An alternative proof of Nyblom’s results
and a generalisation

A. David Christopher

Department of Mathematics, The American College
Tamil Nadu, India
e-mail: davchrame@yahoo.co.in

Received: 2 April 2019 Revised: 27 June 2020 Accepted: 30 June 2020

Abstract: Let \( n \) be a positive integer and \( k \) be a non-negative integer. We define

\[
p(n, k) = \begin{cases} 
  n(n + k), & \text{if } k \equiv 0 \pmod{2}; \\
  \frac{n(n+k)}{2}, & \text{if } k \equiv 1 \pmod{2},
\end{cases}
\]

and \( D(n, k) \) to be the number of ways \( n \) can be expressed as a difference of two elements from the sequence \( p(n, k) \). Nyblom found closed expressions for \( D(n, 0) \) and \( D(n, 1) \) in terms of some restricted number-of-divisors functions. Here we re-establish these two results of Nyblom in a relatively simple way. Along with the other interpretations for \( D(n, k) \), an expression for \( D(n, k) \) is presented in terms of restricted form of \( D(n, 0) \) and \( D(n, 1) \). Also we consider another function due to Nyblom, denoted \( p_D(n) \), which counts the number of partitions of \( n \) with parts in arithmetic progression having common difference \( D \). Nyblom and Evan found a simple expression for \( p_2(n) \) and put \( p_D(n) \) in terms of a divisor-counting functions when \( D \geq 3 \). Here we re-establish Nyblom’s expression for \( p_2(n) \), and find equinumerous expressions for \( p_D(n) \) when \( D \geq 3 \). Finally, we present the following generalised version of \( D(n, k) \): given a set of positive integers say, \( A \), we denote by \( D(n, A) \), the number of ways \( n \) can be written as a difference of two elements from the set \( A \). And we express \( D(n, A) \) in terms of partition enumerations when some restrictions are imposed upon the elements of \( A \). We close with the hint that, boundedness of \( D(n, A) \) together with the divergence of \( \sum_{a \in A} \frac{1}{a} \) disproves Erdős arithmetic progression conjecture.

Keywords: Square numbers, Triangular numbers, Representation as difference of integers.

2010 Mathematics Subject Classification: Primary 11A67, Secondary 11B34.
1 Introduction

Nyblom [6] converted the problem of finding the number of representation of integers as a difference of two squares to a system of linear Diophantine equation and thereby got the following expression:

\[ S(n) = \frac{1}{2} \left( \tau_e(n) + (-1)^{n+1} \tau_o(n) + \frac{1 + (-1)^{\tau(n)+1}}{2} \right), \] (1)

where \( S(n) \) denote the number of ways \( n \) can be expressed as a difference of two square numbers, \( \tau_e(n) \) (resp. \( \tau_o(n) \)) denote the number of even (resp. odd) positive divisors of \( n \) and \( \tau(n) \) denote the number of positive divisors of \( n \). In Section 2, we establish this expression in a simple and different way by counting a particular kind of divisors of \( n \); this counting has already been done in a paper by the author [2].

In one of the other paper of Nyblom [5] following expression was derived:

\[ T(n) = \tau_o(n), \] (2)

where \( T(n) \) denote the number of ways \( n \) can be expressed as a difference of two triangular numbers. Even to derive this expression, Nyblom employed a technique similar to the one used in deriving (1). In Section 2, we establish (2) by counting another kind of divisors of \( n \); essentials of this counting has been derived already in the same paper of the author [2].

Now, we define a sequence which is a common generalisation of square number sequence and triangular number sequence and find the expression of the above kind when elements of this sequence come into play.

**Definition 1.1.** Let \( n \) be a positive integer and \( k \) be a non-negative integer. We define

\[ p(n, k) = \begin{cases} 
  n(n + k), & \text{if } k \equiv 0 \pmod{2}; \\
  \frac{n(n + k)}{2}, & \text{if } k \equiv 1 \pmod{2},
\end{cases} \]

and \( D(n, k) \) to be the number of ways \( n \) can be expressed as a difference of two elements from the sequence \( p(n, k) \).

