On a zero-one law for the norm process of transient random walk

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\textbf{Summary.} A zero-one law of Engelbert–Schmidt type is proven for the norm process of a transient random walk. An invariance principle for random walk local times and a limit version of Jeulin’s lemma play key roles.

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1 Introduction

Let $S = (S_n : n \in \mathbb{Z}_{\geq 0})$ be a random walk in $\mathbb{Z}^d$ starting from the origin. Let $\| \cdot \|$ be a norm on $\mathbb{R}^d$ taking integer values on the integer lattice $\mathbb{Z}^d$. The norm $\| \cdot \|$ cannot be the Euclidean norm denoted by $|x| = \sqrt{|x_1|^2 + \cdots + |x_d|^2}$. By the norm process of the random walk $S$, we mean the process $\| S_n \| = (\| S_n \| : n \in \mathbb{Z}_{\geq 0})$. The purpose of the present paper is to study summability of $f(\| S_n \|)$ for a non-negative function $f$ on $\mathbb{Z}$.

Set $X_n = S_n - S_{n-1}$ for $n \in \mathbb{Z}_{\geq 1}$. Then $X_n$’s are independent identically-distributed random vectors taking values in $\mathbb{Z}^d$. We suppose that $E[X_i] = 0$ and $E[(X_i)^2] < \infty$, $i = 1, 2, \ldots, d$. Let $Q$ denote the covariance matrix of $X_1$, i.e., $Q = (E[X_i X_j])_{i,j}$. We introduce the following assumption:

(A0) $Q = \sigma^2 I$ for some constant $\sigma > 0$, where $I$ stands for the identity matrix.

We write

\begin{align}
B(0;r) &= \{ x \in \mathbb{R}^d : \| x \| \leq r \}, \\
\partial B(0;r) &= \{ x \in \mathbb{R}^d : \| x \| = r \}. 
\end{align}

For $k \in \mathbb{Z}_{\geq 0}$, we set

\begin{equation}
N(k) = \sharp(\partial B(0;k) \cap \mathbb{Z}^d) = \sharp \{ x \in \mathbb{Z}^d : \| x \| = k \}.
\end{equation}

We call $B$ a \textit{d-polytope} if $B$ is a bounded convex region in a $d$-dimensional space enclosed by a finite number of $(d-1)$-dimensional hyperplanes. The
part of the polytope $B$ which lies in one of the hyperplanes is called a cell. (See, e.g., [4] for this terminology.) We introduce the following assumptions:

(A1) $\|x\| \in \mathbb{Z}_{\geq 0}$ for any $x \in \mathbb{Z}^d$.

(A2) For each $k \in \mathbb{Z}_{\geq 1}$, the set $B(0; k)$ is a $d$-polytope whose vertices are contained in $\mathbb{Z}^d$. Consequently, its boundary $\partial B(0; k)$ is the union of all cells of the $d$-polytope $B(0; k)$.

(A3) For any $k \in \mathbb{Z}_{\geq 1}$, there exists a finite partition of $\partial B(0; 1)$, which is denoted by $\{U_j^{(k)} : j = 1, \ldots, M(k)\}$, such that the following statements hold:

(i) $M(k) \leq N(k)$ and $M(k)/N(k) \to 1$ as $k \to \infty$;

(ii) each $U_j^{(k)}$ contains at least one point of $\partial B(0; 1) \cap (k^{-1}\mathbb{Z}^d)$;

(iii) the $U_j^{(k)}$'s for $j = 1, \ldots, M(k)$ have a common area;

(iv) $\max_j \max\{\|x - y\| : x, y \in U_j^{(k)}\} \to 0$ as $k \to \infty$.

Note that these assumptions (A0)-(A3) imply that $N(k) \to \infty$ as $k \to \infty$.

Our main theorem is the following:

**Theorem 1.1** Suppose that $d \geq 3$ and that (A0)-(A3) hold. Then, for any non-negative function $f$ on $\mathbb{Z}_{\geq 0}$, the following conditions are equivalent:

(I) $P\left(\sum_{n=1}^{\infty} f(\|S_n\|) < \infty\right) > 0$;

(II) $P\left(\sum_{n=1}^{\infty} f(\|S_n\|) < \infty\right) = 1$;

(III) $E\left[\sum_{n=1}^{\infty} f(\|S_n\|)\right] < \infty$;

(IV) $\sum_{k=1}^{\infty} k^{2-d} N(k) f(k) < \infty$.

Suppose, moreover, that

(A4) There exists $k_0 \in \mathbb{Z}_{\geq 1}$ such that $N(k)$ is non-decreasing in $k \geq k_0$.

Then the above conditions are equivalent to

(V) $\sum_{k=1}^{\infty} k f(k) < \infty$.

We will prove, in Section 5, that (III) and (IV) are equivalent, by virtue of the asymptotic behavior of the Green function due to Spitzer [29] (see Theorem 5.1). We will prove, in Section 6, that (I) implies (IV), where a key role is played by a limit version of Jeulin’s lemma (see Proposition 3.2). Note that (III) trivially implies (II) and that (II) trivially implies (I).
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The equivalence between (I) and (II) may be considered to be a zero-one law of Engelbert–Schmidt type; see Section 2. However, we remark that this equivalence follows also from the Hewitt–Savage zero-one law (see, e.g., [2, Thm.7.36.5]). In fact, the event \( \{ \sum f(\|S_n\|) < \infty \} \) is exchangeable, i.e., invariant under permutation of any finite number of the sequence \((X_n)\).

If \( d = 1 \) or \( 2 \), the random walk \( S \) is recurrent, and hence it is obvious that the conditions (I)-(III) are equivalent to stating that \( f(k) \equiv 0 \). This is why we confine ourselves to the case \( d \geq 3 \), where the random walk \( S \) is transient so that \( \|S_n\| \) diverges as \( n \to \infty \). In the case \( d \geq 3 \), the summability of \( f(\|S_n\|) \) depends upon how rapidly the function \( f(k) \) vanishes as \( k \to \infty \). Theorem 1.1 gives a criterion for the summability of \( f(\|S_n\|) \) in terms of summability of \( kf(k) \).

