Cardy-Verlinde formula in Taub-NUT/Bolt-(A)dS space

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(Dated: November 14, 2008)

We consider a finite action for a higher dimensional Taub-NUT/Bolt-(A)dS space via the so-called counter term subtraction method. In the limit of high temperature, we show that the Cardy-Verlinde formula holds for the Taub-Bolt-AdS metric and for the specific dimensional Taub-NUT-(A)dS metric, except for the Taub-Bolt-dS metric.

PACS numbers: 11.25.Hf, 11.25.Tq

I. INTRODUCTION

The AdS/CFT duality was first conjectured by [1] in his search for relationship between gauge theories and strings. The AdS/CFT correspondence [2, 3, 4, 5, 6] asserts there is an equivalence between a gravitational theory in the bulk and a conformal field theory in the boundary. According to AdS/CFT, a (d+1)-dimensional S-(A)dS action \( A \) is given by

\[
A = A_B + A_{\partial B} + A_{ct},
\]

where the bulk action \( A_B \), action boundary \( A_{\partial B} \), and counterterm action \( A_{ct} \) are given as

\[
A_B = \frac{1}{16\pi G_{d+1}} \int_M d^{d+1}x \sqrt{-g} (R - 2\Lambda),
\]

\[
A_{\partial B} = -\frac{1}{8\pi G_{d+1}} \int_{\partial M} d^d x \sqrt{-\gamma} \Theta,
\]

\[
A_{ct} = -\frac{1}{8\pi G_{d+1}} \int_{\partial M} d^d x \sqrt{-\gamma} \left\{ -\frac{d-1}{l} - \frac{l R}{2(d-2)} F(d-3) \right. \\
- \left. \frac{2(d-2)^2(d-4)}{l^3} \right. \\
\times \left( \frac{3d+2}{4(d-1)} RR_{ab} R^{ab} \right. \\
\left. \frac{d(d+2)}{10(d-1)^2} R^3 \right. \\
+ \frac{d^2}{4(d-1)} \left( \frac{d^2}{2} \right) R^{ab} \nabla_a R_{ab} - R^{ab} \Box R_{ab} \right. \\
\left. + \frac{1}{2(d-1)} R^{ab} R_{ab} \right\},
\]

where a negative cosmological constant \( \Lambda \) is \( \Lambda = -d(d-1)/2l^2 \), \( \Theta \) is the trace of extrinsic curvature. Here, \( F(d) \) is a step function, 1 when \( d \geq 0 \), 0 otherwise. The boundary action \( A_{\partial B} \) is added to the action \( A \) to obtain equations of motion well behaved at the boundary. Then the boundary energy-momentum tensor is expressed in

\[
\frac{2}{\sqrt{-\gamma}} \frac{\partial}{\partial \gamma} \frac{\partial A_{\partial B}}{\partial \gamma} = \Theta_{ab} - \gamma_{ab} \Theta. \tag{3}
\]

The counterterm action \( A_{ct} \) is added to the action \( A \) to remove the divergence appearing as the boundary goes to infinity [3]. For low dimensional S-AdS, a few terms in the counterterm action \( A_{ct} \) were explicitly evaluated in [3, 4]. Using the universality of the structure of divergences, the counterterm action \( A_{ct} \) for arbitrary dimension is suggested in [10]. This action \( A_{ct} \) leads to the entropy \( S \) via the Gibbs-Duhem relation

\[
S = \frac{E}{T} - A \tag{4}
\]

where \( T \) denotes the temperature and \( E \) is the total energy.

