This paper studies left invertibility of discrete-time linear output-quantized systems. Quantized outputs are generated according to a given partition of the state-space, while inputs are sequences on a finite alphabet. Left invertibility, i.e. injectivity of I/O map, is reduced to left D-invertibility, under suitable conditions. While left invertibility takes into account membership to sets of a given partition, left D-invertibility considers only membership to a single set, and is much easier to detect. The condition under which left invertibility and left D-invertibility are equivalent is that the elements of the dynamic matrix of the system form an algebraically independent set. Our main result is a method to compute left D-invertibility for all linear systems with no eigenvalue of modulus one. Therefore we are able to check left invertibility of output-quantized linear systems for a full measure set of matrices. Some examples are presented to show the application of the proposed method.

Left invertibility, uniform quantization, finite inputs, algebraic independent set, discrete time, control systems

1. Introduction

Left invertibility is an important problem of systems theory, which corresponds to injectivity of I/O map. It deals so with the possibility of recovering unknown inputs applied to the system from the knowledge of the outputs.

We investigate left invertibility of discrete–time linear output-quantized systems in a continuous state-space. In particular, inputs are arbitrary sequences of symbols in a finite alphabet: each symbol is associated to an action on the system. Information available on the system is represented by sequences of output values, generated by the system evolution according to a given partition of the state-space (quantization).

In recent years there has been a considerable amount of work on quantized control systems (see for instance [9, 20, 21] and references therein), stimulated also by the growing number of applications involving “networked” control systems, interconnected through channels of limited capacity (see e.g. [3, 7, 23, 27]). The quantization and the finite cardinality of the input set occur in many communication and control systems. Finite inputs arise because of the intrinsic nature of the actuator, or in presence of a logical supervisor, while output quantization may occur because of the digital nature of the sensor, or if data need a digital transmission.

Applications of left invertibility include fault detection in Supervisory Control and Data Acquisition (SCADA) systems, system identification, and cryptography ([12, 15]). Invertibility of linear systems is a well understood problem, first handled in [6], and then considered with algebraic approaches (see e.g. [22]), frequency domain techniques ([17, 18]), and geometric tools (cf. [19]). Invertibility of nonlinear systems is discussed in [21]. More recent work has addressed the left invertibility for switched systems ([28]), and for quantized contractive systems ([10]).
The main intent of the paper is to show that the analysis of left invertibility can be substituted, under suitable conditions, by an analysis of a stronger notion, called left D-invertibility. The condition under which left invertibility and left D-invertibility are equivalent is that the elements of the dynamic matrix of the system form an algebraically independent set (Theorem 6). Therefore the set of matrices for which left D-invertibility and left invertibility are equivalent is a full measure set. While left invertibility takes into account whether two states are in the same element of a given partition, left D-invertibility considers only the membership of a single state to a single set. For this reason left D-invertibility is much easier to detect. Our main result (Theorem 7) is a method to compute left D-invertibility for all linear systems whose dynamic matrix has no eigenvalue of modulus one.

The main tools used in the paper are the theory of Iterated Function Systems (IFS), and a theorem of Kronecker. The use of IFS in relation with left invertibility is described in [10]. The Kronecker’s theorem has to do with density in the unit cube of the fractional part of real numbers. By means of a particular construction illustrated in section 5 the problem of “turning” left D-invertibility into left invertibility can be handled with a Kronecker-type density theorem.

The paper is organized as follows. Section 2 contains an illustrative example. Section 3 is devoted to the definitions of left invertibility and uniform left invertibility. Section 4 illustrates the background knowledge. In section 5 left D-invertibility is introduced and main results (Theorems 6 and 7) are proved. Section 6 contains a deeper study of unidimensional systems. In section 7 we present some examples and section 8 shows conclusions and future perspectives. The Appendix is devoted to technical proofs. Finally, in section 9, we collect the notations used in the paper.

2. An illustrative example

In this section we give an illustrative example, to clarify methods and the purpose of this paper. Consider the linear output-quantized system, in one dimension, given by

\[
\begin{align*}
    x(k+1) &= ax(k) + u(k) \\
    y(k) &= \lfloor x(k) \rfloor \\
    u(k) &\in \{-1, 0, 1\},
\end{align*}
\]

where \(a\) is the input, \(y\) is the output, \(a\) is a constant and \(\lfloor \cdot \rfloor\) is the floor function. The left invertibility problem consists in reconstructing the unknown input sequence \(u(k)\) of the system from the information available reading only the quantized output sequence \(y(k)\). For the purpose of this example, define left invertibility in a negative way (see also definition 3): if there exist two input sequences \(\{u'(k)\}, \{u''(k)\}\) and two initial conditions \(x'(0), x''(0)\) such that the resulting orbits \(\{x'(k)\}\) and \(\{x''(k)\}\) give rise to the same output, then the system is not invertible (i.e. there is an output sequence that does not allow the reconstruction of the input sequence). No matter here how long are the sequences we are considering. Observe now that \(x'(k), x''(k)\) give rise to the same output if and only if the couple \([x'(k), x''(k)]\) remains in \(Q\), then the system (1) is not left invertible. For many reasons (mainly due to the “complexity” of the shape of the
set \( Q \) it is much easier to check the existence of a sequence of pair of states in \( Q_0 \), the “strip” that includes \( Q \) (see figure 1):

\[
Q_0 = \{ [x', x''] \in \mathbb{R}^2 : |x' - x''| < 1 \} \subset \mathbb{R}^2
\]

The problem is then to deduce the presence of an orbit in \( Q \) from the existence of an orbit in \( Q_0 \). We solve this problem with the help of the following remarks:

- Consider the orbits \( \{ x(k) \}^K_{k=1}, \{ x(k)'' \}^K_{k=1} \) of the system \( \{\} \), obtained respectively with initial conditions \( [x(0)', x(0)'' + \pi] \) instead of \( [x(0)', x(0)'' + \pi] \).

Then easy calculations shows that each \( [x(k)', x(k)''] \) differs from \( [x(k)', x(k)'' - \pi] \) by a translation along a parallel of the bisecting line of \( \mathbb{R}^2 \).

- \( \star \) The length of the translation of the state \( [x(k)', x(k)''] \), up to a constant, is proportional to \( a^k \), since

\[
x(k)'' = a^k (x(0)' + \pi) + \text{terms that depend only on the sequence of } u \text{'s}.
\]

The same holds for \( x(k)'' \).

Therefore the question is the following. Consider an orbit \( \{ [x(k)', x(k)''] \} \) which is included in \( Q_0 \). Does there exist a suitable translation \( [\pi, \pi] \) of the initial states \( [x(0)', x(0)'' + \pi] \) such that the resulting orbit is indeed in \( Q_0 \)?

By \( \star \), as \( \pi \) varies in \( \mathbb{R} \), the initial condition is moving along a parallel of the bisector of \( \mathbb{R}^2 \), but every time it cover a distance of \( \sqrt{2} \), it is in the same relative position with respect to a square of \( Q \) : in other words the property of “being inside \( Q_0 \)” is periodic.

Moreover, by \( \star \), the distance covered by \( [x(k)', x(k)'' - \pi] \) with respect to \( [x(k)', x(k)'''] \) is proportional to \( a^k \), and easy calculations shows that the property of “being inside \( Q_0 \)”, which is periodic, depends on the fractional parts of \( a^k \) (see also figure 2).

