Succinct data structure for dynamic trees with faster queries

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Abstract

Navarro and Sadakane [TALG 2014] gave a dynamic succinct data structure for storing an ordinal tree. The structure supports tree queries in either \(O(\log n / \log \log n)\) or \(O(\log n)\) time, and insertion or deletion of a single node in \(O(\log n)\) time. In this paper we improve the result of Navarro and Sadakane by reducing the time complexities of some queries (e.g., degree and level ancestor) from \(O(\log n)\) to \(O(\log n / \log \log n)\).

1 Introduction

A problem which was extensively studied in recent years is designing a succinct data structure that stores a tree while supporting queries on the tree, like finding the parent of a node, or computing the lowest common ancestor of two nodes. This problem has been studied both for static trees \[2,4,7,9,11,13,17,19,20\] and dynamic trees \[1,6,12,18,19,21\].

For dynamic ordinal trees, Farzan and Munro \[6\] gave a data structure with \(O(1)\) query time and \(O(1)\) amortized update time. However, the structure supports only a limited set of queries, and the update operations are restricted (insertion of a leaf, insertion of a node in the middle of an edge, deletion of a leaf, and deletion of a node with one child). A wider set of queries is supported by the data structure of Gupta et al. \[12\]. This data structure has \(O(\log \log n)\) query time and \(O(n^\epsilon)\) amortized update time. The data structure of Navarro and Sadakane \[19\] supports a large set of queries. See Table \[\] for some of the supported queries. The structure supports the following update operations (1) Insertion of a node \(x\) as a child of an existing node \(y\). The insert operation specifies a (possibly empty) consecutive range of children of \(y\) and these nodes become children of \(x\) after the insertion. (2) Deletion of a node \(x\). The children of \(x\) become children of the parent of \(x\). The time complexity of a query is either

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Table 1: Some of the tree queries supported by the data structure of Navarro and Sadakane [19]. In the table below $x$ is some node of the tree. The queries marked by * take $O(\log n)$ time in the structure of Navarro and Sadakane, and $O(\log n/\log \log n)$ time in our structure. The queries marked by + take $O(\log n)$ time in both structures, and unmarked queries take $O(\log n/\log \log n)$ time in both structures.

| Query                | Description                                           |
|----------------------|-------------------------------------------------------|
| depth($x$)           | The depth of $x$.                                     |
| height($x$)          | The height of $x$.                                    |
| num_descendants($x$)| The number of descendants of $x$.                     |
| parent($x$)          | The parent of $x$.                                    |
| lca($x, y$)          | The lowest common ancestor of $x$ and $y$.            |
| level_ancestor($x, i$)*| The ancestor $y$ of $x$ for which depth($y$) = depth($x$) − $i$. |
| level_next($x$)*    | The node after $x$ in the BFS order                   |
| level_prev($x$)*    | The node before $x$ in the BFS order                  |
| level_lmost($x, d$)*| The leftmost node with depth $d$.                      |
| level_rmost($x, d$)*| The rightmost node with depth $d$.                    |
| degree($x$)*        | The number of children of $x$.                        |
| child_rank($x$)+     | The rank of $x$ among its siblings.                   |
| child_select($x, i$)+| The $i$-th child of $x$.                             |
| first_child($x$)     | The first child of $x$.                               |
| last_child($x$)      | The last child of $x$.                                |
| next_sibling($x$)    | The next sibling of $x$.                              |
| prev_sibling($x$)    | The previous sibling of $x$.                          |

$O(\log n/\log \log n)$ or $O(\log n)$ (see Table 1). Moreover, the time complexity of insert and delete operations is $O(\log n)$. Additionally, by dropping support for degree, child_rank, and child_select queries, the time complexity of insert and delete operations can be reduced to $O(\log n/\log \log n)$.

In this paper, we improve the result of Navarro and Sadakane by reducing the time for the queries level_ancestor, level_next, level_prev, level_lmost, level_rmost, and degree from $O(\log n)$ to $O(\log n/\log \log n)$. The time complexities of the other operations are unchanged. Additionally, by dropping support for degree, child_rank, and child_select queries, we obtain a data structure that handles all queries and update operations in $O(\log n/\log \log n)$ time.

The rest of the paper is organize as follows. In Section 2 we give a dynamic partial sums structure that will be used later in our data structure. In Section 3 we give a short description of the data structure of Navarro and Sadakane. Then, we describe our improved structure in Sections 4 and 5.
2 Dynamic partial sums

In the dynamic partial sums problem, the goal is to store an array $Z$ of integers and support the following queries.

**sum($Z, i$):** Return $\sum_{j=1}^{i} Z[j]$.

**search($Z, d$):** Return the minimum $i$ for which $\text{sum}(Z, i) \geq d$.

Additionally, the following update operations are supported.

