Proximal Stochastic Dual Coordinate Ascent

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Abstract

We introduce a proximal version of dual coordinate ascent method. We demonstrate how the derived algorithmic framework can be used for numerous regularized loss minimization problems, including $\ell_1$ regularization and structured output SVM. The convergence rates we obtain match, and sometimes improve, state-of-the-art results.

1 Introduction

We consider the following generic optimization problem associated with regularized loss minimization of linear predictors: Let $X_1, \ldots, X_n$ be matrices in $\mathbb{R}^{d \times k}$, let $\phi_1, \ldots, \phi_n$ be a sequence of vector convex functions defined on $\mathbb{R}^k$, and $g(\cdot)$ is a convex function defined on $\mathbb{R}^d$. Our goal is to solve $\min_{w \in \mathbb{R}^d} P(w)$ where

$$P(w) = \left[ \frac{1}{n} \sum_{i=1}^{n} \phi_i (X_i^T w) + \lambda g(w) \right],$$

and $\lambda \geq 0$ is a regularization parameter. We will later show how to use a solver for (1) for several popular regularized loss minimization problems including $\ell_1$ regularization and structured output SVM.

Let $w^*$ be the optimum of (1). We say that a solution $w$ is $\epsilon_P$-sub-optimal if $P(w) - P(w^*) \leq \epsilon_P$. We analyze the runtime of optimization procedures as a function of the time required to find an $\epsilon_P$-sub-optimal solution.

The dual coordinate ascent (DCA) method solves a dual problem of (1). Specifically, for each $i$ let $\phi_i^*: \mathbb{R}^k \rightarrow \mathbb{R}$ be the convex conjugate of $\phi_i$, namely, $\phi_i^*(u) = \max_{z \in \mathbb{R}^k} (z^T u - \phi_i(z))$. Similarly we define the convex conjugate $g^*$ of $g$. The dual problem is

$$\max_{\alpha \in \mathbb{R}^{k \times n}} D(\alpha) \text{ where } D(\alpha) = \left[ \frac{1}{n} \sum_{i=1}^{n} -\phi_i^* (-\alpha_i) - \lambda g^* \left( \frac{1}{\lambda n} \sum_{i=1}^{n} X_i \alpha_i \right) \right],$$

where $\alpha_i$ is the $i$'th column of the matrix $\alpha$, which forms a vector in $\mathbb{R}^k$. The dual objective in (2) has a different dual vector associated with each example in the training set. At each iteration of DCA, the dual objective is optimized with respect to a single dual vector, while the rest of the dual vectors are kept intact.

We assume that $g^*(\cdot)$ is continuous differentiable. If we define

$$w(\alpha) = \nabla g^*(v(\alpha)) \quad v(\alpha) = \frac{1}{\lambda n} \sum_{i=1}^{n} X_i \alpha_i,$$
then it is known that \( w(\alpha^*) = w^* \), where \( \alpha^* \) is an optimal solution of (2). It is also known that \( P(w^*) = D(\alpha^*) \) which immediately implies that for all \( w \) and \( \alpha \), we have \( P(w) \geq D(\alpha) \), and hence the duality gap defined as
\[
P(w(\alpha)) - D(\alpha)
\]
can be regarded as an upper bound on the primal sub-optimality \( P(w(\alpha)) - P(w^*) \).

We focus on a stochastic version of DCA, abbreviated by SDCA, in which at each round we choose which dual vector to optimize uniformly at random. We analyze SDCA either for \( L \)-Lipschitz loss functions or for \((1/\gamma)\)-smooth loss functions, which are defined as follows.

**Definition 1.** A function \( \phi : \mathbb{R}^k \rightarrow \mathbb{R} \) is \( L \)-Lipschitz if for all \( a, b \in \mathbb{R}^k \), we have
\[
|\phi_i(a) - \phi_i(b)| \leq L \|a - b\|_P,
\]
where \( \| \cdot \|_P \) is a norm.

A function \( \phi : \mathbb{R}^k \rightarrow \mathbb{R} \) is \((1/\gamma)\)-smooth if it is differentiable and its gradient is \((1/\gamma)\)-Lipschitz. An equivalent condition is that for all \( a, b \in \mathbb{R} \), we have
\[
\phi_i(a) \leq \phi_i(b) + \nabla \phi_i(b) ^\top (a - b) + \frac{1}{2\gamma} \|a - b\|_D^2.
\]

It is well-known that if \( \phi_i(a) \) is \((1/\gamma)\)-smooth, then \( \phi_i^* (u) \) is \( \gamma \) strongly convex w.r.t. the dual norm: for all \( u, v \in \mathbb{R} \) and \( s \in [0, 1] \):
\[
-\phi_i^*)(su + (1-s)v) \geq -s\phi_i^*(u) - (1-s)\phi_i^*(v) + \frac{\gamma s(1-s)}{2} \|u - v\|_D^2,
\]
where \( \| \cdot \|_D \) is the dual norm of \( \| \cdot \|_P \) defined as
\[
\|u\|_D = \sup_{\|v\|_P = 1} u^\top v.
\]

We also assume that \( g(w) \) is 1-strongly convex with respect to another norm \( \| \cdot \|_{P'} \):
\[
g(w + \Delta w) \geq g(w) + \nabla g(w)^\top \Delta w + \frac{1}{2} \|\Delta w\|_{P'}^2,
\]
which means that \( g^*(w) \) is 1-smooth with respect to its dual norm \( \| \cdot \|_{D'} \). Namely,
\[
g^*(v + \Delta v) \leq h(v; \Delta v), \tag{4}
\]
where
\[
h(v; \Delta v) := g^*(v) + \nabla g^*(v)^\top \Delta v + \frac{1}{2} \|\Delta v\|_{D'}^2. \tag{5}
\]

## 2 Main Results

The generic Prox-SDCA algorithm which we analyze in this paper is presented in Figure 1. The ideas are described as follows. Consider the maximal increase of the dual objective, where we only allow to change the \( i \)'th column of \( \alpha \). At step \( t \), let \( v(t-1) = (\lambda n)^{-1} \sum_i X_i \alpha_i^{(t-1)} \) and let \( w(t-1) = \nabla g^*(v(t-1)) \). We will update the \( i \)-th dual variable \( \alpha_i^{(t)} = \alpha_i^{(t-1)} + \Delta \alpha_i \), in a way that will lead to a sufficient increase of the dual
objective. For primal variable, this would lead to the update
\[ v(t) = v(t-1) + (\lambda n)^{-1} X_i \Delta a, \]
and therefore
\[ w(t) = \nabla g^*(v(t)) \]
can also be written as
\[ w(t) = \arg\max_w \left[ w^\top v(t) - g(w) \right] = \arg\min_w \left[ -w^\top \left( n^{-1} \sum_{i=1}^{n} X_i a_i(t) \right) + \lambda g(w) \right]. \]

Note that this particular update is rather similar to the update step of proximal-gradient dual-averaging method in the SGD domain [Xiao 2010]. The difference is on how \( a(t) \) is updated, and as we will show later, stronger results can be proved for the Prox-SDCA method when we run SDCA for \( t > n \) iterations with smooth loss functions.

