Efficiency of higher-order algorithms for minimizing general composite optimization

Yassine Nabou\(^1\) and Ion Necoara\(^1,2\)

\(^1\)Automatic Control and Systems Engineering Department, University Politehnica Bucharest, Spl. Independentei 313, 060042, Bucharest, Romania.

\(^2\)Gheorghe Mihoc-Caius Iacob Institute of Mathematical Statistics and Applied Mathematics of the Romanian Academy, 050711, Bucharest, Romania.

Contributing authors: yassine.nabou@stud.acs.upb.ro; ion.necoara@upb.ro;

Abstract

Composite minimization involves a collection of functions which are aggregated in a nonsmooth manner. It covers, as a particular case, optimization problems with functional constraints, minimization of max-type functions, and simple composite minimization problems, where the objective function has a nonsmooth component. We design a higher-order majorization algorithmic framework for fully composite problems (possibly nonconvex). Our framework replaces each component with a higher-order surrogate such that the corresponding error function has a higher-order Lipschitz continuous derivative. We present convergence guarantees for our method for composite optimization problems with (non)convex and (non)smooth objective function. In particular, we prove stationary point convergence guarantees for general nonconvex (possibly nonsmooth) problems and under Kurdyka-Lojasiewicz (KL) property of the objective function we derive improved rates depending on the KL parameter. For convex (possibly nonsmooth) problems we also provide sublinear convergence rates.

Keywords: Composite optimization, (non)convex minimization, higher-order methods, Kurdyka-Lojasiewicz property, convergence rates.
1 Introduction

In this work, we consider the class of general composite optimization problems:

$$\min_{x \in \text{dom}f} f(x) := g(F(x)) + h(x),$$  \hspace{1cm} (1)

where $F : \mathbb{E} \to \mathbb{R}^m$ and $h : \mathbb{E} \to \bar{\mathbb{R}}$ are general proper lower semicontinuous functions on their domains and $g : \mathbb{R}^m \to \bar{\mathbb{R}}$ is a proper closed convex function on its domain. Here, $\mathbb{E}$ is a finite-dimensional real vector space and $F = (F_1, \cdots, F_m)$. Note that $\text{dom}f = \{x \in \text{dom} F : F(x) \in \text{dom} g\} \cap \text{dom} h$. This formulation unifies many particular cases, such as optimization problems with functional constraints, max-type minimization problems or exact penalty formulations of nonlinear programs, while recent instances include robust phase retrieval and matrix factorization problems $[4, 9, 11, 17]$. Note that the setting where $g$ is the identity function was intensively investigated in large-scale optimization $[1, 18, 21, 26]$. In this paper, we call this formulation simple composite optimization. When $g$ is restricted to be a Lipschitz convex function and $F$ smooth, a natural approach to this problem consists in linearizing the smooth part, leaving the nonsmooth term unchanged and adding an appropriate quadratic regularization term. This is the approach considered e.g., in $[8, 9, 27]$, leading to a proximal Gauss-Newton method, i.e. given the current point $\bar{x}$ and a regularization parameter $t > 0$, solve the subproblem:

$$x^+ = \arg \min_x g\left(F(\bar{x}) + \nabla F(\bar{x})(x - \bar{x})\right) + \frac{1}{2t} \|x - \bar{x}\|^2 + h(x).$$

For such a method it was proved in $[9]$ that $\text{dist}(0, \partial f(x))$ converges to 0 at a sublinear rate of order $O(1/k^{1/2})$, where $k$ is the iteration counter, while convergence of the iterates under KL inequality was recently shown in $[27]$. In $[4]$ a new flexible method is proposed, where the smooth part $F$ is replaced by its quadratic approximation, i.e., given $\bar{x}$, solve the subproblem:

$$x^+ = \arg \min_x g\left(F(\bar{x}) + \nabla F(\bar{x})(x - \bar{x}) + \frac{L}{2} \|x - \bar{x}\|^2\right) + h(x),$$

where $L = (L_1, \cdots, L_m)^T$, with $L_i$ being the Lipschitz constant of the gradient of $F_i$, for $i = 1 : m$. Assuming $F$ and $g$ are convex functions, $h = 0$ and $g$ additionally is componentwise nondecreasing and Lipschitz, $[4]$ derives sublinear convergence rate of order $O(1/k)$ in function values. Finally, in the recent paper $[11]$, a general composite minimization problem of the form:

$$\min_x g(x, F(x)), \hspace{1cm} (2)$$

is considered, where $F = (F_1, \cdots, F_m)$, with $F_i$’s being convex and $p$-smooth functions and having the $p$-derivative Lipschitz, with $p \geq 1$ an integer constant.
Under these settings, \cite{11} replaces the smooth part by its Taylor approximation of order $p$ plus a proper regularization term, i.e., given $\bar{x}$, solve the subproblem:

$$x^+ = \arg \min_x g\left(x, T_p^F(x; \bar{x}) + \frac{L}{(p+1)!}\|x - \bar{x}\|^{p+1}\right),$$

where $L = (L_1, \cdots, L_m)^T$, with $L_i$ being related to the Lipchitz constant of the $p$-derivative of $F_i$ and $T_p^F(x; \bar{x})$ is the $p$-Taylor approximation of $F$ around the current point $\bar{x}$. For such a higher-order method, \cite{11} derives a sublinear convergence rate in function values of order $O\left(1/k^p\right)$.

Note that the optimization scheme in \cite{11} belongs to the class of \textit{higher-order methods}. Such methods are popular due to their performance in dealing with ill conditioning and fast rates of convergence, see e.g., \cite{2, 5, 6, 12, 13, 22–25}. For example, first-order methods achieve convergence rates of order $O(1/k)$ for smooth convex optimization \cite{18, 26}, while higher-order methods of order $p > 1$ have converge rates $O(1/k^p)$ for minimizing $p$ smooth convex objective functions \cite{12, 13, 22–24}. Accelerated variants of higher-order methods were also developed e.g., in \cite{14, 23, 24}. Local convergence results for higher-order methods in convex and nonconvex settings were given in \cite{11, 22}. Recently, \cite{22} provided a unified framework for the convergence analysis of higher-order optimization algorithms for solving simple composite optimization problems using the \textit{majorization-minimization} approach. This is a technique that approximate an objective function by a majorization function, which can be minimized in closed form or its solution computed fast, yielding a solution or some acceptable improvement. Note that papers such as \cite{4, 16, 18, 28} use a first-order majorization-minimization approach to build a model (i.e., use only gradient information), while \cite{22} uses higher-order derivatives to build such a model. However, global complexity bounds for higher-order methods based on the majorization-minimization principle for solving general composite optimization problem (1) are not yet given. This is the goal of this work.

\textbf{Contributions.} In this paper, we provide an algorithmic framework based on the notion of higher-order upper bound approximation of the general composite problem (1). Note that in this optimization formulation we consider very general properties for our objects, e.g., the functions $F$ and $h$ can be smooth or nonsmooth, convex or nonconvex and $g$ is only convex and monotone. Our framework consist of replacing $F$ by a higher-order surrogate, leading to a \textit{General Composite Higher-Order} minimization algorithm, which we call GCHO. This approach yields an array of algorithms, each of which is associated with the specific properties of $F$ and the corresponding surrogate. Note that most of our variants of GCHO were never explicitly considered in the literature before. In particular, algorithms derived from surrogates as in Examples 1 and 2 have not been analyzed before even in the convex case.
Moreover, our new first-, second-, and third-order methods can be implemented in practice using existing efficient techniques from e.g., [24, 30]. We derive convergence guarantees for the GCHO algorithm when the upper bound approximate $F$ from the objective function up to an error that is $p \geq 1$ times differentiable and has a Lipschitz continuous $p$ derivative; we call such upper bounds composite higher-order surrogate functions. More precisely, on general composite (possibly nonsmooth) nonconvex problems we prove for GCHO global assymptotic stationary point guarantees and with the help of a new auxiliary sequence also convergence rates $O\left( \frac{1}{k^{p/(p+1)}} \right)$ in terms of first-order optimality conditions. We also characterize the convergence rate of GCHO algorithm locally, in terms of function values, under the Kurdyka-Lojasiewicz (KL) property. Our result show that the convergence behavior of GCHO ranges from sublinear to linear depending on the parameter of the underlying KL geometry. Moreover, on general (possibly nonsmooth) composite convex problems (i.e., $F, g$ and $h$ are convex functions) our algorithm achieves global sublinear convergence rate of order $O\left( \frac{1}{k^{p}} \right)$ in function values.

Besides providing a general framework for the design and analysis of composite higher-order methods, in special cases, where complexity bounds are known for some particular algorithms, our convergence results recover the existing bounds. For example, from our convergence analysis one can easily recover the convergence bounds of higher-order algorithms from [13, 23] for unconstrained minimization and from [22–24] for simple composite minimization. Furthermore, in the general composite convex case we recover the convergence bounds from [4] for $p = 1$ and from [11] for $p \geq 1$. To the best of our knowledge, this is the first complete work to deal with general composite problems in the nonconvex and nonsmooth settings, and explicitly derive convergence bounds for higher-order majorization-minimization algorithms (including local convergence under the KL property).

Content. The paper is organized as follows. In Section 2 we introduce some notations and preliminaries. Then, in Section 3 we formulate the optimization problem, we present the algorithm and we derive global convergence results in the nonconvex and convex case. Finally, in Section 4 we present some preliminary numerical experiments.

