CHARACTER VARIETIES OF ABELIAN GROUPS

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Abstract. We prove that for every reductive group $G$ with a maximal torus $T$ and the Weyl group $W$, $T^N/W$ is the normalization of the irreducible component, $X_G^0(Z^N)$, of the $G$-character variety $X_G(Z^N)$ of $Z^N$ containing the trivial representation. We also prove that $X_G^0(Z^N) = T^N/W$ for all classical groups. Additionally, we prove that even though there are no irreducible representations in $X_G^0(Z^N)$ for non-abelian $G$, the tangent spaces to $X_G^0(Z^N)$ coincide with $H^1(Z^N, Ad\rho)$. Consequently, $X_G^0(Z^N)$, has the “Goldman” symplectic form for which the combinatorial formulas for Goldman bracket hold.

1. Introduction

Let $G$ will be an affine reductive algebraic group over $\mathbb{C}$. For every finitely generated group $\Gamma$, the space of all $G$-representations of $\Gamma$ forms an algebraic set, $\text{Hom}(\Gamma, G)$, on which $G$ acts by conjugating representations. The categorical quotient of that action

$$X_G(\Gamma) = \text{Hom}(\Gamma, G)//G$$

is the $G$-character variety of $\Gamma$, cf. [LM, S2] and the references within. In this paper we study $G$-character varieties of free abelian groups.

For a Cartan subgroup (a maximal complex torus) $T$ of $G$, the map

$$T^N = \text{Hom}(Z^N, T) \to \text{Hom}(Z^N, G) \to \text{Hom}(Z^N, G)//G = X_G(Z^N)$$

factors through

$$\chi : T^N/W \to X_G(Z^N),$$

where the Weyl group $W$ acts diagonally on $T^N = T \times \ldots \times T$. Thaddeus proved that for every reductive group $G$, $\chi$ is an embedding, [TH]. In this paper we discuss the image of this map and the conditions under which it is an isomorphism. This is known to be a difficult problem. A version of

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1The field of complex numbers can be replaced an arbitrary algebraically closed field of zero characteristic throughout the paper.
it for compact groups is discussed for example in [BFM]. (The connections between the algebraic and compact versions of this problem are discussed in [FL].) The version of this problem for algebraic groups is harder than that for compact ones, since a regular bijective function between algebraic varieties does not have to be an algebraic isomorphism.

Goldman constructed a symplectic form on an open dense subset of the set of equivalence classes of irreducible representations in \( X_G(\pi_1(F)) \), for closed surfaces \( F \) of genus \( g > 1 \), [Go1]. In the second part of the paper, we extend Goldman’s construction to the connected component of the identity of the \( G \)-character variety of \( F \) torus, even though there are no irreducible representations in that component.

This paper was motivated by [Th] and by our work, [SI], in which we relate deformation-quantizations of character varieties of the torus to the \( q \)-holonomic properties of Witten-Reshetikhin-Turaev knot invariants.

2. Main results

Let \( X_G^0(\mathbb{Z}^N) = \chi(\mathbb{T}^N/W) \).

**Theorem 2.1** (Proof in Sec. 7).

1. \( X_G^0(\mathbb{Z}^N) \) is an irreducible component of \( X_G(\mathbb{Z}^N) \).
2. \( \chi : \mathbb{T}^N/W \to X_G^0(\mathbb{Z}^N) \) is a normalization map for every \( G \) and \( N \). (It was proved for \( N = 2 \) in [Th].)
3. \( \chi : \mathbb{T}^N/W \to X_G^0(\mathbb{Z}^N) \) is an isomorphism for classical groups: \( G = GL(n, \mathbb{C}), SL(n, \mathbb{C}), Sp(n, \mathbb{C}), SO(n, \mathbb{C}) \) and for every \( n \) and \( N \). (\( Sp(n, \mathbb{C}) \) denotes the group of \( 2n \times 2n \) matrices preserving a symplectic form.)

**Remark 2.2.** (1) It is easy to show that \( \chi \) is onto for \( G = SL(n, \mathbb{C}) \) and \( GL(n, \mathbb{C}) \), since one can conjugate every \( G \)-representation of \( \mathbb{Z}^N \) arbitrarily close to representations into \( \mathbb{T} \). That does not hold though for some other groups \( G \). For example, a representation sending \( \mathbb{Z}^n \) onto the group of diagonal matrices \( D \) in \( O(n, \mathbb{C}) = \{ A : A \cdot A^T = I \} \) \( (D = \{ \pm 1 \}^n) \) for \( n > 3 \) cannot be conjugated arbitrarily close to a representation into a maximal torus.

(2) \( \chi \) being onto and 1-1 does not imply that it is an isomorphism of algebraic sets. (For example, \( x \to (x^2, x^3) \) from \( \mathbb{C} \) to \( \{ (x, y) : x^3 = y^2 \} \subset \mathbb{C}^2 \) is a bijection which is not an isomorphism.)

**Problem 2.3.** Is \( \chi \) is an isomorphism onto its image for Spin groups and the exceptional ones? (By Theorem 2.1(2), \( \chi \) is an isomorphism if and only if \( X_G^0(\mathbb{Z}^N) \) is normal.)

Here are a few basic facts about irreducible and connected components of \( X_G(\mathbb{Z}^N) \).

**Remark 2.4.** (1) \( X_G(\mathbb{Z}) \) is irreducible, cf. [SI] §6.4].
(2) \( X_G(\mathbb{Z}^2) \) is irreducible for every semi-simple simply-connected group \( G \), cf. [Ric] Thm C].
(3) For every connected $G$, $X^0_G(\mathbb{Z}^2)$ coincides with the connected component of the trivial representation in $X_G(\mathbb{Z}^2)$, cf. [Th]. (For completeness, a proof is enclosed in Sec. 5.)

**Proposition 2.5** (Proof in Sec. 8.)

$X_G(\mathbb{Z}^N)$ is irreducible for $G = \text{GL}(n, \mathbb{C}), \text{SL}(n, \mathbb{C})$ and $\text{Sp}(n, \mathbb{C})$ for all $N$ and $n$.

3. Irreducible representations of $\mathbb{Z}^N$

Following [S2], we say that $\rho : \mathbb{Z}^N \to G$ is irreducible if its image does not lie in a proper parabolic subgroup of $G$. We say that $\rho : \mathbb{Z}^N \to G$ is completely reducible if for every parabolic subgroup $P \subset G$ containing $\rho(\mathbb{Z}^N)$, the image of $\rho$ lies in a Levi subgroup of $P$.

**Proposition 3.1.** For non-abelian $G$ there are no irreducible representations $\rho : \mathbb{Z}^N \to G$ with $[\rho] \in X^0_G(\mathbb{Z}^N)$.

**Proof.** Assume that $\rho$ is irreducible and $[\rho] \in X^0_G(\mathbb{Z}^N)$. Every equivalence class in $X^0_G(\mathbb{Z}^N)$ contains a representation $\phi : \mathbb{Z}^N \to T \subset G$, and such $\phi$ is completely reducible. Since $\rho$ (being irreducible) is completely reducible and each equivalence class in $X_G(\mathbb{Z}^N)$ contains a unique conjugacy class of completely reducible representation, $\rho$ is conjugate to $\phi$. Hence, $\phi$ is irreducible. Therefore, $G = T$, contradicting the assumption of $G$ being non-abelian. \hfill \Box

**Corollary 3.2.** There are no irreducible representations of $\mathbb{Z}^2$ into simply-connected reductive groups.

**Proof.** Every simply connected reductive algebraic Lie group is semi-simple. (That follows for example from two facts: 1. every reductive Lie algebra is a product of a semi-simple one and an abelian one. 2. There are no non-trivial simply-connected abelian reductive algebraic groups.) Now the statement follows from Remark 2.4(2) and (3). \hfill \Box

There are, however, irreducible representations of abelian groups into non-abelian ones.