In order to motivate the proximal SDCA algorithm, we note that the goal of SDCA is to increase the dual objective as much as possible, and thus the optimal way to choose \( \Delta a_i \) would be to maximize the dual objective, namely, we shall let
\[ \Delta a_i = \arg\max_{\Delta a_i \in \mathbb{R}^k} \left[ -\frac{1}{n} \phi^*_i(-(a_i + \Delta a_i)) - \lambda g^*(v(t-1) + (\lambda n)^{-1} X_i \Delta a_i) \right]. \]

However, for complex \( g^*(\cdot) \), this optimization problem may not be easy to solve. We will simplify this optimization problem by relying on (4). That is, instead of directly maximizing the dual objective function, we try to maximize the following proximal objective which is a lower bound of the dual objective:
\[
\begin{align*}
\arg\max_{\Delta a_i \in \mathbb{R}^k} \left[ -\frac{1}{n} \phi^*_i(-(a_i + \Delta a_i)) - \lambda g^*(v(t-1)) - (\lambda n)^{-1} X_i \Delta a_i \right.
&\left. - \frac{1}{2} \lambda \| (\lambda n)^{-1} X_i \Delta a_i \|_{D'}^2 \right]
\end{align*}
\]

However, in general, this optimization problem is not necessarily simple to solve. We will thus also propose alternative update rules for \( \Delta a_i \) of the form \( \Delta a_i = s(u - a_i(t-1)) \) for an appropriately chosen step size parameter \( s > 0 \) and any vector \( u \in \mathbb{R}^k \) such that \(-u \in \partial \phi_i(X_i^\top u(t-1))\). Our analysis shows that an appropriate choice of \( s \) still leads to a sufficient increase in the dual objective.

We analyze the algorithm based on different assumptions on the loss functions. To simplify the statements of our theorems, we always make the following assumptions:

- Assume that the loss functions satisfy
  \[ \frac{1}{n} \sum_{i=1}^{n} \phi_i(0) \leq 1 \quad \text{and} \quad \forall i, a, \phi_i(a) \geq 0. \]

- Assume that \( \max_i \| X_i \| \leq R \), where
  \[ \| X_i \| = \sup_{u \neq 0} \frac{\| X_i u \|_{D'}}{\| u \|_D}. \]

Under the above assumptions, we have the following convergence result for smooth loss functions.
Procedure Prox-SDCA

Parameters scalars \( \lambda, \gamma \) (\( \gamma \) can be 0), \( R \), norms \( \| \cdot \|_D, \| \cdot \|_{D'} \)

Let \( \alpha^{(0)} = 0, w^{(0)} = \nabla g^*(0) \)

Iterate: for \( t = 1, 2, \ldots, T \):
- Randomly pick \( i \)
- Find \( \Delta \alpha_i \) using any of the following options (or achieving larger dual objective than one of the options):
  
  **Option I:**
  \[
  \Delta \alpha_i \in \arg\max_{\Delta \alpha_i} \left[ -\phi_i^* \left( -\alpha_i^{(t-1)} + \Delta \alpha_i \right) - w^{(t-1)}^T X_i \Delta \alpha_i - \frac{1}{2\lambda n} \| X_i \Delta \alpha_i \|_{D'}^2 \right]
  \]

  **Option II:**
  Let \( u \) be s.t. \( -u \in \partial \phi_i (X_i^T w^{(t-1)}) \)
  Let \( z = u - \alpha_i^{(t-1)} \)
  Let \( s = \arg\max_{s \in [0,1]} \left[ -\phi_i^* \left( -\alpha_i^{(t-1)} + sz \right) - s w^{(t-1)}^T X_i z - \frac{s^2}{2\lambda n} \| X_i z \|_{D'}^2 \right] \)
  Set \( \Delta \alpha_i = s z \)

  **Option III:**
  Same as Option II but replace the definition of \( s \) as follows:
  
  Let \( s = \frac{\phi_i(X_i^T w^{(t-1)}) + \phi_i^*(-\alpha_i^{(t-1)}) + w^{(t-1)}^T X_i \alpha_i^{(t-1)} + \frac{2}{\lambda n} \| z \|_{D'}^2}{\| z \|_{D'}^2 (\gamma + \| X_i \|_2^2 / (\lambda n))} \)

  **Option IV:**
  Same as Option III but replace \( \| X_i \|_2^2 \) in the definition of \( s \) with \( R^2 \)
  May also replace \( \| z \|_{D'}^2 \) with an upper bound no larger than \( 4L^2 \) for \( L \)-Lipschitz non-smooth loss

  **Option V (only for smooth losses):**
  Set \( \Delta \alpha_i = \frac{\lambda n \gamma}{R^2 + \lambda n \gamma} \left( -\nabla \phi_i (X_i^T w^{(t-1)}) - \alpha_i^{(t-1)} \right) \)
  \( \alpha_i^{(t)} \leftarrow \alpha_i^{(t-1)} + \Delta \alpha_i e_i \)
  \( w_i^{(t)} \leftarrow w_i^{(t-1)} + (\lambda n)^{-1} X_i \Delta \alpha_i \)
  \( w_i^{(t)} \leftarrow \nabla g^*(v_i^{(t)}) \)

Output (Averaging option):
- Let \( \bar{\alpha} = \frac{1}{T-T_0} \sum_{i=T_0+1}^T \alpha_i^{(t-1)} \)
- Let \( \bar{w} = \nabla g^*(\bar{\alpha}) = \frac{1}{T-T_0} \sum_{i=T_0+1}^T w_i^{(t-1)} \)
- return \( \bar{w} \)

Output (Random option):
- Let \( \alpha = \alpha_i^{(t)} \) and \( \bar{w} = w_i^{(t)} \) for some random \( t \in T_0 + 1, \ldots, T \)
- return \( \bar{w} \)

Figure 1: The Generic Proximal Stochastic Dual Coordinate Ascent Algorithm
Theorem 1. Consider Procedure Prox-SDCA. Assume that $\phi_i$ is $(1/\gamma)$-smooth for all $i$. To obtain an expected duality gap of $\mathbb{E}[P(w(T)) - D(\alpha(T))] \leq \epsilon_P$, it suffices to have a total number of iterations of

$$T \geq \left( n + \frac{R^2}{\lambda} \right) \log((n + \frac{R^2}{\lambda}) \cdot \frac{1}{\epsilon_P}).$$

Moreover, to obtain an expected duality gap of $\mathbb{E}[P(\bar{w}) - D(\bar{\alpha})] \leq \epsilon_P$, it suffices to have a total number of iterations of

$$T_0 \geq \left( n + \frac{R^2}{\lambda} \right) \log((n + \frac{R^2}{\lambda}) \cdot \frac{1}{(T - T_0)\epsilon_P}).$$

The linear convergence result in the above theorem is faster than the corresponding proximal SGD result when $T \gg n$. This indicates the advantage of Proximal SDCA approach when we run more than one pass over the data. Similar results can also be found in Collins et al. [2008], Le Roux et al. [2012], Shalev-Shwartz and Zhang [2012] but in more restricted settings than the general problem considered in this paper. Unlike traditional batch algorithms (such as proximal gradient descent, or accelerated proximal gradient descent) that can only achieve relatively fast convergence when the condition number $1/(\lambda \gamma) = O(1)$, our algorithm allows relatively fast convergence even when the condition number $1/(\lambda \gamma) = O(n)$, which can be a significant improvement for real applications.

For nonsmooth loss functions, the convergence rate for Prox-SDCA is given below.

