The pion-kaon scattering amplitude constrained with forward dispersion relations up to 1.6 GeV

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In this work we provide simple and precise parameterizations of the existing $\pi K$ scattering data from threshold up to 1.6 GeV, which are constrained to satisfy forward dispersion relations as well as three additional threshold sum rules. We also provide phenomenological values of the threshold parameters and of the resonance poles that appear in elastic scattering.

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I. INTRODUCTION

Pion-kaon scattering is a very relevant process for our understanding of Hadron Physics and the strong interaction. The motivation to study it is threefold.

First of all, because pions and/or kaons appear in the final states of all hadronic processes. In particular kaons do so if the process involves net strangeness. Since pions and kaons interact strongly, final state $\pi K$ re-scattering effects are essential to describe and understand such hadronic processes.

Second, the reaction is interesting by itself, because even though we cannot solve QCD at low energies, the identification of pions and kaons as pseudo-Goldstone Bosons of the QCD spontaneous chiral symmetry breaking allows for a rigorous formulation in terms of a low energy effective theory known as Chiral Perturbation Theory [1] (ChPT). In turn, ChPT provides $\pi K$ scattering amplitudes which have been calculated first to one-loop [2] and then to two-loops [3]. Relevant constraints on the ChPT low energy constants can be obtained from sum rules and dispersion relations applied to $\pi K$ scattering [4]. In addition, $\pi K$ scattering was subsequently unitarized to one-loop [5,6] or within the chiral unitary approach [7], providing a simultaneous description of the low-energy and resonant regimes. Moreover, there is a renewed interest in $\pi K$ scattering from Lattice QCD, where the main features, like threshold parameters [8], scattering phases and resonances [9], have already been calculated. Although the pion mass used for these lattice calculations is not physical, one can expect physical values to be within reach soon. Alternative lattice strategies that calculate $\pi K$ scattering from unitarized chiral Lagrangians have also been followed recently in [10].

Third, in pion-kaon scattering appear some of the still controversial light scalar mesons, like the $K_0^*$ (800) or $\kappa$ resonance and the $K_0^*$ (1430). The former has been the subject of a longstanding debate about its very existence and nature. Actually, it is a firm candidate to form the lightest nonet of scalar mesons together with the $f_0$ (500) or $\sigma$-meson, the $f_0$ (980) and the $a_0$ (980). There are strong evidences that these states might form a nonet of non ordinary mesons [8,11], i.e., mesons not predominantly made of a quark and an antiquark. The $\kappa$ resonance has been obtained within different variants of unitarized ChPT in [6,7], it has also been shown to have a mass smaller than 900 MeV [12] and has been found [13] from a rigorous solution [14] of Roy-Steiner dispersion relations [15], which is the best determination so far. However, those pieces of evidence are still not considered enough by the Review of Particle Properties (RPP) [16], which still lists the $K_0^*(800)$ resonance under the “needs confirmation” label. Thus, the $\kappa$-meson is a further motivation for our present study, since any rigorous resonance determination from data (not a solution of dispersion relations or lattice) requires first a consistent knowledge of $\pi K$ scattering, which, in order to control all uncertainties should reach beyond the pure elastic regime. Incidentally, the latter region is also of direct interest for the $K_0^*(1430)$ resonance.

Hence, the goal of this work is to perform an analysis of the existing $\pi K$ scattering data constrained to satisfy Forward Dispersion Relations. The advantage of these relations is that, contrary to other kinds of dispersion relations (like Roy-Steiner equations in their simplest form), they can be easily implemented up to arbitrarily high energies. Here we will apply to $\pi K$ scattering an approach that has been recently followed [17] to obtain a precise description of $\pi\pi$ scattering data, consistent with dispersion relations. Namely, on a first stage one obtains simple fits to different, even conflicting, sets of data for each partial wave up to 1.74 GeV, without any further constraint apart from unitarity. The resulting parameterizations form a set of simple “Unconstrained Fits to Data” that could be easily modified wave by wave in case new data would appear. However, we check later that this set is not consistent with Forward Dispersion Relations up to 1.74 GeV. Then, using this set as a starting point, one refines its parameters by imposing the dispersion relations without spoiling the data description. The resulting “Constrained Fits to Data” will be the main result of this work and provide precise parameterizations describing the existing data, while being simultaneously consistent with Forward Dispersion relations up to 1.6 GeV as well as with three threshold sum rules. Since these parameterizations are rather simple, we expect that they will become a useful tool for further studies, either theoretical and experimental, involving $\pi K$ scattering at some stage and particularly for the precise determination of resonance parameters. This was indeed the case of the
parameters resulting from a similar analysis of $\pi\pi$ scattering.

II. KINEMATICS AND NOTATION

As it is customary we will use the partial wave decomposition of the $\pi K$ scattering amplitudes

$$T^I(s, t, u) = \frac{4}{\pi} \sum_l (2l + 1) P_l(\cos \theta) t^I_l(s),$$

where $s, t, u$ are the standard Mandelstam variables, satisfying $s + t + u = 2(m_1^2 + m_2^2)$ and $\sigma(s) = 2q_K\sqrt{s}$. The center of mass momentum of two particles with mass $m_1$ and $m_2$ is:

$$q_{12}(s) = \frac{1}{2\sqrt{s}} \sqrt{(s - (m_1 + m_2)^2)(s - (m_1 - m_2)^2)}.$$ \hspace{1cm} (2)

For later convenience we also define $\Sigma_{12} = m_1^2 + m_2^2$ and $\Delta_{12} = m_1^2 - m_2^2$. Unless explicitly stated $m_1 = m_K$ and $m_2 = m_\pi$ and $\Sigma = q_{K\pi}$ in this work. Note that we are working in the isospin limit of equal masses for all pions $m_\pi = 139.57$ MeV and equal masses for all kaons $m_K = 496$ MeV. We also use $m_\eta = 547$ MeV.

The elastic unitarity condition $\operatorname{Im} t(s) = \sigma(s)|t(s)|^2$ implies that the elastic partial wave can be recast in terms of a real phase shift

$$t_I(s) = \frac{i(s)}{\sigma(s)} = \frac{\sin \delta_I(s)}{\sigma(s)} = \frac{1}{\sigma(s) \cot \delta(s) - i},$$

where we have introduced the “Argand” partial wave $i(s)$ for later convenience.

In contrast, in the inelastic regime an inelasticity function is also introduced to write:

$$t_I(s) = \frac{i(s)}{\sigma(s)} = \eta(s)e^{2i\delta_I(s)} - 1 - \frac{1}{2\sigma(s)}.$$ \hspace{1cm} (4)

Later on we will also study the scattering at very low energies through the threshold parameters defined as:

$$\operatorname{Re} t^I_l(s) \sim q^{2l+1}(a^I_l + b^I_l q^2 + O(q^4)).$$ \hspace{1cm} (5)

Throughout this work we will also use the traditional spectroscopic notation, naming the partial waves with isospin $I$ and angular momentum $l = 0, 1, 2, 3...$ as $S^I$, $P^I$, $D^I$ and $F^I$-waves..., respectively.

III. UNCONSTRAINED FITS TO DATA

A. The Data

Data on $\pi K$ scattering were obtained mostly during the 70’s and the 80’s, measured indirectly from $K N \rightarrow K \pi N$ reactions assuming they are dominated by the exchange of a single pion.

On the one hand, data on the $I = 3/2$ $\pi K$-scattering cross sections was isolated in the early 70’s using different reactions: Early experiments provided cross sections by studying $K^−d → K^-\pi^-pp$ in Y. Cho et al. \cite{22}, $K^-n → K^-\pi^-p$ in A.M. Bakker et al. \cite{23} as well as $K^+p → K^+\pi^-\Delta^{++}$ in B. Jongejaans et al. \cite{24}. Since this $\pi K$ channel seems elastic up to at least 1.8 GeV, it is straightforward to obtain the phase shift. Actually, this was done explicitly by D. Linglin et al. in \cite{25} from their $K^-p → K^-\pi^-\Delta^{++}$ analysis. In general, the experiments in the earlier 70’s have low statistics, which were improved by later experiments. In particular, in 1977 P. Estabrooks et al. \cite{27} performed a relatively high statistics analysis of $K^\pm p → K^{\pm}\pi^\pm n$ and $K^\pm p → K^{\pm}\pi^-\Delta^{++}$ at 13 GeV to obtain the $I = 3/2$ $\pi K$ component, also with no evidence of inelasticity up to 1.8 GeV in $\pi K$ scattering. We will see that the differences between experiments are larger than the statistical uncertainties they quote, which points at the existence of a sizable systematic uncertainty that we will have to estimate separately for each wave.

On the other hand, isospin $I = 1/2$ scattering waves have always been obtained in combination with those with $I = 3/2$. This was done for instance by R. Mercer et al. in \cite{24} using $K^\pm p → K^{\pm}\pi^-\Delta^{++}$ and $K^\mp n → K^0\pi^0\Delta^{++}$ reactions. Due to low statistics, in order to separate different isospins, they needed to combine their results with the so-called “World Data Summary Tape”, an heterogeneous and not very precise collection of data that existed at that time. As a consequence, the results for their $I = 1/2$ and $3/2$ waves have huge uncertainties, which is why they are usually neglected against later and more precise experiments.

As a matter of fact, what was really measured in scattering experiments was the $t_I = t^I_{l/2} + t^{I}_{3/2}/2$ combination. This was already studied with relatively high statistics in \cite{27} up to 1.85 GeV, but also in the experiment with the highest statistics so far that was performed in the 80’s by Aston et al. at the LASS Spectrometer \cite{28} at SLAC. This LASS experiment studied the $K^- p → K^-\pi^+n$ reaction at 11 GeV and obtained the same $\pi K$ partial wave combination up to 2.6 GeV.

The analysis needed to extract $\pi K$ scattering amplitudes from $K N \rightarrow K \pi N$ has several sources of systematic uncertainties, like corrections to the on-shell extrapolation of the exchanged pion or rescattering effects. However, most experimental works only quote statistical uncertainties for each solution and for this reason conflicting data exist. This will be clearly seen in the figures below. Thus, in our fits we sometimes add a systematic uncertainty to different sets or to certain data points which are in conflict with other data points in the same region. In the case of the most delicate and controversial wave, which is the $S^{1/2}$, we have checked that the resulting data set and the fit are consistent with certain statistical tests explained in appendix \cite{11}. 


In addition some ambiguities occur in the determination of the phase that sometimes lead to different solutions for $\pi K$ scattering even within the same $KN \to K\pi N$ experiment. In the case of Aston et al. these ambiguities appear above the region of interest for this work. In contrast, Estabrooks et al. do have four solutions above 1.5 GeV, but we only consider Solution B since it is the one qualitatively closer to Aston et al.

So far we have been discussing scattering data where the $I = 1/2$ state has always been obtained in combination with the $I = 3/2$ one. However, it is also possible to obtain information on $\pi K$ scattering from the decays of heavier particles. In particular, when $\pi K$ are the only strongly interacting particles in the decay, Watson theorem implies that, in the $\pi K$ elastic region, the phase of the global process should be the same as the scattering phase shift. In particular, the phase-shift difference between S and P waves with $I = 1/2$ have been measured from $D^+ \to K^-\pi^+e^+\nu_e$ by the BaBar Collaboration and recently by the BESIII Collaboration. The results are very consistent with the LASS experiment, but their uncertainties are too large and will not be included in our fits, although we will show them for completeness.

Moreover, there are measurements of the $I = 1/2$ phase of the $K \pi$ S-wave amplitude obtained from Dalitz plot analyses of $D^+ \to K^-\pi^+\pi^+$ by the E791, FO-CUS and CLEO-c collaborations, as well as a recent similar analysis of $\eta_c \to K\bar{K}\pi$ by the BaBar Collaboration. These phases (and amplitudes) are not necessarily those of $\pi K$ scattering due to the presence of a third strongly interacting particle, which invalidates the use of Watson’s Theorem. However, a posteriori comparison with the scattering data has shown that, within the large uncertainties and at least in the elastic region, the resulting phase (but not the amplitude) is very similar to that of LASS. This means that the effect of the third particle on the phase is rather constant and almost amounts to a global shift. But these data cannot really be interpreted as a scattering phase beyond this qualitative agreement and are therefore not included in our fits. Nevertheless, we will show and discuss them in comparison with our results.

