REMARKS CONNECTED WITH THE WEAK LIMIT
OF ITERATES OF SOME RANDOM-VALUED FUNCTIONS
AND ITERATIVE FUNCTIONAL EQUATIONS

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Dedicated to Professor Zygfryd Kominek on his 75th birthday

Abstract. The paper consists of two parts. At first, assuming that \((\Omega, \mathcal{A}, P)\)
is a probability space and \((X, \varrho)\) is a complete and separable metric space with
the \(\sigma\)-algebra \(\mathcal{B}\) of all its Borel subsets we consider the set \(\mathcal{R}_c\) of all \(\mathcal{B} \otimes \mathcal{A}\)-measurable and contractive in mean functions \(f: X \times \Omega \to X\) with finite
integral \(\int_\Omega \varrho(f(x, \omega), x) \, P(d\omega)\) for \(x \in X\), the weak limit \(\pi^f\) of the sequence
of iterates of \(f \in \mathcal{R}_c\), and investigate continuity-like property of the function
\(f \mapsto \pi^f, \, f \in \mathcal{R}_c\), and Lipschitz solutions \(\varphi\) that take values in a separable
Banach space of the equation

\[
\varphi(x) = \int_\Omega \varphi(f(x, \omega)) \, P(d\omega) + F(x).
\]

Next, assuming that \(X\) is a real separable Hilbert space, \(\Lambda: X \to X\) is linear
and continuous with \(\|\Lambda\| < 1\), and \(\mu\) is a probability Borel measure on \(X\) with
finite first moment we examine continuous at zero solutions \(\varphi: X \to \mathbb{C}\) of the
equation

\[
\varphi(x) = \hat{\mu}(x)\varphi(\Lambda x)
\]

which characterizes the limit distribution \(\pi^f\) for some special \(f \in \mathcal{R}_c\).

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CC BY (http://creativecommons.org/licenses/by/4.0/).
Fix a probability space $(\Omega, \mathcal{A}, P)$ and a complete and separable metric space $(X, \varrho)$. Let $\mathcal{B}$ denote the $\sigma$-algebra of all Borel subsets of $X$.

We say that $f : X \times \Omega \to X$ is a random-valued function (an rv-function for short) if it is measurable with respect to the product $\sigma$-algebra $\mathcal{B} \otimes \mathcal{A}$. The iterates of such an rv-function are given by

$$f^0(x, \omega_1, \omega_2, \ldots) = x, \quad f^n(x, \omega_1, \omega_2, \ldots) = f(f^{n-1}(x, \omega_1, \omega_2, \ldots), \omega_n)$$

for $x$ from $X$ and $(\omega_1, \omega_2, \ldots)$ from $\Omega^\infty$ defined as $\Omega^\mathbb{N}$. Note that $f^n : X \times \Omega^\infty \to X$ is an rv-function on the product probability space $(\Omega^\infty, \mathcal{A}^\infty, P^\infty)$. More exactly, the $n$-th iterate $f^n$ is $\mathcal{B} \otimes \mathcal{A}_n$-measurable, where $\mathcal{A}_n$ denotes the $\sigma$-algebra of all sets of the form

$$\{(\omega_1, \omega_2, \ldots) \in \Omega^\infty : (\omega_1, \ldots, \omega_n) \in A\}$$

with $A$ from the product $\sigma$-algebra $\mathcal{A}^n$. (See [7, Sec. 1.4], [3].)

A simple criterion for the convergence in law of $(f^n(x, \cdot))_{n \in \mathbb{N}}$ to a random variable independent of $x \in X$ was proved in [1] and applied to the equation

$$\varphi(x) = \int_{\Omega} \varphi(f(x, \omega)) P(d\omega) + F(x)$$

with $\varphi$ as the unknown function. This criterion reads.

