Holonomy and Symmetry in M-theory

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Abstract

Supersymmetric solutions of 11-dimensional supergravity can be classified according to the holonomy of the supercovariant derivative arising in the Killing spinor condition. It is shown that the holonomy must be contained in SL(32, R). The holonomies of solutions with flux are discussed and examples are analysed. In extending to M-theory, account has to be taken of the phenomenon of ‘supersymmetry without supersymmetry’. It is argued that including the fermionic degrees of freedom in M-theory requires a formulation with a local SL(32, R) symmetry, analogous to the need for local Lorentz symmetry in coupling spinors to gravity.

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1 Introduction

In [1], Duff and Liu addressed the two key questions of what the symmetries of M-theory might be, and how to classify the supersymmetric solutions. The concept of \emph{generalised holonomy} played a central role in their discussion. A bosonic solution of 11-dimensional supergravity will preserve $n$ supersymmetries ($0 \leq n \leq 32$) if it admits $n$ spinor fields $\epsilon$ satisfying the condition

$$\tilde{D}_M \epsilon = 0,$$

where $\tilde{D}_M$ is a certain connection (the supercovariant derivative) on the spin-bundle. The number of solutions can then be analysed in terms of the holonomy $\mathcal{H}(\tilde{D})$ of the connection $\tilde{D}$. Such generalised holonomy has been considered by a number of authors, including [2, 3, 4, 5]. In [1], the holonomy groups were analysed for a special class of warped product solutions with a $d/(11 - d)$ split in which the spacetime decomposes locally into a $d$ dimensional piece and one of $11 - d$ dimensions. It was found that for splits in which the holonomy of the Levi-Civita connection $D$ was in $\text{SO}(d - 1, 1) \times \text{SO}(11 - d)$, the holonomy of $\tilde{D}$ was in the enlarged groups $G_{\text{spacelike}}(n)$ and $G_{\text{timelike}}(n)$ are given in table 1. They also considered null splits in which the holonomy of the Levi-Civita connection is in $\text{ISO}(d - 1) \times \text{ISO}(10 - d)$, but that of $\tilde{D}$ is in $G = \text{ISO}(d - 1) \times G_{\text{null}}(10 - d)$, where the groups $G_{\text{null}}(n)$ are also given in table 1.

These same groups $G$ also arise as the local symmetry groups of 11-dimensional gravity dimensionally reduced on a spacelike $n$-torus [6], a timelike $n$-torus [7] or a null $n$-torus [1], respectively, for $n \leq 8$. For example, for a spacelike reduction on $T^n$, the resulting theory has local $\text{SO}(d - 1, 1)$ Lorentz invariance where $d = 11 - n$, together with a local $G_{\text{spacelike}}(n)$ R-symmetry and a global $E_{n(+n)}$ duality symmetry (where $E_{n(+n)}$ is the maximally non-compact form of $E_n$ and for $n \leq 5$, the group is defined by a Dynkin diagram of the E-type, so that $E_5(5) = \text{SO}(5, 5)$, $E_4(4) = \text{SL}(5, \mathbb{R})$ etc). The scalar fields take values in $E_{n(+n)}/G_{\text{spacelike}}(n)$ and in the quantum theory the rigid symmetry is broken to a discrete subgroup $E_{n(+n)}(\mathbb{Z})$ [8]. For a timelike reduction, the theory has local $\text{SO}(d) \times G_{\text{timelike}}(n)$ and global $E_{n(+n)}$ symmetries, with scalars in $E_{n(+n)}/G_{\text{timelike}}(n)$ [7].

Remarkably, it has been shown that the full 11-dimensional supergravity can be rewritten in a form with local $\text{SO}(d - 1, 1) \times G_{\text{spacelike}}(11 - d)$ symmetry for $d = 4$ [9], $d = 3$ [10] and $d = 5, 6$ [11]. These formulations involve making a $d/(11 - d)$ split and gauging away the off-diagonal components of the vielbein. This led Duff and Liu to conjecture that there could be a similar formulation of the full 11-dimensional supergravity theory using any of the groups $G$, in which the field equations have a local $G$ invariance. It could be the case that the spacetime symmetry group $G$ depends on certain features of the spacetime, such as whether it has a product structure, but it might also be the case that it is a larger group, containing all of the
### Table 1: Generalized structure groups.

| $n$ | $G_{\text{spacelike}}(n)$ | $G_{\text{timelike}}(n)$ | $G_{\text{null}}(n)$ |
|-----|----------------------|------------------------|---------------------|
| 1   | $\{1\}$             | $\{1\}$                | $\{1\}$            |
| 2   | $\text{SO}(2)$      | $\text{SO}(1,1)$       | $\mathbb{R}$        |
| 3   | $\text{SO}(3) \times \text{SO}(2)$ | $\text{SO}(2,1) \times \text{SO}(1,1)$ | $\text{ISO}(2) \times \mathbb{R}$ |
| 4   | $\text{SO}(5)$      | $\text{SO}(3,2)$       | $[\text{SO}(3) \times \text{SO}(2)] \times \mathbb{R}^{6}_{(3,2)}$ |
| 5   | $\text{SO}(5) \times \text{SO}(5)$ | $\text{SO}(5,\mathbb{C})$ | $\text{SO}(5) \times \mathbb{R}^{10}_{(10)}$ |
| 6   | $\text{USp}(8)$     | $\text{USp}(4,4)$      | $[\text{SO}(5) \times \text{SO}(5)] \times \mathbb{R}^{16}_{(4,4)}$ |
| 7   | $\text{SU}(8)$      | $\text{SU}^*(8)$       | $\text{USp}(8) \times \mathbb{R}^{27}_{(27)}$ |
| 8   | $\text{SO}(16)$     | $\text{SO}^*(16)$      | $[\text{SU}(8) \times \text{U}(1)] \times \mathbb{R}^{56}_{(28_{1/2},\mathbb{R}_{-1/2})}$ |
| 9   | $\text{SO}(16) \times \text{SO}(16)$ | $\text{SO}(16,\mathbb{C})$ | $\text{SO}(16) \times \mathbb{R}^{120}_{(120)}$ |
| 10  | $\text{SO}(32)$     | $\text{SO}(16,\mathbb{C})$ | $[\text{SO}(16) \times \text{SO}(16)] \times \mathbb{R}^{256}_{(16,16)}$ |

For spacelike reductions, the holonomy group is in $G = \text{SO}(d-1,1) \times G_{\text{spacelike}}(11-d)$, for timelike ones it is in $G = G_{\text{timelike}}(d-1,1) \times \text{SO}(11-d)$ while for null ones it is in $G = \text{ISO}(d-1) \times G_{\text{null}}(10-d)$.

The purpose here is to consider the general case in which no assumption is made about a product structure of the solution. The holonomy of $\tilde{D}$ must be contained in $\text{GL}(32,\mathbb{R})$ as 11-dimensional Majorana spinors are real and have 32 components. Moreover, it should contain the groups in table 1, so in particular it should contain both $\text{SO}(32)$ and $\text{SO}(16,16)$. As will be seen, the holonomy of $\tilde{D}$ must in fact be in $\text{SL}(32,\mathbb{R})$, and the consequences of this for supersymmetric solutions will be explored. The holonomy is $\text{SL}(32,\mathbb{R})$ for generic backgrounds, and particular classes of background have holonomy restricted to special subgroups, such as backgrounds with a $d/(11-d)$ split which have the special holonomies in table 1. Examples will be considered in which the holonomy is in other subgroups of $\text{SL}(32,\mathbb{R})$ that did not arise in $\text{SL}(10)$, e.g. for static solutions with electric flux, the holonomy is in $\text{SL}(16,\mathbb{C})$ or, with an additional assumption on the ansatz, in $\text{Spin}(10,\mathbb{C})$.

This, together with the arguments of $\text{II}$, motivates the conjecture that there should be a formulation of 11-dimensional supergravity in which there is a local $\text{SL}(32,\mathbb{R})$ symmetry. This would be similar to the formulations of $\text{II}$, $\text{IX}$ and $\text{XI}$ and in this sense it could then be said that the 11-dimensional theory would have a hidden $\text{SL}(32,\mathbb{R})$ spacetime symmetry.

An important issue regarding such reformulations of $D = 11$ supergravity is that there is a sense in which they are simply rewritings of the original theory and have no physical content. After all, it is possible to enlarge the symmetry of any theory by introducing extra degrees of freedom, and then introducing extra symmetries that can be used to eliminate these extra degrees of freedom. For the dimensionally reduced supergravity theory in $11-n$ dimensions...
(\(n \leq 8\)), the physical scalars take values in the coset \(E_{n(+n)}/G(n)\) and the theory has a non-linearly realised \(E_{n(+n)}\) global symmetry, where \(G(n)\) is the appropriate group from table 1. Introducing extra scalars taking values in the group \(G(n)\) leads to a formulation in which the \(E_{n(+n)}\) global symmetry is linearly realised, and in which there is a new local \(G(n)\) symmetry which can be used to gauge away the extra unphysical scalars \[6\]. This local \(G(n)\) symmetry is not an essential part of the classical theory, but it is very convenient to write the theory in a formulation with this symmetry.

As was to be expected, the situation is similar in the 11-dimensional formulations with local \(G(n)\) symmetry. In \[9\], \[10\] and \[11\], extra fields are introduced in \(d = 11\) which can be gauged away by the local \(G(n)\) symmetry in \(d = 11\). In both the reduced and eleven-dimensional supergravities, introducing the extra fields and extra \(G(n)\) symmetry is a matter of convenience leading to a useful way of formulating the theory, but the local symmetry is not an essential part of the theory, although it is suggestive that many of the interactions have such a symmetry.

