Breaking Quadratic Time for Small Vertex Connectivity and an Approximation Scheme

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Abstract

Vertex connectivity a classic extensively-studied problem. Given an integer $k$, its goal is to decide if an $n$-node $m$-edge graph can be disconnected by removing $k$ vertices. Although a linear-time algorithm was postulated since 1974 [Aho, Hopcroft and Ullman], and despite its sibling problem of edge connectivity being resolved over two decades ago [Karger STOC’96], so far no vertex connectivity algorithms are faster than $O(n^2)$ time even for $k = 4$ and $m = O(n)$.

In the simplest case where $m = O(n)$ and $k = O(1)$, the $O(n^2)$ bound dates five decades back to [Kleitman IEEE Trans. Circuit Theory’69]. For higher $m$, $O(m)$ time is known for $k \leq 3$ [Tarjan FOCS’71; Hopcroft, Tarjan SICOMP’73], the first $O(n^2)$ time is from [Kanevsky, Ramachandran, FOCS’87] for $k = 4$ and from [Nagamochi, Ibaraki, Algorithmica’92] for $k = O(1)$. For general $k$ and $m$, the best bound is $\tilde{O}(\min(\min(km^2/n, n^2), nk^\omega))$ [Henzinger, Rao, Gabow FOCS’96; Linial, Lovász, Wigderson FOCS’86].

In this paper, we present a randomized Monte Carlo algorithm with $\tilde{O}(m + k^{7/3}n^{4/3})$ time for any $k = O(\sqrt{n})$. This gives the first subquadratic time bound for any $4 \leq k \leq o(n^{2/7})$ and improves all above classic bounds for all $k \leq n^{0.44}$. We also present a new randomized Monte Carlo $(1 + \epsilon)$-approximation algorithm that is strictly faster than the previous Henzinger’s 2-approximation algorithm [J. Algorithms’97] and all previous exact algorithms. The story is the same for the directed case, where our exact $\tilde{O}(\min\{km^2/n, km^{4/3}\})$-time for any $k = O(\sqrt{n})$ and $(1 + \epsilon)$-approximation algorithms improve classic bounds for small and large $k$, respectively. Additionally, our algorithm is the first approximation algorithm on directed graphs.

The key to our results is to avoid computing single-source connectivity, which was needed by all previous exact algorithms and is not known to admit $o(n^2)$ time. Instead, we design the first local algorithm for computing vertex connectivity; without reading the whole graph, our algorithm can find a separator of size at most $k$ or certify that there is no separator of size at most $k$ “near” a given seed node.

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$\tilde{O}$ hides polylogarithmic terms and $\omega < 2.38$ is the matrix multiplication exponent.

As previously used in the literature (e.g. [RW12, HKN14, AWW16]), subquadratic time refers to $O(m) + o(n^2)$ time.
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1 Introduction

Vertex connectivity is a central concept in graph theory. The vertex connectivity $\kappa_G$ of a graph $G$ is the minimum number of nodes needed to be removed to disconnect some remaining node from another remaining node. (When $G$ is directed, this means that there is no directed path from some node $u$ to some node $v$ in the remaining graph.)

Since 1969, there has been a long line of research on efficient algorithms [Kle69, Pod73, ET75, Eve75, Gal80, EH84, Mat87, BDD+82, LLW88, CT91, NI92, CR94, Hen97, HRG00, Gab06, CGK14] for deciding $k$-connectivity (i.e. deciding if $\kappa_G \geq k$) or computing the connectivity $\kappa_G$ (See Table 1 for details). For the undirected case, Aho, Hopcroft and Ullman [AHU74, Problem 5.30] conjectured in 1974 that there exists an $O(m)$-time algorithm for computing $\kappa_G$ on a graph with $n$ nodes and $m$ edges. However, no algorithms to date are faster than $O(n^2)$ time even for $k = 4$.

On undirected graphs, the first $O(n^2)$ bound for the simplest case, where $m = O(n)$ and $k = O(1)$, dates back to five decades ago:

Kleitman [Kle69] in 1969 presented an algorithm for deciding $k$-connectivity with running time $O(kn \cdot \text{VC}_k(n, m))$ where $\text{VC}_k(n, m)$ is the time needed for deciding if the minimum size $s$-$t$ vertex-cut is of size at least $k$, for fixed $s, t$. Although the running time bound was not explicitly stated, it was known that $\text{VC}_k(n, m) = O(mk)$ by Ford-Fulkerson algorithm [FF56]. This gives $O(k^2nm) = O(n^2)$ when $m = O(n)$ and $k = O(1)$. Subsequently, Tarjan [Tar72] and Hopcroft and Tarjan [HT73] presented $O(m)$-time algorithms when $k = 2$ and $3$ respectively.

All subsequent works improved Kleitman’s bound for larger $k$ and $m$, but none could break beyond $O(n^2)$ time. For $k = 4$ and any $m$, the first $O(n^2)$ bound was by Kanevsky and Ramachandran [KR91]. The first $O(n^2)$ for any $k = O(1)$ (and any $m$) was by Nagamochi and Ibaraki [NI92].

For general $k$ and $m$, the fastest running times are $\tilde{O}(n^\omega + nk^\omega)$ by Linial, Lovász and Wigderson [LLW88] and $O(kn^2)$ by Henzinger, Rao and Gabow [HRG00]. Here, $\tilde{O}$ hides polylog($n$) terms, and $\omega$ is the matrix multiplication exponent. Currently, $\omega < 2.37287$ [Gal14].

For directed graphs, an $O(m)$-time algorithm is known only for $k \leq 2$ [Geo10]. For general $k$ and $m$, the fastest running times are $\tilde{O}(n^\omega + nk^\omega)$ by Cheriyan and Reif [CR94] and $\tilde{O}(mn)$ by Henzinger et al. [HRG00]. All mentioned state-of-the-art algorithms for general $k$ and $m$, for both directed and undirected cases [LLW88, CR94, HRG00], are randomized and correct with high probability.

The fastest deterministic algorithm is by Gabow [Gab06] and has slower running time. Some approximation algorithms have also been developed. The first is the deterministic 2-approximation $O(\min\{\sqrt{n}, k\}n^2)$-time algorithm by Henzinger [Hen97]. The second is the recent randomized $O(\log n)$-approximation $\tilde{O}(m)$-time algorithm by Censor-Hillel, Ghaffari, and Kuhn [CGK14]. Both algorithms work only on undirected graphs.

Besides a few $O(m)$-time algorithms for $k \leq 3$, all previous exact algorithms could not go beyond $O(n^2)$ for a common reason: As a subroutine, they have to solve the following problem. For a pair of nodes $s$ and $t$, let $\kappa(s, t)$ denote the minimum number of nodes (excluding $s$ and $t$) required to be removed so that there is no path from $s$ to $t$ in the remaining graph. In all previous algorithms, there is always some node $s$ such that these algorithms decide if $\kappa(s, t) \geq k$ for all other nodes $t$ (and some algorithms in fact computes $\kappa(s, t)$ for all $t$). We call this problem single-source $k$-connectivity. Until now, there is no $o(n^2)$-time algorithm for this problem even when $k = O(1)$ and $m = O(n)$. 

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example, consider the interesting case the graph is sparse but can still be
in the dense case when
and Reif [Hen97] also return a corresponding separator. \( t_{\text{directed}} = \text{poly}(1/\epsilon) \min(T_{\text{apx}}(k, m, n), n^{\omega}) \), and \( t_{\text{undirected}} = m + \text{poly}(1/\epsilon) \min(k^{4/3}n^{4/3}, k^{2/3}n^{5/3}+o(1), n^{3+o(1)}/k, n^{\omega}) \) where \( T_{\text{apx}} \) is defined in Equation (1).

### 1.1 Our Results

In this paper, we present first algorithms that break the \( O(n^2) \) bound on both undirected and undirected graphs, when \( k \) is small. More precisely:

**Theorem 1.1.** There are randomized (Monte Carlo) algorithms that take as inputs an \( n \)-node \( m \)-edge graph \( G = (V, E) \) and an integer \( k = O(\sqrt{m}) \), and can decide w.h.p.\(^3\) if \( \kappa_G \geq k \). If \( \kappa_G < k \), then the algorithms also return corresponding separator \( S \subset V \), i.e. a set \( S \) where \( |S| = \kappa_G \) and \( G[V-S] \) is not connected if \( G \) is undirected and not strongly connected if \( G \) is directed. The algorithm takes \( \tilde{O}(m + k^{7/3}n^{4/3}) \) and \( \tilde{O}(\min(k^{2/3}n, km^{4/3})) \) time on undirected and directed graphs, respectively.

Our bounds are the first \( O(n^2) \) for the range \( 4 \leq k \leq o(n^{2/7}) \) on undirected graphs and range \( 3 \leq k \leq o(n/m^{2/3}) \) on directed graphs. Our algorithms are combinatorial, meaning that they do not rely on fast matrix multiplication. For all range of \( k \) that our algorithms support, i.e. \( k = O(\sqrt{m}) \), our algorithms improve upon the previous best combinatorial algorithms by Henzinger et al. [HRG00], which take time \( \tilde{O}(kn^2) \) on undirected graphs and \( \tilde{O}(mn) \) on directed graphs\(^4\). Comparing with the \( \tilde{O}(n^{\omega} + nk^{\omega}) \) bound based on algebraic techniques by Linial et al. [LLW88] and Cheriyan and Reif [CR94], our algorithms are faster on undirected graphs when \( k \leq n^{3\omega/7-4/7} \approx n^{0.44} \). For directed graph, our algorithm is faster where the range \( k \) depends on graph density. For example, consider the interesting case the graph is sparse but can still be \( k \)-connected which is when \( m = O(nk) \). Then ours is faster than [CR94] for any \( k \leq n^{0.44} \) like the undirected case. However, in the dense case when \( m = \Omega(n^2) \), ours is faster than [CR94] for any \( k \leq n^{7/3} \approx n^{0.239} \).

\(^3\)We say that an event holds with high probability (w.h.p.) if it holds with probability at least \( 1 - 1/n^c \), where \( c \) is an arbitrarily large constant.

\(^4\)As \( k \leq \sqrt{m} \) and \( m \geq nk \), we have \( k \leq m^{1/3} \). So \( km^{2/3}n \leq mn \).
To conclude, our bounds are lower than all previous bounds when \(4 \leq k \leq n^{0.44}\) for undirected graphs and \(3 \leq k \leq n^{0.44}\) for directed sparse graphs (i.e. when \(m = O(nk)\)). All these bounds [HRG00, LLW88, CR94] have not been broken for over 20 years. In the simplest case where \(m = O(n)\) and \(k = O(1)\), we break the 49-year-old \(O(n^2)\) bound [Kle69] down to \(\tilde{O}(n^{4/3})\) and \(\tilde{O}(n^{5/3})\) for undirected and directed graphs, respectively.

**Approximation algorithms.** We can adjust the same techniques to get \((1 + \epsilon)\)-approximate \(\kappa_G\) with faster running time. In addition, we give another algorithm using a different technique that can \((1 + \epsilon)\)-approximate \(\kappa_G\) in \(\tilde{O}(n^\omega/\epsilon^2)\) time.

We define the function \(T_{\text{apx}}(k, m, n)\) as

\[
T_{\text{apx}}(k, m, n) = \begin{cases} 
\min(m^{4/3}, mn^{2/3}k^{1/2}) & \text{if } k \leq \sqrt{n}. \\
\min(mn^{2/3+o(1)}/k^{1/3}, n^{7/3+o(1)}/k^{1/6}) & \text{if } \sqrt{n} < k \leq n^{4/5}. \\
n^{3+o(1)}/k & \text{if } k > n^{4/5}.
\end{cases}
\]  

(1)

**Theorem 1.2** (Approximation Algorithm). There is a randomized (Monte Carlo) algorithm that takes as input an \(n\)-node \(m\)-edge graph \(G = (V, E)\) and w.h.p. outputs \(\tilde{\kappa}\), where \(\kappa_G \leq \tilde{\kappa} \leq (1 + \epsilon)\kappa_G\), in \(\tilde{O}(m + \text{poly}(1/\epsilon) \min(k^{4/3}n^{4/3}, k^{2/3}n^{5/3+o(1)}, n^{3+o(1)}/k, n^\omega))\) time for undirected graph, and in \(\tilde{O}(\text{poly}(1/\epsilon) \min(T_{\text{apx}}(k, m, n), n^\omega))\) time for directed graph where \(T_{\text{apx}}(k, m, n)\) is defined in Equation (1). The algorithm also returns a pair of nodes \(x, y\) where \(\kappa(x, y) = \tilde{\kappa}\). Hence, with additional \(O(m \min\{\sqrt{n}, \tilde{\kappa}\})\) time, the algorithm can compute the corresponding separator.

As noted earlier, previous algorithms achieve 2-approximation in \(O(\min\{\sqrt{n}, k\}n^2)\)-time [Hen97] and \(O(\log n)\)-approximation in \(\tilde{O}(m)\) time [CGK14]. For all possible values of \(k\), our algorithm is strictly faster than the 2-approximation algorithm of [Hen97].

Our approximation algorithm is also strictly faster than almost all previous exact algorithms. In particular, even when \(\epsilon = 1/n^\gamma\) for small constant \(\gamma > 0\), our algorithms are always polynomially faster than the exact algorithms by [HRG00] with running time \(\tilde{O}(mn)\) and \(\tilde{O}(kn^2)\) on directed and undirected graphs, respectively. Compared with the bound \(\tilde{O}(n^\omega + nk^\omega)\) by [LLW88] and [CR94], our bound is always lower for undirected graph. For directed graph, our bound is always lower except the ties when \(k \in [n^{0.079}, n^{0.58}]\) for dense directed graphs.

To this end, note that our approximation algorithm works on directed graphs while the previous ones [Hen97, CGK14] do not.

### 1.2 The Key Technique

At the heart of our main result in Theorem 1.1 is a new local algorithm for finding minimum vertex cuts. In general, we say that an algorithm is local if its running time does not depend on the size of the whole input.

More concretely, let \(G = (V, E)\) be a directed graph where each node \(u\) has out-degree \(\text{deg}^\text{out}(u)\). Let \(\text{deg}^\text{out}_\text{min} = \min_u \text{deg}^\text{out}(u)\) be the minimum out-degree. For any set \(S \subset V\), the out-volume of \(S\) is \(\text{vol}^\text{out}(S) = \sum_{u \in S} \text{deg}^\text{out}(u)\) and the set of out-neighbors of \(S\) is \(N^\text{out}(S) = \{v \notin S \mid (u, v) \in E\}\). We show the following algorithm (see Theorem 4.1 for a more details):

**Theorem 1.3** (Local vertex connectivity (informal)). There is a deterministic algorithm that takes as inputs a node \(x\) in a graph \(G\) and parameters \(\nu\) and \(k\) where \(\nu, k\) are not too large, and in \(\tilde{O}(\nu^{1.5}k)\) time either

1. returns a set \(S \ni x\) where \(|N^\text{out}(S)| \leq k\), or

2. certifies that there is no set \(S \ni x\) such that \(\text{vol}^\text{out}(S) \leq \nu\) and \(|N^\text{out}(S)| \leq k\).
Our algorithm is the first local algorithm for finding small vertex cuts (i.e. finding small separator $N^{out}(S)$). The algorithm either finds a separator of size at most $k$, or certifies that no separator of size at most $k$ exists “near” some node $x$. Our algorithm has no “gap” this sense.

Previously, there was rich literature on local algorithms for finding low conductance cuts\(^5\), which is a different problem from ours. The study was initiated by Spielman and Teng [ST04] in 2004. Since then, deep techniques have been further developed, such as spectral-based techniques\(^6\) (e.g. [ST13, ACL06, AL08, AP09, GT12]) and newer flow-based techniques [OZ14, HRW17, WFH+17, VGM16]). Applications of these techniques for finding low conductance cuts are found in various contexts (e.g. balanced cuts [ST13, SW19]), edge connectivity [KT15, HRW17], and dynamically maintaining expanders [Wul17, NS17, NSW17, SW19]).

It is not clear a priori that these previous techniques can be used for proving Theorem 1.3. First of all, they were invented to solve a different problem, and there are several small differences about technical input-output constraints. More importantly is the following conceptual difference. In most previous algorithms, there is a “gap” between the two cases of the guarantees. That is, if in one case the algorithm can return a cut $S \ni x$ whose conductance is at most $\phi \in (0,1)$, then in the other case the algorithm can only guarantees that there is no cut “near” $x$ with conductance $\alpha \phi$, for some $\alpha = o(1)$ (e.g. $\alpha = O(\phi)$ or $O(1/\log n))$.\(^7\)

Because of these differences, not many existing techniques can be adapted to design a local algorithm for vertex connectivity. In fact, we are not aware of any spectral-based algorithms that can solve this problem, even when we can read the whole graph. Fortunately, it turns out that Theorem 1.3 can be proved by adapting some recent flow-based techniques. In general, a challenge in designing flow-based algorithms is to achieve the following goals simultaneously.

1. Design some well-structured graph so that finding flows on this graph is useful for our application (proving Theorem 1.3 in this case). We call such graph an augmented graph.

