Conservation and Integrability in TMG

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Abstract

In this work, following the paper by Romain Ruzziconi and Céline Zwikel [4] we extend the questions of conservation, integrability and renormalization in Bondi gauge and in GR to the theory of Topological Massive Gravity (TMG). We construct the phase space and renormalize the divergences arising within the symplectic structure through a holographic renormalization procedure. We show that the charge expressions are generically finite, conserved and can be made integrable by a field–dependent redefinition of the asymptotic symmetry parameters.

1 Introduction

For the sake of simplicity the three dimensions space time is an arena for studying various aspects of gravity. Maximally symmetric vacuum solutions of Einstein field equations in three dimensions are characterized by their global properties which encoded in the asymptotic charges. By specifying the boundary conditions which form the asymptotic symmetries, the charges are computed. To perform the boundary conditions and apply the further analyses, it is customary to fix the gauge. Here, we used the Bondi gauge to study asymptotically AdS spacetimes [1]. The analysis of asymptotically flat spacetimes at null infinity led to the Bondi mass formula, which states that due to the emission of gravitational waves the mass of the system decreases in time [1]. The non–conservation of charges is an important part to explain the dynamics of the system which is related to non–integrability of the charges [2], [3]. Non–integrability is commonly considered as an unpleasant property because it implies that the finite charge expressions rely on the particular path that one chooses to integrate on the solution space, which is a common feature of a dissipative system [4]. One way to solve the matter of non–integrability is to keep the full non–integrable expressions and try to make sense of it. A vital technical result going into this direction is that the Barnich–Troessaert bracket that enables one to derive mathematically consistent charge algebras for non-integrable charges [5]. This bracket has been employed in many different contexts and also the associated charge algebras have been shown to be physically very relevant since they contain all the information concerning the flux-balance laws of the theory [6].

Progress in understanding the relation between non–conservation and non–integrability of the charges has been made. In [7], it was conjectured that no flux passing through the boundary is equivalent to the existence of a specific section of the phase space for which the charges are integrable. A natural however non–trivial question that we would like to explore in this work is whether this conjecture is also applicable for asymptotic boundaries?

To get some insights into a attainable well–behaved quantum gravity, countless number of works

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have been devoted to study the lower dimensional models. Notably, studying cosmological Einstein’s theory in 3-dimensions has attracted large attentions. Although the 3D cosmological Einstein gravity does not possess any local physical degree of freedom (dof), it has appealing global properties. For example, it possesses the Banados-Teitelboim-Zanelli (BTZ) black hole solutions \cite{8} or 2D boundary CFT with the Brown–Henneaux central charge ($c = 3l/2G$) dual to the bulk $AdS_3$ spacetime \cite{9}. Here, the important purpose is to construct a 3D gravity theory that also possess a local propagating dof as it has a viable boundary CFT. In this perspective, the Topologically Massive Gravity (TMG) is particularly interesting because it is a renormalizable theory and also describes a local dynamical dof \cite{10}, \cite{11}. The so-called shortcoming in cosmological TMG, however, is that the signs of energy of boundary graviton and mass of BTZ black hole are in conflict with each others. For the cosmological TMG, the $AdS_3$ solution is dual to a 2D CFT admitting two copies of Virasoro algebra.

When we were working on this paper, the paper \cite{12} on the subject has been published. In the mentioned paper the authors addressed the question of conservation and integrability of charges in Bondi gauge and in TMG. Unlike our paper, they have perturbatively considered the field equations and obtained an approximated solution.

The paper is organized as follows: in section 2 we apply the covariant phase space methods on topological massive gravity in the Bondi gauge. We discuss the solution space, renormalize the action principle and the symplectic structure, obtain the corresponding finite surface charges associated with the asymptotic symmetries. We conclude with some comments in section 3.

2 Phase space of TMG

Deser, Jackiw and Templeton constructed a $2+1$- dimensional renormalizable dynamical gravity theory with the help of the gravitational Chern–Simons term \cite{10}. The model is called Topologically Massive Gravity (TMG). The Lagrangian density of TMG is presented as follows

$$L_{TMG} = \frac{\sqrt{-g}}{16\pi G} \left[ \mathcal{R} - 2\Lambda + \frac{1}{2\mu} \epsilon^{\lambda\mu\nu} \left( \Gamma^\rho_{\lambda\sigma} \partial_\mu \Gamma^\sigma_{\rho\nu} + \frac{2}{3} \Gamma^\rho_{\lambda\sigma} \Gamma^\sigma_{\mu\tau} \Gamma^\tau_{\nu\rho} \right) \right].$$

Here $\epsilon^{\lambda\mu\nu}$ is a rank–3 tensor. This model describes a topologically massive graviton possessing a single helicity mode and acquires asymptotically AdS black hole solutions called Banados–Teitelboim-Zanelli (BTZ) black holes \cite{8}. Here we briefly review the phase space of TMG \cite{25, 16}. By an infinitesimal variation of the above Lagrangian we obtain

$$\delta L_{TMG} = \frac{\delta L_{TMG}}{\delta g_{\mu\nu}} \delta g_{\mu\nu} + \partial_\mu \Theta_{TMG}^\mu$$

Using the following Euler–Lagrange derivatives

$$\frac{\delta L_{TMG}}{\delta g_{\mu\nu}} = -\frac{\sqrt{-g}}{16\pi G} \left( C^{\mu\nu} + \Lambda g^{\mu\nu} + \frac{1}{\mu} C^{\mu\nu} \right)$$

one can find the TMG equations and the canonical TMG presymplectic potential takes following form

$$\Theta_{TMG}^\mu = \frac{\sqrt{-g}}{16\pi G} \left( g^{\sigma\nu} \delta \Gamma^\mu_{\nu\sigma} - g^{\sigma\mu} \delta \Gamma^\alpha_{\sigma\alpha} + \frac{1}{2\mu} \left( \epsilon^{\lambda\mu\nu} \Gamma^\rho_{\lambda\sigma} \delta \Gamma^\sigma_{\nu\rho} + \epsilon^{\lambda\sigma\nu} R_{\sigma\rho}^{\mu\nu} \delta g_{\lambda\rho} \right) \right)$$
The TMG theory is invariant under diffeomorphisms which act on the metric with a standard Lie derivative $\delta g_{\mu\nu} = 2\nabla_{(\mu}e_{\nu)}$. The weakly–vanishing Noether current $S^\mu_\xi$, is obtained as

$$
\frac{\delta L_{\text{TMG}}}{\delta g_{\mu\nu}} \delta g_{\mu\nu} = \sqrt{-g} \left( \frac{1}{8\pi G} \left( \nabla_\mu G^{\mu\nu} \xi_\nu + \frac{1}{\mu} \nabla_\mu C^{\mu\nu} \xi_\nu \right) + \partial_\mu S^\mu_\xi \right)
$$

(5)

The first and second terms in the right–hand side vanish due to the Bianchi identities which correspond to the Noether identities of the theory. The total derivative term gives $S^\mu_\xi$,

$$
S^\mu_\xi = 2 \frac{\delta L_{\text{TMG}}}{\delta g_{\mu\nu}} = -\sqrt{-g} \left( G^{\mu\nu} + \Lambda g^{\mu\nu} + \frac{1}{\mu} C^{\mu\nu} \right) \xi_\nu.
$$

