PARABOLIC PRINCIPAL HIGGS BUNDLES

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ABSTRACT. In [2], with Balaji and Nagaraj we introduced the ramified principal bundles. The aim here is to introduce the Higgs bundles in the ramified context.

1. Introduction

Parabolic vector bundles were introduced in [12]. The corresponding notion for principal bundles was introduced in [1] and [2]. In [1], the parabolic principal bundles were defined using the Tannakian category theory. We recall that Nori showed that principal bundles have a natural description in Tannakian category theory. The definition of a parabolic principal bundle given in [1] is modeled on that. In [2] it was shown that the parabolic principal bundles have a rather concrete description. More precisely, parabolic principal bundles were identified with what are called in [2] as ramified bundles.

Let $G$ be a complex linear algebraic group. For a principal $G$–bundle $\psi : E_G \to X$, the group $G$ acts freely transitively on each fiber of $\psi$. For a ramified $G$–bundle $E_G$ over $X$, the group $G$ acts transitively on each fiber. However, the action of $G$ on some fibers needs not be free. In other words, points on some fibers may have nontrivial isotropies; the details are in [2], [3], [4]. It was shown that there is a natural bijective correspondence between that parabolic principal $G$–bundle and the ramified $G$–bundles.

Here we define Higgs fields on a ramified $G$–bundle. For a principal $G$–bundle $E_G$ over $X$, we recall that a Higgs field on $E_G$ is a section

$$\theta \in H^0(X, \text{ad}(E_G) \otimes \Omega_X^1),$$

where $\text{ad}(E_G)$ is the adjoint vector bundle of $E_G$ and $\Omega^1_X$ is the cotangent bundle, such that $\theta \wedge \theta = 0$. We also recall that the adjoint bundle $\text{ad}(E_G)$ is the one associated to $E_G$ for the adjoint action of $G$ on its Lie algebra $\mathfrak{g}$.

The definition of the adjoint vector bundle of a principal bundle extends to the context of ramified $G$–bundles. More precisely, for a parabolic $G$–bundle $E_G$ over $X$, its adjoint bundle $\text{ad}(E_G)$ is the parabolic vector bundle over $X$ associated to $E_G$ for the $G$–module $\mathfrak{g}$. However, a Higgs field on a ramified $G$–bundle $E_G$ is not quite a section of $\text{ad}(E_G) \otimes \Omega_X^1$. Rather it is a section of a bigger sheaf over $X$ containing $\text{ad}(E_G) \otimes \Omega_X^1$, which is called $\mathcal{A}_{E_G}$. Outside the parabolic divisor, $\mathcal{A}_{E_G}$ is identified with $\text{ad}(E_G) \otimes \Omega_X^1$.

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2. Higgs bundle with parabolic structure

2.1. Preliminaries. Let $X$ be an irreducible smooth projective variety of dimension $d$ defined over $\mathbb{C}$. Let $D \subset X$ be a simple normal crossing hypersurface. This means that each irreducible component of $D$ is smooth and they intersect transversally. Let $\text{Pvect}(X)$ denote the category of parabolic vector bundles over $X$ with parabolic structure over $D$ and rational parabolic weights (see [4, Section 2] for more details).

Let $G$ be a complex linear algebraic group. Let $\text{Rep}(G)$ denote the category of all finite dimensional rational left representations of $G$. A parabolic $G$–bundle over $X$ with $D$ as the parabolic divisor is a functor from $\text{Rep}(G)$ to $\text{Pvect}(X)$ that is compatible with the operations of taking direct sum, tensor product and dual. (See [1], [3, Section 3] and [4, Section 2]; this approach is based on [9].)

We will now recall the definition of a ramified $G$–bundle over $X$ with ramification over $D$.

A ramified $G$–bundle over $X$ with ramification over $D$ is a smooth complex variety $E_G$ equipped with a right algebraic action of $G$

$$f : E_G \times G \longrightarrow E_G$$

and a surjective algebraic map

$$\psi : E_G \longrightarrow X,$$

such that the following five conditions hold:

