Research Article

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Self-dual DeepBKZ for finding short lattice vectors

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Abstract: In recent years, the block Korkine-Zolotarev (BKZ) and its variants such as BKZ 2.0 have been used as de facto algorithms to estimate the security of a lattice-based cryptosystem. In 2017, DeepBKZ was proposed as a mathematical improvement of BKZ, which calls LLL with deep insertions (DeepLLL) as a subroutine alternative to LLL. DeepBKZ can find a short lattice vector by smaller blocksizes than BKZ. In this paper, we develop a self-dual variant of DeepBKZ, as in the work of Micciancio and Walter for self-dual BKZ. Like DeepBKZ, our self-dual DeepBKZ calls both DeepLLL and its dual variant as main subroutines in order to accelerate to find a very short lattice vector. We also report experimental results of DeepBKZ and our self-dual DeepBKZ for random bases on the Darmstadt SVP challenge.

Keywords: Lattice basis reduction, SVP, BKZ, DeepLLL

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1 Introduction

Since the US National Institute of Standards and Technology (NIST) began a process to develop new standards for post-quantum cryptography (PQC) in 2015 and called for proposals in 2016, it has rapidly accelerated to research lattice-based cryptography as a candidate of PQC. At the submission deadline of November 30, 2017 for the call, NIST received more than 20 proposals of lattice-based cryptosystems (see the web page of “Round 1 Submissions” of [13]). The security of such proposals relies on the hardness of lattice problems such as LWE and NTRU, and it is becoming more important to precisely evaluate the hardness.

Lattice basis reduction is a strong tool in cryptanalysis, and it has been used to estimate the security of lattice-based cryptosystems. In particular, BKZ [16] and its variants such as BKZ 2.0 [3] are de facto algorithms to estimate the security level (see [1]). Given \( \beta \), BKZ repeatedly calls an SVP oracle in a \( \beta \)-dimensional lattice to find a short lattice vector. In security estimation, it is discussed which block sizes \( \beta \) are required for BKZ to find a short lattice vector of target norm. A new improvement of BKZ, called DeepBKZ [18], was proposed, which calls DeepLLL [16, Section 3] before every SVP oracle to find a short lattice vector by smaller block sizes than BKZ. In fact, DeepBKZ with around \( \beta = 40 \) had found new solutions for the SVP challenge [4] in most dimensions from 102 to 127.

In this paper, we develop a self-dual variant of DeepBKZ, emulating the self-dual BKZ by Micciancio and Walter [11]. The original self-dual BKZ calls SVP and dual SVP oracles in forward and backward tours, respectively, to find a reduced basis \( \mathbf{B} = [\mathbf{b}_1, \ldots, \mathbf{b}_n] \) with short \( \mathbf{b}_1 \) and long \( \mathbf{b}_n \), where \( [\mathbf{b}_1', \ldots, \mathbf{b}_n'] \) denotes the Gram-Schmidt orthogonalization of \( \mathbf{B} \). In our self-dual DeepBKZ, DeepLLL and dual DeepLLL [19] are respectively called to reduce \( [\mathbf{b}_1, \ldots, \mathbf{b}_{n-1}] \) and \( [\pi_2(\mathbf{b}_1), \ldots, \pi_2(\mathbf{b}_n)] \) before every SVP and dual SVP oracles, where \( \pi_2 \) denotes the orthogonal projection. Our construction is similar to [11], but its mathematical background is based on classical proofs of Mordell’s inequality as well as the slide reduction algorithm [6] (see also [12]).

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Like DeepBKZ [18], we hope that DeepLLL and its dual variant could help to find a basis with shorter $b_1$ and longer $b_n$ than the original self-dual BKZ [11] (as a result, we expect that a very short lattice vector could be found). To show practicality, we report experimental results of both DeepBKZ and our self-dual variant for random bases on [4].

**Notation 1.1.** The symbols $\mathbb{Z}$ and $\mathbb{R}$ denote the ring of integers and the field of real numbers, respectively. We represent all vectors in column format. For a vector $a = (a_1, \ldots, a_n)^\top \in \mathbb{R}^n$, let $|a|$ denote its Euclidean norm. For $a = (a_1, \ldots, a_n)^\top$ and $b = (b_1, \ldots, b_n)^\top$, let $\langle a, b \rangle$ denote the inner product $\sum_{i=1}^n a_i b_i$.

# 2 Preliminaries

In this section, we review some definitions on lattices. We also introduce typical reduction algorithms and DeepBKZ [18], an improvement of BKZ.

