Detecting Topological Changes in Dynamic Community Networks

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Abstract

The study of time-varying (dynamic) networks (graphs) is of fundamental importance for computer network analytics. Several methods have been proposed to detect the effect of significant structural changes in a time series of graphs.

The main contribution of this work is a detailed analysis of a dynamic community graph model. This model is formed by adding new vertices, and randomly attaching them to the existing nodes. It is a dynamic extension of the well-known stochastic blockmodel. The goal of the work is to detect the time at which the graph dynamics switches from a normal evolution – where balanced communities grow at the same rate – to an abnormal behavior – where communities start merging.

In order to circumvent the problem of decomposing each graph into communities, we use a metric to quantify changes in the graph topology as a function of time. The detection of anomalies becomes one of testing the hypothesis that the graph is undergoing a significant structural change.

In addition to the theoretical analysis of the test statistic, we perform Monte Carlo simulations of our dynamic graph model to demonstrate that our test can detect changes in graph topology.

1 Introduction

The study of time-varying (dynamic) networks (or graphs) is of fundamental importance for computer network analytics and the detection of anomalies associated with cyber crime [20, 22, 25]. Dynamic graphs also provide models for social networks [2, 14], and are used to decode the functional connectivity in neuroscience [16, 21, 46] and biology [5]. The significance of this research topic has triggered much recent work [3, 30, 43]. Several methods have been proposed to detect the effect of significant structural changes (e.g., changes in topology, connectivity, or relative size of the communities in a community graph) in a time series of graphs. We focus on networks that change over time, allowing both edges and nodes to be added or removed. We refer to these as dynamic networks.

A fundamental goal of the study of dynamic graphs is the identification of universal patterns that uniquely couple the dynamical processes that drive the evolution of the connectivity with the specific topology of the network; in essence the discovery of universal spatio-temporal patterns [29, 24]. In this context, the goal of the present work is to detect anomalous changes in the evolution of dynamic graphs. We propose a novel statistical method, which captures the coherence of the dynamics under baseline (normal) evolution of the graph, and can detect switching and regime transitions triggered by anomalies. Specifically, we study a mathematical model of normal and abnormal growth of a community network. Dynamic community networks have recently been the topic of several studies [50, 4, 8, 12, 27, 35, 37]. The simplest incarnation of such models, a dynamic stochastic blockmodel [48, 53, 55, 54, 56, 52, 40, 19], is the

*This work was supported by NSF DMS 1407340.
subject of our study. These graph models have a wide range of applications, ranging from social networks [23, 57, 34, 49, 39, 17, 18, 31] to computer networks [41, 47] and even biology and neuroscience [32].

In order to circumvent the problem of decomposing each graph into simpler structures (e.g., communities), we use a metric to quantify changes in the graph topology as a function of time. The detection of anomalies becomes one of testing the hypothesis that the graph is undergoing a significant structural change. Several notions of similarity have been proposed to quantify the structural similitude without resorting to the computation of a true distance (e.g., [6, 28] and references therein). Unlike a true metric, a similarity is typically not injective (two graphs can be perfectly similar without being the same), and rarely satisfies the triangle inequality. This approach relies on the construction of a feature vector that extracts a signature of the graph characteristics; the respective feature vectors of the two graphs are then compared using a distance, or a kernel. In the extensive review of Koutra et al. [28], the authors studied several graph similarities and distances. They concluded that existing similarities and distances either fail to conform to a small number of well-founded axioms, or suffer from a prohibitive computational cost. In response to these shortcomings, Koutra et al. proposed a novel notion of similarity [28].

Inspired by the work of [28], we proposed in [38] a true metric that address some of the limitations of the DeltaCon similarity introduced in [28]. We emphasize that it is highly preferable to have a proper metric, rather than an informal distance, when comparing graphs; this allows one to employ proof techniques not available in the absence of the triangle inequality. Our distance, coined the resistance-perturbation distance, can quantify structural changes occurring on a graph at different scales: from the local scale formed by the neighbors of each vertex, to the largest scale that quantifies the connections between clusters, or communities. Furthermore, we proposed fast (linear in the number of edges) randomized algorithms that can quickly compute an approximation to the graph metric, for which error bounds are proven (in contrast to the DeltaCon algorithm given in [28], which has a linear time approximate algorithm but for which no error bounds are given).

The main contribution of this work is a detailed analysis of a dynamic community graph model, which we call the dynamic stochastic blockmodel. This model is formed by adding new vertices, and randomly attaching them to the existing nodes. The goal of the work is to detect the time at which the graph dynamics switches from a normal evolution – where two balanced communities grow at the same rate – to an abnormal behavior – where the two communities are merging. Because the evolution of the graph is stochastic, one expects random fluctuations of the graph geometry. The challenge is to detect an anomalous event under normal random variation. We propose an hypothesis test to detect the abnormal growth of the balanced stochastic blockmodel. In addition to the theoretical analysis of the test statistic, we conduct several experiments on synthetic networks, and we demonstrate that our test can detect changes in graph topology.

The remainder of this paper is organized as follows. In the next section we introduce the main mathematical concepts and corresponding nomenclature. In section 3 we recall the definition of the resistance perturbation distance. We provide a straightforward extension of the metric to graphs of different sizes and disconnected graphs. In section 4 we formally define the problem, we introduce the dynamic balanced two-community stochastic blockmodel. We describe the main contributions and the line of attack in Section 5. In Section 6 we present the results of experiments conducted on synthetic dynamic networks, followed by a short discussion in Section 7.

2 Preliminaries and Notation

We denote by $G = (V, E)$ an undirected, unweighted graph, where $V$ is the vertex set of size $n$, and $E$ is the edge set of size $m$. We will often use $u$, $v$, or $w$ to denote vertices in $V$. For an edge $e \in E$, we denote by endpoints $(e)$ the subset of $V$ formed by the two endpoint of $e$. 
We use the standard asymptotic notation; see Appendix A for details. Given a family of probability spaces $\Omega = \{\Omega_n, \text{Prob}_n\}$, and a sequence of events $E = \{E_n\}$, we write that $\Omega$ has the property with high probability ("w.h.p.") if $\lim_{n \to \infty} \text{Prob} (E_n) = 1$.

When there is no ambiguity, we use the following abbreviated summation notation,

$$\sum_{u \leq n} \text{ is short for } \sum_{u=1}^{n} \text{ and } \sum_{u < v \leq n} \text{ is short for } \sum_{u=1}^{n} \sum_{v=u+1}^{n}. $$

Table B in Appendix B provides a list of the main notations used in the paper.

### 2.1 Effective Resistance

We briefly review the notion of effective resistance [26, 10, 13, 11] on a connected graph. The reader familiar with the concept can jump to the next section. There are many different ways to present the concept of effective resistance. We use the electrical analogy, which is very standard (e.g., [10]). Given an unweighted graph $G = (V, E)$, we transform $G$ into a resistor network by replacing each edge $e$ by a resistor with unit resistance.

**Definition 1** (Effective resistance [26]). The effective resistance $\tilde{R}_{uv}$ between two vertices $u$ and $v$ in $V$ is defined as the voltage applied between $u$ and $v$ that is required to maintain a unit current through the terminals formed by $u$ and $v$.

We denote by $\tilde{R}$ the $n \times n$ matrix with entries $\tilde{R}_{uv}$, $u, v = 1, \ldots, n$.

The relevance of the effective resistance in graph theory stems from the fact that it provides a distance on a graph [26] that quantifies the connectivity between any two vertices, not simply the length of the shortest path. Changes in effective resistance reveal structural changes occurring on a graph at different scales: from the local scale formed by the neighbors of each vertex, to the largest scale that quantifies the connections between clusters, or communities.

### 3 Resistance Metrics

#### 3.1 The Resistance Perturbation Metric

The effective resistance can be used to track structural changes in a graph, and we use it to define a distance between two graphs on the same vertex set [38] (see also [45] for a similar notion of distance). Formally, we define the Resistance Perturbation Distance as follows.

**Definition 2** (Resistance Perturbation Distance). Let $G^{(1)} = (V, E^{(1)})$ and $G^{(2)} = (V, E^{(2)})$ be two connected, unweighted, undirected graphs on the same vertex set, with respective effective resistance matrices $\tilde{R}^{(1)}$ and $\tilde{R}^{(2)}$ respectively. The RP-$p$ distance between $G^{(1)}$ and $G^{(2)}$ is defined as the element-wise $p$-norm of the difference between their effective resistance matrices. For $1 \leq p < \infty$,

$$\text{RP}_p(G^{(1)}, G^{(2)}) = \left\| \tilde{R}^{(1)} - \tilde{R}^{(2)} \right\|_p = \left[ \sum_{i,j \in V} \left| \tilde{R}^{(1)}_{ij} - \tilde{R}^{(2)}_{ij} \right|^p \right]^{1/p}. \quad (3.1) $$

In this paper, we will restrict our attention to the RP$_1$ distance (we will omit the subscript $p = 1$), because it is directly analogous to the Kirchhoff index.
3.2 Extending the Metric to Disconnected Graphs

The resistance metric is not properly defined when the vertices are not within the same connected component. To remedy this, we use a standard approach. Letting $\hat{R}_{uv}$ denote the effective resistance between two vertices $u$ and $v$ in a graph, then the conductivity $C_{uv} = \frac{1}{\hat{R}_{uv}}$ can be defined to be zero for vertices in disconnected components. Considering the conductivity as a similarity measure on vertices, a distance is defined by $R_{uv} = \frac{1}{\hat{R}_{uv}}$ if $u$ and $v$ are connected, otherwise, $R_{uv} = 1$, where $\hat{R}_{uv}$ is the effective resistance between $u$ and $v$, and $\beta > 0$ is an arbitrary constant.

We now proceed to extend the notion of resistance perturbation distance.

**Definition 3 (Renormalized Effective Resistance).** Let $G = (V, E)$ be a graph (possibly disconnected).

We define the renormalized effective resistance between any two vertices $u$ and $v$ to be

$$R_{uv} = \begin{cases} \frac{\hat{R}_{uv}}{\hat{R}_{uv} + \beta} & \text{if } u \text{ and } v \text{ are connected,} \\ 1 & \text{otherwise,} \end{cases}$$

where $\hat{R}_{uv}$ is the effective resistance between $u$ and $v$, and $\beta > 0$ is an arbitrary constant.

We define the renormalized resistance distance to be

$$\text{RD}_\beta(G^{(1)}, G^{(2)}) = \sum_{u < v \leq n} \left| R_{uv}^{(1)} - R_{uv}^{(2)} \right|.$$  \hspace{1cm} (3.3)

where the parameter $\beta$ (see (3.2)) is implicitly defined. In the rest of the paper we work with $\beta = 1$, and dispense of the subscript $\beta$ in (3.3). In other words,

$$\text{RD} \overset{\text{def}}{=} \text{RD}_1.$$  \hspace{1cm} (3.4)

**Remark 1.** An additional parameter $\beta$ has been added to the definition. Changing $\beta$ is equivalent to scaling the effective resistance before applying the function $x \rightarrow x/(1 + x)$. Note that when $\hat{R} \ll \beta$, then $R \approx \hat{R}/\beta$, i.e. the renormalized resistance is approximately a rescaling of the effective resistance. Note that in this metric, two graphs are equal if they differ only in addition or removal of isolated vertices.

The following lemma confirms that the distance defined by (3.3) remains a metric when we compare graphs with the same vertex set.

**Lemma 1.** Let $V$ be a vertex set. RD defined by (3.3) as a metric on the space of unweighted undirected graphs defined on the same vertex set $V$.

**Remark 2.** The metric given in Definition 4 can be used to compare graphs of two different sizes, by adding isolated vertices to both graphs until they have the same vertex set (this is why we must form the union $V = V^{(1)} \cup V^{(2)}$ and compare the graphs over this vertex set). This method will give reasonable results when the overlap between $V^{(1)}$ and $V^{(2)}$ is large. In particular, if we are comparing graphs of size $n$ and $n + 1$, then we only need add one isolated vertex to the former so that we can compare it to the latter. This situation is illustrated in Figure 3.1.
Figure 3.1: In order to compare $G_n$ and $G_{n+1}$, we include node $n + 1$ into $G_n$ (see left), and evaluate the renormalized effective resistance on the augmented graph, with vertex set $\{1, \ldots, n\} \cup \{n + 1\}$.

When the graphs $G^{(1)}$ and $G^{(2)}$ have different sizes, the distance $RD$ still satisfies the triangle inequality, and is symmetric. However, $RD$ is no longer injective: it is a pseudo-metric. Indeed, as explained in the following lemmas, if $RD(G^{(1)}, G^{(2)}) = 0$, then the connected components of $G^{(1)}$ and $G^{(2)}$ are the same, but the respective vertex sets may differ by an arbitrary number of isolated vertices.

**Lemma 2.** Let $G = (E, V)$ be an unweighted undirected graph, and let $V^{(i)}$ be a set of isolated vertices, to wit $V^{(i)} \cap V = \emptyset$ and $\forall e \in E, \text{endpoints}(e) \notin V^{(i)}$. Define $G' = (V \cup V^{(i)}, E)$, then we have $RD(G, G') = 0$.

The following lemma shows that the converse is also true.

**Lemma 3.** Let $G^{(1)} = (V, E^{(1)})$ and $G^{(2)} = (V, E^{(2)})$ be two unweighted, undirected graphs, where $|V^{(1)}| > |V^{(2)}|$. If $RD(G^{(1)}, G^{(2)}) = 0$, then $E^{(1)} = E^{(2)}$. Furthermore, there exists a set $V^{(i)}$ of isolated vertices, such that $V^{(1)} = V^{(2)} \cup V^{(i)}$.

In summary, in this work the distance $RD$ will always be a metric since we will only consider graphs that are connected with high probability.

## 4 Graph Models

In our analysis, we will discuss two common random graph models, the classic model of Erdős and Rényi [7] and the stochastic blockmodel [1].

**Definition 5** (Erdős-Rényi Random Graph [7]). Let $n \in \mathbb{N}$ and let $p \in [0, 1]$. We recall that the Erdős-Rényi random graph, $G(n, p)$, is the probability space formed by the graphs defined on the set of vertices $[n]$, where edges are drawn randomly from $\binom{n}{2}$ independent Bernoulli random variables with probability $p$. In effect, a graph $G \sim G(n, p)$, with $m$ edges, occurs with probability

$$\text{Prob}(G) = p^m(1 - p)^{\binom{n}{2} - m}.$$  \hspace{1cm} (4.1)

**Definition 6.** Let $G = (V, E) \sim G(N, p)$. For any vertex $u \in V$, we denote by $d_u$ the degree of $u$; we also denote by $\overline{d}_n = (n - 1)p$ the expected value of $d_u$.

We now introduce a model of a dynamic community network: the balanced, two-community stochastic blockmodel.

