GALOIS THEORY IN BICATEGORIES

J. GÓMEZ-TORRECILLAS AND J. VERCRUYSSE

Abstract. We develop a Galois (descent) theory for comonads within the framework of bicategories. We give generalizations of Beck’s theorem and the Joyal-Tierney theorem. Many examples are provided, including classical descent theory, Hopf-Galois theory over Hopf algebras and Hopf algebroids, Galois theory for corings and group-corings, and Morita-Takeuchi theory for corings. As an application we construct a new type of comatrix corings based on (dual) quasi bialgebras.

Introduction

The classical Galois Theory on field extensions has been generalized in many directions. For instance, it has been extended to a Galois theory for commutative rings by Auslander and Goldman [1] and by Chase, Harrison and Rosenberg [21]. A group action can be generalized to a Hopf algebra (co)action. This leads to the Hopf-Galois theory, developed first for finitely generated and projective Hopf algebras (see [22] and [34]) and later for arbitrary Hopf algebras (see [25] and [43]). During the nineties, the theory of Hopf algebras went through a range of generalizations, such as Doi-Koppinen structures [24], [33] and entwining structures [11] to arrive at the theory of corings and comodules [46], which provides a general framework to explain many results of Hopf algebra theory in a simple and clarifying way. In this respect, it is no surprise that Hopf-Galois theory has a formulation in terms of corings. This was shown in [7], where a Galois theory is developed for corings with a grouplike element. To a ring morphism \( i : B \to A \), we can associate an \( A \)-coring, the so called canonical Sweedler coring. A morphism from this coring to another \( A \)-coring \( \mathcal{C} \) is completely determined by a grouplike element \( g \in \mathcal{C} \). When this morphism is an isomorphism we say that \((\mathcal{C}, g)\) is a Galois coring. We can construct a pair of adjoint functors between the categories \( \mathcal{M}_B \) and \( \mathcal{M}_C \) and formulate sufficient and necessary conditions for this pair to be an equivalence of categories. El Kaoutit and the first author [26] introduced a yet more general version of Galois theory, replacing the grouplike element by a right \( \mathcal{E} \)-comodule \( \Sigma \) which is finitely generated and projective as a right \( \mathcal{A} \)-module. This generalizes the Galois theory of [7], and, in particular, allows to characterize those corings whose category of comodules has a finitely generated projective generator. To this end, an \( A \)-coring, called the comatrix coring, is constructed out of \( \Sigma \). We call \( \Sigma \) a Galois comodule if the canonical homomorphism from the comatrix coring to \( \mathcal{C} \) is bijective (see also [8], [16]). In general, there exists an adjunction

\[ 
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\]
between the categories $\mathcal{M}_T$ and $\mathcal{M}_C$, where $T = \text{End}^C(\Sigma)$, and this adjunction induces an equivalence of categories if and only if $C$ is Galois and $\tau \Sigma$ is faithfully flat [26, Theorem 3.2]. Several attempts were made to drop or weaken the finiteness condition on $\Sigma$ and construct infinite versions of Galois theory for comodules. El Kaoutit and the first author introduced Galois comodules that are (infinite) direct sums of finitely generated and projective right $A$–modules [27]. Caenepeel, De Groot and the second author generalized this to a method to construct corings out of colimits. Both authors developed a theory of Galois comodules over firm rings [31]. Wisbauer introduced a functorial definition for a Galois comodule [51].

In a slightly different direction, there has been a similar evolution in (categorical) descent theory. Both theories are closely related (this connection is discussed in [30]), and some authors would prefer to term the theory of this paper ‘descent theory’. However, we reserve the name ‘descent theory’ for the special situation that is treated in Section 4.4. Our motivation to keep the term Galois theory is founded by the evolution in Hopf-Galois theory described above. Descent theory investigates the extension of scalars functor $- \otimes_B A : \mathcal{M}_B \to \mathcal{M}_A$ associated to a homomorphism of rings $B \to A$, in particular it looks for sufficient and necessary conditions for this functor to be comonadic (cotripleable). A first important theorem in this respect, is Beck’s theorem that gives sufficient and necessary conditions for a functor with a right adjoint to be comonadic (see e.g. [2], [36]). Another interesting and more general result states that the extension of scalars functor for commutative rings $A$ and $B$ is comonadic if and only if $B \to A$ is a pure monomorphism of $B$–modules. Although this theorem is presently known as the theorem of Joyal-Tierney, it was never published as such; a proof of this theorem can be found in [38]. During the last few years, the Joyal-Tierney theorem is generalized in several ways. In [14], a non-commutative version is presented, in [16] a generalization is formulated where the ring extension is replaced by a $B$–$A$–bimodule, where $A$ is non-commutative and $B$ is commutative, in [39] $B$ is allowed to be a (non-commutative) separable algebra. In [32] a categorical version of the theorem is presented.

In this paper we propose a Galois theory for comonads in the general setting of bicategories. Our approach rests upon the set up proposed in [35] in the light of the theory developed in [9] for bimodules and corings. Our work intends to provide not only a transparent view on the interactions between the different approaches in the recent development on Galois comodules, but we also hope to shed new light on the relationship between the coring theory and the theory of comonads [30]. Moreover several other versions and generalizations of Hopf-Galois theory, which have been formulated during the last years (such as equivalences between categories of comodules [9], [52], Galois theory for $C$–rings [12] and Galois theory for group-corings [18]) fit perfectly within our general framework.

One final remark on notation: in any category $\mathcal{C}$, we will denote the identity morphism on an object $X \in \mathcal{C}$ again by $X$.

1. Elementary definitions and notation

Recall from [3] that a bicategory $\mathcal{B}$ consists of the following data.

(i) A class of objects $A, B, \ldots$ which are called 0–cells (or objects).

(ii) For every two objects $A$ and $B$, there exists a category $\text{Hom}_\mathcal{B}(A, B) = \text{Hom}(A, B)$, whose class of objects we denote by $\text{Hom}_1(A, B)$ and which are called 1–cells. We denote $f : A \to B$ for an 1–cell $f \in \text{Hom}_1(A, B)$. Take two 1–cells $f, g \in \text{Hom}_1(A, B)$. The set of morphisms from $f$ to $g$ in the category $\text{Hom}(A, B)$ is denoted by $\text{Hom}_2^A(f, g)$. We call
these morphisms 2–cells and denote them as \( \alpha : f \to g \). We will denote the composition of morphisms in the category \( \text{Hom}(A, B) \) by \( \circ \), i.e. for all \( f, g, h \in \text{Hom}_1(A, B) \) such that \( \alpha : f \to g \) and \( \beta : g \to h \), we have \( \beta \circ \alpha : f \to h \). This composition will now be called the vertical composition of 2–cells.

(iii) For any three objects \( A, B, C \in B \), there exists a functor

\[
c_{ABC} : \text{Hom}(A, B) \times \text{Hom}(B, C) \to \text{Hom}(A, C).
\]

For all \( f \in \text{Hom}_1(A, B) \) and \( g \in \text{Hom}_1(B, C) \), we denote \( c_{ABC}(f, g) = f \bullet_B g \in \text{Hom}_1(A, C) \). For all \( \alpha \in A \text{Hom}_2^B(f, g) \) and \( \beta \in B \text{Hom}_2^C(h, k) \), we denote \( c_{ABC}(\alpha, \beta) = \alpha \bullet_B \beta : f \bullet_B h \to g \bullet_B k \). This composition will be called the horizontal composition of 2–cells.

(iv) For any object \( A \in B \), there exists a functor

\[
\mathbb{1}_A : 1 \to \text{Hom}(A, A),
\]

where 1 denotes the discrete category with one object *. We will denote \( \mathbb{1}_A(*) \) just by \( \mathbb{1}_A \).

(v) For any four objects \( A, B, C, D \in B \), there exists a natural isomorphism

\[
(\alpha_{ABCD} : c_{A,C} \circ (c_{A,B} \times \text{Hom}(C, D)) \to c_{A,B} \circ (\text{Hom}(A, B) \times c_{B,C,D})).
\]

For any two objects \( A, B \in B \), there exist two natural isomorphisms

\[
\lambda_{AB} : \text{Hom}(A, B) \to c_{A,B} \circ (\mathbb{1}_A \times \text{Hom}(A, B)),
\]

\[
\rho_{AB} : \text{Hom}(A, B) \to c_{A,B} \circ (\text{Hom}(A, B) \times \mathbb{1}_B).
\]

All these data are required to satisfy some compatibility (associativity and coherence) conditions. We refer to e.g. [3], where the notion of a bicategory was introduced, or [6] section 7.7.

For all objects \( A, B, C \in B \), we obtain from the functorality of \( c_{ABC} \) the interchange law, i.e.

\[
\alpha \bullet_B \beta \circ (\gamma \bullet_B \delta) = (\alpha \circ \gamma) \bullet_B (\beta \circ \delta),
\]

for \( \alpha \in A \text{Hom}_2^B(a, c) \), \( \beta \in B \text{Hom}_2^C(b, d) \), \( \gamma \in A \text{Hom}_2^B(c, e) \) and \( \delta \in B \text{Hom}_2^C(d, f) \). From [3] one immediately deduces that for all \( \alpha \in A \text{Hom}_2^B(a, c) \) and \( \beta \in B \text{Hom}_2^C(b, d) \),

\[
(\alpha \bullet_B b) \circ (c \bullet_B \beta) = a \bullet_B \beta = (a \bullet_B \beta) \circ (a \bullet_B d).
\]

A 2–category is a bicategory such that the isomorphisms \( \alpha_{ABCD} \), \( \lambda_{AB} \) and \( \rho_{AB} \) are identities for all choices of \( A, B, C, D \). In particular, \( \mathbb{1}_A = A \) for all objects \( A \) of a 2–category.

To any bicategory \( B \) one can associate new bicategories denoted by \( B^{op} \), \( B^{coop} \) and \( B^{coop} \). These are constructed by taking respectively opposite composition for the 1–cells, for vertical composition of 2–cells and for both.

Recall that a comonad \( \mathcal{C} = (A, c, \Delta_c, \varepsilon_c) \) in \( B \) consists of a 0–cell \( A \), an 1–cell \( c \in \text{Hom}_1(A, A) \) and two 2–cells \( \Delta_c \in A \text{Hom}_2^A(c, c \bullet_A c) \) and \( \varepsilon_c \in A \text{Hom}_2^A(c, \mathbb{1}_A) \) such that the
are commutative in \( \text{Hom}(A, A) \). For each object \( \Omega \) of \( \mathcal{B} \), the comonad \( \mathcal{C} \) in \( \mathcal{B} \) induces a comonad on the category \( \text{Hom}(\Omega, A) \) (in the sense of [2, Definition 3.1])

\[
\delta : G \to G^2, \quad \epsilon : G \to 1,
\]

where

\[
G = - \cdot_A c, \quad \delta = \alpha^{-1}_{\Omega AAA} \circ (- \cdot_A \Delta_c), \quad \text{and} \quad \epsilon = \rho^{-1}_{\Omega A} \circ (- \cdot_A \varepsilon_c).
\]

Let \( \mathsf{Rcom}(\Omega, \mathcal{C}) \) denote the Eilenberg-Moore category for this comonad. Its objects will be called right \( \mathcal{C} \)-comodules of \( \Omega \)-type or \( \Omega \)-\( \mathcal{C} \)-comodules, and consist of couples \( (m, \rho^m) \) where \( m \in \text{Hom}_1(\Omega, A) \) for some 0–cell \( \Omega \) in \( \mathcal{B} \) and \( \rho^m \in \Omega \text{Hom}^A(m, m \cdot_A c) \) such that \( \alpha_{\Omega AAA} \circ (\rho^m \cdot_A c) \circ \rho^m = (m \cdot_A \Delta_c) \circ \rho^m \) and \( (m \cdot_A \varepsilon_c) \circ \rho^m = \rho_{\Omega A} \). A morphism of \( \Omega \)-\( \mathcal{C} \)-comodules, \( \psi : (m, \rho^m) \to (n, \rho^n) \) consists of a 2–cell \( \psi \in \Omega \text{Hom}^2(m, n) \) such that \( \rho^n \circ \psi = (\psi \cdot_A c) \circ \rho^m \). We will denote the set of all morphisms of \( \Omega \)-\( \mathcal{C} \)-comodules between \( m \) and \( n \) by \( \Omega \text{Hom}^2(m, n) \). Analogously, our comonad \( \mathcal{C} \) induces, for each 0–cell \( \Omega \), a comonad on \( \text{Hom}(\Omega, A) \), whose corresponding Eilenberg-Moore category will be denoted by \( \mathsf{Lcom}(\mathcal{C}, \Omega) \), and their objects will be referred to as left \( \mathcal{C} \)-comodules of \( \Omega \)-type or \( \mathcal{C} \)-\( \Omega \)-comodules.

Given comonads \( \mathcal{C} = (A, c, \Delta_c, \varepsilon_c) \) and \( \mathcal{D} = (B, d, \Delta_d, \varepsilon_d) \), we define a \( \mathcal{C} \)-\( \mathcal{D} \)-bicomodule as a three-tuple \( (m, \lambda^m, \rho^m) \) where \( (m, \lambda^m) \) is a \( \mathcal{C} \)-\( \mathcal{B} \)-comodule and \( (m, \rho^m) \) is an \( \mathcal{A} \)-\( \mathcal{D} \)-comodule, such that \( \alpha_{\mathcal{A}\mathcal{B}B} \circ (\lambda^m \cdot_B d) \circ \rho^m = (\cdot_B c) \circ \lambda^m \). These bicomodules are the objects of a category \( \text{Bicom}(\mathcal{C}, \mathcal{D}) \), whose morphisms are the 2–cells that are both morphisms of left \( \mathcal{C} \)-comodules and of right \( \mathcal{D} \)-comodules.

A monad and modules over a monad in a bicategory \( \mathcal{B} \) can be defined as a comonad and comodules over a comonad in the co-opposite bicategory \( \mathcal{B}^\circ \), which consists of the same data as \( \mathcal{B} \), except that one considers the opposite composition of the vertical composition of 2–cells in \( \mathcal{B} \).

A pseudo-functor \( F : \mathcal{B} \to \mathcal{C} \) between bicategories \( \mathcal{B} \) and \( \mathcal{C} \) assigns a 0–cell \( FA \) of \( \mathcal{C} \) to each 0–cell \( A \) of \( \mathcal{B} \), and a functor

\[
F_{AB} : \text{Hom}_\mathcal{B}(A, B) \to \text{Hom}_\mathcal{C}(FA, FB);
\]

for every pair of 0–cells \( A, B \) of \( \mathcal{B} \). The pseudo-functor preserves the horizontal composition only up to isomorphisms, in the sense that for every three-tuple of 0–cells \( A, B, C \) in \( \mathcal{B} \), there exist natural isomorphisms

\[
\gamma_{ABC} : c_{FA,FB,FC} \circ (F_{AB} \times F_{BC}) \to F_{AC} \circ c_{ABC} ;
\]

\[
\xi_A : 1_{FA} \to F_{AA} \circ 1_A ;
\]

subject to suitable associativity and coherence axioms (see e.g. [6, Section 7.5]). If \( P \) is a property of functors, then we say that a pseudo functor \( F \) satisfies the local \( P \)-property if
and only if $F_{AB}$ satisfies the property $P$ for all choices of $A$ and $B$ (e.g. $F$ is locally faithful, locally exact).

Given a comonad $\mathcal{C} = (A, c, \Delta, \varepsilon)$ in $\mathcal{B}$, the pseudo-functor $F : \mathcal{B} \to \mathcal{C}$ induces a comonad in $\mathcal{C}$ given by

$$F\mathcal{C} = (FA, FC, \gamma^{-1}_{AA} \circ F\Delta, \xi^{-1}_A \circ F\varepsilon),$$

and, for each object $\Omega$ of $\mathcal{B}$, a functor $F_\Omega : R\text{com}(\Omega, \mathcal{C}) \to R\text{com}(F\Omega, F\mathcal{C})$. If all $F_{AB}$'s are equivalences (e.g. if $F$ is a biequivalence of bicategories), then $F_\mathcal{C}$ is an equivalence for every comonad $\mathcal{C}$. Since every bicategory is biequivalent to a 2–category (see \[45\], \[37\]), we can, without loss of generality, restrict ourselves to the case of comonads in 2–categories when studying categories of comodules. In fact, our former argument could have been replaced by the more general “Coherence Theorem”, which asserts that all diagrams that are placed by the more general “Coherence Theorem”, which asserts that all diagrams that are

In a symmetric way, a $\text{Hom}_\mathcal{B}$-comonad $\Delta$ constructed out of the associativity and identity isomorphisms commute in any bicategory. Those readers familiar enough with the Coherence Theorem may consider that we will rely on the 2–categorical calculus when we are dealing with bicategories.

2. The 2–categories of Comonads

Throughout this section, $\mathcal{B}$ will denote a 2–category. Following \[44\] and \[35\], we will define several 2–categories whose 0–cells are comonads in $\mathcal{B}$. Their 1–cells will be comonad-morphisms, in the sense of Definition 2.1. They will encode certain functors between categories of comodules (see Section 3). In this section, we show that the comonad-morphisms can be understood as bicomodules (Lemma 2.3).

**Definition 2.1.** Let $\mathcal{C} = (A, c, \Delta, \varepsilon)$ and $\mathcal{D} = (B, d, \Delta_d, \varepsilon_d)$ be two comonads in $\mathcal{B}$. A right comonad-morphism from $\mathcal{D}$ to $\mathcal{C}$ is a pair $(q, \alpha)$, consisting of $q \in \text{Hom}_1(B, A)$ and $\alpha \in B\text{Hom}_2^A(d \bullet_B q, q \bullet_A c)$ such that the following diagrams commute

$$\begin{array}{ccc}
\Delta_d \bullet_B q & \overset{\alpha}{\longrightarrow} & q \bullet_A c \\
\downarrow & & \downarrow q \bullet_A \Delta_c \\
d \bullet_B d \bullet_B q & \overset{\varepsilon_d \bullet_B q}{\longrightarrow} & q \bullet_A c \bullet_A c
\end{array}$$

In a symmetric way, a left comonad-morphism from $\mathcal{D}$ to $\mathcal{C}$ is a pair $(p, \beta)$, where $p \in \text{Hom}_1(A, B)$ and $\beta \in A\text{Hom}_2^B(p \bullet_B d, c \bullet_A p)$, such that the following diagrams commute

$$\begin{array}{ccc}
p \bullet_B d & \overset{\beta}{\longrightarrow} & c \bullet_A p \\
\downarrow p \bullet_B \Delta_d & & \downarrow \Delta_c \bullet_A p \\
p \bullet_B d \bullet_B p & \overset{\Delta_c \bullet_A p}{\longrightarrow} & c \bullet_A c \bullet_A p
\end{array}$$

**Examples 2.2.** (i) Let $\varphi : \mathcal{C} = (A, c, \Delta, \varepsilon) \to \mathcal{C}' = (A, c', \Delta', \varepsilon')$ be a homomorphism of $A$-comonads, i.e., a 2–cell $\varphi : c \to c'$ such that $\Delta_c \circ \varphi = (\varphi \bullet_A \Delta_c) \circ \Delta_c$ and $\varepsilon_c \circ \varphi = \varepsilon_c$. Then $(A, \varphi)$ is both a left and a right comonad-morphism from $\mathcal{C}$ to $\mathcal{C}'$.

