DEFINITION AND CHARACTERIZATION
OF
SUPERSMOOTH FUNCTIONS ON SUPERSPACE
BASED ON FRÉCHET-GRASSMANN ALGEBRA

ATSUSHI INOUE

Dedicated to the memory of late Professor N. Suita

Abstract. Preparing the Fréchet-Grassmann (FG-)algebra $\mathcal{A}$ composed with countably infinite Grassmann generators, we introduce the superspace $\mathcal{R}^{m|n}$. After defining Grassmann continuation of smooth functions on $\mathcal{R}^m$ to those on $\mathcal{R}^{m|0}$, we introduce a class of functions which are called supersmooth (alias superfields) and are regarded as one of those with countably infinite independent variables. In this paper, we characterize such supersmooth functions in Gâteaux (but not necessarily Fréchet) differentiable category in Fréchet but not in Banach space. This type of arguments for $G^\infty$-functions is done in the Banach-Grassmann (BG-)algebra, but we find it rather natural to work within FG-algebra when we treat systems of PDE such as Dirac equation. Though we took this point of view in our previous works, but is insufficiently managed.

Contents

1. Introduction 2
2. Preliminaries and Problem 3
2.1. A construction of countable Grassmann generators à la Rogers 3
2.2. Superalgebra and Superspace 4
2.3. A class of functions 6
2.4. Problem 6
3. Elementary Differential calculus on Banach or Fréchet spaces 8
3.1. Gâteaux differentiability 8
3.2. Fréchet differentiability 11
4. The case with finite Grassmann generators 12
4.1. Finite dimensional Grassmann algebras 13
4.2. $\mathcal{B}_L$ is not self-dual 14
4.3. Superdifferentiable functions on $\mathcal{B}_L$ 14
5. The definition and characterization of supersmooth functions on FG-algebra 20
5.1. Remarks on FG-algebras 20
5.2. $\mathcal{C}$-valued functions and superdomains 21
5.3. Differentiability 21
5.4. Examples of superdifferentiable functions 22
5.5. Cauchy-Riemann relation 27
5.6. Proof of Main Theorem 30
References 31

Date: October 20, 2009.
1991 Mathematics Subject Classification. Primary 58C50, 46S10; Secondary 58A50, 17A01.
Key words and phrases. superspace, Cauchy-Riemann equation, Grassmann algebra, infinite independent variables.
1. INTRODUCTION

In order to treat “photon” and “electron” on the same footing as is proposed in Berezin and Marinov [3], there are many trials to extend the fundamental fields $\mathbb{R}$ or $\mathbb{C}$ to those such as De Witt algebra $\Lambda_\infty$, Rogers’ Banach-Grassmann(BG-)algebra $\mathfrak{B}_\infty$ or Fréchet-Grassmann(FG-)algebra (such as [31], [10] but we use $\mathbb{R}$ or $\mathbb{C}$ explained below. We don’t refer $\Lambda(\infty)$ of [32] because the usage of nuclearity there isn’t clear for the time being). On such extended “field”, we need to develop elementary and real analysis for treating what we have done over $\mathbb{R}^m$ or $\mathbb{C}^m$.

Not only above mentioned reason from mathematical physics but also to treat systems of PDE without diagonalizing matrix structures, we need, so-called, odd variables. For example, Feynman proposed the following in p.355 of Feynman and Hibbs [13], since ‘spin’ has been the object outside Feynman’s procedures at that time: (underlined by the author)

\[ \cdots \text{path integrals suffer grievously from a serious defect. They do not permit a discussion of spin operators or other such operators in a simple and lucid way. They find their greatest use in systems for which coordinates and their conjugate momenta are adequate. Nevertheless, spin is a simple and vital part of real quantum-mechanical systems. It is a serious limitation that the half-integral spin of the electron does not find a simple and ready representation. It can be handled if the amplitudes and quantities are considered as quaternions instead of ordinary complex numbers, but the lack of commutativity of such numbers is a serious complication.} \]

On the other hand, a physicist Witten [41] introduced the notion of supersymmetric quantum mechanics to mathematicians by re-interpreting Morse theory. That is, deforming the form Laplacian by Morse function and getting it as the infinitesimal generator of heat flow type corresponding to the Lagrangian represented by “odd variables”, he applied rather naively the asymptotic method to the path-integral representation of the heat flow to get the Morse inequality. Though his procedure is beautiful and persuading, but there exists no mathematical theory to make rigorous his argument directly, because there exists not only no Feynman measure (i.e. roughly speaking, no integration theory based on Lebesgue-like measure in $\infty$-dimensional space) but also the lack of the consistent theory including even and odd variables on equal footing.

In §2, we explain our problem after preparing a concrete countable Grassmann generators à la Rogers. Define FG-algebra $\mathfrak{R}$ or $\mathfrak{C}$ and superspace $\mathfrak{R}^{m|n}$, we introduce Grassmann continuation of ordinary smooth functions and define supersmooth functions (alias superfields by physicists).

It is well-known that the elementary differential calculus in Euclidean spaces is extended straight forwardly to those in Banach spaces but not so in Fréchet spaces. As a typical example, we have, though the dual of a Banach space is again a Banach space, but the dual of a Fréchet space is not necessarily a Fréchet space. Therefore, in §3, following Hamilton [14], we enumerate a part of elementary differential calculus in Banach and Fréchet spaces.

In §4, we recall the results when the number of Grassmann generators is finite. It seems appropriate to mention here that though not only Lemma 2.2 of Vladimirov and Volovich [40] but also Lemma 1.7 of Boyer and Gitler [7] contain unsatisfactory arguments, but their conclusions hold true. This point is
clarified with the aid by Kazuo Masuda. Moreover, Rogers [36, 38] does not remark the Cauchy-Riemann
relation which should be satisfied for her $G^\infty$ functions.

In §5, using the fact that not only $\mathfrak{B}_\infty$ but also $\mathcal{C}$ (or $\mathcal{R}$) are self-dual, we answer affirmatively to our
problem mentioned in §2. Though the precise definitions such as superdifferentiability, supersmoothness,
etc. will be given later, we have

**Theorem 1.1.** Let $\mathcal{U}$ be an open set in $\mathfrak{R}^m|n$ and let a function $f : \mathcal{U} \to \mathcal{C}$ be given. Following conditions
are equivalent:

(a) $f \in \mathcal{G}_{\mathfrak{S}_d}(\mathcal{U} : \mathcal{C})$, i.e. $f$ is superdifferentiable on $\mathcal{U}$,
(b) $f$ is $\infty$-times Gâteaux (G-, in short) differentiable and $f \in \mathcal{G}_{\mathfrak{S}_d}(\mathcal{U} : \mathcal{C})$,
(c) $f$ is $\infty$-times G-differentiable and its G-differential $df$ is $\mathfrak{R}_\infty$-linear,
(d) $f$ is $\infty$-times G-differentiable and its G-differential $df$ satisfies Cauchy-Riemann equations,
(e) $f$ is supersmooth, i.e. it has the following representation, called superfield expansion, such that

$$f(x, \theta) = \sum_{|\alpha| \leq n} \theta^\alpha \tilde{f}_\alpha(x) \quad \text{with} \quad f_\alpha(q) \in C^\infty(\pi_B(\mathcal{U})) \quad \text{and} \quad \tilde{f}_\alpha(x) = \sum_{|\alpha| = 0}^{\infty} \frac{1}{\alpha!} \frac{\partial^\alpha f(q)}{\partial q^\alpha} \bigg|_{q=x_B} x_0^\alpha.$$

**Remark 1.1.** We should mention that this theorem is almost proved in Yagi [32] without (c). Moreover, we remark that the definition of the Z-expansion there is slightly different from our Grassmann
continuation in §2.

**Remark 1.2.** Concerning Feynman’s problem mentioned above, as we need to define the Hamilton function
and to solve Hamilton-Jacobi equation corresponding to the Dirac equation or the systems of PDE. Our solution of these problem is affirmative, see for example, Inoue [17, 18, 19]. This is based on the fact
that any $2^d \times 2^d$-matrix is decomposed by matrices satisfying Clifford relations and the Clifford algebra has the
differential operator representation on the Grassmann algebras. But this decomposition of matrices
doesn’t work directly for systems with sizes $3 \times 3$, $5 \times 5$ etc. Seemingly to treat those cases, we need new
class of non-commutative numbers and analysis on it.

2. Preliminaries and Problem

2.1. A construction of countable Grassmann generators à la Rogers. In this subsection, we use
the lexicographic representation for multiple indeces. Denote by $\mathcal{M}_L$ the set of integer sequences given by

$$\mathcal{M}_L = \{ \mu \mid \mu = (\mu_1, \mu_2, \ldots, \mu_k), 1 \leq \mu_1 < \mu_2 < \cdots < \mu_k \leq L \} \quad \text{and} \quad \mathcal{M}_\infty = \cup_{L=1}^\infty \mathcal{M}_L.
$$

We put $(j) \in \mathcal{M}_\infty$, for any $j \in \mathbb{N}$. For each $r \in \mathbb{N}$, we may correspond a member $\mu \in \mathcal{M}_\infty$ by using

$$r = \frac{1}{2} (2^{\mu_1} + 2^{\mu_2} + \cdots + 2^{\mu_k}) = r(\mu) \leftrightarrow \mu = (\mu_1, \cdots, \mu_k) = \mu(r).
$$

Regarding $\emptyset \in \mathcal{M}_L$, we put $e_\emptyset = 1$ and for each $\mu \in \mathcal{M}_\infty$, we define $e_\mu$ as $e_\mu = e_{(r(\mu))} = (\overbrace{0, \cdots, 0, 1, 0, \cdots}^{r-1}) \in \ell_\infty \cap \ell_1$ where $r$ and $\mu$ are related by (2.2). Then, we identify

$$w \ni w = (w_1, w_2, w_3, w_4, \cdots) = \sum_{r=1}^\infty w_r e_r \leftrightarrow (w(1), w(2), w(1, 2), w(3), \cdots) = \sum_{\mu \in \mathcal{M}} w_\mu e_\mu.$$
Now, we introduce the multiplication by
\[
\begin{align*}
\sigma \sigma = \sigma \sigma & = \sigma \sigma \quad \text{for } \mu \in \mathcal{M}, \\
\sigma (j) \sigma (j) = -\sigma (j) \sigma (i) & \quad \text{for } i, j \in \mathbb{N}, \\
\ell = \sigma (\mu \downarrow) \ell & \quad \text{where } \mu = (\mu_1, \mu_2, \ldots, \mu_k).
\end{align*}
\]
Then, putting \( \sigma (j) = \sigma (j) \) for \( j \geq 1 \), we have a family of Grassmann generators \( \{ \sigma (j) \}_{j=1}^{\infty} \).

2.2. **Superalgebra and Superspace.** Regarding the above constructed one as an example, we prepare a countable number of letters \( \{ \sigma (j) \}_{j=1}^{\infty} \) with multiplication and addition satisfying
\[
\sigma (j) \sigma (j) + \sigma (j) \sigma (i) = 0 \quad \text{for } i, j = 1, 2, \ldots.
\]
Denoting these letters as Grassmann generators, we put form ally
\[
\mathcal{C} = \{ X = \sum_{I \in \mathcal{I}} X_I \sigma (I) \mid X_I \in \mathcal{C} \}
\]
where
\[
\mathcal{I} = \{ I = (I_k) \in \{0, 1\}^\mathbb{N} \mid |I| = \sum_k i_k < \infty \},
\]
with \( \sigma (I) = \sigma (i_1) \sigma (i_2) \sigma (j) \cdots \), \( \sigma (0) = 1 \), \( \mathcal{I} = (i_1, i_2, \cdots) \), \( \mathcal{I} = (0, 0, \cdots) \in \mathcal{I} \).
For the notational simplicity, we put also
\[
\mathcal{I}_0 = \{ I \in \mathcal{I} \mid |I| = \text{ev} \}, \quad \mathcal{I}_1 = \{ I \in \mathcal{I} \mid |I| = \text{od} \}.
\]
Besides trivially defined linear operations of sums and scalar multiplications, we have a product operation in \( \mathcal{C} \): For
\[
X = \sum_{I \in \mathcal{I}} X_I \sigma (I), \quad Y = \sum_{K \in \mathcal{I}} Y_K \sigma (K),
\]
we put
\[
XY = \sum_{I \in \mathcal{I}} (XY)_I \sigma (I) \quad \text{with } (XY)_I = \sum_{I+J+K} (-1)^{\tau(I+J,K)} X_J Y_K.
\]
Here, \( \tau(I,J,K) \) is an integer defined by
\[
\sigma (I) \sigma (K) = (-1)^{\tau(I+J,K)} \sigma (I), \quad I = J + K,
\]
which is not necessary to specify more concretely.

Identifying \( \mathcal{C} \) with the sequence space \( \omega \) of Köthe \[27\], we have

**Proposition 2.1.** \( \mathcal{C} \) forms an \( \infty \)-dimensional FG-algebra over \( \mathbb{C} \), that is, an associative, distributive and non-commutative ring with degree, which is endowed with the Fréchet topology.

**Remark 2.1.** For the proof, see, Inoue and Maeda \[21\]. By the way, I want to know the reason why Pestov wrote in p.278 of \[33\] “The DeWitt supernumber algebra \( \Lambda \infty \) was implicitly topologized, in fact, by DeWitt himself”. Because, DeWitt himself wrote in p.3 of his book \[12\] “In the formal limit \( N \to \infty \), they (i.e. \( \Lambda_N \) etc.) may continue to be regarded as vector spaces, but we shall not give them a norm or even a topology.”

**Remark 2.2.** (1) Degree in \( \mathcal{C} \) is defined by introducing subspaces
\[
\mathcal{C}^{(0)} = \{ X = \sum_{I \in \mathcal{I}_0} X_I \sigma (I) \} \quad \text{for } j = 0, 1, \cdots
\]
which satisfy
\[
\mathcal{C} = \mathcal{C}^{\infty}, \quad \mathcal{C}^{(j)} \cdot \mathcal{C}^{(k)} \subset \mathcal{C}^{(j+k)}.
\]
(2) Define
\[ \text{proj}_I(X) = X_I \quad \text{for} \quad X = \sum_{I \in \mathcal{I}} X_I \sigma^I \in \mathcal{C}. \]

The topology in $\mathcal{C}$ is given by $X \to 0$ in $\mathcal{C}$ if and only if $\text{proj}_I(X) \to 0$ in $\mathcal{C}$ for any $I \in \mathcal{I}$.

This topology is equivalent to the one introduced by the metric $\text{dist}(X, Y) = \text{dist}(X - Y)$ where $\text{dist}(X)$ is defined by
\[ \text{dist}(X) = \sum_{I \in \mathcal{I}} \frac{1}{2^r(I)} \frac{|\text{proj}_I(X)|}{1 + |\text{proj}_I(X)|} \text{ with } r(I) = 1 + \sum_{k=1}^{\infty} 2^{kI_k} \text{ for } I \in \mathcal{I}. \]

(3) We introduce parity in $\mathcal{C}$ by setting
\[ p(X) = \begin{cases} 0 & \text{if } X = \sum_{I \in \mathcal{I}_0} X_I \sigma^I, \\ 1 & \text{if } X = \sum_{I \in \mathcal{I}_1} X_I \sigma^I, \\ \text{undefined} & \text{if otherwise}. \end{cases} \]

We put
\[ \mathcal{C}_{ev} = \bigoplus_{j=0}^{\infty} \mathcal{C}^{[2j]} = \{ X \in \mathcal{C} | p(X) = 0 \}, \]
\[ \mathcal{C}_{od} = \bigoplus_{j=0}^{\infty} \mathcal{C}^{[2j+1]} = \{ X \in \mathcal{C} | p(X) = 1 \}, \]
\[ \mathcal{C} \cong \mathcal{C}_{ev} \oplus \mathcal{C}_{od} \cong \mathcal{C}_{ev} \times \mathcal{C}_{od}. \]

We introduced the body (projection) map $\pi_B$ by
\[ \pi_B X = \text{proj}_0(X) = X_0 = X^{[0]} = X_B \quad \text{for any } X \in \mathcal{C}, \]
and the soul part $X_S$ of $X$ as
\[ X_S = X - X_B = \sum_{|I| \geq 1} X_I \sigma^I. \]

Moreover, we define other projections as
\[ \pi_{ev}(= \pi_0) : \mathcal{C} \ni X \to \pi_{ev}X = \sum_{I \in \mathcal{I}_0} X_I \sigma^I \in \mathcal{C}_{ev}, \]
\[ \pi_{od}(= \pi_1) : \mathcal{C} \ni X \to \pi_{od}X = \sum_{I \in \mathcal{I}_1} X_I \sigma^I \in \mathcal{C}_{od}. \]

Analogous to $\mathcal{C}$, we define, as an alternative of $\mathbb{R}$,
\[ \mathcal{R} = \{ X \in \mathcal{C} | \pi_B X \in \mathbb{R} \}, \quad \mathcal{R}^{[j]} = \mathcal{R} \cap \mathcal{C}^{[j]}, \]
\[ \mathcal{R}_{ev} = \mathcal{R} \cap \mathcal{C}_{ev}, \quad \mathcal{R}_{od} = \mathcal{R} \cap \mathcal{C}_{od} = \mathcal{C}_{od}, \]
\[ \mathcal{R} \cong \mathcal{R}_{ev} \oplus \mathcal{R}_{od} = \mathcal{R}_{ev} \times \mathcal{R}_{od}. \]

We define the (real) superspace $\mathcal{R}^{m|n}$ by
\[ \mathcal{R}^{m|n} = \mathcal{R}^{m}_{ev} \times \mathcal{R}^{n}_{od}. \]

The distance between $X, Y \in \mathcal{R}^{m|n}$ is defined by,
\[ \text{dist}_{m|n}(X, Y) = \text{dist}_{m|n}(X - Y) \]
\[ \text{where } \text{dist}_{m|n}(X) = \sum_{j=1}^{m} \left( \sum_{I \in \mathcal{I}} \frac{1}{2^r(I)} \frac{|\text{proj}_I(x_j)|}{1 + |\text{proj}_I(x_j)|} \right) + \sum_{k=1}^{n} \left( \sum_{I \in \mathcal{I}} \frac{1}{2^r(I)} \frac{|\text{proj}_I(\theta_k)|}{1 + |\text{proj}_I(\theta_k)|} \right). \]

We use the following notation:
\[ X = (X_A)^{m+n}_{A=1} = (x, \theta) \in \mathcal{R}^{m|n} \text{ with } \]
\[ x = (X_A)^{m}_{A=1} = (x_j)_{j=1}^{m} \in \mathcal{R}^{m|0}, \quad \theta = (X_A)^{m+n}_{A=m+1} = (\theta_k)_{k=1}^{n} \in \mathcal{R}^{0|n}. \]
We generalize the body map $\pi_B$ from $\mathcal{R}^{m|n}$ or $\mathcal{R}^{m|0}$ to $\mathbb{R}^m$ by putting,

$$\mathcal{R}^{m|n} \ni X = (x, \theta) \mapsto \pi_B X = X_B = (x_B, 0) \cong x_B = \pi_B x = (\pi_B x_1, \cdots, \pi_B x_m) \in \mathbb{R}^m.$$ 

We call $x_j \in \mathcal{R}_{ev}$ and $\theta_k \in \mathcal{R}_{od}$ as even and odd (alias bosonic and fermionic) variable, respectively.

