THE CLOSURE OF THE SET OF PERIODIC MODULES
OVER A CONCEALED CANONICAL ALGEBRA
IS REGULAR IN CODIMENSION ONE

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Abstract. Let Λ be a concealed canonical algebra and d the
dimension vector of a Λ-module which is periodic respect to the
action of the Auslander–Reiten translation τ. In the paper, we
investigate the union of the closures of the orbits of the τ-periodic
Λ-modules of dimension vector d. We show that this set is closed
and regular in codimension one.

Introduction and the main result
Throughout the paper k is a fixed algebraically closed field. By Z,
N, and N+ we denote the sets of integers, nonnegative integers, and
positive integers, respectively. If i, j ∈ Z, then [i, j] denotes the set of
all l ∈ Z such that i ≤ l ≤ j.

For a finite dimensional k-algebra Λ and a nonnegative element d of
the Grothendieck group, one defines a variety mod^d_d_Λ(k) (see Section 2
for details), whose points parameterize the Λ-modules with dimension
vector d (i.e. modules whose Jordan–Hölder composition factors are
given by d). If M is a Λ-module of dimension vector d, then we denote
by O_M the set of points of mod^d_d_Λ(k), which correspond to modules
isomorphic to M.

The variety mod^d_d_Λ(k) is in general reducible. However, if Λ is tri-
gonal (there is no sequence P_0 → P_1 → · · · → P_n of nonzero
nonisomorphisms between indecomposable projective Λ-modules such
that n > 0 and P_0 ∼= P_n), then the points corresponding to modules
with projective dimension at most 1 form an open and irreducible
(if nonempty) subset P(d) of mod^d_d_Λ(k) ([2, Proposition 3.1]). Con-
sequently, if P(d) ≠ ∅, then P(d) is an irreducible component of mod^d_d_Λ(k).

Our ongoing aim is to study components of the form P(d). We
expect they are normal ([8] provides an evidence for this belief) or
at least regular in codimension one. There are two sources of such
components. First, if M is a directing (see Subsection 6.2 for the
definition) Λ-module of dimension vector d, then O_M = P(d). The

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components of this form were studied in [4] and the main result of [4] states they are regular in codimension one.

We describe now another class of such components. An important role in the representation theory of algebras is played by the Auslander–Reiten translation $\tau = \tau_\Lambda$, which is an operation on the $\Lambda$-modules (see for example [1, Chapter IV] for the definition). A $\Lambda$-module $M$ is called $\tau$-periodic, if $M \cong \tau^n M$ for some $n \in \mathbb{N}_+$. The algebra $\Lambda$ is called concealed canonical if there exists a tilting line bundle $T$ over a weighted projective line such that $\Lambda$ is isomorphic to the opposite algebra of $\text{End}(T)$. If $\Lambda$ is a concealed canonical algebra, then every $\tau$-periodic $\Lambda$-module has projective dimension at most 1. Moreover, if $\mathcal{U}$ is the set of points of $\text{mod}^d_\Lambda(k)$ corresponding to the $\tau$-periodic modules and $\mathcal{U} \neq \emptyset$, then $\overline{\mathcal{U}} = \mathcal{P}(d)$. Geometry of this set and related problems (including study of semiinvariants) are objects of interest, especially in the case of Ringel’s canonical algebras [20] – see [2, 3, 6, 7, 12, 13, 21] for some results. The following theorem is the main result of the paper.

**Theorem A.** Let $\Lambda$ be a concealed canonical algebra and $d$ a dimension vector. If
\[ \mathcal{U} := \{ M \in \text{mod}^d_\Lambda(k) : M \text{ is } \tau\text{-periodic} \}, \]
then $\overline{\mathcal{U}}$ is regular in codimension one.

According to famous Drozd’s Wild–Tame Dichotomy Theorem [14] (see also [11]) the algebras can be divided into two disjoint classes. One class consists of the tame algebras for which the indecomposable modules occur, in each dimension, in a finite number of discrete and a finite number of one-parameter families. The second class is formed by the wild algebras whose representation theory is as complicated as the study of finite dimensional vector spaces together with two (noncommuting) endomorphisms, for which the classification up to isomorphism is a well-known unsolved problem.

If $\Lambda$ is a tame concealed canonical algebra and $d$ a dimension vector of a $\tau$-periodic $\Lambda$-module, then Theorem A is a consequence of [7, Theorem 1]. More precisely, if this is the case, then $\overline{\mathcal{U}} = \text{mod}^d_\Lambda(k)$ and $\text{mod}^d_\Lambda(k)$ is normal. Moreover, results of [7, Section 3] imply, that there exists a convex subalgebra $\Lambda'$ of $\Lambda$, which is also concealed canonical, such that the support algebra of $d$ is contained in $\Lambda'$ and
\[ \text{mod}^d_\Lambda(k) = \bigcup_{M \in \mathcal{U}'} \overline{\sigma}_M, \]
where
\[ \mathcal{U}' := \{ M \in \text{mod}^d_\Lambda(k) : M \text{ is } \tau_\Lambda\text{-periodic} \}. \]
Note that in general $\tau_\Lambda$ and $\tau_\Lambda'$ differ, hence $\mathcal{U}' \neq \mathcal{U}$.

In the course of the proof of Theorem A we obtain the following analogue of the above result in the case of wild concealed canonical algebras, which seems to be of interest on its own.
Theorem B. Let $\Lambda$ be a wild concealed canonical algebra and $d$ a dimension vector. If

$$U := \{ M \in \text{mod}^d_{\Lambda}(k) : M \text{ is } \tau\text{-periodic} \},$$

then

$$\overline{U} = \bigcup_{M \in U} \mathcal{O}_M.$$ 

The paper is organized as follows. In Section 1 we present preliminaries on quivers and their representations. Next, in Section 2 we define geometric objects of interest and their variants, which play a role in the proof. Sections 3 and 4 are crucial: the former contains a nonsingularity criterion which we use in the proof, while the latter shows existence of an exact sequence, which is necessary in order to apply the nonsingularity criterion. Section 5 contains more general versions of Theorems A and B and in final Section 6 we explain how these versions imply Theorems A and B. Moreover, in Section 6 we correct some arguments from [4] and [5].

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1. Preliminaries on quivers and their representations

By a quiver $\Delta$ we mean a finite set $\Delta_0$ of vertices and a finite set $\Delta_1$ of arrows together with two maps $s, t : \Delta_1 \to \Delta_0$, which assign to $\alpha \in \Delta_1$ the starting vertex $s\alpha$ and the terminating vertex $t\alpha$, respectively. If $l \in \mathbb{N}_+$, then by a path in $\Delta$ of length $l$ we mean every sequence $\sigma = \alpha_1 \cdots \alpha_l$ such that $\alpha_i \in \Delta_1$, for each $i \in [1, l]$, and $s\alpha_i = t\alpha_{i+1}$, for each $i \in [1, l - 1]$. In the above situation we put $s\sigma := s\alpha_l$ and $t\sigma := t\alpha_1$. Moreover, for each $x \in \Delta_0$, we introduce the path $1_x$ in $\Delta$ of length $0$ such that $s1_x := x =: t1_x$. We denote the length of a path $\sigma$ in $\Delta$ by $\ell(\sigma)$. If $\sigma'$ and $\sigma''$ are two paths in $\Delta$ such that $s\sigma' = t\sigma''$, then we define the composition $\sigma'\sigma''$ of $\sigma'$ and $\sigma''$, which is a path in $\Delta$ of length $\ell(\sigma') + \ell(\sigma'')$, in the obvious way (in particular, $\sigma 1_{s\sigma} = \sigma = 1_{t\sigma}\sigma$, for each path $\sigma$).

With a quiver $\Delta$ we associate its path algebra $k\Delta$, which as a $k$-vector space has a basis formed by all paths in $\Delta$ and whose multiplication is induced by the composition of paths. If $\rho \in 1_x(k\Delta)1_y$, for $x, y \in \Delta_0$, then we put $s_\rho := y$ and $t_\rho := x$. If additionally $\rho = \lambda_1\sigma_1 + \cdots + \lambda_n\sigma_n$, for $\lambda_1, \ldots, \lambda_n \in k$ and paths $\sigma_1, \ldots, \sigma_n$ such that $\ell(\sigma_i) > 1$, for each $i \in [1, n]$, then we call $\rho$ a relation. A set $\mathfrak{R}$ of relations is called minimal if, for every $\rho \in \mathfrak{R}$, $\rho$ does not belong to the ideal $\langle \mathfrak{R} \setminus \{ \rho \} \rangle$ of $k\Delta$ generated by $\mathfrak{R} \setminus \{ \rho \}$. A pair $(\Delta, \mathfrak{R})$ consisting of a quiver $\Delta$ and a minimal set of relations $\mathfrak{R}$, such that there exists $n \in \mathbb{N}_+$ with the property $\sigma \in \langle \mathfrak{R} \rangle$, for each path $\sigma$ in $\Delta$ of length at least $n$, is called a bound quiver. If $(\Delta, \mathfrak{R})$ is a bound quiver, then the algebra $k\Delta/\langle \mathfrak{R} \rangle$ is called the path algebra of $(\Delta, \mathfrak{R})$. 

