ON A SYSTEM OF NONLINEAR PSEUDOPARABOLIC EQUATIONS WITH ROBIN-DIRICHLET BOUNDARY CONDITIONS

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Abstract. In this paper, we investigate a system of pseudoparabolic equations with Robin-Dirichlet conditions. First, the local existence and uniqueness of a weak solution are established by applying the Faedo-Galerkin method. Next, for suitable initial datum, we obtain the global existence and decay of weak solutions. Finally, using concavity method, we prove blow-up results for solutions when the initial energy is nonnegative or negative, then we establish the lifespan for the equations via finding the upper bound and the lower bound for the blow-up times.

1. Introduction. In this paper, we consider the initial-boundary value problem for the system of nonlinear pseudoparabolic equations with Robin-Dirichlet conditions

\[
\begin{align*}
  u_t - \lambda_1 u_{txx} - \frac{\partial}{\partial x} \left( \mu_1(x,t)u_x \right) + \int_0^t g_1(t-s)u_{xx}(x,s)ds &= f_1(x,t,u,v,u_t,v_t,u_x,v_x,u_{xt},v_{xt}), \\
  v_t - \lambda_2 v_{txx} - \frac{\partial}{\partial x} \left( \mu_2(x,t)v_x \right) + \int_0^t g_2(t-s)v_{xx}(x,s)ds &= f_2(x,t,u,v,u_t,v_t,u_x,v_x,u_{xt},v_{xt}),
\end{align*}
\]

(1.1)

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\[ f_2(x, t, u, v, u_t, v_t, u_x, v_x, u_{xt}, v_{xt}) \]

\[
\begin{aligned}
&\frac{\partial u}{\partial t} - u_{xx} = F(x, t, u, v, u_x, v_x, u_t, v_t), \quad 0 < x < 1, \quad 0 < t < T, \\
&\begin{cases}
  u_x(0, t) - h_0 u(0, t) = u(1, t) = 0, \\
  v(0, t) = v_x(1, t) + h_1 v(1, t) = 0,
\end{cases}
\end{aligned}
\]  

(1.2)

where \( h_0 \geq 0, \ h_1 \geq 0; \ \lambda_1, \lambda_2 > 0 \) are given constants and \( \mu_i, \ g_i, \ f_i, \ (i = 1, 2), \ \tilde{u}_0, \ \tilde{v}_0 \) are given functions satisfying conditions specified later.

Equation of type (1.1) is a form of the Sobolev-type differential equations, which are characterized by having mixed time and space derivatives appearing in the highest order terms of the equation. Equation of this type with a one time derivative appearing in the highest order term is also called pseudoparabolic equation after Showalter’s works [27, 28, 29] in the seventies, since then, numerous interesting results for a general model of pseudoparabolic equations as follows

\[ u_t - u_{xxt} = F(x, t, u, v, u_x, v_x, u_t, u_{xt}), \quad 0 < x < 1, \quad t > 0, \]

(1.4)

with different initial - boundary conditions have been obtained, see for example [1, 3, 4, 5, 6, 8, 10, 11, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 27, 31, 32, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43] among others and the references given therein. In (1.4) with \( \nu > 0, \ \alpha = 1, \ x \in \mathbb{R}, \ t \geq 0, \) in which the solution of (1.5) with initial data in \( L^1 \cap H^2 \) decays to zero in \( L^2 \) norm as \( t \to +\infty. \) With \( \nu > 0, \ \alpha > 0, \ x \in [0, 1], \ t \geq 0, \) the model has the form (1.5) was also investigated earlier by Bona and Dougalis [3], where uniqueness, global existence and continuous dependence of solutions on initial and boundary data were established and the solutions were shown to depend continuously on \( \nu \geq 0 \) and on \( \alpha > 0. \) The results obtained in [1] were developed by many authors, such as by Zhang [41] for equations of the form

\[ u_t - \nu u_{xx} - u_{xxt} - u_x + u^m u_x = 0, \]

where \( m \geq 0, \) see Meyvaci [19].

It is well known that the Sobolev-type differential equations or the pseudoparabolic equations as above appear in the study of various problems of hydrodynamics, thermodynamics and filtration theory, see [2, 31] and the references given therein. In the absence of the memory terms in (1.1), i.e. \( g_1 = g_2 = 0, \) the nonlinear pseudoparabolic problem of the type (1.1) is arisen in the investigations about second-grade or third-grade fluid flows, see [10, 11, 34] and references therein. In [10], a mathematical model describing the unsteady flow of second-grade fluid in a circular cylinder is considered as follows

\[
\begin{cases}
  \frac{\partial w}{\partial t} = (\nu + \alpha \frac{\partial}{\partial r}) \left( w_{rr} + \frac{1}{r} w_r \right) - N w, \quad 0 < r < a, \quad t > 0, \\
  w(a, t) = W, \quad t > 0, \\
  w(r, 0) = 0, \quad 0 \leq r < a,
\end{cases}
\]

where \( w(r, t) \) is the velocity along the z-axis, \( \nu \) is the kinematic viscosity, \( \alpha \) is the material parameter, and \( N \) is the imposed magnetic field. In the boundary and initial conditions, \( W \) is the constant velocity at \( r = a \) and \( a \) is the radius of the cylinder. In the presence of the memory term in (1.1), i.e \( g_i \neq 0, \) the
problems of the type (1.1) are also studied in the theory of viscoelasticity, see [26]. Besides, it is also well known that pseudoparabolic equations with different kinds of boundary conditions have been studied and many interesting results have been obtained such as stability, global existence and finite time blow-up, for example, we refer to [4, 8, 32, 43] and the references therein. The obtained results in the above mentioned works show that, the reciprocal effects between the mixed time and space derivatives term (it is seen as a strong damping term) and the source term can cause the decayed property or the blow-up phenomena of solutions in some cases.

In [4], Bouziani studied the solvability of solutions for the nonlinear pseudoparabolic equation

\[ u_t - \frac{\partial}{\partial x} \left( a(x,t)u_x \right) - \eta \frac{\partial^2}{\partial t \partial x} \left( a(x,t)u_x \right) = f(x,t,u,u_x), \alpha < x < \beta, \ 0 < t < T, \]

subject to the initial condition and the nonlocal boundary condition.

In [8], Dai and Huang studied the solvability and the well-posedness of solutions for the nonlinear pseudoparabolic equation

\[ u_t + (a(x,t)u_x)_x = F(x,t,u,u_x,u_{xx}), \alpha < x < \beta, \ 0 < t < T, \]

with the nonlocal moment boundary conditions.

As for the initial-boundary value problems for multi-dimensional pseudoparabolic equations, we introduce the following form which usually appears in many works

\[ u_t - k \Delta u_t - \Delta u = f(u). \] (1.6)

In (1.6), the source term often has a concrete form, almost in form of exponential nonlinearity. An important method to study the global existence and blow up of solutions to the pseudoparabolic equation (1.6) (or to the parabolic equation when \( k = 0 \)) is the potential wells method, which was introduced by Payne and Sattinger in [23]. Cao et. al. [5] studied the Cauchy problem for the \( n \)-dimensional equation (1.6) with \( f(u) = u^p \). By using the integral representation and the contraction-mapping principle, they proved the existence and uniqueness for all \( p > 0 \), and established some related comparison principles. Furthermore, if \( 0 < p < 1 \), the solution of (1.6) exists globally and if \( 1 < p < 1 + \frac{2}{n} \), it blows up in a finite time for any positive nontrivial initial data and if \( p \geq 1 + \frac{2}{n} \), also blows up in a finite time for sufficiently large positive initial data. A relative general result of [5] has been given by Xu and Hu in [37], for \( 1 < p < \infty \) if \( n = 1, 2; 1 < p < \frac{n+2}{n} \) if \( n \geq 3 \), they used the potential well method and the comparison principle to obtain global existence and finite-time blow-up results for the solutions with initial data at high energy level. Especially, in [15], Luo used a differential inequality technique to get a lower bound for blow-up time if the power \( p \) and the initial value satisfy some conditions, and establish a blow-up criterion and an upper bound for blow-up time under some conditions as well as a nonblow-up and exponential decay under some other conditions. Very recent, in [42], the properties of global existence and blow up in fine time corresponding to the different domains of initial energy have been also established for the equation (1.6) with a general form of power source given by \( f(u) = |x|^\sigma |u|^{p-1} u \). In recent years, a great deal of attention has been paid to the pseudoparabolic equations with memory or viscoelastic term. For instance, Sun et. al. [32] considered the Dirichlet problem for the nonlinear pseudoparabolic
equation with a power source term and a memory term as follows
\[
\begin{aligned}
& u_t - \Delta u - \Delta u + \int_0^T g(t - \tau) \Delta u(\tau) d\tau = |u|^{p-2} u, \text{ in } \Omega \times (0, T), \\
& u = 0, \text{ on } \partial \Omega \times (0, T), \\
& u(0) = u_0, \text{ in } \Omega,
\end{aligned}
\]
where \( \Omega \) is a bounded domain of \( \mathbb{R}^n \) \( (n \geq 1) \) with smooth boundary \( \partial \Omega \), \( p > 2 \), \( T \in (0, \infty) \), \( u_0 \in H^1(\Omega) \) and \( g : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) is a positive nonincreasing function. The authors used the Levine’s classical concavity method and the improved potential well method to obtain the global existence and the finite time blow-up phenomena of solutions. For more results of global existence and blow up of solutions to the pseudoparabolic equations with memory or viscoelastic terms, we refer to [18, 40] in which the pseudoparabolic problems with nonlinearities of variable-exponent type were investigated. Moreover, the results given in [18, 40] can be considered as a generalization of the above problems with exponential nonlinearity when variable exponent is identified as a constant.

To the best of our knowledge, there are many publications on properties of solutions to single parabolic/pseudoparabolic equations, but it seems that few results of system of these types are investigated. We refer here to some results as in [9, 12, 13, 33]. For a modelling interruption, some more simple initial-boundary value problems in form of (1.1)-(1.3) (or corresponding to the initial-boundary value problems for system of parabolic equations when \( \lambda_1 = \lambda_2 = 0 \)) appear in several branches of applied mathematics and have been used to model systems arising in biological dynamic in the soil [9, 13], in dynamics of a diffusive predator-prey system [7], or in chemotaxis-Stokes system with nonlinear cell diffusion [33] and and the references therein.

Inspired and motivated by the idea of the above mentioned works, we study the existence, uniqueness, blow-up and general decay of solutions for Prob. (1.1)-(1.3). With the effect of the mixed time and space derivatives terms \( u_{txx}, v_{txx} \) (strong damping terms), the decay or blow up of solutions of Prob. (1.1)-(1.3) can be obtained in some cases of the initial energy and datum. Specifically, if the initial energy is positive and small enough then we get here the decay. And we also have blow up result with any initial conditions and the negative initial energy, or with the lower bounded initial conditions and the bounded positive initial energy.

This paper consists of five sections. Section 2 is devoted to the presentation of preliminaries. In Section 3, by using the linear approximating method together with the Galerkin method, we establish the local existence and uniqueness of weak solution. In Section 4, we consider Prob. (1.1)-(1.3) with \( f_i(x, t, u, v, u_t, v_t, u_x, v_x, u_{xt}, v_{xt}) = f_i(u, v) + F_i(x, t), i = 1, 2 \), and prove a sufficient condition for the global existence and decay of weak solutions via the energy method. Finally, Section 5 is devoted to the study of the blow-up property for Prob. (1.1)-(1.3) in the special case \( f_i(x, t, u, v, u_t, v_t, u_x, v_x, u_{xt}, v_{xt}) = f_i(u, v), i = 1, 2 \). Based on the concavity method and the improved potential method, we describe the blow-up phenomenon of weak solutions of the problem considered here with respect to different kinds of initial energy. This section also derives the lifespan for the equations considered via finding the upper bound and the lower bound for the blow-up times.

2. Preliminaries. First, we put \( \Omega = (0, 1), Q_T = \Omega \times (0, T), T > 0 \) and denote the usual function spaces used in this paper by the notations \( L^p = L^p(\Omega), H^m = H^m(\Omega) \). Let \( \langle \cdot, \cdot \rangle \) be either the scalar product in \( L^2 \) or the dual pairing of a continuous linear functional and an element of a function space. The notation \( \| \cdot \| \)
The imbedding $H^1 \hookrightarrow C^0(\bar{\Omega})$ is compact, and
\[
\|v\|_{C^0(\bar{\Omega})} \leq \sqrt{2} \|v\|_{H^1}, \quad \forall v \in H^1.
\]

Lemma 2.2. With $i = 1, 2$, the imbedding $V_i \hookrightarrow C^0(\bar{\Omega})$ is compact. Moreover, we have
\[
\begin{align*}
(i) \quad & \|v\|_{C^0(\bar{\Omega})} \leq \|v_x\|, \quad \forall v \in V_i, \\
(ii) \quad & \frac{1}{\sqrt{2}} \|v\|_{H^1} \leq \|v_x\| \leq \|v\|_{H^1}, \quad \forall v \in V_i.
\end{align*}
\]

Next, let $\mu_1, \mu_2 \in C^1(\bar{\Omega} \times [0, T])$ with $\mu_i(x, t) \geq \mu_{i*} > 0$ for all $(x, t) \in \bar{\Omega} \times [0, T]$ and $(i = 1, 2)$ and $h_0 \geq 0, h_1 \geq 0$. On $V_i \times V_i$, we consider the symmetric bilinear forms $a_i(\cdot, \cdot)$, and the families of symmetric bilinear forms \{a_i(t; \cdot, \cdot)\}_{t \in [0, T]}.

\[
a_1(u, \varphi) = \langle u_x, \varphi_x \rangle + h_0 u(0) \varphi(0), \\
a_1(t; u, \varphi) = \langle \mu_1(t) u_x, \varphi_x \rangle + h_0 \mu_1(0, t) u(0) \varphi(0), \\
\tilde{a}_1(t; u, \varphi) = \langle \mu'_1(t) u_x, \varphi_x \rangle + h_0 \mu'_1(0, t) u(0) \varphi(0), \quad \forall (u, \varphi) \in V_i \times V_i, \quad t \in [0, T]; \\
a_2(v, \psi) = \langle v_x, \psi_x \rangle + h_1 v(1) \psi(1), \\
\tilde{a}_2(t; v, \psi) = \langle \mu_2(t) v_x, \psi_x \rangle + h_1 \mu_2(1, t) v(1) \psi(1), \\
a_2(t; v, \psi) = \langle \mu'_2(t) v_x, \psi_x \rangle + h_1 \mu'_2(1, t) v(1) \psi(1), \quad \forall (v, \psi) \in V_2 \times V_2, \quad t \in [0, T].
\]
Lemma 2.3. Let $\mu_1, \mu_2 \in C^0(\Omega \times [0, T])$ with $\mu_i(x, t) \geq \mu_{i*} > 0$ for all $(x, t) \in \Omega \times [0, T]$ ($i = 1, 2$) and $h_0 \geq 0$, $h_1 \geq 0$. Then, the symmetric bilinear forms $a_i (\cdot, \cdot)$, and the families of symmetric bilinear forms $\{\tilde{a}_i (t; \cdot, \cdot)\}_{t \in [0, T]}$ defined by (2.2) are continuous on $V_i \times V_i$ and coercive in $V_i$ for all $(i = 1, 2)$.

Moreover, we have

$$|a_i (u, v)| \leq (1 + h_{i-1}) \|u\|_2 \|v\|_2, \quad \forall u, v \in V_i, \quad i = 1, 2,$$

$$a_i (v, v) \geq \|v\|^2, \quad \forall v \in V_i, \quad i = 1, 2,$$

$$|\tilde{a}_i (t; u, v)| \leq \tilde{K}_i \|u\|_{a_i} \|v\|_{a_i}, \quad \forall u, v \in V_i, \quad t \in [0, T], \quad i = 1, 2,$$

$$\tilde{a}_i (t; v, v) \geq \mu_{i*} \|v\|^2_{a_i}, \quad \forall v \in V_i, \quad t \in [0, T], \quad i = 1, 2,$$

where

$$\|v\|_{a_i} = \sqrt{a_i (v, v)}, \quad \forall v \in V_i, \quad \tilde{K}_i = \tilde{K}_i (\mu_i) = \sup_{(x, t) \in \Omega \times [0, T]} \mu_i (x, t), \quad i = 1, 2.$$

On the other hand, if $\mu_1, \mu_2 \in C^0 (\Omega \times [0, T])$ such that $\mu_1', \mu_2' \in C^0 (\Omega \times [0, T])$, with $\mu_i (x, t) \geq \mu_{i*} > 0$ for all $(x, t) \in \Omega \times [0, T]$ ($i = 1, 2$), then we have

$$|\tilde{a}_i (t; u, v)| \leq \tilde{K}_i \|u\|_{a_i} \|v\|_{a_i}, \quad \forall u, v \in V_i, \quad t \in [0, T], \quad i = 1, 2,$$

with

$$\tilde{K}_i = \tilde{K}_i (\mu_i) = \sup_{(x, t) \in \Omega \times [0, T]} \mu_i (x, t) + \sup_{(x, t) \in \Omega \times [0, T]} |\mu_i' (x, t)|, \quad i = 1, 2.$$

Lemma 2.4. Let $h_0 \geq 0$, $h_1 \geq 0$. Then, there exists the Hilbert orthonormal base $\{w_j^{(i)}\}$ of $L^2$ consisting of the eigenfunctions $w_j^{(i)}$ corresponding to the eigenvalue $\lambda_{ij}$ such that

$$0 < \bar{\lambda}_{i1} \leq \bar{\lambda}_{i2} \leq \cdots \leq \bar{\lambda}_{ij} \leq \bar{\lambda}_{i,j+1} \leq \cdots, \quad \lim_{j \to +\infty} \bar{\lambda}_{ij} = +\infty,$$

$$a_i (w_j^{(i)}, \varphi) = \bar{\lambda}_{ij} (w_j^{(i)}, \varphi) \quad \text{for all } \varphi \in V_i, \quad j = 1, 2, \cdots, \quad i = 1, 2. \quad (2.3)$$

Furthermore, the sequence $\{w_j^{(i)} / \bar{\lambda}_{ij}\}$ is also the Hilbert orthonormal base of $V_i$ with respect to the scalar product $a_i (\cdot, \cdot)$, and $w_j^{(i)}$ satisfies the following boundary value problem

$$\begin{cases} -\Delta w_j^{(i)} = \bar{\lambda}_{ij} w_j^{(i)}, \quad \text{in } (0, 1), \quad (i = 1, 2), \\
 w_0^{(j)} (0) = h_0 w_0^{(j)} (1) = w_2^{(j)} (1) = 0, \\
 w_1^{(j)} (0) = w_1^{(j)} (1) + h_1 w_2^{(j)} (1) = 0, \\
 w_j^{(i)} \in C^\infty (\bar{\Omega}), \quad (i = 1, 2). \quad (2.4) \end{cases}$$

The proof of Lemma 2.4 can be found in [[30], p.87, Theorem 7.7], with $H = L^2$, $V = V_i$, and $a_i (\cdot, \cdot)$ are defined as in (2.1), (2.2). □

Remark 1. On $V_i$ ($i = 1, 2$), three norms $v \mapsto \|v\|_2$, $v \mapsto \|v\|_{H^1}$, $v \mapsto (\|v\|_2^2 + h_{i-1} v^2 (i))^{1/2}$, with $h_0 \geq 0$, $h_1 \geq 0$, are equivalent. And on $H^2 \cap V_i$ ($i = 1, 2$), three norms $v \mapsto \|v\|_2$, $v \mapsto (\|v\|_2^2 + \|\Delta v\|^2)^{1/2}$, $v \mapsto \|v\|_{H^2 \cap V_i} = (\|v\|_2^2 + \|\Delta v\|^2)^{1/2}$ are also equivalent.
3. The existence and uniqueness theorem. Consider $T^* > 0$ fixed and for each $M > 0$ given, we put $\bar{\Omega}_M = [0, 1] \times [0, T^*] \times [-M, M]^2 \times [-\sqrt{2}M, \sqrt{2}M]^2$. We make the following assumptions:

For each $t > h_0 \bar{v}_0 = (H^2 \cap V_1) \times (H^2 \cap V_2)$, $\bar{\omega}_{02}(0) = h_0 \bar{v}_0(0) = \bar{v}_{02}(1) + h_1 \bar{v}_0(1) = 0$;

such that $\mu_i(x,t) \geq \mu_\alpha$, $\forall \alpha \in [0, 1] \times [0, T^*]$, $i = 1, 2$;

$\mu_i(x,t) = \mu_i(x,t) = \mu_i(x,t)$, $i = 1, 2$;

$\bar{\omega}_i \in C^1([0, 1] \times [0, T^*] \times \mathbb{R}^8)$, $i = 1, 2$, such that

(3.1) $\mu_i \in \mathbb{R}^8$ together with the initial condition

(3.2) $u(0), v(0) = (u_0, \bar{v}_0)$,

where

$\bar{f}_i(x, t, u, v, u_t, v_t, u_{xt}, \bar{v}_0, v_{xt})$, $i = 1, 2$.