- $\psi \circ f = \psi \circ p_1$, where $p_1$ is the projection of $E_G \times G$ to $E_G$,
- for each point $x \in X$, the action of $G$ on the reduced fiber $\psi^{-1}(x)_{\text{red}}$ is transitive,
- the restriction of $\psi$ to $\psi^{-1}(X \setminus D)$ makes $\psi^{-1}(X \setminus D)$ a principal $G$–bundle over $X \setminus D$, meaning the map $\psi$ is smooth over $\psi^{-1}(X \setminus D)$ and the map to the fiber product

$$\psi^{-1}(X \setminus D) \times G \longrightarrow \psi^{-1}(X \setminus D) \times_{X \setminus D} \psi^{-1}(X \setminus D)$$

defined by $(z, g) \mapsto (z, f(z, g))$ is an isomorphism,
- for each irreducible component $D_i \subset D$, the reduced inverse image $\psi^{-1}(D_i)_{\text{red}}$ is a smooth divisor and

$$\widehat{D} := \sum_{i=1}^{\ell} \psi^{-1}(D_i)_{\text{red}}$$

is a normal crossing divisor on $E_G$, and
- for any smooth point $z \in \widehat{D}$, the isotropy group $G_z \subset G$, for the action of $G$ on $E_G$, is a finite cyclic group that acts faithfully on the quotient line $T_zE_G/T_z\psi^{-1}(D)_{\text{red}}$.

There is a natural bijective correspondence between parabolic $G$–bundles and ramified $G$–bundles. (See [2, 4].) We will interchange without any further explanation the two terminologies: ramified $G$–bundle and parabolic $G$–bundle.
2.2. Definition of a parabolic principal Higgs bundle. Let
\[ (2.1) \quad \psi : E_G \rightarrow X \]
be a ramified \( G \)-bundle over \( X \) with ramification over \( D \). The Lie algebra of \( G \) will be denoted by \( \mathfrak{g} \). Let
\[ (2.2) \quad \mathcal{K} \subset T E_G \]
be the subbundle defined by the orbits of the action of \( G \) on \( E_G \). The action of \( G \) on \( E_G \) identifies \( \mathcal{K} \) with the trivial vector bundle over \( E_G \) with fiber \( \mathfrak{g} \). Let
\[ (2.3) \quad \eta : E_G \times \mathfrak{g} \rightarrow \mathcal{K} \]
be the isomorphism of \( \mathcal{K} \) with the trivial vector bundle \( E_G \times \mathfrak{g} \). This homomorphism takes the Lie algebra structure of \( \mathfrak{g} \) to the Lie bracket of globally defined vector fields.

Let \( \mathcal{Q} \) denote the quotient vector bundle \( TE_G/\mathcal{K} \). So we have a short exact sequence of vector bundles
\[ (2.4) \quad 0 \rightarrow \mathcal{K} \rightarrow TE_G \xrightarrow{\theta} \mathcal{Q} \rightarrow 0 \]
over \( E_G \). The action of \( G \) on \( E_G \) induces an action of \( G \) on the tangent bundle \( TE_G \). This action of \( G \) on \( TE_G \) clearly preserves the subbundle \( \mathcal{K} \). It may be mentioned that the isomorphism \( \eta \) in Eq. (2.3) intertwines the action of \( G \) on \( \mathcal{K} \) and the diagonal action of \( G \) constructed using the adjoint action of \( G \) on \( \mathfrak{g} \). Therefore, we have an induced action of \( G \) on the quotient bundle \( \mathcal{Q} \).

Let \( \theta_0 \in H^0(E_G, \mathcal{Q}) \) be an algebraic section. We note that the actions of \( G \) on \( \mathcal{K} \) and \( \mathcal{Q} \) together define an action of \( G \) on the complex vector space \( H^0(E_G, \mathcal{Q}) \).

Combining the exterior algebra structure of \( \bigwedge \mathcal{Q}^* \) and the Lie algebra structure on the fibers of the vector bundle \( \mathcal{K} = E_G \times \mathfrak{g} \) (see Eq. (2.3)), we have a homomorphism
\[ (2.6) \quad \tau : (\mathcal{K} \bigotimes \mathcal{Q}^*) \bigotimes (\mathcal{K} \bigotimes \mathcal{Q}^*) \rightarrow \mathcal{K} \bigotimes (\bigwedge^2 \mathcal{Q}^*) . \]
So \( \tau((A_1 \bigotimes \omega_1) \bigotimes (A_2 \bigotimes \omega_2)) = [A_1, A_2] \bigotimes (\omega_1 \bigwedge \omega_2) \). We will denote \( \tau(a, b) \) also by \( a \bigwedge b \).

**Definition 2.1.** A Higgs field on a parabolic \( G \)-bundle \( E_G \) is a section
\[ \theta_0 \in H^0(E_G, \mathcal{K} \bigotimes \mathcal{Q}^*) \]
as in Eq. (2.5) satisfying the following two conditions:

1. the action of \( G \) on \( H^0(E_G, \mathcal{K} \bigotimes \mathcal{Q}^*) \) leaves \( \theta_0 \) invariant, and
2. \( \theta_0 \bigwedge \theta_0 = 0 \) (see Eq. (2.6)).
Definition 2.2. A parabolic Higgs $G$–bundle is a pair $(E_G, \theta_0)$, where $E_G$ is a parabolic $G$–bundle, and $\theta_0$ is a Higgs field on $E_G$.

