On estimating the output entropy of a tensor product of the quantum phase-damping channel with an arbitrary channel

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Abstract

We obtained the estimation from below for the output entropy of a tensor product of the quantum phase-damping channel with an arbitrary channel. It is shown that from this estimation immediately follows that the strong superadditivity of the output entropy holds for this channel as well as for the quantum depolarizing channel.

1 Introduction

Let $\mathcal{S}(H)$ denote the set of all states, i.e. positive unit trace operators, in a Hilbert space $H$, $\dim H < +\infty$. By a quantum channel we mean a completely positive linear map $\Phi : \mathcal{S}(H) \to \mathcal{S}(K)$ preserving the trace. A quantum channel $\Phi$ is said to be unital if $\Phi(I_H) = I_K$. Here and in the following we denote $I_L$ the identity operator in a Hilbert space $L$.

Put

$$ S_{\min}(\Phi) = \min_{\rho \in \mathcal{S}(H)} S(\Phi(\rho)), $$

where $S(\rho) = -\text{Tr}(\rho \log \rho)$ is the von Neumann entropy of a state $\rho$. In [8] the quantity

$$ \hat{S}_{\Phi}(\rho) = \min_{\rho = \sum \pi_i \rho_i} \sum \pi_i S(\Phi(\rho_i)) $$

was introduced. We shall say that the strong superadditivity of the output entropy of the channel $\Phi : \mathcal{S}(H) \to \mathcal{S}(H)$ holds if

$$ \hat{S}_{\Phi\otimes\Omega}(\rho) \geq \hat{S}_\Phi(\text{Tr}_K(\rho)) + \hat{S}_\Omega(\text{Tr}_H(\rho)) \quad (1) $$
for any quantum channel $\Omega : \mathcal{G}(K) \to \mathcal{G}(K)$ and states $\rho \in \mathcal{G}(H \otimes K)$. In particular, if the strong superadditivity holds for the channel $\Phi$, then the minimal output entropy is additive with respect to tensor product of channels, i.e.

$$S_{\text{min}}(\Phi \otimes \Omega) = S_{\text{min}}(\Phi) + S_{\text{min}}(\Omega)$$

(2)

is satisfied for all quantum channels $\Omega$. Unfortunately the additivity of minimal output entropy (2) is not valid in general [6]. Nevertheless, it was proved for many significant cases [7,9,10,12]. The strong superadditivity holds for the noiseless channel and for the entanglement-breaking channels [8].

Notice that the prove of (2) in [9,10] is based upon the estimation of the Schatten-von Neumann trace $p$-norms [1]. Let us define a quantum relative entropy as follows

$$S(\rho || \sigma) = Tr(\rho \log \rho) - Tr(\rho \log \sigma),$$

where $\rho, \sigma \in \mathcal{G}(H)$. Then, the quantum H-theorem reads [11]

$$S(\Phi(\rho) || \Phi(\sigma)) \leq S(\rho || \sigma)$$

(3)

for $\rho, \sigma \in \mathcal{G}(H)$ and for all (not only unital in general) quantum channels $\Phi$. In [2,3] it was introduced the method based upon the property (3). Using this method the additivity in the known cases was proved without estimation of $p$-norms. The same method allowed to prove the strong superadditivity for the quantum depolarizing channel [4], quantum-classical channels and quantum erasure channels [5]. Here we will improve the method introduced in [2,3]. It results in the strong estimation from below for the output entropy of a tensor product of the quantum phase-damping channel with an arbitrary quantum channel.

Pick up an arbitrary orthonormal basis $(e_j)$ in the Hilbert space $H$, $\text{dim}H = n < +\infty$. Suppose that non-negative numbers $\lambda_j \geq 0$ form the probability distribution on $(e_j)$ such that $\sum_{j=0}^{n-1} \lambda_j = 1$. Let us take the discrete Fourier transforms of $(e_j)$ and $(\lambda_j)$ as follows

$$\hat{\lambda}_j = \sum_{m=0}^{n-1} e^{\frac{2\pi i}{n} jm} \lambda_m,$$

$$f_j = \sum_{m=0}^{n-1} e^{\frac{2\pi i}{n} jm} e_m,$$

(4)

$0 \leq j \leq n - 1$. Let us define a linear map $\Psi$ by the formula

$$\Psi(|f_j><f_k|) = \hat{\lambda}_{j-k}|f_j><f_k|.$$
\[ 0 \leq j, k \leq n - 1. \] The map \( \Psi \) determines a quantum channel in \( \mathcal{S}(H) \) for which only the phases of a state are damped. Thus, \( \Psi \) is said to be a phase damping channel. Notice that the phase-damping channel introduced in [10] is a particular case of phase-damping channels of our definition.