An important question for M-theory then is whether the symmetry \(G(n)\) is a convenience leading to a useful way of formulating the theory, as in supergravity, or whether it is an essential part of the theory. It will be argued here that for M-theory extra symmetries such as those in table 1 play a crucial role and that they do act non-trivially on physical degrees of freedom, so that the theory cannot be written in a form without these symmetries. In particular, it will be seen that certain physical degrees of freedom of M-theory arise as sections of bundles with transition functions in the structure group \(G\) and which cannot be regarded as sections of e.g. the spin bundle. These arise in situations where there is ‘supersymmetry without supersymmetry’ \[12, 13, 14\], corresponding to M-theory vacua which are known to be supersymmetric but for which the corresponding supergravity solution does not have Killing spinors.

It is interesting to compare with gravity. General relativity in \(d\)-dimensions can be formulated in terms of a metric, and the holonomy is in \(SO(d-1,1)\). It can instead be formulated in terms of a vielbein with local \(SO(d-1,1)\) Lorentz symmetry. This involves introducing extra fields (the extra components of the vielbein) together with an extra local Lorentz gauge symmetry that can be used to eliminate them, so that the number of degrees of freedom remains the same. For pure gravity, this is just a convenient rewriting of the theory with a tangent space group that is the same as the holonomy group. However, for coupling to spinor fields, the formulation with local Lorentz symmetry is essential. A similar story seems to be true for M-theory. In classical supergravity (reduced from 11 to \(d\) dimensions), one can work in physical gauge (with no extra scalars) and the generalised holonomy is in the appropriate group \(G\) from table 1, or one can introduce extra scalars so that the structure group \(G\) is the same as the holonomy group; the two formulations are equivalent. However, it will be argued that to describe the fermionic degrees of freedom in M-theory, local \(G\) symmetry is essential and so
a formulation with such symmetry is required. A formulation with local \( SL(32, \mathbb{R}) \) symmetry allows the coupling to all such degrees of freedom that can arise, and the extra symmetry is independent of the choice of background.

The plan of the paper is as follows. In section 2, the generalised holonomy of \( \tilde{\mathcal{D}} \) will be reviewed, and it will be shown that in general it is in \( SL(32, \mathbb{R}) \). In section 3, the holonomy will be discussed further and the number of supersymmetries will be discussed. In section 4, the holonomies of certain examples will be considered. Section 5 extends the discussion to other 11-dimensional supergravities. Section 6 discusses M-theory, and in particular the phenomenon of supersymmetry without supersymmetry, and argues that M-theory requires fermions which are sections of an \( SL(32, \mathbb{R}) \) bundle, rather than the spin bundle. It concludes with some speculative remarks. Further details of the structure groups that arise in gauged supergravities are given in an appendix.

## 2 Killing Spinors and Generalized Holonomy

The fields of \( D = 11 \) supergravity are a graviton \( g_{MN} \), a gravitino \( \Psi_M \) and 3-form gauge field \( A_{MNP} \), where \( M = 0, 1, \ldots 10 \). All spinors are in the Majorana spinor representation of \( \text{Spin}(10, 1) \), and a vielbein \( e_M^A \) is used to convert coordinate indices \( M, N \) to tangent space indices \( A, B \). The bosonic field equations (setting \( \Psi_M = 0 \)) are

\[
R_{MN} = \frac{1}{12} \left( F_{MPQR} F_N^{PQR} - \frac{1}{12} g_{MN} F^{PQRS} F_{PQRS} \right),
\]

and

\[
d * F^{(4)} + \frac{1}{2} F^{(4)} \wedge F^{(4)} = 0, \]

where \( F^{(4)} = dA^{(3)} \). The supersymmetry transformation rule of the gravitino in a bosonic background is

\[
\delta \Psi_M = \tilde{\mathcal{D}}_M \epsilon,
\]

with spinor parameter \( \epsilon \), where

\[
\tilde{\mathcal{D}}_M = D_M - \frac{1}{288} (\Gamma^N_{MNPQR} - 8 \delta^N_M \Gamma^{PQR} ) F_{NPQR},
\]

The \( \Gamma_A \) are \( D = 11 \) Dirac matrices and \( \Gamma_{AB...C} \) are antisymmetrised products of gamma matrices, \( \Gamma_{AB...C} = \Gamma_{[A} \Gamma_{B}...\Gamma_{C]} \). The signature is \( (-++\ldots+) \) and a Majorana representation is used in which the spinors have 32 real components and the gamma-matrices are real. Here \( D_M \) is the usual Riemannian covariant derivative involving the Levi-Civita connection \( \omega_M \) taking values in the tangent space group \( \text{Spin}(10, 1) \)

\[
D_M = \partial_M + \frac{1}{4} \omega_M^{AB} \Gamma_{AB}.
\]
In the quantum theory, the field equations and supersymmetry transformations receive higher derivative corrections; these will not be considered explicitly here. Note that a space admitting Killing spinors does not necessarily satisfy the field equations.

Each solution of
\[ \tilde{D}_M \epsilon = 0, \]  \( (7) \)
is a Killing spinor field that generates a supersymmetry leaving the background invariant, so that the number of supersymmetries preserved by a supergravity background depends on the number of supercovariantly constant spinors satisfying (7). Any commuting Killing spinor field \( \epsilon \) defines a Killing vector \( v_A = \tau \Gamma_A \epsilon \), which is either timelike or null, together with a 2-form \( \tau \Gamma_{AB} \epsilon \) and a 5-form \( \tau \Gamma_{ABCDE} \epsilon \).

If \( F_{(4)} = 0 \), then \( \tilde{D} = D \) and the Killing spinors are covariantly constant with respect to the Levi-Civita connection. If the holonomy group of this connection is \( \mathcal{H}(D) \subseteq \text{Spin}(10, 1) \), then the covariantly constant spinors are the singlets of the holonomy group \( \mathcal{H}(D) \) under the decomposition of the 32 of \( \text{Spin}(10, 1) \) under \( \mathcal{H}(D) \). For Euclidean signature, the holonomy groups have been classified by Berger \[15\], while in Lorentzian signature the holonomies of spacetimes with parallel spinors were analysed by Bryant \[16\]. Suppose there is at least one Killing spinor \( \epsilon \). If the Killing vector \( v_A = \tau \Gamma_A \epsilon \) is timelike, then \( \mathcal{H}(D) \subseteq SU(5) \subset \text{Spin}(10, 1) \) and the allowed values for the number of Killing spinors are 2, 4, 6, 8, 16, 32 \[17, 18\]. If on the other hand the Killing vector \( v_A \) is null, then \( \mathcal{H}(D) \subseteq \mathbb{R} \times (\text{Spin}(7) \rtimes \mathbb{R}^8) \subset \text{Spin}(10, 1) \) and the allowed values for the number of Killing spinors are 1, 2, 3, 4, 8, 16, 32 \[17, 18\].

If \( F_{(4)} \neq 0 \), then \( \tilde{D} \) is a connection on the spin bundle. The Clifford algebra \( Cl(10, 1) \) is spanned by the matrices \( \{1, \Gamma_A, \Gamma_{AB}, \Gamma_{ABC}, \Gamma_{ABCD}, \Gamma_{ABCDE}\} \) and is the algebra of real 32 \( \times \) 32 matrices, \( \text{Mat}(32, \mathbb{R}) \). In particular, the commutation relations of these matrices are those of the algebra \( GL(32, \mathbb{R}) \), and the holonomy of \( \tilde{D} \) must be contained in \( GL(32, \mathbb{R}) \). Note that \( \Gamma_{AB} \) generate \( \text{Spin}(10, 1) \), \( \{\Gamma_A, \Gamma_{AB}\} \) generate the subalgebra \( \text{Spin}(10, 2) \) and \( \{\Gamma_A, \Gamma_{AB}, \Gamma_{ABCDE}\} \) generate the subalgebra \( \text{Sp}(32, \mathbb{R}) \).

In \[11\], spaces with a product structure were considered, with Riemannian holonomy \( \mathcal{H}(D) \subseteq \text{SO}(d - 1, 1) \times \text{SO}(11 - d) \), allowing a \( d/(11 - d) \) split. Attention was restricted to cases that allow a dimensional reduction to \( d \) dimensions. The system was truncated to one in which only the metric and scalars in \( d \) dimensions were kept, with the ansatz
\[
g^{(11)}_{MN} = \begin{pmatrix} \Delta^{-1/(d-2)} g_{\mu\nu} & 0 \\ 0 & g_{ij} \end{pmatrix}, \quad A^{(11)}_{ijk} = \phi_{ijk}, \tag{8} \]
where \( \Delta = \det g_{ij} \). The \( d \)-dimensional fermion fields were defined as
\[
\psi_\mu = \Delta^{\frac{1}{d-2}} \left( \psi^{(11)}_\mu + \frac{1}{d-2} \gamma_\mu \Gamma^i \psi^{(11)}_i \right), \quad \lambda_i = \Delta^{\frac{1}{d-2}} \psi^{(11)}_i, \tag{9} \]
with the gravitino transforming as
\[ \delta \psi_\mu = \hat{D}_\mu \epsilon, \quad \hat{D}_\mu = \partial_\mu + \frac{1}{4} \Omega_\mu, \] (10)
with a certain generalised connection \( \Omega_\mu \). The supersymmetry was analysed in [1] in terms of the holonomy of the supercovariant derivative \( \hat{D}_\mu \) of the reduced system, and an important role was played by the fact that \( \Omega_\mu \) involves only \( \gamma_{\alpha\beta} \) together with the algebra generated by \( \{ \tilde{\Gamma}_{ab}, \tilde{\Gamma}_{abc} \} \). Here \( \gamma_\alpha \) are SO\((d - 1, 1)\) Dirac matrices, while \( \tilde{\Gamma}_a \) are SO\((11 - d)\) Dirac matrices. The holonomy group is then SO\((d - 1, 1) \times \text{Gspacelike}(11 - d)\) where \( \text{Gspacelike}(11 - d) \) is the group generated by \( \{ \tilde{\Gamma}^{(2)}, \tilde{\Gamma}^{(3)}, \tilde{\Gamma}^{(6)}, \tilde{\Gamma}^{(7)}, \tilde{\Gamma}^{(10)} \} \), with the notation that \( \tilde{\Gamma}^{(n)} \) represents the antisymmetric product of \( n \) gamma matrices, \( \tilde{\Gamma}_a_{1...a_n} \). (For \( n > 11 - d \), \( \tilde{\Gamma}^{(n)} = 0 \).) This gives the groups in table 1, with the exception that for \( d = 4 \), the generator \( \tilde{\Gamma}^{(7)} \) which could have occurred in the algebra in fact does not arise on the right hand side of any commutators of the algebra generated by \( \{ \tilde{\Gamma}^{(2)}, \tilde{\Gamma}^{(3)}, \tilde{\Gamma}^{(6)} \} \), so that the algebra is \( \text{Gspacelike}(7) = SU(8) \) rather than the \( U(8) \) that would have arisen on adding \( \tilde{\Gamma}^{(7)} \).