Let
\begin{equation}
\mathcal{A}_{E_G} := (\psi_* (\mathcal{K} \bigotimes Q^*))^G
\end{equation}
be the invariant direct image, where $\psi$ is the projection in Eq. (2.1). Therefore,
\begin{equation}
H^0(X, \mathcal{A}_{E_G}) = H^0(E_G, \mathcal{K} \bigotimes Q^*)^G.
\end{equation}

For $i \geq 0$, let
\begin{equation}
\tilde{\mathcal{K}}_i := (\psi_* (\mathcal{K} \bigotimes (\bigwedge^i Q^*)))^G
\end{equation}
be the invariant direct image. So, $\tilde{\mathcal{K}}_1 = \mathcal{A}_{E_G}$. The homomorphism $\tau$ in Eq. (2.6) yields a homomorphism
\begin{equation}
\tilde{\tau} : \tilde{\mathcal{K}}_1 \bigotimes \tilde{\mathcal{K}}_1 \rightarrow \tilde{\mathcal{K}}_2.
\end{equation}

The following lemma is an immediate consequence of the above constructions.

Lemma 2.3. A Higgs field on $E_G$ is a section
\[ \theta \in H^0(X, \mathcal{A}_{E_G}) \]
such that
\[ \tilde{\tau}(\theta, \theta) = 0, \]
where $\tilde{\tau}$ is constructed in Eq. (2.10).

2.3. The adjoint vector bundle. We noted earlier that there is a natural bijective correspondence between parabolic $G$–bundles and ramified $G$–bundles (see \cite{2}, \cite{4}). Let $E^P_G$ denote the parabolic $G$–bundle corresponding to a ramified $G$–bundle $E_G$. We also recall that $E^P_G$ associates a parabolic vector bundle over $X$ to each object in $\text{Rep}(G)$. Let
\begin{equation}
ad(E_G) := E^P_G(\mathfrak{g})
\end{equation}
be the parabolic vector bundle over $X$ associated to the parabolic $G$–bundle $E^P_G$ for the adjoint action of $G$ on its Lie algebra $\mathfrak{g}$. This parabolic vector bundle $ad(E_G)$ will be called the adjoint vector bundle of $E_G$. The vector bundle underlying the parabolic vector bundle $ad(E_G)$ will also be denoted by $ad(E_G)$. From the context it will be clear which one is being referred to.

Consider the vector bundle $\mathcal{K} \rightarrow E_G$ constructed in Eq. (2.2). We noted that $\mathcal{K}$ is equipped with a natural action of $G$. It is straight forward to check that the invariant direct image $(\psi_* \mathcal{K})^G$ is identified with the vector bundle underlying the parabolic vector bundle $ad(E_G)$ constructed in Eq. (2.11). Indeed, this follows from the fact that for a usual principal bundle, its adjoint vector bundle coincides with the invariant direct image of the relative tangent bundle. Therefore, we have
\begin{equation}
ad(E_G) = (\psi_* \mathcal{K})^G.
\end{equation}
There is a natural $\mathcal{O}_X$–linear homomorphism
\begin{equation}
\text{ad}(E_G) \otimes \Omega^1_X \longrightarrow A_{E_G},
\end{equation}
where $A_{E_G}$ is constructed in Eq. (2.7). This homomorphism is an isomorphism over the complement $X \setminus D$. To prove these, note that
\[\Omega^1_{X \setminus D} = \Omega^1_X|_{X \setminus D} = (\psi_*\mathcal{Q}^*)^G|_{X \setminus D},\]
where $\mathcal{Q}$ is the vector bundle in Eq. (2.4). The isomorphism $\Omega^1_{X \setminus D} \longrightarrow (\psi_*\mathcal{Q}^*)^G|_{X \setminus D}$ extends to a $\mathcal{O}_X$–linear homomorphism
\begin{equation}
\Omega^1_X \longrightarrow (\psi_*\mathcal{Q}^*)^G
\end{equation}
over $X$. The homomorphism in Eq. (2.13) is not an isomorphism over $X$ in general.

