A hierarchy of integrable PDEs in 2+1 dimensions associated with 2-dimensional vector fields

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Abstract

We study a hierarchy of integrable partial differential equations in 2+1 dimensions arising from the commutation of 2-dimensional vector fields, and we construct the formal solution of the associated Cauchy problems using the Inverse Scattering Transform for one-parameter families of vector fields recently introduced in [1, 2]. Due to the ring property of the space of eigenfunctions, the inverse problem can be formulated in three distinguished ways; in particular, one formulation corresponds to a linear integral equation for a Jost eigenfunction, and another formulation is a scalar nonlinear Riemann problem for suitable analytic eigenfunctions.

1 Introduction

In this paper we study a hierarchy of integrable partial differential equations (PDEs) in 2+1 dimensions arising from the commutation of 2-dimensional vector fields, and we construct the formal solution of the associated Cauchy problems using the Inverse Scattering Transform (IST) for vector fields recently introduced in [1, 2].

The first nontrivial member of such hierarchy is the equation

$$v_{xt} + v_{yy} = v_y v_{xx} - v_x v_{xy}, \quad v = v(x, y, t) \in \mathbb{R}, \quad x, y, t \in \mathbb{R},$$

introduced in [3] and studied in [3, 4, 5] from different points of view, which admits the operator representation

$$[\hat{L}, \hat{M}] = 0$$

(2)
in terms of the 2-dimensional vector fields [5]

\[ \hat{L} \equiv \partial_y + (p + v_x)\partial_x, \]
\[ \hat{M} \equiv \partial_t + (p^2 + pv_x - v_y)\partial_x, \]  

(3)

where \( p \) is a constant parameter.

Equation (1) is the \( u = 0 \) reduction of the recently introduced [1] integrable system:

\[ u_{xt} + u_{yy} = -(uu_x)_x - v_xu_{xy} + v_yu_{xx}, \quad u, v \in \mathbb{R}, \quad x, y, t \in \mathbb{R}, \]
\[ v_{xt} + v_{yy} = -uv_{xx} - v_xv_{xy} + v_yv_{xx} \]  

(4)

admitting the operator representation \([\hat{L}, \hat{M}] = 0\) in terms of the following 3-dimensional vector fields

\[ \tilde{L} \equiv \partial_y + (p + v_x)\partial_x - u_x\partial_p, \]
\[ \tilde{M} \equiv \partial_t + (p^2 + pv_x + u - v_y)\partial_x + (-pu_x + u_y)\partial_p. \]  

(5)

We remark that the \( v = 0 \) reduction of the system (4) is the celebrated dispersionless Kadomtsev-Petviashvili (dKP) (or Khokhlov-Zabolotskaya) equation:

\[ u_{tx} + u_{yy} + (uu_x)_x = 0, \quad u = u(x, y, t) \in \mathbb{R}, \quad x, y, t \in \mathbb{R}. \]  

(6)

arising in various problems of Mathematical Physics, and intensively studied in the recent literature (see, f.i., [6, 7, 8, 9, 10, 11]).

The inverse scattering problem for one - parameter families of multi-dimensional vector fields has been recently introduced in [1], and used to construct the formal solution of the Cauchy problem for a wide class of nonlinear PDEs in arbitrary dimensions [1], including the heavenly equation of Plebanski [12], and for the system (4) [2]. The Hamiltonian constraint on the associated spectral data leading to the heavenly equation and to the dKP reduction of (4) was also found in [1] and [2]. We remark that, in the case of Hamiltonian vector fields, an elegant and alternative integration scheme was already known in the literature [7].

We also remark that, since the hierarchy of PDEs investigated in this paper arises from the commutation of 2 - dimensional vector fields, according to the theory developed in [1, 2], the associated inverse problem can be formulated in terms of a scalar nonlinear Riemann problem.
2 Hierarchy of PDEs associated with $\hat{L}$

Equation (1) is the first nontrivial member (corresponding to $n = 2$) of the following hierarchy of PDEs in 2+1 dimensions arising from the commutation of 2-dimensional vector fields:

$$v_t + \hat{Q}^nv_x = 0, \quad n \in \mathbb{N},$$

(7)

where $\hat{Q}$ is the recursion operator

$$\hat{Q} \equiv \partial_x^{-1}(v_{xx} - \partial_y - v_x\partial_x) = v_x - \partial_x^{-1}\partial_y - 2\partial_x^{-1}v_x\partial_x.$$

