Alternative approximations using the similarity theory for turbulent moments of the atmospheric surface layer

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Abstract. Two limiting cases can be discerned within the atmospheric surface layer by the Monin-Obukhov similarity theory: the dynamic limit with a logarithmic profile of wind velocity and a zero buoyancy flux at the underlying surface; and the free-convection limit with zero wind velocity and the positive buoyancy flux. In this article, a two sublayer model of the convective surface layer is presented. It identifies the lower dynamic sublayer and the upper forced-convection sublayer. The turbulent moments can be determined separately for each of these sublayers by the linear approximations, which are proposed for the first and second turbulent moments of vertical velocity and potential temperature variance. The free-convection limits of the Monin-Obukhov theory corresponds to the first-order expansion terms of the developed approximations. The second-order expansion terms describe profiles of the turbulent moments under conditions with moderate wind. A comparison between the experimental data sets and the proposed approximations shows that the accepted approximations are correct within a forced convection sublayer.

1. Introduction
The classical similarity theory of the atmospheric surface layer was first formulated in [1–3] for the approximation of the first-order turbulent moments.

The Monin-Obukhov similarity theory for the convective surface layer distinguishes two limiting cases: a dynamic limit and a free-convection limit.

The dynamic limit of the convective surface layer is defined as a flow with a logarithmic profile of wind and a zero buoyancy flux at the underlying surface. In the dynamic limit \( z/L_\ast = 0 \), where \( z \) is the height above the surface and \( |L_\ast| = \infty \) is the Monin-Obukhov length parameter.

The free-convection limit of the surface layer characterizes a flow with a zero wind speed and a positive buoyancy flux at the underlying surface. In the free convection limit \( z/L_\ast = \infty \), where \( z \) is the height above the surface and \( |L_\ast| = 0 \) is the Monin-Obukhov length parameter.

For the limiting cases, the generalized Monin-Obukhov similarity theory makes it possible to calculate higher order turbulent moments as well.
A direct generalization of Monin-Obukhov similarity theory for the approximation of the second order turbulent moments in the dynamic regime was suggested in [4,5]. For the free-convection regime, the Monin-Obukhov similarity theory was first generalized in [6,7].

In this paper, it is supposed that the convective surface layer consists of two sublayers: the lower dynamic sublayer adjacent to the surface and the upper forced-convection sublayer. Therefore, independent approximations can be used for describing turbulent moments of each sublayer.

The limiting regimes of the convective surface layer can be used as first-order approach in constructing linear approximations of the turbulent moments of each of the sublayers.

Second-order linear approximations of turbulent moments in the lower dynamic sublayer, which use the dynamic limits as a first-order approximation, are described in [3,4].

This paper examines the possibility of using second-order linear approximations of turbulent moments of vertical velocity and buoyancy in the upper forced-convection sublayer. The first-order expansion terms of the linear approximations correspond to the free-convection limits of Monin-Obukhov similarity theory under no-wind conditions. The second-order expansion terms of the linear approximations describe the profiles of turbulent moments under wind conditions. A presented comparison with experimental data strongly suggests that linear approximations are correct within the range of $2 \cdot 10^{-2} < z / L \times \infty$.

2. Local parameters of friction velocity and buoyancy flux

Let $t$ be time and $x$, $y$, $z$ be a Cartesian coordinate system placed at the underlying surface $z = 0$ so that the $z$ axis is opposite to gravitational acceleration $g$.

Let us assume that $u = u(x, y, z, t)$ and $w = w(x, y, z, t)$ are the velocity vector components along the horizontal and vertical axes; $\bar{u} = \bar{u}(z)$ is the mean value of the horizontal wind velocity vector along the $x$ axis; and $u'(x, y, z, t) = u(x, y, z, t) - \bar{u}$ is perturbations of the horizontal velocity.

