On third homologies of groups and of quandles
via the Dijkgraaf-Witten invariant and Inoue-Kabaya map

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Abstract

We propose a simple method to produce quandle cocycles from group cocycles, as a modification of Inoue-Kabaya chain map. We further show that, in respect to “universal central extended quandles”, the chain map induces an isomorphism between their third homologies. For example, all Mochizuki’s quandle 3-cocycles are shown to be derived from group cocycles of some non-abelian group. As an application, we calculate some \(Z\)-equivariant parts of the Dijkgraaf-Witten invariants of some cyclic branched covering spaces, via some cocycle invariant of links.

Keywords quandle, group homology, 3-manifolds, link, branched covering, Massey product

1 Introduction

A quandle, \(X\), is a set with a binary operation whose definition was partially motivated from knot theory. Fenn-Rourke-Sanderson \cite{FRS1,FRS2} defined a space \(BX\) called rack space, in analogy to the classifying spaces of groups. Furthermore, Carter et. al \cite{CJKLS,CKS} introduced quandle cohomologies \(H^*_Q(X;A)\) with local coefficients, by slightly modifying the cohomology of \(BX\); they further defined combinatorially a state-sum invariant \(I_\psi(L)\) of links \(L\) constructed from a cocycle \(\psi \in H^*_Q(X;A)\). The construction can be seen as an analogue of the Dijkgraaf-Witten invariant \cite{DW} of closed oriented 3-manifolds \(M\) constructed from a finite group \(G\) and a 3-cocycle \(\kappa \in H^3_{gr}(G;A)\): To be specific, the invariant is defined as the formal sum of pairings expressed by

\[
\text{DW}_\kappa(M) := \sum_{f \in \text{Hom}_{gr}(\pi_1(M),G)} \langle f^*(\kappa), [M] \rangle \in \mathbb{Z}[A], \tag{1}
\]

where \([M]\) is the fundamental class in \(H_3(M;\mathbb{Z})\). Inspired by this analogue, for many quandles \(X\), the author \cite{No3} gave essentially topological meanings of the cocycle invariants with using the Dijkgraaf-Witten invariant and the homotopy group \(\pi_2(BX)\).

We mainly focus on a relation between quandle homology and group one. There are several such studies. For example, the second quandle homology is well-studied by Eisenmann \cite{Eis} from first group homologies. In addition, the author \cite{No3} roughly computed some third quandle homologies from the group homologies of \(\pi_1(BX)\) with some ambiguity. Furthermore, for any quandle \(X\), Inoue-Kabaya \cite{IK} constructed a chain map \(\varphi_{IK}\) from the quandle complex to a certain complex. Although the latter complex seems far from something familiar, Kabaya \cite{Kab} modified the \(\varphi_{IK}\) mapping to a group homology under a certain strong condition of \(X\). Furthermore, for certain special quandles, the author \cite{No2} proposed a method to construct quandle cocycles from invariant theory via the chain map.
This paper demonstrates a relation between third homologies of groups and those of quandles via the Inoue-Kabaya map, with respect to a broad subclass of quandles. Here a quandle in the subclass is defined as a group \(G\) with an operation \(g \triangleleft h := \rho(gh^{-1})h\) for \(g, h \in G\), where \(\rho : G \to G\) is a fixed group isomorphism (Definition 2.1). Denote such a quandle by \(X = (G, \rho)\). Furthermore, denote by \(H^\text{gr}_{n}(G; \mathbb{Z})\) a quotient of the group homology of \(G\) subject to the action by the \(\rho\), called the \(\mathbb{Z}\)-coinvariant. In \(\S 2.2\), we reformulate the Inoue-Kabaya map, \(\Phi_{n}\), which induces a homomorphism

\[
(\Phi_{n})_{\ast} : H^Q_{n}(X; \mathbb{Z}) \longrightarrow H^\text{gr}_{n}(G; \mathbb{Z}).
\]

Furthermore, we lift this map \(\Phi_{n}\) to being a chain map \(\varphi_{n}\) from \(C^Q_{n}(X; \mathbb{Z})\) to the usual group homology \(H^\text{gr}_{n}(G; \mathbb{Z})\); see Proposition 2.6. As a corollary, if found a presentation of a group \(n\)-cocycle \(\kappa\) of \(G\), we easily obtain that of the induced quandle \(n\)-cocycle \(\varphi_{n}^{\ast}(\kappa)\). Hence, this approach is expected to be useful of computing the quandle cocycle invariant constructed from such a quandle \((G, \rho)\).

This paper moreover investigates properties of the chain map \(\Phi_{n}\) above. To begin with, we focus on a class of universal quandle coverings \(\widetilde{Y}\), introduced by Eisemann [Eis]. Roughly speaking about this \(\widetilde{Y}\), given a “connected” quandle \(Y\) of finite order, we can set the quandle \(\widetilde{Y}\) and an epimorphism \(p_{Y} : \widetilde{Y} \to Y\) (as a quandle covering); further the quandle \(\widetilde{Y}\) is of the form \((\text{Ker}(\epsilon_{Y}), \rho)\) for some group \(\text{Ker}(\epsilon_{Y})\); see Example 2.3 for details. We then show that the associated chain map \(\tilde{\Phi}_{3}\) induces an isomorphism

\[
(\tilde{\Phi}_{3})_{\ast} : H^Q_{3}(\widetilde{Y}) \cong H^\text{gr}_{3}(\text{Ker}(\epsilon_{Y}));\quad \text{up to } t_{Y}\text{-torsion,}
\]

where \(t_{Y} \in \mathbb{N}\) is the minimal satisfying \(\rho^{t_{Y}} = \text{id}\) (Theorem 2.11).

Needless to say, the \(\Phi_{3}\) is not always isomorphic for such quandles \((G, \rho)\); However, by the help of the universality of coverings, in some cases we can analyse the map \(\Phi_{3}\) as follows.

Next, we will demonstrate Mochizuki quandle 3-cocycles [Moc], which are most known quandle 3-cocycles so far. Consider quandles \(Y\) of the forms \((\mathbb{F}_{q}, \times \omega)\) with \(\omega \in \mathbb{F}_{q}\), called Alexander quandle usually, where we regard the finite field \(\mathbb{F}_{q}\) as an additive group and the symbol \(\times \omega\) is a \(\omega\)-multiple of \(\mathbb{F}_{q}\). He found all 3-cocycles of \(Y\) by solving a certain differential equations over \(\mathbb{F}_{q}\) and his statement was a little complicated (see 4.1). However, in this paper we easily obtain and explain his all 3-cocycles from some group 3-cocycles via the map \(\Phi_{n}^{\ast}\) (see (20) and Lemma 4.6). Moreover, we show that the third quandle cohomology \(H^3_Q(Y; \mathbb{F}_q)\) is isomorphic to a sum of some group homologies via the maps \(\Phi_{2}, \Phi_{3}\) and \(\tilde{\Phi}_{3}\) (see Theorem 2.14 in details). In conclusion, all the Mochizuki 3-cocycles stems from some group 3-cocycles via the three maps.

Furthermore, we propose a relation to a partial sum of some Dijkgraaf-Witten invariants of \(\hat{C}_{L}\), where \(\hat{C}_{L}\) denotes the \(t\)-fold cyclic covering space of \(S^3\) branched over a link \(L\). See [9] for the detailed definition of the partial sum, and denote it by \(DW^\ast_{n}(\hat{C}_{L}) \in \mathbb{Z}[A]_{\ast}\). To be specific, we show (Theorem 2.16) that if the induced map \(p_{Y}^{\ast} : H^3_{Q}(Y; A) \to H^3_{Q}(\widetilde{Y}; A)\) is surjective, and if \(Y\) is connected and of finite order, then any group 3-cocycle \(\kappa\) of the
above group $\text{Ker}(\epsilon_Y)$ admits some quandle 3-cocycle $\psi$ of $Y$ for which the equality
\[
\text{DW}^\mathbb{Z}_\kappa(\widehat{C}_L) = I_\psi(L) \in \mathbb{Z}[A]
\]
holds. Here the right side $I_\psi(L)$ is the quandle cocycle invariant of links $L$ [CKS] (see Remark 2.17 for some quandles satisfying the assumption on $p_Y$). While the equivalence of the two invariants was implied in the previous paper [No3] by abstract nonsense, the point is that the cocycle $\psi$ is definitely obtained from the chain map $\tilde{\Phi}_3$.

We here emphasize that our theorem serves as computing some parts of the Dijkgraaf-Witten invariants $\text{DW}^\mathbb{Z}_\kappa(\widehat{C}_L)$ via the right invariant $I_\psi(L)$. A known standard way to compute the invariant is to find a fundamental class from a triangulation of $M$ (see [DW, Wa]). However, in computing them via the right invariant $I_\psi(L)$, we use no triangulation of $M$ and many quandle 3-cocycles are simpler than group ones (in our experience).

In fact, in [5] we succeed in some computations of the formal sums $\text{DW}^\mathbb{Z}_\kappa(\widehat{C}_L)$ by using the Mochizuki 3-cocycles, which are derived from triple Massey products of a meta-abelian group $G_X$ (see Proposition 4.7). For example, we will calculate the cocycle invariants of the torus knots $T(m, n)$ (see Theorem 5.1); hence we obtain the partial sum $\text{DW}^\mathbb{Z}_\kappa(\widehat{C}_K)$ of the Brieskorn manifold $\Sigma(m, n, t)$, which is the covering space branched over the knot $T(m, n)$. Furthermore, as a special case $\omega = -1$, we compute the cocycle invariant of some knots $K$, and, hence, obtain some values $\text{DW}^\mathbb{Z}_\kappa(\widehat{C}_K)$ for the double covering spaces branched along $K$ (see Table 1 in §5.1).

This paper is organized as follows. In §2, we introduce a lift of Inoue-Kabaya chain map and state theorems. In §3, we prove Theorems 2.11 and 2.16. In §4, we show that Mochizuki 3-cocycles are derived from some group 3-cocycles. In §5, we calculate some partial sum of the Dijkgraaf-Witten invariants.

**Notation and convention** A symbol $\mathbb{F}_q$ is a finite field of characteristic $p > 0$. Denote $H^n_G(G)$ the group homology of a group $G$ with trivial integral coefficients. Furthermore we assume that manifolds are smooth, connected, oriented.

## 2 Results

In §2.3 and 2.4, we state theorems. For this, we briefly review quandle homologies and their properties in §2.1, and we modify the Inoue-Kabaya map in 2.2.

### 2.1 Review of quandles and quandle cohomologies

We start by recalling basic concepts about quandles. A **quandle**, $X$, is a set with a binary operation $(x, y) \rightarrow x \triangleleft y$ such that, for any $x, y, z \in X$, $x \triangleleft x = x$, $(x \triangleleft y) \triangleleft z = (x \triangleleft z) \triangleleft (y \triangleleft z)$ and there exists uniquely $w \in X$ such that $w \triangleleft y = x$. A quandle $X$ is said to be of type $t_X$, if $t_X > 0$ is the minimal $N$ number satisfying $a = (\cdots (a \triangleleft b) \cdots) \triangleleft b$ $[N$-times on the
right with \(b\) for any \(a, b \in X\). Furthermore, the associated group of \(X\), \(\text{As}(X)\), is defined to be the group presented by

\[
\text{As}(X) := \langle e_x \mid (x \in X) \mid e_x^{-1} e_y^{-1} e_x e_y \quad (x, y \in X) \rangle.
\]

The group \(\text{As}(X)\) acts on \(X\) by the formula \(x \cdot e_y := x < y\) for \(x, y \in X\). If the action is transitive, \(X\) is said to be connected. Furthermore, take a homomorphism \(\epsilon_X : \text{As}(X) \to \mathbb{Z}\) sending \(e_x\) to 1; so we have a group extension

\[
0 \longrightarrow \text{Ker}(\epsilon_X) \overset{\iota}{\longrightarrow} \text{As}(X) \overset{\epsilon_X}{\longrightarrow} \mathbb{Z} \longrightarrow 0 \quad \text{(exact)}.
\]

Next, we introduce a subclass of quandles which we mainly use in this paper.

**Definition 2.1** ([Joy] §4). Fix a group \(G\) and a group isomorphism \(\rho : G \to G\). Equip \(X = G\) with a quandle operation by setting

\[
g \triangleleft h := \rho(gh^{-1})h.\]

Note that the quandle \((G, \rho)\) is of type \(t_X\), if and only if the \(t_X\)-th power of \(\rho\) is the identity, i.e., \(\rho^{t_X} = \text{id}_G\).

Although this class of such quandles \((G, \rho)\) is a subclass of quandles, it includes interesting quandles as follows:

**Example 2.2** (Alexander quandle). Let \(X = G\) be an abelian group. Denoting \(\rho\) by \(T\) instead, we can regard \(X\) as a \(\mathbb{Z}[T^{\pm 1}]\)-module. Then the quandle operation is rewritten in

\[
x \triangleleft y := Tx + (1 - T)y,
\]

called Alexander quandle. Given a finite field \(\mathbb{F}_q\) and \(\omega \in \mathbb{F}_q^*\) with \(\omega \neq 1\), the quandle of type \(X = \mathbb{F}_q[T]/(T - \omega)\) is called Alexander quandle on \(\mathbb{F}_q\) with \(\omega\).

The type \(t_X\) of \(X\) equals the minimal \(n\) satisfying \(T^n = 1\) in \(X\). We easily check that \(X\) is connected if and only if \((1 - T)\) is invertible.

**Example 2.3** (Universal quandle covering). Given a connected quandle \(X\), consider the kernel \(G = \text{Ker}(\epsilon_X)\) in \([3]\). Fix \(a \in X\). Using a group homomorphism \(\rho_a : \text{Ker}(\epsilon_X) \to \text{Ker}(\epsilon_X)\) define by \(\rho_a(g) = e_a^{-1} g e_a\), we have a quandle \(\widetilde{X} = (\text{Ker}(\epsilon_X), \rho_a)\), called extended quandle of \(X\). We easily see the independence of the choice of \(a \in X\) up to quandle isomorphisms.

