Dirac quasinormal modes of Chern-Simons and BTZ black holes with torsion

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We study Chern-Simons black holes in $d$-dimensions and we calculate analytically the quasinormal modes of fermionic perturbations. Also, we consider as background the five-dimensional Chern-Simons black hole with torsion and the BTZ black hole with torsion. We have found that the quasinormal modes depend on the highest power of curvature present in the Chern-Simons theory, such as occurs for the quasinormal modes of scalar perturbations. We also show that the effect of the torsion is to modify the real part of the quasinormal frequencies, which modify the oscillation frequency of the field for the five-dimensional case. However, for the BTZ black hole with torsion, the effect is to modify the imaginary part of these frequencies, that is, the relaxation time for the decay of the black hole perturbation. The imaginary part of the quasinormal frequencies is negative which guaranties the stability of these black holes under fermionic field perturbations.

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I. INTRODUCTION

The quasinormal modes (QNMs) and their quasinormal frequencies (QNFs) are an important property of black holes and have a long history, [1–6]. It is known that the presence of event horizons dampens the vibration modes of a matter field that evolves perturbatively in the exterior region. In this way, the system is intrinsically dissipative, i.e., there is no temporary symmetry. In general, the oscillation frequencies are complex therefore, the system is not Hermitian. Nevertheless, the oscillation frequency of these modes is independent of the initial conditions and it only depends on the parameters (mass, charge and angular momentum) and the fundamental constants (Newton constant and cosmological constant) that describe a black hole just like the parameters that define the test field.

The study of the QNMs gives information about the stability of black holes under matter fields that evolves perturbatively in the exterior region of them, without backreacting on the metric, [7–11]. Also, the QNMs determine how fast a thermal state in the boundary theory will reach thermal equilibrium according to the AdS/CFT correspondence [12], where the relaxation time of a thermal state of the boundary thermal theory is proportional to the inverse of the imaginary part of the QNMs of the dual gravity background [13].

In the context of black hole thermodynamics, the QNMs allow the quantum area spectrum of the black hole horizon to be studied, as well as the mass and the entropy spectrum. In this regard, Bekenstein [14] was the first to propose the idea that in quantum gravity the area of black hole horizon is quantized, leading to a discrete spectrum which is evenly spaced. Then, Hod [15] conjectured that the asymptotic QNF is related to the quantized black hole area, by identifying the vibrational frequency with the real part of the QNFs. However, it is not universal for every black hole background. Then, Kunstatter [16] propose that the black hole spectrum can be obtained by imposing the
Bohr-Sommerfeld quantization condition to an adiabatic invariant quantity involving the energy and the vibrational frequency. Furthermore, Maggiore [17] argued that in the large damping limit the identification of the vibrational frequency with the imaginary part of the QNF could lead to the Bekenstein universal bound. Then, the consequences of these proposals were studied in several spacetimes.

Commonly the analysis of the QNMs has been carried out on gravitational theories with a Riemannian geometry. However, the incorporation of torsion in the geometry has acquired great interest (for instance, Poincare Gauge Theory of Gravity, Teleparallel Gravity, f(T)-Gravity and f(R,T)-Gravity). In this sense, the simplest gravitational theory that allows spacetime to have torsion is the Einstein-Cartan theory of gravity [18, 19]. Where, the source of torsion should be the spin of matter fields \(^\text{1}\) and the curvature and torsion represent independent degrees of freedom of the gravitational field. In four dimensions, the motion equations give that the torsion is non-propagating, and null in the absence of sources. However, in higher dimensions the Lovelock theories allow a first order formulation [21] and the equations of motion admit non null torsion solutions even in the absence of sources, in such a way that it becomes a new propagating degree of freedom [22, 23]. Some black holes solutions with torsion in these theories have been studied in [24]. In three dimensions, a theory of gravity that includes torsion is known with the name of Mielke-Baekler theory [25], and this model is itself a Chern-Simons theory which includes (along with the Einstein-Hilbert) the gravitational Chern-Simons term and a translational Chern-Simons term. This model admits as solution a generalization of the BTZ black hole with torsion [26]. Three-dimensional gravity with torsion was also considered in Ref. [27], where the supersymmetric extension in the Chern-Simons formulation was investigated. Also, exact solutions with torsion were analyzed recently in Ref. [28]. Other interesting spacetimes with torsion may be found in [29–33]. In regarding the applications to physics it is well known that the introduction of torsion often induces new physical effects and changes the local degrees of freedom of the theory, for example, if the Einstein-Hilbert Lagrangian is the reduction of a higher dimensional model, as the Chern-Simons theories, one can obtain a non null torsion that propagates [34]. Moreover, gravity with torsion in 2+1 dimensions has been related with the continuum theory of lattice defects in solid physics [35, 36]. On the other hand, the relationship between neutrino oscillation and the torsion was proposed in [37]. Furthermore, it was discussed in [24] along with other observables effects of torsion in high energy physics and in solid state physics.

