A GENERALIZATION OF FULTON-MACPHERSON CONFIGURATION SPACES

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ABSTRACT. We construct a wonderful compactification of the variety parameterizing \( n \) distinct labeled points in \( X \) away from \( D \), where \( X \) is a nonsingular variety and \( D \) is a nonsingular proper subvariety. When \( D \) is empty, it coincides with the Fulton-MacPherson configuration space.

1. INTRODUCTION

1.1. Let \( X \) be a complex connected nonsingular algebraic variety \( X \) and let \( D \) be a nonsingular closed proper subvariety of \( X \). The goal of this paper is to construct the following two spaces:

- A compactification \( X_D^{[n]} \) of the configuration space of \( n \) labeled points in \( X \) away from \( D \), “not allowing the points to meet \( D \).”
- A compactification \( X_D[n] \) of the configuration space of \( n \) distinct labeled points in \( X \) away from \( D \), “not allowing the points to meet each other as well as \( D \).”

To describe the constructions, we introduce some notation. Let \( D = \bigcup_c D_c \) where \( D_c \) are irreducible components of \( D \). For a subset \( S \) of \( N := \{1, 2, ..., n\} \) denote by \( D_{c,S} \) the collection of points \( x \) in \( X^n \) whose \( i \)-th component \( x_i \) is in \( D_c \) if \( i \in S \). For a subset \( I \) (with \( |I| \geq 2 \)) of \( N \) let \( \Delta_I \subset X^n \) be the diagonal consisting of \( x \) satisfying \( x_i = x_j \) whenever \( i, j \in I \). We denote by \( \text{Bl}_Z X \) the blowup of a variety \( X \) along a closed subvariety \( Z \).

Then:

- Define \( X_D^{[n]} \) to be the closure of \( X^n \setminus \bigcup_{c,S} D_{c,S} \) diagonally embedded in
  \[
  X^n \times \prod_{c, S \subset N, |S| \geq 1} \text{Bl}_{D_{c,S}} X^n.
  \]
- Define \( X_D[n] \) to be the closure of \( (X \setminus D)^n \setminus \bigcup_{|I| \geq 2} \tilde{\Delta}_I \) in the product
  \[
  X_D^{[n]} \times \prod_{I \subset N, |I| \geq 2} \text{Bl}_{\tilde{\Delta}_I} X_D^{[n]},
  \]
  where \( \tilde{\Delta}_I \) is a proper transform of \( \Delta_I \).

These spaces satisfy wonderful properties as follows.

Theorem 1. (1) The variety \( X_D^{[n]} \) is nonsingular.
There is a “universal” family $X_D^{[n]^+} \to X_D^{[n]}$: It is a flat family of stable degenerations of $X$ with $n$ smooth labeled points away from $D$.

(3) The boundary $X_D^{[n]} \setminus (X^n \setminus \bigcup_{c,S} D_{c,S})$ is a union of divisors $\bar{D}_{c,S}$ corresponding to $D_{c,S}$, $|S| \geq 1$. Any set of these divisors intersects transversally.

(4) The intersection of boundary divisors $\bar{D}_{c_1,S_1}, \ldots, \bar{D}_{c_a,S_a}$ is nonempty if and only if they are nested in the sense that each pair $S_i$ and $S_k$ is:

- disjoint if $c_i \neq c_k$;
- one is contained in the other if $c_i = c_k$.

**Theorem 2.**

(1) The variety $X_D[n]$ is nonsingular.

(2) There is a “universal” family $X_D[n]^+ \to X_D[n]$; It is a flat family of stable degenerations of $X$ with $n$ distinct smooth labeled points away from $D$.

(3) The boundary $X_D[n]^+ \setminus ((X \setminus D)^n \setminus \cup_I \Delta_I)$ is a union of divisors $\bar{D}_{c,S}$ and $\bar{\Delta}_I$, corresponding to $D_{c,S}$, $|S| \geq 1$, and $\Delta_I$ with $|I| \geq 2$. Any set of these divisors intersects transversally.

