EXTRA-TWISTED CONNECTED SUM $G_2$-MANIFOLDS

JOHANNES NORDSTRÖM

Abstract. We present a construction of closed 7-manifolds of holonomy $G_2$, which generalises Kovalev’s twisted connected sums by taking quotients of the pieces in the construction before gluing. This makes it possible to realise a wider range of topological types, and Crowley, Goette and the author use this to exhibit examples of closed 7-manifolds with disconnected moduli space of holonomy $G_2$ metrics.

The twisted connected sum construction pioneered by Kovalev [21] is a way to construct closed 7-dimensional Riemannian 7-manifolds with holonomy $G_2$ from algebraic geometric data. Corti, Haskins, Pacini and the author [7] employed the construction to exhibit many examples of $G_2$-manifolds whose topology can be understood in great detail. The aim of this paper is to present a variation of the twisted connected sum construction that removes some restrictions on the topology of the resulting 7-manifolds and $G_2$-structures. In particular, it is proved by Crowley, Goette and the author in [9] that this construction can be used to produce examples of 7-manifolds such that the moduli space of $G_2$ metrics is disconnected.

7-dimensional manifolds with holonomy $G_2$ appear as an exceptional case in Berger’s classification of possible holonomy groups of Riemannian manifolds [2]. The first complete examples of manifolds with holonomy $G_2$ were found by Bryant and Salamon [4], and have large symmetry group. In contrast, closed $G_2$-manifolds can never have continuous symmetries, because $G_2$-metrics are always Ricci-flat. The first examples of holonomy $G_2$ metrics on closed manifolds were found by Joyce [20], gluing together reducible pieces to resolve quotients of flat orbifolds.

The twisted connected sum construction developed later by Kovalev [21] works by gluing together two pieces, each of which is a product of a circle $S^1$ and a complex 3-fold with an asymptotically cylindrical Calabi-Yau metric. Each piece thus has holonomy $SU(3)$, a proper subgroup of $G_2$. The asymptotically cylindrical Calabi-Yau 3-folds can be obtained from algebraic geometry data, e.g. starting from Fano 3-folds. The cross-section of the asymptotic cylinder is of the form $S^1 \times \Sigma$ for a K3 surface $\Sigma$. In the gluing, the asymptotic cylinders of the pieces—each with cross-section $S^1 \times S^1 \times \Sigma$—are identified by an isomorphism that swaps the $S^1$ factors in order to produce a simply-connected 7-manifold $M$, admitting metrics with holonomy exactly $G_2$. This relies on finding a so-called hyper-Kähler rotation between the K3 factors in the cross-sections, see Definition 1.2.

Corti, Haskins, Pacini and the author [6, 7] extended the supply of algebraic geometric building blocks to which the twisted connected sum construction can be applied, and analysed the topology of millions of the resulting $G_2$-manifolds. While the $G_2$-manifolds constructed by Joyce typically have non-zero second Betti number $b_2$, many twisted connected sums—indeed, the ones that can be constructed with the least effort—are 2-connected, making it possible to apply the classification theory of Wilkens [29, 30], Crowley [8] and Crowley and the author [11] (see Theorem 7.14) to completely determine the diffeomorphism type of the underlying 7-manifold.

Twisted connected sum $G_2$-manifolds $M$ always have the following topological properties.

(i) $b_2(M) + b_3(M)$ is odd [21 (8.56)].

(ii) The torsion subgroup $\text{Tors} H^4(M)$ equipped with the linking form splits as $G \times \text{Hom}(G, \mathbb{Q}/\mathbb{Z})$ for some finite group $G$ [12 Proposition 3.8]. In particular, the size of $\text{Tors} H^4(M)$ is a square integer.

(iii) The invariant $\nu \in \mathbb{Z}/48$ takes the value 24 [10 Theorem 1.7], and the refinement $\bar{\nu} \in \mathbb{Z}$ vanishes [9 Corollary 3].

Here $\nu$ and $\bar{\nu}$ are invariants not of the 7-manifold, but of the $G_2$-metric. A metric with holonomy exactly $G_2$ is equivalent to a torsion-free $G_2$-structure. A $G_2$-structure means a reduction of the
structure group of the frame bundle from $GL(7, \mathbb{R})$ to $G_2$, but is simplest described in terms of a smooth pointwise stable 3-form $\varphi \in \Omega^3(M)$. The torsion-free condition corresponds to a first-order partial differential equation for the 3-form $\varphi$.

Now, given a $G_2$-structure $\varphi$ on any closed 7-manifold, we may define $\nu(\varphi) \in \mathbb{Z}/48$ in terms of a spin coboundary [9] Definition 3.1. This is invariant under both diffeomorphisms and homotopies (continuous deformations of the $G_2$-structure, ignoring the torsion-free condition). Further, [9] Definition 1.4 introduces a refinement $\bar{\nu}(\varphi) \in \mathbb{Z}$ in terms of eta invariants. It is a refinement in the sense that $\bar{\nu}$ determines $\nu$, for $G_2$-structures of holonomy $G_2$ metrics by the relation $\nu(\varphi) \equiv \bar{\nu}(\varphi) + 24 \mod 48$. While $\bar{\nu}(\varphi)$ too is invariant under diffeomorphisms, it is not invariant under arbitrary homotopies of $G_2$-structures. However, $\bar{\nu}$ is invariant under deformations through torsion-free $G_2$-structures.

Remark. There is a parity constraint

$$\nu(\varphi) = \chi_2(M) \mod 2, \quad (1)$$

where $\chi_2(M)$ is the semi-characteristic $\sum_{i=0}^{3} b_i(M) \in \mathbb{Z}/2$, reducing to $1 + b_2(M) + b_3(M)$ for a simply-connected 7-manifold. Thus (iii) formally entails (i).

These invariants give a potential method to distinguish connected components of the $G_2$ moduli space on a closed 7-manifold. However, even though there are many pairs of twisted connected sums whose underlying 7-manifolds can be shown to be diffeomorphic by the classification theory, (iii) means that $\nu$ and $\bar{\nu}$ fail to distinguish their components in the moduli space in this case.

In this paper we modify the twisted connected sum construction by dividing either or both of the two pieces in the construction by an involution before gluing. This maintains many of the attractive features of the twisted connected sum construction: examples can be generated starting from algebraic geometry data, topological invariants can be computed from the algebraic inputs, and the resulting 7-manifolds are often 2-connected and simple enough to apply diffeomorphism classification theory. On the other hand, the topology of the result is less restrictive.

(i') There is no constraint on the parity of $b_2(M) + b_3(M)$.

(ii') The size of $\text{Tor} H^4(M)$ need not be a square integer, and in particular the linking form need not split.

(iii') $\nu$ and $\bar{\nu}$ can be non-trivial.

The drawback compared with the ordinary twisted connected sum construction is that requiring an involution limits the range of algebraic building blocks to which the construction can be applied. Also, the topological computations are more involved.

We exhibit a selection of 49 explicit examples of 7-manifolds with holonomy $G_2$ obtained from the new construction. All except Example 8.11 are 2-connected, 7 of those have odd $b_3$ and torsion-free $H^4(M)$, and 5 of those are diffeomorphic to some ordinary twisted connected sum. The $\bar{\nu}$-invariant of extra-twisted connected sums is computed in [9, Corollary 2] (see Theorem 7.13), and used there to prove that these lead to examples of closed 7-manifolds with disconnected moduli space of holonomy $G_2$ metrics.

Among the examples in this paper, we also find

- A 7-manifold whose $G_2$ moduli space has at least 3 components (see Examples 8.2 and 8.18 using the formula for $\bar{\nu}$ from [9]).
- A pair of $G_2$-manifolds whose diffeomorphism types are distinguished only by the type of the torsion linking form (Examples 8.3 and 8.4).
- A pair of $G_2$-manifolds with equal $\bar{\nu}$-invariant, such that the underlying manifolds are diffeomorphic, but (due to order 3 torsion in $H^4$) only by an orientation-reversing diffeomorphism; thus the fact that $\bar{\nu}$ changes sign under reversing orientation can be used to distinguish connected components of the $G_2$ moduli space on this 7-manifold (Examples 8.11 and 8.12).

Organisation. The paper consists of two strands. The first is to set up the general machinery of the extra-twisted connected sum construction. The procedure for gluing ACyl Calabi-Yau manifolds (possibly with involution) is made precise in [1] while [2] describes the closed Kähler 3-fold “building blocks” from which we obtain ACyl Calabi-Yau 3-folds, and what data of these
blocks is important. The matching problem, i.e. how to find hyper-Kähler rotations between pairs of ACyl Calabi-Yau 3-folds, is addressed in [6] and [7] explains how to compute key invariants of the resulting $G_2$-manifolds.

The second strand is producing examples. Two methods of producing building blocks are provided in [3] and [5] starting from semi-Fano 3-folds and K3s with non-symplectic involution, respectively. In [8] we exhibit a number of examples of matchings of those blocks and compute the topology of the extra-twisted connected sums.

Some of the machinery we set up—in particular the discussion of the matching problem in [6]—would work in the same way in a more general setting where one allows to divide by automorphisms of order greater than 2. However, we only explore examples involving involutions. The case of automorphisms of higher order will be studied further by Goette and the author [15].

Acknowledgements. The author thanks Alessio Corti, Diarmuid Crowley, Sebastian Goette, Mark Haskins, Jesus Martinez Garcia, and Dominic Wallis for valuable discussions, and the Simons Foundation for its support under the Simons Collaboration on Special Holonomy in Geometry, Analysis and Physics (grant #488631, Johannes Nordström).

1. The basics of the construction

1.1. Reducible $G_2$-manifolds. For $\zeta > 0$, let $S_\zeta^1$ denote $\mathbb{R}/\zeta\mathbb{Z}$, and $u$ its coordinate (with period $\zeta$); the parameter $\zeta$ affects the geometric meaning of the coordinate expressions for metrics below.

Theorem 1.1 ([14 Theorem D]). Let $Z$ be a compact Kähler 3-fold containing a smooth anticanonical K3 surface $\Sigma$ with trivial normal bundle. Let $V := Z \setminus \Sigma$, and consider it as a manifold with a cylindrical end of cross-section $S_\zeta^1 \times \Sigma$. Let $I$ be the complex structure on $\Sigma$ induced by $Z$, and let $(\omega^I, \omega^J, \omega^K)$ be a hyper-Kähler K3 structure on $\Sigma$ such that $\omega^J + i\omega^K$ is $(2,0)$ with respect to $I$ while $[\omega^I]$ is the restriction of some Kähler class $k \in H^2(Z; \mathbb{R})$. For any $\zeta > 0$ there is a unique ACyl Calabi-Yau structure $(\Omega, \omega)$ on $V$, with $\omega \in k|_V$ and asymptotic limit

$$
\omega_\infty := dt \wedge du + \omega^J, \\
\Omega_\infty := (du - idt) \wedge (\omega^J + i\omega^K).
$$

(In this metric, the $S_\zeta^1$ factor in the cross-section has circumference $\zeta$.)

Given $\zeta > 0$, define a product $G_2$-structure $\varphi$ on $S_\zeta^1 \times V$ by

$$
\varphi := dv \wedge \omega + \text{Re} \Omega,
$$

where $v$ denotes the coordinate on the external circle factor $S_\zeta^1$ (whose circumference with respect to the induced metric is $\zeta$). The asymptotic limit of $\varphi$ is

$$
\varphi_\infty = dv \wedge dt \wedge du + dv \wedge \omega^J + du \wedge \omega^J + dt \wedge \omega^K.
$$

Letting

$$
z = v + iu,
$$

we can rewrite the limit as

$$
\varphi_\infty = \text{Re} \left( dz \wedge (\omega^J - i\omega^K) \right) + dt \wedge (\omega^K - \frac{i}{2} dz \wedge dz). \tag{3}
$$

Note that $\zeta$ and $\xi$ are the side lengths of the rectangular $T^2$ factor in the cross-section of $S_\zeta^1 \times V$. If $\partial_u, \partial_v \in \mathbb{R}^2$ is the orthonormal frame dual to $du, dv$, then we can think of $\zeta \partial_u$ and $\xi \partial_v$ as the generators of the lattice defining the $T^2$. Let $\varphi^{u0}$ be the $G_2$-structure obtained by setting $\zeta = \xi = 1$, as we do in the ordinary twisted connected sum construction; then the $T^2$ factor is simply the quotient of $\mathbb{C}$ by the unit square lattice as illustrated in Figure 1 (Note that real axis (in red) $\leftrightarrow u = 0 \leftrightarrow$ external circle factor.)

Suppose now that there is a holomorphic involution $\tau$ on $Z$ such that $\Sigma$ is a component of the fixed set; cf. Definition 2.6 Then the restriction of $\tau$ to $V$ is asymptotic to the involution $a \times \text{Id}$ on $S_\zeta^1 \times \Sigma$, where $a : S_\zeta^1 \to S_\zeta^1$ denotes the antipodal map $v \mapsto v + \frac{i}{2} \zeta$. If we choose the Kähler
class $k$ in Theorem 1.1 to be $\tau$-invariant, then so is the resulting Calabi-Yau structure $(\Omega, \omega)$. The product $G_2$-structures above then descend to ones on the quotient $S^1_\xi \tilde{\times} V := S^1_\xi \times V / a \times \tau$. The cross-section is $T^2 \times \Sigma$ for $T^2 := S^1_\xi \times S^1_\zeta / a \times a$. Note that $T^2$ is still a flat 2-torus, but not a metric product of circles unless $\xi = \zeta$. Let $\varphi^s$, $\varphi^h$ and $\varphi^{h1}$ be the $G_2$-structures on $S^1_\xi \tilde{\times} V$ corresponding to $(\zeta, \xi) = (\sqrt{2}, \sqrt{2}), (\sqrt{3}, 1)$ and $(1, \sqrt{3})$ respectively. As illustrated in Figures 2–4, the $T^2$ factor in the cross-section is a unit square torus with respect to $\varphi^s$, and a hexagonal torus with side length 1 with respect to $\varphi^h$ and $\varphi^{h1}$.

1.2. Gluing. Let $(M^+, \varphi^+)$ and $(M^-, \varphi^-)$ be a pair of reducible ACyl $G_2$-manifolds, such that either each is of the form $(S^1_\xi \times V, \varphi^0)$ or $(S^1_\xi \tilde{\times} V, \varphi^s)$, or each is of the form $(S^1_\xi \tilde{\times} V, \varphi^h)$ or $(S^1_\xi \tilde{\times} V, \varphi^{h1})$ above. Let $a \in \{s, h\}$ and $b_\pm \in \{0, 1\}$ accordingly; we strive to treat the cases as uniformly as possible. Let $(\omega^+_A, \omega^+_B, \omega^+_K)$ be the corresponding hyper-Kähler structures, and define $z_\pm$ by (2).

Let $\vartheta \in \mathbb{R}$ such that the isometry $\mathbb{C} \to \mathbb{C}$, $z_+ \mapsto z_- := e^{i \vartheta} z_+$ descends to a well-defined isometry $t : T^+_2 \to T^-_2$ of the torus factors in the cross-sections of $M^+$ and $M^-$. This means that

$$\vartheta = \begin{cases} \frac{k\pi}{2} & \text{if } a = s, \\ \frac{3k\pi}{2} & \text{if } a = h, \end{cases}$$

for some $k \in \frac{1}{2}\mathbb{Z}$ with $k \equiv \frac{b_+ + b_-}{2} \mod \mathbb{Z}$. We call $\vartheta$ the gluing angle of $t$. 

![Figure 1. $\varphi^s$](image1.png)

![Figure 2. $\varphi^h$](image2.png)

![Figure 3. $\varphi^h$](image3.png)

![Figure 4. $\varphi^{h1}$](image4.png)
Let \( r : \Sigma_+ \to \Sigma_- \) be a diffeomorphism, and
\[
F := (-I_3) \times t \times r : \mathbb{R} \times T^2_+ \times \Sigma_+ \to \mathbb{R} \times T^2_\mathcal{R} \times \Sigma_-. \tag{6}
\]
From (3), we see that (6) is an isomorphism of the asymptotic limits of \( \varphi_\pm \) if and only if
\[
r^* \omega^K_+ = -\omega^K_+ \quad \text{and} \quad r^*(\omega^I_+ + i\omega^J_+) = e^{i\theta}(\omega^I_+ - i\omega^J_+). \tag{7}
\]

**Definition 1.2.** Call \( r : \Sigma_+ \to \Sigma_- \) a \( \vartheta \)-hyper-Kähler rotation if (7) holds.

We consider the problem of finding such hyper-Kähler rotations in [3]. The special case of a \( \frac{\pi}{2} \)-hyper-Kähler rotation coincides with the notion of a hyper-Kähler rotation from previous work on twisted connected sums, e.g. [4, Definition 3.10].

In these terms, suppose we can find a pair of reducible ACyl \( G_2 \)-manifolds \( (M_\pm, \varphi_\pm) \) of the above form, with asymptotic cross-sections \( T^2_\mathcal{R} \times \Sigma_\pm \). Suppose further we can find an isometry \( t : T^2_+ \to T^2_\mathcal{R} \) as in [4], and a \( \vartheta \)-hyper-Kähler rotation \( r : \Sigma_+ \to \Sigma_- \) where \( \vartheta \) is the gluing angle of \( t \).

**Theorem 1.3.** For \( \ell \gg 0 \), let \( M_{\pm}[\ell] \) be the truncation of \( M_\pm \) at \( t = \ell \), and form a closed 7-manifold \( M \) by gluing \( M_\pm[\ell] \) to \( M_{\mp}[\ell] \) along their boundaries by the diffeomorphism \( t \times r : T^2_+ \times \Sigma_+ \to T^2_\mathcal{R} \times \Sigma_- \). Use a cut-off function to patch \( \varphi_+ \) and \( \varphi_- \) to a closed \( G_2 \)-structure \( \varphi_\ell \) on \( M \) such that \( \|\tilde{\varphi}_{M_\pm}[\ell] - \varphi_\pm[M_{\mp}[\ell]]\| = O(e^{-\delta t}) \). Then there exists a unique torsion-free \( G_2 \)-structure \( \varphi \) in the cohomology class of \( \tilde{\varphi}_\ell \) such that \( \|\varphi - \tilde{\varphi}\| = O(e^{-\delta t}) \).

**Proof.** Analogous to [21, Theorem 5.34]. \( \square \)

**Construction 1.4.** We call the 7-manifold \( M \) from Theorem 1.3 a \( \vartheta \)-twisted connected sum.

When \( a = s \) and \( b_+ = b_- = 0, \vartheta = \frac{\pi}{2} \) recovers the usual notion of a twisted connected sum (and \( \vartheta \in \pi \mathbb{Z} \) gives an “untwisted” connected sum, with \( b_1(M) = 1 \)).

### 1.3. Angles.

Before we enumerate the possible combinations of \( (a, b_+, b_-, \vartheta) \), let us discuss briefly the geometric meaning of \( \vartheta \). We can think of \( \vartheta \) as the angle in \( T^2 \) between the external circle factors in \( M_+ \) and \( M_- \), but that leaves an ambiguity of sign and complementary angles. However, because the definition of the \( G_2 \)-structures involves an orientation of the external circle factors the direction of the tangent vectors \( \partial_{\alpha_+} \) and \( \partial_{\alpha_-} \) have some meaning, and the angle between them is \( |\vartheta| \in (0, \pi) \).

The sign can be described in terms of the complex structure on the cross-section induced by the \( G_2 \)-structure on \( M_+ \) (vector multiplication by \( \partial_\nu \)); because the \( T^2 \) factor is a complex curve, it makes sense to consider the oriented angle from \( \partial_{\alpha_+} \) to \( \partial_{\alpha_-} \).

If we swap the roles of \( M_+ \) and \( M_- \) then the complex structure on the cross-section is conjugated, so even though \( \partial_{\alpha_+} \) and \( \partial_{\alpha_-} \) are swapped the oriented angle \( \vartheta \) is unchanged. More formally, note that if \( r : \Sigma_+ \to \Sigma_- \) is a \( \vartheta \)-hyper-Kähler rotation, then so is \( -r \). Let \( (M', \varphi') \) be the corresponding \( \vartheta \)-twisted connected sum of \( M_+ \) and \( M_- \). Then there is a tautological (oriented) diffeomorphism \( M \to M' \), and that pulls back \( \varphi' \) to \( \varphi \).

Here is another symmetry to bear in mind. We obtained the product \( G_2 \)-structures \( \varphi_\pm \) on \( M_\pm \) from ACyl Calabi-Yau structures \( (\Omega_\pm, \omega_\pm) \) on \( V_\pm \). Phase rotation by \( \pi \) gives an equally good Calabi-Yau structure \(-\Omega_\pm, \omega_\pm\), and another product \( G_2 \)-structure \( \varphi'_\pm \). The asymptotic limit of \( \varphi'_\pm \) is encoded by the hyper-Kähler structure \( (\omega^K_\pm, -\omega^I_\pm, -\omega^J_\pm) \). Inspecting (7) we see that a \( \vartheta \)-hyper-Kähler rotation for \( \varphi_\pm \) and \( \varphi_- \) is the same thing as a \( (-\vartheta) \)-hyper-Kähler rotation for \( \varphi'_\pm \) and \( \varphi'_- \). Let \( (M', \varphi') \) be the resulting \( (-\vartheta) \)-twisted connected sum. Now \( (v_\pm, x) \to (-v_\pm, x) \) defines an orientation-reversing diffeomorphism of \( M_\pm \), pulling back \( \varphi'_\pm \) to \(-\varphi'_\pm \). These match up to define an orientation-reversing diffeomorphism \( M \to M' \) that pulls back \( \varphi' \) to \(-\varphi \).

Taking these symmetries into account, we can restrict our attention to cases where \( b_+ \geq b_- \), and \( \vartheta \in (0, \pi) \).

We find below that there is essentially a single interesting type of \( \vartheta \)-twisted connected sum for each
\[
\vartheta \in \left\{ \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6} \right\}. \tag{8}
\]
Remark 1.5. Finally, one can also argue that every \( \vartheta \)-twisted connected sum is diffeomorphic to some \( \vartheta + \pi \)-twisted connected sum. Let \( V'_+ \) be \( V_+ \) with the orientation reversed, equipped with the ACyl Calabi-Yau structure \((\Omega_+, -\omega_+)\). Then a \( \vartheta \)-hyper-Kähler rotation for \( M_+ \) and \( M_- \) is also a \( \vartheta + \pi \)-hyper-Kähler rotation for \( M'_+ \) and \( M_- \). The orientation-preserving diffeomorphism \( S^2_{\xi_+} \times V_+ \rightarrow S^2_{\xi_+} \times V'_+ \), \((v_+, x) \mapsto (-v_+, x)\) descends to \( M_+ \rightarrow M'_+ \), and pulls back \( \varphi'_+ \) to \( \varphi_+ \). It patches up with the identity map on \( M_- \) to define an isomorphism from \( M \) to the \( \vartheta + \pi \)-twisted connected sum of \( M'_+ \) and \( M_- \).

So in a sense we can obtain all extra-twisted connected sum by considering just \( \vartheta \in (0, \frac{\pi}{2}) \). On the other hand, this remark does not say that a \( \vartheta \)-twisted connected sum from \( V_+ \) and \( V_- \) is a priori isomorphic, or even homeomorphic, to \( \vartheta + \pi \)-twisted connected sum of some deformations of \( V_+ \) and \( V_- \). While the complex conjugation preserves all the topological invariants we can compute (see Remark 2.13), it might in principle affect the topology, and we keep distinguishing between \( \vartheta \) and \( \pi - \vartheta \) below.

- **Square**, \( b_+ = b_- = 0 \), \( \vartheta = \frac{\pi}{2} \).

  As we have already explained, this corresponds to the usual twisted connected sums. \( \vartheta = -\frac{\pi}{2} \) is the same up to orientation. We illustrate the action on the \( T^2 \) factor in both cases in Figures 5 and 6.

- **Square**, \( b_+ = 1 \), \( b_- = 0 \), \( \vartheta = \frac{\pi}{4} \) or \( \frac{3\pi}{4} \).

  The action on \( T^2 \) is illustrated in Figures 7 and 8. The figures also help us understand the fundamental group. Note that \( \sqrt{2}\partial_{b_+} \) and \( \partial_{b_-} \) generate \( \pi_1 T^2 \). On the other hand, we can picture \( \pi_1 M_\pm \) as the projection of the lattice onto the line spanned by \( \partial_{\pm} \) (this uses that \( V_\pm \) is simply connected, which is a consequence of our definition of what it means for \( Z_\pm \) to be a building block, cf. Lemma 2.3(i)). Thus we see that \( \sqrt{2}\partial_{b_+} \) is in the kernel of the push-forward to \( \pi_1 M_+ \), while its image in \( \pi_1 M_- \) is a generator. Similarly \( \partial_{b_-} \) is in the kernel of the push-forward to \( \pi_1 M_- \), while its image in \( \pi_1 M_+ \) is a generator. Van Kampen implies that the resulting extra-twisted connected sums are simply-connected.

- **Hexagonal**, \( b_+ = b_- = 1 \), \( \vartheta = \frac{\pi}{3} \) or \( \frac{2\pi}{3} \).

  See Figures 9 and 10. The resulting extra-twisted connected sums are simply-connected by the same reasoning as in the previous case.

- **Hexagonal**, \( b_+ = 1 \), \( b_- = 0 \), \( \vartheta = \frac{\pi}{6} \) or \( \frac{5\pi}{6} \).

  See Figures 11 and 12. Once more, the resulting extra-twisted connected sums are simply-connected.

The remaining possibilities do not give simply-connected extra-twisted connected sums, and are in fact quotients of twisted connected sums of the types above. By a “\( \vartheta \)-twisted connected sum” for \( \vartheta \) as in 8, we will therefore usually mean one of the types above.
The lattice in Figure 13 has index 2 in the direct sum of the projections onto the $\partial_{v_{\pm}}$ axes, so the fundamental group of the extra-twisted connected sum $M$ is $\mathbb{Z}_2$. The universal cover is the ordinary twisted connected sum $\overline{M}$ of $S^1_{\sqrt{2}} \times V_+$ and $S^1_{\sqrt{2}} \times V_-$ (where $M_{\pm} = S^1_{\sqrt{2}} \times V_{\pm}/a \times \tau_{\pm}$); the involutions $a \times \tau_{\pm}$ patch up to an involution on $\overline{M}$ with quotient $M$. 

Square, $b_{\pm} = b_{\mp} = 1$, $\vartheta = \frac{\pi}{6}$.

Figure 7. $\vartheta = \frac{\pi}{4}$

Figure 8. $\vartheta = \frac{3\pi}{4}$

Figure 9. $\vartheta = \frac{\pi}{3}$

Figure 10. $\vartheta = \frac{2\pi}{3}$

Figure 11. $\vartheta = \frac{\pi}{6}$

Figure 12. $\vartheta = \frac{5\pi}{6}$
• Hexagonal, \( b_+ = 1, b_- = 0, \vartheta = \frac{\pi}{2} \).

See Figure 14. Clearly this configuration is essentially the same as the previous one, up to some squashing of the \( T^2 \) factor.

• Hexagonal, \( b_+ = b_- = 0, \vartheta = \frac{\pi}{3} \) or \( 2\frac{\pi}{3} \).

See Figures 15 and 16. Using \( \{\partial v_+, \partial v_-\} \) as a basis for \( \pi_1 T^2 \), and \( \frac{1}{2} \partial v_\pm \) as generators for \( \pi_1 M_\pm \), the push-forward \( \pi_1 T^2 \to \pi_1 M_+ \times \pi_1 M_- \) is represented by \( (\frac{2}{3}, \frac{2}{3}) \). Since the determinant is 3, we find \( \pi_1 M \cong \mathbb{Z}_3 \).

Up to scale, the universal cover of \( M \) is a \( \vartheta \)-twisted connected sum \( \overline{M} \) of the form above, \( \text{i.e.} \) with \( b_+ = b_- = 1 \). Note that \( M_\pm = S^1_{\sqrt{3}} \times V_\pm / a \times \tau_\pm \) has an innocuous order 3 automorphism \( \rho_\pm : (v_\pm, x) \mapsto (v_\pm + \frac{1}{\sqrt{3}}, x) \). The quotient \( M_\pm / \rho_\pm \) is diffeomorphic to \( M_\pm \), but the covering map pulls back product \( G_2 \)-structures of the form \( \varphi^{h_1}_\pm \) to ones of the form \( \varphi^{h_0}_\pm \) (up to a scale factor \( \sqrt{3} \)). The automorphisms \( \rho_\pm \) patch up to an order 3 automorphism of the \( \vartheta \)-twisted connected sum \( \overline{M} \), whose quotient is \( M \).

2. Building blocks

In [11] we started off by using Theorem 1.1 to produce ACyl Calabi-Yau 3-folds \( V \) from closed Kähler 3-folds \( Z \). We now discuss how the topology of the ACyl Calabi-Yau 3-folds is related to the topology of these building blocks, especially in the presence of an involution. Further we discuss the second Chern class of the blocks, and the moduli space of K3s that appear as anticanonical divisors blocks, as these will also prove relevant for finding matchings and computing the topology of the resulting extra-twisted connected sums.
2.1. Ordinary building blocks. We begin by reviewing the results from \([6, \S 5]\) in the absence of an involution. Like there, we incorporate into our notion of building block some conditions beyond those needed to apply Theorem [14] in order to simplify the topological calculations later.

**Definition 2.1.** A building block is a nonsingular algebraic 3-fold \(Z\) together with a projective morphism \(f: Z \to \mathbb{P}^1\) satisfying the following assumptions:

(i) the anticanonical class \(-K_Z \in H^2(Z)\) is primitive.

(ii) \(\Sigma = f^*(\infty)\) is a nonsingular K3 surface and \(\Sigma \sim -K_Z\).

Identify \(H^2(\Sigma)\) with the K3 lattice \(L\) (i.e. choose a marking for \(\Sigma\)), and let \(N\) denote the image of \(H^2(Z) \to H^2(\Sigma)\).

(iii) The inclusion \(N \hookrightarrow L\) is primitive, that is, \(L/N\) is torsion-free.

(iv) The group \(H^3(Z)\) — and thus also \(H^4(Z)\) — is torsion-free.

**Lemma 2.2.** If \(Z\) is a building block then

(i) \(\pi_1(Z) = (0)\). In particular, \(H^*(Z)\) and \(H_*(Z)\) are torsion-free.

(ii) \(H^2,0(Z) = 0\), so \(N \subseteq \text{Pic } \Sigma\).

We regard \(N\) as a lattice with the quadratic form inherited from \(L\). In examples, \(N\) is almost never unimodular, so the natural inclusion \(N \hookrightarrow N^*\) is not an isomorphism. We write \(T = N^\perp = \{ l \in L \mid \langle l, n \rangle = 0 \ \forall n \in N\}\).

\((T\) stands for “transcendental”; in examples, \(N\) and \(T\) are the Picard and transcendental lattices of a lattice polarised K3 surface.) Using \(N\) primitive and \(L\) unimodular we find \(L/T \simeq N^*\).

Let \(V = Z \setminus \Sigma\). Since the normal bundle of \(\Sigma\) in \(Z\) is trivial, there is an inclusion \(\Sigma \hookrightarrow V\) well-defined up to homotopy. We let

\[
\rho: H^2(V) \to L \quad \text{the natural restriction map, and } K = \ker(\rho).
\]

It follows from (ii) of the following lemma that the image of \(\rho\) equals \(N\).

**Lemma 2.3.** Let \(f: Z \to \mathbb{P}^1\) be a building block. Then:

(i) \(\pi_1(V) = (0)\) and \(H^2(V) = (0)\);

(ii) the class \([\Sigma] \in H^2(Z)\) fits in a split exact sequence

\[
(0) \to Z \overset{\delta}{\to} H^2(Z) \to H^2(V) \to (0),
\]

hence \(H^2(Z) \simeq \mathbb{Z}[\Sigma] \oplus H^2(V)\), and the restriction homomorphism \(H^2(Z) \to L\) factors through \(\rho: H^2(V) \to L\);

(iii) there is a split exact sequence

\[
(0) \to H^3(Z) \to H^3(V) \to T \to (0),
\]

hence \(H^3(V) \simeq H^3(Z) \oplus T\);

(iv) there is a split exact sequence

\[
(0) \to N^* \to H^4(Z) \to H^4(V) \to (0),
\]

hence \(H^4(Z) \simeq H^4(V) \oplus N^*\);

(v) \(H^5(V) = (0)\).