2. At the same time, design a local flow-based algorithm which is fast when running of the augmented graph.

For the first task, the design of the augmented graph require some careful choices (see Section 2.2 for the high-level ideas and Section 4.1 for details). For the second task, it turns out that previous flow-based local algorithms [OZ14, HRW17, WFH+17, VGM16] can be adjusted to give useful answers for our applications when ran on our augmented graph. However, these previous algorithms only give slower running time of at least $O((\nu k)^{1.5})$. To obtain the $O(\nu k)$ bound, we first speed up Goldberg-Rao max flow algorithm [GR98] from running time $O(m \min\{\sqrt{m}, n^{2/3}\})$ to $O(m\sqrt{n})$ when running on a graph with certain structure. Then, we “localize” this algorithm in a similar manner as in [OZ14], which completes our second task (see Section 2.2 for more discussion).

As a byproduct, our modification of Goldberg-Rao algorithm in fact gives the fastest weakly-polyalgorithm for computing $s$-$t$ vertex connectivity in node-weighted graphs:

**Theorem 1.4 (Weighted $s$-$t$ vertex connectivity).** Let $G = (V,E)$ be a directed graph with $n$ nodes and $m$ edges where each node has integer weight from $[1,U]$. For any $s,t \in V$, in time $O(m \sqrt{n} \log n \log U)$, we can compute deterministically the minimum weight $s$-$t$ separator $S \subset V$, i.e., $s,t \notin S$ and there is no path from $s$ to $t$ in $G[V-S]$.

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\(^5\)The conductance of a cut $(S,V-S)$ is defined as $\Phi(S) = \frac{|E(S,V-S)|}{\min\{\deg(S),\deg(V-S)\}}$.

\(^6\)They are algorithms based on some random-walk or diffusion process.

\(^7\)In [KT15, HRW17], in the second case the algorithm does not even guarantee non-existence of some low conductance cuts, but instead about min-cuts.
The previous fastest algorithm is by using the general max flow algorithm by [LS14], giving an $O(m\sqrt{n}\text{polylog}(nU))$ running time. Our algorithm is simpler and slightly faster.

Given the key local algorithm in Theorem 1.3, we obtain Theorems 1.1 and 1.2 by combining our local algorithm with other known techniques including random sampling, Ford-Fulkerson algorithm, Nagamochi Ibaraki’s connectivity certificate [NI92] and convex embedding [LLW88, CR94]. We sketch how everything fits together in Section 2.

2 Overview

2.1 Exact Algorithm

To illustrate the main idea, let us sketch our algorithm with running time $\tilde{O}(m + n^{4/3})$ only on an undirected graph with $m = O(n)$ and $k = O(1)$. This regime is already very interesting, because the best bound has been $\tilde{O}(n^2)$ for nearly 50 years [Kle69]. Throughout this section, $N(C)$ is a set of neighbors of nodes in $C \subseteq V$ that are not in $C$, and $E_G(S,T)$ is the set of edges between (not necessarily disjoint) vertex sets $S$ and $T$ in $G$ (the subscript is omitted when the context is clear). A vertex partition $(A, S, B)$ is called a separation triple if $A, B \neq \emptyset$ and $N(A) = S$.

Given a graph $G = (V, E)$ and a parameter $k$, our goal is to either return a set $C \subseteq V$ where $|N(C)| < k$ or certify that $\kappa_G \geq k$. Our first step is to find a sparse subgraph $H$ of $G$ where $\kappa_G = \min\{\kappa_G, k\}$ using the algorithm by Nagamochi and Ibaraki [NI92]. The nice property of $H$ is that it is formed by a union of $k$ disjoint forests, i.e. $H$ has arboricity $k$. In particular, for any set of nodes $C$, we have $|E_H(C,C)| \leq k|C|$. As the algorithm only take linear time, from now, we treat $H$ as our input graph $G$.

The next step has three cases. First, suppose there is a separation triple $(A, S, B)$ where $|S| < k$ and $|A|, |B| \geq n^{2/3}$. Here, we sample $\tilde{O}(n^{1/3})$ many pair $(x, y)$ of nodes uniformly at random. With high probability, one of these pairs is such that $x \in A$ and $y \in B$. In this case, it is well known (e.g. [Eve75]) that one can modify the graph and run a max xy-flow algorithm. Thus, for each pair $(x, y)$, we run Ford-Fulkerson max-flow algorithm in time $O(km) = O(n)$ to decide whether $\kappa(x, y) < k$ and if so, return the corresponding cut.

So w.h.p. the algorithm returns set $C$ where $|N(C)| < k$ in total time $\tilde{O}(n^{1+1/3})$.

The next case is when all separation triples $(A, S, B)$ where $|S| < k$ are such that either $|A| < n^{2/3}$ or $|B| < n^{2/3}$. Suppose w.l.o.g. that $|A| < n^{2/3}$. By a binary search trick, we can assume to know the size $|A|$ up to a factor of 2. Here, we sample $\tilde{O}(n/|A|)$ many nodes uniformly at random. For each node $x$, we run the local vertex connectivity subroutine from Theorem 1.3 where the parameter $k$ in Theorem 1.3 is set to be $k - 1$.

Note that the volume of $A$ is $$\text{vol}(A) = 2|E(A,A)| + |E(A,S)| = O(k|A|) = O(|A|)$$ where the second equality is because $G$ has arboricity $k$ and $|S| < k$ (also recall that we only consider $m = O(n)$ and $k = O(1)$ in this subsection). We set the parameter $\nu = \Theta(|A|)$. With high probability, we have that one of the samples $x$ must be inside $A$. Here, the local-max-flow cannot be in the second case, and will return a set $C$ where $|N(C)| < k$, which implies that $\kappa_G < k$. The total running time is $\tilde{O}(n/|A|) \times \tilde{O}(|A|^{1.5}) = \tilde{O}(n^{1+1/3})$ because $|A| < n^{2/3}$.

The last case is when $\kappa_G \geq k$. Here, both of Ford-Fulkerson algorithm and local max flow algorithm will never return any set $C$ where $|N(C)| < k$. So we can correctly report that $\kappa_G \geq k$. All of our techniques generalize to the case when $\kappa_G$ is not constant. For directed graphs, only Nagamochi and Ibaraki’s [NI92] algorithm does not generalize and so we cannot assume that the graph has low arboricity. This explains why we obtain a slower algorithm for directed graphs.
We remark that our algorithm for undirected graphs is the first vertex connectivity algorithm which exploits the fact that Nagamochi Ibaraki’s connectivity certificate has low arboricity. Previous algorithms only uses the fact that it is a sparse graph.

2.2 Local Vertex Connectivity

In this section, we give a high-level idea how to obtain our local vertex connectivity algorithm in Theorem 1.3. Recall from the introduction that there are two tasks which are to design an augmented graph and to devise a local flow-based algorithm running on such augmented graph. We have two goals: 1) the running time of our algorithm is local; i.e., it does not depend on the size of the whole graph and 2) the local flow-based algorithm’s output should be useful for our application.

The local time principles. We first describe high-level principles on how to design the augmented graph and the local flow-based algorithm so that the running time is local\(^8\).

1. Augmented graph is absorbing: Each node \(u\) of the augmented graph is a sink that can “absorb” flow proportional to its degree \(\text{deg}(u)\). More formally, each node \(u\) is connected to a super-sink \(t\) with an edge \((u, t)\) of capacity \(\alpha \text{deg}(u)\) for some constant \(\alpha\). In our case, \(\alpha = 1\).

2. Flow algorithm tries to absorb before forward: Suppose that a node \(u\) does not fully absorb the flow yet, i.e. \((u, t)\) is not saturated. When a flow is routed to \(u\), the local flow-based algorithm must first send a flow from \(u\) to \(t\) so that the sink at \(u\) is fully absorbed, before forwarding to other neighbors of \(u\). Moreover, the absorbed flow at \(u\) will stay at \(u\) forever.

We give some intuition behind these principles. The second principle resembles the following physical process. Imagine pouring water on a compartment of an ice tray. There cannot be water flowing out of an unsaturated compartment until that compartment is saturated. So if the amount of initial water is small, the process will stop way before the water reaches the whole ice tray. This explains in principle why the algorithm needs not read the whole graph.

The first principle allows us to argue why the cost of the algorithm is proportional to the part of the graph that is read. Very roughly, the total cost for forwarding the flow from a node \(u\) to its neighbors depends on \(\text{deg}(u)\), but at the same time we forward the flow only after it is already fully absorbed at \(u\). This allows us to charge the total cost to the total amount of absorbed flow, which in turn is small if the initial amount of flow is small.

Augmented graph. Let us show how to design the augmented graph in the context of edge connectivity in undirected graphs first. The construction is simpler than the case of vertex connectivity, but already captures the main idea. We then sketch how to extend this idea to vertex connectivity.

Let \(G = (V, E)\) be an undirected graph with \(m\) edges and \(x \in V\) be a node. Consider any numbers \(\nu, k > 0\) such that

\[2\nu k + \nu + 1 \leq 2m.\] (2)

We construct an undirected graph \(G'\) as follows. The node set of \(G'\) is \(V(G') = \{s\} \cup V \cup \{t\}\) where \(s\) and \(t\) is a super-source and a super-sink respectively. For each node \(u\), add \((u, t)\) with capacity \(\text{deg}_G(u)\). (So, this satisfied the first local time principle.) For each edge \((u, v) \in E\), set the capacity to be \(2\nu\). Finally, add an edge \((s, x)\) with capacity \(2\nu k + \nu + 1\).

\(^8\)In fact, these are also principles behind all previous local flow-based algorithms. To the best of our knowledge, these general principles have not been stated. We hope that they explain previous seemingly ad-hoc results.
Theorem 2.1. Let $F^*$ be the value of the s-t max flow in $G'$. We have the following:

1. If $F^* = 2\nu k + \nu + 1$, then there is no vertex partition $(S, T)$ in $G$ where $S \ni x$, $\operatorname{vol}(S) \leq \nu$ and $|E(S, V - S)| \leq k$.

2. If $F^* \leq 2\nu k + \nu$, then there is a vertex partition $(S, T)$ in $G$ where $S \ni x$ and $|E(S, V - S)| \leq k$.

Proof. To see (1), suppose for a contradiction that there is such a partition $(S, T)$ where $S \ni x$. Let $(S', T') = (\{s\} \cup S, T \cup \{t\})$. The edges between $S'$ and $T'$ has total capacity

$$c(E_{G'}(S', T')) = 2\nu|E_G(S, V - S)| + \operatorname{vol}_G(S) \leq 2\nu k + \nu.$$ 

So $F^* \leq 2\nu k + \nu$, a contradiction. To see (2), let $(S', T') = (\{s\} \cup S, T \cup \{t\})$ be a min st-cut in $G'$ corresponding to the max flow, i.e. by the min-cut max-flow theorem, the edges between $S'$ and $T'$ has total capacity

$$c(E_{G'}(S', T')) \leq 2\nu k + \nu. \quad (3)$$

Observe that $S' \neq \{s\}$ and $S \ni x$ because the edge $(s, x)$ has capacity with capacity strictly more than $2\nu k + \nu$. Also, $T' \neq \{t\}$ because edges between $\{s\} \cup V$ and $\{t\}$ has total capacity $\operatorname{vol}(V) = 2m > 2\nu k + \nu$ (the inequality is because of Equation (2)). So $(S, T)$ gives a cut in $G$ where $S \ni x$. Suppose that $|E_G(S, T)| \geq k + 1$, then $c(E_{G'}(S', T')) \geq 2\nu(k + 1) = 2\nu k + 2\nu > 2\nu k + \nu$ which contradicts Equation (3).

Observe that the above theorem is similar to Theorem 1.3 except that it is about edge connectivity. To extend this idea to vertex connectivity, we use a standard transformation as used in [ET75, HRG00] by constructing a so-called split graph. In our split graph, for each node $v$, we create two nodes $v_{in}$ and $v_{out}$. For each edge $(u, v)$, we create an edge $(u_{out}, v_{in})$ with infinite capacity. There is an edge $(v_{in}, v_{out})$ for each node $v$ as well. Observe that a cut set with finite capacity in the split graph corresponds to a set of nodes in the original graph. Then, we create the augmented graph of the split graph in a similar manner as above, e.g. by adding nodes $s$ and $t$ and an edge $(s, x)$ with $2\nu k + \nu + 1$. The important point is that we set the capacity of each $(v_{in}, v_{out})$ to be $2\nu$. The proof of Theorem 1.3 (except the statement about the running time) is similar as above (see Section 4.1 for details).

Local flow-based algorithm. As discussed in the introduction, we can in fact adapt previous local flow-based algorithms to run on our augmented graph and they can decide the two cases in Theorem 1.3 (i.e. whether there is a small vertex cut “near” a seed node $x$). Theorem 2.1 in fact already allows us to achieve this with slower running time than the desired $\tilde{O}(\nu^{1.5}k)$ by implementing existing local flow-based algorithms. For example, the algorithm by [OZ14], which is a “localized” version of Goldberg-Rao algorithm [GR98], can give a slower running time of $\tilde{O}((\nu k)^{1.5})$. Other previous local flow-based algorithms that we are aware of (e.g. [OZ14, HRW17, WFH+17, VGM16]) give even slower running time (even after appropriate adaptations).

We can speed up the time to $\tilde{O}(\nu^{1.5}k)$ by exploiting the fact that our augmented graph is created from a split graph sketched above. To begin with, we first observe that, when running Goldberg-Rao algorithm on split graphs (which are weighted), the running time can be sped up from $\tilde{O}(m \min\{\sqrt{m}, n^{2/3}\})$ to $\tilde{O}(m \sqrt{n})$. This already gives us the new fastest algorithm for computing $s$-$t$ weighted vertex connectivity as stated in Theorem 1.4. This improvement resembles the technique by Even and Tarjan [ET75] which yields an $\tilde{O}(m \sqrt{n})$-time algorithm for computing $s$-$t$ unweighted vertex connectivity. They showed that Dinic’s algorithm with running time $O(m \min\{\sqrt{m}, n^{2/3}\})$ on
a general unit-capacity graph can be sped up to $\tilde{O}(m\sqrt{n})$ when run on a special graph called “unit network”. It turns out that unit networks share some structures with our split graphs, allowing us to apply a similar idea. Although our improvement is based on a similar idea, it is more complicated to implement this idea on our split graph since it is weighted.

Finally, we “localize” our improved improved algorithm by enforcing the second local time principle. Our way to localize the algorithm goes hand in hand with the way Orecchia and Zhu [OZ14] did to the standard Goldberg-Rao algorithm (see Section 4.3 for details).

We remark that our local-flow technique can be augmented to provide an approximation algorithm by essentially using “less” amount of flow injected to the augmented graph.

### 2.3 Approximation Algorithm

Our new approximation algorithm is based on a tool called *convex embedding* introduced by Linial, Lovász, and Wigerson [LLW88] for finding vertex connectivity in undirected graphs, and extended to directed graphs by Cheriyan and Reif [CR94].

Our key insight is to view the convex embedding as a data structure on graphs the following property: Given an $n$-node graph $G = (V,E)$, a node $y \in V$, and parameter $k$, we can preprocess in $O(n^\omega)$, i.e., build a convex embedding w.r.t. $(G,y,k)$, so that, given any query node $x$, we can return in time $O(k^\omega)$ an answer which equals to $\min\{\kappa(x,y),k\}$ with high probability. This can be derived directly from the work of [LLW88, CR94]. Intuitively, the convex embedding is a way for preprocessing a graph $G$ and a node $y$ so that information about $\kappa(x,y)$ can computed “locally” without reading the whole graph. From now, we assume for simplicity that all queries to this data structure return correct answers.

With this data structure, our algorithm is very simple. Given an $n$-node graph $G = (V,E)$ and an error parameter $\epsilon \in (0,1)$, we compute in time $O(n^\omega)$ the convex embedding w.r.t. $(G,y,k)$ where $y$ is a random node, and $\delta$ is a minimum degree of $G$. Next, we sample $\tilde{O}(n/\epsilon \delta)$ many nodes $x$ and query $x$ to the data structure. We return the minimum answer among all the queries. The total time is clearly $\tilde{O}(n^\omega + \frac{m}{n} \cdot \delta^\omega) = \tilde{O}(n^\omega/\epsilon)$. We claim that this is correct with probability $\Omega(\epsilon)$.

By repeating $\tilde{O}(1/\epsilon)$ times, our algorithm takes $\tilde{O}(n^\omega/\epsilon^2)$ time and is correct with high probability.

To see the correctness, let $\tilde{\kappa}$ be the answer of the algorithm. Observe that $\delta \geq \tilde{\kappa} \geq \kappa(G)$. If $\kappa(G) \geq (1-\epsilon)\delta$, then $\tilde{\kappa}$ is trivially a $(1+\epsilon)$-approximate answer. So we suppose that $\kappa(G) \leq (1-\epsilon)\delta$. Let $(A,B)$ be a vertex cut where $|A| = \kappa(G) \leq (1-\epsilon)\delta$ and $|A| \leq |B|$. Here, we know that $|A| \geq \epsilon \delta$. Indeed, for any vertex $x \in A$ with neighbor set $N(x)$, we have $\delta \leq |N(x)| \leq |A| + |S|$. So $|A| \geq \epsilon \delta$. We also claim that $|B| = \Omega(\epsilon n)$. Indeed, if $\delta < n/2$, then $2|B| \geq |A| + |B| = n - |S| \geq n/2$. Else, if $\delta > n/2$, then $|B| \geq |A| = \Omega(\epsilon n)$.

