DYNAMICS IN THE ISOTOPY CLASS OF A
PSEUDO-ANOSOV MAP

FEDERICO RODRIGUEZ HERTZ, JANA RODRIGUEZ HERTZ, AND RAÚL URES

ABSTRACT. Despite its homotopical stability, new relevant dynamics appear in the isotopy class of a pseudo-Anosov homeomorphism. We study these new dynamics by identifying homotopically equivalent orbits, obtaining a more complete description of the topology of the corresponding quotient spaces, and their stable and unstable sets. In particular, we get some insight on how new periodic points appear, among other corollaries. A list of further questions and problems is added at the end of the paper.

1. Introduction

In this paper we consider a pseudo-Anosov homeomorphism $f : M^2 \to M^2$ on a closed surface, and homeomorphisms $g : M^2 \to M^2$ in its isotopy class. The reader not familiar with these concepts may consult [PLP]. Following [H], we shall use Katok’s global shadowing to compare orbits of $f$ with orbits of $g$. According to this definition the $f$-orbit of $x$ is globally shadowed by the $g$-orbit of $y$ if $\sup_{n \in \mathbb{Z}} D(F^n(\tilde{x}), G^n(\tilde{y})) < \infty$, for some adequate lifts $F, G, \tilde{x}, \tilde{y}$ of $f, g, x, y$, respectively, to the universal cover ($D$ is any equivariant metric). For simplicity we denote $(f, x) \sim (g, y)$.

This equivalence relation permitted to state a kind of homotopical stability of $f$ in its isotopy class, in the sense that for all $g \simeq f$ each $f$-orbit $(f, x)$ is globally shadowed by (at least) one $g$-orbit $(g, y_x)$ [H]. That is, the dynamics of $f$ is repeated all over its isotopy class. However, unlike the Anosov case, new dynamics may appear for $g$, which are not globally shadowed by $f$.

In this work we study these new dynamics using the following approach. Given a fixed $g$, we identify all $(g, x) \sim (g, x')$, obtaining a quotient space $M_L(g)$. The map $g$ naturally induces a homeomorphism $g_L$ on $M_L(g)$. Observe that, in general $M_L(g)$ is no longer a surface, though $g_L$ is still expansive.

The core of this paper is based on an observation, due to Fathi [F], of the existence of a “universal torus” $Z$ and a hyperbolic extension $\Phi$ of $f$ in $Z$, for which $M$ is the smallest non trivial invariant sub-continuum. The universal cover of $Z$ is the product of two trees obtained from the stable and unstable foliations, respectively.

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Our first theorem states that all $M_L(g)$ are naturally embedded in $Z$, and that $g_L$ is nothing but $\Phi$ restricted to $M_L(g)$

**Theorem 1.1.** Let $f : M^2 \to M^2$ be a pseudo-Anosov homeomorphism on a closed surface. Then there exists $Z$ and a homeomorphism $\Phi : Z \to Z$, such that:

1. $\Phi : Z \to Z$ is a metrically split hyperbolic homeomorphism.
2. $\Phi$ has the pseudo-orbit tracing property on $Z$. Additionally, the pseudo orbits of any lift of $\Phi$ are uniquely shadowed in the universal cover of $Z$.
3. For all $g : M^2 \to M^2$ homotopic to $f$, there exists a $\Phi$-invariant subset $L(g) \subset Z$ on which $g_L$ is conjugate to $\Phi|_{L(g)}$.

**Remark.** A hyperbolic homeomorphism is to be understood in the sense of Bowen, that is, one having hyperbolic canonical coordinates [B2]. The definition of metrically splitting is, mutatis mutandis, as in Franks ([Fr], p.67).

Theorem 1.1 provides a unified scenario for all $M_L(g)$ and $g_L$, endowed with a hyperbolic structure. From now on, we shall write $L(g)$ and $\Phi_g = \Phi|_{L(g)}$ instead of $M_L(g)$ and $g_L$. We find that:

**Theorem 1.2.** The application $g \mapsto L(g) \in 2^Z$ is continuous in the isotopy class of $f$, $2^Z$ endowed with the Hausdorff topology. The set $\{L(g) : g \simeq f\} \subset 2^Z$ is arcwise connected.