B. General form of our parameterizations

Each partial wave will now be fitted to the existing data up to $\sim 1.7$ GeV, which means that we will only fit $S$, $P$, $D$ and $F$ waves, since there are no data for $G$, $H$ and higher waves below 1.8 GeV. In this first stage, the fit to a wave with a given angular momentum will be performed independently of other waves with different angular momentum, by means of simple functions, without imposing any dispersive constraint. For this reason the resulting set of partial waves will be called “Unconstrained Fit to Data” (UFD). When possible, as in waves which are elastic in the whole energy range, a single functional form will be used throughout the whole energy region. However, for more complicated waves different functional forms will be used in different regions. Typically these piecewise functions will be matched at thresholds demanding continuity.

We would like to add a word of caution here. The data are not precise nor numerous enough to exclude large fluctuations between successive data points, particularly in certain energy regions. One could devise complicated parameterizations that would pass through every single data point, or even produce fluctuations between points. In this work we are assuming that such fluctuations do not occur and that the data can be correctly fitted with simple and relatively smooth parameterizations. The size of the uncertainties thus depends on this assumption. The parameterizations we describe below are the ones we have finally chosen because they satisfy the above assumption and yield uncertainty bands which do not show wild fluctuations or become too large in a region where the data spread does not require so. In particular, we have explored different kinds of conformal parameterizations (with different centers and more terms in the expansion, see Appendix A), we have tried simple polynomials in different variables, including orthogonal polynomials in a given region, adding or removing resonant shapes, etc. Since all them fit the data, their central result is not too different from our final choice. Except in some few relevant cases, we spare the reader from explaining the caveats that affect these many other parameterizations we tried. We just present below our final choice. Moreover, for a given parameterization, and once the systematic uncertainty that affects the data has been estimated, we decide to stop adding parameters when the $\chi^2/\text{dof}$ is close or less than one. Of course, the size of the final uncertainties depend on our educated guess of systematic uncertainties, which, as we will see, dominate the final error bands in many cases.

1. Partial waves in elastic regions

For the elastic regions, in which a partial wave can be recast in terms of just a phase-shift, we will use a conformal expansion of the type:

$$\cot \delta_l(s) = \frac{\sqrt{s}}{2q^{2l+1}} F(s) \sum_n B_n \omega(s)^n,$$

where $F(s) = 1$ except for scalar waves that have an Adler zero at $s_{Adler}$, in which case $F(s) = 1/(s-s_{Adler})$, or for waves that exhibit a clear narrow resonance and whose phase shift crosses $\pi/2$ at $m_\pi$, in which case $F(s) = (s-m_\pi^2)$. In addition, the conformal variable is defined as:

$$\omega(y) = \sqrt{y - \alpha \sqrt{y_0 - y}} / \sqrt{y + \alpha \sqrt{y_0 - y}}, \quad y(s) = \left( \frac{s - \Delta K\pi}{s + \Delta K\pi} \right)^2.$$
The energy where the expansion is centered. Note that for the S waves, we will set to its leading order within Chiral Perturbation Theory, i.e., $s_{Adler} = \Sigma_{K\pi}$. For this wave, the constants that define the conformal variable $\omega$ in Eq. (11) are fixed to $\alpha = 1.4$, $s_0 = (1.84 \text{ GeV})^2$. The existing data are shown in Fig. 1. There is a relatively fair agreement between different experiments below 1.1 GeV. However we can already notice some incompatible points between the Bakker et al. [28] and Estabrooks et al. [27] data sets, mostly due to the very small uncertainty of some points in the latter set. Note also the large variations between the uncertainties of successive data points in the Estabrooks et al. set. Above 1.1 GeV the two data sets that exist are largely incompatible. It is clear that some systematic uncertainty exists.

Therefore, we have fitted the data in Fig. 1 in two ways, either adding a constant systematic uncertainty of 1σ or multiplying the existing statistical uncertainties by a factor of 2, which is chosen so that the resulting $\chi^2/d.o.f.$ is
Since no dispersion relation has been imposed yet, this line whose uncertainties are covered by the gray band. The result of our fit, with the systematic uncertainty added to the statistical one. In addition, the second approach is similar, but we have preferred the uncertainty band of the data on S\(^{3/2}\)-wave. The result provides the final S\(^{3/2}\)-wave parameterization, which is also shown in Fig. 1 as a thick continuous line whose uncertainties are covered by the gray band. Since no dispersion relation has been imposed yet, this result will be called Unconstrained Fit to Data (UFD), whose parameters are found in Table I. The Constrained Fit to Data (CFD) in that table will be discussed later in Sec. V. In the Figure it can be noticed that this UFD result is similar to the fit to the S\(^{3/2}\)-wave data alone that has been described in this subsection.

2. \(I=1/2\) S-wave

For this wave, inelasticity has been measured above 1.3 GeV and for the most part it is due to the \(K\eta\) state rather than to states with more than two mesons. Hence, we are going to parameterize the amplitude using the elastic formalism of Subsec. III B 1 below \(K\eta\) threshold, and with the inelastic formalism of Subsec. III B 2 above that threshold.

Thus, for \((m_K + m_\eta)^2 \leq s \leq (m_K + m_\eta)^2\) we will use a conformal expansion of the type in Eq. (14), namely:

\[
cot \delta_0^1/s = \frac{\sqrt{s}}{2q(s - s_{Adler})}(B_0 + B_1 \omega).
\]

Once again we have explicitly factorized the Adler zero, which we have set to its leading order within Chiral Perturbation Theory value:

\[
s_{Adler} = \left( \Sigma_{K\pi} + 2\sqrt{\Delta_{K\pi}^2 + m_{K一本}^2m_{\pi一本}^2} \right)/5 \approx 0.236\text{ GeV}^2.
\]

As explained in Appendix A, for this wave it is convenient to fix the constants that define the center of the conformal variable \(\omega\) in Eq. (17) to the following values:

\[
\alpha = 1.15, \quad s_0 = (1.1\text{ GeV})^2.
\]

The parameters obtained for the best Unconstrained Fit to Data (UFD) are given in the first column of Table II.

\[
\begin{array}{|l|c|c|}
\hline
\text{Parameter} & \text{UFD} & \text{CFD} \\
\hline
B_0 & 2.25 \pm 0.04 & 2.27 \pm 0.04 \\
B_1 & 4.21 \pm 0.17 & 3.94 \pm 0.17 \\
B_2 & 2.45 \pm 0.50 & 3.36 \pm 0.50 \\
\hline
\end{array}
\]

Had we considered only two \(B_0\) parameters, the fit would yield an 80% larger \(\chi^2/d.o.f.,\) whereas with four it would decrease by 15%. Since three parameters as in Eq. (14) already provide a \(\chi^2/d.o.f. < 1\) we do not consider necessary to have a fourth parameter. We show this fit as a dashed line in Fig. 1 where the uncertainty band is delimited by the dotted lines.

Still this is not our final fit because there is also experimental information on the \(t_5 \equiv t_0^{3/2} + t_0^{1/2}/2\) combination. In the next subsection we will explain how the fit to the \(t_5\) data produces a small modification on the \(S\)^{3/2}-wave. The result provides the final \(S\)^{3/2}-wave parameterization, which is also shown in Fig. 1 as a thick continuous line whose uncertainties are covered by the gray band. Since no dispersion relation has been imposed yet, this result will be called Unconstrained Fit to Data (UFD), whose parameters are found in Table I. The Constrained Fit to Data (CFD) in that table will be discussed later in Sec. V. In the Figure it can be noticed that this UFD result is similar to the fit to the \(S\)^{3/2}-wave data alone that has been described in this subsection.

\[
\begin{array}{|l|c|c|}
\hline
\text{Parameter} & \text{UFD} & \text{CFD} \\
\hline
B_0 & 0.411 \pm 0.007 & 0.411 \pm 0.007 \\
B_1 & 0.181 \pm 0.034 & 0.162 \pm 0.034 \\
\hline
\end{array}
\]

In contrast, in the \(s \geq (m_K + m_\eta)^2\) region we will implement the inelastic formalism of Eqs. (8), (9), (10) as follows:

\[
t_0^{1/2}(s) = \frac{S_0^bS_0^cS_0^c - 1}{2i\sigma(s)},
\]

where

\[
S_0^b = \exp[2iq_0K(\phi_0 + \phi_1q_0^2K)].
\]

For \(S_1^c\) we use Eq. (10) with

\[
P_i(s) = (s_1 - s)b + e_1G_i \frac{p_1(q_\pi K)q_\pi K - \hat{q}_\pi K}{p_1(q_\pi K)q_\pi K - \hat{q}_\pi K}.
\]

\[
Q_i(s) = (1 - e_1)G_i \frac{p_1(q_\pi K)q_\pi K}{p_1(q_\pi K)q_\pi K} \Theta_{\pi K}(s),
\]

\[
\Delta_{K\pi} = \left( \Sigma_{K\pi} + 2\sqrt{\Delta_{K\pi}^2 + m_{K一本}^2m_{\pi一本}^2} \right)/5 \approx 0.236\text{ GeV}^2.
\]
where \( p_1(x) = 1 + ax^2 + bx^4 \), \( q^2_{ij} = q_{ij}(s_{ij}) \), \( q_{ij} = q_{ij}((m_\eta + m_K)^2) \) and \( \Theta_{\eta K}(s) = \Theta(s - (m_K + m_\eta)^2) \) is the step function at the \( K\eta \) threshold. In addition, for \( S_2 \) we use Eq. (13) with

\[
P_2(s) = e_2 G_2 \frac{p_2(q_\pi K) q_\pi K - \hat{q}_\pi K}{p_2(q_\pi K) q_\pi K - q_\pi K},
\]

\[
Q_2(s) = (1 - e_2) G_2 \frac{p_2(q_\pi K) q^0_\eta K \Theta_{\eta K}(s)}{p_2(q_\pi K) q^0_\eta K},
\]

with \( p_2(x) = 1 + cx^2 \).

By matching the elastic and inelastic parameterizations at the \( K\eta \) threshold we only need to demand continuity, which is ensured by defining \( \beta \equiv 1/\cot \delta_0^{1/2}((m_K + m_\eta)^2) \), where \( \delta_0^{1/2} \) is calculated here with the elastic parameterization in Eq. (13).

3. \( t_S \)-wave

Nevertheless, as already explained above, we do not fit the \( S^{1/2} \)-wave alone, but in the \( t_S = t_0^{1/2} + t_0^{3/2} / 2 \) combination that was originally measured. Let us then define

\[
t_S(s) = |t_S(s)| e^{i\phi_S(s)},
\]

and remark that, since \( |t_S| \) and \( \phi_S \) were measured independently, we will fit them both. In order to compare with data is convenient to use the normalization

\[
\hat{t}_S(s) = t_S(s) \sigma(s).
\]

Thus, in Fig. 2 we show the data on \( \hat{t}_S \) and the result of our Unconstrained Fit Data (UFD). The upper panel shows \( |\hat{t}_S| \) whereas the lower one shows \( \phi_S \). The combined \( \chi^2/d.o.f. \) of the \( S^{1/2} \) and \( S^{3/2} \) data fits is \( \chi^2/d.o.f. = 168/(182 - 15 + 1) \) with the UFD parameters provided in Table II. The \( e_1 \) parameter was initially left free but it comes out practically indistinguishable from 1, with tiny uncertainties, and has been later fixed to 1 for practical purposes.

From Fig. 2 it can be easily noted that there are data points which are largely incompatible with one another, not only between the two different experiments \([27, 28]\), but even among the successive data points of Estabrooks et al. \([27]\). Thus, it seems clear that there are some systematic errors not covered by the experimental uncertainties. Since these are the most controversial waves, here we have followed a more elaborated procedure to estimate the uncertainties of the resulting fit. In particular, we follow one of the techniques suggested in \([33]\), which, in brief, consists of running Gaussianity tests on the data with respect to the fit and enlarge the uncertainties of those data points that spoil the test. This yields a new fit upon which the procedure is iterated until the Gaussianity test is satisfied. The details of this method are given in Appendix B. We show in Fig. 2 as a gray band, the resulting uncertainty of our fit to those data and the \( I = 3/2 \) data already discussed in the previous section.

In the literature it is rather usual to neglect the Estabrooks et al. data, although it is not always the case \([39]\). To be able to compare with this choice, we have thus considered a fit to the \( I = 3/2 \) data together with only the \( I = 1/2 \) data set of Aston et al. \([28]\), which is much smoother than that of Estabrooks et al. \([27]\), particularly below 1.5 GeV. In this case we have not added any systematic uncertainty and the result is shown in Fig. 2 as a dashed line, which almost overlaps with our previous fit up to 1.5 GeV, and has a very similar uncertainty band represented as the area between dotted lines.

