In Search of a Data Driven Symbolic Multi-Fluid Closure

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The inclusion of kinetic effects into fluid models has been a long standing problem in magnetic reconnection and plasma physics. Generally the pressure tensor is reduced to a scalar which is an approximation used to aid in the modeling of large scale global systems such as the Earth’s magnetosphere. This unfortunately omits important kinetic physics which have been shown to play a crucial role in collisionless regimes. The multi-fluid model on the other-hand retains the full pressure tensor. Use of the multi-fluid model requires a closure which truncates the cascading system of equations, resulting in either the 5 or 10 moment models. Here we look to leverage data-driven methodologies to seek a closure which may improve performance of the 10 moment multi-fluid model in collisionless regimes. Specifically we use the SINDy method for symbolic equation discovery to seek the truncating closure from fully kinetic particle-in-cell simulation data, which inherently retains the relevant kinetic physics. We verify our method by reproducing the 10-moment model from the PIC particle data and use the method to generate a closure truncating the 10-moment model which is analyzed through the nonlinear phase of reconnection.

1. Introduction

Magnetic reconnection is the process in which a magnetic field embedded in plasma undergoes a topological restructuring. This process, which converts energy stored in magnetic fields into plasma kinetic and thermal energy is ubiquitous throughout the universe and plays an important role in such diverse events as sawtooth crashes in fusion plasmas (Yamada\textsuperscript{[2010]} Zweibel\textsuperscript{[2009]}), magnetic substorms in the earth’s magnetosphere (Hastie\textsuperscript{[1997]}), and solar coronal mass ejections (Masuda \textit{et al.}\textsuperscript{[1994]}).

Magnetic reconnection was first studied using magnetohydrodynamics (MHD). In ideal MHD, which describes a plasma as a single fluid of infinite conductivity, the Ohm’s Law $\mathbf{E} + \mathbf{u} \times \mathbf{B} = 0$ means that magnetic flux is frozen into the plasma flows and hence magnetic reconnection is topologically prohibited.

In real plasmas, however, the right-hand-side (r.h.s.) of the Ohm’s Law is not zero. In resistive MHD, finite conductivity is introduced as the resistive term $\eta \mathbf{J}$ on the r.h.s. of Ohm’s Law. Sweet and Parker (Parker\textsuperscript{[1957]} used this model to develop the first self-consistent description of magnetic reconnection. However, many plasmas of interest are nearly collisionless, and the reconnection rates predicted by Sweet-Parker do not match observations. Petschek’s model (Petschek\textsuperscript{[1964]} addressed the shortcomings of the Sweet-Parker model by realizing that the geometry of the reconnection region is crucial and that reducing the aspect ratio of that region can explain the faster reconnection rates observed. This means that the reconnection region does not extend all the way on the global length scale but is much shorter and connected to the global geometry by slow shocks. Petschek’s model, however is not realized in numerical simulations unless one

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triggers a shorter reconnection region by anomalous resistivity or other modifications of the physics.

It was later realized that the one-fluid MHD description of plasmas is too limited to incorporate all the essential reconnection physics. The next steps were Hall-MHD and extended MHD models that take into account that electrons and ions do not move together at the small scales of a reconnection region. From the electron momentum equation, one can derive a generalized Ohm’s Law (Vasyliunas 1975; Birn et al. 2001):

\[
E + u \times B = \eta J + \frac{J \times B}{n|e|} - \frac{\nabla \cdot P_e}{n|e|} + \frac{m_e}{n|e|^2} \left( \partial_t J + \nabla \cdot \left( uJ + Ju - \frac{JJ}{n|e|} \right) \right)
\]  

The generalized Ohm’s law (1.1) describes the various non-ideal contributions to the electric field in a plasma. The resistive dissipation term, sometimes denoted as the collisional term, is given by \(\eta J\). The second term on the right hand side is called the Hall term. In itself, it cannot effect magnetic reconnection since it describes the magnetic field being frozen into the electron fluid flow, but it is known to change the geometry of the reconnection process to make "fast", i.e., of the order of the Alfven velocity. The third term is the divergence of the electron pressure tensor, and the last term is the electron inertial term.

