ANALYTIC RANKS OF ELLIPTIC CURVES OVER NUMBER FIELDS

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Abstract. Let $E$ be an elliptic curve over $\mathbb{Q}$. Then, we show that the average analytic rank of $E$ over cyclic extensions of degree $l$ over $\mathbb{Q}$ with $l$ a prime not equal to 2, is at most $2 + r_{\mathbb{Q}}(E)$, where $r_{\mathbb{Q}}(E)$ is the analytic rank of the elliptic curve $E$ over $\mathbb{Q}$. This bound is independent of the degree $l$. Also, we also obtain some average analytic rank results over $S_d$-fields.

1. Introduction

Let $E$ be an elliptic curve defined over $\mathbb{Q}$ with conductor $Q_E$. For a number field $F$, let $E(F)$ be the group of $F$-rational points of the elliptic curve $E$. Let $L_F(s, E)$ be the normalized $L$-function of $E$ over the field $F$ so that its central point is $\frac{1}{2}$. We omit the subscript $F$ from $L_F(s, E)$ when $F$ is the field of rational numbers. We are interested in the behavior of the analytic ranks of the $L$-functions $L_F(s, E)$ when $F$ is a cyclic extension of prime degree $l$ over $\mathbb{Q}$. For a prime $l \geq 2$, we denote the family of all cyclic extensions $F$ of degree $l$ over $\mathbb{Q}$ by $C_l$. Then, for a number field $F$ in $C_l$ we have

$$L_F(s, E) = L(s, E) \prod_{\chi} L(s, E \times \chi),$$

where $\chi$ runs over the $(l - 1)$ primitive $l$-th order Dirichlet characters corresponding to the field $F$.

For quadratic fields $F$, the average analytic rank of $L(s, E \times \chi)$ is expected to be $\frac{1}{2}$ regardless of the analytic rank $r_{\mathbb{Q}}(E)$ of $L(s, E)$ by Goldfeld’s conjecture. So we have the following statement equivalent to Goldfeld’s conjecture [12].

Conjecture 1.1 (Goldfeld’s conjecture). Let $E$ be an elliptic curve over $\mathbb{Q}$. Then, the average analytic rank of $E$ over quadratic fields $F$ is $\frac{1}{2} + r_{\mathbb{Q}}(E)$.

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Goldfeld’s conjecture says that a half of the twisted \( L(s, E \times \chi) \) do not vanish at the central point and the other half of them vanish to order 1 at the central point. However, the story seems different for cyclic extensions of prime degree \( l \geq 3 \). David, Fearnley and Kisilevsky [10] conjectured that for a fixed elliptic curve \( E \) and a fixed prime \( l \geq 7 \), there are only finitely many primitive Dirichlet characters \( \chi \) of order \( l \) for which \( L(s, E \times \chi) \) vanishes at the central point \( s = 1/2 \). Also, they conjectured that only a small number of twisted \( L \)-functions \( L(s, E \times \chi) \) vanish for \( l = 3 \) and 5. As a direct consequence of the conjecture, we have the following conjecture.

**Conjecture 1.2.** Let \( l \geq 3 \) be a prime and \( E \) an elliptic curve over \( \mathbb{Q} \). Then, the average analytic rank of \( E(F) \) over the family \( C_l \) is \( r_{\mathbb{Q}}(E) \).

We can relate this conjecture with Diophantine Stability introduced by Mazur and Rubin [10] recently. Let \( K \) be a number field. Suppose \( V \) is an irreducible algebraic variety over \( K \), if \( L \) is a field containing \( K \), we say that \( V \) is diophantine-stable for \( L/K \) if \( V(L) = V(K) \). For a given elliptic curve \( E \) over \( \mathbb{Q} \), if \( r_F(E) = r_{\mathbb{Q}}(E) \) for a number field \( F \), then under the Birch and Swinnerton-Dyer conjecture, the algebraic rank \( r_{\mathbb{Q}}^{\text{alg}}(E) \) of \( E(F) \) is equal to the algebraic rank \( r_{\mathbb{Q}}^{\text{alg}}(E) \) of \( E(\mathbb{Q}) \). By Merel’s uniform bound [20, Theorem 7.5.1] on the size of \( E(F)_{\text{tor}} \), we can see that there are only finitely many number fields \( F \) of degree \( l \) for which \( E(F)_{\text{tor}} > E(\mathbb{Q})_{\text{tor}} \). Even if \( r_{\mathbb{Q}}^{\text{alg}}(E) = r_{\mathbb{Q}}^{\text{alg}}(E) \) and \( E(F)_{\text{tor}} = E(\mathbb{Q})_{\text{tor}} \), there could be a \( F \)-rational point not belonging to \( E(\mathbb{Q}) \) unfortunately. However, these results still give strong conjectural evidence that an elliptic curve \( E \) over \( \mathbb{Q} \) is diophantine-stable for \( L/\mathbb{Q} \) for most cyclic fields \( L \) of prime degree \( l \geq 3 \).