(ii) From the definition one can immediately see that a right comonad-morphism $(m, \rho)$ from $\mathcal{B}$ to $\mathcal{C}$, where $\mathcal{B} = (B, B, B, B)$ is the trivial $B$-comonad, is nothing else than a right $\mathcal{C}$-comodule of $B$–type. Similarly, left comonad-morphisms from $\mathcal{B}$ to $\mathcal{C}$ are exactly left $\mathcal{C}$-comodules. More generally, we have the following result.
Lemma 2.3. Let \( \mathcal{C} = (A, c, \Delta_c, \varepsilon_c) \) and \( \mathcal{D} = (B, d, \Delta_d, \varepsilon_d) \) be comonads in \( \mathcal{B} \). Then the following statements hold

(i) \((q, \alpha)\) is a right comonad-morphism from \( \mathcal{D} \) to \( \mathcal{C} \) if and only if \( d \bullet_B q \) is a \( \mathcal{D}-\mathcal{C} \)-bicomodule.

(ii) \((p, \beta)\) is a left comonad-morphism from \( \mathcal{D} \) to \( \mathcal{C} \) if and only if \( p \bullet_B d \) is a \( \mathcal{C}-\mathcal{D} \)-bicomodule.

Proof. Suppose first that \((q, \alpha)\) and \((p, \beta)\) are comonad-morphisms. The coactions on \( d \bullet_B q \) and \( p \bullet_B d \) are given by the following formulas,

\[
\rho^{d \bullet_B q} : \quad d \bullet_B q \xrightarrow{\Delta_d \bullet_B q} d \bullet_B d \bullet_B q \xrightarrow{d \bullet_B \alpha} d \bullet_B q \bullet_A c ;
\]

\[
\lambda^{d \bullet_B q} : \quad d \bullet_B q \xrightarrow{\Delta_d \bullet_B q} d \bullet_B d \bullet_B q ;
\]

\[
\rho^{p \bullet_B d} : \quad p \bullet_B d \xrightarrow{p \bullet_B \Delta_d} p \bullet_B d \bullet_B d ;
\]

\[
\lambda^{p \bullet_B d} : \quad p \bullet_B d \xrightarrow{p \bullet_B \Delta_d} p \bullet_B d \bullet_B d \xrightarrow{p \bullet_B \beta} c \bullet_A p \bullet_B d .
\]

We will only check the coassociativity of \( \rho^{d \bullet_B q} \) and leave other verifications (the coassociativity and counit conditions, as well as the compatibility between left and right coaction) to the reader, since they are all similar computations. Consider the following diagram, of which the outer quadrangle expresses the coassociativity of \( \rho^{d \bullet_B q} \).

The lower quadrangle of this diagram commutes by the definition of a right comonad-morphism, and the upper quadrangle by application of (4).

Now suppose that \( d \bullet_B q \) is a \( \mathcal{D}-\mathcal{C} \)-bicomodule. We will prove that \((q, \alpha)\) is a comonad-morphism, where

\[
\alpha : d \bullet_B q \xrightarrow{d \bullet_B \rho} d \bullet_B q \bullet_A c \xrightarrow{\varepsilon_d \bullet_B q \bullet_A c} q \bullet_A c .
\]
We have to check the commutativity of the outer quadrangle of the following diagram.

The upper inner quadrangle commutes since \( d \bullet_B q \) is a bicomodule. The upper triangle commutes by the counital property of the comonad \( D \). The lower triangle commutes on the image of the incoming arrow \( d \bullet_B \rho \) on the left as an application of the coassociativity condition on \( d \bullet_B q \) as a right \( C \)–comodule. Finally, applying (4), we find that the lower quadrangle commutes.

The proof for \( p \bullet_B d \) is similar.

The 2–category of comonads in \( B \) introduced in [41] is the following:

**Definition 2.4.** By \( RCOM(B) \) we denote the right 2–category of comonads in \( B \). This 2–category is defined as follows.

(i) A 0–cell in \( RCOM(B) \) is a comonad in \( B \);
(ii) a 1–cell in \( RCOM(B) \) is a right comonad-morphism;
(iii) a 2–cell in \( RCOM(B) \) between two right comonad-morphisms \( (q, \alpha) \) and \( (q', \alpha') \) from \( D \) to \( E \) is a 2–cell \( \sigma : q \to q' \) in \( B \) that makes the following diagram commute

\[
\begin{CD}
   d \bullet_B q @>d \bullet_B \sigma>> d \bullet_B q' \\
   @V\alpha VV @VV\alpha' V \\
   q \bullet_A c @>>> q' \bullet_A c;
\end{CD}
\]

(iv) the composition of one-cells is defined as follows. Consider comonads \( C = (A, c, \Delta_c, \varepsilon_c) \), \( D = (B, d, \Delta_d, \varepsilon_d) \) and \( E = (C, e, \Delta_e, \varepsilon_e) \). Let \( (q, \alpha) : C \to D \) and \( (q', \alpha') : D \to E \) be two comonad-morphisms. Then we define a new comonad-morphism \( (q, \alpha) \bullet_D (q', \alpha') = (q' \bullet_B q, (q' \bullet_A \alpha) \circ (\alpha' \bullet_B \varepsilon)) : C \to E \).

Considering left comonad-morphisms, one obtains the left 2–category of comonads \( LCOM(B) \).

In [35], an alternative 2–category of comonads is proposed, which contains the same 0–cells and 1–cells, but different 2–cells. In [35, p. 249] there were given two equivalent descriptions of the 2–cells, called the reduced and the unreduced form. In [9, Proposition 2.2], where the dual case for \( B = \text{Bim} \), the bicategory of bimodules, was considered, the unreduced form is being interpreted as a morphism of bicomodules. We think that the treatment of 2–cells as
Lemma 2.5. Let \((q, \alpha)\) and \((q', \alpha')\) be two right comonad-morphisms from \(\mathcal{D}\) to \(\mathcal{C}\) in \(\mathcal{B}\). There exists a bijective correspondence between the following objects:

(i) a 2–cell \(\sigma : d \bullet_B q \rightarrow q'\) in \(\mathcal{B}\) making the following diagram commutative

\[
\begin{array}{ccc}
  d \bullet_B q & \xrightarrow{\Delta \bullet_B q} & d \bullet_B d \bullet_B q \\
\end{array}
\]

(ii) a 2–cell \(\tilde{\sigma} : d \bullet_B q \rightarrow q' \bullet_A c\) in \(\mathcal{B}\) making the following two diagrams commutative

\[
\begin{array}{ccc}
  d \bullet_B q & \xrightarrow{\tilde{\sigma}} & q' \bullet_A c \\
\end{array}
\]

(iii) a \(\mathcal{D}\)-\(\mathcal{C}\)–bicomodule morphism \(\tilde{\sigma} : d \bullet_B q \rightarrow d \bullet_B q'\), where the \(\mathcal{D}\)-\(\mathcal{C}\)–bicomodule structures on \(d \bullet_B q\) and \(d \bullet_B q'\) are obtained from Lemma 2.3.

Proof. Let us just give the corresponding formulae. The remaining part of the proof is just a computation. The equivalence of (i) and (ii) can also be deduced as a dualization of [35, page 249], and the equivalence of (i) and (iii) as a formalization of [9, Proposition 2.2]. Take \(\sigma\) as in statement (i), then define

\[
\tilde{\sigma} = (\sigma \bullet_A c) \circ (d \bullet_B \alpha) \circ (\Delta \bullet_B q), \quad \tilde{\sigma} = (d \bullet_B \sigma) \circ (\Delta \bullet_B q).
\]

Conversely, if \(\tilde{\sigma}\) or \(\tilde{\sigma}\) are given, we can define

\[
\sigma = (\varepsilon \bullet_B q') \circ \tilde{\sigma}, \quad \sigma = (q' \bullet_A \varepsilon) \circ \tilde{\sigma}.
\]

Definition 2.6. We denote by REM(\(\mathcal{B}\)) the 2–category whose 0–cells and 1–cells are precisely those of RCOM(\(\mathcal{B}\)) and whose 2–cells are the 2–cells \(\sigma\) in \(\mathcal{B}\) that satisfy the condition of Lemma 2.5 (i). Similarly one introduces the left-handed version \(\text{LEM}(\mathcal{B})\).

There exist a locally faithful pseudo functor

\[
(7) \quad U : \text{REM}(\mathcal{B}) \rightarrow \mathcal{B};
\]

\[
\mathcal{C} = (A, c, \Delta, \varepsilon) \mapsto A
\]

\[
(q, \alpha) \mapsto d \bullet_B q
\]

\[
\sigma \mapsto \tilde{\sigma} = (d \bullet_B \sigma) \circ (\Delta \bullet_B q)
\]

Of course we can as well introduce the left hand versions of this functor, \(U_L : \text{LEM}(\mathcal{B})^{\text{op}} \rightarrow \mathcal{B}\).
Theorem 2.7. The pseudo functor $U$ from (7) has locally a right adjoint.

Proof. Consider two comonads $\mathcal{C} = (A, c, \Delta_c, \varepsilon_c)$ and $\mathcal{D} = (B, d, \Delta_d, \varepsilon_d)$ in $\mathcal{B}$. The pseudo functor $U$ evaluated at $\mathcal{D}$ and $\mathcal{C}$ induces a functor

$$U_{\mathcal{D}, \mathcal{C}}: \hom_{\mathcal{REM}(\mathcal{B})}(\mathcal{D}, \mathcal{C}) \rightarrow \hom_{\mathcal{B}}(B, A)$$

which has a right adjoint defined as follows

$$V_{\mathcal{D}, \mathcal{C}}: \hom_{\mathcal{B}}(B, A) \rightarrow \hom_{\mathcal{REM}(\mathcal{B})}(\mathcal{D}, \mathcal{C})$$

$$m \mapsto (m \bullet_A c, \varepsilon_d \bullet_B m \bullet_B \Delta_c)$$

The unit $\theta$ and counit $\epsilon$ of the adjunction are given as follows

$$\theta_q = (d \bullet_B \alpha) \circ (\Delta_d \bullet_B q) : d \bullet_B q \rightarrow d \bullet_B q \bullet_c$$

for all $(q, \alpha) \in \hom_{\mathcal{REM}(\mathcal{B})}(\mathcal{D}, \mathcal{C})$

$$\epsilon_m = m \bullet_A \varepsilon_c : m \bullet_A c \rightarrow m$$

for all $m \in \mathcal{Rcom}(B, A)$

In view of Lemma 2.3, $\theta_q$ is given exactly by the right $\mathcal{C}$-comodule structure on $d \bullet_B q$.

Considering a trivial comonad $(\Omega, \Omega, \Omega, \Omega)$, as in Example 2.2, we obtain from Theorem 2.7 immediately the following well-known result.

Corollary 2.8. Let $\mathcal{C} = (A, c, \Delta_c, \varepsilon_c)$ be a comonad in $\mathcal{B}$ and $\Omega$ any 0-object in $\mathcal{B}$. Then the forgetful functor $\mathcal{Rcom}(\Omega, \mathcal{C}) \rightarrow \hom(\Omega, A)$ has a right adjoint given by $- \bullet_A \varepsilon : \hom(\Omega, A) \rightarrow \mathcal{Rcom}(\Omega, \mathcal{C})$.

Although the definition of a comonad is perfectly left-right symmetric, we have two different possibilities for the definition of the 2-category of comonads, both in the original (LCOM and RCOM) and in the modified case (LEM and REM). This is due to the asymmetry in the definition of comonad-morphisms. The following proposition shows that there exists a ‘local’ duality between the left and right versions, however in general no duality can be obtained for the whole 2-categories.

Recall that an adjoint pair in $\mathcal{B}$ is a sextuple $p = (A, B, p, q, \mu, \eta)$ where $A$ and $B$ are 0-cells, $p \in \hom_1(A, B)$, $q \in \hom_1(B, A)$, $\mu \in A\hom_2(p \bullet_B q, A)$ and $\eta \in B\hom_2(B, q \bullet_A p)$, such that the following diagrams commute

\begin{align*}
(8) & \quad q \quad \cong \quad B \bullet_B q \\
& \quad q \bullet_A A \quad \cong \quad q \bullet_A p \bullet_B q
\end{align*}

\begin{align*}
(9) & \quad p \quad \cong \quad p \bullet_B B \\
& \quad A \bullet_A p \quad \cong \quad p \bullet_B q \bullet_A p
\end{align*}

If $(A, B, p, q, \mu, \eta)$ is an adjoint pair in $\mathcal{B}$, then we have the following monad and comonad

$$(B, q \bullet_A p, q \bullet_A \mu \bullet_A p, \eta), \quad (A, p \bullet_B q, q \bullet_B \eta \bullet_B p, \mu)$$

More generally, given a monad $\mathfrak{r} = (A, r, \mu_r, \eta_r)$ and a comonad $\mathfrak{D} = (B, d, \Delta_d, \varepsilon_d)$, we can construct from the adjoint pair a new monad

$$(B, q \bullet_A r \bullet_A p, q \bullet_A \mu_r \bullet_A p \circ (q \bullet_A \eta_r \bullet_A p), (q \bullet_A \eta_r \bullet_A p) \circ \eta)$$

and comonad

$$(A, p \bullet_B d \bullet_B q, (p \bullet_B \eta_d \bullet_B d \bullet_B q) \circ p \bullet_B \Delta_d \bullet_B q, \mu \circ (p \bullet_B \varepsilon_d \bullet_B q)).$$

The following proposition generalizes \cite[Proposition 2.1]{30}. The equivalence between (ii) and (iii) is already stated in \cite[Theorem 9]{44}.
Proposition 2.9. Suppose \( p = (A, B, p, q, \mu, \eta) \) is an adjoint pair, \( \mathcal{C} = (A, c, \Delta_c, \varepsilon_c) \) and \( \mathcal{D} = (B, d, \Delta_d, \varepsilon_d) \) are two comonads in \( \mathcal{B} \). Then the following statements are equivalent

(i) There exists a homomorphism of \( A \)-comonads \( \varphi \in A \text{Hom}_2^A(p \bullet_B d \bullet_B q, c) \);
(ii) there exists an \( \alpha \in B \text{Hom}_2^A(d \bullet_B q, q \circ_A c) \), such that \((q, \alpha)\) is a right comonad-morphism from \( \mathcal{D} \) to \( \mathcal{C} \);
(iii) there exists a \( \beta \in A \text{Hom}_2^B(p \bullet_B d, c \circ_A p) \), such that \((p, \beta)\) is a left comonad-morphism from \( \mathcal{D} \) to \( \mathcal{C} \).

When these equivalent conditions hold, we say that \((p, q)\) is a comonad-morphism with adjunction from \( \mathcal{D} \) to \( \mathcal{C} \).

Proof. (i) \( \Rightarrow \) (ii) Suppose there exists a morphism of comonads \( \varphi \) as in the statement, then we define \( \alpha : q \bullet_B d \rightarrow c \circ_A q \) as follows

\[
\alpha : d \bullet_B q \xrightarrow{\eta_B d \bullet_B q} q \circ_A p \bullet_B d \bullet_B q \xrightarrow{\eta_A \varphi} q \circ_A c.
\]

Consider the following diagram.

The outer diagram expresses the first condition for \( \alpha \) to be a right comonad-morphism. The upper part of the diagram commutes by application of (4), the lower part expresses that \( \varphi \) is a morphism of comonads. The second condition can be computed as follows

\[
(g \circ_A \varepsilon_c) \circ \alpha = (q \circ_A \varepsilon_c) \circ (q \circ_A \varphi) \circ (\eta_B d \bullet_B q)
\]

\[
= (q \circ_A \mu) \circ (q \circ_A \mu) \circ (\varepsilon_B d \bullet_B q)
\]

\[
= (q \circ_A \mu) \circ (\varepsilon_B q) \circ (\varepsilon_B d \bullet_B q)
\]

\[
= \varepsilon_B d \bullet_B q.
\]

Here we used in the second equality the counit condition of the morphism of comonads \( \varphi \), third equation is again an application of (4) and the last equation follows from the fact that \( p \) is an adjoint pair in \( \mathcal{B} \).

(ii) \( \Rightarrow \) (i) Suppose \( \alpha \) exists as in the statement, then we define

\[
\varphi : p \bullet_B d \bullet_A q \xrightarrow{\eta_B^A} p \bullet_B q \circ_A c \xrightarrow{\eta_A^c} c.
\]
We have to check that $\varphi$ is a morphism of comonads. The following quadrangle expresses the counit condition on $\varphi$.

The inner triangle commutes because of the counit condition on $\varphi$, while the inner quadrangle commutes by (4). Next, we verify the compatibility of $\alpha$ with the comultiplication maps.

The second, third and penultimate equality are applications of (4), the fourth one follows from the condition on the adjoint pair $p$. Since every adjoint pair $(q, \alpha)$ determines in $\text{Hom}_B(\Omega, A)$ a natural isomorphism $\theta : A \rightarrow B$, we find that $\theta_A : p \rightarrow p'$ is an isomorphism. Thus, the adjoint 1–cell of a comonad-morphism $(q, \alpha)$ is, up to isomorphism, uniquely determined in $\text{Hom}_B(A, B)$.

The equivalence $(i) \Leftrightarrow (iii)$ follows by left-right duality from $(i) \Leftrightarrow (ii)$. \hfill $\square$

**Definition 2.10.** The 2–category $\text{fREM}(B)$ consists of the following objects

- 0–cells are comonads;
- a 1–cell from $\mathcal{C}$ (over $A$) to $\mathcal{D}$ (over $B$) consists of an adjoint pair $(A, B, p, q, \mu, \eta)$ in $\mathcal{B}$ together with a right comonad-morphism from $\mathcal{D}$ to $\mathcal{C}$ of the form $(q, \alpha)$;
- a 2–cell in $\text{fREM}(B)$ is a 2–cell $\sigma$ in $\mathcal{B}$ that satisfies the conditions of Lemma 2.9 (i).

To define the composition of 1–cells, we use the composition of comonad-morphisms as in $\text{RCOM}(\mathcal{B})$ together with the usual composition of adjoint pairs.

And similar for $\text{fLEM}(B)$.

**Remark 2.11.** Consider two 1–cells in $\text{fREM}(B)$ from $\mathcal{D}$ to $\mathcal{C}$, consisting of the same comonad-morphism $(q, \alpha)$ together with an adjoint pair $(A, B, p, q, \mu, \eta)$, respectively $(A, B, p', q, \mu', \eta')$. Since every adjoint pair $(A, B, p, q, \mu, \eta)$ induces a pair of adjoint functors $- \otimes_B q \dashv - \otimes_A p$ between the categories $\text{Hom}_B(\Omega, A)$ and $\text{Hom}_B(\Omega, B)$ for each 0–cell $\Omega$ in $\mathcal{B}$, we find by the uniqueness of adjoint functors a natural isomorphism $\theta : - \otimes_A p \rightarrow - \otimes_A p'$ of functors from $\text{Hom}_B(\Omega, A)$ to $\text{Hom}_B(\Omega, B)$. Taking $\Omega = A$, we find that $\theta_A : p \rightarrow p'$ is an isomorphism. Thus, the adjoint 1–cell of a comonad-morphism $(q, \alpha)$ is, up to isomorphism, uniquely determined in $\text{Hom}_B(A, B)$.

From Proposition 2.9 we obtain the following generalization of [9 Proposition 2.5] to our setting.

