Quaternary quadratic forms with prime discriminant

by

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In memory of Professor Andrzej Schinzel

1. Introduction and statement of results. The study of which integers are represented by a positive-definite quadratic form is an old one. After initial predominantly algebraic results from Euler, Lagrange and Legendre, in the 19th century beginning with Jacobi’s formula for the number of representations of an integer by the sum of four squares, analytic techniques of increasing complexity have been applied. Towards the classification specifically of universal positive-definite quadratic forms (those which represent all \( n \in \mathbb{N} \)) were results from Ramanujan [17], Dickson, and eventually the Conway–Schneeburger 15-Theorem (see [2]) and the Bhargava–Hanke 290-Theorem [3]. With almost universal forms (forms which are not universal and which fail to represent finitely many \( n \in \mathbb{N} \)) or with forms which represent entire congruence classes are recent works respectively of Barowsky et al. [1] and Rouse [19]. Additionally, in the development of the theory of modular forms (and in particular bounds on cusp forms) are results by Tartakowsky [25], Ono–Soundararajan [16], and Schulze-Pillot [22].

To give specific context to the results in the present paper, we highlight a few predecessors. First, recall that a quadratic form \( Q \) in \( r \) variables is anisotropic at a prime \( p \) if the only solution to \( Q(\vec{x}) = 0 \) with \( \vec{x} \in \mathbb{Z}_p^r \) is \( \vec{x} = \vec{0} \). If \( Q = \frac{1}{2} \vec{x}^T A \vec{x} \), where \( A \) is an \( r \times r \) symmetric matrix with integer entries and even diagonal entries, we let \( D(Q) = \det(A) \) be the discriminant of \( Q \). We let \( N(Q) \) denote the level of \( Q \); this is the smallest positive integer \( N \) such that \( NA^{-1} \) has integer entries and even diagonal entries. With that, we begin with the following result:
Theorem (Tartakowsky, 1929). Let \( Q \) be a positive-definite quadratic form in four variables and fix an integer \( k \geq 0 \). Then there is a constant \( C(Q, k) \) such that if \( n \) is a positive integer with \( n \geq C(Q, k) \) and \( \text{ord}_p(n) \leq k \) for all anisotropic primes \( p \), then \( n \) is represented by \( Q \).

A number of more recent projects including [6] and [21] have given bounds on the quantity \( C(Q, k) \) for forms in four or more variables. In [19], the first author proved the following result regarding forms with fundamental discriminant (which do not have any anisotropic primes):

**Theorem (Rouse, 2014).** If \( D(Q) \) is a fundamental discriminant and \( \epsilon > 0 \), there is a constant \( C_\epsilon \) such that all positive integers \( n \geq C_\epsilon D(Q)^{2+\epsilon} \) are represented by \( Q \).

The goal of the present paper is to prove a stronger result and make it completely effective in the case that \( D(Q) = p \) is a prime number. If \( Q = \frac{1}{2} \vec{x}^T A \vec{x} \) is such a form, the dual form \( Q^* = \frac{1}{2} \vec{x}^T p A^{-1} \vec{x} \) has level \( p \) as well. Let \( \min Q^* \) denote the smallest positive integer represented by \( Q^* \). Also, let \( M(n) = \max_{1 \leq m \leq n} \tau(m) \). Note that \( M(n) = O(n^{\frac{1.538 \log 2}{\log \log n}}) \) for all \( \epsilon > 0 \). More precisely, the bound \( M(n) \leq n^{1.538 \log 2 / \log \log n} \) for \( n \geq 3 \) is proven in [15].

**Theorem 1.** Let \( Q \) be a positive-definite quaternary quadratic form with prime discriminant \( p \). Let \( n \) be a positive integer, and denote by \( \phi(n) \) Euler’s totient, and by \( \tau(n) \) the number of divisors of \( n \). If \( p \geq 101 \), then

\[
r_Q(n) \geq \frac{24}{p^{3/2}} (p - 1)\phi(n) - 3.75 \sqrt{p} \left( \frac{1}{\min Q^*} + \frac{3216.66 M(25.09 p^{35/6})}{p^{1/4}} \right) \tau(n) n^{1/2},
\]

where \( r_Q(n) := \{ \vec{x} \in \mathbb{Z}^4 \mid Q(\vec{x}) = n \} \).

**Remark.** The inequality above shows that \( r_Q(n) > 0 \) provided that \( n \gg \max \{ \frac{p^{2+\epsilon}}{\min Q^*}, p^{7/4+\epsilon} \} \).

The method of proof is to take \( \theta_Q(z) := \sum_{n=0}^{\infty} r_Q(n) q^n, q = e^{2\pi i z} \), and decompose it as \( \theta_Q(z) = E(z) + C(z) \) into an Eisenstein series and a cusp form. The cusp form in turn may be decomposed as

\[
C(z) = \sum_i c_i g_i(z),
\]

where each \( g_i(z) \) is a newform. The Deligne bound on the Fourier coefficients of \( g_i(z) \) implies that the \( n \)th coefficient is bounded above by \( \tau(n) \sqrt{n} \). It suffices then to estimate the coefficients \( c_i \). We do this by using the orthogonality of the Petersson inner product and finding an upper bound on
\langle C, C \rangle$, and estimating $\sum \frac{1}{\langle g_i, g_i \rangle}$ using the Petersson formula. We obtain the following:

**Theorem 2.** Suppose that $Q$ is a positive-definite form in four variables with prime discriminant $p$. Then

$$\langle C, C \rangle \leq \frac{1}{\min Q^*} + \frac{321.66M(25.09p^{35/6})}{p^{1/4}}.$$ 

When $\min Q^*$ is small ($\ll p^{1/4}$), our upper bound on $\langle C, C \rangle$ matches the lower bound given by Waibel in [26].

**Theorem (Waibel, 2022).** Suppose that $Q$ is a positive-definite form in $r$ variables. Then

$$\langle C, C \rangle \gg \frac{1}{[\text{SL}_2(\mathbb{Z}) : \Gamma_0(N(Q))]} \left( \frac{N(Q)^{r/2}}{D(Q)} (\min Q^*)^{1-r/2} + O(N(Q)^\epsilon) \right).$$

The result above is stated using our normalization of the Petersson norm, which differs from that used by Waibel [26]. In our case, $r = 4$, $N(Q) = D(Q) = p$.

As a consequence of Theorems 1 and 2, we obtain a nontrivial bound on the sum of the integers not represented by $Q$.

**Corollary 3.** Assume the notation above. Then

$$\sum_{r_Q(n)=0} n \ll \max \left\{ \frac{p^{3+\epsilon}}{(\min Q^*)2}, p^{5/2+\epsilon} \right\}.$$

The remainder of the paper is organized as follows: In Section 2, we introduce notation and review necessary background. In Section 3, we prove Theorem 2 and use that to prove Theorem 1. In Section 4, we prove Corollary 3. Last, in Section 5 we provide an example of an infinite family of forms of prime discriminant along with the complete classification of their excepted values.

**2. Background.** Throughout, we use the word “form” to denote a positive-definite, integer-valued, quaternary quadratic form $Q(\vec{x}) = \frac{1}{2} \vec{x}^T A \vec{x}$. The determinant $D(Q)$ of the form is the determinant of $A$. The level $N = N(Q)$ of the form is the smallest integer such that $NA^{-1}$ is an integral matrix with even diagonal entries. Let

$$r_Q(n) := \# \{ \vec{x} \in \mathbb{Z}^4 \mid Q(\vec{x}) = n \}$$

denote the representation number of $n$ by $Q$ and define the theta series of $Q$ by

$$\Theta_Q(z) = \sum_{n \geq 0} r_Q(n) q^n, \quad q = e^{2\pi i z}.$$
It is well-known that $\Theta_Q$ is in $M_2(\Gamma_0(N(Q)), \chi_D(Q))$, and it decomposes as

$$\Theta_Q(z) = E(z) + C(z) = \sum_{n=0}^{\infty} a_E(n)q^n + \sum_{n=1}^{\infty} a_C(n)q^n,$$

where $E(z)$ is the Eisenstein series contribution and $C(z)$ is the cusp form contribution.

Siegel expressed $a_E(n)$ as a product of local densities:

**Theorem 4 (Siegel).** We have

$$a_E(n) = \prod_{p \leq \infty} \beta_p(n),$$

where the product is over primes $p$ and, for finite $p$,

$$\beta_p(n) = \lim_{v \to \infty} \frac{\# \{ \vec{x} \in (\mathbb{Z}/p^v\mathbb{Z})^4 \mid Q(\vec{x}) \equiv n \pmod{p^v} \}}{p^{3v}},$$

while

$$\beta_\infty(n) = \frac{4\pi^2n}{\sqrt{\det(A)}}.$$

**Proof.** The formula for $\beta_\infty$ is a special case of [23, Hilfssatz 72]. Note that Siegel’s normalization of $A$ is different from ours, hence our correction with the factor of 4. ■

In practice, for the finite places we rely on computational methods of Hanke, outlined in [9]. Throughout, we consider $Q$ interchangeably with its local Jordan splitting

$$Q(\vec{x}) = \sum_j p^{v_j} Q_j(\vec{x}_j)$$

where $\dim Q_j \leq 2$ and when $p \neq 2$, $\dim Q_j = 1$. Let

$$R_{p^v}(n) := \{ \vec{x} \in (\mathbb{Z}/p^v\mathbb{Z})^4 \mid Q(\vec{x}) \equiv n \pmod{p^v} \}$$

and set $r_{p^v}(n) := \# R_{p^v}(n)$. We say $\vec{x} \in R_{p^v}(n)$ is

- of Zero type if $\vec{x} \equiv \vec{0} \pmod{p}$, in which case we write $\vec{x} \in R_{p^v}^{\text{Zero}}(n)$ with $r_{p^v}^{\text{Zero}}(n) := \# R_{p^v}^{\text{Zero}}(n)$;
- of Good type if $p^{v_j}x_j \not\equiv 0 \pmod{p}$ for some $j \in \{1, 2, 3, 4\}$, in which case we write $\vec{x} \in R_{p^v}^{\text{Good}}(n)$ with $r_{p^v}^{\text{Good}}(n) := \# R_{p^v}^{\text{Good}}(n)$;
- of Bad type otherwise, in which case we write $\vec{x} \in R_{p^v}^{\text{Bad}}(n)$ with $r_{p^v}^{\text{Bad}}(n) := \# R_{p^v}^{\text{Bad}}(n)$.