Considering the restriction of the action \(X \curvearrowright \text{As}(X)\) to \(\text{Ker}(\epsilon_X)\), a map \(p_X : \widetilde{X} \to X\) sending \(g\) to \(a \cdot g\) is known to be a quandle homomorphism (see [Joy] Theorem 4.1)], and called (universal quandle) covering \([12]\). It can easily be seen that if \(X\) is of type \(t_X\) and of finite order, so is \(\widetilde{X}\). Furthermore, the \(\widetilde{X}\) is shown to be connected \([No3, \text{Lemma 6.8}]\).

Finally, we briefly review the quandle complexes introduced by \([CJKLS]\). Let \(X\) be a quandle. Let us construct a complex by putting the free \(\mathbb{Z}\)-module \(C^R_n(X)\) spanned by \((x_1, \ldots, x_n) \in X^n\) and letting its boundary \(\partial^R_n(x_1, \ldots, x_n) \in C^R_{n-1}(X)\) be

\[
\sum_{2 \leq i \leq n} (-1)^i ((x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) - (x_1 \triangleleft x_i, \ldots, x_{i-1} \triangleleft x_i, x_{i+1}, \ldots, x_n)).
\]
The composite \( \partial^R_{n-1} \circ \partial^R_n \) is zero. The pair \((C^R_n(X), \partial^R_n)\) is called rack complex. Let \(C^D_n(X)\) be a submodule of \(C^*_n(X)\) generated by \(n\)-tuples \((x_1, \ldots, x_n)\) with \(x_i = x_{i+1}\) for some \(i \in \{1, \ldots, n-1\}\) if \(n \geq 2\); otherwise, let \(C^D_1(X) = 0\). Since \(\partial^R_n(C^D_n(X)) \subset C^D_{n-1}(X)\), we can define a complex \((C^*_n(X), \partial_*)\) by the quotient \(C^R_n(X)/C^D_n(X)\). The homology \(H^R_n(X)\) is called quandle homology of \(X\). Dually, we can define the cohomologies \(H^n_R(\mathbb{A}; A)\) and \(H^n_Q(\mathbb{A}; A)\) with a commutative ring \(A\).

However, the second term of the differential \(\partial^R_n\) seems incomprehensible from the definition. In the next subsection, for a quandle of the form \((\mathbb{G}, \rho)\), we give a simple formula of the \(\partial^R_n\).

### 2.2 A lift of Inoue-Kabaya chain map

We now construct a chain map \((\bigcirc)\) with respect to a class of quandles in Definition 2.1. Our construction is a modification of Inoue-Kabaya map \([IK, \S 3]\) (see Remark 2.8).

In this subsection, we often denote \(\rho(x)\) by \(x^\rho\) and \(\rho^2(x)\) by \(x^{\rho^2}\) for short, respectively.

For quandles \(X\) of the forms \((\mathbb{G}, \rho)\) in Definition 2.1, we will reformulate the rack complex \(C^R_n(X)(\cong \mathbb{Z}(G^n))\) in non-homogeneous coordinates. Define another differential \(\partial^R_G : C^R_n(X) \to C^R(X)\) by setting

\[
\partial^R_G(g_1, \ldots, g_n) := \sum_{1 \leq i \leq n-1} (-1)^i \left( (g_1, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_n) - (g_1^\rho, \ldots, g_{i-1}^\rho, g_i^\rho g_{i+1}, g_{i+2}, \ldots, g_n) \right).
\]

We easily check \(\partial^R_{n-1} \circ \partial^R_G = 0\), and can further see

**Lemma 2.4.** Take a bijection \(\mathbb{G}^n \to \mathbb{G}^n\) defined by setting

\[
(x_1, \ldots, x_n) \mapsto (x_1 x_2^{-1}, x_2 x_3^{-1}, \ldots, x_{n-1} x_n^{-1}, x_n).
\]

This map yields a chain isomorphism \(\Upsilon : (C^R_n(X), \partial^R_n) \cong (C^R_n(X), \partial^R_G)\).

**Proof.** By direct calculation. \(\square\)

Furthermore, we define a subcomplex \(D_n(G)\) generated by \(n\)-tuples \((g_1, \ldots, g_n)\) such that \(g_i = 1\) for some \(i \leq n - 1\). We denote the quotient complex by \(C^Q_n(X)\). By Lemma 2.4 this homology, \(H^Q_n(X)\), is isomorphic to the quandle homology \(H^R_n(X)\) in \(\S 2.1\).

We briefly review normalized chain complexes of groups, \(C^\text{gr}_n(G)\), in non-homogeneous terms (see, e.g. \([Bro]\)) as follows: Let \(\mathcal{C}^\text{gr}_n(G)\) denote the free \(\mathbb{Z}\)-module generated by \(\mathbb{G}^n\), and let its boundary map \(\partial^\text{gr}_n(g_1, \ldots, g_n) \in \mathcal{C}^\text{gr}_{n-1}(G)\) be

\[
(g_2, \ldots, g_n) + \sum_{1 \leq i \leq n-1} (-1)^i \left( (g_1, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_n) - (g_1^\rho, \ldots, g_{i-1}^\rho, g_i^\rho g_{i+1}, g_{i+2}, \ldots, g_n) \right).
\]

Furthermore, concerning the submodule \(D_n(G)\) mentioned above, we easily check \(\partial^\text{gr}_n(D_n(G)) \subset D_{n-1}(G)\). We denote by \(C^\text{gr}_n(G)\) the quotient complex of \(\mathcal{C}^\text{gr}_n(G)\) modulo \(D_n(G)\). As is well-known, this homology coincides with the usual group homology of \(G\) (see \([Bro, \S I.5]\)).

We next construct a chain map \(\varphi_n\) from the complex \(C^R_n(X)\) to another \(C^\text{gr}_n(G)\).
Definition 2.5. Assume that a quandle \(X\) of the form \((G, \rho)\) is of type \(t_X\). Take a set

\[ \mathcal{K}_n := \{(k_1, \ldots, k_n) \in \mathbb{Z}^n \mid 0 \leq k_i - k_{i-1} \leq 1, \ 0 \leq k_n \leq t_X - 1 \}. \]

of order \(t_X 2^{n-1}\). We define a homomorphism \(\varphi_n : C^R_n(G) \to C^\Delta_n(G)\) by setting

\[ \varphi_n(g_1, g_2, \ldots, g_n) = \sum_{(k_1, \ldots, k_n) \in \mathcal{K}_n} (-1)^{k_1}(g_1^{k_1 \rho}, g_2^{k_2 \rho}, \ldots, g_n^{k_n \rho}) \in C^\Delta_n(G). \]

For example, when \(n = 3\), the \(\varphi_3(x, y, z)\) is written in

\[ \sum_{0 \leq i \leq t_X-1} (x^i y^i z^i) - (x^{i+1} y^{i+1} z^{i+1}) - (x^{i+1} y^i z^i) + (x^i y^{i+1} z^i). \]

Proposition 2.6. Let \(X\) be a quandle of the form \((G, \rho)\). If \(X\) is of type \(t_X < \infty\), then the homomorphism \(\varphi_n : C^R_n(G) \to C^\Delta_n(G)\) is a chain map. Namely, \(\partial^\Delta_n \circ \varphi_n = \varphi_{n-1} \circ \partial^R_n\).

Furthermore the image of \(D_n(G)\) is zero. In particular, the \(\varphi_n\) induces a chain map from the quotient \(C^{Q\Delta}_n(X)\) to \(C^\Delta_n(G)\), and a homomorphism \(H^{Q\Delta}_n(X) \to H^\Delta_n(G)\).

Proof. The identity \(\partial^\Delta_n \circ \varphi_n = \varphi_{n-1} \circ \partial^R_n\) can be proven by direct calculation similar to \([IK\] Lemma 3.1\) or \([No2\] Appendix\), so we omit the details. It is not hard to check the latter part directly.

Accordingly, we obtain an easy method to quandle cocycles from group cocycles:

Corollary 2.7. Let a quandle \(X = (G, \rho)\) be of type \(t_X\). For a normalized group \(n\)-cocycle \(\kappa\) of \(G\), then the pullback \(\varphi_n^*(\kappa)\) is a quandle \(n\)-cocycle.

Remark 2.8. We now roughly compare our map \(\varphi_n\) with a chain map \(\varphi_{IK}\) introduced by Inoue and Kabaya \([IK\]. For any quandle \(Q\), they constructed a complex “\(C^\Delta_n(Q)\)” from a simplicial object, and formulated the map \(\varphi_{IK} : C^R_n(Q) \to C^\Delta_n(Q)\) in its homogeneous coordinate system (see \([IK\ §3]\) for details).

To see this in some detail, we define a module, \(C^\Delta_n(G)_\mathbb{Z}\), to be the quotient of \(C^\Delta_n(G)\) modulo the relation \((g_1, \ldots, g_n) = (\rho(g_1), \ldots, \rho(g_n))\), called \(\mathbb{Z}\)-coinvariant of \(C^\Delta_n(G)\). We denote by \(\pi_\rho\) the projection \(C^\Delta_n(G) \to C^\Delta_n(G)_\mathbb{Z}\). We can see that, if \(Q\) is a quandle of the form \((G, \rho)\) and connected, then the above complex \(C^\Delta_n(G)\) is isomorphic to the coinvariant \(C^\Delta_n(G)_\mathbb{Z}\); further, we can check the equality \(t_X \cdot \varphi_{IK} = \pi_\rho \circ \varphi_n\). In summary, our map \(\varphi_n\) is of a lift of the Inoue-Kabaya map \(\varphi_{IK}\) in connected cases, and is relatively simple. So we fix a notation:

Definition 2.9. Let \(\Phi_n\) denote the composite chain map \(\pi_\rho \circ \varphi_n : C^{Q\Delta}_n(X) \to C^\Delta_n(G)_\mathbb{Z}\), i.e.,

\[ \Phi_n : C^{Q\Delta}_n(X) \xrightarrow{\varphi_n} C^\Delta_n(G) \xrightarrow{\text{proj}} C^\Delta_n(G)_\mathbb{Z}. \]

Incidentally, we prepare a ‘reduced map’ of the \(\Phi_n\), which is used temporarily in Theorem 2.16. Consider a homomorphism \(\mathcal{P} : C^R_n(X) \to C^R_{n-1}(X)\) derived from a map \(X^n \to X^{n-1}\) sending \((x_1, \ldots, x_n)\) to \((x_1, \ldots, x_{n-1})\). We discuss the composite \(\Phi_{n-1} \circ \mathcal{P}\) as follows:
2.3 Results on the chain map $\Phi$

**Proposition 2.10.** Let $X$ be a quandle $(G, \rho)$ of type $t_X$. The composite $\Phi_{n-1} \circ P : C_{n-1}^{QG}(X) \rightarrow C_{n-1}^{gr}(G)_\mathbb{Z}$ is a chain map. Furthermore it induces a chain map from the quotient $C_n^{QG}(X) \rightarrow C_n^{gr}(G)_\mathbb{Z}$.

*Proof. By direct calculation (cf. Proposition 2.6 and [IK, Lemma 3.1]).*

2.3 Results on the chain map $\Phi_3$

In this paper, we study the chain map $\Phi_n$ with $n = 3$ (Theorems 2.11, 2.14).

We first study the maps $\Phi_n$ with respect to extended quandles in Example 2.3.

**Theorem 2.11.** Let $X$ be a connected quandle of type $t_X$, and $\tilde{X} = (\text{Ker}(\epsilon_X), \rho_n)$ be the extended quandle in Example 2.3. Let $\tilde{\Phi}_n$ denote the chain map in Definition 2.9. Assume that the $H^3_3(\text{As}(X))$ is finitely generated, e.g., $X$ is of finite order. Then the induced map $(\tilde{\Phi}_3)_* : H^3_3(\tilde{X}) \rightarrow H^3_3(\text{Ker}(\epsilon_X))_\mathbb{Z}$

is an isomorphism modulo $t_X$-torsion.

**Remark 2.12.** We here compare this theorem with the result [No3, Theorem 3.18] which stated an existence of an isomorphism $H^3_3(\tilde{X}) \cong H^3_3(\text{As}(X))$ modulo $t_X$. So this theorem says that the chain map $(\tilde{\Phi}_3)_*$ gives an explicit presentation of this isomorphism. Indeed we late get a canonical isomorphism $H^3_3(\text{As}(X)) \cong H^3_3(\text{Ker}(\epsilon_X))_\mathbb{Z}$ modulo $t_X$; see Lemma 3.4.

Next, as a special case, we focus on the Alexander quandles on $\mathbb{F}_q$ in Example 2.2. Using the maps $\Phi_n$, we will characterize the third quandle cohomology from group homologies. Identifying $X = \mathbb{F}_q$ with $(\mathbb{Z}_p)^h$ as an additive group, let $\rho : \mathbb{F}_q \rightarrow \mathbb{F}_q$ be the $\omega$-multiple. We then have a chain map $\Phi^n_* : C_{gr}^n((\mathbb{Z}_p)^h)_\mathbb{Z} \rightarrow C_{Q}^n(X)$, and later show the following:

**Proposition 2.13.** Let $X$ be an Alexander quandle on $\mathbb{F}_q$ with $\omega$ in Example 2.2. Then the induced map $\Phi^3_* : H^3_{gr}(\mathbb{F}_q, (\mathbb{Z}_p)^h)_\mathbb{Z} \rightarrow H^3_Q(X; \mathbb{F}_q)$ is injective.

Furthermore, if $H^3_{g}(X)$ vanishes, then this $\Phi^3_*$ is an isomorphism.

