On the Number of Distinct Legendre, Jacobi and
Hessian Curves

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Abstract
We give explicit formulas for the number of distinct elliptic curves over a finite field, up to isomorphism, in the families of Legendre, Jacobi, Hessian and generalized Hessian curves.

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1 Introduction

A nonsingular absolutely irreducible projective curve of genus 1 defined over a field \( \mathbb{F} \) with at least one \( \mathbb{F} \)-rational point is called an elliptic curve over \( \mathbb{F} \), see [1, 24] for a general background on elliptic curves. Koblitz [19] and Miller [22] were the first to show that the group of rational points on an elliptic curve over a finite field can be used for the discrete logarithm problem in a public-key cryptosystem.

In particular, an elliptic curve \( E \) over \( \mathbb{F} \) can be given by the so-called Weierstrass equation

\[
E: \quad Y^2 + a_1XY + a_3Y = X^3 + a_2X^2 + a_4X + a_6,
\]

where the coefficients \( a_1, a_2, a_3, a_4, a_6 \in \mathbb{F} \), which traditionally has been most commonly used. For a field \( \mathbb{F} \) of characteristic \( p \neq 2, 3 \), the Weierstrass equation (1) can be transformed to the so called short Weierstrass equation given by

\[
E_{W,u,v}: \quad Y^2 = X^3 + uX + v,
\]

where the coefficients \( u, v \in \mathbb{F} \) (see [1, 24]).
There are many other forms of equations to represent elliptic curves such as Legendre equation, Hessian equation, quartic equation and intersection of two quadratic surfaces [26, Chapter 2]. Any equation given by each of the latter forms over a field \( \mathbb{F} \) can be transformed to a Weierstrass equation by change of variables that uses rational functions with coefficient in \( \mathbb{F} \). Usually, any elliptic curve over an algebraically closed field \( \mathbb{F} \) can be defined by each of the latter equations.

A Legendre equation is a variant of Weierstrass equation with one parameter. Any elliptic curve defined over an algebraically closed field \( \mathbb{F} \) of characteristic \( p \neq 2 \) can be expressed by an elliptic curve by the Legendre equation

\[
E_{L,u} : \quad Y^2 = X(X - 1)(X - u),
\]

for some \( u \in \mathbb{F} \). Furthermore, an elliptic curve in Legendre form is birationally equivalent to a Jacobi quartic curve that is given by the equation

\[
E_{JQ,u} : \quad Y^2 = X^4 + 2uX^2 + 1.
\]

for some \( u \in \mathbb{F} \) with \( u \neq \pm 1 \). Moreover, an elliptic curve in Legendre form is birationally equivalent to a so-called Jacobi intersection that is defined by the intersection of two quadratics given by

\[
E_{JI,u} : \quad X^2 + Y^2 = 1 \quad \text{and} \quad uX^2 + Z^2 = 1,
\]

where \( u \in \mathbb{F} \) and \( u \neq 0, 1 \). See [7] for more background on Jacobi curves. The latter two forms over finite fields are used for cryptographic interest in [21]. Also, for the recent improvements on their arithmetic see [10, 17].

A Hessian curve over a field \( \mathbb{F} \) is given by the cubic equation

\[
E_{H,u} : \quad X^3 + Y^3 + 1 = uXY,
\]

for some \( u \in \mathbb{F} \) with \( u^3 \neq 27 \) (see [7]). For the cryptographic interests on Hessian curves over finite fields see [3, 13, 17, 18, 25]. Recently, Farashahi and Joye have considered the generalization of Hessian curves to the so-called generalized Hessian family, [13], given by

\[
E_{GH,u,v} : \quad X^3 + Y^3 + v = uXY,
\]

where \( u, v \in \mathbb{F}, v \neq 0 \) and \( u^3 \neq 27v \). Moreover, Bernstein, Kohel and Lange, [3], have also considered the twisted Hessian form that is similar to the latter form up to the order of the coordinates.

We note that, above families do not cover all distinct curves over finite fields. Accordingly, a natural question arise about the number of isomorphism classes of these curves.

Lenstra [20] gave explicit estimates for the number of isomorphism classes of elliptic curves over a prime field \( \mathbb{F}_p \) with order divisible by a prime \( l \neq p \). After that, Howe [16] extended Lenstra’s work to arbitrary integers \( l \). Moreover, Castryck and Hubrechts [8] generalized these results giving explicit estimates for the number of isomorphism classes.
of elliptic curves over a finite field \( \mathbb{F}_q \) having order with a fix remainder divided by an integer \( l \).

Furthermore, Farashahi and Shparlinski [15], using the notion of the \( j \)-invariant of an elliptic curve, see [1, 24, 26], gave exact formulas for the number of distinct elliptic curves over a finite field (up to isomorphism over the algebraic closure of the ground field) in the families of Edwards curves [11] and their generalization due to Bernstein and Lange [4] as well as the curves introduced by Doche, Icart and Kohel [9]. Moreover, the open question of [15] is whether there are explicit formulas for the number of distinct elliptic curves over a finite field in the families of Hessian curves, Jacobi quartic and Jacobi intersections.

In this paper, we give precise formulas for the number of distinct \( j \)-invariants of elliptic curves over a finite field in the families of Legendre, Jacobi and Hessian curves. The next interesting and more challenging step is to study isomorphism classes over the ground field of these families. Moreover, we give exact formulas for the number of isomorphism classes over the ground field of above families.

Throughout the paper, for a field \( \mathbb{F} \), we denote its algebraic closure by \( \overline{\mathbb{F}} \) and its multiplicative subgroup by \( \mathbb{F}^* \). The letter \( p \) always denotes a prime number and the letter \( q \) always denotes a prime power. As usual, \( \mathbb{F}_q \) is a finite field of size \( q \). Let \( \chi_2 \) denote the quadratic character in \( \mathbb{F}_q \), where \( p \geq 3 \). So, for any \( q \) where \( p \geq 3 \), \( u = w^2 \) for some \( w \in \mathbb{F}_q^* \) if and only if \( \chi_2(u) = 1 \). The cardinality of a finite set \( S \) is denoted by \( \# S \).

2 Background on isomorphisms and outline of our approach

An elliptic curve \( E \) over \( \mathbb{F} \) given by the Weierstrass equation (1) can be transformed to the elliptic curve \( \tilde{E} \) over \( \mathbb{F} \) given by the Weierstrass equation

\[
\tilde{E} : \quad \tilde{Y}^2 + \tilde{a}_1 \tilde{X} \tilde{Y} + \tilde{a}_3 \tilde{Y} = \tilde{X}^3 + \tilde{a}_2 \tilde{X}^2 + \tilde{a}_4 \tilde{X} + \tilde{a}_6,
\]

via the invertible maps \( X \mapsto a^2 \tilde{X} + \beta \) and \( Y \mapsto a^3 \tilde{Y} + a^2 \gamma \tilde{X} + \delta \) with \( \alpha, \beta, \gamma, \delta \in \mathbb{F} \) and \( \alpha \neq 0 \). In this case, the elliptic curves \( E \) and \( \tilde{E} \) are called \textit{isomorphic over} \( \mathbb{F} \) or \textit{twists} of each other. In case \( \alpha, \beta, \gamma, \delta \in \mathbb{F} \), the elliptic curves \( E \) and \( \tilde{E} \) are called \textit{isomorphic over} \( \mathbb{F} \). We use \( E \cong_F \tilde{E} \) to denote \( E \) and \( \tilde{E} \) are \( F \)-isomorphic.

The elliptic curve \( E \) over \( \mathbb{F} \) given by the Weierstrass equation (1) has the non-zero discriminant

\[
\Delta_E = -b_2^2 b_8 - 8b_4^3 - 27b_6^2 + 9b_2 b_4 b_6,
\]

\[
b_2 = a_1^2 + 4a_2, \quad b_4 = a_1 a_3 + 2a_4, \quad b_6 = a_3^2 + 4a_6, \quad b_8 = a_1^2 a_6 - a_1 a_3 a_4 + 4a_2 a_6 + a_2 a_3^2 - a_4^2.
\]

Also, its \( j \)-invariant is explicitly defined as

\[
j(E) = (b_2^2 - 24b_4)^3 / \Delta_E.
\]

It is known that two elliptic curves \( E, \tilde{E} \) over a field \( \mathbb{F} \) are isomorphic over \( \mathbb{F} \) if and only if \( j(E_1) = j(E_2) \), see [24, Proposition III.1.4(b)].
Note that over a finite field \( \mathbb{F}_q \) each of the \( q \) values appears as \( j \)-invariant of at least one curve. So, the number of distinct elliptic curves over \( \mathbb{F}_q \) (up to isomorphism over \( \mathbb{F}_q \)) is equal to \( q \). The same is true for the family \( \mathcal{J}_2 \). Furthermore, for a finite field \( \mathbb{F}_q \) with characteristic \( p \neq 2, 3 \), the number of \( \mathbb{F}_q \)-isomorphism classes of the family \( \mathcal{J}_2 \) is equal to \( 2q + 6, 2q + 2, 2q + 4, 2q \) if \( q \equiv 1, 5, 7, 11 \mod 12 \) respectively (e.g. see [20]).

In the following, we use \( J_L(q), J_{JQ}(q), J_H(q), J_{H}(q) \) and \( J_{GH}(q) \) to denote the number of distinct \( j \)-invariants of the curves defined over \( \mathbb{F}_q \) in the families \( \mathcal{J}_3, \mathcal{J}_4, \mathcal{J}_5, \mathcal{J}_6 \) and \( \mathcal{J}_7 \), respectively.

Moreover, we use \( I_L(q), I_{JQ}(q), I_H(q), I_{H}(q) \) and \( I_{GH}(q) \) to denote the number of \( \mathbb{F}_q \)-isomorphism classes of the families \( \mathcal{J}_3, \mathcal{J}_4, \mathcal{J}_5, \mathcal{J}_6 \) and \( \mathcal{J}_7 \) respectively.

