Non-linear Laplace equation, de Sitter vacua and information
geometry

Farhang Loran

Department of Physics, Isfahan University of Technology (IUT)
Isfahan, Iran

Abstract

Three exact solutions say \( \phi_0 \) of massless scalar theories on Euclidean space, i.e. \( D = 6 \phi^3 \), \( D = 4 \phi^4 \) and \( D = 3 \phi^6 \) models are obtained which share similar properties. The information geometry of their moduli spaces coincide with the Euclidean AdS\(_7\), AdS\(_5\) and AdS\(_4\) respectively on which \( \phi_0 \) can be described as a stable tachyon. In \( D = 4 \) we recognize that the SU(2) instanton density is proportional to \( \phi_0^4 \). The original action \( S[\phi] \) written in terms of new scalars \( \tilde{\phi} = \phi - \phi_0 \) is shown to be equivalent to an interacting scalar theory on \( D\)-dimensional de Sitter background.

AdS/CFT correspondence [1], as a bulk/boundary correspondence, is a quantitative realization of the holographic principle. In [2] Witten showed that the metric on the boundary of the AdS space is well-defined only up to a conformal transformation and the correlation functions of the CFT on the boundary are given by the dependence of the supergravity action on the asymptotic behavior at infinity, see also [3]. Using the metric

\[
ds^2 = \frac{1}{x_0^2} (dx_0^2 + dx_1^2 + \cdots + dx_d^2)
\]

for the Euclidean AdS\(_{d+1}\) Witten showed that the generating function for CFT correlators,

\[
I[\phi] = \ln \langle \exp \int \phi \mathcal{O} \rangle
\]

is

\[
I[\phi] = \int d^d y d^d z \frac{\phi_0(\vec{y}) \phi_0(\vec{z})}{|\vec{y} - \vec{z}|^{2(d+\lambda_+)}},
\]

where \( \phi_0 \) here, is some scalar field on the boundary, determined by the asymptotic behavior of scalar fields \( \Phi \) in the bulk: \( \Phi \sim x_0^{-\lambda_+} \phi_0 \) as \( x_0 \to 0 \). Here, \( \lambda_+ \) is the larger root of the equation \( \lambda(\lambda + d) = m^2 \). These results led us to a classical interpretation for EAdS/CFT correspondence as the relation between the solutions of the Klein-Gordon equation \( \phi(x_0, \vec{x}) \) (bulk fields) and the Cauchy data \( \phi(0, \vec{x}) \) (boundary fields) [5]. In fact under the conformal transformation

\[
d_{\mu\nu} \to g_{\mu\nu} = x_0^{-2}\delta_{\mu\nu}
\]

that gives the EAdS\(_{d+1}\) metric mentioned above in terms of \( \delta_{\mu\nu} \), the metric of the \( D\)-dimensional flat Euclidean space \( R^{d+1} \), massless fields \( \phi \) on \( R^{d+1} \) transform to massive scalars \( \Phi = x_0^{-\lambda_+} \phi \) with mass \( -\frac{d^2-1}{4} \). From the classical equation of motion \( \delta S[\phi] = 0 \) one can determine \( \phi(x_0, \vec{x}) \) in terms of the Cauchy data \( \phi_0(\vec{x}) = \phi(0, \vec{x}) \). Inserting the solution in \( S[\phi] \) one obtains \( I[\phi] \) given in Eq.(1). By the same method, though only for scalars with
specific mass \( m^2 = \frac{d^2 - 1}{4} \), the correlation functions of the boundary operators in dS_{d+1}/CFT\(d \) correspondence that Strominger [4] explicitly calculated for \( d = 2 \) and proposed for general \( d \) can be obtained [5]. By generalizing the method to free spinors the boundary term to be added to the bulk Dirac action necessary for AdS/CFT and dS/CFT correspondence [6] are obtained for general free massive spinors in (A)dS space [7].