The definition of \( \psi_\mu \) in (9) eliminates the terms involving \( \Gamma^{(5)} \) in \( \delta \psi^{(11)} \) from \( \delta \psi_\mu \), which now appear in \( \delta \lambda^i \). The conditions for supersymmetry considered in [1] that there be spinors satisfying \( \hat{D}_\mu \epsilon = 0 \), restricting the holonomy of \( \hat{D}_\mu \), are then necessary but not sufficient, as they must be supplemented by the conditions \( \delta \lambda^i = 0 \), and the holonomy of \( \hat{D} \) is in general different from that of \( \tilde{D} \). Here, the emphasis will be on the holonomy of \( \tilde{D} \), giving necessary and sufficient conditions for supersymmetry. Examples will be considered in the next section.

In general, from the form of \( \tilde{D} \), the holonomy for \( \tilde{D} \) must be in the subalgebra of GL\((32, \mathbb{R})\) generated by \( \{ \Gamma^{(2)}, \Gamma^{(3)}, \Gamma^{(5)} \} \). From above, closing the algebra generated by \( \{ \Gamma^{(2)}, \Gamma^{(3)} \} \) leads to the set \( \{ \Gamma^{(2)}, \Gamma^{(3)}, \Gamma^{(6)}, \Gamma^{(7)}, \Gamma^{(10)} \} \) which, using the fact that \( \Gamma^{(n)} \propto *\Gamma^{(11-n)} \), is the algebra generated by \( \{ \Gamma^{(1)}, \Gamma^{(2)}, \Gamma^{(3)}, \Gamma^{(4)}, \Gamma^{(5)} \} \). In particular, adding \( \Gamma^{(5)} \) to this does not enlarge the algebra. The issue is then whether \( \Gamma^{(11)} = 1 \) occurs on the right hand side of any commutators. A calculation shows that it does not (the situation is similar to the absence of \( \Gamma^{(7)} \) for the case \( 11 - d = 7 \) discussed above) so that the algebra is indeed SL\((32, \mathbb{R})\), not the full GL\((32, \mathbb{R})\). As a result, the holonomy of \( \tilde{D} \) must be contained in SL\((32, \mathbb{R})\).

3 Supersymmetric Backgrounds and Special Holonomies

3.1 Holonomies and Structures

The key question is which subgroups of SL\((32, \mathbb{R})\) actually occur as holonomies of supergravity backgrounds. This is the analogue of the question of which holonomies can occur for Riemannian manifolds, which was answered by Berger [15].
It is often useful to write the supercovariant derivative as

$$\tilde{D}_M = \hat{D}_M + X_M$$

(11)

for some other connection $\hat{D}_M$ on the spin-bundle, and some covariant $32 \times 32$ matrix $X_M$. Then one can make the ansatz in which one seeks backgrounds admitting Killing spinors satisfying the algebraic constraints

$$X_M \epsilon = 0$$

(12)

These should also satisfy $\hat{D}_M \epsilon = 0$, and so can be analysed in terms of the holonomy of the associated derivative $\hat{D}_M$. Clearly $\mathcal{H}(\hat{D}_M) \subseteq \mathcal{H}(\tilde{D}_M)$, but often $\mathcal{H}(\hat{D}_M)$ is easier to analyse. For example, as reviewed in the last section, Duff and Liu analysed the holonomy of the connection $\hat{D}$ arising from requiring $\delta \psi_{\mu} = 0$ where $\psi_{\mu}$ is defined in (9), which must be supplemented by the condition $\delta \lambda_i = 0$, which is algebraic of the form (12) for their ansatz.

As another example, consider the case in which $X_M = \Gamma_M f$ where

$$f = \frac{1}{24} F_{MPQR} \Gamma^{MPQR}$$

(13)

and note that the derivative (5) can be rewritten as

$$\tilde{D}_M = D_M + \frac{1}{24} \Gamma^{PQR} F_{MPQR} - \frac{1}{12} \Gamma_M f$$

(14)

Then for backgrounds in which the Killing spinor satisfies

$$f \epsilon = 0$$

(15)

the Killing spinor condition simplifies to

$$\tilde{D}_M \epsilon \equiv (D_M + \frac{1}{24} \Gamma^{PQR} F_{MPQR}) \epsilon = 0$$

(16)

and the analysis of supersymmetric backgrounds in terms of the holonomy $\mathcal{H}(\tilde{D})$ will be explored in the next section.

It will be useful to refer to the maximal holonomy group for a class of configurations as the structure group. Thus the structure group associated with $\tilde{D}$ for a generic configuration is the group $\text{SL}(32, \mathbb{R})$ generated by $\{ \Gamma^{(2)}, \Gamma^{(3)}, \Gamma^{(5)} \}$. As we have seen, for special classes of configuration, the structure group for a related operator $\hat{D}$ is the group generated by $\{ \Gamma^{(2)}, \Gamma^{(3)} \}$. In 11 dimensions, this is the same group $\text{SL}(32, \mathbb{R})$, but in lower dimensions or for product spaces, this leads to the groups in table 1, as will be explored further in the next section. The particular subgroup of the structure group that arises as the holonomy will determine the number of Killing spinors.
Note that $\tilde{D}$ is not the most general $\text{SL}(32, \mathbb{R})$ connection one could write down, because of the particular way that $F$ enters into the expression, so in principle it could have been that it would have led to a structure group smaller than $\text{SL}(32, \mathbb{R})$. However, as the structure group has to contain both $\text{SO}(32)$ and $\text{SO}(16, 16)$, together with the other groups found in the next section, the structure group must in fact be $\text{SL}(32, \mathbb{R})$.

The connection $\tilde{D}$ on the spin bundle extends to tensor products, so that one can define the supercovariant derivative of multi-spinors $\chi^{\alpha\beta\ldots\gamma}$. The only invariant of $\text{SL}(32, \mathbb{R})$ is the 32nd rank alternating tensor, while for the subgroup $\text{Sp}(32, \mathbb{R})$ there is an invariant anti-symmetric 2-form $C^{\alpha\beta}$, the charge conjugation matrix. For the subgroup $\text{Spin}(10, 1)$ (or $\text{Spin}(10, 2)$, $\text{Spin}(6, 5)$, $\text{Spin}(6, 6)$) there are invariant gamma matrices $(\Gamma_M)^{\alpha\beta}$. A bi-spinor can be related to a set of forms using gamma-matrices, so that $\chi^{\alpha\beta}$ can be written as a linear combination of the $n$-forms $\chi_{M_1\ldots M_n} = (C^{-1}\Gamma_{M_1\ldots M_n})^{\alpha\beta} \chi^{\alpha\beta}$ for $n = 0, 1, 2, 3, 4, 5$, where $C$ is the charge conjugation matrix. If $H(\tilde{D})$ is in $\text{Spin}(10, 1)$, then there is a natural extension of $\tilde{D}$ to a metric connection on the tangent bundle so that the gamma-matrices are supercovariantly constant, and a bi-spinor satisfying $\tilde{D}\chi^{\alpha\beta} = 0$ will then define forms that are supercovariantly constant, $\tilde{D}\chi_{M_1\ldots M_n} = 0$. In this case, one can consider $\tilde{D}$ as a connection on the tangent bundle, and analyse its holonomy. However, for holonomies not in $\text{Spin}(10, 1)$ (or one of the other Spin subgroups) there is no natural definition of $\tilde{D}$ on the tangent bundle. Nonetheless, if a space admits Killing spinors, then the tangent bundle will have a $G$-structure, i.e. it can be regarded as a bundle with transition functions in some group $G \subset \text{Spin}(10, 1)$, with the group related to the number of Killing spinors, which are singletons under $G$; this has been used to analyse the geometry associated with Killing spinors in [23]. There are many more subgroups of $\text{SL}(32, \mathbb{R})$ that can arise as holonomies than there are subgroups $G \subset \text{Spin}(10, 1)$ that can arise in $G$-structures; for example, all of the solutions with $16 < n \leq 32$ supersymmetries have trivial $G$-structures (with $G=1$) but each of the different values of $n$ corresponds to a different holonomy. Then the generalised holonomy may be more useful in classifying supersymmetric spaces, while the $G$-structure approach of [23] is more useful in the construction of explicit solutions.