In general, this $\Phi^3_*$ is not surjective. To solve the obstruction $H^3_Q(X)$, we consider the chain map $\bar{\Phi}_n : C_{R}^n(\tilde{X}) \rightarrow C_{gr}^n(\text{Ker}(\epsilon_X))_\mathbb{Z}$ with respect to the extended quandle (Example 2.3). We then obtain a commutative diagram

$$
\begin{array}{ccc}
H^3_{g}(\text{Ker}(\epsilon_X); \mathbb{F}_q)_\mathbb{Z} & \xrightarrow{\bar{\Phi}_n^*} & H^3_{Q}(\tilde{X}; \mathbb{F}_q)_\mathbb{Z} \\
\downarrow \text{Proj}^* & & \downarrow \rho^* \\
H^3_{g}(X; \mathbb{F}_q)_\mathbb{Z} & \xrightarrow{\Phi^*_n} & H^3_{Q}(X; \mathbb{F}_q)_\mathbb{Z}
\end{array}
$$

Remark that, when $n = 3$, the bottom map $\bar{\Phi}_3^*$ is an isomorphism by Theorem 2.11. Denote $\text{res}(\bar{\Phi}_3^*)$ the isomorphism restricted on the cokernel $\text{Coker}(\text{Proj}^*)$. In addition, we take the chain map $\Phi_{n-1} \circ P : C_{n-1}^{QG}(X) \rightarrow C_{n-1}^{gr}(G)_\mathbb{Z}$ in Proposition 2.10.

To summarize these homomorphisms, we characterize the third quandle cohomology of $X$:
Theorem 2.14. Let \( X \) be an Alexander quandle on \( \mathbb{F}_q \). Let \( q \) be odd. Then there is a section \( s : H^3_Q(\tilde{X}; \mathbb{F}_q) \to H^3_Q(X; \mathbb{F}_q) \) of \( p_X^* \) such that the following homomorphism is an isomorphism:

\[
(\Phi_2 \circ \mathcal{P})^* \oplus \Phi_3^* + (s \circ \text{res}(\Phi_3^*)) : \\
H^2_{gr}((\mathbb{Z}_p)^h; \mathbb{F}_q)^Z \oplus H^3_{gr}((\mathbb{Z}_p)^h; \mathbb{F}_q)^Z \oplus \text{Coker}((\text{Proj}^*) \longrightarrow H^3_Q(X; \mathbb{F}_q).
\]

(7)

The proof will appear in \[\text{[4]}\]. In conclusion, all the Mochizuki 3-cocycles are derived from group 3-cocycles of \( (\mathbb{Z}_p)^h \) and of \( \text{Ker}(\epsilon_X) \) via the chain map \( \Phi_n \).

Incidentally, in higher degree, we now observe that the induced map \( (\varphi_n)_* : H^Q_n(X) \to H^g_n(G) \) is far from injective and surjective.

Example 2.15. To see this, letting \( q = p \), we observe the chain map \( \varphi_n \) with respect to an Alexander quandle \( X \) on \( \mathbb{F}_p \) in Example \[\text{[23]}\]. Then, we easily see \( \text{Ker}(\epsilon_X) \cong \mathbb{Z}_p \) (cf. \[\text{[13]}\]). The cohomology \( H^g_n(\mathbb{Z}_p; \mathbb{F}_p) \) is \( \mathbb{F}_p \) for any \( n \in \mathbb{N} \). On the other hand, in \[\text{[No1]}\], the integral quandle homology \( H^Q_n(X) \) was shown to be \( (\mathbb{Z}_p)^{b_n} \), where \( b_n \in \mathbb{Z} \) is determined by the recurrence formula

\[
b_{n+2t} = b_n + b_{n+1} + b_{n+2}, \quad b_1 = b_2 = \cdots = b_{2t-2} = 0, \quad \text{and} \quad b_{2t-1} = b_{2t} = 1,
\]

and \( t > 0 \) is the minimal number satisfying \( \omega^t = 1 \). In conclusion, since the \( b_n \) is an exponentially growing, the map \( (\varphi_n)_* \) is not bijective.

2.4 Shadow cocycle invariant and Dijkgraaf-Witten invariant

Furthermore, we address a relation between shadow cocycle invariant \[\text{[CKS]}\] and the Dijkgraaf-Witten invariant \[\text{[DW]}\]. We now review the both invariants, and state Theorem 2.16.

First, to describe the former invariant, we begin reviewing \( X \)-colorings. Given a quandle \( X \), an \( X \)-coloring of an oriented link diagram \( D \) is a map \( \mathcal{C} : \{\text{arcs of } D\} \to X \) satisfying the condition in the left of Figure \[\text{[1]}\] at each crossing of \( D \). Denote by \( \text{Col}_X(D) \) the set of \( X \)-colorings of \( D \). Note that two diagrams \( D_1 \) and \( D_2 \) related by Reidemeister moves admit a 1:1-correspondence \( \text{Col}_X(D_1) \leftrightarrow \text{Col}_X(D_2) \); see \[\text{[CJKLS]} \text{[CKS]}\] for details.

We further define a shadow coloring to be a pair of an \( X \)-coloring \( \mathcal{C} \) and a map \( \lambda \) from the complementary regions of \( D \) to \( X \) such that, if regions \( R \) and \( R' \) are separated by an arc \( \alpha \) as shown in the right of Figure \[\text{[1]}\], the equality \( \lambda(R) \triangleleft \mathcal{C}(\alpha) = \lambda(R') \) holds. Let \( \text{Col}_X(D) \) denote the set of shadow colorings of \( D \). Given an \( X \)-coloring \( \mathcal{C} \), we put \( x_0 \in X \) on the region containing a point at infinity. Then, by the rules in Figure \[\text{[1]}\] colors of other regions are uniquely determined, and ensure a shadow coloring \( \mathcal{S} \) denoted by \( (\mathcal{C}; x_0) \). We thus obtain a bijection \( \text{Col}_X(D) \times X \simeq \text{Col}_X(D) \) sending \( (\mathcal{C}, x_0) \) to \( \mathcal{S} = (\mathcal{C}; x_0) \).

We briefly formulate (shadow) quandle cocycle invariants \[\text{[CKS]}\]. Let \( D \) be a diagram of a link \( L \), and \( \mathcal{S} \in \text{Col}_X(D) \) a shadow coloring. For a crossing \( \tau \) shown in Figure \[\text{[2]}\], we define a weight of \( \tau \) to be \( \epsilon_\tau(x, y, z) \in C_3^Q(X; \mathbb{Z}) \), where \( \epsilon_\tau \in \{\pm 1\} \) is the sign of \( \tau \). Further the
fundamental class of \( S \) is defined to be \([S] := \sum \epsilon_+(x, y, z) \in C_3^Q(X; \mathbb{Z})\), and is known to be a 3-cycle. For a quandle 3-cocycle \( \psi \in C_3^Q(X; A) \), we consider the pairing \( \langle \psi, [S] \rangle \in A \). The formal sum \( I_\psi(L) := \sum_{S \in \text{Col}_X(D)} 1_{\mathbb{Z}}\{\langle \psi, [S] \rangle\} \) in the group ring \( \mathbb{Z}[A] \) is called quandle cocycle invariant of \( L \), where a symbol \( 1_{\mathbb{Z}}\{a\} \in \mathbb{Z}[A] \) means the basis represented by \( a \in A \).

By construction, in order to calculate the invariant concretely, it is important to find explicit formulas of quandle 3-cocycles.

On the other hand, we will briefly formulate a Dijkgraaf-Witten invariant below \([9]\). For this, for a link \( L \), denote by \( \tilde{C}_L^m \) the \( m \)-fold cyclic covering space of \( S^3 \) branched over \( L \). Note that \( \mathbb{Z} \) canonically acts on the space \( \tilde{C}_L^m \) by the covering transformations. According to \([\text{No}3]\), when \( X \) is connected and of type \( t \), for an \( X \)-coloring of \( L \), we can construct a \( \mathbb{Z} \)-equivariant homomorphism \( \Gamma_C : \pi_1(\tilde{C}_L^t) \to \text{Ker}(\epsilon_X) \), where \( \mathbb{Z} \) acts on \( \text{Ker}(\epsilon_X) \) via the split surjection \([3]\); see \([3.1]\) for the definition of \( \Gamma_C \). In summary, given a link-diagram \( D \), we have a map

\[
\Gamma_* : \text{Col}_X(D) \longrightarrow \text{Hom}_{\mathbb{Z}}^\#(\pi_1(\tilde{C}_L^t), \text{Ker}(\epsilon_X)),
\]

where the right side is the set of the \( \mathbb{Z} \)-equivariant group homomorphisms \( \pi_1(\tilde{C}_L^t) \to \text{Ker}(\epsilon_X) \). Furthermore, consider the pushforward of the fundamental class \( [\tilde{C}_L^t] \in H_3(\tilde{C}_L^t) \) via the homomorphism \( \Gamma_C \). Using this, with respect to a \( \mathbb{Z} \)-invariant 3-cocycle \( \kappa \) of \( \text{Ker}(\epsilon_X) \), we define a \( \mathbb{Z} \)-equivariant part of Dijkgraaf-Witten invariant of \( \tilde{C}_L^t \) by the formula

\[
\text{DW}^Z_\kappa(\tilde{C}_L^t) = \sum_{C \in \text{Col}_X(D)} \langle \kappa, (\Gamma_C)_*[\hat{[\tilde{C}_L^t]}] \rangle \in \mathbb{Z}[A].
\]

We will show that, with respect to a connected quandle with a certain assumption, the two invariants explained above are equivalent (see \([3]\) for its proof). Precisely,

**Theorem 2.16.** Let \( X \) be a finite connected quandle of type \( t_X \). Let an abelian group \( A \) contain no \( t_X \)-torsion. Assume that the induced map \( p_X^* : H_3^Q(X; A) \to H_3^Q(\tilde{X}; A) \) is surjective. Then any \( \mathbb{Z} \)-invariant 3-cocycle \( \kappa \) of \( \text{Ker}(\epsilon_X) \) admits a quandle 3-cocycle \( \psi \) of \( X \) and the equality

\[
I_\psi(L) = |X| \cdot \text{DW}^Z_\kappa(\tilde{C}_L^{t_X}) \in \mathbb{Z}[A].
\]
Moreover, conversely, given a quandle 3-cocycle $\psi$ of $X$, there is a $\mathbb{Z}$-invariant group 3-cocycle $\kappa$ of $\text{Ker}(\epsilon_X)$ for which the equality holds.

**Remark 2.17.** As is seen in the proof in §3.2 for some quandles, we can obtain the quandle cocycle $\psi$ in Theorem 2.16 concretely from a group 3-cocycle $\kappa$. For instance, if $p_X : \tilde{X} \to X$ is isomorphic, then the $\psi$ is given by $\varphi_3^q(\kappa)$. As another example, for Alexander quandles on $\mathbb{F}_q$, the relations between $\psi$ and $\kappa$ are given by explicit formulas (see §4.2 in details).

To conclude, under the assumption on the $p^*_X$, the invariant $\text{DW}^Z_{\kappa}(\hat{C}_L^t)$ constructed from any $\mathbb{Z}$-invariant 3-cocycle $\kappa$ of $\text{Ker}(\epsilon_X)$ is can be computed from the quandle cocycle invariants via link-diagrams. Fortunately, there are some quandles with the assumption of the surjectivity of $p^*_X$. For example, connected Alexander quandles $X$ which satisfy either oddness of $|X|$ or evenness of $t_X$ ([No3, Lemma 9.15]), and “symplectic quandles $X$ over $\mathbb{F}_q$” with $g = 1$ ([No3, §3.3]).

However, on the contrary, other quandles do not satisfy the assumption, e.g., with respect to symplectic quandles $X$ with $g > 1$ in a stable range, which are of type $p$, as is shown in [No3, §3], the $H^3_{gr}(\text{Ker}(\epsilon_X)) \cong H^3_{gr}(\text{Sp}(2g; \mathbb{F}_q)) \cong \mathbb{Z}/q^2 - 1$ and $H^3_Q(X) \cong 0$. Hence, in general, the invariant $\text{DW}^Z_{\kappa}(\hat{C}_L^t)$ is not always interpreted from shadow cocycle invariants.

# 3 Proofs of Theorems 2.11 and 2.16

Our objectivity in this section is to prove Theorems 2.11 and 2.16. This outline is, roughly speaking, a translation from some homotopical results in [No3] to terms of the group homology $H^3_{gr}(\text{Ker}(\epsilon_X))$: Actually, using homotopy groups, the author studied the group homology and a relation to Dijkgraaf-Witten invariant. So Section 3.1 reviews a group, $\Pi_2(X)$, and two homomorphisms from the group. In §3.2 we prove the theorems using a key lemma as such a translation. In §3.3 we give a proof of the key lemma. Readers who are interested in only Theorem 2.14 may skip to §4.

## 3.1 Review of two homomorphisms $\Delta_{X,x_0}$ and $\Theta_X$

In order to construct the two homomorphisms in (10), (11), we first review the group $\Pi_2(X)$ defined in [FRS1, FRS2]. Consider the set of all $X$-colorings of all link-diagrams. Then we define $\Pi_2(X)$ to be the quotient set subject to Reidemeister moves and concordance relations illustrated in Figure 3. Then disjoint unions of $X$-colorings make $\Pi_2(X)$ into an abelian group. For a connected quandle $X$ of finite order, the group $\Pi_2(X)$ was well-studied (see also Theorem 3.1 below).

