Blow-up and lifespan estimates for solutions to the weakly coupled system of nonlinear damped wave equations outside a ball

TUAN ANH DAO and MASAHIRO IKEDA

Abstract. In this paper, we consider the initial-boundary value problems with several fundamental boundary conditions (the Dirichlet/Neumann/Robin boundary condition) for the multi-component system of semi-linear classical damped wave equations outside a ball. By applying a test function approach with a judicious choice of test functions, which approximates the harmonic functions being subject to these boundary conditions on \( \partial \Omega \), simultaneously we have succeeded in proving the blow-up result in a finite time as well as in catching the upper bound of lifespan estimates for small solutions in all spatial dimensions. Moreover, such kind of these results, which become sharp in the subcritical cases for one-dimensional case, will be discussed at the end of this paper.

1. Introduction

This paper is mainly concerned with investigating upper bound of lifespan estimates for small solutions to the following weakly coupled system of semi-linear classical damped wave equations in an exterior domain:

\[
\begin{cases}
\partial_t^2 u_1(t, x) - \Delta u_1(t, x) + \partial_t u_1(t, x) = |u_k(t, x)|^{p_1}, & (t, x) \in (0, T) \times \Omega, \\
\partial_t^2 u_2(t, x) - \Delta u_2(t, x) + \partial_t u_2(t, x) = |u_1(t, x)|^{p_2}, & (t, x) \in (0, T) \times \Omega, \\
\vdots \\
\alpha \frac{\partial u_k}{\partial n^+}(t, x) + \beta u_\ell(t, x) = 0, & (t, x) \in (0, T) \times \partial \Omega, \\
\end{cases}
\]

where \( k \geq 2, p_\ell > 1 \) with \( \ell = 1, 2, \ldots, k \) and \( T > 0 \). The domain \( \Omega \) is given by \( \Omega := \{ x \in \mathbb{R}^d : |x| > 1 \} \) with \( d \geq 2 \), and \( u_\ell : (0, T) \times \Omega \to \mathbb{R} \) with \( \ell = 1, 2, \ldots, k \) denote unknown functions to the problem (1). In addition, \( n^+ \) stands for the outward unit normal on \( \partial \Omega \). The positive constant \( \varepsilon \) presents the size of initial data. The given functions \( u_{0, \ell} \) and \( u_{1, \ell} \) with \( \ell = 1, 2, \ldots, k \) represent the shape of the initial data. Let

Mathematics Subject Classification: 35B44, 35A01, 35L15, 35L05

Keywords: Blow-up, Lifespan, Damped wave equations, Weakly coupled system, Boundary conditions, Exterior domain.
\(\alpha, \beta \in \mathbb{R}\) be real constants satisfying \((\alpha, \beta) \neq (0, 0)\). The boundary condition is called the Dirichlet boundary if \(\alpha = 0\), the Neumann boundary condition if \(\beta = 0\) and the Robin boundary condition otherwise. Here \(\Delta = \Delta_{\alpha, \beta}\) denotes the Laplace operator dependent on the boundary condition in the open subset \(\Omega\) of \(\mathbb{R}^d\) (see Notations at the end of this section for the precise definition).

To get started, let us make some attention to the Cauchy problem of (1) in the whole space, i.e., \(\Omega = \mathbb{R}^d\). Concerning \(k = 1\), the single classical semi-linear damped wave equation

\[
\begin{aligned}
&\partial_t^2 u - \Delta u + \partial_t u = |u|^p, \\
&(t, x) \in (0, T) \times \mathbb{R}^d, \\
&u(0, x) = \varepsilon u_0(x), \\
&\partial_t u(0, x) = \varepsilon u_1(x), \\
&x \in \mathbb{R}^d,
\end{aligned}
\]

(2)

Todorova-Yordanov [42] introduced the so-called Fujita exponent \(p_{\text{Fuj}}(d) := 1 + \frac{2}{d}\) (see more [9, 13, 44], the well-known Fujita exponent for the semi-linear heat equation), which classifies between the global (in time) existence of small solutions to (2) for \(p > p_{\text{Fuj}}(d)\) and a small data blow-up result in the inverse case \(1 < p < p_{\text{Fuj}}(d)\). Especially, the treatment of the critical case \(p = p_{\text{Fuj}}(d)\), verified by Zhang [46] later, is also to conclude nonexistence of global solutions to (2) even for small data. When the blow-up phenomenon in finite time occurs, the maximal existence time of solutions to (2), which is the so-called lifespan, can be estimated by a series of previous works [12, 19, 24, 27, 29, 30, 34, 35] and references therein as follows:

\[
\text{LifeSpan}(u) \sim \begin{cases} 
\varepsilon^{-\frac{2(p-1)}{2d(p-1)}} & \text{if } 1 < p < p_{\text{Fuj}}(d), \\
\exp(C \varepsilon^{-(p-1)}) & \text{if } p = p_{\text{Fuj}}(d) .
\end{cases}
\]

Regarding \(k = 2\) of (1) in \(\mathbb{R}^d\), the authors in [7, 32, 36, 40] described the following critical curve in the \(p_1 - p_2\) plane:

\[
\gamma_{\max}(p_1, p_2) := \max\{p_1, p_2\} + 1 = \frac{d}{2}.
\]

More specifically, they proved that the global (in time) small data energy solutions exist if \(\gamma_{\max}(p_1, p_2) < d/2\), meanwhile, every non-trivial local (in time) weak solution blows up in finite time if \(\gamma_{\max}(p_1, p_2) \geq d/2\). One point worth to be noticed in [36, 37] is that the sharp upper bound estimate for the lifespan of solutions in the subcritical case \(\gamma_{\max}(p_1, p_2) > d/2\) was given by

\[
\text{LifeSpan}(u_1, u_2) \leq C \varepsilon^{-\frac{1}{\gamma_{\max}(p_1, p_2)} - \frac{d}{4}} .
\]

Quite recently, for the purpose of making the study of lifespan self-contained, Chen-Dao [4] have found out the sharp lifespan estimates in the critical case \(\gamma_{\max}(p_1, p_2) = d/2\), namely

\[
\text{LifeSpan}(u_1, u_2) \sim \begin{cases} 
\exp(C \varepsilon^{-(p_1-1)}) & \text{if } p_1 = p_2, \\
\exp(C \varepsilon^{-\left(p_1 p_2 - p_{\text{Fuj}}(d)\right)}) & \text{if } p_1 \neq p_2 .
\end{cases}
\]
To demonstrate this, the authors have applied a suitable test function method linked to the technical estimates for nonlinear differential inequalities and have constructed polynomial-logarithmic type time-weighted Sobolev spaces as well in terms of deriving upper bound estimates and lower bound estimates for the lifespan, respectively. For the more general cases $k \geq 3$ of (1) in $\mathbb{R}^d$, Takeda [43] obtained both the global existence and the finite time blow-up result for small Sobolev solutions to establish the critical condition

$$\gamma_{\text{max}} := \max\{\gamma_1, \gamma_2, \ldots, \gamma_k\} = \frac{d}{2}$$

for any $d \leq 3$, which was further extended by Narazaki [33] for any $d \geq 4$ by using weighted Sobolev spaces. Here we note that the aforementioned parameters $\gamma_\ell$ with $\ell = 1, 2, \ldots, k$ are introduced as in (6). Not much later, Nishihara-Wakasugi [37] improved these results for any $d \geq 1$ by the application of a weighted energy method. One may see that the authors in the latter paper also showed the following lifespan estimates for solutions from both the above and the below:

$$c\varepsilon^{-\frac{1}{\gamma_{\text{max}}-d/2}} + \delta \leq \text{LifeSpan}(u_1, u_2, \ldots, u_k) \leq C\varepsilon^{-\frac{1}{\gamma_{\text{max}}-d/2}}$$

for any small number $\delta > 0$. In other words, they gave an almost optimal estimate for the lifespan, but it seems to be far from lower bound one to upper bound one (see also [14]).

Turning back to our models (1), as far as there have been a lot of investigations in the study of exterior problems for semi-linear classical damped wave equation. It is significant to recognize that the essential difference to the initial problem originates from the influence of reflection at the boundary and the lack of symmetric properties including scale-invariance, rotation-invariance and so on. Let us recall several previous literatures involving the initial-boundary value problem for the single equation in an exterior domain, i.e., (1) with $k = 1$, as follows:

$$\left\{ \begin{array}{ll}
\partial_t^2 u(t, x) - \Delta u(t, x) + \partial_t u(t, x) = |u(t, x)|^p, & (t, x) \in (0, T) \times \Omega, \\
u(t, x) = 0, & (t, x) \in (0, T) \times \partial \Omega, \\
u(0, x) = \varepsilon u_0(x), \quad \partial_t u(0, x) = \varepsilon u_1(x), & x \in \Omega.
\end{array} \right. \tag{3}$$

Particularly, Ikehata [15,17,18] and Ono [38] succeeded in proving the existence of global energy solutions to (3) if $p > p_{\text{Fuj}}(2)$ for $d = 2$ and $1 + \frac{4}{d+2} < p \leq 1 + \frac{2}{d-2}$ for $d = 3, 4, 5$. Meanwhile, one can find in the paper of Ogawa-Takeda [39] that the energy solutions to (3) cannot exist globally if $1 < p < p_{\text{Fuj}}(d)$ for any $d \geq 1$ by using Kaplan-Fujita method. Afterward, the critical case $p = p_{\text{Fuj}}(d)$ was independently filled by Fino-Ibrahim-Wehbe [10] and Lai-Yin [28], where the finite time blow-up phenomena also occurs. To demonstrate these blow-up results, the authors employed a suitable test function method essentially, which was originally developed by Baras-Pierre [3] (see also [31,46]). More precisely, the aid of the first eigenfunction of $-\Delta$ as well as the corresponding first eigenvalue over $\Omega$ was taken into consideration.
in [10,39], whereas the employment of some properties on the Riemann–Liouville fractional derivative and the harmonic function in $\Omega$ came into play in [28]. However, their approaches seem to be difficult to apply directly in catching the lifespan estimates for solutions to (3) due to technical issues. More recently, Sobajima [41] (see also Ikeda-Sobajima [20]) established the following upper bound estimates for lifespan of solutions to (3):

$$\text{LifeSpan}(u) \lesssim \begin{cases} 
\varepsilon^{-\frac{p-1}{2-p}} \left( \log(\varepsilon^{-1}) \right)^{\frac{p-1}{2-p}} & \text{if } 1 < p < 2 \text{ and } d = 2, \\
\exp\exp\left( C\varepsilon^{-1} \right) & \text{if } p = 2 \text{ and } d = 2, \\
\exp\left( C\varepsilon^{-(p-1)} \right) & \text{if } 1 < p < p_{\text{Fuj}}(d) \text{ and } d \geq 3, \\
\varepsilon^{-\frac{2(p-1)}{2-d(p-1)}} & \text{if } p = p_{\text{Fuj}}(d) \text{ and } d \geq 3. 
\end{cases}$$

(4)

At first sight, these estimates are exactly the same as those for the corresponding Cauchy problem (2) in higher dimensions $d \geq 3$ but the difference comes from the case of $d = 2$, especially, the double exponential type appears in the critical exponent $d = 2$. This different behavior can be interpreted as the influence of recurrency of the Brownian motion in two-dimensional case. Regarding the Robin initial-boundary value problem for the single equation in an exterior domain, i.e., (1) with $k = 1$, very recently Ikeda-Jleli-Samet [25] have investigated both the existence and nonexistence of global weak solutions to the following problem:

$$\begin{cases} 
\partial_t^2 u(t, x) - \Delta u(t, x) + \partial_t u(t, x) = |u(t, x)|^p, & (t, x) \in (0, T) \times \Omega, \\
\frac{\partial u}{\partial n^+} (t, x) + \beta u(t, x) = f(x), & (t, x) \in (0, T) \times \partial\Omega, \\
u(0, x) = u_0(x), & \partial_t u(0, x) = u_1(x), \quad x \in \Omega, 
\end{cases}$$

(5)

with a nontrivial Robin boundary condition (see also [26] and the references therein), i.e., $\beta > 0$ and a function $f \neq 0$. In their paper, one should recognize that the information about estimating lifespan of solutions has been not obtained to (5) even for a zero Robin boundary condition. Additionally, the fact is that at present there is not any result for blow-up of damped wave equation with the Neumann boundary condition, in particular, for (5) with $\beta = 0$ as far as we know.

To the best of the authors’ knowledge, no work in terms of the study of the lifespan estimates for solutions to (1) exists in the literature so far even when this system consists of two equations, i.e., (1) with $k = 2$. Motivated strongly by [4,20,25], our main goal of this paper is to indicate not only the blow-up of weak solutions but also the sharp upper bound of lifespan to (1) for any $k \geq 2$ in two and higher spatial dimensions. For this purpose, we appropriately apply the test function method with a dedicated choice of test functions, which approximate the harmonic functions enjoying the Dirichlet boundary condition or the Robin boundary condition on $\partial\Omega$ for $d = 2$ and for any $d \geq 3$ individually, as well as effectively employ the technical derivation of lifespan estimates modified from [4,20,21]. Nevertheless, as we can see later (Sect. 4.2), our strategy dealing with $d \geq 3$ does not work so well to explore
the special case $d = 2$. Hence, considering $d = 2$ as an exceptional case of (1) we will discuss the treatment of this case by another approach. Moreover, we would say that this paper seems to be the first result to investigate the blow-up phenomenon for the system (1), even for the single damped wave equation, with Neumann boundary condition. For this kind of boundary condition, we want to point out that it is enough to use the same harmonic function for treating all spatial dimensions. Finally, such kind of results in one-dimensional case will also be remarked at the end of this paper.