We compute the number of distinct \( j \)-invariants of a family of elliptic curves \( E_u \) over a finite field \( \mathbb{F}_q \) with a parameter \( u \), using the general approach mentioned in [15]. In this approach, the \( j \)-invariant of \( E_u \) is given by a rational function \( F(U) \in \mathbb{F}_q(U) \) of small degree. Next, we consider the bivariate rational function

\[
F(U) - F(V) = g(U,V)/l(U,V)
\]

with two relatively prime polynomials \( g \) and \( l \). We factor \( g(U,V) \). Then, studying the number of distinct roots of the polynomials \( g_u(V) = g(u,V) \), for \( u \in \mathbb{F}_q \), provides the necessary information, which is the cardinality of the set \( J_u \) of all curves \( E_v \) with \( j(E_v) = j(E_u) \). Then, for several small integers \( k \), we count the number of elements \( u \) of \( \mathbb{F}_q \) with \( \#J_u = k \). Therefore, we obtain the number of different sets \( J_u \), i.e., the number of distinct \( j(E_u) \) in the family.

We propose an analogous approach to count the number of \( \mathbb{F}_q \)-isomorphism classes of elliptic curves \( E_u \), for \( u \in \mathbb{F}_q \). Considering the set \( J_u \), we study the set \( I_u \) of curves \( E_v \) which are \( \mathbb{F}_q \)-isomorphic to the curve \( E_u \). Then, counting the number of distinct sets \( I_u \) provides our results.

3 Legendre curves

We consider the curves \( E_{L,u} \) given by Legendre equation \( \mathcal{J}_3 \) over a finite field \( \mathbb{F}_q \) with characteristic \( p \geq 3 \). We note that \( u \neq 0, 1 \), since the curve \( E_{L,u} \) is nonsingular. The Legendre curve \( E_{L,u} \) over \( \mathbb{F}_q \) of characteristic \( p > 3 \) is isomorphic to the Weierstrass curve \( E_{W,a_u,b_u} \) given by

\[
Y^2 = X^3 + a_uX + b_u,
\]

where

\[
a_u = - (u^2 - u + 1)/3, \quad b_u = -(u + 1)(u - 2)(2u - 1)/27.
\]

The \( j \)-invariant of \( E_{L,u} \) is given by \( j(E_{L,u}) = F(u) \) where

\[
F(U) = \frac{2^8(U^2 - U + 1)^3}{(U^2 - U)^2}.
\]
Here, we study the cardinality of preimages of \( F(u) \), for all elements \( u \in \mathbb{F}_q \setminus \{0, 1\} \), under the map \( u \mapsto F(u) \). In particular, we see this map is 6 : 1, for almost all \( u \in \mathbb{F}_q \).

We consider the bivariate rational function \( F(U) - F(V) = g(U, V)/l(U, V) \) with two relatively prime polynomials \( g \) and \( l \). We see that

\[
g(U, V) = 2^8(U - V)(U + V - 1)(UV - 1)(UV + V + 1)(UV - U + 1)(UV - U - V).
\]

Then, we need to study the number of roots of the polynomial \( g_u(V) = g(u, V) \), for \( u \in \mathbb{F}_q \setminus \{0, 1\} \). Moreover, for \( u \in \mathbb{F}_q \setminus \{0, 1\} \), we let

\[
 J_{L,u} = \left\{ v : v \in \mathbb{F}_q, \ E_{L,u} \cong_{\mathbb{F}_q} E_{L,u} \right\}.
\]

We note that, for all \( v \in J_{L,u} \), the curves \( E_{L,u} \) and \( E_{L,v} \) have the same \( j \)-invariants. But, these curves may not be isomorphic over \( \mathbb{F}_q \). Next, for a fixed value \( u \in \mathbb{F}_q \setminus \{0, 1\} \), we let

\[
 I_{L,u} = \left\{ v : v \in \mathbb{F}_q, \ E_{L,u} \cong_{\mathbb{F}_q} E_{L,v} \right\}.
\]

Clearly, for all \( u \in \mathbb{F}_q \setminus \{0, 1\} \), we have \( I_{L,u} \subseteq J_{L,u} \). We see that \( j(E_{L,u}) = j(E_{W,a_u,b_u}) = 0 \) if and only if \( a_u = 0 \). Furthermore, \( j(E_{L,u}) = 1728 \) if and only if \( b_u = 0 \) (see Equation (8)). Let

\[
 B = \{ u : u \in \mathbb{F}_q, a_u b_u = 0 \}.
\]

The following lemma gives the cardinality of \( J_{L,u} \), for all \( u \in \mathbb{F}_q \setminus \{0, 1\} \).

**Lemma 1.** For all \( u \in \mathbb{F}_q \setminus \{0, 1\} \), we have

\[
 \# J_{L,u} = \begin{cases} 
 1, & \text{if } u = -1 \text{ and } p = 3, \\
 3, & \text{if } u \in \{-1, 2, 2^{-1}\} \text{ and } p > 3, \\
 2, & \text{if } u^2 - u + 1 = 0 \text{ and } p > 3, \\
 6, & \text{if } u \notin B.
\end{cases}
\]

**Proof.** For a fixed value \( u \in \mathbb{F}_q \setminus \{0, 1\} \), let \( g_u(V) = g(u, V) \) and

\[
 Z_u = \{ v : v \in \mathbb{F}_q \setminus \{0, 1\}, g_u(v) = 0 \}.
\]

Then,

\[
 Z_u = \left\{ u, \frac{1}{u}, 1 - u, \frac{1}{1 - u}, \frac{u - 1}{u}, \frac{u}{u - 1} \right\}.
\]

We note that, for all \( v \in J_{L,u} \), the curves \( E_{L,u} \) and \( E_{L,v} \) have the same \( j \)-invariants, so \( J_{L,u} = Z_u \). Next, we consider several cases depending on the value of \( u \) in \( \mathbb{F}_q \).

- If \( u = -1 \) and \( p = 3 \), then \( J_{L,-1} = \{-1\} \).
- If \( u \in \{-1, 2, 2^{-1}\} \) and \( p > 3 \), then \( J_{L,u} = \{-1, 2, 2^{-1}\} \).
- If \( u^2 - u + 1 = 0 \) and \( p > 3 \), then \( J_{L,u} = \{ u, \frac{1}{u} \} \).
- If \( u \neq -1, u \neq 2, u \neq 2^{-1} \) and \( u^2 - u + 1 \neq 0 \), then all 6 elements in \( J_{L,u} \) are distinct.
So, the proof of the lemma is complete. □

Let \( \chi_2 \) denote the quadratic character in \( \mathbb{F}_q \), where \( p \geq 3 \). So, for any \( q \) where \( p \geq 3 \), \( u = w^2 \) for some \( w \in \mathbb{F}_q^* \) if and only if \( \chi_2(u) = 1 \).

**Lemma 2.** For all elements \( u, v \in \mathbb{F}_q \setminus \{0, 1\} \), we have \( E_{L,u} \cong_{\mathbb{F}_q} E_{L,v} \) if and only if \( u, v \) satisfy one of the following:

1. \( v = u \),
2. \( v = \frac{1}{u} \) and \( \chi_2(u) = 1 \)
3. \( v = 1 - u \) and \( \chi_2(-1) = 1 \)
4. \( v = \frac{1}{1-u} \) and \( \chi_2(u - 1) = 1 \)
5. \( v = \frac{u-1}{u} \) and \( \chi_2(-u) = 1 \)
6. \( v = \frac{u}{u-1} \) and \( \chi_2(1-u) = 1 \).

**Proof.** For \( u, v \in \mathbb{F}_q \setminus \{0, 1\} \), we have \( E_{L,u} \cong_{\mathbb{F}_q} E_{L,v} \) if and only if there exist elements \( \alpha, \beta \) in \( \mathbb{F}_q \), where \( \alpha \neq 0 \) and

\[
\left\{ \frac{-\beta}{\alpha^2}, \frac{1-\beta}{\alpha^2}, \frac{u-\beta}{\alpha^2} \right\} = \{0, 1, v\}.
\]

This is equivalent to one of the following cases:

- \( \beta = 0, \alpha^2 = 1 \) and \( v = u \),
- \( \beta = 0, \alpha^2 = u \) and \( v = \frac{1}{u} \),
- \( \beta = 1, \alpha^2 = -1 \) and \( v = 1 - u \),
- \( \beta = 1, \alpha^2 = u - 1 \) and \( v = \frac{1}{1-u} \),
- \( \beta = u, \alpha^2 = -u \) and \( v = \frac{u-1}{u} \),
- \( \beta = u, \alpha^2 = 1 - u \) and \( v = \frac{u}{u-1} \).

which concludes the proof. □

Furthermore, the following lemma gives the cardinality of \( \mathcal{L}_{L,u} \), for all \( u \in \mathbb{F}_q \setminus \{0, 1\} \).
Lemma 3. For all \( u \in \mathbb{F}_q \setminus \{0, 1\} \), we have

\[
\#I_{L,u} = \begin{cases} 
1, & \text{if } u = -1 \text{ and } p = 3, \\
3, & \text{if } u \in \{-1, 2, 2^{-1}\}, \ q \equiv 1, 3, 7 \pmod{8} \text{ and } p > 3, \\
2, & \text{if } u \in \{-1, 2\}, \ q \equiv 5 \pmod{8}, \\
1, & \text{if } u = 2^{-1}, \ q \equiv 5 \pmod{8}, \\
2, & \text{if } u^2 - u + 1 = 0, \ q \equiv 1 \pmod{12} \text{ and } p > 3, \\
1, & \text{if } u^2 - u + 1 = 0, \ q \not\equiv 1 \pmod{12}, \\
3, & \text{if } \chi_2(-1) = -1 \text{ and } u \notin \mathcal{B}, \\
2, & \text{if } \chi_2(-1) = 1, \chi_2(u) = \chi_2(1-u) = -1 \text{ and } u \notin \mathcal{B}, \\
4, & \text{if } \chi_2(-1) = 1, \chi_2(u)\chi_2(1-u) = -1 \text{ and } u \notin \mathcal{B}, \\
6, & \text{if } \chi_2(-1) = 1, \chi_2(u) = \chi_2(1-u) = 1 \text{ and } u \notin \mathcal{B}.
\end{cases}
\]

