INDECOMPOSABILITY FOR DIFFERENTIAL ALGEBRAIC GROUPS

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Abstract. We study a notion of indecomposability in differential algebraic groups. Then several examples are given and basic propositions are established. We compare this notion to other notions of indecomposability from both model theory and differential algebra. We prove an indecomposability theorem for differential algebraic groups. This theorem is used for various definability results. For instance, we show every nonabelian almost simple differential algebraic group is perfect, answering a question of Cassidy and Singer.

The indecomposability theorem of Zilber generalized a known theorem for algebraic groups [17] to the setting of weakly categorical groups [23]. Zilber’s theorem was generalized to the superstable (possibly infinite rank) setting by Berline and Lascar [3]. Because $DCF_{0,m}$ is $\omega$-stable, their results apply to differential algebraic groups. For differential algebraic groups over a partial differential field, the existing indecomposability theorems are not suitable for some applications because, at present, there is no known lower bound for Lascar rank in terms of several important differential birational invariants [21]. Though the superstable version of the indecomposability theorem given by Berline and Lascar applies in this context, both the hypotheses and the conclusions the theorem are not clear from the perspective of differential algebra because of the lack of control of Lascar rank in partial differential fields. With that in mind, we give a version of the indecomposability theorem in which both the hypotheses and conclusions are purely differential algebraic in nature. Additionally, we provide applications and examples of the ideas. These include the definability of commutators of differential algebraic groups with appropriate hypotheses. The notion of indecomposability we consider is related to the notion of strongly connected studied by Cassidy and Singer [5]. We also discuss open problems which could connect these results to those of Berline and Lascar [3].

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1. Definitions and notation

The model theory of partial differential fields of characteristic zero with finitely many commuting derivations was developed in [12]. In this setting, we have a model
companion, which we denote $DCF_{0,m}$. For a recent alternate (geometric) axiomatization of partial differentially closed fields, see \cite{18}. For a reference in differential algebra, we suggest \cite{8} and \cite{11}. $DCF_{0,m}$ has quantifier elimination, so the definable groups are precisely differential algebraic groups. $DCF_{0,m}$ has elimination of imaginaries, so quotients of differential algebraic groups are again differential algebraic groups. Throughout this paper, $K \models DCF_{0,m}$ will be a field over which our sets are defined and $\mathcal{U}$ is a very saturated model of $DCF_{0,m}$. All tuples in differential field extensions of $K$ can be assumed to come from $\mathcal{U}$. Quantifier elimination also gives a correspondence between varieties, types, and radical differential ideals. We will make this explicit and fix notation next. Given a type $p \in S(K)$, we have a corresponding differential radical ideal via

$$p \mapsto I_p = \{ f \in K\{y\} \mid f(y) = 0 \in p \}$$

Of course, the corresponding variety is simply the zero set of $I_p$. We will use this correspondence implicitly throughout including in the notation of Kolchin polynomials, which we will define next.

Let $\Theta$ be the free commutative monoid generated by $\Delta$. For $\theta \in \Theta$, if $\theta = \delta_1^{\alpha_1} \ldots \delta_m^{\alpha_m}$, then $\text{ord}(\theta) = \alpha_1 + \ldots + \alpha_m$. The order gives a grading on the monoid. $\Theta(s) = \{ \theta \in \Theta : \text{ord}(\theta) \leq s \}$.

The next theorem is due to Kolchin, \cite{8} Theorem 6, page 115.

**Theorem 1.1.** Let $\eta = (\eta_1, \ldots, \eta_n)$ be a finite family of elements in some extension of $k$. There is a numerical polynomial $\omega_{\eta/k}(s)$ with the following properties.

1. For sufficiently large $s \in \mathbb{N}$, the transcendence degree of $k((\theta \eta_j)_{\theta \in \Theta(s), 1 \leq j \leq n})$ over $k$ is equal to $\omega_{\eta/k}(s)$.
2. $\deg(\omega_{\eta/k}(s)) \leq m$
3. One can write

$$\omega_{\eta/k}(s) = \sum_{0 \leq i \leq m} a_i \binom{s+i}{i}$$

In this case, $a_m$ is the differential transcendence degree of $k(\eta)$ over $k$.

4. If $p$ is the defining differential ideal of $\eta$ in $k\{y_1, \ldots, y_n\}$ and $\Lambda$ is a characteristic set of $p$ with respect to an orderly ranking of $(y_1, \ldots, y_n)$, and if for each $y_j$ we let $E_j$ denote the set of all points $(e_1, \ldots, e_m) \in \mathbb{N}^m$ such that $\delta_1^{e_1} \ldots \delta_m^{e_m} y_j$ is a leader of an element of $\Lambda$, then

$$\omega_{\eta/k}(s) = \sum_{1 \leq j \leq n} \omega_{E_j} - b$$

where $b \in \mathbb{N}$.

