Scalar perturbations on the background of Kerr black holes in the quadratic dynamical Chern-Simons gravity

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Abstract

We study the scalar perturbation on the background of Kerr black hole in the dynamical Chern-Simons modified gravity with a quadratic coupling between scalar field and Chern-Simons term. In particular, the late-time tails of scalar perturbations are investigated numerically in time domain by using hyperboloidal layer method. It is found that Kerr black hole become unstable under linear perturbations in a certain region of the parameter space, which depends on the harmonic azimuthal index $m$ of the perturbation’s mode. This may indicate that some Kerr black holes in this theory will get spontaneous scalarised into a non-Kerr black hole.
I. INTRODUCTION

Einsteins theory of general relativity (GR) has passed all precision tests performed so far with flying colors \cite{1}. However, due to its incompatibility with quantum theory and motivations from cosmology, it is believed that GR may not be the final theory to describe the gravitational physics, but just an effective description of an unknown fundamental theory of gravity and should be modified at both low and high energies \cite{2}. In order to go beyond GR, plenty of alternative theories of gravity have been proposed, see Ref. \cite{3} for a comprehensive review.

Amongst various modifications or extensions of GR, a class of theories, so-called quadratic gravity whose action contains terms quadratic in the curvature, are in particular of interest. It is known that a major obstacle on the road to quantum gravity is that GR can not be perturbatively renormalized as the standard model of particle physics. The situation changes if the Einstein-Hilbert action is assumed to be only the first term in an expansion containing high order curvature invariants. In fact, Stelle showed long time ago that including quadratic curvature terms in the action makes the theory renormalizable \cite{4}. These extra quadratic curvature terms, lead to new effects in the strong-field regime, manifesting themselves most naturally in the black hole (BH) solutions of these models.

Chern-Simons (CS) modified gravity \cite{5} is a special kind of quadratic gravity where an additonal CS invariant (i.e. the contraction of the Riemann tensor and its dual, also called Pontryagin density) coupled to a scalar field is added in the action, which captures leading-order, gravitational parity violation \cite{6}. Such a term is not only reduced from the Green-Schwarz anomaly canceling mechanism in heterotic string theory \cite{7, 8}, but also appears naturally in loop quantum gravity \cite{9}, especially when the Barbero-Immirzi parameter is promoted to a scalar field coupled to the Nieh-Yan invariant \cite{10–12}. For a review of CS modified gravity, we refer to Ref. \cite{13}.

The CS modified gravity was at first investigated in the nondynamical formulation, in which there is no kinetic term for the scalar field in the action, and hence it is assumed to be an a priori prescribed spacetime function. However, nondynamical CS theory is quite contrived because a valid solution for the spacetime must satisfy the condition that the Pontryagin density vanishes. It has been shown from different aspects that nondynamical CS modified gravity is theoretically problematic \cite{14–16}. Therefore, it can only be considered
as a toy model used to gain some insight in parity-violating theories of gravity.

In the last decade, much attention has been payed on the so-called dynamical Chern-Simons (dCS) modified gravity [17], which is a more natural formulation, where the scalar field is treated as a dynamical field. It is worth noting that although the action of the non-dynamical CS gravity can be obtained as a certain limit of that of dCS gravity, the non-dynamical CS gravity and dCS gravity are inequivalent and independent theories.

When the spacetime has spherical symmetry, the parity-violating Pontryagin density vanishes, and then the Schwarzschild solution with vanishing scalar field is an exact solution of dCS gravity. The perturbations of Schwarzschild BHs in dCS modified gravity were first investigated by Cardoso and Gualtieri [18]. Later, Garfinkle et al. found that dCS modified gravity is linearly stable on Schwarzschild and other of physically relevant backgrounds by performing a linear stability analysis in the geometric optics approximation and discussed the the speed of gravitational waves in this theory [19]. The linear mode stability for a generic massive scalar in the background of a Schwarzschild BH in dCS gravity was proved recently [20].