Theorem 2. Consider Procedure Prox-SDCA. Assume that $\phi_i$ is $L$-Lipschitz for all $i$. To obtain an expected duality gap of $\mathbb{E}[P(\bar{w}) - D(\bar{\alpha})] \leq \epsilon_P$, it suffices to have a total number of iterations of

$$T \geq T_0 + n + \frac{4(RL)^2}{\lambda \epsilon_P} \geq \max(0, \left\lceil n \log(0.5\lambda n(RL)^{-2}) \right\rceil) + n + \frac{20(RL)^2}{\lambda \epsilon_P}.$$

Moreover, when $t \geq T_0$, we have dual sub-optimality bound of $\mathbb{E}[D(\alpha^*) - D(\alpha(t))] \leq \epsilon_P/2$.

The result shown in the above theorem for nonsmooth loss is comparable to that of proximal SGD. However, one advantage of our result is that the convergence is in duality gap, which can be easily checked during the algorithm to serve as a stopping criterion. In comparison, SGD does not have an easy to implement stopping criterion. Moreover, as discussed in Shalev-Shwartz and Zhang [2012], faster convergence (such as linear convergence) can be obtained asymptotically when the nonsmooth loss function is nearly everywhere smooth, and in such case, the practical performance of the algorithm will be superior to SGD when we run more than one pass over the data.

3 Applications

There are numerous possible applications of our algorithmic framework. Here we list three applications.

3.1 $\ell_1$ regularization assuming instances of low $\ell_2$ norm

Suppose our interest is to solve $\ell_1$ regularization problem of the form

$$\min_w \left[ \frac{1}{n} \sum_{i=1}^{n} \phi_i(x_i^Tw) + \sigma \|w\|_1 \right], \quad (6)$$

5
with a positive regularization parameter $\sigma \in \mathbb{R}_+$. Assume also that $R = \max_i \|x_i\|_2$ is not too large. This would be the case, for example, in text categorization problems where each $x_i$ is a bag-of-words representation of some short document.

Let $w^*$ be an optimal solution of (6) and assume that $\|w^*\|_2 \leq B$. Choose $\lambda = \frac{\epsilon}{2B^2}$ and

$$g(w) = \frac{1}{2} \|w\|^2 + \frac{\sigma}{\lambda} \|w\|_1. \quad (7)$$

Consider the problem:

$$\min_w P(w) := \left[ \frac{1}{n} \sum_{i=1}^{n} \phi_i(x_i^\top w) + \lambda g(w) \right]. \quad (8)$$

Then, if $\hat{w}$ is an $(\epsilon/2)$-approximated solution of the above it holds that

$$\frac{1}{n} \sum_{i=1}^{n} \phi_i(x_i^\top \hat{w}) + \sigma \|\hat{w}\|_1 \leq P(\hat{w}) \leq P(w^*) + \frac{\epsilon}{2} \leq \frac{1}{n} \sum_{i=1}^{n} \phi_i(x_i^\top w^*) + \sigma \|w^*\|_1 + \epsilon.$$

It follows that $\hat{w}$ is an $\epsilon$-approximated solution to the problem (6). Hence, we can focus on solving (8) based on the Prox-SDCA framework. Note that if our goal is to solve a general $L_1-L_2$ regularization problem with a fixed $\lambda$ independent of $\epsilon$, then linear convergence can be obtained from our analysis when the loss functions are smooth. However, this section focuses on the case that our interest is to solve (6), and thus $\lambda$ is chosen according to $\epsilon$. The reason to introduce an extra $\ell_2$ regularization in (7) is because our theory requires $g(w)$ to be 1-strongly convex, which is satisfied by (7) with respect to the $\ell_2$-norm.

To derive the actual algorithm, we first need to calculate the gradient of the conjugate of $g$. We have

$$\nabla g^*(v) = \arg\max_w \left[ w^\top v - \frac{1}{2} \|w\|^2 - \frac{\sigma}{\lambda} \|w\|_1 \right]$$

$$= \arg\min_w \left[ \frac{1}{2} \|w - v\|^2 + \frac{\sigma}{\lambda} \|w\|_1 \right].$$

A sub-gradient of the objective of the optimization problem above is of the form $w - v + \frac{\sigma}{\lambda} z = 0$, where $z$ is a vector with $z_i = \text{sign}(w_i)$, where if $w_i = 0$ then $z_i \in [-1, 1]$. Therefore, if $w$ is an optimal solution then for all $i$, either $w_i = 0$ or $w_i = v_i - \frac{\sigma}{\lambda} \text{sign}(w_i)$. Furthermore, it is easy to verify that if $w$ is an optimal solution then for all $i$, if $w_i \neq 0$ then the sign of $w_i$ must be the sign of $v_i$. Therefore, whenever $w_i \neq 0$ we have that $w_i = v_i - \frac{\sigma}{\lambda} \text{sign}(v_i)$. It follows that in that case we must have $|v_i| > \frac{\sigma}{\lambda}$. And, the other direction is also true, namely, if $|v_i| > \frac{\sigma}{\lambda}$ then setting $w_i = v_i - \frac{\sigma}{\lambda} \text{sign}(v_i)$ leads to an objective value of

$$\left(\frac{\sigma}{\lambda}\right)^2 + \frac{\sigma}{\lambda} (|v_i| - \frac{\sigma}{\lambda}) \leq |v_i|^2,$$

where the right-hand side is the objective value we will obtain by setting $w_i = 0$. This leads to the conclusion that

$$\nabla_i g^*(v) = \text{sign}(v_i) \left[ |v_i| - \frac{\sigma}{\lambda} \right] = \begin{cases} v_i - \frac{\sigma}{\lambda} \text{sign}(v_i) & \text{if } |v_i| > \frac{\sigma}{\lambda} \\ 0 & \text{o.w.} \end{cases}$$

The resulting algorithm is as follows:

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1 We can always take $B = 1/\sigma$ since by the optimality of $w^*$ we have $\|w^*\|_2 \leq \|w^*\|_1 \leq 1/\sigma$. 

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Procedure Prox-SDCA for minimizing (6) using g as in (7)

Parameters
- regularization σ
- target accuracy ε
- \( B \geq \lVert w^* \rVert_2 \) (default value \( B = 1/\sigma \))

Run Prox-SDCA with:
- \( \lVert \cdot \rVert_D = \lVert \cdot \rVert, \lVert \cdot \rVert_D' = \lVert \cdot \rVert_2 \), and \( R \geq \max_i \lVert x_i \rVert_2 \)
- \( \lambda = \frac{\epsilon}{B^2} \)
- \( \nabla_i g^*(v) = \text{sign}(v_i) \left[ |v_i| - \frac{\sigma}{\lambda} \right]_+ \)

In terms of runtime, we obtain the following result from the general theory, where the notation \( \tilde{O}(\cdot) \) ignores any log-factor.

**Corollary 1.** The number of iterations required by Prox-SDCA, with \( g(\cdot) \) as in (7), for solving (6) to an accuracy \( \epsilon \) is

\[
\tilde{O} \left( n + \frac{R^2 B^2}{\epsilon \gamma} \right) \quad \text{if } \forall i, \phi_i \text{ is } \left( \frac{1}{\gamma} \right) -\text{smooth}
\]

\[
\tilde{O} \left( n + \frac{L^2 R^2 B^2}{\epsilon^2} \right) \quad \text{if } \forall i, \phi_i \text{ is } \left( L \right) -\text{Lipschitz}
\]

In both cases, \( R \) is an upper bound of \( \max_i \lVert x_i \rVert_2 \) and \( B \) is an upper bound on \( \lVert w^* \rVert_2 \).

