The post-Minkowskian limit and gravitational wave solutions for general fourth-order gravity theories are discussed. Specifically, we consider a Lagrangian with a generic function of curvature invariants \( f(R, R_{\alpha \beta} R^{\alpha \beta}, R_{\alpha \beta \gamma \delta} R^{\alpha \beta \gamma \delta}) \). It is well known that when dealing with General Relativity such an approach provides massless spin-two waves as propagating degrees of freedom of the gravitational field while this theory implies other additional propagating modes in the gravity spectra. We show that, in general, fourth order gravity, besides the standard massless graviton is characterized by two further massive modes with a finite-distance interaction. We find out the most general gravitational wave solutions in terms of Green functions in vacuum and in presence of matter sources. If an electromagnetic source is chosen, only the modes induced by \( R_{\alpha \beta} R^{\alpha \beta} \) are present, otherwise, for any \( f(R) \) gravity model, we have the complete analogy with tensor modes of General Relativity. Polarizations and helicity states are classified in the hypothesis of plane wave.

PACS numbers: 04.30.Nk, 98.70.Vc, 04.50.Kd

Keywords: Post-Minkowskian limit, gravitational radiation, alternative theories of gravity

I. INTRODUCTION

Identifying the correct theory of gravity is a crucial issue of modern physics due to the fact that General Relativity, in its standard formulation, presents shortcomings at ultraviolet and infrared limits. For the first issue, we need a theory that should deal with gravity under the same standard of the other fundamental interactions (Quantum Gravity) [1, 2]. In the other case, modifications of gravity are required to deal with the vast phenomenology coming from astrophysics and cosmology, generally addressed as dark matter and dark energy issues [3, 4]. This matter is rather controversial due to the fact that the ambiguity comes out from the fact that phenomenology could be explained successfully both considering new material ingredients (dark matter particles addressing the problem of structure formation and light scalar fields giving rise to the acceleration of the Hubble fluid) or modifying gravity that, at scales larger than Solar System, could behave in different way with respect to the weak field limit related to General Relativity [3, 4]. Furthermore, there are very few investigations and experimental constraints probing the gravitational field in very strong regimes. Several times, extrapolations of General Relativity are simply assumed without considering corrections and alternatives that could strongly affect theoretical and experimental results.

With this situation in mind, it is urgent to find out some experimentum crucis or some test bed capable of discriminating among concurring gravitational theories, that, in any case, should reproduce the well-founded theoretical and experimental results of General Relativity. At astrophysical level, discriminations could come from anomalous stellar systems whose structures and parameters do not find room in the constraints and limits imposed by General Relativity. For example, extremely massive neutron stars, magnetars or compact objects like quark stars could be independent signatures for modified theories of gravity considered as extensions of General Relativity in the strong field regime [11, 12].

Besides, discrimination could happen in the realm of gravitational wave physics. This sector of physics, practically unexplored from the point of view of modified gravity, deserves a lot of attention since the large part of efforts has been devoted to the study of gravitational radiation in the realm of General Relativity discarding the fact that modified gravity presents a huge amount of new phenomenology and features. For example, only General Relativity strictly forecasts massless gravitons with two polarizations. In general, modified gravity and, in particular Extended
Gravity, allows also massive and ghost modes and then further polarizations\[13, 14\]. These possibilities are partially studied from a theoretical point of view and practically ignored from the experimental point of view due to the enormous difficulties related to the detection of gravitational waves. However, the forthcoming experimental facilities like VIRGO–LIGO collaboration, LISA etc. could be suitable, in principle, for detecting these further modes.

In this paper, we propose a systematic study of gravitational wave solutions in theories where generic functions of curvature invariants are considered. These are a straightforward generalization of $f(R)$ gravity where the full budget of degrees of freedom, related to the curvature invariants, is considered. It is interesting to see that relaxing the hypothesis that gravitational interaction is derived only from the Hilbert-Einstein action, linear in the Ricci curvature scalar $R$, further gravitational modes, polarizations and helicity states come out. This new features are directly derived from the post-Minkowskian limit of the theory and points out a new rich phenomenology that deserves investigation.

This paper is organized as follows. In Section II we report briefly the field equations of fourth order gravity. In Section III we discuss the post-Minkowskian limit and the linearized field equations while, in Section IV, the gravitational wave solutions are reported. Section V is devoted to the discussion of all possible polarization and helicity states of the wave solutions. Conclusions are reported in Section VI.

II. THE FIELD EQUATIONS OF FOURTH ORDER GRAVITY

The most general class of gravitational theories involving curvature invariants in four dimensions is given by the action

$$A = \int d^4x\sqrt{-g}\left[f(X,Y,Z) + \mathcal{L}_m\right]$$

(1)

where $f$ is an unspecified function of curvature invariants $X = R$, $Y = R_{\alpha\beta}R^{\alpha\beta}$, and $Z = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$. The term $\mathcal{L}_m$ is the minimally coupled ordinary matter contribution. In the metric approach, the field equations are obtained by varying (1) with respect to $g_{\mu\nu}$. We get the fourth-order differential equations

$$H_{\mu\nu} = f_X R_{\mu\nu} - \frac{f}{2}g_{\mu\nu} - f_X g_{\mu\nu} + f_{Y} R_{\mu}^{\alpha}R_{\alpha\nu} - 2[f_{Y} R_{\mu}^{\alpha\beta};\alpha] + \Box[f_Y R_{\mu\nu}] + [f_Y R_{\alpha\beta}]^{\alpha\beta}g_{\mu\nu}$$

$$+ 2f_Z R_{\mu\alpha\beta\gamma}R_{\nu}^{\alpha\beta\gamma} - 4[f_Z R_{\mu}^{\alpha\beta};\alpha\beta] = \mathcal{X} T_{\mu\nu}$$