**Example 3.3.** Let $n \geq 3$ and let $\rho : \mathbb{Z}^N \to \text{SO}(n, \mathbb{C})$ be a representation whose image contains all diagonal orthogonal matrices with entries $\pm 1$ in the diagonal. Then $\rho$ is irreducible, cf. [S2, Eg. 21].

Another example was suggested to us by Angelo Vistoli:

**Example 3.4.** Let $g \in \text{PSL}(n, \mathbb{C})$ be represented by the diagonal matrix with $1, \omega^1, \ldots, \omega^n$ on the diagonal, where $\omega = e^{2\pi i/n}$, and let $h$ be represented by the permutation matrix associated with the cycle $(1, 2, \ldots, n)$. Then it is easy to see that $g$ and $h$ commute and to prove that $\rho : \mathbb{Z}^2 \to \text{PSL}(n, \mathbb{C})$ sending the generators of $\mathbb{Z}^2$ to $g$ and $h$ is irreducible.
4. Étale Slices and Chevalley sections

Let $\text{Hom}^0(\mathbb{Z}^N, G)$ be the preimage of $X^0_G(\mathbb{Z}^N)$ under $\pi : \text{Hom}(\mathbb{Z}^N, G) \to X_G(\mathbb{Z}^N)$.

**Theorem 4.1** (Proof in Sec. 9). If $\rho : \mathbb{Z}^N \to \mathbb{T} \subset G$ has a Zariski dense image in $\mathbb{T}$ then

1. $X_G(\mathbb{Z}^N)$ is smooth at $\rho$ and $\rho$ belongs to a unique irreducible component of $X_G(\mathbb{Z}^N)$.
2. $d\chi : T_\rho \mathbb{T}^N/W \to T_\rho X_G(\mathbb{Z}^N)$ is an isomorphism.
3. A Zariski open neighborhood of $\rho$ in $\mathbb{T}^N = \text{Hom}(\mathbb{Z}^N, \mathbb{T})$ is an étale slice at $\rho$ with respect to the $G$ action on $\text{Hom}^0(\mathbb{Z}^N, G)$ by conjugation.

With $\text{Hom}(\Gamma, G)$ and $X_G(\Gamma)$, there are naturally associated algebraic schemes $\text{Hom}(\Gamma, G)$ and $X_G(\Gamma) = \text{Hom}(\Gamma, G) // G$ such that the coordinate rings, $\mathbb{C}[\text{Hom}(\Gamma, G)]$ and $\mathbb{C}[X_G(\Gamma)]$, are nil-radical quotients of the algebras of global sections of $\text{Hom}(\Gamma, G)$ and of $X_G(\Gamma)$, cf. [S2].

For every completely reducible $\rho : \mathbb{Z}^N \to G$ there exists a natural linear map

$$\phi : H^1(\mathbb{Z}^N, Ad\rho) \to T_{\rho}X_G(\mathbb{Z}^N)$$

defined explicitly in [S2] Thm. 53, where the cohomology group has coefficients in the Lie algebra $\mathfrak{g}$ of $G$, twisted by $\rho$ composed with the adjoint representation of $G$. Although this map is not an isomorphism in general, it is known to be one for good $\rho$, cf. [S2] Thm. 53. As we have seen in the previous section, there are no irreducible representations in $X^0_G(\mathbb{Z}^N)$. Nonetheless, Theorem 4.1 implies the following result which will be used in Section 6:

**Corollary 4.2** (Proof in Sec. 9).

For every $\rho : \mathbb{Z}^N \to \mathbb{T} \subset G$ such that $\rho(\mathbb{Z}^N)$ is Zariski dense in $\mathbb{T}$, the map (2) is an isomorphism.

One says that a subvariety $S$ of an algebraic variety $X$ is a Chevalley section with respect to a $G$-action on $X$, if the natural map $S//N(S) \to X/G$ is an isomorphism, where $N(S) = \{g \in G : gS = S\}$, cf. [PV] Sec 3.8. For example, any maximal torus in $G$ is a Chevalley section of $G$ with respect to the $G$-action by conjugation.

The crucial question of whether $\chi : \mathbb{T}^N/W \to X^0_G(\mathbb{Z}^N)$ is an isomorphism is equivalent to the question whether $\text{Hom}(\mathbb{Z}^N, \mathbb{T})$ is a Chevalley section of $\text{Hom}^0(\mathbb{Z}^N, G) = \pi^{-1}(X^0_G(\mathbb{Z}^N))$ under the $G$ action by conjugation.

5. More on connected components of $X_G(\mathbb{Z}^N)$ for semi-simple $G$

Assume now that $G$ is semi-simple. Then $\pi_1(G)$ is finite and the central extension

$$\{e\} \to \pi_1(G) \to \bar{G} \to G \to \{e\},$$
where $\tilde{G}$ is the universal cover of $G$, defines an element $\tau \in H^2(G, \pi_1(G))$, cf. [Lm] Thm IV.3.12. (Since the extension is central, the action of $G$ on $\pi_1(G)$ is trivial.) Hence, every representation $\rho : \mathbb{Z}^N \to G$ defines $\rho^*(\tau) \in H^2(\mathbb{Z}^N, \pi_1(G))$. By the universal coefficient theorem,

$$H^2(\mathbb{Z}^N, \pi_1(G)) = \text{Hom}(H_2(\mathbb{Z}^N), \pi_1(G)) = \text{Hom}(\mathbb{Z}^2(\mathbb{Z}^N), \pi_1(G)) = \pi_1(G)(\mathbb{Z}^N).$$

The map $\rho \to \rho^*(\tau)$ is continuous on $\text{Hom}(\mathbb{Z}^N, G)$ and it is invariant under the conjugation by $G$. Therefore, its restriction to completely reducible representations $\text{Hom}^r(\mathbb{Z}^N, G) \subset \text{Hom}(\mathbb{Z}^N, G)$ factors through a continuous map $\text{Hom}^r(\mathbb{Z}^N, G)/G = X_G(\mathbb{Z}^N) \to \pi_1(G)(\mathbb{Z}^N)$. Since this map is constant on connected components of $X_G(\mathbb{Z}^N)$, it yields

$$\Psi : \pi_0(X_G(\mathbb{Z}^N)) \to H^2(\mathbb{Z}^N, \pi_1(G)).$$

**Proposition 5.1.** $\Psi$ is a bijection for $G$ connected and $N = 2$.

Following [BFM], we say that $(g_1, g_2) \in \tilde{G}^2$ is a c-pair if $[g_1, g_2] = c \in C(G)$, the center of $G$.

**Proof of Proposition 5.1.** Since $G$ is connected, the connected components of $\text{Hom}(\mathbb{Z}^2, G)$ are in a natural bijection with those of $X_G(\mathbb{Z}^N)$. Let $K$ be the compact form of $G$. By [FL], the map $\text{Hom}(\mathbb{Z}^2, K) \to \text{Hom}(\mathbb{Z}^2, G)$ induced by the embedding $K \to G$ is a bijection on connected components. By Cartan decomposition, $K$ is a deformation retract of $G$ and, consequently, $\pi_1(K) = \pi_1(G)$. Therefore, it is enough to show that the corresponding map $\pi_0(\text{Hom}(\mathbb{Z}^2, K)) \to \pi_1(K)$ is a bijection. The representations $\rho : \mathbb{Z}^2 \to K$ with $\Psi(\rho) = c \in H^2(\mathbb{Z}^2, \pi_1(K)) = \pi_1(K) \subset C(K)$ correspond to c-pairs in $K$, cf. [BFM]. By [BFM Thm. 1.3.1], the space of c-pairs for any given $c \in \pi_1(K)$ is non-empty and connected.

