ENTRAINMENT OF AN INERTIAL REFERENCE FRAME BY AN ACCELERATED GRAVITATING SHELL

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Abstract

For the example of an accelerated shell we show that omission of the energy-momentum tensor (EMT) of the body that causes the acceleration and the tensions due to this acceleration can lead to a paradoxical result; Namely, the entrainment of an inertial frame by the accelerated shell in the direction opposite to that of the acceleration. We consider several models and demonstrate that the correct result can be obtained only if all components of the full EMT are adequately taken into account, and the problem statement is physically correct.

1 Introduction

Before the theory of general relativity was conceived, Einstein published the paper Is there a gravitational interaction similar to electromagnetic induction? [1], in which the effect of an accelerated shell on a test body placed inside it was discussed. The problem was analyzed using semi-classical concepts (here we use the term classical in the sense of Newton’s gravitation), and Einstein had come to the following result (in linear approximation with respect to the acceleration): the test body is driven by the shell with acceleration $a_{\text{body}} = \frac{3M}{2R}a_{\text{shell}}$, where $M$ is the shell mass and $R$ is its radius.

After the formulation of the theory of general relativity, Lense and Tirring [2] considered a similar problem. They studied the entrainment of inertial reference frames by material currents on the basis of Einstein’s linearized equations. The results were applied to analyzing a rotating shell, and they obtained the well-known solution in which the reference frame inside the shell rotates with the frequency $\omega_{i.f.} = \frac{4M}{3R}\omega_{\text{shell}}$. Later Brill and Cohen [3] calculated a metric for a rotating shell assuming a linear approximation in $\omega$, and this metric was identical to Kerr’s one outside the shell. In the case $\frac{M}{R} \ll 1$, it reduced to that calculated by Lense and Thirring.

Here we analyze an accelerated shell having at our disposal Einstein’s equations. In that sense we have much more advantage than Einstein had. Using the linear approximation of general relativity we shall consider several different statements of the problem. In one case, we shall demonstrate the well-known and methodologically important result that in the case...
of the general relativity one should take into account the full energy-momentum tensor of the all objects relevant to given problem. In the other case, the incorrect and mainly paradoxical results will be received.

The paper is organized as follows. Section 2 takes into account only the EMT of the shell. Section 3 analyzes a closed system of a shell + a massless string driving the shell. It is shown that tension in the string and shell lead to the additional terms in the expression for the acceleration of the probe body. But in this case again we have the incorrect result. The reason for that is quite different from the first example and we shall investigate it for the methodological purposes. Finally, Section 4 analyses a consistent model, namely a charged shell in a DC electric field. It is shown that the EMT of the electromagnetic field also contributes to the acceleration of the reference frame.

2 Uniformly Accelerated Shell

Let’s choose a frame system \( x^i = (t, x, y, z) \) build on the base of “the rigid rod lattice”, which is stationary with respect to the far stars. Here \( t \) is time synchronized with the remote clocks. Let a rigid infinitesimally thin shell of the radius \( R \) be accelerated along the \( z \)-axis at the constant acceleration \( a \). (All limitations on \( R \) and \( a \) will be written below.) Now our frame is influenced by the shell and as a result it is not inertial. A free falling probe will move with some acceleration \( \mathbf{a}_{\text{body}} \) with respect to ”the rigid rod lattice”. We can use the system of such probe to construct an inertial frame system (obviously, it is a question of the local inertial frame systems). So we can say that the inertial frame systems are intrained by the accelerated shell.

It’s obvious that in the given case it is practically impossible to find the exact solution of Einstein’s equations. So we shell work in the weak field limit and with all velocities less than light: \( R_g \ll R \) and \( R a \ll 1 \), where \( R_g = 2M \) is the shell gravitational radius and \( a \) is its acceleration \( (c = G = 1) \). In this case, Einstein’s equations are reduced to the following form

\[
\Box h_{ik} = 16\pi T_{ik},
\]

where \( h_{ik} = \bar{h}_{ik} - \frac{1}{2} \eta_{ik} \bar{h}, \bar{h} = \bar{h}_{ik} \bar{\eta}^{ik}, g_{ik} = \eta_{ik} + h_{ik}, \) and \( \eta_{ik} = \text{diag}(1, -1, -1, -1) \).