### 2.3. A class of functions.

Since we introduce rather weak topology in $\mathcal{C}$, we may define $\tilde{f}$ for any function $f \in C^\infty(\mathbb{R}^m : \mathbb{C})$ as follows.

**Lemma 2.2** (Proposition 2.2 of [21]). For $f \in C^\infty(\mathbb{R}^m : \mathbb{C})$, we define

$$\tilde{f}(x) = \sum_{|\alpha| = 0}^\infty \frac{1}{\alpha!} \partial^\alpha f(q) |_{q = x_B} x_S^\alpha$$

for $x = x_B + x_S \in \mathcal{R}^{m|0}$

which is called the Grassmann continuation of $f$, denoted by $\tilde{f} \in \mathcal{C}_{SS}(\mathcal{R}^{m|0})$.

**Remark 2.3.** (i) Because of this definition, not only functions in $C^\infty(\mathbb{R}^m : \mathbb{C})$ but also those in $\mathcal{D}'(\mathbb{R}^m : \mathbb{C})$, we may define the Grassmann continuation by using the duality between the test sequence space $c_0$ and $\omega$. This generalization is necessary to introduce Sobolev spaces on superspace, but this point is not discussed here.

(ii) If $f \in C^\infty(\mathbb{R}^m : \mathbb{C})$, i.e. $f(q) = \sum f_1(q) \sigma^I$ with $f_1 \in C^\infty(\mathbb{R}^m : \mathbb{C})$, we may define analogously $\tilde{f}(x) = \sum \tilde{f}_1(x) \sigma^I$, which is denoted by $\tilde{f} \in \mathcal{C}_{SS}(\mathcal{R}^{m|0})$. Concerning these, see Proposition 3.7 below.

**Definition 2.1** (Supersmooth functions on FG-algebra). We define a function $u : \mathcal{R}^{m|n} \rightarrow \mathcal{C}$ by

$$u(X) = u(x, \theta) = \sum_{|\alpha| \leq n} \theta^a \tilde{u}_a(x) \text{ where } u_a(q) \in C^\infty(\mathbb{R}^m : \mathbb{C})$$

which is called a supersmooth function on $\mathcal{R}^{m|n}$ and denoted by $u \in \mathcal{C}_{SS}(\mathcal{R}^{m|n})$.

Analogously, we put

$$\mathcal{C}_{SS}(\mathcal{R}^{m|n} : \mathbb{C}) = \{ u(X) = \sum_{|\alpha| \leq n} \theta^a \tilde{u}_a(x) | u_a(q) = \partial_0^a u(q, 0) \in C^\infty(\mathbb{R}^m : \mathbb{C}) \text{ for any } a \}.$$

**Remark 2.4.** In the above, we put $\theta^a \tilde{u}_a(x)$ but not $\tilde{u}_a(x) \theta^a$, because we prefer the left derivative w.r.t. odd variables, i.e. after putting the variable in question to the most left, we contract it with the derivation.

### 2.4. Problem.

Though we introduce supersmooth functions as a polynomial of odd variables with a special class of coefficient functions, we have

**Problem 2.3.** How do we characterize a supersmooth function $u(X) \in \mathcal{C}_{SS}(\mathcal{R}^{m|n})$ defined above?

(1) Is it possible to say that Gâteaux infinitely differentiability with superdifferentiability is necessary and sufficient for supersmoothness? Here, $u$ is said to be superdifferentiable if for any $X = (X_A) = (x, \theta), Y = (Y_A) = (y, \omega) \in \mathcal{R}^{m|n}$, there exist $(\gamma_A(X)) = (\gamma_J(X), \gamma_{m+k}(X)) \in \mathcal{C}^{m+n}$ such that

$$\Phi(X; Y) = u(X + Y) - u(X) - \sum_{j=1}^m y_j \gamma_j(X) - \sum_{k=1}^n \omega_k \gamma_{m+k}(X)$$

is “horizontal” w.r.t. $Y$.

(2) How about the Cauchy-Riemann relation?

**Remark 2.5.** (i) For example, Jadczyk and Pilch [23] proves, in BG-algebra category, the Fréchet infinitely differentiability with its differential being “$Q_0$” linear is necessary and sufficient for $G^\infty$-superdifferentiability of Rogers.
(ii) Since we work in the Fréchet space category, if we work within Fréchet differentiability, we need the notion “horizontal” following Schwartz [39], which will be explained in §3.

(iii) In order to make clear our problem, we recall what is well-known for analytic functions:

Let a function \( f(z) \) from \( \mathbb{C} \) to \( \mathbb{C} \) be given, which is decomposed as

\[
f(z) = u(x, y) + iv(x, y), \quad u(x, y) = \Re f(z) \in \mathbb{R}, \quad v(x, y) = \Im f(z) \in \mathbb{R},
\]

where \( z = x + iy, \ |z| = \sqrt{x^2 + y^2} \) with \( z_0 = x_0 + iy_0 \).

- \( f \) is said to be \( F \)-differentiable in \( \mathbb{C} \) at \( z = z_0 \) if the following limit exists in \( \mathbb{C} \):

\[
\lim_{w \to 0} \frac{f(z_0 + h) - f(z_0)}{w} = \gamma \in \mathbb{C}.
\]

In other word, there exists a number \( \gamma \in \mathbb{C} \equiv L(\mathbb{C} : \mathbb{C}) \) such that

\[
|f(z_0 + w) - f(z_0) - \gamma w| = o(|w|) \quad (|w| \to 0).
\]

- \( f(z) \) is called analytic in \( D = \{z \in \mathbb{C} \mid |z - z_0| < R\} \) if one of the following conditions is satisfied:

(a) For any \( z \in D, f(z) \) is differentiable in the above sense (2.8).

(b) Identifying \( D \subset \mathbb{C} \) with \( \tilde{D} = \{ (x, y) \in \mathbb{R}^2 \mid |x - x_0|^2 + |y - y_0|^2 < R^2 \} \), we have

(b-1) \( u, v \in C^1(\tilde{D} : \mathbb{R}^2) \) and \( df(x, y) \) is not only linear w.r.t. \( \mathbb{R} \) but also linear w.r.t. \( \mathbb{C} \), or

(b-2) \( u, v \in C^1(\tilde{D} : \mathbb{R}^2) \) and \( u, v \) satisfies Cauchy-Riemann equation.

(c) \( f(z) \) has the convergent power series expansion \( f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n \) for \( z \in D \).

**Remark 2.6** (The meaning of (b-1) and (b-2)). For \( f : \mathbb{C} \to \mathbb{C} \) given above, we define a map

\[
\Phi : \mathbb{R}^2 \ni \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix} \in \mathbb{R}^2.
\]

Since \( u, v \in C^1(\tilde{D} : \mathbb{R}^2) \), \( \Phi \) is \( F \)-differentiable at \( (x_0, y_0) \), that is, there exists \( \Phi_F(x_0, y_0) \in L(\mathbb{R}^2 : \mathbb{R}^2) \) such that

\[
\| \Phi(x_0 + h, y_0 + k) - \Phi(x_0, y_0) - \Phi_F(x_0, y_0) \begin{pmatrix} h \\ k \end{pmatrix} \| = o(\| \begin{pmatrix} h \\ k \end{pmatrix} \|).\]

Here, we have

\[
\Phi_F(x, y) = \begin{pmatrix} u_x(x, y) & u_y(x, y) \\ v_x(x, y) & v_y(x, y) \end{pmatrix},
\]

and if we require \( \Phi_F(x_0, y_0) \in L(\mathbb{R}^2 : \mathbb{R}^2) \) is not only \( \mathbb{R} \)-linear but also \( \mathbb{C} \)-linear, that is, for any \( a, b \in \mathbb{R} \), especially for \( b \neq 0 \),

\[
\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \Phi_F(x_0, y_0) = \Phi_F(x_0, y_0) \begin{pmatrix} a & -b \\ b & a \end{pmatrix}
\]

holds, we need \( u_x(x, y) = v_y(x, y) \) and \( u_y(x, y) = -v_x(x, y) \).

**Remark 2.7.** In order to prove the equivalence to (a) to (c), one uses the Cauchy’s integral representation in general. But to prove the equivalence of (b) and (c) without integral representation, it seems useful to recall the notion of Pringsheim regularity as follows, see [35]:

A function \( f \) is said to be Pringsheim regular in \( U \) if the Taylor series of \( f \) converges in a neighborhood of every point of \( x \in U \) (though not necessarily to the function \( f \) itself).
3. Elementary Differential calculus on Banach or Fréchet spaces

Since we use rather weak topology on the “number-field” $\mathcal{R}$ or $\mathcal{C}$, in §5 we need to develop the analysis on Fréchet but not necessarily Banach spaces with multiplication structure. In order to fix the notation, we enumerate here known facts for elementary differential calculus on Banach or Fréchet spaces.

Remark 3.1 (The difference between Banach and Fréchet space calculus - p.1 of Yamanuro [13]). Let $X$ be a locally convex space and let $L(X) = L(X : X)$ be the algebra of all continuous linear mappings.

(a) If the multiplication in $L(X)$ is jointly continuous, then $X$ is normable.

(b) If $L(X)$ is continuous inverse algebra, then $X$ is normable.

Therefore, if $X$ is not normable, then neither good chain rule nor good inverse mapping theorem are available in general because of (a) or (b), respectively. To have those, we need additional structure like “tame” in Hamilton [14] and need to regard the differential as continuous from $X \times X \to X$ but not from $X \to L(X)$, etc.

3.1. Gâteaux differentiability.

3.1.1. Gâteaux derivatives in one variable.

Definition 3.1 (Gâteaux differentiability). (i) Let $X$, $Y$ be Fréchet spaces with countable seminorms $\{p_n\}$, $\{q_n\}$, respectively. Let $U$ be an open subset of $X$. For a function $f : U \to Y$, we say that $f$ has the Gâteaux (or G-)differential $df(x,y) \in Y$ at $x \in U$ in the direction $y \in X$ if there exists limit in $Y$

\[
\lim_{t \to 0} \frac{f(x + ty) - f(x)}{t} = \left. \frac{d}{dt} f(x + ty) \right|_{t=0}
\]

which is denoted by $df(x,y)$, $d_G f(x,y)$. If we may write it as $df(x)y$ or $f'(x)y$, $df(x)$ or $f'(x)$ is called G-derivative. That is, for given $x \in U$ and $y \in X$ there exists $df(x,y)$ such that $\forall \epsilon > 0, \forall n$, there exists $\delta = \delta(x,y,\epsilon,n) > 0$ satisfying

\[
q_n\left( \frac{f(x + ty) - f(x)}{t} - df(x,y) \right) \leq \epsilon \quad \text{if } |t| \leq \delta.
\]

We call this the Gâteaux (or G-)differential of $f$ at $x$ in the direction $y$.

(ii) If $X$, $Y$ are Banach spaces with norms $\|\cdot\|_X$, $\|\cdot\|_Y$, respectively, then, $f$ has $G$-differential $df(x,y) \in Y$ at $x \in U$ in the direction $y \in X$ iff

\[
\|f(x + ty) - f(x) - tdf(x,y)\|_Y = o(|t|) \quad \text{as } t \to 0.
\]

Moreover, $f$ is said to be Gâteaux (or G-)differentiable in $U$ if $f$ has the Gâteaux (or G-)derivative for every $x \in U$ and any direction $h \in X$.

Definition 3.2 (Gâteaux-differentiability). (i) Let $X$, $Y$ be Fréchet spaces and let $U$ be an open subset of $X$. We call a map $f : U \to Y$ continuously differentiable on $U$, denoted by $f \in C^1_G(U : Y)$, if $f$ is Gâteaux (G-) differentiable in $U$ and if $df : U \times X \ni (x,y) \to df(x,y) \in Y$ is continuous.

(ii) In case $X$, $Y$ are Banach spaces, $f \in C^1_G(U : Y)$ iff $f$ is $G$-differentiable at $x$ and $df$ is continuous from $U \ni x$ to $df(x) \in L(X : Y)$. 


Remark 3.2. Even in Banach spaces case, there is a large difference in what it means to be continuous for \( L: U \times Y \to Z \), as opposed to \( L: U \to L(Y : Z) \) be continuous. See, p.70 and Example 1.3.1 of [14].

Proposition 3.3 (Lemma 3.2.3, 3.2.4 and Theorem 3.2.5 of [14]). Let \( X \) and \( Y \) be Fréchet spaces and let \( U \) be an open subset of \( X \). Let \( f \) be \( C^1_G(U : Y) \) and \( x + ty \in U \) for any \( t \in [0, 1] \). Then
(i) \( f(x + ty) \) is a path w.r.t. \( t \) in \( C^1([0, 1] : Y) \), and moreover
\[
f'(x + ty) = \frac{d}{dt}f(x + ty) = df(x + ty)y.
\]
(ii) Applying the Riemann integral for Fréchet space valued functions, we have
\[
(3.1)
f(x + y) - f(x) = \int_0^1 df(x + ty)ydt.
\]

Lemma 3.3 (Lemma 3.3.1 of [14]). Let \( f \) be in \( C^1_G(U : Y) \).
(i) If \( c \) is a scalar, then
\[
df(x)cy = df(x, cy) = c df(x, y) = c df(x)y \quad \text{for} \quad y \in X.
\]
(ii) If \( x \in U \) and \( y_1, y_2 \in X \), then
\[
df(x)(y_1 + y_2) = df(x, y_1 + y_2) = df(x, y_1) + df(x, y_2) = df(x)y_1 + df(x)y_2.
\]

Remark 3.3. In (ii) of Proposition 3.3 and in (ii) of Lemma 3.2, the following is crucial. For any continuous seminorm \( p : Y \to \mathbb{R} \), we have
\[
p\left( \int_a^b g(t)dt \right) \leq \int_a^b p(g(t))dt
\]
for any \( g \in C([a, b] : Y) \). Here, the Riemannian integral with value in Fréchet space \( Y \) is defined analogously as Banach space valued case.

Lemma 3.4 (Lemma 3.3.1 of [14]). Let \( f : U \to Y \) be continuous and let suppose for simplicity that \( U \) is convex. Then \( f \) is continuously differentiable iff there exists a continuous map \( L : U \times U \times X \to Y \) with \( L(x_0, x_1)y \) linear in the last variable \( y \) such that for all \( x_0 \) and \( x_1 \) in \( U \)
\[
df(x_1) - df(x_0) = L(x_0, x_1)(x_1 - x_0).
\]

Proposition 3.5 (Theorem 3.4.5 of [14]). If \( L(x)y \) is \( C^1_G(X \times Y : Y) \) and linear in \( y \), then
\[
dL(x)\{y, z\} = \lim_{t \to 0} \frac{L(x + tz)y - L(x)y}{t}
\]
is bilinear w.r.t. \( y \) and \( z \).
3.1.2. Higher order derivatives.

**Definition 3.3** (p.80 of [14]). If the following limit exists, we put
\[
d^2 f(x)\{y, z\} = \lim_{t \to 0} \frac{df(x + ty + sz) - df(x + ty) - df(x + sz) + df(x)}{ts}.
\]

\(f\) is said to be \(C^2_G(U : Y)\) if \(df\) is \(C^1_G(U \times X : Y)\), which happens iff \(d^2f\) exists and is continuous. If \(f : U \to Y\), we require \(d^2f\) to be continuous jointly on the product as a map
\[
d^2 f : U \times X \times X \to Y.
\]

**Proposition 3.6** (Theorem 3.5.2 of [14]). If \(f \in C^2_G(U : Y)\), then \(d^2 f(x)\{y, z\}\) is bilinear w.r.t. \(y\) and \(z\).

**Proposition 3.7** (Theorem 3.5.3 of [14]). If \(f \in C^2_G(U : Y)\), then
\[
d^2 f(x)\{y, z\} = \lim_{t, s \to 0} \frac{f(x + ty + sz) - f(x + ty) - f(x + sz) + f(x)}{ts}.
\]

**Corollary 3.8** (Corollary 3.5.4 of [14]). If \(f \in C^2_G(U : Y)\), then the second derivative is symmetric, i.e.
\[
d^2 f(x)\{y, z\} = d^2 f(x)\{z, y\} \quad \text{for} \quad x \in U, \; y, z \in X.
\]

**Proposition 3.9** (Theorem 3.5.5 of [14]). If \(f \in C^2_G(U : Y)\) and \(g \in C^2_G(V : Z)\) with \(f(U) \subset V\), then \(g \circ f \in C^2_G(U : Z)\) and
\[
d^2 (g \circ f)(x)\{y, z\} = d^2 g(f(x))\{df(x)y, df(x)z\} + dg(f(x))d^2 f(x)\{y, z\} \quad \text{for} \quad x \in U, \; y, z \in X.
\]

**Proposition 3.10** (Theorem 3.5.6 of [14]). If \(f \in C^2_G(U : Y)\) and if the path \([0, 1] \ni t \mapsto x + ty \in U\), then
\[
f(x + y) = f(x) + df(x)y + \int_0^1 (1 - t)d^2 f(x + ty)\{y, y\}dt.
\]

**Definition 3.4** (Higher order derivatives, Definition 3.6.1 of [14]). The third derivative is defined by
\[
d^3 f(x)\{y_1, y_2, y_3\} = \lim_{t \to 0} \frac{d^2 f(x + ty_3)\{y_1, y_2\} - d^2 f(x)\{y_1, y_2\}}{t}.
\]

Analogously, we define
\[
d^n f : U \times \overline{X} \times \cdots \times \overline{X} \ni (x, y_1, \ldots, y_n) \mapsto d^n f(x)\{y_1, \ldots, y_n\} \in Y.
\]

\(f\) is \(C^n_G(U : Y)\) iff \(d^n f\) exists and is continuous. We put \(C^n_G(U : Y) = \cap_{n=0}^{\infty} C^n_G(U : Y)\).

**Proposition 3.11** (Theorem 3.6.2 of [14]). If \(f \in C^2_G(U : Y)\), then \(d^2 f(x)\{y_1, \ldots, y_n\}\) is completely symmetric and linear separately in \(y_1, \ldots, y_n\).

**Proposition 3.12** (Taylor’s theorem, p.101 of Keller [29] and Theorem 2.1.31 of [2]). (i) Let \(f \in C^p_G(U : Y)\), we have
\[
f(x + y) = \sum_{k=0}^{p} \frac{1}{k!} d^k f(x)\{g, \ldots, y\} + R_p f(x, y) \quad \text{with} \quad \lim_{t \to 0} t^{-p} R_p f(x, ty) = 0 \quad \text{for} \quad y \in X.
\]

\(R_p f(x, y) = \int_0^1 \frac{(1 - s)^{p-1}}{(p-1)!} ds^p f(x + sy)dy\)ds.