Note they are in fair agreement with other existing values in the literature, also provided in the table.

Once we have fitted the data on the two \( S \)-waves, we can use our UFD parameterizations to obtain the scattering lengths defined in Eq. (4), which we show in Table IV. Note they are in fair agreement with other existing values in the literature, also provided in the table.
FIG. 2: Data on $|\hat{t}_S(s)|$ from Estabrooks et al. [27] and Aston et al. [28]. The upper panel shows $|\hat{t}_S(s)|$ whereas the lower one shows $\phi_S(s)$, which were measured independently. The continuous line is our unconstrained fit (UFD), whose uncertainties are covered by the gray band. For comparison we show, as a dashed line, a fit only to the data in [28], whose corresponding uncertainties are delimited by the dotted lines.

TABLE IV: S-wave scattering lengths from our UFD and CFD sets, in $m_\pi^{-1}$ units, compared to other values in the literature.

|                  | $m_\pi a^{1/2}_0$ | $m_\pi a^{3/2}_0$ |
|------------------|------------------|------------------|
| Büttiker et al. Ref.[14] | 0.224±0.022     | -0.0448±0.0077   |
| Dobado & Peláez Ref.[40] | 0.155±0.012     | -0.049±0.004     |
| Jamin et al. Ref.[39]   | 0.18             | -0.12            |
| Bugg Ref.[36]           | 0.195±0.006      | -                |
| Zhou & Zheng Ref.[37]   | 0.219±0.034      | -0.042±0.002     |
| UFD, this work          | 0.22±0.01        | -0.10^{+0.03}_{-0.05} |
| CFD, this work          | 0.22±0.01        | -0.05^{+0.010}_{-0.014} |

There is a renewed interest on these quantities due to recent lattice calculations [8] and also due to the experimental measurement by the DIRAC collaboration [41]

$$\frac{1}{3} \left( a^{1/2}_0 - a^{3/2}_0 \right) = 0.11^{+0.09}_{-0.04} m_\pi^{-1}, \quad \text{(DIRAC)} \quad (24)$$

which was not determined from scattering experiments, but from the formation of $\pi K$ atoms. From our UFD set we find:

$$\frac{1}{3} \left( a^{1/2}_0 - a^{3/2}_0 \right) = 0.108^{+0.018}_{-0.010} m_\pi^{-1}. \quad \text{(UFD)} \quad (25)$$

Note that our uncertainties are smaller, by roughly an order of magnitude, than the present direct experimental knowledge. We have explicitly checked that including or not the DIRAC value does not change the result of our fits.
systematic effects into account the resulting from Fig.4. If we make a naive fit without taking these have set of the region where data exists. Thus, for this wave we center of the conformal expansion lies on the center formal variable $\omega$ form ($\text{Eq.}(7)$) has been chosen so that $\alpha$ parameter that defines the con-

D. $P$-waves

1. $I = 3/2$ $P$-wave

Only Estabrooks et al. $[27]$ provide data for the $I = 3/2$ P-wave phase-shift up to 1.74 GeV, which we show in Fig.4. As it can be noticed in the figure, this wave is rather small. Namely, below 1.1 GeV its phase shift is less than $1^\circ$, below 1.4 GeV is less than $2^\circ$ and below 1.74 GeV it is less than $3^\circ$. There is no inelasticity measured up to 1.74 GeV so that we will parameterize this partial wave with a conformal expansion as in Eqs. (6) and (7):

$$\cot \delta^ {3/2} (s) = \frac{\sqrt{S}}{2q^2} (B_0 + B_1 \omega). \quad (26)$$

Let us remark that the $\alpha$ parameter that defines the conformal variable $\omega$ (see Eq. (7)) has been chosen so that the center of the conformal expansion lies on the center of the region where data exists. Thus, for this wave we have set

$$\alpha = 1.45, \quad s_0 = (1.84 \text{ GeV})^2. \quad (27)$$

The existence of systematic uncertainties is evident from Fig.4. If we make a naive fit without taking these systematic effects into account the resulting $\chi^2/d.o.f. \approx 2$. Hence, we have included an estimation of the systematic uncertainty in our fits by multiplying the data statistical uncertainties by $\sqrt{2}$. Two conformal parameters are enough to describe this wave and no significant improvement is obtained by considering a third one.

As it happened with the $S^{3/2}$ wave, our final fit for the $I = 3/2$ $P$-wave is obtained by fitting simultaneously the data for this wave alone together with the data on the $t_p = t_1^{1/2} + t_1^{3/2}/2$ combination obtained by Estabrooks et al. $[27]$ and Aston et al. $[28]$. The resulting unconstrained fit to data (UFD) is shown in Fig.4 where the gray band covers its uncertainty. The corresponding UFD parameters are listed in Table V.

| Parameter | UFD | CFD |
|-----------|-----|-----|
| $B_0$     | $-14.8 \pm 2.6$ | $-15.6 \pm 2.6$ |
| $B_1$     | $2.7 \pm 7.4$    | $-2.2 \pm 7.4$ |

Table V: Parameters of the $P^{3/2}$-wave.

Also in Figure 4 we show, as a dashed line, the result when fitting only the data on that wave. Its corresponding uncertainty band is delimited by dotted lines. As we can see it is almost indistinguishable from our UFD result, for which we have also fitted data on $t_p = t_1^{1/2} + t_1^{3/2}/2$, as we will see next.

2. $I = 1/2$ $P$-wave

The $I=1/2$-wave is only measured in scattering experiments together with the $I=3/2$-wave in the $t_p$ combination defined just above. Although in the literature it is frequent to neglect the $P^{3/2}$-wave, because as we have just seen is very small, we will keep it in our fits for completeness.
Let us then discuss the $P^{1/2}$-wave in the elastic region, i.e. $s \leq (m_\eta + m_K)^2$, for which we use a conformal fit to describe the data, namely,

$$\cot \delta_1^{1/2}(s) = \frac{\sqrt{s}}{2q^4}(m_r^2 - s)(B_0 + B_1\omega + B_2\omega^2). \tag{28}$$

Note we have explicitly extracted an $(m_r^2 - s)$ factor so that the phase crosses $\pi/2$ at the energy of the peak associated to the $K^*(892)$ resonance, which is the dominant feature of this wave in the elastic region. As explained in Appendix A the $\alpha$ and $s_0$ parameters, which define the conformal variable $\omega$ in Eq. (7), are fixed from the choice of the center of the expansion and the highest energy of the fit to be

$$\alpha = 1.15, \quad s_0 = (1.1 \text{ GeV})^2. \tag{29}$$

For $s \geq (m_\eta + m_K)^2$, we will use once more the inelastic formalism of Eqs. (8), (10). Thus, we write

$$t_1^{1/2}(s) = \frac{S_1^r S_2^r S_3^r - 1}{2i\sigma(s)}, \tag{30}$$

where all the $S_r^i$ are of the form in Eq. (11), with

$$P_1 = (s_r - s)^\beta + e_1 G_1 \frac{p_1(q_{1K}) q_{1K}^2 - \bar{q}_{1K}^2 q_{1K} q_{1K}}{p_1(q_{1K}) q_{1K}^2 - q_{1K}^2 q_{1K}} \tag{31}$$

$$P_{2,3} = e_{2,3} G_{2,3} \frac{p_{2,3}(q_{2K}) q_{2K}^2 - \bar{q}_{2K}^2 q_{2K} q_{2K}}{p_{2,3}(q_{2K}) q_{2K}^2 - q_{2K}^2 q_{2K}}$$

$$Q_{1,2,3} = (1 - e_{1,2,3}) G_{1,2,3} \frac{p_{1,2,3}(q_{1K}) q_{1K}^3}{p_{1,2,3}(q_{1K}) q_{1K}^3} \Theta_{\eta K}(s). \tag{32}$$

In addition,

$$p_i(q_{\eta K}) = 1 + a_i q_{\eta K}^2 \tag{33}$$

and $\Theta_{\eta K}(s) = \Theta(s - (m_K + m_\eta)^2)$ is the step function at the $K\eta$ threshold. Again, in order to impose continuity at $K\eta$ threshold we have defined $\beta = 1/\cot \delta_1^{1/2}((m_K + m_\eta)^2)$, with $\delta_1^{1/2}$ calculated from the elastic expression in Eq. (28).

3. $t_P$-wave

Thus, now that we have the functional forms for the $I = 1/2$ and $I = 3/2$ $P$-waves, we can perform the fit to all the $P$-wave data. As we did for the $S$-wave we first define

$$t_P(s) = |t_P(s)| e^{i\phi_P(s)}, \tag{34}$$

which is sometimes used with the alternative normalization

$$\hat{t}_P(s) = t_P(s)\sigma(s). \tag{35}$$

As commented before, we fit simultaneously the $I = 3/2$ data in Fig. 4 and the data on both $|t_P|$ and $\phi_P$ that we show in Figs. 3 and 5. Note that once again there are clear systematic deviations of certain points, particularly from the Estabrooks et al. data set. In this case we have proceeded as follows: we have performed a first fit, then we have added a systematic uncertainty to the isolated incompatible data points, which is half of their distance to the central value of the fit. In regions where the two sets of data are incompatible a systematic uncertainty is also added to each set, which corresponds to half of the average difference from the fit to the data set in that region. With these additional systematic uncertainties we have performed a final fit, which we call Unconstrained Fit to Data (UFD), with $\chi^2/d.o.f. = 76.4/(78 - 12 + 1)$. The resulting curves are also shown in Figs. 4 and 5 together with a fit in which we have only fitted the Aston et al. data for the $I = 1/2$ wave. It can be noticed that in such case the result would still be compatible with our UFD.

Once all $P$-wave data have been fitted, we can separate the different isospin components. The $I = 3/2$ UFD result was already discussed in a Subsec. III.D.1 and its parameters were given in Table IV.

Concerning the $P^{1/2}$-wave, let us first look at the elastic region. When restricted below the $K\eta$ threshold the UFD result has $\chi^2/d.o.f. = 27/34 - 4 + 1$ and the corresponding parameters are listed in Table VI. The resulting curve for the $P^{1/2}$-wave can be seen in Fig. 5, where the distinct shape of the $K^*(892)$ is nicely observed. We are also showing a fit where only the data of Aston et al. has been fitted and how the results are hard to distinguish from our UFD line, except for the somewhat larger uncertainty band of the latter, particularly at higher en-
The UFD parameters for the $P^{1/2}$-wave inelastic parameterization are given in Table VII. Note that to describe the inelastic region we still need to take into account the high energy tail of the $K^*(892)$ resonance, which is elastic, so that we set $e_1 = 1$. In addition its mass is fixed to the neutral case, 896 MeV, since this is the one measured in the LASS [28] and Estabrooks et al. [27] experiments. The other resonance shapes of the $K^*(1410)$ and $K^*(1680)$ are also very nicely described.

Let us remark that there is a recent fit to the $t_\tau$ data [12], neglecting the $I = 3/2$ wave as usual, in which the authors also consider three poles for the $I = 1/2$ partial wave within a two-channel K-matrix approach, the channels being $\pi K$ and $\pi K^*(892)$. In [12] only the central value of the fit is given and, since it is a fit to basically the same data we fit here, the results are relatively similar to ours within uncertainties, actually, around 1 GeV it is slightly closer to our CFD result, that we will discuss later on, than to the UFD result discussed here. Note also that the parameterization in [12] is a fit to data up to 1.8 GeV and that, in principle, it could be extrapolated up to 2.3 GeV.

