Single- and multi-phonon excitations of $\gamma$ vibration in rotating odd-$A$ nuclei

Masayuki Matsuzaki

Department of Physics, Fukuoka University of Education,
Munakata, Fukuoka 811-4192, Japan

(Dated: June 16, 2014)
Abstract

Background  Collective motions in quantum many-body systems are described as bosonic excitations. Multi-phonon excitations in atomic nuclei, however, were observed very rarely. In particular, the first two-phonon low-lying $\gamma$ vibration ($2\gamma$) in odd-$A$ nuclei was reported in 2006 and only a few have been known so far. Two theoretical calculations for the data on $^{103}$Nb were done, one of which was done by the present author within a limited model space up to $2\gamma$. Quite recently, conspicuously enhanced $B(E2)$s feeding the $2\gamma$ were observed in $^{105}$Nb and conjectured that their mother states, called band (4), are candidates of $3\gamma$.

Purpose  In the present work, the model space is enlarged up to $4\gamma$ states. The purposes are two-folds: One is to see how the description of $2\gamma$ in the previous work is improved, and the other is to look into the existence of $3\gamma$ states, and when they exist, study their collectivity through calculating interband $B(E2)$s.

Method  The particle-vibration coupling model based on the cranking model and the random-phase approximation is used to calculate the vibrational states in rotating odd-$A$ nuclei. The interband $B(E2)$s are calculated by adopting the method of the generalized intensity relation.

Results  The present model reproduces the energy spectra and $B(E2)$s of $0\gamma - 2\gamma$ in $^{103}$ and $^{105}$Nb well. For $3\gamma$, calculated spectra indicate that the most collective state with the highest $K$ at zero rotation feels strong Coriolis force after rotation sets in and consequently is observed with lowered $K$, where $K$ is the projection of the angular momentum to the $z$ axis. The calculated states account for the observed enhanced $B(E2)$s within factors of $2 - 3$.

Conclusions  The present calculation with the enlarged model space reproduces the observed $0\gamma - 2\gamma$ well and gives collective $3\gamma$ states. The most collective one is thought to be the main component of the observed band (4) from the analyses of the energy spectra and interband $B(E2)$s although some mixing with states that are not included in the present model would be possible.

PACS numbers: 21.10.Re, 21.60.Jz, 27.60.+j
I. INTRODUCTION

Collective motions in quantum many-body systems are formed as coherent superpositions of many individual degrees of freedom, and are described as bosonic excitations. In atomic nuclei, one of finite many-body systems, the representative is vibrations of the surface of the average potential produced selfconsistently to the nucleon distribution. However, repeated excitations, the characteristic of bosons, are not always observed. Even when observed, their strengths spread over many eigenstates because collective and individual non-collective excitations have similar energy scale. Therefore, it has been a longstanding subject to study whether multiple excitations exist or not and their properties when exist, both theoretically and experimentally.

Famous examples are known in high-lying giant resonances. Double excitations of the one with the same multipole and of different types have been observed, see for example, a review [1]. Among low-lying vibrations, the multi-phonon quadrupole vibrational states in spherical nuclei have long been studied around $^{110}$Cd [2, 3]. In axially symmetric or weakly triaxial deformed nuclei, the two-phonon $\gamma$ vibration, denoted as $2\gamma$ hereafter, has been studied for three decades as concisely reviewed in the preceding paper of the present author [4], named Paper I hereafter, but observed only in $^{168}$Er, $^{164}$Dy, $^{232}$Th, and $^{106}$Mo in even-even nuclei. Recently, another candidate was reported in the weakly deformed, $\gamma$-soft nucleus, $^{138}$Nd [5].

In odd-$A$ nuclei, vibrations of the average potential alter the particle motion and the change thus caused affects back the average potential. Consequently the particle motion and the vibration more or less couple to each other. This coupling makes the excitation spectrum complex in general. From another point of view, however, a stretched parallel coupling of the high-$j$ particle and $K = 2$ phonons can produce high-$K$ states that can hardly mix with other states with lower $K$. Here $j$ is the single-particle angular momentum, and $K = 2$ the projection to the $z$ axis of the angular momentum carried by the $\gamma$ vibration. Accordingly there can be more opportunities to observe multi-phonon $\gamma$-vibrational states.

Prior to experimental observations, numerical predictions for odd-$A$ nuclei were made in Ref. [6]. The first observation was made ten years later in $^{105}$Mo [7]. Soon after this, similar bands were observed in $^{103}$Nb [8] and in $^{107}$Tc [9]. The first realistic theoretical calculation in terms of the triaxial projected shell model was reported for $^{103}$Nb [10] in 2010. The
calculation in terms of the particle-vibration coupling (PVC) model based on the cranking model and the random-phase approximation (RPA) was done in 2011 [4].

As another type of multi-phonon state in strongly triaxial deformed nuclei, two-phonon wobbling bands were observed in $^{163}$ and $^{165}$Lu [11, 12] and analyzed in Ref. [13]. Further, the rotational bands built on the high-$K$ multi-quasiparticle states can be interpreted theoretically as a multi-phonon excitation of the precession mode, which is the axially-symmetric limit of the wobbling mode [14].

Quite recently, a $2\gamma$ band in $^{105}$Nb bearing much resemblance to that in $^{103}$Nb was reported [15]. The characteristic feature common to these two isotopes but beyond the scope of Paper I is that the observed band (4) is interpreted as a candidate of $3\gamma$ band. In particular in $^{105}$Nb, $B(E2)$s from this band to the $2\gamma$ band are conspicuously enhanced.