In fluid simulations, the electron pressure tensor is usually approximated by a scalar pressure, however it is known from fully kinetic simulations that off-diagonal terms can support reconnection electric fields. It has generally become clear that kinetic physics, being the first-principles description of a plasma, are important for a complete description of magnetic reconnection, in particular in the collisionless regime. Advances in computing power have enabled comparatively large fully kinetic simulations using the particle-in-cell (PIC) method, even in 3-d, that can achieve a good separation between the global size of the simulation and ion and electron scales, even though they usually require a reduced ion/electron mass ratio to keep the computational cost manageable. These kinds of simulations are limited to local simulations of reconnection, although their extent may be 10s or 100s of ion inertial scales \(d_i\). In contrast, many physical systems of interest, e.g., Earth’s magnetosphere or the Sun’s corona are many orders of magnitude larger and will remain inaccessible to fully kinetic simulations for the foreseeable future.

Fluid simulations are orders of magnitude cheaper, since they evolve 3-d fields rather than 6-d distribution functions, so they are computationally much more attractive, but they by definition lose part of the information that describes a kinetic plasma.

An alternative fluid formulation employed in this context is the multi-fluid moment model. This model is derived without any approximations by taking moments of the Vlasov equation and keeps the full set of Maxwell’s Equations.

1.1. Multi-Fluid and Ten Moment Model

\[
d_t f_s = \partial_t f_s + v \cdot \nabla_r f_s + \frac{q}{m} (E + v \times B) \cdot \nabla_v f_s = 0
\]  

The multi-fluid model is constructed by taking sequentially increasing velocity space moments of the Vlasov equation (1.2) for each species \(s\). Following (Wang et al. 2015),
the following moments are defined:

\[ n \equiv \int f d\mathbf{v} \quad (1.3a) \]
\[ u_j \equiv \frac{1}{n} \int v_j f d\mathbf{v} \quad (1.3b) \]
\[ P_{ij} \equiv m \int v_i v_j f d\mathbf{v} \quad (1.3c) \]
\[ Q_{ijk} \equiv m \int v_i v_j v_k d\mathbf{v} \quad (1.3d) \]

The evolution equations for these moments are derived by multiplying the Vlasov equation by consecutive powers of \( v \) and integrating out velocity space. Truncating the hierarchy at 2nd order gives the so-called 10-moment equations (1.4), which describe the evolution of density, the three components of momentum, and each of the 6 unique components of the symmetric pressure tensor. Each species is maintained as a separate set of fluid moments, but a species index has been left out above for brevity.

\[ \partial_t n + \partial_j (nu_j) = 0 \quad (1.4a) \]
\[ m \partial_t (nu_i) + \partial_j P_{ij} = nq (E_i + \epsilon_{ijk} u_j B_k) \quad (1.4b) \]
\[ \partial_t P_{ij} + \partial_k Q_{ijk} = nqu_{[i} E_{j]} + \frac{q}{m} \epsilon_{[ikl} P_{kj]} B_l \quad (1.4c) \]

The square bracket notation above indicates the minimal sum over free indices that yields a completely symmetric tensor. The electromagnetic fields evolve according to the Maxwell equations

\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (1.5a) \]
\[ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} \quad (1.5b) \]

Noticeably, each moment equation contains the next higher order moment. This trend continues ad-infinum resulting in an open system of equations. While this system of equations is exact, it needs to be truncated, which in the ten-moment model is achieved by seeking a closure that removes the unknown \( Q \) from the second order equation. This closure may be a function of the known ten moments or may be zeroed out completely for the adiabatic case, resulting in the five-moment model. In this work, though, we are interested in the pressure tensor term’s impact on reconnection and hence choose to keep it.

The above moments (1.3) are generally used for the derivation of the ten moment model, however the centered moments (1.6) are frequently used for the closure because of their convenient physical interpretation as the pressure stress tensor and heat flux tensor.

\[ P_{ij} \equiv m \int (v_i - u_i)(v_j - u_j) f d\mathbf{v} \quad (1.6a) \]
\[ Q_{ijk} \equiv m \int (v_i - u_i)(v_j - u_j)(v_k - u_k) f d\mathbf{v} \quad (1.6b) \]
The centered and uncentered moments are related according to

\[ P_{ij} = P_{ij} + 2nmu_iu_j \]  
\[ Q_{ijk} = Q_{ijk} + u_i[P_{jk}] - 2nmu_iu_ju_k \]

Previous closures have been proposed such as the CGL family of closures (Chust & Belmont 2006) and the Hammet-Perkins closure (Hammett & Perkins 1990).