We can understand these two seemingly different phenomena through Katz and Sarnak’s \( n \)-level density conjecture for families of \( L \)-functions. Their philosophy is that the distribution of low-lying zeros of \( L \)-functions in a natural family is governed by one of the five classical matrix groups \( O, SO(\text{even}), SO(\text{odd}), USp, \) and \( U \), which we call the symmetry type of the family. We refer to [18] for the introduction of the conjecture.

From a work of Rubinstein [18], when \( \chi \) is quadratic, we can see that the symmetry type for the family of \( L \)-functions \( L(s, E \times \chi) \) is \( O \) and the average of analytic ranks \( r_F(E) \) is at most \( 2.5 = 2 + 0.5 \). Heath-Brown [13] lowered the bound to 1.5. If Katz and Sarnak’s one-level

\[ [1] \text{For example, consider an elliptic curve } E \text{ which is given by the Weierstrass equation } y^2 = x^3 + 9. \text{ Then } E(\mathbb{Q}) \cong \mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}. \text{ Let } K \text{ be the field by adjoining a root } \alpha \text{ of } x^4 + 8x^3 - 72x + 72 \text{ to } \mathbb{Q}. \text{ Then } E(K) \cong \mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \text{ and } E(K) \text{ contains a new point } P = (\alpha, -\frac{1}{2}\alpha^3 - 3\alpha^2 + 9) \text{ of infinite order such that } 2P = (-2, 1). \]
conjecture is true for a test function with arbitrarily large compact support, the average analytic rank would be \(1/2\), which is Goldfeld’s conjecture.

In [7], the author and Park computed the one-level density for families of \(L\)-functions \(L(s, \pi \times \chi)\) for a cuspidal representation \(\pi\) of \(GL_M(A)\). From it, when \(\chi\) is a primitive character of prime degree \(l > 2\), we can determine that the symmetry type for the family of \(L(s, E \times \chi)\) is \(U\). Under the one-level density conjecture for the symmetry type \(U\), the average analytic rank becomes \(r_Q(E)\).

In this article, we make partial progress toward Conjecture 1.2. Let \(E\) be an elliptic curve defined over \(\mathbb{Q}\), and \(F\) a field in \(\mathbb{C}\). By (1.1), we have

\[
r_F(E) = r_Q(E) + \sum_{\chi} \text{ord}_{s=1/2} L(s, E \times \chi).
\]

Hence the average of \(r_F(E)\) is given by

\[
r_Q(E) + \text{the average of } \text{ord}_{s=1/2} L(s, E \times \chi),
\]

where the average is taken over some subfamily of \(C_l\) which we describe now. We consider primitive characters with conductor \(q_\chi\) coprime to \(Q_E\). The condition \((q_\chi, Q_E) = 1\) determines the (not analytic but ordinary) conductor \(q(E \times \chi)\) of \(L(s, E \times \chi)\) completely, which is \(q_\chi^2 Q_E\) by a work of Barthel and Ramakrishnan [1].

Define

\[
C_{l,Q_E} = \{ F \in C_l | (q_F, Q_E) = 1 \}
\]

where \(q_F\) is the conductor of the field \(F\). Now let \(\omega\) be a non-negative Schwartz class function. We define

\[
\mathcal{C}_{l,Q_E}(X) = \sum_{F \in C_{l,Q_E}} \omega\left(\frac{q_F}{X}\right).
\]

We show that the average analytic rank has a nice uniform upper bound independent of the degree \(l\).

**Theorem 1.3.** Assume GRH[2] Let \(E\) be an elliptic curve over \(\mathbb{Q}\), \(l\) be a prime \(\geq 3\). Then,

\[
\lim_{X \to \infty} \frac{\sum_{F \in C_{l,Q_E}}(X) r_F(E) \omega\left(\frac{q_F}{X}\right)}{\mathcal{C}_{l,Q_E}(X)} \leq 2 + r_Q(E).
\]

[2] We need GRH for the following \(L\)-functions: \(\zeta(s)\), Dirichlet \(L\)-functions with \(\chi \mod 2l\), Hecke \(L\)-functions over \(K = \mathbb{Q}(\zeta_l)\) with characters of order \(l\), \(\zeta_{\chi}(s)\) and \(L(s, E \times \chi)\) for primitive Dirichlet characters of order \(l\).
In Section 5 we also give an upper bound on the average analytic rank over some non-abelian fields. A number field \( F \) of degree \( d \) is an \( S_d \)-field if its normal closure \( \hat{F} \) over \( \mathbb{Q} \) is an \( S_d \) Galois extension. For example, quadratic fields are \( S_2 \)-fields.

For an \( S_d \)-field \( F \) we have

\[
L_F(s, E) = L(s, E)L(s, E \times \rho),
\]

where \( \rho \) is the \((d - 1)\)-dimensional standard representation of the symmetry group \( S_d \).

Let \( S_{d, Q_E} \) be the family of \( S_d \)-fields \( F \) with discriminant \( D_F \) coprime to \( Q_E \) and \( S_d \) be the family of \( S_d \)-fields with no restriction on discriminant. For a positive number \( X \), let

\[
S_{d, Q_E}(X) = \{ F \in S_{d, Q_E} | |D_F| \leq X \},
\]

where \( D_F \) is the discriminant of the field \( F \). In [15], Lemke Oliver and Thorne showed that there is a constant \( c_d > 0 \) such that in \( S_d(X) \), for any \( \epsilon > 0 \), there are \( \gg_{E, \epsilon} X^{c_d - \epsilon} \) \( S_d \)-fields \( F \) with \( r_F(E) > r_Q(E) \).