(8)

The first few flows are:

$$v_{t0} + v_x = 0,$$
$$v_{t1} - v_y = 0,$$
$$v_{xt2} + v_{yy} - v_yv_{xx} + v_xv_{xy} = 0,$$
$$v_{xxt3} - (v_{y} + v_{x}^{2})_{yy} + [v_{xy}(v_{y} - v_{x}^{2}) + v_{xx}\partial_x^{-1}(v_{y} + v_{x}^{2})]_y = 0.$$

(9)

This hierarchy admits the operator representation

$$[\hat{L}, \hat{M}_n] = 0, \quad n \in \mathbb{N},$$

(10)

where $\hat{L}$ is defined in (3) and the 2-dimensional vector fields $\hat{M}_n$ are defined by

$$\hat{M}_n \equiv \partial_{n} + \left(p^n + \sum_{k=0}^{n-1} p^k A_{n-k}^{(n)}\right)\partial_x,$$

(11)

with:

$$A_{k+1}^{(n)} = \hat{Q}A_k^{(n)}, \quad 1 \leq k \leq n - 1, \quad A_1^{(n)} = v_x.$$

(12)

To show it, we replace (3a) and (11) into (10), obtaining the following polynomial (in $p$) equation:

$$v_{xt} + v_{xx} \left(p^n + \sum_{k=0}^{n-1} p^k A_{n-k}^{(n)}\right) = \sum_{k=0}^{n-1} p^k(A_{n-k}^{(n)})_y + (p + v_x)\sum_{k=0}^{n-1} p^k(A_{n-k}^{(n)})_x,$$

(13)

which must be satisfied $\forall p$. Equating to zero all powers of $p$, we obtain equations (12) and (7).
Since $\hat{Q}$ does not depend explicitly on $x$, its Lie derivative along $v_x$, the first vector field of the hierarchy, is zero; in addition, it is also possible to verify that $\hat{Q}$ is a Nijenhuis (hereditary) operator. Therefore all the flows (7) commute with each other [13, 14].

We remark that the algebraic theory for the 2-dimensional recursion operator $\hat{Q}$ is conceptually more similar to the theory for the 1-dimensional recursion operators associated with systems like the Korteweg-de Vries and nonlinear Schrödinger equations [13], rather than to the theory for the 2-dimensional recursion operators associated with systems like the Kadomtsev-Petviashvili and Davey Stewartson equations [15]. No results are known though, at the moment, on the Hamiltonian and bi-Hamiltonian [14] character of the hierarchy (7).

3 Inverse scattering transform

Now we consider the Cauchy problem for the hierarchy (7) and, in particular, for equation (1), within the class of rapidly decreasing real potentials $v$:

$$v(x, y, t_n) \to 0, \quad (x^2 + y^2) \to \infty, \quad v \in \mathbb{R}, \quad (x, y) \in \mathbb{R}^2, \quad t_n > 0,$$

interpreting $t_n$ as time and the other two variables $x, y$ as space variables. To solve such a Cauchy problem by the IST method developed in [1, 2], we construct the IST for the operator $\hat{L}$, within the class of rapidly decreasing real potentials $v$, interpreting the operators $\hat{M}_n$ as time operators.

The operator representation (10) implies the existence of common eigenfunctions $f(x, y, t, p)$ of $\hat{L}, \hat{M}_n$ (the Lax pair):

$$\hat{L}f = 0, \quad \hat{M}_n f = 0, \quad n \in \mathbb{N}.$$  (15)

Since the Lax pair (15) is made of vector fields, the space of eigenfunctions is a ring: if $f_1, f_2$ are two solutions of the Lax pair (15), then an arbitrary differentiable function $F(f_1, f_2)$ of them is a solution of (15).

3.1 Direct problem

The localization (14) of the potential $v$ implies that, if $f$ is a solution of $\hat{L}f = 0$, then

$$f(x, y, p) \to f_\pm(\xi, p), \quad y \to \pm \infty,$$

$$\xi := x - py;$$

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i.e., asymptotically, $f$ is an arbitrary function of $\xi = x - py$ and $p$.