In Paper I, the model, which was equipped to study the signature dependence in $0\gamma$ (1qp) bands in the rare-earth region [16, 17] and utilized later to study the two sequences with $K = \Omega \pm 2$ of the $1\gamma$ band in $^{165}$Ho [18], was applied to the $2\gamma$ band for the first time. Here $\Omega$ is the projection of the single-particle angular momentum to the $z$ axis. From a microscopic many-body theoretical point of view, the mechanism that leads to anharmonicity of the spectrum beyond the RPA has been discussed in the studies of $2\gamma$ bands in even-even nuclei [19], however, the present model concentrates on that peculiar to odd-$A$ systems. The result was that the $0\gamma$ and $1\gamma$ bands were reproduced perfectly but the calculated $2\gamma$ band was somewhat higher in energy than the observed one as in the other calculation [10].

In the present paper, the model space is enlarged up to $4\gamma$ states and the interband $B(E2)$s are also calculated utilizing the method of the generalized intensity relation (GIR) [2, 20]. The purposes are two-folds: One is to see how the description of $2\gamma$ in Paper I is improved, and the other is to look into the existence of $3\gamma$ states, and when they exist, study their collectivity through calculating vibrational $B(E2)$s. This is the first attempt to study realistic $3\gamma$ states in deformed nuclei, to the author’s knowledge.

Throughout this paper the $\hbar = 1$ unit is used.
II. THE MODEL

A. Particle-vibration coupling

Eigenstates of the odd-$A$ nucleus in a rotating frame are calculated in the particle-vibration coupling model based on the cranking model and the RPA. The adopted Hamiltonian is the same as in Paper I, therefore recapitulated only briefly here. We begin with the cranked Nilsson plus BCS one-body Hamiltonian,

\[ h' = h - \omega_{\text{rot}} J_x, \]
\[ h = h_{\text{Nil}} - \Delta_\tau (P_\tau + P_\bar{\tau}) - \lambda_\tau N_\tau, \]
\[ h_{\text{Nil}} = \frac{p^2}{2M} + \frac{1}{2} M (\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2) + v_{lt} \mathbf{s} + v_{lt} (l^2 - \langle l^2 \rangle_{\text{osc}}). \]

In these expressions, the standard notations are understood. This Hamiltonian gives quasiparticle states created by \( a_\mu^\dagger \) and \( a_\bar{\mu}^\dagger \) with signature \( r = \exp (-i\pi \alpha) = -i \) and \( +i \), respectively. Then we apply the RPA to the residual pairing plus doubly-stretched quadrupole-quadrupole interaction between quasiparticles. The interaction Hamiltonian is given by

\[ H_{\text{int}} = - \sum_{\tau=1,2} G_\tau \tilde{P}_\tau^\dagger \tilde{P}_\tau - \frac{1}{2} \sum_{K=0,1,2} \kappa^{(+)\dagger}_K Q^{n(+)\dagger}_K Q^{n(+)}_K - \frac{1}{2} \sum_{K=1,2} \kappa^{(-)\dagger}_K Q^{n(-)\dagger}_K Q^{n(-)}_K, \]

where \( Q^{n(\pm)}_K \) are the signature-coupled form of the quadrupole operators defined by the doubly-stretched coordinates.

Among the RPA modes, \( X^\dagger_n \), determined by \( H_{\text{int}} \), we choose the \( \gamma \)-vibrational phonons, \( n = \gamma(\pm) \) with signature \( r = \pm 1 \), which have outstandingly large \( K = 2 \) transition amplitudes. In terms of the quasiparticles and the \( \gamma \)-vibrational phonons thus determined, the particle-vibration coupling Hamiltonian takes the form

\[ H_{\text{couple}}(\gamma) = \sum_{\mu\nu} \Lambda_{\gamma(+)}(\mu\nu) \left( X^\dagger_{\gamma(+)} a^\dagger_\mu a_\nu + X_{\gamma(+)} a^\dagger_\nu a_\mu \right) \]
\[ + \sum_{\mu\bar{\nu}} \Lambda_{\gamma(-)}(\mu\bar{\nu}) \left( X^\dagger_{\gamma(-)} a^\dagger_\mu a_\bar{\nu} + X_{\gamma(-)} a^\dagger_\bar{\nu} a_\mu \right) \]
\[ + \text{sig. conj.} \]
The coupling vertices are given by

\[ \Lambda_{\gamma(+)}(\mu\nu) = - \sum_{K=0,1,2} \kappa_{K}^{(+)} T_{K}^{\mu(+)} Q_{K}^{\mu(+)}(\mu\nu), \]

\[ \Lambda_{\gamma(-)}(\mu\bar{\nu}) = - \sum_{K=1,2} \kappa_{K}^{(-)} T_{K}^{\mu(-)} Q_{K}^{\mu(-)}(\mu\bar{\nu}), \]

and sig. conj.,

where \( T_{K}^{\mu(\pm)} \) are the doubly-stretched quadrupole transition amplitudes associated with the \( \gamma \)-vibrational phonons and \( Q_{K}^{\mu(\pm)}(\alpha\beta) \) denote quasiparticle scattering matrix elements. Eigenstates of the total Hamiltonian at each \( \omega_{\text{rot}} \) take the form

\[ |\chi_{i}\rangle = \sum_{\mu} \psi_{i}^{(1)}(\mu) a_{\mu}^{\dagger}|\phi\rangle \]