In this work we aim to use data-driven methodologies to derive a symbolic closure relation truncating the cascading system of equations at the ten moment model.

We build off the work of (Wang et al. 2015), investigating the collisionless regime using data from a fully kinetic Harris Sheet PIC simulation. We then apply the data-driven technique, Sparse Identification of Nonlinear Dynamics (Brunton et al. 2016) to distill the raw particle output into a symbolic closure which is compared to the approximate local closure (1.8).

\[ \partial_m Q_{ijm} \approx v_t|k_0| (P_{ij} - p\delta_{ij}) \]

In the above, \( v_t \) is the thermal velocity, \( k_0 \) is a typical scale defining wave number, \( p \) is the scalar pressure attained by averaging the diagonal of the pressure stress tensor, and the moments are centered.

In the literature various options for closures have been investigated (Ng et al. 2017) by implementing a given proposed closure into the multi-fluid moment model, choosing parameters like a typical wavelength of kinetic instability \( k_0 \) and running prototypical reconnection problems like the Harris sheet or island coalescence. The time evolution and snapshots of the fields are then compared to results of a particle-in-cell simulation.

In this work, on the other hand, we exploit the additional information available from PIC simulations to test and derive closures. Since we do have the fully kinetic particle information available, we can for example calculate the third order heat flux moment and compare the actual heat flux observed in the simulation to the assumptions of a given closure. This process has the advantage that we can directly compare terms at any given time, whereas when comparing multi-fluid and PIC simulation, the state at any given time is an accumulation of all differences that have occurred up to this time.

2. Background

2.1. Particle-In-Cell Method

The Particle Simulation Code (PSC) (Germaschewski et al. 2016) is a modern, load-balanced, GPU accelerated, fully-kinetic particle-in-cell code. The PIC method is a numerical method which discretizes the electromagnetic fields on a 3D grid and advanced them using Maxwell’s equations while approximating the particle distribution function through macroparticles which are advanced in continuous phase space.

As a fully kinetic algorithm, the PIC method retains the full structure of the electron pressure tensor. The in-situ moments calculated by PSC (1.3) retain the relevant kinetic physics and satisfy the 10-moment model which we will demonstrate in §3.3.

2.2. Sparse Identification of Non-Linear Dynamics

The Sparse Identification of Non-Linear Dynamics, or SINDy, is a framework for the discovery of symbolic equation. Developed by (Brunton et al. 2016), it looks to construct parsimonious solutions from a library containing candidate terms which are calculated from raw data. It may be used to take data from either experiment or simulation and generate symbolic governing equations which describe a dynamical system.
This is accomplished by solving the sparsity-promoting regression problem
\[
\mathcal{L} = \min_{\Theta w} ||\Theta w - y||_2 + \lambda R(w)
\]  
(2.1)

Where \( \Theta \) is the library containing candidate terms, \( w \) is the vector of coefficients associated with each term in the library, \( y \) is the regression target, and \( R(w) \) is some regularization function on \( w \) scaled by constant \( \lambda \). Commonly \( R \) is the \( L_1 \)-norm in which case the above becomes Lasso regression. For a detailed description of the framework, see (Zheng et al. 2019; Champion et al. 2020).

Our experiments approximate (2.1) by solving the least squares problem with thresholding. This involves iteratively solving the least squares problem \( y = \Theta w \) and applying an upper and lower bounded threshold to the discovered \( w \), thus restricting the coefficients to a pre-specified range.

The success of SINDy discovering a parsimonious equation is dependent on selecting an upper and lower bounds which maximally encourages sparse solutions while minimally allowing error. Solutions to problems of this sort are known as pareto-optimal if the family of solutions can not improve on one measure without hindering the other. In other-words, the optimal bounds for the thresholded least squares problem are associated with the solution that lives on the pareto-front where the number of included terms may not be further minimized without increasing the solution error.

Thus, for each instance in which we apply SINDy, we first solve the thresholded least squares problem with various bounds to construct the pareto curve. The optimal bounds are then selected from the pareto-front and the problem is re-solved with those bounds.