2 Notations and preliminaries

We denote a finite-dimensional real vector space with $\mathbb{E}$ and $\mathbb{E}^*$ its dual space composed of linear functions on $\mathbb{E}$. Using a self-adjoint positive-definite operator $D : \mathbb{E} \to \mathbb{E}^*$, we endow these spaces with conjugate Euclidean norms:

$$
\|x\| = \langle Dx, x \rangle, \quad x \in \mathbb{E}, \quad \|g\|_* = \langle g, D^{-1}g \rangle^{\frac{1}{2}}, \quad g \in \mathbb{E}^*.
$$
For a twice differentiable function $f$ on a convex and open domain $\text{dom } f \subseteq \mathbb{E}$, we denote by $\nabla f(x)$ and $\nabla^2 f(x)$ its gradient and hessian evaluated at $x \in \text{dom } f$, respectively. Throughout the paper, we consider $p$ a positive integer. In what follows, we often work with directional derivatives of function $f$ at $x$ along directions $h_i \in \mathbb{E}$ of order $p$, $\nabla^p f(x)[h_1, \cdots, h_p]$, with $i = 1 : p$. If all the direction $h_1, \ldots, h_p$ are the same, we use the notation $\nabla^p f(x)[h]$, for $h \in \mathbb{E}$.

Note that if $f$ is $p$ differentiable, then $\nabla^p f(x)$ is a symmetric $p$-linear form. Then, its norm is defined as:

$$\|\nabla^p f(x)\| = \max_{h \in \mathbb{E}} \{\nabla^p f(x)[h]^p : \|h\| \leq 1\}.$$ 

Further, the Taylor approximation of order $p$ of the function $f$ at $x \in \text{dom } f$ is denoted with:

$$T^f_p(y;x) = f(x) + \sum_{i=1}^{p} \frac{1}{i!} \nabla^i f(x)[y - x]^i \quad \forall y \in \mathbb{E}.$$ 

A function $g : \mathbb{R}^m \rightarrow \mathbb{R}$ is said to be nondecreasing if for all $i = 1 : m$, $g$ is nondecreasing in its $i$th argument, i.e., the univariate function:

$$z \mapsto g(z_1, \cdots, z_{i-1}, z, z_{i+1}, \cdots, z_m),$$

is nondecreasing. In what follows, if $x$ and $y$ are in $\mathbb{R}^m$, then $x \geq y$ means that $x_i \geq y_i$ for all $i = 1 : m$. Similarly, we define $x > y$. Let $f : \mathbb{E} \rightarrow \mathbb{R}$ be a $p$ differentiable function on the open domain $\text{dom } f$. Then, the $p$ derivative is Lipschitz continuous if there exist a constant $L^f_p > 0$ such that the following relation holds:

$$\|\nabla^p f(x) - \nabla^p f(y)\| \leq L^f_p \|x - y\| \quad \forall x, y \in \text{dom } f. \quad (3)$$

It is known that if (3) holds, then the residual between the function and its Taylor approximation can be bounded [23]:

$$|f(y) - T^f_p(y;x)| \leq \frac{L^f_p}{(p+1)!}\|y - x\|^{p+1} \quad \forall x, y \in \text{dom } f. \quad (4)$$

If $p \geq 2$, we also have the following inequalities valid for all $x, y \in \text{dom } f$:

$$\|\nabla f(y) - \nabla T^f_p(y;x)\|_* \leq \frac{L^f_p}{p!}\|y - x\|^p, \quad (5)$$

$$\|\nabla^2 f(y) - \nabla^2 T^f_p(y;x)\| \leq \frac{L^f_p}{(p-1)!}\|y - x\|^{p-1}. \quad (6)$$

In the convex case, Nesterov proved in [23] a remarkable result on the convexity of a proper regularized Taylor approximation, see next lemma.
Lemma 1 Let $f$ be convex and $p$ differentiable function having the $p$ derivative Lipschitz with constant $L_p^f$. Then, the regularized $p$ Taylor approximation:

$$
\phi(y; x) = T_p^f(y; x) + \frac{M_p}{(p+1)!}\|y - x\|^{p+1}
$$

is also a convex function in $y$ provided that the constant $M_p \geq pL_p^f$.

Usually in higher-order (tensor) methods one needs to minimize at each iteration a regularized higher-order Taylor approximation. Then, previous lemma allows us to use a large number of powerful methods from convex optimization to solve the subproblem in these tensor methods [6, 7, 24]. Next, we provide few definitions and properties concerning subdifferential calculus (see [19, 29] for more details).

Definition 1 (Subdifferential): Let $f: \mathbb{E} \to \overline{\mathbb{R}}$ be a proper and lower semicontinuous function. For a given $x \in \text{dom } f$, the Fréchet subdifferential of $f$ at $x$, written $\hat{\partial} f(x)$, is the set of all vectors $g_x \in \mathbb{E}^*$ satisfying:

$$
\lim_{x \neq y, y \to x} \frac{f(y) - f(x) - \langle g_x, y - x \rangle}{\|y - x\|} \geq 0.
$$

When $x \notin \text{dom } f$, we set $\hat{\partial} f(x) = \emptyset$. The limiting-subdifferential, or simply the subdifferential, of $f$ at $x \in \text{dom } f$, written $\partial f(x)$, is defined through the following closure process [19]:

$$
\partial f(x) := \left\{ g_x \in \mathbb{E}^* : \exists x^k \to x, f(x^k) \to f(x) \text{ and } \exists g_x^k \in \hat{\partial} f(x^k) \text{ such that } g_x^k \to g_x \right\}.
$$

Note that we have $\hat{\partial} f(x) \subseteq \partial f(x)$ for each $x \in \text{dom } f$. In the previous inclusion, the first set is closed and convex while the second one is closed, see e.g., [29](Theorem 8.6). Let us recall a generalization of the chain rule for the general composite problem (1), where $F$ can be nondifferentiable. A function $F$ is called regular at $x \in \text{dom } F$ if the directional derivatives of $F$ exist at $x$.

Lemma 2 [15](Theorem 6) Let $F = (F_1, \cdots, F_m)$ and $g$ be locally Lipschitz. Then, we have the following inclusion:

$$
\partial (g \circ F)(x) \subseteq \text{co} \left\{ \sum_{i=1}^m u_i v_i : (u_1, \cdots, u_m) \in \partial g(F(x)), v_i \in \partial F_i(x), i = 1 : m \right\}.
$$

Moreover, if the functions $F_i$’s are regular at $x$, $g$ is regular at $F(x)$ and $\partial g(F(x)) \subseteq \mathbb{R}^m_+$, then the inclusion holds with equality.

If $F_1$ and $F_2$ are regular functions, then from previous lemma we have:

$$
\partial (F_1 + F_2)(x) = \partial F_1(x) + \partial F_2(x).
$$

Throughout the paper we assume that the functions $g$, $F$ and $h$ from problem (1) are such that the previous chain rules hold (with equality).
For any $x \in \text{dom } f$ let us define:

$$S_f(x) = \text{dist}(0, \partial f(x)) := \inf_{g_x \in \partial f(x)} \|g_x\|.$$ 

If $\partial f(x) = \emptyset$, we set $S_f(x) = \infty$. Let us also recall the definition of a function satisfying the Kurdyka-Lojasiewicz (KL) property (see [3] for more details).

**Definition 2** A proper lower semicontinuous function $f : E \to \overline{\mathbb{R}}$ satisfies Kurdyka-Lojasiewicz (KL) property if for any compact set $\Omega \subseteq \text{dom } f$ on which $f$ takes a constant value $f^*$ there exist $\delta, \epsilon > 0$ such that one has:

$$\kappa'(f(x) - f^*) \cdot S_f(x) \geq 1 \quad \forall x : \text{dist}(x, \Omega) \leq \delta, f^* < f(x) < f^* + \epsilon,$$

where $\kappa : [0, \epsilon] \to \mathbb{R}$ is concave differentiable function satisfying $\kappa(0) = 0$ and $\kappa' > 0$.

When $\kappa$ takes the form $\kappa(t) = \sigma_q t^{q-1}$, with $q > 1$ and $\sigma_q > 0$ (which is our interest here), the KL property establishes the following local geometry of the nonconvex function $f$ around a compact set $\Omega$:

$$f(x) - f^* \leq \sigma_q S_f(x)^q \quad \forall x : \text{dist}(x, \Omega) \leq \delta, f^* < f(x) < f^* + \epsilon. \quad (7)$$

Note that the relevant aspect of the KL property is when $\Omega$ is a subset of critical points for $f$, i.e. $\Omega \subseteq \{x : 0 \in \partial f(x)\}$, since it is easy to establish the KL property when $\Omega$ is not related to critical points. The KL property holds for a large class of functions including semi-algebraic functions (e.g., real polynomial functions), vector or matrix (semi)norms (e.g., $\| \cdot \|_p$ with $p \geq 0$ rational number), logarithm functions, exponential functions and uniformly convex functions, see [3] for a comprehensive list.

Let us also recall the following lemma, whose proof is similar to the one in [1](Theorem 2). For completeness, we give the proof in Appendix.