The general solution of Eq.(1) (which satisfies given the boundary conditions) is expressed as

\[
\bar{h}_{ik}(\mathbf{r}, t) = -4 \int \frac{T_{ik}(\mathbf{r}', t')}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}',
\]

where \( t' = t - |\mathbf{r} - \mathbf{r}'| \) is ”retarded time”.

The points of the rigid rod frame system move along the trajectories \( x^\alpha = \text{const} \), where \( \alpha = 1, 2, 3 \). As we explained above, that motion is noninerial (nongeodesic) and characterized by the acceleration 4-vector \( u^i = \frac{\mathbf{D}u^i}{dS} \approx (0, -\mathbf{a}_{\text{body}}) \), where \( u^i = \frac{d\mathbf{r}'}{dS} = (u^0, 0, 0, 0) \) is the velocity 4-vector. So we have

\[
(\mathbf{a}_{\text{body}})_\alpha = -\frac{\mathbf{D}u^i}{dS} = -\Gamma^\alpha_{ik} u^i u^k = -\Gamma^\alpha_{00}(u^0)^2 \approx \Gamma^\alpha_{00} = \dot{h}_{00} - \frac{1}{2} \nabla_\alpha h_{00}.
\]
Here \( \nabla_\alpha \equiv \frac{\partial}{\partial x_\alpha} \), \( \dot{h}_\alpha \equiv \frac{\partial h_\alpha}{\partial t} \). From Eq.(2) and the relations \( h_\alpha \dot{h}_\alpha = h_\alpha h_\alpha = \frac{1}{2} \sum_i \delta_\alpha^i \) we have

\[
(a_{\text{body}})_\alpha = -4 \int \frac{T_{\alpha\alpha}(r', t')}{|r - r'|} d^3 r' + \nabla_\alpha \Phi ,
\]

where \( \Phi = \int \sum_i T_{ii}(r', t') |r - r'| d^3 r' \). It is obvious from the axial symmetry that \( (a_{\text{body}})_x = (a_{\text{body}})_y = 0 \) and \( (a_{\text{body}})_z \equiv a_{\text{body}} \neq 0 \). Note that the expression for \( \Phi \) includes the non-covariant summation over \( i \) but not the covariant trace.

In (3) we have the \( T_{00}, T_{\alpha\alpha} \) and \( T_{0\alpha} \) components of the EMT that describe mass distribution, tensions and energy-momentum currents respectively in all bodies. One cannot drop any of these terms without detailed consideration. Let’s try to calculate \( a_{\text{body}} \) from the assumption that the acceleration is determined only by the mass distribution, i.e. \( T_{00} \) corresponding to the shell (which we regard as the most massive body in our system). This assumption is based on the Newtonian gravity and finiteness of the gravity propagation speed only. So, let’s keep only \( T_{00} \) of the shell in (3).

In our calculations, we retain only the terms linear in \( a \). This means that we should ignore relativistic effects proportional to \( \frac{v^2}{c^2} \sim a^2 \). In the approximation of isotropic matter distribution the energy-momentum tensor is given by

\[
T = (\rho + p)u \otimes u - pg ,
\]

where \( \rho \) is the density of matter and \( p \sim a \) is pressure. Therefore we obtain

\[
T^{00} = (\rho + p)(u^0)^2 - pg^{00} \approx \rho (u^0)^2 ,
\]

\[
u_0 = \frac{dt}{dS} \approx \frac{1}{1 + \beta a_{\text{body}} z} \approx 1 - \beta a_{\text{body}} z ,
\]

where \( \beta \) is of order one. In the last expression we have to drop the \( \beta a_{\text{body}} z \) term because it leads to a correction of order \( \left( \frac{R}{R} \right)^2 a \). This is beyond the accuracy of our approximation.

