Casimir energy and variational methods in AdS spacetime

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Abstract

Following the subtraction procedure for manifolds with boundaries, we calculate by variational methods, the Schwarzschild-Anti-de Sitter and the Anti-de Sitter space energy difference. By computing the one loop approximation for TT tensors we discover the existence of an unstable mode at zero temperature, which can be stabilized by the boundary reduction method. Implications on a foam-like space are discussed.

I. INTRODUCTION

The problem of computing vacuum fluctuations in a field theory can be considered as the first step to probe the validity of a theory. An example is given by the zero point energy (ZPE) responsible of the Casimir effect. This one was predicted by Casimir [1] and experimentally confirmed in the Philips laboratories [2]. This is induced when the presence of electrical conductors distorts the zero point energy of the quantum electrodynamics vacuum. Two parallel conducting surfaces, in a vacuum environment, attract one another by a very weak force that varies inversely as the fourth power of the distance between them. This kind of energy is a pure quantum effect; no real particles are involved, only virtual ones. The difference between the stress-energy computed in presence and in absence of the plates with the same boundary conditions gives
\( \Delta \langle T^{\mu \nu} \rangle = \langle T^{\mu \nu} \rangle^{p \text{ vac}} - \langle T^{\mu \nu} \rangle^{\text{ vac}} = \frac{\pi^2}{720a^4} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} \). \hspace{1cm} (1)

It is evident that separately, each contribution coming from the summation over all possible resonance frequencies of the cavities is divergent and devoid of physical meaning but the difference between them in the two situations (with and without the plates) is well defined. Note that the energy density
\[ \rho = E/V = \Delta \langle T^{00} \rangle = -\frac{\pi^2}{720a^4} \] is negative \cite{3,4}. One can in general formally define the Casimir energy as follows
\[ E_{\text{Casimir}} [\partial \mathcal{M}] = E_0 [\partial \mathcal{M}] - E_0 [0], \] where \( E_0 \) is the zero-point energy and \( \partial \mathcal{M} \) is a boundary. In General Relativity, at the classical level, there exists a subtraction procedure related to the Arnowitt-Deser-Misner (ADM) approach \cite{3}, namely the ADM energy or mass, which can be improperly thought as the classical aspect of the Casimir energy. In a recent paper, the problem of computing the Casimir energy in presence of the Schwarzschild metric for the gravitational field has been considered \cite{6}. The classical energy associated to the related gravitational Casimir energy is represented by the ADM mass
\[ M = \lim_{r \to \infty} \int_{\partial \Sigma} \sqrt{\tilde{g}} \tilde{g}^{ij} \left[ \tilde{g}_{ik,j} - \tilde{g}_{ij,k} \right] dS^k, \] where \( \tilde{g}^{ij} \) is the metric induced on a spacelike hypersurface \( \partial \Sigma \) which has a boundary at infinity like \( S^2 \). An equivalent definition of the classical energy is given by the quasilocal energy defined by
\[ E_{q.l.} = \frac{1}{8\pi G} \int_{S^2} d^2 x \sqrt{\sigma} \left( k - k^0 \right), \] where \( k \) is the extrinsic curvature referred to the Schwarzschild space and \( k^0 \) is the extrinsic curvature referred to flat space. \( \sigma \) is the two-dimensional determinant coming from the...
induced metric $\sigma_{ab}$ on the boundaries $S^2$. It is relevant to observe that the Schwarzschild space is asymptotically flat, namely when $r \to \infty$ we recover the flat metric. In this case to correctly compute the classical energy term a subtraction procedure is involved as widely discussed in Refs. [11,12]. When we transpose this procedure to one loop calculations, we get the zero point energy (ZPE) for gravitons embedded in flat space

$$2 \cdot \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \sqrt{k^2}. \quad (6)$$

This term has a quartic ultra-violet (UV) divergence. The same kind of divergence is present when the Schwarzschild background is considered. However their difference has a divergence degree of a logarithmic type. Since boundary conditions are the same, this ZPE’s difference at one loop represents a Casimir-like computation. In this paper we would like to extend the same evaluation reported in Ref. [6,7] to the computation of the Casimir-like energy for a Schwarzschild-Anti-de Sitter (S-AdS) space at zero temperature discussing the possible existence of an unstable mode. The reason to compute such a correction comes from an analogy between the Schwarzschild metric and the S-AdS metric. Indeed both metrics are spherically symmetric and are characterized by only one root in the gravitational potential. Moreover, no natural outer boundary is present. The rest of the paper is structured as follows, in section II we define the S-AdS line element, in section III we compute the quasilocal energy and the quasilocal mass for the S-AdS space, in section IV we give some of the basic rules to perform the functional integration and we define the Hamiltonian approximated up to second order, in section V we look for stable modes of the spin-two operator acting on transverse traceless tensors, in section VI we show the existence of only one negative mode, in section VII we find a critical radius below which we have a stabilization of the system. We summarize and conclude in section VIII.

II. THE SCHWARZSCHILD-ANTI-DE SITTER METRIC

The S-AdS line element is defined as
\[ ds^2 = -f(r) \, dt^2 + f(r)^{-1} \, dr^2 + r^2 d\Omega^2, \]  

(7)

where

\[ f(r) = \left(1 - \frac{2MG}{r} + \frac{r^2}{b^2}\right), \]  

(8)

\[ b^2 = \sqrt{-\frac{3}{\Lambda_c}} \] and \( \Lambda_c < 0 \) is the negative cosmological constant. For \( \Lambda_c = 0 \) the metric describes the Schwarzschild metric, while for \( M = 0 \), we obtain

\[ ds^2 = -\left(1 + \frac{r^2}{b^2}\right) \, dt^2 + \left(1 + \frac{r^2}{b^2}\right)^{-1} \, dr^2 + r^2 d\Omega^2, \]  

(9)

i.e. the Anti-de Sitter metric (AdS). The gravitational potential \( f(r) \) of (7) has only one root located at

\[ \bar{r} = \sqrt[3]{\frac{3MG}{\Lambda_c} + \sqrt{\frac{1}{\Lambda^2_c} \left[9(MG)^2 + \frac{1}{\Lambda^2_c}\right]}} \]  

and the gravitational potential can be written as

\[ f(r) = \frac{(r - \bar{r}) (r^2 + \bar{r} r + \bar{r}^2 + b^2)}{r b^2}. \]  

(10)

(11)

From Eq.(8), evaluated at \( \bar{r} \), the parameter \( M \) can be written as

\[ MG = \frac{\bar{r} (\bar{r}^2 + b^2)}{2b^2}. \]  

(12)