(4) The intersection of boundary divisors $\bar{D}_{c_1,S_1}, \ldots, \bar{D}_{c_a,S_a}, \bar{\Delta}_{I_1}, \ldots, \bar{\Delta}_{I_b}$ are nonempty if and only if they are nested. Here the collection $\{\bar{D}_{c_i,S_i}, \bar{\Delta}_{I_j}\}_{1 \leq i \leq a, 1 \leq j \leq b}$ is called nested if $\{\bar{D}_{c_i,S_i}\}_{1 \leq i \leq a}$ is nested; for each pair $I_j$ and $I_k$, either they are disjoint or one is contained in the other; and for each pair $S_i$ and $S_k$, either they are disjoint or $I_k$ is contained in $S_i$.

When $D$ is empty, then the construction of $X_D[n]$ is exactly the Fulton-MacPherson compactification $X[n]$ of the configuration space of $n$ distinct labeled points in $X$ ([8]). The meaning of the statements (2) in Theorems will be explained in subsection 3.1. For the definitions of $\bar{D}_{c,S}$ and $\bar{\Delta}_I$, see subsection 1.2.

To prove Theorems 1 and 2 we use L. Li’s general work on wonderful compactifications ([3] [10], [5] [9]). For the history of wonderful compactifications, we refer the reader to [1]. One may show our Theorems also by the conical wonderful compactification ([11]). The Chow rings and motives of the spaces constructed here are described in [11].

Our motivation for the construction of the spaces $X_D^{[n]}$ and $X_D[n]$ is their use in the study of stable relative maps and stable relative (un)ramified maps, respectively. This will be studied in detail elsewhere; here we give only a rough explanation of this application. First note that one can interpret the stable relative maps of $[9]$ as maps from curves to the fibers of the universal family $X_D^{[n]^+}$. Next, the paper [7] constructs a compactification of maps from curves to $X$ without allowing any domain component collapse to points. There, the targets are the fibers of $X[n]^+$, the universal family over the Fulton-MacPherson configuration spaces. Precisely, modify $X$ by blowing up points $x$ where the components collapse and then gluing copies of $\mathbb{P}(T_x \oplus \mathbb{C})$ along the exceptional divisors $\mathbb{P}(T_x)$ to obtain a new target. For the relative version of [7] with respect to $D$, it is natural to use the fibers of $X_D[n]^+$ as targets. The statement (1), (2), and (3) of Theorem 2 will be
some key ingredients for establishing the properness and the perfect obstruction theory of the moduli space of such maps.

1.2. Notation.

- As in [4], for a subset $I$ of $N := \{1, 2, ..., n\}$, let
  \[ I^+ := I \cup \{n + 1\}. \]
- Let $Y_1$ be the blowup of a nonsingular complex variety $Y_0$ along a nonsingular closed subvariety $Z$. If $V$ is an irreducible subvariety of $Y_0$, we will use $\bar{V}$ or $V(Y_1)$ to denote
  - the total transform of $V$, if $V \subset Z$;
  - the proper transform of $V$, otherwise.
  If there is no risk to cause confusion, we will use simply $V$ to denote $\bar{V}$. The space $\text{Bl}_ZY_1$ will be called the iterated blowup of $Y_0$ along centers $Z, V$ (with the order).
- For a partition $I = \{I_0, I_1, ..., I_l\}$ of $N$, $\Delta_I$ denotes the polydiagonal associated to $I$. We will also consider the binary operation $I \land J$ on the set of all partitions defined by
  \[ \Delta_I \cap \Delta_J = \Delta_{I \land J} \]
  as in [12] (page 143). We use $\Delta_{I_0}$ instead of $\Delta_I$ when $I = \{I_0, I_1, ..., I_l\}$ such that $|I_i| = 1$ for all $i \geq 1$.
- We say that a collection $\mathcal{C}$ of closed subvarieties in a variety meets or intersects transversely if, for every pair of two disjoint nonempty subsets $C_1$ and $C_2$ of $\mathcal{C}$, the two subvarieties $\bigcap C_1 := \bigcap_{Z \in C_1} Z$ and $\bigcap C_2$ meet transversely (this includes the case that they are disjoint).

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2. Proof of Main Theorems

2.1. Wonderful Compactifications. We recall some results in [9] which are needed in this paper.

A finite collection $\mathcal{G}$ of nonsingular, proper, nonempty subvarieties of a nonsingular algebraic variety $Y$ is called a building set if the following two conditions are satisfied.

