Threshold resummation beyond leading eikonal level

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The modified evolution equation for parton distributions of Dokshitzer, Marchesini and Salam is extended to non-singlet Deep Inelastic Scattering coefficient functions and the physical evolution kernels which govern their scaling violation. Considering the $x \to 1$ limit, it is found that the leading next-to-eikonal logarithmic contributions to the momentum space physical kernels at any loop order can be expressed in term of the one loop cusp anomalous dimension, a result which can presumably be extended to all orders in $(1 - x)$. Similar results hold for fragmentation functions in semi-inclusive $e^+e^-$ annihilation. The method does not work for subleading next-to-eikonal logarithms, but, in the special case of the $F_1$ and $F_T$ structure and fragmentation functions, there are hints of the possible existence of an underlying Gribov-Lipatov like relation.
1. Threshold resummation in physical evolution kernels

Consider a generic deep inelastic scattering (DIS) non-singlet structure function $F(x, Q^2) = \{2F_1(x, Q^2), F_2(x, Q^2)/x\}$ at large $Q^2 \gg \Lambda^2$. We shall be interested in the elastic limit $x \to 1$ where the final state mass $W^2 \sim (1-x)Q^2 < Q^2$. In this limit, large threshold $\ln(1-x)$ logarithms appear. Their resummation is by now standard [1, 2], but usually performed in moment space. However, the result can also be expressed analytically in momentum space at the level of so-called “physical evolution kernels” which account for the physical scaling violation:

$$\frac{\partial F(x, Q^2)}{\partial \ln Q^2} = \int_x^1 \frac{dz}{z} K(z, a_s(Q^2)) F(z, Q^2) \equiv K \otimes F,$$  

where the “physical evolution kernel” $K(x, a_s)$ ($a_s = \alpha_s/4\pi$ is the $\overline{MS}$ coupling) embodies all the perturbative information about $F$. For $x \to 1$ threshold resummation yields [3]:

$$K(x, a_s(Q^2)) \sim \mathcal{J} \left( \frac{(1-x)Q^2}{1-x} \right) + B^\text{DIS}(a_s(Q^2)) \delta(1-x),$$

where $\mathcal{J}(Q^2)$ is a “physical anomalous dimension” (a renormalization scheme invariant quantity), related to the standard “cusp” $A(a_s) = \sum_{i=1}^\infty A_i a_i^s$ and final state “jet function” $B(a_s) = \sum_{i=1}^\infty B_i a_i^s$ anomalous dimensions by:

$$\mathcal{J}(Q^2) = A(a_s(Q^2)) + \beta(a_s(Q^2)) \frac{dB(a_s(Q^2))}{da_s} = \sum_{i=1}^\infty j_i a_i^s(Q^2).$$

The renormalization group invariance of $\mathcal{J}(Q^2)$ yields the standard relation:

$$\mathcal{J} \left( \frac{(1-x)Q^2}{1-x} \right) = j_1 a_s + a_s^2 \left[ -j_1 b_0 L_x + j_2 \right] + a_s^3 \left[ j_1 b_0^2 L_x^2 - (j_1 b_1 + 2j_2 b_0) L_x + j_3 \right] + O(a_s^4),$$

where $L_x \equiv \ln(1-x)$ and $a_s = a_s(Q^2)$, from which the structure of all the eikonal logarithms in $K(x, a_s(Q^2))$, which can be absorbed into the single scale $(1-x)Q^2$, can thus be derived.

However, no analogous result holds at the next-to-eikonal level (except [4] at large-$\beta_0$). Indeed, expanding

$$K(x, a_s) = K_0(x) a_s + K_1(x) a_s^2 + K_2(x) a_s^3 + O(a_s^4),$$

the $K_i$’s can be determined as combinations of splitting and coefficient functions. One gets:

$$K_0(x) = P_0(x) = k_{10} p_{qq}(x) + \Delta_1 \delta(1-x),$$

with $k_{10} = A_1$ and $p_{qq}(x) = \frac{x}{1-x} + \frac{1}{2}(1-x)$. Moreover for $x \to 1$ one finds [5, 6], barring delta function contributions:

$$K_1(x) = \frac{x}{1-x} (k_{21} L_x + k_{20}) + (h_{21} L_x + h_{20}) + O((1-x)L_x),$$

$$K_2(x) = \frac{x}{1-x} (k_{32} L_x^2 + k_{31} L_x + k_{30}) + (h_{32} L_x^2 + h_{31} L_x + h_{30}) + O((1-x)L_x^2).$$
Despite the similar logarithmic structure, the next-to-eikonal logarithms $h_{ij}$ cannot \[5\] be obtained from a standard renormalization group resummation analogous to the one used (eq.(1.4)) for the eikonal logarithms $k_{ij}$.