If $r_{p^v}(n) > 0$ for all primes $p$ and for all $v \in \mathbb{N}$, we say that $n$ is locally represented. If $Q(\vec{x}) = 0$ has only the trivial solution over $\mathbb{Z}_p$, we say that $p$ is an anisotropic prime for $Q$. 


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For the explicit calculations we carry out in this paper, an analysis of the good type solutions is sufficient. For more detail about counting zero type and bad type solutions, see [9, pp. 359–360].

To estimate $a_C(n)$, we use techniques from [19] as well as the Petersson formula. Assume that $D(Q)$ is a fundamental discriminant and decompose $C(z) = \sum_{i=1}^{s} c_i g_i(z)$ as a linear combination of newforms. Here $s = \dim S_2(\Gamma_0(N(Q)), \chi_{DQ})$. The Deligne bound gives

$$|a_C(n)| \leq \left( \sum_{i=1}^{s} |c_i| \right) \tau(n) \sqrt{n},$$

where $\tau(n)$ denotes the number of positive divisors of $n$. The Petersson inner product of two cusp forms $f, g \in S_2(\Gamma_0(N(Q)), \chi_{DQ})$ is given by

$$\langle f, g \rangle = \frac{3}{\pi |\text{SL}_2(\mathbb{Z}) : \Gamma_0(N(Q))|} \int_{\mathbb{H}/\Gamma_0(N(Q))} f(x + iy) \overline{g(x + iy)} \, dx \, dy.$$

Distinct newforms are orthogonal, and using the Cauchy–Schwarz inequality we obtain

$$\sum_{i=1}^{s} |c_i| \leq \sqrt{\sum_{i=1}^{s} \frac{1}{\langle g_i, g_i \rangle}} \sqrt{\sum_{i=1}^{s} |c_i|^2 \langle g_i, g_i \rangle} = \sqrt{\sum_{i=1}^{s} \frac{1}{\langle g_i, g_i \rangle}} \sqrt{\langle C, C \rangle}.$$

We use the Petersson formula to give a bound on $\sum_{i=1}^{s} |a_i(n)|$. This formula (see for example [10, Theorem 3.6]) states that if $h_1, \ldots, h_s$ is an orthonormal basis for $S_k(\Gamma_0(N), \chi)$, with $h_i(z) = \sum_{n=1}^{\infty} a_i(n) q^n$, then

$$\frac{\Gamma(k-1)}{(4\pi \sqrt{mn})^{k-1}} \sum_{i=1}^{s} a_i(m) a_i(n)$$

$$= \frac{\pi}{3 |\text{SL}_2(\mathbb{Z}) : \Gamma_0(N)|} \left( \delta_{mn} + 2\pi i^{-k} \sum_{c > 0 \atop \chi \equiv 1 \pmod{c}} \frac{S_{\chi}(m, n; c)}{c} J_{k-1} \left( \frac{4\pi \sqrt{mn}}{c} \right) \right).$$

Note that our normalization of the inner product differs from that in [10]. Also,

$$J_{k-1}(x) = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell (x/2)^{2\ell+k-1}}{\ell! \Gamma(\ell + k)}$$

is the usual $J$-Bessel function, and

$$S_{\chi}(m, n; c) = \sum_{a \in 1 \pmod{c}} \chi(a) e \left( \frac{ma + nd}{c} \right) \quad e(x) = e^{2\pi i x},$$

is a generalized Kloosterman sum. We apply the Petersson formula with
$k = 2$ and $m = n = 1$. We have $a_j(1) = 1/\sqrt{\langle g_j, g_j \rangle}$. Hence,

$$\sum_{i=1}^{s} \frac{1}{\langle g_i, g_i \rangle} = \frac{4\pi^2}{3} \left[ \text{SL}_2(\mathbb{Z}) : \Gamma_0(N) \right] \left( 1 - 2\pi \sum_{c \equiv 0 \pmod{N}} S_{\chi}(m, n; c) J_1 \left( \frac{4\pi c}{c} \right) \right).$$

We derive an explicit upper bound on the right hand side in Lemma 7.

The last task is to find an upper bound on $\langle C, C \rangle$. For this task, we let $Q^* = \frac{1}{2} x^T N(Q) A_Q^{-1} x$ be the dual form to $Q$. Let $\theta_{Q^*} = E^* + C^*$ be the decomposition into an Eisenstein series and a cusp form. The Fricke involution $W_N$ sends $\theta_Q$ to $\sqrt{N(Q)} \theta_{Q^*}$ and is an isometry for the Petersson inner product and therefore $\langle C, C \rangle = N \langle C^*, C^* \rangle$. Moreover, the form $C^* = \sum a_{C^*}(n) q^n$ has the property that $a_{C^*}(n) = 0$ if $\chi_D(n) = 0$ or 1 and so $C^* \in S_2(\Gamma_0(N), \chi_D)$ (see [19, Proposition 15]). Proposition 14 of [19] then gives

$$\langle C^*, C^* \rangle = \frac{1}{\left[ \text{SL}_2(\mathbb{Z}) : \Gamma_0(N) \right]} \sum_{n=1}^{\infty} \frac{2 \omega(gcd(n, N(Q))) a_{C^*}(n)^2}{n} \sum_{d=1}^{\infty} \psi \left( d \sqrt{\frac{n}{N(Q)}} \right),$$

where

$$\psi(x) = -\frac{6}{\pi} x K_1(4\pi x) + 24 x^2 K_0(4\pi x),$$

with $K_0$ and $K_1$ are the usual $K$-Bessel functions. We use these formulas to estimate $\langle C, C \rangle$.

3. The largest excepted value. In this section we explicitly find $a_{E}(m)$ and give bounds on $a_{C}(m)$ to prove Theorems [1 and 2].

3.1. The Eisenstein series. There are many papers devoted to understanding the Eisenstein contribution to a theta series. We use the results of [9] and [30] to give formulas for local densities. In another direction, Lynne Walling has given more explicit formulas for the Eisenstein coefficient $a_{E}(n)$ (see [27]) and a formula for $E(z)$ as a linear combination of “basic Eisenstein series” (see [28]).

In the special case of weight 2 and prime level, the Eisenstein subspace of $M_2(\Gamma_0(p), \chi_p)$ is 2-dimensional (see the details in [7, proof of Theorem 5]) and is spanned by

$$G(z) = 1 + \frac{2}{L(-1, \chi_p)} \sum_{n=1}^{\infty} \left( \sum_{d|n} d \chi_p(d) \right) q^n,$$

$$H(z) = \sum_{n=1}^{\infty} \left( \sum_{d|n} d \chi_p(n/d) \right) q^n.$$
Therefore, to express $E(z)$ as a linear combination of $G(z)$ and $H(z)$, it suffices to note that $a_E(0) = 1$ due to $Q$ being positive-definite, and then:

**Theorem 5.** The exact value of the first Eisenstein coefficient is

$$a_E(1) = \frac{-2(p - 1)}{L(-1, \chi_p)}.$$

**Proof.** Note: the following could also be viewed as a special case of [25, equation (I) on p. 116].

Siegel’s local density formula (Theorem 4) gives

$$a_E(1) = \prod_{q \leq \infty} \beta_q(1) = \beta_\infty(1)\beta_2(1)\beta_p(1)\left(\prod_{q \mid 2p} \beta_q(1)\right).$$

We have $\beta_\infty(1) = 4\pi^2/\sqrt{p}$. For $\beta_2(1)$, noting via the language of [9] that all solutions are of Good type, one can see that $\beta_2(1) = (4 - \chi_p(2))/4$. The local density at $p$ can be computed using [30] which gives $\beta_p(1) = 1 - 1/p$. Last, for the remaining infinitely many primes, taking a restricted look at Theorem 4 gives

$$\prod_{q \mid 2p} \beta_q(1) = \prod_{q \mid 2p} \left(1 - \frac{\chi_p(q)}{q^2}\right) = L(2, \chi_p)^{-1} \frac{4}{4 - \chi_p(2)}.$$

Techniques outlined in [11, p. 104] show that

$$L(2, \chi_p) = -\frac{2\pi^2}{p\sqrt{p}}L(-1, \chi_p)$$

and hence

$$\prod_{q \mid 2p} \beta_q(1) = -\frac{p\sqrt{p}}{2\pi^2L(-1, \chi_p)} \cdot \frac{4}{4 - \chi_p(2)}.$$ 

Putting everything together, we have

$$a_E(1) = \beta_\infty(1)\beta_2(1)\beta_p(1)\left(\prod_{q \mid 2p} \beta_q(1)\right)$$

$$= \frac{4\pi^2}{\sqrt{p}} \cdot \frac{4 - \chi_p(2)}{4} \left(1 - \frac{1}{p}\right) \left(-\frac{p\sqrt{p}}{2\pi^2L(-1, \chi_p)} \cdot \frac{4}{4 - \chi_p(2)}\right)$$

$$= \frac{-2(p - 1)}{L(-1, \chi_p)},$$

as originally claimed.

As a result of Theorem 5, we can now write

$$E(z) = G(z) - \frac{2p}{L(-1, \chi_p)}H(z).$$
Moreover, for \( n \in \mathbb{N} \) the \( n \)th coefficient of the Eisenstein series is
\[
a_E(n) = -\frac{2}{L(-1, \chi_p)} \sum_{d|n} d(p\chi_p(n/d) - \chi_p(d)).
\]
Replacing \( d \) with \( n/d \) simplifies this to
\[
a_E(n) = -\frac{2}{L(-1, \chi_p)} \sum_{d|n} \left( \frac{pn}{d} - d \right) \chi_p(d).
\]