**Definition 7** (Dynamic Stochastic Blockmodel). Let $n \in \mathbb{N}$, and let $p, q \in [0, 1]$. We denote by $G(n, p, q)$ the probability space formed by the graphs defined on the set of vertices $[n]$, constructed as follows.
We split the vertices \([n]\) into two communities \(C_1\) and \(C_2\), formed by the odd and the even integers in \([n]\) respectively. We denote by \(n_1 = \lfloor (n + 1)/2 \rfloor\) and \(n_2 = \lfloor n/2 \rfloor\) the size of \(C_1\) and \(C_2\) respectively.

Edges within each community are drawn randomly from independent Bernoulli random variables with probability \(p\). Edges between communities are drawn randomly from independent Bernoulli random variables with probability \(q\). For \(G \in \mathcal{G}(n, p, q)\), with \(m_1\) and \(m_2\) edges in communities \(C_1\) and \(C_2\) respectively, we have

\[
\text{Prob}(G) = p^{m_1}(1-p)^{m_1-m_1}q^{m_2}(1-q)^{m_2-m_2}.
\]

(4.2)

**Remark 3.** Although we use \(\mathcal{G}\) for both random graph models, the presence of two or three parameters prevents ambiguity in our definitions.

**Definition 8.** Let \(G \sim \mathcal{G}(n, p, q)\). We denote by \(\overline{d_{n_1}} = pn_1\) the expected degree within community \(C_1\), and by \(\overline{d_{n_2}} = pn_2\) the expected degree within community \(C_2\).

We denote by \(k_n\) the binomial random variables that counts the number of cross-community edges between \(C_1\) and \(C_2\).

Because asymptotically, \(n_1 \sim n_2\), we ignore the dependency of the expected degree on the specific community when computing asymptotic behaviors for large \(n\). More precisely, we have the following results.

**Lemma 4.** Let \(G_n \in \mathcal{G}(n, p, q)\). We have

1. \(\overline{d_{n_1}} = \overline{d_{n_2}} + \epsilon p\), where \(\epsilon = 0\) if \(n\) is even, or \(\epsilon = 1\) otherwise.
2. \(\overline{d_{n_1}}^2 = \overline{d_{n_2}}^2(1 + o(1))\).
3. \(\frac{1}{\overline{d_{n_1}}} = \frac{1}{\overline{d_{n_2}}} (1 + o(1))\).
4. \(\frac{1}{\overline{d_{n_1}}} = \frac{1}{\overline{d_{n_2}}} + \Theta\left(\frac{1}{\overline{d_n}}\right)\) where \(\overline{d_n} = \overline{d_{n_1}}\), or \(\overline{d_n} = \overline{d_{n_2}}\).

In summary, in the remaining of the text we loosely write \(1/\overline{d_{n}}\) when either \(1/\overline{d_{n_1}}\) or \(1/\overline{d_{n_2}}\) could be used, and the error between the two terms is no larger than \(\Theta\left(1/\overline{d_{n}}^2\right)\).

**Remark 4.** In this work, we study nested sequences of random graphs, and we use sometime the subscript \(n\) to denote the index of the corresponding element \(G_n\) in the graph process.

**Remark 5.** While our model assumes that the two communities have equal size, or differ at most by one vertex, the model can be extended to multiple communities of various sizes.
5 Main Results

5.1 Informal Presentation of our Results

Before carefully stating the main result in the next subsection, we provide a back of the envelope analysis to help understand under what circumstances the resistance metric can detect an anomalous event in the dynamic growth of a stochastic blockmodel. In particular, we aim to detect whether cross-community edges are formed at a given timestep. In graphs with few cross-community edges, the addition of such an edge changes the geometry of the graph significantly. We will show that the creation of such edges can be detected with high probability when the average in-community degree dominates the number of cross-community edges.

Figure 4.1 illustrates the statement of the problem. As a new vertex (shown in magenta) is added to the graph $G_n$, the connectivity between the communities can increase, if edges are added between $C_1$ and $C_2$, or the communities can remain separated, if no cross-community edges are created. If the addition of the new vertex promotes the merging of $C_1$ and $C_2$, then we consider the new graph $G_{n+1}$ to be \textit{structurally different} from $G_n$, otherwise $G_{n+1}$ remains \textit{structurally the same} as $G_n$ (see Fig. 4.1).

The goal of the present work is to detect the fusion of the communities without identifying the communities. We show that the effective resistance yields a metric that is sensitive to changes in pattern of connections and connectivity structure between $C_1$ and $C_2$. Therefore it can be used to detect structural changes between $G_n$ and $G_{n+1}$ without detecting the structure present in $G_n$.

The informal derivation of our main result relies on the following three ingredients:

1. each community in $\mathcal{G}(n, p, q)$ is approximately a “random graph” (Erdős-Rényi), $\mathcal{G}(n/2, p)$;
2. the effective resistance between two vertices $u, v$ within $\mathcal{G}(n/2, p)$ is concentrated around $2/d_n = 2/(p(n/2 - 1))$;
3. the effective resistance between $u \in C_1$ and $v \in C_2$ depends only on the bottleneck formed by the $k_n$ cross-community edges, $\tilde{R}_{uv} \approx 1/k_n$. 

Figure 4.1: Left: the dynamic stochastic blockmodel $G_n$ is comprised of two communities ($C_1$: red and $C_2$: blue). As a new (magenta) vertex is added, the new graph $G_{n+1}$ can remain structurally the same – if no new edges are created between $C_1$ and $C_2$ (top right) – or can become structurally different if the communities start to merge with the addition of new edges between $C_1$ and $C_2$ (bottom right).
We now proceed with an informal analysis of the changes in effective resistance distance when the new vertex, \( n + 1 \), is added to the stochastic blockmodel \( G_n \) (see Fig. 4.1).

We first consider the "null hypothesis" where no cross-community edges is formed when vertex \( n + 1 \) is added to the graph. All edges are thus created in the community of \( n + 1 \), say \( C_1 \) (without any loss of generality). Roughly \( pn/2 \) new edges are created, and thus about \( \Theta(n) \) vertices are affected by the addition of these new edges to \( C_1 \).

Because the effective resistance between any two vertices \( u, v \) in \( C_1 \) is concentrated around \( 2/\lfloor p(n_1 - 1 + 1) \rfloor \geq 2/(d_n + 1) \), the changes in resistance after the addition of vertex \( n + 1 \) is bounded by

\[
\Delta \hat{R}_{uv} \leq \frac{2}{d_n} - \frac{2}{d_n + 1} = \Theta\left(\frac{1}{d_n^2}\right).
\]

(5.1)

Although, one would expect that only vertices in community \( C_1 \) (wherein \( n + 1 \) has been added) be affected by this change in effective resistance, a more detailed analysis shows that vertices in \( C_2 \) slightly benefit of the increase in connectivity within \( C_1 \).

We now consider the alternate hypothesis, where at least one cross-community edge is formed after adding \( n + 1 \) (see Fig. 4.1-bottom right). This additional cross-community edge has an effect on all pairwise effective resistances. Nevertheless, the most significant perturbation in \( \hat{R}_{uv} \) occurs for the \( n/2 \times n/2 \) pairs of vertices in \( C_1 \times C_2 \). Indeed, if \( u \in C_1 \) and \( v \in C_2 \), the change in effective resistance becomes

\[
\Delta \hat{R}_{uv} \approx \frac{1}{k_n} - \frac{1}{k_n + 1} = \Theta\left(\frac{1}{k_n^2}\right).
\]

(5.2)

In summary, we observe asymptotic separation of the two regimes precisely when \( k_n/d_n \rightarrow 0 \), which occurs with high probability when \( n \cdot q_n = o(p_n) \). We should therefore be able to use the renormalized resistance distance to test the null hypothesis that no edge is added between \( C_1 \) and \( C_2 \), and that \( G_n \) and \( G_{n+1} \) are structurally the same.

We will now introduce the main character of this work: the dynamic stochastic block model, and we will then provide a precise statement of the result. In particular, we hope to elucidate our model of a dynamic community graph, in which at each time step a new vertex joins the graph and forms connections with previous vertices. The idea of graph growth as a generative mechanism is commonplace for models such as preferential attachment, but is less often seen in models such as Erdős-Rényi and the stochastic blockmodel.

### 5.2 The Growing Stochastic Blockmodel

We have described in Definition 7 a model for a balanced stochastic block model, where the probabilities of connections \( p \) and \( q \) are fixed. However, we are interested in the regime of large graphs \( n \rightarrow \infty \), where \( p \) and \( q \) cannot remain constant. In fact the probabilities of connection, within each community and across communities go to zero as the size of the graph, \( n \), goes to infinity.

The elementary growth step, which transforms \( G_n = (V_n, E_n) \) into \( G_{n+1} = (E_{n+1}, V_{n+1}) \) proceeds as follows: one adds a vertex \( n + 1 \) to \( V_n \) to form \( V_{n+1} \), assigns this new vertex to \( C_1 \) or \( C_2 \) according to the parity of \( n \). One then connects \( n + 1 \) to each member of its community with probability \( p \) and each member of the opposite community with probability \( q \). This leads to a new set of vertices, \( E_{n+1} \).
Table 1: Each row depicts the growth sequence that leads to the construction of \( G_{n+1} \equiv G_n^{(n)} \). The distance \( D_n \) is always defined with respect to the subgraph \( G_n^{(n)} \) on the vertices 1, ..., \( n \) that led to the construction of \( G_{n+1} \).

The actual sequence of graphs \( \{G_n\} \) is created using this elementary process with a twist: for each index \( n \), the graph \( G_{n+1} \) is created by iterating the elementary growth process, starting with a single vertex and no edges, \( n + 1 \) times with the fixed probabilities of connections \( p_n \) and \( q_n \). Once \( G_{n+1} \) is created, the growth is stopped, the probabilities of connections are updated and become \( p_{n+1} \) and \( q_{n+1} \). A new sequence of graphs is initialized to create \( G_{n+2} \).

Table 1 illustrates the different sequences of growth, of increasing lengths, that lead to the creation of \( G_1, G_2, \ldots \). This growth process guarantees that \( G_{n+1} \) is always a subgraph of \( G_n \), and that both \( G_n \) and \( G_{n+1} \) have been created with the same probabilities. Furthermore, each \( G_n \) is distributed according to Definition 7, and the \( G_n \) are independent of one another.

In order to study the dynamic evolution of the graph sequence, we focus on changes between two successive time steps \( n \) and \( n + 1 \). These changes are formulated in the form of the distance \( D_n = \text{RD}(G_n^{(n)}, G_{n+1}) \) between \( G_{n+1} \) and the subgraph \( G_n^{(n)} \) on the vertices 1, ..., \( n \), which led to the construction of \( G_{n+1} \). The subgraph \( G_n^{(n)} \) is the graph on the left of the boxed graph \( G_{n+1} \) on each row of Table 1. The definition of \( D_n \) is the only potential caveat of the model: \( D_n \) is not the distance between \( G_n \) and \( G_{n+1} \); this restriction is necessary since in general \( G_n \) is not a subgraph of \( G_{n+1} \).

This model provides a realistic prototype for the separation of scales present in the dynamics of large social networks. Specifically, the time index \( n \) corresponds to the slow dynamics associated with the evolution of the networks over long time scale (months to years). In contrast, the random realizations on each row of Table 1 embody the fast random fluctuations of the network over short time scales (minutes to hours).

In this work, we are interested in examining fluctuations over fast time scales (minutes to hours). We expect that the probabilities of connection, \( (p_n, q_n) \), remain the same when we study the distance between \( G_n \) and \( G_{n+1} \). As \( n \) increases, the connectivity patterns of members of the network evolve, and we change accordingly the probabilities of connection, \( (p_n, q_n) \). Similar dynamic stochastic block models have been proposed in the recent years (e.g., [19, 40, 48, 52, 53, 55, 54, 56], and references therein).

In the stochastic blockmodel, each vertex \( u \) belongs to a community within the graph. If vertex \( u \) forms no cross-community edges, then the geometry of the graph is structurally the same. However, if \( u \) forms at least one cross-community edge, then (depending on the geometry of the preceding graphs in the sequence) the geometry may change significantly. We examine in what regimes of \( p_n \) and \( q_n \), we can differentiate between the two situations with high probability. We phrase the result in terms of a hypothesis test, with the null hypothesis being that no cross-community edges have been formed in step \( n + 1 \).

| Probabilities of connection | Growth sequence to generate \( G_n \) | Definition of \( G_n \) | Definition of \( D_n \) |
|-----------------------------|--------------------------|-----------------|-----------------|
| \( \{p_1, q_1\} \)         | \( G_1^{(1)} \)          | \( G_2 \equiv G_2^{(1)} \) | \( D_1 \equiv \text{RD}(G_1^{(1)}, G_2^{(1)}) \) |
| \( \{p_2, q_2\} \)         | \( G_1^{(2)} \)          | \( G_3 \equiv G_3^{(2)} \) | \( D_2 \equiv \text{RD}(G_2^{(2)}, G_3^{(2)}) \) |
| \( \{p_3, q_3\} \)         | \( G_1^{(3)} G_2^{(3)} G_3^{(3)} \) | \( G_4 \equiv G_4^{(3)} \) | \( D_3 \equiv \text{RD}(G_3^{(3)}, G_4^{(3)}) \) |
| ...                         | ...                      | ...             | ...             |

Table 1: Each row depicts the growth sequence that leads to the construction of \( G_{n+1} \equiv G_n^{(n)} \). The distance \( D_n \) is always defined with respect to the subgraph \( G_n^{(n)} \) on the vertices 1, ..., \( n \) that led to the construction of \( G_{n+1} \).
Figure 5.1: A typical time series of $D_n$ for a growing stochastic blockmodel. The red curve is the distance between time steps $D_n$ and the blue vertical lines mark the formation of cross-community connections. Two different regimes are compared. On the left, the formation of cross-community edges is easily discernible, while on the right, such an event is quickly lost in the noise.

Figure 5.1 shows a time series of distances $D_n$ for a growing stochastic blockmodel. We see that when the in-community connectivity is much greater than the cross community connectivity, the formation of cross-community edges is easily discernible (left figure). However, when the level of connectivity is insufficiently separated, then the formation of cross-community edges is quickly lost in the noise. Our result clarifies exactly what is meant when we say that the parameters $p_n$ and $q_n$ are “well separated.”

Our main result is given by the following theorem.

**Theorem 1.** Let $G_{n+1} \sim G(n + 1, p_n, q_n)$ be a stochastic blockmodel with $p_n = \omega \left( \frac{\log n}{n} \right)$, $q_n = \omega \left( \frac{1}{n^2} \right)$, $q_n = o \left( \frac{p_n}{n} \right)$, and $p_n = O \left( \frac{1}{\sqrt{n}} \right)$. Let $G_n$ be the subgraph induced by the vertex set $[n]$, with $m_n$ edges. Let $D_n = RD(G_n, G_{n+1})$ be the normalized effective resistance distance, $RD$, defined in (3.3).

