Remark on Pauli–Villars Lagrangian on the Lattice

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ABSTRACT

It is interesting to superimpose the Pauli–Villars regularization on the lattice regularization. We illustrate how this scheme works by evaluating the axial anomaly in a simple lattice fermion model, the Pauli–Villars Lagrangian with a gauge non-invariant Wilson term. The gauge non-invariance of the axial anomaly, caused by the Wilson term, is remedied by a compensation among Pauli–Villars regulators in the continuum limit. A subtlety in Frolov–Slavnov’s scheme for an odd number of chiral fermions in an anomaly free complex gauge representation, which requires an infinite number of regulators, is briefly mentioned.

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It seems interesting to put a Pauli–Villars type Lagrangian level regularization on the lattice. The interest is twofold: The Pauli–Villars regularization [1] for fermion one-loop diagrams can be expressed as a Lagrangian of regulators (bosonic and fermionic spinors). In actual perturbative calculations however, the Lagrangian has to be supplemented with additional prescriptions, such that the momentum of propagators have to be assigned in the same way for all the fields, and the integrand in the momentum integral has to be summed before the integration. Once the Lagrangian is put on the lattice, no prescription are needed and one is free to choose any momentum assignment. (To get a finite gauge invariant result in the continuum limit one has to assign the momenta of all fields on the lattice in the same way.)

More interestingly and importantly, “superimposing” a different kind of regularization on the lattice regularization may give some clue to the lattice regularization of chiral gauge theories. No manifestly gauge invariant lattice formulation of the chiral gauge theory, being consistent with the unitarity and the locality, is yet known [2]. In particular, for a chiral fermion in a complex gauge representation, it is impossible to introduce in a gauge invariant way the Wilson term [3] to eliminate unwanted species doublers.† The difficulty of a manifestly gauge invariant lattice formulation of chiral gauge theories is highlighted by the No-Go theorem [4].

The basic idea of “superimposing” is quite simple. Let us consider, for example, the naive momentum cutoff regularization applied to fermion one-loop diagrams in QED. This regularization breaks the gauge invariance, generating gauge non-invariant contributions. However we may use in addition say, the gauge invariant dimensional regularization. With this superimposed regularization, the infinite momentum cutoff limit can be taken and we are left with gauge invariant expressions in the dimensional regularization. Of course there is no real need to break the gauge invariance by introducing the momentum cutoff in this example, but with the

† For chiral fermions in a real-positive gauge representation, and for even number of chiral fermions in a pseudoreal representation, it is possible to introduce a gauge invariant (Majorana-type) Wilson term.
lattice regularization, it is not obvious how to treat chiral fermions in a manifestly
gauge invariant manner. To perform this program it is clearly crucial that there
exists a regularization which preserves the gauge symmetry and simultaneously is
congenial to the lattice regularization.

In fact a proposal based on this idea has been made by Frolov and Slavnov [5]
(see also [6]). They used the gauge invariant generalized Pauli–Villars regular-
ization [7,8] for chiral fermions in an anomaly free complex representation, and
discussed that taking the continuum limit $a \to 0$ ($a$ is the lattice spacing) with
an appropriately scaled regulator mass $M(a) \ll 1/a$, the regulator fields compen-
sate the effect of gauge non-invariant Wilson term and that the gauge invariant
regularized continuum theory [7] is reproduced.

We illustrate in this letter how this scheme works by evaluating the axial
anomaly [9,10,11] in a simpler lattice fermion model, the Pauli–Villars Lagrangian
with a gauge non-invariant Wilson term in the lattice vector gauge theory. The
relation to the proposal in [5] and the possible implication will be commen-
ted on later.

