Solutions of Second-Order PDEs with First-Order Quotients

E. Schneider*

(Submitted by V. V. Lychagin)

Faculty of Science, University of Hradec Králové, Hradec Králové, 50003 Czech Republic
Received May 30, 2020; revised June 18, 2020; accepted June 22, 2020

Abstract—We investigate a general approach for solving a partial differential equation by using the differential invariants of its point symmetries. By first solving its quotient PDE, which is given by the differential syzygies in the algebra of differential invariants, we obtain new differential constraints which are compatible with the PDE under consideration. Adding these constraints to our system makes it overdetermined, and easier to solve. We focus on second-order scalar PDEs on a function of two variables whose quotients are first-order scalar PDEs. This situation occurs only when the Lie algebra of symmetries of the second-order PDE is infinite-dimensional. We apply this idea to various second-order PDEs with infinite-dimensional symmetry Lie algebras, one of which is the Hunter–Saxton equation.

DOI: 10.1134/S1995080220120367

Keywords and phrases: Nonlinear differential equations, differential invariants, quotient PDE, differential syzygies, Hunter–Saxton equation

1. INTRODUCTION

For a PDE $\mathcal{E}$ with a Lie algebra $\mathfrak{g}$ of (point) symmetries, the differential algebra of scalar differential invariants is generated by a finite number of differential invariants and invariant derivations. The differential syzygies in this algebra are algebraic equations on the generators and their invariant derivatives. When the differential syzygies are written in terms of Tresse derivatives, the interpretation of the differential syzygies as the defining equations for the quotient PDE becomes clear. Each solution of the quotient determines a set of additional differential constraints that can be added to the defining equations for $\mathcal{E}$, resulting in an overdetermined system which is, in general, easier to solve than $\mathcal{E}$ itself.

We note that this description gives a transparent view of the quotient and allows for efficient use of computer algebra systems. We also point out the global nature of the quotient [9].

This idea can be traced back to Lie [12] and Vessiot [2], and is commonly referred to as group splitting or the group foliation method. It was made popular by Ovsiannikov [15] and has recently been applied to several nonlinear PDEs (see for example [1, 18, 5]). In [20], the method was formulated in terms of moving frames and applied to several examples. We also note that similar ideas have been explored in the context of exterior differential systems [2, 16].

The main challenges in the above approach are to solve the quotient PDE to obtain the additional differential constraints and to solve the resulting overdetermined PDE. Finding the expressions for the quotient is not trivial in general, but it depends solely on solving linear differential equations and doing algebraic manipulations.

In this paper we restrict ourselves to the case where $\mathcal{E}$ is a second-order scalar PDE on a function of two variables. In general, we can not expect the quotient PDE to be easier to solve than the original PDE. Therefore we will focus on PDEs with a first-order scalar quotient PDE, which can, at least in principle, be solved by the method of characteristics. We will see that such quotients appear only when the symmetry Lie algebra is infinite-dimensional. The significance of infinite-dimensional Lie algebras in the application of these ideas was pointed out by Lie [12].

*E-mail: eivind.schneider@uhk.cz
With several examples, some of which are well known, we apply the ideas outlined above with the goal of finding general solutions of second-order scalar PDEs on functions of two variables.

Section 2 contains a comprehensive overview of the theory of jet spaces and differential invariants, with the aim of making the paper self-contained and accessible to a wide audience. We illustrate the concepts using Burgers’ equation as a running example and provide Maple code in order to show how the computations in this paper are done and to demonstrate that they are well-suited for computer algebra systems. In Section 2.7, we show that a necessary condition for the quotient to be a first-order scalar PDE is that the Lie algebra of symmetries is infinite-dimensional.

A reader with some knowledge in the theory of differential invariants, or one who is mostly interested in seeing how the ideas work in practice, may wish to jump directly to Section 3, where we describe in detail how to find the general solution to the Hunter–Saxton equation. It illustrates the ideas in detail and gives a new perspective on the general solution found by Hunter and Saxton. We also use this example to discuss how the method can be used when initial data is given.

In Section 4, we consider five different infinite-dimensional Lie algebras of vector fields. We find invariant PDEs and compute their quotients. Details here are sparse, as the purpose is to illustrate the general method with many examples rather than getting lost in the details of each of them. For some of the equations, we compute their quotient with respect to a finite-dimensional Lie subalgebra of the infinite-dimensional one. In this case, the quotient can be presented as a system of two, partially uncoupled, first-order PDEs on two functions instead of a single first-order PDE on one function. By solving these, in sequence, we obtain two additional differential constraints that, together with the original second-order PDE, determine a five-dimensional manifold in \( J^2(\mathbb{R}^2) \). The Cartan distribution restricted to this manifold is completely integrable, and the three-dimensional symmetry Lie algebra is transversal to it.

2. SYMMETRIES OF PDES AND DIFFERENTIAL INVARIANTS

We will focus our attention on differential equations of the form

\[
F(t, x, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}) = 0
\]

on a function \( u(t, x) \), with a nontrivial Lie algebra (sheaf) \( g \) of (infinitesimal point) symmetries.

In this section we introduce the necessary concepts from the geometric theory of PDEs and differential invariants. For a more detailed treatment of these topics we recommend \([8–10, 14]\). We will see how solutions of the quotient PDE give rise to additional differential constraints of the form

\[
G(t, x, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}) = 0
\]

which are compatible with (1) and \( g \)-invariant. At the end of this section we explain how imposing particular requirements on the algebra of differential invariants (by requiring the quotient to be a scalar first-order PDE) puts restrictions on \( g \).

2.1. The PDE as a Manifold

Fix a point \( a \in \mathbb{R}^2 \), and let \( J^k_a(\mathbb{R}^2) \) denote the set of \( k \)-degree Taylor polynomials of smooth functions on \( \mathbb{R}^2 \) centered at the point \( a \). Let \( J^k(\mathbb{R}^2) = \bigcup_{a \in \mathbb{R}^2} J^k_a(\mathbb{R}^2) \), so that \( J^k(\mathbb{R}^2) \) is a bundle over \( \mathbb{R}^2 \). Denote the projection \( J^k(\mathbb{R}^2) \to \mathbb{R}^2 \) by \( \pi_k \). For any function \( f \in C^\infty(\mathbb{R}^2) \), its \( k \)-jet \( [f]^k_a \in J^k_a(\mathbb{R}^2) \) at \( a \in \mathbb{R}^2 \) is the \( k \)-degree Taylor polynomial of \( f \) centered at \( a \).

We will use coordinates \( t, x, u, u_t, u_x, ..., u_{txk-1}, u_{xk} \) on \( J^k(\mathbb{R}^2) \). If \( a \in \mathbb{R}^2 \) is given by \( (t_0, x_0) \), then the coordinates of \( \theta = [f]^k_a \) are

\[
\begin{align*}
t(\theta) &= t_0, & x(\theta) &= x_0, & u(\theta) &= f(t_0, x_0), & u_t(\theta) &= \frac{\partial f}{\partial t}(t_0, x_0), \\
u_x(\theta) &= \frac{\partial f}{\partial x}(t_0, x_0), & ..., & u_{txk-1}(\theta) &= \frac{\partial^k f}{\partial t \partial x^{k-1}}(t_0, x_0), & u_{xk}(\theta) &= \frac{\partial^k f}{\partial x^k}(t_0, x_0).
\end{align*}
\]

By varying \( a \), we see that any function \( f \) gives rise to a section of \( \pi_k \) which we denote by \( j^k f \). It is given by \( j^k f(t, x) = [f]^{k}_{(t, x)} \).

LOBACHEVSKII JOURNAL OF MATHEMATICS Vol. 41 No. 12 2020
There is additional geometric structure on $J^k(\mathbb{R}^2)$ responsible for filtering out, from the set of all sections of $E_k$, those that are of the form $j^k f$. It is called the Cartan distribution, denoted $\mathcal{C}^k$. At each point $\theta \in J^k(\mathbb{R}^2)$, it defines a subspace $\mathcal{C}^k_\theta \subset T_\theta J^k(\mathbb{R}^2)$. It is the span of tangent planes of sections of the form $j^k f$ with the property $\theta = j^k f(\pi_k(\theta))$. In $J^1(\mathbb{R}^2)$ the Cartan distribution is the kernel of the one-form $\omega_0 = du - u dt - u dx$, and in $J^2(\mathbb{R}^2)$ it is the kernel of the three one-forms $\omega_0, du_t - u_t dt - u_t dx, du_x - u_x dt - u_x dx$. In a similar way we may define the Cartan distribution on $J^k(\mathbb{R}^2)$ as the kernel of the following differential constraint (which are obtained by differentiating $F = 0$, we get $\dim \mathcal{E}_k = \dim J^k(\mathbb{R}^2)$, we expect the solution space to be parametrized by two functions of one variable.

By interpreting (1) as an equation on $J^2(\mathbb{R}^2)$, we obtain a submanifold $E_2 \subset J^2(\mathbb{R}^2)$. We will assume that $dF|_{E_2} \neq 0$. The significance of this manifold comes from the following fact: If $f$ is a solution to (1) defined on $D \subset \mathbb{R}^2$, then $J^2 f(D)$ is a two-dimensional submanifold of $E_2$. Moreover it is an integral manifold of the restriction of the Cartan distribution to $E_2$.

