Modeling and Control for Spacecraft Hovering Considering $J_2$–$J_4$ Perturbation

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Abstract. In this paper, the required control with the $J_2$–$J_4$ perturbation is derived for the spacecraft hovering. The $J_2$–$J_4$ perturbation is introduced into spacecraft relative motion model by Gauss's variation equations, and the orbit elements affected by the $J_2$–$J_4$ perturbation are obtained. Combining the nonlinear relative motion model of the spacecraft with the obtained orbit elements, the analytic expression of the hovering control is completely derived. The numerical verification results indicate that the derived control considering the $J_2$–$J_4$ perturbation can ensure the high accuracy for the spacecraft hovering.

Keywords: Space hovering; $J_2$–$J_4$ perturbation; Gauss's variation equations.

1. Introduction

The spacecraft hovering is a special space configuration in which the spacecraft must keep a relatively continuous stationary state to the space target. Undoubtedly, the spacecraft hovering can provide a stable work environment for close space missions, especially the on-orbit serving (OOS), such as urgent rescue, on-orbit injection, operation on uncooperative targets and so on.

Initially, Scheeres [1] held the spacecraft relatively stationary in the exploration of small bodies, and this state is defined as hovering. Then Sawai et al. [2] gave some ideal control methodologies for the uniformly rotating asteroid. Broschart and Scheeres [3] designed a minimum dimension dead-band control for the spacecraft hovering in time-invariant Lagrangian dynamical systems. Taking advantage of the non-canonical Hamiltonian structure, Wang and Xu [4] design a feedback control of hovering based on the coupled orbit-attitude dynamics.

In fact, the key to the realization of the spacecraft hovering is modeling and control. Lin and Li [5] utilized polar coordinate form to deduce a hovering control method in a circular reference orbit. Then Yan [6] gave a control algorithm by using a co-called geometric method. For the elliptical orbit, the similar control algorithm was extended by Zhu and Yan [7]. Further, the required control of the spacecraft hovering was provided according to TH (Tschauner-Hempel) equation in Wang et al.’s paper [8]. Similarly, Zhang et al. [9] discussed the feasibility of uncontrolled hovering in some special region. Further, Dang et al. [10] took the $J_2$ perturbation into the relative hovering control by using the method of the mean orbit element.

This paper introduces the $J_2$–$J_4$ perturbation into the spacecraft hovering to improve the control accuracy. If the $J_2$–$J_4$ perturbation is dealt with by using the mean orbit element method, the specific form of the
orbit element changes will be extremely complicated and cumbersome. In view of these problems, the numerical method will be adopted to calculate the $J_2$~$J_4$ perturbation. By using the Runge-Kutta method, Gauss's variation equations (GVE) describing the orbit elements affected by the $J_2$~$J_4$ perturbation is solved, and the variation of the orbit elements and the corresponding orbit parameters are obtained. Based on these results, the required control of the spacecraft hovering considering $J_2$~$J_4$ perturbation can be derived. The calculation results indicate the feasibility of the derived control for the spacecraft hovering.

2. Nonlinear Dynamics Model for Spacecraft Hovering

In Fig.1, ECI (Earth-centered inertial) frame is denoted by $O$-$XYZ$ and $O$ represents the center of Earth. $M$-$xyz$ is the LVLH (local vertical local horizontal) frame in which the target spacecraft is the center, $x$ is consistent with the orbit radius direction, $z$ is normal to the spacecraft orbital plane, and $y$ forms the right-handed Cartesian frame.

![Figure 1. Diagram of ECI and LVLH coordinates.](image)

The relative dynamics equation for spacecraft hovering can be expressed in the $M$-$xyz$

$$\dot{\rho} = -2w \times \rho - w \times (w \times \rho) - w \times \rho + \frac{\mu}{r_m^3} r_m - \frac{\mu}{r_c^3} r_c + f_c - f_m + f_u$$

where the subscripts $m$ and $c$ represent the target spacecraft and the chaser spacecraft, $r_m$ and $r_c$ represent the orbital radius. $\rho$ is the position vector of the chaser spacecraft, and $\mu$ is the gravitational constant, $w$ and $w$ are the orbital angular velocity and angular acceleration of the target spacecraft. $f_c$ and $f_u$ represent the external perturbation. $f_u$ denotes the control acceleration of the chaser spacecraft.