### 3.2 Number of Supersymmetries

The superalgebra in 11-dimensions with tensorial charges [20] allows any number of supersymmetries $0 \leq n \leq 32$ to be preserved by a state [19]. This is a non-trivial statement, as other superalgebras in other dimensions place restrictions on the allowed number of supersymmetries. Many of the supersymmetric solutions are not asymptotically flat, and so are difficult to analyse in terms of a global superalgebra. For this reason, it seems more useful to address the problem through the Killing spinor equation and generalised holonomy.
Until recently, no solutions with $16 < n < 32$ supersymmetries were known, but now solutions are known preserving $0, 1, 2, 3, 4, 5, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 32$ supersymmetries [21] – [30]. These values can be discussed in terms of the holonomy groups of Table 1 [1]. (Note that the conditions for supersymmetry from the holonomy of $\hat{D}$ are necessary but not sufficient in [1], as one also needs to take into account the conditions from the supersymmetry variations of the spin-half fermions arising in the dimensional reduction.)

In [31], it was suggested that there could be ‘preon’ states in M-theory preserving 31 supersymmetries from which all other BPS states might arise as bound states. Although their conjecture does not require that there be classical supergravity solutions preserving 31 supersymmetries, it is nonetheless interesting to ask whether such backgrounds exist. In [1], it was shown that $n = 31$ cannot arise from holonomies contained in any of the structure groups in Table 1. However, it could be that such solutions could occur without being of the type arising from the ansatz of [1].

For a configuration preserving $n$ supersymmetries, the holonomy must be in $\text{SL}(32 - n, \mathbb{R}) \ltimes \mathbb{R}^{n(32 - n)}$. All values of $n$ are then possible in principle, including $n = 31$, which would require a non-trivial $\mathbb{R}^{31}$ holonomy. The issue is then whether this holonomy can actually arise from a supergravity configuration. The maximally supersymmetric solutions have been classified in [5].

4 Examples of Supersymmetric Backgrounds and their Holonomies

4.1 Examples with Cosmological Constant

Consider the Freund-Rubin ansatz of a product space with a 4/7 split in which the holonomy group of the Levi-Civita connection is in $\text{Spin}(3,1) \times \text{Spin}(7)$ and with

$$F_{\mu \nu \rho \sigma} = 3 \mu \epsilon_{\mu \nu \rho \sigma} \tag{17}$$

where $\mu, \nu, \rho = 0, 1, 2, 3$ and $\mu$ is a constant with the dimensions of mass. Then the space is the product of a four-dimensional spacetime with negative curvature

$$R_{\mu \nu} = -12 \mu^2 g_{\mu \nu} \tag{18}$$

and a seven-dimensional space of positive curvature

$$R_{ij} = 6 \mu^2 g_{ij} \tag{19}$$

where $i, j = 1, 2, \ldots, 7$. The supercovariant derivative is then given by

$$\tilde{D}_\mu = D_\mu + im\gamma_\mu \gamma_5 \tag{20}$$
and

\[ \tilde{D}_i = D_i - \frac{i}{2} \mu \Gamma_i \]  

The holonomy of these two connections are Spin(3, 2) and Spin(8) respectively, so in this case the structure group is Spin(3, 2) × Spin(8) and the holonomy must be a subgroup of this. Note that both factors in Spin(3, 1) × Spin(7) have been enlarged. The maximally supersymmetric solution AdS_4 × S^7 has trivial holonomy. A similar ansatz with a product of a Euclidean four-dimensional space and a Lorentzian seven-dimensional one has the Riemannian structure group Spin(4) × Spin(6, 1) enhanced to Spin(5) × Spin(6, 2), with maximally supersymmetric solution AdS_7 × S^4 with trivial holonomy.

This is an example of something that occurs in other contexts and so it is worth considering more generally. Consider first a d dimensional space X with positive definite metric, so that \( \mathcal{H}(\omega) \subseteq \text{Spin}(d) \), and consider Killing spinors satisfying

\[ \tilde{D}_i \epsilon = 0 \]  

where the derivative is

\[ \tilde{D}_i = D_i - i m \Gamma_i \quad \text{if } d \text{ odd,} \quad \tilde{D}_i = D_i - m \Gamma_i \Gamma_* \quad \text{if } d \text{ even} \]  

where \( \Gamma_* \) is the chirality operator in d dimensions, \( \Gamma_* \propto \prod_i \Gamma_i \). The geometries that give rise to different number of spinors have been classified in [32]. As \( i \Gamma_i \), \( \Gamma_{ij} \) together generate Spin(d + 1) if d is odd, while \( \Gamma_i \Gamma_*, \Gamma_{ij} \) generate Spin(d + 1) if d is even, the holonomy of \( \tilde{D}_i \) will in either case be in the structure group Spin(d + 1), and the number of Killing spinors will depend on this holonomy. The geometries arising for various holonomies \( \mathcal{H}(\tilde{D}) \subseteq \text{Spin}(d + 1) \) are given in table 2, and these holonomies are a useful way of categorising these spaces. For odd d, the number of Killing spinors depends on the sign of \( m \), and the table gives the numbers \( n_+, n_- \) of Killing spinors for either sign. For odd d, changing the orientation of X effectively changes the sign of \( m \).

The holonomy \( \mathcal{H}_X(\tilde{D}) \) is also the holonomy of the Levi-Civita connection \( \mathcal{H}_{C(X)}(D) \) on the cone over X, C(X), which has metric \( ds^2_{C(X)} = dr^2 + r^2 ds^2_X \) where \( ds^2_X \) is the metric on X. For example the cone over a 3-Sasaki space is hyperkähler. The round sphere \( S^d \) with isometry \( \text{SO}(d + 1) \) has maximal supersymmetry and trivial holonomy \( \mathcal{H}(\tilde{D}) \), and the cone \( C(S^d) = \mathbb{R}^{d+1} \) is flat space with \( \mathcal{H}(D) = 1 \).

If \( m \) is replaced with \( im \) in (23), so that

\[ \tilde{D}_i = D_i - m \Gamma_i \quad \text{if } d \text{ odd,} \quad \tilde{D}_i = D_i - im \Gamma_i \Gamma_* \quad \text{if } d \text{ even} \]  

then the cosmological constant changes sign to become negative and the holonomy group \( \mathcal{H}(\tilde{D}) \) is in Spin(d, 1). The subgroups of Spin(d, 1) that give rise to Killing spinors have been classified in
| $d = \dim X$ | Generalised Holonomy | Geometry of $X$ | $(n_+, n_-)$ |
|-------------|-----------------|----------------|-------------|
| $d$         | $\{1\}$        | round sphere   | $(2^{[d/2]}, 2^{[d/2]})$ |
| $4k - 1$    | $\text{Sp}(k)$  | 3-Sasaki       | $(k + 1, 0)$ |
| $4k - 1$    | $\text{SU}(2k)$ | Sasaki–Einstein | $(2, 0)$     |
| $4k + 1$    | $\text{SU}(2k + 1)$ | Sasaki–Einstein | $(1, 1)$     |
| $6$         | $G_2$           | nearly Kähler  | $(1, 1)$     |
| $7$         | $\text{Spin}(7)$ | weak $G_2$ holonomy | $(1, 0)$     |

Table 2: Manifolds admitting real Killing spinors

In this case the generalised holonomy $\mathcal{H}(\hat{D})$ is the holonomy of the Levi-Civita connection on a timelike cone $\tilde{C}(X)$ over $X$, with metric $ds^2_{\tilde{C}(X)} = -dt^2 + t^2 ds^2_X$. This Killing spinor equation arises in the supergravity theories of $\cite{33}$ and hyperbolic space $H^d = SO(d,1)/SO(d)$ is maximally supersymmetric with trivial holonomy, and the cone $\tilde{C}(H^d)$ is $d + 1$ dimensional Minkowski space.

Similarly, one can consider the Killing spinor equations for Lorentzian spaces $X$ with the Riemannian holonomy contained in $\text{Spin}(d-1,1)$. In this case, with supercovariant derivatives given by $\cite{23}$, the generalised holonomy is contained in $\text{Spin}(d-1,2)$, so that the maximally supersymmetric case with trivial holonomy is anti-de Sitter space; this again arises in many supergravities. The holonomy $\mathcal{H}(\hat{D})$ is the holonomy of the Levi-Civita connection on a timelike cone $\tilde{C}(X)$ over $X$, with metric of signature $(d-1,2)$. If $m$ is replaced with $im$ to give $\cite{24}$, the generalised holonomy is contained in $\text{Spin}(d,1)$ instead, with maximally supersymmetric solution given by de Sitter space, as in the supergravity theories of $\cite{33,37}$. The corresponding cone is the spacelike cone $C(X)$ over $X$, with Lorentzian signature $(d,1)$.

### 4.2 Direct Products with Flux

We will seek backgrounds admitting Killing spinors satisfying

$$\frac{1}{24} F_{MPQR} \Gamma^{MPQR} \epsilon = 0 \tag{25}$$

Such a constraint was used in $\cite{40,41,42}$. Then for such solutions, the Killing spinors satisfy $\hat{D} \epsilon = 0$ where $\hat{D}$ is the associated derivative

$$\hat{D}_M = (D_M + \frac{1}{24} \Gamma^{PQR} F_{MPQR}) \tag{26}$$

and we will analyse the holonomy of $\hat{D}$. This holonomy group is in the group generated by $\Gamma_{(2)}, \Gamma_{(3)}$. In general these generate the whole of $\text{SL}(32, \mathbb{R})$, but further assumptions about the configuration lead to interesting restrictions.
Consider product spaces $M_d \times M_{\bar{d}}$ with a $d/\bar{d}$ split, so that the coordinates can be split into $x^\mu, y^i$ with $\mu, \nu = 1, ..., d$ and $i, j = 1, ..., \bar{d} = 11 - d$, and a product metric of the form

$$g^{(11)}_{MN} = \begin{pmatrix} g_{\mu \nu}(x) & 0 \\ 0 & g_{ij}(y) \end{pmatrix}$$

(27)

with one of the metrics $g_{\mu \nu}(x), g_{ij}(y)$ having Lorentzian signature and the other Euclidean signature. One of these spaces must be even dimensional; suppose it is $M_d$. A convenient realisation of the gamma matrices $\Gamma_M$ in terms of the gamma matrices $\gamma_\mu$ on $M_d$ and the ones $\tilde{\Gamma}_i$ on $M_{\bar{d}}$ is

$$\Gamma_\mu = \gamma_\mu \otimes \tilde{\Gamma}_*, \quad \Gamma_i = 1 \otimes \tilde{\Gamma}_i$$

(28)

where $\tilde{\Gamma}_*$ is the chirality operator on $M_{\bar{d}}$, $\tilde{\Gamma}_* \propto \prod_i \tilde{\Gamma}_i$.

