Noetherian intersections of regular local rings of dimension two

William Heinzer\textsuperscript{a}, Bruce Olberding\textsuperscript{b}

\textsuperscript{a}Department of Mathematics, Purdue University, West Lafayette, Indiana 47907-1395 U.S.A.
\textsuperscript{b}Department of Mathematical Sciences, New Mexico State University, Las Cruces, NM 88003-8001 U.S.A.

Abstract

Let $D$ be a 2-dimensional regular local ring and let $Q(D)$ denote the quadratic tree of 2-dimensional regular local overrings of $D$. We examine the Noetherian rings that are intersections of rings in $Q(D)$. To do so, we describe the desingularization of projective models over $D$ both algebraically in terms of the saturation of complete ideals and order-theoretically in terms of the quadratic tree $Q(D)$.

Keywords: regular local ring, quadratic transform, quadratic tree, Noetherian ring, complete ideal, projective model

2000 MSC: 13A15, 13C05, 13E05, 13H15

1. Introduction

Let $D$ be a 2-dimensional regular local ring with quotient field $F$. This article concerns the structure of the Noetherian rings that are intersections of 2-dimensional regular local rings between $D$ and $F$. As an intersection of normal rings, such rings are necessarily normal. We show these rings have the property that every maximal ideal has height 2. Conversely, it follows from Lipman’s work \cite{12} on rational singularities that every normal Noetherian overring $R$ of $D$ with height 2 maximal ideals has the form $R = \bigcap_{T \in \mathcal{U}} T$, where $\mathcal{U}$ is a set of 2-dimensional regular local overrings of $D$.

In a paper in preparation \cite{11}, we show the existence of subsets $\mathcal{U}$ of the set of 2-dimensional regular local overrings of $D$ such that $R = \bigcap_{T \in \mathcal{U}} T$ is not Noetherian. The question arises as to which sets $\mathcal{U}$ correspond to normal Noetherian overrings of $D$. This question is the main focus of the article. To address it we situate the problem in the context of the quadratic tree of $D$, that is, the partially ordered set $Q(D)$ of 2-dimensional regular local rings that birationally dominate $D$.

---

Email addresses: heinzer@purdue.edu (William Heinzer), olberdin@nmsu.edu (Bruce Olberding)
Each ring $R$ in $Q(D)$ is obtained via Abhyankar-Zariski factorization by a sequence $D = D_0 \subseteq D_1 \subseteq \cdots \subseteq D_n = R$ of quadratic transformations of $D$. The set $\{D_0, \ldots, D_n\}$ is precisely the set of 2-dimensional regular local rings between $D$ and $R$. In Section 3 we recall basic properties of the tree $Q(D)$, and in Section 4 we recall the concept of a projective model over $D$. A nonsingular projective model $X = \text{Proj } D[I]$, with $I$ a complete $\mathfrak{m}_D$-primary ideal, can be expressed (via its closed points) in terms of $Q(D)$. We do this in Theorem 4.6 using the Rees valuation rings and base points of the ideal $I$.

Nonsingular projective models over $D$ are central to our approach for describing the sets $U$ in $Q(D)$ that give rise to Noetherian rings $R = \bigcap_{T \in U} T$. We recall in Proposition 4.4 that every normal projective model $X = \text{Proj } D[I]$ over $D$ has a desingularization, and in Theorem 5.2 we use Zariski’s structure theorem for complete ideals to describe how to obtain the unique minimal desingularization via saturation of the ideal $I$. This leads in Theorem 5.3 to a strictly order-theoretic description of the closed points of the minimal desingularization of $X$ in terms of the partially ordered set $Q$.

Theorem 6.3 describes properties of the intersection of the 2-dimensional regular local rings in an affine component of $X$. Corollary 6.5 asserts the following description of an intersection of many rings in $Q(D)$: If $n$ is a positive integer and $R$ is an irredundant intersection of $n$ elements in $Q(D)$, then $R$ is a Noetherian regular domain with precisely $n$ maximal ideals, each maximal ideal of $R$ is of height 2, and the localizations of $R$ at its maximal ideals are the $n$ elements in $Q(D)$ that intersect irredundantly to give $R$.

Let $R$ be a normal overring of $D$ such that each maximal ideal of $R$ has height 2. Theorem 7.2 asserts: (i) $R$ is Noetherian if and only if $R$ is a flat overring of a finitely generated $D$-subalgebra of $R$, and (ii) if $R$ is Noetherian and local, then $R$ is a spot over $D$.

Theorem 7.4 asserts that the normal Noetherian overrings of $D$ with height 2 maximal ideals are precisely the rings $R$ for which there exists a nonsingular projective model $X$ over $D$ and a subset $U$ of the closed points of $X$ such that $R = \bigcap_{T \in U} T$.

In Section 8 we consider irredundant intersections of rings in $Q(D)$. We prove in Theorem 8.3 that the representation of $D$ as the intersection of its first neighborhood rings is irredundant, and that if $U$ is a proper subset of the set of all such rings, then the intersection of the rings in $U$ is a flat extension of a regular finitely generated $D$-subalgebra of $F$ and hence $U$ is an essential irredundant representation of the ring $\bigcap_{T \in U} T$.

When $D$ is Henselian we obtain our strongest result regarding irredundance. Let $U$ be a set of pairwise incomparable rings in $Q(D)$. Theorem 8.6 establishes that if $D$ is Henselian, then the representation $\bigcap_{R \in U} R$ is irredundant. In Corollary 8.7 we
use this to show that for $D$ Henselian, every Noetherian normal overring $R$ of $D$ for which each maximal ideal has height 2 is an irredundant intersection of the regular local rings in $Q(D)$ that are minimal with respect to containing $R$.

2. Preliminaries

Our notation is as in Matsumura [15]. Thus a local ring need not be Noetherian. We refer to Swanson and Huneke [20] for material on Rees valuation rings and blowing up of ideals. We refer to an extension ring $B$ of an integral domain $A$ as an overring of $A$ if $B$ is a subring of the quotient field of $A$. A local ring $B$ is said to be a spot over $A$, if $B$ is a localization of a finitely generated $A$-algebra.

We use the following definitions.

**Definition 2.1.** Let $R$ be a Noetherian local integral domain and let $S$ be a local overring of $R$.

1. The center of $S$ on $R$ is the prime ideal $m_S \cap R$ of $R$, where $m_S$ denotes the maximal ideal of $S$.
2. $S$ is said to dominate $R$ if the center of $S$ on $R$ is the maximal ideal of $R$, that is, $m_S \cap R = m_R$, where $m_R$ is the maximal ideal of $R$.
3. If $\dim R \geq 2$, a valuation overring $V$ of $R$ centered on $m_R$ is said to be a prime divisor of the second kind on $R$ if the field $V/m_V$ has transcendence degree $\dim R - 1$ over the field $R/m_R$.
4. $V$ is said to be a minimal valuation overring of $R$ if $V$ is minimal with respect to set-theoretic inclusion in the set of valuation overrings of $R$.

**Remark 2.2.** Assume notation as in Definition 2.1.

1. If $W$ is a valuation overring of $R$ and the center $m_W \cap R$ of $W$ on $R$ is a nonmaximal prime ideal of $R$, then by composite construction [21, p. 43], there exists a valuation overring $V$ of $R$ such that $V \subset W$ and $m_V \cap R = m_R$. Therefore every valuation overring of $R$ contains a valuation overring of $R$ that is centered on the maximal ideal of $R$.
2. If $W$ is a valuation overring of $R$ that dominates $R$ and the field $W/m_W$ is transcendental over $R/m_R$, then by composite construction, there exists a valuation overring $V$ of $R$ such that $V \subset W$.
3. Every valuation overring of $R$ contains a minimal valuation overring of $R$.

---

1 See Zariski-Samuel [21, p. 95]. Valuation overrings of $R$ centered on height 1 primes are prime divisors of the first kind. Prime divisors are necessarily DVRs.
(4) Let $V$ be a valuation overring of $R$. Then $V$ is a minimal valuation overring of $R$ if and only if $V$ dominates $R$ and the field $V/\mathfrak{m}_V$ is algebraic over the field $R/\mathfrak{m}_R$.

Abhyankar in Proposition 3 of [1] characterizes prime divisors of a regular local domain centered on the maximal ideal. The characterization is as follows.

**Theorem 2.3.** Let $R$ be a regular local domain with $\dim R = n \geq 2$ and let $\mathfrak{m}_R$ denote the maximal ideal of $R$. Let $V$ be a prime divisor of $R$ centered on $\mathfrak{m}_R$. There exists a unique finite sequence

$$R = R_0 \subset R_1 \subset \cdots \subset R_h \subset R_{h+1} = V$$

(1)

of regular local rings $R_j$, where $\dim R_h \geq 2$ and $R_{j+1}$ is the first local quadratic transform of $R_j$ along $V$ for each $j \in \{0, \ldots, h\}$, and $\text{ord}_{R_h} = V$. \footnote{For the definition of quadratic transforms, see for example [3, pp. 569–577] and [21, p. 367]. The powers of the maximal ideal of a regular local domain $S$ define a rank one discrete valuation domain denoted $\text{ord}_S$. If $\dim S = d$, then the residue field of $\text{ord}_S$ is a pure transcendental extension of the residue field of $S$ of transcendence degree $d - 1$.}

It follows from Theorem 2.3 that the residue field $V/\mathfrak{m}_V$ of $V$ is a pure transcendental extension of the field $R_h/\mathfrak{m}_{R_h}$ of transcendence degree one less than $\dim R_h$. Therefore the residue field of $V$ is ruled as an extension field of the residue field of $R$. \footnote{A field extension $F \subset L$ is said to be ruled if $L$ is a simple transcendental extension of a subfield $K$ such that $F \subset K$.}

The association of the prime divisor $V$ with the regular local ring $R_h$ in Equation 1, and the uniqueness of the sequence in Equation 1 establishes a one-to-one correspondence between the prime divisors $V$ dominating the regular local ring $R$ and the regular local rings $S$ of dimension at least 2 that dominate $R$ and are obtained from $R$ by a finite sequence of local quadratic transforms as in Equation 1. The regular local rings $R_j$ with $j \leq h$ displayed in Equation 1 are called the infinitely near points to $R$ along $V$. In general, a regular local ring $S$ of dimension at least 2 is called an infinitely near point to $R$ if there exists a sequence

$$R = R_0 \subset R_1 \subset \cdots \subset R_h = S, \quad h \geq 0$$

of regular local rings $R_j$ of dimension at least 2, where $R_{j+1}$ is the first local quadratic transform of $R_j$ for each $j$ with $0 \leq j \leq h - 1$ [14, Definition 1.6].