Consider the max norm
\[
\|x\|^{(d)}_{\infty} = \max_{i=1,\ldots,d} |x^i|, \quad x = (x^1,\ldots,x^d) \in \mathbb{R}^d
\]  
(1.4)
and the \( \ell^1 \)-norm
\[
\|x\|^{(d)}_1 = \sum_{i=1}^d |x^i|, \quad x = (x^1,\ldots,x^d) \in \mathbb{R}^d.
\]  
(1.5)
We will show in Section 4 that these norms satisfy (A1)-(A4). Thus we obtain the following corollary:

**Corollary 1.2** Let \( S \) be a simple random walk of dimension \( d \geq 3 \) and take \( \| \cdot \| \) as the max norm or the \( \ell^1 \)-norm. Then, for any non-negative function \( f \) on \( \mathbb{Z}_{\geq 0} \), the conditions (I)-(V) are equivalent.

The organization of this paper is as follows. In Section 2, we give a brief summary of known results of zero-one laws of Engelbert–Schmidt type. In Section 3, we recall Jeulin’s lemma. We also state and prove its limit version in discrete time. In Section 4, we present some examples of norms which satisfy (A1)-(A4). Sections 5 and 6 are devoted to the proof of Theorem 1.1. In Section 7, we present some results about Jeulin’s lemma obtained by Shiga [28].

2 Zero-one laws of Engelbert–Schmidt type

Let us give a brief summary of known results of zero-one laws concerning finiteness of certain integrals, which we call zero-one laws of Engelbert–Schmidt type.

Let \((B_t: t \geq 0)\) be a one-dimensional Brownian motion starting from the origin. The following theorem, which originates from Shepp–Klauder–Ezawa [27] with motivation in quantum theory, is due to Engelbert–Schmidt
[5, Thm.1] with motivation in construction of a weak solution of a certain stochastic differential equation by means of time-change method.

**Theorem 2.1** ([27],[5]) Let \( f \) be a non-negative Borel function on \( \mathbb{R} \). Then the following conditions are equivalent:

(B1) \( P \left( \int_0^t f(B_s)ds < \infty \text{ for every } t \geq 0 \right) > 0 \);

(B2) \( P \left( \int_0^t f(B_s)ds < \infty \text{ for every } t \geq 0 \right) = 1 \);

(B3) \( f(x) \) is integrable on all compact subsets of \( \mathbb{R} \).

The proof of Theorem 2.1 was based on the formula

\[
\int_0^t f(B_s)ds = \int_\mathbb{R} f(x)L^B_t(x)dx
\]  

(2.1)

where \( L^B_t(x) \) stands for the local time at level \( x \) by time \( t \) (see [15]).

Engelbert–Schmidt [6, Thm.1] proved that a similar result holds for a Bessel process of dimension \( d \geq 2 \) starting from a positive number.

2°). Let \( (R_t : t \geq 0) \) be a Bessel process of dimension \( d > 0 \) starting from the origin, i.e., \( R_t = \sqrt{Z_t} \) where \( Z_t \) is the unique non-negative strong solution of

\[
Z_t = td + 2 \int_0^t \sqrt{Z_s}dB_s.
\]  

(2.2)

The following theorem is due to Pitman–Yor [24, Prop.1] and Xue [30, Prop.2].

**Theorem 2.2** ([24],[30]) Suppose that \( d \geq 2 \). Let \( f \) be a non-negative Borel function on \([0, \infty)\). Then the following conditions are equivalent:

(R1) \( P \left( \int_0^t f(R_s)ds < \infty \text{ for every } t \geq 0 \right) > 0 \);

(R2) \( P \left( \int_0^t f(R_s)ds < \infty \text{ for every } t \geq 0 \right) = 1 \);

(R3) \( f(r) \) is integrable on all compact subsets of \((0, \infty)\) and

(R3a) \( \int_0^c f(r)(\log \frac{1}{r})_+dr < \infty \text{ if } d = 2 \);

(R3b) \( \int_0^c f(r)rdr < \infty \text{ if } d > 2 \)

where \( c \) is an arbitrary positive number.

The proof of Theorem 2.2 was done by applying Jeulin’s lemma (see Theorem 3.1 below) to the total local time, where the assumption of Jeulin’s lemma was assured by the Ray–Knight theorem (see Le Gall [19, pp.299]).

3°). Xue [30, Cor.4] generalized Engelbert–Schmidt [6, Cor. on pp.227] and proved the following theorem.
Theorem 2.3 ([30]) Suppose that $d > 2$. Let $f$ be a non-negative Borel function on $[0, \infty)$. Then the following conditions are equivalent:

(RI) $P \left( \int_0^\infty f(R_t)dt < \infty \right) > 0$;

(RII) $P \left( \int_0^\infty f(R_t)dt < \infty \right) = 1$;

(RIII) $E \left[ \int_0^\infty f(R_t)dt \right] < \infty$;

(RIV) $\int_0^\infty rf(r)dr < \infty$.

The proof of Theorem 2.3 was based on Jeulin’s lemma and the Ray–Knight theorem. Our results (Theorem 1.1 and Corollary 1.2) may be considered to be random walk versions of Theorem 2.3. Note that, in Theorem 2.3, the condition (RIII), which is obviously stronger than (RII), is in fact equivalent to (RII). We remark that, in Theorem 2.3, we consider the perpetual integral $\int_0^\infty f(R_t)dt$ instead of the integrals on compact intervals.

4°). Höhnle–Sturm [13],[14] obtained a zero-one law about the event

$$\left\{ \int_0^t f(X_s)ds < \infty \text{ for every } t \geq 0 \right\}$$

where $(X_t : t \geq 0)$ is a symmetric Markov process which takes values in a Lusin space and which has a strictly positive density. Their proof was based on excessive functions. As an application, they obtained the following theorem ([14, pp.411]).