For the rest of the section we fix the path algebra $\Lambda$ of a bound quiver $(\Delta, \mathcal{R})$.

Let $R$ be a commutative $k$-algebra. An $R$-representation $M$ of $\Delta$ associates with each vertex $x \in \Delta_0$ a free $R$-module $M_x$ of finite rank and with each arrow $\alpha \in \Delta_1$ an $R$-linear map $M_\alpha : M_{s\alpha} \to M_{t\alpha}$. If $M$ is an $R$-representation of $\Delta$ and $\sigma = \alpha_1 \cdots \alpha_n$ is a path in $\Delta$ with $\alpha_1, \ldots, \alpha_n \in \Delta_1$, then we put

$$M_\sigma := M_{\alpha_1} \cdots M_{\alpha_n}.$$  
Similarly, if $\rho = \lambda_1 \sigma_1 + \cdots + \lambda_n \sigma_n$ is a relation, for $\lambda_1, \ldots, \lambda_n \in k$ and paths $\sigma_1, \ldots, \sigma_n$, then

$$M_\rho := \lambda_1 M_{\sigma_1} + \cdots + \lambda_n M_{\sigma_n}.$$  

An $R$-representation $M$ of $\Delta$ is called an $R$-representation of $(\Delta, \mathcal{R})$ if $M_\rho = 0$, for each $\rho \in \mathcal{R}$. By a morphism $f : M \to N$ of $R$-representations we mean a collection $(f_x)_{x \in \Delta_0}$ of $R$-linear maps $f_x : M_x \to N_x$, $x \in \Delta_0$, such that $f_{t\alpha} M_\alpha = N_\alpha f_{s\alpha}$, for each $\alpha \in \Delta_1$. The category of $R$-representations of $(\Delta, \mathcal{R})$ is equivalent to the full subcategory $\text{mod}_\Lambda(R)$ of the category of $\Lambda$-$R$-bimodules formed by the bimodules $M$ such that $M_x := 1_x M$ is a free $R$-module, for each $x \in \Delta_0$ (see for example [1, Theorem III.1.6] for this statement in the case $R = k$). We will identify such $\Lambda$-$R$-bimodules and the $R$-representations of $(\Delta, \mathcal{R})$.

We write $\text{mod}\Lambda$ for $\text{mod}_\Lambda(k)$. For $M, N \in \text{mod}\Lambda$, we denote by $[M, N]_\Lambda$ the $k$-dimension of $\text{Hom}_\Lambda(M, N)$. Similarly, for $n \in \mathbb{N}_+$, we denote by $[M, N]_\Lambda^n$ the $k$-dimension of $\text{Ext}^n_\Lambda(M, N)$. For $M \in \text{mod}\Lambda$, we denote by $\dim M$ the dimension vector of $M$, which is an element of $\mathbb{N}^{\Delta_0}$ such that, for $x \in \Delta_0$, $(\dim M)_x$ is the $k$-dimension of $M_x$.

If $\text{gldim} \Lambda < \infty$, then one defines the bilinear form $\langle -, - \rangle_\Lambda : \mathbb{Z}^{\Delta_0} \times \mathbb{Z}^{\Delta_0} \to \mathbb{Z}$ by the condition

$$\langle \dim M, \dim N \rangle_\Lambda = \sum_{n \in \mathbb{N}} (-1)^n [M, N]_\Lambda^n,$$

for all $\Lambda$-modules $M$ and $N$. We denote the associated quadratic form by $\chi_\Lambda$, i.e. $\chi_\Lambda(d) := \langle d, d \rangle_\Lambda$, for $d \in \mathbb{Z}^{\Delta_0}$. If additionally $\Lambda$ is triangular and $\text{gldim} \Lambda \leq 2$, then

$$\langle d_1, d_2 \rangle_\Lambda = \sum_{x \in \Delta_0} d_1(x) d_2(x) - \sum_{\alpha \in \Delta_1} d_1(s\alpha) d_2(t\alpha) + \sum_{\rho \in \mathcal{R}} d_1(s\rho) d_2(t\rho),$$

for all $d_1, d_2 \in \mathbb{Z}^{\Delta_0}$ (see [9, Section 1]).

2. Geometric preliminaries

Let $(\Delta, \mathcal{R})$ be a bound quiver and $\Lambda$ its path algebra. We define geometric objects associated with $(\Delta, \mathcal{R})$ and $\Lambda$. 


If $d_1$ and $d_2$ are dimension vectors, then we define affine schemes
$\mathbb{V}^{d_1,d_2}$ and $\mathbb{A}^{d_1,d_2}$ by

$\mathbb{V}^{d_1,d_2} := \prod_{x \in \Delta_0} M_{d_2(x),d_1(x)}$ and $\mathbb{A}^{d_1,d_2} := \prod_{\alpha \in \Delta_1} M_{d_2(\alpha),d_1(\alpha)}$,

i.e. if $R$ is a commutative $k$-algebra, then

$\mathbb{V}^{d_1,d_2}(R) := \prod_{x \in \Delta_0} M_{d_2(x),d_1(x)}(R)$

and

$\mathbb{A}^{d_1,d_2}(R) := \prod_{\alpha \in \Delta_1} M_{d_2(\alpha),d_1(\alpha)}(R)$.

If $R$ is a commutative $k$-algebra, $M \in \mathbb{A}^{d_1,d_2}(R)$ and $h \in \mathbb{V}^{d_2,d_3}(R)$, then we define $h \circ M \in \mathbb{A}^{d_3}(R)$ by

$$(h \circ M)_\alpha := h_{t\alpha} M_\alpha \quad (\alpha \in \Delta_1).$$

Analogously, we define $M \circ h \in \mathbb{A}^{d_1}(R)$, for $h \in \mathbb{V}^{d_1,d_2}(R)$ and $M \in \mathbb{A}^{d_2,d_3}(R)$.

If $d$ is a dimension vector and $R$ is a commutative $k$-algebra, then we view $M \in \mathbb{A}^{d,d}(R)$ as an object of $\text{mod}_k^{d}(R)$, where $M_x := R^{d(x)}$, for each $x \in \Delta_0$. Consequently, we define $\text{mod}_k^d$ as the subscheme of $\mathbb{A}^{d,d}$ such that, for a commutative $k$-algebra $R$, $\text{mod}_k^d(R)$ consists of $M \in \mathbb{A}^{d,d}(R)$ which are $R$-representations of $(\Delta, \mathfrak{R})$. Then $\text{mod}_k^d$ is an affine scheme, which is called the scheme of $\Lambda$-modules of dimension vector $d$, as for each $N \in \text{mod} \Lambda$ with dimension vector $d$, there exists $M \in \text{mod}_k^d(k)$ such that $M \simeq N$. The scheme $\text{mod}_k^d$ is described within $\mathbb{A}^{d,d}$ by $\sum_{\rho \in \mathfrak{R}} d(s\rho)d(tp)$ equations. Consequently, if

$$a_\Lambda(d) := \sum_{\alpha \in \Delta_1} d(s\alpha)d(t\alpha) - \sum_{\rho \in \mathfrak{R}} d(s\rho)d(tp)$$

and $Z$ is an irreducible component of $\text{mod}_k^d(k)$, then $\dim Z \geq a_\Lambda(d)$. Note that

$$a_\Lambda(d) = \dim_k \mathbb{V}^{d,d}(k) - \chi_\Lambda(d),$$

provided $\Lambda$ is triangular and $\text{gldim} \Lambda \leq 2$.

For a dimension vector $d$, let $GL_d$ be the affine group scheme defined by

$$GL_d := \prod_{i \in \Delta_0} GL_{d_i}.$$ 

Then $GL_d$ acts on $\mathbb{A}^{d,d}$ by

$$(g \ast M)_\alpha := g_{t\alpha} M_\alpha g_{s\alpha}^{-1},$$

for $g \in GL_d(R)$, $M \in \mathbb{A}^{d,d}(R)$, and $\alpha \in \Delta_1$, where $R$ is a commutative $k$-algebra. Note that $\text{mod}_k^d$ is a $GL_d$-invariant subscheme of $\mathbb{A}^{d,d}$.
mod\(^d\)_\(\Lambda\)(k), and by \(\mathcal{O}_M\) the closure of \(\mathcal{O}_M\). If \(\mathcal{Z}\) is a \(\text{GL}_d(k)\)-invariant subset of \(\text{mod}\^d\_\(\Lambda\)(k)\) and \(M \in \mathcal{Z}\), then we say that \(\mathcal{O}_M\) is maximal in \(\mathcal{Z}\) if there is no \(N \in \mathcal{Z}\) such that \(\mathcal{O}_M \subseteq \mathcal{O}_N\) and \(\mathcal{O}_M \neq \mathcal{O}_N\). Note that \(\mathcal{O}_M = \mathcal{O}_N\) if and only if \(M \simeq N\).

