FREE BRAIDED DIFFERENTIAL CALCULUS,
BRAIDED BINOMIAL THEOREM
AND THE BRAIDED EXPONENTIAL MAP

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ABSTRACT Braided differential operators $\partial^i$ are obtained by differentiating the addition law on the braided covector spaces introduced previously (such as the braided addition law on the quantum plane). These are affiliated to a Yang-Baxter matrix $R$. The quantum eigenfunctions $\exp_R(x|v)$ of the $\partial^i$ (braided-plane waves) are introduced in the free case where the position components $x_i$ are totally non-commuting. We prove a braided $R$-binomial theorem and a braided-Taylors theorem $\exp_R(a|\partial)f(x) = f(a + x)$. These various results precisely generalise to a generic $R$-matrix (and hence to $n$-dimensions) the well-known properties of the usual 1-dimensional $q$-differential and $q$-exponential. As a related application, we show that the $q$-Heisenberg algebra $px - qxp = 1$ is a braided semidirect product $\mathbb{C}[x] \rtimes \mathbb{C}[p]$ of the braided line acting on itself (a braided Weyl algebra). Similarly for its generalization to an arbitrary $R$-matrix.

1 Introduction

In recent times there has been a lot of interest in the programme of $q$-deforming physics. This programme is usually phrased in the language on non-commutative geometry, namely one works with non-commutative algebras such as $q$-deformed planes, $q$-Minkowski space etc and tries to proceed as if these algebras are like the commutative algebra of functions on planes, Minkowski space etc. One would like to make all basic constructions needed for physics in this $q$-deformed context. Here we develop further our new and fully systematic approach to this problem. Instead

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of working in a usual way but with non-commutative algebras, our new approach is to introduce $q$ directly into the notion of tensor product. In the simplest case $\otimes$ acquires a factor $q$ in a similar way to the $\pm 1$ factors inserted in super-symmetry, while more generally the modification of the $\otimes$ is given by an $R$-matrix (which could depend on one or more parameters). We will see that this approach recovers such things as the usual $q$-differential and $q$-exponential, and the $q$-Heisenberg algebra in the simplest case, but works just as well for any $R$-matrix.

Mathematically, our approach to $q$-deformations consists in shifting the category in which we work from the category of vector spaces to a braided tensor category. We have introduced groups and other structures in such braided categories in [1][2][3][4][5] and we build on this work here. This line of development is a little different from non-commutative geometry in that it is the tensor product that is being primarily deformed (other deformations follow from this). Rather it extends the philosophy of super-geometry to some kind of braided geometry.

A second physical motivation for braided geometry is the existence in low dimensions of particles with braid statistics, whose symmetries might be described in such a way. We will not develop this here, limiting ourselves to the motivation from q-deforming physics. In q-deformed physics $q$ is not a physical parameter but a parameter introduced to help regularise infinities[6], and we can set $q = 1$ after intelligent renormalization (using identities from $q$-analysis). Perhaps in Planck-scale physics we might also keep $q \neq 1$ as a model of quantum corrections to the geometry. In the first case then the usual spin-statistics theorem which fails in the $q$-deformed setting, does so as an artifact of the regularization process.

Since our results generalise these ideas to any generic $R$-matrix obeying the Quantum Yang-Baxter Equations (QYBE) they are not of course tied to this deformation point of view. There are plenty of non-standard solutions of the QYBE to which our results apply equally well. We hope in this context to provide some kind of differential calculus for the handling and computation of the knot invariants and 3-manifold invariants associated to the chosen $R$ matrix (cf Fox’s free calculus). This is a long-term pure-mathematical motivation for the paper.

The paper begins in the preliminary Section 2 by recalling our approach to braided-differential calculus announced in [6, Sec. 7.3]. We begin with an addition law on the quantum planes associated to an $R$-matrix, and define differentiation as an infinitesimal translation. There is more than one plane associated to an $R$-matrix. As well as ones associated to non-zero ‘eigenvalues’
of $R$ there is the free one in which the coordinates $x_i$ obey no commutation relations at all. The first main result is in Section 3 where we prove a braided-binomial theorem precisely generalising the familiar (and indispensable) $q$-binomial theorem. We then use this in Section 4 when we introduce and study an $R$-exponential map in the free case. The problem of an $R$-exponential map in the non-free case is left open.

In section 5 we turn to another application of the braided group technology. Because Hopf algebras in braided categories (like familiar quantum groups) are group-like objects one can make semidirect products\[8\]. In particular, the differential operators $\partial^i$ act on the $x_j$ and hence we can make a semidirect product. In the usual setting this is precisely the definition of the canonical commutation relations algebra (or Weyl algebra). Thus we have at once from the general theory in \[8\] a braided Weyl algebra. In the one-dimensional case we recover the $q$-Heisenberg algebra proposed for $q$-deformed quantum mechanics in \[9\]. See also \[10\][11]. In the higher-dimensional case we recover something like the algebra studied in \[12\] and elsewhere for the $SL_q(n)$ $R$-matrix (though with $x, p$ rather than $a, a^\dagger$). All of these are thereby generalised to the case of a general quantum plane based on a general $R$-matrix and its projectors. Most importantly, these algebras are obtained now in a systematic way by a standard semidirect product construction, requiring us only to remember the braid statistics.

We will not discuss it explicitly, but the category in which all our constructions take place can be understood as the category of (co)representations of a quantum group, at least for nice $R$. This means that all our constructions are quantum-group covariant under this background quantum group symmetry. Related to this, there is also a matrix braided group $B(R)$\[1\] acting on our braided covectors and braided vectors. See \[4\] for an elaboration of these points.