**Theorem 1.** If \( 1, a, a^2, \ldots, a^K \) are linearly independent over \( \mathbb{Z} \), then, for every \( \alpha_0, \alpha_1, \ldots, \alpha_K \in \mathbb{R} \) the set of points

\[
\left\{ \left[ \frac{\alpha_0 + l}{K}, \frac{\alpha_1 + l}{K}, \ldots, \frac{\alpha_K + l}{K} \right] : l \in \mathbb{R} \right\}
\]

is dense in the unit cube of \( \mathbb{R}^{K+1} \). \( \diamondsuit \)

The Theorem (see also Theorem 5) indeed assures that there exists a suitable translation \( [\pi, \pi] \) of the initial states \( [x(0)'', x(0)''] \) such that the resulting orbit is inside \( Q \). Referring to figure 3, each arrow represents the “period” 1 (in terms of fractional parts, to be multiplied by \( \sqrt{2} \)), and in particular, the values of fractional parts going from 0 to the intersection of the square with the arrow correspond to a point inside \( Q \). Therefore arbitrary small fractional parts, whose existence is assured by the Theorem, means that every point of the orbit can be positioned inside \( Q \) simply modifying the initial conditions in the way we showed. Indeed it is well known also that the numbers \( a \in \mathbb{R} \) such that \( 1, a, a^2, \ldots, a^K \) are linearly independent over \( \mathbb{Z} \) are a set of full measure.

That’s the point of this paper. In this work we solve the problem of checking the left invertibility of a system in arbitrary dimension, by investigating the presence of an orbit in the “strip” \( Q_0 \), and then deducing the presence of an orbit in \( Q \). This strategy works for a full measure set of parameters.

### 3. Basic setting

Throughout this paper we use the following notations:

- \( \pi_p \) is the canonical projection on the first \( p \) coordinates;
- \( \pi_W \) is the canonical projection on a subspace \( W \);
Definition 1. The uniform partition of rate $\delta$ of $\mathbb{R}^p$ is

$$\mathcal{P} = \bigcup_{i \in \mathbb{Z}^p} \mathcal{P}_i = \bigcup_{i_1, \ldots, i_p \in \mathbb{Z}} [i_1 \delta, (i_1 + 1)\delta] \times \ldots \times [i_p \delta, (i_p + 1)\delta],$$

where $i = i_1, \ldots, i_p$. 

We consider systems of the form

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) \\ y(k) = q(Cx(k)) \end{cases}$$

(2)

where $A \in \mathbb{R}^{d \times d}$, $B \in \mathbb{R}^{m \times d}$, $C \in \mathbb{R}^{p \times d}$, $x(k) \in \mathbb{R}^d$ is the state, $y(k) \in \mathbb{Z}^p$ is the output, and $u(k) \in \mathcal{U} \subset \mathbb{R}^m$ is the input. The map $q : \mathbb{R}^p \to \mathbb{Z}^p$ is induced by the uniform partition $\mathcal{P} = \bigcup_{i \in \mathbb{Z}^p} \mathcal{P}_i$ of $\mathbb{R}^p$ of rate $\delta$ through $q : (x \in \mathcal{P}_i) \mapsto i$ and will be referred to as the output quantizer. We assume that $\mathcal{U}$ is a finite set of cardinality $n$.

Remark 1. Suitably changing bases, without loss of generality in the system (2) we can suppose $\delta = 1$ and $C = \pi_p$. 
Figure 2. The distance covered by \( \{ [x(k), x(k')] \} \) with respect to \( \{ [x(k'), x(k'')] \} \) is proportional to \( a^k \), along a line parallel to the bisecting line of \( \mathbb{R}^2 \). The property of “being inside \( Q \)” is related to fractional parts of \( a^k \) which are associated with dashed line inside the squares.

So we consider only systems of the form

\[
\begin{align*}
  x(k+1) &= Ax(k) + Bu(k) = f(u(k), x(k)) \\
  y(k) &= \lfloor \pi_p x(k) \rfloor.
\end{align*}
\]

(3)

Definition 2. A pair of input strings \( \{ u(i) \}_{i \in \mathbb{N}}, \{ u'(i) \}_{i \in \mathbb{N}} \) is uniformly distinguishable in \( k \) steps if there exists \( l \in \mathbb{N} \) such that \( \forall x(0), x'(0) \in \mathbb{R}^d \) and \( \forall m > l \) the following holds for the correspondent orbits:

\[
u(m) \neq u'(m) \Rightarrow [y(m+1), \ldots, y(m+k)] \neq [y'(m+1), \ldots, y'(m+k)],
\]

(outputs \( y(i) \) are referred to the system with initial condition \( x(0) \) and inputs \( u(i) \), while outputs \( y'(i) \) are referred to the system with initial condition \( x'(0) \) and inputs \( u'(i) \)). In this case, we say that the strings are uniformly distinguishable with waiting time \( l \).

Definition 3. A system of type (4) is uniformly left invertible (ULI) in \( k \) steps if every pair of distinct input sequences is uniformly distinguishable in \( k \) steps after a finite time \( l \), where \( k \) and \( l \) are constant.

For a ULI system, it is possible to recover the input string until instant \( m \) observing the output string until instant \( m+k \). For applications, it is important to obtain an algorithm to reconstruct the input symbol used at time \( m > l \) by processing the output symbols from time \( m \) to \( m+k \).
Figure 3. Every arrow’s length is $\sqrt{2}$. The first part of the arrow, corresponding to small fractional parts, is the part inside a square of $Q$.

Definition 4. Define the quantization set relative to the system (3) to be

$$Q = \bigcup \{ [i_1, i_1 + 1] \times \ldots \times [i_p, i_p + 1] \times \langle e_{p+1}, \ldots, e_d \rangle \times [e_1, \ldots, e_d] \} \subset \mathbb{R}^{2d}$$

i.e. $Q$ contains all pairs of states that are in the same element of the partition $P$.

To address left invertibility, we are interested in the following system on $\mathbb{R}^{2d}$:

Definition 5. Define the doubled system relative to the system (3) to be

$$X(k+1) = \begin{bmatrix} Ax_1(k) + Bu(k) \\ Ax_2(k) + Bu'(k) \end{bmatrix}$$

where $X(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$, $U(k) = \begin{bmatrix} u(k) \\ u'(k) \end{bmatrix}$.

If there exist sequences $\{u(k)\}, \{u'(k)\}$, and an initial state in $Q$ such that the corresponding orbit of (4) remains in $Q$, then the two strings of inputs generate the same output for the system (3). So conditions ensuring that the state is outside $Q$ for some $k$ will be investigated to guarantee left invertibility.
4. Background: attractors and left invertibility

In this section we recall some results coming from Iterated Function System theory (see [3, 13] for general theory about IFS), in connection with the notions of left invertibility.

**Definition 6.** An output-quantized linear system of type (3) is joint contractive if $|\lambda| < 1$ for every eigenvalue $\lambda$ of the matrix $A$. It is joint expansive if $|\lambda| > 1$ for every eigenvalue $\lambda$ of the matrix $A$. ♦

**Definition 7.** Consider an output-quantized system.

- A set $A$ is an attractor if for all orbits $\{x(k)\}_{k \in \mathbb{N}}$ it holds
  $$\lim_{k \to \infty} \text{dist}(x(k), A) = 0.$$ Here $\text{dist}(x, A)$ is the inf of distances between $x$ and points of $A$.
- A set $I$ is an invariant set if $I = \bigcup_{u \in U} A(I) + Bu$. ♦

**Theorem 2.** [3, 14] Let a system be joint contractive. Then, for every $u \in U^\mathbb{N}$ the limit $\phi(u) = \lim_{k \to \infty} A^k x + A^{k-1} u(k-1) + \ldots + Bu(0)$ exists for every $x$ and is independent of $x$. The set $\phi(U^\mathbb{N})$ is the unique compact attractor and invariant set of the system. ♦

Consider now an output-quantized linear system, together with two sets, namely $H$ and $S$: $H$ plays the role of an attractive and invariant set, in which the dynamic of the system is confined, and $S$ plays the role of a quantization set, i.e. a set that the state has to exit to guarantee an invertibility property. In [11] a necessary and sufficient condition for left invertibility of joint contractive systems is given, but here we state the same condition in a more abstract setting: $H$ and $S$ are an attractor and a quantization set, not the attractor and the quantization set of the system (3). That’s because in the following we will use these results for another attractor and quantization set, i.e. those ones of the difference systems.