We present now an interpretation of the extension spaces, which will play an important role later. Let \(R\) be a commutative \(k\)-algebra, \(d_1\) and \(d_2\) dimension vectors, \(M \in \text{mod}\^d\_\(\Lambda\)(R)\) and \(N \in \text{mod}\^d\_\(\Lambda\)(R)\), and fix \(Z \in \text{A}_{d_2,d_1}(R)\). If \(\sigma = \alpha_1 \cdots \alpha_n\) is a path in \(\Delta\) with \(\alpha_1, \ldots, \alpha_n \in \Delta_1\), then we put

\[
Z_{\sigma}^{N,M} := \sum_{i=1}^{n} M_{\alpha_1} \cdots M_{\alpha_{i-1}} Z_{\alpha_i N_{\alpha_{i+1}}} \cdots N_{\alpha_n}.
\]

Moreover, if \(\rho = \lambda_1 \sigma_1 + \cdots + \lambda_n \sigma_n\) is a relation, for \(\lambda_1, \ldots, \lambda_n \in k\) and paths \(\sigma_1, \ldots, \sigma_n\), then

\[
Z_{\rho}^{N,M} := \lambda_1 Z_{\sigma_1}^{N,M} + \cdots + \lambda_n Z_{\sigma_n}^{N,M}.
\]

We denote by \(Z^{N,M}\) the set of \(Z \in \text{A}_{d_2,d_1}(R)\) such that \(Z_{\rho}^{N,M} = 0\), for all \(\rho \in R\). If \(Z \in Z^{N,M}\), then we define \(W^{Z} \in \text{mod}\^d\_\(\Lambda\)(k)\) by

\[
W^{Z} := \begin{bmatrix} M & Z \\ 0 & N \end{bmatrix} \quad \text{(i.e. \(W^{Z\_\alpha} := \begin{bmatrix} M_{\alpha} & Z_{\alpha} \\ 0 & N_{\alpha} \end{bmatrix}\) (\(\alpha \in \Delta_1\))}.
\]

Then we have the canonical exact sequence

\[
\xi^{Z}: 0 \to M \to W^{Z} \to N \to 0.
\]

The assignment

\[
Z^{N,M} \ni Z \mapsto [\xi^{Z}] \in \text{Ext}^1_{\Lambda}(N, M)
\]

is a linear epimorphism. We denote its kernel by \(B^{N,M}\). Then \(Z \in B^{N,M}\) if and only if there exists \(h \in \text{V}_{d_2,d_1}(R)\) such that

\[
Z = h \circ N - M \circ h.
\]

Consequently, we have exact sequences

\[
0 \to \text{Hom}_{\Lambda}(N, M) \to \text{V}_{d_2,d_1}(R) \to B^{N,M} \to 0
\]

and

\[
0 \to B^{N,M} \to Z^{N,M} \to \text{Ext}^1_{\Lambda}(N, M) \to 0.
\]

(2.2)

In particular, if \(M \in \text{mod}_{\Lambda}^d(k)\) and \(N \in \text{mod}_{\Lambda}^{d_2}(k)\), then

\[
\dim_k B^{N,M} = \dim_k \text{V}_{d_2,d_1}(k) - [N, M]_{\Lambda}
\]

and

\[
(2.3) \quad \dim_k Z^{N,M} = \dim_k \text{V}_{d_2,d_1}(k) - [N, M]_{\Lambda} + [N, M]_{\Lambda}^1.
\]
The next thing we need is a relationship of elements of $\text{mod}^d_A(k[t]/t^n)$ with exact sequences. Fix $M \in \text{mod}^d_A(k[t]/t^n)$. Put $U := M/tM \in \text{mod}^d_A(k)$ and $V := M/t^{n-1}M \in \text{mod}^d_A(k[t]/t^{n-1})$. If we write

$$M = M_0 + tM_1 + \cdots + t^{n-1}M_{n-1},$$

for $M_0, \ldots, M_{n-1} \in A^{d,d}(k)$, then $M_0 = U$ and $V = M_0 + tM_1 + \cdots + t^{n-2}M_{n-2}$. We may view $M$ and $V$ as $A$-modules of dimension vectors $n \mathbf{d}$ and $(n - 1) \mathbf{d}$, respectively. Then we have isomorphisms

$$M \simeq \begin{bmatrix} U & M_1 & \cdots & M_{n-1} \\ 0 & U & \cdots & \vdots \\ \vdots & \ddots & M_1 \\ 0 & \cdots & 0 & U \end{bmatrix}$$

and

$$V \simeq \begin{bmatrix} U & M_1 & \cdots & M_{n-2} \\ 0 & U & \cdots & \vdots \\ \vdots & \ddots & M_1 \\ 0 & \cdots & 0 & U \end{bmatrix},$$

of $A$-modules, and consequently exact sequences

$$0 \to U \to M \to V \to 0 \quad \text{and} \quad 0 \to V \to M \to U \to 0.$$

The above gives us an interpretation of the tangent spaces. Let $k[\varepsilon]$ be the algebra of dual numbers and $\pi: k[\varepsilon] \to k$ the canonical projection. Recall that if $M \in \text{mod}^d_A(k)$, then the tangent space $T_M \text{mod}^d_A$ is $(\text{mod}^d_A(\pi))^{-1}(M)$. If $M + \varepsilon Z \in T_M \text{mod}^d_A$, then the above shows that $Z \in Z^M, M$, and we identify $T_M \text{mod}^d_A$ with $Z^M, M$. Under this identification, $B^{M, M} = T_M O_M$ and (2.2) takes the form

$$0 \to T_M O_M \to T_M \text{mod}^d_A \to \text{Ext}^1_A(M, M) \to 0,$$

which is the famous Voigt’s result [23].

We finish this section with the following variant of the above constructions. Let $d_1$ and $d_2$ be dimension vectors and $\mathbf{d} := d_1 + d_2$. Then we have a natural decomposition

$$A^{d,d} = A^{d_1,d_1} \oplus A^{d_2,d_2} \oplus A^{d_1+d_2,d_1+d_2}.$$

We define $E^d_A$ to be the intersection of $\text{mod}^d_A$ and $[\Lambda^{d_1,d_1} \Lambda^{d_2,d_2} \Lambda^{d_1+d_2,d_1+d_2}]$. Consequently, if $R$ is a commutative $k$-algebra, then

$$E^d_A(R) = \left\{ \begin{bmatrix} U & Z \\ 0 & V \end{bmatrix} : \begin{array}{c} U \in \text{mod}^{d_1}_A(R), V \in \text{mod}^{d_2}_A(R), \text{ and } Z \in \mathbb{Z}^{V,U} \end{array} \right\}.$$
If additionally $h, e \in \mathbb{N}$, then one defines (see [4] for details) the scheme $E_{d_1,d_2}^{d_1}$ such that

$$E_{d_1,d_2}^{d_1}(k) = \left\{ \begin{bmatrix} U & Z \\ 0 & V \end{bmatrix} \in E_{d_1,d_2}^{d_1} : \right. [V,U] = h \text{ and } [V,U]_A^1 = e \left. \right\}.$$ 

Recall that if $Z \in \mathbb{Z}^{M,N}$, then $\xi^Z$ denotes the corresponding exact sequence. Moreover, if $\xi_1 \in \text{Ext}_{A_{\lambda}}^n(M,N)$ and $\xi_2 \in \text{Ext}_{A_{\lambda}}^m(N,L)$, then $\xi_2 \circ \xi_1 \in \text{Ext}_{A_{\lambda}}^{n+m}(M,L)$ is the Yoneda product. The following is a reformulation of [4, Propositions 3.2 and 3.3].

**Proposition 2.1.** Let $d_1$ and $d_2$ be dimension vectors, $U \in \text{mod}_{A_{\lambda}}^{d_1}(k)$, $V \in \text{mod}_{A_{\lambda}}^{d_2}(k)$, and $N := U \oplus V$. If

$$h := \dim_k \text{Hom}_{A_{\lambda}}(V,U) \quad \text{and} \quad e := \dim_k \text{Ext}_{A_{\lambda}}^1(V,U),$$

and

$$Z = \begin{bmatrix} Z_{1,1} & Z_{2,1} \\ 0 & Z_{2,2} \end{bmatrix} \in T_N E_{A_{\lambda},h,e}^{d_2,d_1},$$

then $Z \in T_N E_{A_{\lambda},h,e}^{d_2,d_1}$ if and only if

1. $[\xi_{2,1}] \circ f = f \circ [\xi_{2,2}]$, for each $f \in \text{Hom}_{A_{\lambda}}(V,U)$, and
2. $[\xi_{2,1}] \circ \xi + \xi \circ [\xi_{2,2}] = 0$, for each $\xi \in \text{Ext}_{A_{\lambda}}^1(V,U)$.

### 3. Criterion for nonsingularity

We first prove the following result, which is crucial for the proof of the main result.