**update($Z, i, \Delta$):** Set $Z[i] \leftarrow Z[i] + \Delta$.

**merge($Z, i$):** Replace the entries $Z[i]$ and $Z[i+1]$ by a new entry that is equal to $Z[i] + Z[i+1]$.

**divide($Z, i, t$):** Replace the entry $Z[i]$ by the entries $t$ and $Z[i] - t$.

Note that a partial sums structure also supports access to $Z$ since $Z[i] = \text{sum}(Z, i) - \text{sum}(Z, i-1)$.

**Lemma 1.** (Bille et al. [3]) There is a dynamic partial sums structure for an array $Z$ containing $k = O(\log n / \log \log n) \cdot O(\log n)$-bit non-negative integers. The structure uses $O(k \log n)$ bits and supports all queries and update operations in $O(1)$ time. The update($Z, i, \Delta$) operation is supported for values of $\Delta$ satisfying $|\Delta| = \log^{O(1)} n$.

In the rest of this section we describe structures for storing an array $Z$ with negative integers. These structures support only subsets of the operations defined above.

**Lemma 2.** (Dietz [5]) There is a structure for an array $Z$ containing $k = O(\log n / \log \log n) \cdot O(\log n)$-bit integers. The structure uses $O(k \log n)$ bits and supports the following operations in $O(1)$ time: (1) $\text{sum}(Z, i)$ queries. (2) $\text{update}(Z, i, \Delta)$ operations, where $|\Delta| = \log^{O(1)} n$.

**Corollary 3.** There is a structure for an array $Y$ containing $k = O(\log n / \log \log n) \cdot O(\log n)$-bit integers. The structure uses $O(k \log n)$ bits and supports the following operations in $O(1)$ time: (1) Access $Y[i]$. (2) Add $\Delta$ to the entries of $Y[i..k]$, where $|\Delta| = \log^{O(1)} n$.

**Proof.** Define an array $Z[i..k]$ in which $Z[i] = Y[i] - Y[i-1]$ and store the structure of Lemma 2 on $Z$. ■

**Lemma 4.** There is a structure for an array $Z$ containing $k = O(\log n / \log \log n) \cdot O(\log n)$-bit integers. The structure uses $O(k \log n)$ bits and supports the following operations in $O(1)$ time: (1) $\text{sum}(Z, i)$ queries. (2) $\text{search}(Z, d)$ queries, where $d > 0$. (3) $\text{update}(Z, i, \Delta)$ operations, where $\Delta \in \{-1, 1\}$.
Table 2: An example showing the arrays of the data structure of Lemma 2. The left table gives the values of the array $Z$ and the corresponding arrays $Y$, $I$, $D$, $Z'$ and $Y'$ (the arrays $Y$ and $I$ are shown without entries 0 and $k+1$). The table on the right shows the array $Z$ and the corresponding arrays after an update($Z$, 2, 1) operation. Changed entries appear in bold.

Proof. Define $Y[0..k+1]$ to be an array in which $Y[0] = 0$, $Y[k+1] = \infty$, and $Y[i] = \text{sum}(Z, i)$ for $1 \leq i \leq k$. Let $I[0..k+1]$ be a binary string in which $I[0] = 1$, and for $i \geq 1$, $I[i] = 1$ if $\max(Y[0..i-1]) < Y[i]$. Let $Y'$ be an array containing the entries $Y[i]$ for all indices $i \neq 0, k+1$ for which $I[i] = 1$, and let $Z'$ be an array of size $|Y'|$ in which $Z'[i] = Y'[i] - Y'[i - 1]$ ($Z'[1] = Y'[1]$). Note that by definition, $Z'[i] \geq 1$ for all $i$. Our data structure consists of the following structures:

- The structure of Lemma 2 on $Z$.
- The string $I$.
- The structure of Lemma 1 on $Z'$.
- An array $D[1..k]$ in which $D[i] = Y[\text{prev}_1(I, i)] - Y[i]$, where $\text{prev}_1(I, i)$ is the maximum index $i' \leq i$ such that $I[i'] = 1$.

See Table 2 for an example.

To answer a search($Z$, $d$) query, compute $i = \text{search}(Z', d)$ and return $\text{select}_1(I, i)$. The computation of $\text{select}_1(I, i)$ is done in $O(1)$ time using a lookup table. Therefore, the query is handled in $O(1)$ time.

We next describe how to handle an update($Z$, $i$, $\Delta$) operation (recall that $\Delta \in \{-1, 1\}$).

