On the Rank Decoding Problem Over Finite Principal Ideal Rings

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Abstract

The rank decoding problem has been the subject of much attention in this last decade. This problem, which is at the base of the security of public-key cryptosystems based on rank metric codes, is traditionally studied over finite fields. But the recent generalizations of certain classes of rank-metric codes from finite fields to finite rings have naturally created the interest to tackle the rank decoding problem in the case of finite rings. In this paper, we show that solving the rank decoding problem over finite principal ideal rings is at least as hard as the rank decoding problem over finite fields. We also show that computing the minimum rank distance for linear codes over finite principal ideal rings is equivalent to the same problem for linear codes over finite fields. Finally, we provide combinatorial type algorithms for solving the rank decoding problem over finite chain rings together with their average complexities.

Keywords: Rank Decoding Problem; Finite Principal Ideal Rings; Rank Metric Codes.

1 Introduction

Rank metric codes are subspaces whose elements can be seen as matrices and the distance between two elements is the rank of their difference [1,2]. These codes have received a lot of attentions these recent years, especially for their applications in space time coding [3], network coding [4] and cryptography [5]. One could see, among others: the definition of a new family of structured codes equipped with the rank metric together with efficient decoding algorithms [6,7], the generalizations of several known classes of structured rank metric codes from finite fields to other poorer structures like Galois rings [8,9] or finite principal ideal rings [10,11], each of these generalizations coming with efficient decoding algorithms for the new underlined code families. It is important to note that the question of whether one can decode a given code or not is quite fundamental in coding theory and code-based cryptography. The answer of this question is generally obvious when dealing with a code that has a known structure, since each structured code generally comes with a decoding algorithm. But if the structure of the code is unknown, this question is well known as the “problem of decoding a random linear code”.

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Decoding Problem for a Random Linear Code. The general framework for setting up a code based cryptosystem \cite{12, 13} is to take a generator matrix $G$ of a structured linear code that will undergo some transformations, giving place to a new generator matrix $G_{pub}$ that is, by assumption, indistinguishable from a generator matrix of a random linear code \cite{14}. The matrix $G_{pub}$ is then published by Alice together with a correction capacity $t$ that depends of the transformations applied. To send a message $m$ to Alice, Bob generates a random error $e$ of weight $t$ and send the cryptogram $y = mG_{pub} + e$. An attacker that intercepts $y$ and wants to find $m$ must then solve the problem of decoding a “random” linear code. Concretely, the problem of decoding a random linear code is, given a linear code $C$ (or its generator matrix), a vector $y$ of the ambient space and an integer $t$, to find a word $c$ in $C$ such that the distance from $y$ to $c$ is at most $t$. This problem is well known as being NP-complete in the Hamming metric \cite{15}, and was also shown recently to be NP-complete for the Lee metric \cite{16}. However, the case that interests us in this article is when the rank metric is used. In that specific case of rank metric codes, a randomized reduction from the same problem in the Hamming metric was proposed in \cite{17}.

Solving the Rank Decoding Problem. From an algorithmic point of view, there are two main techniques for solving the Rank Decoding Problem. The oldest one is of combinatorial type and was introduced in \cite{18}. This technique can be seen as a generalization to the rank metric of information set decoding algorithms \cite{19} in which one has to look for a set of positions that contains the error support. Note that in the Hamming metric, the support of an error is the set of non-zero positions of that error and, given the support or a slightly larger set containing the support of an error, one can find the associated error in polynomial time by solving a linear system. The situation is quiet the same in the rank metric with the difference that each coordinate of a vector is seen as a vector with coefficients in a base field and, the support of the vector is then the vector subspace generated by its coefficients.

The combinatorial algorithms thus works by guessing the support of the error and then solve a system of linear equations to find the coordinates of the error components in a basis of that support. The complexity of such an algorithm is then dominated by the inverse of the probability that a vector subspace chosen randomly is the good one. These algorithms have undergone several improvements, first in \cite{20} and very recently in \cite{21, 22} where the authors guess a slightly bigger vector subspace containing the support of the error.

Besides combinatorial techniques, there are also algebraic techniques for which the main idea is to translate the notion of rank into an algebraic setting. The first approach from \cite{23} first reduces the rank decoding problem to the search of minimum rank codewords in an extended linear code. This approach has been the subject of several recent improvements in \cite{24, 25}. Another approach based on linearized polynomials was also proposed in \cite{21}. It should be noted that the algorithms cited above only apply to codes whose alphabets are finite fields.

Structured Rank Metric Codes Over Finite Rings. An important point for setting up a code-based cryptosystem like in \cite{12} is to have a structured family of codes (that is to say code families with efficient decoding algorithms). This last decade has seen the birth of several works going in that direction, in particular for rank metric codes over finite rings. Tchatchiem and Mouaha \cite{10} first proposed a generalisation of the well known family of Gabidulin codes to finite principal ideal rings. This work was followed by \cite{26} where the authors provide an iterative decoding algorithm for Gabidulin codes over Galois rings with provable quadratic complexity in the code length. Note that the previous algorithm is similar to the iterative algorithm presented in \cite{10} for interleaved Gabidulin codes. Although Gabidulin codes over finite rings have many applications in network coding and space time coding like in the case
of finite fields, they are not too much attractive in cryptography when thinking about the story of their use in the Gabidulin, Paramonov and Tretjakov (GPT) cryptosystem \[5\]. To put it in a nutshell, due to the rich algebraic structure of Gabidulin codes, the original GPT cryptosystem was drastically broken in a series of structural attacks from Gibson \[27, 28\] and Overbeck \[29, 30, 31\]. Even if several variants where proposed to avoid these attacks \[32, 33, 34, 35, 36\] almost all of them are now found to be vulnerable \[37, 38, 39\].

In \[9\], the recent and promising family of Low-Rank Parity-Check (LRPC) codes \[6\] was also generalized to the rings of integers modulo a prime power. This work was followed by the paper of Renner, Neri, and Puchinger \[8\] that defined LRPC codes over Galois rings, the paper of Kamwa, Tale, and Fouotsa \[40\] that generalized LRPC codes to the ring of integers modulo a positive integer and finally the work from \[41\] where the authors generalize LRPC codes to finite commutative rings. Note that LRPC codes is known as having a very poorer algebraic structure and, as a consequence, their use in code-based cryptography closes the door to structural attacks and in this case, a cryptanalysis must focus on the problem of solving the rank decoding problem. This is why they are very attractive in code-based cryptography and consequently, their recent generalizations to finite rings have naturally highlighted the possibility of doing cryptography using rank metric codes over finite rings.

The Rank Decoding Problem Over Finite Rings. The existence of interesting code families over finite rings is certainly a determining element for doing McEliece-like cryptography over finite rings, but it is essential to make sure that the task of a cryptanalyst facing the rank decoding problem over finite rings will not be facilitated compared to finite fields. One could also wonder what more one gains by moving from finite fields to finite rings. Note that some properties of rank metric over finite fields do not apply to rank metric over finite rings due to zero divisors (see Example \[4, 6\]) and, as a consequence, the main technique used to translate the notion of rank into an algebraic setting like in \[20\], is not directly applicable in the case of finite commutative principal ideal rings (see Example \[4, 6\]). Remark that this modelling way is at the kernel of the recent improvements of algebraic algorithms for solving the rank decoding problem over finite fields \[24, 25\]. Thus, the existence of zero divisors over finite rings could help to avoid some existing attacks over finite fields. All these elements make the rank decoding problem over finite rings very attractive for code-based cryptography and give rise to several questions around this problem. A natural one is its difficulty compared to the same problem over finite fields. It would also be interesting to provide practical algorithms for solving this problem as well as their complexities. Another question which is generally related to the decoding problem is the calculation of the minimum distance for a rank metric code over finite rings.