$\Psi$ is a bijection between $\pi_0(X_G(\pi_1(F)))$ and $\pi_1(G)$ for closed orientable surfaces $F$ of genus $> 1$ as well, cf. [LM].

$\Psi$ is generally not 1-1 for $N > 2$. For example, $X_G(\mathbb{Z}^N)$ is disconnected for $N > 2$ and for all simply-connected groups $G$ other than the products of $\text{SL}(2, \mathbb{C})$ and of $\text{Sp}(n, \mathbb{C})$, [FL] [KS].

Denote by $X_G(\mathbb{Z}^2)$ the connected component of $X_G(\mathbb{Z}^2)$ with the $\Psi$-value $c \in \pi_1(G)$. Identify $\pi_1(G)$ with a subgroup of the center of $G$, $C(G)$. The group $G$ acts on on the set of all c-pairs, $M^c_G \subset \tilde{G}^2$, and it is easy to see that the natural map

$$\phi_c : M^c_G/\tilde{G} \to X_G(\mathbb{Z}^2)$$

is a finite algebraic map. Let us analyze $M^c_G/\tilde{G}$ further following the approach of [BFM]: Any element $c \in C(\tilde{G})$ acts on $\mathbb{T}$. Let $S^c \subset \mathbb{T}$ be the connected component of identity in the invariant part of $c$ action on $\mathbb{T}$. Let $S'$ be the subtorus of $\mathbb{T}$ determined by the orthogonal component of the Lie algebra of $S$ in the Lie algebra of $\mathbb{T}$, with respect to the Killing form. Then $F_S = S \cap S'$ is a finite group. Following [BFM Thm 1.3.1], it is easy to
show that there is a regular map,
\[ \theta_c : ((S/F) \times (S/F))/W \to M^c_G, \]
where \( W \) is the quotient of the normalizer of \( S \) by its centralizer in \( G \).

**Problem 5.2.** Is \( M^c_G \) is irreducible? Is \( \theta_c \) a normalization map? Is it an isomorphism?

6. Symplectic nature of the character varieties of the torus

Goldman constructed a symplectic form on the set of equivalence classes of “good” representations in \( X_G(\pi_1(F)) \) for every reductive \( G \) and for every closed orientable surface \( F \). Motivated by applications to quantum topology, \([S1]\), we are going to extend his construction to tori.

Goldman’s approach relies on identifying the tangent space, \( T[\rho] X_G(\pi_1(F)) \), at an irreducible \( \rho \) with \( H^1(F, \mathfrak{g}) \), where \( \mathfrak{g} \) is the Lie algebra of \( G \), and the coefficients in this cohomology are twisted by \( \text{Ad}\rho \), cf. \([Go1, Go2]\). (One needs an additional assumption that the stabilizer of \( \rho(\pi_1(F)) \subset G \) coincides with the center of \( G \), \([S2]\).) Although his construction does not extend to \( X^0_G(\mathbb{Z}_2) \) (i.e. torus), since, as shown in Sec. 3, no representation in that component of character variety is irreducible for non-abelian \( G \), we resolved that difficulty with our Corollary 4.2.

Let \( X'_G(\mathbb{Z}^N) = \text{Hom}'(\mathbb{Z}^N,G) // G \), where \( \text{Hom}'(\mathbb{Z}^N,G) \) is the space of \( G \)-representations of \( \mathbb{Z}^N \) with a Zariski dense image in a maximal torus of \( G \). (Since all representations in \( \text{Hom}'(\mathbb{Z}^N,G) // G \) are completely reducible, it is the set-theoretic quotient.) Let \( N = 2 \). The composition of the cup product
\[ H^1(\mathbb{Z}^2, \text{Ad}\rho) \times H^1(\mathbb{Z}^2, \text{Ad}\rho) \to H^1(\mathbb{Z}^2, \text{Ad}\rho \otimes \text{Ad}\rho) \]
with the map
\[ H^1(\mathbb{Z}^2, \text{Ad}\rho \otimes \text{Ad}\rho) \to H^2(\mathbb{Z}^2, \mathbb{C}) = \mathbb{C} \]
induced by an \( \text{Ad}G \)-invariant symmetric, non-degenerate bilinear form,
\[ \mathfrak{B} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}, \]
defines a skew-symmetric pairing
\[ \omega : H^1(\mathbb{Z}^2, \text{Ad}\rho) \times H^1(\mathbb{Z}^2, \text{Ad}\rho) \to \mathbb{C}. \]
By Corollary 4.2 this pairing defines a differential 2-form on \( X'_G(\mathbb{Z}^N) \). We claim that \( \omega \) is symplectic. Let us precede the proof with a construction of another closely related form: Let \( \omega' \) be the 2-form on \( T(e,e)T \times T = t \times t \) defined by
\[ \omega'((v_1, w_1), (v_2, w_2)) = \mathfrak{B}(v_1, w_2) - \mathfrak{B}(v_2, w_1). \]
It defines an invariant 2-form on \( T \times T \) which descends to a non-degenerate skew-symmetric 2-form on \( (T \times T)/W \). (Recall that \( W \) acts on \( T \) and, by extension, it acts diagonally on \( T \times T \).)
Proposition 6.1 (Proof in Sec. [9]).
(1) The pullback of $\omega$ through $\chi : \mathbb{T}^2/W \to X^0_G(\mathbb{Z}^2)$ coincides with $\omega'$.
(2) Both $\omega$ and $\omega'$ are symplectic.

The most obvious choice for $\mathcal{B}$ is the Killing form, $\mathcal{R}$. However, it is also useful to consider the trace form $\mathcal{T}(A, B) = Tr(AB)$ for classical Lie algebras $\mathfrak{g}$ with their natural representations by matrices, $\mathfrak{sl}(n, \mathbb{C}), \mathfrak{so}(n, \mathbb{C}) \subset M(n, \mathbb{C})$, and $\mathfrak{sp}(n, \mathbb{C}) \subset M(2n, \mathbb{C})$. In that case, $\mathcal{R} = c_{\mathfrak{g}} : \mathcal{T}$, where

$$c_{\mathfrak{sl}(n, \mathbb{C})} = 2n, \quad c_{\mathfrak{so}(n, \mathbb{C})} = n - 2, \quad c_{\mathfrak{sp}(n, \mathbb{C})} = 2n + 2.$$

Our construction of $\omega$ is an exact analogue of that of Goldman’s symplectic form for character varieties of surfaces of higher genera, except for the fact that it is a holomorphic form (defined using the form $\mathcal{B}$) rather than a real form (defined by the real part of $\mathcal{B}$), cf. [2]. Therefore, it is not surprising to see most of the methods and results of [Go2] apply to character varieties of tori as well, cf. our Proposition 10.1 For example, here is a version of Goldman’s combinatorial formulas for Poisson brackets for the character varieties of the torus.

Proposition 6.2 (Proof in Sec. [10]). Let $\{\cdot, \cdot\}$ be the Poisson bracket on $\mathbb{C}[X^0_G(\mathbb{Z}^2)]$ induced by $\omega$ defined by a form $\mathcal{B} = c \cdot \mathcal{T}$, where $\mathcal{T}$ is the trace form and $c \in \mathbb{C}^*$. Let $\tau_g : X^0_G(\mathbb{Z}^2) \to \mathbb{C}$ be defined by $\tau_g([\rho]) = Tr p(g)$. Then for any $p, q, r, s \in \mathbb{Z}$,

$$\{\tau_{(p,q)}, \tau_{(r,s)}\} = \frac{1}{c} \begin{vmatrix} p & q & r & s \end{vmatrix} \left( \tau_{(p+r,q+s)} - \frac{\tau_{(p,q)} \tau_{(r,s)}}{n} \right), \text{ for } G = SL(n, \mathbb{C}),$$

and

$$\{\tau_{(p,q)}, \tau_{(r,s)}\} = \frac{1}{2c} \begin{vmatrix} p & q & r & s \end{vmatrix} \left( \tau_{(p+r,q+s)} - \tau_{(p-r,q-s)} \right) \text{ for } G = SO(n, \mathbb{C}), Sp(n, \mathbb{C}).$$

7. Proof of Theorem 2.1

Proposition 7.1 (cf. [Th]). (1) $\chi$ is 1-1
(2) $\chi$ is finite.