3. The quadratic tree of $D$

Let $D$ be a 2-dimensional regular local ring. The Zariski-Abhyankar Factorization Theorem [1, Theorem 3] implies that every 2-dimensional regular local ring
R that birationally dominates $D$ is an infinitely near point to $D$. Because we will often be treating such rings as points in what follows, we follow Lipman [14] and denote the infinitely near points to $D$ with Greek letters. We record in Theorem 3.1 implications of [1, Theorem 3 and Lemma 12].

**Theorem 3.1.** Let $D$ be a 2-dimensional regular local ring, and let $\alpha$ be a 2-dimensional regular local ring that birationally dominates $D$.

1. If $D \neq \alpha$, then $m_D$ extends to a proper principal ideal of $\alpha$. Therefore $\alpha$ dominates a unique local quadratic transform $\alpha_1$ of $D$.

2. There exists for some positive integer $\nu$ a sequence $D = \alpha_0 \subset \alpha_1 \subset \cdots \subset \alpha_\nu = \alpha$, where $\alpha_i$ is a local quadratic transform of $\alpha_{i-1}$ for each $i \in \{1, \ldots, \nu\}$. The rings $\alpha_i$ are precisely the regular local domains that are subrings of $\alpha$ and contain $D$.

3. If $V$ is a minimal valuation ring of $D$, then $V$ is the union of the infinite quadratic sequence of $D$ along $V$.

**Definition 3.2.** Let $D$ be a regular local ring of dimension 2 and let $F$ denote the quotient field of $D$.

1. The quadratic tree $Q(D)$ of $D$ is the partially ordered set (ordered by inclusion) defined as the set of all iterated quadratic transforms of $D$. Theorem 3.1 implies that $Q(D)$ is the set of all 2-dimensional regular local rings that birationally dominate $D$.

2. $Q(D)$ is the disjoint union of sets $Q_j(D)$ for $j \geq 0$, where $Q_j(D)$ denotes the set of all 2-dimensional regular local rings that are obtained by making precisely $j$ quadratic transforms starting at $D$, where $Q_0(D) = \{D\}$, see [1] and [2]. We refer to the elements in $Q_j(D)$ as infinitely near points at level $j$ to $D$.

**Remark 3.3.**

1. The elements in $Q_1(D)$ are in one-to-one correspondence with the points on a projective line over the residue field $\kappa(D) = D/m_D$ of $D$. Therefore the infinitely near points at level 1 to $D$ are in one-to-one correspondence with the irreducible homogeneous polynomials in $\kappa(D)[x, y]$. For each irreducible homogeneous polynomial $f \in \kappa(D)[x, y]$ there exists an infinitely near point $\alpha_f \in Q_1(D)$. Irreducible homogeneous polynomials $f$ and $g$ in $\kappa(D)[x, y]$ are such that $\alpha_f = \alpha_g$ if and only if $f$ and $g$ are associates in $\kappa(D)[x, y]$.

---

4 This is the notation used by Abhyankar in several papers such as [3].
(2) Assume that $\kappa(D)$ is algebraically closed and that $D$ has a coefficient field, that is, there exists a subfield $k$ of $D$ that maps onto $\kappa(D)$ under the natural surjection $D \rightarrow D/m_D$. Then each infinitely near points at level 1 to $D$ is uniquely determined by a nonzero homogeneous linear polynomial in $k[x, y]$. For $a, b \in k$ with $a \neq 0$, the polynomial $ay + bx$ is associated to $D[y,x]/(ay + bx)$. If $a = 0$, then $b \neq 0$ and we may assume $b = 1$ and associate $y$ to the local quadratic transform $D[y,x]/(y)$.

For future reference, we collect here notation we will use throughout the article.

**Notation 3.4.** We use the following notation.

1. $D$ is a 2-dimensional regular local ring with quotient field $F$ and maximal ideal $m_D = (x, y)D$.
2. $Q(D)$ is the quadratic tree of $D$ as in Definition 3.2.
3. For each $\alpha \in Q(D)$ and $j \geq 0$, $Q_j(\alpha)$ is the set of infinitely points at level $j$ to $\alpha$.
4. For each subset $\mathcal{U}$ of $Q(D)$, let $O_\mathcal{U} = \bigcap_{R \in \mathcal{U}} R$.
5. Let $R(D)$ denote the set of rings of the form $O_\mathcal{U}$ for some subset $\mathcal{U}$ of $Q(D)$.

**Remark 3.5.** The Noetherian rings in $R(D)$ are all Krull domains. Associated to a Krull domain $A$ is a unique set of DVRs, the set $\mathcal{E}(A)$ of essential valuation rings of $A$; $\mathcal{E}(A) = \{A_p\}$, where $p$ varies over the height 1 prime ideals of $A$. Two useful properties related to $\mathcal{E}(A)$ are:

1. $A = \bigcap\{A_p | A_p \in \mathcal{E}(A)\}$ and the intersection is irredundant.
2. The set $\mathcal{E}(A)$ defines an essential representation of $A$.

One of our motivations for this article and [11] is to examine the extent to which there are similarities between the intersections of elements in $Q(D)$ with the representation of a Krull domain $A$ as an intersection of its essential valuation rings.

### 4. Projective models over $D$

Let $D$ with quotient field $F$ be as in Notation 3.4. In this section we relate the geometry of $Q(D)$ to nonsingular projective models over $D$. We use the following terminology as in Section 17, Chapter VI of Zariski-Samuel [21]. If $A$ is a finitely generated $D$-subalgebra of $F$, the *affine model* over $D$ associated to $A$ is the set of local rings $A_p$, where $p$ varies over the set of prime ideals of $A$. A *model* $M$ over $D$ is a subset of the local overrings of $D$ that has the properties: (i) $M$ is a finite union of affine models over $D$, and (ii) each valuation overring of $D$ dominates at most one of the local rings in $M$. This second condition is called *irredundance*. A model
M over D is said to be complete if each valuation overring of D dominates a local ring in M.

A model M is said to be projective over D if there exists a finite set of nonzero elements \(a_0, a_1, \ldots, a_n\) in D such that \(J = (a_0, \ldots, x_n)D\) is an \(m\)-primary ideal of D and M is the union of the affine models defined by the rings \(A_i = D[\frac{a_0}{a_i}, \frac{a_1}{a_i}, \ldots, \frac{a_n}{a_i}], i = 0, 1, \ldots, n\). This projective model is the blowup\(^5\) \(\text{Proj } D[Jt]\) of the ideal J. In the language of schemes, the projective and affine models we consider correspond to the projective schemes over Spec D and the affine schemes over Spec D of finite type.

A basis for the Zariski topology on a model M is given by the sets of the form \(\{R \in M : x_1, \ldots, x_n \in R\}\), where \(x_1, \ldots, x_n \in F\). The closed points in this topology are the local rings in M that are maximal with respect to set inclusion. If M is a projective model over D, then the closed points are precisely the local rings in M of dimension 2.

A model M over D is normal if every local ring in M is a normal domain. The normalization of a projective model M over D is the projective model over D obtained by normalizing each affine component in M; i.e., if M is the union of the affine models defined by the rings \(A_0, \ldots, A_n\), then the normalization of M is the union of the affine models defined by the normalization of the \(A_i\). If \(M = \text{Proj } D[It]\) is a projective model over D for an ideal I of D, then the normalization of M is \(\text{Proj } D[Jt]\), where J is the integral closure of the ideal I\(^6\). Thus a projective model \(M = \text{Proj } D[It]\) is normal if and only if J is a complete ideal\(^7\).

Classical results proved by Zariski on the structure of complete ideals of a 2-dimensional regular local ring D simplify the structure of projective models birational over D. Complete ideals of D are closed with respect to ideal multiplication, and there is a marvelous unique factorization theorem: every nonzero complete ideal can be written uniquely as a finite product of simple complete ideals, cf. [21, Appendix 5] or [20, Chapter 14].

We use the following terminology.

**Definition 4.1.** Assume Notation 3.4 and let I be an \(m_D\)-primary ideal.

1. The base points of I are the points \(\alpha \in Q(D)\) for which the transform of I in \(\alpha\) is a proper ideal of \(\alpha\). Let \(\mathcal{B}(I)\) denote the set of base points of I. Then \(\mathcal{B}(I)\) is a finite subset of \(Q(D)\) [14]. A base point \(\alpha\) of J is called a maximal or terminal base point of J if \(\alpha\) is a maximal element of the partially ordered set \(\mathcal{B}(J)\), cf. [9, Remark 2.9].

\(^5\)We are identifying the projective scheme \(\text{Proj } D[It]\) with the model \(\bigcup_{p=0}^n \{A_i\}p | p \in \text{Spec } A_i\).

\(^6\)Here we are using that the powers of J are also integrally closed, and \(D[Jt]\) is a normal domain.

\(^7\)Also called integrally closed ideals.
(2) The set Rees $I$ of Rees valuation rings of $I$ is the smallest set $\{V_1, \ldots, V_n\}$ of valuation overrings of $D$ such that for each $k > 0$, the integral closure of $I^k$ is $I^k V_1 \cap \cdots \cap I^k V_n \cap D$. The set with this property is unique, and each $V \in \text{Rees } I$ is the order valuation ring $\text{ord}_\alpha$ of a unique point $\alpha \in Q(D)$.

(3) Let $J$ be a simple complete $m_D$-primary ideal. Then $\text{Rees } J = \{\text{ord}_\alpha\}$ for a unique point $\alpha \in Q(D)$, cf. [20, Prop. 14.4.8]. As in Theorem 2.3, there exists a unique chain $D = \alpha_0 \subset \alpha_1 \subset \cdots \subset \alpha_n = \alpha$ of infinitely near points from $D$ to $\alpha$. Then $\mathcal{B}(J) = \{\alpha_0, \ldots, \alpha_n\}$ is the set of base points of $J$. There exists a descending sequence $m_D = J_0 \supset J_1 \supset \cdots \supset J_n = J$ of simple complete ideals of $D$, where $\text{Rees } J_i = \{\text{ord}_{\alpha_i}\}$ for each $i$. The saturation of $J$ is the ideal $L = \prod_{i=0}^n J_i$.

(4) To define the saturation of an $m_D$-primary ideal $I$, let $J$ be the integral closure of $I$. For each simple complete factor $J_i$ of $J$, let $L_i$ denote the saturation of $J_i$. The saturation $L$ of $I$ is the product of the ideals $L_i$ as we vary over all the distinct simple complete factors of $J$.

(5) If $L$ is the saturation of $I$, then $L$ is also the saturation of $L$, and we say that $L$ is a saturated ideal.

We summarize in Remark 4.2 properties of saturated ideals that follow from the definition.

