Narasimhan–Simha-type metrics on strongly pseudoconvex domains in $\mathbb{C}^n$

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ABSTRACT

For a bounded domain $D \subset \mathbb{C}^n$, let $K_D = K_D(z) > 0$ denote the Bergman kernel on the diagonal and consider the reproducing kernel Hilbert space of holomorphic functions on $D$ that are square integrable with respect to the weight $K_D^{-d}$, where $d \geq 0$ is an integer. The corresponding weighted kernel $K_{D,d}$ transforms appropriately under biholomorphisms and hence produces an invariant Kähler metric on $D$. Thus, there is a hierarchy of such metrics starting with the classical Bergman metric that corresponds to the case $d = 0$. This note is an attempt to study this class of metrics in much the same way as the Bergman metric has been with a view towards identifying properties that are common to this family. When $D$ is strongly pseudoconvex, the scaling principle is used to obtain the boundary asymptotics of these metrics and several invariants associated with them. It turns out that all these metrics are complete on strongly pseudoconvex domains.

1. Introduction

For a bounded domain $D \subset \mathbb{C}^n$ and a non-negative measurable function $\phi$ on it, consider the space $L^2_\phi (D)$ of all measurable functions on $D$ that are square integrable with respect to the measure $\phi \, dV$, where $dV$ is the Lebesgue measure on $\mathbb{C}^n$. It is known that if $1/\phi \in L^\infty_{\text{loc}} (D)$ then the weighted Bergman space

$$A^2_\phi (D) = \left\{ f : D \to \mathbb{C} \text{ holomorphic and} \int_D |f|^2 \phi \, dV < \infty \right\}$$

is a closed subspace of $L^2_\phi (D)$ (see [1]), and therefore, there is a reproducing kernel denoted by $K_{D,\phi}(z, w)$ that makes $A^2_\phi (D)$ a reproducing kernel Hilbert space. The kernel $K_{D,\phi}(z, w)$ is uniquely determined by the following properties: $K_{D,\phi}(\cdot, w) \in A^2_\phi (D)$ for each $w \in D$, it is anti-symmetric, i.e. $K_{D,\phi}(z, w) = \overline{K_{D,\phi}(w, z)}$, and it reproduces $A^2_\phi (D)$:

$$f(z) = \int_D K_{D,\phi}(z, w)f(w)\phi(w) \, dV(w), \quad f \in A^2_\phi (D).$$
It also follows that for any complete orthonormal basis \( \{ \phi_k \} \) of \( A^2_{\phi}(D) \),

\[
K_{D,\phi}(z, w) = \sum_k \phi_k(z)\overline{\phi_k(w)},
\]

where the series converges uniformly on compact subsets of \( D \times D \).

Let \( K_D(z, w) \) be the usual Bergman kernel for \( D \), and \( K_D(z) = K_D(z, z) \) its restriction to the diagonal. Since \( D \) is bounded, \( K_D(z) > 0 \) everywhere, and hence, for an integer \( d \geq 0 \), we may consider \( \phi = K_D^{-d} \) as a weight to which all the considerations mentioned above can be clearly applied. In this case, the weighted Bergman space will be denoted by \( A^2_d(D) \), i.e.

\[
A^2_d(D) = \left\{ f : D \to \mathbb{C} \text{ holomorphic and } \int_D |f|^2 K_D^{-d} \, dV < \infty \right\},
\]

and the corresponding weighted Bergman kernel by \( K_{D,d}(z, w) \) to emphasize the dependence on \( d \geq 0 \). The fact \( K_D(z) \geq 1/\text{Vol}(D) \) everywhere for a bounded domain \( D \) implies that \( K_{D,d}(z) = K_{D,d}(z, z) > 0 \), and that \( \log K_{D,d} \) is strongly plurisubharmonic in the same way as for the usual Bergman kernel can also be established (see [2]). The transformation rule for \( K_{D,d}(z, w) \) under a biholomorphism \( F : D \to D' \) is

\[
K_{D,d}(z, w) = \left( \det F'(z) \right)^{d+1} K_{D',d}(F(z), F(w)) \left( \det F'(w) \right)^{d+1}
\]

as will be seen later. Here, \( F'(z) \) is the complex Jacobian matrix of \( F \) at \( z \). Therefore, \( K_{D,d}(z) \) induces an invariant Kähler metric

\[
ds^2_{D,d} = \sum_{\alpha, \beta = 1}^n g^{D,d}_{\alpha\overline{\beta}}(z) \, dz_\alpha \, d\overline{z}_\beta,
\]

where

\[
g^{D,d}_{\alpha\overline{\beta}}(z) = \frac{\partial^2}{\partial z_\alpha \partial \overline{z}_\beta} \log K_{D,d}(z).
\]

Several aspects of weighted Bergman spaces have been considered even with weight \( K_D^{-d} \).
However, the observation that the choice of \( K_D^{-d}, d \geq 0 \), as a weight gives rise to a kernel that transforms correctly under a change of coordinates, and hence induces an invariant metric, goes back to the work of Narasimhan–Simha [3] on the moduli space of compact complex manifolds with ample canonical bundle. Adapting a special case of their construction on a bounded domain in \( \mathbb{C}^n \) leads to the consideration of \( A^2_d(D) \) and hence \( ds^2_{D,d} \) as in (2). Henceforth, we will refer to \( K_{D,d} \) as a weighted kernel of order \( d \) and \( ds^2_{D,d} \) as a Narasimhan–Simha-type metric of order \( d \). It must be mentioned that some general properties of this metric on pseudoconvex domains in complex manifolds were studied by Chen [2] and more recent applications of this metric can be found in [4–6], for example.

The purpose of this note is to study some aspects of these metrics on \( \mathbb{C}^2 \)-smoothly bounded strongly pseudoconvex domains. In what follows, the integer \( d \geq 0 \) will be fixed once and for all. Let \( \tau_{D,d}(z, \nu) \) be the length of a vector \( \nu \in \mathbb{C}^n \) at \( z \in D \) in the metric \( ds^2_{D,d} \).
The Riemannian volume element of this metric will be denoted by

\[ g_{D_d}(z) = \det G_{D,d}(z) \quad \text{where} \quad G_{D,d}(z) = \left( g_{\alpha \beta}^{(D,d)}(z) \right), \]

and note that

\[ \beta_{D,d}(z) = g_{D,d}(z) \left( K_{D,d}(z) \right)^{-1/(d+1)} \]

is a biholomorphic invariant. The holomorphic sectional curvature of \( ds_{D,d}^2 \) at \( z \in D \) along the vector \( v \in \mathbb{C}^n \setminus \{0\} \) is given by

\[ R_{D,d}(z,v) = \frac{\sum_{\alpha, \beta, \gamma, \delta} R_{\alpha \beta \gamma \delta}^{D,d}(z) v^\alpha v^\beta v^\gamma v^\delta}{\left( \sum_{\alpha, \beta} g_{\alpha \beta}^{D,d}(z) v^\alpha v^\beta \right)^2}, \]

where

\[ R_{\alpha \beta \gamma \delta}^{D,d} = -\frac{\partial^2 g_{\alpha \beta}^{D,d}}{\partial z^\gamma \partial \bar{z}^\delta} + \sum_{\mu, \nu} g_{\alpha \mu}^{D,d} \frac{\partial g_{\mu \nu}^{D,d}}{\partial z^\gamma} g_{\nu \beta}^{D,d}, \]

\( g_{\alpha \beta}^{D,d}(z) \) being the \((\nu, \mu)\)th entry of the inverse of the matrix \( G_{D,d}(z) \). The Ricci curvature of \( ds_{D,d}^2 \) at \( z \in D \) along the vector \( v \in \mathbb{C}^n \setminus \{0\} \) is given by

\[ \text{Ric}_{D,d}(z,v) = \frac{\sum_{\alpha, \beta} \text{Ric}_{\alpha \beta}^{D,d}(z) v^\alpha v^\beta}{\sum_{\alpha, \beta} g_{\alpha \beta}^{D,d}(z) v^\alpha v^\beta}, \]

where

\[ \text{Ric}_{\alpha \beta}^{D,d} = -\frac{\partial^2}{\partial z^\alpha \partial \bar{z}^\beta} \log g_{D,d}. \]

We emphasize that \( \text{Ric}_{D,d}(z,v) \) is not the standard Ricci curvature tensor (acting as a bilinear form), but rather its restriction to the unit sphere, and in particular, independent of the norm of \( v \). Finally, for \( p \in D \), let \( \delta_D(p) \) be the Euclidean distance from \( p \) to the boundary \( \partial D \). For \( p \) close to the boundary \( \partial D \), let \( \tilde{p} \in \partial D \) be such that \( \delta_D(p) = |p - \tilde{p}| \), and for a tangent vector \( v \in \mathbb{C}^n \) based at \( p \), write \( v = v_t(p) + v_n(p) \) where this decomposition is taken along the tangential and normal directions, respectively, at \( \tilde{p} \). We will also use the standard notation

\[ z = (z_1, \ldots, z_n), \quad \dot{z} = (z_1, \ldots, z_{n-1}). \]

**Theorem 1.1:** Let \( D \subset \mathbb{C}^n \) be a \( C^2 \)-smoothly bounded strongly pseudoconvex domain and \( p^0 \in \partial D \). Denote by

\[ D_{\infty} = \left\{ z \in \mathbb{C}^n : 2 \Re z_n + |\dot{z}|^2 < 0 \right\}, \]

the unbounded realization of the unit ball in \( \mathbb{C}^n \) and \( b^* = (0, -1) \in D_{\infty} \). Then there are local holomorphic coordinates near \( p^0 \) in which we have, as \( p \to p^0 \),

\[ (a) \quad \delta_D(p)^{(d+1)(n+1)} K_{D,d}(p) \to K_{D_{\infty},d}(b^*) = c \left( \frac{n!}{2^{n+1} \pi^n} \right)^{d+1}, \]
\( (b) \quad \delta_D(p)^{n+1}g_{D,d}(p) \to g_{D,\infty,d}(b^*) = \frac{1}{2n+1}(d+1)^n(n+1)^n, \)

\( (c) \quad \beta_{D,d}(p) \to \beta_{D,\infty,d}(b^*) = (d+1)^n(n+1)^n \left( \frac{1}{c} \right)^{1/(d+1)} \frac{\pi^n}{n!}, \)

\( (d) \quad \delta_D(p)\tau_{D,d}(p,v) \to \tau_{D,\infty,d}(b^*,(0,v_n)) = \frac{1}{2} \sqrt{(d+1)(n+1)} |v_N(p^0)|, \)

\( (e) \quad \sqrt{\delta_D(p)}\tau_{D,d}(p,v_H(p)) \to \tau_{D,\infty,d}(b^*,(v,0)) = \frac{1}{2} \sqrt{(d+1)(n+1)} L_{\partial D}(p^0,v_H(p^0)), \)

\( (f) \quad R_{D,d}(p,v) \to -\frac{2}{(d+1)(n+1)}, \quad v \in \mathbb{C}^n \setminus \{0\}, \)

\( (g) \quad \text{Ric}_{D,d}(p,v) \to -\frac{1}{d+1}, \quad v \in \mathbb{C}^n \setminus \{0\}. \)

Here

\[ c = \frac{\Gamma((d+1)(n+1))}{n!\Gamma(d(n+1)+1)}, \]

and \( L_{\partial D} \) denotes the Levi form of \( \partial D \) with respect to some defining function for \( D \).