For each $M > 0$ given, we set the constants $\bar{K}_M = \bar{K}_M(1, 2)$, $\bar{K}_M = \bar{K}_M(1, 2)$ as follows

\begin{align*}
\bar{K}_M = \bar{K}_M(1, 2) &= \max_{i=1,2} \| f_i \|_{C^1(\bar{\Omega}_M)} \, ,
\end{align*}

(3.3)

with

\begin{align*}
\| f_i \|_{C^1(\bar{\Omega}_M)} &= \max_{\| \alpha \| \leq 1} \| D^\alpha f_i \|_{C^0(\bar{\Omega}_M)} = \bar{K}_M (f_i) ,
\end{align*}

(3.4)

where

\begin{align*}
\| f_i \|_{C^0(\bar{\Omega}_M)} &= \max_{\| \alpha \| \leq 1} \| D^\alpha f_i \|_{C^0(\bar{\Omega}_M)} = \bar{K}_M (f_i) ,
\end{align*}

(3.5)

$W_T = \{(u, v) \in L^\infty (0, T; (H^2 \cap V_1) \times (H^2 \cap V_2)) :$

\begin{align*}
(u', v') \in L^\infty (0, T; (H^2 \cap V_1) \times (H^2 \cap V_2)) \}
\end{align*}

is a Banach space with respect to the norm

\begin{align*}
\| (u, v) \|_{W_T} &= \max \left\{ \| (u, v) \|_{L^\infty (0, T; (H^2 \cap V_1) \times (H^2 \cap V_2))} : \| (u', v') \|_{L^\infty (0, T; (H^2 \cap V_1) \times (H^2 \cap V_2))} \right\} ,
\end{align*}

(3.6)

(3.7)
For every $M > 0$, we put
\[ B_T(M) = \{(u, v) \in W_T : \|(u, v)\|_{W_T} \leq M\}. \] (3.8)
Now, we establish the following recurrent sequence $\{(u_m, v_m)\}$. The first term is chosen as $(u_0, v_0) \equiv (0, 0)$, suppose that
\[ (u_{m-1}, v_{m-1}) \in B_T(M), \] (3.9)
we associate (1.1)-(1.3) with the following problem. Find $(u_m, v_m) \in B_T(M)$ $(m \geq 1)$ which satisfies the linear variational problem
\[
\begin{cases}
\langle u_m'(t), \varphi \rangle + \lambda_1 a_1(u_m'(t), \varphi) + \bar{a}_1(t; u_m(t), \varphi) \\
= \int_0^t g_1(t-s) a_1(u_m(s), \varphi)ds + \langle F_{1m}(t), \varphi \rangle,
\end{cases}
\] (3.10)
where
\[
\begin{cases}
\langle v_m'(t), \psi \rangle + \lambda_2 a_2(v_m'(t), \psi) + \bar{a}_2(t; v_m(t), \psi) \\
= \int_0^t g_2(t-s) a_2(v_m(s), \psi)ds + \langle F_{2m}(t), \psi \rangle, \quad \forall (\varphi, \psi) \in V_1 \times V_2, \text{ a.e. } t \in (0, T),
\end{cases}
\] (3.11)
Then we have the following theorem.

**Theorem 3.1.** Let $(H_1)-(H_4)$ hold. Then there exist constants $M, T > 0$ such that, for $(u_0, v_0) \equiv (0, 0)$, there exists a recurrent sequence $\{(u_m, v_m)\} \subset B_T(M)$ defined by (3.9)-(3.11).

**Proof of Theorem 3.1.** This proof consists of three steps.

**Step 1.** The Faedo-Galerkin approximation (introduced by Lions [14]).

Considering the basis $\{w_j^{(i)} / \sqrt{\lambda_{ij}}\}$ for $V_i$ as in Lemma 2.4. Put
\[ u_m^{(k)}(t) = \sum_{j=1}^k c^{(k)}_{mj}(t)w_j^{(1)}(t), \quad v_m^{(k)}(t) = \sum_{j=1}^k d^{(k)}_{mj}(t)w_j^{(2)}(t), \] (3.12)
where the coefficients $c^{(k)}_{mj}, d^{(k)}_{mj}$ satisfy the system of linear integrodifferential equations
\[
\begin{cases}
\langle u_m^{(k)}(t), w_j^{(1)} \rangle + \lambda_1 a_1(u_m^{(k)}(t), w_j^{(1)}) + \bar{a}_1(t; u_m^{(k)}(t), w_j^{(1)}) \\
= \int_0^t g_1(t-s) a_1(u_m^{(k)}(s), w_j^{(1)})ds + \langle F_{1m}(t), w_j^{(1)} \rangle,
\end{cases}
\] (3.13)
\[
\begin{cases}
\langle v_m^{(k)}(t), w_j^{(2)} \rangle + \lambda_2 a_2(v_m^{(k)}(t), w_j^{(2)}) + \bar{a}_2(t; v_m^{(k)}(t), w_j^{(2)}) \\
= \int_0^t g_2(t-s) a_2(v_m^{(k)}(s), w_j^{(2)})ds + \langle F_{2m}(t), w_j^{(2)} \rangle, \quad 1 \leq j \leq k,
\end{cases}
\] (3.14)
in which
\[
\begin{cases}
\tilde{u}_{0k} = \sum_{j=1}^k \alpha^{(k)}_{mj}w_j^{(1)} \to \tilde{u}_0 \text{ strongly in } H^2 \cap V_1, \\
\tilde{v}_{0k} = \sum_{j=1}^k \beta^{(k)}_{mj}w_j^{(2)} \to \tilde{v}_0 \text{ strongly in } H^2 \cap V_2.
\end{cases}
\] (3.15)
By using the Banach’s contraction principle, we verify that the system (3.13) has a unique solution $c^{(k)}_{mj}(t), d^{(k)}_{mj}(t), 1 \leq j \leq k$ on interval $[0, T]$.

**Step 2.** _A priori estimate._ We put
\[
\begin{align*}
S^{(k)}_m(t) &= \bar{a}_1(t; u_m^{(k)}(t), u_m^{(k)}(t)) + \bar{a}_2(t; v_m^{(k)}(t), v_m^{(k)}(t)) + \left\| \sqrt{\mu_1(t)} \Delta u_m^{(k)}(t) \right\|^2 \\
&+ \left\| \sqrt{\mu_2(t)} \Delta v_m^{(k)}(t) \right\|^2 + \left\| \dot{u}_m^{(k)}(t) \right\|^2_{a_1} + \left\| \dot{v}_m^{(k)}(t) \right\|^2_{a_2}
\end{align*}
\]
Then, it follows from (3.13), (3.15), that

\[
S^{(k)}_m(t) = a_1(0; \bar{u}_{0k}, \bar{v}_{0k}) + a_2(0; \bar{v}_{0k}, \bar{\bar{v}}_{0k}) + \left\| \sqrt{\mu_1(0)} \Delta \bar{u}_{0k} \right\|^2 + \left\| \sqrt{\mu_2(0)} \Delta \bar{v}_{0k} \right\|^2 \tag{3.16}
\]

\[
+ 2(\mu_{1x}(0)\bar{u}_{0kx}, \Delta \bar{u}_{0k}) + 2(\mu_{2x}(0)\bar{v}_{0kx}, \Delta \bar{v}_{0k})
\]

\[
+ \int_0^t \left[ a_1'(s; u^{(k)}_m(s), u^{(k)}_m(s)) + \int_0^1 \mu_1'(x, s) \left| \Delta u^{(k)}_m(x, s) \right|^2 \right] ds
d
\]

\[
+ \int_0^t \left[ a_2'(s; v^{(k)}_m(s), v^{(k)}_m(s)) + \int_0^1 \mu_2'(x, s) \left| \Delta v^{(k)}_m(x, s) \right|^2 \right] ds
d
\]

\[
- 2g_1(0) \int_0^t \left[ \left\| \bar{u}^{(k)}_m(s) \right\|_{a_1}^2 + \left\| \Delta u^{(k)}_m(s) \right\|^2 \right] ds
\]

\[
- 2g_2(0) \int_0^t \left[ \left\| \bar{v}^{(k)}_m(s) \right\|_{a_2}^2 + \left\| \Delta v^{(k)}_m(s) \right\|^2 \right] ds
\]

\[
+ \int_0^t g_1(t - s) \left[ 2a_1(u^{(k)}_m(s), u^{(k)}_m(t)) + \left\langle \Delta u^{(k)}_m(s), 2\Delta u^{(k)}_m(t) + \Delta \bar{u}^{(k)}_m(t) \right\rangle \right] ds
\]

\[
+ \int_0^t g_2(t - s) \left[ 2a_2(v^{(k)}_m(s), v^{(k)}_m(t)) + \left\langle \Delta v^{(k)}_m(s), 2\Delta v^{(k)}_m(t) + \Delta \bar{v}^{(k)}_m(t) \right\rangle \right] ds
\]

\[
- 2 \int_0^t d\tau \int_0^\tau g_1'(\tau - s) \left[ a_1(u^{(k)}_m(s), u^{(k)}_m(\tau)) + \left\langle \Delta u^{(k)}_m(s), \Delta u^{(k)}_m(\tau) \right\rangle \right] ds
\]

\[
- 2 \int_0^t d\tau \int_0^\tau g_2'(\tau - s) \left[ a_2(v^{(k)}_m(s), v^{(k)}_m(\tau)) + \left\langle \Delta v^{(k)}_m(s), \Delta v^{(k)}_m(\tau) \right\rangle \right] ds
\]

\[
- 2(\mu_{1x}(t)\bar{u}^{(k)}_{mx}(t), \Delta \bar{u}^{(k)}_m(t)) - 2(\mu_{2x}(t)v^{(k)}_{mx}(t), \Delta \bar{v}^{(k)}_m(t))
\]

\[
+ 2 \int_0^t \left[ \frac{\partial}{\partial s} \left( \mu_{1x}(s)u^{(k)}_{mx}(s) \right), \Delta u^{(k)}_m(s) \right] + \left( \frac{\partial}{\partial s} \left( \mu_{2x}(s)v^{(k)}_{mx}(s) \right), \Delta v^{(k)}_m(s) \right] ds
\]

\[
- \frac{\partial}{\partial x} \left( \mu_1(t)u^{(k)}_{mx}(t) \right), \Delta \bar{u}^{(k)}_m(t) \right] - \frac{\partial}{\partial x} \left( \mu_2(t)v^{(k)}_{mx}(t) \right), \Delta \bar{v}^{(k)}_m(t) \right] ds
\]

\[
+ 2 \int_0^t \left[ \{ F_1_m(s), \bar{u}^{(k)}_m(s) - \Delta \bar{u}^{(k)}_m(s) \} + \{ F_2_m(s), \bar{v}^{(k)}_m(s) - \Delta \bar{v}^{(k)}_m(s) \} \right] ds
\]

\[
+ 2 \int_0^t \left[ \{ F_1_m(s), \bar{u}^{(k)}_m(s) - \Delta \bar{u}^{(k)}_m(s) \} + \{ F_2_m(s), \bar{v}^{(k)}_m(s) - \Delta \bar{v}^{(k)}_m(s) \} \right] ds
\]

\[
= a_1(0; \bar{u}_{0k}, \bar{v}_{0k}) + a_2(0; \bar{v}_{0k}, \bar{\bar{v}}_{0k}) + \left\| \sqrt{\mu_1(0)} \Delta \bar{u}_{0k} \right\|^2 + \left\| \sqrt{\mu_2(0)} \Delta \bar{v}_{0k} \right\|^2
\]

\[
+ 2(\mu_{1x}(0)\bar{u}_{0kx}, \Delta \bar{u}_{0k}) + 2(\mu_{2x}(0)\bar{v}_{0kx}, \Delta \bar{v}_{0k}) + \sum_{j=1}^{16} f_j.
\]
We shall estimate the terms $I_j$ on the right-hand side of (3.16) as follows. Using the inequality
\[
S_m^{(k)}(t) \geq \bar{\mu} S_m^{(k)}(t),
\]
where $\bar{\mu} = \min \{1, \mu_1, \mu_2, \lambda_1, \lambda_2\}$ and
\[
\bar{S}_m^{(k)}(t) = \left\| u_m^{(k)}(t) \right\|_{H^2 \cap V_1}^2 + \left\| u_m^{(k)}(t) \right\|_{H^2 \cap V_2}^2 + \left\| \dot{u}_m^{(k)}(t) \right\|_{H^2 \cap V_1}^2 + \left\| \dot{u}_m^{(k)}(t) \right\|_{H^2 \cap V_2}^2 + \int_0^t \left[ \left\| \dot{u}_m^{(k)}(s) \right\|_{H^2 \cap V_1}^2 + \left\| \dot{u}_m^{(k)}(s) \right\|_{H^2 \cap V_2}^2 \right] ds,
\]
and the following inequalities
\[
\bar{a}_1'(t; u_m^{(k)}(t), u_m^{(k)}(t)) \leq \bar{K}_1 \left\| u_m^{(k)}(t) \right\|_{a_1}, \\
\bar{a}_2'(t; v_m^{(k)}(t), v_m^{(k)}(t)) \leq \bar{K}_2 \left\| v_m^{(k)}(t) \right\|_{a_2},
\]
\[
\left| a_1(u_m^{(k)}(s), u_m^{(k)}(\tau)) + \langle \Delta u_m^{(k)}(s), \Delta u_m^{(k)}(\tau) \rangle \right| \leq \sqrt{\bar{S}_m^{(k)}(s)} \sqrt{\bar{S}_m^{(k)}(\tau)}, \\
\left| a_2(v_m^{(k)}(s), v_m^{(k)}(\tau)) + \langle \Delta v_m^{(k)}(s), \Delta v_m^{(k)}(\tau) \rangle \right| \leq \sqrt{\bar{S}_m^{(k)}(s)} \sqrt{\bar{S}_m^{(k)}(\tau)},
\]
we shall estimate respectively the following terms $I_1 - I_8$ on the right-hand side of (3.16) as follows
\[
I_1 + I_2 = \int_0^t \left[ \bar{a}_1'(s; u_m^{(k)}(s), u_m^{(k)}(s)) + \int_0^s \mu_1'(x, s) \left\| \Delta u_m^{(k)}(x, s) \right\| dx \right] ds \\
+ \int_0^t \left[ \bar{a}_2'(s; v_m^{(k)}(s), v_m^{(k)}(s)) + \int_0^s \mu_2'(x, s) \left\| \Delta v_m^{(k)}(x, s) \right\| dx \right] ds \\
\leq \bar{K} \int_0^t \bar{S}_m^{(k)}(s) ds, \quad \bar{K} = \max \{\bar{K}_1, \bar{K}_2\}, \tag{3.19}
\]
\[
I_3 + I_4 = \left[-2g_1(0) \int_0^t \left( \left\| \dot{u}_m^{(k)}(s) \right\|_{a_1}^2 + \left\| \Delta u_m^{(k)}(s) \right\|_{a_1}^2 \right) ds \\
- 2g_2(0) \int_0^t \left( \left\| \dot{v}_m^{(k)}(s) \right\|_{a_2}^2 + \left\| \Delta v_m^{(k)}(s) \right\|_{a_2}^2 \right) ds \right] \\
\leq 2g_{\max}(0) \int_0^t \bar{S}_m^{(k)}(s) ds, \quad g_{\max}(0) = \max_{i=1,2} |g_i(0)|,
\]
\[
I_5 = \int_0^t g_1(t - s) \left[ 2a_1(u_m^{(k)}(s), u_m^{(k)}(t)) + \langle \Delta u_m^{(k)}(s), 2\Delta u_m^{(k)}(t) + \Delta \dot{u}_m^{(k)}(t) \rangle \right] ds \\
\leq \beta \bar{S}_m^{(k)}(t) + \frac{5}{4\beta} T^* \|g_1\|_{L^2(0, T^*)} \int_0^t \bar{S}_m^{(k)}(s) ds,
\]
\[
I_6 = \int_0^t g_2(t - s) \left[ 2a_2(v_m^{(k)}(s), v_m^{(k)}(t)) + \langle \Delta v_m^{(k)}(s), 2\Delta v_m^{(k)}(t) + \Delta \dot{v}_m^{(k)}(t) \rangle \right] ds \\
\leq \beta \bar{S}_m^{(k)}(t) + \frac{5}{4\beta} T^* \|g_2\|_{L^2(0, T^*)} \int_0^t \bar{S}_m^{(k)}(s) ds,
\]
\[
I_7 = -2 \int_0^t d\tau \int_0^\tau g_1'(\tau - s) \left[ a_1(u_m^{(k)}(s), u_m^{(k)}(\tau)) + \langle \Delta u_m^{(k)}(s), \Delta u_m^{(k)}(\tau) \rangle \right] ds
\]
We have the following estimates

\[ \frac{\partial}{\partial x} \left( \mu_1(t) u_{m_1}^{(k)}(t) \right) \leq \frac{\partial}{\partial x} \left( \mu_1(0) \tilde{u}_{0k} \right) + 2 \tilde{K}_1 \int_0^t \tilde{S}_m^{(k)}(s) ds, \]

\[ \frac{\partial}{\partial x} \left( \mu_2(t) v_{m_1}^{(k)}(t) \right) \leq \frac{\partial}{\partial x} \left( \mu_2(0) \tilde{v}_{0k} \right) + 2 \tilde{K}_2 \int_0^t \tilde{S}_m^{(k)}(s) ds, \]

\[ \frac{\partial}{\partial x} \left( \mu_1(t) v_{m_1}^{(k)}(t) \right) \leq \frac{\partial}{\partial x} \left( \mu_1(0) \tilde{v}_{0k} \right) + 2 \tilde{K}_1 \int_0^t \tilde{S}_m^{(k)}(s) ds, \]

\[ \frac{\partial}{\partial x} \left( \mu_2(t) u_{m_1}^{(k)}(t) \right) \leq \frac{\partial}{\partial x} \left( \mu_2(0) \tilde{u}_{0k} \right) + 2 \tilde{K}_2 \int_0^t \tilde{S}_m^{(k)}(s) ds. \]