It is supposed that $\Theta = \Theta(x, y, z, t)$ is the potential air temperature and $\Theta_0$ is a constant mean value of the potential temperature at the upper boundary of the surface layer [8]. Following [8], we introduce a potential temperature fluctuation $\Theta'(x, y, z, t) = \Theta(x, y, z, t) - \Theta_0$ and a dimensionless potential-temperature fluctuation $\theta(x, y, z, t) = \Theta'(x, y, z, t) / \Theta_0$. The quantity $g \theta(x, y, z, t)$ will denote local buoyancy [9].

Let us assume that

$$U_* = \lim_{z \to 0} \left( -\bar{u}'w' \right)^{1/2} > 0, \quad gS_\Theta = \lim_{z \to 0} g \bar{\theta}w > 0$$

(1)

Here, $U_*$ and $gS_\Theta$ have dimensions $[U_*] = \text{m/s}$ and $[gS_\Theta] = \text{m}^2/\text{s}^3$.

To implement the similarity theory in the convective surface layer, three key parameters: $z$, $gS_\Theta$ and $U_*$ are used. The parameter of height $z$ is a variable; the buoyancy flux $gS_\Theta$ and friction velocity $U_*$ parameters are constants.

The parameters $U_* > 0$ and $gS_\Theta > 0$ make it possible to introduce constant parameters of length $L_* < 0$ and buoyancy $g \Theta_*$, so that

$$L_* = -\frac{U_*^2}{gS_\Theta}, \quad g \Theta_* = \frac{gS_\Theta}{U_*},$$

(2)

where $k_* = 0.4$ is the von Kármán constant.

Let us assume that $H$ is the mean heat flux entering the atmosphere from the underlying surface, $\rho_0$ is the mean air density at the underlying surface, $c_p$ is the specific heat capacity of dry air, and $T_*$ is the temperature parameter of the Monin-Obukhov theory.

With (2) taken into account, we then obtain
\[ gS_0 = \left( \frac{g}{\Theta_0} \right) \frac{H}{\varepsilon P_0} \quad \text{and} \quad g\Theta_0 = -\left( \frac{g}{\Theta_0} \right) T_e \quad T_e = -\Theta_0 \Theta_0 \]  

Equations (3) suggest the proportionality of the key parameters \( gS_0 \) and \( g\Theta_0 \) to the conventional Monin-Obukhov similarity theory parameters \( H \) and \( T_e \).

3. Monin-Obukhov similarity theory and free-convection limits of the surface layer

Let us consider a convective surface layer under the free convection conditions \( U_e = 0 \). In this case, there are only two key parameters \( z \) and \( gS_0 \). In accordance with [3], the first moments of the vertical velocity and buoyancy have the form

\[ \bar{w} = 0, \quad \frac{\partial g\theta}{\partial z} = -\lambda_0 z^{-4/3} (gS_0)^{2/3} \]  

where \( \lambda_0 = 1 \) from [6].

According to [6], for the second moments of vertical velocity and buoyancy have the form,

\[ \bar{w}^2 = \lambda_{w w} (gS_0)^{2/3} z^{2/3}, \quad \bar{g\theta w} = gS_0, \quad (g\theta)^2 = \lambda_{\theta \theta} (gS_0)^{4/3} z^{-2/3} \]  

where \( \lambda_{w w} = \lambda_{\theta \theta} = 1.8 \) from [10].

In accordance with [11–13], the equations (5) are formed by a stochastic ensemble of convective jets.

Similar relationships can also be deduced for the third moments (see [14]).

Let us consider the convective surface layer in forced convection with low winds \( 0 \neq U_e, << 1 \). In this case, all three key parameters \( z, gS_0, \) and \( U_e \), are finite.

To construct the Monin-Obukhov free-convection limit, let us assume that the moments of vertical velocity and buoyancy in low wind \( 0 \neq U_e, << 1 \) are the same as those when there is no wind \( U_e = 0 \).