Proof. For \( p = 3 \), we have \( \mathcal{B} = \{-1\} \). Then, from Lemma 1, we obtain

\[
\mathcal{J}_{L,-1} = \{-1\}, \quad \text{if } p = 3.
\]

Next, we assume that \( p > 3 \). From Lemma 2 (part 5), we see \( E_{L,-1} \cong \mathbb{F}_q E_{L,2} \). Furthermore, \( E_{L,-1} \cong \mathbb{F}_q E_{L,2} \) if and only if \( \chi_2(2) = 1 \) or \( \chi_2(-2) = 1 \), which is equivalent to the cases where \( q \equiv 1, 3, 7 \pmod{8} \). From the proof of Lemma 1, if \( u \in \{-1, 2, 2^{-1}\} \), we have \( \mathcal{J}_{L,u} = \{-1, 2, 2^{-1}\} \). Therefore,

\[
\mathcal{I}_{L,u} = \begin{cases} 
\{-1, 2, 2^{-1}\}, & \text{if } u \in \{-1, 2, 2^{-1}\}, \ q \equiv 1, 3, 7 \pmod{8}, \\
\{-1, 2\}, & \text{if } u \in \{-1, 2\}, \ q \equiv 5 \pmod{8}, \\
\{2^{-1}\}, & \text{if } u = 2^{-1}, \ q \equiv 5 \pmod{8}.
\end{cases}
\]

Now, we assume that \( u \in \mathbb{F}_q \) with \( u^2 - u + 1 = 0 \). This happens if \( \chi_2(-3) = 1 \). Then, \( u = \frac{1+\zeta}{2} \), where \( \zeta \) is a square root of \(-3\) in \( \mathbb{F}_q \). Furthermore, \( u \) can be written as \( u = -(\frac{1-\zeta}{2})^2 \).

Then, from Lemma 2 we see that \( E_{L,u} \cong \mathbb{F}_q E_{L,\frac{u}{u}} \) if and only if \( \chi_2(-1) = 1 \). Moreover, \( \chi_2(-3) = \chi_2(-1) = 1 \) if and only if \( q \equiv 1 \pmod{12} \). From the proof of Lemma 1 we have \( \mathcal{J}_{L,u} = \{u, \frac{1}{u}\} \). Hence,

\[
\mathcal{I}_{L,u} = \begin{cases} 
\{u, \frac{1}{u}\}, & \text{if } u^2 - u + 1 = 0, \ q \equiv 1 \pmod{12}, \\
\{u\}, & \text{if } u^2 - u + 1 = 0, \ q \not\equiv 1 \pmod{12}.
\end{cases}
\]

From now on, we let \( u \in \mathbb{F}_q \setminus \{0, 1\} \) with \( u \notin \mathcal{B} \). By the the proof of Lemma 1 we have

\[
\mathcal{J}_{L,u} = \left\{ u, \frac{1}{u}, 1-u, \frac{1}{1-u}, \frac{u-1}{u}, \frac{u}{u-1} \right\}.
\]

We distinguish the following cases.

- First, we assume that \( \chi_2(-1) = -1 \). From Lemma 2, we see that exactly one of \( \frac{1}{u} \) and \( \frac{u-1}{u} \) belongs to \( \mathcal{I}_{L,u} \). Also, exactly one of \( \frac{1}{1-u} \) and \( \frac{u}{u-1} \) belongs to \( \mathcal{I}_{L,u} \). Furthermore, \( 1-u \notin \mathcal{I}_{L,u} \). Therefore, if \( \chi_2(-1) = -1 \), we obtain \( \#\mathcal{I}_{L,u} = 3 \).
Now, we assume that \( \chi_2(-1) = 1 \). So, \( 1 - u \in \mathcal{I}_{L,u} \).

- If \( \chi_2(u) = \chi_2(1-u) = -1 \), then \( \chi_2(-u) = \chi_2(u-1) = -1 \). Next, from Lemma \( \ref{lemma2} \), we have \( \#\mathcal{I}_{L,u} = 2 \).

- If \( \chi_2(u)\chi_2(1-u) = -1 \), then \( \chi_2(-u)\chi_2(u-1) = -1 \). So, we have \( \mathcal{I}_{L,u} = \{1 - u, \frac{1}{u}, \frac{u-1}{u}, \frac{u}{u-1}\} \) if \( \chi_2(u) = 1 \) and \( \mathcal{I}_{L,u} = \{1 - u, \frac{1}{u}, \frac{u}{u-1}\} \) if \( \chi_2(u) = -1 \). Thus, \( \#\mathcal{I}_{L,u} = 4 \).

- If \( \chi_2(u) = \chi_2(1-u) = 1 \), then \( \chi_2(-u)\chi_2(u-1) = 1 \). Hence, \( \mathcal{I}_{L,u} = \mathcal{J}_{L,u} \) (see Lemma \( \ref{lemma2} \)).

So, the proof of this lemma is complete. \qed

The following Lemma is used in the proof of Theorem \( \ref{theorem6} \), which can be of independent interest.

**Lemma 4.** Let \( \chi_2 \) be the quadratic character of a finite field \( \mathbb{F}_q \) of characteristic \( p \neq 2 \). For \( i, j \in \{-1, 1\} \), let

\[
S_{i,j} = \{u : u \in \mathbb{F}_q, \chi_2(u) = i, \chi_2(1-u) = j\}.
\]

We have

\[
\#S_{i,j} = \begin{cases} 
(q - 5)/4, & \text{if } (i, j) = (1, 1) \text{ and } q \equiv 1 \pmod{4}, \\
(q - 3)/4, & \text{if } (i, j) \neq (1, 1) \text{ and } q \equiv 1 \pmod{4}, \\
(q + 1)/4, & \text{if } (i, j) = (-1, -1) \text{ and } q \equiv 3 \pmod{4}.
\end{cases}
\]

**Proof.** Consider the set \( S_{i,j} \), for fixed \( i, j \) in \( \{-1, 1\} \). Let \( C_{i,j} \) be an affine conic over \( \mathbb{F}_q \) given by the equation

\[
C_{i,j} : \alpha X^2 + \beta Y^2 = 1,
\]

where \( \chi_2(\alpha) = i, \chi_2(\beta) = j \). Let \( C_{i,j}(\mathbb{F}_q) \) be the set of affine \( \mathbb{F}_q \)-rational points on \( C_{i,j} \). We note that, \( \#C_{i,j}(\mathbb{F}_q) = q - 1 \) if \( \chi_2(-1) = ij \) and \( \#C_{i,j}(\mathbb{F}_q) = q + 1 \) if \( \chi_2(-1) = -ij \). Let

\[
T_{i,j} = C_{i,j}(\mathbb{F}_q) \setminus \{(x, y) \in C_{i,j}(\mathbb{F}_q) : xy = 0\}.
\]

Then, we see

\[
\#T_{i,j} = \begin{cases} 
q - 5, & \text{if } \chi_2(-1) = 1, \ i = j = 1, \\
q - 1, & \text{if } \chi_2(-1) = 1, \ i = -1 \text{ or } j = -1, \\
q - 3, & \text{if } \chi_2(-1) = -1, \ i = 1 \text{ or } j = 1, \\
q + 1, & \text{if } \chi_2(-1) = -1, \ i = j = -1.
\end{cases}
\]

We note that \( \chi_2(-1) = 1 \) if and only if \( q \equiv 1 \pmod{4} \). Next, we consider the map \( \tau : T_{i,j} \rightarrow S_{i,j} \) given by \( x \mapsto \alpha x^2 \). The map \( \tau \) is surjective. Moreover, it is a \( 4 : 1 \) map. So, \( \#S_{i,j} = \frac{\#T_{i,j}}{4} \), which completes the proof of this lemma. \qed

In the following, we give a precise formula for the number of distinct curves over \( \mathbb{F}_q \), up to isomorphism classes over \( \mathbb{F}_q \), of the family \( \ref{family3} \).
Theorem 5. For any prime $p \geq 3$, for the number $J_L(q)$ of distinct values of the $j$-invariant of the family $(3)$, we have

$$J_L(q) = \left\lfloor \frac{q + 5}{6} \right\rfloor.$$

Proof. We note that

$$J_L(q) = \sum_{u \in \mathbb{F}_q \setminus \{0,1\}} \frac{1}{\#J_L,u}.$$

Let

$$N_k = \# \{ u : u \in \mathbb{F}_q \setminus \{0,1\}, \#J_L,u = k \}, \quad k = 1, 2, \ldots.$$

From Lemma 1, we see that $N_k = 0$ for $k > 6$. Therefore,

$$J_L(q) = \frac{6}{k} \sum_{k=1}^{6} N_k.$$

First, we let $p = 3$. Form Lemma 2, we have $N_1 = 1$, $N_2 = N_3 = N_4 = N_5 = 0$ and $N_6 = q - 3$. Then, using (10), we obtain

$$J_L(q) = (q + 3)/6.$$

Now, we assume that $p > 3$. From Lemma 2, we have $N_1 = N_4 = N_5 = 0$ and $N_3 = 3$. Furthermore, $N_2$ is the number of roots of the polynomial $U^2 - U + 1$ in $\mathbb{F}_q$. Since, $-3$, the discriminant of this polynomial, is a quadratic residue in $\mathbb{F}_q$ if and only if $q \equiv 1 \pmod{3}$, we have $N_2 = 2$ if $q \equiv 1 \pmod{3}$ and $N_2 = 0$ if $q \equiv 2 \pmod{3}$. Because $\sum_{k=1}^{6} N_k = q - 2$, we obtain $N_6 = q - 5 - N_2$. Next, using (10), we have

$$J_L(q) = \begin{cases} (q + 5)/6, & \text{if } q \equiv 1 \pmod{3}, \\ (q + 1)/6, & \text{if } q \equiv 2 \pmod{3}. \end{cases}$$

$\square$

Now, we give an exact formula for the number of $\mathbb{F}_q$-isomorphism classes of elliptic curves over $\mathbb{F}_q$ of the family $(3)$.

