NOTE ON K-STABILITY OF PAIRS

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Abstract. We prove that a pair $(X, D)$ with $X$ Fano and $D$ an anti-canonical divisor is K-unstable for negative angles, and is K-semistable for zero angle.

1. Introduction

Let $X$ be a Fano manifold. It was first proposed by Yau [20] that finding Kähler-Einstein metrics on $X$ should be related to a certain algebro-geometric stability. In [17], the notion of K-stability was introduced by Tian. This has been conjectured to be equivalent to the existence of a Kähler-Einstein metric. One direction is essentially known, in a wider context of constant scalar curvature Kähler metrics [3]. Namely, it is proved by Donaldson [4] that the existence of a constant scalar curvature metric implies K-semistability. This was later strengthened by Stoppa [15] to K-stability in the absence of continuous automorphism group, and by Mabuchi [9] to K-polystability in general.

Recently in [6] (see also, [10], [7]) K-stability has been defined for a pair $(X, D)$, where $X$ is a Fano manifold and $D$ is a smooth anti-canonical divisor. The definition involves a parameter $\beta \in \mathbb{R}$. At least when $\beta \in (0, 1]$, the K-stability of a pair $(X, D)$ with parameter $\beta$ is conjectured to be equivalent to the existence of a Kähler-Einstein metric on $X$ with cone singularities of angle $2\pi\beta$ transverse to $D$. This generalization grew out of a new continuity method for dealing with the other direction of the above conjecture, as outlined in [5]. Note heuristically the case $\beta = 0$ corresponds to a complete Ricci flat metric on the complement $X \setminus D$. By the work of Tian-Yau [18] such a metric always exists if $D$ is smooth. In this short article we prove the following theorem, which may be viewed as an algebraic counterpart of the differential geometric result of Tian-Yau.

Theorem 1.1. Any pair $(X, D)$ is strictly K-semistable with respect to angle $\beta = 0$, and K-unstable with respect to angle $\beta < 0$.

By the definition of K-stability for pairs which will be recalled in the next section, the Futaki invariant depends linearly on the angle $\beta$. Thus Theorem 1.1 leads immediately to the following

Corollary 1.2. If $X$ is K-stable(semi-stable), then for any smooth anticanonical divisor $D$, the pair $(X, D)$ is K-stable(semi-stable) with respect to angle $\beta \in (0, 1]$. 

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This corollary provides evidence to the picture described in [5] that a smooth Kähler-Einstein metric on $X$ should come from a complete Calabi-Yau metric on $X \setminus D$ by increasing the angle from 0 to $2\pi$. The relevant definitions will be given in the next section. The strategy to prove K-unstability for negative angles is by studying a particular test configuration, namely the deformation to the normal cone of $D$. To deal with the zero angle case we shall construct “approximately balanced” embeddings using the Calabi-Yau metric on $D$. In [11], Odaka proved that a Calabi-Yau manifold is K-stable, by a purely algebro-geometric approach. It is very likely that his method can give an alternative proof of the above theorem, but the one we take seems to be more quantitative.

2. K-stability for pairs

We first recall the definition of K-stability.

**Definition 2.1.** Let $(X, L)$ be a polarized manifold. A *test configuration* for $(X, L)$ is a $\mathbb{C}^*$ equivariant flat family $(\mathcal{X}, \mathcal{L}) \to \mathbb{C}$ such that $(\mathcal{X}_1, \mathcal{L}_1)$ is isomorphic to $(X, L)$. $(\mathcal{X}, \mathcal{L})$ is called *trivial* if it is isomorphic to the product $(X, L) \times \mathbb{C}$ with the trivial action on $(X, L)$ and the standard action on $\mathbb{C}$.