**Proposition 2.12.** There is a biequivalence between $\text{fREM}(B)$ and $\text{fLEM}(B)^{co}$. 

\[\begin{array}{c}
\text{p} \bullet_B d \bullet_B q \overset{\mu \circ \eta}{\longrightarrow} p \bullet_B q \bullet_A c \overset{\mu \circ \eta}{\longrightarrow} c \\
\begin{array}{c}
\text{p} \bullet_B q \\
\mu \\
\end{array}
\end{array}\]
Proof. Consider a 1–cell from \( \mathcal{D} \) to \( \mathcal{C} \) in \( \text{fREM}(\mathcal{B}) \) consisting of a right comonad-morphism \((q, \alpha)\), together with an adjoint pair \((A, B, p, q, \mu, \eta)\). Then we know by Proposition 2.9 that we can construct a left comonad-morphism \((p, \beta)\), hence a 1–cell in \( \text{fLEM}(\mathcal{B}) \).

If we take now a 2–cell \( \sigma \) in \( \text{fREM}(\mathcal{B}) \) connecting the 1–cell \((q, \alpha)\) with \((A, B, p, q, \mu, \eta)\) and \((q', \alpha')\) with \((A, B, p', q', \mu', \eta')\), then \( \sigma \) is a 2–cell in \( \mathcal{B} \) of the form \( d \cdot_B q \rightarrow q' \). We define now \( \tau \) as a 2–cell in \( \text{fLEM}(\mathcal{B}) \) from the left comonad-morphism \((p, \beta)\) to \((p', \beta')\) as follows:

\[
\tau : p' \cdot_B d \cong p \cdot_B d \cdot_B B \xrightarrow{p' \cdot_B d \cdot_B p' \cdot_B \rho} p' \cdot_B d \cdot_B q \cdot_A p \xrightarrow{p' \cdot_B \sigma \cdot_B A p} A \cdot_A p \cong p.
\]

The remaining details are left to the reader. \(\square\)

Consider 1–cells \( m, n : A \rightarrow B \) and 2–cells \( \alpha, \beta : m \rightarrow n \). Suppose that the equalizer \((e, \epsilon)\) of \( \alpha \) and \( \beta \) exists in \( \text{Hom}(A, B) \),

\[
e \xrightarrow{\epsilon} m \xrightarrow{\alpha} n \xrightarrow{\beta} n.
\]

Let \( a : B \rightarrow C \) be any 1–cell. We say that the equalizer \((e, \epsilon)\) is right \( a \)-pure if and only if the following diagram is an equalizer in \( \text{Hom}(A, C) \):

\[
e \cdot_B a \xrightarrow{\epsilon \cdot_B a} m \cdot_B a \xrightarrow{\alpha \cdot_B a} n \cdot_B a.
\]

If an equalizer is right \( a \)-pure for all choices of \( a \), then we just say that the equalizer is right pure. Similarly, let \( b : C \rightarrow A \) be a 1–cell. We say that the equalizer \((e, \epsilon)\) is left \( b \)-pure if and only if the following diagram

\[
b \cdot_A e \xrightarrow{b \cdot_A \epsilon} b \cdot_A m \xrightarrow{b \cdot_A \alpha} b \cdot_A n
\]

is an equalizer in \( \text{Hom}(C, A) \).

Lemma 2.13. Consider a comonad \( \mathcal{D} = (B, d, \Delta_d, \varepsilon_d) \) and a 0–cell \( \Omega \) in \( \mathcal{B} \). Take \( f, g : (m, \rho^m) \rightarrow (n, \rho^n) \in \text{Rcom}(\Omega, \mathcal{D}) \). Suppose that the equalizer \((q, \epsilon)\) of the pair \((f, g)\) exists in \( \text{Hom}(\Omega, B) \) (i.e. after applying the forgetful functor \( U_{\Omega, \mathcal{D}} : \text{Rcom}(\Omega, \mathcal{D}) \rightarrow \text{Hom}(\Omega, B) \)).

Then the following statements are equivalent

(i) \((q, \epsilon)\) is right \( d \cdot_B d \)-pure;

(ii) the following equalizers exist in \( \text{Rcom}(\Omega, \mathcal{D}) \) and are preserved by the forgetful functor \( U_{\Omega, \mathcal{D}} : \text{Rcom}(\Omega, \mathcal{D}) \rightarrow \text{Hom}(\Omega, B) \).

\[
q \xrightarrow{\epsilon} m \xrightarrow{f} n
\]

\[
q \cdot_B d \xrightarrow{\epsilon \cdot_B d} m \cdot_B d \xrightarrow{f \cdot_B d} n \cdot_B d
\]

\[
q \cdot_B d \cdot_B d \xrightarrow{\epsilon \cdot_B d \cdot_B d} m \cdot_B d \cdot_B d \xrightarrow{f \cdot_B d \cdot_B d} n \cdot_B d \cdot_B d
\]
Proof. We only prove the implication $(i) \Rightarrow (ii)$, the converse is easy. First remark that if an equalizer \( q \rightarrow m \xrightarrow{f \circ g} n \) is right \( d \bullet_B d \)-pure, then this equalizer is also right \( d \)-pure. This follows easily from the following diagram, once we observe that \( \Delta \) is right \( d \)-pure.

\[
\begin{array}{c}
q \bullet_B d \xrightarrow{\epsilon_B d} m \bullet_B d \xrightarrow{f \circ g_B d} n \bullet_B d \\
\end{array}
\]

We know by assumption that the equalizer \( \text{Rcom}(\Omega, \mathcal{D}) \) exists in \( \text{Hom}(\Omega, B) \). Moreover, the functor \( - \bullet_B d : \text{Hom}(\Omega, B) \rightarrow \text{Rcom}(\Omega, \mathcal{D}) \) preserves equalizers, since it is the right adjoint to the forgetful functor \( U_{\Omega, \mathcal{D}} \). Therefore the equalizer \( \text{Rcom}(\Omega, \mathcal{D}) \) exists in \( \text{Rcom}(\Omega, \mathcal{D}) \) and is preserved by the forgetful functor. Applying this argument a second time we obtain the same result for the equalizer \( \text{Rcom}(\Omega, \mathcal{D}) \).

From the \( d \)-purity we obtain the following commutative diagram, where horizontal rows are equalizers.

\[
\begin{array}{c}
q \xrightarrow{\epsilon} m \xrightarrow{f} n \\
q \bullet_B d \xrightarrow{\epsilon_B d} m \bullet_B d \xrightarrow{f \circ g_B d} n \bullet_B d \\
\end{array}
\]

From the universal property of the equalizer, we therefore obtain a 2–cell \( \rho : q \rightarrow q \bullet_B d \). One can easily prove that \( (q \bullet_B d \circ \rho) = q \) and the coassociativity of this coaction follows from the right \( d \bullet_B d \)-purity.

Let us prove that \( ((q, \rho), \epsilon) \) is an equalizer in \( \text{Rcom}(\Omega, \mathcal{D}) \). Suppose there exists a morphism \( \kappa : (q', \rho') \rightarrow (m, \rho^m) \in \text{Rcom}(\Omega, \mathcal{D}) \) such that \( f \circ \kappa = g \circ \kappa \). Then we know that there exists a 2–cell \( \lambda \in \text{Hom}_B(q', q) \) such that \( \kappa = \epsilon \circ \lambda \). Let us check that \( \lambda \) is right \( \mathcal{D} \)-colinear. We use the defining property of \( \rho \) in the first equation, the relation between \( \kappa \) and \( \lambda \) in the second and last equality and the right \( \mathcal{D} \)-colinearity of \( \kappa \) in the third one.

\[
(\epsilon_B d) \circ \rho \circ \lambda = \rho^m \circ \epsilon \circ \lambda = \rho^m \circ \kappa
\]

From this we obtain that \( \rho \circ \lambda = (\lambda \bullet_B d) \circ \rho' \) since \( \epsilon_B d \) is a monomorphism in \( \text{Hom}(\Omega, B) \), as \( (q \bullet_B d, \epsilon_B d) \) is an equalizer.

\[\square\]

**Theorem 2.14.** Consider a comonad \( \mathcal{D} = (B, d, \Delta_d, \varepsilon_d) \) and a 0–cell \( \Omega \) in \( \mathcal{B} \). If \( \text{Hom}(\Omega, B) \) has all equalizers, then the following statements are equivalent.

(i) \( T = - \bullet_B d : \text{Hom}(\Omega, B) \rightarrow \text{Hom}(\Omega, B) \) preserves equalizers;

(ii) \( \text{Rcom}(\Omega, \mathcal{D}) \) has all equalizers, and they are preserved by the forgetful functor \( U_{\Omega, \mathcal{D}} : \text{Rcom}(\Omega, \mathcal{D}) \rightarrow \text{Hom}(\Omega, B) \);

(iii) the forgetful functor \( U_{\Omega, \mathcal{D}} : \text{Rcom}(\Omega, \mathcal{D}) \rightarrow \text{Hom}(\Omega, B) \) preserves equalizers.

**Proof.** (i) \( \Rightarrow \) (ii). Consider two parallel morphisms \( f, g : (m, \rho^m) \rightarrow (n, \rho^n) \) in \( \text{Rcom}(\Omega, \mathcal{D}) \), and construct the equalizer \( (q, \epsilon) \) of \( (f, g) \) in \( \text{Hom}(\Omega, \mathcal{D}) \). Since \( T \) preserves all equalizers, this equalizer is \( d \bullet_B d \)-pure. Therefore, we can apply Lemma 2.13 and we obtain that \( (q, \epsilon) \)
is also an equalizer in $\text{RCom}(\Omega, \mathcal{D})$.

$(ii) \Rightarrow (iii)$. Trivial.

$(iii) \Rightarrow (i)$. Consider 2–cells $f, g \in \Omega\text{Hom}^B_\Omega(m, n)$, and let $(q, e)$ be the equalizer of $(f, g)$ in $\text{Hom}(\Omega, B)$. The functor $- \cdot_B d : \text{Hom}(\Omega, B) \to \text{RCom}(\Omega, \mathcal{D})$ preserves equalizers, since it is the right adjoint to the forgetful functor $U_{\Omega, \mathcal{D}}$. As the forgetful functor preserves equalizers as well, we find that $T = U_{\Omega, \mathcal{D}} \circ (- \cdot_B d)$ preserves equalizers. □

Consider now a comonad $\mathcal{C} = (A, c, \Delta_c, \varepsilon_c)$, a right $\mathcal{C}$–comodule of $C$–type $(m, \rho^m)$ and a left $\mathcal{C}$–comodule of $B$–type $(n, \lambda^n)$ for all 0–cells $B$ and $C$. If it exists, then we will denote the equalizer of $\rho^m \bullet_A n$ and $m \bullet_A \Delta_c \lambda^n$ (in $\text{Hom}(C, B)$) by $m \bullet^e n$ and we call this the cotensor product,

$$m \bullet^e n \longrightarrow m \bullet n \xrightarrow{\rho^m \bullet_A n} m \bullet_A \Delta_c \lambda^n.$$  

Consider now two other comonads $\mathcal{D} = (B, d, \Delta_d, \varepsilon_d)$ and $\mathcal{E} = (C, e, \Delta_e, \varepsilon_e)$. Suppose that $(n, \lambda^n, \rho^n)$ is a $\mathcal{C}-\mathcal{D}$–bicomodule and $(m, \lambda^m, \rho^m)$ is an $\mathcal{E}-\mathcal{C}$–bicomodule.

**Corollary 2.15.** With notation as introduced above, suppose that the equalizer $m \bullet^e n$ exists in $\text{Hom}(C, B)$.

$(i)$ If the cotensor product $m \bullet^e n$ is right $d \cdot_B \bullet^e d$–pure, then $m \bullet^e n$ is a right $\mathcal{D}$–comodule.

$(ii)$ If the cotensor product $m \bullet^e n$ is left $e \bullet_C \cdot e$–pure, then $m \bullet^e n$ is a left $\mathcal{E}$–comodule.

$(iii)$ If the cotensor product $m \bullet^e n$ is both right $d \cdot_B \bullet^e d$–pure and left $e \bullet_C \cdot e$–pure, then $m \bullet^e n$ is a $\mathcal{C}-\mathcal{D}$–bicomodule.

Consider a 2–category $\mathcal{B}$ such that for every pentuple of comonads $\mathcal{C}$ (over $A$), $\mathcal{D}$ (over $B$), $\mathcal{E}$ (over $C$), $\mathcal{F}$ (over $D$) and $\mathcal{G}$ (over $E$) and for all $\mathcal{E}-\mathcal{C}$–bicomodule $m$ and every $\mathcal{C}-\mathcal{D}$–bicomodule $n$, the cotensor product $m \bullet^e n$ exists and is left $p$–pure and right $q$–pure for any $\mathcal{F}-\mathcal{E}$–bicomodule $p$ and any $\mathcal{D}-\mathcal{G}$–bicomodule $q$. We can now construct a new bicategory $\text{Bic}(\mathcal{B})$ out of $\mathcal{B}$:

- A 0–cell in $\text{Bic}(\mathcal{B})$ is a comonad in $\mathcal{B}$;
- a 1–cell $\mathcal{C} \rightarrow \mathcal{D}$ in $\text{Bic}(\mathcal{B})$ is a $\mathcal{C}-\mathcal{D}$–bicomodule;
- a 2–cell $\varphi$ in $\text{Bic}(\mathcal{B})$ between two $\mathcal{C}-\mathcal{D}$–bicomodules, is a left $\mathcal{C}$– and a right $\mathcal{D}$–colinear morphism.

Composition of 1–cells is given by the cotensor product, unit elements are the comonads, considered as bicomodules over themselves.

**Remark 2.16.** It follows from Corollary 2.15 that the composition of 1–cells in $\text{Bic}(\mathcal{B})$ is well-defined. Moreover the composition of 1-cells is also associative (up to isomorphism), as we can see as follows. Consider four comonads in $\mathcal{B}$: $\mathcal{C}$ (over $A$), $\mathcal{D}$ (over $B$), $\mathcal{E}$ (over $C$), $\mathcal{F}$ (over $D$) together with a $\mathcal{C}-\mathcal{D}$–bicomodule $m$, a $\mathcal{D}-\mathcal{E}$–bicomodule $n$ and a $\mathcal{E}-\mathcal{F}$–bicomodule $p$. Then we need an isomorphism of the form

$$(m \bullet^d n) \bullet^e p \cong m \bullet^d (n \bullet^e p).$$
Consider the following diagram

\[
\begin{array}{ccc}
(m \cdot d n) \cdot^e p & \xrightarrow{\psi_1} & (m \cdot_B n) \cdot^e p \\
\downarrow & & \downarrow \\
(m \cdot d \cdot_B n) \cdot^e p & \xrightarrow{\psi_2} & (m \cdot_B d \cdot_B n) \cdot^e p
\end{array}
\]

The bottom row is exact by the definition of the cotensor product. The exactness of the top row can be deduced from the fact that \( m \cdot^e n \) is right \( p^- \), right \( e^- \) and right \( e \cdot_C p^- \)-pure. Furthermore we know by the left \( m^- \) and left \( m \cdot_B d^- \)-purity of \( n \cdot^e p \) that \( \psi_2 \) and \( \psi_3 \) are isomorphisms, hence \( \psi_1 \) is an isomorphism as well by the universal property of the equalizer.

An example of a bicategory \( \mathcal{B} \) as above is given by the bicategory of bimodules over division algebras.

**Theorem 2.17.** Let \( \mathcal{B} \) be a 2–category as above. Then there exist locally full and locally faithful pseudo functors

\[
F : \text{REM}(\mathcal{B}) \to \text{Bic}(\mathcal{B}), \quad G : \text{LEM}(\mathcal{B})^{co} \to \text{Bic}(\mathcal{B}).
\]

**Proof.** Since both the 0–cells in \( \text{REM}(\mathcal{B}) \) and \( \text{Bic}(\mathcal{B}) \) are comonads, we can define \( F \) to be the identity on 0–cells. A 1–cell in \( \text{REM}(\mathcal{B}) \) is a right comonad-morphism \((q, \alpha) : \Delta \to \mathcal{C}\). By Lemma 2.3 \( d \cdot_B q \) is a \( \Delta \cdot \mathcal{C} \)–bicomodule, thus we define \( F(q, \alpha) = d \cdot_B q \). A 2–cell in \( \text{REM}(\mathcal{B}) \) is a comonad transformation \( \sigma : (q, \alpha) \to (q', \alpha') \). We know from Lemma 2.5 that the second unreduced form \( \hat{\sigma} \) of \( \sigma \) is a bicomodule morphism from \( F(q, \alpha) = d \cdot_B q \) to \( F(q', \alpha') = d \cdot_B q' \), thus we define \( F(\sigma) = \hat{\sigma} \).

Finally we have to find natural isomorphisms \( \gamma (5) \) and \( \delta (6) \). To this end, consider a comonad-morphism \((q, \alpha) : \mathcal{C} \to \Delta \) and a comonad-morphism \((q', \alpha') : \mathcal{C} \to \mathcal{C}\). We need to prove that \( F(q) \cdot^d F(q') \cong F(q \cdot_B q') \). Indeed:

\[
(e \cdot_C q) \cdot^d (d \cdot_B q') \cong ((e \cdot_C q) \cdot^d d) \cdot_B q' \cong e \cdot_C q \cdot_B q'
\]

where the first isomorphism is a consequence of the purity conditions. The property for \( \delta \) is trivial, the unit objects in both categories are comonads.

That the pseudo functors \( F \) and \( G \) are locally full and locally faithful follows from the description of 2–cells in \( \text{REM}(\mathcal{B}) \) as bicomodule-morphisms in Lemma 2.5. \( \square \)

### 3. Equivalences between Comodule categories

In this section we will show that the theory developed in [9] for the bicategory \( \text{Bim} \) of bimodules, can be established in any 2–category \( \mathcal{B} \), and, since every bicategory is biequivalent to a suitable 2–category, it is already extended to any bicategory (see Section [1]). In fact, the results of [9, §5] are improved in what follows even in the case \( \mathcal{B} = \text{Bim} \).

#### 3.1. Push-out and pull-back functors

Given comonads \( \Delta = (B, d, \Delta_d, \varepsilon_d) \) and \( \mathcal{C} = (A, c, \Delta_c, \varepsilon_c) \) in the 2–category \( \mathcal{B} \), every comonad-morphism \((q, \alpha) \) from \( \Delta \) to \( \mathcal{C} \) defines, for each object \( \Omega \) of \( \mathcal{B} \), a functor

\[
\mathcal{D} : \text{Rcom}(\Omega, \Delta) \to \text{Rcom}(\Omega, \mathcal{C}),
\]

given by \( \mathcal{D}(m, \rho) = (m \cdot_B q, (m \cdot_B \alpha) \circ (\rho \cdot_B q)) \), for any comodule \((m, \rho) \in \text{Rcom}(\Omega, \Delta) \) and \( \mathcal{D}(\phi) = \phi \cdot_B q \) for any morphism of comodules \( \phi \in \text{Rcom}(\Omega, \Delta) \). This functor makes commute...
the diagram

\[
\begin{array}{c}
\text{Rcom}(\Omega, \mathcal{D}) \xrightarrow{\mathcal{Q}} \text{Rcom}(\Omega, \mathcal{C}) \\
\downarrow \text{U}_{\Omega, \mathcal{D}} \quad \quad \quad \quad \quad \quad \quad \quad \downarrow \text{U}_{\Omega, \mathcal{C}} \\
\text{Hom}(\Omega, B) \xrightarrow{\bullet_B q} \text{Hom}(\Omega, A).
\end{array}
\]

Conversely, if \( \mathcal{Q} : \text{Rcom}(B, \mathcal{D}) \to \text{Rcom}(B, \mathcal{C}) \) is a functor making commute (15) for \( \Omega = B \) and some 1–cell \( q : B \to A \), then there is a comonad-morphism \( (q, \alpha) \) which induces \( \mathcal{Q} \). In fact, \( \mathcal{Q}(d) = d \bullet_B q \) as an object in \( \text{Hom}(B, A) \). Therefore, we find that \( d \bullet_B q \) possesses a right \( \mathcal{C} \)-comodule structure and, from the functoriality of \( \mathcal{Q} \), it follows that \( \mathcal{Q}(\Delta_d) \) must be a homomorphism of right \( \mathcal{C} \)-comodules, hence \( d \bullet_B q \) is even an \( \mathcal{D} \)-\( \mathcal{C} \)-bicomodule. By Lemma 2.3 we obtain a comonad-morphism from \( \mathcal{D} \) to \( \mathcal{C} \) of the form \( (q, \alpha) \).