2. An alternative approach: the modified physical kernel

Instead, consider \[7\] a modified physical evolution equation, similar to the one used in \[8\] (see also \[9\]) for parton distributions:

$$
\frac{\partial \mathcal{F}(x, Q^2)}{\partial \ln Q^2} = \int_x^1 \frac{dz}{z} K(z, a_s(Q^2), \lambda) \mathcal{F}(x/z, Q^2/z^{\lambda}) \, ,
$$

(2.1)

where the arbitrary parameter $\lambda$ shall be set to 1 at the end. Expanding $\mathcal{F}(y, Q^2/z^{\lambda})$ around $z = 1$, one can relate $K(x, a_s, \lambda)$ to $K(x, a_s)$:

$$
K(x, a_s, \lambda) = K(x, a_s) + \lambda [\ln x K(x, a_s, \lambda)] \otimes K(x, a_s) + \ldots .
$$

(2.2)

Solving perturbatively, one finds that for $x \to 1$ the corresponding expansion coefficients $K_i(x, \lambda)$ satisfy the analogue of eq.(1.7), with the same coefficients $k_{ji}$'s of the eikonal logarithms, but with the coefficients of the leading next-to-eikonal logarithms given by:

$$
h_{21}(\lambda) = h_{21} - \lambda k_{10}^2
$$

(2.3)

$$
h_{32}(\lambda) = h_{32} - \frac{3}{2} \frac{1}{2} \lambda k_{21} k_{10} .
$$

Setting now $\lambda = 1$, one observes that both $h_{21}(\lambda = 1)$ and $h_{32}(\lambda = 1)$ vanish, which means that $h_{21} = k_{10}^2 = 16 C_F^2$ and $h_{32} = \frac{1}{2} k_{21} k_{10} = - \frac{25}{9} \beta_0 A_1^2 = - 24 \beta_0 C_F^2$, which agree with the exact results in \[3, \mathcal{E}\]. It should be stressed that, whereas $h_{21}$ is contributed by the two loop splitting function alone (and thus one simply recovers in this case the result of \[3\]), $h_{32}$ is instead contributed only by the one and two loop coefficient functions, which represents a new result. Similar results are obtained for the coefficients $h_{ji}$ ($j = i + 1$) of the leading next-to-eikonal logarithms at any loop order, which can all be expressed in term of the one loop cusp anomalous dimension assuming the corresponding $h_{ji}(\lambda)$ vanish for $\lambda = 1$. In particular, one predicts $h_{43} = \frac{1}{3} k_{10} k_{32} + \frac{1}{4} k_{21}^2 = \frac{11}{9} \beta_0 A_1^2 = \frac{11}{9} \beta_0 C_F^2$, which is correct \[3, \mathcal{E}\], and $h_{54} = \frac{1}{3} k_{10} k_{43} + \frac{1}{4} k_{21} k_{32} = - \frac{25}{12} \beta_0^2 A_1^2 = - \frac{100}{3} \beta_0^2 C_F^2$, which remains to be checked.

Similar results are obtained for the coefficients $f_{ji}$ ($j = i + 1$) of the leading next-to-next-to eikonal logarithms, defined by:

$$
K_i(x)_{LL} = L_i^{[p_{qq}(x) k_{ji} + h_{ji} + (1 - x)f_{ji} + O((1 - x)^2])} ,
$$