**Remark 4.2.** Assume Notation 3.4, and let $J$ be a complete $m_D$-primary ideal.

1. Assume $V \in \text{Rees } J$. Then $V$ is the order valuation ring of $\alpha_n \in Q_n(D)$, for some integer $n \geq 0$. Let $D \subset \alpha_1 \subset \alpha_2 \subset \cdots \subset \alpha_n$ be the unique chain of regular local rings from $D$ to $\alpha_n$. If $J$ is saturated, then the order valuation rings for $D, \alpha_1, \ldots, \alpha_{n-1}$ are in the set $\text{Rees } J$.

2. $\text{Rees } J \subseteq \{\text{ord}_\alpha \mid \alpha \in \mathcal{B}(J)\}$.

3. $J$ is saturated $\iff$ $\text{Rees } J = \{\text{ord}_\alpha \mid \alpha \in \mathcal{B}(J)\}$.

In [8, Definition 5.11] the following equivalent formulation to Definition 4.1 of a saturated ideal is given.

**Remark 4.3.** A complete $m_D$-primary ideal $J$ is saturated if for each simple complete ideal $I$ with $J \subseteq I$ and $I = IV \cap D$ for some $V \in \text{Rees } J$, the ideal $I$ is a factor of $J$.

---

*Two ideals with the same simple complete factors define the same blowup. Thus in defining the saturation of an ideal, it does not matter if a given complete simple factor occurs more than once.*
Proof. The equivalence follows because if $V \in \text{Rees } J$ and $V = \text{ord}_\alpha$, then $V$ dominates $\alpha$ and therefore $V$ dominates each of the infinitely near points in the chain from $D$ to $\alpha$. Let $I_n$ be the simple complete ideal corresponding to $\alpha$. Then $V \in \text{Rees } J$ implies that $I_n$ is a factor of $J$ by the unique factorization theorem of Zariski [20, Theorem 14.4.9]. Moreover, the simple complete ideals corresponding to points in the chain from $D$ to $\alpha$ are contracted from $V$. The condition in Remark 4.3 implies that all these simple complete ideals are also factors of $J$. Hence $J$ is saturated.

Conversely, if $J$ is saturated, then $\text{Rees } J = B(J)$, and the condition in Remark 4.3 holds.

Saturation has an important geometric interpretation. A model $M$ over $D$ is nonsingular if every ring in $M$ is a regular local ring. We record in Proposition 4.4 a result given in [8, Proposition 5.12].

**Proposition 4.4.** A normal projective model $M = \text{Proj } D[Jt]$ over $D$, where $J$ is a complete $m_D$-primary ideal, is nonsingular if and only if $J$ is saturated.

Facts 4.5 records known properties of a nonsingular projective model $X$ over $\text{Spec } D$. In our description of $X$ it is useful that $X = \text{Proj } D[Jt]$, where $J$ is a saturated complete $m_D$-primary ideal. We are interested in relating properties of $X$ to the quadratic tree of $D$. The finite set $B(J)$ of base points of $J$ plays an important role in this connection.

**Facts 4.5.** Assume Notation 3.4. Let $X$ be a nonsingular projective model over $\text{Spec } D$ and let $J$ be a saturated complete $m_D$-primary ideal such that $X = \text{Proj } D[Jt]$. The closed points of $X = \text{Proj } D[Jt]$ are a subset of $Q(D)$ of a special form that is related to the set $\text{Rees } J$ as follows:

1. All but finitely many of the points in $Q_1(D)$ are in $X$.
2. The following are equivalent:
   a. All the points in $Q_1(D)$ are in $X$.
   b. $Q_1(D)$ is the set of closed points of $X$.
   c. $B(J) = \{D\}$.
   d. $\{\text{ord}_D\} = \text{Rees } J$.
   e. $J = m_D^k$ for some $k > 0$.
3. Let $\alpha \in B(J)$. Then:
   a. Each of the points of $Q(D)$ in the chain from $D$ to $\alpha$ is in $B(J)$.
   b. $\alpha \notin X$.
   c. $\text{ord}_\alpha \in \text{Rees } J$.
   d. All but finitely many of the points of $Q_1(\alpha)$ are in $X$.  

9
(4) \( \alpha \in B(J) \) is a terminal base point of \( J \) as in Definition 4.1 \( \iff \) \( Q_1(\alpha) \subset X \).

(5) The finite set of terminal base points of \( J \) uniquely determines the nonsingular projective model \( X = \text{Proj} \, D[Jt] \).

(6) There exists an integer \( s \geq 0 \) such that the terminal base points of \( J \) are contained in \( Q_0(D) \cup Q_1(D) \cup \cdots \cup Q_s(D) \).

(7) If the terminal base points of \( J \) are contained in \( Q_0(D) \cup \cdots \cup Q_s(D) \), then the closed points of \( X \) are contained in \( Q_1(D) \cup Q_2(D) \cup \cdots \cup Q_{s+1}(D) \).

Facts 4.5 implies the following characterization of the closed points of a nonsingular projective model \( X \) over \( \text{Spec} \, D \) in terms of elements of the quadratic tree \( Q(D) \), and the base points \( B(J) \) of \( J \).

**Theorem 4.6.** Let \( X = \text{Proj} \, D[Jt] \) be a nonsingular projective model over \( D \) as in Facts 4.5. The set \( U \) of closed points of \( X \) has the following form: Either

1. \( U = Q_1(D) \) in which case \( B(J) = \{ D \} \) and \( X = \text{Proj} \, D[xt, yt] \), or
2. \( B(J) \setminus \{ D \} = \{ \alpha_1, \ldots, \alpha_n \} \), and \( U = (Q_1(D) \cup Q_1(\alpha_1) \cup \cdots \cup Q_1(\alpha_n)) \setminus \{ \alpha_1, \ldots, \alpha_n \} \),

in which case \( X = \text{Proj} \, D[Jt] \) for a saturated complete \( m_D \)-primary ideal \( J \) such that \( \text{Rees} \, J \) is the set of order valuation rings of the rings in \( B(J) \).

Conversely, if \( S = \{ \alpha_1, \ldots, \alpha_n \} \) is a finite subset of \( Q(D) \setminus \{ D \} \) having the property that \( \alpha \in S \) implies each point in the chain for \( D \) to \( \alpha \) is in the set \( S \cup \{ D \} \), then there exists a saturated complete ideal \( J \) such that

\[
U = (Q_1(D) \cup Q_1(\alpha_1) \cup \cdots \cup Q_1(\alpha_n)) \setminus \{ \alpha_1, \ldots, \alpha_n \}
\]

is the set of closed points of \( X = \text{Proj} \, D[Jt] \).

**Proof.** Apply Facts 4.5 \( \Box \)

5. Desingularization of projective models

As Proposition 4.4 suggests, saturation is the algebraic analogue of desingularization. We formalize this connection in Theorem 5.2. We recall the desingularization of a projective model, as defined in [12, p. 199].

**Definition 5.1.** Let \( M \) and \( N \) be models over \( D \). Then \( N \) dominates \( M \) if each valuation overring \( V \) of \( D \) centered on a ring in \( N \) dominates the center of \( V \) on \( M \); equivalently, each local ring in \( N \) dominates a local ring in \( M \).

Let \( R \) be a Noetherian overring of \( D \). Let \( M \) be a projective model over \( R \).

---

9Viewing \( N \) and \( M \) as projective schemes over \( \text{Spec} \, D \), this implies there is a birational morphism \( N \to M \).
(1) A desingularization\textsuperscript{10} of $M$ is a nonsingular projective model $N$ over $D$ that dominates $M$.

(2) A desingularization $N$ of $M$ is a minimal desingularization\textsuperscript{11} if every desingularization of $M$ dominates $N$.

The Zariski theory of complete ideals along with the Zariski-Abhyankar factorization theorem yields the following result.

**Theorem 5.2.** Let $J$ be a complete ideal of $D$, let $M = \text{Proj } D[Jt]$ and let $L$ denote the saturation of $J$. Then $N = \text{Proj } D[Lt]$ is a minimal desingularization of $M$. The converse also holds: if $L'$ is a complete ideal such that $\text{Proj } D[L't]$ is a minimal desingularization of $M$, then $L$ and $L'$ have the same simple complete factors and $\text{Proj } D[L't] = \text{Proj } D[Lt]$.

**Proof.** The model $N$ dominates $M$. Also, $N = \text{Proj } D[Lt]$ is a nonsingular model over $D$ by Proposition \[4.4\]. Hence $\text{Proj } D[Lt]$ is a desingularization of $\text{Proj } D[Jt]$.

Since $\text{ord}_\alpha \in \text{Rees } J$ for each terminal base point $\alpha \in B(J)$, the set $B(J)$ of base points of $J$ is the same as the set $B(L)$ of base points of $L$. Theorem \[4.6\] implies that $\text{Proj } D[Lt]$ is the unique nonsingular projective model over $D$ having the set $B(L) = B(J)$ as base points.

Let $Y$ be a nonsingular projective model over $D$ that dominates $\text{Proj } D[Jt]$. There exists a complete saturated ideal $I$ of $D$ such that $Y = \text{Proj } D[It]$.

If $V \in \text{Rees } J$, then $V \in \text{Proj } D[Jt]$. Since $\text{Proj } D[It]$ dominates $\text{Proj } D[Jt]$ and the only local ring birationally dominating a valuation ring is the valuation ring itself, it follows that $V \in \text{Proj } D[It]$ for each $V \in \text{Rees } J$. Therefore the set $B(I)$ of base points of $I$ contains $B(J) = B(L)$. Proposition \[4.4\] implies that each of the simple complete factors of $L$ is a factor of $I$. Therefore $Y = \text{Proj } D[It]$ dominates $\text{Proj } D[Lt]$, and the domination map $Y \to \text{Proj } D[Jt]$ factors through $Y \to \text{Proj } D[Lt]$.

For the converse, if $L'$ is a complete ideal such that $\text{Proj } D[L't]$ is a minimal desingularization of $M$, then $B(L') = B(J) = B(L)$, and the complete ideals $L$ and $L'$ have the same complete simple factors. Therefore $\text{Proj } D[Lt] = \text{Proj } D[L't]$ and $L'$ is also the saturation of $J$.

\textsuperscript{10}Our definition differs from Lipman’s but is equivalent in our context. Following [12, p. 199], a desingularization of a projective model $M$ of $D$ is a proper birational map of surfaces, $N \to M$, such that $N$ is nonsingular. In this case, $N \to \text{Spec } D$ is also a proper birational map of surfaces. This fact, along with the assumptions that $N$ is nonsingular and $D$ is a Noetherian ring, implies $N$ is projective [12, Corollary 27.2]. Thus our definition is equivalent to Lipman’s in our setting.

\textsuperscript{11}If there exists a desingularization $N$ of $M$, then there exists a unique minimal desingularization of $M$ [12, Corollary 27.3].
While Theorem 5.2 characterizes the desingularization of a normal projective model \( M = \text{Proj} \ D[Jt] \) in terms of the saturation of the ideal \( J \), Theorem 5.3 characterizes the desingularization of \( M \) strictly in terms of order-theoretic properties of \( Q(D) \).