Our Contribution. In this paper, we use the structure theorem for finite commutative rings \[42\] to show that solving the rank decoding problem over finite principal ideal rings is equivalent to solve the same problem over finite chain rings. We then use the socle and the injective envelope of modules over finite chain rings to show that the rank decoding problem over finite chain rings is at least as hard as the rank decoding problem over finite fields. We also show that computing the minimum rank distance for linear codes over finite principal ideal rings is equivalent to the same problem for linear codes over finite fields as in the case of hamming metric \[43, 44, 45\]. Furthermore, we provide combinatorial type algorithms similar to \[21, 22\] for solving the rank decoding problem over finite chain rings. To evaluate the average complexity of our algorithms, we use the shape of modules to give a formula that allows to count the number of submodules of fixed rank for a finitely generated module over a finite chain ring.
2 Preliminaries

We denote by $\mathbb{N}$ the set of positive integers including 0, and $\mathbb{N}^*$ the set $\mathbb{N}$ excluding 0. Let $m$ and $n$ be two elements of $\mathbb{N}^*$ and $R$ a finite commutative principal ideal ring, that is to say, a finite ring in which each ideal is generated by one element. The set of all $m \times n$ matrices with entries from $R$ will be denoted by $R^{m \times n}$. In general, we will use bold uppercase letters for matrices and bold lowercase letters for vectors.

In [2], Gabidulin used Galois extensions of finite fields to give vector representations of matrices and thus defined the notion of rank for vectors. Thanks to [10], this notion can be extended to finite principal ideal rings by defining Galois extension for finite principal ideal rings. In this section, we recall how to construct such a Galois extension. Note that by the structure theorem for finite commutative rings [42, Theorem VI.2], any finite commutative principal ideal ring can be decomposed as a direct sum of finite commutative local principal ideal rings, that is to say, finite chain rings. Galois extensions of finite chain rings can then be used to construct Galois extensions of finite principal ideal rings.

2.1 Galois Extensions of Finite Chain Rings

A chain ring is a ring whose ideals are linearly ordered by inclusion [42]. A finite chain ring have exactly one maximal ideal with is generated by one element. As an example, given $k \in \mathbb{N}^*$ and a prime number $p$, the ring $\mathbb{Z}_{p^k} = \mathbb{Z}/p^k\mathbb{Z}$ of integers modulo $p^k$ is a finite chain ring with $p\mathbb{Z}_{p^k}$ as the unique maximal ideal.

In this subsection, we assume that $R$ is a finite commutative chain ring with maximal ideal $\mathfrak{m}$ and residue field $\mathbb{F}_q = R/\mathfrak{m}$. Let $\pi$ be a generator of $\mathfrak{m}$ and $\nu$ the nilpotency index of $\pi$, i.e., $\nu$ is the smallest element of $\mathbb{N}^*$ such that $\pi^\nu = 0$. Then, any element $a$ in $R$ can be decomposed into $a = \pi^i u$ where $u$ is a unit of $R$ and $i$ is a unique element in $\{0, \ldots, \nu\}$. The natural projection $R \to R/\mathfrak{m}$ is denoted by $\Psi$ and can be extended coefficient-by-coefficient to polynomials over $R$.

Let $h \in R [X]$ be a monic polynomial of degree $m$ such that $\Psi (h)$ is irreducible in $\mathbb{F}_q [X]$. Set $S = R [X] / (h)$, where $(h)$ denotes the ideal of $R [X]$ generated by $h$. Then, $S$ is a local Galois extension of $R$ of degree $m$, with maximal ideal $\mathfrak{M} = \mathfrak{m}S$ and residue field $\mathbb{F}_{q^m} = S/\mathfrak{M}$. Also note that $S$ can be seen as a free $R$–module of rank $m$. Since $R$ is a finite chain ring, $S$ is also a finite chain ring and $\pi$ is a generator of $\mathfrak{M}$. A Galois extension of $\mathbb{Z}_{p^k}$ is called a Galois ring. We refer the reader to [42] for more details about Galois extensions of finite chain rings, where a characterization of finite chain rings using Galois rings is also given in [42, Theorem XVII.5]. The following example provides a construction of a Galois extension of $\mathbb{Z}_8$ of degree 4.

Example 2.1 Let $R = \mathbb{Z}_8$ and $h = X^4 + 4X^3 + 6X^2 + 3X + 1 \in R [X]$. Then $\Psi (h) = X^4 + X + 1$ irreducible in $\mathbb{F}_2 [X]$. Therefore, $S = R [X] / (h)$ is a Galois extension of $R$ of degree 4.
2.2 Galois Extension of Finite Principal Ideal Rings

As previously said, a principal ideal ring $R$ is isomorphic to a product of finite chain rings. That is to say, there exists a positive integer $\rho$ such that $R \cong R_{(1)} \times \cdots \times R_{(\rho)}$, where each $R_{(j)}$ is a finite chain ring. Using this isomorphism, we identify $R$ with $R_{(1)} \times \cdots \times R_{(\rho)}$. As an example, if $\eta = p_1^{k_1} \times \cdots \times p_d^{k_d}$ where $p_1, \ldots, p_d$ are prime numbers and $k_1, \ldots, k_d$ belonging to $\mathbb{N}^*$, then the ring $\mathbb{Z}_\eta$ is isomorphic to the product of finite chain rings $\mathbb{Z}_{p_1^{k_1}} \times \cdots \times \mathbb{Z}_{p_d^{k_d}}$, that is to say $\mathbb{Z}_\eta \cong \mathbb{Z}_{p_1^{k_1}} \times \cdots \times \mathbb{Z}_{p_d^{k_d}}$.

For any $j \in \{1, \ldots, \rho\}$, since $R_{(j)}$ is a finite chain rings, let $S_{(j)}$ be a Galois extension of $R_{(j)}$ of degree $m$. $S := S_{(1)} \times \cdots \times S_{(\rho)}$ is a free $R$–module of degree $m$. Since each $S_{(j)}$ is a finite principal ideal ring, $S$ is also a finite principal ideal ring. Furthermore, as specified in [10] pp.7720, $S$ is a Galois extension of $R$ and there exists a monic polynomial $h \in R[X]$ of degree $m$ such that $S \cong R[X]/(h)$.

Example 2.2 Let us construct a Galois extension of $R = \mathbb{Z}/40\mathbb{Z}$ of degree 4. Let $R_{(1)} = \mathbb{Z}/5\mathbb{Z}$ and $R_{(2)} = \mathbb{Z}/8\mathbb{Z}$. The map $\Phi : R \to R_{(1)} \times R_{(2)}$ given by $x + 40\mathbb{Z} \mapsto (x + 5\mathbb{Z}, x + 8\mathbb{Z})$ is a ring isomorphism and its inverse $\Phi^{-1}$ is defined by $(x + 5\mathbb{Z}, y + 8\mathbb{Z}) \mapsto xe_1 + ye_2$, where $e_1 = 16 + 40\mathbb{Z}$ and $e_2 = 25 + 40\mathbb{Z}$. Consider $h_{(1)} = X^4 + 24X^3 + 33X^2 + 4X + 2 \in R_{(1)}[X]$, $h_{(2)} = X^4 + 4X^3 + 6X^2 + 3X + 1 \in R_{(2)}[X]$, $S_{(1)} = R_{(1)}[X]/(h_{(1)})$, and $S_{(2)} = R_{(2)}[X]/(h_{(2)})$. Since $R_{(1)}$ is a finite field and $h_{(1)}$ is irreducible over $R_{(1)}$, then $S_{(1)}$ is a Galois extension of $R_{(1)}$ of degree 4. Furthermore, from Example 2.7, $S_{(2)}$ is also a Galois extension of $R_{(2)}$ of degree 4 and then, $S_{(1)} \times S_{(2)}$ is a Galois extension of $R_{(1)} \times R_{(2)}$ of degree 4. If we extend $\Phi^{-1}$ coefficient-by-coefficient to $R_{(1)}[X] \times R_{(2)}[X]$, by taking $h = \Phi^{-1}(h_{(1)}, h_{(2)}) = X^4 + 20X^3 + 14X^2 + 19X + 17$ we have $S_{(1)} \times S_{(2)} \cong R[X]/(h)$ and so, $R[X]/(h)$ is a Galois extension of $R$ of degree 4.