From this viewpoint we get a natural generalization of the concept of solution to (1), namely a two-dimensional integral manifold of the Cartan distribution. We will see in some of the examples below that we get solutions that are not given globally by a function on $\mathbb{R}^2$.

A smooth solution to (1) is also a solution to the third-order equations $D_t(F) = 0, D_x(F) = 0$. We define $E_3 = \{ F = 0, D_t(F) = 0, D_x(F) = 0 \} \subset J^3(\mathbb{R}^2)$, and similarly, by repeated differentiation, $E_k \subset J^k(\mathbb{R}^2)$. We will also use the notation $E_0 = J^0(\mathbb{R}^2)$ and $E_1 = J^1(\mathbb{R}^2)$ when convenient.

Since $\dim J^k(\mathbb{R}^2) = 2 + (k+2)$ and $E_k$ is given by $\binom{k}{2}$ independent differential constraints (which are obtained by differentiating $F = 0$, we get $\dim E_k = 3 + 2k$. Naively, considering formal solutions of $F = 0$, we may use this count to estimate the size of the solution space of (1). Since $\dim E_k = \dim J^k(\mathbb{R}^2)$, we expect the solution space to be parametrized by two functions of one variable.

**Burgers’ equation:** We will use Burgers’ equation as a running example to illustrate the concepts in this section. Burger’s equation is defined by $F = u_{xx} - u_t - uu_x = 0$. It defines a seven-dimensional submanifold $E_2 \subset J^2(\mathbb{R}^2)$. Its prolongation $E_3$ is defined by $D_x(F) = u_{xxx} - u_{tx} - u_x^2 - uu_{xx} = 0, D_t(F) = u_{txx} - u_{tt} - uu_{tx} - u_t u_x = 0$ in addition to $F = 0$, and is a nine-dimensional submanifold in $J^3(\mathbb{R}^2)$.

### 2.2. Point Symmetries

Let $X$ be a vector field on $J^0(\mathbb{R}^2)$. In coordinates it takes the form $X = a(t, x, u) \partial_t + b(t, x, u) \partial_x + c(t, x, u) \partial_u$.

There is a unique vector field $X^{(k)}$ on $J^k(\mathbb{R}^2)$ that projects to $X$ and preserves the Cartan distribution on $J^k(\mathbb{R}^2)$. We call it the $k$th prolongation of $X$. The flow of $X^{(k)}$ takes integral manifolds of the Cartan distribution on $J^k(\mathbb{R}^2)$ to integral manifolds. The formula for $X^{(k)}$ can be found in many introductory treatments of this topic. See for example [8, 10]. The computations in this paper are done with the help of the DifferentialGeometry and JetCalculus packages in Maple, where the Prolongation procedure can compute the prolongations of $X$.

**Definition 1.** A vector field $X = a(t, x, u) \partial_t + b(t, x, u) \partial_x + c(t, x, u) \partial_u$ is a (point) symmetry of $F = 0$ (or $E$) if $X^{(2)}$ is tangent to $E_2 \subset J^2(\mathbb{R}^2)$, i.e.

$$X^{(2)}(F)|_{E_2} = 0.$$  \hfill (2)

It follows that $X^{(k)}$ is tangent to $E_k$ for every $k$. The set of symmetries forms a Lie algebra, which may be infinite-dimensional. Since $X^{(k)}$ preserves the Cartan distribution, its flow acts on the space of integral manifolds of the Cartan distribution on $E_k$ and thus on the space of solutions of $F = 0$.

Equation (2) is a polynomial in $u_t, u_x, u_{tt}, u_{tx}, u_{xx}$, and restricting to $E_2$ can be done by using $F = 0$ to write one of these coordinates in terms of the others. The vanishing of the remaining polynomial is
equivalent to the vanishing of each of its coefficients, which are linear differential equations on $a, b$ and $c$. This system of PDEs is often highly overdetermined and not difficult to solve.

**Symmetries of Burgers' equation:** The Lie algebra of point symmetries of Burgers' equation is spanned by

$$\partial_t, \quad \partial_x, \quad t \partial_x + \partial_u, \quad t^2 \partial_t + tx \partial_x + (x - tu) \partial_u, \quad 2t \partial_t + x \partial_x - u \partial_u.$$  

We show how the symmetries of Burgers’ equation can be found with a few lines of Maple code.

```maple
restart: with(DifferentialGeometry):
with(JetCalculus): DGsetup([t,x], [u], E, 2):
F := u[2,2] - u[1]*u[2] - u[1]:
```

We see that the number of independent differential invariants is three.

**2.3. Differential Invariants**

We continue to consider the arbitrary, but fixed differential equation $F = 0$ and its corresponding submanifolds $E_k \subset J^k(\mathbb{R}^2)$. Let $\mathfrak{g}$ be a Lie algebra of symmetries, possibly a Lie subalgebra of the full symmetry Lie algebra.

**Definition 2.** A differential invariant of order $k$ is a function on $E_k$ which is constant on $\mathfrak{g}$-orbits.

By $\mathfrak{g}$-orbits in $E_k$ we mean orbits of the Lie algebra consisting of $k$th prolongations of all vector fields from $\mathfrak{g}$. This use of language simplifies notation, and should not lead to confusion.

By this definition, a differential invariant $I \in C^\infty_{\text{loc}}(E_k)$ satisfies the PDE

$$X^{(k)}(I)|_{E_k} = 0 \quad (3)$$

for every $X \in \mathfrak{g}$. In all computations below this is the system of linear PDEs we will solve in order to find a generating set of differential invariants. It is sufficient to check (3) on basis elements, and even when $\mathfrak{g}$ is infinite-dimensional the system will consist of finitely many independent equations (for any fixed order $k$).

We will exclusively consider invariants whose restrictions to fibers of $E_k \rightarrow J^0(\mathbb{R}^2)$ are rational functions, as this is, in most cases of interest, sufficient to separate orbits in general position (see [9]). There are some technical requirements for this, concerning algebraicity of $E$ and the symmetry Lie algebra (or Lie pseudogroup) under consideration. We direct the interested reader to [9], as we will not go deep into this topic here.

Let $A_k$ denote the algebra of differential invariants of order $k$. We have $A_i \subset A_{i+1}$ for $i > 0$. If $s_k$ denotes the number of functionally independent elements in $A_k$, then $s_k$ is equal to the codimension of a $\mathfrak{g}$-orbit in $E_k$ in general position. Define $H_k = s_k - s_{k-1}$ and $H_0 = s_0$. The function $H_k$ of $k$ is called the Hilbert function of the algebra of differential invariants. Since $\dim E_k = 3 + 2k$ we get $H_k \leq 2$ for $k \geq 0$. ($H_0 = 3$ only if $\mathfrak{g}$ is trivial.)

We see that the number of independent differential invariants of order $k$ can, and usually will, increase without bound as $k$ increases. In the next section we introduce invariant derivations, which turn the algebra of differential invariants into a differential algebra, which can be generated by a finite number of differential invariants.
If $s$ is a solution to $\mathcal{E}$, given by a function $f$ on $D \subset \mathbb{R}^2$, we denote the restriction of a $k$th order invariant $I$ to $s$ by $I_s$. It determines a function $I \circ j^k f$ on $D$.

**Differential invariants of Burgers’ equation:** We compute second-order differential invariants of Burgers’ equation with respect to the three-dimensional symmetry Lie algebra $\mathfrak{h} = \langle \partial_t, \partial_x, t \partial_x + \partial_u \rangle$ using Maple.

```maple
sym := [D_t, D_x, t*D_x + D_u[]):
pdsolve(Pullback(phi, map(LieDerivative, map(Prolong, sym, 3), f(t, x, u[], u[1], u[2], u[1, 1], u[1, 2]))));
```

Notice that this is a situation in where Maple’s `pdsolve` can be safely used. Since we have other ways of knowing how many independent invariants exist, `pdsolve`’s output can easily be checked.

There are four second-order invariants:

$$u_x, \quad u_t + uu_x, \quad u_{tx} + u(u_t + uu_x), \quad u_{tt} + 4u_t u_x + 2(2u_x + u_{tx}) + u^2(u_t + uu_x)$$

Here we have used the variable $t, x, u, u_t, u_x, u_{tt}, u_{tx}$ as coordinates on $\mathcal{E}_2$. Notice that these invariants can also be given by

$$I = u_x, \quad J = u_{xx}, \quad H = u_{xxx}, \quad K = u_{xxxx}.$$  

The rewriting may be done by using $F = 0$ and its derivatives. In Maple the rewriting can be done like this:

```maple
A := u[2]: B := u[2, 2]: H := u[2, 2, 2]: K := u[2, 2, 2, 2]:
Pullback(phi, [A, B, H, K]);
```

We named the invariants $A$ and $B$ in Maple, instead of $I$ and $J$, because Maple’s $I$ is the imaginary unit. Since $\dim \mathcal{E}_2 = 7$, and $\mathfrak{h}$ orbits are three-dimensional (the action is free already on $J^0(\mathbb{R}^2)$), the transcendence degree of the field of second-order differential invariants is 4. In the next section we will show how to generate the rest of the differential invariants.