The transformation of the first buoyancy moment (5) by taking (2) and (4) into account leads to the equality

\[ k_z \frac{\partial^2 g\theta}{\partial z^2} = -k_z \frac{\partial^2 \theta}{\partial z^2} = \alpha_0 \left( z / |L_e| \right)^{4/3} \]  

where \( \alpha_0 = k_z^4 \lambda_{\theta \theta} \approx 0.3 \) is a positive constant (see [5,15]).

The transformation of the second moments (6), with (2) and (4) taken into account, leads to equalities

\[ \frac{\bar{w}^2}{U_e^2} = \alpha_{w w}^2 \left( z / |L_e| \right)^{2/3}, \quad \frac{(g\theta)^2}{\Theta_e^2} = \frac{(\Theta)^2}{T_e^2} = \alpha_{\theta \theta} \left( z / |L_e| \right)^{4/3} \]  

where \( \alpha_{w w}^2 = k_z^{-2/3} \lambda_{w w} \approx 3.3 \) and \( \alpha_{\theta \theta} = k_z^{2/3} \lambda_{\theta \theta} \approx 1 \) are constant coefficients (see [7,15,16]).

4. Linear approximations of the Monin-Obukhov universal similarity functions, the first and second moments of the surface layer

Following [17], let us assume that the convective surface layer consists of two sublayers: the lower dynamic sublayer adjacent to the surface and the upper sublayer of forced convection. A narrow transition area forms a diffuse boundary between these sublayers. This area is not considered as a separate sublayer.

An alternative model of the convective surface layer, that includes three-sublayer structure, was developed in [5,18].

Obviously, the turbulent moments of vertical velocity and buoyancy can be approximated uniformly in the entire convective layer. In the two-sublayer model of the convective surface layer, the turbulent moments can be approximated separately for each sublayer. This approach is quite
justifiable, because turbulence in the dynamic sublayer and turbulence in the forced-convection sublayer are of different hydrodynamic natures.

Length parameter \( L \leq 0 \) is used to construct the dimensionless height \( z / |L| \). Let us suppose that region \( 0 < z / |L| \leq 2 \cdot 10^{-2} \), adjacent to the underlying surface, corresponds to the dynamic sublayer and the transition sublayer. The region \( 2 \cdot 10^{-2} < z / |L| < \infty \) is denoted as a forced-convection sublayer with moderate winds. In the forced-convection sublayer, let us also distinguish a free-convection sublayer \( 1 << z / |L| \).

Let us construct linear approximations of the moments of vertical velocity and potential temperature fluctuations in the forced-convection sublayer. The first-order expansion terms of the linear approximations is chosen in accordance with the free-convection limits of the Monin-Obukhov similarity theory in no-wind conditions \( U_* = 0 \). The second-order expansion terms of linear approximations will then describe profiles of turbulent moments under wind condition \( U_* \neq 0 \).

It can be assumed, with no restriction of generality, that the dimensionless form of the equation for first buoyancy moment (6), and the forms of the equations for the second moments of vertical velocity and buoyancy (7) in the forced-convection sublayer with \( 2 \cdot 10^{-2} < z / |L| < \infty \) can be written as

\[
\frac{-k_z}{|T_*|} \frac{\partial}{\partial z} \Theta = \alpha_0 \left( \frac{z}{|L|} \right)^{-1/3} \frac{\partial}{\partial (z / |L|)} F_h \left( \frac{z}{|L|} \right), \quad \lim_{z / |L| \to \infty} F_h \left( \frac{z}{|L|} \right) = 1
\]

\[
\frac{w^2}{U_*^2} = \sigma_{ww} \left( \frac{z}{|L|} \right)^{2/3} \frac{\partial^2}{\partial (z / |L|)} F_{ww} \left( \frac{z}{|L|} \right), \quad \lim_{z / |L| \to \infty} F_{ww} \left( \frac{z}{|L|} \right) = 1
\]

\[
\frac{(\Theta)^2}{|T_*|^2} = \sigma_{\Theta \Theta} \left( \frac{z}{|L|} \right)^{2/3} \frac{\partial^2}{\partial (z / |L|)} F_{\Theta \Theta} \left( \frac{z}{|L|} \right), \quad \lim_{z / |L| \to \infty} F_{\Theta \Theta} \left( \frac{z}{|L|} \right) = 1
\]