**Proposition 3.1.** Let $d_1$ and $d_2$ be dimension vectors, $d := d_1 + d_2$, $U \in \text{mod}_{A_{\lambda}}^{d_1}(k)$ and $V \in \text{mod}_{A_{\lambda}}^{d_2}(k)$. Let $N := U \oplus V$ and $Z$ be a $\text{GL}_d(k)$-invariant irreducible closed subset of $\text{mod}_{A_{\lambda}}^{d_1}(k)$, considered as an affine scheme with its reduced structure, such that $N \in Z$. If $[U,V]_A^1 = 0$, then the canonical map

$$\text{GL}_d \times (Z \cap E_{A_{\lambda}}^{d_2,d_1}) \to Z$$

is smooth at $(\text{Id}, N)$. In particular, the scheme $Z \cap E_{A_{\lambda}}^{d_2,d_1}$ is reduced at $N$ and

$$\dim_k T_N Z = \dim_k T_N (Z \cap E_{A_{\lambda}}^{d_2,d_1}) + \dim_k \mathbb{B}^{U,V}.$$

**Proof.** Fix a (linear) complement $C_0$ of $\mathbb{B}^{U,V}$ in $A_{d_1,d_2}(k)$. Put

$$C := \begin{bmatrix} A_{d_1,d_1}(k) & A_{d_2,d_1}(k) \\ C_0 & A_{d_2,d_2}(k) \end{bmatrix},$$

which we consider as an affine scheme with the reduced structure (given by the equations describing $C_0$). Using general properties of smooth morphisms (compare [22, Section 5.1]), it follows that the canonical map

$$\text{GL}_d \times (Z \cap C) \to Z$$
is smooth at (Id, N). Obviously, \(E^{d_2,d_1}_A \subseteq C\), hence \(Z \cap E^{d_2,d_1}_A \subseteq Z \cap C\) (both inclusions are inclusions of schemes). We prove that there exists an open neighbourhood \(U\) of \(N\) in \(Z \cap C\) such that \(U \subseteq E^{d_2,d_1}_A(k)\). This will imply that the schemes \(Z \cap C\) and \(Z \cap E^{d_2,d_1}_A\) coincide in a neighborhood of \(N\), hence finish the proof.

We show that every irreducible component \(\mathcal{X}\) of \(Z \cap C\) containing \(N\) is contained in \(E^{d_2,d_1}_A(k)\). Assume this is not the case and let \(\mathcal{X}\) be an irreducible component of \(Z \cap C\) containing \(N\) such that \(\mathcal{X} \nsubseteq E^{d_2,d_1}_A(k)\). Note that \(\mathcal{X}\) is \(GL_d(k) \times GL_d(k)\)-invariant, since both \(Z\) and \(C\) are \(GL_d(k) \times GL_d(k)\)-invariant, where we embed \(GL_d \times GL_d\) in \(GL_d\) diagonally. If \(U := \mathcal{X} \setminus E^{d_2,d_1}_A(k)\), then \(U\) is a nonempty open subset of \(\mathcal{X}\), hence \(\mathcal{X} = \overline{U}\). Using basic facts from algebraic geometry one shows there exists a nonsingular curve \(\Lambda\) and a regular map \(\varphi: \Lambda \to \mathcal{X}\) such that \(\varphi(c_0) = N\), for some \(c_0 \in \Lambda\), and \(\varphi^{-1}(U)\) is a cofinite subset of \(\Lambda\) (see for example [25, Lemma 5.3]). Let \(R\) be the local ring of \(\Lambda\) at \(c_0\) and \(t\) its uniformizing element. A map \(\varphi\) is represented by \(A \in \text{mod}^d_A(R)\) such that \(A/tA = N\).

Write 
\[
A = \begin{bmatrix} A_{1,1} & A_{2,1} \\ A_{1,2} & A_{2,2} \end{bmatrix},
\]
for \(A_{i,j} \in A^{d_1,d_2}(R)\), \(i, j \in \{1, 2\}\). In particular,
\[
A_{1,1}/tA_{1,1} = U, \quad A_{1,2}/tA_{1,2} = 0,
\]
\[
A_{2,1}/tA_{2,1} = 0 \quad \text{and} \quad A_{2,2}/tA_{2,2} = V.
\]
Since \(\varphi^{-1}(U)\) is a cofinite subset of \(\Lambda\), \(A_{1,2} \neq 0\). Let \(n \in \mathbb{N}\) be maximal such that \(t^n \mid A_{1,2}\). Then \(n > 0\). Let \(g \in GL_d(K)\), where \(K\) is the field of fractions of \(R\), be given by 
\[
g := \begin{bmatrix} t^{n-1} \text{Id} & 0 \\ 0 & \text{Id} \end{bmatrix}.
\]
Then
\[
B := g \ast A = \begin{bmatrix} A_{1,1} & t^{n-1}A_{2,1} \\ t^{n-1}A_{1,2} & A_{2,2} \end{bmatrix}.
\]
By shrinking \(\Lambda\) if necessary, we get from \(B\) a regular map \(\psi: \Lambda \to \mathcal{X}\) (we use here that \(\mathcal{X}\) is \(GL_d(k) \times GL_d(k)\)-invariant and closed) such that \(\psi(c_0) = N\). The maximality of \(n\) implies that 
\[
\text{Im}(T_{c_0}\psi) \nsubseteq \left[ A^{d_1,d_1}_A, A^{d_1,d_1}_A, A^{d_2,d_2}_A, A^{d_2,d_2}_A \right].
\]
On the other hand,
\[
T_N(Z \cap C) \subseteq (T_N \text{mod}^d_A) \cap C = \left[ Z^{U,U}_A, Z^{U,U}_A, Z^{V,V}_A, Z^{V,V}_A \right] \cap C \subseteq \left[ A^{d_1,d_1}_A, A^{d_1,d_1}_A, A^{d_2,d_2}_A, A^{d_2,d_2}_A \right],
\]
since \(Z^{U,V} = B^{U,V}\) (as \([U, V]^1_A = 0\)). This gives a contradiction, which finishes the proof. \(\square\)
If $\mathcal{V}$ is a subset of $\mathcal{E}_{\lambda}^{d_2,d_1}(k)$, for dimension vectors $d_1$ and $d_2$, then
\[ \text{hom}(\mathcal{V}) := \min\{[V,U]_\lambda : (U,V) \in \pi(\mathcal{V})\} \]
and
\[ \text{ext}(\mathcal{V}) := \min\{[V,U]_{\lambda}^1 : (U,V) \in \pi(\mathcal{V})\}, \]
where $\pi : \mathcal{E}_{\lambda}^{d_2,d_1} \to \text{mod}_\lambda^{d_1} \times \text{mod}_\lambda^{d_2}$ is the canonical projection. We will use the following consequence of Proposition 3.1.

**Corollary 3.2.** Assume $\Lambda$ is triangular and $\text{gldim} \Lambda \leq 2$. Let $d_1$ and $d_2$ be dimension vectors, $U \in \text{mod}_\lambda^{d_1}(k)$, $V \in \text{mod}_\lambda^{d_2}(k)$, $d := d_1 + d_2$, and $N := U \oplus V$. Let $Z$ be an irreducible component of $\text{mod}_\lambda^d(k)$, considered as an affine scheme with its reduced structure, such that $N \in Z$. Put 
\[ h := [V,U]_\lambda \quad \text{and} \quad e := [V,U]_{\lambda}^1. \]
Assume $\text{Ext}^1_\Lambda(U,V) = 0$ and there exists an open subset $\mathcal{V}$ of $Z \cap \mathcal{E}_{\lambda}^{d_2,d_1}(k)$ such that $N \in \mathcal{V}$ and 
\[ \text{hom}(\mathcal{V}) = h \quad \text{and} \quad \text{ext}(\mathcal{V}) = e. \]
If $\text{idim}_\lambda V \leq 1$ and there exists an exact sequence 
\[ \xi : 0 \to U \to M \to V \to 0 \]
with $\text{pdim}_\lambda M \leq 1$, then $N$ is a nonsingular point of $Z$.

**Proof.** We want to show that $\text{dim}_k T_N Z \leq \text{dim} Z$. Since $\text{dim} Z \geq a_\lambda(d)$, it is enough to show that $\text{dim}_k T_N Z \leq a_\lambda(d)$. Using Proposition 3.1 it is sufficient to find an appropriate upper bound for the dimension of $T_N(Z \cap \mathcal{E}_{\lambda}^{d_2,d_1})$. Since the scheme $Z \cap \mathcal{E}_{\lambda}^{d_2,d_1}$ is reduced at $N$ by Proposition 3.1, our assumptions imply that $T_N(Z \cap \mathcal{E}_{\lambda}^{d_2,d_1}) = T_N \mathcal{V} \subseteq T_N \mathcal{E}_{\lambda,h,e}^{d_2,d_1}$.