The holonomy of the Levi-Civita connection is in the group generated by $\Gamma_{\mu \nu}, \Gamma_{ij}$, and so $\mathcal{H}(D) \subseteq \text{Spin}(d - 1, 1) \times \text{Spin}(\bar{d})$ if $M_d$ is Lorentzian, or $\mathcal{H}(D) \subseteq \text{Spin}(d) \times \text{Spin}(d - 1, 1)$ if $M_{\bar{d}}$ is Lorentzian. If the only non-vanishing components of $F$ are $F_{ijkl}$, then the holonomy of $\tilde{D}$ is in the group generated by $\Gamma_{\mu \nu}, \Gamma_{ij}, \Gamma_{ijk}$, so $\mathcal{H}(\tilde{D}) \subseteq \text{Spin}(d - 1, 1) \times G_{\text{spacelike}}(\bar{d})$ or $\mathcal{H}(\tilde{D}) \subseteq \text{Spin}(d) \times G_{\text{timelike}}(\bar{d})$. Similarly, if the only non-vanishing components of $F$ are $F_{\mu \nu \rho \sigma}$, then the holonomy of $\tilde{D}$ is in the group generated by $\Gamma_{\mu \nu}, \Gamma_{ij}, \Gamma_{\mu \nu \rho}$, which is $G_{\text{timelike}}(d) \times \text{Spin}(\bar{d})$ or $G_{\text{spacelike}}(d) \times \text{Spin}(\bar{d} - 1, 1)$.

Next suppose both $F_{ijkl}$ and $F_{\mu \nu \rho \sigma}$ are non-zero, and all other components are zero. This requires a $7/4$ or $5/6$ split. Then the holonomy is in the group generated by $\Gamma_{\mu \nu}, \Gamma_{ij}, \Gamma_{ijk}, \Gamma_{\mu \nu \rho}$ and so contains both $G_{\text{spacelike}}(n)$ and $G_{\text{timelike}}(11 - n)$ (where $n$ is the dimension of the spacelike factor), but for $7/4$ and $5/6$ splits, these two subgroups do not commute. For example, the commutator $[\Gamma_{ij}, \Gamma_{\mu \nu \rho}]$ includes the term $\Gamma_{ijk\mu \nu \rho}$ and the holonomy in this case is in general in $SL(32, \mathbb{R})$.

Consider further the example of a $7/4$ split, with a Lorentzian 4-space. On the four-dimensional factor, the maximal structure group is $SO(3, 2)$ generated by $\Gamma_{ij}, \Gamma_{ijk}$. The maximal subgroup of $SL(32, \mathbb{R})$ commuting with this is $SU(8)$, generated by $\Gamma_{\mu \nu}, \Gamma_{\mu \nu \rho \sigma}, \Gamma_{\mu_1 \mu_2 \ldots \mu_6}$. However, the 4-gamma term $\Gamma_{\mu \nu \rho \sigma}$ does not occur in the supercovariant derivative, nor does it occur in the commutators of terms that do, so that the maximal structure group containing $SO(3, 2)$ that is a proper subgroup of $SL(32, \mathbb{R})$ and can arise as a holonomy of the supercovariant derivative is $SO(3, 2) \times SO(8)$, and $SO(3, 2) \times SU(8) \subset SL(32, \mathbb{R})$ does not occur.

Similarly, for a $5/6$ split with a Lorentzian 5-space the structure groups $SO(4, 1) \times USp(8)$ and $SO(5, \mathbb{C}) \times SO(6)$ are possible (from table 1) but $SO(5, \mathbb{C}) \times USp(8)$ is not as it is not contained in $SL(32, \mathbb{R})$. Although $SL(32, \mathbb{R})$ has subgroups $SO(4, 2) \times USp(8)$ and $SO(5, \mathbb{C}) \times SO(6, \mathbb{C})$, neither of these arise as structure groups as the extra generators are products of four gamma matrices, which do not occur in the supercovariant derivative.
4.3 Warped Products with Flux

Consider now a warped product with a 3/8 split, with metric of the form (8) with \( d = 3 \), and 4-form field strength with \( F_{ijkl} \), \( F_{\mu \nu \rho i} \) the only non-vanishing components of the 4-form field strength, as in [41,42]. Then the Killing spinor \( \eta \) can be decomposed into a 2-component Spin(2,1) spinor \( \epsilon \) and a 16-component Spin(8) spinor \( \xi \)

\[
\eta = \epsilon \otimes \xi
\]  

(29)

and \( \xi \) can be decomposed into 8-component chiral spinors \( \xi = \xi_+ + \xi_- \), with \( \tilde{\Gamma}_s \xi_\pm = \pm \xi_\pm \). As in [41,42], we consider configurations with

\[
F_{\mu \nu \rho i} = \epsilon_{\mu \nu} \partial_i \Delta^{-3/2},
\]  

(30)

and

\[
F_{mnpq} \gamma^{mnpq} \xi = 0
\]  

(31)

Then the condition \( \tilde{D}_\mu \eta = 0 \) gives

\[
D_\mu \eta + \frac{1}{4} \partial_n (\log \Delta) \left[ \gamma_\mu \otimes \tilde{\Gamma}_n (1 - \tilde{\Gamma}_s) \right] \eta = 0
\]  

(32)

As \( \tilde{D}_\mu \) commutes with \( 1 \otimes \tilde{\Gamma}_s \), one can decompose \( \eta = \eta_+ + \eta_- \), with \( \eta_\pm = \epsilon \otimes \xi_\pm \) and consider the action of \( \tilde{D}_\mu \) separately on \( \eta_\pm \). On \( \eta_+ \), the holonomy is \( \mathcal{H}(\tilde{D}_\mu)^+ \subseteq \text{Spin}(2,1) \times \text{Spin}(8) \) while on \( \eta_- \), the term involving \( \partial \Delta \) leads to the structure group generated by \( \gamma_\mu \otimes 1, 1 \otimes \tilde{\Gamma}_m \) and \( \gamma_\mu \otimes \tilde{\Gamma}_n (1 - \tilde{\Gamma}_s) \), which is the semi-direct product \([\text{Spin}(2,1) \times \text{Spin}(8)] \rtimes \mathbb{R}^{24,8}\). Then \( \mathcal{H}(\tilde{D}_\mu)^- \) is contained in this group. The Killing spinor condition \( \tilde{D}_\mu \eta = 0 \) is satisfied if \( \eta = \epsilon \otimes \xi_+ \) with chiral \( \xi \) and \( D_\mu \epsilon = 0 \). If \( \partial \Delta = 0 \), there is no warping and the space is a direct product with \( F_{\mu \nu \rho i} = 0 \) and the holonomy for \( \eta^+, \eta^- \) is in \( \text{Spin}(2,1) \times \text{Spin}(8) \).

The remaining conditions \( \tilde{D}_i \eta = 0 \) give

\[
D_m \xi + \frac{1}{24} \Delta^{3/2} \tilde{\Gamma}_{mnpq} F_{mnpq} \xi + \frac{1}{4} \partial_m (\log \Delta) \xi - \frac{3}{8} \partial_n (\log \Delta) \gamma_m \gamma^n \xi = 0
\]  

(33)

For this to have a solution with chiral \( \xi \) requires

\[
\tilde{\Gamma}_{mnpq} F_{mnpq} \xi = 0
\]  

(34)

Then the holonomy of \( \tilde{D}_i \) is in \( \text{CSpin}(8) = \text{Spin}(8) \times \mathbb{R}^+ \) with a conformal piece \( \mathbb{R}^+ \), and a Weyl transformation \( g_{ij} \rightarrow \hat{g}_{ij} = \Delta^{-1/2} g_{ij} \) brings this to

\[
\hat{D}_i \hat{\xi} = 0
\]  

(35)

where \( \hat{D}_i \) is the Levi-Civita connection for \( \hat{g}_{ij} \) and \( \hat{\xi} = \Delta^{1/4} \xi \), and \( \mathcal{H}(\hat{D}_i) \subseteq \text{Spin}(8) \). This requires that \( \hat{\xi}_+ \) is covariantly constant with respect to \( \hat{D}_i \), and so \( \hat{g}_{ij} \) must be a special holonomy.
metric. For one parallel spinor on the eight-manifold $\mathcal{H}(\hat{D}_i) \subseteq \text{Spin}(7)^+$ and for two $\mathcal{H}(\hat{D}_i) \subseteq \text{SU}(4)$.

