Quantum Gravity and Black Hole†

KEN-JI HAMADA AND ASATO TSUCHIYA

Institute of Physics, University of Tokyo
Komaba, Meguro-ku, Tokyo 153, Japan

Abstract

The quantum theory of the spherically symmetric gravity in 3+1 dimensions is investigated. The functional measures are explicitly evaluated and the physical state conditions are derived by using the technique developed in two dimensional quantum gravity. Then the new features which are not seen in ADM formalism come out. If $\kappa_s > 0$, where $\kappa_s = (N - 27)/12\pi$ and $N$ is the number of matter fields, a singularity appears, while for $\kappa_s < 0$ the singularity disappears. The quantum dynamics of black hole seems to be changed by the sign of $\kappa_s$.

1. Introduction

Since the original work of Hawking,[1] many authors study the quantum dynamics of black holes. Almost all of works are done within the semi-classical approximation.[2,3,4,5] In this talk we discuss how the quantum gravity will influence the dynamics of black holes.[6,7] As a model of gravity we consider the spherically symmetric gravity in 3+1 dimensions.

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As a quantization method of gravitation, Arnowitt-Deser-Misner (ADM) formalism is well-known. This method, however, has some serious problems, which are the issues of measures and orderings. Here we explicitly evaluate the contributions from measures. Following the procedure developed in two dimensional quantum gravity we determine the measures in conformal gauge. From the gauge fixed theory the physical state conditions are derived. Then the new features which are not seen in ADM formalism appear.

The spherically symmetric gravity in 3+1 dimensions is defined by reducing the Einstein-Hilbert action to two dimensional one as

$$I_{EH} = \frac{1}{16\pi G} \int d^4x \sqrt{-g}^{(4)} R^{(4)} = \frac{1}{4} \int d^2x \sqrt{-g} \left( R + 2g^{\alpha\beta} \partial_\alpha \varphi \partial_\beta \varphi + \frac{2}{G} \right).$$  \hfill (1)

The fields $g_{\alpha \beta}$ and $\varphi$ are defined through the four dimensional metric $(ds^{(4)})^2 = g_{\alpha \beta} dx^\alpha dx^\beta + G \varphi^2 d\Omega^2$, where $\alpha, \beta = 0, 1$ and $d\Omega^2$ is the volume element of a unit 2-sphere. $G$ is the gravitational constant. In the following we set $G = 1$. We couple $N$ two dimensional conformal matter fields

$$I_M(g, f) = -\frac{1}{2} \sum_{j=1}^N \int d^2x \sqrt{-g} g^{\alpha \beta} \partial_\alpha f_j \partial_\beta f_j.$$  \hfill (2)

Some classical solutions of this system are known. For $f = 0$ the Schwarzschild geometry is well-known. The gravitationnal collapse geometry is given by \[^8\]

$$ds^2 = -\left( 1 - \frac{2M \vartheta(\bar{v})}{\bar{u}} \right) \frac{\bar{u}}{\bar{u} + 4M \vartheta(\bar{v})} d\bar{u} d\bar{v}, \quad \varphi = r,$$  \hfill (3)

where $ds^2 = g_{\alpha \beta} dx^\alpha dx^\beta$ and the coordinate $(\bar{u}, \bar{v})$ is defined through the relations, $d\bar{u} = d\bar{u}^* (\bar{u} + 4M) / \bar{u}$, $u^* = v - 2r^*$, $r^* = r + 2M \log(\frac{r}{2M} - 1)$ and $\bar{v} = v$. This geometry is derived by sewing the flat space time and the Schwarzschild black hole geometry along the shock wave line $\bar{v} = 0$, where the infalling matter flux is given by $T_{\bar{u}\bar{v}}^f = M \delta(\bar{v})$. In $(\bar{u}, \bar{v})$ coordinate the horizon locates at $\bar{u} = -4M$. 

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2. Quantization of Spherically Symmetric Gravity

Let us define the quantum theory of the spherically symmetric gravity. The partition function is expressed in terms of the path integral over the two dimensional metric $g_{\alpha\beta}$, the scale field $\phi$ and the matter fields $f$ as

$$Z = \int \frac{Dg(g)Dg(\phi)Dg(f)}{\text{Vol(Diff.)}} e^{iI_{SSG}(g,\phi,f)}, \quad I_{SSG} = I_{EH} + I_{M},$$

where $\text{Vol(Diff.)}$ is the gauge volume of diffeomorphism. The functional measures are defined by the following norms

$$< \delta g, \delta g >_g = \int d^2x \sqrt{-g} g^{\alpha\beta} \epsilon^{\gamma\delta} (\delta g_{\alpha\gamma} \delta g_{\beta\delta} + \delta g_{\alpha\beta} \delta g_{\gamma\delta}),$$

$$< \delta \phi, \delta \phi >_g = \int d^2x \sqrt{-g} \delta \phi \delta \phi,$$

$$< \delta f_j, \delta f_j >_g = \int d^2x \sqrt{-g} \delta f_j \delta f_j \quad (j = 1, \ldots N).$$