### 2.4. Invariant Derivations

The only invariant derivations we will encounter in this paper are the Tresse derivatives. They are a commuting pair of invariant derivations that play the roles of partial derivatives with respect to a pair of independent differential invariants. In order to construct them, it will be useful to have the notion of horizontal differential. The horizontal differential $\hat{d}$ on a function $\psi$ on $\mathcal{E}_k$ is given in coordinates by

$$\hat{d}\psi = D^k_t(\psi)dt + D^k_x(\psi)dx,$$

where $D^k_t$ and $D^k_x$ are the restrictions of the total derivatives

$$D_t = \partial_t + u_t \partial_u + u_{tt} \partial_{u_t} + u_{tx} \partial_{u_x} + \cdots, \quad D_x = \partial_x + u_x \partial_u + u_{tx} \partial_{u_t} + u_{xx} \partial_{u_x} + \cdots$$

to $\mathcal{E}$. If $s$ is a solution of $F = 0$ given by the function $f$ defined on $D \subset \mathbb{R}^2$, then

$$\hat{d}\psi \circ j^k f = d(\psi \circ j^k f)$$

on $D$. Let $I, J$ be two differential invariants of order $k$ with the property

$$\hat{d}I \wedge \hat{d}J \neq 0. \quad (4)$$

In general, this inequality will hold on a Zariski-open set in $\mathcal{E}_{k+1}$, and all subsequent computations we do are restricted to this Zariski-open set, even if it is not mentioned explicitly. In particular, when we are going to compute solutions of $F = 0$, we can not expect to find solutions whose $(k + 1)$-jets lie outside this set.

Assuming that (4) holds, we call the pair $\hat{d}I, \hat{d}J$ a horizontal coframe (for a solution $s$ in general position, the pair $(\hat{d}I)_s, (\hat{d}J)_s$ will give a coframe on the two-dimensional manifold $s$). Its dual horizontal frame consists of derivations, which we denote by $\hat{\partial}_t, \hat{\partial}_j$, that are of the form $\alpha D_t + \beta D_x$ and satisfy

$$\hat{d}(\hat{\partial}_t) = 1, \quad \hat{d}(\hat{\partial}_j) = 0, \quad \hat{d}J(\hat{\partial}_t) = 0, \quad \hat{d}J(\hat{\partial}_j) = 1.$$  

Here $\alpha, \beta$ are functions on $\mathcal{E}_{k+1} \subset J^{k+1}(\mathbb{R}^2)$. 
The derivations \( \partial_I \) and \( \partial_J \) commute, and they are invariant with respect to \( \mathfrak{g} \), in the sense that
\[
[X^{(\infty)}, \partial_I] = 0, \quad [X^{(\infty)}, \partial_J] = 0
\]
for every \( X \in \mathfrak{g} \). In general, if \( H \) is an invariant of order \( l \geq k \), then \( \partial_I(H) \) and \( \partial_J(H) \) will be invariants of order \( l + 1 \), and if we apply the Tresse derivatives to \( I \) and \( J \) we get
\[
\partial_I(I) = 1, \quad \partial_I(J) = 0, \quad \partial_J(I) = 0, \quad \partial_J(J) = 1.
\]
This explains the interpretation of the Tresse derivatives as partial derivatives with respect to \( I \) and \( J \), respectively: When restricted to a solution \( s \), they become the partial derivatives with respect to \( I_s \) and \( J_s \). Notice also that both \( \partial_I \) and \( \partial_J \) depends on the pair \( (I,J) \). If one of the invariants \( I \) or \( J \) are changed, both derivations will change.

**Theorem 1.** The algebra of differential invariants is generated by a finite number of differential invariants \( I, J, H_1, ..., H_q \) together with the invariant derivations \( \partial_I, \partial_J \).

Again we refer to [9] for the general theory. The number \( q \) is not uniquely determined. One may always add more generators, but that also increases the number of differential syzygies. In this paper \( q \) will always be 1 or 2.

**Tresse derivatives for Burgers’ equation:** We compute the Tresse derivatives with Maple (remember that we defined the invariants \( A \) and \( B \) in the previous subsection):

```maple
T:=proc(f) a*TotalDiff(f,t)+b*TotalDiff(f,x); end proc:
cf1:=eval([a,b],solve(map(T,[A,B],a,b)-[1,0],[a,b])):
cf2:=eval([a,b],solve(map(T,[A,B],a,b)[-0,1],[a,b])):
T1:=proc(f)cf1[1]*TotalDiff(f,t)+cf1[2]*TotalDiff(f,x); end proc:
T2:=proc(f)cf2[1]*TotalDiff(f,t)+cf2[2]*TotalDiff(f,x); end proc:
simplify([T1(A),T1(B),T2(A),T2(B)]);
```

They are given by
\[
\partial_I = \frac{u_{xx}D_t - u_{tx}D_x}{u_{tx}u_{xxx} - u_{xx}u_{txx}}, \quad \partial_J = \frac{-u_{xx}D_t + u_{tx}D_x}{u_{tx}u_{xxx} - u_{xx}u_{txx}}.
\]

Note that the coefficients can be considered as functions on \( E_2 \), but their expressions are simpler when written like this.

2.5. The Quotient PDE

Assume that the algebra of differential invariants is generated by the invariants \( I, J, H_1, ..., H_q \) and the Tresse derivatives \( \partial_I, \partial_J \). In general this algebra will not be freely generated: there are differential syzygies, i.e. algebraic relations between \( I, J, H_1, \partial_I H_1, \partial_J H_1 \) and higher order derivatives. The differential syzygies define what we call the quotient PDE. Its meaning can be explained as follows.

Let us restrict the invariants to a solution \( s \) of \( F = 0 \), and denote the obtained functions by \( I_s, J_s, (H_i)_s \), respectively. These can be viewed (locally) as functions on \( \mathbb{R}^2 \). Since we have \( 2 + q \) functions on a two-dimensional manifold, there must be at least \( q \) independent relations between them. We will consider only such solutions that \( I_s \) and \( J_s \) are independent (the \( (k+1) \)-jets of \( s \) satisfy (4), where \( k \) is the order of \( I \) and \( J \)). Locally we may solve for \( (H_i)_s \), so that \( (H_i)_s = h_i(I_s, J_s) \) for some set of functions \( h_i \) of two variables. Since \( I, J, H_1 \) are invariant, \( h_i \) will be the same for two solutions that are related by a symmetry. Moreover, \( s \) is a solution of both the PDE \( F = 0 \) and the PDE system \( H_1 = h_1(I,J), ..., H_q = h_q(I,J) \).

The differential syzygies imply that the functions \( h_1, ..., h_q \) are not completely arbitrary. Instead, they satisfy a system of differential equations. By differentiating \( H_i = h_i(I,J) \) with respect to \( \partial_I \) and \( \partial_J \) we get \( \partial_I(H_i) = (h_i)_1(I,J), \partial_J(H_i) = (h_i)_2(I,J) \), where \( (h_i)_j \) denotes the partial derivative of \( h_i \) with respect to its \( j \)th argument. The right-hand sides are just functions of \( I \) and \( J \), while left-hand sides are new differential invariants that are related by the differential syzygies. In this way the differential syzygies can be identified with differential equations on the functions \( h_i \) (it may be necessary to differentiate
We refer to [19] for a treatment of quotients of evolutionary PDEs. The quotient PDE is given by the two equations

$$\frac{\partial^2 H}{\partial x \partial u} + \frac{\partial}{\partial u} \left( \frac{\partial H}{\partial \mu} \right) = 0,$$

$$\frac{\partial^2 H}{\partial u \partial \mu} - \frac{\partial}{\partial \mu} \left( \frac{\partial H}{\partial x} \right) = 0.$$
Example: The quotient and solution of an ODE. We chose the following example for its complete transparency, and because it shows a subtle detail of quotients of differential equations that can be useful to keep in mind. Consider the ODE defined by \( u_3 = u_2 \) and the Lie algebra \( \mathfrak{g} = \langle \partial_x, \partial_u \rangle \) of symmetries. The four-dimensional ODE \( \mathcal{E}_3 \subset \mathcal{J}^3(\mathbb{R}) \) is foliated by two-dimensional orbits. The field of rational differential invariants is generated by \( I = u_1 \) and \( H = u_2 \), and these two invariants completely describe all orbits. The Tresse derivative is given by \( \hat{\partial}_t = \frac{1}{u_2} D_x \), and the quotient is given by \( \hat{\partial}_t(H) = 1 \). We assume that \( u_2 \neq 0 \).

The solution of the quotient is given by \( H = I - A \), which gives us a new ODE: \( u_2 = u_1 - A \). Its derivative is the ODE we started with. The solution of this second-order ODE is \( u(x) = Ax + B + Ce^x \).

