Detailed characterization of unconditional convergence and invertibility of multipliers

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Abstract

In this paper we investigate the possibility of unconditional convergence and invertibility of multipliers $M_{\psi, \Phi}$ depending on the properties of the sequences $\Psi, \Phi$ and $\psi$. We characterize a complete set of conditions for the invertibility and the unconditional convergence of multipliers, and collect those results in tables. We either prove that unconditional convergence and invertibility is not possible, that one or both of these conditions are always the case for the given parameters, or we give examples for the feasible combinations. We give a full list of examples for all conditions.

Keywords: multiplier, invertibility, unconditional convergence, Riesz basis, frame, Bessel sequence, non-Bessel sequence

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1 Introduction

In [16], R. Schatten provided a detailed study of ideals of compact operators using their singular decomposition. He investigated the operators of the form $\sum_k \lambda_k \varphi_k \otimes \psi_k$ where $(\varphi_k)$ and $(\psi_k)$ are orthonormal families. Later such operators were investigated for Gabor frames [10]. In [3] abstract Bessel and frame sequences were used to define Bessel and frame multipliers, where several basic properties of frame multipliers were investigated. Recently this concept was extended to $p$-frames in Banach spaces [14] and generalized frames [15].

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While multipliers are interesting from a theoretical point of view [5, 4, 9] there are also interesting for applications, in particular in the fields of audio and acoustics. Time-invariant filters are often used for audio applications. These systems can be described by the multiplication of the frequency spectrum of the signal by a fixed function. Using the Fourier transform to calculate the spectrum, such an operator can be called a Fourier multiplier [11, 6]. As a particular way to implement time-frequency filters or time-variant filtering, Gabor (frame) multipliers [10] can be used. In signal processing they are used under the name ‘Gabor filters’ [13].

In the present table we investigate the unconditional convergence and the invertibility of multipliers $M_{m,\Phi,\Psi}$. In [17] the focus was on existence results and formulas for the inversion. Here we give a complete set of examples varying the type of the sequences $\Phi = (\phi_n)$, $\Psi = (\psi_n)$ (non-Bessel, Bessel non-frames, frames non-Riesz bases, Riesz bases; norm-semi-normalized, non-norm-semi-normalized with all possible combinations) and varying the symbol $m$ (semi-normalized, $\in L^{\infty}$ but non-semi-normalized, $\notin L^{\infty}$). We list all possible combinations. We give a full classification, if multipliers under those conditions can be (‘POSSIBLE’), have to be (‘ALWAYS’) or never can be (‘NOT POSSIBLE’) unconditionally convergent and invertible (resp. non-invertible) on the given Hilbert space. Please note that examples for multipliers that are identical to the identity give examples for those cases, where sequences can be dual to each other. We only consider sequences with non-zero elements, as in this case, for example, the invertible identity operator and the zero operator can be described as multiplier, if zeros are put at appropriate places (see [17] for details).

The paper is organized as follows. In Section 2 we fix the notation in this paper, summarize known and prove some new results needed for the rest of the paper. In Section 3 we determine if unconditional convergence and invertibility is always, sometimes or never possible for the complete set of possibilities for the sequences $\Phi = (\phi_n)$ and $\Psi = (\psi_n)$: non-Bessel, Bessel non-frames, frames non-Riesz bases, Riesz bases, combined with all the possibilities for norm-boundedness; and varying the symbol $m$ to be semi-normalized, bounded above or bounded below or non-bounded. These results are collected in tables, which are linked to Section 4. There we give concrete examples for all the combinations in the tables.

Please note that in the electronic version of this paper, the examples and the tables are connected with hyperlinks.

## 2 Preliminaries and Notations

Throughout the whole paper, $\mathcal{H}$ denotes an (infinite-dimensional) Hilbert spaces. The range of an operator $G$ is denoted by $\mathcal{R}(G)$. The identity operator on $\mathcal{H}$ is denoted by $I_{\mathcal{H}}$. The operator $G : \mathcal{H} \to \mathcal{H}$ is called invertible on $\mathcal{H}$ if there exists bounded operator $G^{-1} : \mathcal{H} \to \mathcal{H}$ such that $GG^{-1} = G^{-1}G = I_{\mathcal{H}}$.

The notation $\Phi$ (resp. $\Psi$) is used to denote the sequence $(\phi_n)$ (resp. $(\psi_n)$) with elements from $\mathcal{H}$; $m$ denotes a complex scalar sequence $(m_n)$ and $\overline{m}$ denotes the sequence of the complex conjugates of $m_n$. The sequence $(e_n)$ denotes an orthonormal basis of $\mathcal{H}$. The index set of sequences will be often omitted, in such cases the set $\mathbb{N} = \{1, 2, 3, \ldots \}$ is assumed implicitly.

Recall that $\Phi$ is called a Bessel sequence (in short, Bessel) for $\mathcal{H}$ with bound $B_\Phi$ if $B_\Phi > 0$ and $\sum |\langle h, \phi_n \rangle|^2 \leq B_\Phi \|h\|^2$ for every $h \in \mathcal{H}$. A Bessel sequence $\Phi$ with bound $B_\Phi$ is called a frame for $\mathcal{H}$ with bounds $A_\Phi, B_\Phi$, if $A_\Phi > 0$ and $A_\Phi \|h\|^2 \leq \sum |\langle h, \phi_n \rangle|^2 \leq B_\Phi \sum |c_n|^2$, $\forall (c_n) \in \ell^2$. Every Riesz basis for $\mathcal{H}$ with bounds $A, B$ is a frame for $\mathcal{H}$ with bounds $A, B$. For standard references for frame theory and related topics see [7, 8, 12].

For a given Bessel sequence $\Phi$, the mapping $U_\Phi : \mathcal{H} \to \ell^2$ given by $U_\Phi f = (\langle f, \phi_n \rangle)$ is called the analysis operator for $\Phi$ and the mapping $T_\Phi$ given by $T_\Phi(c_n) = \sum c_n \phi_n$ is called the synthesis operator for $\Phi$.

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1 For future work such sequences including zeros could be linked to the excess of frames [1, 2].

2 All results will be valid for finite-dimensional spaces also, but some classifications might not be useful, as e.g. every finite sequence in a finite-dimensional vector space is a frame on its span.
Given sequences $m$, $\Phi$, and $\Psi$, the operator $M_{m,\phi,\psi}$ given by

$$M_{m,\phi,\psi}f = \sum_{n=1}^{\infty} m_n(f, \psi_n) \phi_n, \quad f \in \mathcal{H},$$

is called multiplier. The multiplier $M_{m,\phi,\psi}$ is called unconditionally convergent on $\mathcal{H}$ (resp. surjective) if $\sum m_n(f, \psi_n) \phi_n$ converges unconditionally for every $f \in \mathcal{H}$ (resp. $R(M_{m,\phi,\psi}) = \mathcal{H}$). The sequence $m$ is called semi-normalized if $0 < \inf_n |m_n| \leq \sup_n |m_n| < \infty$. The sequence $\Phi$ is called norm-bounded below (resp. norm-bounded above) if $\inf_n \|\phi_n\| > 0$ (resp. $\sup_n \|\phi_n\| < \infty$) and $\Phi$ is called norm-semi-normalized if $0 < \inf_n \|\phi_n\| \leq \sup_n \|\phi_n\| < \infty$.

To shorten the file we use the following abbreviations: R.b. - Riesz basis, fr. - frame, B. - Bessel sequence, $SN$ - semi-normalized, $NBB$ - norm-bounded below, $NBA$ - norm-bounded above, un. conv. - unconditionally convergent on $\mathcal{H}$, INV. - invertible on $\mathcal{H}$. Recall that a Riesz basis is always $\|\cdot\|_{SN}$ and a Bessel sequence is always $NBA$. That is why the tables do not include the impossible cases “non-$NBA$ Bessel sequence” and “non-$\|\cdot\|_{SN}$ Riesz basis”.

### 2.1 Concerning Well-definedness and Unconditional Convergence

First note that the sequence $m\Phi$ is Bessel, frame, Riesz basis or satisfies the lower frame condition for $\mathcal{H}$, if and only if $m\Phi$ is Bessel, frame, Riesz basis or satisfies the lower frame condition for $\mathcal{H}$, respectively [17]. Furthermore, if $m$ is $SN$, then the sequence $\Psi$ is Bessel (resp. frame, Riesz basis) for $\mathcal{H}$ if and only if $m\Phi$ is Bessel (resp. frame, Riesz basis) for $\mathcal{H}$.

We will often use the following result:

**Proposition 2.1.** [3, Theorem 6.1 (1)] Let $\Phi$ and $\Psi$ be Bessel sequences for $\mathcal{H}$. If $m \in \ell^\infty$, then $M_{m,\phi,\psi}$ is well defined from $\mathcal{H}$ into $\mathcal{H}$, bounded and unconditionally convergent on $\mathcal{H}$.

To shorten notation, for $\nu = (\nu_n), \Theta = (\theta_n), \Xi = (\xi_n)$, we will write $M_{m,\phi,\psi} \equiv M_{\nu,\Xi,\Theta}$ if there exist scalar sequences $(c_n), (d_n)$ so that $\xi_n = c_n \phi_n, \theta_n = d_n \psi_n, \text{ and } m_n = \nu_n c_n d_n$ for every $n$. This means that in the series the summands are the same element-wise. We need this equality for conclusions of unconditional convergence. If $M_{\nu,\Xi,\Theta}$ is unconditionally convergent on $\mathcal{H}$ and $M_{m,\phi,\psi} \equiv M_{\nu,\Xi,\Theta}$, then $M_{m,\phi,\psi}$ is clearly also unconditionally convergent on $\mathcal{H}$. Note that if $M_{\nu,\Xi,\Theta}$ is unconditionally convergent on $\mathcal{H}$ and $M_{m,\phi,\psi} = M_{\nu,\Xi,\Theta}$, then $M_{m,\phi,\psi}$ might not be unconditionally convergent on $\mathcal{H}$. Consider for example the sequences $\Phi$ and $\Psi$ in Remark 4.1.1(a) - the multiplier $M_{m,\phi,\psi}$ is equal to the identity operator and thus it can be written as $M_{(1),(c_n),(d_n)}$, which is unconditionally convergent on $\mathcal{H}$; however, $M_{m,\phi,\psi}$ is not unconditionally convergent on $\mathcal{H}$.

**Proposition 2.2.** [17, Prop. 3.3 and Lemma 3.5] Let $M_{m,\phi,\psi}$ or $M_{m,\phi,\psi}$ be unconditionally convergent on $\mathcal{H}$.

(i) If $\Phi$ (resp. $m\Phi$) is $NBB$, then $m\Psi$ (resp. $\Psi$) is Bessel for $\mathcal{H}$.

(ii) If both $\Phi$ and $\Psi$ are $NBB$, then $m \in \ell^\infty$.

**Proposition 2.3.** [17, Prop. 3.9] Let $\Phi$ be a Riesz basis and let $M_{m,\phi,\psi}$ (resp. $M_{m,\phi,\psi}$) be well defined on $\mathcal{H}$. Then the following holds.

(i) The sequence $m\Psi$ is Bessel for $\mathcal{H}$.

(ii) If $\Psi$ is $NBB$, then $m \in \ell^\infty$. 
2.2 Concerning invertibility

Proposition 2.4. [17, Theorem 4.3] Let $M_{m, \Phi, \Psi}$ be invertible on $\mathcal{H}$. If $\Psi$ and $\Phi$ are Bessel sequences for $\mathcal{H}$ and $m \in \ell^\infty$, then $\Psi$ and $\Phi$ are frames for $\mathcal{H}$.

Proposition 2.5. [3, Prop. 7.7] Let $\Phi$ and $\Psi$ be Riesz bases for $\mathcal{H}$. If $m$ is SN, then $M_{m, \Phi, \Psi}$ is invertible on $\mathcal{H}$.

Proposition 2.6. [17, Corollary 4.12] Let $\Phi$ be a Riesz basis for $\mathcal{H}$. Then $M_{m, \Phi, \Psi}$ (resp. $M_{m, \Psi, \Phi}$) is invertible on $\mathcal{H}$ if and only if $m\Psi$ is a Riesz basis for $\mathcal{H}$. This may happen only in the following cases:

- $\Psi$ is a Riesz basis for $\mathcal{H}$ and $m$ is SN;
- $\Psi$ is non-NBB and Bessel for $\mathcal{H}$, which is not a frame for $\mathcal{H}$, and $m$ is NBB, but not in $\ell^\infty$;
- $\Psi$ is non-NBA, NBB, and non-Bessel for $\mathcal{H}$, $m$ is non-NBB and $m \in \ell^\infty$;
- $\Psi$ is non-NBA, non-NBB, and non-Bessel for $\mathcal{H}$, $m$ is non-NBB and $m \notin \ell^\infty$.

Proposition 2.7. Let $\Phi$ be a Riesz basis for $\mathcal{H}$, $\Psi$ be an overcomplete frame for $\mathcal{H}$, and $m$ be SN. Then

(a) $M_{m, \Phi, \Psi}$ is injective, but not surjective.

(b) $M_{m, \Psi, \Phi}$ is surjective, but not injective.

Proof: First note that $m\Psi$ and $\Psi \Phi$ are also overcomplete frames for $\mathcal{H}$.

(a) Let $M_{n, \Phi, \Psi} f = 0$ for some $f \in \mathcal{H}$. Since $\Phi$ is a Riesz basis for $\mathcal{H}$, it follows that $(f, \Phi_n) = 0$ for all $n$, see [8, Theorem 6.1.1]. Since $\Phi \Phi$ is complete in $\mathcal{H}$, we obtain that $f = 0$. Therefore, $M_{m, \Phi, \Psi}$ is injective. It is proved in [17, Theorem 4.11] that $M_{m, \Phi, \Psi}$ is not surjective.

(b) Since $\Phi$ is a Riesz basis for $\mathcal{H}$, $U_\Phi$ is a bijection of $\mathcal{H}$ onto $\ell^2$. Since $m\Psi$ is a frame for $\mathcal{H}$, $T_{m\Psi}$ is surjective from $\ell^2$ onto $\mathcal{H}$. Therefore, $M_{m, \Psi, \Phi} = T_{m\Psi} U_\Phi$ is surjective. It is proved in [17, Theorem 4.11] that $M_{m, \Psi, \Phi}$ is not injective. □

Proposition 2.8. Let $\Phi$ be a NBB Bessel for $\mathcal{H}$, which is not a frame for $\mathcal{H}$. Then, for any $\Psi$ and any $m$, the multiplier $M_{m, \Phi, \Psi}$ (resp. $M_{m, \Psi, \Phi}$) can not be both unconditionally convergent on $\mathcal{H}$ and invertible on $\mathcal{H}$.

Proof: Assume that $M_{m, \Phi, \Psi}$ (resp. $M_{m, \Psi, \Phi}$) is unconditionally convergent on $\mathcal{H}$. Write $M_{m, \Phi, \Psi} = M_{1, \Phi, \Psi, \Phi}$ (resp. $M_{m, \Psi, \Phi} = M_{1, m, \Psi, \Phi}$). By Proposition 2.2(i), the sequence $\Phi \Phi$ (resp. $m\Psi$) is Bessel for $\mathcal{H}$. Now Proposition 2.4 implies that $M_{1, \Phi, \Psi, \Phi}$ (resp. $M_{1, m, \Psi, \Phi}$) is not invertible on $\mathcal{H}$. □

Lemma 2.9. Let $G_k$ denote the multiplier $M_{\Phi, \Psi}(\frac{1}{n^k})$, $k \in \mathbb{N}$. Then $G_k$ is unconditionally convergent on $\mathcal{H}$ and not invertible on $\mathcal{H}$.

Proof: By Proposition 2.1, $G_k$ is well defined from $\mathcal{H}$ into $\mathcal{H}$ and unconditionally convergent on $\mathcal{H}$. The multiplier $G_k$ is injective, but not surjective - for example, the element $\sum \frac{1}{n^k} e_n \in \mathcal{H}$ does not belong to the range of $G_k$. □
3 Classification Tables

Table 1: two non-Bessel sequences

| φ - not B. | ψ - not B. | $m$ - SN | $m \in \ell^\infty$, but non-SN | $m \notin \ell^\infty$ |
|------------|------------|----------|---------------------------------|--------------------------|
| $M_m, \phi, \psi, M_m, \phi, \psi$ | $M_m, \phi, \psi, M_m, \phi, \psi$ | unc. conv. & INV. | UNC. conv. & NON-INV. |  |
| $M_m, \phi, \psi, M_m, \phi, \psi$ | $M_m, \phi, \psi, M_m, \phi, \psi$ | unc. conv. & INV. | UNC. conv. & NON-INV. |  |

||-|\|--SN  
|----------|----------|---------------------------------|--------------------------|
| NOT POSSIBLE | NOT POSSIBLE | POSSIBLE Example 4.1.3(i) | NOT POSSIBLE Example 4.1.3(ii) |
| $M_m, \phi, \psi, M_m, \phi, \psi$ - not unc. conv., see Proposition 2.2(i) | $M_m, \phi, \psi, M_m, \phi, \psi$ - not unc. conv., see Proposition 2.2(ii) |  |
| Remark 4.1.1(a) | Remark 4.1.1(b) |  |  |

||-|\|--SN  
|----------|----------|---------------------------------|--------------------------|
| NOT POSSIBLE | NOT POSSIBLE | POSSIBLE Example 4.1.4(i) | POSSIBLE Example 4.1.4(ii) |
| $M_m, \phi, \psi, M_m, \phi, \psi$ - not unc. conv., see Proposition 2.2(i) | $M_m, \phi, \psi, M_m, \phi, \psi$ - not unc. conv., see Proposition 2.2(ii) |  |
| Remark 4.1.2(a) | Remark 4.1.2(b) |  |  |

||-|\|--SN  
|----------|----------|---------------------------------|--------------------------|
| NOT POSSIBLE | NOT POSSIBLE | POSSIBLE Example 4.1.6(i) | POSSIBLE Example 4.1.6(ii) |
| $M_m, \phi, \psi, M_m, \phi, \psi$ - not unc. conv., see Proposition 2.2(i) | $M_m, \phi, \psi, M_m, \phi, \psi$ - not unc. conv., see Proposition 2.2(ii) |  |
|  |  |  |  |

||-|\|--SN  
|----------|----------|---------------------------------|--------------------------|
| NOT POSSIBLE | NOT POSSIBLE | POSSIBLE Example 4.1.7(i) | POSSIBLE Example 4.1.7(ii) |
| $M_m, \phi, \psi, M_m, \phi, \psi$ - not unc. conv., see Proposition 2.2(i) | $M_m, \phi, \psi, M_m, \phi, \psi$ - not unc. conv., see Proposition 2.2(ii) |  |
|  |  |  |  |