What can one learn about AdS/CFT correspondence if one uses this method for interacting scalar theories instead of free scalars? The scalar field theories that can be considered are massless \( D = 6 \) \( \phi^3 \), \( D = 4 \) \( \phi^4 \) and \( D = 3 \) \( \phi^6 \) models [7, 8] given by the action,

\[
S[\phi] = \int d^{D}x \left( \frac{1}{2} \delta^{\mu \nu} \partial_{\mu} \phi \partial_{\nu} \phi - \frac{g}{(2D-2)} \phi^{2D-2} \right), \quad D = 3, 4, 6, \tag{2}
\]

which is classically invariant under rescaling \( x \rightarrow \lambda x, \phi \rightarrow \lambda^{\frac{2-D}{2}} \phi \). Using the conformal transformation from \( R^{D} \) (D-dimensional Minkowski space-time) to EAdS\(D \) (dS\(D \)) one obtains the same interacting theory, i.e. \( \Phi^{2D-2} \)-model but for scalars with mass \( m^2 = \pm \frac{d^2 - 1}{4} \) on \( D = d + 1 \)-dimensional (A)dS space. The corresponding generating function for boundary CFT correlators can be obtained by solving (by perturbation) \( \phi(x^0, \vec{x}) \) in terms the Cauchy data and inserting the solution in \( S[\phi] \) [8]. As is discussed in [8] finding the solution of the equation of motion as power series in \( g \) the coupling constant, is necessary for applicability of the above mentioned method. As we will see, there are exact solutions of the corresponding non-linear Laplace equation which although are not useful for that purpose but open a new window to the AdS/CFT correspondence for critical scalar field theories (2).

**The Information Geometry**

As is shown in [8], critical scalar field theories (2) are particular in the sense that the \( SO(D) \)-invariant (in flat Euclidean space) nonlinear Laplace equation, \( \nabla^2 \phi + g\phi^n = 0 \), \( (g > 0) \), in which \( \nabla^2 = \delta^{\mu \nu} \partial_{\mu} \partial_{\nu} \), has solutions like

\[
\phi_0(s) = \frac{\alpha}{(\beta^2 + (s-a)^2)^{\gamma}}, \quad (s-a)^2 = \delta_{\mu \nu}(x-a)^\mu (x-a)^\nu. \tag{3}
\]

for some constants \( a^\mu, \alpha, \beta \) and \( \gamma \) only if \( n = \frac{2D}{D-2} - 1 \) and \( D = 3, 4, 6 \). In these cases \( \gamma = \frac{D-2}{2} \), and

\[
\beta^2 = g \frac{1}{2(D-2)\alpha^{\frac{1}{n-1}}}, \tag{4}
\]

which for example for \( D = 4 \) gives,

\[
\phi_0 \sim \frac{\beta}{\beta^2 + (x-a)^2} \tag{5}
\]

We note that \( \phi_0(x; \beta, a_\mu) \) is invariant under rescaling, see appendix A and is a stable classical solution of an unstable model \( (V(\phi) \sim -\phi^4) \). An interesting observation is that in \( D = 4 \), the SU(2) instanton density is [10]

\[
\text{tr} F^2 = 96 \frac{\beta^4}{(\beta^2 + (x-a)^2)^4} \sim \phi_0^4. \tag{6}
\]
In [10] $\beta$ in Eq.(6) is considered as the size of the instanton, suggesting to call $\beta$ in Eq.(3) the size of $\phi_0$. Considering $\theta^I = \beta, a^a, I = 0, \cdots, D$ in Eq.(3) as moduli, the Hitchin information metric of the moduli space, defined as follows [9]:
\[
\mathcal{G}_{IJ} = \frac{1}{N(D)} \int d^D x \mathcal{L}_0 \partial_I (\log \mathcal{L}_0) \partial_J (\log \mathcal{L}_0),
\]
Equation (7) can be shown to describe Euclidean AdS$_{D+1}$ space:
\[
\mathcal{G}_{IJ} d\theta^I d\theta^J = \frac{1}{\beta^2} \left(d\beta^2 + da^2\right).
\]
Equation (8)

$N(D)$ is a normalization constant,
\[
N(D) = \frac{D^3}{D+1} \int d^D x \mathcal{L}_0,
\]
Equation (9)
and
\[
\mathcal{L}_0 = -\frac{1}{2} \phi_0 \nabla^2 \phi_0 - \frac{g}{(D-2)^2} \phi_0^{\frac{D}{D-2}} = \frac{g}{D} \phi_0^{\frac{D}{D-2}},
\]
Equation (10)
is the Lagrangian density calculated at $\phi = \phi_0$. See appendix B for details. Similar results are obtained for the information geometry of instantons on $R^4$, for $N = \frac{1}{2}$ U(N) theories and for instantons on noncommutative space. See for example [10, 11].