![Figure 3: The concordance relations](image-url)
Next, we explain the homomorphism in (10) below. Recall from [2.4] that, given an $X$-coloring $C$ and $x_0 \in X$, we can construct a shadow coloring of the form $(C; x_0)$, and the fundamental class $[(C; x_0)]$ contained in $H^Q_3(X)$. We easily see that, if two $X$-colorings $C$, $C'$ are related by Reidemeister moves and concordance relations, then the associated classes $[(C; x_0)]$, $[(C'; x_0)]$ are equal in $H^Q_3(X)$ by definition. Hence we obtain a homomorphism

$$\Delta_{X,x_0} : \Pi_2(X) \longrightarrow H^Q_3(X), \quad C \longmapsto [(C; x_0)].$$

On the other hand, we will explain the homomorphism $\Theta_X$ below (11). For this end, we first observe the fundamental group of the $t$-fold cyclic covering space $\tilde{C}_t^L$. Given a link-diagram $D$ of $L$, let $\gamma_0, \ldots, \gamma_n$ be the arcs of $D$. Consider Wirtinger presentation of $\pi_1(S^3 \setminus L)$ generated by $\gamma_0, \ldots, \gamma_n$. For $s \in \mathbb{Z}$, we take a copy $\gamma_{i,s}$ of the arc $\gamma_i$. Then, by Reidemeister-Schreier method (see, e.g., [Kab] §3 for the details), the group $\pi_1(\tilde{C}_t^L)$ can be presented by

- generators: $\gamma_{i,s}$ ($0 \leq i \leq n, \ s \in \mathbb{Z}$),
- relations: $\gamma_{k,s} = \gamma_{j,s}^{-1} \gamma_{i,s-1} \gamma_{j,s} \gamma_{i,s}$ for each crossings in the figure below, and

\[
\gamma_i \quad \gamma_j \quad \gamma_k
\]

Let $X$ be a connected quandle of type $t$. Given an $X$-coloring $C \in \text{Col}_X(D)$, we denote the color on the arc $\gamma_i$ by $x_i \in X$. Define a group homomorphism $\Gamma_C : \pi_1(\tilde{C}_t^L) \rightarrow \text{Ker}(\epsilon_X)$ by setting $\Gamma_C(\gamma_{i,s}) := e_{x_s}^{s_0} e_{x_s}^{-s}$ (see [No3] §4 for the well-definedness). Furthermore, consider the pushforward $(\Gamma_C)_* : \pi_1(\tilde{C}_t^L) \rightarrow H^Q_3(\text{Ker}(\epsilon_X))$, where $[\tilde{C}_t^L]$ is the fundamental class in $H_3(\tilde{C}_t^L)$. We thus obtain a map

$$\theta_{X,D} : \text{Col}_X(D) \longrightarrow H^Q_3(\text{Ker}(\epsilon_X)), \quad C \longmapsto (\Gamma_C)_*([\tilde{C}_t^L]).$$

As is shown [No3], if two $X$-colorings $C, C'$ can be related by Reidemeister moves and the concordance relations, then the identity $\theta_{X,D}(C) = \theta_{X,D'}(C')$ holds. Therefore the maps $\theta_{X,D}$ with respect to all diagrams $D$ yield a homomorphism

$$\Theta_X : \Pi_2(X) \longrightarrow H^Q_3(\text{Ker}(\epsilon_X)).$$

This $\Theta_X$ plays an important role to study the group $\Pi_2(X)$ up to $t$-torsion. Indeed,

**Theorem 3.1** ([No3] Theorems 3.4 and 3.18). Let $X$ be a connected quandle of type $t_X$. Put the inclusion $\iota : \text{Ker}(\epsilon_X) \rightarrow \text{As}(X)$ in (3). If the homology $H^Q_3(\text{As}(X))$ is finitely generated, then the composite $\iota_* \circ \Theta_X : \Pi_2(X) \rightarrow H^Q_3(\text{As}(X))$ is a split surjection modulo $t_X$-torsion, whose kernel is isomorphic to $H^Q_3(\text{As}(X))$ modulo $t_X$-torsion.

Furthermore, the induced map $(p_X)_* : \Pi_2(\tilde{X}) \rightarrow \Pi_2(X)$ is a split injection modulo $t_X$-torsion, and this cokernel equals the kernel of the composite $\iota_* \circ \Theta_X$.

---

1In [No3], the maps $\Theta_X$, $\iota_* \circ \Theta_X$ were denoted by $\Theta_{H^Q}$, $\Theta_X$, respectively.
3.2 A key lemma and Proofs of Theorems 2.11 and 2.16

For the proofs, we state a key lemma. We here fix a terminology: A (shadow) \( \tilde{X} \)-coloring of \( D \) is said to be based, if an arc \( \gamma_0 \) of \( D \) is colored by the identity \( 1_{\ker(\epsilon_X)} \in \tilde{X} \).

**Lemma 3.2** (cf. [Kab] Theorem 9.1). Let \( X \) be a connected quandle of type \( t_X < \infty \), and \( p_X : \tilde{X} \to X \) the projection in Example 2.3. Let \( \Upsilon : C_3^R(\tilde{X}) \to C_3^R(\tilde{X}) \) be the chain isomorphism in [5]. Let \( S \in \text{Col}_X(D) \) be a based shadow coloring. Define an \( \tilde{X} \)-coloring \( \tilde{C} \in \text{Col}_{\tilde{X}}(D) \) to be the \( X \)-coloring as the restriction of \( S \in \text{Col}_X(D) \). Then

\[
\Theta_X([p_X(\tilde{C})]) = \varphi_3 \circ \Upsilon([S]) \in H_3^\gr(\ker(\epsilon_X)).
\]

Before going to the proof, we will complete the proofs of Theorems 2.11 and 2.16.

**Proof of Theorem 2.11.** As mentioned in Remark 2.12, there is an isomorphism \( H_3^Q(\tilde{X}) \cong H_3^\gr(\text{As}(X)) \) up to \( t_X \)-torsion, as finitely generated \( \mathbb{Z} \)-modules. Hence, in order to prove that the map \( \tilde{\Phi}_3 \) is an isomorphism, it suffices to show the surjectivity.

To this end, we set the composite of the three homomorphisms mentioned above:

\[
\Pi_2(\tilde{X}) \xrightarrow{(p_X)_*} \Pi_2(X) \xrightarrow{\Theta_X} H_3^\gr(\ker(\epsilon_X)) \xrightarrow{\iota_*} H_3^\gr(\text{As}(X)).
\]

It follows from Theorem 3.1 above that the composite is an isomorphism up to \( t_X \)-torsion.

Note that, Lemma 3.2 below ensures an isomorphism \( \xi : H_3^Q(\text{As}(X)) \to H_3^\gr(\ker(\epsilon_X)) \) such that \( \xi \circ \iota_* = t_X \cdot (\pi_\rho)_* \), where \( \pi_\rho \) is the projection \( C_3^\gr(\ker(\epsilon_X)) \to C_3^\gr(\ker(\epsilon_X)) \) explained in §2.2. Hence, the following composite is an isomorphism up to \( t_X \)-torsion as well:

\[
(\pi_\rho)_* \circ \Theta_X \circ (p_X)_* : \Pi_2(\tilde{X}) \longrightarrow H_3^\gr(\ker(\epsilon_X)) \mathbb{Z}.
\]

Therefore, for any 3-cycle \( \mathcal{K} \in H_3^\gr(\ker(\epsilon_X)) \mathbb{Z} \) which is annihilated by \( t_X \), we choose some based \( \tilde{X} \)-coloring \( \tilde{C} \) such that \( \mathcal{K} = (\pi_\rho)_* \circ \Theta_X \circ (p_X)_*([\tilde{C}]) \). We here set a shadow coloring \( \sigma \) of the form \( (\tilde{C} ; 1, \tilde{\chi}) \). Then, by the lemma 3.2, we notice the equalities

\[
\tilde{\Phi}_3((\iota_*)(S)) = (\pi_\rho)_* \circ (\varphi_3 \circ \Upsilon)([S]) = (\pi_\rho)_* \circ \Theta_X(p_X([\tilde{C}])) = \mathcal{K}.
\]

Noting the left element \( \Upsilon([S]) \in H_3^Q(\tilde{X}) \), we obtain the surjectivity of \( \tilde{\Phi}_3 \) as required, \( \Box \).

**Proof of Theorem 2.16.** We first construct two homomorphisms (12), (13). Let \( X \) be a finite connected quandle of type \( t_X \). Given a \( \mathbb{Z} \)-invariant group 3-cocycle \( \kappa \), consider a composite homomorphism from \( \Pi_2(X) \):

\[
\Pi_2(X) \xrightarrow{\Theta_X} H_3^\gr(\ker(\epsilon_X)) \xrightarrow{(\kappa \bullet)} A.
\]

On the other hand, by the assumption of the surjectivity of \( p_X^* : H_3^Q(X; A) \to H_3^Q(\tilde{X}; A) \), we can choose a quandle cocycle \( \psi \in H_3^Q(X; A) \) such that \( p_X^*(\psi) = (\varphi_3 \circ \Upsilon)^*(\kappa) \). We then set a composite homomorphism

\[
\Pi_2(X) \xrightarrow{[\bullet, 0]} H_3^Q(X) \xrightarrow{(\psi \bullet)} A.
\]
We will prove Lemma 3.2 as a modification of \([\text{Kab}, \text{Theorem} 9.1]\). Next, we claim the equivalence of the two maps \((12)\) and \((13)\). For this, we put some \(\tilde{X}\)-colorings \(\tilde{C}_1, \ldots, \tilde{C}_n\), which generate the \(\Pi_2(\tilde{X})\); here we may assume that these colorings are based by Lemma 3.3 below. Notice that, by Theorem 3.1, the group \(\Pi_2(X)\) is generated by the kernel \(\text{Ker}(\varepsilon_X)\) and the elements \(p_X(\tilde{C}_1), \ldots, p_X(\tilde{C}_n)\). Therefore, the claimed equivalence results from the following equalities:

\[
\langle \kappa, \Theta_X(p_X(\tilde{C}_i)) \rangle = \langle \kappa, \varphi_3 \circ \Upsilon([\tilde{C}_i; \tilde{x}_0]) \rangle = \langle p_X(\psi), ([\tilde{C}_i; \tilde{x}_0]) \rangle = \langle \psi, [p_X(\tilde{C}_i); p_X(\tilde{x}_0)] \rangle,
\]

where the first equality is obtained from Lemma 3.2.

We will show the equivalence of the two invariants as stated in Theorem 2.16. By definitions, these invariants are reformed as

\[
\text{DW}_{\kappa}^Z(\tilde{C}_L^t) = \sum_{\mathcal{C} \in \text{Col}_X(D)} 1_Z\{\langle \kappa, \Theta_X(\mathcal{C}) \rangle\}, \quad I_{\psi}(L) = \sum_{x \in X} \sum_{\mathcal{C} \in \text{Col}_X(D)} 1_Z\{\langle \psi, [\mathcal{C}; x] \rangle\} \in \mathbb{Z}[A].
\]

Further, it is shown \([\text{IK}, \text{Theorem} 4.3]\) that the right invariant \(I_{\psi}(L) = |X| \sum_{\mathcal{C} \in \text{Col}_X(D)} 1_Z\{\langle \psi, [\mathcal{C}; x_0] \rangle\} \) for any \(x_0 \in X\). In conclusion, since the homomorphisms \((12), (13)\) are equal as claimed above, so are the two invariants.

Finally, to prove the latter part of Theorem 2.16 recall that the map \(\tilde{\Phi}_3\) is an isomorphism after tensoring \(A\) (Theorem 2.11). So, given a quandle 3-cocycle \(\psi\), we define a \(\mathbb{Z}\)-invariant group 3-cocycle \(\kappa\) of \(\text{Ker}(\varepsilon_X)\) to be \((\Upsilon \circ \tilde{\Phi}_3^{-1})^{-1}(p_X(\psi))\). Hence, by a similar argument as above, we have the desired equality \(I_{\psi}(L) = |X| \cdot \text{DW}_{\kappa}^Z(\tilde{C}_L^t)\). \(\square\)

### 3.3 Proof of the key lemma

We will prove Lemma 3.2 as a modification of \([\text{Kab}, \text{Theorem} 9.1]\).

We will review descriptions in \([\text{Kab}, \text{§4}]\), in order to formulate concretely the orientation class \([\tilde{C}_L^t] \in H_3(\tilde{C}_L; \mathbb{Z})\) of the branched covering space \(\tilde{C}_L^t\). Let \(N(L) \subset S^3\) denote a tubular neighborhood of \(S^3 \setminus L\). Let \(\gamma_0, \ldots, \gamma_n\) be the oriented arcs of the diagram \(D\) such as \([\text{§3.1}]\). We may assume that each arc \(\gamma_i\) has boundaries. For each arc \(\gamma_i\), we can construct 4 tetrahedra \(T_i^{(u)} \subset \tilde{C}_L^t\) with \(1 \leq u \leq 4\), and further decompose \(S^3 \setminus N(L)\) into these \(4(n+1)\) tetrahedra as a triangulation, which is commonly referred as to “a standard triangulation” (see, e.g., \([\text{Wee}]\) or \([\text{Kab}, \text{§4}]\)). Furthermore, Kabaya concretely constructed 4t tetrahedra \(T_i^{(u)}\) in the branched covering space \(\tilde{C}_L\), where \(0 \leq s \leq t - 1\) and \(1 \leq u \leq 4\) (see Figures 7 and 14 in \([\text{Kab}]\)). Roughly speaking, the tetrahedron \(T_i^{(u)}\) corresponds with the \(s\)-th connected component of the preimage of the \(T_i^{(u)} \subset S^3\) via the branched covering \(\tilde{C}_L \rightarrow S^3\). Let us fix the orderings of \(T_i^{(u)}\) following Figure 8 in \([\text{Kab}]\). Then he showed the union \(\bigcup_{i,s,u} T_i^{(u)} = \tilde{C}_L^t\) and that the formal sum \(\sum_{i,s} \epsilon_i(T_i^{(1)} - T_i^{(2)} - T_i^{(3)} + T_i^{(4)})\) represents the orientation class \([\tilde{C}_L^t]\), where \(\epsilon_i \in \{\pm 1\}\) is the sign of the crossing at the endpoint of \(\gamma_i\).

We next discuss labelings of the tetrahedron \(T_i^{(u)}\) by a group \(G\). Put a map \(L : \{T_i^{(u)}\}_{i,s,u} \rightarrow G^3\). Let \(I : \{T_i^{(u)}\}_{i,s,u} \rightarrow G\) be a constant map taking elements to the
the identity of $G$. We would regard the product $\mathcal{I} \times \mathcal{L}$ as a labeling of vertices in $T_{i,s}^{(u)}$ according to the ordering. Fix a homomorphism $f : \pi_1(\hat{C}_L^t) \rightarrow G$ and recall the generators $\gamma_{i,s} \in \pi_1(\hat{C}_L^t)$ in 3.1. He showed [Kab] §4 that, if the labeling is globally compatible and the $\mathcal{L}(T_{i,s}^{(u)})$'s satisfy the following condition:

\[
\bullet \quad \mathcal{L}(T_{i,s}^{(1)}) \cdot f(\gamma_{i,s}) = \mathcal{L}(T_{i,s+1}^{(2)}) \quad \text{and} \quad \mathcal{L}(T_{i,s}^{(2)}) \cdot f(\gamma_{i,s}) = \mathcal{L}(T_{i,s+1}^{(3)}) \in G^3, \tag{14}
\]

where the action $\cdot$ is diagonal, then the product $\mathcal{I} \times \mathcal{L}$ satisfies a group 1-cocycle condition in the union $\bigcup_{i,s,u} T_{i,s}^{(u)}$, and that the induced homomorphism coincides with $f$. In the sequel, the push-forward $f_*([\hat{C}_L^t])$ is represented by the formula

\[
\Upsilon(\sum_{0 \leq i \leq n} \sum_{0 \leq s < t} \mathcal{L}(\epsilon_i(T_{i,s}^{(1)} - T_{i,s}^{(2)} + T_{i,s}^{(3)}))) \in C_n^{gr}(G), \tag{15}
\]

where $\Upsilon : C_n^{gr}(G) \rightarrow C_n^{gr}(G)$ is an isomorphism given in Lemma 2.4.