Before considering the Pauli–Villars regulators, let us study for a while a mas-
sive Dirac fermion coupled to a background gauge field and the axial U(1) Ward
identity. The naive fermion action is

$$I \equiv \sum_x \overline{\psi}(x)[iD \psi(x) - m] \psi(x) = \sum_x \overline{\psi}(x)[-i \overleftarrow{D} \psi(x) - m] \psi(x),$$

(1)

where the covariant derivative on the lattice $D_\mu(x)$ has been defined by

$$D_\mu(x) \equiv \frac{1}{2a} \left[ U_\mu(x)e^{a\partial_\mu} - e^{-a\partial_\mu}U^\dagger_\mu(x) \right],$$

$$\overleftarrow{D}_\mu(x) \equiv -\frac{1}{2a} \left[ U_\mu(x)e^{-a\partial_\mu} - e^{a\partial_\mu}U^\dagger_\mu(x) \right].$$

(2)

In the continuum limit $a \to 0$, we parameterize the link variable as $U_\mu(x) = e^{iagA_\mu(x)}$. As is well known, the naive action (1) contains unwanted species dou-
blers. Therefore we add the Wilson term to decouple them in the continuum limit,
but an artificially chosen gauge non-invariant one:

\[ I_W \equiv \sum_x \bar{\psi}(x) R(x) \psi(x) = \sum_x \bar{\psi}(x) (-\leftarrow R(x)) \psi(x), \]  

(3)

with

\[ R(x) \equiv \frac{r}{2a} \sum_\mu \left( e^{a \partial_\mu} + e^{-a \partial_\mu} - 2 \right), \quad \leftarrow R(x) \equiv -\frac{r}{2a} \sum_\mu \left( e^{-a \partial_\mu} + e^{a \partial_\mu} - 2 \right). \]  

(4)

Although \( I_W \) is irrelevant in the naive continuum limit, the effect of hard breaking of chiral and gauge symmetries survives in the axial anomaly as we will see below.

The axial U(1) Ward identity for the lattice action \( I + I_W \) is derived by performing a change of variable \( \psi(x) \rightarrow e^{i\alpha(x)\gamma_5} \psi(x) \) and \( \bar{\psi}(x) \rightarrow \bar{\psi}(x)e^{i\alpha(x)\gamma_5} \) in the partition function. For an infinitesimal \( \alpha(x) \), the action changes as

\[ I \rightarrow I + \sum_x \alpha(x) \left[ \partial_\mu J_5^\mu(x) - 2im\bar{\psi}(x)\gamma_5\psi(x) \right], \]

\[ I_W \rightarrow I_W - \sum_x \alpha(x)B(x), \]  

(5)

where the divergence of the axial U(1) current has been defined by

\[ \partial_\mu J_5^\mu(x) \equiv \bar{\psi}(x) \gamma_5 \psi(x) - \bar{\psi}(x) \gamma_5 \psi(x) \]

\[ = \sum_\mu \partial_\mu \left[ \bar{\psi}(x) \gamma_\mu \gamma_5 \psi(x) \right] + O(a), \]  

(6)

(the second line is the naive continuum limit) and

\[ B(x) \equiv -\bar{\psi}(x)i\gamma_5 R(x) \psi(x) + \bar{\psi}(x)i\gamma_5 \leftarrow R(x) \psi(x), \]  

(7)

the explicit axial U(1) breaking part. Since the functional integration measure is invariant under the change of variable with the lattice regularization, the Ward
identity reads
\[ \langle \partial_\mu J_5^\mu(x) \rangle = \langle 2i m \bar{\psi}(x) \gamma_5 \psi(x) \rangle + \langle B(x) \rangle. \] (8)

Let us evaluate the right hand side of (8). We first concentrate on the vacuum expectation value of \( B(x) \),
\[ \langle B(x) \rangle = \mathrm{tr} \left[ i \gamma_5 R(x) \langle \psi(x) \bar{\psi}(y) \rangle \right]_{x=y} - \mathrm{tr} \langle \psi(x) \bar{\psi}(y) \rangle i \gamma_5 \overset{\leftarrow}{R}(y) \right|_{x=y} \equiv b(x) + \overset{\leftarrow}{b}(x). \] (9)

