The integrated density of states of the random graph Laplacian

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We analyse the density of states of the random graph Laplacian in the percolating regime. A symmetry argument and knowledge of the density of states in the nonpercolating regime allows us to isolate the density of states of the percolating cluster (DSPC) alone, thereby eliminating trivially localised states due to finite subgraphs. We derive a nonlinear integral equation for the integrated DSPC and solve it with a population dynamics algorithm. We discuss the possible existence of a mobility edge and give strong evidence for the existence of discrete eigenvalues in the whole range of the spectrum.

The eigenvalue spectrum of sparse random matrices is a fascinating and largely unsolved problem with widespread applications ranging from transport in disordered systems, graph theory and optimization problems to nuclear physics and QCD \cite{1,2}. In this paper we consider a prototype of such a random matrix, the Laplace operator on a mean-field random graph with \( N \) nodes, i.e., a graph where a link between two arbitrary sites is either present with probability \( p = 2c/N \) or not present with probability \( 1 - p \). The constant \( 2c \) is the mean connectivity of the graph. The Laplace operator on this graph is a matrix \( \Gamma_{ij} \) where for \( i \neq j \) \( \Gamma_{ij} = -1 \) if the nodes \( i \) and \( j \) of the graph are connected and \( \Gamma_{ij} = 0 \) otherwise, while on the diagonal \( \Gamma_{ii} = -\sum_{j \neq i} \Gamma_{ij} \). The entries on the diagonal are thus correlated to the random entries outside the diagonal.

Even though the computation of the density of states for a mean-field random graph has been reduced to an integral equation \cite{3,5}, a complete solution is still missing. In the limit of infinite coordination, \( c \to \infty \), Wigner’s semi-circle law is recovered. For any finite \( c \), Lifshitz tails were shown to exist in the integrated density of states \cite{6}. Beyond these asymptotic results a number of approximations have been used to compute the spectrum approximately, such as effective medium theory, single defect approximation, moment expansions and numerical diagonalisation \cite{7,4}.

In this paper we show first that the density of states of the percolating cluster (DSPC) can be isolated using a symmetry argument and our knowledge of the density of states below the percolation threshold. Second, we derive an integral equation for the integrated density of states which can be solved reliably with a population algorithm. The numerical solution reveals jumps in the integrated DSPC in the whole range of the spectrum, calling in question the existence of a mobility edge.

**Model and symmetry** The model shows a percolation transition at \( c_{\text{crit}} = 1/2 \). Below this concentration there is no macroscopic cluster and almost all finite clusters are trees. The average number of tree clusters \( T_n \) with \( n \) nodes is given in the macroscopic limit by \cite{10}

\[
\lim_{N \to \infty} \frac{T_n(2c)}{N} = \tau_n(2c) = \frac{n^{n-2}(2c e^{-2c})^n}{2c n!}.
\]

In particular the total number of clusters per particle is \( \tau_0(2c) = 1 - c \). For \( c < 1/2 \), the spectrum consists of a very complicated, but countable set of \( \delta \)-peaks which can be calculated iteratively \cite{11}. Above the percolation threshold \( c > 1/2 \) a percolating cluster coexists with many finite clusters, which are also trees. The fraction of sites in the macroscopic cluster, \( Q(c) \), is the solution of \( 1 - Q(c) = \exp(-2c Q(c)) \). Alternatively we rewrite \( Q(c) = 1 - \frac{1 - c}{2c} \) and obtain \( x(c) \) as the solution of

\[
x(c)e^{-x(c)} = 2ce^{-2c}.
\]

This equation has two solutions, a trivial one with \( x(c) = 2c \) and a nontrivial one with \( x(c) = 2c^* \) such that \( c^* > \frac{1}{2} \) if \( c < \frac{1}{2} \) and vica versa. This nontrivial solution allows one to establish a symmetry for the number of trees above and below the percolation threshold. Using Eq. \( (2) \), we can rewrite Eq. \( (1) \) according to

\[
\tau_n(x(c)) = \frac{2c}{x(c)} \tau_n(2c) \quad \text{or} \quad \tau_n(2c^*) = \frac{c}{c^*} \tau_n(2c).
\]

Hence the number of trees above the percolation threshold is simply related to the number of trees below the percolation threshold.

