Quasi-isometric embedding of the fundamental group of an orthogonal graph-manifold into a product of metric trees

Alexander Smirnov*

Abstract

In every dimension $n \geq 3$ we introduce a class of orthogonal graph-manifolds and prove that the fundamental group of any orthogonal graph-manifold quasi-isometrically embeds into a product of $n$ trees. As a consequence, we obtain that asymptotic and linearly-controlled asymptotic dimensions of such group are equal to $n$.

1 Introduction

We introduce a class $\mathcal{O}$ of orthogonally glued higher-dimensional graph-manifolds (that we call throughout this paper orthogonal graph-manifolds; see section 2.2 for the definition). Using the ideas of the paper [7], we generalize the results of that paper to the case of the class $\mathcal{O}$.

**Theorem 1.** For every $n$-dimensional orthogonal graph-manifold its fundamental group supplied with an arbitrary word metric admits a quasi-isometric embedding into a product of $n$ metric trees. As a consequence, asymptotic and linearly-controlled asymptotic dimensions of such group are equal to $n$.

In the paper [7] this result was obtained in the 3-dimensional case for every graph-manifold in the sense of the definition in section 2.1. In fact, according to the paper [8], the fundamental group of any 3-dimensional graph-manifold is quasi-isometric to the fundamental group of some flip-manifold, which is precisely an orthogonal graph-manifold in the dimension 3. Also note that the inequality $\text{asdim} \pi_1(M) \leq n$ for the fundamental group of an orthogonal graph-manifold $M$ follows from the result obtained in the Bell – Dranishnikov [3].

---

*Supported by RFFI Grant 11-01-00302-a
2 Preliminaries

2.1 Graph-manifolds

**Definition.** A higher-dimensional graph-manifold is a closed, orientable, $n$-dimensional, $n \geq 3$, manifold $M$ that is glued from a finite number of blocks $M_v$, $M = \bigcup_{v \in V} M_v$. These should satisfy the following conditions (1)–(3).

1. Each block $M_v$ is a trivial $T^{n-2}$-bundle over a compact, orientable surface $\Phi_v$ with boundary (the surface should be different from the disk and the annulus), where $T^{n-2}$ is a $(n-2)$-dimensional torus;

2. the manifold $M$ is glued from blocks $M_v$, $v \in V$, by diffeomorphisms between boundary components (the case of gluing boundary components of the same block is not excluded);

3. gluing diffeomorphisms do not identify the homotopy classes of the fiber tori.

For brevity, we use the term “graph-manifold” instead of the term “higher-dimensional graph-manifold”.

Let $G$ be a graph dual to the decomposition of $M$ into blocks. The set of blocks of the graph-manifold coincides with the vertex set $V = V(G)$ of the graph $G$. The set of (non-oriented) edges $E = E(G)$ of $G$ consists of pairs of glued components of blocks. We denote the set of the oriented edges of $G$ by $W$.

For more information about the graph-manifolds see [4].

2.2 Orthogonal graph-manifolds

In this section we define a class of graph-manifolds that admit an orthogonally glued metric of a special form. For brevity, we will call them orthogonal graph-manifolds.

Fix a graph $G$ and for each vertex $v \in V(G)$ consider a surface $\Phi_v$ of nonnegative Euler characteristic with $|\partial_v|$ boundary components, where $\partial_v$ is the set of all edges adjacent to the vertex $v$. Moreover, we assume that there is a bijection between the set of boundary components of the surface $\Phi_v$ and the set of all oriented edges adjacent to $v$. For the block $M_v$, corresponding to a vertex $v$ we fix a trivialization $M_v = \Phi_v \times S^1 \times \cdots \times S^1$, where $\Phi_v$ is the base surface, i.e. we fix simultaneously a trivialization $M_v = \Phi_v \times T^{n-2}$ of $M_v$ and a trivialization $T^{n-2} = S^1 \times \cdots \times S^1$ of the fiber torus. For each block $M_v$, we fix a coordinate system $(x, x_1, \ldots, x_{n-2})$ compatible with this decomposition, where $x \in \Phi_v$ and $x_i \in [0, 1)$ for each $1 \leq i \leq n-2$. For each oriented edge $w$ adjacent to the vertex $v$, we define the coordinate system $(x_0, x_1, \ldots, x_{n-2})$ on the corresponding component of the boundary $\partial \Phi_v$ of the surface $\Phi_v$. It defines the coordinate system $(x_0, \ldots, x_{n-2})$ on the boundary
torus $T_w$ of the block $M_v$. Similarly, for the edge $-w$ inverse to the edge $w$ on the boundary torus $T_{-w}$, we define the coordinate system $(x'_0, \ldots, x'_{n-2})$. For each oriented edge $w$, we consider a permutation $s_w$ of a well-ordered $(n - 1)$-element set $(x_0, \ldots, x_{n-2})$ such that $s_w(x_0) \neq x_0$. Furthermore, we assume that for mutually inverse edges $w$ and $-w$ the permutations $s_w$ and $s_{-w}$ are inverse ($s_{-w} \circ s_w = \text{id}$). We define the gluing map $\eta_w: T_w \to T_{-w}$ by $\eta_w((x_0, \ldots, x_{n-2})) = (s_w(x_0), \ldots, s_w(x_{n-2}))$. Note that this map is a well-defined gluing, as permutations $s_w$ and $s_{-w}$ are selected to be mutually inverse. Also, the map $\eta_w$ does not identify the homotopy classes of fiber tori.

**Definition.** The above described graph-manifold is called an orthogonal graph-manifold.

**Remark 1.** As mentioned above, in the case $n = 3$ the class of all orthogonal graph-manifolds coincides with the class of all flip graph-manifolds considered in [8].

### 2.3 Metric trees

A tripod in a geodesic metric space $X$ is a union of three geodesic segments $xt \cup yt \cup zt$ which have only one common point $t$. A geodesic metric space $X$ is called a metric tree if each triangle in it is a tripod (possibly degenerate).