In complete analogy with the Schwarzschild case, we will consider a constant time slice \( \Sigma \) of the S-AdS manifold \( M_4 \). Even if there is a cosmological constant term we generalize the terminology by saying that the hypersurface \( \Sigma \) is an Einstein-Rosen bridge with wormhole topology \( S^2 \times R^1 \). The Einstein-Rosen bridge defines a bifurcation surface dividing \( \Sigma \) in two parts denoted by \( \Sigma^+ \) and \( \Sigma^- \). Our purpose is to consider perturbations at \( \Sigma \) with \( t \) constant in absence of matter fields, which naturally define quantum fluctuations of the Einstein-Rosen bridge. The explicit expression of the Hamiltonian can be calculated by means of the following line element

\[ \text{In Appendix A, we will report the details concerning the Kruskal-Szekeres description of the S-AdS manifold.} \]
\[ ds^2 = -N^2 \left( dx^0 \right)^2 + g_{ij} \left( N^i dx^0 + dx^i \right) \left( N^j dx^0 + dx^j \right), \]  

(13)

where \( N \) is called the lapse function and \( N_i \) is the shift function. When \( N = \sqrt{1 - \frac{2MG}{r} + \frac{r^2}{b^2}} \), \( N_i = 0 \) and

\[ g_{ij}dx^i dx^j = \left( 1 - \frac{2MG}{r} + \frac{r^2}{b^2} \right)^{-1} dr^2 + r^2 d\Omega^2, \]

(14)

we recover the S-AdS line element. On the slice \( \Sigma \), deviations from the S-AdS metric spatial section of the form

\[ g_{ij} = \bar{g}_{ij} + h_{ij} \]

(15)

will be considered with \( N_i = 0 \) and \( N \equiv N(r) \). Then the line element (13) becomes

\[ ds^2 = -N^2(r) \left( dx^0 \right)^2 + g_{ij} dx^i dx^j \]

(16)

and the total Hamiltonian is

\[ H_T = H_\Sigma + H_{\partial \Sigma} = \int_\Sigma d^3 x (N \mathcal{H} + N_i \mathcal{H}^i) + H_{\partial \Sigma}, \]

(17)

where

\[
\begin{align*}
\mathcal{H} &= G_{ijkl} \pi^{ij} \pi^{kl} \left( \frac{16 \pi G}{\sqrt{g}} \right) - \left( \frac{\sqrt{g}}{16 \pi G} \right) \left( R^{(3)} + \frac{6}{b^2} \right) \quad \text{(Super Hamiltonian)} \\
\mathcal{H}^i &= -2 \pi^{ij} \quad \text{(Super Momentum)}
\end{align*}
\]

(18)

and \( H_{\partial \Sigma} \) represents the energy stored into the boundary. According to Witten [8], to discuss the existence of an unstable sector, we have to compare spaces with the same boundary conditions. An instability appears when we consider the Euclidean S-AdS spacetime with a periodically identified time representing the equilibrium temperature of a S-AdS black hole with the environment [9]. The same boundary conditions on the AdS spacetime can be imposed if the temperature on the boundary is the same. Indeed the AdS spacetime has no natural temperature and this seems to suggest that only the “hot” AdS spacetime will be unstable. However, by applying the same method of Ref. [4], it is possible to discuss if the instability appears even when we have the \( T = 0 \) temperature case. To this purpose the expression we need to evaluate is
$E^{\text{S-AdS}}(M, b) = E^{\text{AdS}}(b) + \Delta E^{\text{S-AdS}}(M, b)_{\text{classical}} + \Delta E^{\text{S-AdS}}(M, b)_{\text{1-loop}}.$ (19)

$E^{\text{AdS}}(b)$ represents the reference space energy which is zero for the flat space. $\Delta E^{\text{S-AdS}}(M, b)_{\text{classical}}$ represents the energy difference stored in the boundaries due to the presence of the hole and $\Delta E^{\text{S-AdS}}(M, b)_{\text{1-loop}}$ is the quantum correction to a classical term.

III. QUASILOCAL ENERGY AND QUASILOCAL MASS FOR THE S-ADS SPACE

In this section we fix our attention on the classical part of Eq.(19). We consider the outer boundary located at some radius $R$. Thus the total energy at the classical level is

$E^{\text{S-AdS}}(M, b) = E^{\text{AdS}}(b) + \Delta E^{\text{S-AdS}}(M, b)_{\text{classical}}.$ (20)

We begin by looking at the “outside region” of the Kruskal manifold associated to the S-AdS spacetime. We will use the quasilocal energy to evaluate $\Delta E^{\text{S-AdS}}(M, b)_{\text{classical}}$. Quasilocal energy is defined as the value of the Hamiltonian that generates unit time translations orthogonal to the two-dimensional boundary,

$\Delta E^{\text{S-AdS}}(M, b)_{\text{classical}} = \frac{1}{8\pi G} \int_{S^2} d^2 x \sqrt{\sigma} \left( k - k^0 \right),$ (21)

where $|N| = 1$ at $S^2$ and $k$ is the trace of the extrinsic curvature corresponding to the S-AdS space and $k^0$ is the trace of the extrinsic curvature referred to the AdS space. Following Refs. [10–12] and by means of Eq.(8), we obtain

$\Delta E^{\text{S-AdS}}(M, b)_{\text{classical}} = -\frac{1}{8\pi G} \int_{S^2} d^2 r^2 \left[ \frac{-2\sqrt{f(r)}}{r} + \frac{2\sqrt{f(r)|_{M=0}}}{r} \right] |_{r=R}$

$\approx \frac{R}{G} \left[ \sqrt{1 - \frac{2MG}{r} + \frac{r^2}{b^2}} - \sqrt{1 + \frac{r^2}{b^2}} \right] \approx \frac{Mb}{R}.$ (22)

See Appendix A for details.
When the boundary is pushed to infinity $\Delta E_{\text{AdS}}^{S-\text{AdS}}(M, b)|_{\text{classical}} \to 0$. The same happens to the temperature defined by

$$T = \left( \frac{\partial E}{\partial S} \right)_{r=R} = \frac{1}{2\pi} \frac{\kappa}{N(R)},$$

where $N(R) = \sqrt{f(R)}$ is the redshift factor and $\kappa$ is the surface gravity defined by

$$\lim_{r \to \bar{r}} \frac{1}{2} \left| g_{00}(r) \right| = \frac{3r^2 + b^2}{2rb^2}. \quad (24)$$