**Theorem 5.3.** Let \( M \) be a normal projective model over \( D \). The closed points of the minimal desingularization of \( M \) are the points in \( Q(D) \) that are minimal with respect to dominating a closed point in \( M \).

**Proof.** We may assume \( M \) has singularities since otherwise the theorem is clear. By Theorem 5.2 there is a unique minimal desingularization \( N \) of \( M \). Since \( M \) is a normal surface that can be desingularized, \( M \) has finitely many singularities [12, Theorem, p. 151]. By [12, Propositions 1.2 and 8.1] and [13, B, p. 155], there is a sequence \( M_n \to \cdots \to M_1 \to M_0 = M \) of normal projective models over \( D \) such that \( M_n \) is nonsingular and for each \( i, M_{i+1} \) is obtained from \( M_i \) by blowing up the finitely many singular points of \( M_i \). Let \( U \) be the set of points in \( Q(D) \) that are minimal with respect to dominating a closed point in \( M \). We claim that \( U \) is the set of closed points of \( M_n \).

Let \( \alpha \in U \), and let \( R \) be the center of \( \alpha \) in \( M \). If \( \alpha \neq R \), then \( R \) is a singular point in \( M \). Since \( \alpha \) dominates \( R \) and \( R \) is a singular point in \( M \), \( \alpha \) dominates a point in \( M_1 \) [12, (*), p. 203]. If \( \alpha \notin M_1 \), then \( \alpha \) dominates a singular point in \( M_1 \) since \( \alpha \in U \). Continuing in this manner, we obtain either that \( \alpha \in M_i \) for some \( i \) or \( \alpha \notin M_n \) and \( \alpha \) dominates a point in \( M_n \). The latter property is contrary to the fact that \( M_n \) is nonsingular and \( \alpha \) is minimal among points in \( Q(D) \) dominating \( R \). Thus \( \alpha \in M_i \) for some \( i \), and since \( \alpha \) is a nonsingular point in \( M_i \) and in the sequence \( M_n \to \cdots \to M_1 \) we have only blown up singular points, we have \( \alpha \in M_n \). This shows that every point in \( U \) is a closed point in \( M_n \).

Conversely, let \( \alpha \) be a closed point in \( M_n \), and let \( R \) be the center of \( \alpha \) in \( M \). Let \( \beta \in Q(D) \) such that \( \beta \subseteq \alpha \) and \( \beta \) is minimal with respect to dominating \( R \). If \( \beta \in M \), then \( \beta \) is a nonsingular point in \( M \). By the construction of the \( M_i \), it follows then that \( \beta \in M_n \), so that \( \beta = \alpha \). Otherwise, \( \beta \notin M \) and so \( R \) is a singular point in \( M \). Thus \( \beta \) dominates a point \( R_1 \) in \( M_1 \) [12, (*), p. 203]. Continuing in this manner and using the fact that \( \alpha \in M_n \) dominates \( \beta \), we obtain that \( \beta \in M_i \) for some \( i \) and hence \( \beta = \alpha \). Therefore, \( \alpha \) is minimal with respect to dominating its center in \( M \), which proves that every closed point in \( M_n \) is in \( U \).

Now consider the minimal desingularization \( N \) of \( M \). Each closed point in \( N \) dominates a point in \( U \), so \( N \) dominates \( M_n \). Since \( N \) is a minimal desingularization of \( M \), we conclude that \( N = M_n \), which complete the proof.

**Remark 5.4.** Let \( M \) and \( M' \) be projective models over \( D \). We refer to [21, p. 120] for the definition of the join \( M'' \) of \( M \) and \( M' \). \( M'' \) is a projective model over \( D \).
that dominates both $M$ and $M'$. If $I$ and $I'$ are nonzero ideals of $D$ such that
$M = \text{Proj } D[It]$ and $M' = \text{Proj } D[I't]$, then $M'' = \text{Proj } D[II't]$.

Assume that $M$ and $M'$ are nonsingular projective models over $D$. There exist
saturated complete $m_D$-primary ideals $I$ and $I'$ such that $M = \text{Proj } D[It]$ and
$M' = \text{Proj } D[I't]$. The ideal $II'$ is also complete and saturated: $B(II') = B(I) \cup
B(I')$ and Rees $(II') = \text{Rees } I \cup \text{Rees } I'$, cf. [20, Prop. 10.4.8]. Hence the join
$M'' = \text{Proj } D[II't]$ is also nonsingular. The set $\mathcal{U}''$ of closed points of $M''$ has the
following description in terms of the sets $\mathcal{U}$ and $\mathcal{U}'$ of closed points of $M$ and $M'$
and the base points $\mathcal{B}(I)$ and $\mathcal{B}(I')$: $\mathcal{U}'' = (\mathcal{U} \cup \mathcal{U}') \setminus (\mathcal{B}(I) \cup \mathcal{B}(I'))$.

By Fact 4.5.5, a finite set $\alpha_1, \ldots, \alpha_n$ of incomparable rings in $Q(D) \setminus \{D\}$ uniquely
determines a nonsingular projective model $M$ over $D$ that has the $\alpha_i$ as precisely
the terminal base points of $M$. Since the points $\alpha_i$ are base points of $M$, they are
not points of $M$. On the other hand, Theorem 5.5 describes nonsingular projective
models $M$ in terms of finitely many points $\alpha \in M$.

As an application of saturation and desingularization, Theorem 5.5 records properties
of nonsingular models over $D$ in terms of finite subsets of incomparable points
of $Q(D) \setminus \{D\}$.

**Theorem 5.5.** Assume Notation 3.4.

1. Let $\alpha \in Q(D) \setminus \{D\}$. Then:
   1. There exists a unique nonsingular projective model $M = \text{Proj } D[It]$ such
      that $\alpha \in M$ and every nonsingular projective model over $D$ that contains $\alpha$
dominates $M$.
   2. The set $\mathcal{U}$ of closed points of $M$ has the property that $\mathcal{U} \setminus \{\alpha\}$ is the set of
      points of $Q(D)$ minimal with respect to being incomparable to $\alpha$.

2. Let $S = \{\alpha_1, \ldots, \alpha_n\}$ be a finite set of incomparable points in $Q(D) \setminus \{D\}$.
   Then:
   1. There is a unique nonsingular projective model $M$ over $D$ such that $\alpha_1, \ldots,
      \alpha_n \in M$ and every nonsingular projective model over $D$ containing $\alpha_1, \ldots, \alpha_n$
dominates $M$.
   2. The set $\mathcal{U}$ of closed points of $M$ has the property that $\mathcal{U} \setminus S$ is the set of
      points of $Q(D)$ minimal with respect to being incomparable to every $\alpha \in S$.

**Proof.** For item 1, let $\gamma \in Q(D)$ be the unique point such that $\alpha \in Q_1(\gamma)$. Let
$J$ be the simple complete ideal of $D$ having $\text{ord}_\gamma$ as its Rees valuation ring, and let
$L$ be the saturation of $J$. Let $M = \text{Proj } D[Lt]$. By Fact 4.5.5 and Theorem 5.2,
$M = \text{Proj } D[Lt]$ satisfies item1.a.

Notice that $\gamma$ is the unique terminal base point of $M$, and $\gamma \in Q_d(D)$ for some
integer $d \geq 0$. If $d = 0$, then $\gamma = D$ and $M = \text{Proj } D[xt, yt]$. It is clear in this case
that \( U = Q_1(\gamma) \setminus \{ \alpha \} \) is the set of points in \( Q(D) \) minimal with respect to being incomparable to \( \alpha \).

Assume that \( d \geq 1 \) and let \( D = \gamma_0 \subseteq \gamma_1 \subseteq \cdots \subseteq \gamma_d = \gamma \) be the unique chain of infinitely near points from \( D \) to \( \gamma \). As in Definition 4.1.3, \( B(L) = \{ \gamma_0, \ldots, \gamma_d \} \), and the set \( U \) of closed points of \( M = \Proj D[L] \) is

\[
U = (Q_1(D) \cup Q_1(\gamma_1) \cup \cdots \cup Q_1(\gamma_n)) \setminus \{ \gamma_1, \ldots, \gamma_d \}.
\]

It follows also in this case that \( U \setminus \{ \alpha \} \) is the set of points in \( Q(D) \) minimal with respect to being incomparable to \( \alpha \). This verifies item 1.

For item 2, for each \( \alpha \in S \), there exists a point \( \gamma_i \) such that \( \alpha \in Q_1(\gamma_i) \). By item 1, for each \( \alpha \), there exists a nonsingular projective model \( M_i = \Proj D[L_i] \) such that \( M_i \) has a unique terminal base point \( \gamma_i \) and \( \alpha \in Q_1(\gamma_i) \subseteq U_i \), where \( U_i \) is the set of closed points of \( M_i \).

Let \( L \) be the product of the saturated complete ideals \( L_i \). Then \( L \) is a saturated complete ideal and \( M = \Proj D[L] \) is the join of the models \( M_i \). Theorem 5.2 and Fact 4.5.5 imply that \( M \) satisfies item 2.a.

The set \( U \) of closed points of \( M \) is \( \bigcup_{i=1}^n U_i \setminus B(L) \), where \( B(L) = \bigcup_{i=1}^n B(L_i) \) is the set of base points of \( L \). Then \( S \subseteq U \) and \( U \setminus S \) is the set of points in \( Q(D) \) minimal with respect to being incomparable to every \( \alpha \in S \).

6. Intersections of closed points in affine models

In this section we consider the intersection of closed points in affine components of nonsingular projective models over \( D \). We save the more subtle non-affine case for Section 7.

We use the following terminology in Theorem 6.3.

**Definition 6.1.** Let \( U \) be a subset of \( Q(D) \). A ring \( \alpha \in U \) is **essential** for \( U \) if \( \alpha \) is a localization of \( \mathcal{O}_U = \bigcap_{\alpha \in U} \alpha \). We say \( U \) defines an **essential representation** of \( \mathcal{O}_U \) if each \( \alpha \in U \) is essential for \( U \).

**Remark 6.2.** Let \( U' \subseteq U \) be subsets of \( Q(D) \). If \( U \) defines an essential representation of \( \mathcal{O}_U \), then \( U' \) defines an essential representation of \( \mathcal{O}_{U'} \).

**Proof.** Let \( B = \mathcal{O}_U \) and \( C = \mathcal{O}_{U'} \). Then \( U' \subseteq U \) implies \( B \subseteq C \). Let \( \alpha \in U' \) and let \( m_{\alpha} \) denote the maximal ideal of \( \alpha \). Since \( U \) defines an essential representation of \( B \), we have \( \alpha = B_{m_{\alpha} \cap B} \). Then \( \alpha = B_{m_{\alpha} \cap B} \subseteq C_{m_{\alpha} \cap C} \subseteq \alpha \) implies \( C_{m_{\alpha} \cap C} = \alpha \). Therefore \( U' \) defines an essential representation of \( C \).