Using Lemma 3.2, the terms \( I_9 - I_{14} \) are estimated as follows

\[ I_9 = -2 \left( \mu_1(x(t)) u_{m_1}^{(k)}(t), \Delta u_{m_1}^{(k)}(t) \right) \]

\[ \leq 2 \left( \| \mu_1(0) \tilde{u}_{0k} \| + \| \Delta \tilde{u}_{0k} \|^2 \right) + 1 + 2 \tilde{K}_1 \int_0^t \tilde{S}_m^{(k)}(s) ds, \]

\[ I_{10} = -2 \left( \mu_2(x(t)) v_{m_1}^{(k)}(t), \Delta v_{m_1}^{(k)}(t) \right) \]

\[ \leq 2 \left( \| \mu_2(0) \tilde{v}_{0k} \|^2 + \| \Delta \tilde{v}_{0k} \|^2 \right) + 1 + 2 \tilde{K}_2 \int_0^t \tilde{S}_m^{(k)}(s) ds, \]

\[ I_{11} = 2 \int_0^t \left[ \left( \frac{\partial}{\partial x} \left( \mu_1(s) u_{m_1}^{(k)}(s) \right), \Delta u_{m_1}^{(k)}(s) + \left( \frac{\partial}{\partial x} \left( \mu_2(s) v_{m_1}^{(k)}(s) \right), \Delta v_{m_1}^{(k)}(s) \right) \right) \right] ds \]

\[ \leq 4 \sqrt{2} \tilde{K} \int_0^t \tilde{S}_m^{(k)}(s) ds, \]

\[ I_{12} = - \left( \frac{\partial}{\partial x} \left( \mu_1(t) u_{m_1}^{(k)}(t) \right), \Delta \dot{u}_{m_1}^{(k)}(t) \right) \]

\[ \leq \beta \tilde{S}_m^{(k)}(t) + \frac{1}{2 \beta} \left\| \frac{\partial}{\partial x} \left( \mu_1(0) \tilde{u}_{0k} \right) \right\|^2 + \frac{2 \tilde{K}_1}{\beta} \int_0^t \tilde{S}_m^{(k)}(s) ds, \]

\[ I_{13} = - \left( \frac{\partial}{\partial x} \left( \mu_2(t) v_{m_1}^{(k)}(t) \right), \Delta \dot{v}_{m_1}^{(k)}(t) \right) \]

\[ \leq \beta \tilde{S}_m^{(k)}(t) + \frac{1}{2 \beta} \left\| \frac{\partial}{\partial x} \left( \mu_2(0) \tilde{v}_{0k} \right) \right\|^2 + \frac{2 \tilde{K}_2}{\beta} \int_0^t \tilde{S}_m^{(k)}(s) ds, \]

\[ I_{14} = 2 \int_0^t \left[ \left( F_{1m}(s), \dot{u}_{m_1}^{(k)}(s) - \Delta \dot{u}_{m_1}^{(k)}(s) \right) + \left( F_{2m}(s), \dot{v}_{m_1}^{(k)}(s) - \Delta \dot{v}_{m_1}^{(k)}(s) \right) \right] ds \]

\[ \leq 8 T \tilde{K}_M + \int_0^t \tilde{S}_m^{(k)}(s) ds. \]

In order to estimate the term \( I_{15} = \langle F_{1m}(t), -\Delta \dot{u}_{m_1}^{(k)}(t) \rangle \), we note that

\[ I_{15} = \langle F_{1m}(t), -\Delta \dot{u}_{m_1}^{(k)}(t) \rangle \leq \| F_{1m}(t) \| \| \Delta \dot{u}_{m_1}^{(k)}(t) \| \leq \beta \tilde{S}_m^{(k)}(t) + \frac{1}{4 \beta} \| F_{1m}(t) \|^2. \]
with $\beta = \frac{\rho^2}{16}$.

The function $F_{1m}(x, t)$ can be written as follows

$$F_{1m}(x, t) = f_1(x, t, u_{m-1}, v_{m-1}, 0, 0, \nabla u_{m-1}, \nabla v_{m-1}, 0, 0)$$

Applying mean value theorem to the function $f_1$, we obtain that

$$F_{1m}(x, t) - f_1(x, t, u_{m-1}, v_{m-1}, 0, 0, \nabla u_{m-1}, \nabla v_{m-1}, 0, 0)$$

$$= u'_{m-1}D_5f_1[\zeta^*_m] + v'_{m-1}D_6f_1[\zeta^*_m] + \nabla u'_{m-1}D_9f_1[\zeta^*_m] + \nabla v'_{m-1}D_{10}f_1[\zeta^*_m],$$

where $\zeta^*_m = (x, t, u_{m-1}, v_{m-1}, \theta u'_{m-1}, \theta v'_{m-1}, \nabla u_{m-1}, \nabla v_{m-1}, \theta \nabla u'_{m-1}, \theta \nabla v'_{m-1})$, $0 < \theta < 1$.

Note that

$$\max\{\|D_jf_1\|_{C^0(\Omega_M)} : j = 1, 2, 3, 4, 7, 8, \forall M > 0, \forall j = 1, 2, 3, 4, 7, 8, \forall M > 0,$$

hence

$$\|F_{1m}(t) - f_1(\cdot, t, u_{m-1}, v_{m-1}, 0, 0, \nabla u_{m-1}, \nabla v_{m-1}, 0, 0)\|$$

$$\leq 2\sigma \left( \|u'_{m-1}(t)\| + \|v'_{m-1}(t)\| + \|\nabla u'_{m-1}(t)\| + \|\nabla v'_{m-1}(t)\| \right)$$

$$\leq 4\sigma M.$$
\[ \leq \| f_1 (\cdot, 0, \tilde{u}_0, \tilde{v}_0, 0, 0, \tilde{u}_{0x}, \tilde{v}_{0x}, 0, 0) \| + T (1 + 4M) \tilde{K}_M. \]

It follows that
\[ \| F_{1m}(t) \| \leq 4\sigma M + \| f_1 (\cdot, 0, \tilde{u}_0, \tilde{v}_0, 0, 0, \tilde{u}_{0x}, \tilde{v}_{0x}, 0, 0) \| + T (1 + 4M) \tilde{K}_M. \quad (3.27) \]
hence
\[ I_{15} = \langle F_{1m}(t), -\Delta \hat{u}_{m}^{(k)}(t) \rangle \]
\[ = \beta \hat{u}_{m}^{(k)}(t) + \frac{12\sigma^2 M^2}{\beta} + \frac{3}{4\beta} \| f_1 (\cdot, 0, \tilde{u}_0, \tilde{v}_0, 0, 0, \tilde{u}_{0x}, \tilde{v}_{0x}, 0, 0) \|^2 \]
\[ + \frac{3T^2}{4\beta} (1 + 4M)^2 \tilde{K}_M^2. \quad (3.28) \]

Similarly to \( I_{16} \), we have
\[ I_{16} = \langle F_{2m}(t), -\Delta \hat{v}_{m}^{(k)}(t) \rangle \]
\[ \leq \beta \hat{v}_{m}^{(k)}(t) + \frac{12\sigma^2 M^2}{\beta} + \frac{3}{4\beta} \| f_2 (\cdot, 0, \tilde{u}_0, \tilde{v}_0, 0, 0, \tilde{u}_{0x}, \tilde{v}_{0x}, 0, 0) \|^2 \]
\[ + \frac{3T^2}{4\beta} (1 + 4M)^2 \tilde{K}_M^2. \quad (3.29) \]

Combining (3.19), (3.20), (3.28) and (3.29), it implies from (3.16)-(3.18) that
\[ \tilde{S}_{m}^{(k)}(t) \leq \tilde{S}_{0k} + \sigma_* M^2 + T D_1(M) + D_2 \int_0^t \tilde{S}_{m}^{(k)}(s) ds, \quad (3.30) \]

where
\[
\begin{align*}
\sigma_* &= \frac{576}{\mu_*^2}, \\
\tilde{S}_{0k} &= \frac{2}{\mu_*} \left[ \bar{a}_1(0; \tilde{u}_{0k}, \tilde{v}_{0k}) + \tilde{a}_2(0; \tilde{v}_{0k}, \tilde{v}_{0k}) + \left( \sqrt{\mu_1(0)} \Delta \tilde{u}_{0k} \right)^2 + \left( \sqrt{\mu_2(0)} \Delta \tilde{v}_{0k} \right)^2 \right] \\
&+ \frac{4}{\mu_*} \left[ \langle \mu_{1x}(0) \tilde{u}_{0xk}, \Delta \tilde{u}_{0k} \rangle + \langle \mu_{2x}(0) \tilde{v}_{0xk}, \Delta \tilde{v}_{0k} \rangle \right] \\
&+ \frac{4}{\mu_*} \left[ \| \mu_{1x}(0) \tilde{u}_{0xk} \|^2 + \| \mu_{2x}(0) \tilde{v}_{0xk} \|^2 + \| \Delta \tilde{u}_{0k} \|^2 + \| \Delta \tilde{v}_{0k} \|^2 \right] \\
&+ \frac{12}{\mu_*^2} \left( \left\| \frac{\partial}{\partial x} (\mu_1(0) \tilde{u}_{0xk}) \right\|^2 + \left\| \frac{\partial}{\partial x} (\mu_2(0) \tilde{v}_{0xk}) \right\|^2 \right) \\
&+ \frac{18}{\mu_*^2} \left( \| f_1 (\cdot, 0, \tilde{u}_0, \tilde{v}_0, 0, 0, \tilde{u}_{0x}, \tilde{v}_{0x}, 0, 0) \|^2 \right) \\
&+ \| f_2 (\cdot, 0, \tilde{u}_0, \tilde{v}_0, 0, 0, \tilde{u}_{0x}, \tilde{v}_{0x}, 0, 0) \|^2 \right), \\
D_1(M) &= \frac{4}{\mu_*} \left[ 4 + \frac{9}{\mu_*} T^* (1 + 4M)^2 \right] \tilde{K}_M^2, \\
D_2 &= \frac{4}{\mu_*} \left[ g_{\text{max}}(0) + \frac{15T^*}{2\mu_*} \left( \| g_1 \|_{L^2(0,T^*)}^2 + \| g_2 \|_{L^2(0,T^*)}^2 \right) \right] \\
&+ \frac{4\sqrt{T^*}}{\mu_*} \left( \| g_1 \|_{L^2(0,T^*)}^2 + \| g_2 \|_{L^2(0,T^*)}^2 \right) \\
&+ \frac{2}{\mu_*} \left( 5 + (1 + 4\sqrt{2}) \tilde{K} + 8 \left( 1 + \frac{6}{\mu_*} \right) \tilde{K}^2 \right) T^*. \\
\end{align*}
\]
Note that $0 < \sigma < \frac{\mu_* - 4\sigma}{4(7 + 4\sqrt{2})}$, it yields

$$\sigma_* = \frac{576\sigma^2}{\mu_*^2} \leq \frac{36}{(7 + 4\sqrt{2})^2} < 1.$$  \hspace{1cm} (3.32)

The convergence in (3.14) shows that there exists a constant $M > 0$ independent of $k$ and $m$ such that

$$\bar{S}_{0k} \leq \frac{1 - \sigma_*}{2} M^2, \text{ for all } m, k \in \mathbb{N}.$$  \hspace{1cm} (3.33)

By $0 < \sigma < \frac{\mu_* - 4\sigma}{4(7 + 4\sqrt{2})}$, we have the following lemma

**Lemma 3.2.** Let $\gamma = \frac{\mu_* - 4\sigma}{4(7 + 4\sqrt{2})}$. For every $T \in (0, T^*)$, we put

$$k_T = \left(1 + \sqrt{2}\right)\sqrt{\frac{2}{\gamma} \left(\frac{2}{\gamma} T K^2 M + \sigma\right)} \exp\left(\frac{T}{4\gamma} \left[\bar{K} + \frac{T^*}{\gamma} \max_{i=1, 2} \|g_i\|^2_{L^2(0, T^*)}\right]\right).$$  \hspace{1cm} (3.34)

Then, we can choose $T \in (0, T^*)$, such that

1. $\left(1 + \frac{\sigma_*}{2} M^2 + TD_1(M) \right) \exp(TD_2) \leq M^2$,
2. $k_T < 1$. \hspace{1cm} (3.35)

**Proof.** By the fact that the inequality $\sigma < \frac{\mu_* - 4\sigma}{4(7 + 4\sqrt{2})}$ is equivalent to the equality

$$2\sqrt{2}(1 + \sqrt{2})\sqrt{\frac{\sigma}{\mu_* - 4\sigma}} < 1, \text{ and } \sigma_* < 1, \text{ we deduce that}$$

$$\lim_{T \to 0^+} \left(1 + \frac{\sigma_*}{2} M^2 + TD_1(M) \right) \exp(TD_2) = \frac{1 + \sigma_*}{2} M^2 < M^2,$$ \hspace{1cm} (3.36)

and

$$\lim_{T \to 0^+} k_T$$

$$= \lim_{T \to 0^+} \left(1 + \sqrt{2}\right)\sqrt{\frac{2}{\gamma} \left(\frac{2}{\gamma} T K^2 M + \sigma\right)} \exp\left(\frac{T}{4\gamma} \left[\bar{K} + \frac{T^*}{\gamma} \max_{i=1, 2} \|g_i\|^2_{L^2(0, T^*)}\right]\right)$$

$$= \left(1 + \sqrt{2}\right)\sqrt{\frac{\sigma}{\gamma} = 2\sqrt{2}(1 + \sqrt{2})\sqrt{\frac{\sigma}{\mu_* - 4\sigma}} < 1.$$  \hspace{1cm} (3.37)

Therefore, Lemma 3.3 is proved. \hfill \Box

**Proof.** By (3.30), (3.33) and (3.35), we obtain

$$\bar{S}^{(k)}_m(t) \leq M^2 e^{-TD_2} + D_2 \int_0^t \bar{S}^{(k)}_m(s) ds.$$  \hspace{1cm} (3.38)

By using Gronwall’s Lemma, we deduce from (3.38) that

$$\bar{S}^{(k)}_m(t) \leq M^2 e^{-TD_2 e^{TD_2}} \leq M^2,$$ \hspace{1cm} (3.39)

for all $t \in [0, T]$, for all $m, k \in \mathbb{N}$. Therefore, we have

$$(u_m^{(k)}, v_m^{(k)}) \in B_T(M), \text{ for all } m \text{ and } k \in \mathbb{N}.$$ \hspace{1cm} (3.40)
Step 3. Limit procedure. From (3.40), there exists a subsequence of the sequence \((u_m^{(k)}, v_m^{(k)})\), with the same notation, such that

\[
\begin{align*}
(u_m^{(k)}, v_m^{(k)}) & \rightarrow (u_m, v_m) \quad \text{in} \quad L^\infty (0, T; (H^2 \cap V_1) \times (H^2 \cap V_2)) \quad \text{weak}, \\
(\bar{u}_m^{(k)}, \bar{v}_m^{(k)}) & \rightarrow (\bar{u}_m', \bar{v}_m') \quad \text{in} \quad L^\infty (0, T; (H^2 \cap V_1) \times (H^2 \cap V_2)) \quad \text{weak}, \quad (3.41)
\end{align*}
\]

Passing to limit in (3.13), (3.14), we have \((u_m, v_m)\) satisfying (3.10), (3.11) in \(L^2(0, T)\). Theorem 3.1 is proved.

By using Theorem 3.1 and the compact imbedding theorems, we shall prove the existence and uniqueness of weak local solution to the problem (1.1)-(1.3). We first note that the following space

\[
W_1(T) = \{(u, v) \in C^{0}([0, T]; V_1 \times V_2) : (u', v') \in L^2(0, T; V_1 \times V_2)\},
\]

is a Banach space with respect to the norm (see Lions [14])

\[
\|(u, v)\|_{W_1(T)} = \|(u, v)\|_{C^{0}([0, T]; V_1 \times V_2)} + \|(u', v')\|_{L^2(0, T; V_1 \times V_2)}.
\]

Theorem 3.3. Suppose that the hypotheses \((H_1) - (H_4)\) are satisfied. Then, the recurrent sequence \((u_m, v_m)\) defined by (3.9)-(3.11) converges strongly to a function \((u, v)\) in \(W_1(T)\) and \((u, v) \in B_T(M)\) is the unique weak solution of Prob. (1.1)-(1.3). Moreover, we have the following estimate

\[
\|(u_m, v_m) - (u, v)\|_{W_1(T)} \leq C_T k_T^m, \quad \text{for all} \ m \in \mathbb{N},
\]

where \(k_{T} \in [0, 1)\) is defined as in (3.34) and \(C_T\) is a constant depending only on \(T, f_1, f_2, g_1, g_2, \mu_1, \mu_2, \bar{u}_0, \bar{v}_0\) and \(k_{T}\).