3. Control of Spacecraft Hovering Considering Nonspherical $J_2$~$J_4$ Perturbation

According to the characteristics of the spacecraft hovering, $\rho$, $\dot{\rho}$ and $\ddot{\rho}$ of the chaser spacecraft are

$$\rho = [x \ y \ z] = \text{const}, \quad \dot{\rho} = \ddot{\rho} = \theta$$

Substituting Eq.(2) into Eq.(1) yields the control to maintain the hovering state of the chaser spacecraft

$$f_u = w \times (w \times \rho) + \dot{w} \times \rho + \frac{\mu}{r_c^3} r_c - \frac{\mu}{r_m^3} r_m + f_m - f_c$$

There are many types of the external perturbations, such as the nonspherical gravitational perturbation, the atmospheric drag perturbation, the solar radiation pressure perturbation, the third-body perturbation and so on. In terms of the magnitude of perturbation, the $J_2$~$J_4$ perturbation is taken into consideration for improving the accuracy. $f_{u_{J_2~J_4}}$ denotes the required control considering $J_2$~$J_4$ perturbation

$$f_{u_{J_2~J_4}} = w_{J_2~J_4} \times (w_{J_2~J_4} \times \rho) + \dot{w}_{J_2~J_4} \times \rho + \frac{\mu}{r_{J_2~J_4}^3} r_{J_2~J_4} - \frac{\mu}{r_{m_{J_2~J_4}}^3} r_{m_{J_2~J_4}} + f_{m_{J_2~J_4}} - f_{J_2~J_4}$$

where $f_{m_{J_2~J_4}}$ and $f_{J_2~J_4}$ are the $J_2$~$J_4$ perturbation. Gauss's variation equations with the $J_2$~$J_4$ perturbation of the target spacecraft is given as follows

$$f_{u_{J_2~J_4}} = w_{J_2~J_4} \times (w_{J_2~J_4} \times \rho) + \dot{w}_{J_2~J_4} \times \rho + \frac{\mu}{r_{J_2~J_4}^3} r_{J_2~J_4} - \frac{\mu}{r_{m_{J_2~J_4}}^3} r_{m_{J_2~J_4}} + f_{m_{J_2~J_4}} - f_{J_2~J_4}$$
where the subscript $J2-4$ represents that the relevant orbital parameters are affected by the $J_2 \sim J_4$ perturbation. And $a_{J2-4}, e_{J2-4}, i_{J2-4}, \Omega_{J2-4}, \omega_{J2-4}$ and $\theta_{J2-4}$ are the orbit semi-major axis, eccentricity, orbit inclination, right ascension of ascending node, argument of perigee, and true anomaly, respectively. The $J_2 \sim J_4$ perturbation of the target spacecraft can be expressed as

$$f_{J2-4} = f_{J2} + f_{J3} + f_{J4}$$  \hspace{1cm} (6)

