The Hopf algebra structure of the $Z_3$-graded quantum supergroup $\text{GL}_{q,j}(1|1)$

Salih Celik$^1$, Ergün Yasar$^2$

Yildiz Technical University, Department of Mathematics, 34210 Davutpasas-Esenler, Istanbul, Turkey.

In this work, we give some features of the $Z_3$-graded quantum supergroup.

I. INTRODUCTION

Recently, there have been many attempts to generalize $Z_2$-graded constructions to the $Z_3$-graded case$^{1-3}$. The $Z_3$-graded quantum space that generalizes the $Z_2$-graded space called a superspace$^4$, was studied using the methods of Ref. $^5$. The first author studied the noncommutative geometry of the $Z_3$-graded quantum superplane$^6$. In this work, we have investigated the Hopf algebra structure of the $Z_3$-graded quantum supergroup $\text{GL}_{q,j}(1|1)$.

Let us shortly investigate a general $Z_3$-graded algebraic structure. Let $z$ be a $Z_3$-graded variable. Then we say that the variable $z$ satisfies the relation

$$z^3 = 0.$$  

If $f(z)$ is an arbitrary function of the variable $z$, then the function $f(z)$ becomes a polynomial of degree two in $z$, that is,

$$f(z) = a_0 + a_1 z + a_2 z^2,$$

where $a_0$, $a_2$, $a_1$ denote three fixed numbers whose grades are $\text{grad}(a_0) = 0$, $\text{grad}(a_2) = 1$ and $\text{grad}(a_1) = 2$, respectively.

The cyclic group $Z_3$ can be represented in the complex plane by means of the cubic roots of 1: let $j = e^{2\pi i/3}$ ($i^2 = -1$). Then one has

$$j^3 = 1 \quad \text{and} \quad j^2 + j + 1 = 0.$$  

$^1$Electronic mail: sacelik@yildiz.edu.tr  
$^2$Electronic mail: ergyasmat@gmail.com
One can define the $Z_3$-graded commutator $[A, B]$ as

$$[A, B]_{Z_3} = AB - j^{ab}BA,$$

where $\text{grad}(A) = a$ and $\text{grad}(B) = b$. If $A$ and $B$ are $j$-commutative, then we have

$$AB = j^{ab}BA.$$

II. REVIEW OF THE ALGEBRA OF FUNCTIONS ON THE $Z_3$-GRADED QUANTUM SUPERPLANE

The $Z_3$-graded quantum superplane is defined as an associative unital algebra generated by $x$ and $\theta$ satisfying

$$x\theta = q\theta x, \quad \theta^3 = 0 \quad (1)$$

where $q$ is a nonzero complex deformation parameter. Here, the coordinate $x$ with respect to the $Z_3$-grading is of grade 0 and the coordinate $\theta$ with respect to the $Z_3$-grading is of grade 1. This associative algebra over the complex numbers is known as the algebra of polynomials over the quantum superplane and we shall denote it by $\mathcal{R}_q(1|1)$, that is,

$$\mathcal{R}_q(1|1) \ni \begin{pmatrix} x \\ \theta \end{pmatrix} \iff x\theta = q\theta x, \quad \theta^3 = 0.$$  

If we denote the dual of the $\mathcal{R}_q(1|1)$ by $\mathcal{R}^*_{q,j}(1|1)$, one has

$$\mathcal{R}^*_{q,j}(1|1) \ni \begin{pmatrix} \varphi \\ y \end{pmatrix} \iff \varphi y = qjy\varphi, \quad \varphi^3 = 0. \quad (2)$$

Here,

$$[\mathcal{R}_q(1|1)]^* = \mathcal{R}^*_{q,j}(1|1).$$

We define the extended quantum superplane to be the algebra that contains $\mathcal{R}_q(1|1)$, the unit and $x^{-1}$, the inverse of $x$, which obeys

$$xx^{-1} = 1 = x^{-1}x.$$

We denote the extended algebra by $\mathcal{A}$. We know that the algebra $\mathcal{A}$ is a $Z_3$-graded Hopf algebra.\(^6\)
III. A PERSPECTIVE TO $Z_3$-GRADED $h$-DEFORMATION

We know that the commutation relation between the coordinate $x'$ and the coordinate $\theta'$ of the $Z_3$-graded quantum superplane is in the form

\[ x'\theta' - q\theta'x' = 0. \]

We now introduce new coordinates $x$ and $\theta$, in terms of $x'$ and $\theta'$ as

\[ x = x', \quad \theta = \theta' - \frac{h}{q - 1}x' \quad (3) \]

as in Ref. 7. This transformation is singular in the $q \to 1$ limit. Using relation (1), it is easy to verify that

\[ x\theta = q\theta x + hx^2 \quad (4) \]

where the new deformation parameter $h$ commutes with the coordinate $x$. Also, since the grassmann coordinate $\theta'$ satisfies

\[ \theta'^3 = 0 \]

one obtains

\[ \theta^3 = 0 \quad (5) \]

provided that

\[ \theta h = qjh\theta, \quad h^3 = 0. \quad (6) \]

Taking the $q \to 1$ limit we obtain the following relations which define the $Z_3$-graded $h$-superplane

\[ x\theta = \theta x + hx^2, \quad \theta^3 = 0. \quad (7) \]

Also, it can be obtained the $Z_3$-graded $h$-supergroup with Aghamohammadi’s approach in Ref. 7. So, it can be investigated the differential geometry of this group. This work is in progress.

