Some characterization of coprime graph of dihedral group $D_{2n}$

A G Syarifudin, Nurhabibah, D P Malik, dan I G A W Wardhana*

Department of Mathematics, Universitas Mataram, Jln. Majapahit No. 62, Mataram, 83125, Indonesia

*adhitya.wardhana@unram.ac.id

Abstract. The Coprime graph of group G denoted $\Gamma_G$ is a graph with vertices is an element of G, and two distinct vertices are adjacent when its order relative prime. In 2020, Gazir et al. give some characterizations of $\Gamma_{D_{2n}}$ for n a prime power. The method that uses in this paper is deductive proof by taking some example of a coprime graph of $D_{2n}$, then generalized the characterization of example. This paper gives some characteristics of the coprime graph of a dihedral group for more general cases. One of the result, $\Gamma_{D_{2n}}$ is a multipartite graph with girth 3, radius 1, and diameter 2.

1. Introduction

In recent years, mathematicians construct a graph from a mathematical system such as commuting and non-commuting graph, cycle graph, identity graph, and zero divisor graph. In 2014, X. Ma et al. defined a new graph called the coprime graph. The Coprime graph of finite group $G$ denoted by $\Gamma_G$ is a graph with vertices are elements of $G$ and two distinct vertices $x$ and $y$ are adjacent if and only if $(\text{ord}(x), \text{ord}(y)) = 1$[1]. In 2017, Abdussakir researched the commuting and non-commuting graph of a dihedral group and got some characteristics of that graph such as radius, diameter, cycle multiplicity, and matrix dimensions [2]. In 2020, Gazir S. et al. found the form of the coprime graph of a dihedral group that is complete tripartite when $n$ odd prime number and if $n = 2^k$ then $\Gamma_{D_{2n}}$ is a complete bipartite graph [3].

From the above description, we will determine the characteristics of the coprime graph of a dihedral group with $n$ odd composite numbers, such as form, girth, radius, and diameter.

2. Method

This research’s methods are deductive proof that makes conjectures based on properties and then proves them with rigorous proof. The first step is studying definitions and theories about the coprime graph of the dihedral group, then studying examples of characterizations of the coprime graph of the dihedral group. The last step is to make conjectures and prove them.

3. Result and Discussion

This section will discuss about the dihedral group and its representation in the coprime graph and some characteristics.
3.1 Coprime Graph of $D_{2n}$

**Definition 1** ([4]) Group $G$, named dihedral group with order $2n$, $n \geq 3$ and $n \in \mathbb{N}$, is a group generated by $a, b \in G$ with properties

$$G = \langle a, b | a^n = e, b^2 = e, bab^{-1} = a^{-1} \rangle.$$

A dihedral group with order $2n$ denoted by $D_{2n}$.

**Definition 2** ([5]) If $(G, \ast)$ any group. Let $a$ any elements of $G$. The least positive integer $m$ with $a^m = e$ (e identity in G) then $m$ named order of $a$, and denoted by $|a| = m$ or $ord(a) = m$.

**Definition 3** ([11]) Let $G$ finite group, the coprime graph of $G$ denoted by $\Gamma_G$ is a graph with vertices are elements of $G$ and two distinct vertices $x$ and $y$ are adjacent if and only if $(ord(x), ord(y)) = 1$.

These are three theorems given by Gazir S. et al. about the form of the coprime graph. The first result about the form of the coprime graph of $D_{2n}$, with $n$ odd prime number explained in the next theorem.

**Theorem 1** ([6]) Let if $n$ is an odd prime number, then the coprime graph of $D_{2n}$ is complete tripartite.

In addition, the coprime graph of $D_{2n}$ is a complete bipartite graph when $n = 2^k$, for some $k \in \mathbb{N}$.

**Theorem 2** ([6]) Let $n = 2^k$, for some $k \in \mathbb{N}$ then the coprime graph of $D_{2n}$ is a complete bipartite.

The next theorem explains that the coprime graph of $D_{2n}$ is complete tripartite for some $n = p^k$.

**Theorem 3** ([6]) Let $n = p^k$ for some $k \in \mathbb{N}$ and $p$ is a prime number, $p \neq 2$, then the coprime graph of $D_{2n}$ is complete tripartite.

The last theorem about the form of the coprime group of $D_{2n}$ where $n$ that the following Theorem gives more generalize.

**Theorem 4** Let $n = p_1^{k_1}p_2^{k_2}p_3^{k_3} \ldots p_m^{k_m}$ where $1 \leq i \leq m$, $p_i$ are distinct prime number, and $p_i \neq 2$ then the coprime graph of $D_{2n}$ is $(m + 2)$-partite.

