NONPARAMETRIC INFERENCE ON LÉVY MEASURES OF LÉVY-DRIVEN ORNSTEIN-UHLENBECK PROCESSES UNDER DISCRETE OBSERVATIONS

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Abstract. In this paper, we study nonparametric inference on a stationary Lévy-driven Ornstein-Uhlenbeck (OU) process \( X = (X_t)_{t \geq 0} \) with a compound Poisson subordinator. We propose a new spectral estimator for the Lévy measure of the Lévy-driven OU process \( X \) under macroscopic observations. We derive multivariate central limit theorems for the estimator over a finite number of design points. We also derive high-dimensional central limit theorems for the estimator in the case that the number of design points increases as the sample size increases. Built upon these asymptotic results, we develop methods to construct confidence bands for the Lévy measure and propose a practical method for bandwidth selection.

1. Introduction

Given a positive number \( \lambda \) and an increasing Lévy process \( J = (J_t)_{t \geq 0} \) without drift component, an Ornstein-Uhlenbeck (OU) process \( X = (X_t)_{t \geq 0} \) driven by \( J \) is defined by a solution to the following stochastic differential equation

\[
\frac{dX_t}{dt} = -\lambda X_t dt + dJ_t, \quad t \geq 0.
\]

(1.1)

We refer to [Sato (1999)] and [Bertoin (1996)] as standard references on Lévy processes. In this paper, we consider nonparametric inference on the Lévy measure \( \nu \) of the back-driving Lévy process \( J \) in (1.1) from discrete observations of \( X \). The Lévy measure \( \nu \) is a Borel measure on \([0, \infty)\) such that

\[
\int_0^\infty (1 \wedge x^2) \nu(dx) < \infty.
\]

We will assume that \( X \) is stationary. If \( \int_{(2, \infty)} \log x \nu(dx) < \infty \), then the unique stationary solution of (1.1) exists (see Theorem 17.5 and Corollary 17.9 in [Sato (1999)]), and the stationary distribution \( \pi \) of \( X \) is self-decomposable with the characteristic function

\[
\varphi(t) = \int_R e^{itx} \pi(dx) = \exp \left( \int_0^\infty \left( e^{itx} - 1 \right) \frac{k(x)}{x} dx \right),
\]

(1.2)

where \( k(x) = \nu((x, \infty)) 1_{[0, \infty)} \).

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In this paper, we focus on the case that the Lévy process \( J \) in (1.1) is a compound Poisson process, that is, \( J \) is of the form
\[
J_t = \sum_{j=1}^{N_t} U_j, \quad t \geq 0,
\]
where \( N = (N_t)_{t \geq 0} \) is a Poisson process with intensity \( \alpha > 0 \) and \( \{U_j\}_{j \geq 1} \) is a sequence of independent and identically distributed positive-valued random variables with common distribution \( F \). In this case, \( J_t \) has a characteristic function of the form
\[
\phi_{J_t}(u) = E[e^{iuJ_t}] = \exp \left( t\alpha \int_0^\infty (e^{iux} - 1)F(dx) \right)
\]
and the Lévy measure is given by \( \nu(dx) = \alpha F(dx) \). We also work with macroscopic observation set up, that is, we have discrete observations \( X_{\Delta}, X_{2\Delta}, \ldots, X_{n\Delta} \) at frequency \( 1/\Delta > 0 \) with \( \Delta = \Delta_n \to \infty \) and \( \Delta_n/n \to 0 \) as \( n \to \infty \). This is a technical condition to make the dependence among observations \( \{X_{j\Delta}\}_{j=1}^n \) asymptotically negligible. Masry (1993b) investigates deconvolution problems for mixing sequences. The author assumes that for a mixing sequence \( \{\tilde{X}_j\}_{j \geq 0} \), the joint densities \( p(x_1, x_{j+1}) \) of \( \tilde{X}_1 \) and \( \tilde{X}_{j+1} \) are uniformly bounded for any \( j \geq 1 \) and \( x_1, x_{j+1} \in \mathbb{R} \) to show the asymptotic independence of their estimators at different design points. Although we also observe a \( \beta \)-mixing sequence \( \{X_{j\Delta}\} \) (see Remark 3.1 for details on the \( \beta \)-mixing property of \( \{X_{j\Delta}\} \)), we cannot assume such a condition in the paper directly in our situation. Since the transition probability \( P_t(x,dy) \) of \( X \) has a point mass at \( y = e^{-\lambda t}x \), \( P_t(x,\cdot) \) does not have a transition density function. Therefore, to avoid such a problem, we consider the macroscopic regimes in this paper.

The goal of this paper is to develop nonparametric inference on the Lévy measure of Lévy-driven Ornstein-Uhlenbeck process (1.1). For this, we propose a spectral (or Fourier-based) estimator for the \( k \)-function and derive a multivariate central limit theorem for the estimator over finite design points. As extension of the result, we also derive high-dimensional central limit theorems for the estimator in the case that design points over a compact interval included in \( (0,\infty) \) increases as the sample size \( n \) goes to infinity. Built upon those limit theorems, we develop methods for implementing confidence bands for the \( k \)-function. This kind of method to construct “asymptotic” uniform confidence bands is also proposed in Horowitz and Lee (2012). Since confidence bands provide a simple graphical description of the accuracy of a nonparametric curve estimator quantifying uncertainties of the estimator simultaneously over design points, they are practically important in statistical analysis. Moreover, we propose a practical method for bandwidth selection inspired by the idea developed in Bissantz et al. (2007) on bandwidth selection in density deconvolution. To the best of our knowledge, this is the first paper to establish limit theorems for nonparametric estimators for the Lévy measure of Lévy-driven OU processes.

Lévy-driven OU processes are widely used in modeling phenomena where random events occur at random discrete times. See, for example, Albrecher et al. (2001), Kella and Stadje (2001).
and Noven et al. (2015) for applications to insurance, dam theory, and rainfall models. Several authors investigate parametric inference on Lévy-driven OU processes driven by subordinators. We refer to Hu and Long (2009), Masuda (2010) and Mai (2014) under the high-frequency set up (i.e., $\Delta = \Delta_n \to 0$ and $n\Delta_n \to \infty$ as $n \to \infty$) and Brockwell et al. (2007) under the low-frequency set up (i.e., $\Delta > 0$ is fixed). There are also a large number of studies on parametric and nonparametric estimation and inference on Lévy processes. We refer to Buchmann and Gründl (2003) and Brouste and Masuda (2018) as important and recent contribution in the literature on parametric inference on Lévy processes. We can also find an overview on recent developments on parametric inference on Lévy processes in Masuda (2015). Some authors have studied statistical inference on Lévy process under macroscopic observations. Duval and Hoffmann (2011) investigates statistical inference on a compound Poisson process under three kinds of time scales: high-frequency scale, low-frequency scale, and macroscopic scale. Duval (2014) studies statistical inference on compound Poisson processes under macroscopic observations. Duval and Kappus (2018) is also a recent paper on nonparametric estimation on compound Poisson processes under macroscopic observations. Kappus and Reiβ (2010) studies nonparametric estimation on Lévy densities under the high-frequency set up. As recent contributions on nonparametric inference on Lévy process under high-frequency set up, we refer to Figueroa-López (2011), Vetter (2014), Konakov and Panov (2016), Nickl et al. (2016), and Kato and Kurisu (2017). We mention van Es et al. (2007), Gugushvili (2009, 2012), Neumann and Reiβ (2009), Kappus and Reiβ (2010), Belomestny (2011), Duval (2013), Kappus (2014), Belomestny and Reiβ (2015), and Belomestny and Schoenmakers (2016) as recent papers on nonparametric estimation of Lévy densities. We also mention Nickl and Reiβ (2012) and Coca (2018) as papers on inference on Lévy measures under the low-frequency set up. Belomestny et al. (2017) studies nonparametric estimation on Lévy measures of moving average Lévy processes under low-frequency observations. Bücher and Vetter (2013), Bücher et al. (2017), and Hoffmann and Vetter (2017) study nonparametric inference on Lévy measures of an Itô semimartingales with Lévy jumps under high-frequency observations. Jongbloed et al. (2005) and Ilhe et al. (2015) investigate nonparametric estimation of the Lévy-driven OU processes. Jongbloed et al. (2005) derives consistency of their estimator for a class of Lévy-driven OU processes which include compound Poisson-driven OU processes. Ilhe et al. (2015) establishes consistency of their estimator of the Lévy density of (1.1) with compound Poisson subordinator in uniform norm at a polynomial rate. However, they do not derive limit distributions of their estimators.

The analysis of the present paper is related to deconvolution problems for mixing sequence. Masry (1991, 1993a,b) investigate probability density deconvolution problems for $\alpha$-mixing sequences and they derive convergence rates and asymptotic distributions of deconvolution estimators. Since the Lévy-driven OU process (1.1) is $\beta$-mixing under some conditions (see Masuda (2004) for details), our analysis can be interpreted as a deconvolution problem for a $\beta$-mixing
sequence. However, we need non-trivial analysis since we have to take account of additional structures which come from the properties of the compound Poisson-driven OU process.

The estimation problem of Lévy measures is generally ill-posed in the sense of inverse problems and the ill-posedness is induced by the decay of the characteristic function of a Lévy process. We refer to Neumann and Reiß (2009) as the seminal work in which such explanation is given for the first time. In our case, the ill-posedness is induced by the decay of the characteristic function of the stationary distribution $\pi$ of the Lévy-driven OU (1.1). In this sense the problem in this paper is a (nonlinear) inverse problem. Trabs (2014a) investigates conditions that a self-decomposable distribution is nearly ordinary smooth, that is, the characteristic function of the self-decomposable distribution decays polynomially at infinity up to a logarithmic factor. Trabs (2014b) applies those results to nonparametric calibration of self-decomposable Lévy models from option prices. As a refinement of a result for a special case in Trabs (2014a), we will show that the characteristic function of a self-decomposable distribution is regularly varying at infinity with some index $\alpha > 0$. This enables us to derive asymptotic distributions of the spectral estimator proposed in this paper.

Kato and Sasaki (2018) is a recent contribution in the literature on the construction of uniform confidence bands in probability density deconvolution problems for independent and identically distributed observations. They develop methods for constructing uniform confidence bands built on applications of intermediate Gaussian approximation theorems developed in Chernozhukov et al. (2014a,b, 2015, 2016) and provides multiplier bootstrap methods for implementing uniform confidence bands. Kato and Kurisu (2017) also develops confidence bands for Lévy densities based on intermediate Gaussian and multiplier bootstrap approximation theorems. Our analysis is related to these papers but we adopt different methods for the construction of confidence bands. We derive high-dimensional central limit theorems based on intermediate Gaussian approximation for $\beta$-mixing process. We can show that the variance-covariance matrix of the Gaussian random vector appearing in multivariate and high-dimensional central limit theorems is the identity matrix. Therefore, we do not need bootstrap methods to compute critical values of confidence bands.

The rest of the paper is organized as follows. In Section 2, we define a spectral estimator for the $k$-function. We give a multivariate central limit theorem of the spectral estimator in Section 3 and high-dimensional central limit theorems for the estimator are also given in Section 4. Procedures for implementing confidence bands are described in Section 5. In Section 6, we propose a practical method for bandwidth selection and report simulation results to study finite sample performance of the spectral estimator. All proofs are collected in Appendix.

1.1. Notation. For any non-empty set $T$ and any (complex-valued) function $f$ on $T$, let $\|f\|_T = \sup_{t \in T} |f(t)|$, and for $T = \mathbb{R}$, let $\|f\|_p = (\int_{\mathbb{R}} |f(x)|^p dx)^{1/p}$ for $p > 0$. For any positive sequences $a_n, b_n$, we write $a_n \lesssim b_n$ if there is a constant $C > 0$ independent of $n$ such that $a_n \leq C b_n$ for
all $n$, $a_n \sim b_n$ if $a_n \leq b_n$ and $b_n \leq a_n$, and $a_n \ll b_n$ if $a_n/b_n \to 0$ as $n \to \infty$. For $a, b \in \mathbb{R}$, let $a \vee b = \max(a, b)$. For $a \in \mathbb{R}$ and $b > 0$, we use the shorthand notation $[a \pm b] = [a - b, a + b]$. The transpose of a vector $x$ is denoted by $x^\top$. We use the notation $d$ as convergence in distribution. For random variables $X$ and $Y$, we write $X \overset{d}{=} Y$ if they have the same distribution. $N(\mu, \Sigma)$ denotes a (multivariate) normal distribution with a mean $\mu$ and a variance-covariance matrix $\Sigma$.

2. Estimation of the $k$-function

In this section, we introduce a spectral estimator for the Lévy measure ($k$-function) of the Lévy-driven Ornstein-Uhlenbeck process (I.1). First, we consider a symmetrized version of the $k$-function, that is,

$$k^+_2(x) = \begin{cases} k(x) & \text{if } x \geq 0, \\ k(-x) & \text{if } x < 0, \end{cases}$$

A simple calculation yields that

$$\frac{1}{\varphi(-t)} = \exp \left( \int_{-\infty}^0 (e^{itx} - 1) \frac{k(-x)}{x} dx \right).$$

Therefore, we have that

$$\varphi^+_2(t) := \frac{\varphi(t)}{\varphi(-t)} = \exp \left( \int_{\mathbb{R}} (e^{itx} - 1) \frac{k^+_2(x)}{x} dx \right),$$

$$\varphi'_2(t) = \frac{\varphi'(t) \varphi(-t) + \varphi(t) \varphi'(-t)}{\varphi^2(-t)} = \frac{1}{\varphi(-t)} \varphi'(t) - \left( \frac{1}{\varphi(-t)} \right)' \varphi(t) = i \left( \int_{\mathbb{R}} e^{itx} k^+_2(x) dx \right) \varphi_2(t).$$

This formally yields that

$$k^+_2(x) = \frac{-i}{2\pi} \int_{\mathbb{R}} e^{-itx} \frac{\varphi'_2(t)}{\varphi_2(t)} dt.$$ 

Let

$$\hat{\varphi}(u) = \frac{1}{n} \sum_{j=1}^n e^{jua} \varphi^\Delta_j, \quad \varphi_\theta(u) = \frac{i}{n} \sum_{j=1}^n X_j^\Delta e^{jua} 1\{|X_j^\Delta| \leq \theta_n\}.$$ 

Here, $\theta_n$ is a sequence of constants such that $\theta_n \to \infty$ as $n \to \infty$ (in the rest of this paper, we set $\theta_n \sim n^{1/2} (\log n)^{-3}$). Let $W : \mathbb{R} \to \mathbb{R}$ be an integrable (kernel) function such that $\int_{\mathbb{R}} W(x) dx = 1$ and its Fourier transform $\varphi_W$ is supported in $[-1, 1]$ (i.e., $\varphi_W(u) = 0$ for all $|u| > 1$). Then the spectral estimator for $k$ at $x > 0$ is defined by

$$\hat{k}_2(x) = \frac{-i}{2\pi} \int_{\mathbb{R}} e^{-itx} \frac{\hat{\varphi}'_2(t)}{\hat{\varphi}_2(t)} \varphi_W(th) dt,$$

where $h = h_n$ is a sequence of positive constants (bandwidths) such that $h_n \to 0$ as $n \to \infty$, and

$$\hat{\varphi}_2(t) = \frac{\hat{\varphi}(t)}{\hat{\varphi}(-t)}; \quad \hat{\varphi}'_2(t) = \frac{1}{\hat{\varphi}(-t)} \hat{\varphi}'_\theta(t) + \frac{\hat{\varphi}'_\theta(-t)}{\hat{\varphi}(-t)} \hat{\varphi}(t).$$

In the following sections we develop central limit theorems for $\hat{k}$. 
Remark 2.1. We need the truncation in $\hat{\varphi}_{n,a}$ to show Lemma A.2 in Appendix A by applying an exponential inequality for bounded mixing sequences. See also Remark 3.2 in Belomestny (2010) and the proof of Proposition 9.4 in the same paper.