The entropy of the (1+1)-dimensional CFT is expressed in terms of the Virasoro operator \( L_0 \) and the central charge \( c \), the so-called the Cardy formula [11]. Using conformal invariance, the generalized Cardy formula in arbitrary dimension is shown to be given universal form as [12] (for the review articles of the issue, see, e.g., [13, 14, 15])

\[
S_{\text{CFT}} = \frac{2\pi R}{\sqrt{ab}} \sqrt{E_c(2E - E_c)}, \tag{5}
\]

where \( a \) and \( b \) are certain constants. \( R \) denotes the radius of the universe at a given time and \( E_c \) is the Casimir energy defined by

\[
E_c = dE - (d - 1)TS. \tag{6}
\]

Employing AdS/CFT dual picture, \( \sqrt{ab} \) is fixed to \( (d-1) \) exactly, in particular, for a \( d \)-dimensional CFT on \( R \times S^{d-1} \) [12]. Then, the entropy is given as

\[
S_{\text{CFT}} = \frac{2\pi R}{d-1} \sqrt{E_c(2E - E_c)}, \tag{7}
\]

which is shown to hold for Schwarzschild (A)dS (S-(A)dS) [12, 16], charged (A)dS [17, 18], Kerr-(A)dS [18, 19], and Taub-Bolt-AdS [20]. There are many other relevant papers on the subject [21, 22, 23, 24, 25]. Thus,
II. TAUB-NUT/BOLT-ADS BLACK HOLE

When the total number of dimension of the spacetime is even, \((d+1) = 2u + 2\), for some integer \(u\), the Euclidean section of the arbitrary \((d+1)\)-dimensional-Taub-NUT-Ads metric, for a \(U(1)\) fibration over a series of the space \(\mathcal{M}^2\) as the base space \(\bigotimes_{i=1}^{u} \mathcal{M}^2\), is given by [27, 28, 29, 30, 31, 32, 33] (for the generalized versions of the issue, see, e.g., [34, 35]).

One may naively expect that the CFT entropy is given as the form [17, 26]. Therefore, one intriguing question is whether this formula is valid for higher dimensional Taub-NUT-(A)dS at high temperature. In this Letter, we will endeavor to do this.

The metric function \(f(r)\) has the general form

\[
d s^2_{AdS} = f(r) \left[ d\tau + 2N \sum_{i=1}^{u} \cos(\theta_i) d\phi_i \right]^2 + \frac{dr^2}{f(r)} + (r^2 - N^2) \sum_{i=1}^{u} \left[ d\theta_i^2 + \sin^2(\theta_i) d\phi_i^2 \right],
\]

where \(N\) represents a NUT charge for the Euclidean section, and \(f(r)\) is taken to be

\[
f(r) = \frac{r}{(r^2 - N^2)^{u}} \int_{r}^{(2u + 2)(p^2 - N^2)^{u+1}} dp - \frac{2m r}{(r^2 - N^2)^{u}}.
\]

with a cosmological parameter \(l\) and a geometric mass \(m\).

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f(r) = \frac{r}{(r^2 - N^2)^{u}} \int_{r}^{(2u + 1)(p^2 - N^2)^{u+1}} dp - \frac{2m r}{(r^2 - N^2)^{u}}.
\]

Employing the thermal relation \(E = \partial_\beta I\) the total energy can also be written by

\[
E = \frac{(4\pi)^{\frac{d}{2}}(d-1)N^{d-2}(d+1)N^2 - l^2)}{32\pi^2 q l^2} \times \Gamma \left( \frac{2-d}{2} \right) \Gamma \left( \frac{d+1}{2} \right) \beta.
\]

and the entropy is given as

\[
S_{NUT,AdS} = \frac{(4\pi)^{\frac{d}{2}}N^{d-2}(d(d-1)N^2 - (d-2)l^2)}{32\pi^2 q l^2} \times \Gamma \left( \frac{2-d}{2} \right) \Gamma \left( \frac{d+1}{2} \right) \beta,
\]

by the Gibbs-Duhem relation \(S = \beta M - I\) where \(M\) denotes the conserved mass

\[
M = \frac{(d-1)(4\pi)^{\frac{d}{2}}}{16\pi^2} m.
\]

Substituting [10], [12], and [13] into (9), one gets the Casimir energy [12]

\[
E_{c} = \frac{(4\pi)^{\frac{d}{2}}(d-1)N^{d-2}(dN^2 - l^2)}{16\pi^2 q l^2} \times \Gamma \left( \frac{2-d}{2} \right) \Gamma \left( \frac{d+1}{2} \right) \beta.
\]