(ii) Let \(X, Y\) be Banach spaces. If \(f\) is \((N - 1)\)-times \(F\)-differentiable in a neighborhood \(U\) of \(x\) and \(f^{(N)}(x)\) exists, then
\[
\|f(x + y) - f(x) - f'(x)y - \cdots - \frac{1}{N!} f^{(N)}(x)y^N\|_Y = o(\|y\|^N).
\]
3.1.3. Many variable case. Now, we put

\[ \partial_{z_1} f(x_1, x_2, z) = \lim_{t \to 0} \frac{f(x_1 + tz_1, x_2, z) - f(x_1, x_2, z)}{t}, \quad \partial_{z_2} f(x_1, x_2, z) = \lim_{t \to 0} \frac{f(x_1, x_2 + tz_2) - f(x_1, x_2, z)}{t}. \]

Lemma 3.13 (Lemma 3.4.2 of [14]). The partial derivative \( \partial_{z_1} f(x_1, x_2, z) \) exists and is continuous iff there exists a continuous function \( L(u_0, u_1, u_2) \) linear in \( u \) with

\[ f(u_1, y) - f(u_0, y) = L(u_0, u_1, u_2)(u_1 - u_0). \]

In this case, \( \partial_{z_1} f(x_1, x_2, z) = L(x_1, x_2)(z_1, z_2) \).

We define the total derivative as

\[ df(x_1, x_2)(z_1, z_2) = \lim_{t \to 0} \frac{f(x_1 + t z_1, x_2 + t z_2) - f(x_1, x_2)}{t}. \]

Proposition 3.14 (Theorem 3.4.3 of [14]). The partial derivatives \( \partial_{z_1} f(x_1, x_2, z) \) and \( \partial_{z_2} f(x_1, x_2, z) \) exist and are continuous iff the total derivative \( df \) exists and is continuous. In that case,

\[ df(x_1, x_2)(z_1, z_2) = \partial_{z_1} f(x_1, x_2, z_1) + \partial_{z_2} f(x_1, x_2, z_2). \]

Lemma 3.15 (Corollary 3.4.4 of [14]). If \( L(x, y) \) is jointly continuous, \( C^1_1 \) separately in \( x \) and linear separately in \( y \), then it is \( C^1_1 \) jointly in \( x \) and \( y \), and

\[ dL(x, y)(z_1, z_2) = \partial_y L(x, y)z_1 + L(x, z_2). \]

3.2. Fréchet differentiability.

Definition 3.5 (Definition 1.8. of Schwartz [39]). (i) Let \( X, Y \) be Fréchet spaces, and let \( U \) be an open subset of \( X \). For a function \( \phi : U \to Y \), we say that \( \phi \) is horizontal (or tangential) at \( 0 \) iff for each neighbourhood \( V \) of \( 0 \) in \( Y \) there exists a neighbourhood \( U' \) of \( 0 \) in \( X \), and a function \( o(t) : (-1, 1) \to \mathbb{R} \) such that

\[ \phi(U') \subset o(t)V \quad \text{with} \quad \lim_{t \to 0} \frac{o(t)}{t} = 0, \]

i.e. for any seminorm \( q_n \) on \( Y \) and \( \epsilon > 0 \), there exists a seminorm \( p_m \) on \( E \) and \( \delta > 0 \) such that

\[ q_n(\phi(x)) \leq \epsilon t \quad \text{for} \quad p_m(x) < 1, \quad |t| \leq \delta \quad \text{where} \quad V = \{ z \in Y \mid q_n(z) < 1 \}, \quad U' = \{ x \in X \mid p_m(x) < 1 \}. \]

(ii) For Banach spaces \( (X, \|\cdot\|_X) \) and \( (Y, \|\cdot\|_Y) \), “horizontal” implies

\[ \|\phi(x)\|_Y \leq \|x\|_X \psi(x) \quad \text{with} \quad \psi : X \to \mathbb{R}, \quad \lim_{x \to 0} \psi(x) = 0 \quad \text{i.e.} \quad \|\phi(x)\|_Y = o(\|x\|_X) \quad \text{as} \quad \|x\|_X \to 0. \]

Definition 3.6 (Fréchet differentiability: Definition 1.9. of [39]). (i) Let \( x \in U \). We say that \( f \) has a Fréchet (or \( F \)- or strong) differentiable at \( x \), if there exists a continuous linear map \( A : X \to Y \) such that if we define

\[ \phi(y) = f(x + y) - f(x) - Ay, \]

then \( \phi \) is horizontal at \( 0 \). We denote \( f \in C^1_F \) if \( f \) is \( F \)-differentiable. Analogously, we define \( f \in C^0_F, \ C^\infty_F \).

(ii) For Banach spaces, \( f \) is \( F \)-differentiable at \( x \) if there exists a continuous linear map \( A : X \to Y \) satisfying

\[ \|f(x + y) - f(x) - Ay\|_Y = o(\|y\|_X) \quad \text{as} \quad \|y\|_X \to 0. \]

We call \( A \) the derivative of \( f \) at \( x \), and we write \( Ay \) as \( df(x, y) \) as above.
Lemma 3.16 (Lemma 1.13 of [39]). If \( f \) has a F-derivative at \( x \), it also has a G-derivative at \( x \), and they are equal.

Lemma 3.17. Let \( X \) and \( Y \) be Fréchet spaces. Let \( U \) be open in \( X \) and \( f : U \to Y \). If \( f \) has a Gâteaux derivative \( f'(x, y) \) in \( U \), which is linear in \( y \) and continuous from \( U \times X \to Y \), then \( f \) is F-differentiable in \( U \).

Proof. Put
\[
\psi(x, y) = f(x + y) - f(x) - f'(x)y = \int_0^1 [f'(x + sy) - f'(x)]yds.
\]
Since \( f'(x)y \) is continuous from \( U \times X \to Y \), for any \( \epsilon > 0 \), any \( p_n \), there exists a seminorm \( q_m \) of \( X \) such that
\[
p_n([f'(x + sy) - f'(x)]y) \leq \epsilon \quad \text{for} \quad \forall s \in [0, 1], y \in U' = \{ y \in X \mid q_m(y) \leq 1 \}.
\]
Therefore there exists a function \( o(t) : (-1, 1) \to \mathbb{R} \) such that
\[
\psi(tU') \subset o(t)V \quad \text{with} \quad \lim_{t \to 0} \frac{o(t)}{t} = 0 \quad \text{with} \quad V = \{ z \in Y \mid p_n(z) \leq 1 \}.
\]
In fact,
\[
\psi(x, ty) = \int_0^1 [f'(x + sty) - f'(x)]yds = \int_0^t [f'(x + \tau y) - f'(x)]yd\tau \Rightarrow p_n(\psi(x, ty)) \leq t\epsilon. \quad \Box
\]

Lemma 3.18 (Lemma 1.14 of [39]). If \( f : U \to V \) is F-differentiable at \( x \), and \( g : V \to W \) is F-differentiable at \( f(x) \), then \( g(f(x)) \) is F-differentiable at \( x \) and its derivative is given by:
\[
d(g \circ f)(x, y) = dg(f(x), df(x, y)).
\]

Proof. We have
\[
g[f(x + y)] = g[f(x) + df(x, y) + \phi(y)] = g[f(x)] + dg[f(x, 0), df(x, y)]
\]
\[
+ dg[f(x), \phi(y)] + \psi[df(x, y) + \phi(y)]
\]
where \( \phi \) and \( \psi \) are horizontal at 0. The last term above is horizontal at 0 as a function from \( X \) to \( Z \). As is easily proved, if \( \phi \) is horizontal at 0 and if \( \ell \) is linear and continuous, then \( \ell \circ \phi \) is also horizontal at 0. Thus \( dg[f(x, 0), \phi(y)] \) is horizontal at \( x \) and its derivative is \( dg[f(x, 0), df(x, y)] \). \( \Box \)

Remark 3.4. Here, we don’t mention the implicit or inverse function theorems in Fréchet spaces, but see [14, 21, 39].

4. The case with finite Grassmann generators

To clarify the problem, we first gather results when the number \( L \) of Grassmann generators is finite, which are mainly treated by Vladimirov and Volovich [40] and Boyer and Gitler [7], though the terminology is slightly modified from them.
4.1. **Finite dimensional Grassmann algebras.** Preparing Grassmann generators \( \{\sigma\}_{j=1}^{L} \) for finite \( L \), we put

\[
\mathfrak{B}_L = \{ X = \sum_{\mathbf{I} \in I_L} X_\mathbf{I} \sigma^\mathbf{I} \mid X_\mathbf{I} \in \mathbb{R} \text{ and } \|X\| = \sum_{\mathbf{I} \in I_L} |X_\mathbf{I}| < \infty \},
\]

\[
\mathfrak{B}_{L,ev} = \{ X = \sum_{|\mathbf{I}|=\text{ev},\mathbf{I} \in I_L} X_\mathbf{I} \sigma^\mathbf{I} \mid X_\mathbf{I} \in \mathbb{R} \text{ and } \|X\| = \sum_{|\mathbf{I}|=\text{ev},\mathbf{I} \in I_L} |X_\mathbf{I}| < \infty \},
\]

\[
\mathfrak{B}_{L,od} = \{ X = \sum_{|\mathbf{I}|=\text{od},\mathbf{I} \in I_L} X_\mathbf{I} \sigma^\mathbf{I} \mid X_\mathbf{I} \in \mathbb{R} \text{ and } \|X\| = \sum_{|\mathbf{I}|=\text{od},\mathbf{I} \in I_L} |X_\mathbf{I}| < \infty \},
\]

where \( I_L = \{ \mathbf{I} = (i_1, \ldots, i_L) \in \{0,1\}^L \} \), \( I_L \ni \mathbf{I} \mapsto (i_1, \ldots, i_L, 0, \ldots) \in \mathcal{I} \).

Regarding \( \mathfrak{B}_L \) as a vector space \( \mathbb{R}^{2^L} \), we introduce its topology as Euclidian space, or \( \|X\| = \sum_{\mathbf{I} \in I_L} |X_\mathbf{I}| \) for \( X \in I_L \). We define a superspace as

\[
\mathfrak{B}_L^{m|n} = \mathfrak{B}_{L,ev}^m \times \mathfrak{B}_{L,od}^n,
\]

identified with \( \mathbb{R}^{2^{(L-1)}(m+n)} \) as vector space.

**Proposition 4.1** (Proposition 2.4 of [36]). If \( X = (x, \theta) \in \mathfrak{B}_L^{m|n} \) satisfies

\[
\langle Y|X \rangle = \langle y|x \rangle + \langle \omega|\theta \rangle = 0 \quad \text{for any } Y = (y, \omega) \in \mathfrak{B}_L^{m|n}
\]

with \( x = (x_1, \ldots, x_m), y = (y_1, \ldots, y_m) \in \mathfrak{B}_L^{m|n}, \theta = (\theta_1, \ldots, \theta_n), \omega = (\omega_1, \ldots, \omega_n) \in \mathfrak{B}_L^{n|n}, \)

\[
\langle y|x \rangle = \sum_{j=1}^{m} y_j x_j, \quad \langle \omega|\theta \rangle = \sum_{k=1}^{n} \omega_k \theta_k,
\]

then,

\[
\begin{cases}
(i) & x_j = 0 \\
(ii) & \begin{array}{l}
\theta_k = \lambda_k \sigma_1 \cdots \sigma_L \text{ with some } \lambda_k \in \mathbb{R} \text{ if } L \text{ is finite, for } k = 1, \ldots, n, \\
\theta_k = 0 \text{ if } L = \infty, \text{ for } k = 1, \ldots, n.
\end{array}
\end{cases}
\]

**Remark 4.1.** We put \( \mathfrak{B}_L = \{ \alpha \sigma_1 \cdots \sigma_L \mid \alpha \in \mathbb{R} \} = \{ X \in \mathfrak{B}_L \mid \sigma_i X = 0 \text{ for } \forall i = 1, \ldots, L \} \).

Though the following is mentioned in [7] as Lemma 1.7, it is necessary to modify it as follows:

**Lemma 4.2.** If \( A \) is any algebra and \( I_1, \ldots, I_n \) are ideals in \( A \) satisfying

\[
(KM) \quad \cap_{j=1}^{k-1} (I_j + I_k) = \cap_{j=1}^{k-1} I_j + I_k \quad \text{for any } k,
\]

then there exists an exact sequence

\[
\cap_k I_k \xrightarrow{i} \cap_{j=1}^{k-1} I_j \xrightarrow{\alpha} \oplus_k A/I_k \xrightarrow{\beta} \oplus_{(k,j)} A/(I_k + I_j)
\]

where \( i \) is the injection, \( \alpha \) is the diagonal followed by the natural projection, \( (k,j) \) runs over all pairs \( n \geq k > j \geq 1 \) and

\[
\beta([a_1], \ldots, [a_n]) = ([a_2 - a_1], [a_3 - a_1], [a_4 - a_2], \ldots, [a_n - a_{n-1}]).
\]

**Corollary 4.3** (Lemma 2.2 of [40]). Suppose that there exist elements \( \{A_i\}_{i=1}^{L} \subset \mathfrak{B}_L \) satisfying

\[
(4.1) \quad \sigma_j A_i + \sigma_i A_j = 0 \quad \text{for any } i, j = 1, \ldots, L.
\]

Then there exists an element \( F \in \mathfrak{B}_L \) such that \( A_i = \sigma_i F \) for \( i = 1, \ldots, L \).
Proof. Putting $I_i = \langle \sigma_i \rangle = \{ \sigma_iX \mid X \in \mathcal{B}_L \}$, we have $I_i \cap I_j = \langle \sigma_i\sigma_j \rangle$, etc. Then it is clear that these $\{I_i\}$ satisfy [KM]. Taking $i = j$ in (4.1), we have $\sigma_iA_i = 0$ which implies $\exists B_i \in \mathcal{B}_L$ such that $A_i = \sigma_iB_i$. Putting this into (4.1), we have $\sigma_i\sigma_j(B_i - B_j) = 0$, i.e., $B_i - B_j \in I_i + I_j =$Kernel of multiplication $\sigma_i\sigma_j$. Therefore, $(B_1, \cdots, B_L) \in$Kernel of $\beta$. This implies, by Lemma 4.2, there exists $F$ such that $F = B_1 + a_i (a_i \in I_i)$, then $\sigma_iF = \sigma_iB_i = A_i$. \qed

Remark 4.2. (i) The above condition [KM], the proofs and the following facts are due to Kazuo Masuda.

(ii) Without the condition [KM], there exists a counter-example for Lemma 4.2.

Adding to $\mathbb{R}^2$ the trivial multiplication, $a \cdot b = 0$ for any $a, b \in \mathbb{R}^2$, we take this as $A$. Since $I_1 = \mathbb{R} + 0, I_2 = 0 + \mathbb{R}, I_3 = \{(x, x) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$ are ideals satisfying $I_1 + I_2 = I_1 + I_3 = I_2 + I_3 = A$, we have $\cap_{i=1}^3 I_i = 0$. Let $A = \mathbb{R}^2, \oplus_{j=1}^3 A/I_j = \mathbb{R}^3, \oplus_{(i, j)} A/(I_i + I_j) = 0$. Hence, the sequence $0 \to \mathbb{R}^2 \to \mathbb{R}^3 \to 0$ is never exact.

(iii) The proof of Lemma 2.2 of [10] contains the following statements:

Suppose the above claim of Lemma 2.2 of [10] holds for all Grassmann algebras $\mathcal{B}_{L-1}$ with generators $\{\sigma_1, \cdots, \sigma_L\} \setminus \{\sigma_j\} \ (j = 1, \cdots, L)$, which are subalgebras of $\mathcal{B}_L$. Then from $\sigma_i\sigma_j(B_j - B_j) = 0$ with the induction hypothesis, there exists $F_1, \cdots, F_L \in \mathcal{B}_L$ such that

$$
\begin{align*}
\sigma_iB_i &= \sigma_iF_{L-i} \quad i \in \{1, \cdots, L-1\} = \{1, \cdots, L\} \setminus \{L\}, \\
\sigma_iB_i &= \sigma_iF_{L-1} \quad i \in \{1, \cdots, L-2, L\} = \{1, \cdots, L\} \setminus \{L-1\}, \\
\cdots & \\
\sigma_iB_i &= \sigma_iF_1 \quad i \in \{2, \cdots, L\} = \{1, \cdots, L\} \setminus \{1\}.
\end{align*}
\tag{4.2}
$$

Not only in [10] but also in [23], they denote the left-hand side $\sigma_iA_i = \sigma_iF_L$ instead of $\sigma_iB_i = \sigma_iF_L$. Those seem to be the misprints and its copy. But any way this statement is not correct! Because the left hand $\sigma_iB_i = \sigma_iF_L$, in the first line of (4.2) does contain $\sigma_L$, but the right hand side doesn’t contain $\sigma_L$, or more precisely we should give reasoning why the left hand side doesn’t contain $\sigma_L$, etc.

4.2. $\mathcal{B}_L$ is not self-dual. From the context of [24], it seems natural to have,

Conjecture 4.1. For $f \in \mathcal{L}(\mathcal{B}_{L,od} : \mathcal{B}_L)$, does there exist an element $u_f \in \mathcal{B}_L$ satisfying

$$
f(X) = Xu_f \quad \text{for} \quad X \in \mathcal{B}_{L,od}.
$$

Here, we denote the set of maps $f : \mathcal{B}_{L,od} \to \mathcal{B}_L$ which are continuous and $\mathcal{B}_{L,od}$-linear (i.e. $f(\lambda X) = \lambda f(X)$ for $\lambda \in \mathcal{B}_{L,od}$ and $X \in \mathcal{B}_{L,od}$) by $f \in \mathcal{L}(\mathcal{B}_{L,od} : \mathcal{B}_L)$.

Though K. Masuda gives a following counter example for this conjecture, but it implies also that to be self-dual, we need the countable number of Grassmann generators.

Let $L = 2$. Define a map $f$ as

$$
f(X_1\sigma_1 + X_2\sigma_2) = X_1\sigma_2 \quad \text{for any} \quad X_1, X_2 \in \mathbb{R}.
$$

Then, remarking that $b_0 + b_1\sigma_1\sigma_2)(X_1\sigma_1 + X_2\sigma_2) = b_0(X_1\sigma_1 + X_2\sigma_2)$, we have readily $f \in \mathcal{L}(\mathcal{B}_{2,od} : \mathcal{B}_2)$, but $\sigma_1f(\sigma_1) = \sigma_1\sigma_2 \neq 0$, therefore, there exists no $u_f \in \mathcal{B}_2$ such that $f(X) = Xu_f$.