(2)

where $T_{\mu\nu} = \frac{\delta\sqrt{-g}}{\delta g_{\mu\nu}}$ is the energy-momentum tensor of matter, $f_X = \frac{\delta f}{\delta X}$, $f_Y = \frac{\delta f}{\delta Y}$, $f_Z = \frac{\delta f}{\delta Z}$, $\Box = \sigma\sigma$, and $\mathcal{X} = 8\pi G$. The conventions for Ricci’s tensor is $R_{\mu\nu} = R^{\alpha}{}_{\mu\nu}{}^{\alpha}$ and for the Riemann tensor is $R^{\alpha}{}_{\beta\mu\nu} = R^{\alpha}{}_{\beta\nu\mu} + \ldots$. The affinities are the usual Christoffel’s symbols of the metric: $\Gamma_{\alpha\beta}^{\nu} = \frac{1}{2}g^{\nu\sigma}(\partial X_{\alpha\sigma} + \partial Y_{\alpha\sigma} - \partial Z_{\alpha\sigma}).$

The adopted signature is $(- + + -)$ (see for the details \[13\]). The trace of field Eqs. (2) is the following

$$H = f_X X + 2f_Y Y + 2f_Z Z - 2f + \Box[3f_X X + f_Y X] + 2[(f_Y + 2f_Z)R^{\alpha\beta};\alpha\beta] = \mathcal{X} T$$

(3)

where $T = T_{\sigma}^\sigma$ is the trace of energy-momentum tensor and $H = H_{\sigma}^\sigma$. Some authors considered a linear Lagrangian containing not only $X, Y$ and $Z$ but also the first power of curvature invariants $\Box R$ and $R^{\alpha\beta}_{\gamma\delta}$. Such a choice is justified because all curvature invariants have the same dimension \(L^{-2}\) \[16\]. Furthermore, this dependence on the two last invariants is only formal, since from the contracted Bianchi identity \((2R^{\alpha\beta}_{\gamma\delta} - \Box R = 0)\) we have only one independent invariant. In any linear theory of gravity (the function $f$ is linear) the terms $\Box R$ and $R^{\alpha\beta}_{\gamma\delta}$ give us no contribution to the field equations, because they are four-divergences. However if we consider a function of $\Box R$ or $R^{\alpha\beta}_{\gamma\delta}$ by varying the action, we still have four-divergences but we would have the contributions of sixth order differential terms. In this paper, we consider only fourth order differential field equations. This means that the most general fourth-order theory is \[11\]. However, as we will see below, one needs only two of the three curvature invariants, due to the Gauss-Bonnet topological invariant which fixes a constraint among the curvature terms.

\footnote{Here we use the convention $c = 1.$}
III. THE POST MINKOWSKI LIMIT

Any theory of gravity has to be discussed in the weak field limit approximation. This prescription is needed to test if the given theory is consistent with the well-established Newtonian theory and with the Special Relativity as soon as the the gravitational field is weak or is almost null. Both requirements are fulfilled by General Relativity and then they can be considered two possible paradigms to confront a given theory, at least in the weak field limit, with the General Relativity itself. The Newtonian limit of \( f(R) \)-gravity and \( f(R, R_{\alpha\beta}R^{\alpha\beta}, R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}) \)-gravity can be investigated always remaining in the Jordan frame [17–22] while a preliminary study of the post-Minkoskian limit for the \( f(R) \)-gravity is provided in Ref. [23]. Here we want to derive the post-Minkowskian limit of \( f(R, R_{\alpha\beta}R^{\alpha\beta}, R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}) \) gravity, that is for generic fourth-order theory of gravity, with the aim to investigate the gravitational radiation.

The post-Minkowskian limit of any theory of gravity arises when the regime of small field is considered without any prescription on the propagation of the field. This case has to be clearly distinguished with respect to the Newtonian limit which, differently, requires both the small velocity and the weak field approximations. Often, in literature, such a distinction is not clearly remarked and several cases of pathological analysis can be accounted. The post-Minkowskian limit of General Relativity gives rise to massless gravitational waves and reproduce the Special Relativity. An analogous study can be pursued considering, instead of the Hilbert-Einstein Lagrangian, linear in the standard matter energy momentum tensor \( \Pi_{\mu\nu} \).