The rotating BH solutions in dCS gravity have been obtained in the small-coupling and/or slow-rotation limit by many authors [14, 21–25]. The null geodesics and shadow of a slowly rotating BH in dCS gravity with a small coupling constant are also studied [26]. Chen and Jing investigate the geodetic precession and the strong gravitational lensing in the slowly rotating BH in the dCS gravity and find the effects of the CS coupling parameter on the geodetic precession angle for the timelike particles and the coefficients of gravitational lensing in the strong field limit [27]. The perturbative BH solutions has the advantage of having analytic expressions, leading to some insights on the effect of the CS coupling. However, some important features, occurring in the fast spinning and/or large coupling regimes, cannot be captured by them. Recently, spinning BHs in dCS gravity are constructed by directly solving the field equations, without resorting to any perturbative expansion [28].

The possible signatures of dCS gravity in the gravitational-wave emission produced in the inspiral of stellar compact objects into massive BHs are investigated in Ref. [29], both for intermediate- and extreme-mass ratios. By applying the effective field theory method, Loutrel et al. [30] recently derived the leading post-Newtonian order spin-precession equations for binary BHs in dCS gravity. It is worth mentioning that the detection of gravitational waves has ruled out a lot of alternative theories [31–34] (see [35, 36] for some earlier works),
however, as is mentioned in Ref. [37], gravitational waves in dCS gravity propagate at the speed of light on conformally flat background spacetimes, such as Friedmann-Robertson-Walker spacetime [19, 38].

Up to now, most of the literature on dCS modified gravity considers the linear coupling between the scalar field and the CS term. In fact, the coupling in principle may be arbitrary and there exists no known natural constraints on the choice of the coupling function. In this paper, inspired by the phenomenon of spontaneous scalarisation recently discussed in the so-called quadratic scalar-Gauss-Bonnet gravity [39–42], we want to consider a dCS modified theory of gravity where the CS invariant is coupled to the quadratic function of the dynamical scalar field. In this theory, it is easy to find that, different from that in the theory with linear coupling, Kerr black hole solution holds when the scalar field becomes trivial. What we are concerned about are stability of the Kerr black hole in this theory and how a perturbation of the scalar field coupled with the CS invariant evolves in the Kerr background.

The paper is organized as follows. In Sec.II a brief review on the dCS theory of gravity with a quadratic coupling is given. Next, in Sec.III we derive the equation of motion for the scalar perturbations. The numerical method employed is described in Sec.IV and the main results are presented in Sec.V. Finally, we conclude in Sec.VI. Throughout the paper, we use geometric units in which $G = c = 1$.

II. DYNAMICAL CHERN-SIMONS GRAVITY WITH A QUADRATIC COUPLING

A general model of dCS modified gravity can be described by the following action

$$S = \int d^4x \sqrt{-g} \left[ \kappa R - \frac{1}{2} \nabla_\mu \Phi \nabla^\mu \Phi - V(\Phi) + \hat{\alpha} f(\Phi) * R R + \mathcal{L}_m \right]$$  \hspace{1cm} (1)

where $\kappa = (16\pi)^{-1}$, $\Phi$ is a real scalar field with a self-interaction potential $V(\Phi)$, $\hat{\alpha}$ is a dimensional parameter, $f(\Phi)$ is an arbitrary function of the scalar field and $\mathcal{L}_m$ denotes the action for matter that minimal coupled to gravity. As usual, $g$ denotes the determinant of the metric $g_{\mu\nu}$ and $R$ is the Ricci scalar. The CS invariant is defined by

$$* R R = \frac{1}{2} \epsilon^{\alpha\beta\gamma\delta} R^\mu_{\nu\gamma\delta} R^\nu_{\mu\alpha\beta},$$  \hspace{1cm} (2)
where $\epsilon^{\alpha\beta\gamma\delta}$ and $R^\mu_{\nu\gamma\delta}$ are the Levi-Civita tensor and Riemann curvature tensor, respectively. Note that the CS invariant itself is a topological term and can be expressed as a total divergence \[13\]. For simplicity, in this work we omit the contribution of matter and only consider the vacuum solution in the case that the scalar field is massless and has no self-interaction, i.e. we set $V(\Phi) = 0$.