2.3 Rank Metric Codes Over Finite Principal Ideal Rings

An introduction to rank metric codes over finite principal ideal ring can be found in [10]. Here we give some fundamental notions needed for the sequel of the paper. Let us start by recalling the following definitions of the rank for a module over a finite principal ideal ring, the rank for a matrix, and a vector with coefficients in a finite principal ideal ring. Note that the notion of rank for a module is a generalization of the well known notion of dimension for a vector space. So we have the following definition.

Definition 2.3 (Rank of a Module) Let $M$ be a finitely generated $R$–module. The rank of $M$, denoted by $rk_R(M)$ or simply $rk(M)$, is the smallest number of elements in $M$ generating $M$ as an $R$–module. The rank of the module $\{0\}$ is by convention 0.

Since the columns (or the rows) of a matrix with coefficients in $R$ generate an $R$–module, the previous notion of rank for a $R$–module naturally extends to matrices with coefficients in $R$.

Definition 2.4 (Rank of a matrix) Let $A \in R^{m \times n}$. The rank of $A$, denoted by $rk_R(A)$, or simply $rk(A)$, is the rank of the $R$–submodule generated by the column vectors (or row vectors) of $A$.

A simple way to compute the rank of a matrix from $R^{m \times n}$ is to compute its Smith normal form and count the number of non-zero elements on the diagonal (see [10] Proposition 3.4). Also remark that thanks to the notion of Galois extension for finite principal ideal rings, $R^m$ is isomorphic to a Galois extension $S$ of $R$ so that each element of $S$ can be considered as a vector of the $R$–module $R^m$. Consequently, we have the following definition that defines the rank for vectors in $S^m$. 

5
Definition 2.5 (Rank of a vector) Let \( \mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{S}^n \).

1) The support of \( \mathbf{u} \), denoted \( \text{supp}(\mathbf{u}) \), is the \( R \)-submodule of \( \mathbb{S} \) generated by \( \{u_1, \ldots, u_n\} \).

2) The rank of \( \mathbf{u} \), denoted \( \text{rk}_R(\mathbf{u}) \), or simply \( \text{rk}(\mathbf{u}) \), is the rank of the support of \( \mathbf{u} \).

So, as in the case of fields, the map \( \mathbb{S}^n \times \mathbb{S}^n \rightarrow \mathbb{N} \) given by \((\mathbf{u}, \mathbf{v}) \mapsto \text{rk}_R(\mathbf{u} - \mathbf{v}) \) is a metric [10]. Also note that the rank of a vector can be computed using its matrix representation. Indeed, since \( S \) is also a free \( R \)-module, let \((b_1, \ldots, b_m)\) be a basis of \( S \) and consider \( \mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{S}^n \). For \( j = 1, \ldots, n \), \( a_j \) can be written as \( a_j = \sum_{1 \leq i \leq m} a_{i,j} b_i \), where \( a_{i,j} \in R \).

The matrix \( \Phi = (a_{i,j})_{1 \leq i \leq m, 1 \leq j \leq n} \) is then the matrix representation of \( \mathbf{a} \) in the \( R \)-basis \((b_1, \ldots, b_m)\) and \( \text{rk}_R(\mathbf{a}) = \text{rk}_R(\Phi) \).

It is important to underline the fact that some properties of the rank for matrices over finite fields do not generalize for matrices over finite rings due to zero divisors. As an example, the rank of a matrix \( A \) with entries in a field \( F \) is the order of a highest order non-vanishing minor of \( A \) and for any non-zero element \( \alpha \) from \( F \), both \( A \) and \( \alpha A \) have the same rank. However, those properties are not always true in finite rings.

Example 2.6 Let \( A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \) be a matrix with entries in \( \mathbb{Z}_4 \). Since \( A \) is in the Smith normal form, \( \text{rk}(A) = 2 \) while \( \text{rk}(2A) = 0 \). Moreover, the order of a highest order non-vanishing minor of \( A \) is 1, which is different from the rank of \( A \).

Since \( R = R(1) \times \cdots \times R(\rho) \) and \( S = S(1) \times \cdots \times S(\rho) \), for any \( j \in \{1, \ldots, \rho\} \), we denote by \( \Phi(j) \) the \( j \)-th projection map from \( S \) to \( S(j) \) in the following. We will also extend \( \Phi(j) \) coefficient-by-coefficient as a map from \( \mathbb{S}^n \) to \( S(\rho) \) and, the restriction of \( \Phi(j) \) to \( R^n \) will also be denoted by \( \Phi(j) \).

Then we have the following result from [15]:

\[ \text{rk}_R(N) = \max_{1 \leq j \leq \rho} \left\{ \text{rk}_{R(j)}(\Phi(j)(N)) \right\}. \]

Proof. See [15] Corollary 2.5. □

The above proposition shows that computing the rank of a submodule \( N \) over a finite principal ideal ring is equivalent to compute the highest rank for the projections of \( N \) as submodules over finite chain rings. This result does apply also to vectors from \( \mathbb{S}^n \) as they can be viewed as \( R \)-submodules when computing their ranks.

Corollary 2.8 For any \( \mathbf{a} \in \mathbb{S}^n \),

\[ \text{rk}(\mathbf{a}) = \max_{1 \leq j \leq \rho} \left\{ \text{rk}(\Phi(j)(\mathbf{a})) \right\}. \]

Let us recall that an \( S \)-submodule \( C \) of \( \mathbb{S}^n \) is also called a linear code of length \( n \) over \( S \). Its rank will be denoted by \( \text{rk}(C) \) and, its minimum rank distance is \( d(C) := \min \{ \text{rk}(\mathbf{u} - \mathbf{v}) : \mathbf{u}, \mathbf{v} \in C, \mathbf{u} \neq \mathbf{v} \} \). A generator matrix of \( C \) is any \( k(C) \times n \) matrix over \( S \) whose rows generate \( C \). The dual of \( C \) denoted by \( C^\perp \) is the orthogonal of \( C \) with respect to the usual Euclidean inner product on \( \mathbb{S}^n \) and, a parity-check matrix of \( C \) is a generator matrix of \( C^\perp \). By [10] Proposition 2.9], if \( C \) is a free module, then \( C^\perp \) is also a free module of rank \( n - k(C) \).