Suppose $D$ is a smooth divisor in $X$, then any test configuration $(\mathcal{X}, \mathcal{L})$ induces a test configuration $(\mathcal{D}, \mathcal{L})$ by simply taking the flat limit of the $\mathbb{C}^*$ orbit of $D$ in $X_1$. We call $(\mathcal{X}, \mathcal{D}, \mathcal{L})$ a test configuration for $(X, D, L)$. Given any test configuration $(\mathcal{X}, \mathcal{D}, \mathcal{L})$ for $(X, D, L)$, we denote by $A_k$ and $A_0$ the infinitesimal generators for the $\mathbb{C}^*$ action on $H^0(X_0, \mathcal{L}_0^k)$ and $H^0(D_0, \mathcal{L}_0^k)$ respectively. By general theory for $k$ large enough we have the following expansions

$$d_k := h^0(X_0, \mathcal{L}_0^k) = a_0 k^n + a_1 k^{n-1} + O(k^{n-2}),$$

$$w_k := \text{tr}(A_k) = b_0 k^{n+1} + b_1 k^n + O(k^{n-1}),$$

$$\tilde{d}_k := h^0(D_0, \mathcal{L}_0^k) = \tilde{a}_0 k^{n-1} + \tilde{a}_1 k^{n-2} + O(k^{n-3}),$$

$$\tilde{w}_k := \text{tr}(\tilde{A}_k) = \tilde{b}_0 k^n + \tilde{b}_1 k^{n-1} + O(k^{n-2}).$$

**Definition 2.2.** For any real number $\beta$, the Futaki invariant of a test configuration $(\mathcal{X}, \mathcal{D}, \mathcal{L})$ with respect to angle $\beta$ is

$$\text{Fut}(\mathcal{X}, \mathcal{D}, \mathcal{L}, \beta) = \frac{2(a_1 b_0 - a_0 b_1)}{a_0} + (1 - \beta)(\tilde{b}_0 - \tilde{a}_0 b_0).$$

When $\beta = 1$ we get the usual Futaki invariant of a test configuration $(\mathcal{X}, \mathcal{L})$

$$\text{Fut}(\mathcal{X}, \mathcal{L}) = \frac{2(a_1 b_0 - a_0 b_1)}{a_0}.$$

**Definition 2.3.** A polarized manifold $(X, L)$ is called *K-stable(semistable)* if $\text{Fut}(\mathcal{X}, \mathcal{L}) > 0(\geq 0)$ for any nontrivial test configuration $(\mathcal{X}, \mathcal{L})$. Similarly, $(X, D, L)$ is called *K-stable(semistable) with respect to angle $\beta$* if $\text{Fut}(\mathcal{X}, \mathcal{D}, \mathcal{L}, \beta) > 0(\geq 0)$ for any nontrivial test configuration $(\mathcal{X}, \mathcal{D}, \mathcal{L})$. 


When the central fiber \((X_0, D_0)\) is smooth, by Riemann-Roch the Futaki invariant then has a differential geometric expression as
\[
\text{Fut}(X, D, L, \beta) = \int_{X_0} (S-S) H \frac{\omega^n}{n!} - (1-\beta) \left( \int_{D_0} H \frac{\omega^{n-1}}{(n-1)!} \frac{\text{Vol}(D_0)}{\text{Vol}(X_0)} \int_{X_0} H \frac{\omega^n}{n!} \right),
\]
where \(\omega\) is an \(S^1\) invariant Kähler metric in \(2\pi c_1(L_0)\) and \(H\) is the Hamiltonian function generating the \(S^1\) action on \(L_0\). This differs from the usual Futaki invariant by an extra term which reflects the cone angle.