The equation of motion for the metric $g_{\mu\nu}$ derived from the action with vanishing $V(\Phi)$ and $\mathcal{L}_m$ in Eq.(1) are the following modified Einstein’s equation

$$G^{\mu\nu} + 4\frac{\alpha}{\kappa}C^{\mu\nu} = \frac{1}{2\kappa}T^{\mu\nu}_{(\Phi)}$$

(3)

where $G^{\mu\nu}$ is the Einstein tensor. The tensor $C^{\mu\nu}$ and the stress-energy tensor for the scalar field $T^{\mu\nu}_{(\Phi)}$ are defined by

$$C^{\mu\nu} = \nabla_\sigma f(\Phi)\epsilon^{\sigma\alpha\beta}(\mu\nabla_\nu R^\nu_{\alpha\beta} + \nabla_\alpha \nabla_\beta f(\Phi)\ast R^{\alpha(\mu\nu}\beta$$

(4)

and

$$T^{\mu\nu}_{(\Phi)} = \nabla^\mu \Phi \nabla^\nu \Phi - \frac{1}{2}g^{\mu\nu}(\nabla^\lambda \Phi)(\nabla^\lambda \Phi),$$

(5)

respectively. On the other hand, the Klein-Gordon equation for the scalar field $\Phi$ is modified to be

$$\Box \Phi + \hat{\alpha} \ast RR \frac{df(\Phi)}{d \Phi} = 0.$$  

(6)

By taking the covariant divergence of Eq.(4), it is not difficult to find that

$$\nabla_\mu C^{\mu\nu} = \frac{\ast RR}{8} \nabla^\nu f(\Phi).$$

(7)

Therefore, from Eq.(3), the evolution of $\Phi$ is also determined by

$$\hat{\alpha} \ast RR \nabla^\nu f(\Phi) = \nabla_\mu T^{\mu\nu}_{(\Phi)},$$

(8)

which is just the requirement of energy-momentum conservation and equivalent to Eq.(6).

Clearly, Eq. (6) or (8) does not admit the solution $\Phi = \Phi_0$ where $\Phi_0$ is a constant, unless the following condition

$$\frac{df(\Phi)}{d \Phi} |_{\Phi=\Phi_0} = 0$$

(9)

is satisfied. Eq. (9) is an existence condition for GR solutions \[11\] and we will focus on such theories.
As is mentioned in the introduction, in the present paper, we simply choose
\[ f(\Phi) = \Phi^2, \] (10)
so that GR solutions for the metric, together with \( \Phi = 0 \), are guaranteed. When \( f(\Phi) \) is specified like this, the coupling parameter \( \hat{\alpha} \) has the dimension of length squared.

III. EQUATION OF MOTION FOR SCALAR PERTURBATIONS ON THE KERR BACKGROUND

Among the vacuum solutions in GR, Kerr solution, which describes a stationary, axisymmetric spacetime, such as that around a Kerr black hole, is of most interest. In Boyer-Lindquist coordinates, the line element of Kerr spacetime reads
\[ ds^2 = -\frac{\Delta}{\rho^2}(dt - a \sin^2 \theta d\phi)^2 + \frac{\sin^2 \theta}{\rho^2}[(r^2 + a^2) d\phi - a dt]^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2, \] (11)
where
\[ \Delta \equiv r^2 - 2Mr + a^2, \]
\[ \rho^2 \equiv r^2 + a^2 \cos^2 \theta, \]
\[ a \equiv \frac{J}{M}, \]
and \( M, J \) are the Arnowitt-Deser-Misner (ADM) mass and ADM angular momentum, respectively.

