Finite-time control of formation system for multiple flight vehicles subject to actuator saturation

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Abstract: This paper investigates the problem of formation tracking control for multiple flight vehicle (MFV) system considering actuator saturation constraints. First, the formation tracking control model is established. Then, the problem of formation control of the MFV system is converted to the convergence of a dynamical system, which is obtained by using the differential geometry theory. A class of saturation functions is introduced, and on this basis a second-order finite-time formation control protocol is developed. With the help of the homogeneous theory and Lasalle's invariance principle, it is theoretically proved that the designed formation protocol could complete the formation task in finite time, and the control inputs are shown to remain within their available actuating limits. Finally, simulations are performed to verify the effectiveness of the scheme.

Keywords: formation control, multiple flight vehicle (MFV), finite-time, actuator saturation.

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1. Introduction

In recent years, the area of cooperative control has become an important research field motivated by the requirement to study engineering issues with new and efficient methods. In contrast to the traditional control ways, cooperative control is always distributed and does not depend on the unique central control unit only. Therefore, the cooperative control systems have the benefits of being robust, efficient, and flexible, allowing the control systems to have better performance in the new control strategies.

With the quick development of the aerospace technology, the conventional combat strategy using a single flight vehicle cannot satisfy the demands of the combat mission in complicated combat fields. In order to improve the hitting accuracy and enlarge the defense area, the cooperative combat system using multiple flight vehicles (MFVs) has been a research focus in recent years. The cooperative combat system can be applied in a variety of fields, including cooperative path planning [1], formation control [2], cooperative guidance [3], multiple sensor and data fusion [4], etc. Formation control is the key technology in cooperative attack, which plays an important role in the cooperative combat.

There are several methods for formation control design in general, such as virtual leader, leader-follower and behavioral strategies [5], of which the leader-follower strategy is widely used because of its significant advantage [6–8]. In the virtual leader strategy [9,10], the whole formation system is regarded as an individual rigid body. The virtual leader formation system can be developed into a unit in a given direction and keep a rigid relative position between multiple vehicles. In the behavioral strategy [11,12], the control effect is determined by a weighted average of control corresponding to every vehicle expected behavior. Furthermore, three formation control strategies have their own strengths and weaknesses. In this paper, the leader-follower strategy is adopted due to its simplicity and efficiency.

In the past years, lots of results with the consensus method can be found in various applications [13–15]. A class of consensus methods for systems with second-order dynamics was introduced in [16], which expanded the first-order consensus algorithms. Lin et al. [17] introduced a new technique taking advantage of the complex Laplacian to solve the problems of which the formation figures are appointed by inter-agent relative positions. Rezaee et al. [18] managed the problem of cyclic pursuit for the multi-agent system (MAS) in polygon formations. In all above works, the control inputs are unbounded. It means that the control inputs maintaining in a clearly defined bound cannot be ensured, which is determined by the agents actual
control constraints.

Since the ability of any physical actuator is limited, actuator saturation is one of the most common and crucial nonlinearities in practical control systems [19,20]. By comparing various performances of systems with saturation constraints, input saturation and actuator saturation, it can be concluded that input saturation only moderates long-term actuator saturation and avoiding actuator saturation is impossible in practical control systems [21]. The control system subject to actuator saturation in the MAS has received considerable attention since actuator saturation cannot be avoided and it universally exists in various application areas [22].

Based on the existing results and motivated by the aforementioned statements, the finite-time control problems for MFV systems subject to actuator saturation is investigated. Apart from taking into consideration the actuator saturation, this paper solves the problem with a finite-time control tool.

The remainder of this paper is organized as follows. Section 2 gives a review on the mathematical tool that is necessary for the research. The formation control systems subject to actuator saturation is presented in Section 3. The finite-time formation controller is presented and developed in Section 4. Simulation results are provided in Section 5. The concluding statements are given in Section 6.

2. Preliminaries

2.1 Graph theory

The graph $G = (v, \varepsilon, A)$ is introduced to describe the information sharing structure, $v = \{1, 2, 3, \ldots, N\}$ is called the node set, $\varepsilon \subseteq v \times v = \{(i, j) : i, j \in v\}$ is called the edge set. The weighted adjacency matrix $A = [a_{ij}] \in \mathbb{R}^{N \times N}$ ($i, j = 1, 2, \ldots, N$) is defined as $a_{ii} = 0, a_{ij} > 0$ when $i \neq j, a_{ij}$ is the weight of edge from $i$ to $j$. Besides, a diagonal matrix $D = \text{diag}\{d_1, d_2, \ldots, d_N\} \in \mathbb{R}^{N \times N}$ with $d_i = \sum_{j \in N_i} a_{ij}$ ($i = 1, 2, \ldots, N$) is a degree matrix of $G$. The matrix $L = D - A \in \mathbb{R}^{N \times N}$ is the Laplacian of the weighted directed graph $G$.

2.2 Homogeneous theory

Before presenting the main results of this paper, some important lemmas are given first for further analysis.