The above abstract notion of K-stability is closely related to Chow stability for projective varieties, which we now recall. Given a \(\mathbb{C}^*\) action on \(\mathbb{CP}^N\), and suppose the induced \(S^1\) action preserves the Fubini-Study metric. Then the infinitesimal generator is given by a Hermitian matrix, say \(A\). The Hamiltonian function for the \(S^1\) action on \(\mathbb{CP}^N\) is
\[
H_A(z) = \frac{z^*Az}{|z|^2}.
\]
Given a projective manifold \(V\) in \(\mathbb{CP}^N\), we define the center of mass of \(V\) with respect to \(A\) to be
\[
\mu(V; A) = -\text{Tr}(\mu(V) \cdot A) = -\int_V H_A d\mu_{FS} + \frac{\text{Vol}(V)}{N+1} \text{Tr} A.
\]
Notice this vanishes if \(A\) is a scalar matrix. The definition is not sensitive to singularities of \(V\) so one may define the Chow weight of any algebraic cycles in a natural way. It is well-known that the \(CH(e^{tA}.V, A)\) is a decreasing function of \(t\), see for example [4]. So
\[
CH(V, A) \leq CH(V_\infty, A),
\]
where \(V_\infty\) is the limiting Chow cycle of \(e^{tA}.V\) as \(t \to -\infty\). \(V_\infty\) is fixed by the \(\mathbb{C}^*\) action and then \(CH(V_\infty, A)\) is an algebraic geometric notion, i.e. independent of the Hermitian metric we choose on \(\mathbb{CP}^{N+1}\).

This well-known theory readily extends to pairs, see [3, 4]. We consider a pair of varieties \((V, W)\) in \(\mathbb{CP}^N\) where \(W\) is a subvariety of \(V\). Given a parameter \(\lambda \in [0, 1]\), we define the center of mass of \((V, W)\) with parameter \(\lambda\)
\[
\mu(V, W; \lambda) = \lambda \int_V \frac{z^*zd\mu_{FS}}{|z|^2} + (1-\lambda) \int_W \frac{z^*zd\mu_{FS}}{|z|^2} - \lambda \frac{\text{Vol}(V) + (1-\lambda)\text{Vol}(W)}{N+1} \text{Id},
\]
and the Chow weight with parameter \(\lambda\):
\[
CH(V, W; A, \lambda) = -\text{Tr}(\mu(V, W; \lambda) \cdot A).
\]
A pair \((V, W)\) with vanishing center of mass with parameter \(\lambda\) is called a \(\lambda\)-balanced embedding.

Now given a test configuration \((X, D, L)\), it is explained in [13] and [4] (see also [12]) that for \(k\) large enough one can realize it by a family of projective
schemes in \( \mathbb{P}(H^0(X, L^k)^*) \) with a one parameter group action. Moreover one could arrange that the fiber \((X_1, D_1, L_1)\) is embedded into \( \mathbb{P}(H^0(X, L^k)^*) \) with a prescribed Hermitian metric, and the \( \mathbb{C}^* \) action is generated by a Hermitian matrix \(-A_k\) (negative sign because we are taking the dual). Then as in [14] the Futaki invariant is the limit of Chow weight:

\[
\lim_{k \to \infty} k^{-n}CH_k(X_0, D_0, -A_k, \lambda) = Fut(X, D, L, \beta),
\]

with \( \beta = \frac{3\lambda - 2}{\chi} \).

3. Proof of the main theorem

From now on we assume \( X \) is a Fano manifold of dimension \( n \), \( D \) is a smooth anti-canonical divisor and the polarization is given by \( L = -K_X \). We first prove the part of unstability in theorem 1.1, by considering the \( \nu \) of the normal bundle blow up \( D \) smooth anti-canonical divisor and the polarization is given by \( X \) we get test configurations \((\pi_0, D_0, \beta)\) with \( \beta = \frac{3\lambda - 2}{\chi} \). Using the short exact sequence

\[
0 \to H^0(X, L^{i-1}) \to H^0(X, L^i) \to H^0(D, L^i) \to 0,
\]

we obtain

\[
H^0(X_0, L^k) = H^0(X, L^k)/tH^0(X, L^k) = H^0(X, L^{(1-c)k}) \oplus \bigoplus_{i=0}^{ck-1} t^{ck-i}H^0(D, L^{k-i}).
\]

