EMBEDDING TOPOLOGICAL SPACES INTO
HAUSDORFF $\kappa$-BOUNDED SPACES

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Abstract. Let $\kappa$ be an infinite cardinal. A topological space $X$ is $\kappa$-bounded if the closure of any subset of cardinality $\leq \kappa$ in $X$ is compact. We discuss the problem of embeddability of topological spaces into Hausdorff (Urysohn, regular) $\kappa$-bounded spaces, and present a canonical construction of such an embedding. Also we construct a (consistent) example of a sequentially compact separable regular space that cannot be embedded into a Hausdorff $\omega$-bounded space.

1. Introduction

It is well-known that a topological space $X$ is homeomorphic to a subspace of a compact Hausdorff space if and only if the space $X$ is Tychonoff. In this paper we address the problem of characterization of topological spaces that embed into Hausdorff (Urysohn, regular, resp.) spaces possessing some weaker compactness properties.

One of such properties is the $\kappa$-boundedness, i.e., the compactness of closures of subsets of cardinality $\leq \kappa$. It is clear that each compact space is $\kappa$-bounded for any cardinal $\kappa$. Any ordinal $\alpha := [0, \alpha)$ of cofinality $\text{cf}(\alpha) > \kappa$, endowed with the order topology, is $\kappa$-bounded but not compact. More information on $\kappa$-bounded spaces can be found in [9], [10], [13]. Embedding of topological spaces into compact-like spaces was also investigated in [1, 2, 5, 6].

In this paper we discuss the following:

Problem 1.1. Which topological spaces are homeomorphic to subspaces of $\kappa$-bounded Hausdorff (Urysohn, regular) spaces?

In Theorem 3.3 (and Theorem 3.5) we shall prove that the necessary and sufficient conditions of embeddability of a $T_1$-space $X$ into a Hausdorff (Urysohn) $\kappa$-bounded space are the (strong) $\pi$-regularity and the (strong) $\pi$-normality of $X$, respectively. In Theorem 3.6 we shall prove that a sufficient condition of embeddability of a $T_1$-space $X$ into a regular $\kappa$-bounded space is the total $\pi$-normality of $X$. The above mentioned separation axioms are introduced and studied in Section 2. In Section 3 we shall present a canonical construction of an embedding a (strongly or totally) $\pi$-normal space into a Hausdorff (Urysohn or regular) $\kappa$-bounded space. In Section 4 we construct a space that is totally $\pi$-normal but not functionally Hausdorff, and a (consistent) example of a sequentially compact separable regular space which is not Tychonoff and hence does not embed into a Hausdorff $\omega$-bounded space. Also, for each cardinal $\kappa$ we construct a topological space which is $\kappa$-bounded, $\pi$-normal, $H$-closed, but not Urysohn.

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2. Useful separation axioms

Let $\mathcal{F}$ be a family of closed subsets of a topological space $X$. The topological space $X$ is called

- $\mathcal{F}$-regular if for any set $F \in \mathcal{F}$ and point $x \in X \setminus F$ there exist disjoint open sets $U, V \subset X$ such that $F \subset U$ and $x \in V$;
- strongly $\mathcal{F}$-regular if for any set $F \in \mathcal{F}$ and point $x \in X \setminus F$ there exist open sets $U, V \subset X$ such that $F \subset U$, $x \in V$ and $\overline{U} \cap \overline{V} = \emptyset$;
- $\mathcal{F}$-Tychonoff if for any set $F \in \mathcal{F}$ and point $x \in X \setminus F$ there exist a continuous function $f : X \to [0, 1]$ such that $f(F) \subset \{0\}$ and $f(x) = 1$;
- $\mathcal{F}$-normal if for any disjoint sets $A, B \in \mathcal{F}$ there exist disjoint open sets $U, V \subset X$ such that $A \subset U$ and $B \subset V$;
- strongly $\mathcal{F}$-normal if for any disjoint sets $A, B \in \mathcal{F}$ there exist open sets $U, V \subset X$ such that $A \subset U$, $B \subset V$ and $\overline{U} \cap \overline{V} = \emptyset$;
- totally $\mathcal{F}$-normal if for any disjoint closed sets $A \in \mathcal{F}$ and $B \subset X$ there exist disjoint open sets $U, V \subset X$ such that $A \subset U$ and $B \subset V$.

For families $\mathcal{F}$ containing all one-point subsets of $X$, these properties relate as follows:

\[
\text{totally } \mathcal{F}\text{-normal} \iff \text{strongly } \mathcal{F}\text{-normal} \iff \mathcal{F}\text{-normal}
\]

\[
\mathcal{F}\text{-Tychonoff} \iff \text{strongly } \mathcal{F}\text{-regular} \iff \mathcal{F}\text{-regular}.
\]

However, the total $\mathcal{F}$-normality does not imply the $\mathcal{F}$-Tychonoff property; see Example 2.4 below.

**Proposition 2.1.** If a topological space $X$ is $\mathcal{F}$-regular for some family $\mathcal{F}$ of closed Lindelöf subspaces of $X$, then $X$ is $\mathcal{F}$-normal.

**Proof.** To show that $X$ is $\mathcal{F}$-normal, fix any two disjoint closed sets $A, B \in \mathcal{F}$. By the $\mathcal{F}$-regularity, for every $a \in A$ there exists an open neighborhood $V_a \subset X$ of $a$ whose closure $\overline{V_a}$ in $X$ does not intersect the set $B$. By the Lindelöf property of $A$ the open cover $\{V_a : a \in A\}$ of $A$ has a countable subcover $\{V_{a_n}\}_{n \in \omega}$. By analogy, for every $b \in B$ there exists an open neighborhood $U_b \subset X$ of $b$ whose closure $\overline{U_b}$ in $X$ does not intersect the set $A$. By the Lindelöf property of $B$, the open cover $\{U_b : b \in B\}$ of $B$ has a countable subcover $\{U_{b_n}\}_{n \in \omega}$. For every $n \in \omega$ let

\[
V_A = \bigcup_{n \in \omega} V_{a_n} \setminus \bigcup_{k \leq n} \overline{U_{b_k}} \quad \text{and} \quad U_B = \bigcup_{n \in \omega} U_{b_n} \setminus \bigcup_{k \leq n} \overline{V_{a_k}}.
\]

Then $V_A, U_B$ are two disjoint open neighborhoods of the sets $A, B$, witnessing that the space $X$ is $\mathcal{F}$-normal.