**Lemma 1** [23] (LaSalle’s invariance principle) Let $x(t)$ be a solution of $\dot{x} = f(x), x(0) = x_0 \in \mathbb{R}^n$, where $f : U \to \mathbb{R}^n$ is continuous with an open subset $U$ of $\mathbb{R}^n$, and $V : U \to \mathbb{R}$ be a locally Lipschitz function such that $D^+V(x(t)) \leq 0$, where $D^+$ denotes the upper Dini derivative. Then $\Theta^+(x_0) \cap U$ is contained in the union of all the solutions that remain in $S = \{x \in U : D^+V(x) = 0\}$, where $\Theta^+(x_0)$ denotes the positive limit set.

**Definition 1** Consider the system $\dot{x} = f(x), f(0) = 0$, where $x \in \mathbb{R}^n$, and $f(x) = [f_1(x), \ldots, f_n(x)]^T$ is a continuous vector field. Let $(r_1, \ldots, r_n) \in \mathbb{R}^n$ with $r_i > 0$, $(i = 1, 2, \ldots, n)$. $f(x)$ is said to be homogeneous of degree $\kappa \in \mathbb{R}$ with respect to $(r_1, \ldots, r_n)$ if, for any given $\varepsilon > 0$, $f_i(\varepsilon^{\kappa} x_1, \ldots, \varepsilon^{\kappa} x_n) = \varepsilon^{\kappa + r_i} f_i(x)$ $(i = 1, 2, \ldots, n), \forall x \in \mathbb{R}^n$. The system $\dot{x} = f(x)$ is said to be homogeneous if $f(x)$ is homogeneous.

**Lemma 2** [24] Consider the following system:

$$\dot{x} = f(x) + \hat{f}(x), \quad f(0) = 0; \quad x \in \mathbb{R}^n$$

where $f(x)$ is a continuous homogeneous vector field of degree $\kappa < 0$ with respect to $(r_1, \ldots, r_n)$, and $\hat{f}(x)$ satisfies $f(0) = 0$. Assume $x = 0$ is an asymptotically stable equilibrium of the system $\dot{x} = f(x)$. Then, $x = 0$ is a locally finite-time stable equilibrium of the system if

$$\lim_{\varepsilon \to 0} \hat{f}(\varepsilon^{r_1} x_1, \ldots, \varepsilon^{r_n} x_n) = 0, \quad i = 1, \ldots, n; \quad \forall x \neq 0.$$

In addition, if the stable equilibrium $x = 0$ of the original system (1) is globally asymptotically stable, then $x = 0$ is a globally finite-time stable equilibrium of the system (1).

2.3 Accurate linearization by differential geometry

For the nonlinear system as

$$\dot{x} = f(x) + \sum_{i=1}^{m} c_i(x)u_i \triangleq f(x) + C(x)u$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $f(x) \in \mathbb{R}^n$, $C(x) = [c_1, \ldots, c_m]$ and $g_i$ ($i = 1, 2, \ldots, m$) $\in \mathbb{R}^m$ are smooth vector fields, and

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}, \quad f(x) = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}, \quad c_i(x) = \begin{bmatrix} c_{i1} \\ \vdots \\ c_{in} \end{bmatrix}.$$

In the condition that the rank of $C(x)$ is $m$, (2) is linearizable if and only if it meets the following requirements:

(i) the dimension of $C_1$ $(0 \leq i \leq n - 1)$ is constant nearby $x_0$;
(ii) the dimension of $C_{n-1}$ is $n$;
(iii) $C_i (0 \leq i \leq n - 2)$ is involutive. Moreover,

$$C_i = \text{span}\{\text{ad}_{\gamma}e_j : 0 \leq r \leq i, 1 \leq j \leq m\}$$
ad_j c_j of f and c_j is called the \( r \)th Lie bracket, and \( \{r_1, \ldots, r_m\} \) is
\[
r_i = \text{card}\{m_j|m_j \geq i, j \geq 0\}, \quad 1 \leq i \leq m.
\] (4)

Since
\[
\begin{align*}
    m_0 &= \text{rank} \ C_0 \\
m_1 &= \text{rank} \ C_1 - \text{rank} \ C_0 \\
    \vdots \\
m_{n-1} &= \text{rank} \ C_{n-1} - \text{rank} \ C_{n-2}
\end{align*}
\]
there exists a \( \varphi(x) = [\varphi_1(x) \ \cdots \ \varphi_m(x)]^T \), such that
\[
    < d\varphi_i, C_{r-2} >= 0, \quad j \geq i,
\] (5)

and
\[
\begin{bmatrix}
    < d\varphi_1, ad_r^{-1}c_1 > \\
    \vdots \\
    < d\varphi_m, ad_r^{-1}c_m >
\end{bmatrix}
\]
is nonsingular on the neighborhood \( V_0 \) of \( x_0 \), \( V_0 \subseteq \mathbb{R}^n \). Therefore, (2) can be converted into
\[
    \dot{z} = H_c z + M_c v.
\] (7)

\( (H_c, M_c) \) is in the form of the standard Brunovsky controller, which includes \( \{r_1, \ldots, r_m\} \).

3. Formation control systems subject to actuator saturation

In this section, the formation control systems subject to actuator saturation is constructed.