1. For every $V$ and $W$ in $\mathcal{G}$, they intersect cleanly, that is, the tangent bundle $T(V \cap W)$ of the intersection coincides with the intersection of tangent bundles $TV$ and $TW$ in $TY$.
2. For the intersection $\bigcap \mathcal{C}$ of a subset $\mathcal{C}$ of $\mathcal{G}$, an element $V$ in $\mathcal{G}$ is called a $\mathcal{G}$-factor of $\bigcap \mathcal{C}$ if
   - $V$ contains $\bigcap \mathcal{C}$ and
there is no other $V'$ in $\mathcal{G}$, contained in $V$ and containing $\bigcap \mathcal{C}$.

Then the second condition is as follows. The collection $\mathcal{C}'$ of all $\mathcal{G}$-factors of $\bigcap \mathcal{C}$ meets transversely and the intersection $\bigcap \mathcal{C}'$ is exactly $\bigcap \mathcal{C}$.

Define the so-called wonderful compactification $Y_{\mathcal{G}}$ of $Y$ with respect to $\mathcal{G}$ to be the closure of $Y \setminus \bigcup_{V \in \mathcal{G}} V$ diagonally embedded in

$$ Y \times \prod_{V \in \mathcal{G}} \text{Bl}_V Y. $$

It has the following wonderful properties.

**Theorem 3.** (9)

1. The variety $Y_{\mathcal{G}}$ coincides with the iterated blowup of $Y$ along all $V$ in $\mathcal{G}$ whenever the order of centers $V$ is an inclusion order, or a building set order.

2. The boundary $Y_{\mathcal{G}} \setminus (Y \setminus \bigcup_{V \in \mathcal{G}} V)$ is the union of divisors $\widetilde{V}$, corresponding to $V \in \mathcal{G}$. The divisors intersect transversally.

3. A subset $\mathcal{C}$ of $\mathcal{G}$ is nested if and only if the intersection of all divisors $\widetilde{V}$, for $V \in \mathcal{C}$, is nonempty.

We explain terminologies used in Theorem 3. An inclusion order (resp. a building set order) above is by definition a total order $V_1, ..., V_l$ of $\mathcal{G} = \{V_1, ..., V_l\}$ if $i < j$ whenever $V_i \subset V_j$ (resp. if $V_1, ..., V_k$ form a building set for any $k = 1, ..., l$).

Hence, $Y_{\mathcal{G}} \cong \text{Bl}_{V_1} ... \text{Bl}_{V_l} Y$ as $Y$-varieties. Here one should recall the convention on the centers. A subset $\mathcal{C}$ of a building set $\mathcal{G}$ is called nested if there are a positive integer $k$ and a flag $(W_1 \subset W_2 \subset ... \subset W_k)$ such that every element of $\mathcal{C}$ is a $\mathcal{G}$-factor of some $W_i$. Here $W_i$ is an intersection of elements of $\mathcal{G}$.

For example, the Fulton-MacPherson configuration space $X[n]$ is the wonderful compactification of $X^n$ with respect to the building set $\{\Delta_I \subset X^n \mid I \subset N, |I| \geq 2\}$.

2.2. Proof of Theorem 1 and Inductive Construction of $X_D^{[n]}$. Note that the collection of all subsets $D_{c,S}$ in $X^n$ is a building set. Hence parts (1), (3) and (4) of Theorem 1 follow from Theorem 3. In particular, $X_D^{[n]}$ can be constructed by iterated blowups of $X^n$ along nonsingular centers (and the proper transforms of)

$$ D_S := \bigcup_{c \in S} D_{c,S} $$

arrayed by an inclusion order. We may reshuffle centers as:

$$ D_{\{1\}}; D_{\{1,2\}}; D_{\{1,2,3\}}; D_{\{1,3\}}; D_{\{2,3\}}; D_{\{3\}}; ...; D_{\{1,2, ..., n\}}; ...; D_{\{n\}}, $$

keeping the same result $X_D^{[n]}$ after the blowup along the centers with this building set order.