Then with this ansatz the structure group is $\mathcal{G} = C\text{Spin}(8)^+ \times ([\text{Spin}(2,1) \times \text{Spin}(8)^-] \times \mathbb{R}^{24}_{(3,8)})$, where $\text{Spin}(8)^\pm$ act on positive or negative chirality spinors, and the holonomy is contained in this. There will be Killing spinors if the 3-space is 3-dimensional Minkowski space, and the 8-manifold is conformally related to a manifold with special holonomy $H \subseteq \text{Spin}(7)^+$. Then the holonomy group is in $\mathcal{H} = H \times \mathbb{R}^+ \times (\text{Spin}(8)^- \times \mathbb{R}^{24})$.

If $\Delta$ is constant so that there is no warping, the structure group reduces to $\mathcal{G} = \text{Spin}(2,1) \times \text{Spin}(8)^+ \times \text{Spin}(8)^-$, which is contained in the group $\mathcal{G} = \text{Spin}(2,1) \times \text{Spin}(16)$ of table 1. The holonomy would then be in the subgroup $1 \times H \times \text{Spin}(8)^- \subseteq \text{Spin}(2,1) \times \text{Spin}(8)^+ \times \text{Spin}(8)^-$. However, for non-trivial warping, one obtains a holonomy and structure group not contained in any of the groups in table 1.

In general there are no negative chirality Killing spinors. If $H = \text{Spin}(7)$, one can take $F_{ijkl}$ to be proportional to the Spin(7)-invariant 4-form and there are 2 positive chirality Killing spinors, and the background preserves 1/16 supersymmetry [12]. If $H = \text{SU}(4)$, the space is conformal to a Calabi-Yau space and one take $F_{ijkl}$ to be a $(2,2)$ form satisfying $J^{ij}F_{ijkl} = 0$, where $J_{ij}$ is the Kahler form, and there are 4 positive chirality Killing spinors, so the background preserves 1/8 supersymmetry [21].

### 4.4 Static Spaces

Consider static spacetimes of the form

$$ds^2 = -\Delta(x)dt^2 + g_{ij}dx^i dx^j$$

For the ansatz of [1], the structure group for $\hat{D}_i$ is in $\text{SO}(32)$, generated by $\{\Gamma_2, \Gamma_3\}$, which closes on the set of generators $\{\Gamma_2, \Gamma_3, \Gamma_6, \Gamma_{10}\}$ where $\Gamma_n = \Gamma_{i_1...i_n}$ are products of spatial gamma-matrices (so that $\Gamma_n \propto \Gamma_{(10-n)} \Gamma_0$). The subset $\{\Gamma_2, \Gamma_6\}$ generates $\text{SU}(16)$. Consider general electric and magnetic fluxes $E_{ijk} = F_{bijk}$ and $B_{ijkl} = F_{ijkl}$. With general $E, B$, the structure group is $\text{SL}(32, \mathbb{R})$. For the purely magnetic case, $E = 0$, the structure group for $\hat{D}_i$ is generated by $\{\Gamma_2, \Gamma_3, \Gamma_5\}$, giving for generic cases a holonomy the full $\text{SL}(32, \mathbb{R})$. However, for those configurations in which the Killing spinor in addition satisfies

$$F_{ijkl} \Gamma^{ijkl} \epsilon = 0$$

the $\Gamma_5$ term is absent for the corresponding associated derivative $\hat{D}$ and the holonomy group $\mathcal{H}(\hat{D})$ is in $\text{SO}(32)$. For $n$ supersymmetries, the holonomy must be in the subgroup $\text{SO}(32 - n)$.

For electric configurations with $B = 0$, the structure group is generated by $\{\Gamma_2, \Gamma_0 \Gamma_2, \Gamma_0 \Gamma_4\}$, which closes on the generators $\{\Gamma_2, \Gamma_4, \Gamma_6, \Gamma_8\}$ of $\text{SL}(16, \mathbb{C})$. For $2n$ supersymmetries, the
holonomy must be in $\text{SL}(16 - n, \mathbb{C})$. If in addition the Killing spinors satisfy

$$F_{ijkl} \Gamma^{ijkl} \epsilon = 0$$  \hspace{1cm} (38)

the $\Gamma_0 \Gamma_{(4)} \sim \Gamma_{(6)}$ generator is absent from the associated connection $\tilde{D}$, so that the holonomy $\mathcal{H}(\tilde{D})$ must be in the group $\text{Spin}(10, \mathbb{C})$ generated by $\{\Gamma_{(2)}, \Gamma_0 \Gamma_{(2)}\}$.

In addition one needs to consider the condition $\tilde{D}_0 \epsilon = 0$. For non-trivial warpings with $\partial_i \Delta \neq 0$, the holonomy $\mathcal{H}(\tilde{D}_M)$ will in general be strictly larger than $\mathcal{H}(\tilde{D}_i)$, but in the case of trivial warping in which $\Delta$ is constant, the structure groups corresponding to $\tilde{D}_M$ and $\tilde{D}_i$ are the same.

If the Killing spinor is time-independent, $D_0 \epsilon = 0$, then $\tilde{D}_0 \epsilon = 0$ becomes the algebraic equation.

$$F_{ijkl} \Gamma^{ijkl} \epsilon = -8 F_{ijkl} \Gamma^0 \Gamma^{ijkl} \epsilon$$  \hspace{1cm} (39)

If $E = 0$, this implies (37) while if $B = 0$ it implies (38). Thus the conditions (37), (38) naturally arise from requiring $D_0 \epsilon = 0$. If both $E, B$ are non-zero, the holonomy is generic in general, but if the 10-space has a product structure, then the holonomy is further restricted and the analysis is similar to that in section 4.2.

5 Other $D = 11$ Supergravities

The classical $D = 11$ supergravity field equations are invariant under the scaling transformations

$$g_{MN} \rightarrow \lambda^2 g_{MN}, \quad \Psi_M \rightarrow \lambda^{3/2} \Psi_M, \quad A_{MNP} \rightarrow \lambda^3 A_{MNP}$$  \hspace{1cm} (40)

In [34], Howe found a generalisation of the usual $D = 11$ supergravity theory in which this symmetry is made local by coupling to a conformal connection $k_M$, which is a gauge field transforming under the scaling transformations as $\delta k = d\lambda$. This requires that the conformal connection be flat, $dk = 0$, so that locally it is pure gauge and introduces no new degrees of freedom. However, this generalisation allows new solutions in which $k$ has non-trivial holonomy.

A circle compactification with conformal holonomy around the circle (i.e. with a Wilson line for $k$) gives a new massive $D = 10$ supergravity [35] which has de Sitter solutions [36]. In particular, the condition for supersymmetry of a bosonic background becomes

$$(\tilde{D}_M + k_M) \epsilon = 0, \hspace{1cm} (41)$$

which can be analysed in terms of the holonomy of the connection $\tilde{D}_M + k_M$. Whereas $\tilde{D}$ takes values in $\text{SL}(32, \mathbb{R})$, adding the conformal connection means that $\tilde{D} + k$ takes values in $\text{GL}(32, \mathbb{R})$, and the holonomy is a subgroup of $\text{GL}(32, \mathbb{R})$. Thus including the conformal connection allows
more general configurations. For a configuration preserving $n$ supersymmetries, the holonomy of $\tilde{D} + k$ must be in $\text{GL}(32 - n, \mathbb{R}) \ltimes \mathbb{R}^{n(32-n)}$. All values of $n$ are possible in principle, as a non-trivial holonomy of $GL(1, \mathbb{R}) \ltimes \mathbb{R}^{31}$ allows $n = 31$ supersymmetries, as would a holonomy in the subgroup $GL(1, \mathbb{R})$.

In [36], it was suggested that this modified $D = 11$ supergravity might arise as a limit of a modified M-theory, referred to as MM-theory. However, the classical scaling symmetry of the usual $D = 11$ supergravity is not a symmetry of the quantum theory. For example, the supergravity field equations receive higher derivative corrections in the quantum theory, and these break the scaling symmetry as each higher derivative term will scale according to the number of derivatives. Then it would be inconsistent to gauge the scaling symmetry in the quantum theory, so that it would seem that M-theory could not be a part of an MM-theory. (However, such a structure could be of interest if M-theory had a scale invariant phase.)

In [37], it was shown that in addition to the classical supergravity in $10+1$ dimensions, there are supergravities in $9+2$ or $6+5$ dimensions, and these signatures with 1,2 or 5 times are the only possibilities that can arise in eleven dimensions (together with the mirror theories in $1+10$, $2+9$ or $5+6$ dimensions). Chains of dualities involving solutions with periodic time [37] lead to phases of M-theory in $9+2$ or $6+5$ dimensions, and the supergravity theories arise as limits of these. The arguments leading to these exotic phases are formal and assume that the quantum theory is consistent in configurations with periodic time. The supergravities are similar in structure to the usual one, and supersymmetric solutions have been found in [38], [39]. The conditions for Killing spinors can be analysed in terms of the holonomy of a supercovariant derivative of the same form as $\tilde{D}$. Similar groups to those in table 1 arise as possible holonomies (they are different real forms of the same complex groups), but the general holonomy is again $\text{SL}(32, \mathbb{R})$. The subgroup generated by $\Gamma_{(1)}, \Gamma_{(2)}$ is $\text{Spin}(10, 2)$ or $\text{Spin}(6, 6)$ in the two cases. It is interesting that a formulation of M-theory with local $\text{SL}(32, \mathbb{R})$ symmetry could be a natural framework to incorporate the conjectured phases with signatures $9+2$ and $6+5$, together with the theory in $10+1$.

