SIMPLE CONNECTIVITY OF FARGUES–FONTAINE CURVES

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ABSTRACT. We show that the Fargues–Fontaine curve associated to an algebraically closed field of characteristic \( p \) is geometrically simply connected; that is, its base extension from \( \mathbb{Q}_p \) to any complete algebraically closed overfield admits no nontrivial connected finite étale covering. This is a key step in the proof of Drinfeld’s lemma for perfectoid spaces.

Let \( F \) be an algebraically closed field of characteristic \( p \) which is complete with respect to a nontrivial multiplicative norm. The construction of Fargues–Fontaine [5] associates to \( F \) a pair of geometric objects, a scheme over \( \mathbb{Q}_p \) and an adic space over \( \mathbb{Q}_p \) in the sense of Huber, which together play a central role in \( p \)-adic Hodge theory. The two spaces are related by a morphism from the adic space to the scheme which has some formal features of an analytification map, such as a form of the GAGA principle for coherent sheaves.

Various aspects of the geometry of the spaces constructed by Fargues–Fontaine justifies their use of the term curves with reference to these spaces. At the level of local geometry, the scheme is noetherian and regular of dimension 1, while (as shown by the present author [18]) the adic space is noetherian and admits a neighborhood basis consisting of the adic spectra of principal ideal domains. At the level of global geometry, if one defines the degree of an effective divisor to be its length as a scheme, then every principal divisor has degree 0 and the degree map defines a bijection of the Picard group with \( \mathbb{Z} \). One also has an interpretation of \( p \)-adic representations of the Galois group of \( F \) over a subfield in terms of vector bundles over a corresponding quotient of the curve, in analogy with the theorem of Narasimhan–Seshadri [25] on unitary representations of the fundamental group of a compact Riemann surface.

Going further, one finds indications that the correct analogy is not with arbitrary curves, but with curves of genus 0. Most notably, the celebrated theorem of Grothendieck that every vector bundle on a projective line splits into line bundles [6] has a close analogue for Fargues–Fontaine curves, on which every vector bundle splits into summands each of which is the pushforward of a line bundle along a finite étale morphism. (This is materially a reformulation of a prior result of the present author [11,12], but with a much more transparent statement and an independent proof.) Using this result, Fargues–Fontaine showed (see Lemma 4.7) that the curves satisfy a form of geometric simple connectivity: if one performs a base extension from \( \mathbb{Q}_p \) to a completed algebraic closure \( \mathbb{C}_p \), then the profinite fundamental group of the curve becomes trivial. (A similar observation had previously been made by Weinstein [31].)

The purpose of this paper is to extend the result of Fargues–Fontaine by establishing the following result.

**Theorem 0.1.** The Fargues–Fontaine curve associated to \( F \) is geometrically simply connected: for any complete algebraically closed overfield \( K \) of \( \mathbb{Q}_p \), the base extension of the curve from \( \mathbb{Q}_p \) to \( K \) admits no nontrivial connected finite étale covering.

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This result is an input into the discussion of Drinfeld’s lemma for analytic spaces in [20], Lecture 4], expanding on a line of inquiry initiated by Scholze [27]; we summarize this point briefly at the end of this paper.

We do not know of a proof of Theorem 0.1 that proceeds by a direct adaptation of the proof in the case \( K = \mathbb{C} \), as the classification of vector bundles on a Fargues–Fontaine curve makes crucial use of the discreteness of the valuation on finite extensions of \( \mathbb{Q}_p \). (An exception to this occurs when \( F \) is the completed algebraic closure of \( \mathbb{F}_p((t)) \), as then one can deduce the claim from Lemma 4.7 using a certain symmetry between \( F \) and \( K \); see Remark 4.8 and Remark 10.3.) We address this issue with a two-pronged approach. First, we give a direct argument using the Artin–Schreier construction to rule out the existence of nontrivial \( \mathbb{Z}/p\mathbb{Z} \)-covers (Lemma 5.3). We then use this to establish an inductive statement: if simple connectivity holds for a particular coefficient field \( K \), then it is also true for any completed algebraic closure of \( K(t) \). In light of the abelian argument, this reduces to the construction of “ramification filtrations” on representations of the étale fundamental group using tools from the theory of \( p \)-adic differential equations [16]; this is in the spirit of previous work of the author [13, 14] and Xiao [33, 34] on differential Swan conductors. (It should be possible to replace the use of \( p \)-adic differential equations with a somewhat more elementary argument using ramification theory of covers of Berkovich curves, as described in [4, 30]; we did not attempt to do this.)