From now on, for convenience we use \(l/z\) instead of the universe radius \(R\) in [7] since the AdS metric is always asymptotically taken to be [36]

\[
d s^2 = \frac{l^2}{z^2} d z^2 + \frac{l^2}{z^2} g_{ab}(x, z) d x^a d x^b,
\]

where the \(r = \infty\) put to \(z = 0\), and the roman indexes \(a\) and \(b\) refer to boundary coordinates. When \(1/\sqrt{\beta a^2}\) in the formula [7] is taken to be \(2/(d+1)(d-1)(d-2)\), the CFT entropy is given as

\[
S_{CFT} = \frac{4\pi l}{(d+1)(d-1)(d-2)} \frac{|E_{c}|}{|E_{c} - E_{c}|} = \frac{(4\pi)^{\frac{d}{2}}|dN^2 - l^2|(-1)^{\frac{d}{2}}\Gamma \left( \frac{d+1}{2} \right) \Gamma \left( \frac{2-d}{2} \right)}{4\pi \frac{(d+1)(d-2)q}{\beta}}.
\]

where \([x]\) is the Gauss number (greatest integer less than or equal to \(x\)). Here it seems that the difference from the standard Cardy-Verlinde formula [7] is due to the distinctive nature of NUT solution in AdS space like asymptotically locally AdS (ALAdS) metric. In the limit of high temperature, \(N \to 0\), leading term in the entropy of CFT can be expressed as

\[
S_{CFT} = \frac{(4\pi)^{\frac{d}{2}}(d+1)(d-2)N^{d-1}}{16\pi^2 q} \times (-1)^{\frac{d}{2}} \Gamma \left( \frac{d+1}{2} \right) \Gamma \left( \frac{2-d}{2} \right) = (-1)^{\frac{d}{2}} S_{NUT,AdS}.
\]
This result shows that the entropy of the Taub-NUT-AdS space suffices to be the generalized Cardy-Verlinde formula \[ S_{\text{Bolt,AdS}} = \frac{(4\pi)^u}{16\pi l^2} \left[ \frac{(2u-1)(2u+1)}{r_B^2} + \sum_{k=0}^{u} \left( \begin{array}{c} u \\ k \end{array} \right) (-1)^k N^{2k} r_B^{2u-2k} \right] \]

Using parallel way as in the previous case, the inverse of the temperature, the total energy, the entropy, and the Casimir energy are obtained

\[
\beta = \frac{4\pi}{g'(r)} \bigg|_{r = \infty} = \frac{2(d+1)\pi |N|}{q},
\]

where \( g(r) \) is given as

\[
g(r) = \frac{r}{(r^2 - N^2)^u} \int [\frac{(p^2 - N^2)^u}{p^2} + (2u + 1)(p^2 - N^2)^{u+1}] dp + \frac{2mr}{r^2 - N^2} u
\]

Using parallel way as in the previous case, the inverse of the temperature, the total energy, the entropy, and the Casimir energy are obtained

\[
\beta = \frac{4\pi}{g'(r)} \bigg|_{r = \infty} = \frac{2(d+1)\pi |N|}{q},
\]

where \( r_B = q \sqrt{\frac{q^2 l^4 + (2u+1)(2u+2)N^2 (2u+1)(2u+2)l^2}{(2u+1)(2u+2)N}} \). The CFT entropy is written as

\[
S_{\text{CFT}} = \frac{2\pi l \sqrt{E_c(2E - E_c)}}{2u \sqrt{2u - 1}},
\]

where \( 1/\sqrt{ab} \) is fixed to \( 1/(2u \sqrt{2u - 1}) \). In the high temperature limit, the CFT entropy well suffices to be Cardy-Verlinde formula as the following

\[
S_{\text{CFT}} = \frac{(4\pi)^u}{4N^{2u}} \left( \frac{q l^2}{2u^2 + 3u + 1} \right)^{2u} = S_{\text{Bolt,AdS}}.
\]

Note that the higher dimensional Taub-Bolt-AdS space follows the generalized Cardy-Verlinde formula even if Taub-Bolt-AdSdS space \( u = 1 \) exactly satisfies the Cardy-Verlinde formula.