## 2.1 Lattices

### (Primal) lattices and bases

Let $b_1, \ldots, b_n$ be linearly independent vectors in $\mathbb{R}^n$. The set of all integral linear combinations of the $b_i$’s is a (full-rank) lattice

$$L = L(b_1, \ldots, b_n) = \left\{ \sum_{i=1}^n x_i b_i : x_i \in \mathbb{Z} \text{ for all } 1 \leq i \leq n \right\}$$

dimension $n$ with basis $B = [b_1, \ldots, b_n] \in \mathbb{R}^{n \times n}$. Every lattice has infinitely many bases if $n \geq 2$; If two bases $B_1$ and $B_2$ span the same lattice, there exists a unimodular matrix $V \in \text{GL}_n(\mathbb{Z})$ with $B_1 = B_2 V$. The **volume** of $L$ is defined as $\text{vol}(L) = |\det(B)| > 0$, which is independent of the choice of bases. The Gram-Schmidt orthogonalization (GSO) for an ordered basis $B$ is the orthogonal family $B^* = [b_1^*, \ldots, b_n^*]$, recursively defined by $b_1^* := b_1$ and

$$b_i^* := b_i - \sum_{j=1}^{i-1} \mu_{i,j} b_j^* \text{ with } \mu_{i,j} := \frac{\langle b_i, b_j^* \rangle}{\|b_j^*\|^2}$$

for $2 \leq i \leq n$. Then $\text{vol}(L) = \prod_{i=1}^n \|b_i^*\|$. For $2 \leq \ell \leq n$, let $\pi_{\ell}$ denote the orthogonal projection over the orthogonal supplement of the $\mathbb{R}$-vector space $(b_1, \ldots, b_{\ell-1})_\mathbb{R}$ (note that $\pi_{\ell}$ depends on a basis, and set $\pi_1 = \text{id}$). For $1 \leq i \leq j \leq n$, we denote by $B_{i,j}$ the local projected block $[\pi_i(b_j), \pi_i(b_{i+1}), \ldots, \pi_i(b_j)]$, and by $L_{i,j}$ the lattice spanned by $B_{i,j}$. The **first successive minimum** is the length of a shortest non-zero vector in a lattice $L$, denoted by $\lambda_1(L)$.

### Dual lattices and dual bases

The **dual** of a lattice $L$ is defined as

$$\hat{L} = \left\{ x \in \text{span}_\mathbb{R}(L) : \langle x, y \rangle \in \mathbb{Z} \text{ for all } y \in L \right\},$$

where $\text{span}_\mathbb{R}(L)$ denotes the $\mathbb{R}$-vector space spanned by the vectors in $L$. The dual of a full-rank lattice with basis $B$ has a basis $D = (B^{-1})^\top$. In other words, the relation $D^\top B = I$ is always maintained, where $I$ is the identity matrix. This tells how the dual basis $D$ changes with respect to changes of the primal basis $B$.

## 2.2 Lattice Basis Reduction

Here we introduce some notions of reduction and algorithms to achieve them.
LLL
For a parameter $\frac{1}{4} < \delta < 1$, a basis $B = \{b_1, \ldots, b_n\}$ is called $\delta$-LLL-reduced if it satisfies the two conditions; (i) It is size-reduced, namely, the GSO coefficients satisfy $|\mu_{i,j}| \leq \frac{1}{\delta}$ for all $j < i$. (ii) It satisfies Lovász’ condition, namely, $\delta \|b_{k-1}\|^2 \leq \|\pi_k(b_k)\|^2$ for all $k$. An LLL-reduced basis can be found by the LLL algorithm [10], in which adjacent basis vectors $b_{k-1}$ and $b_k$ are swapped if Lovász’ condition does not hold.

DeepLLL
It is a straightforward generalization of LLL, in which non-adjacent vectors can be changed; If the deep exchange condition $\|\pi_i(b_k)\|^2 < \delta \|b_i\|^2$ is satisfied for some $i < k$, the vector $b_k$ is inserted between $b_{i-1}$ and $b_i$ as

$$B \leftarrow [b_1, \ldots, b_{i-1}, b_k, b_i, \ldots, b_{k-1}, b_{k+1}, \ldots, b_n],$$

This is called a deep insertion. Every output basis of DeepLLL satisfies the following condition: For $\frac{1}{4} < \delta < 1$, a basis $B = \{b_1, \ldots, b_n\}$ is called $\delta$-DeepLLL-reduced if it is size-reduced and it also satisfies $\delta \|b_i\|^2 \leq \|\pi_i(b_k)\|^2$ for all $i < k$.

BKZ
A basis $B = \{b_1, \ldots, b_n\}$ of a lattice $L$ is called HKZ-reduced if it is size-reduced and it satisfies $\|b_i^*\| = \lambda_1(\pi_i(L))$ for every $1 \leq i \leq n$. The notion of BKZ-reduction is a local block version of HKZ-reduction, defined as follows [14–16]; For $\beta \geq 2$, a basis $B = \{b_1, \ldots, b_n\}$ of a lattice $L$ is called $\beta$-BKZ-reduced (simply called $\beta$-reduced in [14, 16]) if it is size-reduced and every local block $B_{[i,i+\beta-1]}$ is HKZ-reduced for $1 \leq j \leq n-\beta+1$. The second condition means $\|b_j^*\| = \lambda_1(L_{[j,j+k]})$ for every $1 \leq j \leq n-1$ with $k = \min(j+\beta-1, n)$. The original BKZ algorithm [16] finds an almost $\beta$-BKZ-reduced basis, and it calls LLL to reduce every local block before enumeration of a shortest vector over the block lattice. Efficient variants of BKZ have been proposed such as BKZ 2.0 [3], and some of them have been implemented in software (e.g., [5]).