The minimum rank distance \( d(C) \) is an essential parameter for the code \( C \). It allows to evaluate the error correction capacity of \( C \) which is given by \( [(d(C) - 1) / 2] \). The Singleton bound in rank metric is given by

\[ \log_{\vert R \vert} \vert C \vert \leq \max\{m, n\}(\min\{m, n\} - d(C) + 1). \]
Codes that achieve this bound are called Maximum Rank Distance (MRD) codes. Note that if \( C \) is a free \( S \)-submodule of \( S^n \) then, \( \log_{|R|}|C| = k(C)m \) holds. Similar to [45] Lemmas 6.1 and 6.2, we have the following proposition:

**Proposition 2.9** Consider a linear code \( C \) of length \( n \) over \( S \) and set \( C(j) := \Phi(j)(C) \) for \( j = 1, \ldots, \rho \). We have

\[
k(C) = \max_{1 \leq j \leq \rho} \{ k(C(j)) \} \tag{1}
\]

and

\[
d(C) = \min_{1 \leq j \leq \rho} \{ d(C(j)) \}. \tag{2}
\]

**Proof.** Relation (1) is a direct consequence of Proposition 2.7. For relation (2), let \( j_0 \in \{1, \ldots, \rho\} \) such that \( d(C(j_0)) = \min_{j \leq \rho} \{ d(C(j)) \} \), and \( c \in C \) such that \( rk(\Phi(j_0)(c)) = d(C(j_0)) \). Consider \( \alpha = (\alpha_1, \ldots, \alpha_\rho) \in S \) such that \( \alpha_{j_0} = 1 \) and \( \alpha_j = 0 \) if \( j \in \{1, \ldots, \rho\} \setminus \{j_0\} \). Then, \( \Phi(j_0)(\alpha c) = \Phi(j_0)(c) \) and \( \Phi(j)(\alpha c) = 0 \) if \( j \in \{1, \ldots, \rho\} \setminus \{j_0\} \). Therefore, by Corollary 2.8, \( rk(\alpha c) = d(C(j_0)) \). So, \( d(C) \leq d(C(j_0)) \).

Let \( x \in C \) such that \( rk(x) = d(C) \), then there is \( j_1 \in \{1, \ldots, \rho\} \) such that \( \Phi(j_1)(x) \neq 0 \). Since \( rk(x) \geq rk(\Phi(j_1)(x)) \), we have \( d(C) \geq d(C(j_1)) \) and finally, \( d(C) \geq d(C(j_0)) \).

By Proposition 2.9, the problem of computing the minimum rank distance of linear codes over finite principal ideal rings is reduced to the same problem for codes over finite chain rings. In the next section, we will use the socle and the injective envelope of modules over finite chain rings to show that this problem reduces to finite fields.

3 Some Properties of Linear Codes Over Finite Chain Rings

In this section, we assume as in Subsection 2 that \( R \) is a finite commutative chain ring with residue field \( \mathbb{F}_q \) and maximal ideal \( m \) generated by \( \pi \) has \( \nu \) as its nilpotency index. Remark that \( S \) is also a finite chain ring with residue field \( \mathbb{F}_{q^m} \). The natural projection \( S \to \mathbb{F}_{q^m} \) is also denoted by \( \Psi \) and we extend \( \Psi \) coefficient-by-coefficient as a map from \( S^n \) to \( \mathbb{F}_{q^m} \).

3.1 Socle and Injective Envelope of Modules Over Finite Chain Rings

Let \( M \) be a finitely generated \( R \)-module. We recall that the socle of \( M \) denoted by \( soc_R(M) \) or simply \( soc(M) \), is the sum of the minimum nonzero submodules of \( M \); while the injective envelope \( E(M) \) of \( M \) is the smallest injective module containing \( M \). We refer the reader to [17] for more details about socles and injective envelopes.

**Proposition 3.1** For a finitely generated \( R \)-module \( M \), we have

\[
soc(M) = soc(E(M)) = \pi^{\nu-1}E(M). \]

**Proof.** From [17], \( soc(M) = soc(E(M)) \). By [48] Theorem 2.3, \( E(M) \) is a free module and \( soc(E(M)) = \pi^{\nu-1}E(M) \).

Proposition 3.1 provides a relation between the socle and the envelope of a module. Assume for example that \( M \) is a rank \( k \) submodule of a free \( R \)-module \( V \) of rank \( n \). Using the Smith normal form, one can compute a basis \( \{b_1, \ldots, b_n\} \) of \( V \) and \( k \) elements \( d_1, \ldots, d_k \) in \( R \) such that \( \{d_1b_1, \ldots, d_kb_k\} \) generates \( M \). Consequently, \( E(M) \) is generated by \( \{b_1, \ldots, b_k\} \) and \( soc(M) \) is generated by \( \{\pi^{\nu-1}b_1, \ldots, \pi^{\nu-1}b_k\} \).

The following proposition continues by showing that any linear code \( C \) over \( S \) shares the same rank and the same minimum distance with its socle and its injective envelope.
Proposition 3.2 For a linear code $\mathcal{C}$ of length $n$ over $S$, we have

$$k(\mathcal{C}) = k(\text{soc}(\mathcal{C})) = k(\pi(\mathcal{C})) \quad \text{and} \quad d(\mathcal{C}) = d(\text{soc}(\mathcal{C})) = d(\pi(\mathcal{C}))$$

Proof. The proof of the equality $d(\mathcal{C}) = d(\text{soc}(\mathcal{C}))$ is similar to the proof given in [11 Proposition 3.1]. Furthermore, since $\text{soc}(\mathcal{C}) = \text{soc}(\pi(\mathcal{C}))$, we also have $d(\pi(\mathcal{C})) = d(\text{soc}(\mathcal{C}))$ thanks to [18 Theorem 2.3], $k(\mathcal{C}) = k(\pi(\mathcal{C})) = k(\pi(\mathcal{C}))$. ■

By [11 Lemma 9] we also have the following:

Lemma 3.3 A subset $\{b_i\}_{1 \leq i \leq t}$ of $S$ is $R$-linearly independent if and only if $\{\Psi(b_i)\}_{1 \leq i \leq t}$ is $\mathbb{F}_q$-linearly independent.

Lemma 3.3 states that, showing the $R$-linear independence of a family of elements in $S$ is equivalent to show the $\mathbb{F}_q$-linear independence of its projection on the residue field. This result is very useful as it will help to proof several other results starting from the following lemma.

Lemma 3.4 For any $a \in S^n$, $rk(\pi^{\nu-1}a) = rk(\Psi(a))$.

Proof. By [10 Proposition 3.2], there exist a basis $\{b_i\}_{1 \leq i \leq m}$ of $S$ and $r = rk(a)$ integers $k_1, k_2, \ldots, k_r \in \mathbb{N}$ such that $\{\pi^{k_i}b_i\}_{1 \leq i \leq r}$ generates $\text{supp}(a)$ with $k_1 \leq k_2 \leq \cdots \leq k_r$. If $k_1 \neq 0$, then $\pi^{\nu-1}a = 0$ and $\Psi(a) = 0$. Assume $k_1 = 0$ and let $t$ be the maximum integer in $\{1, \ldots, r\}$ such that $k_t = 0$. Then, $\{\pi^{\nu-1}b_i\}_{1 \leq i \leq t}$ is a minimal generating family of $\text{supp} (\pi^{\nu-1}a)$, that is to say $rk(\pi^{\nu-1}a) = t$. Moreover $\{\Psi(b_i)\}_{1 \leq i \leq t}$ is a generating family of $\text{supp} (\Psi(a))$ and since $\{b_i\}_{1 \leq i \leq t}$ is $R$-linearly independent, thanks to Lemma 3.3, $rk(\Psi(a)) = t$. ■

Remark 3.5 Considering $S$ as an $R$-module, one can remark that the socle of $S$ is given by $\text{soc}_R(S) = \pi^{\nu-1}S$. Furthermore, the map $\phi : \text{soc}_R(S) \rightarrow S/\mathfrak{m}S$ given by $\phi(\pi^{\nu-1}u) = u+\mathfrak{m}S$ is an isomorphism of $R/\mathfrak{m}$-vector spaces and, extending $\phi$ coefficient-by-coefficient from $\pi^{\nu-1}S^n$ to $\mathbb{F}_q^n$ provides an isometry between the normed spaces $(\pi^{\nu-1}S^n, rk_R)$ and $(\mathbb{F}_q^n, rk_{\mathbb{F}_q})$ thanks to Lemma 3.4.