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1. SOME ALGEBRAIC PRELIMINARIES

We start with some assorted algebraic preliminaries.

**Lemma 1.1.** Let \( k \) be an algebraically closed field. Let \( A \) and \( B \) be two integral \( k \)-algebras. Then \( A \otimes_k B \) is again integral.

**Proof.** Since \( k \) is algebraically closed, \( A \) is an absolutely integral \( k \)-algebra in the sense of Bourbaki [2, §V.17, Proposition 8, Corollaire 2]. By [2, §V.17, Proposition 2], this forces \( A \otimes_k B \) to be integral. \( \square \)

**Definition 1.2.** By a nonarchimedean field \( F \), we will mean a field equipped with a specified nontrivial multiplicative norm with respect to which \( F \) is complete. Let \( \mathfrak{o}_F, \kappa_F, \Gamma_F \) denote the valuation ring, residue field, and value group of \( F \) (written additively), respectively.

**Definition 1.3.** For \( k \) a field and \( \Gamma \) a totally ordered subgroup of \( \mathbb{R} \) (written additively), the field of Hahn–Mal’cev–Neumann generalized power series \( k((t^\Gamma)) \) is defined to be the set of formal sums \( \sum_{i \in \Gamma} x_i t^i \) with coefficients in \( k \) having well-ordered support. For \( x \) such a sum, we write \( x_{\leq}, x_0, x_{>\ } \) for the sums of \( x_i t^i \) over indices \( i \) with respectively \( i < 0, i = 0, i > 0 \). Some key facts about this construction are the following; these include results of Kaplansky [9, 10].

- For the \( t \)-adic absolute value, the field \( k((t^\Gamma)) \) is not only complete but spherically complete (every decreasing sequence of balls has nonempty intersection).
- The field \( k((t^\Gamma)) \) is algebraically closed if and only if \( k \) is algebraically closed and \( \Gamma \) is divisible.
- Any nonarchimedean field \( L \) of equal characteristic which is spherically complete is isomorphic to \( \kappa_L((t^\Gamma_L)) \).
• A nonarchimedean field is spherically complete if and only if it is \textit{maximally complete}, that is, it admits no nontrivial algebraic extension with the same value group and residue field.

\textbf{Definition 1.4.} Let \( D \) be a one-dimensional affinoid space over an algebraically closed nonarchimedean field \( K \), in the sense of Huber’s category of adic spaces \cite{Huber}. We classify points of \( D \) into \textit{types 1,2,3,4,5} as in \cite{26} Example 2.20]; note that this classification is preserved by lifting a point along a finite morphism. We recall also the following additional features of the classification.

• Each point of type 1 corresponds to a classical rigid-analytic point of \( D \).
• Each point of type 2 is the generic point on some one-dimensional affinoid space of good reduction. The reduction of this affinoid space is a curve \( C \) over the residue field \( \kappa \) of \( K \); if \( C \) is proper over \( \kappa \), we say that \( D \) is a \textit{strict neighborhood} of the original point. The genus of the smooth compactification of \( C \) is called the \textit{residual genus} of the original point.
• Each point of type 3 is the intersection of a descending sequence of annuli.
• Each point of type 4 is the intersection of a descending sequence of discs.
• Each point of type 5 is a specialization of a type 2 point; we may thus again speak of the \textit{residual genus} of a type 5 point. The points of type 5 are the only points whose associated valuations are of rank 2 rather than rank 1.

2. Drinfeld’s lemma for schemes

We next recall from \cite{20} §4.2 the formulation of Drinfeld’s lemma for schemes.

\textbf{Definition 2.1.} Let \( X \) be a scheme or adic space, and let \( \Gamma \) be a group of automorphisms of \( X \). We say that \( X \) is \( \Gamma \)-\textit{connected} if \( X \) is nonempty and its only \( \Gamma \)-stable closed-open subsets are itself and the empty set.

Let \( \text{FEt}(X/\Gamma) \) be the category of finite étale coverings of \( X \) equipped with \( \Gamma \)-actions (i.e., finite étale coverings of the stack-theoretic quotient of \( X \) by \( \Gamma \)). This is a Galois category in the sense of \cite{29} Tag 0BMQ.

For \( X \) \( \Gamma \)-connected and \( \overline{x} \) a geometric point of \( X \), let \( \pi_{1}^{\text{prof}}(X/\Gamma, \overline{x}) \) be the automorphism group of the fiber functor \( Y \mapsto |Y_{\overline{x}}| \) on \( \text{FEt}(X/\Gamma) \). As usual, the choice of the basepoint \( \overline{x} \) is needed to resolve a conjugation ambiguity in the definition; when this ambiguity is not an issue, we may omit the choice of \( \overline{x} \) from the notation.