### 2.4 Finitely generated groups

Let $G$ be a finitely generated group and $S \subset G$ a finite symmetric generating set for $G$ ($S^{-1} = S$). Recall that a word metric on the group $G$ (with respect to $S$) is the left-invariant metric defined by the norm $|| \cdot ||_S$, where for each $g \in G$ its norm $||g||_S$ is the smallest number of elements of $S$ whose product is $g$. It is known that all such metrics for the group $G$ are bi-Lipschitz equivalent (see [2]). In this paper we will consider only finitely generated groups with a word metric.

### 2.5 Quasi-isometric maps

A map $f: X \to Y$ is said to be quasi-isometric if there exist $\lambda \geq 1, C \geq 0$ such that

$$\frac{1}{\lambda} |xy| - C \leq |f(x)f(y)| \leq \lambda |xy| + C$$

for each $x, y \in X$. Metric spaces $X$ and $Y$ are called quasi-isometric if there is a quasi-isometric map $f: X \to Y$ such that $f(X)$ is a net in $Y$. In this case, $f$ is called a quasi-isometry.
2.6 The metric on the universal cover

Let us recall the famous Milnor–ˇSvarc Lemma.

**Lemma 1.** Let $Y$ be a compact length space and let $X$ be the universal cover of $X$ considered with the metric lifted from $Y$. Then $X$ is quasi-isometric to the fundamental group $\pi_1(Y)$ of the space $Y$ considered with an arbitrary word metric.

It follows from this lemma that to prove Theorem 1 it is sufficient to construct a quasi-isometric embedding of the universal cover into a product of $n$ trees.

2.6.1 Metrics of non-positive curvature

Define a metric on the orthogonal graph-manifold $M$ as follows: for each edge $e \in E(G)$ take a flat metric on its corresponding torus $T_e$ such that any base circle of the coordinate system described above has length 1, and any two of these circles are perpendicular.

In particular, for each vertex $v \in V(G)$ there is a metric on the boundary surface $\Phi_v$ in which every boundary component has length 1. This metric can be extended to a metric of nonpositive curvature on the surface $\Phi_v$ so that its boundary is geodesic. Therefore, the metric from the boundary tori extends to the metric on the block $M_v$, which is locally a product metric (in general, a metric on the block may not be a product metric and it can have nontrivial holonomy along some loops on the base $\Phi_v$).

Further we consider only those metrics on orthogonal graph-manifolds. If we lift the above metric in the universal cover $\tilde{M}$, it follows from the Reshetnyak gluing theorem (see [2]) that the obtained metric space is nonpositively curved (or Hadamard) space.

We fix an orthogonal graph-manifold $M$ with the metric described above.

2.6.2 The standard hyperbolic surface with boundary $H_0$

Consider the hyperbolic plane $H^2_\kappa$ having a curvature $-\kappa$ ($\kappa > 0$) such that the side of a rectangular equilateral hexagon $\theta$ in the plane $H^2_\kappa$ has length 1. Let $\rho$ be the distance between the middle points of sides, which have a common adjacent side, $\delta$ the diameter of $\theta$. We mark each second side of $\theta$ (so we have marked three sides) and consider a set $H_0$ defined as follows. Take the subgroup $G_\theta$ of the isometry group of $H^2_\kappa$ generated by reflections in (three) marked sides of $\theta$ and let $H_0$ be the orbit of $\theta$ with respect to $G_\theta$. Then $H_0$ is a convex subset in $H^2_\kappa$ divided into hexagons that are isometric to $\theta$. Furthermore, the boundary of $H_0$ has infinitely many connected components each of which is a geodesic $H^2_\kappa$. The graph $T_{bin}$ dual to the decomposition of $H_0$ into hexagons is the standard binary tree whose vertices all have degree three. Any metric space isometric to $H_0$ will be
called a \( \theta \)-tree. Given a vertex \( p \) of \( T_{\text{bin}} \), we denote by \( \theta_p \) the respective hexagon in \( H_0 \).

**Remark 2.** In what follows, we will consider \( T_{\text{bin}} \) as the metric space with a metric such that the length of each edge is equal to \( 2p \). Then these metric spaces are metric trees. We will denote the set of vertices in \( T_{\text{bin}} \) by \( V(T_{\text{bin}}) \).

### 2.6.3 Standard metrics and bi-Lipschitz homeomorphisms between bases

Consider a simplicial tree with the degree 3 of each vertex, and the length 1 of each edge. We replace each edge by the rectangle \( 1 \times \frac{1}{3^{100}} \) and each vertex by the equilateral Euclidean triangle with each side equal to \( \frac{1}{3^{100}} \). Then we glue them in a natural way.

**Definition.** The obtained metric space is called a fattened tree or a standard surface and denoted by \( X_0 \).

**Remark 3.** Note that the standard surface \( X_0 \) is bi-Lipschitz homeomorphic to the \( \theta \)-tree \( H_0 \). So we fix an arbitrary bi-Lipschitz homeomorphism \( h_0: H_0 \to X_0 \).

**Definition.** The **standard block** is defined to be a metric product of the \( \theta \)-tree \( H_0 \) and \( n - 2 \) copies of the Euclidean line \( \mathbb{R} \), \( B = H_0 \times \mathbb{R} \times \ldots \times \mathbb{R} \).

The partition of the \( \theta \)-tree by hexagons induces a partition of each boundary component of the \( \theta \)-tree by unit segments. For each boundary component such a partition is called a **grid** on this component. For each Euclidean factor \( \mathbb{R} \) of the standard block, an arbitrary partition by unit segments is called a **grid** on this factor. Finally for each boundary hyperplane \( \sigma \) of the standard block a partition by unit cubes induced by grids on each factor is called a **grid** on this hyperplane.

Recall the following theorem.

