CAN SIGMA MODELS DESCRIBE FINITE TEMPERATURE CHIRAL TRANSITIONS?

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Abstract

Large-$N$ expansions and computer simulations indicate that the universality class of the finite temperature chiral symmetry restoration transition in the 3D Gross-Neveu model is mean field theory. This is a counterexample to the standard 'sigma model' scenario which predicts the 2D Ising model universality class. We trace the breakdown of the standard scenario (dimensional reduction and universality) to the absence of canonical scalar fields in the model. We point out that our results could be generic for theories with dynamical symmetry breaking, such as Quantum Chromodynamics.

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When studying the finite temperature chiral restoration transition in QCD one is usually guided by the concepts of dimensional reduction and universality. A compelling idea, first put forward in [1] and later elaborated in [2], is that in four-dimensional QCD with \(N_f\) light quarks the physics near the chiral transition can be described by the three-dimensional \(\sigma\)-model with the same global symmetry. The reasoning behind this proposal is based on counting the light degrees of freedom and can be phrased as follows. The transition region is dominated by the longitudinal and transverse fluctuations of the order parameter, \(\sigma\) and \(\pi\), which go soft at the transition temperature. Being bosonic, \(\sigma\) and \(\pi\) have zero modes, \(\omega_n = 0\), in their finite-temperature Matsubara decomposition. These zero modes are the only relevant degrees of freedom in the scaling region and at low energies the \(n \neq 0\) modes decouple. Therefore, in the context of a \(d\)-dimensional theory, one concludes that the phase transition is described by an effective scalar theory in \(d - 1\) dimensions. As a consequence, the chiral transition of four-dimensional QCD, with \(N_f = 2\) flavors, should lie in the same universality class as a three-dimensional \(O(4)\) magnet [1,2]. Similarly, other models e.g. four-fermi theories in \(d\)-dimensions like Gross-Neveu [3] with discrete or Nambu-Jona-Lasinio [4] with continuous chiral symmetries, are expected to be in the universality class of a \(d - 1\)-dimensional Ising or Heisenberg magnet, respectively.

It is the purpose of this paper to discuss the assumptions underlying this analysis. As an illustration we will study two examples: a purely bosonic theory, \(O(N)\) \(\sigma\)-model where the ideas of dimensional reduction apply, and a Gross-Neveu model with composite scalars where they fail. We discuss the generic features of the models that might apply to other field theories at finite temperature. At the end we comment on the implications these two examples have on QCD.

To illustrate how the idea of dimensional reduction is realized in scalar theories, we start with the \(N\)-component scalar theory and consider the large-\(N\) limit [5] for simplicity. To avoid complications due to Goldstone bosons, we work in the symmetric phase. At zero temperature, the susceptibility is given by the single tadpole contribution. Defining the critical curvature \(\mu_c^2\) as the point where the susceptibility diverges \((\mu_c^2 + \lambda \int_q 1/q^2 = 0)\), the expression for the inverse susceptibility can be recast into

\[
\chi^{-1} \left( 1 + \lambda \int_q \frac{1}{q^2(q^2 + \chi^{-1})} \right) = \mu^2 - \mu_c^2
\]

(1)

where \(\int_q = \int d^d q/(2\pi)^d\), and we absorb the combinatorial factor in \(\lambda\). The extraction of the critical index \(\gamma\) reduces to counting powers of the infrared (IR) singularities on the
left hand side (LHS) of eq.(1). Above four dimensions, both terms are IR finite and the scaling is mean field \((\gamma = 1)\). Below four dimensions the second term in eq.(1) dominates the scaling region – the integral diverges as \(\chi^{(4-d)/2}\). This gives the zero-temperature susceptibility exponent \(\gamma = 2/(d - 2)\) [5].