### E. D-waves

#### 1. $I=3/2$ D-wave

Once again, only Estabrooks et al. [27] provide data for the $I = 3/2$ D-wave phase shift up to 1.74 GeV, which we show in Fig. 8. Note it is very small in the whole energy region. No inelasticity has been measured so that we can use the elastic formalism parameterized with the conformal expansion in Eqs. (10) and (11), as follows:

$$\cot \delta_1^{3/2}(s) = \frac{\sqrt{s}}{2q^3}(B_0 + B_1 \omega + B_2 \omega^2).$$  (35)

Three conformal parameters are enough to describe this wave. As we did for the $P^{1/2}$-wave, the $\omega$ parameter that defines the conformal variable $\omega$ in Eq. (11) has been

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**TABLE VI: $P^{1/2}$-wave parameters in the elastic region.**

| Parameter  | UFD       | CFD       |
|------------|-----------|-----------|
| $B_0$      | 0.97 ± 0.02 | 0.97 ± 0.02 |
| $B_1$      | 0.98 ± 0.30 | 0.55 ± 0.30 |
| $B_2$      | 0.79 ± 0.95 | 0.75 ± 0.95 |
| $m_e$      | 0.8957 ± 0.0004GeV | 0.8957 ± 0.0004GeV |

**TABLE VII: $P^{1/2}$-wave parameters in the inelastic region.**

| Parameters | UFD                   | CFD                   |
|------------|-----------------------|-----------------------|
| $a_0$      | $-1.90 ± 0.10 GeV^{-2}$ | $-1.76 ± 0.10 GeV^{-2}$ |
| $a_2$      | $-2.14 ± 0.23 GeV^{-2}$ | $-2.33 ± 0.23 GeV^{-2}$ |
| $a_3$      | $-1.34 ± 0.07 GeV^{-4}$ | $-1.41 ± 0.07 GeV^{-4}$ |
| $\sqrt{\sigma_{a1}}$ | 0.896 GeV (fixed) | 0.896 GeV (fixed) |
| $\sqrt{\sigma_{a2}}$ | 1.346 ± 0.012GeV | 1.347 ± 0.012GeV |
| $\sqrt{\sigma_{a3}}$ | 1.644 ± 0.005GeV | 1.645 ± 0.005GeV |
| $\epsilon_1$ | 1 (fixed) | 1 (fixed) |
| $\epsilon_2$ | 0.052 ± 0.007 | 0.055 ± 0.007 |
| $\epsilon_3$ | 0.295 ± 0.016 | 0.306 ± 0.016 |
| $G_1$      | 0.044 ± 0.003GeV | 0.044 ± 0.003GeV |
| $G_2$      | 0.217 ± 0.041GeV | 0.231 ± 0.041GeV |
| $G_3$      | 0.295 ± 0.018GeV | 0.306 ± 0.018GeV |

**FIG. 6:** Data on $\phi_P(s)$ from Estabrooks et al. [27] and Aston et al. [28]. The continuous line is our unconstrained fit (UFD), whose uncertainties are covered by the gray band. For comparison we show as a dashed line a fit only to the data from [28], whose corresponding uncertainties are delimited by the dotted lines.

**FIG. 7:** $P^{1/2}$-wave phase shift below $K\eta$ threshold. The continuous line is our UFD parameterization whose uncertainty is covered by the gray band. Data from Estabrooks et al. [27] and Aston et al. [28]. The dashed curve is the fit to Aston et al. data alone and the dotted lines cover its corresponding uncertainty.
TABLE VIII: Parameters of the D$^{3/2}$-wave.

| Parameter | UFD       | CFD       |
|-----------|-----------|-----------|
| $B_0$     | -1.70 ±0.12 | -1.67 ±0.12 |
| $B_1$     | -6.5 ±1.7   | -7.0 ±1.7   |
| $B_2$     | -36 ±9      | -38±9      |

In Fig. 8 we also show, as a dashed line, the result when fitting only the data in that figure and not data on the $t_D$ combination. The corresponding uncertainty band is delimited by dotted lines. As we can see it is very similar to our UFD curve.

2. $I=1/2$ D-wave

As it happened with the $S$ and $P$-waves, the $I=1/2$ D-wave is only measured together with the $I=3/2$-wave in the $t_D \equiv t_2^{1/2} + t_2^{3/2}/2$ combination. In the literature it is usual to neglect the $D^{3/2}$-wave, because as we have just seen is very small, but we will keep it in our fits for completeness.

Let us then describe our fit to the $D^{3/2}$-wave, which is dominated by the $K_2^*(1430)$ resonance, whose branching ratio to $\pi K$ is approximately 50\%, so that we have to use an inelastic formalism as in Eqs. (38), (39), (40). In practice, it is enough to consider a non-resonant background and a resonant-like form, as follows:

$$t_2^{1/2} = \frac{S_0^b G_{0}^r}{2i\sigma(s)},$$

with a background term

$$S_0^b = e^{2i(p(s))},$$

where

$$p(s) = \phi_0q_{\pi K}^5\Theta_{\eta K}(s) + q_{5\pi K}^5(\phi_1 + \phi_2q_{\pi K}^2)\Theta_{\eta K}(s),$$

and $\Theta_{ab} = \Theta(s - (m_a + m_b)^2)$. A resonant term is also considered in order to describe easily the $K_2^*(1430)$ shape, namely

$$S_1^r = \frac{s_{r1} - s + i(P_1 - Q_1)}{s_{r1} - s - i(P_1 + Q_1)},$$

$$P_1 = e_1G_1 \frac{p_1(q_{\pi K})}{p_1(q_{\pi K})} \left(\frac{q_{\pi K}}{q_{\pi K,r}}\right)^5,$$

$$Q_1 = (1 - e_1)G_1 \frac{p_1(q_{\pi K})}{p_1(q_{\pi K})} \left(\frac{q_{\pi K}}{q_{\pi K,r}}\right)^5 \Theta_{\eta K}(s),$$

with $p_1(q_{\pi K}) = 1 + a q_{\pi K}^2$.

3. $t_D$-wave

Once more we define

$$t_D(s) \equiv |t_D(s)e^{i\sigma(s)}|, \quad \hat{t}_D(s) = t_D(s)\sigma(s).$$

Thus, in Figs. 9 and 10 we show the data on $|\hat{t}_D|$ and $\Phi_D$, respectively. As we did for the $P$-wave, we have treated the systematic uncertainties as follows: we have performed a first fit and added a systematic uncertainty to those isolated data points that are incompatible with it. This systematic uncertainty is half of their distance to the central value of the fit. In regions where the two sets of data are incompatible the systematic uncertainty is half of the average difference from the fit to each set in that region. With these additional systematic uncertainties we have performed a final fit, called Unconstrained Fit to Data (UFD), which is shown as a continuous line in...
For this wave there are no observations of an $I = 3/2$ channel, which is neglected in the literature as will be done here too. In addition, the threshold suppression is so large that there are no data below 1.5 GeV as can be seen in Figs.11 and 12. In the latter we can observe that there are only two data points with very large uncertainties for the phase $\phi_F$ below 1.85 GeV. Thus, in order to stabilize our fit we will extend our data sample to 2 GeV, although later on we will only make use of our partial wave parameterizations up to 1.74 GeV.

The most salient feature of this wave is the $K_1^*(1780)$ resonance, whose branching ratio to $\pi K$ is slightly less than 20%. Therefore we will need the usual inelastic formalism explained in the introduction to this section:

$$t_3^{1/2} = \frac{S_1^r - 1}{2i\sigma(s)},$$

with

$$S_1^r = \frac{s_{r1} - s + i(P_1 - Q_1)}{s_{r1} - s - i(P_1 + Q_1)},$$

$$P_1 = e_1G_1\frac{p_1(q_{\pi K})}{p_1(q_{\eta K})} \left(\frac{q_{\pi K}}{q_{\eta K}}\right)^7,$$

$$Q_1 = (1 - e_1)G_1\frac{p_1(q_{\pi K})}{p_1(q_{\eta K})} \left(\frac{q_{\eta K}}{q_{\pi K}}\right)^7 \Theta_{\eta K}(s).$$

In addition, $p_1(q_{\pi K}) = 1 + a(q_{\pi K})^2$ and $\Theta_{\eta K}(s) = \Theta(s - (m_\eta + m_K)^2)$.

No background term is needed for this wave because its behavior is well described using the resonant-like form only, as it can be observed in Figs.11 and 12. The fit yields $\chi^2/d.o.f. = 16/(21 - 4 + 1)$ and the UFD parameters listed in Table IX.

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**TABLE IX: Parameters of the $D^{1/2}$ fit.**

| Parameters | UFD (GeV) | CFD (GeV) |
|------------|-----------|-----------|
| $\phi_0$   | $2.17 \pm 0.26$ | $3.00 \pm 0.26$ |
| $\phi_1$   | $-12.1 \pm 1.7$ | $-9.3 \pm 1.7$ |
| $\sqrt{\gamma_1}$ | $1.446 \pm 0.002$ | $1.445 \pm 0.002$ |
| $c_1$      | $0.466 \pm 0.006$ | $0.465 \pm 0.006$ |
| $G_1$      | $0.220 \pm 0.009$ | $0.222 \pm 0.009$ |
| $a$        | $-0.53 \pm 0.16$ | $-0.72 \pm 0.16$ |

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**F. $F^{1/2}$-wave**

Once more we define

$$t_F(s) = \left|t_F(s)\right| e^{i\phi_F(s)}, \quad \hat{t}_F(s) = t_F(s)\sigma(s).$$

For this wave there are no observations of an $I = 3/2$ channel, which is neglected in the literature as will be done here too.
TABLE X: Parameters of the $P^{1/2}$-wave.

| Parameters | UFD | CFD |
|------------|-----|-----|
| $\sqrt{s_{\pi\pi}}$ | 1.801 $\pm$ 0.013GeV | 1.804 $\pm$ 0.013GeV |
| $c_1$ | 0.181 $\pm$ 0.006 | 0.184 $\pm$ 0.006 |
| $G_1$ | 0.47 $\pm$ 0.05GeV | 0.50 $\pm$ 0.05GeV |
| $a$ | $-0.88 \pm 0.10$GeV$^{-2}$ | $-0.97 \pm 0.10$GeV$^{-2}$ |

FIG. 11: Data on $\hat{f}_P(s)$ from [23, 25]. The continuous line is our unconstrained fit (UFD), whose uncertainties are covered by the gray band. For comparison we show, as a dashed line, a fit to the data from [28] whose corresponding uncertainties are delimited by the dotted lines.

G. Regge Parameterization

There are no data on $I = 3/2$ above 1.74 GeV, thus above that energy we will make use of the Regge parameterization for $\pi K$ scattering in [17, 19], which was obtained from factorization after fitting data on $NN$, $N\pi$, $NK$ and $\pi\pi$ high energy scattering. Note that for $\pi K$ scattering only the exchange of isospin 0 and 1 can occur in the $t$ channel.

For the isoscalar exchange there are two contributions: the Pomeron, called $P(s)$ here, and the subleading $f_2$ trajectory, called $P'(s)$, so that we write:

$$\text{Im} T_{\pi K}^{(I=0)}(s,t) = f_{K/\pi} [P(s,t) + \rho P'(s,t)],$$

where

$$P(s,t) = \beta_P \psi_P(t) \alpha_P(t) \frac{1 + \alpha_P(t)}{2} e^{bt} \left( \frac{s}{s'} \right)^{\alpha_P(t)},$$

$$P'(s,t) = \beta_P' \psi_P'(t) \alpha_P'(t) (1 + \alpha_P(t)) e^{bt} \left( \frac{s}{s'} \right)^{\alpha_P'(t)},$$

$$\alpha_P(t) = 1 + t \alpha_P', \psi_P = 1 + c_P t,$$

$$\alpha_P'(t) = \alpha_P(0) + t \alpha_P', \psi_P = 1 + c_P t.$$  \hspace{1cm} (45)

Note that, by using factorization, the substitution of the $\pi\pi$-Pomeron vertex by the $KK$-Pomeron vertex is taken into account by the $f_{K/\pi}$ constant, whereas $\rho f_{K/\pi}$ takes care of the similar factorization for $P'$.

Since in this work we are interested in Forward Dispersion Relations, we will only use the above Regge formulas with $t = 0$, but we provide the full expressions for completeness.

For the isovector exchange only the $\rho$ trajectory is needed and we use

$$\text{Im} T_{\pi\pi}^{(I=1)}(s,t) = g_{K/\pi} \text{Im} T_{\pi\pi}^{(I=1)}(s,t),$$

with

$$\text{Im} T_{\pi\pi}^{(I=1)}(s,t) = \frac{\beta_\rho}{1 + \alpha_\rho(0)} \varphi(t) e^{bt} \left( \frac{s}{s'} \right)^{\alpha_\rho(t)},$$

$$\alpha_\rho(t) = \alpha_\rho(0) + t \alpha_\rho' + \frac{1}{2} t^2 \alpha_\rho'',$$

$$\varphi(t) = 1 + d_\rho t + c_\rho t^2.$$  \hspace{1cm} (47)

Once again, the replacement of the $\pi\pi\rho$ vertex by the $KK\rho$ one is described by the $g_{K/\pi}$ constant, assuming factorization. Note that in [17] the value of $g_{K/\pi} = 1.1 \pm 0.1$ was used together with $\beta_\rho = 0.94 \pm 0.2$ to provide a description of $\pi K$. However, the same group [19] updated their $\pi\pi$ scattering analysis using dispersion relations and $\pi\pi$ scattering data at high energies to find $\beta_\rho = 1.48 \pm 0.14$. Consequently, if we want to use this latter value, we also have to update $g_{K/\pi} = 0.70 \pm 0.07$. One should nevertheless take into account that the information on this parameter is relatively scarce, since, in contrast to $\pi\pi$ scattering, there are no high energy data on $\pi K$ scattering. Thus it has to be determined only from factorization of $KN$ scattering. Note that $\beta_\rho$, which is the equivalent value for $\pi\pi$, suffered a large revision when taking into account dispersion relations. Thus,
large deviations in \( q_{K/\pi} \) should not come as a surprise and they actually do occur when imposing our dispersive constraints on \( \pi K \) scattering.