IV. $Z_3$-GRADED QUANTUM SUPERGROUPS

A. Quantum matrices in $Z_3$-graded superspace

In this section, we shall consider the $Z_3$-graded structures of the quantum 2x2 supermatrices. We know, from section 2, that the $Z_3$-graded quantum superplane $R_q(1|1)$ is generated by coordinates $x$ and $\theta$, with the commutation rules (1), and the dual $Z_3$-graded quantum superplane $R^*_q(1|1)$ as generated by $\varphi$ and $y$ with the relations (2).

Let $T$ be a 2x2 (super)matrix in $Z_3$-graded space,

\[ T = \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix} \quad (8) \]
where $a$ and $d$ with respect to the $\mathbb{Z}_3$-grading are of grade 0, and $\beta$ and $\gamma$ with respect to the $\mathbb{Z}_3$-grading are of grade 2 and of grade 1, respectively. We now consider linear transformations with the following properties:

$$T : \mathcal{R}_q(1|1) \longrightarrow \mathcal{R}_q(1|1), \quad T : \mathcal{R}^*_{q,j}(1|1) \longrightarrow \mathcal{R}^*_{q,j}(1|1).$$ \hspace{1cm} (9)

We assume that the entries of $T$ are $j$-commutative with the elements of $\mathcal{R}_q(1|1)$ and $\mathcal{R}^*_{q,j}(1|1)$, i.e. for example,

$$ax = xa, \quad \theta \beta = j^2 \beta \theta,$$

etc. As a consequence of the linear transformations in (9) the elements

$$\tilde{x} = ax + \beta \theta, \quad \tilde{\theta} = \gamma x + d \theta$$ \hspace{1cm} (10)

should satisfy the relations (1):

$$\tilde{x} \tilde{\theta} = q \tilde{\theta} \tilde{x}, \quad \tilde{\theta}^3 = 0.$$

Using these relations, one has

$$a \gamma = q \gamma a, \quad d \gamma = q \gamma d,$$

$$d \beta = q^{-1} j \beta d, \quad \gamma^3 = 0.$$

Similarly, the elements

$$\tilde{\varphi} = a \varphi + j^2 \beta y, \quad \tilde{y} = j \gamma \varphi + dy$$ \hspace{1cm} (11)

must be satisfy the relations (2). Using these relations, one has

$$a \beta = q^{-1} j^{-1} \beta a, \quad \beta^3 = 0.$$

Also, if we use the following relation in Ref. 6 (see, page 4262, eq. (19))

$$\tilde{x} \tilde{y} = q \tilde{y} \tilde{x} + (j^2 - 1) \tilde{\varphi} \tilde{\theta},$$

we have

$$ad = da + q^{-1} (1 - j) \beta \gamma, \quad \beta \gamma = q^2 \gamma \beta.$$
\[ \beta^3 = 0, \quad \gamma^3 = 0. \]  

(12)

The \( Z_3 \)-graded quantum (super)determinant is defined by

\[ D_{q,j}(T) = ad^{-1} + ad^{-1} \gamma a^{-1} \beta d^{-1} + ad^{-1} \gamma a^{-1} \beta d^{-1} \gamma a^{-1} \beta d^{-1} \]  

(13)

provided \( d \) and \( a \) are invertible. The commutation relations between the matrix elements of \( T \) and its (super)determinant:

\[ aD_{q,j}(T) = D_{q,j}(T)a, \quad \beta D_{q,j}(T) = j^2 D_{q,j}(T) \beta \]

\[ \gamma D_{q,j}(T) = D_{q,j}(T)\gamma, \quad da D_{q,j}(T) = D_{q,j}(T) ad \]  

(14)

If we take

\[ \Delta_1 = ad - q^{-1} \beta \gamma \quad \Delta_2 = da - q j \beta \gamma \]

the \( Z_3 \)-graded quantum (super)inverse of \( T \) becomes

\[ T^{-1} = \begin{pmatrix} d \Delta_1^{-1} & -q j \beta \Delta_2^{-1} \\ -q^{-1} \gamma \Delta_1^{-1} & a \Delta_2^{-1} \end{pmatrix}. \]