From this definition, it follows that:

\[ S(n) = \begin{cases} 
  D(n, 0) - 1, & \text{if } \delta(n) = 1; \\
  D(n, 0), & \text{if } \delta(n) = 0,
\end{cases} \] (3)

where \( \delta \) denotes the characteristic function of square numbers. Moreover, we have

\[ T(n) = D(n, 1). \] (4)

The sequences \( p(n, k) \) for the initial cases \( k = 2, 3, 4, 5, 6 \) (with comments) can be found, respectively, in A005563, A000096, A028347, A055998 and A028560 of Online Encyclopedia of Integer Sequences. Following result concerning \( p(n, k) \) is straightforward.
Lemma 1.2. Let \( n \) be a positive integer and let \( k \) be a non-negative integer. Then
\[
p(n, k) = \begin{cases} 
  p(n + 1, k - 2) - (k - 1), & \text{if } k \equiv 0 \pmod{2}; \\
  p(n + 1, k - 2) - \frac{k - 1}{2}, & \text{if } k \equiv 1 \pmod{2}.
\end{cases}
\] (5)

This lemma is instrumental in deriving expressions (which are presented in Section 2) for \( D(n, k) \) in a restricted form of \( D(n, 0) \) and \( D(n, 1) \).

Next we are concerned with another result of Nyblom and Evan. The following definition is essential for mentioning that result.

Definition 1.3. Let \( n \) and \( D \) be two positive integers. Then the function \( p_D(n) \) is defined to be the number of partitions of \( n \) with parts in arithmetic progression having common difference \( D \).

Note that, in the above definition, we consider the partition \((n)\) of \( n \) as a partition in an arithmetic progression.

R. Cook and D. Sharp [1] found a necessary and sufficient condition for an integer \( n \) to be written as a sum of arithmetic progression. In this sequence, Nyblom and Evans [7] defined the function \( p_D(n) \) and obtained that:
\[
p_2(n) = \frac{1}{2} \left( \tau(n) - 2 + \frac{(-1)^{\tau(n) + 1} + 1}{2} \right) + 1, \tag{6}
\]
and also they have obtained an interpretation for the term \( p_D(n) \) in terms of a divisor-counting function. A. O. Munagi [4] pointed out that such an expression for \( p_D(n) \) by Nyblom and Evans is a complicated one. In Section 2 we re-establish (6) and also we find alternative interpretation for \( p_D(n) \) which is relatively simple to that of Nyblom and Evans.

Section 3 is concerned with a generalisation of the above problem, that is, we are concerned with the number of ways a positive integer \( n \) can be expressed as a difference of two elements of a given set of positive integers. Following definition forms a basis for Section 3.

Definition 1.4. Let \( n \) be a positive integer and let \( A \) be a set of positive integers. Then we define the function \( D(n, A) \) to be the number of ways \( n \) can be expressed as a difference of two elements from the set \( A \).

In Section 3 we found that the function \( D(n, A) \) is equinumerous with a kind of partitions involving the set \( A \) with some additional constraints. We close by mentioning a connection between boundedness of \( D(n, A) \) and Erdős arithmetic progression conjecture.

Notations and definitions in this section bears the same meaning throughout the article when used.

2 Proof and an Extension of Nyblom’s results

2.1 Alternative proof of Nyblom’s results

In this section we give an alternative and simple proofs of Nyblom’s expressions [5, 6].

Let \( n \) be a positive integer. Suppose that \( n \) can be written as a difference of two squares; both non-zero. Then we have the following equalities:
\[ n = y^2 - x^2 = (x + d)^2 - x^2 = 2dx + d^2. \]

This gives the following congruence:

\[ \frac{n}{d} - d \equiv 0 \pmod{2} \quad (7) \]

with \( \frac{n}{d} - d \geq 1. \)

Conversely, every solution to the congruence above gives a representation of \( n \) as a difference of two squares; both non-zero.

