Parameterized Coloring Problems on Threshold Graphs

I. Vinod Reddy
Indian Institute of Technology Bhilai, India

Abstract. In this paper, we study several coloring problems on graphs from the viewpoint of parameterized complexity. We show that Pre-coloring Extension and Equitable Coloring problems are fixed-parameter tractable (FPT) parameterized by the distance to threshold graphs. We also study the List $k$-Coloring and show that the problem is NP-complete on split graphs and it is FPT parameterized by solution size on split graphs.

Keywords: Parameterized complexity · Precoloring extension · Equitable coloring · List coloring.

1 Introduction

Given a graph $G$, and a positive integer $k$, the $k$-coloring problem is to color the vertices of $G$ with at most $k$ colors such that adjacent vertices receive different colors. This is a well studied problem in computer science due to its theoretical and practical applications. The problem is NP-complete for every fixed $k \geq 3$. The problem is well studied from the perspective of parameterized complexity. For example, it is FPT when parameterized by vertex cover [4], tree-width [8] and distance to clique [24]. On the other hand, it is W[1]-hard when parameterized by clique-width [10] and distance to split graphs [6].

In this paper we study the parameterized complexity of several graph coloring problems (Precoloring Extension, Equitable Coloring, List $k$-Coloring), with respect to distance parameters [5,18], where we take the parameter to be the distance to threshold graphs. It is an intermediate parameter between vertex cover and clique-width. Many variants of graph coloring are fixed-parameter tractable when parameterized by the vertex cover. However, the parameter vertex cover is very restrictive in the sense that the class of graphs with bounded vertex cover is small. The parameter distance to threshold graphs generalizes vertex cover in the sense that the class of graphs with bounded vertex cover contains in the class of graphs with bounded distance to threshold graphs. Thus the existence of FPT algorithms for any problem parameterized by distance to threshold graphs supplants the results obtained by parameterizing with vertex cover. Hence this reduces the gap between tractability and intractability.
**Problems considered.** The problems we consider in this paper are as follows.

- **Precoloring Extension**
  - **Input:** A graph $G$, an integer $r$, a subset $W \subseteq V(G)$ and a precoloring $C_W : W \rightarrow [r]$
  - **Question:** Is there a proper coloring $c : V(G) \rightarrow [r]$ of $G$ such that $c(u) = C_W(u)$ for every $u \in W$?

- **Equitable Coloring**
  - **Input:** A graph $G$ and an integer $r$
  - **Question:** Is there a proper coloring of $G$ using at most $r$ colors such that the sizes of any two color classes differ by at most one?

- **List $k$-Coloring**
  - **Input:** A graph $G$ and an assignment $L : V(G) \rightarrow S \subseteq [k]$ of color lists to the vertices of $G$.
  - **Question:** Is there a proper coloring $c : V(G) \rightarrow [k]$ such that $c(u) \in L(u)$ for every $u \in V(G)$?

**Related work.** Both Precoloring Extension and Equitable Coloring are FPT when parameterized by the vertex cover number \cite{12} and \W{1}-hard when parameterized by tree-width \cite{14}. However they can be solved in polynomial time on graphs of bounded tree-width \cite{22}. Ganian \cite{17} showed that both problems are FPT parameterized twincover. Doucha et al. \cite{12} showed that Precoloring Extension is FPT parameterized by bounded cluster vertex deletion and \W{1}-hard parameterized by unbounded cluster vertex deletion. They also showed that Equitable Coloring is FPT parameterized by unbounded cluster vertex deletion. D. Marx \cite{22} showed that Precoloring Extension is \W{1}-hard parameterized by either distance to interval graphs or distance to chordal graphs. List $k$-Coloring is \W{1}-hard when parameterized by the vertex cover number. The problem remains hard even for split graphs. Banik et al. \cite{1} showed that $(n - k)$-regular list coloring is FPT parameterized by $k$. Jansen et al. \cite{19} showed that $q$-regular list coloring is FPT parameterized by combined parameter $q + k$.