Since the normal bundle of \(\Sigma\) in \(Z\) is trivial, we get a natural inclusion \(\Sigma \times S_\xi^1 \subset V\) up to homotopy. Since we have not introduced any metric yet the notation \(S_\xi^1\) does not carry much meaning beyond serving to distinguish this “internal” circle factor from the “external” one that will soon be introduced. Let \(u \in H^1(S_\xi^1)\) denote the integral generator (\(u = \zeta^{-1}[du]\) in terms of the coordinate \(u\) on \(S_\xi^1\)).

**Lemma 2.4.** Let \(f: Z \to \mathbb{P}^1\) be a building block. The natural restriction homomorphisms:

\[
\beta^m: H^m(V) \to H^m(\Sigma \times S_\xi^1) = H^m(\Sigma) \oplus uH^{m-1}(\Sigma)
\]

are computed as follows:

(i) \(\beta^1 = 0\);
(ii) $\beta_2^2 : H^2(V) \to H^2(\Sigma \times S_1^3) = H^2(\Sigma)$ is the homomorphism $\rho : H^2(V) \to L$;
(iii) $\beta_3^2 : H^3(V) \to H^3(\Sigma \times S_1^3) = uH^2(\Sigma)$ is the composition $H^3(V) \to T \subset L$;
(iv) the natural surjective restriction homomorphism $H^4(Z) \to H^4(\Sigma) = \mathbb{Z}$ factors through $\beta_4^4 : H^4(V) \to H^4(\Sigma \times S_1^3) = H^4(\Sigma) = \mathbb{Z}$, and there is a split exact sequence:

\[(0) \to K^* \to H^4(V) \xrightarrow{\beta_4^4} H^4(\Sigma) \to (0).\]

The Mayer-Vietoris computation uses the boundary maps from the cohomology of $M := S_3^1 \times V$ to its cross-section $W := S_3^1 \times S_1^1 \times \Sigma$, which are trivial to write down in terms of the maps in Lemma 2.4. Letting $v \in H^1(S_1^3)$ denote the generator $\xi^{-1}[dv]$ of the “external” circle factor, we can write

\[H^m(M) = H^m(V) \oplus vH^{m-1}(V)\]
\[H^m(W) = H^m(\Sigma) \oplus uH^{m-1}(\Sigma) \oplus vH^{m-1}(\Sigma) \oplus uvH^{m-2}(\Sigma).\]

**Corollary 2.5.** The homomorphisms $\gamma^m : H^m(M) \to H^m(W)$ are computed as follows:

(i) $H^1(M) = vH^0(V)$,
$H^1(W) = vH^0(\Sigma) \oplus uH^0(\Sigma)$, and

\[\gamma^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} : H^0(V) \to H^0(\Sigma) \oplus H^0(\Sigma)\]

is the natural isomorphism.

(ii) $H^2(M) = H^2(V)$,
$H^2(W) = H^2(\Sigma) \oplus uvH^0(\Sigma) = L \oplus \mathbb{Z}[\Sigma]$, and

\[\gamma^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} : H^2(V) \to L \oplus \mathbb{Z}[\Sigma].\]

(iii) $H^3(M) = H^3(V) \oplus vH^2(V)$,
$H^3(W) = uH^2(\Sigma) \oplus vH^2(\Sigma)$, and

\[\gamma^3 = \begin{pmatrix} 0 \\ \beta_3^3 \\ 0 \end{pmatrix} : H^3(V) \oplus H^2(\Sigma) \to L \oplus L;\]

(iv) $H^4(M) = H^4(V) \oplus vH^3(V)$,
$H^4(W) = H^4(\Sigma) \oplus uvH^2(\Sigma) = H^4(\Sigma) \oplus L$, and

\[\gamma^4 = \begin{pmatrix} 0 \\ \beta_3^4 \\ 0 \end{pmatrix} : H^4(V) \oplus H^3(V) \to H^4(\Sigma) \oplus L.\]

**2.2. Building blocks with involution.** Next we consider involutions of the type required in [11].

**Definition 2.6.** Call $(Z, f, \Sigma, \tau)$ a building block with involution if $(Z, f, \Sigma)$ is a building block in the sense of Definition 2.1, and $\tau : Z \to Z$ is a holomorphic involution such that $\Sigma$ is a connected component of the fixed set of $\tau$. As before, let $V := Z \setminus \Sigma$. Let $b_3^z(Z)$ and $b_3^z(V)$ denote the rank of the $\pm 1$-eigenlattice of the action of $\tau$ on $H^3(Z)$ and $H^3(V)$ respectively,

\[b_3^z(Z) := \text{rk } H^3(Z)^{\pm \tau}, \quad b_3^z(V) := \text{rk } H^3(V)^{\pm \tau}\]

(which will not be confused with (anti-)self-dual parts since the degree is odd) and let $s$ be the dimension of the 2-elementary group $H^2(V) / H^2(\Sigma)$. We call the involution block pleasant if $K = 0$, i.e. the restriction map

\[H^2(V) \to H^2(\Sigma)\]

is injective, and

\[s = b_3^z(V).\]
Remark 2.7. \( f \circ \tau : Z \to \mathbb{P}^1 \) is a fibration with \( f^*(\infty) = \Sigma \), so must be equal to \( f \). Thus \( \tau \) covers an involution of \( \mathbb{P}^1 \), and WLOG that \( \tau = -1 \). Thus there is precisely one other fibre \( \Sigma' := f^*(0) \) mapped to itself by \( \tau \). Since \( Z \) has a unique (up to scale) holomorphic 3-form with pole along \( \Sigma \), that must be preserved by \( \tau \). The action of \( \tau \) on \( \Sigma' \) must therefore be by a non-symplectic involution in the sense described in [5, 13].

Example 3.21 shows that other fibres of \( f \), in particular \( \Sigma' \), need not admit a non-symplectic involution.

The fixed set of \( \tau \) in \( \Sigma' \) is a smooth holomorphic curve \( C \). The quotients \( Z^0 := Z/\tau \) and \( V^0 := V/\tau = Z^0 \setminus \Sigma \) have orbifold singularities along the image of \( C \). On the other hand, \( Y := \Sigma'/\tau \) is a smooth rational surface; \( \Sigma' \to Y \) is a double cover branched over \( C \), and \( C \in (-2K_Y) \). In particular, because \( C \) is even in \( H_2(Y) \), the image of the restriction map

\[
H^2(Z^0; \mathbb{Z}_2) \to H^2(C; \mathbb{Z}_2)
\]

is contained in the kernel of the integration map, and its rank \( m \) is at most \( k \), where \( k + 1 \) is the number of connected components of \( C \).

Lemma 2.8. If \( K = 0 \) then

\[
b_3^+(V) - s = \dim_{\mathbb{Z}_2} T_2 H^3(Z^0) + k - m.
\]

In particular, an involution block is pleasant if and only if \( K = 0 \), \( H^3(Z^0) \) is torsion-free and \( m = k \).
Remark. We can also readily compute the integral cohomology of $M$, that is fixed by the non-symplectic involution, which is totally even when the fixed locus $C$ is empty. As a consequence of this, the polarising Gysin sequence of the double cover $\pi : V \to V^0$ regarded as a real line bundle.

$H^3(V^0)$ is trivial. Further $b_4(V^0) = 1$, so by universal coefficients the rank of $H^4(V^0;\mathbb{Z}) \cong H^4(V^0;\mathbb{C})$ is one more than that of $T_2H^4(V^0)$. By the exactness of (14), we must have that in fact $T_2H^4(V^0) = 0$, and $I_4 : H^4(V;\mathbb{Z}) \to H^4(V^0;\mathbb{C};\mathbb{Z})$ is an isomorphism.

Next argue that the composition of $I_3$ with the push-forward $p : H^3(V^0;\mathbb{C};\mathbb{Z}) \to H^3(V^0;\mathbb{Z})$ is surjective. Since $p$ is surjective, it suffices to prove that $\cup w_1$ maps $\ker p = \Im(H^2(C;\mathbb{Z}) \to H^3(V^0;C;\mathbb{Z})$ onto $H^4(V^0;\mathbb{Z})$, which we can achieve by comparing with the long exact sequence for the branched cover $\Sigma' \to Y$ (and noting that $H^4(V^0;\mathbb{Z}) \cong H^4(V;\mathbb{Z})$).

Because $\pi^* \circ p = \Id + \pi^*$, it follows that $\Im \pi^* = \Im(Id + \pi^*)$. Hence $s = \rk \pi^* = \dim \ker I_3$. The dimension of $H^2(V^0;\mathbb{Z})$ is $b_3(V^0) + (k + 1 - m) + \dim \ker I_2H^2(C^0)$. Hence $b_3(V^0) = (b_3(V) - b_3(V^0)) = (b_3(V) - \ker I_2H^2(C^0)) = (k - m + \dim I_2H^2(C^0))$ as desired.

\begin{proof}

Note that $b_4(V) = b_4(V^0)$. If $K = 0$ then $\tau$ acts trivially on $H^4(V) \cong \mathbb{Z}$, so $b_4(V^0) = 1$.

By Lee–Weintraub [23] Theorem 1 there exists a long exact sequence

$$H^k(V^0;\mathbb{Z}) \xrightarrow{\partial} H^k(V;\mathbb{Z}) \xrightarrow{\partial} H^{k-1}(V^0;\mathbb{Z}),$$

where $I$ fibre-wise integration, and the connecting map $H^k(V^0;\mathbb{C};\mathbb{Z}) \to H^{k+1}(V^0;\mathbb{Z})$ is the cup product with $w_1 \in H^1(V^0 \setminus C;\mathbb{Z})$ of the double cover. (If $C$ were empty, this would just be the Gysin sequence of the double cover $\pi : V \to V^0$ regarded as a real line bundle.

$H^3(V^0)$ is trivial. Further $b_4(V^0) = 1$, so by universal coefficients the rank of $H^4(V^0;\mathbb{Z}) \cong H^4(V^0;\mathbb{C})$ is one more than that of $T_2H^4(V^0)$. By the exactness of (14), we must have that in fact $T_2H^4(V^0) = 0$, and $I_4 : H^4(V;\mathbb{Z}) \to H^4(V^0;\mathbb{C};\mathbb{Z})$ is an isomorphism.

Next argue that the composition of $I_3$ with the push-forward $p : H^3(V^0;\mathbb{C};\mathbb{Z}) \to H^3(V^0;\mathbb{Z})$ is surjective. Since $p$ is surjective, it suffices to prove that $\cup w_1$ maps $\ker p = \Im(H^2(C;\mathbb{Z}) \to H^3(V^0;C;\mathbb{Z})$ onto $H^4(V^0;\mathbb{Z})$, which we can achieve by comparing with the long exact sequence for the branched cover $\Sigma' \to Y$ (and noting that $H^4(V^0;\mathbb{Z}) \cong H^4(V;\mathbb{Z})$).

Because $\pi^* \circ p = \Id + \pi^*$, it follows that $\Im \pi^* = \Im(Id + \pi^*)$. Hence $s = \rk \pi^* = \dim \ker I_3$. The dimension of $H^2(V^0;\mathbb{Z})$ is $b_3(V^0) + (k + 1 - m) + \dim \ker I_2H^2(C^0)$. Hence $b_3(V^0) = (b_3(V) - b_3(V^0)) = (b_3(V) - \ker I_2H^2(C^0)) = (k - m + \dim I_2H^2(C^0))$ as desired.

\end{proof}

\begin{remark}

In this paper, we will only apply Lemma 2.8 in cases where $C$ is connected, so the condition $m = k$ is automatically satisfied (both are 0). As a consequence of this, the polarising lattice $N$ of the resulting building blocks with involution will always be completely even, in the sense that the product of any two elements is even. For $N$ embeds into the sublattice of $H^2(\Sigma')$ that is fixed by the non-symplectic involution, which is totally even when the fixed locus $C$ is connected (see Lemma 5.1).

From now on we assume (10). This implies in particular that $\tau$ acts trivially on $H^2(V) \cong N$ and $H^4(V) \cong H^4(\Sigma) \cong \mathbb{Z}$, so $V^0$ has the same Betti numbers as $V$ except in the middle degree. Since $\pi : V \to V^0$ is a double cover branched over $C$, we find

$$\chi(V) = 2\chi(V^0) - \chi(C),$$

from which we deduce

$$b_3(V) = 2b_3(V^0) - 2 - \rho + \chi(C).$$

Similarly

$$\chi(Z) = 2\chi(Z^0) - \chi(C) - \chi(\Sigma)$$

implies (using $\chi(Z) = 4 + 2\rho - b_3(Z)$ etc) that

$$b_3(Z) = 2b_3(Z^0) + 20 - 2\rho - \chi(C).$$

\begin{equation}
(15)
\end{equation}

Now let $M := S^1_1 \times V/\alpha \times \tau$. The rational cohomology of $M$ is simply the $\tau$-invariant part of $H^*(S^1_1 \times V)$. We see from Lemma 2.3 and our description of $\tau^*$ that

$$b_1(M) = 0 \quad b_2(M) = b_2(V) \quad b_3(M) = b_3(V) + b_4^*(V)$$

$$b_4(M) = b_3(V) + 1 \quad b_5(M) = 1 \quad b_6(M) = 0$$

We can also readily compute the integral cohomology of $M$ from the Mayer-Vietoris sequence

$$\cdots \to H^{k-1}(V) \to H^k(M) \to H^k(V) \xrightarrow{\Id - \tau^*} H^k(V) \to \cdots$$

(16)

\begin{lemma}

(i) $\mathbb{Z} \cong H^1(M)$

(ii) $H^3(M) \cong H^2(V) = N$

(iii) $0 \to H^2(V) \to H^3(M) \to H^3(V) \tau^* \to 0$

(iv) $0 \to H^3(V)/\ker(\Id - \tau^*) \to H^3(V) \to H^4(M) \to \mathbb{Z} \to 0$

(v) $\mathbb{Z} \cong H^5(M)$

\end{lemma}
(vi) $H^6(M) = 0$

Note that the only torsion in $H^*(M)$ is

$$\text{Tor} H^4(M) \cong \frac{H^3(V)^-}{(\text{Id} - \tau^*) H^3(V)} \cong \mathbb{Z}_2^h(V)^-;$$

thus $H^*(M)$ is torsion-free when the involution block $Z$ is pleasant.

We also need to understand the restriction map to the cross-section of the cylindrical end, $H^*(M) \to H^*(T^2 \times \Sigma)$, where $T^2 := S^1_1 \times S^1_\xi/a \times a$. By a bit of abuse of notation, denote classes in $H^*(T^2)$ by their pull-backs to $H^*(S^1_1 \times S^1_\xi)$; thus $2v$ and $2u \in H^1(T^2)$ are primitive classes, but they generate a subgroup of index 2, and $H^2(T^2)$ is generated by $2vu$.

In a sense it’s obvious what the maps are: the tricky (and relevant) part is to describe the image, especially in $H^3$. Over $\mathbb{Q}$, the image is the same as for the maps in Corollary 2.3 e.g. $H^3(M; \mathbb{Q}) \to H^3(T^2 \times \Sigma; \mathbb{Q})$ has image $vN \oplus uT$.

**Lemma 2.11.**

(i) $H^2(M) \to H^2(T^2 \times \Sigma)$ is an isomorphism onto $N$.

(ii) $H^3(M) \to H^3(T^2 \times \Sigma)$ has image contained in

$$I^3 := \{vn + ut : n \in N, t \in T, n + t = 0 \mod 2L\}.$$

If $s = b_3^{-1}(V)$ then equality holds.

(iii) $H^4(M) \to H^4(T^2 \times \Sigma)$ has image $2vuT \oplus H^4(\Sigma)$.

**Proof.** First part is obvious because $H^2(M) \to H^2(V)$ is an isomorphism. Last part is obvious because the Mayer-Vietoris boundary map in the computation of $H^*(T^2 \times \Sigma)$ maps $H^k(S^1_1 \times \Sigma) \to H^{k+1}(T^2 \times \Sigma)$ by $x \to 2vx$.

$I^3$ is precisely the set of integral classes in the rational image $vN \oplus uT \subseteq H^3(T^2 \times \Sigma; \mathbb{Q})$, so the image of $H^3(M)$ is a finite index subgroup of $I^3$. The long exact sequence of cohomology of $M$ relative to $T^2 \times \Sigma$ gives $I^3/\text{Im } H^3(M) \hookrightarrow H^3_{\text{cpt}}(M) \cong H_3(M)$. Thus $I^3/\text{Im } H^3(M) \to \text{Tor } H_3(M) \cong \text{Tor } H^4(M)$, which is trivial if $s = b_3^{-1}(V)$.

2.3. **The second Chern class.** It will prove convenient to present the second Chern class of a building block with $K = 0$ in the following form:

$$c_2(Z) = g(\tilde{c}_2(Z)) + 24h,$$  \hspace{1cm} (17)

for some $\tilde{c}_2(Z) \in N^*$ and $h \in H^4(Z)$ such that the restriction of $h$ to $\Sigma$ is the positive generator of $H^4(\Sigma)$, where $g : N^* \to H^4(Z)$ is dual to the restriction $H^2(\Sigma) \to N \subset H^2(\Sigma)$. Alternatively, we can describe $g$ as follows: for $\tilde{c} \in N^*$ and any preimage $x$ of $\tilde{c}$ under $\partial : H^2(\Sigma) \to N^*$,

$$g(\tilde{c}) = \iota_\Sigma \partial(ux),$$

where $\partial : H^3(S^1_1 \times \Sigma) \to H^3_{\text{cpt}}(V)$ is the snake map in the long exact sequence of the cohomology of $V$ relative to its boundary, and $\iota_\Sigma : H^3_{\text{cpt}}(V) \to H^3(Z)$ is the push-forward of the inclusion $V \to Z$.

For a building block with $K = 0$, Lemma 2.3.[iv] and 2.4.[iv] give exactness of

$$0 \to N^* \delta H^4(Z) \to H^4(\Sigma) \to 0.$$ 

Since the image of $c_2(Z)$ in $H^4(\Sigma)$ is $\chi(\Sigma) = 24$ times the generator, $c_2(Z)$ can then always be written in the form $\tilde{c}_2(Z)$. This presentation is not unique, but we will make convenient choices for $\tilde{c}_2(Z)$ and $h$ for each class of building blocks. (If $K \neq 0$ then we cannot in general write $c_2(Z)$ in the form $\tilde{c}_2(Z)$, and would need to make some further arbitrary choices to capture the components in a direct summand isomorphic to $K^*$.)

In the case of a building block $Z$ with involution $\tau$, we describe the second Chern class in the same way, but impose one further condition on $h$. Let $Z^0$ denote the (singular) quotient $Z/\tau$, and let $\pi : Z \to Z^0$ be the quotient map. Then any even element in $H^4(Z)$ has a pre-image in $H^4(Z_0)$, and in particular $2h = \pi^{-1}x$ for some $x \in H^4(Z^0)$. There is also a projection map $S^1_1 \times Z \to Z^0$, and the image of $x$ in $H^4(S^1_1 \times Z)$ may or may not be even. We demand that $h$ is chosen so that this class in $H^4(S^1_1 \times Z)$ is even.
2.4. Moduli of lattice-polarised K3s. The final property of building blocks that we will wish to study concerns the relation to moduli spaces of K3s. Because a K3 surface $\Sigma$ is simply connected, its Picard group $\text{Pic} \Sigma$ is isomorphic to $H^2(\Sigma; \mathbb{Z}) \cap H^{1,1}(\Sigma; \mathbb{C})$. The Picard lattice is $\text{Pic} \Sigma$ equipped with the restriction of the intersection form of $H^2(\Sigma; \mathbb{Z})$.

Fix a non-singular lattice $L$ of signature $(3, 19)$. A marking of a K3 surface $\Sigma$ is an isomorphism $h : H^2(\Sigma; \mathbb{Z}) \to L$. The Picard lattice of a marked K3 is thus identified with a (primitive) sublattice of $L$. Meanwhile, the period of the marked K3 is the image in $\mathbb{P}(L_{\mathbb{C}})$ of the 1-dimensional subspace $H^{2,0}(\Sigma; \mathbb{C}) \subset H^2(\Sigma; \mathbb{C})$. It lies in the subset $\{ \Pi \in \mathbb{P}(L_{\mathbb{C}}) : \Pi^2 = 0, \Pi \Pi > 0 \}$. By the Torelli theorem, the moduli space of marked K3s is (modulo some niceties about choice polarisations that do not concern us) isomorphic to an open subset of this period domain.

Crucially, the K3 surfaces $\Sigma$ that appear in a building block $Z$ always belong to a more restricted moduli space. According to Lemma 2.12, the Picard lattice of $\Sigma$ must contain the polarising lattice $N$ of $Z$. Therefore the period $\Pi$ of the marked K3 must be orthogonal to $N$. We say that $\Sigma$ is “$N$-polarised”.

Equivalently, we can think of the period as the positive definite 2-plane $\Pi \subset L_{\mathbb{R}}$ spanned by the images of real and imaginary parts of $H^{2,0}(\Sigma; \mathbb{C})$. If $\Sigma$ is $N$-polarised, then $\Pi$ belongs to the Griffiths domain

$$D_N := \{ \text{positive-definite 2-planes} \Pi \in Gr(2, N^+) \}.$$  \hspace{1cm} (18)

A principle that is valid for all building blocks we consider in this paper is that they come in families, such that a generic $N$-polarised K3 appears as an anticanonical divisor in some element of the family, and moreover we have some control on the size of the ample cone (see Proposition 3.7). In $\mathbb{P}$, we find on the one hand that this genericity property is often enough for producing matchings between some elements of a pair of families. On the other hand, we find also that in some cases one needs to know that even generic elements of a more restricted moduli space of K3s (with a larger polarising lattice $\Lambda \supset N$) appear as anticanonical divisors. We capture these conditions in the following definition.

**Definition 2.12.** Let $N \subset L$ be a primitive sublattice, $\Lambda \subset L$ a primitive overlattice of $N$, and $\text{Amp}_\Sigma$ an open subcone of the positive cone in $N_{\mathbb{R}}$. We say that a family of building blocks $Z$ with polarising lattice $N$ is $(\Lambda, \text{Amp}_\Sigma)$-generic if there is a subset $U_{\mathbb{R}}$ of the Griffiths domain $D_{\Lambda}$ with complement a countable union of complex analytic submanifolds of positive codimension with the property that: for any $\Pi \in U_{\mathbb{R}}$ and $k \in \text{Amp}_\Sigma$ there is a building block $(Z, \Sigma) \in Z$ and a marking $h : L \to H^2(\Sigma; \mathbb{Z})$ such that $h(\Pi) = H^{2,0}(\Sigma)$, and $h(k)$ is the image of the restriction to $\Sigma$ of a Kähler class on $Z$.

2.5. Presentation of data. To finish the section, let us summarise what we consider to be the key pieces of data of a building block, which will be sufficient to compute the topological invariants of the resulting extra-twisted connected sums that we are interested in.

- The kernel $K$ of $H^2(V) \to H^2(\Sigma)$ and (for involution blocks) whether the block is pleasant
- $b_3(Z)$ and—in the case of blocks with involution—$b_3^+ (Z)$
- The form on the polarising lattice $N$
- An element $c_2(Z) \in N^*$ encoding information about $c_2(Z)$ as in $17$.
- An open cone $\text{Amp} \subset N_{\mathbb{R}}$ such that the family of blocks is $(N, \text{Amp})$-generic in the sense of Definition 2.12.

Tables 1, 2, and 3 will include this and some auxiliary data. In fact, all the ordinary blocks included in the tables will have $K = 0$, and all the involution blocks will be pleasant.

We always use the same basis of $N$ for describing the form on $N$, $c_2(Z)$ and $\text{Amp}$. For all blocks we consider, it turns out to be possible to choose a basis for $N$ that consists of the edges of $\text{Amp}$, and in the tables we always use such a basis.

Note that this means that the sign of $c_2(Z)$ is meaningful. Multiplying all elements of the basis by $-1$ preserves the intersection form, but reverses the signs of $c_2(Z)$ and $\text{Amp}$ together. For instance, if $N$ has rank 1, choosing $\text{Amp}$ amounts to designating one of the two generators of $N$ to be positive. Whether $\check{c}$ evaluates to, say, 2 or $-2 \mod 24$ on the positive generator then has an
invariant meaning, and can affect the homeomorphism class of the extra-twisted connected sums built from the block.

Remark 2.13. If $Z$ is a building block then so is its complex conjugate $\overline{Z}$, i.e. the same smooth manifold, but with the complex structure $J$ replaced by $-J$. This reverses the orientation of $Z$, but preserves it on $\Sigma$, so the sign of the dual map $g : N^* \to H^4(Z)$ is reversed. At the same time, the Kähler cone of $\Sigma$ is multiplied by $-1$, so $Z$ and $\overline{Z}$ are indistinguishable by our topological data. This is quite reasonable, since we would expect it to be possible to deform $Z$ to a building block with a real structure, and hence to its complex conjugate.

3. Building blocks from semi-Fano 3-folds

The main method we use in this paper for producing examples of building blocks is to blow up Fano 3-folds or semi-Fano 3-folds. Let us briefly recall some terminology. A projective 3-fold $Y$ is weak Fano if the anticanonical bundle $-K_Y$ is big and nef, i.e. if the sections of a sufficiently high power of $-K_Y$ define a morphism $\phi$ of $Y$ to projective space, whose image $X$ (the anticanonical model) is 3-dimensional. If $\phi$ is an embedding, then $Y$ is Fano, i.e. $-K_Y$ is ample. In the terminology from [6], Definition 4.11, for $Y$ to be semi-Fano means that the fibres of $\phi$ have dimension at most 1.

3.1. Ordinary building blocks from Fano 3-folds. Let us first summarise the results from [6] concerning how to construct building blocks (without involution) from Fano or semi-Fano 3-folds, along with some previously studied examples of applying this to Fano 3-folds mainly of Picard rank 1 or 2.

Proposition 3.1 ([6] Prop 4.24). Let $Y$ be a closed Kähler 3-fold with an anticanonical pencil $|\Sigma_0 : \Sigma_1|$ with smooth base locus $C$. Let $Z$ be the blow-up of $Y$ along $C$, and let $\Sigma \subset Z$ be the proper transform of $\Sigma_0$. Then the image $N$ of $H^2(Y) \to H^2(\Sigma)$ equals the image of $H^2(Y) \to H^2(\Sigma_0)$, while $\ker(H^2(Y) \to H^2(\Sigma)) \cong \mathbb{Z} \oplus \ker(H^2(\Sigma) \to H^2(\Sigma))$. Further $\Tor H^3(Y) \cong \Tor H^3(\Sigma)$, and the image of the Kähler cone of $Z$ in $H^{1,1}(\Sigma; \mathbb{R})$ contains the image of the Kähler cone of $Y$.

Construction 3.2. Let $Y$ be a closed Kähler 3-fold such that

(i) the image $N$ of $H^2(Y) \to H^2(\Sigma_0)$ is primitive,
(ii) $H^3(Y)$ torsion-free, and
(iii) an anticanonical pencil $|\Sigma_0 : \Sigma_1|$ with smooth base locus $C$.

Let $Z$ be the blow-up of $Y$ along $C$, and let $\Sigma \subset Z$ be the proper transform of $\Sigma_0$. Then $(Z, \Sigma)$ is a building block, with polarisation lattice $N$, and $K \cong \ker H^2(Y) \to H^2(\Sigma_0)$.

Proposition 3.3. If $Y$ is a semi-Fano 3-fold whose anti-canonical ring is generated in degree 1 then conditions (i) and (iii) in Construction 3.2 are satisfied, and $K = 0$.

Proof. See [6] Remark 4.10 and Proposition 5.7. \qed

For the anticanonical ring of $Y$ to be generated in degree 1 is equivalent to the anticanonical model $X$ of $Y$ to have very ample $-K_X$. The only Fano 3-folds $Y$ for which $-K_Y$ fails to be very ample are number 1 in the Mori-Mukai list of rank 2 Fanos, and the product of $\mathbb{P}^1$ with a degree 1 del Pezzo surface. The possible singular anticanonical models $X$ for which $-K_X$ fails to be very ample are listed by Jahnke-Radloff [19] Theorem 1.1].

Meanwhile, all known examples of semi-Fano 3-folds $Y$ have torsion-free $H^3(Y)$. Thus we can justifiably say that Construction 3.2 can be applied to produce a building block from almost any semi-Fano 3-fold.

Now let us proceed to explain how to obtain the other data listed in [6].

Lemma 3.4 ([6] Lemma 5.6). $b_3(Z) = b_3(Y) + b_1(C) = b_3(Y) - \chi(C) + 2 = b_3(Y) - K_Y^3 + 2$.

Lemma 3.5 ([6] Proposition 5.11). Let $Z$ be a building block obtain from a closed Kähler 3-fold $Y$ as in Construction 3.2, and let $\pi : Z \to Y$ denote the blow-up map. Let $h \in H^4(Z)$ be the Poincaré dual to a $\mathbb{P}^1$ fibre of $\pi$, and let $\pi : H^4(Z) \to H^4(Y)$, $g : N^* \to H^4(Z)$ and $g_Y : N^* \to H^4(Y)$
be the Poincaré dual to \( \pi^* : H^2(Y) \to H^2(Z) \) and the restrictions \( H^2(Z) \to N \) and \( H^2(Y) \to N \) respectively. Then \( c_2(Z) = g(\bar{c}_2(Z)) + 24 h, \) for 
\[
\bar{c}_2(Z) = g_Y^{-1} \pi c_2(Z).
\]

This description of \( c_2(Z) \) is convenient when coupled with the following claim.

**Lemma 3.6 ([6, (5-13)])**. If \( \pi : Z \to Y \) is the blow-up of some closed Kähler 3-fold \( Y \) along a curve \( C \) contained in an anticanonical divisor \( \Sigma \), then 
\[
\pi(c_2(Z) + c_1(Z)^2) = c_2(Y) + c_1(Y)^2.
\]

Finally, for the matching problem it is an important principle that our blocks come in families, such that a generic \( N \)-polarised K3 surface appears as an anticanonical divisor in some element of the family.

**Proposition 3.7.** Let \( Y \) be a semi-Fano 3-fold with Picard lattice \( N \) (i.e. \( N \) is the image of \( H^2(Y) \to H^2(\Sigma) \) for an anticanonical \( \Sigma \subset Y \)), and let \( \mathcal{Y} \) be the set of Fano 3-folds in the deformation type of \( Y \). Then there is an open cone \( \text{Amp}_Y \subset N_R \) such that \( \mathcal{Y} \) is \((N, \text{Amp}_Y)\)-generic.

In particular, the set of building blocks produced from \( \mathcal{Y} \) by Construction 3.2 is also \((N, \text{Amp}_Y)\)-generic.

Note however that Proposition 3.7 is limited in that it does not tell us what \( \text{Amp}_Y \) is. In the examples we can work it out from the explicit description of the Fanos.