When \( \Gamma \) is the cyclic group generated by a single automorphism \( \varphi \), we typically write \( X/\varphi \) in place of \( X/\Gamma \). When \( \Gamma \) is the trivial group, we typically write \( X \) in place of \( X/\Gamma \); this recovers the usual definition of the profinite (étale) fundamental group.

We now restrict to the case of schemes. For the corresponding discussion for analytic spaces, see \cite{10}.

\textbf{Definition 2.2.} Let \( X_{1}, \ldots, X_{n} \) be schemes over \( \mathbb{F}_{p} \) and put \( X = X_{1} \times_{\mathbb{F}_{p}} \cdots \times_{\mathbb{F}_{p}} X_{n} \). Write \( \varphi_{i} \) as shorthand for \( \varphi_{X_{i}} \), the automorphism of \( X \) induced by the absolute \((p\text{-power})\) Frobenius on \( X_{i} \). Define the category

\[
\text{FEt}(X/\Phi) := \text{FEt}(X/(\varphi_{1}, \ldots, \varphi_{n})) \times_{\text{FEt}(X/\varphi_{X})} \text{FEt}(X);
\]

for each \( i \in \{1, \ldots, n\} \), there is a canonical equivalence

\[
\text{FEt}(X/\Phi) \cong \text{FEt}(X/(\varphi_{1}, \ldots, \check{\varphi}_{i}, \ldots, \varphi_{n})).
\]
We say that $X$ is $\Phi$-connected if $X$ is $\langle \varphi_1, \ldots, \varphi_i, \ldots, \varphi_n \rangle$-connected for some (and hence any) $i \in \{1, \ldots, n\}$.

**Theorem 2.3** (Drinfeld’s lemma for schemes). Let $X_1, \ldots, X_n$ be connected qcqs (quasicompact quasiseparated) schemes over $\mathbb{F}_p$ and put $X = X_1 \times_{\mathbb{F}_p} \cdots \times_{\mathbb{F}_p} X_n$.

(a) The scheme $X$ is $\Phi$-connected.
(b) For any geometric point $\overline{x}$ of $X$, the map

$$\pi_1^{\text{prof}}(X/\Phi, \overline{x}) \to \prod_{i=1}^n \pi_1^{\text{prof}}(X_i, \overline{x})$$

is an isomorphism of topological groups.

**Proof.** See [20, Lemma 4.2.11] for (a) and [20, Theorem 4.2.12] for (b). □

### 3. Fargues–Fontaine curves

We continue with various notations and statements about Fargues–Fontaine curves. We use standard notation for Huber rings and pairs and their adic spectra, as in [32].

**Hypothesis 3.1.** For the remainder of the paper (except as specified), let $L$ be an algebraically closed nonarchimedean field of characteristic $p$. Fix a power $q$ of $p$ and an embedding $\mathbb{F}_q \hookrightarrow \kappa_L$ (which lifts uniquely to an embedding $\mathbb{F}_q \hookrightarrow L$) and let $E$ be a local field with residue field $\mathbb{F}_q$. Let $\varpi$ be a uniformizer of $E$. Let $F$ be an algebraically closed nonarchimedean field containing $E$.

**Remark 3.2.** In [18, Hypothesis 2.1] it is only required that $E$ be a complete discretely valued field whose residue field contains $\mathbb{F}_q$, but almost every subsequent statement requires this containment to be an equality (as in [5]). For instance, in [18, Definition 2.2], the expression of a general element of $W(\mathfrak{a}_L) \otimes_{W(\mathbb{F}_q)} \mathfrak{o}_E$ as a sum $\sum \varpi^n [\overline{x}_n]$ with $\overline{x}_n \in L$ depends on $E$ having residue field $\mathbb{F}_q$.