1. Perform an update($Z$, $i$, $\Delta$) operation on the structure of Lemma 2
2. $j \leftarrow \text{next}_1(I, i)$ (namely, $j \geq i$ is the minimum index such that $I[j] = 1$).
3. $i' \leftarrow \text{rank}_1(I, j)$.
4. update($Z'$, $i'$, $\Delta$).
5. If $i < j$ and $\Delta = 1$:
(a) Let \( l \) be the minimum index such that \( D[l] = 0 \). If no such index exists, \( l = k + 1 \).

(b) Add \(-1\) to the entries of \( D[i..l-1] \).

(c) If \( l \neq j \), set \( I[l] \leftarrow 1 \) and perform divide\((Z', i', 1)\).

6. If \( i < j \) and \( \Delta = -1 \):

(a) Add \( 1 \) to the entries of \( D[i..j-1] \).

(b) If \( Z'[i'] = 0 \), set \( I[j] \leftarrow 0 \) and perform merge\((Z', i')\).

We now show the correctness of the above algorithm. We will only prove correctness for the case \( \Delta = 1 \). The proof for \( \Delta = -1 \) is similar.

Consider an update\((Z, i, 1)\) operation. The update operation causes the entries of \( Y[i..k] \) to increase by 1. Recall that for an index \( p, I[p] = 1 \) if \( \max(Y[0..p-1]) < Y[p] \). By definition, \( \max(Y[0..p-1]) = Y[\text{prev}_1(I, p-1)] \). If \( p > j \) then \( \text{prev}_1(I, p-1) \geq j \geq i \). Therefore, the update operation causes both \( \max(Y[0..p-1]) \) and \( Y[p] \) to increase by 1. Therefore, the condition \( \max(Y[0..p-1]) < Y[p] \) is satisfied after the update if and only if it was satisfied before the update. In other words, the value of \( I[p] \) does not change due to the update operation. For \( p < i \), both \( \max(Y[0..p-1]) \) and \( Y[p] \) do not change, and thus \( I[p] \) does not change. For the index \( p = j \), \( I[j] = 1 \), and thus \( \max(Y[0..j-1]) < Y[j] \) before the update. The update increases \( Y[j] \) by 1, and either increases by 1 or does not change \( \max(Y[0..j-1]) \). Therefore, \( \max(Y[0..j-1]) < Y[j] \) after the update, so \( I[j] \) does not change. If \( i = j \) we have shown that \( I[p] \) does not change for every index \( p \). Therefore, the algorithm correctly updates the array \( I \) in this case.

Suppose now that \( i < j \). For \( p \in [i, l-1] \), \( \max(Y[0..p-1]) = Y[\text{prev}_1(I, p-1)] \) before the update. Since \( \text{prev}_1(I, p-1) \geq l \), we have that \( \max(Y[0..p-1]) \) does not change and \( Y[p] \) increases by one. Therefore, \( \max(Y[0..p-1]) \geq Y[p] \) after the update. It follows that \( I[p] \) does not change. Due to the same arguments, \( \max(Y[l..l-1]) = Y[l] \) before the update and \( \max(Y[0..l-1]) < Y[l] \) after the update. Thus, \( I[l] \) changes from 0 to 1. Finally, for \( p \in [l + 1, j - 1] \), \( \max(Y[0..p-1]) \geq Y[p] \) before the update. The update increases both \( \max(Y[0..p-1]) \) and \( Y[p] \) by one. Therefore, \( I[p] \) does not change. We obtained again that the algorithm updates \( I \) correctly. It is easy to verify that the algorithm also updates \( D \) correctly.

The above algorithm takes \( O(k) \) time due to lines 5a, 5b, and 6a. To reduce the time to \( O(1) \) we use the following approach from Navarro and Sadakane. Instead of storing \( D \), the data structure stores an array \( \hat{D} \) that has the following properties: (1) \( \hat{D}[i] = 0 \) if and only if \( D[i] = 0 \). (2) \( 0 \leq \hat{D}[i] \leq k \) for all \( i \). Due to the first property, we can use \( \hat{D} \) instead of \( D \) in line 6a above. Moreover, due to the second property, the space for storing \( D \) is \( k[\log(k + 1)] = O(\log n) \) bits. Thus, line 6a can be performed in \( O(1) \) time using a lookup table.
The array \( \hat{D} \) is updated as follows. The structure keeps an index \( \alpha \). If \( \Delta = -1 \), instead of line 6a above, first perform \( \hat{D}[p] \leftarrow \min(k, \hat{D}[p] + 1) \) for all \( i \leq p \leq j - 1 \). This takes \( O(1) \) time using a lookup table. Additionally, set \( \hat{D}[\alpha] \leftarrow \text{sum}(Z, \text{prev}_1(I, \alpha)) - \text{sum}(Z, \alpha) \) (so \( \hat{D}[\alpha] = D[\alpha] \) after this step). Finally, update \( \alpha \) by \( \alpha \leftarrow \alpha + 1 \) if \( \alpha < k \) and \( \alpha \leftarrow 1 \) otherwise. Handling the case \( \Delta = 1 \) is similar. It is easy to verify that \( \hat{D} \) satisfies the two properties above.