The structure of this paper is organized as follows: We state the main results including local well-posedness in the energy space, small data blow-up result and upper bound estimate for lifespan of solutions in Sect. 2. Section 3 is to present the proof of local well-posedness result in the energy space. We give some of preliminary calculations of test functions in Sect. 4.1 that are used in the sequel. Finally, Sect. 4.2 is devoted to the proof of small data weak solutions blow-up and upper bound estimate for lifespan of solutions simultaneously.

Notations: We give the following notations which are used throughout this paper.

- We write $f \lesssim g$ when there exists a constant $C > 0$ such that $f \leq Cg$, and $f \sim g$ when $g \lesssim f \lesssim g$.
- Let us introduce the matrix

$$P := \begin{pmatrix}
0 & 0 & \cdots & 0 & p_1 \\
p_2 & 0 & \cdots & 0 & 0 \\
0 & p_3 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & p_k & 0
\end{pmatrix}$$

and the column vector

$$\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_k)^t := (P - I_k)^{-1}(1, 1, \ldots, 1)^t,$$

where $I_k$ and $(\alpha_1, \alpha_2, \ldots, \alpha_k)^t$ stand for the identity matrix and the transposition vector of vector $(\alpha_1, \alpha_2, \ldots, \alpha_k)$, respectively. We want to point out that due to the assumption $p_\ell > 1$ with $\ell = 1, 2, \ldots, k$, it is obvious to recognize that

$$\det(P - I_k) = (-1)^{k+1} \left( \prod_{\ell=1}^{k} p_\ell - 1 \right) \neq 0.$$ 

So, the inverse matrix $(P - I_k)^{-1}$ exists. This means that the above vector $\gamma$ is well-defined. Then, we set the element $\gamma_{\text{max}} := \max\{\gamma_1, \gamma_2, \ldots, \gamma_k\}$.

- Let $m \in \mathbb{N} \cup \{0\}$ and $p \in [1, \infty]$. We introduce the Sobolev space $W^{m,p}(\Omega)$, which is a Banach space of measurable functions $f : \Omega \to \mathbb{C}$ such that $D^\alpha f \in L^p(\Omega)$ in the sense of distributions, for every multi-index $\alpha \in (\mathbb{N} \cup \{0\})^n$ with
$|\alpha| \leq m$. Here $D := (\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_n})$ is a partial differential operator. The space $W^{m,p}(\Omega)$ is equipped with the norm $\| \cdot \|_{W^{m,p}(\Omega)}$ given by

$$
\|f\|_{W^{m,p}(\Omega)} := \sum_{|\alpha| \leq m} \|D^\alpha f\|_{L^p(\Omega)}.
$$

Let us denote by $W^{m,p}_0(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$, where $C_0^\infty(\Omega)$ is the space of functions in $C^\infty(\Omega)$ having a compact support in $\Omega$.

- For $m \in \mathbb{N} \cup \{0\}$, $H^m(\Omega)$ stands for $W^{m,2}(\Omega)$, where $H^m(\Omega)$ is equipped with the equivalent norm $\| \cdot \|_{H^m(\Omega)}$ given by

$$
\|f\|_{H^m(\Omega)} := \left( \sum_{|\alpha| \leq m} \int_\Omega |D^\alpha f(x)|^2 dx \right)^{\frac{1}{2}}.
$$

Then $H^m(\Omega)$ is a Hilbert space endowed with the scalar product $\langle \cdot, \cdot \rangle_{H^m(\Omega)}$ given by

$$
\langle f, g \rangle_{H^m(\Omega)} := \sum_{|\alpha| \leq m} \Re \int_\Omega D^\alpha f(x) \overline{D^\alpha g(x)} dx.
$$

Furthermore, $H^m_0(\Omega) := W^{m,2}_0(\Omega)$.

- For $\alpha, \beta \in \mathbb{R}$ satisfying $(\alpha, \beta) \neq (0, 0)$, we introduce a closed subspace $H^{1}_{\alpha,\beta}(\Omega)$ of the Sobolev space $H^1(\Omega)$ associated with the boundary condition given by

$$
H^{1}_{\alpha,\beta}(\Omega) := \begin{cases}
H^1_0(\Omega), & \text{if } \alpha = 0, \\
\left\{ f \in H^1(\Omega) : \alpha \int_\Omega |\nabla f(x)|^2 dx + \beta \int_{\partial\Omega} |f(x)|^2 dv(x) < \infty \right\}, & \text{if } \alpha \neq 0,
\end{cases}
$$

where $dv$ is the standard surface measure on $\partial\Omega$ (see [8], for example).

- The precise definition of the Laplace operator depending on the boundary condition in the domain $\Omega$ is as follows: We introduce a linear operator $A := A_{\alpha,\beta}$ with the domain $D(A)$ associated with the boundary condition in $L^2(\Omega)$ defined by

$$
D(A) := \left\{ f \in H^{1}_{\alpha,\beta}(\Omega) : \Delta f \in L^2(\Omega) \right\},
$$

$A f := \Delta f$ for $f \in D(A)$.

We write $A$ as $\Delta = \Delta_{\alpha,\beta}$ and call $A$ to be the Dirichlet Laplacian if $\alpha = 0$, the Neumann Laplacian if $\beta = 0$ and the Robin Laplacian otherwise. It is known that $A$ generates a quasi-contractive $C_0$-semigroup $\{e^{tA}\}_{t \geq 0}$ on $L^2(\Omega)$ if $(\alpha, \beta) \neq (0, 0)$ (see [5, Definition 2.2.2 and Proposition 2.6.1] or Chapter 7 in [2], for example).
• We define a Hilbert space $E$ as
  
  $$E = E_{\alpha, \beta} = E_{\alpha, \beta}(\Omega) := \left( H^1_{\alpha, \beta}(\Omega) \right)^k \times \left( L^2(\Omega) \right)^k$$

  with $k \geq 2$, which is called the energy space to the initial-boundary value problem (1).

• Let $m \in \mathbb{N} \cup \{0\}$. For an interval $I \subset \mathbb{R}$, we introduce the space $C^m(I, X)$ of $m$-times continuously differentiable functions from $I$ to $X$ with respect to the topology in $X$.

• Let $p \in [1, \infty]$. We denote by $p'$ the Hölder conjugate of $p$. Moreover, for an interval $I = [0, T) \subset \mathbb{R}$ with $T > 0$, we define the Lebesgue space $L^p(0, T; X)$ of all measurable functions from $I$ to $X$ endowed with the norm $\| \cdot \|_{L^p(0, T; X)}$

2. Main results

In this section, we state our two main results. The first one (Theorem 1) gives local well-posedness to the initial-boundary value problem (1) in the energy space $E$ for suitable data in $E$ in the case of $p_\ell \in [1, d/(d - 2)]$ with $d \geq 3$ and $p_\ell \in [1, \infty)$ with $d = 1, 2$ for $\ell = 1, 2, \ldots, k$. Here we say that well-posedness to (1) holds if existence, uniqueness of the solution and continuous dependence on the initial data are valid. The second one (Theorem 2) gives a small data blow-up result and upper estimates for the lifespan of small data solutions to the problem (1) for suitable data when $\gamma_{\text{max}} \geq d/2$ and $d \geq 2$.

2.1. Large data local well-posedness in the energy space

In order to state the large data local well-posedness result to the initial-boundary value problem (1), we convert the original problem (1) into the following form:

$$\begin{align*}
\partial_t U(t, x) - B_{\alpha, \beta} U(t, x) &= N(U)(t, x), \quad (t, x) \in (0, T) \times \Omega, \\
U(0, x) &= \varepsilon U_0(x), \quad x \in \Omega,
\end{align*}$$

(8)

where $u : (0, T) \times \Omega \to u(t, x) := (u_1(t, x), u_2(t, x), \ldots, u_k(t, x))$ stands for the $k$-tuple of the unknown functions, and

$$U : (0, T) \times \Omega \to U(t, x) := (u(t, x), \partial_t u(t, x))^\varepsilon = (u_1(t, x), u_2(t, x), \ldots, u_k(t, x), \partial_t u_1(t, x), \partial_t u_2(t, x), \ldots, \partial_t u_k(t, x))^\varepsilon$$

is a new unknown function. Moreover,

$$u_0 : \Omega \to u_0(x) := (u_{0, 1}(x), u_{0, 2}(x), \ldots, u_{0, k}(x))$$
denotes the \( k \)-tuple of the initial displacement and
\[
\mathbf{u}_1 : \Omega \to \mathbf{u}_1(x) := (u_{1,1}(x), u_{1,2}(x), \ldots, u_{1,k}(x))
\]
denotes the \( k \)-tuple of the initial velocity, and
\[
U_0 : \Omega \to U_0(x) := (u_0(x), \mathbf{u}_1(x))\text{.}
\]
is a new given initial function. The linear operator \( \mathcal{B}_{\alpha, \beta} \) on the energy space \( E_{\alpha, \beta} \) is defined by
\[
\mathcal{B} = \mathcal{B}_{\alpha, \beta} := \begin{pmatrix} 0 & 1 \\ \Delta_{\alpha, \beta} & -1 \end{pmatrix}
\]
with the dense domain
\[
D(\mathcal{B}) := \left\{ (\mathbf{u}, \mathbf{v}) \in E_{\alpha, \beta} : \Delta_{\alpha, \beta} \mathbf{u} \in \left( L^2(\Omega) \right)^k, \mathbf{v} \in \left( H^1_{\alpha, \beta}(\Omega) \right)^k \right\},
\]
where \( \Delta_{\alpha, \beta} \) is the Laplace operator defined by (7) dependent on the boundary condition. Finally, the nonlinear mapping \( \mathcal{N} \) is given by
\[
\mathcal{N}(U) := \left( 0, 0, \ldots, 0, |u_k|^{p_1}, |u_1|^{p_2}, \ldots, |u_{k-1}|^{p_k} \right)^\mathbb{Z}.
\]
We remark that the original problem (1) is equivalent to the problem (8) through the relation
\[
U = \left( u_1, u_2, \ldots, u_k, \partial_t u_1, \partial_t u_2, \ldots, \partial_t u_k \right)^\mathbb{Z}.
\]
The \( m \)-dissipativity of the operator \( \mathcal{B} \) is known by the following lemma.

**Lemma 1.** (\( m \)-dissipativity of \( \mathcal{B} \)) Let \( \alpha \beta \geq 0 \). Then the operator \( -\frac{1}{2} I + \mathcal{B} \) defined by (9) is \( m \)-dissipative with the dense domain \( D(\mathcal{B}) \) in \( E \). Moreover, \( \mathcal{B} \) generates a quasi-contractive \( C_0 \) semigroup \( \{ e^{t\mathcal{B}} \}_{t \geq 0} \) on \( E \).

This lemma can be proved in the similar manner as the proof of [5, Proposition 2.6.9] and [5, Theorem 3.4.4].

For \( T \in (0, \infty) \), we define a solution space \( E(T) \) to the problem (8) as \( E(T) := L^\infty(0, T; E) \), which is a Banach space endowed with the norm \( \| \cdot \|_{L^\infty(0, T; E)} \). Next we introduce the following notions of mild solution and lifespan of solutions to the problems (1) as well as (8).

**Definition 1.** (Mild solution) We say that a function \( \mathbf{u} = (u_1, u_2, \ldots, u_k) \) is a mild solution to (1) if \( \mathbf{u} \) possesses the regularity
\[
\mathbf{u} \in \left( \mathcal{C}(\left[ 0, T \right), H^1_{\alpha, \beta}(\Omega)) \cap \mathcal{C}^1(\left[ 0, T \right), L^2(\Omega)) \right)^k
\]
and it satisfies the following integral equation:
\[
U(t) = e^{t\mathcal{B}} U_0 + \int_0^t e^{(t-\tau)\mathcal{B}} \mathcal{N}(U)(\tau) d\tau,
\]
which is associated with (8), for \( U_0 \in E \) and for any \( t \in (0, T) \). Additionally, we call \( U \) belonging to the class \( E(T) \) a mild solution to (8) if \( \mathbf{u} \) is a mild solution to (1).
Theorem 1. (Large data local well-posedness in the energy space) Let \( \alpha, \beta \in \mathbb{R} \) satisfying \( \alpha \beta \geq 0, d \in \mathbb{N}, p_\ell \geq 1 \) for \( \ell = 1, 2, \ldots, k \), \( \varepsilon > 0 \) and \( U_0 \in E \). We assume that if \( d \geq 3 \), then \( p_\ell \leq d/(d-2) \) for \( \ell = 1, 2, \ldots, k \). Then the initial-boundary value problem (1) is locally well-posed in the energy space \( E \). More precisely, the following statements hold:

- **Existence:** There exists a positive time \( T = T (\varepsilon, \|U_0\|_E, \{p_\ell\}_{\ell=1}^k, d) > 0 \) such that there exists a mild solution \( U \in E(T) \cap \bar{C}([0, T); E) \) to the problem (8) on \([0, T)\). Moreover, there exist positive constants

\[
e_0 = e_0 (\|U_0\|_E, \{p_\ell\}_{\ell=1}^k, d) \in (0, 1) \quad \text{and} \quad C = C (\|U_0\|_E, \{p_\ell\}_{\ell=1}^k, d)\]

such that for any \( \varepsilon \in [0, e_0] \), the following estimate holds:

\[
T_\varepsilon \geq Ce^{-(\min[p_1, p_2, \ldots, p_k] - 1)}. \tag{11}
\]

- **Uniqueness:** Let \( U \in E(T) \) be the solution to (8) obtained in the Existence part. Let \( T_1 \in (0, T) \) and \( V \in E(T_1) \) be another mild solution to (8) with the same initial data \( \varepsilon U_0 \). Then the identity \( U(t) = V(t) \) holds for any \( t \in [0, T_1] \).