**Our contributions.** We summarize our results below.

- In Section 3, we show that Precoloring Extension is FPT parameterized by distance to threshold graphs.
- In Section 4, we show that Equitable Coloring is FPT parameterized by distance threshold graphs.
- In Section 5, we show that List $k$-Coloring is (a) NP-complete on split graphs and (b) FPT parameterized by $k$ on split graphs.
2 Preliminaries

In this section, we introduce some basic notation and terminology related to graph theory and parameterized complexity. For \( n \in \mathbb{N} \), we use \([n]\) to denote the set \( \{1, 2, \ldots, n\} \).

**Graph theory.** All graphs we consider in this paper are undirected, connected, finite, and simple. For a graph \( G = (V,E) \), by \( V(G) \) and \( E(G) \) we denote the vertex set and edge set of \( G \) respectively. We use \( n \) to denote the number of vertices and \( m \) to denote the number of edges of a graph. An edge between two vertices \( x \) and \( y \) is denoted as \( xy \) for simplicity. For a subset \( X \subseteq V(G) \), the graph \( G[X] \) denotes the subgraph of \( G \) induced by vertices of \( X \). Also, for simplicity, we use \( G \setminus X \) to refer to the graph obtained from \( G \) after removing the vertex set \( X \).

For a vertex \( v \in V(G) \), by \( N(v) \) we denote the set \( \{u \in V(G) | vu \in E(G)\} \) and we use \( N[v] \) to denote the set \( N(v) \cup \{v\} \). The neighborhood of a vertex subset \( S \subseteq V(G) \) is \( N(S) = (\bigcup_{v \in S} N(v)) \setminus S \). A vertex is called *universal vertex* if it is adjacent to every other vertex of the graph. A graph \( G \) has deletion distance \( d \) to a graph class \( \mathcal{F} \) if there exists a set \( X \subseteq V(G) \) of \( d \) vertices such that \( G \setminus X \in \mathcal{F} \). We say that \( X \) is an \( \mathcal{F} \)-modulator of graph \( G \).

A graph is a *split graph* if its vertices can be partitioned into a clique and an independent set. A graph is a *threshold graph* if it can be constructed from the one-vertex graph by repeatedly adding either an isolated vertex or a universal vertex. The class of threshold graphs is the intersection of split graphs and cographs [21]. We denote a threshold graph as \( G = (C,I) \), where \( (C,I) \) denotes the partition of \( G \) into a clique and an independent set, respectively. It is easy to see that every induced subgraph of a threshold graph is also a threshold graph. We have the following characterization of threshold graphs: A graph \( G \) is a threshold graph if and only if it is \((P_4, C_4, 2K_2)\)-free. For any two vertices \( x, y \) in a threshold graph \( G \) we have either \( N(x) \subseteq N[y] \) or \( N(y) \subseteq N[x] \) (neighborhood containment property). For more details on standard graph-theoretic notation and terminology, we refer the reader to [11].

As threshold graphs are \((P_4, C_4, 2K_2)\)-free, checking whether a given graph \( G \) has vertex deletion distance \( d \) to the class of threshold graphs is fixed-parameter tractable. Therefore without loss of generality, in this paper, we assume that threshold graph modulator is given as a part of the input.

**Parameterized complexity.** A parameterized problem denoted as \((I,k) \subseteq \Sigma^* \times \mathbb{N} \), where \( \Sigma \) is fixed alphabet and \( k \) is called the parameter. We say that the problem \((I,k) \) is *fixed parameter tractable* with respect to parameter \( k \) if there exists an algorithm which solves the problem in time \( f(k)|I|^{O(1)} \), where \( f \) is a computable function. A *kernel* for a parameterized problem \( \Pi \) is an algorithm which transforms an instance \((I,k) \) of \( \Pi \) to an equivalent instance \((I',k') \) in polynomial time such that \( k' \leq k \) and \( |I'| \leq f(k) \) for some computable function \( f \). For more details on parameterized complexity, we refer the reader to the texts [9,13].
Fig. 1: An example of a threshold graph $G = (C, I)$, where $C = \{v_1, v_2, v_3, v_4\}$ and $I = \{u_1, u_2, u_3\}$. The vertex $v_1$ is universal in $G$. We can also see that $N[v_4] \subseteq N[v_3] \subseteq N[v_2] \subseteq N[v_1]$ and $N(u_1) \subseteq N(u_2) \subseteq N(u_3)$.