To test the hypothesis

$$H_0 : \quad k_n = k_{n+1} \tag{5.3}$$

versus

$$H_1 : \quad k_n < k_{n+1} \tag{5.4}$$

we use the test based on the statistic $Z_n$ defined by

$$Z_n \overset{def}{=} \frac{16m_n^2}{n^4} (D_n - n), \tag{5.5}$$

where we accept $H_0$ if $Z_n < z_\varepsilon$ and accept $H_1$ otherwise. The threshold $z_\varepsilon$ for the rejection region satisfies

$$\text{Prob}_{H_0} (Z_n \geq z_\varepsilon) \leq \varepsilon \quad \text{as} \quad n \to \infty, \quad (5.6)$$

and

$$\text{Prob}_{H_1} (Z_n \geq z_\varepsilon) \to 1 \quad \text{as} \quad n \to \infty. \quad (5.7)$$

The test has therefore asymptotic level $\varepsilon$ and asymptotic power 1.

**Proof.** The proof of the theorem can be found in Appendix C. \(\square\)
Remark 6. In practice, it would be desirable to have an analytical expression for the constant $z_\epsilon$ such that we can compute a level $\epsilon$ test,

$$\Pr(Z_n \geq z_\epsilon) \leq \epsilon \quad \text{under the null hypothesis.}$$

Unfortunately, our technique of proof, which is based on the asymptotic behavior of $Z_n$ does not yield such a constant. A more involved analysis, based on finite sample estimates of the distance, would be needed, and would yield an important extension of the present work. The results shown in Figure 5.2 suggest that one could numerically estimate a $1 - \epsilon$ point wise confidence interval for $Z_n$ with a bootstrapping technique; the details of such a construction are the subject of ongoing investigation.

6 Experimental Analysis of Dynamic Community Networks

Figure 5.2 shows numerical evidence supporting Theorem 1. The empirical distribution of $Z_n$ is computed under the null hypothesis (solid line) and the alternate hypothesis (dashed line). The data are scaled so that the empirical distribution of $Z_n$ under $H_0$ has zero mean and unit variance. In the left and right figures, the density of edges remains the same within each community, $p_n = \log^2 n/n$.

The plot on the left of Fig. 5.2 illustrates a case where the density of cross-community edges remains sufficiently low – $q_n = \log n/n^2$ – and the test statistic can detect the creation of novel cross-community edges (alternate hypothesis) without the knowledge of $k_n$, or the identification of the communities.

On the right, the density of cross-community edges is too large – $q_n = \log^2 n/n^{3/2}$ – for the statistic to be able to detect the creation of novel cross-community edges. In that case the hypotheses of Theorem 1 are no longer satisfied.

In addition to the separation of the distributions of $Z_n$ under $H_0$ and $H_1$ guaranteed by Theorem 1 when $q_n/p_n = o(1/n)$, we start observing a separation between the two distributions when $q_n/p_n = \Theta(1/n)$ (not shown) suggesting that the hypotheses of Theorem 1 are probably optimal. In the regime where $q_n = \omega(p_n/n)$, shown in Fig. 5.2-right, the two distributions overlap.
7 Discussion

At first glance, our result may seem restrictive compared to existing results regarding community detection in the stochastic blockmodel. However, such a comparison is ill-advised, as we do not propose this scheme as a method for community detection. For example, Abbe et al. have shown that communities can be recovered asymptotically almost surely when \( p_n = a \log(n)/n \) and \( q_n = b \log(n)/n \), provided that \((a + b)/2 - \sqrt{ab} > 1\) [1]. Their method uses an algorithm that is designed specifically for the purpose of community detection, whereas our work provides a very general tool, which can be applied on a broad range of dynamic graphs, albeit without the theoretical guarantees that we derive for the dynamic stochastic block model.

Furthermore, the “efficient” algorithm proposed by Abbe et al. is only proven to be polynomial time, whereas resistance matrices can be computed in near-linear or quadratic time, for the approximate [44] and exact effective resistance respectively. This allows our tool to be of immediate practical use, whereas results such as those found in [1] are of a more theoretical flavor.

Some argue against the use of the effective resistance to analyze connectivity properties of a graph. In [51] it is shown that

\[
\left| \frac{R_{uv} - \frac{1}{2} \left( \frac{1}{d_u} + \frac{1}{d_v} \right)}{1 - \lambda_2} \right| \leq \left( \frac{1}{1 - \lambda_2} + 2 \right) \frac{w_{max}}{\delta_n^2},
\]

(7.1)

where \( \lambda_2 \) is the second largest eigenvalue of the normalized graph Laplacian, \( d_v \) is the degree of vertex \( u \), \( \delta_n \) is the minimum degree, and \( w_{max} \) is the maximum edge weight. If the right-hand side of (7.1) converges to 0, then the effective resistance will converge to the average inverse degree of \( u \) and \( v \).

Luxburg et al. argue that the result (7.1) implies that when the bound converges to zero, the resistance will be uninformative, since it depends on local properties of the vertices and not global properties of the graph. Fortunately, this convergence can coexist peacefully alongside our result. In particular, if both expected degree and minimum degree approach infinity with high probability, as they will when both \( q_n \) and \( p_n \) are \( \omega \left( 1/n^2 \right) \), then such convergence will itself occur with high probability (see (C.14) for the relevant spectral gap bound). Since we only care about relative changes in resistance between \( G_n \) and \( G_{n+1} \) in cases where cross-community edges are and are not formed, this is no problem for us. That said, the warning put forth by Luxburg et al. is well taken; we must be careful to make sure that we understand the expected behavior of the distance \( RD(G_n, G_{n+1}) \) and compare the observed behavior to this expected behavior rather than evaluate it on an absolute scale, since in many situations of interest this distance will converge to zero as the graph grows.

Luxburg et al. have pointed out that the resistance can be fickle when used on graphs with high connectivity, which is to say a small spectral bound \((1 - \lambda_2)^{-1}\). We now know in which circumstances this will become an issue when looking at simple community structure. Further investigation is needed to know when other random graph models such as the small-world or preferential attachment model will be susceptible to analysis via the renormalized resistance metric. We are currently investigating the application of this analysis to a variety of real-world data sets.

A Asymptotic Notations

If \( \{a_n\}_{n=1}^{\infty} \) and \( \{b_n\}_{n=1}^{\infty} \) are infinite sequences. The notations on the left have the interpretation on the right,

\[
\begin{align*}
\text{Asymptotic Notations} & \quad \text{(O, o, \Omega, \omega)} \\
\quad a_n = O(b_n) & \quad \exists n_0 > 0, \exists c > 0, \forall n \geq n_0, \quad 0 \leq |a_n| \leq c|b_n|, \\
\quad a_n = o(b_n) & \quad \forall c > 0, \exists n_0 \geq 0, \forall n \geq n_0, \quad 0 \leq |a_n| \leq c|b_n|, \\
\quad a_n = \omega(b_n) & \quad \forall c > 0, \exists n_0 \geq 0, \forall n \geq n_0, \quad 0 \leq c|b_n| < |a_n|, \\
\quad a_n = \Theta(b_n) & \quad \exists c_1, c_2 > 0, \exists n_0 \geq 0, \forall n \geq n_0, \quad 0 \leq c_1|b_n| \leq |a_n| \leq c_2|b_n|.
\end{align*}
\]
We can adapt any of the above statements to doubly-indexed sequences $a_{n,k}$ and $b_{n,k}$ by requiring that there exist an $n_0 \geq 0$ such that the conditions on the right hold for all $n, k \geq n_0$.

## B Notation

| Symbol | Definition | Definition or Equation Number |
|--------|------------|------------------------------|
| $[n]$  | The subset of natural numbers $\{1, \ldots, n\}$. |  |
| $G(n, p)$ | Erdős-Rényi random graph with parameters $n$ and $p$. | 5 |
| $G(n, p, q)$ | Stochastic blockmodel with parameters $n, p, q$. | 7 |
| $G_n$ | Subgraph of a graph $G$ induced by the vertex set $[n]$. |  |
| $\hat{R}_{uv}$ | Effective resistance between $u$ and $v$. | 1 |
| $R_{uv}$ | Renormalized effective resistance between $u$ and $v$. | (3.2) |
| $\hat{R}_{uv}^{(\beta)}$ | Renormalized effective resistance between $u$ and $v$ in $G_n$. |  |
| $d_u$ | Degree of vertex $u$ (random variable). | 6 & 8 |
| $\bar{d}_n$ | Mean degree (Erdős-Rényi) or expected in-community degree (stochastic blockmodel). |  |
| $k_n$ | Number of cross-community edges (random variable). | 8 |
| $E[k_n]$ | Expectation of the number of cross-community edges. |  |
| $m_n$ | Total number of edges. |  |
| $\text{RP}(\cdot, \cdot)$ | Resistance-perturbation distance. | (3.1) |
| $\text{RD}(\cdot, \cdot)$ | Renormalized resistance distance (with $\beta = 1$). | 4 |

## C Proof of Main Result

We begin by proving a lemma that allows us to transfer bounds on changes in effective resistance into bounds on changes in renormalized resistances.

**Lemma 5.** Suppose that $\hat{R}_1$ and $\hat{R}_2$ are two effective resistances. If

$$C_1 \leq |\hat{R}_1 - \hat{R}_2| \leq C_2,$$

then the corresponding renormalized resistances obey

$$\frac{C_1}{(\hat{R}_1 + 1)(\hat{R}_2 + 1)} \leq |R_1 - R_2| \leq C_2.$$  \hspace{1cm} (C.1)

**Proof.** Recall that the renormalized resistance corresponding to $\hat{R}$ is given by $R = f(\hat{R})$ where $f(x) = x/(x+1)$. The mean value theorem thus implies that

$$|R_1 - R_1| \leq \sup_{x \in \hat{R}} |f'(x)| |\hat{R}_1 - \hat{R}_2| \leq |\hat{R}_1 - \hat{R}_2| \leq C_2.$$

To obtain the lower bound, we compute:

$$|R_1 - R_1| = \left| \frac{\hat{R}_1}{\hat{R}_1 + 1} - \frac{\hat{R}_2}{\hat{R}_2 + 1} \right| = \frac{|\hat{R}_1 - \hat{R}_2|}{(\hat{R}_1 + 1)(\hat{R}_2 + 1)} \geq \frac{C_1}{(\hat{R}_1 + 1)(\hat{R}_2 + 1)}.$$  \hspace{1cm} (C.2)

In our calculation above, we used the fact that $\hat{R}_1$ and $\hat{R}_2$ are non negative. \hfill $\square$
C.1 Resistance Deviations in Erdős-Rényi

We begin by analyzing the perturbations of the distance \( \text{RD}(G_n, G_{n+1}) \), defined by (3.4), when \( G_n \sim G(n, p_n) \) is an Erdős-Rényi random graph. Our ultimate goal is to understand a stochastic blockmodel, and we will leverage our subsequent understanding of the Erdős-Rényi model to help us in achieving this goal.

Lemma 6. Let \( G \sim G(n, p_n) \) be fully connected, with \( p_n = \omega(\log n / n) \). For any two vertices \( u, v \) in \( G \), we have

\[
\left| \hat{R}_{uv} - \left( \frac{1}{d_u} + \frac{1}{d_v} \right) \right| = O \left( \frac{1}{d_n^2} \right) \quad \text{with high probability.} \tag{C.3}
\]

Remark 7. The authors in [51] derive a slightly weaker bound,

\[
\left| \hat{R}_{uv} - \left( \frac{1}{d_u} + \frac{1}{d_v} \right) \right| = o \left( \frac{1}{\delta_n} \right). \tag{C.4}
\]

We need the tighter factor \( O \left( \frac{1}{d_n^2} \right) \); and thus we derive the bound (C.3) using one of the key results (Proposition 5) in [51].

Proof. Define \( D \) to be the diagonal matrix with entries \( d_1, \ldots, d_n \). Since all degrees are positive (\( G \) is fully connected with high probability), we denote by \( D^{-1/2} \) the diagonal matrix with entries \( 1/\sqrt{d_1}, \ldots, 1/\sqrt{d_n} \). Let \( A \) be the adjacency matrix of \( G \). Define \( B = D^{-1/2} A D^{-1/2} \), with eigenvalues \( 1 = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \). As explained above, we use Proposition 5 in [51] to bound the deviation of \( \hat{R}_{uv} \) away from \( 1/d_u + 1/d_v \),

\[
\left| \hat{R}_{uv} - \left( \frac{1}{d_u} + \frac{1}{d_v} \right) \right| \leq \frac{2}{\delta_n^2} \left( 2 + \frac{1}{1 - \lambda_2} \right), \tag{C.5}
\]

where \( \delta_n \) is the minimum degree. Define

\[
\varepsilon_n = \sqrt{6 \log(n)d_n}. \tag{C.6}
\]

We apply Chernoff’s bound on the degree distribution,

\[
\text{Prob} \left( |d_v - d_n| \geq \varepsilon_n \right) \leq 2 \exp \left( -\frac{\varepsilon_n^2}{3d_n} \right).
\]

Now,

\[
\frac{\varepsilon_n^2}{3d_n} = \frac{6d_n \log n}{3d_n} = \log n^2, \tag{C.7}
\]

and thus

\[
\text{Prob} \left( |d_v - d_n| \geq \varepsilon_n \right) \leq \frac{2}{n^2}.
\]

In the end, applying a union bound on all \( n \) vertices yields

\[
\text{Prob} \left( \forall v \in [n], |d_v - d_n| \geq \varepsilon_n \right) \leq \frac{2}{n}. \tag{C.8}
\]

We consider \( d_v \) in the interval \([d_n - \varepsilon_n, d_n + \varepsilon_n]\). The mean value theorem implies that there exists \( \tilde{d} \in (d_n, d_v) \), or \( \tilde{d} \in (d_v, d_n) \), such that

\[
\left| \frac{1}{d_v^2} - \frac{1}{d_n^2} \right| = \frac{2}{d^3} \left| d_n - d_v \right|. \tag{C.9}
\]
Now,
\[
\frac{2}{d^3} \left| d_n - d \right| \leq \frac{2\varepsilon_n}{(d_n - \varepsilon_n)^3} = \frac{2\varepsilon_n}{d_n^3 \left( 1 - \varepsilon_n/d_n \right)^3}.
\] (C.10)

At last, we use the following elementary fact
\[
0 \leq \frac{1}{(1 - x)^3} \leq 1 + 12x, \quad \text{if } x < 1/4,
\] (C.11)
to conclude that
\[
\forall u \in [n], \quad \left| \frac{1}{d_u^2} - \frac{1}{d_n^2} \right| \leq \frac{2\varepsilon_n}{d_n^3} \left( 1 + 12 \frac{\varepsilon_n}{d_n} \right) \quad \text{with probability greater than } 2/n.
\] (C.12)

This eventually yields an upper bound on the inverse of the minimum degree squared,
\[
\frac{1}{\delta_n^2} \leq \frac{1}{d_n^2} + \frac{2\varepsilon_n}{d_n^3} \left( 1 + 12 \frac{\varepsilon_n}{d_n} \right) \quad \text{with probability greater than } 2/n.
\] (C.13)

To complete the proof of the lemma, we use a lower bound on the spectral gap $1 - \lambda_2$. Because the density of edges is only growing faster than $\log n/n$, we use the optimal bounds given by [9]. Applied to the eigenvalue $\lambda_2$ of $B$, Theorem 1.2 of [9] implies that

**Theorem 2 ([9]).** If $d_n > c \log n/n$, then with high probability,
\[
1 - \frac{c}{\sqrt{d_n}} \leq 1 - \lambda_2 \leq 1 + \frac{c}{\sqrt{d_n}}.
\] (C.14)

The lower bound in (C.14) yields the following upper bound, with high probability,
\[
\frac{1}{1 - \lambda_2} \leq 1 + \frac{c}{\sqrt{d_n}}.
\] (C.15)

Using the bounds given by (C.13) with (C.15), which happen both with high probability, in (C.5) yields the advertised result. □

An important corollary of lemma 6 is the concentration of $\hat{R}_{uv}$ around $2/d_n$ (see also [51] for similar results),

**Corollary 1.** Let $G \sim G(n, p_n)$ be fully connected, with $p_n = \omega(\log n/n)$. With high probability,
\[
\left| \hat{R}_{uv} - \frac{2}{d_n} \right| = \frac{16}{d_n^2} + o \left( \frac{1}{d_n^2} \right).
\] (C.16)

The lemma provides a confidence interval for the effective resistance $\hat{R}_{uv}$ centered at $d_u^{-1} + d_v^{-1}$, for pairs of vertices present in the graph $G$ at time $n$. We now use this result to bound the change in (renormalized) resistance between a pair of vertices $u$ and $v$ present in $G_n$, when the graph grows from $G_n$ to $G_{n+1}$.