3.1 Dynamic and kinematic model of flight vehicle

Assuming that \( N \) flight vehicles in the formation system have the same structure, the dynamics of the flight vehicle are
\[
\begin{align*}
    \dot{V}_i &= g(n_{x_i} - \sin \theta_i) \\
    \dot{\theta}_i &= \frac{g}{V_i} (n_{y_i} - \cos \theta_i) \\
    \dot{\psi}_V &= -\frac{g}{V_i \cos \theta_i} n_{z_i},
\end{align*}
\] (11)
where \( i = 1, \ldots, N \) is the index number of the flight vehicle. \( V_i, \theta_i, \psi_V, n_{x_i}, n_{y_i}, n_{z_i} \) and \( g \) represent speed, flight-path angle, heading angle, overload components in three axes, and acceleration of gravity, respectively.

The kinematic equations are
\[
\begin{align*}
    \dot{x}_i &= V_i \cos \theta_i \cos \psi_V \\
    \dot{y}_i &= V_i \sin \psi_V \\
    \dot{z}_i &= -V_i \cos \theta_i \sin \psi_V
\end{align*}
\] (12)
where \((x_i, y_i, z_i)\) denotes the position of the flight vehicle in the inertial coordinate system.

3.2 Accurate feedback linearization

Select the state and control vector as follows, respectively:
\[
\begin{align*}
    x_i &= [x_i \quad y_i \quad z_i \quad V_i \quad \theta_i \quad \psi_V]^T, \\
    u_i &= [n_{x_i} \quad n_{y_i} \quad n_{z_i}]^T
\end{align*}
\] (13) (14)
(11) and (12) can be rewritten as
\[
\dot{x}_i = f(x_i) + C(x_i) u_i
\] (15)
where
\[
\begin{bmatrix}
    V_i \cos \theta_i \cos \psi_V \\
    V_i \sin \psi_V \\
    -V_i \cos \theta_i \sin \psi_V \\
    -g \sin \theta_i \\
    -g \cos \theta_i \\
    \frac{g}{V_i} \\
    0 \\
    0
\end{bmatrix}
\]
\[
\begin{bmatrix}
    0 & 0 & 0 \\
    0 & 0 & 0 \\
    0 & 0 & 0 \\
    g & 0 & 0 \\
    g & 0 & 0 \\
    0 & -g & 0 \\
    0 & 0 & \frac{g}{V_i \cos \theta_i}
\end{bmatrix}
\]
\[
\begin{bmatrix}
    L_{g_1} L_{f_1}^{-1} \varphi_1 \\
    \vdots \\
    L_{g_m} L_{f_m}^{-1} \varphi_m
\end{bmatrix}
\]
\[
\begin{bmatrix}
    L_{g_1} L_{f_1}^{-1} \varphi_1 \\
    \vdots \\
    L_{g_m} L_{f_m}^{-1} \varphi_m
\end{bmatrix}
\]
\[
\begin{bmatrix}
    L_{g_1} L_{f_1}^{-1} \varphi_1 \\
    \vdots \\
    L_{g_m} L_{f_m}^{-1} \varphi_m
\end{bmatrix}
\]

The nonsingular state feedback is
\[
    u = \alpha(x) + \beta(x) v
\] (10)
where \( \alpha(x) \) is a smooth function from \( V_0 \) to \( \mathbb{R}^n \),
\[
    \alpha(x) = \begin{bmatrix}
        L_{f_1} \varphi_1 \\
        \vdots \\
        L_{f_m} \varphi_m
    \end{bmatrix},
\]
\( \beta(x) \) is a nonsingular matrix on \( V_0 \):
\[
    \beta(x) = \begin{bmatrix}
        L_{g_1} L_{f_1}^{-1} \varphi_1 \\
        \vdots \\
        L_{g_m} L_{f_m}^{-1} \varphi_m
    \end{bmatrix}.
\]
If and only if the demand in Section 2.3 is achieved, (15) is full-state-feedback linearizable. The distribution \( C_i \) \((i = 0, 1, \ldots, 5)\) is involutive in this paper.

A nonsingular smooth vector function is selected as follows, which satisfies (5) and (6):

\[
\varphi(x_i) = [\varphi_1 \varphi_2 \varphi_3]^T = [x_i \ y_i \ z_i]^T,
\]

so

\[
z_i = \begin{bmatrix}
\varphi_1 \\
\varphi_2 \\
L_f^2 \varphi_3
\end{bmatrix}
= \begin{bmatrix}
x_i \\
y_i \\
z_i
\end{bmatrix}.
\]

Then the state feedback transformation is obtained:

\[
v_i = \alpha(x_i) + \beta(x_i)u_i
\]

where

\[
\alpha(x_i) = \begin{bmatrix}
0 \\
-g
\end{bmatrix}^T,
\]

\[
\beta(x_i) = \begin{bmatrix}
g \cos \theta_i \cos \psi_i \\
g \sin \theta_i \\
g \sin \theta_i \sin \psi_i
\end{bmatrix}.
\]

Therefore, (15) can be converted into

\[
\dot{z}_i = Hz_i + Mv_i
\]

where

\[
H = \begin{bmatrix}
0_{3 \times 3} & I_{3 \times 3} \\
0_{3 \times 3} & 0_{3 \times 3}
\end{bmatrix},
\]

\[
M = \begin{bmatrix}
0_{3 \times 3}
\end{bmatrix}.
\]

### 3.3 Coordination scheme for formation flying

In this section, the leader coordinate \( o_i - x_i y_i z_i \) is established, so the leader-follower relative coordinate can be obtained in Fig. 1.