**Density of states** This relation allows us to compute the density of states of the percolating cluster alone, which is the quantity of primary interest. It is known that the density of states, \( D(\Omega) \), of the infinite cluster contains at least some \( \delta \)-peaks, so that a population dynamics algorithm \cite{9} cannot be applied. In this paper we instead compute the integrated density of states, \( \Delta(\Omega) = \int_0^\Omega d\Omega' D(\Omega') \), which according to measure theory \cite{12}, may be decomposed into a singular part and an absolutely continuous part. The absolutely continuous part may itself consist of two contributions, eigenvalues stemming from localised eigenvectors which happen to lie
dense and each have vanishing weight in the thermodynamic limit, and the eigenvalues stemming from nonlocalised eigenvectors. We may thus write

\[ \Delta(\Omega) = \Delta_{\text{loc,disc}}(\Omega) + \Delta_{\text{loc,cont}}(\Omega) + \Delta_{\text{nonloc}}(\Omega). \]  

(4)

Often in random matrix problems there is a mobility edge, i.e. a value \( \Omega_0 \) such that all eigenvectors corresponding to eigenvalues \( \Omega < \Omega_0 \) are localised and all eigenvectors corresponding to \( \Omega > \Omega_0 \) are nonlocalised (or vice versa). We will show here that such a sharp edge does not seem to exist in our problem as we find discrete eigenvalues even in the region where nonlocalised eigenvectors lie. This shows that a naive approach using the inverse participation ratio will not drop down to 0 as it would at a true mobility edge.

The density of eigenvalues, \( \{\Omega_k\}_{k=1}^N \), of the Laplacian matrix \( \Gamma \) is defined by

\[ D(\Omega, c) = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^N \delta(\Omega - \Omega_i) = \lim_{N \to \infty} \frac{1}{N} \text{Tr} \delta(\Omega - \Gamma) \]  

(5)

Here \( \text{Tr} \) denotes the average over all realizations of connectivity for a given \( c \). To compute the density of eigenvalues we introduce the resolvent

\[ G(\Omega, c) = \lim_{N \to \infty} \frac{1}{N} \text{Tr} \frac{1}{\Gamma - \Omega} \]  

(6)

\[ g_{c,\Omega}(\rho) = 2c \exp\left\{ -\frac{i\rho^2}{2} \right\} + 2ie^{-2c} \int_{0}^{\infty} dx \rho I_1(i\rho x) \exp\left\{ -\frac{i}{2} \rho^2 + x^2 \right\} + \frac{i\Omega}{2} x^2 + g_{c,\Omega}(x) \]  

(9)

with \( g_{c,\Omega}(0) = 2c \). Here \( I_n(z) \) are the modified Bessel functions of the first kind. The solution of Eq. (9) yields the resolvent \[ 3 \]

\[ G(\Omega, c) = -1 + \frac{i}{2c} \int_{0}^{\infty} d\rho \rho g_{c,\Omega}(\rho) \]  

(10)

\[ \textbf{Nonpercolating regime}: \] The analytical solution of Eq. (9) for \( c < 1/2 \) was given in \[ 11 \]. Briefly, it is

\[ g_{c,\Omega}(\rho) = 2c \sum_k a_k \exp\left\{ -\frac{i}{2} z_k \rho^2 \right\} = 2c \int_{-\infty}^{\infty} d\lambda \rho a_{c,\Omega}(\lambda) e^{-\frac{i}{2} \lambda \rho^2} \]  

(11)

with coefficients \( a_k \) and \( z_k \) to be defined below. The sum can also be expressed as a Riemann-Stieltjes integral with weight function \( a_{c,\Omega}(\lambda) = \sum_k a_k \delta(\lambda - z_k) \) (\( \delta(\cdot) \) is the Heaviside function). This formulation will be useful later. for complex argument \( \Omega = \gamma + i\epsilon, \epsilon > 0 \). In the limit \( \epsilon \to 0 \), we recover the spectrum from the imaginary part of the resolvent according to

\[ D(\Omega, c) = \frac{1}{\pi} \lim_{\epsilon \downarrow 0} \Im \text{Tr} G(\Omega + i\epsilon, c). \]  

(7)

In the percolating regime \( c \geq \frac{1}{2} \) the macroscopic cluster coexists with many finite ones. In order to study localised states of the macroscopic cluster it is essential to isolate the density of states of the macroscopic cluster only. We use Eqs. (2) and (3) to decompose the resolvent into two contributions, one from the percolating cluster and one from the finite clusters:

\[ G(\Omega, c) = G_{\text{perc}}(\Omega, c) + \frac{c^*}{c} G(\Omega, c^*) \]  

(8)

The above relation allows us to compute the density of states of the percolating cluster alone from the full density of states above, \( G(\Omega, c) \), and the density of states below the percolation threshold, \( G(\Omega, c^*) \) for \( c^* < \frac{1}{2} \), which is known \[ 11 \].