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Theorem 3. Let $G_{n+1} \sim G(n, p_n)$ be an Erdős-Rényi random graph with $p_n = \omega(\log n/n)$. Let $G_n$ be the subgraph induced by the vertices $[n]$ in $G_{n+1}$, and let $\bar{d}_n = (n-1)p_n$ be the expected degree in $G_n$.

The change in renormalized effective resistance, when the graph $G_n$ becomes $G_{n+1}$, is given by

$$\max_{u < v \leq n} \left| \overline{R}^{(n+1)}_{u,v} - \overline{R}^{(n)}_{u,v} \right| = \Theta \left( \frac{1}{\bar{d}_n} \right) \quad \text{with high probability.} \quad \text{(C.17)}$$

Remark 8. It is important to note that the bound on changes in $R_{u,v}$ from time $n$ to $n+1$ only holds for the nodes $u, v \in [n]$ that are already present in $G(n, p_n)$. Indeed, for the new node $n+1$ that is added at time $n+1$, we have $\overline{R}^{(n)}_{u,n+1} = \infty$, and thus $\overline{R}^{(n)}_{u,n+1} = 1$. In this case, the bound in (C.17) is replaced by

$$\left| \overline{R}^{(n+1)}_{u,n+1} - \overline{R}^{(n)}_{u,n+1} \right| = \overline{R}^{(n)}_{u,n+1} - \overline{R}^{(n+1)}_{u,n+1} = 1 - \overline{R}^{(n+1)}_{u,n+1} \leq 1 \quad \text{for any vertex } u \in [n]. \quad \text{(C.18)}$$

Proof. By Lemma 5, it suffices to prove the inequality with respect to the effective resistance, rather than the renormalized resistance.

Let $d_u^{(n)}$ denote the degree of vertex $u$ in $G_n$, and similarly define $d_u^{(n+1)}$. Using the triangle inequality,

$$\left| \overline{R}^{(n+1)}_{u,v} - \overline{R}^{(n)}_{u,v} \right| \leq \overline{R}^{(n)}_{u,v} - \left( \frac{1}{d_u^{(n)}} + \frac{1}{d_v^{(n)}} \right) + \left( \frac{1}{d_u^{(n+1)}} + \frac{1}{d_v^{(n+1)}} \right) - \left( \frac{1}{d_u^{(n+1)}} + \frac{1}{d_v^{(n+1)}} \right) \quad \text{(C.19)}$$

From lemma 6, we obtain bounds on the first and third terms, of order $\Theta \left( 1/d_n^2 \right)$. Since $|d_u^{(n+1)} - d_u^{(n)}| \leq 1$, the middle term is of order $\Theta \left( 1/d_n^2 \right)$, which can in turn be bounded by a term of order $\Theta \left( 1/d_n^2 \right)$ with high probability using C.13. Putting everything together, we get

$$\left| \overline{R}^{(n+1)}_{u,v} - \overline{R}^{(n)}_{u,v} \right| = \Theta \left( \frac{1}{d_n^2} \right).$$

The inequality is proven for the effective resistance, and using lemma 5 it also holds for the renormalized resistance. \hfill \Box

C.2 Effective resistances in the stochastic blockmodel

We first recall that the number of cross-community edges, $k_n$, is a binomial distribution, and thus concentrates around its expectation $\mathbb{E} \left[ k_n \right]$ for large $n$, as explained in the following lemma.

Lemma 7. Let $G_n \sim G(n, p_n, q_n)$ be a (balanced two-community) stochastic blockmodel with $p_n = \omega(\log n/n)$ and $q_n = \omega(1/n^2)$. There exists $n_0$, such that

$$\forall n \geq n_0, \quad \frac{3}{4} < \frac{\mathbb{E} \left[ k_n \right]}{k_n} < \frac{3}{2}, \quad \text{with probability} \geq 0.9. \quad \text{(C.20)}$$

Proof. The random variable $k_n$ is binomial $B(n_1 n_2, q_n)$, where $n_1 = \lfloor (n + 1)/2 \rfloor$, and $n_2 = \lfloor n/2 \rfloor$. We have $\mathbb{E} \left[ k_n \right] = n_1 n_2 q_n$. We apply a Chernoff’s bound on $k_n$ to get

$$\text{Prob} \left( |k_n - \mathbb{E} \left[ k_n \right]| > \epsilon \right) < 2e^{-\epsilon^2/(3\mathbb{E} \left[ k_n \right])}. \quad \text{(C.21)}$$
Using $\varepsilon = 3\sqrt{\mathbb{E}[k_n]}$, we get

$$\text{Prob}\left(\left|k_n - \mathbb{E}[k_n]\right| > 3\sqrt{\mathbb{E}[k_n]}\right) < 2\varepsilon^{-3} < 0.1$$  \hfill (C.22)

or

$$1 - 3\frac{1}{\sqrt{\mathbb{E}[k_n]}} < \frac{k_n}{\mathbb{E}[k_n]} < 1 + 3\frac{1}{\sqrt{\mathbb{E}[k_n]}} \quad \text{with probability} \ > 0.9.$$  \hfill (C.23)

Now, $\mathbb{E}[k_n] = q_n n_1 n_2 = q_n \Theta(n^2) = \omega(1)$, and thus $\lim_{n \to \infty} \mathbb{E}[k_n] = \infty$. Consequently $\exists n_0$ such that

$$\forall n \geq n_0, \quad \mathbb{E}[k_n] > 81,$$  \hfill (C.24)

and thus

$$\forall n \geq n_0, \quad \frac{3}{4} < \frac{\mathbb{E}[k_n]}{k_n} < \frac{3}{2}, \quad \text{with probability} \ > 0.9.$$  \hfill (C.25)

We now translate our understanding of the Erdős-Rényi random graph to the analysis of the stochastic blockmodel. As explained in lemma 4, in the following we write $1/d_n$ when either $1/d_{n1}$ or $1/d_{n2}$ could be used, and the error between the two terms is no larger than $\Theta(1/d_n^2)$.

**Lemma 8** (Cross-community resistance bounds). Let $G_n \sim \mathcal{G}(n, p_n, q_n)$ be a (balanced two-community) stochastic blockmodel with $p_n = \omega(\log n/n)$ and $q_n = \omega(1/n^2)$. Let $u$ and $v$ be vertices in the communities $C_1$ and $C_2$ respectively. Let $\bar{d}_n$ be the expected in-community degree of $C_1$. Let $k_n$ be the (random) number of cross-community edges. With high probability, the effective resistance $\hat{R}_{uv}$ is bounded according to

$$\frac{1}{k_n} \leq \hat{R}_{uv} \leq \frac{1}{k_n} + \frac{4}{\bar{d}_n} + \Theta\left(\frac{1}{\bar{d}_n^2}\right).$$  \hfill (C.26)

**Remark 9.** We recall (see lemma 4) that when we write $\bar{d}_n$, in C.32, it either means the expected degree of $C_1$ or $C_2$.

**Remark 10.** The requirement that $p_n = \omega(\log(n)/n)$ guarantees that we are in a regime where resistances in the Erdős-Rényi graph converge to $2/d_n$. The requirement that $q_n = \omega(1/n^2)$ guarantees that $\mathbb{E}[k_n] \to \infty$, and because of lemma 7, $k_n \to \infty$ with high probability. Finally, $q_n = o(p_n/n)$ guarantees that $\mathbb{E}[k_n] = o\left(\bar{d}_n\right)$, and using lemma 7 we have $k_n = o\left(\bar{d}_n\right)$ with high probability.

**Proof of Lemma 8.** Without loss of generality, we assume that $u \in C_1$, and $v \in C_2$ (see Fig. C.1). To obtain the lower bound on $\hat{R}_{uv}$, we use the Nash-Williams inequality [36], which we briefly recall here. Let $u$ and $v$ be two distinct vertices. A set of edges $E_c$ is an edge-cutset separating $u$ and $v$ if every path from $u$ to $v$ includes an edge in $E_c$.

**Lemma 9** (Nash-Williams, [36]). If $u$ and $v$ are separated by $K$ disjoint edge-cutsets $E_k$, $k = 1, \ldots, K$, then

$$\sum_{k=1}^{K} \left[ \sum_{(v_n, v_m) \in E_k} R_{n,m}^{-1} \right]^{-1} \leq \hat{R}_{uv}, \quad \text{where} \ (v_n, v_m) \ \text{is an edge in the cutset} \ E_k.$$  \hfill (C.27)
We note that the expression of Thomson’s principle in the following way: we construct a unit \( f_{\low} \) from which is a unit \( f_{\low} \) from \( u \) to \( v \). Since the set of cross-community edges is a cutset for all pairs of vertices \( u \) and \( v \) in separate communities, and since the size of this set is precisely \( k_n \), we immediately obtain the desired lower bound.

The upper bound is obtained using the characterization of the effective resistance based on Thomson principle [33], which we recall briefly in the following. Let \( f \) be a flow along the edges \( E \) from \( u \) to \( v \), and let

\[
\mathcal{E}(f) = \sum_{e \in E} f^2(e) R_e,
\]

be the energy of the flow \( f \), where each undirected edge \( e \) in the sum is only counted once. A unit flow has strength one,

\[
\text{div}(f)(u) = -\text{div}(f)(v) = 1.
\]

Thomson’s principle provides the following characterization of the effective resistance \( \hat{R}_{uv} \),

\[
\hat{R}_{uv} = \min \{ \mathcal{E}(f), f \text{ is a unit flow from } u \text{ to } v \}.
\]

We use Thomson’s principle in the following way: we construct a unit flow \( f \) from \( u \) to \( v \). For this flow, the energy \( \mathcal{E}(f) \) yields an upper bound on \( \hat{R}_{uv} \).

First, consider the case where neither \( u \) nor \( v \) are incident with any of the \( k_n \) cross-community edges, \( e_i = (u_i, v_i), i = 1, \ldots, k_n \); where \( u_i \in C_1 \) and \( v_i \in C_2 \). Denote by \( f_i^u \) the unit flow associated with the effective resistance between \( u \) and \( u_i \) when only the edges in \( C_1 \) are considered. Similarly define \( f_i^v \) to be the unit flow associated with the effective resistance between \( v \) and \( v_i \) when only the edges in \( C_2 \) are considered. Using the corollary 1, given by \( (C.16) \), we have with high probability,

\[
\mathcal{E}(f_i^u) = \frac{2}{d_n} + \Theta \left( \frac{1}{d_n^2} \right), \quad \text{and} \quad \mathcal{E}(f_i^v) = \frac{2}{d_n} + \Theta \left( \frac{1}{d_n^2} \right).
\]

We note that the expression of \( \mathcal{E}(f_i^v) \) should involve the expected degree in \( C_2 \). As explained in lemma 4, we can use \( \overline{d}_n \) since the difference between the two terms is absorbed in the \( \Theta \left( \frac{1}{d_n^2} \right) \) term. Finally, let \( f_i^e \) be the flow that is 1 on edge \( e_i \) and 0 elsewhere. To conclude, we assemble the three flows and define

\[
f(e) = \frac{1}{k_n} \sum_{i=1}^{k} \left\{ f_i^u(e) + f_i^e(e) + f_i^v(e) \right\},
\]

which is a unit flow from \( u \) to \( v \). Since \( \mathcal{E} \) is a convex function, we can bound the energy of \( f \) via

\[
\mathcal{E}(f) = \mathcal{E} \left( \frac{1}{k_n} \sum_{i=1}^{k} \left\{ f_i^u(e) + f_i^e(e) + f_i^v(e) \right\} \right) \leq \mathcal{E} \left( \frac{1}{k_n} \sum_{i=1}^{k} f_i^u(e) \right) + \mathcal{E} \left( \frac{1}{k_n} \sum_{i=1}^{k} f_i^e(e) \right) + \mathcal{E} \left( \frac{1}{k_n} \sum_{i=1}^{k} f_i^v(e) \right)
\leq \frac{1}{k_n} \sum_{i=1}^{k} \mathcal{E}(f_i^u) + \frac{1}{k_n} \sum_{i=1}^{k} \mathcal{E}(f_i^e) + \frac{1}{k_n} \sum_{i=1}^{k} \mathcal{E}(f_i^v) = \frac{4}{d_n} + \frac{1}{k_n} + \Theta \left( \frac{1}{d_n^2} \right).
\]
The final line holds with high probability. Note that we calculate the energy of the flow in the center term directly, whereas convexity is used to estimate the energy in the first and third term.

This upper bound also holds when either \( u \) or \( v \) is incident with any of the cross-community edges. In this case, \( u = u_i \) for some \( i \). For this \( i \), we can formally define the flow \( f_{i}^{u} \) between \( u \) and \( u_i \) to be the zero flow, which minimizes the energy trivially and has energy equal to the resistance between \( u \) and \( u_i \) (which is zero). Then the above calculation yields a smaller upper bound for the first and third terms.

**Remark 11.** Lemma 8 provides a first attempt at analysing the perturbation of the effective resistance under the addition of edges in the stochastic blockmodel. The upper bound provided by (C.26) is too loose to be useful, and we therefore resort to a different technique to get a tighter bound. The idea is to observe that the effective resistance is controlled by the bottleneck formed by the cross-community edges. We can get very tight estimates of the fluctuations in the effective resistance using a detailed analysis of the addition of a single cross-community edge. We use the Sherman–Morrison–Woodbury theorem [15] to compute a rank-one perturbation of the pseudo-inverse of the normalized graph Laplacian [38], \( L^1 \). The authors in [42] provide us with the exact expression that is needed for our work, see (C.36). The proof proceeds by induction on the number of cross-community edges, \( k_n \).