**Theorem 6.** For all \( n \in \mathbb{N} \),
\[
a_E(n) \geq \frac{24}{p^{3/2}} (p-1) \phi(n).
\]

**Proof.** Throughout we write \( n^* = n/p^{\text{ord}_p(n)} \). We first note
\[
a_E(n) = -\frac{2}{L(-1, \chi_p)} \sum_{d|n^*} \left( \frac{pn}{d} - d \right) \chi_p(d)
\]
\[
= -\frac{2}{L(-1, \chi_p)} (pn - n^* \chi_p(n^*)) \sum_{d|n^*} \frac{\chi_p(d)}{d}
\]
\[
\geq -\frac{2}{L(-1, \chi_p)} (pn - n^* \chi_p(n^*)) \sum_{d|n^*} \frac{\mu(d)}{d}
\]
\[
\geq -\frac{2}{L(-1, \chi_p)} (pn - n^*) \frac{\phi(n^*)}{n^*}.
\]
Let \( \text{ord}_p(n) = k \). Then
\[
-\frac{2}{L(-1, \chi_p)} (pn - n^*) \frac{\phi(n^*)}{n^*} = -\frac{2}{L(-1, \chi_p)} (p^{k+1}n^* - n^*) \frac{\phi(n^*)}{n^*}
\]
\[
= -\frac{2}{L(-1, \chi_p)} (p^{k+1} - 1) \phi(n^*).
\]
When \( k = 0 \), this equals \( \frac{-2}{L(-1, \chi_p)} (p - 1) \phi(n) \), while if \( k \geq 1 \), we obtain
\[
-\frac{2}{L(-1, \chi_p)} (pn - n^*) \frac{\phi(n^*)}{n^*} = \frac{-2}{L(-1, \chi_p)} \cdot \frac{p^{k+1} - 1}{p^k - p^{k-1}} \phi(n)
\]
\[
\geq \frac{-2}{L(-1, \chi_p)} (p - 1) \phi(n).
\]
As \( L(-1, \chi_p) = -\frac{p^{3/2}}{2\pi^2} L(2, \chi_p) \) and \( |L(2, \chi_p)| \leq \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \),
\[
-\frac{2}{L(-1, \chi_p)} \geq \frac{4\pi^2}{p^{3/2}} \cdot \frac{6}{\pi^2} = \frac{24}{p^{3/2}}
\]
and the claim holds. \( \blacksquare \)
3.2. The cusp series. As noted in the introduction, we write
\[ C(z) = \sum_{i=1}^{s} c_i g_i \]
where \( c_i \in \mathbb{C} \), the \( g_i(z) \) are newforms in \( S_2(\Gamma_0(p), \chi_p) \) and
\[ s = \dim S_2(\Gamma_0(p), \chi_p) = 2 \left\lfloor \frac{p-5}{24} \right\rfloor. \]
Bounds of Deligne give
\[ |a_C(n)| \leq C_Q \tau(n) \sqrt{n}, \]
where \( C_Q = \sum_{i=1}^{s} |c_i| \). We have
\[ C_Q \leq \sqrt{\sum_{i=1}^{s} \frac{1}{\langle g_i, g_i \rangle} \sqrt{\langle C, C \rangle}}. \]

**Lemma 7.** For any prime \( p \equiv 1 \pmod{4} \), we have
\[ \sum_{i=1}^{s} \frac{1}{\langle g_i, g_i \rangle} \leq \frac{4\pi^2}{3} (p + 1) \left( 1 + \frac{54.6}{p^{3/2}} \right). \]

**Proof.** Applying the Petersson formula gives
\[ \sum_{i=1}^{s} \frac{1}{\langle g_i, g_i \rangle} = \frac{4\pi^2}{3} [\text{SL}_2(\mathbb{Z}) : \Gamma_0(p)] \left( 1 - 2\pi \sum_{r=1}^{\infty} \frac{S_{\chi}(1,1;rp) \cdot J_1 \left( \frac{4\pi}{rp} \right)}{\tau(rp) \cdot 2} \right). \]
The bound \( S_{\chi}(m,n;c) \leq (m,n,c)^{1/2} c^{1/2} \tau(c) \) is due to Estermann [8] and Weil [29]. (Their proofs assume that \( \chi \) is trivial, but a slight modification handles the case when \( \chi \) is nontrivial.) The power series for \( J_1(x) \) is
\[ J_1(x) = \frac{x}{2} \left( \sum_{\ell=0}^{\infty} \frac{(-1)^\ell (x/2)^{2\ell}}{\ell! (\ell + 1)!} \right). \]
The inner sum is an alternating series if \( |x| < 2 \), and it follows that if \( |x| < 2 \), then \( |J_1(x)| \leq |x|/2 \). These bounds show that the right hand side of the Petersson formula is bounded by
\[ \frac{4\pi^2(p + 1)}{3} \left( 1 + 2\pi \sum_{r=1}^{\infty} \frac{\tau(rp) \cdot 2\pi}{\tau(rp)^{1/2} \cdot rp} \right). \]
We use \( \tau(rp) \leq 2\tau(r) \) to get the desired upper bound:
\[ \frac{4\pi^2(p + 1)}{3} \left( 1 + \frac{8\pi^2}{p^{3/2}} \sum_{r=1}^{\infty} \frac{\tau(r)}{r^{3/2}} \right) = \frac{4\pi^2(p + 1)}{3} \left( 1 + \frac{8\pi^2 \zeta(3/2)^2}{p^{3/2}} \right) \]
\[ \leq \frac{4\pi^2(p + 1)}{3} \left( 1 + \frac{54.6}{p^{3/2}} \right). \]
3.3. Computing an upper bound for $\langle C, C \rangle$. Recall that

$$\langle C, C \rangle = \frac{p}{p+1} \sum_{n=1}^{\infty} \frac{2^{\omega(\gcd(n,p))} a_{C^*}(n)^2}{n} \sum_{d=1}^{\infty} \psi\left( d \sqrt{\frac{n}{p}} \right),$$

where $\omega$ counts the number of distinct prime divisors and where

$$\psi(x) = -\frac{6}{\pi} x K_1(4\pi x) + 24x^2 K_0(4\pi x).$$

Note that $\psi(x) \leq 24x^2 K_0(4\pi x)$ and that (by [19, equation (9), p. 1724]), $K_0(x) \leq \sqrt{\frac{\pi}{2x}} e^{-x}$. Therefore $\psi(x) \leq 6\sqrt{2} x^{3/2} e^{-4\pi x}$. We begin by bounding the sum in $d$:

**Lemma 8.** The function $\sum_{d=1}^{\infty} \psi(dx)$ is decreasing for $x > 0$. Also, for all $x$,

$$\sum_{d=1}^{\infty} \psi(dx) \leq \frac{3}{4\pi^2},$$

and if $x > 0.5$ then

$$\sum_{d=1}^{\infty} \psi(dx) \leq 9x^{3/2} e^{-4\pi x}.$$

**Proof.** Choosing $\psi(0) = \frac{3}{2\pi^2}$ and $\psi(-x) = \psi(x)$ if $x < 0$ makes $\psi$ into a continuous, even function. Let $f(x) = \sum_{d=1}^{\infty} \psi(dx)$. The Weierstrass $M$-test shows that $\sum_{n=1}^{\infty} n\psi'(nx)$ converges uniformly on compact subsets of $(0, \infty)$ and thus $f'(x) = \sum_{n=1}^{\infty} n\psi'(nx)$. We have

$$\psi'(x) = 72x K_1(4\pi x) \left( \frac{K_0(4\pi x)}{K_1(4\pi x)} - \frac{4}{3\pi} \right).$$

So $K_1(4\pi x) > 0$ for all $x > 0$, and [24] shows that $\frac{K_1(x)}{K_0(x)} > 1$ for $x > 0$. This implies that $f'(x) < 0$ if $x > \frac{3}{4\pi}$.

The Poisson summation formula applied to $x\psi'(x)$ gives

$$\sum_{n=1}^{\infty} n\psi'(nx) = \sum_{n=1}^{\infty} \frac{18n^2 x(n^2 - 6x^2)}{\pi^2(n^2 + 4x^2)^{7/2}}.$$

If $x < 1/\sqrt{6}$, every term on the right hand side is negative and therefore $f'(x) < 0$. Since $1/\sqrt{6} > \frac{3}{4\pi}$, we have $f'(x) < 0$ for all $x$, and this proves that $\sum_{d=1}^{\infty} \psi(dx)$ is decreasing.

Similarly, applying Poisson summation to $\sum_{d=1}^{\infty} \psi(dx)$ we see

$$\sum_{d=1}^{\infty} \psi(dx) = \frac{3}{4\pi^2} + \sum_{d=1}^{\infty} \frac{1}{x} \psi(d/x),$$
where $\hat{\psi}$ is the Fourier transform of $\psi$ and

$$\hat{\psi}(y) = -\frac{9y^2}{\pi^2(4 + y^2)^{5/2}}.$$ 

Thus, we have

$$\sum_{d=1}^{\infty} \psi(d) = \frac{3}{4\pi^2} - \frac{9}{\pi^2} \sum_{d=1}^{\infty} \frac{(d/x)^2}{(4 + (d/x)^2)^{5/2}} \cdot \frac{1}{x}$$

$$= \frac{3}{4\pi^2} - \frac{9}{\pi^2} \sum_{d=1}^{\infty} \frac{d^2x^2}{(4x^2 + d^2)^{5/2}} \leq \frac{3}{4\pi^2}.$$ 

At the same time

$$\sum_{d=1}^{\infty} \psi(d) \leq 6\sqrt{2} \sum_{d=1}^{\infty} (d/x)^{3/2} e^{-4\pi dx}$$

$$\leq 6\sqrt{2} x^{3/2} \sum_{d=1}^{\infty} d^2 e^{-4\pi dx}$$

$$\leq \frac{6\sqrt{2} x^{3/2}(1 + e^{-4\pi x})e^{-4\pi x}}{(1 - e^{-4\pi x})^3}.$$ 

Hence, if $x \geq 0.5$ we have

$$\sum_{d=1}^{\infty} \psi(d) \leq 9x^{3/2} e^{-4\pi x},$$ 

as desired. 