## 2 Derivation of Braided-Differential Calculus

We begin by recalling the notion of a braided group or Hopf algebra in a braided category as introduced previously in \[1\][3][5][6] and elsewhere. This is $(B, \Delta, S, \epsilon, \Psi)$ where $B$ is an algebra, $\Psi : B \otimes B \to B \otimes B$ is a braiding (and obeys the QYBE) and $\Delta : B \to B \otimes B$ the braided-coproduct, $\epsilon : B \to k$ the braided counit and $S : B \to B$. Everything we do in this paper works at an algebraic level over any field $k$ (one can take $k = \mathbb{R}, \mathbb{C}$). For full details of the axioms see the above works or \[7\]. They are just the same as the usual Hopf algebra axioms except that $\Delta$...
is an algebra homomorphism when $B \otimes B$ has the braided tensor product algebra structure

$$ (a \otimes c)(b \otimes d) = a\Psi(c \otimes b)d $$

(1)
as determined by $\Psi$. The braiding on the identity is always trivial. In more formal terms, the construction takes place in a general braided tensor category [13][14].

The example which will concern us here is the quantum plane and its higher-dimensional brethren. These have been extensively studied as algebras but without any linear addition law. This is because to formulate the latter one really needs the notion of a braided-Hopf algebra. In fact, there are two flavours, contravariant and covariant (covector and vector) associated to any general $R$ matrix. Let $R \in M_n \otimes M_n$ obey the QYBE. We suppose it is invertible. We suppose also that we are given an affiliated matrix $R'$ such that

$$ (PR+1)(PR'-1) = 0, \quad R_{12}R_{13}R'_{23} = R'_{23}R_{13}R_{12}, \quad R_{23}R_{13}R'_{12} = R'_{12}R_{13}R_{23}, \quad R_{21}R'_{12} = R'_{21}R_{12} $$

(2)

where $P$ is the usual permutation matrix. Here $PR' - 1$ corresponds to the projection operators in the usual approach to the algebra in [13][14] and elsewhere. One choice (which we call the free case) is $R' = P$. Other $R'$ are associated to each non-zero $\lambda_i$ in the characteristic equation $\Pi_i(PR - \lambda_i) = 0$ (normalise $R$ so that the chosen eigenvalue becomes $-1$ and construct $R'$ from the remaining products and $P$).

With this data the braided covectors $V^\ast(R')$ defined by generators $1, x = (x_i)$ and relations $x_2x_1R'_{12} = x_1x_2$ form a braided-Hopf algebra with

$$ \Delta x_i = x_i \otimes 1 + 1 \otimes x_i, \quad \Psi(x_1 \otimes x_2) = x_2 \otimes x_1R_{12} $$

(3)

extended multiplicatively with braid statistics. The counit is $\epsilon(x_i) = 0$. The antipode is $Sx_i = -x_i$ extended as a braided-antialgebra map. Likewise there are braided vectors $V(R')$ defined by $R'_{12}v_2v_1 = v_1v_2$ form a braided-Hopf algebra with

$$ \Delta v^i = v^i \otimes 1 + 1 \otimes v^i, \quad \Psi(v_1 \otimes v_2) = R_{12}v_2 \otimes v_1. $$

(4)

We refer to [3] for full details. Throughout the paper we use the standard compact notation in which numerical suffices on vectors, covectors and matrices refer to the position in a matrix tensor product. In more old-fashioned notation the covector and vector relations are

$$ x_{i_1}x_{i_2} = x_{j_1}x_{j_2}R_{i_1,j_1}^{j_2, i_2}, \quad v^i_1v^i_2 = v^{i_1}_1v^{i_2}_2R_{i_1,j_1}^{j_2, i_2}. $$

(5)
We consider \( V^\prime(R') \) as something like the functions on \( \mathbb{R}^n \) with \( x_i \) something like the coordinate function that picks out the \( i \)-th component. The linear braided-coproduct then corresponds to addition on \( \mathbb{R}^n \). In practice of course we work with \( x_i \) directly. Also, the braided-coproduct \( V^\prime(R') \to V^\prime(R') \otimes V^\prime(R') \) can be regarded just as well as a braided-coaction of one copy on the other. If we write \( a \equiv x \otimes 1 \) and \( x \equiv 1 \otimes x \) for the generators of the two copies of \( V^\prime(R') \) then \( V^\prime(R') \otimes V^\prime(R') \) is generated by the two copies and cross relations
\[
x_1 a_2 = a_2 x_1 R_{12} = a_1 x_2 (PR)_{12}
\]
as determined by \( \Psi \) from (6). The braided coproduct or coaction is
\[
x \mapsto a + x
\]
and says that we can add braided covectors (the sum also realises \( V^\prime(R') \)) provided we remember the braid statistics (6).

**Definition 2.1** We define the braided-differential operators \( \partial^i : V^\prime(R') \to V^\prime(R') \) as
\[
\partial^i f(x) = \left( a_i^{-1}(f(a + x) - f(x)) \right)_{a=0} \equiv \text{coeff of } a^i \text{ in } f(a + x).
\]

In other words, the \( \partial^i \) are the generators of infinitesimal translations. Remembering the braid statistics we have on the monomials
\[
\partial^i(x_1 \cdots x_m) = \text{coeff}_{a} \left( (a_1 + x_1)(a_2 + x_2) \cdots (a_m + x_m) \right)
\]
\[
= \text{coeff}_{a} \left( a_1 x_2 \cdots x_m + x_1 a_2 x_3 \cdots x_m + \cdots + x_1 \cdots x_{m-1} a_m \right)
\]
\[
= \text{coeff}_{a} \left( a_1 x_2 \cdots x_m (1 + (PR)_{12} + (PR)_{12}(PR)_{23} + \cdots + (PR)_{12} \cdots (PR)_{m-1,m}) \right)
\]
\[
= e^i_1 x_2 \cdots x_m [m; R]_{1 \cdots m}
\]
where \( e^i \) is a basis covector \( (e^i)_j = \delta^i_j \) and
\[
[m; R] = 1 + (PR)_{12} + (PR)_{12}(PR)_{23} + \cdots + (PR)_{12} \cdots (PR)_{m-1,m}
\]
is a certain matrix living in the \( m \)-fold matrix tensor product. We call such matrices *braided integers* for reasons that will become apparent in the next section. In explicit terms we have
\[
\partial^i x_{i_1} \cdots x_{i_m} = \delta^i_{j_1} x_{j_2} \cdots x_{j_m} [m; R]_{j_1 \cdots j_m}^{i_1 \cdots i_m}.
\]