A subset $Y$ of a topological space $X$ is defined to be (countably) paracompact in $X$ if for each (countable) cover $\mathcal{U}$ of $Y$ by open subsets of $X$, there exists a locally finite family $\mathcal{V}$ of open subsets of $X$ such that $Y \subset \bigcup \mathcal{V}$ and each set $V \in \mathcal{V}$ is contained in some set $U \in \mathcal{U}$.

The proof of the following proposition is straightforward.

**Proposition 2.2.** If a subset $Y$ of a topological space $X$ is (countably) paracompact in $X$, then each closed subset of $Y$ also is (countably) paracompact in $X$. 
**Proposition 2.3.** Let $\mathcal{F}$ be a family of closed Lindelöf subsets of $X$, which are countably paracompact in $X$. If the space $X$ is strongly $\mathcal{F}$-regular, then $X$ is strongly $\mathcal{F}$-normal.

**Proof.** To show that $X$ is strongly $\mathcal{F}$-normal, fix any two disjoint closed sets $A, B \in \mathcal{F}$. By the strong $\mathcal{F}$-regularity of $X$, for every $a \in A$ and $b \in B$ there exist open sets $V_a, W_a, V_b, W_b$ in $X$ such that $a \in V_a \subset \overline{V_a} \subset W_a \subset X \setminus B$ and $b \in V_b \subset \overline{V_b} \subset W_b \subset X \setminus A$.

By the Lindelöf property of the space $X$, the open cover $\{V_a : a \in A\}$ of $A$ has a countable subcover $\{V_{a_n}\}_{n \in \omega}$. By analogy, the open cover $\{V_b : b \in B\}$ of the Lindelöf space $B$ has a countable subcover $\{V_{b_n}\}_{n \in \omega}$.

It is easy to see that $\{V_a \setminus \bigcup_{k \leq n} \overline{W_{b_k}}\}_{n \in \omega}$ is a cover of $A$ by open subsets of $X$. By the countable paracompactness of $X$, there exists a locally finite family $\mathcal{U}_A$ of open sets in $X$ such that $A \subset \bigcup \mathcal{U}_A$ and each set $U \in \mathcal{U}_A$ is contained in some set $V_{a_n} \setminus \bigcup_{k \leq n} \overline{W_{b_k}}$ and hence $U \subset \overline{V_{a_n}} \setminus \bigcup_{k \leq n} W_{b_k} \subset W_{a_n} \setminus \bigcup_{k \leq n} W_{b_k}$.

Consider the open neighborhood $U_A = \bigcup \mathcal{U}_A$ of $A$. The local finiteness of the family $\mathcal{U}_A$ ensures that $\overline{U_A} = \bigcup_{U \in \mathcal{U}_A} U \subset \bigcup_{n \in \omega} W_{a_n} \setminus \bigcup_{k \leq n} W_{b_k}$.

By analogy, we can find an open neighborhood $U_B$ of the countably paracompact subset $B$ in $X$ such that $\overline{U_B} \subset \bigcup_{n \in \omega} W_{b_n} \setminus \bigcup_{k \leq n} W_{a_k}$.

It remains to observe that

$$\overline{U_A} \cap \overline{U_B} \subset \left( \bigcup_{n \in \omega} W_{a_n} \setminus \bigcup_{k \leq n} W_{b_k} \right) \cap \left( \bigcup_{n \in \omega} W_{b_n} \setminus \bigcup_{k \leq n} W_{a_k} \right) = \emptyset.$$  

\[\square\]

**Proposition 2.4.** Each regular topological space $X$ is totally $\mathcal{F}$-normal for the family $\mathcal{F}$ of closed subsets of $X$ that are paracompact in $X$.

**Proof.** To show that $X$ is totally $\mathcal{F}$-normal, fix any two disjoint closed sets $A \in \mathcal{F}$ and $B \subset X$. By the regularity of $X$, for every $a \in A$ there exists an open neighborhood $U_a \subset X$ such that $a \in U_a \subset X \setminus B$. Since $A$ is paracompact in $X$ there exists a locally finite family $\mathcal{V}$ of open subsets of $X$ such that $A \subset \bigcup \mathcal{V}$ and each set $V \in \mathcal{V}$ is contained in some set $U_a, a \in A$. The locally finiteness of $\mathcal{V}$ implies that $\overline{\bigcup \mathcal{V}} = \bigcup_{V \in \mathcal{V}} \overline{V} \subset \bigcup_{a \in A} U_a \subset X \setminus B$. Then $\overline{\mathcal{V}}$ and $X \setminus \overline{\mathcal{V}}$ are disjoint open neighborhoods of the sets $A$ and $B$, respectively.  

\[\square\]

Let $\kappa$ be a cardinal. A topological space $X$ is called **totally $\pi$-normal** (resp. **strongly $\pi$-normal, $\overline{\pi}$-normal, strongly $\overline{\pi}$-regular, $\overline{\pi}$-regular, $\overline{\pi}$-Tychonoff) if it is totally $\mathcal{F}$-normal (resp. strongly $\mathcal{F}$-normal, $\mathcal{F}$-normal, strongly $\mathcal{F}$-regular, $\mathcal{F}$-regular, $\mathcal{F}$-Tychonoff) for the family $\mathcal{F}$ of closed subsets of $X$ of cardinality $\leq \kappa$ in $X$. Simple examples show that the family $\mathcal{F}$ can be strictly larger than the family of closures of subsets of cardinality $\leq \kappa$ in $X$.