**Related Work**

Standard SGD requires \( O(R^2 B^2/\epsilon^2) \) even in the case of smooth loss functions. Several variants of SGD, that leads to sparser intermediate solutions, have been proposed (e.g. [Langford et al. 2009], Shalev-Shwartz and Tewari [2011], Xiao [2010], Duchi and Singer [2009], Duchi et al. [2010]). However, all of these variants share the iteration bound of \( O(R^2 B^2/\epsilon^2) \), which is slower than our bound when \( \epsilon \) is small.

Another relevant approach is the FISTA algorithm of Beck and Teboulle [2009]. The shrinkage operator of FISTA is the same as the gradient of \( g^* \) used in our approach. It is a batch algorithm using Nesterov’s accelerated gradient technique. For smooth loss functions, FISTA enjoys the iteration bound of

\[
O \left( \frac{RB}{\sqrt{\epsilon \gamma}} \right).
\]

However, each iteration of FISTA involves all the \( n \) examples rather than just a single example, as our method. Therefore, the runtime of FISTA would be

\[
O \left( d n \frac{RB}{\sqrt{\epsilon \gamma}} \right).
\]

In contrast, the runtime of Prox-SDCA is

\[
\tilde{O} \left( d \left( n + \frac{R^2 B^2}{\epsilon \gamma} \right) \right),
\]
which is better when $n \gg \frac{RB}{\sqrt{\epsilon \gamma}}$. This happens in the statistically interesting regime where we usually choose $\epsilon$ larger than $\Omega(1/n^2)$ for machine learning problems. In fact, since the generalization performance of a learning algorithm is in general no better than $O(1/n)$, there is no need to choose $\epsilon = o(1/n)$. This means that in the statistically interesting regime, Prox-SDCA is superior to FISTA.

Another approach to solving (6) when the loss functions are smooth is stochastic coordinate descent over the primal problem. Shalev-Shwartz and Tewari [2011] showed that the runtime of this approach is $O\left(\frac{dnB^2}{\epsilon}\right)$, under the assumption that $\|x_i\|_\infty \leq 1$ for all $i$. Similar results can also be found in Nesterov [2012].

For our method, each iteration costs runtime $O(d)$ so the total runtime is $\tilde{O}\left(d\left(n + \frac{R^2B^2}{\epsilon}\right)\right)$, where $R = \max_i \|x_i\|_2$. Since the assumption $\|x_i\|_\infty \leq 1$ implies $R^2 \leq d$, this is similar to the guarantee of Shalev-Shwartz and Tewari [2011] in the worst-case. However, in many problems, $R^2$ can be a constant that does not depend on $d$ (e.g. when the instances are sparse). In that case, the runtime of Prox-SDCA becomes $\tilde{O}\left(\frac{d(n + B^2/\epsilon)}{\epsilon}\right)$, which is much better than the runtime bound for the primal stochastic coordinate descent method given in Shalev-Shwartz and Tewari [2011].

### 3.2 $\ell_1$ regularization with low $\ell_\infty$ instances

Next, we consider (6) but now we assume that $R = \max_i \|x_i\|_\infty$ is not too large (but $\max_i \|x_i\|_2$ might be large). This is the situation considered in Shalev-Shwartz and Tewari [2011].

Let $w^*$ be an optimal solution of (6) and assume that $\|w^*\|_1 \leq B$. Choose $\lambda = \frac{\epsilon}{3\log(d)B^2}$ and

$$g(w) = \frac{3\log(d)}{2} \|w\|_q^2 + \frac{\sigma}{\lambda} \|w\|_1,$$

where $q = \frac{\log(d)}{\log(d)-1}$. The function $g(w)$ is 1-strongly convex with respect to the norm $\|\cdot\|_1$ over $\mathbb{R}^d$ (see for example Kakade et al. [2012]). Consider the problem (8) with $g(\cdot)$ being defined in (9). As before, if $\hat{w}$ is an $(\epsilon/2)$-approximated solution of the above problem then it is also an $\epsilon$-approximated solution to the problem (6). Hence, we can focus on solving (8) based on the Prox-SDCA framework.

To derive the actual algorithm, we need to calculate the gradient of the conjugate of $g$. We have

$$\nabla g^*(v) = \arg\min_w \left[-w^T v + \frac{3\log(d)}{2} \|w\|_q^2 + \frac{\sigma}{\lambda} \|w\|_1\right].$$

The $i$'th component of a sub-gradient of the objective of the optimization problem above is of the form

$$-v_i + \frac{3 \log(d) \text{sign}(w_i)|w_i|^{q-1}}{\|w\|_q^{q-2}} + \frac{\sigma}{\lambda} z_i,$$

\footnote{We can always take $B = 1/\sigma$ since by the optimality of $w^*$ we have $\|w^*\|_1 \leq 1/\sigma$.}
where \( z_i = \text{sign}(w_i) \) whenever \( w_i \neq 0 \) and otherwise \( z_i \in [-1, 1] \). Therefore, if \( w \) is an optimal solution then for all \( i \), either \( w_i = 0 \) or

\[
|w_i|^{q-1} = \text{sign}(w_i) \frac{\|w\|_q^{q-2}}{3 \log(d)} \left( v_i - \frac{\sigma}{\lambda} \text{sign}(w_i) \right) = \frac{\|w\|_q^{q-2}}{3 \log(d)} \left( \text{sign}(w_i) v_i - \frac{\sigma}{\lambda} \right).
\]

Furthermore, it is easy to verify that if \( w \) is an optimal solution then for all \( i \), if \( w_i \neq 0 \) then the sign of \( w_i \) must be the sign of \( v_i \). Therefore, whenever \( w_i \neq 0 \) we have that

\[
|w_i|^{q-1} = \frac{\|w\|_q^{q-2}}{3 \log(d)} \left( |v_i| - \frac{\sigma}{\lambda} \right).
\]

It follows that in that case we must have \( |v_i| > \frac{\sigma}{\lambda} \). And, the other direction is also true, namely, if \( |v_i| > \frac{\sigma}{\lambda} \) then \( w_i \) must be non-zero. This is true because if \( |v_i| > \frac{\sigma}{\lambda} \), then the \( i \)'th coordinate of any sub-gradient of the objective function at any vector \( w \) s.t. \( w_i = 0 \) is \( -v_i + \frac{\sigma}{\lambda} z_i \neq 0 \). Hence, \( w \) can’t be an optimal solution. This leads to the conclusion that an optimal solution has the form

\[
\nabla_i g^*(v) = \begin{cases} 
\text{sign}(v_i) \left( |v_i| - \frac{\sigma}{\lambda} \right) \frac{1}{q-1} & \text{if } |v_i| > \frac{\sigma}{\lambda}, \\
0 & \text{otherwise}
\end{cases}
\]

where

\[
a = \frac{\|\nabla g^*(v)\|_q^{q-2}}{3 \log(d)} = \frac{1}{3 \log(d)} \left( \sum_{i:|v_i| > \frac{\sigma}{\lambda}} \left( a \left( |v_i| - \frac{\sigma}{\lambda} \right) \right)^{\frac{q-2}{q}} \right)^{\frac{q}{q-1}} = \frac{a^{\frac{q-2}{q}}}{3 \log(d)} \left( \sum_{i:|v_i| > \frac{\sigma}{\lambda}} \left( |v_i| - \frac{\sigma}{\lambda} \right)^{\frac{q}{q-1}} \right)^{\frac{q-2}{q}},
\]

which yields

\[
a = \left( \frac{1}{3 \log(d)} \left( \sum_{i:|v_i| > \frac{\sigma}{\lambda}} \left( |v_i| - \frac{\sigma}{\lambda} \right)^{\frac{q}{q-1}} \right)^{\frac{q-2}{q}} \right)^{q-1}.
\]