Remark 2.2. For a complex value $a$, let $\overline{a}$ be the complex conjugate of $a$. We note that $\hat{k}_2$ is real-valued. In fact, since $\hat{\varphi}'_2(t) = -\hat{\varphi}'_2(-t)$ and $\hat{\varphi}_2(t) = \hat{\varphi}_2(-t)$, by a change of variables, we have that

$$\hat{k}_2(x) = \frac{i}{2\pi} \int_{\mathbb{R}} e^{itx} \frac{\hat{\varphi}'_2(t)}{\hat{\varphi}_2(t)} \varphi_W(th)dt = -\frac{i}{2\pi} \int_{\mathbb{R}} e^{itx} \frac{\hat{\varphi}'_2(-t)}{\hat{\varphi}_2(-t)} \varphi_W(-th)dt = \hat{k}_2(x).$$

Remark 2.3. Another natural estimator for $k$ at $x > 0$ would be

$$\hat{k}_0(x) = \frac{-i}{2\pi} \int_{\mathbb{R}} e^{-itx} \frac{\varphi'(t)}{\varphi(t)} \varphi_W(th)dt$$

but this estimator have large bias than $\hat{k}_2$. We need to symmetrize the $k$-function to use (global) regularity of $k_2$ around the origin for suitable bound of the deterministic bias. Note that $k_2$ is continuous at the origin and if $k_2$ has bounded $r$th derivative on $\mathbb{R}$ for some $r \geq 0$, the deterministic bias of $\hat{k}_2$ is $O(h^r)$ (Lemma A.9 in Appendix A). However, the deterministic bias of $\hat{k}_0$ is $O(h)$ because of the discontinuity of $k$ at the origin. See also Remark 3.1 for discussion on the regularity condition of $k_2$.

3. Multivariate Central Limit Theorem

In this section we present a multivariate central limit theorem for $\hat{k}_2$.

Assumption 3.1. We assume the following conditions.

(i) $\int_0^\infty (1 \vee |x|^{2+\epsilon})k(x)dx < \infty$ for some $\epsilon > 0$.

(ii) $k(0) = \nu((0,\infty)) = \alpha$ and $2 < \alpha < \infty$.

(iii) Let $r > 1/2$, and let $p$ be the integer such that $p < r \leq p + 1$. The function $k_2$ is $p$-times differentiable, and $k_2^{(p)}$ is $(r-p)$-Hölder continuous, that is,

$$\sup_{x,y \in \mathbb{R}, x \neq y} \frac{|k_2^{(p)}(x) - k_2^{(p)}(y)|}{|x-y|^{r-p}} < \infty.$$

(iv) $|\varphi_k(u)| \lesssim (1 + |u|)^{-1}$ and $|\varphi'_k(u)| \vee |\varphi''_k(u)| \lesssim (1 + |u|)^{-2}$, where $\varphi_k(= \varphi/(i\varphi))$ is the Fourier transform of $k$.

(v) Let $W : \mathbb{R} \to \mathbb{R}$ be an integrable function such that

$$\begin{cases}
\int_{\mathbb{R}} W(x)dx = 1, \int_{\mathbb{R}} |x|^{p+1}|W(x)|dx < \infty, \\
\int_{\mathbb{R}} x^\ell W(x)dx = 0, \ell = 1, \ldots, p, \\
\varphi_W(u) = 0, \forall |u| > 1,
\end{cases}$$

$\varphi_W$ is three-times continuously differentiable,
where $\varphi_W$ is the Fourier transform of $W$.

(vi) $\Delta = \Delta_n \geq \frac{5C_0}{4\beta_1(2+2\alpha-\delta)} \log n$, $n/\Delta \to \infty$, and

\[
\left(\frac{\log n}{n}\right)^{1/(2+2\alpha-\delta)} \ll h \ll \left(\frac{1}{n \log n}\right)^{1/(1+2r+2\alpha-\delta)}
\]

for some positive constant $C_0$ and $\delta \in (0, 1/12)$ as $n \to \infty$. Here, $\beta_1$ is a positive constant which appears in the mixing coefficient of $X = (X_t)_{t \geq 0}$ (Conditions (i) and (ii) implies that $X$ is exponentially $\beta$-mixing with $\beta$-mixing coefficient $\beta_X(t) = O(e^{-\beta_1 t})$ for some $\beta_1 > 0$. See also the following remark).

**Remark 3.1.** Conditions (i) and (ii) imply that the stationary distribution $\pi$ has a bounded continuous density (we also denote the density by $\pi$) such that $\|\pi\|_{\mathbb{R}} \lesssim 1$ and $\int_{\mathbb{R}} |x| \pi(dx) < \infty$ (see Lemma A.1). In this case, the stationary Lévy-driven Ornstein-Uhlenbeck process defined by (1.1) is exponentially $\beta$-mixing (Theorem 4.3 in [Masuda (2004)]), that is, the $\beta$-mixing coefficients for the stationary continuous-time Markov process $X$

\[
\beta_X(t) = \int_{\mathbb{R}} \|P_t(x, \cdot) - \pi(\cdot)\|_{TV} \pi(dx), \; t > 0
\]

(this representation follows from Proposition 1 in [Davydov (1973)]) satisfies $\beta_X(t) = O(e^{-\beta_1 t})$ for some $\beta_1 > 0$. Here, $P_t(x, \cdot)$ is the transition probability of the Lévy-driven OU (1.1) and $\| \cdot \|_{TV}$ is the total variation norm.

Condition (iii) is concerned with smoothness of $k_z$ and this condition is used to obtain a suitable bound of the deterministic bias of the estimator $\| [k_z \ast (h^{-1} W(\cdot/h))] - k_z \|_{\mathbb{R}}$. For a special case $(1/2 < r \leq 2)$, we can replace Condition (iii) in Assumption 3.1 (global Hölder continuity) with a local Hölder continuity of $k_z$ on $I^{\epsilon_0} = \{ y \in \mathbb{R} : |x - y| < \epsilon_0, \forall x \in I \}$ which does not include the origin. In fact, if $1/2 < r \leq 2$, by taking symmetric second order kernel function $W$, we have that for any $x \in I$

\[
\left| \int_{\mathbb{R}} \{ k_2(x - yh) - k_2(x) \} W(y)dy \right|
\]

\[
= \int_{|y| \leq \epsilon_0 h^{-1}} \{ k_2(x - yh) - k_2(x) \} W(y)dy + \int_{|y| > \epsilon_0 h^{-1}} \{ k_2(x - yh) - k_2(x) \} W(y)dy
\]

\[
\leq \int_{|y| \leq \epsilon_0 h^{-1}} \left[ k_2(x - yh) - k_2(x) - \sum_{\ell=1}^p \frac{k_2^{(\ell)}(x)}{\ell!} (-yh)^\ell \right] W(y)dy + 2\|k\|_R \int_{|y| > \epsilon_0 h^{-1}} |W(y)|dy
\]

\[
\leq H_0 h^r \int_{\mathbb{R}} |y|^r |W(y)|dy + \frac{2h^2 \|k\|_R}{\epsilon_0^2} \int_{\mathbb{R}} |y|^2 |W(y)|dy \lesssim h^r,
\]

where $H_0 := \sup_{x, y \in I^{\epsilon_0}, x \neq y} \frac{|k^{(\ell)}(x) - k^{(\ell)}(y)|}{|x - y|^{r - \ell}} < \infty$, $\sum_{\ell=1}^0 = 0$ and $0! = 1$ by convention. We note that $k_z = k$ on $I^{\epsilon_0}$. So we can bound $\| [k_2 \ast (h^{-1} W(\cdot/h))] - k_2 \|_R = \| [k \ast (h^{-1} W(\cdot/h))] - k \|_R$. However, for $r > 2$, if would be difficult to weaken the global Hölder continuity assumption on
Since higher order properties of the (symmetric) kernel function \( W \) is not satisfied in general. Therefore we assume Condition (iii) in Assumption 3.1 in this paper.

Condition (iv) is satisfied if \( k \) is two-times continuously differentiable on \((0, \infty)\) and \( \int_0^\infty \{|k(x)| + |xk'(x)| + |x^2k''(x)|\}dx < \infty \). In fact, by Condition (i), we have that \( |\varphi^{(p)}(u)| \lesssim 1 \) for \( p = 0, 1, 2, \) and by integration-by-parts and the Riemann-Lebesgue theorem, we also have that

\[
|\varphi_k(u)| = \left| \int_0^\infty e^{iux}k(x)dx \right| = \left| k(0+) - \frac{1}{iu} \int_0^\infty e^{iux}k'(x)dx \right| \lesssim \frac{1}{|u|},
\]

\[
|\varphi'_k(u)| = \frac{1}{u} \varphi_k(u) + \frac{1}{iu^2} \int_0^\infty e^{iux}(k'(x) + xk''(x))dx \lesssim \frac{1}{u^2},
\]

\[
|\varphi''_k(u)| \leq \frac{2}{u^2} |\varphi_k(u)| + \frac{1}{u^2} \int_0^\infty e^{iux}(4xk'(x) + x^2k''(x))dx \lesssim \frac{1}{u^2}
\]
as \( |u| \to \infty \).

Condition (v) is concerned with the kernel function \( W \). We assume that \( W \) is a \((p + 1)\)-th order kernel, but allow for the possibility that \( \int_{\mathbb{R}} x^{p+1}W(x)dx = 0 \). Note that since the Fourier transform of \( W \) has compact support, the support of the kernel function \( W \) is necessarily unbounded (see Theorem 4.1 in Stein and Weiss (1971)).

Condition (vi) is concerned with the sampling frequency, bandwidth, and the sample size. The condition \( \Delta \gtrsim \log n \) implies that we work with macroscopic observation scheme and this is a technical condition for the inference on \( k \). We assume this condition to guarantee that the dependence among \( \{X_j\}_{j=1}^n \) can be ignored asymptotically. We note that to estimate \( k \) uniformly on an interval \( I \subset (0, \infty) \), we do not need the condition and we can work with low-frequency set up (i.e., \( \Delta > 0 \) is fixed).

From a practical point of view, our methods could be applied to low-frequency data and would work well if we suitably rescale the time scale of the data and the sample size \( n \) is sufficiently large. In our simulation study, we consider the case when \( (n, \Delta) = (500, 1) \) and our method works well in this case. We also need Condition (vi) to derive the lower bound of \( h \) for the uniform consistency of \( \hat{k}_2(x) \) for \( x = x_\ell, j = 1, \ldots, N \) with \( 0 < x_1 < \cdots < x_N < \infty \). We need the upper bound of \( h \) for the undersmoothing condition. See also Remark 3.4 in the present paper for comments on the condition on \( h \).

To state a multivariate central limit theorem for \( \hat{k}_2 \), we introduce the notion of regularly varying functions.

**Definition 3.1** (Regularly varying function). A measurable function \( U_0 : [0, \infty) \to [0, \infty) \) is regularly varying at \( \infty \) with index \( \rho \) (written as \( U_0 \in RV_\rho \)) if for \( x > 0 \),

\[
\lim_{t \to \infty} \frac{U_0(tx)}{U_0(t)} = x^\rho.
\]
We say that a function $U$ is slowly varying if $U_0 \in RV_0$. We refer to Resnick (2007) for details of regularly varying functions. The following lemma plays an important role in the proof of Theorem 3.1.

**Lemma 3.1.** Assume Condition (ii) in Assumption 3.1. There exist a function $L : (1, \infty) \rightarrow [0, \infty)$ slowly varying at $\infty$ and a constant $B > 0$ such that

$$\lim_{|t| \rightarrow \infty} \frac{|t|^\alpha |\varphi(t)|}{L(|t|)} = B.$$ 

**Remark 3.2.** Condition (ii) in Assumption 3.1 is concerned with smoothness of the stationary distribution $\pi$ of the Lévy-driven OU process. Condition (ii) implies that the stationary distribution $\pi$ is nearly ordinary smooth, that is, the characteristic function (1.2) decays polynomially fast as $|u| \rightarrow \infty$ (Lemma 3.1) up to a slowly varying function. Since $k(x) = \nu((x, \infty))$, the finiteness of $k(0)$ is equivalent to the finiteness of the total mass of the Lévy measure of the Lévy process $J$ and this means that the Lévy process $J$ is of finite activity, i.e., it has only finitely many jumps in any bounded time intervals. It is known that a Lévy process with a finite Lévy measure is a compound Poisson process. If $k(0) = \infty$, the Lévy process $J$ is of infinite activity, i.e., it has infinitely many jumps in any bounded time intervals. In this case, the characteristic function (1.2) decays faster than polynomials. In particular, it decays exponentially fast as $|u| \rightarrow \infty$ if the Blumenthal-Getoor index of $J$ is positive, that is, if

$$\rho_{BG} = \inf \left\{ p > 0 : \int_{|x| \leq 1} |x|^p \nu(dx) < \infty \right\} > 0.$$ 

Inverse Gaussian, tempered stable, normal inverse Gaussian processes are included in this case, for example. Condition (ii) rules out those examples since we could not construct confidence bands based on Gaussian approximation under our observation scheme (see also the comments after Assumption 10 in Kato and Sasaki (2018)). Kato and Sasaki (2018) develops some methods to construct uniform confidence bands for density deconvolution problem by using intermediate Gaussian approximation. In their paper, when the density of a measurement error is supersmooth (this case corresponds to the case that BG-index is positive in our framework), they assume that the effect of the estimation of the characteristic function of measurement error is asymptotically negligible, that is, $m_n/n \rightarrow \infty$ as $n \rightarrow \infty$ in their notation. On the other hand, we can use $n$ observations to estimate $\varphi$ (this function corresponds to the characteristic function of a measurement error in deconvolution problems). So $m = n$ in our situation and in this case, we can apply intermediate Gaussian approximation results developed in Chernozhukov et al. (2013) to the case that the density of a measurement error is ordinary smooth (or BG-index is 0). However, to the best of our knowledge, such a result when the density of a measurement error is supersmooth (or BG-index is positive) is not known in the literature of deconvolution problems. Therefore,
we assume the nearly ordinary smoothness of $\pi$ in our situation to obtain practical asymptotic theorems for the inference on $k$.

**Remark 3.3.** Lemma 3.1 implies that $|\varphi(u)|$ is a regularly varying function at $\infty$ with index $\alpha$. A slowly varying function $L(u)$ may go to $\infty$ as $u \to \infty$ but it does not grow faster than any power functions, that is,

$$\lim_{u \to \infty} \frac{L(u)}{u^\delta} = 0$$

for any $\delta > 0$. In fact, if $k(0) = \alpha > 0$, from Proposition 1 in Trabs (2014a), we have that

$$(1 + |u|)^{-\alpha} \lesssim |\varphi(u)| \lesssim (1 + |u|)^{-\alpha+\delta}.$$ 

for any $\delta > 0$. Such a tail behavior of $\varphi$ is related to Condition (vi) in Assumption 3.1. If the stationary distribution $\pi$ is ordinary smooth, that is, $\varphi$ satisfies the relation

$$(1 + |u|)^{-\alpha} \lesssim |\varphi(u)| \lesssim (1 + |u|)^{-\alpha}$$

for some $\alpha > 0$, we can set $\delta = 0$ in Condition (vi). However, we need to introduce $\delta > 0$ to take into account the effect of the slowly varying function $L$.

**Remark 3.4.** As shown in (A.7) and the comments below, if we do not assume the condition

$$h \ll \left( \frac{1}{n \log n} \right)^{1/(1+2r+2\alpha-\delta)},$$

we have that

$$\max_{1 \leq \ell \leq N} |\hat{k}_x(x_\ell) - k(x_\ell)| = O_P((nh^{2\alpha+1-\delta})^{-1/2} \sqrt{\log n}) + O(h^r)$$

as $n \to \infty$ where the second term of the right hand side comes from the deterministic bias. For central limit theorems to hold, we have to choose a bandwidth so that the bias term is asymptotically negligible relative to the first term or “variance” term. The right hand side is optimized if we take $h \sim (\log n/n)^{1/(1+2r+2\alpha-\delta)}$.