The above ordering of centers provides an inductive construction of $X_D^{[n]}$. Define $X_D^{[n]+}$ to be the iterated blowups of $X_D^{[n]} \times X$ along centers $D_{T+}$, arrayed
by an inclusion order, where $T^+ = T \cup \{n + 1\}$, $T \subset N$, and $|T| \geq 1$. (This space is not isomorphic to $X_D^{[n+1]}$ unless $D$ is a divisor.) Note that the flatness of the natural projection $X_D^{[n+1]} \to X_D^{[n]}$ in Theorem 1 holds since it is a map between nonsingular varieties with equi-dimensional fibers. The projection is equipped with sections provided by $I(i) = \subset X_D^{[n]}, i = 1, \ldots, n.$

2.3. Proof of Theorem 2 and Inductive Construction of $X_D[n]$. We would like to take a sequence of blowups starting from $X^n$ along centers $D_S$ and $\Delta_I$, $S, I \subset N, |S| \geq 1, |I| \geq 2.$ However they do not form a building set. (See Remark 3.3 for an example.) Hence we cannot apply Theorem 3 directly to $Y = X^n$. Instead, we use the wonderful compactification in a two-step process. We will show in Proposition 4 that altogether the proper transforms $\Delta_i$ of $\Delta_I$ in $X_D^{[n]}$ form a building set. Therefore we can apply Theorem 3 to $Y = X_D^{[n]}$ with the building set $\{\Delta_i\}$ where $I \subset N, |I| \geq 2$. The technical lemma on blowups will be deferred to Lemma 5 at the end of this subsection.

The inductive construction starting from $X^n$ is given by the iterated blowup with the order:

$$D_{\{1\}};$$

$$D_{\{1,2\}}, D_{\{2\}}, \Delta_{\{1,2\}};$$

$$D_{\{1,2,3\}}, D_{\{1,3\}}, D_{\{2,3\}}, D_{\{3\}}, \Delta_{\{1,2,3\}}, \Delta_{\{1,3\}}, \Delta_{\{2,3\}};$$

$$\vdots$$

$$D_{\{1,2,\ldots,n\}}, \ldots, D_{\{n-1,n\}}, D_{\{n\}}, \Delta_{\{1,2,\ldots,n\}}, \ldots, \Delta_{\{1,n\}}, \ldots, \Delta_{\{n-1,n\}}.$$

One can achieve this sequence from the sequence of the building set orders:

$$D_{\{1\}}, D_{\{1,2\}}, D_{\{2\}}, D_{\{1,3\}}, D_{\{2,3\}}, D_{\{3\}}; \ldots; D_{\{1,2,\ldots,n\}}; \ldots; D_{\{n\}};$$

$$\Delta_{\{1,2\}}, \Delta_{\{1,2,3\}}, \Delta_{\{1,3\}}, \Delta_{\{2,3\}}; \ldots; \Delta_{\{1,2,\ldots,n\}}, \ldots, \Delta_{\{n-1,n\}}.$$

To see it, first note that all the centers $D_T$ and $\Delta_I$ are etale locally linearized simultaneously in $X^n$, and hence in an iterated blowup of $X^n$ along any set of the centers, by Lemma 5 (2). In particular this shows that the divisor $D_T$ is transversal to $\Delta_I$ in any iterated blowup of $X_D^{[n]}$ along any set of all the centers. Now we may rearrange the centers from the initial order using the reordering of two transversal centers (Lemma 6 (1)).

Define $X_D[n]^+$ as the blowup of $X_D[n] \times X$ along $D_S^+, \Delta_I^+$, more precisely, along $D_S^+$ with the inclusion order first, then along $\Delta_I^+$, also with the inclusion order, where $S, I \subset N$ and $|S| \geq 1, |I| \geq 2$. As before, the projection $X_D[n]^+ \to X_D[n]$ has the sections provided by $I(i) = \subset X_D[n]^+, i = 1, \ldots, n.$

**Proposition 4.**

1. Let $I_1$ and $I_2$ be partitions of $N$. Then the intersection of proper transforms $\Delta_{I_1}$ and $\Delta_{I_2}$ in $X_D^{[n]}$ is the proper transform $\Delta_{I_1 \wedge I_2}$ of the intersection $\Delta_{I_1} \cap \Delta_{I_2} = \Delta_{I_1 \wedge I_2}$.

2. The collection of all diagonals $\Delta_I, I \subset N, |I| \geq 2$, is a building set in $X_D^{[n]}$. 
Proof. Note that $\Delta_I$ in $X_D^{[n]}$ coincides with the variety defined by equations

$$\sigma_a = \sigma_b, \ \forall a, b \in I_i, \ I_i \in I$$

where $\sigma_a$ is the section of $X_D^{[n]} \rightarrow X_D^{[n]}$, induced by $\Delta_{(a)+}$. This can be seen by considering the imposed equation at general points. Now the proof is straightforward. □

Proof of Theorem 3 (4). For simplicity assume that $D$ is connected.