---

\(^{12}\)For distinct \( \alpha_i \) and \( \alpha_j \), it may happen that \( \gamma_i = \gamma_j \).
Theorem 6.3. Assume that $X = \text{Proj } D[Jt]$ is a nonsingular projective model over $D$. Let $b \in J$ be a part of a minimal set of generators of $J$ and let $A = D[J/b]$ be the associated affine component of $\text{Proj } D[Jt]$. Then $A$ is a regular Noetherian domain. Let $U \subset Q(D)$ denote the set of closed points of $X$ that contain $A$, and let $B = \mathcal{O}_U$.

(1) Each $\alpha \in U$ is a localization of $A$ at a height 2 maximal ideal.
(2) $A_m \in U$ and $A_m = B_{\langle mA_m \cap B \rangle}$ for each height 2 maximal ideal $m$ of $A$.
(3) $U$ defines an essential representation of $\mathcal{O}_U$.
(4) If $q$ is a maximal ideal of $B$, then $\text{ht } q = 2$ and $B_q = A_q\cap A$.
(5) $\mathcal{O}_U = B$ is a flat overring of $A$.

Proof. Since $A$ is an affine component of $\text{Proj } D[Jt]$, $A$ is a regular Noetherian domain, and each $\alpha \in U$ is a localization of $A$ at a height 2 maximal ideal. Let $m$ be a maximal ideal of $A$. Then $\text{ht } m = 2 \iff m \cap D = m_D \iff A_m \in U$. If $A_m = \alpha \in U$, then $\alpha$ is a localization of $B = \mathcal{O}_U$. This proves items 1, 2 and 3.

Let $q$ be a maximal ideal of $B$ and let $p = q \cap A$. Let $m$ be a maximal ideal of $A$ with $p \subseteq m$. If $\text{ht } m = 2$, then $A_m = B_n$, where $n = mA_m \cap B$. It follows that $A_p = B_q$. Since $q$ is a maximal ideal of $B$, $q = n$ and $m = p$ in this case.

If $\text{ht } m = 1$, then $p = m$ is a maximal ideal of $A$. Since $A$ is Noetherian and $A_p$ is a DVR, it follows that $p$ is an invertible ideal of $A$. Let $p^{-1}$ denote the inverse of $A$. Since $B = \bigcap\{A_m \mid m \in \text{Spec } A, \text{ht } m = 2\}$, we have $p^{-1} \subset B$ and $pB = B$. Therefore this case does not occur. This proves item 4. Theorem 2 of [19] now proves item 5.

Discussion 6.4. We illustrate Theorem 6.3 in a special case. Assume Notation 3.4. Then $\text{Proj } D[xt, yt]$ is the nonsingular projective model over $D$ obtained by blowing up $m_D$. The closed points of $\text{Proj } D[xt, yt]$ are precisely the points of $Q_1(D)$. Let $\beta = D[x/y]((x,y)/D[x/y])$ denote the point in $Q_1(D)$ in the $y$-direction. Then $D[y/x]$ is the affine component of $\text{Proj } D[xt, yt]$ that omits the point $\beta$. The localizations of $D[y/x]$ at its height 2 maximal ideals describe the set $U = Q_1(D) \setminus \{\beta\}$. Let $A = \mathcal{O}_U$. Then $A$ is a localization of $D[y/x]$ at the multiplicatively closed set generated by principal generators of the height one maximal ideals of $D[y/x]$. There are infinitely many height one maximal ideals of $D[y/x]$. For each integer $n \geq 2$, let $V_n = D[\langle y^n-x \rangle]D$. The center of $V_n$ on $D[y/x]$ is a height one maximal ideal of $D[y/x]$. Notice that $xD[y/x] = m_D D[y/x]$ is a principal height one prime ideal that is contained in every height 2 maximal ideal of $D[y/x]$. It follows that $xA$ is the Jacobson radical of $A$ and $A = (1-xD[y/x])^{-1}D[y/x]$. Also $D[y/x]/xD[y/x] = A/xA$ is a polynomial ring in one variable over the field $\kappa(D) = D/m_D$.

Assume Notation 3.4 and let $U$ be a finite subset of $Q(D)$. The ring $\mathcal{O}_U = \bigcap_{\alpha \in U} \alpha$ is the irredundant intersection of the regular local rings in $U$ that are minimal with
respects to inclusion. Therefore in Corollary 6.5 we consider finite subsets \( U \) of \( Q(D) \) for which the intersection \( \bigcap_{\alpha \in U} \alpha \) is irredundant.

**Corollary 6.5.** Let \( U = \{\alpha_1, \ldots, \alpha_n\} \) be a finite subset of \( Q(D) \) such that there are no inclusion relations among the \( \alpha_i \). Then \( \mathcal{O}_U = \bigcap_{i=1}^n \alpha_i \) is a regular Noetherian domain with precisely \( n \) maximal ideals. The maximal ideals of \( \mathcal{O}_U \) may be labeled as \( \{m_i\}_{i=1}^n \) such that \( (\mathcal{O}_U)_{m_i} = \alpha_i \) for each \( i \in \{1, \ldots, n\} \). Therefore \( U \) defines an irredundant essential representation of \( \mathcal{O}_U \).

**Proof.** By Theorem 5.5, there exists a complete \( m_D \)-primary ideal \( J \) such that \( \text{Proj } D[J]t \) is a nonsingular projective surface over \( D \) and each of the \( \alpha_i \) is a closed point on the surface \( \text{Proj } D[J]t \). By homogeneous prime avoidance, there exists an affine component \( A \) of \( \text{Proj } D[J]t \) such that each \( \alpha_i \) is a localization of \( A \). By Remark 6.2 and Theorem 6.3, \( U \) defines an irredundant essential representation of \( \mathcal{O}_U \).

Example 6.6 illustrates Corollary 6.5.

**Example 6.6.** Assume Notation 3.4. Let \( \alpha := D[y/x](x,y/x) \) and \( \beta = D[x/y](y,x/y) \) be the infinitely near points to \( D \) in the \( x \)-direction and \( y \)-direction. Define \( D^* \) to be the local quadratic transform of \( \alpha \) in the \( y \)-direction, and \( D^{**} \) to be the local quadratic transform of \( \beta \) in the \( x \)-direction. Thus

\[
D^* = D[y/x,x^2/y](y/x,x^2/y) \quad \text{and} \quad D^{**} = D[x/y,y^2/x](x/y,y^2/x)
\]

Let \( V \) denote the order valuation ring, \( \text{ord}_D \), of \( D \), and let \( V_\alpha \) and \( V_\beta \) denote the order valuation rings of \( \alpha \) and \( \beta \), respectively. Observe that \( D^* \subset V \) with \( D^*_{(x^2/y)} = V \), and \( D^{**} \subset V \) with \( D^{**}_{(y^2/x)} = V \)  fuzzy. We also have \( V_\alpha = D^*_{(x/y)}D^* \), and \( V_\beta = D^{**}_{(x/y)}D^{**} \).

Define \( E = D^* \cap D^{**} \). Corollary 6.5 implies that \( E \) has precisely 2 maximal ideals and that \( D^* \) and \( D^{**} \) are localizations of \( E \). We give the following direct proof:

Let \( J = (x^4, x^2y, xy^2, y^4)D \) denote the product of the simple complete ideals

\[
(x,y)D, \quad (x^2,y)D, \quad (x,y^2)D
\]

associated to the infinitely near points \( D, \alpha, \beta \), respectively.

\[\text{In classical terminology, } D \text{ is proximate to both } D^* \text{ and } D^{**}.\]
The center of $D$ and $D^*$ are both on this model. Define
\[ A = D\left[\frac{J}{x^3 y + x y^2}\right] = D\left[\frac{x^3}{y(x + y)}, \frac{x}{x + y}, \frac{y}{x}, \frac{y^3}{x(x + y)}\right]. \]

Then $A$ is an affine component of the projective model $X$. We observe that $A$ is a subring of both $D^*$ and $D^{**}$. Notice that the $x$-adic and $y$-adic valuation rings of $D$ do not contain either $D^*$ or $D^{**}$. Moreover, the $(x + y)$-adic valuation ring $W = D_{(x + y)D}$ of $D$ does not contain either $D^*$ or $D^{**}$. The transform of $x + y$ from $D$ to $D[y/x]$ is computed by setting $y_1 = y/x$. Then $x + y = x + xy_1 = x(1 + y_1)$. Hence $1 + y_1$ is the transform of $x + y$, and $W = D[y/x](1 + y_1)D[y/x]$ does not contain $D^* \subset D[y/x](y/x)D[y/x]$. Similarly, $W$ does not contain $D^{**}$.

The height one prime ideals of $D^*$ that contain $y(x + y)$ are the centers of $V$ and $V_\alpha$ on $D^*$, and the height one prime ideals of $D^{**}$ that contain $x(x + y)$ are the centers of $V$ and $V_\beta$. That $A \subset D^*$ and $A \subset D^{**}$ follows because
\[ x^3 V \subset y(x + y)V, \quad y^3 V \subset x(x + y)V, \quad x V = (x + y)V, \quad y V = (x + y)V, \]
\[ x^3 V_\alpha = y(x + y)V_\alpha, \quad y^3 V_\alpha \subset x(x + y)V_\alpha, \quad x V_\alpha = (x + y)V_\alpha, \quad y V_\alpha \subset (x + y)V_\alpha, \]
and
\[ x^3 V_\beta \subset y(x + y)V_\beta, \quad y^3 V_\beta = x(x + y)V_\beta, \quad x V_\beta \subset (x + y)V_\beta, \quad y V_\beta = (x + y)V_\beta. \]

The center of $D^*$ on $A$ is $(x, y, \frac{x^3}{y(x + y)}, \frac{y^3}{x(x + y)}, \frac{y}{x + y})A$, while the center of $D^{**}$ on $A$ is $(x, y, \frac{x^3}{y(x + y)}, \frac{y^3}{x(x + y)}, \frac{y}{x + y})A$. Since these are distinct prime ideals, it follows that $E = D^* \cap D^{**}$ has two distinct maximal ideals and $D^*$ and $D^{**}$ are the localizations of $E$ at these maximal ideals.

**Remark 6.7.** It can happen that $R$ and $S$ are 2-dimensional regular local rings with the same quotient field $F$, and $R \cap S$ is local and is properly contained in both $R$ and $S$. Corollary 6.5 implies this cannot happen if $R$ and $S$ are both overrings of a 2-dimensional regular local ring $D$.