Taking into account the shell acceleration and that at \( t = 0 \) its center of mass is at rest at the origin, we have an expression for \( T^{00}(r, t) \)

\[
T^{00}(r, t) = \frac{M}{4\pi R^2} \delta \left[ r - \frac{1}{2} a t^2 \right] - R \equiv \rho \left( r - \frac{1}{2} a t^2 \right) \approx \rho (r) - \frac{1}{2} a t^2 \nabla \rho (r) . \tag{5}
\]

It is obvious that the first summand does not contribute to the acceleration \( a_{\text{body}} \). (This follows from the spherical symmetry and it can also be seen from (3) where the differentiation with respect to \( z \) would yield zero). By combining Eqs. (3) and (5), we obtain

\[
a_{\text{body}} = -\frac{1}{2} a \partial_z \int |r - r'| \nabla' \rho (r') d^3 r' .
\]

We are interesting in the acceleration of the free body at \( t = 0 \) when shell comes to rest.

So, everywhere and particularly in Eq.(5), we will set \( t = 0 \), i.e. \( t' = -|r - r'| \). Close to the point of origin we have

\[
|r - r'| \approx r' - r' r ,
\]
where $\mathbf{r}' = \frac{\mathbf{r}'}{r'}$. For the same reason that we have dropped the first summand in Eq. (5) we should omit the first term independent of $\mathbf{r}$ in the last expression. Finally, we have

$$a_{\text{body}} = \frac{1}{2} \mathbf{a} \int (\hat{\mathbf{r}}')_z \nabla' \rho(\mathbf{r}') d^3 \mathbf{r}' = \frac{1}{2} \mathbf{a} \int d^3 \mathbf{r}' \rho(\mathbf{r}') \nabla'(\hat{\mathbf{r}}')_z$$

$$= -\frac{1}{2} \mathbf{a} \int d^3 \mathbf{r}' \rho(\mathbf{r}') \left( \frac{\hat{z}'}{r'} - \frac{\hat{r}' z'}{r'^2} \right).$$

After integration over the angular variables (which is trivial owing to the spherical symmetry of the configuration) we obtain an absurd result

$$a_{\text{body}} = -\frac{1}{3} \frac{M a}{R}.$$ (6)

On the basis of general arguments one could assume that $a_{\text{body}}$ should be of the order of $(R_g/R)a$, but there is no reason whatsoever why $a_{\text{body}}$ should point in the direction opposite to that of $\mathbf{a}$.

It is obvious, that the paradox appeared because our system is not closed. Bodies causing shell acceleration and tensions and energy-momentum currents in those bodies were not taken into consideration. The intuition based on the Newton’s law of gravity obviously let us down.

Contribution of Eq. (6) to the probe body acceleration is due to retarding effect of the gravity signals emitted by the shell. Because of analogous reason there is electric field $\mathbf{E} = -\frac{2qa}{3c^2 R}$ in the center of an accelerated sphere with charge $q$.

3 A Shell Driven by a String

To correct the above situation, let us start with the following “simple” model: the shell accelerated by an infinitely long, massless string (Fig. 1). The word “simple” is in quotation marks because, as will be shown below, this model lacks a clear physical sense: there is no such string (at least in the frame of classical physics).

Now $a_{\text{body}} = a_0 + a_1 + a_2$, where $a_0$ is determined by the first summand in Eq. (3), $a_1$ is the acceleration of the probe body calculated in the previous section considering only the shell motion, and $a_2$ takes into account all other factors: tensions in the shell and string, “the spool” on which string is wound, and also the distant body that is located at a distance $L$ from the shell and on which “the spool” is fixed. Let the mass of that body along with “the spool” be $m$. Then its contribution to the $a_{\text{body}}$ is of the order of $m/L^2$ and one can neglect it if $L$ is large enough. So, $a_2 = a_2^{\text{tension}} + a_2^{\text{string}}$. To begin with consider the term due to tension in the shell:

$$a_2^{\text{tension}} = \partial_z \int \frac{\sum_\alpha T_{\alpha \alpha}(\mathbf{r}', 0)}{|\mathbf{r} - \mathbf{r}'|} d^3 \mathbf{r}'$$, (7)

where $\alpha = 1, 2, 3$. It was taken into account because $T_{\alpha \alpha}$ is of first order in $a$ we can put in it (and due to the same reason, in the first summand in Eq. (3)) $t' = 0$. 