$(\Rightarrow)$. The condition on the pair $S_i$ and $S_k$ ($I_j$ and $I_l$, respectively) is a direct consequence of Theorem 2. Suppose that both $S(4)$ and $I(5)$ are not disjoint. Then Lemma 5 shows that $\bar{D}_S \cap \Delta_I$ is empty. □

$(\Leftarrow)$. Let $\{D_{S_i}, \Delta_{I_j}\}_{i,j}$ be a nested set and let $V$ be the transversal intersection $\bigcap_i D_{S_i}(X_D^{[n]})$. Then an argument similar to the proof of Proposition 3 shows that the collection

$$\mathcal{G} := \{ V \cap \Delta_I(X_D^{[n]}) \mid I \subseteq N, \ |I| \geq 2, \ \{S_i, I\}_i \text{ is nested} \}$$

is a building set of $V$. According to Lemma 5, $\bar{V}$ in $X_D^{[n]}$ coincides with the wonderful compactification $\bar{V}$ of $V$. Now since $\{V \cap \Delta_I(X_D^{[n]})\}_j$ is nested, we conclude that $\bar{V} \cap \bigcap_j \Delta_{\tilde{I}_j}$ in $X_D^{[n]}$ is nonempty and transversal by Theorem 3. Also, $\bar{V}$ is $\bigcap D_{S_i}(X_D^{[n]})$ due to Lemma 5 and $\bigcap \tilde{D}_{S_i} \subset \bigcap \tilde{D}_{S_i}$ in $X_D^{[n]}$. This completes the proof. □

Note that the above proof of 4 shows the statement (3) of Theorem 2 is also true.

Lemma 5. Let $Z, Z_i, V, V_i, i = 1, \ldots, k$ be nonsingular subvarieties of a nonsingular variety $X$, let $\pi : \text{Bl}_Z X \rightarrow X$ be the blowup map along $Z$ and let $E$ be the exceptional divisor.

1. If $Z_1$ and $Z_2$ intersect transversely, then $\text{Bl}_{Z_2} \text{Bl}_{Z_1} Y = \text{Bl}_{Z_2} \text{Bl}_{Z_1} Y$.
2. If $Z, V_i, i = 1, \ldots, k$ are étale locally linearized in $X$ simultaneously, then so are their transforms in $\text{Bl}_Z X$, and in particular $V_i$ and $V_j$ for any $i, j$ intersect cleanly.
3. If $V$ meets $Z$ transversally, then $\bar{V} = \pi^{-1}(V)$.
4. If $V$ and $Z$ intersect cleanly and $V$ is not contained in $Z$, then $\bar{V}$ is the blowup of $V$ along $Z \cap V$.
5. Assume that $V_1$ and $V_2$ intersect cleanly. If $V_1 \cap V_2 \subset Z \subsetneq V_1$, then $\bar{V_1}$ and $\bar{V_2}$ are disjoint.

Proof. The only nonstandard result is 5, which we prove here. Assume that they are not disjoint. Then for some point $p \in V_1 \cap V_2$, there are $v_1 \in T_p V_1$ such that in the normal bundle $N_Z/X$, $[v_1] = [v_2] \neq 0$. Since $TZ \subset TV_1$, $v_2$ is an element of $TV_1$ as well as $TV_2$. It implies that $[v_2] = 0$ in $N_Z/X$ since $TV_1 \cap TV_2 = T(V_1 \cap V_2) \subset TZ$. This is a contradiction. □
3. Some more properties

