Naturality and definability II

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In two papers [3] and [4] we noted that in common practice many algebraic constructions are defined only ‘up to isomorphism’ rather than explicitly. We mentioned some questions raised by this fact, and we gave some partial answers. The present paper provides much fuller answers, though some questions remain open. Our main result, Theorem 4, says that there is a transitive model of Zermelo-Fraenkel set theory with choice (ZFC) in which every explicitly definable construction is ‘weakly natural’ (a weakening of the notion of a natural transformation). A corollary is that there are models of ZFC in which some well-known constructions, such as algebraic closure of fields, are not explicitly definable. We also show (Theorem 2) that there is no transitive model of ZFC in which the explicitly definable constructions are precisely the natural ones.

Most of this work was done when the second author visited the first at Queen Mary, London University under SERC Visiting Fellowship grant GR/E9/639 in summer 1989, and later when the two authors took part in the Mathematical Logic year at the Mittag-Leffler Institute in Djursholm in September 2000. The second author proposed the approach of section 3 on the first occasion and the idea behind the proof of Theorem 4 on the second. Between 1975 and 2000 the authors (separately or together) had given some six or seven false proofs of versions of Theorem 4 or its negation.

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# Constructions up to isomorphism

To make this paper self-contained, we repeat or paraphrase some definitions from [4].

Let $M$ be a transitive model of ZFC (Zermelo-Fraenkel set theory with choice). By a *construction* (in $M$) we mean a triple $C = \langle \phi_1, \phi_2, \phi_3 \rangle$ where

1. $\phi_1(x)$, $\phi_2(x)$ and $\phi_3(x)$ are formulas of the language of set theory, possibly with parameters from $M$;
2. $\phi_1$ and $\phi_2$ respectively define first-order languages $L$ and $L^-$ in $M$; every symbol of $L^-$ is a symbol of $L$, and the symbols of $L \setminus L^-$ include a 1-ary relation symbol $P$;
3. the class $\{ a : M \models \phi_3(a) \}$ is in $M$ a class of $L$-structures, called the *graph* of $C$;
4. if $B$ is in the graph of $C$ then $P_B$, the set of elements of $B$ satisfying $P_x$, forms the domain of an $L^-$-structure $B^-$ inside $B$; the class of all such $B^-$ as $A$ ranges over the graph of $C$ is called the *domain* of $C$;
5. the domain of $C$ is closed under isomorphism; and if $A, B$ are in the graph of $C$ then every isomorphism from $A^-$ onto $B^-$ extends to an isomorphism from $A$ onto $B$.

A typical example is the construction whose domain is the class of fields, and the structures $B$ in the graph are the algebraic closures of $B^-$, with $B^-$ picked out by the relation symbol $P$. The algebraic closure of a field is determined only up to isomorphism over the field; in the terminology below, algebraic closures are ‘representable’ but not ‘uniformisable’. (What we called definable in [4] we now call uniformalisable; the new term is longer, but it is less misleading because it agrees better with the common mathematical use of these words.)

We say that the construction $C$ is $X$-representable (in $M$) if $X$ is a set in $M$ and all the parameters of $\phi_1, \phi_2, \phi_3$ lie in $X$. We say that $C$ is small if the domain of $C$ (and hence also its graph) contains only a set of isomorphism types of structures.

An important special case is where the domain of $C$ contains exactly one isomorphism type of structure; in this case we say $C$ is unitype.
The map $B^\rightarrow \to B$ on the domain of a construction $C$ is in general not single-valued; but by clause (5) it is single-valued up to isomorphism over $B^\rightarrow$. We shall say that $C$ is uniformisable if its graph can be uniformised, i.e. there is a formula $\phi_4(x,y)$ of set theory (the uniformising formula) such that

for each $A$ in the domain of $C$ there is a unique $B$ such that $M \models \phi_4(A,B)$, and this $B$ is an $L$-structure in the graph of $C$ with $A = B^\rightarrow$.

We say that $C$ is $X$-uniformisable (in $M$) if there is such a $\phi_4$ whose parameters lie in the set $X$.

## 2 Splitting, naturality and weak naturality

Let $\nu : G \to H$ be a surjective group homomorphism. A splitting of $\nu$ is a group homomorphism $s : H \to G$ such that $\nu s$ is the identity on $H$. We say that $\nu$ splits if it has a splitting.

For our Theorem 4 we shall need a weakening of these notions. A stronger version of Theorem 4 would make this unnecessary, but we do not know whether the stronger version is true.

Let $\nu : G \to H$ be as above. By a weak splitting of $\nu$ we mean a map $s : H \to G$ such that

(a) $\nu s$ is the identity on $H$;

(b) there is a commutative subgroup $G_0$ of $G$ such that if $f_1, \ldots, f_k$ are elements of $H$ for which $f_1^{\varepsilon_1} \cdots f_k^{\varepsilon_k} = 1$ (where $\varepsilon_i$ is each either 1 or $-1$), then $s(f_1)^{\varepsilon_1} \cdots s(f_k)^{\varepsilon_k} \in G_0$.

If we strengthened this definition by requiring $G_0$ to be $\{1\}$, it would say exactly that $s$ is a splitting of $\nu$. In particular every splitting is a weak splitting. We say that $\nu$ weakly splits if it has a weak splitting.

Suppose $s$ is a weak splitting of $\nu$. Then there is a smallest group $G_0$ as in (b); it is the group consisting of the words $s(f_1)^{\varepsilon_1} \cdots s(f_k)^{\varepsilon_k}$ as in (b). This group $G_0$ has the property that if $g$ is in $G_0$ and $f$ is in $H$ then $s(f)^{-1}gs(f)$ is also in $G_0$. So the normaliser of $G_0$ in $G$ contains the image of $s$.

**Example 1.** Let $G$ be the multiplicative group of $3 \times 3$ upper unitriangular matrices over the ring $\mathbb{Z}/(8\mathbb{Z})$. Let $H$ be the corresponding group over
\( \mathbb{Z}/(2\mathbb{Z}) \), and let \( \nu : G \rightarrow H \) be the canonical surjection. We show that \( \nu \) doesn’t weakly split.

Suppose for contradiction that \( s \) is a weak splitting of \( \nu \). Let \( g_1, g_2 \) be the two matrices

\[
g_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}
\]

in \( G \), and write \( f_1 = \nu(g_1), f_2 = \nu(g_2) \). Now \( f_1^2 = f_2^2 = 1 \) in \( H \), so the weak splitting property tells us that \( s(f_1)^2 \) and \( s(f_2)^2 \) commute in \( G \). But it is easily checked (using the fact that all entries of \( s(f_i) - f_i \) are divisible by \( 2 \)) that \( s(f_1)^2 \) and \( s(f_2)^2 \) don’t commute.

**Example 2.** Let \( m \) and \( n \) be positive integers with \( n \geq 3 \), and let \( p \) be a prime with \( p^m > 3 \). Let \( G \) (resp. \( H \)) be the multiplicative group of invertible \( n \times n \) matrices over the ring \( \mathbb{Z}/(p^m\mathbb{Z}) \) (resp. \( \mathbb{Z}/(p^m\mathbb{Z}) \)), and let \( \nu : G \rightarrow H \) be the canonical surjection. We write \( I \) for the identity element in \( G \) and in \( H \). The kernel of \( \nu \) is the group of matrices of the form \( I + p^m f \) where \( f \) is in \( G \). For any \( i, j \) with \( 1 \leq i < j \leq n \) let \( \delta_{ij} \) be the \( n \times n \) matrix which has 1 in the \( ij \)-th place and 0 elsewhere; then \( I + \delta_{ij} \) is an element of \( G \) and \( \nu(I + \delta_{ij}) \) has order \( p^m \). The liftings of \( \nu(I + \delta_{ij}) \) to \( G \) are the matrices of the form \( I + \delta_{ij} + p^m f \) with \( f \) in \( G \). Now we repeat a calculation from Evans, Hodges and Hodkinson [4] Prop. 3.7. The element \( (I + \delta_{ij} + p^m f)^{p^m} \) is

\[
I + \left( \frac{p^m}{1}\right)(\delta_{ij} + p^m f) + \left( \frac{p^m}{2}\right)(\delta_{ij} + p^m f)^2 + \left( \frac{p^m}{3}\right)(\delta_{ij} + p^m f)^3 + \ldots
\]