In [2] to conclude Theorem 5, the number of divisors \( d \) of \( n \) satisfying the congruence (7) was counted. Now in view of the above observations, we equate that counting with \( D(n, 0) \).

\[
D(n, 0) = \begin{cases} 
\frac{\tau(n)}{2} & \text{if } n \equiv 1 \pmod{2} \text{ and } \delta(n) = 1; \\
\tau(n) - 1 & \text{if } n \equiv 1 \pmod{2} \text{ and } \delta(n) = 0; \\
\frac{2}{(\beta - 1)\tau \left( \frac{n}{2^\beta} \right) - 1} & \text{if } n \equiv 0 \pmod{2} \text{ and } \delta(n) = 1; \\
\frac{2}{(\beta - 1)\tau \left( \frac{n}{2^\beta} \right)} & \text{if } n \equiv 0 \pmod{2} \text{ and } \delta(n) = 0,
\end{cases}
\]

where \( \beta \) is the highest power of 2 that divides \( n \).

Now we observe that (8) is a disguised form of (1).

**Case i.** When \( n \equiv 1 \pmod{2} \) and \( \delta(n) = 0 \), in accordance with (1), we can write

\[
D(n, 0) = \frac{1}{2} \left( \tau_e(n) + \tau_o(n) + \frac{1 + 1}{2} \right) - 1 = \frac{1}{2} (\tau(n) + 1) - 1 = \frac{\tau(n) - 1}{2}.
\]

**Case ii.** When \( n \equiv 1 \pmod{2} \) and \( \delta(n) = 1 \), in accordance with (1), we can write

\[
D(n, 0) = \frac{1}{2} \left( \tau_e(n) + \tau_o(n) + \frac{1 - 1}{2} \right) = \frac{1}{2} \tau(n).
\]

**Case iii.** When \( n \equiv 0 \pmod{2} \) and \( \delta(n) = 1 \), by (1), we have

\[
D(n, 0) = \frac{1}{2} \left( \tau_e(n) - \tau_o(n) + 1 \right) - 1 = \frac{1}{2} \left( \beta \tau \left( \frac{n}{2^\beta} \right) - \tau \left( \frac{n}{2^\beta} \right) + 1 \right) - 1 = \frac{1}{2} \left( (\beta - 1)\tau \left( \frac{n}{2^\beta} \right) - 1 \right).
\]
Case iv. When \( n \equiv 0 \pmod{2} \) and \( \delta(n) = 0 \), as before, we have
\[
D(n, 0) = \frac{1}{2}(\tau_e(n) - \tau_o(n)) = \frac{1}{2}\left((\beta - 1)\tau\left(\frac{n}{2^\beta}\right)\right).
\]

Thus we have obtained a proof of (1).

Now we turn to the derivation of (2). Suppose that \( n \) can be written as a difference of two triangular numbers. Then we have
\[
n = \frac{y(y + 1)}{2} - \frac{x(x + 1)}{2} = \frac{(x + d)(x + d + 1)}{2} - \frac{x(x + 1)}{2}.
\]

This gives
\[
\frac{2n}{d} - d = 2x + 1
\]
for some non-negative integer \( x \).

From this, we observe that \( T(n) \) equals the number of divisors \( d \) of \( 2n \) which satisfies the following:

1. \( d < \sqrt{2n} \);
2. Either \( d \) is odd or \( d = 2^{\beta + 1}k \) for some integer \( k \), where \( \beta \) as before denotes the highest power of 2 that divides \( n \).

Now again from the lines of proof of Theorem 5 in [2] we can write
\[
T(n) = \begin{cases} 
\frac{\tau(2n)}{2} - \frac{(\beta + 1 - 1)\tau\left(\frac{2n}{2^{\beta+1}}\right)}{2} & \text{if } \delta(2n) = 0; \\
\frac{\tau(2n) - 1}{2} - \frac{(\beta + 1 - 1)\tau\left(\frac{2n}{2^{\beta+1}}\right) - 1}{2} & \text{if } \delta(2n) = 1.
\end{cases}
\]

Above equality is equivalent to
\[
T(n) = \begin{cases} 
\frac{\tau_e(2n) + \tau_o(2n) - (\tau_e(2n) - \tau_o(2n))}{2} & \text{if } \delta(2n) = 0; \\
\frac{\tau_e(2n) + \tau_o(2n) - 1}{2} - \frac{\tau_e(2n) - \tau_o(2n) - 1}{2} & \text{if } \delta(2n) = 1.
\end{cases}
\]

Consequently, we get
\[
T(n) = \tau_o(2n) = \tau_o(n).
\]

Thus Nyblom’s expression for \( T(n) \) is re-established.