**Example 3.8.** Table 1 summarises the key data of Fano 3-folds of rank 1 and the resulting building blocks (cf. [6, Table 1]). Apart from the data highlighted in \([2,5]\) we include in the table the index \( r \) (i.e. the largest integer such that \(-K_Y = rH \) for some \( H \in \text{Pic}(Y) \)), the anticanonical degree \(-K^3_Y\), and \( b_3(Y) \).

\[
\begin{array}{cccccc}
  r & -K^3_Y & b_3(Y) & b_3(Z) & N & \bar{c}_2(Z) \\
  4 & 4^3 & 0 & 66 & (4) & 22 \\
  3 & 3^3 \cdot 2 & 0 & 56 & (6) & 26 \\
  2 & 2^3 & 42 & 52 & (2) & 16 \\
  2 & 2^3 \cdot 2 & 20 & 38 & (4) & 20 \\
  2 & 2^3 \cdot 3 & 10 & 36 & (6) & 24 \\
  2 & 2^3 \cdot 4 & 4 & 38 & (8) & 28 \\
  2 & 2^3 \cdot 5 & 0 & 42 & (10) & 32 \\
  1 & 2 & 104 & 108 & (2) & 26 \\
  1 & 4 & 60 & 66 & (4) & 28 \\
  1 & 6 & 40 & 48 & (6) & 30 \\
  1 & 8 & 28 & 38 & (8) & 32 \\
  1 & 10 & 20 & 32 & (10) & 34 \\
  1 & 12 & 14 & 28 & (12) & 36 \\
  1 & 14 & 10 & 26 & (14) & 38 \\
  1 & 16 & 6 & 24 & (16) & 40 \\
  1 & 18 & 4 & 24 & (18) & 42 \\
  1 & 22 & 0 & 24 & (22) & 46 \\
\end{array}
\]

**Table 1. Rank 1 Fano blocks**

In the rank 1 case, \( \bar{c} \) is also easily determined as follows: For any Fano one has \( c_2(Y)(-K_Y) = 24 \), so if \(-K_Y = rH \) then
\[
(c_2(Y) + c_1(Y)^2)H = \frac{24 - K^3_Y}{r}.
\]
Table 2. Blocks of rank 2 and 3 from Construction 3.2

| Ex | \( r \) | \(-K_Y^3\) | \( b_3(Y)\) | \( b_3(Z)\) | \( N \) | \( \tilde{c}_2(Z) \) |
|-----|-----|-----|-----|-----|-----|-----|
| 3.9 | 1  | 8  | 22 | 32 | \(\frac{4}{5}\) | \(\frac{6}{7}\) | \(20\) \(12\) |
| 3.9 | 0  | 16 | 6  | 24 | \(\frac{3}{4}\) | \(\frac{6}{7}\) | \(28\) \(12\) |
| 3.9 | 1  | 24 | 2  | 28 | \(\frac{4}{5}\) | \(\frac{6}{7}\) | \(22\) \(26\) |
| 3.9 | 1  | 38 | 0  | 40 | \(\frac{5}{6}\) | \(\frac{7}{8}\) | \(18\) \(22\) |
| 3.9 | 2  | 2 \(\cdot\) 6 | 0  | 50 | \(\frac{2}{3}\) | \(\frac{4}{5}\) | \(18\) \(18\) |
| 3.9 | 2  | 2 \(\cdot\) 7 | 0  | 58 | \(\frac{4}{5}\) | \(\frac{6}{7}\) | \(22\) \(18\) |
| 3.9 | 2  | 2 \(\cdot\) 6 | 0  | 50 | \(\frac{10}{11}\) | \(\frac{12}{13}\) | \(20\) \(12\) |
| 3.9 | 2  | 2 \(\cdot\) 5 | 0  | 42 | \(\frac{8}{9}\) | \(\frac{10}{11}\) | \(28\) \(24\) |
| 3.9 | 2  | 2 \(\cdot\) 4 | 0  | 34 | \(\frac{6}{7}\) | \(\frac{9}{10}\) | \(32\) \(28\) |
| 3.9 | 2  | 2 \(\cdot\) 4 | 0  | 18 | \(\frac{4}{5}\) | \(\frac{6}{7}\) | \(24\) \(20\) |
| 3.9 | 2  | 2 \(\cdot\) 4 | 0  | 26 | \(\frac{2}{3}\) | \(\frac{4}{5}\) | \(24\) \(18\) |
| 3.9 | 2  | 2 \(\cdot\) 4 | 0  | 34 | \(\frac{8}{9}\) | \(\frac{10}{11}\) | \(28\) \(18\) |
| 3.9 | 2  | 2 \(\cdot\) 5 | 0  | 42 | \(\frac{10}{11}\) | \(\frac{12}{13}\) | \(32\) \(28\) |
| 3.9 | 2  | 2 \(\cdot\) 5 | 0  | 33 | \(\frac{2}{3}\) | \(\frac{4}{5}\) | \(20\) \(12\) |
| 3.9 | 2  | 2 \(\cdot\) 4 | 0  | 36 | \(\frac{8}{9}\) | \(\frac{10}{11}\) | \(28\) \(18\) |
| 3.9 | 2  | 2 \(\cdot\) 5 | 0  | 42 | \(\frac{10}{11}\) | \(\frac{12}{13}\) | \(32\) \(12\) |

So Lemma 3.5 implies that with respect to the basis of \(H^4(Z)\) dual to \(H\), \(\tilde{c}\) is represented by the coordinate \(\frac{24-K_Y^3}{r}\). The self-intersection of the generator of \(N\) (which is not mentioned in the table) is simply \(\frac{-K_Y^3}{r}\).

We now proceed with a selection of building blocks obtained from Fanos and semi-Fanos of rank 2 or 3. For later use we prioritise ones with index 2. The data for these blocks are displayed in Table 2 along with the index \(r\), the anticanonical degree \(-K_Y^3\) and the Betti number \(b_3(Z)\) of the (semi-)Fano \(Y\) used. (Table 2 also includes one block from §3.4 that results from applying Construction 3.2 to a 3-fold that is not semi-Fano.)

Example 3.9. Construction 3.2 can be applied to all but the first of the 36 entries in the Mori-Mukai list of classes of rank 2 Fano 3-folds. We will refer to blocks resulting from the \(k\)th entry as Example 3.9. The invariants of the resulting blocks can be found in [12, Table 3]. Let us briefly describe those classes that we will make use of later.

\(k = 3\) Double cover of \(\mathbb{P}^3\) branched over a quartic, blown up in the pre-image of a line (which is an elliptic curve).
$k = 10$ Complete intersection of two quadrics in $\mathbb{P}^5$, blown up in the intersection of two hyperplanes.

$k = 17$ Blow-up of a smooth quadric in $\mathbb{P}^4$ along an elliptic curve of degree 5.

$k = 27$ Blow-up of $\mathbb{P}^3$ along a twisted cubic.

$k = 32$ A $(1,1)$ divisor in $\mathbb{P}^2 \times \mathbb{P}^2$.

$k = 35$ The blow-up of $\mathbb{P}^3$ in a point.

The last two cases (i.e. $k = 32$ and 35) are the only rank 2 Fanos of index 2.

**Example 3.10.** The only rank 3 Fano 3-fold of index 2 is $Y = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. It has

$$N \cong \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}$$

$b_3(Z) = 50$ and $c_2(Z) = (12, 12, 12)$.

### 3.2. Semi-Fano 3-folds of rank 2.

Smooth weak Fano 3-folds must have Picard rank at least 2, and there is a classification programme for Picard rank exactly 2, see e.g. Jahnke-Peternell [18], Blanc-Lamy [3], Arap-Cutrone-Marshburn [1], Cutrone-Marshburn [13] and Fukuoka [14]. We will not explore this fully, but focus on the cases that will prove most relevant later.

As seen in Examples 3.9 and 3.10, rank 2 Fano 3-folds are often obtained by blowing up curves of small genus and degree in $\mathbb{P}^3$. Blanc-Lamy study cases where the degree is a little larger relative to the genus, and produce many semi-Fano 3-folds this way.

**Example 3.11.** Let $Y$ be the blow-up of $\mathbb{P}^3$ in an elliptic curve of degree 7. Then $Y$ is semi-Fano—indeed, $-K_Y$ is a small contraction according to Blanc-Lamy [3, Table 1]. In the basis formed by the pull-back of the hyperplane class from $\mathbb{P}^3$ and $-K_Y$ (which also span the nef cone) the Picard lattice is

$$N \cong \begin{pmatrix} 4 & 9 \\ 9 & 8 \end{pmatrix}.$$  

Compute as above that $b_3(Y) = 2$ and $b_3(Z) = 12$. Since $Z$ can be viewed as the result of performing two blow-ups, we can apply Lemma 3.6 and [19] twice to find $c_2(Z) = (22, 32)$.

We could produce blocks from 21 further cases in [3, Table 1] in a similar way, but let us instead restrict attention to the case of rank 2 “semi del Pezzo 3-folds” (i.e. semi Fanos of index 2), where Jahnke-Peternell [18] have provided a complete classification.

Example 3.9 produced a Fano 3-fold of index 2 by blowing up $\mathbb{P}^3$ at a point. It is true more generally that the canonical bundle being even is preserved by blowing up a point, but the Fano condition is not. However, for 4 of the 5 families of index 2 Fanos the blow-up has small anticanonical morphism.

**Example 3.12.** For $2 \leq d \leq 5$, let $X'$ be a Fano of rank 1, index 2 and degree $d$ as in Example 3.22. Blowing up $X'$ at a generic point $p$ yields a semi-Fano $X$ [18, Theorem 3.7].

$H' := \pi^*(-\frac{1}{2}K_{X'})$ clearly spans one edge of the nef cone of $X$ (the corresponding morphism is just the blow-down $X \to X'$), and $X$ being semi-Fano means that $H := -\frac{1}{2}K_X = H' - E$ spans the other (where $E$ is the class of the exceptional divisor). In the basis $H, E$ the Picard form of $X$ is simply \( \begin{pmatrix} 2d & 0 \\ 0 & 2d - 2 \end{pmatrix} \), so with respect to the basis $H, H'$ for the nef cone we get

$$N \cong \begin{pmatrix} 2d & 2d \\ 2d & 2d - 2 \end{pmatrix}.$$  

See from [19] that $c_2(X) + c_1(X)^2$ evaluates to $24 + 8d - 8$ on $-K_X$. On the other hand, since $-K_X$ can be represented by a divisor that does not contain the blow-up point, [3, Lemma 5.15] gives $(c_2(X) + c_1(X)^2)\pi^*(-K_{X'}) = (c_2(X') + c_1(X')^2)(-K_{X'}) = 24 - K_Y^2 = 24 + 8d$. Hence $c_2(Z) = c_2(X) + c_1(X)^2$ is represented by $(12 + 4d, 8 + 4d)$ with respect to the basis of $N^*$ dual to $H, H'$. By Jahnke-Peternell [18], the remaining classes of rank 2 weak del Pezzos with small anticanonical morphism fall into two categories: conic bundles over $\mathbb{P}^2$ and quadric bundles over $\mathbb{P}^1$. 
Example 3.13. For $2 \leq d \leq 5$, according to [13] Theorem 3.7 there are degree $d$ weak del Pezzos with small anticanonical morphism of the form $Y = \mathbb{P}(E)$, where $E \to \mathbb{P}^2$ is a rank 2 holomorphic vector bundle with $c_1(E) = -1$ and $c_2(E) = 7 - d$.

Then $-K_Y = \det E - 2T + 3F = 2(-T + F)$, where $F$ is the pull-back of the hyperplane class from $\mathbb{P}^2$ and $T$ is the tautological bundle of $\mathbb{P}(E)$. As basis for the Picard lattice, we take $-T + F$ and $F$, which also span the nef cone. Note that $T^2 = c_1(E)T - c_2(E) = -TF + (d - 7)F^2$ and $F^3 = 0$ to find that the Picard lattice is represented with respect to our chosen basis by

$$N = \begin{pmatrix} 2d & 6 \\ 6 & 2 \end{pmatrix}.$$

Patently $b_3(Y) = 0$, so $b_3(Z) = -K_Y^3 + 2 = 8d + 2$.

To compute $c_2(Y)$, note that $TY$ is stably isomorphic to $(-T) \otimes E \oplus F \oplus 3$. We have $c_2((-T) \otimes E) = c_2(E) - Tc_1(E) + T^2 = 0$, so

$$c_2(Y) = 3F^2 + 3Fc_1(E) + c_2(E) = -6FT.$$

Hence

$$c_2(Y) + c_1(Y)^2 = -6FT + 4(-T + F)^2 = -18FT + (4d - 24)F^2.$$

This evaluates to 18 on $F$ and to $4d + 12$ on $-T + F$, i.e. $c_2(Z)$ is represented with respect to our chosen basis by the row vector $(4d+12 \ 18)$.

We refer to the building blocks arising from these semi del Pezzos as Example 3.13.

Example 3.14. For each $1 \leq d \leq 5$, according to [13] Theorem 3.5 there are semi del Pezzo 3-folds $Y$ of degree $d$ that are divisors in the projectivisation of a rank 4 bundle $E$ of $c_1 = 2 - d$ over $\mathbb{P}^1$. The class of the divisor $Y$ is $-2T + (4 - d)F$, where $F$ is the pull-back of the hyperplane class of the $\mathbb{P}^1$ base, and $T$ is the class of the tautological bundle of $\mathbb{P}(E)$—so the generic fibres of $Y \to \mathbb{P}^1$ are quadric surfaces in $\mathbb{P}^3$.

The anticanonical class of $Y$ is

$$-K_Y = -K_{\mathbb{P}^1} + \det E - 4T - (-2T + (4 - d)F) = -2T - F$$

and $F$ form a basis for the Picard lattice. Noting that on $\mathbb{P}(E)$ we have $F^2 = 0$ and $T^4 = T^3c_1(E) = d - 2$, we see that the intersection form is represented in this basis by

$$N \cong \begin{pmatrix} 2d & 4 \\ 4 & 0 \end{pmatrix}.$$

Compute the Chern classes of $Y$ from tangent bundle of $\mathbb{P}(E)$ being stably isomorphic to $(-T) \otimes E \oplus F \oplus F$, and hence find $b_3(Z) = 12 + 6d$ and $\bar{c}_2(Z) = (12+4d \ 12)$.

We refer to the building blocks arising from these semi del Pezzos as Example 3.13.

Remark 3.15. In [13] Theorem 3.5, there are actually two different classes with $d = 2$, corresponding to $E = \mathcal{O}(-1,0,0,1)$ or $E$ being trivial over $\mathbb{P}^1$ (i.e. in the latter case $Y$ is a $(2,2)$-divisor on $\mathbb{P}^1 \times \mathbb{P}^1$). However, these bundles can be deformed to each other, and so can the semi del Pezzos, so as far as we are concerned they form a single family of building blocks, cf. [6] Example 6.11(i).

Remark 3.16. Any rank 2 semi-Fano whose anticanonical morphism is a small contraction can be flopped, i.e. the anticanonical model has another small resolution that is also a rank 2 semi-Fano. In some cases the flop is in the same class as the original semi-Fano, but in some cases it can belong to a different family.

Consider for instance Example 3.14, the blow-up $X$ of the complete intersection $X'$ of two quadrics in $\mathbb{P}^5$ at a point $p \in X$. The morphism defined by $-\frac{1}{2}K_X$ can be interpreted as the projection from $p$ to a hyperplane; it contracts the 4 lines passing through $p$, and the image (i.e. the anticanonical model) is a cubic hypersurface $X''$ that contains a plane $\Pi$. The pre-image of $\Pi$ in $X$ is the exceptional divisor of the blow-up $X \to X'$, whose intersection number with the contracted lines is 1. We therefore find that $X$ is the small resolution of $X''$ obtained by blowing up a quadric surface in $X''$ that intersects $\Pi$ in the singularities of $X''$.

If we instead resolve $X''$ by blowing up $\Pi$ itself, then we obtain a semi del Pezzo from the class in Example 3.14. Indeed we can see in Table 2 that Examples 3.12 and 3.14 have equal $b_3(Y)$ and $-K_Y^3$ and isometric polarising lattices. However, the nef cones and $\bar{c}_2(Z)$ are not identified.
by that lattice isometry, so these blocks will produce different extra-twisted connected sums (see Examples 3.18 and 3.19).

Similarly Examples 3.13 and 3.12, are both small resolutions of a singular intersection of two quadrics in $\mathbb{P}^5$, while Examples 3.13 and 3.14, are both small resolution of a singular del Pezzo 3-fold of degree 5.

3.3. Involution blocks from index 2 Fanos. We now wish to construct building blocks with involution, essentially by applying Construction 5.2 to Kähler 3-folds $Y$ that already admit an involution. Now suppose that $Y$ is Fano, and that the fixed set of the involution is exactly an anticanonical divisor $\Sigma$. Then by Lemma 5.2 in fact $X$ is semi-Fano too, with $-K_X$ even, and $Y$ is branched over an anticanonical divisor. Similarly, $Y$ is semi-Fano if and only if $X$ is. It is expedient for us to set up the construction starting from $X$.

**Construction 3.17.** Let $X$ be a simply-connected non-singular complex 3-fold with $-K_X$ even, and suppose there are smooth divisors $\Sigma \in \{-K_X\}$ and $H \in \{-\frac{1}{2}K_X\}$ with transverse intersection $C$.

Let $Y$ be the double cover of $X$ branched over $\Sigma$, and $Z$ the blow-up of $Y$ in $C$. Because $C$ is contained in the branch set of $Y$, we can lift the branch-switching involution $\tau$ on $Y$ to an involution on $Z$. The proper transform in $Z$ of $\Sigma$ is an anticanonical divisor. Note that $H^*(Y)^-\tau$ has trivial image in $H^*(\Sigma)$. In particular, $H^2(Y)$ and $H^2(X)$ have the same image $N$ in $H^2(\Sigma) = L$.

Under the conditions below, $(Z, \tau)$ is a building block with involution.

**Remark 3.18.** There are usually Fano deformations of $Y$ that are not double covers. Example 3.21 is one case where they are not.

**Proposition 3.19.** If $N \subset L$ is primitive and $H^3(X)$ is torsion-free then $(Z, \tau)$ is an involution block in the sense of Definition 2.6. The image in $H^{1,1}(\Sigma)$ of the $\tau$-invariant Kähler cone of $Z$ contains the image of the Kähler cone of $X$.

**Proof.** That $Z$ is a building block in the sense of Definition 2.1 follows from [6] Proposition 4.14, and the claim about Kähler cones is also analogous. The proper transform of $\Sigma$ is a fixed component of $\tau$, so $Z$ is a building block with involution in the sense of Definition 2.6. (The other fibre preserved by $\tau$ is the pre-image of $H$.)

If $Y$ is semi-Fano then $H^2(Y) \to L$ is injective. We already used in [6] Proposition 5.7 that this implies $K = 0$, the first of the conditions for the involution block to be pleasant. Crucially, it implies the second condition [14] too. Let $\rho := b_2(X) = \text{rk} \, N$.

**Proposition 3.20.** If, in addition to the hypotheses of Proposition 3.19, $H^2(Y) \to L$ is injective then so is $H^2(X) \to L$ (i.e. the building block $Z$ has $K = 0$), and

1. $b_3(Z) - 1 = b_2(V) = b_2(Y) = \rho$.
2. $b_3(Z) = b_1(C) + b_3(Y) = b_1(C) + 2b_3(X) + 22 - 2\rho$.
3. $b_3^+(Z) = b_3(C) + b_3(X)$.
4. $s = b_3^+(V)$.

In particular, $Z$ is pleasant.

**Proof.** Since $H^2(Y)$ and $H^2(X)$ have the same image in $L$, assuming $H^2(Y) \to L$ injective implies that $H^2(X) \cong H^2(Y)$.

Let $W := Y \setminus \Sigma$ and $U := X \setminus \Sigma$. Then

$$\chi(W) = \chi(Y) - 24 = 2\rho - b_3(Y) - 22,$$
$$\chi(U) = \chi(X) - 24 = 2\rho - b_3(X) - 22.$$

Therefore $\chi(W) = 2\chi(U)$ implies

$$b_3(Y) = 2b_3(X) + 22 - 2\rho.$$

$b_3^+(Z) = b_3(Z^0)$, where $Z^0$ is the singular quotient $Z/\tau$. Let $E \subset Z^0$ be the image of the exceptional divisor in $Z$, so that $Z^0 \setminus E \cong X \setminus C$. Comparing the long exact sequences of $X$ relative to $C$ and $Z^0$ relative to $E$ gives an exact sequence $0 \to H^3(X) \to H^3(Z^0) \to H^3(E) \to H^4(Z^0, E)$. The
kernel of the last map is free of rank equal to $b_3(E) = b_1(C)$, so $b_3(Z^0) = b_3(X) + b_1(C)$. Moreover, this shows $H^3(Z^0)$ to be free, so $Z$ is pleasant by Lemma 2.8.

To compute the Chern class data, it is convenient to use that $TY \oplus \pi^*(-K_X) \cong \pi^*(TX \oplus (-\frac{1}{2}K_X))$ implies $c_2(Y) = \pi^*(c_2(X))$. If we have already computed $c_2(X) + c_1(X)^2$ then we can use

$$c_2(Y) + c_1(X)^2 = \pi^*(c_2(X) + c_1(X)^2) - 3c_1(Y)^2 \quad (20)$$

to say that $\bar{c}_2(Y) = 2\bar{c}_2(X) - 3\nu(-K_Y) \in N^*.$

We now apply Construction 3.17 to the various index 2 Fano 3-folds and semi-Fano 3-folds that we have already considered in 2.1–3.2. We collect the data for the resulting pleasant involution blocks in Table 3 for convenience the table also includes a few blocks from 5. The table displays the key data discussed in 2.5 along with the Euler characteristic of the fixed curve $C \subset Z$ of the involution (corresponding to $-K_Y^3$ for semi-Fano type blocks). Note that all the blocks in the table could equally well be used as ordinary blocks if we choose to forget about the involution (but then there is some redundancy with Table 1).

**Example 3.21.** Perhaps the simplest example does not in fact use an index two Fano, but rather the unique one of index 4. Take $X = \mathbb{P}^3$, and let $Y$ be the double cover branched over a smooth quartic $\Sigma$. (In this case, all deformations of the Fano $Y$ are in fact branched double covers of $\mathbb{P}^3.)$

\[ \rho = 1 \text{ and } b_3(X) = 0, \text{ and } C \text{ is a degree 8 curve so has } b_1(C) = 18. \]

Hence $b_3(Z) = 38, \ b_3^+(Z) = 18.$

The Picard lattice of $Y$ is $N \cong \langle 4 \rangle$. Because $Y$ has index 2, $(-K_Y)^3 = 16$ and $\bar{c}_2(Z) = \frac{24+16}{2} = 20 \in N^* \cong \mathbb{Z}$ by (19). (Some of this simply recovers the data for Example 3.8 in Table 1). Note that the other preserved fibre of $\tau$ on $Z$ is a double cover of a quadric, branched over a bidegree $(4,4)$ curve in $\mathbb{P}^1 \times \mathbb{P}^1$, or equivalently a K3 with non-symplectic involution and Picard lattice $\langle 3,0 \rangle$ (cf. Example 5.14). So the other preserved fibre is more special than $\Sigma$.

**Example 3.22.** There are 5 families of Fano 3-folds $X$ of rank 1 and index 2, and the computation of the invariants of a double cover $Y$ branched over an anticanonical K3 divisor $\Sigma$, blow-up $Z$ in an an anticanonical curve $C \subset \Sigma$ follow the same pattern. We refer to the resulting building blocks as Example 3.22a, where $d = 1, \ldots, 5$ is the degree of $X$. Let us provide some varying amounts of additional detail in the 5 cases.

(i) $X$ is a smooth sextic hypersurface in $\mathbb{P}^4(3,2,1,1,1)$, such that the anticanonical section $\Sigma := \{X_1 = 0\}$ is smooth (where $X_1$ is the weight 2 coordinate). The double cover $Y$ of $X$ branched over $\Sigma$ is a sextic hypersurface in $\mathbb{P}^4(3,1,1,1,1)$; it is a double cover of $\mathbb{P}^3$ branched over a sextic surface.

Let $C \subset \Sigma$ be the intersection with a hyperplane (of weight 1, like $\{X_2 = 0\}$). $C$ is a double cover of $\mathbb{P}^1$ branched over 6 points, so has $b_1(C) = 4$. Let $Z$ be the blow-up of $Y$ at $C$. $\rho = 1$ and $b_3(X) = 42$, so

\[ b_3(Z) = 108, \ b_3^+(Z) = 46. \]

The Picard lattice of $Y$ is $N \cong \langle 2 \rangle$, and $\bar{c}_2(Z) = 26$ by (19).

The other fixed fibre is a double cover of a hyperplane section of $X$, which is a degree 1 del Pezzo surface; that fibre is therefore a K3 with non-symplectic involution and diagonal Picard lattice $\langle 2 \rangle \oplus \langle -2 \rangle^8$.

(ii) $X$ is a double cover of $\mathbb{P}^3$, as appeared in Example 3.21. In this case the branched double cover $Y$ of $X$ is isomorphic to a quartic 3-fold in $\mathbb{P}^4$. Note, however, that a generic quartic in $\mathbb{P}^4$ is not a double cover of $\mathbb{P}^3$, but only those in the form $X_0^4 + X_2^2Q_2(X_1, \ldots X_4) + Q_4(X_1, \ldots, X_4)$.

(iii) Let $X \subset \mathbb{P}^4$ be a smooth cubic (which has $b_3(X) = 10$) and $\Sigma \subset X$ smooth section by a quadric. The double cover $Y$ over $X$ branched over $\Sigma$ can be identified with the complete intersection of a cubic and a quadric in $\mathbb{P}^3$. Let $C$ be a hyperplane section of $\Sigma$ (a genus 4 curve), and $Z$ the blow-up of $Y$ in $C$. Then $b_3(Z) = 48, b_3^+(Z) = 18, N \cong \langle 6 \rangle$ and $\bar{c}_2(Z) = 30.$
Table 3. Examples of pleasant involution blocks

| Ex | $-\chi(C)$ | $b_3(Z)$ | $b_3^+(Z)$ | $N$ | $\bar{c}_2(Z)$ |
|----|------------|----------|-------------|-----|----------------|
| 3.21 | 16 | 38 | 18 | (4) | 20 |
| 3.22 | 2 | 108 | 46 | (2) | 26 |
| 3.22 | 4 | 66 | 26 | (4) | 28 |
| 3.22 | 6 | 48 | 18 | (6) | 30 |
| 3.22 | 8 | 38 | 14 | (8) | 32 |
| 3.22 | 10 | 32 | 12 | (10) | 34 |
| 3.23 | 12 | 32 | 14 | $\binom{2}{4}$ | (18 18) |
| 3.23 | 14 | 34 | 16 | $\binom{4}{4}$ | (20 18) |
| 3.24 | 12 | 30 | 14 | $\binom{0}{2} \binom{2}{2} \binom{2}{0} \binom{2}{2}$ | (12 12 12) |
| 3.25 | 0 | 104 | 44 | $\binom{2}{0} \binom{2}{0}$ | (26 24) |
| 3.25 | 2 | 62 | 24 | $\binom{4}{4}$ | (28 26) |
| 3.25 | 4 | 44 | 16 | $\binom{6}{6} \binom{6}{6}$ | (30 28) |
| 3.25 | 6 | 34 | 12 | $\binom{8}{8} \binom{8}{8}$ | (32 30) |
| 3.25 | 8 | 28 | 10 | $\binom{10}{10} \binom{10}{10} \binom{10}{10}$ | (34 32) |
| 3.26 | 4 | 24 | 6 | $\binom{4}{2} \binom{4}{0} \binom{4}{4}$ | (28 18) |
| 3.26 | 6 | 26 | 8 | $\binom{6}{6} \binom{6}{6}$ | (30 18) |
| 3.26 | 8 | 28 | 10 | $\binom{8}{8} \binom{8}{8}$ | (32 18) |
| 3.26 | 10 | 30 | 12 | $\binom{10}{6} \binom{6}{6} \binom{6}{0}$ | (34 18) |
| 3.27 | 2 | 38 | 12 | $\binom{4}{0} \binom{4}{0}$ | (26 22) |
| 3.27 | 4 | 36 | 12 | $\binom{4}{0} \binom{4}{0}$ | (28 12) |
| 3.27 | 6 | 34 | 12 | $\binom{4}{4} \binom{4}{0} \binom{4}{0}$ | (30 12) |
| 3.27 | 8 | 32 | 12 | $\binom{4}{4} \binom{4}{4}$ | (32 12) |
| 3.27 | 10 | 30 | 12 | $\binom{10}{4} \binom{4}{4}$ | (34 12) |
| 5.14 | 16 | 96 | 32 | $\binom{8}{2} \binom{8}{2} \binom{8}{8}$ | (12 12) |
| 5.15 | 18 | 108 | 36 | (2) | 18 |
| 5.15 | 16 | 96 | 32 | $\binom{2}{0} \binom{2}{0} \binom{2}{2}$ | (18 12) |
| 5.15 | 14 | 84 | 28 | $\binom{2}{2} \binom{2}{2} \binom{2}{2}$ | (18 12) |

(iv) Let $X \subset \mathbb{P}^5$ be a complete intersection of two quadrics, $\Sigma \subset X$ smooth section by another quadric. The double cover $Y$ of $X$ branched over $\Sigma$ embeds as a complete intersection of 3 quadrics in $\mathbb{P}^6$.

\[ b_3(X) = 4, \quad b_1(C) = 10, \quad b_4(Y) = 28, \quad \text{so} \quad b_3(Z) = 38 \quad \text{and} \quad b_3^+(Z) = 14. \quad N \cong (8), \quad \text{and} \quad \bar{c}_2(Z) = 32. \]

(v) $X$ is a section of the Grassmannian $Gr(2, 5) \subset \mathbb{P}^9$ by a codimension 3 plane.

Example 3.23. In the Mori-Mukai list of rank 2 Fano 3-folds, two entries are double covers of index 2 Fanos.

\[ k = 6 \quad \text{A branched double cover of a (1,1) divisor} \quad X \subset \mathbb{P}^2 \times \mathbb{P}^2 \quad (\text{cf. Example 3.9.2}) \]

\[ k = 8 \quad \text{A branched double cover of the blow-up of} \quad \mathbb{P}^3 \quad \text{in a point} \quad (\text{cf. Example 3.9.3}). \]
In both cases we can read off the topological data from [12, Table 3].

**Example 3.24.** Let \( X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \).

\[
N \cong \begin{pmatrix}
0 & 2 & 2 \\
2 & 0 & 2 \\
2 & 2 & 0
\end{pmatrix}
\]
\( b_3(Y) = 16, b_1(C) = 14, b_3(Z) = 30, \) and \( b_1^+(Z) = 14. \)

**Example 3.25.** Let \( Z \) be the building block obtained by applying Construction 3.17 to the blow-up of a degree \( d \) del Pezzo 3-fold of rank 1 (cf. Example 3.12). We work out \( b_3(Z) \) and \( b_1^+(Z) \) from \( b_3(X) = b_3(X') \) and \( b_1(C) = -K_Y^3 = 2d - 2. \) By (20), \( c_2(Y) + c_1(Y)^2 \) is represented by \((24 + 8d + 6d) - 3(2d - 2) = (24 + 2d 22 + 2d).\)

We refer to these involution blocks as Example 3.25.

**Example 3.26.** For \( 2 \leq d \leq 5 \), let \( Z \) be the building block resulting from applying Construction 3.17 to the conic-fibred semi del Pezzo 3-fold of degree \( d \) (cf. Example 3.13). (20) yields \( c_2(Z) = (26 + 2d 18) \).

\( b_3(Z) = 20 + 2d, b_1^+(Z) = 2 + 2d. \)

**Example 3.27.** For \( 1 \leq d \leq 5 \), let \( Z \) be the building block resulting from applying Construction 3.17 to the quadric-fibred semi del Pezzo 3-fold of degree \( d \) (cf. Example 3.14). (20) yields \( c_2(Z) = (24 + 2d 12) \).

\( b_3(Z) = 40 - 2d, \) while \( b_1^+(Z) = 12. \)

**3.4. An ad hoc block.** As we have seen, classes of semi-Fano 3-folds often come in sequences. Sometimes these will be part of a bigger sequence, where the borderline case fails to be semi-Fano, yet satisfies the hypotheses of Construction 3.2. However, not being able to apply Propositions 3.3 or 3.7 means it takes a bit more work to employ such blocks. We carry this out in one case that leads to an involution block with 2-elementary polarisation.

Extrapolating from the classes in Example 3.12 consisting of one-point blow-ups of rank 1 del Pezzo 3-folds of degree \( d = 2, \ldots, 5 \), leads us to consider \( X \) a rank 1 del Pezzo 3-fold of degree 1, i.e., a smooth sextic hypersurface in \( \mathbb{P}^4(3, 2, 1, 1, 1) \) (this is the family appearing in Example 3.22), and let \( X \) be the blow-up of \( X' \) at a point \( p \), say \( p = (0:0:0:0:1) \). Then \( X \) fails to be weak Fano—indeed, generically \(-K_X\) does not even have any irreducible sections: \( H^0(-K_X)\) is spanned by \( X_2, X_3X_2 \) and \( X_3^2 \).