The battlefield environment is changing rapidly, and the MFV system should complete the formation task in finite time. The leader of the system is equipped with a high-performance seeker, and the followers are equipped with low-performance seekers. In general, there is only one leader, and other flight vehicles serve as followers. The configuration of the formation varies with the task. The process of formation control for the MFV system with actuator saturation can be described in Fig. 2. The task of the MFV system is to intercept a moving target, and the detection and guidance accuracy can be improved by MFVs. Wedge-shape formation is suitable for the study of this paper.

3.4 Formation tracking control with actuator saturation

In this paper, the MFV system composed of \( N \) flight vehicles is considered. Each flight vehicle exchanges its formation information with its neighbors by communication topology. The dynamics of the \( i \)th flight vehicle which is subject to actuator saturation are described as

\[
\dot{z}_i = Hz_i + M\text{sat}(v_i)
\]

where \( H \in \mathbb{R}^{6 \times 6} \) and \( M \in \mathbb{R}^{6 \times 3} \) are system matrices, which are the same as in (18), \( z_i \in \mathbb{R}^6 \) is the state of flight vehicle \( i \), \( v_i \in \mathbb{R}^3 \) is the control input acting on the flight vehicle \( i \), and sat: \( \mathbb{R}^3 \to \mathbb{R}^3 \) is a saturation function defined as

\[
\text{sat}(v_i) = \begin{bmatrix}
\text{sat}(v_{x_i}) \\
\text{sat}(v_{y_i}) \\
\text{sat}(v_{z_i})
\end{bmatrix}.
\]

where

\[
\text{sat}(v_i) = \begin{cases}
v_{x_i}^m, & v_i > v_{x_i}^m \\
v_i, & -v_{x_i}^m \leq v_i \leq v_{x_i}^m \\
-v_{x_i}^m, & v_i < -v_{x_i}^m
\end{cases}
\]

Fig. 1 Leader-follower relative coordinate

Fig. 2 Formation control based on consensus theory

Desired configuration

Nonlinear flight vehicles

Feedback linearization

Formation control \( u_i \)

Consensus protocol \( v_i \) with actuator saturation

Linearized dynamics \( z_i = H z_i + M \text{sat}(v_i) \)
Equation (20) can be rewritten as
\[
sat(v_i) = \chi_i(v_i)v_i \tag{21}
\]
and
\[
\chi_i(v_i) = \begin{cases} 
\frac{v_i^m}{v_i^m}, & v_i > v_i^m \\
1, & v_i^m \leq v_i \leq v_i^m \\
-\frac{v_i^m}{v_i^m}, & v_i < -v_i^m 
\end{cases}
\tag{22}
\]
\[
\chi_i(v_i) \in (0, 1] \text{ can be denoted as a pointer for the level of saturation of the control input. If } \chi_i(v_i) \text{ is close to } 0,
\]
\[
\text{it means there is nearly no feedback from input } v_i, \text{ and } \chi_i(v_i) = 1 \text{ means that does not saturate.}
\]
\[
\text{As } 0 < \chi_i(v_i) \leq 1, \text{ there exists a constant } \delta \text{ satisfying}
\]
\[
0 < \delta \leq \min(\chi_i(v_i)) \leq 1. \tag{23}
\]

**Definition 2** Finite-time formation problem of MFV with actuator saturation. Considering the MFV systems subject to actuator saturation described in (19), the subscript \( l \) represents the leader, and subscript \( 1, \ldots, N - 1 \) denote the followers. Expected formation with actuator saturation can be achieved in finite time, if there exists \( T_0 \in [0, +\infty) \), such that the MFV system in the leader-following strategy satisfies
\[
\lim_{t \to T_0} \|p_i(t) - p_{fl}(t) - d_i\| = 0,
\]
\[
\lim_{t \to T_0} \|q_i(t) - q_{fl}(t)\| = 0,
\]
\[
\begin{cases} 
p_i(t) = p_i(t) + d_i, & \forall t \geq T_0; i \in \{1, 2, \ldots, N - 1\} \\
q_i(t) = q_i(t)
\end{cases}
\]
in any initial states, where \( p_i \) and \( q_i \) represent the position vector and the velocity vector of the \( i \)th follower, respectively. \( d_i \) denotes the desired relative position vector. \( p_l \) and \( q_l \) denote the position vector and the velocity vector of the leader, respectively.

In this work, the finite-time formation control problem with actuator saturation is concerned, and the block diagram is shown in Fig. 3. A distributed finite-time formation controller is designed, which provides the control command based on mission requirements and neighborhood information.

**4. Formation controller design based on consensus theory**

In this subsection, the focus is put on solving the finite-time formation control problem for the MFV systems subject to actuator saturation. The tracking error dynamics of the flight vehicles are derived, in which the desired positions of the flight vehicles are described. Then a formation control protocol is developed via the finite-time consensus theory. With the aid of the homogeneous theory, the stability of the finite-time formation control system is analyzed.