4.3. Superdifferentiable functions on $\mathcal{B}_L$. 

4.3.1. Superdifferentiability.

**Definition 4.1** (Definition 2.5 of [BB]). Let \( \mathcal{U}_L \) be an open set in \( \mathcal{B}_L^{m|n} \) and \( f : \mathcal{U}_L \rightarrow \mathcal{B}_L \). Then,
(a) \( f \) is said to be \( G^0 \) on \( \mathcal{U}_L \), denoted by \( f \in G^0(\mathcal{U}_L) \), if \( f \) is continuous on \( \mathcal{U}_L \).
(b) \( f \) is said to be \( G^1 \) (or superdifferentiable) on \( \mathcal{U}_L \), denoted by \( f \in G^1(\mathcal{U}_L) \), if there exist \( m+n \) functions \( G_k f : \mathcal{U}_L \rightarrow \mathcal{B}_L \), \((k = 1, \cdots, m+n)\) and a function \( \rho : \mathcal{B}_L^{m|n} \rightarrow \mathcal{B}_L \) such that, if \( X = (x, \theta) \) and \( Y = (y, \omega) \) are in \( \mathcal{U}_L \) with \( X + Y \in \mathcal{U}_L \),

\[
(4.3) \quad f(X + Y) = f(X) + \sum_{j=1}^{m} Y_j (G_j f)(X) + \sum_{k=1}^{n} Y_{m+k} (G_{m+k} f)(X) + \|Y\| \rho(Y; X)
\]

with \( \rho(Y; X) \rightarrow 0 \) when \( \|Y\| \rightarrow 0 \).
(c) For any positive integer \( p \), \( f \in G^p(\mathcal{U}_L) \) or \( G^p \) on \( \mathcal{U}_L \), if \( f \) is \( G^1 \) on \( \mathcal{U}_L \) and \( G_k f \) are \( G^{p-1} \) on \( \mathcal{U}_L \).
\( G_{m+k} f \) is not necessarily unique, but unique up to \( \mathcal{B}_L \).
(d) \( f \in G^\infty(\mathcal{U}_L) \) or \( f \) is said to be \( G^\infty \) (or superdifferentiable) on \( \mathcal{U}_L \), if \( f \) is \( G^p \) on \( \mathcal{U}_L \) for any positive integer \( p \).
(e) \( f \in G^\omega(\mathcal{U}_L) \), or \( f \) is said to be \( G^\omega \) (or superanalytic) on \( \mathcal{U}_L \), if \( f \) is expanded as absolutely convergent power series in Banach topology;

\[
f(X + Y) = \sum_{|a| = 0}^\infty Y^a f_a \quad \text{with} \quad a = (\alpha, \alpha) \in \mathbb{N}^m \times \{0, 1\}^n, \quad |a| = |\alpha| + |\alpha|,
\]

\[
Y^a = y^\alpha \theta^\alpha, \quad \theta^\alpha = y_1^{\alpha_1} \cdots y_m^{\alpha_m}, \quad \theta^\alpha = \theta_1^{\alpha_1} \cdots \theta_n^{\alpha_n}, \quad f_a = f_{\alpha, \alpha} \in \mathcal{B}_L.
\]

(f) Let \( g : \mathcal{U}_L \rightarrow \mathcal{B}_L^{m|n} \). Then, \( g \) is said to be \( G^\infty \) (or \( G^\omega \)) on \( \mathcal{U}_L \) if each of the \( r + s \) components of \( g \) is \( G^\infty \) (or \( G^\omega \)).

**Proposition 4.4.** Let \( \mathcal{U}_L \) be an open set in \( \mathcal{B}_L^{m|n} \) and let \( f : \mathcal{U}_L \rightarrow \mathcal{B}_L \) be \( \infty \)-times Gâteaux differentiable. \( f \) is \( G^\infty(\mathcal{U}_L) \) iff it is 1-time Fréchet differentiable and its differential \( df \) is \( \mathcal{B}_{L, \text{ev}} \)-linear.

**Proof.** \( \implies \) From (b), we have, for \( Y \in \mathcal{U}_L, X \in \mathcal{U}_L, \)

\[
df(X)(Y) = \sum_{j=1}^{m} y_j (G_j f)(x, \theta) + \sum_{k=1}^{n} \omega_k (G_{m+k} f)(x, \theta) = Y df(X)
\]

with \( df f(X) = (G_j f(X), G_{m+k} f(X)) \) which is clearly \( \mathcal{B}_{L, \text{ev}} \)-linear.

\( \iff \) As \( df(X) \) is \( \mathcal{B}_{L, \text{ev}} \)-linear from \( \mathcal{B}_L \) to \( \mathcal{B}_L \), for \( y_j \in \mathcal{B}_{L, \text{ev}} \) and \( e_j = (\underbrace{0, \cdots, 0, 1, 0, \cdots, 0}_{j}) \in \mathbb{R}^{m+n}, \)

\( df(X) : \mathcal{B}_{L, \text{ev}} \ni y_j \rightarrow df(X)(y_j e_j) \in \mathcal{B}_L \) is \( \mathcal{B}_{L, \text{ev}} \)-linear. Therefore, by the self-duality, there exists an element \( F_j \in \mathcal{B}_L \) such that \( df(X)(y_j e_j) = y_j F_j \). Analogously, \( df(X)(\omega_k e_{m+k}) = \omega_k F_{m+k} \). That means

\[
df(X)(Y) = df(X)(\sum_{j=1}^{m} y_j e_j + \sum_{k=1}^{n} \omega_k e_{m+k}) = \sum_{j=1}^{m} y_j F_j + \sum_{k=1}^{n} \omega_k F_{m+k} = Y \cdot F
\]

with \( F = (F_1, \cdots, F_m, F_{m+1}, \cdots, F_{m+n}) \in \mathcal{B}_L^{m+n} \), which implies \( f \in G^1(\mathcal{U}_L) \). Now, we prove \( f \in G^p \) by induction w.r.t. \( p \). Suppose that \( df^{p-1}(X) \in \mathcal{L}_{\mathcal{B}_{L, \text{ev}}}^{m|n}(\mathcal{B}_L^{m|n} : \mathcal{B}_L) \) for all \( X \in \mathcal{U}_L \). Then, given \( X_0 \in \mathcal{U}_L \), we have \( df^p f(X_0) = df df^{p-1}(f(X_0)) \), and therefore \( df^p f(X_0) \) is continuous linear map from \( \mathcal{B}_L^{m|n} \) to \( \mathcal{L}_{\mathcal{B}_{L, \text{ev}}}^{m|n}(\mathcal{B}_L^{m|n} : \mathcal{B}_L) \). Since \( df^p f(X_0) \) is symmetric, it is automatically \( \mathcal{B}_{L, \text{ev}} \)-linear in all of its \( p \) variables. \( \square \)

**Remark 4.3.** In the above, we use the abbreviation \( \mathcal{L}_{\mathcal{B}_{L, \text{ev}}}^p(E : F) \) for \( \mathcal{L}_{\mathcal{B}_{L, \text{ev}}}^p(E, \cdots, E : F) \).
Proposition 4.5 (Theorem 4.2.1 of [38]). Let $\mathcal{U}_L$ be an open set in $\mathfrak{B}^m_L$. Assume $f \in C^\infty(\mathcal{U}_L)$. Take any $X = (x, \theta), Y = (y, \omega) \in \mathfrak{B}^m_L$ such that $X + tY \in \mathcal{U}_L$ for all $t \in [0, 1]$. Then, for $N$,

$$f(X + Y) = \sum_{p=0}^{N} \sum_{|\alpha|+|\beta|=p} \frac{1}{\alpha!} y^\alpha \omega^\beta \partial_x^\alpha \partial_\theta^\beta f(X) + \sum_{|\alpha|+|\beta|=N+1} \frac{1}{\alpha!} y^\alpha \omega^\beta \int_0^1 dt \partial_x^\alpha \partial_\theta^\beta f(x + ty, \theta + t\omega)$$

with

$$\omega^\alpha = \omega_1^{\alpha_1} \cdots \omega_n^{\alpha_n}, \partial_x^\alpha f = \partial_x^{\alpha_n} (\partial_x^{\alpha_{n-1}} \cdots (\partial_x f)).$$

Proof. For any $p$ and $u \in C^{p+1}((0, 1])$, we have

$$u(1) = \sum_{\ell=0}^{p} \frac{1}{\ell!} u^{(\ell)}(0) + \int_0^1 dt \frac{(1-t)^p}{p!} u^{(p+1)}(t).$$

Take $u(t) = f(X + tY)$, since $a! = 1$ for $a \in \{0, 1\}^n$, we have

$$u'(0) = \sum_{j=1}^m y_j \partial_x f + \sum_{k=1}^n \omega_j \partial_\theta f = \sum_{|\alpha|=0}^1 \sum_{|\beta|=1} \partial_x^\alpha \partial_\theta^\beta f,$$

$$u''(0) = \sum_{j,k=1}^m y_j y_k \partial_x^2 \partial_x \omega_no f + \sum_{j,k=1}^m y_j \omega_k \partial_x \omega_n \partial_x f + \sum_{j,k=1}^n \omega_j \omega_k \partial_x \omega_n \partial_x f$$

$$= \sum_{|\alpha|=2} \frac{2!}{\alpha!} y^\alpha \partial_x^2 f + \sum_{|\alpha|=1,|\beta|=1} y^\alpha \omega^\beta \partial_x^\alpha \partial_\theta^\beta f + \sum_{|\alpha|=2} \omega^\alpha \partial_\theta^\alpha f,$$

$$\vdots$$

$$u^{(\ell)}(0) = \sum_{|\alpha|=0}^n \sum_{|\alpha|+|\beta|=\ell} \frac{(\ell - |\alpha|)!}{\alpha!} y^\alpha \omega^\beta \partial_x^\alpha \partial_\theta^\beta f.$$}

Therefore, we have

$$f(x+y, \theta + \omega) = \sum_{\ell=0}^{p} \sum_{|\alpha|+|\beta|=\ell} \frac{1}{\alpha!} y^\alpha \omega^\beta \partial_x^\alpha \partial_\theta^\beta f(x, \theta) + \sum_{|\alpha|+|\beta|=p+1} \frac{1}{\alpha!} y^\alpha \omega^\beta \int_0^1 dt \partial_x^\alpha \partial_\theta^\beta f(x + ty, \theta + t\omega). \boxed{}$$

Inspired by the Taylor’s formula above, we put

Definition 4.2 (Definition 4.2.2 of [38]). Let $V$ be an open set of $\mathbb{R}^m$ and let $\mathfrak{M}_L$ be an open set in $\mathfrak{B}^{m,0}_L$ such that $V = \pi_B(\mathfrak{M}_L)$. For any $f \in C^\infty(V : \mathfrak{M}_L)$, we define the Grassmann continuation of $f$ to $\mathfrak{M}_L$ as

$$\hat{f}(x) = \sum_{|\alpha|=0}^{\infty} \frac{1}{\alpha!} \partial_x^\alpha f(q) |_{q=x_B} x_S^\alpha \quad \text{where} \quad x = x_B + x_S.$$ 

Remark 4.4. Since $x_S^\alpha = 0$ if $|\alpha| \geq \log_2 (L+1)$, the summation above is finite for fixed $L < \infty$.

Proposition 4.6 (Theorem 4.2.4 of [38]). Let $\mathcal{U}_L$ be an open set in $\mathfrak{B}^m_L$.

$$\mathcal{G}^\infty(\mathcal{U}_L : \mathfrak{M}_L) = \{ f : \mathcal{U}_L \to \mathfrak{M}_L \mid f(x, \theta) = \sum_{|\alpha| \leq n} \theta^\alpha \hat{f}_a(x) \quad \text{with} \quad f_a \in C^\infty(\pi_B(\mathcal{U}_L) : \mathfrak{M}_L) \}. $$
Proof. Apply (4.6) to \( f(x, \theta) \) at \((x_B, 0)\). Putting \( u(t) = f(x_B + tx_S, t\theta) \), when \( p \geq n \), we get
\[
\begin{align*}
  u(0) &= f(x_B, 0), \quad u(1) = f(x_B + x_S, \theta) = f(x, \theta), \\
  u'(0) &= \frac{d}{dt}f(x_B + tx_S, t\theta)|_{t=0} = \sum_{|a|=0}^{1} \sum_{|\alpha|+|\alpha|=1} \frac{1}{\alpha!} x_3^3 \theta^a \partial_x^\alpha \partial_\theta^\alpha f(x_B, 0), \\
  u''(0) &= \sum_{|a|=0}^{2} \sum_{|\alpha|+|\alpha|=2} \frac{(2 - |a|)!}{\alpha!} x_3^3 \theta^a \partial_x^\alpha \partial_\theta^\alpha f(x_B, 0), \\
  &\quad \vdots, \\
  u^{(p)}(0) &= \sum_{|a|=0}^{p} \sum_{|\alpha|+|\alpha|=p} \frac{(p - |a|)!}{\alpha!} x_3^3 \theta^a \partial_x^\alpha \partial_\theta^\alpha f(x_B, 0).
\end{align*}
\]
Rearranging above as
\[
f(x, \theta) = f(x_B, 0) + \sum_{j=1}^{m} x_j S \partial_x f(x_B, 0) + \sum_{k=1}^{n} \theta_k \partial_{\theta_k} f(x_B, 0)
\]
\[
\quad + \frac{1}{2!} \sum_{|a|=2}^{n} \sum_{|\alpha|+|\alpha|=2} \frac{(2 - |a|)!}{\alpha!} x_3^3 \theta^a \partial_x^\alpha \partial_\theta^\alpha f(x_B, 0)
\]
\[
\quad + \cdots
\]
\[
\quad + \frac{1}{p!} \sum_{|a|=p}^{n} \sum_{|\alpha|+|\alpha|=p} \frac{(p - |a|)!}{\alpha!} x_3^3 \theta^a \partial_x^\alpha \partial_\theta^\alpha f(x_B, 0) + \int_{0}^{1} dt \frac{1}{p!} (1 - t)^p u^{(p+1)}(t),
\]
\[
f(x_B, 0) + \sum_{j=1}^{m} x_j S \partial_x f(x_B, 0) + \frac{1}{2!} \sum_{|a|=2}^{n} \frac{1}{\alpha!} x_3^3 \theta^a \partial_x^\alpha \partial_\theta^\alpha f(x_B, 0) + \cdots
\]
\[
\quad + \sum_{k=1}^{n} \theta_k \partial_{\theta_k} f(x_B, 0) + \frac{1}{2!} \sum_{|a|=2}^{n} \frac{1}{\alpha!} x_3^3 \theta^a \partial_x^\alpha \partial_\theta^\alpha f(x_B, 0) + \cdots.
\]
Taking \( p \) sufficiently large but finite and remarking the nilpotency, we have
\[
\hat{f}_n(x) = \partial_\theta^n f(x, 0) = \sum_{|n|=0}^{\infty} \frac{1}{\alpha!} x_3^3 \partial_x^\alpha \partial_\theta^\alpha f(x_B, 0) \quad \text{with} \quad f_n(q) = \partial_\theta^n f(x, 0)|_{x=q},
\]
which is the finite sum and we have the desired expression.
Conversely, if \( f(x, \theta) = \sum_{|n|=n} \theta^n \hat{f}_n(x) \), then \( f \) is \( G^\infty(\mathcal{U}_L) \).
\( \square \)

Remarking \( f'_P(X; H) = (f'_{X_A}(X; H_A))_{A=1}^{m+n} \), we get readily

**Proposition 4.7** (Lemma 2.1 of [40]). Let \( \mathcal{U}_L \) be an open subset of \( \mathcal{B}_L^{m+n} \). Then, \( f \in G^1(\mathcal{U}_L) \) iff \( f \) is Fréchet differentiable and there exist functions \( F_A(X) \) defined in \( \mathcal{U}_L \) such that
\[
f'_X(X; H_A) = H_A F_A(X), \quad A = 1, \ldots, m + n \quad \text{for any} \quad H \in \mathcal{B}_L^{m+n}.
\]

Following characterization of superdifferentiability is announced as Theorem 2.1 of [40], but seemingly with insufficient reasoning.

**Proposition 4.8** (Theorem 2.1 of [40]). Let \( \mathcal{U}_L \) be an open subset of \( \mathcal{B}_L^{m+n} \). Then, \( f \in G^1(\mathcal{U}_L) \) iff \( f \) is Fréchet differentiable and its derivatives satisfy the following equations:
\[
\begin{cases}
  G \cdot f'_P(X; H) - (-1)^{p(H) - p(G)} H \cdot f'_P(X; G) = 0 \quad \text{for} \quad H, G \in \mathcal{B}_L^{m+n}, \\
  f'_P(X; H \cdot G) = H \cdot f'_P(X; G) \quad \text{for} \quad H \in \mathcal{B}_L^{m+n}, \quad G \in \mathcal{B}_L^{m+n}.
\end{cases}
\]
Or, in components, we have, for $A = 1, \cdots, m + n$,

\[
\begin{aligned}
G_A f_{X_A}^t(X; H_A) - (-1)^{p(H_A)p(G_A)} H_A f_{X_A}^t(X; G_A) &= 0 \quad \text{for} \quad p(X_A) = p(G_A) = p(H_A), \\
\od_X f_{X_A}^t(X; H_A G_A) &= H_A f_{X_A}^t(X; G_A) \quad \text{for} \quad p(H_A) = 0, \ p(X_A) = p(G_A).
\end{aligned}
\] (4.8)

Proof. \(\Rightarrow\) Multiplying $G_A$ to (4.3) and changing the role of $G_A$ and $H_A$ in the obtained equality, we have the first equation of (4.8). The second equation in (4.8) is derived from (4.6) by

\[
f_{X_A}^t(X; H_A G_A) = H_A G_A f_A(X) = H_A f_{X_A}^t(X; G_A).
\]

\(\Leftarrow\) Putting $G_A = 1$ in (4.8) and defining $F_A(X) = f_{X_A}^t(X; 1)$ which yields (4.9) for $A = 1, \cdots, m$. To define $F_A$ for $A = m + 1, \cdots, m + n$, we use the self-duality. From the second equality of (4.7), for each $A = m + 1, \cdots, m + n$ and $X$, $f_{X_A}^t(X; Y_A)$ gives $\mathfrak{B}_{L,e^v}$-linear map. Therefore from Corollary 4.3 there exists $F_A(X)$ such that $f_{X_A}^t(X; Y_A) = Y_A F_A(X)$ for any $Y_A \in \mathfrak{B}_{L,od}$.

4.3.2. Cauchy-Riemann relation. To understand the meaning of supersmoothness, we may give the dependence with respect to the “coordinate” more precisely. Let $\mathfrak{U}_L$ be an open set in $\mathfrak{B}_L^{m|n}$ and let a function $f : \mathfrak{U}_L \ni X \to f(X) = \sum_i f_i(X)\sigma^i \in \mathfrak{B}_L$ be given such that $f_i(X + tY) \in C^\infty([0, 1] : \mathfrak{B}_L)$ for each fixed $X, Y \in \mathfrak{U}_L$. Let $X = (X_A) = (x_j, \theta_k)$ be represented by $X_A = \sum_j X_{A,1}\sigma^j$ with $X_{A,1} \in \mathbb{R}$ where $A = 1, \cdots, m + n$.