**Definition 3.1.** Inspired by the terminology of [9], we will call the functor \( \mathcal{Q} \) the *pushout* functor associated to the comonad morphism \( (q, \alpha) \).

It is a natural question to pose whether the pushout functor has a right adjoint. Generalizing results from [9] and [30], we find a criterion for this to hold in the ‘finite case’. Consider a left comonad-morphism \( (p, \beta) \) from \( \mathcal{D} \) to \( \mathcal{C} \). We know from Lemma 2.3 that \( p \bullet B d \) is a \( \mathcal{C} \)-\( \mathcal{D} \)-bicomodule. In this note we will say that the category \( \text{Rcom}(\Omega, \mathcal{D}) \) satisfies the *equalizer condition for \( p \) if for all comodules \( (n, \rho) \) in \( \text{Rcom}(\Omega, \mathcal{C}) \) the equalizer of \( \rho_{n, c} p_B d \) and \( \rho_{n = c}(p \bullet_B d) \) exists in \( \text{Rcom}(\Omega, \mathcal{D}) \). In view of Corollary 2.15, a sufficient condition for this to be satisfied is that the following cotensor product exists in \( \text{Hom}(\Omega, B) \) and is right \( d \bullet B d \)-pure.

\[
(16) \quad n \bullet^c (p \bullet_B d) \xrightarrow{\text{eq}} n \bullet_A (p \bullet_B d) \xrightarrow{\rho_{n, c} p_B d} n \bullet_A c \bullet_A (p \bullet_B d).
\]

For this reason, we will denote the equalizer of \( \rho_{n, c} p_B d \) and \( \rho_{n = c}(p \bullet_B d) \) in \( \text{Rcom}(\Omega, \mathcal{D}) \) by \( n \bullet^c (p \bullet_B d) \), \( \rho_{n = c}(p \bullet_B d) \), if \( \text{Rcom}(\Omega, \mathcal{D}) \) satisfies the equalizer condition.

**Proposition 3.2.** Consider comonads \( \mathcal{D} \) and \( \mathcal{C} \) in \( \mathcal{B} \) and take \( p \in \text{Hom}_1(A, B) \).

(1) Let \( \Omega \) be a 0–cell in \( \mathcal{B} \) and \( (p, \beta) \) a left comonad-morphism from \( \mathcal{D} \) to \( \mathcal{C} \). If \( \text{Rcom}(\Omega, \mathcal{D}) \) satisfies the equalizer condition for \( p \), then there exists a functor

\[ \mathcal{P} : \text{Rcom}(\Omega, \mathcal{C}) \to \text{Rcom}(\Omega, \mathcal{D}), \]

such that the following diagram commutes

\[
\begin{array}{c}
\text{Rcom}(\Omega, \mathcal{C}) \xrightarrow{\mathcal{P}} \text{Rcom}(\Omega, \mathcal{D}) \\
\downarrow \text{Hom}(\Omega, A) \quad \quad \quad \quad \quad \quad \quad \quad \downarrow \text{Hom}(\Omega, B) \\
\text{Rcom}(\Omega, \mathcal{C}) \xrightarrow{\mathcal{P}} \text{Rcom}(\Omega, \mathcal{D})
\end{array}
\]

(2) Conversely, if a functor \( \mathcal{P} \) as in part (1) exists for the case \( \Omega = A \), then there exists a left comonad-morphism from \( \mathcal{D} \) to \( \mathcal{C} \) of the form \( (p, \beta) \).

**Proof.** (1). We define the functor \( \mathcal{P} \) by \( \mathcal{P}(n, \rho) = (n \bullet^c (p \bullet_B d), \rho_{n = c}(p \bullet_B d)) \) for all comodules \( (n, \rho) \) in \( \text{Rcom}(\Omega, \mathcal{C}) \).

(2). Consider the element \( A \in \text{Hom}(A, A) \), by the commutativity of diagram (17) we can
compute \( \mathcal{P}(A \bullet_A c) \cong \mathcal{P}(c) \cong p \bullet_B d \) as an object in \( \text{Hom}(A, B) \). Moreover, we find that \( p \bullet_B d \) has a right \( \mathcal{D} \)-comodule structure induced by \( \mathcal{D} \) and a left \( \mathcal{C} \)-comodule structure induced by \( \mathcal{C} \) and the functoriality of \( \mathcal{P} \). It is even a \( \mathcal{C} \)-\( \mathcal{D} \)-bicomodule. Hence we find by Lemma 2.3 a comonad-morphism of the form \((p, \beta)\).

**Definition 3.3.** The functor \( \mathcal{P} \) is called the pullback functor associated to the left comonad-morphism \((p, \beta)\).

Consider the situation where \((p, q) : \mathcal{D} \to \mathcal{C} \) is a comonad-morphism with adjunction. By Proposition 2.9, the existence of the comonad-morphism with adjunction \((p, q)\) implies the existence of right and left comonad-morphisms \((q, \alpha)\) and \((p, \beta)\). Then we obtain for any 0–cell \( \Omega \) in \( \mathcal{B} \) the following diagram of functors

\[
\begin{array}{ccc}
\mathcal{Rcom}(\Omega, \mathcal{C}) & \xleftarrow{\beta} & \mathcal{Rcom}(\Omega, \mathcal{D}) \\
\mathcal{U}_{\Omega, c} & \xleftarrow{\bullet_A c} & \mathcal{U}_{\Omega, d} \\
\mathcal{Hom}(\Omega, A) & \xleftarrow{\bullet_A p} & \mathcal{Hom}(\Omega, B),
\end{array}
\]

where the lower and the vertical pairs of functors are adjoint. The dotted arrow in the upper row only exists if the equalizer condition is satisfied, in that situation the upper pair is also a pair of adjoint functors, as we will show in the next proposition. Moreover, the diagram commutes in the following sense: the outer square and the inner square both commute.

**Proposition 3.4.** Consider a comonad-morphism with adjunction \((p, q) : \mathcal{D} \to \mathcal{C} \). If \( \mathcal{Rcom}(\Omega, \mathcal{D}) \) satisfies the equalizer condition for \( p \), then the pullback functor \( \mathcal{P} \) associated to \( p \) is a right adjoint to the pushout functor \( \mathcal{Q} \) associated to \( q \).

**Proof.** The adjoint pair \((A, B, p, q, \mu, \eta)\) induces an adjoint pair of functors

\[
\begin{array}{ccc}
\mathcal{Hom}(\Omega, B) & \xleftarrow{\bullet_B q} & \mathcal{Hom}(\Omega, A) \\
\mathcal{Hom}(\Omega, A) & \xleftarrow{\bullet_A p} & \mathcal{Hom}(\Omega, B),
\end{array}
\]

Let us consider, for any \( x \in \mathcal{Hom}_1(\Omega, B) \) and \( y \in \mathcal{Hom}_1(\Omega, A) \), the adjointness isomorphism

\[
\Phi^0_{x,y} : \Omega \mathcal{Hom}^A(x \bullet_B q, y) \to \Omega \mathcal{Hom}^B(x, y \bullet_A p).
\]

On the other hand, \( \bullet_B d \) defines a comonad on the category \( \mathcal{Hom}(\Omega, B) \), so that we obtain an adjoint pair

\[
\begin{array}{ccc}
\mathcal{Rcom}(\Omega, \mathcal{D}) & \xleftarrow{U} & \mathcal{Hom}(\Omega, B),
\end{array}
\]

where \( U \) denotes the forgetful functor. The adjunction \( U \dashv - \bullet_B d \) entails a second natural isomorphism

\[
\Phi^1_{x,y} : \Omega \mathcal{Hom}^B(x, z) \to \Omega \mathcal{Hom}^D(x, z \bullet_B d),
\]

where \( x, z \in \mathcal{Hom}_1(\Omega, B) \).
Finally we can construct the following diagram.

\[
\begin{array}{c}
\Omega \text{Hom}^c(x \bullet_B q, y) \\
\downarrow \\
\Omega \text{Hom}^2_A(x \bullet_B q, y) \\
\downarrow \phi_{x,y}^A \\
\Omega \text{Hom}^2_B(x, y \bullet_A p) \\
\downarrow \phi_{x,y}^B \\
\Omega \text{Hom}^2(x, y \bullet_A p \bullet_B d) \\
\|
\end{array}
\]

Now check that the vertical lines are equalizers and that the diagram commutes (following the equally aligned vertical lines). Since both lower horizontal arrows are isomorphisms, we find the existence of an isomorphism in the upper horizontal line as well, by the universal property of the equalizers. This implies the adjunction between the pushout functor \(- \bullet_B q\) and the pullback functor \(- \bullet^c (p \bullet_B d)\).

We state the explicit form of the unit \(\zeta\) and counit \(\nu\) of the adjunction \(\mathcal{D} \dashv \mathcal{P}\). To obtain \(\nu\), consider the equalizer from (16),

\[
y \bullet^c (p \bullet_B d) \xrightarrow{eq} y \bullet_A (p \bullet_B d) \xrightarrow{\nu_y} y \bullet_A c \bullet_A (p \bullet_B d)
\]

and apply the pushout functor \(- \bullet_B q\) on this exact row, then we obtain

\[
(\pi_{p \bullet_B d}) \xrightarrow{eq_{\pi \bullet_B q}} \pi_A (p \bullet_B d) \xrightarrow{\nu_y} \pi_A c \bullet_A (p \bullet_B d) \bullet_B q.
\]

So \(\nu\) is given by the formula \(\nu_y = (y \bullet_A \epsilon) \circ (eq \bullet_B q)\), where \(\epsilon\) denotes the counit of the comonad \(p \bullet_B d \bullet_B q\), i.e. \(\epsilon = \mu \circ (p \bullet_B \varepsilon_d \bullet_B q)\).

To obtain a formula for \(\zeta\), we calculate (16) for \(n = x \bullet_B q\), where \((x, \rho) \in Rcom(\Omega, \mathcal{D})\), then we find the following diagram.

\[
(\pi_{x \bullet_B q}) \xrightarrow{\zeta_x} (x \bullet_B q) \bullet_A (p \bullet_B d) \xrightarrow{(x \bullet_B q \bullet_B d) \circ \rho} (x \bullet_B q) \bullet_A c \bullet_A (p \bullet_B d)
\]

We obtain \(\zeta_x\) by the universal property of the equalizer.

3.2. The canonical morphisms. Consider 0–cells \(\Omega, \Omega'\) and \(B\), two 1–cells \(p \in \text{Hom}_1(\Omega, B)\) and \(q \in \text{Hom}_1(B, \Omega')\), and let \(\mathcal{D} = (B, d, \Delta_d, \varepsilon_d)\) be a comonad in \(\mathcal{B}\). Then \((p \bullet_B d, p \bullet_B \Delta_d) \in Rcom(\Omega, d)\) and \((d \bullet_B q, \Delta_d \bullet_B q) \in Rcom(d, \Omega')\). With this notation, we can state the following lemma.

**Lemma 3.5.** The following isomorphism holds, in particular, the following equalizer (cotensor product) always exists in \(\text{Hom}(\Omega, \Omega')\),

\[
(p \bullet_B d) \bullet^d (d \bullet_B q) \cong p \bullet_B d \bullet_B q.
\]
Proof. We claim that the equalizer of \( p_B \Delta_B \bullet_B \varepsilon_B q \) and \( p_B \Delta_B \bullet_B \eta_B q \) is given by \( (p \bullet_B d \bullet_B q, p_B \Delta_B \bullet_B d \bullet_B q) \). Consider any 2-cell \( \kappa \in \Omega \text{Hom}^B_2 (x, p_B \bullet_B d \bullet_B q) \), such that \( (p_B \Delta_B \bullet_B d \bullet_B q) \circ \kappa = (p_B \bullet_B \Delta_B d \bullet_B q) \circ \kappa \). Then we find that

\[
(p_B \Delta_B \bullet_B q) \circ (p_B \bullet_B d \bullet_B q) \circ \kappa = (p_B \bullet_B d \bullet_B \varepsilon_B q) \circ (p_B \Delta_B \bullet_B d \bullet_B q) \circ \kappa = (p_B \bullet_B d \bullet_B \eta_B q) \circ (p_B \bullet_B \Delta_B d \bullet_B q) \circ \kappa = \kappa,
\]

where we applied (3) in the first equality and the condition on \( \kappa \) in the second one. Consequently, the following diagram commutes

\[
\begin{array}{ccc}
p \bullet_B d \bullet_B q & \xrightarrow{p_B \Delta_B \bullet_B q} & p \bullet_B d \bullet_B q \\
\downarrow \kappa & & \downarrow \kappa \\
x & & x
\end{array}
\]

where \( \lambda = (p_B \bullet_B d \bullet_B \varepsilon_B q) \circ \kappa \). Moreover, for any other \( \lambda' \in \Omega \text{Hom}^B_2 (x, p_B \bullet_B d \bullet_B q) \) such that \( \kappa = (p_B \bullet_B \Delta_B d \bullet_B q) \circ \lambda' \), we find that

\[
\lambda = (p_B \bullet_B d \bullet_B \varepsilon_B q) \circ \kappa = (p_B \bullet_B d \bullet_B \eta_B q) \circ (p_B \bullet_B \Delta_B d \bullet_B q) \circ \lambda' = \lambda',
\]

i.e. \( \lambda \) is the unique 2-cell satisfying this property. \( \square \)

Consider now a comonad-morphism with adjunction \( (p, q) : \mathcal{D} \rightarrow \mathcal{C} \). Denote the associated adjoint pair by \( (A, B, p, q, \eta, \mu) \). We know that \( p \bullet_B d \in \text{Bicom}(\mathcal{C}, \mathcal{D}) \) and \( d \bullet_B q \in \text{Bicom}(\mathcal{D}, \mathcal{C}) \) (see Lemma 2.3). With this notation the following lemma holds.

**Lemma 3.6.** The map \( \gamma = (d_B \bullet_B p_B d) \circ \Delta_B : d \rightarrow d \bullet_B q \bullet_A p \bullet_B d \) satisfies the following equality

\[
(d_B \bullet_B q \bullet_A \lambda_B p_B d) \circ \gamma = (d_B \bullet_B q \bullet_A \mu_B p_B d) \circ \gamma.
\]

**Proof.** Let us denote by \( (q, \alpha) : \mathcal{D} \rightarrow \mathcal{C} \) the right comonad-morphism associated to the comonad-morphism with adjunction \( (p, q) \). Then we know that

\[
p B \bullet_B q = (d_B \bullet_B \alpha) \circ (\Delta_B \bullet_B q);
\]

\[
\lambda_B p_B d = (\mu_B \bullet_B \alpha_B \bullet_B q_B d) \circ (p_B \bullet_B p_B \bullet_B d) \circ (p_B \bullet_B d \bullet_B \eta_B q_B d) \circ (p_B \bullet_B \Delta_B d).
\]

Then we can compute on one hand

\[
(p_B \bullet_B d \bullet_B q_B d) \circ \gamma = (d_B \bullet_B \alpha_B p_B d_B d) \circ (\Delta_B \bullet_B q_B \bullet_A p_B d_B d) \circ (d_B \bullet_B \eta_B q_B d_B d) \circ \Delta_B d = (d_B \bullet_B \alpha_B p_B d_B d) \circ (d_B \bullet_B d \bullet_B \eta_B q_B d_B d) \circ (\Delta_B \bullet_B d_B d) \circ \Delta_B d.
\]
where we used (4) in the second equality. On the other hand, we find
\[(d \bullet_B q_A \mu_A p_B d) \circ \gamma = (d \bullet_B q_A \mu_A p_B d) \circ (d \bullet_B q_A p_B p_B d) \circ (d \bullet_B \eta_B d) \circ \Delta_d = (d \bullet_B q_A \mu_A p_B p_B d) \circ (d \bullet_B \eta_B d) \circ \Delta_d = (d \bullet_B \alpha_A p_B B d) \circ (d \bullet_B \eta_B d) \circ \Delta_d \circ \Delta_d.
\]

Here we used (4) in the second and third equality, the coassociativity of \(D\) as well in the third equality and (8) in the last equality. □

**Definition 3.7.** Consider a comonad-morphism with adjunction \((p, q) : D \to C\). Suppose that the cotensor product \((d \bullet_B q) \circ c \circ (p \bullet_B d)\) exists in \(\text{Hom}(B, B)\). The following canonical 2–cell is well-defined by Proposition 2.9 and Lemma 3.5
\[
\text{can} : (p \bullet_B d) \bullet_B (d \bullet_B q) \cong p \bullet_B d \bullet_B q \rightarrow c.
\]

Lemma 3.6 together with the universal property of the equalizer, induces a well-defined 2–cell \(\text{can} : d \rightarrow (d \bullet_B q) \circ c \circ (p \bullet_B d)\).

If both \(\text{can}\) and \(\overline{\text{can}}\) are isomorphisms, then we say that \((p, q)\) is a \(D\-C\) Galois comonad-morphism.

### 3.3. Structure theorems

We will now give necessary and sufficient conditions for the pullback and pushout functor to be full and faithful, and then to obtain an equivalence between the categories \(\text{Rcom}(\Omega, C)\) and \(\text{Rcom}(\Omega, D)\).

Consider the following diagram in \(\text{Rcom}(\Omega, C)\).

\[
\begin{array}{ccc}
(y \circ c \circ (p \bullet_B d)) \bullet_B q & \xrightarrow{\epsilon_{yB}d_B q} & y \bullet_A (p \bullet_B d) \bullet_B q \\
\downarrow{\nu_y} & & \downarrow{\nu_y} \\
y \bullet_A c & \xrightarrow{y \circ c} & y \bullet_A c \bullet_A c \\
\end{array}
\]

As at the end of Section 3.1, we have denoted \(\epsilon = \mu \circ (p \bullet_B \varepsilon_B \bullet_B q)\).

**Lemma 3.8.** Consider the diagram (21) and any \(\psi : x \rightarrow y \bullet_A p \bullet_B d \bullet_B q\) such that \((\rho^y \circ y \bullet_A \lambda^{p_B d_B q}) \circ \psi = (y \circ A \lambda^{p_B d_B q}) \circ \psi\), i.e. \(\psi\) equalizes the two upper horizontal arrows. Then
\[
(y \circ A \text{can}) \circ \psi = \rho^y \circ (y \circ A \epsilon) \circ \psi.
\]

Consequently the left square of diagram (21) commutes and moreover the right square commutes in a serial way.