This is indeed the weight decomposition of \( H^0(X_0, L^k) \) under the \( \mathbb{C}^* \) action. Note the weight is \(-1\) on \( t \). So

\[
\dim H^0(X_0, L^k) = \dim H^0(X, L^{(1-c)k}) + \sum_{i=0}^{ck-1} \dim H^0(D, L^{k-i}) = \dim H^0(X, L^k).
\]
This actually shows the flatness of the family \((X, D, L)\). Thus by Riemann-Roch,

\[
a_0 = \frac{1}{n!} \int_X c_1(L)^n,
\]

and

\[
a_1 = \frac{1}{2(n-1)!} \int_X c_1(-K_X) \cdot c_1(L)^{n-1} = \frac{n a_0}{2}.
\]

The weight is given by

\[
w_k = - \sum_{i=0}^{ck-1} (ck - i) \dim H^0(D, L^{k-i})
\]

\[
= - \sum_{i=0}^{ck-1} (ck - i) \frac{(k - i)^{n-1}}{(n - 1)!} \int_D c_1(L)^{n-1} + O(k^{n-3})
\]

\[
= -n a_0 \int_0^c (c - x)(1 - x)^{n-1} dx \cdot k^{n+1} = -\frac{nca_0}{2} k^n + O(k^{n-1}).
\]

So

\[
b_0 = \left( \frac{1 - (1 - c)^{n+1}}{n + 1} - c \right) a_0,
\]

and

\[
b_1 = -\frac{nca_0}{2}.
\]

Thus the ordinary Futaki invariant for the test configuration \((X, L)\) is given by

\[
Fut_c(X, L) = \frac{2(a_1 b_0 - a_0 b_1)}{a_0} = n(\frac{1 - (1 - c)^{n+1}}{n + 1})a_0.
\]

Note

\[
H^0(D, L^k_c) = H^0(D \times \mathbb{C}, L^k \otimes (t)^ck) = t^{ck} \mathbb{C}[t] H^0(D, L^k).
\]

So

\[
H^0(D_0, L^k_c) = H^0(D, L^k_c) / t H^0(D, L^k_c) = t^{ck} H^0(D, L^k).
\]

Thus we see

\[
\tilde{a}_0 = \int_D \frac{c_1(L)^{n-1}}{(n - 1)!} = na_0,
\]

and

\[
\tilde{b}_0 = -c \int_D \frac{c_1(L)^{n-1}}{(n - 1)!} = -nca_0.
\]

Therefore,

\[
Fut_c(X, D, \beta) = Fut_c(X, L) + (1 - \beta)(\tilde{b}_0 - \frac{\tilde{a}_0}{a_0} b_0)
\]

\[
= [n(\frac{1 - (1 - c)^{n+1}}{n + 1}) + (1 - \beta)(-nc + n(c - \frac{1 - (1 - c)^{n+1}}{n + 1}))]a_0
\]

\[
= n\beta \frac{1 - (1 - c)^{n+1}}{n + 1} a_0.
\]

Therefore for \(\beta < 0\) this particular test configuration gives rise to unstability, and for \(\beta = 0\) the pair \((X, D)\) can not be stable.
Choosing an orthonormal basis of orthogonal. We can put an arbitrary metric on $H^0(X,L^{s-1})$, and make the splitting (3.1) orthogonal. We also identify the vector spaces for $D \rightarrow f$ arbitrary embedding of $(X,D)$, let $C$ be the union of all these $N(D_{j-1},D_j)$ together with $f_{k-1}(X)$. Then it is not hard to see that as a pair of Chow cycles $(X_k,D_k)$ lies in the closure of the $PGL(d_k;\mathbb{C})$ orbit of a smooth embedding of $(X,D)$ in $\mathbb{P}(H^0(X,L^k))$. We want to estimate its center of mass. The following two lemmas involve some calculation and the proof will be deferred to the end of this section.

