Generalized Heisenberg algebra coherent states for Power-law potentials

K. Berrada∗, M. El Baz† and Y. Hassouni‡

Laboratoire de Physique Théorique, Département de Physique
Faculté des sciences, Université Mohammed V - Agdal
Av. Ibn Battouta, B.P. 1014, Agdal, Rabat, Morocco

Abstract

Coherent states for power-law potentials are constructed using generalized Heisenberg algebras. Klauder’s minimal set of conditions required to obtain coherent states are satisfied. The statistical properties of these states are investigated through the evaluation of the Mandel’s parameter. It is shown that these coherent states are useful for describing the states of real and ideal lasers.

1 Introduction

Ever since the introduction of coherent states theory, several works have been focused on these states and they have turned out to be a very useful tool for the investigation of various problems in physics and have offered surprisingly rich structures [1, 2, 3, 4]. Coherent states were first introduced by Schrödinger [5] in the context of the harmonic oscillator, and was interested in finding quantum states which provide a close behavior to the classical one. Later, the notion of coherent had become very important in quantum optics thanks to the works of Glauber [6]. He defined these states as eigenstates of the annihilation operator \( \hat{a} \) of the harmonic oscillator, while

∗berradakamal2006@gmail.com
†elbaz@fsr.ac.ma
‡y-hassou@fsr.ac.ma
demonstrating that they have the interesting property of minimizing the Heisenberg uncertainty relation. The notion of coherent states was extended afterwards to different systems other the harmonic oscillator for which they were originally built. Coherent states associated to $SU(2)$ and $SU(1,1)$ algebras were introduced by Peremolov [7,8]. These states describe several systems and also have many applications in quantum optics, statistical mechanics, nuclear physics, and condensed matter physics [9,10,11]. Generally, there are several definitions of coherent states. The first is to define the coherent states as eigenstates of an annihilation group operator with complex eigenvalues. This definition is called Barut-Girardello coherent states [12]. The second definition, often called Peremolov coherent states [7,13], is to construct the coherent states by acting with a unitary $z-$displacement operator on the ground state of the system, with $z \in \mathbb{C}$. The last definition, often called intelligent coherent states [14,15], is to consider that the coherent states give the minimum-uncertainty value $\Delta x \Delta p = \hbar/2$. These three definitions are equivalent only in the special case of the Heisenberg–Weyl group (Harmonic oscillator). Due to this multitude of definitions of the notion of coherent states, Klauder [9] proposed a set of conditions to be obeyed by any set of coherent states. These conditions are in fact the common properties satisfied by coherent states in all the definitions presented before. This way this set of condition constitutes the minimum set of requirements on a set of states in order to deserve the nomenclature coherent states.

Generally, it is difficult to build coherent states for arbitrary quantum mechanical systems. But for systems with discrete spectrums, there is an elegant method of building coherent states which has been recently proposed in [1] basing on generalized Heisenberg algebra. These coherent states are defined as eigenstates of the annihilation operator of the generalized Heisenberg algebra having infinite-dimensional representations and satisfying the minimum set of conditions required to construct Klauder’s coherent states.

Coherent states for harmonic oscillator and other systems have attracted much attention for the past years [16,17,18,19,20,21,22] and play important roles in different contexts. The power-law potentials (a general class of potentials) have many applications in theoretical and experimental physics, which can be used to describe a large class of quantum mechanical systems [23,24,25,26]. The coherent states of these potentials could be very helpful and bring more insights on these subjects. In this letter, we are going to construct the coherent states for these potentials following the formalism used in [1]. We, then, investigate the statistical properties of these coherent states. It will be shown that these states can exhibit a sub-Poissonian, Poissonian or a super-Poissonian distribution depending on the parameters of the system.
The organization of the letter is as follows. In section 2 we describe the construction of generalized Heisenberg algebra coherent states; in section 3 we build coherent states for power-law potentials and investigate their statistical properties using Mandel’s $Q$ parameter; in the last section, we present our conclusion.

