) such that 
\[
d_{TV}(S_n, S_{\text{uni}}) \leq \varepsilon
\]

We give the proof of Lemma 5.1 in the next subsection. This lemma immediately implies the desired result.

**Theorem 5.1** The cardinal complexity of the maximum guessing game is at most \( O \left( \frac{1}{\varepsilon} \right)^{n-1} \).

**Proof:** Let the integers \( S \) be sampled from the distribution \( F(\varepsilon) \) of Lemma 5.1. We show that the expected reward of any cardinal algorithm is at most \( 1 + \varepsilon \). Let \( \text{ALG} : \binom{[N]}{n-1} \to \{\text{yes, no}\} \) be
arbitrary cardinal algorithm. We use \( f(S) \) to denote the probability mass function of \( \mathcal{F}(\varepsilon) \). Then, the expected reward of \( \text{ALG} \) is as follows.

\[
\mathbb{E}[R(\text{ALG})] = \sum_{S \in \binom{[N]}{n}} f(S) \cdot \left( \Pr[s_n \text{ is deleted}] \cdot 1 \left[ \text{ALG}(S_n) = \text{yes} \right] \cdot n + \sum_{i \neq n} \Pr[s_i \text{ is deleted}] \cdot 1 \left[ \text{ALG}(S_i) = \text{no} \right] \cdot \frac{n}{n-1} \right)
\]

\[
= \sum_{S \in \binom{[N]}{n}} f(S) \cdot \left( 1 \left[ \text{ALG}(S_n) = \text{yes} \right] + \sum_{i \neq n} 1 \left[ \text{ALG}(S_i) = \text{no} \right] \cdot \frac{1}{n-1} \right)
\]

\[
= \sum_{\tilde{S} \in \binom{[N]}{n-1}} \left( 1 \left[ \text{ALG}(\tilde{S}) = \text{yes} \right] \cdot \sum_{S \in \binom{[N]}{n}} f(S) \cdot 1 \left[ S_n = \tilde{S} \right] + 1 \left[ \text{ALG}(\tilde{S}) = \text{no} \right] \cdot \sum_{S \in \binom{[N]}{n}} f(S) \cdot 1 \left[ S_i = \tilde{S} \right] \cdot \frac{1}{n-1} \right)
\]

\[
\leq \sum_{\tilde{S} \in \binom{[N]}{n-1}} \max \left( \Pr \left[ S_n = \tilde{S} \right], \Pr \left[ S_{\text{uni}} = \tilde{S} \right] \right) = 1 + d_{TV}(S_n, S_{\text{uni}}) \leq 1 + \varepsilon.
\]

5.1 Construction of the distribution \( \mathcal{F} \)

Our construction relies on a set of distributions \( \{L_i \overset{\text{def}}{=} \text{Uni}[N_i]\} \), where \( N_i = O \left( \frac{1}{\varepsilon} \right)^{i-1} \), and a distribution \( p \) over \( \text{Sym}(n-1) \), which is the stationary distribution of a Markov chain that we shall specify later. The construction of \( S \sim \mathcal{F}(\varepsilon) \) is as follows.

1. Consider equivalent representation of \( S \) as \((s_1, d_1, \ldots, d_{n-1})\), where \( d_i = s_{i+1} - s_i \).
2. Sample \( \sigma \in \text{Sym}(n-1) \) from \( p \).
3. Sample \( s_1 \in [N_n] \) from \( L_n \), and \( d_i \in [N_{\sigma(i)}] \) from \( L_{\sigma(i)} \) independently.

Cardinal complexity \( N \). We first calculate the cardinal complexity, i.e., the largest integer \( s_n \) used in \( S \sim \mathcal{F}(\varepsilon) \). For arbitrary \( \sigma \in \text{Sym}(n-1) \) used in the construction, we have

\[
s_n = s_1 + \sum_{i \in [n-1]} d_i \leq N_n + \sum_{i \in [n-1]} N_{\sigma(i)} = \sum_{i \in [n]} N_i \leq \sum_{i \in [n]} O \left( \frac{1}{\varepsilon} \right)^{i-1} = O \left( \frac{1}{\varepsilon} \right)^{n-1}.
\]

Total variation distance. Next, we prove the desired bound on total variation distance. We refer to the index \( i \) of \( L_i \) as a gap level. First, notice that by Lemma 2.4, the sum of two uniformly distributed random variables drawn from different levels \( L_i, L_j \), is close to the random variable
drawn from the higher level $L_{\text{max}(i,j)}$. I.e., two consecutive gap levels $i < j$ effectively merge into level $j$ if we delete a number between them.