The average over all realizations of connectivity is performed with the replica trick, resulting in a nonlinear integral equation for \( g_{c,\Omega}(\rho) \) (cf. Eqs. (16) and (17) in Ref. \[ 3 \])

\[ a_k^{(n+1)} = e^{-2c} \prod_k \left( \frac{2ca_k^{(n)}}{k!} \right) \]  

(12)

\[ z_k^{(n+1)} = \frac{\Omega - \sum_k k z_k^{(n)}}{\Omega - 1 - \sum_k k z_k^{(n)}} \]  

(13)

The coefficients \( a_k \) and \( z_k \) can be grouped in infinitely many “classes.” The coefficients of class \( n+1 \) are given recursively by the relations

(12)

(13)

(12)

(13)
iteration. Such a mapping is possible because the set of sequences \( \{(l_k)\} \) is countably infinite. It was shown in [11] that the correct way to do the mapping is to choose \( m = \sum_k l_k M^k \) and to let \( M \) (formally) tend to infinity at an appropriate point. Note that class \( n + 1 \) also contains all coefficients from class \( n \). See [11] for details.

These classes give us an infinite hierarchy of coefficients, each recursion step adding infinitely many coefficients to the previous ones. Note that the coefficients in class \( n \) constructed in this way are not an approximation but constitute (part of) the exact solution of Eq. (9).

Only the total weight of coefficients \( \sum_k a_k \) falls short of 1 when stopping the recursion at a finite \( n \). This solution of Eq. (9) leads to a density of states consisting of \( \delta \) peaks which are located at those \( \Omega \) where \( z_k = 0 \).

**Percolating regime:** In complete analogy to the re-

\[
\Omega \frac{\delta}{\theta} = b_{c,\Omega} = 2c^* \sum_{n=0}^{\infty} a_n \sum_{M=1}^{\infty} \frac{(2c - 2c^*)^{M-1}}{M!} \int_{-\infty}^{\infty} db_{c,\Omega}(\lambda_1) \cdots \int_{-\infty}^{\infty} db_{c,\Omega}(\lambda_M) = \int_{-\infty}^{\infty} db_{c,\Omega}(\lambda) \times \theta \left( \lambda - \left[ 1 - \frac{1}{1 - \frac{1}{1 - z_n} + \sum_{i=1}^{M} \lambda_i} \right] \right). \quad (15)
\]

This equation can be solved numerically by running a population dynamics algorithm for the coefficients \( z_k \) at \( c^* \) below the critical point in parallel to a population dynamics for a \( \lambda \)-population at \( c \) above the critical point,

\[
2\Delta_{\text{perc}}^{\Omega, c} = \left( 1 + \int_{-\infty}^{\infty} db_{c,\Omega}(\lambda) \left( \text{sgn} \left( \frac{\lambda}{\lambda - 1} \right) + c \text{sgn}(\lambda - 1) \right) \right) - (c - c^*) \int_{-\infty}^{\infty} db_{c,\Omega}(\lambda)db_{c,\Omega}(\lambda')\text{sgn} \left( \frac{\lambda}{\lambda - 1} - \lambda' \right) + c^* \int da_{c^*,\Omega}(\lambda) \text{sgn} \left( \frac{1}{\lambda - 1} \right) - c^* \int (da_{c,\Omega}(\lambda)db_{c,\Omega}(\lambda') + db_{c,\Omega}(\lambda)da_{c^*,\Omega}(\lambda')) \text{sgn} \left( \frac{\lambda}{\lambda - 1} - \lambda' \right). \quad (16)
\]

**Discussion:** We have developed a systematic approach to compute the integrated DSPC. We stress that the (nonintegrated) DSPC could not be reliably obtained from a population dynamics algorithm. Naively it is given in terms of \( b_{c,\Omega}(\lambda) \) by

\[
D_{\text{perc}}^{\Omega, c} = \frac{1}{\pi} \int_0^\infty \int_{-\infty}^{\infty} db_{c,\Omega}(\lambda) e^{-2\lambda \rho^2}dn \int_{-\infty}^{\infty} db_{c,\Omega}(\lambda) \delta(\lambda) \quad (17)
\]