Proof of Theorem 3.4. First, we prove the local existence of Prob. (1.1)-(1.3). It is necessary to prove that \((u_m, v_m)\) (in Theorem 3.1) is a Cauchy sequence in \(W_1(T)\). Let \(\bar{u}_m = u_{m+1} - u_m, \bar{v}_m = v_{m+1} - v_m\). Then \((\bar{u}_m, \bar{v}_m)\) satisfies the variational problem

\[
\begin{align*}
(\bar{u}_m(t), \varphi) + \lambda_1 a_1(\bar{u}_m(t), \varphi) + \bar{a}_1(t; \bar{u}_m(t), \varphi) \\
+ \int_0^t g_1(t - s) a_1(\bar{u}_m(s), \varphi) ds + \langle \bar{F}_1(t, \varphi), \rangle, \\
(\bar{v}_m(t), \psi) + \lambda_2 a_2(\bar{v}_m(t), \psi) + \bar{a}_2(t; \bar{v}_m(t), \psi) \\
+ \int_0^t g_2(t - s) a_2(\bar{v}_m(s), \psi) ds + \langle \bar{F}_2(t, \psi), \rangle,
\end{align*}
\]

\(\forall (\varphi, \psi) \in V_1 \times V_2, \text{a.e.} \ t \in (0, T),\) (3.45)

where

\[
\bar{F}_{im}(t) = F_{im+1}(t) - F_{im}(t), \quad i = 1, 2.
\]

Taking \((\varphi, \psi) = (\bar{u}_m'(t), \bar{v}_m'(t))\) in (3.45)\(_{1,2}\) and then integrating in \(t\), we get

\[
S_m(t) = \int_0^t \left[ a_1'(s; \bar{u}_m(s), \bar{u}_m(s)) + a_2'(s; \bar{v}_m(s), \bar{v}_m(s)) \right] ds \quad (3.47)
\]

\[
+ 2 \int_0^t d\tau \int_0^\tau \left[ g_1(\tau - s) a_1(\bar{u}_m(s), \bar{u}_m(\tau)) + g_2(\tau - s) a_2(\bar{v}_m(s), \bar{v}_m(\tau)) \right] ds
\]

\[
+ 2 \int_0^t \left[ \langle \bar{F}_{1m}(s, \bar{u}_m(s)) \rangle + \langle \bar{F}_{2m}(s, \bar{v}_m(s)) \rangle \right] ds
\]

\[
= \sum_{j=1}^3 \bar{I}_j,
\]
where
\[ S_m(t) = \bar{a}_1(t; \bar{u}_m(t), \bar{u}_m(t)) + \bar{a}_2(t; \bar{v}_m(t), \bar{v}_m(t)) \]
(3.48)
\[ + 2 \int_0^t \left( \| \bar{u}'_m(s) \|^2 + \| \bar{v}'_m(s) \|^2 \right) ds \]
\[ + 2\lambda_1 \int_0^t \| \bar{u}'_m(s) \|^2_{a_1} ds + 2\lambda_2 \int_0^t \| \bar{v}'_m(s) \|^2_{a_2} ds. \]

We note that
\[ S_m(t) \geq \bar{\mu}_* \bar{S}_m(t), \tag{3.49} \]
where \( \bar{\mu}_* = \min \{1, \mu_1, \mu_2, \lambda_1, \lambda_2\} \) and
\[ \bar{S}_m(t) = \| \bar{u}_m(t) \|_{a_1}^2 + \| \bar{v}_m(t) \|_{a_2}^2 + \int_0^t \left( \| \bar{u}'_m(s) \|_{a_1}^2 + \| \bar{v}'_m(s) \|_{a_2}^2 \right) ds. \tag{3.50} \]

Next, with \( \gamma = \frac{\bar{\mu}_* - \delta}{\mu} \) as in Lemma 3.3, we have to estimate the integrals on right-hand side of (3.47) as follows
\[ \bar{I}_1 = \int_0^t \left[ a_1 (s; \bar{u}_m(s), \bar{u}_m(s)) + a_2 (s; \bar{v}_m(s), \bar{v}_m(s)) \right] ds \leq \bar{K} \int_0^t \bar{S}_m(s) ds, \tag{3.51} \]
\[ \bar{I}_2 = 2 \int_0^t d\tau \int_0^\tau \left[ g_1 (\tau - s) a_1 (\bar{u}_m(s), \bar{u}'_m(\tau)) + g_2 (\tau - s) a_2 (\bar{v}_m(s), \bar{v}'_m(\tau)) \right] ds \]
\[ \leq \gamma \bar{S}_m(t) + \frac{1}{\gamma} T^* \max_{i=1,2} \| g_i \|_{L^2(0,T^*)} \int_0^t \bar{S}_m(s) ds. \]

With the integral \( \bar{I}_3 \), applying mean value theorem to the function \( f_1 \), we get
\[ \bar{F}_{1m}(t) = F_{1m+1}(t) - F_{1m}(t) \]
\[ = D_3 f_1 (\zeta^*_m(t)) \bar{u}_m-1(t) + D_4 f_1 (\zeta^*_m(t)) \bar{v}_m-1(t) \]
\[ + D_5 f_1 (\zeta^*_m(t)) \bar{u}'_m-1(t) + D_6 f_1 (\zeta^*_m(t)) \bar{v}'_m-1(t) \]
\[ + D_7 f_1 (\zeta^*_m(t)) \nabla \bar{u}_m-1(t) + D_8 f_1 (\zeta^*_m(t)) \nabla \bar{v}_m-1(t) \]
\[ + D_9 f_1 (\zeta^*_m(t)) \nabla \bar{u}'_m-1(t) + D_{10} f_1 (\zeta^*_m(t)) \nabla \bar{v}'_m-1(t), \]
where
\[ \zeta^*_m = (x, t, u_{m-1} + \theta \bar{u}_{m-1}, v_{m-1} + \theta \bar{v}_{m-1}, u'_{m-1} + \theta \bar{u}'_{m-1}, v'_{m-1} + \theta \bar{v}'_{m-1}), \]
\[ \nabla u_{m-1} + \theta \nabla \bar{u}_{m-1}, \nabla v_{m-1} + \theta \nabla \bar{v}_{m-1}, \nabla u'_{m-1} + \theta \nabla \bar{u}'_{m-1}, \nabla v'_{m-1} + \theta \nabla \bar{v}'_{m-1}, \]
\[ 0 < \theta < 1. \]

Thus
\[ \| \bar{F}_{1m}(t) \| \leq \bar{K}_M (f_1) \left( \| \bar{u}_{m-1}(t) \| + \| \bar{v}_{m-1}(t) \| \right) + \sigma \left( \| \bar{u}'_{m-1}(t) \| + \| \bar{v}'_{m-1}(t) \| \right) \]
\[ + \bar{K}_M (f_1) \left( \| \nabla \bar{u}_{m-1}(t) \| + \| \nabla \bar{v}_{m-1}(t) \| \right) \]
\[ + \sigma \left( \| \nabla \bar{u}'_{m-1}(t) \| + \| \nabla \bar{v}'_{m-1}(t) \| \right) \]
\[ \leq 2 \bar{K}_M \| (\bar{u}_{m-1}, \bar{v}_{m-1}) \|_{W^1(T^*)} \]
\[ + 2\sigma \sqrt{2} \left( \| \nabla \bar{u}_{m-1}(t) \|^2 + \| \nabla \bar{v}_{m-1}(t) \|^2 \right)^{1/2}. \tag{3.52} \]

Similarly
\[ \| \bar{F}_{2m}(t) \| \]
Therefore, we deduce from (3.53) that
\[ \leq 2K_M \| (\bar{u}_{m-1}, \bar{v}_{m-1}) \|_{W_1(T)} + 2\sigma \sqrt{2} \left( \| \nabla \bar{u}'_{m-1}(t) \| + \| \nabla \bar{v}'_{m-1}(t) \| \right) \]  

Hence
\[ I_3 = 2 \int_0^t \left[ (\bar{F}_{1m}(s), \bar{u}'_m(s)) + (\bar{F}_{2m}(s), \bar{v}'_m(s)) \right] ds \]
\[ \leq 2 \int_0^t \left[ \| \bar{F}_{1m}(s) \| \| \bar{u}'_m(s) \| + \| \bar{F}_{2m}(s) \| \| \bar{v}'_m(s) \| \right] ds \]
\[ \leq \frac{8}{\gamma} T K_M^2 \| (\bar{u}_{m-1}, \bar{v}_{m-1}) \|_{W_1(T)}^2 \]
\[ + 4\sigma \int_0^t \left( \| \nabla \bar{u}'_{m-1}(s) \| + \| \nabla \bar{v}'_{m-1}(s) \| \right) ds + (\gamma + 4\sigma) S_m(t). \]
From the estimates for \( I_1, I_2, I_3 \) we deduce that
\[ \bar{S}_m(t) \leq \frac{2}{\gamma} \left( 2\gamma T K_M^2 + \sigma \right) \| (\bar{u}_{m-1}, \bar{v}_{m-1}) \|_{W_1(T)}^2 \]
\[ + \frac{1}{2\gamma} \left( K + \frac{1}{\gamma} \max_{i=1,2} |g_i| \right) \left[ K + \frac{1}{\gamma} \max_{i=1,2} |g_i| \right] \int_0^t \bar{S}_m(s) ds. \]
Using Gronwall’s Lemma, we deduce from (3.55) that
\[ \bar{S}_m(t) \leq \frac{2}{\gamma} \left( 2\gamma T K_M^2 + \sigma \right) \| (\bar{u}_{m-1}, \bar{v}_{m-1}) \|_{W_1(T)}^2 \exp \left( \frac{T}{2\gamma} \left[ K + \frac{1}{\gamma} \max_{i=1,2} |g_i| \right] \right). \]
This deduce that
\[ \| (\bar{u}_m, \bar{v}_m) \|_{W_1(T)} \leq k_T \| (\bar{u}_{m-1}, \bar{v}_{m-1}) \|_{W_1(T)}, \quad \forall m \in \mathbb{N}, \]  
where the constant \( k_T \in [0, 1) \) is defined as in Lemma 3.3, which implies that
\[ \| (u_{m+p}, v_{m+p}) - (u_m, v_m) \|_{W_1(T)} \leq \frac{M}{1 - k_T} k_T^m, \quad \forall m, p \in \mathbb{N}. \]
It follows that \( \{(u_m, v_m)\} \) is a Cauchy sequence in \( W_1(T) \). Then there exists \( (u, v) \in W_1(T) \) such that
\[ (u_m, v_m) \rightarrow (u, v) \text{ strongly in } W_1(T). \]
Note that \( (u_m, v_m) \in B_T(M) \), then there exists a subsequence \( \{(u_{m_j}, v_{m_j})\} \) of \( \{(u_m, v_m)\} \) such that
\[ (u_{m_j}, v_{m_j}) \rightarrow (u, v) \text{ in } L^\infty (0, T; (H^2 \cap V_1) \times (H^2 \cap V_2)) \text{ weakly*}, \]
\[ (u'_{m_j}, v'_{m_j}) \rightarrow (u', v') \text{ in } L^\infty (0, T; (H^2 \cap V_1) \times (H^2 \cap V_2)) \text{ weakly*}, \]
\[ (u, v) \in B_T(M). \]
We note that
\[ \| F_{1m} - f_1 [u, v] \|^2_{L^2(Q_T)} \leq 8 \left( T K_M^2 + 2\sigma^2 \right) \| (u_{m-1} - u, v_{m-1} - v) \|^2_{W_1(T)}. \]
Therefore, we deduce from (3.59), (3.61) that
\[ F_{1m} \rightarrow f_1 [u, v] \text{ strongly in } L^2 (Q_T). \]
Similarly
\[ F_{2m} \rightarrow f_2 [u, v] \text{ strongly in } L^2 (Q_T). \]
Letting \( m = m_j \rightarrow \infty \) in (3.10), (3.11) and using (3.59), (3.60), (3.62), and (3.63), we get that there exists \( (u, v) \in B_T(M) \) satisfying (3.1)-(3.3). The proof of existence is completed.
Proof. It remains to prove the uniqueness. Let \((u_1, v_1), (u_2, v_2) \in B_T(M)\) be two weak solutions of Prob. \((1.1)-(1.3)\). Then \((u, v) = (u_1, v_1) - (u_2, v_2) = (u_1 - u_2, v_1 - v_2)\) satisfies the variational problem

\[
\begin{aligned}
\left\{ \begin{array}{l}
(u'(t), \varphi) + \lambda_1 a_1(u'(t), \varphi) + \bar{a}_1(t; u(t), \varphi) \\
= \int_0^t g_1(t - s) a_1(u(s), \varphi) ds + \langle \bar{F}_1(t), \varphi \rangle, \\
(v'(t), \psi) + \lambda_2 a_2(v'(t), \psi) + \bar{a}_2(t; v(t), \psi) \\
= \int_0^t g_2(t - s) a_2(v(s), \psi) ds + \langle \bar{F}_2(t), \psi \rangle, \\
\forall (\varphi, \psi) \in V_1 \times V_2, \text{ a.e. } t \in (0, T), \\
(u(0), v(0)) = (0, 0),
\end{array} \right.
\]

where

\[
\bar{F}_i(t) = f_i[u_1, v_1](t) - f_i[u_2, v_2](t), \quad i = 1, 2.
\]

Taking \((\varphi, \psi) = (u'(t), v'(t))\) in \((3.64)_{1, 2}\) and integrating in time from 0 to \(t\), we get

\[
\bar{\mu}_Z \bar{Z}(t) \leq \int_0^t \left[ \bar{a}_1' s; u(s), u(s) \right] + \bar{a}_2'(s; v(s), v(s)) \right] ds
\]

\[
+ 2 \int_0^t \int_0^r \left[ g_1(\tau - s) a_1(u(s), u'(\tau)) + g_2(\tau - s) a_2(v(s), v'(\tau)) \right] ds
\]

\[
+ 2 \int_0^t \left[ \langle \bar{F}_1(s), u'(s) \rangle + \langle \bar{F}_2(s), v'(s) \rangle \right] ds,
\]

where

\[
\bar{Z}(t) = \|u(t)\|_{a_1}^2 + \|v(t)\|_{a_2}^2 + \int_0^t \left( \|u'(s)\|_{a_1}^2 + \|v'(s)\|_{a_2}^2 \right) ds.
\]

Put \(\gamma = \frac{1}{\gamma} (K + \frac{2}{\gamma} K_{\gamma}^2 + \frac{1}{\gamma} T^* \max_{i=1,2} \|g_i\|_{L_2((0, T)^2)}\), with \(\bar{\gamma} = \frac{\bar{\mu}_s - 16\alpha}{2} > 0\), from \((3.66), (3.67)\), we obtain

\[
\bar{Z}(t) \leq \bar{\gamma} \int_0^t \bar{Z}(s) ds.
\]

Using Gronwall lemma, it leads to \(\bar{Z}(t) \equiv 0\), i.e., \((u, v) = (u_1, v_1) - (u_2, v_2) = 0\).

The uniqueness is proved. Therefore, Theorem 3.4 is proved completely. \(\square\)

**Remark 2.** We have the following remark for the sequence \(\{(u_m, v_m)\}\) defined by \((3.9)-(3.11)\), as in the statement of Theorems 3.1, 3.4.

In the case \((f_1[0, 0], f_2[0, 0]) \neq (0, 0)\), the non-triviality of \((u_m, v_m)\) can be obtained, in spite of \((\bar{u}_0, \bar{v}_0) = (0, 0)\). This is verified as follows.

We first note that, if \((f_1[0, 0], f_2[0, 0]) \neq (0, 0)\) then the weak solution \((u, v)\) of Prob. \((1.1)-(1.3)\) obtained in Theorem 3.4 is not equal to \((0, 0)\). Indeed, if \((u, v) = (0, 0)\) then we derive from \((3.10)_{1, 2}\) that

\[
\langle f_1[0, 0](t), \varphi \rangle = \langle f_2[0, 0](t), \psi \rangle = 0, \text{ for all } (\varphi, \psi) \in V_1 \times V_2 \text{ and a.e. } t \in (0, T).
\]

Therefore, \((f_1[0, 0], f_2[0, 0]) = (0, 0)\), it is in contradiction with the above assumption.

Now, suppose, by contradiction that, \((u_m, v_m) = (0, 0)\), for all \(m = 1, 2, \ldots\). Then, the sequence \(\{(u_m, v_m)\}\) converges to \((0, 0)\), it means that the weak solution \((u, v)\) is equal to \((0, 0)\). This is a contradiction.

Therefore, we conclude that \((u_m, v_m) \neq (0, 0)\), for all \(m = 1, 2, \ldots\).
4. General decay of the solution. In this section, Prob. (1.1)-(1.3) is considered with 
\[ f_i(x, t, u, v, u_t, v_t, u_x, v_x, u_{xt}, v_{xt}) = f_i(u, v), \] 
and \( t > 0 \), then it becomes

\[
\begin{align*}
\nu_t - \lambda_1 \nu_{txx} - & \frac{\partial}{\partial x} \left( \mu_1(x, t) \nu_x \right) + \int_0^t g_1(t-s) \nu_{xx}(x, s) \, ds \\
= & \, f_1(u, v) + F_1(x, t), 0 < x < 1, t > 0, \\
\nu_t - \lambda_2 \nu_{txx} - & \frac{\partial}{\partial x} \left( \mu_2(x, t) \nu_x \right) + \int_0^t g_2(t-s) \nu_{xx}(x, s) \, ds \\
= & \, f_2(u, v) + F_2(x, t), 0 < x < 1, t > 0, \\
\nu_{x}(0, t) &= u(1, t) = v(0, t) = v_x(1, t) + h_1 v_1(1, t) = 0, \\
\nu(0, 0) = & \nu(0, 0) = (\tilde{u}_0(x), \tilde{v}_0(x)),
\end{align*}
\]

where \( h_0 \geq 0, h_1 \geq 0; \lambda_1, \lambda_2 > 0 \) are given constants and \( \mu_i, g_i, f_i, F_i, (i = 1, 2) \), 
\( \tilde{u}_0, \tilde{v}_0 \) are given functions satisfying conditions specified later.

a. Local existence and uniqueness

Based on the results obtained in Section 3, we can propose the following assumptions to get the local existence and uniqueness of a weak solution for Prob. (4.1).

(\( \tilde{H}_1 \)) \((\tilde{u}_0, \tilde{v}_0) \in V_1 \times V_2; \)

(\( \tilde{H}_2 \)) \( \mu_1, \mu_2 \in C^1([0, 1] \times \mathbb{R}_+), \) and there exist the positive constants \( \mu_{1*}, \mu_{2*} \) such that \( \mu_i(x, t) \geq \mu_{i*}, \forall (x, t) \in [0, 1] \times \mathbb{R}_+, i = 1, 2; \)

(\( \tilde{H}_3 \)) \( g_i \in C^1(\mathbb{R}_+; \mathbb{R}_+) \cap L^1(\mathbb{R}_+); \)

(\( \tilde{H}_4 \)) There exist the function \( F \in C^2(\mathbb{R}^2; \mathbb{R}) \) and the positive constants \( \tilde{d}_2 > 0, \alpha > 2, \beta > 2, \) such that

(i) \( \frac{\partial F}{\partial u} = f_1(u, v), \frac{\partial F}{\partial v} = f_2(u, v), \forall (u, v) \in \mathbb{R}^2, \)

(ii) \( F(u, v) \leq \tilde{d}_2 \left(1 + |u|^\alpha + |v|^\beta \right), \forall (u, v) \in \mathbb{R}^2; \)

(\( \tilde{H}_5 \)) \( F_i \in L^2(\mathbb{R}_+; L^2), i = 1, 2. \)

Combining Theorem 3.4 and using the standard arguments of density, we obtain the following theorem.