**Definition 8.** The graph $G_k$ associated to the attractor $H$ is given by:

- The set of vertices $V = \{H_{u(0)\ldots u(k)} = A^{k+1}(H) + A^k Bu(0) + \ldots + Bu(k) : u(i) \in U\}$.
- There is an edge from $H_{u(0)\ldots u(k)}$ to $H_{u'(0)\ldots u'(k)}$ if and only if $u(i+1) = u'(i)$, for $i = 0, \ldots, k-1$. In this case we say that the edge is induced by the input $u'(k)$. ♦

**Definition 9.** Consider the graph $G_k$, and delete all vertices (together with all starting and arriving edges) $H_{u(0)\ldots u(k)}$ such that $H_{u(0)\ldots u(k)} \cap \{\mathbb{R}^d \setminus S\} \neq \emptyset$. This new graph is called internal invertibility graph, and denoted with $IG_k$. The union of vertices (which are sets) of $IG_k$ is denoted by $V_{IG_k}$.

**Theorem 3.** [10] Denote with $\partial S$ the boundary of $S$. Suppose that $H \cap \partial S = \emptyset$. Then there exists a (computable) $k$ such that $V_{IG_k} = V_{IG_k} \cap S$. ♦
If instead the system $[3]$ is joint expansive, then the map $x(k) \mapsto x(k+1)$ admits an inverse for every $u \in \mathcal{U}$. Therefore it is possible to define a correspondent inverse system:

**Definition 10.** If the system $[3]$ is joint expansive, the inverse system is

$$
\hat{x}(k+1) = A^{-1}[\hat{x}(k) - B u(k)]
$$

The inverse doubled system, relative to the doubled system $[4]$, is defined in a similar way.

In the case of joint contractive systems, the inverse systems give rise to attractors, since they are joint contractive: such attractors can be described also as the set of initial conditions that can start a bounded orbit of the system $[3]$ or of the doubled system $[4]$.

**Theorem 4.** Suppose that the system $[3]$ is joint expansive. If there is an infinite bounded orbit of the system $[3]$ or of doubled system $[4]$, then this orbit is entirely contained in the attractor of the inverse system or in the attractor of the inverse doubled system, respectively. Consequently, if we restrict to bounded orbits, Theorem $[5]$ applies to these attractors.

5. **Difference system and D-invertibility**

**Definition 11.** The difference system associated with the system $[3]$ is

$$
z(k+1) = Az(k) + B v(k)
$$

where $z(k) \in \mathbb{R}^d$, $v(k) \in V = \mathcal{U} - \mathcal{U} = \{u - u' : u \in \mathcal{U}, u' \in \mathcal{U}\}$. ♦

**Remark 2.** The difference system represents at any instant the difference between the two states $x(k) - x'(k) = z(k)$ when the input symbols $u(k) - u'(k) = v(k)$ are performed. So we are interested in understanding the conditions under which

$$
\{z(k)\} \cap \{ \mid -1,1 \}^p \times \{e_{p+1},\ldots,e_d\} = \emptyset.
$$

Indeed, this implies that $y(k) \neq y'(k)$. The converse is obviously not true. ♦

**Definition 12.** Consider the difference system. If $z(0)$ is an initial condition and $(v(1),\ldots,v(k_2))$ a sequence of inputs of the difference system, we let $D_{k_1}^{k_2}(z(0), v(1),\ldots,v(k_2))$ denote the sequence $(\pi_1 v(z(k_1)),\ldots,\pi_p z(k_2))$ generated by the difference system with initial condition $z(0)$ and input string $(v(1),\ldots,v(k_2))$. ♦

**Definition 13.** A pair of input strings $\{u(i)\}_{i \in \mathbb{N}}, \{u'(i)\}_{i \in \mathbb{N}}$ is uniformly D-distinguishable in $k$ steps if there exists $l \in \mathbb{N}$ such that for all $x(0), x'(0) \in \mathbb{R}^d$ and $\forall m > l$ the following holds:

$$
v(m) \neq 0 \Rightarrow D_{m+k}^{n+l}(z(0), v(1),\ldots,v(m+k)) \not\subseteq \underbrace{[-1,1]^p \times \ldots \times [-1,1]^p}_{k \text{ times}},
$$

where $z(0) = x(0) - x'(0)$ and $v(i) = u(i) - u'(i)$. In this case, we say that the strings are uniformly D-distinguishable with waiting time $l$. ♦

**Definition 14.** A system of type $[3]$ is uniformly left D-invertible (ULDI) in $k$ steps if every pair of distinct input sequences is uniformly D-distinguishable in $k$ steps after a finite time $l$, where $k$ and $l$ are constant. ♦
Remark 3. Thanks to Remark 3, uniform left $D$-invertibility implies uniform left invertibility.

The first main result is based on a density theorem of Kronecker.

Definition 15. The numbers $\vartheta_1, \ldots, \vartheta_M \in \mathbb{R}$ are linearly independent over $\mathbb{Z}$ if the following holds:

$$k_1, \ldots, k_M \in \mathbb{Z} : k_1 \vartheta_1 + \ldots + k_M \vartheta_M = 0 \Rightarrow k_1 = \ldots = k_M = 0.$$  

Theorem 5 (Kronecker). If $\vartheta_1, \ldots, \vartheta_M, 1 \in \mathbb{R}$ are linearly independent over $\mathbb{Z}$, then, for every $\alpha_0, \alpha_1, \ldots, \alpha_M \in \mathbb{R}$ the set of points

$$\{\frac{\alpha_0 + l}{\vartheta_1}, \frac{\alpha_1 + l \vartheta_1}{\vartheta_2}, \ldots, \frac{\alpha_M + l \vartheta_M}{\vartheta_1}\} : l \in \mathbb{R}$$

is dense in the unit cube of $\mathbb{R}^{M+1}$.

Considering the difference system (Definition 11), we are interested in orbits completely included in $([-1,1]^d \times \langle e_{p+1}, \ldots, e_d \rangle)$. The following proposition shows that under a very weak condition orbits completely included in $([-1,1]^d \times \langle e_{p+1}, \ldots, e_d \rangle)$ must be bounded.

Proposition 1. Suppose that the matrix $A$ does not have an invariant subspace included in $\langle e_{p+1}, \ldots, e_d \rangle$. Then there exists a bounded set $I$ such that, if $\{z(k)\}_{k \in \mathbb{N}} \subset [-1,1]^d \times \langle e_{p+1}, \ldots, e_d \rangle$ is an orbit of the difference system, then $\{z(k)\}_{k \in \mathbb{N}} \subset I$.

Proof: See Appendix.

Note that the set of matrices $A \in \mathbb{R}^{d \times d}$ that have an invariant subspace in $\langle e_{p+1}, \ldots, e_d \rangle$ is a zero measure set. Define now $S_D(B,U)$ to be the set of matrices $A \in \mathbb{R}^{d \times d}$ such that the system (3) is uniformly left $D$-invertible, and $S(B,U)$ to be the set of matrices $A \in \mathbb{R}^{d \times d}$ such that the system (4) uniformly left invertible.