**Theorem 2** ([1], Theorem 1.2). Let \( X_0 \) be as above with a chosen boundary component \( \partial_0 X_0 \). Then there exists \( K > 0 \) and a function \( \psi: \mathbb{R} \to \mathbb{R} \) such that for any \( K_0 > 0 \) and any \( K_0 \)-bi-Lipschitz homeomorphism \( P_0 \) from \( \partial_0 X_0 \) to a boundary component \( \partial_1 X_0 \), \( P_0 \) extends to a \( \psi(K_0) \)-bi-Lipschitz homeomorphism \( P: X_0 \to X_0 \) which is \( K \)-bi-Lipschitz on every other boundary component.

**Corollary 1.** Let \( \tilde{\Phi}_v \) be the universal cover of the surface \( \Phi_v \) supplied with the metric described in sect. 2.6.1 with a chosen boundary component \( \partial_0 \tilde{\Phi}_v \). Then there exists \( K > 0 \) and a function \( \psi: \mathbb{R} \to \mathbb{R} \) such that for any \( K_0 > 0 \) and any \( K_0 \)-bi-Lipschitz homeomorphism \( P_0 \) from \( \partial_0 \tilde{\Phi}_v \) to a boundary component \( \partial_0 H_0 \), \( P_0 \) extends to a \( \psi(K_0) \)-bi-Lipschitz homeomorphism \( P: \tilde{\Phi}_v \to H_0 \) which is \( K \)-bi-Lipschitz on every other boundary component.
Proof. Rename the $K_0$, $K$ and $\psi$ from Theorem 2 to the $\bar{K}_0$, $\bar{K}$ and $\bar{\psi}$ respectively. Note that there is a bi-Lipschitz homeomorphism $\psi_v : X_0 \to \tilde{\Phi}_v$.

\[ X_0 \longrightarrow X_0 \]
\[ \psi_v \downarrow \quad \uparrow h_0 \]
\[ \tilde{\Phi}_v \longrightarrow H_0 \]

Let $\psi_v$ be $M_1$-bi-Lipschitz and $h_0$ be $M_2$-bi-Lipschitz. We set $M := \max\{M_1, M_2\}$. Consider $P_0 : \partial_0 \tilde{\Phi}_v \to \partial_0 H_0$. Denote the boundary component $h_0(\partial_0 H_0)$ by $\partial_1 X_0$. Then if $\partial_0 X_0 = \psi_v^{-1}(\partial_0 \tilde{\Phi}_v)$ the map

\[ h_0 \circ P_0 \circ \psi_v : \partial_0 X_0 \to \partial_1 X_0 \]

is $M^2 \cdot K_0$-bi-Lipschitz homeomorphism. By Theorem 2 it extends to a $\bar{\psi}(M^2 \cdot K_0)$-bi-Lipschitz homeomorphism that is $\bar{K}$-bi-Lipschitz on each remaining boundary component of the space $X_0$. Therefore, the homeomorphism $P = h_0^{-1} \circ P \circ \psi_v^{-1}$ is $\psi(K_0) = M^2 \tilde{\psi}(M^2 \cdot K_0)$-bi-Lipschitz. Moreover, on every other boundary component it is $K = M^2 \cdot \bar{K}$-bi-Lipschitz. Therefore, for $\psi(x) = \bar{\psi}(M^2 \cdot x)$, $x \in \mathbb{R}$ and $K = M^2 \cdot \bar{K}$ Corollary 1 is proved. \qed

Remark 4. Since the graph $G$ is finite, we can assume that the number $K$ and the function $\psi$ are independent of $v \in V(G)$.

2.6.4 The special metric on the universal cover

In this section we inductively construct for each orthogonal graph-manifold $M$ a special metric on its universal covering $\tilde{M}$ so that $\tilde{M}$ with such a metric is quasi-isometric to the fundamental group $\pi_1(M)$ of the graph-manifold $M$. Afterwards it will be sufficient to construct a quasi-isometric embedding of $\tilde{M}$ into a product of $n$ trees.

The decomposition of $M$ into blocks lifts to a decomposition of $\tilde{M}$ into universal cover blocks, see [4]. We denote the tree dual to this decomposition by $T_0$. Note that the degree of every vertex of $T_0$ is infinite. On $T_0$, we consider an intrinsic metric with length 1 edges. Choose a vertex $o \in V(T_0)$ in the tree $T_0$ and call it the root. For each vertex $v \in V(T_0)$, we define its rank $r(v)$ as the distance to $o$. In particular, $r(o) = 0$. For each vertex $v \in V(T_0)$, we denote by $M_v$ the corresponding block of the space $\tilde{M}$. This block is isometric to the product $\tilde{\Phi}_v \times \mathbb{R}^{n-2}$. Recall that on each boundary hyperplane of the block $M_v$ we have fixed a coordinate system. We call the axes of this system the selected axes.

By induction on the rank of vertices of the tree $T_0$, we construct on the space $\tilde{M}$ a metric of special type, which is bi-Lipschitz homeomorphic to the metric lifted from the graph-manifold $M$. 

6
Base: Let $\psi_o : \widetilde{\Phi}_o \to H_0$ be a bi-Lipschitz homeomorphism. Consider an isometric copy of the standard block $B_o$ and consider the map $\psi'_o : \widetilde{M}_o \to B_o$ which is a direct product of the map $\psi_o : \widetilde{\Phi}_o \to H_0$ and the identity map id: $\mathbb{R}^{n-2} \to \mathbb{R}^{n-2}$.