NBA & non-NBB  

|----------|----------|---------------------------------|--------------------------|
| POSSIBLE Example 4.1.9(i) | POSSIBLE Example 4.1.9(ii) |  |  |

continued on the next page
### Table 1: Possible and Impossible Cases

| Condition | NBA & non-NBB | non-NBA & NBB | non-NBA & NBB | non-NBA & NBB |
|-----------|---------------|---------------|---------------|---------------|
| \( \phi \)-not B. | \( \psi \)-not B. | \( M - \text{SN} \) | \( m \in \ell^\infty \), but non-SN | \( m \notin \ell^\infty \) |
| \( M_{m,\phi,\psi}, M_{m,\psi,\phi} \) | \( M_{m,\phi,\psi}, M_{m,\psi,\phi} \) | \( M_{m,\phi,\psi}, M_{m,\psi,\phi} \) | \( M_{m,\phi,\psi}, M_{m,\psi,\phi} \) |
| unc. conv. & INV. | unc. conv. & NON-INV. | unc. conv. & INV. | unc. conv. & NON-INV. |
| \( \text{NOT POSSIBLE} \) | \( \text{NOT POSSIBLE} \) | \( \text{NOT POSSIBLE} \) | \( \text{NOT POSSIBLE} \) |
| Example 4.1.18(i) | Example 4.1.19(iii) | Example 4.1.18(iii) | Example 4.1.19(iii) |
| \( \text{NOT POSSIBLE} \) | \( \text{NOT POSSIBLE} \) | \( \text{NOT POSSIBLE} \) | \( \text{NOT POSSIBLE} \) |
| Example 4.1.20(i) | Example 4.1.20(ii) | Example 4.1.20(ii) | Example 4.1.20(ii) |
| \( \text{NOT POSSIBLE} \) | \( \text{NOT POSSIBLE} \) | \( \text{NOT POSSIBLE} \) | \( \text{NOT POSSIBLE} \) |
| Example 4.1.20(i) | Example 4.1.20(ii) | Example 4.1.20(ii) | Example 4.1.20(ii) |

### Notes
- Example 4.1.12: \( \phi \)-not B. and \( \psi \)-not B.
- Example 4.1.13: \( M - \text{SN} \).
- Example 4.1.14: \( M \in \ell^\infty \), but non-SN.
- Example 4.1.15: \( m \notin \ell^\infty \).
Table 2: one Bessel not a frame, one non-Bessel sequence

| $\phi$-B. | $\psi$-not B. | $m$ - SN  | $m \in \ell^\infty$, but non-SN | $m \notin \ell^\infty$ |
|-----------|---------------|-----------|---------------------------------|------------------------|
| not fr.   | $M_{m,\phi,\psi}, M_{m,\psi,\phi}$ unc. conv. & INV. | $M_{m,\phi,\psi}, M_{m,\psi,\phi}$ unc. conv. & INV. | $M_{m,\phi,\psi}, M_{m,\psi,\phi}$ unc. conv. & INV. | $M_{m,\phi,\psi}, M_{m,\psi,\phi}$ unc. conv. & INV. |
| ||-SN | ||-SN | NOT POSSIBLE | NOT POSSIBLE | NOT POSSIBLE | NOT POSSIBLE | see Prop. 2.8 | Example 4.2.1(i) | NOT POSSIBLE | NOT POSSIBLE | see Prop. 2.8 | Example 4.2.1(ii) |
| ||-SN | NBA & non-NBB | NOT POSSIBLE | NOT POSSIBLE | NOT POSSIBLE | NOT POSSIBLE | see Prop. 2.8 | Example 4.2.1(ii) | NOT POSSIBLE | POSSIBLE | see Prop. 2.8 | Example 4.2.2 |
| ||-SN | non-NBA & NBB | NOT POSSIBLE | NOT POSSIBLE | NOT POSSIBLE | NOT POSSIBLE | see Prop. 2.8 | Example 4.2.3 | NOT POSSIBLE | POSSIBLE | see Prop. 2.8 | Example 4.2.4 |
| ||-SN | non-NBA & non-NBB | NOT POSSIBLE | NOT POSSIBLE | NOT POSSIBLE | NOT POSSIBLE | see Prop. 2.8 | Example 4.2.4 | NOT POSSIBLE | POSSIBLE | see Prop. 2.8 | Example 4.2.5 |
| non-NBB | ||-SN | POSSIBLE | POSSIBLE | POSSIBLE | POSSIBLE | POSSIBLE | POSSIBLE | POSSIBLE | POSSIBLE | POSSIBLE | POSSIBLE | Example 4.2.9(i) | Example 4.2.9(ii) |
| non-NBB | NBA & non-NBB | POSSIBLE | POSSIBLE | POSSIBLE | POSSIBLE | POSSIBLE | POSSIBLE | POSSIBLE | POSSIBLE | POSSIBLE | POSSIBLE | Example 4.2.13(ii) | Example 4.2.14 |

continued on the next page
| continued from the previous page | \( \phi \)-B. | \( \psi \)-not B. | \( m \) - SN | \( m \in \ell^\infty \), but non-SN | \( m \notin \ell^\infty \) |
|----------------------------------|---------------|----------------|-----------------|-------------------|-------------------|
| \( \phi \)-B. | \( \psi \)-not B. | \( m \) - SN | \( m \in \ell^\infty \), but non-SN | \( m \notin \ell^\infty \) |
| non-\( NBB \) | non-\( NBA \) & \( NBB \) | \( M_{m,\psi,\phi}, M_{m,\psi,\phi} \) & \( M_{m,\psi,\phi}, M_{m,\psi,\phi} \) & \( M_{m,\psi,\phi}, M_{m,\psi,\phi} \) & \( M_{m,\psi,\phi}, M_{m,\psi,\phi} \) & \( M_{m,\psi,\phi}, M_{m,\psi,\phi} \) |
| & | unc. conv. & non-INV. | unc. conv. & non-INV. | unc. conv. & non-INV. | unc. conv. & non-INV. |
| non-\( NBB \) | non-\( NBA \) & non-\( NBB \) | \( M_{m,\psi,\phi}, M_{m,\psi,\phi} \) & \( M_{m,\psi,\phi}, M_{m,\psi,\phi} \) & \( M_{m,\psi,\phi}, M_{m,\psi,\phi} \) & \( M_{m,\psi,\phi}, M_{m,\psi,\phi} \) |
| & | Example 4.2.15 & Example 4.2.16 | Example 4.2.17 & Example 4.2.18 | Example 4.2.19 & Example 4.2.20 |
| non-\( NBB \) | non-\( NBA \) & non-\( NBB \) | \( M_{m,\psi,\phi}, M_{m,\psi,\phi} \) & \( M_{m,\psi,\phi}, M_{m,\psi,\phi} \) & \( M_{m,\psi,\phi}, M_{m,\psi,\phi} \) & \( M_{m,\psi,\phi}, M_{m,\psi,\phi} \) |
| & | Example 4.2.21 & Example 4.2.22 | Example 4.2.23(i) & Example 4.2.23(ii) | Example 4.2.24(i) & Example 4.2.24(ii) |
Table 3: two Bessel sequences which are not frames

| φ-B. | ψ-B. | m - SN | m ∈ ℓ∞, but non-SN | m ∉ ℓ∞ |
|------|------|--------|----------------------|--------|
| not fr. | not fr. | unc. conv. & INV. | unc. conv. & NON-INV. | unc. conv. & NON-INV. |
| Mm,ϕ,ψ, Mm,ϕ,ψ | Mm,ϕ,ψ, Mm,ϕ,ψ | Mm,ϕ,ψ, Mm,ϕ,ψ | Mm,ϕ,ψ, Mm,ϕ,ψ |

| ||-SN | ||-SN | NOT POSSIBLE | ALWAYS apply Prop. 2.1, 2.4 Example 4.3.1(i) | NOT POSSIBLE | ALWAYS apply Prop. 2.1, 2.4 Example 4.3.1(ii) | NOT POSSIBLE | NOT POSSIBLE |
|------|------|--------|------------------|------------------|------------------|------------------|
| non-NBB | see Prop. 2.4 Apply Prop. 2.4 Example 4.3.2(i) | NOT POSSIBLE | ALWAYS apply Prop. 2.1, 2.4 Example 4.3.2(ii) | NOT POSSIBLE | POSSIBLE Example 4.3.3(iii) |
| non-NBB | see Prop. 2.4 Apply Prop. 2.1, 2.4 Example 4.3.3(i) | NOT POSSIBLE | ALWAYS apply Prop. 2.1, 2.4 Example 4.3.3(ii) | POSSIBLE | POSSIBLE Example 4.3.3(iv) |
Table 4: one overcomplete frame, one non-Bessel sequence

| $\phi$-fr. | $\psi$-not B. | $m$ - SN | $m \in \ell^\infty$, but non-SN | $m \notin \ell^\infty$ |
|-----------|-------------|---------|-----------------|-------------------|
| not R.b.  |             |         |                 |                   |
|           |             | $M_{m,\psi,\psi}, M_{m,\psi,\psi}$ | $M_{m,\psi,\psi}, M_{m,\psi,\psi}$ | $M_{m,\psi,\psi}, M_{m,\psi,\psi}$ |
|           |             | unc. conv. & NON-INV. | unc. conv. & NON-INV. | unc. conv. & NON-INV. |
| $||-||$-SN | $||-||$-SN | NOT POSSIBLE | NOT POSSIBLE | POSSIBLE |
|           |           | $M_{m,\psi,\psi}, M_{m,\psi,\psi}$ - not unc. conv., see Proposition 2.2(i) | Example 4.4.1 | Example 4.4.2 |
| $||-||$-SN | $NBA$ & non-$NBB$ | NOT POSSIBLE | NOT POSSIBLE | POSSIBLE |
|           |           | $M_{m,\psi,\psi}, M_{m,\psi,\psi}$ - not unc. conv., see Proposition 2.2(i) | Example 4.4.3 | Example 4.4.4(i) |
| $||-||$-SN | non-$NBA$ & $NBB$ | NOT POSSIBLE | NOT POSSIBLE | POSSIBLE |
|           |           | $M_{m,\psi,\psi}, M_{m,\psi,\psi}$ - not unc. conv., see Proposition 2.2(i) | Example 4.4.6(i) | Example 4.4.6(ii) |
| $||-||$-SN | non-$NBA$ & non-$NBB$ | NOT POSSIBLE | NOT POSSIBLE | POSSIBLE |
|           |           | $M_{m,\psi,\psi}, M_{m,\psi,\psi}$ - not unc. conv., see Proposition 2.2(i) | Example 4.4.7 | Example 4.4.8 |
| $||-||$-SN | non-$NBB$ | NOT POSSIBLE | NOT POSSIBLE | POSSIBLE |
|           |           | $M_{m,\psi,\psi}, M_{m,\psi,\psi}$ - not unc. conv., see Proposition 2.2(i) | Example 4.4.11 | Example 4.4.12 |
| $||-||$-SN | $NBA$ & non-$NBB$ | POSSIBLE | POSSIBLE | POSSIBLE |
|           |           | Example 4.4.11(i) | Example 4.4.12 | Example 4.4.13 | Example 4.4.14(i) |
|           |           | Example 4.4.15 | Example 4.4.16(i) | Example 4.4.17 | Example 4.4.16(ii) | Example 4.4.19 | Example 4.4.18 | Example 4.4.17 | Example 4.4.18 | Example 4.4.16(ii) | Example 4.4.19 |

continued on the next page
| continued from the previous page | $\phi$-fr. not R.b. | $\psi$-not B. | $m \in \ell^\infty$, but non-SN | $m \notin \ell^\infty$ |
|---------------------------------|---------------------|----------------|---------------------------------|------------------|
|                                 | $m$ - SN            | $M_{m,\phi,\psi}, M_{m,\phi,\psi}$ | $M_{m,\phi,\psi}, M_{m,\phi,\psi}$ | $M_{m,\phi,\psi}, M_{m,\phi,\psi}$ |
|                                 | unc. conv. & INV.   | unc. conv. & NON-INV. | unc. conv. & INV. | unc. conv. & NON-INV. |
| non-$NBB$                       | non-$NBB$ & $NBB$   | POSSIBLE Example 4.4.20 | POSSIBLE Example 4.4.21 | POSSIBLE Example 4.4.23(i) | POSSIBLE Example 4.4.23(ii) |
| non-$NBB$                       | non-$NBB$ & non-$NBB$ | POSSIBLE Example 4.4.24 | POSSIBLE Example 4.4.25 | POSSIBLE Example 4.4.26(i) | POSSIBLE Example 4.4.26(ii) | POSSIBLE Example 4.4.27 | POSSIBLE Example 4.4.28 |
Table 5: one overcomplete frame, one Bessel sequence which is not a frame

| φ-fr. | ψ-B. | $m \in \ell^\infty$, but non-SN | $m \not\in \ell^\infty$ |
|-------|------|-------------------------------|--------------------------|
| not R.b. | not fr. | $M_{m,\phi,\psi}, \Phi, M_{m,\phi,\psi}, \Phi$ | $M_{m,\phi,\psi}, M_{m,\phi,\psi}, \phi$ |
|       |       | unc. conv. & INV. | unc. conv. & NON-INV. |
|       |       | $M_{m,\phi,\psi}, \Phi, M_{m,\phi,\psi}, \Phi$ | $M_{m,\phi,\psi}, M_{m,\phi,\psi}, \phi$ |
|       |       | unc. conv. & INV. | unc. conv. & NON-INV. |
|       |       | NOT POSSIBLE | NOT POSSIBLE |
|       |       | ALWAYS apply Prop. 2.4, Example 4.5.1(i) | ALWAYS apply Prop. 2.1, 2.4, Example 4.5.1(ii) |
|       |       | NOT POSSIBLE | ALWAYS apply Prop. 2.4, Example 4.5.2(ii) |
|       |       | ALWAYS apply Prop. 2.4, Example 4.5.4(i) | NOT POSSIBLE |
|       |       | POSSIBLE | POSSIBLE |
|       |       | Example 4.5.5 | Example 4.5.7(i) |
|       |       | NOT POSSIBLE | ALWAYS apply Prop. 2.4, Example 4.5.6(ii) |
|       |       | NOT POSSIBLE | ALWAYS apply Prop. 2.1, 2.4, Example 4.5.7(ii) |
Table 6: two overcomplete frames

| $\phi$-fr. not R.b. | $\psi$-fr. not R.b. | $m$ - SN | $m \in \ell^\infty$, but non-SN | $m \notin \ell^\infty$ |
|----------------------|----------------------|----------|----------------------------------|-----------------------|
|                      |                      | $M_{m,\phi,\psi}, M_{m,\psi,\phi}$ unc. conv. & INV. | $M_{m,\phi,\psi}, M_{m,\psi,\phi}$ unc. conv. & NON-INV. | $M_{m,\phi,\psi}, M_{m,\psi,\phi}$ unc. conv. & INV. |
| $\| \|_{-SN}$       | $\| \|_{-SN}$       | POSSIBLE Example 4.6.1(i)   | POSSIBLE Example 4.6.1(ii) | NOT POSSIBLE NOT POSSIBLE |
|                      |                      | POSSIBLE Example 4.6.2   | POSSIBLE Example 4.6.1(iii) | $M_{m,\phi,\psi}, M_{m,\psi,\phi}$ - not unc. conv., see Proposition 2.2(ii) |
| $\| \|_{-SN}$       | non-$NBB$            | POSSIBLE Example 4.6.3(i) | POSSIBLE Example 4.6.3(ii) | POSSIBLE Example 4.6.3(iv) |
| non-$NBB$            | non-$NBB$            | POSSIBLE Example 4.6.5   | POSSIBLE Example 4.6.7(i) | POSSIBLE Example 4.6.8(i) |
|                      |                      | POSSIBLE Example 4.6.6   | POSSIBLE Example 4.6.7(ii) | POSSIBLE Example 4.6.8(ii) |
Table 7: one Riesz basis, one non-Bessel sequence

| $\phi$-R.b. | $\psi$-not B. | $m \cdot \text{SN}$ | $m \in \ell^\infty$, but non-$\text{SN}$ | $m \notin \ell^\infty$ |
|-------------|--------------|------------------|---------------------------------|---------------------|
|             |              | $M_{m,\varphi,\psi}, M_{m,\varphi,\psi}$ $\text{unc. conv.} \& \text{INV.}$ | $M_{m,\varphi,\psi}, M_{m,\varphi,\psi}$ $\text{unc. conv.} \& \text{INV.}$ | $M_{m,\varphi,\psi}, M_{m,\varphi,\psi}$ $\text{unc. conv.} \& \text{INV.}$ |
| $\|\|\|\cdot\|\| \cdot \text{SN}$ | $\|\|\|\cdot\|\| \cdot \text{SN}$ | NOT POSSIBLE NOT POSSIBLE $M_{m,\varphi,\psi}, M_{m,\varphi,\psi}$ - not well defined, see Proposition 2.3(i) | NOT POSSIBLE POSSIBLE see Prop. 2.6 Example 4.7.1 | NOT POSSIBLE NOT POSSIBLE $M_{m,\varphi,\psi}, M_{m,\varphi,\psi}$ - not well defined, see Proposition 2.3(ii) |
| $\|\|\|\cdot\|\| \cdot \text{SN}$ | $\|\|\|\cdot\|\| \cdot \text{SN}$ | NOT POSSIBLE NOT POSSIBLE $M_{m,\varphi,\psi}, M_{m,\varphi,\psi}$ - not well defined, see Proposition 2.3(i) | NOT POSSIBLE POSSIBLE see Prop. 2.6 Example 4.7.2 | NOT POSSIBLE POSSIBLE see Prop. 2.6 Example 4.7.3 |
| $\|\|\|\cdot\|\| \cdot \text{SN}$ | $\|\|\|\cdot\|\| \cdot \text{SN}$ | NOT POSSIBLE NOT POSSIBLE $M_{m,\varphi,\psi}, M_{m,\varphi,\psi}$ - not well defined, see Proposition 2.3(i) | POSSIBLE POSSIBLE Example 4.7.4 Example 4.7.5 | NOT POSSIBLE NOT POSSIBLE $M_{m,\varphi,\psi}, M_{m,\varphi,\psi}$ - not well defined, see Proposition 2.3(ii) |
| $\|\|\|\cdot\|\| \cdot \text{SN}$ | $\|\|\|\cdot\|\| \cdot \text{SN}$ | NOT POSSIBLE NOT POSSIBLE $M_{m,\varphi,\psi}, M_{m,\varphi,\psi}$ - not well defined, see Proposition 2.3(i) | NOT POSSIBLE POSSIBLE see Prop. 2.6 Example 4.7.6 | POSSIBLE POSSIBLE Example 4.7.7 Example 4.7.8 |
Table 8: one Riesz basis, one Bessel which is not a frame