The set of Regge parameters used before imposing any \( \pi K \) dispersion relation will be labeled as “Unconstrained Fit to Data” (UFD) values, similarly to what we have been doing so far with our partial wave parameterizations. Correspondingly, we will refer to “Constrained Fit to Data” (CFD) values when in the next sections Forward Dispersion Relations will be imposed on our fits. Those Regge parameters that could be determined without \( K\pi \) input are shown in Table XI and their values are fixed both for the UFD and CFD parameterizations. They just correspond to the values in the original works \cite{17,19}.

**TABLE XI:** Values of Regge parameters obtained in \cite{17,19}. Since these could be fixed using reactions other than \( \pi K \) scattering, they will be fixed both in our UFD and CFD parameterizations.

| Parameters | UFD and CFD |
|------------|-------------|
| \( s \)    | 1 GeV\(^2\) |
| \( b \)    | 2.4 ± 0.5 GeV\(^{-2}\) |
| \( \alpha'_{\rho} \) | 0.9 GeV\(^{-2}\) |
| \( \alpha''_{\rho} \) | -0.3 GeV\(^{-4}\) |
| \( d_{\rho} \) | 2.4 ± 0.5 GeV\(^{-2}\) |
| \( c_{\rho} \) | 2.7 ± 2.5 |
| \( \alpha'_{\rho} \) | 0.2 ± 0.1 GeV\(^{-2}\) |
| \( \alpha''_{\rho} \) | 0.9 GeV\(^{-2}\) |
| \( c_{\rho'} \) | 0.6 ± 1 GeV\(^{-2}\) |
| \( c_{\rho''} \) | -0.38 ± 0.4 GeV\(^{-2}\) |
| \( \beta_{\rho} \) | 1.48 ± 0.14 |
| \( \alpha_{\rho}(0) \) | 0.53 ± 0.02 |
| \( \beta_{\rho} \) | 2.50 ± 0.04 |
| \( c_{\rho}(0) \) | 0 ± 0.04 |
| \( \beta_{\rho'} \) | 0.80 ± 0.05 |
| \( c_{\rho'}(0) \) | -0.4 ± 0.4 |
| \( \beta_{\rho''} \) | 0.08 ± 0.2 |
| \( \alpha_{\rho''}(0) \) | 0.53 ± 0.02 |

In contrast, the values \( f_{K/\pi}, g_{K/\pi} \) and \( r \), which are directly related to \( \pi K \) scattering, are listed in Table XII and in this work they are allowed to vary from the UFD to the CFD parameterization, although in practice \( r \) stays the same.

**TABLE XII:** Values of Regge parameters directly related to \( \pi K \) scattering. In practice \( r \) does not change from the UFD to CFD parameterization.

| Parameters | UFD | CFD |
|------------|-----|-----|
| \( f_{K/\pi} \) | 0.67 ± 0.01 | 0.66 ± 0.01 |
| \( r \) | 5·10\(^{-2}\) | 5·10\(^{-2}\) |
| \( g_{K/\pi} \) | 0.70 ± 0.07 | 0.53 ± 0.07 |

**IV. FORWARD DISPERSION RELATIONS AND SUM RULES**

The aim of this work is to provide a simple set of partial waves which are consistent with basic requirements of analyticity (or causality) and crossing. These features impose stringent constraints on the scattering amplitude, which translate into integral equations that relate the \( \pi K \) scattering amplitude at a given energy with an integral over the whole physical energy region. In this section we introduce a complete set of Forward Dispersion Relations that will be used first to check the consistency of our parameterizations and next as constraints on our fits.

**A. Forward dispersion relations**

Forward Dispersion Relations (FDR), i.e., calculated at \( t = 0 \), are useful because forward scattering is relatively easy to measure in the whole energy region, since it is related to the total cross section by the optical theorem. Moreover, this is the only fixed value of \( t \) for which the integrands in the dispersion relation will be given directly in terms of the imaginary part of a physical amplitude. They are applicable at any energy, in contrast to Roy-like equations which, in practice, have a limited applicability energy range due to the projection in partial waves.

Fixed-\( t \) dispersion relations for \( \pi K \) have been frequently used in the literature as an intermediate step for the derivation of more elaborated dispersion relations for partial waves \cite{14,17,44,45}, or of sum rules for low energy parameters \cite{14}, but not directly as constraints on the amplitudes, as will be done here.

For the sake of simplicity, given that \( s + t + u = 2(m_{K}^2 + m_{\pi}^2) \) and \( t = 0 \), it is customary to use an abbreviated notation \( T(s,t=0,u) = T(s) \). It is also very convenient to make use of \( s \leftrightarrow u \) crossing to change the amplitudes from the isospin basis to the \( s \leftrightarrow u \) symmetric and antisymmetric amplitudes. These are defined, respectively, as

\[
T^+(s) = \frac{T^{1/2}(s) + 2T^{3/2}(s)}{3} = \frac{T_{t=0}(s)}{\sqrt{6}},
\]

\[
T^-(s) = \frac{T^{1/2}(s) - T^{3/2}(s)}{3} = \frac{T_{t=1}(s)}{2}. \tag{48}
\]

In the last step we have indicated that these \( T^+ \) and \( T^- \) correspond, by crossing, to the exchange of isospin 0 or 1 in the \( t \)-channel, respectively. This is relevant because it means that \( T^+ \) is dominated at high energies by the \( t \)-channel exchanges of the Pomeron and \( P' \) trajectories, with no \( \rho \) trajectory contribution, whereas the opposite occurs for \( T^- \).

Since dispersive integrals extend to infinity, naively one would need two subtractions to ensure the convergence of the Pomeron contribution and one for that of the \( \rho \).
trajectory. For this reason, even if only used as intermediate steps for the derivation of other dispersion relations, the fixed-\(t\) dispersion relations for \(T^+\) are customarily written with two subtractions and those for \(T^-\) at least with one. However, this is not necessary, because the \(T^\pm\) FDR have integrals over the right-hand and left-hand cuts whose leading terms multiplying the Regge trajectories cancel against each other due to the symmetry properties. As a consequence, one subtraction is enough to ensure the convergence of the \(T^+\) FDR and no subtraction is needed for the \(T^-\) FDR. This kind of cancellations have been recently used for \(\pi\pi\) scattering FDR in \([12, 19, 21]\). Having more subtractions implies that the dispersion relation is determined up to a polynomial of higher order. Thus, generically, less subtractions are convenient to avoid the propagation of the uncertainties in the subtraction constants to become too large in the resonance region, whereas more subtractions are useful when concentrating on the threshold region. Since in this work we will deal with scattering up to 1.74 GeV, we will make use of FDR with the minimum number of subtractions needed, which also makes the equations slightly simpler.

Thus, for \(T^+\) the once-subtracted FDR reads:

\[
\text{Re } T^+(s) = T^+(s_{th}) + \frac{(s - s_{th})}{\pi} P \int_{s_{th}}^{\infty} ds' \frac{\text{Im } T^+(s')}{(s' - s)(s' - s_{th})} - \frac{\text{Im } T^+(s')}{(s' + s - 2\Sigma_{\pi K})(s' + s_{th} - 2\Sigma_{\pi K})}, \tag{49}
\]

where \(s_{th} = (m_\pi + m_K)^2\) and \(P\) stands for the principal part of the integral. In contrast, the unsubtracted FDR for \(T^-\) reads:

\[
\text{Re } T^-(s) = \frac{(2s - 2\Sigma_{\pi K})}{\pi} P \int_{s_{th}}^{\infty} ds' \frac{\text{Im } T^-(s')}{(s' - s)(s' + s - 2\Sigma_{\pi K})}, \tag{50}
\]

We have evaluated these two FDRs at 50 values of \(\sqrt{s_i}\) equally spaced between a minimum energy \(\sqrt{s_{\text{min}}} = 0.56\) GeV, below threshold, because the \(T^-\) FDR has no subtractions and thus provides strong constraints on a combination of scalar scattering lengths. The figure shows that the separation between both calculations is slightly above \(\Delta d_i\) for energies above 0.8 GeV. Above this energy, the \(T^-\) FDR is nicely satisfied within uncertainties up to 1.2 GeV, where the difference between the two curves starts growing, becoming much larger than \(\Delta d_i\). As we will see, the deviation at energies above 1.2 GeV is mainly caused by the \(\rho\)-exchange Regge contribution.

In order to provide a quantitative measure of the fulfillment of each FDRs, we have defined an averaged squared distance for each FDR,

\[
d^2_{T^\pm} = \frac{1}{N} \sum_{i=1}^{N} \left( \frac{d_i}{\Delta d_i} \right)^2, \tag{51}
\]

which is rather similar to the usual definition of a \(\chi^2\) function. Consistency of the data parameterizations with FDRs would demand \(d^2_{T^\pm} \lesssim 1\).

In Table XIII we show the values of \(d^2_{T^\pm}\) in different energy regions. For the UFD set it is clear that the consistency with FDRs is not very satisfactory, particularly at higher energies. The results for \(T^-\) are plotted in the lower panel of Fig[13] using the same conventions. In this case we have set \(\sqrt{s_{\text{min}}} = 0.56\) GeV, below threshold, because the \(T^-\) FDR has no subtractions and thus provides strong constraints on a combination of scalar scattering lengths. The figure shows that the separation between both calculations is slightly above \(\Delta d_i\) up to 0.8 GeV. Above this energy, the \(T^-\) FDR is nicely satisfied within uncertainties up to 1.2 GeV, where the difference between the two curves starts growing, becoming much larger than \(\Delta d_i\).
Threshold parameters of partial waves, defined in Eq. (24), are of interest for our understanding of the lowest energy Physics and particularly for studies within ChPT \[4\]. In this section we present three sum rules (SR) that provide a more accurate determination of certain combinations of threshold parameters, in terms of integrals, than would be achieved directly from the partial wave parameterizations. These SR will be used first as tests of our UFD parameterizations and in Sec.\[V\] will be used as constraints.

Other sum rules have been derived for determining the ChPT low energy constants \[3\], but here we will use our own sum rules for threshold parameters. In \[14\] a sum rule for a combination of scattering lengths is given, but it needs $\pi\pi \rightarrow K\bar{K}$ scattering input, and here we only want to use data on $\pi K$ scattering.

Thus, the first of our sum rules yields precisely the combination of scalar slope parameters and vector scattering lengths. Both the direct and integral results are combined of threshold parameters, in terms of integrals, thus, the first of our sum rules yields precisely the combination of scalar slope parameters and vector scattering lengths. Both the direct and integral results are calculated "directly" from our parameterizations. In practice, since $D_a$ and $SR_a$ are obtained from data the sum rule will not be exactly zero, but consistency requires it to cancel within uncertainties.

In Table XIV we show the results of this sum rule calculation using our UFD parameterizations. Note that it is not very well satisfied, since the $\Delta_a$ is slightly above 1.2 deviations from zero. This small disagreement suggests that there is room for improvement at low energies in the S-waves. Both the direct and integral results are compatible with the experimental value obtained in DIRAC \[11\], see Eq. (24) above. It is basically the $T^-$ FDR evaluated at threshold and for convenience we will write it as follows:

$$0 = \Delta_a \equiv D_a - SR_a,$$

where

$$D_a = \frac{1}{3} \left( a_0^{1/2} - a_0^{3/2} \right)$$

is calculated “directly” from our parameterizations. In principle it should be equal to the following integral expression:

$$SR_a = \frac{2 \cdot m_K m_{\pi}}{\sqrt{s} \cdot t_{th}} \int_{s_{th}}^{\infty} \frac{\text{Im}T^{-}(s')}{s'(s' - s_{th})} ds'.$$

In practice, since $D_a$ and $SR_a$ are obtained from data the sum rule will not be exactly zero, but consistency requires it to cancel within uncertainties.