Several remarks are in order. First, the Cayley transform

\[ \Phi(z) = \left( \frac{\sqrt{2}z, z_n+1}{z_n-1, z_n-1} \right) \]

maps \( D_\infty \) biholomorphically onto the unit ball \( \mathbb{B}^n \subset \mathbb{C}^n \) with \( \Phi(b^*) = 0 \). A computation, which is given later, shows that \( K_{\mathbb{B}^n,d} = cK_{\mathbb{B}^n}^{d+1} \) with \( c \) as defined in the theorem above, and this is what enables the explicit limiting expressions above to be computed. Thus, Theorem 1.1(a) shows that

\[ K_{D,d}(z) \sim \left( K_D(z) \right)^{d+1}, \]

where \( z \) is close to the boundary of a \( C^2 \)-smooth strongly pseudoconvex domain. Thus for a fixed \( d \geq 0 \), the weighted Bergman kernel on the diagonal \( K_{D,d}(z) \) behaves like the \((d+1)\)th power of the usual Bergman kernel \( K_D(z) \) as \( z \) varies near \( \partial D \).

The question of determining the behaviour of \( K_{D,d}(z) \) at a given fixed \( z \in D \) is known and can be read off from several existing theorems [7–9] of which Theorem 1.1 in Chen [7] is a convenient prototype. Indeed, the application of this theorem (set \( \phi = K_D^{-1} \) and \( \psi = 1 \) in the notation therein) shows that

\[ \lim_{d \to \infty} \left( K_{D,d}(z) \right)^{1/d} = K_D(z) \]

for each fixed \( z \in D \), and this translates to \( K_{D,d}(z) \sim (K_D(z))^d \) for \( d \) large.

To summarize, if \( K_{D,d}(z) \) is regarded as a function of \( z \) and \( d \), its behaviour as a function of \( z \) is like \( (K_D(z))^{d+1} \) and is furnished by Theorem 1.1 (a), whereas its behaviour as a function of \( d \) is like \( (K_D(z))^d \) and is already known. A very recent result that is relevant
to Theorem 1.1(a) is Theorem 1.1 in [10] where the boundary asymptotic for a weighted Bergman kernel on the unit ball is obtained.

Second, Theorem 1.1(d) and (e) are direct analogues of Graham’s result [11] for the Kobayashi and Carathéodory metrics. Further, Theorem 1.1(f) shows that the holomorphic sectional curvature of $ds_2^2_D$ along a sequence $(p_j, v_j) \in D \times \mathbb{C}^n$ approaches $-2/((d + 1)(n + 1))$ as $(p_j, v_j) \to (p, v) \in \partial D \times \mathbb{C}^n$. A verbatim application of Theorem 1.17 in [12] shows that if $D, D' \subset \mathbb{C}^n$ are $C^2$-smooth strongly pseudoconvex domains equipped with the metrics $ds_2^2_D, ds_2^2_{D'}$, then every isometry $f : (D, ds_2^2_D) \to (D', ds_2^2_{D'})$ is either holomorphic or conjugate holomorphic.

**Theorem 1.2:** Let $D \subset \mathbb{C}^n$ be a $C^2$-smoothly bounded strongly pseudoconvex domain. Then, $ds_2^2_D$ is complete for every $d \geq 0$.

**2. Some examples**

Transformation rule (1) and the invariance of the Narasimhan–Simha metric can be proved by repeating the arguments used for the usual Bergman kernel and metric [13] (see also [2]), and hence, we omit the details noting only the following for our reference later

$$G_{D,d}(z) = F'(z)^T G_{D',d}(F(z)) F'(z).$$

(5)

Here, $F'(z)^T$ is the transpose of the matrix $F'(z)$. We also emphasize that throughout this article, vectors in $\mathbb{C}^n$ will be regarded as column vectors. We now compute the weighted kernel of order $d \geq 0$ for the unit ball and polydisc in $\mathbb{C}^n$ by working with a complete orthonormal basis and recall Selberg’s observation about this kernel on homogeneous domains. For this computation, we will need the following:

**Lemma 2.1:** Let $f$ be a continuous function on $[0, 1]$. Then for $r \in \mathbb{R}^n$ with $r_i \geq 0$, and multiindex $\alpha$,

(a) $\int_{r_1 + \cdots + r_n < 1} f(r_1 + \cdots + r_n)r_1^{\alpha_1} \cdots r_n^{\alpha_n} \, dr = \frac{\prod \Gamma(\alpha_i)}{\Gamma(|\alpha|)} \int_0^{1} f(\tau)\tau^{|\alpha|-1} \, d\tau,$

(b) $\int_{r_1^2 + \cdots + r_n^2 < 1} f(r_1^2 + \cdots + r_n^2)r_1^{2\alpha_1} \cdots r_n^{2\alpha_n} \, dr = \frac{1}{2^n} \frac{\prod \Gamma(\alpha_i)}{\Gamma(|\alpha|)} \int_0^{1} f(\tau)\tau^{|\alpha|-1} \, d\tau.$

For (a), see Section 12.5 of [14] and (b) follows by substituting $r_i^2 = t_i$ in (a).

**Proposition 2.2:** For the unit ball $\mathbb{B}^n \subset \mathbb{C}^n$ and $d \geq 0$

$$K_{\mathbb{B}^n,d} = \frac{\Gamma((d + 1)(n + 1))}{n!\Gamma(d(n + 1) + 1)} K_{\mathbb{B}^n}^{d+1}.$$
**Proof:** A complete orthonormal basis for $A_d^2(D)$ is given by \( \{z^\alpha / \|z^\alpha\|^2_{A_d^2(B^n)} \} \), and using Lemma 2.1, we compute

\[
\|z^\alpha\|^2_{A_d^2(B^n)} = \frac{\pi^{(d+1)n}}{(n!)^d} \frac{\Gamma(d(n+1)+1)}{\Gamma((d+1)(n+1)+|\alpha|)}.
\]

The proposition now follows immediately from the basis expansion of $K_{B^n,d}(z, w)$ and the identity

\[
\frac{1}{(1 - \langle z, w \rangle)^3} = \sum_{\alpha} \frac{\Gamma(s + |\alpha|)}{\Gamma(s) \prod \Gamma(\alpha_i + 1)} (z \bar{w})^\alpha, \quad z, w \in B^n,
\]

with $s = (d + 1)(n + 1)$.

**Corollary 2.3:** For the unit ball $B^n \subset \mathbb{C}^n$, we have

\[
K_{B^n,d}(z) = c \left( \frac{n!}{\pi^n} \right)^{d+1} \frac{1}{(1 - |z|^2)^{(d+1)(n+1)}},
\]

\[
g_{\alpha \beta}^{B^n,d}(z) = (d + 1)(n + 1) \left( \frac{\delta_{\alpha \beta}}{1 - |z|^2} + \frac{z_\alpha \bar{z}_\beta}{(1 - |z|^2)^2} \right),
\]

\[
g_{B^n,d}(z) = \frac{(d + 1)^n}{(1 - |z|^2)^{n+1}},
\]

\[
\beta_{B^n,d}(z) = (d + 1)^n \left( \frac{1}{c} \right)^{1/(d+1)} \frac{\pi^n}{n!},
\]

\[
R_{B^n,d}(z, v) = -\frac{2}{(d + 1)(n + 1)},
\]

\[
\text{Ric}_{B^n,d}(z, v) = -\frac{1}{d + 1},
\]

for $z \in B^n$ and $v \in \mathbb{C}^n \setminus \{0\}$, and where $c$ is a constant in the statement of Theorem 1.1.

**Proof:** The expression for $K_{B^n,d}(z)$ follows from the previous proposition. From this expression,

\[
g_{\alpha \beta}^{B^n,d}(z) = (d + 1)(n + 1) \frac{\partial^2}{\partial z_\alpha \partial \bar{z}_\beta} \log \frac{1}{1 - |z|^2}, \quad (6)
\]

and the required expression follows upon differentiation. Now observe that

\[
g_{D,d}(z) = \frac{(d + 1)^n}{(1 - |z|^2)^{2n}} \det \left( (1 - |z|^2) I + A_z \right),
\]

where $I = I_n$ is the identity matrix of size $n$ and $A_z$ is the matrix $A_z = (\bar{z}_\alpha z_\beta) = \bar{z}z^T$. Since the rank of $A_z$ is $\leq 1$, at least $n-1$ eigenvalues of $A_z$ are 0, and hence the remaining eigenvalue is trace $A = |z|^2$. Therefore, its characteristic polynomial is $P_{A_z}(\lambda) = \det(\lambda I - A_z) = \lambda^n - |z|^2 \lambda^{n-1}$. From this, we obtain

\[
\det \left( (1 - |z|^2) I + A_z \right) = (-1)^n P_{A_z}(-|z|^2) = (1 - |z|^2)^{n-1},
\]

and hence, the desired expression for $g_{B^n,d}(z)$ follows. The expression for $\beta_{B^n,d}(z)$ is now immediate from its definition. Since the automorphism group of $B^n$ acts transitively on
and contains the unitary rotations, the holomorphic sectional curvature of $ds^2_{B^n,d}$ is constant, and hence, it is enough to compute

$$R_{B^n,d}(0, (0, 1)) = \frac{1}{g_{mn}(0)^2} R_{B^n(0)}. $$

Observe that all the first-order partial derivatives of $g_{\alpha\overline{\beta}}$ vanish at 0 and so

$$R_{mn}(0) = -\frac{\partial g_{\alpha\overline{\beta}}}{\partial z_n \partial \overline{z}_n}(0) = -2(d + 1)(n + 1).$$

Also, $g_{mn}(0) = (d + 1)(n + 1)$. Therefore,

$$R_{B^n,d}(z, v) = R_{B^n,d}(0, (0, 1)) = -\frac{2}{(d + 1)(n + 1)}. $$

Finally, from the expression of $g_{B^n,d}(z)$, we obtain

$$\text{Ric}_{a\overline{\beta}}(z) = -(n + 1) \frac{\partial^2}{\partial z_\alpha \partial \overline{z}_\beta} \log \frac{1}{1 - |z|^2}. \quad (7)$$

From (6) and (7),

$$\text{Ric}_{a\overline{\beta}}(z) = -\frac{1}{d + 1}g_{\alpha\overline{\beta}}(z),$$

which implies that $\text{Ric}_{B^n,d}(z, v)$ is the constant $-1/(d + 1)$.

