Phase covariant qubit dynamics and divisibility

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Phase covariant qubit dynamics describes an evolution of a two-level system under simultaneous action of pure dephasing, energy dissipation, and energy gain with time-dependent rates $\gamma_+ (t)$, $\gamma_- (t)$, and $\gamma_z (t)$, respectively. Non-negative rates correspond to completely positive divisible dynamics, which can still exhibit such peculiarities as non-monotonicity of populations for any initial state. We find a set of quantum channels attainable in the completely positive divisible phase covariant dynamics and show that this set coincides with the set of channels attainable in semigroup phase covariant dynamics. We also construct new examples of eternally indivisible dynamics with $\gamma_z (t) < 0$ for all $t > 0$ that is neither unital nor commutative. Using the quantum Sinkhorn theorem, we for the first time derive a restriction on the decoherence rates under which the dynamics is positive divisible, namely, $\gamma_+ (t) \geq 0, \sqrt{\gamma_+ (t) \gamma_- (t)} + 2 \gamma_z (t) > 0$. Finally, we consider phase covariant convolution master equations and find a class of admissible memory kernels that guarantee complete positivity of the dynamical map.

I. INTRODUCTION

Quantum theory has a well defined statistical structure [1], where quantum states are associated with density operators $\varrho$ in a Hilbert space $\mathcal{H}$, i.e., positive semidefinite operators with unit trace. Hereafter we consider a finite-dimensional Hilbert space $\mathcal{H}_d$ of dimension $d$. We denote the set of bounded (or linear) operators acting on $\mathcal{H}_d$ by $\mathcal{B} (\mathcal{H}_d)$. Provided the system state is decoupled from its environment at time $t = 0$, any physical evolution of the quantum system is described by a linear quantum dynamical map $\Phi (t) : \mathcal{B} (\mathcal{H}_d) \to \mathcal{B} (\mathcal{H}_d)$, $t \geq 0$, satisfying the properties of complete positivity (CP), trace preservation, and the initial condition $\Phi (0) = \text{Id}$, where $\text{Id}$ stands for the identity transformation [2–5]. In the Schrödinger picture of the system-environment evolution, $\varrho (t) = \Phi (t) [\varrho (0)] = \text{tr}_{\text{env}} \left[ U (t) \varrho (0) \otimes \xi (0) U^\dagger (t) \right]$, (1)

where $\xi (0) \in \mathcal{B} (\mathcal{H}_{\text{env}})$ is the initial density operator of the environment, $\text{tr}_{\text{env}}$ is a partial trace over environmental degrees of freedom, and $U (t) \in \mathcal{B} (\mathcal{H}_d \otimes \mathcal{H}_{\text{env}})$ is a unitary evolution operator. The dimension of the effective reservoir for a general dynamics is estimated in Ref. [4]. Eq. [1] automatically guarantees that $\Phi (t)$ is completely positive, i.e., the map $\Phi (t) \otimes \text{Id}_k$ is positive for all identity transformations $\text{Id}_k : \mathcal{B} (\mathcal{H}_k) \to \mathcal{B} (\mathcal{H}_k)$, and $\Phi (t)$ is trace preserving, i.e., $\text{tr} [\Phi (t) [X]] = \text{tr} [X]$ for all $X \in \mathcal{B} (\mathcal{H}_d)$.

An important class of quantum dynamical maps is Markov semigroups $\Phi (t) = e^{tL}$, Ref. [7]. Complete positivity of $\Phi (t)$ forces the generator $L : \mathcal{B} (\mathcal{H}_d) \to \mathcal{B} (\mathcal{H}_d)$ to be of a special Gorini-Kossakovski-Sudarshan-Lindblad (GKSL) form [8–11]:

$$L [\varrho] = -i [H, \varrho] + \sum_k \gamma_k \left( A_k \varrho A_k^\dagger - \frac{1}{2} \{ \varrho, A_k^\dagger A_k \} \right),$$

where $H = H^\dagger$, $\gamma_k \geq 0$ are the decoherence rates, $A_k : \mathcal{H}_d \to \mathcal{H}_d$ are jump operators, $\{ \cdot , \cdot \}$ and $\{ \cdot , \cdot \}$ denote the commutator and anticommutator, respectively. Markov semigroups turned out to be an adequate description of open quantum dynamics in the weak-coupling limit [10–11], the singular-coupling limit [12–13], the stochastic limit [14], the low-density limit and monitoring approach for gas environment [15–19], the stroboscopic limit in the collision model [20–23]. However, it is worth mentioning that the Markov semigroup $e^{tL}$ cannot be exactly reproduced by Eq. [1] with unitary operator $U (t) = e^{-iH^\dagger t}$ unless all $\gamma_k = 0$ or the spectrum of the system-environment Hamiltonian $H'$ is unbounded from below [21].

Covariance of a dynamical quantum map $\Phi (t)$ with respect to a unitary representation $g \to V_g \in \mathcal{B} (\mathcal{H}_d)$ of group $G$, $g \in G$, means that there exists a unitary representation $g \to W_g, g \in G$, in $\mathcal{H}_d$ such that [25–26]

$$\Phi [V_g \varrho V_g^\dagger] = W_g \Phi [\varrho] W_g^\dagger$$

for all $g \in G$ and all density operators $\varrho$. Covariance implies some particular structure on the dynamical map $\Phi (t)$ [27] and, in the special case of the Markov semigroup, on the generator $L$ [25–26–28].

In this paper, we consider phase covariant quantum dynamical maps for two-level systems (qubits, $d = 2$) that satisfy the relation $\exp (-i \sigma_z \varphi) \Phi [\varrho] \exp (i \sigma_z \varphi) = \Phi [\exp (-i \sigma_z \varphi) \varrho \exp (i \sigma_z \varphi)]$ for all real $\varphi$. Hereafter, $\sigma_x, \sigma_y, \sigma_z \in \mathcal{B} (\mathcal{H}_2)$.
The action of parameterized by a Bloch vector $\Lambda(t)$ is contracted into an ellipsoid of revolution with the principal semi-axes $\rho$ are the conventional Pauli operators and $\sigma_0 = I$ is the identity operator on $\mathcal{H}_2$. Up to an irrelevant transformation $\rho \to \exp(-i\sigma_z \theta) \rho \exp(i\sigma_z \theta)$, $\theta \in \mathbb{R}$, the phase covariant qubit dynamical map $\Phi(t)$ reads