Notice that even though \( I \) and \( H \) completely separate \( \mathfrak{g} \)-orbits on \( \mathcal{E}_3 \), the relation \( H = I - A \) is satisfied by both of the two inequivalent solutions \( Ax + e^x \) and \( Ax - e^x \). They imply \( H > 0 \) and \( H < 0 \), respectively, and are thus separated by the singular hypersurface given by \( H = u_2 = 0 \).

2.7. First-Order Scalar Quotients

The method described above shows that solving the PDE \( F = 0 \) can be broken down into two steps: solving the quotient PDE, and solving the overdetermined system made by adding the differential constraints corresponding to a solution of the quotient. The hope is that each of these two steps will be significantly easier than solving the original PDE directly. We saw that in the case of Burgers’ equation, the last step is easy, but the first is not.

The goal of this subsection is to pin down a special case in which we are able to solve the quotient PDE. Since first-order scalar PDEs can be solved by using the method of characteristics, at least in principle (see for example [8]), we seek PDEs with a first-order scalar quotient.

Assume that the pair \( (\mathcal{E}, \mathfrak{g}) \) has a first-order scalar quotient PDE. Reformulating this in terms of the algebra of differential invariants, we require it to be generated by three independent differential invariants \( I, J, K \) and the Tresse derivatives \( \hat{\partial}_I, \hat{\partial}_J \). The quotient is of desired type if and only if there is one syzygy of the form \( S(I, J, K, \hat{\partial}_I(K), \hat{\partial}_J(K)) = 0 \) and all other differential syzygies are generated by it and its Tresse derivatives.

If we differentiate one more time, we get three new invariants \( \hat{\partial}_I^2(K), \hat{\partial}_I \hat{\partial}_J(K) \) and \( \hat{\partial}_J^2(K) \), and two new syzygies:

\[
\hat{\partial}_I(S) = S_1 + S_3 \hat{\partial}_I(K) + S_4 \hat{\partial}_I^2(K) + S_5 \hat{\partial}_I \hat{\partial}_J(K) = 0,
\]
\[
\hat{\partial}_J(S) = S_2 + S_3 \hat{\partial}_J(K) + S_4 \hat{\partial}_I \hat{\partial}_J(K) + S_5 \hat{\partial}_J^2(K) = 0.
\]

Now we have eight differential invariants, and three syzygies between them. Continuing this we see that we obtain at each step exactly one differential invariant that is independent of all the invariants from the previous step. At some point we must obtain an invariant \( L \) that has order \( l \) higher than \( I \) and \( J \), so that at least one of \( \hat{\partial}_I(L) \) or \( \hat{\partial}_J(L) \) is of order \( l + 1 \). Then further differentiation will give invariants of strictly increasing order at each step.

We saw in Section 2.3 that, for the Hilbert function \( \mathcal{H}_k \), we have \( \mathcal{H}_k \leq 2 \) for \( k \geq 0 \). When the differential invariants are generated by three invariants and one differential syzygy, we see that \( \mathcal{H}_k = 1 \) for every \( k \) above some integer. This implies that the dimension of a \( \mathfrak{g} \)-orbit in \( \mathcal{E}_k \) grows without bound, as \( k \) grows. Thus \( \mathfrak{g} \) is infinite-dimensional.

Theorem 2. Let \( \{ F = 0 \} \subset \mathcal{J}^2(\mathbb{R}^2) \) be a second-order PDE, and let \( \mathfrak{g} \) be a Lie algebra of symmetries. Its quotient PDE is a first-order PDE on one function of two variables only if \( \mathfrak{g} \) is infinite-dimensional.

With this in mind, we look for second-order PDEs whose symmetry Lie algebra has infinite dimension. In all of the examples we are going to consider, we will need to solve two first-order PDEs, one of which is the quotient. Thus, in this case the picture is very similar to that of ODEs.
3. THE HUNTER–SAXTON EQUATION

In this section we will find exact solutions of the Hunter–Saxton equation to show in detail how the ideas outlined above can be applied. We will find a formula for the general solution, similar to that found in [6], and we will look at a few concrete solutions, also with respect to initial data. Here, and in the rest of the paper, we will continually recycle the notation used above. The second-order PDE under consideration will be denoted by \( E \) and given by \( F = 0 \), the symmetry Lie algebra under consideration is \( \mathfrak{g} \), and \( I, J, H \) are generators for the algebra of differential invariants, and so on.

The Hunter–Saxton equation is given by \( F = (u_t + uu_x)_x - u_x^2/2 = 0 \). It was derived in [6] in order to describe nonlinear instability in the director field of a nematic liquid crystal.

The Lie algebra of symmetries of the Hunter–Saxton equation is spanned by

\[
X_1 = \partial_t, \quad X_2 = t\partial_t + x\partial_x, \quad X_3 = x\partial_x + u\partial_u, \\
X_4 = t^2\partial_t + 2tx\partial_x + 2x\partial_u, \quad Y_f = f(t)\partial_x + f'(t)\partial_u,
\]

where \( f \) runs through all (locally defined) smooth functions on \( \mathbb{R} ([13]) \). We will consider only the infinite-dimensional Lie subalgebra \( \mathfrak{g} = \langle Y_f \mid f \in \mathcal{C}^\infty_{\text{loc}}(\mathbb{R}) \rangle \).

3.1. Differential Invariants and the Quotient PDE

By choosing an infinite-dimensional Lie algebra of symmetries, we get a first-order scalar quotient PDE. At the same time, by not considering a larger symmetry Lie algebra, we only have to look to \( E_2 \) to find a generating set of invariants.

**Theorem 3.** The algebra of differential invariants is generated by \( I = t, J = u_x, H = u_{xx} \) together with the Tresse-derivatives

\[
\hat{\partial}_I = D_t - \frac{u_{tx}}{u_{xx}}D_x, \quad \hat{\partial}_J = \frac{1}{u_{xx}}D_x.
\]

**Proof.** It is easy to verify that \( I, J, H \) are invariant. What remains to show is that they generate the whole algebra.

Differentiation of \( H \) with respect to Tresse derivatives clearly results in one new independent differential invariant of each order. This shows that \( \mathcal{H}_k \geq 1 \), which gives an upper bound on dimension of \( \mathfrak{g} \)-orbits in \( E_k \). We show that this upper bound is obtained.

Let \( Z_i \) denote the restriction of \( Y^{(i)}_{t+i} \) to the subset in \( E_i \) given by \( t = 0, u_x = 0, \) and \( Z_{-1} = \partial_x \). Then \( Z_i = (i + 1)!\partial_{u^{(i)}} \). This shows that at every point under consideration in \( E_k \), the vectors \( \{Y^{(k)}_{t+k} \mid i = -1, \ldots, k \} \) are independent. It follows that orbits in general position in \( E_k \) have dimension greater than or equal to \( k + 2 \), confirming that \( \mathcal{H}_k = 1 \) for every \( k \).

The quotient PDE is given by \( 2\hat{\partial}_I(H) - J^2\hat{\partial}_J(H) + 4JH = 0 \).

**Remark 2.** The differential invariants \( I \) and \( J \) are independent for generic solutions. Thus each solution gives a relation of the form \( G = H - h(I, J) = 0 \). The condition that \( F = 0 \) and \( G = 0 \) are compatible, puts a restriction on \( h \) which is found by differentiating the system \( \{F = 0, G = 0\} \), and then eliminating \( u_{tx}, u_{txx}, u_{xxx} \). The result is a linear first-order PDE on the function \( h \):

\[
2h_I - J^2h_J + 4Jh = 0.
\]

The general solution of the quotient is easily found to be given implicitly by

\[
16g \left( \frac{2t}{2-4t} \right)^4 H - (2 - IJ)^4 = 0.
\]

By inserting the expressions for \( I, J, H \), we get

\[
G = 16g \left( \frac{2u}{2-tu_x} \right) u_{xx} - (2 - tu_x)^4 = 0.
\]
3.2. The General Solution

The equation $G = 0$ can be viewed as a first-order PDE on $u_x$ which we can solve (in this case as a first-order separable ODE). Its solution is given implicitly by

$$\hat{G} = \left( \int (tw + 2)^2 g(w) dw \right) \bigg|_{w = \frac{2u}{2 - tu_x}} + 4C(t) - 4x = 0.$$  

The Lie algebra $\mathfrak{g}$ acts transitively on the set of integration “constants” $C(t)$.

The four equations $F = 0, G = 0, \hat{G} = 0, D_t \hat{G} = 0$ defines a two-dimensional surface in the space $\mathbb{R}^6(t, x, u, u_x, u_{tx}, u_{xx})$. Its projection to $\mathbb{R}^3(t, x, u)$ is a solution to the Hunter–Saxton equation. Notice that due to the symmetries, $u_t$ and $u_{tt}$ do not appear in any of these equations.