Self-dual BKZ
Motivated from the slide reduction algorithm [6], an elegant generalization of LLL (see also Section 3.3 below), the self-dual BKZ algorithm was proposed by Micciancio and Walter [11]. Their algorithm is based on a new notion [11, Definition 1] of block reduction using lattice duality (cf., the slide reduction algorithm is based on classical proofs of Mordell’s inequality, see also [12] for details). The output quality of self-dual BKZ in the worst case is proven to be at least as the worst case behavior of BKZ [11, Theorem 1]. The self-dual BKZ algorithm calls primal SVP and dual SVP oracles over local blocks in forward and backward tours, respectively, and it calls forward and backward tours alternately. Like BKZ, self-dual BKZ is a proper block generalization of the LLL algorithm.

DeepBKZ
It was proposed in [18], which calls DeepLLL as a subroutine alternative to LLL in BKZ. Every output basis of DeepBKZ satisfies the following condition of reduction; For $\frac{1}{4} < \delta < 1$ and $\beta \geq 2$, a basis is called $(\delta, \beta)$-DeepBKZ-reduced if it is both $\delta$-DeepLLL-reduced and $\beta$-BKZ-reduced. In Algorithm 1, we give a detailed algorithm of DeepBKZ with early-abort strategy, adopted in BKZ 2.0; For an input basis $B = \{b_1, \ldots, b_n\}$ of a lattice $L$, we terminate DeepBKZ with blocksize $\beta$ after

$$N = C \left( \frac{n}{\beta} \right)^2 \left( \log n + \log \log \max_{\text{latt}\in\text{vol}(L)\setminus\text{vol}(L)^{1/n}} \frac{\|b_i\|}{\lambda_1(L_{[j,j+k]})} \right)$$

tours, where $C$ is a small constant. Since it is proven in [9] that the output basis of BKZ after $N$ tours has an enough quality, we expect that a similar result would hold for DeepBKZ (we took different values for $C$ in our experiments).
Dual DeepBKZ
It is a dual version of DeepBKZ proposed in [19]. It consists of the dual enumeration in self-dual BKZ [11] and a dual version of DeepLLL to reduce the dual basis of an input basis. In particular, in dual DeepLLL [19, Algorithm 1], a basis transformation is performed as
\[ B \leftarrow [b_1, \ldots, b_{i-1}, b_i, b_{i+1}, \ldots, b_n], \]  
(3)
called a dual deep insertion (this is opposite to the primal deep insertion (1)).

3 Self-dual DeepBKZ
In this section, we develop a self-dual variant of DeepBKZ.

3.1 Overview of algorithm
Algorithm 2 is our self-dual DeepBKZ. It consists of two parts of a forward tour and a backward tour, as in self-dual BKZ by Micciancio and Walter [11]. Let \( B = [b_1, \ldots, b_n] \) be an input basis of self-dual DeepBKZ with blocksize \( 3 \leq \beta \leq n - 1 \). We describe an overview for each part as follows:

Forward tour
As in the original self-dual BKZ [11], we call SVP oracles in dimension \( \beta \) to reduce every local block \( B[j_1, j_1+\beta-1] \) from \( j = 1 \) to \( n - \beta \). In our self-dual DeepBKZ, we additionally call DeepLLL for the sub-basis \( [b_1, \ldots, b_{n-1}] \) before enumeration to find a shortest vector over every block lattice \( L[j, j+\beta-1] \). Note that this part does not change the last basis vector \( b_n \).

Backward tour
As in [11], we call dual SVP oracles in dimension \( \beta \) to reduce the dual basis of every local block \( B[j-\beta+1, j] \) from \( j = n \) down to \( \beta + 1 \). Similarly to the above part, we call dual DeepLLL [19, Algorithm 1], a dual variant of DeepLLL, before the dual enumeration [11, Algorithm 2] (see also Appendix A for the dual enumeration). As in the above part, we restrict the index \( i \) in every dual deep insertion (3) from \( i = n - 1 \) to \( 2 \) in order not to change the first basis vector \( b_1 \) by dual DeepLLL. This means that it reduces the local block \( B[2, n] \) by dual DeepLLL, equivalently, it reduces its dual basis by (primal) DeepLLL.

3.2 Terminating condition
A mathematical terminating condition is given in [11, Lemma 1] for the original self-dual BKZ, but it is mentioned in [11] that such condition might never be met in practice. As in [11], we use the early-abort strategy (cf., the termination of the original self-dual BKZ depends on the behavior of GSA slopes). We count a pair of forward and backward tours as one tour of our self-dual DeepBKZ. After
\[ M = C \left( \frac{n}{2\beta} \right)^2 \left( \log n + \log \log \max_{1 \leq i \leq n} \frac{\|b_i\|}{\text{vol}(L)^{1/n}} \right) \]  
(4)
tours with the same constant \( C \) as in (2), we terminate self-dual DeepBKZ with blocksize \( \beta \) for an input basis \( B = [b_1, \ldots, b_n] \) of a lattice \( L \) (we replace the denominator of (2) by \( 2\beta \), see [11, Section 4] for details).
3.3 Mathematical background and our motivation