We first observe that $\text{pdim}_\lambda U \leq 1$. Indeed, if $X$ is a $\Lambda$-module and we apply $\text{Hom}_\lambda(-,X)$ to $\xi$, then we get an exact sequence 
\[ 0 = \text{Ext}^2_\Lambda(M,X) \to \text{Ext}^2_\Lambda(U,X) \to \text{Ext}^2_\Lambda(V,X) = 0, \]
as $\text{pdim}_\lambda M \leq 1$ and $\text{gldim} \Lambda \leq 2$. Since $\text{pdim}_\lambda U \leq 1$ and $\text{idim}_\lambda V \leq 1$, $\text{Ext}^2_\Lambda(N,N) = \text{Ext}^2_\Lambda(V,U)$. Let 
\[ Z = \begin{bmatrix} Z_{1,1} & Z_{2,1} \\ 0 & Z_{2,2} \end{bmatrix} \in T_N \mathcal{E}_{\lambda,h,e}^{d_1,d_2} \]
Then $Z_{1,1} \in \mathcal{Z}^{U,U}$, $Z_{2,1} \in \mathcal{Z}^{V,U}$, $Z_{2,2} \in \mathcal{Z}^{V,V}$, and $\pi(Z_{1,1}) \circ [\xi] + \pi(Z_{2,2}) = 0$, by Proposition 2.1. If we apply $\text{Hom}_\lambda(-,U)$ to $\xi$, then we get an exact sequence 
\[ \text{Ext}^1_\Lambda(U,U) \to \text{Ext}^2_\Lambda(V,U) \to \text{Ext}^2_\Lambda(M,U) = 0. \]
Consequently, the map 
\[ \mathcal{Z}^{U,U} \times \mathcal{Z}^{V,V} \to \text{Ext}^2_\Lambda(V,U), \ (Z_{1,1}, Z_{2,2}) \mapsto [\xi(Z_{1,1})] \circ [\xi] + [\xi] \circ [\xi(Z_{2,2})], \]

is an epimorphism, hence
\[ \dim_k T_N \mathcal{Z} \leq \dim_k \mathcal{Z}^U + \dim_k \mathcal{Z}^V + \dim_k \mathcal{Z}^{V/U} - \dim_k \text{Ext}_\Lambda^2(V, U). \]

Since \( \mathcal{Z}^U = B^U \) and \( \text{Ext}_\Lambda^2(V, U) = \text{Ext}_\Lambda^2(N, N) \), Proposition 3.1 implies
\[ \dim_k T_N \mathcal{Z} \leq \dim_k \mathcal{Z}^{N,N} - \dim_k \text{Ext}_\Lambda^2(N, N). \]

Using (2.3), we get
\[ \dim_k T_N \mathcal{Z} \leq \dim_k \mathcal{Z}^{d_1, d_2} - \chi_\Lambda(d) = a_\Lambda(d), \]
where the latter equality follows from (2.1), and the claim follows. \( \square \)

4. Existence of an exact sequence

In view of Corollary 3.2, the following result is important.

**Proposition 4.1.** Let \( d_1 \) and \( d_2 \) be dimension vectors, \( U \in \text{mod}_{\Lambda}^{d_1}(k) \), \( V \in \text{mod}_{\Lambda}^{d_2}(k) \), \( d := d_1 + d_2 \) and \( N := U \oplus V \). Let \( Z \) be a closed irreducible subset of \( \text{mod}_{\Lambda}^{d}(k) \) and let \( \mathcal{U} \) be a nonempty open subset of \( Z \) such that \( N \in Z \setminus \mathcal{U} \) and \( O_N \) is maximal in \( Z \setminus \mathcal{U} \). Assume \( \text{Ext}_\Lambda^1(U, V) = 0 \) and there exists a \( \Lambda \)-module \( H \) such that
\[ (4.1) \quad [H, V]_\Lambda > \min\{[H, M]_\Lambda : M \in Z\} \]
and \( \text{Ext}_\Lambda^1(H, V) = 0 \). Then there exist an exact sequence
\[ 0 \to U \to M \to V \to 0 \]
with \( M \in \mathcal{U} \).

The rest of the section is devoted to the proof of Proposition 4.1. We divide the proof into steps.

Let
\[ \mathcal{U}_0 := \{ M \in Z : [H, M]_\Lambda = d \}, \]
where \( d := \min\{[H, M]_\Lambda : M \in Z\} \). Then \( \mathcal{U}_0 \) is an open subset of \( Z \) and \( Z = \overline{\mathcal{U}_0} \). Since \( N \in Z \setminus \overline{\mathcal{U}_0} \), there exists a nonsingular curve \( C \) and a regular map \( \varphi : C \to Z \) such that \( \varphi(c_0) = N \), for some \( c_0 \in C \), and \( \varphi^{-1}(\mathcal{U}_0) \) is a cofinite subset of \( C \). Let \( R \) be the local ring of \( C \) at \( c_0 \) and \( t \) its uniformizing element. A map \( \varphi \) is represented by \( A \in \text{mod}_{\Lambda}^{d}(R) \) such that \( A/tA = N \). The next observation is inspired by [24, Lemma 3.3].

**Step 1.** There exist \( n \in \{2, 3, \ldots\} \cup \{\infty\} \) and \( g_i \in \text{GL}_d(R) \), \( 0 \leq i < n \), such that the following conditions are satisfied:

1. If \( 1 \leq i < n \) and \( A_i := g_{i-1} \ast A \), then
   \[ A_i/t^iA_i = U_i \oplus V_i, \]
   for some \( U_i \in \text{mod}_{\Lambda}^{d_1}(R/t^i) \) and \( V_i \in \text{mod}_{\Lambda}^{d_2}(R/t^i) \). Moreover, \( U_1 = U \), \( V_1 = V \), and if \( 1 < i < n \), then
   \[ U_i/t^{i-1}U_i = U_{i-1} \quad \text{and} \quad V_i/t^{i-1}V_i = V_{i-1}. \]
(2) If \( n < \infty \) and \( A_n := g_{n-1} * A \), then
\[
A_n/t^n A_n = \begin{bmatrix} U_n & t^{n-1} Z \\ 0 & V_n \end{bmatrix},
\]
for some \( U_n \in \text{mod}_{A}^{d_1}(R/t^n) \), \( V_n \in \text{mod}_{A}^{d_2}(R/t^n) \), and \( Z \in \mathbb{Z}^{V,U} \setminus \mathbb{B}^{V,U} \), such that
\[
U_n/t^{n-1} U_n = U_{n-1} \quad \text{and} \quad V_n/t^{n-1} V_n = V_{n-1}.
\]

**Proof.** We prove the claim by induction on \( i \). We take \( g_0 := \text{Id} \), hence the claim follows for \( i = 1 \). Assume \( n \) is not yet defined, \( i > 1 \), \( g_0, \ldots, g_{i-1} \) are already constructed, and \( A_1, \ldots, A_{i-1} \), defined as above, satisfy the above conditions. Then
\[
A_{i-1}/t^i A_{i-1} = \begin{bmatrix} U'_i & t^{i-1} Z' \\ 0 & V'_i \end{bmatrix},
\]
for \( U'_i \in \text{mod}_{A}^{d_1}(R/t^i) \), \( V'_i \in \text{mod}_{A}^{d_2}(R/t^i) \), \( Z \in \mathbb{A}^{d_2,d_1}(k) \) and \( Z' \in \mathbb{A}^{d_1,d_2}(k) \), such that
\[
U'_i/t^{i-1} U'_i = U_{i-1} \quad \text{and} \quad V'_i/t^{i-1} V'_i = V_{i-1}.
\]

If \( \rho \) is a relation in \( \Delta \), then one calculates that the lower-left coefficient of \( (A_{i-1}/t^i A_{i-1})_\rho \) is \( t^{i-1} Z_{\rho,U,V} \). Consequently, \( Z' \in \mathbb{Z}^{U,V} \). Since \( \text{Ext}_1^A(U,V) = 0 \), \( \mathbb{Z}^{U,V} = \mathbb{B}^{U,V} \), hence there exists \( h' \in \mathbb{V}^{d_1,d_2}(k) \) such that
\[
Z' = h' \circ U - V \circ h'.
\]

We put
\[
g' := \begin{bmatrix} \text{Id} & 0 \\
-t^{i-1} h' & \text{Id} \end{bmatrix} \in \text{GL}_d(R).
\]

If \( A' := g' * A_{i-1} \), then
\[
A'/t^i A' = \begin{bmatrix} U'_i & t^{i-1} Z' \\ 0 & V'_i \end{bmatrix}.
\]

Again \( Z \in \mathbb{Z}^{V,U} \). If \( Z \in \mathbb{B}^{V,U} \), then similarly as above we find \( g'' \in \text{GL}_d(R) \) such that the claim follows for \( i \), provided \( g_{i-1} := g'' g_{i-2} \), hence we may proceed by induction. Otherwise, we put \( n := i \) and \( g_{i-1} := g' g_{i-2} \), and finish the proof. \( \square \)

If we are in the latter case, we obtain our claim.

**Step 2.** Assume there exists \( g \in \text{GL}_d(R) \) and \( n \in \mathbb{N}_+ \) such that, if \( B := g * A \), then
\[
B/tB = U \oplus V \quad \text{and} \quad B/t^{n+1} B = \begin{bmatrix} U' & t^n Z \\ 0 & V' \end{bmatrix},
\]
for some \( U' \in \text{mod}_{A}^{d_1}(R/t^{n+1}) \), \( V' \in \text{mod}_{A}^{d_2}(R/t^{n+1}) \), and \( Z \in \mathbb{Z}^{V,U} \setminus \mathbb{B}^{V,U} \). There exists an exact sequence
\[
0 \rightarrow U \rightarrow M \rightarrow V \rightarrow 0
\]
Then there exists an exact sequence

\[ 0 \to U \to M \to V \to 0. \]

In particular, \(O_N \subseteq O_M\) (see for example [10, Section 1.1]). Moreover, \(M \not\cong N\) (since \(Z \not\subseteq B^U\)), hence \(O_N \neq O_M\).