In this case we are able to write the two-dimensional manifold as a parametrized surface in $\mathbb{R}^3$. We solve $D_t \hat{G} = 0, G = 0$ for $u_{tx}$ and $u_{xx}$ and eliminate second-order derivatives from $F = 0$ to obtain an equation $\hat{F} = 0$ depending only on $t, x, u, u_x$. Since $x$ appears only in $\hat{G} = 0$ and $u$ appears only in $\hat{F} = 0$, both in a linear way, may solve $\hat{G} = 0, \hat{F} = 0$ for $x$ and $u$ to obtain the solution in $\mathbb{R}^3(t, x, u)$, parametrized by $I = t, J = u_x$, or, in order to simplify the expression, by $t$ and $w = \frac{2u}{2 - tu_x}$.

$$
\begin{align*}
t &= t, \\
x &= \frac{1}{4} \int_0^w (tv + 2)^2 g(v) dv + C(t), \\
u &= \frac{1}{2} \int_0^w (tv + 2)g(v) dv + C'(t).
\end{align*}
\tag{7}
$$

The solution given by this two-dimensional manifold will in general be multivalued and have points where it is not differentiable, even when $g$ is smooth. Notice also that the general solution depends on two arbitrary functions, $g$ and $C$, of one variable.

Let $s$ be any of these parametrized solutions, and consider its intersection $s_{t_0}$ with the plane in $\mathbb{R}^3$ given by $t = t_0$. Since $u_x$ is a parameter, the function $u_x$ restricted to the curve $s_{t_0}$ is injective. In other words: at any point in time, the slope $u_x(t_0, x)$ will be different for every $x$ where it is defined.

3.3. Comparing to the Solution of Hunter and Saxton

Hunter and Saxton solved their equation in [6]:

**Theorem 4** (Hunter–Saxton). Every smooth solution of the Hunter–Saxton equation with the initial data $u(0, x) = \alpha(x)$ is given implicitly by

$$u = \alpha(\xi) + t\beta(\xi) + \gamma'(t), \quad x = \xi + t\alpha(\xi) + \frac{1}{2} t^2 \beta(\xi) + \gamma(t),$$

where $\gamma$ is any function with $\gamma(0) = \gamma'(0) = 0$ and $\beta$ satisfies $\beta'(\xi) = \frac{1}{2} \alpha'(\xi)^2$.

We see the relation between our formula for the general solution and this one by writing $\gamma(t) = C(t)$ and

$$\xi = \int_0^w g(v) dv, \quad \alpha(\xi) = \int_0^w g(v) dv, \quad \beta(\xi) = \frac{1}{2} \int_0^w g(v) v^2 dv.$$

It follows that $w = \alpha'(\xi)$. 


3.4. Two Examples

Let us look at solutions given by two different choices of \( g \), and \( C \equiv 0 \). When \( g(w) = e^w \) the solution is given by

\[
\begin{align*}
x &= \frac{(t(2u_x - 2) + 2)^2 + 4}{2(2 - tu_x)^2} \exp \left( \frac{2u_x}{2 - tu_x} \right), \\
u &= \frac{t(2 - u_xt)^2 + t^2u_x^2 + 4u_x - 4}{(2 - tu_x)^2} \exp \left( \frac{2u_x}{2 - tu_x} \right).
\end{align*}
\]

We notice that the solution is defined for positive \( x \) only and it is not smooth everywhere. The function \( g(w) = w(1 + w)(1 - w) \) determines the solution

\[
\begin{align*}
x &= \frac{u_x^2(5t^4u_x^4 - 60t^3u_x^3 - 44t^2u_x^2 + 300t^2u_x^2 + 48tu_x^2 - 640tu_x - 240u_x^2 + 480)}{15(tu_x - 2)^6}, \\
u &= -\frac{2u_x^3(5t^4u_x^3 - 60t^2u_x^3 - 8t^2u_x^3 + 180tu_x - 96u_x^2 - 160)}{15(tu_x - 2)^6}.
\end{align*}
\]

This solution is multivalued, has nonsmooth points and, for fixed \( t \), it is not defined for every \( x \). It is possible to eliminate \( u_x \) from the two equations above to obtain a 16-degree algebraic equation in \( t, x, u \) (in which the highest power of \( u \) is 6).

3.5. Initial Data

Consider the initial data \( u(t_0, x) = u_0(x) \). If there exists a function \( g \) such that (7) satisfies the initial data, it follows from (6) that \( g \) satisfies the functional equation

\[
g \left( \frac{2u_0'(x)}{2 - t_0u_0(x)} \right) = \frac{(2 - t_0u_0(x))^4}{16u_0''(x)}.
\]

A solution \( u = \varphi(t, x) \) of the Hunter–Saxton equation is transformed by the flow of \( Y_t \) to \( u = \varphi(t, x + f(t)) - f'(t) \). Thus, for a function \( g \) satisfying (8), we must choose an appropriate function \( C(t) \) to make the solution fit with the initial data. Moreover, there are many solutions that satisfy the same initial data. In particular, if \( f(t_0) = 0 \) and \( f'(t_0) = 0 \), the transformation above will not affect the initial data. Thus in order to specify the initial value problem completely, additional restrictions must be given, determining the function \( C \). Hunter and Saxton \((6)\) determined \( C \) by imposing the boundary condition \( \lim_{x \to \infty} u(x, t) \to 0 \).

Let us find a solution to the initial data \( u(1, x) = e^{-x} \). Solving (8) with this condition gives \( g(w) = \frac{8}{w(w+2)^2} \). This gives the solution

\[
\begin{align*}
x &= \frac{2(t - 1)^2}{(w + 2)^2} + \frac{2(t^2 - 1)}{w + 2} - \ln(-w) + \ln(w + 2) + C(t), \\
u &= \frac{4(1 - t)}{(w + 2)^2} + \frac{4t}{w + 2} + C'(t).
\end{align*}
\]

For \( t = 1 \) we have \( u(x, t) \to 0 \) as \( x \to \infty \), and we impose this condition for every \( t \) in order to determine the constant \( C(t) = -t^2/2 - t + 2 - \ln 2 \). There is a nonsmooth point on the solution, moving along the curve \( 2u = e^{2-x} \). Let us now find a solution to the initial data \( u_0(x) = u(1, x) = x^2 \). The functional equation (8) gives \( g(w) = \frac{8}{(2 + w)^2} \). By integrating and eliminating \( w \) from (7), we get a solution with

\[
u(1, x) = (x + C(1) + 1)^2 - 1 - C'(1).
\]

So let \( C(t) = -t \) (the choice is not unique). Then we get the implicit solution

\[
(t - 1)^4u^3 - 3(2tx - 2x + 1)(t - 1)^2u^2 + 3(2tx - 2x + 1)^2u \\
+ (2 - 8x^3)(t - 1) - 3x^2 + 6(t - 1)^2x + (t - 1)^4 = 0.
\]
If we solve for $u$, we get

$$u(t, x) = \frac{2x(t - 1) + 1 - ((t - 1)^3 + 3x(t - 1) + 1)^{2/3}}{(t - 1)^2}. $$

Considering these as curves in the $(x, u)$-plane, parametrized by $t$, we can find singular points. They are given by

$$x = \frac{(t^2 - 3t + 3)t}{3(t - 1)}, \quad u = \frac{2(t - 1)^3 - 1}{3(t - 1)^2}. $$

As $t \to 1$ we see that $(x, u) \to (\pm \infty, \infty)$. Eliminating $t$ from the equations above results in $3x^2u^2 + 4x^3 - u^3 + 1 = 0$.

### 3.6. The Action of the Remaining Symmetries on the Quotient

We chose a particular Lie subalgebra of the symmetry Lie algebra of the Hunter—Saxton equation, and computed the quotient PDE. Each solution of the quotient is given by a function $g$. The PDE $\{F = 0, G = 0\}$ is symmetric with respect to the vector fields $Y_f$, but not with respect to the rest of the symmetries of the Hunter—Saxton equation. The flows of the remaining symmetries will act on the function $g$.

$$\begin{align*}
\partial_t &\mapsto g(w) \mapsto \frac{16g \left(\frac{2w}{2sw}\right)}{(2 - sw)^3} \\
\partial_x &\mapsto g(w) \mapsto g(e^{-s}w) \\
\partial_u &\mapsto g(w) \mapsto e^{-s}g(w) \\
t^2\partial_t + 2tx\partial_x + 2x\partial_u &\mapsto g(w) \mapsto g(w + 2s)
\end{align*}$$

### 4. SOLVING PDES WITH FIRST ORDER QUOTIENT

In this section we consider several different second-order PDEs with infinite-dimensional symmetry Lie algebra. The PDEs will be invariant under one of five different infinite-dimensional Lie algebras of the form $\mathfrak{g} = \langle X_f | f(t) \in C_\text{loc}^\infty(\mathbb{R})\rangle$. The generators we consider are the following:

1. $X_f = f(t)\partial_x + f'(t)\partial_u$,
2. $X_f = f(t)\partial_t - f'(t)\partial_u$,
3. $X_f = f(t)\partial_t$,
4. $X_f = f(t)\partial_u$,
5. $X_f = f(t)\partial_t + f'(t)x\partial_x + f''(t)x\partial_u$.

They are considered in sections 4.1, 4.2, 4.3, 4.4 and 4.6, respectively.

**Remark 3.** Lie studied (point-equivalent versions of) the second and third of these Lie algebras, and computed their differential invariants in [12]. An English translation of this paper can be found in [7]. Vessiot computed differential invariants of the fifth one ([21]).