The above density of states would be very simple if \( b_{c,\Omega}(\lambda) \) was differentiable with respect to \( \lambda \), as it would then be equal to \( (b_{c,\Omega})'(0) \). It is however known that the density of states of the percolating cluster contains \( \delta \) peaks also for \( c > 1/2 \). Hence the position and weight of the \( \delta \) peaks can not reliably be obtained from a population dynamics algorithm – a problem already encountered in the nonpercolating regime. It is thus essential to obtain the integrated density of states directly from population dynamics without going to the density of states first and integrating, and this is what Eqs. (15) and (16) provide.

The final equation for the integrated DSPC, Eq. (16), reveals much about the eigenvalues of the percolating cluster. We can, for example, track down the origin of the \( \delta \) peaks in the density of states of the percolating cluster. Suppose for the moment that \( b_{c,\Omega}(\lambda) \) is continuous as a function of \( \Omega \). Then the integrals over \( db_{c,\Omega}(\lambda) \) certainly do not generate any jumps in \( \Delta_{\text{perc}}^{\Omega, c} \) due to the continuity in \( \Omega \). However, we know that \( a_{c^*,\Omega}(\lambda) = \sum_n a_n \theta(\lambda - z_n) \) is discontinuous as a function of \( \Omega \) since the \( z_n \) depend on \( \Omega \), and in-

\[
\int_{-\infty}^{\infty} db_{c,\Omega}(\lambda) e^{-2\lambda \rho^2}dn \int_{-\infty}^{\infty} db_{c,\Omega}(\lambda) \delta(\lambda) \quad (17)
\]
Figure 1: The integrated density of states at $c = \frac{1}{2}$ and $c = 2$. The bottom figure shows an enlargement of the curve for $c = 2$ around $\Omega = 1$.

spectation of Eq. [16] shows that this leads to a jump in $\Delta_{\text{perc}}(\Omega)$ if the location $z_n$ of a jump moves from $-\infty$ to $+\infty$ when $\Omega$ is increased infinitesimally (it does not lead to a jump if $z_n$ moves across 0 or 1, as could be suspected at first sight, since the various contributions cancel in these cases). Eq. [13] shows that $z_n$ can and does indeed pass $\infty$ as $\Omega$ is varied. While the peaks of the density of states of the finite clusters are located at those $\Omega$ for which $z_n = 0$, the peaks for the percolating cluster are located where $z_n = \infty$. Since the zeros and the poles of $z_n$ necessarily alternate when $\Omega$ is varied, it follows that the peaks in the percolating cluster lie dense if the peaks in the finite clusters lie dense. In this sense there is no mobility edge in the percolating cluster since isolated and thus localized eigenvalues exist throughout the whole range of $0 \leq \Omega < \infty$.

Unfortunately, this argument only strictly holds if $b_{c,\Omega}(\lambda)$ is indeed continuous in $\Omega$. If it is not, cancellations might occur which could reduce (or even remove) the peaks from the percolating cluster. It is shown in [13] that indeed $b_{c,\Omega}(\lambda)$ is not continuous but the argument presented there also shows that a complete removal of peaks would seem an extremely fortuitous cancellation. In order to check these results we have performed population dynamics simulations of the integrated DSPC according to Eq. [15]. Fig. 1 shows the integrated DSPC directly at the critical point $c = \frac{1}{2}$ and deep inside the percolating regime at $c = 2$. Some jumps are clearly observed for $c = \frac{1}{2}$, and they are located at the predicted positions. The most prominent ones occur at $\Omega = 1$ where the coefficient $z = \frac{\Omega}{\Omega - 1} = \infty$, or at $\Omega = \frac{3 \pm \sqrt{5}}{2}$ where $z = \frac{\Omega - \Omega}{\Omega - 1} = \infty$. For $c = 2$ the integrated density of states in Fig. 1 looks smooth. This is, however, not the case as the close-up in the bottom part of Fig. 1 reveals. The jump at $\Omega = 1$ which is very pronounced at $c = \frac{1}{2}$ is also present at $c = 2$.

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