Denoting $\epsilon_A = \begin{pmatrix} 0, \cdots, 0, 1, 0, \cdots, 0 \end{pmatrix} \in \mathfrak{N}_n$, we put

\[
\begin{aligned}
\frac{\partial}{\partial X_{A,1}} f(X) &= \frac{d}{dt} f(X + tE_{A,1}) \bigg|_{t=0} \quad \text{with} \quad E_{A,1} = \sigma^1 \epsilon_A \in \mathfrak{N}_n, \\
\frac{\partial}{\partial X_{A,(j)}} f(X) &= \frac{d}{dt} f(X + tE_{A,(j)}) \bigg|_{t=0} \quad \text{with} \quad E_{A,(j)} = \sigma_j \epsilon_A \in \mathfrak{N}_n
\end{aligned}
\] (4.9)

where $|I| =$ even for $1 \leq A \leq m$, $|I| =$ odd for $m + 1 \leq A \leq m + n$ and $\bar{\sigma} = 1$ with $\bar{\sigma} = (\bar{0}, 0, \cdots) \in \{0, 1\}^n$.

Using above coordinate, we rewrite Proposition 4.7 as

**Proposition 4.9** (Proposition 1.5 of [7]). $f \in \mathcal{G}^1(\mathfrak{U}_L)$ iff there exists continuous functions $g_A : \mathfrak{U}_L \to \mathfrak{B}_L$, $1 \leq A \leq m + n$, such that

\[
\frac{\partial}{\partial X_{A,I}} f(X) = \sigma^I g_A(f)(X) \quad \text{for} \quad |I| = \begin{cases} 0 & \text{for} \quad 1 \leq A \leq m, \\
1 & \text{for} \quad m + 1 \leq A \leq m + n.
\end{cases}
\]

(4.10)

Proof. Since $f \in \mathcal{G}^1(\mathfrak{U}_L)$, it is G-differentiable and (4.9) holds, therefore

\[
\frac{d}{dt} f(X + tE_{A,1}) \bigg|_{t=0} = \frac{\partial}{\partial X_{A,1}} f(X) = \sigma^1 F_A(X).
\]

Put $g_A(f)(X) = F_A(X)$, then the assertion holds.

Conversely, multiplying $H_{A,1} \in \mathbb{R}$ to both side of (4.10) and adding w.r.t. $I$, we have

\[
f_{X_A}^t(X; H_A) = \frac{d}{dt} f(X + tH_A) \bigg|_{t=0} = \sum_I H_{A,1} \frac{\partial}{\partial X_{A,I}} f(X) = \sum_I H_{A,1} \sigma^I g_A(f)(X) = H_{A} g_A(f)(X).
\] □
Proposition 4.10 (Theorem 2.2 of [40] and Theorem 11.6 of [7]). Let \( \mathcal{U}_L \subset \mathcal{B}^{m,n}_L \) be open and let \( f \in C^1(\mathcal{U}_L : \mathcal{B}_L) \) be considered as a function of \( 2^{(L-1)}(m+n) \) variables \( \{ X_{A,I} \} \) with values in \( \mathcal{B}_L \). \( f(X) \) is \( \mathcal{G}^1(\mathcal{U}_L) \)-differentiable iff \( f(X) \) is of \( C^1 \) and satisfies the following (Cauchy-Riemann type) equations.

\[
(4.11) \quad \begin{cases} 
\frac{\partial f(X)}{\partial X_{A,I}} = (\sigma_1 \cdots \sigma_{A-I}) \tau^{I-J} \frac{\partial f(X)}{\partial X_{A,J}} & \text{for } |I-J| = \text{ev}, \ 1 \leq A \leq m + n, \\
\sigma_i \frac{\partial f(X)}{\partial X_{A,(j)}} + \sigma_j \frac{\partial f(X)}{\partial X_{A,(i)}} = 0 & \text{for } i, j = 1, \ldots, L, \ m + 1 \leq A \leq m + n.
\end{cases}
\]

Here, integer \( \tau(I, I-J, J) \) is defined in (2.6).

Proof. \( \Rightarrow \) Replacing \( Y \) with \( E_{A,I} \) with \( 1 \leq A \leq m \) and \( |J| = \text{even in } (4.9) \), we get readily the first equation of (4.11). Here, we have used (5.7). Considering \( E_{A,I} \) or \( E_{A,J} \) for \( m + 1 \leq A \leq m + n \) and \( |I| = \text{odd} = |J| \) in (4.9) and multiplying \( \sigma^I \) or \( \sigma^J \) from left, respectively, we have the second equality in (4.11) readily for \( I = (i) = (0, \ldots, 0, 1, 0, \ldots, 0), \ J = (j) = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathcal{I}_L \).

\( \Leftarrow \) To prove the converse statement, we have to construct functions \( F_A(1 \leq A \leq m + n) \) such that

\[
d\frac{d}{dt} f(X + tH) \bigg|_{t=0} = \sum_{A=1}^{m+n} H_A F_A(X)
\]

for \( X \in \mathcal{U}_L \) and \( H \in \mathcal{B}^{m,n}_L \).

Putting \( J = \tilde{0}, \ |J| = \text{even} \) and multiplying \( H_{A,J} \) to both sides of the first equation of (4.11), we have

\[
H_A f'_{X_A}(X) = \frac{d}{dt} f(X) \bigg|_{t=0} = \sum_I H_{A,I} \frac{\partial f(X)}{\partial X_{A,I}} = \sum_I H_{A,I} \sigma^I \frac{\partial}{\partial X_{A,0}} f(X) = H_A \frac{\partial}{\partial X_{A,0}} f(X).
\]

Therefore, for \( A = 1, \ldots, m \), we get

\[
F_A(X) = \frac{\partial}{\partial X_{A,0}} f(X) \quad \text{for } 1 \leq A \leq m, \ X \in U.
\]

To define \( F_A(X) \) for \( m + 1 \leq A \leq m + n \), we need to use the equation of (4.11). From the second one, applying Lemma 4.3 we know there exists an element \( F_A(X) \) satisfying

\[
\frac{\partial f(X)}{\partial X_{A,(i)}} = \sigma_i F_A(X) \quad \text{for } m + 1 \leq A \leq m + n.
\]

Applying the first equation of (4.11) with \( |J| = \text{odd} \), we have, when \( i_1 = 1 \),

\[
\frac{\partial f(X)}{\partial X_{A,1}} = \sigma_1 \frac{\partial f(X)}{\partial X_{A,(i_1)}} = \sigma_1 \sigma_{i_1} \frac{\partial f(X)}{\partial X_{A,(i_1)}} = \sigma^T F_A(X) \quad \text{with } \bar{I} = (0, i_2, i_3, \ldots) \in \{0, 1\}^N. \quad \square
\]

Theorem 4.11. Let \( \mathcal{U}_L \) be an open set in \( \mathcal{B}^{m,n}_L \) and let a function \( f : \mathcal{U}_L \to \mathcal{B}_L \) be given. Following conditions are equivalent:

(a) \( f \) is superdifferentiable on \( \mathcal{U}_L \), i.e. \( f \in \mathcal{G}^\infty(\mathcal{U}_L) \),

(b) \( f \) is \( \infty \)-times Gâteaux differentiable and \( f \in \mathcal{G}^1(\mathcal{U}_L) \),

(c) \( f \) is \( \infty \)-times Gâteaux differentiable and its (Gâteaux-) differential \( df \) is \( \mathcal{B}_{L,\text{ev}} \)-linear,

(d) \( f \) is \( \infty \)-times Gâteaux differentiable and its (Gâteaux-) differential \( df \) satisfies Cauchy-Riemann equations,

(e) \( f \) is supersmooth, i.e. it has the following representation, called superfield expansion, such that

\[
f(x, \theta) = \sum_{|a| \leq n} \theta^a \tilde{f}_a(x) \quad \text{with } \tilde{f}_a(q) \in C^\infty(\pi_B(\mathcal{U}_L)) \quad \text{and} \quad \tilde{f}_a(x) = \sum_{|a| = 0}^{\infty} \frac{1}{a!} \left. \frac{\partial^a f(q)}{\partial q^a} \right|_{q = x} x^a,
\]
5. The definition and characterization of supersmooth functions on FG-algebra

5.1. Remarks on FG-algebras. Though we introduced FG-algebras in §2 by using the sequence space \( \omega \), we prepare another definition using projective limits in order to clarify the relation to §4. We define index sets as

\[
\mathcal{I} = \{ \mathbf{I} = (i_1, i_2, \cdots) \in \{0, 1\}^\infty \mid |\mathbf{I}| = \sum_{j=1}^\infty i_j < \infty \} = \bigcup_{d=0}^\infty \mathcal{I}^{(d)} = \bigoplus_{d=0}^\infty \mathcal{I}^{[d]},
\]

where \( \mathcal{I}^{(d)} = \{ \mathbf{I} = (i_1, i_2, \cdots) \in \mathcal{I} \mid |\mathbf{I}| \leq d \} \), \( \mathcal{I}^{[d]} = \{ \mathbf{I} = (i_1, i_2, \cdots) \in \mathcal{I} \mid |\mathbf{I}| = d \} \),

\[
\mathcal{I}_L = \{ \mathbf{I} = (i_1, i_2, \cdots, i_L) \in \{0, 1\}^L \} = \bigcup_{d=0}^L \mathcal{I}_L^{(d)} = \bigoplus_{d=0}^L \mathcal{I}_L^{[d]},
\]

where \( \mathcal{I}_L^{(d)} = \{ \mathbf{I} = (i_1, i_2, \cdots, i_L) \mid |\mathbf{I}| \leq d \} \), \( \mathcal{I}_L^{[d]} = \{ \mathbf{I} = (i_1, i_2, \cdots, i_L) \mid |\mathbf{I}| = d \} \).

Clearly, we have

\[
\mathcal{I}_L^{(d)} \rightarrow \mathcal{I}^{(d)} \quad \text{and} \quad \mathcal{I}_L \rightarrow \mathcal{I} \quad \text{in} \, \omega \quad \text{when} \, L \rightarrow \infty.
\]

Besides \( \mathcal{C} \), for any \( L \) and \( d \leq L \), we put

\[
\mathcal{E}_L = \{ X = \sum_{\mathbf{I} \in \mathcal{I}_L} X_I \sigma^\mathbf{I} \mid X_I \in \mathbb{C} \}
\]

\[
\cong \bigwedge_{\mathbb{C}} (\mathbb{R}^L) = \text{the exterior algebra of forms on} \, \mathbb{R}^L \, \text{with coefficients in} \, \mathbb{C} \cong \mathbb{C}^{2^L},
\]

\[
\mathcal{E}_L^{(d)} = \left\{ X = \sum_{\mathbf{I} \in \mathcal{I}_L^{(d)}} X_I \sigma^\mathbf{I} \mid X_I \in \mathbb{C} \right\} \quad \text{and} \quad \mathcal{E}_L^{[d]} = \left\{ X = \sum_{\mathbf{I} \in \mathcal{I}_L^{[d]}} X_I \sigma^\mathbf{I} \mid X_I \in \mathbb{C} \right\}.
\]

\( \mathcal{C}_L \) is called \( L \)-skelton of \( \mathcal{C} \), etc. Since the family \( \{ \mathcal{E}_L \}_{L \geq 0} \) and the natural projections \( \{ \psi_{L,K} \} \) for \( K > L \), defined by \( \psi_{L,K} : \mathcal{E}_K \rightarrow \mathcal{E}_L \) with \( \psi_{L,K}(\sum_{\mathbf{I} \in \mathcal{I}_L} X_I \sigma^\mathbf{I}) = \sum_{\mathbf{I} \in \mathcal{I}_L} X_I \sigma^\mathbf{I} \), we have the set \( \{ \mathcal{E}_L, \psi_{L,K} \} \) which forms a projective system and yields a projective limit \( \mathcal{C}_\infty \). More precisely, the topology of \( \mathcal{C}_\infty \) is defined as follows: Elements \( X^{(n)} \) converges to \( X \) in \( \mathcal{C}_\infty \) if and only if for any \( \epsilon > 0 \) and \( \mathbf{I} \), there exists an integer \( n_0 = n_0(\epsilon, \mathbf{I}) \) such that \( |X_I^{(n)} - X_I| < \epsilon \) when \( n > n_0 \).

Claim 5.1. When \( L \rightarrow \infty \), we have the projective limits

\[
\mathcal{E}_L \rightarrow \mathcal{E}_{[\infty]} = \mathcal{C}, \quad \mathcal{E}_{L, ev} = \left\{ X = \sum_{\mathbf{I} \in \mathcal{I}_L, |\mathbf{I}| = ev} X_I \sigma^\mathbf{I} \mid X_0 \in \mathbb{R}, \, X_I \in \mathbb{C} \right\} \rightarrow \mathcal{C}_{ev},
\]

\[
\mathcal{E}_{L, od} = \left\{ X = \sum_{\mathbf{I} \in \mathcal{I}_L, |\mathbf{I}| = od} X_I \sigma^\mathbf{I} \mid X_I \in \mathbb{C} \right\} \rightarrow \mathcal{C}_{od} \quad \text{and} \quad \mathcal{E}_L^{m,n} = \mathcal{E}_{L, ev} \times \mathcal{E}_{L, od} \rightarrow \mathcal{E}_L^{m,n}.
\]

\[
\mathcal{E}_L^{(d)} = \left\{ X = \sum_{\mathbf{I} \in \mathcal{I}_L^{(d)}} X_I \sigma^\mathbf{I} \mid X_I \in \mathbb{C} \right\}, \quad \mathcal{E}_L^{[d]} = \left\{ X = \sum_{\mathbf{I} \in \mathcal{I}_L^{[d]}} X_I \sigma^\mathbf{I} \mid X_I \in \mathbb{C} \right\} \rightarrow \mathcal{E}^{(d)}, \quad \mathcal{E}^{[d]} \quad \text{(respectively)}.
\]

Claim 5.2. We denote the natural projection from \( \mathcal{C} \) to \( \mathcal{E}_L \) as \( p_L \) defined by

\[
p_L : \mathcal{C} \ni X = \sum_{\mathbf{I} \in \mathcal{I}} X_I \sigma^\mathbf{I} \rightarrow X_L = p_L(X) = \sum_{\mathbf{I} \in \mathcal{I}_L} X_I \sigma^\mathbf{I}.
\]

The projection \( p_L \) from \( \mathcal{C}_\infty \) onto \( \mathcal{E}_L \) is continuous and open for any \( L \geq 0 \).

Remark 5.1. Same holds for \( \mathcal{R}, \mathcal{R}_L, \mathcal{R}_{L, ev}, \mathcal{R}_{L, od}, \mathcal{R}_L^{(d)} \) and \( \mathcal{R}_L^{[d]} \).

Lemma 5.1. Suppose that there exist elements \( \{ A_i \}_{i=1}^\infty \subset \mathcal{R} \) satisfying

\[
(\sigma) A_i + \sigma_i A_j = 0 \quad \text{for any} \quad i, j \in \mathbb{N}
\]

Then there exists an element \( F \in \mathcal{R} \) such that \( A_i = \sigma_i F \) for \( i = 1, \cdots, \infty \).
Proof. We follow the argument in Lemma 4.4 of [42]. Since $A_i$ is represented by $A_i = \sum_{j \in \mathbb{T}} a^j_i \sigma^j$ with $a^j_i \in \mathbb{C}$ and $\sigma, A_i = 0$, we have $\sum_{j \in \mathbb{T}} a^j_i \sigma^j = 0$. Therefore, each $A_i$ can be written uniquely as $A_i = (\sum_{j \in \mathbb{T}} b^j_i \sigma^j) \sigma$, for some $b^j_i \in \mathbb{C}$. From the condition (5.1), we have $b^j_i = b^j_i$ for $j_i = j_j = 0$. Letting $b_j = b^j_i$ for $\{ j | j_i = 0 \}$, we put $F = \sum_{j \in \mathbb{T}} b_j \sigma^j$ which is well-defined and further more $A_i = \sigma_i F$ holds for each $i$. □

Definition 5.1. We denote the set of maps $f : \mathcal{R}_{od} \to \mathbb{R}$ which are continuous and $\mathcal{R}_{ev}$-linear (i.e. $f(\lambda X) = \lambda f(X)$ for $\lambda \in \mathcal{R}_{ev}$ and $X \in \mathcal{R}_{od}$) by $f \in \mathcal{L}_{\mathcal{R}_{ev}}(\mathcal{R}_{od} : \mathbb{R})$.

Corollary 5.2 (The self-duality of $\mathcal{R}$). For $f \in \mathcal{L}_{\mathcal{R}_{ev}}(\mathcal{R}_{od} : \mathbb{R})$, there exists an element $u_f \in \mathcal{R}$ satisfying $f(X) = X u_f$ for $X \in \mathcal{R}_{od}$.

Proof. Since $f : \mathcal{R}_{od} \to \mathbb{R}$ is $\mathcal{R}_{ev}$-linear, we have $f(X Y Z) = XY f(Z) = -XZ f(Y)$ for any $X, Y, Z \in \mathcal{R}_{od}$. By putting $X = \sigma_k, Y = \sigma_j, Z = \sigma_i$ and $f_i = f(\sigma_i) \in \mathbb{R}$ for $i = 1, \cdots, \infty$, we have $\sigma_k(\sigma_j f_i + \sigma_i f_j) = 0$ for any $k$. Therefore, $\sigma_i f_i + \sigma_j f_j = 0$, and by Lemma above, there exists $u_f \in \mathcal{R}$ such that $f_i = \sigma_i u_f$ for $i = 1, \cdots, \infty$. For $I = (i_1, \cdots) \in \{ 0, 1 \}^\infty$ with $|I|$ is odd, for any $k$ such that $i_k = 1$, by $\mathcal{R}_{ev}$-linearity of $f$, we have $f(\sigma^I) = \sigma_k f(\sigma^I_k)$. Therefore, we define a map $\tilde{f}(\sigma^I) = (-1)^{i_1 + \cdots + i_{k-1}} \sigma_k f(\sigma^I_k)$.

For $k = (i_1, \cdots, i_{k-1}, 0, i_{k+1}, \cdots)$. This map is well-defined because of $\sigma_j f_i + \sigma_i f_j = 0$. We extend $\tilde{f}$ as $\tilde{f}(X) = \sum_{I \in \mathbb{T}} X_I \tilde{f}(\sigma^I)$ for $X = \sum_{I \in \mathbb{T}} X_I \sigma^I \in \mathcal{R}_{od}$. Then, since $X_{I} \in \mathbb{C}$, $\tilde{f}(X) = \sum_{I \in \mathbb{T}} X_I \sigma^I u_f = X u_f$. In fact, if $I$ with $|I|$-odd contains $i_k \neq 0$, then

$$\tilde{f}(\sigma^I) = (-1)^{i_1 + \cdots + i_{k-1}} \sigma_k f(\sigma_k) = (-1)^{i_1 + \cdots + i_{k-1}} \sigma^I f(\sigma_k) \sigma_k u_f = \sigma^I u_f.$$ Clearly $\tilde{f}(X) = f(X)$. □

5.2. $\mathcal{C}$-valued functions and superdomains.