They are evaluated by the lattice propagator in the presence of background gauge field,
\[ b(x) \equiv - \mathrm{tr} \left[ i \gamma_5 R(x) \frac{1}{i \overset{\leftarrow}{D}(x) - m + R(x)} \delta(x, y) \right]_{x=y}, \]
\[ \overset{\leftarrow}{b}(x) \equiv - \mathrm{tr} \left[ \delta(x, y) \frac{1}{i \overset{\leftarrow}{D}(y) + m + R(y)} \overset{\leftarrow}{i} \gamma_5 \overset{\leftarrow}{R}(y) \right]_{x=y}, \] (10)

where the delta function on the lattice is defined by \( \delta(x, y) \equiv \delta_{x,y}/a^4 = \int_{-\pi/a}^{\pi/a} d^4 k \times e^{ik(x-y)/(2\pi)^4} \) and hence
\[ b(x) = - \mathrm{tr} \left[ \frac{1}{a^4} \int_{-\pi}^{\pi} \frac{d^4 k}{(2\pi)^4} e^{-ikx/a} i \gamma_5 R(-i \overset{\leftarrow}{D} - m + R) \right. \]
\[ \left. \times \left[ - \sum_\mu D_\mu^2 + (m - R)^2 + \sum_{\mu,\nu} [\gamma^\mu, \gamma^\nu][D_\mu, D_\nu]/4 + i[\overset{\leftarrow}{D}, R] \right] \right] e^{ikx/a}. \] (11)

In deriving the above expression, we have multiplied \((-i \overset{\leftarrow}{D} - m + R)\) on the numerator and on the denominator, and used a relation \( \overset{\leftarrow}{D}^2 = - \sum_\mu D_\mu^2 + \sum_{\mu,\nu} [\gamma^\mu, \gamma^\nu] \times [D_\mu, D_\nu]/4 \). Next noting
\[ e^{-ikx/a} D_\mu e^{ikx/a} = i a k_\mu + \overset{\leftarrow}{D}_\mu, \quad e^{-ikx/a} R e^{ikx/a} = \frac{r}{a} \sum_\mu (\cos k_\mu - 1) + \overset{\leftarrow}{R}, \] (12)

* Our calculation method is similar to that of [11], but seems rather simpler.
where
\[
\widetilde{D}_\mu \equiv \frac{1}{2a} \left[ e^{ik_\mu} (U_\mu e^{a \partial_\mu} - 1) + e^{-ik_\mu} (1 - e^{-a \partial_\mu} U_\mu) \right]
\]
\[= \cos k_\mu (\partial_\mu + igA_\mu) + O(a), \tag{13}\]
and
\[
\widetilde{R} \equiv \frac{r}{2a} \sum_\mu \left[ e^{ik_\mu} (e^{a \partial_\mu} - 1) - e^{-ik_\mu} (1 - e^{-a \partial_\mu}) \right]
\]
\[= ir \sum_\mu \sin k_\mu \partial_\mu + O(a), \tag{14}\]
we have
\[
b(x) = -\operatorname{tr} \frac{1}{a^4} \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} i\gamma_5 \left[ r \sum_\mu (c_\mu - 1) + a\widetilde{R} \right] \left[ \hat{k} + r \sum_\nu (c_\nu - 1) - ia\tilde{D} - am + a\widetilde{R} \right]
\]
\[\times \left\{ -\sum_\rho (is_\rho + a\tilde{D}_\rho)^2 + \left[ r \sum_\rho (c_\rho - 1) - am + a\tilde{R} \right]^2 + \frac{a^2}{4} \sum_\rho,\sigma [\gamma_\rho, \gamma_\sigma][\tilde{D}_\rho, \tilde{D}_\sigma] + ia^2[\tilde{D}, \tilde{R}] \right\}^{-1} \cdot 1, \tag{15}\]
where the trigonometric functions have been abbreviated as \( s_\mu \equiv \sin k_\mu \) and \( c_\mu \equiv \cos k_\mu \).

In (15), the expansion with respect to \( a \) is straightforward because the trace of gamma matrices requires at least four of them (\( \operatorname{tr} \gamma_5 \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma = -4\varepsilon^{\mu\nu\rho\sigma} \)). Finally using
\[
[\tilde{D}_\mu, \tilde{D}_\nu] = igc_\mu c_\nu F_{\mu\nu} + O(a),
\]
\[
[\tilde{D}_\mu, \tilde{R}] = rg \sum_\nu c_\mu s_\nu (\partial_\nu A_\mu) + O(a), \tag{16}\]
where \( F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu] \),