As a Corollary we obtain

**Corollary 1.3.** $\Phi_g$ is never topologically stable. If $g$ and $g'$ are semiconjugate by a semiconjugacy homotopic to the identity then $L(g) = L(g')$.

Corollary 1.3 shows that each $\Phi_g$ has a distinct dynamics. Let us point out that always when a pair $g, g'$ are semiconjugate, a semiconjugacy can be chosen to be homotopic to a symmetry of $f$ (see comments in [Fr]). Although previous statements give some light to the new dynamics appearing in the isotopy class of $f$, they mainly open many questions and problems, in §4 we state some of them.

It is expectable that the study, and specially the local analysis of the sets $L(g)$ will bring information about the dynamics of the $\Phi_g$, and are an interesting topic in its own. Following result describes the local properties of $L(g)$, new periodic points, and local stable and unstable sets.

**Theorem 1.4.** Let $g : M^2 \to M^2$ be a homeomorphism on a closed surface, $g \simeq f$. Then:

1. The topological dimension of $L(g)$ is 2.
2. There exists $\varepsilon > 0$ such that $CW^s_\varepsilon(x)$ and $CW^u_\varepsilon(x)$ are nontrivial dendrites for all $x \in L(g)$, with diameters bounded from below.
(1.3) If \( p \in \text{Per}(\Phi_g) \setminus \text{Per}(\Phi_f) \) then \( p \) is an endpoint of both \( CW^s_\varepsilon(p) \) and \( CW^u_\varepsilon(p) \).

The \( \varepsilon \)-stable component of \( x \), \( CW^s_\varepsilon(x) \), denotes the component of \( x \) in \( W^s_\varepsilon(x) \), the \( \varepsilon \)-local stable set of \( x \). A dendrite is a uniquely arcwise connected curve.

Item 1.2 improves [CS] where, using delicate arguments of plane topology, it was shown arcwise connectedness of \( CW^s_\varepsilon(x) \) for \( g C^0 \)-close to an expansive homeomorphism. It also extends the result over the isotopy class of \( f \).

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2. Global Properties

From now on, \( f : M \to M \) will be a pseudo-Anosov homeomorphism and \( \hat{F} : \tilde{M} \to \tilde{M} \) any lift of \( f \) to the universal covering of \( M \). Associated to \( \hat{F} \) there are two transversal invariant foliations with singularities, \( \hat{F}^u \) and \( \hat{F}^s \), with transverse measures contracted and expanded by \( F \) with rates \( \lambda > 1 \) and \( \lambda^{-1} \), respectively.

We briefly describe some remarkable properties of the construction used by A. Fathi to embed a pseudo-Anosov homeomorphism in a hyperbolic dynamics. The reader interested in the details should consult [F].

(2.1) The set of \( \sigma \)-leaves \( T^\sigma = \tilde{M} / \hat{F}^\sigma \) is a non locally compact, not complete tree, when the distance of two points in \( T^\sigma \) is defined as the minimum over the transverse measures of arcs joining their associated leaves \( (\sigma = s, u) \) [MS].

(2.2) The completion \( \hat{T}^\sigma \) of \( T^\sigma \) is still a non locally compact tree [MS], on which \( \hat{F} \) and the fundamental group \( \Gamma \) of \( M \) naturally act, since the covering transformations of \( \hat{F} \) are isometries of the metrics defined in 2.1.

(2.3) The metric associated to \( \hat{T}^s \) is \( \lambda \)-expanded under the action of \( \hat{F} \), while the metric of \( \hat{T}^u \) is \( \lambda^{-1} \)-contracted.