with

$$f_{J2} = \begin{bmatrix} \frac{3}{2} J \frac{\mu R^4}{r^9} (3 \sin \delta_{J2-4} \sin \iota_{J2-4} - 3) \\ - \frac{1}{2} J \frac{\mu R^4}{r^9} \sin \delta_{J2-4} \sin 2 \iota_{J2-4} \\ - \frac{3}{2} J \frac{\mu R^4}{r^9} \sin 2 \delta_{J2-4} \sin \iota_{J2-4} \end{bmatrix}, \quad f_{J3} = \begin{bmatrix} \frac{3}{2} J \frac{\mu R^4}{r^9} (10 \sin \delta_{J2-4} \sin \iota_{J2-4} - 6 \sin \delta_{J2-4} \sin \iota_{J2-4}) \\ - \frac{3}{2} J \frac{\mu R^4}{r^9} (5 \cos \iota_{J2-4} \sin \delta_{J2-4} + \cos \iota_{J2-4} \sin \delta_{J2-4}) \\ - \frac{3}{2} J \frac{\mu R^4}{r^9} (5 \sin \iota_{J2-4} \cos \delta_{J2-4} \sin \iota_{J2-4} - \cos \iota_{J2-4}) \end{bmatrix}, \quad f_{J4} = \begin{bmatrix} \frac{3}{4} J \frac{\mu R^4}{r^9} (35 \sin \delta_{J2-4} \sin \iota_{J2-4} - 30 \sin \delta_{J2-4} \sin \iota_{J2-4} + 3) \\ - \frac{3}{4} J \frac{\mu R^4}{r^9} (14 \sin \delta_{J2-4} \sin \iota_{J2-4} \cos \iota_{J2-4} - 3 \sin \delta_{J2-4} \sin \iota_{J2-4}) \\ - \frac{3}{4} J \frac{\mu R^4}{r^9} (4 \sin \delta_{J2-4} \sin \iota_{J2-4} \cos \iota_{J2-4} - 3 \sin \iota_{J2-4} \sin \delta_{J2-4}) \end{bmatrix}$$  \hspace{1cm} (7)

where $u_{J2-4} = \omega_{J2-4} + \theta_{J2-4}$ denotes the argument of latitude. After substituting Eq.(6) and Eqs.(7) into Eq.(5), Eq.(5) became the ordinary differential equations which can be solved by using the Runge-Kutta method. In the O-XYZ, the $J_2 \sim J_4$ perturbation for the chaser spacecraft can be given directly

$$f''_{C} = \begin{bmatrix} \frac{3}{2} J \frac{\mu R^4}{r^9} (1 - 5 \frac{Z'}{c^2}) \\ - \frac{5}{2} J \frac{\mu R^4}{r^9} (1 - 5 \frac{Z'}{c^2}) \\ \frac{3}{2} J \frac{\mu R^4}{r^9} (1 - 5 \frac{Z'}{c^2}) \end{bmatrix}, \quad f_{C} = \begin{bmatrix} 5 \frac{\mu R^4}{r^9} (1 - 5 \frac{Z'}{c^2}) + 6 \frac{Z'}{c^2} \\ 5 \frac{\mu R^4}{r^9} (1 - 5 \frac{Z'}{c^2}) + 6 \frac{Z'}{c^2} \\ 5 \frac{\mu R^4}{r^9} (1 - 5 \frac{Z'}{c^2}) + 6 \frac{Z'}{c^2} \end{bmatrix}$$  \hspace{1cm} (8)

where $[X' Y' Z']$ represents the position vector of the chaser spacecraft in the O-XYZ. The superscript $O$ represents the O-XYZ. $R_e$ denotes the Earth radius, and $r_C = \sqrt{(r_{J2-4} + x)^2 + y^2 + z^2}$. The coordinate transformation matrix $S$ from O-XYZ to M-xyz are expressed as

$$S = \begin{bmatrix} \cos \Omega_{J2-4} & \cos u_{J2-4} - \sin \Omega_{J2-4} \cos i_{J2-4} & \sin \Omega_{J2-4} \sin u_{J2-4} + \cos \Omega_{J2-4} \cos i_{J2-4} \\ -\cos \Omega_{J2-4} & \sin i_{J2-4} \sin \Omega_{J2-4} + \cos i_{J2-4} \cos \Omega_{J2-4} & \sin i_{J2-4} \cos \Omega_{J2-4} - \cos i_{J2-4} \sin \Omega_{J2-4} \\ -\sin \Omega_{J2-4} & \cos i_{J2-4} \sin u_{J2-4} + \sin i_{J2-4} \cos u_{J2-4} & \sin i_{J2-4} \cos u_{J2-4} - \cos i_{J2-4} \sin u_{J2-4} \end{bmatrix}$$  \hspace{1cm} (9)