The following theorem is a rank metric version of [18 Theorem 3.4].

Theorem 3.6 Given a linear code $\mathcal{C}$ of length $n$ over $S$ such that $\Psi(\mathcal{C}) \neq \{0\}$,

(i) $d(\mathcal{C}) \leq d(\Psi(\mathcal{C}))$.

(ii) If $\mathcal{C}$ is free, then $d(\mathcal{C}) = d(\Psi(\mathcal{C}))$.

Proof. (i) Let $a \in \mathcal{C}$ such that $rk(\Psi(a)) = d(\Psi(\mathcal{C}))$. By Lemma 3.3, we have $rk(\pi^{\nu-1}a) = d(\Psi(\mathcal{C}))$ and since $\pi^{\nu-1}a \in \mathcal{C}$, $d(\mathcal{C}) \leq d(\Psi(\mathcal{C}))$ holds.

(ii) Assume that $\mathcal{C}$ is free. According to Propositions 3.1 and 3.2 respectively, $\text{soc}(\mathcal{C}) = \pi^{\nu-1}\mathcal{C}$ and $d(\mathcal{C}) = d(\text{soc}(\mathcal{C}))$ hold. Hence, there exists $a \in \mathcal{C}$ such that $rk(\pi^{\nu-1}a) = d(\mathcal{C})$ and thanks to Lemma 3.4, $rk(\Psi(a)) = d(\mathcal{C})$ holds. Therefore, $d(\Psi(\mathcal{C})) \leq d(\mathcal{C})$. ■

A direct consequence of Theorem 3.6 is the following:

Corollary 3.7 Let $\mathcal{C}$ be a linear rank metric code of length $n$ over $\mathbb{F}_q^n$, with rank $k$, minimum rank distance $d$ and generated by $g_1, \ldots, g_k$. For each $j$ in $\{1, \ldots, k\}$, let $g'_j$ in $S^n$ such that $\Psi(g'_j) = g_j$ and $\mathcal{C}'$ be the linear code generated by $g'_1, \ldots, g'_k$. Then, $\mathcal{C}'$ is a free linear rank metric code over $S$ of length $n$, rank $k$, and minimum rank distance $d$. ■
**Proof.** By Lemma 3.3 \( C' \) is a free code of rank \( k \). So, by Theorem 3.6 the minimum rank distance of \( C' \) is \( d \). ■

Corollary 3.7 shows that the problem of computing the minimum distance for a linear rank metric code over finite chain rings is at least as hard as the same problem for rank metric codes over finite fields. Note that the latter is considered as being NP-hard [17]. Another consequence of Corollary 3.7 is that one can construct MRD codes over \( S \) from MRD codes over \( \mathbb{F}_{q^m} \). A kind of converse of Corollary 3.7 is given in the following corollary :

**Corollary 3.8** Let \( C \) be a linear code of length \( n \) over \( S \). Then \( \Psi (E(C)) \) and \( C \) have the same rank and the same minimum rank distance.

**Proof.** By Proposition 3.2 \( d(C) = d(E(C)) \) and \( k(C) = k(E(C)) \) hold. Furthermore, we have \( d(E(C)) = d(\Psi(E(C))) \) according to Theorem 3.6 and, thanks to Lemma 3.3 \( k(E(C)) = k(\Psi(E(C))) \) holds and we have the result. ■

Thanks to Corollaries 3.8 and 3.7 the problem of computing the minimum rank distance for linear codes over finite chain rings is equivalent to the same problem for linear codes over finite fields. Nevertheless, from an algorithmic point of view, this problem over finite rings can be worse in practice since the size of the alphabet is naturally bigger than its projection (which is the residue field).

**Example 3.9** Consider the rings \( R = \mathbb{Z}_8 \) and \( S = R[X] / (h) \) defined in Example 2.1. For \( a = X + (h) \) and \( \overline{a} = \Psi(a) = S/R[a] \), let \( g_1 = (1,0,6a^3 + 5a^2 + 5,7a^3 + 5a^2 + a + 4) \), \( g_2 = (0,1,5a^3 + 5a^2 + 2a + 1,5a^3 + 2a^2 + 4a) \) and \( C = \langle g_1, 2g_2 \rangle \), that is to say the linear code generated by \( g_1 \) and \( 2g_2 \). Then, \( C \) is of length \( 4 \) and rank \( 2 \). We have \( \text{soc}(C) = \langle 4g_1, 4g_2 \rangle \), \( E(C) = \langle g_1, g_2 \rangle \) and \( \Psi(E(C)) = \langle \Psi(g_1), \Psi(g_2) \rangle \), with \( \Psi(g_1) = (1,0,\overline{a}^4 + 1,\overline{a}^3 + 4a) \) and \( \Psi(g_2) = (0,1,\overline{a}^3 + 2 + 1,\overline{a}^3) \). By [48], \( C \) has \( 2^20 \) codewords while \( \Psi(E(C)) \) has only \( 2^8 \) codewords. So, it is algorithmically better to compute the minimum rank distance of \( C \) via \( \Psi(E(C)) \). Using SageMath [49], we compute the minimum distance of \( \Psi(E(C)) \) and get \( 3 \). Thus, by Corollary 3.2, the minimum rank distance of \( C \) is \( 3 \).

### 3.2 Shapes for Modules Over Finite Chain Rings

A partition of a positive integer \( n \) is a decreasing sequence of positive integers whose sum is \( n \). For \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k, 0, \ldots) \) we will use the notation \( \lambda \vdash n \) to say that \( \lambda \) is a partition of \( n \) and, will only keep the non-zero components of \( \lambda \), that is to say \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \). The conjugate of a partition \( \lambda \) is the partition \( \lambda' \) defined by \( \lambda'_i = |\{ j : \lambda_j \geq i \}| \). By [45], Theorem 2.2], we have the following proposition :

**Proposition 3.10** Let \( M \) be a finitely generated \( R \)-module. Then, there exists a uniquely determined partition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \vdash \log_q |M| \), with \( \nu \geq \lambda_1 \) and \( \lambda_r \neq 0 \) such that

\[
M \cong R/m^{\lambda_1} \times \cdots \times R/m^{\lambda_r}.
\]

Moreover, \( rk(M) = \lambda'_1 = r \).

**Definition 3.11 (Shape of a module)** The partition \( \lambda \) defined in Proposition 3.10 is called the shape of \( M \).

**Example 3.12 (Shape of a free module)** Let \( \gamma \) be the shape of a free \( R \)-module of rank \( n \). Then

\[
\gamma = (\underbrace{\nu, \ldots, \nu}_n) \quad \text{and} \quad \gamma' = (\underbrace{n, \ldots, n}_\nu).
\]
One can remark that the shape of an $R$–module is very related to its cardinality and its rank. The importance of introducing this notion also comes from the fact that we use it to give the number of submodules of a given module over a finite chain ring. Recall that the number of subspaces of dimension $k$ in a vector space of dimension $n$ over a finite field with $q$ elements is given by the Gaussian binomial coefficient:

$$\left[ \begin{array}{c} n \\ k \end{array} \right]_q := \prod_{i=0}^{k-1} \frac{q^n - q^i}{q^k - q^i}.$$  

Note that from [50], we have

$$q^{k(n-k)} \leq \left[ \begin{array}{c} n \\ k \end{array} \right]_q \leq 4q^{k(n-k)}.$$  

(3)

When dealing with modules over finite chain rings, it is also possible to count the number of submodules of fixed rank. Thanks to [48, Theorem 2.4.], we have the following proposition.