Theorem 2.4 ([14]) Suppose that $0 < d < 2$. Let $f$ be a non-negative Borel function on $[0, \infty)$. Then the following conditions are equivalent:

(Ri) $P \left( \int_0^t f(R_s)ds < \infty \text{ for every } t \geq 0 \right) > 0$;

(Rii) $P \left( \int_0^t f(R_s)ds < \infty \text{ for every } t \geq 0 \right) = 1$;

(Riii) $f(x)$ is integrable on all compact subsets of $[0, \infty)$ and $\int_0^1 f(x)x^{d-1}dx < \infty$.

See also Cherny [3, Cor.2.1] for another approach.

5°). Engelbert–Senf [7] studied integrability of $\int_0^\infty f(Y_s)ds$ where $(Y_t : t \geq 0)$ is a Brownian motion with constant drift. See Salminen–Yor [26] for a generalization of this direction. See also Khoshnevisan–Salminen–Yor [18] for a generalization of the case where $(Y_t : t \geq 0)$ is a certain one-dimensional diffusion process.
3 Jeulin’s lemma and its limit version in discrete time

3.1 Jeulin’s lemma

Jeulin [16, Lem.3.22] gave quite a general theorem about integrability of a function of a stochastic process. He gave detailed discussions in [17] about his lemma. Among the applications presented in [17], let us focus on the following theorem:

Theorem 3.1 ([16],[17]) Let \((X(t) : 0 < t \leq 1)\) be a non-negative measurable process and \(\varphi\) a positive function on \((0,1]\). Suppose that there exists a random variable \(X\) with

\[
E[X] < \infty \quad \text{and} \quad P(X > 0) = 1
\]  

such that

\[
\frac{X(t)}{\varphi(t)} \overset{\text{law}}{=} X \quad \text{holds for each fixed } 0 < t \leq 1.
\]

Then, for any non-negative Borel measure \(\mu\) on \((0,1]\), the following conditions are equivalent:

\(\text{(JI)}\) \(P\left(\int_0^1 X(t) \mu(dt) < \infty\right) > 0\);

\(\text{(JII)}\) \(P\left(\int_0^1 X(t) \mu(dt) < \infty\right) = 1\);

\(\text{(JIII)}\) \(E\left[\int_0^1 X(t) \mu(dt)\right] < \infty\);

\(\text{(JIV)}\) \(\int_0^1 \varphi(t) \mu(dt) < \infty\).

A good elementary proof of Theorem 3.1 can be found in Xue [30, Lem.2].

For several applications of Jeulin’s lemma (Theorem 3.1), see Yor [31], Pitman–Yor [23], [24], Xue [30], Peccati–Yor [22], Funaki–Hariya–Yor [12],[11], and Fitzsimmons–Yano [9].

We cannot remove the assumption \(E[X] < \infty\) from Theorem 3.1; see Proposition 7.1.

3.2 A limit version of Jeulin’s lemma in discrete time

For our purpose, we would like to replace the assumption (3.2) which requires identity in law by a weaker assumption which requires convergence in law. The following proposition plays a key role in our purpose (see also Corollary 7.3).

Proposition 3.2 Let \((V(k) : k \in \mathbb{Z}_{\geq 1})\) be a non-negative measurable process and \(\Phi\) a positive function on \(\mathbb{Z}_{\geq 1}\). Suppose that there exists a random variable \(X\) with
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\[ P(X > 0) = 1 \]  \hspace{1cm} (3.3)

such that

\[ \frac{V(k)}{\Phi(k)} \xrightarrow{\text{law}} X \quad \text{as} \ k \to \infty. \]  \hspace{1cm} (3.4)

Then, for any non-negative function \( f \) on \( \mathbb{Z}_{\geq 1} \), it holds that

\[ P\left( \sum_{k=1}^{\infty} f(k)V(k) < \infty \right) > 0 \implies \sum_{k=1}^{\infty} f(k)\Phi(k) < \infty. \]  \hspace{1cm} (3.5)

The following proof of Proposition 3.2 is a slight modification of that of [30, Lem.2].

Proof. Suppose that \( P\left( \sum f(k)V(k) < \infty \right) > 0 \). Then there exists a number \( C \) such that the event

\[ B = \left\{ \sum_{k=1}^{\infty} f(k)V(k) \leq C \right\} \]  \hspace{1cm} (3.6)

has positive probability. Since \( P(X \leq 0) = 0 \), there exists a positive number \( u_0 \) such that \( P(X \leq u_0) < P(B)/4 \). By assumption (3.4), we see that there exists \( u_1 \) with \( 0 < u_1 < u_0 \) such that

\[ P(V(k)/\Phi(k) \leq u_1) \xrightarrow{k \to \infty} P(X \leq u_1) < \frac{1}{4} P(B). \]  \hspace{1cm} (3.7)

Then, for some large number \( k_0 \), we have

\[ P(V(k)/\Phi(k) \leq u_1) \leq \frac{1}{2} P(B), \quad k \geq k_0. \]  \hspace{1cm} (3.8)

Now we obtain

\[ C \geq E\left[ 1_B \sum_{k=1}^{\infty} f(k)V(k) \right] \]
\[ = \sum_{k=1}^{\infty} f(k)\Phi(k) E\left[ 1_B \cdot \frac{V(k)}{\Phi(k)} \right] \]
\[ = \sum_{k=1}^{\infty} f(k)\Phi(k) \int_0^{\infty} P(B \cap \{ V(k)/\Phi(k) > u \}) du \]
\[ \geq \sum_{k=k_0}^{\infty} f(k)\Phi(k) \int_{0}^{u_1} [P(B) - P(V(k)/\Phi(k) \leq u)]_+ du \]
\[ \geq \frac{1}{2} P(B)u_1 \sum_{k=k_0}^{\infty} f(k)\Phi(k). \]  \hspace{1cm} (3.13)

Since \( P(B)u_1 > 0 \), we conclude that \( \sum f(k)\Phi(k) < \infty \).
4 Examples of norms

Let us introduce several notations. For an index set $A$ (we shall take $A = \mathbb{Z}_{\geq 0}$ or $\mathbb{Z}^d \setminus \{0\}$ later), we denote $\mathcal{M}(A)$ the set of all non-negative functions on $A$. For three functions $f, g, h \in \mathcal{M}(A)$, we say that

$$f(a) \sim g(a) \quad \text{as} \quad h(a) \to \infty$$

if $f(a)/g(a) \to 1$ as $h(a) \to \infty$. For two functions $f, g \in \mathcal{M}(A)$, we say that

$$f(a) \preceq g(a) \quad \text{for} \quad a \in A$$

if there exist positive constants $c_1, c_2$ such that

$$c_1 f(a) \leq g(a) \leq c_2 f(a) \quad \text{for} \quad a \in A.$$  

For two functionals $F, G$ on $\mathcal{M}(A)$, we say that

$$F(f) \preceq G(f) \quad \text{for} \quad f \in \mathcal{M}(A)$$

if there exist positive constants $c_1, c_2$ such that

$$c_1 F(f) \leq G(f) \leq c_2 F(f) \quad \text{for} \quad f \in \mathcal{M}(A).$$

Now let us present several examples of norms which satisfy (A1)-(A4).