In the context of the AdS/CFT correspondence there are some works, where the authors have incorporated the torsion in the geometry. In this regard, the holographic currents in the five-dimensional Chern-Simons gravity with torsion has been studied in [38]. The holographic structure of the Mielke-Baekler model has been studied in [33] and the holographic aspects of four dimensional gravity with a negative cosmological constant deformed by the Nieh-Yan torsional topological invariant with a spacetime-dependent coefficient have been examined in [39, 40]. Also, basic aspects of the correspondence have been studied in the framework of 3-dimensional gravity with torsion such as holographic energy-momentum, spin currents and the associated (anomalous) Noether-Ward identities [41].

The particular motivation of this work is to calculate the QNMs for fermionic field perturbations by considering some black holes with torsion, in order to study the stability and the effect of the torsion on Dirac QNMs. Here, we consider as background the BTZ black hole with torsion [24], and the five-dimensional Chern-Simons black hole with torsion, [42]. Also, we consider the d-dimensional Chern-Simons black hole without torsion, in order to establish the effect of the torsion on the Dirac QNMs. Chern-Simons black holes are very interesting static solutions of gravity theories which asymptotically approach spacetimes of constant negative curvature (AdS spacetimes). They can be considered as generalizations of the (2+1)-dimensional black holes in higher-dimensional gravity theories containing higher powers of curvature. The Chern-Simons black holes of spherical topology have the same causal structure as the BTZ black holes, and these solutions have a thermodynamical behavior which is unique among all possible black holes in competing Lanczos-Lovelock theories with the same asymptotics. The specific heat of these black holes is positive therefore, they can always reach thermal equilibrium with their surroundings and are thus stable against thermal fluctuations. Scalar perturbations, mass and area spectrum and greybody factors of Chern-Simons black holes have been studied in [43]. Here, the authors showed that the rate at which a scalar field in the background of a Chern-Simons black hole will decay or the rate at which the boundary thermal theory will reach thermal equilibrium depends on the value of the curvature parameter \(k\). The mass and area spectrum of these black holes have a strong dependence on the topology of the transverse space and are not evenly spaced. On the other hand, at a low frequency limit there is a range of modes which contributes to the absorption cross-section. However, beyond a certain frequency value, the mass of the black hole does not affect the absorption cross-section, [44].

The paper is organized as follows. In Sec. II we give a brief review of the Chern-Simons theory. In Sec. III we calculate the exact QNMs of fermionic perturbations with spherical, hyperbolic and plane topology for Chern-

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1 Generically, the spin of matter fields is considered to be the source of torsion. However, it can emerges from other sources, see for example [25].
Simons black holes in $d$-dimensions. In Sec. IV we calculate the exact QNMs of fermionic perturbations for the five-dimensional Chern Simons black hole with torsion and for the BTZ black hole with torsion. Finally, conclusions are presented in Sec. V.

II. CHERN-SIMONS BLACK HOLES

In 1915, David Hilbert proposed an action based on the metrics and derivatives of this and considered the metric as a unique fundamental field. The Lagrangian given by the Ricci scalar is invariant under general transformations of coordinates and it is the only one that provides second-order equations for the metric. The requirement that the action be stationary for small variations of the metric gives the field equations of Einstein (in vacuum). Then, in 1971, D. Lovelock \cite{23} generalized the problem and found, for an arbitrary dimension, all tensors $A_{\mu\nu}$ such as: $A_{\mu\nu}$ is symmetrical, $A_{\mu\nu}$ is a function of the metric tensor and of its first and second derivatives, and $A_{\mu\nu}$ has null divergence. In this formulation, the field equations for gravitation (in vacuum) are $A_{\mu\nu} = 0$. It is worth mentioning that in four dimensions, the Einstein tensor and the metric are the only solutions, and the field equation is equal to the Einstein equations with constant cosmology. However, in higher dimensions there are tensors with the higher power of the Riemann curvature tensor that satisfies the requirements established, which can be obtained from the Lagrangian of Lanczos-Lovelock, acknowledging a similar proposition made by C. Lanczos in 1938 for $d = 5$, with $d$ being the number of spacetime dimensions. Using differential forms, specifically the vielbein 1-form and spin connection 1-form, which are related to the 2-form curvature and torsion, and demanding that the Lagrangian be: a $d$-form invariant under local Lorentz transformations, a local polynomial of vielbein, the spin connection and its exterior derivative and built not using the Hodge dual. A. Mardones and J. Zanelli \cite{46} found that in the absence of torsion the only possible solution is Lovelock’s excepting of Pontryagin’s densities, which exist in even dimensions and are different from zero only if $d = 4k$, where $k$ is an integer. But, being close forms, they can be written locally as total derivatives, not contributing to the motion equations. The coefficients of Lovelock’s action are arbitrary; however, these can be fixed in such a way that theories have a unique cosmological constant. The Lanczos-Lovelock (LL) action \cite{45} in $d$-dimensions can be written as follow