Theorem 4.1. Let \((\tilde{H}_1)-(\tilde{H}_5)\) hold. Then, there exist \( T > 0 \) and a unique solution of Prob. (4.1) such that

\[ (u, v) \in C^0([0, T]; V_1 \times V_2), \quad (u', v') \in L^2(0, T; V_1 \times V_2). \]

b. Global existence and General decay of the solution

In the following, we prove that if \( \tilde{a}_1 ⊕ \tilde{a}_2 + 2 \tilde{a}_2 - p \int_0^1 F(\tilde{u}_0(x), \tilde{v}_0(x)) \, dx \)

> 0, with \( p > 2, \) and if the initial energy and \( \|F_1(t)\|^2 + \|F_2(t)\|^2 \) are small enough, then global existence is obtained and the energy of the solution decays as \( t \to +\infty \).

For this purpose, we strengthen the following assumptions.

(\( \tilde{H}_{1\infty} \)) \((\tilde{u}_0, \tilde{v}_0) \in V_1 \times V_2; \)

(\( \tilde{H}_{2\infty} \)) \( \mu_1, \mu_2 \in C^1([0, 1] \times \mathbb{R}_+), \) and there exist the positive constants \( \mu_{1*}, \mu_{2*} \) such that \( \mu_i(x, t) \geq \mu_{i*}, \forall (x, t) \in [0, 1] \times \mathbb{R}_+, i = 1, 2; \)

(\( \tilde{H}_{3\infty} \)) \( g_i \in C^1(\mathbb{R}_+; L^1(\mathbb{R}_+)); \)

and there exist the function \( \zeta \in C^1(\mathbb{R}_+) \) such that

(i) \( \zeta'(t) \leq 0 < \zeta(t), \forall t \geq 0, \int_0^\infty \zeta(t) \, dt = +\infty, \)

(ii) \( g_i'(t) \leq -\zeta(t) g_i(t), 0 < g_i(t) \leq g_i(0), \forall t \geq 0, i = 1, 2, \)

(iii) \( L_i \equiv \mu_i \int_0^\infty g_i(s) \, ds > 0, i = 1, 2; \)
We give an example of the functions $F \in C^2(\mathbb{R}^2;\mathbb{R})$ and the positive constants $d_2 > p$, $d_2 > 0$, $q_i > 2$, $q_i > 2$, $(i = 1, \ldots, N)$, such that

(i) $\frac{\partial F}{\partial u} = f_1(u, v)$, $\frac{\partial F}{\partial v} = f_2(u, v)$, $\forall (u, v) \in \mathbb{R}^2$, $i = 1, 2$,

(ii) $uf_1(u, v) + vf_2(u, v) \leq d_2 F(u, v)$, $\forall (u, v) \in \mathbb{R}^2$,

(iii) $\mathcal{F}(u, v) \leq d_2 \sum_{i=1}^{N} (|u|^{q_i} + |v|^{q_i})$, $\forall (u, v) \in \mathbb{R}^2$.

$(\tilde{H}_3^\infty)$ $F \in L^2(\mathbb{R}^+; L^2) \cap L^1(\mathbb{R}^+; L^2)$ such that there exist two constants $\tilde{C}_1 > 0$, $\tilde{\gamma}_1 > 0$, satisfying $\|F_1(t)\|^2 + \|F_2(t)\|^2 \leq \tilde{C}_1 e^{-\tilde{\gamma}_1 t}$, $\forall t \geq 0$.

**Remark 3.** We give an example of the functions $g_1(t)$, $g_2(t)$, $f_1(u, v)$, $f_2(u, v)$ satisfying $(\tilde{H}_3^\infty)$, $(\tilde{H}_4^\infty)$ as below

$g_i(t) = \sigma_i \exp \left(-\int_0^t \zeta(s) ds\right)$, $i = 1, 2$,

$f_1(u, v) = \alpha k_1 |u|^{\alpha - 2} u + \alpha_1 k_3 |u|^{\alpha_1 - 2} u |v|^{\beta_1}$,

$f_2(u, v) = \beta k_2 |v|^{\beta - 2} v + \beta_1 k_3 |u|^{\alpha_1} |v|^{\beta_1 - 2} v$,

where $\sigma_i, \alpha, k_1, k_2, k_3 > 0$, $\alpha, \beta, \alpha_1, \beta_1 > 2$, are constants, $\zeta \in C^1(\mathbb{R}_+)$ such that $\zeta'(t) \leq 0 < \zeta(t)$, $\forall t \geq 0$. $\int_0^\infty \zeta(t) dt = +\infty$. It is obvious that $(\tilde{H}_3^\infty)$ holds, because of the following

$g'_i(t) = -\sigma_i \zeta(t) \exp \left(-\int_0^t \zeta(s) ds\right) = -\zeta(t) g_i(t)$, $i = 1, 2$.

Assumption $(\tilde{H}_4^\infty)$ also holds, indeed, there exists the function $F \in C^2(\mathbb{R}^2;\mathbb{R})$ defined by

$\mathcal{F}(u, v) = k_1 |u|^{\alpha} + k_2 |v|^{\beta} + k_3 |u|^{\alpha_1} |v|^{\beta_1}$,

satisfying

$D_1 \mathcal{F}(u, v) = f_1(u, v)$, $D_2 \mathcal{F}(u, v) = f_2(u, v)$, $\forall (u, v) \in \mathbb{R}^2$,

$uf_1(u, v) + vf_2(u, v) \leq d_2 \mathcal{F}(u, v)$, $\forall (u, v) \in \mathbb{R}^2$,

where $d_2 = \max \{\alpha, \beta, \alpha_1 + \beta_1\}$.

Moreover, using the inequality $|u|^{\alpha_1} |v|^{\beta_1} \leq \frac{|u|^{2\alpha_1} + |v|^{2\beta_1}}{2}$, we have

$\mathcal{F}(u, v) \leq \tilde{d}_2 \left(|u|^{\alpha} + |u|^{2\alpha_1} + |v|^{\beta} + |v|^{2\beta_1}\right) = \tilde{d}_2 \sum_{i=1}^{2} \left(|u|^{q_i} + |v|^{q_i}\right)$, $\forall (u, v) \in \mathbb{R}^2$,

where $\tilde{d}_2 = \max \{k_1, k_2, k_3\}$, $q_1 = \alpha$, $q_2 = 2\alpha_1$, $q_1 = \beta$, $q_2 = 2\beta_1$.

Consider the Lyapunov functional defined as follows

$L(t) = E(t) + \Psi(t)$, \hspace{1cm} (4.4)

where

$\Psi(t) = \frac{1}{2} \|u(t)\|^2 + \frac{1}{2} \|v(t)\|^2 + \frac{\lambda_1}{2} \|u(t)\|^2_{a_1} + \frac{\lambda_2}{2} \|v(t)\|^2_{a_2}$, \hspace{1cm} (4.5)

$E(t) = \frac{1}{2} \left(\tilde{a}_1(t; u(t), u(t)) - \tilde{g}_1(t) \|u(t)\|^2_{a_1}\right)$

$+ \frac{1}{2} \left(\tilde{a}_2(t; v(t), v(t)) - \tilde{g}_2(t) \|v(t)\|^2_{a_2}\right)$

$+ \frac{1}{2} (g_1 * u)(t) + \frac{1}{p} (g_2 * v)(t) - \mathcal{F}(t)$,

$= \left(\frac{1}{2} - \frac{1}{p}\right) E(t) + \frac{1}{p} I(t)$.
Proof. On the other hand we obtain
\[
E(t) = E(0) + \int_0^t (\tilde{g}_1(t) ||u(t)||_{a_1}^2 + \tilde{g}_2(t) ||v(t)||_{a_2}^2 + (g_1 \ast u)(t) + (g_2 \ast v)(t)),
\]
then it follows from (4.8) that the inequality (4.7) is valid. Lemma 4.2 is proved.

We shall use the inequality (4.7) in Lemma 4.2 to prove the following lemma related to the global existence.
Lemma 4.3. Assume that \((\tilde{H}_1^\infty) - (\tilde{H}_5^\infty)\) hold. Let \((\tilde{u}_0, \tilde{v}_0) \in V_1 \times V_2\) such that \(I(0) > 0\) and the initial energy \(E(0)\) satisfy
\[
\eta_* = L_* - p\bar{d}_2 \max \left\{ \sum_{i=1}^{N} R_i^{\eta_* - 2}, \sum_{i=1}^{N} R_i^{\bar{g}_i - 2} \right\} > \max_{i=1,2} \left\{ \mu_i^* - \frac{p}{d_2} \mu_i^* \right\},
\]
\[
0 < \bar{g}_i(\infty) < \frac{d_2}{p} \left[ \eta_* - \left( \mu_i^* - \frac{p}{d_2} \mu_i^* \right) \right], \quad i = 1, 2,
\]
where
\[
R_* = \sqrt{\frac{2pE_*}{(p-2)L_*}}, \quad E_* = E(0) + \frac{1}{2} \int_0^\infty \left( \|F_1(t)\|^2 + \|F_2(t)\|^2 \right) dt,
\]
\[
\bar{g}_i(\infty) = \int_0^\infty \bar{g}_i(s) ds, \quad \mu_i^* = \max_{0 \leq x \leq 1} \mu_i(x, 0), \quad i = 1, 2,
\]
\[
L_* = \min \{ L_1, L_2 \}, \quad L_i \equiv \mu_i - \bar{g}_i(\infty) > 0, \quad i = 1, 2,
\]
\[
D_i = \sup_{0 \neq \varphi \in V_1} \frac{\|v\|^2_{L^\alpha}}{\|v\|^2_{a_1}}, \quad \bar{D}_i = \sup_{0 \neq \varphi \in V_2} \frac{\|v\|^2_{L^\alpha}}{\|v\|^2_{a_2}}, \quad i = 1, \cdots, N.
\]
Then \(I(t) > 0\), for all \(t \geq 0\).

Proof. By the continuity of \(I(t)\) and
\[
I(0) = \bar{a}_1(0; \tilde{u}_0, \tilde{v}_0) + \bar{a}_2(0; \tilde{v}_0, \tilde{v}_0) - p\bar{F}(0) > 0,
\]
there exists \(T_1 > 0\) such that
\[
I(t) = I(u(t)) > 0, \quad \forall t \in [0, T_1].
\]
From (4.5), (4.12), we get
\[
E(t) \geq \left( \frac{1}{2} - \frac{1}{p} \right) \dot{E}(t)
\]
\[
\geq \frac{(p-2)}{2p} L_* \left( \|u(t)\|^2_{a_1} + \|v(t)\|^2_{a_2} \right) + (g_1 * u)(t) + (g_2 * v)(t)
\]
\[
\geq \frac{(p-2)L_*}{2p} \left( \|u(t)\|^2_{a_1} + \|v(t)\|^2_{a_2} \right), \quad \forall t \in [0, T_1].
\]
On the other hand, it follows from (4.7) that
\[
E(t) \leq E(0) + \frac{1}{2} \int_0^\infty \left( \|F_1(s)\|^2 + \|F_2(s)\|^2 \right) ds = E_*.
\]
Therefore, from (4.13), we deduce that
\[
\|u(t)\|^2_{a_1} + \|v(t)\|^2_{a_2} \leq \frac{2p}{p-2} E(t) \leq \frac{2pE_*}{(p-2)L_*} \equiv R_*^2, \quad \forall t \in [0, T_1].
\]
Since \((\tilde{H}_1)_\{iii\}\) and \((4.15)\), we have
\[
p\bar{F}(t) = \int_0^t F(u(x, t), v(x, t)) dx \leq p\bar{d}_2 \sum_{i=1}^{N} \left( \|u(t)\|^2_{L^\alpha_i} + \|v(t)\|^2_{L^\alpha_i} \right)
\]
\[
\leq p\bar{d}_2 \max \left\{ \sum_{i=1}^{N} R_i^{\eta_* - 2}, \sum_{i=1}^{N} R_i^{\bar{g}_i - 2} \right\} \left( \|u(t)\|^2_{a_1} + \|v(t)\|^2_{a_2} \right).
\]
Therefore
\[
I(t) \geq (g_1 * u)(t) + (g_2 * v)(t) + \eta_* \left( \|u(t)\|^2_{a_1} + \|v(t)\|^2_{a_2} \right) \geq 0, \quad \forall t \in [0, T_1].
\]
where the constant $\eta_* > 0$ is defined as in (4.10).

Put $T_\infty = \sup \{ T_1 > 0 : I(t) > 0, \forall t \in [0, T_1] \}$, we need to show that $T_\infty = +\infty$.

Suppose that $T_\infty < +\infty$, by the continuity of $I(t)$, we have $I(T_\infty) \geq 0$. But, one cannot have $I(T_\infty) = 0$. Indeed, if $I(T_\infty) = 0$, by (4.17) we obtain $\langle g_1 \ast u \rangle(T_\infty) + \langle g_2 \ast v \rangle(T_\infty) = \| u \|^2_{a_1} = \| v \|^2_{a_2} = 0$, which leads to $\int_{0}^{T_\infty} g_1(T_\infty - s) \| u(s) \|^2_{a_1} ds = 0$, by the continuity of function $s \mapsto g_1(T_\infty - s) \| u(s) \|^2_{a_1}$ on $[0, T_\infty]$ and $g_1(T_\infty - s) > 0, \forall s \in [0, T_\infty]$, we imply that $\| u(s) \| = 0, \forall s \in [0, T_\infty]$, i.e., $u(0) = 0$. Similarly $v(0) = 0$, so $I(0) = 0 < I(0)$. This is a contradiction, so $I(T_\infty) > 0$. By the same arguments as above, we can deduce that there exists $T_\infty > T$ such that $I(t) > 0$ for all $t \in [0, T_\infty]$. This is a contradiction to the definition of $T_\infty$.

Thus $T_\infty = +\infty$, i.e. $I(t) > 0$, $\forall t \geq 0$. Lemma 4.3 is proved. \hfill $\square$

Next, we establish the decay of the solution of (4.1). For this goal, we put

$$
E_1(t) = (g_1 \ast u)(t) + \langle g_2 \ast v \rangle(t) + \| u(t) \|^2_{a_1} + \| v(t) \|^2_{a_2} + I(t),
$$

$$
\tilde{E}_1(t) = (g_1 \ast u)(t) + \langle g_2 \ast v \rangle(t) + \| u(t) \|^2_{a_1} + \| v(t) \|^2_{a_2},
$$

and we prove two lemmas (Lemmas 4.4, 4.5) below.

**Lemma 4.4.** There exist the positive constants $\beta_1, \beta_2, \beta_1^*, \beta_2^*$ such that

(i) $\beta_1 E_1(t) \leq \mathcal{L}(t) \leq \beta_2 E_1(t), \forall t \geq 0,$

(ii) $\beta_1^* \tilde{E}_1(t) \leq \tilde{E}(t) \leq \beta_2^* \tilde{E}_1(t), \forall t \geq 0.$

(iii) $\beta_1^* E(t) \leq \mathcal{L}(t) \leq \beta_2^* E(t), \forall t \geq 0.$

**Proof.** (i) It is not difficult to see that

$$
\mathcal{L}(t) \geq E(t) = \left( \frac{1}{2} - \frac{1}{p} \right) \tilde{E}(t) + \frac{1}{p} I(t)
$$

$$
\geq \left( \frac{1}{2} - \frac{1}{p} \right) \left[ L_* \left( \| u(t) \|^2_{a_1} + \| v(t) \|^2_{a_2} \right) + (g_1 \ast u)(t) + \langle g_2 \ast v \rangle(t) \right] + \frac{1}{p} I(t)
$$

$$
\geq \beta_1 E_1(t),
$$

where $\beta_1 = \min \left\{ \frac{(p-2) L_*}{2p}, \frac{(p-2)}{2p}, \frac{1}{p} \right\}.$

Similarly, it follows from

$$
0 < \mu_i(x, t) \leq \mu_i(x, 0) \leq \max_{0 \leq x \leq 1} \mu_i(x, 0) = \mu^*_i \leq \mu^* \equiv \max_{i=1, 2} \mu^*_i \equiv \mu^*,
$$

that

$$
\mathcal{L}(t) = \left( \frac{1}{2} - \frac{1}{p} \right) \tilde{E}(t) + \frac{1}{p} I(t) + \frac{1}{2} \| u(t) \|^2 + \frac{1}{2} \| v(t) \|^2 + \frac{\lambda_1}{2} \| u \|^2_{a_1} + \frac{\lambda_2}{2} \| v \|^2_{a_2}
$$

$$
\leq \left[ \frac{1}{2} - \frac{1}{p} \right] \mu^* + \frac{1 + \lambda^*}{2} \left( \| u(t) \|^2_{a_1} + \| v(t) \|^2_{a_2} \right)
$$

$$
+ \left( \frac{1}{2} - \frac{1}{p} \right) \left[ (g_1 \ast u)(t) + \langle g_2 \ast v \rangle(t) \right] + \frac{1}{p} I(t)
$$

$$
\leq \beta_2 E_1(t),
$$

where $\beta_2 = \left( \frac{1}{2} - \frac{1}{p} \right) \mu^* + \frac{1 + \lambda^*}{2}, \lambda^* = \max \{ \lambda_1, \lambda_2 \}.$

(ii) By $\mu^*_i(x, t) \leq 0, \forall t \geq 0$, we have

$$
| \mu_i(x, t) | = \mu_i(x, t) \leq \mu_i(x, 0) \leq \max_{0 \leq x \leq 1} \mu_i(x, 0) \equiv \mu^*_i,
$$
\[ \alpha_1(t; u(t), u(t)) - \bar{\alpha}_1(t) \| u(t) \|_{a_1}^2 + \alpha_2(t; v(t), v(t)) - \bar{\alpha}_2(t) \| v(t) \|_{a_2}^2 \leq \alpha_1(t; u(t), u(t)) + \bar{\alpha}_2(t; v(t), v(t)) \leq \mu^* \left( \| u(t) \|_{a_1}^2 + \| v(t) \|_{a_2}^2 \right), \]

where \( \mu^* = \max_{i=1,2} \mu_i^* \).