The Kolchin polynomial of a differential variety is not a $\Delta$-birational invariant, but the leading coefficient and the degree are $\Delta$-birational invariants. We call the degree the differential type or $\Delta$-type of $V$. We will use the notation $\tau(V)$ for the the
differential type. Noting the above correspondence between tuples in field extensions (realizations of types) and varieties, we will occasionally write $\tau(p)$ or $\tau(a)$. The leading coefficient is called the typical differential dimension or the typical $\Delta$-dimension. We will write $\alpha_V$ for the typical differential dimension of $V$. Of course, because there is a correspondence between complete types, tuples in differential field extensions, and constructible sets, we will also write $\tau(a)$ and $\alpha_a$ for a tuple of elements $a$ in a field extension. Similarly, we write $\tau(p)$ and $\alpha_p$ for a complete type $p$. Similar notation will be used with typical differential dimension. When we wish to emphasize that we are calculating $\tau$ or $\alpha$ over a specific differential field, we will use the following notation: $\tau(a/F)$ is the differential type of $a$ over $F$. $\tau(a/F \cup \{b\})$ is the differential type of $a$ over the differential field generated by $b$ and $F$.

The following is elementary to prove, see [14].

**Lemma 1.2.** For $a, b$ in a field extension of $K$.

$$\tau(a, b) = \max\{\tau(a), \tau(tp(a/b))\}$$

If $\tau(a) = \tau(tp(b/a))$, then

$$\alpha_{(a, b)} = \alpha_a + \alpha_{b/\{a\} \cup K}$$

If $\tau(a) > \tau(tp(b/a))$, then

$$\alpha_{(a, b)} = \alpha_a$$

For many additional results on the properties of differential type and typical differential dimension, see [5] and [8].

2. **Stabilizers**

Next, we develop the notion of stabilizers of types in the differential algebraic setting. The basic setup is taken from [16], but the proofs of the results are easier or sometimes give more information in the setting of differential algebraic groups (as opposed to general stable groups or even $\omega$-stable groups). In this section, $G$ will be a differential algebraic group, that is, a definable group over $K \models DCF_{0,m}$. For discussion of the categories, see [15].

**Definition 2.1.** Let $p(x) \in S(K)$ be a complete type containing the formula $x \in G$. All of the complete type we deal with will contain this formula. And we let $f(v, w)$ be a formula. Then, we let

$$\text{stab}_G(p, f) = \{a \in G \mid \forall b, c \in G, f(bx, c) \in p \rightarrow f(bx, c) \in ap\}$$

And,

$$\text{stab}_G(p) = \{a \in G \mid ap = p\}$$

For someone that wishes not to think of types, $p$ can be thought of as the isomorphism class of the differential field extension generated by tuple (which is a realization of the type) over $K$ which is in the group $G$. For details on this correspondence in
differential fields, see [11]. The next three results are standard for $\omega$-stable groups [16] (only the second needs $\omega$-stability; for the others superstability suffices).

**Lemma 2.2.** $stab_G(p, f)$ is definable.

**Lemma 2.3.** $stab_G(p)$ is definable.

**Lemma 2.4.** $RU(p) \geq RU(stab_G(p))$.

The first task is to put the last lemma into the differential algebraic context, with differential type and typical differential dimension playing the role that Lascar rank plays in the model theoretic context. The next two lemmas are preparation for this result.

Again, we are working in some fixed differential algebraic group $G$; throughout the paper, all types, elements and tuples in the following lemma are assumed to be in the differential algebraic group in which we are working. Unless specially noted, multiplication of two elements or above as in the case of a type occurs with respect to the group operation in $G$. In differential fields, canonical bases correspond to fields of definition.

**Lemma 2.5.** Suppose that $c \downarrow_A b$, where we assume that $A$ contains the canonical base of the differential algebraic group $G$. Then $\tau(c/A) \leq \tau(cb/A)$ and in the case of equality, $\alpha(c/A) \leq \alpha(cb/A)$.

**Proof.** $c$ is definable over $A \cup \{b, cb\}$ and $cb$ is definable over $A \cup \{c, b\}$. Definable closure is the same as differential field closure in our setting. So,

$$\tau(c/A \cup \{b\}) = \tau(cb/A \cup \{b\}) \leq \tau(cb/A)$$

and in the case of equality in the previous line,

$$\alpha(c/A \cup \{b\}) = \alpha(cb/A \cup \{b\}) \leq \alpha(cb/A).$$

But, we know that

$$\omega_{c/A \cup \{b\}}(t) = \omega_{c/A}(t)$$

by McGrail’s characterization of forking in partial differential fields [12].

**Remark 2.6.** As pointed out to me by W. Sit and P. Cassidy, the difficulty with proving the previous lemma for the Kolchin polynomial itself (instead of the coarser differential type and typical differential dimension) is that the Kolchin polynomial is not a differential birational invariant. So, in the above lemma, when multiplying by group elements, we may be taking differential rational functions of those elements which might not preserve the Kolchin polynomials. While it is true that differential algebraic groups may be embedded in algebraic groups [17], relieving this potential problem, such an embedding (starting from our given differential algebraic group) need not preserve the Kolchin polynomial. One could state the above result with Kolchin polynomials, but only after assuming a specific embedding into an algebraic group.
group. Another possible solution would be to use Sit’s result [20] on the well-ordering of Kolchin polynomials to state everything in terms of the minimal Kolchin polynomial associated with a differential field extension.