The following example is given in [7]: Let $x$ and $y$ be variables over a field $k$ and let $R = k[x, x^2y]$ localized at the maximal ideal $(x, x^2y)R$ and let $S = k[xy^2, y]$ localized at the maximal ideal $(xy^2, y)$. Then $R$ and $S$ both have quotient field $k(x, y)$ and both are subrings of the formal power series ring $k[[x, y]]$. Every element in $k[[x, y]]$ has a unique expression as an infinite sum of monomials in $x$ and $y$ with coefficients from $k$. Every element in $R$ regarded as a formal power series in $k[[x, y]]$ has the property that the $x$-degree of each monomial is greater than the $y$-degree, and for any element in $S$ the $y$-degree of each monomial is greater than the $x$-degree. Hence $R \cap S = k$.}

17
7. Noetherian intersections

As in Notation 3.4, let $\mathcal{R}(D)$ denote the overrings of $D$ obtained as intersections of rings in $Q(D)$. As an intersection of normal rings, each ring in $\mathcal{R}(D)$ is a normal overring of $D$. In Lemma 7.1 and Theorem 7.2 we make some general observations about normal overrings of $D$.

**Lemma 7.1.** Let $R$ be a normal overring of a 2-dimensional Noetherian domain. Then

1. $\dim R \leq 2$. If $R$ has a 2-dimensional Noetherian overring, then $\dim R = 2$.
2. If $p$ is a nonzero nonmaximal prime ideal of $R$, then $R_p$ is a DVR and $R/p$ is a Noetherian domain.
3. $R$ is a Krull domain if and only if $R$ is a Noetherian domain.

**Proof.** For item 1, as an overring of the 2-dimensional Noetherian domain $D$, it follows from the dimension inequality [15, Theorem 15.5] that every finitely generated $D$-algebra overring of $D$ has dimension at most 2, and this implies $\dim R \leq 2$. Assume that $R$ has a Noetherian overring $A$ with $\dim A = 2$. Let $m$ be a maximal ideal of $A$ with $\text{ht } m = 2$. Since $A$ is Noetherian, a nonzero element of $A$ is contained in only finitely many height 1 primes $p$ of $A$, and $p \cap R \neq 0$, since $A$ is an overring of $R$. There exist infinitely many height 1 prime ideals $p$ of $A$ with $p \subset m$. It follows that $(0) \subset p \cap R \subset m \cap R$, for some $p$ and $\dim R \geq 2$.

For item 2, see [16, Proposition 2.3], and for item 3, see [6, Theorem 9].

**Theorem 7.2.** Let $D$ be a 2-dimensional regular local ring, and let $R$ be a normal overring of $D$.

1. Assume that each maximal ideal of $R$ has height 2. Then $R$ is Noetherian $\iff$ $R$ is a flat overring of a finitely generated $D$-subalgebra of $R$.
2. If $R$ is local, Noetherian and $\dim R \geq 2$, then $R$ is a spot over $D$, that is, $R$ is essentially finitely generated over $D$ in the sense that $R$ is the localization of a finitely generated $D$-algebra.
3. A 2-dimensional normal local Noetherian overring $T$ of $R$ is a localization of $R \iff$ each height 1 prime ideal of $T$ contracts to a height 1 prime ideal of $R$.

**Proof.** In item 1, the $\iff$ direction is clear because every ideal in a flat overring is an extended ideal. To prove $\implies$ assume $R$ is Noetherian and let $p$ be a height 1 prime ideal of $R$ and let $q = p \cap D$. Either $q = m_D$ or $\text{ht } q = 1$. If $\text{ht } q = 1$, then $D_q$ is a DVR, and $D_q = R_p$. A nonzero element in the Noetherian domain $R$ is contained in only finitely many height 1 primes of $R$. Hence there exists only a finite set, say $\{p_1, \ldots, p_n\}$, of height 1 prime ideals of $R$ that contain $m_D$. 

18
Since each maximal ideal of $R$ has height 2, the DVRs $V_i = D_{p_i}, i \in \{1, \ldots, n\}$ are prime divisors of the second kind over $D$. Hence there exist elements $a_i \in R$ such that the image of $a_i$ in the residue field of $V_i$ is algebraically independent over $D/m_R$.

Let $A$ denote the integral closure of $D[\{a_1, \ldots, a_n\}]$. A classical result of Rees [18] implies that $A$ is a finitely generated $D$-algebra. Thus $A$ is a normal Noetherian subring of $R$ such that for each height 1 prime $p$ of $R$, then $ht(p \cap A) = 1$.

Let $m$ be a maximal ideal of $R$ and let $n = m \cap A$. If $ht m = 1$, then $ht n = 1$ and $A_n = R_m$ is a DVR. If $ht m > 1$, then $A_n \subseteq R_m$ are 2-dimensional normal Noetherian local domains with $R_m$ dominating $A_n$. Let $q \in \text{Spec } A_n$ with $ht q = 1$. Since $A_n$ dominates $D$, $A_n$ has a rational singularity. By [12, Proposition 17.1], $A_n$ has a finite divisor class group. Hence $q$ is the radical of a principal ideal of $A_n$. It follows that $qR_m$ is contained in a height 1 prime of $R_m$. Therefore the set of essential valuation rings for $A_n$ is the same as the set of essential valuation rings for $R_m$, and thus $A_n = R_m$. By Theorem 2 of [19], $R$ is a flat overring of $A$. Item 2 follows from item 1.

For item 3, it is clear that if $T$ is a localization of $R$, then each height 1 prime ideal of $T$ contracts to a height 1 prime ideal of $R$. Suppose $T$ is not a localization of $R$. By replacing $R$ with $R_{m_T \cap R}$ we may assume without loss of generality that $R$ is a normal local ring with $R \subseteq T$. By item 1, there exist $x_1, \ldots, x_n \in T$ such that $T$ is a localization of $D[x_1, \ldots, x_n]$. Hence $T$ is a localization of $A := R[x_1, \ldots, x_n]$. Since $R$ is integrally closed, $R$ is integrally closed in $A$. Peskine’s version of Zariski’s Main Theorem [17, Proposition 13.4, p. 174] implies that there is a height one prime ideal $p$ of $T$ such that $p \cap R = m_T \cap R$. Since $\dim T = 2$, Lemma [7, 1.2 implies $m_T \cap R$ is a height 2 prime ideal of $R$. Therefore if $T$ is not a localization of $R$, there is a height 1 prime ideal of $T$ that does not contract to a height 1 prime ideal of $R$. This proves item 3.

Let $R$ be a normal Noetherian overring of $D$. By a desingularization of Spec $R$ we mean a desingularization $Y$ of the model $X = \{R_p : p \in \text{Spec } R\}$ over $R$; i.e., $Y$ is a nonsingular projective model over $R$ that dominates $X$.

**Theorem 7.3.** Let $R$ be a normal Noetherian overring of $D$ for which every maximal ideal has height 2. Then

1. There exists a desingularization $Y$ of Spec $R$ such that $Y$ is a subset of a nonsingular projective model $X = \text{Proj } D[Li]$ over $D$.
2. Spec $R$ has finitely many singularities.
3. Each desingularization of Spec $R$ is a product of quadratic transformations.

**Proof.** For item 1, by Theorem [7.2] there exists a finitely generated $D$-subalgebra $A$ of $R$ such that $R$ is flat over $A$. We may assume that $a_1, \ldots, a_n, b$ are nonzero
elements in $D$ such that $A = D[a_1/b, \ldots, a_n/b]$. For each maximal ideal $m$ of $R$, the local ring $R_m$ is a flat overring of $A$ and hence $A_{m \cap A} = R_m$.

Let $I = (a_1, \ldots, a_n, b)D$. Then each $R_m$ is on the model $\text{Proj } D[It]$. Let $J$ denote the integral closure of $I$. Since $R$ is integrally closed, each $R_m$ is on the normal model $\text{Proj } D[Jt]$. Let $L$ be the saturation of $J$. Then $X = \text{Proj } D[It]$ is a nonsingular projective model over $\text{Spec } D$ that dominates $\text{Proj } D[Jt]$, and the map $f : X \to \text{Proj } D[Jt]$ is a desingularization of $\text{Proj } D[Jt]$.

Since $\text{Spec } R \subset \text{Proj } D[Jt]$, the inverse image in $X$ of $\text{Spec } R$ with respect to the map $f$ is a desingularization of $\text{Spec } R$. This proves item 1.

To prove item 2, by Theorem 7.2.2, each localization of $R$ at a maximal ideal is a normal spot over $D$. By [13, Proposition, p. 160], each normal spot over $D$ is analytically normal. That $\text{Spec } R$ has finitely many singularities follows now from [13, Theorem, p. 151].

To prove item 3, we may assume that $\text{Spec } R$ has singularities. Since each localization of $R$ at a maximal ideal is a spot over $D$ by Theorem 7.2.2, it follows from [12, Proposition 1.2] that each normal Noetherian local overring of $D$ has a rational singularity. Thus each of the finitely many singularities of $\text{Spec } R$ is a rational singularity. A result of Lipman [12, Theorem 4.1] implies that any desingularization of $\text{Spec } R$ is a product of quadratic transformations.

With Theorem 7.3, we obtain a characterization of the Noetherian rings in $\mathcal{R}(D)$.

**Theorem 7.4.** Assume Notation 3.4. The following are equivalent for an overring $R$ of $D$.

1. $R$ is a Noetherian domain in $\mathcal{R}(D)$.
2. $R$ is a normal Noetherian domain for which every maximal ideal has height 2.
3. There exists a nonsingular projective model $X$ over $D$ and a subset $\mathcal{U}$ of the closed points of $X$ such that $R = \mathcal{O}_X$.

**Proof.** (1) $\implies$ (2): Since $R \in \mathcal{R}(D)$, there exists a subset $\mathcal{U}$ of $Q(D)$ such that $R = \mathcal{O}_\mathcal{U} = \bigcap_{\alpha \in \mathcal{U}} \alpha$. Let $m_\alpha$ denote the maximal ideal of $\alpha$. Lemma 7.1.1 implies that $\text{ht}(m_\alpha \cap R) = 2$ for each $\alpha \in \mathcal{U}$. Therefore

$$R \subseteq \bigcap_{\alpha \in \mathcal{U}} R_{m_\alpha \cap R} \subseteq \bigcap_{\alpha \in \mathcal{U}} \alpha = R.$$  

Suppose there exists a height 1 maximal ideal $p$ of $R$. Then $p$ is invertible and $R \subseteq p^{-1}$. But $p^{-1} \subseteq R_m$ for each maximal ideal $m$ of $R$ with $\text{ht } m = 2$, and hence
$p^{-1} \subseteq R$, a contradiction. Therefore if $R \in \mathcal{R}(D)$ is Noetherian, then every maximal ideal of $R$ has height 2.