So $y \in B$ with probability $\Omega(\epsilon)$, and one of the sampled nodes $x \in A$ with high probability, as we sampled $\tilde{O}(n/\epsilon \delta)$ many nodes. Given that $y \in B$ and $x \in A$, the answer $\tilde{\kappa} = \kappa(G)$ exactly. This concludes the analysis. Note that the algorithm above only returns $x$ and $y$ where $\kappa(x,y) = \kappa_G$. To get the vertex cut itself, we simply run max flow algorithms in $\tilde{O}(m\{k,\sqrt{n}\})$ time additionally.

### 3 Preliminaries

#### 3.1 Directed Graph

Let $G = (V,E)$ be a directed graph where $|V| = n$ and $|E| = m$. We assume that $G$ is strongly-connected. Otherwise, we can list all strongly connected components in linear time by a standard textbook algorithm. We also assume that $G$ is simple. That is, $G$ does not have a duplicate edge. Otherwise, we can simplify the graph in linear time by removing duplicate edges. For any
edge \((u,v)\), we denote \(e^R = (v,u)\). For any directed graph \(G = (V,E)\), the reverse graph \(G^R = (V,E^R)\) is \(E^R = \{ e^R : e \in E \}\).

**Definition 3.1** (\(\delta, \text{deg}, \text{vol}, N\)). Definitions below are defined for any vertex \(v\) on graph \(G\) and subset of vertices \(U \subseteq V\).

- \(\delta_G^\text{in}(v) = \{(u,v) \in E\}\) and \(\delta_G^\text{out}(U) = \{(x,y) \in E : x \notin U, y \in V\}\); i.e. they are the sets of edges entering \(v\) and \(U\) respectively.
- Analogously, \(\delta_G^\text{out}(v)\) and \(\delta_G^\text{out}(U)\) are the sets of edges leaving \(v\) and \(U\) respectively.
- \(\text{deg}_G^\text{in}(v) = |\delta_G^\text{in}(v)|\) and \(\text{deg}_G^\text{out}(v) = |\delta_G^\text{out}(v)|\); i.e. they are the numbers of edges entering and leaving \(v\) respectively.
- \(\text{vol}_G^\text{in}(U) = \sum_{v \in U} \text{deg}_G^\text{out}(v)\) and \(\text{vol}_G^\text{in}(U) = \sum_{v \in U} \text{deg}_G^\text{in}(v)\). Note that \(\text{vol}_G^\text{in}(V) = \text{vol}_G^\text{in}(V) = m\).
- \(N_G^\text{in}(v) = \{u : (u,v) \in E\}\) and \(N_G^\text{out}(v) = \{u : (v,u) \in E\}\); i.e. they are sets of in- and out-neighbors of \(v\), respectively.
- \(N_G^\text{in}(U) = \bigcup_{v \in U} N_G^\text{in}(v) \setminus U\) and \(N_G^\text{out}(U) = \bigcup_{v \in U} N_G^\text{out}(v) \setminus U\). Call these sets external in-neighborhood of \(U\) and external out-neighborhood of \(U\), respectively.

**Definition 3.2** (Subgraphs). For a set of vertices \(U \subseteq V\), we denote \(G[U]\) as a subgraph of \(G\) induced by \(U\). Denote for any vertex \(v\), any subset of vertices \(U \subseteq V\), any edge \(e \in E\), and any subset of edges \(F \subseteq E\),

- \(G \setminus v = (V \setminus \{v\}, E)\),
- \(G \setminus U = (V \setminus U, E)\),
- \(G \setminus e = (V, E \setminus \{e\})\), and
- \(G \setminus F = (V, E \setminus F)\).

We say that these graphs arise from \(G\) by deleting \(v, U, e, \) and \(F\), respectively.

**Definition 3.3** (Paths and reachability). For \(s,t \in V\), we say a path \(P\) is an \((s,t)\)-path if \(P\) is a directed path starting from \(s\) and ending at \(t\). For any \(S,T \subseteq V\), we say \(P\) is an \((S,T)\)-path if \(P\) starts with some vertex in \(S\) and ends at some vertex in \(T\). We say that a vertex \(t\) is reachable from a vertex \(s\) if there exists a \((s,t)\)-path \(P\). Moreover, if a node \(v\) is in such path \(P\), then we say that \(t\) is reachable from \(s\) via \(v\).

**Definition 3.4** (Edge- and Vertex-cuts). Let \(s\) and \(t\) be any distinct vertices. Let \(S,T \subseteq V\) be any disjoint non-empty subsets of vertices. We call any subset of edges \(C \subseteq E\) (respectively any subset of vertices \(U \subseteq V\)):

- an \((S,T)\)-edge-cut (respectively an \((S,T)\)-vertex-cut) if there is no \((S,T)\)-path in \(G \setminus C\) (respectively if there is no \((S,T)\)-path in \(G \setminus U\) and \(S \cap U = \emptyset, T \cap U = \emptyset\)),
- an \((s,t)\)-edge-cut (respectively \((s,t)\)-vertex-cut) if there is no \((s,t)\)-path in \(G \setminus C\) (respectively if there is no \((s,t)\)-path in \(G \setminus U\) and \(s,t \notin U\)),
- an \(s\)-edge-cut (respectively \(s\)-vertex-cut) if it is an \((s,t)\)-edge-cut (respectively \((s,t)\)-vertex-cut) for some vertex \(t\), and
- an edge-cut (respectively vertex-cut) if it is an \((s,t)\)-edge-cut (respectively \((s,t)\)-vertex-cut) for some distinct vertices \(s\) and \(t\). In other words, \(G \setminus C\) (respectively \(G \setminus U\)) is not strongly connected.

If the graph has capacity function \(c : E \to \mathbb{R}_{\geq 0}\) on edges, then \(c(C) = \sum_{e \in C} c_e\) is the total capacity of the cut \(C\).

**Definition 3.5** (Edge set). We define \(E(S,T)\) as the set of edges \(\{(u,v) : u \in S, v \in T\}\).

**Definition 3.6** (Vertex partition). Let \(S,T \subseteq V\). We say that \((S,T)\) is a vertex partition if \(S\) and \(T\) are not empty, and \(S \cup T = V\). In particular, \(E(S,T)\) is an \((x,y)\)-edge-cut for some \(x \in S, y \in T\).
Definition 3.7 (Separation triple). We call \((L, S, R)\) a separation triple if \(L, S,\) and \(R\) partition the vertex \(V\) in \(G\) where \(L\) and \(R\) are non-empty, and there is no edge from \(L\) to \(R\).

Note that \(S\) is an \((x, y)\)-vertex-cut for any \(x \in L\) and \(y \in R\).

Definition 3.8 (Shore). We call a set of vertices \(S \subseteq V\) an out-vertex shore (respectively in-vertex shore) if \(N_{out}(S)\) (respectively \(N_{in}(S)\)) is a vertex-cut.

Definition 3.9 (Vertex connectivity \(\kappa\)). We define vertex connectivity \(\kappa_G\) as the minimum cardinality vertex-cut or \(n - 1\) if no vertex cut exists. More precisely, for distinct \(x, y \in V\), define \(\kappa_G(x, y)\) as the smallest cardinality of \((x, y)\)-vertex-cut if exists. Otherwise, we define \(\kappa_G(x, y) = n - 1\). Then, \(\kappa_G = \min\{\kappa_G(x, y) \mid x, y \in V, x \neq y\}\). We drop the subscript when \(G\) is clear from the context.

Let \(d_{\min}^{out} = \min_v \deg_{G}^{out}(v)\) and let \(v_{\min}\) be any vertex whose out-degree is \(d_{\min}^{out}\). If \(d_{\min}^{out} = n - 1\), then \(G\) is complete, meaning that \(\kappa_G = n - 1\). Otherwise, \(\delta_G^{out}(v_{\min})\) is a vertex-cut. Hence, \(\kappa_G \leq |\delta_G^{out}(v_{\min})| = d_{\min}^{out}\). So, we have the following observation.

Observation 3.10. \(\kappa_G \leq d_{\min}^{out}\)

Proposition 3.11 ([HRG00]). There exists an algorithm that takes as input graph \(G\), and two vertices \(x, y \in V\) and an integer \(k > 0\) and in \(\tilde{O}(\min(km))\) time outputs either an out-vertex shore \(S\) containing \(x\) with \(|N_{out}(S)| = \kappa_G(x, y) \leq k\) and \(y\) is in the corresponding in-vertex shore, or an “\(\perp\) symbol indicating that no such shore exists and thus \(\kappa_G(x, y) > k\).

### 3.2 Undirected Graph

Let \(G = (V, E)\) be an undirected graph. We assume that \(G\) is simple, and connected.

Theorem 3.12 ([NI92]). There exists an algorithm that takes as input undirected graph \(G = (V, E)\), and in \(O(m)\) time outputs a sequence of forests \(F_1, F_2, \ldots, F_n\) such that each forest subgraph \(H_k = (V, \bigcup_{i=1}^k F_i)\) is \(k\)-connected if \(G\) is \(k\)-connected. \(H_k\) has aboriciity \(k\). For any set of vertices \(S\), we have \(E_{H_k}(S, S) \leq k|S|\). In particular, the number of edges in \(H_k\) is at most \(kn\).

To compute vertex connectivity in an undirected graph, we turn it into a directed graph by adding edges in forward and backward directions and run the directed vertex connectivity algorithm.

### 4 Local Vertex Connectivity

Recall that a directed graph \(G = (V, E)\) is strongly connected where \(|V| = n\) and \(|E| = m\).

Theorem 4.1. There is an algorithm that takes as input a pointer to any vertex \(x \in V\) in an adjacency list representing a strongly-connected directed graph \(G = (V, E)\), positive integer \(\nu\) (“target volume”), positive integer \(k\) (“target \(x\)-vertex-cut size”), and positive real \(\epsilon\) satisfying

\[
\nu/\epsilon + \nu < m, \quad (1 + \epsilon)(\frac{2\nu}{\epsilon k} + k) < n \quad \text{and} \quad \deg_{\min}^{out} \geq k \tag{4}
\]

or,

\[
\nu/\epsilon + (1 + \epsilon)nk < m, \quad \text{and} \quad \deg_{\min}^{out} \geq k \tag{5}
\]

and in \(\tilde{O}(\frac{\nu^{3/2}}{\epsilon^{3/2}\epsilon k^{1/2}})\) time outputs either
• a vertex-cut \( S \) corresponding to the separation triple \((L, S, R), x \in L\) such that

\[
|S| \leq (1 + \epsilon)k \quad \text{and} \quad \text{vol}_G^\text{out}(L) \leq \nu / \epsilon + \nu + 1, \text{or}
\]

\[
|S| \leq k \quad \text{and} \quad \text{vol}_G^\text{out}(L) \leq \nu.
\]  

(6)

By setting \( \epsilon = 1/(2k) \), we get the exact version for the size of vertex-cut. Observe that Equation (6) is changed to \( |S| \leq (1 + 1/(2k))k = k + 1/2 \leq k \) since \( |S| \) and \( k \) are integers.

**Corollary 4.2.** There is an algorithm that takes as input a pointer to any vertex \( x \in V \) in an adjacency list representing a strongly-connected directed graph \( G = (V, E) \), positive integer \( \nu \) (“target volume”), and positive integer \( k \) (“target \( x \)-vertex-cut size”) satisfying Equation (4), or Equation (5) where \( \epsilon = 1/(2k) \), and in \( \tilde{O}(\nu^{3/2}k) \) time outputs either

• a vertex cut \( S \) corresponding to the separation triple \((L, S, R), x \in L\) such that

\[
|S| \leq k \quad \text{and} \quad \text{vol}_G^\text{out}(L) \leq 2\nu k + \nu + 1, \text{or}
\]

\[
|S| \leq k \quad \text{and} \quad \text{vol}_G^\text{out}(L) \leq \nu.
\]  

(8)

The rest of this section is devoted to proving the above theorem. For the rest of this section, fix \( x, \nu, k \) and \( \epsilon \) as in the theorem statement.

### 4.1 Augmented Graph and Properties

**Definition 4.3** (Augmented Graph \( G' \)). Given a directed uncapacitated graph \( G = (V, E) \), we define a directed capacitated graph \( (G', c_{G'}) = ((V', E'), c_{G'}) \) where

\[
V' = V_\text{in} \sqcup V_\text{out} \sqcup \{s, t\} \quad \text{and} \quad E' = E_\nu \sqcup E_\infty \sqcup E_\text{deg} \sqcup \{(s, x_\text{out})\},
\]

(10)

where \( \sqcup \) denotes disjoint union of sets, \( s \) and \( t \) are additional vertices not in \( G \), and sets in Equation (10) are defined as follows.

- For each vertex \( v \in V \setminus \{x\} \), we create vertex \( v_\text{in} \) in set \( V_\text{in} \) and \( v_\text{out} \) in set \( V_\text{out} \). For the vertex \( x \), we add only \( x_\text{out} \) to \( V_\text{out} \).
- \( E_\nu = \{(v_\text{in}, v_\text{out}): v \in V \setminus \{x\}\} \).
- \( E_\infty = \{(v_\text{out}, w_\text{in}): (v, w) \in E\} \).
- \( E_\text{deg} = \{(v_\text{out}, t): v \in V_\text{out}\} \).

Finally, we define the capacity function \( c_{G'} : E' \to \mathbb{R}_{\geq 0} \cup \{\infty\} \) as:

\[
c_{G'}(e) = \begin{cases} 
\nu/(ek) & \text{if } e = (v_\text{in}, v_\text{out}) \in E_\nu \\
\deg_G^\text{out}(v) & \text{if } e = (v_\text{out}, t) \in E_\text{deg} \\
\nu / \epsilon + \nu + 1 & \text{if } e = (s, x_\text{out}) \\
\infty & \text{otherwise}
\end{cases}
\]
Lemma 4.4. Let $C^*$ be the minimum-capacity $(s, t)$-cut in $G'$. Recall that $c_{G'}(C^*)$ is its capacity and $\nu$ and $k$ satisfy Equation (4) or Equation (5).

(I) If there exists a separation triple $(L, S, R), x \in L$ in $G$ satisfying Equation (7), then $c_{G'}(C^*) \leq \nu / \epsilon + \nu$.

(II) If $c_{G'}(C^*) \leq \nu / \epsilon + \nu$, then there exists a separation triple $(L, S, R), x \in L$ in $G$ satisfying Equation (6).

We prove Lemma 4.4 in the rest of this subsection.

We define useful notations. For $U \subseteq V$ in $G$, define $V_{\text{out}}(U) = \{v_{\text{out}} \mid v \in U\} \subseteq V_{\text{out}}$ in $G'$. Similarly, we define $V_{\text{in}}(U) = \{v_{\text{in}} \mid v \in U\} \subseteq V_{\text{in}}$ in $G'$.

We first introduce a standard split graph $SG$ from $G'$.

Definition 4.5 (Split graph $SG$). Given $G'$, a split graph $SG$ is an induced graph $SG = G'[W]$ where

$$W = V_{\text{in}} \sqcup V_{\text{out}} \sqcup \{x\},$$

with capacity function $c'_{G}(e)$ restricted to edges in $G'[W]$ where the edge set of $G'[W]$ is $E_{\nu} \sqcup E_{\infty}$.

Proof of Lemma 4.4(I). We fix a separation triple $(L, S, R)$ given in the statement. Since $x \in L$, $S$ is an $(x, y)$-vertex-cut for some $y \in R$ by Definition 3.7.

Let $C = \{(u_{\text{in}}, u_{\text{out}}) \mid u \in S\}$. It is easy to see that $C$ is an $(x_{\text{out}}, y_{\text{in}})$-edge-cut in the split graph $SG$. Since $S$ is an $(x, y)$-vertex-cut, there is no vertex-disjoint paths from $x$ to $y$ in $SG$. By transforming from $G$ to $G'$, vertex $y$ in $G$ becomes $y_{\text{in}}$ and $y_{\text{out}}$ in $G'$. Since $S$ separates $x$ and $y$ in $G'$, by construction of $C$, $C$ must separate $x$ and $y_{\text{in}}$ in $G'$ and thus in $SG$. Therefore there is no $(x_{\text{out}}, y_{\text{in}})$-path in $SG \setminus C$, and the claim follows.

In $G'$, we define an edge-set $C' = C \cup \{(v, t) \mid v \in V_{\text{out}}(L)\}$. It is easy to see that $C'$ is an $(s, t)$-edge-cut in $G'$. Since $C$ is an $(x_{\text{out}}, y_{\text{in}})$-edge-cut in the split graph $SG$. The graph $G' \setminus C$ has no $(s, V_{\text{out}}(S \sqcup R))$-paths. Since $G$ is strongly connected, the sink vertex $t$ in $G' \setminus C$ is reachable from $s$ via only vertices in $V_{\text{out}}(L)$. Hence, it is enough to remove the edge-set $\{(v, t) \mid v \in V_{\text{out}}(L)\}$ to disconnect all $(s, t)$-paths in $G' \setminus C$, and the claim follows.