Theorem 6. For any prime $p \geq 3$, for the number $I_L(q)$ of $\mathbb{F}_q$-isomorphism classes of the family $(3)$, we have

$$I_L(q) = \begin{cases} \frac{7q + 29}{24}, & \text{if } q \equiv 1 \pmod{12}, \\ \frac{q + 2}{3}, & \text{if } q \equiv 3, 7 \pmod{12}, \\ \frac{7q + 13}{24}, & \text{if } q \equiv 5, 9 \pmod{12}, \\ \frac{q - 2}{3}, & \text{if } q \equiv 11 \pmod{12}. \end{cases}$$
Proof. We note that
\[ I_L(q) = \sum_{u \in \mathbb{F}_q \setminus \{0, 1\}} \frac{1}{\#I_{L,u}}. \]
From Lemma 3, we see that \(1 \leq \#I_{L,u} \leq 6\). Let
\[ M_k = \# \left\{ u : u \in \mathbb{F}_q \setminus \{0, 1\}, \#I_{L,u} = k \right\}, \quad k = 1, 2, \ldots, 6.\]
Then,
\[ I_L(q) = \sum_{k=1}^{6} \frac{M_k}{k}. \quad (11) \]
We partition \(\mathbb{F}_q \setminus \{0, 1\}\) into the following sets:
\[ \mathcal{A} \cup \mathcal{B}, \]
where as before \(\mathcal{B}\) is given by (12) and
\[ \mathcal{A} = \left\{ u \in \mathbb{F}_q \setminus \{0, 1\} : u \neq -1, 2, 2^{-1}, u^2 - u + 1 \neq 0 \right\}. \]
Moreover, we write
\[ \mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2, \]
where
\[ \mathcal{B}_1 = \left\{ u \in \mathbb{F}_q : u = -1, 2, 2^{-1} \right\} \quad \text{and} \quad \mathcal{B}_2 = \left\{ u \in \mathbb{F}_q : u^2 - u + 1 = 0 \right\}. \]
We note that the sets \(\mathcal{B}_1\) and \(\mathcal{B}_2\) are not necessarily disjoint. In Table 1, we show the cardinalities of above sets, where we let \(q \equiv r \pmod{3}\). For the case \(q \equiv 0 \pmod{3}\), we have \(\mathcal{B}_1 = \mathcal{B}_2 = \{2\}\).

| \(r\) | \#\(\mathcal{A}\) | \#\(\mathcal{B}_1\) | \#\(\mathcal{B}_2\) |
|---|---|---|---|
| 0 | \(q - 3\) | 1 | 1 |
| 1 | \(q - 7\) | 3 | 2 |
| 2 | \(q - 5\) | 3 | 0 |

Table 1: Cardinalities of the sets \(\mathcal{A}, \mathcal{B}_1, \mathcal{B}_2\).

If \(q \equiv 1 \pmod{4}\), then \(\chi_2(-1) = 1\). In this case, we partition \(\mathcal{A}\) into the following set
\[ \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3, \]
where
\[ \mathcal{A}_1 = \left\{ u \in \mathbb{F}_q \setminus \{0, 1\} : u \notin \mathcal{B}, \ \chi_2(u) = \chi_2(1 - u) = -1 \right\}, \]
\[ \mathcal{A}_2 = \left\{ u \in \mathbb{F}_q \setminus \{0, 1\} : u \notin \mathcal{B}, \ \chi_2(u)\chi_2(1 - u) = -1 \right\}, \]
\[ \mathcal{A}_3 = \left\{ u \in \mathbb{F}_q \setminus \{0, 1\} : u \notin \mathcal{B}, \ \chi_2(u) = \chi_2(1 - u) = 1 \right\}. \]
We note that, \( \chi_2(2) = \chi_2(2^{-1}) = 1 \) if and only if \( q \equiv \pm 1 \pmod{8} \). Furthermore, for all \( \mu \in \mathcal{B}_2 \), we have \( \chi_2(\mu) = \chi_2(1-\mu) = \chi_2(-1) \); see the proof of Lemma 3. Next, from Lemma 4 we compute the the cardinalities of the sets \( \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3 \), where \( q \equiv 1 \pmod{4} \); see Table 2 where we let \( q \equiv r \pmod{24} \).

| \( r \) | \( \#A_1 \) | \( \#A_2 \) | \( \#A_3 \) |
|---|---|---|---|
| 1 | \( \frac{q-1}{4} \) | \( \frac{q-1}{2} \) | \( \frac{q-5}{4} - 5 \) |
| 5 | \( \frac{q-1}{4} - 1 \) | \( \frac{q-1}{2} - 2 \) | \( \frac{q-5}{4} \) |
| 9 | \( \frac{q-1}{4} \) | \( \frac{q-1}{2} \) | \( \frac{q-5}{4} - 1 \) |
| 13 | \( \frac{q-1}{4} - 1 \) | \( \frac{q-1}{2} - 2 \) | \( \frac{q-5}{4} - 2 \) |
| 17 | \( \frac{q-1}{4} \) | \( \frac{q-1}{2} \) | \( \frac{q-5}{4} - 3 \) |

Table 2: Cardinalities of the sets \( \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3 \), for \( q \equiv 1 \pmod{4} \)

From Lemma 3 for all \( \mu \in \mathbb{F}_q \setminus \{0, 1\} \), we have

\[
\#I_{\mu, u} = \begin{cases} 
1, & \text{if } \mu = -1 \text{ and } p = 3, \\
1, & \text{if } \mu = 2^{-1}, \ q \equiv 5 \pmod{8}, \\
1, & \text{if } \mu \in \mathcal{B}_2 \ and \ q \equiv 7 \pmod{12}, \\
2, & \text{if } \mu = -1, 2, \ q \equiv 5 \pmod{8}, \\
2, & \text{if } \mu \in \mathcal{B}_2 \ and \ q \equiv 1 \pmod{12} \text{ and } p > 3, \\
2 & \text{if } \mu \in \mathcal{A}_1 \ and \ q \equiv 1 \pmod{4}, \\
3, & \text{if } \mu = -1, 2, 2^{-1}, \ q \equiv 1, 3, 7 \pmod{8} \text{ and } p > 3, \\
3, & \text{if } \mu \in \mathcal{A} \ and \ q \equiv 3 \pmod{4}, \\
4, & \text{if } \mu \in \mathcal{A}_2 \ and \ q \equiv 1 \pmod{4}, \\
6, & \text{if } \mu \in \mathcal{A}_3 \ and \ q \equiv 1 \pmod{4}. 
\end{cases}
\]  

(12)

Then, we obtain the values of \( M_k \); see Table 3 (where \( q \equiv r \pmod{24} \)).

Next, using (11), we compute:

\[
I_1(q) = \begin{cases} 
(7q + 17)/24, & \text{if } q \equiv 1 \pmod{24}, \\
q/3, & \text{if } q \equiv 3 \pmod{24}, \\
(7q + 13)/24, & \text{if } q \equiv 5 \pmod{24}, \\
(q + 2)/3, & \text{if } q \equiv 7, 19 \pmod{24}, \\
(7q + 9)/24, & \text{if } q \equiv 9 \pmod{24}, \\
(q - 2)/3, & \text{if } q \equiv 11, 23 \pmod{24}, \\
(7q + 29)/24, & \text{if } q \equiv 13 \pmod{24}, \\
(7q + 1)/24, & \text{if } q \equiv 17 \pmod{24}, 
\end{cases}
\]

which completes the proof.

\( \square \)
| $r$ | $M_1$ | $M_2$ | $M_3$ | $M_4$ | $M_5$ | $M_6$ |
|-----|------|------|------|------|------|------|
| 1   | 0    | $q^2$ | 3    | $q^{-1}$ | 0    | $q^{-25}$ |
| 3   | 1    | 0    | $q - 3$ | 0    | 0    | 0    |
| 5   | 1    | $q^2$ | 0    | $q^{-5}$ | 0    | $q^{-5}$ |
| 7, 19 | 2    | 0    | $q - 4$ | 0    | 0    | 0    |
| 9   | 1    | $q^2$ | 0    | $q^{-1}$ | 0    | $q^{-9}$ |
| 11, 23 | 0    | 0    | $q - 2$ | 0    | 0    | 0    |
| 13  | 1    | $q^2$ | 0    | $q^{-5}$ | 0    | $q^{-13}$ |
| 17  | 0    | $q^{-1}$ | 3    | $q^{-1}$ | 0    | $q^{-17}$ |

Table 3: $M_k$, for $k = 1, \ldots, 6$.

4 Jacobi curves

First, we consider the Jacobi quartic curves $E_{JQ,u}$ given by (4) over a field $\mathbb{F}$ with characteristic $p \geq 3$. Note that $u \neq \pm 1$, since the curve $E_{JQ,u}$ is nonsingular. The change of variable $(X, Y) \mapsto (\tilde{X}, \tilde{Y})$ defined by $\tilde{X} = 2(u + \frac{Y}{X^2})$ and $\tilde{Y} = \frac{2X}{X}$, is a birational equivalence from $E_{JQ,u}$ to the elliptic curve $\tilde{E}_{JQ,u}$ defined by

$$\tilde{Y}^2 = \tilde{X}^3 - 4u\tilde{X}^2 + 4(u^2 - 1)\tilde{X}$$

with $j$-invariant $j(E_{JQ,u}) = F(u)$ where

$$F(U) = \frac{64(U^2 + 3)^3}{(U^2 - 1)^2}.$$

**Lemma 7.** For all $u \in \mathbb{F}$ with $u \neq \pm 1$, the Jacobi curve $E_{JQ,u}$ is birationally equivalent over $\mathbb{F}$ to the Legendre curve $E_{L,\frac{1-\sqrt{u}}{2}}$.