(2) \implies (3): By Theorem 7.2.1, there exists a projective model $\text{Proj } D[It]$ such that $\{R_\mathfrak{p} : \mathfrak{p} \in \text{Spec } R\} \subset \text{Proj } D[It]$. By Theorem 5.2, there is a desingularization $X$ of $\text{Proj } D[It]$. Let $\mathcal{U}$ be the closed points in $X$ that contain $R$. Then $R = \mathcal{O}_\mathcal{U}$.

(3) \implies (1): Each of the local rings in $\mathcal{U}$ is an intersection of exceptional prime divisors of $X$ and prime divisors of the first kind. Since there are only finitely many exceptional prime divisors of $X$ and the set of prime divisors of $D$ of the first kind has finite character, it follows that $R = \mathcal{O}_\mathcal{S}$ is a finite character intersection of DVRs, hence a Krull domain. Lemma 7.1 implies that as a Krull overring of a 2-dimensional Noetherian domain, $R$ is a normal Noetherian domain.

**Corollary 7.5.** Let $X$ be a nonsingular projective model over $D$, and let $L$ be a saturated complete ideal of $D$ such that $X = \text{Proj } D[It]$.

1. There exist only finitely many local domains $R$ that are not regular and have the form $R = \mathcal{O}_\mathcal{U}$, where $\mathcal{U}$ is a subset of the closed points of $X$.
2. Each $R$ is normal Noetherian with $\dim R = 2$.
3. Each $R$ in item 1 is on a normal projective model $N = \text{Proj } D[It]$ over $D$, where $J$ is a complete ideal of $D$ that divides $L$.

**Proof.** If $R = \mathcal{O}_\mathcal{U}$, where $\mathcal{U}$ is a subset of the closed points of $X$, then Theorem 7.5 implies that $R$ is normal Noetherian with $\dim R = 2$, and $R$ is a point on a normal projective model that is dominated by $X$. Every normal projective model dominated by $X$ has the form $\text{Proj } D[It]$, where $J$ is a complete ideal that is the product of a subset of the simple complete factors of $L$. Two subsets with the same simple complete factors define the same model. Hence there exist only finitely many normal projective models over $D$ that are dominated by $X$. Theorem 7.3 implies that each of these normal projective models has only finitely many singular points. Therefore there are only finitely many $R$ of this form that are not regular.

**Example 7.6.** Assume Notation 3.4. Let $R = D[y^2/x, (x, y, y^2/x)D[y^2/x]]$. Then the maximal ideal $\mathfrak{m}_R$ of $R$ is $(x, y, y^2/x)R$. Since $\mathfrak{m}_R$ requires 3 generators, $R$ is not regular. $R$ is a normal domain, and is called an ordinary double point singularity. We give an explicit representation of $R$ as an intersection of rings in $Q(D)$.

We show that the quadratic transform $\text{Proj } R[\mathfrak{m}_R]$ is a nonsingular model over $R$. As a normal domain, $R$ is an intersection of its minimal valuation overrings. Moreover, each minimal valuation overring of $R$ dominates a closed point in $\text{Proj } R[\mathfrak{m}_R]$. We use this fact in describing the closed points in $\text{Proj } R[\mathfrak{m}_R]$.

---

14It is shown in [11] that there exist non-Noetherian $R \in \mathcal{R}(D)$ that have maximal ideals of height 1.
Let $V$ be a minimal valuation overring of $R$. Then $V$ dominates $R$, and $m_R V$ is either $xV$, or $yV$ or $(y^2/x) V$.

If $m_R V = xV$, then $y/x \in V$ and $R[y/x] = D[y/x] \subset V$. The affine component $D[y/x]$ of $\text{Proj } R[m_R t]$ is nonsingular, and $V$ is centered on a height 2 maximal ideal of $D[y/x]$. Since $y^2/x \in yD[y/x]$ and $y \in xD[y/x]$, it follows that $xD[y/x] \cap R = (x, y, y^2/x) R$.

Every maximal ideal $p$ of $D[y/x]$ of height 2 contains $x$, and the map $\text{Spec } D[y/x] \to \text{Spec } R$ maps each maximal ideal $p$ of height 2 of $D[y/x]$ to $p \cap R = m_R$. Therefore all the elements of $Q_1(D)$ other than $\alpha = D[x/y](y,x/y)D[x/y]$ dominate $R$, and are dominated by a minimal valuation overring of $R$.

If $m_R V = yV$, then both $x/y$ and $y/x$ are in $V$. Hence the affine component of $\text{Proj } R[m_R t]$ obtained by dividing by $y$ gives nothing that is not already obtained in the affine component dividing by $x$.

If $m_R V = (y^2/x) V$, the affine component obtained by dividing by $y^2/x$ is $R[x/y] = D[x/y, y^2/x]$ and is nonsingular. If $xV \neq yV$, then $V$ is centered on the maximal ideal $(x/y, y^2/x)D[x/y, y^2/x]$. The localization at this maximal ideal is the point $\beta = \alpha[y^2/x](x/y, y^2/x)\alpha[y^2/x] \in Q_2(D)$. Notice that $\beta$ dominates $\alpha \in Q_1(D)$.

Let $A = \bigcap \{ D[y/x]_p \mid \text{ht } p = 2 \}$. Then $A$ is the intersection of the elements in $Q_1(D)$ other than $\alpha$. The centers on $\text{Proj } R[m_R t]$ of the minimal valuation overrings of $R$ are the localizations of $A$ at its maximal ideals and $\beta$. Therefore $\text{Proj } R[m_R t]$ is nonsingular, and $R = A \cap \beta$.

Since $R$ is proper subring of $A$, the point $\beta$ is irredundant in this representation. Let $\gamma$ be a point in $Q_1(D) \setminus \{ \alpha \}$. Theorem 5.3 implies that there exists an element $f \notin D$ such that $f$ is in each point of $Q_1(D) \setminus \{ \gamma \}$. Since $\beta$ dominates $\alpha$, $f \in \beta$. Since $R \cap \alpha = D$, it follows that $f \notin R$. Therefore the representation $R = A \cap \beta$ is irredundant.

$R$ is the unique singular point of the normal projective model $\text{Proj } D[xt, y^2 t]$. The quadratic transform $\text{Proj } R[m_R t]$ described above is in the nonsingular model $\text{Proj } D[J_1]$, where $J = (x^2, xy, y^3) D$.

The saturated complete ideal $J = (x, y)(x, y^2) R$ has two complete simple factors. Corollary 7.5 implies that $R$ is the unique non-regular local domain of the form $O_U$, where $U$ is a subset of the closed points of $\text{Proj } D[J_1]$.

8. Irredundant representations

Each ring $R$ in $\mathcal{R}(D)$ is an intersection of rings in $Q(D)$. In this section we consider settings in which $R$ can be represented by an irredundant intersection of rings from $Q(D)$; i.e., there is a set $U$ in $Q(D)$ such that $R = O_U$ but $R \subset O_{U \setminus \{ \alpha \}}$ for each $\alpha \in U$. 

22
Definition 8.1. A subset $U$ of $Q(D)$ is said to be complete if $\mathcal{O}_U = \bigcap_{R \in U} R = D$. For a point $\alpha \in Q(D)$, a subset $U$ of $Q(D)$ of points that dominate $\alpha$ is said to be complete over $\alpha$ if $\mathcal{O}_U = \alpha$.

Remark 8.2. Let $X = \text{Proj } D[xt]$ be a normal projective model over $D$. Then the set $U$ of closed points of $X$ is complete since every minimal valuation ring $V$ is centered on a closed point of $X$ and $D$ is the intersection of the minimal valuation overrings of $D$. It is natural to ask if a proper subset of $U$ can be complete. This is equivalent to asking if the representation $D = \bigcap_{\alpha \in U} \alpha = D$ is irredundant.

Theorem 8.3. Assume notation as in Discussion 6.4.

(1) If $U = Q_1(D)$, then $U$ is complete, and the representation $D = \bigcap_{R \in Q_1(D)} R$ is irredundant.

(2) Assume $U$ is a nonempty proper subset of $Q_1(D)$, then
   (1) $\mathcal{O}_U$ is a flat extension of a regular finitely generated $D$-subalgebra of $F$.
   (2) $B := \mathcal{O}_U$ is a regular Noetherian domain and the representation $B = \bigcap_{\alpha \in U} \alpha$ is an irredundant essential representation of $B$.

Proof. The set $U = Q_1(D)$ is complete by Remark 8.2. To see that the representation $D = \bigcap_{R \in Q_1(D)} R$ is irredundant, we use as in Remark 3.3 that the points of $Q_1(D)$ are in a natural one-to-one correspondence with the maximal homogeneous relevant prime ideals of $D[xt, yt]$. The relevant homogeneous maximal ideals of $D[xt, yt]$ all contain $m_D$, and $D[xt, yt]/m_D D[xt, yt] \cong \kappa(D)[\overline{x}, \overline{y}]$, where $xt \mapsto \overline{x}$ and $yt \mapsto \overline{y}$, and $\overline{x}, \overline{y}$ are algebraically independent over $\kappa(D)$. Hence to each point $\gamma \in Q_1(D)$ there corresponds an irreducible homogeneous polynomial $f(\overline{x}, \overline{y}) \in D[xt, yt]$ such that the image $\overline{f}$ of $f$ in $\kappa(D)[\overline{x}, \overline{y}]$ is irreducible. Then $\deg f = \deg \overline{f} = d$ for some positive integer $d$, and the degree zero component $A = D[xt, yt][\overline{x}, \overline{y}]/f^d$ is an affine component of $\text{Proj } D[xt, yt]$. Then $A$ is a regular finitely generated $D$-subalgebra of $F$.

Let $U = Q_1(D) \setminus \{\gamma\}$. Each of the points of $U$ is a localization of $A$. Therefore $A \subseteq \mathcal{O}_U$. Since $A$ properly contains $D$, it follows that $\gamma$ is irredundant in the representation $D = \bigcap_{R \in Q_1(D)} R$. Since this is true for each $\gamma \in Q_1(D)$, the representation $D = \bigcap_{R \in Q_1(D)} R$ is irredundant. This proves item 1.

---

15 A homogeneous prime ideal of the graded domain $D[xt, yt]$ is said to be relevant if it does not contain the maximal graded ideal $(x, y, xt, yt)D[xt, yt]$.
16 The local ring $\gamma$ does not contain $A$. 

23
For item 2, assume that $\mathcal{U}$ is a nonempty proper subset of $Q_1(D)$. Then $\mathcal{U}$ is a subset of $Q_1(D) \setminus \{\gamma\}$, for some $\gamma \in Q_1(D)$. Hence $B = \mathcal{O}_\mathcal{U}$ is a flat extension of the regular finitely generated $D$-algebra $A = D[\{x,y\}]$ and the representation $B = \bigcap_{\alpha \in \mathcal{U}} \alpha$ is an irredundant essential representation of $B$. 