**Proposition 2.2** The operators \( \partial^i \) obey the relations of the braided vectors \( V(R') \). Thus there is an action \( V(R') \otimes V^\prime(R') \to V^\prime(R') \) given by \( v^i \otimes f(x) \mapsto \partial^i f(x) \).
Proof We show the identity

\[ [m - 1; R]_{2 \ldots m} [m; R]_{1 \ldots m} = (PR')_{12} [m - 1; R]_{2 \ldots m} [m; R]_{1 \ldots m}. \]  

(11)

To do this we use the definition of the braided-integers in (8), and (12) to compute

\[
((PR')_{12} - 1) [m - 1; R]_{2 \ldots m} [m; R]_{1 \ldots m} = ((PR')_{12} - 1) \left( (1 + (PR)_{12}) [m - 1; R]_{2 \ldots m} + ([m - 1; R]_{2 \ldots m} - 1)(PR)_{12} [m - 1; R]_{2 \ldots m} \right)
\]

\[
= ((PR')_{12} - 1)(PR)_{23} [m - 2; R]_{3 \ldots m} (PR)_{12} [m - 1; R]_{2 \ldots m}
\]

\[
= (PR)_{23}(PR)_{12}((PR')_{23} - 1) [m - 2; R]_{3 \ldots m} [m - 1; R]_{2 \ldots m}.
\]

Hence if we assume (11) for \( m - 1 \) as an induction hypothesis, the last expression vanishes. Hence the result also holds for \( m \). The start of the induction is provided by (3) itself.

From (8) we have

\[
\partial^i \partial^j x_1 \cdots x_m = e^i_1 e^j_2 x_3 \cdots x_m [m - 1; R]_{2 \ldots m} [m; R]_{1 \ldots m}
\]

\[
R^{i_i}_{a_a} R^{j_j}_{b_b} \partial^a x_1 \cdots x_m = R^{i_i}_{a_a} R^{j_j}_{b_b} e^i_1 e^j_2 x_3 \cdots x_m [m; R]_{1 \ldots m}
\]

\[
= R^{i_i}_{a_a} e^i_1 e^j_2 x_3 \cdots x_m (PR')_{12} [m - 1; R]_{2 \ldots m} [m; R]_{1 \ldots m}.
\]

These are equal due to our identity (11). Hence \( \partial^i \) obey the relations for \( v^i \) in (3) and are therefore an operator realization of \( V(R') \). \( \square \)

Lemma 2.3 The operators \( \partial^i \) obey the braided-Leibniz rule

\[
\partial^i (ab) = (\partial^i a)b + \Psi^{-1}(\partial^i \otimes a)b
\]

Proof We need here the inverse braiding between braided-vectors and braided-covectors. This is computed from (4) and comes out as \( \Psi^{-1}(\partial_2 \otimes x_1) = x_1 \otimes R_{12} \partial_2 \) and extends to products by functoriality as

\[
\Psi^{-1}(\partial^i \otimes x_1 \cdots x_r) = e^i_1 x_2 \cdots x_r x_{r+1}(PR)_{12} \cdots (PR)_{r,r+1} \partial_{r+1}.
\]  

(12)
Taking this in the proposition, we compute the right hand side on monomials as

\[
(\partial^i x_1 \cdots x_r)x_{r+1} \cdots x_m + \Psi^{-1}(\partial^i \otimes x_1 \cdots x_r)x_{r+1} \cdots x_m \\
= e^i_1 x_2 \cdots x_r [r; R]_{1 \cdots r} x_{r+1} \cdots x_m + e^i_1 x_2 \cdots x_r x_{r' + 1} (PR)_{12} \cdots (PR)_{r,r'+1} \partial_{r'+1} x_{r+1} \cdots x_m \\
= e^i_1 x_2 \cdots x_r x_{r+1} \cdots x_m [r; R]_{1 \cdots r} + e^i_1 x_2 \cdots x_r x_{r'+1} (PR)_{12} \cdots (PR)_{r+1} x_{r+2} \cdots x_m [m - r; R]_{r+1 \cdots m}
\]

where we use (8) to evaluate the differentials. The primed \(r' + 1\) labels a distinct matrix space from the existing \(r + 1\) index. These are then identified by the \(e_{r+1}\) brought down by the action of \(\partial_{r'+1}\). The resulting expression coincides with \(e^i_1 x_2 \cdots x_r x_{r+1} \cdots x_m [m; R]_{1 \cdots m}\) from the left hand side of the proposition, since

\[
[r; R]_{1 \cdots r} + (PR)_{12} \cdots (PR)_{r,r'+1} [m - r; R]_{r+1 \cdots m} = [m; R]_{1 \cdots m}.
\]

This is evident from the definition in (8). \(\Box\)

**Example 2.4** Let

\[
R = \begin{pmatrix} q^2 & 0 & 0 & 0 \\ 0 & q & q^2 - 1 & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & q^2 \end{pmatrix}, \quad R' = q^{-2} R
\]

(the \(SL_q(2)\) R-matrix). Then \(V^{-1}(R')\) is the standard quantum plane \(x = (x, y)\) with relations \(yx = qxy\) and \(\partial^i\) are

\[
\frac{\partial}{\partial x} x^n y^m = [n; q^2] x^{n-1} y^m, \quad \frac{\partial}{\partial y} x^n y^m = q^n x^m [m; q^2] y^{m-1}, \quad [m; q^2] = \frac{q^{2m} - 1}{q^2 - 1}.
\]

This is similar to the results of another approach based on differential forms in [15] and elsewhere. Also, for any \(R\)-matrix we can take \(R' = P\). Then \(V^{-1}(P) = k < x_i >\) is the free non-commutative algebra in \(n\) indeterminates. \(V(P)\) is likewise free and \(\partial^i\) are given from (8).