Under Assumption 3.1, we can show that $\hat{k}_x(x) - k(x)$ has the following asymptotically linear representation:

$$\hat{k}_x(x) - k(x) = \frac{-i}{2\pi} \int_{\mathbb{R}} e^{itx} \left( \frac{\varphi'_\theta(t) - \varphi'_\theta(t)}{\varphi(t)} \right) \varphi_W(th)dt + o_P\left((nh^{2\alpha+1-\delta})^{-1/2} \sqrt{\log n}\right), \quad (3.1)$$

where $\varphi'_\theta(t) = E[\varphi'_\theta(t)]$. By a change of variables, we may rewrite the first term in (3.1) as

$$Z_n(x) = \frac{1}{nh} \sum_{j=1}^n \left\{ X_j \Delta 1\{|X_j\Delta| \leq \theta_n\} K_n\left( \frac{x - X_j\Delta}{h} \right) - E \left[ X_1 \Delta 1\{|X_1| \leq \theta_n\} K_n\left( \frac{x - X_1}{h} \right) \right] \right\}, \quad (3.2)$$

where $K_n$ is a function defined by

$$K_n(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-itx} \frac{\varphi_W(t)}{\varphi(t/h)} dt.$$
Note that $K_n$ is well-defined and real-valued. To construct a confidence interval for $k(x)$, we estimate the variance of $\sqrt{nh}Z_n(x)$, which is $\sigma_n^2(x)$, by

$$
\hat{\sigma}_n^2(x) = \frac{1}{n} \sum_{j=1}^{n} \left\{ X_j \Delta 1 \{|X_j| \leq \theta_n\} \hat{K}_n \left( \frac{x - X_j \Delta}{h} \right) \right\}^2 - \left\{ \frac{1}{n} \sum_{j=1}^{n} X_j \Delta 1 \{|X_j| \leq \theta_n\} \hat{K}_n \left( \frac{x - X_j \Delta}{h} \right) \right\}^2,
$$

(3.3)

where

$$
\hat{K}_n(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-itx} \frac{\hat{\varphi}(t)}{\hat{\varphi}(t/h)} dt.
$$

Remark 3.5. We use conditions (ii), (iv), and (v) in Assumption 3.1 to show that

$$
h^\alpha(|K_n(x)| + h|xK_n(x)|) \lesssim \min(1, 1/x^2).
$$

(3.4)

See the proof of Lemma A.5 in Appendix A for details. Combining this bound on $K_n$ and Condition (vi) in Assumption 3.1, we can show that the asymptotic variance-covariance matrix appearing in Theorem 3.1 is diagonal.

Remark 3.6. Propositions A.1 and A.2, and Lemma A.6 (see Appendix A) yield that

$$
\sigma_n^2(x) = \text{Var}(\sqrt{nh}Z_n(x)) \sim \text{Var}(Z_{n,1}(x)) \gtrsim h^{-2\alpha+1-\delta}
$$

uniformly in $x \in I \subset (0, \infty)$ where $I$ is a compact set and $Z_{n,j}(x) = X_j \Delta 1 \{|X_j| \leq \theta_n\} K_n \left( \frac{x - X_j \Delta}{h} \right)$.

Then we can estimate $\sigma_n^2(x)$ by $\hat{\sigma}_n^2(x)$ (see Lemma 4.1 and the proof in Appendix A for details).

Now we give the next multivariate central limit theorem.

**Theorem 3.1.** Assume Assumption 3.1. Then for any $0 < x_1 < \ldots < x_N < \infty$, we have that

$$
\sqrt{nh} \left( \frac{\hat{k}_x(x_1) - k_x(x_1)}{\hat{\sigma}(x_1)}, \ldots, \frac{\hat{k}_x(x_N) - k_x(x_N)}{\hat{\sigma}(x_N)} \right) \xrightarrow{d} N(0, I_N),
$$

where $I_N$ is the $N$ by $N$ identity matrix and $\hat{\sigma}(x) = \sqrt{\hat{\sigma}_n^2(x)}$.

4. **High-dimensional Central Limit Theorems**

In Section 3 we give multivariate (or finite-dimensional) central limit theorems for $\hat{k}_x$. In this Section, we give high-dimensional central limit theorems as an refinement of Theorem 3.1. Moreover, we propose some methods for constructing confidence bands for the $k$-function in Section 4.2 as an application of those results.
4.1. High-dimensional central limit theorems for $\hat{k}_\Delta$. For $1 \leq j \leq n$ and $1 \leq \ell \leq N$, let

$$Z_{n,j}(x_\ell) = X_j \Delta 1\{|X_j \Delta| \leq \theta_n\} K_n \left( \frac{x_\ell - X_j \Delta}{h} \right),$$

$$W_n(x_\ell) = \frac{1}{\sigma_n(x_\ell) \sqrt{n}} \sum_{j=1}^{n} (Z_{n,j}(x_\ell) - E[Z_{n,1}(x_\ell)]) = \frac{\sqrt{n} h}{\sigma_n(x_\ell)} Z_n(x_\ell),$$

and let $I \subset (0, \infty)$ be an interval with finite Lebesgue measure $|I|$, $0 < x_1 < \cdots < x_N < \infty$, $x_j \in I$, $\ell = 1, \ldots, N$. We assume that

$$\min_{1 \leq k \neq \ell \leq N} |x_k - x_\ell| \gg h^{1-2\delta} \quad (4.1)$$

and this implies that $N \ll h^{2\delta-1}$. Therefore, $N$ is allowed to go to infinity as $n \to \infty$.

**Lemma 4.1.** Under Assumption [3.1] and (4.1), we have that

$$\max_{1 \leq \ell \leq N} \left| \frac{\sigma_n^2(x_\ell)}{\sigma_n^2(x_\ell)} - 1 \right| = o_P((\log n)^{-1}).$$

**Remark 4.1.** Since

$$\left| \frac{\hat{\sigma}_n^2(x)}{\sigma_n^2(x)} - 1 \right| = \left| \frac{\hat{\sigma}_n(x)}{\sigma_n(x)} - 1 \right| \left| \frac{\hat{\sigma}_n(x) + 1}{\sigma_n(x) + 1} \right| \geq \left| \frac{\hat{\sigma}_n(x) - 1}{\sigma_n(x) - 1} \right|$$

for any $0 < x < \infty$, Lemma 4.1 implies that

$$\max_{1 \leq \ell \leq N} \left| \frac{\hat{\sigma}_n(x_\ell)}{\sigma_n(x_\ell)} - 1 \right| = o_P((\log n)^{-1}).$$

**Theorem 4.1.** Under Assumption [3.1] and (4.1), we have that

$$\sup_{t \in \mathbb{R}} \left| P\left( \max_{1 \leq \ell \leq N} |W_n(x_\ell)| \leq t \right) - P\left( \max_{1 \leq \ell \leq N} |Y_\ell| \leq t \right) \right| \to 0,$$

where $Y = (Y_1, \ldots, Y_N)^T$ is the standard normal random vector in $\mathbb{R}^N$.

**Remark 4.2.** Theorem 4.1 can be shown in two steps. As a first step, we approximate the distribution of $\max_{1 \leq \ell \leq N} |W_n(x_\ell)|$ by that of $\max_{1 \leq \ell \leq N} |\hat{Y}_n,\ell|$. Here, $\hat{Y}_n = (\hat{Y}_{n,1}, \ldots, \hat{Y}_{n,N})^T$ is a centered normal random vector with covariance matrix $E[\hat{Y}_n \hat{Y}_n^T] = q^{-1} E[W_{I_1} W_{I_1}^T]$ where $q = q_n$ is a sequence of integers with $q_n \to \infty$ and $q_n = o(n)$ as $n \to \infty$, and

$$W_{I_1} = \left( \sum_{k=1}^{q} \left( \frac{Z_{n,k}(x_1) - E[Z_{n,1}(x_1)]}{\sigma_n(x_1)} \right), \ldots, \sum_{k=1}^{q} \left( \frac{Z_{n,k}(x_N) - E[Z_{n,1}(x_N)]}{\sigma_n(x_N)} \right) \right)^T.$$

As a second step, we approximate the distribution of $\max_{1 \leq \ell \leq N} |\hat{Y}_n,\ell|$ by that of $\max_{1 \leq \ell \leq N} |Y_\ell|$. For this, we compare the variance-covariance matrices $E[\hat{Y}_n \hat{Y}_n^T]$ and $E[YY^T] = I_N$ of two Gaussian random vectors $\hat{Y}_n$ and $Y$ to establish

$$\sup_{t \in \mathbb{R}} \left| P\left( \max_{1 \leq \ell \leq N} |\hat{Y}_n(x_\ell)| \leq t \right) - P\left( \max_{1 \leq \ell \leq N} |Y_\ell| \leq t \right) \right| \to 0,$$

as $n \to \infty$. See proofs of Theorem A.1 and Proposition A.4 in Appendix A.
As a well known result in extreme value theory, \( \max_{1 \leq \ell \leq N} |Y_{\ell}| = O_P(\sqrt{\log N}) \) for independent standard normal random variables \( Y_{\ell}, \ell = 1, \ldots, N \) (see Example 1.1.7 in de Haan and Ferreira (2006)). Then, Theorem 4.1 implies that \( \max_{1 \leq \ell \leq N} |W_n(x_{\ell})| = O_P(\sqrt{\log n}) \) since \( \log N \leq \log(h^{2\gamma-1}) \leq \log n \) under Assumption 3.1. We can also show that

\[
\frac{\sqrt{n}h(\hat{k}_2(x_{\ell}) - k_2(x_{\ell}))}{\hat{\sigma}_n(x_{\ell})} = W_n(x_{\ell}) + o_P(\sqrt{\log n}) \tag{4.2}
\]

uniformly in \( x \in \{x_1, \ldots, x_N\} \). Therefore, together with Lemma 4.1 and (4.2), we have that

\[
\frac{\sqrt{n}h(\hat{k}_2(x) - k_2(x))}{\hat{\sigma}_n(x)} = \frac{\sigma_n(x)}{\hat{\sigma}_n(x)} \frac{\sqrt{n}h(\hat{k}_2(x) - k_2(x))}{\sigma_n(x)} = \frac{\sigma_n(x)}{\hat{\sigma}_n(x)} \{W_n(x) + o_P((\log n)^{-1/2})\} \quad \text{(from (4.2))}
\]

\[
= \{1 + o_P((\log n)^{-1})\} \{W_n(x) + o_P((\log n)^{-1/2})\} \quad \text{(from Lemma 4.1)}
\]

\[
= W_n(x) + o_P((\log n)^{-1/2}) \quad \text{(from } \max_{1 \leq \ell \leq N} |W_n(x_{\ell})| = O_P(\sqrt{\log n}))
\]

uniformly in \( x \in \{x_1, \ldots, x_N\} \). This yields the following theorem.

**Theorem 4.2.** Under Assumption 3.1 and (4.1), we have that

\[
\sup_{t \in \mathbb{R}} \left| P \left( \max_{1 \leq \ell \leq N} \left| \frac{\sqrt{n}h(\hat{k}_2(x_{\ell}) - k_2(x_{\ell}))}{\hat{\sigma}_n(x_{\ell})} \right| \leq t \right) - P \left( \max_{1 \leq \ell \leq N} |Y_{\ell}| \leq t \right) \right| \to 0, \; \text{as } n \to \infty,
\]

where \( Y = (Y_1, \ldots, Y_N)^\top \) is the standard normal random vector in \( \mathbb{R}^N \).

### 4.2. Confidence bands for the \( k \)-function

In this section, we discuss methods for constructing confidence bands for the \( k \)-function over \( I = [a, b] \subset (0, \infty) \). Let \( \xi_1, \ldots, \xi_N \) be i.i.d. standard normal random variables, and for \( \tau \in (0, 1) \), let \( q_{\tau} \) satisfy

\[
P \left( \max_{1 \leq j \leq N} |\xi_j| > q_{\tau} \right) = \tau.
\]

Then,

\[
\hat{C}_{1-\tau}(x_{\ell}) = \left[ \hat{k}_2(x_{\ell}) \pm \frac{\hat{\sigma}_n(x_{\ell})}{\sqrt{nh}} q_{\tau} \right], \; \ell = 1, \ldots, N
\]

are joint asymptotic 100(1 - \( \tau \))% confidence intervals for \( k_2(x_1), \ldots, k_2(x_N) \). Theorem 4.2 implies that we can construct confidence bands by linear interpolation of simultaneous confidence intervals \( \{\hat{C}_{1-\tau}(x_{\ell})\}_{\ell=1}^N \). If the sample size \( n \) is sufficiently large, we can take the number of design points \( N \) sufficiently large. Therefore, proposed confidence bands can be arbitrary close to uniform confidence bands in such cases.
4.3. **Discussion on the confidence bands.** Our method can be seen as an alternative method for constructing confidence bands based on a functional central limit theorem (FCLT) if the FCLT for the Lévy measure \( \nu \) is available (but to the best of our knowledge, such a result is not known in the literature on nonparametric inference of Lévy-driven SDEs). Moreover, it is not difficult to see from the proofs that if we strengthen the condition

\[
h \ll \left( \frac{1}{n \log n} \right)^{1/(1+2r+2\alpha-\delta)}
\]

in Assumption 3.1 (vi) to \( h^r \sqrt{n} h^{2\alpha+1-\delta} (\log n) = o(n^{-c}) \) for some (sufficiently small) constant \( c > 0 \), then there exists a positive constant \( c' \) such that the approximation of the high-dimensional central limit theorem hold at the rate \( n^{-c'} \). This shows an advantage of our method to construct confidence bands based on the intermediate Gaussian approximation compared with a method based on the Gumbel approximation since the coverage error of the latter is known to be logarithmically slow because of the slow convergence of normal extrema; see Hall (1991). The proposed method is inspired by the idea developed in Horowitz and Lee (2012). If we take \( x_\ell \in I, \ell = 1, \ldots, N \) to satisfy \( \min_{1 \leq k \neq \ell \leq N} |x_k - x_\ell| = O(h^{1/2}) \) (in this case, the condition (4.1) is satisfied), then \( |x_\ell - x_{\ell-1}| \to 0 \) uniformly for \( \ell = 2, \ldots, N \). Therefore, for \( x \) in \( I \),

\[
c_L(x) \leq k(x) \leq c_U(x)
\]

where

\[
c_L(x) = \left( \frac{\hat{k}_2(x_\ell) - \hat{k}_2(x_{\ell-1})}{x_\ell - x_{\ell-1}} - \frac{\hat{\sigma}_n(x_\ell) - \hat{\sigma}_n(x_{\ell-1})}{\sqrt{nh}} \right) (x - x_{\ell-1}) + \frac{\hat{\sigma}_n(x_{\ell-1})}{\sqrt{nh}} q_r,
\]

\[
c_U(x) = \left( \frac{\hat{k}_2(x_\ell) - \hat{k}_2(x_{\ell-1})}{x_\ell - x_{\ell-1}} + \frac{\hat{\sigma}_n(x_\ell) - \hat{\sigma}_n(x_{\ell-1})}{\sqrt{nh}} \right) (x - x_{\ell-1}) + \frac{\hat{\sigma}_n(x_{\ell-1})}{\sqrt{nh}} q_r
\]

(if \( x_{\ell-1} \leq x \leq x_\ell (\ell = 2, \ldots, N) \)) can be interpreted as an “asymptotic” \( 100(1-\tau) \%) \) uniform confidence band for \( k \) on \( I \). In fact, we can show that as \( n \to \infty \),

\[
P \left( \max_{1 \leq \ell \leq N} \left| \frac{\sqrt{nh}(\hat{k}_2(x_\ell) - k_2(x_\ell))}{\hat{\sigma}_n(x_\ell)} \right| \leq q_r \right) \to 1 - \tau.
\]

See Appendix B for the asymptotic validity of the proposed confidence bands.

5. Simulations

5.1. **Simulation framework.** In this section, we present simulation results to see the finite-sample performance of the central limit theorems and the proposed confidence bands in Sections 3 and 4. We consider the following data generating process.

\[
dX_t = -\lambda X_t dt + dJ_t
\]

(5.1)
where \( J_t = \sum_{j=1}^{N_t} U_j \) is a compound Poisson process with intensity \( \alpha \) and Gamma jump distribution with shape parameter 2 and rate parameter 1. In particular, we consider three models, that is, \((\alpha, \lambda) = (2.1, 0.5), (3, 0.5) \) and \((3, 0.75)\).