Proof of Lemma 3.3. The proof will be shown by expressing the left side $\Theta_X(\mathcal{C})$ in details:

For this, given a shadow coloring $S$ by $\tilde{X} = \text{Ker}(\epsilon_X)$, we will define a labeling $\mathcal{L}$ compatible with the homomorphism $\Gamma_c : \pi_1(\hat{C}_L^t) \rightarrow G$ with $G = \text{As}(X)$ as follows. Let $(g, h, k) \in \tilde{X}^3 = \text{Ker}(\epsilon_X)^3$ be the weight of the endpoint of the arc $\gamma_i$. Using the quandle structure on $\tilde{X}$, we then define a map $\mathcal{L} : \{T_{i,s}^{(u)}\}_{s,i,u} \rightarrow (\text{Ker}(\epsilon_X))^3 = \tilde{X}^3$ by

\[
\mathcal{L}(T_{i,s}^{(1)}) := (g_{s-1}, h_{s-1}, k_{s-1}), \quad \mathcal{L}(T_{i,s}^{(2)}) := (g_{s-1} < h_{s-1}, h_{s-1}, k_{s-1}),
\]

\[
\mathcal{L}(T_{i,s}^{(3)}) := (g_{s} < k_{s}, h_{s} < k_{s}, k_{s}), \quad \mathcal{L}(T_{i,s}^{(4)}) := ((g_{s} < h_{s}) < k_{s}, h_{s} < k_{s}, k_{s}),
\]

where we put $g_s := e^a g_a e^{-a} \in \text{Ker}(\epsilon_X)$ for short.

We will verify the equalities (14) on $\mathcal{L}$. From the definition of the action $X \acts \text{As}(X)$, we notice an equality $e_{p_X(k)} = e_{a,k} = k^{-1}e_a k \in \text{As}(X)$ for any $k \in \tilde{X}$. In addition, we note $(p_X)_*(S(\gamma_0)) = p_X(1_X) = a \in X$ since the $S$ is based by assumption. Hence, using the notation $\gamma_{i,s} \in \pi_1(\hat{C}_L^t)$, we have

\[
\Gamma_c(\gamma_{i,s}) = (e_a)^{s-1}e_a S(\gamma_i) e_a^{s} = (e_a)^{s-1}e_{p_X(k)} e_a^{s} = e_a^{s-1}k^{-1}e_a k e_a^{s}. \tag{16}
\]

Therefore, for any $b \in X$, using (16), we have the equality

\[
(e_a^{s-1}b e_a^{-s}) \cdot \Gamma_c(\gamma_{i,s}) = e_a^{s}(b < k) e_a^{-s} \in \text{Ker}(\epsilon_X).
\]

Consequently, applying $b = g$, $b = h$ or $b = g < h$ to this identity concludes the condition (14). Furthermore it is not hard to see that the labelling is globally compatible with all the triangulations. Hence, the labeling $\mathcal{L}$ induces the homomorphism $\Gamma_c : \pi_1(\hat{C}_L^t) \rightarrow \text{As}(X).

Finally, we discuss the push-forward of the orientation class $(\Gamma_c)_*([\hat{C}_L^t]) \in C_n^{gr}(\text{Ker}(\epsilon_X); \mathbb{Z})$. We first check that, for $x, y, z \in \tilde{X} = \text{Ker}(\epsilon_X)$, the following equality holds:

\[
\Upsilon^{-1} \circ \varphi_3 \circ \Upsilon(x, y, z) = \sum_{1 \leq s \leq t} (x, y_s, z_s) - (x < y_s, y_s, z_s) = (x < z_s, y_s < z_s, z_s) + (x < y_s) < z_s, y_s < z_s, z_s).
\]
Here this verification is easily obtained from recalling the definitions of $\varphi_3$ in Definition 2.5 and $\Upsilon$ in [5]. Hence, compared with the map $L$, we have the equality
\[
\Upsilon^{-1} \circ \varphi_3 \circ \Upsilon([S]) = \sum_i \sum_s L(\epsilon_i(T_{i,s}^{(1)} - T_{i,s}^{(2)} - T_{i,s}^{(3)} + T_{i,s}^{(4)})) \in C_3^{gr}(\text{Ker}(\epsilon_X))
\]
extactly. Since the right side is the push-forward $\Upsilon^{-1}(\langle \Gamma C \rangle_3([\tilde{C}_L^i]))$ by [15], we conclude the desired equality.

We now provide proofs of two lemmas above.

**Lemma 3.3.** Let $X$ be a connected quandle. If an element in $\Pi_2(\tilde{X})$ is represented by an $\tilde{X}$-coloring of $D$, then the class equals some based $\tilde{X}$-coloring of the $D$ in $\Pi_2(\tilde{X})$.

**Proof.** Let the arc $\gamma_0$ be colored by $h \in \tilde{X}$. Since the extended quandle $\tilde{X}$ is also connected [No3, Lemma 9.15], we have $g_1, \ldots, g_n \in \tilde{X}$ such that $(\cdots (h \prec g_1) \prec \cdots) \prec g_n = 1_{\tilde{X}}$. Then by observing the following picture, we can change the $C$ to another $\tilde{X}$-coloring $\tilde{C}'$ of $D$ such that the arc $\gamma_0$ is colored by $h \prec g_1$ and that $[C] = [C'] \in \Pi_2(\tilde{X})$.

\[
\begin{array}{cccc}
\gamma_0 & h & \circlearrowright & g_1 & \Rightarrow & \begin{array}{c}
\gamma_0 \\
\hline
\end{array} & h & \circlearrowright & g_1 & = & \begin{array}{c}
\gamma_0 \\
\hline
\end{array} & h & \circlearrowright & g_1 & \in \Pi_2(X).
\end{array}
\]

Here the first and forth equalities are obtained from the concordance relation, and in the second (resp. third) equality the loop colored by $g_1$ passes under (resp. over) the all arc of $D$. Note that we here only use Reidemeister moves. Hence, iterating this process, we have a based $\tilde{X}$-coloring $\tilde{C}^{(n)}$ of the $D$ such that $[C] = [C^{(n)}] \in \Pi_2(\tilde{X})$.

**Lemma 3.4.** Let $X$ be a connected quandle of type $t_X$. Let $i : \text{Ker}(\epsilon_X) \rightarrow \text{As}(X)$ be the inclusion [3], and $\pi_\rho : C_3^{gr}(\text{Ker}(\epsilon_X)) \rightarrow C_3^{gr}(\text{Ker}(\epsilon_X))_{\mathbb{Z}}$ be the projection. Then there is an isomorphism $\xi : H_3^{gr}(\text{As}(X)) \rightarrow H_3^{gr}(\text{Ker}(\epsilon_X))_{\mathbb{Z}}$ modulo $t_X$-torsion such that $\xi \circ i_* = t_X \cdot (\pi_\rho)_*$.

**Proof.** Fix $x \in X$, and consider the subgroup $\langle e_{nX}^n \rangle_{n \in \mathbb{Z}}$ of $\text{As}(X)$, which is contained in the center (see [No3, Lemma 4.1]). Put the quotient $Q_X := \text{As}(X)/\langle e_{nX}^n \rangle_{n \in \mathbb{Z}}$. By the Lyndon-Hochshild spectral sequence, the projection induces an isomorphism $P_* : H_2^{gr}(\text{As}(X)) \cong H_2^{gr}(Q_X)$ up to $t_X$-torsion, since the $H_2^{gr}(\text{As}(X))$ is shown to be annihilated by $t_X$ [No3 Corollary 6.4]. Furthermore, noting the group extension $\text{Ker}(\epsilon_X) \rightarrow G_X \rightarrow \mathbb{Z}/t_X$, the transfer gives an isomorphism $\mathcal{T} : H_3^{gr}(Q_X) \rightarrow H_3^{gr}(\text{Ker}(\epsilon_X))_{\mathbb{Z}}$ modulo $t_X$; see [Bro] [§III.10]. Hence, denoting $\mathcal{T} \circ P_*$ by $\xi$, we have the equality $\xi \circ i_* = t_X \cdot (\pi_\rho)_*$ by construction.

### 4 Proofs of Proposition 2.13 and Theorem 2.14

This section proves Proposition 2.13 and Theorem 2.14. The outline of the proof is as follows. With respect to an Alexander quandle over $\mathbb{F}_q$, a basis of the third cohomology over $\mathbb{F}_q$ was found by Mochizuki [Moc], which we review in §4.1. Furthermore, we will see that it is enough for the isomorphisms stated in Theorem 2.14 to show these surjectivity.
So, we will construct group 3-cocyles of $\text{As}(X)$ as preimages of the basis via the chain maps $\Phi_3$ and $\tilde{\Phi}_3$ (see [12]).

To accomplish this outline, we start reviewing a simple presentation of $\text{As}(X)$ of a connected Alexander quandle $X$, shown by Clauwens [Cla]. Set a homomorphism $\mu_X : X \otimes X \to X \otimes X$ defined by

$$\mu_X(x \otimes y) = x \otimes y - Ty \otimes x.$$ 

Using this $\mu_X$, let us equip $\mathbb{Z} \times X \times \text{Coker}(\mu_X)$ with a group operation given by

$$(n, a, \kappa) \cdot (m, b, \nu) = (n + m, T^m a + b, \kappa + \nu + [T^m a \otimes b]).$$  \hspace{1cm} (17)

Then a homomorphism $\text{As}(X) \to \mathbb{Z} \times X \times \text{Coker}(\mu_X)$ sending the generators $e_x$ to $(1, x, 0)$ is isomorphic [Cla, Theorem 1]. The lower central series of $\text{As}(X)$ are then described as

$$\text{As}(X) \supset X \times \text{Coker}(\mu_X) \supset \text{Coker}(\mu_X) \supset 0.$$  \hspace{1cm} (18)

In particular, the kernel $\text{Ker}(e_X)$ in [3] is the subgroup on the set $X \times \text{Coker}(\mu_X)$. Incidentally, an isomorphism $H^2_Q(X) \cong \text{Coker}(\mu_X)$ is shown [Cla].

**Notation** Denote by $G_X$ the subgroup $\text{Ker}(e_X)$ on $X \times \text{Coker}(\mu_X)$. From now on, in this section, we let $X$ be an Alexander quandle on $\mathbb{F}_q$ with $\omega \in \mathbb{F}_q$. Let $X$ be of type $t_X$. That is, $t_X$ is the minimal satisfying $\omega^{t_X} = 1$. Note that $t_X$ is coprime to $q$ since $\omega^{q-1} = 1$.

### 4.1 Review of Mochizuki 3-cocycles

We will review Mochizuki 2-, 3-cocycles of $X = \mathbb{F}_q$. We here regard polynomials in the ring $\mathbb{F}_q[U_1, \ldots, U_n]$ as functions from $X^n$ to $\mathbb{F}_q$, and as being in the complex $C_n^{Q_2}(X; \mathbb{F}_q)$ in [2,2].

**Theorem 4.1** ([Moc Lemma 3.7]). The following set represents a basis of $H^2_Q(X; \mathbb{F}_q)$.

$$\{ U_1^{q_1} U_2^{q_2} | \omega^{q_1 + q_2} = 1, 1 \leq q_1 < q_2 < q, \text{ and } q_i \text{ is a power of } p. \}.$$ 

Next, we describe all the quandle 3-cocycles of $X$. To see this, recall the following three polynomials over $\mathbb{F}_q$ ([Moc §2.2]):

$$\chi(U_j, U_{j+1}) := \sum_{1 \leq i \leq p-1} (-1)^{i-1} U_j^{p-i} U_{j+1} = ((U_{i-1} + U_i)^p - U_{i-1}^p - U_i^p) / p,$$

$$E_0(a \cdot p, b) := (\chi(U_1, U_2) - \chi(U_1, U_2))^a \cdot U_3^b, \quad E_1(a, b \cdot p) := U_1^a \cdot (\chi(U_2, U_3) - \chi(U_1, U_3))^b.$$

Define the following set $I^+_{q, \omega}$ consisting of the polynomials under some conditions:

$$I^+_{q, \omega} := \{ E_0(q_1 \cdot p, q_2) | \omega^{p q_1 + q_2} = 1, \ q_1 < q_2 \} \cup \{ E_1(q_1, q_2 \cdot p) | \omega^{q_1 + p q_2} = 1, \ q_1 \leq q_2 \}$$

$$\cup \{ U_1^{q_1} U_2^{q_2} U_3^{q_3} | \omega^{q_1 + q_2 + q_3} = 1, \ q_1 < q_2 < q_3 \}.$$  \hspace{1cm} (19)

Here the symbols $q_i$ range over powers of $p$ with $q_i < q$.

Furthermore, we review polynomials denoted by $\Gamma(q_1, q_2, q_3, q_4)$. For this, we define a set $Q_{q, \omega} \subset \mathbb{Z}^4$ consisting of quadruples $(q_1, q_2, q_3, q_4)$ such that
Lemma 4.5. First, to prove Proposition 2.13, we prepare a lemma for a study of the quandle 3-cocycles

4.2 Proof of Theorem 2.14

slight errors, which had however been corrected by Mandemaker \[\text{[Man]}\]. Unfortunately the original statement and his proof of this theorem contained

Remark 4.4. composed of non-trivial 3-cocycles. Here

\[\text{q}\]

\[\text{group cohomology}\]

\[\text{\([Moc]\)}\]

The 3-cocycle in Case 3 (resp. 4 and 5) is obtained from that in Case 1 after

changing the indices \((1, 2, 3, 4)\) to \((1, 3, 4, 2)\) (resp. to \((3, 1, 2, 4)\)).