**Proof.** We note that, the Jacobi-quartic curve $E_{JQ,u}$ is birationally equivalent over $\mathbb{F}$ to the elliptic curve $\tilde{E}_{JQ,u}$ defined by (4). Moreover, the map $(\tilde{X}, \tilde{Y}) \mapsto (X, Y)$ defined by

$$X = \frac{\tilde{X} - 2u + 2}{4} \quad \text{and} \quad Y = \frac{-\tilde{Y}}{8},$$

is an isomorphism over $\mathbb{F}$ from the curve $\tilde{E}_{JQ,u}$ to the Legendre curve $E_{L,\frac{1-\sqrt{u}}{2}}$. \hfill $\square$

Now, we consider the curves $E_{JL,u}$ given by (5) over a field $\mathbb{F}$ with characteristic $p \geq 3$. We note that $u \neq 0, 1$, since the curve $E_{JL,u}$ is nonsingular. The change of variable
(X, Y, Z) ↦ (˜X, ˜Y) defined by ˜X = \frac{u(Y - Z)}{uY - Z + 1 - u} and ˜Y = \frac{u(1 - u)X}{uY - Z + 1 - u}, is a birational equivalence from E_{JI,u} to the elliptic curve ˜E_{JI,u} defined by

\[ ˜Y^2 = ˜X^3 - (u + 1)\bar{X}^2 + u\bar{X}. \]

We note that ˜E_{JI,u} = E_{L,u}. Moreover, the Jacobi intersection curve E_{JI,u} is birationally equivalent over \( \mathbb{F} \) to the the Jacobi curve E_{JQ,1−2a}.

From Theorem 6, the following lemma gives the numbers of distinct curves over a finite field \( \mathbb{F}_q \) of the families (4) and (5).

**Theorem 8.** For any prime \( p \geq 3 \), for the numbers \( I_L(q) \), \( I_{JQ}(q) \) and \( I_{JI}(q) \) of \( \mathbb{F}_q \)-isomorphism classes of the families (3), (4) and (5) respectively, we have

\[ I_L(q) = I_{JQ}(q) = I_{JI}(q). \]

**Proof.** From Lemma 7 for all \( u \in \mathbb{F}_q \setminus \{0, 1\} \), the Legendre curve \( E_{L,u} \) in the family (3) is birationally equivalent to the curve \( E_{JQ,1−2a} \) in the family (4). Clearly, this correspondence is bijective. Moreover, they are birationally equivalent to the curve \( E_{JI,u} \) in the family (5). Hence, the proof is complete.

**Theorem 9.** For any prime \( p \geq 3 \), for the numbers \( J_{JQ}(q) \) and \( J_{JI}(q) \) of distinct values of the \( j \)-invariant of the families (4) and (5) respectively, we have

\[ J_{JQ}(q) = J_{JI}(q) = \left\lfloor \frac{q + 5}{6} \right\rfloor. \]

**Proof.** This theorem is a direct consequence of Theorems 5 and 8.

### 5 Hessian curves

We consider the curves \( E_{H,u} \) given by (6) over a finite field \( \mathbb{F}_q \) of characteristic \( p \). We note that \( u^3 \neq 27 \), since the curve \( E_{H,u} \) is nonsingular. For \( p \geq 3 \), the curve \( E_{H,u} \) is birationally equivalent to the elliptic curve

\[ \tilde{E}_{W, A_u, B_u} : \tilde{Y}^2 = \tilde{X}^3 + A_u \tilde{X} + B_u \]

where

\[ A_u = -u(u^3 + 6^3)/3 \quad \text{and} \quad B_u = (u^6 - 540u^3 - 18^3)/27. \]

Furthermore, we have \( j(E_{H,u}) = \left( \frac{u(u^3 + 6^3)}{u^3 - 3^3} \right)^3 \). Therefore, the curve \( E_{H,u} \) has the \( j \)-invariant

\[ j(E_{H,u}) = (F(u))^3 \]

where

\[ F(U) = \frac{U(U^3 + 216)}{U^3 - 27}. \]

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We consider the bivariate rational function $F(U) - F(V) = g(U, V)/l(U, V)$ with two relatively prime polynomials $g$ and $l$. We see that

$$g(U, V) = (U - V)(UV - 3U - 3V - 18)h(U, V),$$

where $h(U, V) = (U^2 + 3U + 9)V^2 + 3(U^2 + 12U - 18)V + 9(U^2 - 6U + 36)$.

For a fixed value $u \in \mathbb{F}_q$ with $u^3 \neq 27$, let $g_u(V) = g(u, V)$. Next, for $u \in \mathbb{F}_q$ with $u^3 \neq 27$, we investigate the number of roots of the polynomial $g_u(V) = g(u, V)$. Moreover, for $u \in \mathbb{F}_q$ with $u^3 \neq 27$, we let

$$\mathcal{J}_{H,u} = \left\{ v : v \in \mathbb{F}_q, \ E_{H,u} \sim_{\mathbb{F}_q} E_{H,v} \right\}, \ \mathcal{I}_{H,u} = \left\{ v : v \in \mathbb{F}_q, \ E_{H,u} \sim_{\mathbb{F}_q} E_{H,v} \right\}.$$

In the following, we give the cardinalities of $\mathcal{J}_{H,u}$ and $\mathcal{I}_{H,u}$, for all $u \in \mathbb{F}_q$ with $u^3 \neq 27$.

**Lemma 10.** For all $u \in \mathbb{F}_q$ with $u^3 \neq 27$, we have

$$\#\mathcal{J}_{H,u} = \begin{cases} 
1, & \text{if } p = 2 \text{ and } u = 0, \text{ or } p = 3, \\
4, & \text{if } q \equiv 1 \pmod{3}, \text{ } p \neq 2 \text{ and } A_u = 0, \\
6, & \text{if } q \equiv 1 \pmod{3}, \text{ } p \neq 2 \text{ and } B_u = 0, \\
12, & \text{if } q \equiv 1 \pmod{3} \text{ and } A_uB_u \neq 0, \\
2, & \text{if } q \equiv 2 \pmod{3}, \text{ } p \neq 2 \text{ or } u \neq 0.
\end{cases}$$

**Proof.** For a fixed value $u \in \mathbb{F}_q$ with $u^3 \neq 27$, let $g_u(V) = g(u, V)$ and

$$\mathcal{Z}_u = \{ v : v \in \mathbb{F}_q, v^3 \neq 27, \ g_u(v) = 0 \}.$$

We note that, $v \in \mathcal{J}_u$ if and only if $F(u)^3 = F(v)^3$, which is equivalent to $F(u) = \zeta F(v)$ for some third root of unity $\zeta$ in $\mathbb{F}_q$. Moreover, we have $F(\zeta v) = \zeta F(v)$ for all $\zeta \in \mathbb{F}_q$ with $\zeta^3 = 1$. Therefore, for all third roots of unity $\zeta \in \mathbb{F}_q$, we have $v \in \mathcal{J}_u$ if and only if $\zeta v \in \mathcal{Z}_u$. In other words,

$$\mathcal{J}_{H,u} = \{ \zeta v : \zeta \in \mathbb{F}_q \text{ with } \zeta^3 = 1, v \in \mathcal{Z}_u \}.$$

We note that, all third roots of unity in $\mathbb{F}_q$ are in $\mathbb{F}_q$ if and only if $q \equiv 1 \pmod{3}$. Otherwise, $\mathbb{F}_q$ has only unity as the trivial third root of unity. Moreover, if $F(u) \neq 0$, i.e. $A_u \neq 0$, then for all elements $v_1, v_2 \in \mathcal{Z}_u$ and for all distinct third roots of unity $\zeta_1, \zeta_2 \in \mathbb{F}_q$, we see $\zeta_1 v_1 \neq \zeta_2 v_2$. If $F(u) = 0$, then for all $v \in \mathcal{Z}_u$ and for all third roots of unity $\zeta \in \mathbb{F}_q$, we have $F(\zeta v) = 0$, so, $\zeta v \in \mathcal{Z}_u$. Therefore,

$$\#\mathcal{J}_{H,u} = \begin{cases} 
\#\mathcal{Z}_u, & \text{if } q \equiv 0, 2 \pmod{3}, \\
\#\mathcal{Z}_u, & \text{if } q \equiv 1 \pmod{3} \text{ and } A_u = 0, \\
3\#\mathcal{Z}_u, & \text{if } q \equiv 1 \pmod{3} \text{ and } A_u \neq 0.
\end{cases} \tag{16}$$

Here, we study the set $\mathcal{Z}_u$ for all possibilities for $u$ and $q$. 



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If $p = 3$, then we see $Z_u = \{u\}$ and $J_{H,u} = \{u\}$. Next, we assume that $p \neq 3$. We write

$$g_u(V) = -(u^3 - 27)(V - u)(V - v)h_u(V),$$

where $v = \frac{2(u+6)}{u-3}$ and $h_u(V) = V^2 + \frac{\beta(u)}{\alpha(u)}V + \frac{\gamma(u)}{\alpha(u)}$, where

$$\alpha(U) = U^2 + 3U + 9, \quad \beta(U) = 3(U^2 + 12U - 18), \quad \gamma(U) = 9(U^2 - 6U + 36)$$

(see Equation (15)). We write,

$$h_u(V) = (V - v_1)(V - v_2),$$

where $v_i = \frac{3i(u+6)\zeta^i}{u-3\zeta^i}$, for $i = 1, 2$, $\zeta \in \overline{F}_q$ with $\zeta^3 = 1$ and $\zeta \neq 1$.

For $p \neq 3$, the polynomial $g_u$ is square free if $\Delta_u$ the discriminant of $g_u$ is nonzero. We have $\Delta_u = -3^2B_u^4$, so $g_u$ has distinct roots with multiplicity one if $B_u \neq 0$.

We note that, if $v^3 = 27$ then $g_u(v) = -3^5v(u^3 - 27)$. Thus, for all $v \in Z_u$, we have $v^3 \neq 27$.

Now, we distinguish the following possibilities for $q$.