Lemma 5.3. Let $\phi(t)$ and $\Phi(t)$ be continuous $\mathcal{C}$-valued functions on an interval $[a, b] \subset \mathbb{R}$. Then,

(1) $\int_{a}^{b} dt \phi(t)$ exists,

(2) if $\Phi'(t) = \phi(t)$ on $[a, b]$, then $\int_{a}^{b} dt \phi(t) = \Phi(b) - \Phi(a)$,

(3) if $\lambda \in \mathcal{C}$ is a constant, then

$$\int_{a}^{b} dt (\phi(t) \cdot \lambda) = \left( \int_{a}^{b} dt \phi(t) \right) \cdot \lambda \quad \text{and} \quad \int_{a}^{b} dt (\lambda \cdot \phi(t)) = \lambda \cdot \int_{a}^{b} dt \phi(t).$$

Moreover, we may generalize above lemma for a $\mathcal{C}$-valued function $\phi(t)$ on an open set $\Omega \subset \mathbb{R}^m$.

Definition 5.2. For a set $U \subset \mathbb{R}^m$, we define $\pi_B^{-1}(U) = \{ X \in \mathcal{R}^m | \pi_B(X) \in U \}$. A set $\mathcal{U}_{ev} \subset \mathcal{R}^m$ is called an even superdomain if $\mathcal{U}_{ev,B} = \pi_B(\mathcal{U}_{ev}) \subset \mathbb{R}^m$ is open and connected and $\pi_B^{-1}(\mathcal{U}_{ev,B}) = \mathcal{U}_{ev}$. When $\mathcal{U} \subset \mathcal{R}^m$ is represented by $\mathcal{U} = \mathcal{U}_{ev} \times \mathcal{R}^m$ with a even superdomain $\mathcal{U}_{ev} \subset \mathcal{R}^m$, $\mathcal{U}$ is called a superdomain in $\mathcal{R}^m$.

5.3. Differentiability.

Definition 5.3. Let $f$ be a $\mathcal{C}$-valued function on a superdomain $\mathcal{U} \subset \mathcal{R}^m$. Then,

(i) a function $f$ is said to be super $C^1_{\mathcal{C}}$-differentiable, denoted by $f \in \mathcal{G}^1_{SD}(\mathcal{U} : \mathcal{C})$ or simply $f \in \mathcal{G}^1_{SD}$ if
there exist \( \mathcal{C} \)-valued continuous functions \( F_A \) \( (1 \leq A \leq m + n) \) on \( \mathcal{U} \) such that
\[
\frac{d}{dt} f(X + tH) \bigg|_{t=0} = f'_G(X, H) = \sum_{A=1}^{m+n} H_A F_A(X)
\]
for each \( X \in \mathcal{U} \) and \( H \in \mathcal{R}^{m|n} \) where \( f(X + tH) \) is considered as a \( \mathcal{C} \)-valued function w.r.t. \( t \in \mathbb{R} \). We denote \( F_A(X) \) by \( f_{X,A}(X) \). Moreover, for \( r \geq 2 \), \( f \) is said to be in \( \mathcal{G}^r_{SD} \), if \( F_A \) are \( \mathcal{G}^{r-1}_{SD} \). \( f \) is said to be \( \mathcal{G}^\infty_{SD} \) or superdifferentiable if \( f \) is \( \mathcal{G}^r_{SD} \) for all \( r \geq 1 \).

(ii) A function \( f \) is said to be super \( C^1_{SD} \)-differentiable, denoted by \( f \in \mathcal{F}^{1}_{SD}(\mathcal{U} : \mathcal{C}) \) or simply \( f \in \mathcal{F}^{1}_{SD} \) if there exist \( \mathcal{C} \)-valued continuous functions \( F_A \) \( (1 \leq A \leq m + n) \) on \( \mathcal{U} \) and functions \( \rho_A : \mathcal{U} \times \mathcal{R}^{m|n} \to \mathcal{C} \) such that
\[
\begin{align*}
(a) & \quad f(X + H) - f(X) = \sum_{J=1}^{m+n} H_A F_A(X) + \sum_{A=1}^{m+n} H_A \rho_A(X; H) \quad \text{for} \quad X \in \mathcal{R}^{m|n}, \\
(b) & \quad \rho_A(X, H) \to 0 \quad \text{in} \quad \mathcal{C} \quad \text{when} \quad H \to 0 \quad \text{in} \quad \mathcal{R}^{m|n},
\end{align*}
\]
for each \( X \in \mathcal{U} \) and \( X + H \in \mathcal{U} \). \( f \) is said to be super \( C^1_{SD} \)-differentiable, when \( F_A \in \mathcal{F}^{1}_{SD}(\mathcal{R}^{m|n} : \mathcal{C}) \) \( (1 \leq A \leq m + n) \). Analogously, we may define super \( C^r_{SD} \)-differentiability and we say it superdifferentiable if it is super \( C^\infty_{SD} \)-differentiable, denoted by \( \mathcal{F}^{\infty}_{SD} \).

**Problem 5.4.** What is the difference between \( \mathcal{G}^{1}_{SD} \) and \( \mathcal{F}^{1}_{SD} \)?

**Remark 5.2.** Though the notion of \( \mathcal{G}^{1}_{SD} \) in Definition 5.3 is of Gâteaux type, but by Lemma 5.10 it implies Fréchet type differentiability. Moreover, Gâteaux type definition of differentiability is easier to check.

**Remark 5.3.** Let \( \mathfrak{U} \) be an open set \( \mathcal{C}^{m|n} \). When \( f : \mathfrak{U} \to \mathcal{C} \) is in \( \mathcal{F}^{\infty}_{SD} \), \( f \) is also said to be superanalytic.

### 5.4. Examples of superdifferentiable functions.

**Lemma 5.5** ((1.1.17) of [12], Theorem 1 of [31] for \( m = 1, n = 0 \)). Let \( f \) be a analytic function on an open set \( V \subset \mathbb{C} \) to \( \mathbb{C} \). Then, we may extend \( f \) uniquely to a function \( \hat{f} : \mathcal{C}^{0|0} \to \mathcal{C} \) as
\[
(5.2) \quad \hat{f}(z) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(z_B) z_B^n \quad \text{for} \quad z = z_B + z_S \quad \text{with} \quad z_B \in V,
\]
which is superanalytic.

**Proof.** See the proof of Lemma 5.3 below, more precise than that in Theorem 1 of [31], and it gives also that \( \hat{f} \) is superanalytic on \( \mathfrak{V} = \pi_B^{-1}(V) \).

To prove the uniqueness, in the proof of Theorem 1 of [31], they use the fact:

“for a superanalytic function \( g \) on \( \mathfrak{V} \), if \( g(z_B) = 0 \) on \( \pi_B(\mathfrak{V}) \) implies \( g(z) = 0 \) on \( z \in \mathfrak{V} \).

As this fact is generalized to any function in \( \mathcal{G}^{\infty}_{SD} \), we omit its proof. See, Lemma 5.10 below.

**Lemma 5.6** (Theorem 1 of [31]). Let \( f \) be real analytic on \( \mathbb{R}^m \). Then, there exist \( f_J(x) \in \mathcal{C} \) and \( \rho_J(x, y) \in \mathcal{C} \) such that
\[
\hat{f}(x + y) = \hat{f}(x) + \sum_{J=1}^{m} y_J f_J(x) + \sum_{J=1}^{m} y_J \rho_J(x, y) \quad \text{with} \quad \rho_J(x, y) \to 0 \quad \text{in} \quad \mathcal{C} \quad \text{when} \quad y \to 0 \quad \text{in} \quad \mathcal{R}^{m|0}.
\]
Proof. (Following is a slight modification of [31]) For the sake of simplicity, we consider only the case $m = 1$. Then, we have
\[
\tilde{f}(x + y) = \sum_{n=0}^{\infty} \frac{1}{n!} \tilde{f}^{(n)}(x_B + y_B)(x_S + y_S)^n
\]
\[
= \sum_{n=0}^{\infty} \frac{1}{n!} \left( \sum_{\ell=0}^{\infty} \frac{1}{\ell!} f^{(\ell+n)}(x_B) y_B^\ell \right) \left( \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} x_S^{n-k} y_S^k \right) \text{ (real analyticity of } f(q))
\]
\[
= \sum_{n=0}^{\infty} \left[ \sum_{\ell=0}^{\infty} \frac{1}{\ell!} f^{(\ell+j+k)}(x_B) y_B^{\ell+j+k} \right] \left( \sum_{k=0}^{j+k} \frac{n!}{k!(j+k)!} x_S^{j+k} y_S^k \right) \text{ (renumbering)}
\]
\[
= \sum_{n=0}^{\infty} \frac{1}{n!} \left( \sum_{j=0}^{\infty} \frac{1}{j!} f^{(n+j)}(x_B) x_S^j \right) \left( \sum_{k=0}^{\infty} \frac{n!}{k! j!} y_B^k y_S^j \right) \text{ (rearranging)}
\]
\[
= \sum_{n=0}^{\infty} \frac{1}{n!} \left( \sum_{j=0}^{\infty} \frac{1}{j!} f^{(n+j)}(x_B) x_S^j (y_B + y_S)^n \right) = \sum_{n=0}^{\infty} \frac{1}{n!} \tilde{f}^{(n)}(x) y^n.
\]
Therefore, $\epsilon(x, y)$ defined below is horizontal w.r.t. $y$, i.e.
\[
\tilde{f}(x + y) - \tilde{f}(x) = \sum_{n=1}^{\infty} \frac{1}{n!} \tilde{f}^{(n)}(x) y^n = \tilde{f}^{(1)}(x) y + \epsilon(x, y)
\]
with $\epsilon(x, y) = \sum_{n=2}^{\infty} \frac{1}{n!} \tilde{f}^{(n)}(x) y^{n-1} \to 0$ in $\mathcal{C}$ when $y \to 0$. \(\square\)

Remark 5.4. Since $f$ is real analytic on $\mathbb{R}^m$, there exists a function $\delta(q) > 0$ such that when $|q'| \leq \delta(q)$, $f(q + q')$ has the Taylor series expansion at $q$. From above proof, $\tilde{f}(x + y)$ is Pringsheim regular for $|y_B| \leq \delta(x_B)$.

Following proposition exhibits the reason why we introduce a weaker topology than Rogers’ one:

Proposition 5.7 (Proposition 2.2 of [31]). Let $U \subset \mathbb{R}^m$ be an open set and let $f \in C^\infty(U : \mathcal{C})$ be represented by
\[
f(q) = \sum_{J \in \mathcal{I}} f_J(q) \sigma^J \quad \text{with} \quad f_J(q) \in C^\infty(\mathcal{U}_{ev,B} : \mathcal{C}) \quad \text{for each } J \in \mathcal{I}.
\]
Then, we may define a mapping $\tilde{f}$ from $\mathcal{U}_{ev}$ into $\mathcal{C}$, called the Grassmann continuation of $f$, by
\[
\tilde{f}(x) = \sum_{|\alpha| \geq 0} \frac{1}{\alpha!} \partial_\alpha^2 f(x_B) x_\mathcal{S}^\alpha \quad \text{where} \quad \partial_\alpha^2 f(x_B) = \sum_J \partial_\alpha^2 f_J(x_B) \sigma^J.
\]
Here, we put, $\mathcal{U}_{ev} = \pi_B^{-1}(U) \subset \mathfrak{m}^{n|0}, x = (x_1, \ldots, x_m), x = x_B + x_S \quad \text{with} \quad x_B = (x_{1,B}, \ldots, x_{m,B}) = (q_1, \ldots, q_m) = q \in \mathcal{U}_{ev,B}, x_S = (x_{1,S}, \ldots, x_{m,S})$ and $x^\alpha = x_1^{\alpha_1} \cdots x_m^{\alpha_m}$.

Proof. [Since this proposition uses essentially the weak topology with algebraic manipulation, we restate it fully here. Main point of this proposition is to see whether this mapping (5.4) is well-defined. Therefore, by using the degree of Grassmann generators, we need to define $\tilde{f}^{[k]}$, the $k$-th degree component of $\tilde{f}$,]

Denoting by $x_{1,S}^{[k_1]}$, the $k_1$-th degree component of $x_{1,S}$, we get
\[
(x_{1,S}^{[k_1]})^{[k_1]} = \sum (x_{1,S}^{[r_1]} p_{1,1} \cdots (x_{1,S}^{[r_]} p_{1,\ell}).
\]
Here, the summation is taken for all partitions of an integer $\alpha_1$ into $\alpha_1 = p_{1,1} + \cdots + p_{1,\ell}$ satisfying $\sum_{i=1}^{\ell} r_i p_{1,i} = k_1, r_i \geq 0$. Using these notations, we put
\[
\tilde{f}^{[k]}(x) = \sum_{|\alpha| \geq 0} \frac{1}{\alpha!} (\partial_\alpha^2 f)(x_B) (x_{1,S}^{[k_1]} \cdot \cdots (x_{m,S}^{[k_m]})^{[k_m]}
\]
where
\[(\partial_q^\alpha f)^{[k_0]}(x_B) = \sum_{|J|=k_0} \partial_q^\alpha f_J(x_B) \sigma^J.
\]

Or more precisely, we have
\[
\begin{align*}
\tilde{f}^{[0]}(x) &= f^{[0]}(x_B), \\
\tilde{f}^{[1]}(x) &= f^{[1]}(x_B), \\
\tilde{f}^{[2]}(x) &= f^{[2]}(x_B) + \sum_{J=1}^m (\partial_q f)^{[0]}(x_B)(x_{j,S})^2, \\
\tilde{f}^{[3]}(x) &= f^{[3]}(x_B) + \sum_{J=1}^m (\partial_q f)^{[1]}(x_B)(x_{j,S})^2, \\
\tilde{f}^{[4]}(x) &= f^{[4]}(x_B) + \sum_{J=1}^m (\partial_q f)^{[2]}(x_B)(x_{j,S})^2
\end{align*}
\]

\[+ \frac{1}{2} \sum_{J=1}^m \sum_{k \neq k} (\partial_q^2 f)^{[0]}(x_B)(x_{j,S})^4, \quad \tilde{f}^{[k]}(x) = \tilde{f}^{[k]}(x_B) \quad \text{for} \quad k = 4, 5, \ldots.
\]

Since \(\tilde{f}^{[i]}(x) \neq \tilde{f}^{[k]}(x) (j \neq k)\) in \(C\), we may take the sum \(\sum_{i=0}^\infty \tilde{f}^{[i]}(x) \in C = \bigoplus_{k=0}^\infty C^{[k]}\), which is denoted by \(\tilde{f}(x)\). Therefore, rearranging the above ‘summation’, we get rather the ‘familiar’ expression as in (5.3).

\[\square\]

**Corollary 5.8** (Corollary 2.3 of [21]). If \(f\) and \(\tilde{f}\) be given as above, then (i) \(\tilde{f}\) is continuous, (ii) \(\tilde{f}(x) = 0\) in \(U_{ev}\) implies \(f(x_B) = 0\) in \(U_{ev,B}\) and (iii) if we define the partial derivatives of \(\tilde{f}\) by
\[
\partial_x \tilde{f}(x) = \frac{d}{dt} \tilde{f}(x + t\epsilon_j) \bigg|_{t=0} \quad \text{where} \quad \epsilon_j = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^{m[0]},
\]
then we get
\[
\partial_x \tilde{f}(x) = \tilde{\partial}_q f(x) \quad \text{for} \quad j = 1, \ldots, m.
\]

Analogously, we have
\[
\partial^\alpha q \tilde{f}(x) = \tilde{\partial}^\alpha q \tilde{f}(\cdot)(x).
\]
(iv) Moreover, for \(y = (y_1, \ldots, y_m) \in \mathbb{R}^{m[0]},\)
\[
\frac{d}{dt} \tilde{f}(x + ty) \bigg|_{t=0} = \sum_{J=1}^{m} y_j \sum_{\alpha = 0}^{m} \frac{1}{\alpha!} \partial^\alpha q \partial q_j f(x_B) x^\alpha_J \sum_{J=1}^{m} y_j \partial_x \tilde{f}(x).
\]

**Proof.** Let \(y_j = y_j,B + y_j,S \in \mathbb{R}_{ev}\). For \((y,J) = y_j,\epsilon_j = y_j,B \epsilon_j + y_j,S \epsilon_j = y_j,B + y_j,S \in \mathbb{R}^{m[0]}\), as
\[
\frac{d}{dt} \tilde{f}(x + ty(J)) = \frac{d}{dt} \left\{ \sum_{|\alpha|=0}^{\infty} \frac{1}{\alpha!} \left( \sum_J \partial^\alpha_q f_J(x_B + ty(J,B)) \sigma^J \right) (x_S + ty(J,S))^\alpha \right\},
\]
we get easily,
\[
\frac{d}{dt} \tilde{f}(x + ty(J)) \bigg|_{t=0} = y_j,B \sum_{|\alpha|=0}^{\infty} \frac{1}{\alpha!} \left( \sum_J \partial^\alpha_q f_J(x_B) \sigma^J \right) x^\alpha_S + y_j,S \sum_{|\alpha|=0}^{\infty} \frac{1}{\alpha!} \left( \sum_J \partial^\alpha_q f_J(x_B) \sigma^J \right) x^\alpha_S
\]
\[= y_j \sum_{\alpha}^{m} \frac{1}{\alpha!} \partial^\alpha q \partial q_j f(x_B) x^\alpha_S = y_j \tilde{\partial}_q f(x).
\]

Here \(\alpha = (\alpha_1, \ldots, \alpha_j - 1, \ldots, \alpha_m)\). Putting \(y_j = 1\) in the above, we have (5.7). Last equality is proved by induction with the length \(|\alpha|\). \(\square\)
Definition 5.5. A mapping $f : C \rightarrow \mathbb{R}$ is assumed to be homogeneous (i.e.,

$$f(a \cdot x) = a^p f(x)$$

for each $a > 0$).

More generally,

**Lemma 5.9.** Let $f(q) \in C^\infty(\mathbb{R}^m)$, we have the Taylor expansion for $\tilde{f}$: For any $N$, there exists $\tilde{\tau}_N(x, y) \in C$ such that

$$\tilde{f}(x + y) = \sum_{|\alpha| = 0}^N \frac{1}{\alpha!} \partial_x^\alpha \tilde{f}(x) y^\alpha + \tilde{\tau}_N(x, y),$$

with

$$\tilde{\tau}_N(x, y) = \sum_{|\alpha| = N + 1} y^\alpha \int_0^1 dt \frac{1}{N!} (1 - t)^N \partial_x^\alpha \tilde{f}(x + ty).$$

**Proof.** Substituting $q = x_B$ and $q' = y_B$ in

$$f(q + q') = \sum_{|\alpha| = 0}^N \frac{1}{\alpha!} \partial_x^\alpha f(q) q^\alpha + \sum_{|\alpha| = N + 1} q'^\alpha \int_0^1 dt \frac{1}{N!} (1 - t)^N \partial_x^\alpha f(q + t q'),$$

and extending both sides, we have the desired result by (5.17).