3 The min-max tree

In this section we describe the data structure of Navarro and Sadakane \cite{NavarroSadakane19} for dynamic trees. Let \( T \) be an ordinal tree. The balanced parentheses string of \( T \) is a string \( P \) obtained by performing a DFS traversal on \( T \). When reaching a node for the first time an opening parenthesis is appended to \( P \), and when the traversal leaves a node, a closing parenthesis is appended to \( P \). We will assume \( P \) is a binary string, where the character 1 encodes an opening parenthesis and 0 encodes a closing parenthesis. We also assume that a node \( x \) in \( T \) is represented by the index of its opening parenthesis in \( P \). For example, consider a tree \( T \) with 3 nodes in which the root has 2 children. The balanced parenthesis string of \( T \) is \( P = 110100 \), and the second child of the root is represented by the index 4.

For a binary string \( P \) and a function \( f: \{0, 1\} \to \{-1, 0, 1\} \), the following queries are called base queries.

\[
\begin{align*}
\text{sum}(P, f, i, j) &= \sum_{k=i}^{j} f(P[k]) \\
\text{fwd_search}(P, f, i, d) &= \min\{j \geq i : \text{sum}(P, f, i, j) = d\} \\
\text{bwd_search}(P, f, i, d) &= \max\{j \leq i : \text{sum}(P, f, j, i) = d\} \\
\text{rmq}(P, f, i, j) &= \min\{\text{sum}(P, f, 1, k) : i \leq k \leq j\} \\
\text{rmqi}(P, f, i, j) &= \min\{i \leq k \leq j : \text{sum}(P, f, 1, k) = \text{rmq}(P, f, i, j)\} \\
\text{min_count}(P, f, i, j) &= |\{i \leq k \leq j : \text{sum}(P, f, 1, k) = \text{rmq}(P, f, i, j)\}| \\
\text{min_select}(P, f, i, j, d) &= \text{The } d\text{-th smallest element of} \\
&\{i \leq k \leq j : \text{sum}(P, f, 1, k) = \text{rmq}(P, f, i, j)\} \\
\text{RMQ}(P, f, i, j) &= \max\{\text{sum}(P, f, 1, k) : i \leq k \leq j\} \\
\text{RMQi}(P, f, i, j) &= \min\{i \leq k \leq j : \text{sum}(P, f, 1, k) = \text{RMQ}(P, f, i, j)\}
\end{align*}
\]

Navarro and Sadakane showed that in order to support queries on the tree \( T \), it suffices to support the following base queries, where \( P \) is the balanced parentheses string of \( T \).

- All base queries on a function \( \pi \) defined by \( \pi(1) = 1 \) and \( \pi(0) = -1 \).
- sum and fwd_search queries on a function \( \phi \) defined by \( \phi(1) = 1 \) and \( \phi(0) = 0 \).
• sum and fwd_search queries on a function \( \psi \) defined by \( \psi(1) = 0 \) and \( \psi(0) = 1 \).

For example, level_ancestor\((x, d)\) = bwd_search\((P, \pi, x, d + 1)\). As noted in Tsur [22], the base queries fwd_search and bwd_search can be replaced by the following queries:

\[
\begin{align*}
\text{fwd_search}_\geq(P, f, i, d) &= \min\{j \geq i : \text{sum}(P, f, i, j) \geq d\} \\
\text{bwd_search}_\geq(P, f, i, d) &= \max\{j \leq i : \text{sum}(P, f, j, i) \geq d\}
\end{align*}
\]

We now need to support the base query bwd_search\(_\geq\) on the functions \( \pi \) and \( \pi' = -\pi \) (namely, \( \pi'(1) = -1 \) and \( \pi'(0) = 1 \)) and the base query fwd_search\(_\geq\) on the functions \( \pi, \pi', \phi, \) and \( \psi \).

To support the base queries, it is convenient to use an equivalent formulation of these queries. For an array of integers \( A \), let

\[
\text{fwd_search}_\geq(A, i, d) = \min\{j \geq i : A[j] \geq d\}.
\]

For a binary string \( P \), let \( f(P) \) be an array of length \( |P| \), where \( f(P)[i] = \text{sum}(P, f, 1, i) \). Then,

\[
\text{fwd_search}_\geq(P, f, i, d) = \text{fwd_search}_\geq(f(P), i, d + f(P)[i - 1])
\]

The other base queries on \( f \) can also be rephrased accordingly.