We can however restrict attention to the case when \( X' \subset \mathbb{P}^4(3, 2, 1, 1, 1) \) is tangent to \( \{X_1 = 0\} \) at \( p \). Then the section \( \Sigma' := \{X_1 = 0\} \cap X' \) has a double point at \( p \); generically it is an ordinary double point, and the proper transform \( \Sigma \subset X \) is a smooth section of \(-K_X\). Now \(-K_X\) is spanned by \( X_3, X_2X_3 \) and \( X_2^3 \), and defines a morphism onto a quadric cone in \( \mathbb{P}^3 \) (mapping \( p \) to the vertex of the cone); it is defined everywhere because the conditions \( p \in X' \) and tangency with \( \{X_1 = 0\} \) at \( p \) imply that the defining polynomial of \( X' \) has no \( X_1^2 \) or \( X_0X_1X_2 \) coefficients, so that \( p \) is the only point on \( X' \) with \( X_1 = X_2 = X_3 = 0. \) (Geometrically, the morphism resolves the projection of \( X' \) onto \( \{X_0 = X_4 = 0\} \cong \mathbb{P}^2(2, 1, 1) \subset \mathbb{P}^4(3, 2, 1, 1, 1) \).

Since \(-K_X\) is evidently not big, even this non-generic blow-up fails to be weak Fano. We can nevertheless apply Construction 3.2 to construct a building block from \( X \), or Construction 3.17 from the double cover \( Y \) branched over \( \Sigma \).

But it takes more work since we now have to check some properties, which are automatic if \( Y \) is semi-Fano, by hand:
- \( H^2(Y) \to L \) is injective with image \( N \) primitive. Then the hypotheses of Propositions 3.19 and 3.20 hold, so that \( Z \) is a pleasant involution block.
- Any generic \( N \)-polarised K3 appears as an anticanonical divisors in some member of the family of blocks.

**Example 3.28.** Note that there exist sections of \( \mathcal{O}(-1) \) passing through \( p \) that meet \( X' \) transversely, defining smooth \( H' \subset |-\frac{1}{3}K_X|. \) The proper transform \( H \subset X \) of such a divisor is in \( |-\frac{1}{3}K_X| \). Let \( C \subset \Sigma \) be the intersection with such a section. It is a double cover of \( \mathbb{P}^2 \) branched over 4 points, so
C is an elliptic curve (and $b_1(C) = 2$). The nef cone of $X$ is spanned by $H$ and $\pi^*H' = H + E$, where $E$ is the exceptional $\mathbb{P}^2$.

Let $Y$ be the double cover of $X$ branched over $\Sigma$, and let $Z$ be the blow-up of $Y$ at $C$. The pre-image $\tilde{H} \subset Y$ of $H$ is a smooth anticanonical divisor. The pencil $|\tilde{H} : \Sigma| \subseteq |-K_Y|$ has base locus $C$, and yields an anticanonical fibration of $Z$.

$$\rho = 2 \text{ and } b_3(X) = 42,$$

so

$$b_3(Z) = 104, \quad b_3^+(Z) = 44.$$  

The Picard lattice of $Y$ is $N \cong \langle \frac{1}{2}, 0, -2 \rangle$ with respect to the basis $\{\tilde{H} + \tilde{E}, \tilde{E}\}$, where $\tilde{E}$ is the exceptional $\mathbb{P}^1 \times \mathbb{P}^1 \subset Y$. Meanwhile the Picard group of $\Sigma$ is generated by the hyperplane section and the exceptional $\mathbb{P}^1$. Thus we see directly that $H^2(Y) \to L$ is injective with primitive image.

The other fixed fibre $\tilde{H}$ has diagonal Picard lattice $\langle 2 \rangle \oplus \langle -2 \rangle^9$, since it is a branched double cover of $H$, which is a blow-up of a degree 1 del Pezzo $H'$ at a point. The non-genericity of the choice of blow-up point $p \in X'$ is reflected in the fact that $H$ is the result of blowing up $H'$ in the nodes of a sextic with 9 nodes rather than 9 generic points; $\tilde{H}$ is a K3 with non-symplectic involution whose fixed set is single elliptic curve (the proper transform of the nodal sextic) isomorphic to $C$.

In the basis for $N$ given by the edges $\tilde{H} + \tilde{E}, \tilde{H}$ of the nef cone

$$N = \begin{pmatrix} 2 & 2 \\ 2 & 0 \end{pmatrix}.$$  

Analogously to Example 3.25 we find that $\tilde{e}_2(Z) = (26 \quad 24)$ with respect to this basis.

**Example 3.29.** Without taking double cover, we get an ordinary block with $b_3(Z) = b_3(X) + (-K_X)^3 + 2 = 42 + 0 + 2 = 44$, and $\tilde{e}_2(Z) = (16 \quad 12)$.

(Now the blow-up curve is just a fibre of the morphism to the quadric cone—which is generically smooth as required.)

### 4. Genericity results

In [5] we will exhibit examples of extra-twisted connected sums using blocks constructed in [3]. To match pairs of blocks in the required way, we will apply Theorem 6.8. In some of the examples, this requires stronger genericity properties than that provided by Proposition 3.7. We therefore collect here the genericity results that will prove necessary for our selected examples.

Given a family of building blocks $\mathcal{Z}$ with polarising lattice $N$, the problem is basically to establish sufficient conditions for an overlattice $\Lambda \subset L$ of $N$ that ensure that any $\Sigma$ with $\text{Pic } K3 \cong \Lambda$ embeds as an anticanonical divisor in some element of $Z$. If the conclusion holds, then elements of $N \subset \Lambda$ are given some geometric meaning, e.g. if elements of $\mathcal{Z}$ are described in terms of some embedding into projective space, then there is an element $H \in N$ corresponding to the hyperplane class. The general strategy to reconstruct these embeddings into projective space from knowing that $\text{Pic } \Sigma \cong \Lambda$.

The first step is to recall that the positive cone of a complex K3 has a chamber structure, where walls are planes orthogonal to $(-2)$-classes in $\text{Pic } \Sigma$, and the chambers are possible nef cones. Thus for a marked K3 with $\text{Pic } \Sigma = \Lambda$ and $H \in \Lambda$ such that has $H^2 > 0$ and $H$ is orthogonal to all $(-2)$-classes in $\Lambda$, we can always choose a different marking (composing the original choice with reflections in $(-2)$-classes) to assume WLOG that $H$ is a nef class for the marked K3.

Once we have a nef class $H$, we can try to apply results of Saint-Donat [27] to prove that $H$ is very ample, i.e. that its sections define an embedding $\Sigma \hookrightarrow \mathbb{P}(H^0(H)) \cong \mathbb{P}^{n^2+1}$.

**Lemma 4.1** ([26] Chapter 3). Let $\Sigma$ be a K3 surface, and $H \in \text{Pic } \Sigma$ a nef class.

(a) If $H^2 \geq 4$, $H$ is not twice an element of square 2, and

(i) there is no $v \in \text{Pic } \Sigma$ such that $v.H = 2$ and $v^2 = 0$ then $|H|$ defines a birational morphism to $\mathbb{P}^{n^2+1}$, which is an isomorphism away from a set of contracted $(-2)$-curves. If in addition

(ii) there is no $v \in \text{Pic } \Sigma$ such that $v.H = 0$ and $v^2 = -2$ then $\Lambda$ is very ample.
(b) If $A^2 = 2$ and $[A]$ holds then $|A|$ defines a double cover of $\mathbb{P}^2$, branched over a sextic curve. (i) $v.H = 2$ and $v^2 = 0$, or
(ii) $v.H = 0$ and $v^2 = -2$.

Then for any $K3$ with Picard lattice exactly $\Lambda$, we can choose a marking such that the linear system $|H|$ defines a double cover $\Sigma \to \mathbb{P}^2$, branched over a smooth sextic curve.

In particular, the families of blocks from Examples 3.8 and 3.22 (essentially same as 3.8) are $(\Lambda, H^\vee \oplus)$-generic.

Proof. That $\Sigma$ is branched over a smooth sextic is a restatement of Lemma 4.1(b).

Now let $F$ be the polynomial defining the sextic curve. Then for a generic homogeneous quadric $Q$ and quartic $C$ in three variables, the sextic hypersurface

$$X := \{X_2^2 + X_1C(X_2, X_3, X_4) + X_1^2Q(X_2, X_3, X_4) + F(X_2, X_3, X_4 = 0) \subset \mathbb{P}^4(3, 2, 1, 1, 1)$$

is a smooth degree 1 del Pezzo 3-fold, with $\{X_1 = 0\} \cong \Sigma$ as anticanonical divisor. Blowing up a curve on $X$ yields an building block in the family of Example 3.8. Taking a double cover $Y$ of $X$ branched over $\Sigma$ and then blowing up yields an element of the family Example 3.22.

Thus a generic $A$-polarised $K3$ embeds as an anticanonical divisor in Examples 3.8 and 3.22 as required.

Lemma 4.3 ([12, Lemma 7.7]). Let $N \subset L$ be a primitive rank 2 lattice, with quadratic form represented wrt a basis $G, H$ by $(\frac{1}{2}, \frac{1}{2})$, let $\text{Amp} \subset N_\mathbb{R}$ be the open cone spanned by $G$ and $H$. Let $\Lambda \subset L$ be an overlattice of $N$, and suppose that

(i) there is no $v \in \Lambda$ such that $v.H = 1$ and $v^2 = 0$, and
(ii) there is no $v \in \Lambda$ other than $\pm (H - G)$ such that $v.H = 0$ and $v^2 = -2$.

Then for any $K3$ with Picard lattice exactly $\Lambda$, we can choose a marking such that the linear system $|H|$ defines a morphism $\Sigma \to \mathbb{P}^2$, contracting a $(-2)$-curve $E \subset \Sigma$ to a point $p \in \mathbb{P}^2$, which is 2-to-1 except over a sextic curve $C \subset \mathbb{P}^2$ that is smooth apart from an ordinary double point at $p$.

In particular, the family of building blocks from Example 3.28 is $(\Lambda, \text{Amp})$-generic.

Proof. The first part is immediate from Lemma 4.1(b).

Let $F$ be the sextic polynomial that defines the curve with ordinary double point at $p$. Then a generic sextic hypersurface of the form (21) is a smooth degree 1 del Pezzo 3-fold tangent to the hyperplane $\{X_1 = 0\}$ at $p$, so we can proceed to construct building blocks as in Examples 3.28 and 3.29.

4.2. Quartic K3s.

Lemma 4.4 ([12, Lemma 7.7]). Let $N \subset L$ be a primitive rank 2 lattice, with quadratic form represented wrt a basis $G, H$ by $(\frac{1}{2}, \frac{1}{2})$, let $\text{Amp} \subset N_\mathbb{R}$ be the open cone spanned by $G$ and $H$. Let $\Lambda \subset L$ be an overlattice of $N$, and suppose that there is no $v \in \Lambda$ such that

(i) $v.H = 2$ and $v^2 = 0$, or
(ii) $v.H = 0$ and $v^2 = -2$,
(iii) $v.H = 1$ and $v^2 \geq -2$.

Then for any $K3$ with Picard lattice exactly $\Lambda$, we can choose a marking such that the linear system $|H|$ defines an embedding $\Sigma \to \mathbb{P}^3$, whose image is a smooth quartic hypersurface and $2G - H$ is represented by a twisted cubic curve $C$.

In particular, the family of building blocks from Example 3.9 is $(\Lambda, \text{Amp})$-generic.
4.3. Sextic K3s.

Proposition 4.5. Let $\Lambda \subset L$ be a primitive lattice, with $H \in \Lambda$ such that $H^2 = 6$. Suppose that there is no $v \in \Lambda$ such that

(i) $v \cdot H = 2$ and $v^2 = 0$, or
(ii) $v \cdot H = 0$ and $v^2 = -2$.

Then for any $K3$ with Picard lattice exactly $\Lambda$, we can choose a marking such that the linear system $|H|$ defines an $\Sigma \to \mathbb{P}^4$, whose image is the intersection of a quadric (which may be singular) and a smooth cubic.

In particular, the families of blocks from Examples 3.8 and 3.22 (essentially same as 3.8) are $(\Lambda, H \mathbb{R}^+)$-generic.

Proof. Lemma 4.1 gives that $H$ is very ample. It is well known that the image is then a complete intersection of a quadric and a cubic, and that the cubic may be taken to be smooth. \hfill $\square$

Proposition 4.6. Let $N \subset L$ be a primitive rank 2 lattice, with quadratic form represented wrt a basis $H, \Gamma$ by $(\frac{1}{2}, \frac{1}{2})$. Let $\Lambda \subset L$ be an overlattice of $N$, and suppose that there is no $v \in \Lambda$ such that

(i) $v \cdot H = 2$ and $v^2 = 0$, or
(ii) $v \cdot H = 0$ and $v^2 = -2$.

Then for any $K3$ with Picard lattice exactly $\Lambda$, we can choose a marking such that the linear system $|H|$ defines an embedding $\Sigma \to \mathbb{P}^4$, whose image is the intersection of a quadric $Q$ and a cubic $C$, and contains a conic representing the class $\Gamma$.

The cubic $C$ can be chosen so that it contains the plane $\Pi$ of the conic, and so that it has no singularities other than 4 ordinary double points along $\Pi$.

Further if $\text{Amp}_+ \subset N_{\mathbb{R}}$ is the open cone spanned by $H$ with $H \pm \Gamma$, then Examples 3.12 and 3.25 are $(\Lambda, \text{Amp}_+)$-generic, and Examples 3.14 and 3.27 are $(\Lambda, \text{Amp}_-)$-generic.

Proof. Using (i) and (ii), Lemma 4.1 implies that the class $H$ is very ample, so $\Sigma$ embeds as a degree 6 surface in $\mathbb{P}^4$, which is known to be a complete intersection of a quadric $Q$ and a cubic $C$.

Since the $(-2)$-class $\Gamma$ has positive intersection with $H$ it is effective. (iii) implies that $\Gamma$ is irreducible, so represented by a smooth rational curve. The image in $\mathbb{P}^4$ is a smooth rational curve of degree 2, so a conic as required.

Recall from Remark 3.16 that the semi-Fano 3-folds in Examples 3.12 and 3.14 (whose double covers are used in Examples 3.25 and 3.27) are small resolutions of a cubic containing a plane. Let us therefore consider the unique plane $\Pi \subset \mathbb{P}^4$ that contains the conic $\Gamma$.

As a variety in $\Pi$, $C$ is defined by the vanishing of $q := Q|_\Pi$. Since $C \cap \Pi$ contains $\Gamma$, we can write $C|_\Pi = q\ell$ for a line $\ell$ on $\Pi$. If we take $L$ to be any hyperplane in $\mathbb{P}^4$ intersecting $\Pi$ in $\ell$, then by replacing $C$ with $C - LQ$ we can assume without loss of generality that $C$ contains $\Pi$ as well as $\Sigma$.

Without loss of generality, $\Pi = \{x_0 = x_1 = 0\}$. We obtain a 3-dimensional space of cubic polynomials of the form $(a_0x_0 + a_1x_1)Q + a_2C$ with base locus exactly $\Sigma \cup \Pi$. By Bertini’s theorem a generic element of this linear system is smooth away from the base locus. On the other hand, it must also be smooth along the smooth Cartier divisor $\Sigma$, so any singularities must lie on $\Pi$.

If we write $C = x_0R_0 + x_1R_1$ for some quadrics $R_0, R_1$, then the singularities of $(a_0x_0 + a_1x_1)Q + a_2C = x_0(a_0Q + a_2R_0) + x_1(a_1Q + a_2R_1)$ in $\Pi$ correspond to the intersection points of $a_0q + a_2r_0$ and $a_1Q + a_2r_1$, where $r_i := R_i|_\Pi$. The smoothness of $Q \cap C$ implies that $r_0, r_1$ and $q$ have no common zeros, i.e. the linear system that they span is basepoint-free. Therefore for generic $a_0, a_1, a_2$, the quadrics $a_0q + a_2r_0$ and $a_1Q + a_2r_1$ intersect transversely in 4 points, and $(a_0x_0 + a_1x_1)Q + a_2C$ is smooth except for ordinary double points at those 4 points.

Blowing up $C$ in $\Pi$—or equivalently blowing up $\mathbb{P}^4$ in $\Pi$ and taking the proper transform of $C$—gives a semi-Fano del Pezzo $Y_\Sigma$ of the class from Example 3.14, with $\Sigma$ as an anticanonical divisor. The nef cone of the blow-up of $\mathbb{P}^4$ is spanned by $H_\Sigma$ and $H_\Sigma - E_\Sigma$, where $H_\Sigma$ is the pull-back of the hyperplane class and $E_\Sigma$ is the exceptional divisor. The restriction to $\Sigma$ corresponds
to Amp_., so Example 3.14 is $(\Lambda, \text{Amp}_.)$-generic. Since Examples 3.14 and 3.27 have the same anticanonical divisors, Example 3.27 is $(\Lambda, \text{Amp}_.)$-generic too.

Finally, consider the intersection of C with a generic hyperplane that contains $\Pi$. This intersection will be the union of $\Pi$ and a smooth quadric surface $S$ that passes through the singularities of $C$. Blowing up $C$ in $S$ yields another semi del Pezzo $Y_+$, which belongs to the class from Example 3.12. If $E_+$ is the exceptional divisor of the corresponding blow-up of $P^4$, then the nef cone is generated by $H_+$ and $2H_+ - E_+$. The restriction of $E_+$ to $\Sigma$ is $H - \Gamma$, so the image of the nef cone of $Y_+$ in $H^2(\Sigma; \mathbb{R})$ is spanned by $H$ and $2H - (H - \Gamma) = H + \Gamma$. Thus Example 3.12 is $(\Lambda, \text{Amp}_.)$-generic, as is Example 3.25.

Lemma 4.7 ([12, Lemma 7.7]). Let $N \subset L$ be a primitive rank 2 lattice, with quadratic form represented wrt a basis $G, H$ by $(\frac{1}{2} I_0)$, let $\text{Amp} \subset \mathbb{R}$ be the open cone spanned by $G$ and $H$. Let $\Lambda \subset L$ be an overlattice of $N$, and suppose that there is no $v \in \Lambda$ such that

(i) $v.H = 2$ or $3$ and $v^2 = 0$, or
(ii) $v.H = 0$ and $v^2 = -2$, or
(iii) $v.H = 1$ or $2$, and $v^2 \geq -2$.

Then for any $K3$ with Picard lattice exactly $\Lambda$, we can choose a marking such that the linear system $|H|$ defines an embedding $\Sigma \rightarrow \mathbb{P}^4$, whose image is contained in a smooth quadric 3-fold, and $2H - G$ is represented by an elliptic curve of degree 5.

In particular, the family of building blocks from Example 3.9 is $(\Lambda, \text{Amp})$-generic.

5. Building blocks from $K3$s with non-symplectic involution

Since involution blocks always contain a $K3$ fibre with non-symplectic involution, it is natural to consider the construction of Kovalev and Lee [21] of building blocks starting from $K3$s with non-symplectic involution. We find that these do indeed also lead to building blocks with involution. Moreover, by modifying their construction we can also find some pleasant building blocks with involution.

5.1. $K3$s with non-symplectic involution. Let $\Sigma$ be a $K3$ surface with a non-symplectic involution, i.e. a holomorphic involution $\tau$ which acts as $-1$ on $H^2(\Sigma; \mathbb{Z})$. Such involutions are classified by Nikulin in terms of the fixed part $N$ of $H^2(\Sigma, \mathbb{Z})$ under the action of $\tau$. The discriminant group of $N$ is $2$-elementary, i.e. $N^*/N$ is of the form $\mathbb{Z}_2^k$. The discriminant form of $N$ is the symmetric $\mathbb{Q}/\mathbb{Z}$-valued form $b$ on $N^*/N$ induced by the form on $N$; because $N^*/N$ is $2$-elementary, $b$ takes values in $\frac{1}{2} \mathbb{Z}/\mathbb{Z}$. (It also has a $\frac{1}{2} \mathbb{Z}/\mathbb{Z}$-valued quadratic refinement, which is unimportant to us.)

The primitive lattice $N$, and hence the deformation family of $(\Sigma, \tau)$, is characterised by the rank $r$, the discriminant rank $a$, and a further invariant $\delta \in \{0, 1\}$ defined by

$$\delta := \begin{cases} 0 & \text{if } b(a, a) = 0 \text{ for all } a \in N^*/N, \\ 1 & \text{otherwise.} \end{cases}$$

The quotient $Y = \Sigma/\tau$ is a smooth complex surface, which is rational when the fixed set $C$ of $\tau$ is non-empty (by Castelnuovo’s theorem [2]; if $C$ is empty then $Y$ is an Enriques surface, but this case is of no further interest to us). $\Sigma$ is a double cover of $Y$, branched over a smooth reduced divisor $C \in \{-2K_Y\}$, and $\tau$ corresponds to the branch-switching involution. With a few exceptions, $C$ has $k + 1$ components, where one has genus $g$ and the other $k$ are $P^1$s, for

$$k = \frac{r - a}{2}, \quad g = \frac{22 - r - a}{2}.$$ 

The pull-back of the quotient map gives an inclusion $H^2(Y) \rightarrow H^2(\Sigma)$, but its image $N'$ is not in general primitive, but a finite index sublattice of $N$. Note that since the quotient map has degree $2$, the intersection form on $N'$ is exactly twice the unimodular form on $H^2(Y)$. Its discriminant group is therefore $\mathbb{Z}_2^k$. Since $N$ is an overlattice with discriminant group $\mathbb{Z}_2^k$, the index must $2^k$. (Can also see this from the long exact sequence [14].)

The quotient $N/N' \cong \mathbb{Z}_2^k$ is generated by the Poincaré duals of the $k + 1$ components $C_i$ of the fixed set of $\tau$; the sum of these classes is contained in $N'$ (as it is the image of $-K_Y \in H^2(Y)$), but (when $k > 0$) the individual classes are not.
Lemma 5.1. Let $P \in N$ be the Poincaré dual of the fixed set $C$; equivalently, $P := \pi^*(-K_Y) \in N' \subseteq N$. Then

(i) $P.x = x^2 \mod 4$ for any $x \in N'$

(ii) $\alpha(P) = 2b(\alpha, \alpha) \mod 2$ for all $\alpha \in N^*$, where $b$ is the discriminant form. In particular, $P$ has even product with all elements of $N$, and $P$ is an even element of $N$ if and only if $\delta = 0$.

Proof. (i) By Wu’s theorem, $-K_Y = c_1(Y) = w_2(Y) \in H^2(Y)$ is characteristic for the intersection form, i.e.

$$-K_Y.x = x^2 \mod 2$$

for any $x \in H^2(Y)$. Hence for any $\pi^*x \in N'$,

$$P.\pi^*x = -2K_Y.x = 2x^2 = (\pi^*x)^2 \mod 4.$$  

(ii) Any $\alpha \in (N')^*$, and hence also any $\alpha \in N^* \subseteq (N')^*$, can be written as $\frac{1}{2}b(y)$, for some $y = \pi^*x \in N'$, where $b : N \to N^*$ is induced by the intersection form. Then

$$\alpha(P) = \frac{1}{2}y.P = \frac{1}{2}y^2 \mod 2,$$

while by definition of the discriminant form,

$$b(\alpha, \alpha) = (\frac{1}{2}y)^2 = \frac{1}{2}y^2 \in \mathbb{Q}/\mathbb{Z}.$$

Let us now make some remarks on Picard lattices and ample cones, needed later in the context of genericity of families of building blocks in the technical sense of Definition 2.12. For any K3 surface $\Sigma$ with non-symplectic involution, the fixed set $N \subset H^2(\Sigma)$ is contained in Pic $\Sigma$. The intersection of the ample cone of $\Sigma$ with $N\mathbb{R}$ is simply the image of the ample cone of $Y := \Sigma/\tau$.

Lemma 5.2. Let $\Sigma \to Y$ be a branched double cover. A class $k \in \text{Pic} Y$ is ample if and only if its image $\pi^*k \in \text{Pic} \Sigma$ is ample.

In particular, the orthogonal complement of $N$ is Pic $\Sigma$ cannot contain any $(−2)$-classes. Conversely

Proposition 5.3. For any K3 surface $\Sigma$ such that Pic $\Sigma$ contains a primitive 2-elementary sublattice $N$, and the orthogonal complement of $N$ in Pic $\Sigma$ contains no $(−2)$-classes, there exists a non-symplectic involution on $\Sigma$ with fixed lattice $N$.

As one deforms $\Sigma$ and $Y$, the ample cone of $Y$ can jump due to the appearance of exceptional curves, e.g. the Hirzebruch surface $\mathbb{F}_2$, which has a $(−2)$-curve, can be deformed to $\mathbb{P}^1 \times \mathbb{P}^1$, which does not. This corresponds to the appearance of a $(−2)$-class in Pic $\Sigma \setminus N$ (since the orthogonal complement of $N$ in Pic $\Sigma$ a priori cannot contain any $(−2)$-classes, such a class must be a half the sum of two classes of square $−4$, contained in $N$ and its orthogonal complement in Pic $\Sigma$ respectively). We call a K3 surface with involution degenerate if Pic $\Sigma \setminus N$ contains a $(−2)$-class. In the moduli space of K3 surfaces with involution with a fixed $N$, the non-degenerate ones form a connected moduli space, with essentially constant ample cone.

Lemma 5.4 (cf. Nikulin-Saito [24 page 5 (P)]). Let $N \subset L$ be a primitive 2-elementary lattice. Then there exists an open cone $\text{Amp}_N \subset N\mathbb{R}$ such that for any non-degenerate K3 surface with non-symplectic involution $(\Sigma, \tau)$ and a marking $H^2(\Sigma) \to L$ mapping the fixed set of $\tau$ to $N$, such that the intersection of the image of the ample cone of $\Sigma$ with $N\mathbb{R}$ equals $\text{Amp}_N$.

If $Y$ is a del Pezzo surface, then $N \subset L$ is a totally even primitive sublattice of rank $≤ 9$. Because $Y$ does not contain any $(−2)$-curves, $(\Sigma, \tau)$ must be non-degenerate. The converse also holds.

Lemma 5.5. Let $(\Sigma, \tau)$ be a K3 surface with non-symplectic involution. Then the quotient $\Sigma/\tau$ is a del Pezzo surface if and only if $(\Sigma, \tau)$ is non-degenerate and $N$ is totally even of rank $≤ 9$.

Proof. The intersection forms of del Pezzo surfaces are precisely the unimodular lattices of rank $≤ 9$. For a del Pezzo surface $Y$, a smooth section of $−2K_Y$ is connected, so the resulting K3 surface with involution has $N = N'$ totally even.

Conversely, if $(\Sigma, \tau)$ is non-degenerate with $N$ totally even of rank $r ≤ 9$, then $P = \pi^*(-K_Y)$ has a smooth connected section, and $P^2 = 20 - 2r ≥ 2$, so $P$ is nef.
If we set \( H = 3P \), then condition (i) of Lemma 4.1 certainly holds. By Lemma 4.3, there can be no \((-2)\)-classes in \( N \) that are orthogonal to \( P \). The non-degeneracy condition means that there are no other \((-2)\)-classes in \( \text{Pic} \Sigma \), so condition (ii) holds too. Hence Lemma 4.1 shows that \( H = 3P \) is very ample. By Lemma 5.2, \( -K_Y \) must therefore be ample. \( \square \)

5.2. Kovalev-Lee blocks. Let \( \Sigma \) be a K3 with non-symplectic involution \( \tau \), and let \( \psi : \mathbb{P}^1 \to \mathbb{P}^1 \) be the holomorphic involution \( \psi : (x : y) \mapsto (y : x) \). Kovalev and Lee \([22, \S 4]\) use the following complex 3-folds \( Z \) as blocks in the twisted connected sum construction. The quotient \( Z_0 = \mathbb{P}^1 \times \mathbb{P}^1 \) by \( \tau \times \psi \) has orbifold singularities along the \( 2k + 2 \) components of \( C \times \{ (1 : 1), (1 : -1) \} \).

**Construction 5.6.** Let \( Z \) be the blow-up of \( Z^0 \) along its singular locus.

Kovalev and Lee computed the rational cohomology of 3-folds. By computing integral cohomology, we find that \( Z \) are indeed building blocks also in the sense of Definition 2.1. Moreover, if we let \( \sigma : \mathbb{P}^1 \to \mathbb{P}^1 \) be the involution \( (x : y) \mapsto (x : -y) \), which commutes with \( \psi \), then \( \Delta_2 \times \sigma \) induces an involution on \( Z \), making it a building block with involution in the sense of Definition 2.6.

**Proposition 5.7.** Let \( \Sigma \) be a K3 surface with non-symplectic involution \( \tau \), and non-empty fixed set \( C \). Then

\[
\begin{align*}
b_2(Z) &= r + 2k + 3 = 2r - a + 3, \\
b_3(Z) &= 4g = 44 - 2r - 2a, \\
\text{rk } K &= 2k + 2 = 2 + r - a.
\end{align*}
\]

Further \( H^3(Z) \) is torsion-free, and the image of \( H^2(Z) \to H^2(\Sigma) \) is the fixed lattice \( N \) of \( \tau \) (which is primitive). In particular, \( Z \) is a building block in the sense of Definition 2.1.

**Proof.** The Betti numbers were computed in \([22, \text{Proposition 4.3, and (4.3)}]\).

\( Z_0 \) can be viewed as the result of gluing two copies of \( U_0 = (\Sigma \times \Delta)/\langle \tau, -1 \rangle \), along their common boundary which is the mapping torus \( T \). \( \pi_1(T) \cong T \), and by a Mayer-Vietoris sequence

\[
H^2(T) \cong \ker(1 - \tau^*) = N, \quad H^3(T) \cong \text{coker}(1 - \tau^*) \cong N^* \times \mathbb{Z}^2.3.\]

The restriction map \( H^2(T) \to H^2(\Sigma) \) for the slices \( \Sigma \subset T \) is the natural inclusion \( N' \hookrightarrow N \).

\( U_0 \) deformation retracts to the simply-connected rational surface \( Y = \Sigma/\tau \). The restriction map \( H^2(U_0) \to H^2(T) \) corresponds to the inclusion \( N' \hookrightarrow N \).

Let \( U \) be the blow-up of \( U_0 \) at its singular locus. Comparing the long exact sequences of \( U \) and \( U_0 \) relative to neighbourhoods of the exceptional divisor \( E \) and singular set \( C \), respectively, shows that the difference between \( H^*(U) \) and \( H^*(U_0) \) is the same as the difference between \( H^*(E) \cong H^*(C) \otimes H^*(\mathbb{P}^1) \) and \( H^*(\Sigma) \), i.e.

\[
H^2(U) \cong N' \times \mathbb{Z}^{k+1}, \quad H^3(U) \cong \mathbb{Z}^2.3.\]

However, the added factors are not simply generated by duals of cycles in the exceptional set, so it does not follow that \( H^*(U) \) and \( H^*(U_0) \) have the same image in \( H^*(T) \) (though this is the case with real coefficients). For example, for a component \( C_i \) of \( C \), consider the proper transform in \( U \) of the image of \( C_i \times \Delta \) in \( U_0 \), and let \( c_i \in H^2(U) \) be the class it represents. Then the image of \( c_i \) in \( H^2(T) \cong N \subset H^2(\Sigma) \) corresponds to the dual of \( C_i \) in \( H^2(\Sigma) \), which is precisely one of the generators for \( N/N' \) described before. So \( H^2(U) \to H^2(T) \) is surjective. The class in \( H^2(U) \) represented by the exceptional set over \( C_i \) is \( 2c_i \mod \text{image of } H^2(U_0) \in H^2(U) \).