(H) There exists a $\lambda \in (0, 1)$ such that

$$\int_{\Omega} \varrho(f(x, \omega), f(z, \omega)) P(d\omega) \leq \lambda \varrho(x, z) \quad \text{for } x, z \in X$$

and

$$\int_{\Omega} \varrho(f(x, \omega), x) P(d\omega) < \infty \quad \text{for } x \in X.$$

Thus, denoting by $\pi^f_n(x, \cdot)$ the distribution of $f^n(x, \cdot)$, i.e.,

$$\pi^f_n(x, B) = P^\infty \left(f^n(x, \cdot) \in B\right) \quad \text{for } n \in \mathbb{N} \cup \{0\}, \ x \in X \text{ and } B \in \mathcal{B},$$

hypothesis (H) guarantees the existence of a probability Borel measure $\pi^f$ on $X$ such that

$$\lim_{n \to \infty} \int_X u(z) \pi^f_n(x, dz) = \int_X u(z) \pi^f(dz)$$
for $x \in X$ and for any continuous and bounded $u: X \to \mathbb{R}$; moreover, as observed in [3] (see also [6]),

$$\int_X \varrho(x,z)\pi^f(dz) < \infty \quad \text{for } x \in X.$$  

In [2] we considered continuity-like property of the function $f \mapsto \pi^f$. In [4] we characterized the limit distribution $\pi^f$ via a functional equation for its characteristic function for some special rv-functions in Hilbert spaces. In the present paper we are strengthening the result of [2], apply it to equation (1) and consider also the equation used for the above mentioned characterization of the limit distribution.

1. Assuming that $(\Omega, \mathcal{A}, P)$ is a probability space and $(X, \varrho)$ is a complete and separable metric space, consider the set $\mathcal{R}_c$ of all rv-functions $f: X \times \Omega \to X$ such that

$$\int_{\Omega} \varrho(f(x,\omega), f(z,\omega))P(d\omega) \leq \lambda_f \varrho(x,z) \quad \text{for } x, z \in X$$

with a $\lambda_f \in [0,1)$ and (2) holds. Put also

$$d(f,g) = \sup \left\{ \int_{\Omega} \varrho(f(x,\omega), g(x,\omega))P(d\omega) : x \in X \right\} \quad \text{for } f, g \in \mathcal{R}_c.$$  

The theorem in [2] says that if $f, g \in \mathcal{R}_c$, then

$$\left| \int_X u d\pi^f - \int_X u d\pi^g \right| \leq \frac{1}{1 - \min\{\lambda_f, \lambda_g\}} d(f,g)$$

for every non-expansive $u: X \to [-1,1]$. In fact the above inequality was proved there for every non-expansive and bounded $u: X \to \mathbb{R}$. But if $f \in \mathcal{R}_c$, then (3) holds and so every Lipschitz function mapping $X$ into a separable Banach space is Bochner integrable with respect to $\pi^f$. Therefore we can ask whether (4) holds also for such a function. The theorem reads as follows.

**Theorem 1.** If $f, g \in \mathcal{R}_c$, then

$$\left\| \int_X u d\pi^f - \int_X u d\pi^g \right\| \leq \frac{1}{1 - \min\{\lambda_f, \lambda_g\}} d(f,g)$$

for every non-expansive $u$ mapping $X$ into a separable Banach space.
Proof. Let $u$ be a non-expansive mapping of $X$ into a separable Banach space $Y$. To show that (5) holds we may assume that $Y$ is a real space.

Fix $y^* \in Y^*$ such that $\|y^*\| \leq 1$ and

\[
\left\| \int_X u \, d\pi^f - \int_X u \, d\pi^g \right\| = y^* \left( \int_X u \, d\pi^f - \int_X u \, d\pi^g \right).
\]

For every $k \in \mathbb{N}$ the function $\tau_k: \mathbb{R} \to \mathbb{R}$ given by $\tau_k(t) = -k$ for $t \in (-\infty, -k)$, $\tau_k(t) = t$ for $t \in [-k, k]$, $\tau_k(t) = k$ for $t \in (k, \infty)$ is non-expansive and $|\tau_k(t)| \leq |t|$ for $t \in \mathbb{R}$. Consequently, since (4) holds for every non-expansive and bounded $u: X \to \mathbb{R}$, for every $k \in \mathbb{N}$ we have

\[
\left| \int_X \tau_k \circ y^* \circ u \, d\pi^f - \int_X \tau_k \circ y^* \circ u \, d\pi^g \right| \leq \frac{1}{1 - \min \{\lambda_f, \lambda_g\}} d(f, g)
\]

and

\[
|\tau_k(y^* u(z))| \leq \|u(z)\| \quad \text{for } z \in X \text{ and } k \in \mathbb{N}.
\]

Hence, applying the Lebesgue dominated convergence theorem and passing with $k$ to the limit in (7) we get

\[
\left| \int_X y^* \circ u \, d\pi^f - \int_X y^* \circ u \, d\pi^g \right| \leq \frac{1}{1 - \min \{\lambda_f, \lambda_g\}} d(f, g)
\]

and (5) follows now from (6).