2 Coherent states of generalized Heisenberg algebra

In this letter, we are going to use the generalized Heisenberg algebra ($GHA$) which has been introduced in Ref. [27] and construct the coherent states associated. The version of the $GHA$ we are going to describe is defined by the following commutation relations

$$HA^\dagger = A^\dagger f(H),$$

$$AH = f(H)A,$$

$$[A, A^\dagger] = f(H) - H,$$

where its generators $(H, A, A^\dagger)$ obey the Hermiticity properties $A = A^\dagger$, $H = H^\dagger$. The Casimir element of the $GHA$ is given by

$$C = A^\dagger A - H = AA^\dagger - f(H).$$

Here $H$ is the Hamiltonian of the physical system under consideration and $f(H)$ is an analytic function of $H$, called the characteristic function of the algebra. It was shown in Ref. [27] that a large class of Heisenberg algebras can be obtained by choosing the appropriate function $f(H)$. For example, by choosing the characteristic function as $f(H) = H + 1$, the algebra described by Eqs. (1)–(3) becomes the harmonic oscillator algebra and for $f(H) = qH + 1$ we obtain the deformed Heisenberg algebra.

The irreducible representation of the $GHA$ is given through a general vector $|m\rangle$ which is required to be an eigenstate of the Hamiltonian. It is described as follows

$$H|m\rangle = E_m|m\rangle$$

$$A^\dagger|m\rangle = N_m|m + 1\rangle$$

$$A|m\rangle = N_{m-1}|m - 1\rangle$$

with

$$N_m^2 = E_{m+1} - E_0,$$
where $E_m = f^m(E_0)$, the $m^{th}$ iterate of $E_0$ under $f$, are the eigenvalues of the Hamiltonian $H$. $A^\dagger$ and $A$ are the generalized creation and annihilation operators of the GHA. This GHA describes a class of quantum systems having energy spectra written as

$$E_{n+1} = f(E_n)$$

where $E_{n+1}$ and $E_n$ are successive energy levels.

The coherent states for the GHA were previously investigated in Ref. [1]. They are defined as the eigenvalues of the annihilation operator of the GHA

$$A|z\rangle = z|z\rangle \quad z \in \mathcal{C},$$

and can be expressed as

$$|z\rangle = \mathcal{N}(z) \sum_{n=0}^{\infty} \frac{z^n}{N_{n-1}!} |n\rangle,$$

The normalization factor is given by

$$\mathcal{N}(z) = \left( \sum_{n=0}^{\infty} \frac{|z|^n}{N_{n-1}^2} |n\rangle \right)^{-\frac{1}{2}},$$

where by definition $N_n! = N_0N_1 \cdots N_n$ and by consistency $N_{-1}! = 1$.

According to Klauder [9, 28], the minimal set of requirements to be imposed on a state $|z\rangle$ to be a coherent state is:

(i) Normalizability

$$\langle z|z\rangle = 1;$$

(ii) Continuity in the label

$$|z - z'| \to 0 \quad \Rightarrow \quad |\langle z| - |z'\rangle| \to 0;$$

(iii) Resolution of unity

$$\int \int_{\mathcal{C}} d^2 z \langle z|W(|z|^2) = 1.$$

In general, the first two conditions are easily satisfied. This is not the case for the third. In fact, this condition restricts considerably the choice of the states to be considered.

Now, we are going to analyze the above minimal set of conditions to obtain a Klauder’s coherent states for power-law potentials.
3 Coherent states for Power-law potentials

The general expression of a one-dimensional power-law potential is

\[
\hat{V}(x, k) = V_0 \left| \frac{x}{a} \right|^k,
\]

where \( V_0 \) and \( a \) are constants with the dimensions of energy and length, respectively. Here \( k \) is the power-law exponent.

These power-law potentials can be used to describe a large class of quantum systems by the proper choice of the exponent \( k \). For \( k > 2 \) we find tightly binding potentials, \( k = 2 \) we have harmonic oscillator potentials and \( k < 2 \) expresses loosely binding potentials.

The Hamiltonian corresponding to different potentials can be written in the form

\[
\hat{H} = \frac{\hat{p}^2}{2m} + \hat{V}(k, x),
\]

where the eigenvalue equations are given by

\[
\hat{H}(k)|n\rangle = E_{n,k}|n\rangle, \quad n \geq 0.
\]

The energies \( E_{n,k} \) may be obtained with the help of WKB approximation

\[
E_{n,k} = \omega(k) \left( n + \frac{\gamma}{4} \right)^{\frac{2}{k+2}},
\]

where

\[
\omega(k) = \left[ \frac{\pi V_0^{\frac{1}{2}} \Gamma \left( \frac{1}{k+1} + \frac{3}{2} \right)}{2a\sqrt{2m}} \Gamma \left( \frac{1}{k+1} + \frac{1}{2} \right) \Gamma \left( \frac{3}{2} \right) \right]^{\frac{2}{k+2}}
\]

is the effective frequency. Here \( \gamma \) defines the Maslov index which accounts for boundary effects at the classical turning points.