For \( S_3 \)-fields, using a recent result of Bhargava, Taniguchi and Thorne [3] we have our second main result.

**Theorem 1.4.** Assume GRH. Let \( E \) be an elliptic curve over \( \mathbb{Q} \). The average analytic rank \( r_F(E) \) over \( S_{3, Q_E} \) is bounded by \( 7.5 + r_Q(E) \).

**Remark 1.5.** We also have an analogue of Theorem 1.4 for \( S_4 \)-fields and \( S_5 \)-fields, which are mentioned at the end of Section 5. These poor bounds are due to the poor error term of the counting functions (5.1) for \( S_4 \)-fields and \( S_5 \)-fields. See [6].

In Section 2 we introduce an explicit formula we use which is one of the main tools for one-level density. In Section 3 we recall some preliminaries on primitive Dirichlet characters and lemmas for proof of Theorem 1.3. Sections 4 and 5 are devoted to the proof of Theorems 1.3 and 1.4.

2. **Explicit Formula**

Let \( L(s, f) \) be an entire \( L \)-function with conductor \( q(f) \) and gamma factor \( \gamma(f, s) \) which satisfies the standard functional equation:

\[
\Lambda(s, f) = q(f)^{\frac{s}{2}} \gamma(f, s)L(s, f) = \omega_f \Lambda(1 - s, \overline{f}),
\]

where \( \omega_f \) is the root number of modulus 1. Let \( \Lambda_f(n) = a_f(n)\Lambda(n) \) be the \( n \)-th coefficient of the Dirichlet series \( -L(s, f) = \sum_{n=1}^{\infty} \frac{\Lambda_f(n)}{n^s} \). If the Euler factor of \( L(s, f) \) at the place \( p \) is
\[
\prod_{i=1}^{d} \left(1 - \frac{\alpha_i(p)}{p^s}\right)^{-1}, \text{ then } a_f(p^k) = \sum_{i=1}^{d} \alpha_i(p)^k \text{ and } \Lambda_f(p^k) = a_f(p^k) \log p. \] By [14, Theorem 5.12], we have the following explicit formula.

**Lemma 2.1.** Let \( \phi \) be an even Schwartz class function such that its Fourier transform \( \hat{\phi} \) is compactly supported. Let \( L(s, f) \) be an \( L \)-function as above. For a parameter \( L > 0 \), we have

\[
\sum_{\gamma = \frac{1}{2} + i\gamma} \phi \left( \frac{\gamma \log L}{2\pi} \right) = \hat{\phi}(0) \frac{\log q(f)}{\log L} - \frac{1}{\log L} \sum_n \left( \frac{\Lambda_f(n)}{\sqrt{n}} + \frac{\Lambda_f(n)}{\sqrt{n}} \right) \hat{\phi} \left( \frac{\log n}{\log L} \right) 
\]

\[
+ \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( \frac{\gamma'}{\gamma} (f, 1 + it) + \frac{\gamma'}{\gamma} (f, 1 - it) \right) \phi \left( \frac{t \log L}{2\pi} \right) dt,
\]

where the sum is over non-trivial zeros \( \rho = \frac{1}{2} + i\gamma \) of \( L(s, f) \) with multiplicity.

We can show using [17, Lemma 12.14] that

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} \left( \frac{\gamma'}{\gamma} (f, 1/2 + it) + \frac{\gamma'}{\gamma} (f, 1/2 - it) \right) \phi \left( \frac{t \log L}{2\pi} \right) dt \ll \frac{1}{\log L}.
\]

We assume that \( f \) satisfies the Ramanujan-Petersson conjecture. This assumption is true for the \( L \)-functions we consider. Then we have

\[
\frac{1}{\log L} \sum_{n = p^k, k \geq 3} \left( \frac{\Lambda_f(n)}{\sqrt{n}} + \frac{\Lambda_f(n)}{\sqrt{n}} \right) \hat{\phi} \left( \frac{\log n}{\log L} \right) \ll \frac{1}{\log L},
\]

by absolute convergence of the Dirichlet series. Therefore,

\[
\sum_{\gamma} \phi \left( \frac{\gamma \log L}{2\pi} \right) = \hat{\phi}(0) \frac{\log q(f)}{\log L} - \frac{1}{\log L} \sum_p \left( \frac{\Lambda_f(p)}{\sqrt{p}} + \frac{\Lambda_f(p)}{\sqrt{p}} \right) \hat{\phi} \left( \frac{\log p}{\log L} \right) 
\]

\[
- \frac{1}{\log L} \sum_p \left( \frac{\Lambda_f(p^2)}{p} + \frac{\Lambda_f(p^2)}{2p} \right) \hat{\phi} \left( \frac{2 \log p}{\log L} \right) + O \left( \frac{1}{\log L} \right).
\]

We will use (2.1) for our one-level density computation.