**Corollary 5.10.** For $f(q) \in C^\infty(\mathbb{R}^m)$, $\tilde{f} \in F^1_{SD}$.

**Proof.** For $N = 1$ in the above, we have

$$\tilde{f}(x + y) = \tilde{f}(x) + \sum_{j=1}^m y_j \partial_{x_j} \tilde{f}(x) + \sum_{j=1}^m y_j \rho_j(x, y) \quad \text{with} \quad \rho_j(x, y) = \sum_{k=1}^m y_k \int_0^1 (1 - t) \partial_{x_k x_j} \tilde{f}(x + ty) dt.$$ 

Clearly, if $y \to 0$ in $\mathbb{R}^m_{ev}$, then for each $x \in \mathbb{R}^m_{ev}$, $\rho_j(x, y) \to 0$.

**Definition 5.4.** For a given even superdomain $\mathcal{U}_{ev} \subset \mathbb{R}^{m|0}$, a mapping $\tilde{f}$ from $\mathcal{U}_{ev}$ into $C$ is called a supersmooth function if $\tilde{f}$ is the Grassmann continuation of a smooth mapping $f$ from $\mathcal{U}_{ev,B} = \pi_B(\mathcal{U}_{ev})$ into $C$. We denote by $C_{SS}(\mathcal{U}_{ev} : C)$, the set of supersmooth functions on $\mathcal{U}_{ev}$.

**Definition 5.5.** (1) A mapping $f$ from a superdomain $\mathcal{U} \subset \mathbb{R}^{m|n}$ to $C$ is called supersmooth, if it has the following form:

$$f(x, \theta) = \sum_{|\alpha| \leq n} \theta^\alpha f_\alpha(x)$$

with $a = (a_1, \cdots, a_n) \in \{0, 1\}^m$, $\theta^\alpha = \theta_1^{a_1} \cdots \theta_n^{a_n}$ and $f_\alpha(x) \in C_{SS}(\mathcal{U}_{ev} : C)$. In the following, supersmooth functions are assumed to be homogeneous (i.e., $f_\alpha(x)$ is homogeneous for each $a$), unless otherwise mentioned and we denote the set of them by $C_{SS}(\mathcal{U} : C)$. Moreover, we put

$$\mathcal{X}_{SS} = \{ f(x, \theta) \in C_{SS}(\mathcal{U} : C) | f_\alpha(x) \in \mathcal{C}_{SS} \}. $$

(2) For $f \in C_{SS}(\mathcal{U} : C)$, $j = 1, 2, \cdots, m$ and $s = 1, 2, \cdots, n$, we put

$$f_j(X) = \sum_{|\alpha| \leq n} \theta^\alpha \partial_{x_j} f_\alpha(x),$$

$$F_{s+m}(X) = \sum_{|\alpha| \leq n} (-1)^{|\alpha|} \theta_1^{a_1} \cdots \theta_s^{a_s-1} \cdots \theta_n^{a_n} \cdot f_\alpha(x).$$
where \( l(a) = \sum_{j=1}^{s-1} a_j \) and \( \theta_s^{-1} = 0 \). \( F_A(X) \) are called the partial derivatives of \( f \) with respect to \( X_A \) at \( X = (x, \theta) \) and are denoted by

\[
\begin{aligned}
f_A(X) &= \frac{\partial}{\partial x_j} f(x, \theta) = \partial_{x_j} f(x, \theta) = f_{x_j}(x, \theta) \quad \text{for} \quad j = 1, 2, \cdots, m, \\
F_m+s(X) &= \frac{\partial}{\partial \theta_s} f(x, \theta) = \partial_{\theta_s} f(x, \theta) = f_{\theta_s}(x, \theta) \quad \text{for} \quad s = 1, 2, \cdots, n
\end{aligned}
\]

or simply by

\[
F_A(X) = \partial_{X_A} f(X) = f_{X_A}(X) \quad \text{for} \quad A = 1, \cdots, m + n.
\]

**Remark 5.5.**

(1) We only use the derivatives defined above which are called the left derivatives with respect to odd variables. Because, after bringing the variable \( \theta_k \) to the left in each monomial, we replace it with \( 1 \). (Some people call these as right derivatives, cf. Vladimirov and Volovich [40], etc.) Similarly, we define the right derivatives with respect to odd variables as follows: Put

\[
C^{(r)}_{SS}(U : \mathcal{C}) = \{ f(x, \theta) = \sum_{|a| \leq n} f_a(x)\theta^a \mid f_a \in \mathcal{C} \}.
\]

For \( f \in C^{(r)}_{SS}(U : \mathcal{C}), j = 1, 2, \cdots, m \) and \( s = 1, 2, \cdots, n \), we put

\[
\begin{aligned}
F^{(r)}_j(X) &= \sum_{|a| \leq n} \partial_{x_j} f_a(x)\theta^a, \\
F^{(r)}_{m+s}(X) &= \sum_{|a| \leq n} (-1)^{r(a)} f_a(x)\theta_{a_1}^s \cdots \theta_{a_s}^s \cdots \theta_{a_n}
\end{aligned}
\]

where \( r(a) = \sum_{j=s+1}^n a_j \). \( F^{(r)}_A(X) \) are called the (right) partial derivatives of \( f \) with respect to \( X_A \) at \( X = (x, \theta) \) and are denoted by

\[
F_j^{(r)}(X) = \frac{\partial}{\partial x_j} f(x, \theta) = \partial_{x_j} f(x, \theta), \quad F_{m+s}^{(r)}(X) = f(x, \theta) \frac{\partial}{\partial \theta_s} = f(x, \theta) \partial_{\theta_s}
\]

for \( j = 1, 2, \cdots, m \) and \( s = 1, 2, \cdots, n \).

(2) As we use the infinite dimensional Grassmann algebras, the expression [5.14] is unique. In fact, \( \sum a_j f_a(x) \equiv 0 \) on \( U \) implies \( f_a(x) \equiv 0 \) (see, p. 322 in Vladimirov and Volovich [40].)

(3) The higher derivatives are defined analogously and we use the following notations.

\[
\partial^a \equiv \partial_{a_1} \cdots \partial_{a_m} \quad \text{and} \quad \partial^a \equiv \partial_{a_n} \cdots \partial_{a_1},
\]

for multi-indices \( \alpha = (\alpha_1, \cdots, \alpha_m) \in (\mathbb{N} \cup \{0\})^m \) and \( a = (a_1, \cdots, a_n) \in \{0, 1\}^n \).

Repeating the argument in proving Corollary 5.8 we get

\[
f \in C_{SS}(U : \mathcal{C}) \implies \left. \frac{d}{dt} f(X + tY) \right|_{t=0} = \sum_{j=1}^{m} y_j \frac{\partial}{\partial x_j} f(X) + \sum_{s=1}^{n} \omega_s \frac{\partial}{\partial \theta_s} f(X)
\]

where \( X = (x, \theta), Y = (y, \omega) \in \mathbb{R}^{m+n} \) such that \( X + tY \in U \) for any \( t \in [0, 1] \). That is,

**Corollary 5.11.** \( C_{SS}(U : \mathcal{C}) \implies G_{SD}^{1}(U : \mathcal{C}) \).

To relate the definitions \( C_{SS} \) and \( G_{SD}^{1} \), we need the following notion.

**Definition 5.6** (p. 246 of [42]). Let \( U \) be an open set in \( \mathbb{R}^{m+n} \) and \( f : U \to \mathbb{R} \) (or \( \mathbb{C} \)). \( f \) is said to be admissible on \( U \) if there exists some \( L \geq 0 \) and a \( \mathbb{R} \) (or \( \mathbb{C} \))-valued function \( \phi \) defined on \( U_L = p_L(U) \) such that \( f(X) = \phi \circ p_L(X) = \phi(p_L(X)) \). For \( r \geq 0 \) \( (0 \leq r \leq \infty) \), \( f \) is said to be admissible \( C^r \) (or simply \( f \in C^r(U : \mathcal{C}) \)) if \( \phi \in C^r(U_L : \mathbb{R}) \) or \( \in C^r(U_L : \mathbb{C}) \).
Let \( f(X) = \sum_{I \in \mathcal{I}} \sigma^I f_I(X) \) with \( f_I \) is admissible \( C^r \) (or simply \( f \in C^r \)) on \( \mathcal{U} \), \( f \) is called admissible \( C^r \) \((\mathcal{U} : \mathcal{C})\) on \( \mathcal{U} \). More precisely, there exists some \( L_I \geq 0 \) and a \( \mathcal{R}(or \mathcal{C})\)-valued function \( \phi_I \) defined on \( \mathcal{U}_L = p_L(\mathcal{U}) \) such that \( f_I(X) = \phi_I p_L(X) = \phi_I(p_L(X)). \) Moreover, we define its partial derivatives by

\[
\frac{\partial f}{\partial X_{A,K}} = \sum_J \sigma^I \frac{\partial f_J}{\partial X_{A,K}} = \begin{cases} |K| = ev & \text{if } 1 \leq A \leq m, \\
|K| = od & \text{if } m + 1 \leq A \leq m + n. 
\end{cases}
\]

**Definition 5.7** (p.246 of [42]). A \( \mathcal{R}(or \mathcal{C})\)-valued function \( f \) on \( \mathcal{U} \) is said to be projectable if for each \( L \geq 0 \), there exists a \( \mathcal{R}(or \mathcal{C})\)-valued function \( f_L \) defined on \( \mathcal{U}_L = R^m_{[n]} \) such that \( p_L \circ f = f_L \circ p_L \) on \( \mathcal{U} \).

**Claim 5.3.** A projectable function on \( \mathcal{U} \) is also admissible on \( \mathcal{U} \).

**Proof.** We use the map \( \text{proj} : \mathcal{R} \ni X = \sum_{I \in \mathcal{I}} X_I \sigma^I \rightarrow X_I \in \mathcal{R}(or \mathcal{C}) \) introduced in §2. Then, for each \( I \in \mathcal{I} \), taking \( L \) such that \( I \in \mathcal{I}_L \), we have

\[
\begin{array}{ccc}
\mathcal{U} & \xrightarrow{f} & \mathcal{R} \\
\downarrow p_L & & \downarrow p_L \\
\mathcal{U}_L & \xrightarrow{f_L} & \mathcal{R}_L \\
\downarrow \text{proj} & & \downarrow \text{proj} \\
\mathcal{U} & \xrightarrow{\text{Id}} & \mathcal{R} \\
\end{array}
\]

**Theorem 5.12** (Theorem 1 of [42]). Let \( \mathcal{U} \) be a convex open set in \( \mathcal{R}^m_{[n]} \). If \( f : \mathcal{U} \rightarrow \mathcal{R} \) is in \( G_{SD}^m \), then \( f \) is projectable and \( C^r \) on \( \mathcal{U} \).

**Proof.** Since \( \frac{d}{dt} f(X + tH) = \sum_{A=1}^{m+n} H_A F_A(X + tH) \), we have

\[
f(X + H) - f(X) = \sum_{A=1}^{m+n} H_A \int_0^1 \frac{d}{dt} f(X + tH) dt = \sum_{A=1}^{m+n} H_A \int_0^1 F_A(X + tH) dt.
\]

This means that if \( p_L(H_A) = 0 \), then \( p_L(f(X + H) - f(X)) = 0 \). Therefore if we define \( f_L : \mathcal{U}_L \rightarrow \mathcal{R}_L \) by \( f_L(p_L(Z)) = p_L(f(Z)) \), then it implies that \( f \) is projectable and so admissible. For \( E_{A,K} = \sigma^K \varepsilon_A \in \mathcal{R}^m_{[n]} \) with \( \varepsilon_A = (0, \ldots, 0, 1, 0, \ldots, 0) \), we have \( \partial f / \partial X_{A,K} = \sum_{A=1}^{m+n} H_A \int_0^1 F_A(X + tH) dt \).

\( f_L \) is \( C^1 \) on \( \mathcal{U}_L \), thus the function \( f \) is admissible \( C^1 \) on \( \mathcal{U} \).

5.5. **Cauchy-Riemann relation.** To understand the meaning of supersmoothness, we consider the dependence with respect to the “coordinate” more precisely.

**Proposition 5.13** (Theorem 2 of [42]). Let \( f(X) = \sum_I f_I(X) \sigma^I \in G_{SD}^m(\mathcal{U} : \mathcal{C}) \) where \( \mathcal{U} \) is a superdomain in \( \mathcal{R}^m_{[n]} \). Let \( X = (X_A) \) be represented by \( X_A = \sum_I X_{A,I} \sigma^I \) where \( A = 1, \ldots, m + n \), \( X_{A,} \in \mathcal{C} \) for \( I \neq 0 \) and \( X_{A,0} \in \mathcal{R} \). Then, \( f(X) \), considered as a function of countably many variables \( \{X_{A,} \} \) with values in \( \mathcal{C} \), satisfies the following (Cauchy-Riemann type) equations.

\[
\begin{align*}
\partial f / \partial X_{A,K} &= \sigma^I \partial f / \partial X_{A,0} \quad \text{for } 1 \leq A \leq m, \ |I| = ev, \\
\sigma^K \partial f / \partial X_{A,J} + \sigma^I \partial f / \partial X_{A,K} &= 0 \quad \text{for } m + 1 \leq A \leq m + n, \ |J| = od = |K|.
\end{align*}
\]
Here, we define
\begin{equation}
\frac{\partial}{\partial X_{A_{i}j}} f(X) = \left. \frac{d}{dt} f(X + tE_{A_{i}j}) \right|_{t=0} \quad \text{with} \quad E_{A_{i}j} = \sigma^{j}e_{A_{i}} = (0, \cdots, 0, \sigma^{j}, 0, \cdots, 0) \in \mathfrak{R}^{m|n}.
\end{equation}

Conversely, let a function \( f(X) = \sum_{i} f_{i}(X)\sigma^{i} \) be given such that \( f_{i}(X + tY) \in C^{\infty}([0, 1] : \mathbb{C}) \) for each fixed \( X, Y \in U \) and \( f(X) \) satisfies above \( \text{(5.16)} \) with \( \text{(5.17)} \). Then, \( f \in G_{SD}^{\infty}(U : \mathbb{C}) \).

Proof. Replacing \( Y \) with \( E_{A_{i}j} \) with \( 1 \leq A \leq m \) and \( |J| = \text{even} = |K| \) in \( \text{(5.16)} \), we get readily the first equation of \( \text{(5.16)} \). Here, we have used \( \text{(5.7)} \). Considering \( E_{A_{i}j} \) or \( E_{A_{i}K} \) for \( m + 1 \leq A \leq m + n \) and \( |J| = \text{odd} = |K| \) in \( \text{(5.15)} \) and multiplying \( \sigma^{K} \) or \( \sigma^{J} \) from left, respectively, we have the second equality in \( \text{(5.16)} \) readily.

To prove the converse statement, we have to construct functions \( F_{A}(1 \leq A \leq m + n) \) which satisfies
\begin{equation}
\frac{d}{dt} f(X + tH) \bigg|_{t=0} = \sum_{A=1}^{m+n} H_{A}F_{A}(X)
\end{equation}
for \( X \in U \) and \( H = (H_{A}) \in \mathfrak{R}^{m|n} \).

For \( 1 \leq A \leq m \), we put \( F_{A}(X) = \frac{\partial}{\partial X_{A_{i}j}} f(X) \quad \text{for} \quad X \in U \).

On the other hand, from the second equation of \( \text{(5.16)} \) and Lemma \( \text{5.1} \), we have an element \( F_{A}(X)(m + 1 \leq A \leq m + n) \) such that \( \sigma^{j}F_{A}(X) = \frac{\partial}{\partial X_{A_{i}j}} f(X) \).

Using these \( \{ F_{A}(X) \} \) defined above, we claim that \( \text{(5.18)} \) holds following Yagi’s argument.

Since \( f \) is admissible, for any \( L \geq 0 \), \( p_{L} \circ f \) is so also, therefore there exist some \( N \geq 0 \) and a \( \mathfrak{R}_{L} \)-valued \( C^{\infty} \) function \( f_{N} \) such that \( p_{L} \circ f(X) = f_{N} \circ p_{N}(X) \) on \( X \in U \). By natural imbedding from \( \mathfrak{R}_{L} \) to \( \mathfrak{R}_{N} \), we may assume \( N \geq L \). Then, we can show that
\[ \frac{\partial}{\partial X_{A_{i}K}} f_{N}(p_{N}(X)) = \begin{cases} p_{L} \left( \frac{\partial}{\partial X_{A_{i}K}} f(X) \right) & \text{if} \quad K \in \mathcal{I}_{N}, \\ 0 & \text{if otherwise}. \end{cases} \]

Therefore, for any \( L \geq 0 \),
\[ p_{L} \left. \frac{d}{dt} f(X + tH) \right|_{t=0} = \frac{d}{dt} p_{L}(f(X + tH)) \bigg|_{t=0} = \left( \vdots \right) p_{L} \left( \frac{d}{dt} g(t) \right) \bigg|_{t=0} = \frac{d}{dt} (p_{L}(g(t))) \bigg|_{t=0}, \]
\[ = \frac{d}{dt} f_{N}(p_{N}(X + tH)) \bigg|_{t=0} = \left( \vdots \right) p_{L}(f(X)) = f_{N}(p_{N}(X)) \]
\[ = \sum_{A} \sum_{K} (p_{N}(H))_{A_{i}K} \frac{\partial}{\partial X_{A_{i}K}} f_{N}(p_{N}(X)) \left( \vdots \right) \text{finite dimensional case} \]
\[ = \sum_{A} \sum_{K} (p_{N}(H))_{A_{i}K} p_{L} \left( \frac{\partial}{\partial X_{A_{i}K}} f(X) \right) \left( \vdots \right) p_{L}(g(X)) = g_{N}(p_{N}(X)) \]
\[ = \sum_{A} \sum_{K} (p_{N}(H))_{A_{i}K} p_{L}(\sigma^{K}F_{A}(X)) \left( \vdots \right) \text{by} \quad \text{(5.16)} \]
\[ = \sum_{A} \sum_{K} (p_{N}(H))_{A_{i}K} p_{L}(\sigma^{K}) \cdot p_{L}(F_{A}(X)) \]
\[ = \sum_{A} \left( \sum_{K} (p_{N}(H))_{A_{i}K} p_{L}(\sigma^{K}) \right) p_{L}(F_{A}(X)) \]
\[ = \sum_{A} p_{L} \left( (p_{N}(H))_{A_{i}K} \right) p_{L}(\sigma^{K}) p_{L}(F_{A}(X)) \]
Thus, we have (5.13). The continuity of $F_A(X)$ is clear. \qed

Remark 5.6. For function with finite number of independent variables, it is well-known how to define its partial derivatives. But when that number is infinite, it is not so clear whether the change of order of differentiation affects the result, etc. Therefore, we reduce the calculation to the cases with finite number of variables and making the number to infinity.