Define the unitary shift operator \( V \) as follows

\[ Ve_j = e_{j + 1 \mod n}, \quad 0 \leq j \leq n. \]

Then,

\[ \Psi(\rho) = \sum_{j=0}^{n-1} \lambda_j V^j \rho V^{*j}, \quad (5) \]

\( \rho \in \mathcal{S}(H) \).

Given two orthonormal bases \( (f_j) \) in \( H \) and \( (g_j) \) in \( K \) the unit vector \( e \in H \otimes K \) can be represented in two different ways, namely

\[ |e> = \sum_j \mu_j |f_j> \otimes |h_j>, \quad (6) \]

and

\[ |e> = \sum_j \nu_j |\tilde{h}_j> \otimes |g_j>, \quad (7) \]

where \( \mu_j, \nu_j \in \mathbb{C}, \quad \sum_j |\mu_j|^2 = \sum_j |\nu_j|^2 = 1 \) and \( (h_j), (\tilde{h}_j) \) are (non-orthogonal in general) unit vectors. In the following theorem we suppose that \( (f_j) \) in (6) are the same as in (4). On the other hand, an orthonormal basis \( (g_j) \) in (7) will be chosen arbitrarily.

**Theorem 1.** For a unit vector \( e \in H \otimes K \) represented in (6 - 7) the following estimation holds

\[ S((\Psi \otimes \Omega)(|e><e|)) \geq \sum_j |\nu_j|^2 S(\Psi(|\tilde{h}_j><\tilde{h}_j|)) + \sum_j |\mu_j|^2 S(\Omega(|h_j><h_j|)), \]

where \( \Omega \) is an arbitrary quantum channel. Moreover,

\[ Tr_H(|e><e|) = \sum_j |\mu_j|^2 |h_j><h_j|, \]

\[ Tr_K(|e><e|) = \sum_j |\nu_j|^2 |\tilde{h}_j><\tilde{h}_j|. \]

**Corollary 2.** The strong superadditivity of the output entropy holds for quantum phase-damping channel \( \Psi \).
Remark. The phase-damping channel is known to be complementary to the entanglement-breaking channel. Thus, the strong superadditivity for this channel follows from [8].

Now, let us consider the quantum depolarizing channel \( \Upsilon \) defined by the formula
\[
\Upsilon(\rho) = (1 - p)\rho + \frac{p}{n}I_H,
\]
where \( \rho \in \mathcal{S}(H) \), \( 0 < p \leq \frac{n^2}{n^2 - 1} \).

Corollary 3. The strong superadditivity of the output entropy holds for the quantum depolarizing channel \( \Upsilon \).

2 Estimation of output entropy

To prove Theorem 1 we need the following two Propositions.

Proposition 4. Denote \( d \) a number of non-zero terms in (6) such that \( d \leq n \). Then, the orthogonal projection
\[
P = \sum_{j: \mu_j \neq 0} |f_j><f_j| \otimes |h_j><h_j|
\]
on the subspace \( \mathcal{L} \subset H \otimes K \) of the dimension \( \dim \mathcal{L} = d \) has the property
\[
(\Psi \otimes Id)(|e><e|)P = P(\Psi \otimes Id)(|e><e|) = (\Psi \otimes Id)(|e><e|), \quad (8)
\]
i.e. \( \mathcal{L} \) contains the support of the state \( (\Psi \otimes Id)(|e><e|) \).

Proof of Proposition 4.
It follows from (6) that
\[
P|e><e| = |e><e|P = |e><e|.
\]
On the other hand,
\[
VPV^* = P
\]
because \( V|f_j><f_j|V^* = |f_j><f_j| \) for any \( j \). Thus, the result follows from the representation [12]. \( \square \)

Proposition 5. Given \( e \in H \otimes K \) represented as (6) the following estimation holds
\[
S((\Psi \otimes \Omega)(|e><e|)) \geq S((\Psi \otimes Id)(|e><e|)) + \sum_j |\mu_j|^2 S(\Omega(|h_j><h_j|)),
\]
where \( \Omega \) is an arbitrary quantum channel and
\[
Tr_H((|f_j><f_j| \otimes I_K)|e><e|) = |\mu_j|^2 |h_j><h_j|
\]
such that
\[ T_{RH}(|e><e|) = \sum_{j} |\mu_j|^2 |h_j><h_j| \]

with the vectors \((f_j)\) defined by \((\mathbf{4})\).

Proof of Proposition 5.