We call this the free braided differential calculus associated to an \(R\)-matrix. It is in a certain sense universal.

**Example 2.5** Let \(R = (q)\) the one-dimensional solution of the QYBE. The free braided-line introduced in [3] is \(V^{-1}(R') = k[x]\) (polynomials in one variable) with braiding \(\Psi(x \otimes x) = qx \otimes x\). In this case we have

\[
\partial x^m = x^{m-1} (1 + q + \cdots + q^{m-1}) = [m; q] x^{m-1} \quad \Rightarrow \quad \partial f(x) = \frac{f(qx) - f(x)}{(q - 1)x}.
\]
One can verify the various properties in Proposition 2.2 and Lemma 2.3 easily enough in these examples. In the braided-line case the braided-Leibniz rule is easily verified in the form
\[ \partial(x^n x^m) = (\partial x^n)x^m + q^n x^n(\partial x^m) \]
which is just as in the case of a super-derivation, but with \( q \) in the role of \((-1)\) and a \( \mathbb{Z} \)-grading in the role of \( \mathbb{Z}_2 \)-grading. The degree of \( \partial \) here is \(-1\) and the degree of \( x^n \) is \( n \). Similarly in Example 2.4 the \( q^n \) factor can be thought of as arising from the braided Leibniz rule as \( \frac{\partial}{\partial y} \) passes \( x^n \). The general case based on an \( R \)-matrix should be viewed in just the same way. Of course products of the \( \partial^i \) do not obey the same Leibniz rule, but rather one deduced from the one shown for the generators. We also note that for a fully categorical picture we need the all possible braidings \( \Psi, \Psi^{-1} \) between vectors and covectors, not only the limited combinations used above. For this one needs that \( R^{t_2} \) is invertible as well as \( R \), where \( t_2 \) denotes transposition in the second matrix factor. We come to this full picture in Section 5.

3 Braided Binomial Theorem

In this section we prove an \( R \)-matrix version of the familiar \( q \)-binomial theorem. This tells us how to expand \((a+x)^m\) when \( xa = qax \) say. The \( q \)-binomial theorem has \( q \)-integers \([r; q]\) in place of usual integers. The \( R \)-matrix generalization covers both the non-commutativity which takes the form (3) and the fact that we deal with the multidimensional case. We need this theorem in later section. We let \( R \in M_n \otimes M_n \) be an invertible solution of the QYBE and \([m; R]\) the braided-integers introduced in (3).

**Definition 3.1** Let \([m; R]\) be the matrix defined recursively by
\[
\begin{align*}
[m; R]_{1\ldots m} &= (PR)_{r,r+1}\cdots(PR)_{m-1,m} [m-1; R]_{1\ldots m-1} + [m-1; R]_{1\ldots m-1} \\
[m; R]_0 &= 1, \\
[m; R]_r &= 0 \text{ if } r > m.
\end{align*}
\]
The suffices here refer to the matrix position in tensor powers of \( M_n \) in the standard way.

This defines in particular
\[
[m; R] = [m-1; R] = \cdots = [1; R] = 1
\]
(14)
\[
\left[ \begin{array}{c} m \\ 1 \\ R \end{array} \right] = (PR)_{12} \cdots (PR)_{m-1,m} + \left[ \begin{array}{c} m-1 \\ 1 \\ R \end{array} \right] = \cdots = [m; R].
\]

A similar recursion defines \([m/2; R]_{1\ldots m}\) in terms of \([m/1; R]\) (which is known) and \([m-1/2; R]\), and similarly (in succession) up to \(r = m\).

**Proposition 3.2** Suppose that \(x_1 a_2 = a_1 x_2 (PR)_{12}\) (as in the braided tensor product algebra in Section 2). Then

\[
(a_1 + x_1) \cdots (a_m + x_m) = \sum_{r=0}^{r=m} a_1 \cdots a_r x_{r+1} \cdots x_m \left[ \begin{array}{c} m \\ r \\ R \end{array} \right]_{1\ldots m}. 
\]