At finite temperature, apart from the replacement of the frequency integral with the Matsubara sum, modifications are minimal [6]. For a given value of \(\mu^2\) we define the critical temperature, \(T_c\), by \(\mu^2 + \lambda T_c \sum_n \int \frac{1}{\omega_{nc}^2 + q^2} = 0\), where \(\omega_{nc} = 2\pi n T_c\). The momentum integrals are now performed over \(d-1\) dimensional space. Separating the \(n = 0\) mode \((\omega_0 = 0)\) from the rest of the sum, we get the leading singular behavior

\[
\chi^{-1} \left( 1 + \lambda T_c \int \frac{1}{q^2(\chi^2 + 1)} + \sum_{n \neq 0} \ldots \right) = \lambda T_c \int \frac{T/T_c - 1}{q^2 + \chi^{-1}} + \sum_{n \neq 0} \ldots
\]

The \(n = 0\) piece dominates the scaling region. It resembles the zero-temperature expression, eq.(1), except that now, the integrals are performed in \(d - 1\) dimensions, instead of \(d\). The power counting is the same as before and it yields the thermal exponent \(\gamma_T = 2/(d - 3)\) which is the same as the zero-temperature \(\gamma\) in \(d - 1\) dimensions [6]. It is easy to obtain the other critical exponents; they show the same type of behavior as \(\gamma\).

To illustrate how compositeness affects the physics near the phase transition, we analyze the problem of chiral symmetry restoration in a Gross-Neveu model given by the lagrangian \(L = \bar{\psi}(i\partial + m + g\sigma)\psi - \frac{1}{2}\sigma^2\), where notation is standard [3]. Besides being an interesting theoretical model, it is also believed that, when properly extended to incorporate continuous chiral symmetry, four-fermi models are more realistic as effective theories of \(QCD\) than the linear sigma model, especially at scales where quark substructure is important. When fermions are integrated out of the Gross-Neveu model, the Ising symmetry, \(\sigma \rightarrow -\sigma\), of the effective action becomes manifest. If the dimensional reduction + universality arguments hold [1], the finite temperature transition of the \(d\)-dimensional model would lie in the universality class of the \(d - 1\) dimensional Ising model. In the remainder of the paper we explain how and why this argument fails.

First, we start with the zero-temperature gap equation and corresponding critical exponents. The model can be treated in the large-\(N\) limit. To leading order, the fermion self-energy, \(\Sigma\), comes from the \(\sigma\)-tadpole: \(\Sigma = m - g^2 < \bar{\psi}\psi >\). To obtain the scaling properties of the theory, we define the critical coupling as \(1 = 4g_c^2 \int q 1/q^2\). Combining this definition with the gap equation leads to
\[ \frac{m}{\Sigma} + \left( \frac{g^2}{g_c^2} - 1 \right) = 4g^2 \int_q \frac{\Sigma^2}{q^2(q^2 + \Sigma^2)} \]  

(3)

Like the scalar example, this form is especially well suited for extracting critical indices since the problem reduces again to the counting of the infra-red divergences on the right hand side [7,8]. The critical indices are defined by \( \langle \bar{\psi}\psi \rangle |_{m=0} \sim t^\beta, \langle \bar{\psi}\psi \rangle |_{t=0} \sim m^{1/\beta}, \Sigma|_{m=0} \sim t^\nu \), etc.. Here, \( t = \frac{g^2}{g_c^2} - 1 \) is the deviation from the critical coupling. Since \( \Sigma \sim \langle \bar{\psi}\psi \rangle \), \( \beta = \nu \) to leading order. Above four dimensions the integral in eq.(3) is finite in the limit of vanishing \( \Sigma \) and the scaling is mean-field.

Below four dimensions, the \( \Sigma \to 0 \) limit is singular – the integral scales as \( \Sigma^{d-2} \). Thus, in the chiral limit \( t \sim \Sigma^{d-2} \), and at the critical point, \( t = 0 \), away from the chiral limit, \( m \sim \Sigma^{d-1} \). The resulting exponents are non-gaussian: \( \beta = 1/(d-2) \) and \( \delta = d - 1 \). The remaining exponents are obtained easily: \( \eta = 4 - d, \gamma = 1 \) [7,8] and one can check that they obey hyperscaling.