For the polydisc $\Delta^n \subset \mathbb{C}^n$, observe that if $D_1 \subset \mathbb{C}^n$ and $D_2 \subset \mathbb{C}^m$ are bounded domains, then for $(z_1, z_2), (w_1, w_2) \in D_1 \times D_2$

$$K_{D_1 \times D_2,d}((z_1, z_2), (w_1, w_2)) = K_{D_1,d}(z_1, w_1)K_{D_2,d}(z_2, w_2) \quad (8)$$

as the reproducing property shows. By combining this with Proposition 2.2, we obtain

$$K_{\Delta^n,d} = (2d + 1)^n K_{\Delta^n}^{d+1}. \quad (9)$$

In general, for a bounded domain $\Omega \subset \mathbb{C}^n$ with a transitive holomorphic automorphism group, it is known that (see [15] and note that our $K_{\Omega,d}$ coincides with $K_{d+1}$ of this paper)

$$K_{\Omega,d}(z, w) = c(d + 1)(K_{\Omega}(z, w))^{d+1},$$

where

$$\frac{1}{c(d)} = \int_{\Omega} \frac{|K_{\Omega}(z, w)|^{2d}}{(K_{\Omega}(z, z)K_{\Omega}(w, w))^{d+1}} K_{\Omega}(z, w) \, dV.$$ 

3. Localization

The localization of the usual Bergman kernel and metric are well known (see, e.g. [16] for a qualitative estimate on bounded domains of holomorphy and [17] for a quantitative estimate near holomorphic peak points of such domains). We will now show that the analogues of these results hold for $K_{\Delta,d}$ and $\tau_{\Delta,d}$ as well. The first step in doing so is to understand their relation with certain minimum integrals.
3.1. Minimum integrals

Let $D \subset \mathbb{C}^n$ be a bounded domain. For $p \in D$ and a nonzero vector $v \in \mathbb{C}^n$, consider the following minimum integrals:

$$I_{D,d}^0(p,v) = \inf \left\{ \|f\|_{A_d^2(D)}^2 : f \in A_d^2(D), f(p) = 1 \right\},$$

$$I_{D,d}^1(p,v) = \inf \left\{ \|f\|_{A_d^2(D)}^2 : f \in A_d^2(D), f(p) = 0, f'(p)v = 1 \right\},$$

$$I_{D,d}^2(p,v) = \inf \left\{ \|f\|_{A_d^2(D)}^2 : f \in A_d^2(D), f(p) = 0, f'(p)v = 0, v^Tf''(p)v = 1 \right\},$$

$$\lambda_{D,d}^k(p,v) = \inf \left\{ \|f\|_{A_d^2(D)}^2 : f \in A_d^2(D), f(p) = 0, \frac{\partial f}{\partial z_j}(p) = 0 \text{ for } 1 \leq j < k, \frac{\partial f}{\partial z_k}(p) = 1 \right\}, \quad 1 \leq k \leq n,$$

$$I_{D,d}(p,v) = \inf \left\{ \|f\|_{A_d^2(D)}^2 : f \in A_d^2(D), f(p) = 0, f'(p) = 0, v^Tf''(p)\overline{G_{D,d}^{-1}(p)f''(p)v} = 1 \right\},$$

and

$$M_{D,d}(p,v) = \inf \left\{ \|f\|_{A_d^2(D)}^2 : f \in A_d^2(D), f(p) = 0, f'(p) = 0, K_{D,d}^{n-1}v^Tf''(p)\overline{adjG_{D,d}(p)f''(p)v} = 1 \right\}. \quad (10)$$

Here $f''(p)$ is the symmetric matrix

$$\left( \frac{\partial^2 f}{\partial z_i \partial z_j}(p) \right)_{n \times n},$$

and $\text{adj} G_{D,d}(p)$ is the adjugate of the matrix $G_{D,d}(p)$. Note that though $I^0$ and $\lambda^k$ do not depend on $v$, the notations $I_{D,d}^k(p,v)$ and $\lambda_{D,d}^k(p,v)$ will be used purely for convenience. By Montel’s theorem, note that these infimums are realized. Also, the following homogeneity property is evident from the definition of these minimum integrals:

$$I_{D,d}^k(p,\alpha v) = |\alpha|^{-2k}I_{D,d}^k(p,v), \quad k = 1, 2,$$

$$I_{D,d}(p,\alpha v) = |\alpha|^{-2}I_{D,d}(p,v), \quad M_{D,d}(p,\alpha v) = |\alpha|^{-2}M_{D,d}(p,v). \quad (11)$$

Proposition 3.1: The minimum integrals $I$ and $M$ are related by

$$I_{D,d}(p,v) = K_{D,d}^{n-1}(p)g_{D,d}(p)M_{D,d}(p,v) = K_{D,d}^{n-1}\frac{1}{\pi^{n+1}}(p)\beta_{D,d}(p)M_{D,d}(p,v). \quad (12)$$

Proof: The proof follows immediately from $\text{adj} G_{D,d}(p) = g_{D,d}(p)G_{D,d}(p)^{-1}$. ■
\textbf{Proposition 3.2:} If $F : D \to D'$ is a biholomorphism and $\mathcal{I}$ is any of the minimum integrals $I^k$ and $I$, then
\[
\mathcal{I}_{D,d}(p,v) = \left| \det F'(p) \right|^{2d-2} \mathcal{I}_{D',d}(F(p), F'(p)v).
\] (13)

\textbf{Proof:} If $g \in A_d^2(D')$ is a function realizing $\mathcal{I}_{D',d}(F(p), F'(p)v)$, then a routine calculation shows that the function
\[
f(z) = \left( \det F'(p) \right)^{-d-1} g \circ F(z) \left( \det F'(z) \right)^{d+1}
\]
is a candidate for $\mathcal{I}_{D,d}(p,v)$ and so
\[
\mathcal{I}_{D,d}(p,v) \leq \left| \det F'(p) \right|^{2d-2} \mathcal{I}_{D',d}(F(p), F'(p)v).
\]
The analogue of this inequality must hold for $F^{-1} : D' \to D$ as well and hence (13) follows. \hfill \blacksquare

\textbf{Proposition 3.3:} The minimum integrals $I^k_{D,d}$, $k = 0, 1, 2$, are related to $K_{D,d}$, $\tau_{D,d}$, and $R_{D,d}$ by
\[
K_{D,d}(p) = \frac{1}{I^0_{D,d}(p,v)},
\]
\[
\tau^2_{D,d}(p,v) = \frac{I^0_{D,d}(p,v)}{I^1_{D,d}(p,v)},
\]
and
\[
R_{D,d}(p,v) = 2 - \frac{\left( I^1_{D,d}(p,v) \right)^2}{I^0_{D,d}(p,v)I^2_{D,d}(p,v)}.
\]

These formulas can be derived in the same way as for the usual Bergman kernel and metric [18–21], and so, we omit the details. In particular, the first relation when combined with
\[
K_{D,d}(p) = \left\| K_{D,d}(\cdot,p) \right\|_{A_d^2(D)^*}^2,
\]
which is a consequence of the reproducing property, implies that
\[
\left\| \frac{K_{D,d}(\cdot,p)}{K_{D,d}(p)} \right\|^2 = \frac{1}{K_{D,d}(p)} = I^0_{D,d}(p,v),
\]
and so $K_{D,d}(\cdot,p)/K_{D,d}(p)$ uniquely realizes the minimum integral $I^0_{D,d}(p,v)$. The third relation in Proposition 3.3 implies that $R_{D,d} \leq 2$.

The minimum integrals, $\lambda^k_{D,d}(p,v)$, $I_{D,d}(p,v)$, and $M_{D,d}(p,v)$ for $d = 0$, are due to Krantz-Yau [17]. More precisely, for the usual Bergman metric, these are the reciprocals of the extremal domain functions in Section 2 of the above-mentioned paper which were introduced for localizing the Bergman invariant and the Ricci curvature. It is to be noted the notational difference that in the above paper a vector in $\mathbb{C}^n$ is regarded as a row vector. To obtain their relation to $\beta_{D,d}$ and $\text{Ric}_{D,d}$, define
\[
\lambda_{D,d}(p) = \lambda^1_{D,d}(p) \lambda^2_{D,d}(p) \cdots \lambda^n_{D,d}(p).
\]
Proposition 3.4: We have
\[\lambda_{D,d}(p) = \frac{1}{K_{D,d}^n(p)g_{D,d}(p)}.\]

Proof: Solving the minimization problem associated with \(\lambda_{D,d}^k(p)\) exactly as in the proof of 7(a) in Section 2, Chapter II of [19], we obtain
\[\lambda_{D,d}^k(p) = \frac{\det(K_{ij}(p))_{i,j=0}^{k-1}}{\det(K_{ij}(p))_{i,j=0}^k},\]
where
\[K_{ij}(p) = \frac{\partial^2 K_{D,d}(p)}{\partial z_i \partial \overline{z}_j}, \quad 0 \leq i, j \leq n.\]

Note that
\[\det(K_{ij}(p))_{i,j=0}^n = \det\left(\frac{\partial^2 K_{D,d}(p)}{\partial z_i \partial \overline{z}_j}\right) = K_{D,d}^{n+1}(p) \det(\partial \overline{\partial} \log K_{D,d}(p)) = K_{D,d}^{n+1}(p) g_{D,d}(p).\]

Thus,
\[\lambda_{D,d}(p) = \frac{K_{00}(p)}{\det(K_{ij}(p))_{i,j=0}^1} \cdots \frac{\det(K_{ij}(p))_{i,j=0}^{n-1}}{\det(K_{ij}(p))_{i,j=0}^n} = \frac{1}{K_{D,d}^n(p)g_{D,d}(p)},\]
as required. ■

From Propositions 3.3 and 3.4, we immediately obtain

Proposition 3.5: We have
\[g_{D,d}(p) = \left(\frac{r_{{D,d}}(p,v)}{\lambda_{D,d}(p)}\right)^n\]
and
\[\beta_{D,d}(p) = \left(\frac{r_{{D,d}}^0(p,v)}{\lambda_{D,d}(p,v)}\right)^{n+\frac{1}{n+2}}.\]

Finally, we derive the following relation between the Ricci curvature and the minimum integral \(M\).