As a kernel function, we use a flat-top kernel which is defined by its Fourier transform

\[
\varphi_W(u) = \begin{cases} 
1 & \text{if } |u| \leq c \\
\exp \left\{ -b \exp\left( -\frac{b}{(|u|-c)^2} \right) \right\} & \text{if } c < |u| < 1 \\
0 & \text{if } 1 \leq |u| 
\end{cases}
\]  

where \(0 < c < 1\) and \(b > 0\). Note that \(\varphi_W\) is infinitely differentiable with \(\varphi_W^{(\ell)}(0) = 0\) for all \(\ell \geq 1\), so that its inverse Fourier transform \(W\) is of infinite order, i.e., \(\int_{\mathbb{R}} x^\ell W(x) dx = 0\) for all integers \(\ell \geq 1\) (cf. McMurry and Politis (2004)). In our simulation study, we set \(b = 1\) and \(c = 0.05\). We also set the sample size \(n\) and the time span \(\Delta\) as \(n = 500\) and \(\Delta = 1\).

Now, we discuss bandwidth selection. We use a method which is similar to that proposed in Kato and Kurisu (2017). They adopt an idea of Bissantz et al. (2007) on bandwidth selection in density deconvolution. From a theoretical point of view, for our confidence bands to work, we have to choose bandwidths that are of smaller order than the optimal rate for estimation under the loss function (or a “discretized version” of \(L^\infty\)-distance) \(\max_{1 \leq \ell \leq N} |\hat{k}_\ell(x_\ell) - k(x_\ell)|\). At the same time, choosing a too small bandwidth results in a too wide confidence band. Therefore, we should choose a bandwidth “slightly” smaller than the optimal one that minimizes \(\max_{1 \leq \ell \leq N} |\hat{k}_\ell(x_\ell) - k(x_\ell)|\). We employ the following rule for bandwidth selection. Let \(\hat{k}_h\) be the spectral estimate with bandwidth \(h\).

1. Set a pilot bandwidth \(h^P > 0\) and make a list of candidate bandwidths \(h_j = jh^P/J\) for \(j = 1, \ldots, J\).

2. Choose the smallest bandwidth \(h_j \) \((j \geq 2)\) such that the adjacent value \(\max_{1 \leq \ell \leq N} |\hat{k}_{h_j}(x_\ell) - \hat{k}_{h_{j-1}}(x_\ell)|\) is smaller than \(\kappa \times \min\{\max_{1 \leq \ell \leq N} |\hat{k}_{h_k}(x_\ell) - \hat{k}_{h_{k-1}}(x_\ell)| : k = 2, \ldots, J\}\) for some \(\kappa > 1\).

In our simulation study, we set \(h^P = 1, J = 20, \) and \(\kappa = 1.5\). This rule would choose a bandwidth “slightly” smaller than a bandwidth which is intuitively the optimal bandwidth for the estimation of \(k\) (as long as the threshold value \(\kappa\) is reasonably chosen).

Figure 1 shows five realizations of the discretized \(L^\infty\)-distance between the true \(k\)-function and estimates \(\hat{k}_\ell\) for different bandwidth values (left) and between the estimates of \(k\) with adjacent bandwidth values (right) when \((\alpha, \lambda) = (2.1, 0.5)\). We find that the discretized \(L^\infty\)-distance between the estimates of \(k\) with adjacent bandwidth values behave similarly to that between the true \(k\)-function and estimates \(\hat{k}_\ell\) for different bandwidth values. So we can expect that by using the proposed method for bandwidth selection, we can choose a “good” bandwidth for the construction of confidence bands.
Figure 1. Discrete $L^\infty$-distance between the true $k$-function and estimates $\hat{k}_j^\#$ (left), and between estimates of $k^\#_j$ (right) for different bandwidth values when $(\alpha, \lambda) = (2.1, 0.5)$. We set $(n, \Delta) = (500, 1)$, $I = [1, 3]$, and $x_\ell = 1 + 0.2(\ell - 1)$, $\ell = 1, \ldots, 11$.

Remark 5.1. In practice, it is also recommended to make use of visual information on how $\max_{1 \leq \ell \leq N} |\hat{k}_{h_j}(x_\ell) - \hat{k}_{h_{j-1}}(x_\ell)|$ behaves as $j$ increases when determining the bandwidth.

Figure 2 shows the normalized empirical distributions of $\hat{k}_j^\#(x_\ell)$ at $x = 1.5$ (left), $x = 2$ (center), and $x = 2.5$ (right) when $(\alpha, \lambda) = (2.1, 0.5)$. The number of Monte Carlo iteration is 1,000 for each case. As seen from these figures, the central limit theorem implied by Theorem 3.1 holds true.

Table 1 presents simulation results of the cases when $(\alpha, \lambda) = (2.1, 0.5), (3, 0.5)$, and $(3, 0.75)$. We find that the cases when $\alpha = 3$ tends to give more accurate results compared with the case when $\alpha = 2.1$. In general, the empirical coverage probabilities could be more accurate as the intensity of the Poisson process increases (see also the comments on Figure 3). We can also find that the empirical coverage probabilities are reasonably close to the nominal coverage probabilities overall.

Figure 3 shows the 85%(dark gray), 95%(gray), and 99%(light gray) confidence bands for the $k$-function when $(\alpha, \lambda) = (2.1, 0.5)$. We find that the proposed confidence bands capture the monotonicity of the $k$-function and the width of confidence bands tend to increase as the design point is distant from the origin. The latter point partly comes from the property of the Lévy measure $\nu$: For any (Borel) set $A \subset [0, \infty)$, $\nu(A)$ coincides with the expected number of jumps falling in $A$ in the unit time, that is, $\nu(A) = E[\sum_{0 < t < 1} 1(J_t - J_{t-} \in A)]$, where $J_{t-} = \lim_{s \uparrow t} J_s$. Therefore, jumps of larger size are less frequently observed since $\nu([0, \infty)) < \infty$ in our simulation study.
Figure 2. Normalized empirical distributions of estimates at $x = 1.5$ (left), $x = 2$ (center) and $x = 2.5$ (right) when $(\alpha, \lambda) = (2.1, 0.5)$. The red line is the density of the standard normal distribution. We set $(n, \Delta) = (500, 1)$.

| Cov. Prob. $1 - \tau$ | Model $(\alpha, \lambda) = (2.1, 0.5)$ | $(3, 0.5)$ | $(3, 0.75)$ |
|------------------------|--------------------------------------|------------|------------|
| $I_1$ 0.85             | 0.768                                | 0.892      | 0.848      |
| $I_2$                   | 0.806                                | 0.904      | 0.888      |
| $I_1$ 0.95             | 0.896                                | 0.976      | 0.964      |
| $I_2$                   | 0.908                                | 0.972      | 0.980      |
| $I_1$ 0.99             | 0.952                                | 0.988      | 0.992      |
| $I_2$                   | 0.958                                | 0.984      | 0.996      |

Table 1. Empirical coverage probabilities of the confidence bands on $I_1 = [1.5, 3.5]$ with $x_\ell = 1.5 + 0.2(\ell - 1)$ and $I_2 = [2, 4]$ with $x_\ell = 2 + 0.2(\ell - 1)$, $\ell = 1, \ldots, 11$, based on 250 Monte Carlo repetitions.

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Appendix A. Proofs

A.1. Proofs for Section 3.
Figure 3. Estimates of $k$ together with 85%(dark gray), 95%(gray), and 99%(light gray) confidence bands. The solid line corresponds to the true $k$-function. We set $(n, \Delta) = (500, 1)$, $I = [1, 3]$, and $x_\ell = 1 + 0.2(\ell - 1)$, $\ell = 1, \ldots, 11$.

Proof of Lemma 3.1. Observe that
\[
|\varphi(u)| = |\varphi(-u)| = \exp \left( \int_0^\infty (\cos(ux) - 1) \frac{k(x)}{x} dx \right).
\]
For $x > 1$, define
\[
L(x) = \exp \left( \int_{1/x}^1 (\alpha - k(y)) \frac{dy}{y} \right).
\]
For any $\lambda > 0$,
\[
\frac{L(\lambda x)}{L(x)} = \exp \left( \int_{1/\lambda x}^{1/x} (\alpha - k(y)) \frac{dy}{y} \right) = \exp \left( \int_{1/\lambda}^1 (\alpha - k(z/x)) \frac{dz}{z} \right) \to 1, \text{ as } x \to \infty.
\]
Therefore, $L$ is a slowly varying function at $\infty$. Consider the following decomposition of $I(u) := \int_0^\infty (\cos(ux) - 1)k(x)x^{-1}dx$.
\[
I(u) = \left( \int_0^{1/u} + \int_{1/u}^1 + \int_1^\infty \right) (\cos(ux) - 1) \frac{k(x)}{x} dx
\]
\[
=: I_1(u) + I_2(u) + I_3(u).
\]
Now we evaluate three terms $I_j(u)$, $j = 1, 2, 3$. First, by Riemann-Lebesgue theorem,
\[
I_3(u) \to - \int_1^\infty \frac{k(x)}{x} dx, \text{ as } u \to \infty.
\]
Moreover,
\[
I_1(u) = \int_0^1 (\cos(y) - 1) \frac{k(y/u)}{y} dy \to \alpha \int_0^1 (\cos(y) - 1) \frac{dy}{y}, \text{ as } u \to \infty.
\]
Lemma A.1. Assume the conditions (i), (ii) and (iv) in Assumption 3.1. Then we have that

We also have that

\[ I_2(u) + \alpha \log u - \log L(u) = \int_{1/u}^{1} \cos(ux) \frac{k(x)}{x} \, dx \]
\[ = \int_{1}^{u} \cos(y) \frac{k(y/u)}{y} \, dy =: \overline{I}_2(u). \]

Since \( \int_{1}^{u} \cos(y) y^{-1} dy \) is convergent as \( u \to \infty \) and \( k \) is monotone decreasing function, we have that

\[ \limsup_{u \to \infty} |\overline{I}_2(u)| \lesssim \left| \int_{1}^{\infty} \frac{\cos(y)}{y} \, dy \right| < \infty. \]

So, we complete the proof. \( \square \)

For the proof of Theorem 3.1, we prepare some auxiliary results.

Lemma A.2. Assume conditions (i) and (vi) in Assumption 3.1. Then we have that the measure \( \pi \) and \( x^3 \pi(dx) \) has \( \pi \) has a bounded continuous Lebesgue density on \( \mathbb{R} \).

Proof. By Theorem 28.4 in [Sato 1999], \( \pi \) has a bounded continuous Lebesgue density on \( \mathbb{R} \). Also from the relation

\[ \varphi''(u) = \varphi(u) \varphi_k^2(u) + \varphi(u) \varphi_k'(u), \]
\[ \varphi'''(u) = \varphi(u) \varphi_k^2(u) + 3 \varphi(u) \varphi_k(u) \varphi_k'(u) + \varphi(u) \varphi_k''(u) \]
\[ = (\varphi(u) \varphi_k^2(u)) \varphi_k(u) + 3(\varphi(u) \varphi_k'(u)) \varphi_k(u) + \varphi(u) \varphi_k''(u), \]

we see that

\[ x^2 \pi = (k \ast \pi) \ast k + (xk) \ast \pi, \]
\[ x^3 \pi = ((x^2 \pi) \ast (xk) \ast \pi) \ast k + 3((xk) \ast \pi) \ast k + (x^2 k) \ast \pi. \]

Therefore \( x^2 \pi \) has a Lebesgue density \( x^2 \pi(x) \) with

\[ \|x^2 \pi\|_\mathbb{R} \lesssim \|k\|_\mathbb{R} \|k\|_{L^1} + \|xk\|_{L^1} \lesssim 1. \]

Here, \( \|f\|_{L^p} = (\int_\mathbb{R} |f(x)|^p \, dx)^{1/p} \). Moreover, \( x^3 \pi \) has a Lebesgue density \( x^3 \pi(x) \) with

\[ \|x^3 \pi\|_\mathbb{R} \lesssim (\|x^2 \pi\|_\mathbb{R} + \|xk\|_{L^1})\|k\|_{L^1} + 3\|xk\|_{L^1}\|k\|_{L^1} + \|x^2 k\|_{L^1} \lesssim 1. \]

\( \square \)
We have that
\[
|\varphi'_{\theta_n}(u) - \varphi'(u)| \leq E \left[ |X_1| \mathbb{1}\{|X_1| > \theta_n\} \right]
\leq E[|X_1|(|X_1|/\theta_n)^2] \leq \theta_n^{-2} \ll n^{-1/2} \log n.
\]

We can also evaluate \(\|\varphi'_{\theta_n} - \varphi'\|_{[-h^{-1},h^{-1}]}\) in a similar way.

**Lemma A.3.** Assume Condition (ii) in Assumption 3.1. Then we have \(\inf_{|u| \leq h^{-1}} |\varphi(u)| \gtrsim h^\alpha\).

**Proof.** This result immediately follows from Remark 3.3.

If we take \(h\) sufficiently small, then Lemmas A.2 and A.3 imply that
\[
\inf_{|u| \leq h^{-1}} |\hat{\varphi}(u)| \geq \inf_{|u| \leq h^{-1}} |\varphi(u)| - o_P(h^\alpha) \gtrsim h^\alpha - o_P(h^\alpha),
\]
so that with probability approaching one, \(\inf_{|u| < h^{-1}} |\hat{\varphi}(u)| \gtrsim h^\alpha\).

**Lemma A.4.** Assume the conditions (i), (iv) and (v) in Assumption 3.1. Then we have that
\[
\left\| \left( \frac{\varphi'_{\theta_n} - \varphi'}{\varphi'} \right) \right\|_{[-h^{-1},h^{-1}]} = O_P(h^{-2\alpha} n^{-1}(\log n)^2 + h^{1-\alpha} n^{-1/2} \log n).
\]

**Proof.** (Step 1): First, we show that
\[
\left\| \left( \frac{\varphi'_{\theta_n} - \varphi'}{\varphi'} \right) - \left( \frac{1}{\varphi'} \right) (\varphi'_{\theta_n} - \varphi') \right\|_{[-h^{-1},h^{-1}]} = O_P(h^{-2\alpha} n^{-1}(\log n)^2 + h^{1-\alpha} n^{-1/2} \log n).
\]

Consider the following decomposition.
\[
\frac{\varphi'_{\theta_n}(u)}{\varphi_{\theta_n}(u)} - \frac{\varphi'(u)}{\varphi'(u)} = \left( \frac{1}{\varphi'(u)} \right)' (\varphi'_{\theta_n}(u) - \varphi'(u)) + \left( \frac{1}{\varphi'(u)} \right) (\varphi'_{\theta_n} - \varphi') + R_2(u),
\]
where
\[
R_2(u) = \left( 1 - \frac{\varphi'_{\theta_n}(u)}{\varphi'(u)} \right) \left( \frac{\varphi'_{\theta_n}(u)}{\varphi'(u)} - \frac{\varphi'(u)}{\varphi'(u)} \right).
\]
We have that
\[
\left\| \left( \frac{1}{\varphi'} \right)' (\varphi'_{\theta_n} - \varphi') \right\|_{[-h^{-1},h^{-1}]} \lesssim \left\| \left( \frac{1}{\varphi'} \right) \right\|_{[-h^{-1},h^{-1}]} \left\| \varphi'_{\theta_n} - \varphi' \right\|_{[-h^{-1},h^{-1}]}
\]
and
\[
\left\| R_2 \right\|_{[-h^{-1},h^{-1}]}
\leq \left\| \left( \frac{1}{\varphi'} \right) \right\|_{[-h^{-1},h^{-1}]} \left\| \varphi'_{\theta_n} - \varphi' \right\|_{[-h^{-1},h^{-1}]}
\times \left( \left\| \left( \frac{1}{\varphi'} \right) \right\|_{[-h^{-1},h^{-1}]} \left\| \varphi'_{\theta_n} - \varphi' \right\|_{[-h^{-1},h^{-1}]} + \left\| \frac{\varphi'_{\theta_n}}{\varphi'(\varphi')^2} \right\|_{[-h^{-1},h^{-1}]} \left\| \varphi'_{\theta_n} - \varphi' \right\|_{[-h^{-1},h^{-1}]} \right).
\]
In the rest of the proof, we write $\| \cdot \|_{[-h^{-1}, h^{-1}]}$ as $\| \cdot \|$ for simplicity. We observe that

$$
\left\| \frac{1}{\varphi_z} \right\| \lesssim 1, \quad \left\| \frac{1}{\varphi_z^\prime} \right\| \lesssim 1, \quad \left\| \left( \frac{1}{\varphi_z^\prime} \right)^{\prime} \right\| \lesssim h^{1-\alpha}.
$$  \tag{A.1}