$$I_k = \kappa \int \sum_{q=0}^{k} \epsilon_q^k L^q,$$

with

$$L^q = \epsilon_{a_1...a_d} R^{a_1a_2}...R^{a_{2q-1}a_{2q}} e^{a_{2q+1}}...e^{a_d},$$

where $e^a$ and $R^{ab}$ stand for the vielbein and the curvature two-form respectively, and $\kappa$ and $l$ are related to the gravitational constant $G_k$ and the cosmological constant $\Lambda$ through

$$\kappa = \frac{1}{2(d-2)!\Omega_{d-2}G_k},$$

$$\Lambda = -\frac{(d-1)(d-2)}{2l^2},$$

and $\alpha_q := \epsilon_q^k$ where $\epsilon_q^k = \frac{\Gamma(q-k)}{d-2q} (\frac{k}{q})$ for $q \leq k$ and vanishes for $q > k$, with $1 \leq k \leq \left\lfloor \frac{d-1}{2} \right\rfloor$ ($\lfloor x \rfloor$ denotes the integer part of $x$) and $\Omega_{d-2}$ corresponds to the volume of a unit $(d-2)$-dimensional sphere. Static black hole-like geometries with spherical topology were found in $\cite{17}$ possessing topologically non-trivial AdS asymptotics. These theories and their corresponding solutions were classified by a $k$ integer, which corresponds to the highest power of curvature in the Lagrangian. If $d - 2k = 1$, the solutions are known as Chern-Simons black holes (for a review on the Chern-Simons theories see $\cite{22}$). These solutions were further generalized to other topologies $\cite{48}$ and can be described in general by a non-trivial transverse spatial section $\sum_{\gamma}$ of $(d-2)$-dimensions labelled by the constant $\gamma = +1, -1, 0$ that represents the curvature of the transverse section, corresponding to a spherical, hyperbolic or plane section, respectively. The solution describing a black hole in a free torsion theory can be written as $\cite{48}$

$$ds^2 = -\left(\gamma + \frac{r^2}{l^2} - \alpha \left(\frac{2G_k \sigma}{d-2k-1}\right)^{\frac{1}{2}}\right)dt^2 + \frac{dr^2}{\left(\gamma + \frac{r^2}{l^2} - \alpha \left(\frac{2G_k \sigma}{d-2k-1}\right)^{\frac{1}{2}}\right)} + r^2 d\sigma_{\gamma}^2,$$
where $\alpha = (\pm 1)^{k+1}$ and the constant $\sigma$ is related to the horizon $r_+$ through

$$\sigma = \frac{r_+^{d-2k-1}}{2G_k} \left( \gamma + \frac{r_+^2}{l^2} \right)^k,$$

and to the mass $M$ by

$$\sigma = \frac{\Omega_{d-2}}{\Sigma_{d-2}} M + \frac{1}{2G_k} \delta_{d-2k, \gamma},$$

where $\Sigma_{d-2}$ denotes the volume of the transverse space. As can be seen in (5), if $d-2k \neq 1$ the $k$ root makes the curvature singularity milder than the corresponding black hole of the same mass. At the exact Chern-Simons limit $d-2k = 1$, the solution has a structure similar to that of like the (2+1)-dimensional BTZ black hole with a string-like singularity.

### III. FERMIONIC QUASINORMAL MODES OF D-DIMENSIONAL CHERN-SIMONS BLACK HOLES

Quasinormal modes of fermionic perturbations are governed by the Dirac equation. We will determine the quasi-normal modes by imposing modes ingoing at the horizon, since nothing can escape from the black hole (classically) and, as the black hole is asymptotically AdS, we will consider that the flux of the field vanishes at infinity. The metric of Chern-Simons theories is

$$ds^2 = -f(r)^2 dt^2 + \frac{1}{f(r)^2} dr^2 + r^2 d\sigma^2,$$

with

$$f(r)^2 = \frac{r^2}{l^2} - \mu,$$

where we have defined

$$\mu = \alpha (2\sigma G_k)^{\frac{1}{2k}} - \gamma,$$

and the horizon is located at

$$r_+ = l \sqrt{\mu}.$$

A minimally coupled fermionic field to curvature in the background of the Chern-Simons black hole in $d-$dimensions is given by the Dirac equation

$$(\gamma^\mu \nabla_\mu + m) \psi = 0,$$

where the covariant derivative is defined as

$$\nabla_\mu = \partial_\mu + \frac{1}{2} \omega_{ab}^{\mu} J_{ab},$$

and the generators of the Lorentz group $J_{ab}$ are

$$J_{ab} = \frac{1}{4} [\gamma_a, \gamma_b].$$

The gamma matrices in curved spacetime $\gamma^\mu$ are defined by

$$\gamma^\mu = e^\mu_a \gamma^a,$$

where $\gamma^a$ are the gamma matrices in flat spacetime. In order to solve the Dirac equation we use the diagonal vielbein