**Lemma 3** Let $\theta > 0$, $C_1, C_2 \geq 0$ and $(\lambda_k)_{k \geq 0}$ be a nonnegative, nonincreasing sequence, satisfying the following recurrence:

$$\lambda_{k+1} \leq C_1 (\lambda_k - \lambda_{k+1})^\theta + C_2 (\lambda_k - \lambda_{k+1}). \quad (8)$$

If $\theta \leq 1$, then there exists an integer $k_0$ such that:

$$\lambda_k \leq \left(\frac{C_1 + C_2}{1 + C_1 + C_2}\right)^{k-k_0} \lambda_0 \quad \forall k \geq k_0.$$ 

If $\theta > 1$, then there exist $\alpha > 0$ and integer $k_0$ such that:

$$\lambda_k \leq \frac{\alpha}{(k-k_0)^{\frac{1}{1-\theta}}} \quad \forall k \geq k_0.$$ 

Further, let us introduce the notion of a higher-order surrogate, see also [22].
Definition 3 Let $f : \mathbb{E} \rightarrow \mathbb{R}$ be a proper lower semicontinuous nonconvex function. We call an extended valued function $s(\cdot ; x) : \mathbb{E} \rightarrow \mathbb{R}$, with dom $s(\cdot ; x) = \text{dom } f$, a $p$-higher-order surrogate of $f$ at $x \in \text{dom } f$ if it has the following properties:

(i) the surrogate is bounded from below

\[ s(y; x) \geq f(y) \quad \forall y \in \text{dom } f. \tag{9} \]

(ii) the error function

\[ e(y; x) = s(y; x) - f(y) \tag{10} \]

with dom $f \subseteq \text{int(dom e)}$ is $p$ differentiable and the $p$ derivative is smooth with Lipschitz constant $L^e_p$ on dom $f$.

(iii) the derivatives of the error function $e$ satisfy

\[ \nabla^i e(x; x) = 0 \quad \forall i = 0 : p, \ x \in \text{dom } f, \tag{11} \]

and there exist a positive constant $R_p > 0$ such that

\[ e(y; x) \geq \frac{R_p}{(p + 1)!} \| y - x \|^{p+1} \quad \forall x, y \in \text{dom } f. \tag{12} \]

Next, we give two nontrivial examples of higher-order surrogate functions, see also [22] for more examples.

Example 1 (Composite functions) Let $f_1 : \mathbb{E} \rightarrow \mathbb{R}$ be a proper closed convex function and let $f_2 : \mathbb{E} \rightarrow \mathbb{R}$ be $p$ times differentiable and the $p$ derivative is Lipschitz with constant $L^{f_2}_p$ on dom $f_1 \subseteq \text{int(dom } f_2)$. Then, for the composite function $f = f_1 + f_2$ one can consider the following $p$ higher-order surrogate function:

\[ s(x; y) = f_1(y) + T^f_p(y; x) + \frac{M_p}{(p + 1)!} \| x - y \|^{p+1} \quad \forall x, y \in \text{dom } f, \]

where $M_p > L^{f_2}_p$. Indeed, from the definition of the error function, we get:

\[ e(x; y) = T^f_p(y; x) - f_2(y) + \frac{M_p}{(p + 1)!} \| x - y \|^{p+1}, \]

thus $e(\cdot; x)$ has the $p$ derivative Lipschitz. Moreover, since $f_2$ has the $p$ derivative Lipschitz, it follows from the inequality (4) that:

\[ T^f_p(y; x) - f_2(y) \geq \frac{-L^{f_2}_p}{(p + 1)!} \| x - y \|^{p+1}. \]

Combining the last two inequalities, we get:

\[ e(x; y) \geq \frac{M_p - L^{f_2}_p}{(p + 1)!} \| x - y \|^{p+1}. \]

Hence, the error function $e$ has $L^e_p = M_p + L^{f_2}_p$ and $R_p = M_p - L^{f_2}_p$.

Example 2 (proximal higher-order) Let $f : \mathbb{E} \rightarrow \mathbb{R}$ be a proper lower semicontinuous function. Then, we can consider the following higher-order surrogate function:

\[ s(y; x) = f(y) + \frac{M_r}{(r + 1)!} \| y - x \|^{r+1}, \]

where $r$ is a positive integer. Indeed, the error function becomes:

\[ e(y; x) = s(y; x) - f(x) = \frac{M_r}{(r + 1)!} \| y - x \|^{r+1}, \]

which has the $r$ derivative Lipschitz with $L^e_r = M_r$ and $R_r = M_r$. 
3 General composite higher-order algorithm

In this work, we consider the following assumptions for the general composite optimization problem (1).

Assumption 1.1. The functions $F_i$, with $i = 1 : m$, and $h$ are proper and lower semicontinuous on their domains.

2. The function $g$ is proper, lower semicontinuous, convex and nondecreasing, satisfying additionally the following property:

\[ g(\alpha x) \leq \alpha g(x) \quad \forall x, \alpha x \in \text{dom } g, \forall \alpha \geq 0. \]  

(13)

3. Problem (1) has a solution and thus $f^* := \inf_{x \in \text{dom } f} f(x) > -\infty$.

If Assumption 1.2 holds, then from [11] (Theorem 4) it follows that:

\[ g(x + ty) \leq g(x) + tg(y) \quad \forall x, y, x + ty \in \text{dom } g, t \geq 0. \]  

(14)

Moreover, since $g$ is convex and increasing, then for any $x \in \text{dom } g$ and $u \in \partial g(x)$, we have $u \in \mathbb{R}^m_+$. Indeed, for any $t > 0$ such that $x - te_i \in \text{dom } g$, using the convexity of $g$, we have $-t \langle u, e_i \rangle + g(x) \leq g(x - te_i)$. Since $g$ is nondecreasing function, it follows that $t \langle u, e_i \rangle \geq g(x) - g(x - te_i) \geq 0$ for all $i = 1 : m$, proving our statement. Next, we provide several examples of optimization problems that can be written as (1) and satisfy our Assumption 1.

Example 3 (Constrained minimization problems) Consider a nonlinear problem with general functional constraints:

\[ \min_{x \in Q} F_1(x) \quad \text{s.t.} \quad F_i(x) \leq 0 \quad \forall i = 2 : m, \]

where $Q$ is closed convex set and $F_i$ are proper lower semicontinuous functions (possibly nonconvex). Let $1_X : \mathbb{E} \mapsto \{0, +\infty\}$ be the indicator function of the set $X \subseteq \mathbb{E}$. Using the following reformulation:

\[ \min_{x \in Q} F_1(x) + 1_{\mathbb{R}^{m-1}}(F_2(x), \ldots, F_m(x)) \]

and setting $g(y_1, y_2, \ldots, y_m) = y_1 + 1_{\mathbb{R}^{m-1}}(y_2, \ldots, y_m)$, $F(x) = [F_1(x), \ldots, F_m(x)]^T$ and $h(x) = 1_Q(x)$, the previous constrained minimization problem with functional constraints can be written as problem (1). Let us prove that for this choice of $g$ Assumption 1.2 holds. Indeed, for all $y \in \text{dom } g$ and $\alpha \geq 0$, we have $g(\alpha y) = \alpha y_1 = \alpha g(y)$. Additionally, the following properties hold for this choice of $g$: $0 \in \text{dom } g = \mathbb{R} \times \mathbb{R}_+^{m-1}$ and for all $y \in \text{dom } g$ such that $y < 0$ we have that $g(y) = y_1 < 0$.

Example 4 (Min-max problems) Let us consider the following min-max problem:

\[ \min_{x \in Q} \max_{i=1:m} F_i(x). \]
This type of problem is classical in optimization but also in game theory. Note that if we define \( g(y_1, \cdots, y_m) = \max_{i=1:m} y_i \) and \( h = 1_Q \), then, the previous min-max problem can be written as problem (1). Note that in this case Assumption 1.2 also holds. Indeed, for all \( \alpha \geq 0 \) we have \( g(\alpha y) = \max_{i=1:m} \alpha y_i = \alpha \max_{i=1:m} y_i = \alpha g(y) \). Additionally, the following properties hold for this choice of \( g \): \( 0 \in \text{dom } g = \mathbb{R}^m \) and for all \( y < 0 \) we have \( g(y) = \max_{i=1:m} y_i < 0 \).

**Example 5** (Simple composite functions) Let consider the following simple composite minimization problem:

\[
\min_{x \in \mathbb{R}^n} F_0(x) + h(x).
\]

By considering \( F(x) = F_0(x) \) and \( g \) the identity function, we can clearly see that \( g(F(x)) + h(x) = F_0(x) + h(x) \). It is easy to see that Assumption 1.2 holds for \( g \) taken as identity function, \( 0 \in \text{dom } g \) and for all \( y < 0 \), we have \( g(y) = y < 0 \).

In the following, we assume for problem (1) that each function \( F_i \), with \( i = 1 : m \), admits a \( p \) higher-order surrogate as in Definition 3. Then, we propose the following General Composite Higher-Order algorithm, called GCHO.

**Algorithm GCHO**

Given \( x_0 \in \text{dom } f \). For \( k \geq 1 \) do:

1. Compute surrogate \( s(x; x_k) := (s_1(x; x_k), \cdots, s_m(x; x_k)) \) of \( F \) near \( x_k \).
2. Compute \( x_{k+1} \) a stationary point of the following subproblem:

\[
x_{k+1} \in \arg \min_{x \in \text{dom } f} \left( g(s(x; x_k)) + h(y) \right),
\]

satisfying the following descent:

\[
g(s(x_{k+1}; x_k)) + h(x_{k+1}) \leq f(x_k).
\]

Note that since we assume \( x_{k+1} \) to be a stationary point, then we have:

\[
0 \in \partial \left( g(s(x; x_k)) + h(x) \right)|_{x=x_{k+1}}.
\]

However, our stationary condition can be relaxed to a condition of the form:

\[
\|g x_{k+1}\| \leq \theta \|x_{k+1} - x_k\|^p,
\]

where \( g x_{k+1} \in \partial \left( g(s(x_{k+1}; x_k)) + h(x_{k+1}) \right) \) and \( \theta > 0 \). For simplicity of the exposition, in our convergence analysis below we assume however that \( x_{k+1} \) satisfies the exact stationary condition (17), although our results can be extended to the relaxed stationary point condition from above. Note that our algorithmic framework is quite general and yields an array of algorithms, each of which is associated with the specific properties of \( F \) and the corresponding
surrogate. For example, if \( F \) is a sum between a smooth term and a nonsmooth one we can use a surrogate as in Example 1; if \( F \) is fully nonsmooth we can use a surrogate as in Example 2. This is the first time such an analysis is performed, and most of our variants of GCHO were never explicitly considered in the literature before. In particular, algorithms derived from surrogates as in Examples 1 and 2 have not been analyzed before even in the convex case.