Proof. (1) The proof is an extension of the arguments of [B1] and of [Th]: Let $\rho, \rho' : \mathbb{Z}^N \to G$ be equivalent in $X_G(\mathbb{Z}^N)$. We prove first that $\rho$ and $\rho'$ are conjugate. Since the algebraic closures of $\rho(\mathbb{Z}^N)$ and of $\rho'(\mathbb{Z}^N)$ are finite extensions of tori, they are linearly reductive and, hence, by [S2 Prop. 8], $\rho$ and $\rho'$ are completely reducible representations of $\mathbb{Z}^N$ into $G$. Since the orbit of the $G$-action by conjugation on a completely reducible representation in $Hom(\mathbb{Z}^N, G)$ is closed, cf. [S2 Thm. 30], we see that $\rho' = g g^{-1}$, for some $g \in G$. The centralizer of $\rho(\mathbb{Z}^N)$, $Z(\rho(\mathbb{Z}^N)) \subset G$ is a reductive group by [Hu 26.2A] since the proof there is valid not only for a subtorus but for any subset. Clearly $\mathbb{T} \subset Z(\rho(\mathbb{Z}^N))$. Since the elements of $\mathbb{T}$ commute with elements of $\rho'(\mathbb{Z}^N) = g p(\mathbb{Z}^N) g^{-1}$, the elements of $g^{-1} \mathbb{T} g$ commute with those of $\rho(\mathbb{Z}^N)$ and, hence, $g^{-1} \mathbb{T} g \subset Z(\rho(\mathbb{Z}^N))$. Since $\mathbb{T}$ and $g^{-1} \mathbb{T} g$ are maximal tori in $Z(\rho(\mathbb{Z}^N))$, there is $h \in Z(\rho(\mathbb{Z}^N))$ such that $h^{-1} g^{-1} \mathbb{T} g = \mathbb{T}$. This
conjugation on $T$ coincides with the action of an element $w$ of the Weyl group on $T$. Since

$$w \cdot \rho(x) = h^{-1} g^{-1} \rho'(x) gh = h^{-1} \rho(x) h = \rho(x)$$

for every $x \in \mathbb{Z}^N$, the statement follows.

(2) We follow [Th]: The map $T^N/W \to T/W \times \cdots \times T/W$ is finite. Since it factors through $T^N/W \to X_G(\mathbb{Z}^N) \to X_G(\mathbb{Z}) \to \cdots \times X_G(\mathbb{Z}) \to T/W \times \cdots \times T/W$, and the right map is an isomorphism, the map $T^N/W \to X_G(\mathbb{Z}^N)$ is finite, cf. [Ka, Lemma 2.5]. □

**Lemma 7.2.** (1) For every non-trivial homomorphism $\psi : \mathbb{Z}^N \to \mathbb{C}^*$, $H^1(\mathbb{Z}^N, \psi) = 0$.

(2) Let $g$ and $t$ be the Lie algebras of $G$ and of a maximal torus $T$ in $G$, respectively. If $\rho : \mathbb{Z}^N \to T$ is a representation whose image does not lie in $\text{Ker} \alpha$, for any root $\alpha$ of $g$, then the embedding $t \subset g$ induces an isomorphism $t^N = H^1(\mathbb{Z}^N, t) \to H^1(\mathbb{Z}^N, \text{Ad}\rho)$.

**Proof.** (1) The first cohomology group is the quotient of the space of derivations

$$\sigma : \mathbb{Z}^N \to \mathbb{C}, \quad \sigma(a + b) = \sigma(a) + \psi(a)\sigma(b)$$

by the principal derivations, $\sigma_m(a) = (\psi(a) - 1) \cdot m$,

for some $m \in \mathbb{C}$.

If $\psi(v) \neq 0$, for some $v \in \mathbb{Z}^N$ then for every $w \in \mathbb{Z}^N$,

$$\sigma(v) + \psi(v)\sigma(w) = \sigma(v + w) = \sigma(w) + \psi(w)\sigma(v).$$

Hence

$$\sigma(w) = (\psi(w) - 1)\sigma(v)/(\psi(v) - 1)$$

and $\sigma$ is the principal derivation $\sigma_m$ for $m = \sigma(v)/(\psi(v) - 1)$.

(2) Consider a root decomposition of $g$,

$$g = t \oplus \bigoplus_{\alpha} g_\alpha,$$

where the sum is over all roots of $g$ relative to $t$ and $g_\alpha$’s are root subspaces of $g$. [B2, 8.17]. Since the image of $\rho$ lies in $T$, this root decomposition is $\text{Ad}\rho$ invariant. Therefore,

$$H^1(\mathbb{Z}^N, \text{Ad}\rho) = H^1(\mathbb{Z}^N, (t)_{\text{Ad}\rho}) \oplus \bigoplus_{\alpha} H^1(\mathbb{Z}^N, (g_\alpha)_{\text{Ad}\rho}).$$

The $\text{Ad}\rho$ action on $t$ is trivial. On the other hand, every $v \in \mathbb{Z}^N$ acts on $g_\alpha$ by the multiplication by $\alpha(\rho(v))$. Now the statement follows from (1). □
Proof of Theorem 2.1. (1) By Prop 7.1(2), $\chi$ is finite and, hence, its image is closed. Since $T^N/W$ is irreducible, also $X^0_G(Z^N) = \chi(T^N/W)$ is irreducible and, consequently, it is contained in an irreducible component $Z$ of $X_G(Z^N)$. It is enough to show that $\dim Z = \dim X^0_G(Z^N)$.

Consider a representation $\rho : Z^N \to T \subset G$ with a Zariski dense image in $T$. We have

$$T_\rho \text{Hom}(Z^N, G) \simeq Z^1(Z^N, Ad \rho) \simeq H^1(Z^N, Ad \rho) \oplus B^1(Z^N, Ad \rho),$$

by [S2 Thm 35]. The first summand has dimension $N \cdot \text{rank } G$, by Lemma 7.2. The second, composed of functions $\sigma_v : Z^N \to g$ of the form

$$\sigma_v(w) = (Ad \rho(w) - 1)v,$$

for some $v \in g$, has dimension $\dim g - \text{rank } g$. (Indeed, $\sigma_v = 0$ for $v \in \mathfrak{t}$ while $\sigma_v$'s are linearly independent for basis elements $v$ of a subspace of $g$ complementary to $\mathfrak{t}$.) As before, let $\text{Hom}^0(Z^N, G) = \pi^{-1}(X^0_G(Z^N))$. Then

$$(5) \quad \dim \text{Hom}^0(Z^N, G) \leq \dim T_\rho \text{Hom}(Z^N, G) = N \cdot \text{rank } G + \dim G - \text{rank } G.$$ 

$X_G(Z^N)$ is the quotient of $\text{Hom}^0(Z^N, G)$ by the action of $G$ with the stabilizer of dimension $\text{rank } G$ at $\rho$. Since the stabilizer dimension is an upper semi-continuous function, cf. [PV] Sec. 7], the stabilizers near $\rho$ have dimensions at least $\text{rank } G$. Therefore,

$$(6) \quad \dim Z \leq \dim \text{Hom}^0(Z^N, G) - (\dim G - \text{rank } G) \leq N \cdot \text{rank } G.$$ 

However, since $\chi$ is an embedding of a variety of dimension $N \cdot \text{rank } G$ into $X^0_G(Z^N)$,

$$N \cdot \text{rank } G \leq \dim X^0_G(Z^N) \leq \dim Z.$$ 

This inequality together with (6) implies the statement.