### 3. Cyclic extensions of degree \( l \)

For a prime \( l \geq 3 \), let \( F \) be a cyclic extension of degree \( l \) over \( \mathbb{Q} \). There is an \((l - 1)\)-to-1 correspondence between primitive Dirichlet characters \( \chi \) of order \( l \) and cyclic extensions \( F \) of degree \( l \) over \( \mathbb{Q} \). Thus, counting cyclic extensions of degree \( l \) over \( \mathbb{Q} \) can be reduced to counting primitive Dirichlet characters.

In [7], the author and Park summarize the following well-known results for primitive Dirichlet characters of prime order \( l \).
Proposition 3.1. Assume that $l$ is a prime.

(1) When $l = 2$, $q_\chi$ is the conductor of a primitive quadratic character $\chi$ if and only if $q_\chi = 2^b m$ where $m$ is an odd square-free integer and $b = 0, 2$ or $3$.

(2) When $l > 2$, $q_\chi$ is the conductor of a primitive character of order $l$ if and only if

$$q_\chi = l^b \prod_{q=1 \mod l} q, \quad b = 0 \text{ or } 2.$$ 

(3) Let $q$ be the conductor of a primitive character of order $l$ with $\gcd(q, l) = 1$. Then, the number of primitive characters of order $l$ with conductor $q$ is $(l - 1)\omega(q)$, where $\omega(n)$ is the number of distinct prime divisors of $n$.

Remark 3.2. Since there are $(l - 1)$ primitive Dirichlet character of order $l$ with conductor $l^2$, we can see that the number of primitive characters of order $l$ with conductor $q$ is also $(l - 1)\omega(q)$.

Recall that $E$ is an elliptic curve over $\mathbb{Q}$ with conductor $Q_E$. We want to count the fields in $C_{Q_E}$ by considering the primitive Dirichlet characters of order $l$ with conductor $q_\chi$ coprime to $Q_E$. This can be achieved by the following generating series:

$$\left(1 + \frac{(l - 1)}{l^{2s}}\right)^{1-\delta_{l|Q_E}} \prod_{p \equiv 1 \mod l, p \nmid Q_E} \left(1 + \frac{(l - 1)}{p^s}\right) = \sum_{q=1}^{\infty} \frac{a(q)}{q^s},$$

where $a(q)$ is the number of primitive Dirichlet characters $\chi$ of order $l$ with conductor $q_\chi$ coprime to $Q_E$, which is $(l - 1)\omega(q)$. In [7], we showed that the Dirichlet series $\prod_{p \equiv 1 \mod l} \left(1 + \frac{(l - 1)}{p^s}\right)$ has meromorphic continuation for $\Re(s) > 1/4$ with a simple pole at $s = 1$ and a pole of a finite order at $s = 1/3$. Since the term $\frac{p^s}{p^s + (l - 1)}$ for a prime divisor $p$ of $Q_E$ congruent to $1$ modulo $l$ has poles on the line $Re(s) = \log_p(l - 1) < 1$, the Dirichlet series $\sum_{q=1}^{\infty} a(q)q^{-s}$ is meromorphic with a simple pole at $s = 1$ for $\Re(s) > H_{Q_E}$ for some constant $H_{Q_E}$ with $\frac{1}{3} < H_{Q_E} < 1$.

We count the primitive characters with a weight. Let $\omega$ be a non-negative Schwartz class function. Then, we define

$$W_{Q_E}(X) = \sum_{\chi} \omega\left(\frac{q_\chi}{X}\right) = \sum_q \omega\left(\frac{q}{X}\right) a(q),$$

where the first sum is over all primitive characters of order $l$ with conductors $q_\chi$ coprime to $Q_E$.

Lemma 3.3. Under GRH, for any $\varepsilon > 0$

$$W_{Q_E}(X) = R_{\omega, l, Q_E} X + O_{\omega, l, Q_E, \varepsilon}(X^{H_{Q_E} + \varepsilon})$$

for some constant $R_{\omega, l, Q_E}$.
Proof. The proof is essentially the same as that of [7, Lemma 3.8]. □

Lemma 3.4. Under GRH,

\[ \sum_{\chi} \omega \left( \frac{q\chi}{X} \right) \log q\chi = W_{Q_E}(X) \log X + O_{\omega,l,Q_E}(X). \]

Proof. The proof is essentially the same as that of [7, Lemma 3.5]. □

Lemma 3.5. Under GRH when \( n \) is not a \( l \)-th power,

\[ \sum_{\chi} \omega \left( \frac{q\chi}{X} \right) \chi(n) \ll_{\omega, \epsilon} n^{\epsilon} X^{1/2+\epsilon}. \]

Proof. The proof is essentially the same with that of [7, Lemma 3.9]. □

4. Proof of Theorem 1.3

Let \( f \) be the modular form of weight 2 with level \( Q_E \) which corresponds to the elliptic curve \( E \) and \( \chi \) a primitive Dirichlet character of order \( l \) with conductor \( q\chi \) coprime to \( Q_E \). Then, the conductor \( q(f \times \chi) \) of \( L(s, f \times \chi) = L(s, E \times \chi) \) is exactly \( Qq_{\chi}^2 \) by a work of Barthel and Ramakrishnan [1].