The following lemma has undoubtedly appeared in various places (though, to the author’s knowledge, not precisely in this form). For instance, [2] proves this in the case of ordinary differential fields. The proof in the partial case is not any harder, but we include it for convenience.

**Lemma 2.7.** Suppose that $G$ is an irreducible differential algebraic group. Then a type is generic in the sense of the Kolchin topology if and only if it is of maximal Lascar rank.

*Proof.* We will refer to types which are generic in the Kolchin topology (in the sense that there is a realization of the type of Kolchin polynomial equal to the differential algebraic group) as a topological generic. We will refer to the types of maximal Lascar rank as RU-generics. We will refer to types which can cover the group $G$ by finitely many left translates as group generics. In any superstable group, being group generic is equivalent to being RU-generic [16].

Suppose that $p(x)$ is RU-generic but not topological generic. Then finitely many left translates of any formula in $p(x)$ cover the group, but $p(x)$ is not topological generic, so the type is contained in a proper Kolchin closed subset of $G$. Take the formula witnessing this, $\phi(x)$. Now, finitely many left translates of $\phi(x)$ cover the group $G$, and each of these is clearly closed in the Kolchin topology (if $a$ is a topological generic in $\phi(x)$ then $g\phi(x)$ is simply the zero set of the ideal of differential polynomials vanishing at $ag$). But, this is a problem. Now $G$ is the finite union of proper closed subsets.

Now, assume that $p(x)$ is a type such that any realization $a$ is topological generic. Then take any differential polynomial $P(x)$ vanishing at $a$. As $a$ is topological generic, $P(x)$ vanishes everywhere in $G$. So, by quantifier elimination, then only possible non-group generic formula in $p(x)$ is the negation of a differential polynomial equality. Suppose that $P(x) \neq 0$ is not group generic. Then $P(x) = 0$ is group generic, so finitely many translates cover $G$, which is again a contradiction if $Z(P) \cap G$ is a proper closed subset of $G$. Thus $P(x) \neq 0$ is group generic. \hfill $\Box$

So, from now on, we will simply refer to these types as generics. Note that this argument also shows that for a differential algebraic group, irreducibility in the Kolchin topology implies that Morley degree is one. In a general differential algebraic variety (with no group structure), there are examples in which the topological generics are disjoint from the RU-generics and Morley degree (and even Morley rank) of a definable (constructible) set is not preserved by taking the closure of the set in the Kolchin topology [7].
Proposition 2.8. For any complete type which includes the formula “x ∈ G”, \( \tau (p(x)) \geq \tau (\text{stab}_G(p(x))) \) and in the case of equality, \( \alpha_{p(x)} \geq \alpha_{\text{stab}_G(p(x))} \).

Proof. Suppose that \( s(x) \) is a generic type of the stabilizer of \( p(x) \). Take \( b \models p(x) \) and \( c \models s(x) \) such that \( b \downarrow_K c \).

\[
\tau (bc/G(K)) \geq \tau (c/G(K))
\]

and if equality holds, then

\[
\alpha_{bc/K} \geq \alpha_{c/K}
\]

by Lemma 2.5. One needs only to argue that \( tp(bc/G) = p(x) \). To see this, simply take an elementary extension of \( K \) containing \( c \) over which \( b \) does not fork. Then, \( c \) is in the stabilizer of \( p(x) \), so the result follows.

In the next lemma, we will discuss a tuple called a canonical base of a type, see [11]. Essentially, the result is to choose the canonical base carefully enough to make sure that it has a small Kolchin polynomial.

Lemma 2.9. Suppose that \( p(x) \in S(K) \) with “x ∈ G” \( \in p(x) \). Then, suppose, for some \( A \subseteq K \), that

\[
\omega_{p|A}(t) < \omega_p(t) + t^n
\]

Then there is a tuple \( \bar{c} \in K \) such that \( \omega_p(t) = \omega_{p|\bar{c}}(t) \) and \( \omega_{c/A}(t) < t^n \).

Proof. Let \( \langle b_i \rangle_{i \in \mathbb{N}} \) be a sequence of \( K \)-indiscernibles in the type of \( p \). By the characterization of forking in \( DCF_{0,m} \) this simply means that

\[
\omega_p(t) = \omega_{b_i/K}(t) = \omega_{b_i/K \cup \{b_0, \ldots, b_{i-1} \}}(t)
\]

We do not know, however, that the same holds over the (arbitrary) subset \( A \subseteq K \). In general, we simply know that \( \omega_{b_i/A \cup \{b_0, \ldots, b_{i-1} \}} \) is a decreasing sequence of polynomials, again, ordered by eventual domination. By the well-orderedness of Kolchin polynomials we know that the sequence is eventually constant. Alternatively, this fact can be seen by noting the superstability of \( DCF_{0,m} \) and the fact that decreases in Kolchin polynomial correspond to forking extensions. So, for the rest of the proof, we fix a \( k \) such that if \( n \geq k \), the sequence is constant. That is, above \( k \), we know that we have a sequence of \( A \)-indiscernibles in the type of \( p \). Now, fix a model \( K' \models DCF_{0,m} \) with \( K' \) containing \( K \) and \( \{b_0, \ldots, b_{k-1} \} \). We let \( p' \) be the (unique) nonforking extension of \( p \) to \( K' \).