Thus except the limiting case of pushing the boundary to infinity, the temperature $T$ and the classical energy $\Delta E_{\text{AdS}}^{S-\text{AdS}}(M, b)|_{\text{classical}}$ do not vanish and therefore we cannot consider the problem of searching for unstable modes at zero temperature. Nevertheless if we look at the whole S-AdS manifold, the total classical energy can be written as

$$E_{\text{S-AdS}}^S(M, b) = E_{\text{AdS}}^S(b) + E_{\text{tot}}(M, b)$$

$$= E_{\text{AdS}}^S(b) + \Delta E_{\text{AdS}}^{S-\text{AdS}}(M, b)|_{\text{classical}}^+ + \Delta E_{\text{AdS}}^{S-\text{AdS}}(M, b)|_{\text{classical}}^- \quad (25)$$

with

$$\Delta E_{\text{AdS}}^{S-\text{AdS}}(M, b)|_{\text{classical}}^+ = \frac{1}{8\pi G} \int_{S^2_+} d^2 x \sqrt{\sigma} (k - k^0),$$

$$\Delta E_{\text{AdS}}^{S-\text{AdS}}(M, b)|_{\text{classical}}^- = -\frac{1}{8\pi G} \int_{S^2_-} d^2 x \sqrt{\sigma} (k - k^0), \quad (26)$$

and $|N| = 1$ at both $S^2_+$ and $S^2_-$. $E_{\text{tot}}(M, b)$ is the quasilocal energy of a spacelike hypersurface $\Sigma = \Sigma_+ \cup \Sigma_-$ bounded by two boundaries $S^2_+$ and $S^2_-$ located in the two disconnected regions $\mathcal{M}_+$ and $\mathcal{M}_-$ respectively. To evaluate $\Delta E_{\text{AdS}}^{S-\text{AdS}}(M, b)|_{\text{classical}}^\pm$ we can use Eq.(22) or more pictorially by looking at the static Einstein-Rosen bridge associated to the S-AdS space, whose metric is

$$ds^2 = -N^2(r) dt^2 + g_{xx} dx^2 + r^2(x) d\Omega^2, \quad (27)$$

where $N$, $g_{xx}$, and $r$ are functions of the radial coordinate $x$ continuously defined on $\mathcal{M}$, with
\[ dx = \pm \frac{dr}{\sqrt{1 - \frac{2MG}{r} + \frac{r^2}{b^2}}}, \]  

(28)

where the plus sign is relative to \( \Sigma_+ \), while the minus sign is related to \( \Sigma_- \). If we make the identification \( N^2 = 1 - \frac{2MG}{r} + \frac{r^2}{b^2} \), the line element reduces to the S-AdS metric written in another form. The boundaries \( S^2_{\pm} \) are located at coordinate values \( x = \bar{x}^\pm \). The normal to the boundaries is \( n^\mu = (h^{xx})^{\frac{1}{2}} \delta^\mu_y \). By using the expression of the trace

\[ k = -\frac{1}{\sqrt{h}} \left( \sqrt{hn^\mu} \right)_\mu, \]

(29)

we obtain

\[ k_{S-AdS} = \begin{cases} 
-2r_{,x}/r & \text{on } \Sigma_+ \\
2r_{,x}/r & \text{on } \Sigma_- 
\end{cases}. \]

(30)

Thus the computation of \( E_+ \) gives exactly the result of Eq.(22). On the other hand the computation of \( E_- \) gives the same value but with the reversed sign. Thus one gets

\[ (E_{S-AdS} - E_{AdS})_\pm = \begin{cases} 
Mb/R & \text{on } S^2_+ \\
-Mb/R & \text{on } S^2_- 
\end{cases}, \]

(31)

where for \( E_- \) we have used the conventions relative to \( \Sigma_- \) and \( S^2_- \). Therefore for every value of the boundary \( R \), (provided we take symmetric boundary conditions with respect to the bifurcation surface, even for the limiting value \( R \to \infty \)), we have

\[ E_{S-AdS} (M,b) = E_{AdS} (b) + M b/R - M b/R = E_{AdS} (b), \]

(32)

namely the energy is conserved for every choice of the boundary location.

3Note that if we take as a reference space the flat space, then the trace is taken to be \( k_{\text{flat}} = -2/r \) and

\[ (E_{S-AdS} - E_{\text{flat}})_\pm = \begin{cases} 
-R^2/Gb & \text{on } S^2_+ \\
R^2/Gb & \text{on } S^2_- 
\end{cases}. \]

When \( R \to \infty \), \( (E_{S-AdS} - E_{\text{flat}})_\pm \to -\infty \).
IV. ENERGY DENSITY CALCULATION IN SCHRÖDINGER REPRESENTATION

In previous section we have fixed our attention to the classical part of Eq. (19). In this section, we apply the same calculation scheme of Refs. [6,7] to compute one loop corrections to the classical S-AdS term. Like the Schwarzschild case, there appear two classical constraints for the Hamiltonian

\[ \begin{cases} \mathcal{H} = 0 \\ \mathcal{H}^i = 0 \end{cases} \] (33)

which are satisfied both by the S-AdS and AdS metric and two quantum constraints

\[ \begin{cases} \mathcal{H} \tilde{\Psi} = 0 \\ \mathcal{H}^i \tilde{\Psi} = 0 \end{cases} \] (34)

\( \mathcal{H} \tilde{\Psi} = 0 \) is known as the Wheeler-DeWitt equation (WDW). Nevertheless, our purpose is the computation of

\[ \Delta E_{S-AdS}^{S-AdS} (M, b)_{1-loop} = \frac{\langle \Psi | H_{S-AdS} - H_{AdS} | \Psi \rangle}{\langle \Psi | \Psi \rangle} \] (35)

where \( H_{S-AdS} \) and \( H_{AdS} \) are the total Hamiltonians referred to the S-AdS and AdS spacetimes respectively for the volume term \[3\] and \( \Psi \) is a wave functional obtained following the usual WKB expansion of the WDW solution. In this context, the approximated wave functional will be substituted by a trial wave functional of the gaussian form according to the variational approach we shall use to evaluate \( \Delta E_{S-AdS}^{S-AdS} (M, b)_{1-loop} \). Following the same procedure of Refs. [6,7], we expand the three-scalar curvature \( \int d^3x \sqrt{g} \left( R^{(3)} + 6/b^2 \right) \) up to \( o(h^2) \) and we get

\[ \int_{\Sigma} d^3x \sqrt{g} \left[ -\frac{1}{4} h \triangle h + \frac{1}{4} h_{ii} \triangle h_{i} - \frac{1}{2} h^{ij} \nabla_i \nabla_j h_{i} + \frac{1}{2} h \nabla_i \nabla_i h_{i} - \frac{1}{2} h^{ij} R_{ia} h^{a}_{j} + \frac{1}{2} h R_{ij} h^{ij} \right] 
- \int_{\Sigma} d^3x \sqrt{g} \left[ \frac{1}{4} h_{i} \left( 6/b^2 \right) h_{ii} \right]. \] (36)
To explicitly make calculations, we need an orthogonal decomposition for both \( \pi_{ij} \) and \( h_{ij} \) to disentangle gauge modes from physical deformations. We define the inner product

\[
\langle h, k \rangle := \int_{\Sigma} \sqrt{g} G^{ijkl} h_{ij}(x) k_{kl}(x) \, d^3x,
\]

by means of the inverse WDW metric \( G^{ijkl} \), to have a metric on the space of deformations, i.e. a quadratic form on the tangent space at \( h \), with

\[
G^{ijkl} = (g^{ik}g^{jl} + g^{il}g^{jk} - 2g^{ij}g^{kl}).
\]