Inductive step: Suppose that for all vertices $v \in V(T_0)$ such that $r(v) \leq m$ we built a bi-Lipschitz homeomorphism $\psi'_o : \widetilde{M}_o \to B_o$ where $B_o$ is an isometric copy of the standard block. Consider a vertex $u \in V(T_0)$ such that $r(u) = m + 1$. There is a unique vertex $v \in V(T_0)$ adjacent to it such that $r(v) = m$. Consider the blocks $\widetilde{M}_u$ and $\widetilde{M}_v$ of the universal cover $\widetilde{M}$. Denote the covering map by pr: $\widetilde{M} \to M$. Recall that the gluing of the blocks pr($\widetilde{M}_v$) and pr($\widetilde{M}_u$) is obtained by the permutation $s$ of the coordinate system on the torus $T_e$. Consider an isometric copy of the standard block. Denote it by $B_u$ and glue it to the block $B_v$ by the permutation $s^{-1}$ of the coordinates along the corresponding hyperplanes thus matching the grid on them. By the induction, the map $\psi'_o : \widetilde{M}_v \to B_v$ is the direct product of the map $\widetilde{\Phi}_v \to H_0$ and $n-1$ maps $\mathbb{R} \to \mathbb{R}$. Moreover, the restriction of each of these maps to the intersection with the common boundary hyperplane of the blocks $\widetilde{M}_v$ and $\widetilde{M}_u$ is a $K$-bi-Lipschitz homeomorphism onto its image. It follows from the orthogonality of the gluing that these restrictions induce a $K$-bi-Lipschitz homeomorphism $\partial_l \psi_u : \partial_l \widetilde{\Phi}_u \to \partial_l H_0$ from the boundary component $\partial_l \widetilde{\Phi}_u$ of the surface $\widetilde{\Phi}_u$ adjacent to the block $\widetilde{M}_u$ to the boundary component $\partial_l H_0$ of the $\theta$-tree adjacent to the block $B_u$. Also, these restrictions induce a collection of $K$-bi-Lipschitz homeomorphisms $\partial_l : \mathbb{R} \to \mathbb{R}$, each of which maps the corresponding $\mathbb{R}$-factor of the decomposition $\widetilde{M}_u = \Phi_u \times \mathbb{R} \times \ldots \times \mathbb{R}$ to the corresponding $\mathbb{R}$-factor of the decomposition $B_u = H_0 \times \mathbb{R} \times \ldots \times \mathbb{R}$.

By Corollary[1] the homeomorphism $\partial_l \psi_u$ extends to a $\psi([K])$-bi-Lipschitz homeomorphism $\psi_v : \widetilde{\Phi}_u \to H_0$, which is $K$-bi-Lipschitz on every other boundary component. We define the homeomorphism $\psi'_o$ as the direct product of the homeomorphism $\psi_u$ and $n - 3$ homeomorphisms $\partial_l$ ($i = 2, \ldots, n - 2$).

Let us construct a map $\psi_M : \widetilde{M} \to X$, where $X$ is a metric space obtained by gluing blocks described above. Namely, if the point $x$ lies in the block $\widetilde{M}_v$ for some vertex $v \in V(T_0)$ we define $\psi_M(x) := \psi_v'(x)$. The map $\psi_M$ is well defined, since the maps $\psi'_o$ are compatible with each other.

**Proposition 1.** The map constructed above is a bi-Lipschitz homeomorphism.

**Proof.** Let $C = \max\{K, \psi(K)\}$. It follows from the construction that for each vertex $u \in V(T_0)$ the map $\psi'_u$ is $C$-bi-Lipschitz. Suppose that $x \in \widetilde{M}_v$ for some vertex $v \in V(T)$ and $y \in \widetilde{M}_u$ for some vertex $u \in V(T)$. Denote $x' := \psi_M(x)$ and $y' := \psi_M(y)$.

Let $\gamma$ be a geodesic between vertices $v$ and $u$ in the tree $T$. Denote its
consecutive edges by $e_1, \ldots, e_k$. Note that a geodesic $xy \subset \tilde{M}$ consecutively intersects hyperplanes $\sigma_1, \ldots, \sigma_k$ in the space $\tilde{M}$ that correspond to these edges. Similarly, a geodesic $x'y' \subset X$ consistently intersects hyperplanes $\sigma'_1, \ldots, \sigma'_k$ in the space $X$. Moreover, $\sigma'_i = \psi_M(\sigma_i)$. Let $z_i$ be an intersection point of the geodesic $xy$ and the hyperplane $\sigma_i$. (We assume that $z_0 = x$, $z_{k+1} = y$.)

Let $z'_i = \psi_M(z_i)$. Since for each vertex $v$ the restriction of the map $\psi_M$ on the block $\tilde{M}_v$ is $C$-bi-Lipschitz and the points $z_i$ and $z_{i+1}$ ($i = 0, \ldots, k$) lie in the same block, we have $|z'_i z'_{i+1}| \leq C |z_i z_{i+1}|$. Combining all these inequalities, we find that $\sum_{i=0}^k |z'_i z'_{i+1}| \leq C|xy|$. On the other hand, by the triangle inequality we have $|x'y'| \leq \sum_{i=0}^k |z'_i z'_{i+1}|$. This implies that $|x'y'| \leq C|xy|$. Similarly, we have $|xy| \leq C|x'y'|$. \hfill $\square$

Thus, we define a metric of special type on the universal cover of the orthogonal graph-manifold. Such a metric has nonpositive curvature in the sense of Alexandrov. Moreover, for every vertex $v \in V(T_0)$ the corresponding block $\tilde{M}_v$ is isometric to the direct product of the $\theta$-tree $H_0$ and $n-2$ factors $\mathbb{R}$.

Let us introduce some technical notations that will be needed later. For each vertex $v$ and the corresponding block of $\tilde{M}_v$, denote by $X_v$ a copy of the corresponding $\theta$-tree. Moreover, denote a copy of the tree $T_{bin}$ naturally (isometrically) embedded in the surface $X_v$, considered with the above described metric, by $T_v$. Let $G'_\theta$ be the isometry group of the $\theta$-tree. Note that for each vertex $v$ there exists $2\delta$-Lipschitz retraction $r_v: X_v \to T_v$ equivariant under the action of $G'_\theta$.