(6)

The Barnich–Brandt codimension 2 form is given by

$$
k^{\mu\nu}_{BB,\xi} = \frac{1}{2} \delta \phi^j \frac{\delta}{\delta \phi^\nu} S^\mu_\xi + \left( \frac{2}{3} \partial_\sigma \delta \phi^j - \frac{1}{3} \delta \phi^j \partial_\sigma \right) \frac{\delta}{\delta \phi^\nu} S^\mu_\xi - (\mu \leftrightarrow \nu)
$$

(7)

this 2–form in our interesting case takes following form

$$
k^{\mu\nu}_{BB,\xi} = k^E_{\mu\nu}(\xi) + e^{\mu\nu\beta} g^{(l)}_{\alpha\beta} \xi^\alpha + e^{\mu\nu\alpha} G^{(l)}_{\beta\alpha} \xi_\beta + e^{\mu\nu\alpha} \xi_\beta + k^E_{\mu\nu}(\xi \nabla \xi)
$$

(8)

where

$$
k^E_{\mu\nu}(\xi) = \xi^\alpha \nabla^\mu h_{\alpha\nu} - \xi^\alpha \nabla^\nu h_{\alpha\mu} + \xi^\mu \nabla^\nu h - \xi^\nu \nabla^\mu h + h_{\mu\sigma} \nabla^\nu \xi_\sigma - h_{\nu\sigma} \nabla^\mu \xi_\sigma + \xi^\nu \nabla^\alpha h_{\mu\alpha} - \xi^\mu \nabla^\alpha h_{\nu\alpha} + h \nabla^\mu \xi^\nu,
$$

(9)

where

$$
G^{(l)}_{\mu\nu} = R^{(l)}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R^{(l)} - 2 \Lambda h_{\mu\nu},
$$

(10)

with

$$
R^{(l)}_{\mu\nu} = \frac{1}{2} \left[ - \nabla^2 h_{\mu\nu} - \nabla_\mu \nabla_\nu h + \nabla_\mu \nabla_\sigma h^\sigma_{\nu} + \nabla_\nu \nabla_\sigma h^\sigma_{\mu} \right]
$$

(11)

$$
R^{(l)} = - \nabla^2 h + \nabla_\rho \nabla_\sigma h^{\rho\sigma} - 2 \Lambda h
$$

(12)

and $\delta g_{\mu\nu} = h_{\mu\nu}$ is a metric perturbation and $h = g^{\mu\nu} h_{\mu\nu}$.

### 2.1 Bondi gauge in three dimensions

In the following, we will work in the Iyer–Wald approach that allows us to renormalize the symplectic structure.

#### 2.1.1 Solution space

Asymptotically AdS spacetimes solving field equation are of the form

$$
ds^2 = \frac{V}{r} e^{2\beta} du^2 - 2 e^{2\beta} du dr + r^2 e^{2\varphi} \left( d\phi - U du \right)^2,
$$

(13)

with coordinates $(x^\mu) = (u, r, \phi)$. In the metric, $V$, $\beta$ and $U$ are functions of $(u, r, \phi)$ and $\varphi$ is a function of $(u, \phi)$. The Bondi gauge is obtained by requiring the following three gauge–fixing conditions

$$
g_{rr} = 0, \quad g_{r\phi} = 0, \quad g_{\phi\phi} = r^2 e^{2\varphi}.
$$

(14)
Applying the TMG’s field equations determine the metric for $\Lambda^2 = -1$, $\mu l \neq 1$ as follows:

$$\beta(u, r, \phi) = \beta_0(u, \phi)$$
$$U(u, r, \phi) = U_0(u, \phi) + \frac{U_1(u, \phi)}{r} + \frac{U_2(u, \phi)}{r^2}$$
$$V(u, r, \phi) = -\frac{U_2^2(u, \phi)}{r} e^{2\varphi(u, \phi) - 2\beta_0(u, \phi)} - 2U_1(u, \phi)U_2(u, \phi)e^{2\varphi(u, \phi) - 2\beta_0(u, \phi)} + V_0(u, \phi)r + V_1(u, \phi)r^2 + V_2(u, \phi)r^3$$

where

$$U_1(u, \phi) = 2e^{2\beta_0(u, \phi) - 2\varphi(u, \phi)\partial_\phi\beta_0(u, \phi)},$$
$$U_2(u, \phi) = -e^{2\beta_0(u, \phi) - 2\varphi(u, \phi)}N(u, \phi),$$
$$V_1(u, \phi) = \frac{1}{3U_2e^{4\varphi+2\beta_0} + 2\mu r^2 e^{4\beta_0+3\varphi}} \left[ (12U_2\partial_u \varphi - 12U_2\partial_u \beta_0 + 6\partial_\phi (U_0U_2) + 12U_2(U_0\partial_u \varphi - \partial_u \beta_0) + 6\partial_\phi U_2)e^{4\varphi+2\beta_0} - 4\mu r^2(\partial_u \varphi + U_0\partial_\phi \varphi + \partial_\phi U_0)e^{4\beta_0+3\varphi} + (6V_0\partial_u \beta_0 + 3\partial_\phi V_0)e^{4\beta_0+2\varphi} + 48\partial_\phi \beta_0 e^{6\beta_0}(2(\partial_u \beta_0)^2 - \partial_\phi \varphi \partial_\phi \beta_0 + \partial_\phi^2 \beta_0) \right]$$
$$V_2(u, \phi) = \frac{1}{2\mu l^2 r^3 e^{4\beta_0+3\varphi}} \left[ -2l^2 e^{4\varphi+2\beta_0} [\partial_\phi (U_2U_0) + 2U_2U_0\partial_\phi \varphi + 2U_2 \partial_u \varphi - 2U_2U_0\partial_u \beta_0 - 2U_2 \partial_u \beta_0 + \partial_u U_2 - \frac{1}{2}U_2 V_1 - l^2 e^{4\beta_0+2\varphi}[2V_0\partial_\phi \beta_0 + \partial_\phi V_0] - 2\mu l^2 e^{6\beta_0+3\varphi} + 8l^2 \partial_\phi \beta_0 e^{6\beta_0}[-2(\partial_u \beta_0)^2 + \partial_\phi \varphi \partial_\phi \beta_0 - \partial_\phi^2 \beta_0) \right]$$

We also have two constrained field equations which determine the time evolution of $N$ and $V_0$ (see appendix A). The expansion of the metric around the AdS spacetime is

$$\frac{V}{r} = \left[ -\frac{r^2 e^{2\beta_0}}{l^2} - 2r (U_0 \partial_u \varphi + \partial_u \varphi + \partial_\phi U_0) + V_0 + \frac{4N \partial_\phi \beta_0 e^{2\beta_0 - 2\varphi}}{r} - \frac{N^2 \beta_0 e^{2\beta_0 - 2\varphi} r}{r^2} \right]$$
$$+ \frac{2e^{-\varphi}}{\mu r} \left[ e^{2\beta_0 - 2\varphi} \left( 4\partial_\phi \beta_0 \partial_u^2 \beta_0 + 8(\partial_\phi \beta_0)^3 - 4\partial_\phi \varphi (\partial_u \beta_0)^2 \right) + V_0 \partial_\phi \beta_0 - NU_0 \partial_\phi \varphi - U_0 \partial_\phi N - N \partial_u \varphi - 2N \partial_u U_0 - \partial_u N + \frac{1}{2} \partial_\phi V_0 \right] + \mathcal{O}(\mu^{-2})$$