**4.1 Tracking error dynamics**

It can be shown that the dynamics of the \( i \)th flight vehicle can be rewritten as
\[
\begin{bmatrix}
\dot{x}_i \\
\dot{y}_i \\
\dot{z}_i \\
\end{bmatrix} = 
\begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
x_i \\
y_i \\
z_i \\
\end{bmatrix}
+ 
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
sat(v_{x_i}) \\
sat(v_{y_i}) \\
sat(v_{z_i}) \\
\end{bmatrix}.
\tag{24}
\]

Then (24) can be described as
\[
\begin{bmatrix}
\dot{x}_i \\
\dot{y}_i \\
\dot{z}_i \\
\end{bmatrix} = 
\begin{bmatrix}
sat(v_{x_i}) \\
sat(v_{y_i}) \\
sat(v_{z_i}) \\
\end{bmatrix}.
\tag{25}
\]

Therefore, the nonlinear system composed of (11) and (12) is transformed into a linear double-integrator system (25) with actuator saturation, so the research and design processes are greatly simplified.

**Assumption 1** The position and velocity of flight vehicles can be measured.

**Assumption 2** The leader could plan its trajectory autonomously.

\[
p_{fl_i}^d \text{ represents the desired position in the coordinate frame } o_l - x_l y_l z_l \text{ for the } i \text{th follower flight vehicle.}
\]
\[
p_{fl_i}^d = p_l + \delta(\theta_l, \psi_{fl})p_{fl_i}^d
\tag{26}
\]
where \( p_l \) represents the leader’s position, \( \delta(\theta_l, \psi_{fl}) \) denotes the coordinate transformation matrix from \( o_l - x_l y_l z_l \)
to \( o_l - x_l y_l z_l \). \( p_{fi}^d \) represents the desired position in \( o_l - x_l y_l z_l \) for the \( i \)th follower.

\[
p_l = \begin{bmatrix} x_l \\ y_l \\ z_l \\ \end{bmatrix},
\]

\[
p_{fi}^d = \begin{bmatrix} x_{fi}^d \\ y_{fi}^d \\ z_{fi}^d \\ \end{bmatrix},
\]

\[
\delta(\theta_l, \psi_{V_l}) = \begin{bmatrix} \cos \theta_l \cos \psi_{V_l} & -\sin \theta_l \cos \psi_{V_l} & \sin \psi_{V_l} \\ \sin \theta_l & \cos \psi_{V_l} & 0 \\ -\cos \theta_l \sin \psi_{V_l} & \sin \theta_l \sin \psi_{V_l} & \cos \psi_{V_l} \end{bmatrix}.
\]

The tracking error \( e_i \) can be shown as

\[
e_i = \begin{bmatrix} e_{x_i} \\ e_{y_i} \\ e_{z_i} \end{bmatrix}^T = p_{fi}^d - p_{fi}^d.
\]  

(27)

Then \( \dot{e}_i \) and \( \ddot{e}_i \) can be obtained, so the error dynamics can be described as

\[
\dot{E}_i = HE_i + Mv_{fi} - M\ddot{p}_{fi}^d
\]

where \( v_{fi} \) is the control input which needs to be designed to stabilize error dynamics, \( E_i = [e_i \ \dot{e}_i]^T \), \( H \) and \( M \) are the same as those in (18). Furthermore,

\[
\ddot{p}_{fi}^d = \ddot{p}_i + \delta(\theta_l, \psi_{V_l})p_{fi}^d.
\]

(29)

4.2 Formation tracking law based on finite-time control

Suppose that there are \( n \) followers (labelled as 1 to \( n \)) and one leader (labelled as 0) denoted as in the group. The objective of this subsection is to design a bounded control law \( v_{fxi} \) for all the followers such that the desired formation configuration in the presence of actuator saturation can be achieved in finite time.

**Assumption 3** For a communication graph which contains a directed spanning tree with the leader as the root, its subgraph associated with the followers is undirected.

For convenience, let

\[
x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T \in \mathbb{R}^n
\]

and \( \text{sign}(x) = |x|^\alpha \text{sign}(x) \), where \( \text{sign}(\cdot) \) denotes the sign function and \( |x| \) denotes the absolute value of the real number \( x \).

**Theorem 1** When Assumption 3 holds, considering the formation tracking control system (19), the control variable \( v_{fxi} \) in the \( x \) direction can be given as

\[
v_{fxi} = \text{sat}(x_{fi}^d + k_1 s_{xi} + k_2 h_{xi})
\]

(31)

where

\[
s_{xi} = \varphi_1[\text{sign}(x_{fi}^d - x_i)^{\alpha_1}] + \varphi_2\left\{ \text{sign}\left( \sum_{j=1}^{n} a_{ij}[(x_j - x_{fi}^d) - (x_i - x_{fi}^d)]^{\alpha_1} \right) \right\},
\]

\[
h_{xi} = \varphi_3[\text{sign}(x_{fi}^d - \dot{x}_i)^{\alpha_2}] + \varphi_4\left\{ \text{sign}\left( \sum_{j=1}^{n} a_{ij}[(\dot{x}_j - x_{fi}^d) - (\dot{x}_i - x_{fi}^d)]^{\alpha_2} \right) \right\},
\]

\[
v_{fxi} \text{ can realize } x_i \rightarrow x_{fi}^d, \dot{x}_i \rightarrow \dot{x}_{fi}^d \text{ in finite-time, if there exists a continuous odd functions } \varphi_r \text{ satisfying } x\varphi_r(x) > 0 \ (\forall x \neq 0), \text{ and } \varphi_r(x) = \varepsilon_r x + o(x) \text{ around } x = 0 \text{ for some constant } \varepsilon_r > 0 (r = 1, 2, 3, 4). \text{ Moreover, } \|v_i\|_\infty \leq v_{\max}. \text{ } k_1 \text{ and } k_2 \text{ are the feedback control gains, } 0 < \alpha_1 < 1, \alpha_2 = \frac{2\alpha_1}{1 + \alpha_1}.
\]