Since \( \delta_{ij}\delta_{ij} = 0 \), \( p^{3m}x = 0 \) in \( \mathbb{Z}/(p^{3m}\mathbb{Z}) \) and \( p^m > 3 \), this multiplies out to

\[
I + p^m \delta_{ij} + p^{2m} f + \frac{p^{2m}(p^m - 1)}{2}(\delta_{ij} f + f\delta_{ij}) + \frac{p^{2m}(p^m - 1)(p^m - 2)}{6} \delta_{ij} f \delta_{ij}.
\]

To apply these calculations to a concrete example, take

\[
g_1 = I + \delta_{12}, \quad g_2 = I + \delta_{23}
\]

in \( H \), and let \( s \) be a weak splitting of \( \nu \). Then

\[
s(g_1) = I + \delta_{12} + p^m f_1, \quad s(g_2) = I + \delta_{23} + p^m f_2
\]
for some $f_1, f_2$ in $G$. Since $s$ is a weak splitting,

$$s(g_1)^p s(g_2)^p = s(g_2)^p s(g_1)^p.$$  

But our calculations show at once that

$$s(g_1)^p s(g_2)^p - s(g_2)^p s(g_1)^p = p^{2m} \delta_{13} \neq 0.$$  

This contradiction proves that $\nu$ doesn’t weakly split.

Now suppose $C$ is a construction in the model $M$, and $B$ is a structure in the graph of $C$. Let $A$ be $B^-$. We write $\text{Aut}(A)$ and $\text{Aut}(B)$ for the automorphism groups of $A$ and $B$ respectively. By (4) in the definition of constructions, each automorphism $g$ of $B$ restricts to an automorphism $\nu_B(g)$ of $A$. This map $\nu_B : \text{Aut}(B) \rightarrow \text{Aut}(A)$ is clearly a homomorphism; by (5) in the definition of constructions it is surjective.

We say that $B$ is (weakly) natural over $A$ if the map $\nu_B$ (weakly) splits. We say that the construction $C$ is (weakly) natural if for every $B$ in the graph of $C$, $\nu_B$ (weakly) splits. (Our paper [4] explained how this terminology connects with the notion of a natural transformation. In a related context Harvey Friedman [2] used the term ‘naturalness’ in a weaker sense.)

**Example 3.** Let $G$ and $H$ be as in Example 1. Since $n \times n$ upper triangular matrix groups are nilpotent of class $n - 1$, $G$ is a finite soluble group. So by Shafarevich [3] there is a Galois extension $K$ of the field $\mathbb{Q}$ of rationals such that $G$ is the Galois group of $K/\mathbb{Q}$. Let $k$ be the fixed field of the kernel $G_0$ of $\nu : G \rightarrow H$. Then $H$ is the Galois group of the extension $k/\mathbb{Q}$. One can write a set-theoretic description of these fields—up to isomorphism—as a construction $C$ where $K$ is in the graph and $k$ is picked out within $K$ by the relation symbol $P$. This construction $C$ is small (in fact unitype) and $\emptyset$-representable in any model of set theory, and it is not weakly natural.

**Example 4.** Let $G$ and $H$ be as in Example 2. Let $B$ (resp. $A$) be the direct sum of $n$ copies of the abelian group $\mathbb{Z}/(p^{3m} \mathbb{Z})$ (resp. $\mathbb{Z}/(p^m \mathbb{Z})$), and identify $A$ with $p^{2m}B$. Let the relation symbol $P$ pick out $A$ within $B$. Then $G$ (resp. $H$) is the automorphism group of $B$ (resp. $A$), and $\nu : G \rightarrow H$ is the map induced by restriction. By the result of Example 2, the construction of $B$ over $A$, which is again unitype and $\emptyset$-representable in any model of set theory, is not weakly natural.
In [4] we conjectured that there are models of set theory in which each representable construction is uniformisable if and only if it is natural. See section 7 below for some of the background to this. Section 3 will show that no reasonable version of this conjecture is true. Sections 4–6 will show that there are models in which uniformisability implies weak naturality. Section 7 solves some of the problems raised in [3] and [4].

3 Uniformisability

A structure \( B \) is said to be \emph{rigid} if it has no nontrivial automorphisms. We shall say that a construction \( C \) is \emph{rigid-based} if for every structure \( B \) in the graph of \( C \), \( B^{-} \) has no nontrivial automorphisms. A rigid-based construction is trivially natural.

Let \( M \) be a transitive model of set theory. We shall use a device that takes any construction \( C \) in \( M \) to a construction \( C^{r} \), called its \emph{rigidification}. Each structure \( B^{-} \) in the domain is replaced by a two-part structure \( B^{r-} \), where the first part is \( B^{-} \) and the second part consists of the transitive closure of the set \( P^{B} \) with a membership relation \( \varepsilon \) copying that in \( M \). Now \( B^{r} \) is defined to be the amalgam of \( B \) and \( B^{r-} \), so that \( B^{r-} \) is \((B^{r})^{-}\). Then \( C^{r} \) is the closure of the class

\[ \{ B^{r} : B \text{ in the graph of } C \} \]

under isomorphism in \( M \). It is clear that \( C^{r} \) and the map \( B \mapsto B^{r} \) are definable in \( M \) using no parameters beyond those in the formulas representing \( C \).

**Lemma 1** If \( C \) is any construction, then \( C^{r} \) is rigid-based, natural and not small.

**Proof.** If \( B^{-} \) is in the domain of \( C \), then \( B^{r-} \) is rigid because its set of elements is transitively closed; so \( C^{r} \) is rigid-based. Naturality follows at once. Since the domain of \( C \) is closed under isomorphism, the relevant transitive closures are arbitrarily large. \( \square \)

**Theorem 2** There is no transitive model of ZFC in which both the following are true:

(a) Every rigid-based construction in \( M \) is uniformisable.
(b) Every unitype uniformisable construction in $M$ is weakly natural.

In particular there is no transitive model of ZFC in which the natural constructions are exactly the uniformisable ones.

**Proof.** Suppose $M$ is a counterexample. Let $C$ be a unitype non-weakly-natural construction in $M$, such as Example 2 in section 2. Then $C^r$ is rigid-based and hence uniformisable by assumption.

But we can use the uniformising formula of $C^r$ to uniformise $C$ with the same parameters. So by the assumption on $M$ again, $C$ is weakly natural; contradiction. $\square$

The next result gives some finer information about small constructions.

**Theorem 3** Let $M$ be a transitive model of ZFC, $Y$ a set in $M$ and $\bar{c}$ a well-ordering of $Y$ in $M$. Assume:

In $M$, if $X$ is any set, then every unitype $X$-representable rigid-based construction is $X \cup Y$-uniformisable.

Then

In $M$, every small $\emptyset$-representable construction is $\{\bar{c}\}$-uniformisable, and hence there are unitype $\{\bar{c}\}$-uniformisable constructions that are not weakly natural.

**Proof.** Let $\gamma$ be the length of $\bar{c}$. Write $\bar{v}$ for the sequence of variables $(v_i : i < \gamma)$. In $M$ we can well-order (definably, with no parameters) the class of pairs $\langle j, \psi \rangle$ where $j$ is an ordinal and $\psi(x,y,z,\bar{v})$ is a formula of set theory. We write $H_j$ for the set of sets hereditarily of cardinality less than $\aleph_j$ in $M$.

Let $C$ be a small $\emptyset$-representable construction in $M$. Then $C^r$ is an $\emptyset$-representable rigid-based construction. It is not small; but if $B$ is any structure in the graph of $C$, let $C_B$ be the construction got from $C^r$ by restricting the graph to structures isomorphic to $B^r$. Then $C_B$ is a unitype and $\{B\}$-representable rigid-based construction, so by assumption it is $\{B\} \cup Y$-uniformisable, say by a formula $\psi_B(-,-,B,\bar{c})$ where $B, \bar{c}$ are the parameters.

