A SOLUTION OPERATOR FOR $\bar{\partial}$ ON THE HARTOGS TRIANGLE AND $L^p$ ESTIMATES

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Abstract. An integral solution operator for $\bar{\partial}$ is constructed on product domains that include the punctured bidisc. This operator is shown to satisfy $L^p$ estimates for all $1 \leq p < \infty$, though with non-standard – relative to strongly pseudoconvex domains – bounding term. These estimates imply $L^p$ estimates for $\bar{\partial}$ on the Hartogs triangle, with greater range of $p$ than the canonical solution satisfies.

Introduction

Let $\Omega$ be a domain in $\mathbb{C}^n$. If $\alpha$ is a $\bar{\partial}$-closed $(0,1)$-form on $\Omega$, consider solutions to the Cauchy-Riemann system $\bar{\partial}v = \alpha$. A fundamental problem is to determine whether particular solutions satisfying norms estimates exist for various norms. Such solutions lead to construction of non-trivial holomorphic functions on $\Omega$.

Solving $\bar{\partial}$ with estimates depends both on the geometry of $\Omega$ and the norms considered. In this paper results on two classes of domains are established – product domains, especially with non-smooth factors, and the Hartogs triangle – with estimates in $L^p$ norms, $1 \leq p < \infty$. Obtaining $L^p$ estimates for $\bar{\partial}$ on the Hartogs triangle motivated our investigation and is achieved in Theorem 6.3. However this result is proved by transferring the $\bar{\partial}$ problem to a 2-dimensional product domain, so $L^p$ estimates for $\bar{\partial}$ on product spaces are established first. The results on product domains are new and of independent interest. That such estimates were not previously established is unusual, given the success of integral formulas on domains with more complicated geometry. The study of $\bar{\partial}$ on product spaces accounts for most of the paper’s length.

A successful method for obtaining non-$L^2$ estimates on $\bar{\partial}$ starts by establishing integral representation formulas with holomorphic kernels for forms. This approach was inaugurated by [Hen69] and [GL70] on strongly pseudo convex domains and was intensely pursued in the two decades after [Hen69], [GL70]; see [Ker71], [Lie70], and [Ovr71] for some early foundational results. There are many significant results in this direction, too numerous to survey; see [Ran86] and [HL84] for references to the main results prior to 1985.

Integral formulas follow from a general procedure, the Cauchy-Fantappié method, once a generating form is constructed; see [Ran86], [RS73], and [LS13]. However Cauchy-Fantappié integral formulas have almost exclusively been derived for domains with smooth boundary (plus additional, restrictive geometric conditions) because Stokes’ theorem is freely applied during the construction. Two notable exceptions are [RS73], on strongly pseudoconvex domains with piecewise smooth boundary, and [Hen71], on analytic polyhedra.

For a product domain, the boundary is not smooth nor strongly pseudoconvex away from its boundary singularities. So while many techniques used below are well-known, modifications of the “standard recipe” are also required to establish our integral formulas. The first goal is to obtain the abstract integral formula (1.24) on a 2-dimensional product domain with smoothly bounded factors; the derivation crucially uses an idea from [RS73].
A formula for products with higher-dimensional factors is also obtained, see Remark 1.21. The 2-dimensional formula is then converted into an explicit solution operator using the Cauchy generating form, when the data is sufficiently smooth:

**Proposition 0.1.** Suppose $D_1, D_2 \subset \mathbb{C}$ are domains with $C^1$ boundary. If $f \in C^1_{0,1}(D_1 \times D_2)$ satisfies $\overline{\partial} f = 0,$ the function

\begin{equation}
T(f) = -\frac{1}{2\pi i} \int_{D_2} \frac{f_z(z_1, z_2)}{z_2 - z_1} d\bar{z}_1 \wedge dz_2 + \frac{1}{2\pi i} \int_{D_1} \frac{f(z_1, z_2)}{z_1 - z_1} d\bar{z}_1 \wedge dz_2
\end{equation}

solves $\partial(Tf) = f$. In (0.2), $\partial f = \frac{\partial f_1}{\partial z_2} = \frac{\partial f_2}{\partial z_1}$.

This is proved as Proposition 2.3 below. Even in the case of the bidisc, i.e., when $D_j = \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\},$ Proposition 0.1 is new. Since the kernels in the integrands are Cauchy kernels, or iterated Cauchy kernels, mapping properties of the operator $T$ are easy to derive. Additionally, a re-expression of (0.2) – see Remark 2.5 – corrects a minor error in a formula displayed on page 212 of [Hen71], given without proof, and reproduced in [FLZ11]. (The error is inconsequential for the estimates proved in [FLZ11].)

The solution operator (0.2) is extended to non-$C^1$ bounded domains – including $\mathbb{D} \times \mathbb{D}^*$, where $\mathbb{D}^* = \{0 < |z| < 1\}$ is the punctured disc – and to forms not necessarily smooth up to the boundary in Section 3.2. The $L^p$ boundedness of the integral operators is also proved in Section 3. In contrast to results on strongly pseudoconvex domains, non-standard $L^p$ boundedness of the data is needed to obtain an $L^p$ bound on the solution. Define the norm $\|f\|_B := \|f_1\|_{L^p(D_1 \times D_2)} + \|f_2\|_{L^p(D_1 \times D_2)} + \|\partial(f)\|_{L^p(D_1 \times D_2)}$ on (0,1)-forms $f = f_1 d\bar{z}_1 + f_2 d\bar{z}_2$, see Definition 3.2. The main $L^p$ result on product domains, Theorem 3.4, says the ordinary $L^p$ norm of $Tf$ is dominated by $\|f\|_B$.

In Section 4, the condition $f \in L^p(D_1 \times D_2)$ alone is shown not to be sufficient to conclude $Tf \in L^p$. More dramatically, the example there shows that $Tf$ can fail to exist for $f \in L^1(D_1 \times D_2)$, or more generally for $f \in L^p, 1 \leq p < 2$. This contrasts sharply with results on the Henkin-Cauchy-Fantappié operator known on strongly pseudoconvex domains [Ker71], [Ovr71], some finite type domains [CNS92], [FK88], and even some infinite type domains [HKR14]. The contrast is interesting and should be understood more fully. The observation in Section 4 merely inaugurates this new phenomena; finding actual necessary conditions on $f$ that follow from $Tf$ existing or satisfying $L^p$ estimates remains open. Such conditions, beyond $f \in L^p(D_1 \times D_2)$, obviously have consequences when using the estimates in application. To be clear: our computations in Section 4 are made only on the Henkin solution operator.

The main previous result on $\bar{\partial}$-estimates for the bidisc are the $L^\infty$ estimates in [FLZ11]. There is a point connecting [FLZ11], the undetermined necessary conditions mentioned above, and the older literature on the Henkin solution. Norm control of derivatives of the data $f$ is assumed in [FLZ11], though somewhat obliquely. In that paper the estimate $\|Tf\|_{L^\infty(\mathbb{D} \times \mathbb{D})} \leq C\|f\|_{L^\infty(\mathbb{D} \times \mathbb{D})}$ is proved, but only under the assumption that $f \in C^1_{0,1}(\mathbb{D} \times \mathbb{D})$. Thus the question posed by Kerzman in 1971 – does there exist a solution operator satisfying $\|Sf\|_{L^\infty} \leq C\|f\|_{L^\infty}$ for all $\bar{\partial}$-closed forms in $L^\infty$, [Ker71] remark on pages 311–312 – is still unresolved on the bidisc.

Finally in Section 6, the biholomorphism between the Hartogs triangle $\mathbb{H}$ and $\mathbb{D} \times \mathbb{D}^*$ transfers the $\partial$ problem on $\mathbb{H}$ to a $\bar{\partial}$ problem on $\mathbb{D} \times \mathbb{D}^*$ with different data. $L^p$ estimates for a solution operator on $\mathbb{H}$ are then inferred from those on $\mathbb{D} \times \mathbb{D}^*$, cf. Theorem 6.3. There are previous results about solving $\bar{\partial}$ with estimates in Hölder spaces $C^{k,\alpha}(\mathbb{H})$, including the...
A generating form

Definition 1.1. A generating form \( w \) on \( bD \times D \) is a \( C^1_{1,0} \)-form in \( \zeta \) and a \( C^\infty \) function in \( z \),

\[
  w(\zeta, z) = \sum_{l=1}^{n} w^l(\zeta, z) d\zeta_l,
\]

with the following property

\[
  \langle w(\zeta, z), \zeta - z \rangle = \sum_{l=1}^{n} w^l(\zeta, z)(\zeta_l - z_l) = 1 \quad \text{for } (\zeta, z) \in U^*.
\]

When applying a differential operator to forms depending on multiple sets of independent variables (like \( w \)), subscripts will be used to indicate which variables are differentiated. For instance, \( \partial_{\zeta}^k w(\zeta, z) = \sum_{l=1}^{n} \frac{\partial w^l}{\partial \zeta_k} d\zeta_l \land d\zeta_l \), while \( \partial_{\zeta}^k w(\zeta, z) = \sum_{l=1}^{n} \frac{\partial w^l}{\partial \zeta_k} d\zeta_l \land d\zeta_l \). The same convention is used on functions.

The universal form

\[
  w_0(\zeta, z) = \frac{\partial_{\zeta}(|\zeta - z|^2)}{|\zeta - z|^2} = \sum_{l=1}^{n} \frac{\partial}{\partial \zeta_l} (\zeta_l - z_l) d\zeta_l
\]

is called the Bochner-Martinelli generating form. This form satisfies Definition 1.1 for any domain \( D \). Let \( \mu \in I = [0, 1] \). If \( w \) is a generating form on \( bD \times D \), define the homotopy between \( w \) and \( w_0 \) by

\[
  \hat{w}(\zeta, z, \mu) = \mu w(\zeta, z) + (1 - \mu) w_0(\zeta, z).
\]

Note that for each fixed \( \mu \in I \), the form \( \hat{w} \) also satisfies Definition 1.1.