We can get elements \( \bar{c} \subseteq acl(A \cup \{b_0, \ldots, b_{k-1} \}) \) such that \( p' \downarrow_{\bar{c}} K' \). In fact, by [19] (page 132) and the fact that \( DCF_{0,m} \) eliminates imaginaries, we can assume that \( \bar{c} \in K \). We know that

\[
\omega_{p|A}(t) = f(t) + h(t)
\]

where all of the terms of \( f(t) \) are larger than degree \( n \) in \( t \) and all of the terms \( h(t) \) are less than or equal to degree \( n \) in \( t \). By assumption, \( \omega_{p|A}(t) < \omega_p(t) + t^n \). Thus, \( f(t) \leq \omega_p(t) \).
By construction \( \langle b_i \rangle \) was an indiscernible sequence, so if we define \( \bar{b} := (b_0, \ldots, b_{k-1}) \), then
\[
k \cdot f(t) \leq \omega_{b/K}(t)
\]
Then we know that
\[
k \cdot f(t) \leq \omega_{\bar{b}/A \cup \bar{c}}(t)
\]
So, for all \( i = 0, 1, \ldots, k - 1 \), we have that
\[
\omega_{b_i/A \cup \{b_0, \ldots, b_{i-1}\}}(t) \leq \omega_{p|A}(t) = f(t) + h(t)
\]
Further,
\[
\omega_{\bar{b}/A}(t) \leq \omega_{b_0/A}(t) + \omega_{b_1/A \cup \{b_0\}}(t) + \ldots \omega_{b_{k-1}/A \cup \{b_0, \ldots, b_{k-1}\}}(t)
\]
But, this means that
\[
\omega_{\bar{b}/A}(t) \leq k f(t) + k h(t)
\]
By assumption, \( \bar{c} \in acl(A \cup \bar{b}) \) so \( \omega_{\bar{b}/A}(t) = \omega_{\bar{c}/A}(t) \). Then
\[
\omega_{\bar{b}/A \cup \bar{c}}(t) + \omega_{\bar{c}/A}(t) \leq \omega_{\bar{b}/A}(t).
\]
Now, using (1) and (2) we know that
\[
\omega_{\bar{c}/A}(t) < t^n.
\]

The next lemma appears in [3]:

**Lemma 2.10.** Let \( p(x) \in S(K) \) with \( "x \in G" \) \( \in p(x) \). Suppose \( p \) does not fork over the empty set. Let \( b \) be an element of \( G(K) \). Let \( A \subset K \). If \( \bar{b} = b \mod \text{stab}_G(p) \) is not algebraic over \( A \), then \( bp \) forks over \( A \).

**Proposition 2.11.** Suppose that \( a \downarrow_K b \) with \( a, b \in G \). Let \( p = tp(a/K) \). If
\[
\omega_{ba/K}(t) < \omega_{a/K}(t) + t^n
\]
and \( \bar{b} \) is the class of \( b \mod \text{stab}_G(p) \) then
\[
\omega_{\bar{b}/K}(t) < t^n
\]

**Proof.** We let \( K' \) be an elementary extension of \( K \) containing \( b \) such that \( a \downarrow_K K' \). That is, \( tp(a/K') \) is the unique nonforking extension of \( tp(a/K) \). Then,
\[
\omega_{a/K}(t) = \omega_{a/K'}(t).
\]
Further, we know from Proposition 2.3 that \( \tau(a/K) = \tau(ba/K) \) and \( \alpha_{a/K} = \alpha_{(ba/K)} \). But, the same holds over \( K' \), since \( ba \) is interdefinable with \( a \) over \( K' \). Then we know that
\[
\omega_{ba/K}(t) < \omega_{ba/K'}(t) + t^n
\]
By lemma 2.9 we can get $\bar{c} \in K'$ with $\omega_{\bar{c}/K}(t) < t^n$ such that $ba \downarrow_{K \cup \bar{c}} K'$. Then, applying lemma 2.10 we can see that $\bar{b}$ is algebraic over $K \cup \bar{c}$. We know that $\omega_{\bar{c}/K}(t) < t^n$ and so $\omega_{\bar{b}/K}(t) < t^n$.

Roughly, the next result says that if an element in a differential field extension of the base field lying in the differential algebraic group is sufficiently generic (in the differential algebraic sense), then the stabilizer of this element is large (again, in the differential algebraic sense). One might regard this as a sort of converse statement to Proposition 2.8.

**Proposition 2.12.** Let $p(x) \in S(K)$, with “$x \in G$” a formula in $p(x)$. Let $n$ be such that

$$\omega_G(t) < \omega_p(t) + t^n.$$  

Then

$$\omega_G(t) < \omega_{\text{stab}_G(p)}(t) + t^n$$

Proof. Choosing $b$ to be a generic point on $G$ over $K$ and applying Proposition 2.11 gives that

$$\omega_G(t) \leq \omega_{\text{stab}_G(p)}(t) + \omega_{\bar{b}/K}(t)$$

so

$$\omega_G(t) < \omega_{\text{stab}_G(p)}(t) + t^n$$

completing the proof.