Let us define a state \(\rho\) as follows
\[ \rho = (\Psi \otimes \text{Id})(|e><e|) = \sum_{j=0}^{n-1} \lambda_j (V^j \otimes I_K)|e><e|(V^{*j} \otimes I_K), \]

Put
\[ \sigma = \frac{1}{d} \Pi, \quad (\mathbf{9}) \]
where \(d = \dim P\) and \(P\) was defined in Proposition 4.

Then, the quantum H-theorem implies
\[ S((\text{Id} \otimes \Omega)(\rho) || (\text{Id} \otimes \Omega)(\sigma)) \leq S(\rho || \sigma). \quad (\mathbf{10}) \]
Notice that
\[ S(\rho || \sigma) = Tr(\rho \log \rho) - Tr(\rho \log \sigma) = -S((\Psi \otimes \text{Id})(|e><e|)) \]
\[ -Tr\left( \frac{1}{n} \sum_{j=0}^{n-1} \lambda_j (V^j \otimes I_K)|e><e|(V^{*j} \otimes I_K) \log \frac{1}{d} \Pi \right) = \]
\[ -S((\Psi \otimes \text{Id})(|e><e|)) + \log d \quad (\mathbf{11}) \]
because \((\mathbf{8})\) is valid in virtue of Proposition 4.

Now we need to calculate \(S((\text{Id} \otimes \Omega)(\rho) || (\text{Id} \otimes \Omega)(\sigma))\). Taking into account \((\mathbf{8})\) we can conclude that the state \((\Psi \otimes \text{Id})(|e><e|)\) can be represented as the sum
\[ (\Psi \otimes \text{Id})(|e><e|) = \sum_{k,l} |f_k><f_l| \otimes y_{kl}, \quad (\mathbf{12}) \]
where \(y_{kl}\) are operators in \(K\) defined by the formula
\[ y_{kl} = \hat{\lambda}_{k-l} \mu_k \mu_l |h_k><h_l| \quad (\mathbf{13}) \]
supported by \(L\) from Proposition 4 and satisfying the relation
\[ y_{kl} |h_l><h_l| = |h_k><h_k|y_{kl} = y_{kl} \]

and \( y_{kl} \neq 0 \) only if \( \mu_k \neq 0, \mu_l \neq 0 \) in (6). In particular,
\[
y_{kk}|h_k><h_k| = |h_k><h_k|y_{kk} = y_{kk}.
\] (14)

Following the definition of \( P \) we get
\[
-Tr((\Psi \otimes \Omega)(|e><e|) \log \frac{1}{d} (\text{Id} \otimes \Omega)(\sigma)) = 

-Tr\left( \sum_{k,l} |f_k><f_l| \otimes \Omega(y_{kl}) \log \frac{1}{d} \sum_k |f_k><f_k| \otimes \Omega(|h_k><h_k|) \right) = 

-\sum_k Tr(\Omega(y_{kk}) \log(\Omega(|h_k><h_k|)) + \log d). \tag{15}
\]

Definition (13) implies that
\[
y_{kk} = |\mu_k|^2 |h_k><h_k| 
\]
and
\[
|\mu_k|^2 = Tr(y_{kk}).
\]

It follows that the terms of a sum in (15) can be rewritten as follows
\[
-Tr((\Psi \otimes \Omega)(|e><e|) \log \frac{1}{d} (\text{Id} \otimes \Omega)(\sigma)) = 

-\sum_k |\mu_k|^2 Tr(\Omega(|h_k><h_k|) \log(\Omega(|h_k><h_k|)) + \log d). \tag{16}
\]

Substituting in (11) equalities (11) and (16) we complete the proof.

□

Proof of Theorem 1.

The noiseless channel \( \text{Id} \) is a partial case of the phase-damping channel. Hence, applying Proposition 5 to the quantity \( S((\Psi \otimes \text{Id})(|e><e|)) \) we obtain the result.

□

Proof of Corollary 2.