We now compute the capacity of the cut $C'$.

$$c_{G'}(C') = c_{G'}(C \cup \{(v, t) \mid v \in V_{\text{out}}(L)\})$$
$$= c_{G'}(C) + c_{G'}(\{(v, t) \mid v \in V_{\text{out}}(L)\})$$
$$= \nu |S| / (\epsilon k) + \sum_{v \in S} \deg_{G'}(v)$$
$$= \nu |S| / (\epsilon k) + \text{vol}_{G'}(S)$$
$$\leq \nu / \epsilon + \nu$$

The last two equations follow from the construction of $G'$, and the last inequality follows from the given condition of the statement.

Hence, the capacity of the minimum $(s, t)$-cut $C^*$ is $c_{G'}(C^*) \leq c_{G'}(C') \leq \nu / \epsilon + \nu$. \hfill \qed

Before proving Lemma 4.4(II), we observe structural properties of an $(s, t)$-edge-cut in $G'$.

Definition 4.6. Let $C$ be the set of $(s, t)$-cuts of finite capacities in $G'$. We define three subsets of $C$ as,

- $C_1 = \{C : C \in C$, and one side of vertices in $G' \setminus C$ contains $s$ or $t$ as a singleton $\}$.
- $C_2 = \{C : C \in C \setminus C_1$, and $C$ is an $(\{s\} \sqcup V_{\text{in}}, \{t\})$-edge-cut $\}$.
- $C_3 = \{C : C \in C \setminus C_1$, and $C$ is an $(\{s\}, \{v_{\text{in}}, t\})$-edge-cut for some $v_{\text{in}} \in V_{\text{in}}$ $\}$. 

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Observe that three partitions in Definition 4.6 formed a complete set \( C \) and are pairwise disjoint by Definition 3.4, and by the construction of \( G' \).

**Observation 4.7.**

\[ C = C_1 \cup C_2 \cup C_3 \]

**Proposition 4.8.** We have the following lower bounds on cut capacity for cuts in \( C_1 \cup C_2 \).

- For all \( C \in C_1 \), \( c_{G'}(C) \geq \min(\nu/\epsilon + \nu + 1, m) \)
- For all \( C \in C_2 \), \( c_{G'}(C) \geq \min(\nu/\epsilon + \nu + 1, \max((n - (1 + \epsilon)k)k, m - (1 + \epsilon)nk)) \)

**Proof.** By Definition 4.6, any \( C \in C_1 \) contains \((s, x_{\text{out}})\) or \( E_{\text{deg}} \). So, \( C \) has capacity \( c_{G'}(C) \geq \min(\nu/\epsilon + \nu + 1, \sum_{v \in V} \deg_{G'}(v)) = \min(\nu/\epsilon + \nu + 1, m) \).

Next, we show that if \( C \in C_2 \), then \( c_{G'}(C) \geq \min(\nu/\epsilon + \nu + 1, \max((n - (1 + \epsilon)k)k/2, m - (1 + \epsilon)nk)) \).

By Definition 4.6, \( C \) has finite capacity. We can write \( C = E^*_\nu \cup E^*_{\text{deg}} \) where \( E^*_\nu \subseteq E_\nu \) and \( E^*_{\text{deg}} \subseteq E_{\text{deg}} \). If \( |E^*_\nu| > (1 + \epsilon)k \), then, by construction of \( G' \), \( c_{G'}(C) \geq |E^*_\nu| > \nu/\epsilon + \nu \).

From now, we assume that \( |E^*_\nu| \leq (1 + \epsilon)k \). We show two inequalities:

\[ c_{G'}(C) \geq (n - (1 + \epsilon)k)k \tag{11} \]

and

\[ c_{G'}(C) \geq m - nk(1 + \epsilon). \tag{12} \]

We claim that \( |E^*_{\text{deg}}| \geq n - (1 + \epsilon)k \). Consider \( G' \setminus C \). Let \( S = \{x\} \cup V_\text{in} \). Observe that any \( w \in S \) cannot reach \( t \) in \( G' \setminus C \) since \( C \) is an \((\{s\} \cup V_\text{in}, \{t\})\)-edge-cut. So, for all \( v_\text{in} \in V_\text{in} \), we have \((v_\text{in}, v_{\text{out}}) \in C \) or \((v_{\text{out}}, t) \in C \). Since \( |E^*_\nu| \leq (1 + \epsilon)k \), this means we can include edges of type \((v_\text{in}, v_{\text{out}})\) at most \((1 + \epsilon)k \) edges. Hence, the rest of the edges must be of the form \((v, t)\), and thus \( |E^*_{\text{deg}}| \geq n - (1 + \epsilon)k \).

We now show Equation (11). By Equation (4) or Equation (5), \( \deg_{\text{min}} \geq k \). Since \( C = E^*_\nu \cup E^*_{\text{deg}} \), we have \( c_{G'}(C) \geq c_{G'}(E^*_\nu) \geq |E^*_\nu| \deg_{\text{min}} \geq (n - (1 + \epsilon)k)k \).

Finally, we show Equation (12). Since \( |E^*_{\text{deg}}| \geq n - (1 + \epsilon)k \), \( |E_{\text{deg}} \setminus E^*_{\text{deg}}| \leq (1 + \epsilon)k \), and thus \( c_{G'}(E_{\text{deg}} \setminus E^*_{\text{deg}}) \leq (1 + \epsilon)nk \), (recall each vertex has degree at most \( n - 1 \)). Therefore,

\[ c_{G'}(C) \geq c_{G'}(E^*_{\text{deg}}) = \sum_{v \in E^*_{\text{deg}}} \deg_{G'}(v) - c_{G'}(E_{\text{deg}} \setminus E^*_{\text{deg}}) \geq m - (1 + \epsilon)nk \]

\[ \square \]

**Corollary 4.9.** For all \( C \in \mathcal{C} \), if \( c_{G'}(C) \leq \nu/\epsilon + \nu \), then \( C \in C_3 \)

**Proof.** By Observation 4.7 and Proposition 4.8, it is enough to show that \( C \notin C_1 \) and \( C \notin C_2 \) using Equation (4), or Equation (5). By either Equation (4) or Equation (5), \( \nu/\epsilon + \nu < m \), and thus \( C \notin C_1 \). Next, we show that \( C \notin C_2 \). It is enough to show that \( \nu/\epsilon + \nu \) is smaller than one of two terms in max. If Equation (4) is satisfied, then \((1 + \epsilon)(2\nu/(\epsilon k) + k) < n \). This implies \( \nu/\epsilon + \nu < (n - (1 + \epsilon)k)k \). If Equation (5) is satisfied, then we immediately get \( \nu/\epsilon + \nu < m - (1 + \epsilon)nk \). \[ \square \]

We now ready to prove Lemma 4.4(II).
Proof of Lemma 4.4(II). In $G$, we show the existence of a separation triple $(L, S, R)$ where $x \in L$, $|S| \leq (1 + \epsilon)k$.

The minimum $(s, t)$-cut in $G'$, $C^*$, is an $(s, \{v_{in}, t\})$-edge-cut (with finite capacity) for some $v_{in} \in V_{in}$. Since $c_{G'}(C^*) \leq \nu/k + \nu$, by Corollary 4.9, $C^* \in C_4$.

We can write $C^* = C_{\nu}^* \cup E_{\nu}^*$ where $\emptyset \neq E_{\nu}^* \subseteq E_{\nu}$ and $\emptyset \neq E_{\nu}^* \subseteq E_{\nu}$ in $G'$. To see that $E_{\nu}^* \neq \emptyset$, suppose otherwise, then $C^*$ must be in $C_4$, a contradiction.

It is easy to see that $E_{\nu}^*$ is an $(x_{out}, v_{in})$-edge-cut in $SG$. First of all, $E_{\nu}^*$ is the subset of edges in $SG$ by Definition 4.5. Since $v_{in}$ is not reachable from $s$ in $G'$ \ $C^*$ and $(s, x) \notin C^*$, $x$ cannot reach $v_{in}$ in $G' \ C^*$. Observe that edges in $E_{\nu}^*$ (and in particular, $E_{\nu}^*$) have no effect for reachability of the $(x_{out}, v_{in})$ path in $G'$. Since $C^* = C_{\nu}^* \cup E_{\nu}^*$, only edges in $E_{\nu}^*$ can affect the reachability of the $(x, v_{in})$ path in $G'$. So, when restricting $G'$ to $SG$, $x_{out}$ cannot reach $v_{in}$ in $SG \ E_{\nu}^*$. Therefore, $E_{\nu}^*$ is an $(x_{out}, v_{in})$-edge-cut in $SG$, and the claim follows.

To show a separation triple $(L, S, R)$, it is enough to define $S$, and show that $S$ is an $(x, y)$-vertex-cut where $x \in L$ and $y \in R$. This is because $L$ and $R$ can be found trivially when we remove $S$ from $G$.

Let $S = \{u \in V: (u_{in}, u_{out}) \in E_{\nu}^*\}$. It is easy to see that $S$ is an $(x, y)$-vertex-cut in $G$ for some $y \in V$. Since $E_{\nu}^* \neq \emptyset$ is an $(x, y_{in})$-edge-cut in $SG$, $y_{in}$ is not reachable by $x$ in $SG \ E_{\nu}^*$. By construction of $G'$ (Definition 4.3), the corresponding out-vertex pair of $y_{in}$, $y_{out}$, has in-degree one from $y_{in}$. So, $y_{out}$ in $SG \ E_{\nu}^*$ is also not reachable by $x$. Hence, in $SG \ E_{\nu}^*$, both $y_{in}$ and $y_{out}$ are not reachable from $x$. So, by the construction of $G'$, and in $G \ U$, $y$ is not reachable from $x$. Therefore, $U$ is an $(x, y)$-vertex-cut in $G$.

Next, $|S| \leq (1 + \epsilon)k$ since otherwise $c_{G'}(C^*) > (1 + \epsilon)k(\nu/(\epsilon k)) = \nu/\epsilon + \nu$, a contradiction to the capacity of $C^*$.

We next show that $\text{vol}_{G'}^*(L) \leq \nu/\epsilon + \nu + 1$.

Let $E_{\nu}^*(L) = \{(v_{out}, t): v \in L\}$. We claim that $E_{\nu}^*(S) = E_{\nu}^*(L)$. Since $E_{\nu}^*$ is an $(x_{out}, v_{in})$-cut in $SG$, the graph $G' \ E_{\nu}^*$ has no $(s, V_{out}(S \cup R))$-paths. Since $G$ is strongly connected, the sink vertex $t$ in $G' \ E_{\nu}^*$ is reachable from $s$ via only vertices in $V_{out}(L)$. Since $C^*$ is the minimum $(s, t)$-cut in $G'$, $E_{\nu}^*$ only contains edges in $E_{\nu}^*(L)$. The claim follows.

We now show that $\text{vol}_{G'}^{out}(L) \leq \nu/\epsilon + \nu + 1$. By the previous claim, $c_{G'}(E_{\nu}^*) = \sum_{v \in S} \deg_{G}^\text{out}(v) = \text{vol}_{G'}^*(S)$. Also, denote $F^*$ as the value of the maximum $(s, t)$-flow in $G'$. By strong duality (max-flow min-cut theorem), $c_{G'}(C^*) = F^*$. Note that $F^* \leq \nu/\epsilon + \nu + 1$ since this corresponds to an $(s, t)$-edge cut that contains edge $(s, x_{out})$. Hence,

$$\text{vol}_{G'}^{out}(S) + c_{G'}(E_{\nu}^*) = c_{G'}(E_{\nu}^*) + c_{G'}(E_{\nu}^*) = c_{G'}(E_{\nu}^* \cup E_{\nu}^*) = c_{G'}(C^*) = F^* \leq \nu/\epsilon + \nu + 1.$$ 

Therefore, $\text{vol}_{G'}^{out}(S) + c_{G'}(E_{\nu}^*) \leq \nu/\epsilon + \nu + 1$, and thus $\text{vol}_{G'}^{out}(S) \leq \nu/\epsilon + \nu + 1$ as desired.

\[\Box\]

4.2 Preliminaries for Flow Network and Binary Blocking Flow

We define notations related flows on a capacitated directed graph $G = (V, E, c)$. We fix vertices $s$ as source and $t$ as sink.

**Definition 4.10** (Flow). For a capacitated graph $G = (V, E, c)$, a flow $f$ is a function $f: E \rightarrow \mathbb{R}$ satisfying two conditions:
• For any \((v, w) \in E, f(v, w) \leq c(v, w)\), i.e., the flow on each edge does not exceed its capacity.
• For any vertex \(v \in V \setminus \{s, t\}\), \(\sum_{w:(u,v)\in E} f(u,v) = \sum_{w:(v,w)\in E} f(v,w)\), i.e., for each vertex except for \(s\) or \(t\), the amount of incoming flow is equal to the amount of outgoing flow.

We denote \(|f| = \sum_{(v, t) \in E} f(v, t)|\) as the value of flow \(f\).

**Definition 4.11 (Residual graph).** Given a capacitated graph \(G = (V, E, c)\) and a flow function \(f\), we define the residual graph with respect to \(f\) as \((G, c, f) = (V, E_f, c_f)\) where \(E_f\) contains all edges \((v, w) \in E\) with \(c(v, w) - f(v, w) > 0\). Note that \(f(v, w)\) can be negative if the actual flow goes from \(w\) to \(v\), and \(E_f\) may contain reverse edge \(e^R\) to the original graph \(G\). We call an edge in \(E_f\) as residual edge with residual capacity \(c_f(v,w) = c(v,w) - f(v,w)\). An edge in \(E\) is not in \(E_f\) when the amount of flow through this edge equals its capacity. Such an edge is called an saturated edge. We sometimes use notation \(G_f\) as the shorthand for the residual graph \((G, c, f)\) when the context is clear.

**Definition 4.12 (Blocking flow).** Given a capacitated graph \(G = (V, E, c)\), a blocking flow is a flow that saturates at least one edge on every \((s, t)\)-path in \(G\).

We will use Definition 4.12 mostly on the residual graph \(G_f\).

Given a binary length function \(\ell\) on \((G, c, f)\), we define a natural distance function to each vertex in \((G, c, f)\) under \(\ell\).

**Definition 4.13 (Distance function).** Given a residual graph \(G_f\), and binary length function \(\ell\), a function \(d_\ell : V \to \mathbb{Z}_{\geq 0}\) is a distance function if \(d_\ell(v)\) is the length of the shortest \((s, v)\)-path in \(G_f\) under the binary length function \(\ell\).

For any \((v, w) \in E_f\), \(d_\ell(v) + \ell(v, w) \geq d_\ell(w)\) by Definition 4.13. If \(d_\ell(v) + \ell(v, w) = d_\ell(w)\), then we call \((v, w)\) admissible edge under length function \(\ell\).

We denote \(E_a\) to be the set of admissible edges of \(E_f\) in \((G, c, f)\) under length function \(\ell\).

**Definition 4.14 (Admissible graph).** Given a residual graph \((G, c, f)\), and a length function \(\ell\), we define an admissible graph \(A(G, c, f, \ell) = (G[E_a], c, f)\) to be an induced subgraph of \((G, c, f)\) that contains only admissible edges under length function \(\ell\).

**Definition 4.15 (\(\Delta'\)-or-blocking flow).** For any \(\Delta' > 0\), a flow is called a \(\Delta'\)-or-blocking flow if it is a flow of value exactly \(\Delta'\), or a blocking flow.

**Definition 4.16 (Binary length function \(\hat{\ell}\)).** Given \(\Delta > 0\), a capacitated graph \((G, c)\) and a flow \(f\), we define binary length functions \(\hat{\ell}\) and \(\tilde{\ell}\) for any edge \((u, v)\) in a residual graph \((G, c, f)\) as follows.

\[
\hat{\ell}(u, v) = \begin{cases} 
0 & \text{if residual capacity } c(u, v) - f(u, v) \geq \Delta \\
1 & \text{otherwise}
\end{cases}
\]

Let \(\hat{d}(v)\) be the shortest path distance between \(s\) and \(v\) under the length function \(\hat{\ell}\). We define special edge \((u, v)\) to be an edge \((u, v)\) such that \(\hat{d}(u) = \hat{d}(v), \Delta/2 \leq c(u, v) - f(u, v) < \Delta\), and \(c(v, u) - f(v, u) \geq \Delta\). We define the next length function \(\tilde{\ell}\).

\[
\tilde{\ell}(u, v) = \begin{cases} 
0 & \text{if } (u, v) \text{ is special} \\
\hat{\ell}(u, v) & \text{otherwise}
\end{cases}
\]

Classic near-linear time blocking flow algorithm by [ST83] works only for acyclic admissible graph. Note that an admissible graph \(A(G, c, f, \tilde{\ell})\) may contain cycles since an edge-length can be...
zero. To handle this issue, the key idea by [GR98] is to contract all strongly connected components, and run the algorithm by [ST83]. To route the flow, they construct a routing flow network inside each strongly connected component using two directed trees for a fixed root in the component network. One tree is for routing in-flow, the other one is for routing out-flow from the component. The internal routing network ensures that each edge inside is used at most twice. Hence, by restricting at most $\Delta/4$ amount of flow, each edge is used at most $\Delta/2$. Since each edge in the component has length zero, it has residual capacity at least $\Delta$. So, the result of flow augmentation respects the edge capacity. Finally, special edges (with the condition related to $\Delta/2$) play an important role to ensure that blocking flow augmentation strictly increases the distance $d_B(t)$.