Inspired by classical proofs of Mordell’s inequality, a reduction notion is introduced in [12]; A basis \( B = [b_1, \ldots, b_n] \) of a lattice \( L \) is called Mordell-reduced with factor \( \varepsilon \geq 0 \) if it satisfies both

\[
\|b_1\| = \lambda_1(L(b_1, \ldots, b_{n-1})) \quad \text{and} \quad \frac{1}{\|b_n^*\|} \leq (1 + \varepsilon)\lambda_1(\pi_2(L)).
\]

Mordell’s reduction satisfies some important properties such as [12, Lemma 10], and the slide reduction algorithm [6] is designed to achieve a blockwise version of Mordell’s reduction. In particular, the output quality of the slide reduction algorithm in the worst case is proven to be slightly better than that of BKZ for a fixed blocksize \( \beta \). (However, from [11, Figure 2], the output quality of the slide reduction algorithm is worse than both BKZ and self-dual BKZ in practice.)

In order to describe the basic idea of our self-dual DeepBKZ, consider

\[
\begin{bmatrix}
  b_1 & b_2 & \cdots & b_{n-1} \\
  \pi_2(b_2) & \pi_2(b_3) & \cdots & \pi_2(b_n)
\end{bmatrix}
\]

for a basis \( B = [b_1, \ldots, b_n] \) of a lattice \( L \). Our self-dual DeepBKZ reduces the sub-basis \([b_1, \ldots, b_{n-1}]\) by DeepBKZ [18] in forward tours. It also reduces the basis \([\pi_2(b_2), \ldots, \pi_2(b_n)]\) of the projected lattice \( \pi_2(L) \) by dual DeepBKZ [19] in backward tours, equivalently, it reduces the dual basis by DeepBKZ. With this construction, we expect that our self-dual DeepBKZ could find a basis satisfying a condition close to Mordell’s reduction. In particular, like DeepBKZ [18], we hope that our self-dual DeepBKZ could find a shorter \( b_1 \) (resp., longer \( b_n^* \)) with help of DeepLLL (resp., dual DeepLLL [19]) than the original self-dual BKZ [11].

3.4 Implementation

We implemented DeepBKZ (Algorithm 1) and self-dual DeepBKZ (Algorithm 2) in C++ programs with the NTL library [17]. We used the g++ compiler with -O3 -std=c++11 option. We set a triple of \( B = [b_1, \ldots, b_n] \), \( \mu = (\mu_{i,j})_{1 \leq i < j \leq n} \) and \( (B_i)_{1 \leq i \leq n} \) as a class, where \( B \) is a basis, \( \mu \) its GSO coefficients, and \( B_i = \|b_i^*\|^2 \). We used the int data type for \( B \), and long double for both \( \mu \) and \( (B_i)_{1 \leq i \leq n} \). To keep track of the GSO information in DeepLLL and dual DeepLLL, we implemented pseudo-codes of [18, Algorithm 4] and [19, Algorithm 2], respectively. We also implemented [8, Algorithm 2] and Algorithm 3 in Appendix A for primal and dual enumerations, respectively. We took a full enumeration setting as in [18] for both enumerations. Our experiments ran on an Intel Xeon CPU E3-1225 v6@3.30GHz with 32.0 GB RAM (we run every reduction algorithm over a single thread).

3.5 Experimental results

Here we report experimental results of DeepBKZ and self-dual DeepBKZ for random bases on [4] in terms of both the output quality and performance.

Output quality

The Hermite factor is a good index to measure the (practical) output quality of a reduction algorithm [7]. The factor of an algorithm for a basis of an \( n \)-dimensional lattice \( L \) is defined as

\[
\gamma = \frac{|b|}{\mathrm{vol}(L)^{1/n}}
\]

where \( b \) is a shortest non-zero basis vector output by the algorithm. Smaller \( \gamma \) means that the algorithm can find a shorter lattice vector. For practical algorithms such as LLL and BKZ, Gama and Nguyen [7] experimentally showed that the root Hermite factor \( \sqrt[n]{\gamma} \) converges a constant for high dimensions. In Table 1 (resp., Table 2), we show the average of the root Hermite factor of DeepBKZ (resp., self-dual DeepBKZ), for random
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Table 1: The average of the root Hermite factor $\gamma$ of DeepBKZ with early-abort for some constants $C$, for the SVP challenge in dimensions $n$ with seeds 0–9 ($C = \infty$ means the average factor without early-abort, which data are from [18]).