The resulting algorithm is as follows:

**Procedure Prox–SDCA for minimizing (6) using \( g \) as in (9)**

**Parameters**
- regularization \( \sigma \)
- target accuracy \( \epsilon \)
- dimension \( d \)
- \( B \geq \|w^*\|_1 \) (default value \( B = 1/\sigma \))

Run Prox–SDCA with:
- \( \|\cdot\|_D = \|\cdot\|, \|\cdot\|_{D'} = \|\cdot\|_\infty \), and \( R \geq \max_i \|x_i\|_\infty \)
- \( \lambda = \frac{\epsilon}{3 \log(d) B^2} \)
- \( \nabla g^*(v) \) according to (10) and (11)

In terms of runtime, we obtain the following
Corollary 2. The number of iterations required by Prox-SDCA, with $g$ as in (9), for solving (6) to accuracy $\epsilon$ is
\[
\tilde{O}\left( n + \frac{R^2B^2\log(d)}{\epsilon^2} \right) \quad \text{if } \forall i, \phi_i \text{ is } (1/\gamma) - \text{smooth}
\]
\[
\tilde{O}\left( n + \frac{L^2R^2B^2\log(d)}{\epsilon^2} \right) \quad \text{if } \forall i, \phi_i \text{ is } (L) - \text{Lipschitz}
\]
In both cases, $R = \max_i \|x_i\|_\infty$ and $B$ is an upper bound over $\|w^*\|_1$.

Related work

The algorithm we have obtained is similar to the Mirror Descent framework Beck and Teboulle [2003] and its online or stochastic versions (see for example Shalev-Shwartz [2011] and the references therein). It is also closely related to the SMIDAS and COMID algorithms Shalev-Shwartz and Tewari [2011] as well as to dual averaging Xiao [2010]. Comparing the rates of these algorithms to Prox-SDCA, we obtain similar differences as in the previous subsection, only now $B$ is a bound on $\|w^*\|_1$ rather than $\|w^*\|_2$ and $R$ is a bound on $\max_i \|x_i\|_\infty$ rather than $\max_i \|x_i\|_2$.

3.3 Multiclass categorization and structured prediction

In structured output problems, there is an instance space $X$ and a large target space $Y$. There is a function $\psi : X \times Y \to \mathbb{R}^d$. We assume that the range of $\psi$ is in the $\ell_2$ ball of radius $R$ of $\mathbb{R}^d$. The prediction of a vector $w \in \mathbb{R}^d$ is
\[
\arg\max_{y \in Y} w^T \psi(x, y)
\]
There is also a function $\delta : Y \times Y \to \mathbb{R}^+$ which evaluates the cost of predicting a label $y'$ when the true label is $y$. We assume that $\delta(y, y) = 0$ for all $y$. The generalized hinge-loss defined below is used as a convex surrogate loss function
\[
\max_{y'} \left[ \delta(y', y) - w^T \psi(x, y) + w^T \psi(x, y') \right].
\]
The optimization problem associated with learning $w$ is now
\[
\min_w \left[ \frac{\lambda}{2} \|w\|_2^2 + \frac{1}{n} \sum_{i=1}^n \left( \max_{y'} \delta(y', y_i) - w^T \psi(x_i, y_i) + w^T \psi(x_i, y') \right) \right]. \quad (12)
\]
The above optimization problem can be cast in our setting as follows. W.l.o.g. assume that $Y = \{1, \ldots, k\}$. For each $i$ and each $j$, let the $j$'th column of $X_i$ be $\psi(x_i, j)$. Define,
\[
\phi_i(v) = \max_j (\delta(j, y_i) - v_{y_i} + v_j).
\]
Finally, let $g(w) = \frac{1}{2} \|w\|_2^2$. Then, (12) can be written in the form of (1).

To apply the Prox-SDCA to this problem, note that $g$ is 1-strongly convex w.r.t. $\|\cdot\|_2$ and that $\phi_i$ is 2-Lipschitz w.r.t. norm $\|\cdot\|_\infty$. Indeed, given vectors $u, v$, let $j$ be the index that attains the maximum in the definition of $\phi_i(v)$, then
\[
\phi_i(v) - \phi_i(u) \leq (\delta(j, y_i) - v_{y_i} + v_j) - (\delta(j, y_i) - u_{y_i} + u_j) \leq 2\|v - u\|_\infty.
\]
Therefore $\| \cdot \|_D = \| \cdot \|_1$ and $\| \cdot \|_{D'} = \| \cdot \|_2$. If we let

$$R = \max_j \| \psi(x_i, j) \|_2,$$

then we have that

$$\| X_i \| = \sup_{u \neq 0} \frac{\| X_i u \|_2}{\| u \|_1} = \sup_{u: \| u \|_1 = 1} \| X_i u \|_2 = \max_j \| \psi(x_i, j) \|_2 \leq R.$$

To calculate the dual of $\phi_i$, note that we can write $\phi_i$ as

$$\phi_i = \max_{\beta \in \Delta^k} \sum_j \beta_j (\delta(j, y_i) - v_{y_i} + v_j),$$

where $\Delta^k = \{ \beta: \sum_j \beta_j \leq 1; \beta_j \geq 0 \}$ is the non-negative simplex of $\mathbb{R}^k$. Hence, the dual of $\phi_i$ is

$$\phi_i^*(\alpha) = \max_v \left[ v^\top \alpha - \phi_i(v) \right]$$

$$= \max_v \min_{\beta} \left[ v^\top \alpha - \sum_j \beta_j (\delta(j, y_i) - v_{y_i} + v_j) \right]$$

$$= \min_{\beta} \max_v \left[ v^\top \alpha - \sum_j \beta_j (\delta(j, y_i) - v_{y_i} + v_j) \right]$$

$$= \min_{\beta} \left[ \beta^\top \delta(\cdot, y_i) + \max_v \left[ v^\top (\alpha - \beta) + v_{y_i} \sum_j \| \beta_j \|_1 \right] \right].$$

The inner maximization over $v$ would be $\infty$ if for some $j \neq y_i$ we have $\alpha_j \neq \beta_j$. Otherwise, if for all $j \neq y_i$ we have $\alpha_j = \beta_j$ the inner objective becomes

$$v_{y_i} (\alpha_{y_i} - \beta_{y_i} + \sum_j \beta_j) = v_{y_i} (\alpha_{y_i} + \sum_{j \neq y_i} \alpha_j).$$

Therefore, the objective would again be $\infty$ if $\alpha_{y_i} \neq -\sum_{j \neq y_i} \alpha_j$. In all other cases, the objective is zero. Overall, this implies that:

$$\phi_i^*(\alpha) = \begin{cases} \sum_j \alpha_j \delta(j, y_i) & \text{if } \sum_j \alpha_j = 0 \land \forall j \neq y_i, \alpha_j \geq 0 \land \sum_{j \neq y_i} \alpha_j \leq 1 \\ \infty & \text{o.w.} \end{cases}$$

Finally, we specify Prox-SDCA (using Option IV with 2 as an upper bound of $\| z \|_D$, and the random output option), and rely on the fact that a sub-gradient of $\phi_i(v)$ is a vector $e_j - e_{y_i}$ with $j \in \arg\max_j (\delta(j, y_i) - v_{y_i} + v_j)$. 