$$e^0 = f(r) \, dt, \quad e^1 = \frac{1}{f(r)} \, dr, \quad e^m = r e^m,$$
where \( \tilde{e}^m \) denotes a vielbein for the base manifold \( \sigma_\gamma \). From the null torsion condition
\[
de^a + \omega^a_b e^b = 0 ,
\]
we obtain the spin connection
\[
\omega^{01} = f'(r) e^0 , \quad \omega^{m1} = f(r) \tilde{e}^m , \quad \omega^{mn} = \tilde{\omega}^{mn} ,
\]
(17)
Now, by means of the change of coordinates \( r = r_+ \cosh \rho \), and using the following representation of the gamma matrices
\[
\gamma^0 = i \sigma^2 \otimes 1 , \quad \gamma^1 = \sigma^1 \otimes 1 , \quad \gamma^m = \sigma^3 \otimes \tilde{\gamma}^m ,
\]
(18)
where \( \sigma^i \) are the Pauli matrices, and \( \tilde{\gamma}^m \) are the Dirac matrices in the base manifold \( \sigma_\gamma \), along with the following ansatz for the fermionic field
\[
\psi = e^{-i \omega t} \left( \frac{\psi_1}{\psi_2} \right) \otimes \varsigma ,
\]
(19)
where \( \varsigma \) is a two component fermion. Also, doing the substitutions
\[
\psi_1 \pm \psi_2 = \sqrt{\frac{\cosh \rho \pm \sinh \rho}{\sinh \rho \cosh \frac{d}{2}}} (\psi_1' \pm \psi_2') ,
\]
(20)
and performing the change of variables \( x = \tanh^2 \rho \), we obtain the following equations
\[
2 (1 - x) x^{1/2} \partial_x \psi_1' + i \left( \frac{\omega l}{\sqrt{\mu}} x^{-1/2} - \frac{i \xi}{\sqrt{\mu}} x^{1/2} \right) \psi_1' + \left( i \left( \frac{\omega l}{\sqrt{\mu}} - \frac{i \xi}{\sqrt{\mu}} \right) + \frac{1}{2} + ml \right) \psi_2' = 0 ,
\]
\[
2 (1 - x) x^{1/2} \partial_x \psi_2' - i \left( \frac{\omega l}{\sqrt{\mu}} x^{-1/2} - \frac{i \xi}{\sqrt{\mu}} x^{1/2} \right) \psi_2' + \left( -i \left( \frac{\omega l}{\sqrt{\mu}} - \frac{i \xi}{\sqrt{\mu}} \right) + \frac{1}{2} + ml \right) \psi_1' = 0 ,
\]
(21)
where \( i \xi \) is the eigenvalue of the Dirac operator of the base manifold \( \sigma_\gamma \). By decoupling the system of equations and using
\[
\psi_1' = x^\alpha (1 - x)^\beta F(x) ,
\]
(22)
with
\[
\alpha = -\frac{i \omega l}{2 \sqrt{\mu}} ,
\]
(23)
\[
\beta = -\frac{1}{2} \left( \frac{1}{2} + ml \right) ,
\]
(24)
we obtain the following equation for \( F(x) \)
\[
x (1 - x) F''(x) + (c - (1 + a + b) x) F'(x) - ab F(x) = 0 ,
\]
(25)
whose solution is given by
\[
\psi_1' = C_1 x^\alpha (1 - x)^\beta \, _2 F_1(a, b, c, x) + C_2 x^{1/2 - \alpha} (1 - x)^\beta \, _2 F_1(a - c + 1, b - c + 1, 2 - c, x) ,
\]
(26)
which has three regular singular point at \( x = 0, x = 1 \) and \( x = \infty \). Here, \( _2 F_1(a, b, c, x) \) is a hypergeometric function and \( C_1, C_2 \) are constants and
\[
a = \frac{1}{2} + \alpha + \beta - \frac{i \xi}{2 \sqrt{\mu}} ,
\]
(27)
\[
b = \alpha + \beta + \frac{i \xi}{2 \sqrt{\mu}} ,
\]
(28)
\[ c = \frac{1}{2} + 2\alpha . \]  

(30)

Now, imposing boundary conditions at the horizon, i.e., that there is only ingoing modes, implies that \( C_2 = 0 \). Thus, the solution can be written as

\[ \psi'_1 = C_1 x^\alpha (1-x)^\beta 2F_1 (a, b, c, x) , \]  

(31)

and by using the integrating factor \( e^{-\int \frac{\omega l}{\sqrt{\mu}} \left( \frac{x^\alpha}{\sqrt{\mu}} + \frac{x}{\sqrt{\mu}} \right) dx} \), in Eq. (22), we get the solution

\[ \psi'_2 = -\frac{C_1}{2} \left( -i \left( \frac{\omega l}{\sqrt{\mu}} - \frac{\xi}{\sqrt{\mu}} \right) + \frac{1}{2} + ml \right) x^{-\alpha} (1-x)^{\alpha+\gamma} \int x^{\alpha-1} (1-x')^{\alpha-c-1} 2F_1 (a, b, c, x') dx' . \]  