$\phi_0$ as a function of $\theta^I$'s is a free stable-tachyon field on EAdS$_{D+1}$ as it satisfies the Klein-Gordon equation given in terms of the metric (8),
\[
\left(\beta^2 \partial_\beta^2 + (1 - D) \beta \partial_\beta + \beta^2 \partial_a^2 + \frac{D^2 - 4}{4}\right) \phi_0 = 0.
\]
Equation (11)
The tachyon is stable as far as $-\frac{D^2}{4} < m^2 < 0$ [2].

$\phi_0$ as a function of $g$ the coupling constant (or $\beta^2$), can not be analytically continued to $g = 0$. For $g = 0$, $\phi_0$ is the Green function of the Laplacian operator i.e. $\nabla^2 \phi(x, a) = \delta^D(x - a)$ and does not satisfy the Klein Gordon equation $\nabla^2 \phi = 0$ for free scalar theory. This shows that $\phi_0$ can not be obtained by perturbation around $g = 0$. In [10], the same asymptotic behavior for the instanton density is observed and tr$F^2$ is interpreted as the boundary to bulk propagator of a massless scalar field on AdS$_5$.

**$\phi_0$ as classical de Sitter vacua**

Rewrite the action (2) in terms of new fields $\tilde{\phi} = \phi - \phi_0$, one obtains
\[
S[\tilde{\phi}] = S[\phi_0] + S_{\text{free}}[\tilde{\phi}] + S_{\text{int}}[\tilde{\phi}],
\]
Equation (12)
where $S[\phi_0] = \int d^D x \mathcal{L}_0$ (see Eq.(10)),
\[
S[\phi_0] = \frac{D+2}{2D-1} \frac{\pi^{D+1} g^{2-\frac{D}{2}}}{\Gamma(D+1)} \quad \text{with}
\begin{align*}
&\text{for } D = 6, \\
&\quad \frac{192\pi^3}{3g^2} \\
&\text{for } D = 4, \\
&\quad \frac{8\pi^2}{3g} \\
&\text{for } D = 3, \\
&\quad \frac{\pi^2}{4}\sqrt{\frac{2}{g}}.
\end{align*}
\]
Equation (13)
and
\[ S_{\text{free}}[\phi] = \int d^D x \left( \frac{1}{2} \delta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{1}{2} M^2(x) \phi^2 \right) \]  
(14)
in which,
\[ M^2(x) = -g \left( \frac{D+2}{D-2} \right) \phi^2 = -\left( 2 + D \right) \frac{\beta^2}{\left( \beta^2 + (x-a)^2 \right)^2}. \]  
(15)

Defining \( \bar{\phi} = \Omega^{\frac{D+2}{D-2}} \phi \), one can show that \( S_{\text{free}}[\phi] \) given in Eq.(14) is the action of the scalar field \( \phi \) on some conformally flat background with metric \( g_{\mu\nu} = \Omega \delta_{\mu\nu} \):
\[ S_{\text{free}}[\phi] = \int d^D x \sqrt{|g_{\mu\nu}|} \left( \frac{1}{2} \left[ g^\mu\nu \partial_\mu \phi \partial_\nu \phi + \frac{1}{2} (\xi R + m^2) \phi^2 \right] \right). \]  
(16)

Here, \( m^2 \Omega = M^2(x) \), where \( m^2 \) is the mass of \( \phi \) (undetermined) and \( M^2(x) \) is given in Eq.(15). \( R \) is the curvature scalar and \( \xi = \frac{D-2}{4(D-1)} \) is the conformal coupling constant. This result is surprising as one can show that the Ricci tensor \( R_{\mu\nu} = \Lambda_D g_{\mu\nu} \), where
\[ \Lambda_D = -m^2 \frac{4(D-1)}{D(D+2)}. \]  
(17)

Since \( -m^2 > 0 \) as far as \( \Omega > 0 \), one verifies that \( \Lambda_D > 0 \) which means that \( \phi \) lives on \( D \)-dimensional de Sitter space which radius is proportional to \( -m^{-2} \). See Appendix C for details.