The inverse metric is defined on co-tangent space and it assumes the form

\[
\langle p, q \rangle := \int_{\Sigma} \sqrt{g} G_{ijkl} p^{ij}(x) q^{kl}(x) \, d^3x,
\]

so that

\[
G^{ijnm} G_{nmkl} = \frac{1}{2} \left( \delta_i^k \delta_j^l + \delta_i^l \delta_j^k \right).
\]

Note that in this scheme the “inverse metric” is actually the WDW metric defined on phase space. The desired decomposition on the tangent space of 3-metric deformations [13,14] is:

\[
h_{ij} = \frac{1}{3} h g_{ij} + (L\xi)_{ij} + h_{ij}^\perp
\]

where the operator \( L \) maps \( \xi_i \) into symmetric tracefree tensors

\[
(L\xi)_{ij} = \nabla_i \xi_j - \nabla_j \xi_i - \frac{2}{3} g_{ij} (\nabla \cdot \xi).
\]

Thus the inner product between three-geometries becomes

\[
\langle h, h \rangle := \int_{\Sigma} \sqrt{g} G^{ijkl} h_{ij}(x) h_{kl}(x) \, d^3x = \int_{\Sigma} \sqrt{g} \left[ -\frac{2}{3} h^2 + (L\xi)^{ij} (L\xi)_{ij} + h^{ij} h_{ij}^\perp \right].
\]

With the orthogonal decomposition in hand we can define the trial wave functional

\[
\Psi [h_{ij}(x)] = \mathcal{N} \exp \left\{ -\frac{1}{4F_p^2} \left[ \langle h K^{-1} h \rangle_{x,y}^\perp + \langle (L\xi) K^{-1} (L\xi) \rangle_{x,y}^\parallel + \langle h K^{-1} h \rangle_{x,y}^{Trace} \right] \right\},
\]
where $N$ is a normalization factor. Since we are interested only to the perturbations of the physical degrees of freedom, we will fix our attention only to the TT tensor sector reducing therefore the previous form into

$$
\Psi [h_{ij} (\vec{x})] = N \exp \left\{ - \frac{1}{4l_p^2} \left\langle hK^{-1}h \right\rangle_{x,y} \right\}.
$$

(45)

This restriction is motivated by the fact that if an instability appears this will be in the physical sector referred to TT tensors, namely a non conformal instability. This means that does not exist a gauge choice that can eliminate negative modes. This choice seems also corroborated by the action decomposition of Ref. [15], where only TT tensors contribute to the partition function[4]. Therefore to calculate the energy density, we need to know the action of some basic operators on $\Psi [h_{ij}]$. The action of the operator $h_{ij}$ on $|\Psi\rangle = \Psi [h_{ij}]$ is realized by

$$
h_{ij} (x) |\Psi\rangle = h_{ij} (\vec{x'}) \Psi [h_{ij}] .
$$

(46)

The action of the operator $\pi_{ij}$ on $|\Psi\rangle$, in general, is

$$
\pi_{ij} (x) |\Psi\rangle = -i \frac{\delta}{\delta h_{ij} (\vec{x'})} \Psi [h_{ij}] .
$$

(47)

The inner product is defined by the functional integration:

$$
\langle \Psi_1 | \Psi_2 \rangle = \int [Dh_{ij}] \Psi_1^{*} \{h_{ij}\} \Psi_2 \{h_{kl}\} ,
$$

(48)

and by applying previous functional integration rules, we obtain the expression of the one-loop-like Hamiltonian form for TT (traceless and transverseless) deformations

$$
H^\perp = \frac{1}{4l_p^2} \int_{\mathcal{M}} d^3x \sqrt{g}G^{ijkl} \left[ K^{-1 \perp} (x, x)_{ijkl} + (\Delta_2)^a_j \left( K^\perp (x, x)_{iakl} \right) \right] .
$$

(49)

The propagator $K^\perp (x, x)_{iakl}$ comes from a functional integration and it can be represented as

$$
K^\perp (\vec{x'}, \vec{y'})_{iakl} := \sum_{N} \frac{h^{\perp}_{ia} (\vec{x'}) h^{\perp}_{kl} (\vec{y'})}{2\lambda_N (p)} ,
$$

(50)

where $h^{\perp}_{ia} (\vec{x'})$ are the eigenfunctions of $\Delta'^a_2$ and $\lambda_N (p)$ are infinite variational parameters.

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[4] See also Ref. [16] for another point of view.
V. THE SCHWARZSCHILD-ANTI-DE SITTER METRIC SPIN 2 OPERATOR
AND THE EVALUATION OF THE ENERGY DENSITY

The Spin-two operator for the S-AdS metric is defined by

\[(\triangle_2)^a_j := -\triangle \delta^a_j + 2R^a_j + 6/b^2 \delta^a_j\]  \hfill (51)

where \(\triangle\) is the curved Laplacian (Laplace-Beltrami operator) on a S-AdS background and \(R^a_j\) is the mixed Ricci tensor whose components are:

\[R^a_i = \left\{ -\frac{2MG}{r^3} - 2/b^2, \frac{MG}{r^3} - 2/b^2, \frac{MG}{r^3} - 2/b^2 \right\}. \hfill (52)\]

Note that the form of the mixed Ricci tensor for the S-AdS space is the same of the mixed Ricci tensor computed in the Schwarzschild space, except for the presence of the negative cosmological term. We are led to study the following eigenvalue equation

\[\left( -\triangle \delta^a_j + 2R^a_j + 6/b^2 \delta^a_j \right) h^a_i = E^2 h^i_j\]  \hfill (53)

where \(E^2\) is the eigenvalue of the corresponding equation. In doing so, we follow Regge and Wheeler in analyzing the equation as modes of definite frequency, angular momentum and parity [17]. The quantum number corresponding to the projection of the angular momentum on the z-axis will be set to zero. This choice will not alter the contribution to the total energy since we are dealing with a spherical symmetric problem. In this case, Regge-Wheeler decomposition shows that the even-parity three-dimensional perturbation is

\[h^\text{even}_{ij}(r, \vartheta, \phi) = \text{diag} \left[ H(r) \left( 1 - \frac{2MG}{r} + \frac{r^2}{b^2} \right)^{-1}, r^2 K(r), r^2 \sin^2 \vartheta K(r) \right] Y_{l0}(\vartheta, \phi). \hfill (54)\]