In order to support the base queries, the string \( P \) is partitioned into blocks of sizes \( \Theta(\log^2 n / \log \log n) \). The blocks are kept in a B-tree, called a min-max tree, where each leaf stores one block. The degrees of the internal nodes of the min-max tree are \( \Theta(\sqrt{\log n}) \), and therefore the height of the tree is \( \Theta(\log n / \log \log n) \). Each internal node stores local structures that are used for answering the base queries. A base query is handled by going down from the root of the min-max tree to one or two leaves of the tree, while performing queries on the local structures of the internal nodes that are traversed.

The tree queries level_ancestor, level_next, level_prev, level_lmost, and level_rmost are handled by performing a \text{fwd_search}_\geq(f(P), i, d) or a \text{bwd_search}_\geq(f(P), i, d) query. These queries take \( O(\log n) \) time in the data structure of Navarro and Sadakane. In Section 4 we will show how to reduce the time of \text{fwd_search}_\geq queries to \( O(\log n / \log \log n) \) (the handling of \text{bwd_search}_\geq queries is similar and thus omitted). In Section 5 we will show how to support degree queries in \( O(\log n / \log \log n) \) time.

4 fwd_search queries

In this section we describe how to support \text{fwd_search}_\geq(f(P), i, d) queries in \( O(\log n / \log \log n) \) time. We first describe how these queries are handled in
$O(\log n)$ time in the structure of Navarro and Sadakane. For each node $v$ in the min-max tree, let $P_v$ be the substring of $P$ obtained by concatenating the blocks of the descendant leaves of $v$. Suppose $v$ is an internal node of the min-max tree and the children of $v$ are $v_1, \ldots, v_k$. We partition $f(P_v)$ into blocks $f(P_{v_1}), \ldots, f(P_{v_k})$ where the size of $i$-th block is $|P_{v_i}|$. Note that $f(P_{v_i})[i] = f(P_{v_i})[i] + \delta_i$ for all $i$, where $\delta_i$ is the last element of $f(P_{v_i})_{i-1}$.

In data structure of Navarro and Sadakane, each internal node $v$ of the min-max tree stores the following local structures.

- The structure of Lemma 1 on an array $S_v[1..k]$ in which $S_v[i]$ is the size of $P_v$.
- A structure supporting $fwd_{\text{search}}_{\geq}$ queries on an array $M^f_v[1..k]$ in which $M^f_v[i] = \max(f(P_v))_{i}$.
- The structure of Corollary 3 on an array $L^f_v[1..k]$ in which $L^f_v[i]$ is the last entry of $f(P_v)_{i-1}$.

We now give a recursive procedure $\text{compute}_fwd_{\text{search}}(v, i, d)$ that computes $fwd_{\text{search}}_{\geq}(f(P_v), i, d)$.

1. If $v$ is a leaf in the min-max tree, compute the answer using a lookup table and return it.

2. If $i = 1$
   
   (a) $t \leftarrow 0$.
   
   else
   
   (b) $t \leftarrow \text{search}(S_v, i)$.
   
   (c) $j' \leftarrow \text{compute}_fwd_{\text{search}}(v, i - \sum(S_v, t - 1), d - L^f_v[t])$.
   
   (d) If $j' \neq \infty$, return $j' + \sum(S_v, t - 1)$.

3. $t' \leftarrow \text{fwd_{search}}_{\geq}(M^f_v, t + 1, d)$.

4. If $t' = \infty$ return $\infty$.

5. Return $\text{compute}_fwd_{\text{search}}(v, 1, d - L^f_v[t']) + \sum(S_v, t' - 1)$.

The time of step 1 is $O(\log n / \log \log n)$. Navarro and Sadakane showed that $fwd_{\text{search}}_{\geq}$ queries on $M^f_v$ can be handled in $O(\log k) = O(\log \log n)$ time. During the computation of $fwd_{\text{search}}_{\geq}(f(P), i, d)$, the procedure $\text{compute}_fwd_{\text{search}}$ is called on $O(\log n / \log \log n)$ nodes of the min-max tree. Therefore, the time for a $fwd_{\text{search}}_{\geq}(f(P), i, d)$ query is $O(\log n)$.

When a single character is inserted or deleted from $P$, the local structures of $O(\log n / \log \log n)$ nodes in the min-max tree are updated: If $v$ is the leaf whose
block contains the inserted or deleted character, only the local structures of the ancestors of $v$ are updated, assuming no split or merge operations were used to rebalance the min-max tree. The cost of splitting or merging min-max nodes can be ignored if an appropriate B-tree balancing algorithm is used (Navarro and Sadakane used the balancing algorithm of Fleischer [8], but other balancing algorithms can be used, e.g. the algorithm of Willard [23]). Each update takes $O(1)$ time, and therefore updating all local structures for a single character update on $P$ takes $O(\log n/\log \log n)$ time. An insertion or deletion of a node from $T$ consists of insertion or deletion of two characters from $P$. Therefore, the time to update the local structures is $O(\log n/\log \log n)$.