The one-level density for an \( L \)-function \( L(s, f \times \chi) \) is defined to be

\[ D_X(f \times \chi, \phi) = \sum_{\gamma_{f \times \chi}} \phi \left( \frac{\gamma_{f \times \chi}}{2\pi} \right) \log \frac{L}{2\pi}, \]

where \( \gamma_{f \times \chi} \) denote the imaginary part of a generic non-trivial zero and \( L = X^2 \) for a parameter \( X \). [3] Let \( \phi \) be an even Schwartz class function such that its Fourier transform \( \hat{\phi} \) is compactly

[3] In place of \( L \) in the one-level density, there should be a parameter which is of the same order of magnitude as the analytic conductor of \( L(s, f \times \chi) \). In our case, the conductor \( q(f \times \chi) \) of \( L(s, f \times \chi) \) is of the same order of magnitude as the corresponding analytic conductor and \( L = X^2 \) is of the same order of magnitude as the conductor \( q(f \times \chi) \).
supported. Then, by Weil’s explicit formula \(2.4\), we have

\[
D_X(f \times \chi, \phi) = \hat{\phi}(0) \frac{\log c_{f \chi \chi}}{\log L} - \frac{1}{\log L} \sum_p \frac{\log p}{p^{1/2}} \left( a_{f \chi \chi}(p) \hat{\phi} \left( \frac{\log p}{\log L} \right) + a_{f \chi \chi}(p) \hat{\phi} \left( \frac{\log p}{\log L} \right) \right) \\
- \frac{1}{\log L} \sum_p \frac{\log p}{p^{1/2}} \left( a_{f \chi \chi}(p^2) \hat{\phi} \left( \frac{2 \log p}{\log L} \right) + a_{f \chi \chi}(p^2) \hat{\phi} \left( \frac{2 \log p}{\log L} \right) \right) + O \left( \frac{1}{\log L} \right) \\
= \hat{\phi}(0) \frac{2 \log q_X}{\log L} - \frac{1}{\log L} \sum_p \frac{\log p}{p^{1/2}} \left( a_{f \chi \chi}(p) \hat{\phi} \left( \frac{\log p}{\log L} \right) + a_{f \chi \chi}(p) \hat{\phi} \left( \frac{\log p}{\log L} \right) \right) \\
- \frac{1}{\log L} \sum_p \frac{\log p}{p^{1/2}} \left( a_{f \chi \chi}(p^2) \hat{\phi} \left( \frac{2 \log p}{\log L} \right) + a_{f \chi \chi}(p^2) \hat{\phi} \left( \frac{2 \log p}{\log L} \right) \right) + O \left( \frac{1}{\log L} \right).
\]

Note that \(a_{f \chi \chi}(n) = a_f(n) \times \chi(n)\).

Hence, we have

\[
\frac{1}{W_{Q_E}(X)} \sum_{\chi}^* D_X(f \times \chi, \phi) \omega \left( \frac{q_X}{X} \right) = \frac{\hat{\phi}(0)}{W_{Q_E}(X)} \sum_{\chi}^* \frac{2 \omega(q_X/X) \log q_X}{\log L} + S_1 + S_2 + O \left( \frac{1}{\log L} \right),
\]

where

\[
S_1 = -\frac{2}{W_{Q_E}(X) \log L} \sum_p \frac{\log p}{p^{1/2}} a_f(p) \hat{\phi} \left( \frac{\log p}{L} \right) \left( \Re \sum_{\chi}^* \chi(p) \omega \left( \frac{q_X}{X} \right) \right),
\]

\[
S_2 = -\frac{2}{W_{Q_E}(X) \log L} \sum_p \frac{\log p}{p^{1/2}} a_f(p^2) \hat{\phi} \left( \frac{2 \log p}{L} \right) \left( \Re \sum_{\chi}^* \chi(p^2) \omega \left( \frac{q_X}{X} \right) \right).
\]

By Lemma 3.3, the first sum is \(\hat{\phi}(0) + O \left( \frac{1}{\log L} \right)\). Now we assume that \(\hat{\phi}\) is supported in \((-1/2,1/2)\). Then, by Lemma 3.5

\[
S_1 \ll \frac{1}{X \log X} \sum_{p<X^{1-2\epsilon}} p^{\epsilon} \log p \frac{X^{1/2+\epsilon}}{p^{1/2}} \ll \frac{1}{\log X},
\]

\[
S_2 \ll \frac{1}{X \log X} \sum_{p<X^{1/2-\epsilon}} p^{\epsilon} \log p \frac{X^{1/2+\epsilon}}{p} \ll \frac{1}{\log X}.
\]