**Theorem 4.** Let \( G_n \sim G(n, p_n, q_n) \) be a balanced, two community stochastic blockmodel with \( p_n = \omega(\log n/n) \), \( q_n = \omega(1/n^2) \), and \( q_n = o(p_n/n) \). We assume that \( G_n \) is connected. The effective resistance between two vertices \( u \) and \( v \) is given by

\[
\hat{R}_{u,v} = \frac{2}{d_n} + \begin{cases} \mathcal{O} \left( \frac{1}{d_n^2} \right), & \text{if } u \text{ and } v \text{ are in the same community,} \\ \frac{1}{k_n} + \frac{\alpha(k_n, u, v)}{d_n k_n} + \mathcal{O} \left( \frac{1}{d_n^2} \right), & \text{otherwise.} \end{cases}
\]  

(C.32)

Also, conditioned on \( k_n = k \) the random variable \( \alpha(k_n, u, v) \) is a deterministic function of \( k \), and we have \( \alpha(k_n, u, v) = \mathcal{O}(k_n) \).

**Proof.** First, observe that Lemma 8 immediately implies that

\[-2k_n \leq \alpha(k_n, u, v) \leq 2k_n, \quad \forall u, v,\]

(C.33)

with high probability, so \( \alpha = \mathcal{O}(k_n) \).

Next, let us show that the in-community resistances follow the prescribed form. The proof proceeds as follows: we derive the expression (C.32) conditioned on the random variable \( k_n = k \), and we prove that \( \alpha(k, u, v) \) is indeed a deterministic function in this case; the derivation of (C.32) is obtained by induction on \( k \).

The engine of our induction is the update formula (equation 11) in [42]. This provides an exact formula (equation (C.36) below) for the change in resistance between any pair of vertices in a graph when a single edge is added or removed. The particular motivation of the authors in [42] is to calculate rank one updates to the pseudoinverse of the combinatorial graph Laplacian; however, it conspires that their formula is also very useful to inductively calculate resistances in the stochastic blockmodel.

We first consider the base case, where \( G_n \) is a balanced, two community stochastic blockmodel of size \( n \) with \( k_n = 1 \) cross-community edge. Denote this edge by \( e_1 = (u_1, v_1) \), where \( u_1 \in C_1 \) and \( v_1 \in C_2 \). We will refer to this graph as \( G_n^{(1)} \).

The addition of a single edge connecting otherwise disconnected components does not change the resistance within those components, as it does not introduce any new paths between two vertices within the same component. Because each community is an Erdős–Rényi graph with parameters \( p_n \) and respective sizes
$n_1 = [(n + 1)/2]$ and $n_2 = [n/2]$, corollary 1 provides the expression for the effective resistance between two vertices within each community. A simple circuit argument allows us to obtain the resistance between $u$ and $v$ in separate communities via

$$\hat{R}_{uv} = \hat{R}_{u_1v_1} + \hat{R}_{u_2v_2} + \hat{R}_{v_1u_1}. \quad \text{(C.34)}$$

If $u \neq u_1$ and $v \neq v_1$, then we combine Nash-Williams and corollary 1 to get

$$\hat{R}_{uv} = \frac{1}{k} + 4\frac{\alpha(u, v)}{k} + \Theta\left(\frac{1}{d_n^2}\right).$$

If $u = u_1$ and/or $v = v_1$ then the appropriate resistances are set to zero in (C.34).

In summary, for arbitrary pairs $(u, v)$ in $G^{(1)}_n$, we have

$$\hat{R}_{uv} = \frac{1}{k} + 2\frac{\alpha(u, v)}{d_n} + \Theta\left(\frac{1}{d_n^2}\right),$$

where

$$\alpha(u, v) = \begin{cases} 0 & \text{if } u = u_1 \text{ and } v = v_1, \\ 1 & \text{if } u = u_1 \text{ and } v \neq v_1, \text{ or } u \neq u_1 \text{ and } v = v_1, \\ 2 & \text{if } u \neq u_1 \text{ and } v \neq v_1. \end{cases} \quad \text{(C.35)}$$

This establishes the base case for (C.32).

We now assume that (C.32) holds for any balanced, two community stochastic blockmodel of size $n$ with $k_n = k$ cross-community edges. We consider a balanced, two community stochastic blockmodel $G^{(k+1)}_n$ of size $n$ with $k_n = k + 1$ cross-community edges. We denote the cross-community edges by $e_i = (u_i, v_i), i = 1, \ldots, k + 1$, where $u_i \in C_1$ and $v_i \in C_2$ (see Fig. C.2).

Finally, we denote by $G^{(k)}_n$ the balanced, two community stochastic blockmodel with $k$ cross-community edges obtained by removing the edge $e_{k+1} = (u_{k+1}, v_{k+1})$ from $G^{(k+1)}_n$.

Let $\hat{R}$ denote the effective resistances in $G^{(k)}_n$ and $\hat{R}'$ denote the effective resistances in $G^{(k+1)}_n$.

Since $G^{(k)}_n$ is obtained by removing an edge from $G^{(k+1)}_n$, we can apply equation (11) in [42] to express $\hat{R}'$ from $\hat{R}$,

$$\hat{R}'_{uv} = \hat{R}_{uv} - \frac{\left(\hat{R}_{u_1v_{k+1}} - \hat{R}_{u_{k+1}v_1} - \hat{R}_{v_1u_{k+1}} + \hat{R}_{u_{k+1}v_{k+1}}\right)^2}{4(1 + \hat{R}_{u_{k+1}v_{k+1}})}. \quad \text{(C.36)}$$

![Figure C.2: Balanced, two community stochastic blockmodel $G^{(k+1)}_n$ of size $n$ with $k_n = k + 1$ cross-community edges. Vertices $u$ and $v$ are in the same community $C_1$](image-url)
In the following we use the induction hypothesis to compute \( \hat{R}' \) using (C.36). We first consider the case where the vertices \( u \) and \( v \) belong to the same community, say \( C_1 \) without loss of generality (see Fig. C.2).

We need to consider the following three possible scenarios:

1. \( u \neq u_{k+1} \) and \( v \neq v_{k+1} \),
2. \( u = u_{k+1} \) and \( v \neq v_{k+1} \),
3. \( u \neq u_{k+1} \) and \( v = v_{k+1} \).

We will treat the first case; the last two cases are in fact equivalent, and are straightforward consequences of the analysis done in the first case. From the induction hypothesis we have

\[
\hat{R}_{u \cdot v_{k+1}} = \frac{1}{k} + \frac{2}{dn} + \frac{\alpha(k, u, v_{k+1})}{kd_n} + \Theta\left(\frac{1}{d_n^2}\right),
\]

(C.37)

\[
\hat{R}_{u \cdot u_{k+1}} = \frac{2}{dn} + \Theta\left(\frac{1}{d_n^2}\right),
\]

(C.38)

\[
\hat{R}_{u \cdot v_{k+1}} = \frac{1}{k} + \frac{2}{dn} + \frac{\alpha(k, v, v_{k+1})}{kd_n} + \Theta\left(\frac{1}{d_n^2}\right),
\]

(C.39)

\[
\hat{R}_{u \cdot u_{k+1}} = \frac{2}{dn} + \Theta\left(\frac{1}{d_n^2}\right),
\]

(C.40)

\[
\hat{R}_{u_{k+1} \cdot v_{k+1}} = \frac{1}{k} + \frac{2}{dn} + \frac{\alpha(k, u_{k+1}, v_{k+1})}{kd_n} + \Theta\left(\frac{1}{d_n^2}\right).
\]

(C.41)

Substituting these expression into (C.36), we get

\[
\hat{R}'_{u \cdot v} = \hat{R}_{u \cdot v} - \frac{\left(\frac{1}{k} + \frac{\alpha(k, u, v_{k+1})}{kd_n}\right) - \left(\frac{1}{k} + \frac{\alpha(k, v, v_{k+1})}{kd_n}\right) + \Theta\left(\frac{1}{d_n^2}\right)}{4\left(1 + \frac{1}{k} + \frac{2}{dn} + \frac{\alpha(k, u_{k+1}, v_{k+1})}{kd_n} + \Theta\left(\frac{1}{d_n^2}\right)\right)}
\]

(C.42)

\[
= \hat{R}_{u \cdot v} - \Theta\left(\frac{1}{d_n^2}\right) \cdot \frac{\left(\frac{\alpha(k, u, v_{k+1})}{k} - \frac{\alpha(k, v, v_{k+1})}{k}\right) + \Theta\left(\frac{1}{d_n^2}\right)}{4\left(1 + \frac{1}{k} + \frac{2}{dn} + \frac{\alpha(k, u_{k+1}, v_{k+1})}{kd_n} + \Theta\left(\frac{1}{d_n^2}\right)\right)}
\]

(C.43)

Because \( \alpha \) is bounded with high probability (see (C.33)), we have

\[
\frac{\alpha(k, u, v_{k+1}) - \alpha(k, v, v_{k+1})}{k} + \Theta\left(\frac{1}{d_n^2}\right) = \Theta(1) \quad \text{with high probability},
\]

(C.44)

and also

\[
1 + \frac{1}{k} + \frac{2}{dn} + \frac{\alpha(k, u_{k+1}, v_{k+1})}{kd_n} + \Theta\left(\frac{1}{d_n^2}\right) = 1 + \frac{1}{k} + \Theta\left(\frac{1}{d_n}\right) \quad \text{with high probability},
\]

(C.45)

which implies

\[
\frac{\left(\frac{\alpha(k, u, v_{k+1})}{k} - \frac{\alpha(k, v, v_{k+1})}{k}\right) + \Theta\left(\frac{1}{d_n^2}\right)}{4\left(1 + \frac{1}{k} + \frac{2}{dn} + \frac{\alpha(k, u_{k+1}, v_{k+1})}{kd_n} + \Theta\left(\frac{1}{d_n^2}\right)\right)} = \Theta(1) \quad \text{with high probability}.
\]

(C.46)
Figure C.3: Balanced, two community stochastic blockmodel $G_n^{(k+1)}$ of size $n$ with $k_n = k + 1$ cross-community edges. The vertices $u$ and $v$ are in different communities, $u \in C_1$ and $v \in C_2$.

We conclude that
\[
\hat{R}^{u,v}_{u',v'} = \hat{R}^{u,v}_{u',v'} + O\left(\frac{1}{d_n^2}\right) \quad \text{with high probability.} \tag{C.47}
\]

This completes the induction, and the proof of (C.32) in the case where $u$ and $v$ belong to the same community.

We now consider the case where $u \in C_1$ and $v \in C_2$ (see Fig. C.3). As above, we need to consider the following three possible scenarios:

1. $u \neq u_{k+1}$ and $v \neq v_{k+1}$,
2. $u = u_{k+1}$ and $v \neq v_{k+1}$,
3. $u \neq u_{k+1}$ and $v = v_{k+1}$.

Again, we only prove the first case; the last two equivalent cases are straightforward consequences of the first case. From the induction hypothesis we now have
\[
\hat{R}^{u,u_{k+1}}_{u',u_{k+1}} = \frac{2}{d_n} + O\left(\frac{1}{d_n^2}\right), \tag{C.48}
\]
\[
\hat{R}^{v,v_{k+1}}_{v',v_{k+1}} = \frac{2}{d_n} + O\left(\frac{1}{d_n^2}\right), \tag{C.49}
\]
\[
\hat{R}^{u,v_{k+1}}_{u',v_{k+1}} = \frac{1}{k} + \frac{2}{d_n} + \frac{\alpha(u,v_{k+1})}{kd_n} + O\left(\frac{1}{d_n^2}\right), \tag{C.50}
\]
\[
\hat{R}^{v,u_{k+1}}_{v',u_{k+1}} = \frac{1}{k} + \frac{2}{d_n} + \frac{\alpha(v,u_{k+1})}{kd_n} + O\left(\frac{1}{d_n^2}\right), \tag{C.51}
\]
\[
\hat{R}^{u_{k+1},v_{k+1}}_{u_{k+1}',v_{k+1}'} = \frac{1}{k} + \frac{2}{d_n} + \frac{\alpha(u_{k+1},v_{k+1})}{kd_n} + O\left(\frac{1}{d_n^2}\right). \tag{C.52}
\]

To reduce notational clutter, we use some abbreviated notation to denote the various $\alpha$ terms associated with the vertices of interest (see Fig. C.4),
\[
\begin{align*}
\alpha_0 & \overset{\text{def}}{=} \alpha(k_n^{(k)}, u, v), \\
\alpha_1 & \overset{\text{def}}{=} \alpha(k_n^{(k)}, u_{k+1}, v), \\
\alpha_2 & \overset{\text{def}}{=} \alpha(k_n^{(k)}, u, v_{k+1}), \\
\alpha_3 & \overset{\text{def}}{=} \alpha(k_n^{(k)}, u_{k+1}, v_{k+1}).
\end{align*}
\]
From (C.36), we have

\[ \Delta R = \frac{1}{k} + \frac{\alpha_2}{kd_n} - \left( \frac{1}{k} - \frac{\alpha_1}{kd_n} \right) + O \left( \frac{1}{d_n^2} \right) \]

\[ \Delta R = \frac{1}{4} \left( 1 + \frac{1}{k} + \frac{2}{d_n} + \frac{\alpha_3}{kd_n} + O \left( \frac{1}{d_n^2} \right) \right) \]

\[ = \frac{1}{k^2} \left[ 1 + \frac{\alpha_1 + \alpha_2}{2d_n} + O \left( \frac{1}{d_n^2} \right) \right]^2 \frac{1}{1 + \frac{1}{k} + \frac{2}{d_n} + \frac{\alpha_3}{kd_n} + O \left( \frac{1}{d_n^2} \right)} \]

\[ = \frac{1}{k^2} \left[ 1 + \frac{\alpha_1 + \alpha_2}{2d_n} + O \left( \frac{1}{d_n^2} \right) \right]^2 \frac{1}{1 + \frac{1}{k} + \frac{2}{d_n} + \frac{\alpha_3}{kd_n} + O \left( \frac{1}{d_n^2} \right)} \]

\[ = \frac{1}{k^2} \left[ 1 + \frac{\alpha_1 + \alpha_2}{2d_n} + O \left( \frac{1}{d_n^2} \right) \right]^2 \frac{1}{1 + \frac{1}{k} + \frac{2}{d_n} + \frac{\alpha_3}{kd_n} + O \left( \frac{1}{d_n^2} \right)} \]

At this juncture, we need to expand \( \left( 1 + \frac{1}{k} + \frac{2}{d_n} + \frac{\alpha_3}{kd_n} + O \left( \frac{1}{d_n^2} \right) \right)^{-1} \) using a Taylor series, which is possible when \( \frac{1}{k} + \frac{2}{d_n} + \frac{\alpha_3}{kd_n} < 1 \). Since we are interested in the large \( n \) asymptotic, we can assume that for \( n \) sufficiently large \( \frac{2}{d_n} + \frac{\alpha_3}{kd_n} < 1/2 \), and thus we need to guarantee that \( k \geq 2 \).