In fact, since we have that

$$
\left\| \frac{1}{\varphi_z} \right\| \lesssim \left\| \frac{1}{\varphi(\cdot)} \right\| \| \hat{\varphi} - \varphi \| + \| \varphi^{\prime}(\cdot) - \varphi^{\prime}(\cdot) \| + \| \varphi_z \|
$$

$$
\lesssim \left\| \frac{1}{\varphi} \right\| \| \hat{\varphi} - \varphi \| + \left\| \frac{1}{\varphi} \right\| \| \hat{\varphi} - \varphi \| + \| \varphi_z \| \lesssim h^{-\alpha} n^{-1/2} \log n + 1 \lesssim 1,
$$

we obtain the second inequality. By Lemma \text{[A.2]} we also have that

$$
\| \hat{\varphi}_z - \varphi_z \| \lesssim \left\| \frac{1}{\varphi(\cdot)} \right\| \| \hat{\varphi} - \varphi \| + \left\| \frac{1}{\varphi(\cdot)} \right\| \| \hat{\varphi} - \varphi \| \lesssim h^{-\alpha} n^{-1/2} \log n. \tag{A.2}
$$

Now we evaluate $\| \hat{\varphi}_z' - \varphi_z' \|$.

$$
\| \hat{\varphi}_z' - \varphi_z' \| \leq \| \hat{\varphi}_z' - \hat{\varphi}_z \| + \| \hat{\varphi}_z - \varphi_z' \|,
$$

where

$$
\varphi_z'(t) = \frac{\hat{\varphi}_b(t) \varphi(-t) + \varphi'(-t) \hat{\varphi}(t)}{\varphi^2(-t)}.
$$

We observe that

$$
\| \hat{\varphi}_z' - \varphi_z' \| \lesssim \left\| \frac{\hat{\varphi}_b - \varphi_z'}{\varphi^2(\cdot)} \right\| \times (\| \hat{\varphi} - \varphi \| + \| \hat{\varphi}_b - \varphi_z' \|)
$$

$$
\lesssim \left( \left\| \frac{\hat{\varphi}_b - \varphi_z'}{\varphi^2(\cdot)} \right\| + \left\| \frac{\varphi_z'}{\varphi^2(\cdot)} \right\| \right) \times (\| \hat{\varphi} - \varphi \| + \| \hat{\varphi}_b - \varphi_z' \|)
$$

$$
\lesssim h^{-2\alpha} n^{-1}(\log n)^2 + h^{1-\alpha} n^{-1/2} \log n.
$$

$$
\| \hat{\varphi}_z' - \varphi_z' \| \lesssim \left\| \frac{1}{\varphi(\cdot)} \right\| \| \hat{\varphi} - \varphi \| + \left\| \frac{\varphi_z'-\varphi_z}{\varphi^2(\cdot)} \right\| \times \| \hat{\varphi} - \varphi \| \lesssim h^{-\alpha} n^{-1/2} \log n.
$$

Then we have that

$$
\| \hat{\varphi}_z' - \varphi_z' \| \lesssim h^{-\alpha} n^{-1/2} \log n. \tag{A.3}
$$

Together with \text{[A.1]}, \text{[A.2]}, and \text{[A.3]}, we have that

$$
\left\| \left( \frac{\hat{\varphi}_z - \varphi_z}{\varphi_z} \right) - \left( \frac{1}{\varphi_z} \right) \left( \hat{\varphi}_z - \varphi_z' \right) \right\| = O_P(h^{-2\alpha} n^{-1}(\log n)^2 + h^{1-\alpha} n^{-1/2} \log n).
$$

\text{(Step 2):} Next we show that

$$
\left\| \left( \frac{1}{\varphi_z} \right) \left( \hat{\varphi}_z' - \varphi_z' \right) - \left( \frac{1}{\varphi} \right) \left( \hat{\varphi}_b - \varphi_b \right) \right\| = O_P(h^{1-\alpha} n^{-1/2} \log n).$$
Observe that
\[
\left( \frac{1}{\varphi_n}(u) \right) \left( \hat{\varphi}_n(u) - \varphi'(u) \right) = \frac{\varphi_n'(u)}{\varphi(u)} \left( \varphi(-u) - 1 \right) + \frac{\varphi'(u) - \varphi(-u) \varphi(u)}{\varphi(u)}.
\]

Moreover, we have that
\[
\left\| \varphi_n \left( \frac{\varphi(-u)}{\varphi(-u)} - 1 \right) \right\| \lesssim \left\| \varphi' \right\| \times \| \hat{\varphi} - \varphi \| \lesssim h^{1-\alpha} n^{-1/2} \log n, \tag{A.4}
\]
and
\[
\left\| \varphi \left( \frac{\varphi_n(-u)}{\varphi^2(-u)} - \frac{\varphi'(-u)}{\varphi^2(-u)} \right) \right\| \lesssim \left\| \frac{1}{\varphi} \right\| \times \left\| \varphi_n - \varphi' \right\| + \left\| \frac{\varphi'}{\varphi^2} \right\| \| \hat{\varphi} - \varphi \|
\lesssim h^{1-\alpha} n^{-1/2} \log n. \tag{A.5}
\]

Together with (A.4) and (A.5), we have that
\[
\left\| \left( \frac{1}{\varphi_n} \right) \left( \hat{\varphi}' - \varphi' \right) - \left( \frac{1}{\varphi} \right) \left( \hat{\varphi}'_{\theta_n} - \varphi' \right) \right\| = O_P(h^{1-\alpha} n^{-1/2} \log n). \tag{A.6}
\]

Since \(\| (\varphi_n - \varphi') / \varphi \| \lesssim h^{1-\alpha} n^{-2} \ll h^{1-\alpha} n^{-1/2} \log n\), we can replace \(\varphi'\) by \(\varphi'_{\theta_n}\) in (A.6) and this completes the proof.

\[
\square
\]

With almost the same arguments in the proof of Lemma 4 we can show that
\[
\left\| \left( \frac{\varphi_n}{\varphi} \right) - \left( \frac{\varphi_n}{\varphi} \right) \right\|_{[-h^{-1}, h^{-1}]} = O_P(h^{-2\alpha} n^{-1} (\log n)^2 + h^{1-\alpha} n^{-1/2} \log n).
\]

Therefore, together with the result of Lemma 4 we have that
\[
\left\| \left( \frac{\varphi_n}{\varphi} \right) - \left( \frac{\varphi_n}{\varphi} \right) \right\|_{[-h^{-1}, h^{-1}]} = O_P(h^{-2\alpha} n^{-1} (\log n)^2 + h^{1-\alpha} n^{-1/2} \log n).
\]

**Lemma A.5.** We have that \(h^\alpha(\| K_n(x) \| + h| x K_n(x) |) \lesssim \min(1, 1/x^2)\).

**Proof.** We first show \(h^\alpha(\| K_n(x) \|) \lesssim \min(1, 1/x^2)\). We follow the proof of Lemma 3 in Masry (1991). By integration by parts, we have that
\[
K_n(x) = \frac{1}{2\pi x^2} \int_{\mathbb{R}} e^{-itx} \left( \frac{\varphi W(t)}{\varphi(t/h)} \right)'' dt.
\]
We also observe that
\[
\left( \frac{\varphi W(t)}{\varphi(t/h)} \right)'' = \frac{\varphi W(t)}{\varphi(t/h)} - 2 \frac{\varphi W(t) \varphi'(t/h)}{h \varphi^2(t/h)} + \frac{\varphi W(t)}{h^2} \left( - \frac{\varphi''(t/h)}{\varphi^2(t/h)} + 2 \frac{(\varphi'(t/h))^2}{\varphi^3(t/h)} \right)
=: I_{1,n}(t) + I_{2,n}(t) + I_{3,n}(t).
\]
Since $\varphi_W$ is supported in $[-1,1]$ and two-times differentiable, we can show

$$h^\alpha \int_\mathbb{R} |I_{j,n}(t)| \, dt \lesssim 1$$

for $j = 1, 2, 3$. Indeed,

$$h^\alpha L(h^{-1}) \int_{[-1,-\frac{1}{2}) \cup (\frac{1}{2},1]} |I_{1,n}(t)| \, dt = \int_{[-1,-\frac{1}{2}) \cup (\frac{1}{2},1]} \frac{|t|^{\alpha} |\varphi''_W(t)|}{t/h^\alpha L^{-1}(|t/h| |\varphi(t/h)|)} \frac{L(1/h)}{L(|t/h|)} \, dt \lesssim \int_\mathbb{R} |t|^{\alpha} |\varphi''_W(t)| \, dt \lesssim 1,$$

$$h^\alpha L(h^{-1}) \int_{[-1,-\frac{1}{2}) \cup (\frac{1}{2},1]} |I_{2,n}(t)| \, dt = \int_{[-1,-\frac{1}{2}) \cup (\frac{1}{2},1]} \frac{|t/h| |\varphi_k(t/h)|}{t/h^\alpha L^{-1}(|t/h| |\varphi(t/h)|)} \frac{L(1/h)}{L(|t/h|)} |t|^{\alpha-1} |\varphi'_W(t)| \, dt \lesssim \int_\mathbb{R} |t|^{\alpha-1} |\varphi'_W(t)| \, dt \lesssim 1,$$

$$h^\alpha L(h^{-1}) \int_{[-1,-\frac{1}{2}) \cup (\frac{1}{2},1]} |I_{3,n}(t)| \, dt \lesssim \int_{[-1,-\frac{1}{2}) \cup (\frac{1}{2},1]} \left( \frac{|t/h|^2 |\varphi^3_k(t/h)|}{t/h^\alpha L^{-1}(|t/h| |\varphi(t/h)|)} + \frac{|t/h|^2 |\varphi'_k(t/h)|}{t/h^\alpha L^{-1}(|t/h| |\varphi(t/h)|)} \right) \frac{L(1/h)}{L(|t/h|)} |t|^{\alpha-2} |\varphi_W(t)| \, dt \lesssim \int_\mathbb{R} |t|^{\alpha-2} |\varphi_W(t)| \, dt \lesssim 1.$$

Moreover, we have that

$$h^\alpha \int_{[-\frac{1}{2}, \frac{1}{2}]} |I_{1,n}(t)| \, dt = h^\alpha \int_{[-\frac{1}{2}, \frac{1}{2}]} |\varphi''_W(t)| \, dt \lesssim \int_\mathbb{R} (h + |t|)^\alpha |\varphi''_W(t)| \, dt \lesssim 1,$$

$$h^\alpha \int_{[-\frac{1}{2}, \frac{1}{2}]} |I_{2,n}(t)| \, dt = h^{\alpha-1} \int_{[-\frac{1}{2}, \frac{1}{2}]} \frac{|\varphi_k(t/h)|}{\varphi(t/h)} |\varphi'_W(t)| \, dt \lesssim \int_\mathbb{R} \frac{h^\alpha (1 + |t/h|)^\alpha}{h(1 + |t/h|)^h} |\varphi'_W(t)| \, dt \lesssim \int_\mathbb{R} (h + |t|)^{\alpha-1} |\varphi'_W(t)| \, dt \lesssim 1,$$

$$h^\alpha \int_{[-\frac{1}{2}, \frac{1}{2}]} |I_{3,n}(t)| \, dt \lesssim h^{\alpha-2} \int_{[-\frac{1}{2}, \frac{1}{2}]} \left( \frac{|\varphi^3_k(t/h)|}{|\varphi(t/h)|} + \frac{|\varphi'_k(t/h)|}{|\varphi(t/h)|} \right) |\varphi_W(t)| \, dt \lesssim \int_\mathbb{R} \frac{h^\alpha (1 + |t/h|)^\alpha}{h^2(1 + |t/h|)^2} |\varphi'_W(t)| \, dt \lesssim \int_\mathbb{R} (h + |t|)^{\alpha-2} |\varphi_W(t)| \, dt \lesssim 1.$$

Since $\int_\mathbb{R} I_{j,n}(t) \, dt = \int_{[-1,1]} I_{j,n}(t) \, dt$ for $j = 1, 2, 3$, we obtain the desired result. Next we show $h^{\alpha+1} |x K_n(x)| \lesssim \min(1, 1/x^2)$. Observe that

$$K_n(x) = \frac{i}{2\pi x^2} \int_\mathbb{R} e^{-i tx} \left( \frac{\varphi_W(t)}{\varphi(t/h)} \right)^{\alpha} \, dt.$$
Therefore, we have the desired result.

We can show that $h^{\alpha+1} \int_{\mathbb{R}} |\tilde{I}_{j,n}(t)| dt \lesssim 1$, $j = 1, 2, 3$ and

$$h^{\alpha+1} L(1/h) \int_{[-1, -\frac{1}{2}) \cup (\frac{1}{2}, 1]} |\tilde{I}_{4,n}(t)| dt \lesssim \int_{[-1, -\frac{1}{2}) \cup (\frac{1}{2}, 1]} \left( \frac{h|t|/h|^3|\varphi_3(t/h)|}{|t/h|^\alpha L^{-1}(|t/h)|\varphi(t/h)|} \right) L(1/h) |t|^{\alpha-3} |\varphi_W(t)| dt$$

$$\lesssim \int_{\mathbb{R}} |t|^{\alpha-3} |\varphi(t)| dt \lesssim 1,$$

$$h^{\alpha+1} \int_{[-\frac{1}{2}, \frac{1}{2}] [\tilde{I}_{4,n}(t)| dt \lesssim h^{-\alpha-1} \int_{[-\frac{1}{2}, \frac{1}{2}]} \left( \frac{|\varphi_3(t/h)|}{|\varphi(t/h)|} + \frac{|\varphi_k(t/h)||\varphi'(t/h)|}{|\varphi(t/h)|} + \frac{|\varphi_k'(t/h)|}{|\varphi(t/h)|} \right) |\varphi(t)| dt$$

$$\lesssim \int_{\mathbb{R}} (h + |t|)^{\alpha-3} |\varphi_W(t)| dt \lesssim 1$$

Therefore, we have the desired result.

Since

$$h^\alpha y K_n \left( \frac{x-y}{h} \right) = -h^{\alpha+1} \left( \frac{x-y}{h} \right) K_n \left( \frac{x-y}{h} \right) + h^\alpha x K_n \left( \frac{x-y}{h} \right),$$

Lemma A.5 implies that each term on the right hand side is bounded (as a function of $y$) uniformly in $n$ and $x \in \{x_1, \ldots, x_N\}$.

**Lemma A.6.** Assume the conditions (i), (ii), (iv) and (v) in Assumption 3.1. For any compact set $I$ such that $I \subset (0, \infty)$, we have that

$$\int_{\mathbb{R}} K_n^2(x) dx \gtrsim h^{-2\alpha+\delta}.$$
Now observe that
\[
 h^{2\alpha} \tilde{E}(1/h) \int_{\mathbb{R}} |\varphi W(t)|^2 dt = h^{2\alpha} \int_{[-\frac{1}{2}, \frac{1}{2}]} |\varphi W(t)|^2 dt \\
+ \int_{[-1, -\frac{1}{2}]} \int_{[-1, 1]} |t|^{2\alpha} |\varphi W(t)|^2 \left( \frac{L^2(1/h)}{|t|} \right)^2 dt.
\]
Since \(|t|^{2\alpha} |\varphi W(t)|^2\) is integrable and
\[
\lim_{h \to 0} \frac{|t/h|^\alpha |\varphi(t/h)|}{L(|t/h|)} =: B, \quad \lim_{h \to 0} \frac{L(1/h)}{L(|t/h|)} = 1
\]
for any \(|t| > 0\), by dominated convergence theorem we have the desired result. \(\square\)

**Lemma A.7.** Let \(Z_{n,j}(x) = X_{j,\Delta}1\{|X_{j,\Delta}| \leq \theta_n\}K_n((x-X_{j,\Delta})/h)\). Then \(\max_{1 \leq \ell \leq N} |E[Z_{n,1}(x_{\ell})]| \lesssim h\).