Recall that the block $\tilde{M}_v$ is a product $X_v \times \mathbb{R} \times \ldots \times \mathbb{R}$. Denote the projections to the corresponding factors by $p^1_v, \ldots, p^{n-1}_v$.

For each point $x \in \tilde{M}_v$ consider the map given by

$$\pi_x(y) := (y, p^2_v(x), \ldots, p^{n-1}_v(x))$$

for every point $y \in X_v$.

We call an orthogonal graph-manifold irreducible if its universal cover $\tilde{M}$ is not a product of a Euclidean space and universal cover of the orthogonal graph-manifold of lower dimension. It suffices to prove Theorem 1 for the irreducible case.

### 3 Trees $T_c$ and maps to them

#### 3.1 Construction of trees

Let $\gamma = w_1 \ldots w_k$ be an oriented path in the tree $T_0$. Denote by $s_\gamma$ the permutation $s_{w_k} \circ \ldots \circ s_{w_1}$ of well-ordered $(n-1)$-element set.
Suppose that vertices \( u, v \in T_0 \) are connected by two oriented paths \( \gamma_1 \) and \( \gamma_2 \). Note that \( s_{\gamma_1} = s_{\gamma_2} \), therefore, we can define the permutation \( s_{uv} \) as the permutation \( s_\gamma \) along any path \( \gamma \) between \( u \) and \( v \). Furthermore, \( s_{u_1v_1} = s_{u_2v_2} \circ s_{x_1x_2} \) and \( s_{uv} = s_{v_1u_1}^{-1} \).

We define a relation \( \sim \) on the set of vertices of \( T_0 \) by \( u \sim v \) if and only if the permutation \( s_{uv} \) fixes the smallest element. It is easy to check that the relation \( \sim \) is an equivalence relation.

Let us prove that the relation \( \sim \) divides the set \( V(T_0) \) into not more than \( n - 1 \) equivalence classes. Indeed, if it fails, then we can choose \( n \) pairwise non-equivalent vertices \( v_1, \ldots, v_n \). Then, for some different \( 2 \leq i, j \leq n \), we have \( s_{v_i v_j} (x_0) = s_{v_j v_i} (x_0) \), where \( x_0 \) is the smallest element. Then, since \( s_{v_i v_j} (x_0) = s_{v_j v_i} \circ s_{v_i v_j} (x_0) = x_0 \), \( v_i \sim v_j \). This is a contradiction.

Fix a vertex \( u \) in the tree \( T_0 \). Since the manifold \( M \) is irreducible, the set of permutations \( \{s_{vu} \mid u, v \in V(T_0)\} \) is transitive. That is, for each element \( x \) of a well-ordered \( (n - 1) \)-element set, there is a vertex \( v \in V(T_0) \) that \( s_{uv}(x_0) = x \). It follows that there are at least \( n - 1 \) different equivalence classes. Hence there are exactly \( n - 1 \).

Denote the set of all these classes by \( \mathcal{C} \). We have shown that \( |\mathcal{C}| = n - 1 \). Given \( c \in \mathcal{C} \), note that if \( u, v \in c \) then for any vertex \( v' \in V(T_0) \) we have \( s_{uv'} (x_0) = s_{v'v} (x_0) \).

Fix a vertex \( v \in V(T_0) \) and an equivalence class \( c \in \mathcal{C} \). We construct a tree \( T_{v,c} \) as follows. If the vertex \( v \) belongs to the class \( c \) then we set \( T_{v,c} := T_v \); see the end of section 2. Otherwise, we set \( T_{v,c} := \mathbb{R} \).

Also, for each vertex \( v \in V(T_0) \) we construct a map \( r_{v,c} : \tilde{M}_v \to T_{v,c} \). If the vertex \( v \) belongs to \( c \) then we set \( r_{v,c} := r_v \circ p_v^1 \). Otherwise, if the vertex \( v \) belongs to some class \( c' \neq c \), then we set \( r_{v,c} := p_v^k \), where \( k = s_{uv}(x_0), u \in c \), and \( x_0 \) is the smallest element.

For each class \( c \in \mathcal{C} \), we construct a tree of \( T_c \) as follows. For each pair of adjacent vertices \( u, v \in V(T_0) \), we say that a point \( x \in T_{u,c} \) and a point \( y \in T_{v,c} \) are \( \sim_c \)-equivalent if there exists a point \( z \in \tilde{M}_u \cap \tilde{M}_v \) such that \( x = r_{u,c}(z) = r_{v,c}(z) = y \).

This relation is well defined. Indeed, for every point \( x \in T_{u,c} \) the preimage \( r_{u,c}^{-1}(x) \cap \tilde{M}_u \cap \tilde{M}_v \) is an \( (n - 2) \)-dimensional subspace orthogonal to the coordinate \( s_{v,u}(x_0) \), where the vertex \( v' \) belongs to \( c \). Similarly, for each point \( y \in T_{v,c} \) the preimage \( r_{v,c}^{-1}(x) \cap \tilde{M}_u \cap \tilde{M}_v \) is an \( (n - 2) \)-dimensional subspace orthogonal to the coordinate \( s_{v,v}(x_0) \). But by the definition of coordinates \( s_{v,u}(x_0) \) and \( s_{v,v}(x_0) \), any two such subspaces are either disjoint or coincide. This implies immediately the following lemma.

**Lemma 2.** Let \( u, v \in V(T_0) \) be a pair of adjacent vertices and \( c \in \mathcal{C} \) be an equivalence class. If the points \( x, x' \in T_{u,c} \) and \( y \in T_{v,c} \) are such that \( x \sim_c y \) and \( x' \sim_c y \), then \( x = x' \).