The measures explicitly depend on the dynamical field $g$. Therefore we must extract its contributions from the measures. They are evaluated by using the procedure of David-Distler-Kawai (DDK) in conformal gauge $g = e^{2\rho} \hat{g}$, where $\hat{g}$ is the background metric. The final expression is given by

$$Z = \int D\hat{g}(\Phi) e^{i\hat{I}(\hat{g},\Phi)},$$

where $\Phi$ denotes the fields $\rho, \varphi, f$ and the reparametrization ghosts $b$ and $c$. The gauge-fixed action $\hat{I}$ is

$$\hat{I} = \kappa_s S_L(\rho, \hat{g}) + I_{EH}(e^{2\rho} \hat{g}, \varphi) + I_{M}(\hat{g}, f) + I_{gh}(\hat{g}, b, c)$$

$$= \frac{1}{2} \int d^2x \sqrt{-\hat{g}} \left[ \hat{g}^{\alpha\beta} \partial_\alpha \varphi \partial_\beta \varphi + 2\hat{g}^{\alpha\beta} \varphi \partial_\alpha \varphi \partial_\beta \rho + \frac{1}{2} \hat{R} \varphi^2 + e^{2\rho} \right.

$$

$$+ \kappa_s (\hat{g}^{\alpha\beta} \partial_\alpha \rho \partial_\beta \rho + \hat{R} \rho) - \sum_{j=1}^{N} \hat{g}^{\alpha\beta} \partial_\alpha f_j \partial_\beta f_j \right] + I_{gh}(\hat{g}, b, c)$$

(7)
with
\[ \kappa_s = \frac{1}{12\pi} (1 + c_\varphi + N - 26) = \frac{N - 27}{12\pi}, \]
where \( S_L(\rho, \hat{g}) \) is the Liouville action defined by
\[ S_L(\rho, \hat{g}) = \frac{1}{2} \int d^2 x \sqrt{-\hat{g}} (\hat{g}^\alpha{}^\beta \partial_\alpha \rho \partial_\beta \rho + \hat{R} \rho). \] (8)

The value of \( \kappa_s \) is given by setting \( \xi = 1/2 \) in ref.6. The functional measure of the Liouville field \( \rho \) is defined by the norm on \( \hat{g} \) as
\[ \langle \delta \rho, \delta \rho \rangle_{\hat{g}} = \int d^2 x \sqrt{-\hat{g}} \delta \rho \delta \rho \] (10)
and also the measures for \( \varphi \) and \( f \) is defined by the norms (5) on \( \hat{g} \) instead of \( g \).

The background metric \( \hat{g} \) is very artificial so that the theory should be independent of how to choose it. Really it is proved that the partition function is invariant under the conformal change of the background metric, or \( Z(e^{2\sigma} \hat{g}) = Z(\hat{g}) \), where \( \sigma \) is an arbitrary local function. This means that the theory is considered as a kind of conformal field theory defined on \( \hat{g} \). The Virasoro algebra without central extension should be realized.\(^{*}\) The physical state conditions are derived from the demand that the theory should be independent of how to choose the background metric,
\[ \frac{\delta Z}{\delta g^{\alpha\beta}} \bigg|_{\hat{g} = \eta} = 0, \quad \text{or} \quad \langle \hat{T}_{\alpha\beta} \rangle = 0, \] (11)
where \( \eta_{\alpha\beta} = (-1, 1) \) and the energy-momentum tensor is defined by \( \hat{T}_{\alpha\beta} = -\frac{2}{\sqrt{-\hat{g}}} \frac{\delta \hat{f}}{\delta g^{\alpha\beta}} |_{\hat{g} = \eta} \).