We now consider the Gross-Neveu model at finite-temperature. We choose to stay between two and four dimensions to emphasize how zero-temperature power-law scaling changes at finite temperature. The gap equation is now modified to

\[ \Sigma = m + 4Tg^2 \sum_n \int_q \frac{\Sigma}{\omega_n^2 + q^2 + \Sigma^2} \]  

(4)

where \( \omega_n = (2n + 1)\pi T \). For \( g > g_c \) the critical temperature is determined by: \( 1 = 4Tc/g^2 \sum_n \int_q 1/(\omega_{nc}^2 + q^2) \), where \( \omega_{nc} = (2n + 1)\pi Tc \). This expression defines a critical line in the \( (g,T) \) plane. For every coupling there exists a critical temperature beyond which the symmetry is restored. Conversely, for a fixed temperature there is a critical coupling, defined by the above expression, corresponding to symmetry restoration. At zero temperature, the symmetry is restored at \( g = g_c \). Thus, \( (g = g_c, T = 0) \) is the ultra-violet (UV) fixed point. As the coupling moves away from \( g_c \), a higher restoration temperature results. At infinite coupling the end-point, \( (g = \infty, T = T_c) \), is the IR fixed point. The critical line connects the UV and IR fixed points dividing the \( (g,T) \) plane into two parts.†

† The equation for the critical line can be brought into a compact form by combining the expression for \( T_c \) with the definition of the zero-temperature critical coupling. This results in: \( (g^2/g_c^2 - 1) \sim T_c^{d-2}(g) \), i.e. \( T_c(g) \sim \Sigma(T = 0) \). In this way, for any value of the coupling, the critical temperature remains the same in physical units.
Combining the definition of $T_c$ with the finite-temperature gap equation, we can bring it to a form similar to eq.(3)

$$\frac{m}{\Sigma} = (1 - T/T_c) + 4Tg^2 \sum_n \int_{\vec{q}} \frac{\Sigma^2 + \omega_{nc}(\omega_n + \omega_{nc})(T/T_c - 1)}{(\omega_{nc}^2 + \vec{q}^2)(\omega_n^2 + \vec{q}^2 + \Sigma^2)}$$

(5)

The extraction of the critical exponents proceeds along the same lines as in the zero-temperature case. One difference relative to eq.(3) becomes apparent immediately: the zero modes are absent here and the integrand in eq.(5) is regular in the $\Sigma \to 0$ limit even below four dimensions. Consequently, the IR divergences are absent from all the integrals and the scaling properties are those of mean-field theory: $\beta = \nu = 1/2, \delta = 3$, etc. This is true for any $d$, below or above four. It appears that in this case, contrary to the scalar example, the effect of making the temporal direction finite ($1/T$) is to regulate the IR behavior and suppress fluctuations. This is manifest in other thermodynamic quantities as well. For example, to leading order, the scalar susceptibility, $\chi = \partial <\bar{\psi}\psi>/\partial m$, is given by

$$\chi^{-1} = 8g^2T \sum_n \int_{\vec{q}} \frac{\Sigma^2}{(\omega_n^2 + \vec{q}^2 + \Sigma^2)^2}$$

(6)

Once again, because of the absence of the zero mode ($\omega_0 = \pi T$), the integral in eq.(6) is analytic in $\Sigma$, and the mean field relation $\chi^{-1} \sim \Sigma^2$ follows. This is equivalent to $\gamma = 2\nu = 1$. The explicit calculation of the momentum dependence of the $\sigma$ propagator [9] yields $\eta = 0$.

The scaling laws of the finite temperature transition obtained above are completely different from the predictions of ref.[1]. In fact, even the systematics are opposite. The regulating character of the temperature drives the lower dimensional theory towards an effective theory that has gaussian critical exponents.