A piece of notation simplifies writing formulas below: let \( \bar{\partial}_{\zeta, \mu} = \partial_{\zeta} + \mu \partial_{\zeta} \).

Definition 1.3. The Cauchy-Fantappié kernel of order \( q \) generated by \( \hat{w} \) is

\[
  \Omega_q(\hat{w}) = \frac{(-1)^{(q-1)/2}}{(2\pi i)^n} \left( \begin{array}{c} n - 1 \\ q \end{array} \right) \hat{w} \land (\bar{\partial}_{\zeta, \mu} \hat{w})^{n-q-1} \land (\bar{\partial}_z \hat{w})^q
\]
for $0 \leq q \leq n - 1$, and 0 otherwise ($q = -1$ and $q = n$).

**Remark 1.5.** The kernel $\Omega_q(w)$ associated to an arbitrary generating form $w$ is defined in the same manner:

$$
\Omega_q(w) = \frac{(-1)^{q(q-1)/2}}{(2\pi i)^n} \left( \begin{array}{c} n - 1 \\ q \end{array} \right) w \wedge (\bar{\partial}_z w)^{n-q-1} \wedge (\bar{\partial}_z w)^q
$$

Note that $\bar{\partial}_z^{-\mu}$ has been replaced by $\bar{\partial}_z$. Moreover, if $q \geq 1$ and $w$ is holomorphic in $z$, $\Omega_q(w) = 0$ because of the final factor in (1.6).

**Remark 1.7.** The kernel $\Omega_q(w_0)$ is also denoted $K_q$ and called the Bochner-Martinelli-Koppelman kernel, following [Ran86]. For $D \subset \subset \mathbb{C}^n$ with piecewise $C^1$ boundary, the Bochner-Martinelli-Koppelman representation for $f \in C_{0,q}^1(\bar{D})$ is

$$
f(z) = \int_{bD} f \wedge K_q(\cdot, z) - \int_D \bar{\partial} f \wedge K_q(\cdot, z) - \bar{\partial}_z \int_D f \wedge K_{q-1}(\cdot, z)
$$

where $0 \leq q \leq n$. It holds that $\int_D f \wedge K_{q-1}(\cdot, z) \in C_{0,q-1}^1(\bar{D})$. See [Ran86, Chap. IV, Theorem 1.10] for proofs of these facts.

A more general representation formula than (1.8) uses the following ingredient:

**Definition 1.9.** Let $D \subset \subset \mathbb{C}^n$ be a domain with $C^1$ boundary and $w$ a generating form on $bD \times D$. For $1 \leq q \leq n$, define the integral operator

$$
T^w_q : C_{0,q}^1(\bar{D}) \to C_{0,q-1}^1(\bar{D})
$$

by

$$
T^w_q(f) = \int_{bD \times I} f \wedge \Omega_q(\hat{w}) - \int_D f \wedge K_{q-1}(\cdot, z).
$$

Set $T^w_0 = T^w_{n+1} \equiv 0$.

The following theorem is proved in [Ran86, Chap. IV, Theorem 3.6].

**Theorem 1.10.** Let $D \subset \subset \mathbb{C}^n$ be a domain with $C^1$ boundary and $w$ a generating form on $bD \times D$. For $0 \leq q \leq n$ and $f \in C_{0,q}^1(\bar{D})$,

$$
f = \int_{bD} f \wedge \Omega_q(w) + \bar{\partial} T^w_q(f) + T^w_{q+1}(\bar{\partial} f) \quad \text{on } D.
$$

Moreover, for $k = 0, 1, 2, \ldots, \infty$, if $f \in C_{0,q}^k(D) \cap C_{0,q}^1(\bar{D})$ then $T^w_q(f) \in C_{0,q-k}^1(D)$.

**Remark 1.12.** Suppose $q \geq 1$. If the generating form $w$ is holomorphic in $z$ and $f \in C_{0,q}^1(\bar{D})$ is $\bar{\partial}$-closed, (1.11) implies $u = T^w_q(f)$ solves

$$
\bar{\partial} u = f,
$$

since $\Omega_q(w) = 0$ as noted in Remark 1.5.
1.2. **Product domains.** An idea from [RS73] is used to construct a generating form on a product domain from known generating forms on the factors. In [RS73], only domains with piecewise smooth boundaries that are strongly pseudoconvex away from boundary singularities are considered. However, strong pseudoconvexity is only used to build the integral kernels on smooth pieces of the boundary, not to piece the kernels together to get a solution operator for $\bar{\partial}$. This latter idea is what we extract.

Let $D = D_1 \times D_2 \subset \mathbb{C}^n$, where $D_1 \subset \mathbb{C}^{n_1}$ and $D_2 \subset \mathbb{C}^{n_2}$ are domains with $C^1$ boundary. Let $S_1 = bD_1 \times \bar{D}_2$ and $S_2 = \bar{D}_1 \times bD_2$.

**Definition 1.13.** For $j = 1, 2$, a generating form $w_j(\zeta, z)$ on $S_j \times D$ is a $(1,0)$-form in $\zeta$

$$w_j(\zeta, z) = \sum_{l=1}^{n_j} w_j^l(\zeta, z) d\zeta_l$$

with the following properties

1. for each fixed $\zeta \in S_j$, $w_j^l(\zeta, \cdot)$ is $C^\infty$ in $D$,
2. for each fixed $z \in D$, $w_j^l(\cdot, z)$ is $C^1$ in a neighborhood $U^*_j$ of $S_j$, and
3. for each $z \in D$

$$\langle w_j(\zeta, z), \zeta - z \rangle = \sum_{l=1}^{n_j} w_j^l(\zeta, z)(\zeta_l - z_l) = 1 \quad \text{for all } \zeta \in U^*_j.$$  

**Remark 1.15.** The forms in Definition 1.13 are generating for only part of $bD$, namely $S_j$. To connect this with the previous definition, suppose $\tilde{w}_j$ is a generating form on $bD_j \times \bar{D}_j$ as in Definition 1.1, for $j = 1, 2$. Define

$$w_1(\zeta, z) = \sum_{l=1}^{n_1} \tilde{w}_1^l(\zeta^1, z^1) d\zeta^1_l$$

and

$$w_2(\zeta, z) = \sum_{l=1}^{n_2} \tilde{w}_2^l(\zeta^2, z^2) d\zeta^2_l$$

where $\zeta = (\zeta^1, \zeta^2) \in \mathbb{C}^{n_1} \times \mathbb{C}^{n_2}$ and $z = (z^1, z^2) \in \mathbb{C}^{n_1} \times \mathbb{C}^{n_2}$. Note that $w_1, w_2$ are independent of $(\zeta^2, z^2), (\zeta^1, z^1)$ respectively. Elementary algebra shows that $w_1$ and $w_2$ are generating forms on $S_1 \times D$ and $S_2 \times D$, respectively, as given by Definition 1.13.

Following [RS73], let

$$\Delta = \{ \lambda = (\lambda_0, \lambda_1, \lambda_2) \in \mathbb{R}^3 \mid \lambda_0, \lambda_1, \lambda_2 \geq 0, \lambda_0 + \lambda_1 + \lambda_2 = 1 \},$$

$$\Delta_0 = \{ \lambda \in \Delta \mid \lambda_1 = \lambda_2 = 0 \},$$

$$\Delta_{01} = \{ \lambda \in \Delta \mid \lambda_2 = 0 \},$$

$$\Delta_{02} = \{ \lambda \in \Delta \mid \lambda_1 = 0 \}.$$  

As before, let

$$w_0(\zeta, z) = \frac{\partial_\zeta(|\zeta - z|^2)}{|\zeta - z|^2} = \sum_{l=1}^{n} (\bar{\zeta}_l - \bar{z}_l) d\zeta_l.$$  

Consider the partial convex combination of $w_0, w_1,$ and $w_2$

$$W(\zeta, z, \lambda) = \sum_{j=0}^{2} \lambda_j w_j(\zeta, z),$$

defined only on the following sets

1. $(\zeta, z, \lambda) \in \bar{D} \times D \times \Delta_0$ with $\zeta \neq z$,
Remark 1.17. For $\zeta$ fixed, the form $W$ in (1.16) is $C^\infty$ for $z \in D$. For $z \in D$ fixed, $W$ is $C^1$ in $\zeta$ and satisfies (1.14) in the corresponding neighborhood depending on $z$. Also, the form $W$ is differentiable in $\zeta$ in the interiors of $\Delta, \Delta_{01}$, and $\Delta_{02}$.

When derivatives with respect to the vector $\lambda$ are written, the meaning is that derivatives with respect to $\lambda_0, \lambda_1, \lambda_2$ are taken and the results are added. Notationally

$$\tilde{\partial}_{\zeta,\lambda} = \partial_{\zeta} + d\lambda_0 + d\lambda_1 + d\lambda_2.$$  

Similar to Definition 1.3, a kernel is associated to the form $W$ in (1.16):

Definition 1.18. The Cauchy-Fantappié kernel of order $q$ generated by the form $W$ in (1.16) is defined

$$\Omega_q(W) = \left(\frac{-1}{(2\pi i)^n}\right)^{q-1/2} \left(\begin{array}{c} n-1 \\ q \end{array}\right) W \wedge (\tilde{\partial}_{\zeta,\lambda} W)^{n-q-1} \wedge (\tilde{\partial}_{\zeta} W)^q$$

for $0 \leq q \leq n - 1$, and 0 otherwise ($q = -1$ and $q = n$).