\[
\lim_{a \to 0} b(x) = -\frac{ig^2}{(2\pi)^4} I_1(r) \varepsilon^{\mu\nu\rho\sigma} \operatorname{tr} F_{\mu\nu} F_{\rho\sigma} - \frac{ig^2}{(2\pi)^4} I_2(r) \varepsilon^{\mu\nu\rho\sigma} \operatorname{tr} \partial_\mu A_\nu F_{\rho\sigma}. \tag{17}\]
In (17), $I_1(r)$ and $I_2(r)$ are the well-known lattice integrals [9,11]:

$$I_1(r) = r^2 \int_{-\pi}^{\pi} d^4k \frac{\prod_{\mu} c_\mu \left[ \sum_{\nu} (1 - c_\nu) \right]^2}{\left\{ \sum_{\rho} s_\rho^2 + r^2 \left[ \sum_{\rho} (1 - c_\rho) \right]^2 \right\}^3},$$

$$I_2(r) = -r^2 \int_{-\pi}^{\pi} d^4k \frac{\sum_{\mu} s_\mu^2 \prod_{\nu \neq \mu} c_\nu \sum_{\sigma} (1 - c_\sigma)}{\left\{ \sum_{\rho} s_\rho^2 + r^2 \left[ \sum_{\rho} (1 - c_\rho) \right]^2 \right\}^3},$$

and obey $I_1(r) + I_2(r) = -\pi^2/2$. A similar calculation shows $c^e_b(x) = b(x)$. Thus finally we arrive at

$$\lim_{a \rightarrow 0} \langle \partial_\mu J_5^\mu (x) \rangle = \lim_{a \rightarrow 0} \langle 2im\bar{\psi}(x)\gamma_5\psi(x) \rangle + \frac{ig^2}{16\pi^2} \varepsilon^{\mu\nu\rho\sigma} \text{tr} F_{\mu\nu} F_{\rho\sigma}$$

$$+ \frac{ig^2}{4\pi^2} I_2(r) \varepsilon^{\mu\nu\rho\sigma} \text{tr} \left[ \partial_\mu (A_\nu \partial_\rho A_\sigma + igA_\nu A_\rho A_\sigma) \right].$$

Several comments are in order: It is well-known that, with the gauge invariant Wilson term [3], the species doublers effectively act as the Pauli–Villars regulators (i.e., with an alternative axial charge [9]) and the gauge invariant correct axial anomaly is reproduced in the continuum limit [9,11]. In our present case, the last term on the right hand side of (19) is not gauge invariant (the first term will be shown to be gauge invariant). It would be gauge invariant if the last coefficient was $2/3$ instead of $1$. It also depends on the Wilson parameter $r$, although for an infinitesimal $r$ the term vanishes $\lim_{r \rightarrow 0} I_2(r) = 0$. Therefore the effect of hard breaking of chiral and gauge symmetries in the Wilson term (3) survives in the continuum limit. In other words, our identification of the axial current (6) was not gauge invariant (with the gauge non-invariant Wilson term (3)), and the operator does not coincide with the naive continuum limit. We may redefine the axial current as only the first line in (19) survives and to restore the gauge invariance. Such an
intricacy of an operator identification in the lattice regularization in the view point of the anomaly and the usefulness of superimposing a different regularization are emphasized in [12].

For our purpose, the first term in (19) is important:

\[
\langle 2im\bar{\psi}(x)\gamma_5\psi(x) \rangle
\]

\[
= 2i \text{tr} \frac{1}{a^4} \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \, am\gamma_5 \left[ \not{k} + r \sum_{\nu} (c_\nu - 1) - ia\not{D} - am + a\not{R} \right]
\]

\[
\times \left\{ -\sum_{\rho} (is_\rho + a\not{D}_\rho)^2 + \left[ r \sum_{\rho} (c_\rho - 1) - am + a\not{R} \right]^2
\right. \\
\left. + \frac{a^2}{4} \sum_{\rho,\sigma} [\gamma^\rho, \gamma^\sigma] [\not{D}_\rho, \not{D}_\sigma] + ia^2[\not{D}, \not{R}] \right\}^{-1} \cdot 1.
\]

(20)

A simple expansion by \( a \) is impossible in (20) because of the singular infrared behavior near \( k \sim 0 \). Thus we divide the integration region to the “outer” region \( |k| > \delta \), which is free of the infrared divergence, and the “inner” region \( |k| \leq \delta \) with an infinitesimal \( \delta \ll 1 \) [9]. In the outer region, we may safely expand the integrand with respect to \( a \), yielding

\[
\lim_{a \to 0} \langle 2im\bar{\psi}(x)\gamma_5\psi(x) \rangle_{\text{outer}} = 0.
\]