4
Figure 1: Shell accelerated by an infinitely long, massless string along the $z$-axis. There is a probe, whose “induced acceleration” is measured at the shell center.

The expression for $\sum_a T_{aa}$ of shell is derived in the Appendix:

$$\sum_a T_{aa} = \frac{Ma}{4\pi R} \cos \theta \delta(r - R) = aR \rho(r) \cos \theta .$$ (8)

After expanding $1/|r - r'|$ up to the first order in $r$, one receives from Eqs. (7) and (8):

$$a_{tension}^2 = \frac{Ma}{3R} .$$

Comparing this result to Eq. (6), one can see that contribution to the acceleration of the probe due to the shell alone is zero: tensions in the shell have compensated the effect due to mass.

Now, let’s consider $a_{string}^2$. Since the string has zero mass and is infinitesimally thin, we can write down the only nonzero component of the string EMT, taking into account the axial symmetry, as

$$T_{zz}(x, y, z) = D \theta(z - R) \delta(x) \delta(y) .$$

The constant $D$ is calculated from Eq. (A.3) in the Appendix. Using that formula, one can receive that string acts on shell with the force

$$F_x = F_y = 0 ,$$

$$F_z = - \int T_{zz} df_z = - \int T_{zz} dxdy = -D = Ma .$$
Thus, we have

\[ T_{zz} = -aM\theta(z - R)\delta(x)\delta(y) \ . \]

Now it is easy to calculate the contribution of the string EMT to \( a_{\text{body}} \):

\[ a_{\text{string}}^2 = \frac{-Ma}{R} \ . \tag{9} \]

The easiest way to calculate \( a_0 \) is to use Eq. (4):

\[ T_{0\alpha}(r, t) = (\rho + p)u_0 u_\alpha \approx \rho u_\alpha \approx -\rho(v)_\alpha \ , \]

\[ \dot{T}_{0\alpha}(r, t) \approx -\rho(a)_\alpha \ . \]

Therefore

\[ a_0 = 4\frac{Ma}{R} \ . \tag{10} \]

Hence the total acceleration of the probe body rested at the center of shell which is accelerated by infinitely thin massless string is

\[ a_{\text{body}} = \frac{3M}{R}a \equiv \frac{3R_g}{2R}a \ . \tag{11} \]

If one pays attention to the sign in Eq. (9) which tells us that the probe is repulsed by the stretched massless string, doubts about Eq. (11) appear. From Eq. (3) and the expression for \( \Phi \) one can see that this gravitational repulsion appears in the case

\[ \sum_i T_{ii} < 0 \ . \tag{12} \]

For the homogeneous body this means that \( \epsilon + 3p < 0 \), i.e. \( p < -\frac{\epsilon}{3} \). There are no known bodies or fields with such a property.

For the usual string we have \(|p| \ll \epsilon \approx \rho\), where \(|p| \sim \frac{E}{d} = \frac{Ma}{d} \) and \( \rho \) is string density and \( d \) is its diameter. Such string accelerates a probe body with the usual Newtonian acceleration \( a_N \sim \frac{Ma}{R^2} \). It follows that

\[ a_N \gg \frac{|p|d^2}{R} \sim \frac{Ma}{R} \ . \]

So, in the experiment with the shell driven by a string, it would impossible to detect the effect of inertial frame entrainment because it is hidden under the much more substantial effect from the string gravitational attraction. Though one can speak only about the gedankenexperiment because entrainment effect is so small that it cannot be detected. For example, additional phase incursion for the laser beam past through an accelerated body would be \( \sim 10^{-40} \) in typical laboratory experiment.

Keeping this conclusion in mind, let us proceed to the consistent system, namely a charged shell driven by an electric field.
A charged shell with a charge $q$ is in the field generated by the charge $-Q$. The distance between the charges is selected so that the direct effect of the mass of charge $Q$ and effects quadratic in the electric field could be neglected.