$$\Phi(t)[\rho] = \frac{1}{2} \left\{ \text{tr}[\rho] (I + t_z(t) \sigma_z) + \lambda(t) \text{tr}[\sigma_x \rho] \sigma_x + \lambda(t) \text{tr}[\sigma_y \rho] \sigma_y + \lambda_z(t) \text{tr}[\sigma_z \rho] \sigma_z \right\}$$

and is fully characterized by three real-valued functions $\lambda(t), \lambda_z(t)$, and $t_z(t)$. The trace-preservation condition for $\Phi(t)$ is automatically fulfilled, whereas the complete positivity of $\Phi(t)$ is equivalent to positivity of the Choi state $\Omega_{\Phi(t)}(\lambda) = (\Phi(t) \otimes \text{Id})|\psi_+\rangle\langle \psi_+| \in \mathcal{B}(\mathcal{H}_4)$, where $\mathcal{H}_4 \ni |\psi_+\rangle = \frac{1}{\sqrt{2}} (|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle)$, with $\{|0\rangle, |1\rangle\}$ being an orthonormal basis in $\mathcal{H}_2$, see, e.g., Ref. [20]. Direct calculation shows that $\Phi(t)$ is completely positive, and hence a valid dynamical map, if and only if

$$|\lambda_z(t)| + |t_z(t)| \leq 1 \quad \text{and} \quad 4\lambda^2(t) + t_z^2(t) \leq [1 + \lambda_z(t)]^2.$$  

(5)

For a fixed time $t \geq 0$ the map $\Phi(t) : \mathcal{B}(\mathcal{H}_2) \to \mathcal{B}(\mathcal{H}_2)$ is a phase covariant qubit channel that we will further denote by $\Phi$ for brevity. $\Phi$ is given by three real parameters $\lambda, \lambda_z$, and $t_z$,

$$\Phi[\rho] = \frac{1}{2} \left\{ \text{tr}[\rho] (I + t_z \sigma_z) + \lambda \text{tr}[\sigma_x \rho] \sigma_x + \lambda \text{tr}[\sigma_y \rho] \sigma_y + \lambda_z \text{tr}[\sigma_z \rho] \sigma_z \right\}.$$  

(6)

The action of $\Phi$ on the set of qubit density operators has a clear geometrical meaning. Any qubit density operator $\rho$ is parameterized by a Bloch vector $r \in \mathbb{R}^3$ inside the Bloch ball $|r| \leq 1$, namely, $\rho = \frac{1}{2} (I + r_x \sigma_x + r_y \sigma_y + r_z \sigma_z)$. Action of $\Phi$ on $\rho$ leads to an affine transformation: $r_x \to \lambda r_x, \ r_y \to \lambda r_y, \ r_z \to \lambda_z r_z$ $+$ $t_z$. In other words, the Bloch ball is contracted into an ellipsoid of revolution with the principal semi-axes $|\lambda|, |\lambda_z|$ and shifted by $t_z$ along the $z$-axis, see Fig. 1. Conditions (5) for complete positivity of $\Phi$ are visualized in Fig. 2.

The important feature of the phase covariant qubit dynamical maps is that the concatenation of two such maps $\Phi_1(t)$ and $\Phi_2(t)$ is again phase covariant, however, $\Phi_1(t)\Phi_2(t) \neq \Phi_2(t)\Phi_1(t)$, i.e., phase covariant qubit dynamical maps are not commutative in general. In particular, $\Phi(t)\Phi(s) \neq \Phi(s)\Phi(t)$ for a general phase covariant dynamics $\Phi(t)$ and different time moments $s$ and $t$.

Much attention has been recently paid to divisibility of quantum dynamical maps and the study of its relation with non-Markovianity (see the reviews [25,22]). The notion of divisibility is based on the intermediate propagator map $\Lambda(t_2, t_1)$ between time moments $t_1 \geq 0$ and $t_2 \geq t_1$ such that $\Phi(t_2) = \Lambda(t_2, t_1)\Phi(t_1)$. Hereafter we assume that the quantum dynamical map $\Phi(t)$ is invertible for any $t \geq 0$, i.e., $\lambda(t)\lambda_z(t) \neq 0$ for any finite $t$. Then

$$\Lambda(t_2, t_1) = \Phi(t_2)\Phi^{-1}(t_1).$$  

(7)

$\Phi(t)$ is called positive divisible (P-divisible) if $\Lambda(t_2, t_1)$ is positive for all $t_2 \geq t_1 \geq 0$, i.e., $\Lambda(t_2, t_1)[X] \geq 0$ for all positive semidefinite operators $X \geq 0$. Similarly, $\Phi(t)$ is called completely positive divisible (CP-divisible) if $\Lambda(t_2, t_1)$ is completely positive for all $t_2 \geq t_1 \geq 0$. 

![FIG. 1: Bloch ball transformation under the action of the phase covariant map (6).](image)

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corresponds to a CP-indivisible dynamics, which is one of possible approaches to define non-Markovianity [45–47]. In possible initial states of time $t$

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For differentiable quantum dynamical maps $\Phi(t)$ the condition of CP-divisibility is readily reformulated in terms of the time dependent generator $L(t) = \frac{d\Phi(t)}{dt} - \Phi^{-1}(t)$. In fact, for the phase covariant qubit dynamical map [4] we have

where $\sigma_{\pm} = \frac{1}{2}(\sigma_x \pm i\sigma_y)$ and the real valued decoherence rates $\gamma_{\pm}(t)$ and $\gamma_z(t)$ are expressed through $\lambda(t)$, $\lambda_z(t)$, and $t_{\pm}(t)$ as follows:

Given the time-local generator $L(t)$, the density matrix evolution is given by the master equation

which has a clear physical meaning: the first term in the dissipator describes energy gain, the second term describes energy dissipation, and the third term describes pure dephasing. Therefore, the time-convolutionless master equation [10] is a generalization of the commonly used decoherence model for spin systems and superconducting qubits that involves two characteristic times $T_1$ and $T_2$ [13, 14].

Physically, the propagator $\Lambda(t + dt, t)$ for an infinitesimal time interval $dt$ reads $e^{L(t)dt}$. Therefore, $\Lambda(t + dt, t)$ is completely positive if and only if $L(t)$ is a time-local version of the GKSL generator, i.e., $\gamma_{\pm}(t) \geq 0$ and $\gamma_z(t) \geq 0$. On the other hand, if all infinitesimal propagators are completely positive, then $\Lambda(t_2, t_1)$ is completely positive for all $t_2 \geq t_1 \geq 0$. Hence, $\Phi(t)$ is CP-divisible if and only if $\gamma_{\pm}(t) \geq 0$ and $\gamma_z(t) \geq 0$.