The interacting part of the action, \( S_{\text{int}}[\phi] = \int d^D x \sqrt{|g_{\mu\nu}|} L_{\text{int}} \) is well-defined in terms of \( \phi \) on the corresponding dS\(_D\):
\[ L_{\text{int}} = \begin{cases} 
-\frac{g}{3} \phi^3, & D = 6, \\
-g \sqrt{-m^2} \phi^3 - \frac{4}{3} \phi^4, & D = 4, \\
-\frac{10}{3} g \left( \frac{-m^2}{3g} \right)^\frac{2}{3} \phi^3 - \frac{5}{2} g \left( \frac{-m^2}{3g} \right)^\frac{1}{2} \phi^4 - g \left( \frac{-m^2}{3g} \right)^\frac{1}{2} \phi^5 - \frac{g}{6} \phi^6, & D = 4.
\end{cases} \]  
(18)

It is interesting to note that in \( D = 4 \), by a shift of the scalar field \( \phi \to \phi - \sqrt{-m^2/3g} \) the action (12) can be written in the dS\(_4\) as follows:
\[ S[\phi] = \int d^D x \sqrt{|g_{\mu\nu}|} \left( \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{1}{2} (\xi R) \phi^2 - \frac{g}{4} \phi^4 \right) + \int d^D x \sqrt{|g_{\mu\nu}|} \left( \frac{-m^4}{36g} \right). \]  
(19)

\( \phi_0 \) on Minkowski space-time

After a Wick rotation \( x^0 \to ix^0 \), \( \phi_0 \) given in (3) can be shown to satisfy the corresponding non-linear wave equation on Minkowski space-time. These solutions have a time-like singularity. The singularity is a hypersurface given by the equation \( -(x^0-a)^2 + (x^1-a^1) + \cdots + (x^D-a^D)^2 = -\beta^2 \), which can be considered as a \( D-1 \) dimensional anti de Sitter space.\(^1\) The idea, here, is to some extend similar to the holographic reduction of Minkowski space-time [12] where the Minkowski space-time is sliced in terms of Euclidian AdS and Lorentzian dS slices which correspond to

\(^1\)The singularity becomes space-like i.e. a dS\(_{D-1}\) hypersurface if the coupling \( g \) is negative.
the time-like and space-like regions respectively. Considering only the free part of the scalar action $S_{\text{free}}(\tilde{\phi})$ given in Eq. (14), one can verify that $D$-dimensional free scalar theory given by $S_{\text{free}}(\tilde{\phi})$, induces a free (but unstable) scalar theory on the AdS hypersurface, the singularity. To show this, first note that the equation of motion for the scalars $\tilde{\phi}$ is

$$\left(\Box + \frac{D(D+2)\beta^2}{(\beta^2 + x^2)^2}\right) \tilde{\phi} = 0, \quad x^2 = -x^0^2 + \vec{x}^2,$$

(20)

where without losing the generality we have assumed $a^\mu = 0$. Defining new coordinates

$$x^0 = (R + \beta) \cosh \rho,$$
$$x^i = (R + \beta) \sinh \rho \, z_i, \quad \sum_{i=1}^{D-1} z_i^2 = 1,$$

(21)

which locates the singularity at $R = 0$, Eq. (20) can be written as follows,

$$\left(\Box_{R} + \frac{1}{R^2} \Box_{\rho,z_i} + \frac{D(D+2)\beta^2}{R^2(R+2\beta)^2}\right) \tilde{\phi}(R; \rho, z_i) = 0.
$$