The expression inside the first bracket is the solution of Einstein gravity [23], and the second bracket comes from the TMG. If the $\partial_u N$ equals to the equation (23) of [23], then the second bracket equals zero. Also, the Cotton tensor for our solution does not equal zero. The solution space is paramatized by five functions which three of them ($\beta_0, U_0, \varphi$) characterize the induced boundary metric through

$$\bar{s}_{ab}dx^a dx^b = \lim_{r \to \infty} \left( \frac{1}{r^2} ds^2 \right) = \left( -\frac{e^{4\beta_0(u, \phi)}}{l^2} + U_0^2 e^{2\varphi(u, \phi)} \right) du^2 - 2U_0 e^{2\varphi} dud\phi + e^{2\varphi} d\phi^2 ,$$

where $(x^a) = (u, \phi)$ are the coordinates on boundary. The other two functions $V_0$ and $N$ encode the bulk information on the mass and the angular momentum.

\[^1\partial_u N + N \partial_u \varphi = e^{2\beta_0 - 2\varphi} \left( 4\partial_u \beta_0 \partial_u^2 \beta_0 + 8(\partial_u \beta_0)^3 - 4\partial_u \varphi (\partial_u \beta_0)^2 \right) + V_0 \partial_\phi \beta_0 - NU_0 \partial_\phi \varphi - U_0 \partial_\phi N - 2N \partial_u U_0 + \frac{1}{2} \partial_\phi V_0 \]
2.2 Residual symmetries

The residual gauge diffeomorphisms $\xi = \xi^a \partial_a + \xi^\phi \partial_\phi + \xi^r \partial_r$ preserving the Bondi gauge fixing have to satisfy the conditions

$$\mathcal{L}_\xi g_{rr} = \mathcal{L}_\xi g_{r\phi} = 0, \quad g^{\phi\phi} \mathcal{L}_\xi g_{\phi\phi} = 2w(u, \phi),$$

(22)

which the explicit solutions of them are given by

$$\xi^a = f,$$

$$\xi^\phi = Y - \frac{1}{r} \partial_\phi f e^{2\beta_0 - 2\varphi},$$

$$\xi^r = r(\partial_\phi f U_0 + w - \partial_\phi Y - f \partial_u \varphi - Y \partial_\phi \varphi) + \partial_\phi f U_1 + \frac{\partial_\phi f U_2}{r} + e^{2\beta_0 - 2\varphi} \left( \partial_\phi^2 f + 2\partial_\phi \beta_0 \partial_\phi f - \partial_\phi f \partial_\phi \varphi \right).$$

(23)

So, the residual diffeomorphisms are encoded in $f, Y, w$ which arbitrary functions of the boundary coordinates. Using the modified Lie bracket

$$[\xi_1, \xi_2]_* = [\xi_1, \xi_2] - \delta_{\xi_1} \xi_2 + \delta_{\xi_2} \xi_1,$$

(24)

these vector fields satisfy the commutation relations $[\xi(f_1, Y_1, w_1), \xi(f_2, Y_2, w_2)]_* = \xi(f_{12}, Y_{12}, w_{12})$ where

$$f_{12} = f_1 \partial_u f_2 + Y_1 \partial_\phi f_2 - \delta_{\xi_1} f_2 - (1 \leftrightarrow 2),$$

$$Y_{12} = f_1 \partial_u Y_2 + Y_1 \partial_\phi Y_2 - \delta_{\xi_1} Y_2 - (1 \leftrightarrow 2),$$

$$w_{12} = - \delta_{\xi_1} w_2 - (1 \leftrightarrow 2).$$

(25)

Under the infinitesimal residual gauge diffeomorphisms ($\mathcal{L}_\xi g_{ab} = \delta_\xi g_{ab}$), the boundary structure transforms as

$$\mathcal{L}_\xi g_{\phi\phi} = \delta_\xi g_{\phi\phi}, \quad \rightarrow \quad \delta_\xi \varphi = w,$$

$$\mathcal{L}_\xi g_{ur} = \delta_\xi g_{ur}, \quad \rightarrow \quad \delta_\xi \beta_0 = -(\partial_\phi f U_0 + Y \partial_\phi Y - \partial_u \varphi - Y \partial_\phi \varphi - w),$$

$$\mathcal{L}_\xi g_{\phi a} = \delta_\xi g_{\phi a}, \quad \rightarrow \quad \delta_\xi U_0 = f \partial_u U_0 + Y \partial_\phi U_0 - U_0 \partial_\phi Y - \partial_\phi f + \frac{1}{l^2} e^{4\beta_0 - 2\varphi} \partial_\phi f + U_0 (\partial_\phi f + U_0 \partial_\phi f),$$

(26)

and the variations of the $N$ and $V_0$ can be found in [17].

In Bondi gauge, the conditions

$$\beta_0 = 0, \quad U_0 = 0, \quad \varphi = 0,$$

(27)

corresponds to the Brown–Henneaux boundary condition, i.e. the induced boundary metric is flat. These lead to the additional conditions as follows

$$\partial_u f = \partial_\phi Y, \quad \partial_u Y = \frac{1}{l^2} \partial_\phi f \quad w = 0.$$

(28)
2.3 Renormalization of the phase space

By evaluating the radial component of the TMG presymplectic potential (4) on the solution space, we obtain some $O(r^2), O(\ln(r))$ terms that diverge when $r \to \infty$. Also, there is $O(\ln(r))$ divergence which comes from the Chern–Simons part of on–shell action. Furthermore, the presymplectic potential admits some $O(l^2)$ terms that make an prevention to take the flat limit $l \to \infty$. Therefore, to remove these divergences, we should add some counter-terms which we show below the exact form of them. In asymptotically $AdS_3$ spacetimes, the minimal action principle in Bondi gauge that satisfies the mentioned requirements is given by [22],[4],[25]

$$S = \frac{1}{16\pi G} \int_M \sqrt{-g} \left[ \mathcal{R} - 2\Lambda + \frac{1}{2\mu} \epsilon^{\lambda\mu\nu} \left( \Gamma^\rho_{\lambda\sigma} \partial_\mu \Gamma^\sigma_{\rho\nu} + \frac{2}{3} \Gamma^\rho_{\lambda\sigma} \Gamma^\sigma_{\mu\tau} \Gamma^\tau_{\nu\rho} \right) \right] d^3x + \int_{\partial M} a_1 L_{GHY} d^2x$$
$$+ \int_{\partial M} a_2 L_{ct} d^2x + \int_{\partial M} a_3 L_0 d^2x + \int_{\partial M} a_4 L_b d^2x + \int_{\partial M} a_5 L_R d^2x + \frac{1}{\mu} \int_{\partial M} a_6 L_{ccs} d^2x$$
$$+ \frac{1}{\mu} \int_{\partial M} a_7 L_{bccs} \log(r) d^2x$$