**Proof** We proceed by induction. Suppose the proposition is true for \(m - 1\), then

\[
\begin{align*}
(a_1 + x_1) \cdots (a_m + x_m) &= (a_1 + x_1) \cdots (a_{m-1} + x_{m-1})a_m + (a_1 + x_1) \cdots (a_{m-1} + x_{m-1})x_m \\
&= \sum_{r=0}^{m-1} a_1 \cdots a_r x_{r+1} \cdots x_{m-1} a_m \left[ \begin{array}{c} m-1 \\ r \\ R \end{array} \right]_{1\ldots m-1} + (a_1 + x_1) \cdots (a_{m-1} + x_{m-1})x_m \\
&= \sum_{r=1}^{m} a_1 \cdots a_{r-1} x_r \cdots x_{m-1} a_m \left[ \begin{array}{c} m-1 \\ r-1 \\ R \end{array} \right]_{1\ldots m-1} + \sum_{r=0}^{m-1} a_1 \cdots a_r x_{r+1} \cdots x_m \left[ \begin{array}{c} m-1 \\ r \\ R \end{array} \right]_{1\ldots m-1} \\
&= \sum_{r=1}^{m} a_1 \cdots a_r x_{r+1} \cdots x_m (PR)_{r,r+1} \cdots (PR)_{m-1,m} \left[ \begin{array}{c} m-1 \\ r-1 \\ R \end{array} \right]_{1\ldots m-1} \\
&\quad + \sum_{r=0}^{m-1} a_1 \cdots a_r x_{r+1} \cdots x_m \left[ \begin{array}{c} m-1 \\ r \\ R \end{array} \right]_{1\ldots m-1} \\
&= \sum_{r=1}^{m} a_1 \cdots a_r x_{r+1} \cdots x_m \left[ \begin{array}{c} m \\ r \\ R \end{array} \right]_{1\ldots m} + x_1 \cdots x_m \left[ \begin{array}{c} m-1 \\ 0 \\ R \end{array} \right]_{1\ldots m-1}
\end{align*}
\]

using the induction hypothesis and Definition 3.1. The last term is also the \(r = 0\) term in the desired sum, proving the result for \(m\). \(\Box\)

In explicit notation the last proposition reads

\[
(a_{i_1} + x_{i_1}) \cdots (a_{i_m} + x_{i_m}) = \sum_{r=0}^{r=m} a_{j_1} \cdots a_{j_r} x_{j_{r+1}} \cdots x_{j_m} \left[ \begin{array}{c} m \\ r \\ R \end{array} \right]_{i_1 \ldots i_m}. 
\]

The main theorem is to actually compute these \(R\)-binomial coefficient matrices. We need two lemmas.

**Lemma 3.3** Denote the monodromy or ‘parallel transport’ operator in Definition 3.1 by

\[
[a, b; R] = (PR)_{a,a+1} \cdots (PR)_{b-1,b}
\]

for \(a \leq b\) with the convention \([a, a; R] = 1\). It obeys

\[
[a, b; R][b, c; R] = [a, c; R] \quad \forall a \leq b \leq c
\]

\[
[a, b; R][c, d; R] = [c + 1, d + 1; R][a, b; R] \quad \forall a \leq c \leq d < b
\]
Proof The proof of the second identity follows from repeated use of the QYBE or braid relations, in a standard way. This is easily done by writing the monodromies as braided crossings. Thus \([a, b; R]\) is the braid of strand \(a\) past the strands up to and including strand \(b\), with each crossing represented by an \(R\)-matrix. If the braid \([c, d; R]\) lies entirely inside \([a, b; R]\) then it can be pulled through it in its entirety, giving the commutation relation shown. This is depicted in Figure 1.

\[\equiv\]

**Lemma 3.4** The monodromy commutes with the braided-binomial coefficients in the sense

\[
[1, m; R] \left[ \frac{m - 1}{r} ; R \right]_{1 \cdots m - 1} = \left[ \frac{m - 1}{r} ; R \right]_{2 \cdots m} [1, m; R]
\]

Proof We proceed by induction. Thus using Definition 3.1 and the result for \(m - 2\) we have

\[
[1, m; R] \left[ \frac{m - 1}{r} ; R \right]_{1 \cdots m - 1} = [1, m; R] \left( [r, m - 1; R] \left[ \frac{m - 2}{r - 1} ; R \right]_{1 \cdots m - 2} + \left[ \frac{m - 2}{r} ; R \right]_{1 \cdots m - 2} \right)
\]

\[
= [r + 1, m; R] [1, m; R] \left[ \frac{m - 2}{r - 1} ; R \right]_{1 \cdots m - 2} + [1, m; R] \left[ \frac{m - 2}{r} ; R \right]_{1 \cdots m - 2}
\]

\[
= [r + 1, m; R] \left[ \frac{m - 2}{r - 1} ; R \right]_{2 \cdots m - 1} [1, m; R] + \left[ \frac{m - 2}{r} ; R \right]_{2 \cdots m - 1} [1, m; R]
\]

which equals the right hand side using Definition 3.1 again in reverse. For the second equality we used the preceding lemma. \(\square\)

**Theorem 3.5**

\[
[r; R]_{1 \cdots r} \left[ \frac{m}{r} ; R \right]_{1 \cdots m} = \left[ \frac{m - 1}{r - 1} ; R \right]_{2 \cdots m} [m; R]_{1 \cdots m}
\]

and hence (formally supposing that all the braided-integers are invertible)

\[
\left[ \frac{m}{r} ; R \right]_{1 \cdots m} = [r; R]_{1 \cdots r}^{-1} [r - 1; R]_{2 \cdots r}^{-1} \cdots [2; R]_{r - 1, r}^{-1} [m - r + 1; R]_{r \cdots m} \cdots [m; R]_{1 \cdots m}
\]
Proof We proceed by induction. Suppose the result is true up to \( m - 1 \). Then from Definition 3.1 and the above lemmas we have

\[
[r; R]_{1\ldots r} \left[ \frac{m}{r}; R \right]_{1\ldots m} = [r; R]_{1\ldots r} [r, m; R] \left[ \frac{m-1}{r-1}; R \right]_{1\ldots m-1} + [r; R]_{1\ldots r} \left[ \frac{m-1}{r}; R \right]_{1\ldots m-1}
\]

\[
= [r, m; R] [r-1; R]_{1\ldots r-1} \left[ \frac{m-1}{r-1}; R \right]_{1\ldots m-1} + [1, m; R] \left[ \frac{m-1}{r-1}; R \right]_{1\ldots m-1}
\]

\[
+ [r; R]_{1\ldots r} \left[ \frac{m-1}{r}; R \right]_{1\ldots m-1}
\]