**Proof.** Let \(\tilde{Z}_{n,j}(x) = X_{j,\Delta}K_n((x-X_{j,\Delta})/h)\). By Fubini’s theorem, we have that
\[
\max_{1 \leq \ell \leq N} |E[\tilde{Z}_{n,1}(x_{\ell})]| \leq h \max_{1 \leq \ell \leq N} \int_{\mathbb{R}} k(x - hz)W(z)dz \leq h\|k\|_{\mathbb{R}} \int_{\mathbb{R}} |W(y)|dy \lesssim h,
\]
and
\[
\max_{1 \leq \ell \leq N} E[|\tilde{Z}_{n,1}(x_{\ell}) - Z_{n,1}(x_{\ell})|] \leq \max_{1 \leq \ell \leq N} E[\tilde{Z}_{n,1}^2(x_{\ell})]^{1/2} P(|X_1| > \theta_n)^{1/2} \lesssim (h^{1-2\alpha})^{1/2} \times E(||X_1/\theta_n||^3)^{1/2} = h^{1/2 - 2\alpha} \theta_n^{-3/2} \lesssim h.
\]
Therefore, we have that
\[
\max_{1 \leq \ell \leq N} |E[Z_{n,1}(x_{\ell})]| \leq \max_{1 \leq \ell \leq N} |E[\tilde{Z}_{n,1}(x_{\ell})]| + \max_{1 \leq \ell \leq N} E[|\tilde{Z}_{n,1}(x_{\ell}) - Z_{n,1}(x_{\ell})|] \lesssim h.
\]
\(\square\)

Lemmas A.6 and A.7 yield the following result on the lower bound of the variance of \(Z_{n,1}(x)\).

**Proposition A.1.** For any \(\delta \in (0, 1/12)\), \(\min_{1 \leq \ell \leq N} \text{Var}(Z_{n,1}(x_{\ell})) \gtrsim h^{-2\alpha + \delta + 1}\).

**Proof.** Let \(Z'_{n,j}(x) = X_{j,\Delta}1\{|X_{j,\Delta}| > \theta_n\}K_n((x-X_{j,\Delta})/h)\). Observe that
\[
\min_{1 \leq \ell \leq N} E[(Z'_{n,1})^2(x_{\ell})] \lesssim h^{-2\alpha} \theta_n^{-1} E[|X_1|^3] \lesssim h^{-2\alpha} \theta_n^{-1} \ll h^{-2\alpha + \delta + 1}.
\]
Since \(\min_{1 \leq \ell \leq N} E[Z_{n,1}^2(x_{\ell})] \gtrsim h^{-2\alpha + \delta + 1}\) by Lemma A.6, we have that
\[
\min_{1 \leq \ell \leq N} E[Z_{n,1}^2(x_{\ell})] = \min_{1 \leq \ell \leq N} E[(\tilde{Z}_{n,1}(x_{\ell}) - Z'_{n,1}(x_{\ell}))^2] \sim \min_{1 \leq \ell \leq N} E[\tilde{Z}_{n,1}^2(x_{\ell})].
\]
\(\square\)

** Lemma A.8.** \(\max_{1 \leq k \leq \ell \leq N} \text{Cov}(Z_{n,1}(x_k), Z_{n,1}(x_{k+1})) \lesssim e^{-i\beta_1/3} h^{2/3 - 2\alpha} \).
Proof. Since $x^3 \pi$ has a bounded Lebesgue density on $\mathbb{R}$ by Lemma A.1 and $h^{2\alpha}|K_n|^2$ is integrable by Lemma A.5, we first observe that

$$\max_{1 \leq \ell \leq N} E[Z_n,1^3(x)] \leq \max_{1 \leq \ell \leq N} \int_{\mathbb{R}} |y|^3 K_n \left( \frac{x_{\ell} - y}{h} \right)^3 \pi(y) dy$$

$$\leq h\|y^3 \pi\|_{L^1} \leq h\|y^3 \pi\|_{L^1} K_n^3 \|L^1 \leq h^{-1-3\alpha}.$$ 

Therefore, by Proposition 2.5 in Fan and Yao (2003), we obtain

$$\max_{1 \leq k, \ell \leq N} |\text{Cov}(Z_n,1(x_k), Z_n, j+1(x))| \lesssim e^{-j\Delta_1/3} \max_{1 \leq k, \ell \leq N} E[|Z_n,1(x_k)|^3]^{1/3} \max_{1 \leq \ell \leq N} E[|Z_n, j+1(x)|^3]^{1/3} \lesssim e^{-j\Delta_1/3} h^{2/3 - 2\alpha}.$$ 

Then we have the desired result. \hfill \Box

**Proposition A.2.** Let $\tilde{S}_n(x) = \sum_{j=1}^n Z_n, j(x)$. Then for any $\delta \in (0, 1/12)$, we have that

$$\max_{1 \leq \ell \leq N} \left( \frac{1}{n} \text{Var}(\tilde{S}_n(x)) - \text{Var}(Z_n,1(x)) \right) = o(h^{-2\alpha + \delta + 1}).$$

**Proof.** It is easy to show that

$$\frac{1}{n} \text{Var}(\tilde{S}_n(x)) = \text{Var}(Z_n,1(x)) + 2 \sum_{j=1}^{n-1} (1 - j/n) \text{Cov}(Z_n,1(x), Z_n, j+1(x)).$$

By Lemma A.8, we have that

$$h^{2\alpha - 1 - \delta} \max_{1 \leq \ell \leq N} \left| \sum_{j=1}^{\infty} \text{Cov}(Z_n,1(x), Z_n, j+1(x)) \right| \leq h^{2\alpha - 1 - \delta} \max_{1 \leq \ell \leq N} \sum_{j=1}^{\infty} |\text{Cov}(Z_n,1(x), Z_n, j+1(x))|$$

$$\lesssim h^{2\alpha - 1 - \delta} \times h^{2/3 - 2\alpha} \sum_{j=1}^{\infty} e^{-j\Delta_1/3} \lesssim h^{2\alpha - 1 - \delta} e^{-\Delta_1/3} \lesssim e^{\frac{5}{12} \log(1/h) - \Delta_1/3}.$$ 

Since $\log(1/h) < \frac{C_0}{2 + 2\alpha - \delta} \log n$ for sufficiently large $n$ and $\frac{5C_0}{4\beta_1(2 + 2\alpha - \delta)} \log n \leq \Delta$, we have that

$$\frac{5}{12} \log(1/h) - \Delta_1/3 = -c_0 \log n$$

for some positive constant $c_0$. Therefore, we have the desired result. \hfill \Box

Proposition A.2 implies that the dependence between $Z_n,1(x)$ and $Z_n, j+1(x)$ is negligible. This enables us to estimate $\sigma_n^2(x) = n^{-1} \text{Var}(S_n(x)) = \text{Var}(\sqrt{n}hZ_n(x))$ by the sample variance (3.3). Moreover Propositions A.1 and A.2 and Lemma A.8 yield that $\min_{1 \leq \ell \leq N} \sigma_n^2(x) \gtrsim h^{-2\alpha + \delta + 1}$. 

Observe that
\[
\widehat{k}_2(x) - k_2(x) = -\frac{i}{2\pi} \int_{\mathbb{R}} e^{-iux} \frac{\varphi_2'(u)}{\varphi_2(u)} \varphi_W(\cdot) du - k_2(x)
\]
\[+ \frac{i}{2\pi} \int_{\mathbb{R}} e^{-iux} \left( \frac{\varphi_2'(u)}{\varphi_2(u)} - \frac{\varphi_2'(u)}{\varphi_2(u)} \right) \varphi_W(\cdot) du \]
\[= [k_2 * (h^{-1}W(\cdot))] - k_2(x) \]
\[+ \frac{i}{2\pi} \int_{\mathbb{R}} e^{-iux} \left( \frac{\varphi_2'(u)}{\varphi_2(u)} - \frac{\varphi_2'(u)}{\varphi_2(u)} \right) \varphi_W(\cdot) du \]
\[=: I_n + II_n. \quad (A.7)\]

For the first term, we have that \(\|I_n\| \lesssim h^r\) (by Lemma A.9). For the second term \(II_n\), Lemma A.4 yields that
\[
II_n = -\frac{i}{2\pi} \int_{\mathbb{R}} e^{-iux} \left( \frac{\varphi_2'(t) - \varphi_2'(t)}{\varphi(t)} \right) \varphi_W(\cdot) dt + O_P(h^{-2\alpha-1}n^{-1}(\log n)^2 + h^{-\alpha}n^{-1/2} \log n)
\]
\[= -\frac{i}{2\pi} \int_{\mathbb{R}} e^{-iux} \left( \frac{\varphi_2'(t) - \varphi_2'(t)}{\varphi(t)} \right) \varphi_W(\cdot) dt + o_P((nh^{2\alpha+1-\delta} \log n)^{-1/2})
\]
uniformly in \(x \in \{x_1, \ldots, x_N\}\). Therefore, since \(\min_{1 \leq \ell \leq N} \sigma_n(x_\ell) \gtrsim \sqrt{h^{-2\alpha+\delta+1}}\) (see the comment after Proposition A.2), we have that
\[
\frac{\sqrt{n}h(\widehat{k}_2(x) - k_2(x))}{\sigma_n(x)} = W_n(x) + o_P((\log n)^{-1/2}) \quad (A.8)
\]
uniformly in \(x \in \{x_1, \ldots, x_N\}\).

**Lemma A.9.** Assume the conditions (iii), (v), and (vi) in Assumption 3.1. Then we have that
\[
\| [k_2 * (h^{-1}W(\cdot))] - k_2 \| \lesssim h^r = o((nh^{2\alpha+1-\delta} \log n)^{-1/2}).
\]

**Proof.** Observe that by a change of variables, \( [k_2 * (h^{-1}W(\cdot))] (x) - k_2(x) = \int_{\mathbb{R}} k_2(x-yh) - k_2(x) W(y) dy \). If \( p \geq 1 \), then by Taylor’s theorem, for any \( x, y \in \mathbb{R} \),
\[
k_2(x-yh) - k_2(x) = \sum_{\ell=1}^{p-1} \frac{k_2^{(\ell)}(x)}{\ell!} (-yh)^\ell + \frac{k_2^{(p)}(x-\theta yh)}{p!} (-yh)^p
\]
for some \( \theta \in [0, 1] \), where \( \sum_{\ell=1}^{p-1} \frac{k_2^{(\ell)}(x)}{\ell!} (-yh)^\ell = 0 \) by convention. Since \( k_2^{(p)} \) is \((r-p)\)-Hölder continuous, we have that \( H := \sup_{x, y \in \mathbb{R}, x \neq y} \frac{|k_2^{(p)}(x) - k_2^{(p)}(y)|}{|x-y|^{r-p}} < \infty \). Now, since \( \int_{\mathbb{R}} y^r W(y) dy = 0 \) for \( \ell = 1, \ldots, p \), we have that for any \( x \in \mathbb{R} \),
\[
\left| \int_{\mathbb{R}} k_2(x-yh) - k_2(x) W(y) dy \right| = \left| \int_{\mathbb{R}} \left[ k_2(x-yh) - k_2(x) \right] \sum_{\ell=1}^{p-1} \frac{k_2^{(\ell)}(x)}{\ell!} (-yh)^\ell \right| W(y) dy \right|
\[\leq \frac{H h^r}{p!} \int_{\mathbb{R}} |y|^r |W(y)| dy,
\]
where \(0! = 1\) by convention. This completes the proof.

Let \(Q_n(x) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} Z_{n,j}(x)\) with \(Z_{n,j}(x) = X_j \Delta 1\{|X_j \Delta | \leq \theta_n\} K_n((x - X_j \Delta)/h)\). We use the following result to show that the asymptotic variances which appear in Theorem 3.1 is a diagonal matrix.

**Proposition A.3.** For any \(\delta \in (0,1/12)\), we have that

\[
\max_{1 \leq k \neq \ell \leq N} \left| \text{Cov}(Q_n(x_k), Q_n(x_\ell)) \right| = o(h^{-2\alpha + \delta + 1}).
\]

**Proof.** Since \(\max_{1 \leq \ell \leq N} \left| E[Z_{n,1}(x_\ell)] \right| \leq h\) by Lemma A.7, we have that

\[
\text{Cov}(Q_n(x_1), Q_n(x_2)) = \frac{1}{n} \sum_{j, \ell = 1}^{n} E[Z_{n,j}(x_1)Z_{n,\ell}(x_2)] - E[Z_{n,1}(x_1)]E[Z_{n,1}(x_2)]
\]

\[
= E[Z_{n,1}(x_1)Z_{n,1}(x_2)] + 2 \sum_{j=1}^{n-1} \left(1 - \frac{j}{n}\right) E[Z_{n,1}(x_1)Z_{n,j+1}(x_2)] + O(h^2).
\]

With almost the same arguments in the proof of Proposition A.2 yields that

\[
\max_{1 \leq k \neq \ell \leq N} \left( \sum_{j=1}^{n-1} E[|Z_{n,1}(x_k)Z_{n,j+1}(x_\ell)|] \right) = o(h^{-2\alpha + \delta + 1}).
\]

Hence it is sufficient to show that \(\max_{1 \leq k, \ell \leq N} E[|Z_{n,1}(x_k)Z_{n,1}(x_\ell)|] = o(h^{-2\alpha + \delta + 1})\). Let \(0 < x_1 < x_2 < \infty\). Since \(h^{\alpha} |K_n(x)| \leq \min(1,1/x^2)\) by Lemma A.5,

\[
h^{2\alpha - 1 - \delta} E[|Z_{n,1}(x_1)Z_{n,1}(x_2)|] = h^{2\alpha - 1 - \delta} \int_{\mathbb{R}} y^2 \left| K_n \left( \frac{x_1 - y}{h} \right) \right| \left| K_n \left( \frac{x_2 - y}{h} \right) \right| \pi(y) dy
\]

\[
\leq h^{-\delta} |x^2 \pi| \| \int_{\mathbb{R}} |h^{\alpha} K_n(z)| \left| h^{\alpha} K_n \left( z + \frac{x_2 - x_1}{h} \right) \right| dz
\]

\[
\leq h^{-\delta} \int_{\mathbb{R}} (1 \wedge z^{-2}) \left(1 \wedge \frac{h^2}{(zh + (x_2 - x_1))^2} \right) dz.
\]

If \(|z| \leq h^{-2\delta}\) and take \(h\) sufficiently small, then we have that

\[
\int_{|z| \leq h^{-2\delta}} (1 \wedge z^{-2}) \left(1 \wedge \frac{h^2}{(zh + (x_2 - x_1))^2} \right) dy \leq \int_{|z| \leq h^{-2\delta}} (1 \wedge z^{-2}) \frac{h^2}{(x_2 - x_1)^2} dz
\]

\[
\leq \frac{h^2}{\min_{1 \leq k \neq \ell \leq N} |x_k - x_\ell|^2} \ll h^{4\delta}.
\]

Moreover,

\[
\int_{|z| > h^{-2\delta}} (1 \wedge z^{-2}) \left(1 \wedge \frac{h^2}{(zh + (x_2 - x_1))^2} \right) dy \leq \int_{|z| > h^{-2\delta}} (1 \wedge z^{-2}) dz \ll h^{2\delta}.
\]

Therefore we have that

\[
h^{2\alpha - 1 - \delta} \max_{1 \leq k \neq \ell \leq N} E[|Z_{n,1}(x_k)Z_{n,1}(x_\ell)|] \lesssim h^{-\delta} (h^{4\delta} + h^{2\delta}) \lesssim h^{\delta} \ll 1.
\]
Proof of Theorem 3.1. Now we prove Theorem 3.1. Let $S_n(x) = \sum_{j=1}^{n} Y_{n,j}(x)$ with $Y_{n,j}(x) = (Z_{n,j}(x) - E[Z_{n,1}(x)])$. First we will show that

$$\frac{S_n(x)}{\sigma_n(x)\sqrt{n}} \overset{d}{\to} N(0, 1)$$

for $0 < x < \infty$. We consider the following decomposition of $S_n(x)$.

$$S_n(x) = \sum_{j=1}^{k_n} \xi_{n,j}(x) + \sum_{j=1}^{k_n} \eta_{n,j}(x) + \zeta_n(x),$$

where

$$\xi_{n,j}(x) = \sum_{k=(j-1)(n_j+s_n)+1}^{jl_n+(j-1)s_n} Y_{n,k}(x), \quad \eta_{n,j}(x) = \sum_{k=jl_n+(j-1)s_n+1}^{j(l_n+s_n)} Y_{n,k}(x),$$

$$\zeta_n(x) = \sum_{j=k_n(t_n+s_n)}^{n} Y_{n,j}(x).$$

We take $l_n = \lfloor \sqrt{n h / (\log n)} \rfloor$, $s_n = \lfloor (\sqrt{n h / \log n})^{1/6} \rfloor$. Since $(\log n)^4 \ll nh^{7/5}$, we have that

$$\frac{s_n}{l_n} = O \left( \frac{1}{nh^{7/5}} (\log n)^{5/3} \right) \to 0$$

and $k_n = [n/(l_n + s_n)] = O(\sqrt{n h / \log n})$. We show the desired result in several steps.