The following example shows that both sides of (5) can be equal and non-zero.

Example 1. If $\Omega = \{0, 1\}$, $P(\{\omega\}) = 1/2$ for $\omega \in \{0, 1\}$ and $f_\alpha(x, \omega) = (x + \alpha \omega)/2$ for $x \in \mathbb{R}$, $\omega \in \{0, 1\}$ and $\alpha \in (0, \infty)$, then $f_\alpha \in \mathcal{R}_c$ with $\lambda_{f_\alpha} = \frac{1}{2}$ and (see [4, Example 1])

\[
\pi^{f_\alpha}(B) = \frac{1}{\alpha} \lambda_1(B \cap [0, \alpha]) \quad \text{for Borel } B \subset \mathbb{R} \text{ and } \alpha \in (0, \infty),
\]

where $\lambda_1$ denotes the one-dimensional Lebesgue measure. Hence

\[
\int_\mathbb{R} u \, d\pi^{f_\alpha} = \frac{1}{\alpha} \int_0^\alpha u(x) \, dx
\]
for every $\alpha \in (0, \infty)$ and Lipschitz $u : \mathbb{R} \to \mathbb{R}$. In particular,
\[
\left| \int_{\mathbb{R}} x^{\pi f_\alpha}(dx) - \int_{\mathbb{R}} x^{\pi f_\beta}(dx) \right| = \frac{1}{2} |\alpha - \beta| = \frac{1}{2} \min\{\lambda f_\alpha, \lambda f_\beta\} d(f_\alpha, f_\beta)
\]
for $\alpha, \beta \in (0, \infty)$.

Denoting by $\|F\|_L$ the smallest Lipschitz constant for a Lipschitz function $F$ we have the following corollary concerning Lipschitz solutions $\varphi$ of (1).

**Corollary 1.** Assume $F$ is a Lipschitz mapping of $X$ into a separable Banach space $Y$. If $f, g \in \mathcal{R}_c$ and
\[
\left(8\right) \frac{\|F\|_L}{1 - \min \{\lambda f, \lambda g\}} d(f, g) < \left\| \int_X F d\pi^g \right\|,
\]
then equation (1) has no Lipschitz solution $\varphi : X \to Y$.

**Proof.** It follows from Theorem 1 and (8) that
\[
\left\| \int_X F d\pi^f - \int_X F d\pi^g \right\| \leq \frac{\|F\|_L}{1 - \min \{\lambda f, \lambda g\}} d(f, g) < \left\| \int_X F d\pi^g \right\|,
\]
whence $\int_X F d\pi^f \neq 0$ and according to [4, Theorem 2.1] equation (1) has no Lipschitz solution $\varphi : X \to Y$. □

The following example shows that under the assumptions of Corollary 1 equation (1) may have a continuous solution. (Cf. [1, Example 4.2].)

**Example 2.** Assume $p_1, p_2$ are positive reals, $p_2 < \frac{1}{9}$ and $p_1 + p_2 = 1$, reals $L_1, L_2$ satisfy
\[
p_1 L_1^2 < 1, \quad p_1 |L_1| + 3 \sqrt{p_2(1 - p_1 L_1^2)} < 1, \quad |L_2| = \sqrt{(1 - p_1 L_1^2)/p_2},
\]
and $a \in \mathbb{R} \setminus \{0\}$. Define $F : \mathbb{R} \to \mathbb{R}$ by
\[
F(x) = -a p_2(2L_2x + a).
\]
Putting $\Omega = \{1, 2\}$ and $P(\{k\}) = p_k$ for $k \in \{1, 2\}$ consider the rv-functions $f, g : \mathbb{R} \times \Omega \to \mathbb{R}$ given by
\[
f(x, 1) = L_1x, \quad f(x, 2) = L_2x + a, \quad g(x, k) = L_kx \quad \text{for } k \in \{1, 2\}.
Then \( f, g \in \mathcal{R}_c \) with
\[
\lambda_f = \lambda_g = p_1 |L_1| + p_2 |L_2| = p_1 |L_1| + \sqrt{p_2(1 - p_1 L_1^2)} < 1,
\]
and equation (1) takes the form
\[
\varphi(x) = p_1 \varphi(L_1 x) + p_2 \varphi(L_2 x + a) + F(x).
\]
Since \( p_1 L_1^2 + p_2 L_2^2 = 1 \), the function \( x \mapsto x^2, x \in \mathbb{R} \), solves it. Moreover, \( g(0, \omega) = 0 \) for \( \omega \in \Omega \), whence also \( g^n(0, \omega) = 0 \) for \( \omega \in \Omega^\infty \) and \( n \in \mathbb{N} \). Consequently, \( \pi^g \) is the Dirac measure \( \delta_0 \) and
\[
\int_{\mathbb{R}} F d\pi^g = F(0) = -p_2 a^2.
\]
Finally, \( \|F\|_L = 2p_2 |a L_2| \), \( d(f, g) = p_2 |a| \), and so
\[
\frac{\|F\|_L}{1 - \min\{\lambda_f, \lambda_g\}} d(f, g) = \frac{2p_2 a^2 |L_2|}{1 - (p_1 |L_1| + \sqrt{p_2(1 - p_1 L_1^2)})} < p_2 a^2 = \left| \int_{\mathbb{R}} F d\pi^g \right|.
\]

