Symmetry and spectral properties for viscosity solutions of fully nonlinear equations

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Joint work with Fabiana Leoni and Filomena Pacella.
GOAL Obtain symmetry properties of solutions of fully nonlinear equations related to spectral properties of what, improperly, will be called the linearized operator.
GOAL Obtain symmetry properties of solutions of fully nonlinear equations related to spectral properties of what, improperly, will be called the linearized operator.

Question: which symmetry features of the domain and the operator are inherited by the viscosity solutions of the homogeneous Dirichlet problem

\begin{equation}
\begin{aligned}
- F(x, D^2 u) &= f(x, u) \quad \text{in } \Omega, \\
\quad u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\end{equation}

where \( \Omega \subset \mathbb{R}^n, n \geq 2 \), is a bounded domain and \( F \) is a fully nonlinear uniformly elliptic operator?
We are under the condition that $F$ is \textbf{Lipschitz continuous} in $x$ and uniformly elliptic i.e. $\forall \, x \in \Omega$

\[
\mathcal{M}^-_{\alpha, \beta}(M-N) \leq F(x, M) - F(x, N) \leq \mathcal{M}^+_{\alpha, \beta}(M-N) \quad , \quad M, N \in S_n ,
\]

where $\mathcal{M}^-_{\alpha, \beta}$ and $\mathcal{M}^+_{\alpha, \beta}$ are the Pucci’s extremal operators with ellipticity constants $0 < \alpha \leq \beta$.

i.e.

\[
\mathcal{M}^+_{\alpha, \beta}(M) = \sup_{\alpha l \leq A \leq \beta l} \text{tr} AM = \alpha \sum_{e_i < 0} e_i(M) + \beta \sum_{e_i > 0} e_i(M)
\]

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\]
Invariance with respect to reflection

For a given unit vector $e$, let $H(e)$ be the plane orthogonal to $e$ and $\sigma(e)$ the reflection with respect to $e$. $F$ is invariant with respect to the reflection $\sigma(e)$ if

$$F(\sigma_e(x), (I - 2e \otimes e)M(I - 2e \otimes e)) = F(x, M) \quad \forall x \in \Omega, \; M \in S_n$$

$F$ is invariant with respect to rotation if

$$F(\begin{pmatrix} O & 0 \\ 0 & O \end{pmatrix}x, OMO^T) = F(x, M) \quad \forall x \in \Omega, \; M \in S_n$$

all orthogonal matrix $O$.
For a given unit vector $e$, let $H(e)$ be the plane orthogonal to $e$ and $\sigma(e)$ the reflection with respect to $e$. $F$ is invariant with respect to the reflection $\sigma(e)$ if

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$F$ is invariant with respect to rotation if

$$F(0x, OMO^T) = F(x, M) \quad \forall x \in \Omega, M \in S_n \text{ all orthogonal matrix } O$$

Observe that $M$ and $(I_n - 2e \otimes e)M(I_n - 2e \otimes e)$ and $OMO^T$ have the same eigenvalues. So any operator that depends only on the eigenvalues of the Hessian is invariant with respect to reflection and to rotation e.g. Pucci's operators, the Laplace operator, Monge Ampere....
Eigenfunctions of the Laplacian in the ball

\[ \begin{aligned}
\Delta u + \lambda u &= 0 \quad \text{in } B, \\
u &= 0 \quad \text{on } \partial B,
\end{aligned} \]

It is well known that the principal eigenfunction (which is positive) is radial.
On the other hand, the eigenfunctions corresponding to the second eigenvalue are not radial, they are equal to

$$\phi_2(x) = \frac{x \cdot e}{|x|} J(|x|).$$

In particular, we cannot expect general solutions to inherit "all the symmetries" of the domain and the operator.
Starting with Alexandrov and after the fundamental works of Serrin and Gidas, Ni, Nirenberg most results on symmetry of solutions rely on the moving plane method. It is impossible to even start mentioning all the results obtained via that method, be they for semilinear, quasilinear or fully nonlinear equations. For positive solutions of fully nonlinear equations: Bardi (1985), Badiale, Bardi (1990), Da Lio and Sirakov, B. and Demengel, Silvestre and Sirakov.
The much acclaimed moving plane method is the tool that allows to extend the symmetry of the principal eigenfunction to positive solutions of semi-linear equations.

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-\Delta u &= f(x, u) \quad \text{in } \Omega , \\
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**Can the analogy be continued?** when can one expect solutions of nonlinear equations to share the same symmetry of other eigenfunctions?
The much acclaimed moving plane method is the tool that allows to extend the symmetry of the principal eigenfunction to positive solutions of semi-linear equations.