(32)

So, if we consider the relation

\[ \int x^{\alpha-1} (1-x)^{\alpha-c-1} 2F_1 (a, b, c, x) dx = (1-x)^{\alpha-c} x^{c} 2F_1 (a, b + 1, c + 1, x) , \]  

(33)

and that

\[ -\frac{1}{2} \left( -i \left( \frac{\omega l}{\sqrt{\mu}} - \frac{\xi}{\sqrt{\mu}} \right) + \frac{1}{2} + ml \right) = a - c , \]  

(34)

\( \psi'_2 \) can be rewritten as

\[ \psi'_2 = \frac{a - c}{c} C_1 x^{a+1/2} (1-x)^{\beta} 2F_1 (a, b + 1, c + 1, x) . \]  

(35)

In this manner, the flux

\[ \mathcal{F} = \sqrt{-g} \psi \psi' \]  

where, \( \gamma^r = e^r_1 \gamma^1 \), \( \bar{\psi} = \psi^\dagger \gamma^0 \), \( \sqrt{-g} = (l \sqrt{\mu})^{d-2} (1-x)^{(d-2)/2} \), and \( e^r_1 \propto \frac{x^{1/2}}{(1-x)^{1/2}} \), is given by

\[ \mathcal{F} \propto x^{d/2} (1-x)^{(-d+1)/2} \left( |\psi_1|^2 - |\psi_2|^2 \right) , \]  

(37)

and the flux as a function of \( \psi'_1 \) and \( \psi'_2 \) is given by

\[ \mathcal{F} \propto \left( |\psi'_1|^2 - |\psi'_2|^2 \right) , \]  

(38)

where we have considered

\[ |\psi_1|^2 - |\psi_2|^2 = \frac{(1-x)^{d-1/2}}{x^{1/2}} \left( |\psi'_1|^2 - |\psi'_2|^2 \right) , \]  

(39)

from (21), and doing \( x = \tanh^2 \rho \).

So, for \( ml > -1/2 \) and imposing null flux at infinity \( \rho \to \infty \), we obtain the following sets of quasinormal frequencies

\[ \omega = \frac{\xi}{l} - i \sqrt{\mu} \left( \frac{1}{2l} + \frac{2}{l^2} n + m \right) , \]  

(40)

and, for \( ml < -1/2 \), we obtain

\[ \omega = \frac{\xi}{l} - i \sqrt{\mu} \left( \frac{3}{2l} + \frac{2}{l^2} n - m \right) , \]  

(41)
Where, we have considered the Kummer’s formula, \[50\],

\[
2F1(a, b, c, x) = \frac{\Gamma(c) \Gamma(c - a - b)}{\Gamma(c - a) \Gamma(c - b)} 2F1(a, b, a + b - c, 1 - x) + \\
(1 - x)^{c - a - b} \frac{\Gamma(c) \Gamma(a + b - c)}{\Gamma(a) \Gamma(b)} 2F1(c - a, c - b, c - a - b + 1, 1 - x)
\]

in Eq. (38). It is worth mentioning that these frequencies are similar to that of the BTZ black hole \[51–53\] and that the imaginary part of the quasinormal frequencies is negative, which ensures the stability of the black hole under fermionic perturbations. Also, it is possible to observe that if \(\mu = 1\), we recover the QNMs of the massless topological black holes in \(d\)-dimensions \[54\]. Actually, if \(\mu = 1\) the metric (5) coincides with the metric of a massless topological black hole. It is also interesting to observe that if \(\mu \neq 1\) the QNMs (40) and (41) of fermionic perturbations for Chern-Simons black holes have the imprint of the high curvature of the original theory. According to the AdS/CFT correspondence the relaxation time \(\tau\) for a thermal state to reach thermal equilibrium in the boundary conformal field theory is \(\tau = 1/\omega_I\) where \(\omega_I\) is the imaginary part of QNMs. As can be seen in relation to (40) and (41), \(\omega_I\) scales with \(\sqrt{\mu}\). Then, depending on the sign of \(\sigma\), \(\mu\) can be larger or smaller than one. This means that the fermionic field will decay faster or slower depending on the value of the curvature parameter \(k\).

IV. QUASINORMAL MODES OF FERMIONIC PERTURBATIONS FOR SOME SPACETIMES WITH TORSION

In this section we will study the quasinormal modes of fermionic perturbations for some spacetimes with torsion. First, we will consider the five-dimensional Chern-Simons black hole with torsion and then the BTZ black hole with torsion.