Using the relations (12), one can check that the following relations:

\[ \Delta_1 a = a \Delta_1, \quad \Delta_1 d = d \Delta_1 \]

\[ \Delta_2 a = a \Delta_2, \quad \Delta_2 d = d \Delta_2 \]

\[ \Delta_k \beta = q^{-2} \beta \Delta_k, \quad \Delta_k \gamma = q^2 \gamma \Delta_k, \quad k = 1, 2 \]

\[ \Delta_1 \Delta_2 = \Delta_2 \Delta_1. \]  

(15)

The \( Z_3 \)-graded quantum (super)determinant of \( T \), according to these facts, is given by

\[ D_{q,j}(T) = a^2 \Delta_2^{-1}. \]

Of course, the \( Z_3 \)-graded quantum (super)determinant of \( T^{-1} \) may also be defined and it is of the form

\[ D_{q,j}(T^{-1}) = d^2 \Delta_1^{-1}. \]

Explicitly,

\[ D_{q,j}(T^{-1}) = da^{-1} + da^{-1} \beta d^{-1} \gamma a^{-1} + da^{-1} \beta d^{-1} \gamma a^{-1} \beta d^{-1} \gamma a^{-1}. \]  

(16)

Let’s now consider the multiplication of two \( Z_3 \)-graded quantum (super) matrices. If we take them as

\[ T_1 = \begin{pmatrix} a_1 & \beta_1 \\ \gamma_1 & d_1 \end{pmatrix} \quad \text{and} \quad T_2 = \begin{pmatrix} a_2 & \beta_2 \\ \gamma_2 & d_2 \end{pmatrix} \]
where the matrix elements of $T_1$ and $T_2$ satisfy the relations (12). Then the matrix elements of $T_1T_2$ leave invariant the relations (12), as expected. Here, we assume that the commutation relations between the elements of $T_1$ and $T_2$ are as follows

\[
\begin{align*}
\beta_1\gamma_2 &= j\gamma_2\beta_1, \\
\beta_1\beta_2 &= j^2\beta_2\beta_1, \\
\gamma_1\beta_2 &= j\beta_2\gamma_1, \\
\gamma_1\gamma_2 &= j^2\gamma_2\gamma_1
\end{align*}
\]

and the elements whose gradings are 0 commute with all the other elements.

Also, the $Z_3$-graded quantum (super)inverse of $T$ with

\[
T^{-1} \in GL_{q^{-1}j^{-1}}(1|1).
\]

The relations between the matrix elements of $T$ and (super)inverse of $T$ are important, because we will use them to set up the $Z_3$-graded differential

\[
A = a^{-1} + a^{-1}\beta d^{-1}\gamma a^{-1} - a^{-1}\beta d^{-1}\gamma a^{-1}\beta d^{-1}\gamma a^{-1},
\]

\[
B = -a^{-1}\beta d^{-1} - a^{-1}\beta d^{-1}\gamma a^{-1}\beta d^{-1} - a^{-1}\beta d^{-1}\gamma a^{-1}\beta d^{-1}\gamma a^{-1}\beta d^{-1},
\]

\[
C = -d^{-1}\gamma a^{-1} - d^{-1}\gamma a^{-1}\beta d^{-1}\gamma a^{-1},
\]

\[
D = d^{-1} + d^{-1}\gamma a^{-1}\beta d^{-1} + d^{-1}\gamma a^{-1}\beta d^{-1}\gamma a^{-1}\beta d^{-1}.
\]

Now, using the relations (15) and (12) one has $T^{-1} \in GL_{q^{-1}j^{-1}}(1|1)$. The relations between the matrix elements of $T$ and (super)inverse of $T$ are important, because we will use them to set up the $Z_3$-graded differential
geometric structure of the $\mathbb{Z}_3$-graded quantum supergroup. These relations are as follows:

\[
\begin{align*}
A &= j^2 A + 1 - j^2, & B &= q^{-1} j^2 B, \\
C &= q C, & D &= D,
\end{align*}
\]

Also, it can be investigated some properties of the quantum (super)matrices in the quantum supergroup $GL_{q,j}(1\vert 1)$. So, perhaps any element of $GL_{q,j}(1\vert 1)$ can be expressed as the exponential of a matrix of non-commuting elements, like the group $GL_q(1\vert 1)$. This work is also in progress.