3 Precoloring Extension

In this section, we present an FPT algorithm for Pre-coloring Extension when parameterized by the distance to threshold graphs. First, we show that the problem is FPT when parameterized by the distance to clique and then by using this result, we show that the problem is FPT when parameterized by the distance to threshold graphs.

3.1 Parametrization by distance to clique

**Theorem 1.** Pre-coloring Extension can be solved in $O(2^{k(3\log k + 3)}r(n + \sqrt{nm}))$ time when parameterized by the distance to clique.

**Proof.** Let $X \subseteq V(G)$ of size $k$ such that $G \setminus X = C$ is a clique. Let $f : W \subseteq V(G) \rightarrow [t]$ be the given precoloring, where $t \leq r$ and $f(W) = [t]$. Let $V_C = W \cap C$ and $V_X = W \cap X$ be the set of precolored vertices in the clique $C$ and the modulator $X$ respectively. Let $S_C = f(V_C)$ and $S_X = f(V_X)$. As $C$ is a clique, no color of $S_C$ can be used to color any uncolored vertices of the clique.

First, we check whether it is possible to use some colors of $S_C \cup S_X$ to color vertices of $X \setminus V_X$. For each vertex in $X \setminus V_X$ we assign a list of colors according to their neighborhood in $G$, i.e., for each $v \in X \setminus V_X$, $L(v) := [t] \setminus f(N(v) \cap W)$. Let $X_h = \{v \in X \setminus V_X \mid |L(v)| > 2k\}$. Any vertex $v$ in $X_h$ can be colored at the end: $L(v)$ has at least $k$ colors from $S_C$, and there are at most $k$ uncolored vertices in $X$. Therefore $v$ can be colored greedily with one of the color from $L(v)$ at the end.

We are left with a subset $Y \subseteq X \setminus V_X$ of at most $k$ vertices having at most $2k$ colors in their lists. Partition $Y = Y' \cup Y''$ into two sets such that $Y'$ contains vertices which gets the colors from $[t]$ and $Y''$ contains vertices which gets colors from the set $[r \setminus [t]$. Since $|Y'| \leq k$ and $|L(v)| \leq 2k$ for all $v \in Y'$, we try all possible $O(k^{2k})$ ways to color the vertices of $Y'$. Similarly as the size of $Y''$ is at most $k$, we can assign colors to the vertices of $Y''$ from the set $[r \setminus [t]$ in $O(k^k)$ time.
Now we have a partial coloring of $G$ in which all vertices of $X \setminus X_h$ and some vertices of $C$ are colored. The uncolored vertices in the clique are colored by finding a maximum matching in the following bipartite graph. The vertex set of the bipartite graph contains all $r$ colors as one partition and uncolored clique vertices as the other. A color $c$ is adjacent to a vertex $x$, if $x$ is not adjacent to a vertex colored with $c$ in $G$. If there is a matching saturating all uncolored vertices of $C$, then we assign the colors to clique as per the matching. In the end, we greedily assign colors to the vertices of $X_h$ from their lists.

**Running time.** Computing the threshold graph modulator $X$ takes $O(4^k(m+n))$ time. Trying all possible ways of partitioning $Y$ into $Y'$ and $Y''$ takes $O(2^k)$ time and coloring the vertices of $Y$ requires $O(k^k)$ time. Constructing the bipartite graph takes $O(r(n+m))$ time. Computing the maximum matching in a bipartite graph need $O(n + \sqrt{nm})$ time. Therefore running time of the whole algorithm is $O(2^{k(3\log k + 3)r}(n + \sqrt{nm})).

\[\square\]

### 3.2 Parametrization by distance to threshold graphs

**Theorem 2.** Precoloring Extension problem can be solved in $O(2^{p(3\log p + 3)r}(n + \sqrt{nm}))$ time, where $k$ is the distance to threshold graphs and $p = k + 2^k$.

**Proof.** Let $X \subseteq V(G)$ of size $k$ such that $G \setminus X = (C, I)$ is a threshold graph. Let $f : W \subseteq V(G) \to [t]$ be the given precoloring, where $t \leq r$ and $f(W) = [t]$. We partition the vertices of independent set based on their neighborhood in $X$. For a subset $U \subseteq X$ define $T_U^I = \{x \in I \mid N(x) \cap X = U\}$. There are at most $2^k$ possible subsets in the partition of $I$. For each $U \subseteq X$ there exists a vertex $v_U \in T_U^I$ such that $N(w) \subseteq N(v_U)$ for all $w \in T_U^I$, which implies all uncolored vertices in $T_U^I$ can be colored with the color of $v_U$.