The following lemma summarizes the sketch of aforementioned algorithm.

**Lemma 4.17** ([GR98]). Let $A(G, c, f, \ell)$ be an admissible graph and $m_A$ be its number of edges. Then, there exists an algorithm that takes as input $A$ and $\Delta > 0$, and in $O(m_A \log(m_A))$ time, outputs a $\Delta/4$-or-blocking flow. We call the algorithm as BinaryBlockingFlow($A(G, c, f, \ell), \Delta$).

We now define the notion of shortest-path flow. Intuitively, it is a union of shortest paths on admissible graphs. This is the flow resulting from, e.g., the Binary Blocking Flow algorithm [GR98].

**Definition 4.18** (Shortest-path flow). Given a graph $(G, c)$ with a flow $f$, and length function $\ell$, and let $G_f$ be the residual graph. A flow $f^*$ in $G_f$ is called shortest-path flow if it can be decomposed into a set of shortest paths under length function $\ell$, i.e., $f^* = \sum_{i=1}^b f_i^*$ for some integer $b > 0$ where support($f_i^*$) is a shortest-path in $G_f$ under length function $\ell$.

Observe that BinaryBlockingFlow($A(G, c, f, \ell), \Delta$) always produces a shortest-path flow.

From the rest of this section, we fix an augmented graph $(G', c_{G'})$ (Definition 4.3), and also a flow $f$.

Given residual graph $G'_f$, and $d_f$, we can use BinaryBlockingFlow($A(G', c_{G'}, f, \ell), \Delta$) to compute a $\Delta/4$-or-binary blocking flow in $(G', c_{G'}, f)$ in $O(m)$ time.

[OZ14] provide a slightly different binary length function such that the algorithm in [GR98] has local running time.

Our goal in next section is to output the same $\Delta/4$-or-binary blocking flow in $G'_f$ in $O(\nu k)$ time using a slight adjustment from [OZ14].

### 4.3 Local Augmented Graph and Binary Blocking Flow in Local Time

The goal in this section is to compute binary blocking flow on the residual graph of the augmented graph $(G', c_{G'})$ with a flow $f$ in “local” time. To ensure local running time, we cannot construct the augmented graph $G'$ explicitly. Instead, we compute binary blocking flow from a subgraph of $G'$ based on “absorbed” vertices.

**Definition 4.19** (Split-node-saturated set). Given a residual graph $(G', c_{G'}, f)$, let $B_{out}$ be the set of vertices $v \in V_{out} \cup \{x\}$ in the residual graph $(G', c_{G'}, f)$ whose edge to $t$ is saturated. The split-node-saturated set $B$ is defined as:

$$B = B_{out} \cup N^\text{out}_{G'}(B_{out}) \setminus \{t\}$$

Note that $x$ is a fixed vertex as in Definition 4.3.

**Definition 4.20** (Local binary length function). Fix a parameter $\Delta > 0$ to be selected, let $\ell$ be the length function in Definition 4.16 for the residual graph $(G', c_{G'}, f)$. For vertex $u, v$ in the
residual graph, if \( u, v \in B \), we call residual edge \((u, v)\) \textit{modern}. Otherwise, we call residual edge \((u, v)\) \textit{classical}.

We define \textit{local binary length} function \( \ell \):

\[
\ell(u, v) = \begin{cases} 
1 & \text{if } (u, v) \text{ is classical} \\
\ell(u, v) & \text{otherwise}
\end{cases}
\]

**Definition 4.21** (Distance under local binary length \( \ell \)). Define distance function \( d(v) \) as the shortest path distance between the source vertex \( s \) and vertex \( v \) in the residual graph \((G', c_{G'}, f)\) under the local length function \( \ell \).

The following observations about structural properties of the residual graph \( G' \) follows immediately from the definition of local length function \( \ell \).

**Observation 4.22.** For a given residual graph \((G', c_{G'}, f)\),
- for any residual edge \((u, v) \in E_{\infty, f}\) that is modern, \( \ell(u, v) = 0 \).
- for any residual edge \((u, v) \in E_{\deg, f} \cup (s, x)\), \((u, v)\) is classical.
- any residual edge with length zero is modern.

**Definition 4.23** (Layers). Given distance function \( d \) on residual graph \((G', c_{G'}, f)\), define \( L_j = \{v \in G' : d(v) = j\} \) to be the set of \( j^{\text{th}}\)-layer with respect to distance \( d \). Define \( d_{\max} = d(t) \) to be distance between \( s \) and \( t \) in \((G', c_{G'}, f)\).

The proof of the following Lemma is similar to that from [OZ14], but we focus on the augmented graph \((G', c_{G'}, f)\). Recall split-node-saturated set \( B \) from Definition 4.19.

**Lemma 4.24.** If \( d_{\max} < \infty \) and \((x, t)\) is saturated, then we have:

(I) \( d_{\max} \geq 3 \).

(II) \( L_0 = \{s\} \).

(III) \( L_j \subseteq B \) for \( 1 \leq j \leq d_{\max} - 2 \).

(IV) \( L_j \subseteq B \cup N_{G'}^{\text{out}}(B) \) for \( j = d_{\max} - 1 \).

**Proof of Lemma 4.24(I).** First, \( d_{\max} \geq 2 \) since \((s, x_\text{in})\) and any \((v_\text{out}, t) \in E_{\deg, f}\) is classical by Observation 4.22. This means \( d_{\max} \geq 3 \) or \( d_{\max} = 2 \). We show that \( d_{\max} = 2 \) is not possible. Suppose for the contradiction that \( d_{\max} = 2 \). Then every intermediate edge in any \((s, t)\)-path, i.e., \( s \to x \to v_\text{in} \to v_\text{out} \to \ldots \to w_\text{in} \to w_\text{out} \to t \), must have zero length. Also, the path cannot be of the form \( s \to x \to t \) since \((x, t)\) is assumed to be saturated. In particular, \((w_\text{in}, w_\text{out})\) has zero length. By Observation 4.22, \((w_\text{in}, w_\text{out})\) must be modern. This edge is modern when \( w_\text{out} \in B \) by definition of split-node-saturated set \( B \). Therefore, \((w_\text{out}, t)\) is saturated, a contradiction.

**Proof of Lemma 4.24(II).** The second item follows from the fact that \((s, x)\) is the only outgoing-edge from \( s \) and \((s, x)\) is classical and hence has length 1.

**Proof of Lemma 4.24(III).** For \( 1 \leq j \leq d_{\max} - 2 \), if \( v \in L_j \), then we consider two types of \( v \). If \( v \) is an out-vertex \( v_\text{out} \), then \( d(v_\text{out}) = j \leq d_{\max} - 2 \). Thus, \((v_\text{out}, t)\) must be saturated since \( d(t) = d_{\max} > d_{\max} - 1 \geq d(v_\text{out}) + 1 \). Hence, \( v_\text{out} \in B_\text{out} \), which is in \( B \).

If \( v \) is an in-vertex \( v_\text{in} \), then there must be an out-vertex \( u_\text{out} \) such that

\[
d(v_\text{in}) = d(u_\text{out}) + \ell(u_\text{out}, v_\text{in}) \tag{13}
\]

We consider two cases for \( j \). We show that \( v_\text{in} \in B \) for either case.
• If \( j = 1 \), then \( u_{\text{out}} \) could also be \( x \). Since \( d_{\text{max}} \geq 3 \), \( u_{\text{out}} \) at \( L_1 \) must be saturated, meaning that \( u_{\text{out}} \in B_{\text{out}} \). Hence, \( v_{\text{in}} \) is an out-neighbor of \( u_{\text{out}} \in B_{\text{out}} \).

• If \( j \geq 2 \), then we show that \( 1 \leq d(u_{\text{out}}) \leq d_{\text{max}} - 2 \). For the upper bound \( d(u_{\text{out}}) \leq d_{\text{max}} - 2 \), rearranging Equation (13), and use the fact that \( \ell \) is a binary function, \( \ell(u_{\text{out}}, v_{\text{in}}) \in \{0, 1\} \) to get:

\[
d(u_{\text{out}}) = d(v_{\text{in}}) - \ell(u_{\text{out}}, v_{\text{in}}) = j \leq d_{\text{max}} - 2
\]

The lower bound \( d(u_{\text{out}}) \geq 1 \) follows from \( d(v_{\text{in}}) = j \geq 2 \), Equation (13), and \( \ell \) is binary.

Since \( 1 \leq d(u_{\text{out}}) \leq d_{\text{max}} - 2 \), by the previous discussion, \( u_{\text{out}} \in B_{\text{out}} \). Therefore, \( v_{\text{in}} \in B \) since \( v_{\text{in}} \) is the out-neighbor of \( u_{\text{out}} \in B_{\text{out}} \).

\[\square\]

**Proof of Lemma 4.24 (IV).** For any \( v \in L_d_{\text{max}} - 1 \), if \( v \in B \), then we are done. Now, assume that \( v \notin B \). Then, \( v \) is either an in-vertex or out-vertex. We first show that \( v \) cannot be an in-vertex.

Suppose for contradiction that \( v \) is an in-vertex \( v_{\text{in}} \notin B \), then there must be a vertex \( u_{\text{out}} \) such that \( d(v_{\text{in}}) = d(u_{\text{out}}) + \ell(u_{\text{out}}, v_{\text{in}}) \). Since \( v_{\text{in}} \notin B \), the residual edge \( (u_{\text{out}}, v_{\text{in}}) \) is classical. Then, \( \ell(u_{\text{out}}, v_{\text{in}}) = 1 \). So,

\[
d(u_{\text{out}}) = d(v_{\text{in}}) - \ell(u_{\text{out}}, v_{\text{in}}) = (d_{\text{max}} - 1) - 1 \leq d_{\text{max}} - 2
\]

By Lemma 4.24(III), \( u_{\text{out}} \) is in \( B \), which means \( u_{\text{out}} \in B_{\text{out}} \). Hence, \( v_{\text{in}} \) is an out-neighbor of \( u_{\text{out}} \in B_{\text{out}} \). So, \( v_{\text{in}} \in B \), a contradiction.

Finally, if \( v = v_{\text{out}} \notin B \), then we show that \( v_{\text{out}} \in N_{\text{out}}(B) \). There exists \( u_{\text{in}} \) such that \( d(v_{\text{out}}) = d(u_{\text{in}}) + \ell(u_{\text{in}}, v_{\text{out}}) \). Since \( v_{\text{out}} \notin B \), \( v_{\text{out}} \) is not saturated. Hence, \( (u_{\text{in}}, v_{\text{out}}) \) is classical. Therefore, \( u_{\text{in}} \in L_j \) for \( j \leq d_{\text{max}} - 2 \). So, \( u_{\text{in}} \in B \) by Lemma 4.24(III), and \( v_{\text{out}} \) is the out-neighbor of \( u_{\text{in}} \). Therefore, \( v_{\text{out}} \in N_{\text{out}}(B) \).

**Definition 4.25 (Local graph, LG).** Given the augmented graph \( G' = (V', E') \) and split-node-saturated set \( B \), we define the local graph \( LG(G', B) = G'[V''] = (V'', E'') \) as an induced subgraph of \( G' \) where

\[
V'' = B \sqcup N_{\text{out}}(B) \sqcup \{s, t\} \quad \text{and} \quad E'' = E''_v \sqcup E''_w \sqcup E''_{\text{deg}} \sqcup \{(s, x)\}
\]

where the sets in Equation (14) are defined as follows.

- \( E''_v = \{(v_{\text{in}}, v_{\text{out}}): v_{\text{out}} \in B_{\text{out}} \sqcup N_{\text{out}}(B), (v_{\text{in}}, v_{\text{out}}) \in E_v \} \).
- \( E''_w = \{(v_{\text{out}}, w_{\text{in}}): v_{\text{out}} \in B_{\text{out}} \sqcup N_{\text{out}}(B), (v_{\text{out}}, w_{\text{in}}) \in E_w \} \).
- \( E''_{\text{deg}} = \{(v_{\text{out}}, t): v_{\text{out}} \in B_{\text{out}} \sqcup N_{\text{out}}(B) \} \).

Using the same capacity and flow as in \( G' \), the residual local graph is \( (LG(G', B), c_{LG}, f_{LG}) \) where \( c_{LG} \) and \( f_{LG} \) are the same as \( c_{G'} \) and \( f_{G'} \), but restricted to the edges in \( LG(G', B) \). The local length function \( \ell \) also applies to \( LG(G', B) \).

**Lemma 4.26.** Let \( m' \) be the number of edges in \( LG(G', B) \), and \( n' = |V''| \) be the number of vertices in \( LG(G', B) \). We have

\[
m' \leq 4\nu/\epsilon \quad \text{and} \quad n' \leq 8\nu/(\epsilon k).
\]

**Proof.** For any out-vertex \( v_{\text{out}} \in B_{\text{out}} \), its edge to \( t \) must be saturated before it is included in \( B \) with capacity of \( \deg_{\text{out}}(v) \). The edge \((s, x)\) is also an \((s, t)\)-edge cut in \( G' \) with capacity \( \nu/\epsilon + \nu + 1 \). Hence, the maximum flow \( F^* \) in \( G' \) is at most \( \nu/\epsilon + \nu + 1 \). We have

\[
\sum_{v_{\text{out}} \in B_{\text{out}}} \deg_{\text{out}}(v) \leq F^* \leq \nu/\epsilon + \nu + 1.
\]
By Lemma 4.24 and Definition 4.25, \( m' = |E''_v| + |E''_\infty| + |E''_{\text{deg}}| + 1 \) where \( |E''_v| = |B_{\text{out}}| + |N_{G'}^\text{out}(B)| - 1, |E''_\infty| = \sum_{v_{\text{out}} \in B_{\text{out}}} \deg_{G'}^\text{out}(v) \), and \( |E''_{\text{deg}}| = |B_{\text{out}}| + |N_{G'}^\text{out}(B)| \). By Definition 4.25, \( B_{\text{out}} \cup N_{G'}^\text{out}(B) \subseteq V'' \). Since \( |V''| = n' \) and every out-vertex has a corresponding in-vertex \( (x \text{ has } s, |B_{\text{out}}| + |N_{G'}^\text{out}(B)| \leq n'/2 \leq \sum_{v_{\text{out}} \in B_{\text{out}}} \deg_{G'}^\text{out}(v) \). So,

\[
m' \leq 2 \sum_{v_{\text{out}} \in B_{\text{out}}} \deg_{G'}^\text{out}(v) \leq 2(\nu/\epsilon + \nu + 1) \leq 4\nu/\epsilon.
\]

To compute \( n' \), note that each \( v_{\text{out}} \) has at least \( d_{\text{out}}^\min \geq k \) edges. Therefore, the number of vertices including \( v_{\text{in}} \) is at most \( n' \leq 2(m'/d_{\text{out}}^\min) \leq 2m'/k \leq 8\nu/(ck) \).

**Corollary 4.27.** Given a residual graph \((G', c_{G'}, f)\) and split-node-saturated set \( B \), and a pointer to vertex \( x \), we can construct \((LG(G', B), c_{LJ}, f)\) in \( O(m') = O(\nu/\epsilon) \) time.

The proof of the following Lemma is a straightforward modification from [OZ14].

**Lemma 4.28.** Given the local length function \( \ell \) on both residual augmented graph \((G', c_{G'}, f)\) and residual local graph \((LG, c_{LG}, f_{LG})\) \((V'' \cup E''_v, E''_\infty, c_{LG}, f)\) (Recall \( f_{LG} \) from Definition 4.25). Let \( f_1 \) be the output of \( \text{BinaryBlockingFlow}(A(G', c_{G'}, f, \ell), \Delta) \). Let \( f_2 \) be the output of \( \text{BinaryBlockingFlow}(A(LG, c_{LG}, f_{LG}, \ell), \Delta) \). Then,

- \( f_1 = z(f_2) \) where \( z(f_2)(e) = \begin{cases} 0 & \text{ if } e \notin E''_v, \\ f_2(e) & \text{ otherwise} \end{cases} \)

i.e., \( f_1 \) and \( f_2 \) coincide.

- BinaryBlockingFlow \((A(LG, c_{LG}, f_{LG}, \ell), \Delta)\) takes \( \tilde{O}(\nu/\epsilon) \) time.