**Lemma 5.2** Suppose $\ell_i \sim L_i, \ell_j \sim L_j$ for $1 \leq i < j \leq n$. Then we have

$$d_{\text{TV}}(\ell_i + \ell_j, \ell_j) \leq \varepsilon.$$

**Proof:** Since $0 \leq N_i \leq N_j - 1$ we can apply Lemma 2.4 and get $d_{\text{TV}}(\ell_i + \ell_j, \ell_j) \leq \frac{N_i}{N_j} \leq \varepsilon$. ■

We note that a naive approach of sampling $s_1 \sim L_n$ and $d_i \sim L_i$ for all $i \leq n - 1$ without a random permutation $\sigma$ fails. Indeed, in this case the levels $i$ and $i - 1$ become a single level after deletion of $s_i$. Thus the algorithm can determine levels of every gap, find the missing level $(i - 1)$, and guess with high probability whether $i = n$, or $i \neq n$. The algorithm still can tell which level is missing even when we use a random permutation of levels $\sigma \sim p$. But now the exposure of a missing level is not necessarily problematic, as the algorithm does not know where in the order $\sigma$ the missing gap level was placed.

We now formalize the intuition of “merging levels” of distributions $L_i$ and $L_j$ as an algebraic property of the permutation distribution $\sigma \sim p$. Given a permutation $\sigma = (\sigma(1), \sigma(2), \ldots, \sigma(n-1))$, we define the following operation corresponding to the deletion of the $i$-th number $s_i$, i.e., the merge of the gaps $d_{i-1}$ and $d_i$.

- $s_1$ is deleted: $\sigma_1 \overset{\text{def}}{=} (\sigma(2), \sigma(3), \ldots, \sigma(n-1))$
- $s_i$ is deleted: $\sigma_i \overset{\text{def}}{=} (\sigma(1), \ldots, \sigma(i-2), \max(\sigma(i-1), \sigma(i)), \sigma(i+1), \ldots, \sigma(n-1)), \quad i \in [2, n-1]$
- $s_n$ is deleted: $\sigma_n \overset{\text{def}}{=} (\sigma(1), \sigma(2), \ldots, \sigma(n-2))$

In other words, $\sigma_i$ corresponds to the $(n-2)$ gap levels of the observed set after the deletion of the $i$-th number from $S$ ignoring small error term of $\varepsilon$ from Lemma 5.2.

To specify the distribution $p$ of the random permutation $\sigma$, we consider a $(n-1)! \times (n-1)!$ matrix $M$ indexed by $\sigma, \sigma' \in \text{Sym}(n-1)$:

$$M(\sigma, \sigma') \overset{\text{def}}{=} \frac{|\{i \in [n-1] \mid \sigma_i = \sigma'_i\}|}{n-1}.$$

We can view $M$ as a transition matrix on the state space $\text{Sym}(n-1)$ and let $p$ be its stationary distribution, i.e., $p \cdot M = p$. We give a concrete example below to help understand the idea.
Example. Consider the case when $n = 4$, the transition matrix $M$ is given below:

$$
M = \begin{pmatrix}
12(3) & 13(2) & 21(3) & 23(1) & 31(2) & 32(1) \\
123 & 0 & 1/3 & 0 & 2/3 & 0 \\
132 & 0 & 1/3 & 0 & 0 & 2/3 \\
213 & 0 & 1/3 & 0 & 2/3 & 0 \\
231 & 0 & 0 & 0 & 1/3 & 2/3 \\
312 & 1/3 & 0 & 0 & 0 & 2/3 \\
321 & 0 & 0 & 1/3 & 0 & 1/3 & 1/3 \\
\end{pmatrix}
$$

E.g., consider the first row of the matrix when $\sigma = (1, 2, 3)$. When the first or the second number $s_1, s_2$ is deleted, the gaps observed by the algorithm would be of levels 2 and 3; when the third number $s_3$ is deleted, the gaps observed by the algorithm would be of levels 1 and 3. We don’t consider deletion of the last number $s_n$. The stationary distribution of the above transition matrix is

$$p = \left( \frac{5}{66}, \frac{6}{66}, \frac{7}{11}, \frac{2}{22}, \frac{7}{22} \right)$$

Let $\sigma_{\text{uni}}$ be a random permutation $\sigma_i$ where $\sigma \sim p$ and $i \sim \text{Uni}[n - 1]$. It turns out that $\sigma_{\text{uni}}$ is indistinguishable from $\sigma_n$, for $\sigma \sim p$.