- **Continuous dependence on initial data:** The flow map \( E \to E(T, M), \varepsilon U_0 \mapsto U \) is Lipschitz continuous, where \( U \) is the mild solution to (8) with initial data \( \varepsilon U_0 \) and the subspace \( E(T, M) \) given by

\[
E(T, M) := \{ U \in E(T) : \|U\|_{E(T)} \leq M \quad \text{for some} \ M > 0 \}. \tag{12}
\]

- **Blow-up alternative:** If \( T_\varepsilon < \infty \), then

\[
\lim_{t \to T_\varepsilon - 0} \|U(t, \cdot)\|_E = \infty.
\]

The above existence and uniqueness results imply well-definedness of the lifespan \( T_\varepsilon \) given in Definition 2, that is, \( T_\varepsilon > 0 \).

2.2. Blow-up and lifespan estimates for small data solutions

Before indicating the blow-up result for small data solutions as well as the lifespan estimates, let us state the following definition of weak solutions to (1).

Definition 3. (Weak solution) Let \( (u_{0, \ell}, u_{1, \ell}) \in L^1_{\text{loc}}(\Omega) \times L^1_{\text{loc}}(\Omega), p_\ell > 1 \) with \( \ell = 1, 2, \ldots, k \) and \( T > 0 \). A \( k \)-tuple of functions \( (u_1, u_2, \ldots, u_k) \) is called a weak solution to (1) on \([0, T)\) if
\((u_1, u_2, \ldots, u_k) \in \left( C([0, T), H^1_{\alpha, \beta}(\Omega)) \cap C^1([0, T), L^2(\Omega)) \cap L^p_{\text{loc}}([0, T) \times \overline{\Omega}) \right) \times \left( C([0, T), H^1_{\alpha, \beta}(\Omega)) \cap C^1([0, T), L^2(\Omega)) \cap L^p_{\text{loc}}([0, T) \times \overline{\Omega}) \right) \times \cdots \times \left( C([0, T), H^1_{\alpha, \beta}(\Omega)) \cap C^1([0, T), L^2(\Omega)) \cap L^p_{\text{loc}}([0, T) \times \overline{\Omega}) \right) \)

and, moreover, the following relations hold:

\[
\int_0^T \int_\Omega |u_\kappa(t, x)|^{p_1} \Phi(t, x) \, dx \, dt + \int_\Omega u_{1,1}(x) \Phi(0, x) \, dx
\]

\[
= \int_0^T \int_\Omega \left( \nabla u_1(t, x) \cdot \nabla \Phi(t, x) - \partial_t u_1(t, x) \partial_t \Phi(t, x) \right) \, dx \, dt
\]

\[
+ \partial_t u_1(t, x) \Phi(t, x) \right) \, dx \, dt
\]

(13)

and

\[
\int_0^T \int_\Omega |u_\ell(t, x)|^{p_{\ell+1}} \Phi(t, x) \, dx \, dr + \int_\Omega u_{1,\ell+1}(x) \Phi(0, x) \, dx
\]

\[
= \int_0^T \int_\Omega \left( \nabla u_{\ell+1}(t, x) \cdot \nabla \Phi(t, x) - \partial_t u_{\ell+1}(t, x) \partial_t \Phi(t, x) \right) \, dx \, dr
\]

\[
+ \partial_t u_{\ell+1}(t, x) \Phi(t, x) \right) \, dx \, dr
\]

(14)

with \(\ell = 1, 2, \ldots, k - 1\), for all compactly supported function \(\Phi = \Phi(t, x) \in C^2([0, T) \times \Omega)\) with \(\text{supp}\Phi \subset [0, T) \times \overline{\Omega}\) such that \(\left( a \frac{\partial \Phi}{\partial n^+} + \beta \Phi \right) (t, \cdot) \big|_{\partial \Omega} = 0\).

**Proposition 1.** (Relation between mild solution and weak solution) *We assume the same assumptions as in Theorem 1. Then \((u_1, u_2, \ldots, u_k)\) is a mild solution to \((1)\) on \([0, T)\) in the sense of Definition 1 if and only if it is a weak solution to \((1)\) on \([0, T)\) in the sense of Definition 3, respectively.*

The proof of this proposition is presented in the “Appendix”.

The second main result reads as follows:

**Theorem 2.** (Blow-up and lifespan estimates) *Let \(d \geq 2\). With \(\ell = 1, 2, \ldots, k\), assume that the exponents \(p_\ell > 1\) fulfill the condition

\[
\max \{\gamma_1, \gamma_2, \ldots, \gamma_k\} \geq \frac{d}{2},
\]

(15)

and the initial data \(u_{0,\ell}, u_{1,\ell}\) satisfy

\[
u_{0,\ell}\Psi + u_{1,\ell}\Psi \in L^1(\Omega)
\]

(16)

as well as

\[
\int_\Omega (u_{0,\ell}(x) + u_{1,\ell}(x)) \Psi(x) \, dx > 0,
\]

(17)

where the function \(\Psi = \Psi(x)\) is defined by
\[ \beta \neq 0: \]
\[ \Psi(x) = \begin{cases} 
\log |x| + \frac{\alpha}{\beta} & \text{if } d = 2, \\
1 - |x|^{2-d} + \frac{\alpha}{\beta}(d-2) & \text{if } d \geq 3,
\end{cases} \tag{18} \]

\[ \beta = 0: \]
\[ \Psi(x) = 1 \quad \text{for all } d \geq 2. \tag{19} \]

Then, there exists a positive constant \( \varepsilon_0 > 0 \) such that for any \( \varepsilon \in (0, \varepsilon_0] \) the following upper bound estimates for the lifespan of weak solutions to (1) hold:

- \( \beta \neq 0: \)
\[
\text{LifeSpan}(u_1, u_2, \ldots, u_k) \leq \begin{cases} 
C (\varepsilon^{-1} \log(\varepsilon^{-1}))^{(\max\{\gamma_1, \gamma_2, \ldots, \gamma_k\} - 1)^{-1}} & \text{if } \max\{\gamma_1, \gamma_2, \ldots, \gamma_k\} > 1, \; d = 2, \\
\exp \exp (C \varepsilon^{-1}) & \text{if } \max\{\gamma_1, \gamma_2, \ldots, \gamma_k\} = 1, \; d = 2, \\
\text{and } p_1 = p_2 = \cdots = p_k, \\
C \varepsilon^{-(\max\{\gamma_1, \gamma_2, \ldots, \gamma_k\} - d/2)^{-1}} & \text{if } \max\{\gamma_1, \gamma_2, \ldots, \gamma_k\} = d/2, \; d \geq 3, \\
\exp (C \varepsilon^{-(p_1 p_2 \cdots p_k - 1)}) & \text{if } \max\{\gamma_1, \gamma_2, \ldots, \gamma_k\} = d/2, \; d \geq 3, \\
\text{and } p_{j_1} \neq p_{j_2} \text{ with } j_1, j_2 \in \{1, 2, \ldots, k\}, j_1 \neq j_2, \\
\exp (C \varepsilon^{-(p_1 - 1)}) & \text{if } \max\{\gamma_1, \gamma_2, \ldots, \gamma_k\} = d/2, \; d \geq 3, \\
\text{and } p_1 = p_2 = \cdots = p_k, 
\end{cases} 
\]

- \( \beta = 0: \)
\[
\text{LifeSpan}(u_1, u_2, \ldots, u_k) \leq \begin{cases} 
C \varepsilon^{-(\max\{\gamma_1, \gamma_2, \ldots, \gamma_k\} - d/2)^{-1}} & \text{if } \max\{\gamma_1, \gamma_2, \ldots, \gamma_k\} > d/2, \; d \geq 2, \\
\exp (C \varepsilon^{-(p_1 p_2 \cdots p_k - 1)}) & \text{if } \max\{\gamma_1, \gamma_2, \ldots, \gamma_k\} = d/2, \; d \geq 2, \\
\text{and } p_{j_1} \neq p_{j_2} \text{ with } j_1, j_2 \in \{1, 2, \ldots, k\}, j_1 \neq j_2, \\
\exp (C \varepsilon^{-(p_1 - 1)}) & \text{if } \max\{\gamma_1, \gamma_2, \ldots, \gamma_k\} = d/2, \; d \geq 2, \\
\text{and } p_1 = p_2 = \cdots = p_k, 
\end{cases} 
\]

where \( C \) is a positive constant independent of \( \varepsilon \).
Remark 1. By plugging \( k = 1 \) in Theorem 2, one may realize obviously that our main results from Theorem 2 exactly coincide with (4) derived from [20, 41] when the case of the Dirichlet boundary condition, i.e., \( \alpha = 0 \), is considered.

Remark 2. Speaking about the conditions of power exponents from Theorems 1 and 2, we point out that the local (in time) mild solutions to (1) exist if \( p_\ell \geq 1 \) when \( d \leq 2 \), or \( 1 \leq p_\ell \leq d/(d - 2) \) when \( d \geq 3 \) for \( \ell = 1, 2, \ldots, \kappa \), furthermore, every non-trivial local (in time) weak solution to (1) blows up in finite time if the assumption (15) is satisfied. So it is necessary in the connection of Theorems 1 and 2 to verify the admissible set of \( p_\ell \) with \( \ell = 1, 2, \ldots, \kappa \) enjoying these conditions. Clearly, it suffices to do this in the case \( d \geq 3 \). For simplicity, let us now illustrate the desired result with \( \kappa = 2 \). Then, the assumption (15) becomes

\[
\frac{1 + \max\{p_1, p_2\}}{p_1p_2 - 1} \geq \frac{d}{2}.
\]

In order to observe the whole picture of the admissible range for power exponents in the \( p_1 - p_2 \) plane, we may describe it in the following Fig. 1:

Remark 3. This remark is to underline that in the present paper we have succeeded not only in proving the local well-posedness in the energy space and the blow-up phenomenon of small data solutions but also in catching upper bound estimates for lifespan of solutions to (1) By using an appropriately weighted energy method, we will devote to our concern in investigating global existence results for solutions to (1) in a forthcoming paper.
3. Proof of Theorem 1

In this section, we give a proof of local well-posedness in the energy space to the problems (1). Namely, we prove the Existence part only, since the other parts can be proved in a similar or standard manner. Let $M \geq 2\varepsilon C_0\|U_0\|_E$, where $C_0$ appears in (22) below. We take $T > 0$ such as

$$T \leq \frac{1}{4} \min \left\{ 1, \frac{1}{2C_* C_0 \sum_{\ell=1}^{k} M^{p\ell-1}} \right\},$$

where $C_* > 0$ is defined in (21) and (23) below and independent of $T$. We introduce a closed ball $E(T, M)$ at the origin with radius $M$ in the Banach space $E(T)$, which is determined as in (12) with a metric $d_T : E(T) \times E(T) \to \mathbb{R}_{\geq 0}$ defined by

$$d_T(U_1, U_2) := \|U_1 - U_2\|_{E(T)}.$$ 

For $U_0 \in E$, we introduce a nonlinear mapping $\mathcal{J}$ defined by

$$\mathcal{J}[U](t) := \varepsilon e^{tB}U_0 + \int_0^t e^{(t-\tau)B}N(U)(\tau)d\tau$$

for $t \in [0, T)$. To construct a local mild solution to (8), we will prove that $\mathcal{J}$ is a contraction mapping from $E(T, M)$ into itself.