Let us consider the quantity
\[
\hat{S}_{\Psi \otimes \Omega}(\rho) = \min_{\rho=\sum_s \pi_s \rho_s} \sum_s \pi_s S((\Psi \otimes \Omega)(\rho_s)) \tag{17}
\]

The minimum in (17) is achieved for some set of pure states \( \rho_s = |e^{(s)}><e^{(s)}| \). Applying Theorem 1 to the values \( S((\Psi \otimes \Omega)(|e^{(s)}><e^{(s)}|)) \) we obtain the following estimation
\[
\hat{S}_{\Psi \otimes \Omega}(\rho) \geq \sum_s \pi_s \sum_j |\nu^{(s)}_j|^2 S(\Psi(|\tilde{h}^{(s)}_j><\tilde{h}^{(s)}_j|))
\]
\[ + \sum_s \pi_s \sum_j |\mu_j^{(s)}|^2 S(\Omega(|h_j^{(s)}><h_j^{(s)}|)) \equiv C, \]

where

\[ Tr_H(|e^{(s)}><e^{(s)}|) = \sum_j |\mu_j^{(s)}|^2 |h_j^{(s)}><h_j^{(s)}| \]

and

\[ Tr_K(|e^{(s)}><e^{(s)}|) = \sum_j |\nu_j^{(s)}|^2 |e_j^{(s)}><e_j^{(s)}|. \]

Notice that

\[ \sum_s \pi_s \hat{S}_\Psi(Tr_K(|e^{(s)}><e^{(s)}|)) \geq \hat{S}_\Psi(Tr_K(\rho)), \]

by the definition of \( \hat{H}_\Psi \), and

\[ \sum_s \pi_s \sum_j |\mu_j^{(s)}|^2 Tr_H(|h_j^{(s)}><h_j^{(s)}|) = Tr_H(\rho). \]

Then, notice that

\[ \sum_s \pi_s \sum_j |\mu_j^{(s)}|^2 S(\Omega(|h_j^{(s)}><h_j^{(s)}|)) \geq \hat{S}_\Omega(Tr_H(\rho)). \]

Hence

\[ C \geq \hat{S}_\Psi(Tr_K(\rho)) + \hat{S}_\Omega(Tr_H(\rho)). \]

\[ \square \]

Proof of Corollary 3.

Following [10] let us define the set of orthonormal bases \( (f^k_j) \) as follows

\[ |f^k_j> = \sum_{s=0}^{n-1} e^{\frac{2\pi is}{n}} e^{\frac{i2\pi j}{d}} e_s, \quad 1 \leq k \leq 2n^2. \]

Then, put

\[ U = \sum_{s=0}^{n-1} e^{\frac{2\pi is}{n}} e_s, \quad |e_s><e_s|, \]

\[ V_k = \sum_{s=0}^{n-1} e^{\frac{2\pi is}{n}} |f^k_s><f^k_s|, \]
1 \leq k \leq 2n^2. Now, consider phase-damping channels determined as follows

\[ \Upsilon_k(\rho) = (1 - \frac{n-1}{n}p)\rho + \frac{p}{n} \sum_{s=1}^{n-1} V_k^s \rho V_k^{*s}, \]

1 \leq k \leq 2n^2. It is straightforward to check that

\[ \Upsilon_k([e_j \rangle \langle e_j]) = |e_{j+1 \mod n} \rangle \langle e_{j+1 \mod n}|. \quad (18) \]

It is known that [10]

\[ \Upsilon(\rho) = \frac{1-p}{1+(n-1)(1-p)} \frac{1}{2n^2} \sum_{k=1}^{2n^2} \Upsilon_k(\rho) \]

\[ + \frac{p}{1+(n-1)(1-p)} \frac{1}{2n^3} \sum_{j=1}^{n-1} \sum_{k=1}^{2n^2} U^j \Upsilon_k(\rho) U^{*j}. \quad (19) \]

Let us consider the quantity

\[ \hat{S}_{\Upsilon \otimes \Omega}(\rho) = \min_{\rho = \sum_s \pi_s \rho_s} \sum_s \pi_s S((\Upsilon \otimes \Omega)(\rho_s)) \quad (20) \]

The minimum in (20) is achieved for some set of pure states \( \rho_s = [e^{(s)}] \rangle \langle e^{(s)}| \). Let us consider the quantity \( S((\Upsilon \otimes \Omega)(\rho_s)) \). Below we shall apply Theorem 1 to estimate the quantity \( S((\Upsilon \otimes \Omega)(\rho_s)) \). There are two possible decompositions (6) and (7) of \( e^{(s)} \). The decomposition (6) is fixed, while (7) is depending on a choice of the orthogonal basis \((g_j)\). We will take \((g_j)\) in such a way that the unit vectors \((\tilde{h}_j)\) in (7) would be orthogonal (it is not so in general). For these purposes it is appropriate to take \((g_j)\) from the Schmidt decomposition of \([e^{(s)}] \rangle \langle e^{(s)}| \) defined by the formula

\[ [e^{(s)}] = \sum_j \nu_j^s [\tilde{g}_j^s] \otimes [g_j^s], \quad (21) \]

\[ \nu_j^s \geq 0, \sum_j |\nu_j^s|^2 = 1. \]