### 3.1 Nonconvex convergence analysis

In this section we consider that each \( F_i \), with \( i = 1 : m \), and \( h \) are nonconvex functions (possible nonsmooth). Then, problem (1) becomes a pure nonconvex optimization problem. Now we are ready to analyze the convergence behavior of GCHO algorithm under these general settings.

**Theorem 1** Let \( F \), \( g \) and \( h \) satisfy Assumption 1 and additionally each \( F_i \) admits a \( p \) higher-order surrogate \( s_i \) as in Definition 3 with the constants \( L_p^i \) and \( R_p(i) \), for \( i = 1 : m \). Let \((x_k)_{k \geq 0}\) be the sequence generated by Algorithm GCHO, \( R_p = (R_p(1), \cdots, R_p(m)) \) and \( L_p = (L_p^1, \cdots, L_p^m) \). Then, the sequence \((f(x_k))_{k \geq 0}\) is nonincreasing and satisfies the following descent relation:

\[
f(x_{k+1}) \leq f(x_k) + \frac{g(-R_p)}{(p + 1)!} \|x_{k+1} - x_k\|^{p+1} \quad \forall k \geq 0.
\]

**Proof** Denote \( e(x_{k+1};x_k) = (e_1(x_{k+1};x_k), \cdots, e_m(x_{k+1};x_k)) \). Then, we have:

\[
g\left(e(x_{k+1};x_k) + F(x_{k+1})\right) = g\left(s(x_{k+1};x_k)\right)
= g\left(s(x_{k+1};x_k)\right) + h(x_{k+1}) - h(x_{k+1})
\leq f(x_k) - h(x_{k+1}).
\]

Further, from (14), we have:

\[
g\left(e(x_{k+1};x_k) + F(x_{k+1})\right) \geq -g\left(-e(x_{k+1};x_k)\right) + g\left(F(x_{k+1})\right).
\]

Combining the last two inequalities, we get:

\[-g\left(-e(x_{k+1};x_k)\right) \leq f(x_k) - f(x_{k+1}).\]

Using that \( g \) is nondecreasing function, (12) and (13), we further have:

\[g\left(-e(x_{k+1};x_k)\right) \leq \frac{\|x_{k+1} - x_k\|^{p+1}}{(p + 1)!} g(-R_p).
\]

Finally, combining the last two inequalities, we get:

\[-\frac{g(-R_p)}{(p + 1)!} \|x_{k+1} - x_k\|^{p+1} \leq f(x_k) - f(x_{k+1}),
\]

which yields our statement. \( \square \)
In the sequel, we assume that \( g(-R_p) < 0 \). Note that since the vector \( R_p > 0 \), then for all the optimization problems considered in Examples 3, 4 and 5 this assumption holds. Summing (18) from \( j = 0 \) to \( k \), we get:

\[
\sum_{j=0}^{k} - \frac{g(-R_p)}{(p+1)!} \|x_{j+1} - x_j\|^{p+1} \leq \sum_{j=0}^{k} f(x_j) - f(x_{j+1}) = f(x_0) - f(x_{k+1}) \leq f(x_0) - f^*.
\]

Taking the limit as \( k \to +\infty \), we obtain:

\[
\sum_{k=0}^{+\infty} \|x_k - x_{k+1}\|^{p+1} < +\infty.
\]

Hence \( \lim_{k \to +\infty} \|x_k - x_{k+1}\| = 0 \). In our convergence analysis, we also require the following additional assumption which requires the existence of some auxiliary sequence that must be closed to the sequence generated by GCHO algorithm and some first-order relation holds:

Assumption 2 Given the sequence \( (x_k)_{k \geq 0} \) generated by GCHO algorithm, there exist two constants \( L^1_p, L^2_p > 0 \) and a sequence \( (y_k)_{k \geq 0} \) such that:

\[
\|y_{k+1} - x_k\| \leq L^1_p \|x_{k+1} - x_k\| \text{ and } S_f(y_{k+1}) \leq L^2_p \|y_{k+1} - x_k\|^p \quad \forall k \geq 0.
\]

3.2 Approaching the set of stationary points

Before continuing with the convergence analysis of GCHO algorithm, let us analyze the relation between \( \|x_{k+1} - x_k\|^p \) and \( S_f(x_{k+1}) \) and then give examples when Assumption 2 is satisfied. For simplicity, consider the following simple composite minimization problem:

\[
\min_{x \in \text{dom} f} f(x) := F(x) + h(x).
\]

where \( F \) is \( p \) times differentiable function, having the \( p \) derivative Lipschitz with constant \( L^F_p \) and \( h \) is proper lower semicontinuous function. In this case \( g \) is the identity function and we can take as a surrogate \( s(y; x) = T^F_p(y; y) + \frac{M_p}{(p+1)!} \|x - y\|^{p+1} + h(y) \), with the positive constant \( M_p \) satisfying \( M_p > L^F_p \).

Then, GCHO algorithm becomes:

\[
x_{k+1} \in \arg \min_x T^F_p(x; x_k) + \frac{M_p}{(p+1)!} \|x - x_k\|^{p+1} + h(x).
\]

This algorithm has been also considered e.g., in the recent papers \([22, 24]\), with \( h \) assumed to be a convex function. In this paper we remove this assumption.
Lemma 4 If $g$ is the identity function and $F$ has the $p$ derivative Lipschitz, then Assumption 2 holds with $y_{k+1} = x_{k+1}$, $L^1_p = 1$ and $L^2_p = \frac{M_p + L^F_p}{p!}$.

Proof Since $x_{k+1}$ is a stationary point, then from (21) and Lemma 2, we get:

$$\frac{M_p}{p!} ||x_{k+1} - x_k||^{p-1}(x_k - x_{k+1}) - \nabla T^F_p (x_{k+1}; x_k) \in \partial h(x_{k+1}),$$

or equivalently

$$\frac{M_p}{p!} ||x_{k+1} - x_k||^{p-1}(x_k - x_{k+1}) + \left( \nabla F(x_{k+1}) - \nabla T^F_p (x_{k+1}; x_k) \right) \in \nabla F(x_{k+1}) + \partial h(x_{k+1}) = \partial f(x_{k+1}).$$

Taking into account that $F$ is $p$-smooth, we further get:

$$S_f(x_{k+1}) \leq \frac{M_p}{p!} ||x_{k+1} - x_k||^p + \| \nabla F(x_{k+1}) - \nabla T^F_p (x_{k+1}; x_k) \|^* \quad (22)$$

or equivalently

$$\leq \frac{M_p + L^F_p}{p!} ||x_{k+1} - x_k||^p.$$ 

Hence, Assumption 2 holds. \(\square\)

Combining (22) and (18), we further obtain:

$$S_f(x_{k+1}) \frac{p+1}{p} \leq \left( \frac{M_p + L^F_p}{p!} \right)^{\frac{p+1}{p}} \frac{(p+1)!}{M_p - L^F_p} \left( f(x_k) - f(x_{k+1}) \right)$$

$$= C_{M_p,L^F_p} \left( f(x_k) - f(x_{k+1}) \right),$$

where $C_{M_p,L^F_p} = \left( \frac{M_p + L^F_p}{p!} \right)^{\frac{p+1}{p}} \frac{(p+1)!}{M_p - L^F_p}$. Summing the last inequality from $j = 0 : k - 1$, and using that $f$ is bounded from below by $f^*$, we get:

$$\sum_{j=0}^{k-1} S_f(x_j) \frac{p+1}{p} \leq C_{M_p,L^F_p} \left( f(x_0) - f(x_k) \right)$$

$$\leq C_{M_p,L^F_p} \left( f(x_0) - f^* \right).$$

Hence:

$$\min_{j=0:k-1} S_f(x_j) \leq \left( C_{M_p,L^F_p} \left( f(x_0) - f^* \right) \right)^{\frac{p}{p+1}}.$$

Thus, we have proved convergence for the simple composite problem under slightly more general assumptions than in [22, 24], i.e., $F$ and $h$ are both non-convex functions. Finally, from (22), the inequality $||x_{k+1} - x_k||^p \leq \frac{p^p}{L^F_p + M_p} \epsilon$
guarantees that \( x_{k+1} \) is nearly stationary for \( f \) in the sense that \( \text{dist}(0, \partial f(x_{k+1})) \leq \epsilon \).