Actually, in order to perform the post-Minkowskian limit of field equations, one has to perturb Eqs. (2) on the Minkowski background \( \eta_{\mu\nu} \). In such a case, we obtain

\[
g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{4}\]

with \( h_{\mu\nu} \) small (\( O(h_{\mu\nu}^2) \ll 1 \)). Then the curvature invariants \( X, Y, Z \) become

\[
\begin{align*}
X & \sim X^{(1)} + O(h_{\mu\nu}^2) \\
Y & \sim Y^{(2)} + O(h_{\mu\nu}^3) \\
Z & \sim Z^{(2)} + O(h_{\mu\nu}^3)
\end{align*}
\tag{5}
\]

and the function \( f \) can be developed as

\[
f(X,Y,Z) \sim f(0) + f_X(0)X^{(1)} + \frac{1}{2}f_{XX}(0)X^{(1)}X^{(1)} + f_Y(0)Y^{(2)} + f_Z(0)Z^{(2)} + O(h_{\mu\nu}^3) \tag{6}\]

Analogous relations for partial derivatives of \( f \) are obtained. From lowest order of field Eqs. (2) and (3) we have the condition \( f(0) = 0 \), while at \( O(1) \) - order, we have\(^2\)

\[
\begin{align*}
f_X(0)R_{\mu\nu}^{(1)} & + [f_Y(0) + 4f_Z(0)]\Box_\eta R_{\mu\nu}^{(1)} - \frac{f_{XX}(0)}{2}X^{(1)}\eta_{\mu\nu} + [f_{XX}(0) + \frac{f_{XY}(0)}{2}]\eta_{\mu\nu}\Box_\eta X^{(1)} \\
-f_{XX}(0)X^{(1)}\eta_{\mu\nu} & - [f_Y(0) + 4f_Z(0)]R_{\mu\nu,\alpha}^{(1)} - f_Y(0)R_{\mu\nu,\alpha}^{(1)} = \mathcal{X}T_{\mu\nu}^{(0)} \\
-f_X(0)X^{(1)} & + [3f_{XX}(0) + 2f_Y(0) + 2f_Z(0)]\Box_\eta X^{(1)} = \mathcal{X}T^{(0)}
\end{align*}
\tag{7}\]

where \( \Box_\eta \) is the d’Alambert operator in the flat space. \( T_{\mu\nu} \) is fixed at zero-order in [22] since, in this perturbation scheme, the first order on Minkowski space has to be connected with the zero order, with respect to \( h_{\mu\nu} \), of the standard matter energy momentum tensor\(^3\). Then \( T_{\mu\nu} \) is independent of \( h_{\mu\nu} \), and satisfies the standard conservation conditions \( T^{\mu\nu}_{\mu\nu} = 0 \). By introducing the quantities

\(^2\) We are using the properties: \( 2R_{\alpha\beta}^{\alpha\beta} - \Box R = 0 \) and \( R_{\alpha\beta}^{\alpha\beta} = R^{\alpha\beta} - \Box R_{\alpha\beta} \).

\(^3\) This formalism descends from the theoretical setting of Newtonian mechanics which requires the appropriate scheme of approximation when obtained from a more general relativistic theory. This scheme coincides with a gravity theory analyzed at the first order of perturbation in the curved spacetime metric.
we get two differential equations for curvature invariant $X^{(1)}$ and Ricci tensor $R^{(1)}_{\mu\nu}$

\[
\begin{align*}
(m_1^2 - \frac{f_x(0)}{3f_{xx}(0) + 2f_y(0) + 2f_z(0)}) R_{\mu\nu}^{(1)} &= \partial^2_{\mu\nu} X^{(1)} - \eta_{\mu\nu} \left( \frac{m_2^2 - m_3^2}{3m_1^2} \partial^2_{\mu\nu} + \eta_{\mu\nu} \left( \frac{m_2^2}{2} + \frac{m_2^2 + 2m_3^2}{6m_1^2} \Box \eta \right) \right) \right) \Box X^{(1)} = m_2^2 \mathcal{X} T_{\mu\nu}^{(0)} \\
(m_1^2 + m_2^2) X^{(1)} &= -m_1^2 \mathcal{X} T^{(0)}
\end{align*}
\]

We note that in the case of $f(X)$-theory we obtain only a massive mode (with mass $m_1$) of Ricci scalar $(X)$. In fact if $f_y = f_Z = 0$ from the mass definition $m_1^2 \rightarrow (-3f_{XX}(0))^{-1}$ and $m_2^2 \rightarrow \infty$ we recover the equations of $f(X)$-gravity.

A first consideration regarding the masses $(8)$ induced by $f(X,Y,Z)$-gravity is necessary at this point. The second mass $m_2$ is originated by the presence, in the Lagrangian, of Ricci and Riemann tensor squares, but also a theory containing only Ricci tensor square gives rise to the same outcome. Obviously the same is valid also with the Riemann tensor square alone. Then such a modification of theory enables a massive propagation of Ricci Tensor and, as it is well known in the literature, a substitution of Ricci Scalar with any function of Ricci scalar enables a massive propagation of Ricci scalar. We can conclude that a Lagrangian containing any function of only Ricci scalar and Ricci tensor square is not restrictive. This result is coming from the Gauss- Bonnet invariant defined by the relation $G_{GB} = X^2 - 4Y + Z$.

In fact the induced field equations satisfy, in four dimensions, the following condition

\[
H_{\mu\nu}^{GB} = H_{\mu\nu}^{X^2} - 4H_{\mu\nu}^{Y} + H_{\mu\nu}^{Z} = 0
\]

and by substituting it at post Minkowskian level in Eqs. $(9)$, we find the same Eqs. $(8)$ with a redefinition of the masses $5$. In other words, the topological Gauss- Bonnet invariant is a constraint for the curvature invariants derived from the Riemann tensor and then one of the three can be derived from the other two. For gravity theories containing explicitly such an invariant in the post-Newtonian limit see Ref. $25$.