It was recently noticed in Ref. [34] that the population $p(t) := \frac{1}{2}(1 + \text{tr}[\rho(t)\sigma_z])$ can be a non-monotonic function of time $t$ in a CP-divisible phase covariant dynamics for some initial states $\rho(0)$. In this paper, we revisit this issue and demonstrate a CP-divisible phase covariant dynamics $\Phi(t)$ such that the population $p(t)$ is not monotonic for all possible initial states $\rho(0)$.

The rates $\gamma_{\pm}(t)$ and $\gamma_z(t)$ can temporarily get negative values without violating complete positivity of $\Phi(t)$. This corresponds to a CP-indivisible dynamics, which is one of possible approaches to define non-Markovianity [15, 17]. In fact, an inverse relation to [9] is

where $\Gamma_{\pm}(t) = \int_0^t \gamma_{\pm}(t')dt'$ and $\Gamma_z(t) = \int_0^t \gamma_z(t')dt'$. The only restriction on the rates $\gamma_{\pm}(t)$ and $\gamma_z(t)$ is that $\lambda(t)$, $\lambda_z(t)$, and $t_{\pm}(t)$ must satisfy inequalities [3], so the rates $\gamma_{\pm}(t)$ and $\gamma_z(t)$ can temporarily become negative. Moreover, the
rate $\gamma_z(t)$ can be negative for all $t > 0$, which corresponds to an eternal CP-indivisible dynamics \cite{47,49}. Previously, the examples of eternal CP-indivisible dynamics were constructed only in the case of unital qubit dynamical maps that are commutative \cite{47,49}. In this paper, we construct an extended class of eternal CP-indivisible dynamics for non-commutative phase covariant qubit maps.

As the positivity of a linear map is much more difficult to characterize as compared to complete positivity \cite{50}, it is not surprising that the necessary and sufficient conditions for P-divisibility of the phase covariant qubit dynamics have remained unknown. In this paper, we fill this gap and find a criterion of P-divisibility in terms of decoherence rates $\gamma_\pm(t)$ and $\gamma_z(t)$. A key tool in this study is the quantum Sinkhorn theorem \cite{51,57} that allows to characterize the region of parameters $\lambda, \lambda_z$, and $t_z$ such that Eq. (6) defines a positive map.

The dynamics of a $d$-dimensional open quantum system can be alternatively described by means of the Nakajima-Zwanzig projective techniques \cite{3,61,63} leading to an integro-differential master equation of the form
\[
\frac{d\rho(t)}{dt} = \int_0^t K(t')\rho(t-t')dt'.
\]

with the memory kernel $K(t') : \mathcal{B}(H_d) \to \mathcal{B}(H_d)$. In terms of the dynamical maps, Eq. (13) reads $\frac{d\rho(t)}{dt} = \int_0^t K(t')\Phi(t-t')dt'$. It is an open question what memory kernels $K(t')$ define a legitimate quantum dynamics $\Phi(t)$ in general. The admissible memory kernels are characterized for Pauli dynamical maps \cite{64} and quantum semi-Markov processes \cite{65,67}. In this paper, we give sufficient conditions for admissible memory kernels corresponding to legitimate phase covariant qubit dynamics. It facilitates the analysis of divisibility of such dynamical maps by subjecting them to time deformations \cite{68}.

The paper is organized as follows. In Section II, we characterize channels \cite{6} that are attainable in the phase covariant CP divisible dynamics and semigroup dynamics. In Section III, we revisit the result of Ref. \cite{34} and construct a phase covariant process with non-negative rates $\gamma_\pm(t)$ and $\gamma_z(t) \geq 0$ such that the population $p(t)$ is non-monotonic for all initial states. In Section IV, we construct two families of non-unital eternal CP-indivisible phase covariant processes, one of which is commutative and the other is not. In Section V, we use the quantum Sinkhorn theorem and find parameters $\lambda, \lambda_z, t_z$ for which Eq. (6) defines a positive map. We further use this result and characterize positive divisible dynamical maps \cite{6}. In Section VI, we find a class of admissible memory kernels $K(t)$ leading to legitimate phase covariant qubit dynamics. In Section VII, brief conclusions are given.

II. CHANNELS ATTAINABLE IN SEMIGROUP DYNAMICS AND CP-DIVISIBLE DYNAMICS

Following the classification of Refs. \cite{15,59,70}, consider a class of quantum channels $C^L$ attainable in at least one phase covariant semigroup dynamics, i.e.,
\[
C^L = \{e^{Lt} \mid t \geq 0\} \quad \text{and} \quad L \text{ has the form } \mathcal{L} \text{ with constant rates } \gamma_\pm, \gamma_z \geq 0,\]

where the overline denotes the closure in operator norm for Choi matrices.

Proposition 1. $C^L$ consists of phase covariant qubit channels \cite{6} with $|\lambda_z| + |t_z| \leq 1$, $4\lambda^2 + t_z^2 \leq (1 + \lambda_z)^2$, $\lambda \geq 0$, and $\lambda_z \geq \lambda^2$.

Proof. The relations \cite{11} and \cite{12} imply that in the semigroup dynamics
\[
\lambda(t) = e^{-\frac{t}{2}(\gamma_+ + \gamma_- + 4\gamma_z) t}, \quad \lambda_z(t) = e^{-\frac{t}{2}(\gamma_+ + \gamma_-) t}, \quad t_z(t) = \frac{\gamma_+ - \gamma_-}{\gamma_+ + \gamma_-} \left[1 - e^{-\frac{t}{2}(\gamma_+ + \gamma_-) t}\right]
\]

and necessarily satisfy the claimed conditions. To see the other direction, suppose $|\lambda_z| + |t_z| \leq 1$, $4\lambda^2 + t_z^2 \leq (1 + \lambda_z)^2$, $\lambda \geq 0$, and $1 > \lambda_z > \lambda^2$, so that the channel $\Phi$ with parameters $\lambda, \lambda_z, t_z$ can be expressed as $\Phi = e^L$, where
\[
\gamma_\pm = \frac{(1 - \lambda_z \pm t_z)(-\ln \lambda_z)}{2(1 - \lambda_z)} \geq 0, \quad \gamma_z = \frac{1}{4} \ln \frac{\lambda_z}{\lambda^2} \geq 0.
\]