By the reflection principle in $M$ there is an ordinal $j$ such that
\[ M \models \exists C(C \in C_B \land C^- = B'^- \land C \text{ is the unique set such that} \]
\[ "H_j \models \psi_B(B'^-, C, B, \bar{c})". \]

Hence in \( M \) there is a first pair \( \langle j_B, \psi_B \rangle \), definable from \( B \), such that
\[ M \models \exists C(C \in C_B \land C^- = B'^- \land C \text{ is the unique set such that} \]
\[ "H_{j_B} \models \psi_B(B'^-, C, B, \bar{c})". \]

Since all of this is uniform in \( B \), it follows that the construction \( C \) is \( \{\bar{c}\} \)-uniformisable in \( M \) by the formula
\[ y = C|L \text{ where } H_{j_B} \models \psi_B(B'^-, C, B, \bar{c}). \]

The last clause of the theorem follows by choosing \( C \) suitably, for example as in Example 2 of section 2. \( \square \)

### 4 The model

**Theorem 4** There is a transitive model \( N \) of ZFC in which every uniformisable construction is weakly natural.

The next three sections are devoted to proving this theorem. We use forcing. The central idea is to consider a construction \( C \) whose parameters lie in the ground model, and introduce a highly homogeneous generic structure \( B^\star \) into the graph of \( C \); by homogeneity \( B^\star \) must be highly symmetrical over \( B'^- \). Since the parameters of a construction may lie anywhere in the set-theoretic universe, we have to iterate this idea right up through the universe. So we need to build \( N \) by a proper class iteration.

Our forcing notation is mainly as in Jech [5]. Thus \( p < q \) means that \( p \) carries more information than \( q \). We write \( \dot{x} \) for a boolean name of the element \( x \) of \( N \), and \( \check{x} \) for the canonical name of an element \( x \) of the ground model. If \( y \) is a boolean name and \( G \) a generic set, we write \( y[G] \) for the element named by \( y \) in the generic extension by \( G \). Our notion of forcing is of the kind described in Menas [8] as ‘backward Easton forcing’, and we shall borrow some technical results from Menas’ paper.

We start from a countable transitive model \( M \) of ZFC + GCH. In \( M \), \( \Lambda \) is a definable continuous monotone increasing function from ordinals to infinite cardinals, with the property that for any ordinal \( \alpha \), \( \Lambda(\alpha + 1) > \Lambda(\alpha)^+ \). Our
notion of forcing $\mathbb{R}_\infty$ will be defined by induction on the ordinals. We start with a trivial ordering $\mathbb{R}_0$. At limit ordinals we take inverse limits.

For each ordinal $\alpha$ we shall define an $\mathbb{R}_\alpha$-name $\dot{S}_\alpha$; then $\mathbb{R}_{\alpha+1}$ will be $\mathbb{R}_\alpha \otimes \dot{S}_\alpha$. To define this name, let $\lambda$ be an infinite cardinal. We consider a notion of forcing, $P_\lambda$. In $P_\lambda$, conditions are partial maps $p : \lambda^+ \times \lambda^+ \times \lambda^+ \rightarrow 2$ with domain of cardinality $\leq \lambda$. Write $TP(\lambda)$ for a set-theoretical term which defines the notion of forcing $P_\lambda$. For each ordinal $\alpha$, we choose $\dot{S}_\alpha$ to be an $\mathbb{R}_\alpha$-name such that $||\dot{S}_\alpha = TP(\Lambda(\alpha + 1))||_{\mathbb{R}_\alpha} = 1$.

This defines a proper class notion of forcing, $\mathbb{R}_\infty = \text{direct limit of } \langle \mathbb{R}_\alpha : \alpha \text{ ordinal} \rangle$.

**Lemma 5** For each ordinal $\alpha$, suppose $\Lambda(\alpha)$ is a cardinal in $M_{\mathbb{R}_\alpha}$. Then with $\mathbb{R}_\alpha$-boolean value 1, $\dot{S}_\alpha$ is $\Lambda(\alpha + 1)$-closed and satisfies the $\Lambda(\alpha + 1)^+$-chain condition.

**Proof.** Straightforward. □

**Lemma 6**

(a) For every successor ordinal $\alpha$, $\Lambda(\alpha)$ and $\Lambda(\alpha)^+$ are cardinals with $\mathbb{R}_\infty$-boolean value 1.

(b) For each successor ordinal $\alpha$, $|\mathbb{R}_\alpha| = \Lambda(\alpha)^+$; for each limit ordinal $\delta$, $|\mathbb{R}_\delta| \leq \Lambda(\delta)^+$.

(c) For each ordinal $\alpha$, $\mathbb{R}_\alpha$ satisfies the $\Lambda(\alpha)^+$-chain condition.

**Proof.** We prove all parts simultaneously by induction. Suppose $R_\alpha$ has cardinality $\leq \Lambda(\alpha)^+$ and satisfies the $\Lambda(\alpha)^+$-chain condition. Put $\lambda = \Lambda(\alpha + 1) > \Lambda(\alpha)^+$. All cardinals $\geq \lambda$ are cardinals with $\mathbb{R}_\alpha$-boolean value 1. Let $\dot{q}$ be an element of $\dot{S}_\alpha$. Then $\dot{q}$ has cardinality $\leq \lambda$ with $\mathbb{R}_\alpha$-boolean value 1, and $\mathbb{R}_\alpha$ satisfies the $\lambda^+$-chain condition, so with boolean value 1 the domain of $\dot{q}$ lies within some $\gamma < \lambda^+$. Now $\dot{p}$ can be taken to be a map from the set $\gamma$ to the regular open algebra $RO(R_\alpha)$, which has cardinality $\leq (|\mathbb{R}_\alpha|)^{|\mathbb{R}_\alpha|} = |\mathbb{R}_\alpha|$. The number of such maps is at most $(|\mathbb{R}_\alpha|)^\lambda = \lambda^+$. Therefore $|\mathbb{R}_\alpha \otimes \dot{S}_\alpha| = \lambda^+$. Also $R_{\alpha+1}$ satisfies the $\lambda^+$-chain condition by Lemma 5, using a standard argument on iterated forcing (Menas [7] Proposition 10(i)).

Now with boolean value 1, $\dot{S}_\alpha$ is $\lambda$-closed and satisfies the $\lambda^+$-chain condition, so cardinals are preserved in passing from $\mathbb{R}_\alpha$ to $\mathbb{R}_{\alpha+1}$.

We turn to limit ordinals $\delta$. The cardinality of $\mathbb{R}_\delta$ is at most the product of the cardinalities of $\mathbb{R}_\alpha$ with $\alpha < \delta$, hence at most $\Lambda(\delta)^+$. It follows at once
that $\mathbb{R}_\delta$ satisfies the $\Lambda(\delta)^+$-chain condition and preserves all cardinals from $\Lambda(\delta)^+$ upwards.

It remains to show that for successor ordinals $\alpha + 1$, the cardinals $\Lambda(\alpha + 1)$ and $\Lambda(\alpha + 1)^+$ are not collapsed by $\mathbb{R}_\infty$. Using the next lemma (which doesn’t depend on the clause we are now proving), $\mathbb{R}_\infty$ can be written as $\mathbb{R}_\alpha \otimes \check{\mathbb{S}}_{\alpha+1} \otimes \check{\mathbb{R}}_{\alpha+1,\infty}$. The first factor satisfies the $\Lambda(\alpha+1)$-chain condition and hence preserves these two cardinals. The third factor preserves them since it is $\Lambda(\alpha+1)^+$-closed with boolean value 1. The middle factor is $\Lambda(\alpha+1)$-closed with boolean value 1, so that it preserves $\Lambda(\alpha+1)$ and $\Lambda(\alpha+1)^+. \square$

Lemma 7 For each ordinal $\alpha$ there is a proper class notion of forcing $\check{\mathbb{R}}_{\alpha,\infty}$ such that

(a) $\mathbb{R}_\infty$ is isomorphic to $\mathbb{R}_\alpha \otimes \check{\mathbb{R}}_{\alpha,\infty}$;

(b) In $M^{\mathbb{R}_\alpha}$, $\check{\mathbb{R}}_{\alpha,\infty}$ is the direct limit of iterated notions of forcing $\check{\mathbb{R}}_{\alpha,\beta}$ in such a way that for each $\beta > \alpha$, $\mathbb{R}_\beta$ is isomorphic to $\mathbb{R}_\alpha \otimes \check{\mathbb{R}}_{\alpha,\beta}$;

(c) For each successor ordinal $\alpha$, $\check{\mathbb{R}}_{\alpha,\infty}$ is $\Lambda(\alpha+1)$-closed with $M^{\mathbb{R}_\alpha}$-boolean value 1.