Representation (54) shows a gravitational perturbation decoupling. For a generic value of the angular momentum \(L\), one gets

\[\begin{align*}
\left\{ \begin{array}{l}
\left( -\triangle_l + \frac{4MG}{r^3} + \frac{2}{b^2} \right) H(r) = E^2_l H(r) \\
\left( -\triangle_l + \frac{2MG}{r^3} + \frac{2}{b^2} \right) K(r) = E^2_l K(r).
\end{array} \right. \hfill (55)\end{align*}\]
The Laplacian restricted to $\Sigma$ can be written as

$$\Delta_l = \left(1 - \frac{2MG}{r} + \frac{r^2}{b^2}\right) \frac{d^2}{dr^2} + \left(\frac{2r - 3MG}{r^2} + \frac{3r}{b^2}\right) \frac{d}{dr} - \frac{l(l+1)}{r^2}. \quad (56)$$

Defining reduced fields

$$H (r) = \frac{h (r)}{r}; \quad K (r) = \frac{k (r)}{r}, \quad (57)$$

and passing to the proper geodesic distance from the throat of the bridge defined by Eq.(28), the system (55) becomes

$$\begin{cases} 
- \frac{d^2}{dx^2} h (x) + \left(V^- (x) + \frac{3}{b^2}\right) h (x) = E_l^2 h (x) \\
- \frac{d^2}{dx^2} k (x) + \left(V^+ (x) + \frac{3}{b^2}\right) k (x) = E_l^2 k (x)
\end{cases} \quad (58)$$

with

$$V^\pm (x) = \frac{l(l+1)}{r^2 (x)} \pm \frac{3MG}{r(x)^3}. \quad (59)$$

When $r \to \infty$, $x (r) \simeq b \ln r$ and $V (x) \to 0$. When $r \to r_0$, $x (r) \simeq 0$ and

$$V^\pm (x) \to \frac{l(l+1)}{r_0^2} \pm \frac{3MG}{r_0^3} = const, \quad (60)$$

where $r_0$ satisfies the condition $r_0 > \bar{r}$. The solution of (58), in both cases (S-AdS and AdS one) is

$$h (px) = k (px) = \sqrt{\frac{2}{\pi}} \sin (px). \quad (61)$$

This choice is dictated by the requirement that

$$h (x), k (x) \to 0 \quad \text{when} \quad x \to 0 \text{ (alternatively} \ r \to \bar{r}). \quad (62)$$

Thus the propagator becomes

5The system does not change in form if we make the minus choice in Eq.(28).
\[
K_{\pm}^\perp (x, y) = \frac{V}{2\pi^2} \int_0^\infty dpp^2 \frac{\sin (px) \sin (py)}{r(x) r(y)} \frac{Y_{l0}(\vartheta, \phi) Y_{r0}(\vartheta, \phi)}{\lambda_{\pm}(p)}
\] (63)

\(\lambda_{\pm}(p)\) is referred to the potential function \(V_{\pm}(x)\). Substituting Eq. (63) in Eq. (19) one gets (after normalization in spin space and after a rescaling of the fields in such a way as to absorb \(l^2\))

\[
E(M, b, \lambda) = \frac{V}{8\pi^2} \sum_{l=0}^\infty \sum_{i=1}^2 \int_0^\infty dpp^2 \left[ \lambda_i(p) + \frac{E_i^2(p, M, b, l)}{\lambda_i(p)} \right]
\] (64)

where

\[
E_{1,2}^2(p, M, b, l) = p^2 + \frac{l(l + 1)}{r_0^2} \pm \frac{3MG}{r_0^3} + \frac{3}{b^2}.
\] (65)

\(\lambda_i(p)\) are variational parameters corresponding to the eigenvalues for a (graviton) spin-two particle in an external field and \(V\) is the volume of the system. By minimizing (64) with respect to \(\lambda_i(p)\) one obtains \(\lambda_i(p) = \left[ E_i^2(p, M, b, l) \right]^\frac{1}{2}\) and

\[
E(M, b, \lambda) = \frac{V}{8\pi^2} \sum_{l=0}^\infty \sum_{i=1}^2 \int_0^\infty dpp^2 2\sqrt{E_i^2(p, M, b, l)}
\] (66)

with

\[
p^2 + \frac{l(l + 1)}{r_0^2} + \frac{3}{b^2} > \frac{3MG}{r_0^3}.
\]

For the S-AdS background we get

\[
E(M, b) = \frac{V}{4\pi^2} \sum_{l=0}^\infty \int_0^\infty dpp^2 \left( \sqrt{p^2 + c_+^2} + \sqrt{p^2 + c_-^2} \right)
\] (67)

where

\[
c_+^2 = \frac{l(l + 1)}{r_0^2} + \frac{3MG}{r_0^3} + \frac{3}{b^2}
\]

while when we refer to the AdS space we put \(M = 0\) and \(c^2 = \frac{l(l + 1)}{r_0^2} + \frac{3}{b^2}\). Then

\[
E(b) = \frac{V}{4\pi^2} \sum_{l=0}^\infty \int_0^\infty dpp^2 \left( 2\sqrt{p^2 + c^2} \right)
\] (68)

Now, we are in position to compute the difference between (67) and (68). Since we are interested in the UV limit, we have
\[ \Delta E(M, b) = E(M, b) - E(b^2) \]

\[ = \frac{V}{4\pi^2} \sum_{l=0}^{\infty} \int_0^\infty dp \, p^3 \left[ \sqrt{1 + \left( \frac{c_-}{p} \right)^2} - \sqrt{1 + \left( \frac{c_+}{p} \right)^2} \right] \]

\[ = \frac{V}{4\pi^2} \sum_{l=0}^{\infty} \int_0^\infty dp \, p^3 \left[ \sqrt{1 + \left( \frac{c_-}{p} \right)^2} - \sqrt{1 + \left( \frac{c_+}{p} \right)^2} - 2 \sqrt{1 + \left( \frac{c}{p} \right)^2} \right] \] \quad (69)

and for \( p^2 >> c_\pm^2 \), we obtain

\[ \frac{V}{4\pi^2} \sum_{l=0}^{\infty} \int_0^\infty dp \, p^3 \left[ 1 + \frac{1}{2} \left( \frac{c_-}{p} \right)^2 - \frac{1}{8} \left( \frac{c_-}{p} \right)^4 + 1 + \frac{1}{2} \left( \frac{c_+}{p} \right)^2 - \frac{1}{8} \left( \frac{c_+}{p} \right)^4 \right] \]

\[ - 2 - \left( \frac{c}{p} \right)^2 + \frac{1}{4} \left( \frac{c}{p} \right)^4 \] \quad (70)

where \( c_M^2 = 3MG/r_0^3 \). We will use a cut-off \( \Lambda \) to keep under control the UV divergence