**Proof.** We focus on proving the first item. For notational convenience, denote \( G'_{f} = (G', c_{G'}, f) \), and \( LG_f = (LG, c_{LG}, f_{LG}) \). We show that there is no \((s, t)\)-path in \( G'_{f} \) if and only if there is no \((s, t)\)-path in \( LG_f \). The forward direction follows immediately from the fact that \( LG_f \) is a subgraph of \( G'_{f} \). Next, we show the backward direction. Let \( U \) be a subset of vertices in graph \( LG_f \) such that \( s \in U \) and \( t \notin U \) and there is no edge between \( U \) and \( V_{LG_f} \setminus U \). By Definition 4.25, \( U \subseteq V_{LG_f} = B \cup N_{G'}^\text{out}(B) \cup \{s\} \). In fact, \( U \subseteq B \{s\} \) since vertices in \( N_{G'}^\text{out}(B) \) have residual edges to sink \( t \) with positive residual capacity by the construction of \( LG_f \). Now, we claim that all edges at the boundary of \( B \cup \{s\} \) in \( G' \) and \( LG \) are the same. Indeed, all edges at the boundary of \( B \cup \{s\} \) have the form \((u, v)\) where \( u \in B \) and \( v \in N_{G'}^\text{out}(B) \) and \( B \cup N_{G'}^\text{out}(B) \subseteq V'' \) in \( LG_f \). Furthermore, there is no \((U, t)\) path in \( LG_f \) where \( U \supseteq s \). Therefore, there is no \((s, t)\) path in \( G'_{f} \).

For the rest of the proof, we assume that there is an \((s, t)\) path, i.e., \( d(t) = d_{\text{max}} < \infty \).

We claim that a flow \( f^* \) in \( G'_{f} \) is shortest-path flow if and only if the same flow \( f^* \) when restricting to edges in \( LG_f \) is shortest-path flow.

We show the forward direction. If \( f^* \) in \( G'_{f} \) is shortest-path flow, then by definition of shortest-path flow, the support of \( f^* \) contains only vertices with \( d(v) < d_{\text{max}} \) and \( t \). By Lemma 4.24,

\[
\{s\} \sqcup L_1 \sqcup \ldots \sqcup L_{d_{\text{max}}-1} \sqcup \{t\} \subseteq \{s, t\} \sqcup B \sqcup N_{G'}^\text{out}(B)
\]

We show that support of \( f^* \) form a subgraph of \( LG \). The edges are either between consecutive layers \( L_i, L_{i+1} \) or within a layer. We can limit the edges using Lemma 4.24. From \( s \) to vertices in \( L_1 \), there is only one edge \((s, x)\). Edges from \( L_i \) to \( L_{i+1} \) for \( 1 \leq i \leq d_{\text{max}} - 2 \) must be of the form \( \{(v_{\text{in}}, v_{\text{out}}), (v_{\text{out}}, v_{\text{in}}) : v_{\text{in}}, v_{\text{out}} \in B\} \) or \( \{(v_{\text{in}}, w_{\text{out}}) : w_{\text{out}} \in N_{G'}^\text{out}(B), w_{\text{in}} \in B\} \). From \( L_{d_{\text{max}}-1} \) to \( L_{d_{\text{max}}} \), the edge must be of the form \( \{(v_{\text{out}}, t) : v_{\text{out}} \in B_{\text{out}}\} \). If the edges are within a layer, then they must be modern since their length is zero. This has the form of \( \{(u, v) \in E': u, v \in B\} \).
Since the support of $f^*$ form a subgraph of $LG$, we can restrict $f^*$ to the graph $LG$, and we are done with the forward direction of the claim.

We show the backward direction of the claim. Let $f'$ be a shortest-path flow in $LG_f$. We can extend $f'$ to be the flow in $G'_f$ by the function $z(f')$.

$$z(f')(e) = \begin{cases} 
0 & \text{if } e \notin E'_f. \\
 f'(e) & \text{otherwise}
\end{cases}$$

The support of the flow function $z(f')$ in $G'_f$ contains vertices in contains only vertices with $d(v) < d_{\text{max}}$ and $t$ since $f'$ is the shortest-path flow. Therefore, $z(f')$ is the shortest-path flow in $G'_f$, and we are done with the backward direction of the claim.

Finally, the first item of the lemma follows since BinaryBlockingFlow outputs a shortest-path flow.

The running time for the second item follows from the fact that number of edges $m'$ in $LG(G', B)$ is $O(\nu / \epsilon)$ by Lemma 4.26. By Lemma 4.17, BinaryBlockingFlow($A(LG, c_{LG}, f, \ell), \Delta$) can be computed in $\tilde{O}(m') = \tilde{O}(\nu / \epsilon)$ time.

### 4.4 Local Goldberg-Rao’s Algorithm for Augmented Graph

**Theorem 4.29.** Given graph $G$, we can compute the $(s, t)$ max-flow in $G'$ in $\tilde{O}(\nu^{3/2}/(\epsilon^{3/2} \sqrt{k}))$ time.

#### Algorithm 1: LocalFlow($G, x, \nu, k$)

- **Input:** $x \in V, \nu, k$
- **Output:** maximum $(s, t)$-flow and its corresponding minimum $(s, t)$-edge-cut in $G'$

1. Let $G'$ be an implicit augmented graph on $G$. // No need to construct explicitly.
2. $\Lambda \leftarrow \sqrt{8\nu / (\epsilon k)}$
3. $F \leftarrow 2\nu k + \nu + 1 - \deg_{G}(x)$ // $F$ is an upper bound on $(s, t)$-flow value in $G'$.
4. if $F \leq 0$ then the minimum $(s, t)$-edge-cut is $(s, x)$, and return.
5. $f \leftarrow$ a flow of value $\deg_{G}(x)$ through $s - x - t$ path.
6. $B \leftarrow \{x\} \cup N_{G'}^{\text{out}}(x)$ // a set of saturated vertices and out-neighbors.
7. while $F \geq 1$ do
8. | $\Delta \leftarrow F / (2\Lambda)$
9. | for $i \leftarrow 1$ to $5\Lambda$ do
10. | $LG \leftarrow$ local subgraph of $G'$ given $B$. // see Definition 4.25, Corollary 4.27
11. | $\ell \leftarrow$ local length function on current flow $f$.
12. | $f \leftarrow f + \text{BinaryBlockingFlow}(A(LG, c_{LG}, f, \ell), \Delta)$.
13. | $C \leftarrow$ vertices in $N_{G'}^{\text{out}}(B)$ whose edges to sink are saturated in the new flow.
14. | $B \leftarrow B \cup C \cup N_{G'}^{\text{out}}(C)$
15. | $F \leftarrow F / 2$
16. return maximum $(s, t)$-flow $f$ and its corresponding minimum $(s, t)$-edge-cut $A$ in $G'$.

**Correctness.** We show that $F$ is the upper bound on the maximum flow value in $G'_f$. We use induction on inner loop. Before entering the inner loop for the first time, $F$ is set to be the value of $(s, t)$ edge minus $\deg_{G}(x)$. Since $F$ is positive, then $G_f$ has valid maximum flow upper bound $F$. Now, we consider the inner loop. After $5\Lambda$ times, either
- we find a flow of value $\Delta/4$ at least $4\Lambda$ times, or
- we find a blocking flow at least $\Lambda$ times.
If the first case holds, then we increase the flow by at least \( \geq (\Delta/4)(4\Lambda) = F/2 \). Hence, the flow \( F/2 \) is the valid upper bound. For the second case, we need the following Lemma whose proof is essentially the same as the original proof of Goldberg’s Rao:

**Lemma 4.30.** A flow augmentation does not decrease the distance \( d(t) \). On the other hand, a blocking flow augmentation strictly increases \( d(t) \).

**Proof Sketch.** The only issue for a blocking flow augmentation is that \( s-t \) distance in residual graph may not increase if an edge length decrease from 1 to 0. This happens when such an edge is modern since classical edges have a constant length of 1. The proof that modern edges do not have the issue follows exactly from the classic Gaoberg-Rao algorithm [GR98] using the notion of special edges.

If the second case holds, we claim:

**Claim 4.31.** If we find a blocking flow at least \( \Lambda \) times, then there exists an \((s,t)\)-edge cut of capacity at most \( \Delta \Lambda = F/2 \), which is an upper bound of the remaining flow to be augmented.

**Proof.** Before entering the inner loop for the first time, by Lemma 4.24, \( d(t) = d_{\text{max}} \geq 3 \). After \( \Lambda \) blocking flow augmentation, \( d(t) \geq 3 + \Lambda \) by Lemma 4.30. Since the \( \Delta \)-blocking flow in \( G' \) on \( B \) and \( LG \) coincide by Lemma 4.28, we always get the correct blocking flow augmentation.

Let \( L_0, L_1, \ldots, L_{d_{\text{max}}} \) be the layers of vertices with distance 0, 1, \ldots, \( d_{\text{max}} = d(t) \geq 3 + \Lambda \). We focus on edges between layers \( L_i, L_{i+1} \) for 1 \( \leq i \leq d_{\text{max}} - 2 \). By Lemma 4.24, any two vertices \( v_1 \in L_i, v_2 \in L_{i+1} \) must be in \( B \). Therefore, by definition of local length function \( \ell \), all edges between \( L_i, L_{i+1} \) must be modern. Since any edge between \( L_i, L_{i+1} \) is modern, and has length 1, it must be of the form \((v_{\text{in}}, v_{\text{out}})\) or \((v_{\text{out}}, v_{\text{in}})\) with residual capacity \( \leq \Delta \) by definition of local length function (Definition 4.20).

Since there are at least \( \Lambda = \sqrt{8\nu/(\epsilon k)} \geq \sqrt{n'} \) layers \( L_i \) (By Lemma 4.26) where 1 \( \leq i \leq d_{\text{max}} - 2 \), by counting argument, there must be a layer \( L' \) such that \( |L'| \leq \sqrt{n'} \).

Next, for any vertex \( v \in L' \), \( v \) has either a single outgoing edge or a single incoming edge by construction of \( G' \) since this edge must be of the form \((v_{\text{in}}, v_{\text{out}})\) or \((v_{\text{out}}, v_{\text{in}})\).

Therefore, we find an \((s,t)\)-edge-cut consisting of the single incoming-or-outgoing edge from each node in \( L' \). This cut has capacity at most \( \Delta \sqrt{n'} \leq \Delta \sqrt{8\nu/(\epsilon k)} = \Delta \Lambda = F/2 \).

The correctness follows since at the end of the loop we have \( F < 1 \).

**Running Time.** By Lemma 4.28, we can compute \( \Delta \)-blocking flow in \( LG \) with local binary length function \( \ell \) in \( \tilde{O}(\nu/\epsilon) \) time. The time already includes the time to read \( LG \). The number of such computations is \( O(\Lambda \log(\nu/\epsilon)) = O(\sqrt{\nu/(\epsilon k)} \log(m)) = \tilde{O}(\sqrt{\nu/(\epsilon k)}) \). So the total running time is \( \tilde{O}(\nu^{3/2}/(\epsilon^{3/2}k^{1/2})) \). This completes the proof of Theorem 4.29.

### 4.5 Proof of Theorem 4.1

**Proof of Theorem 4.1.** Given \( G, x, \nu, k, \epsilon \), by Theorem 4.29, we compute the minimum \((s, t)\)-edge-cut \( C^* \) in \( G' \) in \( \tilde{O}(\nu^{3/2}/(\epsilon^{3/2}k^{1/2})) \) time. If the edge-cut \( C^* \) has capacity > \( \nu/\epsilon + \nu \), then by Lemma 4.4(I), we can output \( \bot \). Otherwise, \( C^* \) has capacity at most \( \nu/\epsilon + \nu \), by Lemma 4.4(II), we can output the separation triple \((L, S, R)\) with the properties in Lemma 4.4(II).
5  Vertex Connectivity via Local Vertex Connectivity

Theorem 5.1 (Exact vertex connectivity). There exist randomized (Monte Carlo) algorithms that take as inputs a graph $G$, integer $0 < k < O(\sqrt{n})$, and in $\tilde{O}(m + k^{7/3}n^{4/3})$ time for undirected graph (and in $\tilde{O}(\min(km^{2/3}n, km^{1/3}))$ time for directed graph) can decide w.h.p. if $\kappa_G \geq k$. If $\kappa_G < k$, then the algorithms also return the corresponding vertex-cut.

We define the function $T(k, m, n)$ as

$$T(k, m, n) = \begin{cases} 
\min(m^{4/3}, nm^{2/3}k^{1/2}) & \text{if } k \leq \sqrt{n}. \\
\min(mn^{2/3+o(1)}/k^{1/3}, n^{7/3+o(1)}/k^{1/6}) & \text{if } \sqrt{n} < k \leq n^{4/5}. \\
n^{3+o(1)}/k & \text{if } k > n^{4/5}.
\end{cases}$$  

(15)

Theorem 5.2 (Approximate vertex connectivity). There exist randomized (Monte Carlo) algorithms that take as inputs a graph $G$, an positive integer $k$, and positive real $\epsilon < 1$, and in $\tilde{O}(m + \text{poly}(1/\epsilon) \min(k^{4/3}n^{4/3}, k^{2/3}n^{5/3+o(1)}, n^{3+o(1)}/k))$ time for undirected graph (and in $\tilde{O}(\text{poly}(1/\epsilon)T(k, m, n))$ time for directed graph where $T(k, m, n)$ is defined as in Equation (15)) w.h.p. return a vertex-cut with size at most $(1 + O(\epsilon))\kappa_G$ or certify that $\kappa_G \geq k$.

This section is devoted to proving Theorem 5.1. and Theorem 5.2.
5.1 Vertex Connectivity Algorithms

Algorithm 2: VC(Sampling method, LocalVC, \(\kappa(x, y); G, k, a, \epsilon\))

**Input:** Sampling method, LocalVC, \(G = (V,E)\), \(k, a, \epsilon\)

**Output:** a vertex-cut \(U\) such that \(|U| \leq k\) or a symbol \(\perp\).

1. If undirected, replace \(E = \{(u, v), (v, u): (u, v) \in E(H_{k+1})\}\) where \(H_{k+1}\) as in Theorem 3.12.
2. if Sampling method = vertex then
   3. for \(i \leftarrow 1\) to \(n/(\epsilon a)\) (use \(n/a\) for exact version) do
      4. Sample a random pair of vertices \(x, y \in V\).
      5. if \(k\) is not specified then compute approximate \(\kappa_G(x, y)\).
      6. if \(\kappa_G(x, y) \leq (1 + \epsilon)k\) then return the corresponding \((x, y)\)-vertex-cut \(U\).
3. if Sampling method = edge then
   4. for \(i \leftarrow 1\) to \(m/(\epsilon a)\) (use \(m/a\) for exact version) do
      5. Sample a random pair of edges \((x_1, y_1), (x_2, y_2) \in E\).
      6. if \(k\) is not specified then compute approximate \(\kappa_G(x_1, y_2), \kappa_G(x_1, x_2), \kappa_G(y_1, x_2), \kappa_G(y_1, y_2)\).
      7. if \(\min(\kappa_G(x_1, y_2), \kappa_G(x_1, x_2), \kappa_G(y_1, x_2), \kappa_G(y_1, y_2)) \leq (1 + \epsilon)k\) then return the corresponding \((x, y)\)-vertex-cut \(U\).
8. if LocalVC is not specified then
   9. Let \(x^*, y^*\) be vertices with minimum \(\kappa_G(x^*, y^*)\) computed so far.
   10. Let \(W\) be the vertex-cut corresponding to \(\kappa_G(x^*, y^*)\).
   11. Let \(v_{\min}, u_{\min}\) be the vertex with the minimum out-degree in \(G\) and \(G^R\) respectively.
   12. return The smallest set among \(\{W, N^\text{out}_G(v_{\min}), N^\text{out}_{G^R}(u_{\min})\}\).
13. Let \(L = \{2^\ell: 1 \leq \ell \leq \lceil \log_2 a \rceil\), and \(\ell \in \mathbb{Z}\}\).
14. if Sampling method = vertex then
   15. for \(s \in L\) do
      16. for \(i \leftarrow 1\) to \(n/s\) do
         17. Sample a random vertex \(x \in V\).
         18. Let \(\nu \leftarrow O(s(s + k))\).
         19. if LocalVC\((G, x, \nu, k, \epsilon)\) or LocalVC\((G^R, x, \nu, k, \epsilon)\) outputs a vertex-cut \(U\) then return \(U\).
20. if Sampling method = edge then
   21. for \(s \in L\) do
      22. for \(i \leftarrow 1\) to \(m/s\) do
         23. Sample a random edge \((x, y) \in E\).
         24. Let \(\nu \leftarrow O(s), \) and \(G = \{G, G^R\}\).
         25. for \(H \in G, z \in \{x, y\}\) do
            26. if LocalVC\((H, z, \nu, k, \epsilon)\) outputs a vertex-cut \(U\) then return \(U\).
27. return \(\perp\).
5.2 Correctness

We can compute approximate vertex connectivity by standard binary search on $k$ with the decision problem. We focus on correctness of Algorithm 2 for approximate version. For exact version, the same proof goes through when we use $\epsilon = 1/(2k)$, and $\kappa_G \leq \sqrt{n}/2$. Let $\Delta = \min(n/(1+\epsilon), (m/(1+\epsilon))^{1/2})$. For the purpose of analysis of the decision problem, we assume the followings.

Assumption 5.3. If $k$ is specified in Algorithm 2, then

(I) $\deg_{\min}^{\text{out}} \geq k$.

(II) $k \leq \Delta$. We use $k \leq \sqrt{n}/2$ for exact vertex connectivity.

(III) Local conditions in Theorem 4.1 are satisfied. We use exact version of local conditions for exact vertex connectivity.

We justify above assumptions. For Assumption 5.3(I), if it does not hold, then we can trivially output the neighbors of the vertex with minimum degree and we are done.