1. First, we assume that $p = 2$. We have $\beta(u) = \gamma(u) = u^2$. For $u = 0$, we have $Z_0 = \{0\}$ and $J_{H,0} = \{0\}$. Next, we let $u \neq 0$, i.e., $A_uB_u \neq 0$. We note that, the polynomial $h_u$ is irreducible over $\mathbb{F}_q$ if and only if $\text{Tr}_{\mathbb{F}_q/\mathbb{F}_2}(\frac{\gamma(u)/\alpha(u)}{(\beta(u)/\alpha(u))^2}) = 1$. Moreover,

$$\text{Tr}_{\mathbb{F}_q/\mathbb{F}_2}(\frac{\alpha(u)}{\beta(u)}) = \text{Tr}_{\mathbb{F}_q/\mathbb{F}_2}(1 + \frac{1}{u} + \frac{1}{u^2}) = \text{Tr}_{\mathbb{F}_q/\mathbb{F}_2}(1).$$

Furthermore, $\text{Tr}_{\mathbb{F}_q/\mathbb{F}_2}(1) = 1$ if and only if $q \equiv 2 \pmod{3}$. Therefore, we have the following cases.

- If $q \equiv 1 \pmod{3}$, then $v_1, v_2 \in \mathbb{F}_q$. Hence,

$$Z_u = \{u, v, v_1, v_2\}.$$

- If $q \equiv 2 \pmod{3}$, then $Z_u = \{u, v\}$.

2. Second, we assume that $p > 3$. We distinguish the following possibilities for $u \in \mathbb{F}_q$.

- First, we assume that $B_u = 0$. Then $A_u \neq 0$, since $u^3 \neq 27$. Then, $u$ is a root of one of the following equations.

$$U^2 - 6U - 18 = 0 \quad \text{or} \quad U^4 + 6U^3 + 54U^2 - 108U + 324 = 0. \quad (17)$$

Since $p > 3$, the sets of solutions to the equations in (17) do not intersect and so may be considered separately.

- If $u^2 - 6u - 18 = 0$, then $v = u$ and $v_1 = v_2 = -u + 6$. So,

$$Z_u = \{u, -u + 6\}.$$
- If $u^4 + 6u^3 + 54u^2 - 108u + 324 = 0$, then
  \[ g_u(V) = -(u^3 - 27)(V - u)^2(V - v)^2. \]

Furthermore, $v \neq u$. So,
\[ \mathcal{Z}_u = \{u, v\}. \]

- Next, we assume that $B_u \neq 0$. Let $D_u$ be the discriminant of $h_u$, i.e., $D_u = -3^3(u^2 - 6u - 18)/\alpha^2(u)$. We consider the following cases for $q$.
  - If $q \equiv 1 \pmod{3}$, then $D_u$ is a quadratic residue in $\mathbb{F}_q$. Moreover, $D_u \neq 0$. Thus, $h_u$ has two distinct roots $v_1, v_2$ in $\mathbb{F}_q$. Therefore,
    \[ \mathcal{Z}_u = \{u, v_1, v_2\}. \]
  - If $q \equiv 2 \pmod{3}$, then $D_u$ is a quadratic non-residue in $\mathbb{F}_q$. Thus, $h_u$ has no root in $\mathbb{F}_q$. Therefore,
    \[ \mathcal{Z}_u = \{u, v\}. \]

Therefore, we have
\[ \#\mathcal{Z}_u = \begin{cases} 1, & \text{if } p = 2 \text{ and } u = 0, \text{ or } p = 3, \\ 2, & \text{if } q \equiv 1 \pmod{3}, p \neq 2 \text{ and } B_u = 0, \\ 4, & \text{if } q \equiv 1 \pmod{3} \text{ and } B_u \neq 0, \\ 2, & \text{if } q \equiv 2 \pmod{3}, p = 2 \text{ and } u \neq 0, \\ 2, & \text{if } q \equiv 2 \pmod{3} \text{ and } p \neq 2. \end{cases} \]

Next, using (16), we obtain the cardinality of $\mathcal{J}_{H,u}$. \hfill \Box

In the following Lemma, we study the $\mathbb{F}_q$-isomorphism classes of Hessian curves over $\mathbb{F}_q$.

**Lemma 11.** For all elements $u, v \in \mathbb{F}_q$ with $u^3 \neq 27, v^3 \neq 27$, we have $E_{H,u} \cong_{\mathbb{F}_q} E_{H,v}$ if and only if $u, v$ satisfy one of the following:

1. $v = \zeta_1 u$,
2. $v = \frac{3\zeta_1(u + 6\zeta_2)}{u - 3\zeta_2}$ and $q \equiv 1 \pmod{3},$

for some third roots of unity $\zeta_1, \zeta_2 \in \mathbb{F}_q$.

**Proof.** Let $u, v \in \mathbb{F}_q$ with $u^3 \neq 27, v^3 \neq 27$. First, we assume that $E_{H,u} \cong_{\mathbb{F}_q} E_{H,v}$. So, $v \in \mathcal{J}_{H,u}$. From the proof of Lemma 10, we see that
\[ \mathcal{J}_{H,u} = \left\{ w \in \mathbb{F}_q : w = \zeta_1 u \text{ or } w = \frac{3\zeta_1(u + 6\zeta_2)}{u - 3\zeta_2}, \zeta_1, \zeta_2 \in \mathbb{F}_q, \zeta_1^3 = 1, \zeta_2^3 = 1 \right\}. \]

We consider the following cases for $q$.

- If $p = 3$, then we have $v = u$, which satisfy the property 1.
• If \( q \equiv 1 \pmod{3} \), then \( \mathbb{F}_q \) contains all third roots of unity in \( \overline{\mathbb{F}_q} \). So, \( u, v \) satisfy either the property 1 or 2.

• If \( q \equiv 2 \pmod{3} \), then \( \zeta = 1 \) is the only third root of unity in \( \mathbb{F}_q \). From the proof of Lemma 10, we have

\[
\mathcal{F}_{H,u} = \begin{cases} 
\{u\}, & \text{if } p = 2 \text{ and } u = 0, \\
\{u, \frac{3(u+6)}{u-3}\}, & \text{if } q \equiv 2 \pmod{3}, u^2 - 6u - 18 \neq 0, \\
\{u, -u + 6\}, & \text{if } q \equiv 2 \pmod{3} \text{ and } u^2 - 6u - 18 = 0.
\end{cases}
\]

Then, we distinguish the following cases to show that \( v \equiv -1 \pmod{3} \), then

\[
\mathcal{E}_{H,u} = \begin{cases} 
\{u\}, & \text{if } p = 2 \text{ and } u = 0, \\
\{u, \frac{3(u+6)}{u-3}\}, & \text{if } q \equiv 2 \pmod{3}, u^2 - 6u - 18 \neq 0, \\
\{u, -u + 6\}, & \text{if } q \equiv 2 \pmod{3} \text{ and } u^2 - 6u - 18 = 0.
\end{cases}
\]

Then, we distinguish the following cases to show that \( v = u \).

- We let \( p = 2 \). If \( u = 0 \), then clearly \( v = u \). So, we assume that \( u \neq 0 \). Then, the map \( (X, Y) \mapsto (\tilde{X}, \tilde{Y}) \) defined by

\[
\tilde{X} = \frac{(u^3 + 1)(X + Y)}{u^3(X + Y + u)} \quad \text{and} \quad \tilde{Y} = \frac{(u^3 + 1)(X + (1 + u^3)Y + u)}{u^6(X + Y + u)},
\]

is a birational equivalence from \( \mathcal{E}_{H,u} \) to the elliptic curve

\[
\mathbb{E}_{BW,a_u,b_u} : \tilde{Y}^2 + \tilde{X}\tilde{Y} = \tilde{X}^3 + a_u\tilde{X}^2 + b_u,
\]

where \( a_u = \frac{1}{u^3} \) and \( b_u = \left(\frac{1 + u^3}{u^4}\right)^3 \). Similarly, \( \mathcal{E}_{H,v} \) is birationally equivalent to \( \mathbb{E}_{BW,a_v,b_v} \).

If \( u = \frac{3(u+6)}{u-3} \), i.e., \( v = \frac{u}{u+1} \), then we have \( b_u = b_v \). Next, we note that the elliptic curves \( \mathbb{E}_{BW,a_u,b_u} \) and \( \mathbb{E}_{BW,a_v,b_v} \) are isomorphic over \( \mathbb{F}_q \) if and only if there exists an element \( \gamma \) in \( \mathbb{F}_q \) such that \( a_u + a_v = \gamma^2 + \gamma \). In other words, they are isomorphic if and only if \( \text{Tr}_{\mathbb{F}_q/\mathbb{F}_2}(a_u + a_v) = 0 \). But, for \( v = \frac{u}{u+1} \), we have

\[
\text{Tr}_{\mathbb{F}_q/\mathbb{F}_2}(a_u + a_v) = \text{Tr}_{\mathbb{F}_q/\mathbb{F}_2}(1 + \frac{1}{u}\frac{1}{u^4}) = \text{Tr}_{\mathbb{F}_q/\mathbb{F}_2}(1) = 1.
\]

So, the curves \( \mathcal{E}_{H,u} \) and \( \mathcal{E}_{H,u^{-1}} \) are not isomorphic over \( \mathbb{F}_q \). Hence, we only have \( v = u \).