\[ + \sum_{\mu} \psi_{i}^{(3)}(\mu\gamma) a_{\mu}^{\dagger} X_{\gamma}^{\dagger}|\phi\rangle + \sum_{\bar{\mu}} \psi_{\bar{i}}^{(3)}(\bar{\mu}\bar{\gamma}) a_{\bar{\mu}}^{\dagger} X_{\bar{\gamma}}^{\dagger}|\phi\rangle \]

\[ + \sum_{\mu} \psi_{i}^{(5)}(\mu\bar{\gamma}) \frac{1}{\sqrt{2}} a_{\mu}^{\dagger} (X_{\gamma}^{\dagger})^{2}|\phi\rangle + \sum_{\mu} \psi_{\bar{i}}^{(5)}(\bar{\mu}\gamma) \frac{1}{\sqrt{2}} a_{\bar{\mu}}^{\dagger} (X_{\bar{\gamma}}^{\dagger})^{2}|\phi\rangle \]

\[ + \sum_{\mu} \psi_{i}^{(5)}(\bar{\mu}\gamma) a_{\mu}^{\dagger} X_{\gamma}^{\dagger} X_{\bar{\gamma}}^{\dagger}|\phi\rangle \]

\[ + \sum_{\mu} \psi_{i}^{(7)}(\mu\gamma\gamma) \frac{1}{\sqrt{3!}} a_{\mu}^{\dagger} (X_{\gamma}^{\dagger})^{3}|\phi\rangle + \sum_{\mu} \psi_{\bar{i}}^{(7)}(\bar{\mu}\gamma\bar{\gamma}) \frac{1}{\sqrt{3!}} a_{\bar{\mu}}^{\dagger} (X_{\bar{\gamma}}^{\dagger})^{3}|\phi\rangle \]

\[ + \sum_{\bar{\mu}} \psi_{\bar{i}}^{(7)}(\bar{\mu}\gamma\bar{\gamma}) \frac{1}{\sqrt{2}} a_{\bar{\mu}}^{\dagger} (X_{\bar{\gamma}}^{\dagger})^{2} X_{\gamma}^{\dagger}|\phi\rangle + \sum_{\mu} \psi_{i}^{(7)}(\mu\bar{\gamma}\bar{\gamma}) \frac{1}{\sqrt{2}} a_{\mu}^{\dagger} X_{\gamma}^{\dagger} (X_{\bar{\gamma}}^{\dagger})^{2}|\phi\rangle \]

\[ + \sum_{\mu} \psi_{i}^{(9)}(\mu\gamma\gamma\gamma) \frac{1}{\sqrt{4!}} a_{\mu}^{\dagger} (X_{\gamma}^{\dagger})^{4}|\phi\rangle + \sum_{\bar{\mu}} \psi_{\bar{i}}^{(9)}(\bar{\mu}\gamma\bar{\gamma}\bar{\gamma}) \frac{1}{\sqrt{4!}} a_{\bar{\mu}}^{\dagger} (X_{\bar{\gamma}}^{\dagger})^{4}|\phi\rangle \]

\[ + \sum_{\bar{\mu}} \psi_{\bar{i}}^{(9)}(\bar{\mu}\gamma\bar{\gamma}\bar{\gamma}) \frac{1}{\sqrt{2}} a_{\bar{\mu}}^{\dagger} X_{\gamma}^{\dagger} (X_{\bar{\gamma}}^{\dagger})^{3}|\phi\rangle + \sum_{\mu} \psi_{i}^{(9)}(\mu\bar{\gamma}\bar{\gamma}\bar{\gamma}) \frac{1}{\sqrt{2}} a_{\mu}^{\dagger} (X_{\gamma}^{\dagger})^{3} X_{\bar{\gamma}}^{\dagger}|\phi\rangle \]

\[ + \sum_{\mu} \psi_{i}^{(9)}(\mu\gamma\gamma\gamma) \frac{1}{2} a_{\mu}^{\dagger} (X_{\gamma}^{\dagger})^{2} (X_{\gamma}^{\dagger})^{2}|\phi\rangle, \]

for the \( r = -i \) sector, where \( \gamma \) and \( \bar{\gamma} \) abbreviate \( \gamma(+) \) and \( \gamma(-) \), respectively, and \(|\phi\rangle\) is the rotating vacuum configuration. Those for the \( r = +i \) sector take a similar form. This notation indicates that a limited class of 1, 3, 5, 7, and 9qp states that contribute to multi-phonon \( \gamma \)-vibrational states are taken into account. Among these, the model space was truncated up to the \( \psi_{(5)}^{(5)} \) terms in Paper I.
B. Generalized intensity relation

The rotational effects on the vibrational (interband) transition rates are well described by the generalized intensity relation in terms of the intrinsic matrix elements, see Fig. 4-30 in Ref. [2]. On the other hand, the cranking model and its extensions can provide us with rotationally perturbed intrinsic matrix elements precisely. Therefore, a method to combine these two was proposed in Ref. [20] and applied to the $1\gamma \to 0\gamma$ transitions in $^{165}$Ho in Ref. [18]. In the present study, this method is applied to the $n\gamma \to (n-1)\gamma$ transitions with $n = 1, 2, \text{and} 3$, along the way of this reference. The expressions for the $B(E2)$ are

$$B(E2 : I_iK_i \to I_fK_f) = \langle I_iK_i 2\Delta K|I_fK_f\rangle^2 Q_{\text{out}}^2,$$  \hspace{1cm} (8)

$$Q_{\text{out}} = Q_1 + Q_2[I_f(I_f + 1) - I_i(I_i + 1)],$$  \hspace{1cm} (9)

$$Q_1 = \sqrt{2}Q_{\text{tr}} - \Delta K(K_i + K_f)Q_2,$$  \hspace{1cm} (10)

$$Q_{\text{tr}} = \langle f|Q_2^{(+)}|i\rangle,$$  \hspace{1cm} (11)

$$Q_2 = \frac{1}{\sqrt{2}J} \frac{d\langle f|Q_2^{(+)}|i\rangle}{d\omega_{\text{rot}}},$$  \hspace{1cm} (12)

where $Q_{\text{tr}}$ and $Q_2$ are evaluated at $\omega_{\text{rot}} = 0$, and the moment of inertia, $J$, is extracted from the experimental energy of the first $\Delta I = 2$ excited states in the ground band.