**Proposition 2.5.** Each $\kappa$-bounded Hausdorff space $X$ is $\pi$-normal.

**Proof.** Let $\mathcal{F}$ be the family of closed subspaces of $X$ of cardinality $\leq \kappa$ in $X$. Given two disjoint closed sets $A, B \in \mathcal{F}$, we observe that the sets $A, B$ are compact. By the Hausdorff property of $X$, the disjoint compact sets $A, B$ have disjoint open neighborhoods.  

\[\square\]

In Example 3.3 we shall construct a Hausdorff $\omega$-bounded space which is not strongly $\overline{\pi}$-normal.

**Proposition 2.6.** Each subspace $X$ of a $\kappa$-bounded Hausdorff space $Y$ is $\pi$-regular.
Proof. Let $F$ be a closed subspace of the closure of a set $E \subset X$ of cardinality $|E| \leq \kappa$ in $X$ and let $x \in X \setminus F$ be a point. The $\kappa$-boundedness of $Y$ ensures that the closure $\overline{F}$ of $F$ in $Y$ is compact and so is the closure $\overline{\mathcal{F}}$ of $\mathcal{F} \subset Y$. Since $F = X \cap \overline{\mathcal{F}}$ and $x \in X \setminus F$, $x \not\in \overline{F}$ and so by the Hausdorff property of $Y$ there exist two disjoint open sets $V, U \subset Y$ such that $x \in V$ and $\overline{F} \subset U$. Then $V \cap X$ and $U \cap X$ are two disjoint open sets in $X$ such that $x \in V \cap X$ and $F \subset U \cap X$, which means that the space $X$ is $\kappa$-regular. \hfill $\square$

Let us recall that a topological space $X$ is called Urysohn if any distinct points of $X$ have disjoint closed neighborhoods in $X$. Similarly as above one can prove the following facts.

Proposition 2.7. Each $\kappa$-bounded Urysohn space $X$ is strongly $\kappa$-normal.

Proposition 2.8. Each subspace $X$ of a $\kappa$-bounded Urysohn space $Y$ is strongly $\kappa$-regular.

Proposition 2.9. Each $\kappa$-bounded regular space $X$ is totally $\kappa$-normal.

We recall that the density $d(X)$ of a topological space $X$ is the smallest cardinality of a dense subset in $X$.

Proposition 2.10. Each subspace $X$ of density $d(X) \leq \kappa$ in a $\kappa$-bounded Hausdorff space $Y$ is Tychonoff.

Proof. Let $D$ be a dense subset of $X$ with $|D| = d(X) \leq \kappa$. By definition of a $\kappa$-bounded space, the closure $\overline{D}$ of $D$ in $Y$ is compact and being Hausdorff is Tychonoff. Then $X \subset \overline{D}$ is Tychonoff, too. \hfill $\square$

3. The Wallman $\kappa$-bounded extension of a topological space

We recall [8 §3.6] that the Wallman extension $WX$ of a topological space $X$ consists of closed ultrafilters, i.e., families $\mathcal{U}$ of closed subsets of $X$ satisfying the following conditions:

- $\emptyset \notin \mathcal{U}$;
- $A \cap B \in \mathcal{U}$ for any $A, B \in \mathcal{U}$;
- a closed set $F \subset X$ belongs to $\mathcal{U}$ if $F \cap A \neq \emptyset$ for every $A \in \mathcal{U}$.

The Wallman extension $WX$ of $X$ carries the topology generated by the base consisting of the sets

$$\langle U \rangle = \{ F \in WX : \exists F \in \mathcal{F} (F \subset U) \}$$

where $U$ runs over open subsets of $X$.

By (the proof of) Theorem [8 3.6.21], the Wallman extension $WX$ is compact. By Theorem [8 3.6.22] a $T_1$-space $X$ is normal if and only if its Wallman extension $WX$ is Hausdorff.

If $X$ is a $T_1$-space, then we can consider the map $j_X : X \to WX$ assigning to each point $x \in X$ the principal closed ultrafilter consisting of all closed sets $F \subset X$ containing the point $x$. It is easy to see that the image $j_X(X)$ is dense in $WX$. By [8 3.6.21], the map $j_X : X \to WX$ is a topological embedding.

The following lemma can be easily derived from the definition of a closed ultrafilter and should be known.

Lemma 3.1. For a subset $A$ of a $T_1$-space $X$, a closed ultrafilter $\mathcal{F} \in WX$ belongs to $\overline{\mathcal{F}}$ if and only if $\overline{A} \in \mathcal{F}$.

Given an infinite cardinal $\kappa$, in the Wallman extension $WX$ of a $T_1$-space $X$, consider the subspace

$$\mathcal{W}_\kappa X = \bigcup \{ j_X(C) : C \subset X, |C| \leq \kappa \}$$
of \( WX \). The space \( W_kX \) will be called the Wallman \( \kappa \)-bounded extension of \( X \).

The following proposition justifies the choice of terminology.

**Proposition 3.2.** For any topological space \( X \), the space \( W_kX \) is \( \kappa \)-bounded.

*Proof.* We should prove that for any subset \( \Omega \subset W_kX \) of cardinality \( |\Omega| \leq \kappa \), the closure \( \overline{\Omega} \) is compact. By the definition of \( W_kX \), for every ultrafilter \( u \in \Omega \) there exists a set \( C_u \subset X \) such that \( |C_u| \leq \kappa \) and \( u \in j_X(C_u) \). Consider the set \( C = \bigcup_{u \in \Omega} C_u \) and observe that \( |C| \leq \kappa \) and the closure \( j_X(C) \) in \( WX \) is compact (by the compactness of \( WX \)). Then the closure \( \overline{\Omega} \) of \( \Omega \) in \( W_kX \) coincides with the closure of \( \Omega \) in \( j_X(C) \) and hence is compact. \( \Box \)

The following proposition characterizes some separation properties of the Wallman \( \kappa \)-bounded extension \( W_kX \) of a \( T_1 \)-space.