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\begin{cases}
-\Delta u = f(x, u) & \text{in } \Omega, \\
 u = 0 & \text{on } \partial\Omega.
\end{cases}
\]

**Can the analogy be continued?** when can one expect solutions of nonlinear equations to share the same symmetry of other eigenfunctions? which is the right symmetry to consider?
Introduction

Limits of applications of the moving plane method.

\[
\begin{cases}
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- Sign changing solutions
Limits of applications of the moving plane method.

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- Non convex domains e.g. when \( \Omega \) is an annulus.
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\]

- Sign changing solutions
- Non convex domains e.g. when \( \Omega \) is an annulus.
- If the nonlinear term \( f(x, u) \) does not have the right monotonicity in the \( x \) variable.
Foliated Schwarz Symmetry

Definition

Let $B$ be a ball or an annulus in $\mathbb{R}^n$. A function $u : \overline{B} \to \mathbb{R}$ is foliated Schwarz symmetric if there exists a unit vector $p \in S^{n-1}$ such that $u(x)$ only depends on $|x|$ and $\theta = \arccos \left( \frac{x}{|x|} \cdot p \right)$, and $u$ is non increasing with respect to $\theta \in (0, \pi)$.

Though more telling would be to call it axial Schwarz symmetry.
### Lemma

A function \( u \in C^1(B) \cap C(B) \) is foliated Schwarz symmetric if and only if there exists a direction \( e \in S^{n-1} \) such that \( u \) is symmetric with respect to \( H(e) \) and for any other direction \( e' \in S^{n-1} \setminus \{ \pm e \} \) one has either \( u_{\theta e, e'} \geq 0 \) in \( B(e) \) or \( u_{\theta e, e'} \leq 0 \) in \( B(e) \).

Whose consequence is
**Lemma**

*u* is foliated Schwarz symmetric with respect to the direction $p \in S^{n-1}$ if and only if $u(x) \geq u(\sigma_e(x))$ for all $x \in B(e)$ and for every $e \in S^{n-1}$ such that $e \cdot p \geq 0$.

Whose consequence is

**Proposition**

A function $u \in C^1(B) \cap C(\overline{B})$ is foliated Schwarz symmetric if and only if there exists a direction $e \in S^{n-1}$ such that $u$ is symmetric with respect to $H(e)$ and for any other direction $e' \in S^{n-1} \setminus \{\pm e\}$ one has either $u_{\theta,e,e'} \geq 0$ in $B(e)$ or $u_{\theta,e,e'} \leq 0$ in $B(e)$. 
In the last decades, some work has been devoted to understanding under which conditions solutions of semilinear elliptic equations are foliated Schwarz symmetric. This line of research, which strongly relies on the maximum principle, was started by Pacella and then developed by Pacella and Weth. See also
-Gladiali, Pacella, Weth
-Pacella, Ramaswamy
-Weth.
In order to prove foliated Schwarz symmetry of a solution $u$ of

$$\begin{align*}
\Delta u + f(|x|, u) &= 0 \quad \text{in } B, \\
u &= 0 \quad \text{on } \partial B.
\end{align*}$$

one needs to study the sign of the first eigenvalue $\lambda_1(\mathcal{L}_u, B(e))$ of the linearized operator $\mathcal{L}_u = \Delta + \frac{\partial f}{\partial u}(|x|, u)$ at the solution $u$, in the half domain $B(e) = \{ x \in B : x \cdot e > 0 \}$ for a direction $e \in S^{n-1}$. 
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**Question:** what plays the role of the linearized operator for the fully nonlinear problem?
\begin{align*}
\mathcal{M}^-_{\alpha,\beta}(M-N) &\leq F(x,M)-F(x,N) \leq \mathcal{M}^+_{\alpha,\beta}(M-N), \quad M, N \in S_n, \\
This, in order, will imply that if \( u \) is a solution of
\begin{align*}
\begin{cases}
-F(x, D^2 u) = f(x, u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\end{align*}
any \( v \) "derivative of \( u \)" will satisfy
\begin{align*}
-F(x, D^2 v) &\leq \frac{\partial f}{\partial u}(x,u) v \quad \text{in } \Omega,
\end{align*}
and
\begin{align*}
-F(x, D^2 v) &\geq \frac{\partial f}{\partial u}(x,u) v \quad \text{in } \Omega.
\end{align*}
\[ \mathcal{M}^-_{\alpha,\beta}(M-N) \leq F(x, M) - F(x, N) \leq \mathcal{M}^+_{\alpha,\beta}(M-N), \quad M, N \in S_n, \]