11
For structured prediction problem, SGD enjoys the rate

\[ \tilde{O}\left(\frac{R^2}{\lambda \epsilon}\right), \]

while the most expensive operation at each iteration of SGD also involves solving (13). Therefore, our bound matches the bound of SGD when \( n = \tilde{O}\left(\frac{R^2}{\lambda \epsilon}\right) \). The main advantage of our result is that it bounds

Note that even if \( k \) is very large, the above implementation does not maintain \( \alpha \) explicitly, but only maintains \( d \)-dimensional vectors. Therefore, we can implement the above procedure efficiently whenever the optimization problem involves in finding \( j \) can be performed efficiently. This is the same requirement as in implementing SGD for structured output prediction.

**Corollary 3.** Prox-SDCA can be implemented for structured output prediction. To obtain an expected duality gap of at most \( \epsilon_P \), it suffices to have a total number of iterations of

\[ T \geq \max(0, \lceil n \log(0.5\lambda n(2R)^{-2}) \rceil) + n + \frac{20(2R)^2}{\lambda \epsilon P}, \]

where \( R \) is an upper bound on \( \|\phi(x, j)\|_2 \). The most expensive operation at iteration \( t \) is solving

\[ \arg\max_j \left( \delta(j, y_i) - w^{(t-1)\top} \phi(x_i, y_i) + w^{(t-1)\top} \phi(x_i, j) \right). \quad (13) \]

**Remark 1.** Since for this problem, \( \|z\|_2^2 \) in Option IV can be bounded by \( L^2 = 4 \) instead of \( 4L^2 = 16 \), the proof of Theorem 2 implies that the constant 20 in Corollary 3 can be replaced by 5.
duality gap which can be checked in practice. Moreover, the practical convergence speed can be faster than what is indicated in Corollary 3 when the non-smooth loss function can be approximated by a smooth loss function, as pointed out in Shalev-Shwartz and Zhang [2012].

Recently, Lacoste-Julien et al. [2012] derived a stochastic coordinate ascent for structural SVM based on the Frank-Wolfe algorithm. Their algorithm is very similar to our algorithm and the rate they obtain for the convergence of duality gap matches our rate.

Note that the generality of our framework enables us to easily handle structured output problems with other regularizers, such as $\ell_1$ norm regularization.

4 Proofs

Note that the proof technique follows that of Shalev-Shwartz and Zhang [2012], but with more involved notations of the paper. We prove the theorems for running Prox-SDCA while choosing $\Delta \alpha_i$ as in Option I. A careful examination of the proof easily reveals that the results hold for the other options as well. More specifically, Lemma 1 only requires choosing $\Delta \alpha_i = s(u_i^{(t-1)} - \alpha_i^{(t-1)})$ as in (14), and Option III chooses $s$ to optimize the bound on the right hand side of (16), and hence ensures that the choice can do no worse than the result of Lemma 1 with any $s$. The simplification in Option IV and V employs the specific simplification of the bound in Lemma 1 in the proof of the theorems.

For convenience, we list the following simple facts about primal and dual formulations, which will be used in the proofs. For each $i$, we have

$$-\alpha_i^* \in \partial \phi_i(X_i^T w^*), \quad X_i^T w^* \in \partial \phi_i^*(-\alpha_i^*),$$

and

$$w^* = \nabla g^*(v^*), \quad v^* = \frac{1}{\lambda n} \sum_{i=1}^{n} X_i \alpha_i^*.$$

The key lemma is the following:

**Lemma 1.** Assume that $\phi_i^*$ is $\gamma$-strongly-convex (where $\gamma$ can be zero). Then, for any iteration $t$ and any $s \in [0, 1]$ we have

$$\mathbb{E}[D(\alpha^{(t)}) - D(\alpha^{(t-1)})] \geq \frac{s}{n} \mathbb{E}[P(w^{(t-1)}) - D(\alpha^{(t-1)})] - \left(\frac{s}{n}\right)^2 \frac{G^{(t)}}{2\lambda},$$

where

$$G^{(t)} = \frac{1}{n} \sum_{i=1}^{n} \left(\|X_i\|^2 - \frac{\gamma(1-s)\lambda n}{s}\right) \mathbb{E} \left[\|u_i^{(t-1)} - \alpha_i^{(t-1)}\|^2_B\right],$$

and $-u_i^{(t-1)} \in \partial \phi_i(X_i^T w^{(t-1)})$.

**Proof.** Since only the $i$'th element of $\alpha$ is updated, the improvement in the dual objective can be written as

$$n[D(\alpha^{(t)}) - D(\alpha^{(t-1)})]$$

$$= \left(\sum_{i=1}^{n} \lambda g^*(v_i^{(t-1)} + (\lambda n)^{-1} X_i \Delta \alpha_i)\right) - \left(\sum_{i=1}^{n} \lambda g^*(v_i^{(t-1)})\right)$$

$$\geq \left(\sum_{i=1}^{n} \lambda h_i(v_i^{(t-1)}; (\lambda n)^{-1} X_i \Delta \alpha_i)\right) - \left(\sum_{i=1}^{n} \lambda g^*(v_i^{(t-1)})\right).$$

13
By the definition of the update we have for all \( s \in [0, 1] \) that
\[
A = \max_{\Delta \alpha_i} -\phi^*(-(\alpha_i^{(t-1)} + \Delta \alpha_i)) - \lambda n h(\nu^{(t-1)}; (\lambda n)^{-1} X_i \Delta \alpha_i)
\geq -\phi^*(-(\alpha_i^{(t-1)} + s(u_i^{(t-1)} - \alpha_i^{(t-1)}))) - \lambda n h(\nu^{(t-1)}; (\lambda n)^{-1} s X_i (u_i^{(t-1)} - \alpha_i^{(t-1)})).
\]

(14)