We shall show that the Kerr black hole solution within a certain regime is unstable in the framework of the quadratic dCS gravity introduced above. To this end, it is useful to investigate the linear perturbations of the Kerr spacetime with the trivial scalar field. However, it is expected that, similar to the theory discussed in Ref.\([40]\), the equations governing the perturbations of the metric \( \delta g_{\mu\nu} \) are decoupled from that of the scalar field. Therefore, we focus on the perturbation of the scalar field \( \delta \Phi \), which is governed by the following equation
\[ (\Box + 2\hat{\alpha}^* RR)\delta \Phi = 0. \] (12)
This is a Teukolsky-like equation, and it is obvious that the curvature correction acts as an effective mass. For the Kerr spacetime, the CS invariant reads
\[ *RR = \frac{96aM^2 r \cos \theta (3r^2 - a^2 \cos^2 \theta)(r^2 - 3a^2 \cos^2 \theta)}{(r^2 + a^2 \cos^2 \theta)^6}. \] (13)
Due to the uncertainty of the sign of $\cos \theta$, it is worth noting that the curvature correction term may provide a negative effective mass squared which can cause tachyonic instability.

In order to solve Eq. (12) numerically, it is helpful to rewrite the Kerr metric in the ingoing Kerr-Schild coordinates \{\tilde{t}, r, \theta, \phi\} through the following transformation

\[
\begin{align*}
\tilde{d}t & = dt + \frac{2Mr}{\Delta} dr, \\
\tilde{d}\phi & = d\phi + \frac{a}{\Delta} dr.
\end{align*}
\]

As a result, the line element of Kerr metric and the equation for scalar perturbation (12) can be rewritten as

\[
ds^2 = - \left(1 - \frac{2Mr}{\rho^2}\right) \tilde{d}t^2 - \frac{4aMr}{\rho^2} \sin^2 \theta \tilde{d}t \tilde{d}\phi + \frac{4Mr}{\rho^2} \tilde{d}t \tilde{d}r + \left(1 + \frac{2Mr}{\rho^2}\right) dr d\phi + \rho^2 d\theta^2
\]

\[
+ \left(r^2 + a^2 + \frac{2Ma^2 r \sin^2 \theta}{\rho^2}\right) \sin^2 \theta d\phi^2
\]

and

\[
(\rho^2 + 2Mr) \partial^2_{\tilde{t}} \delta \Phi = 2M \partial_{\tilde{t}} \delta \Phi + 4Mr \partial_{\tilde{t}} \partial_r \delta \Phi + \partial_r (\Delta \partial_r \delta \Phi)
\]

\[
+ 2a \partial_r \partial_\phi \delta \Phi + \frac{1}{\sin^2 \theta} \partial^2_\phi \delta \Phi + \frac{1}{\sin \theta} \partial_\theta (\sin \theta \partial_\theta \delta \Phi) + 2\alpha \rho^2 \ast RR \delta \Phi,
\]

respectively. Given the axial symmetry of the Kerr geometry, the perturbative variable $\delta \Phi$ can be decomposed as

\[
\delta \Phi(\tilde{t}, r, \theta, \phi) = \frac{1}{r} \psi(\tilde{t}, r, \theta) e^{im\phi}.
\]

Inserting the above expression into Eq. (16), we finally obtain that

\[
A^{\tilde{t}\tilde{t}} \partial^2_{\tilde{t}} \psi + A^{\tilde{r}\tilde{t}} \partial_{\tilde{t}} \partial_r \psi + A^{rr} \partial^2_r \psi + A^{\theta \phi} \partial^2_\phi \psi + B^{\tilde{t}} \partial_{\tilde{t}} \psi + B^r \partial_r \psi + B^\theta \partial_\theta \psi + C \psi = 0,
\]
where
\begin{align*}
A^{\bar{t}\bar{t}} &= \rho^2 + 2Mr, \\
A^{\bar{t}r} &= -4Mr, \\
A^{rr} &= -\Delta, \\
A^{\theta\theta} &= -1, \\
B^{\bar{t}} &= 2M, \\
B^{r} &= \frac{2}{r}(a^2 - Mr) - 2ima, \\
B^{\theta} &= -\cot \theta, \\
C &= \frac{m^2}{\sin^2 \theta} - \frac{2(a^2 - Mr)}{r^2} - 2\hat{\alpha}^2\rho^2 RR + \frac{2ima}{r}.
\end{align*}

(19)

IV. NUMERICAL METHOD

Eq. (18) is a modified homogeneous 2+1 Teukolsky equation for spin-0 perturbations. Solving this equation is not a trivial task. In fact, it is not long ago that the behavior of a scalar field on fixed Kerr background in GR was examined by Rcz and Tth \[43\]. In their work, a numerical framework incorporating the techniques of conformal compactification and hyperbolic initial value formulation is employed. Later, numerical solution of the 2 + 1 Teukolsky equation for generic spin perturbations on a hyperboloidal and horizon penetrating foliation of Kerr has also been investigated \[44–46\].