A central role in the theory is played by the real Jost eigenfunction $\varphi(x, y, p)$, the solution of $\hat{L}\varphi = 0$ uniquely defined by the asymptotics

$$\varphi(x, y, p) \to \xi, \quad y \to -\infty,$$

and equivalently characterized by the linear integral equation $\varphi = \xi + \hat{G}(-v_x \varphi_x)$, for the Green’s function $G(x, y, p) = \theta(y)\delta(x - py)$.

The $y = +\infty$ limit of $\varphi$ defines the natural scattering datum $\sigma$ for $\hat{L}$:

$$\lim_{y \to +\infty} \varphi(x, y, p) \equiv S(\xi, p) = \xi + \sigma(\xi, p).$$

The direct problem is the transformation from the real potential $v$, function of the two real variables $(x, y)$, to the real scattering datum $\sigma$, function of the two real variables $(\xi, p)$. Therefore the mapping is consistent. In the small field limit, this mapping reduces to the Radon transform [16]:

$$\sigma(\xi, p) = - \int_{\mathbb{R}} v_\xi(\xi + py, y) dy.$$  \hfill (19)

A crucial role in the IST theory for the vector field $\hat{L}$ is also played by the analytic eigenfunctions $\psi_{\pm}(x, y, p)$, the solution of $\hat{L}\psi_{\pm} = 0$ satisfying the integral equations

$$\psi_{\pm}(x, y, p) = - \int_{\mathbb{R}^2} dx' dy' G_{\pm}(x - x', y - y', p)v_{x'}(x', y')\psi_{\pm}(x', y', p) + \xi,$$

where $G_{\pm}$ are the analytic Green’s functions

$$G_{\pm}(x, y, p) = \pm \frac{1}{2\pi i|x - (p \pm i\epsilon)y|}.$$  \hfill (21)

The analyticity properties of $G_{\pm}(x, y, p)$ in the complex $p$-plane imply that $\psi_{\pm}(x, y, p)$ are analytic, respectively, in the upper and lower halves of the $p$-plane, with the following asymptotics, for large $p$:

$$\psi_{\pm}(x, y, p) = \xi - \frac{v(x, y)}{p} + O\left(\frac{1}{p^2}\right), \quad |p| >> 1.$$  \hfill (22)

It is important to remark that the analytic Green’s functions (21) exhibit the following asymptotics for $y \to \pm\infty$:

$$G_{\pm}(x - x', y - y', p) \to \pm \frac{1}{2\pi i(\xi - \xi' \pm i\epsilon)}, \quad y \to +\infty,$$$$

$$G_{\pm}(x - x', y - y', p) \to \pm \frac{1}{2\pi i(\xi - \xi' \pm i\epsilon)}, \quad y \to -\infty,$$

\hfill (23)
entailing that the \( y = +\infty \) asymptotics of \( \psi_+ \) and \( \psi_- \) are analytic respectively in the lower and upper halves of the complex plane \( \xi \), while the \( y = -\infty \) asymptotics of \( \psi_+ \) and \( \psi_- \) are analytic respectively in the upper and lower halves of the complex plane \( \xi \) (similar features have been observed first in [17] and later in [1, 2]).

The Jost eigenfunction \( \varphi \) and the constant eigenfunction \( p \) form a basis in the ring of solutions of \( \hat{L}f = 0 \); thus any solution \( f \) of \( \hat{L}f = 0 \) is a function of \( \varphi \) and \( p \). The analytic eigenfunctions \( \psi_\pm \), in particular, possess the representations:

\[
\psi_\pm = \mathcal{K}_\pm(\varphi, p) = \varphi + \chi_\pm(\varphi, p),
\]

defining the spectral data \( \chi_\pm \).

Since the \( y \to -\infty \) limit of (24) reads:

\[
\lim_{y \to -\infty} \psi_\pm - \xi = \chi_\pm(\xi, p),
\]

the above analyticity properties of the LHS of (25) in the complex \( \xi \) - plane imply that \( \chi_+(\xi) \) and \( \chi_-(\xi) \) are analytic respectively in the upper and lower halves of the complex plane \( \xi \), decaying at \( \xi \sim \infty \) like \( O(\xi^{-1}) \). Therefore their Fourier transforms \( \tilde{\chi}_+(\omega) \) and \( \tilde{\chi}_-(\omega) \) have support respectively on the positive and negative \( \omega \) semi-axes.