Now we turn to another result of Nyblom and Evans about \( p_2(n) \) mentioned in (6). Our derivation is based upon the following observation: if \( n \) can be written as a sum of the terms in arithmetic progression with common difference \( D \), then we have:
\[ n = a + (a + D) + \cdots + (a + (m-1)D) \]
\[ = ma + \frac{m(m-1)}{2}D. \]

This gives
\[ 2n = 2ma + m^2D - mD. \] (9)

When \( D = 2 \), we have
\[ \frac{n}{m} - m = a - 1. \]

Thus \( p_2(n) \) equals the number of divisors of \( n \) satisfying the above equation. Since \( a \) is positive, \( p_2(n) \) equals the number of divisors \( m \) of \( n \) satisfying the inequality \( m \leq \sqrt{n} \). One can see that if \( n \) is a non-square then the mapping \( m \rightarrow \frac{n}{m} \) is one-one and non-fixed. If \( n \) is a square number then the mapping \( m \rightarrow \frac{n}{m} \) is again one-one and non-fixed when \( m < \sqrt{n} \) and \( m \) fixes to itself under the map \( m \rightarrow \frac{n}{m} \) if \( m = \sqrt{n} \). These observations lead to the conclusion that

\[ p_2(n) = \begin{cases} 
\frac{\tau(n)}{2} & \text{if } \delta(n) = 0; \\
\frac{\tau(n) - 1}{2} + 1 & \text{if } \delta(n) = 1.
\end{cases} \] (10)

Thus the expression for \( p_2(n) \) due to Nyblom and Evans [7] follows.

Next we give an alternative interpretation for \( p_{2D}(n) \) and \( p_{2D+1}(n) \) in terms of a divisor-counting function which is the contention of the following result.

**Theorem 2.1.** Let \( n \) and \( D \) be positive integers. Then we have

1. \( p_{2D}(n) \) equals the number of divisors \( m \) of \( n \) such that \( m < \frac{D + \sqrt{D^2 + 4Dn}}{2D} \).
2. \( p_{2D+1}(n) \) equals the number of divisors \( m \) of \( 2n \) such that \( m < \frac{(D-\frac{1}{2}) + \sqrt{(D-\frac{1}{2})^2 + 2(2D-1)n}}{2D-1} \).

**Proof.** If common difference is \( 2D \) then from (9) it follows that
\[ \frac{n}{m} - (m - 1)D = a > 0. \] (11)

This can be put as a quadratic inequality
\[ m^2D - mD - n < 0. \]

After factoring we get
\[ \left( m - \frac{D - \sqrt{D^2 + 4Dn}}{2D} \right) \left( m - \frac{D + \sqrt{D^2 + 4Dn}}{2D} \right) < 0. \]

Since the first factor is always positive we have
\[ \left( m - \frac{D + \sqrt{D^2 + 4Dn}}{2D} \right) < 0. \]
Also from (9) we conclude that \( m \) must be a divisor of \( n \). Moreover every divisor \( m \) of \( n \) satisfying the above inequality contribute to the equality (11). Hence first part of the result follows. Similar approach will settle the second part.

**Definition 2.2.** Let \( n \) be a positive integer. By unrestricted partition of \( n \), we mean a non-increasing sequence of integers say \((a_1, a_2, \ldots, a_k)\) such that \( a_1 + a_2 + \cdots + a_k = n \). We use \( p_D(n) \) to denote the number of unrestricted partitions of \( n \) whose parts follows arithmetic progression with common difference \( D \).

**Theorem 2.3.** Let \( n \) be a positive integer. Then we have

\[
p_D(n) = \begin{cases} 
\tau(n) & \text{if } D \equiv 0 \pmod{2}; \\
2\tau_o(n) & \text{if } D \equiv 1 \pmod{2}.
\end{cases}
\] (12)

**Proof.** Assume that the common difference is \( 2D \), an even integer. Then from (9) it follows that

\[
\frac{n}{m} - (m - 1)D = a.
\] (13)

Since \( a \) can be any integer, each divisor \( m \) of \( n \) contribute an \( a \). Consequently, corresponding to each divisor \( m \) of \( n \) there exists an unrestricted partition of the said type and vice versa. Hence the first part of the result follows.