For the sum in $n$, we use $|a_{C^*}(n)| \leq r_{Q^*}(n) + a_{E^*}(n)$, which in turn will give the bound

$$|a_{C^*}(n)|^2 \leq r_{Q^*}(n)^2 + 2r_{Q^*}(n)a_{E^*}(n) + a_{E^*}(n)^2 \leq 2r_{Q^*}(n)^2 + 2a_{E^*}(n)^2.$$ 

We now bound the representation number and Eisenstein components separately. We begin with the Eisenstein contribution to $\langle C, C \rangle$:

**Lemma 9.** We have

$$\sum_{n=1}^{p/2} \frac{2^{\omega(lcm(n,p))} \cdot 2a_{E^*}(n)^2}{n} \sum_{d=1}^{\infty} \psi\left(d, \sqrt{\frac{n}{p}}\right) \leq \frac{337.26 \log(p + 2) + 206.64}{p}.$$ 

**Proof.** The proof of Proposition 15 of [19] shows that

$$E^*(z) = \sum_{n=0}^{\infty} a_{E^*}(n)q^n$$
has the property that $a_{E^*}(n) = 0$ if $n$ is a quadratic residue modulo $p$. Equation (14) of [7] gives a formula for the unique Eisenstein series with this property and implies that

$$a_{E^*}(n) = \frac{2}{L(-1, \chi_p)} \sum_{d|n} \left( \chi_p(d) - \chi_p(n/d) \right)$$

and hence

$$a_{E^*}(n) \leq -\frac{4}{L(-1, \chi_p)} \sigma(n) \leq \frac{4\pi^4}{3p^{3/2}} \sigma(n).$$

Robin’s inequality [18] states that for $n \geq 3$,

$$\sigma(n) \leq e^{\gamma} n \log \log n + \frac{0.6483 n \log \log n}{\log \log n}.$$  

This implies that $\frac{\sigma(n)}{n \sqrt{\log(n+2)}}$ is decreasing if $n > 1652$, and in particular

$$\sigma(n) \leq 1.44n \sqrt{\log(n+2)} \quad \text{for } n \geq 1.$$

Thus

$$\frac{p}{p+1} \sum_{n=1}^{\infty} \frac{2^{\omega(\gcd(n,p))} \cdot 2a_{E^*}(n)}{n} \sum_{d=1}^{\infty} \psi \left( d \sqrt{\frac{n}{p}} \right) \leq \frac{32\pi^8}{9p^3} \sum_{n=1}^{\infty} \frac{2^{\omega(\gcd(n,p))} \sigma(n)^2}{n} \sum_{d=1}^{\infty} \psi \left( d \sqrt{\frac{n}{p}} \right) \leq (1.44)^2 \frac{32\pi^8}{9p^3} \sum_{n=1}^{\infty} 2^{\omega(\gcd(n,p))} n \log(n+2) \sum_{d=1}^{\infty} \psi \left( d \sqrt{\frac{n}{p}} \right).$$

For the terms with $n < p$ we use the fact that $\sum_{d=1}^{\infty} \psi(dx)$ is decreasing. For $0 \leq a < b < 1$, the contribution for $ap \leq n \leq bp$ is bounded by

$$(1.44)^2 \frac{32\pi^8}{9p^3} \sum_{n=ap}^{bp} n \log(n+2) \sum_{d=1}^{\infty} \psi \left( d \sqrt{\frac{n}{p}} \right) \leq \frac{69958}{p^3} \cdot [bp \log(bp+2)](b-a)p \sum_{d=1}^{\infty} \psi(d\sqrt{a}).$$

We apply this for $(a, b) = (0, 0.01), (0.01, 0.1), (0.1, 0.2), \ldots, (0.9, 1)$ and obtain a bound of

$$\frac{69958 \log(p+2)}{p} \cdot 0.00150889 \leq \frac{105.56 \log(p+2)}{p}.$$  

Now, for $n \geq p$, we have $2^{\omega(\gcd(n,p))} \leq 2$. We also use the bound

$$\sum_{d=1}^{\infty} \psi(d\sqrt{n/p}) \leq 9(n/p)^{3/4} e^{-4\pi\sqrt{n/p}}$$
from Lemma 8. This shows that the contribution for \( n \geq p \) is bounded by

\[
\frac{(1.44)^264\pi^8}{p^3} \sum_{k=1}^{\infty} \sum_{n=k^2p}^{(k+1)p} \frac{n \log(n+2) \cdot n^{3/4} \cdot p^{-3/4} e^{-4\pi k}}{n} \leq \frac{1259227}{p^{15/4}} \sum_{k=1}^{\infty} \frac{(p(k+1)^2)^{7/4} \log((k+1)^2(p+2))(2k+1)p e^{-4\pi k}}{p} \leq \frac{1259227}{p} \sum_{k=1}^{\infty} \frac{(k+1)^{7/2}(2k+1) \log((k+1)^2(p+2)) e^{-4\pi k}}{p} \leq \frac{1259227}{p} \left[ \sum_{k=1}^{\infty} \frac{2 \log(k+1)(k+1)^{7/2}(2k+1) e^{-4\pi k}}{p} + \log(p+2) \sum_{k=1}^{\infty} \frac{(k+1)^{7/2}(2k+1) e^{-4\pi k}}{p} \right] \leq \frac{1259227}{p} \left[ 0.0001641 + 0.0001184 \log(p+2) \right] \leq 206.64 + 231.8 \log(p+2).
\]

The second last line above comes from numerically evaluating the two convergent series. Combining the contributions for \( n < p \) and \( n \geq p \) gives the desired result.

To estimate the other contribution to \( \langle C, C \rangle \), namely

\[
\frac{p}{p+1} \sum_{n=1}^{\infty} \frac{2^{\omega(gcd(n,p))} \cdot 2 r_{Q^*}(n)^2}{n} \sum_{d=1}^{\infty} \psi\left( d, \sqrt{\frac{n}{p}} \right),
\]

we will need to bound \( \sum_{n \leq x} r_{Q^*}(n)^2 \). We do this via the simple inequality

\[
\sum_{n \leq x} r_{Q^*}(n)^2 \leq \left( \max_{n \leq x} r_{Q^*}(n) \right) \left( \sum_{n \leq x} r_{Q^*}(n) \right).
\]

We will proceed to estimate the two factors on the right hand side. To do so we write \( Q \) as

\[
Q = a_1(x_1 + m_{12}x_2 + m_{13}x_3 + m_{14}x_4)^2 + a_2(x_2 + m_{23}x_3 + m_{24}x_4)^2 + a_3(x_3 + m_{34}x_4)^2 + a_4(x_4)^2.
\]

Here \( a_1 \) is the minimum nonzero value of \( Q \), \( a_{i+1}/a_i \geq 3/4 \), and \( a_1 a_2 a_3 a_4 = \det(A/2) = p/16 \). Also, \( |m_{ij}| \leq 1/2 \) for all \( i \) and \( j \). (That we can write \( Q \) in such a form follows from [13, Theorem 2.1.1].)

This corresponds to writing \( Q = \frac{1}{2} \bar{x}^T A \bar{x} \) with \( \frac{1}{2} A = M^T D M \), where \( D \) is diagonal with entries \( a_1, \ldots, a_4 \). This gives \( \frac{1}{2} p A^{-1} = \frac{p}{4} M^{-1} D^{-1}(M^T)^{-1} \)
and yields the formula
\begin{equation}
Q^* = a_1^* x_1^2 + a_2^* (x_2 + n_{21} x_1)^2 + a_3^* (x_3 + n_{31} x_1 + n_{32} x_2)^2
+ a_4^* (x_4 + n_{41} x_1 + n_{42} x_2 + n_{43} x_3)^2,
\end{equation}
where $a_i^* = \frac{p}{4a_i}$.

The following lemma is an explicit quantitative version of the first part of GH from MO’s answer to a question on MathOverflow [14].

**Lemma 10.** Define $M(n) = \max_{1 \leq m \leq n} \tau(m)$. Then for $p \geq 17$,
\[ r_{Q^*}(n) \leq 2 \left( 2 \sqrt{\frac{n}{a_1^*}} + 1 \right) \left( 2 \sqrt{\frac{n}{a_2^*}} + 1 \right) M(25.09 np^{29/6}). \]

Note that $M(n) = O(n^c)$; more precisely, $M(n) \leq n^{1.538 \log 2 \log \log n}$ for $n \geq 3$ (see [15]).

**Proof of Lemma 10** From (1) we know that the number of choices for $(x_1, x_2)$ so that $Q^*(x_1, x_2, x_3, x_4) = n$ is at most
\[ \left( 2 \sqrt{\frac{n}{a_1^*}} + 1 \right) \left( 2 \sqrt{\frac{n}{a_2^*}} + 1 \right). \]
For each such pair $(x_1, x_2)$, we specialize $Q^*$ at this value, leaving a binary quadratic polynomial in the two variables $x_3, x_4$. More precisely, we wish to count integer solutions to
\[ P(x_3, y_3) := ax_3^2 + bx_3 x_4 + cx_4^2 + dx_3 + ex_4 + f = 0, \]
where
\begin{align*}
a &= a_3^* + a_4^* n_{43}^2, \quad b = 2a_4^* n_{43}, \quad c = a_4^*, \\
d &= 2a_3^* n_{31} + 2a_3^* n_{32} x_2 + 2a_4^* n_{41} n_{43} x_1 + 2a_4^* n_{42} n_{43} x_2, \\
e &= 2a_4^* n_{43} (n_{41} x_1 + n_{42} x_2), \quad f = Q^*(x_1, x_2, 0, 0) - n.
\end{align*}

Lemma 8 of [5] shows that if $\Delta = b^2 - 4ac$, $\xi = (be - 2cd)/\Delta$ and $\eta = (bd - 2ae)/\Delta$, then
\[ P(x, y) = \frac{(2a(x + \xi) + b(y + \eta))^2 - \Delta(y + \eta)^2}{4a} + P(-\xi, -\eta). \]

Note that $\Delta < 0$ since $Q^*$ is positive-definite. It follows that the number of solutions to $P(x, y) = 0$ is at most the number of solutions to $X^2 - \Delta Y^2 = -4a\Delta^2 P(-\xi, -\eta)$. The number of such solutions is at most the number of elements in the ring of integers of $K = \mathbb{Q}(\sqrt{-\Delta})$ with norm $m := -4a\Delta^2 P(-\xi, -\eta)$. If $K$ is neither $\mathbb{Q}(\sqrt{-3})$ nor $\mathbb{Q}(\sqrt{-1})$, the number of elements of norm $m$ is at most $2\tau(m)$ because there are at most $\tau(m)$ ideals with norm $m$ and $\mathcal{O}_K$ has only two units.

If $K = \mathbb{Q}(\sqrt{-1})$ then $-\Delta$ is an even square and it suffices to show that the number of solutions to $X^2 + 4Y^2 = m$ is at most $2\tau(m)$. The fact that
$X^2 + 4Y^2$ is anisotropic at 2 reduces this to proving the result for $m$ odd. The number of elements with norm $m$ in $\mathbb{Z}[i]$ is at most $4\tau(m)$, but $\mathbb{Z}[i]$ contains two sublattices with index 2 (namely, $(2, i), (1, 2i)$) corresponding to the quadratic form $X^2 + 4Y^2$. These two sublattices intersect only in elements with even norm, and this shows that if $m$ is odd, $r_{X^2+4Y^2}(m) \leq \frac{1}{2}r_{X^2+4Y^2}(m) \leq \frac{1}{2} \cdot 4\tau(m)$, giving the desired result. A similar argument applies if $K = \mathbb{Q}(\sqrt{-3})$.