The spectral data \( \chi_\pm \) can be constructed from the scattering datum \( \sigma \) through the following linear integral equations

\[
\begin{align*}
\tilde{\chi}_+(\omega, p) + \theta(\omega) &\left( \tilde{\sigma}(\omega, p) + \int_{\mathbb{R}} d\eta \, \tilde{\chi}_+(\eta, p) Q(\eta, \omega, p) \right) = 0, \\
\tilde{\chi}_-(\omega, p) + \theta(-\omega) &\left( \tilde{\sigma}(\omega, p) + \int_{\mathbb{R}} d\eta \, \tilde{\chi}_-(\eta, p) Q(\eta, \omega, p) \right) = 0,
\end{align*}
\]

involving the Fourier transforms \( \tilde{\sigma} \) and \( \tilde{\chi}_\pm \) of \( \sigma \) and \( \chi_\pm \):

\[
\tilde{\sigma}(\omega, p) = \int_{\mathbb{R}} d\xi \sigma(\xi, p)e^{-i\omega\xi}, \quad \tilde{\chi}_\pm(\omega, p) = \int_{\mathbb{R}} d\xi \chi_\pm(\xi, p)e^{-i\omega\xi}
\]

and the kernel:

\[
Q(\eta, \omega, p) = \int_{\mathbb{R}} \frac{d\xi}{2\pi} e^{i(\eta-\omega)\xi} [e^{i\eta\sigma(\xi, p)} - 1].
\]

To prove this result, one first evaluates (24) at \( y = +\infty \), obtaining

\[
\left( \lim_{y \to \infty} \psi_\pm - \xi \right) = \sigma(\xi, p) + \chi_\pm(\xi + \sigma(\xi, p), p).
\]
Applying the integral operator $\int_{\mathbb{R}} d\xi e^{-i\omega \xi}$, for $\omega > 0$ and $\omega < 0$ respectively to equations (29)$_+$ and (29)$_-$, using the above analyticity properties and the Fourier representations of $\chi_\pm$ and $\sigma$, one obtains equations (26).

The reality of the potential: $v \in \mathbb{R}$ implies that, for $p \in \mathbb{R}$, $\overline{\varphi} = \varphi$, $\overline{\psi}_+ = \psi_-$; consequently: $\overline{\sigma} = \sigma$, $\overline{\chi}_+ = \chi_-.$

3.2 Inverse problem(s)

As in [1, 2], due the ring property of the space of eigenfunctions, it is possible to construct three distinguished versions of the inverse problem, all based on equations (24).

The first version is obtained subtracting $\xi$ from equations (24)$_-$ and (24)$_+$, applying respectively the analyticity projectors $\hat{P}_+$ and $\hat{P}_-$:

$$\hat{P}_\pm \equiv \pm \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{dp'}{p' - (p \pm i\epsilon)}.$$

and adding up the resulting equations, to obtain the following nonlinear integral equation for the Jost eigenfunction $\varphi$:

$$\varphi(x, y, p) + \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{dp'}{p' - (p + i\epsilon)} \chi_-(\varphi(x, y, p'), p') - \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{dp'}{p' - (p - i\epsilon)} \chi_+(\varphi(x, y, p'), p') = x - py. \quad (31)$$

Once $\varphi$ is reconstructed from (31), given $\chi_{\pm}(\xi, p)$, the analytic eigenfunctions follow from (24), and the potential $v$ from equation (22). This inversion procedure was first introduced in [18] and also used in [1, 2].

The second version is the linear analogue of the nonlinear problem (31), obtained exponentiating the Jost and analytic eigenfunctions used so far. Consider the following functions:

$$\Phi(x, y, p; \alpha) \equiv e^{i\alpha \varphi(x, y, p)}, \quad \Psi_{\pm}(x, y, p; \alpha) \equiv e^{i\alpha \psi_{\pm}(x, y, p)}, \quad \alpha \in \mathbb{R}. \quad (32)$$

Due to the ring property of the space of eigenfunctions, also $\Phi(x, y, p; \alpha)$ and $\Psi_{\pm}(x, y, p; \alpha)$ are eigenfunctions; $\Phi(x, y, p; \alpha)$ is a Jost eigenfunction characterized by the asymptotics $\Phi \to \exp(i\alpha \xi)$, $y \to -\infty$, while $\Psi_{\pm}(x, y, p; \alpha)$ are analytic respectively in the upper and lower halves of the $p$ plane, with asymptotics: $\Psi_{\pm} = \exp(i\alpha \xi)[1 - (\alpha v)/p + O(p^{-2})]$.