**Theorem 4.1.** Let \(\phi\) be an even Schwartz class function such that its Fourier \(\hat{\phi}\) is supported in \((-1/2,1/2)\). Then,

\[
\lim_{X \to \infty} \frac{1}{W_{Q_E}(X)} \sum_{\chi}^* \omega \left( \frac{q_X}{X} \right) D_X(f \times \chi, \phi) = \hat{\phi}(0).
\]
Let \( r_{E,\chi} \) denote the analytic rank of \( L(s, f \times \chi) \). If \( \phi \) is a non-negative valued function with \( \phi(0) > 0 \), by a trivial bound

\[
   r_{E,\chi} \phi(0) \leq \sum_{\gamma \chi} \phi \left( \gamma f \times \chi \frac{L}{2\pi} \right),
\]

we have

\[
   \frac{\phi(0)}{W_{Q_E}(X)} \sum_{\chi} r_{E,\chi} \omega \left( \frac{q_{\chi} X}{X} \right) \leq \frac{1}{W_{Q_E}(X)} \sum_{\chi} D_X(f \times \chi, \phi) \omega \left( \frac{q_{\chi} X}{X} \right),
\]

and it implies

\[
   \lim_{X \to \infty} \frac{1}{W_{Q_E}(X)} \sum_{\chi} r_{E,\chi} \omega \left( \frac{q_{\chi} X}{X} \right) = \lim_{X \to \infty} \frac{\sum_{F \in C_{\ell Q_E}(X)} (r_F(E) - r_{\mathbb{Q}}(E)) \omega \left( \frac{q_{\chi} X}{X} \right)}{c_{\ell Q_E}(X)} \leq \frac{\hat{\phi}(0)}{\phi(0)}.
\]

In particular, we take \( \phi(x) = \frac{\sin^2(2\pi x/2)}{(2\pi x)^2} \). Then,

\[
   \hat{\phi}(u) = \frac{1}{2} \left( \frac{1}{2\sigma} - \frac{1}{2} |u| \right) \quad \text{for} \quad |u| \leq \sigma, \quad \phi(0) = \frac{\sigma^2}{4}, \quad \text{and} \quad \hat{\phi}(0) = \frac{\sigma}{4}.
\]

By choosing \( \sigma = 1/2 \), Theorem 1.3 follows.

5. \( S_d \)-fields

The main tool for the one-level density of the family of elliptic curve \( L \)-functions over \( S_d \)-fields is counting number fields with a finite number of local conditions. First, we introduce some notation and known results. Let \( C \) denote a conjugacy class of the group \( S_d \), and \( r_1, r_2, \ldots, r_w \) be the possible splitting types of a prime in a \( S_d \)-field. We say that an \( S_d \)-field \( F \) satisfies the local condition \( S_p, C \) if \( p \) is unramified in \( F \) and the conjugacy class of Frobenius automorphism at \( p \) is \( C \). An \( S_d \)-field \( F \) is said to satisfy the local condition \( S_p, r_i \) if \( p \) is ramified in \( F \) and its splitting type is \( r_i \).

Let \( S = (LC_{p_i})_{i=1}^k \) be a finite set of local conditions. Define the density of the set \( S \) by

\[
   |S_{p, C}| = \frac{|C|}{|S_d|(1 + f(p))}, \quad |S_{p, r_i}| = \frac{c_i(p)}{1 + f(p)}, \quad |S| = \prod_{i=1}^k |LC_{p_i}|
\]

for some positive-valued functions \( f(p) \) and \( c_i(p) \) on the set of primes with \( \sum_i c_i(p) = f(p) \). Note that the functions \( f(p) \) and \( c_i(p) \) depend on the group \( S_d \). For \( S_3 \)-fields, there are two splitting types for a ramified prime in a \( S_d \)-field, which are partial ramification and total ramification and we denote them by \( r_1 \) and \( r_2 \) respectively. Then \( c_1(p) = \frac{1}{p}, \ c_2(p) = \frac{1}{p^2} \), and \( f(p) = \frac{1}{p} + \frac{1}{p^2} \).
Let

$$S_d(X, S) = \{ F \in S_d \mid |D_F| \leq X, F \text{ satisfies } S \}.$$ 

For $d = 3, 4,$ and $5,$ the cardinality of $S_d(X, S)$ can be estimated with a power saving error term

$$|S_d(X, S)| = c_d|S|X + O \left( \left( \prod_{i=1}^k p_i \right)^{\alpha_d} X^{1-\delta_d} \right),$$

for some positive constants $c_d, \alpha_d$ and $0 < \delta_d < 1$ which depends on $S_d.$

Then, we can compute the cardinality of $S_{d,Q_E}(X)$ by forcing all the prime divisors $p$ of $Q_E$ not to ramify. From now on, we focus on the case $d = 3.$ For $S_3$-fields, due to a recent work of Bhargava, Taniguchi and Thorne [3], we have

$$|S_3(X, S)| = c_3|S|X + O_{Q_E} \left( X^{\frac{2}{3}} + X^{\frac{2}{3}+\epsilon} \right).$$

By (5.2), we have

$$|S_{3,Q_E}| = c_{3,Q_E}X + O_{Q_E} \left( X^{\frac{2}{3}} + X^{\frac{2}{3}+\epsilon} \right)$$

where $c_{3,Q_E} = \left( \prod_{q|Q_E} \frac{1}{1+f(p)} \right) c_3.$

We define the one-level density for $L(s, f \times \rho)$ by

$$D_X(f \times \rho, \phi) = \sum_{\gamma_f \times \rho} \phi \left( \gamma_f \times \rho \frac{\log L}{2\pi} \right)$$

where $\gamma_f \times \chi$ denote the imaginary part of a generic non-trivial zero and $L = X^2.$