Put \(g' := \begin{bmatrix} \id & 0 \\ 0 & t_{\lambda} \end{bmatrix} \in \text{GL}_d(K)\), where \(K\) is the field of fractions of \(R\), and \(B' := g' \cdot B\). Then \(B' \in \text{mod}^d_\Lambda(R)\) (although, \(g' \not\in \text{GL}_d(R)\)) and \(B' / tB' = M\). Hence, there exists a cofinite subset \(C'\) of \(C\) such that \(c_0 \in C\) and \(B'\) defines a regular map \(\psi: C' \to Z\) (we use that \(Z\) is \(\text{GL}_d(k)\)-invariant and closed) with \(\psi(c_0) = M\). In particular, \(M \in Z\). Since \(O_N\) is maximal in \(Z \setminus U\), \(M \in U\), and the claim follows. \(\square\)

The following observation finishes the proof.

**Step 3.** There do not exist \(g_i \in \text{GL}_d(R), i \in \mathbb{N}_+,\) such that the following conditions are satisfied, where, for \(i \in \mathbb{N}_+,\) \(A_i := g_i \cdot A:\)

1. if \(i \in \mathbb{N}_+,\) then \(A_i / t^i A_i = U_i \oplus V_i,\) for some \(U_i \in \text{mod}^d_\Lambda(R / t^i)\) and \(V_i \in \text{mod}^d_\Lambda(R / t^i),\) and
2. \(U_1 = U, V_1 = V,\) and, if \(i \in \mathbb{N}_+,\) then \(U_{i+1} / t^{i+1} U_{i+1} = U_i\) and \(V_{i+1} / t^{i+1} V_{i+1} = V_i.\)

**Proof.** Assume that such \(g_i, i \in \mathbb{N}_+,\) exist. We show this leads to a contradiction. Recall that \(U_i\) and \(V_i\) can be viewed as \(\Lambda\)-modules of dimension vectors \(d\mathbf{1}_i\) and \(d\mathbf{2}_i,\) respectively. Moreover, for each \(i \in \mathbb{N}_+,\) we have an exact sequence

\[(4.3) \quad 0 \to V \to V_{i+1} \to V_i \to 0.\]

For each \(i \in \mathbb{N}_+,\) let \(\varphi_i: C_i \to \text{mod}^d_\Lambda(k)\) be the regular map defined by \(A_i\), where \(C_i\) is a cofinite subset of \(C\) containing \(c_0\). Then, for each \(i \in \mathbb{N}_+,\) \(\varphi_i(c_0) = N\) and \(\varphi_i^{-1}(U_0)\) is a cofinite subset of \(C_i\).

If \(W \in \text{mod}^d_\Lambda(S),\) for a commutative ring \(S,\) then \(\text{Hom}_\Lambda(H, W)\) is the solution of the set of \(p\) linear homogeneous equations

\[ W_{\alpha} f_{s\alpha} - f_{t\alpha} H_{\alpha} = 0, \quad \alpha \in \Delta_1, \]

with coefficients in \(S\) and in \(q\) indeterminates, which form an element \(f\) of \(\mathbb{V}^{h-d}(S)\), where

\[ p := \sum_{\alpha \in \Delta_1} h(s\alpha) d(t\alpha), \quad q := \sum_{y \in \Delta_0} h(y) d(y), \]

and \(h := \text{dim} H.\) If we associate with \(W\) the matrix \(\Phi(W)\) of this system, we obtain a morphism \(\Phi: \text{mod}^d_\Lambda \to \mathbb{M}_{p,q}\) of schemes. We treat \(\mathbb{M}_{p,q}\) as the scheme of representations of the one arrow quiver (with
no relations) of dimension vector \((p, q)\). The map \(\Phi\) is equivariant in the following sense: there exists a morphism \(\Psi: \text{GL}_d \to \text{GL}_{(p,q)}\) of algebraic groups such that

\[ \Phi(g \ast W) = \Psi(g) \ast \Phi(W), \]

for all \(g \in \text{GL}_d\) and \(W \in \text{mod}^d_{\Lambda}\).

For \(r \in [0, \min(p,q)]\), let \(O_r\) be the \(\text{GL}_{(p,q)}(k)\)-orbit in \(M_{p,q}(k)\) consisting of the matrices of rank \(r\). The equation (4.2) implies that \(\Phi(M) \in O_{q-d}\), if \(M \in U_0\).

For \(i \in \mathbb{N}^+\), let \(\psi_i := \Phi \circ \varphi_i\) and \(B_i := \Phi(A_i)\) be the corresponding element of \(M_{p,q}(k)\). Note that \(B_i \simeq B := B_1\), for all \(i \in \mathbb{N}^+\), since the map \(\Phi\) is equivariant. Moreover \(\psi_i^{-1}(O_{q-d})\) is a cofinite subset of \(C_i\), hence \(B_i \otimes_R K \simeq L \otimes_k K\), where \(L\) is a chosen element of \(O_{q-d}\).

Consequently, by results of \([24, (3.3), (3.5)]\), there exists \(l\) such that

\[ B/t^{j+1}B \simeq (B/t^j B) \oplus L, \]

for all \(j \geq l\).

Fix \(j \geq l\). We calculate \([H, N_j]_\Lambda\), where \(N_j := A/t^j A\). Our assumptions imply \(N_j \simeq A_j/t^j A_j = U_j \oplus V_j\). In particular, \([H, N_j]_\Lambda \geq [H, V_j]_\Lambda\).

By applying \(\text{Hom}_\Lambda(N, -)\) to the sequence (4.3), we get the exact sequence

\[ 0 \to \text{Hom}_\Lambda(H, V) \to \text{Hom}_\Lambda(H, V_{j+1}) \to \text{Hom}_\Lambda(H, V_j) \to 0, \]

since \(\text{Ext}_\Lambda^1(H, V) = 0\). Consequently, by easy induction

\[ [H, N_j]_\Lambda \geq [H, V_j]_\Lambda = j \cdot [H, V]_\Lambda \geq jd + j, \]

since \([H, V]_\Lambda > d\) by assumption (4.1). On the other hand,

\[ [H, N_j] = qj - \text{rank}_k(B/t^j B), \]

where \(\text{rank}_k(B/t^j B)\) denotes the \(k\)-dimension of the image of the map \((R/t^j)^q \to (R/t^j)^p\) determined by \(B/t^j B\). The equation (4.4) implies,

\[ B/t^j B \simeq (B/t^j B) \oplus L^{j-l}. \]

Since \(L \in O_{q-d}\),

\[ [H, N_j]_\Lambda = qj - \text{rank}_k(B/t^j B) - (q-d)(j-l) = ql - \text{rank}_k(B/t^j B) + jd - ld = jd + ([H, N_l] - ld). \]

If \(j > [H, N_l] - ld\), we get a contradiction with (4.5). \(\square\)

5. **Main result**

Now we describe the setup in which we apply results of the previous two sections.

Let \(\Lambda\) be an algebra. Assume we are given full subcategories \(\mathcal{L}\) and \(\mathcal{R}\) of \text{mod}\(\Lambda\) such that the following conditions are satisfied:

1. \(\mathcal{L}\) and \(\mathcal{R}\) are closed under direct sums and direct summands;
(2) if $M \in \text{mod} \Lambda$, then there exist $U \in \mathcal{L}$ and $V \in \mathcal{R}$ with $M \simeq U \oplus V$;
(3) $\text{pdim}_\Lambda U \leq 1$, for each $U \in \mathcal{L}$, and $\text{idim}_\Lambda V \leq 1$, for each $V \in \mathcal{R}$;
(4) $\text{Hom}_\Lambda(V, U) = 0 = \text{Ext}^1_\Lambda(U, V)$, for all $U \in \mathcal{L}$ and $V \in \mathcal{R}$;
(5) if $d$ is a dimension vectors, then
\[
\mathcal{L}(d) := \{ M \in \text{mod}_\Lambda^d(k) : M \in \mathcal{L} \}
\]
and
\[
\mathcal{R}(d) := \{ M \in \text{mod}_\Lambda^d(k) : M \in \mathcal{R} \}
\]
are open and irreducible (if nonempty) subsets of $\text{mod}_\Lambda^d(k)$.

We call such a pair of subcategories of $\text{mod} \Lambda$ a geometric bisection.