For each of these infinite-dimensional Lie algebras we will find the general invariant PDE of the form $F = u_{tx} - \varphi(t, x, u, u_t, u_x, u_{tt}, u_{xx}) = 0$.

**Remark 4.** Notice that PDEs for which $\partial_{xtx}(F) = 0$ holds have, for all our Lie algebras, the additional properties $\partial_{xt}(F) = 0$, $\partial_{ttx}(F) = 0$. They are essentially ODEs, possibly parametrized by $t$. 

SCHNEIDER

LOBACHEVSKII JOURNAL OF MATHEMATICS  Vol. 41  No. 12  2020
In order for the global Lie–Tresse theorem ([9]) to apply we require the fibers of \( E_k \to J^0(\mathbb{R}^2) \) to be (irreducible) algebraic manifolds. This restricts \( \varphi \) even more. We assume that the equations in this chapter satisfy this condition, also when they depend on a not completely determined function.

We will find the differential invariants and quotient PDE for the general symmetric PDE, and then take a closer look at more specific PDEs. We will focus on computations, with the aim of getting a feeling for how the ideas explained above can be used efficiently. For some PDEs we write down the general solution, and for others we will be satisfied with only solving the quotient PDE.

In Section 4.5 we discuss the possibility to consider three-dimensional Lie subalgebras of \( \mathfrak{g} \) to obtain two first-order syzygies, as we did for Burgers’ equation in Section 2.5. Now, because the Lie algebra in these cases is a subalgebra of infinite-dimensional ones, the two differential syzygies are partially uncoupled.

### 4.1. Symmetries of Type 1

Consider the Lie algebra spanned by the vector fields of the form \( f(t)\partial_x + f'(t)\partial_u \). The equation

\[
\frac{\partial}{\partial t} u_x = \varphi(t, x, u, u_t, u_x, u_{tx}, u_{xx})
\]

is invariant if and only if \( \varphi = -\alpha(t, u, u_x) - uu_{xx} \). Thus we consider the PDE

\[
F = u_{tx} + uu_{xx} + \alpha(t, u, u_x).
\]

When \( \alpha = u_x^2/2 \) we get the Hunter–Saxton equation, considered above. The algebra of differential invariants is generated by \( I = t, J = u_x, H = u_{xx} \), and the quotient PDE is given by \( \hat{\partial}_t H = (\alpha - H\alpha_H)\hat{\partial}_J^2 H + (J + \alpha J)H = 0 \).

**Example 1.1:** Consider the PDE given by \( u_{tx} + uu_{xx} + \alpha(u_x) = 0 \). This is the Calogero equation ([4]), which was also solved in [20]. Its quotient PDE is given by \( \hat{\partial}_t H = -\alpha(J)\hat{\partial}_J^2 H + (J + \alpha(J))H = 0 \) which has general solution

\[
H g \left( \int \frac{dJ}{\alpha(J)} + I \right) = e^{\int J \cdot \alpha(J) dJ}.
\]

This gives the additional equation

\[
u_{xx} g \left( \int \frac{du_x}{\alpha(u_x)} + t \right) = e^{\int \frac{u_x + \alpha'(u_x)}{\alpha(u_x)} du_x}
\]

which can be considered as a first-order PDE in \( u_x \). Its solution is given implicitly by

\[
x = \int g \left( \int \frac{du_x}{\alpha(u_x)} + t \right) e^{\int \frac{u_x + \alpha'(u_x)}{\alpha(u_x)} du_x} du_x + C(t).
\]

By solving the original PDE for \( u \), we get

\[
u = -\frac{u_{tx} - \alpha(u_x)}{u_{xx}},
\]

where \( u_{tx} \) and \( u_{xx} \) may be replaced by functions of \( t \) and \( u_x \) (in the same way as for the Hunter–Saxton equation). We end up with a solution parametrized by \( t \) and \( u_x \).

**Example 1.2:** Consider the PDE \( u_{tx} + uu_{xx} + \alpha(t, u_x)u_{xx} = 0 \). The quotient PDE is given by \( \hat{\partial}_t H + (J + \alpha_J H)H = 0 \). It has general solution

\[
H = e^{-IJ} \frac{g(J) + \int \alpha_J(I, J) e^{-IJ} dI}{g(J)}
\]

By considering this as a first-order PDE on \( u_x \) we get the implicit solution

\[
x = \int e^{tu_x} \left( g(u_x) + \int e^{-tu_x} \alpha_u(t, u_x) dt \right) du_x + C(t).
\]
We also have \( u = -\frac{u_t}{u_x} - \alpha(t, u_x) \), where \( u_t \) and \( u_x \) can be eliminated to give a parametrization of the solution by \( t \) and \( u_x \).

**Example 1.3:** Consider the PDE \( u_{tx} + uu_{xx} + u^2_{xx} = 0 \). The quotient PDE is given by \( \hat{\partial}_t(H) + H^2\hat{\partial}_J(H) + JH = 0 \) whose general solution is given implicitly by

\[
(g(J^2 + H^2) - I) \sqrt{J^2 + H^2} + \text{arctanh} \left( \frac{J}{\sqrt{J^2 + H^2}} \right) = 0.
\]

Again, this is a first-order PDE on \( u_x \), but contrary to the previous cases this PDE can not be solved easily as a first-order, separable ODE.

### 4.2. Symmetries of Type 2

Consider the Lie algebra spanned by the vector fields of the form \( f(t)\partial_t - f'(t)\partial_u \). The general invariant second-order PDE, assuming it can be solved for \( u_{tx} \), is of the form \( u_{tx} - \alpha(x, u_x, u_{xx})e^u = 0 \). The differential invariants are generated by

\[
I = x, \quad J = u_x, \quad H = u_{xx}, \quad \hat{\partial}_I = -\frac{u_{xx}}{u_{tx}} D_t + D_x, \quad \hat{\partial}_J = \frac{1}{u_{tx}} D_t
\]

and the quotient PDE is given by

\[
\alpha_H \hat{\partial}_I(H) + (H\alpha_H - \alpha) \hat{\partial}_J(H) + \alpha_J H + \alpha J + \alpha I = 0.
\]

Let \( H = g(I, J) \) be a solution of this equation. It gives an equation \( u_{xx} = g(x, u_x) \), which can be viewed as a first-order PDE on \( u_x \). Now, let \( u_x(t, x) = w(t, x) \) be a solution of this PDE. Inserting it into \( u_{tx} = \alpha e^u \) results in the solution

\[
u(t, x) = \ln \left( \frac{w_t(t, x)}{\alpha(x, w(t, x), w_x(t, x))} \right).
\]

Let us consider a few different choices of \( \alpha \).

**Example 2.1:** Assume that \( \alpha = H \beta(I, J) \). The quotient PDE now takes the form \( \beta \hat{\partial}_I(H) + \beta_J H^2 + \beta JH + \beta I H = 0 \) and has general solution

\[
He^{IJ} \beta(I, J) \left( \int \frac{\beta_I(I, J)}{e^{IJ} \beta(I, J)^2} dI + g(J) \right) = 1.
\]

Inserting the expressions for \( I, J, H \) gives

\[
u_{xx}e^{ux} \beta(x, u_x) \left( \int \frac{\beta_{ux}(x, u_x)}{e^{ux} \beta(x, u_x)^2} dx + g(u_x) \right) = 1.
\]

**Example 2.2:** Let us now assume that \( \alpha = \alpha(I, J) \). Then the quotient reduces to \( -\alpha \hat{\partial}_J(H) + \alpha_J H + \alpha J + \alpha I = 0 \) which has general solution

\[
H = \left( \int \frac{J\alpha(I, J) + \alpha_I(I, J)}{\alpha(I, J)^2} dI + g(I) \right) \alpha(I, J).
\]

This gives

\[
u_{xx} = \left( \int \frac{u_x\alpha(x, u_x) + \alpha_x(x, u_x)}{\alpha(x, u_x)^2} du_x + g(x) \right) \alpha(x, u_x).
\]

**Example 2.3:** We continue the computations here for \( \alpha = -1 \), i.e. the Liouville equation. We note that this equation is point-equivalent to \( uu_{xx} - u_tu_{tx} = u^3 \) (by \( u \mapsto -e^u \)) which was considered in [16] and [20]. In this case the quotient is given by \( \hat{\partial}_J(H) - J = 0 \). Its general solution is \( H = \frac{1}{2} J^2 + g(I) \) which gives the differential constraint \( u_{xx} = \frac{1}{2} u_x^2 + g(x) \). Interpreted as a first-order ODE on \( u_x \) this is a Riccati equation.
4.3. Symmetries of Type 3

Consider the Lie algebra spanned by the vector fields of the form \( f(t) \partial_t \). The general invariant second-order PDE is, assuming it can be solved for \( u_{tx} \), given by

\[
u_{tx} = u_t \alpha(x, u, u_x, u_{xx}). \tag{9}\]

The differential invariants are generated by

\[
I = x, \quad J = u, \quad H = u_x, \quad \hat{\partial}_I = -\frac{u_x}{u_t} D_t + D_x, \quad \hat{\partial}_J = \frac{1}{u_t} D_t
\]

and the quotient PDE is given by \( H_J - \alpha(I, J, H, H_I + HH_J) = 0 \).