Remark 1.19. If $q \geq 1$ and if for $j = 1, 2$ $w_j$ is holomorphic in $z$ for $\zeta$ fixed, then $\Omega_q(W) = 0$ on the set where $\lambda_0 = 0$. This follows since $\lambda_0 = 0$ implies that $W$ defined by (1.16) is holomorphic in $z$.

On the other hand, note that none of the sets (1)-(4) in (1.16) allow $\lambda_0 = 0$. Nevertheless, $\Omega_q(w) = 0$ on this set is needed in order to show that the operator in Definition 1.20 below is solution operator for $\tilde{\partial}$; for a proof see [RS73, §(2.5)].

Parallel to §1.1, an integral operator associated to the form $W$ in (1.16) is defined.

Definition 1.20. For $1 \leq q \leq n$, define the integral operator

$$T^W_q : C_{0,q}(\bar{D}) \to C_{0,q-1}(D)$$

by

$$T^W_q(f) = -\int_{bD_1 \times bD_2 \times \Delta} f \wedge \Omega_{q-1}(W) + \int_{S_1 \times \Delta_0} f \wedge \Omega_{q-1}(W) + \int_{S_2 \times \Delta_0} f \wedge \Omega_{q-1}(W) - \int_{D \times \Delta_0} f \wedge \Omega_{q-1}(W).$$

Set $T^W_0 = T^W_{n+1} \equiv 0$.

Remark 1.21. For $1 \leq q \leq n$, if $w_j$ is holomorphic in $z$ for $j = 1, 2$, then $T^W_q$ is a solution operator to the $\tilde{\partial}$-equations; i.e. $u = T^W_q(f)$ solves

$$\tilde{\partial}u = f$$

when $\tilde{\partial}f = 0$. This follows from Stokes’ theorem, but non-trivially as the different dimensional facets of the simplex $\Delta$ must be handled. As for Remark 1.19, a detailed proof is given in [RS73, §(2.5)].

Remark 1.22. Since $\lambda_0 + \lambda_1 = 1$ on $\Delta_{01}$, $d\lambda_0 = -d\lambda_1$ on this set. By change of variables, it follows that

$$\int_{S_1 \times \Delta_0} f \wedge \Omega_{q-1}(W) = \int_{S_1 \times I} f \wedge \Omega_{q-1}(\bar{w}_1).$$
where $\hat{w}_1$ is the homotopic form as in §1.1 and $\mu \in I$. Similarly,
\[
\int_{S^2 \times \Delta_{02}} f \wedge \Omega_{q-1}(W) = \int_{S^2 \times I} f \wedge \Omega_{q-1}(\hat{w}_2).
\]
Moreover, since $\lambda_0 = 1$ on $\Delta_0$, $w = w_0$ on this singleton. Thus
\[
T^W_q(f) = -\int_{bD_1 \times bD_2 \times \Delta} f \wedge \Omega_{q-1}(W) + \int_{bD_1 \times D_2 \times I} f \wedge \Omega_{q-1}(\hat{w}_1)
+ \int_{D_1 \times bD_2 \times I} f \wedge \Omega_{q-1}(\hat{w}_2) - \int_D f \wedge K_{q-1}
\]
for $f \in C_{0,q}(\bar{D})$ and $1 \leq q \leq n$.

**Remark 1.23.** Of particular importance here, when $D_1$ and $D_2$ are 1-dimensional the first integral in the displayed equation above vanishes. I.e.,
\[
\int_{bD_1 \times bD_2 \times \Delta} f \wedge \Omega_0(W) = 0.
\]
This follows since the degree of the form (with respect to the integration variable) in the integrand must equal the dimension of the set over which it is integrated, otherwise the integral is 0. This argument will be called *dimension-degree counting* when used below.

When $D = D_1 \times D_2$ is 2-dimensional, the operator $T^W_1$ reduces to

\begin{equation}
(1.24) \quad T^W_1(f) = \int_{bD_1 \times D_2 \times I} f \wedge \Omega_0(\hat{w}_1) + \int_{D_1 \times bD_2 \times I} f \wedge \Omega_0(\hat{w}_2) - \int_D f \wedge K_0
\end{equation}

for $f \in C^1_{0,1}(\bar{D})$, where
\[
\Omega_0(\hat{w}_j) = \frac{1}{(2\pi i)^2} w_j \wedge w_0 \wedge d\mu \quad \text{for } j = 1, 2
\]
and
\[
K_0 = \frac{1}{(2\pi i)^2} \frac{(\bar{\zeta}_1 - \bar{z}_1) d\zeta_1 \wedge d\bar{\zeta}_2 \wedge d\zeta_2 + (\bar{\zeta}_2 - \bar{z}_2) d\zeta_2 \wedge d\bar{\zeta}_1 \wedge d\zeta_1}{|\zeta - z|^4}.
\]

The form on the operator $T^W_1$ given by (1.24) is the starting point for the computations in the next section.

By dimension-degree counting, it is also easy to see
\[
T^W_2(f) = -\int_D f \wedge K_1 \quad \text{for } f \in C^1_{0,2}(\bar{D}),
\]
where
\[
K_1 = \frac{1}{(2\pi i)^2} \frac{(\bar{\zeta}_1 - \bar{z}_1) d\zeta_2 \wedge d\zeta_1 \wedge d\bar{z}_2 + (\bar{\zeta}_2 - \bar{z}_2) d\zeta_1 \wedge d\bar{\zeta}_2 \wedge d\bar{z}_1}{|\zeta - z|^4}.
\]

2. **The $\bar{\partial}$-equation on product spaces**

For a two-dimensional product domain, the right hand side of (1.24) can be written as explicit integral operators. This is now derived for arbitrary bounded domains $D_1, D_2 \subset C^1$ with $C^1$ boundary, using the Cauchy generating form $w$. 
2.1. The product space $D_1 \times D_2 \subset \mathbb{C}^2$. Definition 1.1 shows the Cauchy kernel

$$w = \frac{d\zeta}{\zeta - z}$$

is a generating form for any domain $\Omega \subset \mathbb{C}^1$. This form is holomorphic in $z$, away from $z = \zeta$. Set

$$w_j = \frac{d\zeta_j}{\zeta_j - z_j} \quad \text{for } j = 1, 2,$$

the Cauchy kernels on the two domains $D_j$, $j = 1, 2$. Note that Remark 1.15 shows that $w_1, w_2$ give generating forms on $S_1 \times (D_1 \times D_2)$ and $S_2 \times (D_1 \times D_2)$ respectively. For the rest of this section $w_j$ will refer to the Cauchy forms above, and $\hat{w}_j$ is defined via (1.2) relative to these particular $w_j$.

Direct computation from (1.4) gives

$$\Omega_0(\hat{w}_1) = \frac{1}{(2\pi i)^2} \frac{(\bar{\zeta}_2 - \bar{z}_2) d\zeta_1 \wedge d\zeta_2 \wedge d\mu}{(\zeta_1 - z_1)|\zeta - z|^2}$$

and

$$\Omega_0(\hat{w}_2) = \frac{1}{(2\pi i)^2} \frac{(\bar{\zeta}_1 - \bar{z}_1) d\zeta_2 \wedge d\zeta_1 \wedge d\mu}{(\zeta_2 - z_2)|\zeta - z|^2}.$$

Let $f \in C^1_{1,0}(\bar{D})$ and write $f = f_1 d\zeta_1 + f_2 d\zeta_2$. The first term on the right hand side of (1.24) becomes

$$\int_{bD_1 \times D_2 \times I} f \wedge \Omega_0(\hat{w}_1) = \int_{bD_1 \times D_2 \times I} f \wedge \left( \frac{1}{(2\pi i)^2} \frac{(\bar{\zeta}_2 - \bar{z}_2) d\zeta_1 \wedge d\zeta_2 \wedge d\mu}{(\zeta_1 - z_1)|\zeta - z|^2} \right)$$

$$= \frac{1}{(2\pi i)^2} \int_{bD_1 \times D_2 \times I} f \wedge \frac{f_2 \cdot (\bar{\zeta}_2 - \bar{z}_2) d\zeta_2 \wedge d\zeta_1 \wedge d\mu}{(\zeta_1 - z_1)|\zeta - z|^2}$$