**Proof.** We compute
\[
\rho^y \circ (y \circ A \epsilon) \circ \psi = (y \circ A \epsilon) \circ (\rho^y \circ y \bullet_A \lambda^{p_B d_B q}) \circ \psi = (y \circ A \epsilon) \circ (y \circ A \lambda^{p_B d_B q}) \circ \psi = (y \circ A \text{can}) \circ \psi.
\]
Here we used (4) in the first equality, the equalizing condition of $\psi$ in the second equality and one of the alternative formulas for $\mathbf{can}$ in the last equality. We can conclude that the left square of (21) commutes by taking $(x, \psi) = ((y \cdot c \cdot (p \bullet B \ d)) \bullet B \ q, \mathbf{eq}_y \bullet B \ q)$. The last statement is easy. □

The following theorem generalizes [9, Theorem 5.1] and [30, Theorem 2.7]. For its proof, we need Lemma 3.9 that states that “all comonules are equalizers”, which could have been gathered from Beck’s Theorem. However, we find it adequate to include a proof here.

**Lemma 3.9.** Let $\mathcal{C} = (A, c, \Delta_c, \varepsilon_c)$ be a comonad in $\mathcal{B}$ and consider $(m, \rho) \in \mathbf{Rcom}(\Omega, \mathcal{C})$. Then the equalizer of $(m \bullet_A \Delta_c)$ and $(\rho \bullet_A c)$ always exists in $\mathbf{Rcom}(\Omega, \mathcal{C})$ and is isomorphic to $(m, \rho)$, i.e. $m \bullet c \cong m$.

**Proof.** For any pair $(x, \xi)$ such that $\xi$ equalizes $m \bullet_A \Delta$ and $\rho \bullet_A c$, we can define $\zeta = (m \bullet_A \varepsilon) \circ \xi$. We have that $\rho \circ \zeta = \xi$, indeed:

$$(\rho \circ \zeta) = \rho \circ (m \bullet_A \varepsilon) \circ \xi = (m \bullet_A c \varepsilon) \circ (\rho \bullet_A c) \circ \xi = (m \bullet_A c \varepsilon) \circ (m \bullet_A \Delta) \circ \xi = \xi$$

Moreover if there exists a $\zeta'$ such that $\rho \circ \zeta' = \xi$, then we find $(m \bullet_A \varepsilon) \circ \xi = (m \bullet_A \varepsilon) \circ \rho \circ \zeta' = \zeta'$ □

**Theorem 3.10.** Let $\mathcal{D}$ be comonads in $\mathcal{B}$ and consider a comonad-morphism with adjunction $(p, q) : \mathcal{D} \rightarrow \mathcal{C}$ (see Proposition 2.9). Suppose $\mathbf{Rcom}(\Omega, \mathcal{D})$ satisfies the equalizer condition for $p$. Then the pullback functor $\mathcal{P}$ associated to $p$ is fully faithful if and only if $\mathbf{can}$ is an isomorphism and the pushout functor $\mathcal{Q}$ preserves the equalizers of the form (16).

**Proof.** Suppose first that $\mathbf{can}$ is an isomorphism and $\mathcal{Q}$ preserves the equalizers of the form (16). This last condition means that $(y \cdot c \cdot (p \bullet B \ d)) \bullet B \ q \cong y \cdot c \cdot (p \bullet B \ d \bullet B \ q)$. Moreover, since $\mathbf{can}$ is an isomorphism, we can compute further $y \cdot c \cdot (p \bullet B \ d \bullet B \ q) \cong y \cdot c \cdot c$, and by Lemma 3.9 we know that the equalizer $y \cdot c = y$, so we conclude $(y \cdot c \cdot (p \bullet B \ d)) \bullet B \ q \cong y$ and the pullback functor is fully faithful.

To prove the converse, take first $y = c$, then we see that the equalizer $(c \cdot A \ varepsilon \cdot (p \bullet B \ d), \mathbf{eq}_c) \cong (p \bullet B \ d, \lambda \bullet B \ d)$ by Lemma 3.9 and consequently $(c \cdot A \ varepsilon \cdot (p \bullet B \ d)) \bullet B \ q \cong p \bullet B \ d \bullet B \ q$. As a consequence, we find that

$$\nu_c = (c \cdot A \ varepsilon) \circ (\mathbf{eq}_c \bullet B \ q) = (c \cdot A \ varepsilon) \circ (\lambda \bullet B \ d \bullet B \ q) = (c \cdot A \ varepsilon) \circ (\mathbf{can} \cdot A \ p \bullet B \ d \bullet B \ q) \circ (p \bullet B \ d \bullet B \ q) \circ (p \bullet B \ d \bullet B \ q) = \mathbf{can} \circ (p \bullet B \ d \bullet B \ q) \circ (\Delta \bullet B \ d \bullet B \ q) = \mathbf{can}.$$
Lemma 3.11. Let $\mathcal{D} = (B, d, \Delta_d, \varepsilon_d)$ be a comonad in $\mathcal{B}$ and consider $(x, \rho) \in \text{Rcom}(\Omega, \mathcal{D})$ and $q \in \text{Hom}_1(B, A)$. Then $(x \bullet_B q, \rho \bullet_B q)$ is the equalizer of $\rho \bullet_B d \bullet_B q$ and $x \bullet_B \Delta_d \bullet_B q$ in $\text{Hom}(\Omega, A)$, i.e. we have the equalizer

$$x \bullet_B q \cong x \bullet^d (d \bullet_B q) \xrightarrow{x \bullet_B d} x \bullet_B d \bullet_B q \xrightarrow{\rho \bullet_B d \bullet_B q} x \bullet_B d \bullet_B d \bullet_B q.$$  

Moreover, if $\mathcal{C} = (A, c, \Delta_c, \varepsilon_c)$ is another comonad in $\mathcal{B}$ and $d \bullet_B q$ is a $\mathcal{D}$-$\mathcal{C}$-bicomodule with left $\mathcal{D}$-coaction $\Delta_d \bullet_B q$, then the above equalizer exists even in $\text{Rcom}(\Omega, \mathcal{C})$.

Proof. The first part of the lemma can be proven in the same way as Lemma 3.3.

Suppose that $d \bullet_B q$ is a $\mathcal{D}$-$\mathcal{C}$-bicomodule with left $\mathcal{D}$-coaction $\Delta_d \bullet_B q$, and right $\mathcal{C}$-coaction $\rho'$. Then one can verify that $(x \bullet_B q, q) \in \text{Rcom}(\Omega, \mathcal{C})$, where $q = (x \bullet_B \varepsilon_B q \bullet_A c) \circ (x \bullet_B \rho') \circ (\rho_B q)$. Furthermore, one can use the same technique as in the proof of Lemma 2.13 to check that the equalizer (24) exists in $\text{Rcom}(\Omega, \mathcal{C})$.

Theorem 3.12. Let $\mathcal{D}$, $\mathcal{C}$ be comonads in $\mathcal{B}$ and consider a comonad-morphism with adjunction $(p, q) : \mathcal{D} \to \mathcal{C}$ (see Proposition 2.12). Suppose $\text{Rcom}(\Omega, \mathcal{D})$ satisfies the equalizer condition for $p$ and that the equalizer $(d \bullet_B q) \bullet^c (p \bullet_B d)$ is left $d \bullet_B d$-pure. Then the following statements are equivalent:

(i) The pullback functor $\mathcal{D}$ associated to $q$ is fully faithful;

(ii) $\text{can}$ is an isomorphism and

$$(x \bullet_B q) \bullet^c (p \bullet_B d) \cong x \bullet^d ((d \bullet_B q) \bullet^c (p \bullet_B d))$$

for all $x \in \text{Rcom}(\Omega, \mathcal{D})$;

(iii) $\text{can}$ is an isomorphism and the equalizer $(d \bullet_B q) \bullet^c (p \bullet_B d)$ is left $y$-pure for all $y \in \text{Hom}_1(\Omega, B)$.

Proof. (i) $\Rightarrow$ (ii). Since the equalizer $(d \bullet_B q) \bullet^c (p \bullet_B d)$ is left $d \bullet_B d$-pure, $(d \bullet_B q) \bullet^c (p \bullet_B d)$ is a left $\mathcal{D}$-comodule and the equalizer $x \bullet^d ((d \bullet_B q) \bullet^c (p \bullet_B d))$ is well-defined for all $x \in \text{Rcom}(\Omega, \mathcal{D})$.

It follows directly from the definition of $\text{can}$ and the formula for $\zeta$ in (20) that $\zeta_d = \text{can}$. If $\text{can}$ is an isomorphism, then we find

$$x \bullet^d ((d \bullet_B q) \bullet^c (p \bullet_B d)) \cong x \bullet^d d \cong x.$$  

Combining this isomorphism with (25), we find that

$$\mathcal{P} \mathcal{D}(x) = (x \bullet_B q) \bullet^c (p \bullet_B d) \cong x,$$

and from naturality it follows that this combined isomorphism is exactly the counit of the adjunction evaluated in $x$. The converse follows in the same way.

(ii) $\Rightarrow$ (iii). Take any $y \in \text{Hom}_1(\Omega, B)$ and put $x = y \bullet_B d$ in (25), then we find

$$(y \bullet_B d \bullet_B q) \bullet^c (p \bullet_B d) \cong (y \bullet_B d) \bullet^d ((d \bullet_B q) \bullet^c (p \bullet_B d)) \cong y \bullet_B ((d \bullet_B q) \bullet^c (p \bullet_B d),$$

where we used Lemma 2.11 for the last isomorphism. This means exactly that $(d \bullet_B q) \bullet^c (p \bullet_B d)$ is left $y$-pure.

(iii) $\Rightarrow$ (ii). If $(d \bullet_B q) \bullet^c (p \bullet_B d)$ is left $y$-pure for all $y \in \text{Hom}_1(\Omega, B)$, then

$$y \bullet_B ((d \bullet_B q) \bullet^c (p \bullet_B d)) \cong (y \bullet_B d \bullet_B q) \bullet^c (p \bullet_B d).$$
Now take \( x \in \text{Rcom}(\Omega, \mathcal{D}) \) and construct the following commutative diagram.

\[
\begin{array}{c}
(x \bullet_B q) \bullet^c (p \bullet_B d) \xrightarrow{=} (x \bullet_B d \bullet_B q) \bullet^c (p \bullet_B d) \xrightarrow{=} (x \bullet_B d \bullet_B d \bullet_B q) \bullet^c (p \bullet_B d)
\end{array}
\]

The upper row is obtained by applying the functor \( \mathcal{P} \) on the equalizer \( \mathcal{Q} \). Since \( \mathcal{P} \) is a left adjoint, it preserves the equalizers and the upper row is an equalizer. The lower row is an equalizer by definition. The vertical isomorphisms follow from our previous observation. By the universal property of the equalizer we obtain that the left vertical arrow must be an isomorphism as well.

**Theorem 3.13.** Let \( \mathcal{D}, \mathcal{C} \) be comonads in \( \mathcal{B} \) and consider a comonad-morphism with adjunction \( (p, q) : \mathcal{D} \to \mathcal{C} \) (see Proposition 2.9). Suppose \( \text{Rcom}(\Omega, \mathcal{D}) \) satisfies the equalizer condition for \( p \). Then the following statements are equivalent.

(i) The functors \( (\mathcal{P}, \mathcal{Q}) \) establish an equivalence of categories between \( \text{Rcom}(\Omega, \mathcal{C}) \) and \( \text{Rcom}(\Omega, \mathcal{D}) \);

(ii) \( \mathcal{Q} \) is an isomorphism, the pushout functor \( \mathcal{Q} \) reflects isomorphisms and preserves the equalizers of the form \( (16) \).

If \( d \bullet_B q \bullet^c (p \bullet_B d) \) is left \( d \bullet_B d \)-pure, then the previous statements are furthermore equivalent to

(iii) \( (p, q) \) is a Galois comonad-morphism, \( \mathcal{Q} \) preserves the equalizers of the form \( (16) \) and \( (22) \) is satisfied for all \( x \in \text{Rcom}(\Omega, \mathcal{D}) \);

(iv) \( (p, q) \) is a Galois comonad-morphism, \( \mathcal{Q} \) preserves the equalizers of the form \( (16) \) and \( (d \bullet_B q) \bullet^c (p \bullet_B d) \) is left \( y \)-pure for all \( y \in \text{Hom}_1(\Omega, B) \).

**Proof.** We only prove the equivalence between (i) and (ii), the equivalence with the other statements follows directly from Theorem 3.10 and Theorem 3.12.

First suppose that \( (\mathcal{P}, \mathcal{Q}) \) establishes an equivalence of categories. Then obviously \( \mathcal{Q} \) reflects isomorphisms, and the other statements follow from Theorem 3.10.

Conversely, suppose \( \text{can} : p \bullet_B d \bullet_B q \to c \) to be an isomorphism of comonads, then by Lemma 3.9 we find the following equalizer in \( \text{Rcom}(\Omega, \mathcal{C}) \) for any \( (m, \rho^m) \) in \( \text{Rcom}(\Omega, \mathcal{D}) \),

\[
m \bullet_B q \xrightarrow{\varepsilon} m \bullet_B q \bullet_A p \bullet_B d \bullet_B q \xrightarrow{\varepsilon} m \bullet_B q \bullet_A p \bullet_B d \bullet_B q \bullet_A p \bullet_B d \bullet_B q.
\]

Furthermore, if \( \mathcal{Q} \) preserves equalizers, then we can apply \( - \bullet_B q \) on \( (16) \) in the situation \( n = m \bullet_B q \) and we obtain a second equalizer in \( \text{Rcom}(\Omega, \mathcal{C}) \). These two equalizers can be related in the following diagram.
Since an isomorphism, we find by the properties of the equalizers that $\zeta_m \bullet_B q$ is an isomorphism as well, and since $\mathcal{Q}$ reflects isomorphisms, $\zeta_m$ must be an isomorphism. From Theorem 3.10 we know that $\nu$ is a natural isomorphism as well, so we find that $(\mathcal{P}, \mathcal{Q})$ is an equivalence of categories.

3.4. Coseparable comonads. A coalgebra in a monoidal category is called coseparable if its comultiplication splits in the category of bicomodules. In analogy, we will say that a comonad $\mathfrak{C} = (A, c, \Delta_c, \varepsilon_c)$ is coseparable if and only if there exists a 2–cell $\gamma \in \mathcal{A}^{4 \text{Hom}_2}(c \bullet_A c, c)$ which is a morphism of $\mathfrak{C}$–bicomodules and such that $\gamma \circ \Delta_c = c$.

The following proposition has a straightforward proof.

**Proposition 3.14.** Consider a quadruple $\mathfrak{C} = (A, c, \Delta, \varepsilon)$ in $\mathcal{B}$ consisting of a 0–cell $A$, a 1–cell $c : A \to A$ and 2–cells $\Delta : c \to c \bullet c$ and $\varepsilon : A \to c$. The following statements are equivalent.

(i) $\mathfrak{C}$ is a coseparable comonad in $\mathcal{B}$;
(ii) $(c, \Delta, \varepsilon)$ is a coseparable coalgebra in the monoidal category $\text{Hom}(A, A)$;
(iii) $(- \bullet_A c, - \Delta_A \Delta, - \Delta_A \varepsilon)$ is a coseparable comonad on $\text{Hom}(\Omega, A)$ for all $\Omega \in \mathcal{B}$;
(iv) $(- \bullet_A c, - \varepsilon_A \Delta, - \varepsilon_A \varepsilon)$ is a coseparable comonad on $\text{Hom}(A, A)$.

The notion of a separable functor was introduced in [40] and can be used to characterize separable algebras and coseparable coalgebras (see e.g. [20]), and, more generally, of separable morphisms of corings [29]. Separable functors of the second kind were introduced in [19]. A useful characterization of separable functors is given by Rafael’s theorem [42]. Let us state a version of Rafael’s theorem for separable functors of the second kind (the corresponding theorem for separable functors of the first kind can be obtained by taking $\mathfrak{C} = A$ and $H = 1_A$).

**Theorem 3.15** ([19 Theorem 2.7]). Let $F : \mathcal{A} \to \mathcal{B}$ be a covariant functor with a right adjoint $G : \mathcal{B} \to \mathcal{A}$ and consider an additional functor $H : \mathcal{A} \to \mathcal{C}$. Denote by $\eta : 1_A \Rightarrow GF$ the unit of the adjunction $(F,G)$, then $F$ is $H$–separable if and only if there exists a natural transformation $\mu : HGF \Rightarrow H$ such that $\mu \circ H \eta = H$.

Combining Proposition 3.14 with [28 Theorem 1.6], we obtain more characterizations of coseparable comonads in bicategories.

**Theorem 3.16.** Consider a comonad $\mathfrak{C} = (A, c, \Delta, \varepsilon)$ in $\mathcal{B}$. The following statements are equivalent.

(i) $\mathfrak{C}$ is coseparable;
(ii) the forgetful functor $U_{A, \mathfrak{C}} : \text{Rcom}(A, \mathfrak{C}) \to \text{Hom}(A, A)$ is separable;
(iii) the forgetful functor $U_{\Omega, \mathfrak{C}} : \text{Rcom}(\Omega, \mathfrak{C}) \to \text{Hom}(\Omega, A)$ is separable for all 0–cells $\Omega$ in $\mathcal{B}$;
(iv) the forgetful functor $\varepsilon_{A, U} : \text{Lcom}(\mathfrak{C}, A) \to \text{Hom}(A, A)$ is separable;
(v) the forgetful functor $\varepsilon_{\Omega, U} : \text{Lcom}(\mathfrak{C}, \Omega) \to \text{Hom}(A, \Omega)$ is separable for all 0–cells $\Omega$ in $\mathcal{B}$.

**Remark 3.17.** Recall that the unit of the adjunction $U_{\Omega, \mathfrak{C}} \dashv - \bullet_A c$ is given by $\theta_m = \rho^m \in \Omega \text{Hom}^A(m, m \bullet_A c)$, the right $\mathfrak{C}$–coaction for all $(m, \rho^m) \in \text{Rcom}(\Omega, \mathfrak{C})$. Therefore, a comonad $\mathfrak{C}$ is coseparable if and only if there exists, for all 0–cells $\Omega$ in $\mathcal{B}$ and all $(m, \rho^m) \in \text{Rcom}(\Omega, \mathfrak{C})$, a 2–cell $\tilde{\rho}_m \in \Omega \text{Hom}^A(m \bullet_A c, m)$ such that $\tilde{\rho}_m$ is natural in $m$ and $\tilde{\rho}_m \circ \rho^m = m$. This is equivalent to the existence, for all 0–cell $\Omega$ in $\mathcal{B}$ and all $(n, \lambda^n) \in \text{Lcom}(\mathfrak{C}, \Omega)$, of a 2–cell $\lambda_n \in \text{A}^{4 \text{Hom}_2}(c \bullet_A n, n)$ such that $\lambda_n$ is natural in $n$ and $\lambda_n \circ \lambda^n = n$. □
Our first aim is to prove that Galois comonad-morphisms having as codomain a coseparable comonad give rise to equivalences of categories. The following lemma generalizes [51, 2.13] and [48, Lemma 9.1].

**Lemma 3.18.** Let \( \mathcal{C} = (A, c, \Delta_c, \varepsilon_c) \) be a comonad in \( \mathcal{B} \) and 0–cells \( B, \Omega \) and \( \Omega' \) in \( \mathcal{B} \). Take \((m, \rho^m) \in \text{Rcom}(\Omega, \mathcal{C})\) and \((n, \lambda^n) \in \text{Lcom}(\mathcal{C}, B)\). Suppose that the cotensor product \( m \bullet^c n \) exists in \( \text{Hom}(\Omega, B) \), then for all \( p \in \text{Hom}(B, \Omega') \) the functor \(- \bullet_A p : \text{Hom}(\Omega, A) \to \text{Hom}(\Omega, B)\) preserves this equalizer, i.e. natural map

\[
f : (m \bullet^c n) \bullet_B p \to m \bullet^c (n \bullet_B p)
\]

is an isomorphism in each of the following situations:

(i) \( \rho^m : m \to m \bullet_A c \) has a left inverse \( \bar{\rho}_m \) in \( \text{Rcom}(\Omega, \mathcal{C}) \);
(ii) \( \lambda^n : n \to c \bullet_A n \) has a left inverse \( \bar{\lambda}_n \) in \( \text{Lcom}(\mathcal{C}, B) \).