This, in order, will imply that if \( u \) is a solution of

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u = 0 & \text{on } \partial\Omega,
\end{cases}
\]

any \( v \) "derivative of \( u \)" will satisfy

\[
-M^+_{\alpha,\beta}(D^2 v) \leq \frac{\partial f}{\partial u}(x, u) v \quad \text{in } \Omega,
\]

and

\[
-M^-_{\alpha,\beta}(D^2 v) \geq \frac{\partial f}{\partial u}(x, u) v \quad \text{in } \Omega.
\]

\( v \) being only a viscosity subsolution or supersolution.
The fully nonlinear operator

\[ \mathcal{L}_u(v) := \mathcal{M}^+_{\alpha, \beta}(D^2 v) + \frac{\partial f}{\partial u}(x, u) v. \]

will be improperly called "linearized operator at u".
The fully nonlinear operator

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will be improperly called "linearized operator at \( u \)."

**Proposition**

Assume \( F \) is invariant with respect to rotation, that \( \Omega = B \) is radially symmetric and \( \lambda^+_1(\mathcal{L}_u, B) > 0 \) then \( u \) is radial.
The fully nonlinear operator

\[ \mathcal{L}_u(v) := M_{\alpha, \beta}^+(D^2 v) + \frac{\partial f}{\partial u}(x, u) v. \]

will be improperly called "linearized operator at u".

**Proposition**

Assume \( F \) is invariant with respect to rotation, that \( \Omega = B \) is radially symmetric and \( \lambda_1^+(\mathcal{L}_u, B) > 0 \) then \( u \) is radial.

**Proposition**

Assume that \( \Omega \) and \( F \) are symmetric with respect to the hyperplane \( H(e) \), let \( \Omega(e) = \Omega \cap \{x \cdot e \geq 0\} \)

(i) If \( f \) is convex and \( \lambda_1^+(\mathcal{L}_u, \Omega(\pm e)) > 0 \)

or

(ii) If \( f \) is strictly convex and \( \lambda_1^+(\mathcal{L}_u, \Omega(\pm e)) \geq 0 \)

Then, \( u \) is symmetric with respect to the hyperplane \( H(e) \).
Symmetry and spectral properties for viscosity solutions of fully nonlinear equations

Introduction

Definition of the eigenvalue.

Following the ideas of Berestycki, Nirenberg, Varadhan,
Symmetry and spectral properties for viscosity solutions of fully nonlinear equations

Introduction

Definition of the eigenvalue.

Following the ideas of Berestycki, Nirenberg, Varadhan,

\begin{align*}
    \lambda_1^+(F, \Omega) &:= \sup\{\mu \in \mathbb{R}, \ \exists \ \phi > 0 \text{ in } \Omega, \ F[\phi] + \mu \phi \leq 0 \}, \\
    \lambda_1^-(F, \Omega) &:= \sup\{\mu \in \mathbb{R}, \ \exists \ \phi < 0 \text{ in } \Omega, \ F[\phi] + \mu \phi \geq 0 \}.
\end{align*}

with e.g. \( F[\phi] = \mathcal{M}^{+}_{\alpha, \beta}(D^2 \phi) + c(x)\phi. \)
There exists $\phi > 0$ (and $\psi < 0$) in $\Omega$ such that

$$\begin{cases}
F[\phi] + \lambda_1^+(F, \Omega)\phi = 0 & \text{in } \Omega \\
\phi = 0 & \text{on } \partial\Omega
\end{cases}$$

and

$$\begin{cases}
F[\psi] + \lambda_1^-(F, \Omega)\psi = 0 & \text{in } \Omega \\
\psi = 0 & \text{on } \partial\Omega
\end{cases}$$

in the viscosity sense. (See the works of Lions, Busca-Esteban-Quaas, Ishii-Yoshimura, B.-Demengel, Sirakov-Quaas, Armstrong,...)
Symmetry and spectral properties for viscosity solutions of fully nonlinear equations

Introduction

Proposition

With the above notations, the following properties hold:

(i) If $D_1 \subset D_2$ and $D_1 \neq D_2$, then $\lambda_1^+(D_1) > \lambda_1^-(D_2)$.

(ii) For a sequence of domains $\{D_k\}$ such that $D_k \subset D_{k+1}$, then

$$\lim_{k \to +\infty} \lambda_1^+(D_k) = \lambda_1^+(\bigcup_k D_k).$$

(iii) If $F[\phi] = M_{\alpha,\beta}^+(D^2 \phi) + c(x)\phi$ and $\alpha < \beta$ then $\lambda_1^+ < \lambda_1^-$. 