### IV. GRAVITATIONAL WAVE SOLUTIONS

Once developed the post-Minkowskian limit, one can search for gravitational wave solutions. The general solution of field equations $(9)$ is given by

\[
\Box h_{\mu\nu} = \Box h_{\mu\nu} + h_{\mu\nu}
\]

where $h_{\mu\nu}$ is the (homogeneous) solution in the vacuum and $h_{\mu\nu}$ is the (particular) one in the matter. First we try to find the solution $h_{\mu\nu}$. From the second line of $(9)$, by introducing the Green function $\mathcal{G}_{KG,1}(x, x')$ of Klein-Gordon field operator $\Box \eta + m_1^2$, we find

\[
4 \text{ We set } f_X = 1 \text{ i.e. } G \rightarrow f_X(0) G.
\]

\[
5 \text{ In the Newtonian and post-Minkowskian limits, we can consider as Lagrangian in the action } 1 \text{ the quantity } f(X,Y,Z) = a X + b X^2 + c Y \text{ } 22. \text{ Then the masses } (8) \text{ becomes as } m_1^2 = -\frac{a}{b}, m_2^2 = \frac{b}{c}. \text{ For the interpretation of these quantities as real masses, we find the conditions } a > 0, b < 0 \text{ and } 0 < c < -3b.
\]
\[ X^{(1)} = -m_1^2 \mathcal{X} \int d^4x' \mathcal{G}_{KG,1}(x, x') T^{(0)}(x') \] (13)

where \( x = x^\mu = (t, x) = (t, x^1, x^2, x^3) \). The first line of (10) can be recast as follows

\[
(\Box + m_2^2) R^{(1)}_{\mu\nu} = \mathcal{X} \left[ m_2^2 T^{(0)}_{\mu\nu} - \frac{m_1^2 + 2m_2^2}{6} \eta_{\mu\nu} T^{(0)} \right] 
- \frac{(m_1^2 - m_2^2) \mathcal{X}}{3} \int d^4x' \left[ \partial_{\mu\nu}^2 - \frac{m_1^2}{2} \eta_{\mu\nu} \right] \mathcal{G}_{KG,1}(x, x') T^{(0)}(x')
\] (14)

so the solution for the Ricci tensor is obtained

\[
R^{(1)}_{\mu\nu} = \mathcal{X} \int d^4x' \mathcal{G}_{KG,2}(x, x') \left[ m_2^2 T^{(0)}_{\mu\nu}(x') - \frac{m_1^2 + 2m_2^2}{6} \eta_{\mu\nu} T^{(0)}(x') \right] 
- \frac{(m_1^2 - m_2^2) \mathcal{X}}{3} \int d^4x' d^4x'' \mathcal{G}_{KG,2}(x, x') \left[ \partial_{\mu\nu}^2 - \frac{m_1^2}{2} \eta_{\mu\nu} \right] \mathcal{G}_{KG,1}(x', x'') T^{(0)}(x'')
\] (15)

where \( \mathcal{G}_{KG,2}(x, x') \) is the Green function of the field operator \( \Box + m_2^2 \). The Ricci tensor, in terms of the metric \( h \), is given by

\[
R^{(1)}_{\mu\nu} = h^\sigma_{(\mu\nu)} - \frac{1}{2} \Box h_{\mu\nu} - \frac{1}{2} h_{..\mu\nu}
\] (16)

where \( h = h^\sigma_{\sigma} \). Since we can use the harmonic gauge condition \( g^{\rho\sigma} \Gamma^\alpha_{\rho\sigma} = 0 \), we set \( h_{\mu\sigma} - 1/2 h_{..\mu} = 0 \), then the Ricci tensor becomes \( R^{(1)}_{\mu\nu} = -1/2 \Box h_{\mu\nu} \). The solution of Eq. (15) is

\[
h_{\mu\nu} = \frac{2(m_1^2 - m_2^2) \mathcal{X}}{3} \int d^4x' d^4x'' d^4x''' \mathcal{G}_{GR}(x, x') \mathcal{G}_{KG,2}(x', x'') \left[ \partial_{\mu\nu}^2 + \frac{m_1^2}{2} \eta_{\mu\nu} \right] \mathcal{G}_{KG,1}(x'', x''') T^{(0)}(x''')
- 2\mathcal{X} \int d^4x' d^4x'' \mathcal{G}_{GR}(x, x') \mathcal{G}_{KG,2}(x', x'') \left[ m_2^2 T^{(0)}_{\mu\nu}(x'') - \frac{m_1^2 + 2m_2^2}{6} \eta_{\mu\nu} T^{(0)}(x'') \right]
\] (17)

where \( \mathcal{G}_{GR}(x, x') \) is the Green function of the field operator \( \Box \). Considering the expressions of the Green functions \( \mathcal{G}_{GR}(x, x') \) and \( \mathcal{G}_{KG,1,2}(x, x') \) in terms of plane waves it is possible to rewrite the solution (17) as follows

\[
h_{\mu\nu}(x) = \mathcal{X} \int d^4x' \left[ \mathcal{Z}_s(x, x') T^{(0)}_{\mu\nu}(x') + \left( \mathcal{Z}(x, x') \eta_{\mu\nu} + \mathcal{Z}_{\mu\nu}(x, x') \right) T^{(0)}(x') \right]
\] (18)