Proof. As Menas [7] Propositions 11 and 10(i), using the previous lemma. \square

The model $N$ for the theorem will be an $R_\infty$-generic extension of $M$.

Lemma 8 $N$ is a model of ZFC.

Proof. Menas [7] Proposition 14 derives this from the previous lemma. \square

Lemma 9 If $\dot{x}$ is an $\mathbb{R}_\infty$-name of a subset of $\Lambda(\alpha+1)$, and $r \in \mathbb{R}_\infty$, then there are $s \in \mathbb{R}_\infty$ and an $\mathbb{R}_\alpha$-name $\check{y}$ such that $s \leq r$ and $s \models_{\mathbb{R}_\infty} \"\dot{x} = \check{y}\"$.

Proof. This follows from the fact that $\check{R}_{\alpha,\infty}$ is $\Lambda(\alpha+1)$-closed. \square

Lemma 10 Let $\alpha$ be an ordinal. Then if $\lambda$ is $\Lambda(\alpha+1)$ or $\Lambda(\alpha+1)^+$, we have $2^\lambda = \lambda^+$ in $N$. 

10
Proof. Suppose $\lambda = \Lambda(\alpha + 1)$. Then $2^\lambda \leq \mu$ with $R_\alpha$-boolean value 1, where

$$\mu = |RO(R_\alpha)|^\lambda \leq (|R_\alpha|^{\Lambda(\alpha)^+})^\lambda = \lambda^+.$$ 

With $\mathbb{R}_\alpha$-boolean value 1, $\mathbb{R}_{\alpha,\infty}$ is $\lambda$-closed and hence adds no new subsets of $\lambda$. Similar calculations apply to the other cases. □

A notion of forcing $\mathbb{R}$ is said to be homogeneous if for any two elements $p, q$ of $\mathbb{R}$ there is an automorphism $\sigma$ of $\mathbb{R}$ such that $\sigma(p)$ and $q$ are compatible.

Lemma 11 The notion of forcing $\mathbb{R}_\infty$ is homogeneous.

Proof. Menas [7] Proposition 13 proves this under the assumption that each step of the iteration is homogeneous with boolean value 1. That assumption holds here. □

If $\alpha$ is an automorphism of the notion of forcing $\mathbb{R}$, then $\alpha$ induces an automorphism $\alpha^*$ of the boolean universe $M^P$. Also $\alpha$ takes any $\mathbb{R}$-generic set $G$ over $M$ to the $\mathbb{R}$-generic set $\alpha G$.

Lemma 12 For every element $\dot{x}$ of $M^\mathbb{R}$ we have

$$\dot{x}[G] = (\alpha^*)\dot{x}[\alpha G]$$

Proof. Immediate. □

To save notation we write $\alpha^*$ as $\alpha$. We note that $(\alpha \beta)^* = \alpha^* \beta^*$, which removes one possible source of ambiguity.

5 The generic copies of $A, B$

As explained earlier, our model $N$ in the theorem will be $M[G]$ where $G$ is an $\mathbb{R}_\infty$-generic class over $M$. Henceforth $C$ is a construction which is uniformisable in $N$ with uniformising formula $\phi$; we want to show that $C$ is weakly natural. Let $B$ be any structure in the graph of $C$. At the cost of adding $B$ as a parameter, we can assume without loss that $C$ is unitype and its graph consists of structures isomorphic to $B$. We put $A = B^-$. Choose an ordinal $\alpha$ so that $A, B$ and the parameters of the formulas representing $C$
all lie in $M[G \cap R_\alpha]$, and $B, \text{Aut}(B)$ both have cardinality $\leq \Lambda(\alpha + 1)$. We can decompose $N$ as a two-stage extension $M[G_\alpha][G_{\alpha,\infty}]$, where $G_\alpha = G \cap R_\alpha$ and $G_{\alpha,\infty}$ is $\bar{R}_{\alpha,\infty}[G_\alpha]$-generic over $M[G_\alpha]$.

At this point we adjust our notation. We put $\lambda = \Lambda(\alpha + 1)$, and we rename $M[G_\alpha]$ as $M$. By Lemma 1, $\lambda$ and $\lambda^+$ in the old $M$ are still cardinals in the new $M$. By Lemma 2, $N$ is constructed from the new $M$ by an iterated forcing notion $\bar{R}_{\alpha,\infty} = \bar{R}_{\alpha,\infty}[G_\alpha]$ with the same properties as the forcing notion $\bar{R}_{\infty}$, with two differences. First, the function $\Lambda$ must now be replaced by the function $\Lambda_\alpha$ where $\Lambda_\alpha(\beta) = \Lambda(\alpha + \beta)$. Second, $M$ need not satisfy the GCH everywhere; but this never matters. (In fact it would be possible to make the GCH hold in the new $M$ and in $N$, by adding extra factors in $\bar{R}_{\infty}$ to collapse the cardinalities of power sets.) One can check that all the preliminary lemmas to still hold for this notion of forcing $\bar{R}_{\alpha,\infty}$.

We now write $\mathbb{P}, \mathbb{Q}$ for $\mathcal{S}_\alpha[G_\alpha], \mathcal{R}_{\alpha,\infty+1}[G_\alpha]$ respectively. Thus

$$\bar{R}_{\alpha,\infty} = \mathbb{P} \otimes \mathbb{Q}.$$ 

We shall not need to refer to $G_\alpha$ again, and so we start afresh with our notation for generic sets.

We shall write $N$ as $M[G_1][G_2]$ where $G_1$ is $\mathbb{P}$-generic over $M$ and $G_2$ is $\mathbb{Q}[G_1]$-generic over $M[G_1]$.

We shall write $G$ for the $\mathbb{P} \otimes \mathbb{Q}$-generic set $G_1 \otimes G_2$ over $M$, so that $N = M[G]$. If $x$ is an element of $N$, we write $\dot{x}$ for a boolean name for $x$ in the forcing language for $\mathbb{P} \otimes \mathbb{Q}$. Note that every $\mathbb{P}$-name over $M$ can be read as a $\mathbb{P} \otimes \mathbb{Q}$-name too, so that there is no need for a separate symbol for $\mathbb{P}$-names.

The set $\bigcup G_1$ is a total map from $\lambda^+ \times \lambda^+ \times \lambda^+$ to 2. For each $i < \lambda^+$ and $j < \lambda^+$, we define $a_{ij} = \{k < \lambda^+ : \bigcup G_1(i,j,k) = 1\}$ and $a'_{ij} = \{a_{ij} : j < \lambda^+\}$, so that $a'_{ij}$ is a set of $\lambda^+$ independently generic subsets of $\lambda^+$. If $a$ and $b$ are (in $M_1[G_2]$) sets of subsets of $\lambda^+$, we put $a \equiv b$ iff the symmetric difference of $a$ and $b$ has cardinality $\leq \lambda$. We write $a_i$ for the equivalence class $(a'_i)^{=\equiv}$. The boolean names $\dot{a}_{ij}, \dot{a}_i$ can be chosen in $M^{\mathbb{P}}$ independently of the choice of $G$.

Consider again the structures $A$ and $B$ in $M$. Without loss we can suppose that $\text{dom}(A)$ is an initial segment of $\lambda$. In $M[G_1]$ there is a map $e$ which takes each element $i$ of $A$ to the corresponding set $a_i = \dot{a}_i[G_1]$; by means of $e$ we can define a copy $A^*$ of $A$ whose elements are the sets $a_i$ ($i \in \text{dom}(A)$). Again the boolean names $\dot{A}^*, \dot{e}$ can be chosen to be independent of the choice of $G$. 