**Proposition 3.13** Let $M$ be a finitely generated $R$–module of shape $\lambda$. Let $\mu$ be a partition satisfying $\mu \leq \lambda$, that is to say $\mu_j \leq \lambda_j$ for all $j$. The number of submodules of $M$ of shape $\mu$ is

$$\prod_{i=1}^{\nu} q^{\mu'_{i+1}(\lambda'_{i}-\mu'_{i})} \left[ \begin{array}{c} \lambda'_{i} - \mu'_{i+1} \\ \mu'_{i} - \mu'_{i+1} \end{array} \right]_q.$$  

Also note that according to Proposition [51, the rank of an $R$–module of shape $\mu$ is $k$ if and only if $\mu'_{1} = k$. So, we have the following corollary.

**Corollary 3.14** Let $M$ be a finitely generated $R$–module of shape $\lambda$. For any $k \in \mathbb{N}$ such that $k \leq rk(M)$, the number of submodules of $M$ of rank $k$ is

$$\sum_{0=\mu'_{1}+1 \leq \mu'_{2} \leq \cdots \leq \mu'_{\nu} = k} \prod_{i=1}^{\nu} q^{\mu'_{i+1}(\lambda'_{i}-\mu'_{i})} \left[ \begin{array}{c} \lambda'_{i} - \mu'_{i+1} \\ \mu'_{i} - \mu'_{i+1} \end{array} \right]_q.$$  

We now end this section by the following proposition expressing the number of submodules of fixed rank for a given free module over a finite chain ring and also providing upper and lower bounds.

**Proposition 3.15** Let $F$ be a free $R$–module of rank $n$. The number of submodules of $F$ of rank $k$ is given by

$$\beta(q, \nu, k, n) := \sum_{0=l_{\nu+1} \leq l_{\nu} \leq \cdots \leq l_1 = k} \prod_{i=1}^{\nu} q^{l_{i+1}(n-l_i)} \left[ \begin{array}{c} n - l_{i+1} \\ l_i - l_{i+1} \end{array} \right]_q.$$  

Moreover, if $k \leq n/2$ then

$$q^{\nu(k(n-k))} \leq \beta(q, \nu, k, n) \leq 4^{\nu} \binom{k+\nu-1}{\nu-1} q^\nu q^{\nu(k(n-k))}$$  

where $\binom{k+\nu-1}{\nu-1}$ is a binomial coefficient.

**Proof.** The number $\beta(q, \nu, k, n)$ is obtained using Example [51,2 and Corollary 3.14. Applying (3), we have

$$q^{\sum_{i=1}^{\nu} l_i(n-l_i)} \leq \prod_{i=1}^{\nu} q^{l_{i+1}(n-l_i)} \left[ \begin{array}{c} n - l_{i+1} \\ l_i - l_{i+1} \end{array} \right] q \leq 4^{\nu} q^{\sum_{i=1}^{\nu} l_i(n-l_i)}.$$  

If $k \leq n/2$, then $\sum_{i=1}^{\nu} l_i(n-l_i)$ is maximal when $l_i = k$ for $i = 1, \ldots, \nu$. By [51, Theorem 2.5.1], the number of partitions $(l_1, l_2, \ldots, l_{\nu+1})$ such that $l_1 = k$ and $l_{\nu+1} = 0$ is $\binom{k+\nu-1}{\nu-1}$. So, the result follows.  


4 Rank Decoding Problem

The Rank Decoding Problem over finite principal ideal rings is an extension of the well known Rank Decoding Problem from finite fields to finite principal ideal rings. So the main difference is the change of the alphabet which of course impacts the metric properties and several other aspects of the problem. For simplicity, this problem will be sometimes called “Rank Decoding Problem” without specification of the alphabet we are working with. Recall that $R$ is a finite principal ideal ring and $S$ is a Galois extension of $R$ as in Section 2. We have the following definitions.

**Definition 4.1 (Rank Decoding Problem $\text{RD}$)** Let $C$ be an $S$-submodule of $S^n$, $y$ an element of $S^n$ and $t \in \mathbb{N}^*$. The Rank Decoding Problem is to find $e$ in $S^n$ and $c$ in $C$ such that $y = c + e$ with $rk(e) \leq t$.

The dual version of this problem uses parity-check matrices and can be defined as follows.

**Definition 4.2 (Rank Syndrome Decoding Problem $\text{RSD}$)** Let $H \in S^{t \times n}$, $s$ an element of $S^t$ and $t \in \mathbb{N}^*$. The Rank Syndrome Decoding Problem is to find $e$ in $S^n$ such that $eH^\top = s$ with $rk(e) \leq t$.

As in the case of finite fields, solving the $\text{RD}$ problem is equivalent to solve the $\text{RSD}$ problem. Applying Proposition 2.7 we have the following:

**Proposition 4.3** Let $C$ be an $S$-submodule of $S^n$, $y$ an element of $S^n$ and $t \in \mathbb{N}^*$. Then there exist $e$ in $S^n$ and $c$ in $C$ such that $y = c + e$ with $rk_R(e) \leq t$ if and only if for all $j$ in $\{1, \ldots, \rho\}$, there exist $e(j)$ in $S^{t(j)}$ and $c(j)$ in $\Phi(j)(C)$ such that $\Phi(j)(y) = c(j) + e(j)$ with $rk_{R(j)}(e(j)) \leq t$.

By Proposition 4.3 solving the $\text{RD}$ problem over finite principal ideal rings is equivalent to solve the same problem over finite chain rings. Furthermore, according to Proposition 3.2 solving the $\text{RD}$ problem over finite chain rings for a linear code $C$ reduces to solving the same problem for the free module $E(C)$. So, solving the $\text{RD}$ problem over finite principal ideal rings reduces to solving the same problem for free modules over finite chain rings. The following proposition gives a relation between the $\text{RD}$ problem over finite chain rings and the $\text{RD}$ problem over finite fields.

**Proposition 4.4** Assume as in Section 3 that $R$ is a finite chain ring and $\Psi$ is the natural projection $S \to F_q^\omega$. Let $C$ be a linear rank metric code of length $n$ over $F_q^\omega$, with rank $k$, minimum rank distance $d$ and generated by $g_1, \ldots, g_k$. Let $g_j'$ in $S^n$ such that $\Psi(g_j') = g_j$ for $j$ in $\{1, \ldots, k\}$. Let $C'$ be a linear code generates by $g_1', \ldots, g_k'$ and $C'' = \text{soc}(C')$. Then,

(a) $C''$ is a linear rank metric code over $S$ of length $n$, rank $k$, and minimum rank distance $d$.

(b) Let $t \in \mathbb{N}^*$, $y$ an element of $F_q^n$, and $y'$ in $S^n$ such that $\Psi(y') = y$. For $y'' = \pi^{n-1}y'$, the following statements are equivalent.

(i) There exist $e$ in $F_q^n$ and $c$ in $C$ such that $y = c + e$ with $rk(e) \leq t$.

(ii) There exist $e''$ in $S^n$ and $c''$ in $C''$ such that $y'' = c'' + e''$ with $rk(e'') \leq t$. 

Proof. (a) By Corollary 3.7, \( C' \) is a free code of rank \( k \) and minimum rank distance \( d \). Thus, thanks to Proposition 3.2, the result follows.

(b) This result is a direct consequence of Remark 3.5. 

According to Proposition 4.3 and Proposition 3.2, the RD problem for a linear code \( C \) over the finite field \( \mathbb{F}_{q^m} \) reduces to solving the same problem for the linear code \( C'' \) over the finite chain ring \( S \). This reduction shows that the RD problem over finite chain rings is at least as hard as its finite fields version.