(2.4)

where the full one loop prefactor $p_{qq}(x)$ should be used in the leading term to define the $f_{ji}$'s. The corresponding $f_{ji}(\lambda)$ coefficients in $K_i(x, \lambda)$ are given by:

$$
f_{21}(\lambda) = f_{21} + \lambda \frac{1}{2} k_{10}^2
$$

(2.5)
\[ f_{32}(\lambda) = f_{32} - \lambda \left( -\frac{3}{4}k_{10}k_{21} + k_{10}h_{21} \right) + \lambda^2 \frac{1}{2}k_{10}^3 \]
\[ f_{43}(\lambda) = f_{43} - \lambda \left( -\frac{2}{3}k_{10}k_{32} + \frac{1}{2}(h_{21} - \frac{1}{2}k_{21})k_{21} + k_{10}h_{32} \right) + \lambda^2 k_{10}^2k_{21} , \]

where one notes the presence of contributions quadratic in \( \lambda \). Assuming the \( f_{ji}(\lambda)'s \) vanish for \( \lambda = 1 \), the resulting predictions for the \( f_{ji}'s \) \( (j = i+1) \) are again found to agree with the exact results of \([8]\).

### 3. Fragmentation functions

Similar results hold for physical evolution kernels associated to fragmentation functions in semi-inclusive \( e^+e^- \) annihilation (SIA), provided one sets \( \lambda = -1 \) in the modified evolution equation:

\[
\frac{\partial \mathcal{F}_{SIA}(x, Q^2)}{\partial \ln Q^2} = \int_1^\infty \frac{dz}{z} K_{SIA}(z, a_s(Q^2), \lambda) \mathcal{F}_{SIA}(x/z, Q^2/z^\lambda) , \tag{3.1}
\]

where \( \mathcal{F}_{SIA} = \{ \mathcal{F}_T, \mathcal{F}_{T+L} \} \) denotes a generic non-singlet fragmentation function (I use the notation of \([6]\)). At the leading eikonal level, threshold resummation \([10]\) can be summarized in the standard SIA physical evolution kernel by:

\[
K_{SIA}(x, a_s(Q^2)) \sim \left[ \mathcal{F} \left( \frac{(1-x)Q^2}{1-x} \right) \right] + B_{\delta}^{SIA}(a_s(Q^2)) \delta(1-x) , \tag{3.2}
\]

where the “physical anomalous dimension” \( \mathcal{F} \) (hence the \( k_{ji}'s \) are the same for DIS and SIA, as follows from the results in \([11]\). Assuming the leading threshold logarithms vanish beyond the leading eikonal level in the modified SIA evolution kernel for \( \lambda = -1 \), and setting \( \lambda = -1 \) in eqs. \((3.1)\) and \((3.2)\), one derives predictions for \( h_{ji}^{SIA} \) and \( f_{ji}^{SIA} \) \( (j = i+1) \) which again agree with the exact results of \([8]\). In particular, one finds that \( h_{ji}^{SIA} = -h_{ji} \).

### 4. Subleading next-to-eikonal logarithms

The previous approach does not work for subleading next-to-eikonal logarithms, namely the latter do not vanish in the modified physical evolution kernels for \( \lambda = \pm 1 \). The following facts are nevertheless worth quoting:

- At large \( \beta_0 \), we have a generalization \([8]\) of the leading eikonal single scale ansatz (which takes care of all subleading logarithms) to any eikonal order:

\[
K(x, Q^2)\big|_{\text{large } \beta_0} = \left[ \frac{x}{1-x} \mathcal{F}(W^2)\big|_{\text{large } \beta_0} \right] + \left( \delta(1-x)\text{term} \right) + \mathcal{F}_0(W^2)\big|_{\text{large } \beta_0} + (1-x)\mathcal{F}_1(W^2)\big|_{\text{large } \beta_0} + \ldots \tag{4.1}
\]

where \( W^2 = (1-x)Q^2 \), and the \( \mathcal{F}_j's \) (except the leading eikonal one) are structure function dependent. A similar result holds for \( K_{SIA}(x, Q^2)\big|_{\text{large } \beta_0} \).
• There are remarkable relations between the momentum space next-to-leading threshold logarithms of the (DIS) \( F_1 \) and the corresponding (SIA) \( F_T \) transverse fragmentation function physical evolution kernels at the next-to-eikonal level. Namely, using the moment space results of [8], one can derive the following momentum space relations:

1) At two loop for the \( \mathcal{O}(L_0^0) \) next-to-eikonal constant term:

\[
\begin{align*}
  h_{20}^{(F_1)} &= h_{20}^{(F_1)} \bigg|_{\text{large } \beta_0} + \Delta h_{20}, \\
  h_{20}^{(F_T)} &= h_{20}^{(F_T)} \bigg|_{\text{large } \beta_0} - \Delta h_{20},
\end{align*}
\]  

with \( h_{20}^{(F_1)} \bigg|_{\text{large } \beta_0} = -11\beta_0 C_F, \ h_{20}^{(F_T)} \bigg|_{\text{large } \beta_0} = 7\beta_0 C_F, \) and \( \Delta h_{20} = A_1\Delta_1 = 12C_F^2. \)

2) At three loop for the single \( \mathcal{O}(L_A) \) next-to-eikonal logarithms:

\[
\begin{align*}
  h_{31}^{(F_1)} &= h_{31}^{(F_1)} \bigg|_{\text{large } \beta_0} + \Delta h_{31}, \\
  h_{31}^{(F_T)} &= h_{31}^{(F_T)} \bigg|_{\text{large } \beta_0} - \Delta h_{31},
\end{align*}
\]  

with \( h_{31}^{(F_1)} \bigg|_{\text{large } \beta_0} = -2\beta_0 \ h_{20}^{(F_1)} \bigg|_{\text{large } \beta_0} = 22C_F \beta_0^2, \ h_{31}^{(F_T)} \bigg|_{\text{large } \beta_0} = -2\beta_0 \ h_{20}^{(F_T)} \bigg|_{\text{large } \beta_0} = -14C_F \beta_0^2, \) and:

\[
\Delta h_{31} = 2A_1A_2 - 20\beta_0 C_F C_A + 20\beta_0 C_F^2. \]  

3) At four loop for the double \( \mathcal{O}(L_A^2) \) next-to-eikonal logarithms:

\[
\begin{align*}
  h_{42}^{(F_1)} &= h_{42}^{(F_1)} \bigg|_{\text{large } \beta_0} + \Delta h_{42}, \\
  h_{42}^{(F_T)} &= h_{42}^{(F_T)} \bigg|_{\text{large } \beta_0} - \Delta h_{42},
\end{align*}
\]  

with \( h_{42}^{(F_1)} \bigg|_{\text{large } \beta_0} = 3\beta_0^2 \ h_{20}^{(F_1)} \bigg|_{\text{large } \beta_0} = -33C_F \beta_0^3, \ h_{42}^{(F_T)} \bigg|_{\text{large } \beta_0} = 3\beta_0^2 \ h_{20}^{(F_T)} \bigg|_{\text{large } \beta_0} = 21C_F \beta_0^3, \) and:

\[
\Delta h_{42} = -24\beta_1 C_F^2 + 45\beta_0^2 C_F C_A - 178\beta_0^2 C_F^2 - (47 - 10\xi_2)\beta_0 C_F C_A^2 - (60 - 140\xi_2)\beta_0 C_F^2 C_A - 16\beta_0 C_F^3. \]  

The large-\( \beta_0 \) parts are consistent with eq. \([4.1]\), while the remaining \( \pm\Delta h_{ij} \) corrections are suggestive of an underlying (yet to be discovered) Gribov-Lipatov like relation \([14]\).
• No such relations exist between the DIS $F_2$ structure function and the corresponding total angle-integrated $F_{T+L}$ fragmentation function. This fact suggests to focus instead on the momentum space physical evolution kernels of the longitudinal structure function and fragmentation functions. Indeed, some observations in [8] do suggest that the $\mathcal{O}(1/(1-x))$ part of the spacelike and timelike longitudinal evolution kernels might actually be identical to any logarithmic accuracy.

5. Conclusions

• Using a kinematically modified [8] physical evolution equation, evidence has been given that the leading threshold logarithms at any eikonal order in the momentum space DIS and SIA non-singlet physical evolution kernels can be expressed in term of the one loop cusp anomalous dimension $A_1$, which represents the first step towards threshold resummation beyond the leading eikonal level. This result also explains the observed universality [5, 6] of the leading logarithmic contributions to the physical kernels of the various non-singlet structure functions at any order $[6]$ in $1-x$.

• The present approach does not work for subleading next-to-eikonal logarithms. However, there are hints of the possible existence of an underlying (yet to be understood) Gribov-Lipatov like relation in the special case of the $F_1$ DIS structure function and the corresponding $F_T$ SIA transverse fragmentation function.

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