12
Since $A, B$ and the parameters of the uniformising formula $\phi$ lie in $M$, and the notion of forcing $\mathbb{P} \otimes \dot{\mathbb{Q}}$ is homogeneous by Lemma 11, the statement "$\phi$ defines a construction on the class of structures isomorphic to $A$, which takes $A$ to $B$" is true in $N$ independently of the choice of $G$. In particular there are $\mathbb{P} \otimes \dot{\mathbb{Q}}$-boolean names $\dot{B}^*$, $\dot{\varepsilon}$ such that

\[ ||\dot{B}^*|\dot{\varepsilon}| \text{ is the unique structure such that } \phi(\dot{A}^*, \dot{B}^*) \text{ holds,} \]

The set of maps from $\text{Aut}(A)$ to $\text{Aut}(B)$ is the same in $M$ as it is in $M[G_1]$ and $M[G]$.\]

**Proof.** Using Lemma 5 and Lemma 6(c), $\mathbb{P}$ is $\lambda$-closed over $M$, and $\dot{\mathbb{Q}}$ is $\lambda$-closed over $M[G_1]$. Hence no new permutations of $A$ or $B$ are added since $|A| \leq |B| \leq \lambda$ in $M$; this proves (a), (b). Likewise (c) holds since $|\text{Aut}(A)| \leq |\text{Aut}(B)| \leq \lambda$ in $M$. \(\square\)

We regard $\text{Aut}(A)$ as a permutation group on $\lambda^+$ by letting it fix all the elements of $\lambda^+$ which are not in $\text{dom}(A)$.

By a neat map we mean a map $\alpha : \lambda^+ \to \text{Aut}(A)$ in $M$ which is constant on a final segment of $\lambda^+$; we write $\mathcal{N}$ for the set of neat maps. We write $\pi$ for the map from $\mathcal{N}$ to $\text{Aut}(A)$ which takes each neat map to its eventual value. We write $\mathcal{N}^-$ for the set of all neat maps $\alpha$ with $\pi(\alpha) = 1$. For each ordinal $i < \lambda^+$ we write $\mathcal{N}_i$ for the set of neat maps $\alpha$ such that $\alpha(j) = 1$ for all $j < i$. We write $\mathcal{N}_i^-$ for $\mathcal{N}^- \cap \mathcal{N}_i$.

We can regard $\alpha$ as a permutation of the set $\lambda^+ \times \lambda^+ \times \lambda^+$ by putting

\[ \alpha(i, j, k) = (\alpha(j)(i), j, k). \]

Then $\alpha$ induces an automorphism of $\mathbb{P}$.

**Lemma 14** If $\alpha$ and $\beta$ are distinct neat maps then they induce distinct automorphisms of $\mathbb{P}$. Identifying each neat map with the automorphism it induces, $\mathcal{N}$ forms a group with subgroups $\mathcal{N}^-, \mathcal{N}_i$ ($i < \lambda^+$); the map $\pi : \mathcal{N} \to \text{Aut}(A)$ is a group homomorphism.

13
**Proof.** From the definitions. □

The automorphism \( \alpha \) can be extended to an automorphism of \( P \otimes \hat{Q} \) in many different ways, by induction on \( R \) as an iterated notion of forcing. Each factor of \( R \) is with boolean value 1 the set of all maps from \( X \) to 2 of cardinality \( \leq \mu \), where \( X \) is \( \mu^+ \times \mu^+ \times \mu^+ \) for some cardinal \( \mu \). If \( \Sigma' \) is (in \( M \)) the group of permutations of \( X \), then an automorphism of the factor of \( R \) is determined by an element of \( \Sigma' \) and a permutation of the boolean values. For each ordinal \( i \) let \( \Sigma_i \) be in \( M_1 \) the product of the permutation groups \( \Sigma' \) for the first \( i \) factors of \( \hat{Q} \), and let \( \Sigma \) be the direct limit of the \( \Sigma_i \) in \( M_i \).

Then an automorphism \( \alpha \) of \( P \) and an element \( \sigma \) of \( \Sigma \) together determine an automorphism \( \langle \alpha, \sigma \rangle \) of \( P \otimes \hat{Q} \), and hence of \( M_{P \otimes \hat{Q}} \).

**Lemma 15** The actions of the group \( N \) of neat maps and the group \( \Sigma \) on \( P \otimes \hat{Q} \) commute with each other.

**Proof.** The class \( P \otimes \hat{Q} \) is \( P \times \hat{Q} \) where \( \hat{Q} \) is a class of boolean-valued subsets of a class \( X \) which is definable in \( M \); the action of \( \Sigma \) is through its action on \( X \). Thus each element of \( \hat{Q} \) is essentially a set of ordered pairs \( \langle x, y \rangle \) where \( x \in X \) and \( y \in P \). Since \( \Sigma \) and \( N \) act respectively on the first and second coordinates, the actions on \( \hat{Q} \) commute. The group \( \Sigma \) keeps \( P \) fixed. □

**Lemma 16** Suppose \( \alpha : \lambda^+ \to \text{Aut}(A) \) is neat and \( \alpha' \) is an automorphism of \( P \otimes \hat{Q} \) extending \( \alpha \). Then the action of \( \alpha' \) on \( M_{P \otimes \hat{Q}} \) setwise fixes the set \( \{ \hat{a}_i : i \in \text{dom}(A) \} \) of canonical names of the elements of \( A^*[G] \), and it acts on this set in the way induced by \( \pi(\alpha) \) and the map \( i \mapsto \hat{a}_i \). Thus \( \alpha'(\hat{a}_i) = \hat{a}_{\pi(\alpha)(i)} \).

**Proof.** Write out the names! (They lie in \( M_P \), so that the extension from \( M_P \) to \( M_{P \otimes \hat{Q}} \) is irrelevant.) □

If \( G \) is \( P \otimes \hat{Q} \)-generic over \( M \), then so is \( \langle \alpha, \sigma \rangle G \) for every neat map \( \alpha \) and every \( \sigma \in \Sigma \), since \( \alpha, \sigma \in M \).

**Lemma 17** For each element \( i \) of \( A \), each neat map \( \alpha \) and each \( \sigma \in \Sigma \), \( \hat{a}_{\pi(\alpha)(i)}[\langle \alpha, \sigma \rangle G] = \hat{a}_i[G] \). In particular \( A^*[\langle \alpha, \sigma \rangle G] = A^*[G] \).
Proof. By Lemma 16, \( \dot{a}_\pi(\alpha(i))[(\alpha, \sigma)G] = (\alpha \dot{a}_i)[(\alpha, \sigma)G] \). Then by Lemma 22 and the fact that \( \alpha \dot{a}_i \) lies in \( M^p \),

\[
(\alpha \dot{a}_i)[(\alpha, \sigma)G] = (\alpha \dot{a}_i)[\alpha G_1] = \dot{a}_i[G_1] = \dot{a}_i[G].
\]

\( \square \)

We write \( \hat{\varepsilon}^{-1} \) for a boolean name such that \( \hat{\varepsilon}^{-1}[G] = (\hat{\varepsilon}[G])^{-1} \) for all generic \( G \).

Lemma 18 Suppose \( \alpha \) is a neat map, \( \sigma \in \Sigma \) and \( G \) is \( P \otimes \dot{Q} \)-generic over \( M_1 \). Then \( \dot{B}^*[\langle \alpha, \sigma \rangle^{-1}G] = \dot{B}^*[G] \), and the map \( (\hat{\varepsilon}^{-1} \circ \langle \alpha, \sigma \rangle \hat{\varepsilon})|G \) is an automorphism of \( B \) which extends \( \pi(\alpha) \).