where

\[
\mathcal{Z}_s(x, x') = 2m_2^2 \left( -1 \right)^{1+s} \int \frac{d^4k}{(2\pi)^4} e^{ik(x-x')}
\]
\[
\mathcal{Z}(x, x') = \int \frac{d^4k}{(2\pi)^4} \frac{(m_1^2 + 2m_2^2)k^2 - 3m_1^2m_2^2}{3k^4(k^2 - m_1^2)(k^2 - m_2^2)} e^{ik(x-x')}
\]
\[
\mathcal{Z}_{\mu\nu}(x, x') = \int \frac{d^4k}{(2\pi)^4} \frac{2m_1^2m_2^2k_{\mu}k_{\nu}}{3k^4(k^2 - m_1^2)(k^2 - m_2^2)} e^{ik(x-x')}
\] (19)
\(k = k^\mu = (\omega, \mathbf{k}) = (\omega, k_1^2, k_2^2, k_3^2), k x = k_\sigma x^\sigma = \omega t - \mathbf{k} \cdot \mathbf{x}, k^2 = k_\sigma k^\sigma = \omega^2 - |\mathbf{k}|^2\) and \(s = 1, 2\). We note that in the case \(f \to X\) we find \(\mathcal{Z}(k) \to 2(-1)^s k^{-2}, \mathcal{Z}(k) \to -k^{-2}, \mathcal{Z}_{\mu\nu}(k) \to 0\) and the solutions [18] become those of General Relativity:

\[
h_{\mu\nu}(x) = -2\mathcal{X} \int d^4x' \mathcal{G}_{GR}(x, x') S_{\mu\nu}^{(0)}(x'),
\]

where \(S_{\mu\nu}^{(0)} = T_{\mu\nu}^{(0)} - \eta_{\mu\nu} T^{(0)}/2\).

Finally we can recast the propagators [19] in terms of the Green functions \(\mathcal{G}_{GR}(x, x')\) and \(\mathcal{G}_{KG,s}(x, x')\) and we can see immediately the propagation of the massless interaction and of two massive interactions. We get

\[
\mathcal{Z}(x, x') = 2(-1)^{1+s} \left[ \mathcal{G}_{GR}(x, x') - \mathcal{G}_{KG,s}(x, x') \right]
\]

\[
\mathcal{Z}(x, x') = \mathcal{G}_{GR}(x, x') - \frac{1}{2} \mathcal{G}_{KG,1}(x, x') - \frac{2}{3} \mathcal{G}_{KG,2}(x, x')
\]

\[
\mathcal{Z}_{\mu\nu}(x, x') = \frac{2}{3} \eta_{\mu\nu} \left[ \frac{m_1^2 - m_2^2}{m_1^2 m_2^2} \mathcal{G}_{GR}(x, x') + \frac{\mathcal{G}_{KG,1}(x, x')}{{m_1^2}} - \frac{\mathcal{G}_{KG,2}(x, x')}{{m_2^2}} \right]
\]

A very interesting feature is achieved when we consider a traceless source with \(T^{(0)} = 0\). In this case, solutions [18] are unaffected by the functions of the Ricci scalar but only by the presence of the Ricci tensor square terms into the interaction Lagrangian. Moreover, from the second line of [18], we have also that the gravitational waves must be traceless. A prototype of traceless source is the electromagnetic field, so we have

\[
h_{\mu\nu}^{em}(x) = \mathcal{X} \int d^4x' \mathcal{Z}_2(x, x') T_{\mu\nu}^{(0), em}(x') \quad \text{with} \quad h^{em}(x) = 0.
\]

In other words, this means that, if we consider only tensor modes, all \(f(X)\) non-linear gravities are equivalent to the General Relativity also if they present a further massive scalar mode [20]. If we want to find some differences we must include the contributions generated by the Ricci tensor square. A similar behavior can be found for the propagation of photon lensed by a pointlike source [27].

Finally, the solution of the first field equation (19) in the vacuum \((T_{\mu\nu} = 0)\) can be derived. By using again the hypothesis of harmonic gauge and the principle of plane wave superposition, we get

\[
h_{\mu\nu} = \frac{m_1^2 - m_2^2}{3m_2^2} \int d^4x' \mathcal{Z}_2(x, x') \left[ \frac{\partial^2_{\mu\nu}}{m_1^2} - \frac{\eta_{\mu\nu}}{2} \right] X_{(hs)}^{(1)}(x') - 2 \int d^4x' \mathcal{G}_{GR}(x, x') R_{\mu\nu}^{(1)}(x') + h_{\mu\nu}(hs)
\]

where \(X_{(hs)}^{(1)}, R_{\mu\nu}^{(1)}, h_{\mu\nu}(hs)\) are respectively the homogeneous solutions of the Klein-Gordon equation for the Ricci scalar and tensor and the solution of wave equation for the metric. The homogeneous solutions are chosen in such a way that they satisfy the boundary conditions and the gauge harmonic condition \(h^{\mu\rho,\rho} - 1/2 h^{\mu\mu} = 0\). Below we will consider polarizations and helicity states in vacuum starting from this result.