### III. TAUB-NUT/BOOLT-DS BLACK HOLE

Taub-NUT-ds metric is obtained from the Taub-NUT-AdS metric by replacing \( l^2 \to -l^2 \), and one has \[ S_{\text{CFT}} = \frac{(4\pi)^u}{4N^{2u}} \left( \frac{q l^2}{2u^2 + 3u + 1} \right)^{2u} = S_{\text{Bolt,AdS}}. \]
For the Bolt solution in dS space, the inverse of the temperature, the total energy, and the entropy are respectively

\[
\beta = \frac{4\pi}{f'(r)} \bigg|_{r=r_B} = -\frac{4\pi^2 r_B}{f'(r)} \bigg|_{r=r_B}
\]

\[
E = -\frac{(4\pi)^u u}{8\pi} \left( \sum_{k=0}^{u} \frac{u}{k} \left( \frac{1}{k} \right) N^{2k} r_B^{2u-2k-1} \right)
\]

\[
S_{\text{Bolt,dS}} = \frac{(4\pi)^u \beta}{16\pi^2} \left[ -\frac{(2u-1)(2u+1)(-1)^u N^{2u+2}}{r_B} \right.
\]

\[
+ \sum_{k=0}^{u} \frac{u}{k} \left( \frac{1}{k} \right) N^{2k} r_B^{2u-2k-1} \times \left( -\frac{(2u-1)^2}{(2u-2k-1)} r_B \right)
\]

\[
+ \frac{(2u+1)(2u+3u-2k+1)}{(2u-2k+1)(u+k+1)} \right]
\]

(33)

where \( r_B = \sqrt{4l^2 + \sqrt{4l^2 + (2u+1)^2}} N^{2u+2} \). The CFT entropy is given as

\[
S_{\text{CFT}} = \frac{(4\pi)^u}{4N^{2u}} \left( \frac{q l^2}{2u^2 + 3u + 1} \right)^{2u} = -S_{\text{Bolt,dS}},
\]

(35)

where \( 1/\sqrt{ab} \) is fixed to \( 1/2u\sqrt{2u-1} \). As the NUT charge goes to 0, the CFT entropy becomes

\[
S_{\text{CFT}} = \frac{(4\pi)^u}{4N^{2u}} \left( \frac{q l^2}{2u^2 + 3u + 1} \right)^{2u} = -S_{\text{Bolt,dS}},
\]

(36)

which shows that no entropy of the Taub-Bolt-dS metric satisfies the Cardy-Verlinde formula. This means that any Bolt solution in dS space is thermodynamically unstable at high temperature limit.

IV. CONCLUSION

We have considered that the Taub-NUT/Bolt-(A)dS metric in general even dimension, and have checked that its metric suffices to be the the Cardy-Verlinde formula. In the limit of high temperature, we showed that the Taub-Bolt-AdS space well follows the generalized Cardy-Verlinde formula (7) rather than the Cardy-Verlinde formula (5). It seems that the modification of the standard Cardy-Verlinde formula (5) is due to the distinctive property of the Taub-NUT solution such as the ALAdS metric. It was proven that the leading term of the CFT entropy at the boundary for all even \( u \) is exactly matched with that of the entropy in the Taub-NUT-(A)dS space by using the generalized Cardy-Verlinde formula at high temperature. Thermal stability of the Taub-NUT-(A)dS solution for all odd \( u \) and Taub-Bolt-dS solution for all \( u \) is determined by the magnitude of the NUT charge so that the negative entropy occurs as the NUT charge goes to 0. Finally, the the breaking of Cardy-Verlinde formula in the Taub-Bolt-dS metric seems to reflect the fact that there is no Bolt solution in dS space due to the absence of hyperbolic NUT in AdS space (30).

Acknowledgements

We are grateful to Cristian Stela for useful comments. This work was supported by the BK 21 project of the Ministry of Education and Human Resources Development, Korea (C.O.L.).

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