Proof. Since \( M[(\alpha, \sigma)^{-1}G] = M_1[G] \) and \( \dot{A}^*[\langle \alpha, \sigma \rangle^{-1}G] = \dot{A}^*[G] \), (1) (before Lemma 13) tells us that \( \dot{\varepsilon}[(\alpha, \sigma)^{-1}G](i) = \dot{a}_i[(\alpha, \sigma)^{-1}G] \) for each \( i \in \text{dom}(A) \), and that \( \dot{B}^*[\langle \alpha, \sigma \rangle^{-1}G] = \dot{B}^*[G] \) and \( \dot{\varepsilon}[G]^{-1} \circ \hat{\varepsilon}[(\alpha, \sigma)^{-1}G] \) extends \( \dot{\varepsilon}[G]^{-1} \circ \dot{\varepsilon}[(\alpha, \sigma)^{-1}G] \). Now using Lemma 17,

\[
\dot{\varepsilon}[G]^{-1} \circ \dot{\varepsilon}[(\alpha, \sigma)^{-1}G](i) = \dot{\varepsilon}[G]^{-1}(\dot{a}_i[(\alpha, \sigma)^{-1}G])
\]

\[= \dot{\varepsilon}[G]^{-1}(\dot{a}_\pi(\alpha)(i)[G]) = \pi(\alpha)(i).\]

\( \square \)

Lemma 19 For every neat map \( \alpha \), each \( \sigma \in \Sigma \) and all \( \langle p, q \rangle \in P \otimes \dot{Q} \) there are \( \langle p', q' \rangle \preceq \langle p, q \rangle \) and \( g \in \text{Aut}B \) such that

\[
\langle p', q' \rangle \Vdash_{P \otimes \dot{Q}} \sigma(\hat{\varepsilon}^{-1}) \circ \alpha \sigma(\hat{\varepsilon}) = \dot{g}.
\]

Proof. Let \( f \) be \( \pi(\alpha) \). By Lemma 18 we have

\[
||\sigma \hat{\varepsilon}^{-1} \circ \alpha \sigma \hat{\varepsilon} \text{ is an automorphism of } B \text{ extending } \hat{f}||_{P \otimes \dot{Q}} = 1.
\]

Unpacking the existential quantifier in “an automorphism of \( B \)” gives the lemma. \( \square \)

Consider any \( i < \lambda^+ \). Given \( \langle p, \dot{q} \rangle \in P \otimes \dot{Q} \), define \( t_{\langle p, \dot{q} \rangle} \) to be the set of all triples \( (f, g, \sigma) \), with \( f \in \text{Aut}(A) \), \( g \in \text{Aut}(B) \) and \( \sigma \in \Sigma \), such that for some \( \alpha \in N_i \), \( \pi(\alpha) = f \) and

\[
\langle p, \dot{q} \rangle \Vdash_{P \otimes \dot{Q}} \sigma(\hat{\varepsilon}^{-1}) \circ \alpha \sigma(\hat{\varepsilon}) = \dot{g}.
\]
Clearly if \( \langle p', q' \rangle \leq \langle p, \dot{q} \rangle \) then \( t_{p', q', i} \supseteq t_{p, \dot{q}, i} \). Since there are only a set of values for \( \sigma(\dot{\varepsilon}) \) and \( \sigma(\dot{\varepsilon}^{-1}) \) with \( \sigma \in \Sigma \), it follows that there is \( \langle p_i, \dot{q}_i \rangle \) such that for all \( \langle p', q' \rangle \leq \langle p, \dot{q} \rangle \),

\[
t_{p', q', i} = t_{p_i, \dot{q}_i}.
\]

We fix a choice of \( p_i, \dot{q}_i \), and we write \( t_i \) for the resulting value \( t_{p_i, \dot{q}_i} \). If \( (f, g, \sigma) \) is in \( t_i \), we write \( \alpha^i_{f, g, \sigma} \) for some \( \alpha \in \mathcal{N}_i \) such that

\[
\langle p_i, \dot{q}_i \rangle \vdash \mathbf{P} \otimes \dot{Q} \sigma(\dot{\varepsilon}^{-1}) \circ \alpha \sigma(\dot{\varepsilon}) = \dot{g}
\]

and \( \pi(\alpha) = f \).

**Lemma 20** For each \( i < \lambda^+ \), \( t_i \) is a subclass of \( \text{Aut}(A) \times \text{Aut}(B) \times \Sigma \) such that

1. For each \( (f, g, \sigma) \) in \( t_i \), \( g|A = f \);
2. For each \( f \) in \( \text{Aut}(A) \) and \( \sigma \) in \( \Sigma \) there is \( g \) with \( (f, g, \sigma) \) in \( t_i \).

(So \( t_i(-, -, \sigma) \) is a first attempt at a lifting of the restriction map from \( \text{Aut}(B) \) to \( \text{Aut}(A) \).)

**Proof.** By Lemma 19. \( \square \)

We write \( t_{p', \dot{q}, i}^- \) for the set of pairs \( (g, \sigma) \) such that \( (1, g, \sigma) \) is in \( t_{p, \dot{q}, i} \). We write \( \alpha^i_{g, \sigma} \) for \( \alpha^i_{1, g, \sigma} \); note that \( \alpha^i_{g, \sigma} \) is in \( \mathcal{N}_i^- \) by Lemma 18.

**Lemma 21** For each \( i < \lambda^+ \) there are \( \sigma_i \) in \( \Sigma \), a condition \( p'_i \leq p_i \) and a boolean name \( \dot{v}_i \) such that

1. For each \( i < \lambda^+ \), \( p'_i \models \mathbf{P} \text{ dom}(\sigma_i^{-1} \dot{q}_i) \subseteq \dot{v}_i \);
2. For all \( i < j < \lambda^+ \), \( ||\dot{v}_i \cap \dot{v}_j = \emptyset||_p = 1 \);
3. For all \( i < j < \lambda^+ \), \( \sigma_i \sigma_j = \sigma_j \sigma_i \).

**Proof.** By induction on \( i < \lambda^+ \). As we choose the \( p'_i \), \( \sigma_i \) and \( \dot{v}_i \), we also choose an eventually zero sequence of ordinals \( \gamma_{\mu, i} < \mu^+ \) in \( M_i \) so that

\[
||\dot{v}_i \subseteq \prod_{\mu} (\gamma_{\mu, i} \times \mu^+ \times \mu^+)||_p = 1.
\]

16
Then when we have made our choices for all $i < j$, we first extend $p_i$ to $p'_i$ forcing the domain of $q_i$ to lie within some set

$$X = \prod_{\mu < \mu'} (\gamma'_\mu \times \gamma'_\mu \times \mu^+)$$

lying in $M_1$, and we choose $\hat{w}_i$ to be a canonical boolean name for this set $X$. Then we choose $\sigma_i$ so that $\sigma_i^{-1}$ moves $X$ to

$$\prod_{\mu < \mu'} \left( \left[ \bigcup_{j < i} \gamma_{\mu,i}, \bigcup_{j < i} \gamma_{\mu,i} + \gamma'_\mu \right] \times \gamma'_\mu \times \mu^+ \right),$$

(the product of products of three intervals), and we put $\dot{v}_i = \sigma_i^{-1} \dot{w}_i$ and $\gamma_{\mu,j} = \bigcup_{j < i} \gamma_{\mu,i} + \gamma'_\mu$. □

We fix the choice of $\sigma_i$ and $\dot{v}_i$ ($i < \lambda^+$) given by this lemma. Without loss we extend the conditions $p_i$ to be equal to $p'_i$.

**Lemma 22** There is a stationary subset $S$ of $\lambda^+$ such that:

(a) for each $i \in S$ and $j < i$, the domain of $p_i$ is a subset of $i \times \text{dom}A$;

(b) for each $i \in S$ and $j < i$, every map $\alpha_{f,g}^i : \lambda^+ \to \text{Aut}(A)$ is constant on $[i, \lambda^+]$;

(c) for all $i, j \in S$,

$$\{(f, g) : (f, g, \sigma_i) \in t_i\} = \{(f, g) : (f, g, \sigma_j) \in t_j\};$$

(d) there is a condition $p^* \in \mathbb{P}$ such that for all $i \in S$, $p_i|(i \times \text{dom}A) = p^*$.