(21)

In the inner region we expand it by the external gauge field:

\[
\langle 2im\bar{\psi}(x)\gamma_5\psi(x) \rangle_{\text{inner}} = 2im\Gamma_5
\]

\[
+ 2im \sum_{n=1}^{\infty} \frac{1}{n!} \prod_{i=1}^{n} \left[ \sum_{x_i, \mu_i} A_{\mu_i}(x_i) \int \frac{d^4p_i}{(2\pi)^4} \, e^{ip_i(x-x_i)} e^{-iap_i/2} \right] \Gamma_{\mu_1\mu_2\cdots\mu_n}(p_1, p_2, \cdots, p_n).
\]

(22)
For example, $\Gamma_5 = \Gamma_5^\nu(p) = 0$, and

\[
\Gamma_5^{\mu\nu}(p,q) = -\text{tr} \gamma_5 \int_{-\delta/a}^{\delta/a} \frac{d^4 k}{(2\pi)^4} \left[ S(k + p + q) V^\mu(k + p + q, k + q) S(k + q) V^\nu(k + q, k) S(k) + (\mu \leftrightarrow \nu, p \leftrightarrow q) \right],
\]

(23)

where

\[
\begin{align*}
S(k) & \equiv \left[ \sum_\mu \gamma_\mu \frac{1}{a} \sin a k_\mu + m + \frac{r}{a} \sum_\mu (1 - \cos a k_\mu) \right]^{-1}, \\
V^\mu(k_1, k_2) & \equiv g \left[ \gamma_\mu \cos a \left( \frac{1}{2} k_1^\mu + \frac{1}{2} k_2^\mu \right) + r \sin a \left( \frac{1}{2} k_1^\mu + \frac{1}{2} k_2^\mu \right) \right].
\end{align*}
\]

(24)

Since what to be worried about is the infrared divergence, we may simply expand the numerator by $a$ as $V^\mu(k + p + q, k + q) = g \gamma^\mu + O(a)$ etc. In the denominator, we may expand the propagator as, $S(k + p) = [\sum_\mu (k_\mu + p_\mu) + m + O(\delta^2)]^{-1}$ because $\delta \ll 1$. Note that the subleading contributions from the denominator always give a ultraviolet convergent integral. From these arguments, we have

\[
\begin{align*}
\lim_{a \to 0} \Gamma_5^{\mu\nu}(p,q) & = -g^2 \text{tr} \gamma_5 \lim_{a \to 0} \int_{-\delta/a}^{\delta/a} \frac{d^4 k}{(2\pi)^4} \\
& \times \left[ \frac{1}{k + p + q + m} \gamma^\mu \frac{1}{k + q} + m + r \sum_\mu (1 - \cos a k_\mu) \right] + O(\delta^2) \\
& = -\frac{g^2}{4\pi^2} m \sum_{\rho,\sigma} \varepsilon^{\mu\nu\rho\sigma} p_\rho q_\sigma \int_0^1 \int_0^1 \frac{dy}{y} \ \frac{dx}{x} \\
& \times \frac{1}{m^2 - y(1 - y)q^2 - 2xy(1 - y)p \cdot q - xy(1 - xy)p^2} + O(\delta^2).
\end{align*}
\]

(25)

After the safe limit $\delta \to 0$, (25) is nothing but the expression in the continuum

* The so-called anomalous vertex [9] does not contribute to this function.
theory. The same consideration can be repeated for higher point functions, and we may summarize this fact compactly as

$$\lim_{a \to 0} \langle 2i m \bar{\psi}(x) \gamma_5 \psi(x) \rangle = 2i \lim_{y \to x} m \gamma_5 \frac{1}{i \mathcal{D}_c - m} \delta(x - y), \quad (26)$$

where $\mathcal{D}_c \equiv \gamma^\mu(\partial_\mu + igA_\mu)$ is the covariant derivative in the continuum theory. Note (26) is finite without any further regularization as the last line of (25) shows.