The above equation (19) shows that the energy difference between adjacent levels, \( \Delta E_{n,k} = E_{n,k} - E_{n-1,k} \propto (n + \frac{\gamma}{4})^{\frac{2}{k+2}} \), increases for \( k > 2 \) as \( n \) increases (tightly binding potentials). In contrast, for \( k < 2 \) it decreases with the increase of \( n \) (loosely binding potentials), while for \( k = 2 \) it does not depend on \( n \), so the energy spectrum is equally spaced.

The question now is to find the characteristic function of the generalized Heisenberg algebra associated with these spectrums. From the above expression, we have

\[
n = (E_{n,k})^{\frac{2}{k+2}} - \frac{\gamma}{4},
\]

leading us to the expression

\[
E_{n+1,k} = \left( n + 1 + \frac{\gamma}{4} \right)^{\frac{2}{k+2}}.
\]
The substitution of Eq. (20) in Eq. (21) allows us to obtain the characteristic equation

$$E_{n+1, k} = \left( (E_{n, k})^{k+2} + 1 \right)^{\frac{2k}{k+2}}. \quad (22)$$

The characteristic function, then, is

$$f(x, k) = \left( (x)^{k+2} + 1 \right)^{\frac{2k}{k+2}}. \quad (23)$$

According to the scheme prescribed in the previous section, the coherent states associated to power-law potentials are given by

$$|z, k\rangle = N(|z|^2, k) \sum_{n=0}^\infty \frac{z^n}{\sqrt{g(n, k)}} |n\rangle \quad (24)$$

where

$$g(n, k) = \prod_{i=1}^n \left( \left( i + \frac{\gamma}{4} \right)^{\frac{2k}{k+2}} - \left( \frac{\gamma}{4} \right)^{\frac{2k}{k+2}} \right), \quad g(0, k) = 1, \quad (25)$$

and the normalization function

$$N(|z|^2, k) = \left( \sum_{n=0}^\infty \frac{|z|^{2n}}{g(n, k)} \right)^{-\frac{1}{2}}. \quad (26)$$

Now, we can investigate the overcompleteness (or resolution of unity operator) of the GHA coherent states for power-law potentials given by Eq. (24). To this end, we assume the existence of a positive Weight function $W(|z|^2)$ so that an integral over the complex plane exists and gives the result

$$\int \int d^2 z |z, k\rangle \langle z, k| W(|z|^2) = 1. \quad (27)$$

Indeed, substituting Eq. (24) into Eq. (27), adopting the polar coordinate representation $z = r \exp(i\theta)$ of complex numbers and using the completeness of the states $|n\rangle$, the resolution of unity can be expressed by

$$\int_0^\infty x^n W(x) \, dx = g(n, k), \quad n = 0, 1, 2, \cdots \quad (28)$$

where $x = r^2$ and $W(x) = \pi W(x)N^2(x, k)$.

The positive quantities $g(n, k)$ are the power moments of the function $W(x)$ and the problem stated in Eq. (28) is the Stieltjes moment problem which can be solved by means of Mellin and inverse Mellin transforms. To determine the form of the function $W(x)$, we extend the natural integers $n$ to complex values $s$ ($n \to s - 1$) and we rewrite Eq. (28) as

$$\int_0^\infty x^{s-1} W(x) \, dx = g(s - 1, k) = M[|W(x)|^2; s], \quad (29)$$
on the other hand
\[ W(x) = \frac{1}{2\pi i} \int_{C-\infty}^{C+i\infty} g(s - 1)x^{-s}ds = \mathcal{M}^{-1}[g(s - 1, k); x] \]  
(30)
where \( \mathcal{M}[W(x); s] \) is the Mellin transform of \( W(x) \) and \( \mathcal{M}^{-1}[g(s - 1, k); x] \) is the inverse Mellin transform of \( g(s - 1, k) \).

Finally, the Weight function may be written as
\[ W(x) = \frac{\mathcal{M}^{-1}[g(s - 1, k); x]}{\pi N^2(x, k)}, \]  
(31)
and Eq. (27) holds.