Letting $\tau \to 0^+$ gives

$$\lim_{\tau \to 0^+} \int_{D_1 \setminus D(z_1; \tau)} \frac{\partial f_2}{\partial \zeta_1} \frac{\bar{\zeta}_2 - \bar{z}_2}{(z_1 - \zeta_1)|\zeta - z|^2} d\zeta \wedge d\zeta_1 = \int_{D_1} \frac{\partial f_2}{\partial \zeta_1} \frac{\bar{\zeta}_2 - \bar{z}_2}{(z_1 - \zeta_1)|\zeta - z|^2} d\zeta \wedge d\zeta_1.$$
and
\[
\lim_{\tau \to 0^+} \int_{|z_1 - z_2| = \tau} \frac{f_2(z_1, \zeta_2 - \bar{z}_2)}{(z_1 - \zeta_1)^2} d\zeta_1 = -2\pi i \frac{f_2(z_1, \zeta_2)}{\zeta_2 - \bar{z}_2}.
\]
Substituting these terms in the previous equation, the first term on the right hand side of (1.24) can be written
\[
\int_{bD_1 \times D_2 \times I} f \wedge \Omega_0(\hat{w}_1) = -\frac{1}{2\pi i} \int_{D_2} \frac{f_2(z_1, \zeta_2)}{\zeta_2 - \bar{z}_2} d\zeta_2 \wedge d\zeta_1 + \frac{1}{(2\pi i)^2} \int_{D_1 \times D_2} \frac{f_2(z_1, \zeta_2)}{|\zeta - z|^4} d\zeta_1 \wedge d\zeta_2 \wedge d\zeta_2 + \frac{1}{(2\pi i)^2} \int_{D_1 \times D_2} \frac{\partial f_2}{\partial \zeta_1} \cdot \frac{\zeta_2 - \bar{z}_2}{(|\zeta_2 - z_1)| |\zeta - z|^2} d\zeta_1 \wedge d\zeta_1 \wedge d\zeta_2 \wedge d\zeta_2.
\]
Similarly, the second term on the right hand side in (1.24) can be written
\[
\int_{D_1 \times bD_2 \times I} f \wedge \Omega_0(\hat{w}_2) = -\frac{1}{2\pi i} \int_{D_1} \frac{f_1(z_1, \zeta_2)}{\zeta_1 - z_1} d\zeta_1 \wedge d\zeta_1 + \frac{1}{(2\pi i)^2} \int_{D_1 \times D_2} \frac{f_1(z_1, \zeta_2)}{|\zeta - z|^4} d\zeta_1 \wedge d\zeta_2 \wedge d\zeta_2 + \frac{1}{(2\pi i)^2} \int_{D_1 \times D_2} \frac{\partial f_1}{\partial \zeta_2} \cdot \frac{\zeta_1 - \bar{z}_1}{(|\zeta_2 - z_2)| |\zeta - z|^2} d\zeta_1 \wedge d\zeta_1 \wedge d\zeta_2 \wedge d\zeta_2.
\]
Note that the last term on the right hand side of (1.24) is
\[
\int_{D_1 \times D_2} f \wedge K_0 = \frac{1}{(2\pi i)^2} \int_{D_1 \times D_2} \frac{f_1(z_1, \zeta_2)}{|\zeta - z|^4} d\zeta_1 \wedge d\zeta_1 \wedge d\zeta_2 \wedge d\zeta_2.
\]
Hence, formula (1.24) can be expressed
\[
T_1^W(f) = \int_{bD_1 \times D_2 \times I} f \wedge \Omega_0(\hat{w}_1) + \int_{D_1 \times bD_2 \times I} f \wedge \Omega_0(\hat{w}_2) + \int_{D_1 \times D_2} f \wedge K_0
\]
\[
= -\frac{1}{2\pi i} \int_{D_2} \frac{f_2(z_1, \zeta_2)}{\zeta_2 - \bar{z}_2} d\zeta_2 \wedge d\zeta_1 + \frac{1}{(2\pi i)^2} \int_{D_1} \frac{f_1(z_1, \zeta_2)}{\zeta_1 - z_1} d\zeta_1 \wedge d\zeta_2 \wedge d\zeta_1 + \frac{1}{(2\pi i)^2} \int_{D_1 \times D_2} \frac{\partial f_2}{\partial \zeta_1} \cdot \frac{\zeta_2 - \bar{z}_2}{(|\zeta_2 - z_1)| |\zeta - z|^2} d\zeta_1 \wedge d\zeta_1 \wedge d\zeta_2 \wedge d\zeta_2
\]
\[
+ \frac{1}{(2\pi i)^2} \int_{D_1 \times D_2} \frac{\partial f_1}{\partial \zeta_2} \cdot \frac{\zeta_1 - \bar{z}_1}{(|\zeta_2 - z_2)| |\zeta - z|^2} d\zeta_1 \wedge d\zeta_1 \wedge d\zeta_2 \wedge d\zeta_2.
\]
\[
(2.1)
\]
**Definition 2.2.** If \( f = f_1 d\bar{z}_1 + f_2 d\bar{z}_2 \) is a \((0,1)\)-form, let
\[
\partial f = \frac{1}{2} \left( \frac{\partial f_1}{\partial \bar{z}_2} + \frac{\partial f_2}{\partial \bar{z}_1} \right),
\]
with derivatives taken in the distributional sense. If \( \partial f = 0 \), note \( \partial f = \frac{\partial f_1}{\partial \bar{z}_2} = \frac{\partial f_2}{\partial \bar{z}_1} \).

Using Definition 2.2, combine the last two terms in (2.1). The following expression for a strong solution operator on \( D_1 \times D_2 \) is obtained:
Proposition 2.3. Suppose $D_1, D_2 \subset \mathbb{C}$ are domains with $C^1$ boundary. If $f \in C^1_{0,1}(D_1 \times D_2)$ satisfies $\partial f = 0$, define

$$T(f) = -\frac{1}{2\pi i} \int_{D_2} \frac{f_2(z_2, \zeta_2)}{\zeta_2 - z_2} d\zeta_2 \wedge d\zeta_1 + \frac{1}{2\pi i} \int_{D_1} \frac{f_1(\zeta_1, z_2)}{\zeta_1 - z_1} d\zeta_2 \wedge d\zeta_1$$

\begin{equation}
(2.4)
+ \frac{1}{(2\pi i)^2} \int_{D_1 \times D_2} \frac{\partial f(\zeta_1, \zeta_2)}{(\zeta_1 - z_1)(\zeta_2 - z_2)} d\zeta_2 \wedge d\zeta_1 \wedge d\zeta_2 \wedge d\zeta_1.
\end{equation}

Then $\partial(Tf) = f$.

Remark 2.5. Consider the third term on the right hand side of (2.1). If the idea from [FLZ11, Hen71] is followed and Stokes' theorem is applied in the $\zeta_2$ variable, this term becomes

$$\int_{D_1 \times D_2} \frac{\partial f_1}{\partial \zeta_2} \cdot \frac{\bar{\zeta}_2 - \bar{z}_2}{(\zeta_1 - z_1)|\zeta - z|^2} d\zeta_1 \wedge d\zeta_2 \wedge d\zeta_2 \wedge d\zeta_1 = \int_{D_1} \left[ \int_{bD_2} \frac{f_1 \cdot (\bar{\zeta}_2 - \bar{z}_2)}{(\zeta_1 - z_1)|\zeta - z|^2} d\zeta_2 \right.$$

$$\left. - \int_{D_2} \frac{f_1 \cdot (\bar{\zeta}_1 - \bar{z}_1)}{|\zeta - z|^4} d\zeta_2 \right] d\zeta_1 \wedge d\zeta_1.$$

Similarly, the last term in (2.1) can be rewritten as

$$\int_{D_1 \times D_2} \frac{\partial f_2}{\partial \zeta_1} \cdot \frac{\bar{\zeta}_1 - \bar{z}_1}{(\zeta_2 - z_2)|\zeta - z|^2} d\zeta_1 \wedge d\zeta_1 \wedge d\zeta_2 \wedge d\zeta_2 = \int_{D_2} \left[ \int_{bD_1} \frac{f_2 \cdot (\bar{\zeta}_1 - \bar{z}_1)}{(\zeta_2 - z_2)|\zeta - z|^2} d\zeta_2 \right.$$

$$\left. - \int_{D_1} \frac{f_2 \cdot (\bar{\zeta}_2 - \bar{z}_2)}{|\zeta - z|^4} d\zeta_1 \right] d\zeta_2 \wedge d\zeta_2.$$

Thus an alternative expression for the operator $T = T^W_1$ is

$$T(f) = -\frac{1}{2\pi i} \int_{D_2} \frac{f_2(z_1, \zeta_2)}{\zeta_2 - z_2} d\zeta_2 \wedge d\zeta_1 - \frac{1}{(2\pi i)^2} \int_{D_1 \times bD_2} \frac{f_1 \cdot (\bar{\zeta}_2 - \bar{z}_2)}{(\zeta_1 - z_1)|\zeta - z|^2} d\zeta_1 \wedge d\zeta_1 \wedge d\zeta_2$$

$$+ \frac{1}{2\pi i} \int_{D_1} \frac{f_1(\zeta_1, z_2)}{\zeta_1 - z_1} d\zeta_2 \wedge d\zeta_1 - \frac{1}{(2\pi i)^2} \int_{bD_1 \times D_2} \frac{f_2 \cdot (\bar{\zeta}_1 - \bar{z}_1)}{(\zeta_2 - z_2)|\zeta - z|^2} d\zeta_2 \wedge d\zeta_2 \wedge d\zeta_2$$

$$+ \frac{1}{(2\pi i)^2} \int_{D_1 \times D_2} \frac{f_1 \cdot (\bar{\zeta}_1 - \bar{z}_1) + f_2 \cdot (\bar{\zeta}_2 - \bar{z}_2)}{|\zeta - z|^4} d\zeta_1 \wedge d\zeta_1 \wedge d\zeta_2 \wedge d\zeta_2.$$

This formula corrects a small error in [Hen71, FLZ11], where a different constant appears before the last term.

3. The $L^p$ estimate of the solution operator

For the rest of the paper, $T$ denotes the operator defined by (2.4).

3.1. The $L^p$ estimate of $T$. As a integral operator, $T$ is first shown to be well-defined and bounded between particular Banach spaces. The following lemma is used.

Lemma 3.1. Let $D_1, D_2 \subset \mathbb{C}$ be bounded domains and $1 \leq p < \infty$. If $g \in L^p(D_1 \times D_2)$, the functions

$$-\frac{1}{2\pi i} \int_{D_1} \frac{g(\zeta_1, z_2)}{\zeta_1 - z_1} d\zeta_1 \wedge d\zeta_1 \quad \text{and} \quad -\frac{1}{2\pi i} \int_{D_2} \frac{g(z_1, \zeta_2)}{\zeta_1 - z_1} d\zeta_1 \wedge d\zeta_1$$

belong to $L^p(D_1 \times D_2)$. Their $L^p$-norms are bounded by $C\|g\|_{L^p(D_1 \times D_2)}$, for a constant $C > 0$ independent of $g$. 
Proof. The symmetry of the functions show it suffices to prove the result for either one; consider the second function. Let \(g^{*2}(\zeta_2) = g(z_1, \zeta_2)\chi_{D_2}(\zeta_2)\), where \(\chi_{D_2}\) is the characteristic function over \(D_2\). Since \(g \in L^p(D_1 \times D_2)\), \(g^{*2} \in L^p(\mathbb{C})\).