(22)

The ansatz for scalar fields living on the singularity is $\phi^*(\rho, z_i) = \phi(0; \rho, z_i)$, which from Eq. (22) satisfy the following Klein-Gordon equation:

$$\left(\Box_{\rho,z_i} - m^*^2\right) \phi^* = 0.
$$

(23)

Since $-m^*^2 = -\frac{D(D+2)}{4} < -\frac{(D-1)^2}{4}$, the scalar theory is not stable. The interacting theory in terms of $\tilde{\phi} = \phi - \phi_0$ is still well-defined and the corresponding conformally flat background is a $D$-dimensional de Sitter space which horizon is located at the singularity. It is interesting to note that $m^*^2 = m^2\ell^2$, where $m$ is the mass of scalars $\phi$ on $dS_D$ with radius $\ell$, see appendix C.

**Acknowledgement**

The author gratefully thanks S. J. Rey and F. Shahbazi for useful discussions. The financial support of Isfahan University of Technology is acknowledged.

**Appendix A**

In this appendix we show that $\phi_0(x)$ is scale-invariant. We assume for simplicity that $D = 4$. Under rescaling $x \to x' = \lambda x$ the scalar field changes as $\phi(x) \to \phi'(x') = \lambda^{-1} \phi(x)$. By a scale-invariant object we mean a field that satisfies the relation $\delta_\epsilon \phi = 0$ where $\delta_\epsilon \phi(x) = \phi'(x) - \phi(x)$ is the infinitesimal scale transformation given by $\lambda = 1 + \epsilon$ for some infinitesimal $\epsilon$. To this aim we first note that under a general rescaling $\phi(0) \to \phi'(0) = \lambda^{-1} \phi(0)$, thus, in fact, $\phi(x; \phi(0)) \to \phi'(x; \phi(0)) = \lambda^{-1} \phi(\lambda^{-1} x; \lambda \phi(0))$. Defining $\beta^{-1} = \beta(0)$, one can show that $\delta_\epsilon \phi(x) = -\epsilon(1 + x'\partial_1)\phi(x)$, where $x' \in \{x^u, \beta\}$. The $SO(D)$ invariant solutions of equation $\delta_\epsilon \phi = 0$ satisfying the condition $\phi(0; \phi(0)) = \phi(0)$ are $\phi_k = \beta^{-1} \left(\frac{\beta}{\sqrt{\beta^2 + x^2}}\right)^{k+2}$. It is easy to see that the action (2) is invariant under the variation generated by $\delta_\epsilon$ and $\phi_0$, among the others, is the solution of classical equation of motion.
Appendix B

Here we give a detailed calculation of Hitchin information metric on the moduli space of $\phi_0$ (3):

$$
\phi_0 = \left( \frac{(D(D - 2)}{g} \right)^{\frac{D-2}{2}} \left( \frac{\beta}{\beta^2 + (x - a)^2} \right)^{\frac{D-2}{2}}.
$$

From Eq.(10) one verifies that

$$
L_0 = \frac{g}{D} \left( \frac{D(D - 2)}{g} \right)^{\frac{D}{2}} \left( \frac{\beta}{\beta^2 + (x - a)^2} \right)^{D}.
$$

Therefore

$$
\partial_\beta \log L_0 = D \left( \frac{1}{\beta} - \frac{2\beta}{\beta^2 + (x - a)^2} \right),
$$

$$
\partial_a \log L_0 = \frac{2D(x - a)_i}{\beta^2 + (x - a)^2}.
$$

Using these results and after some elementary calculations one can show that,

$$
G_{ij} = \frac{1}{N(D)} \int d^Dx L_0 \partial_{a_i} \log L_0 \partial_{a_j} \log L_0 = \frac{4K(D)}{N(D)\beta^2} \delta_{ij} \int d^Dy \frac{y^2}{(1 + y^2)^{D+2}},
$$

$$
G_{i\beta} = \frac{1}{N(D)} \int d^Dx L_0 \partial_{a_i} \log L_0 \partial_{\beta} \log L_0 = 0,
$$