III. RESULTS AND DISCUSSION

Numerical calculations are performed for two isotopes, $^{103}$ and $^{105}$Nb, in which candidates of $3\gamma$ states were observed [8, 15]. The former was investigated in Paper I within a limited model space, and the latter, on which the data were reported quite recently, is newly studied. In both isotopes, the ground band is based on the $\pi[4225/2^+]$ asymptotic state, and the single- and multi-phonon $\gamma$-vibrational excitations on top of it were observed. Cranking and RPA calculations are done in the five major shells, $N_{\text{osc}} = 2 - 6$ for the neutron and 1 - 5 for the proton. The indices $\mu$ and $\bar{\mu}$ in Eq. (7) for the PVC eigenstates run from 1 to 15, the number of quasiparticle states with $N_{\text{osc}} = 4$. In the following, the results for the favored signature, $r = -i$, are mainly presented while those for the unfavored $r = +i$ are also shown when necessary.
A. $0\gamma - 2\gamma$ in $^{103}$Nb

All the parameters entering into the calculation are the same as those adopted in Paper I. The pairing gaps $\Delta_n = 1.05$ MeV, $\Delta_p = 0.85$ MeV and the deformation $\epsilon_2 = 0.31$ are adopted conforming to the experimental analyses [8, 21]. The triaxiality $\gamma = -7^\circ$ is chosen to reproduce the measured signature splitting of the ground band in the PVC calculation. The quadrupole force strengths are determined to reproduce in the axially symmetric limit the $\gamma$-vibrational energy observed in $^{104}$Mo [22], see Paper I for the detail.

1. Distribution of collective states

In Fig. 1, the distribution of collective states are shown. In the following, we denote the favored signature of the lowest quasiparticle state as $1qp$, and its signature partner as $\overline{1qp}$. The lowest PVC eigenstate whose main component is this $1qp$ is often denoted also as $0\gamma$. Then the fraction of the $1\gamma$ (green dashed in the figure) means the sum of the probabilities of $1qp \otimes \gamma(+) \text{ and } \overline{1qp} \otimes \gamma(-)$. The conventions for multi-phonon states are understood straightforwardly.

At $\omega_{rot} = 0$, $0\gamma - 3\gamma$ are almost harmonic, whereas the $4\gamma$ strength is located higher than the harmonic location. In addition, the main peaks of $0\gamma - 4\gamma$ are almost of the same height although the collective strength spreads as the number of phonon increases. As discussed later, the main peaks have rather pure $K$.

As soon as rotation sets in, the heights of the main peaks decrease approximately in proportional to the number of phonon. Two sequences of $1\gamma$ and three sequences of $2\gamma$ states survive up to high spins as discussed in Paper I. In addition, three or four sequences of $3\gamma$ look to keep their collective character to some extent. Energies of $n\gamma$ in the rotating frame are $e'_2\gamma < e'_0\gamma + 2\Delta_p$, but $e'_0\gamma + 2\Delta_p < e'_3\gamma < e'_0\gamma + 2\Delta_n$. Therefore, the collectivity of the calculated $3\gamma$ states other than high-$K$ ones, which are hard to mix with other states, should be considered with reservations, since in the present model the $1qp \otimes 3\gamma$ states couple only with $(1qp)' \otimes 2\gamma$ and $(1qp)' \otimes 4\gamma$, whereas direct couplings to $3qp$ states are not included, where $(1qp)'$ denotes other one-quasiparticle states including those of the opposite signature.

In Fig. 2, calculated eigenstates in the energy regions of (a) $1\gamma$ and (b) $2\gamma$ are shown. Among them, those with more than 50% collective $(1\gamma \text{ or } 2\gamma)$ fraction are emphasized with
FIG. 1: (Color online) Distribution of the collective fraction (probability in the wave function) of the $0\gamma$, $1\gamma$, $2\gamma$, $3\gamma$, and $4\gamma$ components in the favored signature ($r = -i$) sector of $^{103}$Nb, $|\psi^{(1)}(1)|^2$ (blue longer dotted), $|\psi^{(3)}(1\gamma)|^2 + |\psi^{(3)}(1\bar{\gamma})|^2$ (green dashed), $|\psi^{(5)}(1\gamma\gamma)|^2 + |\psi^{(5)}(1\bar{\gamma}\bar{\gamma})|^2$ (red solid), $|\psi^{(7)}(1\gamma\gamma\gamma)|^2 + |\psi^{(7)}(1\bar{\gamma}\bar{\gamma}\bar{\gamma})|^2 + |\psi^{(7)}(1\bar{\gamma}\bar{\gamma})|^2$ (magenta dotted), and $|\psi^{(9)}(1\gamma\gamma\gamma\gamma)|^2 + |\psi^{(9)}(1\bar{\gamma}\bar{\gamma}\bar{\gamma}\bar{\gamma})|^2 + |\psi^{(9)}(1\bar{\gamma}\bar{\gamma}\bar{\gamma})|^2$ (blue dot-dashed), respectively, at $\omega_{\text{rot}} = 0 - 0.3$ MeV.