Indeed, let $U \in E(T, M)$. By the definitions of the nonlinear function $N$ and the energy space $E$ and the Sobolev embedding $H^1(\Omega) \hookrightarrow L^{2p}(\Omega)$ from Lemma 3 of the “Appendix” for $\ell = 1, 2, \ldots, k$, the following estimates

$$\|N(U)(t, \cdot)\|_E = \left\| (u_{k}(t, \cdot)|^{p_1}, |u_{1}(t, \cdot)|^{p_2}, \ldots, |u_{k-1}(t, \cdot)|^{p_k}) \right\|_{L^2(\Omega)^k} \leq \sum_{\ell=1}^{k} \|u_{\ell-1}(t, \cdot)\|_{L^{2p}(\Omega)}^{p_{\ell}} (\text{where we set } u_0 := u_k) \leq C_* \sum_{\ell=1}^{k} M^{p_{\ell}}$$

hold for some positive constant $C_*$ independent of $T$. By Lemma 1 and the estimates (21), one derives

$$\|\mathcal{J}[U](t)\|_E \leq \varepsilon \|e^{tB}U_0\|_E + \int_0^t \|e^{(t-\tau)B}N(U)(\tau, \cdot)\|_E d\tau \leq \varepsilon C_0 \|U_0\|_E + TC_0 \sup_{t \in [0, T)} \|N(U)(t, \cdot)\|_E$$

for $t \in [0, T)$.
\[ \leq \frac{M}{2} + C C_0 T \sum_{\ell=1}^{k} M^{p_{\ell}} \leq \frac{M}{2} + \frac{M}{8} < M, \quad (22) \]

which implies that the mapping \( J \) from \( E(T, M) \) to itself is well-defined. Let \( U, V \in E(T, M) \). We set
\[
U := (u_1, u_2, \ldots, u_k, \partial_t u_1, \partial_t u_2, \ldots, \partial_t u_k)
\]
and
\[
V := (v_1, v_2, \ldots, v_k, \partial_t v_1, \partial_t v_2, \ldots, \partial_t v_k).
\]
By the definitions of the nonlinear function \( \mathcal{N} \) and the energy space \( E \), the Hölder inequality and the Sobolev embedding \( H^1(\Omega) \hookrightarrow L^{2p_t}(\Omega) \) from Lemma 3 of the “Appendix” for \( \ell = 1, 2, \ldots, k \), the following chain of inequalities
\[
\| \mathcal{N}(U)(t, \cdot) - \mathcal{N}(V)(t, \cdot) \|_E
\leq \sum_{\ell=1}^{k} \left\| |u_{\ell-1}(t, \cdot)|^{p_{\ell}} - |v_{\ell-1}(t, \cdot)|^{p_{\ell}} \right\|_{L^2(\Omega)} \quad \text{(where we set } v_0 := v_k \)
\leq C \sum_{\ell=1}^{k} \left( \| u_{\ell-1}(t, \cdot) \|_{L^{2p_t}(\Omega)}^{p_{\ell}-1} + \| v_{\ell-1}(t, \cdot) \|_{L^{2p_t}(\Omega)}^{p_{\ell}-1} \right)
\times \| u_{\ell-1}(t, \cdot) - v_{\ell-1}(t, \cdot) \|_{L^{2p_t}(\Omega)}
\leq C \sum_{\ell=1}^{k} \left( \| U \|_{E(T)}^{p_{\ell}-1} + \| V \|_{E(T)}^{p_{\ell}-1} \right) \| U - V \|_{E(T)}
\leq 2C \sum_{\ell=1}^{k} M^{p_{\ell}-1} d_T(U, V) \quad (23)
\]
holds. By Lemma 3 of the “Appendix” and the inequality (23), we obtain
\[
d_T(\mathcal{J}[U], \mathcal{J}[V]) = \sup_{t \in [0, T]} \| \mathcal{J}[U](t) - \mathcal{J}[V](t) \|_E
\leq \sup_{t \in [0, T]} \int_0^T \left\| e^{(t-\tau)E} \left( \mathcal{N}(U)(\tau, \cdot) - \mathcal{N}(V)(\tau, \cdot) \right) \right\|_E d\tau
\leq T \sup_{t \in [0, T]} \| \mathcal{N}(U)(t, \cdot) - \mathcal{N}(V)(t, \cdot) \|_E
\leq 2C \sum_{\ell=1}^{k} M^{p_{\ell}-1} d_T(U, V) \leq \frac{1}{2} d_T(U, V),
\]
which implies that the mapping \( \mathcal{J} \) is a contraction mapping. By the contraction mapping principle, we see that there exists a unique function \( U \in E(T, M) \) such that the identity \( \mathcal{J}[U](t) = U(t) \) holds for any \( t \in [0, T] \). By a standard argument, we can prove that \( U \in C([0, T]; E) \), which completes the proof of the theorem.
Next we prove the lower estimate (11) of the lifespan $T_\varepsilon$ following the argument of the proof of [19, Theorem 2.6]. Let $\varepsilon \in (0, \varepsilon_0)$, where $\varepsilon_0 < 1$ will be fixed below. If $T_\varepsilon = \infty$, then the conclusion holds. Thus we may assume $T_\varepsilon < \infty$. We define a continuous function $F : (0, T_\varepsilon) \to \mathbb{R}_{\geq 0}$ as $F(t) := \|U\|_{L^\infty(0,t;E)}$. Let $L > 1$ be a sufficiently large constant, which will be determined later. We set

$$T^* := \sup\{T \in [0, T_\varepsilon) : F(T) \leq L\varepsilon\}.$$ 

By the local existence theorem above, we find $T^* > 0$. Moreover, we can see $T^* < T_\varepsilon$. Indeed, if we assume $T^* = T_\varepsilon$, then by $\|U\|_{L^\infty(0,T^*;E)} \leq L\varepsilon$ and the local existence theorem again, we can construct a mild solution to (1) beyond $T_\varepsilon$, which contradicts the definition of $T_\varepsilon$. In the similar manner as the proof of the estimate (22), the following estimates hold:

$$F(T^*) \leq \varepsilon C_0 \|U_0\|_E + C_* C_0 T \sum_{\ell=1}^{k} (L\varepsilon)^p \ell$$

$$\leq \varepsilon C_0 \|U_0\|_E + CL \max\{p_1,\ldots,p_k\} T\varepsilon^{\min\{p_1,\ldots,p_k\}}$$

$$\leq \left\{ C_0 \|U_0\|_E L^{-1} + CL \max\{p_1,\ldots,p_k\}-1 T\varepsilon^{\min\{p_1,\ldots,p_k\}-1} \right\} L\varepsilon.$$

We choose $L > 1$ such that $C_0 \|U_0\|_E L^{-1} \leq 1/4$. We assume that

$$CL \max\{p_1,\ldots,p_k\}-1 T\varepsilon^{\min\{p_1,\ldots,p_k\}-1} < 1/4.$$

Then we see that $F(T^*) \leq (1/2)L\varepsilon$. Noting that $T^* \leq T_\varepsilon$, by the continuity of $F(t)$ with respect to $t \in (0, T_\varepsilon)$, we find that there exists $\tilde{T} \in (T^*, T_\varepsilon)$ such that $F(\tilde{T}) \leq L\varepsilon$. This contradicts to the definition of $T^*$. Therefore, the estimate

$$CL \max\{p_1,\ldots,p_k\}-1 T\varepsilon^{\min\{p_1,\ldots,p_k\}-1} \geq 1/4,$$

which completes the proof of the estimate (11).

4. Proof of Theorem 2

4.1. Test function method

At first, let us introduce a test function $\varphi = \varphi(\rho)$ having the following properties:

$$\varphi \in C_0^\infty([0, \infty)) \text{ and } \varphi(\rho) := \begin{cases} 1 & \text{if } \rho \in [0, 1/2], \\ \text{decreasing} & \text{if } \rho \in (1/2, 1), \\ 0 & \text{if } \rho \in [1, \infty). \end{cases}$$

Then, another test function $\varphi^* = \varphi^*(\rho)$ is given by

$$\varphi^*(\rho) := \begin{cases} 0 & \text{if } \rho \in [0, 1/2), \\ \varphi(\rho) & \text{if } \rho \in [1/2, \infty). \end{cases}$$
Let $R \in (0, \infty)$ be a large parameter. We introduce two test functions $\phi_R = \phi_R(t, x)$ and $\phi_R^* = \phi_R^*(t, x)$ as follows:

$$\phi_R(t, x) := \left( \varphi \left( \frac{t^2 + (|x| - 1)^4}{R^4} \right) \right)^{\lambda + 2}$$

and

$$\phi_R^*(t, x) := \left( \varphi^* \left( \frac{t^2 + (|x| - 1)^4}{R^4} \right) \right)^{\lambda + 2}$$

with a positive constant $\lambda$, which will be fixed later. In addition, we define two notations

$$Q_R := \left\{ (t, x) \in (0, T) \times \Omega : t^2 + (|x| - 1)^4 < R^4 \right\},$$

$$Q_R^* := \left\{ (t, x) \in (0, T) \times \Omega : \frac{R^4}{2} < t^2 + (|x| - 1)^4 < R^4 \right\}.$$ 

The following useful lemma comes into play in our proof in the next section.

**Lemma 2.** The following estimates hold for any $(t, x) \in Q_R$:

(i) \[ |\partial_t \phi_R(t, x)| \lesssim R^{-2} (\phi_R^*(t, x))^{\frac{\lambda+1}{\lambda+2}}, \]

(ii) \[ |\partial_t^2 \phi_R(t, x)| \lesssim R^{-4} (\phi_R^*(t, x))^{\frac{\lambda+1}{\lambda+2}}, \]

(iii) \[ |\Delta \phi_R(t, x)| \lesssim R^{-2} (\phi_R^*(t, x))^{\frac{\lambda+1}{\lambda+2}}. \]

Moreover, by taking $\Psi = \Psi(x)$ as in (18) or (19) we have the further estimate

(iv) \[ |\Delta (\Psi(x) \phi_R(t, x))| \lesssim R^{-2} \Psi(x) (\phi_R^*(t, x))^{\frac{\lambda+1}{\lambda+2}}. \]

**Proof.** First of all, a direct calculation leads to

\[
\partial_t \phi_R(t, x) = \frac{2(\lambda + 2)}{R^4} t \left( \varphi \left( \frac{t^2 + (|x| - 1)^4}{R^4} \right) \right)^{\lambda+1} \varphi' \left( \frac{t^2 + (|x| - 1)^4}{R^4} \right),
\]

\[
\partial_t^2 \phi_R(t, x) = \frac{2(\lambda + 2)}{R^4} \left( \varphi \left( \frac{t^2 + (|x| - 1)^4}{R^4} \right) \right)^{\lambda+1} \varphi' \left( \frac{t^2 + (|x| - 1)^4}{R^4} \right)
\]

\[
+ \frac{4(\lambda + 1)(\lambda + 2)}{R^8} t^2 \left( \varphi \left( \frac{t^2 + (|x| - 1)^4}{R^4} \right) \right)^\lambda \varphi' \left( \frac{t^2 + (|x| - 1)^4}{R^4} \right)^2
\]

\[
+ \frac{4(\lambda + 2)}{R^8} t^2 \left( \varphi \left( \frac{t^2 + (|x| - 1)^4}{R^4} \right) \right)^{\lambda+1} \varphi'' \left( \frac{t^2 + (|x| - 1)^4}{R^4} \right),
\]

and
\[ \nabla \phi_R(t, x) = \frac{4(\lambda + 2)}{R^4} (|x| - 1)^3 \frac{x}{|x|} \left( \varphi \left( \frac{t^2 + (|x| - 1)^4}{R^4} \right) \right)^{\lambda+1} \varphi' \left( \frac{t^2 + (|x| - 1)^4}{R^4} \right), \]

\[ \Delta \phi_R(t, x) = \frac{12(\lambda + 2)}{R^4} (|x| - 1)^2 \left( \varphi \left( \frac{t^2 + (|x| - 1)^4}{R^4} \right) \right)^{\lambda+1} \varphi' \left( \frac{t^2 + (|x| - 1)^4}{R^4} \right) \]

\[ + \frac{16(\lambda + 1)(\lambda + 2)}{R^8} (|x| - 1)^6 \left( \varphi \left( \frac{t^2 + (|x| - 1)^4}{R^4} \right) \right)^{\lambda} \left( \varphi' \left( \frac{t^2 + (|x| - 1)^4}{R^4} \right) \right)^2 \]

\[ + \frac{16(\lambda + 2)}{R^8} (|x| - 1)^6 \left( \varphi \left( \frac{t^2 + (|x| - 1)^4}{R^4} \right) \right)^{\lambda+1} \varphi'' \left( \frac{t^2 + (|x| - 1)^4}{R^4} \right). \]

Thanks to the auxiliary properties

\[ \varphi' \left( \frac{t^2 + (|x| - 1)^4}{R^4} \right) \neq 0, \quad \varphi'' \left( \frac{t^2 + (|x| - 1)^4}{R^4} \right) \neq 0 \text{ and } |x| - 1 \leq R \]

for any \((t, x) \in Q^*_R\), we may conclude the estimates from (i) to (iii). Next, in order to verify the last estimate, one observes that

\[ \Delta (\Psi(x) \phi_R(t, x)) = \Delta \Psi(x) \phi_R(t, x) + \nabla \Psi(x) \cdot \nabla \phi_R(t, x) + \Psi(x) \Delta \phi_R(t, x) \]

\[ = \nabla \Psi(x) \cdot \nabla \phi_R(t, x) + \Psi(x) \Delta \phi_R(t, x) \]

because of the fact \(\Delta \Psi(x) = 0\). Let us divide our consideration into two cases as follows:

- If \(\beta \neq 0\), then we take \(\Psi(x)\) as in (18). By noticing that

\[ \nabla \Psi(x) = \begin{cases} \frac{x}{|x|^2} \quad & \text{if } d = 2, \\ (d - 2) \frac{x}{|x|^{d-1}} & \text{if } d \geq 3, \end{cases} \]

we derive

\[ \nabla \Psi(x) \cdot \nabla \phi_R(t, x) = \begin{cases} \frac{4(\lambda + 2)}{R^4} \left( \frac{t^2 + (|x| - 1)^4}{R^4} \right)^{\lambda+1} \times \varphi' \left( \frac{t^2 + (|x| - 1)^4}{R^4} \right) & \text{if } d = 2, \\ \frac{4(d - 2)(\lambda + 2)}{R^4} \left( \frac{|x| - 1}{|x|^{d-1}} \right) \left( \frac{t^2 + (|x| - 1)^4}{R^4} \right)^{\lambda+1} \times \varphi' \left( \frac{t^2 + (|x| - 1)^4}{R^4} \right) & \text{if } d \geq 3. \end{cases} \]

Thus, it implies

\[ \left| \nabla \Psi(x) \cdot \nabla \phi_R(t, x) \right| \lesssim R^{-2} \Psi(x) \left( \phi_R^*(t, x) \right)^{\frac{\lambda+1}{2+\lambda}} \]
by applying the elementary inequalities
\[
\begin{cases}
1 - \frac{1}{|x|} \leq \log |x| & \text{if } d = 2, \\
|x|^{d-1} + 1 \geq 2|x| & \text{if } d \geq 3,
\end{cases}
\]
for any $|x| \geq 1$.