Extend the relation \( \sim_c \) by transitivity. This means that we set \( x \sim_c y \)
if and only if there exists a chain \( x = x_0, \ldots, x_l = y \) that \( x_i \sim_c x_{i+1} \) and \( x_i \in T_{v_i,c} \) for each \( 0 \leq i \leq l-1 \), and the vertices \( v_i \) and \( v_{i+1} \) are adjacent in the tree \( T_0 \). From Lemma 2 it follows that the relation \( \sim_c \) is an equivalence relation.

**Lemma 3.** Let \( c \in \mathcal{C} \) be an equivalence class. Fix any pair of vertices \( u, v \in c \), and consider points \( x, y \in T_{u,c} \) and points \( x', y' \in T_{v,c} \) such that \( x \sim_c x' \) and \( y \sim_c y' \). Then \( |xy| = |x'y'| \).

**Proof.** In fact, let \( u = v_0, \ldots, v_l = v \) be consecutive vertices of the geodesic between vertices \( u \) and \( v \) in the tree \( T_0 \). Note that it suffices to consider the case when the vertices \( v_1, \ldots, v_{l-1} \) are not in the class \( c \). For each \( 0 \leq i \leq l-1 \) denote the common hyperplane of the blocks \( \overline{M}_{v_i} \) and \( \overline{M}_{v_{i+1}} \) by \( \sigma_i \). Let \( \partial X_u \) be the common boundary component of the \( \theta \)-tree \( X_u \) corresponding to the hyperplane \( \sigma_0 \). Well as, let \( \partial X_v \) be the common boundary component of the \( \theta \)-tree \( X_v \) corresponding to the hyperplane \( \sigma_l \). By the construction of the metric on the universal cover \( \tilde{M} \) for each interval \( I_0 \) of the grid on the boundary component \( \partial X_u \), there are segments \( I_1, \ldots, I_{l-2} \) of the grids on the on the corresponding \( \mathbb{R} \)-factors and the segment \( I_{l-1} \) of the grid on the boundary component \( \partial X_u \) such that for each \( 1 \leq i \leq l-1 \) we have \( p^\theta_{v_i}(I_{i-1}) = p^\theta_{v_i}(I_i) \), where \( j = s_{uv_i}(x_0) \). Note that it is sufficient to prove the Lemma for arbitrary points \( x, y \in r_{u,v}(I_0) \subset T_{u,v} \).

But for such points we have \( x', y' \in r_{v,c}(I_{l-1}) \), hence, by the equivariance of retractions \( r_v \) and \( r_u \), the required equality \( |xy| = |x'y'| \) is satisfied. \( \square \)

Fix \( c \in \mathcal{C} \). Define the space \( T_c \) as a factor \( \{ \bigcup_{v \in V(T_0)} T_{v,c} \}/ \sim_c \). From Lemma 3 it follows that the resulting space is a metric tree. Moreover, for each vertex \( v \in c \) the natural embedding \( \text{proj}_{v,c} : T_{v,c} \to T_c \) is isometric.

Thus, for each \( c \in \mathcal{C} \) we constructed a tree \( T_c \), which is naturally divided into blocks \( T_{v,c} \).

### 3.2 Maps to trees

The remainder of this paper, we consider the product of \( |\mathcal{C}| + 1 = n \) of constructed trees \( T_0 \times \prod_{c \in \mathcal{C}} T_c \) as a metric space with the sum metric. It means that the distance between two points \( x, y \in T_0 \times \prod_{c \in \mathcal{C}} T_c \) defined as the sum of the distances between their projections in the trees \( T_0, T_c \), for each \( c \in \mathcal{C} \).

\[
|xy| = |x_0y_0|_{T_0} + \sum_{c \in \mathcal{C}} |x_cy_c|_{T_c}.
\]

We define a map \( \varphi_c : \tilde{M} \to T_c \) by the formula:

\[
\varphi_c(x) := (\text{proj}_{v,c} \circ r_{v,c})(x),
\]
where $x \in \tilde{M}_v$ and $\text{proj}_{v,c}$ is the natural embedding of the tree $T_{v,c}$ in the tree $T_c$. It follows from the definition of the maps $r_{v,c}$ and from the construction of the tree $T_c$ that the map $\varphi_c$ is well defined on the intersection $\tilde{M}_u \cap \tilde{M}_v$ of each pair of adjacent blocks $\tilde{M}_u$ and $\tilde{M}_v$.

From the definition of the map $r_{v,c}$, we have that $\varphi_c$ is $2\delta$-Lipschitz. To prove this, it suffices to show that

$$|\varphi_c(x)\varphi_c(y)| \leq 2\delta|xy|,$$

where $x$ and $y$ belong to the same block $\tilde{M}_v$. This fact follows from the definition of $\varphi_c$.

Define a map $\varphi_0 : \tilde{M} \to T_0$ as follows. If $x \in \tilde{M}_v$ and for any other vertex $u \in V(T_0)$, $x \notin \tilde{M}_u$ set $\varphi_0(x) = v$. Otherwise if $x \in \tilde{M}_u \cap \tilde{M}_v$ and $r(u) < r(v)$ set $\varphi_0(x) = u$. It is clear that $|\varphi_0(x)\varphi_0(y)| \leq |xy| + 1$.

Define a map $\varphi : \tilde{M} \to T_0 \times \prod_{c \in C} T_c$ by the equality $\varphi := \varphi_0 \times \prod_{c \in C} \varphi_c$.

Then

$$|\varphi(x)\varphi(y)| = |\varphi_0(x)\varphi_0(y)| + \sum_{c \in C} |\varphi_c(x)\varphi_c(y)| \leq (2\delta(n - 1) + 1)|xy| + 1.$$

### 4 Special curves

For a curve $\gamma$ in a metric space $X$ by $|\gamma|$ denote its length. For further proof we need the following lemma.

**Lemma 4.** Let $x,y \in \tilde{M}$ be a pair of points. Then there exists a curve $\gamma \subset \tilde{M}$ between them such that $|\gamma| \leq (2\delta + 1)|\varphi(x)\varphi(y)| + 2\delta$.