Once we show that for $\text{supp}(\hat{\phi}) \subset [-\sigma, \sigma]$ with $\sigma < \frac{1}{10}$

$$\frac{1}{|S_{d,Q_E}(X)|} \sum_{F \in S_{d,Q_E}(X)} D_X(f \times \rho, \phi) = \hat{\phi}(0) + \frac{\phi(0)}{2} + O \left( \frac{1}{\log X} \right),$$

we have

$$\frac{\phi(0)}{|S_{d,Q_E}(X)|} \sum_{F \in S_{d,Q_E}(X)} r_{E \times \rho} = \frac{\phi(0)}{|S_{d,Q_E}(X)|} \sum_{F \in S_{d,Q_E}(X)} (r_F(E) - r_Q(E)) \leq \hat{\phi}(0) + \frac{\phi(0)}{2} + O \left( \frac{1}{\log X} \right).$$

By taking

$$\phi(x) = \frac{\sin^2 \left( \frac{2\pi}{2} \sigma x \right)}{(2\pi x)^2}, \quad \hat{\phi}(u) = \frac{1}{2} \left( \frac{1}{2} \sigma - \frac{1}{2} |u| \right) \text{ for } |u| \leq \sigma$$

with $\sigma = \frac{1}{10},$ Theorem 1.4 follows.
Remark 5.1. The one-level density in (5.4) is different from that of Theorem 4.1, which means that the symmetry types for the two families are different. The symmetry type of the former one is $U$ and the symmetry type of the latter one is $O$.

Now, it is left to show (5.4). Since the conductor $q(f \times \rho)$ of $L(s, f \times \rho)$ is $|DF|^2 Q_E$, by the Explicit formula (2.1), we have

$$D_X(f \times \rho, \phi) = \hat{\phi}(0) \frac{2 \log |DF|}{\log L} - \frac{2}{\log L} \sum_p \frac{\log p}{p^{1/2}} a_{f \times \rho}(p) \hat{\phi} \left( \frac{\log p}{\log L} \right)$$

$$- \frac{2}{\log L} \sum_p \frac{\log p}{p} a_{f \times \rho}(p^2) \hat{\phi} \left( \frac{2 \log p}{\log L} \right) + O \left( \frac{1}{\log L} \right).$$

Since

$$a_{f \times \rho}(p) = a_f(p)a_{\rho}(p), \quad a_{f \times \rho}(p^2) = a_f(p^2)a_{\rho}(p^2),$$

we have

$$\frac{1}{|S_{d,Q_E}(X)|} \sum_{F \in S_{d,Q_E}(X)} D_X(f \times \rho, \phi) = \frac{2\hat{\phi}(0)}{|S_{d,Q_E}(X)| \log L} \sum_{F \in S_{d,Q_E}(X)} \log |DF|$$

$$+ S_1 + S_2 + O \left( \frac{1}{\log L} \right),$$

where

$$S_1 = -\frac{2}{|S_{d,Q_E}(X)| \log L} \sum_p \frac{\log p}{p^{1/2}} a_f(p) \hat{\phi} \left( \frac{\log p}{\log L} \right) \left( \sum_{F \in S_{d,Q_E}(X)} a_{\rho}(p) \right),$$

$$S_2 = -\frac{2}{|S_{d,Q_E}(X)| \log L} \sum_p \frac{\log p}{p} a_f(p^2) \hat{\phi} \left( \frac{2 \log p}{\log L} \right) \left( \sum_{F \in S_{d,Q_E}(X)} a_{\rho}(p^2) \right).$$

We can determine $a_{\rho}(p)$ and $a_{\rho}(p^2)$ by the corresponding conjugacy class $C$ and it is summarized in the table below.

| Conjugacy class | $a_{\rho}(p)$ | $a_{\rho}(p^2)$ |
|----------------|--------------|----------------|
| (1)            | 2            | 2              |
| (12)           | 0            | 2              |
| (123)          | -1           | -1             |
Note that for \( a_f(p) = \alpha(p) + \overline{\alpha}(p) = \alpha + \overline{\alpha} \), we have the following relations:

\[
\begin{align*}
  a_f(p^2) &= \alpha^2 + \overline{\alpha}^2 = \alpha^2 + 1 + \overline{\alpha}^2 - 1 = a_{\text{Sym}^2 f}(p) - a_{\Lambda^2 f}(p) \\
  a_{f \times f}(p) &= a_f(p)^2 = \alpha^2 + 2 + \overline{\alpha}^2 = \alpha^2 + 1 + \overline{\alpha}^2 + 1 = a_{\text{Sym}^2 f}(p) + a_{\Lambda^2 f}(p).
\end{align*}
\]

Since \( f \) is self-dual, \( L(s, f \times f) \) has a simple pole at \( s = 1 \) and \( L(s, f, \Lambda^2) = \prod_{q \mid Q_{E}} \left( 1 - \frac{1}{p^s} \right) \zeta(s) \) also has a simple pole at \( s = 1 \), from the relations above, \( L(s, \text{Sym}^2 f) \) is entire. Hence, under GRH we have

\[
\theta_f(x) = \sum_{n \leq x} a_f(p^2) \log p = -x + O \left( x^{\frac{1}{2}} (\log x) (\log(x^3 c_f)) \right) \tag{5.7}
\]

for some constant \( c_f > 0 \) [14, Theorem 5.15].