(2.4) The product action \( \hat{\Phi} \) on \( \hat{Z} = \hat{T}^s \times \hat{T}^u \) is a **metrically split hyperbolic homeomorphism**. \( \hat{Z} = \tilde{Z} / \Gamma \) is a metric space having \( \tilde{Z} \) as universal cover (the product metric is \( \Gamma \)-equivariant). \( \hat{\Phi} \) trivially induces a homeomorphism \( \Phi \) on \( Z \).

(2.5) The set \( \tilde{L}(f) = \{(l^s, l^u) \in \tilde{Z} : l^s \cap l^u \neq \emptyset \} \) is homeomorphic to \( \tilde{M} \). Its projection to \( Z \), \( L(f) \), is \( \Phi \)-invariant and \( \Phi_f = \Phi_{|L(f)} \) is conjugate to \( f \). Moreover, \( L(f) \) is contained in all non trivial \( \Phi \)-invariant closed connected subsets of \( Z \) [F].
Bowen’s proof of the shadowing lemma \[ B1 \] works also in this setting, both for \( \Phi \) and for \( \tilde{\Phi} \), since \( Z \) and \( \tilde{Z} \) are complete. Being \( \tilde{Z} \) metrically split, the fact that \( \tilde{\Phi} \) is a hyperbolic homeomorphism easily implies that every \( \tilde{\Phi} \)-pseudo-orbit is uniquely shadowed.

Let us prove that each \( g_L \) is a restriction of \( \Phi \) to a set homeomorphic to \( M_L(g) \):

In view of \( Z \) \( L(f) \) is a universal cover of \( M \) and \( \tilde{\Phi} \) is a lift of \( f \) when restricted to this set. Take a lift \( G \) of \( g \), equivariantly homotopic to \( \tilde{\Phi}|_{L(f)} \). Then each \( G \)-orbit is a pseudo orbit of \( \tilde{\Phi} \). Shadowing Lemma provides a continuous map \( \tilde{H} : L(f) \to \tilde{Z} \) that is a \( \Gamma \)-invariant semiconjugacy between \( G \) and \( \tilde{\Phi}|_{\text{Im}(\tilde{H})} \). This induces a semiconjugacy \( H : L(f) \to \text{Im}(H) \subset Z \), with the property that \( x' \in H^{-1}(x) \) if and only if \( (g, x) \sim (g, x') \). Hence \( \text{Im}(H) = L(g) \) is homeomorphic to \( M_L(g) \) and \( g_L \) is conjugate to \( \Phi|_{L(g)} \), ending proof of Theorem \[ 1.1 \]

It is easy to prove that semiconjugacies that send \( M \) into \( Z \) vary continuously with respect to homeomorphisms \( g \) of \( M \), in the \( C^0 \) topology. This implies that \( \mathcal{L} = \{ L(g) \subset Z : g \text{ homeomorphism of } M \} \) is an arcwise-connected set in the Hausdorff topology. In particular, no \( L(g) \) is an isolated set of \( \Phi \). This implies they have not the shadowing property, which in this context means they are not topologically stable.

For the proof of Corollary \[ 1.3 \] take a continuous \( h : M \to M \) homotopic to the identity such that \( h \circ g = g' \circ h \). Consider a lift \( \tilde{h} \) of \( h \) to \( L(f) \), and the \( \Gamma \)-invariant semiconjugacies \( \tilde{H}, \tilde{H}' : L(f) \to \tilde{Z} \) obtained in the proof of Theorem \[ 1.3 \] for lifts \( G, G' \) of \( g, g' \) respectively. We claim that \( \tilde{H} = \tilde{H}' \circ \tilde{h} \) for a suitable \( \tilde{h} \). Indeed, as \( \tilde{h} \) can be chosen to be at a bounded distance of the identity, we easily obtain that the distance between \( \tilde{\Phi}^{n} \circ \tilde{H}(x) (= \tilde{H} \circ G^n(x)) \) and \( \tilde{\Phi}^{n} \circ \tilde{H}' \circ \tilde{h}(x) (= \tilde{H}' \circ \tilde{h} \circ G^n(x)) \) is uniformly bounded over \( n \in \mathbb{Z} \). We conclude \( \tilde{H}(x) = \tilde{H}' \circ \tilde{h}(x) \) due to hyperbolicity of \( \tilde{\Phi} \). The fact that \( \tilde{h} \) is onto implies \( \text{Im}(H) = \text{Im}(H') \) and the Corollary follows.