For Assumption 5.3(II), if we can verify that $\kappa_G \geq k = \Delta$, then in Section 5.2.1 we show that the out-neighbors of the vertex with minimum out-degree is an approximate solution. For exact vertex connectivity, we either find a minimum vertex-cut or verify that $\kappa_G \geq \sqrt{n}/2$. For Assumption 5.3(III), we can easily verify the parameters $a$ and $\nu, k, \epsilon$ supplied to the LocalVC.

Ignoring running time, we classify Algorithm 2 into four algorithms depending on sampling edges or vertices, and using LocalVC or not. We omit edge-sampling without LocalVC since the running time is subsumed by vertex-sampling counterpart. We now prove the correctness for each of them.

5.2.1 High Vertex Connectivity

We show that if we can verify that the graph has high vertex connectivity, then we can simply output the out-neighbors of the vertex with minimum out-degree to obtain an $(1+\epsilon)$-approximate solution.

Proposition 5.4. If $\kappa_G \geq \Delta$, then $|\deg_{\min}^{\text{out}}| \leq (1+\epsilon)\kappa_G$.

Proof. We first show that if $\kappa_G \geq (m/(1+\epsilon))^{1/2}$, then $\kappa_G \geq n/(1+\epsilon)$. Since $\kappa_G \geq (m/(1+\epsilon))^{1/2}$, we have $\kappa_G^2 \geq m/(1+\epsilon)$. Therefore, we obtain $\kappa_G \deg_{\min}^{\text{out}} \geq \kappa_G^2 \geq m/(1+\epsilon) \geq n \deg_{\min}^{\text{out}}/(1+\epsilon)$.

The first inequality follows from Observation 3.10, which is $\deg_{\min}^{\text{out}} \geq \kappa_G$. The second inequality follows from above discussion. The third inequality follows from each vertex has at least $\deg_{\min}^{\text{out}}$ edges. Therefore, $\kappa_G \geq n/(1+\epsilon)$.

Now, we show that if $\kappa_G \geq n/(1+\epsilon)$, then $\deg_{\min}^{\text{out}} \leq (1+\epsilon)\kappa_G$. We have $(1+\epsilon)\kappa_G \geq n \geq \deg_{\min}^{\text{out}} \geq \kappa_G$. The first inequality follows from the condition above. The second inequality follows from size of vertex cut is at most $n$. The third inequality follows from Observation 3.10. \qed

5.2.2 Edge-Sampling with LocalVC

Lemma 5.5. Algorithm 2 with edge-sampling, and LocalVC outputs correctly w.h.p. a vertex-cut of size $\leq (1+\epsilon)k$ if $\kappa_G \leq k$, and a symbol $\bot$ if $\kappa_G > k$.

We describe notations regarding edge-sets from a separation triple $(L, S, R)$ in $G$. Let $E^*(L, S) = E(L, L) \sqcup E(L, S) \sqcup E(S, L)$, and $E^*(S, R) = E(R, R) \sqcup E(S, R) \sqcup E(R, S)$.

Definition 5.6. $(L$-volume, and $R$-volume of the separation triple). For a separation triple $(L, S, R)$, we denote $\text{vol}^*_G(L) = \sum_{v \in L} \deg_{G}^{\text{out}}(v) + |E(S, L)|$ and $\text{vol}^*_G(R) = \sum_{v \in R} \deg_{G}^{\text{out}}(v) + |E(S, R)|$. 

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It is easy to see that \( \text{vol}_G^*(L) = |E^*(L, S)| \) and \( \text{vol}_G^*(R) = |E^*(S, R)| \).

The following observations follow immediately from the definition of \( E^*(L, S) \) and \( E^*(S, R) \), and a separation triple \((L, S, R)\).

**Observation 5.7.** We can partition edges in \( G \) according to \((L, S, R)\) separation triple as

\[
E = E^*(L, S) \cup E(S, S) \cup E^*(S, R)
\]

And,

- For any edge \((x, y) \in E^*(L, S), x \in L \) or \( y \in L \).
- For any edge \((x, y) \in E^*(S, R), x \in R \) or \( y \in R \).

Furthermore,

\[
m = \text{vol}_G^*(L) + |E(S, S)| + \text{vol}_G^*(R)
\]

We proceed the proof. There are three cases for the set of all separation triples in \( G \). The first case is there exists a separation triple \((L, S, R)\) such that \(|S| \leq k\), \(\text{vol}_G^*(L) \geq a\), \(\text{vol}_G^*(R) \geq a\). We show that w.h.p. Algorithm 2 outputs a vertex-cut of size at most \((1 + \epsilon)k\).

**Lemma 5.8.** If \( G \) has a separation triple \((L, S, R)\) such that \(|S| \leq k\), \(\text{vol}_G^*(L) \geq a\), \(\text{vol}_G^*(R) \geq a\), then w.h.p. Algorithm 2 outputs a vertex-cut of size at most \((1 + \epsilon)k\).

**Proof.** We show that the first loop (with edge-sampling method) of Algorithm 2 finds a vertex-cut of size at most \((1 + \epsilon)k\).

We sample two edges randomly \( e_1 = (x_1, y_1), e_2 = (x_2, y_2) \in E \). The probability that \( e_1 \in E^*(L, S) \) and \( e_2 \in E^*(S, R) \) is

\[
P(e_1 \in E^*(L, S), e_2 \in E^*(S, R)) = P(e_1 \in E^*(L, S))P(e_2 \in E^*(S, R))
\]

This follows from the two events are independent.

By Assumption 5.3(II), \( k \leq \Delta \), which means \( k^2 \leq m/(1 + \epsilon) \). For exact vertex connectivity, we have \( k^2 \leq n/4 \leq m/4 \). For generality, we denote \( k^2 \leq m/c \). We use \( c = 1 + \epsilon \) for the approximate vertex connectivity, and \( c = 4 \) for exact version.

We claim \( \text{vol}_G^*(L) + \text{vol}_G^*(R) = \Omega((1 - 1/c)m) \). Indeed, by Observation 5.7, \( \text{vol}_G^*(L) + \text{vol}_G^*(R) = m - |E(S, S)| \), and we have \(|E(S, S)| \leq k^2 \leq m/c \).

If \( \text{vol}_G^*(R) = \Omega((1 - 1/c)m) \), then \( P(e_1 \in E^*(L, S), e_2 \in E^*(S, R)) = P(e_1 \in E^*(L, S))P(e_2 \in E^*(S, R)) \geq P(e_1 \in E^*(L, S)) \text{vol}_G^*(R)/m^2 = \Omega((1 - 1/c)a/m) \). Otherwise, \( \text{vol}_G^*(L) = \Omega((1 - 1/c)m) \). Similarly, we get \( P(e_1 \in E^*(L, S), e_2 \in E^*(S, R)) = \Omega((1 - 1/c)a/m) \).

Therefore, it is enough to sample \( O(m/(ca)) \) times \( O(m/a) \) times for the exact vertex connectivity) to get w.h.p. at least one trial where \( e_1 = (x_1, y_1) \in E^*(L, S), e_2 = (x_2, y_2) \in E^*(S, R) \). From now we assume, \( e_1 = (x_1, y_1) \in E^*(L, S), e_2 = (x_2, y_2) \in E^*(S, R) \).

Finally, we show that the first loop of Algorithm 2 outputs a vertex-cut of size at most \( k \). By Observation 5.7, at least one vertex in \((x_1, y_1)\) is in \( L \), and at least on vertex in \((x_2, y_2)\) is in \( R \). Therefore, we find a separation triple corresponding to \( \min(\kappa_G(x_1, y_2), \kappa_G(x_1, x_2), \kappa_G(y_1, x_2), \kappa_G(y_1, y_2)) \leq (1 + \epsilon)k \).

The second case is there exists a separation triple \((L, S, R)\) such that \(|S| \leq k\) and \(\text{vol}_G^*(L) < a\) or \(\text{vol}_G^*(R) < a\). We show that w.h.p. Algorithm 2 outputs a vertex-cut of size at most \((1 + \epsilon)k\).

**Lemma 5.9.** If \( G \) has a separation triple \((L, S, R)\) such that \(|S| \leq k\) and \(\text{vol}_G^*(L) < a\) or \(\text{vol}_G^*(R) < a\), then w.h.p. Algorithm 2 outputs a vertex-cut of size at most \((1 + \epsilon)k\).
Proof. We show that the second loop (LocalVC with edge-sampling mode) of Algorithm 2 finds a vertex-cut of size at most \((1 + \epsilon)k\).

We focus on the case \(\text{vol}_G^*(L) < a\). The case \(\text{vol}_G^*(R) < a\) is similar, except that we need to compute local vertex connectivity on the reverse graph instead.

We show that w.h.p. there is an event \(e = (x, y) \in E^*(L, S)\). Since \(\text{vol}_G^*(L) < a\), there exists an integer \(\ell \) in range \(1 \leq \ell \leq \lfloor \log_2 a \rfloor\) such that \(2^{\ell - 1} \leq \text{vol}_G^*(L) \leq s^\ell\). That is, \(s/2 \leq \text{vol}_G^*(L) \leq s\) for \(s = 2^\ell\). The probability that \(e \in E^*(L, S)\) is \(\text{vol}_G^*(L)/m \geq s/(2m)\). Hence, it is enough to sample \(O(m/s)\) edges to get an event \(e \in E^*(L, S)\) w.h.p.

From now we assume that \(\text{vol}_G^*(L) \leq s\) and that \(e = (x, y) \in E^*(L, S)\). By Definition 5.6, \(\text{vol}_G^*(L) = \sum_{v \in L} \deg_{G}^\text{out}(v) + |E(S, L)| \leq s\). Therefore, \(\text{vol}_G^*(L) = \sum_{v \in L} \deg_{G}^\text{out}(v) \leq s\).

By Observation 5.7, \(x \in L\) or \(y \in L\). We assume WLOG that \(x \in L\) (Algorithm 2 runs LocalVC on both \(x\) and \(y\)).

Hence, we have verified the following conditions for the parameters \(x, \nu, k\) for LocalVC\((G, x, \nu, k)\):

- Local conditions are satisfied by Assumption 5.3(III).
- \(x \in L\).
- \(|S| \leq k\).
- \(\text{vol}_G^*(L) \leq \nu\) and we use \(\nu = s\).

By Theorem 4.1, LocalVC outputs a vertex-cut of size at most \((1 + \epsilon)k\).

The final case is when every separation triple \((L, S, R)\) in \(G\), \(|S| > k\). In other words, \(\kappa_G > k\). If Algorithm 2 outputs a vertex-cut, then it is a \((1 + \epsilon)\)-approximate vertex-cut. Otherwise, Algorithm 2 outputs \(\perp\) correctly.

The proof of Lemma 5.5 is complete since Lemma 5.5 follows from Lemma 5.8, Lemma 5.9, and the case \(\kappa_G > k\) corresponding the three cases of the set of separation triples in \(G\).

5.2.3 Vertex-Sampling with LocalVC

Lemma 5.10. Algorithm 2 with vertex-sampling, and LocalVC outputs correctly w.h.p. a vertex-cut of size \(\leq (1 + \epsilon)k\) if \(\kappa_G \leq k\), and a symbol \(\perp\) if \(\kappa_G > k\).

We consider three cases for the set of all separation triples in \(G\). The first case is there exists a separation triple \((L, S, R)\) such that \(|S| \leq k, |L| \geq a, \) and \(|R| \geq a\). We show that w.h.p. Algorithm 2 outputs a vertex-cut of size at most \((1 + \epsilon)k\).

Lemma 5.11. If \(G\) has a separation triple \((L, S, R)\) such that \(|S| \leq k, |L| \geq a, \) and \(|R| \geq a\) Then w.h.p. Algorithm 2 outputs a vertex-cut of size at most \((1 + \epsilon)k\).

Proof. We show that the first loop of Algorithm 2 finds a vertex-cut of size at most \((1 + \epsilon)k\).

We sample two vertices independently \(x, y \in V\). Since two events \(x \in L\) and \(y \in R\) are independent, the probability that \(x \in L\) and \(y \in R\) is \(P(x \in L, y \in R) = P(x \in L)P(y \in R)\).

By Assumption 5.3(II), \(k \leq \Delta\), which means \(k \leq n/(1 + \epsilon)\). For exact vertex connectivity, we have \(k \leq \sqrt{n}/2 \leq n/2\). For generality, we denote \(k \leq n/c\). We use \(c = 1 + \epsilon\) for the approximate vertex connectivity, and \(c = 2\) for exact version.

Since \(k \leq n/c\), we have \(|L| + |R| = n - |S| \geq n - k \geq n - n/c = (1 - 1/c)n\). If \(|R| = \Omega((1 - 1/c)n)\), then \(P(x \in L, y \in R) = P(x \in L)P(y \in R) \geq |R|a/n^2 = \Omega((1 - 1/c)a/n)\). Otherwise, \(|L| = \Omega((1 - 1/c)n)\), and with similar argument we get \(P(x \in L, y \in R) = \Omega((1 - 1/c)a/n)\).

Therefore, it is enough to sample \(O(n/(ae))\) times (and \(O(n/a)\) times for exact version) to get at least one trial corresponding to the event \(x \in L\) and \(y \in R\) w.h.p. With that event, we can find a separation triple corresponding to \(\kappa(x, y) \leq (1 + \epsilon)k\). 

\(\square\)
The second case is there exists a separation triple \((L, S, R)\) such that \(|S| \leq k\) and \(|L| < a\) or \(|R| < a\). We show that w.h.p. Algorithm 2 outputs a vertex-cut of size at most \((1 + \epsilon)k\).

**Lemma 5.12.** If \(G\) has a separation triple \((L, S, R)\) such that \(|S| \leq k\) and \(|L| < a\) or \(|R| < a\), then w.h.p. Algorithm 2 outputs a vertex-cut of size at most \((1 + \epsilon)k\).

**Proof.** We show that the second loop (LocalVC with vertex sampling method) of Algorithm 2 finds a vertex-cut of size at most \((1 + \epsilon)k\).

We focus on the case \(|L| < a\). The case \(|R| < a\) is similar, except that we need to compute local vertex connectivity on the reverse graph instead.

We show that w.h.p, there is an event \(x \in L\). Since \(|L| < a\), there exists \(\ell \) in range \(1 \leq \ell \leq \lceil \log_2 a \rceil\) such that \(2^{\ell - 1} \leq |L| \leq s^\ell\). In other words, for \(s = 2^{\ell}\), we have \(s/2 \leq |L| \leq s\). Since \(x\) is independently and uniformly sampled, the probability that \(x \in L\) is \(|L|/n\), which is at least \(|L|/n \geq s/(2n)\). Therefore, by sampling \(O(n/s)\) rounds, w.h.p. there is at least one event where \(x \in L\).

From now we assume that \(|L| \leq s\) and that \(x \in L\). We show that \(vol_G^{out}(L) = s(s + k)\). Since \(|L| \leq s\), \(vol_G^{out}(L) = |E_G(L, L)| + |E_G(L, S)| \leq |L|^2 + |L||S| \leq s^2 + sk\).

We have verified the following conditions for the parameters \(x, \nu, k\) for LocalVC\((G, x, \nu, k)\):

- **Local conditions** are satisfied by Assumption 5.3(III).
- \(x \in L\).
- \(|S| \leq k\).
- \(vol_G^{out}(L) \leq \nu\) since \(vol_G^{out}(L) \leq s^2 + sk\), and we use \(\nu = s^2 + sk\).

By Theorem 4.1, LocalVC outputs a vertex-cut of size at most \((1 + \epsilon)k\). □

The final case is when every separation triple \((L, S, R)\) in \(G\), \(|S| > k\). In other words, \(\kappa_G > k\). If Algorithm 2 outputs a vertex-cut, then it is a \((1 + \epsilon)\)-approximate vertex-cut. Otherwise, Algorithm 2 outputs \(\perp\) correctly. The proof of Lemma 5.13 is complete since Lemma 5.13 follows from Lemma 5.11, Lemma 5.12, and the case \(\kappa_G > k\) corresponding the three cases of the set of separation triples in \(G\).

### 5.2.4 Vertex-Sampling without LocalVC

We do not specify \(k\) and LocalVC algorithm.

**Lemma 5.13.** Algorithm 2 with vertex-sampling, but without LocalVC w.h.p. outputs a vertex-cut of size \(\leq (1 + \epsilon)\kappa_G\)

**Proof.** Let \(\bar{\kappa}\) be the answer of our algorithm. By design, we have \(\bar{\kappa} \leq \min(d_{\text{min}}^{\text{out}}, d_{\text{min}}^{\text{in}})\). Also, \(\bar{\kappa} \geq \kappa\) since the answer corresponds to some vertex-cut. It remains to show \(\bar{\kappa} \leq (1 + O(\epsilon))\kappa\).

Let \((L, S, R)\) be an optimal separation triple. We assume without loss of generality that \(|L| \leq |R|\). The other case is symmetric, where we use \(d_{\text{min}}^{\text{in}}\) instead.

We first show the inequality \(|L| \geq d_{\text{min}}^{\text{out}} - \kappa\). Since \((L, S, R)\) is a separation triple where \(|S| = \kappa|\), the number of out-neighbors of a fixed vertex \(x \in L\) that can be included in \(S\) is at most \(\kappa\). By definition of separation triple, neighbors of \(x\) cannot be in \(R\), and so the rest of the neighbors must be in \(L\).