Proposition 3.6: We have
\[\text{Ric}_{D,d}(p,v) = (n + 1) - \frac{\lambda_{D,d}(p,v)}{\tau_{D,d}^2(p,v)M_{D,d}(p,v)}.\]
\[ = (n + 1) - \frac{I_{D,d}^1(p,v)\lambda_{D,d}(p)}{I_{D,d}^0(p,v)M_{D,d}(p,v)}. \]

**Proof:** The arguments in the proof of Proposition 2.1 of [17] applied to \(K_{D,d}\) yield

\[ \text{Ric}_{D,d}(p,v) = (n + 1) - \frac{1}{K_{D,d}(p)\tau_{D,d}^2(p,v)I_{D,d}(p,v)}. \]

From Propositions 3.1, 3.3, and 3.4, we have

\[ I_{D,d}(p,v) = \frac{M_{D,d}(p,v)}{K_{D,d}(p)\lambda_{D,d}(p)} = \frac{I_{D,d}^0(p,v)M_{D,d}(p,v)}{\lambda_{D,d}(p,v)}. \]

Combining the above two equations, the proposition follows. ■

### 3.2. Monotonicity of the minimum integrals

Let \(D \subset D', p \in D, \) and \(v \in \mathbb{C}^n\) be a nonzero vector. It is evident from their definitions that

\[ I_{D,d}^k(p,v) \leq I_{D',d}^k(p,v), \quad k = 0, 1, 2, \]
\[ \lambda_{D,d}^k(p) \leq \lambda_{D',d}^k(p), \quad k = 1, \ldots, n. \]

We note that the minimum integral \(I_{D,d}(p,v)\) is not monotonic in general (see the remark on page 236 of [17]) but \(M_{D,d}(p,v)\) is. Indeed, the first and second equations in Proposition 3.3 give

\[ K_{D,d}(p)\tau_{D,d}^2(p,v) = \frac{1}{I_{D,d}^1(p,v)}, \]

and hence by (16),

\[ K_{D,d}(p)\tau_{D,d}^2(p,v) \geq K_{D',d}(p)\tau_{D',d}^2(p,v). \]

This can be written as

\[ K_{D,d}(p)v^iG_{D,d}(p)v \geq K_{D',d}(p)v^iG_{D',d}v. \]

Now the arguments as in the proof of Proposition 2.2 in [17] give

\[ \frac{\nu^iG_{D,d}^{-1}(p)v}{K_{D,d}(p)} \leq \frac{\nu^iG_{D',d}^{-1}(p)v}{K_{D',d}(p)}. \]

and

\[ M_{D,d}(p,v) \leq M_{D',d}(p,v). \]
3.3. Localization on domains of holomorphy

**Lemma 3.7:** Let \( D \subset \mathbb{C}^n \) be a bounded domain of holomorphy and \( U_0 \subset \subset U \) be neighbourhoods of a point \( p^0 \in \partial D \). Then, there exist constants \( c, C > 0 \) such that if \( I \) is any of the minimum integrals in (10),

\[
 cI_{U \cap D, d}(p, v) \leq I_{D, d}(p, v) \leq CI_{U \cap D, d}(p, v)
\]

for all \( p \in U_0 \) and \( v \in \mathbb{C}^n \).

**Proof:** We begin with the following:

**Claim 1:** There exists a constant \( C > 0 \) such that for any \( p \in U_0 \) and \( f \in A_2^2(\mathbb{U} \cap D) \), we can find an \( F_p \in A_2^2(D) \) that agrees with \( f \) at \( p \) up to second derivatives and

\[
 \|F_p\|_{A_2^2(D)}^2 \leq C\|f\|_{A_2^2(\mathbb{U} \cap D)}^2.
\]

To prove this claim, we choose a neighbourhood \( U_1 \) of \( p \), where \( U_0 \subset \subset U_1 \subset \subset U \), and a cut-off function \( \chi \in C_0^\infty(U) \), with \( 0 \leq \chi \leq 1 \) and \( \chi \equiv 1 \) on \( U_1 \). Let \( p \in U_0 \) and \( f \in A_2^2(\mathbb{U} \cap D) \). Define the \((0, 1)\) form

\[
 \alpha_f = \overline{\partial}(\chi f)
\]

which is smooth, \( \overline{\partial} \)-closed on \( D \), and vanishes on \( U_1 \cap D \). Also, consider the plurisubharmonic function \( \phi_p \) on \( D \) defined by

\[
 \phi_p(z) = (2n + 6) \log |z - p| + d \log K_D(z).
\]

Now apply Theorem 4.2 of [22] to obtain a solution \( u_p \) to the equation \( \overline{\partial} u = \alpha_f \) on \( D \) satisfying

\[
 \int_D \frac{|u_p(z)|^2}{|z - p|^{2n+6}(1 + |z|^2)^2} K_D^{-d}(z) \, dV(z) \leq \int_D \frac{|\alpha_f|^2}{|z - p|^{2n+6} K_D^{-d}(z)} \, dV(z).
\]  

But \( \text{supp} \alpha_f \subset (U \setminus U_1) \cap D \) and there \( \alpha_f \) satisfies

\[
 |\alpha_f|^2 \leq c_1 |f|^2
\]

for some constant \( c_1 > 0 \) independent of \( p \) and \( f \). Moreover, there exists a constant \( c_2 > 0 \) such that

\[
 |z - p|^{2n+6} \geq c_2
\]

for all \( z \in (U \setminus U_1) \cap D \) and \( p \in U_0 \). Hence, it follows from (21) that

\[
 \int_D \frac{|u_p(z)|^2}{|z - p|^{2n+6}(1 + |z|^2)^2} K_D^{-d}(z) \, dV(z) \leq c_3 \|f\|_{A_2^2(\mathbb{U} \cap D)}^2
\]

where \( c_3 > 0 \) is a constant independent of \( f \) and \( p \). On the other hand, since \( D \) is bounded, there exists a constant \( c_4 > 0 \) such that

\[
 |z - p|^{2n+6}(1 + |z|^2)^2 \leq c_4
\]
for all \( z, p \in D \). Consequently,

\[
\int_D \frac{|u_p(z)|^2}{|z - p|^{2n + 6}(1 + |z|^2)^2} K_D^{-d}(z) \, dV(z) \geq c_5 \|u_p\|^2_{A^2_d(D)},
\]

where \( c_5 > 0 \) is a constant independent of \( f \) and \( p \). Combining (22) and (23), there exists a constant \( c_6 \) independent of \( f \) and \( p \), such that

\[
\|u_p\|_{A^2_d(D)} \leq c_6 \|f\|_{A^2_d(U \cap D)}.
\]

Also, note that \( u_p \) is holomorphic on \( U_1 \cap D \) and if \( \overline{B}(p, \epsilon) \subset U_1 \cap D \) then by (22)

\[
\int_{B(p, \epsilon)} \frac{|u_p(z)|^2}{|z - p|^{2n + 6}} \, dV < \infty.
\]

This implies that \( u_p \) vanishes at \( p \) up to the second derivative. Finally, set \( F_p = \chi f - u_p \). Then \( F_p \) agrees with \( f \) at \( p \) up to the second derivative. Moreover,

\[
\|F_p\|_{A^2_d(D)} \leq \|\chi f\|_{A^2_d(D)} + \|u_p\|_{A^2_d(D)} \leq \|f\|_{A^2_d(U \cap D)} + c_6 \|f\|_{A^2_d(U \cap D)}
\]

using (24), which proves the claim with \( C = (1 + c_6) \).

To complete the proof of the lemma, let \( \beta \in U \cap D, \nu \in \mathbb{C}^n \) be a nonzero vector, and \( f \in A^2_d(U \cap D) \) be a minimizing function for \( \mathcal{I}_{U \cap D}(p, \nu) \). Let \( C \) and \( F_p \) be as given by the claim. Then observe that \( F \) is a candidate for \( \mathcal{I}_D(p, \nu) \) and so

\[
\mathcal{I}_D(p, \nu) \leq C \mathcal{I}_{U \cap D}(p, \nu).
\]

For the reverse inequality, note that if \( \mathcal{I} \neq I \), then by the monotonicity of \( \mathcal{I} \)

\[
\mathcal{I}_{U \cap D}(p, \nu) \geq \mathcal{I}_{U \cap D, \lambda}(p, \nu)
\]

and so we can take \( c = 1 \). For \( \mathcal{I} = I \), we write \( I_{U \cap D, \lambda}(p, \nu) \) in terms of \( I_{U \cap D, \lambda}(p, \nu), \lambda_{U \cap D}(p, \nu), \) and \( M_{U \cap D, \lambda}(p, \nu) \) as given by (15), and then using (25) and (26) for the latter minimum integrals,

\[
\mathcal{I}_{U \cap D, \lambda}(p, \nu) \geq \frac{I_{0 \cap D, \lambda}(p, \nu) M_{U \cap D, \lambda}(p, \nu)}{C_{U \cap D}(p, \nu)} = \frac{1}{C} \mathcal{I}_{U \cap D}(p, \nu).
\]

This completes the proof.

In view of Propositions 3.3, 3.5, and 3.6, this lemma immediately gives

**Theorem 3.8:** Let \( D \subset \mathbb{C}^n \) be a bounded domain of holomorphy and \( U_0 \subset \subset U \) be neighbourhoods of a point \( p^0 \in \partial D \). Then, there exist constants \( c, C > 0 \) such that

\[
cK_{U \cap D, \lambda}(p) \leq K_{D, \lambda}(p) \leq K_{U \cap D, \lambda}(p),
\]

\[
cg_{U \cap D, \lambda}(p) \leq g_{D, \lambda}(p) \leq g_{U \cap D, \lambda}(p),
\]

\[
c\beta_{U \cap D, \lambda}(p) \leq \beta_{D, \lambda}(p) \leq \beta_{U \cap D, \lambda}(p),
\]

\[
c\tau_{U \cap D, \lambda}(p, \nu) \leq \tau_{D, \lambda}(p, \nu) \leq C\tau_{U \cap D, \lambda}(p, \nu),
\]

where \( c_4 > 0 \) is a constant independent of \( f \) and \( p \). Combining (22) and (23), there exists a constant \( c_6 \) independent of \( f \) and \( p \), such that
$$c R_{U \cap D, d}(p, v) \leq 2 - R_{D, d}(z, v) \leq C R_{U \cap D, d}(p, v),$$
$$c \left( (n + 1) - \text{Ric}_{U \cap D, d}(p, v) \right) \leq n + 1 - R_{D, d}(z, v)$$
$$\leq C \left( (n + 1) - \text{Ric}_{U \cap D, d}(p, v) \right),$$
for all $p \in U_0$ and $v \in \mathbb{C}^n$.

### 3.4. Localization near peak points

While the localization of the weighted kernel and the Narasimhan–Simha-type metric in Theorem 3.8 are in terms of small–large–constants, more precise estimates can be given near a holomorphic peak point as was observed for the Bergman kernel and metric in [17].

