CONTINUUMIZATION OF REGULARLY ARRANGED RIGID BODIES

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This paper proposes continuumnization for a set of regularly arranged spherical rigid bodies of identical radius. Motion of each rigid body, which is assumed to be infinitesimally small, is converted to a smooth function, and tensor quantities that correspond to material mechanical properties are derived from spring constants; one of them is a fourth-order elasticity tensor. For simplified two-dimensional setting, the characteristic equation of the continuumnized motion is studied. It is observed that there are high-frequency modes of local spin. Such high-frequency modes of local spin induce damping effects on translation.

\textbf{Key Words : }rigid body, continuum mechanics, wave velocity, spin and rotation, continuumnization

1. INTRODUCTION

A material consists of its own microstructure. Soil is a typical example of such materials, as it consists of soil particles of various sizes\textsuperscript{1); rock, which is not intact, is another example\textsuperscript{3). Each particle behaves like a rigid body, although the assembly of the particles exhibit quite a complicated non-linear behavior. Therefore, it is natural to use a set of rigid bodies as a simplified model of the material microstructure, in which a rigid body translates and spins, subjected to forces and torques that are transmitted by neighboring rigid bodies\textsuperscript{3);4). In this paper, we use spin, rather than rotation, to designate rotating motion of a rigid body; rotation is used as an antisymmetric part of displacement gradient in this paper.

As far as we survey, a set of mass points are often used as a model of theoretical studies of material microstructures; see, for instance, an exceptional case\textsuperscript{6);7). This is due to the difficulty in rigorous treatment of spin of a rigid body, which is much more complicated than translation. Moreover, continuum mechanics does not require local spin since symmetry of stress tensor does not allow the presence of local torque that is associated with local spin. Therefore, it is useless to consider a spinning rigid body, which requires tedious mathematical treatment.

In this paper, we seek to provide a new perspective of a rigid body set as a model of material microstructure. It starts from providing a new mathematical treatment of the rigid body set, so that effects of local spin upon the global response, which could be interpreted as overall material properties, are studied. This treatment is called continuumnization, which transforms motion of each rigid body, translation and spin, to a smooth function that corresponds to continuum deformation. We will show that continuumnization leads to the following:

- local spin produces deformation modes of high frequencies.

We will also show that local spin could be a source of damping for waves of large wave lengths.

The objective of this paper is to provide basic formulation of continuumnization of a rigid body set when it is used as a model of material microstructure. The above finding will be rigorously derived as the consequence of continuumnization. The contents
of this paper are organized as follows: First, we make a brief literature survey about microstructure models which use a rigid body set in Section 2. We explain continuumization in Section 3. A set of regularly arranged spherical rigid bodies of identical radius at dynamics state are used to this end. The vanishing of continuumized spin and associated local torque is discussed in Section 4. Characteristic equation of continuumized motion is studied in Section 5. The study findings are presented in this section.

2. LITERATURE SURVEY

A rigid body set is often used for a microstructure of granular materials such as soil\(^7\). Similar models are used for damaged rock\(^2,8\). This model is straightforward since the material consists of granules. As mentioned, some consider spin of granules\(^7\). Considering spin usually results in micropolar materials, in which the presence of nonsymmetric stress, coupled stress or local torque, is assumed\(^9,10,11,12\); Cosserat theory\(^13\) is usually used for such micropolar materials. We accept the presence of the local torque that is related to the local spin of granules. However, we will show the vanishing of such spin and torque in the limit as the size of granules vanishes in later sections.

As for concrete, a remarkably large collection of papers are found that use an assembly of particles as a microstructure model; see, for instance, a list of related papers\(^14,15\). The major concern is to treat microcracks of mortar in which aggregates are embedded\(^16,17,18\). Indeed, it is natural to use a particle assembly in which local failure is induced at contact points of particles. Failure analysis that takes advantage of readiness of an assembly model is found in literature\(^19\); this type of analysis\(^20\) is similar to the discrete element method.