In Table XIV we show the results of this sum rule calculation using our UFD parameterizations. Note that it is not very well satisfied, since the $\Delta_a$ is slightly above 1.2 deviations from zero. This small disagreement suggests that there is room for improvement at low energies in the S-waves. Both the direct and integral results are compatible with the experimental value obtained in DIRAC \[11\], see Eq. (24) above. It is basically the $T^-$ FDR evaluated at threshold and for convenience we will write it as follows:

$$0 = \Delta_I \equiv D_I - SR_I,$$

where

$$D_I = \frac{1}{3} \left( a_0^{1/2} - a_0^{3/2} \right)$$

is calculated “directly” from our parameterizations. In principle it should be equal to the following integral expression:

$$SR_I = \frac{2 \cdot m_K m_{\pi}}{\sqrt{s} \cdot t_{th}} \int_{s_{th}}^{\infty} \frac{\text{Im}T^{-}(s')}{s'(s' - s_{th})} ds'.$$
is more than 30 times smaller than $b_0^{3/2}$, then $D_{3/2} \sim b_0^{3/2}$.

|               | UFD          | CFD          |
|---------------|--------------|--------------|
| $D_{1/2}$     | 0.108±0.018  | 0.091±0.005  |
| $SR_{1/2}$    | 0.093±0.004  | 0.091±0.003  |
| $\Delta_{1/2}$| 0.015±0.012  | 0.000±0.006  |
| $D_{3/2}$     | 0.205±0.039  | 0.187±0.023  |
| $SR_{3/2}$    | 0.186±0.006  | 0.182±0.005  |
| $\Delta_{3/2}$| 0.019±0.038  | 0.005±0.022  |

with $I = 1/2, 3/2$. On the one hand

$$D_I \equiv b_0^{I} + 3a_1^I$$  \quad (55)

will be calculated “directly” from the parameterizations. Note that $a_1^{3/2}$ is more than 30 times smaller than $b_0^{3/2}$, so that $D_{3/2} \sim b_0^{3/2}$ is a very good approximation. On the other hand, the $SR_I$ are calculated with the following integral expressions:

$$SR_{1/2} \equiv \frac{\sqrt{s_{th}}}{2m_\pi m_K} \times \left[ P \int_{s_{th}}^{\infty} ds' \frac{\text{Im} T^+(s') + 2\text{Im} T^-(s')}{(s' - s_{th})^2} \right. - \frac{\text{Im} T^+(s') - 2\text{Im} T^-(s')}{(s' + s_{th} - 2\Sigma_{\pi K})^2} \left. \right]$$  \quad (56)

and

$$SR_{3/2} \equiv \frac{\sqrt{s_{th}}}{2m_\pi m_K} \times \left[ P \int_{s_{th}}^{\infty} ds' \frac{\text{Im} T^+(s') - \text{Im} T^-(s')}{(s' - s_{th})^2} \right. - \frac{\text{Im} T^+(s') + \text{Im} T^-(s')}{(s' + s_{th} - 2\Sigma_{\pi K})^2} \left. \right]$$  \quad (57)

In Table XIV we show the results of these sum rules when the UFD set is used as input. As expected, the integral result, $SR_I$, is more accurate than the direct evaluation, $D_I$, for both sum rules. Although the fulfillment of these sum rules by our UFD set is fairly good, this is mostly due to the large and very asymmetric uncertainties in $D_I$, not to a very good agreement in the central values.

In summary, the UFD set leaves room for improving the fulfillment of the sum rules just discussed. Hence in Sec [X] they will be considered, together with the FDRs, as constraints for our parameterizations.

V. CONSTRAINED FITS TO DATA

So far we have used the FDRs and sum rules as checks of our UFD set. We have seen that there is room for improvement and therefore in this section we will use them as constraints to obtain a new set of parametrizations, that we will call “Constrained Fit to Data” (CFD) set.

In particular we will minimize the following quantity:

$$W^2(d^2_{T^+} + d^2_{T^-}) + \sum_{I=\frac{1}{2}, \frac{3}{2}} \left( \frac{\Delta_I}{\delta I} \right)^2 + \sum_k \left( \frac{p^U_k - p^F_k}{\delta p_k} \right)^2,$$  \quad (58)

where $d^2_{T^\pm}$ are the average square distances between the FDR input and output defined in Eq.(51). $\Delta_I$ are the sum rules defined in Eq.(52) and $\delta I$ are their associated uncertainties. Note that the $\Delta_I$ sum rule of Eq.(52) is included in the $d^2_{T^-}$ term. Finally, to avoid large deviations from the UFD data description, we add a $\chi^2$-like penalty function for each UFD parameter. Generically we have denoted these UFD parameters by $p^U_k$ and their uncertainty by $\delta p^U_k$. The $W^2 = 12$ constant stands for the number of degrees of freedom observed naively from the shape of $Re T\pm$ which as seen in Fig.14 is roughly 12. This approach is the same already followed for $\pi\pi$ scattering in [17, 19–21].

With this minimization procedure we have arrived to a Constrained Fit to Data (CFD) set, whose parameters can be found in Tables [I] to [III] and [V] to [X]. Most CFD parameters are consistent within one deviation with their UFD counterparts. Actually, we have allowed 46 parameters to vary, of which 38 lie within 1 deviation, and only three lie beyond 1.6 deviations. These are the $\Phi_0$ parameter of the $D^{1/2}$-wave, which changes by 3 deviations, the Regge $g_{K/\pi}$ parameters, changing by 2.5 deviations and the $B_2$ parameter of the $S^{3/2}$-wave that changes by 1.8 deviations.

Before discussing in detail the changes from the UFD to the CFD set, let us discuss first how well this new CFD set satisfies the FDRs and sum rules.

1. FDRs and sum rules for the CFD set

In Fig.[E] we show the FDR results for the $T^+$ and $T^-$ amplitudes using the CFD set as input. These have to be compared with the corresponding results for the UFD set in Fig.[D]. Note that, in contrast to what happened when using the UFD set as input, the CFD input and its dispersive output now agree within uncertainties. The only exception is still the $T^+$ FDR above 1.6 GeV, where the agreement has nevertheless improved compared to the UFD result. For this reason in this work we only claim to have precise and consistent parameterizations up to 1.6 GeV. It seems that improving the agreement above this region would require our parameterizations to depart from data. This could be due to the existence of some large systematic uncertainties in some waves or to
In Table XIV we have also provided the CFD result for the sum rules. The central value of all $\Delta_I$ are now closer to zero and the uncertainties are much smaller and much more symmetric.

Thus, once we have seen that the consistency of the description has improved, let us study in detail the changes in the partial waves from the UFD to the CFD set, and check that they still provide a good description of data up to 1.6 GeV.

A. S-waves

1. $S^{3/2}$-wave

The $S^{3/2}$-wave CFD parameters can be found in Table I. In Fig.14 we show as a continuous line the CFD phase shift whose uncertainties are covered by the gray band, whereas the UFD phase is represented by a dashed line. We do not plot the uncertainty band of the UFD curve because it was already given in Fig.1 and it overlaps with the CFD band. From 1 to 1.74 GeV, the UFD and CFD fits are almost identical, although they differ at lower energies. In particular the central value of the CFD scattering length is about a half of that obtained for the UFD, as seen in Table IV.

That some changes were needed at low energies in the scalar waves was to be expected since we already saw that the $\Delta_a$ sum rule was not satisfied very well by the UFD set. Moreover, in Fig.13 a deviation in the low energy region of the FDRs was also observed for the UFD set. It turns out that the FDRs and sum rule constraints tend to correct these small deviations by modifying only the $S^{3/2}$ wave at low energies. Actually, note that both the $B_1$ and $B_2$ parameters of the $S^{3/2}$ wave change from their UFD values by 1.5 and 1.8 deviations, respectively. In contrast, imposing the FDR and sum rule constraints barely changes the $S^{1/2}$-wave in the elastic region, as we will see next. Note also that the CFD result strongly disfavors the Estabrooks et al. data at low energies. This may serve as a posteriori justification for those works that neglect these data from the start.

2. $S^{1/2}$-wave

As it can be seen in Fig.15 the CFD $S^{1/2}$-wave in the elastic region (continuous line) is almost indistinguishable from the UFD parameterization. Actually, as seen in Table II the $B_0$ parameter does not change at all, whereas the CFD $B_1$ central value lies within less than one deviation of the UFD parameter.

3. $t_S$-wave

As explained in previous sections, the data in the inelastic region is presented in terms of the modulus and
The CFD parameterization of the $S^{3/2}$-wave is shown as a continuous line and its uncertainty as a gray band. For comparison the UFD parameterization is shown as a dashed line. See Fig.1 for data references.

FIG. 15: The CFD parameterization of the $S^{3/2}$-wave is shown as a continuous line and its uncertainty as a gray band. For comparison the UFD parameterization is shown as a dashed line. See Fig.1 for data references.

The CFD solution for this wave is shown as a continuous line in Fig.15 where its uncertainty is covered by the gray band. Note that the UFD solution is compatible in the whole energy region with the new CFD parameterization. Moreover, in Table it can be seen that the two CFD $B_k$ parameters lie well within the uncertainties of their UFD counterparts. Therefore the data description is still acceptable.

Let us recall that although the absolute value of this phase shift is smaller than 2.5 degrees in the whole energy region, this wave still has some sizable effect in our calculations. This is partly due to the $(2l + 1)$ factors in Eq.(1), but particularly because all other waves become relatively small around 1.5 GeV.

2. $P^{1/2}$-wave

As seen in Fig.19 the CFD (continuous line) and UFD (dashed) fits are almost indistinguishable up to 930 MeV despite the very small uncertainty (gray band). Around that energy, the CFD result starts deviating towards slightly lower values of the phase, although it is still compatible with the UFD thanks to the fact that the uncertainty band becomes larger in that region.

This means that describing the data around the $K^+(892)$ resonance, whose mass is $\approx 896$ MeV and its width is $\approx 49$ MeV, requires the phase in the 930 MeV to 1 GeV region to be somewhat below the existing data. We emphasize this remark because in the solution of the Roy-Steiner equations in [14], the $K^+(892)$ phase comes out somewhat incompatible with the data (we show the result as a dotted line in Fig.19). To obtain such a solution the authors use as a boundary condition the value of the phase (and its derivative) at $\sqrt{s} = \sqrt{0.935}$ GeV, which they take as $155.8 \pm 0.4^\circ$. However, at that energy, our CFD result yields $152.5 \pm 2.0^\circ$. This could suggest that the mismatch between the Roy-Steiner solution of [14] and the scattering data around the $K^+(892)$ resonance could be due in part to the choice of matching phase and that it might be improved by lowering it by roughly $3^\circ$, as our CFD prefers.
In the threshold region we have calculated the scattering length directly from the CFD parameterization:

\[ m_\pi a_{1/2}^{1/2} = 0.024^{+0.008}_{-0.005}, \]

(59)

to be compared with the UFD result \[ m_\pi a_{1/2}^{1/2} = 0.031^{+0.013}_{-0.008}. \] Note that since our UFD and CFD fits describe the data in Fig. 19, the resulting scattering lengths are larger than the one obtained in \[ 14, m_\pi a_{1/2}^{1/2} = 0.019 \pm 0.001. \]

3. \( t_P \)-wave

Once we have described both the isospin 1/2 and 3/2 \( P \)-waves, we show the modulus and phase of the \( t_P = t_{1/2}^{1/2} + t_{3/2}^{3/2}/2 \) amplitude in Fig. 20 and Fig. 21, respectively. In the inelastic region both the phase and the modulus obtained for the CFD solution are compatible with the UFD parameterizations. Actually, by looking at Table VII one can check that the CFD parameters are almost identical to their UFD counterparts, varying by less than one deviation, except the \( a \) parameter, which changes by 1.4 deviations.

Our CFD solution describes fairly well the three resonances observed in this partial wave, namely the \( K^*(892) \) the \( K^*(1410) \) and the \( K^*(1680) \).

Let us remark that although the two parameterizations are compatible, the CFD result prefers, for the modulus, the data of Estabrooks et al. [27] between 1 and 1.5 GeV.

C. \( D \)-waves

1. \( D^{3/2} \)-wave

In Fig. 22 we show the CFD result for the \( D^{3/2} \)-wave, whose structure is relatively simple and its size and influence are rather small, but not completely negligible, particularly in the inelastic regime. As seen in the fig-
FIG. 18: Phase shift of the $P_{3/2}$ wave. Data from [27]. We show as a continuous line the CFD fit and the gray band cover its uncertainties. The UFD result lies right on the border of this uncertainty band. Note that this phase is rather small up to 1.74 GeV.