| \( \phi \)-B.b. | \( \psi \)-B. | \( m \) - \( \mathcal{SN} \) | \( m \in \ell^\infty \), but non-\( \mathcal{SN} \) | \( m \notin \ell^\infty \) |
|-----------------|----------------|----------------|----------------|----------------|
|                 |                 |                 |                 |                 |
| \( M_{m,\psi,\phi},M_{m,\psi,\phi} \) unc. conv. & INV. | \( M_{m,\psi,\phi},M_{m,\psi,\phi} \) unc. conv. & INV. | \( M_{m,\psi,\phi},M_{m,\psi,\phi} \) unc. conv. & INV. | \( M_{m,\psi,\phi},M_{m,\psi,\phi} \) unc. conv. & INV. | \( M_{m,\psi,\phi},M_{m,\psi,\phi} \) unc. conv. & INV. |
| \( \|\cdot\|_\mathcal{SN} \) NOT POSSIBLE Always see Prop. 2.6 | \( \|\cdot\|_\mathcal{SN} \) NOT POSSIBLE Always see Prop. 2.6 | \( \|\cdot\|_\mathcal{SN} \) NOT POSSIBLE Always see Prop. 2.6 | \( \|\cdot\|_\mathcal{SN} \) NOT POSSIBLE Always see Prop. 2.6 | \( \|\cdot\|_\mathcal{SN} \) NOT POSSIBLE Always see Prop. 2.6 |
| \( \|\cdot\|_\mathcal{SN} \) non-NBB | \( \|\cdot\|_\mathcal{SN} \) non-NBB | \( \|\cdot\|_\mathcal{SN} \) non-NBB | \( \|\cdot\|_\mathcal{SN} \) non-NBB | \( \|\cdot\|_\mathcal{SN} \) non-NBB |
|                 |                 |                 |                 |                 |
| \( \|\cdot\|_\mathcal{SN} \) NOT POSSIBLE Always see Prop. 2.6 | \( \|\cdot\|_\mathcal{SN} \) NOT POSSIBLE Always see Prop. 2.6 | \( \|\cdot\|_\mathcal{SN} \) NOT POSSIBLE Always see Prop. 2.6 | \( \|\cdot\|_\mathcal{SN} \) NOT POSSIBLE Always see Prop. 2.6 | \( \|\cdot\|_\mathcal{SN} \) NOT POSSIBLE Always see Prop. 2.6 |
| \( \|\cdot\|_\mathcal{SN} \) NOT POSSIBLE Always see Prop. 2.6 | \( \|\cdot\|_\mathcal{SN} \) NOT POSSIBLE Always see Prop. 2.6 | \( \|\cdot\|_\mathcal{SN} \) NOT POSSIBLE Always see Prop. 2.6 | \( \|\cdot\|_\mathcal{SN} \) NOT POSSIBLE Always see Prop. 2.6 | \( \|\cdot\|_\mathcal{SN} \) NOT POSSIBLE Always see Prop. 2.6 |
| \( \|\cdot\|_\mathcal{SN} \) NOT POSSIBLE Always see Prop. 2.6 | \( \|\cdot\|_\mathcal{SN} \) NOT POSSIBLE Always see Prop. 2.6 | \( \|\cdot\|_\mathcal{SN} \) NOT POSSIBLE Always see Prop. 2.6 | \( \|\cdot\|_\mathcal{SN} \) NOT POSSIBLE Always see Prop. 2.6 | \( \|\cdot\|_\mathcal{SN} \) NOT POSSIBLE Always see Prop. 2.6 |
| \( \|\cdot\|_\mathcal{SN} \) NOT POSSIBLE Always see Prop. 2.6 | \( \|\cdot\|_\mathcal{SN} \) NOT POSSIBLE Always see Prop. 2.6 | \( \|\cdot\|_\mathcal{SN} \) NOT POSSIBLE Always see Prop. 2.6 | \( \|\cdot\|_\mathcal{SN} \) NOT POSSIBLE Always see Prop. 2.6 | \( \|\cdot\|_\mathcal{SN} \) NOT POSSIBLE Always see Prop. 2.6 |

Example 4.8.1 Example 4.8.2 Example 4.8.3 Example 4.8.4 Example 4.8.5 Example 4.8.6
Table 9: one Riesz basis, one overcomplete frame

| $\phi$-R.b. | $\psi$-fr. not R.b. | $m$ - SN | $m \in \ell^\infty$, but non-SN | $m \notin \ell^\infty$ |
|-------------|-------------------|--------|-------------------------------|-------------------|
| $M_{m,\phi,\psi}, M_{m,\psi,\phi}$ | unc. conv. & INV. | $M_{m,\phi,\psi}, M_{m,\psi,\phi}$ | unc. conv. & NON-INV. | $M_{m,\phi,\psi}, M_{m,\psi,\phi}$ |
| $\|\|\| - SN$ | $\|\|\| - SN$ | NOT POSSIBLE | ALWAYS apply Prop. 2.1, 2.5 | NOT POSSIBLE |
| | see Prop. 2.7 | ALWAYS apply Prop. 2.1, 2.5 | see Prop. 2.6 | see Proposition 2.3(ii) |
| | $M_{m,\phi,\psi}$-inj., non-surj. Example 4.9.1 | $M_{m,\phi,\psi}$-inj., non-surj. Example 4.9.4 | $M_{m,\phi,\psi}$-inj., non-surj. Example 4.9.5 |

Table 10: two Riesz bases

| $\phi$-R.b. | $\psi$-R.b. | $m$ - SN | $m \in \ell^\infty$, but non-SN | $m \notin \ell^\infty$ |
|-------------|-------------|--------|-------------------------------|-------------------|
| $M_{m,\phi,\psi}, M_{m,\psi,\phi}$ | unc. conv. & INV. | $M_{m,\phi,\psi}, M_{m,\psi,\phi}$ | unc. conv. & NON-INV. | $M_{m,\phi,\psi}, M_{m,\psi,\phi}$ |
| $\|\|\| - SN$ | $\|\|\| - SN$ | ALWAYS apply Prop. 2.1, 2.5 | NOT POSSIBLE | NOT POSSIBLE |
| | see Prop. 2.7 | NOT POSSIBLE apply Prop. 2.1, 2.5 | see Prop. 2.6 | see Proposition 2.3(ii) |
| | $M_{m,\phi,\psi}$-inj., non-surj. Example 4.10.1 | $M_{m,\phi,\psi}$-inj., non-surj. Example 4.10.2 | $M_{m,\phi,\psi}$-inj., non-surj. Example 4.10.3 |
4 Examples

4.1 Examples for two non-Bessel sequences; Table 1 on page 5

Remark 4.1.1. When $\Phi$ and $\Psi$ are $||\cdot||$-$SN$ non-Bessel sequences and $m$ is $SN$, then both $M_{m,\Phi,\Psi}$ and $M_{m,\Psi,\Phi}$ can never be unconditionally convergent on $H$ due to Proposition 2.2(i). However, they can be conditionally convergent and invertible (resp. conditionally convergent and non-invertible). Examples:

(a) Let $\Phi = (e_1, e_2, e_2, e_2, e_3, e_3, e_3, e_3, e_4, e_4, e_4, e_4, \ldots)$.
$\quad \Psi = (e_1, e_2, e_2, -e_2, e_3, e_3, -e_3, e_4, e_4, -e_4, e_4, -e_4, \ldots)$.

Then $M_{(1),\Phi,\Psi} = M_{(1),\Psi,\Phi} = I$.

(b) Let $\Phi = (e_1, e_2, e_2, e_2, e_3, e_3, e_3, e_3, e_4, e_4, e_4, e_4, e_4, \ldots)$.
$\quad \Psi = (e_1, e_2, -e_2, e_3, -e_3, e_4, e_4, -e_4, e_4, -e_4, e_4, -e_4, \ldots)$.

Then $M_{(1),\Phi,\Psi}f = f + (f, e_1 - e_2)e_2$ and $M_{(1),\Psi,\Phi}f = f + (f, e_2)(e_1 - e_2)$, $f \in H$. Both $M_{(1),\Phi,\Psi}$ and $M_{(1),\Psi,\Phi}$ are not injective - for example, $M_{m,\Phi,\Psi}e_2 = 0$ and $M_{m,\Psi,\Phi}e_1 = e_1 = M_{m,\Psi,\Phi}e_2$.

Remark 4.1.2. When $\Phi$ is $||\cdot||$-$SN$ non-Bessel for $H$, $\Psi$ is NBA non-$NBB$ non-Bessel for $H$ and $m$ is $SN$, then both $M_{m,\Phi,\Psi}$ and $M_{m,\Psi,\Phi}$ can not be unconditionally convergent on $H$ due to Proposition 2.2(i), but they can be conditionally convergent and invertible (resp. conditionally convergent and non-invertible). Examples:

(a) Let $\Phi = (e_1, e_2, e_2, e_2, e_3, e_3, e_3, e_3, e_4, e_4, e_4, e_4, e_4, \ldots)$.
$\quad \Psi = (e_1, e_2, \frac{1}{2}e_2, -\frac{1}{2}e_2, e_3, e_3, -e_3, \frac{1}{2}e_3, -\frac{1}{2}e_3, e_4, e_4, -e_4, e_4, -\frac{1}{2}e_4, -\frac{1}{2}e_4, \ldots)$.

Then $M_{(1),\Phi,\Psi} = M_{(1),\Psi,\Phi} = I$.

(b) Let $\Phi = (e_1, e_2, e_2, e_2, e_3, e_3, e_3, e_3, e_4, e_4, e_4, e_4, e_4, e_4, \ldots)$.
$\quad \Psi = (e_1, \frac{1}{2}e_2, e_2, -e_2, \frac{1}{2}e_3, e_3, -e_3, \frac{1}{2}e_3, -e_3, \frac{1}{2}e_4, e_4, -e_4, e_4, -e_4, \ldots)$.

Then $M_{(1),\Phi,\Psi} = M_{(1),\Psi,\Phi} = G_1$ - non-invertible on $H$ (see Lemma 2.9).

There are also other places in the table, where the multipliers can not be unconditionally convergent on $H$, but they can be conditionally convergent an invertible (resp. conditionally convergent and non-invertible). We will not list such examples any more, as our main aim is to consider the possibilities for the combination of unconditional convergence and invertibility on $H$.

Example 4.1.3. Let $\Phi = (e_1, e_2, e_2, e_3, e_3, e_3, e_4, e_4, e_4, e_4, \ldots)$.

(i) Let $m = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \ldots)$.
Then $M_{m,\Phi,\Phi} = I$. The convergence is unconditional on $H$, because $M_{m,\Phi,\Phi} = M_{(1),\sqrt{m_n} \phi_n},(\sqrt{m_n} \phi_n)$ and the sequence $(\sqrt{m_n} \phi_n)$ is Bessel for $H$ (apply Prop. 2.1).
(ii) Let \( m = (\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \ldots) \).

Then \( M_{m,\psi,\psi} = G_1 \) - non-invertible on \( \mathcal{H} \) (see Lemma 2.9). The unconditional convergence on \( \mathcal{H} \) follows as in (i).

**Example 4.1.4.** Let \( \Phi = (e_1, e_2, e_2, e_1, e_3, e_3, e_3, e_1, e_4, e_4, \ldots) \),
\[ \Psi = (e_1, e_2, \frac{1}{2} e_2, -\frac{1}{2} e_2, e_1, e_3, \frac{1}{3} e_3, -\frac{1}{3} e_3, e_1, e_4, \frac{1}{4} e_4, -\frac{1}{4} e_4, \ldots). \]

(i) Let \( m = (\frac{1}{2}, 1, 1, 1, 1, \frac{1}{2\pi}, 1, 1, 1, 1, \ldots) \).

Then \( M_{m,\psi,\psi} = M_{m,\psi,\psi} = I \). The convergence is unconditional on \( \mathcal{H} \), because \( M_{m,\psi,\psi} \cong M_{(1), (\sqrt{m_n} \phi_n), (\sqrt{m_n} \phi_n)} \).

Example 4.1.5. Let \( \Phi = (e_1, e_2, e_2, e_1, e_3, e_3, e_3, e_1, e_4, e_4, \ldots) \),
\[ \Psi = (e_1, \frac{1}{2} e_2, e_1, \frac{1}{3} e_3, e_1, \frac{1}{4} e_4, \ldots). \]

Then \( M_{m,\psi,\psi} = G_1 \) - non-invertible on \( \mathcal{H} \) (see Lemma 2.9). The unconditional convergence follows as in Example 4.1.4(i).

Example 4.1.6. Let \( \Phi = (e_1, e_2, e_2, e_1, e_3, e_3, e_1, e_4, e_4, \ldots) \),
\[ \Psi = (e_1, 2e_2, e_1, 3e_3, e_1, 4e_4, \ldots). \]

(i) Let \( m = (\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \ldots) \).

Then \( M_{m,\psi,\psi} = M_{m,\psi,\psi} = I \). The convergence is unconditional on \( \mathcal{H} \), because \( M_{m,\psi,\psi} \cong M_{(1), \psi, \psi} \), where
\[ \Theta = (\sqrt{2} e_1, e_2, \frac{1}{\sqrt{2}} e_2, e_1, \frac{1}{\sqrt{3}} e_3, e_3, \frac{1}{\sqrt{4}} e_4, e_4, \ldots) \] is Bessel for \( \mathcal{H} \) (apply Prop. 2.1).

(ii) Let \( m = (\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \ldots) \).

Then \( M_{m,\psi,\psi} = G_1 \) - non-invertible on \( \mathcal{H} \) (see Lemma 2.9). The convergence is unconditional on \( \mathcal{H} \), because \( M_{m,\psi,\psi} \cong M_{m,\psi,\psi} \), where
Example 4.1.11. Let $\Phi = (e_1, e_2, e_2, e_2, e_2, e_3, e_3, e_3, e_3, e_1, e_1, e_4, e_4, e_4, e_1, e_5, e_5, e_5, \ldots)$ and $\Psi = (e_1, e_2, 2e_2, -2e_2, e_3, e_3, 1/3e_3, -1/3e_3, e_1, e_4, 4e_4, -4e_4, e_1, e_5, 5/3e_5, 5/3e_5, \ldots)$.

(i) Let $m = (1/2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, \ldots)$.

Then $M_{m, \Phi, \Psi} = M_{m, \Phi, \Psi} = I$. The convergence is unconditional on $H$, because $M_{m, \Phi, \Psi} \subseteq M_{m, \Phi, \Psi} \subseteq M_{\nu, \Theta, \Theta}$, where

$\Theta = (\sqrt{2} e_1, e_2, e_2, e_2, 1/3 e_3, 1/3 e_3, 1/3 e_3, 1/3 e_3, e_1, e_4, e_4, e_4, 1/3 e_1, e_5, 5/3 e_5, 5/3 e_5, \ldots)$ is Bessel for $H$ and

$\nu = (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, \ldots)$ is $SN$ (apply Prop. 2.1).

(ii) Let $m = (1/2, 1, 1/2, 1/2, 1/3, 1/3, 1/3, 1/3, 1/3, 1/3, 1/3, 1/3, 1/3, 1/3, \ldots)$.

Then $M_{m, \Phi, \Psi} = M_{m, \Phi, \Psi} = I$. The unconditional convergence follows as in Example 4.1.4(ii).

(iii) Let $m = (1/2, 1/2, 1/2, 1/2, 1/3, 1/3, 1/3, 1/3, 1/3, 1/3, 1/3, 1/3, 1/3, 1/3, \ldots)$.

Then $M_{m, \Phi, \Psi} = M_{m, \Phi, \Psi} = G_1$ - non-invertible on $H$ (see Lemma 2.9). The unconditional convergence follows as in Example 4.1.4(iii).

Example 4.1.8. Let $\Phi = (e_1, e_2, e_1, e_3, e_3, e_1, e_1, e_4, e_1, e_5, e_1, e_1, e_6, e_1, e_1, e_7, \ldots)$, $\Psi = (e_1, 2e_2, e_1, 1/3 e_3, e_1, 1/3 e_3, 1/3 e_3, e_1, e_4, e_4, e_4, e_1, 1/3 e_5, e_1, 1/3 e_5, e_1, 1/3 e_7, \ldots)$, and $m = (1/2, 1/2, 1/2, 1/2, 1/3, 1/3, 1/3, 1/3, 1/3, 1/3, 1/3, 1/3, 1/3, 1/3, \ldots)$.

Then $M_{m, \Phi, \Psi} = M_{m, \Phi, \Psi} = G_1$ - non-invertible on $H$ (see Lemma 2.9). The unconditional convergence on $H$ follows in the same way as in Example 4.1.4(i).

Example 4.1.9. Let $\Phi = (1/2 e_1, e_2, e_1, e_3, 1/3 e_3, e_1, e_4, e_1, e_5, 2/3 e_1, e_6, e_1, e_1, e_7, \ldots)$.

(i) Let $\Psi = (e_1, e_2, 1/3 e_1, e_3, e_1, e_4, 1/3 e_1, e_5, e_1, e_6, 1/3 e_1, e_7, \ldots)$.

Then $M_{(1), \Phi, \Psi} = M_{(1), \Phi, \Psi} = I$. The convergence is unconditional on $H$, because $M_{(1), \Phi, \Psi} \subseteq M_{(1), \Phi, \Psi} \subseteq M_{(1), \Theta, \Theta}$, where $\Theta$ is the same as in Example 4.1.6(i).

(ii) Let $\Psi = (e_1, 1/2 e_2, 1/3 e_1, 1/3 e_3, e_1, 1/3 e_4, 1/3 e_3, 1/3 e_5, 1/3 e_1, 1/3 e_5, 1/3 e_6, 1/3 e_6, \ldots)$.

Then $M_{(1), \Phi, \Psi} = M_{(1), \Phi, \Psi} = G_1$ - non-invertible on $H$ (see Lemma 2.9). The convergence is unconditional on $H$, because $M_{(1), \Phi, \Psi} \subseteq M_{(1), \Phi, \Psi} \subseteq M_{(1), \Theta, \Theta}$, where $\Theta$ is the same as in Example 4.1.6(ii) (apply Prop. 2.1).

Example 4.1.10. Let $\Phi = (e_1, e_2, 1/2 e_1, e_3, e_3, e_1, e_4, 1/3 e_1, e_5, e_1, e_6, 1/3 e_1, e_7, \ldots)$, and $m = (1/2, 1, 1, 1, 1/3, 1/3, 1/3, 1/3, 1/3, 1/3, 1/3, 1/3, 1/3, \ldots)$.