**Example 4.1 (Max norms)** Consider $\|x\|_{\infty}^{(d)} = \max_i |x_i|$. It is obvious that the conditions (A1)-(A3) are satisfied. In fact, the partition of $\partial B(0; 1)$ in (A3) can be obtained by separating $\partial B(0; 1)$ by hyperplanes $\{x \in \mathbb{R}^d : x^i = j/k\}$ for $i = 1, \ldots, d$ and $j = -k, \ldots, k$. Let us study $N(k) = N_{\infty}^{(d)}(k)$ and its asymptotic behavior. For $k \in \mathbb{Z}_{\geq 1}$, we have

$$N_{\infty}^{(d)}(k) = \#\{x \in \mathbb{Z}^d : \|x\| \leq k\} - \#\{x \in \mathbb{Z}^d : \|x\| \leq k - 1\}$$

$$= (2k + 1)^d - (2k - 1)^d.\quad (4.6)$$

Now we obtain

$$N_{\infty}^{(d)}(k) \sim d^2 k^{d-1} \quad \text{as} \quad k \to \infty. \quad (4.7)$$

**Example 4.2 ($\ell^1$-norms)** Consider

$$\|x\|_1^{(d)} = \sum_{i=1}^d |x_i|, \quad x \in \mathbb{R}^d.\quad (4.9)$$

It is obvious that the conditions (A1)-(A3) are satisfied. In this case,

$$N(k) = N_1^{(d)}(k) = \#\{x \in \mathbb{Z}^d : \|x\|_1^{(d)} = k\}$$

$$= (2k + 1)^d.\quad (4.10)$$
satisfies the recursive relation

\[ N_1^{(d)}(k) = \sum_{j=0}^{k} N_1^{(1)}(j) N_1^{(d-1)}(k-j), \quad d \geq 2, \; k \geq 0 \]  

with initial condition

\[ N_1^{(1)}(k) = \begin{cases} 1 & \text{if } k = 0, \\ 2 & \text{if } k \geq 1. \end{cases} \quad (4.12) \]

Since the moment generating function may be computed as

\[ \sum_{k=0}^{\infty} s^k N_1^{(d)}(k) = \left( \frac{1 + s}{1 - s} \right)^d, \quad 0 < s < 1, \]  

we see, by Tauberian theorem (see, e.g., [8, Thm.XIII.5.5]), that

\[ N_1^{(d)}(k) \sim \frac{2^d}{(d-1)!} k^{d-1} \quad \text{as } k \to \infty. \]  

Example 4.3 (Weighted ℓ^1-norms) Consider

\[ \|x\|_{w1}^{(d)} = \sum_{i=1}^{d} |x_i|, \quad x \in \mathbb{R}^d. \]  

The conditions (A1)-(A3) are obviously satisfied.

Now let us discuss the asymptotic behavior of \(N_{w1}^{(d)}(k)\). Note that

\[ N(k) = N_{w1}^{(d)}(k) = \sharp \{x \in \mathbb{Z}^d : \|x\|_{w1}^{(d)} = k \} \]  

satisfies the recursive relation

\[ N_{w1}^{(d)}(k) = \sum_{j \in \mathbb{Z}_{\geq 0} : k - dj \geq 0} N_1^{(1)}(j) N_{w1}^{(d-1)}(k - dj), \quad d \geq 2, \; k \geq 0 \]  

with initial condition \(N_1^{(1)}(k) \equiv N_1^{(1)}(k)\). Then, by induction, we can easily see that

\[ |N_{w1}^{(d)}(k) - a^{(d)} k^{d-1}| \leq b^{(d)} k^{d-2}, \quad k \in \mathbb{Z}_{\geq 1}, \; d \geq 2 \]  

for some positive constants \(a^{(d)}, b^{(d)}\) where \(a^{(d)}\) is defined recursively as

\[ a^{(1)} = 2, \quad a^{(d)} = \frac{2}{d(d-1)} a^{(d-1)} (d \geq 2). \]  

In particular, we see that \(N_{w1}^{(d)}(k) \sim a^{(d)} k^{d-1}\) as \(k \to \infty\). For instance, by easy computations, we obtain
\[ N_{w_1}^{(2)}(k) = \begin{cases} 1 & \text{if } k = 0, \\ 2k & \text{if } k \geq 1 \end{cases} \tag{4.20} \]

and
\[ N_{w_1}^{(3)}(k) = \begin{cases} 1 & \text{if } k = 0, \\ \frac{4}{3}k^2 + 2 & \text{if } k \equiv 0 \pmod{3}, k \neq 0, \\ \frac{2}{3}k^2 + \frac{4}{3} & \text{if } k \equiv 1, 2 \pmod{3}. \end{cases} \tag{4.21} \]

Example 4.4 (Transformation by unimodular matrices) Let \( A \) be a unimodular \( d \times d \) matrix, i.e., \( A \) is a \( d \times d \) matrix whose entries are integers and whose determinant is 1 or \(-1\). Let \( \| \cdot \|_0 \) be a norm on \( \mathbb{R}^d \) satisfying (A1)-(A3). Then the norm \( \| Ax \|_0 \) also satisfies (A1)-(A3). Note that
\[ \# \{ x \in \mathbb{Z}^d : \| x \| = k \} = \# \{ x \in \mathbb{Z}^d : \| x \|_0 = k \}, \quad k \in \mathbb{Z}_{\geq 0}. \tag{4.22} \]
For example, the norm on \( \mathbb{R}^3 \) defined as
\[ \| (x^1, x^2, x^3) \| = |x^1 - x^2| + |x^2 - x^3| + |x^1 - x^2 + x^3| \tag{4.23} \]
satisfies (A1)-(A3).