**Proof.** Let us show that the equalizer defining the cotensor product

\[
m \bullet^c n \xrightarrow{i} m \bullet_A n \xrightarrow{\rho^m \bullet_A n} m \bullet_A c \bullet_A n,
\]

is a contractible equalizer. We define 2–cells \( \alpha : m \bullet_A n \to m \bullet^c n \) and \( \beta, \beta' : m \bullet_A c \bullet_A n \to m \bullet_A n \) as follows. Under condition (i) we put \( \alpha = (\bar{\rho}_m \bullet_A n) \circ (m \bullet_A \bar{\lambda}_n) \) and \( \beta = \bar{\rho}_m \bullet_A n \), if (ii) is satisfied, then we define \( \alpha = (m \bullet_A \bar{\lambda}_n) \circ (\rho^m \bullet_A n) \) and \( \beta' = m \bullet_A \bar{\lambda}_n \). Moreover, one can easily verify that \( \alpha \circ i = m \bullet^c n \), \( \beta \circ (\rho^m \bullet_A n) = m \bullet_A n \), \( \beta' \circ (m \bullet_A \bar{\lambda}_n) \) and \( \alpha \circ i = \beta \circ (n \bullet_A \bar{\lambda}_n) = \beta' \circ (\rho^m \bullet_A n) \). By [21 Proposition 3.3.2], any functor preserves a contractible equalizer. Therefore, we obtain the following diagram, applying the functor \(- \bullet_B p : \text{Hom}(\Omega, B) \to \text{Hom}(\Omega, \Omega')\).

\[
\begin{array}{c}
(m \bullet^c n) \bullet_B p \xrightarrow{f} m \bullet_A n \bullet_B p \xrightarrow{\cong} m \bullet_A c \bullet_A n \bullet_B p \\
\downarrow \cong \downarrow \cong \\
(m \bullet^c (n \bullet_B p)) \bullet_B p \xrightarrow{\cong} m \bullet_A (n \bullet_B p) \bullet_B p \xrightarrow{\cong} m \bullet_A c \bullet_A n \bullet_B p
\end{array}
\]

Since we know that both horizontal rows are equalizers, \( f \) is an isomorphism by the universal property of the equalizer.

The following theorem generalizes [48, Theorem 9.2] and [51, 5.7], it should be compared with [16, Theorem 5.8]. Our result is as well related to the split (co)monadicity theorem (see e.g. [32, Theorem 2.2]).

**Theorem 3.19.** Let \( (p, q) : \mathcal{D} \to \mathcal{C} \) be a comonad-morphism with adjunction and suppose that \( \text{Rcom}(\Omega, \mathcal{D}) \) satisfies the equalizer condition for \( p \). If \( \lambda^{\bullet_B d} : p \bullet_B d \to c \bullet_A p \bullet_B d \) has a left inverse in \( \text{Lcom}(\mathcal{C}, B) \) (in particular, if \( \mathcal{C} \) is a coseparable comonad), then the following statements hold.

(i) If \( \text{can} \) is an isomorphism, then \( \mathcal{P} \) is fully faithful;
(ii) if \( \text{can} \) is an isomorphism, then \( \mathcal{D} \) is fully faithful;
(iii) if \( (p, q) \) is a Galois comonad-morphism, then \( (\mathcal{P}, \mathcal{D}) \) is an equivalence of categories.

**Proof.** (i). Since the coaction \( \lambda^{\bullet_B d} : p \bullet_B d \to c \bullet_A p \bullet_B d \) has a left inverse in \( \text{Lcom}(\mathcal{C}, B) \), it follows by Lemma 3.18 that for all \( m \in \text{Rcom}(\Omega, \mathcal{C}) \), the following isomorphism holds

\[
(m \bullet^c (p \bullet_B d)) \bullet_B q \cong m \bullet^c ((p \bullet_B d) \bullet_B q),
\]
i.e. \( \mathcal{D} \) preserves equalizers of the form (16). The statement follows now by Theorem 3.10 (ii). Applying now left-right duality on Lemma 3.18 we find now that \((d \bullet_B q) \bullet^c (p \bullet_B d)\) is left \( y \)-pure for all choices of \( y \in \text{Hom}_1(\Omega, B) \), since \( \lambda^*_{p,d} \) has a left inverse in \( \text{Lcom}(\mathcal{C}, B) \). Therefore, \( \mathcal{D} \) is fully faithful by Theorem 3.12 (iii). Follows directly from (i) and (ii).

**Lemma 3.20.** Let \( F \dashv G \) and \( H \dashv K \) be two pairs of adjoint functors as in the following diagram

\[
\begin{array}{ccc}
A & \xrightarrow{F} & B \\
\downarrow{G} & & \downarrow{K} \\
D & \xleftarrow{Z} & C
\end{array}
\]

Then the following statements hold.

(i) If \( F \) is separable and \( H \) is \( G \)-separable, then \( HF \) is separable;

(ii) if \( H \) is \( Z \)-separable, then \( HF \) is \( ZF \) separable.

**Proof.** Denote the unit of the adjunction \( F \dashv G \) by \( \eta_a : a \to GFa \) and the unit of \( H \dashv K \) by \( \zeta_b : b \to KHb \). Then the unit of the composed adjoint pair \( GK \dashv HF \) is given by

\[
\xi_a = G\zeta_{Fa} \circ \eta_a : a \to GKHFa.
\]

(i). If \( F \) is separable, then there exists a natural transformation \( \mu_a : GFa \to a \) which is a left inverse for \( \eta_a \). If \( H \) is \( G \)-separable, then there exists a natural transformation \( \nu_b : GKHFa \to Kb \) which is a left inverse for \( G\zeta_b \). One can now easily see that \( \mu_a \circ \nu_b \) is a left inverse for \( \xi_a \).

(ii). Let \( \varepsilon_b : FGB \to b \) be the counit of the adjunction \((F, G)\), and denote by \( \nu_b : ZKHb \to Zb \) the inverse for \( Z\zeta_b \), obtained by the \( Z \)-separability of \( H \). Then we can consider the following diagram.

\[
\begin{array}{ccc}
ZF a & \xrightarrow{ZF \eta_a} & ZF G Fa \\
\downarrow{ZF \zeta_a} & & \downarrow{ZF \xi_a} \\
ZF a & \xrightarrow{\xi_{ZF a}} & ZKHFa \\
\downarrow{Z\zeta_{ZF a}} & & \downarrow{\nu_{ZF a}} \\
ZF a & \xleftarrow{\nu_{ZF a}} & ZFa
\end{array}
\]

The upper left triangle in this diagram commutes by the fact that \( F \dashv G \) is an adjunction, the lower right triangle commutes by the fact that \( H \) is \( Z \)-separable, the inner square commutes trivially. Therefore we find that \( ZF \xi_a = ZF G \zeta_{Fa} \circ ZF \eta_a \) has a left inverse given by \( \nu_{ZF a} \circ Z\varepsilon_{KHFa} \), i.e. \( ZH \) is \( ZF \)-separable.

**Remark 3.21.** It follows by Lemma 3.20 (ii) that any functor \( F \) with a right adjoint is \( F \)-separable.

The next Theorem is also related to the split (co)monadicity theorem (see e.g. [32, Theorem 2.2]).

**Theorem 3.22.** Let \( (p, q) : \mathcal{D} \to \mathcal{C} \) be a comonad-morphism with adjunction, and suppose that \( \text{Rcom}(\Omega, \mathcal{D}) \) satisfies the equalizer condition for \( p \). Consider the forgetful functor \( U_{\Omega, \mathcal{D}} : \text{Rcom}(\Omega, \mathcal{D}) \to \text{Hom}(\Omega, B) \) and its right adjoint \( G = - \bullet_B : \text{Hom}(\Omega, B) \to \text{Rcom}(\Omega, \mathcal{D}) \).
If \( \text{can} \) is an isomorphism and the functor \((- \bullet_B q) U_{\Omega, \mathcal{D}}\) is separable (in particular, if the functor \(- \bullet_B q\) is \(G\)-separable and \(\mathcal{D}\) is coseparable), then \(\mathcal{P}\) is fully faithful.

If moreover \(\text{can}\) is an isomorphism, then \((\mathcal{P}, \mathcal{D})\) is an equivalence of categories.

**Proof.** Obviously, \(G(- \bullet_A p)\) is right adjoint to \((- \bullet_B q) U_{\Omega, \mathcal{D}}\). The unit is of this adjunction is given by \(\chi_m = (m \bullet_B \eta_B d) \circ \rho^m : m \to m \bullet_B q \bullet_A p \bullet_B d\) for all \((m, \rho^m) \in R_{com}(\Omega, \mathcal{D})\).

If the functor \((- \bullet_B q) U_{\Omega, \mathcal{D}}\) is separable, then for all \(m \in \text{Hom}(\Omega, \mathcal{D})\), the map \(\chi_m : m \to m \bullet_B q \bullet_A p \bullet_B d\) has a left inverse. Taking \(m = p \bullet_B d\) and taking into account the fact that \(\text{can}\) is an isomorphism, we find that \(\lambda_{p \bullet_B d}\) has a left inverse. Therefore, the statement follows by Theorem 3.19.

It follows by Lemma 3.20 (i) that the functor \((- \bullet_B q) U_{\Omega, \mathcal{D}}\) is separable if the functor \(- \bullet_B q\) is \(G\)-separable and \(\mathcal{D}\) is coseparable.

The following theorem generalizes Theorem 3.22 (taking \(B = \text{Rcom}(\Omega, \mathcal{D}), Y = \text{Hom}(\Omega, A)\), \(\mathcal{D} = - \bullet_B q\) and \(H\) and \(H'\) the identity functors), however the proof of Theorem 3.23 is a lot more involved. Theorem 3.23 generalizes as well [32, Theorem 2.3] and goes back to the Joyal-Tierney theorem for descent theory (see Section 4.4).

**Theorem 3.23.** Let \(\mathcal{C} = (A, c, \Delta_c, \varepsilon_c)\) and \(\mathcal{D} = (B, d, \Delta_d, \varepsilon_d)\) be comonads in \(\mathcal{B}\) and \((p, q) : \mathcal{D} \to \mathcal{C}\) be a comonad-morphism with adjunction such that \(\text{Rcom}(\Omega, \mathcal{D})\) satisfies the equalizer condition for \(p\). Suppose furthermore that there exist functors \(H, H'\) and \(\mathcal{Q}'\) that make the following diagram commutative

\[
\begin{array}{ccc}
R_{com}(\Omega, \mathcal{D}) & \xrightarrow{\mathcal{D}} & R_{com}(\Omega, \mathcal{C}) \\
H & \downarrow & H' \\
\text{Hom}(\Omega, B) & \xrightarrow{\bullet_B q} & \text{Hom}(\Omega, A) \\
\mathcal{B} & \xrightarrow{\mathcal{Q}'} & \mathcal{Y}
\end{array}
\]

(remark that the upper inner square of this diagram always commutes, see (18))

1. The functor \(\mathcal{P}\) is fully faithful if the following conditions hold,
   (i) \(\text{can}\) is an isomorphism
   (ii) the composite functor \((- \bullet_B q) U_{\Omega, \mathcal{D}}\) is \(H\)-separable;
   (iii) \(H\) preserves equalizers of the form (16);
   (iv) \(H'\) reflects isomorphisms.
2. The functor \(\mathcal{D}\) is fully faithful and therefore \((\mathcal{D}, \mathcal{P})\) is an equivalence of categories if in addition to (1) the following conditions hold,
   (i) \(\text{can}\) is an isomorphism (hence \((p, q)\) is a Galois \(\mathcal{D} - \mathcal{C}\) comonad-morphism);
   (ii) \(H\) reflects isomorphisms.

**Proof.** Denote, as in the proof of Theorem 3.22, the unit of the adjoint pair

\[\chi_m = (m \bullet_B \eta_B d) \circ \rho^m : m \to m \bullet_B q \bullet_A p \bullet_B d\]

by

\[\chi_m = (m \bullet_B \eta_B d) \circ \rho^m : m \to m \bullet_B q \bullet_A p \bullet_B d\]
for all \((m, \rho^m) \in \text{Rcom}(\Omega, \mathcal{D})\). Then \((- \bullet_B q)U_{\Omega, \mathcal{D}}\) is \(H\)-separable if and only if there exist morphisms \(\xi_m : H(m \bullet_B q \bullet_A p \bullet_B d) \to H(m)\) for all \((m, \rho^m) \in \text{Rcom}(\Omega, \mathcal{D})\) such that \(\xi\) is natural in \(m\) and \(\xi_m \circ H(\chi_m) = H(m)\).

(1) By Theorem 3.10, we only have to prove that \(\mathcal{Q}\) preserves equalizers of the form (16). Since \(\mathcal{Q}\) is an isomorphism, we will identify \(\mathcal{C}\) with the comonad \((p \bullet_B d \bullet_B g)\), the same applies for the \(\mathcal{C}\)–comodules. Consider the following commutative diagram of equalizers for any \(x \in \text{Rcom}(\Omega, \mathcal{C})\).

The equalizers in both columns are of the form (16), taking \(n = x\) and \(n = x \bullet_A c\) respectively. Consequently, if we apply the functor \(H\) to this diagram, then we obtain again a commutative diagram, with equalizers in the columns. Moreover, we find that the two lower horizontal arrows are split by respectively \(\xi_{x \bullet_A c \bullet_A p \bullet_B d}\) and \(\xi_{x \bullet_A p \bullet_B d}\).

Using the naturality of \(\xi\), we find that the pair \((H(\chi_{m \bullet_A p \bullet_B d}), H(\rho^m \bullet_A p \bullet_B d))\) is equalized by \(\xi_{m \bullet_A p \bullet_B d} \circ H(\rho^m \bullet_A p)\). Hence, from the universal property of the equalizer, we find a morphism \(\kappa : H(x \bullet_A (p \bullet_B d)) \to H(x \bullet^c (p \bullet_B d))\) such that \(H(\text{eq}_x) \circ \kappa = \xi_{m \bullet_A p \bullet_B d} \circ H(\rho^m \bullet_A p)\). Moreover, we can compute

\[
H(\text{eq}_x) \circ \kappa \circ H(\text{eq}_x) = \xi_{m \bullet_A p \bullet_B d} \circ H(\rho^m \bullet_A p) \circ H(\text{eq}_x) = \xi_{m \bullet_A p \bullet_B d} \circ H(\xi_{m \bullet_A p \bullet_B d} \circ H(\text{eq}_x)) = H(\text{eq}_x).
\]

Since the left column of (26) is an equalizer, \(H(\text{eq}_x)\) is a monomorphism, therefore \(H(\text{eq}_x) \circ \kappa\) is the identity. Hence, we find that the equalizer in the left column is contractible by the maps

\[
H(x \bullet_A c \bullet_A p \bullet_B d) \xrightarrow{\xi_{m \bullet_A p \bullet_B d}} H(x \bullet_A p \bullet_B d) \xrightarrow{\kappa} H(x \bullet^c (p \bullet_B d)).
\]

Since a contractible equalizer is preserved by any functor, this equalizer is in particular preserved by \(\mathcal{Q}'\). Therefore, we find that \(\mathcal{Q}'H = H'U_{\Omega, \mathcal{C}, \mathcal{Q}}\) applied to the an equalizer of the form (16) is an equalizer, that is \(H'((x \bullet^c (p \bullet_B d)) \bullet_B q)\) is the equalizer of the pair

\[
H(x \bullet_A c \bullet_A p \bullet_B d) \xrightarrow{\xi_{m \bullet_A p \bullet_B d}} H(x \bullet_A p \bullet_B d) \xrightarrow{\kappa} H(x \bullet^c (p \bullet_B d)).
\]
(H'(\rho^x_{\bullet A p \bullet B d} \bullet B q), H'(\chi_{m \bullet A p \bullet B d} \bullet B q)). Now consider the diagram

(27) \[
\begin{array}{cccccc}
x & \xrightarrow{\rho^x} & x \bullet A c & \xrightarrow{\rho^x_{\bullet A c}} & x \bullet A c \bullet A c & \xrightarrow{\Delta} \\
\downarrow{\cong} & & \downarrow{\cong} & & \downarrow{\cong} \\
x \bullet A p \bullet B d \bullet B q & \xrightarrow{\rho^x_{\bullet A p \bullet B d} \bullet B q} & x \bullet A c \bullet A p \bullet B d \bullet B q.
\end{array}
\]

This diagram defines an equalizer in \( R_{com} (\Omega, \mathcal{C}) \), which is preserved by the forgetful functor \( U_{\Omega, \mathcal{C}} \). Moreover, this equalizer is even split in \( \text{Hom}(\Omega, A) \), by the maps

\[
x \leftarrow x \bullet A^p \leftarrow x \bullet A \leftarrow x \bullet A \bullet A^p \leftarrow x \bullet A c \bullet A c.
\]

Therefore, this equalizer is preserved by the functor \( H' \), and we find in \( \mathcal{Y} \) that also \( H'(x) \) is the equalizer of the pair \((H'(\rho^x_{\bullet A p \bullet B d} \bullet B q), H'(\chi_{m \bullet A p \bullet B d} \bullet B q))\). Therefore \( H'(x) \cong H'((x \cdot \epsilon^x(p \bullet B d)) \bullet B q) \). Since \( H' \) reflects isomorphisms, we obtain that \((x \cdot \epsilon^x(p \bullet B d)) \bullet B q \cong x \cong (x \cdot \epsilon^x(p \bullet B d) \bullet B q) \) in \( \text{Hom}(\Omega, A) \), and therefore as well in \( R_{com} (\Omega, \mathcal{C}) \), since the equalizer (27) was preserved by \( U_{\Omega, \mathcal{C}} \). Hence \( \mathcal{D} \) preserves the equalizers of the form (10).

(2). By Theorem 3.12 we only have to prove that the equalizer \((d \bullet B q) \cdot \epsilon^x(p \bullet B d)\) is left \(-\)pure for all \( y \in \text{Hom}_1(\Omega, B) \). Since \( \overline{\text{can}} \) is an isomorphism, we can identify \( \mathcal{C} \) with the comonad \((p \bullet B d \bullet B q)\), the same apply for the comodules. Therefore, the equalizer \((d \bullet B q) \cdot \epsilon^x(p \bullet B d)\) can be understood as the equalizer of the pair \((d \bullet B q \circ A p \bullet B d, \gamma \circ A p \bullet B d)\), where \( \gamma = (d \bullet B q) \circ \Delta \). We have to prove that the following diagram, where all subscripts have been removed by typography’s needs, is an equalizer in \( R_{com} (\Omega, \mathcal{D}) \).

(28) \[
y \cdot ((d \bullet q) \cdot \epsilon^x(p \bullet d)) \xrightarrow{y \cdot \gamma} y \cdot (d \bullet q) \cdot (p \bullet d) \xrightarrow{y \cdot \gamma \cdot (d \bullet q) \circ \epsilon^x(p \bullet d)} y \cdot d \bullet q \circ p \bullet d \bullet q \circ p \bullet d
\]

Here we used that \( \overline{\text{can}} \) is an isomorphism. If we apply \( H \) to (28), then, taking into account that \( y \cdot ((d \bullet q) \cdot \epsilon^x(p \bullet d)) \cong y \cdot d \), we obtain the following diagram in \( \mathcal{A} \).