These research achievements share the same mathematical treatment called **homogenization**. They consider the volume average of continuum field variables, such as strain and stress, by computing the integration of displacement and force of rigid bodies. Homogenization is different from continuumization presented in this paper, although both relate a rigid body set to a continuum in order to study material properties. As will be shown in the next section, continuumization derives local field equations for smooth functions that correspond to deformation, by computing the derivative of the functions.

We should point out that apart from the engineering field, there are various works which use a rigid body set or a lattice model in modern physics\(^21\). The target of such works\(^22,23\) is so called disordered system which corresponds to heterogeneous material microstructure. Although it does not provide practical engineering solutions, a lattice model\(^24,25,26,27,28\) is used to study elasticity and failure; composites and polycrystals are often a target of the lattice model.

3. CONTINUUMIZATION OF RIGID BODY SET

\section{(1) Motion and force of rigid body set}

Let us consider a rigid body grid, a set of regularly arranged spherical rigid bodies of identical radius; see Fig. 1. We denote by \(\mathbf{x}^a\) the center of the \(a\)-th rigid body in the grid; if it is not confusing, \(\mathbf{x}^a\) is used to designate the rigid body itself. We assume that an infinitesimally short spring connects a rigid body to another neighboring rigid body; see Fig. 2. Since the grid consists of regularly arranged spheres, the number of the spring direction is finite. We denote them by \([\mathbf{n}^l]\). We also denote by \([\mathbf{t}^l]\) and \([\mathbf{s}^l]\) tangential directions associated with \(\mathbf{n}^l\); \((\mathbf{n}^l, \mathbf{t}^l, \mathbf{s}^l)\) forms a right-hand triad. We assume that all springs share the same normal and tangential spring constants, \(k\) and \(h\), respectively; the spring constant in the \(\mathbf{n}^l\)-direction is \(k\), and that in the \(\mathbf{t}^l\)- or \(\mathbf{s}^l\)-direction is \(h\); the spring constants are rigorously computed\(^29\) for two contacting spheres of the same radius.

A rigid body \(\mathbf{x}^a\) is connected to two rigid bodies

![Fig. 1 Rigid body grid.](image1)

![Fig. 2 Springs connecting rigid bodies.](image2)
Thus, the force provided by the spring that connects \( \mathbf{x}^a + (\mathbf{a} \times \mathbf{n}^I) \) and \( \mathbf{x}^a - (\mathbf{a} \times \mathbf{n}^I) \), and the forces are \( \mathbf{F}^+ \) and \( \mathbf{F}^- \). The equation of motion for \( \mathbf{x}^a \) is thus expressed as
\[
M \ddot{\mathbf{x}}^a = \sum_I \mathbf{F}^+ + \mathbf{F}^-,
\]
where \( \dot{\mathbf{u}} \) stands for temporal derivative (\( \dot{\mathbf{u}} = \frac{\partial \mathbf{u}}{\partial t} \)) and \( M \) is the mass of \( \mathbf{x}^a \). Note that even though superscript \( I \) is omitted, \( \mathbf{F}^+ \) is the force provided by the springs in the \( \mathbf{n}^I \)-direction. Euler’s momentum equation of \( \mathbf{x}^a \) is
\[
I \ddot{\mathbf{a}} = \sum_I (\mathbf{a} \times \mathbf{F}^+) + (-\mathbf{a} \times \mathbf{F}^-),
\]
where \( I \) is the moment of inertia of \( \mathbf{x}^a \).