(iv) If $\lambda \neq \lambda_1^\pm$ is an eigenvalue then every corresponding eigenfunction changes sign.
Proposition

With the above notations, the following properties hold:

(i) If $D_1 \subset D_2$ and $D_1 \neq D_2$, then $\lambda_1^\pm(D_1) > \lambda_1^\pm(D_2)$.

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$$\lim_{k \to +\infty} \lambda_1^\pm(D_k) = \lambda_1^\pm(\cup_k D_k).$$

(iii) If $F[\phi] = \mathcal{M}_{\alpha,\beta}^+(D^2 \phi) + c(x)\phi$ and $\alpha < \beta$ then $\lambda_1^+ < \lambda_1^-$.

(iv) If $\lambda \neq \lambda_1^\pm$ is an eigenvalue then every corresponding eigenfunction changes sign.

(v) $\lambda_1^+(D) > 0$ ($\lambda_1^-(D) > 0$) if and only if the maximum (minimum) principle holds for $F$ in $D$.

(vi) $\lambda_1^\pm(D) \to +\infty$ as $\text{meas}(D) \to 0$. 
Proposition

Assume that $F$ is invariant with respect to rotation, $\Omega = B$ is radially symmetric and $\lambda_1^+ (\mathcal{L}_u, B) > 0$ then $u$ is radial.

Proposition

Assume that $\Omega$ and $F$ are symmetric with respect to the hyperplane $H(e)$,

(i) If $f$ is convex and $\lambda_1^+ (\mathcal{L}_u, \Omega(\pm e)) > 0$

or

(ii) If $f$ is strictly convex and $\lambda_1^+ (\mathcal{L}_u, \Omega(\pm e)) \geq 0$

Then, $u$ is symmetric with respect to the hyperplane $H(e)$.

Proof: (On the board)
Theorem

Suppose that $F$ is invariant with respect to any reflection $\sigma_e$ and by rotations. Let $u$ be a viscosity solution of problem

\[
\begin{cases}
-F(x, D^2 u) = f(x, u) & \text{in } B, \\
u = 0 & \text{on } \partial B,
\end{cases}
\]

with $f(x, \cdot) = f(|x|, \cdot)$ convex in $\mathbb{R}$. If there exists $e \in S^{n-1}$ such that

\[
\lambda_1^+(\mathcal{L}u, B(e)) \geq 0,
\]

then $u$ is foliated Schwarz symmetric.
On the other hand one has the following necessary condition.

**Theorem**

Assume that the solution $u$ is not radial but it is foliated Schwarz symmetric with respect to $p \in S^{n-1}$. Then, for all $e \in S^{n-1}$ such that $e \cdot p = 0$, one has

$$\lambda_{\frac{1}{2}}(\mathcal{L}_u, B(e)) \geq 0.$$ 

**Proposition**

Let $u \in C^1(\bar{\Omega})$ be a sign changing viscosity solution, with $F$ invariant with respect to reflection to the hyperplane $H(e)$ (say $e = e_1$) and assume that $u$ is even with respect to $x_1$. Then

$$\lambda_{\frac{1}{2}}(\mathcal{L}_u, \Omega(\pm e_1)) \geq 0 \quad \Rightarrow \quad N(u) \cap \partial\Omega \neq \emptyset,$$

where $N(u)$ is the nodal set of $u$. 


In any bounded domain $\Omega$, one can define

\[ \mu_2^+ (L_u, \Omega) = \inf_{D \subset \Omega} \max \{ \lambda_1^+ (L_u, D), \lambda_1^+ (L_u, \Omega \setminus D) \} \]  

(2)

where the infimum is taken on all subdomains $D$ contained in $\Omega$. When $L_u = \Delta + f'(|x|, u)$, $\mu_2^+ = \Lambda_2$ i.e. it is just the second eigenvalue of $L_u$.

**Corollary**

Under the above assumptions, if $\mu_2^+ (L_u, \Omega) \geq 0$ then $u$ is foliated Schwarz symmetric.
In any bounded domain $\Omega$, one can define
\[
\mu_2^+(L_u, \Omega) = \inf_{D \subset \Omega} \max\left\{ \lambda_1^+(L_u, D), \lambda_1^+(L_u, \Omega \setminus \overline{D}) \right\}
\]  \hspace{1cm} (2)
where the infimum is taken on all subdomains $D$ contained in $\Omega$.