It suffices to bound

$$-4a\Delta^2P(-\xi, -\eta) = 4a(4ac - b^2)((b^2 - 4ac) + ae^2 - bde + cd^2).$$

We have

$$a_3^* = \frac{p}{4a_3} \leq \frac{p}{4(9/16)} = \frac{4p}{9}$$

and

$$\frac{p}{16} = a_1a_2a_3a_4 \leq \left(\frac{4}{3}\right)^6 a_4,$$

which give $a_4 \geq \frac{3\sqrt{3}p^{3/4}}{16}$ and so $a_3^* \leq \frac{4p^{3/4}}{3\sqrt{3}}$. Therefore

$$a = a_3^* + a_4n_43 \leq \frac{4p^{3/4}}{9} + \frac{p^{3/4}}{3\sqrt{3}} \leq 0.54p,$$

$$|b| = 2a_4^*|n_43| \leq \frac{4p^{3/4}}{3\sqrt{3}},$$

$$c = a_4^* \leq \frac{4p^{3/4}}{3\sqrt{3}}, \quad |b^2 - 4ac| = |4a_3^*a_4^*| \leq \frac{64p^{7/4}}{27\sqrt{3}}.$$

To bound $d$ and $e$ we use $\frac{1}{a_1^2} = \frac{4a_1}{p}$ and $\frac{1}{a_2^2} = \frac{4a_2}{p}$ together with the bounds $a_1 \leq \frac{4}{3\sqrt{3}}p^{1/4}$, $a_2 \leq \frac{2^{2/3}}{3}p^{1/3}$. This yields

$$|d| = 2a_3^*|n_31x_1| + 2a_3^*|n_32x_2| + 2a_4^*|n_41n_43x_1| + 2a_4^*|n_42n_43x_2|$$

$$\leq \sqrt{n} \cdot \frac{2p^{3/4}}{3} + 7\frac{p^{3/4}}{6\sqrt{3}} + \sqrt{n} \cdot \frac{4p^{3/4}}{3} + \frac{p^{3/4}}{\sqrt{3}} \leq \sqrt{n} \cdot \frac{24^{3/4}p^{-1/3}}{\sqrt{3}} \cdot \left(\frac{2p}{3} + \frac{p^{3/4}}{\sqrt{3}}\right) \leq 2.941n^{1/2}p^{2/3},$$

$$|e| = 2a_4^*|n_43(n_41x_1 + n_42x_2)| \left(\frac{2p}{3} + \frac{p^{3/4}}{\sqrt{3}}\right)$$

$$\leq 2 \cdot \frac{4p^{3/4}}{3\sqrt{3}} \cdot \left(\frac{7}{8} \sqrt{n} \cdot \frac{1}{a_1^2} + \frac{3}{4} \sqrt{n} \cdot \frac{1}{a_2^2}\right) \leq 0.946n^{1/2}p^{5/12}.$$
Finally, $-n \leq f \leq 0$. Putting everything together we have

$$4a(4ac - b^2)((b^2 - 4ac)f + ae^2 - bde + cd^2)$$

$$\leq 4 \cdot 0.54p \cdot \frac{64p^{7/4}}{27\sqrt{3}} \left( \frac{64p^{7/4}}{27\sqrt{3}}n + 0.484np^{11/6} + 2.142np^{11/6} + 6.659np^{25/12} \right)$$

$$\leq 25.09np^{29/6}.$$  

It follows that for each $(x_1, x_2)$, the number of solutions to $Q^*(x_1, x_2, x_3, x_4) = n$ is $\leq 2M(25.09np^{29/6})$. \[\blacksquare\]

**Lemma 11.** For $p \geq 17$, we have

$$\sum_{n \leq x} r_Q^*(n) \leq 64x^2p^{-3/2} + 75.52x^{3/2}p^{-1} + 30.49xp^{-1/2} + 8.11x^{1/2} + 1.$$  

We note that a similar bound is worked out in [4, Lemma 4.2].

**Proof of Lemma 11.** We begin with by expanding a rather simple bound and perhaps inelegantly moving term by term:

$$\sum_{n \leq x} r_Q^*(n) \leq \prod_{i=1}^{4} \left( 4\sqrt{\frac{x}{a_i}} + 1 \right)$$

$$\leq 256\sqrt{a_1a_2a_3a_4}x^2p^{-2}$$

$$+ 64(\sqrt{a_1a_2a_3} + \sqrt{a_1a_2a_4} + \sqrt{a_1a_3a_4} + \sqrt{a_2a_3a_4})x^{3/2}p^{-3/2}$$

$$+ 16(\sqrt{a_1a_2} + \sqrt{a_1a_3} + \sqrt{a_1a_4} + \sqrt{a_2a_3} + \sqrt{a_2a_4} + \sqrt{a_3a_4})xp^{-1}$$

$$+ 4(\sqrt{a_1} + \sqrt{a_2} + \sqrt{a_3} + \sqrt{a_4})xp^{-1/2} + 1.$$  

Using $a_1a_2a_3a_4 = \frac{p}{16}$ immediately bounds the first term by $64x^2p^{-3/2}$. Combining $a_1a_2a_3a_4 = \frac{p}{16}$ with the fact that $a_i \leq \frac{4}{3}a_{i+1}$ for $1 \leq i \leq 3$ shows that $\sqrt{a_1a_2a_3} \leq \frac{1}{3\sqrt{4}}p^{3/8}$.  

As the minimum of $a_3$ is $\frac{9}{16}$, we have $\sqrt{a_1a_2a_4} \leq \frac{1}{3}p^{1/2}$. Similarly, with a minimum of $a_2$ of $\frac{3}{4}$ we have $\sqrt{a_1a_3a_4} \leq \frac{1}{2\sqrt{3}}p^{1/2}$. Last, considering the minimum of $a_1$ is 1, we see that $\sqrt{a_2a_3a_4} \leq \frac{1}{3}p^{1/2}$.  

For the remaining six terms: Using $a_i \geq \frac{3}{4}a_{i-1}$ gives $a_1 \leq \frac{4}{3\sqrt{3}}p^{1/4}$, $a_2 \leq \frac{1}{\sqrt{3}}p^{1/4}$, which bounds $\sqrt{a_1a_2}$. For $\sqrt{a_3a_4}$ we note that $a_2 \geq \frac{3}{4}a_1$, $a_4 \geq \frac{3}{4}a_3$ and therefore $\frac{p}{16} \geq \frac{9}{16}(a_1a_3)^2$.  

For $\sqrt{a_2a_3}$ we note that the maximum value of $a_2a_3$ occurs when $a_1 = 1$, $a_2 = \frac{2^{2/3}}{3}p^{1/3}$, $a_3 = 2^{-4/3}p^{1/3}$, $a_4 = 3 \cdot 2^{-10/3}p^{1/3}$. This yields $\sqrt{a_2a_3} \leq \frac{1}{2^{1/3}3^{-1/3}}p^{1/3}$.  

We have $a_1a_4 = \frac{p}{16a_2a_3}$ and $a_2a_3$ is minimized when $a_2 = 3/4$ and $a_3 = 9/16$. This gives $\sqrt{a_1a_4} \leq \frac{2}{3\sqrt{3}}p^{1/2}$.  

Similarly, \( \sqrt{a_2a_4} \leq \frac{\sqrt{p}}{\sqrt[4]{16a_3a_4}} \leq \frac{1}{3} \sqrt{p} \) and \( \sqrt{a_3a_4} \leq \frac{\sqrt{p}}{\sqrt[16]{a_1a_2}} \leq \frac{1}{2\sqrt{3}} \sqrt{p} \).

Finally, note that \( \sqrt{a_4} \leq \frac{2}{3\sqrt{3}} p^{1/2} \). These substitutions give

\[
\sum_{n \leq x} r_{Q^*}(n) 
\leq 256 \sqrt{a_1a_2a_3a_4}x^2p^{-2} 
+ 64(\sqrt{a_1a_2a_3} + \sqrt{a_1a_2a_4} + \sqrt{a_1a_3a_4} + \sqrt{a_2a_3a_4})x^{3/2}p^{-3/2} 
+ 16(\sqrt{a_1a_2} + \sqrt{a_1a_3} + \sqrt{a_1a_4} + \sqrt{a_2a_3} + \sqrt{a_2a_4} + \sqrt{a_3a_4})xp^{-1} 
+ 4(\sqrt{a_1} + \sqrt{a_2} + \sqrt{a_3} + \sqrt{a_4})\sqrt{x}p^{-1/2} + 1
\]

\[
\leq 64x^2p^{-3/2} + 64\left(\frac{1}{3^{1/4}}p^{3/8} + \frac{1}{3}p^{1/2} + \frac{1}{2\sqrt{3}}p^{1/2} + \frac{1}{4}p^{1/2}\right)x^{3/2}p^{-3/2}
+ 16\left(\frac{2}{3}p^{1/4} + \frac{1}{3}p^{1/4} + \frac{1}{2\sqrt{3}}p^{1/4} + \frac{3}{2}\sqrt{\frac{1}{3}}p^{1/2} + \frac{1}{3}p^{1/2} + \frac{1}{2\sqrt{3}}p^{1/2}\right)xp^{-1}
+ 4\left(\frac{2}{3\sqrt{3}}p^{1/8} + \frac{1}{2\sqrt{3}}p^{1/6} + \frac{1}{\sqrt{3}}p^{1/4} + \frac{2}{3\sqrt{3}}p^{1/2}\right)x^{3/2}p^{-1/2} + 1
\]

and the result follows.

**Corollary 12.** For \( 1 \leq x \leq p \) we have

\[
\sum_{n \leq x} r_{Q^*}(n) \leq 179.12 \sqrt{x}
\]

and for \( x \geq p \) we have

\[
\sum_{n \leq x} r_{Q^*}(n) \leq 178.37 x^2p^{-3/2}.
\]

In bounding the main term in \( \langle C, C \rangle \) we apply Lemma 10 and for that, we need lower bounds on \( a_1^*, a_2^*, a_3^*, \) and \( a_1^*a_2^* \).

**Lemma 13.** We have

\[
a_1^* \geq \min \left\{ \frac{9}{48/3}p^{3/4}, \frac{4p}{3 \cdot 2^{4/3}(a_4^*)^{1/3}} \right\}, \quad a_3^* \geq \frac{3}{4}p^{1/2},
\]

\[
a_2^* \geq \min \left\{ \frac{3p^{2/3}}{4^{4/3}}, \frac{p}{\sqrt{3}(a_4^*)^{1/2}} \right\}, \quad a_1^*a_2^* \geq \frac{3p^2}{2^{16/3}(a_4^*)^{2/3}}.
\]

**Proof.** Recall that \( a_1 \geq 1, a_i+1 \geq \frac{3}{4}a_i \) for \( i \geq 2 \), \( a_1a_2a_3a_4 = p/16 \) and \( a_i^* = \frac{p}{4a_i} \). These imply that \( a_1^* \geq \frac{3}{4} a_2^* \geq \frac{9}{16} a_3^* \geq \frac{27}{64} a_4^* \) and \( a_1^* \leq p/4 \).