**Proposition 3.3.** For a \( T_1 \)-space \( X \) the following statements hold:

1) \( W_kX \) is Hausdorff iff \( X \) is \( \pi \)-normal;
2) \( W_kX \) is Urysohn iff \( X \) is strongly \( \pi \)-normal;
3) \( W_kX \) is regular iff \( X \) is totally \( \pi \)-normal.

*Proof.* 1. To prove the “if” part of the statement 1), assume that \( X \) is \( \pi \)-normal. Given any distinct closed ultrafilters \( u, v \in W_kX \), use the maximality of \( u, v \) and find two disjoint closed sets \( F \in u \) and \( E \in v \). By definition of \( W_kX \), there exists a subset \( C \subset X \) such that \( |C| \leq \kappa \) and \( u, v \in j_X(C) \). By Lemma 3.1, \( C \in u \cap v \). By the \( \pi \)-normality of \( X \), the disjoint closed sets \( F \cap C \in u \) and \( E \cap C \in v \) have disjoint open neighborhoods \( U \) and \( V \) in \( X \), respectively. Then \( \langle U \rangle \) and \( \langle V \rangle \) are disjoint neighborhoods of the ultrafilters \( u \) and \( v \) in \( WX \), witnessing that the space \( W_kX \) is Hausdorff.

To prove the “only if” part, assume that the space \( W_kX \) is Hausdorff. By Proposition 3.2, the space \( W_kX \) is \( \kappa \)-bounded. To show that the space \( X \) is \( \pi \)-normal, take any subset \( C \subset X \) of cardinality \( |C| \leq \kappa \) and two disjoint closed subsets \( F, E \) of \( \overline{C} \subset X \). Lemma 3.1 implies \( j_X(F) \cap j_X(E) = \emptyset \). Since \( j_X(F) \cup j_X(E) \subset j_X(C) \) and \( |C| \leq \kappa \), the sets \( j_X(F) \) and \( j_X(E) \) are compact and by the Hausdorffness of \( W_kX \), these compact sets have disjoint open neighborhoods \( U \) and \( V \) in \( W_kX \). Then \( j_X^{-1}(U) \) and \( j_X^{-1}(V) \) are disjoint neighborhoods of the sets \( F \) and \( E \) in \( X \), witnessing that the space \( X \) is \( \pi \)-normal.

2. To prove the “if” part of the statement 2), assume that the space \( W_kX \) is strongly \( \pi \)-normal. Given any distinct closed ultrafilters \( u, v \in W_kX \), use the maximality of \( u, v \) and find two disjoint closed sets \( F \in u \) and \( E \in v \). By definition of \( W_kX \), there exists a subset \( C \subset X \) such that \( |C| \leq \kappa \) and \( u, v \in j_X(C) \). By Lemma 3.1, \( C \in u \cap v \). By the strong \( \pi \)-normality of \( X \), the disjoint closed sets \( F \cap C \in u \) and \( E \cap C \in v \) have open neighborhoods \( U \) and \( V \) in \( X \) such that \( \overline{U} \cap \overline{V} = \emptyset \). Then \( \langle U \rangle \) and \( \langle V \rangle \) are disjoint open neighborhoods of the ultrafilters \( u \) and \( v \) in \( WX \). We claim that \( \langle U \rangle \cap \langle V \rangle = \emptyset \). Indeed, given any closed ultrafilter \( w \in WX \), we conclude that either \( U \notin w \) or \( V \notin w \). If \( U \notin w \), then by the maximality of \( w \), the closed set \( \overline{U} \) is disjoint with some set in \( w \) and then \( \langle X \setminus \overline{U} \rangle \) is a neighborhood of \( w \), disjoint with \( \langle U \rangle \). If \( V \notin w \), then \( \langle X \setminus \overline{V} \rangle \) is a neighborhood of \( w \) that is disjoint with \( \langle V \rangle \). In both cases we obtain that \( w \notin \langle U \rangle \cap \langle V \rangle \), which implies \( \langle U \rangle \cap \langle V \rangle = \emptyset \) and witnesses that the space \( W_kX \) is Urysohn.

To prove the “only if” part, assume that the space \( W_kX \) is Urysohn. By Proposition 3.2, the space \( W_kX \) is \( \kappa \)-bounded. To show that the space \( X \) is strongly \( \pi \)-normal, take any subset \( C \subset X \) of cardinality \( |C| \leq \kappa \) and two disjoint closed subsets \( F, E \) of \( \overline{C} \subset X \). Lemma 3.1
implies that \( j_X(F) \cap j_X(E) = \emptyset \). Since \( j_X(F) \cup j_X(E) \subseteq j_X(C) \) and \( |C| \leq \kappa \), the sets \( j_X(F) \) and \( j_X(E) \) are compact. Since the space \( \mathcal{W}_k X \) is Urysohn the compact sets \( j_X(F) \) and \( j_X(E) \) have open neighborhoods \( U \) and \( V \) with disjoint closures in \( \mathcal{W}_k X \). Then \( j_X^{-1}(U) \) and \( j_X^{-1}(V) \) are open neighborhoods with disjoint closures of the sets \( F \) and \( E \) in \( X \), respectively.

3. To prove the “if” part of the statement 3, assume that the space \( X \) is totally \( \kappa \)-normal. Given any closed ultrafilter \( u \in \mathcal{W}_k X \) and a basic open neighborhood \( \langle U \rangle \) of \( u \) in \( \mathcal{W}_k X \), find a closed set \( F \subseteq u \) such that \( F \subseteq U \). Since \( u \in \mathcal{W}_k X \), there exists a subset \( C \subseteq X \) such that \( |C| \leq \kappa \) and \( u \subseteq j_X(C) \). By Lemma 3.1 \( C \subseteq u \). Replacing the set \( F \) by \( F \cap C \), we can assume that \( F \subseteq C \). By the total \( \kappa \)-normality of \( X \), there exists an open neighborhood \( V \) of \( F \) in \( X \) such that \( \langle V \rangle \subseteq U \). Using Lemma 3.1 we can show that \( u \subseteq \langle V \rangle \subseteq \langle U \rangle \), witnessing the regularity of the space \( \mathcal{W}_k X \).