FIG. 2: (Color online) Energies of all calculated PVC states in the $r = -i$ sector of $^{103}$Nb in the energy regions of (a) $1\gamma$ and (b) $2\gamma$ bands are shown by green +s. Those with more than 50% collective fraction are emphasized by red ×s. The labels attached designate the numbers, $i$ in Eq. (7), enumerated from the lowest.
red crosses. The two 1γ sequences are completely isolated from other states. The three 2γ ones are also distinguished from other states but crossed by two upslope (1qp)’ states at around \( \omega_{\text{rot}} = 0.15 - 0.2 \text{ MeV} \).

2. Characterization of the calculated 1γ and 2γ states

On the correspondence between 1γ bands in the calculation in the signature scheme and that in the \( K \) scheme, it was argued in Paper I that the obtained lower band can be identified with the \( K = \Omega - 2 \) band since states with lower \( K \) have lower intrinsic energies than those with higher \( K \) and the same total angular momentum \( I \). In order to look into this correspondence more, the aligned angular momenta around \( \omega_{\text{rot}} = 0 \) are shown in Fig. 3 (a) for 1γ and (b) for 2γ. The labels \( i \) of states refer to those in Fig. 2.

![FIG. 3: (Color online) Expectation values of the projection of the angular momentum to the rotational (x) axis around the bandheads of (a) 1γ and (b) 2γ bands in \(^{103}\text{Nb} \).](image)

The 1γ band should have \( K = |\Omega - 2| = 1/2 \) or \( \Omega + 2 = 9/2 \) at the bandhead aside from weak \( K \) mixing stemming from static triaxial deformation. Figure 3(a) clearly shows that the lower (2nd) pair of states has a strong signature decoupling, non-zero aligned angular momentum with opposite sign at \( \omega_{\text{rot}} = 0 \), while the upper (3rd) pair has practically zero aligned angular momentum at \( \omega_{\text{rot}} = 0 \) and negligible signature splitting. This proves the mapping that the lower band is of low \( K \) and the upper band is of high \( K \).

The characterization is further evidenced by looking at the wave function. Figure 4 graphs the amplitudes of the dominant components, \( D(+) = |\psi_{i}^{(3)}(1\gamma)| \) for \( 1qp \otimes \gamma(+) \) and \( D(-) = |\psi_{i}^{(3)}(\bar{1}\bar{\gamma})| \) for \( \bar{1}\bar{q}p \otimes \gamma(-) \), in (a) upper \( (i = 3\text{rd}) \) and (b) lower \( (i = 2\text{nd}) \) bands. The structure of the \( \gamma(\pm) \) is \((Q_{22} \pm Q_{2-2})/\sqrt{2} \). These two components mix with similar
FIG. 4: (Color online) Amplitudes of the dominant components $D(+) = |\psi_i^{(3)}(1\gamma)|$ and $D(-) = |\psi_i^{(3)}(1\bar{\gamma})|$ in (a) upper ($i = 3$th) and (b) lower ($i = 2$nd) 1\gamma bands in the $r = -i$ sector of $^{103}$Nb.

Magnitudes both in the upper and lower eigenstates and their relative sign (not shown) is always opposite. This means that the two orthogonal combinations of $\gamma(\pm)$ reproduce high-$K$ and low-$K$ states.

The 2\gamma band should have $K = |\Omega - 4| = 3/2$ or $\Omega = 5/2$ or $\Omega + 4 = 13/2$ at the bandhead. An argument for Fig. 3(b) similar to that for 1\gamma in Fig. 3(a) leads to the identification that 9th pair is $K = 13/2$, the 8th and 7th are $K = 3/2$ and $K = 5/2$. The latter two interact with each other in $r = -i$ as soon as rotation sets in. This interaction can also be seen in Fig. 2(b). Note that the 8th state has about 35\% collectivity at $\omega_{\text{rot}} = 0$ but it increases as $\omega_{\text{rot}}$ increases.

These discussion confirms that the calculated upper bands, 3rd for 1\gamma and 9th for 2\gamma, should be compared with the observed bands (2) and (3), respectively.

3. Effect of enlargement of the model space and comparison with experimental data

The first purpose of this paper is to see how the 2\gamma states calculated in Paper I within a smaller model space is affected by the enlargement of the space. The previous calculation was done in the space up to 2\gamma states. This time we examine that up to 3\gamma and 4\gamma. First, by including 3\gamma states, the upper 2\gamma band is pushed down by 0.27 MeV ($\omega_{\text{rot}} = 0$) – 0.23 MeV ($\omega_{\text{rot}} = 0.3$ MeV). Next, by including 4\gamma, this band is pushed down further by 0.06 – 0.03 MeV. The 0\gamma and 1\gamma are almost unaffected.

The final result is presented in Fig. 5. In this nucleus, calculated $\epsilon_{2\gamma}^\prime$ is still higher, by 0.09 – 0.18 MeV, than the data. See the result for $^{105}$Nb below.
FIG. 5: (Color online) Energies of calculated 0γ (blue ∗s), 1γ (green ×s), and 2γ (red +s) states in the \( r = -i \) sector of \(^{103}\text{Nb} \) are compared with the corresponding data (curves) converted to the rotating frame by using the Harris parameters \( J_0 = 15.45 \text{ MeV}^{-1} \) and \( J_1 = 81.23 \text{ MeV}^{-3} \) that fit the yrast band of \(^{104}\text{Mo} \) \[22\]. The observed transition in band (4), the 3γ candidate, converted to the rotating frame is also shown by a large ×.