A. Chern-Simons black hole with torsion

The Chern-Simons black hole with torsion in five dimensions is a solution to the Einstein-Gauss-Bonnet action \[42\], which, in terms of differential forms is given by

\[
I = \int \epsilon_{abcd} \left( \frac{c_0}{5} e^a e^b e^c e^d e^e + \frac{c_1}{3} R^{ab} e^c e^d e^e + c_2 R^{ab} R^{cd} e^e \right),
\]

where \(e^a = e^a_\mu dx^\mu\) is the vielbein, and \(R^{ab} = d\omega^{ab} + \omega^a_\mu \omega^{\mu b}\) is the curvature 2-form for the spin connection \(\omega^{ab} = \omega^{ab}_\mu dx^\mu\). The Gauss-Bonnet coupling is fixed as

\[
c_2 = \frac{c_1^2}{4c_0}.
\]

Thus, the theory possesses the maximum number of degrees of freedom and a unique maximally symmetric vacuum \[22, 47\], and the Lagrangian can be written as a Chern-Simons form. \[55\]. The metric is given by

\[
ds^2 = -f(r)^2 dt^2 + \frac{dr^2}{f(r)^2} + r^2 d\Omega^2,
\]

with

\[
f(r)^2 = \frac{r^2}{l^2} - \mu,
\]

where \(l = \sqrt{\frac{2c_2}{c_1}}\) is the AdS radius and \(\Omega\) is a 3-dimensional base manifold, being this base manifold fixed but arbitrary. The metric (45) represents a black hole for \(\mu > 0\) and the torsion 2-form is given by the expression

\[
T^m = -\frac{\delta}{r} e^{mnp} e_n e_p,
\]

where \(\delta\) is an integration constant. Here, the indices \(m, n, \ldots\) refers to \(\Omega\). Therefore, the non-vanishing components of the torsion have support along the base manifold only. The spin connection results in

\[
\omega^{01} = f’(r) e^0, \quad \omega^{n1} = f(r) \dot{e}^n, \quad \omega^{mn} = \dot{\omega}^{mn} + K^{mn},
\]
where the contorsion reads

$$K^{mn} = \frac{\delta}{r} e^{mp} e_{p} .$$  \hfill (49)$$

In this case, a minimally coupled fermionic field to curvature in the background of the five-dimensional Chern-Simons black hole with torsion is given by the Dirac equation with torsion, \textsuperscript{50}

$$\star \gamma \wedge \left( \nabla - \frac{1}{2} K^{ab} e_{b} \right) \psi + m \psi = 0 .$$ \hfill (50)$$

However, for the solution \textsuperscript{49} \(K^{ab} e = 0\), the Dirac equation (in a coordinate base \(\{d x^\mu\}\)) simplifies to

$$(\gamma^\mu \nabla_\mu + m) \psi = 0 ,$$ \hfill (51)$$

where the effect of the torsion is included in the covariant derivative through the spin connection. As in section III, we consider the change of coordinates \(r = r_c \cosh \rho\) and we use the following ansatz for the fermionic field

$$\psi = e^{-i \omega t} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \otimes \xi .$$ \hfill (52)$$

The substitutions \textsuperscript{21}, with \(d = 5\) and the following representation for the Dirac matrices

$$\gamma^0 = i \sigma^2 \otimes 1, \quad \gamma^1 = \sigma^1 \otimes 1, \quad \gamma^m = \sigma^3 \otimes \bar{\gamma}^m ,$$ \hfill (53)$$

permit us to obtain the equations

$$2 (1 - x) x^{1/2} \partial_x \psi_1' + i \left( \frac{\omega_l}{\sqrt{\mu}} x^{-1/2} - \frac{\left( \xi + \frac{3}{2} \delta \right)}{\sqrt{\mu}} x^{1/2} \right) \psi_1' + \left( i \left( \frac{\omega_l}{\sqrt{\mu}} - \frac{\left( \xi + \frac{3}{2} \delta \right)}{\sqrt{\mu}} \right) + \frac{1}{2} + m l \right) \psi_2' = 0 ,$$ \hfill (54)$$

$$2 (1 - x) x^{1/2} \partial_x \psi_2' - i \left( \frac{\omega_l}{\sqrt{\mu}} x^{-1/2} - \frac{\left( \xi + \frac{3}{2} \delta \right)}{\sqrt{\mu}} x^{1/2} \right) \psi_2' + \left( - i \left( \frac{\omega_l}{\sqrt{\mu}} - \frac{\left( \xi + \frac{3}{2} \delta \right)}{\sqrt{\mu}} \right) + \frac{1}{2} + m l \right) \psi_1' = 0 ,$$

where \(i \xi\) is the eigenvalue of the Dirac operator of the base manifold \(\Omega\), and we have used \(x = \tanh^2 \rho\). So, imposing null flux at infinity, for \(ml > -1/2\) we get the quasinormal frequencies