Now, we are going to build the coherent states of physical systems corresponding to particular choices of the exponent \( k \).

For \( k = 2 \), the \( GHA \) associated to the power-law potential spectrum reduces to the Heisenberg algebra with characteristic function \( f(x, 2) = x + 1 \). In this case, we have \( N^2_{n-1} = n \) and Eq. (24) becomes the standard coherent states for the harmonic oscillator, \( |z, 2\rangle = N(|z|^2, 2) \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle \), with normalization function \( \mathcal{N}(|z|^2, 2) = \exp\left( -\frac{|z|^2}{2} \right) \) and the Weight function \( \frac{1}{\pi} \).

In the limit case \( k \to \infty \), the power-law potential becomes the infinite square-well potential, and the energies defined in Eq. (19) are written as
\[ E_{n,\infty} = (n + 1)^2 \quad n = 1, 2, 3, \ldots \]  
(32)
We can check that
\[ E_{n+1,\infty} = \left( \sqrt{E_{n,\infty}} + 1 \right)^2. \]  
(33)
From Eq. (33) we find that the characteristic function for this physical system is \( f(x, \infty) = (\sqrt{x} + 1)^2 \) and the corresponding \( GHA \) generated by the Hamiltonian \( H \) and the creation and annihilation operators \( A^\dagger, A \) is given by
\[ [H, A^\dagger] = 2A^\dagger\sqrt{H} + A^\dagger, \]  
(34)
\[ [H, A^\dagger] = -2\sqrt{H}A - A, \]  
(35)
\[ [A, A^\dagger] = 2\sqrt{H} + 1. \]  
(36)
Fock space representations of the algebra generated by \( H, A \) and \( A^\dagger \), as in Eqs. (34)−(36), are obtained by considering the eigenstates \( |n\rangle \) of \( H \). We can verify that \( N^2_{n-1} = (n + 1)^2 - 1 \), and the action of this algebra generators on \( |n\rangle \) is given by
\[ H|n\rangle = \sqrt{n^2}|n - 1\rangle, \quad n = 1, 2, \ldots \]  
(37)
\[ A^\dagger |n\rangle = \sqrt{n^2 - 1} |n + 1\rangle, \]  
\[ A |n\rangle = \sqrt{(n + 1)^2 - 1} |n - 1\rangle. \]  

Therefore, the coherent states are

\[ |z, \infty\rangle = \mathcal{N} \left( |z|^2, \infty \right) \sum_{n=0}^{\infty} \frac{z^n}{N_{n-1}} |n\rangle \]  

where

\[ N_{n-1}! = \frac{1}{\sqrt{2}} \sqrt{n! \sqrt{(n + 2)!}}. \]  

The normalization condition requires that

\[ 2\mathcal{N} \left( |z|^2, \infty \right) \sum_{n=0}^{\infty} \frac{|z|^{2n}}{n!(n + 2)!} = 1. \]  

Noting that

\[ \sum_{n=0}^{\infty} \frac{|z|^{2n}}{n!(n + 2)!} = \frac{I_0(2|z|)}{|z|^2}, \]  

for \( 0 \leq |z| < 1 \), where \( I_n(z) \) is the modified Bessel function of the first kind of order \( n \). Thus, the normalization function may be written as

\[ \mathcal{N} \left( |z|^2, \infty \right) = \left( \frac{|z|^2}{2I_0(2|z|)} \right) ^{ \frac{1}{2} } \]  

where \( 0 \leq |z| < \infty \).

To resolve the identity operator in terms of the coherent states corresponding to the square-well potential we need to find the Weight function \( W(x) \) satisfying

\[ \int_{0}^{\infty} x^n W(x) \, dx = n!(n + 2)!, \quad n = 0, 1, 2, \cdots \]  

where \( x = r^2 \) and \( W(x) = \pi W(x) \mathcal{N}^2(x, \infty) \).

From Eq. (30), the function \( W(x) \) becomes

\[ W(x) = \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} \Gamma(s) \Gamma(s + 2) x^{-s} ds. \]  

For determining this function, we shall mainly use the following relations satisfied by the Mellin transform

\[ M \left[ x^b f \left( ax^h \right); s \right] = \frac{1}{a} x^{-\frac{b}{h}} f^\ast \left( \frac{s + b}{h} \right), \quad \{a, h\} > 0, \]  

\[ \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} f^\ast(s) g^\ast(s) x^{-s} ds = \int_{0}^{\infty} f \left( \frac{x}{t} \right) g(t) \frac{dt}{t} \]  

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where \( M[f(x); s] = f^*(s) \) and \( M[g(x); s] = g^*(s) \).