| $n$ | $C$ | $\beta = 20$ | $\beta = 25$ | $\beta = 30$ | $\beta = 35$ | $\beta = 40$ | $\beta = 45$ |
|-----|-----|-------------|-------------|-------------|-------------|-------------|-------------|
| 100 | 1.0 | 1.01077     | 1.01051     | 1.01039     | 1.01033     | 1.01022     | 1.01008     |
| 100 | 4.0 | 1.01077     | 1.01035     | 1.01011     | 1.01005     | 1.00994     | 1.00987     |
| $\infty$ | – | 1.01043 | 1.00999 | 1.00958 | 1.00949 | – |
| 105 | 1.0 | 1.01071     | 1.01028     | 1.01022     | 1.01014     | 1.01005     | 1.01003     |
| 105 | 4.0 | 1.01071     | 1.01023     | 1.00994     | 1.00994     | 1.00991     | 1.00979     |
| $\infty$ | – | 1.01041 | 1.00983 | 1.00937 | 1.00924 | – |
| 110 | 1.0 | 1.01057     | 1.01031     | 1.01023     | 1.01012     | 1.01017     | 1.01004     |
| 110 | 4.0 | 1.01057     | 1.01024     | 1.01009     | 1.01004     | 1.00998     | 1.00983     |
| $\infty$ | – | 1.01040 | 1.00975 | 1.00941 | 1.00916 | – |
| 115 | 1.0 | 1.01065     | 1.01039     | 1.01027     | 1.01020     | 1.01003     | 1.00997     |
| 115 | 4.0 | 1.01065     | 1.01032     | 1.01010     | 1.00995     | 1.00993     | 1.00980     |
| $\infty$ | – | 1.01021 | 1.00964 | 1.00917 | 1.00899 | – |

Table 2: Same as Table 1, but the root Hermite factor of self-dual DeepBKZ

| $n$ | $C$ | $\beta = 20$ | $\beta = 25$ | $\beta = 30$ | $\beta = 35$ | $\beta = 40$ | $\beta = 45$ |
|-----|-----|-------------|-------------|-------------|-------------|-------------|-------------|
| 100 | 1.0 | 1.01078     | 1.01065     | 1.01060     | 1.01056     | 1.01051     | 1.01049     |
| 100 | 4.0 | 1.01038     | 1.01035     | 1.01031     | 1.01025     | 1.01024     | 1.01016     |
| 105 | 1.0 | 1.01053     | 1.01053     | 1.01045     | 1.01042     | 1.01036     | 1.01031     |
| 105 | 4.0 | 1.01033     | 1.01016     | 1.01008     | 1.01008     | 1.01007     | 1.01007     |
| 110 | 1.0 | 1.01060     | 1.01048     | 1.01038     | 1.01036     | 1.01036     | 1.01029     |
| 110 | 4.0 | 1.01038     | 1.01033     | 1.01030     | 1.01015     | 1.01009     | 1.01002     |
| 115 | 1.0 | 1.01054     | 1.01036     | 1.01036     | 1.01036     | 1.01022     | 1.01017     |
| 115 | 4.0 | 1.01026     | 1.01021     | 1.01016     | 1.01014     | 1.01013     | 1.01012     |

bases on the SVP challenge [4] in dimensions from $n = 100$ to 115 with seeds 0–9. We took $C = 1.0$ and 4.0 for two different early-abort constants in (2) and (4) (cf., $C = 0.25$, 2.0 and 8.0 were taken in [20]). As in [18, 19], we set $\delta = 0.99$ as reduction parameters of both DeepLLL and its dual variant. We took as input bases reduced by BKZ with blocksize 20, implemented in the fplll library [5]. We increased blocksizes $\beta$ by every 5 from 20 up to 45 for both DeepBKZ and its self-dual variant (it becomes very slow for $\beta \geq 50$ since we did not use any pruning in [8]). We see from Tables 1 and 2 that the root Hermite factor (i.e., the output quality) of DeepBKZ is slightly better than that of self-dual DeepBKZ for $\beta \geq 30$. In Table 1, we also show the average of the root Hermite factor of DeepBKZ without early-abort, which data are from [18, Table 2]. As seen from Table 1, the root Hermite factor of DeepBKZ with early-abort is pretty worse than without early-abort for $\beta \geq 30$.

Performance

In Tables 3 (resp., Table 4), we show the average of the running time of DeepBKZ (resp., self-dual DeepBKZ) for the SVP challenge in dimensions from $n = 100$ to 115 with seeds 0–9. From Tables 3 and 4, our self-dual DeepBKZ is at least 3 times slower than DeepBKZ for $\beta \geq 30$. In particular, DeepBKZ is much faster for $20 \leq \beta \leq 30$ due to that a DeepBKZ-reduced basis can be found by tours less than the terminating condition (2). Moreover, DeepBKZ with early-abort is much faster than without early-abort; For example, it took 5242 seconds = about 1.5 hours to run DeepBKZ with early-abort constant $C = 4.0$ in $n = 115$ for blocksizes up to $\beta = 45$, while a few days are required to run DeepBKZ without early-abort.
Table 3: The average of the total running time (seconds) of DeepBKZ with blocksizes $\beta$ for the SVP challenge in dimensions $n$ with seeds 0–9 (each time was counted from blocksize 20 to given $\beta$)

| $n$  | $C$ | $\beta = 20$ | $\beta = 25$ | $\beta = 30$ | $\beta = 35$ | $\beta = 40$ | $\beta = 45$ |
|------|-----|--------------|--------------|--------------|--------------|--------------|--------------|
| 100  | 1.0 | 9            | 21           | 52           | 112          | 258          | 439          |
|      | 4.0 | 9            | 43           | 138          | 323          | 750          | 1330         |
| 105  | 1.0 | 10           | 34           | 71           | 193          | 433          | 801          |
|      | 4.0 | 10           | 53           | 202          | 510          | 1177         | 2032         |
| 110  | 1.0 | 27           | 55           | 122          | 298          | 637          | 1006         |
|      | 4.0 | 28           | 59           | 273          | 803          | 1985         | 3529         |
| 115  | 1.0 | 76           | 85           | 173          | 424          | 882          | 1470         |
|      | 4.0 | 78           | 91           | 437          | 1329         | 2916         | 5242         |