Since the Liouville field \( \rho \) is transformed as \( \rho'(x') = \rho(x) - \gamma(x) \) for the conformal coordinate transformation \( x^{\pm'} = x^{\pm}(x^{\pm}) \), where \( \gamma(x) = \frac{1}{2} \log |\frac{\partial x'}{\partial x}|^2 \),

\(^{*}\) Note that in this case the theory does not reduce to the free-like theory. So it is quite different from the usual conformal field theory.
\[ |x|^2 = x^+ x^- \text{ and } x^\pm = x^0 \pm x^1, \]\n\[ \text{the energy-momentum tensor } \hat{T}_{\alpha\beta} \text{ is transformed as} \]
\[
\hat{T}'_{\pm\pm}(x') = \left( \frac{\partial x^\pm}{\partial x'^\pm} \right)^2 \left( \hat{T}_{\pm\pm}(x) + \kappa_s t_\pm(x) \right), \quad \hat{T}'_{+-}(x') = \left| \frac{\partial x}{\partial x'} \right| \hat{T}_{+-}(x),
\] (12)

where \( t_\pm(x) \) is the Schwarzian derivative \( t_\pm(x) = \left( \frac{\partial^2 \gamma}{\partial x^\pm} \right)^2 - \frac{\partial^2 \gamma}{\partial x^\pm}. \) \( t_\pm \) is determined by the boundary condition that the coordinate system which is joined to the Minkowski space time (asymptotically) is considered as the coordinate system with \( t_\pm = 0. \)

3. Black hole dynamics

To derive the black hole dynamics we must solve the physical state conditions (11). But it is a very difficult problem so that we take an approximation. The original actions (1) and (2) are the order of \( 1/\hbar, \) while the Liouville part of \( \hat{I} \) is the zeroth order of \( \hbar. \) However, if \( |\kappa_s| \) is large enough, then it is meaningful to consider the “classical” dynamics of \( \hat{I}. \) This is nothing but the semi-classical approximation, which is valid only in the case of \( M \gg 1 \) and \( |\kappa_s| \gg 1. \) The classical dynamics of \( \hat{I} \) is ruled by the equations \( \hat{T}_{\alpha\beta} = 0 \) and the \( \varphi \) field equation of motion. Then we set \( \hat{T}_{\alpha\beta}^{gh} = 0 \) because the ghost flux should vanish in the flat space time.

The gravitational collapse geometry is given as a solution with non-zero infalling matter flux. Giving the flux \( \hat{T}_{\bar{v}\bar{v}} = M\delta(\bar{v}), \) we can get the exact solution along the shock wave line \( \bar{v} = 0, \)
\[
\partial_\bar{v} \varphi(\bar{v} = 0, \bar{u}) = \frac{1}{2} \left( 1 - \frac{4M}{\sqrt{\bar{u}^2 - 4\kappa_s}} \right), \quad \varphi(\bar{v} = 0, \bar{u}) = r = -\frac{1}{2\bar{u}}.
\] (13)

\[ \dagger \text{ More explicitly, } \hat{T}_{\alpha\beta} \text{ is transformed as} \]
\[
\hat{T}'_{\pm\pm}(x') = \left( \frac{\partial x^\pm}{\partial x'^\pm} \right)^2 \left( \hat{T}_{\pm\pm}(x) + \kappa_s t_\pm(x) \right) + \frac{c_{\text{tot}}}{12\pi} \left( \frac{\partial x^\pm}{\partial x'^\pm} \right)^2 t_\pm(x)
\]

with \( c_{\text{tot}} = 1 - 12\pi\kappa_s + c_\varphi + N - 26 = 0. \) Note that if \( \hat{T}'_{\pm\pm}(x') \) satisfies the usual form of the Virasoro algebra with central charge \( c_{\text{tot}} = 0, \) then in \( x \) coordinate the combination \( \hat{T}_{\pm\pm}(x) + \kappa_s t_\pm(x), \) not \( \hat{T}_{\pm\pm}(x) \) itself, just satisfies the same form of the Virasoro algebra. The importance of \( t_\pm \) in quantum gravity is stressed in ref.10.
The (apparent) horizon, which is defined by the equation \( \partial_{\bar{v}}\varphi = 0 \), locates at

\[
\bar{u} = -4M \sqrt{1 + \frac{\kappa_s}{4M^2}}, \quad \bar{v} = 0.
\]  

If \( \kappa_s > 0 \), the location of the horizon initially shifts to the outside of the classical horizon \( \bar{u} = -4M \) by quantum effects. Then the black hole evaporates and the horizon approaches to the singularity asymptotically. The location of the singularity is determined by the equation \( \varphi^2 = \kappa_s \) (at \( \bar{v} = 0 \), it is \( \bar{u} = -2\sqrt{\kappa_s} \)). Note that at the singularity the curvature is singular, but the metric is finite. If \( \kappa_s < 0 \), the singularity disappears. The location of the horizon initially shifts to the inside of the classical horizon. If the effective mass of the black hole is defined by \( M_{BH} = \frac{1}{2}\varphi|_{\text{horizon}} \), this means that the initial mass of the black hole is less than the infalling matter flux \( M \). After the black hole is formed, the positive flux comes in through the horizon and the black hole mass increases. It seems that the horizon approaches to the classical horizon asymptotically and becomes stable.