Let \(B = B(0; R)\) be the disk centered at 0 of radius \(R\) in \(\mathbb{C}\) and \(h(\zeta) = \frac{1}{|\zeta|}\chi_B(\zeta)\), where \(\chi_B\) is the characteristic function over \(B\) and \(R\) is sufficiently large (say \(R > \text{diam}(D_2)\)). Then \(h \in L^1(\mathbb{C})\).

By Young’s inequality, \(g^{*2} \ast h \in L^p(\mathbb{C})\). Note that, for any \(z_2, \zeta_2 \in D_2\), \(|z_2 - \zeta_2| \leq \text{diam}(D_2) < R\), so \(z_2 - \zeta_2 \in B\). Therefore,

\[
\int_{D_2} \left( \int_{D_2} \frac{g(z_1, \zeta_2) dA(\zeta_2)}{|\zeta_2 - z_2|} \right)^p dA(z_2) \leq C \int_{D_2} \left( \int_{\mathbb{C}} \frac{|g^{*2}(\zeta_2)|\chi_B(\zeta_2 - z_2) dA(\zeta_2)}{|\zeta_2 - z_2|} \right)^p dA(z_2).
\]

The conclusion follows by integrating this inequality in \(z_1\) over \(D_1\).

Lemma 3.1 applies to the first two terms on the right hand side of (2.4). Thus, the \(L^p\)-norms of these terms are bounded by \(C\|f_2\|_{L^p(D_1 \times D_2)}\) and \(C\|f_1\|_{L^p(D_1 \times D_2)}\) respectively, provided \(f_1, f_2 \in L^p(D_1 \times D_2)\). Additionally, if \(\mathcal{G}(f) \in L^p(D_1 \times D_2)\), two applications of Lemma 3.1 show the last term in (2.4) is in \(L^p(D_1 \times D_2)\), with \(L^p\)-norm bounded by \(C\|\mathcal{G}(f)\|_{L^p(D_1 \times D_2)}\).

**Definition 3.2.** For \(1 \leq p < \infty\), define the Banach space of \((0,1)\)-forms on \(D_1 \times D_2\)

\[\mathcal{B} = \{f = f_1 d\bar{z}_2 + f_2 d\bar{z}_1 \mid f_1, f_2 \in L^p\} \text{ with norm } \|f\|_\mathcal{B} := \|f_1\|_{L^p(D_1 \times D_2)} + \|f_2\|_{L^p(D_1 \times D_2)} + \|\mathcal{G}(f)\|_{L^p(D_1 \times D_2)}\).

The argument above Definition 3.2 proves

**Lemma 3.3.** The operator \(T\) given by (2.4) maps \(\mathcal{B}\) to \(L^p(D_1 \times D_2)\) boundedly.

3.2. Passage to a weak solution. Lemma 3.3 shows that \(T : \mathcal{B} \to L^p(D_1 \times D_2)\) is well-defined. By (2.4), \(T\) is a strong solution operator to \(\bar{\partial}(Tf) = f\) if \(T\) is restricted to \(f \in C_{0,1}^1(D_1 \times D_2)\) and \(\bar{\partial}f = 0\). In this section \(Tf\) is shown to be a weak solution to \(\bar{\partial}(Tf) = f\), if \(\bar{\partial}f = 0\) weakly and \(f \in \mathcal{B}\), by a limit argument.

**Theorem 3.4.** For \(j = 1, 2\), let \(D_j \subset \mathbb{C}\) be bounded domains with \(C^1\) boundary. Let \(f\) be a \((0,1)\)-form that is \(\bar{\partial}\)-closed in the weak sense on \(D_1 \times D_2\). For \(1 \leq p < \infty\), assume that \(f \in \mathcal{B}\).

Then \(u = Tf\), defined by (2.4), is a weak solution to the equation \(\bar{\partial}u = f\) on \(D_1 \times D_2\) and satisfies the estimate

\[\|T(f)\|_{L^p(D_1 \times D_2)} \leq C\|f\|_\mathcal{B}\]

for a constant \(C > 0\) independent of \(f\).

**Proof.** For \(j = 1, 2\), let \(\rho_j\) be a defining function for the domain \(D_j\). Let \(D_j^{\delta} = \{z \in \mathbb{C} \mid \rho_j(z) < -\delta\}\) for \(\delta > 0\) sufficiently small. Denote by \(T^{\delta}\) the operator in (2.4) with \(D_j\) replaced by \(D_j^{\delta}\). Then

\[\bar{\partial}T^{\delta}(f) = f \quad \text{on } D_1^{\delta} \times D_2^{\delta} \quad \text{for } \bar{\partial}\text{-closed (0,1)-form } f \in C^1(D_1^{\delta} \times D_2^{\delta})\).

Let \(1 \leq p < \infty\). For \(f \in \mathcal{B}\), the standard mollifier argument (see for example [Eva98, Chap. 5.3, Theorem 2]) gives a sequence \(\{f^{\varepsilon}\} \subset C^1(D_1^{\delta} \times D_2^{\delta})\), so that

\[\bar{\partial}f^{\varepsilon} = 0, \quad f^{\varepsilon} \to f \quad \text{and } \mathcal{G}(f^{\varepsilon}) \to \mathcal{G}(f) \quad \text{in } L^p(D_1^{\delta} \times D_2^{\delta}) \text{ as } \varepsilon \to 0.\]
Thus
\[ \partial T^\delta (f^{\varepsilon_j}) = f^{\varepsilon_j} \quad \text{on } D_1^\delta \times D_2^\delta \]
in the strong sense.

On the other hand, replacing \( D_1 \times D_2 \) by \( D_1^\delta \times D_2^\delta \) in Lemma 3.3 and denoting the Banach space on \( D_1^\delta \times D_2^\delta \) by \( \mathcal{B}^\delta \), it follows that \( T^\delta \) is bounded from \( \mathcal{B}^\delta \) to \( L^p(D_1^\delta \times D_2^\delta) \). So for \( f \in \mathcal{B} \), \( \lim T^\delta (f^{\varepsilon_j}) = T^\delta (f) \) in \( L^p(D_1^\delta \times D_2^\delta) \) as \( \varepsilon_j \to 0 \). Hence \( T^\delta \) weakly solves the \( \bar{\partial} \)-equation on \( D_1^\delta \times D_2^\delta \).

Next, for each \( f \in \mathcal{B} \), extend \( T^\delta (f) \) to a function on \( D_1 \times D_2 \) by setting it equal 0 outside \( D_1^\delta \times D_2^\delta \). Consider \( \| T(f) - T^\delta (f) \|_{L^p(D_1 \times D_2)} \). Note that \( f_2 \) can be replaced by \( f_2 \cdot \chi_{D_2 \setminus D_2^\delta} \) in Lemma 3.1, where \( \chi_{D_2 \setminus D_2^\delta} \) is the characteristic function over \( D_2 \setminus D_2^\delta \). Thus, the \( L^p \)-norm in the conclusion in Lemma 3.1 is bounded by \( C \| f_2 \|_{L^p(D_1 \times (D_2 \setminus D_2^\delta))} \), which tends to 0 as \( \delta \to 0^+ \). A similar argument holds for Lemma 3.3. Therefore, \( \lim T^\delta (f) = T(f) \) in \( L^p(D_1 \times D_2) \) as \( \delta \to 0^+ \). This argument shows that the limit \( T(f) \) is unique, and is independent of the defining functions for \( D_1 \) and \( D_2 \) used.

To show \( T \) weakly solves the \( \bar{\partial} \)-equation on \( D_1 \times D_2 \), argue as follows. For any \( \phi \in C_c^\infty(D_1 \times D_2) \), there is a \( \delta_0 > 0 \) so that \( \text{supp} \phi \subset D_1^{\delta_0} \times D_2^{\delta_0} \). Let \( K = D_1^{\delta_0} \times D_2^{\delta_0} \). Then
\[
(\partial T(f), \phi)_{D_1 \times D_2} = (\bar{\partial} T(f), \phi)_K = \lim_{\delta \to 0^+} (T^\delta(f), \bar{\partial}^\ast \phi)_K
\]
by \( L^p(D_1 \times D_2) \)-norm convergence. Note that \( T^\delta \) weakly solves the \( \bar{\partial} \)-equation on \( D_1^\delta \times D_2^\delta \). Thus for \( 0 < \delta < \delta_0 \),
\[
\lim_{\delta \to 0^+} (T^\delta(f), \bar{\partial}^\ast \phi)_K = \lim_{\delta \to 0^+} (\bar{\partial} T^\delta(f), \phi)_K = (f, \phi)_{D_1 \times D_2}.
\]

Now let \( D_1 \times D_2 = \mathbb{D} \times A \), where \( A = A(0; 1, \delta) = \{ z \in \mathbb{C} \mid \delta < |z| < 1 \} \). Theorem 3.4 directly applies to \( \mathbb{D} \times A \). However the proof of Theorem 3.4 also applies, allowing the limit \( \delta \to 0^+ \) to be taken. This yields the following result.