We build a new graph $H = (C_H, I_H)$ from $G$ as follows. For each $U \subseteq X$, delete all uncolored vertices from $T_U^I$ except $v_U$. The number of uncolored vertices in the independent set $I_H$ of $H$ is at most $2^k$. For each uncolored vertex $v \in V(H)$ assign a list of colors based on their neighborhood colors i.e., $L(v) := [t] \setminus f(N(v))$. Then delete all precolored vertices from the independent set $I_H$. Observe that $H \setminus (X \cup I_H)$ is a clique with size of the modulator $(X \cup I_H)$ at most $k + 2^k$. So now, we have an instance of Precoloring Extension for a distance to clique, which we solve using Theorem 1. In the end, for each $U \subseteq X$, we color the deleted vertices of $T_U^I$ by the color of $v_U$.

\[\square\]

In the following corollary we show that Graph Coloring is FPT parameterized by the distance to threshold graphs. This is an important result due to the fact that the problem is FPT parameterized by vertex cover and $W[1]$-hard parameterized by distance to split graphs [6] and distance to interval graphs [22]. Since threshold graphs are subclass of both split graphs and interval graphs, our result reduces the gap between tractability and intractability.
Corollary 3. Graph Coloring is FPT parameterized by the distance to threshold graphs.

Proof. As Graph Coloring is a special case of Precoloring Extension, when the number of precolored vertices is zero. Therefore the proof of this corollary immediately follows from Theorem 2.

4 Equitable Coloring

In this section, we first show that Equitable Coloring is solvable in polynomial time for the class of threshold graphs. Next, we describe FPT algorithm for the problem when parameterized by distance to threshold graphs.

4.1 Threshold graphs

Chen et al. [7] showed that Equitable Coloring can be solved in polynomial time on split graphs. However, here we present a simple polynomial-time algorithm for Equitable Coloring on threshold graphs. This will give a useful warmup for the next part, where we describe our FPT algorithm for the problem.

Lemma 4. Equitable Coloring can be solved in polynomial time on threshold graphs.

Proof. Let $G = (C, I)$ be a threshold graph. As $G$ has at least one universal vertex, the size of a color class in any equitable coloring of $G$ is at most two. If $r < |C|$ then the given instance is a No instance as we need at least $|C|$ colors to color the clique. Therefore without loss of generality, we assume that $r \geq |C|$.

For each integer $t$ with $|C| \leq t \leq r$, we test whether $G$ has an equitable coloring with $t$ colors. Given an integer $t$, first color the vertices of the clique $C$ using $|C|$ many colors. Now we order the vertices of the independent set according to their degree from highest degree to lowest degree. Assign the $t - |C|$ unused colors to the first $t - |C|$ independent vertices according to the above ordering (this greedy choice works because of neighborhood containment property of the threshold graphs). So far, we have used each of the $t$ colors exactly once. Since the size of the maximum color class in any equitable coloring is at most two, any color can be used at most once to color the rest of the uncolored independent set vertices. We do this by solving a network flow problem as follows.

We add a source vertex and $t$ vertices representing colors and connect source vertex with $t$ vertices with each having the capacity one. Next, we create a vertex to represent every uncolored independent set vertex of $G$, and add edges from these vertices to the sink with capacity one. Finally, we add edges of capacity one from $t$ color vertices to independent set vertices if the color may be assigned to that vertex. Then compute the maximum flow in the above-constructed graph and check whether it is equal to the number of uncolored vertices in $I$ and if this is the case, we immediately obtain a solution in $G$. Since the maximum flow is bounded by the number of vertices, this flow problem can be solved in time $O(mn)$. Altogether we get a polynomial-time algorithm for the problem.

\[\square\]
4.2 Parameterized by distance to threshold graphs

The FPT algorithm is very involved. We give a brief overview of the main ideas in our algorithm. Let $G$ be a graph and $X \subseteq V(G)$ of size at most $k$ such that $G \setminus X = (C, I)$ is a threshold graph. We start by guessing the coloring of $X$ in a solution and then try to extend it to an equitable coloring of $G$. To extend a coloring of $X$ to $G$ we use the following key ideas (a) In any equitable coloring of $G$ the size of any color class is at most $k + 2$. (b) As the size of $X$ is at most $k$, we can guess the color class sizes of colors used to color vertices of $X$. (c) We use the neighborhood containment property of threshold graphs to assign new colors (colors not used in $X$) to color the clique and the independent set of $G \setminus X$ respectively.

