Indecomposable Matrices Defining Plane Cubics

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Kippenhahn’s conjecture on Hermitian pencils (1951)

Let $H, K$ be $n \times n$ complex Hermitian matrices and $F \in \mathbb{C}[x, y, z]$ a homogeneous polynomial such that

$$\det(xH + yK - z\text{Id}) = F(x, y, z).$$

When $F(x, y, z)$ has a repeated factor, then $H$ and $K$ are simultaneously unitarily similar to direct sums: there exists an unitary matrix $U$ and matrices $H_i, K_i \in M_{n_i}(\mathbb{C})$ for some $1 \leq n_i < n$ such that

$$UHU^* = \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix} \quad \text{and} \quad UKU^* = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix}.$$

By Burnside’s theorem on matrix algebras, this is equivalent to $H$ and $K$ not generating the whole $M_n(\mathbb{C})$. 
Known results

- Shapiro (1982) showed that the conjecture holds for $n \leq 5$ and for $n = 6$ in the case that the minimal polynomial is cubic.

- Waterhouse (1984) presented a pair of $6 \times 6$ matrices that generate $M_6(\mathbb{C})$ such that $\det(xH + yK - zId)$ has repeated linear factors, thus disproving the general form of Kippenhahn’s conjecture for $n = 6$.

- Li and Spitkovsky (1998) constructed another class of counterexamples for $n = 6$.

- Laffey (1983) constructed a counterexample for $n = 8$ with quartic minimal polynomial.
The conjecture of Kippenhahn is true for $n = 6$ with cubic minimal polynomial.

Let $H, K$ be $6 \times 6$ complex Hermitian matrices and $F$ a homogeneous polynomial defining a smooth cubic, such that

$$\det(xH + yK - zI) = F(x, y, z)^2.$$ 

Then $H$ and $K$ are simultaneously unitarily similar to direct sums. This means that there exists an unitary matrix $U$ and matrices $H_1, H_2, K_1, K_2 \in M_3(\mathbb{C})$ such that

$$UHU^* = \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix} \quad \text{and} \quad UKU^* = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix}.$$
Weierstrass cubic

Let $C$ be an irreducible curve in $\mathbb{CP}^2$ defined by a polynomial $F(x, y, z)$ of degree 3. Every smooth cubic can be brought by a change of coordinates

$$
\begin{bmatrix}
x \\
y \\
z 
\end{bmatrix} \mapsto P \begin{bmatrix}
x \\
y \\
z 
\end{bmatrix}, \text{ for some } P \in \text{GL}_3(\mathbb{C})
$$

into a Weierstrass form

$$F(x, y, z) = yz^2 - x(x - y)(x - \lambda y) = 0,$$

or equivalently

$$F(x, y, z) = -yz^2 + x^3 + \alpha xy^2 + \beta y^3 = 0,$$

for some $\lambda \neq 0, 1$ and $\alpha, \beta \in \mathbb{C}$.
Consider the following question. For given $C$ find a linear matrix

$$A(x, y, z) = x A_x + y A_y + z A_z$$

such that

$$\det A(x, y, z) = c F(x, y, z)^r,$$

where $A_x, A_y, A_z \in M_{3r}$ and $0 \neq c \in \mathbb{C}$. Here $M_{3r}$ is the algebra of all $3r \times 3r$ matrices over $\mathbb{C}$.

We call $A$ a determinantal representation of $C$ of order $r$.

Determinantal representation $A$ is definite if $A(x_0, y_0, z_0)$ is definite at some point $(x_0, y_0, z_0)$.
Equivalent determinantal representations

Two determinantal representations $A$ and $A'$ are equivalent if there exist $X, Y \in \text{GL}_3(\mathbb{C})$ such that

$$A' = XAY.$$ 

We study:

- self-adjoint representations $A = A^*$ modulo unitary equivalence $Y = X^*$,
- skew-symmetric representations $A = -A^t$ under $Y = X^t$ equivalence.

Obviously, equivalent determinantal representations define the same curve.
Theorem

Consider a linear matrix \( A = xA_x + yA_y + zA_z \) with \( \det A = F(x, y, z)^r \). When \( F \) defines a smooth curve \( C \), the cokernel of \( A \) is a vector bundle of rank \( r \) on \( C \).

This follows from Beauville (2000) using arithmetically Cohen-Macaulay sheaves, or Eisenbud (1980) and Backelin, Herzog, Sanders (2006) using purely algebraic methods for matrix factorizations of polynomials.
The conjecture of Kippenhahn is true for \( n = 6 \) with cubic minimal polynomial.