FIG. 19: Phase shift of the $P_{1/2}$ wave. The CFD and UFD results are almost indistinguishable up to 950 MeV, where the CFD phase becomes somewhat smaller. Note however that the UFD results still lies inside the uncertainty band. In addition we show the solution in [14]. Data from [27, 28].

FIG. 20: Modulus of the $\hat{t}_P = \hat{t}_{1/2} + \hat{t}_{3/2}/2$ amplitude. We show the CFD fit as a continuous line and its uncertainty as a gray band. Note that the UFD result (dashed line) is also compatible within the CFD uncertainties. Data from [27, 28].

FIG. 21: Phase of the $\hat{t}_P = \hat{t}_{1/2} + \hat{t}_{3/2}/2$ amplitude. We show the CFD fit as a continuous line and its uncertainty as a gray band. Note that the UFD result (dashed line) is also compatible within the CFD uncertainties. Data from [27, 28].

Since there are no data in the elastic region for the $I = 1/2$ D-wave partial wave, we directly show the modulus and phase of the $t_D = t_{1/2} + t_{1/2}/2$ combination in Figs. 23 and 24, respectively. For the modulus, the CFD solution is almost indistinguishable from the UFD curve up to 1.6 GeV. However, above that energy the central UFD value lies typically two to three deviations away from the central CFD value. Nevertheless, both fits are still compatible due to the rather large uncertainty band of the UFD set, shown in Fig.[9]. Concerning the phase, this is the curve where, above 1.6 GeV, we find the largest deviation from the data and the UFD set. By comparing the CFD versus the UFD parameters for this wave, given
in Table IX we find that the $\phi_0$ parameter changes by more than 3 deviations. This is the only parameter that changes so dramatically from its UFD to its CFD value. Note it is closely related to the background produced by the opening of the $K\eta$ channel.

This deviation is not too worrisome since it occurs at the very end of our parameterizations and outside the peak of the $K_2^*(1430)$, whose width is roughly 100 MeV. Therefore, the amplitude in that region is relatively small. At this point it is important to recall that the symmetric FDR, shown in Fig 14, is well satisfied by the CFD set only up to 1.6 GeV. Above that energy, it improves the UFD result, but it is not enough to consider it satisfactory. As already commented, this is one of the reasons why in this work we claim to have precise and consistent data parameterizations up to 1.6 GeV and not beyond. Above that region the measured data is hard to reconcile with the dispersive constraints. This might be due to the existence of further systematic uncertainties, not necessarily in this wave, or to the increasingly important contribution from the tower of partial waves to the partial wave expansion.

D. $F^{1/2}$-wave

The CFD result for the $F^{1/2}$-wave is almost indistinguishable from our previous UFD result and describes nicely the $K^*_2(1780)$. This can be seen in Figs 22 and 23 where we show the modulus and the phase of the partial wave, respectively.

E. CFD Regge parameterizations

When imposing dispersive constraints on the amplitude, we have also allowed the $f_{K/\pi}$, $r$ and $g_{K/\pi}$ Regge parameters to vary. The rest of Regge parameters have been kept fixed to the values in the literature, also used for the UFD set and given in Table XI. The reason is that, in principle, these other parameters can be determined without using processes involving kaons or $\pi K$ scattering.
In Table XI we can observe that, in the end, the \( f_{K/\pi} \) and \( r \) parameters barely change. However the CFD value of \( g_{K/\pi} \) changes by 2.5 deviations from its UFD counterpart and is responsible for more than half of the reduction in \( d_{2-}^2 \), particularly at high energies. As we commented before, it is not very surprising that this parameter suffers a large change, since there is little information to determine it reliably. It can be considered that in this work we are making a dispersive determination of this parameter.

### VI. DISCUSSION

Before concluding, let us discuss our results in relation with data obtained from the decay of heavier particles, as well as regarding poles of resonances in the elastic regime and particularly the controversial \( K_0^*(800) \) or \( \kappa \)-meson.

#### A. Data from decays of heavier particles

As already commented in Subsec III A further information on the \( \pi K \) system has been obtained from the decays of heavier particles.

The semileptonic \( D^+ \to K^-\pi^+e^+\nu_e \) decays have been analyzed by the BaBar [29] and BESIII [30] collaborations providing data on the phase difference between the \( S \) and \( P \) components. Since only the \( \pi K \) interact strongly in the final state, Watson’s theorem applies and in the elastic region this measurement is nothing but the difference between the \( S \) and \( P \) scattering phase shifts. In Fig. 27 we show the results for the \( I = 1/2 \) \( S \)-wave phase obtained from semileptonic \( D \) decays, compared to those from scattering experiments. Note that the uncertainties from decays are much larger than those obtained from scattering. Although what is actually measured in these decays is the phase-shift difference between the \( P \) - and \( S \) -waves, the experimental collaborations provide tables for the \( S \)-wave alone, by using a simple \( P \)-wave description, whose uncertainty is much smaller and can be neglected. A similar procedure has been followed with the LASS scattering data of Aston et al. [28] shown in Fig. 27 for comparison, where the \( I = 3/2 \) component has been separated with the Estabrooks et al. model [27]. The above caveats and the very large uncertainties justify our not including data from decays in our fits. All in all, there is a nice qualitative agreement between different data sources and also with our UFD and CFD results that we also show in the figure. Moreover it is reassuring to see the good agreement between our parameterizations and the decay data in the near threshold region, where no scattering data exist.

In addition, from Dalitz plot analyses it has been possible to extract the \( I = 1/2 \) amplitudes and phase of the \( \pi K \) \( S \)-wave component in \( D^+ \to K^-\pi^+\pi^+ \) by the E791 [31], FOCUS [32] and CLEO-c [33] collaborations, as well as in \( \eta_c \to K\bar{K}\pi \) by the BaBar collaboration [34]. As already commented in Subsec III A, in this case Watson’s Theorem does not imply that the phase thus measured should be the same as that of scattering. The reason is the presence of another strongly interacting particle in the final state. This is particularly obvious by noting that the measured amplitudes and phases do not satisfy the elastic scattering unitarity condition. Nevertheless, it has also been noticed in these works that the measured phase shows a qualitative agreement with the scattering phase-shift in the elastic region, once it is appropriately displaced by a constant. We show this qualitative agreement in Fig 25 where once again the data from scatter-
FIG. 27: Phase of the $I = 1/2 \pi K$ $S$-wave obtained from the semileptonic decay $D^+ \rightarrow K^- \pi^+ e^+ \nu_e$ by the BaBar Collaboration [29] and recently by the BESIII Collaboration [31]. These phases are compared to the LASS scattering phase shift of Aston et al. (using their $I = 3/2$ parameterization to separate the $I = 1/2$). Note that the experiments are in fairly good agreement up to 1.6 GeV.

Note, however, that the agreement disappears in the region above 1.6 GeV, which is where we have also found that the scattering data are largely incompatible with Forward Dispersion Relations. It is then tempting to fit in this region the phase from decays instead of the phase from scattering, in the hope that the FDRs may be better satisfied. However, note that we can only try to fit the phase from decays, based on its similarity to the scattering phase, but not the modulus, since the energy dependence observed for the latter is very different from that of scattering. We have performed this exercise and we have checked that the FDRs are satisfied even worse.

B. Pole parameters of elastic resonances

Our partial waves are constructed as piecewise parameterizations which are matched continuously in the real axis. As a consequence, the resulting global amplitude does not provide a rigorous analytic continuation to the whole complex plane. Each one of the pieces may have an analytic continuation of its own, but at most it may only be a good approximation to the amplitude near the part of the real axis where that particular function is used, far from the other pieces of functions. Nevertheless, in the elastic region we have used a conformal mapping which has a well-defined analytic continuation to the complex $s$ plane. As explained in Appendix A, the interesting feature of this mapping is that it places the inelastic singularities at the boundary of the unit circle. Therefore one can expect that it will provide a relatively good representation of the partial wave for complex values of $s$ which are not close to that boundary.

With these caveats in mind, we can obtain a determination of the pole positions of resonances that appear in the elastic region by considering the analytic continuation of just the elastic conformal parameterizations. Two such resonances exist, both with $I = 1/2$, namely the controversal $K^*_0(800)$, or $\kappa$-meson, and the $K^*(892)$ in the scalar and vector partial waves, respectively. Their associated poles are located in the second Riemann sheet of the partial wave, defined as

$$ t^{II}(s) = \frac{t(s)}{1 + 2i\sigma(s)t(s)}, \tag{60} $$

where in the upper half complex $s$-plane $\sigma(s)$ is defined as in Sec. III whereas in the lower half plane $\sigma(s) = -\sigma(s^*)^*$. Therefore the second sheet pole position is a solution of

$$ \cot \delta^I_l(s_{pole}) = -i, \tag{61} $$

where the analytic continuation of the cotangent of the phase-shift is obtained through the conformal expansion in Eqs. (13) and (28) for the $K^*_0(800)$ and $K^*(892)$, respectively.
Customarily, since for narrow resonances isolated from other poles or thresholds the Breit-Wigner formula applies, one identifies the pole position of a resonance with its mass and width as follows: $s_R = (M_R - i\Gamma_R/2)^2$. Despite the $K^*_0(800)$ being a very wide resonance, we keep this convention and the resulting pole parameters for this resonance can be found in Table XV, both for the UFD and CFD parameterization. This is also the convention used in the Review of Particle Physics (RPP) \cite{PDG}. The values we obtain are very compatible with the averaged mass in the RPP, $M_{K^*_0(800)} = 682 \pm 29$ MeV. In contrast, the width is somewhat larger than the value quoted there $\Gamma_{K^*_0(800)} = 547 \pm 24$ MeV. Actually, the most rigorous derivation is that in \cite{Roy-Steiner} by means of a Roy-Steiner analysis, where it is found that $M_{K^*_0(800)} = 658 \pm 13$ MeV and $\Gamma_{K^*_0(800)} = 557 \pm 24$ MeV. Nevertheless, there is a large spread of values listed in the RPP and several other determinations find a width very similar to ours. As a word of caution, when comparing to the RPP one should take into account that our numbers correspond to a pole position, whereas many values there correspond to peak parameterizations through Breit-Wigner formalisms or its variations, whose applicability is dubious due to the large width of this resonance.

The corresponding poles for the vector $K^*(892)$ are found in Table XVI. In this case the pole mass is very similar to the values provided in the RPP, typically obtained from Breit-Wigner parameterizations. In contrast our pole width is about 10 MeV higher than the ones listed in the RPP, $M_{K^*(892)} = 673 \pm 15$ MeV. Actually, the most rigorous derivation is that in \cite{Roy-Steiner} by means of a Roy-Steiner analysis, where it is found that $M_{K^*(892)} = 687 \pm 10$ MeV and $\Gamma_{K^*(892)} = 57 \pm 24$ MeV. Nevertheless, there is a large spread of values listed in the RPP and several other determinations find a width very similar to ours. As a word of caution, when comparing to the RPP one should take into account that our numbers correspond to a pole position, whereas many values there correspond to peak parameterizations through Breit-Wigner formalisms or its variations, whose applicability is dubious due to the large width of this resonance.

| Poles Mass (MeV) | Width (MeV) | Coupling |
|------------------|-------------|----------|
| UFD 673±15       | 674±15      | 5.01±0.07|
| CFD 680±15       | 668±15      | 4.99±0.08|

In this work we have presented a set of pion-kaon scattering parameterizations, which up to 1.6 GeV describe data and simultaneously satisfy a complete set of Forward Dispersion Relations as well as three sum rules for threshold parameters. Our aim has been to make the parameterizations relatively simple and easy to implement in future theoretical or experimental applications.

On a first step we have obtained a set of Unconstrained Fits to Data (UFD), in which partial waves with different angular momentum are fitted independently. Waves with different isospin are fitted together because that is how data was originally obtained. We have paid particular attention to the estimation of uncertainties, particularly to those of a systematic nature, which are not always taken into account in the literature. In addition, for the most controversial wave we have checked some statistical tests for the consistency of our uncertainty estimates. Above 1.74 GeV, since no data on all partial waves exist, we have used Regge parameterizations that were obtained in previous works by applying factorization to other processes involving nucleons, pions and kaons.

However, it is shown that, even within uncertainties, this UFD set does not satisfy well Forward Dispersion Relations and also shows some tension when used inside the threshold sum rules. In particular, above the $K\eta$ threshold the dispersive results lie typically two deviations or more away from the direct calculation when using the UFD parameterizations. Throughout the elastic region the agreement is somewhat better, but still only at the level of 1.5 deviations.