The case \( k = 1 \) needs to be handled separately. Setting \( k = 1 \) into (C.54) yields

\[ \Delta R = \frac{1 + \frac{\alpha_1 + \alpha_2}{d_n} + O \left( \frac{1}{d_n^2} \right)}{2 + \frac{2 + \alpha_3}{d_n} + O \left( \frac{1}{d_n^2} \right)} \]

\[ = \frac{1}{2} \left( 1 + \frac{\alpha_1 + \alpha_2}{d_n} + O \left( \frac{1}{d_n^2} \right) \right) \left( 1 - \frac{2 + \alpha_3}{2d_n} + O \left( \frac{1}{d_n^2} \right) \right) \]

\[ = \frac{1}{2} \left( 1 + \frac{\alpha_1 + \alpha_2 - \alpha_3/2}{2d_n} + O \left( \frac{1}{d_n^2} \right) \right) \]

Let us denote the decrease in effective resistance by \( \Delta \bar{R} \),

\[ \Delta \bar{R} \equiv \bar{R}_{u,v} - \bar{R}'_{u,v}. \]  

(C.53)

Figure C.4: Coefficients \( \alpha_0, \ldots, \alpha_3 \) in (C.32) for several pairs of vertices in \( G_n^{(k+1)} \).
which leads to

\[
\hat{R}'_{\mu \nu} - \hat{R} = 1 + \frac{2}{d_n} + \frac{\alpha_0}{2} - \frac{1}{2} - \frac{\alpha_1 + \alpha_2 - 1 - \alpha_3/2}{2d_n} + \mathcal{O}\left(1/d_n^2\right),
\]

\[= \frac{1}{2} + \frac{2}{d_n} + \frac{2\alpha_0 - \alpha_1 - \alpha_2 + 1 + \alpha_3/2}{2d_n} + \mathcal{O}\left(1/d_n^2\right),
\]

(C.57)

Let

\[
\alpha_0' \overset{\text{def}}{=} 2\alpha_0 - \alpha_1 - \alpha_2 + 1 + \alpha_3/2,
\]

(C.58)

then we have

\[
\hat{R}'_{\mu \nu} = \frac{1}{2} + \frac{2}{d_n} + \frac{\alpha_0'}{2d_n} + \mathcal{O}\left(1/d_n^2\right),
\]

(C.59)

which matches the expression given in (C.32) for \(k = 2\). This completes the induction for \(k = 1\).

We now proceed to the general case where \(k \geq 2\). In that case, we use a Taylor series expansion of

\[
\left(1 + \frac{1}{k} + 2/d_n + \frac{\alpha_3/(kd_n)}{2} + \mathcal{O}\left(1/d_n^2\right)\right)^{-1},
\]

and we get

\[
\left(1 + \frac{1}{k} + 2/d_n + \frac{\alpha_3/(kd_n)}{2} + \mathcal{O}\left(1/d_n^2\right)\right)^{-1} = \sum_{m=0}^{\infty} (-1)^m \left(\frac{1}{k} + \frac{2}{d_n} + \frac{\alpha_3}{kd_n}\right)^m + \mathcal{O}\left(1/d_n^2\right).
\]

(C.60)

Now, most of the term in \(\left(\frac{1}{k} + \frac{2}{d_n} + \frac{\alpha_3}{kd_n}\right)^m\) are of order \(\mathcal{O}\left(1/d_n^2\right)\), and we need to carefully extract the few significant terms. In the expansion of \(\left(\frac{1}{k} + \frac{2}{d_n} + \frac{\alpha_3}{kd_n}\right)^m\) the only terms that do not contain a \(1/d_n^2\) are obtained by choosing systematically \(1/k\) in each of the \(m\) factors, or choosing \(1/k\) in all but one factors and either \(\frac{2}{d_n}\) or \(\frac{\alpha_3}{kd_n}\) in the last factor. There are \(m\) ways to construct these last two terms. In summary, we have for \(m \geq 1\),

\[
\left(\frac{1}{k} + \frac{2}{d_n} + \frac{\alpha_3}{kd_n}\right)^m = k^m + m \frac{\alpha_3}{k^{m-1}d_n} + m \frac{2}{k^{m-1}d_n} + \mathcal{O}\left(1/d_n^2\right).
\]

(C.61)

We can substitute (C.61) into (C.60) to get

\[
\left(1 + \frac{1}{k} + 2/d_n + \frac{\alpha_3/(kd_n)}{2} + \mathcal{O}\left(1/d_n^2\right)\right)^{-1} = \sum_{m=0}^{\infty} (-1)^m \frac{1}{k^m} + \sum_{m=1}^{\infty} m \frac{\alpha_3}{k^{m-1}d_n} + m \frac{2}{k^{m-1}d_n} + \mathcal{O}\left(1/d_n^2\right)
\]

\[= \frac{1}{1 + 1/k} - \frac{\alpha_3 k}{d_n (k + 1)^2} - 2 \frac{k^2}{(k + 1)^2} + \mathcal{O}\left(1/d_n^2\right)
\]

\[= \frac{k}{k + 1} - \frac{(\alpha_3 + 2k)k}{d_n(k + 1)^2} + \mathcal{O}\left(1/d_n^2\right).
\]

(C.62)

We can insert (C.62) into (C.54) to get

\[
\Delta R = \left(\frac{1}{k^2} + \frac{\alpha_1 + \alpha_2}{d_n k^2}\right) \left(\frac{k}{k + 1} - \frac{(\alpha_3 + 2k)k}{d_n(k + 1)^2}\right) + \mathcal{O}\left(1/d_n^2\right)
\]

\[= \frac{1}{k(k + 1)} + \frac{\alpha_1 + \alpha_2}{d_n k(k + 1)} - \frac{(\alpha_3 + 2k)k}{d_n k(k + 1)^2} + \mathcal{O}\left(1/d_n^2\right),
\]

(C.63)
We note that
\[ \alpha_0 k \]
which matches the expression given in (C.32) for \( k \). This leads to
\[ \alpha = \frac{1 + \alpha_1}{\alpha_2} \]
The cases where \( k \) are deterministic functions of \( k \), by the inductive hypothesis.

Let \( G = G(n, p_n, q_n) \) be a stochastic blockmodel with \( p_n = o(\log n/n) \). Assume that \( q_n = o(p_n/n) \). Let \( G_n \) be the subgraph of \( G_{n+1} \) induced by the vertex set \([n]\). Let \( k_n \) and \( k_{n+1} \) be the number of cross-community edges in \( G_n \) and \( G_{n+1} \) respectively.

Let \( u \) and \( v \) be two vertices, and let \( \hat{R}_{uv}^{(n)} \) and \( \hat{R}_{uv}^{(n+1)} \) be the effective resistances measured in \( G_n \) and \( G_{n+1} \) respectively.

Let \( \alpha(k_n, u, v) \) and \( \alpha(k_{n+1}, u, v) \) be the coefficients in the expansion of \( \hat{R}_{uv}^{(n)} \) and \( \hat{R}_{uv}^{(n+1)} \) in (C.32) respectively. We have,
\[ |\alpha(k_{n+1}, u, v) - \alpha(k_n, u, v)| = o(1) \] with high probability,
\[ |\frac{\alpha(k_{n+1}, u, v)}{k_{n+1}} - \frac{\alpha(k_n, u, v)}{k_n}| = o\left(\frac{1}{k_n}\right) \] with high probability.

The proof of the corollary is a consequence of the following proposition which shows that, conditioned on \( k_n = k \), the functions \( \alpha(k, u, v) \) and \( \alpha(k, u, v)/k \), defined for \( k \geq 1 \), have bounded variation.

Proposition 1. Let \( G_n \sim G(n, p_n, q_n) \) be a balanced, two community stochastic blockmodel with \( p_n = o(n^2) \), and \( q_n = o(p_n/n) \). We assume that \( G_n \) is connected. Let \( u \) and \( v \) be two vertices. Given \( k_n = k \), let \( \alpha(k, u, v) \) be the coefficient in the expansion of \( \hat{R}_{uv} \) in (C.32). Similarly, let \( \alpha(k+1, u, v) \)
be the corresponding quantity when \( k_n = k + 1 \). We have

\[
|\alpha(k+1, u, v) - \alpha(k, u, v)| \leq 8, 
\]

(C.71)

and

\[
\left| \frac{\alpha(k+1, u, v)}{k+1} - \frac{\alpha(k, u, v)}{k} \right| \leq \frac{6}{k}. 
\]

(C.73)

**Proof.** The proof is a direct consequence of (C.65), and the fact that \( |\alpha(k, u, v)| \leq 2k \) (see (C.33)). Let us start with the first inequality. From (C.65) we have

\[
\alpha(k+1, u, v) - \alpha(k, u, v) = \frac{\alpha(k, u, v) - \alpha_1 - \alpha_2}{k} + \frac{2}{k + 1} + \frac{\alpha_3}{k(k+1)}, 
\]

(C.74)

and thus

\[
|\alpha(k+1, u, v) - \alpha(k, u, v)| \leq \frac{6k}{k} + \frac{2}{k + 1} + \frac{2k}{k(k+1)} = 6 + \frac{4}{k + 1} \leq 8. 
\]

(C.75)

We now show the second inequality. Again, from (C.65) we have

\[
\frac{\alpha(k+1, u, v)}{k+1} - \frac{\alpha(k, u, v)}{k} = \frac{\alpha(k, u, v) - \alpha(k, u, v) - \alpha_1 - \alpha_2}{k(k+1)} + \frac{2}{(k + 1)^2} + \frac{\alpha_3}{k(k+1)^2} 
\]

\[
= -\frac{\alpha_1 + \alpha_2}{k(k+1)} + \frac{2}{(k + 1)^2} + \frac{\alpha_3}{k(k+1)^2}, 
\]

(C.76)

and thus

\[
\left| \frac{\alpha(k+1, u, v)}{k+1} - \frac{\alpha(k, u, v)}{k} \right| \leq \frac{4k}{k(k + 1)} + \frac{2}{(k + 1)^2} + \frac{2k}{k(k+1)^2} = \frac{4}{k + 1} + \frac{2}{(k + 1)^2} + \frac{2}{(k + 1)^2} 
\]

(C.77)

We now proceed to the proof of corollary 2.

**Proof of corollary 2.** We first verify that with high probability \( k_{n+1} - k_n \) is bounded. We then apply proposition 1.

Let \( \Delta k_n \overset{\text{def}}{=} k_{n+1} - k_n \) be the number of adjacent cross-community edges that have vertex \( n+1 \) as one of their endpoints. \( \Delta k_n \) is a binomial random variable \( B(n_1, q_n) \), where we assume without loss of generality that vertex \( n+1 \in C_2 \). Because \( \mathbb{E}[\Delta k_n] = nq_n/2 = p_n o(1) \), \( \mathbb{E}[\Delta k_n] \) is bounded, and there exists \( \kappa \) such that \( \forall n, \mathbb{E}[\Delta k_n] < \kappa \). Using a Chernoff bound we have

\[
\text{Prob} \left( |\mathbb{E}[\Delta k_n] - \Delta k_n| > 3\sqrt{\kappa} \right) \leq 2 \exp \left( -\frac{1}{3} \right) < 0.01, 
\]

(C.78)

and thus

\[
|k_{n+1} - k_n| = |k_{n+1} - \mathbb{E}[\Delta k_n] + \mathbb{E}[\Delta k_n] - k_n| < 6\sqrt{\kappa} \quad \text{with probability} \ > 0.99 
\]

(C.79)
In other words, with high probability \( k_{n+1} - k_n \) is bounded by \( C = [6\sqrt{k}] \), independently of \( n \).

Finally, we have

\[
\frac{\alpha(k_{n+1}, u, v)}{k_{n+1}} - \frac{\alpha(k_n, u, v)}{k_n} = \frac{1}{k_{n+1}} \sum_{k=k_n}^{k_{n+1} - 1} \frac{\alpha(k+1, u, v)}{k+1} - \frac{\alpha(k, u, v)}{k} \leq \frac{1}{k_{n+1}} \sum_{k=k_n}^{k_{n+1} - 1} \left( \frac{\alpha(k+1, u, v)}{k+1} - \frac{\alpha(k, u, v)}{k} \right) \tag{C.80}
\]

Because of corollary 1, each term in the sum is bounded by \( 6/k \); the largest upper bound being \( 6/k_n \). Also, with high probability there are at most \( C \) terms. We conclude that

\[
\left| \frac{\alpha(k_{n+1}, u, v)}{k_{n+1}} - \frac{\alpha(k_n, u, v)}{k_n} \right| \leq C \frac{6}{k_n} \quad \text{with high probability.} \tag{C.81}
\]

We now combine lemma 4 with theorem 4 to estimate the perturbation created by the addition of an \( n + 1 \)th vertex to \( G_n \). The following lemma shows that adding an additional vertex, with corresponding edges to either one of the communities does not change the effective resistance, as long as no new cross-community edges are created.

**Lemma 10.** Let \( G_{n+1} \sim \mathcal{G}(n+1, p_n, q_n) \) be a stochastic blockmodel with \( p_n = \omega(\log n/n) \). Assume that \( q_n = o(p_n/n) \). Let \( G_n \) be the subgraph of \( G_{n+1} \) induced by the vertex set \( [n] \). Let \( k_n \) and \( k_{n+1} \) be the number of cross-community edges in \( G_n \) and \( G_{n+1} \) respectively.

Let \( u \) and \( v \) be two vertices in \( G_{n+1} \), for which the effective resistance \( \hat{R}^{(n)}_{u v} \), measured in \( G_n \), is properly defined. Let \( \tilde{R}^{(n+1)}_{u v} \) be the corresponding effective resistance measured in \( G_{n+1} \).