Exponentiating the representations (24), one obtains the linear expansions of the analytic eigenfunctions $\Psi_{\pm}$ in terms of the Jost eigenfunction.
\[ \Phi: \]

\[ \Psi_\pm(x, y, p; \alpha) = \Phi(x, y, p; \alpha) + \int_\mathbb{R} d\beta K_\pm(\alpha, \beta, p) \Phi(x, y, p; \beta), \quad (33) \]

where the new spectral data \( K_\pm \) are defined in terms of \( \chi_\pm \) by:

\[ K_\pm(\alpha, \beta, p) \equiv \int_\mathbb{R} \frac{d\xi}{2\pi} e^{i(\alpha - \beta)\xi} \left[ e^{i\alpha \chi_\pm(\xi, p)} - 1 \right]. \quad (34) \]

Multiplying the equations (33)+ and (33)_− by \( \exp(-i\alpha \xi) \), subtracting 1, applying respectively \( \hat{P}_- \) and \( \hat{P}_+ \), and adding the resulting equations, one obtains the following linear integral equation for \( \Phi \):

\[ \Phi(x, y, p; \alpha) + \frac{1}{2\pi i} \int_\mathbb{R} \frac{dp'}{p' - (p + i\epsilon)} \int_\mathbb{R} d\beta K_-(\alpha, \beta, p') \Phi(x, y, p'; \beta) e^{i\alpha(p' - p)y} - \]

\[ - \frac{1}{2\pi i} \int_\mathbb{R} \frac{dp'}{p' - (p - i\epsilon)} \int_\mathbb{R} d\beta K_+(\alpha, \beta, p') \Phi(x, y, p'; \beta) e^{i\alpha(p' - p)y} = e^{i\alpha(x - py)}. \quad (35) \]

Once \( \Phi \) is reconstructed from (35) and, via (33), \( \Psi_\pm \) are also known, the potentials are reconstructed in the usual way from the asymptotics of \( \Psi_\pm \).

The reality constraint \((v = 0)\) implies, for \( \alpha, p \in \mathbb{R} \): \( \Phi(\alpha) = \Phi(-\alpha) \), \( \Psi_+(\alpha) = \Psi_-(\alpha) \). Consequently: \( K_+(\alpha, \beta) = K_-(\alpha, -\beta) \).

The third version of the inverse problem is a Riemann problem. Solving the algebraic system (24)_± with respect to \( \varphi: \varphi = \mathcal{L}(\psi_-, p) \) (assuming local invertibility) and replacing this expression in the algebraic system (24)_+, one obtains the representation of the analytic eigenfunction \( \psi_+ \) in terms of the analytic eigenfunction \( \psi_- \):

\[ \psi_+ = \mathcal{R}(\psi_-, p) = \psi_- + \mathcal{R}(\psi_-, p), \quad p \in \mathbb{R}, \quad (36) \]

where \( \mathcal{R}(\psi_-, p) = K_+(\mathcal{L}(\psi_-, p), p) \), which defines a scalar nonlinear Riemann problem on the real \( p \) axis. The Riemann datum \( \mathcal{R} \) is therefore constructed from the data \( K_\pm \) by algebraic manipulation. Vice-versa, given \( \mathcal{R} \), one constructs the solutions \( \psi_\pm \) of the nonlinear Riemann problem (36) and, via the asymptotics (22), the potential \( v \).

The reality constraint for \( \mathcal{R} \) takes the form: \( \mathcal{R}(\mathcal{R}(\xi, p), p) = \xi, \forall \xi \), for \( p \in \mathbb{R} \).