The non-strict inequalities for $\lambda, \lambda_z$, and $t_z$ then follow from the closure procedure. \hfill $\square$

Remark 1. Restoring rotations around $z$-axis of the Bloch ball $(\varrho \to e^{-i\sigma_z \omega t} \varrho e^{i\sigma_z \omega t}, \omega \in \mathbb{R})$ into a general phase covariant dynamics, we get an extra term $-i\omega[\sigma_z, \varrho]$ in $\mathcal{L}[\varrho]$. Whenever $\sin \omega t = \pm 1$, this results in the change $\lambda(t) \to -\lambda(t)$ in Eq. (4). Hence, the region of parameters $\lambda, \lambda_z, t_z$ attainable in a general phase covariant semigroup dynamics is twice larger, namely, $|\lambda_z| + |t_z| \leq 1$, $4\lambda^2 + t_z^2 \leq (1 + \lambda_z)^2$, and $\lambda_z \geq \lambda^2$. 
The class $C^L$ is illustrated in Fig. 3. Every quantum channel within the class $C^L$ is a convex mixture of a dephasing channel in the basis $\{|0\rangle, |1\rangle\}$, an amplitude damping channel with a stationary point $|0\rangle\langle 0|$, and an amplitude damping channel with a stationary point $|1\rangle\langle 1|$. Generalized amplitude damping semigroups $\Phi(t) = \exp\left[\gamma_+ t (\sigma_+ \rho \sigma_- - \frac{i}{2} \{\sigma_+, \sigma_-\}) + \gamma_- t (\sigma_- \rho \sigma_+ - \frac{i}{2} \{\sigma_-, \sigma_+\})\right]$ are ultimately CP-divisible, which means that an infinitesimal perturbation in the trajectory $(\lambda(t), \lambda_z(t), t_z(t))$ of such a dynamical map may break the CP-divisibility property $^{19}$. Extending the classification of Refs. $^{14, 69}$, consider a class of quantum channels $C_{\text{ph cov}}^{\text{CP}}$ attainable in a CP-divisible phase covariant dynamics, i.e.,

$$C_{\text{ph cov}}^{\text{CP}} = \left\{ T_{\text{ph}} \exp\left(\int_0^t L(t') dt'\right) \mid t \geq 0 \text{ and } L(t) \text{ has the form } (\ref{formula}) \text{ with } \gamma_+ (t), \gamma_z (t) \geq 0 \right\}, $$

where $T_{\text{ph}}$ is the Dyson time ordering operator.

**Proposition 2.** $C_{\text{ph cov}}^{\text{CP}} = C^L$.

**Proof.** Obviously, $C^L \subset C_{\text{ph cov}}^{\text{CP}}$ as a particular case of time-independent decoherence rates. Let us prove that $C_{\text{ph cov}}^{\text{CP}} \subset C^L$. Note that the dynamics with the generator $L(t)$ of the form (8) allows only non-negative values of $\lambda(t)$, see Eq. (11). Consider a valid quantum channel $\Phi'$ of the form (6) with parameter $\lambda \geq 0$. As $\Phi'$ is completely positive, the conditions $|\lambda_z| + |t_z| \leq 1$ and $4\lambda^2 + t_z^2 \leq (1 + \lambda)^2$ are automatically satisfied. By Proposition 1, if $\Phi' \notin C^L$, then $\lambda_z < \lambda^2$. Suppose $\Phi' \notin C^L$ and $\Phi'$ is attained by a phase covariant dynamics (4) with some time dependent rates $\gamma_+ (t)$ and $\gamma_z (t)$, i.e., $\Phi' = \Phi(t_0)$ for some $t_0 \geq 0$. As $\lambda_z < \lambda^2$, Eq. (9) implies that $\gamma_z (t_0) < 0$, i.e., $\Phi(t)$ is not CP divisible. Therefore, a CP divisible phase covariant dynamics with the local generator (8) results in the dynamical maps $\Phi(t)$ such that $\Phi(t_0) \in C^L$ for any fixed $t_0 \geq 0$, i.e., $C_{\text{ph cov}}^{\text{CP}} \subset C^L$.

**Remark 2.** As in Remark 1, restoring rotations around z-axis of the Bloch ball into a general phase covariant dynamics, we get the twice larger region of parameters $\lambda, \lambda_z, t_z$ for channels $\Phi$ attainable by CP-divisible phase covariant dynamics, namely, $|\lambda_z| + |t_z| \leq 1$, $4\lambda^2 + t_z^2 \leq (1 + \lambda)^2$, and $\lambda_z \geq \lambda^2$.

**Remark 3.** If a phase covariant qubit channel $\Phi \neq \text{Id}$ is obtained as a result of a semigroup dynamics, i.e., $\Phi = e^{Lt_0}$ for some $t_0 > 0$, then any channel from the semigroup $e^{Lt}$, $t \geq 0$, is phase covariant. However, if a phase covariant qubit channel $\Phi \neq \text{Id}$ is obtained as a result of a general qubit CP-divisible dynamics $\Theta(t)$, i.e., $\Phi = \Theta(t_0)$ for some $t_0 > 0$, then $\Theta(t)$ does not have to be phase covariant for all $t \geq 0$. In fact, consider a phase covariant qubit channel $\Phi'[\rho] = \frac{1}{2} (\text{tr}[\rho I] - \text{tr}[\sigma_x \rho \sigma_x])$ such that $\Phi' \notin C^L = C_{\text{ph cov}}^{\text{CP}}$. Note that $\Phi'[\rho] = \sigma_x \Phi[\rho] \sigma_x$, where $\Phi[\rho] = \frac{1}{2} (\text{tr}[\rho I] + \text{tr}[\sigma_x \rho \sigma_x])$, i.e., $\Phi \in C^L = C_{\text{ph cov}}^{\text{CP}}$. Therefore, the channel $\Phi'$ can be obtained as a result of a CP-divisible dynamics, however, the intermediate transformation $\rho \rightarrow \sigma_x \rho \sigma_x$ is not phase covariant. This example shows that $\Phi' \notin C_{\text{ph cov}}^{\text{CP}}$ but $\Phi' \in C^{\text{CP}}$,

$$C^{\text{CP}} = \left\{ T_{\text{ph}} \exp\left(\int_0^t L(t') dt'\right) \mid t \geq 0 \text{ and } L(t) \text{ has a time-dependent form } (2) \text{ with } \gamma_k (t) \geq 0 \right\}. $$
By using the results of Refs. [15, 60] and the explicit form of the quantum Sinkhorn theorem for phase covariant qubit channels (Proposition 6) as well as taking into account possible unitary rotations of the Bloch ball, we conclude that a channel (6) with parameters \( \lambda, \lambda_z, t_z \) belongs to the class \( \text{C}^\text{P} \) if and only if \( |\lambda_z| + |t_z| \leq 1, 4\lambda^2 + t_z^2 \leq (1 + \lambda_z)^2, \lambda \geq \lambda^2 \) or \( |\lambda_z| + |t_z| \leq 1, 4\lambda^2 + t_z^2 \leq (1 + \lambda_z)^2, \lambda = 0 \).