*Proof.* Let $D_{2n}$ a dihedral group with $n = p_1^{k_1}p_2^{k_2}p_3^{k_3} \ldots p_m^{k_m}$ where $1 \leq i \leq m$, $p_i$ are distinct prime number, $p_i \neq 2$. We define some set, the first set is a set of elements with order 1, the second set is a set of elements with order 2, or even the third set is a set of elements with order $p_i$ and odd, and the $(m + 2)$ set is a set of elements with order $p_m$ and odd and $p_j$ not divide $p_m$ where $1 \leq j \leq m - 1$, clearly these sets are a partition of $D_{2n}$. Let $x, y \in V_i$, thus $p_i | ord(x)$ and $p_i | ord(y)$, so $(ord(x), ord(y)) \neq 1$, then $x$ and $y$ are not adjacent. So, the coprime graph of $D_{2n}$ is $m + 2$-partite. ■

3.2 Radius and Diameter

**Definition 4** ([7]) Let $u$ and $v$ are vertices in $G$, the $d(u, v)$ denotes the length of the shortest path between $u$ and $v$. The least distance between all pairs of the vertices of $G$ is called the diameter of $G$, and is denoted by $\text{rad}(G)$.

**Definition 5** ([7]) Let $u$ and $v$ are vertices in $G$, the $d(u, v)$ denotes the length of the shortest path between $u$ and $v$. The largest distance between all pairs of the vertices of $G$ is called the diameter of $G$, and is denoted by $\text{diam}(G)$.

The next theorem explains other characteristics of the coprime graph of the dihedral group, like radius and diameter. For any $n$, $\text{rad}(\Gamma_{D_{2n}}) = 1$ and $\text{diam}(\Gamma_{D_{2n}}) = 2$. 


Theorem 5 Let \( \Gamma_{D_{2n}} \) coprime graph then \( \text{rad}(\Gamma_{D_{2n}}) = 1 \) and \( \text{diam}(\Gamma_{D_{2n}}) = 2 \).

Proof. We know that \((ord(x), ord(y)) = 1\), for each \( x \in D_{2n} \). Thus, \( e \) are adjacent with all vertices in \( D_{2n} \) then \( e(e) = 1 \), cause the radius \( D_{2n} \) is the least eccentric in \( D_{2n} \) we get \( \text{rad}(D_{2n}) = 1 \). Next, let \( a \) and \( b \) two distinct vertices in \( D_{2n} \). If \( a \) and \( b \) are adjacent, then \( d(a, b) = 1 \). In other, if \( a \) and \( b \) are not adjacent, then \( d(a, b) = 2 \). Such distances of two distinct vertices in \( D_{2n} \) is 1 or 2. Thus diameter is the largest eccentric then \( \text{diam}(\Gamma_{D_{2n}}) = 2 \).

3.3 Girth

Definition 6 ([8]) The girth of graph \( G \) is the length of the shortest cycle contained in \( G \).

Theorem 6 Let \( D_{2n} \) dihedral group. If \( n = p_1^{k_1}p_2^{k_2}...p_m^{k_m} \) and \( n \neq 2 \), then girth of \( \Gamma_{D_{2n}} \) is 3.

Proof. Let \( D_{2n} \) dihedral group. If \( n = p_1^{k_1}p_2^{k_2}...p_m^{k_m} \) and \( 2 \nmid n \). Based on Theorem 4, \( \Gamma_{D_{2n}} \) is \((m + 2)\)-partite. Thus \((m + 2)\)-partite and always contain an element with order \( p_i \) and \( p_j \) with \( i \neq j \). Let \( \text{ord}(x) = p_l \) and \( \text{ord}(y) = p_j \), so \( x \) and \( y \) are adjacent. Consequently, we have cycle \( e - x - y - e \) then girth of \( \Gamma_{D_{2n}} \) is 3.

4. Conclusion

The obtained result shows that the coprime graph of a dihedral group where \( n \) is a composite number specifically \( n = p_1^{k_1}p_2^{k_2}p_3^{k_3}...p_m^{k_m} \), since it forms a multipartite graph or \((m + 2)\)-partite graph, it has a girth of 3, a radius of 1, and a diameter of 2. For the next result, it is interesting to see other characteristics of \( \Gamma_{D_{2n}} \) like a clique, chromatic numbers, etc.

5. Acknowledgments

Special Thanks to Kemendikbud (PKM AI 2020) for financial support to this paper. We also would like to thank the Algebra Research Group of Universitas Mataram and Gamatika Research Club for their advice and support.

6. References

[1] Ma X L Wei H Q and Yang L Y 2014 *International Journal of Group Theory* 3 (3) 13-23
[2] R. Munir 2010 *Matematika Diskrit* (Bandung: Penerbit Informatika)
[3] Gazir S A Wardhana I G A W Switrayni N W and Aini Q 2020 *Journal of Fundamental Mathematics and Applications* 3 no. 1 34-38
[4] Dummit S D Foote M R 2004 *Abstract Algebra Third Edition* (New York: John Wiley & Sons, Inc.)
[5] Herstein I N 1975 *Topics In Algebra Second Edition* (United States of America: Prentice-Hall, Inc.)
[6] Abdussakir 2017 *Jurnal Matematika “Mantik”* 3 (1) 14.
[7] Chelvam T T Selvakumar K and Raja S 2011 *The Journal of Mathematics and Computer Science* 2 (2) 402-406
[8] Finbow A Hartenell B and Nowakosky R J 1993 *Jurnal of Combinatorial Theory* 2 (1) 44-68