Consider also a special case of (1), viz.
\[
\varphi(x) = \frac{1}{n} \sum_{k=0}^{n-1} \varphi\left(\frac{1}{n} x + a_k\right) + F(x).
\]

**Corollary 2.** Assume \( n \geq 2 \) is an integer and \( F \) is a Lipschitz mapping of \( \mathbb{R} \) into a separable Banach space \( Y \) with \( \int_0^1 F(x) dx \neq 0 \). If reals \( a_0, \ldots, a_{n-1} \) satisfy
\[
\|F\|_L \sum_{k=0}^{n-1} \left| \frac{k}{n} - a_k \right| < (n - 1) \left\| \int_0^1 F(x) dx \right\|,
\]
then equation (9) has no Lipschitz solution \( \varphi: \mathbb{R} \to Y \).

**Proof.** Put \( \Omega = \{0, 1, \ldots, n - 1\} \), \( P(\{k\}) = \frac{1}{n} \) for \( k \in \Omega \) and define \( f, g: \mathbb{R} \times \Omega \to \mathbb{R} \) by
\[
f(x, k) = \frac{1}{n} x + a_k, \quad g(x, k) = \frac{1}{n} x + \frac{k}{n}.
\]
Clearly $f, g \in \mathcal{R}_c$ with $\lambda_f = \lambda_g = \frac{1}{n}$ and (see [4, Example 1]) $\pi^a(B) = \lambda_1(B \cap [0, 1])$ for Borel $B \subset \mathbb{R}$. Moreover,

$$d(f, g) = \frac{1}{n} \sum_{k=0}^{n-1} \left| \frac{k}{n} - a_k \right|.$$ 

By Corollary 1 equation (9) has no Lipschitz solution $\varphi: \mathbb{R} \to Y$. □

2. Assuming now that $X$ is a real separable Hilbert space, $X \neq \{0\}$, $\Lambda: X \to X$ is linear and continuous with $\|\Lambda\| < 1$, and $\mu$ is a probability Borel measure on $X$, consider the equation

$$\varphi(x) = \hat{\mu}(x)\varphi(\Lambda x),$$

where $\hat{\mu}$ denotes the Fourier transform of $\mu$,

$$\hat{\mu}(x) = \int_X e^{i(x|z)} \mu(dz) \text{ for } x \in X.$$ 

**Theorem 2.** If $\mu$ has a finite first moment, then there exists a probability Borel measure $\nu$ on $X$ with a finite first moment such that $\hat{\nu}$ solves (10), and for any continuous at zero solution $\varphi: X \to \mathbb{C}$ of (10) we have

$$\varphi = \varphi(0)\hat{\nu};$$

in particular, every continuous at zero solution $\varphi: X \to \mathbb{C}$ of (10) is of class $C^1$ and Lipschitz.

Remind that a probability Borel measure $\nu$ on $X$ has a finite first moment provided the integral $\int_X \|x\| \nu(dx)$ is finite. We shall prove Theorem 2 later on, together with the next one and with the following remark.

**Remark.** If $\mu$ has a finite first moment and $\Lambda$ is injective, then for every $c \in \mathbb{C}$ the set of all discontinuous at zero solutions $\varphi: X \to \mathbb{C}$ of (10) such that $\varphi(0) = c$ and $\varphi|_{X\{0\}}$ is of class $C^1$ and Lipschitz has the cardinality of the continuum.