**Lemma 3.9:** Let $D$ be a bounded pseudoconvex domain and $p^0 \in \partial D$ be a local holomorphic peak point of $D$. Then for a sufficiently small neighbourhood $U$ of $p^0$,

$$\lim_{p \to p^0} \frac{I_{U \cap D, d}(p, v)}{I_{D, d}(p, v)} = 1,$$

where $I$ is any of the minimum integrals in (10). Moreover, the convergence is uniform on $\{\|v\| = 1\}$.

**Proof:** Let $h$ be a local holomorphic peak function at $p^0$ defined in a neighbourhood $U$ of $p^0$. Take any neighbourhood $U_0$ of $p^0$ such that $U_0 \subset U$ and $h$ is nonvanishing on $U_0$.

**Claim 2:** There exist constants $0 < a < 1$ and $c > 0$ depending only on $U$ with the following property: given any $p \in U_0$, a function $f \in A_d^2(U \cap D)$, and an integer $N \geq 1$; there exists $F_{p, N} \in A_d^2(D)$ satisfying

$$\|F_{p, N}\|_{A_d^2(D)} \leq \frac{1 + ca^N}{|h(p)|^N} \|f\|_{A_d^2(U \cap D)}, \quad (27)$$

and such that (i) $F_{p, N}(p) = f(p)$, (ii) $F_{p, N}^\prime(p) = f'(p)$ if $f(p) = 0$, (iii) $v^\top F_{p, N}'(p) v = v^\top f''(p) v$ if $f(p) = 0$ and $f'(p)v = 0$ where $v \in \mathbb{C}^n$, and (iv) $F_{p, N}''(p) = f''(p)$ if $f(p) = 0$ and $f'(p) = 0$.

To prove this claim, choose a neighbourhood $U_1$ of $p^0$ such that $U_0 \subset U_1 \subset U$. Then there is a constant $a \in (0, 1)$ such that $|h| < a$ on $(U \setminus U_1) \cap D).$ Choose a cut-off function $\chi \in C_0^\infty(U)$ satisfying $0 \leq \chi \leq 1$ on $U$, and $\chi \equiv 1$ on $U_1$. Let $p \in U_0$ and $f \in A_d^2(U \cap D)$. Define the $(0, 1)$ form

$$\alpha_f = \overline{\partial} (\chi fh^N),$$

which is smooth and $\overline{\partial}$-closed on $D$. Moreover, supp $\alpha_f \subset (U \setminus U_1) \cap D$ and there $\alpha_f$ satisfies

$$|\alpha_f|^2 = |\overline{\partial} \chi|^2 |f|^2 |h|^2 \leq c_1 a^{2N} |f|^2,$$

for some constant $c_1 > 0$ independent of $p$ and $f$. Also, consider the plurisubharmonic function $\phi_p$ on $D$ defined in Lemma 3.7 and solve the equation $\overline{\partial} u = \alpha$ with $L^2$-estimates...
as in there. Repeating the same arguments in that lemma, we arrive at a solution $u_p$ which vanishes at $p$ up to the second derivative and satisfies
\[ \|u_p\|_{A^2_d(D)} \leq c|a|^N \|f\|_{A^2_d(U \cap D)}, \] (28)
where $c$ is a constant independent of $p$ and $f$. Define
\[ F_{p,N} = \frac{\chi h^N - u_p}{h^N(p)}. \] (29)

Then $F_{p,N}$ belongs to $A^2_d(D)$ and
\[ |h(p)|^N \|F_{p,N}\|_{A^2_d(D)} \leq \|\chi h^N\|_{L^2_d(D)} + \|u_p\|_{L^2_d(D)} \leq \|f\|_{A^2_d(U \cap D)} + ca^N \|f\|_{A^2_d(U \cap D)} \]
using (28), and thus $F_{p,N}$ satisfies estimate (27). We now proceed to show that $F_{p,N}$ satisfies the other properties in our claim. First note that (i) follows from (29). Now differentiating $F_{p,N}$ in $U_0$ and as $\chi = 1$ there, we obtain
\[ \frac{\partial F_{p,N}}{\partial z_j} = \frac{1}{h^N(p)} \left\{ \frac{\partial f}{\partial z_j} h^N + Nh^{N-1} \frac{\partial h}{\partial z_j} - \frac{\partial u_p}{\partial z_j} \right\}, \quad j = 1, \ldots, n, \]
from which (ii) follows. Differentiating the above equation, we obtain that if $f(p) = 0$, then
\[ \frac{\partial^2 F_{p,N}}{\partial z_i \partial z_j} (p) = \frac{1}{h^N(p)} \left\{ h^N(p) \frac{\partial^2 f}{\partial z_i \partial z_j} (p) + Nh^{N-1}(p) \frac{\partial f}{\partial z_j} (p) \frac{\partial h}{\partial z_i} (p) \right\} + Nh^{N-1}(p) \frac{\partial f}{\partial z_i} (p) \frac{\partial h}{\partial z_j} (p), \]
for $1 \leq i, j \leq n$, from which (iii) and (iv) follow. This proves our claim.

Now to complete the proof of the lemma, fix $p \in U_0$, $v \in \mathbb{C}^n \setminus \{0\}$, and let $f$ be a minimizing function for $I_{U \cap D}(p, v)$. Let the constants $c, a$, and the function $F_{p,N}$ be as given by the claim. Then observe that $F_{p,N}$ is a candidate for $I_D(p, v)$ and so (27) gives
\[ \frac{I_D(p, v)}{I_{U \cap D}(p, v)} \leq \frac{(1 + ca^N)^2}{|h(p)|^{2N}}. \] (30)

First, let $p \to p^0$ to get
\[ \limsup_{p \to p^0} \frac{I_{D_d}(p, v)}{I_{U \cap D}(p, v)} \leq (1 + ca^N)^2, \]
and then let $N \to \infty$ to get
\[ \limsup_{p \to p^0} \frac{I_{D_d}(p, v)}{I_{U \cap D}(p, v)} \leq 1. \] (31)

Now suppose $I \not= I$. Then by the monotonicity of $I$, we have $I_{D_d} \geq I_{U \cap D, d}$, and so
\[ \liminf_{p \to p^0} \frac{I_{D_d}(p, v)}{I_{U \cap D, d}(p, v)} \geq 1. \] (32)
Combining (31) and (32), we obtain

$$\lim_{p \to p^0} \frac{I_{D,d}(p,v)}{I_{U \cup D,d}(p,v)} = 1. \tag{33}$$

Note that the right-hand side of (30) is independent of $v$, and hence, the convergence is uniform in unit vectors $v \in \mathbb{C}^n$.

Now let $I = I$. We express $I$ in terms of $I^0$, $\lambda$, and $M$ as in (15), and then using monotonicity of $I^0$ and $M$, we obtain

$$I_{D,d}(p,v) \geq \frac{\lambda_{U \cup D,d}(p)}{\lambda_{D,d}(p)}.$$ 

Applying (32) to $\lambda$,

$$\liminf_{p \to p^0} \frac{I_{D,d}(p,v)}{I_{U \cup D,d}(p,v)} \geq 1. \tag{34}$$

Combining this with (31) and (34), we obtain

$$\lim_{p \to p^0} \frac{I_{D,d}(p,v)}{I_{U \cup D,d}(p,v)} = 1.$$ 

Again, as the right-hand side of (30) is independent of $v$, the convergence is uniform in unit vectors $v \in \mathbb{C}^n$. This completes the proof of the lemma. ■

In view of this lemma and Propositions 3.3, 3.5, and 3.6, we have

**Theorem 3.10:** Let $D \subset \mathbb{C}^n$ be a bounded pseudoconvex domain and $p^0 \in \partial D$ be a local holomorphic peak point of $D$. Then for a sufficiently small neighbourhood $U$ of $p^0$,

$$\lim_{p \to p^0} \frac{K_{U \cup D,d}(p)}{K_{D,d}(p)} = 1, \quad \lim_{p \to p^0} \frac{g_{U \cup D,d}(p)}{g_{D,d}(p)}, \quad \lim_{p \to p^0} \frac{\beta_{U \cup D,d}(p)}{\beta_{D,d}(p)} = 1,$$

and also

$$\lim_{p \to p^0} \frac{\tau_{U \cup D,d}(p,v)}{\tau_{D,d}(p,v)} = 1, \quad \lim_{p \to p^0} \frac{2 - R_{U \cup D}(p,v)}{2 - R_D(p,v)} = 1,$$

and

$$\lim_{p \to p^0} \frac{n + 1 - \text{Ric}_{U \cup D}(p,v)}{n + 1 - \text{Ric}_D(p,v)} = 1,$$

uniformly on $\{\|v\| = 1\}$.

### 4. A Ramadanov-type theorem

In this section, we observe a stability result for the weighted kernel $K_{D,d}$. This is an analogue of a result that was recently obtained for the Bergman kernel in [23]. Let $D$ be a domain in $\mathbb{C}^n$. For $q \in \mathbb{C}^n$, denote by $D - q$ the image of $D$ under the translation $Tv = v - q$ and for $r > 0$, by $rD$ the image of $D$ under the homothety $Rv = rv$. 

Proposition 4.1: Let $D_j$ be a sequence of domains in $\mathbb{C}^n$ converging to a domain $D$ in $\mathbb{C}^n$ in the following way:

(i) any compact subset of $D$ is eventually contained in each $D_j$,

(ii) there exists a common interior point $q$ of $D$ and all $D_j$ such that for every $\epsilon > 0$ there exists $j_\epsilon$ satisfying

$$D_j - q \subset (1 + \epsilon)(D - q),$$

for all $j \geq j_\epsilon$.

Assume further that $D$ is star-convex with respect to $q$, and both $K_{D_j}$, $K_{D,J_d}$ are nonvanishing along the diagonal. Then $K_{D_j,d} \rightarrow K_{D,J_d}$ uniformly on compact subsets of $D \times D$ together with all the derivatives.

Proof: Note that by Lemma 2.1 in [23], the weights $K_{D_j}^{-d}(z) \rightarrow K_{D,J_d}^{-d}(z)$ locally uniformly on $D$. This allows us to repeat the arguments of this lemma to obtain the proposition. ■

5. Proof of Theorem 1.1

5.1. Change of coordinates

We fix some notations first. For a $C^1$-smooth real-valued function $\rho$ defined on some open subset $U \subset \mathbb{C}^n$, we denote by $\nabla_\rho$ the gradient vector field of $\rho$, by $\nabla_{z^\rho}$ the vector field $(\partial \rho / \partial z_1, \ldots, \partial \rho / \partial z_n)$, and $\nabla_{\bar{z}^\rho} = \nabla_{\bar{z}}\rho$. The identity map of $\mathbb{C}^n$ will be denoted by $I$. For any linear map $L : \mathbb{C}^n \rightarrow \mathbb{C}^n$, its matrix will be denoted by $L$. The following change of coordinates was introduced by Pinchuk – see Lemma 2.1 in [24].