Substituting Eqs. (3) and (4) into Eqs. (5) and (6), we have
\[
M \ddot{\mathbf{u}} = \sum_I k((\Delta \mathbf{u}^a + \Delta \mathbf{u}^a) \cdot \mathbf{n}^I) + h((\Delta \mathbf{u}^a - \Delta \mathbf{u}^a) \cdot \mathbf{t} - a(\theta^a - \theta^a) \cdot s)t + h((\Delta \mathbf{u}^a + \Delta \mathbf{u}^a) \cdot \mathbf{s} + a(\theta^a - \theta^a) \cdot t)s,
\]
and
\[
I \ddot{\mathbf{a}} = \sum_I ah((\Delta \mathbf{u}^a - \Delta \mathbf{u}^a) \cdot \mathbf{t} - a(\theta^a + 2\theta^a) \cdot s)s - ah((\Delta \mathbf{u}^a - \Delta \mathbf{u}^a) \cdot \mathbf{s} + a(\theta^a + 2\theta^a) \cdot t)t.
\]
Equations (7) and (8) form a set of ordinary equations for \( \{\mathbf{u}^a, \theta^a\} \) which are function of time \( t \).

(3) Continuumization of rigid body motion

We consider a smooth function, \( \mathbf{u}(\mathbf{x}, t) \), which satisfies
\[
\mathbf{u}(\mathbf{x}^a, t) = \mathbf{u}^a(t) \quad \text{for all } \mathbf{x}^a \text{'s}.
\]
By definition, we can approximate \( \Delta \mathbf{u}^a \) in terms of this function. That is, \( \Delta \mathbf{u}^a \) is approximated as
\[
\Delta \mathbf{u}^a \approx ((2\mathbf{a} \cdot \nabla) \mathbf{u}^a),
\]
or, in terms of \( \mathbf{u}^a \) and \( \mathbf{u}^a \),
\[
\mathbf{u}^a(t) - \mathbf{u}^a(t) \approx (2\mathbf{a} \cdot \nabla) \mathbf{u}^a(\mathbf{x}^a, t),
\]
where \( \nabla \mathbf{u} \) is the gradient of \( \mathbf{u} \), i.e., \( (\nabla \mathbf{u})_{ij} = \frac{\partial u_i}{\partial x_j} \), and, by definition, \( \mathbf{n} \cdot \nabla \) is interpreted as the derivative in the \( \mathbf{n} \)-direction (i.e., \( \nabla \cdot \mathbf{n} = \sum_i \frac{\partial u_i}{\partial x_i} \)). Note that while the ordinary definition of the gradient is \( (\nabla \mathbf{u})_{ij} = \frac{\partial u_j}{\partial x_i} \), we define the gradient component in this way to make the expression of \( \nabla \cdot \mathbf{n} \) consistent. In the limit as the rigid body radius \( a \) goes to 0, the error of the above approximation tends to vanish. We call this \( \mathbf{u}(\mathbf{x}, t) \) a continuumized displacement that is associated with \( \{\mathbf{u}^a(t)\} \).

In a similar manner, we introduce a continuumized spin, \( \mathbf{\theta}(\mathbf{x}, t) \), which satisfies \( \mathbf{\theta}(\mathbf{x}^a, t) = \mathbf{\theta}^a(t) \); see Fig. 4 for a schematic view of continuumization of motion that consists of translation and spin. Unlike translation, spin is not natural if it is defined as
a smooth function. Indeed, we will show that in the limit as \(a\) goes to 0, this \(\theta(x, t)\) tends to vanish. However, for a non-zero value of \(a\), we can define \(\theta(x, t)\) for \(\{\theta^a\}\), and approximate

\[
\theta^a(t) \approx (4a) \cdot (\nabla \theta(x^a, t)).
\]

Again, the error of this approximation approaches 0 in the limit as \(a\) goes to 0; as will be shown in the next section, \(\theta\) itself vanishes in this limit.

In terms of the continuumized \(u\) and \(\theta\), Eqs. (7) and (8) are approximately expressed as

\[
\dot{M} \ddot{u} \approx a^3 (\nabla \cdot (c : \nabla u)) + q^s : \nabla \theta),
\]

\[
I \ddot{\theta} \approx -a^5 (r \cdot \theta + q : \nabla u)
\]

at \(x^{a}\),

where

\[
c = \sum \frac{k}{a} n \otimes n \otimes n \otimes n
\]

\[
+ \frac{1}{a} (n \otimes t \otimes n \otimes t + n \otimes s \otimes n \otimes s),
\]

\[
q = \sum \frac{h}{a} (t \otimes n \otimes s - s \otimes n \otimes t),
\]

\[
r = \sum \frac{h}{a} (t \otimes t + s \otimes s).
\]