Now Mayer-Vietoris for \( Z \) as a union of two copies of \( U \) shows that

\[
H^2(Z) \cong \mathbb{Z} \times \mathbb{Z}^{2k+2} \times N, \quad H^3(Z) \cong \mathbb{Z}^4.3.\]

So the cohomology is torsion-free, the image of \( H^2(Z) \to H^2(\Sigma) \) is the primitive sublattice \( N \), \( \text{rk } K = 2k + 2 = 2 + r - a \), and \( b_3(Z) = 4g = 44 - 2r - 2a \). \( \square \)

**Proposition 5.8.** Fix a primitive 2-elementary lattice \( N \subset L \), and let \( Z \) be the set of building blocks obtained by applying Construction 5.7 to K3s with non-symplectic involution with fixed lattice \( N \). Then there exists an open cone \( \text{Amp} \subset N_{\mathbb{R}} \) such that if \( \Lambda \subset L \) is primitive sublattice that contains \( N \) and \( \Lambda \setminus N \) does not contain any \((-2)\)-classes, then \( Z \) is \((\Lambda, \text{Amp})\)-generic.
5.3. Smoothing. Let $\Sigma$ be a K3 surface with non-symplectic involution $\tau$, and $Z \doteq \Sigma \times \mathbb{P}^1/\tau \times \psi$ as above. Instead of desingularising $Z$ by blowing up each component of the singular set, we can attempt to smooth those components that have positive genus while blowing up the $\mathbb{P}^1$'s. Further, we can carry out the smoothing in such a way that the involution $\text{Id} \times \sigma$ on $\Sigma$ persists, yielding a building block with involution.

For simplicity, we consider only the cases when the the fixed curve $C \subset \Sigma$ of $\tau$ has no $\mathbb{P}^1$ components. Moreover, we ignore the cases where $C$ consists of elliptic curves ($a = 10$ and $r = 8$ or 10). That leaves precisely the 10 cases where $Y$ is a del Pezzo surface, one each for $a = r \in \{1, 3, 4, \ldots, 9\}$, and two with $a = r = 2$.

We can regard $Z$ as the double cover of $Y \times \mathbb{P}^1$ branched over the zero set of the reducible section $(x^2 + y^2)s$ of $O_{\mathbb{P}^1}(2) - 2K_Y$, where $s$ is a section of $-2K_Y$ cutting out $C$. The normal crossing singularities of the divisor correspond precisely to the orbifold singularities of $Z$.

Considering instead a double cover of $Y \times \mathbb{P}^1$ branched over a smooth divisor in $|O_{\mathbb{P}^1}(2) - 2K_Y|$ we obtain a smoothing of $Z$, which is moreover a building block in the sense of Definition 2.1. It is convenient to consider the following concrete realisation of the double cover.

Construction 5.9. Let $Y$ be a del Pezzo surface, and $z \in \mathbb{P}^1$. Let $f$ be a section of the line bundle $O_{\mathbb{P}^1}(2) - 2K_Y$ over $Y \times \mathbb{P}^1$, such that both its zero locus $D$ and $C \doteq D \cap Y \times \{z\}$ are smooth. Thinking of $f$ as a homogenous quadratic polynomial on $\mathbb{C}^2$, taking values in sections of $-2K_Y$, we can define a smooth subvariety $Z$ of the total space $G$ of the projectivisation of $-K_Y \oplus \mathbb{C}^2 \to Y$ by

$$Z \doteq \{(\alpha : \beta : \gamma) \in G : \alpha^2 = f(\beta, \gamma)\}. \quad (22)$$

The projection map $p : G \to \mathbb{P}^1$, $\alpha : \beta : \gamma \mapsto (\beta : \gamma)$ is defined away from the section $\beta = \gamma = 0$, and hence in particular on $Z$. If $p : G \to Y$ is the bundle projection map, then the restriction $\pi \times p : Z \to Y \times \mathbb{P}^1$ realises $Z$ as the double cover branched over $D$. The fibre

$$\Sigma \doteq p^{-1}(z)$$

is a double cover of $Y$ branched over $C \subset |2K_Y|$, so is a K3 surface with non-symplectic involution.

Proposition 5.10. $(Z, \Sigma)$ is a building block, with the subset invariant under the action of the branch-switching involution of $\Sigma \to Y$. Further $K = 0$.

Proof. The canonical bundle of $G$ is $\pi^*(K_Y - \det(-K_Y \oplus \mathbb{C}^2)) + 3T = 2\pi^*(K_Y) + 3T$, where $T$ is the tautological bundle of $\mathbb{P}^1$. $Z$ is defined by a degree 2 homogeneous polynomial taking values in $-2\pi^*K_Y$, i.e. it is cut out by a section of $-2T - 2\pi^*K_Y$. Therefore its canonical bundle is $T|Z$; this equals the pull-back of the tautological bundle of $\mathbb{P}^1$ by $p : Z \to G$, so the fibres of $p$ are anticanonical divisors. (Each of the fibres is a double cover of $Y$ branched over a divisor in the linear system $\text{Im} f \subseteq \Sigma \subset |2K_Y|$, so they are deformations of $\Sigma$ with non-symplectic involution.)

The fact that $-2K_Y$ is very ample on the del Pezzo surface $Y$ implies that the sections of $-2T - 2\pi^*K_Y$ define a morphism $G \to \mathcal{P}(H^0(-2T - 2\pi^*K_Y)^*)$, and it is easy to see that the only set contracted by this morphism is the section $\{\beta = \gamma = 0\} \subset G$. In particular the morphism is semi-small, and the “relative Lefschetz theorem with large fibres” of Goresky-MacPherson [10, Theorem 1.1, page 150] implies $H^3(Z)$ torsion-free, and $H^2(Z) \cong H^2(G) \cong H^2(Y) \oplus H^2(\mathbb{P}^1)$. Since $\alpha = r$ implies that $H^2(Y) \to H^2(\Sigma)$ has image $N$, the image of $H^2(Z) \to H^2(\Sigma)$ is also precisely $N$. So $Z$ is a building block, with $K = 0$. 

Note that since $\pi^* : H^2(Y \times \mathbb{P}^1) \to H^2(Z)$ is an isomorphism, $\pi^* : H^4(Y \times \mathbb{P}^1) \to H^4(Z)$ must have image exactly $2H^4(Z)$. Let $h \in H^4(Z)$ be half the image of the generator of $H^4(Y)$.

Lemma 5.11. $b_3(Z) = 12(10 - r)$, and $c_4(Z) = 24h + 3\pi^*K_Y$.

Proof. As a complex bundle, $TG = T_{vert\ G} \oplus \pi^*TY$ is stably isomorphic to $T^{-1} \oplus \pi^*(-K_Y \oplus \mathbb{C}^2) \oplus TY$. Using that $\pi^*(-K_Y)^2 = (20 - 2r)h \in H^4(Y)$ and $T^2|Z = 0$ we find

$$c(Z) = \frac{\pi^*c(Y)(1 - T)^2(1 - T - \pi^*K_Y)}{1 - 2T - 2\pi^*K_Y} = 1 - T + (3T\pi^*K_Y + 24h) + (116 - 14r)Th \in H^*(Z).$$
This gives the claimed value of $c_2(Z)$, and also shows $\chi(Z) = -116 + 14r$. This we can determine $b_3(Z)$, since we know the other Betti numbers:

$$\chi(Z) = 2 + 2(1 + r) - b_3(Z).$$

Alternatively, we can compute $\chi(Z)$ from

$$\chi(Z) = 2\chi(Y \times \mathbb{P}^1) - \chi(D).$$

In turn, we can understand $\chi(D)$ by considering the projection $D \to Y$. Generically, the linear system $\text{Im} f \subseteq [-2K_Y]$ is base-point free, so that the projection does not contract any curves. Then the projection is a double cover, whose branch locus $B \subset Y$ is cut out by the discriminant of $f$, which is a section of $-4K_Y$. By adjunction, $K_B = 3K_Y|_B$, so $\chi(B) = (3K_Y)(-4K_Y) = -12(10 - r)$, and

$$\chi(D) = 2\chi(Y) - \chi(B).$$

Hence

$$\chi(Z) = 2\chi(Y) - 12(10 - r),$$

giving the same result as above. \(\square\)

By considering more special smoothings of $Z_0$ we obtain building blocks with involution. The subset of the space of sections of $\mathcal{O}_{\mathbb{P}^1}(2) - 2K_Y$ that is invariant under the action of $\text{Id} \times \sigma$, consists of elements of the form $(x^2 + y^2)s + (x^2 - y^2)s'$, for $s, s' \in \text{sections of } -2K_Y$. This linear system is base-point free, so a general element is smooth.

**Construction 5.12.** Let $Y$ be a del Pezzo surface, and $\infty \in \mathbb{P}^1$ a fixed point of $\sigma$. Let $f$ be a section of $\mathcal{O}_{\mathbb{P}^1}(2) - 2K_Y$ that is invariant under $\text{Id} \times \sigma$, such that both its zero locus $D$ and $C := D \cap Y \times \{\infty\}$ are smooth. Define $G$, $Z$ and $\Sigma$ as in Construction 5.9. Define an involution $\tau : Z \to Z$ as the restriction of the involution $(\alpha : \beta : \gamma) \mapsto (\alpha : -\beta : \gamma)$. Then $\tau$ fixes $\Sigma := p^{-1}(\infty)$, and acts as a non-symplectic involution on $\Sigma' := p^{-1}(0)$. (If we instead lifted $\text{Id} \times \sigma$ to $Z$ as $(\alpha : \beta : \gamma) \mapsto (\alpha : \beta : -\gamma)$, then the lift would fix $\Sigma'$ and map $\Sigma$ to itself by a non-symplectic involution.)

**Proposition 5.13.** $(Z, \Sigma)$ is a pleasant involution block.

*Proof.* We already know from Proposition 6.10 that $K = 0$. Since $C$ is connected, to apply Lemma 2.8 it remains only to check that $H^1(Z^0)$ is torsion-free for $Z^0 := Z/\tau$.

Now observe that the branched double cover $G \to G$, $(\alpha : \beta : \gamma) \mapsto (\alpha : \beta^2 : \gamma^2)$ induced an embedding $Z^0 \hookrightarrow G$. If $f = (x^2 - y^2)s + (x^2 + y^2)s'$, then the image of $Z^0$ in $G$ is

$$\{(\alpha : \beta : \gamma) \in G : \alpha^2 = \beta((\beta - \gamma)s + (\beta + \gamma)s')\}.$$

So $Z^0$ is cut out by a section of the line bundle $-2T - 2\pi^*K_Y$, which we argued to be semi-ample in the proof of Proposition 5.10. While $Z^0$ is singular along the curve $\alpha = \beta = s' = 0$, that is no obstacle to applying Goresky-MacPherson’s Lefschetz theorem with large fibres as in Proposition 5.10 to deduce that $H^3(Z^0)$ is torsion-free. \(\square\)

Applying (15), with $\rho = r$ and $\chi(C) = 2r - 20$, we obtain

$$b_3^+(Z) = \frac{1}{2}(120 - 12r - 20 + 2r + 2r - 20) = 40 - 4r. \quad (23)$$

**Example 5.14.** Consider the del Pezzo $Y = \mathbb{P}^1 \times \mathbb{P}^1$, and a double cover $\Sigma$ branched over a bidegree $(2, 2)$ divisor $C$. The intersection form on the invariant lattice $N \subset H^2(\Sigma)$ is twice that of $H^2(\mathbb{P}^1 \times \mathbb{P}^1)$, i.e. in the obvious basis given by the pull-backs of the generators of $H^2$ of the two $\mathbb{P}^1$ factors,

$$N \cong \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}. $$

These basis vectors also span the nef cone. In terms of this basis, $-\pi^*K_Y = (\frac{3}{2})$, and the image $c_2(Z) \in N^*$ of $-3\pi^*K_Y$ is $(12 12)$.

**Example 5.15.** For $r \in \{1, \ldots, 9\}$, consider the blow-up $Y$ of $\mathbb{P}^2$ in $r - 1$ points in general position.
Remark 5.16. There are two non-symplectic involutions with \( r = a = 10 \), which correspond to \( Y \) being an Enriques surface (which is of no interest to use, since the involution has no fixed points), and \( \mathbb{P}^2 \) blown up in 9 points that are the nodes of a nodal sextic curve. In the latter case, \([-2K_Y]\) is a pencil spanned by the proper transform of the given sextic (which is an elliptic curve) and the square of the unique cubic passing through them. A double cover branched over a generic section of \([-2K_Y]\) therefore gives a K3 with non-symplectic involution whose fixed set is an elliptic curve. We can construct a complex 3-fold \( Z \) as a double cover of \( Y \times \mathbb{P}^1 \) branched over a smooth divisor \( D \in |O_{\mathbb{P}^1} - 2K_Y| \) as above. However, because \(-K_Y\) is not ample, we cannot apply the Lefschetz hyperplane theorem to prove that \( H^3(Z) \) is torsion-free, and moreover, considering \( D \) as a branched double cover of \( Y \) shows that the conclusions of the Lefschetz theorem are in fact false.

Proposition 5.17. Let \( N \subset L \) be a primitive sublattice, isometric to twice the intersection lattice of a del Pezzo surface \( Y \). Let \( \text{Amp} \subset \text{N}_\mathbb{R} \) be the subcone corresponding to the ample cone of \( Y \), and let \( Z \) be the set of building blocks obtained by applying Construction 5.12 to the deformation family of \( Y \). Then \( Z \) is \((\Lambda, \text{Amp})\)-generic for any primitive sublattice \( \Lambda \subset L \) that contains \( N \) such that \( \Lambda \setminus N \) does not contain any \((-2)\)-classes.

Proof. Combine Proposition 5.3 and Lemmas 5.2 and 5.4. \( \square \)

6. The matching problem

To use the extra-twisted connected sum construction to produce closed \( G_2 \)-manifolds it not enough to produce some examples of ACyl Calabi-Yau 3-folds \( V_\pm \)—possibly with involutions—as in \([1, 1, 1]\) and pick a compatible torus isometry \( t \) as in \([1, 3]\) since we also need the asymptotic K3s of \( V_\pm \) to be related by a \( \vartheta \)-hyper-Kähler rotation \( r \). It is helpful to rearrange the problem as: fix a pair \( Z_+, Z_- \) of deformation families of building blocks with automorphism groups \( \Gamma_\pm \), fix \( t \), and then construct the pair \( V_+, V_- \) with the desired \( r \) from elements of \( \Gamma_\pm \).

6.1. Matchings and hyper-Kähler rotations. Let us consider the consequences of the \( \vartheta \)-hyper-Kähler rotation condition for the action of \( r \) on cohomology. Let \( N_{\mathbb{R}}^\pm \subset H^2(\Sigma_{\pm}; \mathbb{R}) \) be the image of \( H^2(V_\pm; \mathbb{R}) \to H^2(\Sigma_{\pm}; \mathbb{R}) \) (generated by the polarising lattice \( N_\pm \) as defined in \([2, 1]\)), and let \( \Pi_{\pm} \subset H^2(\Sigma_{\pm}) \) be period of \( \Sigma_{\pm} \), i.e. the space of classes of type \((2,0) + (0,2)\). Then \( [\omega^\pm_r] \in N_{\mathbb{R}}^\pm \), and it is moreover the restriction of a Kähler class from \( Z_\pm \). Meanwhile \( \Pi_{\pm} \) is orthogonal to \( N_{\mathbb{R}}^\pm \), and \( k_\pm \) is spanned by \( [\omega^+_r] \) and \( [\omega^-_r] \). If we let \( \pi_\pm : H^2(\Sigma_{\pm}; \mathbb{R}) \to N_{\mathbb{R}}^\pm \) be the orthogonal projection, and \( \pi^\pm_\pm = \text{Id} - \pi_\pm \), then \( r : \Sigma_+ \to \Sigma_- \) satisfying \([5, 12]\) implies the following condition also holds.

Definition 6.1. Given building blocks \((Z_+, \Sigma_+)\) and \((Z_-, \Sigma_-)\) and \( \vartheta \neq 0 \), call a diffeomorphism \( r : \Sigma_+ \to \Sigma_- \) a \( \vartheta \)-matching if there are Kähler classes on \( Z_\pm \), whose restrictions \( k_\pm \in H^2(\Sigma_{\pm}; \mathbb{R}) \), satisfy

- \( \pi_+ r^* k_- = (\cos \vartheta) k_- \) and \( \pi_- (r^{-1})^* k_+ = (\cos \vartheta) k_+ \);
- \( \pi^+_+ r^* k_- \in \Pi_+ \) and \( \pi^+_+ (r^{-1})^* k_+ \in \Pi_- \) and moreover
- \( r^* \Pi_- \cap \Pi_+ \) is non-trivial.

Lemma 6.2. Given blocks \((Z_\pm, \Sigma_\pm)\), a diffeomorphism \( r : \Sigma_+ \to \Sigma_- \) is a \( \vartheta \)-matching if and only if there exist hyper-Kähler triples \( \omega^+_r, \omega^+_r, \omega^-_r \) on \( \Sigma_\pm \) such that \([\omega^+_r] \) is the restriction of a Kähler class from \( Z_\pm \), and \( r \) is a \( \vartheta \)-hyper-Kähler rotation with respect to the triples.

Proof. If \( r \) is a \( \vartheta \)-hyper-Kähler rotation then taking \( k_\pm = [\omega^+_r] \) satisfies the first two conditions in Definition 6.1 while \([\omega^+_r] \in r^* \Pi_- \cap \Pi_+ \).

For the converse, note that \( \pi^+_+ r^* k_- \) is a non-zero element of \( \Pi_+ \), but is not in \( r^* \Pi_- \cap \Pi_+ \). Therefore \( \Sigma_+ \) has a holomorphic 2-form \( \omega^+_r + i \omega^+_r + \omega^-_r \) with \([\omega^+_r] \in \pi^+_+ r^* k_- \) and \([\omega^-_r] \in r^* \Pi_- \cap \Pi_+ \). By the Calabi-Yau theorem, there is a Ricci-flat Kähler metric \( \omega^+_r \in k_+ \).

Choosing \( \omega^+_r, \omega^+_r, \omega^-_r \) analogously and normalising ensures that \([r^* \omega^+_r] = (\cos \vartheta) [\omega^+_r] + (\sin \vartheta) [\omega^-_r] \), \([r^{-1} \omega^+_r] = (\cos \vartheta) [\omega^+_r] + (\sin \vartheta) [\omega^-_r] \) and \([r^* \omega^-_r] = [\omega^-_r] \). Uniqueness of Ricci-flat Kähler metrics in their Kähler class implies \([7]\), so \( r \) is a hyper-Kähler rotation. \( \square \)
Remark 6.5. Given a configuration of $N_+$ and $N_-$, we get a well-defined lattice $W := N_+ + N_-$ containing $N_+$ and $N_-$ as primitive sublattices. In general, it is possible for $W$ to fail to be primitive in $L$, but we will never use such configurations. By only using examples of small rank and with $W$ primitively embedded in $L$ in this paper, the configurations are completely characterised just by the lattice $W$ according to the following result of Nikulin [24, Theorem 1.12.4].
Theorem 6.6. Let $W$ be an even non-degenerate lattice of signature $(\ell_+, \ell_-)$, and $L$ an even unimodular lattice of indefinite signature $(\ell_1, \ell_2)$. If $\ell_+ \leq \ell_1$, $\ell_- \leq \ell_2$ and $2 \rk W \leq \rk L$, then there exists a primitive embedding $W \hookrightarrow L$, unique up to $O(L)$.

6.3. Necessary conditions for matching. Let us next consider what necessary conditions Lemma 6.4 imposes on a configuration for it to be realised by a matching of blocks. Note first of all that one must have $k_\pm \in \text{N}_\pm$, while the period $(k_\pm - \cos \vartheta k_\pm, \pm k_0)$ is orthogonal to $\text{N}_\pm$. Hence $k_\pm$ is precisely $\cos \vartheta \varpi \pm k_\pm$, where $\varpi_\pm : L_R \rightarrow \text{N}_\pm(\mathbb{R})$ is the orthogonal projection. Observe that $\varpi_+ \varpi_- : \text{N}_+(\mathbb{R}) \rightarrow \text{N}_+(\mathbb{R})$ is self-adjoint, so $\text{N}_+(\mathbb{R})$ splits as a direct sum of eigenspaces.

Notation 6.7. For $\psi \in \mathbb{R}$, let $\text{N}_\psi^0 \subset \text{N}_\pm(\mathbb{R})$ denote the $(\cos \psi)^2$-eigenspace of $\varpi_+ \circ \varpi_-$. Clearly $\varpi_+$ maps $\text{N}_+(\mathbb{R})^0$ to $\text{N}_-(\mathbb{R})^0$, and is invertible if $\psi \neq 0$. Of course, $\text{N}_+(\mathbb{R})^0 = \text{N}_-(\mathbb{R})^0 = \text{N}_+(\mathbb{R}) \cap \text{N}_-(\mathbb{R})$. For any $x \in \text{N}_+(\mathbb{R})^0$ and $y := \varpi_+ x$ we have
\begin{equation}
(x:y)^2 = (\cos \psi)^2, \quad y.y = (\cos \psi)^2(x:x).
\end{equation}
In particular, it is necessary that $k_\pm \in \text{N}_\psi^0$.

Here is a qualitative difference between the matching problem for rectangular twisted connected sums $(\vartheta = \frac{\pi}{2})$ and extra-twisted connected sums $(\vartheta \neq \frac{\pi}{2})$: in the former case we can choose $k_\pm \in \text{N}_\psi^0 = \text{N}_+(\mathbb{R}) \cap \text{N}_-(\mathbb{R})$ independently of each other, while in the latter case $k_+$ and $k_-$ determine each other.

Remark. If the ambient space $L$ were positive definite then the eigenvalues $\lambda$ of $\varpi_+ \circ \varpi_-$ would all lie in $[0, 1]$. In a space of indefinite signature it could happen both that $\lambda > 1$ (e.g. if $\text{N}_\pm$ both have signature $(1, 0)$ in a space of signature $(1, 1)$) or that $\lambda < 0$ (e.g. if $\text{N}_+$ has signature $(1, 0)$ and $\text{N}_-$ has signature $(0, 1)$). However, for matchings with the given configuration to exist, $\text{N}_+ + \text{N}_-$ must have signature $(2, \rk -2)$, and all eigenvalues of $\varpi_+ \varpi_-$ must be in $[0, 1]$.

The existence of a $\vartheta$-matching with a given configuration may also impose constraints on the Picard lattices of the $\text{K}_3$s $\Sigma_\pm$, beyond the a priori condition that $\text{Pic} \Sigma_\pm$ contains $\text{N}_\pm$. Let $\text{N}_\psi^{\vartheta \neq \vartheta} \subset \text{N}_\pm$ denote the orthogonal complement of $\text{N}_\psi^{\vartheta}$. Then $k_0 \perp \text{N}_\psi^{\vartheta \neq \vartheta}$ because $\text{N}_\psi^{\vartheta \neq \vartheta} \subset \text{N}_\pm$, while $k_+ \perp k_- \perp \text{N}_\psi^{\vartheta \neq \vartheta}$ because $k_\pm \in \text{N}_\psi^{\vartheta}$. Therefore
\begin{equation}
\Lambda_\pm := \text{primitive overlattice of } \text{N}_\pm + \text{N}_\psi^{\vartheta \neq \vartheta} \subset L
\end{equation}
is perpendicular to the period $\langle k_\pm - \cos \vartheta k_\pm, \pm k_0 \rangle$ of $\Sigma_\pm$, i.e. $\Lambda_\pm \subset \text{Pic} \Sigma_\pm$.

In summary, given a pair of families of building blocks $Z_\pm$, to find some pair of elements $(Z_+, Z_-) \in \mathcal{Z}_\pm$ with a $\vartheta$-matching $\varrho : \Sigma_+ \rightarrow \Sigma_-$. it is necessary that we can take the marked $(Z_+, Z_-, h_\pm)$ such that
(i) The intersection of $N_\pm(\mathbb{R})^\vartheta$ with the image $\mathcal{K}_{Z_\pm} \subset L_R$ of the Kähler cone of $Z_\pm$ is non-empty.

Moreover, if $\vartheta \neq \frac{\pi}{2}$ then the intersection of $\epsilon \varpi_-(N_+ \cap \mathcal{K}_{Z_+})$ and $\mathcal{K}_{Z_-}$ is non-empty too, where $\epsilon := (\text{sign of } \cos \vartheta) \in \{\pm 1\}$.

(ii) $\Sigma_\pm$ is $\Lambda_\pm$-polarised.

6.4. Sufficient conditions for existence of matching. On the other hand, for the family $\mathcal{Z}_\pm$ to be $(\Lambda_\pm, \text{Amp}_{Z_\pm})$-generic for some open cone $\text{Amp}_{Z_\pm} \subset \text{N}_\pm(\mathbb{R})$ (Definition 2.12) says roughly that a generic $\Lambda_\pm$-polarised $\text{K}_3$ can be embedded as an anticanonical divisor in some block $Z_\pm \in \mathcal{Z}_\pm$, and moreover in such a way that the Kähler cone of $Z_\pm$ contains $\text{Amp}_{Z_\pm}$. This genericity property is enough to obtain a sufficient condition for the existence of $\vartheta$-matchings.

Theorem 6.8. Let $Z_\pm$ be a pair of families of building blocks with polarising lattices $\text{N}_\pm$, and $\vartheta \in \mathbb{R} \setminus \frac{\pi}{2} \mathbb{Z}$. Let $N_\pm \hookrightarrow L$ be a configuration of the polarising lattices, and define $\Lambda_\pm$ as in (25). Suppose that the family $\mathcal{Z}_\pm$ is $(\Lambda_\pm, \text{Amp}_{Z_\pm})$-generic. If
\begin{equation}
\epsilon \varpi_-(N_+)(\mathbb{R})^\vartheta \cap \text{Amp}_{Z_\pm} \cap \text{Amp}_{Z_-} \neq \emptyset
\end{equation}
then there exist $(Z_\pm, \Sigma_\pm) \in \mathcal{Z}_\pm$ with an angle $\vartheta$ $\text{K}_3$ matching $\varrho : \Sigma_+ \rightarrow \Sigma_-$ with the prescribed configuration.
Proof. The proof uses the same basic idea as the $\vartheta = \frac{\pi}{2}$ case from [27] Proposition 6.18, but the way that $k_+$ and $k_-$ determine each other in this case makes it slightly different.

Let $W_\pm$ be the orthogonal complement of $N_\pm(\mathbb{R})^\vartheta$ in $N_\pm(\mathbb{R})^\vartheta \oplus N_\pm(\mathbb{R})^\vartheta$, and $T$ the orthogonal complement of $N_+(\mathbb{R}) + N_-(\mathbb{R})$ in $L_\vartheta$. $W_\pm$ and $T$ all have signature $(1, \text{rk} - 1)$. Note that $W_\pm \oplus T$ is the orthogonal complement of $\Lambda_\pm$. Thus a pair of real lines in the positive cones of $W_\pm$ and $T$ span a positive-definite 2-plane in $\Lambda_\pm$, so

$$\mathbb{P}(W_\pm^+) \times \mathbb{P}(T^+)$$

can be regarded as a submanifold of $G_{\Lambda_\pm}$. Analogously to [27] Proposition 6.18 it is an analytic, totally real submanifold. Moreover, because $\Lambda_\pm$ is exactly $W_\pm \oplus T$,

$$\dim \mathbb{R}(W_\pm^+) \times \mathbb{P}(T^+) = \dim \mathbb{C} G_{\Lambda_\pm}.$$

Therefore the intersection of the submanifold $\mathbb{P}(W_\pm^+) \times \mathbb{P}(T^+) \subset G_{\Lambda_\pm}$ with the subset $U_{\vartheta Z_\pm} \subset G_{\Lambda_\pm}$ from Definition 2.12 is an open dense subset of $\mathbb{P}(W_\pm^+) \times \mathbb{P}(T^+)$.

Now we wish to find $(\ell_+, \ell_0) \in \mathbb{P}(N_+(\mathbb{R})^\vartheta) \times \mathbb{P}(T^+)$ such that

(i) $\ell_+ \in \text{Amp}_{\vartheta Z_\pm}$,

(ii) $\varepsilon_\vartheta \ell_+ \in \text{Amp}_{\vartheta Z_\pm}$,

(iii) $(w_+ (\ell_+), \ell_0) \in \left( \mathbb{P}(W_+^+) \times \mathbb{P}(T^+) \right) \cap U_{\vartheta Z_\pm}$,

(iv) $(w_-(\ell_+), \ell_0) \in \left( \mathbb{P}(W_-^+) \times \mathbb{P}(T^+) \right) \cap U_{\vartheta Z_\pm}$,

where $w_{\pm} : N_+(\mathbb{R})^\vartheta \to W_{\pm}$ are the orthogonal projections (which are both isomorphisms since $\vartheta \neq \frac{\pi}{2}$). The first two conditions define open subsets whose intersection is non-empty by the hypothesis [26]. The intersection with the open dense subsets defined by the last two conditions is therefore non-empty. Hence there is a pair $(\ell_+, \ell_0)$ satisfying (i)-(iv).

By the definition of $Z_\pm$ being $(\Lambda_\pm, \text{Amp}_{\vartheta Z_\pm})$-generic, this means there exist $(Z_\pm, \Sigma_\pm)$ in $Z_\pm$ with periods $(w_+ (\ell_+), \ell_0)$ such that $\text{Amp}_{\vartheta Z_\pm}$ is contained in the image of Kähler cone of $Z_\pm$. Taking $k_+$, $k_-$ and $k_0$ to be the unit norm representatives of $\ell_+$, $\varepsilon_\vartheta \ell_+$ and $\ell_0$ respectively, we can therefore apply Lemma 6.4 to obtain the desired $\vartheta$-matching $\tau : \Sigma_+ \to \Sigma_-$. \hfill $\square$

6.5. Configuration angles and pure configurations. The following invariants of a configuration turn out to have several uses.

**Definition 6.9.** Given a configuration $N_+, N_- \subset L$, let $A_\pm : L_\vartheta \to L_\vartheta$ denote the reflection of $L_\vartheta := L \oplus \mathbb{R}$ in $N_\pm$ (with respect to the intersection form of $L_\vartheta$; this is well-defined since $N_\pm$ is non-degenerate). Suppose $A_+ \circ A_- \text{ preserves some decomposition } L_\vartheta = L^+ \oplus L^-$ as a sum of positive and negative-definite subspaces. Then the configuration angles are the arguments $\alpha_1^+ \vartheta, \alpha_2^+ \vartheta, \alpha_3^+ \vartheta$ and $\alpha_1^- \vartheta, \alpha_2^- \vartheta, \alpha_3^- \vartheta$ of the eigenvalues of the restrictions $A_+ \circ A_- : L^+ \to L^+$ and $A_+ \circ A_- : L^- \to L^-$ respectively.

Note that if the configuration is to be realised by a $\vartheta$-hyper-Kähler rotation, then $A_+ \circ A_- \text{ preserves the decomposition of } L_\vartheta$ into the subspaces that self-dual and anti-self-dual with respect to the hyper-Kähler metric, so the configuration angles are defined. Further, the necessary condition (i) from [26] can be expressed in terms of the configuration angles as requiring that $\alpha_1^+, \alpha_2^+, \alpha_3^+$ are precisely $0$ and $\pm 2\vartheta$.

In view of Proposition 5.17, the hypothesis that the family $\mathcal{Y}_\pm$ is $(\Lambda_{\pm}, \text{Amp}_{\vartheta Z_\pm})$-generic (for some cone $\text{Amp}_{\vartheta Z_\pm}$) is easiest to verify for configurations where $\Lambda_\pm = N_\pm$. This amounts to requiring that $N_{\pm}^\vartheta \vartheta$ is contained in $N_\pm$, or equivalently that $N_\pm$ is spanned (at least rationally) by $N_{\pm}^\vartheta \vartheta = N_\pm \cap N_\pm$. Noting that for $0 < |\vartheta| < \frac{\pi}{2}$

$$\text{multiplicity of } 2\psi \text{ as a configuration angle } = \dim N_{\pm}^\psi = \dim N_{\pm}^\vartheta,$$

this is in turn equivalent to requiring that the only non-zero configuration angles are $\pm 2\vartheta$. This is in particular the case if $N_{\pm}^\vartheta = N_{\pm}$; we refer to such configurations as having “pure angle $\vartheta$”.

Configurations with pure angle $\frac{\pi}{2}$ are very easy to produce (as long as $\text{rk } N_+ + \text{rk } N_- \leq 11$): simply apply Theorem 6.6 to embed the perpendicular direct sum $N_+ \perp N_-$ primitively in $L$. On the other hand, for $\vartheta \neq \frac{\pi}{2}$, the existence of a pure angle $\vartheta$ configuration of a given pair of
lattices $N_1$, $N_-$ is a non-trivial condition. To be able define a bilinear form on $W := N_+ \oplus N_-$ that restricts to the prescribed one on $N_\pm$ and such that $N^0_\pm \equiv N_\pm$, it is necessary but not sufficient that the ranks be equal.