Claim 5.1 For $\sigma \sim p$ and $i \sim \text{Uni}[n - 1]$, the distributions of $\sigma_{\text{uni}}$ and $\sigma_n$ are the same.

Proof: Fix an arbitrary $\mu \in \text{Sym}(n - 1)$, we have

$$
\Pr[\sigma_{\text{uni}} = \mu] = \Pr[\sigma_i = \mu] = \sum_{\sigma \in \text{Sym}(n-1)} p_{\sigma} \cdot \sum_{i \in [n-1]} \frac{1[\sigma_i = \mu]}{n-1} = \sum_{\sigma \in \text{Sym}(n-1)} p_{\sigma} \cdot M(\sigma, \mu) = p_\mu = \Pr[\sigma = \mu] = \Pr[\sigma_n = \mu],
$$

where the third and forth equalities follow from the definition of $M$ and $p$.

Informally, the Claim 5.1 means that after we ignore small error term of $\epsilon$ from Lemma 5.2, it will be impossible to tell apart only by looking at the gaps whether we deleted $s_i$ with $i \sim \text{Uni}[n-1]$, or if we deleted $s_n$.

We are ready now to conclude the proof of Lemma 5.1. Consider a distribution $T(\sigma_i)$ associated with a fixed permutation $\sigma_i$ over ordered sets $T = (t_1, \ell_1, \ell_2, \ldots, \ell_{n-2})$, where

$$t_1 \sim L_n, \quad \text{and} \quad \ell_j \sim L_{\sigma_i(j)}, \quad \forall j \in [n-2].$$

Similar to $S_{\text{uni}}$ and $S_n$ we define $T_{\text{uni}} = T(\sigma_{\text{uni}})$ and $T_n = T(\sigma_n)$ for $\sigma \sim p$. Since random permutations $\sigma_{\text{uni}}$ and $\sigma_n$ have identical distributions by Claim 5.1, we have

$$0 = d_{\text{TV}}(T_{\text{uni}}, T_n) = d_{\text{TV}}(T_{\text{uni}}, S_n). \quad (5.2)$$

Furthermore, for any fixed $\sigma \in \text{Sym}(n - 1)$ and $i \in [n-1]$ the distributions of the random set $S_{\sigma_i}$ (we denote it $S_{\sigma_i}^\sigma$ to indicate that $\sigma$ is fixed) and $T(\sigma_i)$ are very similar.

Lemma 5.3 For any fixed $\sigma$ and $i \in [n - 1]$, we have $d_{\text{TV}}(S_{\sigma_i}^\sigma, T(\sigma_i)) \leq \epsilon$. 

25
Proof: We consider two cases based on the value of $i$. For $i = 1$ we have,

$$d_{TV}(S^i_{1}, T(\sigma_{-1})) = d_{TV}((s_1 + d_1, d_2, \ldots, d_{n-1}), (t_1, \ell_1, \ldots, \ell_{n-2})) = d_{TV}(s_1 + d_1, t_1) \leq \varepsilon,$$

where the second equality holds since $d_j, \ell_{j-1} \sim L_{\sigma(j)}$ and all random variables $s_1, \{d_i\}_{j=1}^{n-1}, t_1, \{\ell_j\}_{j=1}^{n-2}$ are mutually independent; the inequality holds by Lemma 5.2 for $L$. The second inequality follows from Lemma 2.2 and the fact that 

The first inequality is the triangle inequality for TV distance (Lemma 2.1). The equality follows despite that the size of the distribution

Remark 5.1 One can also strengthen the guarantee of Lemma 5.1 to hold not only for $S$ but also for two uniform distributions over sets of size $n$ and $\frac{n}{e}$, and the gaps from simple uniform distributions $d_i \sim L_{\sigma(i)}$ for different $\sigma, i$. The last inequality follows from Lemma 5.3.