\[
= [r, m; R] \left[ \frac{m-2}{r-2}; R \right]_{2\ldots m-1} [m-1; R]_{1\ldots m-1} + [m-1; R]_{2\ldots m} [1, m; R]
\]

\[
+ \left[ \frac{m-2}{r-1}; R \right]_{2\ldots m-1} [m-1; R]_{1\ldots m-1}
\]

\[
= \left[ \frac{m-1}{r-1}; R \right]_{2\ldots m} [1, m; R] + \left[ \frac{m-1}{r-1}; R \right]_{2\ldots m} [m-1; R]_{1\ldots m-1}
\]

\[
= \left[ \frac{m-1}{r-1}; R \right]_{2\ldots m} [m; R]_{1\ldots m}
\]

as required. Here the first second equality splits \([r; R]_{1\ldots r} = [r-1; R]_{1\ldots r-1} + [1, r; R]\). The \([r, m; R]\) commutes past the first term of this, while with the second term it combines to give \([1, m; R]\). The third equality is our induction hypothesis for the outer terms and Lemma 3.4 for the middle term. We then used Definition 3.1 in reverse to recognise two of the terms to obtain the fourth equality. We then recognise \([1, m; R] + [m-1; R] = [m; R]\) to obtain the result. \(\square\)

Note that one does not really need the braided-integers to be invertible here (just as for the usual binomial coefficients). For example, the recursion relation in the theorem implies that

\[
\left[ \frac{m}{m-1}; R \right]_{1\ldots m} = 1 + (PR)_{m-1,m} + (PR)_{m-1,m} (PR)_{m-2,m-1} + \cdots + (PR)_{m-1,m} \cdots (PR)_{1,2}
\]

which is a right-handed variant of \([m; R]\). This can also be proven directly by using the quantum Yang-Baxter equations a lot of times. One can prove numerous other identities of this type in analogy with usual combinatoric identities. This theorem demonstrates the beginning of some kind of braided-number-theory or braided-combinatorics. Because it holds for any invertible solution of the QYBE, it corresponds to a novel identity in the group algebra of the braid group. Physically, it corresponds to ‘counting’ the ‘partitions’ of a box of braid-statistical particles. These points of view will be developed elsewhere.
4 Braided Exponentials and Braided Taylor’s Theorem

As an application of the braided-binomial theorem we are going to study a braided version of the exponential map. Throughout this section we assume that \( R \) is generic in the sense that all the braided-integers \([m; R]\) are invertible matrices. This corresponds when defining the usual \( q \)-exponential to assuming that \( q \) is generic. On the other hand, looking at (2) this means that \( R' = P \) for the corresponding braided vectors and braided covectors. Thus we necessarily proceed in the case of this free braided differential calculus.

\[
\Psi(P) = k < x_i > \quad \text{and} \quad \Psi(P) = k < v^i >
\]

are free algebras. They have a linear coproduct making them into braided-Hopf algebras with braiding \( \Psi \) defined by \( R \). This seems to be the natural generalization to an \( R \)-matrix of the usual 1-dimensional theory of \( q \)-differential calculus.

**Definition 4.1** Let \( R \) be a generic \( R \)-matrix. We define the \( R \)-exponential to be the formal power-series in \( V^-(P) \otimes V(P) \) defined by

\[
\exp_R(x|v) = \sum_{m=0}^{\infty} x_1 \cdots x_m [m; R]_{1-m}^{-1} [m-1; R]_2^{-1} \cdots [2; R]_{m-1}^{-1} v_m \cdots v_1.
\]

From (8) we see at once that

\[
\partial^i \exp_R(x|v) = \sum_{m=0}^{\infty} \partial^i x_1 \cdots x_m [m; R]_{1-m}^{-1} [m-1; R]_2^{-1} \cdots [2; R]_{m-1}^{-1} v_m \cdots v_1
\]

\[
= \sum_{m=0}^{\infty} e^i_1 \left( x_2 \cdots x_m [m-1; R]_2^{-1} \cdots [2; R]_{m-1}^{-1} v_m \cdots v_2 \right) v_1
\]

\[
= \exp_R(x|v) v^i
\]

(18)

where \( v^i \) is the non-commutative eigenvalue. Recall that the \( \partial^i \) themselves are a realization of the vector algebra.

**Corollary 4.2** (Braided Taylor’s Theorem) in the braided tensor product algebra \( V^-(P) \otimes V^-(P) \) as in Section 2 we have

\[
\exp_R(a|\partial)f(x) = f(a + x) = \Delta f(x).
\]

We see that \( \partial \) is the infinitesimal generator of the translation corresponding to this braided-coproduct.
Proof This follows at once from the braided-binomial theorem in Section 3. On any polynomial the power-series \( \exp_R(a|\partial) \) truncates to a finite sum. Computing from (8) we have

\[
\exp_R(a|\partial)x_1 \cdots x_m = \sum_{r=0}^{r=m} a^{1}_{r'} \cdots a^{r'}_{r'} [r; R]^{-1}_{r_{1},r_{1}'} \cdots [2; R]^{-1}_{r_{1}',r_{1}} \partial_{r'} \cdots \partial_{r'}x_1 \cdots x_m
\]

Here the \( 1', 2' \) etc refer to copies of \( M_n \) distinct from the copies labelled by \( 1 \cdots m \), but they are successively identified by the \( e_i \) (which are Kronecker delta-functions) brought down by the application of \( \partial_i \). The \( x_{r+1} \cdots x_m \) commute to the right and Theorem 3.5 and Proposition 3.2 allow us to identify the result. The result is products of \( \Delta x \) according to (3) in Section 2. But \( \Delta \) is an algebra homomorphism to the braided tensor product, giving the final form shown. \( \square \)

**Corollary 4.3** Let \( a_1x_2 = x_2a_1R_{12} \) as above. Then

\[
: \exp_R(a|v) \exp_R(x|v) := \exp_R(a+x|v)
\]

where : : denotes to keep the components of the second copy of \( v \) to the left of the components from the first. The identity takes place as a formal power-series in \( V^-(P) \otimes V^-(P) \otimes V(P) \).