Consider the case when $\text{rk}N_\pm$ both have rank 1, with generator $n_\pm$ (chosen to be positive, i.e. $n_\pm \in \text{Amp}_{Z_\pm}$). Then there is only a single cross-term to choose in $W$, and by (24) we must set

$$n_+ n_-= (\cos \vartheta) \sqrt{(n_+ n_+)(n_- n_-)}.$$  

(28)

Thus, in this case $W$ exists if and only if the RHS is an integer.

**Example 6.10.** We can make a $\vartheta = \frac{\pi}{4}$ or $\vartheta = \frac{3\pi}{4}$ matching of the involution block from Example 3.22 and a regular block from Example 3.8 using

$$W = \left( 2 \sqrt{2} \epsilon \frac{\sqrt{2}}{4} \right).$$  

(29)

(This leads to a 2-connected $\frac{\pi}{4}$-twisted connected sum with $b_3(M) = 134$ and $p(M)$ divisible by 24, see Table 4.)

**Remark 6.11.** If there does exist a pure angle $\vartheta$ configuration between the polarising lattices, then for $\vartheta \neq \frac{\pi}{4}$ it does not need to be unique, and different pure angle matchings of blocks from the same families can lead to non-diffeomorphic $\vartheta$-twisted connected sums; see Examples 8.16 and 8.17.

Let us think a moment about the meaning of changing the sign of $\vartheta$ or replacing it by a complementary angle. For a start, the condition in Definition 6.1 for $r$ to be a $\vartheta$-matching is actually independent of the sign of $\vartheta$, which is related to the earlier observation that a $\pm \vartheta$-twisted connected sums of phase rotated ACyl Calabi-Yaus are (orientation-reversing) diffeomorphic. So the sign is unimportant.

There are several natural ways to modify a matching in order to complement the angle. We could change the signs of the cross-terms in $W$ like in (29) while keeping everything else the same, or equivalently, we could change the sign of the marking on $(\Sigma_+, I_+)$ (keeping $W$ the same, but multiplying $\text{Amp}_{Z_+}$ by $-1$). Alternatively, we could replace the block $Z_+$ by its complex conjugate; if we keep the marking the same, then $\text{Amp}_{Z_+}$ is multiplied by $-1$. This is precisely the same way of relating extra-twisted connected sums with complementary angles as in Remark 8.5. Any of these changes leaves the cohomology and $p_1$ of the resulting $G_2$-manifolds unchanged, so we will not be concerned with distinguishing between complementary angles in the examples.

7. **Topology**

We now turn to the problem of computing topological properties of extra-twisted connected sums. All our computations will be expressed in terms of data of the building blocks listed in Tables 1, 2, and 3, along with the choice of torus isometry, and the configuration of the hyper-Kähler rotation in the sense of Definition 6.3.

7.1. **Mayer-Vietoris generalities.** It is inevitable that computing the full integral cohomology of an extra-twisted connected sum will involve checking different values of $\vartheta$ case by case. However, some parts of the computation are common to all non-rectangular extra-twisted connected sums.

$H^1(M_\pm) \rightarrow H^1(T^2 \times \Sigma)$ is an isomorphism onto the cyclic subgroup of $H^1(T^2)$ dual to the internal circle factor, i.e. the image is generated by $v_\pm$ or $2v_\pm$ depending on whether $M_\pm$ is an ordinary block or an involution block. The images never intersect, so $H^1(M) = 0$. The sum of the images is primitive precisely for the arrangements when $M$ is simply connected; otherwise the contribution to $H^2(M)$ is (obviously) $\pi_1(M)$, but we ignore this case from now on.

$H^2(M_\pm) \rightarrow H^2(T^2 \times \Sigma)$ is an isomorphism onto $N_\pm \subset H^2(\Sigma)$, regardless of whether $M_\pm$ is an ordinary or an involution block. Thus $H^2(M) = N_+ \cap N_-$, and we get a contribution $\mathbb{Z} \oplus L/(N_+ + N_-)$ to $H^3(M)$. Whether this is torsion-free depends on the choice of push-out $W$ in the matching, and on whether we embed $W$ primitively in $L$ or not.

Since $H^3(M_\pm)$ are torsion-free, there is no other contribution to the torsion in $H^3(M)$. Thus, we get $M$ 2-connected if and only if we use building blocks with $K_\pm = 0$ and a configuration such that $N_+ \cap N_- = 0$ and $N_+ \oplus N_-$ is primitively in $L$. 

To determine $H^3(M)$ we only need to deal with $H^3(M_+) \to H^3(T^2 \times \Sigma)$ rationally; the contribution to the torsion in $H^4(M)$ will have to be dealt with case by case. The image of $H^3(M_+, \mathbb{Q})$ is the Lagrangian $\nu_+ N_+ \oplus u_+ T_+ \subset H^3(T^2 \times \Sigma; \mathbb{Q})$. Since

$$v_+ = \cos \vartheta v_- + \sin \vartheta u_-, \quad u_+ = \sin \vartheta v_+ - \cos \vartheta u_-,$$

for $v_n, u_n, v_{n+1}$ to equal $v_{n-1} + u_{n-1}$ for some $n_+ \in N_+$ and $t_n \in T_+$ implies that $\tau_+ n_+ = \cos \vartheta n_+ \pm \sin \vartheta t_n$, and thus $n_+ \in N^\perp_+$ in Notation 6.7. Hence the dimension of the intersection of the images of $H^3(M_+, \mathbb{Q})$ equals $d_\vartheta = \text{rk} N^\perp_+ = \text{rk} N^\perp_-$ (or the multiplicity of $\vartheta$ as a configuration angle (27)). On the other hand, the kernel of the quotient $H^3(M; \mathbb{Q})$ is $H^3(Z; \mathbb{Q})^\perp$, or just $H^3(Z; \mathbb{Q})$ in the case of an ordinary block. Denoting the dimension of that by $b^3_3(Z_\pm)$, we obtain

$$b_3(M) = 23 - \rho_+ - \rho_- + b_2(M) + b^*_3(Z_+) + b^*_3(Z_-) + d_\vartheta. \quad (31)$$

Remark 7.1. $b^*_3(Z_\pm)$ is always even since $H^3(Z_\pm)^\perp \subset H^3(Z_\pm)$ is symplectic. Therefore

$$1 + b_3(M) + b_3(M) = \rho_+ + \rho_- + d_\vartheta \mod 2.$$

Further, $\rho_+ + \rho_- = \text{rk} N^\perp_+ + \text{rk} N^\perp_- \mod 2$, the rank of the perpendicular parts. Hence the parity of $\nu$ is $d_\vartheta + \text{rk} N^\perp_+ + \text{rk} N^\perp_-$, which is consistent with (1).

Remark 7.2. For $\vartheta = \frac{\pi}{2}$ we should set $d_\vartheta := \text{rk} N^\perp_+ + \text{rk} N^\perp_- = \text{rk}(N_+ \cap T_-) + \text{rk}(N_- \cap T_+)$. In case of orthogonal matching we get $d_\vartheta = \rho_+ + \rho_- - 2b_2(M)$, and (31) recovers the claim from [21] (8.56) that $b_3(M) + b_3(M) = 23 + b_3(Z_+) + b_3(Z_-)$ in this setting.

When the involution blocks are pleasant, then the torsion subgroup of $H^4(M)$ is contained in the image $\delta H^3(T^2 \times \Sigma)$ of the Mayer-Vietoris boundary map. Further below we compute this in detail in the case $\vartheta = \frac{\pi}{4}$ and $\frac{\pi}{6}$. We can make a general statement about the torsion linking form.

**Lemma 7.3.** Let $M^7 = M_+ \cup_X M_-$ be a gluing of manifolds with boundary $X$, and let $I_\pm \subseteq H^3(X)$ be the image of $H^3(M_\pm)$. Let $p_1, p_2 \in H^3(X)$ be classes that are torsion modulo $I_+ + I_-$, so that $\delta(p_1), \delta(p_2) \in H^4(M)$ are torsion classes. Then we can write $mp_1 = p^+_1 - p^-_1$ for some $m \in \mathbb{Z}$ and $p^+_1 \in I_+$, and

$$b(\delta(p_1), \delta(p_2)) = \frac{1}{m} p^+_1 p_2 = \frac{1}{m} p^-_1 p_2 \in \mathbb{Q}/\mathbb{Z}.$$

### 7.2. The spin characteristic class

Apart from the integral cohomology, the main invariant of an extra-twisted connected sum that we are interested in is the spin characteristic class $p(M) \in 2H^4(M)$. Essentially we would like to think of $p(M)$ as the result of patching up the classes $c_2(Z_\pm) \in 2H^4(Z_\pm)$, but making it precise is somewhat complicated. Moreover, even once we have a formula for $p(M)$, one needs to look carefully at the Mayer-Vietoris sequence to understand what it means (e.g. what the greatest divisor in $H^4(M)$ is), which we do in the subsequent sections.

Let $Z$ be a block with involution $\tau$. Let $S^3_\chi \times Z$ be the mapping torus of $\tau$, and let $Z^\alpha$ be the quotient $Z/\tau$—which has singularities along the complex curve $C$ in the fixed set of $\tau$. Let $\pi : S^3_\chi \times Z \to Z^0$ be the obvious projection map, and let $Q := \pi^* H^4(Z^0) \subseteq H^4(S^3_\chi \times Z)$. Below we will define a way to glue pairs of elements of $Q_+$ and $Q_-$ whose images in $H^4(\Sigma)$ agree.

The tangent bundle of $S^3_\chi \times Z$ splits as the sum of a trivial line bundle tangent to the $S^3_\chi$ factor, and the vertical tangent bundle $T_{\chi}(S^3_\chi \times Z)$ of the projection $S^3_\chi \times Z \to Z$. The restriction of the vertical tangent bundle to each fibre is of course just isomorphic to $T_Z$, and the second Chern class of $T_{\chi}(S^3_\chi \times Z)$ is completely determined by that of $T_Z$.

**Lemma 7.4.** $c_2(T_{\chi}(S^3_\chi \times Z))$ is an even element of $H^4(S^3_\chi \times Z)$, belongs to $Q$, and is the unique preimage of $c_2(Z)$ in $Q$.

**Proof.** Because $T_{\chi}(S^3_\chi \times Z)$ is stably isomorphic to the tangent bundle, $c_2(T_{\chi}(S^3_\chi \times Z)) = w_4(S^3_\chi \times Z)$. For any oriented manifold of dimension less than 8 we have $w_4 = w_2^2$. Therefore $c_2(T_{\chi}(S^3_\chi \times Z)) = c_1(T_{\chi}(S^3_\chi \times Z))^2 \mod 2$. But $c_1(T_{\chi}(S^3_\chi \times Z))$ is Poincaré dual to $\Sigma \times S^3_\chi$, which can be deformed off itself, so $c_1(T_{\chi}(S^3_\chi \times Z)) = 0$. □
To define the gluing map, we consider a space \( \tilde{P} \) obtained from \( S^1_\xi \tilde{\times} Z \) by collapsing the external circle factor over \( \Sigma \subset Z \). Let \( \rho : S^1_\xi \tilde{\times} Z \to \tilde{P} \) and \( \pi : \tilde{P} \to Z^0 \) be the projection maps, and let \( \tilde{Q} := \tilde{\pi}^* H^4(Z^0) \).

**Lemma 7.5.** \( \rho^* : H^4(\tilde{P}) \to H^4(S^1_\xi \tilde{\times} Z) \) is surjective, with kernel \( \cong L/N \).

In particular, \( \rho^* : \tilde{Q} \to Q \) is an isomorphism.

**Proof.** We have exact sequences

\[
0 \to H^4_cpt(S^1_\xi \tilde{\times} V) \to H^4(\tilde{P}) \to H^4(\Sigma) \to 0,
H^3(S^1_\xi \times \Sigma) \to H^4_cpt(S^1_\xi \tilde{\times} V) \to H^4(S^1_\xi \tilde{\times} Z) \to H^4(\Sigma) \to 0
\]

and the kernel of \( \rho^* : H^4(\tilde{P}) \to H^4(S^1_\xi \tilde{\times} Z) \) is the image of \( H^3(\Sigma \times S^1_\xi) \to H^4_cpt(S^1_\xi \tilde{\times} V) \). We now consider the following commuting diagram with exact rows and columns.

\[
\begin{array}{cccccc}
H^2(Z) & \to & H^2(\Sigma) & \to & H^2(\Sigma \times S^1_\xi) & \to & H^3(\Sigma) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^3_cpt(V) & \to & H^3_cpt(V) & \to & H^3_cpt(S^1_\xi \tilde{\times} V) & \to & H^4_cpt(V) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^3(Z) & \to & H^3(Z) & \to & H^3(S^1_\xi \tilde{\times} Z) & \to & H^4(\Sigma)
\end{array}
\]

By the conditions for \( Z \) to be a building block, the image \( N \) of \( H^2(\Sigma) \to H^2(\Sigma) \) is primitive, so the image \( L/N \) of \( H^3(\Sigma) \to H^3_cpt(V) \) is free. Since \( H^2(\Sigma) \) is invariant under \( \tau \), so is \( L/N \). As \( H^3_cpt(V) \) is free, \( L/N \) therefore has trivial intersection with the image of \( 1 - \tau : H^3_cpt(V) \to H^3_cpt(V) \). Hence the image of \( H^3(\Sigma \times S^1_\xi) \to H^4_cpt(S^1_\xi \tilde{\times} V) \) is isomorphic to \( L/N \).

Surjectivity of \( \rho^* : \tilde{Q} \to Q \) is obvious. Also, since the only possible kernel of \( \tilde{\pi}^* : H^4(Z^0) \to H^4(S^1_\xi \tilde{\times} Z) \) is 2-torsion, the same is true for \( \rho^* : \tilde{Q} \to Q \). Since the kernel \( L/N \) of \( H^4(\tilde{P}) \to H^4(S^1_\xi \tilde{\times} Z) \) is torsion-free, \( \tilde{Q} \to Q \) must thus be injective. \( \square \)

Now, given a pair of blocks that are used to form an extra-twisted connected sum \( M \), let \( R := \tilde{P}_+ \cup_\Sigma \tilde{P}_- \). We can define a collapsing map \( \gamma : M \to R \). We also have obvious inclusion maps \( j_\pm : \tilde{P}_\pm \hookrightarrow R \). By Mayer-Vietoris,

\[
(j_+^* , j_-^* ) : H^4(R) \to H^4(\tilde{P}_+) \times H^4(\tilde{P}_-)
\]

is an isomorphism onto the subgroup of pairs with equal image in \( H^4(\Sigma) \). In particular, the image contains \( \tilde{Q}_+ \times_0 \tilde{Q}_- \).

**Definition 7.6.** Let \( Y : Q_+ \times_0 Q_- \to H^4(M) \) be the composition of the inverse of \( (\rho^*_+, \rho^*_-) : \tilde{Q}_+ \times_0 \tilde{Q}_- \to Q_+ \times_0 Q_- \), the inverse of \( (j_+^*, j_-^*) \), and \( \gamma^* : H^4(R) \to H^4(M) \).

**Theorem 7.7.** \( p(M) = Y(c_2(T_{vt}S^1_\xi \tilde{\times} Z_+), c_2(T_{vt}S^1_\xi \tilde{\times} Z_-)) \).

**Proof.** We will define an \( SU(3) \)-bundle \( E_\pm \to \tilde{P}_\pm \) and a \( Spin(7) \)-bundle \( T \to R \) such that

(i) \( \rho^*_\pm c_2(E_\pm) = c_2(T_{vt}S^1_\xi \tilde{\times} Z_\pm) \in Q_\pm \subseteq H^4(S^1_\xi \tilde{\times} Z_\pm) \)
(ii) \( j_\pm^* T \) is isomorphic to \( E_\pm \oplus \mathbb{R} \), so in particular \( j_\pm^* p(T) = c_2(E_\pm) \)
(iii) \( \gamma^* T \) is isomorphic to \( TM \)
(iv) \( c_2(E_\pm) \in Q_\pm \).
The claim then follows immediately from the definition of \( Y \).

On a collar neighbourhood \( \mathbb{R}^+ \times S^1_k \times \Sigma \) of the boundary of \( V \), we have an obvious isomorphism \( T(\mathbb{R}^+ \times S^1_k \times \Sigma) \to \mathbb{C} \oplus T \Sigma \). It descends to an isomorphism \( f \) from the restriction of \( T_{\text{rel}}(S^1_k \times \Sigma) \) to a collar neighbourhood to \( \mathbb{C} \oplus T \Sigma \). We construct an \( SU(3) \)-bundle \( E \) over \( \tilde{P} \) by using \( f \) to glue \( T_{\text{rel}}(S^1_k \times \Sigma) \) and the bundle \( \Sigma \oplus \mathbb{C} \) over a neighbourhood \( U \) of \( \Sigma \).

Obviously \( \rho^* E \) over \( P \) can be described by a similar gluing. Over \( P \), we can further define a complex line bundle \( L \) as follows. Let \( u \) be the internal \( S^1 \)-coordinate as usual on the boundary of \( U := S^1_k \times \Sigma \), and along the boundary define a trivialisation \( g : \mathbb{C} \to T_{\text{rel}}(S^1_k \times \Delta) \) by \( g(1) = e^{iu \frac{\partial}{\partial u}} \). \( L \) has a section vanishing precisely along \( \Sigma \times S^1_k \). In particular, \( c_1(L)^2 = PD(\Sigma \times S^1_k)^2 = 0 \).

We may also consider \( T_{\text{rel}}(S^1_k \times Z) \) itself as being obtained by gluing \( T_{\text{rel}}(S^1_k \times V) \) over \( S^1_k \times V \) to \( T_{\text{rel}}U = T_{\text{rel}}(S^1_k \times \Delta) \oplus T \Sigma \) by the derivative of \( \mathbb{R}^+ \times S^1_k \times \Sigma \cong \Delta^* \times \Sigma \), \((t, u) \mapsto z = x + iy = e^{-t - iu} \), which equals precisely \((g \times \Id_{T \Sigma}) \circ f \).

Now let us compare \( \mathbb{C} \oplus T_{\text{rel}}(S^1_k \times Z) \) with \( L \oplus \rho^* E \). By the above, we can regard both of them as the result of gluing \( \mathbb{C} \oplus T_{\text{rel}}(S^1_k \times V) \) to \( \mathbb{C} \oplus T_{\text{rel}}(S^1_k \times \Delta) \oplus T \Sigma \). For the first, the gluing map is the composition of \((\Id \times f) : \mathbb{C} \oplus T_{\text{rel}}(S^1_k \times V) \to \mathbb{C} \oplus \Sigma \oplus T \Sigma \) with \((\Id 0) \times \Id_{T \Sigma} \). For the second, we instead compose with \((0 \Id) \times \Id_{T \Sigma} \). Hence the composition of one gluing map with the other is the automorphism \((0 \Id) \times \Id_{T \Sigma} \) of \( \mathbb{C} \oplus T \Sigma \), which is trivially homotopic to the identity in the space of complex vector bundle automorphisms. Hence

\[
\mathbb{C} \oplus T_{\text{rel}}(S^1_k \times Z) \cong L \oplus \rho^* E,
\]

and since \( c_1(L)^2 \) and \( c_1(E) = 0 \) this proves (i).

Construct \( T \) to satisfy (ii) and (iii) by gluing \( \mathbb{R}^+ \oplus E_+ \) and \( \mathbb{R}^\times E_- \) the right way.

To prove that \( c_2(E) \in \bar{Q} \), we construct a further \( SU(3) \)-bundle \( F_{\pm} \to Z_{\pm} \) as follows. Consider a tubular neighbourhood \( W \) of \( C \) in the symmetric K3 divisor \( \Sigma' \subset Z_{\pm} \) (so \( W \cong \text{unit disc bundle in } N_{C/\Sigma'} \)). Then \( \Delta \times W \) is a tubular neighbourhood of \( C \) in \( Z_{\pm} \). We define \( F_{\pm} \) as a gluing of \( E_{\pm}|Z_{\pm}\backslash C \) and \( T(\Delta \times W) \). The overlap region is basically the unit sphere bundle \( S \) of \( T^* C \oplus \mathbb{C} \to C \) (using some arbitrary hermitian metric on \( T^* C \)) and the restriction of both bundles to the overlap is \( T C \oplus T^* C \oplus \mathbb{C} \). Take the gluing map to be

\[
S \to SU(T C \oplus T^* C \oplus \mathbb{C}), \quad (\alpha, z) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & z & \alpha \\ 0 & \bar{\alpha} & \bar{z} \end{pmatrix}.
\]

Next, define a bundle isomorphism \( \bar{\tau} : F_{\pm} \to F_{\pm} \) covering \( \tau \) by patching up \( D\tau \) on \( T(Z_{\pm} \backslash C) \) and \( \Id \) on \( T(\Delta \times W) \) and \( T U_{\pm} \); this works because on the overlap, \( D\tau \) is the diagonal matrix \((1, -1, -1)\), which equals the difference of the glue map evaluated at \( p \in S \) and \( \tau(p) \). Because \( \bar{\tau} \) acts trivially over the fixed set of \( \tau \), the quotient defines a bundle \( F_{\pm} \to Z_{\pm} \), and \( \bar{\tau}^*F_{\pm} = F_{\pm} \). Hence \( c_2(F_{\pm}) \) is in \( \bar{Q} \).

Now \( F_{\pm} \) and \( E_{\pm} \) are constructed to be isomorphic outside \( C \times S^1_k \). Hence their \( c_2 \)'s differ by a multiple of the Poincare dual of \( C \times S^1_k \). But \( C \), being the fixed set of a non-symplectic involution on the K3 \( \Sigma' \), is necessarily even in \( H_2(\Sigma') \) and hence also in \( H_2(C) \). We may take \( C' \subset Z \) smooth such that \( C = 2C' \) in homology and \( C' \) does not meet \( C \). Then \( PD(C \times S^1_k) = \bar{\tau}^*(\pi(C')) \in \bar{Q} \), completing (iv).

In practice, we are mostly interested in determining the greatest divisor of \( p(M) \). Since that a priori divides \( \chi(K3) = 24 \), we effectively care about the value of \( p(M) \) modulo 24. When \( c_2(Z_{\pm}) \) has been computed in the form \( (17) \), this proves practical to evaluate. Note that \( c_2(Z_{\pm}) \in N'_{\pm} \) is always even, say \( c_2(Z_{\pm}) = 2y_{\pm} \mod T_{\pm} \) for some \( y_{\pm} \in L \). Because the image of \( H^3(M_{\pm}) \to H^3(T^2 \times \Sigma) \) always contains \( 2\mathbf{1}_{\pm} T_{\pm} \) (regardless of whether \( M_{\pm} \) is of the form \( S^1 \times V_{\pm} \) or \( S^1 \times V_{\pm} \)), the value of the Mayer-Vietoris map \( \delta : H^3(T^2 \times \Sigma) \to H^3(M) \) on \( 2\mathbf{1}_{\pm} y_{\pm} \) is independent of the choice of \( y_{\pm} \), and can be interpreted as a well-defined element \( \delta(u_{\pm} c_2(Z_{\pm})) \in H^4(M) \).
Corollary 7.8. Suppose that $c_2(Z_{\pm}) = g_2 \tilde{c}_2(Z_{\pm}) + 24 h_4$ as in (17). Let $x \in H^4(Z_{\pm})$ such that $\pi^* x_{\pm} = 2h_{\pm}$, and let $\bar{x}_{\pm}$ be the pre-image of $x_{\pm}$ in $H^4(S^1 \times (Z_{\pm}))$.

$$p(M) = \delta(u_+ \bar{c}_2(Z_+) - u_- \bar{c}_2(Z_-)) + 12Y(\bar{x}_+, \bar{x}_-).$$

So in particular, if $\bar{x}_{\pm}$ is even (automatic for ordinary blocks, while for involution blocks we have ensured that this is the case when listing the data for our examples), then $p(M) = \delta(u_+ \bar{c}_2(Z_+) - u_- \bar{c}_2(Z_-)) \mod 24$. Moreover, if the involution blocks are pleasant, then $\delta(H^3(T^2 \times \Sigma))$ is a split summand in $H^4(M)$, so we understand the isomorphism class of the pair $(H^4(M), p(M))$ completely.

The detailed study of $\delta(H^3(T^2 \times \Sigma))$ will have to proceed case by case, but let us point out an important qualitative difference between the cases $\vartheta = \pm \frac{\pi}{4}$ and $\vartheta \neq \pm \frac{\pi}{4}$: For rectangular TCS the images of $\delta(u_+ \bar{c}_2(Z_+))$ and $\delta(u_- \bar{c}_2(Z_-))$ belong to two different direct summands in $H^4(M)$ (the image of the push-forward maps $H^4_{\text{cpt}}(M_{\pm}) \to H^4(M)$), so that it suffices to compute the greatest divisors separately and then take their GCD. But for extra-twisted connected sums the images of $H^4_{\text{cpt}}(M_{\pm}) \to H^4(M)$ can overlap, so there can be cancellation between $\delta(u_+ \bar{c}_2(Z_+))$ and $\delta(u_- \bar{c}_2(Z_-))$, and we need to know both terms precisely.

7.3. $\pm \frac{\pi}{4}$-twisted connected sums. Now we describe how to work out the torsion in $H^4(M)$ and the divisibility of $p(M)$ in the case $\vartheta = \pm \frac{\pi}{4}$, and carry it out for some examples.

As described before, we use a block $Z_+$ with involution and an ordinary block $Z_-$. We assume that $Z_+$ is pleasant, in order that $H^4(M_{\pm})$ is torsion-free. Therefore the only contribution to the torsion comes from the Mayer-Vietoris map $\delta : H^3(T^2 \times \Sigma) \to H^4(M)$, whose image is a split summand in $H^4(M)$.

By Lemma 2.11 the assumption that $Z_+$ is pleasant ensures that the image of $H^3(M_{\pm}) \to H^4(T^2 \times \Sigma)$ is exactly

$$I_+ := \{ v_+ n + u_+ t : n \in N_+, t \in T_+, n + t = 0 \mod 2L \}. \quad (32)$$

On the other hand, the image of $H^3(M_-)$ is just

$$I_- := v_- N_- \oplus u_- T_-.$$ The image $\delta(H^3(T^2 \times \Sigma))$ is isomorphic to $H^3(T^2 \times \Sigma)/(I_+ + I_-)$.

To make this more manageable, note that $\{ 2u_+, u_- \}$ is a basis of $H^1(T^2)$, and that we may define a surjective homomorphism

$$H^3(T^2 \times \Sigma) \to N^+_+ \oplus N^+_-, \quad 2u_+ x + u_- y \mapsto (b^+(x), b^-(y)) \quad (33)$$

for $x, y \in L$, where $b^\pm : L \to N^+_\pm$ is defined by the intersection form. Elements in the kernel of (33) have $x \in T_+, y \in T_-$, so definitely lie in $I_+ + I_-$. Hence

$$\delta(H^3(T^2 \times \Sigma)) \cong (N^+_+ \oplus N^+_-) / (I_+ + I_-), \quad (34)$$

where $I_{\pm}$ is the image of $I_{\pm}$ under (33). Using that

$$v_+ = u_+ + u_-, \quad v_- = 2u_+ + u_-$$

in a $\vartheta = \pm \frac{\pi}{4}$ matching, we find

$$I_+ = \{ (\frac{1}{2} b^+(x), b^-(x)) : x \in \mathcal{N}_+ \}, \quad (35)$$

$$I_- = \{ (b^+(y), b^-(y)) : y \in \mathcal{N}_- \},$$

where $\mathcal{N}_+ = \{ x \in N_+ : b^+(x) \in 2N^+_+ \}$.

Proposition 7.9. Let $M$ be a $\pm \frac{\pi}{4}$-twisted connected sum of blocks $Z_+$ and $Z_-$, where $Z_+$ is pleasant, and let

$$\tilde{W} : \tilde{N}_+ \oplus \tilde{N}_- \to N^+_+ \oplus N^+_-, \ (x, y) \mapsto (\frac{1}{2} b^+(x) + b^+(y), b^-(x) + b^-(y)).$$

Then

(i) $\delta(H^3(T^2 \times \Sigma)) \cong \text{coker} \tilde{W}$
(ii) Under the hypothesis of Corollary 7.8, this isomorphism maps \( p_M \mod 24 \) to the image of \( 1/2 \text{even} \)-connected sum \( G_2\)-manifolds.

(iii) Let \( z_1, z_2 \in \text{Tor} \delta(H^3(T^2 \times \Sigma)) \), and let \( (\alpha_1, \beta_1), (\alpha_2, \beta_2) \in N^*_+ \oplus N^*_- \) represent the images of \( z_1 \) and \( z_2 \) in \( \text{coker} \text{Tor} \delta \), \( m(\alpha_1, \beta_1) = \overline{W}(x, y) \). Then \( b(z_1, z_2) = \frac{1}{m}(\alpha_2(x) + \beta_2(y)) \in \mathbb{Q}/\mathbb{Z} \).

Proof. (i) is proved in the preceding discussion, while (ii) is immediate from Corollary 7.8.

For (iii), let \( p_1, p_2 \) be pre-images of \( z_1, z_2 \) in \( H^3(T^2 \times \Sigma) \) under \( \text{Tor} \delta \). According to Lemma 7.3,

\[
 b(z_1, z_2) = \frac{1}{m} p_1^* p_2^* ,
\]

where \( m p_1 = p_1^* - p_1^* \in I_+ + I_- \). Now

\[
 p_1^* = u_+ x + u_+ t_+ = u_+ (x + t_+) + u_- x ,
\]

\[
 -p_1^* = -v_+ y + u_- t_- = 2u_+ y + u_- (y + t_-) ,
\]

for some \( t_{\pm} \in T_{\pm} \) (with \( x + t_+ \) even). In particular, \( y + t_- = -x \mod m \). Hence, writing \( p_2 = 2u_+ w_+ + u_- w_- \) for \( w_{\pm} \in L \) (so that \( \alpha_2 = b^+(w_+) \) and \( \beta_2 = b^-(w_-) \)),

\[
 p_1^* p_2^* = -(v_+ y + u_- t_+)(v_+ w_+ + u_- (w_- - w_+)) = -t_- w_+ + y(w_- - w_+) = yw_- - w_+ y + t_- = yw_- + w_+ x + \beta_2(y) + \alpha_2(x) \mod m .
\]

Now let us assume that \( N_+ \) and \( N_- \) are purely at angle \( \frac{\pi}{4} \). Let \( \pi_{\pm} : N_+ (\mathbb{R}) \to N_+ (\mathbb{R}) \) be the orthogonal projections, and recall that pure angle \( \frac{\pi}{4} \) means that \( \pi_{\pm}(x), \pi_{\pm}(y) = \frac{1}{\sqrt{2}}x, y \) for any \( x, y \in N_+ (\mathbb{R}) \). In particular, note that \( \pi_- N^*_+ \subset N^*_+ (\mathbb{R}) \) equals \( (2\pi_+ N_-)^* \). Therefore \( N^*_+ + 2\pi^*_+ N^*_- = (N_+ \cap 2\pi_+ N_-)^* \), and we get a surjective homomorphism

\[
 N_+^* \oplus N_-^* \to (N_+ \cap 2\pi_+ N_-)^* ,
\]

\[
 (\alpha, \beta) \mapsto \alpha - \pi_-^* \beta .
\]

Note further that \( \pi_-^* \circ b^+ \) equals \( b^+ \) on \( N_- (\mathbb{R}) \) and \( \frac{1}{2}b^+ \) on \( N_+ (\mathbb{R}) \). Therefore \( I_{\pm} \) are both contained in the kernel of \( \text{Tor} \delta \). The kernel in fact

\[
 \{ (\alpha, \beta) : \alpha = \pi_-^* \beta \in (N_+ + 2\pi_+ N_-)^* \} ,
\]

isomorphic to \( (N_+ + 2\pi_+ N_-)^* \) by projection to the first component. The images of \( I_{\pm} \) in there are simply \( b^+ (\frac{1}{2} N_+) \) and \( b^+(N_-) \), respectively. Notice that

\[
 b^+ \left( \frac{1}{2} N_+ \right) + b^+(N_-) = \left\{ \frac{1}{2} b^+(x) : x \in N_+ + 2\pi_+ N_- , \ b^+(x) \in 2(N_+ + 2\pi_+ N_-)^* \right\} .
\]

Hence there is a surjection \( f \) from the discriminant group \( \Delta \) of the even integral lattice \( N_+ + 2\pi_+ N_- \) to the quotient of \( \text{Im} \overline{W} = I_+ + I_- \) in the kernel of \( \text{Tor} \delta \), with kernel precisely the 2-torsion \( T_2 \Delta \); thus \( \text{Tor} \delta(H^3(T^2 \times \Sigma)) \cong \Delta/T_2 \Delta \).