(2) By Proposition 7.1(1) $\chi$ is 1-1. Hence, $\chi : T^N/W \to X^0_G(Z^N)$ is birational, by [Mu] Prop 3.17. Since $\chi$ is finite, $\chi$ is a normalization map, cf. [Sh] II.5.

(3) By Proposition 7.3 $\chi_*$ is onto. Since $T^N/W \to X_G(Z^N)$ factors through $X^0_G(Z^N)$, also $\chi_* : \mathbb{C}[X^0_G(Z^N)] \to \mathbb{C}[T^N/W]$ is onto. Since $\chi$ is onto $X^0_G(Z^N)$, $\chi_*$ is 1-1 and, hence, an isomorphism. 

The proof of Theorem 2.1(3) above relies on the following:

Proposition 7.3. For $G = \text{GL}(n, \mathbb{C}), \text{SL}(n, \mathbb{C}), \text{Sp}(n, \mathbb{C}), \text{SO}(n, \mathbb{C})$ and every $n$ and $N$, the dual map $\chi_* : \mathbb{C}[X_G(Z^N)] \to \mathbb{C}[T^N/W] = \mathbb{C}[T^N]^W$ is onto.

Proof. The coordinate ring of a maximal torus in a reductive algebraic group can be identified with the group ring, $\mathbb{C}A$, of the weight lattice, $\Lambda$, of $G$. Following [PH] §23.2, we have

(a) If $G = \text{GL}(n, \mathbb{C})$ then

$$\mathbb{C}[T^N] = \mathbb{C}[x_{ij}^{\pm 1}, 1 \leq i \leq n, 1 \leq j \leq N].$$
(b) If $G = \text{SL}(n, \mathbb{C})$ then
\[\mathbb{C}[T^N] = \mathbb{C}[x_{ij}^{\pm 1}, 1 \leq i \leq n, 1 \leq j \leq N]/I,\]
where $I$ is generated by $\prod_{i=1}^{n} x_{ij} - 1$ for $1 \leq j \leq N$.

(c) If $G = \text{Sp}(n, \mathbb{C})$ then
\[\mathbb{C}[T^N] = \mathbb{C}[x_{ij}^{\pm 1}, 1 \leq i \leq n, 1 \leq j \leq N]/I.\]

(d) If $G = \text{SO}(2n, \mathbb{C})$ then
\[\mathbb{C}[T^N] = \mathbb{C}[x_{ij}^{\pm 1}, (x_{1j} \cdot \ldots \cdot x_{nj})^{\frac{1}{2}}, 1 \leq i \leq n, 1 \leq j \leq N].\]

(e) If $G = \text{SO}(2n+1, \mathbb{C})$ then
\[\mathbb{C}[T^N] = \mathbb{C}[x_{ij}^{\pm 1}, 1 \leq i \leq n, 1 \leq j \leq N].\]

(Note that $(x_{1j} \cdot \ldots \cdot x_{nj})^{\frac{1}{2}}$ is a weight of $\text{Spin}(2n+1, \mathbb{C})$ but not of $\text{SO}(2n + 1, \mathbb{C})$.)

Hence each monomial in variables $x_{ij}$ is of a form
\[m = \prod_{i=1}^{n} x_{ij}^{\alpha_i},\]
where $\alpha_i = (\alpha_{i1}, ..., \alpha_{in})$ are in $\mathbb{Z}^N$ for $G = \text{GL}(n, \mathbb{C}), \text{SL}(n, \mathbb{C}), \text{Sp}(n, \mathbb{C}), \text{SO}(2n+1, \mathbb{C})$ and in $(\frac{1}{2}\mathbb{Z})^N$ for $G = \text{SO}(2n, \mathbb{C})$. (For $G = \text{SL}(n, \mathbb{C}), \text{Sp}(n, \mathbb{C})$ such presentation of $m$ is not unique.)

We say that $m$ is a monomial of level $l$ if it has a presentation with $l$ non-vanishing alphas, $\alpha_{i1}, ..., \alpha_{in}$, and it has no presentation with $l-1$ non-vanishing alphas. We say that an element of $\mathbb{C}[T^N]$ is of level $l$ if it is a linear combination of monomials of level $\leq l$ but not a linear combination of monomials of level $< l$.

In each of the above cases, the Weyl group is a subgroup of the signed symmetric group, $SS_n = S_n \times (\mathbb{Z}/2)^n$, and the Weyl group action on $\mathbb{C}[T^N]$ extends to that of $SS_n$ on $\mathbb{C}[T^N]$ by permuting the first indices of $x_{ij}$ and negating exponents of these variables, depending on the value of $i$.

Let $\tau_\alpha$, for $\alpha \in \mathbb{Z}^N$, be the function on $\mathbb{C}[X_G(\mathbb{Z}^N)]$ sending the equivalence class of $\rho : \mathbb{Z}^N \to G$ to $Tr(\rho(\alpha))$. (By Theorems 3 and 5 of $[S3]$, $\mathbb{C}[X_G(\mathbb{Z}^N)]$ is generated by functions $\tau_\alpha$ for $G = \text{GL}(n, \mathbb{C}), \text{SL}(n, \mathbb{C}), \text{Sp}(n, \mathbb{C}), \text{SO}(2n+1, \mathbb{C})$, for all $n$.) Note that
\[\chi_*(\tau_\alpha) = \begin{cases} \sum_{i=1}^{n} x_i^\alpha & \text{for } G = \text{GL}(n, \mathbb{C}), \text{SL}(n, \mathbb{C}), \\ \sum_{i=1}^{n} x_i^\alpha + x_i^{-\alpha} & \text{for } G = \text{Sp}(n, \mathbb{C}), \\ \sum_{i=1}^{n} x_i^\alpha + x_i^{-\alpha} + 1 & \text{for } G = \text{SO}(2n+1, \mathbb{C}). \end{cases}\]

Hence $\chi_*(\tau_\alpha)$ is a constant plus a non-zero scalar multiple of $\sum_{w \in W} w \cdot x_i^\alpha$. Consequently, $\chi_*(\mathbb{C}[X_G(\mathbb{Z}^N)])$ contains all elements of level 1 in $\mathbb{C}[T^N]^W$. Therefore, it is enough to prove that $\mathbb{C}[T^N]^W$ is generated by such elements. That follows by induction from Lemma 7.3.

Let $G = \text{SO}(2n, \mathbb{C})$ now. By $[S3]$ Thm 6, $\mathbb{C}[X_G(\mathbb{Z}^N)]$ is generated by functions $\tau_\alpha$ and by the functions $Q_{2n}(\alpha_1, ..., \alpha_n)$, for $\alpha_1, ..., \alpha_n \in \mathbb{Z}^N$. The
homomorphism $\chi_*$ maps $\tau_\alpha$ to $\sum_{i=1}^n (x_i^{\alpha_i} + x_i^{-\alpha_i})$. Therefore, it is enough to prove that $\mathbb{C}[T^N]^W$ is generated by elements of level 1 and by the elements $\chi_*(Q_{2n}(\alpha_1, ..., \alpha_n))$ for $\alpha_1, ..., \alpha_n \in \mathbb{Z}^N$. These latter elements are written explicitly in the lemma below. The statement of Proposition 7.3 follows now by induction from Lemma 7.5. □

**Lemma 7.4.** For every $\alpha_1, ..., \alpha_n \in \mathbb{Z}^N$, the function

$$
\chi_*(Q_n(\alpha_1, ..., \alpha_n)) : T^N/W \rightarrow \mathbb{C}
$$

is given by

$$
\iota^n \cdot \sum_{\sigma \in S_n} sn(\sigma) \prod_{i=1}^n (x_{\sigma(i)}^{\alpha_i} - x_{\sigma(i)}^{-\alpha_i}),
$$

where, $sn(\sigma)$ is the sign of $\sigma$ and, as before, $x_{k}^{\alpha_i} = \prod_{j=1}^n x_{kj}^{\alpha_i}$. 