Then $M_{(1), \Phi, \Psi} = I$. The convergence is unconditional on $H$, because $M_{(1), \Phi, \Psi} \subseteq M_{(1), \sqrt{m_0} \phi_n}, \sqrt{m_0} \phi_n, \sqrt{m_0} \phi_n)$ and $(\sqrt{m_0} \phi_n)$ is Bessel for $H$ (apply Prop. 2.1).

Example 4.1.11. Let $\Phi = (e_1, 1/2 e_2, e_1, 1/3 e_3, e_1, 1/4 e_4, \ldots)$. 

\[ \Theta = (1/\sqrt{2} e_1, 1/\sqrt{2} e_2, 1/\sqrt{2} e_3, 1/\sqrt{2} e_1, 1/\sqrt{2} e_4, \ldots) \] is Bessel for $H$ (apply Prop. 2.1).
(i) Let \( m = (\frac{1}{2}, 1, \frac{1}{2}, 1, \frac{1}{2}, 1, \ldots) \).

Then \( M_{m, \Phi, \Psi} = G_2 \) - non-invertible on \( \mathcal{H} \) (see Lemma 2.9). The unconditional convergence on \( \mathcal{H} \) follows in the same way as in Example 4.1.10.

(ii) Let \( m = (\frac{1}{2}, 2, \frac{1}{2}, 3^2, \frac{1}{2}, 4^2, \ldots) \).

Then \( M_{m, \Phi, \Psi} = I \). The unconditional convergence follows as in Example 4.1.10.

(iii) Let \( m = (\frac{1}{2}, 2, \frac{1}{2}, 3, \frac{1}{2}, 4, \ldots) \).

Then \( M_{m, \Phi, \Psi} = G_1 \) - non-invertible on \( \mathcal{H} \) (see Lemma 2.9). The unconditional convergence on \( \mathcal{H} \) follows in the same way as in Example 4.1.10.

Example 4.1.12. Let \( \Phi = (e_1, \frac{1}{2}e_2, e_1, \frac{1}{3}e_3, e_1, \frac{1}{4}e_4, e_1, \frac{1}{5}e_5, \ldots)\),

\( \Psi = (e_1, 2e_2, e_1, 3e_3, e_1, 4e_4, e_1, 5e_5, \ldots) \).

(i) Let \( m = (\frac{1}{2}, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{2}, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{2}, 1, \ldots) \).

Then \( M_{m, \Phi, \Psi} = M_{m, \Psi, \Phi} = I \). The unconditional convergence follows as in Example 4.1.6(i).

(ii) Let \( m = (\frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{2}, 1, \ldots) \).

Then \( M_{m, \Phi, \Psi} = M_{m, \Psi, \Phi} = G_1 \) - non-invertible on \( \mathcal{H} \) (see Lemma 2.9). The unconditional convergence follows as in Example 4.1.6(ii).

Example 4.1.13. Let \( \Psi = (e_1, 2e_2, e_1, e_3, e_1, 4e_4, e_1, e_5, \ldots) \),

\( m = (\frac{1}{2}, 1, \frac{1}{2}, 3, \frac{1}{2}, 1, \frac{1}{2}, 3, \frac{1}{2}, 1, \ldots) \).

(i) Let \( \Phi = (e_1, \frac{1}{2}e_2, e_1, \frac{1}{3}e_3, e_1, \frac{1}{4}e_4, e_1, \frac{1}{5}e_5, \ldots) \).

Then \( M_{m, \Phi, \Psi} = M_{m, \Psi, \Phi} = I \). The unconditional convergence follows as in Example 4.1.6(i).

(ii) Let \( \Phi = (e_1, \frac{1}{2}e_2, e_1, \frac{1}{3}e_3, e_1, \frac{1}{4}e_4, e_1, \frac{1}{5}e_5, \ldots) \).

Then \( M_{m, \Phi, \Psi} = M_{m, \Psi, \Phi} = G_1 \) - non-invertible on \( \mathcal{H} \) (see Lemma 2.9). The unconditional convergence follows as in Example 4.1.6(ii).

Example 4.1.14. Let \( \Phi = (e_1, \frac{1}{2}e_2, e_1, \frac{1}{3}e_3, e_1, \frac{1}{4}e_4, e_1, \frac{1}{5}e_5, \ldots) \),

\( \Psi = (\frac{1}{2}e_1, 2e_2, \frac{1}{2}e_1, 3e_3, \frac{1}{2}e_1, 4e_4, \ldots) \).

(i) Then \( M_{(1), \Phi, \Psi} = M_{(1), \Psi, \Phi} = I \). The convergence is unconditional on \( \mathcal{H} \), because \( M_{(1), \Phi, \Psi} \cong M_{(1), \Psi, \Phi} \cong M_{(1), \Theta, \Theta} \), where \( \Theta \) is the same as in Example 4.1.6(i) (apply Prop. 2.1).

(ii) Let \( m = (1, \frac{1}{2}, 1, \frac{1}{3}, 1, \frac{1}{4}, \ldots) \).

Then \( M_{m, \Phi, \Psi} = M_{m, \Psi, \Phi} = G_1 \) - non-invertible on \( \mathcal{H} \) (see Lemma 2.9). The unconditional convergence follows as in Example 4.1.6(ii).

Example 4.1.15. Let \( \Phi = (e_1, \frac{1}{2}e_2, e_1, \frac{1}{3}e_3, e_1, \frac{1}{4}e_4, e_1, \frac{1}{5}e_5, \ldots) \),

\( \Psi = (\frac{1}{2}e_1, 2e_2, \frac{1}{2}e_1, 3e_3, \frac{1}{2}e_1, 4e_4, \ldots) \).
(i) Then \(M_{(1),\varphi,\psi} = M_{(1),\varphi,\psi} = G_1\) - non-invertible on \(\mathcal{H}\) (see Lemma 2.9). The convergence is unconditional on \(\mathcal{H}\), because \(M_{(1),\varphi,\psi} \equiv M_{(1),\varphi,\psi} \equiv M_{(1),\Theta,\Theta}\), where \(\Theta\) is the same as in Example 4.1.6(ii) (apply Prop. 2.1).

(ii) Let \(m = (1, 2, 1, 3, 1, 4, \ldots).

Then \(M_{m,\varphi,\psi} = M_{m,\varphi,\psi} = I\). The unconditional convergence follows as in Example 4.1.6(i).

Example 4.1.6. Let \(\Phi = (\sqrt{2}e_1, e_1, \frac{\sqrt{2}}{2} e_2, \sqrt{2}^2 e_1, \frac{\sqrt{2}}{2} e_3, \frac{\sqrt{2}}{2} e_4, \ldots), \Psi = (\frac{\sqrt{2}}{2} e_1, 2e_2, \frac{\sqrt{2}}{2} e_1, 3e_3, \frac{\sqrt{2}}{2} e_4, \frac{\sqrt{2}}{2} e_5, \ldots), \quad m = (\frac{\sqrt{2}}{2}, 1, \frac{\sqrt{2}}{2}, 1, \frac{\sqrt{2}}{2}, 1, \ldots).

Then \(M_{m,\varphi,\psi} = M_{m,\varphi,\psi} = I\). The unconditional convergence follows as in Example 4.1.6(i).

Example 4.1.17. Let \(\Phi = (e_1, \frac{1}{2^2} e_2, e_1, \frac{1}{2} e_3, e_1, \frac{1}{2^3} e_4, \ldots), \Psi = (\frac{1}{2} e_1, 2e_2, \frac{1}{2} e_1, 3e_3, \frac{1}{2} e_4, \frac{1}{2} e_5, \ldots), \quad m = (\frac{1}{2}, 2, \frac{1}{2}, 3, \frac{1}{2}, 4, \ldots).

Then \(M_{m,\varphi,\psi} = M_{m,\varphi,\psi} = G_1\) - non-invertible on \(\mathcal{H}\) (see Lemma 2.9). The unconditional convergence follows as in Example 4.1.6(ii).

Example 4.1.18. Let \(\Phi = (n\varphi_n)\) and \(m = (\frac{1}{n}).\) Then \(M_{m,\varphi,\psi} \equiv M_{(1),\varphi_n,\psi} = I\).

Example 4.1.19. Let \(\Phi = (n\varphi_n)\) and \(m = (\frac{1}{n}).\) Then \(M_{m,\varphi,\psi} \equiv M_{(1),\varphi_n,\psi} = G_1\) - unconditionally convergent and non-invertible on \(\mathcal{H}\) (see Lemma 2.9).

Example 4.1.20. Let \(\Phi = (e_1, 2e_2, e_1, 3e_3, e_1, 4e_4, e_1, 5e_5, \ldots), \Psi = (e_1, \frac{1}{2} e_2, 2e_1, \frac{1}{3} e_3, 2^2 e_1, \frac{1}{4} e_4, 2^3 e_1, \frac{1}{5} e_5, \ldots), \quad (i) \) Let \(m = (\frac{1}{2}, 1, \frac{1}{2}, 1, \frac{1}{2}, 1, \frac{1}{2}, 1, \ldots).

Then \(M_{m,\varphi,\psi} = M_{m,\varphi,\psi} = I\). The unconditional convergence follows as in Example 4.1.6(i).

(ii) Let \(m = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \frac{1}{2^5}, \frac{1}{2^6}, \frac{1}{2^7}, \frac{1}{2^8}, \ldots).

Then \(M_{m,\varphi,\psi} = M_{m,\varphi,\psi} = G_1\) - non-invertible on \(\mathcal{H}\) (see Lemma 2.9). The unconditional convergence follows as in Example 4.1.6(ii).

Example 4.1.21. Let \(\Phi = (e_1, 2e_2, e_1, 3e_3, e_1, 4e_4, e_1, 5e_5, \ldots), \quad m = (\frac{1}{2}, 2, \frac{1}{2}, 3, \frac{1}{2}, 4, \frac{1}{2}, 5, \ldots).

(i) Let \(\Psi = (e_1, \frac{1}{2^2} e_2, 2e_1, \frac{1}{2^3} e_3, 2^2 e_1, \frac{1}{2^4} e_4, 2^3 e_1, \frac{1}{2^5} e_5, \ldots).\) Then \(M_{m,\varphi,\psi} = M_{m,\varphi,\psi} = I\). The unconditional convergence follows as in Example 4.1.6(i).

(ii) Let \(\Psi = (e_1, \frac{1}{2^2} e_2, 2e_1, \frac{1}{2^3} e_3, 2^2 e_1, \frac{1}{2^4} e_4, 2^3 e_1, \frac{1}{2^5} e_5, \ldots).\) Then \(M_{m,\varphi,\psi} = M_{m,\varphi,\psi} = G_1\) - non-invertible on \(\mathcal{H}\) (see Lemma 2.9). The unconditional convergence follows as in Example 4.1.6(ii).
Example 4.1.22. Let $\Phi = (\frac{1}{\sqrt{2}} e_1, \ 2 e_2, \ \frac{1}{\sqrt{2}} e_1, \ \frac{1}{3} e_3, \ \frac{1}{\sqrt{2}} e_1, \ 4 e_4, \ \frac{1}{\sqrt{2}} e_1, \ \frac{1}{3} e_5, \ldots)$.

$\Psi = (\frac{1}{\sqrt{2}} e_1, \ \frac{1}{2} e_2, \ \frac{1}{\sqrt{2}} e_1, \ 3 e_3, \ \frac{1}{\sqrt{2}} e_1, \ \frac{1}{4} e_4, \ \frac{1}{\sqrt{2}} e_1, \ 5 e_5, \ldots)$.

Then $M(1),_{\Phi, \Psi} = M(1),_{\Psi, \Phi} = I$. The convergence is unconditional on $H$, because $M(1),_{\Phi, \Psi} \equiv M(1),_{\Psi, \Phi} \equiv M(1),_{\Theta, \Theta}$, where $\Theta$ is the same as in Example 4.1.6(i).

Example 4.1.23. Let $\Phi = (\frac{1}{\sqrt{2}} e_1, \ 2 e_2, \ \frac{1}{\sqrt{2}} e_1, \ \frac{1}{3} e_3, \ \frac{1}{\sqrt{2}} e_1, \ 4 e_4, \ \frac{1}{\sqrt{2}} e_1, \ \frac{1}{3} e_5, \ldots)$.

$\Psi = (\frac{1}{\sqrt{2}} e_1, \ \frac{1}{2} e_2, \ \frac{1}{\sqrt{2}} e_1, \ 3 e_3, \ \frac{1}{\sqrt{2}} e_1, \ \frac{1}{4} e_4, \ \frac{1}{\sqrt{2}} e_1, \ 5 e_5, \ldots)$.

Then $M(1),_{\Phi, \Psi} = M(1),_{\Psi, \Phi} = G_1$ - non-invertible on $H$ (see Lemma 2.9). The convergence is unconditional on $H$, because $M(1),_{\Phi, \Psi} \equiv M(1),_{\Psi, \Phi} \equiv M(1),_{\Theta, \Theta}$, where $\Theta$ is the same as in Example 4.1.6(ii) (apply Prop. 2.1).

Example 4.1.24. Let $\Phi = (\frac{1}{\sqrt{2}} e_1, \ 2 e_2, \ \frac{1}{\sqrt{2}} e_1, \ 3 e_3, \ \frac{1}{\sqrt{2}} e_1, \ 4 e_4, \ \frac{1}{\sqrt{2}} e_1, \ 5 e_5, \ldots)$.

(i) Let $m = (1, \ \frac{1}{2^2}, \ 1, \ \frac{1}{3^2}, \ 1, \ \frac{1}{4^2}, \ 1, \ \frac{1}{5^2}, \ldots)$.

Then $M_{m,_{\Phi, \Psi}} = I$. The convergence is unconditional on $H$, because $M_{m,_{\Phi, \Psi}} \equiv M_{(1),_{\Theta, \Theta}}$, where $\Theta$ is the same as in Example 4.1.6(i) (apply Prop. 2.1).

(ii) Let $m = (1, \ \frac{1}{2^2}, \ 1, \ \frac{1}{3}, \ 1, \ \frac{1}{4^2}, \ 1, \ \frac{1}{5^2}, \ldots)$.

Then $M_{m,_{\Phi, \Psi}} = G_1$ - non-invertible on $H$ (see Lemma 2.9). The convergence is unconditional on $H$, because $M_{m,_{\Phi, \Psi}} \equiv M_{(1),_{\Theta, \Theta}}$, where $\Theta$ is the same as in Example 4.1.6(ii).

Example 4.1.25. Let $\Phi = (\frac{1}{\sqrt{2}} e_1, \ 2 e_2, \ \frac{1}{\sqrt{2}} e_1, \ \frac{1}{3} e_3, \ \frac{1}{\sqrt{2}} e_1, \ 4 e_4, \ \frac{1}{\sqrt{2}} e_1, \ \frac{1}{3} e_5, \ldots)$.

(i) Let $m = (1, \ \frac{1}{2^2}, \ 1, \ 3^2, \ 1, \ \frac{1}{4^2}, \ 1, \ 5^2, \ldots)$.

Then $M_{m,_{\Phi, \Psi}} = I$. The convergence is unconditional on $H$, because $M_{m,_{\Phi, \Psi}} \equiv M_{(1),_{\Theta, \Theta}}$, where $\Theta$ is the same as in Example 4.1.6(i) (apply Prop. 2.1).

(ii) Let $m = (1, \ \frac{1}{2^2}, \ 1, \ 3, \ 1, \ \frac{1}{4^2}, \ 1, \ 5, \ldots)$.

Then $M_{m,_{\Phi, \Psi}} = G_1$ - non-invertible on $H$ (see Lemma 2.9). The convergence is unconditional on $H$, because $M_{m,_{\Phi, \Psi}} \equiv M_{(1),_{\Theta, \Theta}}$, where $\Theta$ is the same as in Example 4.1.6(ii).

4.2 Examples for one Bessel non-frame and one non-Bessel sequence; TABLE 2 on page 7

Example 4.2.1. Let $\Phi = (e_2, \ e_3, \ e_4, \ e_5, \ e_6, \ e_7, \ldots)$.

$m = (\frac{1}{2^2}, \ 1, \ \frac{1}{2^2}, \ 1, \ \frac{1}{2^2}, \ 1, \ldots)$.

(i) Let $\Psi = (e_1, \ e_2, \ e_3, \ e_4, \ e_5, \ldots)$.
Then $M_{m, \Phi, \Psi}$ and $M_{m, \Psi, \Phi}$ are unconditionally convergent on $\mathcal{H}$, because $\Phi$ and $m\Psi$ are Bessel for $\mathcal{H}$ (apply Prop. 2.1). The multiplier $M_{m, \Phi, \Psi}$ is not surjective on $\mathcal{H}$, because $\Phi$ is not complete in $\mathcal{H}$. The multiplier $M_{m, \Psi, \Phi}$ is not injective, for example $M_{m, \Psi, \Phi}(2e_2) = e_1 = M_{m, \Psi, \Phi}(4e_2)

(ii) Let $\Psi = (e_2, \frac{1}{2}e_2, e_2, \frac{1}{3}e_3, e_2, \frac{1}{4}e_4, \ldots)$. Then $M_{m, \Phi, \Psi}$ and $M_{m, \Psi, \Phi}$ are unconditionally convergent on $\mathcal{H}$, because $\Phi$ and $m\Psi$ are Bessel for $\mathcal{H}$ (apply Prop. 2.1). The multiplier $M_{m, \Phi, \Psi}$ (resp. $M_{m, \Psi, \Phi}$) is not invertible on $\mathcal{H}$, because $\Phi$ (resp. $\Psi$) is not complete in $\mathcal{H}$.

**Example 4.2.2.** Let $\Phi = (e_2, e_3, e_4, e_5, e_6, e_7, \ldots)$, $\Psi = (e_2, 3e_2, 4e_4, 5e_5, \ldots)$, $m = (\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots)$. The conclusion is the same as in Example 4.2.1(ii).

**Example 4.2.3.** Let $\Phi = (e_2, e_3, e_4, e_5, \ldots)$, $\Psi = (2e_2, 3e_2, 4e_4, 5e_5, \ldots)$, $m = (\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots)$. The conclusion is the same as in Example 4.2.1(ii).

**Example 4.2.4.** Let $\Phi = (e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, \ldots)$, $\Psi = (e_2, \frac{1}{2}e_2, 2e_2, \frac{1}{3}e_3, 2^2e_2, \frac{1}{4}e_4, 2^3e_2, \frac{1}{5}e_5, \ldots)$, $m = (\frac{1}{2}, 1, \frac{1}{3}, 1, \frac{1}{4}, 1, \frac{1}{5}, 1, \ldots)$. The conclusion is the same as in Example 4.2.1(ii).