To prove the “only if” part, assume that the space \( \mathcal{W}_k X \) is regular. By Proposition 3.2 the space \( \mathcal{W}_k X \) is \( \kappa \)-bounded. To show that the space \( X \) is totally \( \kappa \)-normal, take any subset \( C \subseteq X \) of cardinality \( |C| \leq \kappa \) and two disjoint closed subsets \( F, E \) of \( X \) such that \( F \subseteq C \). Lemma 3.1 implies that \( j_X(F) \cap j_X(E) = \emptyset \). Since \( j_X(F) \subseteq j_X(C) \) and \( |C| \leq \kappa \), the set \( j_X(F) \) is compact. By the regularity of \( \mathcal{W}_k X \), the sets \( j_X(F) \) and \( j_X(E) \) have disjoint open neighborhoods \( U \) and \( V \) in \( \mathcal{W}_k X \). Then \( j_X^{-1}(U) \) and \( j_X^{-1}(V) \) are disjoint open neighborhood of the sets \( F \) and \( E \) in \( X \), respectively. Hence \( X \) is totally \( \kappa \)-normal.

The following three theorems give a partial answer to Problem 1.1 and are the main results of this paper.

**Theorem 3.4.** For an infinite cardinal \( \kappa \) and a \( T_1 \)-space \( X \) consider the conditions:

1. the space \( X \) is \( \kappa \)-normal;
2. the Wallman \( \kappa \)-bounded extension \( \mathcal{W}_k X \) of \( X \) is Hausdorff;
3. \( X \) is homeomorphic to a subspace of a Hausdorff \( \kappa \)-bounded space;
4. the space \( X \) is \( \kappa \)-regular.

Then \( 1 \iff 2 \Rightarrow 3 \Rightarrow 4 \). If each closed subspace of density \( \leq \kappa \) in \( X \) is Lindelöf, then \( 4 \Rightarrow 1 \) and hence the conditions \( 1 \)–\( 4 \) are equivalent.

**Proof.** The equivalence \( 1 \iff 2 \) was proved in Proposition 3.3(1) and \( 2 \Rightarrow 3 \) follows immediately from Proposition 3.2 and the fact that the canonical map \( j_X : X \to \mathcal{W}_k X \) is a topological embedding. The implication \( 3 \Rightarrow 4 \) follows from Proposition 2.6 If each closed subspace of density \( \leq \kappa \) in \( X \) is Lindelöf, then \( 4 \Rightarrow 1 \) by Proposition 2.1 \( \Box \)

**Theorem 3.5.** For an infinite cardinal \( \kappa \) and a \( T_1 \)-space \( X \) consider the conditions:

1. the space \( X \) is strongly \( \kappa \)-normal;
2. the Wallman \( \kappa \)-bounded extension \( \mathcal{W}_{\kappa} X \) of \( X \) is Urysohn;
3. \( X \) is homeomorphic to a subspace of a Urysohn \( \kappa \)-bounded space;
4. \( X \) is strongly \( \kappa \)-regular.

Then \( 1 \iff 2 \Rightarrow 3 \Rightarrow 4 \). If each closed subspace of density \( \leq \kappa \) in \( X \) is countably paracompact in \( X \) and Lindelöf, then \( 4 \Rightarrow 1 \) and hence the conditions \( 1 \)–\( 4 \) are equivalent.

**Proof.** The equivalence \( 1 \iff 2 \) was proved in Proposition 3.3(2) and \( 2 \Rightarrow 3 \) follows immediately from Proposition 3.2 and the fact that the canonical map \( j_X : X \to \mathcal{W}_k X \) is a topological embedding. The implication \( 3 \Rightarrow 4 \) follows from Proposition 2.8 If each closed subspace of density \( \leq \kappa \) in \( X \) is countably paracompact in \( X \) and Lindelöf, then \( 4 \Rightarrow 1 \) by Propositions 2.2 and 2.3 \( \Box \)
**Theorem 3.6.** For an infinite cardinal $\kappa$ and a $T_1$-space $X$ consider the conditions:

1. the space $X$ is totally $\mathfrak{P}$-normal;
2. the Wallman $\kappa$-bounded extension $W_\kappa X$ of $X$ is regular;
3. $X$ is homeomorphic to a subspace of a regular $\kappa$-bounded space;
4. $X$ is regular.

Then (1) $\iff$ (2) $\implies$ (3) $\implies$ (4). If each closed subspace of density $\leq \kappa$ in $X$ is paracompact in $X$, then (4) $\implies$ (1) and hence the conditions (1)–(4) are equivalent.

**Proof.** The equivalence (1) $\iff$ (2) was proved in Proposition 3.3(3), the implication (2) $\implies$ (3) follows immediately from Proposition 3.2 and the fact that the canonical map $j_X : X \to W_\kappa X$ is a topological embedding, and (3) $\implies$ (4) is trivial. If each closed subspace of density $\leq \kappa$ in $X$ is paracompact in $X$, then (4) $\implies$ (1) by Propositions 2.2 and 2.4. $\square$

**Problem 3.7.** Does each $\mathfrak{P}$-Tychonoff space embed into a Hausdorff $\kappa$-bounded space?

## 4. Some examples

A topological space $X$ is functionally Hausdorff if for any distinct points $x, y \in X$ there exists a continuous function $f : X \to \mathbb{R}$ such that $f(x) \neq f(y)$.

First, we present an example of a first-countable regular space $M$ which is $\mathfrak{P}$-normal but is neither functionally Hausdorff nor strongly $\mathfrak{P}$-normal. The space $M$ is a suitable modification of the famous example of Mysior [11].