The situation is dramatically different for the general composite problem (1). When \( g \) is nonsmooth, the distance \( \text{dist}(0, \partial f(x_{k+1})) \) will typically not even tend to zero in the limit, although we have seen that \( \| x_{k+1} - x_k \|^p \) converges to zero. Indeed, let us consider the minimization of the following univariate function:

\[
f(x) = \max \left( x^2 - 1, 1 - x^2 \right).
\]

For \( p = 1 \), we have \( L_1^F(1) = L_1^F(2) = 2 \). Taking \( x_0 > 1 \) and \( M_1 = M_2 = 4 \), GCHO algorithm becomes:

\[
x_{k+1} = \arg \min_x Q(x, x_k) \left( := \max \left( Q_1(x, x_k), Q_1(x, x_k) - 4xx_k + 2x_k^2 + 2 \right) \right),
\]

where \( Q_1(x, x_k) = 2x^2 - 2xx_k + x_k^2 - 1 \). Let us prove by induction that \( x_k > 1 \) for all \( k \geq 0 \). Assume that \( x_k > 1 \) for some \( k \geq 0 \). We notice that the polynomials \( Q_2(x, x_k) := Q_1(x, x_k) - 4xx_k + 2x_k^2 + 2 \) and \( Q_1(x, x_k) \) are 2-strongly convex functions and they intersect in a unique point \( \tilde{x} = \frac{x_k^2 + 1}{2x_k} \). Also, the minimum of \( Q_2 \) is \( \tilde{x}_2 = \frac{\tilde{x}}{2} \) and the minimum of \( Q_1 \) is \( \tilde{x}_1 := \frac{\tilde{x}}{2} \), satisfying \( \tilde{x}_1 \leq \tilde{x} \leq \tilde{x}_2 \). Let us prove that \( x_{k+1} = \tilde{x} \). Indeed, if \( x \leq \tilde{x} \), then \( Q(x, x_k) = Q_2(x, x_k) \) and it is nonincreasing on \( (-\infty, \tilde{x}] \). Hence, \( Q(x, x_k) \geq Q(\tilde{x}, x_k) \) for all \( x \leq \tilde{x} \). Further, if \( x \geq \tilde{x} \), then \( Q(x, x_k) = Q_1(x, x_k) \) and it is nondecreasing on \( [\tilde{x}, +\infty) \). In conclusion, \( Q(x, x_k) \geq Q(\tilde{x}, x_k) \) for all \( x \leq \tilde{x} \). Finally, we have that: \( Q(x, x_k) \geq Q(\tilde{x}, x_k) \) for all \( x \in \mathbb{R} \). Since \( x_k > 1 \), we also get that \( x_{k+1} = \frac{x_k^2 + 1}{2x_k} > 1 \). Since \( x_k > 1 \), then \( \partial f(x_k) = 2x_k > 2 \) and \( S_f(x_k) \geq 2 > 0 \). Moreover, \( x_{k+1} < x_k \) and bounded below by 1, thus \( (x_k)_{k \geq 0} \) is convergent and its limit is 1. Indeed, assume that \( x_k \to \hat{x} \) as \( k \to \infty \). Then, we get \( \hat{x} = \frac{\hat{x}^2 + 1}{2\hat{x}} \) and thus \( \hat{x} = 1 \) (recall that \( \hat{x} \geq 1 \)). Consequently, \( \| x_{k+1} - x_k \| \) also converges to 0. Therefore, we must look elsewhere for a connection between \( S_f(\cdot) \) and \( \| x_{k+1} - x_k \| \). Let us consider the following subproblem:

\[
\mathcal{P}(x_k) = \arg \min_{y \in \text{dom} f} \mathcal{M}_p(y, x) := f(y) + \frac{\mu_p}{(p+1)!} \| y - x_k \|^{p+1},
\]

where \( \mu_p > g(L_p^f) \). Since \( f \) is assumed bounded from below, then for any fixed \( x \), the function \( y \mapsto \mathcal{M}_p(y, x) \) is coercive and hence the optimal value \( \mathcal{M}_p^* = \inf_y \mathcal{M}_p(y, x) \) is finite. Then, the subproblem (23) is equivalent to:

\[
\inf_{y \in B} f(y) + \frac{\mu_p}{(p+1)!} \| y - x_k \|^{p+1},
\]

for some compact set \( B \). Since \( \mathcal{M}_p \) is proper lower semicontinuous function in the first argument and \( B \) is compact set, then from Weierstrass theorem we have that the infimum \( \mathcal{M}_p^* \) is attained, i.e., there exists \( \bar{y}_{k+1} \in \mathcal{P}(x_k) \) such
that $\mathcal{M}_p(\bar{y}_{k+1}, x_k) = \mathcal{M}_p^*$. Since the level sets of $y \mapsto \mathcal{M}_p(x, y)$ are compact, then the optimal set $\mathcal{P}(x_k)$ is nonempty and compact and one can consider the following point:

$$y_{k+1} = \arg\min_{y \in \mathcal{P}(x_k)} \|y - x_k\|.$$  \hfill (24)

Let us assume that $F_i$ admits a higher-order surrogate as in Definition 3, where the error functions $e_i$ are $p$ smooth with Lipschitz constants $L_p^e(i)$ for all $i = 1 : m$. Denote $L_p^e = (L_p^e(1), \ldots, L_p^e(m))$ and define the following positive constant $C_{L_p^e}^p = \frac{\mu_p}{\mu_p - g(L_p^e)}$ (recall that $\mu_p$ is chosen such that $\mu_p > g(L_p^e)$).

Then, we have the following result for $y_{k+1}$.

**Lemma 5** Let the assumptions of Theorem 1 hold and $x_{k+1}$ be a global optimum of the subproblem (15) in GCHO algorithm. Then, Assumption 2 holds with $y_{k+1}$ given in (24), $L_p^1 = \left( C_{L_p^e}^p \right)^{1/(p+1)}$ and $L_p^2 = \frac{\mu_p}{\mu_p}$.

**Proof** From the definition of $y_{k+1}$ in (24), we have:

$$f(y_{k+1}) + \frac{\mu_p}{(p+1)!} \|y_{k+1} - x_k\|^{p+1} \leq \min_{y \in \text{dom} \, f} f(y) + \frac{\mu_p}{(p+1)!} \|y - x_k\|^{p+1} \leq f(x_{k+1}) + \frac{\mu_p}{(p+1)!} \|x_{k+1} - x_k\|^{p+1}. \hfill (25)$$

Note that since the error functions $e_i$’s have the $p$ derivative Lipschitz with constants $L_p^e(i)$’s, then using (4), we get:

$$|e_i(y; x_k) - T_{p}^{e_i}(y; x_k)| \leq \frac{L_p^e(i)}{(p+1)!} \|y - x_k\|^{p+1} \quad \forall i = 1 : m, \quad \forall y \in \text{dom} \, e_i.$$  

Since the Taylor approximations of $e_i$’s of order $p$ at $x_k$, $T_{p}^{e_i}(y; x_k)$, are zero, we get:

$$|s_i(y; x_k) - F_i(y)| = |e_i(y; x_k)| \leq \frac{L_p^e(i)}{(p+1)!} \|y - x_k\|^{p+1} \quad \forall i = 1 : m. \hfill (26)$$

Further, since $F(x_{k+1}) \leq s(x_{k+1}; x_k)$ and $g$ is a nondecreasing function, we have:

$$f(x_{k+1}) \leq g\left(s(x_{k+1}; x_k)\right) + h(x_{k+1}) \overset{(15)}{=} \min_{y \in \text{dom} \, f} g\left(s(y; x_k)\right) + h(y) \leq \min_{y \in \text{dom} \, f} g(F(y)) + h(y) + \frac{g(L_p^e)}{(p+1)!} \|y - x_k\|^{p+1} \leq f(y_{k+1}) + \frac{g(L_p^e)}{(p+1)!} \|y_{k+1} - x_k\|^{p+1},$$

where the last inequality follows by taking $y = y_{k+1}$. Then, combining the last inequality with (25), we get:

$$\|y_{k+1} - x_k\|^{p+1} \leq \frac{\mu_p}{\mu_p - g(L_p^e)} \|x_{k+1} - x_k\|^{p+1},$$
which is the first statement of Assumption 2. Further, using Lemma 2 and optimality conditions for $y_{k+1}$, we obtain:

$$0 \in \partial f(y_{k+1}) + \frac{\mu_p}{p!} \|y_{k+1} - x_k\|^{p-1}(y_{k+1} - x_k).$$

According to Lemma 2, the subdifferential $\partial f(y_{k+1})$ is well defined. It follows that:

$$S_f(y_{k+1}) \leq \frac{\mu_p}{p!} \|y_{k+1} - x_k\|^p.$$

Hence, the second statement of Assumption 2 follows. □

Define the following constant:

$$D_{R_p, L_{p,1}^2} = \left(\frac{L_{p}^{2} L_{p}^{1} \frac{p + 1}{p} (p + 1)!}{-g(-R_p)}\right).$$

Then, we can derive the following convergence result for GCHO algorithm in the nonconvex case.