□

3. Indecomposability

**Definition 3.1.** Let $G$ be a differential algebraic group defined over $K$. Let $X$ be a definable subset of $G$. For any $n \in \mathbb{N}$, $X$ is $n$-indecomposable if $\tau(X/H) \geq n$ or $|X/H| = 1$ for any definable subgroup $H \leq G$. We use indecomposable to mean $\tau(G)$-indecomposable.

**Remark 3.2.** We should note the relationship of this notion to that of strongly connected considered by Cassidy and Singer [5]. A subgroup of $G$ is strongly connected if and only if it is $n$-indecomposable where $n = \tau(G)$. We will occasionally use $n$-connected to mean $n$-indecomposable, but only in the case that the definable set being considered is actually a subgroup. In the next section, we will show some techniques for constructing indecomposable sets from indecomposable groups.

Cassidy and Singer proved the following, showing the robustness of the notion under quotients,

**Proposition 3.3.** Every quotient $X/H$ of a $n$-connected definable subgroup $X$ by a definable subgroup is $n$-connected.

For many other properties of strongly connected subgroups and numerous examples, see their paper [5].
Theorem 3.4. Let $G$ be a differential algebraic group. Let $X_i$ for $i \in I$ be a family of indecomposable definable subsets of $G$. Assume that $1_G \in X_i$. Then the $X_i$'s generate a strongly connected differential algebraic subgroup of $G$.

Proof. Fix $n = \tau(G)$. Let $\Sigma$ be the set of finite sequences of elements of $I$, possibly with repetition. Then we let $X_\sigma = X_{\sigma(1)} \cdot \ldots \cdot X_{\sigma(n)}$. Let $k_\sigma = \alpha_{X_\sigma}$. We note that $k_\sigma \leq \alpha_G$. For the remainder of the proof, we let $\sigma_1 \in \Sigma$ be such that

$$k_{\sigma_1} = \text{Sup}_{\sigma \in \Sigma} (k_\sigma)$$

is achieved. Now, let $p \in X_{\sigma_1}$ such that $\tau(p) = n$ and $a_p = k_{\sigma_1}$. We consider $\text{stab}_G(p)$.

First, we note that

$$\text{stab}_G(p) \subseteq X_{\sigma_1} X_{\sigma_1}^{-1}$$

To see this, let $b \in \text{stab}_G(p)$ and $c \models p$. Then both $c$ and $bc$ satisfy $X_{\sigma_1}$. Then $b = bcc^{-1} \in X_{\sigma_1} X_{\sigma_1}^{-1}$. Next, we will show, for all $i$,

$$X_i \subseteq \text{stab}_G(p)$$

Let $b \in X_i$ and $c \models p$. We will also assume that $c$ is $K$-independent from $b$. By this we mean simply that $b \not\models_K c$. Then we have, $bc \models X_i \cdot X_{\sigma_1}$. And by assumption, $\tau(bc) = \tau(c) = n$. We claim that $\tau(\bar{b}) < \tau(c) = n$ where $\bar{b}$ is the class of $b$ modulo $\text{stab}_G(p)$.

This follows by applying Proposition 2.11 and noting that $p$ has the properties needed for the hypothesis of that proposition, by the maximality of the differential type of $p \in X_{\sigma_1}$.

But, this holds for all $b \models X_i$ and $X_i$ is indecomposable, which is a contradiction, unless $X_i \subseteq \text{stab}_G(p)$. This completes the proof, except for the strong connectedness. That is actually the result of the next more general proposition. □

Of course, as in the more familiar case of groups of finite Morley rank (see [10], chapter 7, section 3) we know more than simply that the group generated by the family is definable. We have constructed the definition of the group which gives it a very particular form. The next proposition works even if the subgroup is not generated in the manner in the above theorem or the subgroups are of lower type.

Proposition 3.5. Let $X_i$, for $i \in I$, be a family of $k$-indecomposables each containing the identity. Let $H = \langle X_i \rangle$ and suppose that $H$ is definable. Then $H$ is $k$-indecomposable.

Proof. Let $H_1 \leq G$ with $H \not\leq H_1$. Then there exists $i$ such that $X_i \not\leq H_1$. For this particular $i$, we know, by $n$-indecomposability, that the coset space $X_i/H_1$ has high differential type. That is $\tau(X_i/H_1) \geq n$. But, then $\tau(H/H_1) \geq n$. □

Note that there is no assumption in the previous proposition that $\tau(G) = k$. This is assumed in the indecomposability theorem, but the proposition about the $k$-indecomposability of the generated subgroup holds more generally, assuming the group is definable. In general, we do not about the definability of such a subgroup, unless additional assumptions are made.
4. Definability of Commutators of Strongly Connected Groups

In this section, we will show some first applications of the ideas and techniques for constructing indecomposable sets. Any group naturally acts on itself by conjugation, that is $x \mapsto gxg^{-1}$. Analysis of this action provides a way of transferring properties of the group doing the action to the set on which it acts. Now, fix a differential algebraic group $G$ and a differential algebraic subgroup $H$. A subset $X \subseteq G$ is $H$-invariant if for all $h \in H$, conjugation by $h$ is a bijection from $X$ to itself.