- If $\beta = 0$, then we take $\Psi(x)$ as in (19). Obviously, one sees that $\nabla \Psi(x) = 0$, which entails immediately
\[
\Delta \left( \Psi(x)\phi_R(t,x) \right) = \Psi(x)\Delta \phi_R(t,x).
\]
Finally, using again the estimate (iii), it is clear to obtain the estimate (iv). Hence, our proof is complete. \hfill \Box

In the following proof, we will utilize the test functions as well as the notations defined in Sect. 4.1. Moreover, we denote by $C_j$ with $j \in \mathbb{N}$ positive constants independent of $R$ and $\varepsilon$.

4.2. Proof of Theorem 2

At first, we introduce the following test function:
\[
\Phi_R = \Phi_R(t,x) := \Psi(x)\phi_R(t,x),
\]
which enjoys the conditions
\[
\Phi_R \in C^2([0, T) \times \Omega), \quad \text{supp}\Phi_R \subset [0, T) \times \overline{\Omega} \quad \text{and} \quad \left( \alpha \frac{\partial \Phi_R}{\partial n} + \beta \Phi_R \right)(t, \cdot) \big|_{\partial \Omega} = 0.
\]
We define the following functionals with $\ell = 1, 2, \ldots, k - 1$:
\[
\mathcal{I}_R[u_\ell] := \int_0^T \int_\Omega |u_\ell(t,x)|^{p_\ell+1} \Phi_R(t,x) \, dx \, dt = \int_{\mathcal{Q}_R} |u_\ell(t,x)|^{p_\ell+1} \Phi_R(t,x) \, d(t,x)
\]
and
\[
\mathcal{I}_R[u_k] := \int_0^T \int_\Omega |u_k(t,x)|^{p_1} \Phi_R(t,x) \, dx \, dt = \int_{\mathcal{Q}_R} |u_k(t,x)|^{p_1} \Phi_R(t,x) \, d(t,x).
\]
Let us assume that $(u_1, u_2, \ldots, u_k) = (u_1(t,x), u_2(t,x), \ldots, u_k(t,x))$ is a weak solution to (1) in the sense of Definition 3. By plugging $\Phi(t,x) = \Phi_R(t,x)$ into (13) and (14), taking integration by parts then we obtain
\[
\mathcal{I}_R[u_k] + \varepsilon \int_\Omega (u_{0,1} + u_{1,1}) \Phi_R(0,x) dx
\]
\[
= \int_{\mathcal{Q}_R} u_1(t,x) \left( \partial_t^2 \Phi_R(t,x) - \Delta \Phi_R(t,x) - \partial_t \Phi_R(t,x) \right) d(t,x), \quad (24)
\]
and

\[
\mathcal{I}_R[u_\ell] + \varepsilon \int_{\Omega} \left( u_{0,\ell+1}(x) + u_{1,\ell+1}(x) \right) \Phi_R(0, x) \, dx
\]

\[
= \int_{Q_R} u_{\ell+1}(t, x) \left( \partial_t^2 \Phi_R(t, x) - \Delta \Phi_R(t, x) - \partial_t \Phi_R(t, x) \right) \, d(t, x)
\]  

with \( \ell = 1, 2, \ldots, \kappa - 1 \), respectively. By the assumption (16), the Lebesgue convergence theorem implies that

\[
\lim_{R \to \infty} \int_{\Omega} \left( u_{0,\ell}(x) + u_{1,\ell}(x) \right) \Phi_R(0, x) \, dx
\]

\[
= \lim_{R \to \infty} \int_{\Omega} \left( u_{0,\ell}(x) + u_{1,\ell}(x) \right) \psi(x) \phi_R(0, x) \, dx
\]

\[
= \int_{\Omega} \left( u_{0,\ell}(x) + u_{1,\ell}(x) \right) \psi(x) \, dx
\]

for any \( \ell = 1, 2, \ldots, k \). Together with the conditions (16) and (17) for the initial data, we deduce that there exists a sufficiently large constant \( R_0 \) so that it holds

\[
\int_{\Omega} \left( u_{0,\ell}(x) + u_{1,\ell}(x) \right) \Phi_R(0, x) \, dx \geq C_\ell^0
\]

for any \( R > R_0 \), where \( C_\ell^0 \) are suitable positive constants with \( \ell = 1, 2, \ldots, k \). Thus, it entails immediately from (24) and (25) that

\[
\mathcal{I}_R[u_\ell] + C_\ell^0 \varepsilon \leq \int_{Q_R} u_1(t, x) \left( \partial_t^2 \Phi_R(t, x) - \Delta \Phi_R(t, x) - \partial_t \Phi_R(t, x) \right) \, d(t, x)
\]

and

\[
\mathcal{I}_R[u_\ell] + C_{\ell+1}^0 \varepsilon \leq \int_{Q_R} u_{\ell+1}(t, x) \left( \partial_t^2 \Phi_R(t, x) - \Delta \Phi_R(t, x) - \partial_t \Phi_R(t, x) \right) \, d(t, x)
\]

with \( \ell = 1, 2, \ldots, \kappa - 1 \). Employing Lemma 2 gives the following estimate:

\[
\left| \partial_t^2 \Phi_R(t, x) - \Delta \Phi_R(t, x) - \partial_t \Phi_R(t, x) \right|
\]

\[
= \left| \psi(x) \partial_t^2 \phi_R(t, x) - \Delta \left( \psi(x) \phi_R(t, x) \right) - \psi(x) \partial_t \phi_R(t, x) \right|
\]

\[
\lesssim R^{-2} \psi(x) \left( \phi_R^*(t, x) \right)^{\frac{\lambda}{\kappa+2}}
\]

due to the relation \( 0 < \phi_R^*(t, x) < 1 \) for any \( R > R_0 \). Hence, one achieves

\[
\mathcal{I}_R[u_\ell] + C_\ell^0 \varepsilon \lesssim R^{-2} \int_{Q_R} \left| u_1(t, x) \right| \psi(x) \left( \phi_R^*(t, x) \right)^{\frac{\lambda}{\kappa+2}} \, d(t, x).
\]

The application of Hölder’s inequality implies
\[ \mathcal{I}_R[u_k] + C_1^0 \theta \lesssim R^{-2} \left( \int_{Q_R^*} \Psi(x) d(t, x) \right)^{1/p_2} \]

\[ \times \left( \int_{Q_R} |u_1(t, x)|^{p_2} \Psi(x) (\phi_R^*(t, x))^{p \lambda_{2}} d(t, x) \right)^{1/p_2} \]

\[ \lesssim \Theta_{p_2}(R) \left( \int_{Q_R} |u_1(t, x)|^{p_2} \Psi(x) (\phi_R^*(t, x))^{p \lambda_{2}} d(t, x) \right)^{1/p_2}, \]

where the function \( \Theta_{p_2} = \Theta_{p_2}(R) \) is defined by

\[ \Theta_{p_2}(R) = \begin{cases} R^{2 - \frac{4}{p_2} (\log R)^{1 - \frac{1}{p_2}}} & \text{if } d = 2 \text{ and } \beta \neq 0, \\ R^{d - \frac{4 + 2}{p_2}} & \text{if } d \geq 3 \text{ and } \beta \neq 0, \\ R^{d - \frac{4 + 2}{p_2}} & \text{if } d \geq 2 \text{ and } \beta = 0. \end{cases} \] (26)

Here we notice that to derive the previous inequality, we have used the following estimate:

\[ \int_{Q_R^*} \Psi(x) d(t, x) \lesssim \begin{cases} R^2(R + 1)^2 \log(R + 1) \approx R^4 \log R & \text{if } d = 2 \text{ and } \beta \neq 0, \\ R^2(R + 1)^d \approx R^{d + 2} & \text{if } d \geq 3 \text{ and } \beta \neq 0, \\ R^2(R + 1)^d \approx R^{d + 2} & \text{if } d \geq 2 \text{ and } \beta = 0. \end{cases} \]

for any \( R > R_0 \). Carrying out similar calculations, one finds

\[ \mathcal{I}_R[u_1] + C_2^0 \theta \lesssim \Theta_{p_3}(R) \left( \int_{Q_R} |u_2(t, x)|^{p_3} \Psi(x) (\phi_R^*(t, x))^{p \lambda_{3}} d(t, x) \right)^{1/p_3}, \]

\[ \mathcal{I}_R[u_2] + C_3^0 \theta \lesssim \Theta_{p_4}(R) \left( \int_{Q_R} |u_3(t, x)|^{p_4} \Psi(x) (\phi_R^*(t, x))^{p \lambda_{4}} d(t, x) \right)^{1/p_4}, \]

\[ \vdots \]

\[ \mathcal{I}_R[u_{k-1}] + C_{k-1}^0 \theta \lesssim \Theta_{p_k}(R) \left( \int_{Q_R} |u_{k-1}(t, x)|^{p_k} \Psi(x) (\phi_R^*(t, x))^{p \lambda_{k}} d(t, x) \right)^{1/p_k}, \]

where the functions \( \Theta_{p_{\ell}} = \Theta_{p_{\ell}}(R) \) with \( \ell = 1, 2, \ldots, k \) are defined as in (26) by substituting \( p_{\ell} \) for \( p_2 \). Let us now choose the parameter \( \lambda \) fulfilling

\[ \lambda \geq \max_{1 \leq \ell \leq k} \frac{2}{p_{\ell} - 1} = \frac{2}{\min_{1 \leq \ell \leq k} p_{\ell} - 1}, \quad \text{i.e.}, \quad \min_{1 \leq \ell \leq k} \frac{p_{\ell} \lambda}{\lambda + 2} \geq 1 \]

so that we may arrive at the following relations:
Without loss of generality, we assume that \( \gamma \). Let us now separate our considerations into the following two cases:

\[
\mathcal{I}_R[u_k] + C_k^0 \varepsilon \lesssim \Theta_{p_k}(R) \left( \int_{Q_R} |u_1(t, x)|^{p_2} \Psi(x) \phi_R^*(t, x) d(t, x) \right) \frac{1}{p_2},
\]

\[
\mathcal{I}_R[u_1] + C_2^0 \varepsilon \lesssim \Theta_{p_3}(R) \left( \int_{Q_R} |u_2(t, x)|^{p_3} \Psi(x) \phi_R^*(t, x) d(t, x) \right) \frac{1}{p_3},
\]

\[
\vdots
\]

\[
\mathcal{I}_R[u_{k-2}] + C_{k-1}^0 \varepsilon \lesssim \Theta_{p_k}(R) \left( \int_{Q_R} |u_{k-1}(t, x)|^{p_k} \Psi(x) \phi_R^*(t, x) d(t, x) \right) \frac{1}{p_k},
\]

\[
\mathcal{I}_R[u_{k-1}] + C_k^0 \varepsilon \lesssim \Theta_{p_1}(R) \left( \int_{Q_R} |u_{k}(t, x)|^{p_1} \Psi(x) \phi_R^*(t, x) d(t, x) \right) \frac{1}{p_1}. \tag{27}
\]

\[
\mathcal{I}_R[u_k] + C_k^0 \varepsilon \lesssim \Theta_{p_1}(R) \left( \int_{Q_R} |u_k(t, x)|^{p_1} \Psi(x) \phi_R^*(t, x) d(t, x) \right) \frac{1}{p_1}. \tag{28}
\]

Without loss of generality, we assume that \( \gamma_k = \max\{\gamma_1, \gamma_2, \ldots, \gamma_k\} \). A straightforward calculation gives

\[
\gamma_k = \frac{1 + p_k + p_{k-1} p_k + \cdots + p_2 p_3 \cdots p_k}{\prod_{\ell=1}^k p_\ell - 1}.
\]

From the condition (15), one gains

\[
\gamma_k \geq \frac{d}{2}, \quad \text{that is,} \quad \Gamma(d, p_1, p_2, \ldots, p_k) := \gamma_k - \frac{d}{2} \geq 0. \tag{29}
\]

Let us now separate our considerations into the following two cases:

- **Case 1:** \( d \geq 3 \) and \( \beta \neq 0 \). At first, let us turn our attention to the subcritical case \( \Gamma(d, p_1, p_2, \ldots, p_k) > 0 \). Now we plug the above chain of estimates into (28) successively to achieve

\[
\mathcal{I}_R[u_{k-1}] + C_k^0 \varepsilon \lesssim R^{d-\frac{d+2}{p_1}} \left( \mathcal{I}_R[u_k] \right)^{1/p_1} \quad \text{(since} \phi_R^*(t, x) \leq \phi_R(t, x) \text{in} Q_R)\]

\[
\lesssim R^{d-\frac{d+2}{p_1}} \left( R^{d-\frac{d+2}{p_2}} \left( \int_{Q_R} |u_1(t, x)|^{p_2} \Psi(x) \phi_R^*(t, x) d(t, x) \right) \right)^{1/p_2} \frac{1}{p_1} \]

\[
= R^{d-\frac{2}{p_1}-\frac{d+2}{p_1 p_2}} \left( \int_{Q_R} |u_1(t, x)|^{p_2} \Psi(x) \phi_R^*(t, x) d(t, x) \right)^{1/p_1} \frac{1}{p_2}
\]

\[
\lesssim R^{d-2\left( \frac{1}{p_1} + \frac{1}{p_1 p_2} + \ldots + \frac{1}{p_1 p_2 \cdots p_{k-1}} \right)} - R^{d+2\left( \frac{1}{p_1 p_2 \cdots p_k} \mathcal{I}_R[u_{k-1}] \right)} \frac{1}{p_1 p_2 \cdots p_k}
\]

\[
\vdots
\]

\[
\lesssim R^{d-2\left( \frac{1}{p_1} + \frac{1}{p_1 p_2} + \ldots + \frac{1}{p_1 p_2 \cdots p_{k-1}} \right)} - R^{d+2\left( \frac{1}{p_1 p_2 \cdots p_k} \mathcal{I}_R[u_{k-1}] \right)} \frac{1}{p_1 p_2 \cdots p_k}, \tag{30}
\]
Thus, the employment of the elementary inequality

\[ C_{k}^{0} \leq R^{-2\left(\frac{1}{p_{1}} + \frac{1}{p_{1}p_{2}} + \cdots + \frac{1}{p_{1}p_{2} \cdots p_{k}}\right) + d\left(1 - \frac{1}{p_{1}p_{2} \cdots p_{k}}\right) I_{R}[u_{k-1}] \frac{1}{p_{1}p_{2} \cdots p_{k}}} - I_{R}[u_{k-1}]. \] (31)

implies immediately

\[ R^{-2\left(\frac{1}{p_{1}} + \frac{1}{p_{1}p_{2}} + \cdots + \frac{1}{p_{1}p_{2} \cdots p_{k}}\right) + d\left(1 - \frac{1}{p_{1}p_{2} \cdots p_{k}}\right) I_{R}[u_{k-1}] \frac{1}{p_{1}p_{2} \cdots p_{k}}} - I_{R}[u_{k-1}] \leq R^{-2\frac{1 + p_{k} + \cdots + p_{2}p_{3} \cdots p_{k}}{p_{1}p_{2} \cdots p_{k} - 1} + d} \] (32)

for all \( R > R_{0} \). Linking (31) and (32) we deduce

\[ C_{k}^{0} \leq C_{1}R^{-2\frac{1 + p_{k} + \cdots + p_{2}p_{3} \cdots p_{k}}{p_{1}p_{2} \cdots p_{k} - 1} + d} \] (33)

for all \( R > R_{0} \). Because of the assumption \( \Gamma(d, p_{1}, p_{2}, \ldots, p_{k}) > 0 \), letting \( R \to \sqrt{T_{\varepsilon}} \) in (33) we obtain

\[ T_{\varepsilon} \leq C_{2}\varepsilon^{-\Gamma(d, p_{1}, p_{2}, \ldots, p_{k})^{-1}}. \]

This is the third estimate for lifespan of solutions what we wanted to prove. The next step is to focus on the critical case \( \Gamma(d, p_{1}, p_{2}, \ldots, p_{k}) = 0 \), where there exist two exponents \( p_{j_{1}} \neq p_{j_{2}} \) with \( j_{1}, j_{2} \in \{1, 2, \ldots, k\} \) and \( j_{1} \neq j_{2} \). Then, we deduce that there exists \( j \in \{1, 2, \ldots, k-1\} \) such that \( \gamma_{k} > \gamma_{j} \), which follows

\[ p_{k} > 1 + \frac{2}{d}. \] (34)

In an analogous manner to (30), one may arrive at the following estimate:

\[ I_{R}[u_{k-1}] + C_{k}^{0} \leq R^{-2\left(\frac{1}{p_{1}} + \frac{1}{p_{1}p_{2}} + \cdots + \frac{1}{p_{1}p_{2} \cdots p_{k}}\right) + d\left(1 - \frac{1}{p_{1}p_{2} \cdots p_{k}}\right)} \times \left(\int_{Q_{R}} |u_{k-1}(t, x)|^{p_{k}} \Psi(x) \phi_{R}^{*}(t, x) d(t, x)\right) \frac{1}{p_{1}p_{2} \cdots p_{k}} \]

\[ = \left(\int_{Q_{R}} |u_{k-1}(t, x)|^{p_{k}} \Psi(x) \phi_{R}^{*}(t, x) d(t, x)\right) \frac{1}{p_{1}p_{2} \cdots p_{k}}, \] (35)

where we note that the assumption \( \Gamma(d, p_{1}, p_{2}, \ldots, p_{k}) = 0 \) is used. Let us now define the following auxiliary functionals:

\[ h_{p_{k}} = h_{p_{k}}(r) = \int_{Q_{R}} |u_{k-1}(t, x)|^{p_{k}} \Psi(x) \phi_{R}^{*}(t, x) d(t, x) \]
and

\[ \mathcal{H}_{p_k} = \mathcal{H}_{p_k}(R) = \int_0^R h_{p_k}(r)r^{-1}dr. \]

On the one hand, carrying out the change of variable \( \rho = \frac{t^2 + (|x| - 1)^4}{r^4} \) one derives

\[
\mathcal{H}_{p_k}(R) = \int_0^R \left( \int_0^{T_x} \int_\Omega |u_{p_k-1}(t, x)|^{p_k} \Psi(x) \phi^+_R(t, x)d(t, x) \right) r^{-1}dr
\]

\[
= \frac{1}{4} \int_{Q_R} |u_{p_k-1}(t, x)|^{p_k} \Psi(x) \left( \int_{R^4}^\infty \frac{(\varphi^+(\rho))^\lambda + 2}{\rho^1} d\rho \right) d(t, x)
\]

\[
\leq \frac{1}{4} \int_{Q_R} |u_{p_k-1}(t, x)|^{p_k} \Psi(x) \left( \int_{1/2}^1 (\varphi^+(\rho))^\lambda + 2 d\rho \right) d(t, x), \quad (36)
\]

where the support condition for \( \varphi^+(\rho) \) is applied to (36). On the other hand, using the property \( \varphi^+(\rho) \equiv \varphi(\rho) \) for any \( \rho \in [1/2, 1] \) in (36) we have

\[
\mathcal{H}_{p_k}(R) \leq \frac{1}{4} \int_{Q_R} |u_{p_k-1}(t, x)|^{p_k} \Psi(x) \sup_{\rho \in (0, R)} \left( \varphi \left( \frac{t^2 + (|x| - 1)^4}{r^4} \right) \right)^{\lambda + 2}
\]

\[
\times \left( \int_{1/2}^1 \rho^{-1}d\rho \right) d(t, x)
\]

\[
\leq \frac{\log 2}{4} \int_{Q_R} |u_{p_k-1}(t, x)|^{p_k} \Psi(x) \left( \varphi \left( \frac{t^2 + (|x| - 1)^4}{R^4} \right) \right)^{\lambda + 2} d(t, x)
\]

\[
= \frac{\log 2}{4} \int_{Q_R} |u_{p_k-1}(t, x)|^{p_k} \Psi(x) \varphi_R(t, x)d(t, x)d(t, x)
\]

\[
= \frac{\log 2}{4} \mathcal{I}_R[u_{p_k-1}] . \quad (37)
\]

Moreover, it is obvious to recognize the relation

\[ h_{p_k}(R) = R\mathcal{H}'_{p_k}(R). \quad (38) \]

Hence, collecting all the obtained estimates from (35), (37) and (38) we conclude

\[
\frac{4}{\log 2} \mathcal{H}_{p_k}(R) + C^0_k \varepsilon \lesssim \left( R\mathcal{H}'_{p_k}(R) \right)^{\frac{1}{p_1p_2...p_k}},
\]

that is,

\[
R^{-1} \leq C_3 \left( \frac{4}{\log 2} \mathcal{H}_{p_k}(R) + C^0_k \varepsilon \right)^{-p_1p_2...p_k} \mathcal{H}'_{p_k}(R).
\]

By considering \( R > R^2_0 \) and taking integration of both the sides of the above estimate over \([\sqrt{R}, R]\) one finds
\[ \frac{1}{2} \log R \leq C_3 \int_{\sqrt{R}}^R \left( \frac{4}{\log 2} \mathcal{H}_{p_k}(\tau) + C_k^0 \varepsilon \right)^{-p_1 p_2 \cdots p_k} \mathcal{H}_p' (\tau) d\tau \]
\[ \leq \frac{C_3 \log 2}{4(1 - p_1 p_2 \cdots p_k)} \left( \frac{4}{\log 2} \mathcal{H}_{p_k}(\tau) + C_k^0 \varepsilon \right)^{1 - p_1 p_2 \cdots p_k} \bigg|_{\tau=\sqrt{R}} \]
\[ \leq \frac{C_3 \log 2}{4(p_1 p_2 \cdots p_k - 1)} \left( \frac{4}{\log 2} \mathcal{H}_{p_k}(\sqrt{R}) + C_k^0 \varepsilon \right)^{1 - p_1 p_2 \cdots p_k}. \] (39)

Furthermore, combining the estimates (27) and (38) we arrive at
\[ C_{k-1}^0 \varepsilon \lesssim \Theta_{p_k}(R) \left( R \mathcal{H}_p' (R) \right)^{\frac{1}{p_k}}, \]
which is equivalent to
\[ (C_{k-1}^0)^{p_k} \varepsilon^{p_k} R^{1 + d - p_k d} \lesssim \mathcal{H}_p' (R) \]
After the integration of the previous inequality over \([R_0, \sqrt{R}]\), one catches
\[ \mathcal{H}_{p_k}(\sqrt{R}) \geq C_4 (C_{k-1}^0)^{p_k} \varepsilon^{p_k} \int_{R_0}^{\sqrt{R}} \tau^{1 + d - p_k d} d\tau \geq C_5 \varepsilon^{p_k} \] (40)
since we are in the situation \(1 + d - p_k d < -1\) by (34). So, from (39) and (40) we claim that
\[ \log \sqrt{T_\varepsilon} = \lim_{R \uparrow \sqrt{T_\varepsilon}} \log R \leq C_6 \varepsilon^{-(p_1 p_2 \cdots p_k - 1)}. \]

This is to indicate the next desired estimate for lifespan of solutions. Finally, we will give our verification to the critical case \(\Gamma(d, p_1, p_2, \ldots, p_k) = 0\) when \(p_1 = p_2 = \cdots = p_k\). It follows that
\[ p_1 = p_2 = \cdots = p_k = 1 + \frac{2}{d} =: p^*. \]

Obviously, one should realize that the following relation holds:
\[ \partial_t^2 (u_1 + u_2 + \cdots + u_k) - \Delta (u_1 + u_2 + \cdots + u_k) + \partial_t (u_1 + u_2 + \cdots + u_k) \]
\[ = |u_1|^{p^*} + |u_2|^{p^*} + \cdots + |u_k|^{p^*} \geq C_7 |u_1 + u_2 + \cdots + u_k|^{p^*}. \] (41)

For this reason, we consider the treatment of the system (1) as that of the single equations (3) and (5) accompanied with the zero Robin boundary condition. Then, following the same approach as in the paper [20] we may arrive at
\[ T_\varepsilon \leq \exp \left( C_8 \varepsilon^{-(p^* - 1)} \right). \]

- **Case 2**: \(d = 2\) and \(\beta \neq 0\). Repeating an argument as we did in Case 1 we give the following estimates:
\[ I_R[u_{k-1}] + C_0 \varepsilon \]
\[ \lesssim R^{-2} \left( \frac{4}{p} \left( \log R \right)^{1 - \frac{1}{p}} \left( I_R[u_k] \right)^{\frac{1}{p}} \right) \quad \text{(since } \phi_{R}^{*}(t,x) \leq \phi_R(t,x) \text{ in } Q_R) \]
\[ \lesssim R^{-2} \left( \frac{4}{p} \left( \log R \right)^{1 - \frac{1}{p}} \left( \int_{Q_R} \left| u_1(t,x) \right|^{p^2} \Psi(x) \phi_R^{*}(t,x) d(t,x) \right)^{\frac{1}{p^2}} \right) \]
\[ = R^{-2} \left( \frac{4}{p_1 p_2 (\log R)^{1 - \frac{1}{p_1 p_2}}} \left( \int_{Q_R} \left| u_1(t,x) \right|^{p^2} \Psi(x) \phi_R^{*}(t,x) d(t,x) \right)^{\frac{1}{p_1 p_2}} \right) \]
\[ \lesssim R^{-2} \left( \frac{4}{p_1 p_2 (\log R)^{1 - \frac{1}{p_1 p_2}}} \left( \int_{Q_R} \left| u_{k-1}(t,x) \right|^{p_k} \Psi(x) \phi_R^{*}(t,x) d(t,x) \right)^{\frac{1}{p_1 p_2 - p_k}} \right) \]
\[ \quad \vdots \]
\[ \lesssim R^{-2} \left( \frac{4}{p_1 + \frac{1}{p_1 p_2} + \cdots + \frac{1}{p_1 p_2 - p_k}} \left( \int_{Q_R} \left| u_{k-1}(t,x) \right|^{p_k} \Psi(x) \phi_R^{*}(t,x) d(t,x) \right)^{\frac{1}{p_1 p_2 - p_k}} \right) \]
\[ = R^{-2} \left( \frac{4}{p_1 + \frac{1}{p_1 p_2} + \cdots + \frac{1}{p_1 p_2 - p_k}} + 2 \left( 1 - \frac{1}{p_1 p_2 - p_k} \right) \left( \log R \right)^{1 - \frac{1}{p_1 p_2 - p_k}} \right) \]
\[ \int_{Q_R} \left| u_{k-1}(t,x) \right|^{p_k} \Psi(x) \phi_R^{*}(t,x) d(t,x) \right) \frac{1}{p_1 p_2 - p_k} \]. \hspace{1cm} (42)

We pay attention that the subcritical case is equivalent to
\[ \Gamma(2, p_1, p_2, \ldots, p_k) > 0. \]

Then, the immediate employment of Lemma 4 of the “Appendix” to (42) with the function \( \eta(t,x) = \left| u_{k-1}(t,x) \right|^{p_k} \Psi(x) \) and the parameters
\[ \omega = C_k \varepsilon, \quad \sigma = 2 \Gamma(2, p_1, p_2, \ldots, p_k), \quad \mu = 1 \text{ and } p = p_1 p_2 \ldots p_k \]
leads to
\[ \sqrt{T_\varepsilon} \leq C_9 \left( \varepsilon^{-1} \log \varepsilon^{-1} \right)^{\frac{1}{2 \Gamma(2, p_1, p_2, \ldots, p_k)}}, \]
which implies what we wanted to show. When the critical case occurs, i.e., \( \Gamma(2, p_1, p_2, \ldots, p_k) = 0 \), with \( p_1 = p_2 = \cdots = p_k \), we have \( p_1 = p_2 = \cdots = p_k = 2 \). Therefore, noticing again the relation (41) we may apply an analogous strategy used in the paper [20] to conclude
\[ T_\varepsilon \leq \exp \exp \left( C_{10} \varepsilon^{-1} \right). \]

- **Case 3:** \( d \geq 2 \) and \( \beta = 0 \). By carrying out the same procedure as we did in Case 1, one may conclude that all the desired estimates for lifespan in Theorem 2 hold.