B. 0γ − 2γ in \(^{105}\text{Nb} \)

Parameters entering into the calculation are determined in a manner similar to the case of \(^{103}\text{Nb} \). Concretely, the pairing gaps \( \Delta_n = 1.05 \text{ MeV} \) and \( \Delta_p = 0.85 \text{ MeV} \), and the deformation \( \epsilon_2 = 0.3254 \) are adopted conforming to the experimental analyses \[15, 21\]. The triaxiality \( \gamma = -10^\circ \) is chosen to reproduce the observed signature splitting of the ground (0γ) band in the PVC calculation as shown in Fig. 6(a).

FIG. 6: (Color online) (a) Experimental and calculated signature splitting in the \( \pi[4225/2^+] \) one-quasiparticle band in \(^{105}\text{Nb} \). Theoretical curve is the result of the particle-vibration coupling calculation. (b) Excitation energies of \( \gamma \)-vibrational RPA phonons with \( r = \pm 1 \).

The way to determine the quadrupole force strengths is slightly different; those deter-
mined to reproduce $\omega_\gamma = 0.7104$ MeV of $^{106}$Mo in the reference configuration with $\omega_{\text{rot}} = 0$ and $\gamma = 0$ result in a large signature splitting in $\omega_\gamma$ when triaxial deformation is introduced in contrast to the case of $^{103}$Nb. Alternatively, they are adjusted so as to reproduce the above $\omega_\gamma$ at $\omega_{\text{rot}} = 0$ and $\gamma = -10^\circ$, then $\kappa_{0}^{(+)}$ is set equal to $\kappa_{2}^{(+)}$ as in the case of $^{103}$Nb. The values for the residual pairing interaction are set to reproduce the adopted pairing gaps. The obtained $\omega_{\text{rot}}$ dependence and the signature splitting of $\omega_\gamma$ are shown in Fig. 6(b).

1. Distribution of collective states

In Fig. 7, the distribution of collective states is shown. Overall feature is quite similar to the case of $^{103}$Nb but the energies of collective solutions are slightly lower reflecting the input $\omega_\gamma$; this is consistent with the data, see Fig. 7 in Ref. [15]. Other differences from the $^{103}$Nb case are that i) the collectivity of the 3rd strongest $2\gamma$ (5th) state is as low as 30% at around $\omega_{\text{rot}} = 0$ but increases as $\omega_{\text{rot}}$ increases as seen in Fig. 8, and ii) among the $3\gamma$, the lower one is the most collective at $\omega_{\text{rot}} = 0.3$ MeV. The latter feature will be discussed below.

2. Comparison with experimental data

The characterization of calculated states is done in the same manner as in the case of $^{103}$Nb. Then the comparison with the data of $0\gamma - 2\gamma$ states are shown in Fig. 9. In the present case the observed $2\gamma$ is perfectly reproduced in contrast to the $^{103}$Nb case in which some deviation remains.

C. $3\gamma$ states in $^{103}$Nb and $^{105}$Nb

1. Distribution of collective states

An issue beyond the scope of Paper I is to characterize the observed band (4) that is conjectured to be a candidate of the three-phonon state, the first candidate in deformed nuclei.
FIG. 7: (Color online) The same as Fig. 4 but for $^{105}$Nb.

FIG. 8: (Color online) The same as Fig. 2(b) but for $^{105}$Nb.

Calculated eigenstates in the energy region of $3\gamma$ are shown in Fig. 10 (a) $^{103}$Nb, favored, (b) unfavored, and (c) $^{105}$Nb, favored, (d) unfavorable. Contrasting to the previous $1\gamma$ and $2\gamma$ cases, collective bands are not always parallel, and collectivity tends to move to lower energy states as $\omega_{\text{rot}}$ increases. In order to show this tendency clearly, states with 40 – 50% collective fractions are also marked with blue squares in these figures.

This result can be understood as follows. As discussed already, the highest-lying collective states are the most collective and have the highest $K$ due to the parallel coupling $\Omega + 2n$
FIG. 9: (Color online) The same as Fig. 5 but for $^{105}$Nb. The Harris parameters $J_0 = 18.08 \text{ MeV}^{-1}$ and $J_1 = 43.21 \text{ MeV}^{-3}$ that fit the yrast band of $^{106}$Mo [21] were used for the conversion.

for $n\gamma$ at around $\omega_{\text{rot}} = 0$. Since high-$K$ states are effectively of high-$j$, they feel strong Coriolis force when rotation sets in. Then they start to align their angular momenta to the rotational ($x$) axis with reducing $K$, accordingly they reduce their purity and the peak height in Figs. 1 and 7 gradually. This is an aspect of rotational $K$ mixing. Therefore it is expected that the collective $3\gamma$ with $K$ lower than the highest value $\Omega + 6$ would be observed. Actually, the observed band (4) with $K = 9/2$, indicated by large crosses in the figures, sits around the location determined by connecting the most collective states at $\omega_{\text{rot}} = 0$ and 0.3 MeV in the case of $^{105}$Nb. The situation in $^{103}$Nb is to some extent similar, but i) the lowest collective state at high $\omega_{\text{rot}}$ is not collective enough and ii) the calculated states are located higher overall. More sophisticated calculation would be desired because these observations depend on how band crossings occur and in the present model interband interactions occur at the same $\omega_{\text{rot}}$ rather than the same $I$.