In the present work, we shall solve the equation (18) by using the hyperboloidal layer method. For this purpose, we first introduce the compactified radial coordinate $R$ and the suitable time coordinate $T$ with the following definitions following Rcz and Tth \[43\]
\begin{align*}
\tilde{t} &= T + h(R), \quad r = \frac{R}{\Omega(R)},
\end{align*}

(20)

where
\begin{align*}
h(R) &= \frac{1 + R^2}{2\Omega} - 4M \ln(2\Omega) \quad (21)
\end{align*}

and
\begin{align*}
\Omega(R) &= \frac{1 - R^2}{2}.
\end{align*}

(22)

The event horizon $R_+$ in the new radial coordinate $R$ is located at
\begin{align*}
R_+ &= \frac{2\sqrt{2M\sqrt{M^2 - a^2} - a^2 + 2M^2 + 1} - 2}{2(\sqrt{M^2 - a^2} + M)}. \quad (23)
\end{align*}
In addition, we can further define the boost function \( H(R) \), which is useful for later computation
\[
H = \frac{dh}{dr}(R).  \tag{24}
\]
Then, putting these relations into the Teukolsky-like equation (18) and bearing in mind that
\[
\partial_t = \partial_T, \quad \partial_r = -H\partial_T + \frac{2\Omega^2}{1 + R^2}\partial_R,  \tag{25}
\]
we finally obtain that
\[
\partial^2_T \psi + \tilde{A}^{TR}\partial_T \partial_R \psi + \tilde{A}^{RR}\partial^2_R \psi + \tilde{A}^{\theta\theta}\partial^2_\theta \psi + \tilde{B}^T \partial_T \psi \\
+ \tilde{B}^R \partial_R \psi + \tilde{B}^\theta \partial_\theta \psi + \tilde{C} \psi = 0
\tag{26}
\]
where
\[
\tilde{A}^{TR} = \frac{A^{TR}}{A^{TT}}, \quad \tilde{A}^{RR} = \frac{A^{RR}}{A^{TT}}, \quad \tilde{A}^{\theta\theta} = \frac{A^{\theta\theta}}{A^{TT}}, \quad \tilde{B}^T = \frac{B^T}{A^{TT}}, \quad \tilde{B}^R = \frac{B^R}{A^{TT}}, \quad \tilde{B}^\theta = \frac{B^\theta}{A^{TT}}, \quad \tilde{C} = \frac{C}{A^{TT}}
\tag{27}
\]
and
\[
A^{TT} = A^{ii} - HA^{ir} + H^2 A^{rr},  \tag{28}
\]
\[
A^{TR} = \frac{2\Omega^2}{1 + R^2} A^{ir} - \frac{4\Omega^2}{1 + R^2} HA^{rr},  \tag{29}
\]
\[
A^{RR} = \left( \frac{2\Omega^2}{1 + R^2} \right)^2 A^{rr},  \tag{30}
\]
\[
B^T = B^i - HB^r - H' \left( \frac{2\Omega^2}{1 + R^2} \right) A^{rr},  \tag{31}
\]
\[
B^R = \frac{2\Omega^2}{1 + R^2} \left[ B^r + \left( \frac{2\Omega^2}{1 + R^2} \right)' A^{rr} \right],  \tag{32}
\]
where the prime denotes the derivative with respect to \( R \).