We call the set \(Q_{q,\omega}\) Mochizuki quadruples. Then we state the main theorem in \[\text{[Moc]}\]:

Theorem 4.3 (\[\text{[Moc]}\]). The third cohomology \(H^3_Q(X; \mathbb{F}_q)\) is spanned by the following set composed of non-trivial 3-cocycles. Here \(q_i\) means a power of \(p\) with \(q_i < q\).

\[\Gamma^+_Q \cup \{\Gamma(q_1, q_2, q_3, q_4) | (q_1, q_2, q_3, q_4) \in Q_{q,\omega}\} \cup \{U^{q_1} U^{q_2} | \omega^{q_1+q_2} = 1, q_1 < q_2}\] the polynomial \(\Gamma(q_1, q_2, q_3, q_4)\) is defined as follows.

\begin{itemize}
    \item Case 1 \(\Gamma(q_1, q_2, q_3, q_4) := U^{q_1} U^{q_2} U^{q_3} U^{q_4}\).
    \item Case 2 \(\Gamma(q_1, q_2, q_3, q_4) := U^{q_1} U^{q_2} U^{q_3} U^{q_4} - U^{q_2} U^{q_4} U^{q_3} - (\omega^{q_2} - 1)^{-1}(1 - \omega^{q_1+q_2})(U^{q_1} U^{q_2} U^{q_3} U^{q_4} - U^{q_1+q_2} U^{q_4} U^{q_3})\).
    \item Case 3 \(\Gamma(q_1, q_2, q_3, q_4) := U^{q_1} U^{q_2} U^{q_3} U^{q_4}\).
    \item Case 4 and Case 5 \(\Gamma(q_1, q_2, q_3, q_4) := U^{q_1} U^{q_2} U^{q_3} U^{q_4}\).
\end{itemize}

For \((q_1, q_2, q_3, q_4) \in Q_{q,\omega}\) in each cases, the polynomial \(\Gamma(q_1, q_2, q_3, q_4)\) is spanned by the following set

\[\{U^{q_1} U^{q_2} | 1 \leq q_1 < q_2 < q, q_i\ \text{is a power of} \ p, \ \}\] the group \(2\)-cocycles:

\[\{U^{q_1} U^{q_2} U^{q_3} | q_1 < q_2 < q_3\} \cup \{(U_1 + U_2) U^{q_1} - U_1 U^{q_1} - U_2 U^{q_1} U^{q_2} / | q_1 < q_2\} \cup \{U_1 U^3((U_2 + U_3)^2 - U_2^2 - U_3^2) / | q_1 \leq q_2\} \}

where \(q_1, q_2, q_3\) run over powers of \(p\) with \(1 \leq q_1 < q.\) Moreover, regarding the multiplication of \(\omega\) in \(\mathbb{F}_q\) as an action of \(\mathbb{Z}\) on \(\mathbb{F}_q\), the \(\mathbb{Z}\)-invariant parts \(H^3_{gr}(G; \mathbb{F}_q)^\mathbb{Z}\) are generated by the above polynomials of degree \(d\) satisfying \(\omega^d = 1\). Here \(i = 2, 3\).

Remark 4.4. Unfortunately the original statement and his proof of this theorem contained slight errors, which had however been corrected by Mandemaker \[\text{[Man]}\].

4.2 Proof of Theorem 2.14

First, to prove Proposition 2.13, we prepare a lemma for a study of the quandle 3-cocycles in \[\text{[19]}\].

Lemma 4.5. Let us identify \(G = (\mathbb{Z}_q)^h\) with \(\mathbb{F}_q\) as an additive group. Then the second group cohomology \(H^2_{gr}(G; \mathbb{F}_q) \cong \mathbb{F}_q^{\frac{h(h+1)}{2}}\) is generated by the following group 2-cocycles:

\[\{U_1 U_2^q | 1 \leq q_1 < q_2 < q, q_i\ \text{is a power of} \ p, \ \}\] the 2-cocycles:

\[\{U_1 U_2^q U_3^q | q_1 < q_2 < q_3\} \cup \{(U_1 + U_2)^q - U_1^q - U_2^q \cdot U_3^q / | q_1 < q_2\} \cup \{U_1^q ((U_2 + U_3)^2 - U_2^2 - U_3^2) / | q_1 \leq q_2\},\]

where \(q_1, q_2, q_3\) run over powers of \(p\) with \(1 \leq q_1 < q.\) Moreover, regarding the multiplication of \(\omega\) in \(\mathbb{F}_q\) as an action of \(\mathbb{Z}\) on \(\mathbb{F}_q\), the \(\mathbb{Z}\)-invariant parts \(H^3_{gr}(G; \mathbb{F}_q)^\mathbb{Z}\) are generated by the above polynomials of degree \(d\) satisfying \(\omega^d = 1.\) Here \(i = 2, 3.\)
The group cohomologies of abelian groups can be calculated in many ways, e.g., by similar calculations to [Bro, V.§6] or [Moc]; so we omit proving Lemma 4.5.

Returning to our subject, we apply these generators in Lemma 4.5 to the pullback of the chain map \( \varphi \). Then easy computations show the identities

\[
\varphi^*_3(U^q_1 U^q_2 U^q_3) = t_X(1 - \omega^{q_1})(1 - \omega^{q_1 + q_2}) \cdot F(q_1, q_2, q_3),
\]

\[
\varphi^*_3(U^q_1 + U^q_2 - U^q_1 U^q_2 U^q_3) = t_X(1 - \omega^{q_2}) \cdot E_0(q_1, q_2),
\]

\[
\varphi^*_3(U^q_1 ((U_2 + U_3)^q_2 - U^q_2 - U^q_3)/p) = t_X(1 - \omega^{q_1}) \cdot E_1(q_1, q_2) \in C^3_Q(X; \mathbb{F}_q).
\]

(20)

Compared with the way in [Moc] that the right quandle 3-cocycles were found as solutions of a differential equation over \( \mathbb{F}_q \), the three identities via the map \( \varphi^*_3 \) are simple and miraculous.

Using the identities we will prove Proposition 2.13 as follows:

**Proof of Proposition 2.13.** The injectivity of \( \Phi^*_3 = (\pi \circ \varphi)_3 \) follows from that this \( \Phi^*_3 \) gives a 1:1 correspondence between a basis of \( H^3_{gr}((\mathbb{Z}_p)^h; \mathbb{F}_q)^\mathbb{Z} \) and a basis of a subspace of \( H^3_Q(X; \mathbb{F}_q) \) because of the previous three identities (compare Theorem 4.3 with Lemma 4.5).

Next, assume \( H^3_Q(X; \mathbb{F}_q) = 0 \). Then, Theorem 4.1 implies that no pair \((q_1, q_2)\) satisfies \( \omega^{q_1 + q_2} = 1 \) and \( q_1 < q_2 < q \). Hence, by observing Theorem 4.3 carefully, the \( H^3_Q(X; \mathbb{F}_q) \) is generated by the image of \( \Phi^*_3 \). Therefore \( \Phi^*_3 \) is an isomorphism as desired. \( \square \)

Next, we will prove Theorem 2.14. To this end, we now observe the cokernel \( \text{Coker}(\Phi^*_3) \).

To begin, we study the chain map \( (\Phi_2 \circ \mathcal{P})^* : H^3_{gr}((\mathbb{Z}_p)^h; \mathbb{F}_q)^\mathbb{Z} \rightarrow H^3_Q(X) \) stated in Proposition 2.10. Recall from Lemma 4.5 that this domain is generated by polynomials of the form \( U^q_1 U^q_2 \). So, recalling the composite \( \Phi_2 \circ \mathcal{P} \) from Proposition 2.10, we easily see

\[
(\Phi_2 \circ \mathcal{P})^*(U^q_1 U^q_2) = t_X(1 - \omega^{q_1})U^q_1 U^q_2 \in C^3_{Q,g}(X; \mathbb{F}_q).
\]

Hence, the third term in Theorem 1.3 is spanned by the image of this map \( (\Phi_2 \circ \mathcal{P})^* \).

Furthermore, we will discuss the cokernel of \( \Phi^*_3 \oplus (\Phi_2 \circ \mathcal{P})^* \). By observing Theorem 4.3 carefully, we see that a basis of the cokernel consists of the polynomials \( \Gamma \)'s coming from the Mochizuki quadruples \( Q_{q,\omega} \). Let us denote a quadruple \((q_1, q_2, q_3, q_4) \in Q_{q,\omega} \) by \( q \) for short. Case by case, we now introduce a map \( \theta^q_\Gamma : (G \times)^3 \rightarrow \mathbb{F}_q \) by setting the values of \( \theta^q_\Gamma \) at \((x, a \otimes b, y, c \otimes d, z, e \otimes f) \in (X \times \text{Coker}(\mu_X))^3 \) as follows. In Case 1, \( \theta^q_\Gamma \) is defined by the formula

\[
(1 - \omega)^{-q_2} (x^{q_1} y^{q_2 + q_3} + x^{q_1 + q_2} y^{q_2} - (1 - \omega)^{-q_2} (\omega^{q_2 a^{q_1} b^{q_2} + a^{q_2} b^{q_2} - x^{q_1 + q_2}) y^{q_3})
\]

\[+(1 - \omega)^{-q_1} (a^{q_1} b^{q_3} + \omega^{q_1} a^{q_1} b^{q_3} - x^{q_1 + q_3}) y^{q_2}) z^{q_4}.
\]

(21)

In Case 2, the value of \( \theta^q_\Gamma \) is given by the formula

\[
(1 - \omega)^{-q_1 - q_2} (x^{q_1} (y^{q_2 + q_3} z^{q_4} + y^{q_2} z^{q_4}) - (x^{q_1 + q_2} y^{q_4} + x^{q_2} y^{q_1 + q_4}) z^{q_3})
\]

\[+(1 - \omega)^{-q_3} (x^{q_1 + q_3} - \omega^{q_3} a^{q_1} b^{q_3} - a^{q_3} b^{q_3}) y^{q_2} z^{q_4} - (1 - \omega)^{-q_4} (x^{q_2 + q_4} - \omega^{q_4} a^{q_2} b^{q_4} - a^{q_4} b^{q_4}) y^{q_3} z^{q_1}).
\]
Furthermore, for Case 3 (resp. 4 and 5), the value is defined to be that of Case 1 by changing the indices \((1, 2, 3, 4)\) to \((1, 3, 4, 2)\) (resp. to \((3, 1, 2, 4)\)), according to Remark 4.2.

**Lemma 4.6.** For \(q = (q_1, q_2, q_3, q_4) \in \mathbb{Q}_{q, \omega}\), the map \(\theta^q_1\) from \((G_X)^3\) to \(F_q\) is a \(\mathbb{Z}\)-invariant group 3-cocycle of \(G_X\).

Moreover, using the map \(\widetilde{\Phi}_3\), the pullback \(\widetilde{\Phi}_3^*(\theta^q_1)\) equals \(t_X \cdot p_X^*(\Gamma(q))\) in \(C^3_\mathbb{Q}(\widetilde{X}; F_q)\).

**Proof.** Note that a map \(\theta : (G_X)^3 \to A\) is a \(\mathbb{Z}\)-invariant group 3-cocycle, by definition, if and only if it satisfies the two equalities

\[
\theta(b, c, d) - \theta(ab, c, d) + \theta(a, bc, d) - \theta(a, b, cd) + \theta(a, b, c) = 0,
\]

for any \(a = (a, \alpha),\ b = (b, \beta),\ c = (c, \gamma),\ d = (d, \delta) \in G_X = X \times \text{Coker}(\mu_X)\). Then, by elementary and direct computations, it can be seen that the maps \(\theta^q_1\) are \(\mathbb{Z}\)-invariant group 3-cocycles of \(G_X\). Also, the desired equality \(\widetilde{\Phi}_3^*(\theta^q_1) = t_X \cdot p_X^*(\Gamma(q))\) can be obtained by a direct calculation.

**Proof of Theorem 2.14.** Let \(q\) be odd. As is known [No3, Lemma 9.15], the induced map \(p_X^* : H^3_\mathbb{Q}(X; F_q) \to H^3_\mathbb{Q}(\widetilde{X}; F_q)\) is surjective. Hence, according to Lemma 4.6, there exists a section \(s : H^3_\mathbb{Q}(\widetilde{X}; F_q) \to H^3_\mathbb{Q}(X; F_q)\) such that \(s(\widetilde{\Phi}_3^*(\theta^q_1)) = \Gamma(q)\) for any \(q \in \mathbb{Q}_{q, \omega}\). To summarize the above discussion, the sum \(((\Phi_2 \circ \mathcal{P})^* \oplus \Phi_3^*) \oplus (s \circ \text{res}(\widetilde{\Phi}_3^*))\) in (7) is an isomorphism to \(H^3_\mathbb{Q}(X; F_q)\).

Incidentally, we will show that the group 3-cocycles \(\theta^q_1\) above except Case 2 are presented by Massey products. To see this, we consider a group homomorphism

\[
f^{q_i} : G_X \to F_q; \quad (x, \alpha) \mapsto x^{q_i},
\]

which is a group 1-cocycle of \(G_X\). For group 1-cocycles \(f, g\) and \(h\), we denote by \(f \wedge g\) the cup product; further, if \(f \wedge g = g \wedge h = 0 \in H^2_{\text{gr}}(G_X; F_q)\), we denote by \(\langle f, g, h \rangle\) the triple Massey product in \(H^3_{\text{gr}}(G_X; F_q)\) as usual (see, e.g., [Kra] for the definition).