$$\omega = \frac{\left( \xi + \frac{3}{2} \delta \right)}{l} - i \sqrt{\mu} \left( \frac{1}{2l} + \frac{2}{l} n + m \right) ,$$ \hfill (55)$$

$$\omega = - \frac{\left( \xi + \frac{3}{2} \delta \right)}{l} - i \sqrt{\mu} \left( \frac{3}{2l} + \frac{2}{l} n + m \right) ,$$

and for \(ml < -1/2\) we obtain frequencies similar to \textsuperscript{11} making the change \(\xi \to \xi + \frac{3}{2} \delta\). Therefore, we see that the effect of torsion, represented by the term \(\delta\), is modifies the real part of the quasinormal frequencies, which modify the oscillation frequency of the field. Also, this spacetime is stable because the imaginary part is negative and turns out to be the same as the Chern-Simons black hole without torsion. Note that, depending on the value of \(\delta\), a value of \(\xi, \xi = -3/2 \delta\) may exist, for which the quasinormal frequencies are purely imaginary, and therefore the field decay in time without oscillation.

B. BTZ black hole with torsion

Now, we consider the BTZ black hole with torsion \textsuperscript{26}, which is a solution of the topological Mielke-Baekler model for 3D gravity \textsuperscript{25}

$$I = \int 2ae^a R_a - \frac{\Lambda}{3} \epsilon_{abc} e^a e^b e^c + \alpha_3 \left( \omega^a d \omega_a + \frac{1}{3} \epsilon_{abc} \omega^a \omega^b \omega^c \right) + \alpha_4 e^a T_a ,$$ \hfill (56)$$

where, the first term corresponds to the usual Einstein-Cartan action with \(a = \frac{1}{16 \pi G}\), the second term is a cosmological term, the following term is the Chern-Simons action for the Lorentz connection, and the last term is a torsion
The BTZ black hole with torsion is described by the metric

\[ ds^2 = -N(r)^2 dt^2 + \frac{1}{N(r)^2} dr^2 + r^2 (d\phi + N_\phi dt)^2 \]

respectively. We have used the notation of [57] with \( \eta_{ab} = \text{diag} (-,+,+) \), and in the next we will consider \( 8G = 1 \). The difference in these equations with respect to the torsionless BTZ black hole is the term \( \Lambda_{eff} = -\frac{1}{4} \), with \( \Lambda_{eff} = q - \frac{1}{2} p^2 \), where we have \( q = -\frac{\alpha_7 + \alpha_9}{\alpha_3 \alpha_4 - \alpha} \) and \( p = \frac{\alpha_3 \Lambda + \alpha_4 a}{\alpha_3 \alpha_4 - \alpha} \) are constants. For simplicity, we will analyze the case with \( J = 0 \). In this case, the event horizon is located at \( r_+ = \sqrt{M} \). Now, considering the diagonal vielbein

\[ e^0 = N(r) dt \ , \ e^1 = \frac{1}{N(r)} dr \ , \ e^2 = r d\phi \]

the spin connection reads

\[ \omega^{01} = \frac{r}{r^2} dt - \frac{P}{2} r d\phi \ , \ \omega^{02} = \frac{P}{2 N(r)} dr \ , \ \omega^{12} = -N(r) d\phi + \frac{p N(r)}{2} dt \]

For \( p = 0 \), the torsion vanishes and the spin connection becomes the Levi-Civita spin connection. In this case, the Dirac equation with torsion, by considering the change of coordinates \( r = r_+ \cosh \rho \), and the following representation of the gamma matrices

\[ \gamma^0 = i \sigma^2 \ , \ \gamma^1 = \sigma^1 \ , \ \gamma^2 = \sigma^3 \]

where \( \sigma^i \) are the Pauli matrices, along with the following ansatz for the fermionic field

\[ \psi = e^{-i \omega t} e^{i \xi \phi} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \]

and Eq. (24) for \( d = 3 \) and \( x = \tanh^2 \rho \), permit us to write the following equations

\[ 2 (1 - x) x^{1/2} \partial_x \psi_1' + i \left( \frac{\omega_l}{\sqrt{M}} x^{-1/2} - \frac{\xi}{\sqrt{M}} x^{1/2} \right) \psi_1' + \left( i \left( \frac{\omega_l}{\sqrt{M}} - \frac{\xi}{\sqrt{M}} \right) + \frac{1}{2} + m l + \frac{3 m l}{4} \right) \psi_1'' = 0 \ , \]

\[ 2 (1 - x) x^{1/2} \partial_x \psi_2' - i \left( \frac{\omega_l}{\sqrt{M}} x^{-1/2} - \frac{\xi}{\sqrt{M}} x^{1/2} \right) \psi_2' + \left( -i \left( \frac{\omega_l}{\sqrt{M}} - \frac{\xi}{\sqrt{M}} \right) + \frac{1}{2} + m l + \frac{3 m l}{4} \right) \psi_2'' = 0 \ . \]