From now on, we omit the superscripts and subscripts. Since \( \phi^* \) is \( \gamma \)-strongly convex, we have that
\[
\phi^*(-(\alpha + s(u - \alpha))) = \phi^*(s(-u) + (1-s)(-\alpha)) \leq s\phi^*(-u) + (1-s)\phi^*(-\alpha) - \frac{\gamma}{2} s(1-s)\|u - \alpha\|_D^2
\]
(15)
Combining this with (14) and rearranging terms we obtain that
\[
A \geq -s\phi^*(-u) - (1-s)\phi^*(-\alpha) + \frac{\gamma}{2} s(1-s)\|u - \alpha\|_D^2 - \lambda nh(v; (\lambda n)^{-1} s X(u - \alpha))
\]
\[
= -s\phi^*(-u) - (1-s)\phi^*(-\alpha) + \frac{\gamma}{2} s(1-s)\|u - \alpha\|_D^2 - \lambda ng^*(v) - sw^T X(u - \alpha) - \frac{s^2}{2\lambda n} \|X(u - \alpha)\|_D^2
\]
\[
\geq -s\phi^*(-u) - (1-s)\phi^*(-\alpha) + \frac{\gamma}{2} s(1-s)\|u - \alpha\|_D^2 - \lambda ng^*(v) - sw^T X(u - \alpha) - \frac{s^2}{2\lambda n} \|X\|_2^2\|u - \alpha\|_D^2
\]
\[
= -s(\phi^*(-u) + w^T Xu) + (\phi^*(-\alpha) - \lambda ng^*(v)) + \frac{s}{2} \left( \gamma(1-s) - \frac{s\|X\|_2^2}{\lambda n} \right) \|u - \alpha\|_D^2 + s(\phi^*(-\alpha) + w^T X\alpha),
\]
where we used \(-u \in \partial \phi(X^T w)\) which yields \(\phi^*(-u) = -w^T X u - \phi(X^T w)\). Therefore
\[
A - B \geq s \left[ \phi(X^T w) + \phi^*(-\alpha) + w^T X\alpha + \left( \frac{\gamma(1-s)}{2} - \frac{s\|X\|_2^2}{2\lambda n} \right) \|u - \alpha\|_D^2 \right].
\]
(16)
Next note that with \(w = \nabla g^*(v)\), we have \(g(w) + g^*(v) = w^Tv\). Therefore:
\[
P(w) - D(\alpha) = \frac{1}{n} \sum_{i=1}^n \phi_i(X_i^T w) + \lambda g(w) - \left( -\frac{1}{n} \sum_{i=1}^n \phi_i^*(-\alpha_i) - \lambda g^*(v) \right)
\]
\[
= \frac{1}{n} \sum_{i=1}^n \phi_i(X_i^T w) + \frac{1}{n} \sum_{i=1}^n \phi_i^*(-\alpha_i) + \lambda w^Tv
\]
\[
= \frac{1}{n} \sum_{i=1}^n \left( \phi_i(X_i^T w) + \phi_i^*(-\alpha_i) + w^TX_i\alpha_i \right).
\]
Therefore, if we take expectation of (16) w.r.t. the choice of \(i\) we obtain that
\[
\frac{1}{s} \mathbb{E}[A - B] \geq \mathbb{E}[P(w) - D(\alpha)] - \frac{s}{2\lambda n} \cdot \frac{1}{n} \sum_{i=1}^n \left( \|X_i\|_2^2 - \frac{\gamma(1-s)\lambda n}{s} \right) \|u_i - \alpha_i\|_D^2.
\]
\[
\text{We have obtained that}
\]
\[
\frac{n}{s} \mathbb{E}[D(\alpha(i)) - D(\alpha(t-1))] \geq \mathbb{E}[P(w(t-1)) - D(\alpha(t-1))] - \frac{s G(t)}{2\lambda n}.
\]
(17)
Multiplying both sides by \(s/n\) concludes the proof of the lemma.
We also use the following simple lemma:

**Lemma 2.** For all $\alpha$, $D(\alpha) \leq P(w^*) \leq P(0) \leq 1$. In addition, $D(0) \geq 0$.

**Proof.** The first inequality is by weak duality, the second is by the optimality of $w^*$, and the third by the assumption that $n^{-1} \sum_i \phi_i(0) \leq 1$. For the last inequality we use $-\phi_i^*(0) = -\max_z (0 - \phi_i(z)) = \min_z \phi_i(z) \geq 0$, which yields $D(0) \geq 0$. \hfill $\Box$

### 4.1 Proof of Theorem 1

**Proof of Theorem 1** The assumption that $\phi_i$ is $(1/\gamma)$-smooth implies that $\phi_i^*$ is $\gamma$-strongly-convex. We will apply Lemma 1 with $s = \frac{\lambda \gamma}{R^2 + \lambda \gamma} \in [0, 1]$. Recall that $\|X_i\| \leq R$. Therefore, the choice of $s$ implies that

$$\|X_i\|^2 - \frac{\gamma (1 - s) \lambda n}{s} \leq R^2 - \frac{1 - s}{s^2 / (\lambda n \gamma)} = R^2 - R^2 = 0,$$

and hence $C_\alpha(t) \leq 0$ for all $t$. This yields,

$$E[D(\alpha(t)) - D(\alpha(t-1))] \geq \frac{s}{n} E[P(w(t-1)) - D(\alpha(t-1))].$$

But since $\epsilon_D^{(t-1)} := D(\alpha^*) - D(\alpha(t-1)) \leq P(w(t-1)) - D(\alpha(t-1))$ and $D(\alpha(t)) - D(\alpha(t-1)) = \epsilon_D^{(t-1)} - \epsilon_D^{(t)}$, we obtain that

$$E[\epsilon_D^{(t)}] \leq (1 - \frac{s}{n}) E[\epsilon_D^{(t-1)}] \leq (1 - \frac{s}{n}) E[\epsilon_D^{(0)}] \leq (1 - \frac{s}{n})^t \leq \exp(-st/n) = \exp \left( -\frac{\lambda \gamma t}{R^2 + \lambda \gamma n} \right).$$

This would be smaller than $\epsilon_D$ if

$$t \geq \left( n + \frac{R^2}{\lambda \gamma} \right) \log(1/\epsilon_D).$$

It implies that

$$E[P(w(t)) - D(\alpha(t))] \leq \frac{n}{s} E[\epsilon_D^{(t)} - \epsilon_D^{(t+1)}] \leq \frac{n}{s} E[\epsilon_D^{(t)}].$$

(18)

So, requiring $\epsilon_D^{(t)} \leq \frac{\epsilon}{n}$ we obtain a duality gap of at most $\epsilon_P$. This means that we should require

$$t \geq \left( n + \frac{R^2}{\lambda \gamma} \right) \log((n + \frac{R^2}{\lambda \gamma}) \cdot \frac{1}{\epsilon_P}),$$

which proves the first part of Theorem 1.

Next, we sum (18) over $t = T_0, \ldots, T - 1$ to obtain

$$E \left[ \frac{1}{T - T_0} \sum_{t=T_0}^{T-1} (P(w(t)) - D(\alpha(t))) \right] \leq \frac{n}{s(T - T_0)} E[D(\alpha(T)) - D(\alpha(T_0))].$$

Now, if we choose $\bar{w}, \bar{\alpha}$ to be either the average vectors or a randomly chosen vector over $t \in \{T_0 + 1, \ldots, T\}$, then the above implies

$$E[P(\bar{w}) - D(\bar{\alpha})] \leq \frac{n}{s(T - T_0)} E[D(\alpha(T)) - D(\alpha(T_0))] \leq \frac{n}{s(T - T_0)} E[\epsilon_D^{(T_0)}].$$

It follows that in order to obtain a result of $E[P(\bar{w}) - D(\bar{\alpha})] \leq \epsilon_P$, we only need to have

$$E[\epsilon_D^{(T_0)}] \leq \frac{s(T - T_0) \epsilon_P}{n} = \frac{(T - T_0) \epsilon_P}{n + \frac{R^2}{\lambda \gamma}}.$$

This implies the second part of Theorem 1 and concludes the proof. \hfill $\Box$
4.2 Proof of Theorem

Next, we turn to the case of Lipschitz loss function. We rely on the following lemma.