In this paper, suppose that the positions and velocities are measurable, and the flight vehicles are subject to actuator saturation such that \( \|v_i\|_\infty \leq v_{\max}. \text{ It can be seen that the control protocol (31) accounts for the actuator saturation, due to the boundedness of the saturation function sat(\cdot).} \)

Likewise, \( v_{fyi} \) and \( v_{fzi} \) can be derived as

\[
v_{fyi} = \text{sat}(y_{fi}^d + k_1 s_{yi} + k_2 h_{yi})
\]

(32)

where

\[
s_{yi} = \varphi_1[\text{sign}(y_{fi}^d - y_i)^{\alpha_1}] + \varphi_2\left\{ \text{sign}\left( \sum_{j=1}^{n} a_{ij}[(y_j - y_{fi}^d) - (y_i - y_{fi}^d)]^{\alpha_1} \right) \right\},
\]

\[
h_{yi} = \varphi_3[\text{sign}(y_{fi}^d - \dot{y}_i)^{\alpha_2}] + \varphi_4\left\{ \text{sign}\left( \sum_{j=1}^{n} a_{ij}[(\dot{y}_j - y_{fi}^d) - (\dot{y}_i - y_{fi}^d)]^{\alpha_2} \right) \right\},
\]

\[
v_{fzi} = \text{sat}(z_{fi}^d + k_1 s_{zi} + k_2 h_{zi})
\]

(33)

where

\[
s_{zi} = \varphi_1[\text{sign}(z_{fi}^d - z_i)^{\alpha_1}] + \varphi_2\left\{ \text{sign}\left( \sum_{j=1}^{n} a_{ij}[(z_j - z_{fi}^d) - (z_i - z_{fi}^d)]^{\alpha_1} \right) \right\},
\]

\[
h_{zi} = \varphi_3[\text{sign}(z_{fi}^d - \dot{z}_i)^{\alpha_2}] + \varphi_4\left\{ \text{sign}\left( \sum_{j=1}^{n} a_{ij}[(\dot{z}_j - z_{fi}^d) - (\dot{z}_i - z_{fi}^d)]^{\alpha_2} \right) \right\}.
\]

The control law of three channels can be rewritten as

\[
v_{fi} = \text{sat}[p_{fi}^d + k_1 s_i + k_2 h_i]
\]

(34)

where

\[
s_i = \varphi_1[\text{sign}(p_{fi}^d - p_i)^{\alpha_1}] + \varphi_2\left\{ \text{sign}\left( \sum_{j=1}^{n} a_{ij}[(p_j - p_{fi}^d) - (p_i - p_{fi}^d)]^{\alpha_1} \right) \right\},
\]

\[
h_i = \varphi_3[\text{sign}(p_{fi}^d - \dot{p}_i)^{\alpha_2}] + \varphi_4\left\{ \text{sign}\left( \sum_{j=1}^{n} a_{ij}[(\dot{p}_j - p_{fi}^d) - (\dot{p}_i - p_{fi}^d)]^{\alpha_2} \right) \right\}.
\]
4.3 Finite-time stability analysis by homogeneous theory

Denote \( \overline{x}_i = x_i - x_{i+1} \), \( \overline{x}_j = \dot{x}_i - \dot{x}_{i+1} \), and \( \overline{x}_j = x_j - x_{j+1} \), \( \overline{x}_j = \dot{x}_j - \dot{x}_{j+1} \). Then the system (25) can be rewritten as

\[
\overline{\dot{x}}_i = \overline{\dot{x}}_{f,i} = \text{sat}(k_1 \overline{x}_i + k_2 \overline{x}_i),
\]

where \( \overline{x}_i = -\varphi_3[\text{sign}(\overline{x}_i)^\alpha_3] + \sum_{j=1}^{n} a_{ij} \varphi_2[\text{sign}(\overline{x}_j - \overline{x}_i)^\alpha_2] \),

\[
\overline{h}_x = -\varphi_3[\text{sign}(\overline{x}_i)^\alpha_2] + \sum_{j=1}^{n} a_{ij} \varphi_4[\text{sign}(\overline{x}_j - \overline{x}_i)^\alpha_2].
\]

As (21) shows, (36) becomes

\[
\overline{\dot{x}}_{f,i} = \chi_i \overline{\dot{x}}_{f,i} = \chi_i(k_1 \overline{x}_i + k_2 \overline{x}_i).
\]

Therefore,

\[
\overline{\dot{x}}_{f,i} = k_1 \overline{x}_i + k_2 \overline{x}_i =
\]

\[
k_1 \left\{ -\varphi_3[\text{sign}(\overline{x}_i)^\alpha_3] + \sum_{j=1}^{n} a_{ij} \varphi_2[\text{sign}(\overline{x}_j - \overline{x}_i)^\alpha_2] \right\} +
\]

\[
k_2 \left\{ -\varphi_3[\text{sign}(\overline{x}_i)^\alpha_2] + \sum_{j=1}^{n} a_{ij} \varphi_4[\text{sign}(\overline{x}_j - \overline{x}_i)^\alpha_2] \right\}.
\]