III. NON-MONOTONICITY OF POPULATION IN CP-DIVISIBLE DYNAMICS

**Proposition 3.** There exists a CP-divisible phase covariant qubit dynamical map \( \Phi(t) \) such that the population \( p(t) \) is non-monotonic for any initial state \( \rho(0) \).

**Proof.** Let \( \lambda(t) = e^{-\nu t}, \lambda_z(t) = e^{-2\nu t}, \) and \( t_z(t) = \frac{2\nu}{\sqrt{4\nu^2 + \omega^2}} \sin \omega t, \) where \( \nu, \omega > 0 \). Then the population reads

\[
p(t) = \frac{1}{2} \left[ 1 + t_z(t) + \lambda_z(t) \text{tr}[\rho(0) \sigma_z] \right] = \frac{1}{2} \left[ 1 + \frac{2\nu}{\sqrt{4\nu^2 + \omega^2}} \sin \omega t + e^{-2\nu t} \text{tr}[\rho(0) \sigma_z] \right]
\]

and clearly has a non-monotonic behaviour for all initial density operators \( \rho(0) \) if \( t > \frac{1}{2\nu} \ln \frac{\sqrt{4\nu^2 + \omega^2}}{2\nu} \).

To guarantee that \( \Phi(t) \) is a valid CP-divisible dynamical map it suffices to check that the rates \( \gamma_{\pm}(t) \) and \( \gamma_z(t) \) are non-negative for any \( t \geq 0 \). Substituting our particular choice for \( \lambda(t), \lambda_z(t), \) and \( t_z(t) \) into Eq. (9), we get

\[
\gamma_{\pm}(t) = \nu \pm \frac{\nu}{\sqrt{4\nu^2 + \omega^2}} (2\nu \sin \omega t + \omega \cos \omega t) \geq 0, \quad \gamma_z(t) = 0.
\]

Therefore \( \Phi(t) \) is a valid quantum dynamical map enjoying the CP-divisibility property.

The construction used in the proof of Proposition 3 has a clear physical meaning too. As \( \gamma_z(t) = 0 \) for all time moments \( t \geq 0 \), the master equation (10) with the generator (8) defines a generalized amplitude damping dynamics [71], where the time-local stationary state changes in time. Note that the population oscillations (III) do not decay in time. The peak-to-peak amplitude \( \max_{t \geq t_0} p(t) - \min_{t \geq t_0} p(t) \geq \frac{\sqrt{4\nu^2 + \omega^2}}{2\nu} \) for all \( t_0 \geq 0 \) and can be arbitrarily close to 1 if \( \omega \ll \nu \).

IV. ETERNAL CP-INDIVISIBLE DYNAMICS

In this section, we construct a one-parameter family of phase covariant qubit dynamical maps \( \{\Phi_a(t)\}_{|a| < 1} \) such that the intermediate map \( \Lambda(t_2, t_1) \) is not completely positive for any \( t_2 > t_1 > 0 \). Since \( \Phi_a(t) \) is non-unital if \( a \neq 0 \), our construction provides a family of non-unital eternal CP-indivisible dynamical maps.

**Proposition 4.** A phase covariant qubit dynamical map \( \{\Phi_a(t)\} \) of the form (11) with

\[
\lambda(t) = \frac{1}{2} \sqrt{(1 + e^{-2\nu t})^2 - a^2(1 - e^{-2\nu t})^2}, \quad \lambda_z(t) = e^{-2\nu t}, \quad t_z(t) = a(1 - e^{-2\nu t}), \quad \nu > 0,
\]

is eternal CP indivisible for all real \( a \) satisfying \( |a| < 1 \).

**Proof.** \( \{\Phi_a(t)\} \) is a valid quantum dynamical map if \( |a| < 1 \) because the conditions (3) are fulfilled. A direct calculation by Eq. (9) yields

\[
\gamma_{\pm}(t) = \nu(1 \pm a), \quad \gamma_z(t) = -\frac{\nu(1 - a^2) \sinh 2\nu t}{2[1 + a^2 + (1 - a^2) \cosh 2\nu t]}.
\]

If \( |a| < 1 \), then \( \gamma_z(t) < 0 \) for all \( t > 0 \), so \( \Phi_a(t) \) is eternal CP-indivisible.

A feature of the dynamical map \( \Phi_a(t) \) in Proposition 4 is that it is commutative, i.e., \( \Phi_a(t)\Phi_a(s) = \Phi_a(s)\Phi_a(t) \), because the ratio \( \frac{\gamma_{\pm}(t)}{\gamma_z(t)} \) is constant in time. The following proposition shows that there also exist non-commutative eternal CP-indivisible phase covariant qubit processes.