3. Semistable parabolic principal Higgs bundles

3.1. Reduction of structure group. Let
\begin{equation}
\psi : E_G \longrightarrow X
\end{equation}
be a ramified principal $G$–bundle with ramification over $D$. Let
\[H \subset G\]
be a Zariski closed subgroup. Let
\[U \subset X\]
be a Zariski open subset. The inverse image $\psi^{-1}(U)$, where $\psi$ is the projection in Eq. (3.1), will also be denoted by $E_G|_U$.

Definition 3.1. A reduction of structure group of $E_G$ to $H$ over $U$ is a subvariety
\[E_H \subset E_G|_U\]
satisfying the following three conditions:
\begin{itemize}
  \item the action of $H$ on $E_G$ preserves $E_H$,
  \item for each point $x \in U$, the action of $H$ on $\psi^{-1}(x) \cap E_H$ (see Eq. (3.1)) is transitive, and
  \item for each point $z \in E_H$, the isotropy subgroup of $z$, for the action of $G$ on $E_G$, is contained in $H$.
\end{itemize}

For a point $z \in E_G$, let
\[\Gamma_z \subset G\]
be the isotropy subgroup of $z$ for the action of $G$ on $E_G$. It is easy to see that for an element $g \in G$,
\[\Gamma_{zg} = g^{-1}\Gamma_zg.\]
Therefore, for any \( z \in \psi^{-1}(U) \), if \( \Gamma_z \subset H \), then it follows that
\[
\Gamma_z g \subset H
\]
for all \( g \in H \). Consequently, the last one of the three conditions in Definition 3.1 holds if for each point \( x \in U \), there exists some point \( z \in \psi^{-1}(x) \cap E_H \) such that \( \Gamma_z \subset H \).

If
\[
E_H \subset E_G|U
\]
is a reduction of structure group of \( E_G \) to \( H \), then clearly \( E_H \) is a ramified principal \( H \)-bundle over \( U \).

**Remark 3.2.** Consider the quotient map
\[
q_H : E_G \longrightarrow E_G/H.
\]
The projection \( \psi \) in Eq. (3.1) defines a projection
\[
\psi_H : E_G/H \longrightarrow X.
\]
A subvariety
\[
E_H \subset E_G|U
\]
satisfying the first two of the three conditions in Definition 3.1 is given by a section
\[
\sigma : U \longrightarrow E_G/H
\]
of the projection \( \psi_H \). In other words, \( \sigma \) is an algebraic morphism and \( \psi_H \circ \sigma = \text{Id}_U \). Indeed, the inverse image
\[
q_H^{-1}(\sigma(U)) \subset E_G
\]
for any section \( \sigma \) satisfies the first two conditions in Definition 3.1. We note that such a section \( \sigma \) satisfies the third condition in Definition 3.1 if and only if for each point \( x \in U \), there is a point \( z \in q_H^{-1}(\sigma(x)) \) such that the isotropy subgroup \( \Gamma_z \) of \( z \) is contained in \( H \). We also note that if there is one point \( z \in q_H^{-1}(\sigma(x)) \) such that the isotropy subgroup \( \Gamma_z \) is contained in \( H \), then the isotropy subgroup of each point of \( q_H^{-1}(\sigma(x)) \) is actually contained in \( H \). \( \square \)

As before, let \( \psi : E_G \longrightarrow X \) be a ramified \( G \)-bundle. Let
\[
\iota : E_H \hookrightarrow E_G|U
\]
be a reduction of structure group of \( E_G \) to \( H \) over a Zariski open subset \( U \subset X \). Let
\[
K_H \subset TE_H
\]
be the subbundle defined by the orbits of the action of \( H \) on \( E_H \) (see also Eq. (2.2)). Note that \( K_H \) is a subbundle of the pull back \( \iota^*K \) (see Eq. (2.2)), where \( \iota \) is the inclusion map in Eq. (3.2). More precisely,
\[
K_H = TE_H \bigcap \iota^*K.
\]
Let \( \mathfrak{h} \) denote the Lie algebra of \( H \). So \( \mathfrak{h} \) is a submodule of the \( H \)-module \( \mathfrak{g} \). The action of \( H \) on the Lie algebra \( \mathfrak{g} \) of \( G \) is the restriction of the adjoint action. The inclusion of the \( H \)-module \( \mathfrak{h} \) in \( \mathfrak{g} \) evidently induces an inclusion

\[
(3.5) \quad \text{ad}(E_H) \subset \text{ad}(E_G)|_U
\]

of parabolic vector bundles (see Eq. (2.11)).

The inclusion of the underlying vector bundles in Eq. (3.5) also follows from the inclusion of \( \mathcal{K}_H \) in \( \iota^*\mathcal{K} \) (see Eq. (3.4) and Eq. (2.12)).