3. Local Properties

This section is devoted to proving Theorem \[ 1.4 \]

(3.1) Item \[ 1.1 \] is in part a consequence of Theorem \[ 1.3 \] since \( Z \), being the product of two trees, has topological dimension 2. On the other hand, \( L(g) \) contains a copy of \( M \) (item \[ 2.5 \] of Section \[ 2 \] see also \[ 11 \]), hence its topological dimension is 2.

(3.2) Item \[ 1.2 \] follows trivially from the fact that \( CW_s^x(x) \) are compact connected subsets of a tree. These sets have uniform size due to \[ 1 \] (\( L(g) \) are locally connected).

(3.3) Let \( p \) be a periodic point of \( \Phi \). Take \( \tilde{p} = (l^s, l^u) \) a lift of \( p \). If \( l^u \in T^s \), there exist a lift \( \tilde{\Phi} \) of \( \Phi \) and a covering transformation \( \tau \) such that
\( \tau \tilde{\Phi}^n(p) = \tilde{p} \), implying \( l_s \) is periodic for \( f : M \to M \). It follows that \( l^s \) contains an \( n \)-periodic \( q \) of \( f \). There exists a lift \( \tilde{q} \) of \( q \) to \( \tilde{L}(f) \) that is fixed by \( \tau \tilde{\Phi}^n \). Since \( \tau \tilde{\Phi}^n_2 \) has a unique fixed point we obtain that \( p = q \in M \). This shows a \( \Phi \)-periodic point is in \( L(f) \) if and only if its corresponding leaves \( l^s_p \) and \( l^u_p \) are, respectively, in the trees \( T^s \) and \( T^u \).

Now, points in the completion \( \tilde{T}^\sigma \) that are not in \( T^\sigma \), are endpoints of the tree \( \tilde{L} \), so we arrive to the desired conclusion.

4. Questions 

The first set of questions concerns the relation between topological entropy, periodic points, and the sets \( L(g) \).

**Question 4.1.** Is the application \( g \mapsto h_{top}(\Phi_g) \) continuous? Is it strictly increasing with respect to inclusions of \( L(g) \)'s?

**Question 4.2.** Does Bowen’s formula apply for \( h_{top}(\Phi_g) \)?

**Question 4.3.** Are periodic points dense in \( L(g) \)?

Let us point out that, in case \( g \) is Axiom A with the strong transversality condition, it is known from [LU] that question 4.3 has an affirmative answer.

Observe that \( L(g) = L(g') \) does not necessarily imply \( g \) is semiconjugate to \( g' \). However; an equivalence relation can be defined for \( g \) in the isotopy class of \( f \), namely: \( g \approx g' \) iff \( L(g) = L(g') \). Is this relation dynamically relevant in some sense? For example:

**Question 4.4.** Is every \( L(g) \) realized by some Axiom A? That is, does every \( \approx \)-class contain an Axiom A?

**Question 4.5.** Is there a minimal representative (e.g. in the sense of its number of periodic points, topological entropy, etc) in each \( \approx \)-class?

**Question 4.6.** Does any \( \Phi \)-invariant subcontinuum of \( Z \) correspond to a \( L(g) \), for some \( g \) in the isotopy class of \( f \)?

We would like also to observe that this question is still unsolved:

**Question 4.7.** Is \( \dim_p L(g) = 2 \) at all points \( p \in L(g) \)? Does \( L(g) \) have a special topological structure, such as Cantor manifold, etc.?

Finally, though it is not in the scope of our paper, it is interesting to note that the existence of two semiconjugate homeomorphisms in the isotopy class of a pseudo-Anosov map \( f \), whose semiconjugacy cannot be chosen homotopic to the identity indicates the presence of symmetries of \( f \).
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