Here, for simplicity, superscript \(I\) is omitted for \(n, t\) and \(s\), and \(c, q\) or \(r\) is a fourth-, third- or second-order tensor that is determined in terms of \(k\) and \(h\), respectively; superscript \(t\) stands for transpose \((q^{t}_{ijk} = q_{kji})\). The identical equations hold for all \(x^a\)'s of the rigid body grid. That is, continuumization derives coupled partial differential equations for \(\{u, \theta\}\) from a set of ordinary differential equations of \(\{u^a, \theta^a\}\), and the three tensors of Eq. (11) are regarded as mechanical material properties.

4. VANISHING OF LOCAL TORQUE AND SPIN

As briefly explained in Section 1, it is not natural to introduce a field of local spin such as \(\theta(x, t)\). This is because if each point spins itself, it automatically produces discontinuity everywhere. In continuum mechanics, vanishing of local angular momentum is presumed; when a cube of edge length \(a\) is considered, the angular momentum vanishes as the speed of \(a^2\), while the torque caused by traction on the opposite faces vanishes as the speed of \(a^3\). This leads to the symmetry of Cauchy stress tensor. Unlike local spin, it is natural to relate translation of a rigid body set to a smooth function, \(u(x, t)\). While the mathematical procedure of associating \(\{u^a(t)\}\) to \(u(x, t)\) is identical with that of associating \(\{\theta^a(t)\}\) to \(\theta(x, t)\), we observe that there is a difference between translation and spin in the sense that the presence of spin seems strange.

A solution of resolving the difference between translation and spin functions, \(u(x, t)\) and \(\theta(x, t)\), is found in physics. Ignoring the term that does not involve \(\theta(x, t)\), we write Eq. (10) as

\[
\ddot{\theta}(x, t) = -a^5 \frac{r}{I} \frac{1}{a^2} : \theta(x, t).
\]

By definition, \(I/a^5\) and \(r\) correspond to density and elasticity, respectively, and \(a^3r/I\), which corresponds to a constant that corresponds to an elastic wave speed, remains finite in the limit as \(a\) goes to 0. Hence, the right side of Eq. (12) diverges as the speed of \(a^{-2}\). This implies that \(\theta(x, t)\) at each point vibrates at an increasing frequency. The time average of such high-frequency vibration vanishes if a sufficiently large duration is used. Therefore, we can assume that

\[
\lim_{a \to 0} \theta(x, t) = 0.
\]

This is because the natural frequency of \(\theta(x, t)\) diverges in the limit.

The vanishing of \(\theta(x, t)\) in the limit \(a\) goes to 0 implies that the second term in the right side of Eq. (10), \(q : \nabla u(x, t)\), vanishes as well. Indeed, this is true if \(q : \nabla u(x, t)\) corresponds to local torque of ordinary continuum mechanics. In continuum mechanics, stress is symmetric so that local torque vanishes; for instance, around the \(\chi_3\)-axis, local torque is proportional to \(\sigma_{12} - \sigma_{21}\). That is, Euler’s equation becomes trivial in the sense that all the terms involved vanish. As will be shown in the next section, the vanishing
of local torque implies that an antisymmetric part of \( c : \nabla u(x, t) \) in Eq. (8) vanishes as well.

It is worth checking the limit of Eq. (8) as \( a \) goes to 0, since Eq. (10) becomes trivial in the limit. In the absence of \( q : \nabla \theta(x, t) \), this equation is rewritten as

\[
\ddot{u}(x, t) = \frac{\alpha^3}{M} \nabla \cdot (c : \nabla u(x, t)).
\]

By definition, \( M/a^3 \) and \( c \) correspond to density and elasticity; the physical dimensions of \( M/a^3 \) and \( c \) are weight per volume and force per area, respectively. Therefore, in the limit, the coefficient of the right side of the above equation remains finite, and, as expected, \( \alpha^3 c/M \) corresponds to a constant of an elastic wave speed.