The first term in (19) is therefore gauge invariant. As we have seen in (21), only the physical fermion near $k \sim 0$ contributes to the operator (20), and the effect of Wilson term which couples only to the species doublers, is invisible in the continuum limit.

Combining (19) and (26), we have for a single massive Dirac fermion,

$$\lim_{a \to 0} \langle \partial_\mu J_5^\mu(x) \rangle \quad (27)$$

Let us now introduce the Pauli–Villars Lagrangian on the lattice. We introduce the regulator fields $\psi_n$, where $n = 1, 2, \cdots, N$, and $\psi_0$ the original fermion to be regularized. We assign the even number index for fermionic fields and the odd index for bosonic ones. We also denote the mass of those fields as $m_n$ and assume the masses of the regulator fields are of the order of the “cutoff” parameter $\Lambda$. The Pauli–Villars regularization condition [1] requires $\sum_{n=0}^{N} (-1)^n = \sum_{n=0}^{N} (-1)m_n^2 = 0$. For the combined system, we can use (27) for each fermionic field and that with a reversed sign for each bosonic field. For example, by summing up the first term, we have

$$\lim_{a \to 0} \langle \partial_\mu J_5^\mu(x) \rangle = 2i \lim_{y \to x} m \gamma_5 \frac{1}{i \mathcal{D}_c - m} \delta(x - y) \quad (28)$$
where we have defined the regulator function

\[ f(t) \equiv - \sum_{n=1}^{N} \frac{(-1)^n m_n^2 / \Lambda^2}{t + m_n^2 / \Lambda^2}. \]  

(29)

It follows from the definition and the Pauli–Villars condition, \( f(0) = 1 \) and \( f(t) = m_0^2 / (\Lambda^2 t) + O(1/t^2) \). For example, a possible choice is \( m_1^2 = \Lambda^2 \), \( m_2^2 = 2\Lambda^2 \), \( m_3^2 = \Lambda^2 + m_0^2 \), and \( \lim_{\Lambda \to \infty} f(t) = 2/(t + 1)(t + 2) \).

On the other hand, the second line in (27) cancels out among the fermionic and the bosonic fields because it is independent of the mass of the field. Therefore the total axial anomaly is given by

\[ \lim_{\Lambda \to \infty} \lim_{a \to 0} \langle \partial_\mu J_5^\mu(x) \rangle = \langle 2im_0 \bar{\psi}_0 \gamma_5 \psi_0 \rangle + \frac{ig^2}{16\pi^2} \varepsilon^{\mu\nu\rho\sigma} \text{tr} F_{\mu\nu}F_{\rho\sigma}. \]  

(30)

In deriving this, we have used the fact \( f(0) = 1 \) and \( f(\infty) = f'(\infty) = \cdots = 0 \) for \( \Lambda \to \infty \) and the actual calculation is identical to that of [13]. We have thus recovered the correct gauge invariant form of the axial anomaly. The present analysis may be repeated for the conformal anomaly of the Wilson fermion, for which the Wilson term gives rise \(-15\) times the correct coefficient [12]. We expect that the correct coefficient will be reproduced in the present scheme, because the real Pauli–Villars regulators, i.e., with an alternative statistics, always eliminate the effect of Wilson term.

Finally we briefly comment on the implication of above demonstration for the proposal in [5]. By a suitable change of variable,

\[ \chi(x) = \frac{1}{\sqrt{2}} \left[ P_R \psi(x) + P_L C T_{11} C_D \bar{\psi}^T(x) \right], \]

\[ \varphi(x) = \frac{1}{\sqrt{2}} \left[ P_R \phi(x) + P_L C D \bar{\phi}^T(x) \right], \]  

(31)

\[ \tilde{\varphi}(x) = \frac{1}{\sqrt{2}} \left[ P_R \bar{\phi}(x) + P_L C D \bar{\phi}^T(x) \right], \]

(\( \chi(x) \) is a fermionic field and \( \varphi(x) \), \( \tilde{\varphi}(x) \) are bosonic fields), it is possible to rewrite the lattice Pauli–Villars Lagrangian of [5] basically in the form of \( I + I_W \) in (1).
and (3). Note however that the number of the degree of freedom is doubled in (31) because, among four chirality components of $\chi(x)$ and $\overline{\chi}(x)$, only two of them are independent ($P_R \psi(x)$ and $\overline{\psi}(x)P_L$ are independent in the original chiral model). Therefore the above result (27) divided by two with $m = 0$ might be regarded as the fermion number $U(1)$ anomaly [14] of the original massless chiral fermion in the lattice model of [5].