Then $[X' Y' Z']$ can be given.
Substituting Eq.(10) into Eq.(8), $f^\rho_{\psi2-4}$ can be calculated. Then using the transformation matrix $S$ again, the $J_2 \sim J_4$ perturbation in the $M$-$xyz$ is given as

$$f^\rho_{\psi2-4} = S^T f^\rho_{\psi2-4} \quad (11)$$

The expression of $w$ and $\dot{w}$ affected by the $J_2 \sim J_4$ perturbation is given

$$w_{mJ2-4} = \frac{\rho_{mJ2-4}}{\mu p_{wJ2-4}} f_{wJ2-4} \quad (12)$$

where

$$\begin{align*}
\dot{f}_{wJ2-4} &= \dot{f}_{wJ2-4}^o + \dot{f}_{wJ2-4}^o + \dot{f}_{wJ2-4}^o \\
\dot{f}_{wJ2-4}^o &= \frac{\rho_{mJ2-4}}{\mu p_{wJ2-4}} \left[ 2 \frac{\rho_{mJ2-4}}{\mu p_{wJ2-4}} \left( 1 - \cos \theta \right) - 2 \frac{\rho_{mJ2-4}}{\mu p_{wJ2-4}} \sin \theta \right] \\
\dot{f}_{wJ2-4}^o &= \frac{3}{2} \frac{\rho_{mJ2-4}}{\mu p_{wJ2-4}} \left[ 2 \frac{\rho_{mJ2-4}}{\mu p_{wJ2-4}} \left( 1 - \cos \theta \right) - 2 \frac{\rho_{mJ2-4}}{\mu p_{wJ2-4}} \sin \theta \right] \\
\dot{f}_{wJ2-4}^o &= \frac{15}{4} \frac{\rho_{mJ2-4}}{\mu p_{wJ2-4}} \left[ 105 \frac{\sin \theta}{\sin \theta - \cos \theta} \frac{\sin \theta}{\sin \theta - \cos \theta} + \frac{35}{2} \frac{\sin \theta}{\sin \theta - \cos \theta} \frac{\sin \theta}{\sin \theta - \cos \theta} + \frac{15}{2} \frac{\cos \theta}{\sin \theta - \cos \theta} \frac{\cos \theta}{\sin \theta - \cos \theta} \right]
\end{align*}$$

In general, $w_{mJ2-4} \times (w_{mJ2-4} \times \rho)$ and $w_{mJ2-4} \times \rho$ can be expanded as

$$W_{mJ2-4} = w_{mJ2-4} \times (w_{mJ2-4} \times \rho) + w_{mJ2-4} \times \rho = \begin{bmatrix} -w_{mJ2-4} & -w_{mJ2-4} & w_{mJ2-4} & w_{mJ2-4} \\
-w_{mJ2-4} & -w_{mJ2-4} & w_{mJ2-4} & w_{mJ2-4} \\
-w_{mJ2-4} & -w_{mJ2-4} & w_{mJ2-4} & w_{mJ2-4} \\
-w_{mJ2-4} & -w_{mJ2-4} & w_{mJ2-4} & w_{mJ2-4} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad (13)$$

The required control considering the $J_2 \sim J_4$ perturbation can be rewritten as

$$f_{wJ2-4} = W_{mJ2-4} \rho + \frac{\mu}{r_J^3} r_J - \frac{\mu}{r_{mJ2-4}^3} r_{mJ2-4} + f_{mJ2-4} \quad (15)$$