\[ \int_0^\infty dp \, \frac{p}{p} \sim \int_0^{\frac{\Lambda}{c_M}} dx \frac{x}{x} \sim \ln \left( \frac{\Lambda}{c_M} \right), \] \quad (71)

where \( \Lambda \leq m_p \). Thus \( \Delta E(M, b) \) for high momenta becomes

\[ \Delta E(M, b) \sim -\frac{V}{2\pi^2} \frac{c_M^4}{16} \ln \left( \frac{\Lambda^2}{c_M^2} \right) = -\frac{V}{32\pi^2} \left( \frac{3MG}{r_0^3} \right)^2 \ln \left( \frac{r_0^3 \Lambda^2}{3MG} \right). \] \quad (72)

and Eq.(19) to one loop is

\[ E^{\text{S-AdS}}(M, b) - E^{\text{AdS}}(b) = -\frac{V}{32\pi^2} \left( \frac{3MG}{r_0^3} \right)^2 \ln \left( \frac{r_0^3 \Lambda^2}{3MG} \right). \] \quad (73)

Like the Schwarzschild case, we observe that

\[ \lim_{M \to 0} \lim_{r \to r^*} \Delta E(M, b) \neq \lim_{r \to r^*} \lim_{M \to 0} \Delta E(M, b). \] \quad (74)

This behavior seems to confirm that quantum effects come into play when we try to reach the horizon. By means of Eq.(12), \( \Delta E(M, b) \) becomes

\[ \Delta E(\bar{r}, b) = -\frac{V}{32\pi^2} \left( \frac{3\bar{r}^2 (\bar{r}^2 + b^2)}{2b^2 r_0^3} \right)^2 \ln \left( \frac{2b^2 r_0^3 \Lambda^2}{3\bar{r} (\bar{r}^2 + b^2)} \right). \] \quad (75)
If we set \( \bar{r} = b/\sqrt{3} = r_m \), which is the location of the minimum of the surface gravity, we find that \( \Delta E (\bar{r}, b) \) is reduced to

\[
\Delta E (b) = -\frac{V}{32\pi^2} \left( \frac{2}{\sqrt{3}r_0^3} \right)^2 \ln \left( \frac{\sqrt{3}r_0^3\Lambda^2}{2b} \right).
\]

Note that in the terminology of the black hole thermodynamics \( r_m \) corresponds to the unique black hole solution whose temperature reaches its minimum. To better appreciate the result obtained in Eq.(73), we define a scale variable \( x = 3MG/(r_0^3\Lambda^2) \) in such a way that \( \Delta E (M) \) can be cast in the form

\[
\Delta E (x) = -\frac{V}{32\pi^2}\Lambda^4 x^2 \ln x.
\]

A stationary point is reached for \( x = 0 \), namely the AdS space and another stationary point is in \( x = e^{-\frac{1}{2}} \). This last one represents a minimum of \( \Delta E (x) \). This means that there is a probability that the spacetime without the hole will decay into a spacetime with a hole (not black hole). To see if this is really possible, we have to establish if there exist unstable modes. However in case of Eq.(76) the claim that the stationary point \( x = 0 \) represents the AdS space is more delicate. Indeed this corresponds to the vanishing of the parameter \( b \), leading to a diverging negative cosmological constant, saturating the whole space.

VI. SEARCHING FOR NEGATIVE MODES

In this paragraph we look for negative modes of the eigenvalue equation (51). For this purpose we restrict the analysis to the S wave. Indeed, in this state the centrifugal term is absent and this gives the function \( V (x) \) a potential well form, which is different when the angular momentum \( l \geq 1 \). Moreover the potential well appears only for the \( H \) component, whose eigenvalue equation is

\[
\left( -\Delta - \frac{4MG}{r^3} + \frac{2}{b^2} \right) H (r) = -E^2 H (r).
\]

\( \Delta \) is the operator \( \Delta_l \) of Eq.(50) with \( l = 0 \) and \( E^2 > 0 \). By defining the reduced field \( h (r) = H (r) r \), Eq.(78) becomes
\[- \frac{d}{dr} \left( \sqrt{1 - \frac{2MG}{r} + \frac{r^2}{b^2}} \right) + \left( -\frac{3MG}{r^3} + \tilde{E}^2 \right) \frac{h}{\sqrt{1 - \frac{2MG}{r} + \frac{r^2}{b^2}}} = 0, \]  

(79)

where \( \tilde{E}^2 = 3/b^2 + E^2 \). By means of Eq.(28), one gets

\[- \frac{d}{dr} \frac{d}{dx} \left( \sqrt{1 - \frac{2MG}{r} + \frac{r^2}{b^2}} \frac{d}{dx} \right) + \left( -\frac{3MG}{r^3} + \tilde{E}^2 \right) \frac{h}{\sqrt{1 - \frac{2MG}{r} + \frac{r^2}{b^2}}} \]

\[= - \frac{d}{dx} \left( \frac{dh}{dx} \right) + \left( -\frac{3MG}{r^3} + \tilde{E}^2 \right) h = 0. \]  

(80)

Near the throat

\[x(r) \simeq \sqrt{\frac{2r}{\kappa}} \sqrt{\frac{r}{\bar{r}}} - 1, \]  

(81)

where \( \kappa \) is the surface gravity. By defining the dimensionless variable \( \rho = \frac{r}{\bar{r}} \), we obtain \( \rho = 1 + y^2 \) where

\[y = \sqrt{\kappa}x/\sqrt{2\bar{r}} = \tilde{k}x. \]  

(82)

Then Eq.(81) becomes

\[- \frac{d}{dy} \left( \frac{dh}{dy} \right) \tilde{k}^2 + \left( -\frac{3MG}{\bar{r}^2} \rho^2(y) + \tilde{E}^2 \right) h \]

\[= - \frac{d^2h}{dy^2} + \left( -\frac{3MG}{\tilde{k}^2(\bar{r})^3(1 + y^2)} + \lambda \right) h = 0, \]  

(83)

where \( \lambda = \tilde{E}^2/\tilde{k}^2 \). Expanding the potential around \( y = 0 \), one gets

\[- \frac{d^2h}{dy^2} + \left( -\frac{3MG}{\tilde{k}^2(\bar{r})^3(1 - 3y^2)} + \lambda \right) h = 0. \]  

(84)

\[- \frac{d^2h}{dy^2} + \left( \omega^2 y^2 - \frac{3MG}{\tilde{k}^2(\bar{r})^3} + \lambda \right) h = 0, \]  