We now describe how to support $\text{fwd-search}_{\geq}$ queries in $O(\log n/\log \log n)$ time. In addition to the local structures described above, we also store the following local structures in each internal node $v$ of the min-max tree.

- An RMQ structure on $M_f^v$. Like in the structure of Navarro and Sadakane, this RMQ structure consists of the balanced parentheses string of the max-Cartesian tree of $M_f^v$.
- The structure of Lemma 4 on an array $D_f^v[i..k]$ in which $D_f^v[i] = M_f^v[i] - M_f^v[i - 1]$.

The procedure compute$fwd\text{-search}$ is changed by replacing lines 5 and 4 with the following equivalent lines:

5. If RMQ($M_f^v, t + 1, k$) < $d$ return $\infty$.

6. If $i = 1$ then
   (a) $t' \leftarrow \text{search}(D_f^v, d)$.
   
   else
   (b) $t' \leftarrow \text{fwd-search}_{\geq}(M_f^v, t + 1, d)$.

Consider the computation of $i^* = \text{fwd-search}_{\geq}(f(P), i, d)$ using procedure compute$fwd\text{-search}$. Let $w$ (resp., $w^*$) be the leaf in the min-max tree whose block contains $P[i]$ (resp., $P[i^*]$). Let $u_1^*, u_2^*, \ldots, u_h^* = w^*$ be the nodes on the path from the root of the min-max tree to $w^*$, and let $u_s^*$ be the lowest common ancestor of $w$ and $w^*$. Let $u_s = u_s^*, u_{s+1}, \ldots, u_h = w$ be the nodes on the path from $u_s^*$ to $w$. The computation of $\text{fwd-search}_{\geq}(f(P), i, d)$ makes the following calls to compute$fwd\text{-search}$. First, the procedure is called on $u_1^*, u_2^*, \ldots, u_s^*$. Then, the procedure is called on $u_{s+1}, \ldots, u_h$. Finally, the procedure is called on $u_s^*, \ldots, u_h^*$. Note that line 6 holds is executed only when the procedure is called on $u_s^*$. Therefore, line 6 holds contributes $O(\log k) = O(\log \log n)$ time to the total time of the computation. The rest of the recursive calls, except the two calls on $u_h$ and $u_h^*$, take $O(1)$ time each. Therefore, the total time is $O(\log n/\log \log n)$. 

9
Recall that the structure of Lemma 4 supports search(·, d) queries only for d > 0. We therefore need to show that d is non-negative in line 6a. Note that this line is executed only when procedure compute_fwd_search is called on $u_{s+1}^*, \ldots, u_{h-1}^*$. First, in every tree query that is answered by an fwd_search$_>($P, f, i, d) query, the parameter d is non-negative. The fwd_search$_>($P, f, i, d) query is handled by answering an fwd_search$_>($f(P), i, $d'$) query, where $d' = d + f(P)[i-1] = d + \text{sum}(P, f, 1, i-1)$.

Let $l_j, r_j$ be the indices such that $P_{u_j^*} = P[l_j..r_j]$ (note that $i^* \in [l_j, r_j]$ for all j). When procedure compute_fwd_search is called on $u_j^*$, the value of $d'$ is decreased by $\text{sum}(P, f, l_j-1, l_j-1)$. It follows that when the procedure is called on $u_j^*$ for $j > s$, the value of $d'$ is $d' = d - \text{sum}(P, f, i, l_j-1)$. Therefore, $d' > 0$ otherwise fwd_search$_>($P, f, i, d) $\leq l_j-1 - i < i^*$ which contradicts the definition of $i^*$.

**Updating the structures** We now show how to update the additional local structures when a character is inserted or deleted from P. Navarro and Sadakane showed that the RMQ structure on $M_v^f$ can be updated in $O(1)$ time using a lookup table. A single character update on P either does not change $M_v^f$, increases the entries of $M_v^f[i..j]$ by 1 for some i, or decreases the entries of $M_v^f[i..j]$ by 1 for some i. Therefore, either the array $D_v^f$ does not change, or a single entry of $D_v^f$ is either increased by 1 or decreased by 1. Thus, an update($D_v^f, i, \pm 1$) operation updates the structure on $D_v^f$ in $O(1)$ time.

5 degree queries

In this section we show how to handle degree($x$) queries in $O(\log n / \log \log n)$ time. We first describe the handling of these queries in the structure of Navarro and Sadakane.