To suppress the numerical instabilities, we introduce an auxiliary field \( \Pi \)
\[
\Pi \equiv \partial_T \psi + b\partial_R \psi,  \tag{33}
\]
where the coefficient
\[
b \equiv \frac{\tilde{A}^{TR} + \sqrt{(\tilde{A}^{TR})^2 - 4\tilde{A}^{RR}}}{2}.  \tag{34}
\]
It can also convert Eq.(26) to a coupled set of first-order equations in space and time
\[
\partial_T \begin{bmatrix} \psi \\ \Pi \end{bmatrix} + \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \partial_R \begin{bmatrix} \psi \\ \Pi \end{bmatrix} + \begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{bmatrix} \begin{bmatrix} \psi \\ \Pi \end{bmatrix} = 0  \tag{35}
\]
\]
where
\[ \alpha_{11} = b, \quad \beta_{11} = 0, \]
\[ \alpha_{12} = 0, \quad \beta_{12} = -1, \]
\[ \alpha_{21} = \bar{B}^R + (b - \bar{A}^{TR})\partial_R b - b\bar{B}^T, \quad \beta_{21} = \bar{A}^{R\theta} \partial_\theta^2 + \bar{B}^{\theta \theta} \partial_\theta + \bar{C}, \]
\[ \alpha_{22} = \bar{A}^{TR} - b, \quad \beta_{22} = \bar{B}^\tau. \]

When \( \psi \) and \( \Pi \) are splitted into real and imaginary parts as
\[ \psi = \psi_R + i\psi_I \]
and
\[ \Pi = \Pi_R + i\Pi_I, \]
Eq.(35) can be written in the following form
\[ \frac{du}{dT} = Au \]
where
\[ Au = -G \partial_R u - Yu - Xu, \]
\[ u \equiv \{\psi_R, \psi_I, \Pi_R, \Pi_I\}, \]
\[ G \equiv \begin{bmatrix}
    b & 0 & 0 & 0 \\
    0 & b & 0 & 0 \\
    \alpha_{21}^R & -\alpha_{21}^I & \alpha_{22}^R & 0 \\
    \alpha_{21}^I & \alpha_{21}^R & 0 & \alpha_{22}^I
\end{bmatrix}, \]
\[ Y \equiv \begin{bmatrix}
    0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 \\
    l_{31} & 0 & 0 & 0 \\
    0 & l_{31} & 0 & 0
\end{bmatrix}, \]
\[ X \equiv \begin{bmatrix}
    0 & 0 & -1 & 0 \\
    0 & 0 & 0 & -1 \\
    \beta_{21}^R & -\beta_{21}^I & \beta_{22}^R & -\beta_{22}^I \\
    \beta_{21}^I & \beta_{21}^R & \beta_{22}^I & \beta_{22}^R
\end{bmatrix}. \]

Note that all the coefficients are functions of coordinates \( R \) and \( \theta \).

Once the dynamical equation (39) for scalar perturbations is derived, the next task is to solve it numerically by employing a standard fourth-order Runge-Kutta integrator. Taking
into account the computational efficiency, we compute the equation in a domain \((R_+, 1) \times (0, \pi)\) with grids of \(801 \times 67\) points. Before we conduct the numerical computation, it is necessary to define the new dimensionless coupling constant \(\alpha\) by

\[
\alpha = \frac{\dot{\alpha}}{M^2}
\]  

(45)

where \(M\) is the BH’s mass. In the actual process of computation, we set \(M = 1\) for convenience. To discretize the spatial parts, we use the fourth-order accurate finite differences formula \[48\] in both radial and angular directions. We choose a spherically harmonic Gaussian bell centered at \(R_c = 0.8\) in the \(R\) direction as the initial perturbation. That is, function \(\psi\) initially takes the form as

\[
\psi(t = 0, R, \theta) \sim Y_{lm} e^{-\frac{(R-R_c)^2}{2}}
\]  

(46)

where \(Y_{lm}\) represents the \(\theta\)-dependent spherical harmonics. Due to the relations presented in Eqs.\([28]\), the initial form of \(\Pi\) is determined by

\[
\Pi(t = 0, R, \theta) = b \frac{\partial R}{\partial R} \psi(0, R, \theta).
\]  

(47)

As for the boundary conditions, in the \(R\) direction, the use of hyperboloidal foliation and compactification achieves that the transformed system is pure outgoing at the outer boundary, which is just what we expect, so we do not have to exert the boundary conditions at infinity by hand anymore. Similarly, since the foliation is horizon penetrating, we don’t need to specify the inner boundary condition on the horizon either \[46\].