V. POLARIZATIONS AND HELICITY STATES IN VACUUM

We can analyze the propagation in vacuum by performing the Fourier analysis of field Eqs. [19]. In the Fourier space, we have

\[
k^2 \left( m_2^2 - k^2 \right) \tilde{h}_{\mu\nu} - \left[ \frac{m_2^2 - m_1^2}{3m_2^2} k_\mu k_\nu + \eta_{\mu\nu} \left( \frac{m_2^2}{2} - \frac{m_1^2 + 2m_2^2}{6m_1^2} k^2 \right) \right] \tilde{h} = 0
\]

\[
k^2 (m_1^2 - k^2) \tilde{h} = 0
\]

where \(\tilde{h}_{\mu\nu}, \tilde{h}\) are the Fourier representation of \(h_{\mu\nu}, h\). The gauge condition now becomes \(\tilde{h}_{\mu\nu} k^\sigma - 1/2 \tilde{h} k_{\mu\nu} = 0\). The solutions are shown in Table [1]. The case A corresponds to the standard massless gravitational waves. The
solution B is the same of the pure $f(X)$-gravity, i.e. the choice $k^2 = m_1^2$ reduces automatically the field equations of $f(X,Y,Z)$-gravity to those of $f(X)$-gravity. Cases B and C are very different. In fact the solution C is traceless while the solution B satisfies directly the gauge condition. The solution of case B for $h_{\mu\nu}$ is given by the following expression

$$h^B_{\mu
u} = \frac{1}{3} \int \frac{d^4k}{(2\pi)^4} \left[ k_\mu k_\nu \frac{\eta_{\mu\nu}}{m_1^2} + \frac{\eta_{\mu\nu}}{2} \right] h^B(k) e^{ikx} = \frac{1}{3} \left[ \frac{\eta_{\mu\nu}}{2} - \frac{\partial^2_{\mu\nu}}{m_1^2} \right] \int \frac{d^4k}{(2\pi)^4} h^B(k) e^{ikx} = \frac{1}{3} \left[ \frac{\eta_{\mu\nu}}{2} - \frac{\partial^2_{\mu\nu}}{m_1^2} \right] h^B(x), \quad (25)$$

where the trace of metric is a generic Klein-Gordon function with $k = (\omega_1, k)$, where $\omega_1 = \sqrt{|k|^2 + m_1^2}$. We can write the general solution in terms of its Fourier modes which are plane waves, that is

$$h^B(t, x) = \int \frac{d^3k}{(2\pi)^3} C(k) e^{i(\omega_1 t - k \cdot x)} \quad (26)$$

where $C(k)$ is the Fourier representation of the trace $h^B$. If we consider a propagating trace in the $z$-direction\(^6\) $h^B(t, z) = h_0 e^{i(\omega_1 t - k_z z)}$ with $k = (\omega_1, 0, 0, k_z)$, the solution (25) is given by

$$h^B_{\mu\nu}(t, z) = \epsilon^B_{\mu\nu} e^{i(\omega_1 t - k_z z)} = \frac{h_0}{3} \begin{pmatrix} \frac{1}{2} + \frac{\omega_1}{m_1^2} & 0 & 0 & -\frac{\omega_1 k_z}{m_1^2} \\ 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ -\frac{\omega_1 k_z}{m_1^2} & 0 & 0 & -\frac{1}{2} + \frac{k_z^2}{m_1^2} \end{pmatrix} e^{i(\omega_1 t - k_z z)} \quad (27)$$

where $\epsilon^B_{\mu\nu}$ is the polarization tensor. By a change of coordinates $x^\mu \rightarrow x'^\mu = x^\mu + \zeta^\mu(x)$ with $O(\zeta^2) \ll 1$, we can transform the metric $h_{\mu\nu}$ into a new metric $h'_{\mu\nu} = h_{\mu\nu} - \zeta_{\mu\nu} - \zeta_{\nu\mu}$. Let us suppose that we choose $\zeta^\mu(x) = i \theta^\mu e^{ikx}$, the metric $h_{\mu\nu}(x)$ becomes $h'_{\mu\nu}(x) = \epsilon^B_{\mu\nu} e^{ikx}$ where $\epsilon^B_{\mu\nu} = \epsilon^B_{\mu\nu} + k_\mu \theta_\nu + k_\nu \theta_\mu$. By performing a change of coordinates and choosing $\theta_\mu = \theta^B_\mu = \left( -\frac{1}{\omega_1} - \frac{1}{2m_1}, 0, 0, \frac{k_z}{2m_1} - \frac{k_z^2}{4\omega_1} \right)$ the polarization tensor $\epsilon^B_{\mu\nu}$ in Eq. (27) becomes

$$\epsilon^B_{\mu\nu} = \frac{h_0}{6} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} + \frac{k_z^2}{2m_1^2} \end{pmatrix} = -\frac{h_0}{6} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{k_z^2 h_0}{6\omega_1^2} & 0 & 0 & 0 \end{pmatrix}. \quad (28)$$

In the case C, we have

$$h^C_{\mu\nu}(t, x) = \int \frac{d^3k}{(2\pi)^3} C_{\mu\nu}(k) e^{i(\omega_2 t - k \cdot x)} \quad (29)$$

\(^6\) We set $x^3 = z$. 