**Definition 3.3.** For $R$ a perfect $\mathbb{F}_q$-algebra, define $W(R)_E := W(R) \otimes_{W(\mathbb{F}_q)} \mathfrak{o}_E$. For $I \subseteq (0, \infty)$ a closed interval, let $B_{I,E}^t$ denote the Fréchet completion of $W(\mathfrak{o}_L)_{E[\mathbb{F}_q^{-1}]}[\overline{x}] : \overline{x} \in L$ for the family of multiplicative (see Lemma 3.4) norms

$$\lambda_t \left( \sum_{n \in \mathbb{Z}} \varpi^n [\overline{x}_n] \right) = \max \{ p^{-n} |\overline{x}_n|^t \} \quad (t \in I);$$

this ring is a principal ideal domain [18, Theorem 7.11] and a strongly noetherian Huber ring [18, Theorem 4.10]. For any given $x \in B_{I,E}^t$, the function $t \mapsto \log \lambda_t(x)$ is convex on $I$ [18, Lemma 4.4] (see also Lemma 3.4), so $B_{I,E}^t$ is in fact a Banach ring for the norm $\max \{ \lambda_r, \lambda_s \}$ for $I = [r, s]$.

**Lemma 3.4.** Let $\overline{\mathbb{F}}_q$ be the algebraic closure of $\mathbb{F}_q$ in $F$. Choose an extension of the chosen embedding $\mathbb{F}_q \hookrightarrow L$ to an embedding $\overline{\mathbb{F}}_q \hookrightarrow L$.

(a) For $t > 0$, the tensor product norm on $B_{I,E}^t \otimes_{W(\mathbb{F}_q)} \mathfrak{o}_F$ is multiplicative. We again denote this norm by $\lambda_t$.
(b) For $x \in B_{I,E}^t \otimes_{W(\mathbb{F}_q)} \mathfrak{o}_F$, the function $t \mapsto \log \lambda_t(x)$ on $I$ is continuous and convex.
Proof. For any ring $A$ equipped with a submultiplicative norm $\alpha$, define the associated graded ring $\text{Gr} A$ by the formula

$$\text{Gr} A = \bigoplus_{r>0} \text{Gr}^r A,$$

$$\text{Gr}^r A = \left\{ x \in A : \alpha(x) \leq r \right\} \bigcup \left\{ x \in A : \alpha(x) < r \right\}.$$

The ring $\text{Gr} L$ consists of one graded component for each $r$ in the value group $|L^*|$, each of which is a one-dimensional vector space over $\kappa L$. Since $L$ is a nonarchimedean field, $\text{Gr} L$ is an integral domain. We may then write

$$\text{Gr} B_{L,E}^{[t,q]} \cong (\text{Gr} L)[\overline{t}]$$

with $\text{Gr} L$ rescaled by $t$ (that is, place $\text{Gr}^r$ in degree $r^t$ rather than $r$) and $\overline{t}$ placed in degree $p^{-1}$. This is again an integral domain. Finally, by Lemma 3.4

$$\text{Gr}(B_{L,E}^{[t,q]} \widehat{\otimes} W(\overline{q})_E \mathfrak{o}_F) \cong (\text{Gr} L)[\overline{t^\Gamma L}] \otimes_{\overline{q}} \kappa F$$

is integral, and so the tensor product norm on $B_{L,E}^{[t,q]} \widehat{\otimes} W(\overline{q})_E \mathfrak{o}_F$ is multiplicative. This yields (a).

To check (b), we may work locally around a single $t \in I$. In particular, we may write $x$ as a sum of simple tensors, then ignore any of those that do not contribute to the image of $x$ in $\text{Gr}(B_{L,E}^{[t,q]} \widehat{\otimes} W(\overline{q})_E \mathfrak{o}_F)$. From the upper and lower degrees of this image, viewed as a Laurent-Puiseux polynomial in $\overline{t}$, we may read off the slopes of $t \mapsto \log \lambda_t(x)$ on either side of $t$. \qed

Corollary 3.5. With notation as in Lemma 3.4, let $J$ be a (possibly singleton) closed interval contained in the interior of $I$. Then $x \in B_{L,E}^{[t,q]} \widehat{\otimes} W(\overline{q})_E \mathfrak{o}_F$ is a unit in $B_{L,E}^{[t,q]} \widehat{\otimes} W(\overline{q})_E \mathfrak{o}_F$ if and only if $t \mapsto \log \lambda_t(x)$ is an affine function of $t$ on some neighborhood of $J$ in $I$.

Proof. Suppose first that $x$ admits the inverse $y$ in $B_{L,E}^{[t,q]} \widehat{\otimes} W(\overline{q})_E \mathfrak{o}_F$. Let $J'$ be some closed interval contained in the interior of $I$ which contains $J$ in its interior. We can then approximate $y$ by some element $z \in B_{L,E}^{[t,q]} \widehat{\otimes} W(\overline{q})_E \mathfrak{o}_F$ in such a way that $\lambda_t(y - z) < \lambda_t(x)^{-1}$ for all $t \in J$. In particular, $\lambda_t(1 - xz) = \lambda_t(x(y - z)) < 1$ for all $t \in J$; by Lemma 3.4(b), for a suitable choice of $J'$ this remains true for all $t \in J'$. For $t \in J'$, we then have

$$\lambda_t(x) + \lambda_t((xz)^{-1}) = \lambda_t(xz) = 1$$

but by (b) the functions $t \mapsto \lambda_t(x)$ and $t \mapsto \lambda_t(y)$ are both convex. They must therefore both be affine, proving the claim.