(29)

The first term is the TMG bulk action and the second term is the Gibbons–Hawking boundary term as

$$L_{GHY} = \frac{1}{8\pi G} \sqrt{-\gamma} \gamma^{\mu\nu} n_{\nu;\mu}$$

(30)

where $n_{\mu} = \frac{1}{\sqrt{g}} \delta_{\mu}^r = \sqrt{r e^{2\beta_0(u,\phi)}} \delta_{\mu}^r$ and determinant of induced metric $\gamma = r V e^{2(\varphi+\beta)}_{|r=R}$. The induced metric is $\gamma_{ab}$ and obtained by taking $r = R = constant$ as

$$ds^2_B = \frac{V(u,\phi)}{R} e^{2\beta_0} du^2 + R^2 e^{2\varphi} (d\phi - U(u,\phi) du)^2.$$  

(31)

The third term is the counter–term as

$$L_{ct} = - \frac{1}{8\pi G l} \sqrt{-\gamma} = - \frac{1}{8\pi G l} e^{e+\beta} \sqrt{-R V}.$$  

(32)

The fourth term is a corner Lagrangian

$$L_0 = - \frac{1}{8\pi G} \sqrt{-\gamma} D_a v^a, \quad v^a = \frac{R e^\varphi}{\sqrt{-\gamma}} (\delta^a_u + U \delta^a_\phi).$$  

(33)

The vector field $v^a$ appeared in (33) is tangent to the leaves of the foliation and satisfies the two properties[24], [25]

$$v^a \gamma_{ab} v^b = -1, \quad \lim_{r \to \infty} \left( \frac{1}{r} \gamma_{ab} v^a \right) = - \frac{e^{2\beta_0}}{l} \delta^a_u.$$  

(34)

The next term in (29) is a kinetic term for the vector $(v^a)$,

$$L_b = \frac{l}{16\pi G} \sqrt{-\gamma} (D_a v^a)^2.$$  

(35)

The next term in (29) is the Gauss–Bonnet term

$$L_R = - \frac{l}{16\pi G} \sqrt{-\gamma} R[\gamma].$$  

(36)
Finally, the last terms in (29) is a term to remove the divergence correspond to $O(R)$ and $\log(R)$ that are coming from the Chern-Simons term of TMG

$$L_{ccs} = \sqrt{-\gamma} e^{\lambda a} N_{\rho} \bar{\gamma}^{\rho}_{\lambda a}, \quad N_{\rho} = -\frac{1}{2} r e^{-\Phi(u,\phi)} \delta_{\rho}^{u}$$ (37)

$$L_{lccs} = \partial_{a} \bar{m}^{a}, \quad \bar{m}^{a} = \partial_{\phi} \beta_{0}(\delta_{u}^{a} + U_{0} \delta_{\rho}^{a})$$ (38)

The explicit terms of the Lagrangians are given in appendix B. Now we provide some details on how to obtain the values of the coefficients (29) of the different terms in the action.

- By evaluating the action (29) on \(-shell, divergences in $O(R^2)$ arise. The $R^2$-divergences by imposing $2a_1 - a_2 = 1$ are removed, while the $R$-divergences are removed by $2a_1 - a_2 - a_3 = 0$ and $a_6 = 1$.
- The $\ln(R)$-divergences are removed by imposing: $a_7 = 1$.
- The action (29) also exhibits some terms in $O(l^2)$ which can be eliminated by imposing $a_5 = -a_1$ and $2a_1 - a_2 - a_4 = 0$.
- By using the Dirichlet boundary conditions (27) and $a_1 = 1$, the action is stationary on solutions space.

Using all the constraints, we obtain $a_1 = 1$. Sending the cut-off to infinity, $R \to \infty$, the expression of the renormalized action (29) is given by

$$S_{ren} = \frac{1}{16\pi G} \int d^2x \left(-V_0 e^{\varphi} + 4 e^{2\beta_0} \varphi \left(-\partial_{\rho}^{2} \beta_{0} + \partial_{\phi} \beta_{0} \partial_{\phi} \varphi - 2(\partial_{\phi} \beta_{0})^{2}\right)\right) - \Gamma_{bulk}(R_0),$$ (39)

where $\Gamma_{bulk}$ is the finite contribution of the bulk action evaluated on its lower bound.

2.4 Renormalization of the symplectic structure

By inserting the Bondi metric (13) into (4), the radial component of the presymplectic potential gives some divergences. The counter-terms to remove the $O(r^2)$ and $O(l^2)$ divergences are similar to those used in (29) to renormalize the action. We define the renormalized presymplectic potential as [4],[25]

$$\Theta_{ren}^{r} = \Theta_{TMG}^{r} + \delta L_{GHY} + \delta L_{ct} + \delta L_{0} + \delta L_{b} + \delta L_{R} + \frac{1}{\mu} \delta L_{ccs}$$

$$- \partial_{a} \Theta_{0}^{a} - \frac{1}{2} r \partial_{a} \Theta_{0}^{a}$$ (40)

where $g$ is a solution, $\delta g$ a perturbation around it. The first line is the presymplectic potential prescribed by the renormalized action (29). To understand the second term in (40), we define the following quantities induced on the boundary

$$\bar{v}^{a} = \lim_{r \to \infty} (rv^{a}) = le^{-2\beta_{0}}(\delta_{u}^{a} + U_{0} \delta_{\rho}^{a}), \quad \bar{\gamma}_{ab} = \lim_{r \to \infty} \left(\frac{1}{r^{2}} \gamma_{ab}\right)$$ (41)

and $\overline{\gamma} = det(\overline{\gamma}_{ab})$ and $\overline{D}_{a}$ the determinant and the covariant derivative associated with the induced boundary metric $\overline{\gamma}_{ab}$, respectively. The term $\overline{\Theta}_{0}^{a}$ is the presymplectic potential of the boundary Lagrangian

$$\overline{L}_{0} = -\frac{1}{8\pi G} \sqrt{-\overline{\gamma}} \overline{D}_{a} \bar{v}^{a}, \quad \delta \overline{L}_{0} = \partial_{a} \overline{\Theta}_{0}^{a} = -\frac{1}{8\pi G} \sqrt{-\overline{\gamma}} \overline{D}_{a} \delta \bar{v}^{a}$$ (42)
where the variation is taken with respect to $\tilde{v}^a$ by keeping the boundary metric $\gamma_{ab}$ fixed (i.e. $\gamma_{ab}$ is seen as a background). By using above setup one can obtain the presymplectic potential and the associated presymplectic current which explicitly provided as equation (65) and (66) in Appendix (C). When we impose the conditions (27), at leading order this expression vanishes. Hence, the associated charges are conserved and the variational principle (29) is stationary on solutions.