This method was selected to symbolically model our closure because it is fully interpretable, emulates nature by leveraging parsimony, and has previously been used successfully to model MHD systems (Kaptanoglu et al. 2021).

3. Experiments

3.1. Harris Sheet Simulation

This study uses the PSC to simulate a collisionless Harris sheet reconnection problem. Initially in kinetic equilibrium, the magnetic field is initialized as \( B = B_0 \tanh(y/\lambda_B) \hat{x} \) and the densities as \( n_e = n_i = n_0 \text{sech}^2(y/\lambda_B) + n_b \). The magnetic field is then given a small sinusoidal perturbation which initiates the reconnection process.

The run uses a 1280 × 640 grid with a mass ratio of 25 and 16 × 10^9 particles which resolves to 0.676 \( \Delta x/\text{Debye length} \).

3.2. Method

Our method, which relies on SINDy at its core, was used for verifying the ten moment model, analysing the existing locally approximate Hammett-Perkins closure, and searching for an improved closure.

We begin by loading the moment and field data generated by PSC, which we smooth
by applying a spatially moving average. We then calculate relevant spatial and temporal
derivatives on the smoothed data and use domain knowledge to construct a library of
relevant terms. For each order we use a different term library with consistent units. This
acts as an a-priori constraint on which terms may be accepted into the library.

With the constructed library we then calculate the l.h.s of each equation in the ten-
moment model. For consistency, we consider the l.h.s to be the derivative containing terms
of the equation, with the exception of the 0th order where the partial time derivative is
the l.h.s and the particle divergence is the r.h.s. This l.h.s is used as a regression target
wherein we apply the SINDy framework on our term library to find the resulting symbolic
equations which constitute the r.h.s.

The library for the ten-moment verification and analysing the existing closure consisted
entirely of field and source terms, whereas in searching for an improved closure we
included derivative terms which met our unit constraint. This method is summarized
in algorithm 1.

Algorithm 1 Equation Synthesis
1: $E_i, B_i, f^i \leftarrow$ Load field and moment data
2: Calculate $\partial_t f^i$, $\partial_x f^i$, $\partial_y f^i$, $\partial_z f^i$ for relevant moments
3: Construct term library $\Theta$
4: define l.h.s as regression target
5: Apply SINDy with pareto-optimal bounds
6: return symbolic equation

3.3. Ten Moment Model and Method Verification

As the multi-fluid moment equations (1.4) are exact, we can verify them directly from
the Harris sheet PIC data. To show this we individually calculate the left hand side (l.h.s)
and right hand side (r.h.s) of each equation and demonstrate their equivalence.

In the following analysis we use the antisymmetric normalized $L_2$ error (3.1), where
the perfect score is 0, as a method for three separate comparisons. The first is between
between the l.h.s and r.h.s calculated from the directly from the PIC simulation data.
This gives an indicator of how well the data aligns with theory. The second comparison
is between the r.h.s calculated directly from the data and the discovered r.h.s equation
applied to the data. This is an indicator for how well our method is able to reproduce
theory. The third comparison is the error between the l.h.s calculated directly from the data
and the discovered r.h.s equation applied to the data, which gives an indicator for how
well our discovered equation can explain the l.h.s of each moment relation. Finally we
look at the normalized $L_2$ error of the coefficients between those given by the multi-fluid
equations (1.4) and those discovered using the regression method.

$$L_2(x_1, x_2) = \frac{||x_1 - x_2||^2}{||x_1||^2}$$  \hspace{1cm} (3.1)

Using this analysis we can say that the regression method is able to reproduce
the multi-fluid equations if the l.h.s-discovered normalized $L_2$ error is equal to or approximately
equal to the l.h.s-r.h.s normalized $L_2$ error. Ultimately we are looking for symbolic, not
numeric closure approximations. With this in mind the coefficient error is the truest
indicator for the success of the method in reproducing (1.4). We present our findings
in tables 2 to 4 and figs. 1 to 3 below.
3.3.1. 0th Order Verification

The 0th order equation, or continuity equation appears to match, albeit with a significant amount of noise which leads to a higher than expected error on each error metric. This is due to the finite-difference method used in calculating the time and space derivatives. While the 1st and 2nd order equations contain source and field terms, the continuity equation solely relies on derivatives. Noise generated in the finite-difference derivatives introduce systematic error that leads to error in the regression.