We first establish the bounds on \( a_1^* \). We have

\[
\frac{p^3}{16} = a_1^*a_2^*a_3^*a_4^* \leq \left( \frac{4}{3} \right)^6 (a_1^*)^4
\]
and so \( a_1^* \geq \frac{9}{4^8}p^{3/4} \). Also, \( \frac{p^3}{16a_1^*} = a_1^*a_2^*a_3^* \leq \left(\frac{3}{4}\right)3(a_1^*)^3 \) and so \( a_1^* \geq \frac{4}{3^{2/3}}p(a_4^*)^{-1/3} \), as desired.

Next, we establish the bounds on \( a_2^* \). Since \( a_1^* \leq p/4 \), we observe that \( p^3/16 = a_1^*a_2^*a_3^*a_4^* \) gives

\[
\frac{p^2}{4} = \frac{p^3}{16(p/4)} \leq \frac{p^3}{16a_1^*} = a_2^*a_3^*a_4^* \leq \left(\frac{4}{3}\right)^3(a_2^*)^3.
\]

This gives \( a_2^* \geq \frac{3p^2/3}{4^3} \). Again, \( a_1^* \leq p/4 \) implies that

\[
\frac{p^3}{16(p/4)a_4^*} \leq \frac{p^3}{16a_1^*a_4^*} = a_2^*a_3^* \leq \frac{3}{4}(a_2^*)^2.
\]

This gives \( a_2^* \geq \frac{1}{\sqrt[3]{3}}p(a_4^*)^{-1/2} \).

To establish the lower bound on \( a_3^* \), we use \( a_2^* = \frac{p}{4a_2} \leq p/3 \) and so

\[
\frac{3}{4p} = \frac{p^3}{16(p^2/12)} \leq \frac{p^3}{16a_1^*a_2^*} = a_3^*a_4^* \leq \frac{4}{3}(a_3^*)^2,
\]

and this gives \( a_3^* \geq \frac{3}{4}\sqrt{p} \).

To establish the lower bound on \( a_1^*a_2^* \), we next derive an upper bound for \( a_3^* \). We have

\[
\frac{3^3}{4^3}(a_3^*)^3a_4^* \leq a_1^*a_2^*a_3^*a_4^* = \frac{p^3}{16}
\]

and so \( a_3^* \leq \frac{24/3}{3}p(a_4^*)^{-1/3} \). From the bound on \( a_3^* \) we get

\[
a_1^*a_2^* \geq \frac{p^3}{16a_3^*a_4^*} \geq \frac{3p^2}{2^{16/3}(a_4^*)^{2/3}}.
\]

This proves the final claim.

**Lemma 14.** Let \( p \geq 101 \). Then

\[
\frac{p}{p+1} \sum_{n=1}^{\infty} \frac{2^{\omega(gcd(n,p))} \cdot 2r_Q^*(n)^2}{n} \sum_{d=1}^{\infty} \psi\left(d\sqrt{\frac{n}{p}}\right) \leq \frac{1}{\min Q^*} + 3216.6524 \frac{M(25.09p^{35/6})}{p^{1/4}}.
\]

**Proof.** For this proof, we proceed by cases on the size of \( n \) relative to \( p \). In many intervals we use the bound \( \sum_{n \leq x} r_Q^*(n)^2 \leq r_Q^*(x) \sum_{n \leq x} r_Q^*(n) \); this then will use Lemmas 10 and 11.

- \( 1 \leq n \leq \frac{3}{4}p^{1/2} \): First, observe that if \( r_Q^*(n) = 0 \) for \( 1 \leq n \leq \frac{3}{4}p^{1/2} \), then the contribution from this segment is zero. Assume therefore that \( \min Q^* \leq \frac{3}{4}p^{1/2} \).
If
\[ \vec{x} = \begin{bmatrix} x_1 \\
 x_2 \\
 x_3 \\
 x_4 \end{bmatrix} \neq \vec{0} \]
and \( i \) is the smallest positive integer such that \( x_i \neq 0 \), then \( Q^*(\vec{x}) \geq a^*_ix_i^2 \geq a^*_i \). Hence \( \min Q^* \geq \min\{a^*_1, a^*_2, a^*_3, a^*_4\} \). Lemma 13 immediately shows that \( a^*_1, a^*_2 \) and \( a^*_3 \) are all at least \( \frac{3}{4}p^{1/2} \). So if \( r_{Q^*}(n) \neq 0 \), then \( n = a^*_4i^2 \) and \( r_{Q^*}(n) = 2 \). It follows that the contribution from this range is bounded by
\[
\frac{p}{p+1} \sum_{n=1}^{\frac{3}{4}p^{1/2}} 2^{\omega(gcd(n,p))}2r_{Q^*}(n)^2 \sum_{d=1}^{\infty} \psi\left( \frac{d}{\sqrt{p}} \right) \]
\[
\leq \frac{p}{p+1} \sum_{i=1}^{3p^{1/2}} \frac{2 \cdot 2^2}{a^*_i i^2} \cdot \frac{3}{4\pi^2} \]
\[
\leq 8 \cdot \frac{3}{4\pi^2} \sum_{i=1}^{\infty} \frac{1}{a^*_i i^2} \leq \frac{1}{a^*_4} \cdot \frac{6}{\pi^2} \sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{1}{a^*_4} = \frac{1}{\min Q^*}.
\]
Plugging the bounds from Lemma 13 into Lemma 10 we obtain
\[
(2) \quad r_{Q^*}(n) \leq \left( 29.328 \frac{n}{p} (a^*_4)^{1/3} + 10.764 \sqrt{\frac{n}{p}} (a^*_4)^{1/4} + 2 \right) M(25.09np^{29/6}).
\]

• \( \frac{3}{4}p^{1/2} \) \( \leq n \leq \frac{p}{100} \): In this region, and the remaining ones, we use
\[
\frac{p}{p+1} \sum_{n=a}^{beta} 2^{\omega(gcd(n,p))}2r_{Q^*}(n)^2 \sum_{d=1}^{\infty} \psi\left( \frac{d}{\sqrt{p}} \right) \]
\[
\leq 2 \left( \sum_{d=1}^{\infty} \psi\left( \frac{\alpha}{\sqrt{p}} \right) \right) \]
\[
\cdot \left[ \int_{\alpha}^{\beta} \frac{1}{t^2} \sum_{\alpha \leq n \leq t} 2^{\omega(gcd(n,p))}r_{Q^*}(n)^2 dt + \frac{1}{\beta} \sum_{\alpha \leq n \leq \beta} 2^{\omega(gcd(n,p))}r_{Q^*}(n)^2 \right].
\]
We also use the fact that \( gcd(n,p) = 1 \) if \( n < p \), and we apply the above result using inequality 2 to bound \( r_{Q^*}(n) \) and we use Corollary 12 to bound \( \sum_{n \leq x} r_{Q^*}(n) \). For \( a = \frac{3}{4} \sqrt{p} \) and \( \beta = \frac{1}{100} p \), we get an upper bound of
\[
\frac{1}{\sqrt{p}} \left[ 239.52(a^*_4)^{1/3} + 146.52(a^*_4)^{1/4} \log p + 125.74p^{1/4} \right] M(25.09p^{35/6}).
\]

• \( \frac{p}{100} \leq n \leq p - 1 \): We use the same formulas as above and take \( (\alpha, \beta) = (2^k/p/100, 2^{k+1}p/100) \) for \( 0 \leq k \leq 5 \), as well as \( (\alpha, \beta) = (64p/100, p) \).
In total we obtain
\[ \frac{1}{\sqrt{p}} [596.82(a_4^*)^{1/3} + 955.35(a_4^*)^{1/4} + 949.86]. \]

- \( n \geq p \). Using \( a_4^* \geq 2 \), we have

\[ r_{Q^*}(n) \leq 41.1 \cdot \frac{n}{p} (a_4^*)^{1/3} M(25.09np^{29/6}) \]

for \( n \geq p \). We further have
\[ \frac{p}{p+1} \sum_{k=1}^{\infty} \sum_{n=k^2p} (k+1)^2p \cdot 2^{2 \omega(p)} r_{Q^*}(n) \cdot \frac{n}{p} \sum_{d=1}^{\infty} \psi \left( d, \sqrt{\frac{n}{p}} \right). \]

We use \( \sum_{d=1}^{\infty} \psi(dx) \leq 9x^{3/2} e^{-4\pi x} \) and get the bound
\[ \sum_{k=1}^{\infty} 36k^{3/2} e^{-4\pi k} \sum_{n=k^2p} \frac{r_{Q^*}(n)^2}{n} \]
\[ = 36 \sum_{k=1}^{\infty} k^{3/2} e^{-4\pi k} \left[ \int_{k^2p}^{\infty} \frac{1}{t^2} \sum_{n\leq t} r_{Q^*}(n) \left( \max_{n,t} r_{Q^*}(n) \right) dt \right] \]
\[ + \frac{1}{(k+1)^2p} \left( \sum_{n\leq(k+1)^2p} r_{Q^*}(n) \right) \left( \max_{n\leq(k+1)^2p} r_{Q^*}(n) \right). \]

The inequality \( \sum_{n\leq x} r_{Q^*}(n) \leq 178.37x^2p^{-3/2} \) gives the bound
\[ 3665.51 \frac{a_4^*}{\sqrt{p}} (4k^3 + 6k^2 + 4k + 1)M(25.09(k+1)^2p^{35/6}) \]
on the integral. The second term is bounded by
\[ 7331.1(k+1)^4 \frac{a_4^*}{\sqrt{p}} M(25.09(k+1)^2p^{35/6}), \]

for a total of
\[ \frac{a_4^*}{\sqrt{p}} \sum_{k=1}^{\infty} k^{3/2} (131958.36(4k^3 + 6k^2 + 4k + 1) + 263919.6(k+1)^4) \cdot M(25.09(k+1)^2p^{35/6}) e^{-4\pi k}. \]

If \( n \geq 2 \), there is an even integer \( r \) with \( 1 \leq r \leq n \) for which \( \tau(r) \) is a maximum. (If the \( r \) was odd, we could replace an odd prime factor of \( r \) with 2 resulting in a lower number with the same number of divisors.) This implies that for \( n \geq 1 \), \( M(2n) = \max_{r \leq n} \tau(2r) \leq \max_{r \leq n} \tau(2)\tau(r) = \tau(2)M(n) = 2M(n) \). A straightforward induction then shows that \( M(kn) \leq 2kM(n) \).