**Proof** Consider the Lyapunov function as

\[
V = V_1 + V_2 + V_3,
\]

\[
V_1 = \frac{1}{2} \sum_{i=1}^{n} x_i^2,
\]

\[
V_2 = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \int_{0}^{\overline{x}_i - \overline{x}_j} k_1 a_{ij} \varphi_2[\text{sign}(s)^\alpha_1] ds,
\]

\[
V_3 = \sum_{i=1}^{n} \int_{0}^{\overline{x}_i} k_1 \varphi_1[\text{sign}(s)^\alpha_1] ds.
\]

Then

\[
\dot{V}_1 = \sum_{i=1}^{n} \left\{ -\varphi_3[\text{sign}(\overline{x}_i)^\alpha_3] \varphi_2[\text{sign}(\overline{x}_j - \overline{x}_i)^\alpha_2] \right\} +
\]

\[
\sum_{j=1}^{n} a_{ij} \varphi_2[\text{sign}(\overline{x}_j - \overline{x}_i)^\alpha_2] +
\]

\[
\sum_{j=1}^{n} a_{ij} \varphi_4[\text{sign}(\overline{x}_j - \overline{x}_i)^\alpha_2],
\]

\[
\dot{V}_2 = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} k_1 (\overline{x}_i - \overline{x}_j) \varphi_2[\text{sign}(\overline{x}_j - \overline{x}_i)^\alpha_2] -
\]

\[
\sum_{i=1}^{n} k_2 \varphi_3[\text{sign}(\overline{x}_i)^\alpha_2] \leq 0.
\]

When \( \dot{V} = 0 \), combined with (37), we can get \( \overline{x}_i = \overline{x}_j = 0 \), that is \( x_i - x_{i+1} = 0, \dot{x}_i - \dot{x}_{i+1} = 0 \) as \( t \to \infty \). It can be concluded from Lemma 1 that the equilibrium (37) is global asymptotically stable at the origin.

As the odd function satisfying \( \varphi_3(x) = \varepsilon x + o(x) \) \( (r = 1, 2, 3, 4) \), the formation control protocol (31) can be rewritten as \( \overline{x}_{f,i} = \overline{x}_{f,i} + \overline{x}_{f,i} \) with

\[
\overline{x}_{f,i} = \chi_i \left\{ -\varepsilon_1 \text{sign}(\overline{x}_i)^\alpha_1 + \sum_{j=1}^{n} a_{ij} \varepsilon_2 \text{sign}(\overline{x}_j - \overline{x}_i)^\alpha_2 \right\} +
\]

\[
\chi_i \left\{ -\varepsilon_3 \text{sign}(\overline{x}_i)^\alpha_2 + \sum_{j=1}^{n} a_{ij} \varepsilon_4 \text{sign}(\overline{x}_j - \overline{x}_i)^\alpha_2 \right\}
\]

and

\[
\overline{x}_{f,i} = o(\text{sign}(\overline{x}_i)^\alpha_1) + o\left( \sum_{j=1}^{n} \text{sign}(\overline{x}_j - \overline{x}_i)^\alpha_1 \right) +
\]

\[
o(\text{sign}(\overline{x}_i)^\alpha_2) + o\left( \sum_{j=1}^{n} \text{sign}(\overline{x}_j - \overline{x}_i)^\alpha_2 \right).
\]
Also, by choosing \( 0 < \alpha_1 < 1, \alpha_2 = \frac{2\alpha_1}{1+\alpha_1}, \kappa = \alpha_1 - 1, \) the system (36) with variables \((\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n, \vec{\dot{x}}_1, \vec{\dot{x}}_2, \ldots, \vec{\dot{x}}_n)\) is homogeneous with dilation \((2, 2, \ldots, 2, 1+\alpha_1, 1+\alpha_1, \ldots, 1+\alpha_1, \ldots, 1+\alpha_1)\).

Finally, after the analysis and derivation, one can easily get the fact that \(\sigma_{\vec{f}_x}^s\) satisfies \(\lim_{\varepsilon \to 0} \frac{\int_0^\varepsilon \vec{f}_x^s(\varepsilon, x_1, \ldots, \varepsilon^n x_n) \, d\varepsilon = 0.}\)

To sum up, the system (36) is globally finite-time stable and locally homogeneous. Consequently, the origin is a finite-time stable equilibrium point of the system (36) by Lemma 2. Hence, the origin is a globally finite-time stable equilibrium point of the system (8), that means \(x_i - x_{fi}^d \rightarrow 0, \vec{\dot{x}}_i - \vec{\dot{x}}_{fi}^d \rightarrow 0\) in finite time. \(\Box\)

Similarly, \(y_i - y_{fi}^d \rightarrow 0, \vec{\dot{y}}_i - \vec{\dot{y}}_{fi}^d \rightarrow 0\) and \(z_i - z_{fi}^d \rightarrow 0, \vec{\dot{z}}_i - \vec{\dot{z}}_{fi}^d \rightarrow 0\) will be achieved in finite time.