$$
G_{\beta\beta} = \frac{1}{N(D)} \int d^Dx L_0 (\partial_\beta \log L_0)^2 = \frac{DK(D)}{N(D)\beta^2} \int d^Dy \frac{1}{(1 + y^2)^{D}} \left( 1 - \frac{2}{(1 + y^2)^2} \right)^2.
$$

where $K(D) = g^{1-D/2} D^{D/2 + 1} (D - 2)^{D/2}$. By performing the integrations and using Eq.(9), one obtains,

$$
G_{IJ} = \frac{1}{\beta^2} \delta_{IJ},
$$

in which $\delta_{IJ} = 1$ if $I = J$ and vanishes otherwise.

Appendix C

In this appendix we briefly review free scalar field theory in $D + 1$ dimensional (Euclidean) curved space-time [13] and say few words about the classical geometry of de Sitter space [14]. The action for the scalar field $\phi$ is

$$
S = \int d^Dx \sqrt{|g|} \left( g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + (m^2 + \xi R)\phi^2 \right),
$$

for which the equation of motion is

$$
(\Box - m^2 - \xi R) \phi = 0, \quad \Box \phi = |g|^{-1/2} \partial_\mu \left( |g|^{1/2} g^{\mu\nu} \partial_\nu \phi \right).
$$

(With $\hbar$ explicit, the mass $m$ should be replaced by $m/\hbar$.) The case with $m = 0$ and $\xi = \frac{D-2}{4(D-1)}$ is referred to as conformal coupling.

The curvature tensor $R^\mu_{\nu\rho\sigma}$ in term of Levi-Civita connection,

$$
\Gamma^\mu_{\nu\rho} = \frac{1}{2} \eta^{\mu\alpha} (\partial_\rho g_{\alpha\nu} + \partial_\nu g_{\alpha\rho} - \partial_\alpha g_{\nu\rho}),
$$

(31)
is given as follows,

\[ R^\mu_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\nu \Gamma^\mu_{\rho\sigma} + \Gamma^\mu_{\rho\alpha} \Gamma^\alpha_{\nu\sigma} - \Gamma^\mu_{\alpha\sigma} \Gamma^\alpha_{\nu\rho}. \]  

(32)

The Ricci tensor \( R_{\nu\sigma} = R^\mu_{\nu\mu\sigma} \) and the curvature scalar \( R = g^{\nu\sigma} R_{\nu\sigma} \).

The metric of a conformally flat space-time can be given as \( g_{\mu\nu} = \Omega \delta_{\mu\nu} \), where \( \Omega \) is some function of space-time coordinates. One can easily show that,

\[ R_{\mu\nu} = \frac{2 - D}{2} \partial_\mu \partial_\nu (\log \Omega) - \frac{1}{2} \delta_{\mu\nu} \nabla^2 (\log \Omega) \]

\[ + \frac{D - 2}{4} (\partial_\mu (\log \Omega) \partial_\nu (\log \Omega) - \delta_{\mu\nu} \delta^{\rho\sigma} \partial_\rho (\log \Omega) \partial_\sigma (\log \Omega)) \]

\[ \Omega R = (1 - D) \nabla^2 (\log \Omega) + \frac{(1 - D)(D - 2)}{4} \delta_{\mu\nu} \partial_\mu (\log \Omega) \partial_\nu (\log \Omega). \]  

(33)

By inserting \( \tilde{\phi} = \Omega^{\frac{D-2}{2}} \phi \) in the action \( S[\tilde{\phi}] = \int d^Dx \frac{1}{2} \delta_{\mu\nu} \partial_\mu \tilde{\phi} \partial_\nu \tilde{\phi} \), one obtains,

\[ S[\tilde{\phi}] = \int d^Dx \left( \frac{1}{2} \Omega^{\frac{D-2}{2}} \delta_{\mu\nu} \partial_\mu \tilde{\phi} \partial_\nu \tilde{\phi} - \frac{1}{2} \left( \frac{\Omega^{\frac{D-2}{2}} \nabla^2 \Omega^{\frac{D-2}{4}}}{\Omega^{\frac{D-2}{4}}} \right) \tilde{\phi}^2 \right) \]

\[ = \int d^Dx \sqrt{g} \left( \frac{1}{2} g^{\mu\nu} \partial_\mu \tilde{\phi} \partial_\nu \tilde{\phi} + \frac{1}{2} \xi R \tilde{\phi}^2 \right). \]  