The quotient bundle \( (\text{ad}(E_G)|_U)/\text{ad}(E_H) \) in Eq. (3.5) has an induced parabolic structure.

On the other hand, consider the ramified principal \( H \)-bundle \( E_H \) over \( U \). Let \( E'_H(\mathfrak{g}/\mathfrak{h}) \) be the parabolic vector bundle over \( U \) associated to \( E_H \) for the \( H \)-module \( \mathfrak{g}/\mathfrak{h} \). The quotient parabolic vector bundle \( (\text{ad}(E_G)|_U)/\text{ad}(E_H) \) (see Eq. (3.5)) is canonically identified with this parabolic vector bundle \( E'_H(\mathfrak{g}/\mathfrak{h}) \). So, we have

\[
(3.6) \quad (\text{ad}(E_G)|_U)/\text{ad}(E_H) = E'_H(\mathfrak{g}/\mathfrak{h}).
\]

Let

\[
Q_H := TE_H/\mathcal{K}_H
\]

be the quotient bundle (see Eq. (3.3)). From Eq. (3.4) it follows that \( Q_H \) is identified with \( \iota^*Q \), where \( Q \) is the quotient bundle in Eq. (2.4). Therefore, we have a commutative diagram

\[
(3.7)
\begin{array}{ccccccccc}
0 & \rightarrow & \mathcal{K}_H & \rightarrow & TE_H & \rightarrow & Q_H & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \| & & \\
0 & \rightarrow & \iota^*\mathcal{K} & \rightarrow & \iota^*TE_G & \rightarrow & \iota^*Q & \rightarrow & 0
\end{array}
\]

where the bottom exact sequence is pull back of the one in Eq. (2.4) and all the vertical homomorphisms are injective.

Consequently, we have

\[
(3.8) \quad \text{Hom}(Q_H, \mathcal{K}_H) \subset \iota^*\text{Hom}(Q, \mathcal{K}).
\]

Let

\[
\theta \in H^0(E_G, \text{Hom}(Q, \mathcal{K}))
\]

be a Higgs field of \( E_G \) (see Definition 2.1).

**Definition 3.3.** The reduction \( E_H \) in Eq. (3.2) is said to be *compatible* with the Higgs field \( \theta \) if

\[
\theta|_{E_H} \in H^0(E_H, \text{Hom}(Q_H, \mathcal{K}_H)) \subset H^0(E_H, \iota^*\text{Hom}(Q, \mathcal{K}))
\]

(see Eq. (3.8)).
Let 
\[ \hat{\psi} : E_H \longrightarrow U \]
be the natural projection. So, we have \( \hat{\psi} = \psi \circ \iota \), where \( \iota \) is the inclusion map in Eq. (3.2) and \( \psi \) is the projection of \( E_G \) to \( X \). Let

\[ \mathcal{A}_{E_H} := \left( \hat{\psi}^* \text{Hom}(Q_H, K_H) \right)^H \]
be the invariant direct image, where \( Q_H \) and \( K_H \) are as in Eq. (3.7) (see also Eq. (2.7)). From Eq. (3.8) it follows that

\[ \mathcal{A}_{E_H} \subset \mathcal{A}_{E_G}|_U, \]
where \( \mathcal{A}_{E_G} \) is constructed in Eq. (2.7).

A Higgs field on \( E_G \) is an algebraic section of \( \mathcal{A}_{E_G} \) (see Lemma 2.3). It is now easy to see that the reduction \( E_H \) in Eq. (3.2) is compatible with a Higgs field \( \theta \in H^0(X, \mathcal{A}_{E_G}) \) on \( E_G \) if and only if

\[ \theta|_U \in H^0(U, \mathcal{A}_{E_H}) \subset H^0(U, \mathcal{A}_{E_G}) \]
(see Eq. (3.10)).

3.2. Semistable parabolic principal Higgs bundles. Fix a very ample line bundle \( \xi \) over \( X \). The degree of a torsionfree coherent sheaf \( F \) on \( X \) is defined to be the degree of the restriction of \( F \) to a smooth complete intersection curve in \( X \) obtained by intersecting hyperplanes on \( X \) from the complete linear system \( |\xi| \). The parabolic degree of parabolic vector bundle over \( X \) is also defined using \( \xi \) (see [8, p. 81, Definition 1.8(2)] for the details). The parabolic degree of a parabolic vector bundle \( V \) over \( X \) will be denoted by par-deg\((V)\).

In the rest of this section, \( G \) will be a connected reductive linear algebraic group defined over \( \mathbb{C} \).