**Proposition 4.7.** Let \(e \neq 2\). Let \((q_1, q_2, q_3, q_4) \in \mathbb{Q}_{q, \omega}\) satisfy Case \(e\) in (4.4). The group 3-cocycle \(\theta^q_1\) described above is of the followings form in the cohomology \(H^3_{\text{gr}}(G_X; F_q)\).

\[
H^3_{\text{gr}}(G_X; F_q) \ni \theta^q_1 = \begin{cases} 
(1 - \omega^{q_2})^{-1} < f^{q_1}, f^{q_1}, f^{q_2} \wedge f^{q_1} & \text{for } e = 1, \\
(1 - \omega^{q_3})^{-1} < f^{q_1}, f^{q_1}, f^{q_3} \wedge f^{q_2} & \text{for } e = 3, \\
(1 - \omega^{q_3})^{-1} < f^{q_1}, f^{q_2}, f^{q_3} \wedge f^{q_4} & \text{for } e = 4 \text{ or } 5.
\end{cases}
\]

**Proof.** We use notation \((x, a \otimes b, y, c \otimes d, z, e \otimes f) \in (X \times \text{Coker}(\mu_X))^3\) as above. Notice first that the cup product \(f^{q_1} \wedge f^{q_2}\) is the usual product \(x^{q_1} y^{q_2}\) (see [Bro, V. §3]). For Case 1, we now calculate the Massey product \(\langle f^{q_1}, f^{q_1}, f^{q_2} \rangle\). We easily check two equalities

\[
x^{q_1} y^{q_2} = (1 - \omega)^{-q_1} \delta_1 (a^{q_1} b^{q_3} + \omega^{q_1} a^{q_3} b^{q_1} - x^{q_1+q_3}),
\]
Hence, from the definition of Massey products, \(< f^{q_3}, f^{q_1}, f^{q_2} >\) is represented by

\[
(1 - \omega)^{-q_1}(a^{q_1} b^{q_3} + \omega^{q_1} a^{q_3} b^{q_1} - x^{q_1+q_3})y^{q_2} + (1 - \omega)^{-q_2}x^{q_3}(\omega^{q_2} c^{q_1} d^{q_2} + c^{q_2} d^{q_1} - y^{q_1+q_2}).
\]

Furthermore, we set a group 2-cocycle defined by

\[
F := (1 - \omega)^{-q_2}(< f^{q_3}, f^{q_1}, f^{q_2} > + (1 - \omega)^{-q_2}\delta_1(\omega^{q_2} x^{q_3} a^{q_1} b^{q_2} + x^{q_3} a^{q_2} b^{q_1} - x^{q_1+q_2+q_3})).
\]

A direct calculation then shows the equality \(F \cdot z^{q_4} = \theta^4_1\) by definitions, i.e., \(< f^{q_3}, f^{q_1}, f^{q_2} >\wedge f^{q_4} = \theta^4_1 \in H^3_{gr}(G_X; \mathbb{F}_q)\) as desired.

Similarly, the same calculation holds for Cases 3, 4, 5 according to Remark 4.2

However, a geometric meaning of the cocycle \(\theta^4_1\) with Case 2 remains to be open.

5 Some calculations of shadow cocycle invariants

As an application of Theorem 2.16, we will compute some \(\mathbb{Z}\)-equivariant parts of Dijkgraaf-Witten invariants, which is equivalent to a shadow cocycle invariant. In this section, we confirm ourselves to Alexander quandles on \(\mathbb{F}_q\) and \(Witten\) invariants, which is equivalent to a shadow cocycle invariant. In this section, we confirm ourselves to Alexander quandles on \(\mathbb{F}_q\) with \(\omega \in \mathbb{F}_q\). Recall from Lemma 4.6 that the quandle 3-cocycles \(\Gamma(q_1, q_2, q_3, q_4)\) found by Mochizuki (see §4.1 the definition) are derived not from group cohomologies of abelian groups, but from that of the non-abelian group \(G_X\). So we focus on the cocycles, and fix some notation. Let \(q\) denote a Mochizuki quadruple \((q_1, q_2, q_3, q_4)\) in \(Q_{q,\omega}\) for short, and replace \(\Gamma(q_1, q_2, q_3, q_4)\) by \(\Gamma(q)e\), if \(q\) satisfies Case \(e\) in §4.1 \((e \leq 5)\).

Incidentally, the set of \(X\)-colorings was well-studied. In fact, if \(D\) is a diagram of a knot \(K\), then there is a bijection

\[
\text{Col}_X(D) \leftrightarrow X \oplus \bigoplus_{i=1} \mathbb{F}_q[T]/(T - \omega, \Delta_i(T)/\Delta_{i+1}(T)), \tag{22}
\]

where \(\Delta_i(T)\) is the \(i\)-th Alexander polynomial of \(K\) (see [Ino]). Therefore, we shall study weights in the cocycles invariants.

5.1 The cocycle invariants of torus knots constructed from \(\Gamma(q_1, q_2, q_3, q_4)\)

This subsection considers the torus knots \(T(m,n)\). We here remark that \(m\) and \(n\) are relatively prime and the isotopy \(T(m,n) \simeq T(n,m)\); thereby \(n\) may be relatively prime to \(p\) without loss of generality. We determine all of the values of the invariants for \(T(m,n)\) as follows:

Theorem 5.1. Let \(q\) be relatively prime to \(n\). Let \(T(m,n)\) be the torus knot. Let \(q \in Q_{q,\omega}\) be a Mochizuki quadruple, and \(\Gamma(q)e\) be the associated quandle 3-cocycle. Then the quandle cocycle invariant \(I_{\Gamma(q)e}(T(m,n))\) is expressed by one of the following formulas:
(i) If \( e = 1 \), \( \omega^{mn} = 1 \), \( \omega^m \neq 1 \) and \( \omega^n \neq 1 \), then
\[
I_{\Gamma(q)}(T(m, n)) = q^2 \sum_{a \in \mathbb{F}_q} 1\mathbb{Z}\{-2mn(\zeta - \omega)^{q_2+q_3}\omega^{q_4} \cdot a^{q_1+q_2+q_3+q_4} \} \in \mathbb{Z}[\mathbb{F}_q],
\] (23)

where \( \zeta \) is the \( n \)-th primitive root of unity satisfying \( \omega^m = \zeta^n \). Furthermore, if \( e = 3 \) (resp. 4 or 5), then the value of \( I_{\Gamma(q)} \) is obtained from the above value \( I_{\Gamma(q)} \) after changing the indices \( (1, 2, 3, 4) \) to \( (1, 3, 4, 2) \) (resp. to \( (3, 1, 2, 4) \)) such as Remark 4.2.

(ii) Let \( p = 2 \) or 3, and let \( e = 1 \). If \( \omega^m = 1 \) and if \( m \) is divisible by \( p \), then
\[
I_{\Gamma(q)}(T(m, n)) = q^2 \sum_{a \in \mathbb{F}_q} 1\mathbb{Z}\left\{ \frac{mn}{p}(1 - \omega)^{q_1+q_4}a^{q_1+q_2+q_3+q_4} \right\} \in \mathbb{Z}[\mathbb{F}_q].
\] (24)

Furthermore, if \( e = 3 \) (resp. 4 or 5), then the value \( I_{\Gamma(q)} \) is obtained from the value \( I_{\Gamma(q)} \) after changing the indices \( (1, 2, 3, 4) \) to \( (1, 3, 4, 2) \) (resp. to \( (3, 1, 2, 4) \)), similarly.

(iii) Let \( e = 2 \). If \( p = 2 \), \( \omega^m = 1 \) and if \( m \) is divisible by 2, then \( I_{\Gamma(q)}(T(m, n)) \) is equal to
\[
q \sum_{a, \delta \in \mathbb{F}_q} 1\mathbb{Z}\{mn\varepsilon_2(a, \delta)/2\} \in \mathbb{Z}[\mathbb{F}_q].
\] Here \( \varepsilon_2(a, \delta) \) is temporarily defined by
\[
a^{q_2+q_3}(1 + \omega^{q_1})a^{q_3}\delta^{q_4} + (1 + \omega^{q_1})a^{q_3}\delta^{q_1} + a^{q_1+q_4}(1 + \omega^{q_2})a^{q_2}\delta^{q_3} + (1 + \omega^{q_3})a^{q_3}\delta^{q_2}.
\]

(iv) Otherwise, the invariant is trivial. Namely, \( I_{\Gamma(q)}(T(m, n)) \in \mathbb{Z} \).

This is proved in [5.3]. Note that, for \( e = 2 \), the invariant is non-trivial in only the case (iii).

**Remark 5.2.** Asami and Kuga [AK §5.2] partially calculated some values of \( I_{\Gamma(q)}(T(m, n)) \) in the only case \( \mathbb{F}_q = \mathbb{F}_{5^2} \) and \( n = 3 \), by the help of computer.

We consider the \( t \)-fold cyclic covering of \( S^3 \) branched over \( T(m, n) \). This is the Brieskorn manifold \( \Sigma(m, n, t) \); see [Mi]. Hence, we obtain a \( \mathbb{Z} \)-equivariant part of the Dijkgraaf-Witten invariant of \( \Sigma(m, n, t) \).

**Corollary 5.3.** Let \( m, n \) be coprime integers. Let \( X \) be of type \( t \). Let a Mochizuki quadruple \((q_1, q_2, q_3, q_4) \in \mathbb{Q}_{q, \omega} \) satisfy Case 1, and \( \theta_T \in H^3_{gr}(G_X; \mathbb{F}_q) \) be the group 3-cocycle in Lemma 4.6. Let \( p > 2 \) be coprime to \( m \) and to \( t \). If \( \omega^{mn} = 1 \), \( \omega^n \neq 1 \) and \( \omega^m \neq 1 \), then
\[
DW^Z_{\text{gr}}(\Sigma(m, n, t)) = \sum_{a \in \mathbb{F}_q} 1\mathbb{Z}\{-2tmn(\zeta - \omega)^{q_2+q_3}\omega^{q_4} \cdot a^{q_1+q_2+q_3+q_4} \} \in \mathbb{Z}[\mathbb{F}_q].
\]

Proposition 4.7 says that the cocycle \( \theta_T \) forms a Massey product; Hence we clarify partially the Massey product structure of some Brieskorn manifolds. Here remark that there are a few methods to compute Massey products with \( \mathbb{Z}/p \)-coefficients, in a comparison with those with \( \mathbb{Q} \)-coefficients viewed from rational homotopy theory.

Finally, we comment on the interesting result in Theorem 5.1 (ii). For finite nilpotent groups \( G \), the Massey products in \( H^3_{gr}(G; \mathbb{F}_q) \) with \( p = 2, 3 \) are exceptional. For example, when \( q = p^2 \), the group \( G_X \) is isomorphic to the group \( “P(3)” \) in [Le]. See [Le, Theorems 6 and 7] for exceptional phenomenon of the cohomology ring \( H^*_\text{gr}(G_X; \mathbb{F}_p) \) with \( p = 2, 3 \).
5.2 Further examples in the case $\omega = -1$

We change our focus to other knots. However, it is not so easy to calculate the cocycle invariant $I_{\Gamma(q),1}(K)$ of knots, although it is elementary.

We now consider the simplest case $\omega = -1$; hence the quandle $X$ is of type 2. Furthermore, note that, for any Mochizuki quadruple $q = (q_1, q_2, q_3, q_4)$, the associated 3-cocycle forms $U_1^{q_1}U_2^{q_2+q_1}U_3^{q_4}$ by definition; it is not hard to compute the cocycle invariant. However, for many knots whose colorings satisfy Col$_X(D) \cong (F_q)^2$, the invariants are frequently of the form $q^2 \sum_{a \in F_q} a^{q_1 + q_2 + q_3 + q_4}$ up to constant multiples in computer experiments. In order to avoid the case Col$_X(D) \cong (F_q)^2$, recall the bijection \([22]\). Accordingly we shall deal with some knots having non-trivial second Alexander polynomials as follow s:

**Example 5.4.** Let $\omega = -1$. The knots $K$ in Table 1 are those whose crossing numbers are $< 11$ satisfying Col$_X(D) \cong (F_q)^3$ with $p > 3$. We only list results of the invariants without the proofs. Here note that, according to Theorem 2.16 and Proposition 4.7, the cocycle invariant stems from triple Massey products of double branched covering spaces. Refer to the tables in [Kaw, Appendix F] for the correspondences between knots $K$ and double coverings of $S^3$ branched over $K$.

| $K$ | $p$ | $I_{\Gamma(q),1}(K)$ |
|-----|-----|---------------------|
| 9,40 | 5   | $G(q; 1, 5)$        |
| 9,41 | 7   | $G(q; 3, 4)$        |
| 9,49 | 5   | $G(q; 3, 4)$        |

Table 1: The values of $I_{\Gamma(q),1}(K)$.

Here, $q \in \mathbb{Z}$ and $q \in Q_{q,\omega}$ are arbitrary, and, for $n, m \in \mathbb{Z}$, the symbols $G(q; n, m)$ and $G_{155}(q)$ are polynomials expressed by

$$G(q; n, m) := q^2 \sum_{a, b \in F_q} 1_Z \left\{ n(a^{q_1 + q_2}b^{q_3 + q_4} + a^{q_1}b^{q_1 + q_2} + a^{q_1 + q_3}b^{q_2 + q_4} + a^{q_2 + q_4}b^{q_1 + q_3}) 
+ m(a^{q_1 + q_4}b^{q_2 + q_3} + a^{q_2 + q_4}b^{q_1 + q_3}) \right\} \in \mathbb{Z}[F_q],$$

$$G_{155}(q) := q^2 \sum_{a, b \in F_q} 1_Z \left\{ a^{q_1 + q_2 + q_3 + q_4} + a^{q_1}b^{q_2 + q_3 + q_4} \right\} \in \mathbb{Z}[F_q].$$

5.3 Proof of Theorem 5.1

For the proof, we first recall a slight reduction of the cocycle invariant by [IK, Theorem 4.3], that is, we may consider only shadow colorings of the forms $S = (\mathcal{C}; 0)$. More precisely,

$$I_{\psi}(L) = q \cdot \sum_{C \in \text{Col}_X(D)} 1_Z \{ \langle \psi, [(\mathcal{C}; 0)] \rangle \} \in \mathbb{Z}[A]. \tag{25}$$

22
We establish terminologies on the torus knot $T(m,n)$. Regard $T(m,n)$ as the closure of a braid $\Delta^m$, where $\Delta := \sigma_{n-1} \cdots \sigma_1 \in B_n$. Let $\alpha_1, \ldots, \alpha_n$ be the top arcs of $\Delta^m$. For $1 \leq i \leq m$, we let $x_{i,1}, \ldots, x_{i,n-1}$ be the crossings in the $i$-th $\Delta$; see Figure 4.