3.3.2. 1st Order Verification

The inclusion of field terms in the 1st order equations stabilizes the numerics significantly. This is demonstrated by the superb matching between the l.h.s-r.h.s error and the l.h.s-discovered error. Similarly, from the coefficient error we see that the method was able to reproduce the 1st order equations with 1-2% error.

3.3.3. 2nd Order Verification

The 2nd order equations demonstrate different behavior on-diagonal vs off-diagonal. We see very good agreement between theory and data in the on-diagonal terms. Unfortunately we begin to see a departure as we move off-diagonal. All of the xy, xz, and yz, directions omitted terms from the discovered equations which theory says exist and the
| component | r.h.s          | discovered | $L_2$(l.h.s, r.h.s) | $L_2$(r.h.s, discovered) | $L_2$(l.h.s, discovered) | coefficient error |
|-----------|---------------|------------|---------------------|--------------------------|--------------------------|-------------------|
| $x$       | $qnE_x + qn(\vec{u} \times \vec{B})_x$ | $0.98qnE_x + 0.98qn u_y B_z - 1.02qn u_x B_y$ | 0.529 | 0.014 | 0.529 | 0.020 |
| $y$       | $qnE_y + qn(\vec{u} \times \vec{B})_y$ | $0.99qnE_y - 0.99qn u_x B_z + 0.99qn u_y B_z$ | 0.433 | 0.005 | 0.433 | 0.010 |
| $z$       | $qnE_z + qn(\vec{u} \times \vec{B})_z$ | $0.99qnE_z + 0.99qn u_x B_y - 0.99qn u_y B_y$ | 0.077 | 0.007 | 0.077 | 0.010 |

Table 3. The true r.h.s, discovered r.h.s, and systematic errors for each component of the 1st order moment equations.

Figure 2. Top to bottom are the x-component, y-component, and z-component of the regression targets in the top pane, the true r.h.s of the of each component in the middle pane, and the r.h.s discovered by the regression in the bottom pane.
Table 4. The true r.h.s, discovered r.h.s, and systematic errors for each component of the 2nd order moment equations.

| Component | LHS | Discovered | LHSdiscover | Error        |
|-----------|-----|------------|-------------|--------------|
| xx        |     |            |             |              |
| yy        |     |            |             |              |
| zz        |     |            |             |              |
| xy        |     |            |             |              |
| xz        |     |            |             |              |
| yz        |     |            |             |              |

Figure 3. Top to bottom, left to right are the xx, yy, zz, xy, xz, yz components of the 2nd order moment equation. Each contains the regression target in the top pane, the true r.h.s in the center pane, and the discovered relation in the bottom. Visually, the l.h.s, r.h.s, and discovered source term of each component match well and are consistent with theory. This is true of both the on-diagonal and off-diagonal terms which demonstrates the reliability of our data. While some off-diagonal discovered terms may be missing, this is most likely due to their magnitude being small in this particular scenario.

yz direction introduced an additional term which should not be included, as denoted by the * in the coefficient error of table 4.

We believe the systematic error increases with increasing order and presents itself more prominently in the 2nd order equations. PIC methods contain intrinsic noise due to both particle resampling and mapping the particle’s continuous position to discrete space.
Figure 4. The locally approximate Hammett-Perkins closure calculated during the nonlinear phase of reconnection. In each coordinate, the top row shows the divergence of the heat flux based directly on the particle distribution function, while the bottom row represents the prediction of the approximate closure. The unknown factor $k_0$ in the closure has been set to 1 for these plots, so it is expected that the scales differ.

The higher order moments thus contain more particle noise which becomes amplified after subtracting out the lower order moments in order to obtain the centered moments.

3.4. Locally Approximate Hammett-Perkins Closure

The local closure (1.8) replaces the continuous wave-number $k$ in the Fourier space response with just one typical wave-number $k_0$. Thus, omitting $k_0$ when calculating the closure we would expect the l.h.s and r.h.s to differ by a constant factor. Further the closure is most important near the reconnection x-point where we expect important departures from ideal behavior. These trends seem to be verifiable in the kinetic data seen in fig. 4, there is however room for improvement in several areas. One obvious discrepancy is that each direction of the heat-flux divergence tensor scales by a different $k_0$ from the closure. Second is the clear divergence between the heat-flux divergence and the closure at regions of the domain distant from the x-point.