Lemma 5.1: Let $D \subset \mathbb{C}^n$ be a $C^2$-smoothly bounded strongly pseudoconvex domain, $p^0 \in \partial D$, and $\rho$ a $C^2$-smooth local defining function for $D$ defined in a neighbourhood $U$ of $p^0$. Assume further that $\nabla_{z^\rho}(p^0) = (0, 1)$ and $(\partial \rho / \partial z_n)(z) \neq 0$ for all $z \in U$. Then, there is a family of biholomorphic maps $h_\zeta : \mathbb{C}^n \rightarrow \mathbb{C}^n$ depending continuously on $\zeta \in U \cap \partial D$ that satisfies

(i) $h_{p^0} = I$.

(ii) $h_{\zeta}(\zeta) = 0$.

(iii) The local defining function $\rho_\zeta = \rho \circ h_{\zeta}^{-1}$ of the domain $D_\zeta := h_\zeta(D)$, near the origin, has the form

$$\rho_\zeta(z) = 2 \text{Re} \left( z_n + Q_\zeta(z) \right) + H_\zeta(z) + R_\zeta(z),$$

where $Q_\zeta(z) = \sum_{\mu,v=1}^n a_{\mu\nu}(\zeta)z_\mu z_v$, $H_\zeta(z) = \sum_{\mu,v=1}^n b_{\mu\nu}(\zeta)z_\mu \bar{z}_v$ with $Q_\zeta('z,0) \equiv 0$, $H_\zeta('z,0) \equiv |'z|^2$, and $R_\zeta(z) = o(|z|^2)$. The functions $a_{\mu\nu}(\zeta)$, $b_{\mu\nu}(\zeta)$, and $R_\zeta(z)$ depend continuously on $\zeta$.

(iv) The mapping $h_\zeta$ takes the real normal $n_\zeta = \{ z \in \mathbb{C}^n : z = \zeta + 2t\nabla_{\bar{z}}\rho(\zeta), t \in \mathbb{R} \}$ to $\partial D$ at $\zeta$ into the real normal $\{ z \in \mathbb{C}^n : 'z = y_n = 0 \}$ to $\partial D_\zeta$ at the origin.
We will require the derivative of the map $h_\zeta$, and so let us quickly recall its construction. For each $\zeta \in U \cap \partial D$, the map $h_\zeta$ is defined as the composition $h_\zeta = \phi^\zeta_3 \circ \phi^\zeta_2 \circ \phi^\zeta_1$ where the maps $\phi^\zeta_i$ are described below. Fix $\zeta \in U \cap \partial D$.

- The defining function $\rho$ near $\zeta$ has the form

$$\rho(z) = 2 \Re \left( \sum_{\mu=1}^{n} \frac{\partial \rho}{\partial z_\mu}(\zeta)(z_\mu - \zeta_\mu) + \frac{1}{2} \sum_{\mu,v=1}^{n} \frac{\partial^2 \rho}{\partial z_\mu \partial z_v}(\zeta)(z_\mu - \zeta_\mu)(z_v - \zeta_v) \right)$$

$$+ \sum_{\mu,v=1}^{n} \frac{\partial^2 \rho}{\partial z_\mu \partial \bar{z}_v}(\zeta)(z_\mu - \zeta_\mu)(\bar{z}_v - \bar{\zeta}_v) + o(|z - \zeta|^2).$$  \hspace{1cm} (35)

Define $w = \phi^\zeta_1(z)$ by

$$w_v = \frac{\partial \rho}{\partial z_n}(\zeta)(z_v - \zeta_v) - \frac{\partial \rho}{\partial \bar{z}_v}(\zeta)(z_n - \zeta_n), \quad v = 1, \ldots, n - 1,$$

$$w_n = \sum_{v=1}^{n} \frac{\partial \rho}{\partial z_v}(\zeta)(z_v - \zeta_v),$$  \hspace{1cm} (36)

i.e. $w = P^{\zeta}(z - \zeta)$, where $P^{\zeta}$ is the linear map whose matrix is

$$P^{\zeta} = \begin{pmatrix}
\frac{\partial \rho}{\partial z_n}(\zeta) & 0 & \ldots & 0 & -\frac{\partial \rho}{\partial \bar{z}_1}(\zeta) \\
0 & \frac{\partial \rho}{\partial z_n}(\zeta) & \ldots & 0 & -\frac{\partial \rho}{\partial \bar{z}_2}(\zeta) \\
\vdots & \vdots & \ldots & \vdots & \vdots \\
0 & 0 & \ldots & \frac{\partial \rho}{\partial \bar{z}_{n-1}}(\zeta) & -\frac{\partial \rho}{\partial z_n}(\zeta) \\
\frac{\partial \rho}{\partial z_1}(\zeta) & \frac{\partial \rho}{\partial z_2}(\zeta) & \ldots & \frac{\partial \rho}{\partial \bar{z}_{n-1}}(\zeta) & \frac{\partial \rho}{\partial \bar{z}_n}(\zeta)
\end{pmatrix},$$  \hspace{1cm} (37)

and let $D^{\zeta}_1 = \phi^\zeta_1(D)$. Observe that $\phi^\zeta_1(\zeta) = 0$ and

$$\phi^\zeta_1(\zeta + t\nabla_\bar{z}\rho(\zeta)) = tP^{\zeta}\nabla_\bar{z}\rho(\zeta) = (0, t|\nabla_\bar{z}\rho(\zeta)|^2).$$  \hspace{1cm} (38)

Thus $\phi^\zeta$ maps $n_\zeta$ to the $\Re z_n$-axis. The latter is the real normal to $\partial D^{\zeta}_1$ at 0 which can be seen from the Taylor series expansion of the defining function $\rho^{\zeta}_1 = \rho \circ (\phi^\zeta)^{-1}$ for $\partial D^{\zeta}_1$ at 0. Indeed, by writing (35) in terms of $w$ coordinates, and then replacing $w$ by $z$ itself, $\rho^{\zeta}$ has the form

$$\rho^{\zeta}(z) = 2 \Re \left( z_n + Q^{\zeta}_1(z) \right) + H^{\zeta}_1(z) + a^{\zeta}_1(z)$$  \hspace{1cm} (39)
In the current coordinates, where

\[
Q_1^\xi(z) = \sum_{\mu, v=1}^n a_{\mu v}^1(\xi)z_\mu z_v, \quad (a_{\mu v}^1(\xi)) = \frac{1}{2}(P^{-1})^t \left( \frac{\partial^2 \rho}{\partial z_\mu \partial z_v}(\xi) \right) P_{\xi}^{-1},
\]

\[
H_1^\xi(z) = \sum_{\mu, v=1}^n b_{\mu v}^1(\xi)z_\mu z_v, \quad (b_{\mu v}^1(\xi)) = (P_{\xi}^{-1})^* \left( \frac{\partial^2 \rho}{\partial z_\mu \partial z_v}(\xi) \right) P_{\xi}^{-1},
\]

and \(a_1^\xi(z) = o(|z|^2)\). It follows from (39) that the real normal to \(\partial D_1^\xi\) at 0 is \(\text{Re } z_n\)-axis. It is evident from (36) that as \(\zeta \to \zeta_0\), \(\phi_1^\xi(z) \to \phi_1^{\xi_0}(z)\) uniformly on compact subsets of \(\mathbb{C}^n\). Moreover, \((\phi_1^\xi)'(z) = \mathbb{P}_{\xi}\) for all \(\zeta \in \mathbb{C}^n\). Therefore, as \(\zeta \to \zeta_0\), we also have \((\phi_1^\xi)'(z) \to (\phi_1^{\xi_0})'(z)\) in the operator norm and uniformly in \(\zeta \in \mathbb{C}^n\).

- The transformation \(w = \phi_2^\xi(z)\) is a polynomial automorphism defined by

\[
w = \left( z, z_n + \sum_{\mu, v=1}^{n-1} a_{\mu v}^1(\xi)z_\mu z_v \right).
\]

In particular, note that \(\phi_2^\xi\) fixes the points on the \(\text{Re } z_n\)-axis. In these new coordinates \(w\), which we denote by \(z\) itself, the defining function \(\rho_2^\xi = \rho \circ (\phi_1^\xi)^{-1} \circ (\phi_2^\xi)^{-1}\) of the domain \(\phi_2^\xi \circ \phi_1^\xi(D)\), near the origin, has the form

\[
\rho_2^\xi(z) = 2 \text{Re} \left( z_n + Q_2^\xi(z) \right) + H_2^\xi(z) + a_2^\xi(z),
\]

where \(Q_2^\xi(z) = \sum_{\mu, v=1}^n a_{\mu v}^2(\xi)z_\mu z_v\) and \(H_2^\xi(z) = \sum_{\mu, v=1}^n b_{\mu v}^2(\xi)z_\mu z_v\) with

\[
a_{\mu v}^2(\xi) = 0 \quad \text{for } 1 \leq \mu, v \leq n - 1 \quad \text{and } a_{\mu v}^1(\xi) \text{ otherwise},
\]

\[
b_{\mu v}^2(\xi) = b_{\mu v}^1(\xi) \quad \text{for } 1 \leq \mu, v \leq n.
\]

Equation (40) implies that \(a_{\mu v}^1(\xi)\) depends continuously on \(\zeta\), and hence, it follows from (41) that as \(\zeta \to \zeta_0\), we have \(\phi_2^\xi(z) \to \phi_2^{\xi_0}(z)\) uniformly on compact subsets of \(\mathbb{C}^n\). Also,

\[
(\phi_2^\xi)'(z) = \left( \begin{array}{c}
\sum_{\mu=1}^{n-1} a_{\mu \gamma}^1(\xi)z_\mu + \sum_{v=1}^{n-1} a_{\gamma v}^1(\xi)z_v \\
\mathbb{I}_{n-1}
\end{array} \right).
\]

Thus, as \(\zeta \to \zeta_0\), we also have \((\phi_2^\xi)'(z) \to (\phi_2^{\xi_0})'(z)\) in norm and uniformly on a compact subset of \(\mathbb{C}^n\).

- In the current coordinates, \(\partial D\) is strongly pseudoconvex at \(\zeta = 0\) and the complex tangent space to \(\partial D\) at \(\zeta\) is \(H_\zeta(\partial D) = \{z_n = 0\}\). Therefore, the Hermitian form \(H_2^\xi '(z, 0)\) is strictly positive definite. Let \(\lambda_1^\xi, \ldots, \lambda_\nu_{\xi, n-1}\) be its eigenvalues. Thus, we can choose a linear change of coordinates of the form \(w = \phi_2^\xi(z) = \Lambda_\zeta U_\zeta z\), where
$U_\zeta$ is a unitary rotation which keeps the last coordinate unchanged and $\Lambda_\zeta = \text{diag}(\sqrt{\lambda_1}^\zeta, \ldots, \sqrt{\lambda_{n-1}}^\zeta, 1)$, so that in these coordinates, again denoted by $z$ itself, the defining function $\rho^\zeta = \rho \circ (\phi^\zeta_3)_{-1} \circ (\phi^\zeta_2)_{-1} \circ (\phi^\zeta_6)^{-1}$ of the domain $\phi^\zeta_3 \circ \phi^\zeta_2 \circ \phi^\zeta_1 (D)$, near the origin, has the form (ii) in the statement of the lemma. By definition, note that $\phi^\zeta_3$ fixes the points on the Re $z_n$-axis. The linear map $\phi^\zeta_3$ can clearly be chosen so that it depends continuously on $\zeta$, and its derivative at any point being itself also depends continuously on $\zeta$.