**Lemma 3.1.** For $s \leq j \leq k$ we have

$$\pi_{j*}(\omega_{FS}^n) = \sum_{i=0}^{n-1} \omega_j^i \wedge \omega_{j-1}^{n-1-i},$$

where $\omega_j = f_j^*\omega_{FS}$.

This lemma implies that

$$Vol(N(D_{j-1},D_j)) = \frac{1}{n!} \sum_{i=0}^{n-1} j^i (j-1)^{n-1-i} \cdot (n-1)! Vol(D) = (j^n - (j-1)^n) Vol(X).$$

Summing over $j$ we see that $Vol(X_k) = k^n Vol(X)$.

Notice $N(D_{j-1},D_j)$ can only contribute to the $H^0(D,L^{j-1})$ and $H^0(D,L^j)$ components of the center of mass of $X_k$. Denote by $Z_j = (Z_j^1,\cdots,Z_j^n)$ the homogeneous coordinates on $H^0(D,L^j)$ for $s \leq j \leq k$, and by $Z_{s-1}$ the homogeneous coordinate on $H^0(X,L^{s-1})$. Then we have

**Lemma 3.2.** For $s \leq j \leq k$ we have

$$\pi_{j*} \frac{Z_j Z_{j-1}^*}{|Z_j|^2 + |Z_{j-1}|^2} \omega_{FS}^n = 0,$$
The induced metric \( \omega_j \) is related to the original metric \( \omega_0 \) by the “density of state” function:

\[
\omega_j = j \omega_0 + \sqrt{-1} \partial \overline{\partial} \log \rho_j(\omega_0).
\]

It is well-known that we have the following expansion (see [2], [21], [8], [10])

\[
\rho_j(\omega_0) = j^{n-1} + \frac{S(\omega_0)}{2} j^{n-2} + O(j^{n-3}) = j^{n-1} + O(j^{n-3}),
\]

since \( \omega_0 \) is Ricci flat. Thus

\[
\omega_j^{n-1-i} = j^i (j-1)^{n-1-i} \omega_0^{n-1}(1 + O(j^{-3})).
\]

To estimate \( \mu_j \) recall we have chosen an orthonormal basis \( \{ s^j_i \} \) of \( H^0(D, L^j) \) and we can assume \( \mu_j \) is a diagonal matrix. Then for \( s \leq j \leq k-1 \) we obtain

\[
\mu_j(X_k) = \int_D \frac{|s^j_i|^2 (1 + O(j^{-3}))}{j^{n-1} + O(j^{n-3})} \sum_{i=0}^{n-1} \frac{i+1}{n+1} j^i (j-1)^{n-1-i} \frac{n-i}{n+1} (j+1)^i j^{n-1-i} \omega_0^{n-1} \frac{1}{n!}.
\]

It is easy to see that

\[
\sum_{i=0}^{n-1} \left( \frac{i+1}{n+1} j^i (j-1)^{n-1-i} + \frac{n-i}{n+1} (j+1)^i j^{n-1-i} \right) = nj^{n-1} + O(j^{n-3}).
\]

Thus

\[
\mu_j(X_k) = 1 + O(j^{-2}).
\]

For \( j = k \), we have

\[
\mu_k(X_k) = 1/2 + O(k^{-1}).
\]

For \( j = s - 1 \), we have

\[
\mu_{s-1}(X_k) = O(1).
\]
The center of mass of the pair \((X_k, D_k)\) with respect to \(\lambda = 2/3\) is given by

\[
\mu(X_k, D_k, 2/3) = \frac{2}{3} \mu(X_k) + \frac{1}{3} \mu(D_k) - \mu \cdot Id,
\]

where we denote

\[
\mu = \frac{2Vol(X_k) + Vol(D_k)}{3d_k} = \frac{2}{3} + O(k^{-2}).
\]

Thus for \(s \leq j \leq k - 1\) and \(0 \leq l \leq n_j\) we have

\[
\mu_j^l(X_k, D_k, 2/3) = O(j^{-2}) + O(k^{-2}).
\]