Applying this bound in the above formula yields \( \frac{a_4^*}{\sqrt{p}} M(25.09p^{35/6}) \) times a convergent series in \( k \), whose sum is \( \leq 82.525 \), and so we get the contribution
82.525(a_4^*)^{1/3}M(25.09p^{35/6})/\sqrt{p} to the inner product from the portion with \( n \geq p \).

Having completed the analysis of the cases, we use \( a_4^* \leq \frac{4p^{3/4}}{3\sqrt{3}} \) to obtain the bound

\[
\frac{1}{\min Q^*} + \frac{M(25.09p^{35/6})}{p^{1/4}} \left( 918.87 \frac{4}{3\sqrt{3}} + 353.53 \frac{4}{3\sqrt{3}} \cdot \frac{\log(p)}{p^{1/16}} + 125.74 + \frac{949.86}{p^{1/4}} \right).
\]

The maximum value of \( \frac{\log x}{x^{1/16}} \) occurs when \( x = e^{16} \). Using the bound \( \frac{\log p}{p^{1/16}} \leq \frac{16}{e} \) gives the desired result.

Finally, we combine Lemmas 9 and 14 to prove Theorem 2.

**Proof of Theorem 2.** We simplify the formulas for the sums of the right hand sides of Lemmas 9 and 14 by allowing both terms to include \( M(25.09p^{35/6}) \). If \( p = 101 \), then 25.09 \cdot p^{35/6} \approx 12341710124278. We use a tabulated list of record high values of \( \tau(n) \) (see [OEIS sequence A002182](https://oeis.org/A002182)) to determine that the integer with the greatest number of divisors less than 25 \cdot 101^{35/6} is 9316358251200 and so \( M(25.09 \cdot 101^{35/6}) = 10752 \).

Next, we note that

\[
\frac{337.26 \log(p+2)}{p} \leq 0.00457 \frac{10752}{p^{1/4}} \leq 0.00457 \frac{M(25.09p^{35/6})}{p^{1/4}}
\]

for \( p \geq 101 \). This can be seen using the fact that \( \log(x+2)/x^{3/4} \) is decreasing for \( x > e^{4/3} - 2 \). Finally,

\[
\frac{206.67}{p} \leq 0.00061 \frac{10752}{p^{1/4}} \leq 0.00061 \frac{M(25.09p^{35/6})}{p^{1/4}}
\]

for \( p \geq 101 \). Therefore

\[
\langle C, C \rangle \leq \frac{1}{\min Q^*} + (3216.6524 + 0.00457 + 0.00061) \frac{M(25.09p^{35/6})}{p^{1/4}}
\]

\[
\leq \frac{1}{\min Q^*} + 3216.66 \frac{M(25.09p^{35/6})}{p^{1/4}},
\]

as desired.

**Proof of Theorem 1.** We have \( r_Q(n) = a_E(n) + a_C(n) \). A lower bound on \( a_E(n) \) was given in Theorem 6. The bound

\[
C_Q \leq \sqrt{\sum_{i=1}^{s} \frac{1}{\langle g_i, g_i \rangle} \sqrt{\langle C, C \rangle}}
\]
combined with Lemma 7 and Theorem 2 gives

\[ C_Q \leq \sqrt{p} \sqrt{\frac{4\pi^2}{3} \left( 1 + \frac{1}{p} \right) \left( 1 + \frac{54.6}{p^{3/2}} \right)} \sqrt{\frac{1}{\min Q^*} + 3216.66 \frac{M(25.09p^{35/6})}{p^{1/4}}}. \]

For \( p \geq 101 \), we have

\[ \sqrt{\frac{4\pi^2}{3} \left( 1 + \frac{1}{p} \right) \left( 1 + \frac{54.6}{p^{3/2}} \right)} \leq 3.75. \]

4. Sum of exceptions. In this section, we prove Corollary 3. We proceed by using Theorem 2 and relating \( \langle C, C \rangle \) to an integral of the form

\[ \int_{\sigma - 1/2}^{1/2} \int_{-1/2}^{1/2} |C(x + iy)|^2 \, dx \, dy. \]

**Lemma 15.** Let \( \sigma > 0 \) and let \( F(\sigma) \) be the largest number of points in the region \( \{ x + iy \mid -1/2 \leq x \leq 1/2, y \geq \sigma \} \) that can be in a single \( \Gamma_0(p) \)-orbit. Then \( F(\sigma) \leq 1 + \frac{\sqrt{3}}{2p\sigma}. \)

**Proof.** Choose a \( \Gamma_0(p) \)-orbit and let \( w = \alpha + i\beta \) be a point with \( |\alpha| \leq 1/2, \beta \geq \sigma \) that has lowest imaginary part (subject to \( \beta \geq \sigma \)) in this orbit. We will count the number of matrices \( M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) (up to negation) in \( \Gamma_0(p) \) with \( \text{Im}(M(w)) \geq \text{Im}(w) \) and \( |\text{Re}(M(w))| \leq 1/2. \)

We have

\[ \text{Im}(M(w)) = \frac{\text{Im}(w)}{|cw + d|^2}. \]

So in order that \( \text{Im}(M(w)) \geq w \), we need \(|cw + d|^2 \leq 1. \) If \( c = 0 \), then in order for \( M \) to be in \( \text{SL}_2(\mathbb{Z}) \) we must have \( d = \pm 1 \) and so up to negation we have \( d = 1. \) This gives us one choice of \((c, d)\).

Assume then that \( c > 0. \) We have \(|cw + d|^2 = (\alpha + d)^2 + (c\beta)^2 \leq 1 \) and also \( \gcd(c, d) = 1. \) Since \(-1/2 \leq \alpha \leq 1/2, \) there is at most one choice of \( d \) such that \((\alpha + d)^2 \leq 1 \) (either \( d = 1 \) or \( d = -1 \)). In any case, \(|\alpha + d| \geq 1/2 \) and so \((\alpha + d)^2 \geq 1/4. \) Thus, \((c\beta)^2 \leq 3/4 \) and so \( c \leq \frac{\sqrt{3}}{2\beta}. \) Recalling that \( c \) must be a multiple of \( p, \) the total number of choices of \((c, d)\) with \( c > 0 \) is at most \( \frac{\sqrt{3}}{2p\beta}. \)

Finally, for each choice of \((c, d),\) if \( M_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) and \( M_2 = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \) are two different matrices in \( \Gamma_0(p) \) with the same choice of \((c, d),\) then \( M_1 M_2^{-1} = \begin{bmatrix} * & * \\ 0 & 1 \end{bmatrix}. \) If \(-1/2 \leq \text{Re}(M_2(w)) \leq 1/2, \) then \( \text{Re}(M_1(w)) = \text{Re}(M_1 M_2^{-1} M_2(w)) \) is not between \(-1/2 \) and \( 1/2. \) Thus, for each pair \((c, d),\) there is at most one choice of \( a, b \) that gives \(-1/2 \leq \text{Re}(M(w)) \leq 1/2. \) This proves the claim. \( \blacksquare \)
It follows from the claim that
\[
\frac{1}{p+1} \int_{\sigma-1/2}^{\infty} \int_{\sigma-1/2}^{1/2} |C(x+iy)|^2 \, dx \, dy \leq \left(1 + \frac{\sqrt{3}}{2p\sigma}\right) \langle C, C \rangle
\]
since the region $\sigma \leq y \leq \infty$ is contained in at most $F(\sigma) \leq 1 + \frac{\sqrt{3}}{2p\sigma}$ different fundamental domains for $\Gamma_0(p)$.

A straightforward calculation shows that
\[
\frac{1}{p+1} \int_{\sigma-1/2}^{\infty} \int_{\sigma-1/2}^{1/2} |C(x+iy)|^2 \, dx \, dy
\]
\[
= \int_{\sigma-1/2}^{1/2} \sum_{m,n} a_C(m)a_C(n)e^{-2\pi(m+n)y}e^{2\pi i(m-n)x} \, dx \, dy
\]
\[
= \int_{\sigma}^{\infty} \sum_{n=1}^{\infty} a_C(n)^2 e^{-4\pi ny} \, dy = \frac{1}{4\pi} \sum_{n=1}^{\infty} \frac{a_C(n)^2}{n} e^{-4\pi n\sigma}.
\]

If $n$ is an integer such that $r_Q(n) = 0$, then $a_C(n) = -a_E(n)$. We have
\[
a_E(n) \geq \frac{24(p-1)}{p^{3/2}} \phi(n) \gg \frac{1}{\sqrt{p}} \frac{n}{\log \log n}
\]
if $n > 2$.

Let $m$ be the largest positive integer $n$ with $r_Q(n) = 0$ and $\sigma = 1/m$. Theorem 1 shows that $m \ll \max \{p^{2+\epsilon}/\min Q^*, p^{7/4+\epsilon}\}$.

It follows that
\[
\sum_{r_Q(n)=0} n \ll 3 + (\log \log m)^2 \sum_{r_Q(n)=0} \frac{pm}{(\log \log n)^2}
\]
\[
\ll 3 + p(\log \log m)^2 \sum_{r_Q(n)=0} \frac{a_E(n)^2}{n}
\]
\[
\ll 3 + p(\log \log m)^2 \sum_{r_Q(n)=0} \frac{a_C(n)^2}{n}
\]
\[
\ll 3 + p(\log \log m)^2 \sum_{n=1}^{\infty} \frac{a_C(n)^2}{n} e^{-4\pi n\sigma}
\]
\[
\ll 3 + (\log m)^2 (p+1)pF(1/m) \langle C, C \rangle
\]
\[
\ll p^{2+\epsilon} \left(1 + \frac{\sqrt{3} m}{2p}\right) \left(\frac{1}{\min Q^*} + p^{-1/4+\epsilon}\right).
\]
If $\min Q^* \leq p^{1/4-\epsilon}$, we get
\[
\sum_{r_Q(n) = 0} n \ll p^{1+\epsilon} \frac{m}{\min Q^*} \ll \frac{p^{3+\epsilon}}{(\min Q^*)^2}.
\]
If $\min Q^* > p^{1/4-\epsilon}$, we get
\[
\sum_{r_Q(n) = 0} n \ll p^{1+\epsilon} mp^{-1/4+\epsilon} \ll p^{5/2+\epsilon}.
\]
In all cases, we obtain
\[
\sum_{r_Q(n) = 0} n \ll \max \left\{ \frac{p^{3+\epsilon}}{(\min Q^*)^2}, p^{5/2+\epsilon} \right\},
\]
as desired.

5. Family of forms with explicit exceptions. We end this paper with an example of a family of prime discriminant forms, and the classification of the excepted values.