Thus, on a second step, we have imposed the Forward Dispersion relations and the sum rules as constraints on the fit parameters. Note that the parameterizations stay the same and we only change the values of the parameters. Our final result is a set of Constrained Fits to Data (CFD) that satisfies Forward Dispersion Relations remarkably well up to 1.6 GeV while still describing the data. In particular, the deviations between the CFD and
UFD results have been shown to be relatively small and within the uncertainties of the UFD fit. As a consequence, the CFD set still provides a good description of data. Above 1.6 GeV, we have found that the fulfillment of the dispersive constraints would require large modifications of the fits that would spoil the data description. Thus our parameterizations describe the data and are simultaneously consistent with dispersive constraints only up to 1.6 GeV.

Using this CFD set we have provided a precise determination of three combinations of scattering lengths and slope parameters. In addition, given that the conformal map parameterization chosen for the elastic region has very good analytic properties in the complex plane we have obtained the pole parameters of the resonances that appear in that region, namely the vector \( K^* (892) \) and the controversial scalar \( K_0^* (800) \) or \( \kappa \)-meson. The poles and residues come in reasonably good agreement with previous determinations, although of course, the analytic continuation is dependent on our choice of conformal mapping, which is very reasonable, but not entirely model-independent. Nevertheless, we plan to use our CFD results in the real axis as input to extract pole parameters using model-independent analytic approaches.

For the future we also plan to constraint further our parameterizations with a complete set of equations of the Roy-Steiner type. These are much more complicated relations written in terms of partial waves, and parameterizations of different resonances. The vector part can be of use in present and future experimental and theoretical analysis involving pions and kaons in the final state.

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Appendix A: Conformal expansion for elastic waves

Let us recall that elastic partial waves can be written as

\[
t_l(s) = \frac{1}{\sigma(s) \cot \delta_l(s) - i},
\]

where \( \sigma(s) = 2q/\sqrt{s} \) and \( q \) is the CM momentum. In the complex s-plane, partial waves for the scattering of two particles with different masses \( m_1 \) and \( m_2 \) have a distinct analytic structure in the first Riemann sheet, shown in Figure 29a. First of all, there is a right-hand or physical cut extending from the opening of the elastic threshold to infinity. In addition, due to the thresholds in the crossed channels, there is a left-hand cut extending from \((m_1 - m_2)^2 \) to \(-\infty\), as well as a circular cut at \(|s|^2 = (m_1^2 - m_2^2)^2\). Other singularities may appear on the real axis when bound states exist in the direct or crossed channels, but this is not the case in \( \pi K \) scattering. Let us emphasize that there are no poles in the first Riemann sheet. The cut singularities are reproduced in the second Riemann sheet, where poles can now occur anywhere in the complex plane. When poles are sufficiently close to the real axis, they give rise to resonant phenomena.

Now, in order to describe the amplitude in the complex s-plane, it is customary to recast the partial wave as

\[
t_l(s) = \frac{q^{2l}}{\Phi(s) - iq^{2l} \sigma(s)}, \tag{A2}
\]

so that, as shown in Fig. 29b, the effective range function \( \Phi(s) \) does not have elastic cut, but only the left-hand and circular ones, as well as the inelastic cuts. Depending on the dynamics, it might also have poles at the zeros of the amplitude, as we will discuss below. In our case, it has no singularity from \( \pi K \) threshold to the next inelastic threshold \( s_0 \). When the expansion of \( \Phi(s) \) is made in terms of powers of \( q \), the coefficient of the first term of the expansion is known as the scattering length, the second is the slope, etc... But the radius of convergence of this series, centered at \( s = (m_K + m_\pi)^2 \), is small, since the circular cut singularity lies rather close. The best way to use the largest possible domain of analyticity is by changing variables by means of a conformal transformation. In this case, however, it is convenient to perform first another change of variable which maps the circular cut into the left real axis:

\[
y(s) = \left( \frac{s - \Delta_{K\pi}}{s + \Delta_{K\pi}} \right)^2, \tag{A3}
\]

where \( \Delta_{K\pi} = m_K^2 - m_\pi^2 \). The resulting \( \Phi(y(s)) \) function now only has a right-hand “inelastic” cut and a left-hand cut, as shown in Fig. 29c, and then we can use the conformal variable

\[
w(y) = \frac{\sqrt{y} - \alpha \sqrt{y_0 - y}}{\sqrt{y} + \alpha \sqrt{y_0 - y}}, \tag{A4}
\]
FIG. 29: Analytic structure in different variables of a πK scattering partial wave $t(s)$ and effective range function $\Phi(s)$: a) $t(s)$ in the complex $s$-plane. Note the elastic, inelastic, left-hand and circular cuts. b) $\Phi(s)$ in the $s$-plane has the same structure as $t(s)$ except for the absence of the elastic cut. c) In the $y(s)$-plane the circular cut disappears. d) The conformal variable $\omega(y)$ maps the whole analyticity domain of $\Phi(y)$ inside the unit circle, whereas the cut singularities are confined to $|\omega| = 1$. Note that $\omega$ is defined so that the data region is roughly centered around $\omega = 0$ and not too close to the border.

to map the cut $y$-plane into the unit circle in the $\omega$-plane.

With the exception of the minute $P^{3/2}$ and $D^{3/2}$-waves, in this work we have chosen $\alpha$ for each wave so that the center of the conformal expansion $\omega = 0$ corresponds to the intermediate point between the $\pi K$ threshold and the energy of the last data point that is fitted with the conformal formula. The reason for this choice is to ensure that the region where data is to be fitted lies well inside the $\omega$ circle, roughly centered around $|\omega| < 0.45$. However for the $S^{1/2}$-wave the data lie at $|\omega| < 0.6$. The $P^{3/2}$ and $D^{3/2}$-waves are an exception, because their data starts at 1 GeV, far form the $\pi K$ threshold. Thus we have chosen their $\alpha$ parameters so that the center of the conformal expansion corresponds to the intermediate point where data exists. With this choice, the data fitted with this conformal expansion lies at $|\omega| < 0.6$.

Since with these changes of variable the singularities now lie at $|\omega| = 1$, the function has an analytic expansion $\Phi(s) = \sum_n B_n w(s)^n$ convergent in the whole $|\omega| < 1$ circle. In this way, and in terms of $s$, the domain of analyticity of the conformal mapping extends to the whole complex plane outside the circular cut, with a left-hand cut and a right-hand cut above the first inelastic threshold. Thus on the elastic region of the real axis

$$\cot \delta_l(s) = \frac{\sqrt{s}}{2q^{2l+1}} \Phi(s) = \frac{\sqrt{s}}{2q^{2l+1}} \sum_n B_n w(s)^n, \quad (A5)$$

which are the expressions we have used for our elastic fits.

Finally, let us recall that due to chiral symmetry, scalar partial waves have a so-called Adler zero below threshold, which is easily implemented in the partial waves by writing a pole factor in front of the $\Phi(s)$ expansion, as follows:

$$\Phi(s) = \frac{1}{s - s_{Adler}} \sum_n B_n w(s)^n. \quad (A6)$$

In addition, when there is a narrow well-established resonance and the phase crosses $\pi/2$ at $m_r$ it is also convenient to extract a factor out of the conformal expansion as:

$$\Phi(s) = (s - m_r^2) \sum_n B_n w(s)^n, \quad (A7)$$

to accelerate the convergence of the fit.

Appendix B: Statistical test on the S wave

Since the $S^{1/2}$-wave is the most controversial one we have used some statistical tests to check the consistency of our fits and the data obtained from [27] and [28] for the $t_S \equiv t_0^{1/2} + t_0^{3/2}/2$ amplitude. As it has been explained in
the main text, the problem with the data is the existence of large systematic uncertainties that we necessarily had to estimate. Once we had these systematic uncertainties added to the statistical ones, we have performed the fits and obtained, by minimizing the $\chi^2$, the fit parameters and their uncertainties. The $\chi^2$ is based on a Gaussianity assumption and one would like to test that the resulting fit and the data are still consistent with it. For this reason we will check the consistency of our fits by means of the central moment statistical test, which in rather similar conditions was suggested for $\pi\pi$ scattering in [32].

Let $N$ be the number of data points, measured at energies $\sqrt{s_i}, i = 1...N$. We then introduce a set of $N$ residuals $R_i = (P_i - f(\alpha, s_i))/\langle \Delta P_i \rangle$. Here $P_i$ is the experimental value of the $i$-th measurement, $\Delta P_i$ its uncertainty (experimental and systematic) associated to that value, and $f(\alpha, s_i)$ is the theoretical model evaluated at $s_i$. The set of UFD parameters is called $\alpha$.

By assumption, this set of residuals must obey a standardized normal distribution. For this purpose we study the central moments of the residual distribution

$$\mu_{\text{UFD}, n} = \frac{1}{N} \sum_{i=1}^{N} (R_i - \langle R \rangle)^n,$$

where $\langle R \rangle = \sum R_i / N$.

We would like to compare these $\mu_{\text{UFD}, n}$ with the expected value of a set of $N$ data standardized Gaussian points. Thus, we generate $M$ samples of distributions of $N$ data points $R_{ik}, k = 1,...M$, that follow a normal Gaussian distribution, and calculate the central moments $\mu_{nk}$ of each sample. We then define the average central moment $\langle \mu_n \rangle = \sum_k M \mu_{nk} / M$. Similarly, we define the uncertainty in this distribution of residuals as the usual standard deviation: $\Delta \mu_n = \sqrt{(\mu_n^2 - \langle \mu_n \rangle^2)}$.

In order to compare the moments of our UFD result with those of the generated distributions, we have to recall that we have parameterized the S-wave into two regions with different functional forms, and we have fitted two sets of observables, $|f_S|$ and $\phi_S$. Therefore we have 4 different tests, which are presented in Tables XVIII, XIX and XX.

Actually, our procedure to estimate uncertainties has made use of these tests. At first we introduce as systematic uncertainties half of the distance between those points measured at the same energy which are incompatible. Then, we modify the systematic uncertainties of the few data points that cause deviations from the Gaussian behavior of the tests. With these modified systematic uncertainties the fit is performed again, the tests are checked once more and the systematic uncertainties of points that cause deviations from the test are changed again. The procedure is iterated until the Gaussianity tests are well satisfied.

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| TABLE XVII: Normality condition for $\Phi_S$ in the elastic region. |
| --- |
| $n$ | 1 | 2 | 3 | 4 | 5 | 6 |
| $\mu_{\text{UFD}, n}$ | 0 | 0.8 | -0.3 | 1.6 | -1.1 | 4.2 |
| $\mu_{\text{random}, n}$ | 0 ± 0.1 | 1.0 ± 0.2 | 0 ± 0.4 | 2.8 ± 1.6 | 0 ± 4 ± 14 ± 17 |

| TABLE XVIII: Normality condition for $|f_S|$ in the elastic region. |
| --- |
| $n$ | 1 | 2 | 3 | 4 | 5 | 6 |
| $\mu_{\text{UFD}, n}$ | 0 | 1.1 | 0.1 | 2.4 | 0.8 | 6.7 |
| $\mu_{\text{random}, n}$ | 0 ± 0.1 | 1.0 ± 0.2 | 0 ± 0.4 | 2.8 ± 1.6 | 0 ± 4 ± 14 ± 17 |
TABLE XIX: Normality condition for the $\Phi_1$ in the inelastic region.

| $n$ | $\mu_{FD,n}$ | $\mu_{random,n}$ |
|-----|---------------|------------------|
| 1   | 0             | $0 \pm 1.0 \pm 0.2$ |
| 2   | 1             | $0 \pm 0.4$ |
| 3   | 2             | $2.8 \pm 1.6$ |
| 4   | 3             | $0 \pm 1$ |
| 5   | 4             | $0.6$ |
| 6   | 5             | $3.6$ |

TABLE XX: Normality condition for the $\tilde{\mu}_S$ in the inelastic region.

| $n$ | $\mu_{FD,n}$ | $\mu_{random,n}$ |
|-----|---------------|------------------|
| 1   | 0             | $0 \pm 1.0 \pm 0.2$ |
| 2   | 1             | $0 \pm 0.4$ |
| 3   | 2             | $2.8 \pm 1.6$ |
| 4   | 3             | $0 \pm 1$ |
| 5   | 4             | $0.6$ |
| 6   | 5             | $3.6$ |

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