3.1. Stable degenerations. For simplicity assume that \( D \) is connected. Note that \( X_D^{[n]} \rightarrow X_D^{[n]} \) is a flat family of stable degenerations of \( X \) with \( n \) smooth labeled points away from \( D \) (see subsection 2.2). The labeled points may not be distinct. Stability means that every closed fiber \( F \) has no nontrivial automorphism fixing the following data: the natural map \( F \rightarrow X; F \cap \tilde{D}_{(i,n+1)} \); and the marked points \( F \cap \tilde{\Delta}_{(i,n+1)}, i = 1,..., n \). The fibers are normal crossing varieties, étale locally the form \( xy = 0 \). The generic fiber over \( D_S(X_D^{[n]}) \) is the coproduct

\[
\text{Bl}_D X \coprod_{\mathbb{P}(N_D/X)} \mathbb{P}(N_D/X + 1)
\]

of \( \text{Bl}_D X \) and \( \mathbb{P}(N_D/X + 1) \) along \( \mathbb{P}(N_D/X) \). The points labeled by \( a \in S \) are in \( \mathbb{P}(N_D/X + 1) \setminus (\mathbb{P}(N_D/X) \cup \mathbb{P}(1)) \) and the other points are in \( \text{Bl}_D X \setminus \mathbb{P}(N_S/S) \). In general, \( \Delta_{(a)} \) is disjoint from \( D_{(n+1)} \) in \( X_D^{[n]} \) by Lemma 5.5.

Similarly, \( X_D[n] \rightarrow X_D[n] \) is a flat family of stable degenerations of \( X \) with \( n \) distinct smooth labeled points away from \( D \) (see subsection 2.3). It is equipped with sections \( \sigma_i \), which are disjoint to each other. Specifically, the fibers of \( X_D[n] \) over points in the boundary of \( X_D[n] \) are Fulton-MacPherson stable degenerations of fibers of \( X_D[n] \): In a fiber \( F \) of \( X_D[n] \) the labeled points \( F \cap \tilde{\Delta}_{(i,n+1)}, i = 1,..., n \), are away from \( \tilde{D} := F \cap \tilde{D}_{(n+1)} \), but may come together at some points of \( F \setminus \tilde{D} \). Blow up all such points \( x \in F \setminus \tilde{D} \) and then glue copies of \( \mathbb{P}(T_x \oplus C) \) along the exceptional divisors \( \mathbb{P}(T_x) \) to obtain a new modification of \( X \) in which the points in the configuration are now distinct. The stability is similar to the above case.

3.2. Group Action by \( S_n \). Let \( S_n \) be the symmetric group on \( n \) letters. There is a natural \( S_n \)-action on the space \( X_D[n] \) such that the projection \( X_D[n] \rightarrow X^n \) is \( S_n \)-equivariant. By Theorem 5.2 in [1], all stabilizers are solvable.

3.3. Remark. In general, the space \( X_D[n] \) is not isomorphic to the one-step closure of \( (X \setminus D)^n \setminus \bigcup_{|I| \geq 2} \Delta_I \), that is, the closure in the product

\[
X^n \times \coprod_{c \in S \subseteq N} \text{Bl}_{D_{c,S}} X^n \times \coprod_{I \subseteq N, |I| \geq 2} \text{Bl}_{\Delta_I} X^n.
\]

For example, take \( X = \mathbb{C}^2 \) with \( D = \{(x,y) \in \mathbb{C}^2 \mid y = 0 \} \) and consider the limits of \( ((t,at), (2t,bt)) \), as \( t \) goes to 0. Then the limit in \( X_D[2] \) does not depend on \( a,b \). However the limit in the one-step closure depends on \( a,b \).

3.4. Examples.

3.4.1. \( \overline{M}_{0,n} \). Let \( n \geq 3 \). The moduli space \( \overline{M}_{0,n} \) of \( n \)-pointed stable rational curves coincides with \( X_D[\bar{n} - 3] \) where \( X = \mathbb{P}^1 \) and \( D \) consists of three distinct points. Indeed, the inductive construction is exactly the blowup construction of \( \overline{M}_{0,n} \) given by Keel (6).
3.4.2. $T_{d,n}$. Let $n \geq 2$. Take $X = \mathbb{P}^d$ and let $D$ be a hyperplane. Note that the group $G$ of automorphisms of $X$ fixing all points in $D$ is isomorphic to $\mathbb{C}^* \ltimes \mathbb{C}^d$. The natural action of the group $G$ on $X_D[n]$ is free and the quotient $X_D[n]/G$ is isomorphic to the compactification $T_{d,n}$ studied by Chen, Gibney, and Krashen [2]. It compactifies the configuration space of $n$ distinct labeled points in $\mathbb{C}^d$ modulo $\mathbb{C}^* \ltimes \mathbb{C}^d$.

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