Theorem 2 implies that for every Borel and integrable with respect to $\mu$ function $\xi: X \to X$ the equation

$$\varphi(x) = \varphi(\Lambda x) \int_X e^{i(x|\xi(z))} \mu(dz)$$

(11)
has exactly one continuous at zero solution \( \varphi^\xi : X \to \mathbb{C} \) such that \( \varphi^\xi(0) = 1 \), and it is of class \( C^1 \) and Lipschitz. Consequently, we have the operator \( \xi \mapsto \varphi^\xi, \xi \in L^1(\mu, X) \), and a kind of its continuity gives the following theorem.

**Theorem 3.** If \( \xi, \eta : X \to X \) are Borel and integrable with respect to \( \mu \), then

\[
|\varphi^\xi(x) - \varphi^\eta(x)| \leq \frac{\|x\|}{1 - \|\Lambda\|} \int_X \|\xi(z) - \eta(z)\| \mu(dz) \quad \text{for } x \in X.
\]

**Proofs.** Consider the probability space \((X, \mathcal{B}, \mu)\) and, given Borel \( \xi : X \to X \) integrable with respect to \( \mu \), the rv-function \( f \) on it defined by

\[
f(x, \omega) = \Lambda^* x + \xi(\omega) \quad \text{for } (x, \omega) \in X \times X,
\]
as well as the limit distribution \( \pi^f \). Put \( \pi^\xi = \pi^f \). According to [4, Theorem 3.1] \( \hat{\pi}^\xi \) solves (11). Since the first moment of \( \pi^\xi \) is finite, \( \hat{\pi}^\xi \) is of class \( C^1 \) and Lipschitz.

To prove Theorem 2 put \( \nu = \pi^\text{id}_X \) and let \( \varphi : X \to \mathbb{C} \) be a continuous at zero solution of (10). Then

\[
\varphi(x) = \varphi(\Lambda^n x) \prod_{k=0}^{n-1} \int_X e^{i(\Lambda^k x|z)} \mu(dz) \quad \text{for } n \in \mathbb{N}, x \in X,
\]
and \( \lim_{n \to \infty} \Lambda^n x = 0 \) for \( x \in X \). Since

\[
\left| \int_X e^{i(\Lambda^k x|z)} \mu(dz) \right| \leq \int_X \left| e^{i(\Lambda^k x|z)} \right| \mu(dz) = 1
\]
for \( k \in \mathbb{N} \cup \{0\} \) and \( x \in X \), it shows that if \( \varphi(0) = 0 \), then \( \varphi = 0 \), and if \( \varphi(0) \neq 0 \), then

\[
\varphi(x) = \varphi(0) \prod_{n=0}^{\infty} \int_X e^{i(\Lambda^n x|z)} \mu(dz) \quad \text{for } x \in X.
\]

Consequently, for every \( c \in \mathbb{C} \) equation (10) has at most one continuous at zero solution \( \varphi : X \to \mathbb{C} \) satisfying \( \varphi(0) = c \) and by the first part of the proof \( \hat{c} \nu \) is such a solution.

To get Theorem 3 it is enough to observe that since \( \varphi^\xi = \hat{\pi}^\xi, \varphi^\eta = \hat{\pi}^\eta \) and

\[
|e^{i(x|z_1)} - e^{i(x|z_2)}| \leq \|x\| \|z_1 - z_2\| \quad \text{for } x, z_1, z_2 \in X,
\]
by Theorem 1 for every $x \in X$ we have

$$\left| \varphi^\xi(x) - \varphi^\eta(x) \right| = \left| \int_X e^{i(x|z)\pi} \pi^\xi(dz) - \int_X e^{i(x|z)\pi} \pi^\eta(dz) \right|$$

$$\leq \frac{\|x\|}{1 - \|\Lambda\|} \int_X \|\xi(z) - \eta(z)\| \mu(dz).$$

To verify the Remark given $c \in \mathbb{C}$ for every $a \in \mathbb{C}\{c\}$ define $\varphi_a : X \to \mathbb{C}$ by

$$\varphi_a(x) = a\hat{v}(x) \quad \text{for } x \in X \setminus \{0\}, \quad \varphi_a(0) = c,$$

and note that it solves (10), it is discontinuous at zero and $\varphi_a|_{X\setminus\{0\}}$ is of class $C^1$ and Lipschitz. $\square$

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