Definition 16. Indicate with $\mathbb{Q}[\zeta_1, \ldots, \zeta_N]$ the ring of polynomials in the variables $\zeta_i$ with coefficients in $\mathbb{Q}$. The set of numbers $\alpha_1, \ldots, \alpha_N \in \mathbb{C}$ is said to be algebraically independent if

$$0 \neq p(\zeta_1, \ldots, \zeta_N) \in \mathbb{Q}[\zeta_1, \ldots, \zeta_N] \Rightarrow p(\alpha_1, \ldots, \alpha_N) \neq 0.$$  

Theorem 6. Suppose that in the system (3) the set of elements of the matrix $A$ is algebraically independent. Then the system is uniformly left $D$-invertible if and only if it is uniformly left invertible. This in turn implies that $S(B,U) \setminus S_D(B,U)$ has measure zero in $\mathbb{R}^{d \times d}$ for every $B,U$.

Proof: See Appendix.

5.1. $D$-invertibility of output-quantized linear systems. We are going to show how to detect left $D$-invertibility of any linear systems without eigenvalues of modulus one. Suppose that, if $\lambda$ is an eigenvalue of the matrix $A$, then $|\lambda| \neq 1$. Denote with $E_c, E_e$ respectively the contractive and the expansive eigenspaces of the matrix $A$, i.e. the eigenspaces relative to eigenvalues $< 1$ and $> 1$ in modulus, respectively. Because of the hypothesis on the eigenvalues we have $E_c + E_e = \{x + y : x \in E_c, y \in E_e\} = \mathbb{R}^d$. Now consider the following two systems respectively on $E_c, E_e$, that are joint contractive:

$$z_c(k+1) = (\pi_{E_c} A) z_c(k) + \pi_{E_e} (Bv(k)) ;$$

$$z_e(k+1) = (\pi_{E_e} A^{-1})[z_e(k) - \pi_{E_c} (Bv(k))].$$
where with $z_c, z_e$ we indicate the projections of $z$ onto $E_c, E_e$, respectively. $v(k) \in \mathcal{V}$. The above systems must have invariant attractors $T^c, T^e$. Let us denote with $\mathcal{T}$ the attractor

$$\mathcal{T} = T^c + T^e = \{x + y : x \in T^c, y \in T^e\}.$$ 

We can now apply the construction of internal invertibility graph (Definition 9) for the attractor $\mathcal{T}$ (i.e. substituting $\mathcal{H}$ with $\mathcal{T}$), substituting $\mathcal{S}$ with $\{[-1, 1]^p \times \langle e_{p+1}, \ldots, e_d \rangle\}$, and calling a path $\{V_1, \ldots, V_i\}$ on $IG_k$ proper if it is induced by an input $v \in \mathcal{V}, v \neq 0$. Denoting with $\partial([-1, 1]^p \times \langle e_{p+1}, \ldots, e_d \rangle)$ the boundary of $[-1, 1]^p \times \langle e_{p+1}, \ldots, e_d \rangle$ we have the following

**Theorem 7.** Suppose that $|\lambda| \neq 1$ for every eigenvalue $\lambda$ of the matrix $A$, that $\mathcal{T} \cap \partial([-1, 1]^p \times \langle e_{p+1}, \ldots, e_d \rangle) = \emptyset$, and that $A$ does not have an invariant subspace in $\langle e_{p+1}, \ldots, e_d \rangle$. Then the system (3) is uniformly left D-invertible if and only if $IG_k$ does not contain arbitrary long proper paths, where $k$ is the one identified through Theorem 3.

**Proof:** Since $A$ does not have an invariant subspace in $\langle e_{p+1}, \ldots, e_d \rangle$, by Proposition 5 all orbits of the difference system included in $([-1, 1]^p \times \langle e_{p+1}, \ldots, e_d \rangle)$ must be bounded, and, by Theorem 8, must be included in the attractor $\mathcal{T}$. By Theorem 6 the system is uniformly D-invertible if and only if $IG_k$ does not contain arbitrary long proper paths. \(\diamondsuit\)

**Remark 4.** Theorem 7 gives an explicit, algorithmically implementable, way to compute ULDI of a system. By Theorem 6 we are able to compute ULI in the same way for systems with a full measure set of matrices $A$ (the conditions on the eigenspaces do not affect the full measure). \(\diamondsuit\)

**Remark 5.** The technical condition $\mathcal{T} \cap \partial([-1, 1]^p \times \langle e_{p+1}, \ldots, e_d \rangle) = \emptyset$ means, from a practical point of view, that left D-invertibility can be checked up to any finite precision, since the set $([-1, 1]^p \times \langle e_{p+1}, \ldots, e_d \rangle)$ depends only on the partition $\mathcal{P}$, and any small “disturbance” of the rate of the partition $\mathcal{P}$ allows the application of the Theorem 7. Further details on this point are given in [11]. \(\diamondsuit\)

### 6. OUTPUT-QUANTIZED LINEAR SYSTEMS OF DIMENSION 1

Linear systems of dimension 1 assume the following form, deriving from [3]:

$$\begin{align*}
&\begin{cases}
x(k + 1) = ax(k) + u(k) \\
y(k) = \lfloor x(k) \rfloor
\end{cases}
\end{align*}$$

(6)

This is a contractive system if $|a| < 1$ and an expansive system if $|a| > 1$. If $|a| < 1$ the invertibility problem can be solved with the methods of section 3 (see [10]).

The next Theorem shows a necessary condition for the ULI of a system of type (6): if it is not satisfied we construct inductively a pair of strings that gives rise to the same output.

**Theorem 8.** Suppose that in the system (6) $|a| > 2$. If there exist $u_1, u_2 \in \mathcal{U}, u_1 \neq u_2$ such that $|u_1 - u_2| < a$, then the system is not ULI.

**Proof:** We will consider sequences of sets of type

$$\begin{align*}
&S_{i+1} = \{a(S_i) + u(i)\} \cap \{a(S_i) + u'(i)\} \cap \mathcal{P}(i + 1) \\
&S_0 = [0, 1],
\end{align*}$$

(7)

where $u(i), u'(i) \in \{u_1, u_2\}$ and $\mathcal{P}(i + 1) \in \mathcal{P}$ is chosen at each step to maximize the measure of $S_{i+1}$.
In the sequence (7) set \( u(1) = u_1, u'(1) = u_2 \). Since \( |u_1 - u_2| < a \), there exists a \( \mathcal{P}(1) \in \mathcal{P} \) such that \( \mu(S_1) > 0 \) (recall that \( \mu \) indicates the Lebesgue measure). Then, for \( i > 1 \) define
\[
u(i) = u'(i) = u_1.
\]
Since \( |a| > 2 \) there exists an \( i_0 \) such that \( \mu(S_{i_0}) = 1 \), therefore, applying again \( u(i_0 + 1) = u_1 \) and \( u'(i_0 + 1) = u_2 \)
\[
\mu \{ A(S_{i_0}) + Bu_1 \cap A(S_{i_0}) + Bu_2 \} > 0.
\]
So there exists \( x_0, x_0' \in \mathbb{R} \) and \((u(1), \ldots, u(i_0 + 1), u'(1), \ldots, u'(i_0 + 1))\), with \( u(1) \neq u'(1) \) and \( u(i_0 + 1) \neq u'(i_0 + 1) \), such that for the corresponding outputs it holds
\[
(y(0), \ldots, y(i_0 + 1)) = (y'(0), \ldots, y'(i_0 + 1)).
\]
It is then enough to point out that, since we can achieve every pair of states \( x, x' \in S_{i_0} \) in the above described way, we can again go on in the same way and find a new instant \( i_1 \), a pair of initial states \((x_{1,0}, x_{1,0}')\), and control sequences \((u(1), \ldots, u(i_1)), (u'(1), \ldots, u'(i_1))\), with \( u(i_1) \neq u'(i_1) \), such that for the corresponding output it holds
\[
(y(0), \ldots, y(i_1)) = (y'(0), \ldots, y'(i_1)).
\]
Finally, we can achieve by induction an increasing finite sequence, but arbitrarily long, of instants \( i_k \), pairs of initial states \((x_{k,0}, x_{k,0}')\), and sequences of controls \((u(1), \ldots, u(i_k)), (u'(1), \ldots, u'(i_k))\) with \( u(i) \neq u'(i) \) if \( i = i_j + 1 \) for \( j = 1, \ldots, k - 1 \) such that such that for the corresponding output it holds
\[
(y(0), \ldots, y(i_k)) = (y'(0), \ldots, y'(i_k)).
\]
This contradicts the uniform left invertibility property. ◦