**Theorem 2** Let the assumptions of Theorem 1 hold. Additionally, Assumptions 2 holds. Then, for the sequence $(x_k)_{k \geq 0}$ generated by Algorithm GCHO we have the following sublinear convergence rate:

$$\min_{j=0:k-1} S_f(y_j) \leq \left(\frac{D_{R_p, L_{p,1}^2}(f(x_0) - f^*)}{k^{p+1}}\right)^{\frac{p+1}{p}}.$$

**Proof** From Assumptions 2, we have:

$$S_f(y_{k+1}) \leq L_p^2 \|y_{k+1} - x_k\|^p \leq L_p^2 L_{p,1}^1 \|x_{k+1} - x_k\|^p.$$  

Using the descent (18), we get:

$$S_f(y_{k+1}) \leq \frac{(L_{p}^{1} L_{p}^{2})^{\frac{p + 1}{p}} (p + 1)!}{-g(-R_p)} (f(x_k) - f(x_{k+1})).$$

Summing the last inequality from $j = 0 : k - 1$ and taking the minimum, we get:

$$\min_{j=0:k-1} S_f(y_j) \leq \left(\frac{D_{R_p, L_{p,1}^2}(f(x_0) - f^*)}{k^{p+1}}\right)^{\frac{p+1}{p}},$$

which proves the statement of the theorem. □

**Remark 1** To this end, Assumption 2 requires an auxiliary sequence $y_{k+1}$ satisfying:

$$\begin{align*}
&\|y_{k+1} - x_k\| \leq L_{p,1}^1 \|x_{k+1} - x_k\| \\
&S_f(y_{k+1}) \leq L_p^2 \|x_{k+1} - x_k\|^p.
\end{align*}$$

(27)

If $\|x_{k+1} - x_k\|$ is small, the point $x_k$ is near $y_{k+1}$, which is nearly stationary for $f$ (recall that $\|x_{k+1} - x_k\|$ converges to 0). Hence, we do not have approximate stationarity for the original sequence $x_k$ but for the auxiliary sequence $y_k$, which is close to the original sequence. Note that in practice, $y_{k+1}$ does not need to be computed. The purpose of $y_{k+1}$ is to certify that $x_k$ is approximately stationary in the sense of (28). For $p = 1$ a similar conclusion was derived in [9]. For a better understanding of the behavior of the sequence $y_{k+1}$, let us come back to our example...
\( f(x) = \max(x^2 - 1, 1 - x^2) \) and \( p = 1 \). Recall that we have proved \( x_k > 1 \) and choosing \( \mu_p = 4 \), then \( y_{k+1} \) is the solution of the following subproblem:

\[
y_{k+1} = \arg \min_y \max(y^2 - 1, 1 - y^2) + 2(y - x_k)^2.
\]

Then, it follows immediately that:

\[
y_{k+1} = \begin{cases} \frac{2}{3}x_k & \text{if } x_k > \frac{3}{2} \\ 1 & \text{if } 1 \leq x_k \leq \frac{3}{2}. \end{cases}
\]

(28)

Since we have already proved that \( x_k \to 1 \), we conclude that \( |y_{k+1} - x_k| \to 0 \) and consequently \( \operatorname{dist}(0, f(y_{k+1})) \to 0 \) for \( k \to \infty \), as predicted by our theory.

### 3.3 Better rates for GCHO under KL

In this section, we show that improved rates can be derived for GCHO algorithm if the objective function satisfies the KL property. This is the first time when such convergence analysis is derived for the GCHO algorithm on the general composite problem (1). We believe that this lack of analysis comes from the fact that one can’t bound directly the distance \( S_f(x_{k+1}) \) by \( \|x_{k+1} - x_k\| \). However, using the newly introduced (artificial) point \( y_{k+1} \), we can now overcome this difficulty. First, we show that if \( (x_k)_{k \geq 0} \) is bounded, then also \( (y_k)_{k \geq 0} \) is bounded and they have the same limit points.

**Lemma 6** Let \( (x_k)_{k \geq 0} \) generated by Algorithm GCHO be bounded and \( (y_k)_{k \geq 0} \) satisfy Assumption 2. Then, the set of limit points of the sequence \( (y_k)_{k \geq 0} \) coincides with the set of limit points of \( (x_k)_{k \geq 0} \).

**Proof** Indeed, let \( x_* \) be a limit point of the sequence \( (x_k)_{k \geq 0} \). Then, there exists a subsequence \( (x_{k_t})_{t \geq 0} \) such that \( x_{k_t} \to x_* \) for \( t \to \infty \). We have:

\[
\|y_{k_t} - x_{k_t}\| \leq \|y_{k_t} - x_{k_t-1}\| + \|x_{k_t} - x_{k_t-1}\| \\
\leq \left( L_p^1 + 1 \right) \|x_{k_t} - x_{k_t-1}\| \quad \forall k \geq 0,
\]

(29)

which implies that \( y_{k_t} \to x_* \). Hence, \( x_* \) is also a limit point of the sequence \( (y_k)_{k \geq 0} \). Further, let \( y_* \) be a limit point of the sequence \( (y_k)_{k \geq 0} \). Then, there exist a subsequence \( (y_{k_t})_{t \geq 0} \) such that \( y_{k_t} \to y_* \) for \( t \to \infty \). From (29) we have that \( x_{k_t} \to y_* \), which means that \( y_* \) is also a limit point of the sequence \( (x_k)_{k \geq 0} \). \( \square \)

Let us denote the set of limit points of \( (x_k)_{k \geq 0} \) by:

\[
\Omega(x_0) = \{ \bar{x} \in \mathbb{E} : \exists \text{ an increasing sequence of integers } (k_t)_{t \geq 0} \text{, such that } x_{k_t} \to \bar{x} \text{ as } t \to \infty \},
\]

and the set of stationary points of problem (1) by \( \operatorname{crit} f \).
Lemma 7 Let the assumptions of Theorem 1 hold. Assume that either $f$ is continuous on its domain or $x_{k+1}$ is a local minimum of the subproblem (15) in GCHO algorithm. Then, we have: $\emptyset \neq \Omega(x_0) \subseteq \text{crit} f$, $\Omega(x_0)$ is compact and connected set, and $f$ is constant on $\Omega(x_0)$.

Proof First let us show that $f(\Omega(x_0))$ is constant. From (18) we have that $(f(x_k))_{k \geq 0}$ is monotonically decreasing and since $f$ is assumed bounded from below, it converges, let us say to $f_\infty > -\infty$, i.e. $f(x_k) \to f_\infty$ as $k \to \infty$. On the other hand let $x_*$ be a limit point of the sequence $(x_k)_{k \geq 0}$. This means that there exist a subsequence $(x_{k_t})_{t \geq 0}$ such that $x_{k_t} \to x_*$. If $f$ is continuous, then $f(x_{k_t}) \to f(x_*) = f_\infty$. Otherwise, since $f$ is lower semicontinuous, we have:

$$\liminf_{k \to \infty} f(x_{k_t}) \geq f(x_*).$$

Furthermore, if we assume that $x_{k_t}$ is a local minimum of $g(s(\cdot ; x_{k_t-1})) + h(\cdot)$, then there exist $\delta_t > 0$ such that $g(s(x_{k_t};x_{k_t-1})) + h(x_{k_t}) \leq g(s(x;x_{k_t-1})) + h(x)$ for all $\|x-x_{k_t}\| \leq \delta_t$. This implies:

$$f(x_{k_t}) \leq f(x) + \frac{g(L_p^e)}{(p+1)!}\|x-x_{k_t-1}\|^{p+1} \quad \forall x: \|x-x_{k_t}\| \leq \delta_t.$$ 

As $x_{k_t} \to x_*$, then there exist $t_0$ such that $\|x_* - x_{k_t}\| \leq \delta_t$ for all $t \geq t_0$. It follows:

$$f(x_{k_t}) \leq f(x_*) + \frac{g(L_p^e)}{(p+1)!}\|x_* - x_{k_t-1}\|^{p+1}$$

$$\leq f(x_*) + \frac{2^p g(L_p^e)}{(p+1)!}\|x_* - x_{k_t}\|^{p+1} + \frac{2^p g(L_p^e)}{(p+1)!}\|x_{k_t} - x_{k_t-1}\|^{p+1},$$

where in the last inequality we use that $\|x + y\|^{p+1} \leq 2^p \|x\|^{p+1} + 2^p \|y\|^{p+1}$. Taking lim sup and using that $\|x_{k+1} - x_k\| \to 0$, we get:

$$\limsup_{t \to \infty} f(x_{k_t}) \leq \limsup_{t \to \infty} \left[ f(x_*) + \frac{2^{p-1} g(L_p^e)}{(p+1)!}\|x_* - x_{k_t}\|^{p+1} \right. \left. + \frac{2^{p-1} g(L_p^e)}{(p+1)!}\|x_{k_t} - x_{k_t-1}\|^{p+1} \right] = f(x_*).$$

Thus, we conclude that $f(x_{k_t}) \to f(x_*)$. Hence, $f(x_*) = f_\infty$. In conclusion, we have $f(\Omega(x_0)) = f_\infty$. The closeness property of $\partial f$ implies that $\mathcal{S}(x_*) = 0$, and thus $0 \in \partial f(x_*)$. This proves that $x_*$ is a critical point of $f$ and thus $\Omega(x_0)$ is nonempty. By observing that $\Omega(x_0)$ can be viewed as an intersection of compact sets:

$$\Omega(x_0) = \cap_{q \geq 0} \cup_{k \geq q} \{x_k\},$$

so it is also compact. This completes our proof. \qed

In addition, let us consider the following assumption:

Assumption 3 For the sequence $(x_k)_{k \geq 0}$ generated by GCHO algorithm, there exist a positive constant $\theta_p > 0$ such that:

$$f(x_{k+1}) \leq f(y_{k+1}) + \theta_p \|y_{k+1} - x_k\|^{p+1} \quad \forall k \geq 0.$$  (30)
Remark 2 Note that Assumption 3 holds when e.g., \( g \) is the identity function or \( x_{k+1} \) is the global optimum of the subproblem (15). For completeness, we provide a proof for this statement in Appendix.

From previous lemma, all the conditions of the KL property from Definition 2 are satisfied. Then, we can derive the following convergence rates depending on the KL parameter.

**Theorem 3** Let the assumptions of Lemma 7 hold. Additionally, assume that \( f \) satisfy the KL property (7) and Assumption 3 is valid. Then, the following convergence rates hold for the sequence \((x_k)_{k \geq 0}\) generated by GCHO algorithm:

- If \( q \geq \frac{p+1}{p} \), then \( f(x_k) \) converge to \( f_\infty \) linearly for \( k \) sufficiently large.
- If \( q < \frac{p+1}{p} \), then \( f(x_k) \) converge to \( f_\infty \) at sublinear rate of order \( O\left(\frac{1}{k^{pq/p+1-pq}}\right) \) for \( k \) sufficiently large.