This implies
\[ \tilde{E}(t) \leq \alpha_1(t; u(t), u(t)) + \bar{\alpha}_2(t; v(t), v(t)) + (g_1 * u)(t) + (g_2 * v)(t) \leq \max \{ 1, \mu^* \} \tilde{E}_1(t) = \tilde{\beta}_2 \tilde{E}_1(t). \]

On the other hand
\[ \tilde{E}(t) = \left( \alpha_1(t; u(t), u(t)) - \bar{\alpha}_1(t) \| u(t) \|_{a_1} \right) + \left( \bar{\alpha}_2(t; v(t), v(t)) - \tilde{\alpha}_2(t) \| v(t) \|_{a_2} \right) + (g_1 * u)(t) + (g_2 * v)(t) \geq L_\alpha \left( \| u(t) \|_{a_1}^2 + \| v(t) \|_{a_2}^2 \right) + (g_1 * u)(t) + (g_2 * v)(t) \geq \min \{ 1, L_\alpha \} \tilde{E}_1(t) = \tilde{\beta}_1 \tilde{E}_1(t). \]

(iii) We have
\[ E(t) = \left( \frac{1}{2} - \frac{1}{p} \right) \tilde{E}(t) + \frac{1}{p} I(t) \geq \left( \frac{1}{2} - \frac{1}{p} \right) \tilde{\beta}_1 \tilde{E}_1(t) + \frac{1}{p} I(t) \geq \min \left\{ \left( \frac{1}{2} - \frac{1}{p} \right) \tilde{\beta}_1, \frac{1}{p} \right\} E_1(t) \geq \frac{1}{\tilde{\beta}_2} \min \left\{ \left( \frac{1}{2} - \frac{1}{p} \right) \tilde{\beta}_1, \frac{1}{p} \right\} \mathcal{L}(t) = \frac{1}{\tilde{\beta}_2} \mathcal{L}(t), \]

and
\[ E(t) = \left( \frac{1}{2} - \frac{1}{p} \right) \tilde{E}(t) + \frac{1}{p} I(t) \leq \left( \frac{1}{2} - \frac{1}{p} \right) \tilde{\beta}_2 \tilde{E}_1(t) + \frac{1}{p} I(t) \leq \max \left\{ \left( \frac{1}{2} - \frac{1}{p} \right) \tilde{\beta}_2, \frac{1}{p} \right\} E_1(t) \leq \frac{1}{\tilde{\beta}_1} \max \left\{ \left( \frac{1}{2} - \frac{1}{p} \right) \tilde{\beta}_2, \frac{1}{p} \right\} \mathcal{L}(t) = \frac{1}{\tilde{\beta}_1} \mathcal{L}(t). \]

Lemma 4.4 is proved. \( \square \)

**Lemma 4.5.** The functional \( \Psi(t) \) satisfies the following estimation
\[
\Psi'(t) \leq - \left[ \mu_1* - \frac{\varepsilon_1}{2} - \frac{d_2}{p} \left[ \mu_2* - (1 - \varepsilon_2) \eta_2 \right] - \left( \frac{\varepsilon_2}{2} + 1 \right) \bar{g}_1(\infty) \right] \| u(t) \|_{a_1}^2 - \left[ \mu_2* - \frac{\varepsilon_1}{2} - \frac{d_2}{p} \left[ \mu_2* - (1 - \varepsilon_2) \eta_2 \right] - \left( \frac{\varepsilon_2}{2} + 1 \right) \bar{g}_2(\infty) \right] \| v(t) \|_{a_2}^2 \\
+ \frac{d_2}{p} + \frac{1}{2 \varepsilon_2} \right] \| (g_1 * u)(t) + (g_2 * v)(t) \| - \frac{\varepsilon_2 d_2}{p} I(t) + \frac{1}{2 \varepsilon_1} \rho_1(t),
\]
\forall \varepsilon_1, \varepsilon_2 \in (0, 1),

(4.21)

where \( \rho_1(t) = \| F_1(t) \|^2 + \| F_2(t) \|^2 \).
Proof. By multiplying (4.1) by \((u(x,t), v(x,t))\) and integrating over \((0, 1)\), we obtain
\[
\Psi'(t) = -\bar{a}_1(t; u(t), u(t)) - \bar{a}_2(t; v(t), v(t))
\]
\[
+ \int_0^t g_1(t - s)a_1(u(s), u(t))ds + \int_0^t g_2(t - s)a_2(v(s), v(t))ds
\]
\[
+ \langle f_1(u(t), v(t)), u(t) \rangle + \langle f_2(u(t), v(t)), v(t) \rangle
\]
\[
+ \langle F_1(t), u(t) \rangle + \langle F_2(t), v(t) \rangle.
\]
Note that
\[
-\bar{a}_1(t; u(t), u(t)) - \bar{a}_2(t; v(t), v(t)) \leq -\mu_{\bar{a}_1} \|u(t)\|_{a_1}^2 - \mu_{\bar{a}_2} \|v(t)\|_{a_2}^2,
\]
\[
\int_0^t g_1(t - s)a_1(u(s), u(t))ds \leq \frac{1}{2\epsilon_2} (g_1 * u)(t) + \left(\frac{\epsilon_2}{2} + 1\right) \overline{g}_1(\infty) \|u(t)\|_{a_1}^2,
\]
\[
\int_0^t g_2(t - s)a_2(v(s), v(t))ds \leq \frac{1}{2\epsilon_2} (g_2 * v)(t) + \left(\frac{\epsilon_2}{2} + 1\right) \overline{g}_2(\infty) \|v(t)\|_{a_2}^2,
\]
\[
\langle F_1(t), u(t) \rangle + \langle F_2(t), v(t) \rangle \leq \frac{\epsilon_1}{2} \left[\|u(t)\|_{a_1}^2 + \|v(t)\|_{a_2}^2\right] + \frac{1}{2\epsilon_1} \rho_1(t),
\]
\[
\langle f_1(u(t), v(t)), u(t) \rangle + \langle f_2(u(t), v(t)), v(t) \rangle
\]
\[
\leq d_2 \mathcal{F}(t) = \frac{d_2}{p} \left[\dot{E}(t) - I(t)\right] = \frac{d_2}{p} \dot{E}(t) - (1 - \epsilon_2) \frac{d_2}{p} I(t) - \frac{\epsilon_2 d_2}{p} I(t)
\]
\[
\leq \frac{d_2}{p} \left[\dot{E}(t) - (1 - \epsilon_2) \frac{d_2}{p} \eta_\epsilon \left(\|u(t)\|_{a_1}^2 + \|v(t)\|_{a_2}^2\right) - \frac{\epsilon_2 d_2}{p} I(t)\right]
\]
\[
\leq \frac{d_2}{p} \left[\bar{a}_1(t; u(t), u(t)) + \bar{a}_2(t; v(t), v(t)) + (g_1 * u)(t) + (g_2 * v)(t)\right]
\]
\[
- (1 - \epsilon_2) \frac{d_2}{p} \eta_\epsilon \left(\|u(t)\|_{a_1}^2 + \|v(t)\|_{a_2}^2\right) - \frac{\epsilon_2 d_2}{p} I(t)
\]
\[
\leq \frac{d_2}{p} \left[\|u(t)\|_{a_1}^2 + \|v(t)\|_{a_2}^2\right] + \frac{d_2}{p} \left((g_1 * u)(t) + (g_2 * v)(t)\right)
\]
\[
- (1 - \epsilon_2) \frac{d_2}{p} \eta_\epsilon \left(\|u(t)\|_{a_1}^2 + \|v(t)\|_{a_2}^2\right) - \frac{\epsilon_2 d_2}{p} I(t).
\]
Then, it follows from (4.22)-(4.24) that the inequality (4.21) is valid. Lemma 4.5 is proved.

Based on the above results, we can prove the main result in this section as follows.

**Theorem 4.6.** Assume that \((\bar{H}_2^\infty) - (\bar{H}_2^\infty)\) hold. Let \((\bar{u}_0, \bar{v}_0) \in V_1 \times V_2\) such that \(I(0) > 0\) and the initial energy \(E(0)\) satisfy (4.10). Then, there exist positive constants \(\bar{C}, \gamma\) such that
\[
\|u(t)\|_{a_1}^2 + \|v(t)\|_{a_2}^2 \leq \bar{C} \exp \left(-\gamma \int_0^t \zeta(s)ds\right), \forall t \geq 0.
\]

**Proof of Theorem 4.6.** We rewrite (4.21) in Lemma 4.5 as follows
\[
\Psi'(t) \leq -\bar{\theta}_1 \|u(t)\|_{a_1}^2 - \bar{\theta}_2 \|v(t)\|_{a_2}^2 - \frac{\epsilon_2 d_2}{p} I(t)
\]
\[
+ \left(\frac{d_2}{p} + \frac{1}{2\epsilon_2}\right) \left((g_1 * u)(t) + (g_2 * v)(t)\right) + \frac{1}{2\epsilon_1} \rho_1(t),
\]
where
\[ d_3 = \frac{d_2}{p} + \frac{1}{2\varepsilon_2}, \]  
(4.27)
\[ \tilde{\theta}_1 = \tilde{\theta}_1(\varepsilon_1, \varepsilon_2) = \mu_{1*} - \frac{\varepsilon_1}{2} - \frac{d_2 \mu_1^* - (1 - \varepsilon_2)\eta_*}{p} - \left(\frac{\varepsilon_2}{2} + 1\right) \tilde{g}_1(\infty), \]
\[ \tilde{\theta}_2 = \tilde{\theta}_2(\varepsilon_1, \varepsilon_2) = \mu_{2*} - \frac{\varepsilon_1}{2} - \frac{d_2 \mu_2^* - (1 - \varepsilon_2)\eta_*}{p} - \left(\frac{\varepsilon_2}{2} + 1\right) \tilde{g}_2(\infty). \]

By (4.10), (4.27), we deduce that
\[ \lim_{\varepsilon_1 \to 0^+, \varepsilon_2 \to 0^+} \tilde{\theta}_1 = \frac{d_2}{p} \left[ \eta_* - \left( \mu_1^* - \frac{p}{d_2} \mu_{1*} \right) \right] - \tilde{g}_1(\infty) = \tilde{\theta}_1(0, 0) = \tilde{\theta}_{01} > 0, \]
\[ \lim_{\varepsilon_1 \to 0^+, \varepsilon_2 \to 0^+} \tilde{\theta}_2 = \frac{d_2}{p} \left[ \eta_* - \left( \mu_2^* - \frac{p}{d_2} \mu_{2*} \right) \right] - \tilde{g}_2(\infty) = \tilde{\theta}_2(0, 0) = \tilde{\theta}_{02} > 0. \]

Thus, there exists \( \varepsilon_* \in (0, 1) \) such that
\[ \tilde{\theta}_* = \min \{ \tilde{\theta}_1, \tilde{\theta}_2 \} > 0, \]
for all \( \varepsilon_1, \varepsilon_2 \in (0, \varepsilon_*). \) From the following inequality
\[ \bar{E}(t) \leq \mu^* \left( \| u(t) \|_{a_1}^2 + \| v(t) \|_{a_2}^2 \right) + (g_1 * u)(t) + (g_2 * v)(t), \]
we deduce from (4.28), that
\[ \Psi'(t) \leq \tilde{\theta}_* \left[ \bar{E}(t) + (g_1 * u)(t) + (g_2 * v)(t) \right] - \frac{d_2 \varepsilon_2}{p} I(t) \]

\[ + d_3 \left[ (g_1 * u)(t) + (g_2 * v)(t) \right] + \frac{1}{2 \varepsilon_1} \rho_1(t), \]
(4.29)

On the other hand, from the following inequalities
\[ \frac{\tilde{\theta}_*}{\mu^*} \bar{E}(t) + \frac{d_2 \varepsilon_2}{p} I(t) = \frac{\tilde{\theta}_*}{\mu^*} \left( \frac{1}{2} - \frac{1}{p} \right) \bar{E}(t) + d_2 \varepsilon_2 - \frac{1}{p} I(t) \geq \gamma_* \bar{E}(t), \]
(4.30)
with \( \gamma_* = \min \{ \frac{\tilde{\theta}_*}{\mu^*}, \frac{1}{2 \varepsilon_1}, d_2 \varepsilon_2 \}, \) and
\[ E'(t) \leq -\frac{1}{2} \zeta(t) \left[ (g_1 * u)(t) + (g_2 * v)(t) \right] + \frac{1}{2} \rho_1(t), \]
(4.31)
we deduce from (4.29), that
\[ L'(t) = E'(t) + \Psi'(t) \leq -\frac{1}{2} \zeta(t) \left[ (g_1 * u)(t) + (g_2 * v)(t) \right] + \frac{1}{2} \rho_1(t) \]

\[ - \gamma_* \bar{E}(t) + \frac{1}{2} d_4 \left[ (g_1 * u)(t) + (g_2 * v)(t) \right] + \frac{1}{2 \varepsilon_1} \rho_1(t). \]
(4.32)
\[
\leq \frac{1}{2} \left( 1 + \frac{1}{\varepsilon_1} \right) \rho_1(t) - \gamma_s E(t) + \frac{1}{2} d_4 \left( g_1 * u)(t) + (g_2 * v)(t) \right),
\]
with \(d_4 = 2(d_3 + \frac{\beta_3}{\mu_2})\). Thus
\[
\zeta(t) \mathcal{L}'(t) \leq \frac{1}{2} \left( 1 + \frac{1}{\varepsilon_1} \right) \zeta(t) \rho_1(t) - \gamma_s \zeta(t) E(t)
+ \frac{1}{2} d_4 \zeta(t) \left[ (g_1 * u)(t) + (g_2 * v)(t) \right]
\]
\[
\leq \frac{1}{2} \left( 1 + \frac{1}{\varepsilon_1} \right) \zeta(0) \rho_1(t) - \gamma_s \zeta(t) E(t) + \frac{1}{2} d_4 \left[ -2E'(t) + \rho_1(t) \right]
\]
\[
= \frac{1}{2} \left[ (1 + \frac{1}{\varepsilon_1}) \zeta(0) + d_4 \right] \rho_1(t) - \gamma_s \zeta(t) E(t) - d_4 E'(t).
\]

For convenience, we continue to define the new functional
\[
\mathcal{L}_1(t) = \zeta(t) \mathcal{L}(t) + d_4 E(t).
\]

By direct computation, it yields
\[
\mathcal{L}'_1(t) = \zeta'(t) \mathcal{L}(t) + \zeta(t) \mathcal{L}'(t) + d_4 E'(t)
\]
\[
\leq \frac{1}{2} \left[ (1 + \frac{1}{\varepsilon_1}) \zeta(0) + d_4 \right] \rho_1(t) - \gamma_s \zeta(t) E(t)
\]
\[
= \beta_3 \rho_1(t) - \gamma_s \zeta(t) E(t),
\]
with \(\beta_3 = \frac{1}{2}[(1 + \frac{1}{\varepsilon_1}) \zeta(0) + d_4].\)

Note that
\[
\mathcal{L}_1(t) = \zeta(t) \mathcal{L}(t) + d_4 E(t) \leq \left( \zeta(0) \beta_2 + d_4 \right) E(t) = \frac{1}{d_5} E(t),
\]
\[
\rho_1(t) \leq C_1 e^{-\gamma_1 t}, \forall t \geq 0,
\]

hence
\[
\mathcal{L}'_1(t) \leq \beta_3 \bar{C}_1 e^{-\gamma_1 t} - \gamma_s d_5 \zeta(t) \mathcal{L}_1(t).
\]

Thus, we can choose \(\bar{\gamma} > 0\) small enough such that \(\bar{\gamma} < \min\{\gamma_s d_5, \gamma_1 / \zeta(0)\}\).

Based on directly integrating (4.36), we deduce that
\[
L_s d_4 \left( \frac{1}{2} - \frac{1}{p} \right) \left( \|u(t)\|_{a_1}^2 + \|v(t)\|_{a_2}^2 \right) \leq \mathcal{L}_1(t)
\]
\[
\leq C_s \exp \left( -\bar{\gamma} \int_0^t \zeta(s) \, ds \right), \forall t \geq 0,
\]
where \(C_s = \mathcal{L}_1(0) + \frac{\beta_3 \bar{C}_1}{\gamma_1 - \gamma(0)}\). Theorem 4.6 is proved.

5. Blow-up and lifespan of solutions. This section is devoted to the study of the blow-up property for Prob. (4.1) when \(F_1 = F_2 \equiv 0\). We shall state some lemmas (Lemmas 5.1, 5.3, 5.4 below) which are useful for proofs of blow-up results when the initial energy is negative (Theorem 5.2) or nonnegative (Theorem 5.5).

First, we make the following assumptions
\((\check{H}_1)\) \((\check{u}_0, \check{v}_0) \in V_1 \times V_2;\)
\((\check{H}_2)\) \(\mu_1, \mu_2 \in C^1([0, 1] \times \mathbb{R}_+),\) and there exist the positive constants \(\mu_{1*}, \mu_{2*}\) such that \(\mu_i(x, t) \geq \mu_{i*}, \mu_i'(x, t) \leq 0, \forall (x, t) \in [0, 1] \times \mathbb{R}_+, i = 1, 2;\)
\((\check{H}_3)\) \(g_i \in C^1(\mathbb{R}_+) \cap L^1(\mathbb{R}_+),\) such that

(1) \(g_i(t) \geq 0, g_i'(t) \leq 0, \forall t \geq 0, i = 1, 2,\)
The functions
Assume that
Multiplying the equation (\(i\)) have

\[ g_i(s) ds > 0, \quad i = 1, 2, \]

\[ \int_0^\infty g_i(s) ds \leq \frac{e^{(p-2)\mu_2}}{(p-1)^2}, \quad i = 1, 2, \]

where \( \mu_2 = \min\{\mu_1, \mu_2\} \).

(\(\hat{H}_2^\infty\)) There exist the function \( F \in C^2(\mathbb{R}^2; \mathbb{R}) \) and the positive constant \( d_1 > p \), such that

\( \partial_F \frac{\partial}{\partial u} = f_1(u, v), \quad \partial_F \frac{\partial}{\partial v} = f_2(u, v), \) \( \forall (u, v) \in \mathbb{R}^2, \quad i = 1, 2, \)

\[ u f_1(u, v) + v f_2(u, v) \geq d_1 F(u, v) \geq 0, \quad \forall (u, v) \in \mathbb{R}^2. \]

**Remark 4.** The functions \( f_1(u, v), f_2(u, v) \) as in the example of Remark 4.1 also satisfy \((\hat{H}_2^\infty, (ii))\), because of

\[ u f_1(u, v) + v f_2(u, v) = \alpha k_1 |u|^\alpha + \beta k_2 |v|^\beta + (\alpha_1 + \beta_1) k_3 |u|^{\alpha_1} |v|^{\beta_1}, \]

\[ \geq \min \{\alpha, \beta, \alpha_1 + \beta_1\} \left( k_1 |u|^{\alpha} + k_2 |v|^{\beta} + k_3 |u|^{\alpha_1} |v|^{\beta_1} \right), \]

\[ = d_1 F(u, v) \geq 0, \quad \forall (u, v) \in \mathbb{R}^2, \]

where \( d_1 = \min \{\alpha, \beta, \alpha_1 + \beta_1\} \).