**Proof.** First, there is a club $C \subseteq \lambda^+$ on which (a) and (b) hold. Then by Fodor’s lemma there is a stationary subset $S$ of $C$ on which (c) and (d) hold. □

6 The weak lifting

In this section we use the notation $S$, $\sigma_i$, $\dot{v}_i$, $p^*$ from Lemmas 21 and 22. We write $s$ for the constant value of

$$\{(f, g) : (f, g, \sigma_i) \in t_i\} (i \in S)$$
from clause (c) of Lemma 22, and $s^-$ for the set of all $g$ such that $(1, g) \in s$. We write $\nu : \text{Aut}(B) \to \text{Aut}(A)$ for the restriction map.

**Lemma 23**  The relation $s$ is a subset of $\text{Aut}(A) \times \text{Aut}(B)$ that projects onto $\text{Aut}(A)$, and if $(f, g)$ is in $s$ then $\nu(g) = f$.

**Proof.** Lemma 20. $\square$

**Lemma 24**  If $(f_1, g_1)$ and $(f_2, g_2)$ are both in $s$ then $(f_1f_2, g_1g_2)$ is in $s$.

**Proof.** In this and later calculations we freely use the fact (Lemma 13) that the actions of $\mathcal{N}$ and $\Sigma$ on $\mathbb{P} \otimes \mathbb{Q}$ commute. Take any $i, j \in S$ with $i < j$.

Put $\alpha_1 = \alpha^j_{f_1, g_1, \sigma_j}$, $\alpha_2 = \alpha^j_{f_2, g_2, \sigma_i}$, and $\alpha_3 = \alpha_1\alpha_2$. Note that $\alpha_1\alpha_2$ is in $\mathcal{N}_i$ since $i < j$.

We have

$$\langle p_j, q_j \rangle \vdash \sigma_j \hat{\varepsilon}^{-1} \circ \alpha_3 \sigma_j(\hat{\varepsilon}) = \sigma_j \hat{\varepsilon}^{-1} \circ \alpha_1 \sigma_j(\hat{\varepsilon}) \circ (\sigma_j \alpha_1(\hat{\varepsilon}))^{-1} \circ \alpha_3 \sigma_j(\hat{\varepsilon})$$

and by assumption

$$\langle p_j, q_j \rangle \vdash \sigma_j \hat{\varepsilon}^{-1} \circ \alpha_1 \sigma_j(\hat{\varepsilon}) = \hat{g}_1.$$

So

$$\langle p_j, q_j \rangle \vdash \sigma_j \hat{\varepsilon}^{-1} \circ \alpha_3 \sigma_j(\hat{\varepsilon}) = \sigma_j \hat{g}_1 \circ \sigma^j(\alpha_1(\hat{\varepsilon}))^{-1} \circ \alpha_1(\alpha_2 \sigma_j \hat{\varepsilon}).$$

Also by assumption

$$\langle p_i, q_i \rangle \vdash \sigma_i \hat{\varepsilon}^{-1} \circ \alpha_2 \sigma_i(\hat{\varepsilon}) = \sigma_i \hat{g}_2.$$

Acting on this by $\alpha_1 \sigma_j \sigma_i^{-1}$ gives

$$\langle \alpha_1 p_i, \alpha_1 \sigma_j \sigma_i^{-1} q_i \rangle \vdash \alpha_1 \sigma_j(\hat{\varepsilon}^{-1}) \circ \alpha_1 \alpha_2 \sigma_j \hat{\varepsilon} = \alpha_1 \sigma_j \hat{g}_2.$$

Now $g_2$ is in the ground model and hence $\alpha_2 \sigma_j \hat{g}_2 = \hat{g}_2$. Also $\alpha_1 p_i = p_i$ since the support of $p_i$ lies entirely below $j$, and $\alpha_1 = \alpha^j_{g_1, \sigma_j}$ is the identity in this region since it lies in $\mathcal{N}_j$. So we have shown that

$$\langle p_i, \alpha_1 \sigma_j \sigma_i^{-1} q_i \rangle \vdash \alpha_1 \sigma_j \hat{\varepsilon}^{-1} \circ \alpha_1 \alpha_2 \sigma_j \hat{\varepsilon} = \hat{g}_2.$$
Now we note that \( p_i \cup p_j \) is a condition in \( P \), by (a), (d) of Lemma 22. Also \( p_i \cup p_j \) forces that \( \text{dom}(\sigma_i^{-1} \dot{q}_i) \) is disjoint from \( \text{dom}(\sigma_j^{-1} \dot{q}_j) \) by Lemma 21, and hence also that \( \text{dom}\sigma_j \sigma_i^{-1} \dot{q}_i \) is disjoint from \( \text{dom}r_j \). From the action of neat maps on \( Q \), \( \text{dom} \alpha_1 \sigma_j \sigma_i^{-1} = \text{dom} \sigma_j \sigma_i^{-1} \). This shows that \( \langle p_i, \sigma_j \sigma_i^{-1} \dot{q}_i \rangle \) and \( \langle p_j, \dot{q}_j \rangle \) have a common extension \( \langle p', \dot{q}' \rangle \). Putting everything together, we have that

\[
\langle p', \dot{q}' \rangle \models \sigma_j \dot{\epsilon}^{-1} \circ \alpha_3 \sigma_j \dot{\epsilon} = \hat{g}_1 \hat{g}_2.
\]

Since \( \alpha_3 \) is in \( N_i \), this shows that

\[
(f_1f_2, g_1g_2) \in t_{p', q', j}.
\]

Then by the maximality property of \( \langle p_j, \dot{q}_j \rangle \),

\[
(f_1f_2, g_1g_2, \sigma_j) \in t_{p_j, j}
\]

so that \( (f_1f_2, g_1g_2) \) is in \( s \). \( \square \)

**Lemma 25** If \( g_1 \) and \( g_2 \) are in \( s^- \) then \( g_1g_2 = g_2g_1 \).

**Proof.** Apply the proof of Lemma 24 to \((1, g_1)\) and \((1, g_2)\). In the notation of that proof, \( \alpha_1 \) commutes with \( \alpha_2 \), \( \alpha_1 \) is the identity below \( j \) and \( \alpha_2 \) is the identity below \( j \) (since \( i, j \in S \)). But also \( g_2 \) lies in \( s^- \), and this tells us that \( \alpha_2 \) is the identity on \( [j, \lambda^+] \). In particular \( \alpha_1 \) commutes with \( \alpha_2 \).

We follow the proof of Lemma 24 but with \( g_1 \) and \( g_2 \) transposed, starting from the observation that

\[
\langle p_i, \dot{q}_i \rangle \models \sigma_i \dot{\epsilon}^{-1} \circ \alpha_3 \sigma_i \dot{\epsilon} = \sigma_i \dot{\epsilon}^{-1} \circ \alpha_2 \sigma_i \dot{\epsilon} \circ \alpha_2 \sigma_i \dot{\epsilon}^{-1} \circ \alpha_3 \sigma_i \dot{\epsilon}.
\]

As before, we have that

\[
\langle p_i, \dot{q}_i \rangle \models \sigma_i \dot{\epsilon}^{-1} \circ \alpha_2 \sigma_i \dot{\epsilon} = \hat{g}_2
\]

and

\[
\langle \alpha_2 p_j, \alpha_2 \sigma_i \sigma_j^{-1} \dot{q}_j \rangle \models \alpha_2 \sigma_i \dot{\epsilon}^{-1} \circ \alpha_2 \sigma_i \dot{\epsilon} = \alpha_3 \sigma_i \hat{g}_1,
\]