Employing the above relations and using the modified Bessel function of second kind \( K_2(x) \), we find that

\[
W(x) = \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} \Gamma(s)\Gamma(s+2)x^{-s}ds
\]

\[
= \int_0^\infty t \exp(-t) \exp\left(-\frac{x}{t}\right) dt
\]

\[
= 2x K_2(2\sqrt{x}). \tag{49}
\]

The Weight function \( W(x) \) given in Eq. (45), allowing the resolution of the identity operator, is written as

\[
W(x) = \frac{2x K_2(2\sqrt{x})}{\pi N^2(x, \infty)}. \tag{50}
\]

Finally,

\[
W(x) = \frac{4}{\pi} K_2(2\sqrt{x}) J_2(2\sqrt{x}). \tag{51}
\]

In order to complete the discussion, we should investigate the statistical properties of the GHA coherent states for power-law potentials. To this end, we study Mandel’s \( Q \)-parameter defined by

\[
Q = \frac{\langle (\Delta N)^2 \rangle - \langle N \rangle}{\langle N \rangle}, \tag{52}
\]

where

\[
\langle N \rangle = N^2 \langle |z|^2, k \rangle \sum_{n=0}^{\infty} \frac{|z|^{2n}}{g(n, k)}
\]

is the average photon number of the state, and

\[
\langle (\Delta N)^2 \rangle = \langle N^2 \rangle - \langle N \rangle^2. \tag{54}
\]

This parameter is a good indication to determine whether a state has a sub-Poissonian (if \( -1 \leq Q < 0 \)), Poissonian (if \( Q=0 \)) or super-Poissonian photon number distribution (if \( Q > 0 \)).

In Fig. 1, we plot the Mandel’s \( Q \)-parameter in terms of the GHA coherent states amplitude \( |z| \) for loosely binding potentials \( (k < 2) \). From the figure we find that the states exhibit a super-Poissonian behaviour for lower values of \( |z| \) and sub-Poissonian for high \( |z| \). In the case of tightly binding potentials \( (k > 2) \), it is noticed that the Mandel’s parameter is always negative for all values of \( |z| \) (see Fig. 2), so the distribution is sub-Poissonian. However, we observe that the state becomes in some sense less classical as \( |z| \) becomes large. By less classical we mean here that the states in question get farther from the states exhibiting Poissonian statistics in particular Glauber’s coherent states.
It is well known that the study of the statistical properties is an important topic in quantum optics. In this way, our results show that those coherent states may be useful to describe the states of ideal and real lasers by a proper choice of the power-law exponent.

4 Conclusion

In this letter, using an algebraic approach, we constructed coherent states for power-law potentials. The minimum set of conditions required to obtain Klauder’s coherent states is investigated. We have shown that these states describe a large class of quantum systems (harmonic oscillator, loosely binding potentials and tightly binding), which can be used in several branches of quantum physics.

Finally, we investigated the statistical properties of these coherent states using Mandel’s $Q$ parameter. We have shown that they exhibit Poissonian, sub-Poissonian or super-Poissonian distributions. However, these states are useful to describe the states of ideal and real lasers by a proper choice of the different parameters (i.e., $z$ and $k$). These properties, are useful in branches of quantum physics, such as quantum information. Indeed in such studies, we are interested in the quantum behavior of these states. So by a proper choice of the different parameters. One can choose an adequate set of states in order to generate the entanglement of bipartite composite systems using a beam splitter. This may open new perspectives to exploit these entangled states in the context of quantum teleportation [31], dense coding [32] and entanglement swapping [33].

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Figure 1: Mandel’s $Q$ parameter for loosely binding potentials versus $|z|$ for $k = 0.5$ (dotted line), $k = 1$ (dashed line) and $k = 1.5$ (solid line).

Figure 2: Mandel’s $Q$ parameter for tightly binding potentials versus $|z|$ for $k = 5$ (dotted line), $k = 15$ (dashed line) and $k \to \infty$ (solid line).
This figure "Fig1.jpg" is available in "jpg" format from:

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