## 4 Charged Shell

A shell with charge $q$ (Fig. 2) is placed in the field $E_0$ generated by charge $-Q$. The charge is sufficiently far away (at the distance $L$) so that one can consider the field $E_0$ as homogeneous and neglect the gravitational effect of the charge $-Q$ onto a probe. The shell acceleration is

$$a = \frac{qE_0}{M}, \quad E_0 = \frac{Q}{L^2}.$$  \hfill (13)
Let us calculate the tension generated in the shell (Fig. 3). The charge and mass of the dashed section are

\[ q_1 = q \frac{1 + \cos \theta}{2}, \quad M_1 = M \frac{1 + \cos \theta}{2}, \]

respectively, where \( \theta \) is the angle of spherical frame system with a polar axis \( z \) directed along \( a \). This section is acted upon by the inertial force \( F_{in} = -M_1 a \) and electrostatic force \( F_{el} = q_1 E_0 \). Due to the axial symmetry, the tension tensor can be expressed as (see Appendix for detailed calculations)

\[ T_{\alpha\beta} = \left[ A(\theta) \hat{\theta}_\alpha \hat{\theta}_\beta + B(\theta) \hat{\varphi}_\alpha \hat{\varphi}_\beta \right] \delta(r - R), \]

where \( \hat{\theta}, \hat{\varphi} \) are unit vectors tangent to the sphere and directed along meridians and parallels respectively.

Then, from Eq. (A.3) and the formula \( df = -\hat{r}r \sin \theta d\varphi dr \) follows the condition for the force balance along \( z \)-axis:

\[ \frac{qE_0(1 + \cos \theta)}{2} - \frac{ma(1 + \cos \theta)}{2} + F = 0, \]

where \( F = -2\pi A(\theta) R \sin^2 \theta \hat{z} \). It follows from this equation and Eq. (13) that \( A(\theta) = 0 \).

![Diagram](image)

**Figure 4:** Illustration to the calculation of the component \( B(\theta) \) in the shell tension tensor. The forces \( F_1 \) and \( F_2 \) are tangent to the meridian \( \varphi = const \), and the forces \( F_3 \) and \( F_4 \) to the parallel \( \theta = const \). \( \rho \) is the distance between the shell element and \( z \)-axis.

To find \( B(\theta) \), let us consider the balance condition for \( r \)-components (see Fig. 4). Using a relation \( \Delta q E_0 = \Delta M a \), where \( \Delta q \) and \( \Delta M \) are the charge and mass of a shell element and calculations similar to those from Appendix, one can show that \( B(\theta) = 0 \). So, the external electrical field doesn’t cause tensions in the freely moving shell. In fact, this is obvious because the electric forces are exactly balanced by the inertia forces in the accelerated frame system.
The contribution to \( a_{body} \) due to the first summand in Eq. (3) is again determined by Eq. (10): presence of the field doesn’t change this result. Indeed, using a conservation law for the full energy-momentum tensor \( T_{ik} \), for \( k = 0 \) we have

\[
\dot{T}_{0\alpha} = \frac{\partial T_{0\alpha}}{\partial t} = -\frac{\partial T_{\alpha\beta}}{\partial x^\beta} = \frac{\partial T_{\beta\alpha}}{\partial x^\beta} ,
\]

\[
\int \frac{\dot{T}_{0\alpha}(r', t)}{|r - r'|} d^3 r' = \frac{\partial}{\partial x^\beta} \int \frac{T_{\beta\alpha}(r', t)}{|r - r'|} d^3 r' \]

(as we said above, one should neglect signal delay). Now, one should set \( t = 0 \) in Eq. (3), leave only the field contribution and neglect terms of the order of \( v^2/c^2 \), particularly magnetic field. The electrostatic field inside stopped shell is zero, so this field doesn’t contribute to \( a_{body} \).