(Step 1): In this step, we will show that

$$\frac{S_n(x)}{\sigma_n(x)\sqrt{n}} \overset{d}{\to} \sum_{j=1}^{k_n} \xi_{n,j}(x) + o_P(1).$$

Note that $\beta$-mixing coefficients satisfy $n^6 \beta(n) \to 0$ as $n \to \infty$, we have that $k_n \beta(s_n) \to 0$ as $n \to \infty$. By the definition of $\eta_{n,1}(x)$, we have that

$$\frac{1}{s_n \sigma_n^2(x)} \text{Var}(\eta_{n,1}(x)) \leq \frac{\text{Var}(Z_{n,1}(x))}{\sigma_n^2(x)} + \frac{1}{\sigma_n^2(x)} \sum_{j=1}^{s_n} \left( 1 - \frac{j}{s_n} \right) \text{Cov}(Z_{n,1}(x), Z_{n,j+1}(x)) \lesssim 1.$$ 

Since $|\eta_{n,j}(x)|/(s_n h^{-(1+\delta)/2} \sigma_n(x))$ is bounded (see the comment after the proof of Lemma A.5), by Proposition 2.6 in Fan and Yao (2003), $\text{Cov}(\eta_{n,1}(x), \eta_{n,j+1}(x)) \lesssim s_n^2 h^{-(1+\delta)} \sigma_n^2(x) \beta(jl_n \Delta)$. Then we have that

$$\frac{1}{s_n \sigma_n^2(x)} \sum_{j=1}^{k_n} \left| \text{Cov}(\eta_{n,1}(x), \eta_{n,j+1}(x)) \right| \lesssim s_n h^{-(1+\delta)} \sum_{j=1}^{k_n} \beta(jl_n \Delta) \leq s_n h^{-(1+\delta)} \sum_{j=1}^{\infty} \beta(jl_n \Delta) \ll 1.$$
Therefore, we have that
\[
\frac{1}{n\sigma^2_n(x)} \text{Var} \left( \sum_{j=1}^{k_n} \eta_{n,j}(x) \right) \lesssim \frac{k_n \text{Var}(\eta_{n,1})}{n\sigma^2_n(x)} + \frac{2}{n\sigma^2_n(x)} \sum_{j=1}^{k_n-1} \left| \text{Cov}(\eta_{n,1}(x), \eta_{n,j+1}(x)) \right|
\]
\[
\lesssim \frac{k_n s_n}{n} + \frac{2}{n\sigma^2_n(x)} \sum_{j=1}^{k_n-1} \left| \text{Cov}(\eta_{n,1}(x), \eta_{n,j+1}(x)) \right| \to 0, \text{ as } n \to \infty.
\]

Likewise, we have that
\[
\frac{1}{n\sigma^2_n(x)} \text{Var}(\zeta_n(x)) = \frac{l_n + s_n}{n} \frac{1}{(l_n + s_n)\sigma^2_n(x)} \text{Var}(\zeta_n(x)) \to 0, \text{ as } n \to \infty
\]
since \(n - k_n(l_n + s_n) \lesssim (l_n + s_n)\).

(Step 2): We set \(T_n(x) = \sum_{j=1}^{k_n} \xi_{n,j}(x)\). In this step we show that
\[
\frac{T_n(x)}{\sigma_n(x)\sqrt{n}} \xrightarrow{d} N(0, 1).
\]
Define \(M_n = \left| E \left[ \exp \left( itT_n(x)/\sqrt{n\sigma^2_n(x)} \right) \right] - \exp(-t^2/2) \right|\), where \(i = \sqrt{-1}\). Then it is sufficient to show that for any \(\epsilon > 0\), \(\lim_{n \to \infty} M_n < \epsilon\). Note that
\[
M_n \leq \left| E \left[ \exp(itT_n(x)/\sqrt{n\sigma^2_n(x)}) \right] \right| - \prod_{j=1}^{k_n} E \left[ \exp(it\xi_{n,j}(x)/\sqrt{n\sigma^2_n(x)}) \right]
\]
\[
+ \prod_{j=1}^{k_n} \left| E \left[ \exp(it\xi_{n,j}(x)/\sqrt{n\sigma^2_n(x)}) \right] - \exp(-t^2/2) \right|
\]
\[
=: A_{n,1} + A_{n,2}.
\]
By Lemma 2.4 in [Fan and Masry (1992)] and \(k_n \beta(s_n) \to 0\) as \(n \to \infty\), we have that \(A_{n,1} \lesssim k_n \beta(s_n) \to 0\) as \(n \to \infty\).

Finally we show \(\lim_{n \to \infty} A_{n,2} = 0\). This is equivalent to showing that
\[
\frac{1}{\sqrt{n}} \tilde{T}_n(x) \xrightarrow{d} N(0, 1), \quad (A.9)
\]
where \(\tilde{T}_n(x) = \sum_{j=1}^{n} \tilde{\xi}_{n,j}\) and \(\{\tilde{\xi}_{n,j}(x)\}\) are independent random variables such that \(\tilde{\xi}_{n,j}(x) \xrightarrow{d} \xi_{n,j}(x)/\sigma_n(x)\). It is easy to show that \(\{\xi_{n,j}(x)/\sigma_n(x)\}\) is a sequence of bounded random variables. To show \((A.9)\), it is sufficient to check the following Lindeberg condition.
\[
\frac{1}{nh} \sum_{j=1}^{k_n} E[|\tilde{\xi}_{n,j}(x)|^2 1\{|\tilde{\xi}_{n,j}(x)| > \omega\sqrt{n}\}] \to 0, \text{ as } n \to \infty.
\]
for any $\omega > 0$. By Hölder’s inequality, Markov’s inequality and Proposition 2.7 in Fan and Yao (2003), we have that

\[
E[|\tilde{\xi}_{n,j}|^21\{|\tilde{\xi}_{n,j}| > \omega \sqrt{n}\}] \leq E[|\tilde{\xi}_{n,j}|^4]^{1/2}P(|\tilde{\xi}_{n,j}| > \omega \sqrt{n})^{1/2}
\]

\[
\lesssim (l_n^{4/2})^{1/2}E[|\tilde{\xi}_{n,j}|^2]^{1/2} \lesssim l_n \left( \frac{l_n}{\sqrt{nh}} \right)^3 \frac{1}{(nh)^{3/2}}.
\]

Therefore, we have that

\[
\frac{1}{nh} \sum_{j=1}^{k_n} E[|\tilde{\xi}_{n,j}|^21\{\tilde{\xi}_{n,j} > \omega \sqrt{n}\}] \lesssim \frac{k_n l_n}{n} \left( \frac{l_n}{\sqrt{nh}} \right)^3 \left( \frac{1}{nh^3} \right)^{2/3} \to 0, \text{ as } n \to \infty
\]

since $nh^{5/3} \to \infty$.

(Step 3): In this step, we complete the proof. Considering [A.8], Condition (vi) in Assumption 3.1 and Lemma A.9 yields that the bias term $I_n$ is asymptotically negligible since $\sqrt{nh^{2\gamma + 1 - \delta}} \log n \to 0$ as $n \to \infty$. This implies that

\[
\frac{\sqrt{nh}(\hat{k}_2(x) - k_2(x))}{\sigma_n(x)} - \frac{S_n(x)}{\sigma_n(x)\sqrt{n}} = o_P((\log n)^{-1/2})
\]

and the asymptotic distribution of $\sqrt{nh}(\hat{k}_2(x) - k_2(x))$ is the same as that of $S_n(x)$. Moreover, Proposition A.3 implies that asymptotic covariance between $S_n(x_1)/\sqrt{n}$ and $S_n(x_2)/\sqrt{n}$ for different design points $0 < x_1 < x_2 < \infty$ is asymptotically negligible. Therefore, we finally obtain the desired result. □

### A.2. Proofs for Section 4

We note that Lemmas and Propositions in Section A.1 also hold when $0 < x_1 < \cdots < x_N < \infty$, $x_\ell \in I$ for $\ell = 1, \ldots, N$, and $\min_{1 \leq k \neq \ell \leq N} |x_k - x_\ell| \gg h^{1-2\delta}$. In particular, we need to take into account the effect of the separation between points in the proof of Lemmas 4.1 and A.10, and Theorem A.1. In the proof of Theorem A.1, we use the lower bound of $\min_{1 \leq \ell \leq N} \sigma_n(x_\ell)$ to obtain an intermediate Gaussian approximation result. We also need to take care of the effect of the discretization of a compact set $I$ to obtain the consistency of $\hat{\sigma}_n^2(x)$ on the discrete points in Lemma 4.1, that is, $\max_{1 \leq \ell \leq N} |\hat{\sigma}_n^2(x_\ell)/\sigma_n^2(x_\ell) - 1| \overset{P}{\to} 0$. Moreover, in the proof of Lemma A.10, we use the condition $\min_{1 \leq k \neq \ell \leq N} |x_k - x_\ell| \gg h^{1-2\delta}$ to obtain a result that the variance-covariance matrix a random vector $(W_n(x_1), \ldots, W_n(x_N))^\top$ can be approximated by the $N \times N$ identity matrix and this yields a Gaussian comparison result (Proposition A.4).

**Proof of Lemma 4.1** Since $\|K_n\|_\mathbb{R} \lesssim h^{-\alpha}$ and we can show $\|K_n - \hat{K}_n\|_\mathbb{R} \lesssim h^{-2\alpha}n^{-1/2}(\log n)$, we have that

\[
\|\hat{K}_n\|_\mathbb{R} \leq \|K_n\|_\mathbb{R} + \|K_n - \hat{K}_n\|_\mathbb{R} \lesssim h^{-\alpha} + h^{-2\alpha}n^{-1/2}(\log n) \lesssim h^{-\alpha}.
\]
Therefore, we have that \(\|K_n^2 - \hat{K}_n^2\|_\mathbb{R} \leq \|K_n + \hat{K}_n\|_\mathbb{R}\|K_n - \hat{K}_n\|_\mathbb{R} \lesssim h^{-3\alpha}n^{-1/2}(\log n)\). Then we have that

\[
\max_{1 \leq \ell \leq N} \left| \frac{1}{n} \sum_{j=1}^{n} X_j \Delta 1\{|X_j\Delta| \leq \theta_n\} \left\{ \hat{K}_n ((x_\ell - X_j\Delta)/h) - K_n ((x_\ell - X_j\Delta)/h) \right\} \right| \\
\leq \left( \frac{1}{n} \sum_{j=1}^{n} X_j \Delta 1\{|X_j\Delta| \leq \theta_n\} \right) \|\hat{K}_n - K_n\|_\mathbb{R} = O_P(h^{-2\alpha}n^{-1/2}(\log n)),
\]

and likewise,

\[
\max_{1 \leq \ell \leq N} \left| \frac{1}{n} \sum_{j=1}^{n} X_j^2 \Delta 1\{|X_j\Delta| \leq \theta_n\} \left\{ \hat{K}_n^2 ((x_\ell - X_j\Delta)/h) - K_n^2 ((x_\ell - X_j\Delta)/h) \right\} \right| = O_P(h^{-3\alpha}n^{-1/2}(\log n)).
\]

Since \((h^{-2\alpha}n^{-1/2}(\log n))^2/(h^{-3\alpha}n^{-1/2}(\log n)) = h^{-\alpha}n^{-1/2}(\log n) \ll 1\), we have that

\[
\hat{\sigma}_n^2(x) = \frac{1}{n} \sum_{j=1}^{n} Z_{n,j}(x) - \left( \frac{1}{n} \sum_{j=1}^{n} Z_{n,j}(x) \right)^2 + O_P(h^{-3\alpha}n^{-1/2}(\log n))
\]

uniformly \(x = x_\ell, \ell = 1, \ldots, N\). Furthermore, since \(\min_{1 \leq \ell \leq N} \sigma_n^2(x_\ell) \geq h^{-2\alpha+\delta+1}\) and

\[
\frac{h^{-3\alpha}n^{-1/2}(\log n)}{h^{-2\alpha+\delta+1}} = h^{-\alpha-\delta-1}n^{-1/2}(\log n) \ll (\log n)^{-1},
\]

it remains to prove that \(\max_{1 \leq \ell \leq N} |\hat{\sigma}_n^2(x_\ell)/\sigma_n^2(x_\ell) - 1| = o_P((\log n)^{-1})\). Since \(h^\alpha yK_n((x-y)/h)\) is uniformly bounded in \(n\) and \(x_\ell\) for \(\ell = 1, \ldots, N\) (see the comment after the proof of Lemma A.5), we have that

\[
\max_{1 \leq \ell \leq N} E[|X_1|1\{|X_1| > \theta_n\}K_n((x_\ell - X_1)/h)] \lesssim \frac{h^{-\alpha}P(|X_1| > \theta_n)}{h^{-\alpha+\delta/2+1/2}} \lesssim h^{-\delta/2-1/2}\theta_n\lesssim (\log n)^{-1/2},
\]

\[
\max_{1 \leq \ell \leq N} E[X^2_11\{|X_1| > \theta_n\}K_n^2((x_\ell - X_1)/h)] \lesssim \frac{h^{-2\alpha}E[|X_1|1\{|X_1| > \theta_n\}]}{h^{-2\alpha+\delta+1}} \lesssim h^{-\delta-1}E[|X_1|^3/\theta_n^2] \lesssim h^{-\delta-1}\theta_n^{-2} \ll (\log n)^{-1}.
\]

Therefore, to complete the proof, it suffices to prove that

\[
\max_{1 \leq \ell \leq N} \left| \frac{1}{n} \sum_{j=1}^{n} \left( \frac{Z_{n,j}(x_\ell) - E[Z_{n,j}(x_\ell)]}{\sigma_n^2(x_\ell)} \right) \right| = o_P((\log n)^{-1}), \quad \text{and} \quad (A.10)
\]

\[
\max_{1 \leq \ell \leq N} \left| \frac{1}{n} \sum_{j=1}^{n} \left( \frac{Z_{n,j}(x_\ell) - E[Z_{n,j}(x_\ell)]}{\sigma_n(x_\ell)} \right) \right| = o_P((\log n)^{-1/2}). \quad (A.11)
\]
To prove (A.10), we use Theorem 2.18 in Fan and Yao (2003) with $b = h^{-\delta - 1}$, $q = [h^{-\delta - 2}]$ and $[n/2] \ll n$, and $\epsilon = \epsilon_0(\log n)^{-1}$ for any $\epsilon_0 > 0$ in their notations. Here, $[a]$ is the integer part of $a \in \mathbb{R}$. In this case we have that

$$P \left( \max_{1 \leq \ell \leq N} \left| \frac{1}{n} \sum_{j=1}^{n} \left( \frac{Z_{n,j}(x_{\ell}) - E[Z_{n,j}(x_{\ell})]}{\sigma_n(x_{\ell})} \right) \right| > \epsilon_0(\log n)^{-1} \right)$$

$$\leq \sum_{\ell=1}^{N} P \left( \frac{1}{n} \sum_{j=1}^{n} \left( \frac{Z_{n,j}(x_{\ell}) - E[Z_{n,j}(x_{\ell})]}{\sigma_n(x_{\ell})} \right) > \epsilon_0(\log n)^{-1} \right)$$

$$\lesssim h^{-1+2\delta} \left( \exp \left( -\frac{h^{-1}}{8(\log n)^2} \right) + \sqrt{\frac{1 + h^{-\delta - 1}(\log n)}{\epsilon_0} h^{-\delta - 2} e^{-\Delta \beta_1 n h^{\delta + 2}}} \right) \to 0$$

as $n \to \infty$, and likewise, we can show (A.11). Therefore, we complete the proof. 