**Proof.** $Q_n(\alpha_1, ..., \alpha_n)$ is a composition of two functions. The first one sends $(x_{ij})$ to an element in $T^N \subset G^N$ whose $k$th component is $(z_{k1}, ..., z_{kn}) = (x_1^{\alpha_k}, ..., x_n^{\alpha_k}) \in (\mathbb{C}^*)^n = T$. The second one is a complex valued function on $n$-tuples of matrices in $SO(2n, \mathbb{C})$ given by

$$
Q_n(A, ..., Z) = \sum_{\sigma \in S_n} sn(\sigma) (A_{\sigma(1), \sigma(2)} - A_{\sigma(2), \sigma(1)}) \cdot ... \cdot (Z_{\sigma(n-1), \sigma(n)} - Z_{\sigma(n), \sigma(n-1)}).
$$

Since the matrices belonging to the maximal torus $T$ in $SO(n, \mathbb{C})$ are built of diagonal blocks

$$
A_j = \frac{1}{2} \begin{pmatrix}
  x_j + x_j^{-1} & i(x_j - x_j^{-1}) \\
  -i(x_j - x_j^{-1}) & x_j + x_j^{-1}
\end{pmatrix},
$$

for $j = 1, ..., n$, $Q_n$ restricted to $n$-tuples of elements of $T = (\mathbb{C}^*)^n$ sends $(z_{ki})$ to

$$
\iota^n \cdot \sum_{\sigma \in S_n} sn(\sigma) (z_{\sigma(1),1} - z_{\sigma(1),1}^{-1}) \cdot ... \cdot (z_{\sigma(n),n} - z_{\sigma(n),n}^{-1}).
$$

Hence, the statement follows. □

**Lemma 7.5.** (1) For $G = GL(n, \mathbb{C}), SL(n, \mathbb{C}), Sp(n, \mathbb{C}), SO(2n + 1, \mathbb{C})$, every element of $\mathbb{C}[T^N]^W$ of level $> 1$ can be expressed as a polynomial in elements of $\mathbb{C}[T^N]^W$ of lower level. The same is true for $G = SO(2n, \mathbb{C})$, for elements of $\mathbb{C}[T^N]^W$ level $1 < l < n$.

(2) If $G = SO(2n, \mathbb{C})$ then every element of $\mathbb{C}[T^N]^W$ of level $n$ can be expressed as a linear combination of elements

$$
\sum_{\sigma \in S_n} sn(\sigma) \prod_{i=1}^n (x_{\sigma(i)}^{\alpha_i} - x_{\sigma(i)}^{-\alpha_i}),
$$

for $\alpha_1, ..., \alpha_n \in \mathbb{Z}^N$, and of a polynomial in elements of $\mathbb{C}[T^N]^W$ of level $< l$. 

\[\text{□}\]
Proof. (1) Consider an element of $\mathbb{C}[T^N]^W$ of level $l$. Since it is a linear combination of elements
\begin{equation}
\sum_{w \in W} w \cdot m,
\end{equation}
where $m = \prod_{i=1}^{n} x_i^{\alpha_i}$ have level $\leq l$, it is enough to prove the statement for such elements. Since $\sum_{w \in W} w \cdot m$ is invariant under any even permutation of indices $i$, we can assume that $\alpha_i = 0$ for $i > l$. We consider the following three cases separately:

$(A_n)$ If $G = GL(n, \mathbb{C}), SL(n, \mathbb{C})$ then
\begin{equation}
\sum_{w \in W} w \cdot m = \sum_{w \in S_n} \prod_{i=1}^{l} x_{w(i)}^{\alpha_i},
\end{equation}
We have
\begin{equation}
\sum_{k=1}^{n} x_k^{\alpha_i} \cdot \sum_{w \in S_n} \prod_{i=1}^{l-1} x_{w(i)}^{\alpha_i} = A + B,
\end{equation}
where $A$ is the sum of the monomials $x_k^{\alpha_i} \prod_{i=1}^{l-1} x_{w(i)}^{\alpha_i}$ such that $k \in \{w(0), \ldots, w(l-1)\}$ and $B$ is the sum of the remaining ones. Note that
\begin{equation}
B = (n - l) \sum_{w \in S_n} \prod_{i=1}^{l} x_{w(i)}^{\alpha_i}
\end{equation}
is a non-zero multiple of (8). Since $A$ is an element of $\mathbb{C}[T^N]^W$ of level $< l$ and the left hand side of (9) is a product of elements of $\mathbb{C}[T^N]^W$ of level $< l$, the statement follows.

$(B_n+C_n)$ If $G = SO(2n + 1, \mathbb{C})$ or $Sp(n, \mathbb{C})$ then
\begin{equation}
\sum_{w \in W} w \cdot m = \sum_{w \in S_n} \sum_{\varepsilon_1, \ldots, \varepsilon_n \in \{+1, -1\}} \prod_{i=1}^{l} x_{w(i)}^{\varepsilon_i \alpha_i}
\end{equation}
and
\begin{equation}
\sum_{k=0}^{n} \left(x_k^{\alpha_i} + x_k^{-\alpha_i}\right) \cdot \sum_{w \in S_n} \sum_{\varepsilon_1, \ldots, \varepsilon_n \in \{+1, -1\}} \prod_{i=1}^{l-1} x_{w(i)}^{\varepsilon_i \alpha_i} = A + B,
\end{equation}
where $A$ is the sum of the monomials
\begin{equation}
x_k^{\pm \alpha_i} \prod_{i=1}^{l-1} x_{w(i)}^{\pm \alpha_i}
\end{equation}
such that $k \in \{w(0), \ldots, w(l-1)\}$.
and $B$ is the sum of the remaining ones. Note that

$$B = (n - l) \sum_{w \in S_n} \sum_{\varepsilon_1, \ldots, \varepsilon_n \in \{+1, -1\}} x_{w(i)}^{\varepsilon_i \alpha_i}$$

is a non-zero multiple of $(8)$. Since $A$ is an element of $\mathbb{C}[T^N]^W$ of level $< l$ and the left hand side of $(10)$ is a product of elements of $\mathbb{C}[T^N]^W$ of level $< l$, the statement follows.

$(D_n)$ Let $G = SO(2n, \mathbb{C})$. Since $m$ has level $< n$ and the negation of the sign of a missing variable does not affect $m$,

$$\sum_{w \in W} w \cdot m = \frac{1}{2} \sum_{w \in SS_n} w \cdot m.$$

Therefore the statement follows from the argument for the $(B_n)$ case.

(2) As before, since every element of $\mathbb{C}[T^N]^W$ of level $n$ is a linear combination of elements

$$\sum_{w \in W} w \cdot m,$$

where $m = \prod_{i=1}^{n} x_{i}^{\alpha_i}$ have level $\leq n$, it is enough to prove the statement for elements of level $n$.