First, we give the following example, due to Cassidy [9], of a differential algebraic group for which the commutator is not a differential algebraic group.

**Example 4.1.** Let $\Delta = \{\delta_1, \delta_2\}$. Then consider the following group $G$ of matrices of the form:

\[
\begin{pmatrix}
1 & u_1 & u \\
0 & 1 & u_2 \\
0 & 0 & 1
\end{pmatrix}
\]

where $\delta_i(u_i) = 0$. Of course,

\[
\begin{pmatrix}
1 & u_1 & u \\
0 & 1 & u_2 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & v_1 & v \\
0 & 1 & v_2 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & u_1 & u \\
0 & 1 & u_2 \\
0 & 0 & 1
\end{pmatrix}^{-1}
\begin{pmatrix}
1 & u_1 & u \\
0 & 1 & u_2 \\
0 & 0 & 1
\end{pmatrix}^{-1}
\]

Then one can see that the commutator is isomorphic $\mathbb{Q}[C_{\delta_1} \cup C_{\delta_2}]$, where $C_{\delta_i}$ is the field of $\delta_i$-constants. This is not a differential algebraic group. However, $G$ is not almost simple, however, since the subgroup of matrices of the form:

\[
\begin{pmatrix}
1 & 0 & u \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

is a subgroup of $\Delta$-type and typical $\Delta$-dimension equal to $G$. This means that the coset space has $\Delta$-type strictly smaller than $G$. Of course, this means that $G$ is not almost simple.

Cassidy and Singer make the following comment in [5], “We also do not have an example of a noncommutative almost simple linear differential algebraic group whose commutator subgroup is not closed in the Kolchin topology.” In model theoretic terms, they are asking: in an almost simple linear differential algebraic group, is the commutator subgroup definable? The next theorem 4.4 answers this question affirmatively, even in the more general case of the group being strongly connected (with no assumption of linearity or almost simplicity).
The next two lemmas have similar proofs in the context of groups of finite Morley rank (see Chapter 7 of [10]).

**Lemma 4.2.** Let $X$ be $H$-invariant. Suppose that for all $H$-invariant differential algebraic subgroups $H_1 \leq G$, $|X/H_1| = 1$ or $\tau(X/H_1) \geq n$. Then $X$ is $n$-indecomposable.

**Proof.** Suppose that there is a differential algebraic subgroup $H_2 \leq G$ with $\tau(X/H_2) < n$, but $|X/H_2| \neq 1$. Then, by the $H$-invariance of $X$, if $h \in H$ then $x^h \in X$. Thus $h$ defines a map from $X/H_2 \to X/H_2^h$. In particular, $\tau(X/H_2^h) < n$, but $|X/H_2^h| \neq 1$. Then, set

$$H_1 = \bigcap_{h \in H} H_2^h$$

Then, by the Baldwin-Saxl condition, we know that $H_1$ is actually definable and is, in fact, the intersection of finitely many of the subgroups. But then, $H_1$ is clearly $H$-invariant and $\tau(X/H_1) < n$ and $|X/H_1| \neq 1$, contradicting the assumptions on $X$. □

**Lemma 4.3.** If $H$ is an indecomposable differential algebraic subgroup of $G$ and $g \in G$, then $g^H$ is indecomposable.

**Proof.** Using the previous result, it is enough to prove the result for all $N \leq G$ so that $N^h = N$. So, to that end, suppose that $N$ is such that $|g^H/N| \neq 1$ and $\tau(g^H/N) < n$. Now, we get, by the $H$-invariance of $g^H$ and $N$, a transitive action of $H$ on $g^H/N$,

$$h \ast g^{h_1}N \mapsto hg^{h_1}Nh^{-1} = hg^{-1}hNh^{-1} = g^{h_1}N$$

Thus, this is a transitive action of $H$ on a differential algebraic variety of differential type less than $n$. The kernel of the action must be a subgroup of $H$ of differential type $\tau(H)$ and typical differential dimension equal to that of $H$. This is impossible, by the indecomposability of $H$, unless the subgroup in question is simply $H$ itself. If that is the case, then by the transitivity of the action, $|g^H/N| = 1$. □

**Theorem 4.4.** Commutator subgroups of strongly connected differential algebraic groups are differential algebraic subgroups and are strongly connected.

**Proof.** Apply the previous lemma, noting that $g^{-1}g^G$ is indecomposable. As $g$ varies, this family generates the commutator. Now apply Theorem 3.4. □

We should also note the result of Cassidy and Singer which says that if a strongly connected differential algebraic group is not commutative, then the differential type of the differential closure of commutator subgroup is equal to the differential type of the whole group [5]. So, putting this together with the above theorem yields:

**Theorem 4.5.** Let $G$ be a strongly connected nonabelian differential algebraic group. Then the commutator of $G$ is a strongly connected differential algebraic subgroup with $\tau([G, G]) = \tau(G)$. 
Because commutators are characteristic (thus normal), they are candidates to appear in the Cassidy-Singer decomposition of $G$ (see [3]. We also get the following generalization of a theorem of Cassidy and Singer (who proved it in the case of an almost simple linear differential algebraic groups of differential type at most one).