To compute degree($x$), we use the equality degree($x$) = min_count($P, \pi, x+1, \text{enclose}(x)-1$), where enclose($x$) is the index in P of the closing parenthesis of x. As in Section 3, we will use an equivalent formulation of min_count. For arrays of integers A and B define

$$\text{min\_count}(A, i, j) = |\{i \leq k \leq j : A[k] = \min(A[i..j])\}$$

$$\text{min\_count}(A, B, i, j) = \sum_{i \leq k \leq j : A[k] = \min(A[i..j])} B[k]$$

We have that $\text{min\_count}(P, \pi, i, j) = \text{min\_count}(\pi(P), i, j)$. In the following we show a structure for computing $\text{min\_count}(\pi(P), i, j)$.

Consider some internal node v in the min-max tree, and let k be the number of children of v. Recall that the string $\pi(P_v)$ is partitioned into k blocks $\pi(P_v)_1, \ldots, \pi(P_v)_k$. The structure of Navarro and Sadakane stores in v the following local structures.
• The structure of Corollary on an array $m_v[1..k]$ in which $m_v[i] = \min(\pi(P_v)_i)$.

• An rmq structure on $m_v^\pi$ (as before, this structure consists of the balanced parentheses string of the min-Cartesian tree of $m_v^\pi$).

• An array $N_v^\pi[1..k]$ in which $N_v^\pi[i] = \min\text{count}(\pi(P_v)_i, 1, S_v[i])$.

• A structure for answering min_sum queries on $m_v^\pi, N_v^\pi$.

The following procedure compute_min_count($v, i, j$) returns the pair $\min\text{count}(\pi(P_v)_i, j), \text{rmq}(\pi(P_v)_i, i, j)$.

1. If $v$ is a leaf in the min-max tree, compute the answer using a lookup table and return it.

2. $t \leftarrow \text{search}(S_v, i)$ and $t' \leftarrow \text{search}(S_v, j)$.

3. $s \leftarrow \text{sum}(S_v, t - 1)$ and $s' \leftarrow \text{sum}(S_v, t' - 1)$.

4. If $t = t'$:
   (a) $N, m \leftarrow \text{compute_min_count}(v_t, i - s, j - s)$.
   (b) Return $N, m + L_v^\pi[t]$.

5. If $i - s > 1$:
   (a) $N_1, m_1 \leftarrow \text{compute_min_count}(v_t, i - s, S_v[t])$.
   (b) $m_1 \leftarrow m_1 + L_v^\pi[t]$.
   else $m_1 \leftarrow \infty$ and $t \leftarrow t - 1$.

6. If $j - s' < S_v[t']$:
   (a) $N_3, m_3 \leftarrow \text{compute_min_count}(v_{t'}, 1, j - s')$.
   (b) $m_3 \leftarrow m_3 + L_v^\pi[t']$.
   else $m_3 \leftarrow \infty$ and $t' \leftarrow t' + 1$.

7. If $t + 1 \leq t' - 1$:
   (a) $m_2 \leftarrow \text{rmq}(m_v^\pi, t + 1, t' - 1)$.
   (b) $N_2 \leftarrow \min\text{count}(m_v^\pi, N_v^\pi, t + 1, t' - 1)$.
   else $m_2 \leftarrow \infty$.

8. $m \leftarrow \min(m_1, m_2, m_3)$. 

11
9. $N \leftarrow \sum_{l \leq 3: m_l = m} N_l$.

10. Return $N, m$.

The time complexity of line 7b is $O(\log k) = O(\log \log n)$ and therefore the time complexity of a \texttt{min} \texttt{sum} query is $O(\log n)$.

We say that a node $x$ of $T$ is \textit{heavy} if it has at least $D = \lceil \log n \rceil^2$ children. Our approach for handling degree($x$) queries in $O(\log n / \log \log n)$ time is to handle differently heavy nodes and light nodes. In order to handle queries on heavy nodes the data structure stores the following structures.

- A rank-select structure on a binary string $B[1..2n]$ in which $B[x] = 1$ if $P[x]$ is an opening parenthesis and $x$ is a heavy node.
- An array $C$ containing degree($x$) for every $x$ such that $B[x] = 1$, sorted by increasing order of $x$.

For both $B$ and $C$ we use dynamic succinct structures from Navarro and Sadakane [19]. These structure have $O(\log n / \log \log n)$ query and update time. Therefore, checking whether a node is heavy, and computing degree($x$) for a heavy node takes $O(\log n / \log \log n)$ time. To bound the space for $B$ and $C$, we use the fact that there are at most $n/D$ heavy nodes. Therefore, the space for the rank-select structure on $B$ is $n H_0(B) + o(n) = O((n/D) \log \frac{2n}{n/B}) = o(n)$ bits, and the space for the array $C$ is $(1 + o(1)) |C| \log n \leq (1 + o(1)) n/D \cdot \log n = o(n)$ bits.