**Proof** We have:

\[
f(x_{k+1}) - f_\infty \leq f(y_{k+1}) - f_\infty + \theta_p \|y_{k+1} - x_k\|^{p+1}_p.
\]

Using Lemma 3, with \( \theta = \frac{p+1}{pq} \) we get our statements. \( \square \)

Remark 3 Note that when the objective function \( f \) is uniformly convex of order \( p+1 \), [11] proves linear convergence for their algorithm in function values. Our results are more general, i.e., we provide convergence rates for GCHO algorithm depending on a general uniform convexity parameter.

### 3.4 Convex convergence analysis

In this section, we assume that the functions \( F_i \), with \( i = 1 : m \), and \( h \) are convex. Then, it follows that the objective function \( f \) in (1) is also convex. Since the problem (1) is convex, we assume that \( x_{k+1} \) is a global minimum of the subproblem (15). Below, we also assume that the level sets of \( f \) are bounded. Since GCHO algorithm is a descent method, this implies that there exist a positive constant \( R_0 > 0 \) such that \( \|x_k - x^*\| \leq R_0 \) for all \( k \geq 0 \), where \( x^* \) is an optimal solution of (1). Then, we get the following sublinear rate for GCHO algorithm.
**Theorem 4** Let $F$, $g$ and $h$ satisfy Assumption 1 and additionally each $F_i$ admits a $p$ higher-order surrogate $s_i$ as in Definition 3 with the constants $L_{p}^{e}(i)$ and $R_{p}(i)$, for $i = 1 : m$. Additionally, we assume that $F$ and $h$ are convex functions. Let $(x_{k})_{k \geq 0}$ be the sequence generated by Algorithm GCHO, $R_{p} = (R_{p}(1), \ldots, R_{p}(m))$ and $L_{p}^{e} = (L_{p}^{e}(1), \ldots, L_{p}^{e}(m))$. Then, we have the following convergence rate:

$$
f(x_{k}) - f(x^*) \leq \frac{g(L_{p}^{e})R_{0}^{p+1}(p + 1)^{p}}{p!k^{p}}.
$$

**Proof** Using the convexity of $F$, $g$ and $h$, we have:

$$
f(x_{k+1}) \leq g(s(x_{k+1}; x_{k})) + h(x_{k+1})
= \min_{x \in \text{dom} \ f} g(s(x; x_{k})) + h(x)
\leq \min_{x \in \text{dom} \ f} g \left( F(x) + \frac{L_{p}^{e}}{(p + 1)!} \|x - x_{k}\|^{p+1} \right) + h(x)
\leq \min_{\alpha \in [0, 1]} g \left( F(x_{k} + \alpha(x^* - x_{k})) + \alpha^{p+1} \frac{L_{p}^{e}}{(p + 1)!} \|x^* - x_{k}\|^{p+1} \right) + h(x^*) + (1 - \alpha)h(x_{k})
\leq \min_{\alpha \in [0, 1]} g \left( F(x^*) + (1 - \alpha)g(F(x_{k})) + \alpha^{p+1} \frac{g(L_{p}^{e})}{(p + 1)!} \|x^* - x_{k}\|^{p+1} \right) + h(x^*) + (1 - \alpha)h(x_{k})
\leq \min_{\alpha \in [0, 1]} f(x_{k}) + \alpha \left[ (f(x^*) - f(x_{k})) \right] \alpha^{p+1} \frac{R_{0}^{p+1}}{(p + 1)!} g(L_{p}^{e}).
$$

The minimum in $\alpha \geq 0$ is achieved at:

$$
\alpha^* = \left( \frac{f(x_{k}) - f(x^*)p!}{g(L_{p}^{e})R_{0}^{p+1}(p + 1)!} \right)^{\frac{1}{p}}.
$$

We have $0 \leq \alpha^* < 1$. Indeed, since $(f(x_{k}))_{k \geq 0}$ is decreasing, we have:

$$
f(x_{k}) \leq f(x_{1}) \leq g(s(x_{1}; x_{0})) + h(x_{1})
= \min_{x \in \text{dom} \ f} g(s(x; x_{0})) + h(x)
\leq \min_{x \in \text{dom} \ f} g \left( F(x) + \frac{L_{p}^{e}}{(p + 1)!} \|x - x_{0}\|^{p+1} \right) + h(x)
\leq g \left( F(x^*) + \frac{L_{p}^{e}}{(p + 1)!} \|x^* - x_{0}\|^{p+1} \right) + h(x^*)
\leq f(x^*) + \frac{g(L_{p}^{e})R_{0}^{p+1}}{(p + 1)!}.
$$

Hence:

$$
0 \leq \alpha^* \leq \left( \frac{g(L_{p}^{e})R_{0}^{p+1}(p + 1)!}{g(L_{p}^{e})R_{0}^{p+1}} \right)^{\frac{1}{p}} \leq \left( \frac{g(L_{p}^{e})R_{0}^{p+1}(p + 1)!}{g(L_{p}^{e})R_{0}^{p+1}(p + 1)!} \right)^{\frac{1}{p}}.
$$
Thus, we conclude:

\[ f(x_{k+1}) \leq f(x_k) - \alpha^* \left( f(x_k) - f(x^*) - \frac{g(L_p^e)R_0^{p+1}}{(p+1)!} (\alpha^*)^p \right) \]

\[ = f(x_k) - \frac{p \alpha^*}{p+1} [f(x_k) - f(x^*)] . \]

Denoting \( \delta_k = f(x_k) - f(x^*) \), we get the following estimate:

\[ \delta_k - \delta_{k+1} \geq C \delta_k^{p+1} , \]

where \( C = \frac{p}{p+1}\left( \frac{p!}{g(L_p^e)R_0^{p+1}} \right) \). Thus, for \( \mu_k = C^p \delta_k \) we get the following recursive inequality:

\[ \mu_k - \mu_{k+1} \geq \mu_k^{p+1} . \]

Following the same proof as in [23] (Theorem 4), we get:

\[ \frac{1}{\mu_k} \geq \left( \frac{1}{\mu_1} + \frac{k - 1}{p} \right)^p . \]

Since

\[ \frac{1}{\mu_1} = \frac{1}{C \delta_1^p} = \frac{p + 1}{p} \left( \frac{g(L_p^e)R_0^{p+1}}{p!(f(x_1) - f^*))} \right)^{\frac{1}{p}} \geq \frac{1}{p} (p + 1)^{\frac{p+1}{p}} , \]

then

\[ \delta_k = C^{-p} \mu_k = \left( \frac{p + 1}{p} \right)^p \frac{g(L_p^e)R_0^{p+1}}{p!} \mu_k \]

\[ \leq \left( \frac{p + 1}{p} \right)^p \frac{g(L_p^e)R_0^{p+1}}{p!} \left( \frac{1}{p} (p + 1)^{\frac{p+1}{p}} + \frac{k - 1}{p} \right)^{-p} \]

\[ = \frac{g(L_p^e)R_0^{p+1}}{p! (p + 1)^{\frac{1}{p}} + k - 1} \]

\[ \leq \left( \frac{p + 1}{p} \right)^p \frac{g(L_p^e)R_0^{p+1}}{p! k^p} . \]

This proves the statement of the theorem. \( \square \)

Note that in the convex case the convergence results from [4, 9, 11] assume Lipschitz continuity of the \( p \geq 1 \) derivative of the object function \( F \), which may be too restrictive. However, Theorem 4 assume Lipschitz continuity of the \( p \geq 1 \) derivative of the error function \( e(\cdot) \) (note that we may have the error function \( e(\cdot) \) \( p \) times differentiable and with the \( p \) derivative Lipschitz, while the objective function \( F \) may not be even differentiable, see Examples 1 and 2). Hence, our proof is different and more general than in [4, 9, 11]. Moreover, our convergence rate from the previous theorem covers the usual convergence rates \( O\left(\frac{1}{k^p}\right) \) of higher-order Taylor-based methods in the convex
unconstrained case [23], simple composite case [23, 24] and composite case for \( p \geq 1 \) \([4, 11]\). Therefore, Theorem 4 provides a unified convergence analysis for general composite higher-order algorithms, that covers in particular, convex minimization with general functional constraints, min-max convex problems and composite convex problems, under possibly more general assumptions.