Observe that in the above situation, if $d$ is a dimension vector, then $\text{mod}_\Lambda^d(k)$ is the disjoint union of the sets $\mathcal{L}(d_1) \oplus \mathcal{R}(d_2)$, for dimension vectors $d_1$ and $d_2$ such that $d_1 + d_2 = d$. Here and in the sequel, for subsets $\mathcal{Z}_1$ and $\mathcal{Z}_2$ of $\text{mod}_\Lambda^1(k)$ and $\text{mod}_\Lambda^2(k)$, respectively, we put
\[
\mathcal{Z}_1 \oplus \mathcal{Z}_2 := \{ M \in \text{mod}_\Lambda^{d_1 + d_2}(k) : M \simeq M_1 \oplus M_2 \text{ for some } M_1 \in \mathcal{Z}_1 \text{ and } M_2 \in \mathcal{Z}_2 \}.
\]
We stress that in general $\mathcal{Z}_1 \oplus \mathcal{Z}_2$ differs from the set
\[
\mathcal{Z}_1 \times \mathcal{Z}_2 := \{ M \in \text{mod}_\Lambda^{d_1 + d_2}(k) : M = M_1 \oplus M_2 \text{ for some } M_1 \in \mathcal{Z}_1 \text{ and } M_2 \in \mathcal{Z}_2 \} \subseteq \begin{bmatrix} \text{mod}_\Lambda^{d_1}(k) & 0 \\ 0 & \text{mod}_\Lambda^{d_2}(k) \end{bmatrix}.
\]

In addition to the above we assume the following. There exist $\Lambda$-modules $H_x$, $x \in \mathcal{X}$, where $\mathcal{X}$ is an index set, such that the following conditions are satisfied:

(1) if $d$ is a dimension vector and
\[
\mathcal{U}_x(d) := \{ M \in \text{mod}_\Lambda^d(k) : \text{Hom}_\Lambda(H_x, M) = 0 \} \quad (x \in \mathcal{X}),
\]
then
\[
(5.1) \quad \mathcal{L}(d) = \bigcup_{x \in \mathcal{X}} \mathcal{U}_x(d);
\]

(2) if $d$ is a dimension vector and
\[
\mathcal{U}^\prime_x(d) := \{ M \in \text{mod}_\Lambda^d(k) : \text{Ext}^1_\Lambda(H_x, M) = 0 \} \quad (x \in \mathcal{X}),
\]
then
\[
(5.2) \quad \mathcal{R}(d) = \bigcap_{x \in \mathcal{X}} \mathcal{U}^\prime_x(d).
\]

In the above situation we say that the geometric bisection $(\mathcal{L}, \mathcal{R})$ is determined by the modules $H_x$, $x \in \mathcal{X}$.

The following lemma is a direct consequence of Proposition 4.1.
Lemma 5.1. Let $d$ be a dimension vector such that $U := \mathcal{L}(d) \neq \emptyset$. If $Z := \overline{U}$, $N \in Z \setminus U$, and $\mathcal{O}_N$ is maximal in $Z \setminus U$, then there exists an exact sequence
\[ 0 \to U \to M \to V \to 0 \]
such that $N \simeq U \oplus V$, $U \in \mathcal{L}$, $V \in \mathcal{R}$, and $M \in \mathcal{U}$.

Proof. There exist $U \in \mathcal{L}$ and $V \in \mathcal{R}$ such that $N \simeq U \oplus V$. Without loss of generality we may assume $N = U \oplus V$. Condition (5.1) implies that there exists $x \in X$ such that
\[ \min \{ [H_x, M] : M \in Z \} = 0. \]
Put $H := H_x$. Note that $\dim V \neq 0$, since $N \not\in \mathcal{L}(d)$. Consequently, conditions (5.1) and (5.2) imply
\[ [H, V]_\Lambda > 0 = \min \{ [H, M] : M \in Z \} \]
and $[H, V]_\Lambda = 0$. Now the claim follows from Proposition 4.1. □

The following theorem will imply Theorem B.

Theorem 5.2. Let $d$ be a dimension vector such that $U := \mathcal{L}(d) \neq \emptyset$. If $N \in U$, then there exists $M \in U$ such that $N \in O_M$. In other words, $U = \bigcup_{M \in U} O_M$.

Proof. Put $Z := \overline{U}$. We may assume that $\mathcal{O}_N$ is maximal in $Z \setminus U$. By Lemma 5.1 there exists an exact sequence
\[ 0 \to U \to M \to V \to 0 \]
such that $N \simeq U \oplus V$ and $M \in \mathcal{U}$. Then $\mathcal{O}_N \subseteq \overline{O}_M$, hence the claim follows. □

For the proof of the next result we need the following fact. In the proof of this proposition we only use that $(\mathcal{L}, \mathcal{R})$ is a geometric bisection (not necessarily determined by modules $H_x$, $x \in X$). Consequently, it also has its dual version for $\mathcal{U} = \mathcal{R}(d)$.

Proposition 5.3. Let $d$ be a dimension vector such that $U := \mathcal{L}(d) \neq \emptyset$. Let $Z := \overline{U}$, $Z'$ be an irreducible component of $Z \setminus U$, and $d_1$ and $d_2$ be dimension vectors such that $Z' \cap (\mathcal{L}(d_1) \oplus \mathcal{R}(d_2))$ is dense in $Z'$. Then there exist $U \in \mathcal{L}(d_1)$, $V \in \mathcal{R}(d_2)$, and an open subset $V$ of $Z \cap \mathcal{E}(d_2 \cdot d_1)(k)$ such that, if $N := U \oplus V$, then $N \in Z'$, $\mathcal{O}_N$ is maximal in $Z \setminus U$, $N \in \mathcal{V}$, and $\text{ext}(\mathcal{V}) = [V, U]_\Lambda$.

Proof. We first make an auxiliary observation. Let $N' \simeq U' \oplus V'$, for $U' \in \mathcal{L}(d_1)$ and $V' \in \mathcal{R}(d_2)$. Then
\[ (5.3) \quad [N', N']_\Lambda = [V', U']_\Lambda = \langle d_2, d_1 \rangle + [V', U']_\Lambda. \]
For the rest of the proof we fix $U \in \mathcal{L}(d_1)$ and $V \in \mathcal{R}(d_2)$ such that, if $N := U \oplus V$, then $N \in \mathcal{Z}'$, $N$ does not belong to an irreducible component of $\mathcal{Z} \setminus \mathcal{U}$ different from $\mathcal{Z}'$, $O_N$ is maximal in $\mathcal{Z} \setminus \mathcal{U}$, and

$$[N, N]_\Lambda^2 = \min\{[N', N']_\Lambda^2 : N' \in \mathcal{Z}'\}.$$ 

Put $e := [V, U]_\Lambda^1$. Then (5.3) implies that

$$(5.4) \quad [N', N']_\Lambda^2 \geq \langle d_2, d_1 \rangle + e,$$ 

for each $N' \in \mathcal{Z}'$.

Let $\mathcal{E}'$ be the open subset of $\mathcal{E}_{d_2, d_1}(k)$ consisting of $[u'' \ z']$ such that $U'' \in \mathcal{L}(d_1)$ and $V'' \in \mathcal{R}(d_2)$. Then, $\mathcal{Z} \cap \mathcal{E}'$ is an open subset of $\mathcal{Z} \cap \mathcal{E}_{d_2, d_1}(k)$ containing $N$. Let $\mathcal{V}$ be the subset of $\mathcal{Z} \cap \mathcal{E}'$ obtained by subtracting all components of $\mathcal{Z} \cap \mathcal{E}_{d_2, d_1}(k)$, which do not contain $N$. In particular, $N \in \mathcal{V}$ and $\mathcal{V}$ is an open subset of $\mathcal{Z} \cap \mathcal{E}_{d_2, d_1}(k)$. We show that $\text{ext}(\mathcal{V}) = e$, and this will finish the proof.

Assume this is not the case. Let $\mathcal{U}'$ be the set of $[u'' \ z']$ in $\mathcal{V}$ such that $[V, U']_\Lambda^1 < e$. Then there exists an irreducible component $\mathcal{V}_0$ of $\mathcal{V}$ such that $U_0 \in \mathcal{L}(d_1)$ and $V_0 \in \mathcal{R}(d_2)$, then $\mathcal{U}_0$ is an open subset of $\mathcal{V}_0$ and $N \in \mathcal{V}_0 \subseteq \mathcal{U}_0$, hence there exists an irreducible curve $\mathcal{C}$ and a regular map $\varphi: \mathcal{C} \to \mathcal{V}_0$ such that $\varphi(c_0) = N$, for some $c_0 \in \mathcal{C}$, and $\varphi(\mathcal{C}) \cap \mathcal{U}_0 \neq \emptyset$. Write $\varphi = [\varphi_{1,1} \varphi_{2,1}]$ (in particular, $\varphi_{1,1}: \mathcal{C} \to \mathcal{L}(d_1)$ and $\varphi_{2,2}: \mathcal{C} \to \mathcal{R}(d_2)$) and put

$$\psi := \begin{bmatrix} \varphi_{1,1} & 0 \\ 0 & \varphi_{2,2} \end{bmatrix} : \mathcal{C} \to \mathcal{L}(d_1) \times \mathcal{R}(d_2).$$