**Example 4.2.5.** Let $\Phi = (e_2, e_3, e_4, e_5, \ldots)$, $\Psi = (2e_2, \frac{1}{3}e_3, 4e_4, \frac{1}{5}e_5, \ldots)$, $m = (\frac{1}{2}, 3, \frac{1}{4}, 5, \ldots)$. The conclusion is the same as in Example 4.2.1(ii).

**Example 4.2.6.** Let $\Phi = (e_1, \frac{1}{2}e_2, \frac{1}{3}e_3, \frac{1}{4}e_4, \frac{1}{5}e_5, \ldots)$, $\Psi = (e_1, e_2, e_2, e_3, e_3, e_3, \ldots)$. (i) Then $M_{(1), \Phi, \Psi} = M_{(1), \Psi, \Phi} = I$. The convergence is unconditional on $\mathcal{H}$, because $M_{(1), \Phi, \Psi} = M_{(1), \Psi, \Phi} = M_{(1), \Theta, \Theta}$, where $\Theta = (e_1, \frac{1}{\sqrt{2}}e_2, \frac{1}{\sqrt{3}}e_3, \frac{1}{\sqrt{4}}e_4, \frac{1}{\sqrt{5}}e_5, \ldots)$ is Bessel for $\mathcal{H}$ (apply Prop. 2.1).

(ii) Let $m = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots)$. Then $M_{m, \Phi, \Psi} = M_{m, \Psi, \Phi} = G_1$ - non-invertible on $\mathcal{H}$ (see Lemma 2.9). The convergence is unconditional on $\mathcal{H}$, because $M_{m, \Phi, \Psi} = M_{m, \Psi, \Phi} = M_{m, \Theta, \Theta}$, where $\Theta$ is the same as in (i) (apply Prop. 2.1).
Example 4.2.7. Let $\Phi = (e_1, \frac{1}{2}e_2, \frac{1}{3}e_2, \frac{1}{3}e_3, \frac{1}{3}e_3, \frac{1}{4}e_4, \frac{1}{4}e_4, \frac{1}{4}e_4, \ldots)$, $\Psi = (e_1, e_2, e_2, e_3, e_3, e_3, e_4, e_4, e_4, e_4, \ldots)$. Then $M_{(1),\Phi,\Psi} = M_{(1),\Psi,\Phi} = G_1$ - non-invertible on $\mathcal{H}$ (see Lemma 2.9). The convergence is unconditional on $\mathcal{H}$, because $M_{(1),\Phi,\Psi} \subseteq M_{(1),\Psi,\Phi} \subseteq M_{(1),\Theta,\Theta}$, where $\Theta = (e_1, \frac{1}{2}e_2, \frac{1}{3}e_2, \frac{1}{3}e_3, \frac{1}{3}e_3, \frac{1}{4}e_4, \frac{1}{4}e_4, \frac{1}{4}e_4, \frac{1}{4}e_4, \ldots)$ is Bessel for $\mathcal{H}$ (apply Prop. 2.1).

Example 4.2.8. Let $\Phi = (\frac{1}{\sqrt{2}}e_1, \frac{1}{3}e_1, \frac{1}{3}e_2, \frac{1}{3}e_2, \frac{1}{3}e_3, \frac{1}{3}e_3, \frac{1}{4}e_4, \frac{1}{4}e_4, \frac{1}{4}e_4, \frac{1}{4}e_4, \ldots)$, $\Psi = (e_1, e_2, e_2, e_3, e_3, e_3, e_4, e_4, e_4, e_4, \ldots)$, $m = (\frac{1}{\sqrt{2}}, 1, 1, 1, 1, 1, 1, 1, 1, 1, \ldots)$. Then $M_{m,\Phi,\Psi} = M_{m,\Psi,\Phi} = I$. The convergence is unconditional on $\mathcal{H}$, because $M_{m,\Phi,\Psi} \subseteq M_{m,\Psi,\Phi} \subseteq M_{(1),\Theta,\Theta}$, where $\Theta = (\frac{1}{\sqrt{2}}e_1, \frac{1}{\sqrt{2}}e_2, \frac{1}{\sqrt{2}}e_2, \frac{1}{\sqrt{2}}e_3, \frac{1}{\sqrt{2}}e_3, \frac{1}{\sqrt{2}}e_3, \frac{1}{\sqrt{2}}e_4, \frac{1}{\sqrt{2}}e_4, \frac{1}{\sqrt{2}}e_4, \frac{1}{\sqrt{2}}e_4, \ldots)$ is Bessel for $\mathcal{H}$ (apply Prop. 2.1).

Example 4.2.9. Let $\Psi = (e_1, e_2, e_1, e_3, e_1, e_4, \ldots)$, $m = (1, 2, 1, 3, 1, 4, \ldots)$. (i) Let $\Phi = (\frac{1}{2}e_1, \frac{1}{2}e_2, \frac{1}{2}e_2, \frac{1}{2}e_3, \frac{1}{2}e_3, \frac{1}{2}e_4, \frac{1}{2}e_4, \frac{1}{2}e_4, \ldots)$. Then $M_{m,\Phi,\Psi} = M_{m,\Psi,\Phi} = I$. The unconditional convergence follows as in Example 4.1.6(i).

(ii) Let $\Phi = (\frac{1}{4}e_1, \frac{1}{4}e_1, \frac{1}{4}e_1, \frac{1}{4}e_1, \frac{1}{4}e_1, \frac{1}{4}e_1, \frac{1}{4}e_1, \frac{1}{4}e_1, \frac{1}{4}e_1, \frac{1}{4}e_1, \ldots)$. Then $M_{m,\Phi,\Psi} = M_{m,\Psi,\Phi} = G_1$ - non-invertible on $\mathcal{H}$ (see Lemma 2.9). The unconditional convergence follows as in Example 4.1.6(ii).

Example 4.2.10. Let $\Phi = (e_1, \frac{1}{2}e_2, \frac{1}{2}e_2, \frac{1}{2}e_3, \frac{1}{2}e_3, \frac{1}{2}e_4, \frac{1}{2}e_4, \frac{1}{2}e_4, \frac{1}{2}e_4, \ldots)$, $\Psi = (e_1, e_2, \frac{1}{2}e_2, e_3, e_3, e_4, e_4, e_4, e_4, \ldots)$. Then $M_{(1),\Phi,\Psi} = M_{(1),\Psi,\Phi} = I$. The unconditional convergence follows as in Example 4.2.6(i).

Example 4.2.11. Let $\Phi = (\frac{1}{2}e_1, \frac{1}{2}e_1, \frac{1}{2}e_2, \frac{1}{2}e_2, \frac{1}{2}e_3, \frac{1}{2}e_3, \frac{1}{2}e_4, \frac{1}{2}e_4, \frac{1}{2}e_4, \ldots)$, $\Psi = (e_1, e_1, \frac{1}{2}e_2, e_2, e_2, e_3, e_3, e_4, e_4, \ldots)$. Then $M_{(1),\Phi,\Psi} = M_{(1),\Psi,\Phi} = G_1$ - non-invertible on $\mathcal{H}$ (see Lemma 2.9). The convergence is unconditional on $\mathcal{H}$, because $M_{(1),\Phi,\Psi} \subseteq M_{(1),\Psi,\Phi} \subseteq M_{(1),\Theta,\Theta}$, where $\Theta$ is the same as in Example 4.1.6(ii) (apply Prop. 2.1).

Example 4.2.12. Let $\Phi = (e_1, \frac{1}{2}e_2, \frac{1}{2}e_2, \frac{1}{2}e_3, \frac{1}{2}e_3, \frac{1}{2}e_4, \frac{1}{2}e_4, \frac{1}{2}e_4, \frac{1}{2}e_4, \ldots)$, $\Psi = (e_1, e_2, \frac{1}{2}e_2, e_3, e_3, e_4, e_4, e_4, \frac{1}{2}e_4, \ldots)$, $m = (1, 1, \frac{1}{\sqrt{2}}, 1, 1, \frac{1}{\sqrt{2}}, 1, 1, \frac{1}{\sqrt{2}}, \ldots)$. Then $M_{m,\Phi,\Psi} = M_{m,\Psi,\Phi} = I$. The unconditional convergence follows as in Example 4.2.6(i).
Then $M_{m,\Phi,\Psi} = M_{m,\Psi,\Phi} = I$. The unconditional convergence on $\mathcal{H}$ follows in the same way as in Example 4.2.6(i).

Example 4.2.13. Let $\Phi = (e_1, \frac{1}{2}e_2, \frac{1}{2}e_2, e_3, \frac{1}{3}e_3, \frac{1}{4}e_4, \frac{1}{4}e_4, \frac{1}{4}e_4, \ldots)$, $\Psi = (e_1, e_2, \frac{1}{2}e_2, e_3, \frac{1}{3}e_3, e_4, e_4, e_4, \frac{1}{2}e_4, \ldots)$.

(i) Let $m = (1, \frac{1}{2}, 1, \frac{1}{3}, \frac{1}{4}, 1, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 1, \ldots)$. Then $M_{m,\Phi,\Psi} = M_{m,\Psi,\Phi} = G_1$ - non-invertible on $\mathcal{H}$ (see Lemma 2.9). The convergence is unconditional on $\mathcal{H}$, because $M_{m,\Phi,\Psi} \uparrow M_{m,\Psi,\Phi} \uparrow M_{(1),\Theta,\Theta}$, where $\Theta$ is the same as in Example 4.2.7 (apply Prop. 2.1).

(ii) Let $m = (1, 1, 2, 1, 1, 3, 1, 1, 1, 4, \ldots)$. Then $M_{m,\Phi,\Psi} = M_{m,\Psi,\Phi} = I$. The convergence is unconditional on $\mathcal{H}$, because $M_{m,\Phi,\Psi} \uparrow M_{m,\Psi,\Phi} \uparrow M_{(1),\Theta,\Theta}$, where $\Theta$ is the same as in Example 4.2.6(i).

Example 4.2.14. Let $\Phi = (e_1, \frac{1}{2}e_2, \frac{1}{2}e_2, e_3, \frac{1}{3}e_3, \frac{1}{4}e_4, \frac{1}{4}e_4, \frac{1}{4}e_4, \frac{1}{4}e_4, \ldots)$, $\Psi = (e_1, e_2, \frac{1}{2}e_2, e_3, e_3, e_4, e_4, e_4, \frac{1}{2}e_4, \ldots)$, $m = (1, \frac{1}{2}, 2, \frac{1}{3}, \frac{1}{4}, 3, \frac{1}{4}, \frac{1}{4}, 4, \ldots)$. Then $M_{m,\Phi,\Psi} = M_{m,\Psi,\Phi} = G_1$ - non-invertible on $\mathcal{H}$ (see Lemma 2.9). The convergence is unconditional on $\mathcal{H}$, because $M_{m,\Phi,\Psi} \uparrow M_{m,\Psi,\Phi} \uparrow M_{(1),\Theta,\Theta}$, where $\Theta$ is the same as in Example 4.2.7 (apply Prop. 2.1).

Example 4.2.15. Let $\Phi = (\frac{1}{n}e_n)$ and $\Psi = (ne_n)$. Then $M_{(1),\Phi,\Psi} \uparrow M_{(1),\Psi,\Phi} \uparrow M_{(1),e_n,e_n} = I$.

Example 4.2.16. Let $\Phi = (\frac{1}{n}e_n)$ and $\Psi = (ne_n)$. Then $M_{(1),\Phi,\Psi} \uparrow M_{(1),\Psi,\Phi} \uparrow M_{(\Phi),e_n,e_n} = G_1$ - unconditionally convergent and non-invertible on $\mathcal{H}$ (see Lemma 2.9).

Example 4.2.17. Let $\Phi = (\frac{1}{n}e_n)$, $\Psi = (n^2e_n)$, and $m = (\frac{1}{n})$. Then $M_{m,\Phi,\Psi} \uparrow M_{m,\Psi,\Phi} \uparrow M_{(1),e_n,e_n} = I$.

Example 4.2.18. Let $\Phi = (\frac{1}{n}e_n)$, $\Psi = (n^2e_n)$, and $m = (\frac{1}{n^2})$. Then $M_{m,\Phi,\Psi} \uparrow M_{m,\Psi,\Phi} \uparrow M_{(\Phi),e_n,e_n} = G_1$ - unconditionally convergent and non-invertible on $\mathcal{H}$ (see Lemma 2.9).

Example 4.2.19. Let $\Phi = (\frac{1}{n}e_n)$, $\Psi = (ne_n)$, and $m = (n^2)$. Then $M_{m,\Phi,\Psi} \uparrow M_{m,\Psi,\Phi} \uparrow M_{(1),e_n,e_n} = I$.

Example 4.2.20. Let $\Phi = (\frac{1}{n}e_n)$, $\Psi = (ne_n)$, and $m = (n)$. Then $M_{m,\Phi,\Psi} \uparrow M_{m,\Psi,\Phi} \uparrow M_{(\Phi),e_n,e_n} = G_1$ - unconditionally convergent and non-invertible on $\mathcal{H}$ (see Lemma 2.9).

Example 4.2.21. Let $\Phi = (\frac{1}{\sqrt{2}}e_1, \frac{1}{2}e_2, \frac{1}{\sqrt{2}}e_1, \frac{1}{3}e_3, \frac{1}{\sqrt{2}}e_1, \frac{1}{4}e_4, \ldots)$, $\Psi = (\frac{1}{\sqrt{2}}e_1, 2e_2, \frac{1}{\sqrt{2}}e_1, 3e_3, \frac{1}{\sqrt{2}}e_1, 4e_4, \ldots)$. Then $M_{(1),\Phi,\Psi} = M_{(1),\Psi,\Phi} = I$. The convergence is unconditional on $\mathcal{H}$, because $M_{(1),\Phi,\Psi} \uparrow M_{(1),\Psi,\Phi} \uparrow M_{(1),\Theta,\Theta}$, where $\Theta$ is the same as in Example 4.1.6(i).
Example 4.2.22. Let \( \Phi = \left( \frac{1}{\sqrt{2}} e_1, \frac{1}{2} e_2, \frac{1}{\sqrt{2}} e_1, \frac{1}{2} e_3, \frac{1}{\sqrt{2}} e_1, \frac{1}{2} e_4, \ldots \right) \),
\[ \Psi = \left( \frac{1}{\sqrt{2}} e_1, \ 2 e_2, \ \frac{1}{\sqrt{2}} e_1, \ 3 e_3, \ \frac{1}{\sqrt{2}} e_1, \ 4 e_4, \ \ldots \right). \]
Then \( M^{(1)}, \Phi, \Psi = G_1 \) - non-invertible on \( \mathcal{H} \) (see Lemma 2.9). The convergence is unconditional on \( \mathcal{H} \), because \( M^{(1)}, \Phi, \Psi \overset{\psi}{=} M^{(1)}, \Psi, \Phi \overset{\psi}{=} M^{(1)}, \Theta, \Theta \), where \( \Theta \) is the same as in Example 4.1.6(ii) (apply Prop. 2.1).

Example 4.2.23. Let \( \Phi = \left( \frac{1}{\sqrt{2}} e_1, \ \frac{1}{2} e_2, \ \frac{1}{\sqrt{2}} e_1, \ \frac{1}{2} e_3, \ \frac{1}{\sqrt{2}} e_1, \ \frac{1}{2} e_4, \ \ldots \right) \),
\[ \Psi = \left( \frac{1}{\sqrt{2}} e_1, \ 2^2 e_2, \ \frac{1}{\sqrt{2}} e_1, \ 3^2 e_3, \ \frac{1}{\sqrt{2}} e_1, \ 4^2 e_4, \ \ldots \right). \]
(i) Let \( m = \left( \begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ \ldots \end{array} \right) \).
Then \( M_m, \Phi, \Psi = M_m, \Psi, \Phi = I \). The unconditional convergence follows as in Example 4.1.6(i).
(ii) Let \( m = \left( \begin{array}{c} 1 \\ \frac{1}{2} \\ 1 \\ \frac{1}{3} \\ 1 \\ \frac{1}{4} \\ \ldots \end{array} \right) \).
Then \( M_m, \Phi, \Psi = M_m, \Psi, \Phi = G_1 \) - non-invertible on \( \mathcal{H} \) (see Lemma 2.9). The convergence is unconditional on \( \mathcal{H} \), because \( M_m, \Phi, \Psi \overset{\psi}{=} M_m, \Psi, \Phi \overset{\psi}{=} M^{(1)}, m, \Phi, \Psi \) and the sequences \( \Phi \) and \( m, \Psi \) are Bessel for \( \mathcal{H} \) (apply Prop. 2.1).

Example 4.2.24. Let \( \Phi = (e_1, \ \frac{1}{2} e_2, \ \frac{1}{3} e_3, \ \frac{1}{4} e_4, \ \ldots) \),
\[ \Psi = (e_1, \ 2 e_2, \ \frac{1}{3} e_3, \ 4 e_4, \ \frac{1}{5} e_5, \ \ldots) \]
(i) Let \( m = \left( \begin{array}{c} 1 \\ 1 \\ 3^2 \\ 4 \\ 1 \\ 5^2 \\ \ldots \end{array} \right) \).
Then \( M_m, \Phi, \Psi \overset{\psi}{=} M_m, \Psi, \Phi = M^{(1)}, (\epsilon_n), (\epsilon_n) = I \).
(ii) Let \( m = \left( \begin{array}{c} 1 \\ \frac{1}{2} \\ 3 \\ \frac{1}{4} \\ 3 \\ 5 \\ \ldots \end{array} \right) \).
Then \( M_m, \Phi, \Psi \overset{\psi}{=} M_m, \Psi, \Phi = G_1 \) - unconditionally convergent and non-invertible on \( \mathcal{H} \) (see Lemma 2.9).

4.3 Examples for two Bessel non-frame sequences; TABLE 3 on page 9

Example 4.3.1. Let \( \Phi = (e_2, e_3, e_4, e_5, \ldots) \).
(i) Let \( m = (1, 1, 1, 1, \ldots) \). Then \( M_m, \Phi, \Phi \) is clearly unconditionally convergent on \( \mathcal{H} \) and not surjective.
(ii) Let \( m = \left( \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots \right) \). Then \( M_m, \Phi, \Phi \) is clearly unconditionally convergent on \( \mathcal{H} \) and not surjective.