Let us now simplify the second term in Eq. (3). Because one should neglect signal delay in \( T_{\alpha\alpha} \) and in \( T_{00} \) vise versa, we have to take it into account, let us perform expansion in the powers of \( t' = -|r - r'| \):

\[
\begin{align*}
    h_{00}(r, t) &= -2 \int \frac{T_{00}(r', t')}{|r - r'|} d^3 r' - 2 \int \frac{\sum T_{\alpha\alpha}(r', t')}{|r - r'|} d^3 r' \approx \\
    &\approx -2 \int \frac{T_{00}(r', 0)}{|r - r'|} d^3 r' + 2 \int \frac{\partial T_{00}(r', 0)}{\partial t} d^3 r' - \int \frac{\partial^2 T_{00}(r', 0)}{\partial t^2} |r - r'| d^3 r' - \\
    &- 2 \int \frac{\sum T_{\alpha\alpha}(r', 0)}{|r - r'|} d^3 r' .
\end{align*}
\]

Then we have

\[
\int \frac{\partial T_{00}}{\partial t} d^3 r = \int \frac{\partial T_{00}}{\partial x^\beta} d^3 r = - \int \frac{\partial T_{00}}{\partial x^\beta} d^3 r = 0 ,
\]

\[
\frac{\partial^2 T_{00}}{\partial t^2} = \frac{\partial^2 T_{00}}{\partial t^2} = - \frac{\partial}{\partial t} \frac{\partial T_{00}}{\partial x^\beta} = - \frac{\partial}{\partial x^\beta} \frac{\partial^2 T_{00}}{\partial t^2} = \frac{\partial^2 T_{\beta\alpha}}{\partial x^\alpha \partial x^\beta} .
\]

Thus, the second term in Eq. (3) is expressed as follows:

\[
(a_{body})_\alpha = (a_0)_\alpha + \frac{1}{2} \int \frac{T_{\beta\gamma}(r, 0)}{r^3} \left( 3 \delta_{\beta\gamma} x_\alpha + \delta_{\alpha\beta} x_\gamma + \delta_{\alpha\gamma} x_\beta - \frac{3 x_\alpha x_\beta x_\gamma}{r^2} \right) d^3 r + \\
+ \int \frac{\nabla_\alpha T_{00}(r, 0)}{r} d^3 r = \\
= (a_0)_\alpha + \frac{1}{2} \int \frac{d^3 r}{r} \nabla_\alpha T_{\gamma\alpha}(r, 0) \left( 3 \delta_{\beta\gamma} - \frac{x_\beta x_\gamma}{r^2} \right) + \int \frac{\nabla_\alpha T_{00}(r, 0)}{r} d^3 r \equiv \\
\equiv (a_0)_\alpha + (a_1)_\alpha + (a_2)_\alpha ,
\]

where through \((a_\alpha)_\alpha\) we denoted the summand with \( T_{00} \).

The tensor \( T_{\alpha\beta} \) is equal to the sum of a tension tensor of the shell and field. Tensions in a shell are due to acceleration and mutual repulsion of the shell parts. As it was shown above,
the former is zero. One can also neglect the latter because they are of the order of $q^2$ and we are only interested in linear terms (and this means linearity in $q$). Thus, in this approximation we should take into account only $T_{\alpha \beta}$ of the electromagnetic field:

$$T_{\alpha \beta} \approx \frac{1}{4\pi} \left( \frac{1}{2} E^2 \delta_{\alpha \beta} - E_{\alpha} E_{\beta} \right) ,$$

where $E = E_0 + E_1$.

$$\begin{cases} E_0 &= E_0 \hat{z} \\ E_1 &= \theta(r - R) \frac{2\pi}{\hat{r}} \end{cases}$$

is the field generated by the shell.

Since we retain only the terms linear in $q$,

$$E^2 \approx E_0^2 + 2E_0 E_1 , \quad E_{\alpha} E_{\beta} \approx E_{0\alpha} E_{0\beta} + E_{0\alpha} E_{1\beta} + E_{0\beta} E_{1\alpha} .$$

The terms quadratic in $E_0$ can be made infinitesimal by removing the charge $Q$ to infinite distance (Eq. (13)).