Finally, we conclude the proof of the bound on the total variation distance

$$d_{TV}(S_{uni}, S_{n}) \leq d_{TV}(S_{uni}, T_{uni}) + d_{TV}(T_{uni}, T_{n}) + d_{TV}(T_{n}, S_{n}) = d_{TV}(S_{uni}, T_{uni})$$

$$\leq \sum_{\sigma \in Sym(n-1)} \sum_{t \in [n-1]} \frac{p_{\sigma}}{n-1} \cdot d_{TV}(S^i_{\sigma}, T(\sigma_i)) \leq \max_{\sigma \in Sym(n-1)} \frac{d_{TV}(S^i_{\sigma}, T(\sigma_i))}{\varepsilon} \leq \varepsilon.$$

The first inequality is the triangle inequality for TV distance (Lemma 2.1). The equality follows from [5.2]. The second inequality follows from Lemma 2.2 and the fact that $S_{uni}, T_{uni}$ are mixtures of $S^i_{\sigma}, T(\sigma_i)$ for different $\sigma, i$. The last inequality follows from Lemma 5.3.

Remark 5.1 Despite that the size of the distribution $F(\varepsilon)$ is exponential in $n$, one can still sample efficiently (in time polynomial in $n$ and $\frac{1}{\varepsilon}$) from it. Indeed, to construct a random $\sigma \sim p$ we can start from an arbitrary permutation $\pi \in Sym(n-1)$ and implement polynomially many random steps in the Markov chain $M$. Given the structure of $M$ its mixing time cannot be large, and, therefore, we should quickly converge to $M$’s stationary distribution $p$. Next, we sample $s_1 \sim L_n$ and the gaps from simple uniform distributions $d_i \sim L_{\sigma(i)}$.

5.2 Implications for the Game of Googol

The maximum guessing game shares the same task of identifying the maximum element with the well studied game of googol. Besides this obvious connection, our distribution $F$ from the previous section when used for the game of googol turns out to have some interesting implications which we will (informally) discuss here.

First, similar to how we strengthen the statement of Lemma 3.1 to Lemma 3.2 in Section 3 one can also strengthen the guarantee of Lemma 5.1 to hold not only for $S_n$ and $S_{uni}$, but also for two uniform distributions over sets of size $k$ (for all $k \leq n$) that (i) always contain the largest number $s_n$, and (ii) never contain $s_n$. This means that in the game of googol any cardinal algorithm does not get much advantage over the ordinal algorithm in predicting whether the currently largest number $s_{\pi(t)}$ at any step $t$ is the actual maximum.

Second, the cardinal algorithms cannot really improve on the ordinal algorithm at the first $t \in [1, n/e]$ and last $t \in [n/2, n]$ steps. Indeed, any deviation from the best ordinal algorithm at those steps implies that the cardinal algorithm must make a better prediction in the maximum guessing game than the ordinal algorithm for each step $t \in [1, n/e] \cup [n/2, n]$ when $s_{\pi(t)}$ is a current maximum. Namely, for $t \in [1, n/e]$ the cardinal algorithm needs to be correct (that $s_{\pi(t)}$ is the global maximum) with probability at least $1/e$ while the maximum guessing probability of the ordinal algorithm for $t \in [1, n/e]$ is less than $1/e$. For $t \in [n/2, n]$, the cardinal algorithm must
have a success probability more than \( t/n \geq 1/2 \) of the ordinal algorithm. Thus, if it decides to skip \( s_{\pi(t)} \), then \( s_{\pi(t)} \) must not be the global maximum with probability at least \( 1/2 \).

Unfortunately, for \( t \in [n/e, n/2] \) previous intuition does not work, since a deviation of the cardinal algorithm from the ordinal algorithm in the game of googol does not necessarily imply better prediction power for the cardinal algorithm in the maximum guessing game for such \( t \). E.g., if the cardinal algorithm could somehow tell that the current maximum \( s_{\pi(t)} \) is among top two values of \( S \) (and knows that \( s_{\pi(t)} \) is the maximum with probability \( t/n \)), then it would be better off by skipping \( s_{\pi(t)} \) now and winning with probability \( 1 - t/n \) later.

This only shows that the discussed approach of strengthening Lemma 5.1 does not immediately work. The distribution \( F \) might have other useful properties that could allow for a different proof approach. In fact, given the partial success of our construction we conjecture that the distribution \( F \) already works, i.e., it makes cardinal algorithms not better than ordinal algorithms for the game of googol.