**Definition 17.** A number \( \alpha \in \mathbb{C} \) is called algebraic if there exists a polynomial \( p(x) \in \mathbb{Z}[x] \) such that \( p(\alpha) = 0 \). In this case the minimum degree of a polynomial with such a property is called the degree of \( \alpha \). A number \( \alpha \in \mathbb{C} \) is called transcendental if it is not algebraic. ◦

The following Theorem can be deduced from Theorem 6, observing that an algebraically independent set of one element is a transcendental number.

**Theorem 9.** Suppose that \( a \) is transcendental. Then the system (6) is uniformly invertible if and only if it is uniformly D-invertible. ◦

**Proposition 2.** The unidimensional system (6) is either ULTI in time 1, or not ULDI at all.

**Proof:** A sufficient condition for uniform left D-invertibility in one step is
\[
\forall v \in \mathcal{V}, v \neq 0 : |v| \geq |a| + 1:
\]
indeed in this hypothesis \( \forall v \in \mathcal{V}, v \neq 0 \)
\[
|v| \geq |a| + 1 : \quad |v| = |a| + 1
\]
We now prove that if \( \exists v \in \mathcal{V}, v \neq 0 : |v| < |a| + 1 \), then the system is not uniformly left D-invertible. Indeed in this case the system
\[
\begin{align*}
ax_1 + v &= x_2 \\
ax_2 - v &= x_1
\end{align*}
\]
has the solution \( x_1 = \frac{v}{a+1}, x_2 = \frac{v}{a+1} \). Since \( |x_1|, |x_2| < 1 \) the difference system has the infinite orbit \( \{x_1, x_2, x_1, x_2, \ldots \} \subset [-1, 1] \times \{e_{p+1}, \ldots, e_4\} \). Therefore system (6) is not left D-invertible. ◦
Corollary 1. Consider the unidimensional system (6), with transcendent a. Then it is either ULI in one step, or it is not ULI.

Remark 6. It’s easy to see that a system of the form (6) is uniformly D-invertible in one step if for all \( u_1, u_2 \in \mathcal{U} \) it holds \(|u_1 - u_2| > |a| + 1\). Therefore we have this summarizing situation for unidimensional systems:

- \(|a| < 1\): ULI can be detected with methods described in section 3.
- \(|a| > 2\): the system is not ULI;
- \(|a| \leq 2\): the system is ULI in 1 step;
- \(|a| 

\star The unidimensional system (6) is either ULI in time 1, or not ULI at all. With the additional hypothesis of transcendent of a the system is ULI in one step or it is not ULI.

7. Examples

Consider the system (3). As stated in Theorem 6 if the elements of the matrix \( A \) forms an algebraically independent set, then uniform left D-invertibility is equivalent to uniform left invertibility. A standard method to construct algebraically independent sets can be easily deduced from the following Theorem of Lindemann and Weierstrass:

Theorem 10. Suppose that the numbers \( \alpha_1, \ldots, \alpha_N \) are linearly independent over \( \mathbb{Q} \). Then \( e^{\alpha_1}, \ldots, e^{\alpha_N} \) are algebraically independent.

Example 1. Consider the system (3) with

\[
A = \begin{pmatrix} e^{\sqrt{5}} & e^{\sqrt{3}} \\ e^{\sqrt{2}} & e^{\sqrt{7}} \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \mathcal{U} = \{0, \pm 1, 2\} \quad y(k) = [\pi_1 x(k)].
\]

The two eigenvalue of \( A \) are approximately 6.3531 and 17.0974, so system (8) is joint expansive. The difference system is then joint expansive too. The inverse difference system is given by

\[
x(k + 1) = A^{-1}(x(k) - v(k)), \quad v(k) \in \mathcal{V} = \{0, \pm 1, \pm 2, \pm 3\}
\]

which is joint contractive. It is possible to show (see figure 4) that the attractor of the inverse difference system (8) is included in \([-1,1] \times \mathbb{R}\). So the system is not uniformly left D-invertible.

Moreover the elements of the matrix \( A \) by Theorem 10 are an algebraically independent set because \( \sqrt{5}, \sqrt{2}, \sqrt{3}, \sqrt{7} \) are linearly independent over \( \mathbb{Q} \). So by Theorem 6 system (8) is not uniformly left-invertible.

Example 2. Consider the system (3) with

\[
A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & \frac{1}{3} & 1 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}, \quad B = \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}, \quad \mathcal{U} = \{0,1\} \quad y(k) = [\pi_2 x(k)].
\]

We have so \( \mathcal{V} = \{0, \pm 1\} \). The three eigenvalue of the matrix are \( \frac{1}{2}, \frac{1}{3} \) and 2, so we can apply Theorem 7. Therefore we split \( \mathbb{R}^3 \) in \( E_e = \mathbb{R}^2 \) (identified with \( \{0\} \times \mathbb{R}^2 \)) and \( E_c = \mathbb{R} \) (identified with \( \mathbb{R} \times \{0\} \times \{0\} \)). The attractor \( \mathcal{A}_e \) relative to the inverse difference system on \( E_e \) is \([-2,2]\), while the attractor \( \mathcal{A}_c \) relative to the inverse difference system on \( E_c \) is drawn in figure 5. We are interested in orbits of the inverse difference system on \( E_e \) that remains in \([-1,1]\), and in orbits of the difference system on \( E_c \) that remains in \([-1,1] \times \mathbb{R}\). It’s easy to see that
Figure 4. Attractor of system (9).

Figure 5. Attractor of the inverse difference system on $E_e$.

\[
\begin{pmatrix}
  \frac{1}{2} & 1 \\
  0 & \frac{1}{4}
\end{pmatrix} \mathcal{A}_c \cap [1,1[ \times \mathbb{R}) + \begin{pmatrix} 3 \\ 6 \end{pmatrix} \cap \mathcal{A}_c \cap [-1,1[ \times \mathbb{R}) = \emptyset,
\]

so, no matter the behavior of the system on $E_e$, system (10) is uniformly left $D$-invertible in one step, therefore uniformly left invertible in one step.

The last example illustrates the difference between left $D$-invertibility and left invertibility.

**Example 3.** Consider the unidimensional system (6) with

\[ a = 1/2, \quad U = \{-1,0,1\}. \]  

(11)
We are going to show that system (11) is uniformly left invertible but not uniformly left D-invertible.