Let \((E_G, \theta)\) be a parabolic Higgs \( G \)-bundle over \( X \). Consider triples of the form \((H, U, E_H)\), where

- \( H \subset G \) is a maximal proper parabolic subgroup,
- \( U \subset X \) is a nonempty Zariski open subset such that the codimension of the complement \( X \setminus U \) is at least two, and
- \( E_H \subset E_G \) is a reduction of structure group of \( E_G \) to \( H \) compatible with \( \theta \) (see Definition 3.3).

**Definition 3.4.** A parabolic Higgs \( G \)-bundle \((E_G, \theta)\) over \( X \) is called stable (respectively, semistable) if for all triples \((H, U, E_H)\) of the above type, the inequality

\[ \text{par-deg}(\text{ad}(E_G)|_U)/\text{ad}(E_H)) > 0 \]
(respectively, \( \text{par-deg}(\text{ad}(E_G)|_U)/\text{ad}(E_H)) \geq 0 \) holds (see Eq. (3.6)).
Note that since the codimension of the complement of the open subset $U$ is at least two, the degree of a vector bundle over $U$ is well defined.

There is an alternative formulation of the above definition of (semi)stability which we will now explain.

Let $P$ be a parabolic subgroup of the reductive group $G$. Therefore, $G/P$ is a complete variety. The quotient map $G \longrightarrow G/P$ defines a principal $P$–bundle over $G/P$. For any character $\lambda$ of $P$, let

$$L_\lambda = E_P(\lambda) \longrightarrow G/P$$

be the line bundle associated to this principal $P$–bundle for the character $\lambda$.

Let $Z_0(G) \subset G$ be the connected component of the center of $G$ containing the identity element. It is known that $Z_0(G) \subset P$. A character $\lambda$ of $P$ which is trivial on $Z_0(G)$ is called strictly antidominant if the corresponding line bundle $L_\lambda$ over $G/P$ is ample.

Let $(E_G, \theta)$ be a parabolic Higgs $G$–bundle over $X$. Consider quadruples of the form $(H, \lambda, U, E_H)$, where

- $H \subset G$ is a proper parabolic subgroup,
- $\lambda$ is a strictly antidominant character of $H$,
- $U \subset X$ is a nonempty Zariski open subset such that the codimension of the complement $X \setminus U$ is at least two, and
- $E_H \subset E_G$ is a reduction of structure group of $E_G$ to $H$ compatible with $\theta$ (see Definition 3.3).

The character $\lambda$ defines an one-dimensional representation of $H$. Recall that $E_H$ is a parabolic $H$–bundle over $U$. Let $E_H(\lambda)$ denote the parabolic line bundle over $U$ associated to the parabolic $H$–bundle $E_H$ for the $H$–module defined by $\lambda$.

The parabolic Higgs $G$–bundle $(E_G, \theta)$ is stable (respectively, semistable) if and only if for every quadruple $(H, \lambda, U, E_H)$ of the above type,

$$\text{par-deg}(E_H(\lambda)) > 0$$

(respectively, $\text{par-deg}(E_H(\lambda)) \geq 0$).

The above assertion follows by imitating the proof of Lemma 2.1 in [10, pp. 131–132].

Let $E_G$ be a parabolic $G$–bundle over $X$. A reduction of structure group

$$E_H \subset E_G$$

to some parabolic subgroup $H \subset G$ over $X$ is called admissible if for each character $\lambda$ of $H$ trivial on $Z_0(G)$, the associated parabolic line bundle $E_H(\lambda)$ over $X$ satisfies the following condition:

$$\begin{align*}
\text{(3.11) \hspace{1cm} par-deg}(E_H(\lambda)) & = 0
\end{align*}$$

(see [11, p. 307, Definition 3.3] and [6, pp. 3998–3999] for admissible reductions of a principal bundle).
A parabolic Higgs $G$–bundle $(E_G, \theta)$ over $X$ is called \textit{polystable} if either $(E_G, \theta)$ is stable, or there is a proper parabolic subgroup $H$ and a reduction of structure group 

$$E_{L(H)} \subset E_G$$

to a Levi subgroup $L(H)$ of $H$ over $X$ such that the following three conditions hold:

- the reduction $E_{L(H)} \subset E_G$ is compatible with $\theta$,
- the parabolic Higgs $L(H)$–bundle $(E_{L(H)}, \theta)$ is stable (from the first condition it follows that $\theta$ is also a Higgs field on $E_{L(H)}$), and
- the reduction of structure group of $E_G$ to $H$, obtained by extending the structure group of $E_{L(H)}$ using the inclusion of $L(H)$ in $H$, is admissible.