**Proof** We apply the braided Taylor’s theorem just proven,

\[
\exp_R(a+x|v) = \exp_R(a|\partial) \exp_R(x|v)
\]

\[
= \sum_{m=0}^{\infty} x_1 \cdots x_m [m; R]^{-1}_{1,m} [m-1; R]^{-1}_{2,m} \cdots [2; R]^{-1}_{m-1,m} \exp_R(x|v)v_m \cdots v_1
\]

using (18). This expression is what we mean by \( : \exp_R(a|v) \exp_R(x|v) : \). This seems to be the most we can say in the free case (where there are no commutation relations among the components of \( v \)). \( \square \)

**5 Braided Quantum-Mechanics and Braided Weyl Algebras**

Here we develop a corollary of the fact that the vectors \( V(R') \) act on the covectors \( V^-(R') \) by differentiation. Recall that the usual Weyl algebra of quantum mechanics is the semidirect
product of $x$ by $p$ where $p$ acts on $x$ by differentiation. In our situation with data $(R, R')$ as in Section 2 we are in exactly the same situation and hence it is natural to make a (braided) semidirect product and call the result the braided Weyl algebra for some kind of braided quantum mechanics. Remarkably, the algebras we recover are just the ones that have been proposed by other means for this purpose. We understand here only the algebraic structure of these algebras: suitable $*$-structures and inner-products are not yet understood in our systematic braided point of view.

We recall that the data for an ordinary group semidirect product (cross product) is a group acting on an algebra by automorphisms. One can formulate the same concept for any Hopf algebra acting on an algebra. Such covariant systems are called module algebras. There is no problem formulating this just as easily for a Hopf algebra in a braided category. This is what we need here and is depicted in Figure 2 for a braided-Hopf algebra $B$ acting on an algebra $C$. In the diagrammatic notation that we use, the morphisms or maps are written pointing downwards as nodes joining their inputs to their outputs. There is a tri-valent vertex for the product in $B$, another for an action $\alpha : B \otimes C \to C$ and another for the coproduct of $B$. We represent braided transpositions $\Psi$ between any two objects such as $B, C$ by braid crossings

$$
\Psi_{B,C} = \begin{array}{c}
  B \\
  C \downarrow \quad B \\
  C
\end{array} \\
\Psi^{-1}_{C,B} = \begin{array}{c}
  B \\
  C \downarrow \quad C \\
  B
\end{array} = \Psi_{B,C}
$$

There is a symmetry here between positive and negative braid crossings with the result that we can equally well regard $\Psi^{-1}$ as the braiding of another braided category with braiding $\bar{\Psi}$ as shown. Correspondingly, one can formulate braided module-algebras with either $\Psi$ or $\Psi^{-1}$ in Figure 2. If our braided-coproduct $B \to B \bar{\otimes} B$ is an algebra-homomorphism to the braided tensor product with $\Psi$ as in (1) it is appropriate to use the positive braid with $\Psi$ in Figure 2. We have shown it for the negative braid because this is the one which our example will satisfy.

Firstly, we denote the generators of $V(R')$ by $p^i$ rather than $v^i$ in Section 2 (because they are going to become momentum) and we define $B = \tilde{V}(R')$ to be the same as this $V(R')$ and the same coproduct on the generators but reversed braiding

$$
\tilde{\Delta}(p^i) = p^i \otimes 1 + 1 \otimes p^i, \quad \bar{\Psi}(p_1 \otimes p_2) = R_{21}^{-1} p_2 \otimes p_1
$$
when it comes to extending to products. By definition $\bar{\Delta}$ extends as an algebra homomorphism $B \rightarrow B \bar{\otimes} B$ where the latter is defined as in (1) but with $\bar{\Psi} = \Psi^{-1}$ in the role of $\Psi$. It is an equally good braided-Hopf algebra but lives in the category with reversed braiding from our original $V(R')$. From now on we do everything in this category.

**Lemma 5.1** $V^{-}(R')$ is a braided $\bar{V}(R')$-module algebra in the sense of Figure 2, with action $\alpha$ defined by the operators $p_{i} \triangleright a = \partial_{i}(a)$.

**Proof** This follows in fact from general principles as outlined in [7] to do with the identification of $\Delta = \Psi^{-1} \circ \Delta_{\psi}$. It can also be seen by iterating Lemma 2.3. Firstly $\Delta p_{1} \cdots p_{r}$ is given by braided binomial coefficient-matrices similar to those we have seen in Section 3 for braided covectors. For example,

$$\Delta p_{1} p_{2} = p_{1} p_{2} \otimes 1 + 1 \otimes p_{1} p_{2} + p_{1} \otimes p_{2} + R_{21}^{-1} p_{2} \otimes p_{1}. \quad (21)$$

Apply this in the right hand side in Figure 2 gives in this case four terms. On the other hand computing the action of $p_{1} p_{2}$ on the left hand side directly by $\partial_{1} \partial_{2}$ and knowing that these $\partial$ are each derivations from Lemma 2.3 also gives four terms. Comparing them and using functoriality of the braiding one can see that they are equal. Similarly the general form follows from the Leibniz rule for each $\partial$. \(\square\)