Let $q > r$ be positive integers such that

$$q + r \leq n/2, \quad q = q_n \to \infty, \quad q_n = o(n), \quad r = r_n \to \infty, \quad \text{and} \quad r_n = o(q_n) \quad \text{as} \quad n \to \infty,$$

and $m = m_n = \lfloor n/(q+r) \rfloor$. Consider a partition \( \{I_j\}_{j=1}^m \cup \{J_{j}\}_{j=1}^{m+1} \) of \( 1, \ldots, n \) where \( I_j = \{(j-1)(q+r)+1, \ldots, jq+(j-1)r\} \), \( J_j = \{jq+(j-1)r+1, \ldots, j(q+r)\} \) and \( J_{m+1} = \{m(q+r), \ldots, n\} \). First we show the following result on Gaussian approximation.

**Theorem A.1.** Under Assumption 3.1, we have that

$$\sup_{t \in \mathbb{R}} P \left( \max_{1 \leq \ell \leq N} \left| W_n(x_{\ell}) \right| \leq t \right) - P \left( \max_{1 \leq \ell \leq N} \left| \tilde{Y}_{\ell,n} \right| \leq t \right) \to 0, \quad \text{as} \quad n \to \infty,$$

where, \( \tilde{Y}_n = (\tilde{Y}_{n,1}, \ldots, \tilde{Y}_{n,N})^\top \) is a centered normal random vector with covariance matrix \( E[\tilde{Y}_n \tilde{Y}_n^\top] = (mq)^{-1} \sum_{j=1}^{m} E \left[ W_{I_j} W_{I_j}^\top \right] = q^{-1} E \left[ W_{I_1} W_{I_1}^\top \right] \) where \( W_{I_j} = \left( \sum_{k \in I_j} \frac{Z_{n,k}(x_{1}) - E[Z_{n,1}(x_{1})]}{\sigma_n(x_{1})} \right), \ldots, \sum_{k \in I_j} \frac{Z_{n,k}(x_{N}) - E[Z_{n,1}(x_{N})]}{\sigma_n(x_{N})} \right)^\top \) =: \( \left( W_{I_j}(x_{1}), \ldots, W_{I_j}(x_{N}) \right)^\top \).

**Proof.** Since $h^\alpha y K_n((x-y)/h)$ is uniformly bounded in $n$ and $x = x_{\ell}$, $\ell = 1, \ldots, N$ as a function of $y$ (see the comment after the proof of Lemma A.5) and $\min_{1 \leq \ell \leq N} \sigma_n(x_{\ell}) \geq \sqrt{h^{-2\alpha+3+1}}$, we have that

$$\left| \frac{Z_{n,j}(x_{\ell}) - E[Z_{n,j}(x_{\ell})]}{\sigma_n(x_{\ell})} \right| \lesssim h^{-(\delta+1)/2}$$

and $h^{-1/2}(\log N n)^{5/2} \ll n^{1/8}$. Therefore, if we take $q_n = O(n^{q'})$ and $r_n = O(n^{r'})$ with $0 < r' < q' < 3/8$, we have that $q_n h^{-(\delta+1)/2}(\log N n)^{5/2} \ll n^{1/2 - (1/8 + q')}$, $(r_n/q_n)(\log N)^2 \ll n^{-(q'-r')/2}$ and
Since 1
\[ m_n \beta X(r_n) \lesssim m_n e^{-\beta r_n} \lesssim n^{-(q'-r)/2}. \]
Moreover, define
\[ \sigma^2(q) := \max_{1 \leq \ell \leq N} \max_I \text{Var} \left( \frac{1}{\sigma_n(x_\ell)} \sum_{k \in I} (Z_{n,k}(x_\ell) - E[Z_{n,1}(x_\ell)]) \right), \]
\[ \sigma^2(q) := \min_{1 \leq \ell \leq N} \min_I \text{Var} \left( \frac{1}{\sigma_n(x_\ell)} \sum_{k \in I} (Z_{n,k}(x_\ell) - E[Z_{n,1}(x_\ell)]) \right), \]
where \( \max_I \) and \( \min_I \) are taken over all \( I \subset \{1, \ldots, n\} \) of the form \( I = \{j+1, \ldots, j+q\} \). By the stationarity of \( \{X_{j\Delta}\}_{j \geq 0} \) and Proposition A.2, we have that
\[ \sigma^2(q) = \sigma^2 \sim \max_{1 \leq \ell \leq N} (\text{Var}(Z_{n,1}(x_\ell))/\sigma_n(x_\ell)), \]
\[ \sigma^2(q) = \sigma^2 \sim \min_{1 \leq \ell \leq N} (\text{Var}(Z_{n,1}(x_\ell))/\sigma_n(x_\ell)). \]

Then there exists constants \( 0 < c_1, C_1 < \infty \) such that \( c_1 \leq \sigma^2(q) \leq \sigma^2(r) \vee \sigma^2(q) \leq C_1 \). From the above arguments, the conditions of Theorem B.1 in Chernozhukov et al. (2013) are satisfied. So, we have the desired result.

Next we show that the distribution of \( \max_{1 \leq \ell \leq N} |\tilde{Y}_{n,\ell}| \) can be approximated by that of \( \max_{1 \leq \ell \leq N} |Y_\ell| \) where \( Y = (Y_1, \ldots, Y_N)^T \) is a normal random vector in \( \mathbb{R}^N \). For this, we prepare two lemmas.

**Lemma A.10.** Under Assumption [3.1], we have that
\[ \max_{1 \leq 1, \ell \leq N} |\text{Cov}(W_n(x_k), W_n(x_\ell)) - 1_{\{x_k = x_\ell\}}| = O(h^\delta). \]

**Proof.** Since the covariance between \( Z_{n,j}(x_\ell) \) and \( Z_{n,k}(x_\ell) \) for \( j \neq k \) is asymptotically negligible with respect to the variances of each term by the proof of Proposition A.3, it is sufficient to prove
\[ \max_{1 \leq 1, \ell \leq N} \left| \frac{\text{Cov}(Z_{n,1}(x_k), Z_{n,1}(x_\ell))}{\sqrt{\sigma_n^2(x_k) \sigma_n^2(x_\ell)}} - 1_{\{x_k = x_\ell\}} \right| = O(h^\delta). \]

Since \( 1/ \min_{1 \leq 1, \ell \leq N} \sigma_n^2(x) \lesssim h^{2\alpha-\delta-1} \), from the same argument of the proof of Proposition A.3 we have that
\[ \max_{1 \leq 1, \ell \leq N} \left| \frac{\text{Cov}(Z_{n,1}(x_k), Z_{n,1}(x_\ell))}{\sqrt{\sigma_n^2(x_k) \sigma_n^2(x_\ell)}} - 1_{\{x_k = x_\ell\}} \right| \lesssim h^{2\alpha-\delta-1} \max_{1 \leq 1, \ell \leq N} \left| \text{Cov}(Z_{n,1}(x_k), Z_{n,1}(x_\ell)) \right| \]
\[ \lesssim \frac{h^{2\delta}}{\min_{1 \leq 1, \ell \leq N} (x_k - x_\ell)^2} \vee h^{\delta} \lesssim h^\delta \]
since \( \min_{1 \leq 1, \ell \leq N} (x_k - x_\ell)^2 \gg h^{2-\delta} \). Then we have the desired result.

**Lemma A.11.** Under Assumption [3.1], we have that
\[ \max_{1 \leq 1, \ell \leq N} \left| q^{-1} \text{Cov}(W_{11}(x_k), W_{11}(x_\ell)) - 1_{\{x_k = x_\ell\}} \right| = O(h^\delta). \]
Proof. Form the same argument of the proof Propositions A.2 and A.3

\[
\frac{1}{q} \sum_{k, \ell \in I, k \neq \ell} \frac{\operatorname{Cov}(Z_n, k(x_{m_1}), Z_n, \ell(x_{m_2}))}{\sigma_n(x_{m_1})\sigma_n(x_{m_2})}
\]

is asymptotically ignorable for \(1 \leq m_1, m_2 \leq N\). Therefore, the proof of Lemma A.10 yields that

\[
\max_{1 \leq k, \ell \leq N} |q^{-1} \operatorname{Cov}(W_{I_1}(x_k), W_{I_1}(x_\ell)) - 1_{\{x_k = x_\ell\}}| = O(h^\delta).
\]

This completes the proof. \(\square\)

Lemma A.11 and Condition (vi) in Assumption 3.1 yield the following result on Gaussian comparison:

**Proposition A.4.** Under Assumption 3.1, we have that

\[
\sup_{t \in \mathbb{R}} \left| P \left( \max_{1 \leq \ell \leq N} \left| Y_n, \ell \right| \leq t \right) - P \left( \max_{1 \leq \ell \leq N} \left| Y_\ell \right| \leq t \right) \right| \to 0, \text{ as } n \to \infty,
\]

where \(Y = (Y_1, \ldots, Y_N)^\top\) is a standard normal random vector in \(\mathbb{R}^N\).

**Proof.** Let \(\Delta(Y_n, Y) := \max_{1 \leq k, \ell \leq N} |\operatorname{Cov}(Y_n, k, Y_n, \ell) - 1_{\{x_k = x_\ell\}}|\). By Lemma A.11 and Theorem 2 in Chernozhukov et al. (2015), we have that

\[
\sup_{t \in \mathbb{R}} \left| P \left( \max_{1 \leq \ell \leq N} \left| \hat{Y}, \ell \right| \leq t \right) - P \left( \max_{1 \leq \ell \leq N} \left| Y_\ell \right| \leq t \right) \right| \leq \Delta(Y_n, Y)^{1/3}(1 \vee \log(N/\Delta(Y_n, Y)))^{2/3} \to 0
\]
as \(n \to \infty\). Therefore, we obtain the desired result. \(\square\)

**Proof of Theorem 4.1.** Theorem 4.1 immediately follows from Theorem A.1 and Proposition A.4. \(\square\)

**Proof of Theorem 4.2.** The asymptotic linear representation (A.8) yields that

\[
U_n := \max_{1 \leq \ell \leq N} \left| \frac{\sqrt{n}h(\hat{k}_n(x_\ell) - k_0(x_\ell))}{\sigma_n(x_\ell)} \right| = \max_{1 \leq \ell \leq N} |W_n(x_\ell)| + o_P((\log n)^{-1/2})
\]

\(= V_n + o_P((\log n)^{-1/2})\).

This also implies that there exists a sequence of constants \(\epsilon_n \downarrow 0\) such that

\[
P \left( |U_n - V_n| > \epsilon_n(\log n)^{-1/2} \right) \leq \epsilon_n
\]
(which follows from the fact that convergence in probability is metrized by the Ky Fan metric; see Theorem 9.2.2 in Dudley (2002)). Then we have that
\[
P(U_n \leq t) \leq P\left(\{U_n \leq t\} \cap \{|U_n - V_n| \leq \epsilon_n (\log n)^{-1/2}\}\right)
+ P\left(\{U_n \leq t\} \cap \{|U_n - V_n| > \epsilon_n (\log n)^{-1/2}\}\right)
\leq P\left(V_n \leq t + \epsilon_n (\log n)^{-1/2}\right) + \epsilon_n
\]
for any \(t \in \mathbb{R}\). Theorem \ref{thm:anti-concentration} yields that there exists a sequence of constants \(\tilde{\epsilon}_n \downarrow 0\) such that
\[
P\left(V_n \leq t + \epsilon_n (\log n)^{-1/2}\right) \leq P\left(G_n \leq t + \epsilon_n (\log n)^{-1/2}\right) + \tilde{\epsilon}_n
\]
for any \(t \in \mathbb{R}\) where \(G_n = \max_{1 \leq \ell \leq N} |Y_\ell|\). From the anti-concentration inequality for the maxima of Gaussian random vector (Theorem 3 in Chernozhukov et al. (2015)), the right hand side is bounded from above by \(P(G_n \leq t) + 8 \epsilon_n (\log n)^{-1/2} E[|G_n|] + \tilde{\epsilon}_n\). Since \(E[|G_n|] \leq D \log n\) for some positive constant \(D\) which does not depend on \(n\), we have that
\[
P(U_n \leq t) \leq P(G_n \leq t) + 9D \epsilon_n + \tilde{\epsilon}_n = P(G_n \leq t) + o(1) \quad \text{(A.12)}
\]
for any \(t \leq \mathbb{R}\). We also have that
\[
P\left(V_n \leq t - \epsilon_n (\log n)^{-1/2}\right) \leq P\left(\{V_n \leq t - \epsilon_n (\log n)^{-1/2}\} \cap \{|U_n - V_n| \leq \epsilon_n (\log n)^{-1/2}\}\right)
+ P\left(\{V_n \leq t - \epsilon_n (\log n)^{-1/2}\} \cap \{|U_n - V_n| > \epsilon_n (\log n)^{-1/2}\}\right)
\leq P(U_n \leq t) + \epsilon_n
\]
for any \(t \in \mathbb{R}\). Therefore, we can show that
\[
P(U_n \leq t) \geq P(G_n \leq t) - 9D \epsilon_n - \tilde{\epsilon}_n = P(G_n \leq t) + o(1) \quad \text{(A.13)}
\]
for any \(t \in \mathbb{R}\). Combining (A.12) with (A.13), we obtain the desired result. \(\square\)

**Appendix B. On asymptotic validity of confidence bands**

We use the notations used in the proof of Theorem \ref{thm:anti-concentration} here. Let \(q^{U_n}_\tau\) denotes the \((1-\tau)\)-quantile of \(U_n\). Theorem \ref{thm:anti-concentration} implies that there exists a sequence \(\epsilon'_n \downarrow 0\) such that
\[
\sup_{t \in \mathbb{R}} |P(U_n \leq t) - P(G_n \leq t)| \leq \epsilon'_n.
\]
Then we have that
\[
P(U_n \leq q^{U_n}_\tau - \epsilon'_n) \geq P(G_n \leq q^{U_n}_\tau - \epsilon'_n) - \epsilon'_n = 1 - \tau,
\]
where the last inequality holds \(G_n\) has continuous distribution from the anti-concentration inequality (see Theorem 3 in Chernozhukov et al. (2015)). This yields the inequality \(q^{U_n}_\tau \leq q^{U_n}_\tau - \epsilon'_n\).
Therefore, we have that
\[ P(U_n \leq q_\tau) \leq P(U_n \leq q_\tau - \epsilon'_n) \]
\[ \leq P(G_n \leq q_\tau - \epsilon'_n) + \epsilon'_n = 1 - \tau + 2\epsilon'_n. \]
Likewise, we have the inequality \[ q_\tau + \epsilon'_n \leq q^{U_n}_\tau. \] This yields that
\[ P(U_n \leq q_\tau) \geq 1 - \tau - 2\epsilon'_n. \]
Then we obtain \[ P(U_n \leq q_\tau) \to 1 - \tau \] as \( n \to \infty \).

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