Let us show that this is a contractible equalizer, so in particular an equalizer. The identities \( \xi_{y \bullet B d} \circ H(y \bullet B \gamma) = H(y \bullet B \gamma) \) and \( \xi_{y \bullet B d} \bullet B q \circ A p \bullet B d \circ H(y \bullet B d \bullet B q \circ A p \bullet B d) = H(y \bullet B d \bullet B q \circ A p \bullet B d) \) follow directly from the \( H \)-separability of \((- \bullet B q)U_{\Omega, \mathcal{D}} \). Furthermore the naturality of \( \xi \) implies that \( H(y \bullet B \gamma) \circ \xi_{y \bullet B d} = \xi_{y \bullet B d} \bullet B q \circ A p \bullet B d \circ H(y \bullet B \gamma \circ q \circ A p \bullet B d) \).

Now compute the equalizer \((y \bullet B d \bullet B q) \cdot \epsilon^x(p \bullet B d)\) of the pair \((y \bullet B d \bullet B q \circ A p \bullet B d, y \bullet B \gamma \circ q \circ A p \bullet B d)\). Since \( H \) preserves this equalizer, we obtain that \( H(y \bullet B d) \cong H((y \bullet B d \bullet B q) \cdot \epsilon^x(p \bullet B d)) \) in \( \mathcal{A} \). As \( H \) reflects isomorphisms, we obtain \( y \bullet B d \cong (y \bullet B d \bullet B q) \cdot \epsilon^x(p \bullet B d) \) in \( R_{com} (\Omega, \mathcal{D}) \), i.e. \( y \bullet B \) preserves the equalizer (28), or \((d \bullet B q) \cdot \epsilon^x(p \bullet B d)\) is left \(-\)pure.

\[\square\]

Corollary 3.24. Let \( \mathcal{C} = (A, c, \Delta_c, \varepsilon_c) \) and \( \mathcal{D} = (B, d, \Delta_d, \varepsilon_d) \) be comonads in \( \mathcal{B} \) and \((p, q) : \mathcal{D} \to \mathcal{C} \) be a comonad-morphism with adjunction such that \( R_{com} (\Omega, \mathcal{D}) \) satisfies the equalizer condition for \( p \). Suppose there exist functors \( H'' \), \( H' \) and \( \mathcal{D}' \) that make the following diagram
commutative.

\[
\begin{array}{ccc}
\text{Rcom}(\Omega, \mathfrak{D}) & \xrightarrow{\mathcal{P}} & \text{Rcom}(\Omega, \mathfrak{C}) \\
\downarrow U_{\Omega, \mathfrak{D}} & & \downarrow U_{\Omega, \mathfrak{C}} \\
\text{Hom}(\Omega, B) & \xrightarrow{- \cdot_B q} & \text{Hom}(\Omega, A) \\
\downarrow H'' & & \downarrow H' \\
B & \xrightarrow{\mathcal{P}'} & \mathcal{U}
\end{array}
\]

(1) The functor \(\mathcal{P}\) is fully faithful if the following conditions hold.
(i) \(\text{can}\) is an isomorphism;
(ii) the functor \(- \cdot_B q\) is \(H''\)–separable;
(iii) \(H''\) preserves equalizers of the form (16);
(iv) \(H'\) reflects isomorphisms.

(2) \(\mathcal{D}\) is fully faithful and therefore \((\mathcal{D}, \mathcal{P})\) is an equivalence of categories if in addition to (1) the following conditions hold,
(i) \(\text{can}\) is an isomorphism (hence \((p, q)\) is a \(\mathfrak{D}\)-\(\mathfrak{C}\) Galois comonad-morphism);
(ii) \(H\) reflects isomorphisms.

Proof. (1). Since \(\text{Rcom}(\Omega, \mathfrak{D})\) satisfies the equalizer condition for \(p, U_{\Omega, \mathfrak{D}}\) preserves preserves equalizers of the form (16). Put \(H = H'' \circ U_{\Omega, \mathfrak{D}}\), then \(H\) preserves equalizers of the form (16) as well. By Lemma 3.20 (ii), we find that \((- \cdot_B q)U_{\Omega, \mathfrak{D}}\) is \(H\)–separable. The statement follows now from Theorem 3.23 (1).

(2) This follows in the same way from Theorem 3.23(2).

4. Examples

In this section we will briefly describe some situations of current interest where our results apply.

4.1. The bicategories of corings. Consider \(\mathcal{B} = \text{Bim}(k)\) the bicategory of (unital) algebras over a commutative ring, bimodules and homomorphisms of bimodules. Comonads in \(\text{Bim}(k)\) are corings, and were studied, from the point of view of bicategories, in [9]. We will denote the category \(\text{Rcom}(k, \mathfrak{C})\) of right comodules over an \(A\)–coring \(\mathfrak{C}\) by \(\mathcal{M}\), in analogy to the usual notation \(\mathcal{M}_A\) for the category of all right \(A\)–modules over a ring \(A\). An adjoint pair \((A, B, \Sigma, \Sigma^\dagger, \varepsilon, \eta)\) in \(\mathcal{B}\) was termed a comatrix coring context in [10]. Since \(B\) is a ring with unit, we obtain that \(\Sigma\) is finitely generated and projective as a right \(A\)–module and \(\Sigma^\dagger \cong \Sigma^*\).

By [9], we have the \(A\)–coring \(\Sigma^* \otimes_B \mathcal{D} \otimes_B \Sigma\). This construction was considered in [9, Theorem 3.1] and it generalizes that of finite comatrix corings [26].

If we apply the techniques developed in the previous sections to the present situation, then we recover the situation studied in [9], and, in particular, those in [9 Section 5] concerning the pull-back and push-out functors. From Proposition 3.4, Theorem 3.10, Theorem 3.12 and Theorem 3.13 we deduce:

Theorem 4.1. Let \(A\) and \(B\) be unital rings, \(\mathfrak{D}\) a \(B\)–coring that is flat as a left \(B\)–module and \(\mathfrak{C}\) an \(A\)–coring. Consider \(\Sigma \in B \mathcal{M}\) such that \(\Sigma\) is finitely generated and projective as a right \(A\)–module, and let \(\{(e_i, f_i)\} \subset \Sigma \times \Sigma^*\) be a finite dual basis.
(1) We have a pair of adjoint functors \((F,G)\)
\[
F : \mathcal{M}^\mathcal{D} \to \mathcal{M}^\mathcal{e}, \quad F(M) = M \otimes_B \Sigma;
\]
\[
G : \mathcal{M}^\mathcal{e} \to \mathcal{M}^\mathcal{D}, \quad G(N) = N \otimes^\mathcal{e} (\Sigma^* \otimes_B \mathcal{D}).
\]
(2) The functor \(G\) is fully faithful if and only if the canonical map
\[
\text{can} : \Sigma^* \otimes_B \mathcal{D} \otimes_B \Sigma \to \mathcal{C}, \quad \text{can}(\varphi \otimes_B d \otimes_B u) = \varphi(\varepsilon_D(d)u_{[0]}u_{[1]})
\]
is an isomorphism and
\[
(N \otimes^\mathcal{e} (\Sigma^* \otimes_B \mathcal{D})) \otimes_B \Sigma \cong N \otimes^\mathcal{e} (\Sigma^* \otimes_B \mathcal{D} \otimes_B \Sigma)
\]
for all \(N \in \mathcal{M}^\mathcal{e}\).
(3) The functor \(F\) is fully faithful if and only if the map
\[
\overline{\text{can}} : \mathcal{D} \to (\mathcal{D} \otimes_B \Sigma) \otimes^\mathcal{e} (\Sigma^* \otimes_B \mathcal{D}), \quad \overline{\text{can}}(d) = d_{(1)} \otimes_B e_i \otimes_A f_i \otimes_B d_{(2)}
\]
is an isomorphism and the map \(\mathcal{D} \to (\mathcal{D} \otimes_B \Sigma) \otimes_A (\Sigma^* \otimes_B \mathcal{D})\) is a pure morphism in \(B\mathcal{M}\).
(4) \((F,G)\) is an equivalence of categories if and only if \(\text{can}\) is an isomorphism, \(- \otimes_B \Sigma : \mathcal{M}^\mathcal{D} \to \mathcal{M}^\mathcal{e}\) reflects isomorphisms and
\[
(N \otimes^\mathcal{e} (\Sigma^* \otimes_B \mathcal{D})) \otimes_B \Sigma \cong N \otimes^\mathcal{e} (\Sigma^* \otimes_B \mathcal{D} \otimes_B \Sigma)
\]
for all \(N \in \mathcal{M}^\mathcal{e}\).

Remark 4.2. Item (4) in the previous theorem has the following alternative formulation [9 Theorem 5.2]: \((F,G)\) is an equivalence of categories and \(\mathcal{C}\) is flat as a left \(A\)-module if and only if \(\text{can}\) is an isomorphism and \(\mathcal{D} \otimes_B \Sigma\) is faithfully coflat as a left \(\mathcal{D}\)-comodule.

4.2. Coendomorphism corings and Morita-Takeuchi Theory. Let us briefly argue how Morita-Takeuchi Theory on equivalences of categories of comodules over corings (see [52 and 13, Section 23]) might be derived from our set up. Let \(\mathcal{C}\) (resp. \(\mathcal{D}\)) be a coring over a \(k\)-algebra \(A\) (resp. \(B\)). It follows from [29 Proposition 3.4] (see e.g. [52 Theorem 2.1.13]) that for corings flat as left modules over their ground rings, any equivalence between their categories of right comodules is given by a cotensor product functor. Thus, let us consider bicomodules \(\mathcal{D}P\mathcal{D}\) and \(\mathcal{D}Q\mathcal{C}\), and assume that \(- \otimes^\mathcal{D} Q : \mathcal{M}^\mathcal{D} \to \mathcal{M}^\mathcal{e}\) is left adjoint to \(- \otimes^\mathcal{e} P : \mathcal{M}^\mathcal{e} \to \mathcal{M}^\mathcal{D}\). If \(B\mathcal{D}\) is flat, then, by [29 Proposition 4.2], this is equivalent to assume that \(P\) is \((A,\mathcal{D})\)-quasi-finite, that is, \(- \otimes^\mathcal{D} Q : \mathcal{M}^\mathcal{D} \to \mathcal{M}_A\) is left adjoint to the functor \(- \otimes_A P : \mathcal{M}_A \to \mathcal{M}^\mathcal{D}\). This situation is modelled by our theory in the framework of the 2–category \(\text{CAT}\) of categories, functors and natural transformations as follows. Let \(\ast\) be the discrete one-object category. Consider the category \(\mathcal{M}^\mathcal{D}\) as a 0–cell, and the trivial comonad \(\mathcal{D}\mathcal{D}\) (built on the identity functor \(id_{\mathcal{M}^\mathcal{D}} : \mathcal{M}^\mathcal{D} \to \mathcal{M}^\mathcal{D}\)). Then the category \(\text{RCoc}(\ast, id_{\mathcal{M}^\mathcal{D}})\) is isomorphic to \(\mathcal{M}^\mathcal{D}\). By considering the comonad \(- \otimes_A \mathcal{C}\) on the category \(\mathcal{M}_A\), we get an isomorphism of categories \(\text{RCoc}(\ast, - \otimes_A \mathcal{C}) \cong \mathcal{M}^\mathcal{e}\). In this way, we have an adjoint pair \((p,q)\) in \(\text{CAT}\), where \(p = - \otimes_A P, q = - \otimes^\mathcal{D} Q\) (the horizontal composition in \(\text{CAT}\) is the opposite of the composition of functors, thinking that they act on the left on objects and morphisms). Moreover, since \(P\) is a \(\mathcal{C}–\mathcal{D}\)-bicomodule, we obtain that \((p,q)\) is a comonad-morphism from \(id_{\mathcal{M}^\mathcal{D}}\) to \(- \otimes_A \mathcal{C}\), which gives, by Proposition 2.23 (see also 30 Proposition 2.3), a homomorphism of comonads \((- \otimes_A P) \otimes^\mathcal{D} Q \to - \otimes_A \mathcal{C}\). Since \(- \otimes^\mathcal{D} Q\) is a left adjoint, it preserves equalizers, whence, by [29, Lemma 2.2], there is a natural isomorphism \((- \otimes_A P) \otimes^\mathcal{D} Q \cong - \otimes_A (P \otimes^\mathcal{D} Q)\). In this way, the foregoing homomorphism of comonads is determined by a homomorphism of
A–corings $P \otimes^\Sigma Q \rightarrow C$. Now, we can easily deduce from Theorem [8,13] or [30, Theorem 2.7] that, if $B\mathcal{D}$ is flat, then the functors $-\otimes^\Sigma P : \mathcal{M}_C \rightarrow \mathcal{M}_D$ and $-\otimes^\Sigma Q : \mathcal{M}_D \rightarrow \mathcal{M}_C$ give an equivalence of categories if and only if $-\otimes^\Sigma Q$ is exact and faithful. From this, it is easy to deduce [13, 23.12].

**Remark 4.3.** Both the examples given in Section 4.1 and Section 4.2 describe equivalences between two categories of comodules. The relation between the two theories is clarified as follows. Starting with an $A$–coring $\mathcal{C}$ and a $B$–coring $\mathcal{D}$, and a comatrix coring context $(A, B, \Sigma, \Sigma^\dagger, \varepsilon, \eta)$, the comodule $Q = \Sigma^\dagger \otimes_B \mathcal{D}$ is $(A, \mathcal{D})$–quasi-finite and $-\otimes_A Q : \mathcal{M}_A \rightarrow \mathcal{M}_D$ has a left adjoint, represented by the $\mathcal{D}$-$A$–bicomodule $P = \mathcal{D} \otimes_B \Sigma$. In particular, we find the following isomorphism between the two associated comatrix corings occuring in both theories $Q \otimes^\Sigma P = (\Sigma^\dagger \otimes_B \mathcal{D}) \otimes^\Sigma (\mathcal{D} \otimes_B \Sigma) \cong \Sigma^\dagger \otimes_B \mathcal{D} \otimes_B \Sigma$. □

### 4.3. Comatrix Corings over firm rings

A (non-unital) associative ring $R$ is called firm if the multiplication on $R$ induces an isomorphism $R \otimes_R R \cong R$. A right $R$–module over a firm ring $R$ is called firm if and only if $M \otimes_R R \cong R$. One can easily construct a bicategory $\text{Frm}(k)$, whose 0–cells are firm rings, 1–cells are firm bimodules and 2–cells are homomorphisms of bimodules. It was proven in [17] that firm rings can be characterized as corings over their Dorroh-extension. The Galois theory for corings over firm rings has been initiated in [31], [30] and, in a more profound treatment, in [38]. The situation studied in [31] is subsumed by the theory developed in the present paper by taking $\mathcal{B} = \text{Frm}(k)$, and the main results of [31] are obtained as consequences. The version of [9] for corings over firm rings is also recovered in this way. We leave the details of these constructions to the reader.

### 4.4. Descent theory

It was pointed out in [20, Sect. 4.8] and [7, Example 2.1] that the category of descent data associated to a ring extension $B \rightarrow A$ is isomorphic to the category of right comodules over the $A$–coring $A \otimes_B A$. As mentioned in the introduction, some authors would prefer to name all the theory we have developed descent theory, however we reserve this name for the following special situation. Let $A$ and $B$ be $k$–algebras (with unit) and $\Sigma$ a $B$-$A$–bimodule (in the case of the ring extension $B \rightarrow A$, take $\Sigma = B A_A$). Then we can consider the functor

$$- \otimes_B \Sigma : \mathcal{M}_B \rightarrow \mathcal{M}_A.$$ 

The descent problem in this setting is described as the following question

Which right $A$–modules are of the form $N \otimes_B \Sigma$ for some $N \in \mathcal{M}_B$, i.e. for which $M \in \mathcal{M}_A$, can we find an $N \in \mathcal{M}_B$ such that $M \cong N \otimes_B \Sigma$?

A solution to the problem can be formulated if $\Sigma$ is finitely generated and projective as a right $A$–module; then we can construct the comatrix $A$–coring $\mathcal{C} = \Sigma^* \otimes_B \Sigma$ and consider its category of comodules $\mathcal{M}_\mathcal{C}$ as the category of descent data (see [9, §5.2] for the notion of a generalized descent datum). The functor $F$ factorizes in the following way

$$F : \mathcal{M}_B \xrightarrow{- \otimes_B \Sigma} \mathcal{M}_C \xrightarrow{U_\mathcal{C}} \mathcal{M}_A,$$

where $U_\mathcal{C}$ denotes as usual the forgetful functor. Moreover, $- \otimes_B \Sigma$ establishes an equivalence between $\mathcal{M}_B$ and $\mathcal{M}_C$ if $\Sigma$ is faithfully flat as a left $B$–module [26, Theorem 3.10]. Conversely, if $\Sigma^* \otimes_B \Sigma$ is flat as a left $A$–module (e.g., if $B \Sigma$ is flat), and $- \otimes_B \Sigma$ induces an equivalence between $\mathcal{M}_B$ and $\mathcal{M}_C$, then $\Sigma$ is faithfully flat as a left $B$–module [26, Theorem 3.10]. The “classical” descent theorem for a noncommutative ring extension was proved in [23, Teorema 8] (see also [11] and [7, Theorem 3.8]).
Theorem 3.23 can be applied in this situation. A remarkable fact is that in under certain conditions, part (2) of Theorem 3.23 has a converse. Let us first prove the following lemma.

**Lemma 4.4.** Let $A$ and $B$ be rings and $\Sigma \in B\mathcal{M}_A$ finitely generated and projective as right $A$–module with finite dual basis $e = e_i \otimes_A f_i \in \Sigma \otimes_A \Sigma^*$. Consider $H = \text{Hom}_Z(-, \mathbb{Q}/\mathbb{Z}) : \mathcal{M}_B \to B\mathcal{M}_B^{op}$. The following statements are equivalent:

(i) The functor $- \otimes_B \Sigma : \mathcal{M}_B \to \mathcal{M}_A$ is $H$–separable;
(ii) the map $H(\eta) : H(\Sigma \otimes_A \Sigma^*) \to H(B)$ is split epi in $B\mathcal{M}_B$, where $\eta(b) = eb = be$.

**Proof.** We only have to prove that (ii) implies (i). We have to check that for all $M \in \mathcal{M}_B$ the following morphism has a right inverse

$$H(\eta_M) : H(M \otimes_B \Sigma \otimes_A \Sigma^*) \to H(M),$$

where $\eta_M(m) = m \otimes_B e$. Using natural isomorphisms (in the vertical arrows), we see in the following diagram that the right inverse $\xi$ of $\eta$ induces a splitting map $\xi_M$ for $\eta_M$.

\[
\begin{array}{ccc}
H(M \otimes_B \Sigma \otimes_A \Sigma^*) & \xrightarrow{H(\eta_M)} & H(M) \\
\downarrow \cong & & \downarrow \cong \\
\text{Hom}_B(M, H(\Sigma \otimes_A \Sigma^*)) & \xrightarrow{H(\eta)_\circ \eta} & \text{Hom}_B(M, H(B))
\end{array}
\]

\[\square\]

**Theorem 4.5** ([16, Theorem 2.7], [39, Theorem 4.1]). Let $A$ and $B$ be rings, take $\Sigma \in B\mathcal{M}_A$ finitely generated and projective as right $A$–module. Denote by $e \in \Sigma \otimes_A \Sigma^*$ the finite dual basis for $\Sigma$ and put $\mathcal{C} = \Sigma^* \otimes_B \Sigma$. Then (i) implies (ii) implies (iii) $\iff$ (iv).