5. Simulation results

The finite-time formation of three flight vehicles with actuator saturation is investigated in this paper. Three flight vehicles have different roles, one leader and two followers. Two instances are considered to test the performance of the designed formation control law. Triangle formation is selected, and the topology is given in Fig. 4. Obviously, this communication topology satisfies Assumption 3. The initial condition of flight vehicles is shown in Table 1.

![Fig. 4 Communication topology of flight vehicles](image)

**Table 1 Initial condition of flight vehicles**

| Flight vehicle | \((x, y, z)/m\) | \(\theta/(^\circ)\) | \(\psi/(^\circ)\) | \(V/(m/s)\) |
|---------------|----------------|----------------|----------------|------------|
| Leader        | \((2000, 500, 0)\) | 0              | 0              | 240        |
| Follower 1    | \((900, 500, 700)\) | 10             | 10             | 240        |
| Follower 2    | \((900, 500, -700)\) | 10             | 10             | 240        |

The expected formation positions are

\[
\begin{align*}
\{ p_{fl1}^d & = [-1000 -100 500] \\
p_{fl2}^d & = [-1000 -100 -500] 
\end{align*}
\]

The main controller parameters are

\[
\begin{align*}
k_1 & = 5 \\
k_2 & = 3 \\
\alpha_1 & = 0.5 \\
\alpha_2 & = \frac{2\alpha_1}{1+\alpha_1} = 0.67 \\
v_i^n & \text{ is set as } 40 \text{ m/s}^2, \varphi_1(x) = \varphi_2(x) = \varphi_3(x) = \varphi_4(x) = x. \text{ The control parameters in the } y \text{ direction and the } z \text{ direction are the same as in the } x \text{ direction.}
\]

**Case 1** The leader flight in a fix acceleration.

In this case, the leader vehicle flight at a uniform acceleration is

\[
\begin{align*}
V_l & = 240 + 0.8t \\
\theta_l & = 0 \\
\psi_{\psi_l} & = 0
\end{align*}
\]

The trajectory of flight vehicles is plotted in Fig. 5. In Fig. 6, the position track errors \(\|e_i\|\) could decrease, and come to zero rapidly, the convergence time is about 6.5 s. The bounded control input \(v_{fzi}, v_{fyi}\) are also given in Figs. 7–9. It is obvious from these plots that for the given initial state, the control inputs decrease as the values of the tracking errors decrease, and without exceeding their control actuation limits, which indicates that the designed control protocol with actuator saturation is effective. Figs. 10–12 show the origin control input of the followers, and it can be seen from Figs. 13–15 that followers could track states of the leader successfully.
Fig. 7  Bounded control input $v_{fxi}$ in Case 1

Fig. 8  Bounded control input $v_{fyi}$ in Case 1

Fig. 9  Bounded control input $v_{fzi}$ in Case 1

Fig. 10  Origin control input $u_{fxi}$ in Case 1

Fig. 11  Origin control input $u_{fyi}$ in Case 1

Fig. 12  Origin control input $u_{fzi}$ in Case 1

Fig. 13  Velocity $V_i$ in Case 1

Fig. 14  Flight-path angle $\theta_i$ in Case 1
Case 2 The leader flight in a sine law.
In this case, the state of the leader flight is changing continually.

\[
\begin{align*}
V_l &= 240 + 20 \sin(0.2t) \\
\theta_l &= 0.1 \sin(0.1t) \\
\psi_{V_l} &= 0.3 \sin(0.1t)
\end{align*}
\]

For the simulation results of Case 2, clearly, with the leader flight in a sine law, followers should adjust their states to achieve the desired formation configuration. From Fig. 16, it can be seen that the flight vehicles could fly in the expected formation form in the final phase, which guarantees three flight vehicles accomplish the cooperative combat task. It can be seen from Fig. 17 that the position tracking errors come to zero at around 8.5 s. Therefore, the designed formation control method is also effective for the case with a maneuver leader, but it takes a little longer time than Case 1. Figs. 18–20 show that the bounded control inputs of the followers could adjust their states such that the formation can be achieved. However, the bounded control input curves of followers no longer tend to flat, but change continuously with the leader maneuvering flight. The origin control inputs of the followers in Figs. 21–23 also verify it. It can be seen that the velocity $V_i$, the flight-path angle $\theta_i$ and the heading angle $\psi_{V_i}$ could also reach agreement in Case 2 as shown in Figs. 24–26.
From the simulation results, it can be concluded that the designed formation control law with actuator saturation is effective for a maneuver leader, but the tracking time is longer than the instance of the straight flight leader. In addition, the states of the leader change continuously in Case 2, and the MFV system could still maintain the expected formation.

6. Conclusions

In this paper, the finite-time formation control problem subject to actuator saturation has been investigated for MFV systems. For the considered systems, a precise linearization model is obtained by principles of differential geometry for the flight vehicle. A class of saturated finite-time formation control protocols are first proposed based on both relative position and relative velocity measurements under directed communication topologies. The finite-time stability of the formation tracking system is analyzed by the homogeneous theory for network topologies that contain a spanning tree. Finally, simulation results have been shown to illustrate the effectiveness of the proposed method.

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