If \( u \) and \( v \) belong to the same community, then \( \tilde{R}^{(n+1)}_{u v} \) satisfies

\[
\tilde{R}^{(n)}_{u v} - \tilde{R}^{(n+1)}_{u v} = \Theta \left( \frac{1}{d_n} \right) \geq 0. \tag{C.82}
\]

If \( u \) and \( v \) are in different communities, then \( \tilde{R}^{(n+1)}_{u v} \) is controlled by the following inequalities,

\[
\begin{align*}
\tilde{R}^{(n)}_{u v} - \tilde{R}^{(n+1)}_{u v} & = \Theta \left( \frac{1}{d_n} \right) \geq 0, \quad \text{if} \quad k_n = k_{n+1}, \\
\tilde{R}^{(n)}_{u v} - \tilde{R}^{(n+1)}_{u v} & \geq \frac{2}{k_n^2} + \frac{1}{d_n} \Theta \left( \frac{1}{k_n} \right) \geq 0, \quad \text{if} \quad k_{n+1} > k_n.
\end{align*}
\tag{C.83}
\]

**Proof.** Because of lemma 4, we have

\[
\frac{1}{d_{n+1}} = \frac{1}{d_n} + \Theta \left( \frac{1}{d_{n+1}} \right), \tag{C.84}
\]

and

\[
\frac{1}{d_{n+1}^2} = \frac{1}{d_n^2} + \Theta \left( \frac{1}{d_{n+1}^2} \right). \tag{C.85}
\]
If \( u \) and \( v \) are in the same community, the expression for \( \hat{R}_{u,v}^{(n+1)} \) and \( \tilde{R}_{u,v}^{(n)} \), given by (C.32) coincide, up to order \( \Theta \left( 1/d_n^2 \right) \).

When \( u \) and \( v \) are in different communities, we need to consider the values of \( k_n \) and \( k_{n+1} \). If \( k_n = k_{n+1} \) then \( \alpha(k_n, u, v) = \alpha(k_{n+1}, u, v) \), and thus \( \hat{R}_{u,v}^{(n+1)} = \tilde{R}_{u,v}^{(n)} \), up to order \( \Theta \left( 1/d_n^2 \right) \).

If \( k_{n+1} > k_n \), we will show that the decrease in effective resistance is of order \( \Theta \left( 1/k_n^2 \right) \). We first recall that \( \hat{R}_{u,v}^{(n)} \geq \hat{R}_{u,v}^{(n+1)} \), since \( G^{n+1} \) has more edges than \( G_n \), and thus

\[
\left| \hat{R}_{u,v}^{(n)} - \hat{R}_{u,v}^{(n+1)} \right| = \hat{R}_{u,v}^{(n)} - \hat{R}_{u,v}^{(n+1)}. \tag{C.86}
\]

Using the expression for \( \hat{R}_{u,v}^{(n+1)} \) and \( \tilde{R}_{u,v}^{(n)} \), given by (C.32) we have

\[
\begin{align*}
\left| \hat{R}_{u,v}^{(n)} - \hat{R}_{u,v}^{(n+1)} \right| &= \frac{1}{k_n} - \frac{1}{k_{n+1}} + \frac{\alpha(k_n, u, v)}{d_n k_n} - \frac{\alpha(k_{n+1}, u, v)}{d_n k_{n+1}} + \Theta \left( \frac{1}{d_n^2} \right) \\
&\geq \frac{1}{k_n} - \frac{1}{k_{n+1}} + \frac{\alpha(k_n, u, v)}{d_n k_n} - \frac{\alpha(k_{n+1}, u, v)}{d_n (k_n + 1)} + \Theta \left( \frac{1}{d_n} \right) \\
&\geq \frac{1}{k_n (k_n + 1)} + \frac{1}{d_n} \left( \frac{\alpha(k_n, u, v)}{k_n} - \frac{\alpha(k_{n+1}, u, v)}{k_n + 1} \right) + \Theta \left( \frac{1}{d_n} \right) . 
\end{align*}
\tag{C.87}
\]

We recall that corollary 2 implies that

\[
\left| \frac{\alpha(k_n + 1, u, v)}{k_n + 1} - \frac{\alpha(k_n, u, v)}{k_n} \right| = \Theta \left( \frac{1}{k_n} \right) \tag{C.88}
\]

and thus

\[
\left| \hat{R}_{u,v}^{(n)} - \hat{R}_{u,v}^{(n+1)} \right| \geq \frac{1}{2 k_n^2} + \frac{1}{d_n} \Theta \left( \frac{1}{k_n} \right) .
\tag{C.89}
\]

which completes the proof.

\[
\square
\]

### C.3 The Distance \( D_n \) Under the Null Hypothesis

The following theorem provides an estimate of the distance \( D_n \) between \( G_n \) and \( G_{n+1} \) after the addition of node \( n + 1 \). Under the null hypothesis – \( n + 1 \) does not lead to an increase in the number of cross-community edges – the change in the normalized effective resistance distance between \( G_n \) and \( G_{n+1} \) remains negligible (after removing the linear term \( H(n, k_n) \)).

**Theorem 5.** Let \( G_{n+1} \sim Q(n + 1, p_n, q_n) \) be a stochastic blockmodel with \( p_n = o(\log n/n) \), \( q_n = o(1/n^2) \), and \( q_n = o(p_n/n) \). Let \( G_n \) be the subgraph induced by the vertex set \([n]\), and let \( D_n = RD \left( G_n, G_{n+1} \right) \) be the normalized effective resistance distance, RD, defined in (3.3).

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Suppose that the introduction of \( n + 1 \) does not create additional cross-community edges, that is \( k_n = k_{n+1} \), then

\[
D_n - h(n, k_n) = \Theta \left( \frac{n^2}{d_n^2} \right) \geq 0,
\tag{C.90}
\]

where

\[
h(n, k_n) = \left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n}{2} \right\rceil \cdot \frac{k_n}{1 + k_n}.
\tag{C.91}
\]

**Proof.** Since vertex \( n + 1 \) is isolated in \( G_n \), the change in resistance at vertex \( n + 1 \) between \( G_n \) and \( G_{n+1} \) will behave quite differently than as described in Theorem 3. For this reason, we separate the renormalized resistance distance into two portions: the pairs of nodes that do and do not contain vertex \( n + 1 \),

\[
D_n = \sum_{u < v \leq n+1} \left| R_{u,v}^{(n+1)} - R_{u,v}^{(n)} \right| = \sum_{u < v \leq n} \left| R_{u,v}^{(n+1)} - R_{u,v}^{(n)} \right| + \sum_{u \leq n} \left| R_{u,n+1}^{(n+1)} - R_{u,n+1}^{(n)} \right|.
\tag{C.92}
\]

Let us first study the second sum. Because vertex \( n + 1 \) is isolated at time \( n \), \( R_{u,n+1}^{(n)} = 1 \). If \( u \) and \( n + 1 \) are in the same community, then \( R_{u,n+1}^{(n+1)} \leq \hat{R}_{u,v}^{(n)} = \Theta \left( 1/d_n^2 \right) \), and is therefore negligible. We can use the trivial bound \( R_{u,n+1}^{(n+1)} \geq 0 \) that yields

\[
\left| R_{u,n+1}^{(n+1)} - R_{u,n+1}^{(n)} \right| \leq 1.
\]

If \( u \) and \( n + 1 \) are in different communities, then a tighter bound can be derived by considering the bottleneck formed by the cross-community edges. Indeed, a coarse application of Nash-Williams – using only the cross-community cut-set – tells us that the effective resistance between vertices in different communities is greater than \( 1/k_n \), and thus the renormalized effective resistance has the following lower bound,

\[
R_{u,n+1}^{(n+1)} \geq \frac{1}{1 + k_n},
\]

which implies

\[
\left| R_{u,n+1}^{(n+1)} - R_{u,n+1}^{(n)} \right| \leq \frac{k_n}{1 + k_n}.
\]

Observing that there are \( \left\lfloor \frac{n}{2} \right\rfloor \) possible in-community connections and \( \left\lceil \frac{n}{2} \right\rceil \) possible cross-community connections, we have

\[
\sum_{u \leq n} \left| R_{u,n+1}^{(n+1)} - R_{u,n+1}^{(n)} \right| \leq \left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n}{2} \right\rceil \cdot \frac{k_n}{1 + k_n}.
\tag{C.93}
\]

We now consider the first sum in (C.92). Corollary 10 combined with lemma 5 yield

\[
\left| R_{u,v}^{(n+1)} - R_{u,v}^{(n)} \right| \leq \left| \hat{R}_{u,v}^{(n+1)} - \hat{R}_{u,v}^{(n)} \right| = \Theta \left( \frac{1}{d_n^2} \right),
\tag{C.94}
\]

which leads to the following bound on the first sum,

\[
\sum_{u < v \leq n} \left| R_{u,v}^{(n+1)} - R_{u,v}^{(n)} \right| = \Theta \left( \frac{n^2}{d_n^2} \right).
\tag{C.95}
\]

Combining (C.93) and (C.95) yields

\[
0 \leq D_n = \sum_{u < v \leq n+1} \left| R_{u,v}^{(n+1)} - R_{u,v}^{(n)} \right| \leq \left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n}{2} \right\rceil \cdot \frac{k_n}{1 + k_n} + \Theta \left( \frac{n^2}{d_n^2} \right) = h(n, k_n) + \Theta \left( \frac{n^2}{d_n^2} \right),
\tag{C.96}
\]

which implies the advertised result. \( \square \)
The following corollary, which is an immediate consequence of the previous theorem provides the appropriate renormalization of $D_n - h(n, k_n)$ under the null hypothesis $H_0$.

**Corollary 3.** Let $G_{n+1} \sim G(n+1, p_n, q_n)$ be a stochastic blockmodel with the same conditions on $p_n$ and $q_n$ as in Theorem 5, and let $G_n$ be the subgraph induced by the vertex set $[n]$.

Suppose that the introduction of $n + 1$ does not create additional cross-community edges, that is $k_n = k_{n+1}$, then

$$0 \leq p_n^2 (D_n - h(n, k_n)) = \Theta(1) \quad (C.97)$$

**Proof.** As explained in lemma 4, we assume without loss of generality that $n$ is even. We have then

$$\frac{n^2}{d_n^2} = \frac{(n/2 - 1)^2}{p_n^2} \quad (C.98)$$

and thus

$$\frac{n^2}{d_n^2} = \frac{n^2}{p_n^2(n/2 - 1)^2} = \frac{4}{p_n^2} \left( 1 + \Theta\left( \frac{1}{n} \right) \right), \quad (C.99)$$

which leads to

$$\frac{n^2 p_n^2}{d_n^2} = 4 \left( 1 + \Theta\left( \frac{1}{n} \right) \right). \quad (C.100)$$

We recall that theorem 5 gives us the following bound on $(D_n - h(n, k_n))$ under the null hypothesis,

$$(D_n - h(n, k_n)) = \Theta\left( \frac{n^2}{d_n^2} \right), \quad (C.101)$$

we conclude that

$$p_n^2 (D_n - h(n, k_n)) = \Theta(1). \quad (C.102)$$

\[ \square \]

### C.4 The Distance $D_n$ Under the Alternate Hypothesis

We now consider the case where the addition of node $n + 1$ leads to an increase in the number of cross-community edges. Loosening the bottleneck between the two communities creates a significant change in the normalized effective resistance distance between $G_n$ and $G_{n+1}$.

**Theorem 6.** Let $G_{n+1} \sim G(n+1, p_n, q_n)$ be a stochastic blockmodel with $p_n = \omega (\log n / n)$, $q_n = \omega (1/n^2)$, and $q_n = o(p_n/n)$. Let $G_n$ be the subgraph induced by the vertex set $[n]$, and let $D_n = RD(G_n, G_{n+1})$ be the normalized effective resistance distance, RD, defined in (3.3).

Suppose that the introduction of $n + 1$ creates additional cross-community edges, that is $k_{n+1} > k_n$, then

$$0 \leq \frac{1}{16} \left\{ \frac{n^2}{k_n^2} + \frac{n^2}{d_n} \Theta \left( \frac{1}{k_n} \right) \right\} \leq D_n - h(n, k_n), \quad (C.103)$$

where $h(n, k_n)$ is defined in (C.91).
Proof. As before, we split the distance $D_n$ into two terms,
\[
D_n = \sum_{u < v \leq n+1} |R_{u,v}^{(n+1)} - R_{u,v}^{(n)}| = \sum_{u < v \leq n} |R_{u,v}^{(n+1)} - R_{u,v}^{(n)}| + \sum_{1 \leq i \leq n} |R_{u,v}^{(n+1)} - R_{u,v}^{(n)}|.
\] (C.104)

Again, we analyze the second sum, which will generate the same linear contribution,
\[
\left| R_{u+1}^{(n+1)} - R_{u+1}^{(n)} \right| = 1 - \frac{1}{1 + \tilde{R}_{u+1}^{(n+1)}}.
\]

Because we seek an upper bound on $R_{u+1}^{(n+1)}$ to obtain a lower bound on the change $\left| R_{u+1}^{(n+1)} - R_{u+1}^{(n)} \right|$ we have a more refined analysis of $R_{u+1}^{(n+1)}$.

In the case where $u$ and $n + 1$ are in the same community, Theorem 4 tells us that
\[
\tilde{R}_{u+1}^{(n+1)} = \frac{2}{d_{u+1}} + O\left(\frac{1}{d_n}\right).
\]

We use the inequality $\frac{1}{1+x} \geq 1 - x$, which is valid for all $x > -1$, to get a lower bound,
\[
\left| R_{u+1}^{(n+1)} - R_{u+1}^{(n)} \right| \geq 1 - \frac{2}{1 + \tilde{R}_{u+1}^{(n+1)}} + O\left(\frac{1}{d_n}\right) = 1 + O\left(\frac{1}{d_n}\right).
\] (C.105)

We also use the inequality $\frac{1}{1+x} \leq 1 - x/2$, which is valid for all $x \in [0, 1]$, to get an upper bound,
\[
\left| R_{u+1}^{(n+1)} - R_{u+1}^{(n)} \right| \leq 1 - \frac{1}{1 + \tilde{R}_{u+1}^{(n+1)}} + O\left(\frac{1}{d_n}\right) = 1 + O\left(\frac{1}{d_n}\right).
\] (C.106)

If $u$ and $n + 1$ are in separate communities, Lemma 8 tells us that
\[
\frac{1}{k_{n+1}} \leq \tilde{R}_{u+1}^{(n+1)} \leq \frac{1}{k_{n+1}} + \frac{4}{d_{n+1}} + O\left(\frac{1}{d_n}\right) \leq \frac{1}{1 + k_n} + \frac{4}{d_{n+1}} + O\left(\frac{1}{d_n}\right).
\] (C.107)

Using again the inequality $\frac{1}{1+x} \geq 1 - x$, we get a lower bound,
\[
\left| R_{u+1}^{(n+1)} - R_{u+1}^{(n)} \right| \geq 1 - \frac{1}{1 + k_n} - \frac{4}{d_{n+1}} + O\left(\frac{1}{d_n}\right) = \frac{k_n}{1 + k_n} + O\left(\frac{1}{d_n}\right),
\] (C.108)

and using the inequality $\frac{1}{1+x} \leq 1 - x/2$ we get an upper bound,
\[
\left| R_{u+1}^{(n+1)} - R_{u+1}^{(n)} \right| \leq 1 - \frac{1}{2k_{n+1}} \leq 1.
\] (C.109)

Combining (C.105), (C.106),(C.108), and (C.109), we get
\[
\left| R_{u+1}^{(n+1)} - R_{u+1}^{(n)} \right| \leq n + O\left(\frac{n}{d_n}\right).
\] (C.110)

We now consider the first sum in (C.104). To get lower and upper bounds on $\tilde{R}_{u+1}^{(n+1)} - \tilde{R}_{u+1}^{(n)}$ we use lemma 5.