4. Numerical Verification

In Table 1, the initial orbit elements of the target spacecraft are given. Assume that the desirable hovering state of the chaser spacecraft is $\rho_J = [1000 \ 1000 \ 1000]^T \text{m}$ Through the calculation, the variation of $f_{wJ2-4}$ described as Eq.(15) is shown in Fig.2. Then the calculated $f_{wJ2-4}$ will act on the chaser spacecraft to prove its effectiveness for hovering to the target spacecraft. The detailed Verification procedure is: 1)
Transform \( f_{2 \rightarrow 4} \) from the LVLH frame to the ECI frame, and apply \( f_{2 \rightarrow 4} \) on the single dynamics model of the chaser spacecraft in the ECI frame. 2) Integrate the dynamics models of the chaser spacecraft and the target spacecraft including the \( J_2 \sim J_4 \) gravitational perturbation, respectively. 3) Get the actual position vectors of the chaser spacecraft and the target spacecraft in the ECI frame, and transform the actual position vector of the chaser spacecraft from the ECI frame to the LVLH frame.

**Table 1.** Initial orbital element of the target spacecraft

| Orbital elements                  | Values          |
|-----------------------------------|-----------------|
| Orbit semi-major axis (km)        | 7000            |
| Orbit eccentricity                | 0.01            |
| Orbit inclination (deg)           | 45              |
| Orbit right ascension of ascending node (deg) | 0              |
| Orbit argument of perigee (deg)   | 0               |
| Orbit true anomaly (deg)          | 0               |

![Figure 2. Variation of \( f_{2 \rightarrow 4} \)](image)

![Figure 3. Variation of the actual position vectors for the chaser spacecraft.](image)

![Figure 4. Variation of the offset error \( \delta \) under the action of \( f_{2 \rightarrow 4} \)](image)

Fig.3 gives the variation of the actual position vectors. Obviously, the variation amplitudes of the actual position under the action of \( f_{2 \rightarrow 4} \) are very small. The offset distance \( \delta = \rho - \rho_s \) is defined as the offset error. In Fig.4, the magnitude of the offset error \( \delta \) is about \( 10^{-6} \sim 10^{-5} \) m. The offset error \( \delta \) is quite small relative to the desirable hovering state. In fact, the spacecraft is affected by various perturbations in the actual operating environment, among which the nonspherical gravitational perturbation is far greater than other perturbations in the order of magnitude. Therefore, when the nonspherical gravitational \( J_2 \sim J_4 \) perturbation is introduced into the hovering control, the offset error range of hovering state is very small.

![Figure 5. Variation of \( \delta \) in 10 orbital periods](image)
Similarly, the offset error $\delta$ in 10 orbital periods is calculated and display in Fig. 5. During 10 orbital periods, $\delta_x$, $\delta_y$, and $\delta_z$ present a trend of periodic increase, and the magnitude of the offset error $\delta$ is very small relative to the desirable hovering state (about $10^{-5}$ to $10^{-4}$m). The reason for the periodic variation of offset error is that most of the space perturbations are periodic, and the impact on spacecraft will gradually increase with time. Further, the variation of the offset error $\delta$ in 100 orbital periods is shown in Fig 6. It is clear that the change rule of the offset error $\delta$ is consistent with the previous conclusion. And the value of $\delta$ is still very small after 100 periods. The above results show that the accuracy of the derived control $f_{x,y,z}$ considering the $J_2$-$J_4$ perturbation is very high.

5. Conclusion

According to the characteristics of hovering spacecraft, the time-varying parameters of spacecraft relative motion model are calculated by Gaus's variation equations, that is, the orbit parameters under the influence of $J_2$-$J_4$ perturbation are obtained. Furthermore, the relative $J_2$-$J_4$ perturbation between the two spacecraft is obtained by combining the coordinate transformation matrix. Then, the specific hovering control expression is derived by using the obtained orbit parameters and the relative $J_2$-$J_4$ perturbation. The simulation results show that the introduction of $J_2$-$J_4$ perturbation improves the precision of hovering control and ensures the high precision of hovering state in a certain orbit period.

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