**Proposition 5.** A phase covariant qubit dynamical map \( \{\Phi_b(t)\} \) of the form (11) with

\[
\lambda(t) = \frac{1}{2} \sqrt{(1 + e^{-2\nu t})^2 - b^2 e^{-2\nu t}(1 - e^{-2\nu t})^2}, \quad \lambda_z(t) = e^{-2\nu t}, \quad t_z(t) = be^{-\nu t}(1 - e^{-2\nu t}), \quad \nu > 0,
\]

is non-commutative and eternal CP-indivisible for all real \( b \) satisfying \( 0 < |b| \leq 1 \).
Proof. \( \{ \Phi_b(t) \} \) is a valid quantum dynamical map if \( |b| \leq 1 \) because the conditions (5) are fulfilled. A direct calculation by Eq. (4) yields

\[
\gamma_b(t) = \nu \left( 1 \pm be^{-2\nu t} \cosh \nu t \right), \quad \gamma_z(t) = -\frac{\nu(1 - e^{-2\nu t})(e^{3\nu t} \cosh \nu t - b^2)}{4[e^{2\nu t} \cosh^2 \nu t - b^2 \sinh^2 \nu t]}. 
\]

If \( |b| \leq 1 \), then \( \gamma_z(t) < 0 \) for all \( t > 0 \), so \( \Phi_b(t) \) is eternal CP indivisible. Moreover, since \( t_z(s)[1 - \lambda_z(s)] \neq t_z(s)[1 - \lambda_z(t)] \) for all \( s > t > 0 \) and \( b \neq 0 \), the map \( \Phi_b(t) \) is non-commutative for all \( b \) satisfying \( 0 < |b| \leq 1 \).

Trajectories of the processes \( \Phi_a(t) \) and \( \Phi_b(t) \) in the parameter space \( (\lambda, \lambda_z, t_z) \) belong to the surface of the body depicted in Fig. 2. Note that the constructed dynamical maps \( \Phi_a(t) \) and \( \Phi_b(t) \) reduce to the known eternal CP-indivisible unital qubit dynamics [47–49] if \( a = 0 \) and \( b = 0 \), respectively.

V. POSITIVITY AND POSITIVE DIVISIBILITY

Not only completely positive maps but also positive maps have attracted some attention recently for description of quantum systems and their dynamics [72]. For this reason we characterize a region of parameters \( \lambda, \lambda_z, t_z \) within which the map \( \Phi \) is positive, i.e., \( \Phi[X] \geq 0 \) for all \( X \geq 0 \).

Proposition 6. Eq. (6) defines a positive map if and only if

\[
\begin{align*}
2|\lambda| &\leq \sqrt{(1 + \lambda)^2 - t^2_z + \sqrt{(1 - \lambda)^2 - t^2_z}}, \\
4|\lambda| &\leq \sqrt{(1 + \lambda)^2 - t^2_z + \sqrt{(1 - \lambda)^2 - t^2_z}}.
\end{align*}
\]

(14)

Proof. Note that positivity of \( \Phi \) is equivalent to condition \( \Phi[\rho] > 0 \) for all density matrices \( \rho \). We consider two cases.

Suppose \( |\lambda| + |t_z| > 1 \). The image of the Bloch ball under map (6) is an ellipsoid of revolution, which has a common point with the Bloch sphere either at the north pole or the south pole. Geometrically, the image ellipsoid is a subset of the Bloch ball if and only if \( \lambda^2 \leq |\lambda_z| \).

Suppose \( |\lambda| + |t_z| < 1 \), then there exist non-degenerate operators \( A \) and \( B \) such that the map \( \rho \rightarrow \Upsilon[\rho] = A\Phi[B\rho B^\dagger]A^\dagger \) is unital, i.e.,

\[
\Upsilon[\rho] = \frac{1}{2} \left( \text{tr}[\rho]I + \lambda_x \text{tr}[\sigma_x \rho]\sigma_x + \lambda_y \text{tr}[\sigma_y \rho]\sigma_y + \lambda_z \text{tr}[\sigma_z \rho]\sigma_z \right).
\]

The explicit form of operators \( A \) and \( B \) as well as real numbers \( \lambda_x, \lambda_y, \lambda_z \) are derived in Refs. [54–56]. In particular,

\[
\lambda_x = \lambda_y = \frac{2\lambda}{\sqrt{(1 + \lambda)^2 - t^2_z + \sqrt{(1 - \lambda)^2 - t^2_z}}}, \quad \lambda_z = \frac{4\lambda}{\sqrt{(1 + \lambda)^2 - t^2_z + \sqrt{(1 - \lambda)^2 - t^2_z}}^2}.
\]

(15)

As operators \( A \) and \( B \) are non-degenerate, \( \Phi[\rho] = A^{-1}\Upsilon[B^{-1}\rho(B^{-1})^\dagger](A^{-1})^\dagger \). Therefore, \( \Phi[\rho] \geq 0 \) if and only if \( \Upsilon \) is a positive map, i.e., \( |\lambda_x|, |\lambda_y|, |\lambda_z| \leq 1 \).

Finally, the conditions (14) summarize the two cases altogether.

Remark 4. The relation \( \Upsilon[\rho] = A\Psi[B\rho B^\dagger]A^\dagger \) between a unital trace preserving map \( \Upsilon \) and a strictly positive map \( \Psi \) (i.e., \( \Psi[X] > 0 \) for all \( X \geq 0 \)) is known as the quantum Sinkhorn theorem and reviewed in Refs. [51–57].

Conditions (14) are visualized in Fig. 4. The inequalities (14) can be alternatively reformulated in terms of the maximum distance between the center of the Bloch ball and a point in the image ellipsoid. Namely,

\[
r_{\max}(\Phi) = \max_{\rho, n \in \mathbb{R}^3 : |n| = 1} \frac{1}{2} \text{tr} \left[ (n_x \sigma_x + n_y \sigma_y + n_z \sigma_z) \Phi[\rho] \right] = \begin{cases} 
\frac{|\lambda_z| + |t_z|}{\sqrt{1 + t^2_z}} & \text{if } |\lambda| \leq |\lambda_z|, \\
|\lambda| & \text{if } |\lambda| > |\lambda_z|.
\end{cases}
\]

(16)

The map \( \Phi \) is positive if and only if \( r_{\max}(\Phi) \leq 1 \).

Proposition 6 is a powerful tool that can be also applied to the intermediate map (7). This results in the following characterization of P-divisible phase covariant qubit dynamical maps.
Proposition 7. The master equation \( \frac{d\gamma(t)}{dt} = L(t)[\rho(t)] \) with the generator \( L \) defines a P-divisible dynamical map \( \Phi(t) \) if the decoherence rates satisfy
\[
\gamma_\pm(t) \geq 0 \quad \text{and} \quad \sqrt{\gamma_+(t) + 2\gamma_-(t)} > 0.
\] (17)

\( \Phi(t) \) is not P-divisible if either (i) \( \gamma_+(t) < 0 \), or (ii) \( \gamma_-(t) < 0 \), or (iii) \( \gamma_\pm(t) \geq 0 \) and \( \sqrt{\gamma_+(t) + 2\gamma_-(t)} < 0 \).