2.5 Integrability and charge algebra

In this section, we talk about the renormalized charges and present a particular slicing of the phase space in which they are integrable. The renormalized co–dimension 2 form can be derived using

$$\partial_a k^{r_{\text{ren}, \xi}}[g; \delta g] = w^{r_{\text{ren}}}[g; \delta \xi g, \delta g]$$

where $w^{r_{\text{ren}}}[g; \delta_1 g, \delta_2 g]$ given in (66). We obtain the infinitesimal charges by integration on $S^1_{\infty}$,

$$\delta Q_\xi = \int_0^{2\pi} d\phi \ k^{ur_{\text{ren}, \xi}},$$

where $\delta$ indicates that the charge is not integrable. The explicit expression has been provided as equation (67) in Appendix (C).

By using the renormalization procedure (41), the charges are finite. The charges (67) seem to be non–integrable. This obstruction for integrability can be cured by performing field–dependent redefinitions of the symmetry parameters. In our case, we do the redefinition

$$\tilde{f} = f e^{2\beta_0 - \varphi}, \quad \tilde{Y} = Y - U_0 f, \quad \tilde{w} = w$$

where $\tilde{f}, \tilde{Y}$ and $\tilde{w}$ are taken to be field–independent, i.e. $\delta \tilde{f} = \delta \tilde{Y} = \delta \tilde{w} = 0$. In terms of these parameters, the commutation relations (20) become $[\xi(f_1, Y_1, w_1), \xi(f_2, Y_2, w_2)] = \xi(f_{12}, Y_{12}, w_{12})$ with

$$\begin{align*}
\tilde{w}_{12} &= 0 \\
\tilde{f}_{12} &= \tilde{Y}_1 \partial_0 \tilde{f}_2 + \tilde{f}_1 \partial_0 \tilde{Y}_2 - (1 \leftrightarrow 2) \\
\tilde{Y}_{12} &= \tilde{Y}_1 \partial_0 \tilde{Y}_2 + \frac{1}{l^2} \tilde{f}_1 \partial_0 \tilde{f}_2 - (1 \leftrightarrow 2).
\end{align*}$$

The redefinition (41) gives the charges (67) integrable. We have explicitly $\delta Q_\xi \equiv \delta Q_\xi$ with

$$\delta Q_\xi = \frac{1}{8\pi G} \int_0^{2\pi} d\phi \left[ \tilde{Y} \delta(N e^\varphi) + \tilde{f} \delta \left( V_0 \frac{2}{2} - 2 \beta_0 + 4(\partial_0 \beta_0)^2 - 2 \partial_0 \beta_0 \partial_0 \varphi + \frac{1}{2}(\partial_0 \varphi)^2 + \partial^2_0(2\beta_0 - \varphi) \right) \right]$$

$$+ \frac{1}{4\pi G} \left[ \tilde{Y} \delta(V_0 e^{-2\beta_0 + 2\varphi}) + 7(\partial_0 \beta_0)^2 - 3 \partial_0 \varphi \partial_0 \beta_0 + (\varphi - 2\beta_0) \partial_0 \beta_0 \partial_0 \varphi - 2(\varphi - 2\beta_0)(\partial_0 \beta_0)^2 + (\varphi - \beta_0) \partial^2_0(2\beta_0 - \varphi) + 2 \partial^2_0 \partial_0 \varphi + \tilde{w} \delta(\partial_0 \beta_0 + \partial_0 \beta_0 (\varphi - 2\beta_0)) + \tilde{f} \delta(-\partial_0 U_0 \partial^2_0 e^{2\beta_0 - \varphi} + \partial^2_0 U_0 (\varphi - \beta_0) - \varphi \partial_0 U_0 \partial_0 \beta_0) \right]$$

Integrating the expression (47) and using conditions (28), gives the finite charge as

$$Q_\xi[g] = \frac{1}{16\pi G} \int_0^{2\pi} d\phi \left[ \tilde{Y} \tilde{N} + \tilde{f} \tilde{V}_0 \right]$$
where
\[ \tilde{N} = N e^{\varphi} + \frac{2}{\mu} (V_0 e^{-2\beta_0 + 2\varphi} + 7(\partial_\theta \beta_0)^2 - 3 \partial_\phi \varphi \partial_\theta \beta_0 + (\varphi - 2\beta_0) \partial_\theta \beta_0 \partial_\phi \varphi - 2(\varphi - 2\beta_0)(\partial_\phi \beta_0)^2 + (\varphi - \beta_0) \partial_\phi^2 \beta_0 - \varphi + 2 \partial_\phi^2 \beta_0), \]
\[ \tilde{V}_0 = \frac{V_0}{2} e^{2\varphi - 2\beta_0} + 4(\partial_\phi \beta_0)^2 - 2 \partial_\phi \beta_0 \partial_\phi \varphi + \frac{1}{2} (\partial_\phi \varphi)^2 + \partial_\phi^2 (\varphi - \beta_0) + \frac{2}{\mu} (-\partial_\phi U_0 \partial_\phi^2 e^{-2\beta_0 + \varphi} + \partial_\phi^3 U_0 (\varphi - \beta_0) - \varphi \partial_\phi U_0 \partial_\phi \beta_0). \] (49)

As can be seen there are only two independent charges.

According to the theorem of the covariant phase space method, the algebra of charges is the same as the algebra of symmetry generators up to central terms. By using the explicit form of the charges and also using Dirichlet boundary conditions and [19], [20], [21], one can compute the charge algebra as follows:

\[ \delta_{\xi_2} Q_{\xi_1} \equiv \{ Q_{\xi_1}, Q_{\xi_2} \} = Q_{\{\xi_1, \xi_2\}}, \]
where \([\xi_1, \xi_2]\) is given by [40] and \(C_{\xi_1, \xi_2}\) is the central extension
\[ C_{\xi_1, \xi_2} = \frac{1}{8\pi G} \int_0^{2\pi} d\phi \left( \partial_\phi \tilde{f}_1 \partial_\theta \tilde{Y}_2 - \partial_\phi \tilde{f}_2 \partial_\theta \tilde{Y}_1 \right) + \frac{1}{16\pi G \mu d^2} \int_0^{2\pi} d\phi \left( \partial_\phi \tilde{f}_2 \partial_\phi \tilde{f}_1 - \partial_\phi \tilde{f}_1 \partial_\phi \tilde{f}_2 \right). \] (51)

We should mention that this is the first time that the central extension with the gravitational anomaly of TMG appears for such weak boundary conditions we have considered in this paper.

As can be seen from [51], there are two kinds of central extension, the first term arises from the Einstein term and the other one from the CS term and appears in the Virasoro part. Defining
\[ \tilde{f} = \frac{l}{2} (Y^+ + Y^-), \quad \tilde{Y} = \frac{1}{2} (Y^+ - Y^-) \] (52)
and using the decomposition in modes \(Y^\pm = \Sigma Y^\pm_m t^\pm_m, \ t^\pm_m = e^{\pm imx^\pm} \) and writing \(L^\pm = Q_{\xi(t^\pm_m)}\), the charge algebra becomes
\[ i\{L_1^\pm, L_n^\pm\} = (m - n)L_{m+n}^\pm - \frac{c^\pm}{12} m^3 \delta_{m+n,0}, \quad \{L_1^\pm, L_n^\pm\} = 0, \] (53)
where \(c^\pm = \frac{3l}{2G} \pm \frac{3}{2\mu G}\) which is the Brown–Henneaux central charges.