Table 4: Same as Table 3, but the total running time (seconds) of self-dual DeepBKZ

| $n$  | $C$ | $\beta = 20$ | $\beta = 25$ | $\beta = 30$ | $\beta = 35$ | $\beta = 40$ | $\beta = 45$ |
|------|-----|--------------|--------------|--------------|--------------|--------------|--------------|
| 100  | 1.0 | 231          | 340          | 392          | 484          | 664          | 911          |
|      | 4.0 | 539          | 943          | 1293         | 1559         | 2196         | 3198         |
| 105  | 1.0 | 315          | 410          | 599          | 785          | 1104         | 1714         |
|      | 4.0 | 1075         | 1735         | 2437         | 3012         | 4758         | 7548         |
| 110  | 1.0 | 777          | 1182         | 1460         | 1861         | 2647         | 4109         |
|      | 4.0 | 1151         | 2795         | 4443         | 6043         | 8335         | 12154        |
| 115  | 1.0 | 1254         | 1817         | 2565         | 3295         | 4622         | 6606         |
|      | 4.0 | 3263         | 6886         | 9702         | 12899        | 17325        | 24118        |

4 Concluding remarks

In this section, we first compare reduction algorithms in terms of both the output quality and performance. We then conclude this work and give our future work.

4.1 Comparison of algorithms

Output quality
A prediction of limiting value of the root Hermite factor $\gamma_n^{1/2}$ achieved by BKZ is given in Chen’s thesis [2] (it is based on Gaussian Heuristic, and it seems to hold for $\beta \geq 40$ in practice):

$$\lim_{n \to \infty} \gamma_n^{1/2} = \left( \frac{\beta}{2\pi e} \right)^{\frac{2}{\pi(\beta^{1/2})}}. \quad (5)$$

Actually, as seen from [11, Figure 2], the output quality of both BKZ and self-dual BKZ approximately follows the prediction (5) for $\beta \geq 40$. As discussed in Subsection 3.5, the output quality of our self-dual DeepBKZ is slightly worse than that of DeepBKZ for every $\beta \geq 30$. This is the same as the relation between BKZ and self-dual BKZ, shown in [11]. While the prediction (5) implies that $\beta \geq 85$ is required for BKZ to achieve $\gamma_n^{1/2} = 0.01$, it requires only $\beta \geq 40$ for both DeepBKZ and our self-dual DeepBKZ with early-abort from Tables 1 and 2 (cf., only $\beta \geq 30$ is required for DeepBKZ without early-abort).

Performance
Since DeepLLL and dual DeepLLL are somewhat costly, DeepBKZ and our self-dual DeepBKZ are more costly than BKZ and self-dual BKZ. However, for blocksizes $\beta \geq 30$, SVP and dual SVP oracles (i.e., enumerations) are
dominant in DeepBKZ and our self-dual DeepBKZ. Hence DeepBKZ and its variants have comparable running time to BKZ and its variants for high blocksizes.

4.2 Conclusion and future work

We showed by experiments that DeepLLL [16] and dual DeepBKZ [19] can accelerate to find a short lattice vector in the framework of self-dual BKZ [11] even for small blocksizes. Furthermore, our self-dual DeepBKZ has comparable running time to self-dual BKZ for high blocksizes since the enumeration cost is dominant for $\beta \geq 30$. However, DeepBKZ [18] is more efficient than our self-dual DeepBKZ in both the output quality and performance. Therefore, as a future work, we would like to improve DeepBKZ with pruning as in BKZ 2.0 [3], and analyze its practical output quality for high blocksizes $\beta \geq 50$.

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A Dual enumeration with modifications

In Algorithm 3, we show an algorithm of dual enumeration by \[11, \text{Algorithm 2}\] with modifications for efficiency. The modifications are based on \[8, \text{Appendix B}\] for primal enumeration. The strategy of dual enumeration is as follows: Let \(B = [b_1, \ldots, b_n]\) be a basis of a lattice \(L\), and \(v\) a short vector over the dual lattice \(\hat{L}\). By definition of \(\hat{L}\), we have \(x_i := \langle v, b_i \rangle \in \mathbb{Z}\) for all \(1 \leq i \leq n\), that is, we have \(v = \sum_{i=1}^n x_i d_i \in \hat{L}\) for the dual basis \(D = [d_1, \ldots, d_n]\) of \(B\). Since \(x_i = \langle v, b_i \rangle = \sum_{j=1}^{i-1} \mu_{i,j} \langle v, b_j \rangle\) for each \(1 \leq i \leq n\), we can compute \(x_i^* := \langle v, b_i^\ast \rangle\) from \((x_1, \ldots, x_i) \in \mathbb{Z}^i\) as \(x_1^* = x_1\) and \(x_i^* = x_i + \sum_{j=1}^{i-1} \mu_{i,j} x_j^*\) for \(2 \leq i \leq n\). On the other hand, for a search bound \(A > 0\) with \(||v||^2 \leq A\), we have

\[
\left| x_i + \sum_{j=1}^{i-1} \mu_{i,j} x_j^* \right|^2 \leq ||v||^2 \cdot ||b_i^\ast||^2 \leq A \cdot ||b_i^\ast||^2 \text{ for } 1 \leq i \leq n.
\]