Let us note that the coefficient \( \alpha_0 = k_0^2 \beta_0 = 0.3 \) from the data of [5,6,15]. According to [16,19], the coefficient \( \alpha_{ww}^2 \approx 3.25 \), and \( \alpha_{\Theta \Theta}^2 \approx 0.95 \) from the data of [20–22].

Limiting ourselves to a linear Taylor expansion of \( F_h(z / |L|) \), \( F_{ww}(z / |L|) \) and \( F_{\Theta \Theta}(z / |L|) \) in parameter \( (z / |L|)^{-2/3} \), we obtain

\[
\frac{-k_z}{|T_*|} \frac{\partial}{\partial z} \Theta = \alpha_0 \left( \frac{z}{|L|} \right)^{-1/3} \left\{ 1 - \beta_h \left( \frac{z}{|L|} \right)^{-2/3} \right\}
\]

\[
\frac{w^2}{U_*^2} = \frac{\sigma_{ww}^2}{U_*^2} = \alpha_{ww}^2 \left( \frac{z}{|L|} \right)^{2/3} \left\{ 1 + \beta_{ww} \left( \frac{z}{|L|} \right)^{-2/3} \right\}
\]

\[
\frac{(\Theta)^2}{|T_*|^2} = \frac{\sigma_{\Theta \Theta}^2}{|T_*|^2} = \alpha_{\Theta \Theta}^2 \left( \frac{z}{|L|} \right)^{2/3} \left\{ 1 - \beta_{\Theta \Theta} \left( \frac{z}{|L|} \right)^{-2/3} \right\}
\]

where \( \sigma_w, \sigma_h \) are variances of vertical velocity and potential-temperature fluctuation; \( \beta_h, \beta_{ww}, \beta_{\Theta \Theta} \) are constant coefficients.

Fig. 1 presents a comparison of the linear approximation of Monin-Obukhov similarity theory (11) at \( \alpha_0 = 0.3, \beta_h = 2 \cdot 10^{-2} \) with the field data [23]. It shows that the approximation (11) slightly overestimates this field data within the free-convection sublayer.

A comparison between the field data [24] of the dimensionless vertical velocity variance \( \sigma_w / U_* \) and its linear approximation in the form of (12) with \( \alpha_{ww}^2 = 3.25 \) and \( \beta_{ww} = 0.3 \) is shown in Fig. 2. A comparison between the field data [24] of the dimensionless potential temperature variance \( \sigma_\Theta / |T_*| \) and its linear approximation in the form of (13) with \( \alpha_{\Theta \Theta}^2 = 0.95, \beta_{\Theta \Theta} = 0.06 \) is shown in Fig. 3.
The results of comparisons in Fig. 1-3 suggest that linear approximations (11) - (13) are correct within the range of $2 \cdot 10^{-2} < z / L_s < \infty$.

Another expansion, which is similar to (11) - (13), and its comparison with experimental data sets are shown in [25].

Alternative approaches that are using the local similarity theory are presented in [26–29].

Modern comprehensive field data of atmospheric surface layer parameters and techniques of its processing are described in [30,31].

5. Conclusion

The results show that natural approximations of the turbulent moments in the convective surface layer have various forms and are not confined to the classical form of the Monin-Obukhov universal functions.

In the two-sublayer model of the convective surface layer, the linear approximations of turbulent moments are quite efficient in describing the observed data in a forced-convective region.

The proposed version of the linear approximations of the universal functions of similarity theory considers the free-convective limits of Monin-Obukhov theory only as a first-order approach. The second order expansion terms of the linear approximations take into account the existence of wind in the convective surface layer.
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