Let $\varepsilon(w) = \pm 1$ for $w \in SS_n$ depending on whether the number of sign changes in $w$ is even or odd. Then

$$\sum_{w \in W} w \cdot m = \frac{1}{2} \sum_{w \in SS_n} w \cdot m + \frac{1}{2} \sum_{w \in SS_n} w \cdot \varepsilon(w)m.$$

Since the first summand on the right is $SS_n$ invariant, it is a polynomial in elements of $\mathbb{C}[T^N]^W$ of lower level by the argument for $(B_n)$ above. The second summand is equal to

$$\frac{1}{2} \sum_{\sigma \in S_n} s_{\sigma}(\sigma) \prod_{i=1}^{n} (x_{\sigma(i)}^{\alpha_i} - x_{-\sigma(i)}^{\alpha_i}).$$

\[\square\]

8. Proofs of Remark 2.4(3) and of Proposition 2.5

**Proof of Remark 2.4(3)** following [Th]: For any connected reductive group $G$, there is an epimorphism

$$C^c(G) \times [G, G] \rightarrow G,$$

with a finite kernel, where $C^c(G)$ is the connected component of the identity in the center of $G$, cf. [B2], Prop IV.14.2]. Since $[G, G]$ is semi-simple, it has a finite cover $G'$ which is simply-connected. Hence we have a finite extension of $G$:

$$\{1\} \rightarrow K \rightarrow C^c(G) \times G' \xrightarrow{\nu} G \rightarrow \{1\}.$$
By [Ric, Thm. C], \( \text{Hom}(\mathbb{Z}^2, G') \) is irreducible. Since \( C^c(G) \) is irreducible, 
\[
\text{Hom}(\mathbb{Z}^2, C^c(G) \times G') = (C^c(G))^2 \times \text{Hom}(\mathbb{Z}^2, G')
\]
is irreducible as well. Let \( \text{Hom}^c(\mathbb{Z}^2, G) \subset \text{Hom}(\mathbb{Z}^2, G) \) be the connected component of the trivial representation and let 
\[
\nu_\ast : \text{Hom}(\mathbb{Z}^2, C^c(G) \times G') \to \text{Hom}^c(\mathbb{Z}^2, G)
\]
be the morphism induced by \( \nu \). By the lemma below, \( \text{Hom}^c(\mathbb{Z}^2, G) \) is irreducible. Hence, \( X^c_G(\mathbb{Z}^2) \) is irreducible as well and, therefore, it coincides with \( X^0_G(\mathbb{Z}^2) \).

\textbf{Lemma 8.1.} \( \nu_\ast : \text{Hom}(\mathbb{Z}^2, C^c(G) \times G') \to \text{Hom}^c(\mathbb{Z}^2, G) \) is onto.

\textit{Proof.} We need to prove that every representation \( f \) in \( \text{Hom}^c(\mathbb{Z}^2, G) \) lifts to \( \tilde{f} : \mathbb{Z}^2 \to C^c(G) \times G' \) (i.e. \( f = \nu \tilde{f} \)). Since the extension \cite{12} is finite and central, it defines an element \( \alpha \in H^2(G, K) \) such that \( f \) lifts to \( \tilde{f} \) if and only if \( f^\ast(\alpha) = 0 \) in \( H^2(\mathbb{Z}^2, K) \), cf. \cite{GM} Sec. 2. Since \( H^2(\mathbb{Z}^2, K) \) is discrete, the property of \( f \) being “liftable” is locally constant on \( \text{Hom}(\mathbb{Z}^2, G) \) (in complex topology) and, hence, constant on \( \text{Hom}^c(\mathbb{Z}^2, G) \). Since the trivial representation is liftable, the statement follows. \( \square \)

The following will be needed for the proof of Proposition 2.5:

\textbf{Proposition 8.2.} \textit{If the image of \( \rho : \mathbb{Z}^N \to G \) belongs to a Borel subgroup of \( G \) then \( \rho \) is equivalent in \( X_G(\mathbb{Z}^N) \) to a representation with an image in a maximal torus of \( G \).}

\textit{Proof.} Any Borel subgroup \( B \subset G \) is of the form \( \mathbb{T} \cdot U \), where \( \mathbb{T} \) is a maximal torus and \( U \) is a unipotent subgroup of \( G \). Let \( e_1, ..., e_N \) be generators of \( \mathbb{Z}^N \) and let \( t_i, u_i \) be a decomposition of \( \rho(e_i) \). By \cite{Sp} Prop. 8.2.1, \( U \) is generated by rank 1 subgroups \( U_\alpha \), associated with roots \( \alpha \) which are positive with respect to some ordering. Furthermore, it follows from \cite{Sp} Prop. 8.1.1], there exists a sequence \( s_1, s_2, ..., \in \mathbb{T} \) which conjugates \( u_1, ..., u_N \) to elements arbitrarily close to \( \mathbb{T} \). Consequently, \( s_n \rho(e_i) s_n^{-1} \to t_i \) as \( n \to \infty \) for every \( i = 1, ..., n \). Equivalence classes of representations in \( \text{Hom}(\mathbb{Z}^N, G) \) are closed in Zariski topology and, hence, in complex topology as well. Therefore \( \rho \) is equivalent to \( \rho' \) sending \( e_i \) to \( t_i \) for every \( i \). \( \square \)

\textbf{Proof of Proposition 2.5:} Let \( G = \text{GL}(n, \mathbb{C}) \) or \( \text{SL}(n, \mathbb{C}) \). Since the matrices \( \rho(e_1), \rho(e_2), ..., \rho(e_N) \in G \) commute, they can be simultaneously conjugated to upper triangular ones and, hence, they lie in a Borel subgroup of \( G \). Now the statement follows from Proposition 8.2.

The same holds for \( G = \text{Sp}(n, \mathbb{C}) \) : Recall that a subspace \( V \) of a symplectic space \( \mathbb{C}^{2n} \) is isotropic if the symplectic form restricted to \( V \) vanishes. A stabilizer of any complete flag \( \{0\} = V_0 \subset V_1 \subset \ldots \subset V_n \) of isotropic subspaces of \( \mathbb{C}^{2n} \) is a Borel subgroup of \( \text{Sp}(n, \mathbb{C}) \), cf. \cite{GW} Ch. 10. Therefore, to complete the proof, it is enough to show the existence of a complete
isotropic flag preserved by $\rho(\mathbb{Z}^N)$. We construct it inductively. Let $V_0 = \{0\}$. Suppose that $V_k$ is defined already. Then $\rho(\mathbb{Z}^N)$ preserves $V_k^\bot$. Since any number of commuting operators on a complex vector space preserves a 1-dimensional subspace, there is such subspace $W \subset V_k^\bot/V_k$, as long as $V_k$ is not a maximal isotropic subspace. Let $V_{k+1} = \pi^{-1}(W)$ then, where $\pi$ is the projection $V_k^\bot \to V_k^\bot/V_k$. \hfill \Box

9. Proof of Theorem 4.1, Corollary 4.2, and Proposition 6.1

Proof of Theorem 4.1 (1) The argument of the proof of Theorem 2.1(1) shows that (5) is an equality and, therefore, $\rho$ is a simple point of $\text{Hom}(\mathbb{Z}^N, G)$. By [Sh] II §2 Thm 6, $\rho$ belongs to a unique component.

(2) Consider the map $\lambda : X_G(\mathbb{Z}^N) \to X_G(\mathbb{Z}) \times \ldots \times X_G(\mathbb{Z})$, sending $[\rho]$ to the $N$-tuple $([\rho(e_1)], \ldots, [\rho(e_N)])$. Since the composition $T^N \to T^N/W \xrightarrow{\lambda} X_G(\mathbb{Z}^N) \xrightarrow{\lambda} X_G(\mathbb{Z}) \times \ldots \times X_G(\mathbb{Z})$
is the Cartesian product of the maps $T \to X_G(\mathbb{Z}) = T/W$, its differential is onto. Hence $d\chi$ has rank $N \cdot \text{rank } G$, which implies that $d\chi$ is 1-1. By (1) and by Theorem 2.1(1), $\text{dim } T_\rho X_G(\mathbb{Z}^N) = N \cdot \text{rank } G$. Therefore, $d\chi$ is an isomorphism.