Remark. Dressing schemes can be formulated from the three different inverse problems presented in this paper in a straightforward way.
3.3 $t$ - evolution of the spectral data

As $v$ evolves in time according to (1), the $t$-dependence of the spectral data $\sigma, \chi, R$ and $K$ is described by the following explicit formulas:

\[
\begin{align*}
\sigma(\xi,p,t) &= \sigma(\xi - p^2 t, p, 0), \\
S(\xi,p,t) &= \xi + \sigma(\xi - p^2 t, p, 0) ,
\end{align*}
\]

\[
\begin{align*}
\chi_{\pm}(\xi,p,t) &= \chi_{\pm}(\xi - p^2 t, p, 0), \\
K_{\pm}(\xi,p,t) &= \xi + \chi_{\pm}(\xi - p^2 t, p, 0) ,
\end{align*}
\]

\[
\begin{align*}
R(\xi,p,t) &= R(\xi - p^2 t, p, 0), \\
R(\xi,p,t) &= R(\xi - p^2 t, p, 0) ,
\end{align*}
\]

\[
K_{\pm}(\alpha,\beta,p,t) = K_{\pm}(\alpha,\beta,p,0)e^{i(\alpha-\beta)p^2 t}.
\]

To prove it, we first observe that

\[
\phi(x,y,t,p) \equiv \varphi(x,y,t,p) - p^2 t
\]

is a common Jost eigenfunction of $\hat{L}$ and $\hat{M}$. The $y = +\infty$ limit of equation $\hat{M}\phi = 0$ yields $\sigma + p^2 \sigma\xi = 0$, whose solution is (37a). Analogously,

\[
\pi_{\pm}(x,y,t,p) \equiv \psi_{\pm}(x,y,t,p) - p^2 t
\]

are common analytic eigenfunctions of $\hat{L}$ and $\hat{M}$; therefore

\[
\pi_{\pm} = \tilde{K}_{\pm}(\phi,p), \quad \pi_{\pm} = \tilde{R}(\pi_{-},p),
\]

for some functions $\tilde{K}_{\pm}$ and $\tilde{R}$ depending on $x,y,t$ only through $\phi$ and $\pi_{-}$. Comparing at $t = 0$ these equations with equations (24) and (36), one expresses $\tilde{K}_{\pm}$ and $\tilde{R}$ in terms of $K_{\pm}$ and $R$, obtaining equations (37b,c). The $t$-evolution of $K_{\pm}$ is obtained replacing (37b) in (34).

Analogously, one can obtain the explicit $t$ - dependence of the spectral data as $v$ evolves according to the other equations of the hierarchy (7).

4 One-parameter families of commuting dynamical systems

It is well-known (see, f.i., [19]) that linear first order PDEs like (15),(3),(11) are intimately related to systems of ordinary differential equations describing their characteristic curves. For instance, the dynamical systems associated with the one-parameter family of vector fields $\hat{L}, \hat{M}$ in (3) are:

\[
\begin{align*}
\dot{L} : \quad & \frac{dx}{dy} = p + v_x(x,y), \\
\dot{M} : \quad & \frac{dx}{dt} = p^2 + pv_x(x,y) - v_y(x,y).
\end{align*}
\]
Therefore equation (1) characterizes the class of functions $v$ for which the two first order dynamical systems (41) commute $\forall p$.

There is also a deep connection between the above IST and the time $(y)$ - scattering theory for the commuting flows (41). Let $\phi(x, y, t, p)$ be the common eigenfunction of $\hat{L}$ and $\hat{M}$ defined in (38); then, solving the system $\omega = \phi(x, y, t, p)$ with respect to $x$ (assuming local invertibility), one obtains the following common solution of (41):

$$\omega = \varphi(x, y, t; p) - p^2 t \iff x = r(\omega, y, t; p) \sim py + p^2 t + \omega, \ y \sim -\infty. \quad (42)$$

The $y = +\infty$ limit of the solution $r(\omega, y, t; p)$:

$$x \sim py + p^2 t + \Omega(\omega, p), \ y \sim +\infty \quad (43)$$

defines the time $(y)$ - scattering datum $\Delta(\omega, p) = \Omega(\omega, p) - \omega$ of (41a), which is connected with the IST datum $S$ by inverting the system $\omega = S(x - py - p^2 t, p, 0)$ with respect to $x$:

$$\omega = S(x - py - p^2 t, p, 0) \iff x - py - p^2 t = \Omega(\omega, p). \quad (44)$$

As a byproduct of the IST of this paper one can reconstruct, from the scattering datum $\Delta(\omega, p)$ of the one-parameter family of dynamical systems (41a), the function $v$.

Similar considerations can be made for the dynamical systems associated with the whole hierarchy of one-parameter family of operators $\hat{L}, \hat{M}_n, n \in \mathbb{N}$.

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