6 Catalan Construction for Die Guessing

In this section, we focus on the die guessing game. Consider the setting that adversary selects an ordered set of \( n \) distinct elements and one of the numbers will be deleted uniformly at random. The algorithm observes the remaining \( n - 1 \) numbers and guesses the rank of the deleted number. If the algorithm guesses correctly, it receives a reward of \( n \).

Without looking at the \( n - 1 \) numbers, any guessing strategy would lead to the expected reward of 1. In previous sections, we have proved that the cardinal complexity of \( N = O \left( \frac{1}{\varepsilon} \right) \uparrow \uparrow (n - 1) \) is sufficient and necessary (for a constant \( n = O(1) \)) so that the best cardinal algorithm has the expected reward of \( 1 + \varepsilon \).

We relax the \( 1 + \varepsilon \) constraint and ask how large the support size has to be, so that the best cardinal algorithm has the expected reward of at most \( O(1) \).

**Theorem 6.1** For any constant \( \delta > 0 \), there exists \( N = O(n) \) and a distribution of ordered \( n \)-element sets \( S \subseteq [N] \), such that no algorithm achieves an expected reward larger than \( 2 + \delta \).

**Catalan Construction.** We give an explicit construction by providing the probability density function of each ordered set \( S = (s_1 < s_2 < \ldots < s_n) \subseteq [N] \). For notation simplicity, let \( s_0 \equiv 0 \) and \( s_{n+1} \equiv N + 1 \). We are going to use Catalan numbers that satisfy the following conditions:

\[
C_1 \equiv 1 \quad \text{and} \quad C_{\ell+1} \equiv \sum_{k=1}^\ell C_k \cdot C_{\ell+1-k};
\]

\[
\iff C_\ell \equiv \left( \frac{2(\ell - 1)}{\ell - 1} \right) \left( \frac{2(n + 1)}{n} \right) = \frac{(2(\ell - 1))!}{(\ell - 1)! \cdot \ell!}.
\]

We let the probability density function of each ordered-set \( S \) to be

\[
f(S) = \frac{\prod_{i=1}^{n+1} C_{s_i - s_{i-1}}}{C(N, n)}, \quad \text{with } C(N, n) = \sum_\ell \left( \prod_{i=1}^{n+1} C_{\ell_i} \right) \text{ being the normalizing constant,}
\]

\[
\ell \in \mathbb{N}^{n+1} \text{ runs over all partitions of } N + 1 = \sum_{i=1}^{n+1} \ell_i \text{ with } \ell_i \geq 1.
\]

We prove the following mathematical fact that shall be used later in our proof.
Claim 6.1 $C(N, n) = \frac{(n+1)(2N-n)!}{(N+1)!N!}$. 

**Proof:** Fix $\ell_1, \ell_2, \ldots, \ell_{n-1}$ and consider all possible choices of $\ell_n, \ell_{n+1}$ with $\ell_n + \ell_{n+1} = \ell'_n \overset{\text{def}}{=} N + 1 - \sum_{i=1}^{n-1} \ell_i$. By the definition of Catalan number, we have that

$$\sum_{\ell_n + \ell_{n+1} = \ell'_n} C_{\ell_n} \cdot C_{\ell_{n+1}} = C_{\ell'_n}.$$ 

Thus, we have the following recursive formula for $C(N, n)$:

$$C(N, n) = \sum_{\ell_i \geq 1, \forall i \in [n-1], \sum_i \ell_i + \ell_n = N+1} \prod_{i=1}^{n-1} C_{\ell_i} \cdot C_{\ell_n} = \sum_{\ell_i \geq 1, \forall i \in [n-1], \sum_i \ell_i + \ell'_n = N+1} \prod_{i=1}^{n-1} C_{\ell_i} \cdot C_{\ell'_n} - \sum_{\ell_i \geq 1, \forall i \in [n-1], \ell_i = 1} \prod_{i=1}^{n-1} C_{\ell_i} = C(N-1,n-2).$$

Next, we use induction on $n$ to prove the claim. The base case is

$$C(N, 0) = C_{N+1} = \frac{(2N)!}{(N+1)! \cdot N!}$$

and

$$C(N, 1) = \sum_{k=1}^{N} C_k \cdot C_{N+1-k} = C_{N+1} = \frac{(2N)!}{(N+1)! \cdot N!} = \frac{2 \cdot (2N-1)!}{(N+1)! \cdot (N-1)!}.$$