Thus, we have the following expression for $T_{\alpha \beta}$ linear in $q$:

$$T_{\alpha \beta} = \frac{qE_0}{4\pi r^2} \left( \delta_{\alpha \beta} - \delta_{\alpha \gamma} \hat{r}_\gamma \theta(r - R) \right) \hat{r}_\beta \theta(r - R) .$$

Now $a_1$ can be expressed in the form convenient for calculations:

$$(a_1)_\alpha = 2\pi \int_0^\infty dr \frac{qE_0}{4\pi r^2} K_{\alpha \gamma \beta} \left( 3\delta_{\beta \gamma} \hat{r}_\rho \hat{r}_\rho + \delta_{\alpha \beta} \hat{r}_\delta \hat{r}_\gamma + \delta_{\alpha \gamma} \hat{r}_\delta \hat{r}_\beta - 3\hat{r}_\alpha \hat{r}_\beta \hat{r}_\gamma \hat{r}_\delta \right) \theta(r - R) ,$$

where the angular brackets $\langle \cdots \rangle$ denote averaging over angles. After this averaging, we have

$$(a_1)_\alpha = \int_0^\infty dr \frac{2aE_0}{5\pi r} \theta(r - R) \left( K_{\beta \gamma \alpha} + \frac{1}{3} K_{\alpha \beta \gamma} \right) = \int_0^\infty dr \frac{2aE_0}{5\pi r} \theta(r - R) \left[ \hat{z}_\alpha + \frac{1}{3} (-3\hat{z}_\alpha) \right] = 0 .$$

Now let us calculate the contribution due to $T_{00}$, i.e. $a_2$:

$$a_2 = \nabla J , \text{ where } J = \int \frac{T_{00}(r', 0)}{|r - r'|} d^3r' = J_1^{shell} + J_2^{field} .$$

It is obvious that the summand due to the shell, i.e. $J_1^{shell}$, is spherically symmetrical (the Lorentz contraction, proportional to $a^2$, is superfluous in our approximation). Therefore, it is clear that $\nabla J_1 |_{r=0} = 0$. The only remaining contribution is that due to the electric field, $J_2^{field}$.

$$T_{00} \approx \frac{E^2}{8\pi} , \text{ where } E = E_0 + E_1 .$$

The term linear in $q$ and $E_0$ is equal to

$$\frac{qE_0}{4\pi r^2} \left( \hat{r} \cdot b \hat{z} \right) \theta(r - R) .$$
Expanding $1/|\mathbf{r} - \mathbf{r}'|$ in $\mathbf{r}$, we obtain
\[
a_2 = \nabla \int \frac{qE_0}{4\pi r'^2} \theta (\mathbf{r}' - R)(\hat{\mathbf{r}}' \cdot \hat{\mathbf{z}})(\frac{\mathbf{r}'}{r'^2}d^3\mathbf{r}' =
\]
\[
= \frac{qE_0}{4\pi} \int_R^{\infty} \frac{dr'}{r'^2} \langle (\hat{\mathbf{r}}' \cdot \hat{\mathbf{z}}) \hat{\mathbf{r}}' \rangle = \frac{qE_0}{3R} \hat{\mathbf{z}}.
\]
Taking into account Eq. (12), the final result is
\[
a_{\text{body}} = \frac{13M}{3R} a = \frac{13Rg}{6R} a,
\]
i.e. the total effect is the entrainment of the probe in the direction of the acceleration vector.

5 Conclusion

The main aim of this methodological paper was to elucidate the role of the energy-momentum tensor in the general relativity, taking as an example a shell moving at the constant acceleration. The paper would be useful for the development of the “relativistic intuition” of the beginners in gravity. In particular, it was demonstrated that the removal from the full EMT of the parts that seem to be unimportant from the point of view of Newton’s gravity law, leads to incorrect and, usually, paradoxical results.