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
Case & Choices & Gauge condition \\
\hline
A & $k^2 = 0$ any $\tilde{h}_{\mu\nu}$ & $\tilde{h}_{\mu\nu} k^\nu - 1/2 \tilde{h} k_\mu = 0$ \\
\hline
B & $k^2 = m_1^2$ any $\tilde{h}$ with $\tilde{h}_{\mu\nu} = \left[ \frac{\eta_{\mu\nu}}{m_1^2} + \frac{\eta_{\mu\nu}}{2} \right] \frac{\tilde{h}}{3}$ & verified \\
\hline
C & $k^2 = m_2^2$ any $\tilde{h}_{\mu\nu}$ with $\tilde{h} = 0$ & $\tilde{h}_{\mu\sigma} k^\sigma = 0$ \\
\hline
\end{tabular}
\caption{Classification of solutions in the vacuum of field equations.\(\textsuperscript{[24]}\)}
\end{table}
where $C_{\mu\nu}(k)$ is the Fourier representation of the gravitational wave \( h^{C}_{\mu\nu} \) and \( \omega_2 = \sqrt{|k|^2 + m_2^2} \). Also in this case, by considering a propagating wave in the z-direction \( h^{C}_{\mu\nu}(t, z) = e^{i\omega_2 t - k_2 z} \), where \( e^{C}_{\mu\nu} \) satisfies the harmonic gauge \( \epsilon^{C}_{\mu\sigma} k^\sigma = 0 \) and the traceless condition \( \eta^{\mu\sigma} \epsilon^{C}_{\mu\sigma} = 0 \), we find the solution

\[
h^{C}_{\mu\nu}(t, z) = \begin{pmatrix}
\epsilon_{00} & \epsilon_{01} & \epsilon_{02} & \epsilon_{03} \\
\epsilon_{10} & \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\
\epsilon_{20} & \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\
-\frac{\kappa}{k^2} \epsilon_{00} & -\frac{\kappa}{k^2} \epsilon_{01} & -\frac{\kappa}{k^2} \epsilon_{02} & -\frac{\kappa}{k^2} \epsilon_{03}
\end{pmatrix} e^{i(\omega_2 t - k_2 z)}
\]

(30)

where \( \epsilon_{00}, \epsilon_{01}, \epsilon_{02}, \epsilon_{11}, \epsilon_{12} \) are unspecified values. By performing also in this case a change of coordinates \( \xi = \frac{\kappa}{\omega} \) and choosing \( \theta_\mu = \theta^{C}_\mu \) (\( -\frac{\kappa}{2\omega}, \frac{\kappa}{2\omega}, \frac{\kappa}{2\omega}, \frac{\kappa}{2\omega} \)) the polarization tensor \( \epsilon'^{C}_{\mu\nu} \) in Eq. (30) becomes

\[
\epsilon'^{C}_{\mu\nu} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\
0 & -\frac{m^2}{k^2} \epsilon_{01} & -\frac{m^2}{k^2} \epsilon_{02} & -\frac{m^2}{k^2} \epsilon_{03} \\
0 & \frac{m^2}{k^2} \epsilon_{00} & \frac{m^2}{k^2} \epsilon_{02} & \frac{m^2}{k^2} \epsilon_{03}
\end{pmatrix} + \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

(31)

Let us note that, in the case of massless propagation (i.e. \( \omega_2 = |k| \) and obviously \( m_2 = 0 \)) for the choice \( \theta_\mu = \theta^{C}_\mu \), the components \( \epsilon'^{C}_{\mu\nu} \) automatically vanish and we obtain the two standard states of polarization as in General Relativity. But here we have not to impose arbitrarily \( \epsilon_{00} = \epsilon_{01} = \epsilon_{02} = 0 \).

If we introduce an independent basis (see Table (1)) for the polarizations (28) and (31), the general solution of field Eqs. (6) in vacuum is

\[
b_{\mu\nu}(t, z) = \left[ H_1 \epsilon^{(+)\mu\nu} + H_2 \epsilon^{(x)\mu\nu} \right] e^{ik_z(t-z)} + H_3 \left[ \epsilon^{(1)\mu\nu} - \frac{k_z^2}{\sqrt{3} \omega^2} \epsilon^{(a)\mu\nu} \right] e^{i(\omega t - k_z z)}
\]

\[
+ \left[ H_4 \epsilon^{(+)\mu\nu} + H_5 \epsilon^{(x)\mu\nu} + H_6 \left( \sqrt{3} \epsilon^{(1)\mu\nu} - \frac{\sqrt{2}}{\omega^2} \epsilon^{(a)\mu\nu} \right) \right] e^{i(\omega t - k_z z)}
\]

(32)

where \( H_1, H_2 \) are arbitrary constants related to the propagation modes of gravitational waves in General Relativity and the other constants are defined as \( H_3 = -\frac{\sqrt{3} \kappa_0}{6}, H_4 = \epsilon_{11}, H_5 = \epsilon_{12}, H_6 = -\frac{m_2^2 \kappa_2}{2k_z}, H_7 = -\frac{7\kappa^2}{k_z \omega^2}, H_8 = -\frac{7\kappa^2 m_2^2}{k_z \omega^2} \).