- We let \( p > 3 \). Then, the map \( (X, Y) \mapsto (\tilde{X}, \tilde{Y}) \) given by

\[
\tilde{X} = Z - u^2 \quad \text{and} \quad \tilde{Y} = 3Z(Y - X),
\]

where \( Z = \frac{4(u^3 - 27)}{3(u+3X+3Y)} \), is a birational equivalence from \( \mathcal{E}_{H,u} \) to the elliptic curve

\[
\mathbb{E}_{W,A_u,B_u} : \tilde{Y}^2 = \tilde{X}^3 + A_u\tilde{X} + B_u
\]

where

\[
A_u = \frac{-u(u^3 + 6)^3}{3} \quad \text{and} \quad B_u = \frac{u^6 - 540u^3 - 18^3}{27}.
\]

Also, \( \mathcal{E}_{H,v} \) is birationally equivalent to \( \mathbb{E}_{W,A_v,B_v} \). We note that, \( \mathbb{E}_{W,A_u,B_u} \cong \mathbb{E}_{W,A_v,B_v} \) if and only if there exist an element \( \alpha \in \mathbb{F}_q^\star \) such that \( A_v = \alpha^6 A_u \) and \( B_v = \alpha^6 B_u \). We consider the following cases.
Lemma 12. For all \( u \in \mathbb{F}_q \) with \( u^3 \neq 27 \), we have

\[
\# \mathcal{I}_{H,u} = \begin{cases} 
1, & \text{if } q \equiv 0, 2 \pmod{3}, \\
4, & \text{if } q \equiv 1 \pmod{3}, \ p \neq 2 \text{ and } A_u = 0, \\
6, & \text{if } q \equiv 1 \pmod{3}, \ p \neq 2 \text{ and } B_u = 0, \\
12, & \text{if } q \equiv 1 \pmod{3} \text{ and } A_uB_u \neq 0,
\end{cases}
\]

Proof. Let \( u \in \mathbb{F}_q \) with \( u^3 \neq 27 \). From Lemma 11, we see that

\[
\mathcal{I}_{H,u} = \begin{cases} 
\{u\}, & \text{if } q \equiv 0, 2 \pmod{3}, \\
\mathcal{J}_{H,u}, & \text{if } q \equiv 1 \pmod{3}.
\end{cases}
\]

Then Lemma 11 completes the proof. \( \square \)

The following theorems give the number of distinct Hessian curves over \( \mathbb{F}_q \).
Theorem 13. For any prime $p$, for the number $J_H(q)$ of distinct values of the $j$-invariant of the family (6), we have

$$J_H(q) = \begin{cases} 
q - 1 & \text{if } q \equiv 0 \pmod{3}, \\
\left\lfloor \frac{q + 11}{12} \right\rfloor & \text{if } q \equiv 1 \pmod{3}, \\
\left\lfloor \frac{q}{2} \right\rfloor & \text{if } q \equiv 2 \pmod{3}.
\end{cases}$$

Proof. We note that

$$J_H(q) = \sum_{u \in \mathbb{F}_q, u^3 \neq 27} \frac{1}{\#J_{H,u}}.$$

Let

$$N_k = \# \{ u : u \in \mathbb{F}_q, u^3 \neq 27, \#J_{H,u} = k \}, \quad k = 1, 2, \ldots.$$ 

From Lemma 10, we see that $N_k = 0$ for $k > 12$. Then,

$$J_H(q) = \frac{12}{\sum_{k=1}^{12} N_k}. \quad (18)$$

We consider the following possibilities for $q$. First, we assume that $q \equiv 0 \pmod{3}$. From Lemma 10 we see that $N_1 = q - 1$ and $N_k = 0$, for $k \geq 2$. Then, using (18), we have

$$J_H(q) = q - 1, \quad \text{if } q \equiv 0 \pmod{3}.$$

Second, we assume that $q \equiv 1 \pmod{3}$. In this case, the number of third roots of 27 in $\mathbb{F}_q$ equals 3. If $p = 2$, then $N_1 = 1, N_{12} = q - 4$ and $N_k = 0$, for $2 \leq k \leq 11$ (see Lemma 10). Using (18), we have $J_H(q) = (q + 8)/12$.

If $p \neq 2$, then $N_k = 0$ for $k \neq 4, 6, 12$. Also, $N_4 = d_1, N_6 = d_2$ and $N_{12} = q - 3 - d_1 - d_2$, where $d_1, d_2$ are the numbers of $u \in \mathbb{F}_q$ with $A_u = 0, B_u = 0$ respectively (see Lemma 10). Then, using (18), we have

$$J_H(q) = (q + 2d_1 + d_2 - 3)/12.$$

We see that, $d_1$ is the number of solutions in $\mathbb{F}_q$ to the equation $U(U^3 + 6^3) = 0$. Then, $d_1 = 4$, since $q \equiv 1 \pmod{3}$ and $p \neq 2$. Next, we compute $d_2$, i.e., the number of solutions in $\mathbb{F}_q$ to the equations in (17). We note that, the sets of solutions to the equations in (17) are disjoint. Moreover, they do not intersect with the set of the third roots of 27 in $\mathbb{F}_q$.

The discriminant of the quadratic equation $U^2 - 6U - 18 = 0$ in (17) equals 108. So, it has two solutions if 3 is a quadratic residue in $\mathbb{F}_q$ or no solution if 3 is a quadratic non-residue in $\mathbb{F}_q$. We note that $-3$ is a quadratic residue in $\mathbb{F}_q$, since $q \equiv 1 \pmod{3}$. Also, $-1$ is a quadratic residue in $\mathbb{F}_q$ if and only if $q \equiv 1 \pmod{4}$. Hence, the number of solutions to the quadratic equation in (17) is 2 if $q \equiv 1 \pmod{12}$ and 0 if $q \equiv 7 \pmod{12}$.

The quartic equation in (17) can be factored as

$$(U^2 + 3(1 + \omega)U + 9(1 - \omega))(U^2 + 3(1 - \omega)U + 9(1 + \omega)),$$
where $\pm \omega$, are the quadratic roots of $-3$ in $\mathbb{F}_q$. Since, the discriminants of above quadratic factors are $27(1 \pm \omega)^2$, the number of solutions to the quadratic equation in (17) is 4 if $q \equiv 1 \pmod{12}$ and 0 if $q \equiv 7 \pmod{12}$.

Therefore, $d_2 = 6$ if $q \equiv 1 \pmod{12}$ and $d_2 = 0$ if $q \equiv 7 \pmod{12}$. Hence,

$$J_H(q) = \begin{cases} 
\frac{(q + 11)}{12}, & \text{if } q \equiv 1 \pmod{12}, \\
\frac{(q + 8)}{12}, & \text{if } q \equiv 4 \pmod{12}, \\
\frac{(q + 5)}{12}, & \text{if } q \equiv 7 \pmod{12}.
\end{cases}$$

Now, we assume that $q \equiv 2 \pmod{3}$. Then, the number of third roots of $27$ in $\mathbb{F}_q$ equals 1. If $p = 2$, then Lemma 10 shows that $N_1 = 1$, $N_2 = q - 2$ and $N_k = 0$, for $k \geq 3$. Then, using (18), we have $J_H(q) = q/2$. If $p \neq 2$, then from Lemma 10 we have $N_1 = 0$, $N_2 = q - 1$ and $N_k = 0$, for $k \geq 3$. Next, using (18), we obtain $J_H(q) = (q - 1)/2$. Hence,

$$J_H(q) = \begin{cases} 
\frac{q}{2}, & \text{if } q \equiv 2 \pmod{6}, \\
\frac{(q - 1)}{2}, & \text{if } q \equiv 5 \pmod{6}.
\end{cases}$$

In the following theorem, we study the number of $\mathbb{F}_q$-isomorphism classes of Hessian curves.

**Theorem 14.** For any prime $p$, for the number $I_H(q)$ of $\mathbb{F}_q$-isomorphism classes of the family (6), we have

$$I_H(q) = \begin{cases} 
\left\lfloor \frac{q + 11}{12} \right\rfloor & \text{if } q \equiv 1 \pmod{3}, \\
q - 1 & \text{if } q \equiv 0, 2 \pmod{3}.
\end{cases}$$

**Proof.** From Lemma 12, we recall that

$$\#I_{H,a} = \begin{cases} 
1, & \text{if } q \equiv 0, 2 \pmod{3}, \\
\#J_{H,a}, & \text{if } q \equiv 1 \pmod{3}.
\end{cases}$$

Since $\#I_H(q) = \sum_{u \in \mathbb{F}_q, u^3 \neq 27} \frac{1}{\#I_{H,a}}$, if $q \equiv 0, 2 \pmod{3}$, we see $\#I_H(q) = q - 1$. Moreover, if $q \equiv 0, 2 \pmod{3}$, Theorem 13 gives the cardinality of $I_H(q)$. Therefore, we have

$$I_H(q) = \begin{cases} 
\frac{(q + 11)}{12}, & \text{if } q \equiv 1 \pmod{12}, \\
\frac{(q + 8)}{12}, & \text{if } q \equiv 4 \pmod{12}, \\
\frac{(q + 5)}{12}, & \text{if } q \equiv 7 \pmod{12}, \\
q - 1, & \text{if } q \equiv 0, 2 \pmod{3}.
\end{cases}$$

\[\square\]
6 Generalized Hessian curves

We consider the generalized Hessian curves $E_{GH,u,v}$ given by (7) over a finite field $\mathbb{F}_q$ of characteristic $p$. Obviously, a Hessian curve $E_{H,u}$ given by (6) is a generalized Hessian curve $E_{GH,u,v}$ with $v = 1$. Moreover, the curve $E_{GH,u,v}$ over $\mathbb{F}_q$, via the map $(X,Y) \mapsto (\tilde{X},\tilde{Y})$ defined by
\[ \tilde{X} = X/\zeta \quad \text{and} \quad \tilde{Y} = Y/\zeta \]
with $\zeta^3 = v$, is isomorphic over $\overline{\mathbb{F}}_q$ to the Hessian curve $E_{H,\zeta} : \tilde{X}^3 + \tilde{Y}^3 + 1 = \zeta \tilde{X}\tilde{Y}$. Therefore, for the $j$-invariant of $E_{GH,u,v}$, we have
\[ j(E_{GH,u,v}) = j(E_{H,\zeta}) = \frac{1}{v} \left( \frac{u(u^3 + 216v)}{u^3 - 27v} \right)^3. \]

For a fixed element $v$ in $\mathbb{F}_q$, with $v \neq 0$, we let
\[ J_{GH,v} = \{ j \mid j = j(E_{GH,u,v}), \ u \in \mathbb{F}_q, u^3 \neq 27v \}, \]
and let $J_{GH} = \bigcup_{v \in \mathbb{F}_q} J_{GH,v}$. Clearly, we have $J_{GH}(q) = \#J_{GH}$.