**Corollary 3.5.** Let \( f = f_1 d\bar{z}_1 + f_2 d\bar{z}_2 \) be a \((0, 1)\)-form that is \( \bar{\partial} \)-closed in the weak sense on \( \mathbb{D} \times \mathbb{D}^* \). For \( 1 \leq p < \infty \), assume that \( f \in \mathcal{B} \).

Then \( T(f) \) defined
\[
T(f) = \frac{-1}{2\pi i} \int_{\mathbb{D}^*} \frac{f_2(z_1, \zeta_2)}{\zeta_2 - z_2} d\zeta_2 \wedge d\zeta_1 + \frac{-1}{2\pi i} \int_{\mathbb{D}} \frac{f_1(\zeta_1, z_2)}{\zeta_1 - z_1} d\bar{\zeta}_1 \wedge d\zeta_1
+ \frac{-1}{(2\pi i)^2} \int_{\mathbb{D} \times \mathbb{D}^*} \frac{\mathcal{G}(f)(\zeta_1, \zeta_2)}{(\zeta_1 - z_1)(\zeta_2 - z_2)} d\bar{\zeta}_1 \wedge d\zeta_1 \wedge d\bar{\zeta}_2 \wedge d\zeta_2
\]
is well-defined, weakly solves \( \bar{\partial}(T(f)) = f \) on \( \mathbb{D} \times \mathbb{D}^* \), and satisfies the estimate
\[
\| T(f) \|_{L^p(\mathbb{D} \times \mathbb{D}^*)} \leq C \| f \|_{\mathcal{B}}
\]
for a constant \( C > 0 \) independent of \( f \).

---

4. \( f \in L^p \) is not sufficient for existence of \( Tf \)

The assumption \( \| \mathcal{G}(f) \|_p < \infty \) is part of the hypotheses in Theorem 3.4 and Corollary 3.5 via the condition \( f \in \mathcal{B} \). Under this hypothesis and \( f \in L^p(D_1 \times D_2) \), these results imply \( Tf \) exists and belongs to \( L^p \).

In this section we show that only assuming \( f \in L^p(D_1 \times D_2) \) is not enough to conclude that \( Tf \in L^p(D_1 \times D_2) \) for \( 1 \leq p < 2 \) in general. Thus on product domains, estimates on the data beyond \( f \in L^p \) are generally needed for the Henkin solution to belong to \( L^p \), unlike the situation for the Henkin solution on strongly pseudoconvex domains, \([\text{Ker71, Ovr71}]\).
Consider the $\bar{\partial}$-equation on the bidisc $\mathbb{D}^2$. For each $k = 1, 2, \ldots$, let $f^k = f_1^k dz_1 + f_2^k dz_2$ on $\mathbb{D}^2$, where
\[
 f_1^k(z_1, z_2) = \bar{z}_1^{k-1} z_1 z_2^{k-1} \quad \text{and} \quad f_2^k(z_1, z_2) = \bar{z}_1^k z_1^{k-1} z_2^k.
\]
Note each $f^k$ is $\bar{\partial}$-closed. Moreover, direct computation shows
\[
\|f_1^k\|_{L^1(\mathbb{D}^2)} = \int_{\mathbb{D}^2} |\bar{z}_1^{k-1} z_1 z_2^{k-1}| dV(z) = O\left(\frac{1}{k^{2}}\right),
\]
\[
(4.1) \quad \|f_2^k\|_{L^1(\mathbb{D}^2)} = \int_{\mathbb{D}^2} |z_1^{k-1} z_2^{k-1}| dV(z) = O\left(\frac{1}{k^{2}}\right).
\]

An elementary calculation will be used to compute $T(f^k)$.

**Lemma 4.2.** For $z \in \mathbb{D}$ and $k \in \mathbb{Z}^+$,
\[
\int_{\mathbb{D}} \frac{\tilde{z}^{k-1} \zeta d\tilde{\zeta} \wedge d\zeta}{\zeta - z} = \frac{2\pi i}{k} (1 - \bar{z}z^k).
\]

**Proof.** This follows from the generalized Cauchy Integral formula. Details are provided for completeness.

Let $\omega = \frac{1}{k} \frac{\tilde{z}^k \zeta}{\zeta - z} d\zeta$ on $\mathbb{D} \setminus B$, where $B = B(z; \varepsilon)$ is a disk centered at $z$ of radius $\varepsilon$ sufficiently small so that $B \subset \mathbb{D}$. By Stokes’ theorem, $\int_{\mathbb{D}\setminus B} d\omega = \int_{\partial B} \omega - \int_{bB} \omega$. Since $d\omega = (\partial + \bar{\partial}) \omega = \frac{\tilde{z}^{k-1} \zeta}{\zeta - z} d\tilde{\zeta} \wedge d\zeta$, it follows
\[
\int_{\mathbb{D}} \frac{\tilde{z}^{k-1} \zeta d\tilde{\zeta} \wedge d\zeta}{\zeta - z} = \lim_{\varepsilon \to 0^+} \int_{\mathbb{D}\setminus B} d\omega = \lim_{\varepsilon \to 0^+} \left( \int_{bB} \frac{1}{k} \frac{\tilde{z}^k \zeta}{\zeta - z} d\tilde{\zeta} - \int_{bB} \frac{1}{k} \frac{\tilde{z}^{k-1} \zeta}{\zeta - z} d\zeta \right).
\]

By the Cauchy integral formula, the first term on the right hand side is
\[
\int_{bB} \frac{1}{k} \frac{\tilde{z}^k \zeta}{\zeta - z} d\zeta = \int_{bB} \frac{d\zeta}{\zeta - z} = \frac{2\pi i}{k}.
\]

Writing $\zeta = z + \varepsilon e^{i\theta}$ on $bB$, the second term becomes
\[
\lim_{\varepsilon \to 0^+} \int_{bB} \frac{1}{k} \frac{\tilde{z}^{k-1} \zeta}{\zeta - z} d\zeta = \lim_{\varepsilon \to 0^+} \int_{0}^{2\pi} \frac{1}{k} |z + \varepsilon e^{i\theta}|^{2k} \frac{\varepsilon e^{i\theta}}{\varepsilon} d\theta = \frac{2\pi i}{k} |z|^{2k}.
\]

The conclusion follows by combining these terms. $\square$

Compute $T(f^k)$ using the explicit expression (2.4). For the first term,
\[
-\frac{1}{2\pi i} \int_{\mathbb{D}} \frac{f_2^k (z_1, z_2) \bar{\zeta} d\bar{\zeta} \wedge d\zeta}{\zeta - z_2} = -\frac{1}{2\pi i} \int_{\mathbb{D}} \frac{z_1^{k-1} z_2^{-k} \bar{z}_1^{k-1} \zeta d\bar{\zeta} \wedge d\zeta}{\zeta - z_2} = -\frac{1}{2\pi i} \cdot \bar{z}_1^{k} \int_{\mathbb{D}} \frac{\tilde{\zeta}^{k-1} \bar{\zeta} d\tilde{\zeta} \wedge d\zeta}{\zeta - z_2}
\]
\[
= -\frac{1}{2\pi i} \cdot z_1^{k} \cdot \frac{2\pi i}{k} (1 - z_2^{k} z_2^{k}) = \frac{1}{k} |z_1 z_2|^{2k} - \frac{1}{k} |z_2|^{2k}.
\]

The third equality follows from Lemma 4.2. Similarly, the second term on the right hand side in (2.4) is
\[
-\frac{1}{2\pi i} \int_{\mathbb{D}} \frac{f_1^k (z_1, z_2) \bar{\zeta} d\bar{\zeta} \wedge d\zeta}{\zeta - z_1} = \frac{1}{k} |z_1 z_2|^{2k} - \frac{1}{k} |z_2|^{2k}.
\]
For the last term in (2.4), separate the variables in the integral and apply Lemma 4.2 twice to get
\[\begin{align*}
\frac{-1}{(2\pi i)^2} \int_{D^2} \mathcal{D}(f^k)(\zeta_1, \zeta_2) \, d\zeta_1 \wedge d\zeta_2 \wedge d\zeta_2 = & \frac{-1}{(2\pi i)^2} \int_{D^2} k\zeta_1^{-k-1} \zeta_2^{k-1} \zeta_1 \zeta_2 d\zeta_1 \wedge d\zeta_2 \\
& \frac{-k}{(2\pi i)^2} \int_{D} \zeta_1^{-k-1} \zeta_2^{k-1} \zeta_1 d\zeta_1 \int_{D} \zeta_2^{k-1} \zeta_2 d\zeta_2 \\
& = \frac{-k}{(2\pi i)^2} \cdot \frac{2\pi i}{k} \cdot \frac{2\pi i}{k} (1 - \frac{\zeta_1^k}{\zeta_1}) \cdot (1 - \frac{\zeta_2^k}{\zeta_2}) \\
& = -\frac{1}{k} + \frac{1}{k} |z_1|^{2k} + \frac{1}{k} |z_2|^{2k} - \frac{1}{k} |z_1 z_2|^{2k}.
\end{align*}\]

Therefore,
\[T(f^k) = \frac{1}{k} |z_1 z_2|^{2k} - \frac{1}{k}.\]

Now define, for each \(L \in \mathbb{Z}^+\), \(g^L = g^L_1 d\bar{z}_1 + g^L_2 d\bar{z}_2\) with \(g^L_1 = \sum_{k=1}^{L} f_1^k\) and \(g^L_2 = \sum_{k=1}^{L} f_2^k\). Clearly each \(\partial \bar{g}^L = 0\). Since \(\sum \frac{1}{k} < \infty\), (4.1) implies \(g^L \in L^1_{0,1}(\mathbb{D}^2)\) with \(L^1\) norm bounded independent of \(L\). However (4.3) implies
\[T(g^L)(z_1, z_2) = \sum_{k=1}^{L} T(f^k) = \frac{|z_1 z_2| (1 - |z_1 z_2|^{2L})}{1 - |z_1 z_2|^{2L}} - \sum_{k=1}^{L} \frac{1}{k}.
\]

If \(K \subset \mathbb{D}^2\) is a compact set and \((z_1, z_2) \in K\), the last expression tends to \(-\infty\) as \(L \to \infty\), by divergence of the harmonic series. Thus there does not exist a constant \(C\) such that \(\|T(g^L)\|_{L^1} \leq C \|g^L\|_{L^1}\) for all \(L\).