5. CHARACTERISTIC EQUATION OF CONTINUUMIZED MOTION

(1) Simplified 2D setting

We simplify the problem setting as two-dimensional (2D), considering one layer of regularly arranged spherical rigid bodies of identical radius, as shown in Fig. 5. We consider three spring directions, and \([n^1, t^1]\) lie on the \( x_1, x_2 \)-plane and \([s^1, s^2, s^3]\) coincides with the \( x_3 \)-direction. Note that non-zero components of \([x^a, \theta^a]\) (or \([u, \theta]\)) are \( u_1^2 \), \( u_2^2 \) and \( \theta_3^2 \) (or \( u_1, u_2 \) and \( \theta_3 \)); see Fig. 6.

The three tensors defined in Eq. (11) are readily computable. The fourth-order tensor \( c \) consists of an isotropic part,

\[ \frac{3k}{8a} (I^1 + I^2) + \frac{3k}{8a} (-I^1 + I^2), \]

where \( I_{ijkl}^1 = \delta_{ij} \delta_{kl} \) and \( I_{ijkl}^2 = \frac{1}{2} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \), with \( \delta_{ij} \) being Kronecker’s delta. The remaining part, denoted by \( c^* \), has non-zero components of

\[ c_{1212}^* = -c_{2112}^* = -c_{1221}^* = c_{2121}^* = \frac{3h}{4a}. \]

The third-order tensor \( q \) becomes \( \sum_3 \frac{h}{a} e_3 \otimes n^i \otimes t^j \). Denoting by \( q_{ij}^* = q_{3ij} \), we have non-zero components of this \( q^* \) as

\[ q_{12}^* = -q_{21}^* = \frac{3h}{2a}. \]

The second-order tensor \( r \) becomes \( r_{ij} = r^* \delta_{ij} \) with

\[ r^* = \frac{3h}{a}. \]

Note that \( \frac{h}{a} \) corresponds to elasticity, as its physical unit is force per area.

We apply Fourier transform with the kernel of \( \exp(i \vec{\xi} \cdot (x - \omega t)) \) to \( u \) and \( \theta \), and derive the characteristic equation for the transformed function. The \( x_1 \) and \( x_2 \) components of Eq. (9) and the \( x_3 \) component of Eq. (9) yield a three-by-three matrix,

\[
[K] = \begin{bmatrix}
-\frac{Ma^2}{a^3} + \left( \frac{9k}{8a} + \frac{3h}{8a} \right) \xi_1^2 + \left( \frac{3k}{8a} + \frac{9h}{8a} \right) \xi_2^2 \\
\frac{3(k-h)}{4ah} \xi_1 \xi_2 \\
-\frac{3h}{2a} \xi_1 \\
\end{bmatrix},
\]

for the Fourier transform of \([u_1, u_2, \theta_3]'\). The determinant of \([K]\) is

\[
\det[K] = \frac{1}{64a^3} [-8Ma^2 \omega^2 + \left( \frac{9k}{a} + \frac{3h}{a} \right) \xi_1^2] \\
- \frac{8MI}{a^3} \omega^4 + 3 \left( \frac{k}{a^3} \left( \frac{3h}{a} + \frac{8Mh}{a^3} \right) \omega^2 + \frac{9kh}{a^2} + \frac{45h^2}{a^2} \right) \xi_1^2, \tag{14}
\]

which is a polynomial of the sixth order with respect to \( \omega \).
(2) Vanishing of local spin

It is interesting to note that \( e^* \), the remaining of \( c \) subtracted by the fourth-order isotropic part, maps rotation (or an antisymmetric part of displacement gradient) to antisymmetric stress. That is, denoting by

\[
\omega_{12} = -\omega_{21} = \frac{1}{2} \left( \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right)
\]