4 Numerical simulations

In this section we present some preliminary numerical tests for GCHO algorithm. For simulations, we consider the test set from [20]. In this test set one can find 14 systems of nonlinear equations, where one searches for \( x^* \) such that \( F_i(x^*) = 0 \) for all \( i = 1, \cdots, m \). For solving these 14 problems, we implement our GCHO algorithm for \( p = 2 \). We consider two composite minimization formulations: min-max and least-squares problems, respectively. The min-max formulation has the form:

\[
\min_{x \in \mathbb{R}^n} f(x) := \max(F_1^2(x), \cdots, F_m^2(x)).
\]  

Similarly, the least-square formulation can be written as a simple composite minimization problem:

\[
\min_{x \in \mathbb{R}^n} f(x) := \sum_{i=0}^{m} F_i^2(x).
\]  

Note that both formulation fits into our general problem (1). We compare GCHO algorithm for the two formulations, (31) and (32). At each iteration of GCHO algorithm we replace each function \( F_i \) by its Taylor approximation of order 2, i.e. \( p = 2 \), plus a cubic regularization and solve the corresponding subproblem (15) using Ipopt [30]. Since it is difficult to compute the corresponding Lipschitz constants for the hessian \( \nabla^2 F_i \), we use a line search procedure based on the descent inequality (16). In GCHO algorithm, the stopping criterion is \( \|x_{k+1} - x_k\| \leq 10^{-4} \) and the starting point \( x_0 \) is taken from [20]. In Table 1, we summarize our numerical results in terms of cpu time, number of iterations and optimal solution found by GCHO algorithm for the two formulations. From the table, we observe that GCHO algorithm applied to the min-max formulation performs better than the GCHO algorithm applied to the the least-squares problem, both in cpu time and number of iterations. This is due to the fact that the regularization constants for the min-max problem (31), \( M_p^{\max} = (M_p^{\max}(1), \cdots, M_p^{\max}(m)) \), are much smaller than the one for the least-squares formulation (32), \( M_p^{ls} \), i.e., \( M_p^{ls} \approx \sum_{i=1}^{m} M_p^{\max}(i) \). Moreover, we observe that GCHO algorithm for both formulations is able to identify the global optimal points given in [20].
| test functions | $x_0$ | GCHO min – max | GCHO l-s |
|----------------|------|----------------|---------|
|                |      | iter | CPU time | $x^*$ | iter | CPU time | $x^*$ |
| 1 ($m = 2$)    | (-1.2;1) | 16 | 1.09 | (1;1) | 29 | 1.14 | (1;1) |
| 2 ($m = 2$)    | (0.5;-2) | 12 | 0.8 | (11.41;0.89) | 33 | 1.2 | (11.41;0.89) |
|                | (1;3) | 9 | 0.762 | (4.99;4) | 17 | 0.86 | (4.99;3.9) |
| 4 ($m = 3$)    | (1;1) | 20 | 1.3 | (10^6;2.10^{-6}) | 31 | 1.579 | (10^6;2.10^{-6}) |
| 5 ($m = 3$)    | (1;1) | 56 | 2.9 | (3;0.5) | 441 | 12 | (2.99;0.499) |
| 7 ($m = 3$)    | (-1;0;0) | 35 | 2.45 | (1;0;0) | 446 | 21.6 | (0.9;0;0) |
| 8 ($m = 15$)   | (1;1;1) | 58 | 2.52 | (0.05;1.08;2.3) | 97 | 5.48 | (0.08;1.13;2.34) |
| 9 ($m = 15$)   | (0;0;0) | 98 | 4.7 | (0.39;0.9;0) | 153 | 6.05 | (0.39;0.9;0) |
| 12 ($m = 6$)   | (0;10;20) | 198 | 6.2 | (0.9;10;1) | 304 | 7.3 | (0.9;10;1) |
| 13 ($m = 4$)   | (3;1;0;1) | 26 | 5.6 | (2;0.2;0.8)e^{-3} | 43 | 7.04 | (2;0.2;0.8)e^{-3} |
| 14 ($m = 6$)   | (-3;1;3;1) | 17 | 9.4 | (1;1;0;9;0) | 74 | 17.5 | (0.9;0;9;1;1) |
| 15 ($m = 11$)  | (.25;39;41;39) | 15 | 0.85 | (.18;1;0;11) | 78 | 2.17 | (.19;1;0;10;13) |
| 17 ($m = 33$)  | (.5;1.5;-1;01;0.02) | 20 | 10.9 | (.38;1.4;-9;01;02) | 350 | 48.04 | (.37;1.4;-1;01;02) |
| 20 ($m = 31$, $n = 9$) | (0;...;0) | 45 | 36.1 | $x^*$ | 112 | 94.57 | $x^*$ |
| 26 ($m = n = 100$) | ($rac{1}{n};...;rac{1}{n}$) | 28 | 1637.8 | $x^*$ | 69 | 2404 | $x^*$ |

*Both algorithms identify the same global solution $x^*$, but its dimension is too big to be displayed in the table.*
5 Appendix

Proof of Lemma 3. Note that the sequence \( \lambda_k \) is nonincreasing and nonnegative, thus it is convergent. Let us consider first \( \theta \leq 1 \). Since \( \lambda_k - \lambda_{k+1} \) converges to 0, then there exists \( k_0 \) such that \( \lambda_k - \lambda_{k+1} \leq 1 \) and \( \lambda_{k+1} \leq (C_1 + C_2) (\lambda_k - \lambda_{k+1}) \) for all \( k \geq k_0 \). It follows that:

\[ \lambda_{k+1} \leq \frac{C_1 + C_2}{1 + C_1 + C_2} \lambda_k, \]

which proves the first statement. If \( 1 < \theta \leq 2 \), then there exists also an integer \( k_0 \) such that \( \lambda_k - \lambda_{k+1} \leq 1 \) for all \( k \geq k_0 \). Then, we have:

\[ \lambda_{k+1}^\theta \leq (C_1 + C_2)^\theta (\lambda_k - \lambda_{k+1}). \]

Since \( 1 < \theta \leq 2 \), then taking \( 0 < \beta = \theta - 1 \leq 1 \), we have:

\[ \left( \frac{1}{C_1 + C_2} \right)^\theta \lambda_{k+1}^{1+\beta} \leq \lambda_k - \lambda_{k+1}, \]

for all \( k \geq k_0 \). From Lemma 11 in [24], we further have:

\[ \lambda_k \leq \frac{\lambda_{k_0}}{(1 + \sigma (k - k_0))^\frac{1}{\gamma}} \]

for all \( k \geq k_0 \) and for some \( \sigma > 0 \). Finally, if \( \theta > 2 \), then let us define the function \( h(s) = s^{-\theta} \) and let \( R > 1 \) be fixed. Since \( 1/\theta < 1 \), then there exist a \( k_0 \) such that \( \lambda_k - \lambda_{k+1} \leq 1 \) for all \( k \geq k_0 \). Then, we have \( \lambda_{k+1} \leq (C_1 + C_2) (\lambda_k - \lambda_{k+1}) \frac{1}{\theta} \), or equivalently:

\[ 1 \leq (C_1 + C_2) (\lambda_k - \lambda_{k+1}) h(\lambda_{k+1}). \]

If we assume that \( h(\lambda_{k+1}) \leq Rh(\lambda_k) \), then:

\[ 1 \leq R(C_1 + C_2)^\theta (\lambda_k - \lambda_{k+1}) h(\lambda_k) \leq \frac{R(C_1 + C_2)^\theta}{-\theta + 1} \left( \lambda_{k+1}^{-\theta} - \lambda_{k+1}^{-\theta+1} \right). \]

Denote \( \mu = \frac{-R(C_1 + C_2)^\theta}{-\theta + 1} \). Then:

\[ 0 < \mu^{-1} \leq \lambda_{k+1}^{-\theta} - \lambda_k^{-\theta}. \] (1)

If we assume that \( h(\lambda_{k+1}) > Rh(\lambda_k) \) and set \( \gamma = R^{\frac{1}{\theta}} \), then it follows immediately that \( \lambda_{k+1} \leq \gamma \lambda_k \). Since \( 1 - \theta \) is negative, we get:

\[ \lambda_{k+1}^{1-\theta} \geq \gamma^{1-\theta} \lambda_k^{1-\theta} \iff \lambda_{k+1}^{1-\theta} - \lambda_k^{1-\theta} \geq (\gamma^{1-\theta} - 1) \lambda_k^{1-\theta}. \]

Since \( 1 - \theta < 0 \), \( \gamma^{1-\theta} > 1 \) and \( \lambda_k \) has a nonnegative limit, then there exists \( \bar{\mu} > 0 \) such that \( (\gamma^{1-\theta} - 1) \lambda_k^{1-\theta} > \bar{\mu} \) for all \( k \geq k_0 \). Therefore, in this case we also obtain:

\[ 0 < \bar{\mu} \leq \lambda_{k+1}^{1-\theta} - \lambda_k^{1-\theta}. \] (2)

If we set \( \bar{\mu} = \min(\mu^{-1}, \bar{\mu}) \) and combine (1) and (2), we obtain:

\[ 0 < \bar{\mu} \leq \lambda_{k+1}^{1-\theta} - \lambda_k^{1-\theta}. \]

Summing the last inequality from \( k_0 \) to \( k \), we obtain \( \lambda_{k+1}^{1-\theta} - \lambda_{k_0}^{1-\theta} \geq \bar{\mu}(k - k_0) \), i.e.:

\[ \lambda_k \leq \frac{\bar{\mu}^{-\frac{1}{\theta - 1}}}{(k - k_0)^{\frac{1}{\theta - 1}}}. \]
for all $k \geq k_0$. This concludes our proof. □

**Proof of Remark 2.** If $g$ is the identity function, then taking $y_{k+1} = x_{k+1}$ one can see that Assumption 3 holds for all $\theta_p > 0$. Let us also prove that Assumption 3 holds, provided that $x_{k+1}$ is a global optimum of subproblem (15). In this case, we have:

$$f(x_{k+1}) \leq g\left(s(x_{k+1};x_k)\right) + h(x_{k+1})$$

$$\overset{(15)}{=} \min_{y \in \text{dom } f} g\left(s(y;x_k)\right) + h(y)$$

$$\overset{(26),(14)}{\leq} \min_{y \in \text{dom } f} g\left(F(y)\right) + h(y) + \frac{g(L_p^e)}{(p+1)!}\|y - x_k\|^{p+1}$$

$$\leq f(y_{k+1}) + \frac{g(L_p^e)}{(p+1)!}\|y_{k+1} - x_k\|^{p+1},$$

which shows that Assumption 3 also holds in this case with $\theta_p = \frac{g(L_p^e)}{(p+1)!}$. □

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