(3) follows from [DR] Prop. 4.18 (cf. the argument of the proof of Luna Étale Slice Theorem in [DR]). \hfill \Box

Proof of Corollary 4.2 By Lemma 7.2(2), there is a natural identification of $t^N = H^1(\mathbb{Z}^N, t)$ with $H^1(\mathbb{Z}^N, Ad \rho)$. Since the resulting isomorphism $H^1(\mathbb{Z}^N, Ad \rho) \to t^N \to T_\rho T^N/W \to T_\rho X_G(\mathbb{Z}^N)$ coincides with (2), the statement follows. \hfill \Box

Proof of Proposition 6.1 (1) By Lemma 7.2 $H^1(\mathbb{Z}^2, Ad \rho) = H^1(\mathbb{Z}^2, t)$ for $[\rho] \in X'_{G}(\mathbb{Z}^2)$ (i.e. in the domain of $\omega$). Since the cup product $H^1(\mathbb{Z}^2, t) \times H^1(\mathbb{Z}^2, t) \xrightarrow{\cup} H^2(\mathbb{Z}^2, t \otimes t) = t \otimes t$ sends $(v_1, w_1), (v_2, w_2)$ to $v_1 \otimes w_2 - v_2 \otimes w_1$, the statement follows.

(2a) Any triple of vectors in any tangent space to $\mathbb{T} \times \mathbb{T}$ extends to invariant vector fields $X_1, X_2, X_3$ on $\mathbb{T} \times \mathbb{T}$. Since $d\omega'(X_1, X_2, X_3)$ is a linear combination of terms $X_i(\omega'(X_j, X_k))$ and $\omega'([X_1, X_j], X_k)$, it vanishes for such fields. Therefore, $\omega'$ is closed. Being non-degenerate, it is also symplectic.

(2b) Since $\chi'(d\omega) = d(\chi^* \omega) = d\omega' = 0$, and $\chi^*$ (being a normalization map) is an isomorphism of tangent spaces on a Zariski dense subset of $X'_{G}(\mathbb{Z}^N)$, $d\omega = 0$ on $X'_{G}(\mathbb{Z}^N)$. By its construction, $\omega$ is non-degenerate and, hence, symplectic. \hfill \Box
10. Proof of Proposition 6.2

In the statement below, the notion of Goldman bracket refers to the Poisson bracket dual to the holomorphic Goldman symplectic form defined by \( \mathfrak{B} = c \cdot \mathcal{I} \), where \( \mathfrak{B} = c \cdot \mathcal{I} \), \( c \in \mathbb{C}^* \) and \( \mathcal{I} \) is the trace form, as in Sec. 6.

**Proposition 10.1.** The following formulas hold for Goldman brackets for all closed orientable surfaces of genus \( \geq 1 \):

1. For \( G = \text{SL}(n, \mathbb{C}) \),

\[
\{ \tau_\alpha, \tau_\beta \} = \frac{1}{c} \sum_{p \in \alpha \cap \beta} \varepsilon(p, \alpha, \beta) \left( \tau_{\alpha p, \beta p} - \tau_{\alpha p} \tau_{\beta p} \right),
\]

where \( \alpha, \beta \) are any smooth closed oriented loops in \( F \) in general position. (We identify closed oriented loops in \( F \) with conjugacy classes in \( \pi_1(F) \).) \( \alpha \cap \beta \) is the set of the intersection points and \( \alpha p, \beta p \) is the product of \( \alpha \) and \( \beta \) in \( \pi_1(F,p) \), and \( \varepsilon(p, \alpha, \beta) \) is the sign of the intersection:

\[
\begin{array}{c}
\alpha \\
\hline
\beta \quad \alpha
\end{array}
\]

2. For \( G = \text{SO}(n, \mathbb{C}), \text{Sp}(n, \mathbb{C}) \),

\[
\{ \tau_\alpha, \tau_\beta \} = \frac{1}{2c} \sum_{p \in \alpha \cap \beta} \varepsilon(p, \alpha, \beta) \left( \tau_{\alpha p, \beta p} - \tau_{\alpha p} \tau_{\beta p}^{-1} \right).
\]

Since the signed number of the intersection points between any two curves \( (p, q), (r, s) \) in a torus is \( \begin{vmatrix} p & q \\ r & s \end{vmatrix} \), the above statement immediately implies Proposition 6.2.

The proof of Proposition 10.1 uses the notion of variation function introduced in [Go2]. Let \( F : G \to \mathfrak{g} \) be the variation function with respect to \( \mathfrak{B} = c \cdot \mathcal{I} \), \( c \in \mathbb{C}^* \).

**Lemma 10.2.** Consider the standard embeddings \( \text{SL}(n, \mathbb{C}), \text{SO}(n, \mathbb{C}) \subset \text{GL}(n, \mathbb{C}), \text{Sp}(n, \mathbb{C}) \subset \text{GL}(2n, \mathbb{C}) \), and the induced embeddings of Lie algebras. Then

1. \( F : \text{SL}(n, \mathbb{C}) \to \mathfrak{sl}(n, \mathbb{C}) \subset \text{M}(n, \mathbb{C}) \) is given by \( F(A) = \frac{1}{c} (A - \frac{\text{tr}(A)}{n} I) \)

2. \( F : \text{SO}(n, \mathbb{C}) \to \mathfrak{so}(n, \mathbb{C}) \subset \text{M}(n, \mathbb{C}) \) is given by \( F(A) = \frac{1}{2c} (A - A^{-1}) \), and

3. \( F : \text{Sp}(n, \mathbb{C}) \to \mathfrak{sp}(n, \mathbb{C}) \subset \text{M}(2n, \mathbb{C}) \) is given by \( F(A) = \frac{1}{2c} (A - A^{-1}) \).

**Proof.** By its definition, the variation function with respect to \( c \cdot \mathcal{I} \), is \( c^{-1} \) times the variation function with respect to \( \mathcal{I} \). Therefore, it is enough to prove the statement for \( c = 1 \).

It is easy to see that the following “complex” version of [Go2, Sec 1.4] holds: In the above setting, the variation function is given by the composition

\[
G \to \text{GL}(n, \mathbb{C}) \to \text{M}(n, \mathbb{C}) \xrightarrow{pr} \mathfrak{g},
\]
where \( pr \) is the orthogonal projection with respect to \( \mathcal{T} \). Indeed, Goldman’s proof of the “real” version carries over the complex case. Now the statement follows from computations like those of Corollaries 1.8 and 1.9 of \([Go2]\).

**Proof of Proposition 10.1** By Goldman’s Product Formula, \([Go2]\),

\[
\{ \tau_\alpha, \tau_\beta \}(\rho) = \sum_{p \in \alpha \cap \beta} \varepsilon(p; \alpha, \beta) \mathcal{B}(F_{\alpha_p}(\rho_p), F_{\beta_p}(\rho_p)),
\]

where \( \rho_p \) is the \( G \)-representation of \( \pi_1(F, p) \) which belongs to the conjugacy class \([\rho] \).

By Lemma 10.2(1), for \( G = \text{SL}(n, \mathbb{C}) \),

\[
\mathcal{B}(F_{\alpha_p}(\rho_p), F_{\beta_p}(\rho_p)) = \frac{c \cdot Tr}{n} \left( \frac{1}{c} \left( \rho_p(\alpha_p) - \frac{Tr(\rho_p(\alpha_p))}{n} I \right) \left( \rho_p(\beta_p) - \frac{Tr(\rho_p(\beta_p))}{n} I \right) \right) = \frac{1}{c} \left( Tr(\rho_p(\alpha_p \beta_p)) - \frac{Tr(\rho_p(\alpha_p))Tr(\rho_p(\beta_p))}{n} \right)
\]

and Proposition 10.1(1) follows.

An analogous computation using Lemma 10.2(2) and (3) implies part (2).

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