\( e^* \) maps this \( \omega \) to \( \tau \) the non-zero components of which are

\[
\tau_{12} = -\tau_{12} = 2e_{1212}^*\omega_{12}.
\]

Also, \( q^* \) produces stress in the equation of motion which is given as

\[
q_{12}^* \frac{\partial \theta_3}{\partial x_2}, \quad q_{21}^* \frac{\partial \theta_3}{\partial x_1}.
\]

Together with \( \tau \), the gradient of the stress components in the equation of motion are rewritten as

\[
-\frac{\partial (2G^*(\theta_3 - \omega_{12}))}{\partial x_2}, \quad \frac{\partial (2G^*(\theta_3 - \omega_{12}))}{\partial x_1},
\]

where \( G^* = \frac{3h}{2a} \). In Euler’s momentum equation, the stress components are rewritten as

\[4G^*(\theta_3 - \omega_{12}).\]

As expected, \( \theta_3 \) plays the same role as \(-\omega_{12}\). Note that the non-zero components of \( e^*, q^* \) and \( r^* \) satisfy

\[4c_{1212}^* = 2q_{12}^* = r^*.
\]

As mentioned, it is natural to expect the vanishing of local spin (\( \theta_3 \)) in the limit as \( a \) goes to 0. Ignoring the coupling term with translation, we show that \( \theta_3 \) vanishes in the limit. This implies that \( \tau_{12} = 2G^*\omega_{12} \) vanishes as well. In the present analysis, it is not possible to show the vanishing of \( \tau_{12} \) since \( \omega_{12} \), the antisymmetric part of displacement gradient, does not vanish. We should not make any extra condition of \( \omega_{12} = 0 \) for \( u_1 \) and \( u_2 \), since the equation of motion in the absence of \( \theta_3 \) has a unique solution for them.

A possible solution of making \( \tau_{12} = 0 \) in the limit as \( a \) goes to 0 is to consider a certain set of admissible springs with which \( e^* \) and \( q^* \) are computed so that \( e^* : \nabla u \) and \( q^* : \nabla u \) vanish in the limit. We should emphasize that the spring set shown in Fig. 5 is not admissible. We may have to consider a set of heterogeneous spherical rigid bodies for which such a set of admissible springs are found. It should be noted that 6, 4, and 12 springs of identical spring constants, which are equally spaced for one spherical rigid body, produces \( e^* \) and \( q^* \) similar to the case of the present case, 3 equally spaced springs, and are not admissible, either. Also, infinitely many springs (for which \( e^* \) and \( q^* \) are computed by using integration rather than summation) produce similar \( e^* \) and \( q^* \). Table 1 provides non-zero components of \( e^* \) and \( q^* \) for these cases.

Table 1: Comparison of non-zero components of \( c_{1212}^* \), \( q_{12}^* \) and \( r^* \).

| \( N \) | \( c_{1212}^* \) | \( q_{12}^* \) | \( r^* \) |
|-------|-----------------|-------------|-----|
| 3     | \( \frac{3h}{4a} \) | \( \frac{3h}{2a} \) | \( \frac{3h}{a} \) |
| 6     | \( \frac{3h}{2a} \) | \( \frac{3h}{a} \) | \( \frac{6h}{a} \) |
| 12    | \( \frac{h}{a} \)  | \( \frac{2h}{a} \) | \( \frac{4h}{a} \) |
| \( \infty \) | \( \pi h \) | \( \pi h \) | \( \frac{2\pi h}{a} \) |

(3) Study of characteristic equation

For a given \( \xi \), we can solve the characteristic equation for \( \omega \) so that non-trivial Fourier transform exists for \( [u_1, u_2, \theta_3] \). The solution of \( \omega \) is

\[
\omega^2 = \frac{33k + 5h}{8M} \xi^2 \left( 1 \pm \frac{33 M}{16 M (k + 5h) h} \xi^2 \right) \quad (15)
\]

with \( \xi^2 = \xi \cdot \xi \). As is seen, the first case of Eq. (15) gives a wave velocity of

\[
\frac{\omega}{\xi} = \sqrt{\frac{3(3k + h) a^2}{8M}} \quad (16)
\]