3.5. Closure Discovery

Applying our methodology to the improvement or replacement of the above given closure, we decided to restrict ourselves to the nonlinear phase of the reconnection process. This range may be seen in fig. 5. Pareto-optimal bounds are required for SINDy to discover a parsimonious solution. Constructing a separate pareto curve for each element of $\nabla Q_{ij}$, we modified algorithm 1 to that shown in algorithm 2.

In $\Theta$ we include each component of the pressure tensor scaled by the thermal velocity, the energy transfer term $\mathbf{j} \cdot \mathbf{E}$, the Poynting flux terms $\nabla \cdot \mathbf{S}$, each component of the tensor $\left( \mathbf{u} \cdot \nabla \right) \mathbf{P}$, and each term in the scalar product $(P \nabla) \cdot \mathbf{U}$. Table 5 summarizes our findings for a representative step during the nonlinear phase and fig. 6 demonstrates the temporal validity of the closure. The on-diagonal closures were able to reduce the error significantly when compared to the given closure. The off-diagonal terms demonstrated
Algorithm 2 Equation Synthesis

1: $E, B, f^i \leftarrow$ Load field and moment data
2: Calculate $\partial_t f^i, \partial_x f^i, \partial_y f^i, \partial_z f^i$ for relevant moments
3: Construct term library $\Theta$
4: for $i, j$ in $\{x, y, z\} \times \{x, y, z\}$ do
5: calculate pareto-optimal bounds
6: define $\nabla Q_{ij}$ as regression target
7: Apply SINDy
8: end for
return symbolic equation

Table 5. For each component of the heat flux divergence tensor we give the locally approximate Hammett-Perkins closure, the discovered closure, and the relevant $L_2$ errors. The $k_0$ for each component of Hammett-Perkins is calculated by taking the average of $\nabla Q_{ij}/(v_i(P_{ij} - p\delta_{ij}))$ across the domain.

no such improvement over the given Hammett-Perkins closure. However, our proposed closure comes with some caveats to be discussed further in §4

4. Discussion and Conclusion

Fully kinetic data may be used to verify the multi-fluid model as demonstrated up to 2nd order. Calculating the l.h.s and r.h.s of each fluid equation, the kinetic data is in strong agreement with the multi-fluid model, verifying the consistency of our approach. By calculating the l.h.s of each multi-fluid equation and applying the above methodology, we were able to validate SINDy as a candidate data-driven approach to be used in closure modeling.
Applied to heat flux divergence, this data-driven approach demonstrates strong results. The existing locally approximate Hammett-Perkins closure gives us a baseline to measure improvement. The on-diagonal terms of the heat flux divergence were modeled with significantly reduced $L_2$ error over the baseline. Furthermore the discovered closures appear to hold through a significant portion of the nonlinear phase.

The off-diagonal terms were unsuccessfully modeled. We believe this can be attributed to the high data noise, which is demonstrated in table 4. The off-diagonal 2nd order equations were predicted with an order of magnitude higher error than their on-diagonal counterparts. If the data is simply too noisy to accurately represent the 2nd order off-diagonal terms, we cannot expect the off-diagonal closure modeling to succeed.

Other shortcomings of this approach lie in perturbation phase restrictions. While the discovered closure holds well through the nonlinear phase, there is no such guarantees during the linear phase, where due to the particle noise we do not have reliable data. Further work needs to be done in either incorporating the linear phase into the regression or finding a similar set of closures which may be applied during the linear phase.

As a data-driven method we make no attempt to explain the physics of the discovered closures here. Our sole claim is that the discovered closures will out-perform eq. [1.8] during the nonlinear phase. Next steps in this research will involve applying the discovered closure to a multi-fluid simulation, finding a globally applicable closure which is accurate through all phases of reconnection, and pursing a closure which accurately represents the off-diagonal terms.