4. Characteristic classes and connections

4.1. \textbf{Equivariant Higgs $G$–bundles}. Let $Y$ be a complex variety and $\Gamma$ a finite group acting on $Y$ through algebraic automorphisms. So we have a homomorphism

$$h : \Gamma \longrightarrow \text{Aut}(Y),$$

where $\text{Aut}(Y)$ is the group of all automorphisms of the variety $Y$.

Let $G$ be a complex linear algebraic group. A $\Gamma$–linearized principal $G$–bundle over $Y$ is a principal $G$–bundle

$$\phi : F_G \longrightarrow Y$$

and an action of $\Gamma$ on the left of $F_G$

$$\rho : \Gamma \times F_G \longrightarrow F_G$$

such that the following two conditions hold:

- the actions of $\Gamma$ and $G$ on $F_G$ commute, and
- $\phi \circ \rho(\gamma, z) = h(\gamma)(\phi(z))$ for all $(\gamma, z) \in \Gamma \times F_G$, where $h$ is the homomorphism in Eq. (4.1) and $\phi$ is the projection in Eq. (4.2).

Let $\psi : E_G \longrightarrow X$ be a parabolic principal Higgs $G$–bundle. There is a finite Galois covering

$$\varphi : Y \longrightarrow X$$

and a $\Gamma$–linearized principal $G$–bundle $F_G$ over $Y$, where $\Gamma := \text{Gal}(\varphi)$ is the Galois group, such that

$$E_G = \Gamma \backslash F_G$$

(see [4, Section 3]).

Let $\text{ad}(F_G)$ denote the adjoint vector bundle of $F_G$. So $\text{ad}(F_G)$ is the vector bundle over $Y$ associated to $F_G$ for the adjoint action of $G$ on $\mathfrak{g}$. We recall that an Higgs field on $F_G$ is a section

$$\theta \in H^0(Y, \text{ad}(F_G) \boxtimes \Omega_Y^1)$$

such that $\theta \wedge \theta = 0$. 


The actions of $\Gamma$ on $Y$ and $F_G$ together induce an action of $\Gamma$ on the vector bundle $\text{ad}(F_G) \otimes \Omega_Y^1$. A Higgs field $\theta$ on $F_G$ is called $\Gamma$–invariant if the action of $\Gamma$ on $\text{ad}(F_G) \otimes \Omega_Y^1$ leaves the section $\theta$ invariant.

**Proposition 4.1.** There is a natural isomorphism between the Higgs fields on $E_G$ and the $\Gamma$–invariant Higgs fields on $F_G$.

*Proof.* Let

$$\phi : F_G \longrightarrow Y$$

be the natural projection. Using the action of $G$ on $F_G$, the kernel of the differential

$$d\phi : TF_G \longrightarrow \phi^*TY$$

gets identified with the trivial vector bundle $F_G \times \mathfrak{g}$ over $F_G$ with fiber $\mathfrak{g}$. Therefore,

$$(4.5) \quad H^0(Y, \text{ad}(F_G) \otimes \Omega_Y^1)^\Gamma = H^0(F_G, \text{kernel}(d\phi) \otimes \phi^*\Omega_Y^1)^G.$$  

Taking the $\Gamma$–invariants of both sides of Eq. (4.5) we get a $\mathbb{C}$–linear isomorphism

$$H^0(Y, \text{ad}(F_G) \otimes \Omega_Y^1)^\Gamma \xrightarrow{\sim} H^0(X, A_{E_G}),$$

where $A_{E_G}$ is constructed in Eq. (2.7). It is easy to see that the above isomorphism takes the $\Gamma$–invariant Higgs fields on $F_G$ bijectively to the Higgs fields on $E_G$. \hfill \Box

Let $(E_G, \theta)$ be a parabolic Higgs $G$–bundle over $X$. Let $(F_G, \tilde{\theta})$ be the corresponding $\Gamma$–linearized principal Higgs $G$–bundle over $Y$ (see Proposition 4.1). Fix a polarization $\xi$ on $X$, and also fix the polarization $\varphi^*\xi$ on $Y$, where $\varphi$ is the projection in Eq. (4.3).

**Lemma 4.2.** The following three statements are equivalent:

1. The parabolic Higgs $G$–bundle $(E_G, \theta)$ is semistable.
2. The $\Gamma$–linearized principal Higgs $G$–bundle $(F_G, \tilde{\theta})$ is $\Gamma$–semistable.
3. The principal Higgs $G$–bundle $(F_G, \tilde{\theta})$ is semistable.