Then, for $n \geq 2$, suppose the statement holds for $n-1, n-2$. Then we have

$$C(N, n) = C(N, n-1) - C(N-1, n-2) = \frac{n \cdot (2N-n+1)!}{(N+1)! \cdot (N-n+1)!} - \frac{(n-1) \cdot (2N-n)!}{N! \cdot (N-n+1)!}$$

$$= \frac{(2N-n)!}{(N+1)! \cdot (N-n+1)!} \cdot (n \cdot (2N-n+1) - (n-1) \cdot (N+1))$$

$$= \frac{(2N-n)!}{(N+1)! \cdot (N-n+1)!} \cdot (n+1) \cdot (N-n+1) = \frac{(n+1) \cdot (2N-n)!}{(N+1)! \cdot (N-n+1)!}.$$

This concludes the proof of the claim.

Next, we characterize the best guessing strategy for the Catalan construction and calculate its expected reward.

**Lemma 6.1** For the Catalan construction and for arbitrary observed set $\tilde{S} = (\tilde{s}_1 < \tilde{s}_2 < \cdots < \tilde{s}_{n-1})$, the best guessing algorithm is to guess arbitrary $i \in I \overset{\text{def}}{=} \{ j \mid \tilde{s}_j - \tilde{s}_{j-1} > 1 \}$ and the expected reward equals $\frac{\prod_{j=1}^{n} C_{\tilde{s}_j - \tilde{s}_{j-1}}}{C(N,n)}$, where $\tilde{s}_0 \overset{\text{def}}{=} 0$ and $\tilde{s}_n \overset{\text{def}}{=} N+1$.

**Proof:** Fix an arbitrary observed set $\tilde{S}$, it suffices to compare the posterior probability of $i$ being the index of the deleted number from $S$, i.e. $\tilde{S} = S \setminus \{s_i\}$. Notice that if $i \notin I$, the probability must be 0. For $i \in I$, we have

$$\Pr \left[ \tilde{S} = S \setminus \{s_i\} \right] = \sum_{\tilde{s}_{i-1} < s_i < \tilde{s}_i} \Pr \left[ S = (\tilde{s}_1, \ldots, \tilde{s}_{i-1}, s_i, \tilde{s}_i, \ldots, \tilde{s}_{n-1}) \right]$$

$$= \frac{1}{C(N,n)} \cdot \prod_{j \in [n] \setminus \{i\}} C_{\tilde{s}_j - \tilde{s}_{j-1}} \cdot \sum_{\tilde{s}_{i-1} < s_i < \tilde{s}_i} C_{\tilde{s}_i - \tilde{s}_{i-1}} \cdot C_{\tilde{s}_i - s_i} = \frac{1}{C(N,n)} \cdot \prod_{j \in [n]} C_{\tilde{s}_j - \tilde{s}_{j-1}}.$$
where $\tilde{s}_0 = 0, \tilde{s}_n = N + 1$ in the second equality and in the last inequality we use the definition of the Catalan number. Observe that the values of $\Pr[\tilde{S} = S \setminus \{s_i\}]$ are the same for all $i \in I$. Therefore, it makes no difference for an algorithm to guess an arbitrary index $i \in I$ and when we guess it correctly, we have a reward of $n$. Therefore, the expected reward of the best algorithm from seeing $\tilde{S}$ equals the following:

$$
E[\text{ALG} \cdot 1(\tilde{S} \text{ is observed})] = \Pr[\tilde{S} \text{ is observed}] \cdot \frac{n}{|I|}
$$

$$
= \frac{n}{|I|} \cdot \sum_{s \in [n]} \Pr[\tilde{S} = S \setminus \{s_i\} \text{ and } s_i \text{ is deleted}] = \frac{1}{|I|} \cdot \sum_{i \in I} \Pr[\tilde{S} = S \setminus \{s_i\}] = \frac{\prod_{j \in [n]} C_{\tilde{s}_j - \tilde{s}_{j-1}}}{C(N, n)},
$$

that concludes the proof of the lemma.

Finally, we conclude the proof of Theorem 6.1.