Remark 4.5 Let us consider the norm \( 2\| x \|_{(d)}^\infty \). Then the conditions (A1)-(A2) are satisfied, but neither of (A3) nor (A4) is; in fact,
\[ N(k) = \begin{cases} N_{(d)}^\infty(k/2) & \text{if } k \text{ is even}, \\ 0 & \text{if } k \text{ is odd}. \end{cases} \tag{4.24} \]

Nevertheless, we see that the conditions (I)-(IV) are equivalent to each other and also to
\[ \sum_{k=1}^\infty kf(2k) < \infty, \tag{4.25} \]
which is strictly weaker than (V) because there is no restriction on the values of \( f(2k + 1) \).

5 Equivalence between (III) and (IV)

Let us introduce the random walk local times:
\[ L_{n}^S(x) = \# \{ m = 1, 2, \ldots, n : S_m = x \}, \quad x \in \mathbb{Z}^d, \tag{5.1} \]
\[ L_{n}^{\|S\|}(k) = \# \{ m = 1, 2, \ldots, n : \| S_m \| = k \}, \quad k \in \mathbb{Z}_{\geq 0}. \tag{5.2} \]
Then, for any non-negative function \( g \) on \( \mathbb{Z}^d \), we have
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\[ \sum_{n=1}^{\infty} g(S_n) = \sum_{x \in \mathbb{Z}^d} g(x)L^S_{\infty}(x). \]  

(5.3)

Taking the expectations of both sides, we have

\[ E \left[ \sum_{n=1}^{\infty} g(S_n) \right] = \sum_{x \in \mathbb{Z}^d} g(x)E \left[ L^S_{\infty}(x) \right]. \]  

(5.4)

It is obvious by definition that

\[ E \left[ L^S_{\infty}(x) \right] = \sum_{n=1}^{\infty} P(S_n = x) = G(0, x) \]  

(5.5)

where \( G(x, y) \) is the Green function given as

\[ G(x, y) = \sum_{n=1}^{\infty} P_x(S_n = y). \]  

(5.6)

Let \( \| \cdot \| \) denote the Euclidean norm of \( \mathbb{R}^d \), i.e., \( |x|^2 = \sum_{i=1}^{d} (x^i)^2 \). We recall the following asymptotic behavior of the Green function:

**Theorem 5.1 ([29])** It holds that

\[ G(0, x) \sim \frac{\Gamma(d/2 - 1)}{2\pi^{d/2}} \det Q^{-1/2}(x, Q^{-1}x)^{1-d/2} \] as \( |x| \to \infty \).

(5.7)

In particular, if \( Q = \sigma^2 I \), then

\[ |x|^{d-2}G(0, x) \to \frac{\Gamma(d/2 - 1)}{2\pi^{d/2}} \sigma^{-2} \] as \( |x| \to \infty \).

(5.8)

We can prove Theorem 5.1 in the same way as in Spitzer [29, P26.1], so we omit the proof.

**Proposition 5.2** It holds that

\[ E \left[ \sum_{n=1}^{\infty} g(S_n) \right] \asymp g(0) + \sum_{x \in \mathbb{Z}^d \setminus \{0\}} g(x)\|x\|^{2-d} \quad \text{for } g \in \mathcal{M}(\mathbb{Z}^d). \]  

(5.9)

**Proof.** Since \( \|x\| \asymp |x| \) for \( x \in \mathbb{Z}^d \), it follows from Theorem 5.1 that

\[ G(0, x) \asymp \|x\|^{2-d} \quad \text{for } x \in \mathbb{Z}^d \setminus \{0\}. \]  

(5.10)

Combining it with (5.4), we obtain the desired result.
Remark 5.3 It is now obvious from Proposition 5.2 that
\[ \sum_{x \in \mathbb{Z}^d} g(x) \|x\|^{2-d} < \infty \] implies \[ P \left( \sum_{n=1}^{\infty} g(S_n) < \infty \right) = 1. \] (5.11)

But we do not know whether the converse is true or not.

The following proposition proves part of Theorem 1.1.

Proposition 5.4 Suppose that the condition (A1) is satisfied. Then it holds that
\[ E \left[ \sum_{n=1}^{\infty} f(\|S_n\|) \right] \asymp f(0) + \sum_{k=1}^{\infty} kf(k) \] for \( f \in \mathcal{M}(\mathbb{Z}_{\geq 0}) \) (5.12)

and, in particular, that (III) and (IV) are equivalent. If, moreover, the condition (A4) is satisfied, then it holds that
\[ E \left[ \sum_{n=1}^{\infty} f(\|S_n\|) \right] \asymp f(0) + \sum_{k=1}^{\infty} k^2f(k) \] for \( f \in \mathcal{M}(\mathbb{Z}_{\geq 0}) \) (5.13)

and, in particular, that (IV) and (V) are equivalent.

Proof. The former half of Proposition 5.4 is immediate from Propositions 5.2 and 5.7 for \( g(x) = f(\|x\|) \). The latter half is immediate from Proposition 5.7 below.