(85)

where \( \omega = \sqrt{9MG/\left( \tilde{k}^2(\bar{r})^3 \right)} \). In this approximation we have obtained the equation of a quantum harmonic oscillator equation whose spectrum is \( E_n = \hbar \omega \left( n + \frac{1}{2} \right) \). Since we are using natural units, we set \( \hbar = 1 \) and
\[ \lambda_n = 3MG/\left(\tilde{\kappa}^2(\bar{r})^3\right) - \sqrt{9MG/\left(\tilde{\kappa}^2(\bar{r})^3\right)} \left(n + \frac{1}{2}\right). \] (86)

After some algebraic calculation, we obtain
\[ \lambda_n = 6\sqrt{\frac{b^2 + \bar{r}^2}{b^2 + 3\bar{r}^2}} \left(\sqrt{\frac{b^2 + \bar{r}^2}{b^2 + 3\bar{r}^2}} - \frac{1}{\sqrt{2}} \left(n + \frac{1}{2}\right)\right), \] (87)

where we have used the relation \([12]\). We see that
\[ \lambda_0 = 6\sqrt{\frac{b^2 + \bar{r}^2}{b^2 + 3\bar{r}^2}} \left(\sqrt{\frac{b^2 + \bar{r}^2}{b^2 + 3\bar{r}^2}} - \frac{1}{2\sqrt{2}}\right). \] (88)

Since the eigenvalue must be positive, the following inequality must hold
\[ \sqrt{\frac{b^2 + \bar{r}^2}{b^2 + 3\bar{r}^2}} > \frac{1}{2\sqrt{2}} \implies 7b^2 + 5\bar{r}^2 > 0, \] (89)

which is always verified. To proof that there is only one eigenvalue, we look at the second eigenvalue
\[ \lambda_1 = 6\sqrt{\frac{b^2 + \bar{r}^2}{b^2 + 3\bar{r}^2}} \left(\sqrt{\frac{b^2 + \bar{r}^2}{b^2 + 3\bar{r}^2}} - \frac{3}{2\sqrt{2}}\right). \] (90)

The inequality
\[ \sqrt{\frac{b^2 + \bar{r}^2}{b^2 + 3\bar{r}^2}} > \frac{3}{2\sqrt{2}} \implies b^2 + 19\bar{r}^2 < 0, \] (91)

which is never verified since \(b\) and \(\bar{r}\) are real quantities. Thus we can conclude that there is only one eigenvalue and according to Coleman \([18]\), this is a signal of a transition from a false vacuum to a true one. The same unstable mode appears also when we introduce a temperature and we look at the thermodynamic stability of a S-AdS black hole within isothermal cavities \([10,9,19]\). In terms of \(E^2\), the eigenvalue is
\[ E^2 = -3/b^2 - \frac{3b^2 + \bar{r}^2}{2b^2\bar{r}^2} + \frac{3}{4b^2\bar{r}^2} \sqrt{(b^2 + \bar{r}^2)(b^2 + 3\bar{r}^2)} \]
\[ = -3\frac{b^2 + 3\bar{r}^2}{2b^2\bar{r}^2} + \frac{3}{4b^2\bar{r}^2} \sqrt{(b^2 + \bar{r}^2)(b^2 + 3\bar{r}^2)}. \] (92)
VII. BOUNDARY REDUCTION AND STABILITY

An equivalent approach to Eq. (79) can be set up by means of a variational procedure applied on a functional whose minimum represents the solution of the problem. Let us define

$$J(h, E^2) = \frac{1}{2} \int_0^{x(a)} dx \left[ \left( \frac{dh(x)}{dx} \right)^2 - \frac{3MG}{r^3(x)} h^2(x) \right] + \frac{\tilde{E}^2}{2} \int_0^{x(a)} dx h^2(x),$$

where $dx$ is given by Eq.(28). Eq.(79) is equivalent to finding the minimum of

$$\tilde{E}^2 = \frac{\int_0^{x(a)} dx \left[ \left( \frac{dh(x)}{dx} \right)^2 - \frac{3MG}{r^3(x)} h^2(x) \right]}{\int_0^{x(a)} dx h^2(x)}.$$  

(93)

For future purposes, we use the boundary conditions

$$h(x(a)) = 0.$$  

(94)

When $r \to \infty \implies x \to \infty$ and Eq.(80) becomes

$$- \frac{d^2 h}{dx^2} + \tilde{E}^2 h = 0$$

(95)

whose asymptotic solution is

$$h(x) = A \exp\left(-\tilde{E}x\right) + B \exp\left(\tilde{E}x\right).$$

(96)

Since asymptotically $\exp\left(\tilde{E}x\right)$ diverges, we set $B = 0$, then $h(x)$ becomes $A \exp\left(-\tilde{E}x\right)$. If we change the variables in a dimensionless form like Eq.(82), we get

$$\mu = \frac{\tilde{E}^2}{\kappa^2} = \frac{\int_0^{y(a)} dy \left[ \left( \frac{dh(y)}{dy} \right)^2 - \frac{3MG}{r^3\rho(y)} h^2(y) \right]}{\int_0^{y(a)} dy h^2(y)}.$$  

(97)

6Although the asymptotic behaviour is such that

$$x(r) = \pm b \ln r,$$

for practical purposes, we prefer to use the variable $x(r)$. 

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The asymptotic behaviour of $h(x)$ suggests to choose $h(\lambda, y) = \exp(-\lambda y)$ as a trial function, and Eq.(97) becomes

$$
\mu(\lambda) = \lambda^2 - \frac{3MG}{\bar{r}^3 \bar{\kappa}^2} \int_0^{\frac{\bar{y}}{\rho(\bar{y})}} \frac{dy}{\rho(\bar{y})} \exp(-2\lambda y) \left[ \frac{1}{2\lambda} \right].
$$

(98)

Close to the throat $\exp(-2\lambda y) \simeq 1 - 2\lambda y$ and

$$
\mu(\lambda) = \lambda^2 - \frac{3MG}{\bar{r}^3 \bar{\kappa}^2} + \frac{9MG}{\bar{r}^3 \bar{\kappa}^2} \left[ \frac{\bar{y}}{2\lambda} + \bar{y}^2 \right].
$$

(99)

The minimum of $\mu(\lambda)$ is reached for $\bar{\lambda} = \left( \frac{9MG}{4\bar{r}^3 \bar{\kappa}^2} \right)^{\frac{1}{4}}$ with the help of Eq.(12), we can write

$$
\mu(\bar{\lambda}) = 3 \left( \frac{9}{4} \bar{D} \bar{y} \right)^{\frac{3}{4}} - 3\bar{D} + 3\bar{D} \bar{y}^2,
$$

(100)

where

$$
\bar{D} = \frac{MG}{\bar{r}^3 \bar{\kappa}^2} = \frac{2\bar{r}^2 + b^2}{3\bar{r}^2 + b^2}.
$$

(101)

If we set the value of $\bar{r}$ equal to the surface gravity minimum location, then Eq.(100) becomes

$$
\mu(\bar{\lambda}) = 3 (3\bar{y})^{\frac{3}{4}} - 4 + 12\bar{y}^2,
$$

(102)

which is zero for $\bar{y}_c = 0.30915$ corresponding to $\bar{\rho}_c = 1.0956$. This means that the unstable mode persists until the boundary radius $\bar{\rho}$ falls below $\bar{\rho}_c$, to be compared with the value $\rho = 1$ of Refs. [9,19].