(i) $H(\eta) : H(\Sigma \otimes_A \Sigma^*) \to H(B)$ is split epi in $B\mathcal{M}_B$;
(ii) $- \otimes_B \Sigma : \mathcal{M}_B \to \mathcal{M}_C$ is an equivalence of categories;
(iii) $- \otimes_B \Sigma : \mathcal{M}_B \to \mathcal{M}_C$ is fully faithful;
(iv) $\eta : B \to \Sigma \otimes_A \Sigma^*$, $\eta(b) = be = eb$ is pure in $B\mathcal{M}$.

If $B$ is commutative and every $\varphi \in \text{End}_A(\Sigma)$ is left $B$–linear, or $B$ is a separable $k$–algebra, then (iv) implies (i) and all statements become equivalent.

**Proof.** (i) $\Rightarrow$ (ii). From Lemma 4.4 we know that $- \otimes_B \Sigma$ is $H$–separable. Moreover, it is known that the functor $H$ preserves all equalizers and reflects isomorphisms, as $\mathbb{Q}/\mathbb{Z}$ is an injective cogenerator in the category $\Lambda\mathcal{B}$ of abelian groups. Hence, the implication follows by Theorem 3.23 taking $\mathfrak{D} = B$, the trivial comonad, $q = \Sigma$, $p = \Sigma^*$, $\mathcal{C} = \Sigma \otimes_A \Sigma^*$, the comatrix coring associated to $\Sigma$, $\mathcal{A} = \mathcal{B} = B\mathcal{M}_B^{op}$ and $\mathcal{Y} = \Lambda\mathcal{B}_B^{op}$, $H = \text{Hom}_Z(-, \mathbb{Q}/\mathbb{Z}) : \mathcal{M}_B \to B\mathcal{M}_B^{op}$, $H' = \text{Hom}_Z(-, \mathbb{Q}/\mathbb{Z}) : \mathcal{M}_A \to \Lambda\mathcal{B}_A^{op}$ and $\mathcal{Z}' = \text{Hom}_B(\Sigma, -) : B\mathcal{M}_B^{op} \to \Lambda\mathcal{B}_B^{op}$.

(ii) $\Rightarrow$ (iii). Trivial.

(iii) $\iff$ (iv). By Theorem 3.10

(iv) $\Rightarrow$ (i) This last implication is proven in [16] Proposition 2.6] in the case $B$ is commutative, and in [39, Theorem 3.4 (i) $\Rightarrow$ (iii)] in the more general case $B$ is a separable $k$–algebra. \[\square\]

If $B \to A$ is an extension of commutative rings and we take $\Sigma = A$, then Theorem 4.5 reduces to the Joyal-Tierney theorem.
4.5. The Hopf-Galois theories. The Galois theory for corings, originated, with many intermediate steps, from Hopf-Galois theory, which itself initiated as a generalization of classical Galois theory. The connection between corings and entwining structures was made explicit in \([7]\), where it was shown that the category of (right) entwined modules for an algebra \(A\) entwined with a coalgebra \(C\) is isomorphic to the category of right comodules over a suitable \(A\)-coring built on \(A \otimes C\). This in particular covers the case of a comodule algebra \(A\) over the Hopf algebra \(H\), which is an \(H\)-Galois extension of \(A_{\text{co}H} = \{a \in A \mid \rho(a) = a \otimes 1_H\}\) if and only if the canonical map
\[
\text{can} : A \otimes_{A_{\text{co}H}} A \to A \otimes_k H, \quad \text{can}(a \otimes a') = aa'_{[0]} \otimes a'_{[1]},
\]
is an isomorphism, where \(\rho^A(a) = a_{[0]} \otimes a_{[1]}\) denotes the \(H\)-coaction on \(A\). This canonical map is nothing but the canonical map corresponding to the grouplike element \(1 \otimes 1\) of the \(A\)-coring \(A \otimes H\). Therefore, Hopf-Galois theory can be generalized in the framework of corings. This theory can be extended in many ways. One can replace \(H\) by a weak Hopf algebra \([15]\), a Hopf algebroid \([4]\) or a coalgebra \([11]\). We refer to e.g. \([14]\) and \([50]\) for more detailed overviews.

4.6. Comatrix Corings over Quasi-Algebras. In this section we provide a new type of examples where our general theory applies, associated to (dual) quasi-bialgebras. The philosophy is similar to the construction of comatrix corings over firm rings \([31]\), where we constructed comatrix corings, replacing one of the unital rings by a firm ring. Firmness means exactly that the category of bimodules over the firm ring is a monoidal category with the base ring as monoidal unit. In the same way, we can work with a ring (with unit), which is not associative, but inducing a canonical isomorphism \(M \otimes_R (N \otimes_R P) \cong (M \otimes_R N) \otimes_R P\) for all \(R\)-bimodules \(M, N\) and \(P\). Such a framework can be obtained in the following setting.

Let \(k\) be a commutative ring, unadorned tensor products in this section are tensor products over \(k\). A dual quasi-bialgebra is a sextet \((H, \Delta, \epsilon, \mu, \eta, \phi)\), where \(\Delta : H \to H \otimes H\) is a coassociative coproduct and \(\epsilon : H \to k\) is a counit for \(\Delta\). Furthermore \(\phi : H \otimes H \otimes H \to k\) is a unital 3-cocycle that is convolution invertible, this means that the following identities hold for all \(a, b, c, d \in H\)
\[
\begin{align*}
\phi(b_{(1)}, c_{(1)}, d_{(1)}) &\phi(a_{(1)}, b_{(2)} \cdot c_{(2)}, d_{(2)}) \phi(a_{(2)}, b_{(3)}, c_{(3)}) \\
&= \phi(a_{(1)}, b_{(1)}, c_{(1)} \cdot d_{(1)}) \phi(a_{(2)} : b_{(2)}, c_{(2)}, d_{(2)});
\end{align*}
\]
(30) \(\phi(a, 1, b) = \epsilon(a) \epsilon(b)\).

There exists a map \(\phi^{-1} : H \otimes H \otimes H \to k\) such that
\[
\begin{align*}
\phi(a_{(1)}, b_{(1)}, c_{(1)}) \phi^{-1}(a_{(2)}, b_{(2)}, c_{(2)}) &= \epsilon(a) \epsilon(b) \epsilon(c) \\
&= \phi^{-1}(a_{(1)}, b_{(1)}, c_{(1)}) \phi(a_{(2)}, b_{(2)}, c_{(2)}).
\end{align*}
\]
(31) Furthermore, the product \(\mu : H \otimes H \to H, \mu(a \otimes b) = a \cdot b\) is associative up to conjugation with by \(\phi\), i.e.
\[
\begin{align*}
(a_{(1)} \cdot b_{(1)} \cdot c_{(2)}) \phi(a_{(2)}, b_{(2)}, c_{(3)}) &= \phi(a_{(1)}, b_{(1)}, c_{(2)}) (a_{(2)} \cdot b_{(2)}) \cdot c_{(3)}
\end{align*}
\]
for all \(a, b, c \in H\).

A right comodule over a dual quasi-bialgebra, is a right comodule over \(H\), i.e. a \(k\)-module \(M\) together with a map \(\rho^M : M \to M \otimes H\) satisfying \((M \otimes \Delta) \circ \rho^M = (\rho^M \otimes H) \circ \rho^M\) and \((M \otimes \epsilon) \circ \rho^M\).
Recall [5] that, in analogy to $\text{Bim}(k)$, we can construct out of any monoidal category with coequalizers $\mathcal{M}$, a bicategory $\text{Bim}(\mathcal{M})$, whose 0–cells are algebras in $\mathcal{M}$, 1–cells are bimodules over these algebras and 2–cells are bilinear maps. It is known that the category $\mathcal{M}^H$ of right comodules over a dual quasi bialgebra form a monoidal category. The tensor product of $\mathcal{M}^H$ is $\otimes = \otimes_k$, the monoidal unit is $k$, whose coaction is given by $\eta: k \to H$ and the associativity constraint in $\mathcal{M}^H$ is given by the following natural isomorphism,

$$\Phi_{M,N,P} : (M \otimes N) \otimes P \to M \otimes (N \otimes P), \Phi((m \otimes n) \otimes p) = m_{[0]} \otimes (n_{[0]} \otimes p_{[0]})\phi(m_{[1]}, n_{[1]}, p_{[1]}).$$

Since $\mathcal{M}^H$ is a comodule category, it has coequalizers, therefore, we can construct the bicategory $\text{Bim}(\mathcal{M}^H)$. Moreover, we can apply Galois theory as developed in earlier sections to this bicategory, which will be discussed in a forthcoming paper. Let us just state the following remarkable result, that allows the explicit construction of a new type of comatrix corings.

Consider an associative $k$–algebra $A$, an $H$–comodule quasi-algebra $B$ (i.e. $B$ is an algebra in the monoidal category $\mathcal{M}^H$) and let $(A,B,\Sigma, \Sigma', \mu, \eta)$ be an adjoint pair in $\text{Bim}(\mathcal{M}^H)$. This means that $\Sigma \in B \mathcal{M}^A_N$, $\Sigma' \in A \mathcal{M}^H_N$, $\eta: B \to \Sigma \otimes_A \Sigma'$ is a morphism in $B \mathcal{M}^H_N$ and $\mu: \Sigma' \otimes \Sigma \to A$ is a morphism in $A \mathcal{M}^A_N$. Since the map $\eta$ is completely determined by its image on $1_B$, let us denote $\eta(1_B) = e_i \otimes_A f_i \in \Sigma \otimes_A \Sigma'$. The conditions of an adjoint pair then translate into the following identities

$$x = e_i[0] \mu(f_i[0] \otimes_B x_[0]) \phi(e_i[1], f_i[1], x[1]);$$
$$y = \mu(y[0] \otimes_B e_i[0]) f_i[0] \phi^{-1}(e_i[1], f_i[1], x[1]),$$

for all $x \in \Sigma$ and $y \in \Sigma'$. By [9] we immediately obtain the following result, which generalizes the construction of (finite) comatrix corings given in [26] Proposition 2.1.

**Theorem 4.6.** With notation introduced above, $\Sigma' \otimes_A \Sigma$ is an $A$–coring, with counit $\mu$ and comultiplication $\Delta$ defined for all $y \otimes_B x \in \Sigma' \otimes_B \Sigma$ by

$$\Delta(y \otimes_B x) = (y[0] \otimes_B e_i[0]) \otimes_A (f_i[0] \otimes_B x_[0]) \phi(y[1] \cdot e_i[1], f_i[1], x[1]) \phi^{-1}(e_i[2], f_i[2], y[2]).$$

4.7. Galois theory of Group-corings. In [18] a Galois theory for group corings is being developed. We will now show that this theory fits as well in our general framework. To this end, we introduce a new bicategory, that unifies the bicategory of bimodules with the notion of a Turaev category, introduced in [17].

**Definition 4.7.** Let $k$ be a commutative ring, then we define the bicategory $\text{Tur}(k)$ as follows. The 0–cells are $k$–algebras, a 1–cell $M: A \to B$ is a couple $(X, (M_x)_{x \in X})$, where $X$ is a set and all $M_x$ are $A$–$B$–bimodules. A 2–cell $\varphi = (f, \psi): M = (X, M_x) \to N = (Y, N_y)$ consists of a map $f: Y \to X$ together with a collection of $A$–$B$–bilinear maps $\varphi_y: M_{f(y)} \to N_y$. The composition of 1–cells is defined as follows. Take $M = (X, M_x) \in \text{Hom}_1(A, B)$ and $N = (Y, N_y) \in \text{Hom}_1(B, C)$ then we define $M \bullet_B N = (X \times Y, M_x \otimes_B N_y)$. In the same way, we define the horizontal composition $\circ$ of 2–cells. For $\varphi = (f, \psi): M \to N$ and $\psi = (g, \psi): N \to P = (Z, P_z)$ we define the vertical composition $\psi \circ \varphi = (f \circ g, (\psi_z \circ \varphi_{g(z)})_{z \in Z})$.

If we consider $\text{Tur}(k)$ with only one 0–cell $k$, then we recover the notion of a Turaev category $T_k$ of [17].

It is possible to construct another bicategory, with the same 0–cells and 1–cells, but where 2 cells are of the form $\varphi = (f, \varphi): M = (X, M_x) \to N = (Y, N_y)$ consisting of a map $f: X \to Y$ together with a collection of $A$–$B$–bilinear maps $\varphi_x: M_x \to N_{f(y)}$. Considering
such a bicategory with only one 0–cell $k$, we recover the notion of a Zunino category $Z_k$ introduced in [17].

Both bicategories introduced above, admit a locally faithful pseudo functor from $Bim(k)$. More precisely,

$$F : Bim(k) \rightarrow Tur(k)$$

is defined by $F(A) = A$ on 0–cells, $F(M) = \{(*), M\}$ on 1–cells.

As it was shown in [17], the Turaev and Zunino categories lead to a conceptual interpretation of group-algebras, -coalgebras and -Hopf algebras, being algebras, coalgebras and Hopf algebras in a (braided) monoidal category. In a similar way, we can use the Turaev and Zunino bicategories to interpret other constructions of ‘group’-type. For example, mixed distributive laws (entwining structures) can be constructed in any bicategory. Their interpretation in the Turaev bicategory lead to the notion of ‘group entwining structures’ as introduced in [49].

A comonad in $Tur(k)$ coincides with the notion of a group-coring as defined in the recent paper [18]. If we apply the Galois theory of this paper to the Turaev bicategory, we obtain the Galois theory of [18] as a special situation.

4.8. Comonads over CAT. Consider now $B = \mathbf{CAT}$, a category whose 0–cells are categories which are small in some Grothendieck universe, the 1–cells are functors, and the 2–cells are natural transformations between them. Then comonads are cotriples, and the Galois theory that we have developed is tightly linked to the theory of comonadicity (or cotripleability) of functors. In particular, we recover the famous theorem of Beck. Let $(F,G)$ be a pair of adjoint functors with $F : B \rightarrow A$ and $G : A \rightarrow B$. Recall that we can construct a cotriple (comonad) $C = FG$ on $A$ which induces a pair of adjoint functors $(F^C, G^C)$ with $F^C : A^C \rightarrow A$ and $G^C : A \rightarrow A^C$. Moreover there exists a unique functor $K : B \rightarrow A^C$ such that $F = F^C K$ and $KG = G^C$. From Theorem 3.10 and Theorem 3.13 we immediately obtain the following.

**Theorem 4.8** (Beck). If the category $B$ has equalizers, then $K$ has a left adjoint $L : A^C \rightarrow B$, which is fully faithful if and only if $F$ preserves equalizers. If moreover $F$ reflects isomorphisms, then $K$ is an equivalence between the categories $A^C$ and $B$.

This part of the theory and its connection with the theory of Galois comodules over firm rings has been discussed in more detail in [30].

Let us just remark that Galois theory in $\mathbf{CAT}$ applied to the situation of corings and comodules, recovers the theory of Galois comodules in the sense of Wisbauer [44], termed comonadic-Galois comodules in [17]. Consider any firm ring $R$ and an $A$–coring $\mathcal{C}$ and take $\Sigma \in R\mathcal{M}^\mathcal{C}$. Then we say that $\Sigma$ is an $R\mathcal{C}$ comonadic-Galois comodule if the following morphism is an isomorphism for all $M \in \mathcal{M}_A$

$$\text{can}_M : \text{Hom}_A(\Sigma, M) \otimes_R M \rightarrow M \otimes_A \mathcal{C}, \text{ can}_M(\varphi \otimes_R u) = \varphi(u_{[0]}) \otimes_A u_{[1]}.$$  

For any $R\mathcal{C}$–bicomodule $\Sigma$, the functor $- \otimes_R \Sigma : \mathcal{M}_R \rightarrow \mathcal{M}_A$ has a right adjoint given by $\text{Hom}_A(\Sigma, -) \otimes_R R : \mathcal{M}_A \rightarrow \mathcal{M}_R$. Hence we can construct the associated comonad in $\mathbf{CAT}$ and compare it with $\mathcal{C}$ by a canonical comonad-morphism, which becomes exactly the canonical cotriple morphism (33). To apply the Galois theory in $\mathbf{Frm}(k)$ and obtain a firm Galois comodule, we need however a comonad-morphism with adjunction in $\mathbf{Frm}(k)$, which means that there must exist a 2–cell in $\mathbf{Frm}(k)$ that represents the functor $\text{Hom}_A(\Sigma, -) \otimes_R R$. By the Eilenberg-Watt’s theorem over firm rings (see [17, Theorem 3.1]), we know that this means exactly that $\Sigma$ is $R$–firmly projective as right $A$–module. This condition is satisfied.
once $\text{Hom}_A(\Sigma, -) \otimes_R R : M_A \to M_R$ has a right adjoint, so in particular if $\text{Hom}_R^c(\Sigma, -) \otimes_R R : M^c \to M_R$ has a right adjoint or if $- \otimes_R \Sigma : M_R \to M^c$ is an equivalence of categories. In these situations, both theories coincide. A consequence of this reasoning is that any equivalence of categories between a category of comodules and a category of modules over a unital ring reduces to ‘finite’ Galois theory (i.e. in the sense of [26]), this was proven in [47].

4.9. **Galois theory for monads.** Since monads in a bicategory $\mathcal{B}$ are comonads in the bicategory $\mathcal{B}^{co}$, we obtain by direct dualization a Galois theory for monads. The explicit definition of the associated bicategories of Eilenberg-Moore objects have been given in [35] and were one of our major inspirations. However, the Galois theory was not developed there and can therefore be derived from our approach.

An interesting example of this dual theory can be the theory of monadicity of functors, taking $\mathcal{B} = \mathbf{CAT}$. Among many, an interesting paper is [32], where a categorical interpretation of the Joyal-Tierney theorem is proven. However, comparing their result to Theorem 3.23, they only prove a version of part (1) of the theorem.

4.10. **Galois theory of Matrix $C$–rings.** Let $k$ be a field and consider the category $\text{Bic}(k)$. An adjoint pair in this bicategory consists of two $k$–coalgebras $C$ and $D$, a $C$–$D$–bicomodule $M$, a $D$–$C$–bicomodule $N$ and two bicolinear maps $\sigma : C \to N \otimes^D M$ and $\tau : M \otimes^C N \to D$, satisfying $\tau \otimes^D N \cong N \otimes^C \sigma$ and $M \otimes^C \sigma \cong \tau \otimes^D M$. Applying (9), we find that $M \otimes^D N$ is a monad in $\mathcal{B}$ and $N \otimes^C M$ is a comonad in $\mathcal{B}$. If we consider the comonad, then we can apply our general theory (or in fact, the special case of the coendomorphism corings), if we consider the monad, then we have to apply the dual version of the theory. This dual Galois theory is recently developed in [12], the monad $M \otimes^D N$ is termed a matrix $C$–ring.

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DEPARTAMENTO DE ÁLGEBRA, UNIVERSIDAD DE GRANADA, E-18071 GRANADA, SPAIN
E-mail address: gomezj@ugr.es
URL: www.ugr.es/~gomezj

FACULTY OF ENGINEERING, VRIJE UNIVERSITEIT BRUSSEL (VUB), B-1050 BRUSSELS, BELGIUM
E-mail address: jvercruy@vub.ac.be
URL: homepages.vub.ac.be/~jvercruy