Remark 4.4. Taking the full sums, \(g = g_1 d\bar{z}_1 + g_2 d\bar{z}_2\) with \(g_1 = \sum_{k=1}^{\infty} f_1^k\) and \(g_2 = \sum_{k=1}^{\infty} f_2^k\), gives an example where \(Tg\) does not even exist. In this case, (4.1) still shows \(g \in L^1_{0,1}(\mathbb{D}^2)\), while the analogue of the above computation yields
\[T(g)(z_1, z_2) = -\ln(1 - |z_1 z_2|^2) - \sum_{k=1}^{\infty} \frac{1}{k} \equiv \infty.
\]

Remark 4.5. A careful inspection of the integrals shows that \(g \in L^p(\mathbb{D}^2)\) for \(1 \leq p < 2\); details are left to the interested reader. Thus for the \(L^1\) problem, “over-prescribing” integrability by requiring \(g \in L^p\) for \(p < 2\) is still not sufficient to guarantee \(Tg \in L^1\).

5. Non-canonical Solution

The solution \(u = T(f)\) in (2.4) is compared with the \(L^2\)-minimal solution \(u_{\text{can}}\) on \(D_1 \times D_2\). The first observation is that \(u = T(f) \neq u_{\text{can}}\) on \(\mathbb{D}^2\).

Let \(h \in C^1(\overline{\mathbb{D}})\) be a holomorphic function on \(\mathbb{D}\) and let
\[f = z_1^k h(z_2) d\bar{z}_1\]
for some positive integer \(k\). It is easily checked that \(\partial \bar{f} = 0\). Since \(f_2 = 0\), (2.4) becomes
\[u = T(f) = -\frac{1}{2\pi i} \int_{\mathbb{D}} f_1(\zeta_1, z_2) d\bar{\zeta}_1 \wedge d\zeta_1 = -\frac{1}{2\pi i} \cdot h(z_2) \int_{\mathbb{D}} \zeta_1^k d\bar{\zeta}_1 \wedge d\zeta_1.
\]

Let \(\omega = \zeta_1^k \zeta_1/(\zeta_1 - z_1)\) and apply Stokes theorem to the integral \(\int_{\partial B} d\omega\), where \(B = B(z_1, \varepsilon)\) is the disk centered at \(z_1\) of radius \(\varepsilon\) for some \(\varepsilon > 0\) sufficiently small. It follows
that

\[
\frac{1}{2\pi i} \int_{\mathbb{D}} \zeta_1^k d\zeta_1 \wedge d\zeta_1 = \lim_{\varepsilon \to 0^+} \left( \frac{1}{2\pi i} \int_{b\mathbb{D}} \zeta_1^k d\zeta_1 - \frac{1}{2\pi i} \int_{bB} \zeta_1^k d\zeta_1 \right).
\]

By the Cauchy integral formula, the first term in (5.2) is \( z_1^{k-1} \). For the second term of (5.2), write \( \zeta_1 = z_1 + \varepsilon e^{id} \) and note

\[
\lim_{\varepsilon \to 0^+} \frac{1}{2\pi i} \int_{bB} \zeta_1^k d\zeta_1 = z_1^k \bar{z}_1.
\]

Thus, \( u = h(z_2) \left( z_1^k \bar{z}_1 - z_1^{k-1} \right) \).

But the \( L^2 \)-minimal solution of \( \bar{\partial} u = f \) is

\[
u_{can} = h(z_2) \left( z_1^k \bar{z}_1 - \frac{k}{k+1} z_1^{k-1} \right),
\]

since it is easy to verify

\[
\langle \nu_{can}, z_1^m \bar{z}_1^n \rangle = \int_{\mathbb{D} \times \mathbb{D}} z_1^k \bar{z}_1 h(z_2) z_1^m \bar{z}_1^n dV(z) - \frac{k}{k+1} \int_{\mathbb{D} \times \mathbb{D}} z_1^{k-1} h(z_2) z_1^m \bar{z}_1^n dV(z) = 0
\]

for all integers \( m, n \geq 0 \).

**Remark 5.3.** Let \( D_1 \) and \( D_2 \) be bounded simply connected planar domains. Then \( D_1 \times D_2 \) is biholomorphic to \( \mathbb{D}^2 \) under a mapping \( \psi_1 \otimes \psi_2 \), where \( \psi_j \) is the biholomorphism from \( D_j \) to \( \mathbb{D} \) for \( j = 1, 2 \). Now consider the form with the same expression as in (5.1) and let \( h \equiv 1 \), that is \( f = z_1^k d\bar{z}_1 \) on \( D_1 \times D_2 \), for some positive integer \( k \). The solution operator \( T \) gives the same solution \( u = z_1^k \bar{z}_1 - z_1^{k-1} \) as above.

However, to obtain the \( L^2 \)-minimal solution, one must transfer the orthonormal basis on \( A^2(\mathbb{D}^2) \) to one on \( A^2(D_1 \times D_2) \). Thus the expression of \( \nu_{can} \) necessarily involves \( \psi_1' \) and \( \psi_2' \), unlike the expression of \( u = T(f) \).

### 6. The Hartogs Triangle

In this section, consider the \( \bar{\partial} \)-equation on the Hartogs triangle \( \mathbb{H} \):

\[
\mathbb{H} = \{ (z_1, z_2) \in \mathbb{C}^2 : |z_1| < |z_2| < 1 \}.
\]

The first step is to transfer the equation on \( \mathbb{H} \) to the product space \( \mathbb{D} \times \mathbb{D}^* \).

#### 6.1. Transform the \( \bar{\partial} \)-equation. Let

\[
\phi : \mathbb{H} \to \mathbb{D} \times \mathbb{D}^*
\]

\[
\phi(z_1, z_2) = (z_1/z_2, z_2) = (w_1, w_2)
\]

be the usual biholomorphism. Consider

\[
\bar{\partial}_z v = \alpha = \alpha_1 d\bar{z}_1 + \alpha_2 d\bar{z}_2
\]

on \( \mathbb{H} \), where \( \alpha \) is \( \bar{\partial} \)-closed. Using the chain rule

\[
\left\{\begin{array}{l}
d\bar{z}_1 = \frac{\partial \bar{z}_1}{\partial w_1} d\bar{w}_1 + \frac{\partial \bar{z}_1}{\partial w_2} d\bar{w}_2 \\
d\bar{z}_2 = \frac{\partial \bar{z}_2}{\partial w_1} d\bar{w}_1 + \frac{\partial \bar{z}_2}{\partial w_2} d\bar{w}_2
\end{array}\right.
\]
it follows that equation (6.1) is equivalent to
\[
\partial_w u = \left( \partial_{\bar{z}_1} + \partial_{\bar{z}_2} \right) d\bar{w}_1 + \left( \partial_{\bar{z}_1} + \partial_{\bar{z}_2} \right) d\bar{w}_2
\]
\[
= \bar{w}_2 \cdot \partial_{\bar{z}_1} d\bar{w}_1 + (\bar{w}_1 \cdot \partial_{\bar{z}_1} + \partial_{\bar{z}_2}) d\bar{w}_2
\]
\[
= f_1 d\bar{w}_1 + f_2 d\bar{w}_2
\]
\[
= f
\]
on \mathbb{D} \times \mathbb{D}^*, \text{ where } u = v \circ \phi^{-1} \text{ and } \partial \alpha_j = \alpha_j \circ \phi^{-1} \text{ for } j = 1, 2.
\]
Note that
\[
\frac{\partial f_1}{\partial \bar{w}_2} = \frac{\partial}{\partial \bar{w}_2}(\bar{w}_2 \cdot \partial_{\bar{z}_1}) = \partial_{\bar{z}_1} + \bar{w}_2 \partial_{\bar{z}_1} + \bar{w}_2 \partial_{\bar{z}_1}
\]
and
\[
\frac{\partial f_2}{\partial \bar{w}_1} = \frac{\partial}{\partial \bar{w}_1}(\bar{w}_1 \cdot \partial_{\bar{z}_1} + \partial_{\bar{z}_2}) = \partial_{\bar{z}_1} + \bar{w}_1 \partial_{\bar{z}_1} + \bar{w}_1 \partial_{\bar{z}_2}.
\]
Since \( \alpha \) is \( \partial \)-closed on \( \mathbb{H} \), it follows that \( f \) is \( \partial \)-closed on \( \mathbb{D} \times \mathbb{D}^* \).