If \(|L| \leq \epsilon d_{\text{min}}^{\text{out}}\), then \(\kappa = |S| \geq d_{\text{min}}^{\text{out}} - \epsilon d_{\text{min}}^{\text{out}} \geq \bar{\kappa}(1 - \epsilon) \geq \kappa(1 - \epsilon)\). That is, \(\bar{\kappa}\) is indeed an \((1 + O(\epsilon))\)-approximation of \(\kappa\) in this case.

On the other hand, if \(|L| \geq \epsilon d_{\text{min}}^{\text{out}}\), then we claim that \(|R| \geq \epsilon n/4\). To see this, if \(d_{\text{min}}^{\text{out}} \geq n/2\), then \(|R| \geq |L| \geq \epsilon d_{\text{min}}^{\text{out}} \geq \epsilon n/2\). Otherwise, \(d_{\text{min}}^{\text{out}} \leq n/2\). In this case, \(\kappa \leq d_{\text{min}}^{\text{out}} \leq n/2\). Therefore, \(2|R| = |L| + |R| = n - |S| \geq n - \kappa \geq n/2\). In either case, the claim follows.
We show that the probability that two sample vertices \( x \in L \) and \( y \in R \) is at least \( \epsilon^2 \frac{d_{\text{out}}}{\min(4n)} \). First of all, the two events are independent. Recall that \(|L| \geq \epsilon d_{\text{out}} \) and \(|R| \geq \epsilon n/4\). Therefore, 
\[
P(x \in L, y \in R) = P(x \in L)P(y \in R) = (|L|/n)(|R|/n) \geq \epsilon^2 \frac{d_{\text{out}}}{4n}.
\]

Therefore, we sample for \( \tilde{O}(n/(\epsilon^2 d_{\text{out}})) \) many times to get the event \( x \in L \) and \( y \in R \) w.h.p. Hence, we compute approximate \( \kappa(x, y) \) correctly, and so our answer \( \tilde{\kappa} \) is indeed an \((1 + \epsilon)\)-approximation.

5.3 Running Time

Let \( T_1(m, n, k, \epsilon) \) be the time for deciding if \( \kappa(x, y) \leq (1 + \epsilon) \), \( T_2(\nu, k, \epsilon) \) be the running time for approximate LocalVC, and \( T_3(m, n, \epsilon) \) be the time for computing approximate \( \kappa(x, y) \). If \( G \) is undirected, we can replace \( m \) with \( nk \) with additional \( O(m) \) preprocessing time. The running time for exact version is similar except that we do not have to pay \( 1/\epsilon \) factor for the first loop of Algorithm 2.

5.3.1 Edge-Sampling with LocalVC

Lemma 5.14. Algorithm 2 with edge-sampling, and LocalVC terminates in time
\[
\tilde{O}((m/(\epsilon a))(T_1(m, n, k, \epsilon) + T_2(a, k, \epsilon))).
\]

Proof. The first term comes from the first loop of Algorithm 2. That is, we repeat \( O(m/(\epsilon a)) \) times for computing approximate \( \kappa(x, y) \), and each iteration takes \( T_1(m, n, k, \epsilon) \) time.

The second term comes from computing local vertex connectivity. For each \( s \in L \), we repeat the second loop for \( O(m/s) \) times, each LocalVC subroutine takes \( T_2(\nu, k, \epsilon) \) time where \( \nu = s \). Therefore, the total time for the second loop is \( \sum_{s \in L}(m/s)T_2(s, k, \epsilon) = \tilde{O}((m/a)T_2(a, k, \epsilon)) \). \( \square \)

5.3.2 Vertex-Sampling with LocalVC

Lemma 5.15. Algorithm 2 with vertex-sampling, and LocalVC terminates in time
\[
\tilde{O}((n/(\epsilon a))(T_1(m, n, k, \epsilon) + T_2(a^2 + ak, k, \epsilon))).
\]

Proof. The first term comes from the first loop of Algorithm 2. That is, we repeat \( O(n/(\epsilon a)) \) times for computing approximate \( \kappa(x, y) \), and each iteration takes \( T_1(m, n, k, \epsilon) \) time.

The second term comes from computing local vertex connectivity. For each \( s \in L \), we repeat the second loop for \( O(n/s) \) times, each LocalVC subroutine takes \( T_2(\nu, k, \epsilon) \) time where \( \nu = O(s(s + k)) \). Therefore, the total time for the second loop is \( \sum_{s \in L}(n/s)T_2(s, k, \epsilon) = \tilde{O}((n/a)T_2(a^2 + ak, k, \epsilon)) \). \( \square \)

5.3.3 Vertex-Sampling without LocalVC

Lemma 5.16. Algorithm 2 with vertex-sampling, but without LocalVC terminates in time
\[
\tilde{O}(n/(\epsilon^2 k)T_3(m, n, \epsilon)).
\]

Proof. The running time follows from the first loop where we set \( a \) such that the number of sample is \( n/(\epsilon^2 k) \), and computing approximate \( \kappa(x, y) \) can be done in \( T_3(m, n, \epsilon) \) time. \( \square \)
5.4 Proof of Theorems 5.1 and 5.2

For exact vertex connectivity, LocalVC runs in \( n^{1.5}k \) time by Corollary 4.2. We can decide \( \kappa(x, y) \leq k \) in \( O(mk) \) time.

For undirected exact vertex connectivity where \( k < O(\sqrt{n}) \), we first sparsify the graph in \( O(m) \) time. Then, we use edge-sampling with LocalVC algorithm where we set \( a = m^{2/3} \), where \( m' = O(nk) \) is the number of edges of sparsified graph.

For directed exact vertex connectivity where \( k < O(\sqrt{n}) \), we use edge-sampling with LocalVC algorithm where we set \( a = m^{2/3} \) if \( m < n^{3/2} \). If \( m > n^{3/2} \), we use vertex-sampling with LocalVC algorithm where we set \( a = m^{1/3} \).

For approximate vertex connectivity, approximate LocalVC runs in \( \text{poly}(1/\epsilon)n^{1.5}/\sqrt{n} \) by Theorem 4.1. Also, we can decide \( \kappa(x, y) \leq (1 + O(\epsilon))k \) or certify that \( \kappa \geq k \) in time \( \tilde{O}(\text{poly}(1/\epsilon)\min(mk, n^{2+o(1)})) \). The running time \( \text{poly}(1/\epsilon)n^{2+o(1)} \) is due to [CK19].

For undirected approximate vertex connectivity, we first sparsify the graph in \( O(m) \) time. Let \( m' \) be the number of edges of the sparsified graph. For \( k < n^{0.8} \), we use edge-sampling with approximate LocalVC algorithm where we set \( a = m^{\hat{a}} \), where \( \hat{a} = \frac{\min(5\hat{k}+2k+4)}{3k+3} \), and \( \hat{k} = \log n \).

For \( k > n^{0.8} \), we use vertex-sampling without LocalVC.

For directed approximate vertex connectivity, If \( k \leq \sqrt{n} \), we run edge-sampling with \( a = m^{\hat{a}}, \hat{a} = \min(2/3 + \hat{k}, 1) \) where \( \hat{k} = \log n \), or we run vertex-sampling with \( a = m^{1/3+1/2} \). If \( \sqrt{n} < k \leq n^{0.8} \), we run edge-sampling with \( a = m^{\hat{a}} \) where \( \hat{a} = 4\log n/3 + \log n/3 \) or vertex-sampling with \( a = n^\hat{a} \) where \( \hat{a} = (2/3 + (\log n)/6) \). Finally, if \( k > n^{4/5} \), we use vertex-sampling without LocalVC.

6 (1 + \( \epsilon \))-Approximate Vertex Connectivity via Convex Embedding

**Theorem 6.1.** There exists an algorithm that takes \( G \) and \( \epsilon > 0 \), and in \( O(n^{\kappa}/\epsilon^2 + \min(\kappa, \sqrt{n})m) \) time outputs a vertex-cut \( U \) such that \( |U| \leq (1 + \epsilon)\kappa \).

6.1 Preliminaries

**Definition 6.2** (Pointset in \( \mathbb{F}^k \)). Let \( \mathbb{F} \) be any field. For \( k \geq 0 \), \( \mathbb{F}^k \) is \( k \)-dimensional linear space over \( \mathbb{F} \). Denote \( X = \{x_1, \ldots, x_n\} \) as a finite set of points in \( \mathbb{F}^k \). The **affine hull** of \( X \) is \( \text{aff}(X) = \{\sum_{i=1}^k c_ix_i \mid x_i \in X \text{ and } \sum_{i=1}^k c_i = 1\} \). The rank of \( X \) denoted as \( \text{rank}(X) \) is one plus dimension of \( \text{aff}(X) \). In particular, if \( \mathbb{F} = \mathbb{R} \), then we will consider the **convex hull** of \( X \), denoted as \( \text{conv}(X) \).

For any sets \( V, W \), any function \( f : V \to W \), and any subset \( U \subseteq V \), we denote \( f(U) = \{f(u) \mid u \in U\} \).

**Definition 6.3** (Convex directed X-embedding). For any \( X \subset V \), a convex directed X-embedding of a graph \( G = (V, E) \) is a function \( f : V \to \mathbb{R}^{|X|-1} \) such that for each \( v \in V \setminus X \), \( f(v) \in \text{conv}(f(N^\text{out}_G(v))) \).

For efficiency point of view, we use the same method from [LLW88, CR94] that is based on convex-embedding over finite field \( \mathbb{F} \). In particular, they construct the directed X-embedding over the field of integers modulo a prime p, \( \mathbb{Z}_p \) by fixing a random prime number \( p \in [n^5, n^6] \), and choosing a random nonzero coefficient function \( c : E \to (\mathbb{Z}_p \setminus \{0\}) \) on edges. This construction yields a function \( f : V \to (\mathbb{Z}_p^{|X|-1}) \) called **random modular directed X-embedding**.
Definition 6.4. For $X, Y \subseteq V$, $p(X,Y)$ is the maximum number of vertex-disjoint paths from $X$ to $Y$ where different paths have different end points.

Lemma 6.5. For any non-empty subset $U \subseteq V \setminus X$, w.h.p. a random modular directed $X$-embedding $f : V \to \mathbb{Z}_p^{|X|-1}$ satisfies $\text{rank}(f(U)) = p(U,X)$.

Definition 6.6 (Fixed $k$-neighbors). For $v \in V$, let $N_{G,k}^\text{out}(v)$ be a fixed, but arbitrarily selected subset of $N_G^\text{out}(v)$ of size $k$. Similarly, For $v \in V$, let $N_{G,k}^\text{in}(v)$ be a fixed, but arbitrarily selected subset of $N_G^\text{in}(v)$ of size $k$.

Lemma 6.7. Let $\omega$ be the exponent of the running time of the optimal matrix multiplication algorithm. Note it is known that $\omega \leq 2.372$.

- For $y \in V$, a random modular directed $N_{G,k}(y)$-embedding $f$ can be constructed in $O(n^\omega)$ time.
- Given such $f$, for $U \subseteq V$ with $|U| = k$, $\text{rank}(f(U))$ can be computed in $O(k^\omega)$ time.

Lemma 6.8 ([Gab06]). For any optimal out-vertex shore $S$ such that $|N_G^\text{out}(S)| = \kappa_G$, then $\kappa_G \geq d_{\text{out}}^\text{min} - |S|$.

For any set $S, S'$, we denote $\min(S, S')$ as the set with smaller cardinality.

6.2 Algorithm

| Algorithm 3: ApproxConvexEmbedding($G, \epsilon$) |
|-------------------------------------------------|
| **Input:** $G = (V, E)$, and $\epsilon > 0$      |
| **Output:** A vertex-cut $U$ such that w.h.p. $|U| \leq (1 + \epsilon)\kappa_G$. |
| 1 Let $k \leftarrow \max(d_{\text{out}}^\text{min}, d_{\text{in}}^\text{min})$. |
| 2 Let $k' \leftarrow \min(d_{\text{out}}^\text{min}, d_{\text{in}}^\text{min})$. |
| 3 repeat |
| 4 Sample two random vertices $x_2, y_1 \in V$. |
| 5 Let $f$ be a random modular directed $N_{G,k}^\text{in}(y_1)$-embedding. // $O(n^\omega)$ time. |
| 6 Let $f^R$ be a random modular directed $N_{G,R,k}^\text{in}(x_2)$-embedding. |
| 7 repeat |
| 8 Sample two random vertices $y_2, x_1 \in V$. |
| 9 $\text{rank}(x_1, y_1) \leftarrow \text{rank}(f(N_{G,k}^\text{out}(x_1)))$ // $O(k^\omega)$ time. |
| 10 $\text{rank}(x_2, y_2) \leftarrow \text{rank}(f^R(N_{G,R,k}^\text{out}(y_2)))$ |
| 11 until $\Theta(n/(\epsilon k'))$ times |
| 12 until $\Theta(1/\epsilon)$ times |
| 13 Let $x^*, y^*$ be the pair of vertices with minimum rank($x, y$) for all $x, y$ computed so far. |
| 14 Let $W \leftarrow \min(\kappa_G(x^*, y^*), \kappa_{G^R}(x^*, y^*))$ |
| 15 Let $v_{\text{min}}, u_{\text{min}}$ be the vertex with the minimum out-degree in $G$ and $G^R$ respectively. |
| 16 **return** $\min(W, |N_G^\text{out}(v_{\text{min}})|, |N_G^\text{out}(u_{\text{min}})|)$ // Vertex-cut with minimum cardinality. |

6.3 Analysis

Lemma 6.9. Algorithm 3 outputs w.h.p. a vertex-cut $U$ such that $|U| \leq (1 + \epsilon)\kappa_G$. 
Proof. Let \( \tilde{\kappa} \) denote the answer of our algorithm. Clearly \( \tilde{\kappa} \leq d^\text{out}_{\min} \) and \( \tilde{\kappa} \leq d^\text{in}_{\min} \) by design. Observe also that \( \tilde{\kappa} \geq \kappa \) because the answer corresponds to some vertex cut. Let \((A, S, B)\) be the optimal separation triple where \( A \) is a out-vertex shore and \(|S| = \kappa\). W.l.o.g. we assume that \(|A| \leq |B|\), another case is symmetric.

Suppose that \(|A| \leq ed^\text{out}_{\min}\). Then \( \kappa = |S| \geq d^\text{out}_{\min} - ed^\text{out}_{\min} \geq \tilde{\kappa}(1 - \epsilon) \geq \kappa(1 - \epsilon) \). That is, \( \tilde{\kappa} \) is indeed an \((1 + O(\epsilon))\)-approximation of \( \kappa \) in this case.

Suppose now that \(|A| \geq ed^\text{out}_{\min}\). We claim that \(|B| \geq \epsilon n/4\). Indeed, if \( d^\text{out}_{\min} \geq n/2 \), then \(|B| \geq \epsilon n/2\). Else if, \( d^\text{out}_{\min} \leq n/2 \), then we know \(|S| = \kappa \leq n/2\). But \( 2|B| \geq |A| + |B| = n - |S| \geq n/2 \). In either case, \(|B| \geq \epsilon n/4\).

Now, as \(|B| \geq \epsilon n/4\) and we sample \( \tilde{O}(1/\epsilon) \) many \( y_1 \). There is one sample \( y_1 \in B \) w.h.p. and now we assume that \( y_1 \in B \). In the iteration when \( y_1 \) is sampled. As \(|A| \geq \epsilon d^\text{out}_{\min}\) and we sample at least \( \tilde{O}(n/d^\text{out}_{\min}\epsilon) \) many \( x_1 \). There is one sample \( x_1 \in A \) w.h.p.

By Lemma 6.5, w.h.p.,

\[
\rank(x_1, y_1) = \rank(f(N^\text{out}_{G,k}(x_1))) = p(N^\text{out}_{G,k}(x_1), N^\text{in}_{G,k}(y_1)) = \kappa(x_1, y_1) = \kappa.
\]

So our answer \( \tilde{\kappa} = \kappa \) in this case.

\[\square\]

Lemma 6.10. Algorithm 3 terminates in \( O(n\omega/\epsilon^2 + \min(\kappa_G, \sqrt{m})m) \) time.

Proof. By Lemma 6.7, the construction time for a random modular directed \( N^\text{in}_{G,k}(y_1) \)-embedding is \( O(n\omega) \). Given \( y_1 \), we sample \( x_1 \) for \( \Theta(n/(ek)) \) rounds. Each round we can compute \( \kappa(x_1, y_1) \) by computing \( \rank(f(N^\text{out}_{G,k}(x_1))) \) in \( O(k^\omega) \) time. Hence, the total time is to find the best pair \((x, y)\) is \( O(n\omega + nk^{\omega-1}/\epsilon) = O(n\omega/\epsilon) \). Finally, we can compute \( \kappa_G(x, y) \) to obtain the vertex-cut for the best pair in \( O(\min(\kappa_G, \sqrt{m})m) \) time. It takes linear time to compute \( N^\text{out}_{G}(v_\min) \) and \( N^\text{out}_{G}(u_\min) \). Hence, the result follows. \[\square\]

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