However, in the \(\theta\) direction, we need to use a staggered grid to avoid the inherent difficulties of evaluating expressions where \(\csc \theta\) is present \[48\]. In the staggered grid, the values for \(\theta = 0\) and \(\theta = \pi\) are located between two grid points. The points to the left (right) of \(\theta = 0\) (\(\theta = \pi\)) are considered as ghost points, because these points are only used to impose the boundary conditions. In our fourth-order Runge-Kutta integrator, four ghost points are needed, two of which are in the front of the points where \(\theta = 0\) and the other two at the back of the points where \(\theta = \pi\). The values in the ghost points are updated according to the following strategies

\[
\begin{align*}
\psi(T, R, \theta) &= \psi(T, R, -\theta) \\
\psi(T, R, \pi + \theta) &= \psi(T, R, \pi - \theta)
\end{align*}
\]

\(\text{for } m = 0, \pm 2, \ldots\)

(48)
and
\[
\psi(T, R, \theta) = -\psi(T, R, -\theta) \\
\psi(T, R, \pi + \theta) = -\psi(T, R, \pi - \theta)
\]  
\text{for } m = \pm 1, \pm 3, \ldots \tag{49}

V. RESULTS

As a check for the validation of our code, we first consider the degenerated case that the coupling constant \( \alpha \) vanishes. In Table I, the quasinormal frequencies of the \( l = 2, m = 2 \) mode for BHs with different spin are listed. Our results, listed in the middle column, are obtained in the time domain via the Prony method [49, 50]. For comparison, the results in the frequency domain presented in Ref. [51] are also listed in the right column. Clearly, they are very close to each other, which means our code is reliable.

TABLE I. Quasinormal frequencies of the \((l = 2, m = 2)\) mode for the case that the coupling constant \( \alpha \) vanishes.

|       | Our results | Results from Ref. [51] |
|-------|-------------|------------------------|
|       | Re(\(\omega\)) | -Im(\(\omega\)) | Re(\(\omega\)) | -Im(\(\omega\)) |
| \(a = 0.1\) | 0.499473 | 0.096701 | 0.499482 | 0.096666 |
| \(a = 0.5\) | 0.585989 | 0.093495 | 0.585990 | 0.093494 |
| \(a = 0.9\) | 0.781638 | 0.069287 | 0.781638 | 0.069289 |
| \(a = 0.995\) | 0.949513 | 0.023091 | 0.949522 | 0.023104 |

A. The axisymmetric \((m = 0)\) modes

The time-domain profiles of \((l = 2, m = 0)\) mode of scalar perturbation around Kerr BHs in quadratic dCS theory with different values of coupling constant \( \alpha \) are plotted in Fig. [1] As usual in GR, after a period of damping proper oscillations, dominated by quasinormal modes, a stage of power-law tails appears at the very late time. Interestingly, for a Kerr BH with given rotation parameter \( a \), when the coupling constant \( \alpha \) become large enough, the instability will develop after the damping quasinormal oscillations. The larger \( \alpha \) is, the faster the instability grows and the shorter the period of damping lasts. It is obvious that the rapidly rotating BH is more susceptible to \( \alpha \). In addition, although the axisymmetric
FIG. 1. The time-domain profiles of \((l = 2, m = 0)\) mode of scalar perturbation on the Kerr BHs with spin parameter \(a = 0.5\) (left panel) and \(0.998\) (right panel) in the quadratic dCS gravity with different values of coupling constant \(\alpha\). The observing location is at \(r \rightarrow \infty\) \((R = 1)\), \(\theta = \pi/2\).

scalar perturbation is coupled with the CS term, our results are still consistent with the fact that in Kerr spacetime, if any unstable mode occurs, it contains only the imaginary part.