With these equations, we enumerate all coefficient vectors \(x = (x_1, \ldots, x_n)^\top \in \mathbb{Z}^n\) of dual lattice vectors \(v \in \hat{L}\) with \(||v||^2 \leq A\).

**Algorithm 1** DeepBKZ [18] with early-abort

**Input:** A basis \(B = [b_1, \ldots, b_n]\) of a lattice \(L\), a reduction parameter \(1/4 < \delta < 1\), a blocksize \(\beta \in \mathbb{Z}\) with \(2 \leq \beta \leq n\), and the maximum number of tours \(N > 0\)

**Output:** A \((\delta, \beta)\)-DeepBKZ-reduced basis \(B\) of \(L\) (if \(N\) is sufficiently large)

1. \(B \leftarrow \text{DeepLLL}(B, \delta)\) /* Compute \(\mu := \mu_{i,j}^\ast\) and \(B_i^\ast := ||b_i^\ast||^2\) together */
2. \(z \leftarrow 0, j \leftarrow 0, k \leftarrow 0, t \leftarrow 0\) /* \(t\) is the current number of tours */
3. \(\text{while } z < n - 1 \text{ do} \)
4. \(\text{if } j = n - 1 \text{ then} \)
5. \(j \leftarrow 0 \text{ and } t \leftarrow t + 1 \)
6. \(\text{if } t \geq N \text{ then} \)
7. \(\text{break} /* \text{Early-abort} */\)
8. \(\text{end if} \)
9. \(\text{end if} \)
10. \(j \leftarrow j + 1 /* j \mod n - 1 */\)
11. \(k \leftarrow \min(j + \beta - 1, n) \text{ and } h \leftarrow \min(k + 1, n)\)
12. \(v = (v_j, \ldots, v_k) \leftarrow \text{Enum}(\mu_{i,j}^\ast, (B_i^\ast)_{j \leq k}, R)/* \text{Find } v \in \mathbb{Z}^{k-j+1} \text{ such that } \left| \pi_j \left( \sum_{i=1}^k v_i b_i \right) \right| = \lambda_1(L_{[0,k]}) \text{ by enumeration for a search bound } R \text{ (see [8, Algorithm 2], and we took } R = \delta B_j \text{ for our experiments) */}\)
13. \(\text{if } v \neq (\pm 1, 0, \ldots, 0) \text{ then} \)
14. \(z \leftarrow 0\)
15. \(w \leftarrow \sum_{i=1}^k v_i b_i \in L\)
16. \([b_1, \ldots, b_{h-1}, 0] \leftarrow \text{MLLL}([b_1, \ldots, b_{j-1}, w, b_j, \ldots, b_h], \delta) /* \text{Remove the linear dependency after insertion of } w \text{ at the } j\text{-th position */}\)
17. \([b_1, \ldots, b_{h-1}] \leftarrow \text{DeepLLL}([b_1, \ldots, b_h], \delta) \text{ at stage } j\)
18. \(\text{else} \)
19. \(z \leftarrow z + 1\)
20. \([b_1, \ldots, b_{h-1}] \leftarrow \text{DeepLLL}([b_1, \ldots, b_h], \delta) \text{ at stage } h - 1\)
21. \(\text{end if} \)
22. \(\text{end while} \)
Algorithm 2 Self-dual DeepBKZ

Input: A basis $\mathbf{B} = [\mathbf{b}_1, \ldots, \mathbf{b}_n]$ of a lattice $L$, a reduction parameter $\frac{1}{4} < \delta < 1$, a blocksize $3 \leq \beta \leq n - 1$, and the maximum number of tours $M > 0$