Let $Q_1 = \{y \in \mathbb{Q} : 0 < y < 1\}$ be the set of rational numbers in the interval $(0, 1)$ and

$$M = \{-\infty, +\infty\} \cup \mathbb{R} \cup (\mathbb{R} \times Q_1)$$

where $-\infty, +\infty \notin \mathbb{R} \cup (\mathbb{R} \times Q_1)$ are two distinct points. The topology on the space $M$ is generated by the subbase

$$\{\{z\}, M \setminus \{z\} : z \in \mathbb{R} \times Q_1\} \cup \{V_x : x \in \mathbb{R}\} \cup \{U_n : n \in \mathbb{Z}\} \cup \{W_n, n \in \mathbb{Z}\}$$

where

$$V_x = \{x\} \cup \{(z, y) : z \in \mathbb{R} \times Q_1 : z \in \{x, x + y\}\}$$

for $x \in \mathbb{R},$

$$U_n = \{-\infty\} \cup \{x \in \mathbb{R} : x < n\} \cup \{(x, y) : x \in \mathbb{R} \times Q_1 : x < n + 1\}$$

for $n \in \mathbb{Z},$

$$W_n = \{+\infty\} \cup \{x \in \mathbb{R} : x > n\} \cup \{(x, y) : x \in \mathbb{R} \times Q_1 : x > n\}$$

for $n \in \mathbb{Z}.$

**Example 4.1.** The space $M$ has the following properties:

1) $M$ is regular, first-countable and $\mathfrak{P}$-normal;
2) $M$ is neither functionally Hausdorff nor strongly $\mathfrak{P}$-normal.

**Proof.** The definition of the topology of $M$ implies that this space is regular, first-countable and the closure $\overline{C}$ of any countable subset $C \subset M$ is contained in the countable set

$$\{-\infty, +\infty\} \cup C \cup \{y, y - z : (y, z) \in C\}.$$

By Proposition 2.1 the space $M$ is $\mathfrak{P}$-normal.

By analogy with [11] (see also [4] and [8, 1.5.9]), it can be shown that $f(-\infty) = f(+\infty)$ for any continuous real-valued function $f$ which means that the space $M$ is not functionally Hausdorff.

Observe that the unit interval $I = [0, 1]$ is a closed discrete subspace of the space $M$. Besides the discrete topology inherited from $M$, the interval $I$ carries the standard Euclidean topology, inherited from the real line. The interval $I$ endowed with the Euclidean topology.
will be denoted by $\mathbb{I}_E$. To show that the space $M$ is not strongly $\pi$-normal we shall need the following fact.

**Claim 4.2.** For any dense subset $A$ in $\mathbb{I}_E$ and any open neighborhood $U$ of $A$ in $M$ the intersection $U \cap \mathbb{I}$ is a comeager subset of $\mathbb{I}_E$.

**Proof.** To derive a contradiction, assume that the set $B = \mathbb{I} \setminus U$ is not meager in $\mathbb{I}_E$ and hence $B$ is of the second Baire category in $\mathbb{I}_E$. Since $B \cap U = \emptyset$, for every $b \in B$ there exists a finite subset $F_b$ of $\mathbb{Q}_1$ and a basic open neighborhood $V_{F_b} = \{b\} \cup \{(z, y) \in \mathbb{R} \times (\mathbb{Q}_1 \setminus F_b) : z \in \{b, b + y\}\}$ of $b$ such that $V_{F_b} \cap U = \emptyset$. For each finite subset $F \subset \mathbb{Q}_1$ put $B_F = \{b \in B : F_b = F\}$. Since the set of all finite subsets of $\mathbb{Q}_1$ is countable and $B$ is of the second category, there exists a finite subset $F \subset \mathbb{Q}_1$ such that the set $B_F$ is not meager in $\mathbb{I}_E$. Hence there exists an interval $(c, d) \subset \mathbb{I}_E$ such that $B_F$ is dense in $(c, d)$. Recall that $A$ is dense in $\mathbb{I}_E$. At this point it is easy to check that $\emptyset \neq U \cap \bigcup_{b \in B} V_{F_b} \subset U \cap \bigcup_{b \in B} V_{F_b} = \emptyset$, which is a desired contradiction. \(\square\)

Recall that the subspace $\mathbb{I} \subset M$ is discrete. Let $A := \mathbb{Q} \cap \mathbb{I}$ and $B := (\mathbb{Q} + \sqrt{2}) \cap \mathbb{I}$ be two closed countable disjoint subsets of $M$. Assuming that the space $M$ is strongly $\pi$-normal, we can find open sets $U_A$ and $U_B$ in $M$ such that $A \subset U_A$, $B \subset U_B$ and $\overline{U_A} \cap \overline{U_B} = \emptyset$. By Claim 4.2 the sets $\overline{U_A} \cap \mathbb{I}$ and $\overline{U_B} \cap \mathbb{I}$ are comeager in $\mathbb{I}_E$ and hence have nonempty intersection and this is a desired contradiction showing that the space $M$ is not strongly $\pi$-normal. \(\square\)

**Remark 4.3.** The space $M \setminus \{(-\infty, +\infty)\}$ is Tychonoff, zero-dimensional, locally compact, locally countable, $\pi$-normal but not strongly $\pi$-normal.

Now we present an example of a regular, $\omega$-bounded, totally $\pi$-normal space which is not functionally Hausdorff. Let $[0, \alpha]$ be the ordinal $\alpha + 1$ endowed with the order topology. Let $T = [0, \omega_1] \times [0, \omega_2] \setminus \{(\omega_1, \omega_2)\}$ be the subspace of the Tychonoff product $[0, \omega_1] \times [0, \omega_2]$. Observe that $T$ is $\omega$-bounded. Let $\mathbb{Z}$ be the discrete space of integers and $-\infty, +\infty$ be distinct points which do not belong to $T \times \mathbb{Z}$. By $Y$ we denote the set $(T \times \mathbb{Z}) \cup \{-\infty, +\infty\}$ endowed with the topology $\tau$ which satisfies the following conditions:

- the Tychonoff product $T \times \mathbb{Z}$ is an open subspace in $Y$;
- if $-\infty \in U \in \tau$, then there exists $n \in \omega$ such that $\{(t, k) \in T \times \mathbb{Z} : k < -n\} \subset U$;
- if $+\infty \in U \in \tau$, then there exists $n \in \omega$ such that $\{(t, k) \in T \times \mathbb{Z} : k > n\} \subset U$.