The map $h_\zeta$ is the composition $h_\zeta = \phi^\zeta_3 \circ \phi^\zeta_2 \circ \phi^\zeta_1$. It is evident from its construction that $h_{p^0} = I$, $h_\zeta(z) = 0$, and $\rho^\zeta = \rho \circ h_\zeta^{-1}$ has the desired form as in (iii). Moreover, since $\phi^\zeta_1$ maps $n_\zeta$ to the Re $z_n$-axis, and both $\phi^\zeta_2$ and $\phi^\zeta_3$ fix points on the Re $z_n$-axis, it follows that $h_\zeta$ maps $n_\zeta$ to the Re $z_n$-axis which is the real normal to $\partial D_\zeta$ at the origin. More specifically, we have

**Lemma 5.2:** With notations as in Lemma 5.1, and if $p \in n_\zeta \cap D$, then

$$h_\zeta(p) = \left(0, -|p - \zeta||\nabla z \rho(\zeta)|\right)$$  \hspace{1cm} (44)

and

$$h'_\zeta(p) = \Lambda_\zeta U_\zeta P_\zeta.$$  \hspace{1cm} (45)

**Proof:** Note that

$$p = \zeta - |p - \zeta|\frac{\nabla z \rho(\zeta)}{||\nabla z \rho(\zeta)||}.$$  \hspace{1cm}

Therefore, by (38)

$$\phi^\zeta_1(p) = \left(0, -|p - \zeta||\nabla z \rho(\zeta)|\right).$$

The first part of the lemma follows from the fact that $\phi^\zeta_2$ and $\phi^\zeta_3$ fix the points on the Re $z_n$-axis. For the second part, we only need to observe that $(\phi^\zeta_2)'(\phi^\zeta_1(p)) = I$ from (43) and that $(\phi^\zeta_3)'(z) = \Lambda_\zeta U_\zeta$ for any $z$. \hfill \blacksquare

### 5.2. Scaling

We now proceed to prove Theorem 1.1. Let $D \subset \mathbb{C}^n$ be a $C^2$-smoothly bounded strongly pseudoconvex domain and $p^0 \in \partial D$. Then there exist local holomorphic coordinates $z_1, \ldots, z_n$ near $p^0$ in which $p^0 = 0$, a neighbourhood $U$ of $p^0 = 0$ such that $U \cap D = \{z \in U : \rho(z) < 0\}$, where $\rho(z)$ is a smooth function on $U$ of the form

$$\rho(z) = 2 \text{Re} z_n + |z|^2 + o(|z|^2, \text{Im} z_n)$$  \hspace{1cm} (46)

that satisfies $\nabla \rho(z) \neq 0$ for all $z \in U$, and a constant $0 < r < 1$ such that

$$U \cap D \subset \Omega := \{z \in \mathbb{C}^n : 2 \text{Re} z_n + r|z|^2 < 0\}.$$  \hspace{1cm} (47)

Henceforth, we will be working in the above coordinates.
In view of the localization result Theorem 3.10, by shrinking $U$ if necessary, it is enough to establish the theorem for $U \cap D$. We relabel $U \cap D$ as $D$ for convenience. Let $p^j$ be a sequence of points in $D$ converging to $p^0 = 0$. Without loss of generality, we may assume that $p^j \in U$ for all $j$. Choose $\xi^j \in \partial D$ closest to $p^j$, which is unique if $j$ is sufficiently large. Note that $\xi^j \to p^0$. Let $\delta_j = \delta_D(p^j) = |p^j - \xi^j|$. Set $\phi_i^j = \phi_i^{\xi^j}$, $h_j = h_{\xi^j}$, $D_j = D_{\xi^j}$, and $\rho_j = \rho^{\xi^j}$. Then by Lemma 5.1, near 0,

$$
\rho_j(z) = 2 \text{Re} \left( \bar{z}_n + Q_j(z) \right) + H_j(z) + R_j(z),
$$

where $Q_j = Q_{\xi^j}$, $H_j = H_{\xi^j}$, and $R_j = R_{\xi^j}$. Moreover, thanks to the strong pseudoconvexity of $\partial D$ near $p^0 = 0$, shrinking $U$ and taking a smaller $r$ in (47) if necessary, we have

$$
D_j \subset \Omega
$$

for all $j$ large.

Let $q^j = h^j(p^j)$ and $\eta_j = \delta_{D^j}(q^j)$. From Lemma 5.2, we have

$$
q^j = (0, -\delta_j|\nabla_z \rho(\xi^j)|),
$$

and thus, $\eta_j = \delta_j|\nabla_z \rho(\xi^j)|$. Therefore,

$$
\frac{\eta_j}{\delta_j} = |\nabla_z \rho(\xi^j)| \to |\nabla_z \rho(p^0)| = 1
$$

from (46). Also, let $S_j = h_j^\prime(p^j)$. Note that Lemma 5.1 gives

$$
S_j \to \mathbb{I}
$$

in the operator norm.

Now consider the anisotropic dilation map $T_j : \mathbb{C}^n \to \mathbb{C}^n$ defined by

$$
T_j(z, z_n) = \left( \frac{\bar{z}}{\eta_j}, \frac{z_n}{\eta_j} \right).
$$

Set

$$
\tilde{D}_j = T_j(D_j),
$$

$$
\tilde{\rho}_j(z) = \frac{1}{\eta_j} \rho_j(\frac{\bar{z}}{\eta_j}, z, \eta_j z_n).
$$
Note that any compact subset of \( \mathbb{C}^n \) is contained in the domain of \( \tilde{\rho}_j \) for all sufficiently large \( j \) and \( \tilde{\rho}_j \) is a defining function for \( \tilde{D}_j \). By (48)

\[
\tilde{\rho}_j(z) = 2 \Re z_n + |z'|^2 + \frac{1}{\eta_j} A_j(\sqrt{\eta_j} z, \eta_j z_n),
\]

where

\[
A_j(z) = z_n O(|z|) + O(|z|^3).
\]

It follows that \( \tilde{\rho}_j \) converges in \( C^2 \)-topology on compact subsets of \( \mathbb{C}^n \) to

\[
\rho_\infty(z) = 2 \Re z_n + |z'|^2.
\]

This implies that the domains \( \tilde{D}_j \) converge in the local Hausdorff sense to the domain \( D_\infty \) defined in the statement of Theorem 1.1. Also note that since \( T_j \circ h_j(p') = (0, -1) = b^* \), each \( \tilde{D}_j \) contains the point \( b^* \). We now derive a stability result for the Narasimhan–Simha weighted kernel under scaling. First, we note the following properties of the Cayley transform \( \Phi \) defined in (4) that follow by a routine calculation: \( \Phi \) is a biholomorphism of the domain (of \( \Phi \))

\[
D_\Phi := \mathbb{C}^n \setminus \{z : z_n = 1\}
\]

onto itself with \( \Phi^{-1} = \Phi \). The domain \( \Omega \) defined in (47) is mapped by \( \Phi \) onto the bounded domain

\[
\Phi(\Omega) = \left\{ z \in \mathbb{C}^n : r|z'|^2 + |z_n|^2 < 1 \right\},
\]

and the domain \( D_\infty \) is mapped to

\[
\Phi(D_\infty) = \mathbb{B}^n.
\]

Lemma 5.3: For any multi-index \( A = (a_1, \ldots, a_n) \),

\[
\partial^A K_{\tilde{D}_j, d}(z) \to \partial^A K_{D_\infty, d}(z)
\]

uniformly on compact subsets of \( D_\infty \).

Proof: By (49), the domain \( \Omega \) contains \( D_j \) for all sufficiently large \( j \). Also, \( \Omega \) is invariant under the maps \( T_j \). Therefore, \( \Omega \) contains \( \tilde{D}_j \) for all sufficiently large \( j \). Evidently, \( \Omega \) contains \( D_\infty \) (as \( r < 1 \)). We claim that \( \Phi(\tilde{D}_j) \) converges to \( \Phi(D_\infty) = \mathbb{B}^n \) in the way required by the hypothesis of Theorem 4.1 with \( q = 0 \). Indeed, first note that if \( K \) is a compact subset of \( \Phi(D_\infty) \), then \( \Phi^{-1}(K) \) is a compact subset of \( D_\infty \). Since \( \tilde{D}_j \to D_\infty \) in the local Hausdorff sense, \( \Phi^{-1}(K) \) is contained in \( \tilde{D}_j \) for all sufficiently large \( j \) implying that \( K \) is contained in \( \Phi(\tilde{D}_j) \) for all sufficiently large \( j \). Next, assume, if possible, that the second condition in the hypothesis of Theorem 4.1 is not satisfied with \( q = 0 \). Then there exist an \( \epsilon > 0 \), a subsequence of \( \Phi(\tilde{D}_j) \), which we relabel as \( \Phi(\tilde{D}_j) \), and \( \xi_j \in \Phi(\tilde{D}_j) \) such that \( \xi_j \) lies outside \( (1 + \epsilon) \Phi(D_\infty) \), i.e. \( |\xi_j| \geq 1 + \epsilon \). Since \( \Phi(\tilde{D}_j) \subset \Phi(\Omega) \) for all large \( j \), the sequence \( \{\xi_j\} \) is bounded and hence after passing to a subsequence, \( \xi_j \to \xi \) for some \( \xi \in \Phi(\Omega) \). This, first of all, implies that \( |\xi_n|^2 + r|\xi'|^2 \leq 1 \) by (52) as well as
\[ |\xi| \geq 1 + \epsilon, \text{ which together ensures that } \xi_n \neq 1, \text{ i.e. } \xi \in \mathcal{D}_\Phi, \text{ and second, it now also implies that } \Phi^{-1}(\xi_j) \to \Phi^{-1}(\xi). \text{ Since } \Phi^{-1}(\xi_j) \in \hat{D}_j, \text{ we have } \hat{\rho}_j(\Phi^{-1}(\xi_j)) < 0 \text{ for all large } j, \text{ and hence } \rho_{\infty}(\Phi^{-1}(\xi)) \leq 0. \text{ Therefore, } \Phi^{-1}(\xi) \in \hat{D}_\infty \text{ and hence } \xi \in \Phi(D_\infty), \text{ which contradicts the fact that } |\xi| \geq 1 + \epsilon. \text{ This proves our claim. Therefore, by Theorem } 4.1, \text{ } K_{\Phi(D_\infty),d}(z) \text{ converges to } K_{\Phi(D_\infty),d}(z) \text{ uniformly on compact subsets of } \Phi(D_\infty), \text{ together with all derivatives. By the transformation rule, the lemma follows immediately.} \]