6 M-Theory

In $d = 11$ supergravity, the fermion fields are sections of the spin bundle, with transition functions in $\text{Spin}(10, 1)$, and the number of supersymmetries preserved by a background depends on the number of solutions to the Killing spinor condition. In M-theory, there are vacua that do not correspond to supergravity solutions. More surprisingly, there are supergravity solutions that are also solutions of M-theory, and which are known to be supersymmetric vacua of M-theory but for which the supergravity solution has no Killing spinors, or fewer Killing spinors than the
number of expected supersymmetries [12][13][14]. This is the phenomenon of ‘supersymmetry without supersymmetry’, and brane wrapping modes or non-perturbative string states play a crucial role in realising the supersymmetry in such cases.

An example illustrating this is obtained as follows [13]. Consider the $AdS_5 \times S^5$ solution of the type IIB string theory, which is maximally supersymmetric with 32 Killing spinors and has an RR 5-form flux. The 5-sphere admits a Hopf fibration as an $S^1$ bundle over $\mathbb{CP}^2$. The isometry along the $S^1$ can be used to perform a T-duality taking this to a solution of the IIA string theory in which the bundle is untwisted to give the product space $\mathbb{CP}^2 \times S^1$ with the Fubini-Study metric on $\mathbb{CP}^2$, and with a NS-NS 2-form field now turned on. The number of supersymmetries is expected to be preserved by the T-duality, but surprisingly the IIA solution not only does not have any Killing spinors, it does not have any spinors at all, as $\mathbb{CP}^2$ does not admit a spin structure. The IIA theory on $AdS_5 \times \mathbb{CP}^2 \times S^1$ is supposed to give a dual description of the IIB string theory in the $AdS_5 \times S^5$ vacuum, so the question arises as to what happened to the IIB fermions. The resolution is that all the spinors on $S^5$ have non-trivial dependence on the $S^1$ direction and so can be thought of as carrying momentum in that direction, so that in the T-dual picture the spinors of the original theory all now arise in the winding sector, and there are no fermions at all in the zero-winding sector.

This can be understood through the dimensional reduction to $d = 9$ [13]. Reducing the $AdS_5 \times S^5$ solution of the type IIB supergravity along the Hopf fibre direction gives a solution of $d = 9$ supergravity on $AdS_5 \times \mathbb{CP}^2$ with a 2-form flux $F$ proportional to the Kahler 2-form on $\mathbb{CP}^2$ and a 4-form flux proportional to the volume form on $\mathbb{CP}^2$. Here $F = dA$ and $A$ is the Kaluza-Klein vector field coming from the reduction of the metric. The dimensional reduction of the $d = 10$ fermion fields gives $d = 9$ fermions which are all charged with respect to the $U(1)$ and so couple to $A$. However, the fermions are not $d = 9$ spinors, i.e. they are not sections of a Spin$(8, 1)$ bundle over $AdS_5 \times \mathbb{CP}^2$, because $\mathbb{CP}^2$ does not admit a spin structure. However, $\mathbb{CP}^2$ does admit a spin$^c$ structure or generalised spin structure which allows charged spinors, arising as sections of a Spin$(8, 1) \times U(1)$ bundle. Whereas there can be no spinors coupling just to the spin connection $\omega$, there can be spinors coupling to the combined connection $\omega + A$, provided that the $U(1)$ charge is half-integral. The fact that the charge is half-integral instead of the integral charge usually required gives an extra minus sign that cancels the sign inconsistencies that arise in attempting to define spinors on the manifold. The supersymmetry parameters $\epsilon$ are also charged spinors that are sections of the Spin$(8, 1) \times U(1)$ bundle and the Killing spinor condition involves the combined connection $\omega + A$. The IIB Killing spinors in $d = 10$ give rise to charged Killing spinors in $d = 9$.

All spinor fields on $AdS_5 \times S^5$ give rise to charged spinor fields on $AdS_5 \times \mathbb{CP}^2$ coupling to $\omega + A$, and there can be no uncharged spinors. Lifting the $d = 9$ solution up to the $AdS_5 \times \mathbb{CP}^2 \times S^1$
solution of the IIA theory, the vector field $A$ lifts to the NS-NS 2-form $B_2$ and so all the spinor fields of the IIB theory have become winding modes of the fundamental IIA string on the $S^1$ coupling to $B_2$, and again there are no such fields that do not wind. There are no gravitini or spin-half fermions arising as fields on $AdS_5 \times \mathbb{CP}^2 \times S^4$ (as there can be no spinor fields) but the fermionic states arise in the winding sector. Just as the charge in $d = 9$ was half-integral, the charges governing the coupling to $B_2$ are half-integral, so that one might say that the fermion states have half-integral ‘winding number’ or string charge, or that they are fractional strings. In particular, the supersymmetry parameters $\epsilon$ are not spinor fields but have half-integral winding number. As the string charges are all half-integral, there is no zero-winding sector.

Next, the $AdS_5 \times \mathbb{CP}^2 \times S^4$ solution of the IIA theory can be lifted to $AdS_5 \times \mathbb{CP}^2 \times T^2$ solution of M theory [13]. The string winding modes have become modes coupling to $A_3$ and so might be thought of as membrane wrapping modes, ‘wrapping’ the $T^2$, although the ‘wrapping number’ or membrane charge is half-integral, so that they are fractional membranes.

The $AdS_5 \times S^5$ solution has isometry group $SO(4,2) \times SO(6)$, and the $SO(6)$ gives rise to an $SO(6)$ Yang-Mills symmetry on reducing to five dimensions. The $d = 9$ solution $AdS_5 \times \mathbb{CP}^2$ has an internal space with isometry of only $SU(3) \times U(1)$, so that this will be the Kaluza-Klein gauge symmetry on reducing from nine to five dimensions. The remaining gauge fields of $SO(6)$ arise from charged fields in $d = 9$, which lift to winding modes in the IIA theory or membrane wrapping modes in 11-dimensions [13]. On the other hand, the 11-dimensional solution $AdS_5 \times \mathbb{CP}^2 \times T^2$ has internal space isometry $SU(3) \times U(1)^3$, so that this is the Kaluza-Klein gauge symmetry on reduction to five dimensions. The extra $U(1)^2$ gauge fields arise from massive modes on $AdS_5 \times S^5$ in the IIB picture [13].

Dualities can give rise to ‘wrapping modes’ coupling to other form fields. As another example, consider the $AdS_4 \times S^7$ solution of M-theory. The 7-sphere is a Hopf fibration of $S^1$ over $\mathbb{CP}^3$, so reducing on the $S^1$ fibre will give a IIA solution $AdS_4 \times \mathbb{CP}^3$ [12]. The fermions give $d=10$ fields that couple to the Kaluza-Klein vector field, which is the RR gauge field of the IIA theory, so that they carry RR charge. Now a series of $p$ T-dualities give rise to fermion fields coupling to the RR $p + 1$ form gauge field $C_{p+1}$ and so could be said to carry D-brane charge or to be D-brane wrapping modes.

In standard supergravity theories, the fermionic fields are spinors. A conventional viewpoint would be to say that backgrounds without spin structure are forbidden as configurations of the theory. Another view would be to allow non-spin solutions, finding that there are no fermions in the spectrum of fluctuations about the background. Alternatively, the supergravity theory can be modified for backgrounds with a spin$^c$ structure to allow fermions that are charged spinors. The key point here is that we have learnt that in M-theory the supergravity limit is indeed modified in precisely this way and that the fermion fields are charged spinors in general,
so that backgrounds that are not spin but which have a spin\(^c\) structure are indeed allowed.

Let us return to the \(\text{AdS}_5 \times \mathbb{CP}^2\) solution of \(d = 9\) supergravity. There M-theory requires that the supergravity is modified so that the fermions are not spinors but are charged fields arising as sections of a bundle with \(\text{Spin}(8, 1) \times U(1)\) transition functions. However, \(\text{Spin}(8, 1) \times U(1)\) is precisely the structure group \(\mathcal{G} = \text{Spin}(8, 1) \times G_{\text{spacelike}}(2)\) appropriate for a 9/2 split. The local \(\text{Spin}(8, 1) \times U(1)\) symmetry is then crucial for the definition of the theory, and the physical fermionic fields arise as sections of bundles with this structure group. This is an important piece of evidence that enlarged structure groups should play an essential role in M-theory. Similar arguments lead to other structure groups such as those in table 1 arising with ‘charged spinors’ that are sections of \(\mathcal{G}\)-bundles, suggesting that the general picture should involve bundles over spacetime with \(\text{SL}(32, \mathbb{R})\) transition functions. All the examples that arise have transition functions in \(\mathcal{G} \subseteq \text{SL}(32, \mathbb{R})\) for various \(\mathcal{G}\) so all are particular \(\text{SL}(32, \mathbb{R})\)-bundles. Just as in coupling spinors to gravity one needs to use a formulation with local Lorentz symmetry, to encompass charged spinors that are sections of a \(\mathcal{G}\)-bundle requires a formulation with local \(\mathcal{G}\) symmetry, with different \(\mathcal{G}\) for different backgrounds. A formulation with local \(\text{SL}(32, \mathbb{R})\) symmetry would allow all possible \(\mathcal{G} \subseteq \text{SL}(32, \mathbb{R})\) bundles without having to specify a background, and seems to be the minimal requirement for a background-independent formulation. That would mean that the 32-component indices \(\alpha, \beta\) should be regarded not as \(\text{Spin}(10, 1)\) spinor indices but as \(\text{SL}(32, \mathbb{R})\) indices in the fundamental representation.

Some care is needed in this discussion, as there are a number of related but distinct symmetries that play a role here. The \(d = 9\) theory arising from the Hopf reduction can be viewed as a gauged supergravity, and many related constructions lead to gauged supergravities. In such theories, the fermions are typically spinors transforming under the gauge group coupling both to the spin connection and the gauge connection and M-theory allows fermions that are not spinors but are sections of a spin\(^c\)-bundle. A more detailed discussion of the relevant symmetries from the point of view of gauged supergravity is given in an appendix.