*Proof.* From the second part of Proposition 2.4 of [5, p. 26] it follows that the last two statements are equivalent. That the first two statements are equivalent is obvious. \hfill \Box

### 4.2. Characteristic classes.

The adjoint action of $G$ on the Lie algebra $\mathfrak{g}$ gives an action of $G$ on $\mathfrak{g}^*$. This in turn defines an action on the symmetric product $\text{Sym}^n(\mathfrak{g}^*)$ for all $n$.

Fix any invariant

$$(4.6) \quad \beta \in \text{Sym}^n(\mathfrak{g}^*)^G.$$  

For any principal $G$–bundle $F_G$ over $Y$, the invariant element $\beta$ defines a characteristic class

$$(4.7) \quad c_\beta(F_G) \in H^{2n}(Y, \mathbb{C}).$$

(See [7, pp. 113–115] for the details of the construction of $c_\beta(F_G)$.)
Now let $Y$ be as in Eq. (4.3). Let $F_G$ be a $\Gamma$–linearized principal $G$–bundle on $Y$, where $\Gamma = \text{Gal}(\varphi)$. Therefore, the principal $G$–bundle $h(\gamma)^* F_G$ is isomorphic to $F_G$ for all $\gamma \in \Gamma$, where $h$ is the homomorphism in Eq. (4.1). This implies that
\[(4.8)\quad h(\gamma)^* c_\beta(F_G) = c_\beta(F_G)\]
for all $\gamma \in \Gamma$. Consequently, there is a unique cohomology class
\[(4.9)\quad \tilde{c} \in H^{2n}(Y, \mathbb{C})\]
such that $\varphi^* \tilde{c} = c_\beta(F_G)$, where $X$ is the quotient $Y/\Gamma$ (see Eq. (4.3)).

Let $E_G := \Gamma \backslash F_G$ be the corresponding parabolic $G$–bundle (see Eq. (4.4)). The cohomology class $\tilde{c}$ in Eq. (4.9) will be called the characteristic class of $E_G$ for $\beta$. The characteristic class of the parabolic $G$–bundle $E_G$ for $\beta$ will be denoted by $c_\beta(E_G)$. So
\[(4.10)\quad c_\beta(E_G) \in H^{2n}(X, \mathbb{C}).\]
The integer $n$ in Eq. (4.6) will be called the degree of the characteristic class $c_\beta(E_G)$.

4.3. Connections and Higgs bundles. Henceforth, $G$ will be a connected reductive linear algebraic group defined over $\mathbb{C}$.

In [4], we defined holomorphic connections on a parabolic principal $G$–bundle. A holomorphic connection on a parabolic principal $G$–bundle is called flat if its curvature vanishes.

**Theorem 4.3.** There is a canonical bijective correspondence between the flat parabolic principal $G$–bundles over $X$ and the semistable parabolic Higgs $G$–bundles $(E_G, \theta)$ over $X$ such that all the characteristic classes of $E_G$ of degree one and degree two vanish.

**Proof.** Let $E_G$ be a ramified $G$–bundle over $X$. As it was noted in Section 4.1, there is a finite Galois covering
\[\varphi : Y \longrightarrow X\]
and a $\Gamma$–linearized principal $G$–bundle $F_G$ over $Y$, where $\Gamma := \text{Gal}(\varphi)$, such that
\[E_G = \Gamma \backslash F_G\]
(see Eq. (4.3) and Eq. (4.4)).

Higgs fields on $E_G$ are the $\Gamma$–invariant Higgs fields on $F_G$ (see Proposition 4.1). Also, holomorphic connections on the parabolic $G$–bundle $E_G$ are the $\Gamma$–invariant holomorphic connections on $F_G$ (see [3, p. 269, Proposition 3.4] and [4, Theorem 4.4]).

If $\theta$ is a Higgs field on $E_G$ and $\tilde{\theta}$ the corresponding $\Gamma$–invariant Higgs fields on $F_G$, then $(E_G, \theta)$ is semistable if and only if $(F_G, \tilde{\theta})$ is $\Gamma$–semistable (see Lemma 4.2). Also, all the characteristic classes of $E_G$ of degree one and degree two vanish if and only if all the characteristic classes of $F_G$ of degree one and degree two vanish.

Therefore, the theorem follows from [5, p. 20, Theorem 1.1]. Note that since $Y$ is a smooth complex projective variety, $\Gamma$–linearized pseudostable Higgs $G$–bundles on $Y$ are the $\Gamma$–linearized semistable Higgs $G$–bundles on $Y$ (see [5, p. 26, Proposition 2.4]). □
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