### 5.3. Boundary asymptotics

We are now ready to complete the proof of Theorem 1.1. Recall that \( T_j \circ h_j : D \to \hat{D}_j \) sends \( p^j \) to \( b^* = (0, -1) \). The matrix of the linear map \( T_j \) is \( \mathbb{T}_j = \text{diag}(1/\sqrt{n_1}, \ldots, 1/\sqrt{n_1}, 1/\eta_j) \) and

\[
\det \mathbb{T}_j = \eta_j^{-(n+1)/2}. \tag{54}
\]

Also recall that \( S_j = h_j'(p^j) \). Thus \( (T_j \circ h_j)'(p^j) = \mathbb{T}_j S_j \). In the rest of the proof below, we will be repeatedly using Corollary 2.3, (50), (51), Lemma 5.3, and the Cayley transform \( \Phi \) defined in (4) that maps \( D_\infty \) biholomorphically onto \( \mathbb{B}^n \) with \( \Phi(b^*) = 0 \),

\[
\Phi'(b^*) = -\text{diag}(1/\sqrt{2}, \ldots, 1/\sqrt{2}, 1/2), \quad \text{and} \quad \det \Phi'(b^*) = (-1)^n 2^{-(n+1)/2},
\]

without referring to them.

(a) We have

\[
K_{D,d}(p^j) = K_{\hat{D}_j,d}(b^*) \cdot \det \mathbb{T}_j S_j |^{2d+d} = \eta_j^{-(d+1)(n+1)} K_{\hat{D}_j,d}(b^*) \cdot \det S_j |^{2d+2}.
\]

Therefore,

\[
\delta_j^{(d+1)(n+1)} K_{D,d}(p^j) = \left( \frac{\delta_j}{\eta_j} \right)^{(d+1)(n+1)} K_{\hat{D}_j,d}(b^*) \cdot \det S_j |^{2d+2} \to K_{D_\infty,d}(b^*) = K_{\mathbb{B}^n,d}(0) \cdot \det \Phi'(b^*) |^{2d+2} = c \left( \frac{n!}{2^{n+1} \pi^n} \right)^{(d+1)}
\]

(b) By (5), we have

\[
g_{D,d}(p^j) = g_{\hat{D}_j,d} \cdot \det \mathbb{T}_j S_j |^{2} = \eta_j^{-(n+1)} g_{\hat{D}_j,d}(b^*) \cdot \det S_j |^{2},
\]

and therefore,

\[
\delta_j^{n+1} g_{D,d}(p^j) = \left( \frac{\delta_j}{\eta_j} \right)^{n+1} g_{\hat{D}_j,d}(b^*) \cdot \det S_j |^{2} \to g_{D_\infty,d}(b^*) = g_{\mathbb{B}^n,d}(0) \cdot \det \Phi'(b^*) |^{2} = \frac{1}{2^{n+1}} (d + 1)^n (n + 1)^n.
\]

(c) From (a) and (b),

\[
\beta_{D,d}(p^j) = \beta_{\hat{D}_j,d}(b^*) = \frac{g_{\hat{D}_j,d}(b^*)}{K_{\hat{D}_j,d}(b^*)^{1/(d+1)}} \to \frac{g_{D_\infty,d}(b^*)}{K_{D_\infty,d}(b^*)^{1/(d+1)}}
\]
\begin{equation}
\tau_{D,d}(p^j, v_H) = \tau_{D,0}(b^*, (\nabla_z \rho(\xi^j))' (\nabla_z \rho(\xi^j))) = \tau_{D,0}(b^*, (0, v))^n = \frac{1}{n!} \pi^n.
\end{equation}

(d) By invariance of the metric

\begin{equation}
\tau_{D,d}(p^j, v) = \tau_{D,0}(b^*, \nabla_z \rho(\xi^j)) = \tau_{D,0}(b^*, (\nabla_z \rho(\xi^j), \nabla_z \rho(\xi^j)))) = \tau_{D,0}(b^*, (0, v)).
\end{equation}

Therefore,

\begin{align*}
\delta_j \tau_{D,d}(p^j, v) &= \frac{\delta_j}{\eta_j} \tau_{D,0}(b^*, (\nabla_z \rho(\xi^j), \nabla_z \rho(\xi^j)))) \\
&= \tau_{D,0}(0, \nabla_z \rho(\xi^j)) = \tau_{D,0}(0, (\nabla_z \rho(\xi^j))^2)
\end{align*}

as \( \nu_N(p^0) = (0, v) \) by (46).

(e) For brevity, write \( v_H^j = \nabla_z \rho(\xi^j) \) and \( v_H^0 = \nabla_z \rho(\xi^j) = (0, v) \) by (46). We claim that \( (\nabla_z v_H^j)_n = 0 \). Indeed, note that from Lemma 5.2,

\begin{equation}
\nabla_z v_H^j = \Lambda_{\xi^j} \cup_{\xi^j} \nabla_z v_H^j.
\end{equation}

From (38),

\begin{align*}
\nabla_z v_H^j &= \nabla_z v_H^j - \left( v - \frac{\nabla_z \rho(\xi^j)}{\nabla_z \rho(\xi^j)} \right) \left( v - \frac{\nabla_z \rho(\xi^j)}{\nabla_z \rho(\xi^j)} \right) \\
&= \left( \nabla_z v_H^j - \left( v, \nabla_z \rho(\xi^j) \right) \left( \frac{1}{\nabla_z \rho(\xi^j)} \right) \right) \\
&= \left( \nabla_z v_H^j - \left( v, \nabla_z \rho(\xi^j) \right) \right) = (0, (v, \nabla_z \rho(\xi^j))) = (v, 0).
\end{align*}

Also, \( \Lambda_{\xi^j} \) and \( \cup_{\xi^j} \) do not affect the \( z_n \)-coordinate. Hence, it follows that \( \nabla_z v_H^j = (0, v) \) for some \( \nabla_z v_H^j \in \mathbb{C}^{n-1} \) and the claim follows. Now from (55),

\begin{equation}
\tau_{D,d}(p^j, v_H^j) = \tau_{D,0}(b^*, (0, v)),
\end{equation}

and hence,

\begin{align*}
\sqrt{\delta_j} \tau_{D,d}(p^j, v_H^j) &= \sqrt{\frac{\delta_j}{\eta_j}} \tau_{D,0}(b^*, (0, v)) \\
&= \tau_{\mathbb{E}^n,d}(0, \nabla_z \rho(\xi^j)) = \tau_{\mathbb{E}^n,d}(0, (v, 0)) \\
&= \frac{1}{\sqrt{2}} \sqrt{(d + 1)(n + 1)} |v| \\
&= \sqrt{\frac{1}{2}} (d + 1)(n + 1) \lambda_{\rho}(p^0, v_H^j).}
\end{align*}
as \( |v|^2 = |v_0|^2 = L_\rho(p^0, v_0^0) \) by (46).

(f) Let \( \hat{v} = \lim_{j \to \infty} (T_j S_j v) / |T_j S_j v| \). Then

\[
R_{D,d}(p^j, v) = R_{D,j,d}(b^*, T_j S_j v) = R_{D,j,d} \left( b^*, \frac{T_j S_j v}{|T_j S_j v|} \right)
\]

\[
\to R_{D,\infty,d}(b^*, \hat{v}) = R_{\mathbb{B}^n,d}(0, \Phi'(b^*)\hat{v}) = -\frac{2}{(d + 1)(n + 1)}.
\]

(g) Proceeding as in (f), we obtain

\[
\text{Ric}_{D,d}(p^j, v) \to \text{Ric}_{\mathbb{B}^n,d}(0, \Phi'(b^*)\hat{v}) = -\frac{1}{d + 1}.
\]

This completes the proof of Theorem 1.1.

### 6. Proof of Theorem 1.2

First, note that for a bounded domain \( D \subset \mathbb{C}^n \), it is known that (see, e.g. [21]) the Bergman metric always dominates the Carathéodory metric. By using the relationship between \( K_{D,d} \), \( \tau_{D,d} \) with the minimum integrals as discussed in Section 3.1, the same proof applies to show that \( ds^2_{D,d} \) also dominates the Carathéodory metric on \( D \). In particular, if \( D \) is strongly pseudoconvex, it is known that the Carathéodory metric is complete, and hence, the Narasimhan–Simha metric \( ds^2_{D,d} \) must also be complete for every \( d \geq 0 \).

**Remark 6.1:** The proof of the completeness of \( ds^2_{D,d} \) outlined above hinges on another observation of independent interest – namely, that the Carathéodory metric is dominated by each member of the class of metrics \( \{ ds^2_{D,d} : d \geq 1 \} \). That this is true in the classical case \( d = 0 \) is well known as we have already mentioned above (see also [25–28]). The completeness of \( ds^2_{D,d} \) also follows from Remark 2.8 in [2].

### 7. Remarks and questions

(i) It is well known that the paradigm of the scaling principle applies to more general domains. It is, therefore, possible to formulate analogues of Theorem 1.1 for such domains, but to do so, it is essential to understand the weighted kernels on unbounded model domains. As an example, consider the unbounded domain

\[
D_\infty = \{(z_1, z_2) \in \mathbb{C}^2 : 2 \Re z_2 + P(z_1, \bar{z}_1) < 0 \},
\]

where \( P = P(z_1, \bar{z}_1) \) is a subharmonic polynomial without harmonic terms. Such a domain arises as the limiting model domain associated with a smooth weakly pseudoconvex finite-type domain in \( \mathbb{C}^2 \). What can be said about \( K_{D_\infty,d} \) when \( d \geq 1 \)? In particular, is it true that \( K_{D_\infty,d} \) has a positive lower bound everywhere? Knowing this would be helpful in controlling the weighted kernel of the scaled domains and would lead to conclusions similar to those listed in Theorem 1.1 for smooth weakly pseudoconvex finite-type domains. Are there analogues of Catlin’s results ([29]) for \( K_{D,d} \)?

(ii) It is natural to formulate a Lu Qi-Keng-type conjecture for \( K_{D,d}(z, w), d \geq 1 \). Does that also fail generically? (see [30] for the case \( d = 0 \)).
(iii) Are there pseudoconvex domains for which the weighted Bergman spaces $A^2_d(D)$, $d \geq 1$, are nontrivial and finite dimensional? (see [31] for the case $d = 0$).

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