**Theorem 4.6.** Let $G$ be an almost simple differential algebraic group. Then $G$ is either commutative or perfect.

**Proof.** $\tau([G,G]) = \tau(G)$ implies that $[G,G] = G$, since $G$ is almost simple. Otherwise $[G,G] = 1$. □

Explicit calculations of the Kolchin polynomial for linear differential algebraic groups are often easier than for general differential algebraic groups or varieties. We will briefly describe how to perform these calculations, and how they lead to many examples of indecomposable (strongly connected) differential algebraic groups. The techniques are completely covered by Kolchin in [8] and some appear in [5]. The machinery is particularly easy to deal with in the case that $G$ is the zero set of a single linear homogeneous differential polynomial in a single variable, that is, $G$ is given as the zero set of $f(z) \in K\{z\}$. Note that $G$ is a subgroup of the additive group, $\mathbb{G}_a$. Suppose that, for some orderly ranking of the free monoid $\Theta$ generated by $\Delta = \{\delta_1, \ldots, \delta_m\}$, the leader of $f(z)$ is $\delta_1^{i_1} \cdots \delta_m^{i_m} z$. Then the Kolchin polynomial, $\omega_G(t)$, is equal to the number of lattice points of $(n_1, \ldots, n_m) \in \mathbb{N}^m$ with

$$\sum_{i=1}^{m} n_i \leq x$$

and $(n_1, \ldots, n_m)$ not above $(i_1, \ldots, i_m)$ in the (partial) product order. Then we know that the Kolchin polynomial is given by,

$$\omega_G(t) = \left(\frac{t+m}{m}\right) - \left(\frac{t - \sum_{j=1}^{n} i_j + m}{m}\right) + a$$

where $a$ is a constant. Letting $N$ be the sum $\sum i_j$, we have

$$\omega_G(t) = \left(\frac{t+m}{m}\right) - \left(\frac{t - N + m}{m}\right) + a = N t^{m-1} + g(t)$$

where $g(t)$ is lower degree in $t$. Then any subgroup of $H \leq G$ has, in its defining ideal, a differential polynomial $g$ with the leader of $g$ not above the leader of $f$ in the product order. This means that $\tau(H) < \tau(G)$ or the coefficient of $x^{m-1}$ in the Kolchin polynomial of $H$ is less than $N$. In either case, the coset space $G/H$ must have $\Delta$-type $m - 1$. Thus, $G$ is indecomposable.

**Example 4.7.** We will again work over a model of $DCF_{0,2}$. The following example was explored in [5] and was originally given in [4]. Let $G$ be the solution set of

$$(c_2 \delta_1^3 - c_2 \delta_1^2 \delta_2 - 2c_2 \delta_1 \delta_2 + c_2^2 \delta_2^2 + 2\delta_2) x = 0$$
where $\delta_1 c_2 = 1$ and $\delta_2 c_2 = 0$. By the above discussion of linear homogeneous differential equations, this is a strongly connected differential algebraic group. Of course, there are differential algebraic subgroups of $\Delta$-type 1. In fact, since
\[
c_2 \delta_1^3 - c_2 \delta_1^2 \delta_2 - 2c_2 \delta_1 \delta_2 + c_2^2 \delta_2^2 + 2\delta_2
= (c_2 \delta_1 - c_2^2 \delta_2 - 2)(\delta_1^2 - \delta_2)
= (c_2 \delta_1^2 - c_2 \delta_2 - 2\delta_1)(\delta_1 - c_2 \delta_2)
\]
the solution sets to $\delta_1^2 x - \delta_2 x = 0$ and $\delta_1 x - c_2 \delta_1 x = 0$ are differential algebraic subgroups. In [22] Suer showed the solution set of the first equation has Lascar rank $\omega$ by showing that every definable subset has finite transcendence degree. So, this subgroup is indecomposable. The subgroup given by the solutions to $\delta_1 x - c_2 \delta_1 x = 0$ also only has finite transcendence degree definable subsets. This subgroup is irreducible in the Kolchin topology, so the only definable proper subsets correspond to forking extensions of the generic type of subgroup. But, modulo, $\delta_1 x - c_2 \delta_1 x = 0$, any differential polynomial can be expressed as a $\delta_2$-polynomial or a $\delta_1$-polynomial. So, this subgroup is also indecomposable.

5. Another Definability Result

In this section, we prove results inspired by work of Baudisch [1]. As with many of the results of this paper, the relationship between the results here and the existing work on superstable and $\omega$-stable groups would only become clear by getting control (or showing counterexamples) of Lascar rank in terms of differential type. The following lemma is easy to prove, see [5].

**Lemma 5.1.** Suppose there is $H \leq G$ with $\tau(G/H) < n$. Then there is a normal subgroup $L$ of $G$ with $\tau(G/L) < n$.

**Theorem 5.2.** Suppose $\tau(G) = n$ and $H \triangleleft G_n$, the strongly connected component of $G$. Then if $H$ is simple, $H$ is definable.