Example 4.3.2. Let \( \Phi = (e_2, e_3, e_4, e_5, \ldots) \) and \( \Psi = \left( \frac{1}{2} e_2, \ \frac{1}{3} e_3, \ \frac{1}{4} e_4, \ \frac{1}{5} e_5, \ \ldots \right) \).
(i) Let \( m = (1, 1, 1, 1, \ldots) \). Then \( M_m, \Phi, \Psi \) and \( M_m, \Psi, \Phi \) are clearly unconditionally convergent on \( \mathcal{H} \) and not surjective.
(ii) Let \( m = \left( \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots \right) \). Then \( M_m, \Phi, \Psi \) and \( M_m, \Psi, \Phi \) are clearly unconditionally convergent on \( \mathcal{H} \) and not surjective.
(iii) Let \( m = (2, 3, 4, 5, \ldots) \). Then \( M_m, \Phi, \Psi \) and \( M_m, \Psi, \Phi \) are clearly unconditionally convergent on \( \mathcal{H} \) and not surjective.
Example 4.3.3. Let $\Phi = (\frac{1}{m}e_n)$.

(i) Let $m = (1)$. Then $M_{m,\Phi,\Phi} \subseteq M_{(\frac{1}{m^2}),e_n,e_n} = G_2$ - unconditionally convergent and non-invertible on $\mathcal{H}$ (see Lemma 2.9).

(ii) Let $m = (\frac{1}{m})$. Then $M_{m,\Phi,\Phi} \subseteq M_{(\frac{1}{m^2}),e_n,e_n} = G_3$ - unconditionally convergent and non-invertible on $\mathcal{H}$ (see Lemma 2.9).

(iii) Let $m = (n^2)$. Then $M_{m,\Phi,\Phi} \subseteq M_{(1),e_n,e_n} = I$.

(iv) Let $m = (n)$. Then $M_{m,\Phi,\Phi} \subseteq M_{(\frac{1}{m^2}),e_n,e_n} = G_1$ - unconditionally convergent and non-invertible on $\mathcal{H}$ (see Lemma 2.9).

4.4 Examples for one overcomplete frame and one non-Bessel sequence; TABLE 4 on page 10

Example 4.4.1. Let $\Phi = (e_1, e_2, e_2, e_2, e_3, e_3, e_4, e_4, e_4, \ldots)$,
\[ \Psi = (e_1, e_1, -e_1, e_2, e_1, -e_1, e_3, e_1, -e_1, e_4, \ldots) \]
\[ m = (1, \frac{1}{2}, 1, \frac{1}{2}, 1, \frac{1}{2}, 1, \frac{1}{2}, 1, \frac{1}{2}, \ldots) \]
Then $M_{m,\Phi,\Psi} = M_{m,\Phi,\Phi} = I$. The convergence of $M_{m,\Phi,\Psi}$ and $M_{m,\Phi,\Phi}$ is unconditional on $\mathcal{H}$, because $\Phi$ and $m\Psi$ are Bessel for $\mathcal{H}$ (apply Prop. 2.1).

Example 4.4.2. Let $\Phi = (e_1, e_1, e_2, e_3, e_4, e_5, \ldots)$,
\[ \Psi = (e_2, e_2, e_3, e_2, e_4, e_2, \ldots) \]
\[ m = (1, \frac{1}{2}, 1, \frac{1}{2}, 1, \frac{1}{2}, \ldots) \]
The unconditional convergence of $M_{m,\Phi,\Psi}$ and $M_{m,\Phi,\Phi}$ follows as in Example 4.4.1. The multiplier $M_{m,\Phi,\Psi}$ is not injective, for example $M_{m,\Phi,\Psi} e_1 = 0$. The multiplier $M_{m,\Psi,\Psi}$ is not surjective, because $\Psi$ is not complete in $\mathcal{H}$.

Example 4.4.3. Let $\Phi = (e_1, e_1, e_1, e_1, e_2, e_2, e_2, e_2, e_2, e_2, e_3, e_3, e_3, e_3, e_3, e_3, \ldots)$,
\[ \Psi = (e_1, e_1, -e_1, \frac{1}{2}e_1, -\frac{1}{2}e_1, e_2, e_1, -e_1, \frac{1}{2}e_1, -\frac{1}{2}e_1, e_3, e_1, -e_1, \frac{1}{2}e_1, -\frac{1}{2}e_1, \ldots) \]
\[ m = (1, \frac{1}{2}, 1, \frac{1}{2}, 1, 1, \frac{1}{2}, \frac{1}{2}, 1, 1, 1, \frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, \ldots) \]
Then $M_{m,\Phi,\Psi} = M_{m,\Phi,\Phi} = I$. The convergence is unconditional on $\mathcal{H}$, because $\Phi$ and $m\Psi$ are Bessel for $\mathcal{H}$ (apply Prop. 2.1).

Example 4.4.4. Let $\Phi = (e_1, e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, \ldots)$,
\[ \Psi = (e_2, e_3, e_1, e_4, \frac{1}{2}e_5, e_4, \frac{1}{2}e_6, e_4, \frac{1}{2}e_7, e_4, \ldots) \]
(i) Let $m = (1, 1, 1, 1, \frac{1}{2}, 1, \frac{1}{2}, 1, \frac{1}{2}, 1, \frac{1}{2}, \ldots)$. The unconditional convergence of $M_{m,\Phi,\Psi}$ and $M_{m,\Phi,\Phi}$ follows as in Example 4.4.1. The multiplier $M_{m,\Phi,\Psi}$ (resp. $M_{m,\Phi,\Phi}$) is not injective, because $M_{m,\Phi,\Psi} e_1 = e_1 = M_{m,\Phi,\Phi} e_3$ (resp. $M_{m,\Phi,\Phi} e_2 = e_1 = M_{m,\Phi,\Phi} e_3$).

(ii) Let $m = (1, 1, 1, 1, \frac{1}{2}, 5, \frac{1}{2}, 6, \frac{1}{2}, 7, \frac{1}{2}, \ldots)$.}
Then the same conclusion as in (i) holds.
Example 4.4.5. Let $\Phi = (e_1, e_2, e_2, e_3, e_3, e_3, e_4, e_4, e_4, \ldots)$.

$\Psi = (e_1, e_1, -e_1, \frac{1}{2} e_2, e_1, -e_1, \frac{1}{2} e_3, e_1, -e_1, \frac{1}{2} e_4, \ldots)$.

$m = (1, \frac{1}{2}, \frac{1}{3}, 2, \frac{1}{2}, \frac{1}{3}, 3, \frac{1}{2}, \frac{1}{3}, 4, \ldots)$.

Then $M_{m,\Phi,\Psi} = M_{m,\Psi,\Phi} = I$. The convergence is unconditional on $\mathcal{H}$, because $\Phi$ and $m\Psi$ are Bessel for $\mathcal{H}$ (apply Prop. 2.1).

Example 4.4.6. Let $\Phi = (e_1, e_1, e_2, e_3, e_4, e_5, \ldots)$.

$\Psi = (\frac{1}{2} e_1, \frac{1}{2} e_1, 2e_2, 3e_3, 4e_4, 5e_5, \ldots)$.

(i) Let $m = (1, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots)$.

Then $M_{m,\Phi,\Psi} = M_{m,\Psi,\Phi} = I$. The convergence is unconditional on $\mathcal{H}$, because $\Phi$ and $m\Psi$ are Bessel for $\mathcal{H}$ (apply Prop. 2.1).

(ii) Let $m = (1, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots)$.

Then $M_{m,\Phi,\Psi} = M_{m,\Psi,\Phi} = G_1$ - non-invertible on $\mathcal{H}$ (see Lemma 2.9). The convergence is unconditional on $\mathcal{H}$, because $\Phi$ and $m\Psi$ are Bessel for $\mathcal{H}$ (apply Prop. 2.1).

Example 4.4.7. Let $\Phi = (e_1, e_1, e_1, e_2, e_2, e_2, e_3, e_3, e_3, e_4, e_4, e_4, e_5, e_5, e_5, \ldots)$.

$\Psi = (e_1, e_1, e_1, e_2, \frac{1}{2} e_2, \frac{1}{2} e_2, e_3, 3e_3, 3e_3, e_4, \frac{1}{2} e_4, \frac{1}{2} e_4, e_5, 5e_5, 5e_5, \ldots)$.

$m = (1, 1, -1, 1, 1, -1, 1, \frac{1}{2}, -\frac{1}{2}, 1, 1, -1, 1, \frac{1}{2}, -\frac{1}{2}, \ldots)$.

Then $M_{m,\Phi,\Psi} = M_{m,\Psi,\Phi} = I$. The convergence is unconditional on $\mathcal{H}$, because $\Phi$ and $m\Psi$ are Bessel for $\mathcal{H}$ (apply Prop. 2.1).

Example 4.4.8. Let $\Phi = (e_1, e_1, e_2, e_2, e_2, e_3, e_3, e_3, e_4, e_4, e_4, e_5, e_5, e_6, e_6, \ldots)$.

$\Psi = (e_2, e_2, 2e_2, \frac{1}{2} e_2, 3e_3, \frac{1}{3} e_3, 4e_4, \frac{1}{4} e_4, 5e_5, \frac{1}{5} e_5, 6e_6, \frac{1}{6} e_6, \ldots)$.

$m = (1, 1, \frac{1}{2}, 1, \frac{1}{3}, 1, \frac{1}{4}, 1, \frac{1}{5}, 1, \frac{1}{6}, 1, \ldots)$.

Then $M_{m,\Phi,\Psi} = M_{m,\Psi,\Phi}$ are unconditionally convergent on $\mathcal{H}$, because $\Phi$ and $m\Psi$ are Bessel for $\mathcal{H}$ (apply Prop. 2.1). They are not injective - for example, $M_{m,\Phi,\Psi}e_1 = 0$ and $M_{m,\Psi,\Phi}(\frac{2}{3} e_1) = e_2 = M_{m,\Phi,\Psi}(\frac{2}{3} e_2)$.

Example 4.4.9. Let $\Phi = (e_1, e_1, e_2, e_3, e_4, e_5, \ldots)$.

$\Psi = (\frac{2}{3} e_1, \frac{2}{3} e_1, 2e_2, \frac{1}{3} e_3, 4e_4, \frac{1}{3} e_5, \ldots)$.

$m = (1, 1, \frac{1}{2}, 1, \frac{1}{3}, 3, \frac{1}{4}, \frac{1}{5}, 5, \ldots)$.

Then $M_{m,\Phi,\Psi} = M_{m,\Psi,\Phi} = I$. The convergence is unconditional on $\mathcal{H}$, because $\Phi$ and $m\Psi$ are Bessel for $\mathcal{H}$ (apply Prop. 2.1).

Example 4.4.10. Let $\Phi = (e_1, e_1, e_2, e_3, e_4, e_5, \ldots)$.

$\Psi = (\frac{1}{2} e_1, \frac{1}{2} e_1, 2e_2, \frac{1}{2} e_3, 4e_4, \frac{1}{2} e_5, \ldots)$.

$m = (1, 1, \frac{1}{2}, 3, \frac{1}{2}, 5, \ldots)$.
Then $M_{m,\Phi,\Psi} = M_{m,\Psi,\Phi} = G_1$ - non-invertible on $\mathcal{H}$ (see Lemma 2.9). The convergence is unconditional on $\mathcal{H}$, because $\Phi$ and $m\Psi$ are Bessel for $\mathcal{H}$ (apply Prop. 2.1).

**Example 4.4.11.** Let $\Phi = (\frac{1}{2} e_1, \ e_2, \ \frac{1}{\sqrt{2}} e_1, \ e_3, \ \frac{1}{\sqrt{3}} e_1, \ e_4, \ldots)$, 
$\Psi = (e_1, \ e_2, \ e_1, \ e_3, \ e_1, \ e_4, \ldots)$.

(i) Then $M_{(1),\Phi,\Psi} = M_{(1),\Psi,\Phi} = I$. The convergence is unconditional on $\mathcal{H}$, because $M_{(1),\Phi,\Psi} \subseteq M_{(1),\Psi,\Phi} \subseteq M_{(1),\Theta,\Theta}$, where $\Theta$ is the same as in Example 4.1.6(iii) (apply Prop. 2.1).

(ii) Let $m = \left(1, \ \frac{1}{2}, \ 1, \ \frac{1}{3}, \ 1, \ \frac{1}{4}, \ldots \right)$. Then $M_{m,\Phi,\Psi} = M_{m,\Psi,\Phi} = G_1$ - non-invertible on $\mathcal{H}$ (see Lemma 2.9). The convergence is unconditional on $\mathcal{H}$, because $M_{m,\Phi,\Psi} \subseteq M_{m,\Psi,\Phi} \subseteq M_{m,\Theta,\Theta}$, where $\Theta$ is the same as in Example 4.1.6(iii) (apply Prop. 2.1).

**Example 4.4.12.** Let $\Phi = (\frac{1}{2} e_1, \ e_2, \ \frac{1}{\sqrt{2}} e_1, \ e_3, \ \frac{1}{\sqrt{3}} e_1, \ e_4, \ \frac{1}{\sqrt{4}} e_1, \ e_5, \ldots)$, 
$\Psi = (e_1, \ e_3, \ e_1, \ e_3, \ e_1, \ e_4, \ e_1, \ e_5, \ldots)$.

Then $M_{(1),\Phi,\Psi}$ and $M_{(1),\Psi,\Phi}$ are unconditionally convergent on $\mathcal{H}$, because $M_{(1),\Phi,\Psi} \subseteq M_{(1),\Theta,\Xi}$ and $M_{m,\Phi,\Psi} \subseteq M_{m,\Xi,\Theta}$, where $\Theta = (\frac{1}{2} e_1, \ e_2, \ \frac{1}{\sqrt{2}} e_1, \ e_3, \ \frac{1}{\sqrt{3}} e_1, \ e_4, \ \frac{1}{\sqrt{4}} e_1, \ e_5, \ldots)$ and $\Xi = (e_3, \ e_3, \ \frac{1}{\sqrt{2}} e_1, \ e_3, \ \frac{1}{\sqrt{3}} e_1, \ e_4, \ \frac{1}{\sqrt{4}} e_1, \ e_5, \ldots)$ are Bessel for $\mathcal{H}$ (apply Prop. 2.1).

The multiplier $M_{(1),\Phi,\Psi}$ is not injective, for example $M_{(1),\Phi,\Psi} e_2 = 0$. The multiplier $M_{(1),\Phi,\Psi}$ is not surjective, because $\Psi$ is not complete in $\mathcal{H}$.

**Example 4.4.13.** Let $\Phi = (e_1, \ \frac{1}{\sqrt{2}} e_2, \ \frac{1}{\sqrt{2}} e_2, \ \frac{1}{\sqrt{2}} e_3, \ \frac{1}{\sqrt{2}} e_3, \ \frac{1}{\sqrt{3}} e_3, \ \frac{1}{\sqrt{3}} e_3, \ldots)$, 
$\Psi = (e_1, \ e_2, \ e_2, \ e_3, \ e_3, \ e_3, \ldots)$, 
$m = (1, \ \frac{1}{\sqrt{2}}, \ \frac{1}{\sqrt{3}}, \ \frac{1}{\sqrt{2}}, \ \frac{1}{\sqrt{3}}, \ldots)$.

Then $M_{m,\Phi,\Psi} = M_{m,\Psi,\Phi} = I$. The convergence is unconditional on $\mathcal{H}$, because $\Phi$ and $m\Psi$ are Bessel for $\mathcal{H}$ (apply Prop. 2.1).

**Example 4.4.14.** Let $\Phi = (\frac{1}{2\sqrt{2}} e_1, \ e_2, \ \frac{1}{2\sqrt{2}} e_1, \ e_3, \ \frac{1}{2\sqrt{2}} e_1, \ e_4, \ldots)$, 
$\Psi = (\frac{1}{2} e_1, \ e_2, \ e_1, \ e_3, \ e_1, \ e_4, \ldots)$.

(i) Let $m = (2, \ 1, \ 2^2, \ 1, \ 2^3, \ 1, \ldots)$.

Then $M_{m,\Phi,\Psi} = M_{m,\Psi,\Phi} = I$. The unconditional convergence follows as in Example 4.1.6(i).

(ii) Let $m = (2, \ \frac{1}{2}, \ 2^2, \ \frac{1}{3}, \ 2^3, \ \frac{1}{4}, \ldots)$.

Then $M_{m,\Phi,\Psi} = M_{m,\Psi,\Phi} = G_1$ - non-invertible on $\mathcal{H}$ (see Lemma 2.9). The unconditional convergence follows as in Example 4.1.6(ii).
Example 4.4.15. Let \( \Phi = (\frac{1}{2} e_1, \ e_2, \ \frac{1}{3} e_2, \ \frac{1}{2} e_1, \ e_3, \ \frac{1}{2} e_3, \ \frac{1}{3} e_1, \ e_4, \ \frac{1}{2} e_4, \ \frac{1}{3} e_4, \ldots) \),
\( \Psi = (\ e_1, \ e_2, \ \frac{1}{2} e_2, \ -\frac{1}{2} e_2, \ e_3, \ \frac{1}{2} e_3, \ -\frac{1}{2} e_3, \ e_4, \ \frac{1}{2} e_4, \ -\frac{1}{2} e_4, \ldots) \).

Then \( M_{(1),\Phi,\Psi} = M_{(1),\Psi,\Phi} = I \). The convergence is unconditional on \( \mathcal{H} \), because \( M_{(1),\Phi,\Psi} \preceq M_{(1),\Psi,\Phi} \preceq M_{\nu,\Theta,\Theta} \), where
\[
\Theta = (\frac{1}{\sqrt{2}} e_1, \ e_2, \ \frac{1}{2} e_2, \ \frac{1}{2} e_1, \ e_3, \ \frac{1}{2} e_3, \ \frac{1}{2} e_1, e_4, \ \frac{1}{2} e_4, \ \frac{1}{2} e_4, \ldots) \text{ is Bessel for } \mathcal{H} \text{ and }
\nu = (1, \ 1, \ 1, \ -1, \ 1, \ 1, \ 1, \ -1, \ 1, \ 1, \ 1, \ -1, \ldots) \text{ is } SN \text{ (apply Prop. 2.1).}
\]

Example 4.4.16. Let \( \Phi = (\frac{1}{2} e_1, \ e_2, \ \frac{1}{3} e_1, \ e_3, \ \frac{1}{2} e_1, \ e_4, \ldots) \),
\( \Psi = (\ e_1, \ \frac{1}{2} e_2, \ e_1, \ \frac{1}{3} e_3, \ e_1, \ \frac{1}{2} e_4, \ldots) \).