6 Appendix. Tension Tensor in a Rigid Thin Shell Accelerated by a String

One can neglect tensions $T_{\alpha\beta}$, where $\alpha = r, \theta, \varphi$ are components of the spherical frame system with a polar axis directed along the string. Owing to the axial symmetry, only the following components of the tension tensor are nonzero:

\[
T_{\theta\theta}, T_{\varphi\varphi}.
\]
\[\text{(A.1)}\]
Taking the above into account, the most general form of $T_{\alpha\beta}$ is
\[
T_{\alpha\beta} = \left[ A(\theta)\hat{\theta}_{\alpha} \hat{\theta}_{\beta} + B(\theta)\hat{\varphi}_{\alpha} \hat{\varphi}_{\beta} \right] \delta (r - R),
\]
\[\text{(A.2)}\]
where $\hat{\theta}, \hat{\varphi}$ are unit vectors tangent to the arcs $\varphi = \text{const}$ and $\theta = \text{const}$ (see Fig. 3).

Let us consider the balance condition for forces in the accelerated frame system. The mass of shell section corresponding to the polar angles $\theta < \theta' < \pi$ (the dashed section in Fig. 3) is:
\[
M(\theta) = \frac{M}{2} (1 + \cos \theta).
\]
The force acting on this section (it is obvious that in this motion only the \( z \)-components of the force is nonzero) is \([3, 3]\)

\[
F_\alpha = -\int T_{\alpha\beta}(d\mathbf{f})_\beta ,
\]  

(A.3)

where \( d\mathbf{f} \) is an element of the surface restricting given body. In our case \( d\mathbf{f} \) is an element of the sphere on the circle \( \theta = \text{const} \) within the range of the azimuth angle \( (\varphi, \varphi + d\varphi) \):

\[
\hat{\theta}d\mathbf{f} = -r \sin \theta dr d\varphi .
\]

Then, from Eqs. (A.2) and (A.3) we obtain

\[
\mathbf{F} = A(\theta)R \sin \theta \int \hat{\theta}d\varphi = -2\pi R A(\theta)R \sin^2 \theta \hat{z} .
\]

This force is compensated by the force of inertia \( \mathbf{F} = M(\theta)a\hat{z} \). Hence

\[
A(\theta) = \frac{-Ma}{4\pi R} \frac{(1 + \cos \theta)}{\sin^2 \theta} .
\]  

(A.4)

The singularity at \( \theta = 0 \) is related to the fact that the infinitely thin string is attached to the sphere exactly at this point.

In order to calculate \( B(\theta) \), let us consider the equilibrium of an arbitrary shell element in the radial direction (Fig. 3). The forces \( \mathbf{F}_1 \) and \( \mathbf{F}_2 \) are perpendicular to the parallels \( \theta = \text{const} \), and \( \mathbf{F}_3 \) and \( \mathbf{F}_4 \) to the meridians \( \varphi = \text{const} \). Using Eqs. (A.1) and (A.2), we can express these forces as

\[
\left\{ \begin{array}{l}
\mathbf{F}_{1,2} = \pm \hat{\theta} A(\theta)R \sin \theta \Delta \varphi \\
\mathbf{F}_{3,4} = \pm \hat{\varphi} B(\theta) R \Delta \theta
\end{array} \right.
\]

The radial component of the net force is

\[
\hat{\mathbf{r}} \cdot \mathbf{F} = (A(\theta) + B(\theta)) R \sin \theta \Delta \varphi \Delta \theta .
\]

Adding this result with the radial component of the inertia force

\[
\mathbf{F}_{\text{rad}}^{\text{inert}} = -\frac{Ma}{4\pi} \sin \theta \cos \theta \Delta \varphi \Delta \theta
\]

and equating the result with zero, we obtain the equilibrium condition

\[
A(\theta) + B(\theta) = \frac{Ma}{4\pi R} \cos \theta .
\]

From this and from Eq. (A.4), we obtain

\[
B(\theta) = \frac{Ma}{4\pi R} \frac{(1 + \cos \theta + \cos \theta \sin^2 \theta)}{\sin^2 \theta} .
\]

From this result and Eqs. (A.2) and (A.4), we derive Eq. (8).
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