We have to emphasize that continuumnization enables us to compute a wave velocity for a rigid body grid; it is not possible to compute wave velocity for such a discrete system in which spring transfers stress without causing any delay. When the wave length is much larger than \( a \) (or \( a \xi \ll 1 \)), the second equation of the case of sign + gives another wave velocity of

\[
\frac{\omega}{\xi} = \sqrt{\frac{3(k + 5h) a^2}{8M}} \quad (17)
\]

For the case of sign −, \( \omega^2 \) becomes negative. This implies that a solution is not periodic, but decays (or increases) exponentially with respect to time.

For the first case of Eq. (15), the corresponding mode is

\[
[u_1, u_2, \theta_3] = [\xi_1, \xi_2, 0],
\]

i.e., the direction of the displacement vector of this mode is parallel to the wave direction, \( \xi \), just like a primary wave of a linearly isotropic elastic medium. This mode does not accompany spin. For the second case of Eq. (15), when \( a \xi \ll 1 \), the corresponding mode is

\[
[u_1, u_2, \theta_3] = [-\xi_2, \xi_1, \pm 2i \frac{M}{I a^2} (a \xi)^2],
\]

i.e., the direction of the displacement vector of this mode is perpendicular to the wave direction just like
a secondary wave, and spin is accompanied. The amplitude of spin is small, as it includes $1/(a^2)$ in it. It is interesting to note that for a given large wave length (or small wave number $\xi$), the characteristic equation has two positive roots and one negative root, which correspond to two harmonic waves and one exponentially vanishing wave. It is not easy to measure the exponentially vanishing wave; the length scale of this vanishing wave is of the order of $a$. Also, it is not easy to measure spin. The harmonic wave that corresponds to the secondary wave does accompany spin, but its amplitude is small, as previously shown.

These observations suggest difficulty in observing spin for an actual material the microstructure of which is modeled as a rigid body grid. This is because the wave length is much larger than the rigid body size. However, when a body much larger than the wave length is considered, such vanishing small effects of spin on displacement become non-negligible, since the effects are accumulated. Note that the coupling terms that use $\mathbf{q}'$ do not vanish in Eqs. (9) and (10) for $a \neq 0$. Also note that the most rigorous treatment of spin is $\mathbf{u} = \mathbf{\theta} \times \mathbf{x}$, for a point at $\mathbf{x}$ from the center of spin $\mathbf{\theta}$, and an assumption of infinitesimally small $\mathbf{\theta}$ leads to $\mathbf{u} = \mathbf{\theta} \times \mathbf{x}$ when $\mathbf{u}$ is integrated with respect to time. Hence, the coupling terms are replaced by nonlinear terms of $\mathbf{\theta}$, and the resulting nonlinear coupling terms might reduce $\mathbf{u}$. We can presume the effects of the nonlinear coupling terms could be a source of damping for waves which travel in long distance. Further studies on the coupling terms that use $\mathbf{q}'$ are definitely necessary for the case of non-zero $a$.

6. CONCLUDING REMARKS

In this paper, we present continuumization of a rigid body grid. Assuming that spin is infinitesimally small, we derive a set of coupled differential equations for displacement and spin. Spatial derivative of continuumized displacement is included in the equation of motion so that the wave velocity of the rigid body grid can be computed. As the radius of the rigid body sphere becomes smaller, continuumized spin produces deformation modes of higher frequencies, which might produce a damping effect on continuumized displacement.

The above finding is made based on the simplified 2D problem. Further studies are essential to clarify the continuumized spin in the limit as the radius goes to 0 in a general 3D problem setting. Also, more rigorous treatment of spin will be required. At this moment, we think that continuumization will be used as a new tool to analyze a material microstructure model; for instance, continuumization shows a possibility that decoupled spin does not have wave velocity like translation, since spatial derivative of spin is not included in Euler’s momentum equation.

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