To show that the system is not ULDI consider the following orbit with initial condition $x_0 = \frac{1}{2}$:

$$ x_{k+1} = \begin{cases} \frac{1}{2}x(k) + 1, & \text{if } x(k) < 0; \\ \frac{1}{2}x(k) - 1, & \text{if } x(k) > 0. \end{cases} $$

Clearly $x(k) \in [-1, 1]$ for every $k \in \mathbb{N}$, so system (11) is not ULDI.

Nonetheless system (11) is ULI in 1 step. Consider indeed the quantization set (defined in Definition 4)

$$ Q = \bigcup_{i \in \mathbb{Z}} [i, i + 1] \times [i, i + 1] $$

and observe that

$$ \frac{1}{2}Q + \left( \begin{array}{cc} u \\ u' \end{array} \right) \cap Q = \emptyset \quad \forall u \neq u'. $$

This in turn implies that system (11) is ULI in 1 step. ♦

8. Conclusions

In this paper we studied left invertibility of output-quantized linear systems, and we proved that it is equivalent, under suitable conditions, to left D-invertibility, a stronger notion, much easier to detect (Theorem 7). More precisely the condition under which left invertibility and left D-invertibility are equivalent is that the elements of the dynamic matrix of the system form an algebraically independent set. Therefore the set of matrices for which left D-invertibility and left invertibility are equivalent is a full measure set (Theorem 6). Moreover there is a standard way to create matrices whose elements are an algebraically independent set (Theorem 10).

Notice that algebraic conditions play a central role in investigation of left invertibility of quantized systems as well in other fields when a quantization is introduced (see for instance [4, 8]).

Future research will include further investigation on the equivalence between left invertibility and left D-invertibility to matrices whose elements are not algebraically independent.

Appendix

Proof of Proposition 1. Define the following sequence of sets:

$$ \begin{cases} S_0 = \text{Ker}(\pi_{d-p}) \quad (= \langle e_{p+1}, \ldots, e_d \rangle) \\ S_{i+1} = A(S_i) \cap \text{Ker}(\pi_{d-p}). \end{cases} $$

Then $A$ does not have an invariant subspace included in $\text{Ker}(\pi_{d-p})$ if and only if $S_{d-p} = \{0\}$. To prove this, first of all observe that $S_i$ is a subspace of $\text{Ker}(\pi_{d-p})$ for every $i \in \mathbb{N}$. Moreover it holds that $S_i$ is a subspace of $S_{i-1}$ for every $i = 1, \ldots$. To prove this, by induction:

- $S_1 = A(S_0) \cap S_0$ is clearly a subspace of $S_0$;
- Suppose that $S_i$ is a subspace of $S_{i-1}$. Then $S_{i+1} = A(S_i) \cap \text{Ker}(\pi_{d-p})$ is a subspace of $A(S_{i-1}) \cap \text{Ker}(\pi_{d-p}) = S_i$. 


Moreover define

\[ X \]

If

\[ S_{d-p} \neq \{0\}. \]

Viceversa, suppose that \( S_{d-p} \neq \{0\} \). Since the length of the sequence \( \{S_0, \ldots, S_{d-p}\} \) is \( d+1 \), there exists \( j \in \{0, \ldots, d-p\} \) such that \( \text{dim}(S_j) = \text{dim}(S_{j+1}) = \dim[A(S_j) \cap \ker(\pi_{d-p})] \), where we indicate with \( \dim \) the dimension. But \( S_{j+1} \) is a subspace of \( S_j \), and this in turn implies that \( S_j = S_{j+1} \). Moreover both are subspaces of \( \ker(\pi_{d-p}) \). So

\[ S_{j+1} = A(S_j) \cap \ker(\pi_{d-p}) = S_j. \]

Therefore \( A(S_i) = S_j \) because \( \dim[A(S_j)] \leq \dim(S_j) \). So \( S_j \) is an invariant subspace of \( \ker(\pi_{d-p}) \).

Consider now the following succession of sets:

\[
\begin{align*}
\tilde{S}_0 &= \{(i-1,1)^p \times \ker(\pi_{d-p}) \} \\
\tilde{S}_{i+1} &= A(\tilde{S}_i) \cap (i-1,1)^p \times \ker(\pi_{d-p}).
\end{align*}
\]

Since \( S_{d-p} = \{0\} \) (a point) \( \tilde{S}_{d-p} \) must be bounded. Finally, observe that \( \tilde{S}_{d-p} \) is exactly the set of possible states \( z(d-p) \) when every \( z(0), \ldots, z(d-p) \) is in \( (i-1,1)^p \times \ker(\pi_p) \). The Proposition is thus proved. \( \diamond \)

**Proof of Theorem 6**

**Definition 18.** Let us parametrize the possible pairs of states \( (x,x') \in \mathbb{R}^{2d} \) such that \( x' - x \in (i-1,1)^p \times \langle e_{p+1}, \ldots, e_d \rangle \) with the set

\[ Q' = \{(t_1, \ldots, t_{d}, t_1 + s_1, \ldots, t_p + s_p, t_{p+1}, \ldots, t_d) : t_i, \tilde{t}_j \in \mathbb{R}, s_k \in [-1,1]\}. \]

Moreover define

\[ Q_i' = \{(t,t+s) \in \mathbb{R}^2 : t \in \mathbb{R}, s \in [-1,1]\} \subset \langle e_i, e_{d+i} \rangle, \]

\[ Q_i = \bigcup_{j \in \mathbb{Z}} [j, j+1][j, j+1] \cap \langle e_i, e_{d+i} \rangle. \]

If \( X = (t_1, \ldots, t_d, t_1 + s_1, \ldots, t_p + s_p, t_{p+1}, \ldots, t_d) \in Q' \), for \( i = 1, \ldots, p \) define \( d_i(X) \) to be the distance, measured along the line

\[ \{t_1, \ldots, \tau_i, \ldots, t_d, t_1 + s_1, \ldots, \tau_i + s_i, \ldots, t_p + s_p, \tilde{t}_{p+1}, \ldots, \tilde{t}_d : \tau_i \in \mathbb{R}\} \]

from the set \( \Omega_i = \{X \in \mathbb{R}^{2d} : \varpi_j X = 0 \text{ for } j \neq i, i+d\}. \diamond \)

**Lemma 1.** Fix a sequence \( \{U(j)\}_{j \in \mathbb{N}} \) of inputs for the doubled system \( \mathbb{S} \). Suppose that \( \forall \epsilon > 0, \forall m \in \mathbb{N}, \forall s_1, \ldots, s_p \in [-1,1], \) there exists \( t_1, \ldots, t_d, t_{p+1}, \ldots, t_d \in \mathbb{R} \) such that, if \( \{X(j)\}_{j=0}^m \subset Q' \) is the orbit of the doubled system \( \mathbb{S} \) with \( X(0) = (t_1, \ldots, t_d, t_1 + s_1, \ldots, t_p + s_p, \tilde{t}_{p+1}, \ldots, \tilde{t}_d) \), then the following holds

\[ \frac{\text{frac} \left( \frac{d_i(X(j))}{\sqrt{2}} \right)}{< \epsilon}, \]

for every \( i = 1, \ldots, p, j = 1, \ldots, m \). Then the system is not ULI.
Proof: Suppose that an orbit $\{X(i)\}_{i=1}^{\infty}$ of the doubled system is included in $Q'$ and consider the 2-dimensional plane spanned by $(e_i,e_{d+i})$. Observe that
\[
\frac{d_i(X(j))}{\sqrt{2}} = 0 \quad \text{if and only if} \quad X(j) \text{ belongs to some translation of } \Omega_i \text{ along the bisecting line of the 2-dimensional plane } (e_i,e_{d+i}), \text{that is entirely included in } Q_i, \text{i.e. a translation that takes } \Omega_i \text{ to the “bottom-left boundary” of a square of } Q_i.
\]

Suppose now that the relation (12) is satisfied for every $\epsilon$. It’s now easy to see that, for every $X \in Q'$ there exists $\epsilon > 0$ such that, if $\frac{d_i(X)}{\sqrt{2}} < \epsilon$ then the projection of $X$ on $Q'$ is indeed in $Q_i$ (thanks to the periodicity of the property “being in $Q_i$”, see also the illustrative example). Therefore, if the relations (12) are satisfied, then there exists an arbitrary long orbit included in $Q$. \hfill \Box

**Proposition 3.** Suppose that the entries of a matrix $A = \{a_{ij}\} \in \mathbb{R}^{d \times d}$ are algebraically independent, and denote with $a_{ij}^{(k)}$ the entries of the matrix $A^k$. Then the set
\[
\{ a_{ij}^{(k)} : i,j = 1, \ldots, d; k = 1, \ldots, N \}
\]
(13) is a linearly independent set for every $N \in \mathbb{N}$.