**Theorem 5.** Equitable Coloring is fixed-parameter tractable when parameterized by distance to threshold graphs.

**Proof.** Let $X \subseteq V(G)$ of size $k$ such that $G \setminus X = (C, I)$ is a threshold graph. As $G \setminus X$ has at least one universal vertex $u$, in any equitable coloring of $G$, the color of $u$ is unique in $G \setminus X$. Therefore the color of $u$ can appear at most $k + 1$ times in $G$. So the maximum size of a color class is at most $k + 2$ in an equitable coloring of $G$. An $r$-equitable colorable graph may not be $(r + 1)$-equitable colorable, therefore for all possible values $t \in [r]$ we check whether $G$ is $t$-equitable colorable. If the number of colors used in an equitable coloring of $G$ is $t$ then we can find the number of color classes of size $\lfloor n/t \rfloor$ and of size $\lfloor n/t \rfloor + 1$.

We run through all $O(k^k)$ possible proper colorings of $X$. For each of these colorings we check whether they can be extended to an equitable coloring of $G$. For a given coloring of $X$, we guess the size of the color class (either $\lfloor n/t \rfloor$ or $\lfloor n/t \rfloor + 1$) for each color used in $X$. There are at most $2^k$ possibilities for each coloring of $X$.

Given a coloring of $X$ with colors from the set $\{1, 2, \cdots, k'\}$ where $k' \leq k$, we guess the subsets $Q_C, Q_I \subseteq [k']$ of colors which can be used to color clique and independent set vertices of $G \setminus X$ respectively in an equitable coloring of $G$ extending the coloring of $X$. We call colors of $Q_C$ as **compulsory** colors of the clique $C$ and colors of $Q_I$ as compulsory colors of the independent set $I$.

Since $C$ is a clique, each color of $Q_C$ appears exactly once in $C$. However, colors of $Q_I$ may appear more than once in the independent set. As the size of $Q_I$ is at most $k$ and the size of any color class is at most $k + 2$, for each color in $Q_I$ we guess the number times it appears in $I$ in an equitable coloring extending the coloring of $X$.

For each vertex $v$ in the graph $G \setminus X$ we assign a list $L(v)$ of colors from the set $\{1, 2, \cdots, k'\}$ based on their neighborhood in $X$ i.e., $c \in L(v)$ if $v$ is not adjacent to a vertex of color $c$ in $X$.

Next for each vertex $v$ in the graph $G \setminus X$ we refine the list $L(v)$ in two stages. In the first stage for each vertex $v \in C$, delete all colors from $L(v)$ except the colors present in $Q_C \cap L(v)$ and similarly for each vertex $v \in I$, delete all colors from $L(v)$ except the colors present in $Q_I \cap L(v)$. 

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A color $c$ is *eligible* for a vertex $v$ in the clique if the number of non-neighbors of $v$ in $G \setminus X$ having color $c$ in their lists is greater than or equal to the number of vertices which still need to be colored by the color $c$. For example if a color $c$ appears three times in $X$ and the color class size of $c$ is $\lfloor n/t \rfloor$ then $c$ is eligible for a vertex $v \in C$ if the number of non-neighbors of $v$ in $G \setminus X$ having color $c$ in their lists should be at least $\lfloor n/t \rfloor - 4$. For each vertex $v \in C$ remove all non-eligible colors from $L(v)$.

We partition the clique vertices based on their refined list colors, i.e., two vertices $u$ and $v$ belongs to the same set of the partition if $L(u) = L(v)$. Since the size of each list is at most $k$, we can partition vertices of clique into at most $2k$ subsets. For each subset of the partition, we guess the colors of $Q_C$ which appear in that subset in an equitable coloring.