Remark 5.5 Let \( p(x) \) denote the probability that the process visits \( x \) at least once:
\[ p(x) = P(L^{S}_{\infty}(x) \geq 1) = P(T_x < \infty), \quad x \in \mathbb{Z}^d \] (5.14)

where \( T_x = \inf\{ n \geq 1 : S_n = x \} \) is the first hitting time of \( x \). Since \( L^{S}_{\infty}(x) = L^{S}_{\infty}(x)\circ \theta_{T_x} + 1 \) and by translation invariance, we may compute the distribution of the total local time \( L^{S}_{\infty}(x) \) as
\[ P(L^{S}_{\infty}(x) \geq n) = p(x)p(0)^{n-1}, \quad x \in \mathbb{Z}^d, \ n = 1, 2, \ldots \] (5.15)

See [20] for some general discussions for symmetric Markov processes. Note that the Green function \( G(0, x) \) may be expressed as
\[ G(0, x) = E \left[ L^{S}_{\infty}(x) \right] = \sum_{n=1}^{\infty} P(L^{S}_{\infty}(x) \geq n) = \frac{p(x)}{1 - p(0)}. \] (5.16)

Remark 5.6 We do not know any explicit result about the law of the total local time \( L^{\|S\|}_{\infty}(k) \) for the norm process \( \|S\| \).
Proposition 5.7 Let $\| \cdot \|$ be a norm on $\mathbb{R}^d$. Suppose that the condition (A4) is satisfied. Then there exists $k_1 \in \mathbb{Z}_{\geq 1}$ such that $N(k) \asymp k^{d-1}$ for $k \geq k_1$.

Proof. By (4.7), we have

$$\sharp \left\{ x \in \mathbb{Z}^d : \|x\|^{(d)}_{\infty} \leq k \right\} \asymp k^d \quad \text{for } k \in \mathbb{Z}_{\geq 1}. \quad (5.17)$$

Note that $\|x\| \asymp \|x\|^{(d)}_{\infty}$ for $x \in \mathbb{Z}^d$; in fact, any two norms on $\mathbb{R}^d$ are mutually equivalent. This immediately implies that

$$\sum_{j=0}^{k} N(j) = \sharp \left\{ x \in \mathbb{Z}^d : \|x\| \leq k \right\} \asymp k^d \quad \text{for } k \in \mathbb{Z}_{\geq 1}. \quad (5.18)$$

Hence there exist constants $c_1, c_2$ such that

$$c_1 k^d \leq \sum_{j=0}^{k} N(j) \leq c_2 k^d \quad \text{for } k \in \mathbb{Z}_{\geq 1}. \quad (5.19)$$

By the condition (A4), we have

$$kN(k) = \sum_{j=k+1}^{2k} N(k) \leq \frac{2k}{k} \sum_{j=k+1}^{2k} N(j) \leq c_2 (2k)^d \quad \text{for } k \in \mathbb{Z}_{\geq 1}. \quad (5.20)$$

Now we obtain $N(k) \leq c_3 k^{d-1}$ with $c_3 = c_2 2^d$. Again by the condition (A4), we have

$$kN(k) = \sum_{j=1}^{k} N(k) \geq \frac{k}{k} \sum_{j=0}^{k} N(j) \geq c_4 k^d \quad \text{for } k \in \mathbb{Z}_{\geq 1}. \quad (5.21)$$

Now we obtain $N(k) \geq c_1 k^{d-1}$. This completes the proof.

6 Proving that (I) implies (IV)

By the assumption (A2), we may identify each cell of $B(0; r)$ with a subset of $\mathbb{R}^{d-1}$. So we may introduce the area measure $\lambda$ on $\partial B(0; 1)$. We define $\mu(\cdot) = \lambda(\cdot)/\lambda(\partial B(0; 1))$ and call it the uniform measure on $\partial B(0; 1)$.

For $k \in \mathbb{Z}_{\geq 1}$, we define a probability measure on $\mathbb{R}^d$ by

$$\mu_k(A) = \frac{1}{N(k)} \sharp \left\{ x \in k^{-1} \mathbb{Z}^d \cap A : \|x\| = 1 \right\}, \quad A \in \mathcal{B}(\mathbb{R}^d). \quad (6.1)$$

Proposition 6.1 Suppose that (A1)-(A3) are satisfied. Then, as $k \to \infty$, the measure $\mu_k$ converges weakly to $\mu$. 
Proof. Let \( \{U_j^{(k)} : j = 1, \ldots, M(k)\} \) be such as in the assumption (A3). Then we see that \( \mu(U_j^{(k)}) = M(k)^{-1} \) for any \( j \) and any \( k \). For \( j = 1, \ldots, M(k) \), choose \( x_j^{(k)} \in U_j^{(k)} \cap (k^{-1} \mathbb{Z}^d) \). We may choose \( \{x_j^{(k)} : j = M(k) + 1, \ldots, N(k)\} \) so that \( \{x_j^{(k)} : j = 1, \ldots, N(k)\} \) is an enumeration of the points of \( \{x \in k^{-1} \mathbb{Z}^d : \|x\| = 1\} \).

Let \( f : \mathbb{R}^d \to \mathbb{R} \) be a continuous function with compact support. It suffices to prove that

\[
\int_{\mathbb{R}^d} f(x) \mu_k(dx) \xrightarrow{k \to \infty} \int_{\partial B(0;1)} f(x) \mu(dx). \tag{6.2}
\]

Note that

\[
\int_{\mathbb{R}^d} f(x) \mu_k(dx) = \frac{1}{N(k)} \sum_{j=1}^{N(k)} f(x_j^{(k)}). \tag{6.3}
\]

Since \( M(k)/N(k) \to 1 \) as \( k \to \infty \), it suffices to prove that

\[
\frac{1}{M(k)} \sum_{j=1}^{M(k)} f(x_j^{(k)}) \xrightarrow{k \to \infty} \int_{\partial B(0;1)} f(x) \mu(dx). \tag{6.4}
\]

Since \( \partial B(0;1) = \bigcup_j U_j^{(k)} \) and \( \mu(U_j^{(k)}) = M(k)^{-1} \), we obtain

\[
\left| \frac{1}{M(k)} \sum_{j=1}^{M(k)} f(x_j^{(k)}) - \int_{\partial B(0;1)} f(x) \mu(dx) \right| \leq \frac{1}{M(k)} \sum_{j=1}^{M(k)} \int_{U_j^{(k)}} \left| f(x_j^{(k)}) - f(x) \right| \mu(dx) \tag{6.5}
\]

\[
\leq \max_{1 \leq j \leq M(k)} \max_{x, y \in U_j^{(k)}} |f(y) - f(x)|. \tag{6.6}
\]

By uniform continuity of \( f \) and by the assumption (A3), the quantity (6.7) converges to 0 as \( k \to \infty \). Therefore the proof is complete.