Lemma 15. Let $v_1, v_2 \in \mathbb{F}_q^*$ and let $v = v_1/v_2$. If $v$ is a cube in $\mathbb{F}_q$, then we have $J_{GH,v_1} = J_{GH,v_2}$, otherwise we have $J_{GH,v_1} \cap J_{GH,v_2} = \{0\}$.

Proof. Suppose $v = \zeta^3$ is a cube in $\mathbb{F}_q$. For all $u \in \mathbb{F}_q$ with $u^3 \neq 27v$, we have $j(E_{GH,u,v_1}) = j(E_{GH,u/\zeta,v_2})$ and similarly $j(E_{GH,u,v_2}) = j(E_{GH,u,\zeta,v_1})$. Therefore, $J_{GH,v_1} = J_{GH,v_2}$.

Next, suppose that $v$ is not a cube in $\mathbb{F}_q$. Let $j \in J_{GH,v_1} \cap J_{GH,v_2}$. Then,
\[ j = \frac{1}{v_1} \left( \frac{u_1(u_1^3 + 216v_1)}{u_1^3 - 27v_1} \right)^3 = \frac{1}{v_2} \left( \frac{u_2(u_2^3 + 216v_2)}{u_2^3 - 27v_2} \right)^3, \]
for some $u_1, u_2 \in \mathbb{F}_q$. If $j \neq 0$, we see that $v = v_1/v_2$ is a cube in $\mathbb{F}_q$, which is a contradiction. So, $J_{GH,v_1} \cap J_{GH,v_2} = \{0\}$. \qed

Lemma 16. For $q \equiv 1 \pmod{3}$, if $v$ is not a cube in $\mathbb{F}_q$, we have $\#J_{GH,v} = (q + 2)/3$.

Proof. For $u \in \mathbb{F}_q$ with $u^3 \neq 27v$, we let $j(E_{GH,u,v}) = \frac{1}{v} (F(u))^3$ where $F(U) = \frac{U(U^3 + 216v)}{U^3 - 27v}$. We consider the bivariate rational function $F(U) - F(V)$. We obtain
\[ F(U) - F(V) = \frac{U - V}{U^3 - 27v} \prod_{i=1}^3 \left( U - \frac{3\zeta_i(V + 6\zeta_i)}{V - 3\zeta_i} \right), \]
where, $\zeta_1, \zeta_2, \zeta_3$ are three cubic roots of $v$ in $\overline{\mathbb{F}}_q$. For all $u_1, u_2 \in \mathbb{F}_q$ with $u_1^3 \neq 27v, u_2^3 \neq 27v$, we see that $F(u_1) = F(u_2)$ if and only if $u_1 = u_2$. Hence, $F$ is an injective map over $\mathbb{F}_q$ and we have $F(\mathbb{F}_q) = \mathbb{F}_q$. Now, consider the map $\kappa : \mathbb{F}_q^* \to \mathbb{F}_q^*$ by $\kappa(x) = \frac{1}{v}x^3$. This map is $3 : 1$, if $q \equiv 1 \pmod{3}$. Therefore, $\#J_{GH,v} = (q - 1)/3 + 1$. \qed
**Theorem 17.** For any prime $p$, for the number $J_{GH}(q)$ of distinct values of the $j$-invariant of the family (7), we have

$$J_{GH}(q) = \begin{cases} q - 1, & \text{if } q \equiv 0 \pmod{3} \\ \lfloor (3q + 1)/4 \rfloor, & \text{if } q \equiv 1 \pmod{3} \\ \lfloor q/2 \rfloor, & \text{if } q \equiv 2 \pmod{3} \end{cases}.$$ 

**Proof.** If $q \not\equiv 1 \pmod{3}$, every element of $\mathbb{F}_q$ is a cube in $\mathbb{F}_q$. Next, Lemma [15] implies that, for all $v \in \mathbb{F}_q^*$, we have $J_{GH,v} = J_{GH,1}$. Therefore, $J_{GH}(q) = \#J_{GH,1}$. Then, from Theorem [13] we have

$$J_{GH}(q) = \begin{cases} q - 1, & \text{if } q \equiv 0 \pmod{3} \\ \lfloor q/2 \rfloor, & \text{if } q \equiv 2 \pmod{3} \end{cases}.$$ 

For $q \equiv 1 \pmod{3}$, we fix a value $v \in \mathbb{F}_q$ that is not a cube in $\mathbb{F}_q$. Following Lemma [15] we write $J_{GH} = J_{GH,v} \cup J_{GH,v^2} \cup J_{GH,1}$, where $J_{GH,v} \cap J_{GH,v^2} = J_{GH,v} \cap J_{GH,1} = J_{GH,v^2} \cap J_{GH,1} = \{0\}$. By Lemma [13] we have $\#J_{GH,v} = \#J_{GH,v^2} = (q+1)/2$. Moreover, from Theorem [13] we have

$$\#J_{GH,1} = \begin{cases} (q+11)/12, & \text{if } q \equiv 1 \pmod{12} \\ (q+8)/12, & \text{if } q \equiv 4 \pmod{12} \\ (q+5)/12, & \text{if } q \equiv 7 \pmod{12} \end{cases}.$$ 

Therefore, we have

$$J_{GH}(q) = \begin{cases} (3q+1)/4, & \text{if } q \equiv 1 \pmod{12} \\ 3q/4, & \text{if } q \equiv 4 \pmod{12} \\ (3q-1)/4, & \text{if } q \equiv 7 \pmod{12} \end{cases}$$

which completes the proof.

We recall from Theorem [14] that the number of $\mathbb{F}_q$-isomorphism classes of Hessian curves over $\mathbb{F}_q$ is $\lfloor (q+11)/12 \rfloor$ if $q \equiv 1 \pmod{3}$ and $q-1$ if $q \not\equiv 1 \pmod{3}$. The following theorem gives explicit formulas for the number of distinct generalized Hessian curves, up to $\mathbb{F}_q$-isomorphism, over the finite field $\mathbb{F}_q$.

**Theorem 18.** For any prime $p$, for the number $I_{GH}(q)$ of $\mathbb{F}_q$-isomorphism classes of the family (7), we have

$$I_{GH}(q) = \begin{cases} \lfloor (3(q+3)/4 \rfloor, & \text{if } q \equiv 1 \pmod{3} \\ q - 1, & \text{if } q \equiv 0,2 \pmod{3} \end{cases}.$$ 

**Proof.** If $q \equiv 0,2 \pmod{3}$, then every generalized Hessian curve is $\mathbb{F}_q$-isomorphic to a Hessian curve via the map given by Equations (19). So, $I_{GH}(q)$ equals the number of $\mathbb{F}_q$-isomorphism classes of the family of Hessian curves over $\mathbb{F}_q$. Then, from Theorem [14] we have $I_{GH}(q) = q - 1$ if $q \not\equiv 1 \pmod{3}$.
Next, suppose that $q \equiv 1 \pmod{3}$. For $a \in \mathbb{F}_q$, let $i_{GH}(a)$ be the set of $\mathbb{F}_q$-isomorphism classes of generalized Hessian curves $GH_{u,v}$ with $j(E_{GH,u,v}) = a$. So, $#i_{GH}(a)$ is the number of distinct generalized Hessian curves with $j$-invariant $a$ that are twists of each other. Clearly, $#i_{GH}(a) = 0$, if $a \not\in \mathcal{J}_{GH}$. We note that, for all elliptic curve $E$ over $\mathbb{F}_q$, we have $\#E(\mathbb{F}_q) + \#E_t(\mathbb{F}_q) = 2q + 2$, where $E_t$ is the nontrivial quadratic twist of $E$. We also recall that the order of the group of $\mathbb{F}_q$-rational points of a generalized Hessian curve is divisible by 3 (see [13, Theorem 2]). Since $q \equiv 1 \pmod{3}$, if the isomorphism class of $E_{GH,u,v}$ is in $i_{GH}(a)$ then the isomorphism class of the nontrivial quadratic twist of $E_{GH,u,v}$ is not in $i_{GH}(a)$. So, $#i_{GH}(a) = 1$ if $a \in \mathcal{J}_{GH}$ and $a \neq 0, 1728$. Moreover, one can show that $#i_{GH}(a) = 3$ if $a = 0$ and $#i_{GH}(a) = 1$ if $a = 1728$, $a \neq 0$ and $a \in \mathcal{J}_{GH}$. Therefore, we have

$$I_{GH}(q) = \sum_{a \in \mathbb{F}_q} i_{GH}(a) = \sum_{a \in \mathcal{J}_{GH}} i_{GH}(a) = 2 + \sum_{a \in \mathcal{J}_{GH}} 1 = 2 + J_{GH}(q).$$

From the proof of Theorem 17, we have

$$I_{GH}(q) = \begin{cases} (3q + 9)/4, & \text{if } q \equiv 1 \pmod{12} \\ (3q + 8)/4, & \text{if } q \equiv 4 \pmod{12} \\ (3q + 7)/4, & \text{if } q \equiv 7 \pmod{12} \end{cases},$$

which completes the proof.

\section{Comments and Open Questions}

There are also several more interesting families of Elliptic curves such as Montgomery curves [23], Edwards curves [11], and its variants [4, 2, 5]. The numbers of distinct $j$-invariants of the families of Edwards curves in [11, 4] have been studied in [15, Theorems 3 and 5]. Moreover, the explicit formulas for the numbers of $\mathbb{F}_q$-isomorphism classes of these families are given in [12, Theorems 5, 6 and 8]. The proofs of the latter Theorems can be provided by our method. However, we refer to [14] for the other proofs via different techniques.

For future work, we plan to study the exact formulas for the number of distinct elliptic curves $E$ over $\mathbb{F}_q$, where $E(\mathbb{F}_q)$ has a specific small subgroup. In particular, we give the explicit formulas for the number of elliptic curves $E$ over $\mathbb{F}_q$ with a point of small order $n$.

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