Obviously $\psi(c_0) = N$. Moreover, $\psi(\mathcal{C}) \cap \mathcal{U}_0 \neq \emptyset$. Indeed, fix $c \in \mathcal{C}$ such that $\varphi(c) = [u'' \ z'] \in \mathcal{U}_0$. Then $\psi(c) = [U'' \ 0 \ z'] \in \mathcal{U}'$. Moreover,

$$\begin{bmatrix} U'' & \lambda Z \\ 0 & V'' \end{bmatrix} = \begin{bmatrix} \lambda \text{Id} & 0 \\ 0 & \text{Id} \end{bmatrix} \ast \begin{bmatrix} U'' & Z \\ 0 & V'' \end{bmatrix} \in \mathcal{U}_0 \subseteq \mathcal{V}_0,$$

for each $\lambda \in k \setminus \{0\}$, since $\mathcal{U}_0$ is closed, $\psi(c) \in \mathcal{V}_0 \cap \mathcal{U}' = \mathcal{U}_0$. Using that $\mathcal{Z}$ is GL$_d(k)$-invariant and closed, we show similarly that $\psi(\mathcal{C}) \subseteq \mathcal{Z}$. Additionally, $\psi(\mathcal{C}) \cap \mathcal{U} = \emptyset$, hence $\psi(\mathcal{C}) \subseteq \mathcal{Z} \setminus \mathcal{U}$. Since $\psi(\mathcal{C})$ is irreducible, $N \in \psi(\mathcal{C})$, and $\mathcal{Z}'$ is the unique irreducible component of $\mathcal{Z} \setminus \mathcal{U}$ containing $N$, $\psi(\mathcal{C}) \subseteq \mathcal{Z}'$. Consequently, $\mathcal{Z}' \cap \mathcal{U}_0 \neq \emptyset$. However, if $N' \in \mathcal{Z} \cap \mathcal{U}_0$, then (5.3) implies $[N', N']_\Lambda^2 < \langle d_2, d_1 \rangle + e$, which contradicts (5.4). \hfill \Box

The following theorem is a more general version of Theorem A.

**Theorem 5.4.** Assume $\Lambda$ is triangular and gldim $\Lambda \leq 2$. Let $d$ be a dimension vector such that $\mathcal{U} := \mathcal{L}(d) \neq \emptyset$. Then $\overline{\mathcal{U}}$ is regular in codimension one.

**Proof.** Let $\mathcal{Z} := \overline{\mathcal{U}}$. Assume first $N \in \mathcal{U}$. Then pdim$_\Lambda N \leq 1$. Consequently, we may apply Corollary 3.2 with $U = N$ and $V = 0$, and get that $N$ is a nonsingular point of $\mathcal{Z}$. 


Now let $Z'$ be an irreducible component of $Z \setminus U$. In order to finish the proof it is enough to find $N \in Z'$ such that $N$ is a nonsingular point of $Z$. From Proposition 5.3 we know there exist $U \in \mathcal{L}(d_1), V \in \mathcal{R}(d_2)$, and an open subset $V$ of $Z \cap E_{d_2} \Lambda(k)$ such that, if $N := U \oplus V$, then $N \in Z'$, $O_N$ is maximal in $Z \setminus U$, $N \in V$, and $\text{ext}(V) = [V,U]_1^{\Lambda}$. Moreover, by Lemma 5.1 there exists an exact sequence
\[ 0 \to U \to M \to V \to 0 \]
with $\text{pdim}_\Lambda M \leq 1$, hence the claim follows from Corollary 3.2.

6. Applications

In this section we present applications of the results of Section 5, which include Theorems A and B.

6.1. Periodic modules over concealed canonical algebras. Let $\Lambda$ be a concealed canonical algebra. Then $\Lambda$ is triangular and $\text{gldim} \Lambda \leq 2$ [16]. We describe a geometric bisection of $\mod \Lambda$. According to [16] there exists a sincere separating family $T = (T_x)_{x \in \mathbb{P}^1(k)}$. Let $X_0$ be the set of $x \in \mathbb{P}^1(k)$, such that $T_x$ is homogeneous, and $X \supset X_0$ be a set indexing (the isomorphism classes of) the modules lying the mouths of the tubes $T_x$, $x \in \mathbb{P}^1(k)$. Fix, for each $x \in X$, the corresponding module $H_x$. Then there exists a dimension vector $h$ such that $\dim H_x = h$, for each $x \in X_0$. We put:

\[ \mathcal{L} := \{ M \in \mod \Lambda : \langle h, X \rangle \leq 0, \text{ for each indecomposable direct } X \text{ summand of } M \}, \]

and

\[ \mathcal{R} := \{ M \in \mod \Lambda : \langle h, X \rangle > 0, \text{ for each indecomposable direct } X \text{ summand of } M \}. \]

It follows from the representation theory of concealed canonical algebras (we refer to [17, 18]) and [3, Section 3], that $(\mathcal{L}, \mathcal{R})$ is a geometric bisection of $\mod \Lambda$ determined by $H_x$, $x \in X$. Consequently, Theorems 5.2 and 5.4 give the following.

**Theorem 6.1.** If $\Lambda$ is a concealed canonical algebra and $d$ is a dimension vector such that $U := \mathcal{L}(d) \neq \emptyset$, then $\overline{U}$ is regular in codimension one and

\[ \overline{U} = \bigcup_{M \in U} \overline{O}_M. \]

We explain now how the above implies Theorems A and B. If $\Lambda$ is tame, then Theorem A follows from [7, Theorem 1] and there is nothing to prove in Theorem B. Thus assume $\Lambda$ is wild. In this case, let $d$ is a dimension vector and

\[ U := \{ M \in \mod^d \Lambda(k) : M \text{ is } \tau\text{-periodic} \}. \]
If $\mathcal{U} \neq \emptyset$, then $\mathcal{U} = \mathcal{L}(\mathbf{d})$ (by the representation theory of concealed canonical algebras), thus Theorems A and B are direct consequences of Theorem 6.1.

We make one more comment. Let $\Lambda$ be a wild concealed canonical algebra and $\mathcal{U}$ as above. Assume $\mathcal{U} \neq \emptyset$ and put $\mathcal{Z} := \overline{\mathcal{U}}$. Theorem B implies that if $\mathcal{O}_M$ is maximal in $\mathcal{Z}$, then $M$ is $\tau$-periodic. The representation theory of concealed canonical algebras implies that $M$ is a direct sum of modules from $\mathcal{T}$. Then it is standard that we may rewrite the description of maximal orbits in $\mathcal{Z}$ from [7, Proposition 5] (see also [19, Theorem 3.5]).

6.2. Directing modules. A $\Lambda$-module $M$ is called directing if there is no sequence

$$X_0 \to X_1 \to \cdots \to X_n$$

of nonzero maps between indecomposable modules such that $X_0$ and $X_n$ are direct summands of $M$, and there exists $0 < i < n$, such that $\tau X_{i+1} \simeq X_{i-1}$. The following is the main result of [4].

**Theorem 6.2.** If $M$ is a directing, then $\overline{\mathcal{O}}_M$ is regular in codimension one.

The proof of Theorem 6.2 in [4] contains a gap. Roughly speaking, we consider the intersection of $\overline{\mathcal{O}}_M$ with $\text{mod}_{\Lambda}^{d_1} \times \text{mod}_{\Lambda}^{d_2}$ (for suitable dimension vectors $d_1$ and $d_2$) and assume, without proving, it is a reduced scheme. Now we explain how to correct the proof.

The detailed analysis presented in [4] shows that we may assume $\Lambda$ is triangular, $\text{gldim} \Lambda \leq 2$, we have a geometric bisection $(\mathcal{L}, \mathcal{R})$ of $\text{mod} \Lambda$, and $\overline{\mathcal{O}}_M = \overline{\mathcal{R}(\text{dim} M)}$. Moreover, [4, Corollary 4.4] implies that if $\mathcal{O}_N$ is maximal in $\overline{\mathcal{O}}_M \setminus \overline{\mathcal{O}}_M$, then there exists an exact sequence

$$0 \to U \to M \to V \to 0$$

such that $N \simeq U \oplus V$, $\text{pdim}_\Lambda M \leq 1$ and $\text{idim}_\Lambda V \leq 1$. Consequently, Theorem 6.2 follows from the dual of Proposition 5.3 and Corollary 3.2.

6.3. Irreducible components of module varieties for dimension vectors of indecomposable modules over tame quasi-titled algebras. An algebra $\Lambda$ is called quasi-tilted if $\text{gldim} \Lambda \leq 2$ and, for each indecomposable $\Lambda$-module $X$, $\text{pdim}_\Lambda X \leq 1$ or $\text{idim}_\Lambda X \leq 1$. The following theorem is the main result of [5].

**Theorem 6.3.** Let $\Lambda$ be a tame quasi-tilted algebra and $\mathbf{d}$ the dimension vector of an indecomposable $\Lambda$-module. If $\mathcal{Z}$ is an irreducible component of $\text{mod}_\Lambda^n(k)$, then $\mathcal{Z}$ is nonsingular in codimension one.

The proof of this theorem in [5] contains a gap similar to that described in Subsection 6.2. However, in the situation in which the gap occurs we have a geometric bisection $(\mathcal{L}, \mathcal{R})$ determined by a family of modules such that $\mathcal{Z} = \overline{\mathcal{L}(\mathbf{d})}$. Since every quasi-tilted algebra is
triangular [15, Propostion III.1.1], we may use Theorem 5.4 instead of the original arguments.

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