The difference in these equations with respect to the torsionless BTZ black hole is the term \( \frac{3 m l}{4} \). This term redefines the mass of the fermionic field and therefore modifies the quasinormal frequencies of the field. The quasinormal frequencies for \( m l + \frac{3 m l}{4} > -1/2 \) (imposing null flux at infinity \( \rho \rightarrow \infty \)) are given by

\[ \omega = \frac{\xi}{\ell} - i \sqrt{M} \left( \frac{1}{2 \ell} + \frac{2}{\ell} n + m + \frac{3 \mu}{4} \right) \]

\[ \omega = -\xi / \ell - i \sqrt{M} \left( \frac{3}{2 \ell} + \frac{2}{\ell} n + m + \frac{3 \mu}{4} \right) \]

and for \( m l + \frac{3 m l}{4} < -1/2 \) frequencies similar to [11] are obtained by making the changes \( m l \rightarrow m l + \frac{3 m l}{4} \) and \( \mu \rightarrow M \). The effect of the torsion is to modify the imaginary part of these frequencies in contrast with the five-dimensional Chern-Simons case, where the torsion effect was to modify the real part of the quasinormal frequencies. This modification that happens for fermionic fields is absent for scalar fields, where there is no contribution by the torsion to the quasinormal modes. Note that if \( \xi = 0 \), the quasinormal frequencies are purely imaginary, and therefore the field decay in time without oscillation.\(^2\)

\[ ^2 \text{Recently, in Ref. [58] the QNFs obtained in this article for the BTZ black hole with torsion were used to compute the entropy spectrum of this black hole, and was shown that it is equally spaced.} \]
V. CONCLUSIONS

In this work, we have calculated the QNMs of fermionic perturbations for Chern-Simons black holes with spherical, hyperbolic and plane topology. Here, we have considered as boundary conditions (to determine the QNMs) that there are only ingoing modes at the horizon. However, it is known that at infinity their depend on the asymptotic behavior of spacetime. For asymptotically AdS spacetimes the potential diverges and in this manner it is required for the field be null, at infinity. However, by establishing these Dirichlet boundary conditions, not all QNMs are obtained. It is known that the QNMs of the BTZ black hole under Dirichlet boundary conditions permit to obtain only a set of QNMs, for positive masses of the scalar field. However, there is another set of QNMs for a range of imaginary masses which are allowed because the propagation of the scalar field is stable, according to the Breitenlohner-Freedman limit. This set of QNMs, just as the former, can be obtained by requesting the flux to vanish at infinity, which are known as Neumann boundary conditions. Remarkably, for fermionic perturbations there is no Breitenlohner-Freedman limit. However, it is possible to consider Neumann boundary conditions because Dirichlet boundary conditions would lead to the absence of QNMs for a range of masses, without a physical reason for this absence. Here, we have considered Neumann boundary conditions at infinity for Chern-Simons black holes and we have found that the Dirac QNMs depend on the highest power of curvature present in the Lanczos-Lovelock theories, such as the quasimode modes of scalar perturbations. Then, we have found the quasinormal modes of fermionic perturbations for the five-dimensional Chern-Simons black hole with torsion and for the BTZ black hole with torsion analytically.

We have found that these black holes are stable under fermionic perturbations, due to the imaginary part of the Dirac QNFs is negative. Also, we have found that the effect of the torsion on the Dirac QNFs of five-dimensional Chern-Simons black hole is to modify their real part (oscillatory frequency), in comparison with the Dirac QNFs of five-dimensional Chern-Simons black hole without torsion. Therefore, the relaxation time of the perturbation doesn’t depend on the torsion. On the other hand, the effect of the torsion on the Dirac QNFs of BTZ black hole with torsion is to modify the imaginary part (damping). Therefore, in this case, the relaxation time for the decay of the black hole perturbation depend on the torsion value. It is worth mentioning that both metrics are very similar. However, the torsion for the Chern-Simons black hole only exist in the three-dimensional base manifold, which is endowed with a fully antisymmetric torsion. In contrast with the BTZ black hole with torsion, where the torsion exist in the full spacetime. In this case, the torsion cannot be restricted to the base manifold. So, the fact that in the Chern-Simons case the torsion considered is completely antisymmetric and also is restricted to live in the three-dimensional base manifold is the reason why the torsion only enters into the real part of the QNFs. In this manner, in order to determine if the torsion could affect the imaginary part of the QNFs too, as it happens in the BTZ black hole with torsion, would be interesting to study Chern-Simons black holes solutions endowed with a more general torsion.

In the context of AdS/CFT correspondence, we can appreciate that the relaxation time for a thermal state to reach thermal equilibrium in the boundary conformal field theory is modified by the torsion in three dimensions. Which can be established by assuming that the QNFs of the black hole are related with the poles of the retarded correlation function of the corresponding perturbations of the dual conformal field theory as in AdS spacetime without torsion, which is matter of a current investigation.

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