Summarizing, our proof is completed.
5. Concluding remarks

**Remark 4.** In this paper, we have employed the test function method with a special choice of test functions, which approximates the harmonic functions fulfilling some of typical boundary conditions (the Dirichlet/Neumann/Robin boundary condition) on \( \partial \Omega \) for any \( d \geq 2 \) separately, to indicate both the blow-up result in a finite time and the sharp upper bound of lifespan estimates for small solutions to (1). This remark tells us that the used method is also applicable to weakly coupled systems of semi-linear classical wave equations with double damping terms in an exterior domain as follows:

\[
\begin{align*}
\partial_t^2 u_1(t, x) - \Delta u_1(t, x) + \partial_t u_1(t, x) - \Delta \partial_t u_1(t, x) &= |u_k(t, x)|^{p_1}, \\
(t, x) &\in (0, T) \times \Omega,
\end{align*}
\]

\[
\begin{align*}
\partial_t^2 u_2(t, x) - \Delta u_2(t, x) + \partial_t u_2(t, x) - \Delta \partial_t u_2(t, x) &= |u_1(t, x)|^{p_2}, \\
(t, x) &\in (0, T) \times \Omega,
\end{align*}
\]

\[
\vdots
\]

\[
\begin{align*}
\partial_t^2 u_k(t, x) - \Delta u_k(t, x) + \partial_t u_k(t, x) - \Delta \partial_t u_k(t, x) &= |u_{k-1}(t, x)|^{p_k}, \\
(t, x) &\in (0, T) \times \Omega,
\end{align*}
\]

\[
\alpha \left. \frac{\partial u_\ell}{\partial n^+} (t, x) + \beta u_\ell (t, x) \right. = 0, \quad (t, x) \in (0, T) \times \partial \Omega,
\]

\[
u_\ell (0, x) = \varepsilon u_{0, \ell} (x), \quad \partial_t u_\ell (0, x) = \varepsilon u_{1, \ell} (x), \quad x \in \Omega,
\]

with \( \ell = 1, 2, \ldots, k \) and for any \( d \geq 2 \). Following an analogous approach to this paper we may claim that the statements of Theorem 1 and Theorem 2 still remain valid to \( 43 \). Additionally, in term of studying the special case of \( 43 \) when \( k = 1 \) and \( \alpha = 0 \) we refer the interested readers to the recent paper of D’Abbicco-Ikehata-Takeda [6]. However, their paper did not report any information about lifespan estimates for solutions, especially, a blow-up result for the critical case \( p = 2 \) with \( d = 2 \) was also not included. For this reason, we can say that our results are not only to fill this lack of the cited paper in the case of the Dirichlet boundary condition but also to extend the further results for the Neumann and Robin boundary conditions.

**Remark 5.** Speaking about a desired result for estimating upper bound of lifespan of solutions to (1) in one-dimensional case, let’s take the harmonic function \( \Psi = \Psi (x) \) given by

\[
\Psi (x) = \begin{cases} 
|x| - 1 + \frac{\alpha}{\beta} & \text{if } \beta \neq 0, \\
1 & \text{if } \beta = 0,
\end{cases}
\]

in place of (18) and (19), respectively. By carrying out several straightforward calculations as we did in Lemma 2, one sees that all the auxiliary estimates in Lemma 2 are also true for \( d = 1 \). Then, with the same assumptions as in Theorem 2, we may repeat some similar steps to those in the proof of Theorem 2 for Case 1 to conclude
the following estimates for $d = 1$:

\[
\text{LifeSpan}(u_1, u_2, \ldots, u_k) \leq \begin{cases} 
C\varepsilon^{-(\max\{\gamma_1, \gamma_2, \ldots, \gamma_k\} - 1)^{-1}} & \text{if } \max\{\gamma_1, \gamma_2, \ldots, \gamma_k\} > 1, \beta \neq 0, \\
C\varepsilon^{-(\max\{\gamma_1, \gamma_2, \ldots, \gamma_k\} - 1/2)^{-1}} & \text{if } \max\{\gamma_1, \gamma_2, \ldots, \gamma_k\} > 1/2, \beta = 0.
\end{cases}
\]  

(44)

To establish this, we determine the function $\Theta_{p,\ell} = \Theta_{p,\ell}(R)$ by

\[
\Theta_{p,\ell}(R) = \begin{cases} 
R^{2-(\frac{4}{p,\ell}} & \text{if } \beta \neq 0, \\
R^{1-(\frac{3}{p,\ell}} & \text{if } \beta = 0,
\end{cases}
\]

with $\ell = 1, 2, \ldots, k$ instead of (26), which comes from the following observations:

\[
\int_{Q^*_R} \Psi(x) d(t, x) \lesssim \begin{cases} 
R^2 (R + 1)^2 \approx R^4 & \text{if } \beta \neq 0, \\
R^2 (R + 1) \approx R^3 & \text{if } \beta = 0,
\end{cases}
\]

when $d = 1$. One recognizes that if we replace $k = 1$, then the relation $\max\{\gamma_1, \gamma_2, \ldots, \gamma_k\} > 1$ is equivalent to $p_1 < 2$ for $d = 1$. Obviously, there is a gap between 2 and $p_{\text{Fuj}}(1) = 3$ so that the first obtained upper bound in (44) seems to be not sharp when the Dirichlet boundary condition is considered, i.e., $\alpha = 0$ and $\beta \neq 0$. This phenomenon is reasonable due to the fact no manuscript has succeeded in indicating what a critical exponent is in this case so far (see also [16]). However, the remaining upper bound of lifespan in (44) is really sharp from the conclusion of its corresponding lower bound (see Proposition 2 in the “Appendix”).

Remark 6. As we can see in Theorem 2, an open problem, which should be recognized to explore in a further study, is to find out a suitable upper bound in estimating the lifespan of solutions to (1) with $\beta \neq 0$ when $d = 2$, where the critical case is of interest and the exponents $p_1, p_2, \ldots, p_k$ are not necessarily equal. Actually, the applied strategy in this case with $d \geq 3$ fails to $d = 2$. More precisely, in the same way as we estimated (42) one obtains

\[
\mathcal{I}_R[u_{k-1}] + C_k^0 \varepsilon \lesssim (\log R)^{1-p_1p_2\ldots p_k} \left( \int_{Q^*_R} |u_{k-1}(t, x)|^{p_k} \Psi(x) \phi^*_R(t, x)d(t, x) \right)^{\frac{1}{p_1p_2\ldots p_k}}
\]

by observing that $\Gamma(2, p_1, p_2, \ldots, p_k) = 0$. Then, repeating some calculations to achieve (39) and (40) we conclude the following estimates:

\[
\int_{\sqrt{R}}^R \tau^{-1}(\log \tau)^{1-p_1p_2\ldots p_k} d\tau \lesssim \left( \frac{4}{\log 2} \mathcal{H}_{p_k}(\sqrt{R}) + C_k^0 \varepsilon \right)^{1-p_1p_2\ldots p_k},
\]

\[
\mathcal{H}_{p_k}(\sqrt{R}) \gtrsim \varepsilon^{p_k} \int_{R_0}^{\sqrt{R}} \tau^{3-2p_k}(\log \tau)^{1-p_k} d\tau.
\]
It is clear that the condition $p_k > 2$ from (34) leads to the integrability of the functions in the above two integrals. Unfortunately, the combination of the previous two inequalities does not give any information about upper bound estimate for lifespan of solutions.

Acknowledgements

This research of Tuan Anh Dao was partly supported by Vietnam Ministry of Education and Training under Grant Number B2023-BKA-06. Masahiro Ikeda is supported by JST CREST Grant Number JPMJCR1913, Japan and Grant-in-Aid for Young Scientists Research (No. 19K14581), and Japan Society for the Promotion of Science.

Data availability My manuscript has no associated data. No new data were created during the study.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.

Appendix

Lemma 3. (Sobolev embedding) Let $\Omega$ be an open subset of $\mathbb{R}^d$ which has a Lipschitz continuous boundary. Let “$p \in [1, d)$ and $q \in [p, pd/(d - p)]$” or “$p = d$ and $q \in [p, \infty)$”. Then the embedding $W^{1,p} \hookrightarrow L^q(\Omega)$ holds.

For the proof of this lemma, see [1] or [22, Lemma 3.3].

Lemma 4. Let $\omega > 0$, $C_0 > 0$, $R_1 > 0$, $\sigma \geq 0$ and $\mu > 0$. We assume $0 \leq \eta = \eta(t, x) \in L^1_{\text{loc}}([0, T), L^1(\Omega))$ for $T > R_1$ satisfying the following inequality:

$$\omega + \int_{Q_R} \eta(t, x) \phi_R(t, x) \, d(t, x) \leq C_0 \left( \log R \right)^{\frac{\mu}{p'}} \left( \int_{Q_R} \eta(t, x) \phi_R^\sigma(t, x) \, d(t, x) \right)^{\frac{1}{p'}}$$
for any \( R \in \left[R_1, \sqrt{T}\right) \), where \( \phi_R(t, x), \phi^*_R(t, x) \) and \( Q_R \) are introduced as in Sect. 4.1. Then, the following estimates hold:

\[
\sqrt{T} \leq \begin{cases}
C \omega^{-\frac{1}{2}} \left( \log(\omega^{-1})\right)^{\frac{\mu}{\sigma}} & \text{if } \sigma > 0 \text{ and } \mu > 0, \\
\exp \left( C \omega^{\frac{1}{2}} \log(p-1) \right) & \text{if } \sigma = 0 \text{ and } 0 < \mu < \frac{1}{p-1}, \\
\exp \left( C \omega^{-p(1+)} \right) & \text{if } \sigma = 0 \text{ and } \mu = \frac{1}{p-1},
\end{cases}
\]

where \( C \) is a positive constant independent of \( \omega \).

**Proof.** We follow the proof of Lemma 2.2 in [20] with a minor modification to conclude the desired estimates above. \( \square \)

**Proof of Proposition 1.** The arguments in this proof follow from [45, Proposition 9.16] and [23, Proposition 3.1]. Moreover, we are going to use some of the same notations as in Sect. 2.1.

**Proof.** Let \( T > 0 \) and \( u \) be a mild solution on \([0, T)\) in the sense of Definition 1. Let \( \Phi = \Phi(t, x) \in C^2([0, T) \times \Omega) \) be a test function with \( \text{supp} \Phi \subset [0, T) \times \Omega \) such that \( \left( \frac{\alpha}{\partial_n^+} + \beta \Phi \right)(t, \cdot) \right|_{\partial \Omega} = 0 \). Let \( U := (u, \partial_t u)^c \). Since \( U \in C([0, T); E) \), we see that \( \mathcal{N}(U) \in C([0, T); L^2(\Omega)) \) by the Sobolev embedding. Moreover, let \( U_0 := (u_0, u_1) \). Since \( U_0 \in E \), we can take sequences \( \{U^j\}_{j=1}^\infty \subset H^2(\Omega) \times H^1(\Omega) \) and \( \{N^j\}_{j=1}^\infty \subset C([0, T); E) \) such that

\[
U^j_0 \to U_0 \text{ in } E, \quad N^j \to N(U) \text{ in } C([0, T); L^2(\Omega)) \text{ as } j \to \infty.
\]

Let

\[
U^j \in \bigcap_{m=0}^2 C^{2-m}([0, T); H^{2-m}(\Omega))
\]

be the corresponding solution to (8) with \( U^j_0 \) and \( N^j \). Then we can see that \( U^j \to U \) in \( C([0, T); E) \) as \( j \to \infty \). Since each \( U^j \) satisfies the equation

\[
\partial_t U^j(t, x) - B_{u, \beta} U^j(t, x) = N^j(t, x)
\]

in the sense of \( E \). Next multiplying the equivalent equation (1) for \( u^j \) by the test function \( \Phi \), taking integration by parts and then letting \( j \to \infty \), we deduce that \( U \) satisfies the equations (13) and (14). This means that \( u \) becomes a weak solution in the sense of Definition 3.