Let \( (B_t) \) denote a standard Brownian motion of dimension \( d \geq 3 \) starting from the origin. Set

\[
g(x, y) = \int_0^\infty \frac{ds}{(2\pi s)^{d/2}} \exp \left( -\frac{|x-y|^2}{2s} \right), \quad x, y \in \mathbb{R}^d. \tag{6.8}
\]

For the uniform measure \( \mu \) on \( \partial B(0;1) \), we define

\[
g \mu(x) = \int_{\mathbb{R}^d} g(x, y) \mu(dy), \quad x \in \mathbb{R}^d. \tag{6.9}
\]
Then it is well-known (see [21]; see also [10, Thm.5.2.5]) that there exists a unique positive continuous additive functional $(L_\mu^t)$ such that
\[
g_\mu(\sigma B_t) - g_\mu(\sigma B_0) + L_\mu^t
\]
is a martingale with zero mean. The process $(L_\mu^t)$ is called the local time process on the union of cells $\partial B(0; 1)$ for $(\sigma B_t)$. The relation between the measure $\mu$ and the positive continuous additive functional $(L_\mu^t)$ is called the Revuz correspondence (see [25]).

The following theorem is an invariance principle for the random walk local time of the norm process.

**Theorem 6.2** Suppose that (A0)-(A3) are satisfied. Then it holds that
\[
\frac{L_\infty^{\|S\|}(k)}{k^{2-d}N(k)} \xrightarrow{\text{law}} L_\mu^\infty \quad \text{as } k \to \infty.
\]

**Proof.** Note that
\[
\frac{L_\infty^{\|S\|}(k)}{k^{2-d}N(k)} = k^{d-2} \sum_{n=1}^{\infty} \frac{\mu_k \left( \left\{ \frac{S_n}{k} \right\} \right)}{k}.
\]
Hence we obtain the desired result as an immediate consequence of Proposition 6.1 and Bass–Khoshnevisan [1, Prop.6.3].

Now we are in a position to prove that (I) implies (IV) in Theorem 1.1.

**Proof that (I) implies (IV) in Theorem 1.1.** Let us check that the assumptions of Proposition 3.2 are satisfied for $V(k) = L_\infty^{\|S\|}(k)$, $\Phi(k) = k^{2-d}N(k)$ and $X = L_\mu^\infty$.

By Theorem 6.2, assumption (3.4) is satisfied.

Let us show that $P(L_\mu^\infty \leq 0) = 0$. The first hitting place of the union of cells $\partial B(0; 1)$ for the Brownian motion is almost surely an interior point of some cell of the $\mathcal{d}$-polytope $B(0; 1)$ by assumption (A2). Hence it holds that, starting afresh at the first hitting time, the local time on the union of cells $\partial B(0; 1)$ is locally equal to the local time on the hyperplane which contains the cell. Since the local time at the origin for one-dimensional Brownian motion is positive almost surely at any positive time, we see that $L_\mu^\infty$ is positive almost surely.

Thus we may apply Proposition 3.2 (or Corollary 7.3) and we see that (I) implies (IV). The proof is now complete. ∎

### 7 A remark on Jeulin’s lemma

The results of this section are mainly due to Tokuzo Shiga [28].
7.1 Counterexample to Jeulin’s lemma without $E[X] < \infty$

The following proposition gives a counterexample to Jeulin’s lemma (Theorem 3.1) without $E[X] < \infty$.

**Proposition 7.1 ([28])** There exist a non-negative measurable process $(X(t) : 0 < t \leq 1)$, a positive function $\phi$ on $(0, 1]$, a random variable $X$, and a non-negative Borel measure $\mu$ on $(0, 1]$ such that

\[
E[X] = \infty \quad \text{and} \quad P(X > 0) = 1, \tag{7.1}
\]

\[
\frac{X(t)}{\phi(t)} \xrightarrow{\text{law}} X \quad \text{holds for each fixed } 0 < t \leq 1, \tag{7.2}
\]

\[
\int_0^1 \phi(t)\mu(dt) < \infty \quad \tag{7.3}
\]

but

\[
P \left( \int_\varepsilon^1 X(t)\mu(dt) < \infty \quad (\forall \varepsilon > 0), \quad \int_0^1 X(t)\mu(dt) = \infty \right) = 1. \tag{7.4}
\]

**Proof.** Let $(X(t))$ be an $\alpha$-stable subordinator with $0 < \alpha \leq 1/2$. Then we have (7.1) and (7.2) for $\phi(t) \equiv t^{1/\alpha}$. Set

\[
\mu(dt) = t^{-1-1/\alpha}(\log 1/t)^{-1/\alpha}dt \tag{7.5}
\]

so that $\mu((t, 1])^\alpha \sim C t^{-1}(\log 1/t)^{-1}$ as $t \to 0+$ for some positive constant $C$. Thus we obtain (7.3). Since we have

\[
E \left[ \exp - \int_0^1 X(t)\mu(dt) \right] = \exp - \int_0^1 \mu((t, 1])^\alpha dt = 0, \tag{7.6}
\]

we obtain (7.4).

7.2 A limit version of Jeulin’s lemma

**Theorem 7.2 ([28])** Let $(X(t) : 0 < t \leq 1)$ be a non-negative measurable process, $\phi$ a positive function defined on $(0, 1]$, and $\mu$ a non-negative Borel measure on $(0, 1]$. Suppose that there exists a random variable $X$ with $P(X > 0) > 0$ such that

\[
\frac{X(t)}{\phi(t)} \xrightarrow{\text{law}} X \quad \text{as } t \to 0+. \tag{7.7}
\]

Suppose, moreover, that
Then it holds that
\[ P \left( \int_0^1 X(t) \mu(dt) < \infty \right) = 1 \implies \int_0^1 \varphi(t) \mu(dt) < \infty. \] (7.9)

