Interaction-assisted SU(2) adiabatic holonomies in Josephson phase qubits

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We propose a scheme for generating SU(2) adiabatic geometric phases in a circuit consisting of three capacitively coupled flux-biased Josephson phase qubits.

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I. INTRODUCTION

When slowly changing, external controls that govern adiabatic evolution of a quantum system return to their original values, the phase of system’s wave function acquires a purely geometric contribution independent of both system’s energy and duration of the adiabatic process. This $U(1)$ contribution, initially discovered by Berry [1], mathematically interpreted by Simon [2], and later generalized to non-abelian $U(N)$ operations by Wilczek and Zee [3], has been the source of many interesting developments in various fields of chemistry and physics, such as molecular and spin dynamics, optics, fractional quantum Hall effect, various branches of condensed matter physics [4], and others, such as, for example, the field of quantum information processing [5].

Of particular interest to us is the possibility of observing non-abelian geometric phases (SU(2), in our case) in superconducting circuits with Josephson junctions [6]. Such circuits are currently considered as promising candidates for scalable quantum computing architectures [7].

Even though in this paper we consider a definite physical system (the so-called capacitively coupled flux biased Josephson phase qubits [8]), our analysis may apply equally well to any quantum computing architecture whose dynamics is described by the Hamiltonian given in Eq. (1). This very simple Hamiltonian has properties that make the corresponding system ideally suitable for actual realization of non-abelian geometric phases. The most important of these is the presence of two degenerate subspaces, one of which, $V_1$, can support non-abelian phase, and the other, $V_2$, cannot. Experimentally then, $V_2$ may be used as fiducial, reference subspace relative to which various interferometric experiments involving $V_1$ could be performed.

To recall how geometric phases appear in quantum mechanics and to fix our notation, let us consider a quantum system whose Hamiltonian depends on the time $t$ via several controllable parameters $\lambda(t) \equiv \{\lambda^1(t), \lambda^2(t), \lambda^3(t), \ldots\}$, $\mathcal{H} = \mathcal{H}(\lambda(t))$. It is assumed that $\mathcal{H}(\lambda(t))$ forms an iso-degenerate family of Hamiltonians (no level crossings as $\lambda$ varies) [8]. Let the instantaneous energy eigenbasis $\chi_a(\lambda(t))$ within a given degenerate subspace be chosen,

\[\mathcal{H}(\lambda(t))\chi_a(\lambda(t)) = E(\lambda(t))\chi_a(\lambda(t)), \quad a = 1, 2, \ldots, N.\]

At $t = 0$, prepare the system in one of the energy eigenstates, $\psi_a(0) \equiv \chi_a(\alpha(0))$, and let it evolve according to the Schrödinger equation,

\[i\frac{d\psi_a(t)}{dt} = \mathcal{H}(\lambda(t))\psi_a(t).\]

Then, in accordance with the adiabatic theorem, at some later time $t$, the state $\psi_a(t)$ will be a linear superposition of various $\chi_b(\lambda(t))$ belonging to the same degenerate subspace. Therefore, in general,

\[\psi_a(t) \approx e^{-i}\int_0^t E(\lambda(t'))dt' \sum_b \chi_b(\lambda(t))U_{ba}(\lambda(t)),\]

where the first factor on the right is the usual dynamical phase, and $U(\lambda(t))$ is the unitary matrix representing additional, purely geometric “rotation.” The $U(\lambda(t))$ is the sought for non-abelian geometric phase. It is given by the path-ordered exponential,

\[U(\lambda(t)) = \mathcal{P} \exp \left\{ - \int_0^t \sum_i A_i(\lambda(t')) \lambda^i(t') \, dt' \right\},\]

where

\[[A_i(\lambda)]_{ab} = \langle \chi_c(\lambda) | \partial / \partial \lambda^i | \chi_b(\lambda) \rangle\]

is the so-called adiabatic connection, also known as the $a(N)$-valued gauge potential. In general, when $\lambda(t)$ traces a closed loop $C$ in the parameter space, the system picks up a nontrivial geometric phase (the holonomy),

\[U_C = \mathcal{P} \exp \left\{ - \oint_C \sum_i A_i(\lambda) \, d\lambda^i \right\} \neq 0,\]

independent of the speed with which the loop had actually been traversed (as long as it was done adiabatically). In the next section we describe a multi-qubit solid state quantum computing architecture capable of supporting such a phase.