We now identify the clique vertices, which can be colored with colors of $Q_C$. In each subset of the partition, we assign the colors of $Q_C$ according to the degree of the vertices inside the threshold graph $G \setminus X$, starting from highest degree to lowest degree. This greedy choice is correct, as threshold graphs satisfy neighborhood containment property.

For the rest of the clique vertices, we assign new colors from the set $[t] \setminus [k']$. Let $t_1$ be the number of new colors used in the clique, where $t_1 \leq t - k'$. So far, we only know about the sizes of color classes for colors used in $X$ and we don’t know the sizes of color classes for the new colors used in the clique. However, using the neighborhood containment property of threshold graphs we can find the color class size for the new colors used in the clique. Let $p$ and $q$ be the number of color classes of sizes $\lfloor n/t \rfloor$ and $\lfloor n/t \rfloor + 1$ respectively after excluding the colors of $X$. We order the vertices of the clique which are colored with new colors according to their neighborhood in $G \setminus X$ from lowest degree to highest degree. Then for the new colors used to color first $q$ vertices in the above ordering we assign their color class size as $\lfloor n/t \rfloor + 1$ and $\lfloor n/t \rfloor$ for the rest of the colors.

Now we are only left to color the independent set vertices. Let $t_2 = t - t_1 - k'$ be the number of colors which are not used in $X$ and $C$. For these $t_2$ colors we can assign the sizes of color classes based on how many of each size ($\lfloor n/t \rfloor + 1$ or $\lfloor n/t \rfloor$) still need be covered. To color the independent set vertices we reduce it to a network flow problem as follows. We create a source vertex that is connected to $t$ vertices representing colors, and the capacity of these edges is equal to the number of vertices that still need to be colored by that color. Then we add one vertex for each uncolored vertex in independent set $I$, and edges from these vertices to the sink with capacity one. In the end, we add edges of capacity one from color vertices to the independent set vertices if the color is present in that vertex list of colors. Then we compute a maximum flow and check whether it is equal to the size of the independent set and if this is the case then we immediately obtain an equitable coloring of $G$.

**Running time.** For a graph $G$, computing a subset $X$ of size at most $k$ such that $G \setminus X$ is a threshold graph takes $O(4^k (m + n))$ time. We run through all $t \leq r$ number of colors for $G$. Then we run through all possible ways of partitioning the vertices of $X$ into color classes of sizes $\lfloor n/t \rfloor$ and $\lfloor n/t \rfloor + 1$, which takes time at
most \(2^{O(k \log k)}\). We guess the colors of \(X\) which are used to color some vertices of \(G \setminus X\) in time \(O(2^k)\). Guessing the compulsory colors in each partition of clique also takes \(O(2^k)\) time. We can also guess the \(t_1\) and \(t_2\) on \(O(r)\) time and \(p\) and \(q\) in \(O(t_1)\) time. Finally, we use network flow to decide whether the uncolored vertices of independent set vertices can be equitably colored with respect to the coloring of \(X\), and sizes of color classes in this coloring. This takes time \(O(mn)\) and the running time of the entire algorithm is \(O(2^{k \log k} r^3 mn)\).

We showed that Equitable Coloring is FPT parameterized by distance to threshold graphs. However, the problem is unlikely to admit a polynomial kernel \[3\]. In the following, we show that when parameterized by \(r\) and \(k\) the problem admits a polynomial kernel.

**Lemma 6.** Equitable Coloring admits a polynomial kernel parameterized by \(r + k\), where \(k\) is the distance to threshold graphs and \(r\) is the number of colors used.

**Proof.** Let \(X \subseteq V(G)\) such that \(G \setminus X = (C, I)\) is a threshold graph. As \(G \setminus X\) has at least one universal vertex, in any Equitable Coloring of \(G\) the maximum size a color class is at most \(k + 2\). Therefore if \(n > r(k + 2)\) it is a No instance. Otherwise \(n \leq r(k + 2)\) and the input graph contains at most \(r(k + 2)\) vertices and at most \(O(r^2k^2)\) edges. \(\square\)

5 List Coloring

The third variant of graph coloring we study in this paper is List \(k\)-Coloring. Mertzios and Spirakis \[23\] showed that List 3-Coloring is NP-complete for graphs of diameter three. The complexity of List 3-Coloring for graphs of diameter two is open. In this section, we show that List \(k\)-Coloring is NP-complete on split graphs, which is a subclass of diameter at most three graphs. Next, we show that the problem is FPT parameterized by \(k\) on split graphs.