To evaluate the torsion linking form on a pair of elements in \( \text{Tor} \delta \) corresponding to images in \( \Delta \) of \( \alpha_1, \alpha_2 \in (N_+ + 2\pi_+ N_-)^* \), note that the corresponding elements in \( N_+^* \oplus N_-^* \) (are \( \alpha_1, 2\pi_+^* \alpha_1 \)). If \( m \alpha_1 = \frac{1}{2} b^+(x + 2\pi_+ y) \) for \( x \in N_+ \) and \( y \in N_- \), then \( m(\alpha_1, 2\pi_+^* \alpha_1) = \overline{W}(x, y) \) and Proposition 7.9 gives \( b_M(f(\alpha_1), f(\alpha_2)) = \frac{1}{m} (\alpha_2(x) + (2\pi_+^* \alpha_2)(y)) = \frac{1}{m} \alpha_2(x + 2\pi_+ y) \).

In summary

Corollary 7.10. For a pure \( \frac{\pi}{4} \) matching where \( Z_+ \) is pleasant

- There is an isomorphism \( f : \Delta/T_2 \Delta \to \text{Tor} \delta \).
- For \( x, y \in \Delta \), \( b_M(f(x), f(y)) = 2b_\Delta (x, y) \), where \( b_\Delta \) is the discriminant form of \( \Delta \).
- The free part of \( \delta(H^3(T^2 \times \Sigma)) \) is naturally isomorphic to \( (N_+ \cap 2\pi_+ N_-)^* \cong (\pi_- N_- \cap N_-)^* \).
- The image of \( p(M) \mod 24 \) in the free part of \( \delta(H^3(T^2 \times \Sigma)) \) corresponds to \( \frac{1}{2} \text{even} + \pi_-^* \text{even} \in (N_+ \cap 2\pi_+ N_-)^* \), or \( \text{even} \in \pi_- N_- \cap N_-^* \).

Note in particular that if \( N_+ \) has 2-elementary discriminant then automatically \( \Delta = T_2 \Delta \) and \( 2\pi_+ N_- \subseteq N_+ \) (and \( \pi_- N_- \supseteq N_- \)), so \( \text{Tor} \delta \) is torsion-free, and the direct summand \( \delta(H^3(T^2 \times \Sigma)) \subseteq H^4(M) \) is naturally isomorphic to \( N_+^* \).
7.4. $\frac{\pi}{6}$-twisted connected sums. Now we move on to describing the torsion in $H^3(M)$ and the divisibility of $p(M)$ in the case $\vartheta = \frac{\pi}{6}$. The calculations are very similar to the case $\vartheta = \frac{\pi}{4}$, but the details are just sufficiently different to require repetition.

We use a pair of involution blocks $Z_{\pm}$, but recall that there is a basic asymmetry in the set-up (see Figure 11).

We assume that $Z_{\pm}$ are both pleasant, in order that $H^4(M_{\pm})$ are torsion-free. Therefore the only contribution to the torsion comes from the Mayer-Vietoris map $\delta : H^3(T^2 \times \Sigma) \to H^3(M)$, whose image is a split summand in $H^3(M)$. By Lemma 2.11 the image of $H^3(M_{\pm}) \to H^3(T^2 \times \Sigma)$ is exactly

$$I_{\pm} := \{v_{\pm} n + u_{\pm} t : n \in N_{\pm}, t \in T_{\pm}, n + t = 0 \mod 2L\}.$$ (38)

The image $\delta(H^3(T^2 \times \Sigma))$ is isomorphic to $H^3(T^2 \times \Sigma)/(I_+ + I_-)$.

Let \(\{2u_+, 2u_-\}\) be a basis of $H^1(T^2)$, so that we may define a surjective homomorphism

$$H^3(T^2 \times \Sigma) \to N^*_+ \oplus N^*_-,$$

$$2u_+ x + 2u_- y \mapsto (b^+(x), b^-(y))$$ (39)

for $x, y \in L$, where $b^\pm : L \to N^*_{\pm}$ is defined by the intersection form. Elements in the kernel of (39) have $x \in T_+, y \in T_-$, so definitely lie in $I_+ + I_-$. This reduces the problem to understanding the image of the induced isomorphism

$$\delta(H^3(T^2 \times \Sigma)) \cong (N^*_+ \oplus N^*_-)/(I_+ + I_-),$$ (40)

where $\bar{I}_{\pm}$ is the image of $I_{\pm}$ under (39). Using that $v_+ = u_+ + 2u_-, \ v_- = 2u_+ + 3u_-$ in a $\vartheta = \frac{\pi}{6}$ matching, we find

$$\bar{I}_+ = \{(\frac{1}{2}b^+(x), b^-(x)) : x \in \bar{N}_+\},$$

$$\bar{I}_- = \{(b^+(y), \frac{3}{2}b^-(y)) : y \in \bar{N}_-\},$$

where $\bar{N}_{\pm} = \{x \in N_{\pm} : b^\pm(x) \in 2N^*_\pm\}$.

**Proposition 7.11.** Let $M$ be a $\frac{\pi}{6}$-twisted connected sum of pleasant involution blocks $Z_+$ and $Z_-$, and let

$$\bar{W} : \bar{N}_+ \oplus \bar{N}_- \to N^*_+ \oplus N^*_-, \ (x, y) \mapsto \left(\frac{1}{2}b^+(x) + b^-(y), b^-(x) + \frac{3}{2}b^-(y)\right).$$

Then

(i) $\delta(H^3(T^2 \times \Sigma)) \cong \text{coker } \bar{W}$

(ii) Under the hypothesis of Corollary 7.8, this isomorphism maps $p_M \mod 24$ to the image of $(\frac{1}{2}a_+, \frac{3}{4}a_-)$.

(iii) Let $z_1, z_2 \in \text{Tor } \delta(H^3(T^2 \times \Sigma))$, and let $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in N^*_+ \oplus N^*_-$ represent the images in $(N^*_+ \oplus N^*_-)/(I_+ + I_-)$, with $m(\alpha_1, \beta_1) = (\frac{1}{2}b^+(x), b^-(x)) + (b^+(y), \frac{3}{2}b^-(y)) \ (x \in \bar{N}_+, y \in \bar{N}_-)$. Then $b(z_1, z_2) = \frac{1}{m}(\alpha_2(x) + \beta_2(y)) \in \mathbb{Q}/\mathbb{Z}$.

**Proof.** (i) is proved in the preceding discussion, while (ii) is immediate from Corollary 7.8.

For (iii), let $p_1, p_2$ be pre-images of $z_1, z_2$ in $H^3(T^2 \times \Sigma)$. According to Lemma 7.3

$$b(z_1, z_2) = \frac{1}{m}p_1^t p_2,$$

where $mp_1 = p_1^+ - p_1^- \in I_+ + I_-$. Now

$$p_1^+ = v_+ x + u_+ t_+ = u_+(x + t_+) + 2u_- x,$$

$$-p_1^- = v_- y + u_- t_- = 2u_+ y + u_- (3y + t_-),$$

for some $t_\pm \in T_{\pm}$ (with $x + t_+$ and $y + t_-$ both even). In particular, $\frac{3y + t_-}{2} = -x \mod m$. Hence, writing $p_2 = 2u_+ w_+ + 2u_- w_- \text{ for } w_\pm \in L$,

$$p_1^t p_2 = -(v_- y + u_- t_-)(v_+ w_+ + u_+(2w_- - 3w_+)) = \frac{1}{2}(-t_- w_+ + y(2w_- - 3w_+))$$

$$= y w_- - w_+ x = \beta_2(y) \mod m. \quad \Box$$
Now let us assume that $N_+$ and $N_-$ are purely at angle $\pi/2$. Let $\pi_\pm : N_\pm(\mathbb{R}) \rightarrow N_\pm(\mathbb{R})$ be the orthogonal projections, and recall that pure angle $\pi/2$ means that $\pi_\pm(x) \pi_\pm(y) = 3/4 x y$ for any $x, y \in N_\pm(\mathbb{R})$. In particular, see that $3/4 \pi^* N_+ \subset N_+^2(\mathbb{R})$ equals $(2\pi_+ N_-)^*$. We can therefore surjectively map

$$N_+^* \oplus N_-^* \rightarrow (N_+ \cap 2\pi_+ N_-)^*, \quad (\alpha, \beta) \mapsto \alpha - \frac{3}{4} \pi^* \beta. \tag{41}$$

Note further that $\pi^* \circ b^+$ equals $b^+$ on $N_-(\mathbb{R})$ and $\frac{3}{4} b^+$ on $N_+(\mathbb{R})$. Therefore $I_{\pm}$ are both contained in the kernel of $[41]$. The kernel is in fact

$$\{ (\alpha, \beta) : \alpha - \frac{3}{4} \pi^* \beta \in (N_+ + 2\pi_+ N_-)^* \},$$

isomorphic to $(N_+ + 2\pi_+ N_-)^*$ by projection to the first component. The images of $I_{\pm}$ in there are simply $b^+(\frac{3}{4} N_+)$ and $b^+(N_-)$, respectively. Like in the $\vartheta = \frac{3}{4}$ case, their sum is described by [37], implying that the coquotient of $I_+ + I_-$ in the kernel of [41] is isomorphic to the discriminant group $\Delta$ of the even integral lattice $N_+ + 2\pi_+ N_-$ modulo its 2-torsion $T_1 \Delta$.

Similarly to Corollary 7.10, we thus obtain

**Corollary 7.12.** For a pure $\pi/6$ matching where $Z_{\pm}$ are both pleasant

- There is an isomorphism $f : \Delta/T_1 \Delta \rightarrow \ker(H^3(M))$ such that $f(b(x, y)) = 2 b_\Delta(x, y)$, where $b_\Delta$ is the discriminant form of $\Delta$.
- The free part of $H^3(T^2 \times \Sigma)$ is naturally isomorphic to $(N_+ \cap 2\pi_+ N_-)^*$.
- The image of $p(M)$ mod 24 in the free part of $H^3(T^2 \times \Sigma)$ corresponds to $\frac{1}{2} \bar{c}_+ + \frac{1}{2} \pi^* \bar{c}_- \in (N_+ \cap 2\pi_+ N_-)^*$, or $\pi^* \bar{c}_+ + \frac{1}{2} \bar{c}_- \in (\frac{3}{4} \pi^* N_+ \cap N_-)^*$.

Note in particular that if $N_+$ has 2-elementary discriminant then automatically $\Delta = T_1 \Delta$ and $2\pi_+ N_- \subset N_+$ (or $N_- \subset \frac{3}{4} \pi_+ N_+$), so $H^3(M)$ is torsion-free, and $\delta(H^3(T^2 \times \Sigma))$ is naturally isomorphic to $N^*$. (The asymmetry of the construction entails that $N_-$ being 2-elementary is not as helpful: note that $2\pi_+ N_-$ is isometric to $N_-(3)$ which always has some 3-primary discriminant.)

### 7.5. Further Invariants

Any metric of holonomy $G_2$ has an associated torsion-free $G_2$-structure.

To a $G_2$-structure $\varphi$ on closed 7-manifolds, [10] Definition 1.2 associates a value $\nu(\varphi) \in \mathbb{Z}/48$ which is invariant under diffeomorphisms and homotopies, and can thus in particular distinguish components of the moduli space of metrics of holonomy $G_2$.

A stronger invariant $\vartheta(\varphi) \in \mathbb{Z}$ is introduced in [9] Definition 1.4. For extra-twisted connected sums (involving only involutions as in this paper), it can be computed purely in terms of the gluing angle $\vartheta$ and the configuration angles of the matching (Definition 6.9).

**Theorem 7.13** ([9] Corollary 2). Let $(M, \varphi)$ be an extra-twisted connected sum $G_2$-manifold as in Construction 1.1 with gluing angle $\vartheta$. Set $\rho := \pi - 2 \vartheta$. Then

$$\vartheta(\varphi) = -72 \frac{\rho}{\pi} + 3(\text{sign} \rho) \left( \# \{ j \mid \alpha_j \in (\pi - |\rho|, \pi) \} - 1 + 2 \# \{ j \mid \alpha_j \in (\pi - |\rho|, \pi) \} \right),$$

where $\alpha_1, \ldots, \alpha_{19}$ are the configuration angles of the configuration of the hyper-Kähler rotation used in the construction.

There are a number of further invariants of closed 7-manifolds with $G_2$-structure that we do not compute: the quadratic refinement $q$ of the torsion linking form [8] Definition 2.32, the generalised Eells-Kuiper invariant $\mu$ that can detect different smooth structures [11] (26), and the diffeomorphism and homeotopy invariant $\xi(\varphi)$ of the $G_2$-structure. [10] Definition 6.8. The problem is that these invariants are defined in terms of coboundaries, and we have not identified any explicit coboundaries of our extra-twisted connected sums. (The invariant $\nu(\varphi)$ is also defined in terms of coboundaries, but in this case the analytic formula for $\nu$ above gives an alternative method of calculation.) In the case of 2-connected 7-manifolds we have good classification results, but they do in general rely on all of the invariants.
Theorem 7.14 ([11, Theorem 1.2 & 1.3]). Let $M_1$ and $M_2$ be closed 2-connected 7-manifolds, and let $F : H^4(M_2) \to H^4(M_1)$ be a group isomorphism. Then $F$ is realised as $f^*$ for some homeomorphism $f : M_1 \to M_2$ if and only if $F(p(M_2)) = p(M_1)$ and $F$ preserves $b$ and $q$. $F$ is realised as $f^*$ of some diffeomorphism if and only if $F$ is in addition preserves $\mu$.

Theorem 7.15 ([10, Theorem 6.9]). Let $M_1$ and $M_2$ be closed 2-connected 7-manifolds with $G_2$-structures $\varphi_1$ and $\varphi_2$, and let $F : H^4(M_2) \to H^4(M_1)$ be a group isomorphism. Then $F$ is realised as $f^*$ for some diffeomorphism $f : M_1 \to M_2$ such that $f^*\varphi_2$ is homotopic to $\varphi_1$ if and only if $F(p(M_2)) = p(M_1)$, $\nu(\varphi_1) = \nu(\varphi_2)$ and $F$ preserves $b$, $q$ and $\xi$.

However, in many examples the invariants $q$, $\mu$ and $\xi$ are redundant. The quadratic refinement $q$ is uniquely determined by $b$ unless $TH^4(M)$ has 2-torsion. The Eells-Kuiper invariant is vacuous unless $p(M)$ is divisible by 8 modulo torsion, and $\xi$ is completely determined by $\mu$ and $\nu$ when the greatest divisor of $p(M)$ modulo torsion divides 112. Therefore, even though we have not computed $q$, $\mu$ and $\xi$ we can still apply the above classification theorems to many of the examples in [5].

For rectangular twisted connected sums, $q$ and $\mu$ were computed in [12], and $\xi$ by Wallis [28].

8. Examples of extra-twisted connected sums

We now combine the preceding results to produce a number of examples of extra-twisted connected sums. Most of the examples are 2-connected, and their properties are summarised in Tables [4] and [5]. In each case, we describe a configuration of the polarising lattices in terms of a push-out $W$ as described in Remark 6.5 and deduce from Theorem 6.8 that the given configuration is realised by some $\vartheta$-matching.

8.1. Matchings with pure angle $\frac{\pi}{4}$. We begin by considering $\frac{\pi}{4}$-extra twisted connected sums, using configurations where the polarising lattices are at “pure angle” $\frac{\pi}{4}$ as discussed in [6.4] so that Theorem 6.8 can be applied to produce matchings without using any genericity results beyond Proposition 6.7. The topology is also easy to compute using Corollary 7.10.

Matchings among rank 1 blocks are relatively easy to study systematically. We have listed 7 involution blocks of rank 1 (Examples 3.21, 3.22, ..., 3.22, and 5.15), and 18 ordinary rank 1 blocks (17 in Example 3.8 and one in Example 5.15).

If the squares of the generators $x_+$ and $x_-$ of the polarising lattices of the building blocks are $n_+$ and $n_-$ respectively, then as in [28] the necessary and sufficient condition for the existence of a matching is that $2n_+ + n_-$ be a square. An easy computer search identifies that the condition is satisfied for 25 of the 119 pairs of blocks, and to compute the topological invariants from the data in Tables [1]and [3] as follows.

When the condition holds, we can uniquely write $n_+ = 2mq_+^2$ and $n_- = mq_-^2$, for $q_+$ and $q_-$ coprime, and define the configuration by

$$W = \left( \begin{array}{cc} 2mq_+^2 & mq_+q_- \\ mq_+q_- & mq_-^2 \end{array} \right).$$

We can now apply Corollary 7.10 to compute the topological invariants. We find that $\pi_+$ maps $x_-$ to $\frac{q_-}{2q_+}x_+$, so $N_+ + 2\pi_+N_-$ is generated by $\frac{1}{q_-}x_+$, which has square $2m$. Therefore Tor $H^4(M) \cong \mathbb{Z}_m$.

Meanwhile $\pi_-N_+ \cap N_-$ is generated by $q_+x_+$, so the greatest divisor of $p_M$ modulo torsion is $(\pi_+^*, \bar{c}_+)(q_+x_-) = \frac{q_-}{2} \bar{c}_+(x_+) + q_+ \bar{c}_-(x_-)$.

In those cases where the order $m$ of Tor $H^4(M)$ divides the greatest divisor of $p_M$ modulo torsion, the above computation does not suffice to determine $p_M$ up to isomorphisms of $H^4(M)$. However, in all cases it turns out that the greatest divisor of $p_M$ equals the greatest divisor of $p_M$ modulo torsion; then it is possible to choose the isomorphism $H^4(M) \cong \mathbb{Z}_m \times \mathbb{Z}_m$, so that the image of $p_M$ has no $\mathbb{Z}_m$ component. When $m = 2$ there is nothing to check, since $p_M$ is even a priori for any spin 7-manifold. In the remaining 4 cases, we find that $\frac{1}{2} \bar{c}_+$ and $\bar{c}_-$ are both divisible by $m$, so $p_M$ is too.

Finally, $b_3(M)$ is simply $22 + b_3^s(Z_+) + b_3(Z_-)$ by [31]. This is even, so cannot coincide with $b_3$ of any 2-connected ordinary TCS.
These topological invariants of the 25 $\frac{\pi}{4}$-matchings of rank 1 blocks are summarised in Table 4, listing $b_i(M)$, the greatest divisor $d$ of $p(M)$ and the order of $TH^4(M)$. We also list the self-linking of a generator of $TH^4(M)$ when it is not vacuous (i.e., when the order is of the cyclic group $TH^4(M)$ is greater than 2). We have not included the $\bar{\nu}$-invariant in the table, since it is the same in all cases: for a $\frac{\pi}{4}$-matching of rank 1 blocks, the only possibility for the configuration angles is that $\alpha_1 = \cdots = \alpha_{19} = 0$, so Theorem 7.13 gives $\bar{\nu} = -39$.

We now give a number of examples of pure angle $\frac{\pi}{4}$-matchings of blocks of rank 2. In each case we define the desired configuration by writing down a symmetric $4 \times 4$ matrix $W$, where the diagonal $2 \times 2$ blocks are the polarising lattices $N_+$ and $N_-$ of the two building blocks, and the off-diagonal blocks are chosen to ensure that $N_{\pm} = N_{\pm}$: this can be verified by checking that $\pi_+(x), \pi_-(y) = \frac{1}{2} x, y$ for any $x, y \in N_+$. By using bases for $N_+$ and $N_-$ that consist of edges of the respective ample cones (i.e., the bases used in Tables 2 and 3), verifying hypothesis (26) of Theorem 6.8 becomes a simple matter of checking that some element in the positive quadrant of $N_+$ is mapped to the positive quadrant of $N_-$ by $\pi_+$ (or vice versa). Theorem 6.8 then produces a matching with the desired configuration. The resulting $\frac{\pi}{4}$-twisted connected sum $M$ has $b_i(M) = 21 + b_{2i}(Z_+) + b_{2i}(Z_-)$ by (31), and the main remaining topological invariants are easily computed using Corollary 7.10.

We collect the data of these and all remaining 2-connected examples in Table 5. We list for each example the gluing angle, the blocks used, $b_i(M)$, the greatest divisor $d$ of $p(M)$ (which for all of the examples is the same whether we work modulo torsion or not), the order of the torsion subgroup $TH^4(M)$, a description of the torsion linking form $b$, and $\bar{\nu}$. When the torsion $TH^4(M)$ is cyclic we describe the linking form by giving the self-linking of a generator. The only examples of non-cyclic $TH^4(M)$ are $(\mathbb{Z}/2)^2$, where the possibilities for the linking form are that it is diagonalisable $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$, or hyperbolic $\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$.

**Example 8.1.** We match the involution block from Example 3.28 (from one-point blow-up of degree 1 del Pezzo 3-fold) and the regular block from Example 3.9 (from degree 1 del Pezzo 3-fold blown up in an elliptic curve) at pure angle $\frac{\pi}{4}$. The polarising lattices are $N_+ = (\frac{3}{2} \frac{3}{2})$ and $N_- = (\frac{3}{2} \frac{3}{2})$. 

| $Z_+$ | $Z_-$ | $b_3$ | $d$ | $TH^4$ | $b$ |
|-------|-------|-------|-----|--------|-----|
| 3.22  | 3.8   | 60    | 24  | 4      | $\frac{1}{4}$ |
| 3.21  | 3.8   | 64    | 24  | 2      | $\frac{1}{4}$ |
| 3.22  | 3.8   | 68    | 6   | 3      | $\frac{1}{4}$ |
| 3.22  | 3.8   | 72    | 12  | 2      | $\frac{1}{4}$ |
| 3.22  | 3.8   | 74    | 12  | 4      | $\frac{1}{4}$ |
| 3.21  | 3.8   | 78    | 4   | 2      | $\frac{1}{4}$ |
| 3.21  | 3.8   | 78    | 24  | 2      | $\frac{1}{4}$ |
| 5.15  | 3.8   | 82    | 4   | $\frac{1}{4}$ |
| 3.22  | 3.8   | 86    | 8   | 2      | $\frac{1}{4}$ |
| 3.22  | 3.8   | 86    | 12  | 2      | $\frac{1}{4}$ |
| 3.22  | 3.8   | 92    | 4   | $\frac{1}{4}$ |
| 3.21  | 3.8   | 92    | 2   | 2      | $\frac{1}{4}$ |
| 5.15  | 3.8   | 96    | 2   | $\frac{1}{4}$ |

**Table 4.** Extra-twisted connected sums of rank 1 one blocks, with $\vartheta = \frac{\pi}{4}$
and we define the configuration using the matrix

\[ W = \begin{pmatrix} 2 & 2 & 2 & 1 \\ 2 & 0 & 2 & 0 \\ 2 & 2 & 4 & 2 \\ 1 & 0 & 2 & 0 \end{pmatrix}. \]

Actually, because Example 5.28 is not a semi-Fano block, Proposition 3.7 does not provide the genericity result needed for Theorem 6.8 to produce matchings; the required genericity result is instead Lemma 4.3.

The resulting \( \frac{2}{3} \)-twisted connected sum \( M \) is 2-connected, with \( b_3(M) = 21 + 44 + 32 = 97 \). Because \( N_+ \) has 2-elementary discriminant, it is immediate from Corollary 7.10 that \( H^4(M) \) is torsion-free.

In the respective bases for \( N_+^* \), we have \( \hat{c}_+ = (26 \ 24) \) and \( \hat{c}_- = (20 \ 12) \), while \( \pi_+ : N_- \to N_+ \) is represented by \( \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \). In the basis for \( N_+^* \) we thus get \( \pi_+^* c_+ = (26 \ 24) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (26 \ 12) \), and

\[ \pi_+^* \hat{c}_+ + \hat{c}_- = (26 + 24 + 12 = 62) \]

so \( p(M) \) has greatest divisor 2 by Corollary 7.10.

By Theorem 7.14 there is a unique diffeomorphism class of 2-connected 7-manifolds \( M \) with \( b_3(M) = 97 \), torsion-free \( H^4(M) \) and \( d = 2 \). According to [7, Table 3], there are 2 different rectangular twisted connected sums of rank 1 Fano blocks with these invariants, so yield further torsion-free \( G_2 \)-structures on the same manifold. However, the \( \frac{2}{3} \)-twisted connected sum has \( \nu = -12 \) while the rectangular twisted connected sums have \( \nu = 24 \), so the \( G_2 \)-structures cannot be homotopic. In particular, the moduli space of holonomy \( G_2 \) metrics on this manifold is disconnected.

Example 8.2. Match Example 5.14 (from K3 with non-symplectic involution that is a branched double cover of \( \mathbb{P}^1 \times \mathbb{P}^1 \)) and Example 3.9, (from blow-up of complete intersection of two quadrics in an elliptic curve) using the configuration defined by

\[ W = \begin{pmatrix} 0 & 2 & 4 & 0 \\ 2 & 0 & 1 & 1 \\ 2 & 2 & 8 & 4 \\ 2 & 0 & 4 & 0 \end{pmatrix}. \]

Now \( b_3(M) = 21 + 32 + 24 = 77 \). Corollary 7.10 gives that \( H^4(M) \) is torsion-free. Also, \( \pi_+^* \hat{c}_+ + \hat{c}_- = (12 \ 12) \begin{pmatrix} 2 \\ 2 \end{pmatrix} + (28 \ 12) = (30 + 28 \ 6 + 12) = (58 \ 18) \), whose greatest divisor is 2.

These are the same invariants as Example 8.18. Moreover, according to [7, Table 3], there is also a rectangular twisted connected sum of rank 1 Fano blocks (namely Examples 3.8 and 3.8) with these invariants. Thus the smooth 2-connected 7-manifold \( M \) with \( b_3(M) = 77 \), torsion-free \( H^4(M) \) and \( d = 2 \) admits torsion-free \( G_2 \)-structures with \( \varphi = -36, -48 \) and 0, so its moduli space of holonomy \( G_2 \) metrics has at least 3 components.

Example 8.3. Match Examples 3.27 (from double cover of quadric-fibred degree 2 semi del Pezzo 3-fold) and 3.9, using the configuration defined by

\[ W = \begin{pmatrix} 4 & 4 & 4 & 2 \\ 4 & 0 & 4 & 0 \\ 4 & 4 & 8 & 4 \\ 2 & 0 & 4 & 0 \end{pmatrix}. \]

\( b_3(M) = 21 + 12 + 24 = 57. \)

To use Corollary 7.10 to compute \( TH^4(M) \), note that \( 2\pi_+ N_- \) is contained in \( N_+ \), so \( N_+ + 2\pi_+ N_- = N_+ \). The discriminant \( \Delta \cong (\mathbb{Z}/4)^2 \), so \( TH^4(M) \cong (\mathbb{Z}/2)^2 \). The torsion linking form is diagonal. \( TH^4 = (\mathbb{Z}/2)^2 \) with diagonal linking form.

\[ \pi_+^* \hat{c}_+ + \hat{c}_- = (28 \ 12) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + (28 \ 12) = (56 \ 18) \), so Corollary 7.10 implies that the greatest divisor of \( p(M) \) modulo torsion is 2. Since there is only 2-torsion, and \( p(M) \) is even a priori, \( p(M) \) cannot have any interesting torsion component.
Example 8.4. Matching Examples 3.27 (from double cover of quadric-fibred degree 4 semi del Pezzo 3-fold) and 3.9 using

$$W = \begin{pmatrix} 8 & 4 & 6 & 4 \\ 4 & 0 & 2 & 0 \\ 6 & 2 & 8 & 4 \\ 4 & 0 & 4 & 0 \end{pmatrix}.$$  

The calculations are very similar to the previous example. We again find $b_3(M) = 21 + 12 + 24 = 57$ and $TH^4(M) \cong (\mathbb{Z}/2)^4$, but having changed $N_+$ we now find that the torsion linking form is hyperbolic.

$$\pi^*_+ \bar{c}_+ + \bar{c}_- = (32 12) \left( \begin{smallmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} \end{smallmatrix} \right) + (28 12) = (46 24),$$  

so Corollary 7.10 implies that the greatest divisor of $p(M)$ modulo torsion is 2. Again $p(M)$ cannot have any interesting torsion component.

Thus this example is distinguished from Example 8.3 only by the torsion linking form.

Example 8.5. Match Examples 3.26 (from double cover of conic-fibred degree 2 del Pezzo 3-fold) and 3.13 (ordinary block from the conic-fibred degree 2 del Pezzo 3-fold itself) using

$$W = \begin{pmatrix} 4 & 6 & 6 & 2 \\ 6 & 2 & 2 & 3 \\ 6 & 2 & 4 & 6 \\ 2 & 3 & 6 & 2 \end{pmatrix}.$$  

$b_3 = 21 + 6 + 18 = 45. N_+ + 2\pi_+ N_- = N_+$, whose discriminant group $\Delta \cong \mathbb{Z}/14 \times \mathbb{Z}/2$. Thus $TH^4(M) \cong \Delta/T_2 \Delta \cong \mathbb{Z}/7$, and the image of $\alpha := (10) \in \Delta$ is a generator of $TH^4(M)$. Now $b_\Delta(\alpha, \alpha) = (10) \left( \begin{smallmatrix} 1 & -1 \\ 2 & 1 \end{smallmatrix} \right)^{-1} \left( \begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right) = -\frac{1}{7}$, so the image in $TH^4(M)$ has self-linking $-\frac{2}{7}$. Since 2 is a quadratic residue mod 7, another generator has self-linking $-\frac{1}{7}$.

Here is a rank 3 matching.

Example 8.6. Use involution block from Example 5.15 and ordinary block from Example 3.10. Match using

$$W = \begin{pmatrix} 2 & 2 & 2 & 1 & 1 & 2 \\ 2 & 0 & 2 & 1 & 1 & 0 \\ 2 & 2 & 0 & 0 & 2 & 2 \\ 1 & 1 & 0 & 0 & 2 & 2 \\ 1 & 1 & 2 & 2 & 0 & 2 \\ 2 & 0 & 2 & 2 & 2 & 0 \end{pmatrix}.$$  

$b_3(M) = 20 + 28 + 50 = 98. Since $N_+$ is 2-elementary, $H^4(M)$ is torsion-free.

$$\pi^*_+ \bar{c}_+ + \bar{c}_- = (18 12 12) \left( \begin{smallmatrix} 0 & 1 \\ \frac{1}{2} & -\frac{1}{2} \end{smallmatrix} \right) + (12 12 12) = (18 24 24),$$  

so $d = 6$.

For any pure $\pi_4$ matching of rank 3 blocks, exactly 2 each of the configuration angles $\alpha_1, \ldots, \alpha_9$ are $\frac{\pi}{2}$ and $-\frac{\pi}{2}$ while the other 15 are 0. Thus Theorem 7.13 gives $\nu = -33$.

8.2. Other $\pi_4$-matchings. We now consider a few examples of $\pi_4$-extra twisted connected sums where the configuration does not have pure angle $\frac{\pi}{2}$. This involves carrying out some extra work for each example. In addition to checking hypothesis (26) in Theorem 6.8 we also need to compute $\Lambda_\pm$ as in (25), and verify that the families are $\Lambda_{\pm}$-generic (most of the work for the last step has already been carried out in [4]).

Moreover, we cannot use Corollary 7.10 to compute the topology, but instead have to apply the more cumbersome Proposition 7.9. However, we can speed up the calculations a little with the following observation: if $A_+ \in N_+^\pi$ and $A_- = \pi_- A_+ \in N_-^\pi$, then $(A_+, A_-)$ defines a homomorphism $\text{coker} \hat{W} \to \mathbb{Z}$. 