Since \(n_j\) is a polynomial of degree \(n - 1\) in \(j\), we obtain

\[
|\mu_j(X_k, D_k, 2/3)|_2 = \left( \sum_{l=0}^{n_j} |\mu_j^l(X_k, D_k, 2/3)|^2 \right)^{1/2} = O(\frac{n}{\sqrt{2}}),
\]

and

\[
\sum_{j=s}^{k-1} |\mu_j(X_k, D_k, 2/3)|_2 = O(k^{\frac{n}{\sqrt{2}}}).
\]

For \(j = k\), we have

\[
\mu_k^l(D_k) = \int_D \frac{|s_k|^2}{k^{n-1} + O(k^{n-3})} (1 + O(k^{-2})) \frac{k^{n-1} \omega_0^{n-1}}{(n-1)!} = 1 + O(k^{-2}).
\]

So

\[
\mu_k^l(X_k, D_k) = O(k^{-1}),
\]

and

\[
|\mu_k(X_k, D_k)|_2 = O(k^{\frac{n}{\sqrt{2}}}).
\]

Therefore we obtain

\[
|\mu(X_k, D_k)|_2 = O(k^{\frac{n}{\sqrt{2}}}).
\]

So for a smoothly embedded \((X, D)\) in \(\mathbb{P}(H^0(X, L_k))\) we have

\[
\inf_{g \in PGL(d_k; \mathbb{C})} |\mu(g(X, D))|_2 = O(k^{\frac{n}{\sqrt{2}}}).
\]

In particular there are embeddings \(\iota_k : (X, D) \to \mathbb{P}(H^0(X, L_k))\) such that

\[
|\mu(\iota_k(X, D))|_2 = O(k^{\frac{n}{\sqrt{2}}}).
\]

Now any test configuration \((X, D, L)\) can be represented by a family in \(\mathbb{P}(H^0(X, L^k))\) such that the fiber \((X_1, D_1, L_1)\) is embedded by \(\iota_k\) and the \(\mathbb{C}^*\) action is generated by a Hermitian matrix \(A_k\). Again by general theory \(|A_k|^2 = Tr A_k^2 = O(k^{n+2})\). Therefore by monotonicity of the Chow weight we obtain

\[
CH_k(X_0, D_0, -A_k, 2/3) \geq CH_k(X_1, D_1, -A_k, 2/3) \geq -\inf_{g \in PGL(d_k; \mathbb{C})} |\mu(g(X, D))|_2 \cdot |A_k|_2 \geq -O(k^{n+\frac{2}{2}}).
\]

Thus by (2.2)

\[
Fut(X, D, L, 0) = \lim_{k \to \infty} k^{-n} CH_k(X_0, D_0, -A_k, \frac{2}{3}) \geq 0.
\]
This finishes the proof of Theorem 1.1.

Now we prove Lemmas 3.1 and 3.2. In general suppose there are two embeddings \( f_1 : D \to \mathbb{P}^l \) and \( f_2 : D \to \mathbb{P}^m \). As before, let \( N(D) \) be the variety in \( \mathbb{P}^{l+m+1} \) containing all points of the form \((tf_1(x), sf_2(x))\) where \( t, s \in \mathbb{C} \). Intuitively \( N(D) \) is ruled by all lines connecting \( f_1(x) \) and \( f_2(x) \) for \( x \in D \).

Choose a local coordinate chart \( U \) in \( D \) such that the image \( f_1(U) \) and \( f_2(U) \) are contained in a standard coordinate chart for the projective spaces \( \mathbb{P}^l \) and \( \mathbb{P}^m \) respectively. Let \([1 : z]\) and \([1 : w]\) be local coordinates in \( \mathbb{P}^l \) and \( \mathbb{P}^m \). Under unitary transformations we may assume \( f_1(x_0) = [1 : 0] \) and \( f_2(x_0) = [1 : 0] \). The line connecting \( f_1(x_0) \) and \( f_2(x_0) \) is parametrized as \([1 : 0 : t : 0]\) for \( t \in \mathbb{C} \). Along this line we have

\[
\omega_{FS} = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log((1+|t|^2)^2 + |t|^2|w|^2)
= \frac{\sqrt{-1}}{2\pi} (1 + |t|^2) \sum_i dz^i \wedge d\bar{z}^i + |t|^2(1 + |t|^2) \sum_j dw^j \wedge d\bar{w}^j + dt \wedge d\bar{t}.
\]