Considering the functionals \( E(t), \Psi(t) \) defined as in (4.5). On \( V_i \times V_i \), we now consider the symmetric bilinear forms \( \tilde{a}_i (\cdot, \cdot) \), defined by

\[ \tilde{a}_i(u, v) = \langle u, v \rangle + \lambda_i a_i(u, v), \quad (u, v) \in V_i, \quad i = 1, 2, \]

and denote

\[ \|v\|_{\tilde{a}_i} = \sqrt{\tilde{a}_i(v, v)}, \quad v \in V_i, \quad i = 1, 2. \]

We can rewrite \( \Psi(t) \) as follows

\[ \Psi(t) = \frac{1}{2} \left( \|u(t)\|_{\tilde{a}_1}^2 + \|v(t)\|_{\tilde{a}_2}^2 \right). \]

**Lemma 5.1.** Assume that \( (\hat{H}_1), (\hat{H}_2^\infty), (\hat{H}_3^\infty), (\hat{H}_4^\infty) \) and \( (\hat{H}_2^\infty) \) hold. Then we have

\[ \frac{d}{dt} \left[ E(t) + \int_0^t \left( \|u'(s)\|_{\tilde{a}_1}^2 + \|v'(s)\|_{\tilde{a}_2}^2 \right) ds \right] \leq 0. \]

Moreover, the following energy inequality holds

\[ E(t) + \int_0^t \left( \|u'(s)\|_{\tilde{a}_1}^2 + \|v'(s)\|_{\tilde{a}_2}^2 \right) ds \leq E(0). \]

**Proof.** Multiplying the equation (4.1) by \((u'(x, t), v'(x, t))\) and integrating on \((0, 1)\), we obtain

\[ \frac{d}{dt} \left[ E(t) + \int_0^t \left( \|u'(s)\|_{\tilde{a}_1}^2 + \|v'(s)\|_{\tilde{a}_2}^2 \right) ds \right] \]

\[ = \frac{1}{2} \left[ \tilde{a}'_1(u, u(t), u(t)) + \tilde{a}'_2(v, v(t), v(t)) \right] + \frac{1}{2} \left( (g'_1 \ast u)(t) + (g'_2 \ast v)(t) \right) \]

\[ - \frac{1}{2} \left( \tilde{g}_1(t) \|u(t)\|_{\tilde{a}_1}^2 + \tilde{g}_2(t) \|v(t)\|_{\tilde{a}_2}^2 \right) \leq 0, \]

for any regular solution \((u, v)\). We can extend (5.6) to weak solutions by using density arguments. Combining \((\hat{H}_1), (\hat{H}_2^\infty), (\hat{H}_3^\infty), (\hat{H}_4^\infty)\), Lemma 5.1 is proved.

**a. Blow-up solutions with negative initial energy**
Theorem 5.2. Assume that $(\dot{H}_2^\infty)$, $(\dot{H}_3^\infty)$, $(\dot{H}_4^\infty)$ and $(\dot{H}_5^\infty)$ hold. Then, for any initial conditions $(u_0, v_0) \in V_1 \times V_2$ such that $E(0) < 0$, the weak solution of the Prob. (4.1) blows up at finite time and the lifespan $T_\infty$ of the solution $(u, v)$ satisfies

$$T_\infty \leq \frac{-8(p-1)\Psi(0)}{p(p-2)^2E(0)} \equiv T_\infty^{\text{max}}. \tag{5.7}$$

Furthermore, if the following additional assumptions $(\dot{H}_5^\infty)$ There exists the constant $d_2 > p$ such that

(i) $uf_1(u, v) + vf_2(u, v) \leq d_2F(u, v), \forall (u, v) \in \mathbb{R}^2$, for all $(u, v) \in \mathbb{R}^2$,

(ii) $F(u, v) \leq d_2 \sum_{i=1}^N (|u|^{p_i} + |v|^{p_i}), \forall (u, v) \in \mathbb{R}^2$; for all $(u, v) \in \mathbb{R}^2$,

$$(\dot{H}_5^\infty) \int_{\Psi(0)}^{\infty} \frac{dz}{H_1(z)} \leq \frac{-8(p-1)\Psi(0)}{p(p-2)^2E(0)},$$

where

$$H_1(z) = \frac{\tilde{g}_{\max}}{\lambda_*}z + (1 + d_2)d_3 \sum_{i=1}^N \left(z^{q_i/2} + z^{\tilde{g}_i/2}\right), \tag{5.8}$$

$$\tilde{g}_{\max} = \max \{\tilde{g}_1(\infty), \tilde{g}_2(\infty)\}, \quad \lambda_* = \min \{\lambda_1, \lambda_2\},$$

$$d_3 = \tilde{d}_2 \max \left\{\left(\frac{2}{\lambda_*}\right)^{q_i/2}, \left(\frac{2}{\lambda_*}\right)^{\tilde{g}_i/2}, \quad i = 1, \cdots, N\right\},$$

hold, then the blow-up time $T_\infty$ satisfies

$$T_\infty \geq \int_{\Psi(0)}^{\infty} \frac{dz}{H_1(z)} \equiv T_\infty^{\text{min}}. \tag{5.9}$$

Remark 5. One can confirm that the assumption $(\dot{H}_5^\infty)$ is satisfied if $\lambda_* = \min \{\lambda_1, \lambda_2\} > 0$ is small enough.

Indeed, because of

$$\frac{-8(p-1)\Psi(0)}{p(p-2)^2E(0)} \geq \frac{-4(p-1)\left(\|\tilde{u}_0\|^2 + \|\tilde{v}_0\|^2\right)}{p(p-2)^2E(0)} > 0,$$

it gives that

(i) $\frac{-8(p-1)\Psi(0)}{p(p-2)^2E(0)}$ is lower bounded by a positive constant independent of $\lambda_*$. By the following estimations

$$H_1(z) = \frac{\tilde{g}_{\max}}{\lambda_*}z + (1 + d_2)d_3 \sum_{i=1}^N \left(z^{q_i/2} + z^{\tilde{g}_i/2}\right)$$

$$\geq d_4(\lambda_*) \left(z + \sum_{i=1}^N \left(z^{q_i/2} + z^{\tilde{g}_i/2}\right)\right),$$

$$\Psi(0) \geq \frac{1}{2} \left(\|\tilde{u}_0\|^2 + \|\tilde{v}_0\|^2\right) = \bar{\Psi}_0,$$

with $d_3$ and $d_4(\lambda_*)$ satisfying

$$d_3 = d_3(\lambda_*) = \tilde{d}_2 \max \left\{\left(\frac{2}{\lambda_*}\right)^{q_i/2}, \left(\frac{2}{\lambda_*}\right)^{\tilde{g}_i/2}, \quad i = 1, \cdots, N\right\} \rightarrow +\infty, \text{ as } \lambda_* \rightarrow 0_+,$$

$$d_4(\lambda_*) = \min \left\{\frac{\tilde{g}_{\max}}{\lambda_*}, (1 + d_2)d_3(\lambda_*)\right\} \rightarrow +\infty, \text{ as } \lambda_* \rightarrow 0_+,$$

then we derive that
We first prove that the solution (5.12) (5.13) the auxiliary functional 

\[ \int_{\psi(0)}^\infty \frac{dz}{H_1(z)} \leq \frac{1}{d_4(\lambda_\ast)} \int_{\psi_0}^\infty \frac{dz}{z + \sum_{i=1}^{N} (z_{i}',/2 + z_{i}')} \rightarrow 0, \quad \text{as } \lambda_\ast \rightarrow 0. \]

By (i) and (ii), we deduce that \( (H_\infty) \) holds if \( \lambda_\ast > 0 \) is small enough.

**Proof of Theorem 5.2.** We first prove that the solution \( (u, v) \) obtained here is not a global solution in \( \mathbb{R}_+ \). Indeed, by contradiction, we will assume that the weak solution exists in the whole interval \( \mathbb{R}_+ \).

For \( E(0) < 0 \), let \( 0 < \beta \leq -\frac{\mu E(0)}{p-1}, \tau > \frac{2\psi(0)}{\beta(p-2)}, \) and \( T_0 \geq \frac{\beta \tau^2}{(p-2)\beta \tau - 2\psi(0)} \), we define the auxiliary functional \( \Gamma : [0, T_0] \rightarrow \mathbb{R} \) as follows

\[ \Gamma(t) = 2 \int_0^t \psi(s) ds + 2(T_0 - t) \psi(0) + \beta(t + \tau)^2, \quad 0 \leq t \leq T_0. \]

By direct computation, we have

\[ \Gamma'(t) = 2\psi(t) - 2\psi(0) + 2\beta(t + \tau) = 2 \int_0^t \psi'(s) ds + 2\beta(t + \tau) \]

\[ = 2 \int_0^t \bar{a}_1(u'(s), u(s)) ds + 2 \int_0^t \bar{a}_2(v'(s), v(s)) ds + 2\beta(t + \tau), \]

and

\[ \Gamma''(t) = 2\psi'(t) + 2\beta. \]

Since (5.10) and (5.11), we see that \( \Gamma(t) > 0 \) for all \( t \in [0, T_0] \) and \( \Gamma(0) = 2\beta \tau > 0. \)

By multiplying the equation in (4.1) by \((u(x,t), v(x,t)),\) and then integrating over \((0, 1),\) we obtain

\[ \Gamma''(t) \Gamma(t) \]

\[ = 2\psi'(t) + 2\beta \]

\[ = 2 \left[ \beta - \bar{a}_1(t; u(t), u(t)) - \bar{a}_2(t; v(t), v(t)) + \int_0^t g_1(t - s) a_1(u(s), u(t)) ds 

+ \int_0^t g_2(t - s) a_2(v(s), v(t)) ds + \langle f_1(u(t), v(t)), u(t) \rangle + \langle f_2(u(t), v(t)), v(t) \rangle \right]. \]

From (5.12), we have

\[ \Gamma''(t) \Gamma(t) \]

\[ = 2 \int_0^t \left[ \beta - \bar{a}_1(t; u(t), u(t)) - \bar{a}_2(t; v(t), v(t)) + \int_0^t g_1(t - s) a_1(u(s), u(t)) ds 

+ \int_0^t g_2(t - s) a_2(v(s), v(t)) ds + \langle f_1(u(t), v(t)), u(t) \rangle + \langle f_2(u(t), v(t)), v(t) \rangle \right]. \]

By the Cauchy-Schwarz inequality, we get

\[ |\Gamma'(t)| \leq 2 \left[ \int_0^t |\bar{a}_1(u'(s), u(s))| ds + \int_0^t |\bar{a}_2(v'(s), v(s))| ds + \beta(t + \tau) \right] \]

\[ \leq 2 \left[ \int_0^t \|u'(s)\|_{\tilde{a}_1} \|u(s)\|_{\tilde{a}_1} ds + \int_0^t \|v'(s)\|_{\tilde{a}_2} \|v(s)\|_{\tilde{a}_2} ds + \beta(t + \tau) \right] \]

\[ \leq 2 \left( \int_0^t \|u'(s)\|_{\tilde{a}_1}^2 ds \right)^{1/2} \left( \int_0^t \|u(s)\|_{\tilde{a}_1}^2 ds \right)^{1/2} \]
We can easily estimate the third and fourth terms on the right hand side of (5.18) as follows

\[ \int_0^t g_1(t-s)a_1(u(s), u(t))ds + \int_0^t g_2(t-s)a_2(v(s), v(t))ds \geq - \frac{p}{2} \left[ (g_1 * u)(t) + (g_2 * v)(t) \right] + \left( 1 - \frac{1}{2p} \right) \left[ (\bar{g}_1(t) \| u(t) \|_{a_1}^2 + \bar{g}_2(t) \| v(t) \|_{a_2}^2) \right], \]

and

\[ \langle f_1(u(t), v(t)), u(t) \rangle + \langle f_2(u(t), v(t)), v(t) \rangle \geq d_1 \mathcal{F}(t). \]
It implies from (5.5), (5.18)-(5.20) that
\[
D(t) \geq \beta - \bar{a}_1(t; u(t), u(t)) - \bar{a}_2(t; v(t), v(t)) - \frac{p}{2} \left[ (g_1 * u)(t) + (g_2 * v)(t) \right]
+ \left( 1 - \frac{1}{2p} \right) \left[ (\bar{g}_1(t) \| u(t) \|_{\alpha_1}^2 + \bar{g}_2(t) \| v(t) \|_{\alpha_2}^2) \right] + d_1 \bar{F}(t)
- p \left[ E(t) + \int_0^t \left( \| u'(s) \|_{\alpha_1}^2 + \| v'(s) \|_{\alpha_2}^2 \right) ds \right] - p\beta + pE(t)
\geq \beta - \bar{a}_1(t; u(t), u(t)) - \bar{a}_2(t; v(t), v(t)) - \frac{p}{2} \left[ (g_1 * u)(t) + (g_2 * v)(t) \right]
+ \left( 1 - \frac{1}{2p} \right) \left[ (\bar{g}_1(t) \| u(t) \|_{\alpha_1}^2 + \bar{g}_2(t) \| v(t) \|_{\alpha_2}^2) \right] + d_1 \bar{F}(t) - pE(0) - p\beta
+ \frac{1}{2p} \left[ \bar{a}_1(t; u(t), u(t)) + \bar{a}_2(t; v(t), v(t)) - \bar{g}_1(t) \| u(t) \|_{\alpha_1}^2 - \bar{g}_2(t) \| v(t) \|_{\alpha_2}^2 \right]
+ (g_1 * u)(t) + (g_2 * v)(t) - 2\bar{F}(t)
= -pE(0) - (p - 1)\beta + \frac{p - 2}{2} \left[ \bar{a}_1(t; u(t), u(t)) + \bar{a}_2(t; v(t), v(t)) \right]
- \left( \frac{p - 1}{2p} \right) \left[ (\bar{g}_1(t) \| u(t) \|_{\alpha_1}^2 + \bar{g}_2(t) \| v(t) \|_{\alpha_2}^2) \right] + (d_1 - p) \bar{F}(t)
\geq -pE(0) - (p - 1)\beta + \left[ \frac{p - 2}{2} \mu_* - \frac{(p - 1)^2}{2p} \bar{g}_{\max} \right] \left( \| u(t) \|_{\alpha_1}^2 + \| v(t) \|_{\alpha_2}^2 \right)
\geq -pE(0) - (p - 1)\beta + \left[ \frac{p - 2}{2p} \mu_* \right] \left( \| u(t) \|_{\alpha_1}^2 + \| v(t) \|_{\alpha_2}^2 \right)
\geq -pE(0) - (p - 1)\beta \geq 0,
\]
since \( 0 < \beta \leq \frac{pE(0)}{p - 1} \) and \( \bar{g}_{\max} = \max \{ \bar{g}_1(\infty), \bar{g}_2(\infty) \} \leq \frac{p(p - 2)\mu_*}{(p - 1)^2} \).

From (5.17) and (5.21), we have
\[
\Gamma^{\frac{2}{p - 1}}(t) \geq \frac{2\Gamma^p(0)}{(p - 2)\Gamma^p(0)} \frac{1}{T_* - t}, \quad \forall t \in [0, T_*],
\]
where \( T_* = \frac{2\Gamma(0)}{(p - 2)\Gamma(0)} \) and \( T_* = \min \{ T_* , T_0 \} \).

By \( 0 < \beta \leq \frac{pE(0)}{p - 1} \) and \( \tau > \frac{2\Psi(0)}{\beta(p - 2)} \) and \( T_0 \geq \frac{\beta \tau^2}{(p - 2)\beta \tau - 2\Psi(0)} \), we have
\[
T_* = \frac{2\Gamma(0)}{(p - 2)\Gamma^p(0)} = \frac{2T_0 \Psi(0) + \beta \tau^2}{(p - 2)\beta \tau} \in (0, T_0].
\]

From (5.22), we get \( \lim_{t \to T_*} \Gamma(t) = +\infty \). This is a contradiction. Consequently, the solution blows up at finite time.

Now, we find a upper bound for \( T_\infty \). It is clear to see that
\[
T_\infty \leq \frac{2T_\infty \Psi(0) + \beta \tau^2}{(p - 2)\beta \tau}
\]
\[
\iff T_\infty \leq \frac{\beta \tau^2}{(p - 2)\beta \tau - 2\Psi(0)}, \quad \forall (\beta, \tau) \in \hat{D}(\bar{a}_0, \bar{v}_0),
\]
where \( \hat{D}(\bar{a}_0, \bar{v}_0) \) is the domain of \( \bar{a}_0, \bar{v}_0 \).
where
\[
\hat{D} (\bar{u}_0, \bar{v}_0) = \left\{ (\beta, \tau) \in \mathbb{R}^2_+ : 0 < \beta \leq -\frac{pE(0)}{p-1}, \tau > \frac{2\Psi(0)}{\beta(p-2)} \right\}. \tag{5.25}
\]
Consider the function
\[
h(\tau, \beta) = \frac{\beta \tau^2}{(p-2) \beta \tau - 2\Psi(0)} = \frac{\tau^2}{(p-2)(\tau - \tau_0)}, (\beta, \tau) \in \hat{D} (\bar{u}_0, \bar{v}_0),
\]
with \( \tau_0 = \frac{2\Psi(0)}{\beta(p-2)}, \) and fixed \( \beta, 0 < \beta \leq \frac{-pE(0)}{p-1}. \)

By the fact that
\[
\frac{\partial h}{\partial \tau} (\tau, \beta) = \frac{\tau (\tau - 2\tau_0)}{(p-2)(\tau - \tau_0)^2}, \forall \tau > \tau_0,
\]
the function \( \tau \mapsto h(\tau, \beta) \) is decreasing in \((\tau_0, 2\tau_0)\), and it is also increasing in \((2\tau_0, +\infty)\), so
\[
h(\tau, \beta) \geq h(2\tau_0, \beta) = \frac{4\tau_0}{p-2} = \frac{8\Psi(0)}{\beta(p-2)^2} \geq \frac{8\Psi(0)}{p(p-2)^2 E(0)} = \overset{\text{max}}{T_\infty}, \quad \forall (\beta, \tau) \in \hat{D} (\bar{u}_0, \bar{v}_0).
\]

From (5.24) and (5.26), we get \( T_\infty \leq \frac{8(p-1)\Psi(0)}{p(p-2)^2 E(0)} = \overset{\text{max}}{T_\infty}. \) Hence, (5.7) is proved.