II. CAPACITIVELY COUPLED TRIPARTITE SYSTEM AND ITS HOLONOMIES

One system that leads to nontrivial, SU(2) holonomies consists of three capacitively coupled Josephson phase...
qubits [9] whose effective Hamiltonian in the rotating wave approximation is given by (cf. [10])
\[ \mathcal{H}(B_x, B_y) = (1/2)|B_x(\sigma^+_x + \sigma^-_x) + B_y(\sigma^+_y + \sigma^-_y) + g(2\sigma^+_x \sigma^-_y + \sigma^+_y \sigma^-_x + \sigma^+_y \sigma^-_y) + J(\sigma^+_x \sigma^-_y + \sigma^-_y \sigma^+_x)], \]
(7)
where, in polar coordinates, \( B_x = B \cos \phi, B_y = B \sin \phi \). Here, qubits 1 and 2 experience the same magnetic field produced by identical microwave driving on both qubits. This microwave drive is the parameter \( \lambda = \{B, \phi\} \) that will undergo adiabatic change. For future use we define
\[ A \equiv (1/2)(-B^2 + 6g^2)/\sqrt{B^4 + 4g^4}, \]
\[ A' \equiv (1/2)(B^2 + 6g^2)/\sqrt{B^4 + 4g^4} > 0, \]
\[ K \equiv B^4 - 2g^2B^2 + 8g^4 > 0, \]
\[ L \equiv (-B^2 + 4g^2)/\sqrt{B^4 + 4g^4}. \]
(8)
The corresponding Hamiltonian matrix \( \mathcal{H}(B, \phi) \) has six different eigenvalues, two of which are doubly degenerate. The non-degenerate ones are \( E_{5,6,7,8} = (1/2)[J \pm \sqrt{T^2 + 4(2g^2 + B^2 \mp 2Bg)}] \). Their corresponding eigenkets are rather complicated, each leading to the same \( U(1) \) Berry phase \( e^{i\pi n} = e^{-3i\pi} \) on a single precession of the field, and will not be needed in what follows. The degenerate eigenvalues that will be needed are \( E_{3,4} = -J \), with
\[ |\chi_3⟩ = (001) - (010)/\sqrt{2}, \quad |\chi_4⟩ = (110) - (101)/\sqrt{2}, \]
(9)
and \( E_{1,2} = 0 \), with
\[ |\chi_1(B, \phi)⟩ = \{(-B^2 + 4g^2)|000⟩ + B^2e^{2i\phi}|011⟩ - 2gBe^{i\phi}|100⟩\}/\sqrt{2K}, \]
\[ |\chi_2(B, \phi)⟩ = \{2gB^3e^{-3i\phi}|000⟩ - 4g^3Be^{-i\phi}|011⟩ - B^2(2g^2 - 2g^2)e^{-2i\phi}|100⟩ + K|111⟩\}/\sqrt{2K(B^4 + 4g^4)}. \]
(10)
A straightforward calculation based on Eq. [10] then gives the adiabatic connection within the \( |\chi_{1,2}⟩ \) subspace,
\[ dA(B, \phi) = i\{2B^2(2g^2)/K(\sigma^z/2) - (2gB^3A/K)(\sigma^x \cos 3\phi + \sigma^y \sin 3\phi)\} d\phi - i(2g^2A'/K)(\sigma^x \sin 3\phi - \sigma^y \cos 3\phi) dB. \]
(11)
Here, the Pauli matrices operate within the \( |\chi_{1,2}⟩ \) “geometric” subspace and are different from the sigma-matrices originally used to describe the qubits in the circuit.

Notice that in the course of adiabatic evolution we are not allowed to take any of the limits \( J \to 0, g \to 0 \), or \( B \to 2g \), since each breaks the requirement of isodegeneracy. Also notice that \( J \) appears nowhere in \( |\chi_{1,2}⟩ \), \( E_{1,2} \), or \( dA \), so the geometric phase is not sensitive to the precise value of \( J \). Nevertheless, the presence of \( J \neq 0 \) is crucial here: it guarantees separation of the \( |\chi_{1,2}⟩ \) subspace from the rest of the Hilbert space. Such (at least, partial) robustness against possible imperfections in the coupling is important since our ability to implement multidimensional geometric phases relies on the exact degeneracy of the underlying energy subspace. However, in order to have the exact degeneracy of \( E_{1,2} \), we also require that the coupling \( g \) be the same for \( 1 \leftrightarrow 3 \) and \( 2 \leftrightarrow 3 \). This is a very important requirement, not easily achievable experimentally (unless tunable coupling is available). Additionally, since \( B \) does appear explicitly in Eq. [10], we require good control over \( B \), which is not a problem experimentally. In a typical experiment, even for long times, the magnetic field (the microwave drive) can easily be controlled [11].

We also point out that the sequence of operations generating non-abelian phases must be short compared to the decoherence and dephasing times. On the other hand, system’s evolution must remain adiabatic compared to the couplings \( g \) and \( J \) and the magnetic field \( B \). In a typical experiment, \( g \) is on the order of 20 MHz, or \( 1/g = 50 \) ns, so the entire operation would be several hundred nanoseconds which is comparable to the presently achievable decoherence times. However, with stronger couplings up to \( g = 100 \) MHz and \( 1/g = 10 \) ns it is possible to perform an adiabatic operation in several tens of ns [11].