REFERENCES

Birn, J., Drake, J. F., Shay, M. A., Rogers, B. N., Denton, R. E., Hesse, M.,
Kuznetsova, M., Ma, Z. W., Bhattacharjee, A., Otto, A. & Pritchett, P. L. 2001 Geospace environmental modeling (gem) magnetic reconnection challenge. *Journal of Geophysical Research: Space Physics* 106 (A3), 3715–3719, arXiv: https://agupubs.onlinelibrary.wiley.com/doi/pdf/10.1029/1999JA900449.

Brunton, Steven L., Proctor, Joshua L. & Kutz, J. Nathan 2016 Discovering governing equations from data by sparse identification of nonlinear dynamical systems. *Proceedings of the National Academy of Sciences* 113 (15), 3932–3937, arXiv: https://www.pnas.org/content/113/15/3932.full.pdf.

Champion, Kathleen, Zheng, Peng, Aravkin, Aleksandr Y., Brunton, Steven L. & Kutz, J. Nathan 2020 A unified sparse optimization framework to learn parsimonious physics-informed models from data. *IEEE Access* 8, 169259–169271.

Chust, T. & Belmont, G. 2006 Closure of fluid equations in collisionless magnetoplasmas. *Physics of Plasmas* 13 (1), 012506.

Germaschewski, Kai, Fox, William, Abbott, Stephen, Ahmadi, Narges, Maynard, Kristofor, Wang, Liang, Ruhl, Hartmut & Bhattacharjee, Amitava 2016 The plasma simulation code: A modern particle-in-cell code with patch-based load-balancing. *Journal of Computational Physics* 318, 305–326.

Hammett, Gregory W. & Perkins, Francis W. 1990 Fluid moment models for landau damping with application to the ion-temperature-gradient instability. *Phys. Rev. Lett.* 64, 3019–3022.

Hastie, R. J. 1997 Sawtooth Instability in Tokamak Plasmas. *Astrophysics and Space Science* 256 (1), 177–204.

Kaptanoğlu, Alan A., Morgan, Kyle D., Hansen, Chris J. & Brunton, Steven L. 2021 Physics-constrained, low-dimensional models for magnetohydrodynamics: First-principles and data-driven approaches. *Physical Review E* 104 (1).

Masuda, S., Kosugi, T., Hara, H., Tsuneta, S. & Ogawara, Y. 1994 A loop-top hard X-ray source in a compact solar flare as evidence for magnetic reconnection. *Nature* 371, 495–497.

Ng, Jonathan, Hakim, Ammar, Bhattacharjee, A., Stanier, Adam & Daughton, W. 2017 Simulations of anti-parallel reconnection using a nonlocal heat flux closure. *Physics of Plasmas* 24 (8), 082112, arXiv: https://doi.org/10.1063/1.4993195.

Parker, E. N. 1957 Sweet’s mechanism for merging magnetic fields in conducting fluids. *Journal of Geophysical Research (1896-1977)* 62 (4), 509–520.

Petschek, Harry E. 1964 50 magnetic field annihilation. In *AAS-NASA Symposium on the Physics of Solar Flares: Proceedings of a Symposium Held at the Goddard Space Flight Center, Greenbelt, Maryland, October 28-30, 1963*, vol. 50, p. 425. Scientific and Technical Information Division, National Aeronautics and ....

Vasyliunas, Vytenis M. 1975 Theoretical models of magnetic field line merging. *Reviews of Geophysics* 13 (1), 303–336, arXiv: https://agupubs.onlinelibrary.wiley.com/doi/pdf/10.1029/RG013i001p00303.

Wang, Liang, Hakim, Ammar H., Bhattacharjee, A. & Germaschewski, K. 2015 Comparison of multi-fluid moment models with particle-in-cell simulations of collisionless magnetic reconnection. *Physics of Plasmas* 22 (1), 012108.

Yamada, M. 2010 Magnetic reconnection. *Rev. Mod. Phys.* 82, 603.

Zheng, Peng, Askham, Travis, Brunton, Steven L., Kutz, J. Nathan & Aravkin, Aleksandr Y. 2019 A unified framework for sparse relaxed regularized regression: Sr3. *IEEE Access* 7, 1404–1423.

Zweibel, E. G. 2009 Magnetic reconnection in astrophysical and laboratory plasmas. *Annu. Rev. Astron. Astrophys.* 47, 291.