**Proof.** Suppose that
\[ P \left( \int_0^1 X(t) \mu(dt) < \infty \right) = 1 \] (7.10)
but that \( \int_0^1 \varphi(s) \mu(ds) = \infty \). For each \( \varepsilon > 0 \), we define a probability measure \( \mu_\varepsilon \) by
\[ \mu_\varepsilon(dt) = C_\varepsilon^{-1} \mathbf{1}_{(\varepsilon, 1]}(t) \varphi(t) \mu(dt) \quad \text{with} \quad C_\varepsilon = \int_\varepsilon^1 \varphi(t) \mu(dt) \] (7.11)
where \( C_\varepsilon \) is finite by the assumption (7.8). Then \( C_\varepsilon \to \infty \) and \( \mu_\varepsilon \xrightarrow{d} \delta_0 \) as \( \varepsilon \to 0^+ \), where \( \delta_0 \) stands for the unit point mass at 0. Using Jensen’s inequality and changing the order of integration, we have
\[ E \left[ \exp - C_\varepsilon^{-1} \int_\varepsilon^1 X(t) \mu(dt) \right] = E \left[ \exp - \int_\varepsilon^1 \frac{X(t)}{\varphi(t)} \mu_\varepsilon(dt) \right] \leq \int_\varepsilon^1 E \left[ \exp - \frac{X(t)}{\varphi(t)} \right] \mu_\varepsilon(dt). \] (7.12, 7.13)

Hence it follows from (7.10) and (7.7) that
\[ 1 \leq \lim_{\varepsilon \to 0^+} E \left[ \exp - \frac{X(t)}{\varphi(t)} \right] = E \left[ e^{-X} \right], \] (7.14)
which implies \( P(X = 0) = 1 \). This is a contradiction to the assumption that \( P(X > 0) > 0 \).

From Theorem 7.2, we obtain another version of Jeulin’s lemma in discrete time.

**Corollary 7.3** Let \((V(k) : k \in \mathbb{Z}_{\geq 1})\) be a non-negative measurable process and \( \Phi \) a positive function on \( \mathbb{Z}_{\geq 1} \). Suppose that there exists a random variable \( X \) with
\[ P(X > 0) > 0 \] (7.15)
such that
\[ \frac{V(k)}{\Phi(k)} \xrightarrow{\text{law}} X \quad \text{as} \ k \to \infty. \] (7.16)
Then, for any non-negative function $f$ on $\mathbb{Z}_{\geq 1}$, it holds that
\[
P\left(\sum_{k=1}^{\infty} f(k)V(k) < \infty\right) = 1 \quad \text{implies} \quad \sum_{k=1}^{\infty} f(k)\Phi(k) < \infty. \quad (7.17)
\]

Proof. Take
\[
X(t) = V([1/t]), \quad \varphi(t) = \Phi([1/t]) \quad (7.18)
\]
where $[x]$ stands for the smallest integer which does not exceed $x$ and
\[
\mu = \sum_{k=1}^{\infty} f(k)\delta_{1/k}. \quad (7.19)
\]
Then the desired result is immediate from Theorem 7.2.

Proposition 3.2 and Corollary 7.3 cannot be unified in the following sense:

Proposition 7.4 There exist a non-negative measurable process $(V(k) : k \in \mathbb{Z}_{\geq 1})$, a positive function $\Phi$ on $\mathbb{Z}_{\geq 1}$, a random variable $X$, and a non-negative function $f$ on $\mathbb{Z}_{\geq 1}$ such that
\[
P(X > 0) > 0, \quad \frac{V(k)}{\Phi(k)} \xrightarrow{\text{law}} X \quad \text{as} \quad k \to \infty, \quad (7.20)
\]
and
\[
P\left(\sum_{k=1}^{\infty} f(k)V(k) < \infty\right) > 0 \quad (7.21)
\]
but
\[
\sum_{k=1}^{\infty} f(k)\Phi(k) = \infty. \quad (7.22)
\]

Proof. Let $X$ be such that
\[
P(X = 0) = P(X = 1) = \frac{1}{2} \quad (7.23)
\]
and set $V(k) = X$ for $k \in \mathbb{Z}_{\geq 1}$. Then we have (7.20)-(7.22) for $\Phi(k) \equiv 1$ and $f(k) \equiv 1$.

7.3 A counterexample

We give a counterexample to the converse of (7.17) where the assumptions of Corollary 7.3 are satisfied.
Proposition 7.5 ([28]) There exist a non-negative measurable process \( (V(k) : k \in \mathbb{Z}_{\geq 1}) \), a positive function \( \Phi \) on \( \mathbb{Z}_{\geq 1} \), and a non-negative function \( f \) on \( \mathbb{Z}_{\geq 1} \) such that

\[
\frac{V(k)}{\Phi(k)} \xrightarrow{\text{law}} 1 \quad \text{as } k \to \infty \tag{7.24}
\]

and

\[
\sum_{k=1}^{\infty} f(k)\Phi(k) < \infty. \tag{7.25}
\]

but

\[
P\left( \sum_{k=1}^{\infty} f(k)V(k) = \infty \right) = 1. \tag{7.26}
\]

Proof. Let \( 0 < \alpha < 1/2 \). Let \( (V_0(k)) \) be a sequence of i.i.d. random variables such that

\[
E[e^{-\lambda V_0(k)}] = e^{-\lambda^\alpha}, \quad \lambda > 0, \quad k \in \mathbb{Z}_{\geq 1}. \tag{7.27}
\]

Set \( \Phi(k) \equiv k \) and \( f(k) \equiv k^{-1/\alpha} \). Then (7.25) holds and we have

\[
\frac{V_0(k)}{\Phi(k)} \xrightarrow{\text{law}} 0 \quad \text{as } k \to \infty. \tag{7.28}
\]

Since we have

\[
E\left[ \exp - \sum_{k=1}^{\infty} f(k)V_0(k) \right] = \prod_{k=1}^{\infty} E\left[ e^{-f(k)V_0(k)} \right] = \exp - \sum_{k=1}^{\infty} k^{-1} = 0, \tag{7.29}
\]

we obtain

\[
P\left( \sum_{k=1}^{\infty} f(k)V_0(k) = \infty \right) = 1. \tag{7.30}
\]

Since we may take \( V(k) = k + V_0(k) \), we also obtain (7.24) and (7.26). The proof is now complete.

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