Output: A reduced basis $\mathbf{B}$ of $L$

1: $t \leftarrow 0$
2: Compute the GSO information $\mu_{i,j}$ and $B_i = \| \mathbf{b}_i^* \|^2$
3: while $t \leq M$ do
4: /* Part of a forward tour */
5: $[\mathbf{b}_1, \ldots, \mathbf{b}_{n-1}] \leftarrow \text{DeepLLL}([\mathbf{b}_1, \ldots, \mathbf{b}_{n-1}], \delta)$
6: for $j = 1$ to $n - \beta$ do
7: $k \leftarrow j + \beta - 1$
8: $h \leftarrow \min(k + 1, n - 1)$
9: Same as steps from 8 to 21 in Algorithm 1 for DeepBKZ
10: end for
11: /* Part of a backward tour */
12: $\mathbf{B} \leftarrow \text{Dual_DeepLLL}(\mathbf{B}, \delta)$ at stages from $n - 1$ downto 2
13: for $j = n$ downto $\beta + 1$ do
14: $k \leftarrow j - \beta + 1$
15: $\mathbf{x} \leftarrow \text{Dual_Enum}(\mu_{k,j}, (B_j)_{k \leq i \leq j}, A)$ /* Enumerate coefficient vectors $\mathbf{x} \in \mathbb{Z}^{j-k+1}$ of short vectors $\mathbf{v} \in \hat{L}$ (Appendix A) */
16: if $\mathbf{x} \neq (0, \ldots, 0, \pm 1)$ then
17: Insert $\mathbf{v} \in \hat{L}$ into the dual basis of $\mathbf{B}$ at the $j$-th position to obtain a new basis $\mathbf{B}$ /* it can be achieved by LLL, see [11, Section 7] */
18: $\mathbf{B} \leftarrow \text{Dual_DeepLLL}(\mathbf{B}, \delta)$ at stages from $n - 1$ downto 2
19: end if
20: end for
21: $t \leftarrow t + 1$ /* current number of tours */
22: end while
Algorithm 3 Dual\textsubscript{ Enum}: Dual-Enumeration [11] with modifications

Input: GSO information $(\mu_{i,j})$ and $B_i = \|b_i\|^2$ of a basis $B = [b_1, \ldots, b_n]$ of a lattice $L$, and a search bound $A > 0$

Output: The coordinate vector $x = (x_1, \ldots, x_n)^\top \in \mathbb{Z}^n$ of a dual lattice vector $v \in \hat{L}$ satisfying $\|v\|^2 \leq A$ (if such $v$ exists)

1: \textbf{for} $k = 1$ to $n$ \textbf{do}
2: \hspace{1em} $C_k = 1/B_k$ /* inverse of $B_k$ */
3: \hspace{1em} $\tilde{\mu}_{k,k} = 1$; \textbf{for} $j = k + 1$ to $n$: $\tilde{\mu}_{k,j} \leftarrow -\sum_{h=k}^{j-1} \mu_{j,h} \tilde{\mu}_{k,h}$
4: \textbf{end for}
5: \hspace{1em} $\sigma \leftarrow (0)_{(n+1) \times n}$ /* $(n+1) \times n$ matrix with all entries $0$ */
6: \hspace{1em} $r_1 = 0, r_2 = 1, \ldots, r_n = n - 1, r_{n+1} = n$
7: \hspace{1em} $\rho_0 = \rho_1 = \cdots = \rho_n = 0$ /* partial norm */
8: \hspace{1em} $x_1 = 1, x_2 = \cdots = x_n = 0$ /* current coordinate vector $x$ */
9: \hspace{1em} $c_1 = \cdots = c_n = 0; w_1 = \cdots = w_n = 0$ /* centers and jumps */
10: \hspace{1em} $k \leftarrow 1, \text{last\_nonzero} \leftarrow 1$ /* largest $i$ for which $x_i \neq 0$ */
11: \textbf{while} true \textbf{do}
12: \hspace{1em} $\rho_k \leftarrow \rho_{k-1} + (x_k - c_k)^2 C_k$
13: \hspace{1em} \textbf{if} $\rho_k \leq A$ \textbf{then}
14: \hspace{2em} \textbf{return} $x = (x_1, \ldots, x_n)^\top \in \mathbb{Z}^n$ /* solution found; program ends */
15: \hspace{1em} \textbf{else}
16: \hspace{2em} $k \leftarrow k + 1; r_{k+1} \leftarrow \min(r_k, r_{k+1})$
17: \hspace{2em} \textbf{for} $i = r_k$ to $k - 1$ \textbf{do}
18: \hspace{3em} $\sigma_{i+2,k} \leftarrow \sigma_{i+1,k} + x_k \tilde{\mu}_{i,k}$
19: \hspace{2em} \textbf{end for}
20: \hspace{2em} $c_k \leftarrow -\sigma_{k+1,k}; x_k \leftarrow \lfloor c_k \rfloor; w_k \leftarrow 1$
21: \hspace{1em} \textbf{end if}
22: \hspace{1em} \textbf{else}
23: \hspace{2em} $k \leftarrow k - 1$
24: \hspace{2em} \textbf{if} $k = 0$ \textbf{then} \textbf{return} $\emptyset$ /* there is no solution */
25: \hspace{2em} $r_{k+1} \leftarrow k$
26: \hspace{2em} \textbf{if} $k \geq \text{last\_nonzero}$ \textbf{then}
27: \hspace{3em} $\text{last\_nonzero} \leftarrow k; x_k \leftarrow x_k + 1$
28: \hspace{2em} \textbf{else}
29: \hspace{4em} \textbf{if} $x_k > c_k$ \textbf{then} $x_k \leftarrow x_k - w_k$; \textbf{else} $x_k \leftarrow x_k + w_k$
30: \hspace{4em} $w_k \leftarrow w_k + 1$
31: \hspace{2em} \textbf{end if}
32: \hspace{2em} \textbf{end if}
33: \hspace{1em} \textbf{end if}
34: \textbf{end while}