Proof: First, note that all $a_{ij}^{(k)}$ are polynomials of degree $k$ in the $a_{ij}$’s. Since the $a_{ij}$ are algebraically independent, they can be treated formally as the independent variables of polynomials in $d^2$ variables (more precisely there exists a ring isomorphism between $\mathbb{Q}[a_{11}, a_{12}, \ldots, a_{dd}]$ and the ring of polynomials in $d^2$ variables $\mathbb{Q}[x_{11}, \ldots, x_{dd}]$, see [1]). If a nontrivial linear combination of the elements of the set (13) is zero, then there exists a nontrivial polynomial in the $a_{ij}$ which is zero, so there exists a $k_0$ such that a nontrivial linear combination of the $a_{ij}^{(k_0)}$’s, seen as polynomials in the $a_{ij}$’s, which is zero. These are the entries of the matrix $A^{k_0}$, so there would exist a nontrivial linear relation among these entries. Suppose this is the case. If this linear relation results in a linear relation among polynomials which is not identically zero, we are done. Indeed, if there exists $\lambda_{ijk_0} \in \mathbb{Q}$ and $x_{11}, \ldots, x_{dd} \in \mathbb{R}$ such that (note that the entries $a_{ij}^{(k_0)}$ of the matrix $A^{k_0}$ are seen as polynomials in the variables $a_{11}, \ldots, a_{dd}$, renamed as $x_{11}, \ldots, x_{dd}$)
\[
\sum_{i,j} \lambda_{ijk_0} a_{ij}^{(k_0)} (x_{11}, \ldots, x_{dd}) \neq 0,
\]
then, substituting the $a_{ij}$’s to the $x_{ij}$’s, it is not possible that
\[
\sum_{i,j} \lambda_{ijk_0} a_{ij}^{(k_0)} (a_{11}, \ldots, a_{dd}) = 0,
\]
since the $a_{ij}$’s are algebraically independent.

Therefore we only have to show that it is not possible that
\[
\sum_{i,j} \lambda_{ijk_0} a_{ij}^{(k_0)} (x_{11}, \ldots, x_{dd}) \equiv 0,
\]
i.e. that this polynomial cannot be identically zero. Now note that the matrices $M = \{m_{ij}\} \in \mathbb{R}^{d \times d}$ whose entries do not satisfy the (nontrivial) linear relation $\sum_{i,j} \lambda_{ijk_0} m_{ij} = 0$ form a full measure set, dense in $\mathbb{R}^{d \times d}$. On the other hand also the matrices with distinct eigenvalues form a full measure set, dense in $\mathbb{R}^{d \times d}$. Therefore there exists a matrix $\overline{M} = \{\overline{m}_{ij}\}$ having distinct eigenvalues, whose entries do not satisfy the linear relation $\sum_{i,j} \lambda_{ijk_0} \overline{m}_{ij} = 0$.

Since $\overline{M}$ has distinct eigenvalues, there exists a matrix $B$ such that $B^{k_0} = \overline{M}$ (diagonalize and take $k_0$-roots of the eigenvalue). Denote with $b_{ij}^{(k_0)}$ the entries of the
matrix $B^{k_0}$: then the $t_0^{(k_0)}(i,j)$s do not satisfy the linear relation $\sum_{i,j} \lambda_{ij}k_0^{(k_0)}(i,j) = 0$, since $B^{k_0} = \mathbb{M}$. This implies that $\sum_{i,j} \lambda_{ij}k_0^{(k_0)}(x_1, \ldots, x_{dd})$ is not identically zero as a polynomial. ♦

We now prove Theorem 5. Consider the state $X(m)$ of the doubled-system (4) at instant $m$ given by an initial condition $X(0) \in Q'$ and an input sequence

$$U(1) = \left( \begin{array}{c} u(1) \\ u'(1) \end{array} \right), \ldots, U(m) = \left( \begin{array}{c} u(m) \\ u'(m) \end{array} \right) \in \mathcal{U} \times \mathcal{U}.$$  

Then

$$X(m) = \left( \begin{array}{cc} A & 0 \\ 0 & A \end{array} \right)^m X_0 + \ldots + \left( \begin{array}{cc} Bu(m) \\ Bu'(m) \end{array} \right).$$

Suppose that the $(d-\text{dimensional})$ system is not uniformly left D-invertible. So there exists arbitrarily long orbits of the $(2d-\text{dimensional})$ doubled-system included in $Q'$. In the following we provide conditions such that

$$\forall \epsilon > 0, \forall m \in \mathbb{N}, \forall s_1, \ldots, s_p \in [0, 1], \text{there exists } t_1, \ldots, t_d, t_{d+1}, \ldots, t_d \in \mathbb{R} \text{ such that, if } \{X(j)\}_0^m \text{ is the orbit of the doubled-system with } X(0) = (t_1, \ldots, t_d, t_1 + s_1, \ldots, t_p + s_p, t_{p+1}, \ldots, t_d), \text{ then the following holds}$$

$$\frac{d_j(X(j))}{\sqrt{2}} < \epsilon,$$

for every $i = 1, \ldots, p$, $j = 1, \ldots, m$. Therefore the system will be not uniformly left invertible by Lemma 1. These conditions will be verified by a full measure set. Consider the set

$$S' = \{ A \in \mathbb{R}^{d \times d} : \{a_{ij}\}_{i,j=1}^d \text{ is an algebraically independent set} \}.$$

Set $A \in S'$. For $i = 1, \ldots, p$, $\varpi_{i+d}(X(j))$ has the form

$$\varpi_{i+d}(X(j)) = \pi_{i,i+d} \left[ \begin{array}{cccccccc} a_{11} & \cdots & a_{1d} & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
a_{d1} & \cdots & a_{dd} & 0 & \cdots & 0 \\
0 & \cdots & 0 & a_{11} & \cdots & a_{1d} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & a_{d1} & \cdots & a_{dd} \end{array} \right]^j \left( \begin{array}{c} t_1 \\
\vdots \n t_d \\
\vdots \\
t_{d+1} \\
\vdots \\
t_{d} \end{array} \right) + \text{const.}$$

$$= \left( \begin{array}{cc} c_{i,1}^{(j)} t_1 + \cdots + c_{i,d}^{(j)} t_d \\
\vdots \\
c_{i,d}^{(j)} t_1 + \cdots + c_{i,p}^{(j)} t_{p} + c_{i,p+1}^{(j)} t_{p+1} + \cdots + c_{i,d}^{(j)} t_d \end{array} \right) + \text{const.}$$

where $c_{i,l}^{(j)}$ is the entry $(i, l)$ of the matrix $A^j$. By Proposition 3 the set $\{c_{i,l}^{(j)} : i, l = 1, \ldots, d; j = 1, \ldots, N\}$ is a linearly independent set, so, by Kronecker’s Theorem (Theorem 3) there exists a choice of $(t_1, \ldots, t_d, t_{p+1}, \ldots, t_d)$ such that equation (12) is satisfied, and Lemma 1 thus apply.
To prove that the set of matrices with algebraically independent entries are a full measure set, first observe that the set of polynomial $P \in \mathbb{Q}[\zeta_1, \ldots, \zeta_d]$ is countable.