**VIII. SUMMARY AND CONCLUSIONS**

Following Refs. [6,7], in this paper we have computed the Casimir-like energy for a S-AdS space with a AdS space as a reference space. Due to the same asymptotic properties of these spaces and looking at the extended Kruskal S-AdS manifold at constant time, we have found that the classical contribution coming from boundaries disappears. According to Witten [8], since the energy is conserved and since boundary conditions are the same we can discuss the existence of an instability at zero temperature. A proof of instability at
finite temperature has been given by Prestidge in Ref. [9], based on conjectures of Hawking
and Page [19]. The zero temperature one-loop analysis shows a single negative mode in S
wave which is interpreted as a clear signal of a decay form a false vacuum to a true one
[18]. This is also confirmed by one-loop stable modes contribution which shifts the energy
minimum to the S-AdS space. Following the same procedure of Ref. [20], we discover a
critical radius \( r_c \) below which the system becomes stable. In analogy with the flat space
case, the appearance of this instability even at zero temperature is attributed to a neutral
S-AdS black hole pair creation mediated by a three-dimensional S-AdS wormhole with the
holes residing in different universes [7]. The probability of creating such a pair and therefore
the instability appearance at zero temperature is measured by

\[
\Gamma_{1\text{-hole}} = \frac{P_{S\text{-AdS}}}{P_{\text{AdS}}},
\]

where

\[
P \sim |\exp - (\Delta E)(\Delta t)|^2.
\]

In spite of the evident analogy between the S-AdS and the Schwarzschild space it is not
simple at this stage speculate on a possible foam-like structure composed by \( N \) S-AdS
coherent wormholes because the boundary reduction, which is fundamental to have the
stabilization of the system in examination, is not of straightforward application due to
the presence of the negative cosmological factor proportional to the square of the radius.
However, if such a reduction mechanism could work we should discuss what is the meaning
of

\[
\Gamma_{N-S\text{-AdS holes}} = \frac{P_{N-S\text{-AdS holes}}}{P_{\text{AdS}}} \approx \frac{P_{\text{foam}}}{P_{\text{AdS}}},
\]

compared with

\[
\Gamma_{N\text{-holes}} = \frac{P_{N\text{-holes}}}{P_{\text{flat}}} \approx \frac{P_{\text{foam}}}{P_{\text{flat}}}.
\]
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APPENDIX A: KRUSKAL-SZEKERES COORDINATES FOR S-ADS SPACETIME

Before introducing the Kruskal-Szekeres [21–23] type coordinates we recall that the S-AdS line element is defined as

$$ds^2 = -\left(1 - \frac{2MG}{r} + \frac{r^2}{b^2}\right) dt^2 + \left(1 - \frac{2MG}{r} + \frac{r^2}{b^2}\right)^{-1} dr^2 + r^2 d\Omega^2.$$  \hfill (A1)

By looking at the \((t, r)\) coordinates one gets

$$ds^2 = -\left(1 - \frac{2MG}{r} + \frac{r^2}{b^2}\right)\left[dt^2 - dr^*\right] + r^2 d\Omega^2$$

$$= -\left(1 - \frac{2MG}{r} + \frac{r^2}{b^2}\right) dvdu + r^2 (u, v) d\Omega^2.$$  \hfill (A2)

\(v = t + r^*\) is the ingoing radial null coordinate, \(u = t - r^*\) is the outgoing radial null coordinate and

$$dr^* = \frac{rb^2 dr}{(r - \bar{r})(r^2 + \bar{r}r + \bar{r}^2 + b^2)}.$$  \hfill (A3)

The explicit integration gives \(2\kappa r^*\)

$$= \ln\left|\frac{r}{\bar{r}} - 1\right| - \frac{1}{2} \ln\left(\frac{r^2 + \bar{r}r + \bar{r}^2 + b^2}{\bar{r}^2 + b^2}\right) + \frac{3\bar{r}^2 + 2b^2}{\bar{r}\sqrt{3\bar{r}^2 + 4b^2}} \arctan\left(\frac{2r\sqrt{3\bar{r}^2 + 4b^2}}{4\bar{r}^2 + 4b^2 + r\bar{r}}\right),$$  \hfill (A4)

where \(\kappa\) is the surface gravity defined in Eq.(76). We now introduce Kruskal-Szekeres coordinates \((U, V)\) defined (for \(r > \bar{r}\)) by

$$U = -e^{-\kappa u} \quad V = e^{\kappa v}$$  \hfill (A5)

with
\[ UV = - \exp \kappa (v - u) = - \exp (2\kappa r^*) \] (A6)

\[ = - \frac{(r/\bar{r} - 1) \sqrt{\bar{r}^2 + b^2}}{\sqrt{\bar{r}^2 + \bar{r}r + \bar{r}^2 + b^2}} \exp \left( \frac{3\bar{r}^2 + 2b^2}{\bar{r}\sqrt{3\bar{r}^2 + 4b^2}} \arctan \left( \frac{2r\sqrt{3\bar{r}^2 + 4b^2}}{4\bar{r}^2 + 4b^2 + \bar{r}r} \right) \right) \] (A7)

and

\[ \frac{U}{V} = - \exp -\kappa (v + u) = - \exp (-2\kappa t) \] (A8)

In terms of these coordinates Eq. (A2) becomes

\[ ds^2 = - \frac{\bar{r} (r^2 + \bar{r}r + \bar{r}^2 + b^2)^{3/2}}{\sqrt{\bar{r}^2 + b^2 r^2 \kappa^2}} \exp (F (r)) dU dV + r^2 (U, V) d\Omega^2, \] (A9)

where

\[ F (r) = - \frac{3\bar{r}^2 + 2b^2}{\bar{r}\sqrt{3\bar{r}^2 + 4b^2}} \arctan \left( \frac{2r\sqrt{3\bar{r}^2 + 4b^2}}{4\bar{r}^2 + 4b^2 + \bar{r}r} \right). \] (A10)

The only true singularities are at curves \( UV = 1 \), where \( r = 0 \). The region \( \{ U < 0, V > 0 \} \) is the “outside region”, the only region from which distant observers can obtain any information. The line \( V = 0 \), where \( r = \bar{r} \), is the “past horizon”; the line \( U = 0 \) where also \( r = \bar{r} \), is the “future horizon”.

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