Let \( H, K \) be \( 6 \times 6 \) complex Hermitian matrices and \( F \) a homogeneous polynomial defining a smooth cubic, such that

\[
\det(xH + yK - zI) = F(x, y, z)^2.
\]

Then \( H \) and \( K \) are simultaneously unitarily similar to direct sums. This means that there exists an unitary matrix \( U \) and matrices \( H_1, H_2, K_1, K_2 \in M_3(\mathbb{C}) \) such that

\[
UHU^* = \begin{bmatrix}
H_1 & 0 \\
0 & H_2
\end{bmatrix}
\quad \text{and} \quad
UKU^* = \begin{bmatrix}
K_1 & 0 \\
0 & K_2
\end{bmatrix}.
\]
Connection with real algebraic geometry

Note that \( F \) defines a **real cubic curve** in \( \mathbb{CP}^2 \) and that \( zId - xH - yK \) is a **definite determinantal representation** of \( F \).

In the terminology of linear matrix inequalities:

- \( F \) is a **real zero polynomial**;
- point \((0, 0)\) lies inside the convex set of points \( \{(x, y) \in \mathbb{R}^2 : Id - xH - yK \geq 0\} \) called **spectrahedron**;
- spectrahedron is bounded by the compact part of the curve.

More constructions of definite determinantal representations of polynomials are due to Netzer, Thom and Quarez (2012 →).
Every smooth real cubic can be brought into a Weierstrass form by a real change of coordinates

\[
\begin{bmatrix}
  x \\
  y \\
  z
\end{bmatrix} \mapsto P \begin{bmatrix}
  x \\
  y \\
  z
\end{bmatrix}, \text{ for some } P \in \text{GL}_3(\mathbb{R}).
\]

In the new coordinates we get

\[
\det(xA_x + yA_y + zA_z) = \left(-yz^2 + x^3 + \alpha xy^2 + \beta y^3\right)^2, \text{ where } \alpha, \beta \in \mathbb{R}.
\]

- $A_x, A_y, A_z$ are real linear combinations of $H, K, \text{Id}$ and therefore Hermitian;
- $z \text{Id} - xH - yK$ is definite, so $A = xA_x + yA_y + zA_z$ is also definite.
The cokernel of $A$ is a rank 2 bundle

When the cokernel is decomposable $\mathcal{L}_1 \oplus \mathcal{L}_2$, then $A$ is equivalent to a block matrix

$$
\begin{bmatrix}
A_1 & 0 \\
0 & A_2
\end{bmatrix},
$$

where $\mathcal{L}_i$ is the cokernel of $A_i$.

In other words,

$$
\det A_i = -yz^2 + x^3 + \alpha xy^2 + \beta y^3,
$$

and both $A_i$ are determinantal representations of order 1.
Each $3 \times 3$ determinantal representation $A_i$ is unitarily equivalent to one of the two self-adjoint forms

$$\pm \begin{pmatrix} x \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} + z \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + y \begin{bmatrix} \alpha + \frac{3}{4} t_i^2 & i s_i & t_i \\ -i s_i & -t_i & 0 \\ t_i & 0 & -1 \end{bmatrix} \end{pmatrix},$$

where $(s_i, t_i) \in \mathbb{R}^2$ satisfy $-s_i^2 = t_i^3 + \alpha t_i + \beta$.

Moreover, definite self-adjoint determinantal representations of $C$ are exactly those corresponding to the points $(s, t)$ in the compact part of $C(\mathbb{R})$. 
Indecomposable vector bundles

Atiyah (1957) classified indecomposable rank $r$ vector bundles on elliptic curves that correspond to $r$-torsion points.

Rank 2 vector bundles are the cokernels of

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 & \frac{3t^2 - 2t(1 + \lambda) - (1 - \lambda)^2}{4} & 0 & \frac{t - 1 - \lambda}{2} \\ 0 & 0 & 0 & 0 & -t & 0 \\ 0 & 0 & \frac{t - 1 - \lambda}{2} & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

for $t$ satisfying $0 = t(t - 1)(t - \lambda)$. 
\textbf{Remark:} $i$ times the above skew-symmetric representation is self-adjoint.

It is easy to check that these three determinantal representations are \textbf{not} definite and can thus not provide a counterexample to Kippenhahn’s conjecture.
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