Assume that the common difference is \( 2D + 1 \), an odd integer. Then from (9) it follows that

\[
\frac{2n}{m} - (2D + 1)(m - 1) = 2a.
\] (14)

Choosing divisors \( m \) of \( 2n \) such that either \( m \) is odd or \( m \) is of the form \( m = 2^\beta(2r + 1) \), where \( \beta \) is the highest power of 2 that divides \( 2n \) will alone contribute an even integer \( 2a \) at the right side of (14). As one can see, the number of such divisors is \( \tau_o(2n) + \tau_o(2n) \), which is equal to \( 2\tau_o(n) \). Now the result follows.

**Remark 2.4.** It is interesting to note that the function \( p_D(n) \) is independent of the value of \( D \), whereas it is dependent on the parity of \( D \) and the factorisation of \( n \).

### 2.2 Other interpretations of the term \( D(n, k) \)

Now we confine to the expression \( D(n, k) \) when \( k \) exceeds 1.

**Case i.** Assume that \( k \equiv 1 \pmod{2} \). If \( n \) can be written as a difference of two members from the sequence \( p(n, k) \), then we have

\[
n = p(y, k) - p(x, k) \\
= p(x + d, k) - p(x, k) \\
= \frac{(x + d)(x + d + k)}{2} - \frac{x(x + k)}{2}.
\]

This implies that

\[
\frac{2n}{d} - d = 2x + k.
\] (15)

From this observation, we have the following result.
Theorem 2.5. Let $k \equiv 1 \pmod{2}$ be a positive integer and let $n$ be a positive integer. Then $D(n, k)$ counts the divisors $d$ of $2n$ such that $\frac{2n}{d} - d$ is an odd integer not less than $k$.

Corollary 2.6. Let $k \equiv 1 \pmod{2}$ be a positive integer and let $n$ be a positive integer such that: for each divisor $d$ of $2n$, $\frac{2n}{d} - d$ is an odd integer not less than $k$. Then $D(n, k) = D(n, 1)$.

Case ii. Assume that $k \equiv 0 \pmod{2}$. If $n$ can be written as a difference of two members from the sequence $p(n, k)$, then we have

$$n = p(y, k) - p(x, k) = p(x + d, k) - p(x, k) = (x + d)(x + d + k) - x(x + k).$$

This implies that

$$\frac{n}{d} - d = 2x + k. \tag{16}$$

From this observation, we have the following result.

Theorem 2.7. Let $k \equiv 0 \pmod{2}$ be a positive integer and let $n$ be a positive integer. Then $D(n, k)$ counts the divisors $d$ of $n$ such that $\frac{n}{d} - d$ is an even integer not less than $k$.

Corollary 2.8. Let $k \equiv 0 \pmod{2}$ be a positive integer and let $n$ be a positive integer such that: for each divisor $d$ of $n$, $\frac{n}{d} - d$ is an even integer not less than $k$. Then $D(n, k) = D(n, 0)$.

To mention some of the forthcoming results, some basic terminologies from partition theory are required.

Definition 2.9. Let $n$ be a positive integer. By a partition of $n$, we mean a sequence of non-increasing positive integers say $\pi = (a_1, a_2, \ldots, a_k)$ such that $a_1 + a_2 + \cdots + a_k = n$. Each $a_i$ is called a part of $\pi$ and the number of times part $a_i$ occurs is referred as the frequency of $a_i$. Each element in the set of parts of $\pi$ is called a size of $\pi$.

We observe that (15) can be put in the form: $2n = (f_1 + f_2)f_1 + f_1(f_2 + k)$ for some positive integers $f_1$ and $f_2$. This leads to the following result.

Theorem 2.10. Let $n$ be a positive integer and let $k \equiv 1 \pmod{2}$ be a positive integer. Then $D(n, k)$ equals the number of partitions of $2n$ with sizes $f_1 + f_2$ and $f_1$ with their respective frequencies $f_1$ and $f_2 + k$.