Let us now fix a basis \( |\chi_{1,2}(B_0)⟩ \) at \( \phi = 0 \) and initialize the system in some state \( |ϕ₀⟩ = α|\chi_{1}(B_0)⟩ + β|\chi_{2}(B_0)⟩ \). We may then trace a closed loop \( C \) in the parameter space by performing the following sequence of operations:

1. Begin, if needed, by adiabatically changing the field from \( B_0 \) to \( B_1 \) along the radial direction in the \( (x, y) \)-plane (no precession). The system will pick up a phase,
\[ U_Y = e^{-iθ(B_0, B_1)/(σ^y/2)}, \]
\[ θ(B_0, B_1) = 2 \int_{B_0}^{B_1} (2gB^2A'/K) dB. \]
(12)

2. Perform adiabatic precession at \( B_1 \),
\[ B_x(t) = B_1 \cos ω_0 t, \quad B_y(t) = B_1 \sin ω_0 t. \]
(13)
If the field makes \( n \) revolutions of total duration \( T_n = 2πn/ω_0 \), the accumulated phase will be
\[ U_{xz} = e^{-3iπnσ^z} \{ \cos(2Aπn) \]
\[ + i \sin(2Aπn) [(2gB^3/K)σ^x + (L/K)σ^z] \}_{B = B_1}. \]
(14)
which represents a rotation by angle \( α_n = 4Aπn \mid B = B_1 \) about the axis \( \hat{n} = (-1)^{n-1}[2gB^3/K, 0, L/K]|B = B_1 \) that lies in the \( (x, z) \)-plane of the Bloch sphere (constructed with respect to \( |\chi_{1,2}⟩ \)). The set of gates that can be generated this way is shown in Fig. [1].
FIG. 1: The one-parameter family of gates in the $|\chi_{1,2}(B)\rangle$ basis that can be generated in a single precession using non-abelian holonomy $U_{XZ}(B)$ of Eq. (14) at $n = 1$. Panel (a): $B$-dependence of the rotation angle. Panel (b): $B$-dependence of the rotation axis. To implement a given gate (such as a Hadamard, NOT, PHASE, etc.), a specially designed computational basis, different from $|\chi_{1,2}(B)\rangle$, must be chosen. See Sec. III for relevant examples.

(3) Bring the field back to its original value. This will result in additional contribution to the phase, $(U_Y)^{-1}$. The full holonomy is then

$$U_C = (U_Y)^{-1}U_{XZ}U_Y = U_{XZ}(U_Y)^2. \quad (15)$$

III. DESIGNING GEOMETRIC GATES

To demonstrate how this non-abelian scheme works in simple situations, let us find a basis in which a useful gate, such as a Hadamard, can be made in a single revolution of the effective magnetic field. We want $U_Y = 1$, $n = 1$, and $U_C = U_{XZ} \equiv H$, possibly up to an insignificant U(1) factor. First, notice that the choice $|\downarrow, \uparrow\rangle = |\chi_{1,2}(B)\rangle$ does not work as can be seen from Fig. [ill]. We therefore introduce a new basis,

$$|\downarrow\rangle := \cos \beta |\chi_1(B)\rangle + \sin \beta |\chi_2(B)\rangle,$$

$$|\uparrow\rangle := -\sin \beta |\chi_1(B)\rangle + \cos \beta |\chi_2(B)\rangle, \quad (16)$$

with some unknown, $B$-dependent angle $\beta$. Setting $n = 1$ and changing to the new basis gives

$$U_{XZ}(B, \beta) = -\cos(2A\pi) - i \sin(2A\pi) \times$$

$$\left\{ [(L/K) \cos 2\beta + (2gB^3/K) \sin 2\beta]_{\sigma^x} \right.$$  $$\left. - [(L/K) \sin 2\beta - (2gB^3/K) \cos 2\beta]_{\sigma^x} \right\}. \quad (17)$$

It is then straightforward to check that by choosing

$$B = g \left\{ -12 + \sqrt{39 - 4m_1(m_1 + 1)(1 + 2m_1)^2} \right\} / 4m_1(m_1 + 1) - 3$$

$$- \frac{m_2\pi}{2}, \quad (18)$$

with $m_1 \in \{0, 1, 2\}$, and thus $B/g \approx 1.9587, 1.3716, 0.8375$, and $m_2 \in \mathbb{Z}$, we get a Hadamard gate, $U_{XZ} = (\pm i)H$. On the other hand, by choosing same $B$ and setting $\beta_{\text{NOT}} = \beta_H - \pi/8$, we get another basis, in which $U_{XZ} = (\pm i)\text{NOT}$. Other important gates, such as $\sqrt{\text{NOT}}$ and various PHASE gates, can also be generated in a similar manner.

We emphasize that the specially designed basis is only “special” in the sense that it allows a given gate (say, a Hadamard) to be generated by following a particularly simple path in the parameter space — here, by making a single turn of the adiabatically precessing in the $(x, y)$-plane field.

IV. CONCLUSION

In summary, we have shown how capacitively coupled tripartite superconducting qubit system may support non-commuting SU(2) geometric holonomies. The
non-abelian character of the holonomies is due to the inter-qubit coupling. If we treat the relevant degenerate subspace as a “geometric” qubit, then, with a proper choice of the computational basis, any of the standard single-qubit gates used in quantum information processing can be generated by the adiabatic transport alone.

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