Example 8.7. Match the involution block from Example 3.22 and the regular block from Example 3.9 at angle $\frac{\pi}{4}$ (but not pure angle) using the matrix

$$W = \begin{pmatrix} 2 & 3 & 1 \\ 3 & 8 & 4 \\ 1 & 4 & 0 \end{pmatrix}.$$ 

$\pi_-$ maps the positive generator $H_+ \in N_+$ to $\frac{1}{2} A_-$, for $A_- := (\frac{1}{2}) \in N_-$. This is in indeed in the ample cone of the family of Example 3.9, so (26) holds.

Now $\Lambda_- = N_-$, so for the family of Example 3.9 we do not need any genericity result beyond Proposition 3.7. On the other hand, $\Lambda_+$ is generated by $N_+$ and the orthogonal complement of $A_-$ in $N_-$, so

$$\Lambda \cong \begin{pmatrix} 2 & 0 \\ 0 & -16 \end{pmatrix}.$$ 

In particular there are no $(-2)$-classes orthogonal to the degree 2 class $H_+$. In particular, Proposition 4.2 implies that the family of blocks from Example 3.22 is $(\Lambda_+, H_+, \mathbb{R}^+)$-generic. So we can apply Theorem 6.8 to find a matching with this configuration.

The resulting extra-twisted connected sum $M$ is 2-connected, and (31) gives $b_3(M) = 23 - 2 - 2 + 46 + 24 + 1 = 91$. Proposition 7.9 shows that the torsion is isomorphic to the cotorsion of the image of the matrix

$$\hat{W} = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 0 & 4 \\ 3 & 4 & 8 \end{pmatrix}.$$ 

Its image is exactly the kernel of $(4 - 1 - 1)$, so the torsion is in fact trivial.

According to [12, Table 4], there are two rectangular twisted connected sums from Fanois of rank 1 or 2, with the same diffeomorphism invariants.

Example 8.8. Match Examples 5.15 (from K3 with non-symplectic involution branched over one-point blow-up of $\mathbb{P}^2$) and 3.9 (from $\mathbb{P}^3$ blown up in a twisted cubic) using

$$W = \begin{pmatrix} 2 & 2 & 2 & 3 \\ 2 & 0 & 1 & 1 \\ 2 & 1 & 2 & 5 \\ 3 & 1 & 5 & 4 \end{pmatrix}.$$ 

Let $A_+ := (\frac{3}{2}) \in N_+$, and $A_- = (\frac{1}{2}) \in N_-$. Then $A_+^2 = 32$ and $A_-^2 = 16$, and $\pi_- A_+ = A_-^2 + A_+$. Thus $A_+ \in N_+^\perp$, so (26) is satisfied.

The orthogonal complements of $A_\pm$ in $N_\pm$ are spanned by $B_\pm$ for $B_+ := (\frac{2}{3})$ and $B_- := (\frac{2}{3})$. $\Lambda_\pm$ is spanned by $N_\pm$ and $B_\pm$, so

$$\Lambda_+ = \begin{pmatrix} 2 & 5 & -1 \\ 2 & 5 & -1 \\ -3 & 2 & -272 \end{pmatrix}, \quad \Lambda_- = \begin{pmatrix} 2 & 5 & -1 \\ 2 & 5 & -1 \\ -3 & 2 & -272 \end{pmatrix}.$$ 

Then Proposition 5.17 and Lemma 4.4 give the genericity results needed for Theorem 6.8 to yield a matching.

$b_3(M) = 23 - 2 - 2 + 32 + 40 + 1 = 92$. By Proposition 7.9 $TH^4(M)$ is isomorphic to the cotorsion of

$$\hat{W} = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 1 & 0 & 1 & 1 \\ 2 & 1 & 2 & 5 \\ 3 & 1 & 5 & 4 \end{pmatrix}.$$
which is trivial. Indeed, coker $\hat{W}$ is mapped isomorphically to $\mathbb{Z}$ by $(23 - 1 - 1)$. This maps $(\frac{1}{2}\hat{e}_+, -\hat{e}_-) = (9 6 - 18 - 22)$ to 76, so $d = 4$.

To compute $\hat{v}$ we need to determine the configuration angles. Note that $\pi_+B_- = \frac{1}{2}B_+$, whose square is $\frac{1}{4}$ of the square of $B_-$. So $B_\pm$ is in the $\frac{1}{4}\pm$-eigenspace of $\pi_+\pi_\pm$. By (27), two of the configuration angles are $\pm 2\psi$ where $(\cos\psi)^2 = \frac{1}{5}$, and the other 17 configuration angles are 0. Because $2\psi$ is in the interval $(\frac{\pi}{2}, \pi)$, Theorem 7.13 gives $\hat{v} = -33$.

The diffeomorphism classifying invariants coincide with those of the extra-twisted connected sum of Examples 3.22 and 3.8_i in line 11 of Table 4 but the $\hat{v}$-invariants differ.

The next two example illustrate the dependence of $\hat{v}$ on the configuration angles.

Example 8.9. Matching of Examples 3.27 (from double cover of a quadric-fibred degree 3 semi del Pezzo, or equivalently a double cover of a small resolution of cubic 3-fold containing a plane) and 3.9_7 (from the blow-up of a quadric 3-fold in an elliptic curve of degree 5), using

$$W = \begin{pmatrix} 6 & 4 & 4 & 5 \\ 4 & 0 & 2 & 2 \\ 4 & 2 & 4 & 7 \\ 5 & 2 & 7 & 6 \end{pmatrix}.$$ 

The ample class $A_+ = (\frac{3}{4}) \in N_+$ (of square 192) is mapped by $\pi_-$ to $A_- = (\frac{3}{2}) \in N_-$ (of square 96), while $\pi_+ A_- = \frac{1}{2}A_+$. Therefore $A_\pm \in N_0^{\pm}$, so $|\psi| = 34$ is satisfied.

The orthogonal complement of $A_+$ in $N_+$ is spanned by $B_- \pm$ for $B_+ \pm = (\frac{-9}{2}, \frac{-11}{2})$, of square $-192$ and $-600$ respectively.

$$\Lambda_+ = \begin{pmatrix} 6 & 4 & 3 \\ 4 & 0 & -4 \\ 3 & -4 & -600 \end{pmatrix}, \quad \Lambda_- = \begin{pmatrix} 4 & 7 & 4 \\ 7 & 6 & -2 \\ 4 & -2 & -192 \end{pmatrix}.$$ 

Proposition 4.6 and Lemma 4.7 provide the genericity results needed for Theorem 6.8 to yield matchings.

$$b_3(M) = 23 - 2 + 12 + 28 + 1 = 60.$$ 

Cokernel of

$$\hat{W} = \begin{pmatrix} 3 & 2 & 4 & 5 \\ 2 & 0 & 2 & 2 \\ 4 & 2 & 4 & 7 \\ 5 & 2 & 7 & 6 \end{pmatrix}.$$ 

is mapped isomorphically to $\mathbb{Z}$ by $(43 - 2 - 2)$, so $H^4(M)$ is torsion-free. $(\frac{1}{2}\hat{e}_+, -\hat{e}_-) = (15 6 - 22 - 26)$ is mapped to 174, so $d = \gcd(174, 24) = 6$.

$\pi_+ B_- = \frac{1}{2}B_+$, whose square is $-12$. Therefore $B_\pm$ are $\pi_+\pi_\pm$-eigenvectors with eigenvalue $\frac{1}{50}$. Then the non-zero configuration angles are $\pm 2\psi$ for $(\cos\psi)^2 = \frac{1}{50}$. Because $\psi \in (\frac{\pi}{2}, \pi)$, Theorem 7.13 gives $\hat{v} = -33$.

Example 8.10. Match Examples 3.25 (from double cover of one-point blow-up of a complete intersection of two quadrics, or equivalently a flop of the small resolution of a cubic 3-fold containing a plane that was used in the previous example) and 3.9_7, using

$$W = \begin{pmatrix} 8 & 8 & 4 & 6 \\ 8 & 6 & 5 & 4 \\ 4 & 5 & 4 & 7 \\ 6 & 4 & 7 & 6 \end{pmatrix}.$$ 

The ample class $A_+ = (\frac{3}{4}) \in N_+$ (of norm 192) is mapped to $A_- = (\frac{3}{2}) \in N_-$ (of norm 96), while $A_-$ is mapped to $\frac{1}{2}A_+$. So $A_\pm \in N_0^{\pm}$. The orthogonal complements are spanned by $B_+ = (\frac{-9}{2}, \frac{-11}{2}) \in N_+$, and $B_- = (\frac{-13}{2}, \frac{-11}{2}) \in N_-$, of square $-192$ and $-600$ respectively.

$$\Lambda_+ = \begin{pmatrix} 8 & 8 & 14 \\ 8 & 6 & -21 \\ 14 & -21 & -600 \end{pmatrix}, \quad \Lambda_- = \begin{pmatrix} 4 & 7 & 14 \\ 7 & 6 & -14 \\ 14 & -14 & -192 \end{pmatrix}.$$
Proposition 4.6 and Lemma 4.7 provide the genericity results needed for Theorem 6.8 to yield matchings.

\( b_3(M) = 60 \) just as in the previous example. Also, we find again that \( H^4(M) \) is torsion-free, and that \( d = 6 \), so the classifying diffeomorphism invariants all agree.

However, \( \pi_+ B_- = \frac{7}{2} B_+ \), whose square is \(-588\). Therefore the non-trivial configuration angles \( \pm 2\psi \) are in this case given by \((\cos \psi)^2 = \frac{49}{50} \). Since \( 2\psi < \frac{\pi}{2} \), Theorem 7.13 yields \( \nu = -39 \).

The next two examples of \( \frac{3}{4} \)-twisted connected sums are related by an orientation-reversing diffeomorphism. As the underlying manifold has \( TH^4 = \mathbb{Z}/3 \), it does not admit an orientation reversing self-diffeomorphism, components of moduli space distinguished by sign of \( \nu \).

Example 8.11. Match Example 3.22, (from double cover of cubic hypersurface) with Example 3.23, (from double cover of (1,1)-divisor). The polarising lattices are \( N_+ = (6) \) and \( N_- = \left( \frac{3}{4}, \frac{1}{2} \right) \), and we use the configuration defined by

\[
W = \begin{pmatrix} 6 & 3 & 3 \\ 3 & 2 & 4 \\ 3 & 4 & 2 \end{pmatrix}.
\]

If \( H_+ \) is the generator of \( N_+ \) and \( A_- := (1) \in N_- \) then \( \pi_+ A_- = H_+ \) and \( \pi_- H_+ = \frac{1}{2} A_- \), so \( N_+ = N_+^\mathbb{Z} \) and \( A_- \in N_+^\mathbb{Z} \). Thus condition (20) holds.

The orthogonal complement of \( A_- \) in \( N_- \) is generated by \( B_- = \left( \frac{1}{3}, 1 \right) \), and

\[
\Lambda_+ = N_+ \oplus B_- \mathbb{Z} \cong \begin{pmatrix} 6 & 0 \\ 0 & -12 \end{pmatrix}.
\]

The family of blocks from Example 3.8 is \( (\Lambda_+, H_+, \mathbb{R}^+) \)-generic by Proposition 4.5, so Theorem 6.8 yields matchings with the given configuration.

\( b_3(M) = 23 - 1 - 2 + 18 + 32 + 1 = 71 \). By Proposition 7.9, \( \delta(H^1(T^2 \times \Sigma)) \) is isomorphic to the cokernel of

\[
\mathring{W} = \begin{pmatrix} 3 & 3 & 3 \\ 3 & 2 & 4 \\ 3 & 4 & 2 \end{pmatrix}.
\]

The image of \( \mathring{W} \) is an index 3 sublattice of the kernel of \((2-1-1) : \mathbb{Z}^3 \to \mathbb{Z} \), so \( TH^4(M) \cong \mathbb{Z}/3 \). The cotorsion of \( \mathring{W} \) is generated by \( \left( \frac{1}{3} \right) \). Its preimage under \( \mathring{W} \) is \( \frac{1}{2} \left( \frac{1}{6} \right) \), so by Proposition 7.9 the corresponding generator of \( TH^4(M) \) has self linking \( \frac{1}{6} \).

The image of \((\frac{1}{2} e_+, \mathring{c}_-) = (15 - 18 - 18) \in \mathbb{Z} \) is 56, so the greatest divisor of \( p(M) \) modulo torsion is \( \gcd(66, 24) = 6 \). Since this is not coprime to the order of the torsion subgroup, we also need to check the divisibility of \( p(M) \) itself to determine the isomorphism class of the pair \((H^4(M), p(M))\). But the image of \((15 - 18 - 18) \in \text{coker} \mathring{W} \) is divisible by 6 too, so we can choose an isomorphism \( H^4(M) \cong \mathbb{Z}^{21} \times \mathbb{Z}/3 \) such that the image of \( p(M) \) has no \( \mathbb{Z}/3 \) component.

We find \( \nu = -36 \) like in Example 8.7.

Example 8.12. Match Example 3.23, (from double cover of (1,1)-divisor) with Example 8.8, (from cubic 3-fold in \( \mathbb{P}^4 \)). The polarising lattices are the same as in the previous example, except that the roles of \( N_+ \) and \( N_- \) have been swapped, so we can use essentially the same \( W \) as above to define the configuration. The justification for existence of matching is then just the same, and \( \nu = -36 \) by the same calculation as before.

However, the topological computations are different from the previous example, even though most of the final values turn out to be the same. This time \( b_3(M) \) is computed by \( 23 - 1 - 2 + 14 + 36 + 1 = 71 \), while \( TH^4(M) \) etc is controlled by

\[
\mathring{W} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 3 & 6 \end{pmatrix}.
\]
The image of $\hat{W}$ is an index 3 sublattice of the kernel of $(1 \ 1 \ -1) : \mathbb{Z}^3 \to \mathbb{Z}$, so $TH^4(M) \cong \mathbb{Z}/3$. The cotorsion of $\hat{W}$ is generated by $(\frac{1}{2}, \frac{1}{2})$. Its preimage under $\hat{W}$ is $\frac{1}{2}(0 \ 0 \ 1)$, so by Proposition 7.9, the corresponding generator of $TH^4(M)$ has self linking $\frac{3}{2}$.

$\left(\frac{9}{2}s_+, \bar{c}_-\right) = \left(9 \ 9 \ -24\right)$, which is divisible by 6 modulo the image of $\hat{W}$. Thus $p(M)$ is divisible by 6. The image in the free part of the cokernel is $9 + 9 + 24 = 42$, so the greatest divisor of $p(M)$ modulo torsion is 6 too.

Since the torsion-linking form is different from Example 8.11, there is no orientation-preserving diffeomorphism between these $\frac{7}{4}$-twisted connected sums. However, if we reverse the orientation of one, then the sign of the torsion linking form changes (as does $\bar{\nu}$) while the other invariants stay the same, so there does exist an orientation-reversing diffeomorphism.

**Remark 8.13.** Recalling from §8.13 that changing the sign of the gluing angle corresponds to reversing orientation, we could rephrase this as: If we use the configuration in this example to construct a $\left(-\frac{7}{4}\right)$-twisted connected sum, then that is oriented-diffeomorphic to the $\frac{7}{4}$-twisted connected sum from Example 8.11. However, the $\left(-\frac{7}{4}\right)$-twisted connected sum has $\bar{\nu} = 36$, so the two components of the $G_2$ moduli space are distinguished. To emphasise this point, the entry in Table 5 for Example 8.12 lists the $\left(-\frac{7}{4}\right)$-twisted connected sum.

Finally, here is a $\frac{7}{4}$-matching using a configuration where there is a non-trivial intersection between the polarising lattices.

**Example 8.14.** The involution blocks in Example 3.9, (from double cover of one-point blow-up of $\mathbb{P}^3$) have polarising lattice $N_+ = (\frac{1}{2} \frac{1}{2})$, while Example 3.11 (from blow-up of $\mathbb{P}^3$ in an elliptic curve of degree 7) has $N_- = (\frac{1}{2} \frac{1}{2})$. Let $A_+ := (\frac{1}{1}) \in N_+$ and $A_- := (\frac{1}{1}) \in N_-$. The respective orthogonal complements are spanned by $B_+ := (\frac{1}{0}) \in N_+$ and $B_- := (\frac{1}{0}) \in N_-$. We have $A_+^2 = 98$ and $B_+^2 = B_-^2 = -98$. We can thus view $N_+$ as the overlattice extending $\left(\frac{1}{1} \frac{0}{98}\right)$ by adjoining $\frac{1}{1}(196 + 8B_+)$, and $N_-$ as extending $\left(\frac{1}{1} \frac{98}{98}\right)$ by $\frac{1}{1}(5A_- + 3B_-)$. Now extending

$$
\begin{pmatrix}
196 & 0 & 98 \\
0 & -98 & 0 \\
98 & 0 & 98
\end{pmatrix}
$$

by $\frac{1}{25}(9A_+ + 8B_+)$ and $\frac{1}{14}(\frac{1}{2} \frac{3}{5})$ defines an integral lattice $W$ that contains $N_+$ and $N_-$, and can be used to define a configuration where $A_{\pm} \in N_{\pm}$. Alternatively, $W$ can be described as the quotient of the degenerate lattice

$$
\begin{pmatrix}
4 & 4 & 5 & 3 \\
4 & 2 & 2 & 4 \\
5 & 2 & 4 & 9 \\
3 & 4 & 9 & 8
\end{pmatrix}
$$

by its kernel. In any case, although this configuration does not have pure angle $\frac{7}{4}$, because $N_{\pm}$ is spanned by $N_{\pm}$ and $N_+ \cap N_-$ it is still the case that $N_{\pm} = A_{\pm}$. Therefore we do not need any genericity results beyond Proposition 5.7 in order to produce matchings with this configuration from Theorem 6.8.

The resulting $\frac{7}{4}$-twisted connected sums have $\pi_2 M \cong H^2(M) \cong N_+ \cap N_- \cong \mathbb{Z}$, so are not 2-connected. From §31, we get $b_3(M) = 23 - 2 - 2 + 1 + 16 + 12 + 1 = 49$. The cokernel of

$$
\hat{W} = \begin{pmatrix}
2 & 2 & 5 & 3 \\
2 & 1 & 2 & 4 \\
5 & 2 & 4 & 9 \\
3 & 4 & 9 & 8
\end{pmatrix}
$$

is mapped isomorphically to $\mathbb{Z}$ by $(1 \ 8 \ -3 \ -1)$, so $H^4(M)$ is torsion-free. The image of $(\frac{7}{4}s_+, \bar{c}_-) = (10 \ 9 \ -22 \ -32)$ is 186, so the greatest divisor of $p(M)$ is $d = \gcd(186, 24) = 6$.

All 19 of the configuration angles $\alpha_1 = \cdots = \alpha_{19} = 0$, so $\bar{\nu} = -39$ by Theorem 7.13.
8.3. $\pi/6$-matchings.

Example 8.15. We can search for $\pi/6$-matchings of rank 1 involution blocks similarly to how we found the $\pi/4$-matchings of rank 1 blocks in Table 4. If the generators of the polarising lattices square to $n_+$ and $n_-$ respectively, then there is a $\pi/6$-configuration if and only if $3n_+n_-$ is a square integer. Among the 7 rank 1 involution blocks in Table 3, there are 6 such (ordered) pairs.

For instance, we can match the involution blocks from Examples 3.22.1 and 3.22.3 at pure angle $\pi/6$ using the matrix

$$W = \begin{pmatrix} 2 & 3 \\ 3 & 6 \end{pmatrix}.$$

Then

$$b_3(M) = 23 - 1 - 1 + 18 + 46 + 1 = 86.$$
The topological calculations are the same as in the previous example, except that using the configuration defined by non-diffeomorphic extra-twisted connected sums. Leading to $d = 8$ instead. So different pure angle matchings of the same pair of blocks can lead to non-diffeomorphic extra-twisted connected sums.

Since $N_+\quad$ is 2-elementary, $H^4(M)$ is torsion-free. Further we have that $\frac{3}{2} \pi_- N_+ \cap N_-$ is $N_-\quad$, so $\pi_+ \tilde{e}_+ + \frac{3}{2} \tilde{e}_- \in (\frac{3}{2} \pi_- N_+ \cap N_-)^* = N_+^* \cong \mathbb{Z}$ corresponds to $26 \cdot 3 + \frac{3}{2} \cdot 30 = 54$. Hence the greatest divisor of $p(M)$ is 6.

If we swap the roles of those two blocks, then we instead define the configuration by

$$W = \begin{pmatrix} 6 & 3 \\ 3 & 2 \end{pmatrix}.$$ 

$\pi_+$ of the generator of $N_-$ is half the generator of $N_+\quad$, so in particular $N_+ + 2 \pi_+ N_- = N_+\quad$. Its discriminant group is $\Delta = \mathbb{Z}/6\mathbb{Z}$, so Corollary 7.12 gives $\text{Tor} H^4(M) \cong \Delta \cong \mathbb{Z}/3\mathbb{Z}$, and that a generator has self-linking $\frac{3}{2}$.

We still have $\frac{3}{2} \pi_- N_+ \cap N_-$ is $N_-\quad$. In terms of the generator for $N_+^*$ we have $\pi_+ \tilde{e}_+ + \frac{3}{2} \tilde{e}_- = \frac{3}{2} 26 \cdot \frac{3}{2} + \frac{3}{2} 30 = 28$, so the greatest divisor of $p(M)$ is 4.

Similarly we get two examples by matching Example 3.22 to Example 3.22 and another two by matching it to Example 5.15, with invariants as listed in Table 5.

**Example 8.16.** Match the involution block from Example 3.28 with itself at pure angle $\vartheta = \frac{3}{2}$ using the matrix

$$W = \begin{pmatrix} 2 & 2 & 2 & 2 \\ 2 & 0 & 1 & 2 \\ 1 & 2 & 2 & 0 \end{pmatrix}.$$ 

$b_3(M) = 23 - 2.2 + 2.44 + 2 = 109$.

Since $N_+$ has 2-elementary discriminant, $H^4(M)$ is torsion-free, and to determine the greatest divisor of $p(M)$ we just have to consider $\pi_+ \tilde{e}_+ + \frac{3}{2} \tilde{e}_- \in N_+^*$. We compute $\pi_+ \tilde{e}_+ + \frac{3}{2} \tilde{e}_- = (26 \quad 24 \quad \frac{1}{2} \quad \frac{1}{2}) + (13 \quad 12) = (38 \quad 50)$ so the greatest divisor of $p(M)$ is 2.

According to row labelled 86 in Table 3, there are 3 rectangular TCS of rank 1 Fanos with the same classifying invariants.

**Example 8.17.** Example 3.28 with itself at pure angle $\frac{3}{2}$ again, but this time with configuration

$$W = \begin{pmatrix} 2 & 2 & 2 & 1 \\ 2 & 0 & 3 & 0 \\ 1 & 0 & 2 & 0 \end{pmatrix}.$$ 

The topological calculations are the same as in the previous example, except that $p(M)$ is determined from $\pi_+ \tilde{e}_+ + \frac{3}{2} \tilde{e}_- = (26 \quad 24 \quad \frac{3}{2} \quad \frac{1}{2}) + (13 \quad 12) = (40 \quad 24)$ leading to $d = 8$ instead. So different pure angle matchings of the same pair of blocks can lead to non-diffeomorphic extra-twisted connected sums.

**Example 8.18.** Match the involution blocks from Examples 3.28 and 3.27 at pure angle $\vartheta = \frac{3}{2}$ using the configuration defined by

$$W = \begin{pmatrix} 2 & 2 & 4 & 2 \\ 2 & 0 & 3 & 0 \\ 4 & 3 & 10 & 4 \\ 2 & 0 & 4 & 0 \end{pmatrix}.$$ 

$b_3(M) = 21 + 44 + 12 = 77$. $N_+$ is 2-elementary, so $H^4(M)$ is torsion-free $\pi_+ \tilde{e}_+ + \frac{3}{2} \tilde{e}_- = (26 \quad 24 \quad \frac{3}{2} \quad \frac{1}{2}) + (17 \quad 6) = (68 \quad 30)$, with greatest divisor 2.
Example 8.19. We can match the involution blocks from Examples 3.28 and 3.26, with a configuration defined by
\[
W = \begin{pmatrix}
2 & 2 & 4 & 2 \\
2 & 0 & 3 & 3 \\
4 & 3 & 10 & 6 \\
2 & 3 & 6 & 2
\end{pmatrix}.
\]
In fact, instead of applying Theorem 6.8 directly, we can obtain the matchings with this prescribed configuration from the matchings in Example 8.18. This relies on the fact that Example 3.26 is a flop of Example 3.27, and the lattice \( W \) defining the configuration here is isometric to the configuration lattice from Example 8.18. Therefore, for any \( \frac{\pi}{6} \)-matching \( r : \Sigma_+ \to \Sigma_- \) of blocks \( Z_+ \) from Example 3.28 and \( Z_- \) from Example 3.27, as in Example 8.18, flopping \( Z_- \) yields a building block \( \tilde{Z}_- \) in the family of Example 3.26, with the same anticanonical divisor \( \Sigma_- \), so that \( r \) is a \( \frac{\pi}{6} \)-matching of \( Z_+ \) and \( \tilde{Z}_- \). Thus the \( \frac{\pi}{6} \)-twisted connected sums from this Example and Example 8.18 can be regarded as being related by a "\( G_2 \) conifold transition" of the kind discussed in [7, §8].

Flopping does not change the cohomology groups, so just like in the previous example we find that \( b_3(M) = 21 + 44 + 12 = 77 \), and \( H^4(M) \) is torsion-free. On the other hand \( \pi_+^* c_+ + \frac{1}{2} \pi_-^* c_- = (26, 24) \left( \frac{3}{2}, \frac{3}{2} \right) + (17, 9) = (68, 36) \), so the greatest divisor of \( p(M) \) is 4 in the example.

Finally we consider a matching that is not at pure angle \( \frac{\pi}{6} \).

Example 8.20. Match Examples 3.26 and 3.8, using
\[
W = \begin{pmatrix}
4 & 6 & 5 \\
6 & 2 & 4 \\
5 & 4 & 6
\end{pmatrix}
\]
Letting \( A_+ = \left( \frac{1}{2} \right) \in N_+ \) and \( H_- \) be the generator of \( N_- \), we find \( \pi_- A_+ = \frac{3}{2} H_- \) and \( \pi_+ H_- = \frac{1}{2} A_+ \), so \( A_+ \in N_+^{\frac{\pi}{6}} \) and \( N_- = N_-^{\frac{\pi}{6}} \). Thus (26) holds. \( \Lambda_- \) is spanned by \( N_- \) and \( B_+ := \left( \frac{4}{5} \right) \), so
\[
\Lambda_- \cong \left[ \begin{array}{ccc}
6 & 0 \\
0 & -126
\end{array} \right].
\]
The family of blocks from Example 3.8, is \( (\Lambda_-, H_- \mathbb{R}^+) \)-generic by Proposition 4.5, so Theorem 6.8 yields a matching with the prescribed configuration.
\[
b_3(M) = 23 - 1 - 2 + 6 + 18 + 1 = 45.\]
The image of
\[
\widehat{W} = \begin{pmatrix}
2 & 3 & 5 \\
3 & 1 & 4 \\
5 & 4 & 9
\end{pmatrix}
\]
is an index 7 sublattice of the kernel of \( (1 1 -1) : \mathbb{Z}^3 \to \mathbb{Z} \), so \( TH^4(M) \cong \mathbb{Z}/7 \). The image of \( (\frac{1}{2} c_+, -\frac{1}{2} c_-) = (14, 9, -15) \) in \( \mathbb{Z}^3 \) is 38, so the greatest divisor of \( p(M) \) modulo torsion \( d = \gcd(38, 24) = 2 \). As this is coprime to the order of the torsion, \( p(M) \) can have no interesting torsion component.

The data we have computed so far is enough to show that this \( \frac{\pi}{6} \)-twisted connected sum is diffeomorphic to Example 8.5, but to determine the orientedness of the diffeomorphism we also need to determine the torsion-linking form. The cotorsion of \( W \) is generated by \( \left( \frac{1}{2} \right) \). That has \( \frac{1}{2} \left( \frac{1}{2} \right) \) as a preimage under \( \widehat{W} \), so the corresponding generator of \( TH^4(M) \) has torsion self-linking \( \frac{2}{3} \). As \( 3 \) is not a quadratic residue mod 7, another choice of generator has self-linking \( \frac{2}{7} \). Thus the diffeomorphisms between this \( \frac{\pi}{6} \)-twisted connected sum and the one from Example 8.5 is orientation-preserving.

References

[1] M. Arap, J. W. Cutrone, and N. A. Marshburn, On the existence of certain weak Fano threefolds of Picard number two, Math. Scand. 120 (2017), 68–86.

[2] M. Berger, Sur les groupes d’holonomie homogène des variétés à connexion affine et des variétés riemanniennes, Bull. Soc. Math. France 85 (1955), 279–330.
[3] J. Blanc and S. Lamy, Weak Fano threefolds obtained by blowing-up a space curve and construction of Sarkisov links, Proc. London Math. Soc. 105 (2012), 1047–1075.
[4] R. Bryant and S. Salamon, On the construction of some complete metrics with exceptional holonomy, Duke Math. J. 58 (1989), 829–850.
[5] G. Castelnuovo, Sulle superficie di genere zero, Mem. delle Soc. Ital. delle Scienze Ser. III 10 (1895), 103–123.
[6] A. Corti, M. Haskins, J. Nordström, and T. Pacini, Asymptotically cylindrical Calabi-Yau 3-folds from weak Fano 3-folds, Geom. Topol. 17 (2013), 1955–2059.
[7] _____, G2-manifolds and associative submanifolds via semi-Fano 3-folds, Duke Math. J. 164 (2015), 1971–2092.
[8] D. Crowley, On the classification of highly connected manifolds in dimensions 7 and 15, Ph.D. thesis, Indiana University, 2002, arXiv:1207.4470.
[9] D. Crowley, S. Goette, and J. Nordström, An analytic invariant of G2-manifolds, arXiv:1505.02734, 2018.
[10] D. Crowley and J. Nordström, New invariants of G2-structures, Geom. Topol. 19 (2015), 2949–2992.
[11] _____, The classification of 2-connected 7-manifolds, arXiv:1411.0656, 2018.
[12] J. W. Cutrone and N. A. Marshburn, Towards the classification of weak Fano threefolds with $\rho = 2$, Cent. Eur. J. Math. 11 (2013), 1552–1576.
[13] T. Fukuda, On the existence of almost Fano threefolds with del Pezzo fibrations, Math. Nachr. 290 (2017), no. 8-9, 1281–1302.
[14] D. Joyce, Compact Riemannian 7-manifolds with holonomy $G_2$. I, J. Diff. Geom. 43 (1996), 291–328.
[15] A. Kovalev, Twisted connected sums and special Riemannian holonomy, J. reine angew. Math. 565 (2003), 125–160.
[16] A. Kovalev and N.-H. Lee, K3 surfaces with non-symplectic involution and compact irreducible $G_2$-manifolds, Math. Proc. Cambridge Philos. Soc. 151 (2011), no. 2, 193–218.
[17] R. Lee and S. H. Weintraub, On the homology of double branched covers, Proc. Amer. Math. Soc. 123 (1995), no. 4, 1263–1266.
[18] V. Nikulin, Integer symmetric bilinear forms and some of their applications, Izv. Akad. Nauk SSSR Ser. Mat. 43 (1979), 111–177, 238, English translation: Math. USSR Izvestia 14 (1980), 103–167.
[19] V. Nikulin and S. Saito, Real K3 surfaces with non-symplectic involutions and applications, Proc. Lond. Math. Soc. 90 (2005), no. 3, 591–654.
[20] M. Reid, Chapters on algebraic surfaces, Complex algebraic geometry (Park City, UT, 1993), IAS/Park City Math. Ser., vol. 3, Amer. Math. Soc., Providence, RI, 1997, pp. 3–159.
[21] B. Saint-Donat, Projective models of K3 surfaces, Amer. J. Math. 96 (1974), 602–630.
[22] D. Wallis, Disconnecting the moduli space of $G_2$-metrics via $U(4)$-coboundary defects, arXiv:1808.09443, 2018.
[23] D. L. Wilkens, Closed $(s−1)$-connected $(2s+1)$-manifolds, $s = 3, 7$, Bull. Lond. Math. Soc. 4 (1972), 27–31.