**B. Hopf algebra structure of the $\mathbb{Z}_3$-graded $GL_{q,j}(1\vert 1)$**

In this section, we shall build up the Hopf algebra structure of the $\mathbb{Z}_3$-graded quantum supergroup $GL_{q,j}(1\vert 1)$. For this, we shall introduce three operators $\Delta$, $\epsilon$ and $S$ on the $GL_{q,j}(1\vert 1)$, which are called the coproduct (comultiplication), the counit and the coinverse (antipode), respectively. The coproduct $\Delta$ is an algebra homomorphism which is co-associative, that is

\[
(\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta
\]

where $\otimes$ stands for the usual tensor product and the dot refers to the summation over repeated indices and reminds us about the usual matrix multiplication. The coproduct $\Delta$ is an algebra homomorphism which is co-associative, that is

\[
(\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta
\]

where $\circ$ stands for the composition of maps and $\text{id}$ denotes the identity mapping. Also, the multiplication in the algebra of matrix entries of $T$ is defined as

\[
(A \otimes B)(C \otimes D) = j^{\text{grad}(B)\text{grad}(C)} AC \otimes BD.
\]

The action on the generators of the $GL_{q,j}(1\vert 1)$ of $\Delta$ is

\[
\Delta(a) = a \otimes a + \beta \otimes \gamma, \quad \Delta(\beta) = a \otimes \beta + \beta \otimes d.
\]
\[ \Delta(\gamma) = \gamma \otimes a + d \otimes \gamma, \quad \Delta(d) = \gamma \otimes \beta + d \otimes d \]  
(24)

where \( \otimes \) denotes the tensor product. Of course, the coproduct \( \Delta \) leaves invariant the relations (12). The counit

\[ \epsilon : GL_{q,j}(1|1) \longrightarrow \mathcal{C} \]

is defined by

\[ \epsilon(T) = I. \]  
(25)

The action on the generators of \( GL_{q,j}(1|1) \) of \( \epsilon \) is

\[ \epsilon(a) = 1, \quad \epsilon(\beta) = 0, \quad \epsilon(\gamma) = 0, \quad \epsilon(d) = 1. \]  
(26)

The counit \( \epsilon \) is an algebra homomorphism such that

\[ \mu \circ (\epsilon \otimes \text{id}) \circ \Delta = \mu' \circ (\text{id} \otimes \epsilon) \circ \Delta \]  
(27)

where

\[ \mu : \mathcal{C} \otimes GL_{q,j}(1|1) \longrightarrow GL_{q,j}(1|1), \quad \mu' : GL_{q,j}(1|1) \otimes \mathcal{C} \longrightarrow GL_{q,j}(1|1) \]

are the canonical isomorphisms, defined by

\[ \mu(c \otimes a) = ca = \mu'(a \otimes c), \quad \forall a \in GL_{q,j}(1|1), \quad \forall c \in \mathcal{C}. \]

Thus, we have verified that \( GL_{q,j}(1|1) \) is a bialgebra with the multiplication \( m \) satisfying the associativitiy axiom

\[ m \circ (m \otimes \text{id}) = m \circ (\text{id} \otimes m) \]

where \( m(a \otimes b) = ab. \)

A bialgebra with the extra structure of the coinverse is called a Hopf algebra.\(^{10}\)

The coinverse

\[ S : GL_{q,j}(1|1) \longrightarrow GL_{q,j}(1|1) \]

is defined by

\[ S(T) = T^{-1}. \]  
(28)

The coinverse \( S \) is an algebra anti-homomorphism which satisfies

\[ m \circ (S \otimes \text{id}) \circ \Delta = \epsilon = m \circ (\text{id} \otimes S) \circ \Delta. \]  
(29)

The coproduct, counit and coinverse which are specified above supply \( GL_{q,j}(1|1) \) with a Hopf algebra structure. It can be show that \( GL_{q,j}(1|1) \) has a \( \mathbb{Z}_3 \)-graded differential geometric structure.\(^9\)
ACKNOWLEDGMENT

This work was supported in part by TBTAK the Turkish Scientific and Technical Research Council.

1 R. Kerner and V. Abramov, Rep. Math. Phys. 43, 179 (1999).
2 B. Le Roy, J. Math. Phys. 37, 474 (1996).
3 V. Abramov and N. Bazunova, math-ph/0001041, (2001).
4 W. S. Chung, J. Math. Phys. 35, 2497 (1993).
5 J. Wess and B. Zumino, Nucl. Phys. B (Proc. Suppl.) 18 B, 302 (1990).
6 S. Celik, J. Phys. A: Math. Gen. 35, 4257 (2002).
7 A. Aghamohammadi, M. Khorrami and A. Shariati, A. J. Phys. A: Math. Gen. 28, 225 (1995);
   L. Dabrowski and P. Parashar, Lett. Math. Phys. 38, 331 (1996).
8 L. Dabrowski and L. Wang: Phys. Lett. B 266, 51 (1991).
9 E. Yasar, ”Differential geometry of the Z3-graded quantum supergroup ”, Phd Thesis (in preperation).
10 E. Abe, Hopf Algebras, Cambridge Univ. Press, (1980).