3 Conclusion

In this work, we construct the maximal asymptotic symmetry algebra that one will get in topological massive 3D gravity by imposing partial gauge fixing on the components of the metric and very mild falloffs. After a renormalization procedure of the action and the symplectic structure involving covariant counter-terms, we obtain finite charge expressions that we present integrable through a field-dependent redefinition of the symmetry parameters. Then, we find the appropriate field-dependent redefinition of the parameters to exhibit the charges integrable. Finally, we have obtained the charge algebra which involves the Brown–Henneaux central extension in asymptotically AdS_3 spacetimes. This is the first time that the central extension with the gravitational anomaly of TMG appears for such weak boundary conditions we have considered in this paper.
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A Constrained field equations

The constrained components of field equations are presented as follows:

$$E_{u_{\phi}} = \frac{1}{8\mu^4} \left[ 6\mu^5 \partial_u U(u, r, \phi)e^{-6\beta_0 + 3\varphi} (2\partial_r U(u, r, \phi) + r\partial_r^2 U(u, r, \phi)) - 4\mu^4 e^{-4\beta_0 + 2\varphi} (3\partial_r U(u, r, \phi)$$
$$+ r\partial_r^2 U)(6r\partial_r V - 3r^2\partial_r^2 V - 6V + r^3\partial_r^3 V)e^{-4\beta_0 + \varphi} - 8\mu^2 \partial_\phi \beta_0 e^{-2\beta_0} \right] = 0,$$

(54)

and

$$E_{r_{\phi}} = \frac{1}{8\mu^4} \left[ -6r^6 \partial_r U e^{-6\beta_0 + 3\varphi} (U\partial_{\phi}^2 U + \partial_{\phi}^2 U + \partial_r U(-U\partial_{\phi}\beta_0 + \partial_u \phi + U\partial_\phi \phi - 2\partial_u \beta_0 + 2\partial_\phi U))$$
$$+ 4\mu^3 e^{-4\beta_0 + 2\varphi} (U\partial_{r_{\phi}}^2 U + \partial_{r_{\phi}}^2 U + \partial_r U(-2U\partial_{\phi}\beta_0 + 3\partial_u \phi - 2\partial_u \beta_0 + 3U\partial_\phi \phi)) + e^{-2\beta_0 + \varphi} (2\partial_r^3 V$$
$$- 2r\partial_r^2 V(\partial_\phi \phi - \partial_\phi \beta_0) - 6\beta_0^2 V + 6\partial_r V(\partial_\phi \phi - \partial_\phi \beta_0) - 4\mu e^{-2\beta_0}(r - r\partial_r^2 V - 2\partial_\phi \beta_0$$
$$- 2\partial_r \partial_\phi \phi + 2\partial_\phi \beta_0 (U\partial_r^2 \phi + r^2 \partial_\phi \phi + V)) + re^{-4\beta_0 + \varphi} (r(12r^2 \partial_r U(\partial_\phi \phi - \partial_\phi \beta_0) +$$
$$9\partial_\phi V + r^2 U(\partial_\phi \beta_0)^2 + \partial_\phi \beta_0 - 2U^2 \partial_r \phi + r^2 \partial_\phi \beta_0 - 4V) + r^2 U(\partial_\phi \phi)^2 + \partial_\phi \beta_0 (4\mathbf{V} -$$
$$8r^2 \partial_u \beta_0 + 8\partial_u \phi)\partial_\phi \beta_0 U - 12u U - r^2 r_{\phi} V(\partial_\phi \phi - 4U) + 12\partial_r \partial_\phi \phi U(\partial_\phi \phi - 4U^2 \partial_\phi \phi -$$
$$2\partial_\phi \beta_0 + 2\partial_u \phi + 2r^2 \partial_\phi \phi + U^2 \partial_\phi \phi - 2\partial_r \partial_\phi \phi + 2\partial_\phi \beta_0 (U\partial_\phi \phi + r^2 \partial_\phi \phi + V)) + 4^3 \partial_\phi \beta_0 + 2U^3 \partial_\phi \beta_0 V + r^3 \partial_r \partial_\phi \phi$$
$$- 2r^2 \partial_\phi \beta_0 + 2\partial_r \partial_\phi \phi + 2r^2 \partial_\phi \phi + V) + 4r^3 \partial_\phi \beta_0 + 2U^3 \partial_\phi \beta_0 V + r^3 \partial_r \partial_\phi \phi$$
$$\partial_\phi \beta_0 U - 4r^3 \partial_{u_{\phi}} \beta_0 \partial_\phi \beta_0 U + 4r^3 \partial_{u_{\phi}} \beta_0 \partial_\phi \beta_0 + 2U^2 \partial_{r_{\phi}} \partial_\phi \phi - 2r^2 \partial_\phi \phi \partial_\phi \beta_0$$
$$+ \partial_\phi \beta_0 - \partial_\phi \beta_0 + \partial_\phi \beta_0) + r^2 \partial_\phi \beta_0 (3\partial_\phi V + (V(\partial_\phi \phi - \partial_\phi \beta_0)) + 12UV(\partial_\phi \beta_0)\right].$$

(55)

B Explicit terms of Lagrangian

Here we provided the explicit terms of Lagrangian [29].

$$L_{\text{bulk}} = -\frac{e^{2\beta_0 + \varphi} R^2}{8\pi G \mu^2} + \frac{(2e^{2\beta_0 \partial_\phi \beta_0} + (U_0 \partial_\phi U_0 + \phi_u U_0)R^2 e^{-2\beta_0 + 2\varphi})}{8\pi G \mu^2} + \frac{1}{16\pi G \mu} (U_0 \partial_{\phi_{\beta_0}}^2 \beta_0$$
$$+ \partial_\phi U_0 \partial_\phi \beta_0 + \partial_{u_{\phi}}^2 \beta_0) \ln(R) + O \left( \frac{1}{R} \right),$$

(56)

$$L_{\text{G_{\text{HY}}} = -\frac{e^{2\beta_0 + \varphi} R^2}{4\pi G \mu^2} - \frac{e^\varphi}{4\pi} (\partial_\phi U_0 + \partial_\phi \phi + U_0 \partial_\phi \phi) R + \frac{1}{8\pi G} [e^{2\beta_0 - \varphi} (4(\partial_\phi \beta_0)^2 + 2\partial_\phi \beta_0 - 2\partial_\phi \beta_0 \partial_\phi \phi)$$
$$+ l^2 e^{2\beta_0 - \varphi} (-U_0^2 \partial_{u_{\phi}}^2 \phi - U_0 \partial_\phi U_0 - 2\partial_{u_{\phi}}^2 U_0 - \partial_{u_{\phi}} \beta_0) - \partial_\phi \phi + 2\partial_\phi \beta_0 (U_0 \partial_u \phi + U_0 \partial_\phi U_0 + U_0^2 \partial_\phi \phi)$$
$$+ \partial_{u_{\phi}} \phi (2U_0 \partial_u \beta_0 - \partial_u U_0 - U_0 \partial_\phi U_0) + 2\partial_\phi \beta_0 (U_0 \partial_u \phi + \partial_\phi U_0)) + \partial_\phi \phi \phi \phi = \partial_\phi \beta_0) + V_0 e^\varphi] + O \left( \frac{1}{R} \right),$$