The fermionic model discussed above was first analyzed in ref.[9]. Higher order calculations have shown that the results are not artifacts of the large-$N$ limit [10]. In addition, it was explained in [10] how the Ising point is recovered in four dimensional Yukawa models beyond the leading order in $1/N$ and why this does not happen in Gross Neveu models. Lattice simulations of the three dimensional model have verified the predictions of the large-$N$ expansion at zero temperature, at nonzero temperature and at nonzero chemical potential [11]. The results for critical indices have been verified and improved by larger scale simulations enhanced by histogram methods [12]. We have done additional simu-
lations to check the finite temperature results [11] in detail. Lattices of sizes $6 \times 30^2$, $12 \times 36^2$ and $12 \times 72^2$ were simulated at $N = 12$ using the Hybrid Monte Carlo algorithm described in [11]. High statistics runs (several tens of thousands of trajectories for each coupling) were made on a variety of lattices to guarantee that the simulations were probing the physical IR modes at finite temperature. Luckily, our task is to distinguish mean field exponents from those of the two dimensional Ising model, and, as reviewed in Table I, they are dramatically different. We will discuss the exponents $\delta$ and $\beta$ (defined above) and leave other calculations to a lengthier presentation. In Fig.1 we show the square of the order parameter $\sigma$ plotted against $1/g^2$. The data is in excellent agreement with mean field theory where $\beta = 1/2$, and rules out the Ising model value of $1/8$. Note that the statistical error bars in the figure are smaller than the plotting symbols themselves. Since both lattice sizes give the same estimates of $\beta$ while their critical temperatures are quite different, we are confident that the simulation is probing the true continuum behavior of the finite temperature transition and is not corrupted by a sluggish crossover between symmetric and asymmetric lattices. Several runs on the huge $12 \times 72^2$ lattice were made and the $12 \times 36^2$ results were confirmed to a fraction of a percent. We also calculated the susceptibility index $\gamma$ and found $\gamma = 1.0(1)$ in the same runs. The Ising result $\gamma = 7/4$ is decisively ruled out. Next, we read off the critical temperatures for both lattice sizes, measure the response of the order parameter at criticality to an external symmetry breaking field (bare fermion mass) and obtain $\delta$. The data is shown in Fig.2, and for both lattice sizes we find $\delta = 3.1(1)$. The Ising model value of $\delta = 15$ is ruled out. In all of these calculations we carefully visualized the $\sigma$ field to check for nonuniform configurations that would violate the mean field hypothesis [13]. None were found and all the past simulations [11] and the new ones reported here support the contention that the large-$N$ results are reliable for this problem.

An important feature of the exponents corresponding to the finite-temperature transition, is that they violate hyperscaling [14]. Usually, hyperscaling violation occurs above four dimension and is expressed in terms of exponent inequalities [14] e.g. $2\beta\delta - \gamma \leq d\nu$. Strict inequality is applicable only for $d > 4$ and implies factorization of the correlation functions. In our example the above inequality goes in the opposite direction and the breakdown of hyperscaling is not accompanied by the factorization of Green’s functions.

In conclusion, the study of the Gross-Neveu model suggests that arguments invoking dimensional reduction + universality must be used with care. Our results indicate that
an effective scalar model fails to describe the Gross-Neveu model at finite temperature. We believe that the reason for this failure in our example is related to the composite nature of the mesons. Pointlike scalars can not adequately describe the physics in the vicinity of the second order chiral transition. The physical picture behind this failure observes that both the density and the size of the loosely bound sigma meson increase with temperature. Close to the restoration temperature the system is densely populated with overlapping composites. In other words the fluffiness of the mesons can not be ignored – the constituent fermions are essential degrees of freedom even in the scaling region, right before the composites dissociate. Similar discussions of the failure of effective meson theories in a slightly different context have been given in ref.[15].

It is well known that four-fermi models can be used as effective theories of $QCD$ [16]. In addition to having the same global symmetries, the mesons in both theories are composite. Therefore, these models are believed to have common properties over a wide range of scales where the quark substructure of the mesons is relevant. Knowing this, it would be interesting to see what happens in two flavor $QCD$ [17]: does it follow the dimensional reduction scenario, or the Gross-Neveu behavior? Of course, these two alternatives do not exhaust all the possibilities [18], but we believe that the scenario suggested by the Gross-Neveu model is sufficiently compelling to warrant further analyses of $QCD$ simulation data.

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Figure Captions

1. Order parameter squared plotted against temperature on 6×30²(left) and 12×36²(right) lattices.
2. Order parameter response at criticality plotted against bare fermion mass on 6 × 30²(bottom) and 12 × 36²(top) lattices.
Table 1

Critical exponents of the 3D Gross-Neveu and 2D Ising model

|       | $d = 3$ | Gross-Neveu | $d = 2$ |   |
|-------|---------|-------------|---------|---|
|       | $T = 0$ | $T \neq 0$ | Ising   |   |
| $\beta$ | 1   | 1/2 | 1/8 |   |
| $\delta$ | 2   | 3   | 15  |   |
| $\gamma$ | 1   | 1   | 7/4  |   |
| $\nu$  | 1   | 1/2 | 1   |   |
| $\eta$ | 1   | 0   | 1/4  |   |
Order Parameter Squared vs. Temperature, 6x30^2 and 12x36^2
Order Parameter Response at Criticality, 6x30^2 and 12x36^2