For a single polynomial $P$ the set

$$0_P = \{(x_1, \ldots, x_d) \in \mathbb{R}^d : P(x_1, \ldots, x_d) = 0\}$$

is a finite union of manifolds of dimension at most $d^2 - 1$. So the measure of $0_P$ is zero. Moreover

$$S' = \bigcup_{P \in \mathbb{Q}[\zeta_1, \ldots, \zeta_d]} 0_P,$$

i.e. $S'$ is a countable union of sets of measure zero, which in turn implies that the measure of $S'$ is zero. ♦

9. Notations

The authors made an effort to simplify notations, though they are intrisically complex. For this reason in this “special” section we collect the notations used in this paper, ordered as their appearance.

- $\mathcal{P} = \bigcup \mathcal{P}_i$ is the uniform partition (Definition 1);
- $\mathcal{U}$ is the finite alphabet of inputs (just after the Definition 1);
- $q : (x \in \mathcal{P}_i) \mapsto i$ is the output quantizer (just after the Definition 1);
- $Q$ is quantization set (Definition 4);
- $A$ denotes the attractor of a system, (Definition 7);
- $I$ denotes the invariant set of a system (Definition 7);
- $\phi$ indicates the function that associates to each input sequence its limit point (Theorem 2);
- $G_k, IG_k$ are respectively the graph of depth $k$ associated to the attractor $A$, and the internal invertibility graph (Definitions 8, 9);
- $V_{IG_k}$ denotes the union of vertices (which are sets) of the internal invertibility graph (Definition 9);
- $V = \mathcal{U} - \mathcal{U} = \{u - u' : u \in \mathcal{U}, u' \in \mathcal{U}\}$ is the input set of the difference set (Definition 11);
- $D_k^{p_1}(z(0), v(1), \ldots, v(k_2))$ denotes the sequence $(\pi_{p_1}z(k_1), \ldots, \pi_{p_1}z(k_2))$ generated by the difference system with initial condition $z(0)$ and inputs $v(1), \ldots, v(k_2)$ (Definition 12);
- $S_D(B, \mathcal{U})$ denotes the set of matrices $A \in \mathbb{R}^{d \times d}$ such that the system (3) is ULDI (before Theorem 6);
- $S(B, \mathcal{U})$ denote the set of matrices $A \in \mathbb{R}^{d \times d}$ such that the system (3) is ULI (before Theorem 6);
- $Q' = \left\{(t_1, \ldots, t_d, t_1 + s_1, \ldots, t_p + s_p, \tilde{t}_{p+1}, \ldots, \tilde{t}_d)^T : t_i, \tilde{t}_j \in \mathbb{R}, s_k \in [-1, 1]\right\}$.
- $d_i(X)$ is the measure of a distance defined in Definition 18;
- $\Omega_i$ is the union of the two coordinate axes of $\langle e_i, e_{d+i}\rangle$ (Definition 18);

References

[1] Artin M, (1991) Algebra. Prentice-Hall
[2] Baker A, (1993) Transcendental number theory. Camb Univ Press
[3] Barnsley M, (1993) Fractals everywhere. Acad Press inc
[4] Bicchi A, Marigo A, Piccoli B, (1992) On the reachability of quantized control systems. IEEE Trans Autom Control 47(4):546-563
[5] Bobylev N A, Emel’yanov S V, Korovin S K, (2000) Attractor of discrete controlled systems in metric spaces. Comput Math Model 11(4):321-326
[6] Brockett R W, Mesarovic M D, (1965) The reproducibility of multivariable control systems. J Math Anal Appl 11:548-563
Carli R, Fagnani F, Speranzon A, Zampieri S, (2008) Communication constraints in the average consensus problem. Automatica 44:671-684

Chitour Y, Piccoli B, (2001) Controllability for discrete systems with a finite control set. Math Control Signal Syst 14:173-193

Delchamps D F, (1990) Stabilizing a linear system with quantized state feedback. IEEE Trans Autom Control 35(8):916-924

Dubbini N, Piccoli B, Bicchi A, (2008) Left invertibility of discrete systems with finite inputs and quantized output. Proc 47th IEEE Conf Decis Control 4687–4692

Dubbini N, Piccoli B, Bicchi A, (2010) Left invertibility of discrete systems with finite inputs and quantised output. Int J Control, 83(4) 798–809

Edelmayer A, Bokor J, Szabó Z, Szigi E, (2004) Input reconstruction by means of system inversion: a geometric approach to fault detection and isolation in nonlinear systems. Int J Appl Math Comput Sci 14(2):189-199

Falconer K, (2003) Fractal geometry, mathematical foundations and applications. John Wiley and Sons

Gwozdz-Lukawska G, Jaczyński J (2005) The Hutchinson-Barnsely theory for infinite iterated function systems. Bull Aust Math Soc 72:441-454

Inoue E, Ushio T, (2001) Chaos communication using unknown input observer. Electron Commun Jpn pt 3 84(12):21-27

Hardy G H, Wright E M, (1979) An introduction to the theory of numbers. Oxf Sci Publ

Massey J L, Sain M K, (1969) Invertibility of linear time-invariant dynamical systems. IEEE Trans Autom Control, AC-14(2):141-149

Massey J L, Sain M K, (1968) Inverses of linear sequential circuits. IEEE Trans Computers C-17:330-337

Morse A S, Wonham W M, (1971) Status of noninteracting control. IEEE Trans Automat Control, 16(6):568-581

Picasso B, Bicchi A, (2007) On the stabilization of linear systems under assigned I/O quantization. IEEE Trans Autom Control 52(10):1994-2000

Respondek W (1990) Right and Left Invertibility of Nonlinear Control Systems. In: Sussmann (ed) Nonlinear Controllability and Optimal Control, NY pp 133-176

Silverman L M, (1969) Inversion of multivariable linear systems. IEEE Trans Automat Control 14(3):270-276

Sontag E D, (1998) Mathematical control theory: deterministic finite dimensional systems. Springer, NY

Szanier M, Sideris A, (1994) Feedback control of quantized constrained systems with applications to neuromorphic controller design. IEEE Trans Autom Control 39(7):1497-1502

Tanwani A, Liberzon D, (2010) Invertibility of switched nonlinear systems. Automatica, 46:1962-1973

Tatikonda S C, Mitter S, (2004) Control under communication constraints. IEEE Trans Autom Control 49(7):1056-1068

van Schuppen J H, (2004) Decentralized control with communication between controllers. In: Blondel V D, Megretski A (ed) Prince Univ Press, Princeton, pp 144–150

Vu L, Liberzon D, (2008) Invertibility of switched linear systems. Automatica, 44:949-958

Nevio Dubbini: Interdepartmental research center “E. Piaggio”, University of Pisa, Italy, (e-mail: nevio.dubbini@for.unipi.it)

Benedetto Piccoli: Department of Mathematical Sciences, Rutgers University, New Jersey, USA

Antonio Bicchi: Interdepartmental center “E. Piaggio”, University of Pisa, Italy