(57)

$$L_{\text{ct}} = -\frac{e^{2\beta_0 + \varphi} R^2}{8\pi G \mu^2} - \frac{e^\varphi}{8\pi G} (\partial_\phi U_0 + \partial_\phi \phi + \partial_\phi U_0) R + \frac{e^{2\beta_0 - \varphi} l^2 U_0^2 (\partial_\phi \phi)^2 + 2l^2 U_0 \partial_\phi \phi (\partial_\phi \phi + \partial_\phi U_0)$$
$$+ l^2 (\partial_\phi \phi)^2 + 2l^2 \partial_\phi \phi \partial_\phi U_0 + l^2 (\partial_\phi U_0)^2 + V_0 e^{2\beta_0}) + O \left( \frac{1}{R} \right).$$

(58)
The renormalized presymplectic potential as

\[ L_0 = \frac{e^\varphi(\partial_u \varphi + U_0 \partial_u \varphi + \partial_u U_0)}{8\pi G} + e^{2\beta_0 - \varphi}(\partial^2_{\varphi \beta} \beta_0 - \partial_u \varphi \partial_u \beta_0 + 2(\partial_u \beta_0)^2) + O \left( \frac{1}{R} \right), \quad (59) \]

\[ L_b = - \frac{1^2 e^{\varphi - 3\beta_0}(\partial_u \varphi + U_0 \partial_u \varphi + \partial_u U_0)^2}{16\pi G} + O \left( \frac{1}{R} \right), \quad (60) \]

\[ L_R = \frac{1}{8\pi G} \left[ e^{2\beta_0 - \varphi}(-4(\partial_u \beta_0)^2 + 2\partial_u \beta_0 \partial_u \varphi - 2\varphi^2 \beta_0) + \frac{1}{8\pi G}(U_0^2 \partial^2_{\varphi \beta} \beta_0 + U_0 \partial^2_u \varphi + 2U_0 \partial^2_{u \varphi} \varphi + \partial^2_u U_0 + \partial^2_u \varphi + U_0^2 (\partial_u \varphi)^2 + \partial_u \varphi (-2U_0 \partial_u \beta_0 + 3U_0 \partial_u U_0 + 2U_0 \partial_u \varphi - 2\partial_u \beta_0 U_0^2) \right. \]
\[ - 2\partial_u \beta_0 \partial_u \varphi + \partial_u \beta_0 \partial_u U_0) + (\partial_u U_0)^2 + 2\partial_u U_0 (\partial_u \varphi - \partial_u \beta_0) + (\partial_u \varphi)^2 - 2\partial_u \beta_0 \partial_u \varphi \right] + O \left( \frac{1}{R^2} \right); \quad (61) \]

\[ L_{ccs} = 2e^{2\beta_0} \partial_u \beta_0 + l^2 e^{2\varphi - 3\beta_0} (U_0 \partial_u U_0 + \partial_u U_0) \quad (62) \]

\[ L_{ccs} = \partial^2_{u \varphi} \beta_0 + \partial_u \beta_0 \partial_u U_0 + U_0 \partial^2_{\varphi \varphi} \beta_0 \quad (63) \]

The boundary Lagrangian (62) is given as

\[ \bar{L}_0 = \frac{e^{-4\beta_0 - \varphi}}{8\pi G} \left( l^2 e^{2\varphi}((U_0 + U_0^3) \partial_u \varphi + (1 + 3U_0^2) \partial_u U_0 - 4(U_0 + U_0^3) \partial_u \beta_0 - 4(1 + U_0^2) \partial_u \beta_0 \right. \]
\[ + \partial_u U_0^2 + \partial_u \varphi + U_0^2 \partial_u \varphi \right) + e^{4\beta_0} (U_0 \partial_u \varphi - \partial_u U_0). \quad (64) \]

\[ \Theta_{ren}^\varphi(g, \delta g) = \frac{1}{16\pi G} \left[ V_0 e^{\varphi} \delta(\varphi - 2\beta_0) + 2N e^{\varphi} \delta U_0 + 2e^{2\beta_0 - \varphi} (6\partial_u \beta_0 \partial_u \delta \beta_0 - \partial_u \varphi \partial_u \delta \beta_0 + \partial^2_u \delta \beta_0) \right] \]
\[ + \frac{1}{64\pi \mu G} \left[ -2V_0 e^{2\varphi - 2\beta_0} \delta U_0 - 2U_0 \partial^2_{\varphi \beta} \beta_0 + 2U_0 \partial^2_{u \varphi} \varphi - 2\partial_u \delta \beta_0 + 2\partial^2_u \delta U_0 + 2\partial^2_u \delta \varphi - \partial_u \delta \beta_0 (-4U_0 \partial_u \varphi - 2\partial_u \beta_0 + \partial^2_u \beta_0 (8U_0 \delta \varphi - 8U_0 \delta \beta_0 + 6\delta U_0) + \partial^2_u \varphi (-4U_0 \delta \varphi \right. \]
\[ + 4U_0 \delta \beta_0 + 2\delta U_0) + \partial^2_u \varphi (4U_0 \partial_u \beta_0 - 2\delta_u \varphi + 2\delta_u U_0) + 8\partial^2_u \beta_0 \delta(\varphi - \beta_0) - 4\partial^2 U_0 \delta(\varphi - \beta_0) \right. \]
\[ - 4\partial^2_{u \varphi} \varphi (\varphi - \beta_0) + \partial_u \delta \varphi (2\partial_u \varphi + 4\partial_u \beta_0) + \partial_u \delta \beta_0 + \partial_u \delta \beta_0 + \partial_u = \delta(\varphi - \beta_0) \]
\[ + \partial_u \delta(\varphi - 4U_0 \partial_u \beta_0 + 2\delta_u \varphi + 6\delta_u U_0) \right) + \delta \varphi (4U_0 \partial_u \beta_0 - 2\delta_u \varphi - 6\delta_u U_0) + 2\partial_u \beta_0 \delta U_0) - \right. \]
\[ 4\partial_u \beta_0 \delta \beta_0 (\partial_u \varphi + 2\delta_u U_0) + 4\partial_u \beta_0 \delta \varphi (\partial_u \varphi + 2\delta_u U_0) - 16U_0 (\partial_u \beta_0)^2 \right] + O \left( r^{-1} \right), \quad (65) \]
the associated presymplectic current reads as