One can check that the space $Y$ is regular and $\omega$-bounded.

On the space $Y$ consider the smallest equivalence relation $\sim$ such that $(x, \omega_2, 2n) \sim (x, \omega_2, 2n + 1)$ and $(\omega_1, y, 2n) \sim (\omega_1, y, 2n - 1)$ for any $n \in \mathbb{Z}, x \in \omega_1$ and $y \in \omega_2$. Let $X$ be the quotient space $Y/\sim$ of $Y$ by the equivalence relation $\sim$.

**Example 4.4.** The space $X$ is regular, $\omega$-bounded and totally $\pi$-normal, but not functionally Hausdorff and hence is not $\pi$-Tychonoff.

**Proof.** Since the $\omega$-boundedness is preserved by continuous images, the space $X$ is $\omega$-bounded.

Using the classical argument due to Tychonoff (see [12, p.109]), it can be shown that the space $X$ is regular, but for each real-valued continuous function $f$ on $X$, $f(-\infty) = f(+\infty)$. Hence $X$ is not functionally Hausdorff. By Proposition 2.9 $X$ is totally $\pi$-normal. \(\square\)

**Remark 4.5.** For each infinite cardinal $\kappa$ the punctured Tychonoff plank $[0, \kappa] \times [0, \kappa^+] \setminus \{((\kappa, \kappa^+))\}$ is an example of strongly $\pi$-normal space which is not totally $\pi$-normal.
A topological space $X$ is called

- $H$-compact if for any open cover $\mathcal{U}$ of $X$ there exists a finite subfamily $\mathcal{V} \subset \mathcal{U}$ such that $X = \bigcup_{V \in \mathcal{V}} V$;
- $H$-closed if $X$ is Hausdorff and $H$-compact.

It is clear that each compact space is $H$-compact. By [8, 3.12.5], a Hausdorff topological space $X$ is $H$-closed if and only if it is closed in each Hausdorff space containing $X$ as a subspace.

For each infinite cardinal $\kappa$ we shall construct a $\pi$-normal, $\kappa$-bounded, $H$-compact Hausdorff space which is not Urysohn. Given an infinite cardinal $\kappa$, denote by $C$ the set of all isolated points of the cardinal $\kappa^+ = [0, \kappa^+]$ endowed with the order topology. Write $C$ as the union $C = A \cup B$ of two disjoint unbounded subsets of $\kappa^+$. Choose any points $a, b \notin \kappa^+$ and consider the space $X_\kappa = \kappa^+ \cup \{a, b\}$ endowed with the topology $\tau$ satisfying the following conditions:

- $\kappa^+$ with the order topology is an open subspace of $X_\kappa$;
- if $a \not\in U \subset \tau$, then there exists $\alpha \in \kappa^+$ such that $\{\beta \in A : \beta > \alpha\} \subset U$;
- if $b \not\in U \subset \tau$, then there exists $\alpha \in \kappa^+$ such that $\{\beta \in B : \beta > \alpha\} \subset U$.

**Example 4.6.** For each cardinal $\kappa$ the space $X_\kappa$ is $\pi$-normal, $\kappa$-bounded, $H$-compact and Hausdorff, but not Urysohn.

**Proof.** It is straightforward to check that $X_\kappa$ is $\pi$-normal, $\kappa$-bounded, and Hausdorff. The $H$-compactness of $X_\kappa$ follows from the observation that for any open neighborhood $U \subset X_\kappa$ of the doubleton $\{a, b\}$ the closure $\overline{U}$ contains the interval $[\alpha, \kappa^+]$ for some ordinal $\alpha \in \kappa^+$.

To see that $X_\kappa$ is not Urysohn observe that for any open neighborhoods $U_a$ and $U_b$ of $a$ and $b$, respectively, the sets $\overline{U_a} \cap \kappa^+$ and $\overline{U_b} \cap \kappa^+$ are closed and unbounded in $\kappa^+$. Hence $\overline{U_a} \cap \overline{U_b} \neq \emptyset$. □

Next, we are going to present a (consistent) example of a separable sequentially compact scattered space $X$ which is regular but not $\pi$-Tychonoff and hence cannot be embedded into an $\omega$-bounded Hausdorff space.

This example is a combination of van Douwen’s example [7, 7.1] of a locally compact sequentially compact space, based on a regular tower, and the famous example of Tychonoff corkscrew due to Tychonoff, see [12, p.10]. First we recall the necessary definitions related to (regular) towers.

By $[\omega]^{\omega}$ we denote the family of all infinite subsets of $\omega$. For two subsets $A, B \in [\omega]^{\omega}$ we write $A \subseteq^* B$ if $A \setminus B$ is finite. Also we write $A \supseteq^* B$ if $A \subseteq^* B$ but $B \not\subseteq^* A$. A family $\mathcal{T} \subseteq [\omega]^{\omega}$ is called a regular tower if for some regular cardinal $\kappa$ the family $\mathcal{T}$ can be written as $\mathcal{T} = \{T_{\alpha}\}_{\alpha \in \kappa}$ so that

1. $T_\beta \supseteq^* T_\alpha$ for any ordinals $\alpha < \beta$ in $\kappa$, and
2. for any $I \in [\omega]^{\omega}$ there exists $\alpha \in \kappa$ such that $I \not\subseteq^* T_\alpha$.

The first condition implies that the sets $T_\alpha$, $\alpha \in \kappa$, are distinct and hence $\kappa = |\mathcal{T}|$. Also this condition implies that the relation $\supseteq^*$ is a well-order on $\mathcal{T}$.

Consider the uncountable cardinals

$$t = \min\{|\mathcal{T}| : \mathcal{T} \subseteq [\omega]^{\omega} \text{ is a regular tower}\}$$

$$\hat{t} = \sup\{|\mathcal{T}| : \mathcal{T} \subseteq [\omega]^{\omega} \text{ is a regular tower}\}$$

and observe that $t \leq \hat{t} \leq \mathfrak{c}$. It is well-known that Martin’s Axiom implies the equality $t = \hat{t} = \mathfrak{c}$.