From the analyses in [7,8], we know that it is possible to construct a gauge invariant Pauli–Villars Lagrangian, by utilizing a finite number of regulator fields, for chiral fermions in a real-positive representation, and for an even number of chiral fermions in a pseudoreal or an anomaly free complex representation. In these cases the situation would be the same as the above analysis and we would have (30) with $m_0 = 0$ as the twice of the fermion number anomaly, i.e., the correct result, $\lim_{\Lambda \to \infty} \lim_{a \to 0} \langle \partial_\mu J^\mu(x) \rangle = ig^2 \epsilon^{\mu\nu\rho\sigma} \text{tr} F_{\mu\nu} F_{\rho\sigma}/(32\pi^2)$.

For an odd number of chiral fermions in a pseudoreal representation and in an anomaly free complex representation, it is necessary to introduce an infinite number of regulators [7,8]. For the former case, it is possible to first put a finite number of them on the lattice, in a way that $\sum_{n=0}^{N} (-1)^n = 0$, and then to take the limit $N \to \infty$. Thus the above result might hold even in the $N \to \infty$ limit. On the other hand, the situation seems more subtle for the latter case, because the condition $\sum_{n=0}^{N} (-1)^n = 0$ is never satisfied for a finite $N$ [7,5]. The reason is that, while the original chiral fermion belongs to a complex representation, all the regulator fields belong to a “doubled” representation [7,5,8], namely the contribution of one regulator field is twice of the original fermion. For example, if we put the same finite number of fermionic and bosonic regulators first, and then take the limit $N \to \infty$, it would correspond to $\lim_{N \to \infty} \sum_{n=0}^{N} (-1)^n = 1$ in the above notation. On the other hand, the regulator function (29) would be given by $\lim_{N \to \infty} f(t) = \pi \sqrt{t}/\sinh(\pi \sqrt{t})$ [5]. Therefore the fermion number anomaly would be given by
\[
\lim_{\Lambda \to \infty} \lim_{N \to \infty} \lim_{a \to 0} \langle \partial_{\mu} J^\mu(x) \rangle = \frac{ig^2}{32\pi^2} \varepsilon^{\mu\nu\rho\sigma} \text{tr} F_{\mu\nu} F_{\rho\sigma}
\]

\[
+ \frac{ig^2}{32\pi^2} \varepsilon^{\mu\nu\rho\sigma} \text{tr} F_{\mu\nu} F_{\rho\sigma} + \frac{ig^2}{8\pi^2} I_2(r) \varepsilon^{\mu\nu\rho\sigma} \text{tr} \left[ \partial_{\mu} (A_{\nu} \partial_{\rho} A_{\sigma} + ig A_{\nu} A_{\rho} A_{\sigma}) \right],
\]

and the gauge non-invariant piece survives. The conclusion in [5], on the other hand, would imply

\[
\lim_{\Lambda \to \infty} \lim_{N \to \infty} \lim_{a \to 0} \langle \partial_{\mu} J^\mu(x) \rangle = \frac{ig^2}{32\pi^2} \varepsilon^{\mu\nu\rho\sigma} \text{tr} F_{\mu\nu} F_{\rho\sigma}.
\]

The above (admittedly handwaving) argument, (32) and (33), shows that the anomaly is quite sensitive on detailed way of the limit \(a \to 0\) and \(N \to \infty\), in Frolov–Slavnov’s scheme for an odd number of chiral fermions in an anomaly free complex representation.

In conclusion, we have illustrated how the superimposing of the Pauli–Villars regularization on the lattice regularization works, utilizing the axial U(1) identity. Simultaneously, we expect that the scheme also improves non-anomalous chiral symmetric properties of the Wilson fermion in QCD. To pursue this program further, however, we have to treat for example, the fermion self-energy, for which the Pauli–Villars regularization gives no clue. On this point, a superimposing of the higher covariant derivative regularization has been proposed [5].

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