**Example 5.2** In the 1-dimensional case of Example 2.5 the module-algebra property corresponds to the identity

$$\sum_{s=0}^{r} \left[ \begin{array}{c} r \\ s \end{array} \right]_{q^{-1}} \left[ a-r \\ b-s \end{array} \right]_{q} q^{b(r-s)} = \left[ \begin{array}{c} a \\ b \end{array} \right]_{q}, \quad \forall \ r, b \leq a.$$ for $q$-binomial coefficients. These are considered zero outside the usual range.
Proof We examine the braided module-algebra property directly in this 1-dimensional setting. For the left in Figure 2 we have

\[ p^r \triangleright (x^{n+m}) = [n + m; q] \cdots [n + m - r + 1; q] x^{n+m-r} \quad (22) \]

where \( \triangleright \) denotes the action \( \alpha \). On the right hand side by contrast we have, using the usual \( q \)-binomial theorem to compute \( \bar{\Delta} p^r \), the expression

\[
\sum_{s=0}^{r} \left[ \begin{array}{c} r \\ s \end{array} \right] q^{-1} p^r \triangleright \Psi(p^{r-s} \otimes x^n) \triangleright x^m
\]

\[
= \sum_{s=0}^{r} \left[ \begin{array}{c} r \\ s \end{array} \right] q^{-1} (p^s \triangleright x^n)(p^{r-s} \triangleright x^m) q^{n(r-s)}
\]

\[
= \sum_{s=0}^{r} \left[ \begin{array}{c} r \\ s \end{array} \right] q^{-1} [n; q] \cdots [n - s + 1; q] [m; q] \cdots [m - r + s + 1; q] q^{n(r-s)} x^{n+m-r}
\]

We see that the module algebra condition for all \( a = n + m \) and \( b = n \) corresponds to the identity shown. The \( q^{n(r-s)} \) here comes from the braiding of \( p \) past \( x \). \( \Box \)

Now whenever we are in the situation of Figure 2 with \( C \) a braided \( B \)-module algebra, we can make a semidirect product [8]. This is built on the object \( C \otimes B \) but with the multiplication twisted by the action \( \alpha \) as shown in Figure 3.

Proposition 5.3 The braided Weyl algebra \( V^{\gamma}(R') \triangleright V(R') \) defined as the braided semidirect product by the action of braided vectors on braided covectors by \( \partial^i \) is generated by \( \hat{x} = x \otimes 1 \) in \( V^{\gamma}(R') \) and \( \hat{p} = 1 \otimes p \) in \( V(R') \) with cross relations

\[
\hat{p}_1 \hat{x}_2 - \hat{x}_2 R_2 \hat{p}_1 = \text{id.}
\]
Proof. We can compute the expression for the product of a general element $x_1 \cdots x_r \otimes p_{r+1} \cdots p_m$ with another such element from Figure 2. One uses the braided-binomial theorem to compute $\Delta p_{r+1} \cdots p_m$. To know only the relations of the resulting algebra we can concentrate on $\hat{p}$ and $\hat{x}$ as stated. On the one hand we have $(x_2 \otimes 1)(1 \otimes p_1) = x_2 \otimes p_1$ because $\bar{\Psi}$ is trivial on 1. On the other hand we have

$$(1 \otimes p_1)(x_2 \otimes 1) = p_1 \triangleright x_2 + \bar{\Psi}(p_1 \otimes x_2) = \text{id} + x_2 \otimes R_{21}p_1 \quad (23)$$

giving the commutation relations shown. The braiding is as in Lemma 2.3 and taken is from [4]. Note that we do not begin with such relations and afterwards verify that they are consistent, but rather we have a well-defined algebra on the tensor product space from the start. □

Some similar commutation relations in the $SL_q(n)$ case were studied in [12] as a kind of $q$-quantization of harmonic oscillators, but our construction works more generally for any $R$ with associated $R'$ ($R$ need not be of Hecke type). They are also familiar in the theory of quantum inverse scattering in various contexts. The main point however, is that the construction is understood now in a systematic way as a braided-semidirect product.

Example 5.4 Let $R = (q)$ the 1-dimensional $R$-matrix as in Example 2.5. The braided semidirect product $k[x] \triangleright k[p]$ where $p \triangleright f(x) = \partial f(x)$ (the $q$-derivative) is the algebra generated by $\hat{p} = 1 \otimes p$ and $\hat{x} = x \otimes 1$ and relations

$$\hat{p}\hat{x} - q\hat{x}\hat{p} = 1.$$  

Proof. The proof for the commutation relations is just as the previous example. One can also compute the product of general basis elements in closed form using the $q$-binomial theorem, thus

$$\hat{p}^m f(\hat{x}) = (1 \otimes p^m)(f(x) \otimes 1) = \sum_{r=0}^{m} \binom{m}{r}_q p^r \triangleright \Psi(p^{m-r} \otimes f(x))$$

$$= \sum_{r=0}^{m} \binom{m}{r}_q (\partial^r f(q^{m-r}x) \otimes p^{m-r}) = \sum_{r=0}^{m} \binom{m}{r}_q (\partial^r f(q^{m-r}\hat{x})q^{r(m-r)}\hat{p}^{m-r}) \quad (24)$$

For example, $\hat{p} f(\hat{x}) = f(q\hat{x})\hat{p} + (\partial f)(\hat{x})$. □

This is the $q$-Heisenberg algebra as proposed for $q$-deformed quantum mechanics in [3][11]. We see that it too is a semidirect product exactly of the type of the usual Weyl algebra. That
it is built as we have stated on $k[x] \bowtie k[p]$ followed from the non-trivial fact that $\partial$ extended to products as a module-algebra action. The main point of our approach is that all formulae follow as in the usual un-deformed case provided only that we remember the braid-statistics introduced by $q$ or $R$. For example, the $q$-Schroedinger representation of this algebra can be written down immediately along the usual lines, as in principle can other desired data of (deformed) quantum mechanics. This will be explored elsewhere.

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