Let us now assume that \( u \) is a weak solution on \([0, T)\) in the sense of Definition 3. Let \( \tilde{U} \in C([0, T); E) \) be the mild solution in the sense of Definition 1 with initial data \( U_0 \in E \). We see that \( \tilde{U} \) becomes a weak solution in the sense of Definition 3. By considering the difference of the equations, one claims that the identity

\[
\int_0^T \int_{\Omega} (u_\ell - \tilde{u}_\ell)(\partial_t^2 \Phi - \Delta \Phi - \partial_t \Phi) dx dt = 0
\]
holds for any $\ell = 1, \ldots, k$. From this identity, we can see that for any $\chi \in C_0^\infty([0, T) \times \Omega)$, the identity
\[
\int_0^T \int_\Omega (u_{\ell} - \tilde{u}_{\ell}) \chi \, dx \, dt = 0
\]
holds, which implies that $u_{\ell} = \tilde{u}_{\ell}$ on a.e. $[0, T) \times \Omega$. Thus $u$ becomes a mild solution on $[0, T)$. \hfill \Box

**Proposition 2.** Let $\alpha, \beta \in \mathbb{R}$ satisfying $\alpha \beta \geq 0$. Assume that
\[
\min \{ p_1, p_2, \ldots, p_k \} \geq 2, \quad \max \{ \gamma_1, \gamma_2, \ldots, \gamma_k \} > 1/2
\]
and
\[
(u_{0, \ell}, u_{1, \ell}) \in (H^1_{\alpha, \beta}(\Omega) \cap L^1(\Omega)) \times (L^2(\Omega) \cap L^1(\Omega))
\]
for $\ell \in \{1, 2, \ldots, k\}$. Then there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, the following estimate holds:
\[
\text{LifeSpan}(u_1, u_2, \ldots, u_k) \geq C e^{-\left(\max \{ \gamma_1, \gamma_2, \ldots, \gamma_k \} - 1/2\right)^{-1}}.
\]

The proof of this proposition is done by combining the arguments [11, Proposition 1.3] and [22, Lemmas 5.2, 5.3] with the linear estimates [22, Theorem 2.2]. Again, we are going to use some of the same notations as in Sect. 2.1.

**Proof.** Let $u$ be a mild solution on $[0, T_{\varepsilon})$. Then for $\ell \in \{1, \ldots, k\}$, the identity
\[
u_{\ell}(t) = u_{\ell}^{\text{lin}}(t) + \int_0^t \mathcal{D}(t - \tau, -\partial_x^2)(|u_{\ell-1}(\tau, x)|^{p_\ell}) \, d\tau
\]
holds, where $\mathcal{D}(t, -\partial_x^2)$ is the solution operator defined by [22, (1.5)] and $u_{\ell}^{\text{lin}}$ stands for the solution to the corresponding linear equations of (1) with initial data $(u_{\ell}^{\text{lin}}(0), \partial_t u_{\ell}^{\text{lin}}(0)) = (u_{0, \ell}, u_{1, \ell})$. Moreover, we have set $u_0 := u_k$ in the above identity. Let us now introduce
\[
k_{\ell} := \gamma_{\ell} - \max \{ \gamma_1, \gamma_2, \ldots, \gamma_k \} + 1/2 \leq 1/2
\]
for $\ell \in \{1, \ldots, k\}$, and define
\[
M(t) := \sup_{\tau \in [0, t]} \sum_{\ell=1}^k \left\{ (1 + \tau)^{k_{\ell}-1/4} \| u_{\ell}(\tau) \|_{L^2} + (1 + \tau)^{k_{\ell}+1/4} \| \partial_x u_{\ell}(\tau) \|_{L^2}
\right.
\]
\[
+ (1 + \tau)^{k_{\ell}+3/4} \| \partial_t u_{\ell}(\tau) \|_{L^2}
\right\}.
\]

Following the arguments [22, Lemmas 5.2 and 5.3] with a slight modification, we can prove that there exist positive constants $C_0$ and $C_1$ such that the estimate
\[ M(t) \leq C_0 \| (u_0, u_1) \|_{(H^{1, \alpha}_w(\Omega) \cap L^1(\Omega))^k \times (L^2(\Omega) \cap L^1(\Omega))^k} + C_1 \max_{1 \leq \ell \leq k} (1 + t)^{(\max\{\gamma_1, \gamma_2, \ldots, \gamma_k\} - 1/2)(p_\ell - 1)} M(t)^{p_\ell} \]

holds for any \( t \in (0, T_\varepsilon) \). Thus, the conclusion follows from the same argument as the proof of [11, Proposition 1.3]. \( \square \)

**Remark 7.** This remark is to emphasize the following further two points: First, from the proof of Theorem 1 we concluded the lower bound estimate for lifespan (11), which is valid for any spatial dimension. However, there is a gap between (11) and the achieved upper bound estimates from Theorem 2. Secondly, by the other approach used in the proof of Proposition 2 we have arrived at the improved lower bound for lifespan in comparison with (11) to fill this gap when the subcritical case of one dimension and \( \beta \neq 0 \) is interest of. In this case, we would like to say that the estimates for lifespan of solutions are sharp. Unfortunately, the latter approach cannot be applied to treat all remaining cases due to the restriction of \( \min\{p_1, p_2, \ldots, p_k\} \geq 2 \) appearing in Proposition 2. Namely, let us consider the subcritical case with \( d = 2 \) and \( k = 1 \) as an illustrative example. Then, the subcritical condition \( \max\{\gamma_1, \gamma_2, \ldots, \gamma_k\} > d/2 \) is equivalent to \( p_1 < 1 + 2/d = 2 \), which leads to a contradiction to the condition \( \min\{p_1, p_2, \ldots, p_k\} = p_1 \geq 2 \). For these observations, it is challenging to find out sharp estimates for the lower bound of lifespan in the remaining cases.

**REFERENCES**

[1] Adams, R. A.: Sobolev spaces, Academic Press, New York, 1975.

[2] Arendt, W., Urban, K.: Partielle Differenzialgleichungen, Spektrum Akademischer Verlag, 2010. https://doi.org/10.1007/978-3-8274-2237-8

[3] Baras, P., Pierre, M., Critère d’existence de solutions positives pour des équations semi-linéaires non monotones, Ann. Inst. H. Poincaré Anal. Non Linéaire, 2 (1985), 185–212.

[4] Chen, W., Dao, T.A.: Sharp lifespan estimates for the weakly coupled system of semilinear damped wave equations in the critical case, Math. Ann., 385 (2023), 101–130.

[5] Cazenave, T., Haraux, A.: An Introduction to Semilinear Evolution Equations, Oxford University Press, 1998.

[6] D’Abbicco, M., Ikehata, R. and Takeda, H.: Critical exponent for semi-linear wave equations with double damping terms in exterior domains, Nonlinear Differ. Equ. Appl., 26 (2019), 56.

[7] Dao, T.A., Reissig, M.: The interplay of critical regularity of nonlinearities in a weakly coupled system of semilinear damped wave equations, J. Differential Equations, 299 (2021), 1–32.

[8] Fäh, R.: Magnetic perturbations of the robin Laplacian in the strong coupling limit, arXiv:2111.01416.

[9] Fujita, H.: On the blowing up of solutions of the Cauchy problem for \( ut = \Delta u + |u|^{1+\alpha} \), J. Sci. Univ. Tokyo Sec. I, 13 (1966), 109–124.

[10] Fino, A.Z., Ibrahim, H. and Wehbe, A.: A blow-up result for a nonlinear damped wave equation in exterior domain: The critical case, Comput. Math. Appl., 73 (2017), 2415–2420.

[11] Fujiwara, K., Ikeda, M. and Wakasugi, Y.: Lifespan of solutions for a weakly coupled system of semi linear heat equations, Tokyo J. Math., 43 (2020), 163–180.

[12] Fujiwara, K., Ikeda, M. and Wakasugi, Y.: Estimates of lifespan and blow-up rates for the wave equation with a time-dependent damping and a power-type nonlinearity, Funkc. Ekvac., 62 (2019), 157–189.

[13] Hayakawa, K.: On nonexistence of global solutions of some semilinear parabolic differential equations, Proc. Japan Acad., 49 (1973), 503–505.
[14] Hayashi, N., Naumkin, P. I. and Tominaga, M.: Remark on a weakly coupled system of nonlinear damped wave equations, J. Math. Anal. Appl., 428 (2015), 490–501.

[15] Ikehata, R.: Small data global existence of solutions for dissipative wave equations in an exterior domain, Funkcial. Ekvac., 44 (2002), 259–269.

[16] Ikehata, R.: A remark on a critical exponent for the semilinear dissipative wave equation in the one dimensional half space, Differential Integral Equations, 16 (2003), 727–736.

[17] Ikehata, R.: Global existence of solutions for semilinear damped wave equation in 2-D exterior domain, J. Differential Equations, 200 (2004), 53–68.

[18] Ikehata, R.: Two dimensional exterior mixed problem for semilinear damped wave equations, J. Math. Anal. Appl., 301 (2005), 366–377.

[19] Ikeda, M., Ogawa, T.: Lifespan of solutions to the damped wave equation with a critical nonlinearity, J. Differential Equations, 261 (2016), 1880–1903.

[20] Ikeda, M., Sobajima, M.: Remark on upper bound for lifespan of solutions to semilinear evolution equations in a two-dimensional exterior domain, J. Math. Anal. Appl., 470 (2019), 318–326.

[21] Ikeda, M., Sobajima, M.: Sharp upper bound for lifespan of solutions to some critical semilinear parabolic, dispersive and hyperbolic equations via a test function method, Nonlinear Anal., 182 (2019), 57–74.

[22] Ikeda, M., Taniguchi, K. and Wakasugi, Y.: Global existence and asymptotic behavior for nonlinear damped wave equations on measure spaces, arxiv:2106.10322.

[23] Ikeda, M., Wakasugi, Y.: Small data blow-up of $L^2$-solution for nonlinear Schrödinger equation without gauge invariance, Differential Integral Equations, 23 (2013), 1275–1285.

[24] Ikeda, M., Wakasugi, Y.: A note on the lifespan of solutions to the semilinear damped wave equation, Proc. Amer. Math. Soc., 148 (2015), 157–172.

[25] Ikeda, M., Jleli, M. and Samet, B.: On the existence and nonexistence of global solutions for certain semilinear exterior problems with nontrivial Robin boundary conditions, J. Differential Equations, 269 (2020), 563–594.

[26] Jleli, M., Samet, B.: New blow-up results for nonlinear boundary value problems in exterior domains, Nonlinear Anal., 178 (2019), 348–365.

[27] Kirane, M., Qafsaoui, M.: Fujita’s exponent for a semilinear wave equation with linear damping, Adv. Nonlinear Stud., 2 (2002), 41–49.

[28] Lai, N., Yin, S.: Finite time blow-up for a kind of initial-boundary value problem of semilinear damped wave equation, Math. Methods Appl. Sci., 40 (2017), 1223–1230.

[29] Lai, N.A., Zhou, Y.: The sharp lifespan estimate for semilinear damped wave equation with Fujita critical power in higher dimensions, J. Math. Pures Appl. (9), 123 (2019), 229–243.

[30] Li, T. T., Zhou, Y.: Breakdown of solutions to $\Box u + u_t = |u|^{1+\alpha}$, Discrete Contin. Dyn. Syst., 1 (1995), 503–520.

[31] Mitidieri, E., Pohozaev, S.I.: A priori estimates and blow-up of solutions to nonlinear partial differential equations and inequalities, Proc. Steklov. Inst. Math., 234 (2001), 1–383.

[32] Narazaki, T.: Global solutions to the Cauchy problem for the weakly coupled system of damped wave equations, AIMS Proceedings Discrete Contin. Dyn. Syst., (2009), 592–601.

[33] Narazaki, T.: Global solutions to the Cauchy problem for a system of damped wave equations, Differential Integral Equations, 24 (2011), 569–600.

[34] Nishihara, K.: $L^p - L^q$ estimates for the 3-D damped wave equation and their application to the semilinear problem, Sem. Notes Math. Sci., vol 6, Ibaraki Univ. (2003), 69–83.

[35] Ogawa, T., Takeda, H.: Non-existence of weak solutions to nonlinear damped wave equations in exterior domains, Nonlinear Anal., 70 (2009), 3696–3701.
[40] Sun, F., Wang, M.: Existence and nonexistence of global solutions for a nonlinear hyperbolic system with damping, Nonlinear Anal., 66 (2007), 2889–2910.

[41] Sobajima, M.: Remarks on test function methods for blowup of solutions to semilinear evolution equations in sectorial domain, RIMS, Kokyuroku, 2121 (2019), 63–73.

[42] Todorova, G., Yordanov, B.: Critical exponent for a nonlinear wave equation with damping, J. Differential Equations, 174 (2001), 464–489.

[43] Takeda, H.: Global existence and nonexistence of solutions for a system of nonlinear damped wave equations, J. Math. Anal. Appl., 360 (2009), 631–650.

[44] Weissler, F.B.: Existence and nonexistence of global solutions for a semilinear heat equation, Israel J. Math., 38 (1981), 29–40.

[45] Y. Wakasugi, On the diffusive structure for the damped wave equation with variable coefficients, Ph. D thesis, Osaka University, 2014.

[46] Zhang, Q.S.: A blow-up result for a nonlinear wave equation with damping: the critical case, C. R. Acad. Sci. Paris Sér. I Math., 333 (2001), 109–114.

Tuan Anh Dao
School of Applied Mathematics and Informatics
Hanoi University of Science and Technology
No. 1 Dai Co Viet Road
Hanoi
Vietnam
E-mail: anh.daotuan@hust.edu.vn

Masahiro Ikeda
Department of Mathematics, Faculty of Science and Technology
Keio University
3-14-1 Hiyoshi, Kohoku-ku
Yokohama 223-8522
Japan
E-mail: masahiro.ikeda@keio.jp;
masahiro.ikeda@riken.jp

and

Center for Advanced Intelligence Project
RIKEN
Tokyo
Japan

Accepted: 7 February 2023