A Variant of the Wang-Foster-Kakade Lower Bound for the Discounted Setting

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Abstract

Recently, Wang et al. (2020) showed a highly intriguing hardness result for batch reinforcement learning (RL) with linearly realizable value function and good feature coverage in the finite-horizon case. In this note we show that once adapted to the discounted setting, the construction can be simplified to a 2-state MDP with 1-dimensional features, such that learning is impossible even with an infinite amount of data.

Wang et al. (2020) recently showed that in finite-horizon batch RL, the sample complexity of evaluating a given policy $\pi$ has an information-theoretic lower bound that is exponential in the horizon, even if realizable linear features are given (i.e., $\varphi : S \times A \rightarrow \mathbb{R}^d$ such that $Q^\pi(\cdot) = \langle \varphi(\cdot), \theta^\pi \rangle$ for some parameter $\theta^\pi \in \mathbb{R}^d$) and data provides good feature coverage (i.e., $\mathbb{E}[\varphi \varphi^T]$ has lower-bounded eigenvalues under the data distribution). In this document we show that its analogy in the discounted setting has a stronger statement (infinite sample complexity) with a simpler construction (1-d feature, 2 states, and arbitrary discount factor.)

Consider the deterministic MDP in Figure 1 with a discount factor $\gamma \in (0, 1)$, where every state only has 1 action (which we omit in the notations). $s_A$ transitions to $s_B$ with 0 reward, and $s_B$ has a self-loop with $r$ reward per step. The batch data only contains the tuple $(s_A, 0, s_B)$. The feature map is 1-dimensional: $\varphi(s_A) = \gamma$ and $\varphi(s_B) = 1$. Clearly, without data from $s_B$, the learner cannot know the value of $r$, hence cannot determine the value of $s_A$ or $s_B$, even with an infinite amount of data.

We now verify the realizability and coverage assumptions:

- **Realizability:** We show that $V^\pi(\cdot) = \langle \varphi(\cdot), \theta \rangle$ for some $\theta \in \mathbb{R}$. By the Bellman equation, $V^\pi(s_A) = \gamma V^\pi(s_B)$. Therefore, $V^\pi(s_A) = \langle \varphi(s_A), V^\pi(s_B) \rangle$. Similarly, $V^\pi(s_B) = 1 \cdot V^\pi(s_B) = \langle \varphi(s_B), V^\pi(s_B) \rangle$. So $V^\pi(\cdot)$ is always linearly-realizable, with $\theta = V^\pi(s_B) = \frac{r}{1-\gamma}$ being the unknown coefficient.
• **Coverage:** Translating the condition of Wang et al. (2020) to the discounted case, it is required that: (1) $\|\varphi(\cdot)\|_2 \leq 1$ always holds, and (2) $E[\varphi\varphi^\top]$ has polynomially lower bounded eigenvalues. (1) is satisfied in our construction. For (2), since we only have data from $s_A$, the feature covariance matrix under the data distribution is $\varphi(s_A)\varphi(s_A)^\top = \gamma^2$, whose only eigenvalue is $\gamma^2$ and is well above 0 as long as $\gamma$ is.

**Extensions for general $d$ and the controlled setting** We briefly sketch two extensions of the construction. Although it is sufficient to prove the lower bound for $d = 1$, the construction easily scales to arbitrary $d$: we simply make $d$ copies of the construction in Figure 1, and assign a coordinate of $\varphi : \mathcal{S} \to \mathbb{R}^d$ to each copy. Let data be uniform over the $s_A$ of all copies, so the feature covariance matrix is $\gamma^2/d \cdot I$.

The extension to the controlled case is similar. Let $a$ denote the action of $s_A$ in Figure 1. We introduce a second action $a'$ for $s_A$ that transitions to $s_C$ with 0 reward, and $s_C$ is absorbing with reward $r'$. Let the 2-dimensional feature map be: $\varphi(s_A, a) = [\gamma, 0]^\top$, $\varphi(s_A, a') = [0, \gamma]^\top$, $\varphi(s_B) = [1, 0]^\top$, $\varphi(s_C) = [0, 1]^\top$. It is easy to verify that $Q^*$ is realizable$^1$, but $Q^*(s_A, a) = \frac{\gamma}{1-\gamma} r$ and $Q^*(s_A, a') = \frac{\gamma}{1-\gamma} r'$ can independently take arbitrary values between $[0, \gamma/(1-\gamma)]$ (assuming rewards lie in $[0, 1]$), so the learner cannot choose a near-optimal action even with infinite data.

These observations combine to give us the following result:

**Proposition 1** (Informal). For any $d \geq 1, \gamma \in (0, 1)$, given realizable linear features, the value function learned by any batch RL algorithm must have $\Omega(1)$ worst-case error, even with an infinitely large dataset that has $\Theta(1/d)$ feature coverage.

**Final Remark** While the discounted setting allows a very simple construction for the lower bound, this does not imply that the construction for the finite-horizon setting can be simplified in a similar manner. In fact, we believe that the careful construction of Wang et al. (2020) that cleverly exponentiates a negligibly small error is necessary for the finite-horizon setting. Such a difference between the finite-horizon setting and the discounted setting, however, does challenge the conventional wisdom that the results in the finite-horizon setting and the discounted setting are often similar and translate to each other with $H = O(1/(1-\gamma))$ up to minor differences. Are these two lower bounds “essentially the same”, or does their difference imply some fundamental difference between the finite-horizon and the discounted settings? We leave this open question to the readers.

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**References**

Ruosong Wang, Dean P Foster, and Sham M Kakade. What are the statistical limits of offline rl with linear function approximation? *arXiv preprint arXiv:2010.11895*, 2020.

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1. In fact, Q-functions in this MDP do not depend on the policy, since only $s_A$ has multiple actions.