**Proof.** Consider vertices $u,v \in V(T_0)$ such that $x \in \tilde{M}_v$, $y \in \tilde{M}_u$. We prove the lemma by induction on the length of the path $|uv|_{T_0}$.

**Base:** There exists a vertex $v \in V(T_0)$ that $x,y \in M_v$. In this case, the geodesic $xy$ does the job. Indeed,

$$|xy| = \sqrt{|p_1^v(x)p_1^v(y)|^2 + \ldots + |p_{n-1}^v(x)p_{n-1}^v(y)|^2},$$

that does not exceed

$$2\delta + \sum_{c \in C} |\varphi_c(x)\varphi_c(y)| \leq (2\delta + 1)|\varphi(x)\varphi(y)| + 2\delta.$$

**Inductive step:** Fix vertices $u,v \in V(T_0)$. Let $\eta = \eta_0$ be a geodesic between $u$ and $v$ in the tree $T_0$. Choose vertices $u',v' \in \eta_0$ that vertices $u$ and $u'$ as well as vertices $v$ and $v'$ are adjacent. Note that either $r(u) > r(u')$ or $r(v) > r(v')$. Without loss of generality, assume that $r(u) > r(v')$. We can also assume that the point $x$ does not belong to the block $\tilde{M}_{v'}$. In this
case, \( \varphi_0(x) = v \) and and for any point \( x' \in \tilde{M}_v \cap \tilde{M}_{v'} \) we have \( \varphi_0(x') = v' \).

It follows that \( |\varphi_0(x)\varphi_0(x')| = 1 \).

Assume the vertex \( v \) belongs to the class \( c \in \mathcal{C} \). Consider a geodesic \( \eta \) between \( \varphi(x) \) and \( \varphi(y) \). Denote its projection to the tree \( T_v \) by \( \eta_c \). Note that the curve \( \eta_c \) is a geodesic between \( \varphi_c(x) \) and \( \varphi_c(y) \). Divide the curve \( \eta_c \) into two parts \( \eta_c^1 = \eta_c \cap T_{v,c} \) and \( \eta_c^2 = \eta_c \setminus \eta_c^1 \). Let \( z \) be the end of the curve \( \eta_c^1 \) different from \( \varphi_c(x) \).

Recall that \( \pi_x : X_v \to \tilde{M}_v \) is a horizontal embedding such that the image contains the point \( x \). Further, we assume that the tree \( T_{v,c} \) is naturally embedded in the \( \theta \)-tree \( X_v \). Define a map \( \pi_x' \) as the composition of such embedding and the map \( \pi_x \). Denote the image of the curve \( \eta_c^1 \) under the map \( \pi_x' \) by \( \gamma_c \). We set \( x_0 \) := \( \pi_x'(\varphi_c(x)) \) and \( x_1 \) := \( \pi_x'(z) \). Then \( |x_0x_1| < \delta \) and there exists a point \( x_2 \in \tilde{M}_v \cap \tilde{M}_{v'} \) such that \( x_2 \in \pi_x(X_v) \), \( \varphi_c(x_2) = z \) and for any \( c \neq c' \in \mathcal{C} \) we have \( \varphi_{c'}(x_2) = \varphi_{c'}(x) \). Note that \( |x_1x_2| < \delta \) and

\[
|\varphi_c(x)\varphi_{c'}(x_2)| + |\varphi_c(x_2)\varphi_{c'}(y)| = |\varphi_c(x)\varphi_{c'}(y)|.
\]

On the other hand, since \( \varphi_c(x_2) = z \),

\[
|\varphi_c(x)\varphi_{c}(x_2)| + |\varphi_c(x_2)\varphi_{c}(y)| = |\varphi_c(x)z| + |z\varphi_c(y)| = |\varphi_c(x)\varphi_{c}(y)|.
\]

Finally, we note that the point \( \varphi_0(x_2) \) belongs to the geodesic between \( \varphi_0(x) \) and \( \varphi_0(y) \) in the tree \( T_0 \). It follows that

\[
|\varphi_0(x)\varphi_0(x_2)| + |\varphi_0(x_2)\varphi_0(y)| = |\varphi_0(x)\varphi_0(y)|.
\]

It means that

\[
|\varphi(x)\varphi(x_2)| + |\varphi(x_2)\varphi(y)| = |\varphi(x)\varphi(y)|,
\]

and \( x_2 \in \tilde{M}_{v'} \). By induction, for the points \( x_2 \) and \( y \), there exists a curve \( \gamma' \) between \( x_2 \) and \( y \) with

\[
|\gamma'| \leq (2\delta + 1)|\varphi(x_2)\varphi(y)| + 2\delta.
\]

Consider the curve \( \gamma \) which is the union of the geodesic \( xx_0 \), the curve \( \gamma_c \), the geodesic \( x_1x_2 \) and the curve \( \gamma' \). We have

\[
|\gamma| = |xx_0| + |\gamma_c| + |x_1x_2| + |\gamma'| \leq 2\delta + |\gamma_c| + |\gamma'| \leq 2\delta + |\varphi_c(x)\varphi_c(x_2)| + |\gamma'|,
\]

which by induction does not exceed

\[
2\delta + |\varphi(x)\varphi(x_2)| + (2\delta + 1)|\varphi(x_2)\varphi(y)| + 2\delta.
\]

We have shown that \( |\varphi_0(x)\varphi_0(x_2)| = 1 \), therefore,

\[
2\delta + |\varphi(x)\varphi(x_2)| = 2\delta|\varphi_0(x)\varphi_0(x_2)| + |\varphi(x)\varphi(x_2)| \leq (2\delta + 1)|\varphi(x)\varphi(x_2)|.
\]
So
\[ |\gamma| \leq 2\delta + |\varphi(x)\varphi(x_2)| + (2\delta + 1)|\varphi(x_2)\varphi(y)| + 2\delta \leq (2\delta + 1)(|\varphi(x)\varphi(x_2)| + |\varphi(x_2)\varphi(y)|) + 2\delta, \]

hence \( |\gamma| \leq (2\delta + 1)(|\varphi(x)\varphi(y)|) + 2\delta. \)

Corollary 2. For any points \( x, y \in \tilde{M} \) the inequality \( |xy| \leq (2\delta + 1)|\varphi(x)\varphi(y)| + 2\delta \) holds.