**Theorem 16.** Let $p \equiv 5 \pmod{8}$ be prime. Consider the quadratic form
\[
Q_p(\vec{x}) = x^2 + xy + xz + xw + y^2 + yz + yw + z^2 + zw + \frac{p+3}{8} w^2.
\]
This quaternary form is almost universal; more specifically, it represents all $n \in \mathbb{N}$ except those $n < \frac{p}{8}$ of the form $n = 4^k (16\ell + 14)$ for integers $k$ and $\ell$.

We begin with a pair of lemmas regarding the integers represented by a particular ternary subform of $Q_p$ and by a specific ternary form. We begin by discussing the integers represented by the subform $Q_p((x, y, z, 0))$.

**Lemma 17.** The ternary quadratic form $x^2 + xy + xz + y^2 + yz + z^2$ is regular and of class number 1. It represents all positive integers except those of the form $4^k (16\ell + 14)$.

**Proof.** That the form is regular and of class number 1 is readily found in [12]. In particular, this means that $n \in \mathbb{N}$ is represented by the form if and only if $n$ is locally represented by the form. Clearly, over $\mathbb{R}$ all $n \in \mathbb{N}$ are represented. Over $\mathbb{Q}_p$ for $p \neq 2$, Hensel’s Lemma states it is sufficient to check modulo $p$ to guarantee representation. As this form is nondegenerate and of dimension at least 2, that makes it universal over a finite field, and specifically modulo $p$. The only prime left to consider is $p = 2$. Using Siegel’s theory of local densities, one must check whether any quotients of an integer by a square factor are represented modulo $2^5$. A simple Sage computation shows that $n \equiv 14, 30 \pmod{32}$ are not locally represented, which implies that $n = 4^k (16\ell + 14)$ are not represented. Moreover, for all other $n$, there are
Good-type solutions (using the language of [9]), which means \( n \) is represented over \( \mathbb{Q}_2 \).

We will also reference the following result:

**Lemma 18.** The ternary form \( X^2 + 2Y^2 + 6Z^2 \) is regular, and represents every positive integer \( N \equiv 9 \pmod{24} \). Additionally, there is a representation of the form \( X = 4z + 1, Y = 3y + z + 1, \) and \( Z = 2x + y + z + 1 \) for some integers \( x, y, z \).

**Proof.** That the form is regular can again be found in [12]. Clearly, over \( \mathbb{R} \) all \( n \equiv 9 \pmod{24} \) are represented. For \( p \neq 2, 3 \), over \( \mathbb{Q}_p \) Hensel’s Lemma again states it is sufficient to check modulo \( p \), and again in those cases the form is ternary and nondegenerate so it is actually universal. For \( p = 3 \), note that \( (1, 2, 0) \) is a solution with at least one nonzero partial derivative, so it can be lifted to a solution in \( \mathbb{Z}_3 \). For \( p = 2 \), all partial derivatives vanish, and so it suffices to look modulo \( 32 \) to ensure a 2-adic solution. Running Sage [20], we see that there are solutions. By local-global principles, then, \( X^2 + 2Y^2 + 6Z^2 \) represents all positive integers \( n \equiv 9 \pmod{24} \).

Now set \( N = 24m + 9 \) for some nonnegative integer \( m \). For \( 24m + 9 = X^2 + 2Y^2 + 6Z^2 \), \( X \) must be odd. Without loss of generality, by swapping \( X \) and \( -X \) as necessary, \( X = 4z + 1 \) for \( z \in \mathbb{Z} \). Considering now the equation modulo \( 3 \), one has \( X \equiv \pm Y \pmod{3} \). Again, without loss of generality suppose \( X \equiv Y \pmod{3} \). Writing \( X = 3(k + z) + 1 \) for some integer \( k \) and solving for \( Y \) yields \( Y = 3(k + z) + z + 1 \). Last, consider the equation modulo 8. Then \( Y \) and \( Z \) must have the same parity. Therefore \( Z = 2\ell \) for \( \ell \in \mathbb{Z} \). Substituting again, this means \( Z = 2(\ell + k + z) + (k + z) + z + 1 \). And noting the parentheses, letting \( y = k + z \) and \( x = \ell + k + z \), we get the claim.

**Proof of Theorem 16.** By Lemma 17 we need only consider representation of integers \( n = 4^k(16\ell + 14) \) by the form \( Q_p \). We begin by diagonalizing the form over \( \mathbb{Q} \):

\[
Q_p(x) = x^2 + xy + xz + xw + y^2 + yz + yw + z^2 + zw + \frac{p + 3}{8}w^2
\]

\[
= \left( x + \frac{1}{2}(y + z + w) \right)^2 + \frac{3}{4}\left( y + \frac{1}{3}(z + w) \right)^2 + \frac{2}{3}\left( z + \frac{1}{4}w \right)^2 + \frac{p}{8}w^2.
\]

If \( Q_p \) represents \( n \), then \( \frac{p}{8}w^2 \leq n \), or \(|w| \leq 2\left( \sqrt{\frac{2n}{p}} \right) \). When \( \sqrt{\frac{2n}{p}} < \frac{1}{2} \), or equivalently when \( n < \frac{p}{8} \), this forces \( w = 0 \) and Lemma 17 shows that such \( n \) cannot be represented by the subsequent ternary subform.

Next suppose \( n = 4^k(16\ell + 14) \) with \( n \geq \frac{p}{8} \). Set \( m = n - \frac{p + 3}{8} \). By Lemma 18 there exist integers \( x, y, z \) with

\[
24m + 9 = (4z + 1)^2 + 2(3y + z + 1)^2 + 6(2x + y + z + 1)^2.
\]
Expanding and simplifying gives

\[ n = x^2 + xy + xz + y^2 + yz + z^2 + x + y + z + \frac{p + 3}{8} = Q_p((x, y, z, 1)). \]

Hence, \( Q_p \) represents \( n \). ■

References

[1] M. Barowsky, W. Damron, A. Mejia, F. Saia, N. Schock, and K. Thompson, *Classically integral quadratic forms excepting at most two values*, Proc. Amer. Math. Soc. 146 (2018), 3661–3677.

[2] M. Bhargava, *On the Conway–Schneeberger fifteen theorem*, in: Quadratic Forms and Their Applications (Dublin, 1999), Contemp. Math. 272, Amer. Math. Soc., Providence, RI, 2000, 27–37.

[3] M. Bhargava and J. Hanke, *Universal quadratic forms and the 290-Theorem*, preprint.

[4] V. Blomer, *Uniform bounds for Fourier coefficients of theta-series with arithmetic applications*, Acta Arith. 114 (2004), 1–21.

[5] V. Blomer and A. Pohl, *The sup-norm problem on the Siegel modular space of rank two*, Amer. J. Math. 138 (2016), 999–1027.

[6] T. D. Browning and R. Dietmann, *On the representation of integers by quadratic forms*, Proc. London Math. Soc. 96 (2008), 289–416.

[7] J. H. Bruinier and M. Bundschuh, *On Borcherds products associated with lattices of prime discriminant*, Ramanujan J. 7 (2003), 49–61.

[8] T. Estermann, *On Kloosterman’s sum*, Mathematika 8 (1961), 83–86.

[9] J. Hanke, *Local densities and explicit bounds for representability by a quadratic form*, Duke Math. J. 124 (2004), 351–388.

[10] H. Iwaniec, *Topics in Classical Automorphic Forms*, Grad. Stud. Math. 17, Amer. Math. Soc., Providence, RI, 1997.

[11] K. Iwasawa, *Lectures on p-Adic L-Functions*, Princeton Univ. Press, Princeton, NJ, 1972.

[12] W. C. Jagy, I. Kaplansky and A. Schiemann, *There are 913 regular ternary forms*, Mathematika 44 (1997), 332–341.

[13] Y. Kitaoka, *Arithmetic of Quadratic Forms*, Cambridge Tracts in Math. 106, Cambridge Univ. Press, Cambridge, 1993.

[14] https://mathoverflow.net/questions/256576/which-quaternary-quadratic-form-represents-n-the-greatest-number-of-times.

[15] J.-L. Nicolas et G. Robin, *Majorations explicites pour le nombre de diviseurs de N*, Canad. Math. Bull. 26 (1983), 485–492.

[16] K. Ono and K. Soundararajan, *Ramanujan’s ternary quadratic form*, Invent. Math. 130 (1997), 415–454.

[17] S. Ramanujan, *On the expression of a number in the form ax^2 + by^2 + c2^2 + du^2*, Proc. Cambridge Philos. Soc. 19 (1917), 11–21.

[18] G. Robin, *Grandes valeurs de la fonction somme des diviseurs et hypothèse de Riemann*, J. Math. Pures Appl. (9) 63 (1984), 187–213.

[19] J. Rouse, *Quadratic forms representing all odd positive integers*, Amer. J. Math. 136 (2014), 1693–1745.

[20] The Sage Developers, *SageMath, The Sage Mathematics Software System (Version 6.10)*, 2015, http://www.sagemath.org.
Quaternary quadratic forms with prime discriminant

[21] R. Schulze-Pillot, On explicit versions of Tartakovski’s theorem, Arch. Math. (Basel) 77 (2001), 129–137.
[22] R. Schulze-Pillot and A. Yenirce, Petersson products of bases of spaces of cusp forms and estimates for Fourier coefficients, Int. J. Number Theory 14 (2018), 2277–2290.
[23] C. L. Siegel, Über die analytische Theorie der quadratischen Formen III, Ann. of Math. 38 (1937), 212–291.
[24] R. P. Soni, On an inequality for modified Bessel functions, J. Math. and Phys. 44 (1965), 406–407.
[25] W. Tartakowsky, Die Gesamtheit der Zahlen, die durch eine positive quadratische Form $F(x_1x_2\ldots x_s)$ ($s \geq 4$) darstellbar sind. I, II, Izv. Akad. Nauk SSSR 7 (1929), no. 1, 111–122, no. 2, 165–196.
[26] F. Waibel, Uniform bounds for norms of theta series and arithmetic applications, Math. Proc. Cambridge Philos. Soc. 173 (2022), 669–691.
[27] L. Walling, Explicit Siegel theory: an algebraic approach, Duke Math. J. 89 (1997), 37–74.
[28] L. Walling, Explicitly realizing average Siegel theta series as linear combinations of Eisenstein series, Ramanujan J. 47 (2018), 475–499.
[29] A. Weil, On some exponential sums, Proc. Nat. Acad. Sci. U.S.A. 34 (1948), 204–207.
[30] T. Yang, An explicit formula for local densities of quadratic forms, J. Number Theory 72 (1998), 309–356.

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