Over finite fields, given an instance \((C, y)\) of the RD problem, if the rank of the error is less than the error correction capability of the linear code \( C \), then it is always possible to reduce the RD problem to the search of minimum rank codewords in the linear code generated by \( C \cup \{y\} \), see [20]. This technique is at the base of several methods for solving the RD problem over finite fields [20, 21, 22, 24, 25]. When dealing with finite rings, this reduction is generally impossible due to zero divisors. As an illustration, consider the following example.

Example 4.5 For \( R = \mathbb{Z}_4 \), \( S = R[X] / (X^5 + X^2 + 1) \) and \( a = X + (X^5 + X^2 + 1) \), \( S \) is a Galois extension of \( R \). Let \( C \) be the Gabidulin code generated by \( g = (1, a, a^2, a^3, a^4) \). By [10, Theorem 3.24], the error correction capability of \( C \) is 2. Set \( e = (1, 2a, 0, 0, 0) \). By [10], \( \text{rk}(e) = 2 \) and, considering the received word \( y = e \), let \( C_y \) be the linear code generated by \( g \) and \( y \). Then \( 2e = (2, 0, 0, 0, 0) \in C_y \) and \( \text{rk}(2e) = 1 \). So, a solution to the shortest vector problem in the extended code \( C_y \) is not a solution to the associated RD problem as in [20].

5 Solving the Rank Syndrome Decoding Problem

According to Proposition 4.3 and Proposition 3.2, we will restrict the study of the RD problem to free modules over finite chain rings. So in what follows, we assume without loss of generality that \( \rho = 1 \). That is to say, \( R \) is a finite chain ring with residue field \( \mathbb{F}_q \) and \( \nu \) the nilpotency index of its maximal ideal. By [10, Proposition 3.2], we have the following lemma:

Lemma 5.1 Let \( V \) be a free \( R \)-module of rank \( a \), and \( W \) a submodule of \( V \) of rank \( b \). For any integer \( u \) such that \( b \leq u \leq a \), there exists a free submodule \( F \) of \( V \) with rank \( u \) such that \( W \subset F \).

Lemma 5.1 allows to extend the works of [21, 22] to finite principal ideal rings. Indeed, let \((H, s)\) be an instance of the RSD problem where \( eH^\top = s \). Let \( E \) be the matrix representation of \( e \) in an \( R \)-basis of \( S \). To recover \( e \), we have two possibilities. The first possibility is to choose a free \( R \)-submodule \( F \) of \( S \) such that \( \text{supp}(e) \subset F \). This approach is generally used when \( n \geq m \). The second possibility is to choose a free \( R \)-submodule \( F \) of \( R^n \) such that \( \text{row}(E) \subset F \), where \( \text{row}(E) \) is the \( R \)-submodule generated by the row vectors of \( E \). This approach is generally used when \( m \geq n \). In the following, we give more details on these combinatorial approaches.

5.1 First Approach

We recall that this approach is generally used when \( n \geq m \).

Lemma 5.2 Let \( H = (h_{i,j}) \in S^{(n-k) \times n} \) whose row vectors are linearly independent, \( s \) an element of \( S^{n-k} \). Suppose we want to solve an instance \((H, s)\) of the RSD problem with

\[
eH^\top = s\]

(4)
where \(e = (e_1, \ldots, e_n) \in S^n\) and \(rk(e) = r\). Let \(F\) be a free \(R\)–submodule of \(S\) of rank \(u\). Assume that \(\text{supp}(e) \subset F\). Let \(\{f_1, \ldots, f_u\}\) be a basis of \(F\) and \(x_{i,j} \in R\) such that, for all \(j \in \{1, \ldots, n\}\),

\[
e_j = \sum_{i=1}^{u} x_{i,j} f_i.
\]

Then, Equation (4) with unknown \(e\) can be transformed into a system of linear equations over \(R\) (that we denote by \((E_1)\)) with \(m(n-k)\) equations and \(n \times u\) unknowns \(x_{i,j}\).

**Proof.** Set \(X = (x_{i,j})_{1 \leq i \leq u, 1 \leq j \leq n}\) and \(f = (f_1, \ldots, f_u)\). Then, by (3), we have

\[
e = fX.
\]

So, (1) becomes

\[
fXH^\top = s.
\]

Therefore, applying [52, Lemma 4.3.1], we have

\[
(H \otimes f) \text{vec}(X) = \text{vec}(s).
\]

where \(\otimes\) is the Kronecker product and \(\text{vec}(X)\) denotes the vectorization of the matrix \(X\), that is to say the matrix formed by stacking the columns of \(X\) into a single column vector. Since \(S\) is a free \(R\)–module of rank \(m\), \([\mathbb{E}]\) can be expanded over \(R\) into a linear system with \(m(n-k)\) equations and \(n \times u\) unknowns \(x_{i,j}\). ■

**Remark 5.3** Let \(A\) be the \((n-k) \times nu\) matrix which defines Equation \((E_1)\) of Lemma 5.2.

1) If the column vectors of \(A\) are linearly independent, then \((E_1)\) has at most one solution. By [46, Lemma 2.6], if the column vectors of \(A\) are linearly independent, then \(nu \leq m(n-k)\), that is to say \(u \leq m(n-k)/n\). So, in practice, we choose \(u = \lfloor m(n-k)/n \rfloor\).

2) Assume that \(nu \leq m(n-k)\). Then, by Proposition 3.13, the probability that the column vectors of a random \((n-k) \times nu\) matrix with entries from \(R\) are linearly independent is \(\prod_{i=0}^{nu-1} (1 - q^{-m(n-k)})\). So, in practice, the column vectors of \(A\) are linearly independent with high probability.

Lemma 5.2 allows to give Algorithm 1.

**Theorem 5.4** An average complexity of Algorithm 1 is

\[
\mathcal{O} \left( m(n-k) u^2 n^2 \beta(q, \nu, r, m) / \beta(q, \nu, r, u) \right)
\]

operations in \(R\), where \(\beta(q, \nu, r, m)\) and \(\beta(q, \nu, r, u)\) are defined in Proposition 5.15.

**Proof.** Since \(R\) is a finite chain ring, we can use [53, Algorithm 4.2] to solve \((E_1)\). As \(un \leq m(n-k)\), by [52], \((E_1)\) can be solved in \(\mathcal{O}(m(n-k)n^2u^2)\) operations in \(R\). So, an average complexity to recover \(e\) is \(\mathcal{O}(m(n-k)u^2n^2/p)\) where \(p\) is the probability that \(\text{supp}(e) \subset F\). Remark that \(p\) is equal to the number of submodules of \(S\) of rank \(r\) in a free submodule of \(S\) of rank \(u\) divided by the number of submodules of \(S\) of rank \(r\). By Proposition 3.15, \(p = \beta(q, \nu, r, u) / \beta(q, \nu, r, m)\). Thus, the result follows. ■

**Remark 5.5** In practice, we have \(r \leq u/2\). Thus, from Proposition 3.15 we have

\[
\beta(q, \nu, r, m) / \beta(q, \nu, r, u) \approx q^{\nu(r-m-r)} / q^{\nu(u-r)} = |R|^{r(m-u)} / |R|^{r\lfloor mk/n \rfloor}
\]

where \(|R| = q^\nu\) is the cardinality of \(R\). This approximation is analogous to the one given in [21] when \(R\) is a finite field with \(q\) elements.
Algorithm 1: First Syndrome Decoding Algorithm

Input:
- \( r \) the rank of the error;
- \( H \in S^{(n-k) \times n} \) whose row vectors are linearly independent;
- \( s \) an element of \( S^{n-k} \) such that there is \( e \in S^n \) with \( rk(e) = r \leq u \) and \( eH^T = s \),
  where \( u := \lfloor m(n-k)/n \rfloor \).