Applying the above inequalities, we obtain
\[ |xy|/(2\delta + 1) - 2\delta/(2\delta + 1) \leq |\varphi(x)\varphi(y)| \leq (2\delta(n - 1) + 1)|xy| + 1, \]

therefore,
\[ 1/C|xy| - 1 \leq |\varphi(x)\varphi(y)| \leq C|xy| + 1, \]

where \( C = \max\{2\delta + 1, 2\delta(n - 1) + 1\} \). This completes the proof of Theorem 1.

5 Asymptotic dimensions

5.1 Definitions

Recall some basic definitions and notations. Let \( X \) be a metric space. We denote by \( |xy| \) the distance between \( x, y \in X \) and \( d(U,V) := \inf\{|uv| \mid u \in U, v \in V\} \) is the distance between \( U, V \subset X \).

We say that a family \( \mathcal{U} \) of subsets of \( X \) is a covering if for each point \( x \in X \) there is a subset \( U \in \mathcal{U} \) such that \( x \in U \). A family \( \mathcal{U} \) of sets is disjoint if each two sets \( U,V \in \mathcal{U} \) are disjoint. The union \( \mathcal{U} = \cup \{U^\alpha \mid \alpha \in \mathcal{A}\} \) of disjoint families \( U^\alpha \) is said to be \( n \)-colored, where \( n = |\mathcal{A}| \) is the cardinality of \( \mathcal{A} \).

Also, recall that a family \( \mathcal{U} \) is \( D \)-bounded, if the diameter of every \( U \in \mathcal{U} \) does not exceed \( D \), \( \text{diam } U \leq D \). A \( n \)-colored family of sets \( \mathcal{U} \) is \( r \)-disjoint, if for every color \( \alpha \in \mathcal{A} \) and each two sets \( U,V \in \mathcal{U}^\alpha \) we have \( d(U,V) \geq r \).

The linearly-controlled asymptotic dimension is a version of the Gromov’s asymptotic dimension, \( \text{asdim} \).

Definition. (Gromov \[6\]) The asymptotic dimension of a metric space \( X \), \( \text{asdim } X \), is the least integer number \( n \) such that for each sufficiently large real \( R \) there exists a \( (n + 1) \)-colored, \( R \)-disjoint, \( D \)-bounded covering of the space \( X \), where the number \( D > 0 \) is independent of \( R \).

Definition. (Roe \[9\]) The linearly-controlled asymptotic dimension of a metric space \( X \), \( \ell \)-\( \text{asdim } X \), is the least integer number \( n \) such that for each sufficiently large real \( R \) there exists an \( (n + 1) \)-colored, \( R \)-disjoint, \( CR \)-bounded covering of the space \( X \), where the number \( C > 0 \) is independent of \( R \).
It follows from the definition that asdim $X \leq \ell$-asdim $X$ for any metric space $X$.

In the next section we show that the fundamental group of orthogonal graph-manifold satisfies $n \leq \text{asdim} \pi_1(M) \leq \ell$-asdim $\pi_1(M) \leq n$.

5.2 Upper and lower bounds

Recall some properties of the above dimensions.

Let $X$ and $Y$ be metric spaces. If $X$ is quasi-isometric to $Y$ then asdim $X = \text{asdim} Y$ and $\ell$-asdim $X = \ell$-asdim $Y$. If $X \subset Y$ then $\ell$-asdim $X \leq \ell$-asdim $Y$. Also, $\ell$-asdim $X \times Y \leq \ell$-asdim $X + \ell$-asdim $Y$. Let $T$ be a metric tree, then $\ell$-asdim $T \leq 1$. It follows from the above properties, that asdim $\pi_1(M) = \text{asdim} \tilde{M} \leq \ell$-asdim $\tilde{M} \leq \ell$-asdim$(T_0 \times \prod_{c \in C} T_c) \leq n$. On the other hand, the space $\tilde{M}$ is an Hadamard manifold, and hence, see [5] Theorem 10.1.1, asdim $\tilde{M} \geq n$.

References

[1] Behrstock J. A. and Neumann W. D.: Quasi-isometric classification of graph manifold groups, Duke Math. J., Volume 141, Number 2 (2008), 217-240.

[2] Burago D., Burago Y., Ivanov S.: A Course in Metric Geometry, AMS Bookstore, 2001.

[3] Bell G. and Dranishnikov A.: On Asymptotic Dimension of Groups Acting on Trees, Geometriae Dedicata, Volume 103, Number 1 (2004), 89-101.

[4] Buyalo S. V., Kobel’skii V.L.: Generalized graphmanifolds of nonpositive curvature, St. Petersburg Math. J. 11 (2000), 251–268.

[5] Buyalo S. V., Schroeder V.: Elements of asymptotic geometry, EMS monographs in mathematics, European Mathematical Society, 2007.

[6] Gromov M.: Asymptotic invariants of infinite groups, London Mathematical Society Lecture Note Series, Volume 182 (1993), 1-295.

[7] Hume D., Sisto A.: Embedding universal covers of graph manifolds in products of trees, preprint arXiv:math.GT/1112.0263

[8] Kapovich M. and Leeb B.: 3-manifold groups and nonpositive curvature, Geometric Analysis and Functional Analysis, Volume 8 (1998), 841-852.
[9] Roe J.: *Lectures on Coarse Geometry*, University Lecture Series, Volume 31, AMS, 2003.