**Proof of Theorem 6.1:** By the above lemma, the expected reward of the best algorithm for the Catalan construction is:

$$
E[\text{ALG}] = \sum_{\tilde{S}} E[\text{ALG} \cdot 1(\tilde{S} \text{ is observed})] = \sum_{\tilde{S}} \frac{\prod_{j \in [n]} C_{\tilde{s}_j - \tilde{s}_{j-1}}}{C(N, n)} = \frac{C(N, n-1)}{C(N, n)}
$$

$$
= \frac{n \cdot (2N - n + 1)!}{(N+1)! \cdot (N - n + 1)!} \cdot \frac{(n+1) \cdot (2N-n)!}{(N+1)! \cdot (N-n)!} = \frac{n \cdot 2N - n + 1}{n+1} \cdot \frac{N-n+1}{N-n+1} \leq 2 + \delta,
$$

where the third equality follows from Claim 6.1 and the inequality holds for $N = \Omega(\frac{n}{\delta})$. This concludes the proof of the theorem.

**Remark 6.1** If we use $N = O(n^2)$, no algorithm achieves an expected reward larger than 2 for the Catalan construction. Moreover, this is the best possible constant we can have using the Catalan construction, since $\lim_{n \to \infty} \lim_{N \to \infty} \frac{n \cdot 2N - n + 1}{n+1} \cdot \frac{N-n+1}{N-n+1} = \lim_{n \to \infty} \frac{2n}{n+1} = 2$.

**References**

[1] Niv Buchbinder, Kamal Jain, and Mohit Singh. Secretary problems via linear programming. In IPCO, volume 6080 of *Lecture Notes in Computer Science*, pages 163–176. Springer, 2010.

[2] T.-H. Hubert Chan, Fei Chen, and Shaofeng H.-C. Jiang. Revealing optimal thresholds for generalized secretary problem via continuous LP: impacts on online K-item auction and bipartite K-matching with random arrival order. In SODA, pages 1169–1188. SIAM, 2015.

[3] José Correa, Paul Dütting, Felix Fischer, and Kevin Schewior. Prophet inequalities for iid random variables from an unknown distribution. In Proceedings of the 2019 ACM Conference on Economics and Computation, EC, pages 3–17. ACM, 2019.

[4] José R Correa, Andrés Cristi, Boris Epstein, and José A Soto. The two-sided game of gooogle and sample-based prophet inequalities. In SODA, pages 2066–2081. SIAM, 2020.

[5] E. Dynkin. The optimum choice of the instant for stopping a markov process. 1963.
[6] Tomer Ezra, Michal Feldman, Nick Gravin, and Zhihao Gavin Tang. Online stochastic max-weight matching: Prophet inequality for vertex and edge arrival models. In Péter Biró, Jason Hartline, Michael Ostrovsky, and Ariel D. Procaccia, editors, *EC ’20: The 21st ACM Conference on Economics and Computation*, Virtual Event, Hungary, July 13-17, 2020, pages 769–787. ACM, 2020.

[7] Moran Feldman and Rico Zenklusen. The submodular secretary problem goes linear. *SIAM J. Comput.*, 47(2):330–366, 2018.

[8] Thomas S. Ferguson. Who solved the secretary problem? *Statist. Sci.*, 4(3):282–289, 08 1989.

[9] Martin Gardner. *New Mathematical Diversions from Scientific American*, chapter 3, problem 3. Simon and Schuster, 1966. Reprint of the original column published in February 1960 with additional comments.

[10] Alexander V. Gnedin. A solution to the game of goool. *The Annals of Probability*, 22(3):1588–1595, 1994.

[11] Yijie Han. Deterministic sorting in $o(n \log \log n)$ time and linear space. *J. Algorithms*, 50(1):96–105, 2004.

[12] Yijie Han and Mikkel Thorup. Integer sorting in $0(n \sqrt{\log \log n})$ expected time and linear space. In *FOCS*, pages 135–144. IEEE Computer Society, 2002.

[13] Martin Hoefer and Bojana Kodric. Combinatorial secretary problems with ordinal information. In *ICALP*, volume 80 of *LIPIcs*, pages 133:1–133:14. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2017.

[14] Eric Lehman, F. Thomson Leighton, and Albert R Meyer. Mathematics for computer science. 2010.

[15] José A. Soto, Abner Turkieltaub, and Victor Verdugo. Strong algorithms for the ordinal matroid secretary problem. In *SODA*, pages 715–734. SIAM, 2018.