2. Interband $B(E2)$

The observed enhanced $B(E2)$s look to be accounted for primarily by vibrational collectivity, and the above scenario that the collectivity of the 20th state at $\omega_{\text{rot}} = 0$ is observed with a lower $K$ after band crossing(s) can lead to enhanced transitions to the $2\gamma$. Before studying $3\gamma \rightarrow 2\gamma$ transitions, we check the results of the generalization intensity relation on $2\gamma \rightarrow 1\gamma$ and $1\gamma \rightarrow 0\gamma$. Some calculated values for $^{103}$Nb that can be compared with the observed $B(E2)$ ratios are shown in Table II.

First, the calculated values in Table II give

$$\frac{B(E2:15/2^+_2 \rightarrow 11/2^+_1)}{B(E2:11/2^+_1 \rightarrow 7/2^+_0)} = 2.46$$
FIG. 10: (Color online) Energies of all calculated PVC states in the energy regions of $3\gamma$ are shown by green $+$s as in Figs. 2 and 8. Those with collective fraction more than 50% and 40 - 50% are emphasized by red $\times$ s and blue $\Box$ s, respectively. (a) $r = -i$ of $^{103}$Nb, (b) $r = +i$ of $^{103}$Nb, (c) $r = -i$ of $^{105}$Nb, and (d) $r = +i$ of $^{105}$Nb. The observed transitions in band (4), the $3\gamma$ candidate, converted to the rotating frame are also shown by large $\times$ s.

TABLE I: Properties of calculated $n\gamma \rightarrow (n-1)\gamma$ transitions with $n = 1, 2, \text{ and } 3$, designated by the labels of intrinsic states, in $^{103}$Nb. The moment of inertia, $\mathcal{J} = 32.388$ MeV$^{-1}$, extracted from the energy of the $I = 9/2$ state in the ground band through $E(I) = (I(I+1) - K^2)/2\mathcal{J}$, was used.

| $r$  | intr | $I_i$ | $K_i$ | $I_f$ | $K_f$ | $Q_{tr}$ (eb) | $Q_2$ (eb) | $B(E2)$ (e$^2$b$^2$) |
|------|------|------|------|------|------|--------------|------------|----------------------|
| $+i$ 3 $\rightarrow$ 1 | 11/2 | 9/2 | 7/2 | 5/2 | 0.1944 | 0.0044 | 0.02618 |
| $-i$ 3 $\rightarrow$ 1 | 13/2 | 9/2 | 9/2 | 5/2 | 0.1944 | 0.0043 | 0.01768 |
| $+i$ 9 $\rightarrow$ 3 | 15/2 | 13/2 | 11/2 | 9/2 | 0.2704 | 0.0067 | 0.06436 |
| $+i$ 20 $\rightarrow$ 9 | 11/2 | 9/2 | 15/2 | 13/2 | 0.3122 | 0.0116 | 0.19168 |
\[
\frac{B(E2;15/2^+\gamma\rightarrow11/2^+_1)}{B(E2;13/2^+_\gamma\rightarrow9/2^+_0)} = 3.64,
\]
which are very close to the harmonic vibrational values 2.59 and 3.34 in Ref. [8], while the corresponding experimental values are 1.53(16) and 3.45(37), respectively. These shows that our PVC model describes the observed values precisely aside from the fact that the experimental \(B(E2; 11/2^+_1\gamma\rightarrow7/2^+_0)\) is slightly enhanced.

Next, the calculated value, \(B(E2;11/2^+_1\gamma\rightarrow3/2^+_0)\), is smaller than the corresponding experimental value, 13.5(11), but within a factor of 2. The elementary bosonic property, \(\hat{a}\langle n \mid n - 1 \rangle = \sqrt{n}|n-1\rangle\), is reflected in the intrinsic matrix element, \(Q_{tr}\), in the Table, such as \(0.3122/0.1944 = 1.61 \approx 3\). This is modified by the angular-momentum dependence brought by \(Q_2\), and leads to \(Q_{out}\) in Eq. (9). Finally \(B(E2)\) is obtained by multiplying a Clebsch-Gordan coefficient. This means that the calculated value contains some of the enhancement due to the angular-momentum effect.

| \(r\) | intr | \(I_i\) | \(K_i\) | \(I_f\) | \(K_f\) | \(Q_{tr}\) (eb) | \(Q_2\) (eb) | \(B(E2)\) (e\(2b^2\)) |
|------|------|--------|--------|--------|--------|---------------|---------------|---------------|
| +i   | 3    | 1      | 11/2   | 9/2    | 7/2    | 5/2           | 0.1928        | 0.0087        | 0.02065       |
| -i   | 3    | 10     | 13/2   | 9/2    | 9/2    | 5/2           | 0.1928        | 0.0085        | 0.01156       |
| +i   | 8    | 10     | 15/2   | 13/2   | 11/2   | 9/2           | 0.2674        | 0.0125        | 0.05047       |
| +i   | 20   | 10     | 18/2   | 15/2   | 13/2   | 12/2         | 0.3122        | 0.0185        | 0.22386       |
| -i   | 20   | 10     | 18/2   | 13/2   | 17/2   | 13/2         | 0.3122        | 0.0185        | 0.22501       |

TABLE II: The same as Table I but for \(^{105}\)Nb. The moment of inertia \(J = 31.831\) MeV\(^{-1}\) was used.