Similar observation on (16) gives the following result.

Theorem 2.11. Let $n$ be a positive integer and let $k \equiv 0 \pmod{2}$ be a positive integer. Then $D(n, k)$ equals the number of partitions of $n$ with sizes $f_1 + f_2$ and $f_1$ with their respective frequencies $f_1$ and $f_2 + k$.

We observe that: $p(m, 2k - 1) = k + (k + 1) + \cdots + (k + m - 1)$. Therefore, if $n$ can be written as a difference of two elements from $p(m, 2k - 1)$, then this difference gives a partition of $n$ with consecutive integers as parts with least part not less than $k$ and vice versa. This gives the following result.
Theorem 2.12. Let \( n \) be a positive integer. Then \( D(n, 2k - 1) \) equals the number of partitions of \( n \) with consecutive integers as parts with least part not less than \( k \).

Since \( p(m, 2k) = (2k + 1) + (2k + 3) + \cdots + (2k + 2m - 1) \), as in the previous case, we have the following result.

Theorem 2.13. Let \( n \) be a positive integer. Then \( D(n, 2k) \) equals the number of partitions of \( n \) with consecutive odd integers as parts with least part not less than \( 2k + 1 \).

Remark 2.14. From the above results we see that the following enumerations are equivalent and are equal with \( D(n, k) \) when \( k \) is an odd integer:

1. Number of divisors \( d \) of \( 2n \) such that \( \frac{2n}{d} - d \) is an odd integer not less than \( k \).
2. Number of partitions of \( 2n \) with sizes \( f_1 + f_2 \) and \( f_1 \) with their respective frequencies \( f_1 \) and \( f_2 + k \).
3. Number of partitions of \( n \) with consecutive integers as parts with least part not less than \( \frac{k + 1}{2} \).

Similarly following enumerations are equivalent and are equal with \( D(n, k) \) when \( k \) is even:

1. Number of divisors \( d \) of \( n \) such that \( \frac{n}{d} - d \) is an even integer not less than \( k \).
2. Number of partitions of \( n \) with sizes \( f_1 + f_2 \) and \( f_1 \) with their respective frequencies \( f_1 \) and \( f_2 + k \).
3. Number of partitions of \( n \) with consecutive odd integers as parts with least part not less than \( k + 1 \).

To realise the essence of Lemma 1.2 we need the following definition.

Definition 2.15. Let \( n \) and \( r \) be two positive integers and \( k \) be a non-negative integer. Denote by \( D(n, k, r) \), the number of ways \( n \) can be expressed as a difference of two elements from the sequence \( \{p(n, k)\}_{n \geq r+1} \).

As a consequence of Lemma 1.2, the function \( D(n, k) \) can be written in terms of \( D(n, k, r) \) which is the contention of the following result.

Theorem 2.16. Let \( n \) be a positive integer and \( k \) be a non-negative integer. Then we have

\[
D(n, k) = \begin{cases} 
D(n, 1, \frac{k - 1}{2}) & \text{if } k \equiv 1 \pmod{2}; \\
D(n, 0, \frac{k}{2}) & \text{if } k \equiv 0 \pmod{2}.
\end{cases}
\]  

(17)

Proof. Assume \( k \equiv 1 \pmod{2} \). In view of Lemma 1.2 we can write

\[
D(n, k) = D(n, k - 2, 1) = D(n, k - 4, 2)
\]

and so on. Then after the \( \frac{k - 1}{2} \) times of repeated application of Lemma 1.2 as above we get

\[
D(n, k) = D(n, 1, \frac{k - 1}{2}),
\]

which is the expected end. Similar application of Lemma 1.2 serves good in odd case. \( \square \)
3 Generalised version of Nyblom’s results

3.1 An interpretation for $D(n, A)$

Recall from Definition 1.4 that $D(n, A)$ is the number of ways $n$ can be written as a difference of two elements from the set $A$. We presumably take $\gcd(A) = 1$. For otherwise, integer $n$ which are non-multiples of $\gcd(A)$ cannot be expressed as a difference of elements from $A$. Following theorem gives an interpretation for $D(n, A)$ in terms of an integer partition enumeration.