By partial summation, we have

**Lemma 5.2.**

\[
\sum_{F \in S_{d,Q_{E}}(X)} \log |D_F| = |S_{d,Q_{E}}(X)| \log X + O_{Q_{E}}(X).
\]

By Lemma 5.2 we can estimate the first sum in (5.5):

\[
\frac{2\hat{\phi}(0)}{|S_{d,Q_{E}}(X)| \log L} \sum_{F \in S_{d,Q_{E}}(X)} \log |D_F| = \hat{\phi}(0) + O \left( \frac{1}{\log X} \right).
\]

To control the sum \( S_1 \), we need the following lemma.

**Lemma 5.3.**

\[
\sum_{F \in S_{d,Q_{E}}(X)} a_{\rho}(p) = O_{Q_{E}} \left( X^{\frac{5}{6}} + p^{\frac{4}{3}} X^{\frac{2}{3} + \epsilon} \right). \tag{5.8}
\]

**Proof.** By (5.2) and the table,

\[
\sum_{F \in S_{d,Q_{E}}(X)} a_{\rho}(p) = c_3 X \prod_{q \mid Q_{E}} \frac{1}{1 + f(q)} \left[ \frac{1 \times 2 + 0 \times 3 + (-1) \times 2}{|S_3| (1 + f(p))} \right] + O_{Q_{E}} \left( X^{\frac{5}{6}} + p^{\frac{16}{9}} X^{\frac{7}{9} + \epsilon} \right),
\]

\[
= O_{Q_{E}} \left( X^{\frac{5}{6}} + p^{\frac{8}{3}} X^{\frac{2}{3} + \epsilon} \right).
\]

\( \square \)

Again by (5.2) and the table we can show that
Lemma 5.4.

\begin{equation}
\sum_{F \in S_{3,QE}(X)} a_p(p^2) = c_{3,QE} X + O_{QE} \left( \frac{1}{p} X + X^{\frac{5}{6}} + p^{\frac{3}{2}} X^{\frac{1}{3}+\epsilon} \right).
\end{equation}

Assume that support of \( \hat{\phi} \subset [-\sigma, \sigma] \) for some \( \sigma < \frac{1}{7} \). By Lemma 5.3 and (5.3),

\begin{equation}
S_1 \ll \frac{X^{\frac{5}{6}}}{X \log X} \sum_{p \leq X^{2\sigma}} \frac{\log p}{p^{\frac{5}{2}}} + \frac{X^{\frac{4}{3}+\epsilon}}{X \log X} \sum_{p \leq X^{2\sigma}} \frac{p^{\frac{5}{2}-\frac{1}{2} \log p}}{p} \ll \frac{X^{\frac{5}{6}}}{X \log X} + \frac{X^{\frac{4}{3}+\epsilon}}{X \log X} \ll \frac{1}{\log X}.
\end{equation}

For \( S_2 \), we have

\begin{align*}
S_2 &= -\frac{2}{\left| S_{d,QE}(X) \right| \log L} \sum_{p} \frac{\log p}{p} a_f(p^2) \hat{\phi} \left( \frac{2 \log p}{\log L} \right) \left( c_{3,QE} X + O_{QE} \left( \frac{1}{p} X + X^{\frac{5}{6}} + p^{\frac{3}{2}} X^{\frac{1}{3}+\epsilon} \right) \right) \\
&= -\frac{c_{3,QE} X}{\left| S_{d,QE}(X) \right|} \sum_{p} \frac{2a_f(p^2) p \log p \hat{\phi} \left( \frac{2 \log p}{\log L} \right)}{p \log L} + O_{QE} \left( \frac{1}{\log X} X^{\frac{5}{6}} \log X \right) + X^{\frac{4}{3}+\frac{1}{3}+\epsilon}.
\end{align*}

By summation by parts we have

\begin{equation}
\sum_{p} \hat{\phi} \left( \frac{2 \log p}{\log L} \right) \frac{2a_f(p^2) \log p}{p \log L} = \int_{1}^{\infty} \hat{\phi} \left( \frac{2 \log t}{\log L} \right) \frac{2d\theta_f(t)}{t \log L}.
\end{equation}

Using (5.7), we can show that

\begin{equation}
S_2 = -\frac{c_{3,QE} X}{\left| S_{d,QE}(X) \right|} \sum_{p} \hat{\phi} \left( \frac{2 \log p}{\log L} \right) \frac{2a_f(p^2) \log p}{p \log L} + O_{QE} \left( \frac{1}{\log X} \right) = \frac{1}{2} \hat{\phi}(0) + O_{QE} \left( \frac{1}{\log X} \right).
\end{equation}

By Lemma 5.2, (5.10) and (5.11), we establish the one-level density (5.4) for any \( \sigma < \frac{1}{7} \).

For \( d = 4 \) and \( d = 5 \), we choose \( \sigma = 1/864 \) and \( 1/2400 \) by a work of the author and Kim [6] respectively. However, this gives a poor bound.

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