6.2. **An \( L^p \) Assumption on \( \mathbb{H} \).** Based on the transformation (6.2), a vanishing condition on \( \alpha_j \) at the origin,
\[
\|\alpha_j\|_{L^p_{-2}(\mathbb{H})}^p = \int_{\mathbb{H}} |\alpha_j|^p |z_2|^{-2} dV(z) < \infty
\]
for \( j = 1, 2 \), implies that \( f_1, f_2 \in L^p(\mathbb{D} \times \mathbb{D}^*) \). This follows since
\[
\int_{\mathbb{D} \times \mathbb{D}^*} |\alpha_1|^p \cdot |w_2|^p dV(w) \leq \int_{\mathbb{D} \times \mathbb{D}^*} |\alpha_1|^p dV(w) = \|\alpha_1\|_{L^p_{-2}(\mathbb{H})}^p
\]
and
\[
\int_{\mathbb{D} \times \mathbb{D}^*} |\alpha_1| |\bar{w}_1 \cdot \partial_{\bar{z}_1} + \partial_{\bar{z}_2}| dV(w) \leq C_p \int_{\mathbb{D} \times \mathbb{D}^*} |\alpha_1|^p + |\bar{w}_2|^p dV(w) = C_p \left( \|\alpha_1\|_{L^p_{-2}(\mathbb{H})}^p + \|\alpha_2\|_{L^p_{-2}(\mathbb{H})}^p \right).
\]
In addition, a vanishing condition on derivatives of \( \alpha_j \) at the origin,
\[
\left\| \frac{\partial \alpha_1}{\partial z_j} \right\|_{L^p_{-1}(\mathbb{H})}^p = \int_{\mathbb{H}} \left| \frac{\partial \alpha_1}{\partial z_j} \right|^p |z_2|^{-1} dV(z) < \infty
\]
for \( j = 1, 2 \), implies that \( \partial \alpha_j \in L^p(\mathbb{D} \times \mathbb{D}^*) \). This follows since for \( 1 \leq p \leq \infty \)
\[
\int_{\mathbb{D} \times \mathbb{D}^*} \left| \alpha_1 + \bar{w}_1 \bar{w}_2 \frac{\partial \alpha_1}{\partial z_1} + \bar{w}_2 \frac{\partial \alpha_1}{\partial z_2} \right|^p \leq C_p \int_{\mathbb{D} \times \mathbb{D}^*} \left| \alpha_1 + |w_2| \cdot \left| \frac{\partial \alpha_1}{\partial z_1} \right|^p + |w_2| \cdot \left| \frac{\partial \alpha_1}{\partial z_2} \right|^p \right.
\]
\[
\leq C_p \left( \|\alpha_1\|_{L^p_{-2}(\mathbb{H})}^p + \left\| \frac{\partial \alpha_1}{\partial z_1} \right\|_{L^p_{-1}(\mathbb{H})}^p + \left\| \frac{\partial \alpha_1}{\partial z_2} \right\|_{L^p_{-1}(\mathbb{H})}^p \right).
\]
Thus, the following \( L^p \) estimate for a solution of \( \partial \) on \( \mathbb{H} \) holds.

**Theorem 6.3.** Let \( v, \alpha \) be as in (6.1) and \( u, f \) be as in (6.2). Suppose \( \alpha \) is \( \partial \)-closed in the weak sense on \( \mathbb{H} \). For \( 1 \leq p < \infty \), assume that
\begin{enumerate}
  \item \( \alpha_1, \alpha_2 \in L^p_{-2}(\mathbb{H}) \),
  \item \( \partial \alpha_2/\partial z_1 = \partial \alpha_1/\partial z_2 \in L^p_{-2}(\mathbb{H}) \) and \( \partial \alpha_1/\partial z_1 \in L^p(\mathbb{H}) \).
\end{enumerate}
Then there is a weak solution \( v = u \circ \phi = T(f) \circ \phi \), where \( T \) is the solution operator in (3.6), satisfying the \( L^p \) estimate
\[
\|v\|_{L^p(\mathbb{H})} \leq C \left( \sum_{j=1}^{2} \|\alpha_j\|_{L^p_{-2}(\mathbb{H})}^p + \sum_{j=1}^{2} \|\partial \alpha_j/\partial z_j\|_{L^p_{-1}(\mathbb{H})}^p \right)
\]
for some constant \( C > 0 \) independent of \( \alpha \).
6.3. An example. The $L^p$ estimates of the solution $v$ given in Theorem 6.3, and the
canonical solution $v_{\text{can}}$ on $\mathbb{H}$ can be compared via the simple example
$$\alpha = d\bar{z}_2.$$ 
Verifying that $\alpha$ satisfies the conditions in Theorem 6.3 is easy. Thus the solution $v$ in
Theorem 6.3 belongs to $L^p(\mathbb{H})$ for $1 \leq p < \infty$.
On the other hand, we claim the $L^2$-minimal solution of $\bar{\partial}v = \alpha$ is
$$v_{\text{can}} = \bar{z}_2 - cz_2^{-1}$$
for some nonzero constant $c$. Clearly $\bar{\partial}v_{\text{can}} = \alpha$. To see that $v_{\text{can}}$ is orthogonal to holomorphic
functions on $\mathbb{H}$, it suffices to take its inner product with the orthogonal basis \{\( z^n \bar{z}_2^m \)\} on $\mathbb{H}$, for so-called allowable indices $(n, m) \in \mathbb{Z}^+ \times \mathbb{Z}$. See Sections of [EM17] for the
definition of allowable indices and Section 5 of that paper of that paper for a proof that
$$\langle v_{\text{can}}, z^n \rangle = 0$$
for all allowable exponents $\alpha$. Note that $\alpha \in L^p(\mathbb{H})$. On the other hand, the
proof of Proposition 5.5 in [EM17] implies that $v_{\text{can}} \notin L^p(\mathbb{H})$ for $p \geq 4$. Thus, $v$ behaves better than $v_{\text{can}}$ in terms of $L^p$ regularity.
At the operator level, it follows that the canonical solution operator for $\bar{\partial}$ on $\mathbb{H}$ doesn’t
map $L^p$ $\bar{\partial}$-closed $(0,1)$-form to $L^p$ functions for $p \geq 4$. This is consistent with results on the
Bergman projection on $\mathbb{H}$, see [EM16], [CZ16], and [Che17].

6.4. Extra condition. An extra condition on $\alpha$, namely
$$\bar{z}_1 \cdot \alpha_1 + \bar{z}_2 \cdot \alpha_2 = 0,$$
and (6.2) shows that $f = f_1d\bar{w}_1$ on $\mathbb{D} \times \mathbb{D}^*$. By (3.6), $T(f)$ only involves the second term.
Thus a better $L^p$ estimate holds in this case:

**Theorem 6.5.** Let $v, \alpha$ be as in (6.1) and $u, f$ be as in (6.2). Suppose $\alpha$ is $\bar{\partial}$-closed in the
weak sense on $\mathbb{H}$ and satisfies (6.4). For $1 \leq p < \infty$, assume that $\alpha_1 \in L^p(\mathbb{H})$. Then there
is a weak solution $v = u \circ \phi = T(f) \circ \phi$, where $T$ is the solution operator in (3.6), satisfying
the $L^p$ estimate
$$\|v\|_{L^p(\mathbb{H})} \leq C\|\alpha_1\|_{L^p(\mathbb{H})}$$
for a constant $C > 0$ independent of $\alpha$.

**Proof.** Starting with (3.6) and applying Lemma 3.1, it follows that
$$\int_{\mathbb{D} \times \mathbb{D}^*} |T(f)|^p dV(w) \leq C \int_{\mathbb{D} \times \mathbb{D}^*} |f_1|^p dV(w).$$
(i) When $p \geq 2$, it holds that
$$\int_{\mathbb{D} \times \mathbb{D}^*} |\bar{\alpha}_1|^p \cdot |w_2|^p dV(w) \leq \int_{\mathbb{D} \times \mathbb{D}^*} |\bar{\alpha}_1|^p \cdot |w_2|^2 dV(w) = \int_\mathbb{H} |\alpha_1|^p dV(z).$$
Since $f_1 = \bar{w}_2 \cdot \bar{\alpha}_1$ and $u = u \circ \phi = T(f) \circ \phi$, by (6.6), it follows
$$\int_\mathbb{H} |v|^p dV(z) = \int_{\mathbb{D} \times \mathbb{D}^*} |u|^p \cdot |w_2|^2 dV(w) \leq \int_{\mathbb{D} \times \mathbb{D}^*} |T(f)|^p dV(w) \leq C \int_\mathbb{H} |\alpha_1|^p dV(z).$$
(ii) When $1 \leq p < 2$, consider $\bar{\partial}_w(u \cdot w_2) = w_2 \cdot \bar{\partial}_w u = w_2 \cdot f = w_2 \cdot f_1 d\bar{w}_1$. By (3.6),
$T(w_2 \cdot f) = w_2 \cdot T(f) = w_2 \cdot u$. Therefore, replacing $f$ by $w_2 \cdot f$ in (6.6),
$$\int_{\mathbb{D} \times \mathbb{D}^*} |u \cdot w_2|^p dV(w) \leq C \int_{\mathbb{D} \times \mathbb{D}^*} |f_1 \cdot w_2|^p dV(w) = C \int_{\mathbb{D} \times \mathbb{D}^*} |\bar{\alpha}_1|^p \cdot |w_2|^{2p} dV(w).$$
Since $1 \leq p < 2$,
$$\int_\mathbb{H} |v|^p dV(z) = \int_{\mathbb{D} \times \mathbb{D}^*} |u|^p \cdot |w_2|^2 dV(w) \leq \int_{\mathbb{D} \times \mathbb{D}^*} |u|^p \cdot |w_2|^p dV(w)$$
and
\[ \int_{D \times D^*} \left| \tilde{\alpha}_1 \right|^p \cdot |w_2|^{2p} \, dV(w) \leq \int_{D \times D^*} \left| \tilde{\alpha}_1 \right|^p \cdot |w_2|^2 \, dV(w) = \int_{\mathbb{H}} |\alpha_1|^p \, dV(z). \]

Hence,
\[ \int_{\mathbb{H}} |\nu|^p \, dV(z) \leq C \int_{\mathbb{H}} |\alpha_1|^p \, dV(z). \]

\[ \square \]

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