For \( \kappa_s > 0 \), the classically forbidden region \( \kappa_s > \varphi^2 > 0 \) called “Liouville region” extends behind the singularity. To understand this region we must go back to the full quantum gravity. In the canonical quantization the physical state conditions are written as

\[
\hat{T}_{00}\Psi = \hat{T}_{01}\Psi = 0,
\]

where \( \Psi \) is a physical state and

\[
\hat{T}_{00} = \frac{1}{\varphi^2 - \kappa_s} \left( \frac{1}{2} \Pi_{\rho}^2 - \varphi\Pi_{\varphi}\Pi_{\rho} + \frac{\kappa_s}{2} \Pi_{\varphi}^2 \right) + \varphi\varphi'' + \frac{1}{2} \varphi^2 - \varphi\varphi'\rho' - \frac{1}{2} e^{2\rho}
\]

\[
- \frac{\kappa_s}{2} (\rho'' - 2\rho') + \frac{1}{2} \sum_{j=1}^{N} \left( \Pi_{f_j}^2 + f_{j}^2 \right),
\]

\[
\hat{T}_{01} = \rho'\Pi_{\rho} - \Pi_{\rho}' + \varphi'\Pi_{\varphi} + \sum_{j=1}^{N} \Pi_{f_j}f_j'.
\]

These correspond to the Hamiltonian and the momentum constraints. The conju-
gate momentums for $\rho$, $\varphi$ and $f_j$ are defined by

$$
\Pi_\rho = -\kappa_s \dot{\rho} - \varphi \dot{\varphi}, \quad \Pi_{\varphi} = -\dot{\varphi} - \varphi \dot{\rho}, \quad \Pi_{f_j} = \dot{f}_j.
$$

(17)

The notable point is the factor $(\varphi^2 - \kappa_s)^{-1}$ in front of the kinetic term of the Hamiltonian constraint, which does not appear in ADM formalism. The region $\varphi^2 > \kappa_s$ is classically allowed, whereas the Liouville region $\kappa_s > \varphi^2 > 0$ is the classically forbidden region where the sign of the kinetic term changes. There may be some possibility of gravitational tunnelings through this region.

The problem of the information loss seems to come out in the case of $\kappa_s > 0$. Then the black hole evaporates and the information seems to be lost. However in this case the Liouville region extends behind the singularity. So it appears that there is a possibility that the informations run away through this region by gravitational tunneling. On the other hand, if $\kappa_s \leq 0$, the Liouville region disappears. But the black hole seems to be stable. In this case it appears that the problem of the information loss does not exist.

4. Discussions

The quantum model of spherically symmetric gravity discussed in this talk has some problems. Here we adopt the conformal matter described by the action (2). Strictly speaking, however, we should consider the action such as $I_M = -\frac{1}{2} \int d^2x \sqrt{-g} \varphi^2 g^{\alpha\beta} \partial_\alpha f \partial_\beta f$, which is derived by reducing the four dimensional action to the two dimensional one. Ignoring $\varphi^2$-factor corresponds to ignoring the potential which appears when we rewrite the d’Alembertian in terms of the spherical coordinate. The black hole dynamics is determined by the behavior near the horizon so that it seems that this simplification does not change the nature of dynamics.

The other problem is in the definitions of measures. As the actions are derived from the four dimensional ones, the two dimensional measures also should be


derived from the four dimensional one

\[ < \delta g^{(4)}, \delta g^{(4)} >_{g^{(4)}} = \int d^4x \sqrt{-g^{(4)}} g^{(4)ab} g^{(4)cd} (\delta g^{(4)}_{ac} \delta g^{(4)}_{bd} + \delta g^{(4)}_{ab} \delta g^{(4)}_{cd}) . \] (18)

From this definition we get

\[ < \delta g, \delta g >_{g} = \int d^2x \sqrt{-g} \varphi^2 g^{\alpha\beta} g^{\gamma\delta} (\delta g_{\alpha\gamma} \delta g_{\beta\delta} + \delta g_{\alpha\beta} \delta g_{\gamma\delta}) , \]

\[ < \delta \varphi, \delta \varphi >_{g} = \int d^2x \sqrt{-g} \delta \varphi \delta \varphi . \] (19)

And also for the matter fields,

\[ < \delta f_j, \delta f_j >_{g} = \int d^2x \sqrt{-g} \varphi^2 \delta f_j \delta f_j \quad (j = 1, \cdots N) . \] (20)

The difference between (5) and (19-20) is apparent. The factor \( \varphi^2 \) in the measures of \( g \) and \( f \) prevents us from quantizing the spherically symmetric gravity exactly. We expect that this factor also does not change the nature of quantum dynamics drastically.

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