For a light node $x$, we compute degree($x$) using min\_count query. In addition to the local structures described above, each internal node $v$ in the min-max tree stores an array $\hat{N}_v[1..k]$ in which $\hat{N}_v[i] = \min(D, N_v^\pi[i])$. Recall that degree($x$) = min\_count($\pi(P)$, $x + 1$, $\text{enclose}(x) - 1$), and the latter expression can be computed by procedure compute\_min\_count. Since degree($x$) < $D$, we can replace line 7b in procedure compute\_min\_count by $N_2 \leftarrow \text{min}\_\text{count}(m_v^\pi, \hat{N}_v^\pi, t + 1, t' - 1)$. This line can be performed in constant time as follows. Using the balanced parenthesis string of the Cartesian tree of $m_v^\pi$ and a lookup table, obtain in constant time a binary string $X[1..k]$ such that $X[p] = 1$ if $m_v^\pi[p] = \text{rmq}(m_v^\pi, t + 1, t' - 1)$. Since the space for storing the array $\hat{N}_v^\pi$ is $k \log D = o(\log n)$ bits and the space for storing $X$ is $k = o(\log n)$ bits, a lookup table is used to compute in constant time the sum of $\hat{N}_v^\pi[p]$ for every $p$ such that $X[p] = 1$.

**Updating the structures** We first show how to update the local structures in the nodes of the min-max tree. Consider some internal node $v$ in the min-max tree, and let $v_1, \ldots, v_k$ be its children. The structure of Corollary 3 and the rmq structure on $m_v^\pi$ can be updated in $O(1)$ time (see Section 4). An insertion or deletion of a character from $P$ can only change one entry of $N_v^\pi$, namely the entry $N_v^\pi[i]$ where $i$ is the index such that the changed character belongs to $P_{v_i}$. To see why this is true, note that for $j < i$, all values in $\pi(P_{v_j})$ do not change due to
the character update. Therefore, $N^\pi_v[j]$ does not change. Additionally, for $j > i$, either all values in $\pi(v)_j$ are increased by 1 or all these values are decreased by 1, so again, $N^\pi_v[j]$ does not change. Therefore, to update $N^\pi_v$ and $\hat{N}^\pi_v$, we only need to compute the value of $N^\pi_v[i]$. This can be done in $O(\log k) = O(\log \log n)$ time by performing a min_count$(m^\pi_v, N^\pi_v, 1, S_v[i])$ query (recall that the min-max tree nodes have local structures for min-sum queries).

We next show how to update the structures on $B$ and $C$. When a node $x$ is inserted to the tree, perform two character insertions on $B$, and if $x$ is heavy insert its degree to $C$. Additionally, compute the degree of the parent $y$ of $x$. If the insertion of $x$ changes $y$ from light to heavy or from heavy to light, update $B$ and $C$ accordingly. Therefore, the insertion of $x$ causes $O(1)$ changes on $B$ and $C$ which are performed in $O(\log n/ \log \log n)$ time.

We also need to handle the case when insertion or deletions of nodes causes the value of $\lceil \log n \rceil$ to change. Since our definition of a heavy node depends on $\lceil \log n \rceil$, this means that a single node insertion or deletion can cause $\Theta(n/ \log n)$ nodes to change their heavy/light status, requiring a $\Omega(n/ \log n)$ time to update $B$ and $C$. The structure of Navarro and Sadakane already has a mechanism to handle changes to $\lceil \log n \rceil$ since the sizes of the blocks of $P$ and the sizes of the lookup tables used by the structure depend on $\lceil \log n \rceil$. This mechanism works as follows. The string $P$ is partitioned into three parts $P = P_0P_1P_2$. A separate min-max tree is built on each part $P_i$. The tree for $P_1$ uses the current value of $\lceil \log n \rceil$, while the trees for $P_0$ and $P_2$ use $\lceil \log n \rceil - 1$ and $\lceil \log n \rceil + 1$, respectively. When a node is added to or deleted from $T$, the structure changes the partition of $P$ by moving $O(1)$ characters between parts, and updating the min-max trees after each movement. We change the definition of $B$ as follows. For an index $x$, $B[x] = 1$ if $P[x]$ is an opening parenthesis and $\deg(x) \leq (\lceil \log n \rceil - 2 + i)^2$, where $i$ is the index such that $P[x]$ is in $P_i$. When the partition of $P$ changes, for every index $x$ such that $P[x]$ changes its part, we need to compute $\deg(x)$ and then update $B[x]$. If the value of $B[x]$ changes, a corresponding update on the array $C$ is performed. Using this approach, a single node insertion or deletion causes only $O(1)$ changes to $B$ and $C$ (recall that only $O(1)$ characters in $P$ move between parts). Additionally, at all times, if $x$ is heavy then $B[x] = 1$. Thus, the query algorithm remains correct.

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