Proof. \( \Phi(t) \) is P-divisible if and only if the infinitesimal propagator \( \Lambda(t+dt, t) \) is positive for all \( t \geq 0 \). Since \( \Lambda(t+dt, t) = \text{Id} + L(t)dt + o(dt) \), we conclude that \( \Lambda(t+dt, t) \) is positive if \( \text{Id} + L(t)dt \) is strictly positive. The map \( \text{Id} + L(t)dt \) is phase covariant and has the form of Eq. (6) with the generator \( L \) and the terms up to the first order of \( dt > 0 \), we get the equivalent conditions
\[
\gamma_\pm(t) \geq 0 \quad \text{and} \quad \sqrt{\gamma_+(t) + 2\gamma_-(t)} > 0.
\]
In turn, the first inequality is equivalent to \( \gamma_\pm(t) > 0 \). If \( \gamma_+(t) = 0, \gamma_-(t) \geq 0, \gamma_+(t) > 0, \gamma_-(t) > 0, \gamma_\pm(t) \geq 0, \) then the map \( \Lambda(t+dt, t) \) is completely positive and, consequently, positive. Therefore, conditions (17) guarantee positivity of the intermediate map \( \Lambda(t+dt, t) \).

Conversely, suppose \( \gamma_+(t) < 0 \) or \( \gamma_-(t) < 0 \), then \( \Lambda(t+dt, t) \) is not positive as at least one of the operators \( \Lambda(t+dt, t)[\Sigma(t) \pm \sigma_z] \) is not positive semidefinite. Suppose \( \gamma_\pm(t) \geq 0 \) and \( \sqrt{\gamma_+(t) + 2\gamma_-(t)} < 0 \), then Eq. (16) gives \( r_{\text{max}}(\Lambda(t+dt, t)) = 1 + \frac{2\gamma_+(t) - \gamma_-(t)}{\gamma_+(t) + \gamma_-(t) - 2\gamma_z(t)} + o(dt) > 1 \) and \( \Lambda(t+dt, t) \) is not positive.

Remark 5. If \( \gamma_\pm(t) \geq 0 \) and \( \sqrt{\gamma_+(t) + 2\gamma_-(t)} = 0 \), then one should resort to higher order expansions of \( \Lambda(t+dt, t) \) with respect to \( dt \). The second order expansion yields the condition \( \frac{d\gamma_z(t)}{dt} > \gamma_z(t)[\gamma_+(t) + \gamma_-(t)] \) for P-divisibility of \( \Phi(t) \).

The derived condition of P-divisibility (17) is stronger than the condition for monotonically decreasing distinguishability of quantum states, \( \gamma_+ + \gamma_- \geq 0 \) and \( \gamma_+ + \gamma_- + 4\gamma_z \geq 0 \), Ref. [33]. This is trivial if one of the rates \( \gamma_+ \) or \( \gamma_- \) is negative. However, even if \( \gamma_\pm \geq 0 \) we have \( \gamma_+ + \gamma_- \geq 2\sqrt{\gamma_+ \gamma_-} \), so the P-divisibility condition is stronger than the deceasing distinguishability condition. In other words, there exists a phase covariant qubit dynamics such that the trace distance \( \frac{1}{2}[\Phi(t)[g_1] - \Phi(t)[g_2]]_1 \) monotonically decreases for arbitrary two initial states \( g_1 \) and \( g_2 \), however, the dynamics is not P-divisible.

VI. ADMISSIBLE MEMORY KERNELS

In this section, we address the following question: what memory kernels \( K(t') : \mathcal{B}(\mathcal{H}_d) \rightarrow \mathcal{B}(\mathcal{H}_d) \) in the master equation (13) define a legitimate (completely positive and trace preserving) phase covariant qubit dynamics \( \Phi(t) \)? We provide a sufficient condition for memory kernels and illustrate our findings by an example.
The relation between a dynamical map $\Phi(t)$ and the memory kernel $K(t)$ takes the simplest form if we use the Laplace transform $F_s = \int_0^{\infty} F(t) e^{-st} dt$. Indeed, the Laplace transform of the equation $\frac{d\Phi(t)}{dt} = \int_0^t K(t')[\Phi(t-t')]dt'$ yields $s\Phi_s - \text{Id} = K_s\Phi_s$, where we have taken into account that $\Phi(0) = \text{Id}$. Therefore, for a phase covariant qubit dynamics with the memory kernel

$$K(t)\rho = \chi_+(t)\left(\sigma_+\rho\sigma_- - \frac{1}{2}\{\rho, \sigma_+\sigma_-\}\right) + \chi_-(t)\left(\sigma_-\rho\sigma_+ - \frac{1}{2}\{\rho, \sigma_+\sigma_-\}\right) + \chi_z(t)(\sigma_z\rho\sigma_z - \rho)$$

we get the following parameters $\lambda_s, (\lambda_\pm)_s, (t_\pm)_s$ of $\Phi_s$:

$$\lambda_s = \frac{1}{s + \frac{1}{2}(\chi_+)_s + (\chi_-)_s + 4(\chi_z)_s}, \quad (\lambda_\pm)_s = \frac{1}{s + (\chi_\pm)_s + (\chi_\pm)_s}, \quad (t_\pm)_s = \frac{(\chi_\pm)_s - (\chi_\mp)_s}{s + (\chi_\pm)_s + (\chi_\pm)_s}.$$

By Bernstein’s theorem the Laplace transform $f_s$ of the non-negative smooth function $f(t) : [0, \infty) \to [0, \infty)$ is completely monotone, i.e., $(-1)^n \frac{d^n f_s}{dt^n} \geq 0$ for all $s > 0$ and $n = 0, 1, 2, \ldots$. Hence, we can characterize non-negative functions $f(t)$ in terms of $f_s$. Applying this result to Eq. (5), we conclude that $\Phi(t)$ is completely positive if and only if both functions $\frac{1}{s} - |\lambda_\pm|_s - |t_\pm|_s$ and $\frac{1}{s} + (\lambda_\pm)_s + (\lambda_\pm)_s - 4(\lambda_\pm)_s - (t_\pm)_s = \frac{1}{s} + 2(\lambda_\pm)_s + (\lambda_\pm)_s - 4(\lambda_\pm)_s - (t_\pm)_s$ are completely monotone. However, these conditions cannot be further simplified in terms of the parameters $(\chi_\pm)_s, (\chi_\pm)_s, (\chi_\pm)_s$ of the memory kernel. To overcome this difficulty we resort to a subset of legitimate quantum channels $\Phi$ that is given by linear restrictions on parameters $\lambda, \lambda_z, t_z$.