M-theory in a particular background then requires that the local Lorentz symmetry be extended to at least a local \(\mathcal{G}\) symmetry, where the group \(\mathcal{G}\) depends on the background. Extending to a local \(\text{SL}(32, \mathbb{R})\) symmetry removes dependence on the choice of background and includes all possible \(\mathcal{G} \subseteq \text{SL}(32, \mathbb{R})\) symmetries. A reformulation of \(d = 11\) supergravity with local \(\text{SL}(32, \mathbb{R})\) symmetry would be a useful first step towards such a formulation of M-theory.

The \(d = 11\) superalgebra can be written as \(\{Q_{\alpha}, Q_{\beta}\} = \Pi_{\alpha\beta}\) where the symmetric bispinor \(\Pi_{\alpha\beta} = P_M \Gamma^M_{\alpha\beta} + Z_{MN} \Gamma^{MN}_{\alpha\beta} + \ldots\) can be decomposed in terms of the 11-momentum \(P_M\), a membrane charge and other brane charges \cite{20}. If the indices \(\alpha\) are thought of as \(\text{SL}(32, \mathbb{R})\) indices, then there is no invariant way of making this decomposition, which requires choosing a \(\text{Spin}(10, 1)\) subgroup of \(\text{SL}(32, \mathbb{R})\). In other words, an \(\text{SL}(32, \mathbb{R})\) symmetry or enlarged symmetry
$G \subset SL(32, \mathbb{R})$ would mix the momenta with brane and other charges.

However, this cannot be the whole story. In addition to degrees of freedom that are spacetime fields or sections of $G$-bundles over spacetime, there are string winding modes and brane wrapping modes (sometimes with fractional charges or winding numbers) that play a crucial role and which should be taken into account when considering questions of supersymmetry and symmetry of any given vacuum. It is not known what the right formulation is for properly considering such modes in M-theory. One approach for string winding modes might be to replace space with a loop-space (as in string field theory), or with the space of maps from a $p$-dimensional space to spacetime for $p$-brane wrapping modes. Another possibility that has been considered is to extend spacetime with extra coordinates conjugate to brane charges as well as the usual ones that are conjugate to momenta. One could then have 528 coordinates $\Xi^{\alpha\beta}$ conjugate to the charges $\Pi_{\alpha\beta}$, with the usual spacetime arising as an 11-dimensional subspace or 'brane' in this large space. A duality transformation would then take one to a different 11-dimensional subspace, so that what was previously a brane charge becomes a momentum and vice versa. The dualities of \cite{37} could be incorporated in such a picture. In addition, one could introduce a fermionic coordinate $\Theta^a$, with $SL(32, \mathbb{R})$ acting naturally on the superspace with coordinates $\{\Xi^{\alpha\beta}, \Theta^a\}$. Many serious problems arise in attempting such formulations, but they perhaps deserves further exploration.

**Appendix: Charged Spinors and Gauged Supergravity**

It will be useful to compare the $d = 9$ theory appearing in the Hopf reduction of $AdS_5 \times S^5$ with the usual $d = 9$ supergravity. The $d = 9$ supergravity from standard toroidal reduction from 11 dimensions has three abelian vector gauge fields and so a local $U(1)^3$ symmetry, and three scalars taking values in $\mathbb{R}^+ \times SL(2, \mathbb{R})/U(1)$. The theory has a nonlinearly realised $\mathbb{R}^+ \times SL(2, \mathbb{R})$ global symmetry. This becomes linearly realised on introducing an extra scalar and an extra local $U(1)$ symmetry, which will be denoted $U(1)_F$. The theory then has $\mathbb{R}^+ \times SL(2, \mathbb{R})$ global and $U(1)^4$ local symmetry. The fermions are charged only with respect to the extra $U(1)_F$, and couple to a $U(1)_F$ composite connection $B_M$ constructed from the scalar fields; this is not an independent gauge field. On fixing the extra $U(1)_F$ symmetry by setting the extra scalar to zero, the $\mathbb{R}^+ \times SL(2, \mathbb{R})$ acts on the fermions through a compensating $U(1)_F$ transformation, needed to preserve the gauge condition.

The $d = 9$ theory that arises by Hopf reduction is a gauged version of this supergravity in which a $U(1)_G \subset \mathbb{R}^+ \times SL(2, \mathbb{R})$ global symmetry is promoted to a local symmetry, with the corresponding gauge field the Kaluza-Klein vector field $A_M$. (It is presumably one of the gauged supergravities discussed in \cite{33, 11, 15}.) In the formulation with local $U(1)_F$, the fermions do not
transform under $U(1)_G$, but now the $U(1)_F$ connection $B_M$ depends on the gauge connection $A_M$ as well as on the scalars, so that $B_M = A_M + (\text{scalar-dependent terms})$. On going to physical gauge by setting the extra scalar to zero, any local $U(1)_G$ transformation is accompanied by a compensating $U(1)_F$ transformation which does act on the fermions, so that the two $U(1)$’s effectively become identified (the gauge symmetry being now a diagonal subgroup of $U(1)_G \times U(1)_F$). Independently of whether the gauge is fixed or not, the fermions strictly speaking couple to $\omega_M + B_M$, not $\omega_M + A_M$, and these two differ by scalar dependent terms. The fermions are sections of the $\text{Spin}(8,1) \times U(1)_F$ bundle, and the Hopf fibration implies that this must be non-trivial.

A similar situation arises more generally. In dimensionally reduced supergravity theories in $d = 11 - n$ dimensions, there is a global $E_{n(+n)}$ symmetry with physical scalars taking values in $E_{n(+n)}/G(n)$. Introducing extra scalars taking values in $G(n)$, the complete set of scalars now take values in $E_{n(+n)}$ and there is a local $G(n)$ symmetry. The fermions are charged under $G(n)$ but are $E_{n(+n)}$ singlets. When the $G(n)$ symmetry is fixed by eliminating the extra scalars, the $E_{n(+n)}$ symmetry acts on the fermions through a compensating $G(n)$ transformation. In a gauged version of the theory, a subgroup $K \subseteq E_{n(+n)}$ is promoted to a local symmetry, with vector fields from the supergravity theory becoming the $K$ gauge fields.\footnote{In the $d = 9$ example, $E_{2(+2)} = \mathbb{R}^+ \times \text{SL}(2,\mathbb{R})$, $G(2) = U(1)_F$ and $K = U(1)_G$, while for gauged supergravities in $d = 4$ dimensions with $n = 7$, $G_{\text{spacelike}}(7) = SU(8)$ and the gauge group could be $K = SO(8)$ as in \cite{16} or $K = SO(p,8-p)$ as in \cite{17,18}. For a timelike reduction to $d = 4$, $G_{\text{timelike}}(7) = SU^*(8)$, and the ‘natural gauging’ analogous to the $SO(8)$ gauging of \cite{10} is one with gauge group $SO^*(8) = SO(6,2)$ arising from a consistent truncation of the dimensional reduction of the $AdS_7 \times S^4$ solution on $AdS_7$. Other gaugings are also possible.} The fermions couple to a composite $G(n)$ connection $B_M$ which depends on the $K$ gauge connection $A_M$ as well as on the scalars, and so are sections (for a spacelike reduction) of a $\mathcal{G} = \text{Spin}(d-1,1) \times G_{\text{spacelike}}(n)$ bundle. However, typically the transition functions can be taken to be in $K_c \subseteq G(n)$ where $K_c$ is the maximal compact subgroup of $K$, so that the bundle has a $K_c$ structure.\footnote{For a timelike reduction, $K_c$ is again a maximal subgroup of $K$, but is now typically non-compact; it is the maximal subgroup of $K$ that is also a subgroup of $G_{\text{timelike}}(n)$.} The fermions are charged and are sections of a $\mathcal{G} = \text{Spin}(d-1,1) \times G_{\text{spacelike}}(n)$ bundle, and a local $\mathcal{G}$ symmetry (or at least a local $\text{Spin}(d-1,1) \times K_c$ symmetry) is needed to formulate the theory.

Then in a gauged supergravity with a gauge group $K$, fermions are typically charged under the gauge group, which acts on the fermions through $G(n)$ transformations. In general, a spin$^c$ structure is to be expected, with the fermions arising not as spinors but as sections of a bundle with transition functions in $\mathcal{G}$, which is $\text{Spin}(d-1,1) \times G_{\text{spacelike}}(n)$ for spacelike reductions. Non-trivial spin$^c$ structures will often arise from reduction of situations with supersymmetry without supersymmetry, as in the example above. Strictly speaking the transition functions are in $\text{Spin}(d-1,1) \times K$, with $K$ acting through compensating $G(n)$ transformations and $K_c \subseteq G(n)$
acting linearly. However, it is useful to think of these as Spin\((d - 1, 1) \times G_{\text{spacelike}}(n)\) bundles, and this allows the gaugings with different gauge groups \(K\) to be treated on the same footing, although in each case the bundle can be viewed as a Spin\((d - 1, 1) \times K_c\) bundle. In this way, fermions can arise as sections of \(G\)-bundles for various structure groups in table 1, and the theory is naturally formulated with local \(G\) symmetry. If M-theory is to have a background-independent formulation that is independent of \(d\) and the choice of \(K\), then the local symmetry must be one that includes the various groups \(G\) that can arise in this way. This would require a formulation with local \(\text{SL}(32, \mathbb{R})\) symmetry, with fermions arising as sections of \(\text{SL}(32, \mathbb{R})\) bundles. In particular cases, the bundle often reduces to one with transition functions in a subgroup \(G\), such as the group Spin\((d - 1, 1) \times K_c \subset \text{SL}(32, \mathbb{R})\) for the gauged supergravities.

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