Output: an element \( e \in S^n \) such that \( rk(e) = r \) and \( eH^T = s \).

1. \( update \leftarrow false \)
2. while \( update=false \) do
3.  Choose a free \( R \)-submodule \( F \) of \( S \) of rank \( u \).
4.  Choose a basis \( \{ f_1, \ldots, f_u \} \) of \( F \).
5.  Solve Equation \((E_1)\) of Lemma 5.2.
6.  if \((E_1)\) has no solution then
7.    \( update \leftarrow false \)
8.  else
9.    Use a solution of \((E_1)\) to compute \( e \) as in (5).
10.   if \( rk(e) \neq r \) then
11.      \( update \leftarrow false \)
12.   else
13.      \( update \leftarrow true \)
14.   return \( e \)

5.2 Second Approach

We recall that this approach is generally used when \( m \geq n \).

Lemma 5.6 Let \( H = (h_{i,j}) \in S^{(n-k) \times n} \) whose row vectors are linearly independent, \( s \) an element of \( S^{n-k} \). Suppose we want to solve an instance \((H,s)\) of the \( \text{RSD} \) problem with \( eH^T = s \) \hspace{1cm} (7) \n
where \( e \in S^n \) and \( rk(e) = r \). Let \((b_1, \ldots, b_m)\) be a basis of \( S \) as an \( R \)-module and \( E \) a matrix representation of \( e \) in this basis. Let \( F \) be a free \( R \)-submodule of \( R^n \) of rank \( u \). Assume that \( \text{row}(E) \subset F \). Let \( F \) be the \( u \times n \) matrix whose row vectors generate \( F \) and \( X = (x_{i,j}) \in R^{m \times u} \) such that

\[ E = XF \] \hspace{1cm} (8)

Then, Equation \((7)\) with unknown \( e \) can be transformed into a system of linear equations over \( R \) (that we denote by \((E_2)\)) with \( m(n-k) \) equations and \( mu \) unknowns \( x_{i,j} \).

Proof. Set \( b = (b_1, \ldots, b_m) \). We have

\[ e = bE. \] \hspace{1cm} (9)

So, \((7)\) becomes

\[ bXFH^T = s. \]

Therefore, applying [52] Lemma 4.3.1, we have

\[(HH^T \otimes b) \text{vec}(X) = \text{vec}(s).\] \hspace{1cm} (10)

Since \( S \) is a free \( R \)-module of rank \( m \), \[(10)\] can be expanded over \( R \) into a linear system with \( m(n-k) \) equations and \( mu \) unknowns \( x_{i,j} \). \( \blacksquare \)
Remark 5.7 As in Remark 5.3, if the column vectors of the \( m \times u \) matrix which defines Equation (E2) of Lemma 5.6 are linearly independent, then \( mu \leq m(n - k) \). So, in practice, we choose \( u = n - k \).

Lemma 5.6 allows to give Algorithm 2.

**Algorithm 2: Second Syndrome Decoding Algorithm**

**Input:**
- \( r \) the rank of the error;
- \( H \in \mathbb{S}^{(n-k) \times n} \) whose row vectors are linearly independent;
- \( s \) an element of \( S^{n-k} \) such that there is \( e \in S^n \) with \( rk(e) = r \leq n - k \) and \( eH^T = s \).

**Output:** an element \( e \in S^n \) such that \( rk(e) = r \) and \( eH^T = s \).

1. Choose a basis \( \{b_1, \ldots, b_m\} \) of \( S \) as \( R \)-module.
2. update \( \leftarrow \) false
3. while update = false do
4. Choose a free \( R \)-submodule \( F \) of \( R^n \) of rank \( n-k \).
5. Choose a basis \( \{F_1, \ldots, F_{n-k}\} \) of \( F \).
6. Solve Equation (E2) of Lemma 5.6.
7. if (E2) has no solution then
8. update \( \leftarrow \) false
9. else
10. Use a solution of (E2) to compute \( e \) as in (8) and (9).
11. if \( rk(e) \neq r \) then
12. update \( \leftarrow \) false
13. else
14. update \( \leftarrow \) true
15. return \( e \)

**Theorem 5.8** An average complexity of Algorithm 2 is

\[
\mathcal{O}(m^3(n-k)^3 \beta(q, \nu, r, n) / \beta(q, \nu, r, n-k))
\]

operations in \( R \), where \( \beta(q, \nu, r, n) \) and \( \beta(q, \nu, r, n-k) \) are defined in Proposition 3.15.

**Proof.** The proof is similar to that of Theorem 5.4.

**Remark 5.9** As in Remark 5.3, we have

\[
\beta(q, \nu, r, n) / \beta(q, \nu, r, n-k) \approx |R|^{rk}.
\]

**Example 5.10** Consider the linear code \( C \) defined in Example 3.9. Since the minimum rank distance of \( C \) is 3, then the error correction capability of \( C \) is 1. We consider the received word

\[
y = (4a^3 + a^2 + 2a + 3, 4a^3 + 4, 7a + 2, 6a^3 + 4a + 5)
\]

of \( C \). Note that \( C \) is not a free module. Thus, to decode \( y \) we consider that \( y \) is a received word from \( E(C) \). A parity-check matrix of \( E(C) \) is

\[
H = \begin{pmatrix}
6a^3 + 5a^2 + 5 & 5a^3 + 5a^2 + 2a + 1 & 7 & 0 \\
7a^3 + 5a^2 + a + 4 & 5a^3 + 2a^2 + 4a & 0 & 7
\end{pmatrix}
\]
and the syndrome of $y$ is
\[ s = yH^\top = (4a^3 + 4a^2 + 2a, 6a^2 + 6a + 4). \]

We run Algorithm 2 in SageMath [49] with inputs $H$, $s$, and $r = 1$. This algorithm returns
\[ e = (2 + 6a^2, 0, 4 + 4a^2, 6 + 2a^2). \]

So the transmitted codeword is
\[ y - e = (4a^3 + 3a^2 + 2a + 1, 4a^3 + 4a^2 + 7a + 6, 6a^3 + 2a^2 + a + 7). \]

**Remark 5.11** Theorems 5.4 and 5.8 give an average complexities for solving the RD problem over finite chain rings. According to Proposition 4.3, solving the RD problem over the finite principal ideal ring $R = R(1) \times \cdots \times R(\rho)$ is equivalent to solving the same problem over each finite chain ring $R(j)$ for $j$ in $\{1, \ldots, \rho\}$. So, an average complexity for solving the RD problem over the finite principal ideal ring $R$ is the sum of average complexities over $R(j)$ for $j$ in $\{1, \ldots, \rho\}$.

**6 Conclusion**

We have shown that solving the rank decoding problem over finite principal ideal rings is at least as hard as the rank decoding problem over finite fields. We have also shown that computing the minimum rank distance for linear codes over finite principal ideal rings is equivalent to the same problem for linear codes over finite fields as in the case of hamming metric [43, 44, 45]. All these put together with the fact that recent powerful algebraic methods [24, 25] for solving the Rank Decoding Problem over finite fields do not apply directly to finite rings with zero divisors as we have observed in this paper make the RD problem over finite rings very promising for code-based cryptography.

We have also provided combinatorial type algorithms similar to [21, 22] for solving the rank decoding problem over finite chain rings. The average complexities of the underlined algorithms are also given.

An interesting perspective will be to study the cases in which algebraic algorithms do apply. As an example, one can investigate the possibility of using the properties of linearized polynomials generalized in [10] to give an algebraic method as in [21] for solving the rank decoding problem over finite principal ideal rings.

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