B. \(m \neq 0\) modes

When the perturbation is not axisymmetric, there exist \(m \neq 0\) modes. In Fig. 2, we plot the time-domain profiles of \((l = 2, m = 2)\) mode on the background of Kerr BH with spin parameter \(a = 0.9\) in theories with different value of coupling constant \(\alpha\). When \(\alpha\) is large enough, the mode grows rapidly and instability occurs. Different from the growing axisymmetric \((m = 0)\) modes, these unstable modes are oscillating and develops immediately after the initial outburst. This is reasonable, actually, in the case of \(m = 0\), from coefficients of Eq. (39), one can easily find that \(\psi_R\) and \(\psi_I\) are decoupled, there is nothing that can drive the scalar field oscillate, so it just keeps growing when the unstable mode appears. However, for the cases \(m \neq 0\), due to the coupling of \(\psi_R\) and \(\psi_I\), they interact with each other in the process of time evolution, which allows the scalar field to oscillate as it grows under unstable patterns.

In Fig. 3, we show the real and imaginary part of the frequency \(\omega\) at the stage of late-time tail of \(l = m = 1\) and \(l = m = 2\) modes of the perturbation on the background of a Kerr BH with spin parameter \(a = 0.9\), as two functions of the coupling constant \(\alpha\) in the left
FIG. 2. The time-domain profiles of perturbations for $l = 2$, $m = 2$ on the background of Kerr BH with spin parameter $a = 0.9$ in theories with different values of coupling constant $\alpha$.

and right panel, respectively. With the increase of the value of $\alpha$, the real part decreases monotonically and tends to a nonzero limit value, which depends on the harmonic azimuthal index $m$, meanwhile, the imaginary part seems to grow monotonically without limit. It is worth noting that the imaginary part can change its sign from a negative value to a positive one for the coupling constant $\alpha \lesssim 1$.

C. Stable and unstable regions in the parameter space

From the result above, it is found that the occurring of instability depends on the value of spin parameter $a$ and coupling constant $\alpha$. That is to say, this instability occurs only in a certain region of the parameter space. In Fig.4, we plot a dividing line in the parameter space spanned by $\alpha$ and $a$ for the $m = 0$ mode. For the point in the region above this line (shaded), there exit unstable perturbation modes, then the BH becomes unstable, while for the points in the region below the line (blank), there is no growing mode and the BH is stable. The boundary between stable and unstable regions depends also on the value of the harmonic azimuthal index $m$, which is illustrated in Fig.5.
FIG. 3. The real part (left panel) and imaginary part of the frequency $\omega$ at the stage of late-time tail of $l = m = 1$ and $l = m = 2$ modes of the perturbation on the background of a Kerr BH with spin parameter $a = 0.9$ in the quadratic dCS gravity with different values of $\alpha$.

VI. CONCLUSIONS AND DISCUSSIONS

In this paper, the behavior of time evolution for scalar perturbations on the background of Kerr BHs in the quadratic dCS gravity has been numerically investigated in detail. It is found that under scalar perturbations, the Kerr BH becomes indeed unstable at the linear level in some specific region of parameter space, which depends on the value of harmonic azimuthal index $m$.

In recent years, there are lots of studies on spontaneous scalarization, based on the unstable mode of the BH under kinds of perturbations. Most of them were done in the schwarzschild background. Our work is also connected with them, although it is not sure whether the regime of this instability is spontaneous scalarization, we can almost exclude the possibility of superradiance, since we have found that the instability can occur even in the condition that the frequency of the perturbation’s mode is higher than the critical frequency for superradiance.

Finally, what will be the final state of the instability when nonlinear effects are taken into account? It is expected the Kerr BH will undergo spontaneous scalarization, thus forming a non-Kerr BH with scalar hair. Perhaps a fully nonlinear analysis is required to confirm this possibility. Obviously, this deserves a new work in the future.
FIG. 4. The parameter space for the case $l = 2$, $m = 0$. In the region above this line (shaded), there exit unstable perturbation modes, then the BH becomes unstable, while in the region below the line (blank), there is no growing mode and the BH is stable.

FIG. 5. The boundary between stable and unstable regions in parameter space for different values of harmonic azimuthal index $m$. 
ACKNOWLEDGMENTS

This work is supported in part by the Science and Technology Commission of Shanghai Municipality under Grant No. 12ZR1421700 and the Program of Shanghai Normal University.

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