(i) Then \( M_{(1),\Phi,\Psi} = M_{(1),\Psi,\Phi} = G_1 \) - non-invertible on \( \mathcal{H} \) (see Lemma 2.9). The convergence is unconditional on \( \mathcal{H} \), because \( M_{(1),\Phi,\Psi} \preceq M_{(1),\Psi,\Phi} \preceq M_{(1),\Theta,\Theta} \), where \( \Theta \) is the same as in Example 4.4.6(ii).

(ii) Let \( m = (1, \ 2, \ 1, \ 3, \ 1, \ 4, \ldots) \).

Then \( M_{m,\Phi,\Psi} = M_{m,\Psi,\Phi} = I \). The unconditional convergence follows as in Example 4.4.6(i).

Example 4.4.17. Let \( \Phi = (\frac{1}{2} e_1, \ e_2, \ \frac{1}{3} e_1, \ e_3, \ \frac{1}{2} e_1, \ e_4, \ e_5, \ e_6, \ \frac{1}{2} e_1, \ e_7, \ldots) \),
\( \Psi = (\ e_1, \ e_2, \ \frac{1}{\sqrt{2}} e_1, \ e_3, \ e_1, \ e_4, \ e_5, \ e_1, \ e_6, \ \frac{1}{\sqrt{2}} e_1, \ e_7, \ldots) \),
\( m = (1, \ 1, \ \frac{1}{\sqrt{2}}, \ 1, \ \frac{1}{2}, \ 1, \ \frac{1}{\sqrt{2}}, \ 1, \ \frac{1}{2}, \ 1, \ \frac{1}{\sqrt{2}}, \ 1, \ldots) \).

Then \( M_{m,\Phi,\Psi} = M_{m,\Psi,\Phi} = I \). The convergence is unconditional on \( \mathcal{H} \), because \( \Phi \) and \( m\Psi \) are Bessel for \( \mathcal{H} \) (apply Prop. 2.1).

Example 4.4.18. Let \( \Phi = (\frac{1}{2} e_1, \ e_2, \ \frac{1}{\sqrt{2}} e_1, \ e_3, \ \frac{1}{2} e_1, \ e_4, \ldots) \),
\( \Psi = (\ e_1, \ \frac{1}{\sqrt{2}} e_2, \ e_1, \ \frac{1}{\sqrt{3}} e_3, \ e_1, \ \frac{1}{\sqrt{4}} e_4, \ldots) \),
\( m = (1, \ \frac{1}{\sqrt{2}}, \ 1, \ \frac{1}{\sqrt{3}}, \ 1, \ \frac{1}{\sqrt{4}}, \ldots) \).

Then \( M_{m,\Phi,\Psi} = M_{m,\Psi,\Phi} = G_1 \) - non-invertible on \( \mathcal{H} \) (see Lemma 2.9). The unconditional convergence follows as in Example 4.4.6(ii).

Example 4.4.19. Let \( \Phi = (\frac{1}{2} e_1, \ e_2, \ \frac{1}{3} e_1, \ e_3, \ \frac{1}{2} e_1, \ e_4, \ldots) \),
\( \Psi = (\ e_1, \ \frac{1}{2} e_2, \ e_1, \ \frac{1}{3} e_3, \ e_1, \ \frac{1}{2} e_4, \ldots) \),
\( m = (1, \ 2, \ 1, \ 3, \ 1, \ 4, \ldots) \).

Then \( M_{m,\Phi,\Psi} = M_{m,\Psi,\Phi} = G_1 \) - non-invertible on \( \mathcal{H} \) (see Lemma 2.9). The unconditional convergence follows as in Example 4.4.6(ii).

Example 4.4.20. Let \( \Phi = (\frac{1}{2} e_1, \ e_2, \ \frac{1}{3} e_2, \ \frac{1}{3} e_2, \ e_3, \ -\frac{1}{3} e_3, \ \frac{1}{3} e_3, \ e_4, \ \frac{1}{3} e_4, \ \frac{1}{3} e_4, \ldots) \),
\( \Psi = (\ e_1, \ e_2, \ -2 e_2, \ e_3, \ 3 e_3, \ -3 e_3, \ e_4, \ 4 e_4, \ -4 e_4, \ldots) \).

Then \( M_{(1),\Phi,\Psi} = M_{(1),\Psi,\Phi} = I \). The convergence is unconditional on \( \mathcal{H} \), because \( M_{(1),\Phi,\Psi} \preceq M_{(1),\Psi,\Phi} \preceq M_{\nu,\Theta,\Theta} \), where
\[ \Theta = \left( \frac{1}{\sqrt{2}} e_1, e_2, e_2, \frac{1}{\sqrt{2}} e_1, e_3, e_3, e_3, \frac{1}{\sqrt{2}} e_1, e_4, e_4, \ldots \right) \] is Bessel for \( \mathcal{H} \) and
\[ \nu = (1, 1, 1, -1, 1, 1, 1, -1, 1, 1, -1, \ldots) \] is \( SN \) (apply Prop. 2.1).

Example 4.4.21. Let \( \Phi = \left( \frac{1}{2} e_1, e_2, \frac{1}{2} e_2, \frac{1}{2} e_2, \frac{1}{2} e_2, e_1, e_3, e_3, \frac{1}{2} e_3, \frac{1}{2} e_3, e_4, \frac{1}{2} e_4, \frac{1}{2} e_4, \ldots \right) \)
\[ \Psi = (e_2, e_2, 2e_2, -2e_2, e_2, e_3, 3e_3, -3e_3, e_2, e_4, 4e_4, -4e_4, \ldots) \]
Then \( M_{(1),\Phi,\Psi} \) is not injective and \( M_{(1),\Phi,\Psi} \) is not surjective. They are unconditionally convergent on \( \mathcal{H} \), because \( M_{(1),\Phi,\Psi} = M_{(1),\Theta,\Xi} \) and \( M_{(1),\Phi,\Psi} = M_{(1),\Xi,\Theta} \), where
\[ \Theta = \left( \frac{1}{\sqrt{2}} e_1, e_2, e_2, \frac{1}{\sqrt{2}} e_1, e_3, e_3, e_3, \frac{1}{\sqrt{2}} e_1, e_4, e_4, \ldots \right) \] and
\[ \Xi = \left( \frac{1}{\sqrt{2}} e_2, e_2, e_2, -e_2, \frac{1}{\sqrt{2}} e_2, e_3, e_3, -e_3, \frac{1}{\sqrt{2}} e_2, e_4, e_4, -e_4, \ldots \right) \] are Bessel for \( \mathcal{H} \) (apply Prop. 2.1).

Example 4.4.22. Let \( \Phi = \left( \frac{1}{2} e_1, e_2, \frac{1}{2} e_1, e_3, \frac{1}{2} e_1, e_4, \ldots \right) \)
\[ \Psi = (e_1, 2e_2, e_1, 3e_3, e_1, 4e_4, \ldots) \]

(i) Let \( m = (1, 1, 1, 1, 1, \ldots) \). Then \( M_{m,\Phi,\Psi} = M_{m,\Phi,\Psi} = I \). The unconditional convergence follows as in Example 4.1.6(i).

(ii) Let \( m = (1, 1, 1, 1, \ldots) \). Then \( M_{m,\Phi,\Psi} = M_{m,\Phi,\Psi} = G_1 \) - non-invertible on \( \mathcal{H} \) (see Lemma 2.9). The unconditional convergence follows as in Example 4.1.6(ii).

Example 4.4.23. Let \( \Phi = \left( \frac{1}{2} e_1, e_2, \frac{1}{2} e_1, e_3, \frac{1}{2} e_1, e_4, \ldots \right) \)
\[ \Psi = (e_1, 2e_2, e_1, 3e_3, e_1, 4e_4, \ldots) \]

(i) Let \( m = (2, 1, 2, 1, 2, \ldots) \). Then \( M_{m,\Phi,\Psi} = M_{m,\Phi,\Psi} = I \). The unconditional convergence follows as in Example 4.1.6(i).

(ii) Let \( m = (2, 1, 2, 1, 2, \ldots) \). Then \( M_{m,\Phi,\Psi} = M_{m,\Phi,\Psi} = G_1 \) - non-invertible on \( \mathcal{H} \) (see Lemma 2.9). The unconditional convergence follows as in Example 4.1.6(ii).

Example 4.4.24. Let \( \Phi = \left( \frac{1}{2} e_1, e_2, \frac{1}{2} e_2, \frac{1}{2} e_2, e_1, e_3, e_3, \frac{1}{2} e_3, \frac{1}{2} e_3, e_4, \frac{1}{2} e_4, \frac{1}{2} e_4, \ldots \right) \)
\[ \Psi = (e_1, e_2, 2e_2, -2e_2, e_1, e_3, \frac{1}{2} e_3, -\frac{1}{2} e_3, e_1, e_4, 4e_4, -4e_4, e_1, e_5, \frac{1}{2} e_5, -\frac{1}{2} e_5, \ldots) \]
Then \( M_{(1),\Phi,\Psi} = M_{(1),\Phi,\Psi} = I \). The convergence is unconditional on \( \mathcal{H} \), because \( M_{(1),\Phi,\Psi} = M_{(1),\Theta,\Xi} \) and \( M_{(1),\Phi,\Psi} = M_{(1),\Xi,\Theta} \), where
\[ \Theta = \left( \frac{1}{\sqrt{2}} e_1, e_2, e_2, \frac{1}{\sqrt{2}} e_1, e_3, \frac{1}{\sqrt{2}} e_1, e_4, e_4, e_4, \frac{1}{\sqrt{2}} e_1, e_5, \frac{1}{\sqrt{2}} e_5, \ldots \right) \] is Bessel for \( \mathcal{H} \) and
\[ \nu = (1, 1, 1, -1, 1, 1, 1, -1, 1, 1, 1, -1, \ldots) \] is \( SN \) (apply Prop. 2.1).
Example 4.4.25. Let $\Phi = \left( \frac{1}{2} e_1, e_2, e_3, \frac{1}{2} e_2, e_3, e_4, \frac{1}{2} e_3, \frac{1}{2} e_4, \frac{1}{2} e_5, \frac{1}{2} e_6, \ldots \right)$. 

\[ \Psi = \left( e_2, e_2, 2 e_2, -2 e_2, e_2, e_3, \frac{1}{3} e_3, -\frac{1}{3} e_3, e_4, 4 e_4, -4 e_4, e_5, \frac{1}{5} e_5, -\frac{1}{5} e_5, \ldots \right). \]

Then $M_{(1),\Phi,\Psi}$ is not injective, because $M_{(1),\Phi,\Psi} e_1 = 0$. The multiplier $M_{(1),\Phi,\Psi}$ is not surjective, because $\Psi$ is not complete in $\mathcal{H}$. Both $M_{(1),\Phi,\Psi}$ and $M_{(1),\Psi,\Phi}$ are unconditionally convergent on $\mathcal{H}$, because $M_{(1),\Phi,\Psi} \preceq M_{(1),\Psi,\Phi}$ and $M_{(1),\Psi,\Phi} \preceq M_{(1),\Phi,\Psi}$, where

\[ \Theta = \left( \frac{1}{\sqrt{2}} e_1, e_2, e_2, e_2, \frac{1}{\sqrt{2}} e_1, e_3, \frac{1}{\sqrt{2}} e_3, \frac{1}{\sqrt{2}} e_3, e_4, e_4, \frac{1}{\sqrt{2}} e_4, \frac{1}{\sqrt{2}} e_5, \frac{1}{\sqrt{2}} e_5, \ldots \right) \]

and

\[ \Xi = \left( \frac{1}{\sqrt{2}} e_2, e_2, e_2, -e_2, \frac{1}{\sqrt{2}} e_2, e_3, \frac{1}{\sqrt{2}} e_3, -\frac{1}{\sqrt{2}} e_3, e_4, e_4, -e_4, \frac{1}{\sqrt{2}} e_4, \frac{1}{\sqrt{2}} e_5, -\frac{1}{\sqrt{2}} e_5, \ldots \right) \]

are Bessel for $\mathcal{H}$ (apply Prop. 2.1).

Example 4.4.26. Let $\Phi = \left( \frac{1}{2} e_1, e_2, e_2, e_2, \frac{1}{2} e_1, e_3, e_3, \frac{1}{2} e_3, \frac{1}{2} e_4, \frac{1}{2} e_5, \frac{1}{2} e_5, \ldots \right)$. 

\[ \Psi = \left( e_1, e_2, 2 e_2, -2 e_2, e_1, e_3, \frac{1}{2} e_3, -\frac{1}{2} e_3, e_4, 4 e_4, -4 e_4, e_5, \frac{1}{5} e_5, -\frac{1}{5} e_5, \ldots \right). \]

(i) Let $m = \left( 1, 1, \frac{1}{2}, \frac{1}{2}, 1, 1, \frac{1}{3}, \frac{1}{3}, 1, 1, \frac{1}{5}, \frac{1}{5}, \ldots \right)$.

Then $M_{m,\Phi,\Psi} = M_{m,\Psi,\Phi} = I$. The convergence is unconditional on $\mathcal{H}$, because $M_{m,\Phi,\Psi} \preceq M_{m,\Psi,\Phi} \preceq M_{\nu,\Theta,\Theta}$, where $\Theta$ and $\nu$ are the same as in Example 4.4.24.

(ii) Let $m = \left( 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 1, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \ldots \right)$.

Then $M_{m,\Phi,\Psi} = M_{m,\Psi,\Phi} = G_1$ - non-invertible on $\mathcal{H}$ (see Lemma 2.9). The convergence is unconditional on $\mathcal{H}$, because $M_{m,\Phi,\Psi} \preceq M_{m,\Psi,\Phi} \preceq M_{\nu,\Theta,\Theta}$, where

\[ \Theta = \left( \frac{1}{\sqrt{2}} e_1, \frac{1}{\sqrt{2}} e_2, e_2, e_2, \frac{1}{\sqrt{2}} e_1, \frac{1}{\sqrt{2}} e_3, \frac{1}{\sqrt{2}} e_3, \frac{1}{\sqrt{2}} e_3, e_4, e_4, \frac{1}{\sqrt{2}} e_4, \frac{1}{\sqrt{2}} e_5, \frac{1}{\sqrt{2}} e_5, \ldots \right) \]

is Bessel for $\mathcal{H}$ and $\nu = \left( 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, \ldots \right)$ is $SN$ (apply Prop. 2.1).

Example 4.4.27. Let $\Phi = \left( \frac{1}{2} e_1, e_2, \frac{1}{2} e_2, e_3, \frac{1}{2} e_3, e_4, \frac{1}{2} e_4, \frac{1}{2} e_5, \frac{1}{2} e_5, \ldots \right)$. 

\[ \Psi = \left( e_1, \frac{1}{2} e_2, e_1, 3 e_3, e_4, \frac{1}{4} e_4, e_5, 5 e_5, \ldots \right) \]

\[ m = \left( 1, 2, 1, \frac{1}{3}, 1, 4, 1, \frac{1}{5}, \ldots \right). \]

Then $M_{m,\Phi,\Psi} = M_{m,\Psi,\Phi} = I$. The unconditional convergence follows as in Example 4.1.6(i).

Example 4.4.28. Let $\Phi = \left( \frac{1}{2} e_1, e_2, \frac{1}{2} e_2, e_3, \frac{1}{2} e_3, e_4, \frac{1}{2} e_4, e_5, \frac{1}{2} e_5, \ldots \right)$. 

\[ \Psi = \left( e_1, \frac{1}{2} e_2, e_1, 3 e_3, e_4, \frac{1}{4} e_4, e_5, 5 e_5, \ldots \right) \]

\[ m = \left( 1, 2, 1, \frac{1}{3}, 1, 4, 1, \frac{1}{5}, \ldots \right). \]

Then $M_{(1),\Phi,\Psi} = M_{(1),\Psi,\Phi} = G_1$ - non-invertible on $\mathcal{H}$ (see Lemma 2.9). The convergence is unconditional on $\mathcal{H}$, because $M_{(1),\Phi,\Psi} \preceq M_{(1),\Psi,\Phi} \preceq M_{(1),\Theta,\Theta}$, where $\Theta$ is the same as in Example 4.1.6(ii).

4.5 Examples for one overcomplete frame and one Bessel non-frame sequence; TABLE 5 on page 12

Example 4.5.1. Let $\Phi = \left( e_1, e_2, e_3, e_4, e_5, \ldots \right)$ and $\Psi = \left( e_2, e_3, e_4, e_5, e_6, \ldots \right)$.

(i) Then $M_{(1),\Phi,\Psi}$ and $M_{(1),\Psi,\Phi}$ are unconditionally convergent on $\mathcal{H}$ due to Proposition 2.1 and non-invertible on $\mathcal{H}$ due to Proposition 2.4.
(ii) If \( m = \frac{1}{k} \), then \( M_{m,\Phi,\Psi} \) and \( M_{m,\Psi,\Phi} \) are unconditionally convergent on \( \mathcal{H} \) due to Proposition 2.1 and non-invertible on \( \mathcal{H} \) due to Proposition 2.4.

**Example 4.5.2.** Let \( \Phi = (e_1, e_1, e_2, e_3, e_4, e_5, \ldots) \) and \( \Psi = (\frac{1}{2} e_1, \frac{1}{3} e_1, \frac{1}{4} e_2, \frac{1}{5} e_3, \frac{1}{6} e_4, \frac{1}{7} e_5, \ldots) \).

(i) Then \( M_{(1),\Phi,\Psi} = M_{(1),\Psi,\Phi} = G_1 \) - non-invertible on \( \mathcal{H} \) (see Lemma 2.9). The unconditional convergence on \( \mathcal{H} \) follows from Proposition 2.1.

(ii) If \( m = (1, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots) \), then \( M_{m,\Phi,\Psi} = M_{m,\Psi,\Phi} = G_2 \) - non-invertible on \( \mathcal{H} \) (see Lemma 2.9). The unconditional convergence follows from Proposition 2.1.

(iii) If \( m = (1, 1, 2, 3, 4, 5, \ldots) \), then \( M_{m,\Phi,\Psi} = M_{m,\Psi,\Phi} = I \). The convergence is unconditional on \( \mathcal{H} \), because \( \Phi \) and \( m,\Psi \) are Bessel for \( \mathcal{H} \) (apply Prop. 2.1).

**Example 4.5.3.** Let \( \Phi = (\frac{1}{2} e_1, e_2, e_3, e_4, e_5, \ldots) \) and \( \Psi = (e_2, e_3, e_4, e_5, \ldots) \).

(i) Then \( M_{(1),\Phi,\Psi} \) and \( M_{(1),\Psi,\Phi} \) are unconditionally convergent on \( \mathcal{H} \) due to Proposition 2.1 and non-invertible on \( \mathcal{H} \) due to Proposition 2.4.