Theorem 5.14 (Theorem 3 of [12]). Let $f$ be a $\mathcal{C}$-valued $C^\infty$ function on an open set $\mathcal{U}\subset \mathcal{R}^{m|n}$. If $f$ is $G_{\mathcal{SD}}^1(\mathcal{U}: \mathcal{C})$, then $f$ is $G_{\mathcal{SD}}^\infty$ on $\mathcal{U}$.

Proof. Since $f \in G_{\mathcal{SD}}^1$, it satisfies Cauchy-Riemann equation. As $f$ is $C^\infty$ on $\mathcal{U}$, $g(X) = \frac{\partial}{\partial X_A, \theta} f(X)$ also satisfies the C-R equation, for $1 \leq A \leq m$. In fact, for $1 \leq B \leq m$, $|J| = \text{even}$,

$$\frac{\partial}{\partial X_B, \theta} g(X) = \frac{\partial}{\partial X_B, \theta} f(X) = \frac{\partial}{\partial X_A, \theta} f(X) = \frac{\partial}{\partial X_B, \theta} f(X) = \frac{\partial}{\partial X_A, \theta} f(X) = \frac{\partial}{\partial X_B, \theta} f(X) = \frac{\partial}{\partial X_A, \theta} f(X).$$

And for $m+1 \leq A \leq m+n$, $|J| = |K| = \text{odd}$,

$$\sigma^K \frac{\partial}{\partial X_A, \theta} g(X) + \sigma^J \frac{\partial}{\partial X_B, \theta} g(X) = \sigma^K \frac{\partial}{\partial X_A, \theta} f(X) + \sigma^J \frac{\partial}{\partial X_B, \theta} f(X) = \frac{\partial}{\partial X_A, \theta} f(X) = \frac{\partial}{\partial X_B, \theta} f(X) = 0.$$

Hence $\frac{\partial}{\partial X_A} f$ (for $1 \leq A \leq m$) is $G_{\mathcal{SD}}^1$ on $\mathcal{U}$.

Analogously, for $m+1 \leq A \leq m+n$, $\frac{\partial}{\partial X_A, \theta} f = \sigma^J \frac{\partial}{\partial X_A} f$ is also $G_{\mathcal{SD}}^1$ on $\mathcal{U}$. In fact, we have, for $|K| = \text{even}$,

$$\frac{\partial}{\partial X_B, \theta} \sigma^J \frac{\partial}{\partial X_A} f = \sigma^J \sigma^K \frac{\partial}{\partial X_B, \theta} \frac{\partial}{\partial X_A} f = \sigma^K \frac{\partial}{\partial X_B, \theta} f = \sigma^K \frac{\partial}{\partial X_B, \theta} f = 0.$$

And for $|I|, |J|, |K| = \text{odd}$, $m+1 \leq B \leq m+n$,

$$\left(\sigma^K \frac{\partial}{\partial X_B, \theta} + \sigma^J \frac{\partial}{\partial X_B, \theta}\right) \frac{\partial}{\partial X_A} f(X) = -\sigma^J \left(\sigma^K \frac{\partial}{\partial X_B, \theta} + \sigma^J \frac{\partial}{\partial X_B, \theta}\right) = 0.$$

Thus for $m+1 \leq A \leq m+n$, $\frac{\partial}{\partial X_A} f(X)$ also satisfies the C-R equations on $\mathcal{U}$ and hence $G_{\mathcal{SD}}^1$ on $\mathcal{U}$. Therefore $f(X)$ is $G_{\mathcal{SD}}^\infty$ on $\mathcal{U}$. By induction, $f \in G_{\mathcal{SD}}^\infty$.

\qed

Lemma 5.15 (Lemma 5.1 of [12]). Let $f \in G_{\mathcal{SD}}^\infty(\mathcal{R}^{0|n})$. Then

$$f(\theta) = f(\theta_1, \ldots, \theta_n) = \sum_{|\alpha| \leq n} \theta^\alpha f_a \text{ with } f_a \in \mathcal{C}.$$

Proof. For $n = 1$ and $|J| = \text{odd}$, we have,

$$\frac{d}{dt} f(\theta + t\sigma^J) \bigg|_{t=0} = \frac{\partial}{\partial \theta^I} f(\theta) = \sigma^J \frac{d}{d\theta} f(\theta) \text{ with } \theta = \sum_{\gamma \in \Pi_{\text{odd}}} \theta_\gamma \sigma^I, \theta_\gamma \in \mathcal{C}.$$

Hence

$$\frac{\partial}{\partial \theta^K} \frac{\partial}{\partial \theta^J} f(\theta) = \frac{\partial}{\partial \theta^K} \frac{\partial}{\partial \theta^J} f(\theta + t\sigma^J + s\sigma^K) \bigg|_{t=s=0} = \sigma^K \sigma^J \frac{d}{d\theta} \frac{d}{d\theta} f(\theta).$$

Since $|J|, |K|$ are odd, we have $\sigma^K \sigma^J = -\sigma^K \sigma^J$ and therefore

$$\frac{\partial}{\partial \theta^K} \frac{\partial}{\partial \theta^J} f(\theta) = \sigma^K \sigma^J \frac{d}{d\theta} \frac{d}{d\theta} f(\theta) = -\sigma^K \sigma^J \frac{d}{d\theta} \frac{d}{d\theta} f(\theta) = -\frac{\partial}{\partial \theta^K} \frac{\partial}{\partial \theta^J} f(\theta).$$

Since $f$ is $C^\infty$ as a function of infinite variables $\{\theta_J\}$ and its higher derivatives are symmetric, we have therefore

$$\frac{\partial}{\partial \theta_K} \frac{\partial}{\partial \theta_J} f(\theta) = 0.$$  

By representing $f(\theta) = \sum_K \sigma^K f_K(\theta)$, the each component $f_K(\theta)$ is a polynomial of degree 1 with variables $\{\theta_J \mid J \in I_{a0}\}$. Then $\sigma^J \frac{d}{d\theta} f(\theta) = \frac{\partial}{\partial \theta_J} f(\theta)$ is constant for any $|J| = \text{odd}$. Thus $\frac{d}{d\theta} f(\theta)$ is constant denoted by $a \in \mathbb{C}$. Then, $\frac{d}{d\theta} (f(\theta) - \theta a) = 0$. Therefore there exists $b \in \mathbb{C}$ such that $f(\theta) = \theta a + b$.

We proceed by induction w.r.t. $n$. Let $f$ be a $G^\infty_{SD}$ function on an open set $U \subset \mathbb{R}^{m|0}$. Fixing $\theta_1, \ldots, \theta_{n-1}$, $f(\theta_1, \ldots, \theta_{n-1}, \theta_n)$ is a $G^\infty_{SD}$ function with one variable $\theta_n$. Thus, we have

$$f(\theta_1, \ldots, \theta_{n-1}, \theta_n) = \theta_n g(\theta_1, \ldots, \theta_{n-1}) + h(\theta_1, \ldots, \theta_{n-1}) \quad \text{with} \quad \frac{\partial}{\partial \theta_n} f(\theta) = g(\theta_1, \ldots, \theta_{n-1}).$$

Therefore $g$ is $G^\infty_{SD}$ w.r.t. $(\theta_1, \ldots, \theta_{n-1})$, $h$ is also $G^\infty_{SD}$ w.r.t. $(\theta_1, \ldots, \theta_{n-1})$. □

Remark 5.7. Though this Lemma with a sketch of the proof is announced in \[12\] and is cited in \[31\] without proof, but I feel some ambiguity of his proof. This point is ameliorated by \[42\] as above.

Lemma 5.16 (Lemma 5.2 of \[12\]). Let $f \in G^\infty_{SD}(\mathbb{R}^{m|0})$ on a convex open set $U \subset \mathbb{R}^{m|0}$ which vanishes identically on $U_B = \pi_B(U)$. Then, $f$ vanishes identically on $U$.

Proof. It is essential to prove the case $m = 1$. Take an arbitrary point $t \in U_B$ and we consider the behavior of $f$ on $\pi_B^{-1}(t)$. Let $X \in \pi_B^{-1}(t)$ and $X_L = p_L(X)$. Then $\{X_K \mid K \in I_L, |K| = \text{ev} \geq 2\}$ is a coordinate for $(\pi_B^{-1}(t))_L$ as the ordinary space $\mathbb{C}$. Let $f_L$ be the $L$-th projection of $f$. Then,

$$\frac{\partial}{\partial X_K} f_L(X_L) = \sigma^K \frac{\partial}{\partial X_0} f_L(X_L) \quad \text{for} \quad K \in I_L \quad \text{and} \quad |K| = \text{even}. $$

If $K_1, \ldots, K_h \in I_L$, $|K_j| = \text{even} > 0$ and $2h > L$, then $\sigma^{K_1} \cdots \sigma^{K_h} = 0$ and $\frac{\partial}{\partial x_{K_1}} \cdots \frac{\partial}{\partial x_{K_h}} f_L(X_L) = 0$. This implies that $f_L$ is a polynomial on $(\pi_B^{-1}(t))_L$. Moreover, for any $h \geq 0$,

$$\frac{\partial}{\partial x_{K_1}} \cdots \frac{\partial}{\partial x_{K_h}} f_L(t) = \sigma^{K_1} \cdots \sigma^{K_h} \left( \frac{\partial}{\partial X_0} \right)^h f_L(t).$$

Since $f$ vanishes on $U_B$, we have

$$\left( \frac{\partial}{\partial X_0} \right)^h f_L(t) = 0 \quad \text{on} \quad U_B$$

and hence

$$\frac{\partial}{\partial x_{K_1}} \cdots \frac{\partial}{\partial x_{K_h}} f_L(t) = 0 \quad \text{for any} \quad h \geq 0 \quad \text{and} \quad K_1, \ldots, K_h \in I_L \quad \text{with} \quad |K_j| = \text{even} > 0.$$

Thus the polynomial $f_L|_{\pi_B^{-1}(t)}$ must vanish identically and hence $f_L \equiv 0$ on $U_L$. This holds for any $L \geq 0$. Thus $f \equiv 0$ on $U$. □

5.6. Proof of Main Theorem \[13\] It is clear from outset that $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d)$. From Proposition 5.13 $(d) \Rightarrow (a)$. Lastly, the equivalence of $(d)$ and $(e)$ is given by

Theorem 5.17 (Theorem 4 of \[12\]). Let $f$ be a $G^\infty_{SD}$ function on a convex open set $U \subset \mathbb{R}^{m|n}$. Then, there exist $\mathbb{R}$-valued $C^\infty$ functions $u_{a}$ on $U_B$ such that

$$f(x, \theta) = \sum_{|a| \leq n} \theta^a u_{a}(x).$$

Moreover, the expression is unique.
Proof. \( \implies \) For fixed \( x \), by Lemma 5.16, \( f(x, \theta) \) has the representation \( f(x, \theta) = \sum_{|a| \leq n} \theta^a \varphi_a(x) \) with \( \varphi_a(x) \in \mathcal{C} \). Since \( f \in \mathcal{G}_{SD}^{\infty} \), it is clear that for each \( a \), \( \varphi_a(x) \in \mathcal{C} \) is on \( \mathcal{R}^{m|0} \) and moreover \( \varphi_a(x_B) \) is in \( C^\infty(\mathcal{R}^m) \). Denoting the Grassmann continuation of it by \( \tilde{\varphi}_a(x) \), we should have \( \tilde{\varphi}_a(x) = f_a(x) \) by Lemma 5.16.

\( \Longleftarrow \) Since the supersmoothness leads the C-R relation, we get the superdifferentiability. \( \square \)

References

[1] C. Bartocci, U. Bruzzo, D. Hernández Ruipérez and V.G. Pestov, Foundations of supermanifold theory: The axiomatic approach, Diff. Geom. Appl. 3 (1993) 135-155.
[2] M.S. Berger, Nonlinearity and Functional Analysis–Lectures on Nonlinear Problems in Mathematical Analysis, Academic Press, New York, 1977.
[3] F.A. Berezin, The method of second quantization, Academic Press, New York, 1966.
[4] , Introduction to Superanalysis, (ed. A.A. Kirillov) D.Reidel Publishing Company, 1987.
[5] F.A. Berezin and M.S. Marinov, Particle spin dynamics as the Grassmann variant of classical mechanics, Ann.of Physics 104(1977), pp. 336-362.
[6] C. Bochniak and J. Siciak, Analytic functions in topological vector spaces, Studia Math. 39 (1971) 77-112.
[7] C.P. Boyer and S. Gitter, The theory of \( G^\infty - \)supermanifolds, Trans.Amer.Math.Soc. 285(1984), pp. 241-267.
[8] P. Bryant, Supermanifolds, supersymmetry and Berezin integration, in “The mathematical structure of fields theories” Springer-Verlag, Berlin-Heidelberg-New York-Tokyo 1988, pp. 150-167.
[9] , DeWitt supermanifolds and infinite dimensional ground rings, J.London Math.Soc.39(1989), pp. 347-368.
[10] Y. Choquet-Bruhat, Supermanifolds and supermanifolds, in “Topological properties and global structure of space-time” (eds. P. Bergmann and V. de Sabbata), New York, Plenum Press, 1986, pp. 31-48.
[11] , Graded Bundles and Supermanifolds, Monographs and textbooks in Physical Sciences, Bibliopolis, Naples 1991.
[12] B. DeWitt, Supermanifolds, London, Cambridge Univ. Press, 1984.
[13] R. Feynman and A.R. Hibbs, Quantum Mechanics and Path Integrals, McGraw-Hill Book Co., New York, 1965.
[14] R. Hamilton, The inverse function theorem of Nash and Moser, Bulletin of AMS, vol 7. 1982, pp. 65-222.
[15] J. Hoyos, M. Quiros, J. Ramirez Mittelbrunn and F.J. de Urries, Generalized supermanifolds. I,II,III, J.Math.Phys. 25(1984), pp. 833-854.
[16] A. Inoue, Foundations of real analysis on the superspace \( \mathcal{R}^{m|n} \) over \( \infty - \)dimensional Fréchet-Grassmann algebra, J.Fac.Sci.Univ.Tokyo 39(1992), pp. 419-474.
[17] , On a construction of the fundamental solution for the free Weyl equation by Hamiltonian path-integral method – an exactly solvable case with “odd variable coefficients”, Tôhoku J. Math.50(1998), pp. 91-118.
[18] , On a construction of the fundamental solution for the free Dirac equation by Hamiltonian path-integral method – another interpretation of Zitterbewegung, Japanese J.Math.24(1994), pp. 297-334.
[19] , A partial solution for Feynman’s problem – a new derivation of the Weyl equation, Mathematical Physics and Quantum Field Theory, Electron.J.Diff.Eqns., Conf.04, 2000, pp. 121-145.
[20] A. Inoue and Y. Maeda, On integral transformations associated with a certain Lagrangian– as a prototype of quantization, J.Math.Soc.Japan 37(1985), pp. 219-244.
[21] , Foundations of calculus on super Euclidean space \( \mathcal{R}^{m|n} \) based on a Fréchet-Grassmann algebra, Kodai Math.J.14(1991), pp. 72-112.
[22] , On a construction of a good parametrix for the Pauli equation by Hamiltonian path-integral method — an application of superanalysis, Japanese J.Math.29(2003), pp. 27-107.
[23] A. Jadczyk and K. Pilch, Superfields and supersymmetries, Commun.Math.Phys.78(1981), pp. 373-390.
[24] , Classical limit of CAR and self-duality of the infinite-dimensional Grassmann algebra, in Quantum theory of particles and fields, ed b. Jancewicz and J. Lukierski, World Sci. Publ.co. Singapore, 1983, pp. 62-73.
[25] A. Yu. Khrennikov, Superanalysis, Kluwer Acad. Publ.1999.
[26] H.H. Keller, Differential calculus in locally convex spaces, SPLN 417, Berlin-New York-Tokyo-Heidelberg, Springer-Verlag, 1974.
[27] G. Köthe, Topological Linear Spaces I, Berlin-New York-Tokyo-Heidelberg, Springer-Verlag, 1969.
[28] D.A. Leites, Introduction to the theory of supermanifolds, Russian Math. Surveys 35(1980), pp. 1-64.
[29] , Quantization and supermanifolds, in the book The Schrödinger equation, Kluwer Academic Press, Dordrecht/Boston/London, 1991.
[30] Y.I. Manin, New dimensions in geometry, Lecture Notes in Mathematics 1111, pp.59-101, Berlin-New York-Tokyo-Heidelberg, Springer-Verlag, 1985.
[31] S. Matsumoto and K. Kakazu, A note on topology of supermanifolds, J.Math.Phys.27(1986), pp. 2690-2692.
[32] S. Nagamachi & Y. Kobayashi, Usage of infinite-dimensional nuclear algebras in superanalysis, Letters in Math.Phys. 14(1987), pp. 15-23.
[33] V.G. Pestov, Ground algebras for superanalysis, Rep.Math.Phys. 29. No.3(1991) pp. 275-287.
[34] , General construction of Banach-Grassmann algebras, Atti Accad.Naz.Lincet.Cl.Sci.Fis.Mat.Nat.,IX.ser.Rend.Lincei, Mat.Appl.3, No.3 (1992), 223-231.
[35] , Soul expansion of \( G^\infty \) superfunctions, J.Math.Phys. 34(1993), pp. 3316-3323.
[36] A. Rogers, A global theory of supermanifolds, J.Math.Phys. 21(1980), pp. 1352-1365.
[37] ______, *Graded manifolds, supermanifolds and infinite-dimensional Grassmann algebra*, Commun.Math.Phys.105(1986), pp. 375-384.

[38] ______, *Supermanifolds: Theory and Applications*, World Scientific Publishing Company (April 18, 2007)

[39] J.T. Schwartz, *Non-linear Functional Analysis*, Gordon and Breach, 1969.

[40] V.S. Vladimirov and I.V. Volovich, *Superanalysis I. Differential calculus*, Theor. Math. Phys. 59(1983), pp. 317-335.

[41] E. Witten, *Supersymmetry and Morse theory*, J.Diff.Geom.17(1982), pp. 661-692.

[42] K. Yagi, *Superdifferential calculus*, Osaka J. Math. 25(1988) pp. 243-257.

[43] S. Yamamuro, *A theory of differentiation in locally convex spaces*, Memoirs of AMS, vol17, no. 212, 1979.

**Professor Emeritus, Department of Mathematics, Tokyo Institute of Technology**

*Current address: 3-4-10, Kajiwara, Kamakura-city, Kanagawa-prefecture, 247-0063, Japan*

*E-mail address: atlom-inoue60@nifty.com*