![Figure 4: The arcs $\alpha_j$ and crossing points $x_{i,j}$ on the diagram of the torus knot.](image)

**Proof of Theorem 5.1.** Although Asami and Kuga [AK] formulated explicitly $X$-colorings of $T(m,n)$, we will give a reformulation of them appropriate to the 3-cocycle $\Gamma(q)_e$. If given an $X$-coloring $C$ of $T(m,n)$, we define $a_j := C(\alpha_j)$, and put a vector $a = (a_1, \ldots, a_n) \in (\mathbb{F}_q)^n$; Notice that it satisfies the equation $a = aP^m$, where $P$ is given by a companion matrix

$$P := \begin{pmatrix}
0 & \omega & 0 & \cdots & 0 & 0 \\
0 & 0 & \omega & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & \omega \\
1 & 1 - \omega & 1 - \omega & \cdots & 1 - \omega & 1 - \omega
\end{pmatrix} \in \text{Mat}(n \times n; \mathbb{F}_q).$$

Remark that the characteristic polynomial is $(\lambda - 1)(\lambda^n - \omega^n)/(\lambda - \omega)$, and that the roots are $\lambda = \zeta^k \omega$ and 1, where $1 \leq k < n$ and $\zeta$ means an $n$-th primitive root of unity in the algebraic closure $\overline{\mathbb{F}}_p$. Therefore, the proof comes down to the following two cases:

**Case I** $\omega^n \neq 1$. Namely, the roots are distinct.

**Case II** $\omega^n = 1$. Then, $\lambda = 1$ is a unique double root of the characteristic polynomial.

We will calculate the weights coming from such $X$-colorings case by case. While the statement (i) will be derived from Case I, those (ii) and (iii) will come from Case II.

**Case I** Let $\omega^n \neq 1$. We will study the solutions of $a = aP^m$. We easily see that, if $(\zeta^{-k}\omega)^m = 1$ for some $k$, then the solution is of the form

$$a_{j+1} = a((1 - \zeta^{kj})/(1 - \zeta)) + a(\zeta^{kj}/(1 - \omega)) + \delta$$

for some $a, \delta \in \overline{\mathbb{F}}_p$; conversely, if the equation $a = aP^m$ has a non-trivial solution, then there is a unique $k$ satisfying $(\zeta^{-k}\omega)^m = 1$ and $0 < k < n$. It is further verified that such a solution gives rise to an $X$-coloring $C$ if and only if $a, \delta, \zeta$ are contained in $\mathbb{F}_q$. In assumary, we may assume that $a, \delta, \zeta \in \mathbb{F}_p$, and $(\zeta^{-1}\omega)^m = 1$ with $\zeta \neq \omega$. Indeed this assumption justifies a shadow coloring $S$ of the form $(C; 0)$.
Remark 5.5. We give a remark on this assumption. Notice that, for \( s \in \mathbb{Z} \), two equalities \( \omega^m = \zeta^m \) and \( \omega^s = 1 \) imply \( \zeta^{m s} = 1 \) and, hence, \( \zeta^s = 1 \), since \( m \) and \( n \) are coprime. In particular, considering special cases of \( s = q_1 + q_3 \) and \( s = q_1 + q_4 \), we have \( \zeta^{q_1 + q_3} = \zeta^{q_2 + q_4} = 1 \). Similarly we notice that, if \( \omega^{q_1 + q_2} = 1 \), then \( \zeta^{q_1 + q_2} = 1 \).

We will present the weights of \( [\mathcal{S}] = [(\mathcal{C}; 0)] \), where \( \mathcal{C} \) is the \( X \)-coloring as the solution mentioned above. We then can easily check the color of every regions in the link-diagram. After a tedious calculation, the weight of \( x_{i, j} \) is consequently given by

\[
\left( a\zeta^{-i} q^j \frac{1 - \zeta^{-j - 1}}{1 - \zeta} + (1 - \omega^{-2j - 1}) q^j, a\zeta^{-i} q^j \left( \frac{1 - \zeta^{-j - 1}}{1 - \zeta} + \zeta^{-j - 1} + \zeta^{-j - 1} \right) + q^j \right) \in C_3^q(X).
\]

We next compute the pairing \( \langle \Gamma(q)_e, [\mathcal{S}] \rangle \in \mathbb{F}_q \) in turn. To begin with the case \( e = 1 \), recalling \( \Gamma(q)_e = U_1^q U_2^q U_3^q = (x_1 - x_2)^q (x_2 - x_3)^q (x_3 - x_4)^q \), we describe the pairing as

\[
\sum_{i \leq m, j \leq n - 1} \left( a\zeta^{-i} q^j \frac{1 - \zeta^{-j - 1}}{1 - \zeta} + (1 - \omega^{-2j - 1}) q^j \left( \frac{1 - \zeta^{-j - 1}}{1 - \zeta} + \zeta^{-j - 1} + \zeta^{-j - 1} \right) + q^j \right) \in \mathbb{F}_q.
\]

Here we note \( \sum_{i=1}^m (\zeta^{-1} \omega)^i = 0 \) unless \( \zeta^{-s} \omega^s = 1 \). Therefore, several terms in this formula vanish by Remark 5.5 above. It is easily seen that the non-vanishing term in \( \langle \Gamma(q)_1, [\mathcal{S}] \rangle \) is

\[
\frac{a^{q_1 + q_2 + q_3 + q_4} (\zeta - \omega)^{q_2 + q_3 + q_4}}{(1 - \omega)^{q_1 + q_2 + q_3 + q_4}} \sum_{i \leq m, j \leq n - 1} (\zeta^{-1} \omega)^i (1 - \zeta)^{q_1 + q_2 + q_3 + q_4} (1 - \zeta)^{q_2 + q_3} \in \mathbb{F}_q. \quad (26)
\]

Here, by Remark 5.5 again, we notice two equalities

\[
(\zeta^{-1} \omega)^{q_1 + q_2 + q_3 + q_4} = 1, \quad \zeta^{q_1} (1 - \zeta)^{q_2 + q_3} = \zeta^{q_1} + \zeta^{q_2} - 2.
\]

Therefore, since \( \sum_{j=1}^{n-1} \zeta^j = \sum_{j=1}^{n-1} \zeta^{j+2} = -1 \), the sum in the formula \( (26) \) equals \(-2nm\).

By (25), we hence obtain the required formula (23).

Further, by Remark 4.2 the same calculations hold for the cases \( 3 \leq e \leq 5 \).

Next, we deal with \( e = 2 \). For the shadow coloring \( \mathcal{S} = (\mathcal{C}; 0) \), we claim \( \langle \Gamma(q)_2, [\mathcal{S}] \rangle = 0 \). To see this, by a similar calculation to (26), we reduce the pairing \( \langle \Gamma(q)_2, [\mathcal{S}] \rangle \) to

\[
\langle \Gamma(q)_2, [\mathcal{S}] \rangle = -2nm a^{q_1 + q_2 + q_3 + q_4} \cdot A_4,
\]

where \( A_4 \) is temporarily defined by the formula

\[
\frac{(\zeta - \omega)^{q_2 + q_3 + q_4}}{(1 - \zeta)^{q_2 + q_3}} - \frac{(\zeta - \omega)^{q_1 + q_2 + q_3}}{(1 - \zeta)^{q_1 + q_4}} + 1 - \omega^{q_1 + q_2} \left( \frac{(\zeta - \omega)^{q_2 + q_3 + q_4}}{(1 - \zeta)^{q_2}} - \frac{(\zeta - \omega)^{q_4 + q_3}}{(1 - \zeta)^{q_4}} \right).
\]

We now assert that the last term in this formula \( A_4 \) is zero. Indeed, noting \( (1 - \zeta)^{-q_4} = \zeta^{q_4} (1 - \zeta)^{-q_4} \) and \( \zeta^{-q_4} = 1 \) by Remark 5.5, we easily have

\[
\frac{(\zeta - \omega)^{q_2 + q_3 + q_4}}{(1 - \zeta)^{q_2}} - \frac{(\zeta - \omega)^{q_4 + q_3}}{(1 - \zeta)^{q_4}} = \frac{(\zeta - \omega)^{q_2 + q_3 + q_4} + (\zeta - \omega)^{q_4 + q_3} \zeta^{q_2 + q_3}}{(1 - \zeta)^{q_2}} = 0.
\]

Similarly we easily see an equality \( (1 - \zeta)^{-q_1 - q_4} = \zeta^{q_2 + q_3} (1 - \zeta)^{-q_2 - q_3} \); therefore the first and second terms in \( A_4 \) are canceled. Hence \( A_4 = 0 \) as claimed. In conclusion, the cocycle invariants using \( \Gamma(q)_2 \) are trivial as desired.
We claim that if $p \omega q$ it is done by changing the quadruple $(\mathfrak{a}, \mathfrak{e})$ routine reason for the cases.

By Lemma 5.6 (II) below and the definition of $\Gamma(q)$, we obtain the required formula (2 4). Similarly, when $p \omega q$ we run over all shadow colorings, we obtain the required formula (2 4). By Lemma 5.6 (III) below. Furthermore, by the previous change of the indices, we know the second term. In summary, we conclude the desired formula in (iii).

We here consider the sum on $\sum n(\mathfrak{a})$.

Let us calculate the pairing $\langle \Gamma(q), [S] \rangle$. To begin, when $e = 1$, the $\langle \Gamma(q), [S] \rangle$ equals

$$\sum_{i \leq m, j \leq n-1} \left( (aj(\omega - 1) - ai(\omega - 1) - \delta)\omega^{j-1} \right)^{q_1} (a - a\omega^j)^{q_2+q_3} (ai(\omega - 1) + \delta)^{q_4}. \quad (27)$$

We here consider the sum on $i$. However, we notice that $\sum_{i \leq m} i^{q_1+q_4} = \sum_{i \leq m} i^{2} = m(m + 1)(2m + 1)/6$. Hence, since $m$ is divisible by $p$, the pairing vanishes unless $p = 2, 3$.

Similarly, we can see that, in other cases of $e$, the pairings are zero unless $p = 2, 3$. We therefore may devote to the cases $p = 2, 3$ hereafter.

First, assume $p = 3$ and $e = 1$. Note that the non-vanishing term in (27) is only the coefficients of $\sum i^{q_1+q_4}$, and that $\sum_{i \leq m} i^{q_1+q_4} = -m/3$. Then the pairing (27) is reduced to be

$$a^{q_1+q_2+q_3+q_4}(1-\omega)^{q_1+q_4} \sum_{1 \leq j \leq n-1} \omega^{q_1(j-1)}(1-\omega^j)^{q_2+q_3} \sum_{1 \leq i \leq m} i^{q_1+q_4} = \frac{m^n}{3} a^{q_1+q_2+q_3+q_4}(1-\omega)^{q_3+q_4},$$

where $\sum i^{q_1(j-1)}(1-\omega^j)^{q_2+q_3} = 2n\omega^{-q_1}$ in this equality follows from $\omega^n = 1$. Hence, by running over all shadow colorings, we obtain the required formula (24). Similarly, when $p = 2$ and $e = 1$, a calculation using Lemma 5.6 (I) below can show the formula (24).

Furthermore, the same calculation holds for the cases $3 \leq e \leq 5$ and $p = 2, 3$. Actually, it is done by changing the quadruple $(q_1, q_2, q_3, q_4)$ in the previous calculation in Case 1, as a routine reason for the cases.

At last, it is enough for the proof to work out the remaining case $e = 2$ and $p = 2, 3$. By Lemma 5.6 (II) below and the definition of $\Gamma(q)$, the pairing is reduced to

$$\langle \Gamma(q), [S] \rangle = \langle U^{q_1} U^{q_2+q_4} U^{q_3}, [S] \rangle - \langle U^{q_2} U^{q_1+q_4} U^{q_3}, [S] \rangle. \quad (28)$$

We claim that if $p = 3$, $\langle \Gamma(q), [S] \rangle = 0$. The first term is reduced to $2mnq^{q_1+q_2+q_3+q_4}(1-\omega)^{q_3+q_4}/3$, by a similar calculation to (24). The second term is obtained by changing the indices $(1, 2, 3, 4)$ in the first term to $(2, 1, 4, 3)$. Hence the pairing $\langle \Gamma(q), [S] \rangle$ vanishes.

To complete the proofs, we let $p = 2$. The explicit formula of the first term in (28) follows from Lemma 5.6 (III) below. Furthermore, by the previous change of the indices, we know the second term. In summary, we conclude the desired formula in (iii). □

The following lemma used in the above proof can be obtained from the definitions and elementary calculations, although they are a little complicated.
Lemma 5.6. Let $S = (C; 0)$ be the shadow coloring in Case II as above.

(I) If $p = 2$ and $\omega^{q_1+q_2} = 1$, then the pairing $\langle U_1^{q_1} U_2^{q_2} U_3^{q_3}, [S] \rangle$ is equal to $(1 + \omega^{q_1+q_4} a^{q_1+q_2+q_3+q_4} mn/2$.

(II) If $\omega^{q_1+q_2} \neq 1$, and if $p = 2$ or $3$, then $\langle U_1^{q_1} U_2^{q_2} U_3^{q_3+q_4} - U_1^{q_1+q_2} U_2^{q_4} U_3^{q_3}, [S] \rangle = 0$.

(III) If $p = 2$ and $\omega^{q_1+q_2} \neq 1$, then $\langle U_1^{q_1} U_2^{q_2+q_3} U_3^{q_4}, [S] \rangle$ is equal to

$$\frac{mn}{2} \left( a^{q_2+q_3} (1 + \omega a \delta^{q_4} + (1 + \omega^{q_1}) a^{q_4} \delta^{q_1}) + \frac{1 + \omega^{-q_1} + \omega^{-q_2} + \omega^{q_1+q_2}}{1 + \omega^{q_1+q_2}} a^{q_1+q_2+q_3+q_4} \right).$$

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