We proceed to \(^{105}\)Nb, in which more conspicuous enhancement of \(B(E2)\) is observed. The calculated values in Table II give \(\frac{B(E2;15/2^+_\gamma\rightarrow11/2^+_1)}{B(E2;13/2^+_\gamma\rightarrow9/2^+_0)} = 2.44\) and \(\frac{B(E2;13/2^+_\gamma\rightarrow11/2^+_1)}{B(E2;13/2^+_\gamma\rightarrow9/2^+_0)} = 4.37\), while the corresponding experimental values are 2.01(24) and 1.94(25), respectively. The calculated values are similar to those of \(^{103}\)Nb but in the experimental ones the magnitude of the denominator is inverted.

For those from the 3\(\gamma\) candidates, the calculated values are \(\frac{B(E2;11/2^+_\gamma\rightarrow15/2^+_1)}{B(E2;11/2^+_\gamma\rightarrow7/2^+_0)} = 10.8\) and \(\frac{B(E2;13/2^+_\gamma\rightarrow17/2^+_1)}{B(E2;13/2^+_\gamma\rightarrow9/2^+_0)} = 19.5\), while the corresponding experimental values are 27.5(33) and 40.5(55). The degree of enhancement increases from \(^{103}\)Nb both in the calculation and in the data. In the calculation, this is caused by a cooperation of increase of the numerator and decrease of the denominator. By a close looking, the difference in \(Q_2\), magnified by
the angular-momentum factor, $I_f(I_f + 1) - I_i(I_i + 1) - \Delta K(K_i + K_f) = 10$, produces the difference in the $B(E2)$ in the numerator. Again the differences in the ratios from the data are within factors of $2 - 3$. This indicates that the main mechanism of the enhancement of $B(E2)$ is the vibrational collectivity as expected, and the angular-momentum effect is also important for the enhancement. Therefore, from the present analyses, it appears promising that the main component of the observed band (4) is a three-phonon $\gamma$ vibration although some mixing with states that are not included in the present model would be possible.

To summarize, the single- and multi-phonon $\gamma$-vibrational states in $^{103}$ and $^{105}$Nb have been studied by using the particle-vibration coupling model based on the cranking model and the random-phase approximation. The calculations have been done in the model space including up to $4 \gamma$ states. This is an extension of the previous calculations including up to $2 \gamma$ states. By analyzing aligned angular momenta and wave functions, highest-lying ones among the calculated collective states have been shown to be the most collective and have the highest $K = \Omega + 2n$ for $n \gamma$ states at $\omega_{\text{rot}} = 0$. Then, $1 \gamma$ and $2 \gamma$ states have been directly identified with the observed ones and shown to reproduce the observed spectrum precisely. In the case of $3 \gamma$, the energy of the $K = \Omega + 6$ state is lowered by strong Coriolis force.

Since this alignment process reduces $K$, the most collective state is expected to be observed as a band with a lower $K$. Based on this scenario, the interband $B(E2)$s for $n \gamma \rightarrow (n - 1) \gamma$ with $n = 1, 2,$ and 3 have been calculated by adopting the method of the generalized intensity relation. Those with $n = 1$ and 2 have reproduced the observed ones well. The calculated $3 \gamma \rightarrow 2 \gamma$ transition rates have accounted for the observed enhancement of the transitions from the $3 \gamma$ candidates to $2 \gamma$ within factors of $2 - 3$, primarily by the vibrational collectivity and secondly by the angular-momentum effect. These analyses indicate that the main component of the observed band (4) is a three-phonon $\gamma$ vibration.

[1] T. Aumann, P. F. Bortignon, and H. Emling, Nucl. Part. Sci. 48, 351 (1998).
[2] A. Bohr and B. R. Mottelson, Nuclear Structure Vol. II (Benjamin, New York, 1975).
[3] F. Corminboeuf et al., Phys. Rev. Lett. 84, 4060 (2000).
[4] M. Matsuzaki, Phys. Rev. C83, 054320 (2011).
[5] H. J. Li et al., Phys. Rev. C87, 057303 (2013).
[6] J. C. Durand and R. Piepenbring, Phys. Rev. C54, 189 (1996).
[7] H. B. Ding et al., Phys. Rev. C74, 054301 (2006).
[8] J.-G. Wang et al., Phys. Lett. B675, 420 (2009).
[9] G. Long et al., Chin. Phys. Lett. 26, 092502 (2009).
[10] J. A. Sheikh, G. H. Bhat, Y. Sun, and R. Palit, Phys. Lett. B688, 305 (2010).
[11] D. R. Jensen et al., Phys. Rev. Lett. 89, 142503 (2002).
[12] G. Schönwaßer et al., Phys. Lett. B552, 9 (2003).
[13] M. Matsuzaki and S.-I. Ohtsubo, Phys. Rev. C69, 064317 (2004).
[14] Y. R. Shimizu, M. Matsuzaki, and K. Matsuyanagi, Phys. Rev. C72, 014306 (2005).
[15] H. J. Li et al., Phys. Rev. C88, 054311 (2013).
[16] M. Matsuzaki, Y. R. Shimizu, and K. Matsuyanagi, Prog. Theor. Phys. 77, 1302 (1987); ibid. 79, 836 (1988).
[17] M. Matsuzaki, Nucl. Phys. A491, 433 (1989); ibid. A519, 548 (1990).
[18] G. Gervais et al., Nucl. Phys. A624, 257 (1997).
[19] M. Matsuo and K. Matsuyanagi, Prog. Theor. Phys. 74, 1227 (1985); ibid. 76, 93 (1986); ibid., 591 78 (1987).
[20] Y. R. Shimizu and T. Nakatsukasa, Nucl. Phys. A611, 22 (1996).
[21] A. Guessous et al., Phys. Rev. Lett. 75, 2280 (1995).
[22] A. Guessous et al., Phys. Rev. C53, 1191 (1996).