**Proposition 4.7.** The strict inequality $t < \hat{t}$ is consistent. Also $t = \hat{t} = \omega_1 < \omega_2 = \mathfrak{c}$ is consistent.
Proof. The consistency of $t = \hat{t} = \omega_1 < \omega_2 = \mathfrak{c}$ was proved in [3, Theorem 4.1].

To prove the consistency of $t < \hat{t}$, assume that $\text{MA} + \neg \text{CH}$ holds in the ground model $V$ and let $V'$ be the forcing extension of $V$ obtained by adding $\omega_1$ many Cohen reals. Then $t = b = \omega_1$ in $V'$, which yields a regular tower of length $\omega_1$ in $V'$. On the other hand, any maximal tower from $V$ of length $(2^\omega)^V > \omega_1$ (which exists, because in $V$, $t = 2^\omega > \omega_1$) remains regular in $V'$ since it is well-known (and easy to check) that Cohen forcing cannot add infinite pseudointersections to maximal towers. Hence $t < \hat{t}$ in $V'$.

A topological space $X$ is called $\omega$-regular if for any open set $U \subset X$ and point $x \in U$ there exists a sequence $(U_n)_{n \in \omega}$ of open neighborhoods of $x$ such that $\bigcup_{n \in \omega} U_n \subset U$ and $\overline{U}_n \subset U_{n+1}$ for all $n \in \omega$. It is easy to see that each completely regular space is $\omega$-regular.

Example 4.8. If $t < \hat{t}$, then there exists a topological space $X$ such that

1. $X$ is separable, scattered, and sequentially compact;
2. $X$ is regular but not $\omega$-regular and hence not completely regular and not $\omega$-Tychonoff;
3. $X$ does not embed into an $\omega$-bounded Hausdorff space.

Proof. Since $t < \hat{t}$, there are two regular towers $T_1 = \{A_\alpha\}_{\alpha \in \kappa}$ and $T_2 = \{B_\beta\}_{\beta \in \lambda}$ such that $\kappa < \lambda$. For every $\alpha \in \kappa$ and $\beta \in \lambda$ consider the sets $C_\alpha = \omega \setminus A_\alpha$ and $D_\beta = \omega \setminus B_\beta$. Let $T_1 = \{C_\alpha\}_{\alpha \in \kappa}$ and $T_2 = \{D_\beta\}_{\beta \in \lambda}$. Obviously, $\mathbb{C}$ is a well order on $T_1$ and $T_2$. Also, observe that the families $T_1$ and $T_2$ satisfy the following condition: for any infinite subset $I$ of $\omega$ there exist $C_\alpha \in T_1$ and $D_\beta \in T_2$ such that the sets $I \cap C_\alpha$ and $I \cap D_\beta$ are infinite.

For every $i \in \{1, 2\}$, consider the space $Y_i = T_i \cup \omega$ which is topologized as follows. Points of $\omega$ are isolated and a basic neighborhood of $T \in T_i$ has the form

$$B(S, T, F) = \{P \in T_i \mid S \subseteq^* P \subseteq^* T\} \cup (\{T \setminus S\} \setminus F),$$

where $S \subseteq T_i \cup \emptyset$ satisfies $S \subseteq^* T$ and $F$ is a finite subset of $\omega$.

Repeating arguments of Example 7.1 [7] one can check that the space $Y_i$ is sequentially compact, separable, scattered and locally compact for every $i \in \{1, 2\}$.

For every $i \in \{1, 2\}$ choose any point $\infty_i \notin Y_i$ and let $X_i = \{\infty_i\} \cup Y_i$ be the one-point compactification of the locally compact space $Y_i$. It is easy to see that the compact space $X_i$ is scattered, $i \in \{1, 2\}$.

Consider the space $\Pi = (X_1 \times X_2) \setminus \{(\infty_1, \infty_2)\}$. It is easy to check that the space $\Pi$ is separable, scattered and sequentially compact.

Choose any point $\infty \notin \Pi \times \omega$ and consider the space $\Sigma = \{\infty\} \cup (\Pi \times \omega)$ endowed with the topology consisting of the sets $U \subset \Sigma$ satisfying two conditions:

- for any $n \in \omega$ the set $\{z \in \Pi : (z, n) \in U\}$ is open in $\Pi$;
- if $\infty \in U$, then there exists $n \in \omega$ such that $\bigcup_{m \geq n} \Pi \times \{m\} \subset U$.

Taking into account that the space $\Pi$ is separable, scattered and sequentially compact, we conclude that so is the space $\Sigma$. On the space $\Sigma$ consider the smallest equivalence relation $\sim$ such that $(x_1, \infty_2, 2n) \sim (x_1, \infty_2, 2n + 1)$ and $(\infty_1, x_2, 2n + 1) \sim (\infty_1, x_2, 2n + 2)$ for any $n \in \omega$ and $x_i \in X_i \setminus \{\infty_i\}, i \in \{1, 2\}$. Let $X$ be the quotient space $\Sigma/\sim$ of $\Sigma$ by the equivalence relation $\sim$. Observe that the character of the space $X_1$ at $\infty_1$ is equal to the regular cardinal $|T_1| = \kappa$ and is strictly smaller than the pseudocharacter of the space $X_2$ at $\infty_2$, which is equal to the regular cardinal $|T_2| = \lambda$. Using this observation and repeating the classical argument due to Tychonoff (see [12, p.109]), it can be shown that the space $X$ is regular but not $\omega$-regular (at the point $\infty$), and hence not Tychonoff and not $\omega$-Tychonoff (since for separable $T_1$-spaces the Tychonoff property is equivalent to the $\omega$-Tychonoff property).
By Proposition 2.10, the separable space $X$ does not embed into an $\omega$-bounded Hausdorff space.

**Question 4.9.** Does there exist in ZFC an example of a separable regular sequentially compact space which is not Tychonoff?

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