Thus

\[
\omega_{FS}^n = n(\frac{\sqrt{-1}}{2\pi})^n(1 + |t|^2)^{-n-1}(\sum_i dz^i \wedge d\bar{z}^i + |t|^2 \sum_j dw^j \wedge d\bar{w}^j)^n \wedge dt \wedge d\bar{t}.
\]

Hence integrating along the \( \mathbb{P}^1 \) we get

\[
\int_{\mathbb{P}^1} \omega_{FS}^n = \frac{1}{2\pi} \int_{\mathbb{C}} n(\omega_1 + |t|^2 \omega_2)^{n-1} \wedge (1 + |t|^2)^{-n-1} \sqrt{-1} dt \wedge d\bar{t}
= \frac{1}{2\pi} \int_0^\infty n \sum_{j=0}^{n-1} \binom{n-1}{j} \omega_1^j \wedge \omega_2^{n-1-j} x^j (1 + x)^{-n-1} dx
= \sum_{j=0}^{n-1} \omega_1^j \wedge \omega_2^{n-1-j}.
\]

This proves lemma 3.1.

For the center of mass we compute

\[
\int_{\mathbb{P}^1} \frac{1}{1 + |t|^2} \omega_{FS}^n = \sum_{j=0}^{n-1} \frac{j + 1}{n + 1} \omega_1^j \wedge \omega_2^{n-1-j},
\]

and

\[
\int_{\mathbb{P}^1} \frac{|t|^2}{1 + |t|^2} \omega_{FS}^n = \sum_{j=0}^{n-1} \frac{n - j}{n + 1} \omega_1^j \wedge \omega_2^{n-1-j}.
\]

Thus globally we obtain

\[
\int_{N(D)} \frac{zz^*}{|z|^2 + |w|^2} \omega_{FS}^n = \int_D \frac{zz^*}{|z|^2} \sum_{j=0}^{n-1} \frac{j + 1}{n + 1} \omega_1^j \wedge \omega_2^{n-1-j},
\]

and

\[
\int_{N(D)} \frac{ww^*}{|z|^2 + |w|^2} \omega_{FS}^n = \int_D \frac{ww^*}{|w|^2} \sum_{j=0}^{n-1} \frac{n - j}{n + 1} \omega_1^j \wedge \omega_2^{n-1-j}.
\]
Also notice by symmetry of $N(D)$ under the map $w \mapsto -w$ we have

$$\int_{N(D)} \frac{z^w}{|z|^2 + |w|^2} \omega^n_F S = 0.$$ 

Similarly

$$\int_{N(D)} \frac{wz^w}{|z|^2 + |w|^2} \omega^n_F S = 0.$$ 

This proves lemma 3.2.

**Remark 3.3.** In the case when $X$ is $\mathbb{P}^1$ and $D$ consists of two points, one can indeed find the precise balanced embedding for $\lambda = 2/3$. In $\mathbb{P}^k$ let $L$ be the chain of lines $L_i$ connecting $p_i$ and $p_{i+1}$ ($0 \leq i \leq k-1$), where $p_i$ is the $i$-th coordinate point. Then it is easy to see that $L$ is the degeneration limit of a smooth degree $k$ rational curve, and it is exactly $\frac{2}{3}$ balanced. It is well-known that a rational normal curve in $\mathbb{P}^k$ is always Chow polystable, it follows by linearity that it is also Chow polystable for $\lambda \in (2/3, 1]$.

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