\[\begin{align*}
w_{\text{ren}}^r(g, \delta_1 g, \delta_2 g) &= \frac{1}{16\pi G} (\partial_2 (V_0 e^{\varphi}) \delta_1 (\varphi - 2 \beta_0) + 2 \delta_2 (N e^{\varphi}) \delta_1 U_0 - 2 e^{2\beta_0 - \varphi} \partial_\varphi \delta_2 \varphi \partial_\varphi \delta_1 \beta_0 + \\
&2 e^{2\beta_0 - \varphi} \partial_2 (\delta_2 (\varphi - \beta_0)) (6 \partial_\varphi \delta_1 \beta_0 - \partial_\varphi \partial_\varphi \delta_1 \beta_0 + \partial_\varphi^2 \delta_1 \beta_0) + \\
&\frac{1}{64\pi G \mu} [-(2 \delta_2 (V_0 e^{2\varphi - 2\beta_0} \delta_1 U_0) - 2 \delta_2 U_0 \delta_2^2 \delta_1 \beta_0 + 2 \delta_2 U_0 \delta_2^2 \delta_1 \varphi + \\
&(4 \delta_2 (U_0 \partial_\varphi \varphi) - 2 \partial_\varphi \delta_2 \varphi - 2 \partial_\varphi \delta_2 U_0) \partial_\varphi \delta_1 \beta_0 + 8 \delta_2 (U_0 \delta_2^2 \beta_0) \delta_1 \varphi - 8 \delta_2 (U_0 \delta_2^2 \beta_0) \delta_1 \beta_0 \\
&- 6 \delta_2^2 \delta_2 \beta_0 U_0 + 4 \delta_2 (U_0 \delta_2^2 \beta_0) \delta_1 \varphi + 4 \delta_2 U_0 \delta_1 \beta_0 + \partial_\varphi^2 \delta_1 \varphi (4 \delta_2 (U_0 \partial_\varphi \beta_0) - 2 \partial_\varphi (\partial_\varphi \varphi) + \\
&2 \delta_2 (\partial_\varphi U_0) + 8 \delta_2 (\partial_\varphi \beta_0) \delta_1 (\varphi - \beta_0) - 4 \delta_2 (\partial_\varphi \beta_0) \delta_1 (\varphi - \beta_0) - 4 \delta_2 (\partial_\varphi^2 \varphi) \delta_1 (\varphi - \beta_0) \\
&+ \delta_1 (\partial_\varphi \beta_0) \delta_1 U_0 + \delta_1 \delta_2 \varphi \delta_1 \beta_0 (4 \delta_2 U_0 \delta_2 \beta_0 + 2 \partial_\varphi \varphi + 6 \partial_\varphi U_0) + \partial_\varphi \varphi \delta_1 \beta_0 (4 \delta_2 U_0 \delta_2 \beta_0 + \\
&2 \partial_\varphi \varphi + 6 \partial_\varphi U_0) + \partial_\varphi \varphi \delta_1 \beta_0 (4 \delta_2 U_0 \delta_2 \beta_0 - 2 \partial_\varphi \varphi + 6 \partial_\varphi U_0) + 2 \partial_\varphi \varphi \delta_1 \beta_0 (2 \partial_\varphi \varphi + 6 \partial_\varphi U_0) - \\
&4 \partial_\varphi \beta_0 \delta_1 \beta_0 (\partial_\varphi \delta_2 \varphi + 2 \partial_\varphi U_0) + 4 \partial_\varphi \delta_2 \varphi \delta_1 \varphi (\varphi + 2 \partial_\varphi U_0) + 4 \partial_\varphi \beta_0 \delta_1 \varphi (\varphi + 2 \partial_\varphi U_0) \\
&- 16 \delta_1 U_0 \delta_2 ((\partial_\varphi \beta_0)^2)] - (1 \leftrightarrow 2) + \mathcal{O}(r^{-1}).
\end{align*}\]

The explicit expression for the infinitesimal charges \([41]\) reads as

\[\begin{align*}
\delta Q_{\xi}[g] &= \int_0^{2\pi} \frac{d\varphi}{\pi} \mathcal{I}_{\text{ren}}[g; \delta g] = \frac{1}{8\pi G} \int_0^{2\pi} d\varphi [Y \delta (N e^{\varphi}) + \delta (e^{2\beta_0 - \varphi} \partial_\varphi f) (\delta_1 \beta_0 - \varphi) + f \frac{1}{2} \delta V e^{\varphi} \\
&- V e^{\varphi} \delta (\beta_0 - \varphi) - U_0 \delta (N e^{\varphi}) + 2 e^{2\beta_0 - \varphi} (6 \partial_\varphi \beta_0 \partial_\varphi \beta_0 - \partial_\varphi \varphi \partial_\varphi \beta_0 + \partial_\varphi^2 \delta_1 \beta_0)] + \frac{1}{4\pi G \mu} [ \\
&(Y - f U_0) \partial_\varphi^2 \delta_1 \beta_0 + ((7Y - 6f U_0) \partial_\varphi \beta_0 + (f U_0 - \frac{3}{2} Y) \partial_\varphi \varphi + \frac{1}{2} \partial_\varphi f - \partial_\varphi Y - w + 2U_0 \partial_\varphi f \\
&+ 2 \partial_\varphi \beta_0 f - f \partial_\varphi \varphi) \partial_\varphi^2 \beta_0 + (Y - 2f U_0) \partial_\varphi^2 \beta_0 - \frac{1}{2} (Y - 2f U_0) \partial_\varphi^2 \varphi - f \partial_\varphi \beta_0 + f \partial_\varphi U_0 \\
&+ \frac{f}{2} \partial_\varphi^2 \beta_0 - \frac{1}{2} \partial_\varphi^2 \varphi + (2Y (\partial_\varphi \beta_0)^2 - \partial_\varphi \beta_0 (2 \partial_\varphi Y - 2f \partial_\varphi U_0 + Y \partial_\varphi \varphi - \partial_\varphi f + f \partial_\varphi \varphi \\
&- 2 \partial_\varphi \beta_0 - 4U_0 \partial_\varphi f + w) - \partial_\varphi \varphi (U_0 \partial_\varphi f - \frac{1}{2} \partial_\varphi Y - f \partial_\varphi U_0) - \partial_\varphi \varphi \partial_\varphi U_0 - \frac{1}{2} \partial_\varphi w - \frac{1}{2} \partial_\varphi \varphi \partial_\varphi \varphi \\
&+ \partial_\varphi f \partial_\varphi \varphi) \delta_1 \beta_0 + \frac{\delta_1 \beta_0}{2} (-2Y (\partial_\varphi \beta_0)^2 + \partial_\varphi \beta_0 (-2 \partial_\varphi \beta_0 f - 4f \partial_\varphi U_0 + Y \partial_\varphi \varphi + 3 \partial_\varphi Y + w + f \partial_\varphi \varphi \\
&- 6U_0 \partial_\varphi f - \partial_\varphi f + \partial_\varphi \varphi (2U_0 \partial_\varphi f - \partial_\varphi Y + 2f \partial_\varphi U_0) + 2 \partial_\varphi f \partial_\varphi U_0 + \partial_\varphi w + \partial_\varphi f \partial_\varphi \varphi - 2 \partial_\varphi f \partial_\varphi \varphi) + \\
&(- V_0 \delta \beta_0 + \frac{1}{2} \delta V_0 + V_0 \delta \varphi) (Y - f U_0) e^{2\varphi - 2\beta_0}].
\end{align*}\]

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