Let $\alpha \in Q(D)$. Applying Theorem 8.3 to the 2-dimensional regular local domain $\alpha$ gives Corollary 8.4.

**Corollary 8.4.** Let $\alpha \in Q(D)$. Let $\mathcal{U}$ be a subset of $Q_1(\alpha)$. If $\mathcal{U} = Q_1(\alpha)$, then $\mathcal{O}_\mathcal{U} = \alpha$. Otherwise, if $\mathcal{U}$ is a proper subset of $Q_1(\alpha)$, then

1. $\mathcal{O}_\mathcal{U}$ is a regular Noetherian domain such that $\mathcal{U}$ is the set of localizations at a maximal ideal is in $\mathcal{U}$.
2. $\mathcal{O}_\mathcal{U}$ is a flat extension of a regular finitely generated $D$-subalgebra of $F$.
3. The representation of $B = \mathcal{O}_\mathcal{U} = \bigcap_{\alpha \in \mathcal{U}} \alpha$ is an irredundant essential representation.

The rest of the section is devoted to the case where $D$ is Henselian. We first establish a lemma that applies to the Henselian case but whose hypotheses can hold in more general settings for specific choices of height one prime ideals of $D$.

**Lemma 8.5.** Assume Notation 3.4. Let $\gamma \in Q_1(D)$ and let $p$ be a height one prime of $D$ such that $\gamma \subset D_p$.

1. Then $\gamma/(pD_p \cap \gamma)$ is a local quadratic transform of $D/p$.

Suppose in addition that the integral closure of $D/p$ is local.

2. The ring $\gamma$ is the only point of $Q_1(D)$ such that $\gamma \subset D_p$.
3. For each integer $n \geq 2$, there exists a unique point $\gamma_n \in Q_n(D)$ such that $\gamma_n \subset D_p$. Then $V = \bigcup_{n=1}^{\infty} \gamma_n$ is the rank 2 valuation overring of $D$ that is the composite of $D_p$ with the integral closure of $D/p$.

**Proof.** For item 1, since $\gamma \subset D_p$, the canonical surjective map $D_p \to D_p/pD_p$ restricts to a surjective map $\gamma \to \gamma/(pD_p \cap \gamma)$. Since $\gamma$ is a local quadratic transform of $D$, the universal property of blowing up [5, Prop. 7.14 and Cor. 7.15, pp. 164–165] implies that the induced map $D/p \to \gamma/(pD_p \cap \gamma)$ is a local quadratic transform.

For item 2, let $R = D/p$. Then $R$ is a Noetherian local domain with $\dim R = 1$. Since the integral closure of $R$ is local, the integral closure of $R$ is the unique valuation overring of $R$ dominating $R$. It follows that every overring of $R$ is local.

If $\gamma' \in Q_1(D)$ with $\gamma' \neq \gamma$ and $\gamma' \subset D_p$, then Corollary 6.5 implies that $A = \gamma \cap \gamma'$ has two maximal ideals and both of the maximal ideals of $A$ contain $pD_p \cap A$. This
implies that $A/(pD_p \cap A)$ is an overring of $R$ that is not local, a contradiction. Hence $\gamma$ is the unique point in $Q_1(D)$ with $\gamma \subset D_p$.

A similar argument proves for each $n \geq 2$ that there exists a unique point $\gamma_n \in Q_n(D)$ such that $\gamma_n \subset D_p$. This proves item 3. \hfill $\square$

If $D$ is Henselian, then the integral closure of $D/p$ is local for each height 1 prime $p$ of $D$. Thus the statements in items 2 and 3 apply to $D_p$. We use this observation in the proof of Theorem 8.6.

**Theorem 8.6.** Assume that $D$ is Henselian and $\mathcal{U}$ is a set of pairwise incomparable rings in $Q(D)$. Then the representation $\mathcal{O}_{\mathcal{U}} = \bigcap_{\alpha \in \mathcal{U}} \alpha$ is irredundant.

**Proof.** We first make a couple of reductions to simplify the proof. Let $\mathcal{U}^*$ be the union of $\mathcal{U}$ with the set of rings $\alpha \in Q(D)$ such that $\alpha$ is minimal with respect to not containing any of the rings in $\mathcal{U}$. Then $D = \mathcal{O}_{\mathcal{U}^*}$ and to prove that $\mathcal{U}$ is an irredundant representation of $\mathcal{O}_{\mathcal{U}}$ it suffices to prove that $\mathcal{U}^*$ is an irredundant representation of $D$. Thus we may assume that $\mathcal{U} = \mathcal{U}^*$ and hence that $\mathcal{U}$ is complete.

Let $\gamma \in \mathcal{U}$. It suffices to show that $D \not\subseteq \mathcal{O}_{\mathcal{U} \setminus \{\gamma\}}$, and to prove this it suffices to show that $D \not\subseteq \mathcal{O}_{\mathcal{U}'}$, where $\mathcal{U}'$ is the set of rings in $Q(D)$ minimal with respect to not containing $\gamma$. By Theorem 5.5.1 we may assume without loss of generality that $\mathcal{U}$ is the set of closed points in a projective nonsingular model over $D$.

Let $J$ be a complete ideal of $D$ such that $X = \text{Proj} D[J]$. Let $\mathcal{E}(D)$ denote the set of essential valuation rings of $D$. For each $\alpha \in \mathcal{U}$, the set $\mathcal{E}(\alpha)$ of essential valuation rings for $\alpha$ is the union of a subset of $\mathcal{E}(D)$ with a subset of Rees $J$. Since Rees $J$ is a finite set and $\mathcal{E}(\alpha)$ is an infinite set, for each $\alpha \in \mathcal{U}$, there exists a height 1 prime $p_\alpha$ of $D$ such that $D_{p_\alpha} \in \mathcal{E}(\alpha)$.

Since $D$ is Henselian, the integral closure of $D/p_\alpha$ is local. Since there are no inclusion relations among the points in $\mathcal{U}$, Lemma 5.5.3 implies that $\alpha$ is the unique point of $\mathcal{U}$ contained in $D_{p_\alpha}$.

The set $\mathcal{S} := \mathcal{E}(D) \setminus \{D_{p_\alpha} : \alpha \in \mathcal{U} \cup \text{Rees } J\}$ includes the set $\mathcal{E}(\beta)$ for each $\beta \in \mathcal{U}$ with $\beta \neq \alpha$. Therefore $\bigcap\{V \mid V \in \mathcal{S}\} \subseteq \bigcap\{\beta \mid \beta \in \mathcal{U}, \beta \neq \alpha\}$.

Since the set $\mathcal{S}$ has finite character in the sense that a nonzero element of $F$ is a unit in all but finitely many of the elements in $\mathcal{S}$ and since $\{D_{p_\alpha}\} \nsubseteq \mathcal{S}$, it follows that $D \nsubseteq \bigcap\{V \mid V \in \mathcal{S}\}$. Since $\alpha$ is an arbitrary element in $\mathcal{U}$, the representation $D = \mathcal{O}_{\mathcal{U}}$ is irredundant. \hfill $\square$

**Corollary 8.7.** Assume that $D$ is Henselian. Let $R$ be a Noetherian normal overring of $D$ such that every maximal ideal has height 2. Then $R$ is an irredundant intersection of the rings in $Q(D)$ that are minimal with respect to containing $R$.

**Proof.** Apply Theorem 7.4 and Theorem 8.6. \hfill $\square$
Acknowledgment. This paper is an outgrowth of a project the authors are working on with Alan Loper in [11]. We thank Alan for helpful conversations and for motivating us to consider the topics in this paper. We also thank Joe Lipman for several helpful comments on an earlier draft of this paper.

References

[1] S. S. Abhyankar, On the valuations centered in a local domain, Amer. J. Math. 78 (1956), 321–348.

[2] S. S. Abhyankar, Ramification Theoretic Methods in Algebraic Geometry, Princeton University Press, 1959.

[3] S. S. Abhyankar, Lectures on Algebra I, World Scientific, 2006.

[4] S. S. Abhyankar, Generic incarnations of quadratic transforms, Proc. Amer. Math. Soc. 141 (2013), 4103–4117.

[5] R. Hartshorne, Algebraic Geometry, Springer-Verlag New York, 1977.

[6] W. Heinzer, On Krull overrings of a Noetherian domain, Proc. Amer. Math. Soc. 22 (1969), 217–222.

[7] W. Heinzer, Noetherian intersections of integral domains II, Conference in Commutative Algebra, Lecture Notes in Mathematics 311, Springer-Verlag, New York (1973), 107-119.

[8] W. Heinzer, B. Johnston, D. Lantz and K. Shah, Coefficient ideal in and blowups of a commutative Noetherian domain, J. Algebra 162 (1993), 355–391.

[9] W. Heinzer, M-K. Kim and M. Toeniskoetter, Finitely supported $*$-simple complete ideals in a regular local ring, J. Algebra 401 (2014), 76–106.

[10] W. Heinzer, Y. Kim and M. Toeniskoetter, Blowing up finitely supported complete ideals in a regular local ring, J. Algebra 458 (2016), 364–386.

[11] W. Heinzer, K. A. Loper and B. Olberding, Intersections of regular local rings of dimension two, in preparation.

[12] J. Lipman, Rational singularities, with applications to algebraic surfaces and unique factorization, Publ. Math. Inst. Haute Etudes Sci., 36 (1969), 195–279.
[13] J. Lipman, Desingularization of two-dimensional schemes, Ann. Math. 107 (1978), 151–207.

[14] J. Lipman, On complete ideals in regular local rings, in Algebraic Geometry and Commutative Algebra in Honor of Masayoshi Nagata, 1986, 203–231.

[15] H. Matsumura Commutative Ring Theory, Cambridge Univ. Press, Cambridge, 1986.

[16] B. Olberding, Overrings of 2-dimensional Noetherian domains representable by Noetherian spaces of valuation rings, J. Pure Appl. Algebra 212 (2008), 1797–1821.

[17] C. Peskine, An algebraic introduction to complex projective geometry, Cambridge University Press, 1996.

[18] D. Rees, A note on analytically unramified local rings, J. London Math. Soc. 36 (1961), 24–28.

[19] F. Richman, Generalized quotient rings, Proc. Amer. Math. Soc. 16 (1965), 794–799.

[20] I. Swanson and C. Huneke, Integral Closure of Ideals, Rings, and Modules, London Math. Soc. Lecture Note Series 336, Cambridge Univ. Press, Cambridge, 2006.

[21] O. Zariski and P. Samuel, Commutative Algebra, Vol. 2, Van Nostrand, New York, 1960.