(34)

To obtain the last equality the identities \( g_{\mu\nu} = \Omega \delta_{\mu\nu} \) and \( \xi \sqrt{g} R = -\Omega^{\frac{D-2}{4}} \nabla^2 \Omega^{\frac{D-2}{4}} \) are used. Consequently the free massless scalar theory on \( D \)-dimensional Euclidean space, is (classically) equivalent to some conformally coupled scalar theory on the corresponding conformally flat background.

A \( D \)-dimensional de Sitter (dS) space may be realized as the hypersurface described by the equation \(-X_0^2 + X_1^2 + \cdots + X_D^2 = \ell^2\). \( \ell \) is called the de Sitter radius. By replacing \( \ell^2 \) with \(-\ell^2\) the hypersurface is the \( D \)-dimensional anti de Sitter (AdS) space. (A)dS spaces are Einstein manifolds with positive (negative) scalar curvature. The Einstein metric \( G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \), satisfies \( G_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \), where \( \Lambda = \frac{(D-2)(D-1)}{2\ell^2} \) is the cosmological constant. From Eq.(17) one obtains \(-4m^2\ell^2 = D(D + 2)\) which determines the radius of the dS\(_D\) background in terms of the mass of scalar field \( \phi \).
References

[1] J. M. Maldacena, Adv. Theor. Math. Phys. 2, (1998) 231, hep-th/9711200.

[2] E. Witten, Adv. Theor. Math. Phys. 2 (1998) 253, hep-th/9802150.

[3] M. Duetsch and K. H. Rehren, Lett. Math. Phys. 62 (2002) 171-184, hep-th/0204123.

[4] A. Strominger, JHEP 0110 (2001) 034, hep-th/0106113.

[5] F. Loran, Phys. Lett. B601 (2004) 192-196, hep-th/0404067.

[6] M. Henneaux, Boundary terms in the AdS/CFT correspondence for spinor fields, hep-th/9902137;
W. Mück and K. S. Viswanathan, Phys. Rev. D58 (1998) 106006, hep-th/9805145;
M. Henningson and K. Sfetsos, Phys. Lett. B431 (1998) 63-68, hep-th/9803251;
G. E. Arutyunov and S. A. Frolov, Nucl. Phys. B544 (1999) 576-589, hep-th/9806216;
A. M. Ghezelbash, K. Kaviani, S. Parvizi and A. H. Fatollahi, Phys. Lett. B435 (1998) 291-298, hep-th/9805162.

[7] F. Loran, JHEP06(2004)054, hep-th/0404135.

[8] F. Loran, Phys. Lett. B605 (2005) 169-180, hep-th/0409267.

[9] N. J. Hitchin, The Geometry and Topology of Moduli Spaces in Global Geometry and Mathematical Physics 1451, (Springer, Heidelberg, 1988) 1-48.

[10] M. Blau, K. S. Narain and G. Thompson, Instantons, the Information Metric, and the AdS/CFT Correspondence, hep-th/0108122.

[11] R. Britto, B. Feng, O. Lunin and S. J. Rey, Phys. Rev. D69 (2004) 126004, hep-th/0311275;
S. Parvizi, Mod. Phys. Lett. A17 (2002) 341-354, hep-th/0202025.

[12] J. de Boer and S. N. Solodukhin, Nucl. Phys. B665 (2003) 545-593, hep-th/0303006;
S. N. Solodukhin, "Reconstructing Minkowski Space-Time, hep-th/0405252.

[13] T. Jacobson, Introduction to Quantum Fields in Curved Spacetime and the Hawking Effect, gr-qc/0308048.

[14] M. Spradlin, A. Strominger and A. Volovich, Les Houches Lectures on De Sitter Space, hep-th/0110007.