**Lemma 7.** List \(k\)-Coloring coloring is NP-complete on split graphs.

**Proof.** We give a reduction from the Independent Set problem. Given an instance \((G, k)\) of the independent set problem, define a split graph \(H = (C, I)\) as follows. For every vertex \(v \in V(G)\), we introduce a vertex \(c_v \in C\). For every edge \(uv \in E(G)\), we introduce a vertex \(I_{uv} \in I\), and connect it with every vertex of \(C\) except \(c_u\) and \(c_v\). We add edges between every pair of vertices in \(C\), thereby making \(H[C]\) a clique. For every vertex \(c_v \in C\) assign \(L(c_v) = [n]\) and for \(I_{uv} \in I\) assign \(L(I_{uv}) = \{k + 1, k + 2, \ldots, n\}\).

We can easily see that \(H\) is a split graph and it can be constructed in polynomial time. We now show that \(G\) contains an independent set of size \(k\) if and only if \(H\) is list \(k\)-colorable.

Suppose that \(G\) contains an independent set \(X = \{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\}\) of size \(k\). Then we can construct a list \(k\)-colouring of \(H\) as follows.
– Color vertex \( c_{v_i j} \) with color \( j \), for \( 1 \leq j \leq k \).
– Arbitrarily color the uncolored vertices of the clique with the colors from the set \( \{k + 1, \ldots, n\} \).
– For each \( I_{uv} \in I \), at least one of \( u \) or \( v \) is not in \( X \). This implies that at least one of the colors used for vertices \( u \) or \( v \) is from the set \( \{k + 1, \ldots, n\} \). We color the vertex \( I_{uv} \) by the color of \( u \) if \( v \in X \) and by the color of \( v \) otherwise.

Conversely suppose that \( \phi \) is a list \( k \)-coloring of \( H \). Let \( X = \{ u \mid \phi(c_u) \in [k] \} \). We show that \( X \) is an independent set of size \( k \) in \( G \). Clearly the size of \( X \) is \( k \). Suppose there exists two vertices \( u, v \in X \) such that \( uv \in E(G) \). We know that \( \phi(I_{uv}) \) is either \( \phi(c_u) \) or \( \phi(c_v) \). This implies \( \phi(I_{uv}) \in [k] \), which is a contradiction to the fact that \( L(I_{uv}) = \{k + 1, \ldots, n\} \). Hence \( X \) is an independent set of size \( k \) in \( G \).

Lemma 8. List \( k \)-Coloring coloring is fixed parameter tractable on split graphs parameterized by \( k \).

Proof. Given a split graph \( G = (C, I) \) and for each vertex \( v \in V(G) \), a list \( L(v) \) of \( k \) permitted colors. If \( |C| > k \) then the given instance is a NO instance as we need at least \( |C| \) colors to color the clique. Therefore without loss of generality we assume that \( |C| \leq k \). First, we run through the all possible (at most \( k^k \)) ways of coloring clique vertices with colors from their lists and check if each such coloring is proper. Then we try to extend each proper coloring of clique \( C \) to the rest of \( G \) as follows. For each \( v \in I \), color it with any color from \( L(v) \) which is not used to color any vertex of \( N(v) \). Altogether this gives an \( O(k^k(m + n)) \) FPT algorithm.

6 Conclusion

In this paper, we study the parameterized complexity of several graph coloring problems. We showed that (a) PRECOLORING EXTENSION and EQUITABLE COLORING are FPT parameterized by distance to threshold graphs and (b) LIST \( k \)-COLORING is FPT parameterized by \( k \) on split graphs.

The following are some interesting open problems.

1. What is the complexity of LIST \( k \)-COLORING for (a) threshold graphs (b) complete-split graphs (c) diameter two graphs?
2. What is the parameterized complexity of NUMBER COLORING [12] (generalization of EQUITABLE COLORING) parameterized by distance to threshold graphs?
3. It is known that GRAPH COLORING admits polynomial kernel parameterized by distance to clique [10]. Does PRECOLORING EXTENSION and EQUITABLE COLORING admit polynomial kernel parameterized by distance to clique?
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