The different components of polarization tensor can be distinguished if we ask how \( \epsilon^{C}_{\mu\nu} \) changes when the coordinate system undergoes a rotation of a given angle \( \varphi \) about the z-axis. This is a Lorentz transformation of the form

\[
R^{\mu\nu} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos \varphi & \sin \varphi & 0 \\
0 & -\sin \varphi & \cos \varphi & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

(33)

and since it leaves \( k^\mu \) invariant \( (R^{\sigma\tau} k_\sigma = k_\mu) \), the only effect is to transform \( \epsilon^{C}_{\mu\nu} \) into \( \epsilon'^{C}_{\mu\nu} = R^{\mu\rho} R^{\nu\sigma} \epsilon_{\rho\sigma} \). In the case B, the polarization tensor \( \epsilon'^{B}_{\mu\nu} \) is unchanged, then we can state that the helicity is null\(^7\). In the case C, we have

\(^7\) Any plane wave \( \psi \) transforming under a rotation of an angle \( \varphi \) about the direction of propagation into \( \psi' = e^{i\xi \varphi} \psi \) has helicity \( \xi \).
\[ \epsilon^{(+)}_{\mu\nu} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \epsilon^{(\times)}_{\mu\nu} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \epsilon^{(S)}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]

Table II: The basis of the polarizations. Each polarization satisfies the condition \( \epsilon_{\alpha\beta} \epsilon^{\alpha\beta} = 1 \).

\[ \epsilon^{(1)}_{\mu\nu} = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \epsilon^{(2)}_{\mu\nu} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \epsilon^{(3)}_{\mu\nu} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \]

VI. CONCLUSIONS

In this paper, we presented a complete study of post-Minkowskian limit and gravitational wave solutions of fourth-order gravity theories of gravity in four dimensional space-time. We considered a generic action constructed with curvature invariants derived from the Riemann tensor, that are the Ricci curvature scalar, the squared Ricci tensor and the squared Riemann tensor. Thanks to the Gauss-Bonnet topological invariant, it is possible to consider only two curvature invariants since the third is always related to the others by the Gauss-Bonnet constraint. In this sense, all the budget of degrees of freedom can be related to \( R \) and \( R_{\alpha\beta} \).

With respect to standard General Relativity, new features come out in the post-Minkowskian limit. First of all two further massive scalar modes emerge in relation to the non-linearity in the Ricci scalar and tensor terms. This means that massive gravitons are a characteristic of this theories and their effective masses are strictly related to the form of the action \( f \). As supposed by several authors, these particles could play a fundamental role for the dark matter issue that, in this case, could directly come from the gravitational part of cosmic dynamics \cite{28-30}.

Furthermore, the total number of polarizations is six and helicity can come into three distinct states. It is worth noticing that we have not to choose arbitrarily × and + polarizations as in General Relativity but all possible polarizations naturally come out. This fact could be of great interest for the gravitational wave detection since running and forthcoming experiments could take advantage from this theoretical result and investigate new scenarios. In a forthcoming paper, a detailed study of sources compatible with these results will be pursued.

---

[1] Kiefer C., Quantum Gravity, Oxford Univ. Press, Oxford UK (2004)
[2] K.S. Stelle K.S., Phys.Rev. D 16, 953 (1977)
[3] Peebles P.J.E., Ratra B., Bharat, Rev. Mod. Phys. 75 (2), 559 (2003)
[4] Trimble V., Ann. Rev. of Astron. and Astrophys. 25, 425 (1987)
[5] Nojiri S., Odintsov S. D., Int.J.Geom.Meth.Mod.Phys. 4, 115 (2007)
[6] Nojiri S., Odintsov S. D., Phys.Rept. 505, 59 (2011)
[7] Capozziello S., De Laurentis M., Ann. Phys. 324, 545 (2012)
[8] Stelle K.S., Gen.Rel.Grav. 9, 353 (1978)
[9] Schmidt H. J., Phys.Rev. D 78, 023512 (2008)
[10] Schmidt H.J., Astron.Nachr. 307, 339 (1986)
[11] Astashenok A., Capozziello S., Odintsov S.D., JCAP 12, 040 (2013)
[12] Astashenok A., Capozziello S., Odintsov S.D., Phys. Rev. D 89, 103509 (2014)
[13] Bogdanos C., Capozziello S., De Laurentis M., Nesseris S., Astropart. Phys. 34, 236 (2010)
[14] Capozziello S., De Laurentis M., Phys.Rept. 509, 167 (2011)
[15] Landau Lev D., Lifshits E. M. *Theoretical physics* vol. II
[16] Santos E., Phys. Rev. D 81, 064030 (2010)
[17] Capozziello S., Stabile A., Troisi A., Phys. Rev. D 76, 104019 (2007)
[18] Capozziello S., Stabile A., Troisi A., Modern Physics Letters A 24, No 9 659 (2009)
[19] Capozziello S., Stabile A., Class. Quant. Grav. 26, 085019 (2009)
[20] Stabile A., Phys. Rev. D 82, 064021 (2010)
[21] Stabile A., Phys. Rev. D 82, 124026 (2010)
[22] Stabile A., Capozziello S., Phys. Rev. D 87, 064002 (2013)
[23] Capozziello S., Stabile A., Troisi A., Int. Jour. of Theor. Phys. 49, 1251 (2010)
[24] de Witt B.S., *Dynamical Theory of Groups and Fields*, Gordon and Breach, New York (1965)
[25] De Laurentis M., Lopez-Revelles A.J., Int. J. Geom. Meth. Mod. Phys. 11, 1450082 (2014)
[26] Capozziello S., Corda C., De Laurentis M., Phys.Lett. B 669, 255 (2008)
[27] Stabile A., Stabile An., Phys. Rev. D 85, 044014 (2012)
[28] van Dam H., Veltman M. G., Nucl. Phys. B 22, 397 (1970)
[29] Zakharov V. I., JETP Lett. 12, 312 (1970)
[30] Meszaros A., Astroph. and Sp. Sc. 111, 399 (1985)