When $L_u = \Delta + f'(|x|, u)$, $\mu_2^+ = \Lambda_2$ i.e. it is just the second eigenvalue of $L_u$.

**Corollary**

Under the above assumptions, if $\mu_2^+(L_u, \Omega) \geq 0$ then $u$ is foliated Schwarz symmetric.

It turns out that:

**Proposition**

If $\alpha < \beta$, then $\mu_2^+(L_u, \Omega)$ is not an eigenvalue for $L_u$ in $\Omega$ with corresponding sign changing eigenfunctions having exactly two nodal regions.

**Proof:** (On board)
This Proposition leads to believe that a natural candidate for being the second eigenvalue of $L_u$ could be

$$
\gamma_2^+(L_u, B) = \inf_{D \subset B} \max \{ \lambda_1^+(L_u, D), \lambda_1^-(L_u, B \setminus \overline{D}) \} \geq \mu_2^+(L_u, B).
$$
This Proposition leads to believe that a natural candidate for being the second eigenvalue of $\mathcal{L}_u$ could be

$$\gamma_2^+(\mathcal{L}_u, B) = \inf_{D \subset B} \max \{\lambda_1^+(\mathcal{L}_u, D), \lambda_1^- (\mathcal{L}_u, B \setminus \overline{D})\} \geq \mu_2^+(\mathcal{L}_u, B).$$

It would be also interesting to know, at least, whether the non negativity of $\gamma_2^+(\mathcal{L}_u, B)$ implies that $u$ be foliated Schwarz symmetric.
For $F$ uniformly elliptic and positively one homogeneous operator let

$$\Lambda_2(F) = \inf\{\lambda > \max\{\lambda_1^-(F), \lambda_1^+(F)\} : \lambda \text{ is an eigenvalue of } F\}.$$  \hspace{1cm} (3)$$

It was proved by Armstrong, that $\Lambda_2(F) > \max\{\lambda_1^-(F), \lambda_1^+(F)\}$ and that for any $\mu \in (\max\{\lambda_1^-(F), \lambda_1^+(F)\}, \Lambda_2(F))$

and, for any continuous $f$, there exists a solution of the Dirichlet problem

$$\begin{cases}
F(x, D^2 u) + \mu u = f(x) & \text{in } B \\
u = 0 & \text{on } \partial B.
\end{cases}$$
The next result relates the principal eigenvalue of $\mathcal{M}^+_{\alpha,\beta}$ in any half domain $B(e)$ i.e. $\lambda_1^+(\mathcal{M}^+_{\alpha,\beta}, B(e))$ with $\Lambda_2(F, B)$ and $\lambda_2^r(F, B)$ which denotes the smallest radial nodal eigenvalue of $F$ in $B$.

**Theorem**

*Let $F$ be positively one homogeneous, then the following inequalities hold*

$$\lambda_2^r(F, B) > \lambda_1^+(\mathcal{M}^+_{\alpha,\beta}, B(e)) \quad \text{and} \quad \Lambda_2(F, B) \geq \lambda_1^+(\mathcal{M}^+_{\alpha,\beta}, B(e)).$$
To conclude, we observe that an important question which remains open is whether $\lambda_1^+(\mathcal{M}_{\alpha,\beta}^+, B(e))$ is a nodal eigenvalue for $\mathcal{M}_{\alpha,\beta}^+$ in $B$, as for the laplacian, or not.

**Proposition**

Assume that $\lambda_1^+(\mathcal{M}_{\alpha,\beta}^+, B(e))$ is a nodal eigenvalue for $\mathcal{M}_{\alpha,\beta}^+$ in $B$ and that $\psi_2$ is a corresponding eigenfunction, i.e.

\[
\begin{cases}
\mathcal{M}_{\alpha,\beta}^+(D^2\psi_2) + \lambda_1^+(\mathcal{M}_{\alpha,\beta}^+, B(e))\psi_2 = 0 & \text{in } B \\
\psi_2 = 0 & \text{on } \partial B
\end{cases}
\]

Then

(i) $\psi_2$ is not radial;

(ii) $\psi_2$ is foliated Schwarz symmetric;

(iii) the nodal set of $\mathcal{N}(\psi_2)$ does intersect the boundary;

(iv) if $\alpha < \beta$, then, for any $e \in S^{n-1}$, $B^+ := \{x \in B : \psi_2 > 0\} \neq B(e)$. 

Symmetry and spectral properties for viscosity solutions of fully nonlinear equations

Nodal eigenvalues