recalling that \( \alpha_1 \) commutes with \( \alpha_2 \). Now the support of \( p_j \) lies below \( i \) or within \( [j, \lambda^+] \times \text{dom}A \), and \( \alpha_2 \) is the identity in both these regions, and so
\[ \alpha_2(p_j) = p_j. \] 
Also \( p_i \cup p_j \) forces that \( \dot{q}_i \) and \( \alpha_2\sigma_i\sigma_j^{-1}\dot{q}_j \) have disjoint domains.
So as before there is \( \langle p', \dot{q}' \rangle \leq \langle p_i, \dot{q}_i \rangle \) and \( \leq \langle p_j, \alpha_2\sigma_i\sigma_j^{-1}\dot{q}_j \rangle \) such that
\[ \langle p', \dot{q}' \rangle \models \sigma_i\dot{\varepsilon}^{-1} \circ \alpha_3\sigma_i\dot{\varepsilon} = \dot{g}_2\dot{g}_1. \]
As before, it follows that
\[ \langle p_i, \dot{q}_i \rangle \models \sigma_i\dot{\varepsilon}^{-1} \circ \alpha_3\sigma_i\dot{\varepsilon} = \dot{g}_2\dot{g}_1, \]
and so
\[ \langle p_i, \sigma_j\sigma_i^{-1}\dot{q}_j \rangle \models \sigma_j\dot{\varepsilon}^{-1} \circ \alpha_3\sigma_j\dot{\varepsilon} = \dot{g}_2\dot{g}_1. \]
Again there is a condition \( \langle p'', \dot{q}' \rangle \leq \langle p_i, \sigma_j\sigma_i^{-1}\dot{q}_j \rangle \) and \( \leq \langle p_j, \dot{q}_j \rangle \). Recalling that in the proof of Lemma 24 we showed that
\[ \langle p_j, \dot{q}_j \rangle \models \sigma_j\dot{\varepsilon}^{-1} \circ \alpha_3\sigma_j\dot{\varepsilon} = \dot{g}_2\dot{g}_1, \]
we deduce that
\[ \langle p'', \dot{q}' \rangle \models \dot{g}_1\dot{g}_2 = \dot{g}_2\dot{g}_1. \]
But the equation \( g_1g_2 = g_2g_1 \) is about the ground model, and hence it is true.
\[ \square \]

**Corollary 26** If \((f, g_1) \) and \((f, g_2) \) are in \( s \) then \( g_1g_2^{-1} \) is in \( \langle s^- \rangle \).

**Proof.** There is some \( g' \in \text{Aut}(B) \) such that \( \langle f^{-1}, g' \rangle \) is in \( s \). Then by the claim, \( (1, g_1g') \) and \( (1, g_2g') \) are in \( s \) and so \( g_1g', g_2g' \) are in \( s^- \). Hence the element
\[ g_1g_2^{-1} = (g_1g')(g_2g')^{-1} \]
lies in \( \langle s^- \rangle \). \( \square \)

**Corollary 27** Suppose \( g_1, \ldots, g_k \) are elements of \( \text{Aut}(B) \) such that \( (\nu(g_i), g_i) \) is in \( s \) for each \( i \), and let each of \( \varepsilon_1, \ldots, \varepsilon_k \) be either 1 or -1. If
\[ \nu(g_1)^{\varepsilon_1} \cdots \nu(g_k)^{\varepsilon_k} = 1 \]
then
\[ g_1^{\varepsilon_1} \cdots g_k^{\varepsilon_k} \in \langle s^- \rangle. \]
Proof. We write \( f_i \) for \( \nu(g_i) \). First we show the corollary directly in the case \( k = 3 \). Taking inverses, we can assume that \( \varepsilon_2 = 1 \). When \( \varepsilon_1 = \varepsilon_3 = 1 \), the result is immediate from Lemma \( \text{24} \). We consider next the case where \( \varepsilon_1 = 1 \) and \( \varepsilon_3 = -1 \). Here we find \( g \) such that \( (f^{-1}, g) \) is in \( s \). Then both of

\[
g_1^1 g_2 g_3^{-1} \cdot g_3^1 g_1^1
\]

are in \( s^{-} \) by Lemma \( \text{24} \), and so

\[
g_1^1 g_2 g_3^{-1} = (g_1^1 g_2 g_1^1)(g_3^1 g_1)^{-1}
\]

is in \( \langle s^{-} \rangle \). By symmetry this also covers the case where \( \varepsilon_1 = -1 \) and \( \varepsilon_3 = 1 \). Finally when \( \varepsilon_1 = \varepsilon_3 = -1 \), we repeat the same moves, noting that

\[
g_1^{-1} g_2^{-1} g_1^1
\]

is in \( \langle s^{-} \rangle \) by the previous case.

This case also covers the case \( k = 2 \) by adding at the end a factor \( g_3^1 \) where \( (1, g_3^1) \) is in \( s \). The case \( k = 1 \) is trivial.

We prove the remaining cases by induction on \( k \), assuming \( k > 3 \). Choose \( g \) so that \( (g, f_k^{\varepsilon_{k-1}} f_k^{-\varepsilon_k}) \) is in \( s \). Then by induction hypothesis both the elements

\[
g_1^{\varepsilon_1} \cdots g_{k-2}^{\varepsilon_{k-2}} g_1^1
\]

and

\[
g^{-1} g_{k-1}^{\varepsilon_{k-1}} g_k^{\varepsilon_k}
\]

lie in \( \langle s^{-} \rangle \). Hence so does their product, completing the proof. \( \square \)

Consider the subgroup \( \langle s^{-} \rangle \) of \( \text{Aut}(B) \). By Lemma \( \text{25} \), \( \langle s^{-} \rangle \) is commutative. By Lemma \( \text{25} \) and Corollary \( \text{27} \) it follows that \( s \) would be a weak splitting of \( \nu \), with \( \langle s^{-} \rangle \) as \( G_0 \), if for each \( f \) in \( \text{Aut}(A) \) there was a unique \( g \) with \( \langle f, g \rangle \) in \( s \). But we can make this true by cutting down \( s \). So \( \nu \) has a weak splitting, and this concludes the proof of Theorem \( \text{4} \).

7 Answers to questions

The results above answer most of the problems stated in \( \text{4} \). In that paper we showed:
Theorem 3 of [4] If $C$ is a small natural construction in a model of ZFC, then $C$ is uniformisable with parameters.

We asked (Problem A) whether it is possible to remove the restriction that $C$ is small. The answer is No:

Theorem 28 There is a transitive model of ZFC in which some $\emptyset$-representable construction is natural but not uniformisable (even with parameters).

Proof. Let $N$ be the model of Theorem 4. Let $C$ be some construction $\emptyset$-representable in $N$ which is not weakly natural (such as Example 2 in section 2). Then by Theorem 4, $C$ is not uniformisable. The rigidifying construction $C^r$ of section 2 is $\emptyset$-representable, natural and not uniformisable. $\blacksquare$

Problem B asked whether in Theorem 3 of [4] the formula defining $C$ can be chosen so that it has only the same parameters as the formulas chosen to represent $C$. The answer is No:

Theorem 29 There is a transitive model $N$ of ZFC with the following property:

For every set $Y$ there are a set $X$ and a unitype rigid-based (hence small natural) $X$-representable construction that is not $X \cup Y$-uniformisable.

Proof. Take $N$ to be the model given by Theorem 4. Let $Y$ be any set in $N$. If $N$ and $Y$ are not as stated above, then for every set $X$ and every unitype rigid-based $X$-representable construction in $N$, $X$ is $X \cup Y$-uniformisable. So the hypothesis of Theorem 2 holds, and by that theorem there is in $N$ a small $\{c\}$-uniformisable construction that is not weakly natural. But this contradicts the choice of $N$. $\square$

Problem C asked whether there are transitive models of ZFC in which every uniformisable construction is natural. Theorem 4 is the best answer we have for this; the problem remains open.

In [3] one of us asked whether there can be models of ZFC in which the algebraic closure construction on fields is not uniformisable.

Theorem 30 There are transitive models of ZFC in which:

22
(a) no formula (with or without parameters) defines for each field a specific algebraic closure for that field, and

(b) no formula (with or without parameters) defines for each abelian group a specific injective hull of that group.

**Proof.** Let the model \( N \) be as in Theorem 4. In \( N \) the constructions of Examples 3 and 4 in section 2 are not uniformisable, since they are not weakly natural. So these two examples prove (a) and (b) respectively. \( \Box \)

One result in [3] was that there is no primitive recursive set function which takes each field to an algebraic closure of that field. This is an absolute result which applies to every transitive model of ZFC, and so it is not strictly comparable with the consistency results proved above. In this context we note that Garvin Melles showed [6] that there is no “recursive set-function” (he gives his own definition for this notion) which finds a representative for each isomorphism type of countable torsion-free abelian group.

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