**Lemma 3.** Let \( \phi : \mathbb{R}^k \rightarrow \mathbb{R} \) be an \( L \)-Lipschitz function w.r.t. a norm \( \| \cdot \|_p \) and let \( \| \cdot \|_D \) be the dual norm. Then, for any \( \alpha \in \mathbb{R}^k \) s.t. \( \| \alpha \|_D > L \) we have that \( \phi^*(\alpha) = \infty \).

**Proof.** Fix some \( \alpha \) with \( \| \alpha \|_D > L \). Let \( x_0 \) be a vector such that \( \| x_0 \|_P = 1 \) and \( \alpha^T x_0 = \| \alpha \|_D \) (this is a vector that achieves the maximal objective in the definition of the dual norm). By definition of the conjugate we have

\[
\phi^*(\alpha) = \sup_x [\alpha^T x - \phi(x)] \\
\geq -\phi(0) + \sup_x [\alpha^T x - (\phi(x) - \phi(0))] \\
\geq -\phi(0) + \sup_x [\alpha^T x - L\|x - 0\|_P] \\
\geq -\phi(0) + \sup_{c>0} [\alpha^T (cx_0) - L\|cx_0\|_P] \\
= -\phi(0) + \sup_{c>0} (\|\alpha\|_D - L) c = \infty.
\]

A direct corollary of the above lemma is:

**Lemma 4.** Suppose that for all \( i \), \( \phi_i \) is \( L \)-Lipschitz w.r.t. \( \| \cdot \|_P \). Let \( G(t) \) be as defined in Lemma 1 (with \( \gamma = 0 \)). Then, \( G(t) \leq 4R^2 L^2 \).

**Proof.** Using Lemma 3 we know that \( \| \alpha_i^{(t-1)} \|_D \leq L \), and in addition by the relation of Lipschitz and sub-gradients we have \( \|u_i^{(t-1)}\|_D \leq L \). Combining this with the triangle inequality we obtain that \( \|u_i^{(t-1)} - \alpha_i^{(t-1)}\|^2 \leq 4L^2 \), and the proof follows.

We are now ready to prove Theorem 2.

**Proof of Theorem 2.** Let \( G = \max_t G(t) \) and note that by Lemma 4 we have \( G \leq 4R^2 L^2 \). Lemma 1 with \( \gamma = 0 \), tells us that

\[
\mathbb{E}[D(\alpha(t)) - D(\alpha(t-1))] \geq \frac{s}{n} \mathbb{E}[P(w^{(t-1)}) - D(\alpha(t-1))] - \left( \frac{s}{n} \right)^2 \frac{G^2}{2\lambda}, \tag{19}
\]

which implies that

\[
\mathbb{E}[\epsilon_{D}^{(t)}] \leq (1 - \frac{s}{n}) \mathbb{E}[\epsilon_{D}^{(t-1)}] + (\frac{s}{n})^2 \frac{G^2}{2\lambda}.
\]

We next show that the above yields

\[
\mathbb{E}[\epsilon_{D}^{(t)}] \leq \frac{2G}{\lambda(2n + t - t_0)} \tag{20}
\]

for all \( t \geq t_0 = \max(0, [n \log(2\lambda ne_D^{(0)}/G)]) \). Indeed, let us choose \( s = 1 \), then at \( t = t_0 \), we have

\[
\mathbb{E}[\epsilon_{D}^{(t)}] \leq (1 - \frac{1}{n}) t \epsilon_{D}^{(0)} + \frac{G}{2\lambda n^2} (1 - \frac{1}{n}) \leq e^{-t/n} \epsilon_{D}^{(0)} + \frac{G}{2\lambda n} \leq \frac{G}{\lambda n}.
\]
This implies that (20) holds at \( t = t_0 \). For \( t > t_0 \) we use an inductive argument. Suppose the claim holds for \( t - 1 \), therefore
\[
\mathbb{E}[\epsilon_D^{(t)}] \leq (1 - \frac{s}{n}) \mathbb{E}[\epsilon_D^{(t-1)}] + \left( \frac{s}{n} \right)^2 \frac{2G}{2\lambda} \leq (1 - \frac{s}{n}) \frac{2G}{\lambda(2n-t_0-t_0)} + \left( \frac{s}{n} \right)^2 \frac{2G}{2\lambda} .
\]

Choosing \( s = 2n/(2n - t_0 + t - 1) \in [0, 1] \) yields
\[
\mathbb{E}[\epsilon_D^{(t)}] \leq \left( 1 - \frac{2}{2n-t_0+t-1} \right) \frac{2G}{\lambda(2n-t_0+t-1)} + \left( \frac{2n-t_0+t-1}{2n-t_0+t-1} \right)^2 \frac{2G}{2\lambda} .
\]

This provides a bound on the dual sub-optimality. We next turn to bound the duality gap. Summing (19) over \( t = T_0 + 1, \ldots, T \) and rearranging terms we obtain that
\[
\mathbb{E} \left[ \frac{1}{T - T_0} \sum_{t = T_0 + 1}^{T} (P(w^{(t-1)}) - D(\alpha^{(t-1)})) \right] \leq \frac{n}{s(T - T_0)} \mathbb{E}[D(\alpha^{(T)}) - D(\alpha^{(T_0)})] + \frac{sG}{2\lambda n} .
\]

Now, if we choose \( \bar{w}, \bar{\alpha} \) to be either the average vectors or a randomly chosen vector over \( t \in \{T_0 + 1, \ldots, T\} \), then the above implies
\[
\mathbb{E}[P(\bar{w}) - D(\bar{\alpha})] \leq \frac{n}{s(T - T_0)} \mathbb{E}[D(\alpha^{(T)}) - D(\alpha^{(T_0)})] + \frac{sG}{2\lambda n} .
\]

If \( T \geq n + T_0 \) and \( T_0 \geq t_0 \), we can set \( s = n/(T - T_0) \) and combining with (20) we obtain
\[
\mathbb{E}[P(\bar{w}) - D(\bar{\alpha})] \leq \mathbb{E}[D(\alpha^{(T)}) - D(\alpha^{(T_0)})] + \frac{G}{2\lambda(T - T_0)} \leq \mathbb{E}[D(\alpha^{*}) - D(\alpha^{(T_0)})] + \frac{G}{2\lambda(T - T_0)} \leq \frac{2G}{\lambda(2n - t_0 + T_0)} + \frac{G}{2\lambda(T - T_0)} .
\]

A sufficient condition for the above to be smaller than \( \epsilon_p \) is that \( T_0 \geq \frac{4G}{\lambda \epsilon_p} - 2n + t_0 \) and \( T \geq T_0 + \frac{G}{\lambda \epsilon_p} \). It also implies that \( \mathbb{E}[D(\alpha^{*}) - D(\alpha^{(T_0)})] \leq \epsilon_p/2 \). Since we also need \( T_0 \geq t_0 \) and \( T - T_0 \geq n \), the overall number of required iterations can be
\[
T_0 \geq \max\{t_0, 4G/(\lambda \epsilon_p) - 2n + t_0\} , \quad T - T_0 \geq \max\{n, G/(\lambda \epsilon_p)\} .
\]

We conclude the proof by noticing that \( \epsilon_D^{(0)} \leq 1 \) (Lemma 2), which implies that \( t_0 \leq \max(0, \lceil n \log(2\lambda n/G) \rceil) \).
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