**Proof.** Let $h \neq 1, h \in H$. Then we will show $h^G \cup \{1\}$ is indecomposable. Note that by Lemma 5.1 we only need to show the indecomposability conclusions for normal subgroups. So, let $N$ be a normal subgroup of $G$. First, suppose that $N \cap H \neq 1$. Then because $H$ is simple, $H \triangleleft K$. In this case, the coset space $|h^G \cup \{1\}/N| = 1$. Thus, we may assume that $N \cap H = 1$. Now, to verify that $h^G$ is indecomposable, we only need to show that $\tau(h^G) = n$.

There is a bijection between the elements of $h^G$ and the $G$-cosets of $C_G(h)$. So, it would suffice to prove that $\tau(G/C_G(h)) = n$. We know that $H \not\leq C_G(h)$, because $H$ is simple. But, then $|G_n/C_G(h)| \neq 1$. Because $G_n$ is indecomposable, $\tau(G_n/C_G(h)) = n$. But, then $\tau(G/C_G(h)) = n$. Now, we know that the following family of definable sets
$\langle h^G \rangle_h \subseteq H$ is indecomposable. Now we apply Theorem 3.4 to see that $H$ must be definable.

Further definability consequences of indecomposability will be pursued in [6].

6. Generalizations of Strongly Connected

For differential algebraic groups, the notion of indecomposable matches the notion of strongly connected. But, in Definition 3.1 we defined $n$-indecomposable. In this section we will explore the notion in the case that $n \neq \tau(G)$. Consider the following family of proper differential algebraic subgroups,

$$\mathcal{G}_n := \{ H < G \mid \tau(G/H) < n \}$$

We note that this family is closed under finite intersections. Since $G$ is an $\omega$-stable group,

$$\bigcap_{H \in \mathcal{G}_n} H$$

is a definable subgroup, which we will denote $G_n$. We note that $H_n$ is a characteristic subgroup of $G$. We will refer to $G_n$ as the $n$-connected component.

Example 6.1. It is entirely possibly that the subgroups $H_n$ are different for every $n$. The following is a very simple example which readily generalizes. Consider the following group of matrices of the form

$$\begin{pmatrix}
1 & u_{12} & u_1 & u \\
0 & 1 & u_{123} & u_2 \\
0 & 0 & 1 & u_{23} \\
0 & 0 & 0 & 1
\end{pmatrix}$$

where $\delta_1\delta_2u_{12} = 0$, $\delta_1u_1 = 0$, $\delta_1\delta_2\delta_3u_{123} = 0$, $\delta_2u_2 = 0$, and $\delta_2\delta_3u_{23} = 0$.

This is a group since

$$\begin{pmatrix}
1 & u_{12} & u_1 & u \\
0 & 1 & u_{123} & u_2 \\
0 & 0 & 1 & u_{23} \\
0 & 0 & 0 & 1
\end{pmatrix} \cdot \begin{pmatrix}
1 & h_{12} & h_1 & h \\
0 & 1 & h_{123} & h_2 \\
0 & 0 & 1 & h_{23} \\
0 & 0 & 0 & 1
\end{pmatrix} = \begin{pmatrix}
1 & h_{12} + u_{12} & u_1 + u_{12}h_{123} + h_1 & h + u_{12}h_2 + u_1h_{23} + u \\
0 & 1 & u_{123} + h_{123} & u_2 + u_{123}h_{23} + h_2 \\
0 & 0 & 1 & u_{23} + h_{23} \\
0 & 0 & 0 & 1
\end{pmatrix}$$

and the coordinates evidently satisfy the same differential equations as the original matrices. The group is 0-indecomposable. The group is not 1-indecomposable. The
1-connected component is the subgroup of matrices of the form:

\[
\begin{pmatrix}
1 & u_{12} & u_1 & u \\
0 & 1 & 0 & u_2 \\
0 & 0 & 1 & u_{23} \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

The 2-connected component is the subgroup of matrices of the form:

\[
\begin{pmatrix}
1 & 0 & u_1 & u \\
0 & 1 & 0 & u_2 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

The 3-connected (in this case, strongly connected) component is the subgroup of matrices of the form:

\[
\begin{pmatrix}
1 & 0 & 0 & u \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

Though much of the analysis of this paper essentially works in the case of \(n\)-indecomposability with \(n \neq \tau(G)\), by relativizing the appropriate statements (see Proposition 3.5 for instance), there are important pieces which are not immediate. For instance, when seeking definability results for a family of \(n\)-indecomposable subsets, the above techniques are only useful when the subsets can be contained in a strongly connected subgroup of differential type \(n\).

The indecomposability theorem of Berline and Lascar [3] applies in the setting of superstable groups, so, specifically for groups definable in \(DCF_{0,m}\). As we noted in the introduction, there is no known lower bound for Lascar rank in partial differential fields based on differential type and typical differential dimension. In fact, examples of [21] show that any such lower bound can not involve typical differential dimension (Suer constructs differential varieties of arbitrarily high typical differential dimension, differential type 1, and Lascar rank \(\omega\)). It is not currently known if there is an infinite transcendence degree strongly minimal type. One should note that such examples are present in the difference-differential context [13], but have yet to be discovered in the partial differential context.

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