**Theorem 3.1.** Let $n$ be a positive integer and let $P = \{a_1, a_2, \ldots \}$ be a set of positive integers with $\gcd(P) = 1$ and $a_1 < a_2 < \cdots$. Define $s_n = a_1 + a_2 + \cdots + a_n$ for every $n \geq 1$ and $A = \{s_1, s_2, \ldots \}$. Then $D(n, A)$ equals the number of partitions of $n$ with parts as consecutive elements of $P$.

**Proof.** If $n$ can be written as difference of two elements from $A$ then we have

$$n = s_k - s_r = a_{r+1} + a_{r+2} + \cdots + a_k$$

for some $k > r$. Thus this difference gives a representation of $n$ as a sum of consecutive members of $P$.

On the other side, if

$$n = a_{r+1} + a_{r+2} + \cdots + a_k$$

with $k > r$, then we can write

$$n = s_k - s_r.$$ 

Hence, the representation as a sum of consecutive elements of $P$ gives a representation of $n$ as a difference of elements of $A$. This correspondence establishes the result. □

**Corollary 3.2.** Let $n$ be a positive integer. Then we have

1. The number of ways $n$ can be written as a difference of squares of triangular numbers equals the number of ways $n$ can be written as the sum of consecutive cubes.

2. No prime can be expressed as a sum of consecutive cubes.

3. The number of ways $n$ can be expressed as a difference of two Fibonacci numbers equals the number partitions of $n$ with parts as consecutive Fibonacci numbers.

**Proof.** Statements 1. and 2. follows from the identity

$$1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

Statement 3. follows from the identity:

$$F_0 + F_1 + \cdots + F_n = F_{n+2} - 1.$$ □
3.2 Boundedness of $D(n, A)$ and Erdős Arithmetic Progression Conjecture

Erdős conjectured that, if $\sum_{a \in A} \frac{1}{a}$ diverges then $A$ contains arithmetic progression of arbitrary length. Now if we assume that the conjecture is true then the boundedness of $D(n, A)$ implies the convergence of $\sum_{a \in A} \frac{1}{a}$. For if $\sum_{a \in A} \frac{1}{a}$ is diverging, then since we assume the truthness of Erdős conjecture, for any given positive integer $k$ we have

$$\{a, a + d, a + 2d, \cdots, a + kd\} \subset A$$

for some positive integers $a$ and $d$. Consequently, $D(d, A) \geq k$. That is for any given positive integer $k$ we can find a positive integer $d$ such that $D(d, A) \geq k$. Thus $D(n, A)$ is unbounded.

Next we observe that the converse of the above statement need not be true; for we have the converging series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ but $D(n, \{1^2, 2^2, \cdots\})$ is not bounded. Based upon this discussion we state our closing result.

**Theorem 3.3.** If there exist a set of positive integer, say $A$, such that $\sum_{a \in A} \frac{1}{a}$ diverges and $D(n, A)$ is bounded, then the Erdős arithmetic progression conjecture fails.

**Remark 3.4.** If one finds a set of positive integers satisfying the hypothesis of the above theorem, then the Erdős conjecture will be laid to rest.

**References**

[1] Cook, R., & Sharp, D. (1995). Sums of arithmetic progressions, *Fibonacci Quart.*, 33, 218–221.

[2] Christopher, A. D. (2015). Partitions with Fixed Number of Sizes, *J. Integer Seq.*, 18, Article 15.11.5.

[3] Mason, T. E. (1912). On the representation of an integer as the sum of consecutive integers, *Amer. Math. Monthly*, 19 (3), 46–50.

[4] Munagi, A. O. (2010). Combinatorics of integer partitions in arithmetic progression, *Integers*, 10, 73–82.

[5] Nyblom, M. A. (2001). On the Representation of the Integers as a Difference of non-consecutive Triangular numbers, *Fibonacci Quart.*, 39 (3), 256–263.

[6] Nyblom, M. A. (2002). On the Representation of the Integers as a Difference of Squares, *Fibonacci Quart.*, 40 (3), 243–246.

[7] Nyblom M. A., & Evans, C. (2003). On the enumeration of partitions with summands in arithmetic progression, *Australas. J. Combin.*, 28, 149–159.