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We first observe that for \( n \) sufficiently large, we have \( \hat{R}_{u,v}^{(n)} \leq 1 \), and thus \( \hat{R}_{u,v}^{(n+1)} \leq 1 \). Combining this upper bound on the effective resistance with lemma 5 we get

\[
\text{if } C(u, v) \leq \left| \hat{R}_{u,v}^{(n+1)} - \hat{R}_{u,v}^{(n)} \right| \text{ then } \frac{C(u, v)}{4} \leq \frac{C(u, v)}{(1 + \hat{R}_{u,v}^{(n)})(1 + \hat{R}_{u,v}^{(n+1)})} \leq \left| \hat{R}_{u,v}^{(n+1)} - \hat{R}_{u,v}^{(n)} \right|. \tag{C.111}
\]

From corollary 10 we have

\[
\begin{cases}
C(u, v) = \Theta \left( \frac{1}{d_n^2} \right) & \text{if } u \text{ and } v \text{ are in the same community,} \\
C(u, v) \geq \frac{1}{k_n^2} + \frac{1}{d_n} \Theta \left( \frac{1}{k_n} \right) & \text{otherwise,}
\end{cases}
\tag{C.112}
\]

and therefore

\[
4 \sum_{u < v \leq n} \left| R_{u,v}^{(n+1)} - R_{u,v}^{(n)} \right| \geq \sum_{u \neq v \text{ in different communities}} C(u, v) + \sum_{u \neq v \text{ in same community}} C(u, v) \geq \frac{n^2}{4k_n^2} + \frac{n^2}{4d_n} \Theta \left( \frac{1}{k_n} \right) + \frac{n^2}{4} \Theta \left( \frac{1}{d_n^2} \right),
\tag{C.113}
\]

where the differences between \( n/2 \) and the exact size of \( C_1 \) or \( C_2 \) are absorbed in the error terms. Also, we have

\[
\frac{1}{d_n^2} = \frac{n}{d_n} \Theta \left( \frac{1}{k_n} \right),
\tag{C.114}
\]

and thus

\[
\sum_{u < v \leq n} \left| R_{u,v}^{(n+1)} - R_{u,v}^{(n)} \right| \geq \frac{1}{16} \left( \frac{n^2}{k_n^2} + \frac{n^2}{d_n} \Theta \left( \frac{1}{k_n} \right) \right).
\tag{C.115}
\]

Finally, we note that

\[
\frac{n}{d_n} = \frac{n^2}{d_n k_n} - \frac{n^2}{d_n k_n} = \frac{n^2}{d_n k_n} - \frac{n^2}{k_n}\frac{k_n}{n}
\tag{C.116}
\]

Because of lemma 7, we have asymptotically with high probability,

\[
\frac{n}{d_n} = \frac{n^2}{d_n k_n} \Theta \left( \frac{E [k_n]}{n} \right) = \frac{n^2}{d_n k_n} \Theta \left( \frac{nq}{n} \right) = \frac{n^2}{d_n} \Theta \left( \frac{1}{k_n} \right),
\tag{C.117}
\]

and thus we conclude that

\[
\Theta \left( \frac{n}{d_n} \right) = \frac{n^2}{d_n} \Theta \left( \frac{1}{k_n} \right).
\tag{C.118}
\]

Lastly, we add the two sums (C.110) and (C.115) to get

\[
\left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor \frac{k_n}{1 + k_n} + \frac{1}{16} \left( \frac{n^2}{k_n^2} + \frac{n^2}{d_n} \Theta \left( \frac{1}{k_n} \right) \right) \leq \sum_{u < v \leq n+1} \left| R_{u,v}^{(n+1)} - R_{u,v}^{(n)} \right|. \tag{C.119}
\]

The leading term linear term, \( h(n, k_n) \), in (C.119) can be subtracted to arrive at the advertised result. \( \square \)

Using the same normalization described in corollary 3 we obtain a very different growth for \( p_n^2(D_n - h(n, k_n)) \) in the case of the alternate hypothesis.
Corollary 4. Let $G_{n+1} \sim \mathcal{G}(n+1,p_n,q_n)$ be a stochastic blockmodel with the same conditions on $p_n$ and $q_n$ as in Theorem 5, and let $G_n$ be the subgraph induced by the vertex set [n].

Suppose that the introduction of $n+1$ creates additional cross-community edges, that is $k_{n+1} > k_n$, then

$$0 \leq p_n \left(D_n - h(n,k_n)\right) \rightarrow \infty \text{ with high probability.} \quad (C.120)$$

Proof. As explained in lemma 4, we assume without loss of generality that $n$ is even. From (C.103) we have

$$p_n \left(D_n - h(n,k_n)\right) \geq \frac{1}{16} \left\{ \left(\frac{np_n}{k_n^2}\right)^2 + \left(\frac{np_n}{d_n}\right) \Theta\left(\frac{1}{k_n}\right) \right\} \quad (C.121)$$

Without loss of generality we assume $n$ even, and we have

$$\left(\frac{np_n}{d_n}\right) \Theta\left(\frac{1}{k_n}\right) = \frac{p_n}{nq_n} \Theta(1) = \omega(1) \Theta(1). \quad (C.122)$$

Therefore the second term in (C.121) is either bounded, or goes to infinity. We will prove that the first term goes to infinity. We have

$$\frac{np_n}{k_n} = \frac{np_n}{q_n(n/2)^2} \frac{\mathbb{E}[k_n]}{k_n} = \frac{np_n}{q_n(n/2)^2} \mathbb{E}[k_n] = \frac{4p_n}{nq_n} \mathbb{E}[k_n]. \quad (C.123)$$

From lemma 7 we know that asymptotically $\mathbb{E}[k_n]/k_n = \Theta(1)$ with high probability. Also, we have $p_n/(nq_n) = \omega(1)$. This concludes the proof. \hfill \square

The quantity $p_n^2 \left(D_n - h(n,k_n)\right)$ could provide a statistic to test the null hypothesis $k_n = k_{n+1}$ against the alternate hypothesis $k_n < k_{n+1}$. Unfortunately, computing $p_n^2 \left(D_n - h(n,k_n)\right)$ requires the knowledge of the unknown parameter $p_n$, and unknown variable $h(n,k_n)$. We therefore propose two estimates that converge to these unknowns. A simple estimate of $h(n,k_n)$ is provided by $n$. Since we assume that there are much fewer cross-community edges than edges within each community, we can estimate $p_n$ from the total number of edges.

We start with two technical lemmas. The first lemma shows that can replace $h(n,k_n)$ with $n$.

Lemma 11. Let $G_n \sim \mathcal{G}(n,p_n,q_n)$ be a stochastic blockmodel with the same conditions on $p_n$ and $q_n$ as in Theorem 5. If $p_n = \Theta(1/\sqrt{n})$, then we have

$$\lim_{n \rightarrow \infty} p_n^2 \left(n - h(n,k_n)\right) = 0 \text{ with high probability.} \quad (C.124)$$

Proof. We have

$$n - h(n,k_n) = \left\lfloor \frac{n}{2} \right\rfloor \frac{1}{k_n + 1}. \quad (C.125)$$

Because $\left\lfloor n/2 \right\rfloor = (n/2)\Theta(1)$, we have

$$p_n^2 \left(n - h(n,k_n)\right) = \Theta(1) \frac{np_n^2}{2(k_n + 1)} = \frac{np_n^2}{2\mathbb{E}[k_n]} \frac{\mathbb{E}[k_n]}{k_n} \frac{k_n}{k_n + 1} \Theta(1). \quad (C.126)$$

Now, we have $k_n/(k_n + 1) < 1$, $np_n^2 = \Theta(1)$, and $\mathbb{E}[k_n] = \omega(1)$, therefore

$$\lim_{n \rightarrow \infty} \frac{np_n^2}{2\mathbb{E}[k_n]} \frac{k_n}{k_n + 1} \Theta(1) = 0. \quad (C.127)$$

Finally, we recall that $\mathbb{E}[k_n]/k_n = \Theta(1)$ with high probability, which concludes the proof. \hfill \square
We now consider the estimation of $p_n$.

**Lemma 12.** Let $G_n \sim \mathcal{G}(n, p_n, q_n)$ be a stochastic blockmodel with the same conditions on $p_n$ and $q_n$ as in Theorem 5. Let $m_n$ be the total number of edges in $G_n$. Then the probability $p_n$ can be estimated asymptotically from $m_n$ and $n$,

$$\frac{4m_n}{n^2} = p_n \left(1 + O\left(\frac{1}{n}\right)\right), \quad \text{with high probability.} \quad (C.128)$$

**Proof.** The proof proceeds in two steps. We first show that $k_n$ concentrates around its expectation $\mathbb{E}[k_n]$, and then we argue that $\lim_{n \to \infty} 4\mathbb{E}[m_n]/n^2 = p_n$.

The total number of edges, $m_n$, in the graph $G_n$, can be decomposed as

$$m_n = m_{n_1} + m_{n_2} + k_n, \quad (C.129)$$

where $m_{n_1}$ ($m_{n_2}$) is the number of edges in community $C_1$ ($C_2$). The three random variables are binomial (with different parameters), and they concentrate around their respective expectations. Consequently $m_n$ also concentrates around its expectation, and we can combine three Chernoff inequalities using a union bound to show that

$$\frac{m_n}{\mathbb{E}[m_n]} = \Theta(1), \quad \text{with high probability.} \quad (C.130)$$

A quick computation of $\mathbb{E}[m_n]$ shows that

$$\mathbb{E}[m_n] = p_n \frac{n^2}{4} \left(1 - \frac{2}{n} + \frac{q_n}{p_n} + o\left(\frac{1}{n^2}\right)\right). \quad (C.131)$$

Also, $q_n/p_n = o\left(1/n\right)$, and thus

$$\mathbb{E}[m_n] = p_n \frac{n^2}{4} \left(1 + O\left(\frac{1}{n}\right)\right). \quad (C.132)$$

To conclude, we combine (C.130) and (C.132), to get

$$\frac{4m_n}{n^2} = \frac{m_n}{\mathbb{E}[m_n]} \frac{4\mathbb{E}[m_n]}{n^2} = p_n \left(1 - O\left(\frac{1}{n}\right)\right), \quad (C.133)$$

which concludes the proof. \hfill \Box

We define the following statistic that asymptotically converges toward $p_n^2(D_n - h(n,k_n))$ with high probability, as explained in the next theorem.

**Definition 9.** Let $G_{n+1} \sim \mathcal{G}(n, p_n, q_n)$ be a stochastic blockmodel with the same conditions on $p_n$ and $q_n$ as in Theorem 5. Let $G_n$ be the subgraph induced by the vertex set $[n]$. Let $D_n = \text{RD}(G_n, G_{n+1})$ be the normalized effective resistance distance, RD, defined in (3.3).

We define the statistic

$$Z_n \overset{\text{def}}{=} \frac{16m_n^2}{n^4} (D_n - n). \quad (C.134)$$

**Theorem 7.** Let $G_n \sim \mathcal{G}(n, p_n, q_n)$ be a stochastic blockmodel with the same conditions on $p_n$ and $q_n$ as in Theorem 5. If $p_n = O\left(1/\sqrt{n}\right)$, then we have

$$Z_n = p_n^2 (D_n - h(n,k_n)) \left(1 + o(1)\right), \quad \text{with high probability.} \quad (C.135)$$
Proof. The proof is an elementary consequence of the two lemmas 11 and 12. We have
\[
\frac{16m_n^2}{n^2} (D_n - n) = \frac{16m_n^2}{n^4} (D_n - h(n, k_n)) + \frac{16m_n^2}{n^4} (h(n, k_n) - n). \tag{C.136}
\]
Using lemma 12, we have
\[
\frac{16m_n^2}{n^2} (D_n - n) = p_n^2 (1 + O(1/n))^2 (D_n - h(n, k_n)) + (1 + O(1/n))^2 p_n (h(n, k_n) - n). \tag{C.137}
\]
Lemma 11 shows that the second term can be neglected,
\[
\frac{16m_n^2}{n^2} (D_n - n) = p_n^2 (D_n - h(n, k_n)) (1 + O(1/n)) + o(1). \tag{C.138}
\]
Because \( p_n^2 (D_n - h(n, k_n)) \) is either bounded, or goes to infinity, we have
\[
\frac{16m_n^2 / n^2 (D_n - n)}{p_n^2 (D_n - h(n, k_n))} = 1 + o(1), \tag{C.139}
\]
which concludes the proof. \( \square \)

We finally arrive at the main theorem.

**Theorem 8.** Let \( G_{n+1} \sim G(n, p_n, q_n) \) be a stochastic blockmodel with the same conditions on \( p_n \) and \( q_n \) as in Theorem 5. Let \( G_n \) be the subgraph induced by the vertex set \([n]\).

To test the hypothesis
\[
H_0 : \quad k_n = k_{n+1} \tag{C.140}
\]
versus
\[
H_1 : \quad k_n < k_{n+1} \tag{C.141}
\]
we use the test based on the statistic \( Z_n \) defined in (C.134) where we accept \( H_0 \) if \( Z_n < z_\varepsilon \) and accept \( H_1 \) otherwise. The threshold \( z_\varepsilon \) for the rejection region satisfies
\[
\text{Prob}_{H_0} (Z_n \geq z_\varepsilon) \leq \varepsilon \quad \text{as} \quad n \to \infty, \tag{C.142}
\]
and
\[
\text{Prob}_{H_1} (Z_n \geq z_\varepsilon) \to 1 \quad \text{as} \quad n \to \infty. \tag{C.143}
\]
The test has therefore asymptotic level \( \varepsilon \) and asymptotic power 1.

**Proof.** Assume \( H_0 \) to be true. Because of corollary 3 and Theorem 7,
\[
Z_n = O(1), \quad \text{with high probability.} \tag{C.144}
\]
In other words, for every \( 0 < \varepsilon < 1 \) there exists \( z_\varepsilon \) such that
\[
\text{Prob}(Z_n < z_\varepsilon) = 1 - \varepsilon, \quad \text{as} \quad n \to \infty, \tag{C.145}
\]
or
\[
\text{Prob}(Z_n \geq z_\varepsilon) = \varepsilon, \quad \text{as} \quad n \to \infty. \tag{C.146}
\]
Assume now $H_1$ to be true. Because of corollary 4 and Theorem 7,

$$Z_n = \omega(1), \quad \text{with high probability} \quad (C.147)$$

Therefore, for every $0 < \gamma < 1$, there exists $n_0$ such that

$$\forall n \geq n_0, \text{Prob}(Z_n > z_\epsilon) = 1 - \gamma, \quad \text{as} \quad n \to \infty. \quad (C.148)$$

In other words,

$$\text{Prob}_{H_1}(Z_n \geq z_\epsilon) \to 1 \quad \text{as} \quad n \to \infty, \quad (C.149)$$

which concludes the proof. □

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