**Lemma 1.** Eq. (6) defines a completely positive and trace preserving map $\Phi$ if

$$1 + 2\lambda + \lambda_z \pm t_z \geq 0, \quad 1 - 2\lambda + \lambda_z \pm t_z \geq 0, \quad 1 - \lambda_z \pm t_z \geq 0. \quad (20)$$

**Proof.** Conditions (20) define a convex polyhedron in the parameter space $(\lambda, \lambda_z, t_z)$ with vertices (extremal points) that satisfy restrictions [6]. Therefore, conditions (20) define a convex hull of some quantum channels.

Geometrically, conditions (20) define a body depicted in Fig. 5. The edges connecting points $(\lambda = 1, \lambda_z = 1, t_z = 0)$ and $(\lambda = 0, \lambda_z = 0, t_z = \pm 1)$ correspond to families of shifted depolarizing channels. The edge connecting points $(\lambda = 1, \lambda_z = 0, t_z = 0)$ and $(\lambda = 0, \lambda_z = -1, t_z = 0)$ comprises a trajectory of the eternal CP-indivisible unital qubit process $\Phi_{s=0}(t) = \Phi_{s=0}(t)$ from Section IV.

**Proposition 8.** The master equation $\frac{d\Phi(t)}{dt} = \int_0^t K(t')[\Phi(t-t')]dt'$ with the memory kernel (18) defines a completely positive and trace preserving quantum dynamics if six functions

$$\frac{(\chi_m)_s}{s + (\chi_\pm)_s + (\chi_\pm)_s}, \quad \frac{s + (\chi_m)_s}{s + (\chi_\pm)_s + (\chi_\pm)_s} \pm \frac{1}{s + \frac{1}{2}(\chi_\pm)_s + (\chi_\pm)_s + 4(\chi_\pm)_s}, \quad m = \pm,$$

are all completely monotone.
Proof. $\Phi(t)$ is trace preserving by construction. $\Phi(t)$ is completely positive if non-negativity conditions are satisfied. In the Laplace domain, this corresponds to complete monotonicity of six functions \[ \frac{1}{s} + 2\lambda_s + (\lambda_s)_{s} \pm (t_z)_{s}, \quad \frac{1}{s} - 2\lambda_s + (\lambda_s)_{s} \pm (t_z)_{s}. \] Using the relation \[ \lambda, \lambda_z, t_z \] we get functions \[ \lambda, \lambda_z, t_z. \]

Following the idea of Ref. \[ 24 \], we construct an example illustrating Proposition \[ 8 \].

**Example 1.** Let $a > a_\pm > 0$ and $f(t)$ be a real-valued function such that $0 \leq \int_0^t f(t')dt' \leq (a + a_\pm)^{-1}$ for all $t \geq 0$. The memory kernel \[ \Phi(t) \] with coefficients

\[
(\lambda_{\pm})_s = \frac{a_\pm sf_s}{1 - (a_+ + a_-)f_s}, \quad (\lambda_z)_s = \frac{sf_s[2a - a_+ - a_- - a(a_+ + a_-)f_s]}{4[1 - (a_+ + a_-)f_s]} - \frac{a_+ - a_-}{2(a_+ - a_-)f_s}
\]
defines a legitimate master equation \[ \frac{d\hat{\rho}(t)}{dt} = \int_0^t K(t')[g(t-t')]dt'. \] Trajectories of this dynamical process in the parameter space \( \lambda, \lambda_z, t_z \) are given by segments of straight lines because \( \lambda_s = \frac{1}{s}(1 - a f_s), \quad \lambda_z_s = \frac{1}{s}[1 - (a_+ + a_-)f_s], \quad (t_z)_s = \frac{1}{s}(a_+ - a_-)f_s \) and

\[
\lambda(t) = 1 - a \int_0^t f(t')dt', \quad \lambda_z(t) = 1 - (a_+ + a_-) \int_0^t f(t')dt', \quad t_z(t) = (a_+ - a_-) \int_0^t f(t')dt'.
\]

Using the relation \[ 20 \], we conclude that \( \gamma_z(t) < 0 \) and the dynamics is not CP-divisible if (i) \( f(t) < 0 \) or (ii) \( f(t) > 0 \) and \( a(a_+ + a_-) \int_0^t f(t')dt' > 2a - a_+ - a_- \) for some $t > 0$.

**VII. CONCLUSIONS**

We have considered properties of time-local and convolution master equations describing a phase covariant qubit dynamics. We have characterized the parameters $\lambda$, $\lambda_z$, $t_z$ that can be attained in semigroup dynamics with constant decoherence rates (Proposition \[ 1 \] and Remark \[ 1 \]). We have proved that this region of parameters cannot be significantly extended if one allows for time-dependent non-negative rates $\gamma_z(t)$ and $\gamma_z(t)$ in a time-local master equation (Proposition \[ 2 \] and Remarks \[ 2 \] and \[ 3 \]). We have clarified that the population can be a non-monotonic function of time in a CP-divisible phase covariant dynamics without regard to an initial system state (Proposition \[ 5 \]). Then we have extended the class of eternal CP-indivisible dynamics by presenting a family of non-unital commutative dynamical maps (Proposition \[ 1 \] and a family of non-unital non-commutative dynamical maps (Proposition \[ 5 \]). The main results of the paper are the positivity condition for a phase covariant map (Proposition \[ 3 \]) obtained with the help of the quantum Sinkhorn theorem and the condition for positive divisibility (Proposition \[ 4 \]). Finally, we have considered a subset of completely positive phase covariant qubit maps (Lemma \[ 1 \] with linear inequalities on $\lambda$, $\lambda_z$, $t_z$ that we further used to specify a class of admissible memory kernels in the convolution master equation describing a phase covariant qubit dynamics.

The revealed divisibility properties have a close relation to non-Markovianity of the system dynamics and an extended system–ancilla dynamics, however, a discussion of this relation is beyond the scope of this paper. Moreover, the same reduced dynamics of the system $\Phi(t)$ may be caused by completely different physical environments, and such a difference can be revealed by interventions into the system dynamics, e.g., by performing projective measurements on the system during the evolution \[ \Phi(t) \].

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