A solution of the quotient of the form \( H = g(I, J) \) gives an equation \( u_x = g(x, u) \), of which (9) is a differential consequence. Thus, solving the second-order PDE (9) amounts to solving, sequentially, two first-order PDEs. We consider a few different choices of \( \alpha \).

**Example 3.1:** If \( \alpha = \alpha_1(x, u) u_x + \alpha_2(x, u) \), the quotient equation of (9) is \( \hat{\partial}_J(H) - \alpha_1(I, J) H - \alpha_2(I, J) = 0 \) and its general solution is

\[
H = \left( \int \alpha_2(I, J) e^{-\int \alpha_1(I, J) dJ} dJ + g(I) \right) e^{\int \alpha_1(I, J) dJ}.
\]

Inserting the expressions for the invariants gives the PDE

\[
u_x = \left( \int \alpha_2(x, u) e^{\int \alpha_1(x, u) du} du + g(x) \right) e^{\int \alpha_1(x, u) du}.
\]

**Example 3.2:** Consider the PDE \( u_{tx} = u_t (\alpha_1(x) u_x + \alpha_2(x)) \). Its quotient PDE is \( \hat{\partial}_J(H) = \alpha_1(I) H + \alpha_2(I) \). The quotient’s general solution is \( H = \frac{1}{2} \alpha_1(I) J^2 + \alpha_2(I) J + g(I) \), or \( u_x = \frac{1}{2} \alpha_1(x) u_x^2 + \alpha_2(x) u_x + g(x) \). This is a Riccati equation. Choosing the the set of solutions for which \( g \equiv 0 \) lets us write them down explicitly:

\[
u(t, x) = \frac{2 e^{\int \alpha_2(x) dx}}{C(t) - \int \alpha_1(x) e^{\int \alpha_2(x) dx} dx}
\]

In the case when \( \alpha_1 \equiv 0 \) and the PDE is linear, we are able to write down the general solution

\[
u(t, x) = \left( \int g(x) e^{-\int \alpha_2(x) dx} dx + C(t) \right) e^{\int \alpha_2(x) dx}.
\]

**Example 3.3:** Consider the equation \( u_{tx} = u_t u_x \). Its quotient is \( \hat{\partial}_J(H) = H \). This gives \( H = g(I) e^J \), or \( u_x = g(x) e^u \). Thus, the general solution of \( u_{tx} = u_t u_x \) is

\[
u(t, x) = -\ln \left( C(t) - \int g(x) dx \right).
\]

Notice that the PDE is point-equivalent to \( u_{tx} = 0 \).

4.4. Symmetries of Type 4

Consider the Lie algebra spanned by the vector fields of the form \( f(t) \partial_u \). The general invariant second-order PDE is, assuming it can be solved for \( u_{tx} \), given by

\[
u_{tx} = \alpha(t, x, u, u_x, u_{xx}). \tag{10}\]

Note that this is a first-order PDE in \( u_x \). The differential invariants are generated by

\[
I = t, \quad J = x, \quad H = u_x, \quad \hat{\partial}_I = D_t, \quad \hat{\partial}_J = D_x
\]

and the quotient PDE is given by \( \hat{\partial}_I(H) = \alpha(I, J, H, \hat{\partial}_J(H)) \). The quotient is exactly (10) treated as a first-order PDE on \( u_x \).
Remark 5. This shows that all first-order scalar PDEs can be obtained as a quotient of a second-order PDE.

Assume that a solution can be written as \( H = g(I, J) \) for some function \( g \). This gives an equation \( u_x = g(t, x) \) which can be added to (10), and in fact (10) is just a differential consequence of this first-order PDE. The function \( u(t, x) = \int g(t, x) dx + C(t) \) will be a solution to the original equation. We solve some concrete examples.

Example 4.1: Consider the PDE \( u_{tx} = u_x^A \), with constant \( A \neq 1 \). The quotient PDE is given by \( \partial_t (H) = H^A \) and has general solution \( H = (g(J) + (1 - A)I)^{1/(1-A)} \). This gives the PDE \( u_x = (g(x) + (1 - A)t)^{1/(1-A)} \) which is integrated to

\[
u(t, x) = \int \left( g(x) + (1 - A)t \right) \frac{1}{1-A} dx + C(t).
\]

Example 4.2: Consider the PDE \( u_{tx} = u_x^A u_{xx} \). The quotient PDE is given by \( \partial_j (H) = H^A \partial_j (H) \) and has general solution \( J + IH^A - g(H) = 0 \). This gives the PDE \( x + tu_x - g(x) = 0 \). Solve for \( u_x \) and integrate to obtain solutions of (10).

Example 4.3: Consider the PDE \( u_{tx} = \alpha(t, x)u_x^2 + \beta(t, x)u_x + \gamma(t, x) \). The quotient is given by \( \partial_j (H) = \alpha(I, J)H^2 + \beta(I, J)H + \gamma(I, J) \), a Riccati equation. If \( \gamma \equiv 0 \), then

\[H = \frac{e^{\int \beta(I, J) dI}}{g(J) - \int \alpha(I, J) e^{\int \beta(I, J) dI} dI}.
\]

Solving this PDE gives the general solution:

\[u(t, x) = \int \left( \frac{e^{\int \beta(t, x) dI}}{g(x) - \int \alpha(t, x) e^{\int \beta(t, x) dI} dt} \right) dx + C(t).
\]

Some of the equations we solve here may look too trivial to be worth considering. Even though a part of their simplicity is a consequence of their symmetry Lie algebra, one reason they seem trivial is that we have written them down in the right coordinates. Let us illustrate this with an example. Consider the PDE

\[-x^2 u_t + 2x^2 u_t u_x - x^2 u_x^2 + 2x uu_t - 2xu_x - xu_{tx} + xu_{tx} - u^2 + u_t = 0.
\]

Even though the equation looks complicated, its symmetries are easily computed. In particular, we find that all vector fields of the form \( \frac{f(t+x)}{x} \partial_u \) are symmetries. Thus, we may either look for the point-transformation that brings this to \( f(t) \partial_u \) and the PDE to \( u_x = u_x^2 \) which is treated above, or we can find the quotient PDE directly. The algebra of differential invariants, with respect to the Lie algebra spanned by vector fields of the form \( \frac{f(t+x)}{x} \partial_u \), is generated by

\[I = t, \quad J = x, \quad H = u + x(u_x - u_t), \quad \hat{\partial}_I = D_t, \quad \hat{\partial}_J = D_x.
\]

The quotient PDE is given by \( H_I = H^2 \) which has general solution \( H = \frac{1}{g(J) - t} \). It gives \( u + x(u_x - u_t) = \frac{1}{g(x + t - \tau)} \), a new first-order PDE. Its solution is

\[u(t, x) = \frac{1}{x} \left( \int \frac{d\tau}{\tau - g(x + t - \tau)} + C(t + x) \right) \bigg|_{\tau = t}.
\]

4.5. Solving the PDEs Using Finite-Dimensional Lie Subalgebras

We note that all the examples we considered in this and the previous section has a special property: The PDE we get by adding an additional differential constraint \( G = 0 \) (of order 1 or 2) to \( F = 0 \) is of infinite type. Since one would in general expect the result to be a finite type equation ([17]), all our examples are quite special.

This is different from what we got when we found the quotient of Burgers’ equation in Section 2.5. For Burgers’ equation we obtained, after adding the additional differential constraints corresponding to
a solution of the quotient, a finite type equation with a three-dimensional solution space. We argued that any solution to the quotient PDE of Burgers’ equation would determine a five-dimensional submanifold in $J^2(\mathbb{R}^2)$ on which the Cartan distribution was two-dimensional and completely integrable, with the symmetries acting transitively on the set of integral manifolds. Thus, given a solution of the quotient, the Lie-Bianchi theorem would let us find solutions in quadratures. What prohibited us from going through with this was our inability to solve the quotient PDE which consisted of two coupled first-order PDEs.

In this subsection we will see that if we consider three-dimensional Lie subalgebras of the infinite-dimensional ones, we get a situation similar to the one we got for Burgers’ equation, but now with a quotient given by two partially uncoupled first-order PDEs. The uncoupling can be explained by the fact that part of the algebra of differential invariants is the same. If $\mathfrak{h}$ denotes a three-dimensional Lie subalgebra of the infinite-dimensional Lie algebra $g$, we have for the corresponding algebras of differential invariants $\mathcal{A}_g \subset \mathcal{A}_h$. Thus, the same differential syzygy holds among the generators $I, J, H$ of $\mathcal{A}_g$. The algebra $\mathcal{A}_h$ of differential invariants with respect to $\mathfrak{h}$ can be obtained by adding one differential invariant $K$ to the set of generators. It comes together with an additional differential syzygy. We consider two examples.