(ii) If \( m = (\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots) \), then \( M_{m,\Phi,\Psi} \) and \( M_{m,\Psi,\Phi} \) are unconditionally convergent on \( \mathcal{H} \) due to Proposition 2.1 and non-invertible on \( \mathcal{H} \) due to Proposition 2.4.

**Example 4.5.5.** Let \( \Phi = (\frac{1}{2} e_1, e_2, \frac{1}{2} e_1, e_3, \frac{1}{2} e_1, e_4, \frac{1}{2} e_1, e_5, \ldots) \) and \( \Psi = (e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, \ldots) \).

\[
m = (1, 1, 2, 3, 2^2, 1, 2^3, 1, \ldots)
\]

Since \( m,\Phi \) is a frame for \( \mathcal{H} \) and \( \Psi \) is Bessel for \( \mathcal{H} \), which is not a frame for \( \mathcal{H} \), it follows that \( M_{(1),m,\Phi,\Psi} \) and \( M_{(1),\Psi,m,\Phi} \) are unconditionally convergent (see Prop. 2.1), but not invertible on \( \mathcal{H} \) (see Prop. 2.4).

**Example 4.5.6.** Let \( \Phi = (\frac{1}{2} e_1, e_2, \frac{1}{2} e_1, e_3, \frac{1}{2} e_1, e_4, \ldots) \), and \( \Psi = (\frac{1}{n} e_n) \).

(i) Then \( M_{(1),\Phi,\Psi} \) and \( M_{(1),\Psi,\Phi} \) are unconditionally convergent on \( \mathcal{H} \) due to Proposition 2.1 and non-invertible on \( \mathcal{H} \) due to Proposition 2.4.

(ii) If \( m = (\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots) \), then \( M_{m,\Phi,\Psi} \) and \( M_{m,\Psi,\Phi} \) are unconditionally convergent on \( \mathcal{H} \) due to Proposition 2.1 and non-invertible on \( \mathcal{H} \) due to Proposition 2.4.

**Example 4.5.7.** Let \( \Phi = (\frac{1}{2} e_1, e_2, \frac{1}{2} e_1, e_3, \frac{1}{2} e_1, e_4, \ldots) \) and \( \Psi = (\frac{1}{2} e_1, \frac{1}{3} e_2, \frac{1}{4} e_1, \frac{1}{5} e_3, \frac{1}{6} e_4, \ldots) \).

\[
(i) \text{Let } m = (2, 2, 2^2, 3, 2^3, 4, \ldots).
\]

Then \( M_{m,\Phi,\Psi} = M_{m,\Psi,\Phi} = I \). The unconditional convergence follows as in Example 4.1.6(i).

\[
(ii) \text{Let } m = (2, 1, 2^2, 1, 2^3, 1, \ldots).
\]

Then \( M_{m,\Phi,\Psi} = M_{m,\Psi,\Phi} = G_1 \). The unconditional convergence follows as in Example 4.1.6(ii).
4.6 Examples for two overcomplete frames; TABLE 6 on page 13

Example 4.6.1. Let $\Phi = (e_1, e_1, e_2, e_2, e_3, e_3, \ldots)$.

(i) If $m = \left(\frac{1}{2}\right)$, then $M_{m,\Phi,\Phi} = I$.

(ii) If $m = \left(\frac{1}{2}, 1 - \frac{1}{2}, \frac{1}{3}, 1 - \frac{1}{2}, \frac{1}{3}, 1 - \frac{1}{2}, \ldots\right)$, then $M_{m,\Phi,\Phi} = I$.

(iii) If $m = \left(\frac{1}{2}, 1 - \frac{1}{2}, \frac{1}{3}, 1 - \frac{1}{2}, \frac{1}{3}, 1 - \frac{1}{2}, \ldots\right)$, then $M_{m,\Phi,\Phi} = G_1$ - non-invertible on $H$ (see Lemma 2.9).

The convergence of $M_{m,\Phi,\Phi}$ in (i)-(iii) is unconditional on $H$ due to Prop. 2.1.

Example 4.6.2. Let $\Phi = (e_1, e_1, e_2, e_2, e_3, e_3, \ldots)$ and $\Psi = (e_1, e_1, e_2, e_3, e_4, e_5, \ldots)$. By Proposition 2.1, $M_{(1),\Phi,\Psi}$ and $M_{(1),\Psi,\Phi}$ are unconditionally convergent on $H$. However, $M_{(1),\Phi,\Phi}$ is not injective, for example $M_{(1),\Phi,\Phi}e_2 = e_2 = M_{(1),\Psi,\Phi}e_3$. Furthermore, $M_{(1),\Phi,\Psi}$ is injective, but not surjective. Indeed, observe that $M_{(1),\Phi,\Phi} = M_{(1),\Psi,\Phi}$, where $\Theta = (2e_1, e_2, e_3, e_3, \ldots)$ is an overcomplete frame for $H$, and apply Proposition 2.7(a).

Example 4.6.3. Let $\Phi = (e_1, e_1, -e_1, e_2, -e_2, e_3, e_3, -e_3, \ldots)$,

$\Psi = (e_1, e_1, e_2, \frac{1}{2}e_2, e_3, \frac{1}{2}e_3, \frac{1}{2}e_3, \ldots)$.

(i) If $m = (1)$, then $M_{m,\Phi,\Psi} = M_{m,\Phi,\Phi} = I$. The convergence is unconditional on $H$ due to Proposition 2.1.

(ii) If $m = (1, 1, 1, \frac{1}{2}, 1, \frac{1}{2}, \frac{1}{3}, \ldots)$, then $M_{m,\Phi,\Psi} = M_{m,\Phi,\Phi} = I$. The convergence is unconditional on $H$ due to Proposition 2.1.

(iii) If $m = (1, 1, 1, \frac{1}{2}, 1, \frac{1}{2}, 1, \frac{1}{3}, \ldots)$, then $M_{m,\Phi,\Psi} = M_{m,\Phi,\Phi} = G_1$ - non-invertible on $H$ (see Lemma 2.9). The convergence is unconditional on $H$ due to Proposition 2.1.

(iv) If $m = (1, 1, 1, 2, 1, 3, 3, \ldots)$, then $M_{m,\Phi,\Psi} = M_{m,\Phi,\Phi} = I$. The convergence is unconditional on $H$, because $\Phi$ and $m\Psi$ are Bessel for $H$ (apply Prop. 2.1).

(v) If $m = (1, 1, 1, \frac{1}{2}, 2, 2, \frac{1}{3}, 3, 3, \ldots)$, then $M_{m,\Phi,\Psi} = M_{m,\Phi,\Phi} = G_1$ - non-invertible on $H$ (see Lemma 2.9). The convergence is unconditional on $H$, because $\Phi$ and $m\Psi$ are Bessel for $H$ (apply Prop. 2.1).

Example 4.6.4. Let $\Phi = (e_1, e_2, e_3, e_4, e_4, e_5, e_5, e_6, e_6, e_7, e_7, \ldots)$.

$\Psi = (e_2, e_2, e_1, e_4, e_4, \frac{1}{2}e_1, e_5, \frac{1}{2}e_1, e_6, \frac{1}{2}e_1, e_7, \frac{1}{2}e_1, \ldots)$.

Then $M_{(1),\Phi,\Psi}$ and $M_{(1),\Psi,\Phi}$ are unconditionally convergent on $H$ due to Proposition 2.1. However, $M_{(1),\Phi,\Phi}$ and $M_{(1),\Psi,\Psi}$ are not injective, for example $M_{(1),\Phi,\Phi}e_2 = e_1 = M_{(1),\Phi,\Psi}e_3$ and $M_{(1),\Psi,\Phi}e_2 = e_1 = M_{(1),\Psi,\Phi}e_3$.

Example 4.6.5. Let $\Phi = \left(\frac{1}{\sqrt{2}}e_1, e_2, \frac{1}{\sqrt{2}}e_2, e_3, \frac{1}{\sqrt{2}}e_3, e_4, \ldots\right)$. Then $M_{(1),\Phi,\Phi} = I$. The convergence is unconditional on $H$ due to Proposition 2.1.

Example 4.6.6. Let $\Phi = (e_1, e_2, e_1, e_3, e_4, \frac{1}{2}e_1, e_5, \frac{1}{2}e_1, e_6, \frac{1}{2}e_1, e_7, \ldots)$.

$\Psi = (e_2, e_1, e_3, e_4, \frac{1}{2}e_1, e_5, \frac{1}{2}e_1, e_6, \frac{1}{2}e_1, e_7, \ldots)$.

Then $M_{(1),\Phi,\Psi}$ and $M_{(1),\Psi,\Phi}$ are unconditionally convergent on $H$ due to Prop. 2.1. However, $M_{(1),\Phi,\Psi}$ and $M_{(1),\Psi,\Phi}$ are not injective, for example $M_{(1),\Phi,\Psi}e_2 = e_1 = M_{(1),\Phi,\Phi}e_3$ and $M_{(1),\Psi,\Phi}e_2 = e_1 = M_{(1),\Psi,\Phi}e_3$. 


Example 4.6.7. Let $\Phi = (\frac{1}{\sqrt{2}} e_1, e_2, \frac{1}{\sqrt{2}} e_3, e_4, \ldots)$, $\Psi = (\frac{1}{\sqrt{2}} e_1, e_2, \frac{1}{\sqrt{2}} e_3, e_4, \ldots)$.

(i) If $m = (\frac{1}{\sqrt{2}}, 1, \frac{1}{\sqrt{2}}, 1, \ldots)$, then $M_{m,\Phi,\Psi} = M(1),\Psi,\Phi = I$.

(ii) If $m = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1, 1, \ldots)$, then $M_{m,\Phi,\Psi} = G_1$ - non-invertible on $\mathcal{H}$ (see Lemma 2.9).

The convergence of $M_{m,\Phi,\Psi}$ and $M_{m,\Psi,\Phi}$ in (i)-(ii) is unconditional on $\mathcal{H}$ due to Proposition 2.1.

Example 4.6.8. Let $\Phi = (\frac{1}{\sqrt{2}} e_1, e_2, \frac{1}{\sqrt{2}} e_3, e_4, \ldots)$.

(i) If $m = (2, 1, 2^2, 1, 2^3, 1, \ldots)$, then $M_{m,\Phi,\Psi} = I$.

(ii) If $m = (2, 1, 2^2, 1, 2^3, 1, \ldots)$, then $M_{m,\Psi,\Phi} = G_1$ - non-invertible on $\mathcal{H}$ (see Lemma 2.9).

The convergence of $M_{m,\Phi,\Psi}$ in (i)-(ii) is unconditional on $\mathcal{H}$, because $M_{m,\Phi,\Psi} \cong M(1),\Psi,\Phi = (\sqrt{m_{\Phi,\Psi}},(\sqrt{m_{\Psi,\Phi}})$ and $(\sqrt{m_{\Phi,\Psi}})$ is Bessel for $\mathcal{H}$ (apply Prop. 2.1).

4.7 Examples for one Riesz basis and one non-Bessel sequence; TABLE 7 on page 14

Example 4.7.1. Let $\Phi = (e_n), \Psi = (e_1, e_2, e_3, e_4, \ldots)$, and $m = (\frac{1}{2}, 1, \frac{1}{2}, 1, \frac{1}{2}, 1, \ldots)$. The multipliers $M_{m,\Phi,\Psi}$ and $M_{m,\Psi,\Phi}$ are unconditionally convergent on $\mathcal{H}$, because $\Phi$ and $m\Psi$ are Bessel for $\mathcal{H}$ (apply Prop. 2.1). The non-invertibility is due to Proposition 2.6.

Example 4.7.2. Let $\Phi = (e_n), \Psi = (e_1, \frac{1}{2} e_2, e_1, \frac{1}{2} e_3, e_1, \frac{1}{2} e_4, \ldots)$, and $m = (\frac{1}{2}, 1, \frac{1}{2}, 1, \frac{1}{2}, 1, \ldots)$. Then the conclusion is the same as in Example 4.7.1.

Example 4.7.3. Let $\Phi = (e_n), \Psi = (e_1, \frac{1}{2} e_2, e_1, \frac{1}{2} e_3, e_1, \frac{1}{2} e_4, \ldots)$, and $m = (\frac{1}{2}, 2, \frac{1}{2}, 3, \frac{1}{2}, 4, \ldots)$. Then the conclusion is the same as in Example 4.7.1.

Example 4.7.4. Let $\Phi = (e_n), \Psi = (ne_n), and m = (\frac{1}{n})$. Then $M_{m,\Phi,\Psi} \cong M_{m,\Psi,\Phi} \cong M(1),(e_n),(e_n) = I$.

Example 4.7.5. Let $\Phi = (e_n), \Psi = (ne_n), and m = (\frac{1}{n})$. Then $M_{m,\Phi,\Psi} \cong M_{m,\Psi,\Phi} \cong M(1),(e_n),(e_n) = G_1$ - unconditionally convergent and non-invertible on $\mathcal{H}$ (see Lemma 2.9).

Example 4.7.6. Let $\Phi = (e_n), \Psi = (e_1, \frac{1}{2} e_2, e_1, 3 e_3, e_1, \frac{1}{2} e_4, e_1, 5 e_5, \ldots)$, and $m = (\frac{1}{2}, 1, \frac{1}{2}, 1, \frac{1}{2}, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1, \ldots)$. Then the conclusion is the same as in Example 4.7.1.

Example 4.7.7. Let $\Phi = (e_n), \Psi = (e_1, \frac{1}{2} e_2, 3 e_3, \frac{1}{2} e_4, 5 e_5, \ldots)$, and $m = (1, 2, 4, 1, \frac{1}{2}, \ldots)$. Then $M_{m,\Phi,\Psi} \cong M_{m,\Psi,\Phi} \cong M(1),(e_n),(e_n) = I$.

Example 4.7.8. Let $\Phi = (e_n), \Psi = (e_1, \frac{1}{2} e_2, 3 e_3, \frac{1}{2} e_4, 5 e_5, \ldots)$, and $m = (1, 2, 4, \frac{1}{2}, \ldots)$. Then $M_{m,\Phi,\Psi} \cong M_{m,\Psi,\Phi} \cong M(1),(e_n),(e_n) = G_1$ - unconditionally convergent and non-invertible on $\mathcal{H}$ (see Lemma 2.9).
4.8 Examples for one Riesz basis and one Bessel non-frame sequence; TABLE 8 on page 15

Example 4.8.1. Let $\Phi = (e_n)$, $\Psi = (e_2, e_2, e_3, e_4, e_5, \ldots)$, and $m = (1)$. Then $M_{m,\Psi,\Phi}$ and $M_{m,\Psi,\Phi}$ are unconditionally convergent due to Proposition 2.1 and non-invertible on $\mathcal{H}$ due to Proposition 2.6.

Example 4.8.2. Let $\Phi = (e_n)$, $\Psi = (e_2, e_2, e_3, e_4, e_5, \ldots)$, and $m = (\frac{1}{n})$. Then the conclusion is the same as in Example 4.8.1.

Example 4.8.3. Let $\Phi = (e_n)$, $\Psi = (\frac{1}{n}e_n)$, and $m = (1)$. Then $M_{m,\Psi,\Phi} \supseteq M_{m,\Psi,\Phi} \supseteq M_{\frac{1}{n},(e_n),(e_n)} = G_1$ - unconditionally convergent and non-invertible on $\mathcal{H}$ (see Lemma 2.9).

Example 4.8.4. Let $\Phi = (e_n)$, $\Psi = (\frac{1}{n}e_n)$, and $m = (\frac{1}{n})$. Then $M_{m,\Psi,\Phi} \supseteq M_{m,\Psi,\Phi} \supseteq M_{\frac{1}{n^2},(e_n),(e_n)} = G_2$ - unconditionally convergent and non-invertible on $\mathcal{H}$ (see Lemma 2.9).

Example 4.8.5. Let $\Phi = (e_n)$, $\Psi = (\frac{1}{n}e_n)$, and $m = (n)$. Then $M_{m,\Psi,\Phi} \supseteq M_{m,\Psi,\Phi} \supseteq M_{1,(e_n),(e_n)} = I$.

Example 4.8.6. Let $\Phi = (e_n)$, $\Psi = (\frac{1}{n}e_n)$, and $m = (n)$. Then $M_{m,\Psi,\Phi} \supseteq M_{m,\Psi,\Phi} \supseteq M_{\frac{1}{n^2},(e_n),(e_n)} = G_1$ - unconditionally convergent and non-invertible on $\mathcal{H}$ (see Lemma 2.9).

4.9 Examples for one Riesz basis and one overcomplete frame; TABLE 9 on page 16

Example 4.9.1. Let $\Phi = (e_n)$, $\Psi = (e_1, e_1, e_2, e_3, e_4, \ldots)$, and $m = (1)$. Then $M_{m,\Psi,\Phi}$ and $M_{m,\Psi,\Phi}$ are unconditionally convergent due to Proposition 2.1 and non-invertible on $\mathcal{H}$ due to Proposition 2.7.

Example 4.9.2. Let $\Phi = (e_n)$, $\Psi = (e_1, e_1, e_2, e_3, e_4, \ldots)$, and $m = (1, \frac{1}{2}, 1, \frac{1}{3}, 1, \ldots)$. Then the conclusion is the same as in Example 4.9.1.

Example 4.9.3. Let $\Phi = (e_n)$, $\Psi = (\frac{1}{n}e_1, e_2, \frac{1}{n}e_2, e_3, \frac{1}{n^2}e_1, e_4, \ldots)$, and $m = (1, \frac{1}{2}, 1, \frac{1}{3}, 1, \frac{1}{4}, \ldots)$. Then the conclusion is the same as in Example 4.9.1.

Example 4.9.4. Let $\Phi = (e_n)$, $\Psi = (\frac{1}{n}e_1, e_2, \frac{1}{n^2}e_1, e_3, \frac{1}{n^2}e_1, e_4, \ldots)$, and $m = (1, 1, 2, 1, 2, 1, 2^2, 1, \ldots)$. Then $M_{m,\Psi,\Phi}$ and $M_{m,\Psi,\Phi}$ are unconditionally convergent on $\mathcal{H}$, because $\Phi$ and $m\Psi$ are Bessel for $\mathcal{H}$ (apply Prop. 2.1). Both $M_{m,\Psi,\Phi}$ and $M_{m,\Psi,\Phi}$ are non-invertible on $\mathcal{H}$ by Proposition 2.6.

4.10 Examples for two Riesz bases; TABLE 10 on page 16

Example 4.10.1. Consider $M_{(1),(e_n),(e_n)} = I$.

Example 4.10.2. Consider $M_{(\frac{1}{n^2}),(e_n),(e_n)}$ and see Lemma 2.9.

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