Using the covariance property

\[ W \Upsilon(\rho) W^* = \Upsilon(W \rho W^*) \]

for any \( \rho \in \mathcal{G}(H) \) and any unitary operator \( W \) in \( H \), we can change the orthogonal vectors \((\tilde{g}_j^s)\) in (21) to the vectors \((e_j)\) satisfying the property (18). In fact, it suffices to put \( W \tilde{g}_j^s = e_j \). Thus, we obtain

\[ S((\Upsilon \otimes \Omega)(\rho_s)) = S((\Upsilon \otimes \Omega)([\tilde{e}^{(s)}] \rangle \langle \tilde{e}^{(s)}|)), \quad (22) \]
where
\[ |\tilde{e}^{(s)}> = \sum_j \nu_j^s |e_j> \otimes |g_j> \]
like in (7) with \( \tilde{h}_j = e_j \). Using the representation (19) by concavity of entropy we conclude
\[
S((\Upsilon \otimes \Omega)(\rho_s)) \geq \frac{1 - p}{1 + (n - 1)(1 - p)} \frac{1}{2n} \sum_{k=1}^{2n^2} S((\Upsilon_k \otimes \Omega)(|\tilde{e}^{(s)}><\tilde{e}^{(s)}|)) + \\
\frac{p}{1 + (n - 1)(1 - p)} \frac{1}{2n^3} \sum_{j=1}^{n-1} \sum_{k=1}^{2n^2} S((\Upsilon_k \otimes \Omega)(|\tilde{e}^{(s)}><\tilde{e}^{(s)}|)).
\]
(23)

Applying Theorem 1 to the quantities \( S((\Upsilon_k \otimes \Omega)(|\tilde{e}^{(s)}><\tilde{e}^{(s)}|)) \) we get
\[
S((\Upsilon_k \otimes \Omega)(\rho_s)) \geq \sum_j |\nu_j^s|^2 S(\Upsilon_k(|e_j><e_j|)) \\
+ \sum_j |\mu_j^s|^2 S(\Omega(|h_j><h_j|)),
\]
(24)
while
\[
S(\Upsilon_k(|e_j><e_j|)) = -(1 - \frac{n - 1}{n} p) \log(1 - \frac{n - 1}{n} p) - \frac{p(n - 1)}{n} \log \frac{p}{n}
\]
(25)
for all \( k \) due to the property (18) and
\[
Tr_H(|\tilde{e}^{(s)}><\tilde{e}^{(s)}|) = Tr_H(\rho_s) = \sum_j |\mu_j^s|^2 |h_j><h_j|,
\]
(26)
where \( (h_j^s) \) and \( (\mu_j^s) \) are taken from the representation (6) of \( \tilde{e}^{(s)} \). Substituting (25) to (24) and (24) to (23) we obtain
\[
S((\Upsilon \otimes \Omega)(\rho_s)) \geq -(1 - \frac{n - 1}{n} p) \log(1 - \frac{n - 1}{n} p) - \frac{p(n - 1)}{n} \log \frac{p}{n} + \\
\sum_j |\mu_j^s|^2 S(\Omega(|h_j><h_j|)).
\]
(27)

Notice that
\[
-(1 - \frac{n - 1}{n} p) \log(1 - \frac{n - 1}{n} p) - \frac{p(n - 1)}{n} \log \frac{p}{n} = \\
S(\Upsilon(|g><g|)) = \text{const}
\]

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for any unit vector $g \in H$. It follows that

$$\hat{S}_\Upsilon(\rho) = -(1 - \frac{n - 1}{n} p) \log(1 - \frac{n - 1}{n} p) - \frac{p(n - 1)}{n} \log \frac{p}{n} = \text{const} \quad (28)$$

for all states $\rho \in \mathcal{S}(H)$. On the other hand,

$$\sum_j |\mu_j|^2 S(\Omega|\psi_j \rangle \langle \psi_j|) \geq \hat{S}_\Omega(Tr_H(|e^{(s)} \rangle \langle e^{(s)}|)) \quad (29)$$

due to (26). Taking into account (28) and (29) we conclude that (27) gives rise to the inequality

$$S((\Upsilon \otimes \Omega)(\rho_s)) \geq \hat{S}_\Upsilon(Tr_K(\rho_s)) + \hat{S}_\Omega(Tr_H(\rho_s)) \quad (30)$$

Now, to complete the proof it suffices to notice that the minimum in (20) is achieved for the set of states $\rho_s = |e^{(s)} \rangle \langle e^{(s)}|$. □

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