The Hunter–Saxton equation: Consider again the Hunter–Saxton equation $(u_t + uu_x)_x = u_x^2/2$, but now with the three-dimensional symmetry Lie algebra $\mathfrak{h} = \langle \partial_x, t\partial_t + \partial_x, t^2\partial_t + 2t\partial_u \rangle$. It is a Lie subalgebra of the one we already considered in Section 3. In addition to the invariants $I = t$, $J = u_x$, $H = u_{xx}$ we found in Section 3, we now have an additional second-order invariant $K = u_{tt} - u_x^2 u_{xx} + u_t u_x$. With these generators we get two first-order syzygies:

$$2\hat{\partial}_I(H) - J^2 \hat{\partial}_J(H) + 4JH = 0, \quad \hat{\partial}_J(K) = 0.$$ 

The general solution is

$$16g \left( \frac{2J}{2 - IJ} \right) H - (2 - IJ)^4 = 0, \quad K = C(I).$$

This gives two second-order differential constraints:

$$16g \left( \frac{2u_x}{2 - tu_x} \right) u_{xx} - (2 - tu_x)^4 = 0, \quad u_{tt} - u_x^2 u_{xx} + u_t u_x = C(t).$$

Together with the Hunter–Saxton equation $(u_t + uu_x)_x = u_x^2/2$, they determine a five-dimensional submanifold of $J^2(\mathbb{R}^2)$. The restriction of the Cartan distribution to this manifold is a two-dimensional integrable distribution.

Liouville’s equation: Consider now $u_{tx} + e^u = 0$ with its three-dimensional symmetry Lie algebra $\mathfrak{h} = \langle \partial_t, t\partial_t - \partial_x, t^2\partial_t - 2t\partial_u \rangle$. We use the differential invariants $I = x, J = u_x, H = u_{xx}$ as in Section 4.2. In addition we have one more second-order invariant $K = (2u_{tt} - u_t^2)e^{-2u}$. The quotient PDE is given by the first-order system

$$\hat{\partial}_J(H) - J = 0, \quad \hat{\partial}_I(K) + H\hat{\partial}_J(K) + 2JK = 0.$$ 

The first equation gives $H = J^2/2 + g(I)$. Inserting this into the second equation gives a first-order PDE on $K$.

Again, any solution to this system will give two additional differential constraints that together with $u_{tx} + e^u = 0$ determine a five-dimensional manifold in $J^2(\mathbb{R}^2)$ on which the restriction of the Cartan distribution is two-dimensional. The Lie algebra $\mathfrak{h}$ is transversal to the distribution, as in the case for the Hunter–Saxton equation, but this time the Lie algebra is not solvable, and we can not apply the Lie–Bianchi theorem.

4.6. Symmetries of Type 5

Now we consider the Lie algebra spanned by the vector fields of the form $f(t)\partial_t + f'(t)x\partial_x + f''(t)x\partial_u$. The general invariant second-order PDE is, assuming it can be solved for $u_{tx}$, given by

$$u_{tx} + uu_{xx} - \frac{u_t}{x} - \frac{\alpha(u - xu_x, x^2u_{xx})}{x^2} = 0. \quad (11)$$
The differential invariants are generated by \( I = u - xu_x, \ J = x^2u_{xx}, \ H = x^3u_{xxx} \) and the Tresse derivatives \( \hat{\partial}_I, \hat{\partial}_J, \) whose expressions we omit writing due to their size. Finding the quotient is still an easy algebraic problem. The quotient PDE is given by

\[
0 = (\alpha - J\alpha_J)\hat{\partial}_I(H) + (\alpha + \alpha_I - JI)\hat{\partial}_J(H) + \alpha_{IJ}H^2 + ((4\alpha_{IJ} - 2\alpha_{IJ})J - 2\alpha_J - \alpha_I + 2I)H + (4\alpha_{IJ} - 4\alpha_{IJ} + \alpha_{II})J^2 + (2\alpha_I - 4\alpha_J + I - J)J + 3\alpha.
\]

Now, let \( H = h(I, J) \) be a solution of this. It gives a new differential constraint of the form \( x^3u_{xxx} = h(u - xu_x, x^2u_{xx}) \), that can be added to (11). We notice that this additional differential constraint can be solved as a third-order ODE.

Alternatively, we can apply the idea from the previous subsection and consider the three-dimensional Lie subalgebra

\[
h = \langle \partial_t, t\partial_x + x\partial_x, t^2\partial_t + 2tx\partial_x + 2x\partial_u \rangle.
\]

This gives two additional second-order invariants \( K = (u_t + uu_x)x - u^2/2, \) and

\[
L = x^2u_{tt} + \frac{((2uu_{xx} - uu_x^2)x - 2u_t)\alpha(I, J) - ((uu_x^2 - uu_x^2 - u_tu_x)x - 2uu_t)J}{xu_{xx}}.
\]

The differential syzygy from above still holds. In addition, we have

\[
J\hat{\partial}_I(K) - (H + 2J)\hat{\partial}_J(K) + 2K + I^2 + \alpha = 0.
\]

By inserting the solution \( H = h(I, J) \) of the first differential syzygy into the second, we obtain a new scalar PDE. Let \( K = k(I, J) \) be a solution of this. The last second-order invariant \( L \) can be written in terms of \( I, J, H, K, \hat{\partial}_I(K) \) and \( \hat{\partial}_J(K) \). Therefore, the solutions \( H = h(I, J), K = k(I, J) \) determine \( L = l(I, J) \). Adding these differential constraints to (11) gives us a system defined by three second-order conditions, namely (11), \( L = l(I, J) \) and \( K = k(I, J) \). Thus we obtain a five-dimensional submanifold in \( J^2(\mathbb{R}^2) \), with a two-dimensional integrable distribution on which a three-dimensional Lie algebra of symmetries acts transversally. Solving (11) is thus split into three parts. Two of the parts amounts to solving first-order scalar PDEs, and the last one is integration of an integrable distribution with a (nonsolvable) transversal Lie algebra of symmetries.

**FUNDING**

The author acknowledges full support via the Czech Science Foundation (GAČR no. 19-14466Y).

**REFERENCES**

1. S. C. Anco and S. Liu, “Exact solutions of semilinear radial wave equations in n dimensions,” J. Math. Anal. Appl. 297, 317–342 (2004).
2. I. M. Anderson and M. E. Fels, “Exterior Differential Systems with Symmetry,” Acta Appl. Math. 87, 3–31 (2005).
3. G. Bluman, “A reduction algorithm for an ordinary differential equation admitting a solvable Lie group,” SIAM J. Appl. Math. 50, 1689–1705 (1990).
4. F. Calogero, “A solvable nonlinear wave equation,” Stud. Appl. Math. 70, 189–200 (1984).
5. J. J. Early, J. Pohjapello, and R. M. Samelson, “Group foliation of equations in geophysical fluid dynamics,” Discrete Cont. Dyn. Syst. A 27, 1571–1586 (2010).
6. J. K. Hunter and R. Saxton, “Dynamics of director fields,” SIAM J. Appl. Math. 51, 1498–1521 (1991).
7. N. H. Ibragimov, Lie Group Analysis: Classical Heritage (ALGA Publ., Karlskrona, 2008), pp. 725–772.
8. I. S. Krasil’shchik and A. M. Vinogradov, Symmetries and Conservation Laws for Differential Equations of Mathematical Physics (Am. Math. Soc., Providence, RI, 1999).
9. B. Kruglikov and V. Lychagin, “Global Lie-Tresse theorem,” Sel. Math. 22, 1357–1411 (2016).
10. B. Kruglikov and V. Lychagin, “Geometry of Differential equations,” in Handbook of Global Analysis, Ed. by D. Krupka and D. Saunders (Elsevier, Amsterdam, 2008), pp. 725–772.
11. A. Kushner, V. Lychagin, and V. Rubtsov, Contact Geometry and Non-linear Differential Equations (Cambridge Univ. Press, Cambridge, 2007).
12. S. Lie, “Zur allgemeinen Theorie der partiellen Differentialgleichungen beliebiger Ordnung,” Leipz. Ber. 1, 53–128 (1895).
13. M. Nadjaïkhah and F. Ahangari, “Symmetry analysis and conservation laws for the Hunter–Saxton equation,” Commun. Theor. Phys. 59, 335–348 (2013).
14. P. Olver, Equivalence, Invariants, and Symmetry (Cambridge Univ. Press, Cambridge, 1995).
15. L. V. Ovsiannikov, Group Analysis of Differential Equations (Academic, New York, 1982).
16. J. Pohjanpelto, “Reduction of exterior differential systems with infinite dimensional symmetry groups,” BIT Num. Math. 48, 337–355 (2008).
17. W. M. Seiler, “Involution and Symmetry Reductions,” Mathl. Comput. Model. 25, 63–73 (1997).
18. M. B. Sheftel, “Method of group foliation and non-invariant solutions of invariant equations,” Proc. Inst. Math. NAS of Ukraine 43, 215–224 (2002).
19. S. I. Svinolupov and V. V. Sokolov, “Factorization of evolution equations,” Russ. Math. Surv. 47, 127–162 (1992).
20. R. Thompson and F. Valiquette, “Group foliation of differential equations using moving frames,” Forum Math. Sigma 3, e22 (2015).
21. E. Vessiot, “Sur l’intégration des systèmes différentiels qui admettent des groupes continus de transformations,” Acta Math. 28, 307–349 (1904).