Next, we seek a lower bound for the blow-up time \( T_\infty \). We have
\[
\Psi'(t) = -\bar{a}_1(t; u(t), u(t)) - \bar{a}_2(t; v(t), v(t)) \tag{5.27}
\]
\[
+ \int_0^t g_1(t-s)a_1(u(s), u(t))ds + \int_0^t g_2(t-s)a_2(v(s), v(t))ds
\]
\[
+ (f_1(u(t), v(t)), u(t)) + (f_2(u(t), v(t)), v(t)).
\]

We can easily estimate third term on the right hand side of (5.27) as follows
\[
\int_0^t g_1(t-s)a_1(u(s), u(t))ds + \int_0^t g_2(t-s)a_2(v(s), v(t))ds \tag{5.28}
\]
\[
\leq \frac{1}{2} \left[ (g_1 * u)(t) + (g_2 * v)(t) \right] + \frac{3}{2} \left[ \bar{g}_1(t) \| u(t) \|_{a_1}^2 + \bar{g}_2(t) \| v(t) \|_{a_2}^2 \right];
\]
\[
(f_1(u(t), v(t)), u(t)) + (f_2(u(t), v(t)), v(t)) \leq d_2 \mathcal{F}(t); \tag{5.29}
\]
\[
\| u(t) \|_{a_1}^2 + \| v(t) \|_{a_2}^2 \leq \frac{2}{\lambda_s} \Psi(t); \tag{5.30}
\]
\[
\mathcal{F}(t) \leq d_2 \sum_{i=1}^N (\| u(t) \|_{L_{a_1}}^q + \| v(t) \|_{L_{a_2}}^q) \leq d_2 \sum_{i=1}^N (\| u(t) \|_{a_1}^q + \| v(t) \|_{a_2}^q)
\]
\[
\leq d_2 \sum_{i=1}^N \left[ \left( \frac{2}{\lambda_s} \Psi(t) \right)^{q_{i/2}} + \left( \frac{2}{\lambda_s} \Psi(t) \right)^{\bar{q}_{i/2}} \right]
\]
\[
\leq d_3 \sum_{i=1}^N \left[ \Psi(t)^{q_{i/2}} + (\Psi(t))^{\bar{q}_{i/2}} \right], \tag{5.31}
\]
where \( d_3 = d_2 \max\{ (\frac{2}{\lambda_s})^{q_{i/2}}, (\frac{2}{\lambda_s})^{\bar{q}_{i/2}}, i = 1, \cdots, N \}; \)
\[
E(t) + \mathcal{F}(t) - \frac{1}{2} \left( \bar{a}_1(t; u(t), u(t)) - \bar{g}_1(t) \| u(t) \|_{a_1}^2 \right)
\]
\[
- \frac{1}{2} \left( \bar{a}_2(t; v(t), v(t)) - \bar{g}_2(t) \| v(t) \|_{a_2}^2 \right) \tag{5.32}
\]
Combining (5.5), (5.27)-(5.32), it leads to

\[
\Psi'(t) \leq -\bar{a}_1(t; u(t), u(t)) - \bar{a}_2(t; v(t), v(t)) + \frac{1}{2} \left[ (g_1 * u)(t) + (g_2 * v)(t) \right] + \frac{3}{2} \left[ \bar{g}_1(\infty) \|u(t)\|_{a_1}^2 + \bar{g}_2(\infty) \|v(t)\|_{a_2}^2 \right] + d_2 \bar{\mathcal{F}}(t)
\]

(5.33)

\[
= -\frac{3}{2} \bar{a}_1(t; u(t), u(t)) - \frac{3}{2} \bar{a}_2(t; v(t), v(t)) + E(t) + \frac{1}{2} \bar{g}_1(t) \|u(t)\|_{a_1}^2 + \frac{1}{2} \bar{g}_2(t) \|v(t)\|_{a_2}^2
\]

\[
+ \frac{3}{2} \left[ \bar{g}_1(\infty) \|u(t)\|_{a_1}^2 + \bar{g}_2(\infty) \|v(t)\|_{a_2}^2 \right] + (1 + d_2) \bar{\mathcal{F}}(t)
\]

\[
\leq -\frac{3}{2} \bar{a}_1(t; u(t), u(t)) + E(t) + 2 \bar{g}_{\text{max}} \left[ \|u(t)\|_{a_1}^2 + \|v(t)\|_{a_2}^2 \right] + (1 + d_2) \bar{\mathcal{F}}(t)
\]

\[
\leq \bar{g}_{\text{max}} \bar{\lambda}_\ast \Psi(t) + (1 + d_2) d_3 \sum_{i=1}^{N} \left[ \left( \Psi(t) \right)^{q_i/2} + \left( \Psi(t) \right)^{\bar{q}_i/2} \right] = H_1(\Psi(t)),
\]

where \(H_1(z)\) is defined as in (5.8).

From (5.33), we get

\[
t \geq \int_{0}^{t} \frac{\Psi'(s) ds}{H_1(\Psi(s))} = \int_{\Psi(0)}^{\Psi(t)} \frac{dz}{H_1(z)}.
\]

(5.34)

Hence, we derive the lower bound for \(T_\infty\) as follows

\[
T_\infty \geq \int_{\Psi(0)}^{\infty} \frac{dz}{H_1(z)} = T_{\text{min}}^\infty.
\]

(5.35)

We note more that, by \((\bar{H}_\infty)\), the following estimation is fulfilled

\[
T_{\text{min}}^\infty = \int_{\Psi(0)}^{\infty} \frac{dz}{H_1(z)} \leq T_\infty \leq \frac{-8(p - 1)\Psi(0)}{p(p - 2)^2 E(0)} \equiv T_{\text{max}}^\infty.
\]

Theorem 5.2 is proved.

b. Blow-up solutions with nonnegative initial energy

First, by \((\bar{H}_1)\), \((\bar{H}_\infty)\), \((\bar{H}_\infty^0)\), \((\bar{H}_1^\infty)\) and \((\bar{H}_\infty^\infty)\), we have

\[
E(t) = \frac{1}{2} \dot{E}(t) - \bar{\mathcal{F}}(t)
\]

(5.36)

\[
\geq \frac{1}{2} L_* \left( \|u(t)\|_{a_1}^2 + \|v(t)\|_{a_2}^2 \right) - d_2 \bar{a}_2 \sum_{i=1}^{N} \left( \|u(t)\|_{L^{q_i}}^2 + \|v(t)\|_{L^{\bar{q}_i}}^2 \right)
\]

\[
\geq \frac{1}{2} L_* \left( \|u(t)\|_{a_1}^2 + \|v(t)\|_{a_2}^2 \right) - d_2 \bar{a}_2 \sum_{i=1}^{N} \left( \|u(t)\|_{a_1}^{q_i} + \|v(t)\|_{a_2}^{\bar{q}_i} \right)
\]

\[
\geq \frac{1}{2} L_* y(t) - d_2 \bar{a}_2 \sum_{i=1}^{N} \left( y^{q_i}(t) + y^{\bar{q}_i}(t) \right)
\]

\[
= L_* \left[ \frac{1}{2} y(t)^2 - \frac{d_2 \bar{a}_2}{L_*} \sum_{i=1}^{N} \left( y^{q_i}(t) + y^{\bar{q}_i}(t) \right) \right] = \check{H}(y(t)),
\]

where

\[
y(t) = \sqrt{\|u(t)\|_{a_1}^2 + \|v(t)\|_{a_2}^2},
\]

(5.37)
and

\[ \tilde{H}(\lambda) = L_s \left( \frac{\lambda^2}{2} - \frac{d_2 \overline{d}_2}{L_s} \sum_{i=1}^{N} (\lambda^{q_i} + \lambda^{\overline{q}_i}) \right), \quad \forall \lambda \geq 0. \tag{5.38} \]

Then we have the following lemma, the proof here is easy so which is omitted.

**Lemma 5.3.**

(i) The equation \( \tilde{H}'(\lambda) = 0 \) has a unique positive solution \( \lambda_0 \) satisfying

\[ 1 - \frac{d_2 \overline{d}_2}{L_s} \sum_{i=1}^{N} (q_i \lambda_0^{q_i-2} + \overline{q}_i \lambda_0^{\overline{q}_i-2}) = 0; \tag{5.39} \]

(ii) \( \tilde{H}(0) = 0, \lim_{\lambda \to +\infty} \tilde{H}(\lambda) = -\infty; \)

(iii) \( \tilde{H}'(\lambda) > 0 \) if \( \lambda \in (0, \lambda_0) \) and \( \tilde{H}'(\lambda) < 0 \) if \( \lambda > \lambda_0. \)

The next lemma is also very useful, which is similar to the lemma used firstly by E. Vitillaro in [36].

**Lemma 5.3.** Assume that \( E(0) < \tilde{H}(\lambda_0) \). Then

(i) If \( y(0) = \sqrt{\| \tilde{u}_0 \|_{\mathbb{A}_1}^2 + \| \tilde{v}_0 \|_{\mathbb{A}_2}^2} > \lambda_0 \) then there exists \( \bar{\lambda}_2 > \lambda_0 \) such that

\[ y(t) \geq \bar{\lambda}_2, \quad \forall t \in [0, T_{\infty}). \tag{5.40} \]

(ii) If \( y(0) = \sqrt{\| \tilde{u}_0 \|_{\mathbb{A}_1}^2 + \| \tilde{v}_0 \|_{\mathbb{A}_2}^2} < \lambda_0 \) and \( E(0) \geq 0 \) then there exists \( \bar{\lambda}_1 \in (0, \lambda_0) \) such that

\[ y(t) \leq \bar{\lambda}_1, \quad \forall t \in [0, T_{\infty}). \tag{5.41} \]

*Proof.* Let \( 0 < E(0) < \tilde{H}(\lambda_0) \), there are constants \( 0 < \bar{\lambda}_1 < \lambda_0 < \bar{\lambda}_2 \) such that

\[ \tilde{H}(\bar{\lambda}_1) = \tilde{H}(\bar{\lambda}_2) = E(0). \tag{5.42} \]

(i) First, we assume that \( y(0) > \lambda_0 \). We have

\[ \tilde{H}(\bar{\lambda}_2) = E(0) \geq \tilde{H}(y(0)). \tag{5.43} \]

By Lemma 5.3, from (5.43), we get \( y(0) \geq \bar{\lambda}_2 \). We claim that \( y(t) \geq \bar{\lambda}_2 \) for all \( t \in [0, T_{\infty}). \)

Note that \( E(0) \geq E(t) \geq \tilde{H}(y(t)), \) so \( y(t) \notin (\bar{\lambda}_1, \bar{\lambda}_2), \forall t \in [0, T_{\infty}), \) hence \( \forall t \in [0, T_{\infty}), \) we get \( y(t) \leq \bar{\lambda}_1 \) or \( y(t) \geq \bar{\lambda}_2 \). But one can’t have \( y(t) \leq \bar{\lambda}_1, \forall t \in [0, T_{\infty}). \) Indeed, suppose by contradiction that, there exists \( t_0 \in (0, T_{\infty}) \) such that \( y(t_0) \leq \bar{\lambda}_1. \) By \( y(t_0) \leq \bar{\lambda}_1 < \lambda_0 < \bar{\lambda}_2 \leq y(0) \) and the continuity of \( y, \) then there exists \( t_0 \in (0, t_0) \) such that \( y(t_0) = \lambda_0. \) By Lemmas 5.1 and 5.3, we get

\[ E(t_0) = \tilde{H}(y(t_0)) = \tilde{H}(\lambda_0) > E(0), \tag{5.44} \]

this is a contradiction, because of \( E(t) \leq E(0), \forall t \in [0, T_{\infty}). \)

(ii) Now, we assume that \( y(0) < \lambda_0 \). We have

\[ \tilde{H}(\bar{\lambda}_1) = E(0) \geq \tilde{H}(y(0)). \tag{5.45} \]

By Lemma 5.3, from (5.45), we get \( y(0) \leq \bar{\lambda}_1. \) We claim that \( y(t) \leq \bar{\lambda}_1 \) for all \( t \in [0, T_{\infty}). \) By the same arguments as above, suppose by contradiction that, there exists \( t_1 \in (0, T_{\infty}) \) such that \( y(t_1) \geq \bar{\lambda}_2. \) By \( y(0) \leq \bar{\lambda}_1 < \lambda_0 < \bar{\lambda}_2 \leq y(t_1) \) and the continuity of \( y, \) then exists \( t_1 \in (0, t_1) \) such that \( y(t_1) = \lambda_0. \) By Lemmas 5.1 and 5.3, we get

\[ E(t_1) \geq \tilde{H}(y(t_1)) = \tilde{H}(\lambda_0) > E(0), \tag{5.46} \]

this is a contradiction, because of \( E(t) \leq E(0), \forall t \in [0, T_{\infty}). \)

The remaining cases, \( E(0) = 0 \) or \( E(0) < 0, \) are similarly proved.
Lemma 5.4 is proved. \( \square \)

**Theorem 5.4.** Let \((H_2^\infty, \tilde{H}_2^\infty, \tilde{H}_4^\infty, \tilde{H}_4^\infty)\) hold. For any initial conditions \((\tilde{u}_0, \tilde{v}_0) \in V_1 \times V_2\) such that \(\sqrt{\|\tilde{u}_0\|_{a_1}^2 + \|\tilde{v}_0\|_{a_2}^2} > \lambda_0\) and the initial energy satisfying

\[
0 \leq E(0) < \min \left\{ H(\lambda_0), \frac{[p(p-2)\mu_* - (p-1)^2\tilde{g}_{\text{max}}] \lambda_0^2}{2p^2} \right\},
\]

(5.47)

and

\[
\int_0^\infty \frac{dz}{H_2(z)} \leq \frac{16p(p-1)\Psi(0)}{(p-2)^2 \left\{ [p(p-2)\mu_* - (p-1)^2\tilde{g}_{\text{max}}] \lambda_0^2 - 2p^2E(0) \right\}},
\]

(5.48)

where

\[
H_2(z) = E(0) + \frac{\tilde{g}_{\text{max}}}{\lambda_*} z + (1 + d_2)d_3 \sum_{i=1}^N \left( z_i^{\eta_i/2} + z_i^{\eta_i/2} \right),
\]

(5.49)

with the constants \(\tilde{g}_{\text{max}}, \lambda_*\) and \(d_3\) as defined in (5.8). Then the weak solution of the Prob. (4.1) blows up at finite time and the lifespan \(T_\infty\) is estimated by

\[
\int_0^\infty \frac{dz}{H_2(z)} \leq T_\infty \leq \frac{16p(p-1)\Psi(0)}{(p-2)^2 \left\{ [p(p-2)\mu_* - (p-1)^2\tilde{g}_{\text{max}}] \lambda_0^2 - 2p^2E(0) \right\}}.
\]

(5.50)

**Remark 6.** It is similar to Remark 5.2, we can confirm that the condition (5.48) is satisfied if \(\lambda_* = \min\{\lambda_1, \lambda_2\} > 0\) is small enough. This condition ensures that

\[
\int_0^\infty \frac{dz}{H_2(z)} \equiv T_{\text{min}} \leq T_{\text{max}} \leq \frac{16p(p-1)\Psi(0)}{(p-2)^2 \left\{ [p(p-2)\mu_* - (p-1)^2\tilde{g}_{\text{max}}] \lambda_0^2 - 2p^2E(0) \right\}}.
\]

Proof of Theorem 5.5. By the same method in the proof of Theorem 5.2, with

\[
0 < \beta \leq \frac{[p(p-2)\mu_* - (p-1)^2\tilde{g}_{\text{max}}] \lambda_*^2}{2p^2 E(0)},
\]

(5.51)

\[
\tau > \frac{2\Psi(0)}{\beta(p-2)}\text{ and } T_0 \geq \frac{\beta \tau^2}{(p-2)\beta \tau - 2\Psi(0)},
\]

we define the auxiliary functional \(\Gamma : [0, T_0] \rightarrow \mathbb{R}\) as follows

\[
\Gamma(t) = 2 \int_0^t \Psi(s) \, ds + 2(T_0 - t) \Psi(0) + \beta(t + \tau)^2, \quad 0 \leq t \leq T_0.
\]

(5.52)

From (5.21) and Lemma 5.4, we get

\[
D(t) \geq -pE(0) - (p-1)\beta + \left[ \frac{p-2}{2} \mu_* - \frac{(p-1)^2}{2p} \tilde{g}_{\text{max}} \right] (\|u(t)\|_{a_1}^2 + \|v(t)\|_{a_2}^2)
\]

\[
\geq -pE(0) - (p-1)\beta + \frac{(p-1)^2}{2p} \left[ \frac{p(p-2)\mu_*}{(p-1)^2} - \tilde{g}_{\text{max}} \right] \lambda_*^2
\]

\[
= - (p-1)\beta + \frac{[p(p-2)\mu_* - (p-1)^2\tilde{g}_{\text{max}}] \lambda_*^2}{2p} - pE(0).
\]

(5.53)

By \(0 < \beta \leq \frac{[p(p-2)\mu_* - (p-1)^2\tilde{g}_{\text{max}}] \lambda_*^2 - 2p^2E(0)}{2p(p-1)}\), it follows from (5.17) and (5.53) that

\[
\Gamma^2(t) \geq \frac{2G^2(t_0)}{(p-2)\Gamma(0)} \frac{1}{T_* - t}, \quad \forall t \in [0, T_*),
\]

(5.54)

where \(T_* = \frac{2G(t_0)}{(p-2)\Gamma(0)}\) and \(T_* = \min\{T_*, T_0\}\).
By $0 < \beta \leq \frac{\left[ p(p-2)\mu_* - (p-1)^2\bar{g}_{\text{max}} \right] \lambda^2_2 - 2p^2 E(0)}{2p(p-1)}, \quad \tau > \frac{2\Psi(0)}{\beta(p-2)}$
and $T_0 \geq \frac{\beta^2}{(p-2)^2 \beta \tau - 2\psi(0)}$, we will have

$$T_* = \frac{2\Gamma(0)}{(p-2) \Gamma'(0)} = \frac{2T_0\Psi(0) + \beta \tau^2}{(p-2) \beta \tau} \in (0, T_0].$$

(5.55)

From (5.54), we get $\lim_{t \to T_*^-} \Gamma(t) = +\infty$. This is also a contradiction, therefore the solution blows up at finite time.

As in proof of Theorem 5.2, we have

$$T_\infty \leq \frac{16\beta(p-1)\Psi(0)}{(p-2)^2 \left\{ \left[ p(p-2)\mu_* - (p-1)^2\bar{g}_{\text{max}} \right] \lambda^2_2 - 2p^2 E(0) \right\}} = T_{\infty}^{\text{max}}.$$  

(5.56)

Finally, we seek a lower bound for the blow-up time $T_\infty$ for the solution $(u, v)$. We have

$$\Psi'(t) \leq -\frac{3}{2} \mu_* \left[ \left\| u(t) \right\|_{a_1}^2 + \left\| v(t) \right\|_{a_2}^2 \right] + E(t)
+ 2\bar{g}_{\text{max}} \left[ \left\| u(t) \right\|_{a_1} + \left\| v(t) \right\|_{a_2} \right] + (1 + d_2)F(t)
\leq E(0) + \frac{\bar{g}_{\text{max}}}{\lambda_*} \Psi(t) + (1 + d_2) d_3 \sum_{i=1}^N \left[ (\Psi(t))^{q_i/2} + (\Psi(t))^{q_i^/2} \right] = H_2(\Psi(t)),$$

where $H_2(z)$ is defined as in (5.49).

From (5.57), we get

$$T_\infty \geq \int_{\Psi(0)}^{\infty} \frac{dz}{H_2(z)} = T_{\infty}^{\text{min}}.$$  

(5.58)

Theorem 5.5. is proved.  

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