In the other direction, it suffices to check that if $J$ is a singleton interval and $t \mapsto \log \lambda_t(x)$ is an affine function of $t$ on some neighborhood of $J$ in $I$, then $x$ is a unit in $B_{L,E}^{[t,q]} \widehat{\otimes} W(\overline{q})_E \mathfrak{o}_F$ for some closed interval $J'$ containing $J$ in its interior. The image of $x$ in $\text{Gr}(B_{L,E}^{[t,q]} \widehat{\otimes} W(\overline{q})_E \mathfrak{o}_F)$ must then be an element of $(\text{Gr} L) \otimes_{\overline{q}} \mathfrak{o}_F$ times some power of $\overline{t}$. The element of $(\text{Gr} L) \otimes_{\overline{q}} \mathfrak{o}_F$ must be placed in a single degree, and hence must be a unit. For suitable $J'$, we can then construct $y \in B_{L,E}^{[t,q]} \widehat{\otimes} W(\overline{q})_E \mathfrak{o}_F$ for which $\lambda_t(1 - xy) < 1$ for $t \in J'$. Then $xy$ is a unit, as then is $x$. \qed

Remark 3.6. The ring $B_{L,E}^{[t,q]} \widehat{\otimes} W(\overline{q})_E \mathfrak{o}_F$ does not share some of the more refined ring-theoretic properties of $B_{L,E}^{[t,q]}$, essentially due to the value group of $F$ not being discrete. Notably, $B_{L,E}^{[t,q]} \widehat{\otimes} W(\overline{q})_E \mathfrak{o}_F$ is not noetherian.
Remark 3.7. Suppose that $E$ is of characteristic $p$. Then for any Banach $E$-algebra $A$, the ring $W(\omega_L)\otimes_{\mathbb{Z}_p} A$ contains $L \otimes_{\mathbb{Z}_p} A$ as a dense subring. In the case where $A = F$, the restriction to $L \otimes_{\mathbb{Z}_p} A$ of the norm $\lambda_t$ from Lemma 3.4 coincides with the tensor product norm for the given norm on $F$ and the $t$-th power of the given norm on $L$.

Remark 3.8. For any perfectoid $E$-algebra $A$, $B_{1,E}^t \widehat{\otimes} A$ is also perfectoid. Moreover, for any topologically nilpotent unit $t \in A^\circ$ there is a canonical isomorphism

$$(B_{1,E}^t \widehat{\otimes} A) \cong B_{1,E}^t \widehat{\otimes} \left(\mathcal{O}(\varphi_{I})((t)) \otimes F_{q}(yt((t)))\right).$$

Definition 3.9. Let $Y_{L,E}$ be the inductive limit of the adic spaces $\text{Spa}(B_{1,E}^I; B_{1,E}^{I,\circ})$ as $I$ varies over all closed intervals in $(0, \infty)$. The $q$-power Frobenius maps $\varphi_L : B_{1,E}^I \to B_{1,E}^{I,q}$ induces an isomorphism $\varphi_L^* : Y_{L,E} \to Y_{L,E}$. The group $\varphi_L^Z$ acts properly discontinuously on $Y_L$; define the adic Fargues–Fontaine curve $X_L$ to be the quotient by this action.

Remark 3.10. By Lemma 3.4 $B_{1,E}^I \widehat{\otimes} W(\mathbb{F}_p) \otimes_F$ is not connected; its connected components are each isomorphic to $B_{1,E}^I \widehat{\otimes} W(\mathbb{F}_p) \otimes_F$ and, as a topological space, form a principal homogeneous space for $G_{\mathbb{Z}} \cong \widehat{\mathbb{Z}}$. The same description then applies to the connected components of $Y_{L,E} \times_E F$. However, the action of $\varphi_L^Z$ on this space is nontrivial: it is via the action of the dense subgroup $\mathbb{Z} \subset \widehat{\mathbb{Z}}$. As a result, $X_{L,E} \times_E F$ is connected.

Remark 3.11. Suppose that $E$ is of characteristic $p$ and that $A$ is a Banach $E$-algebra. Let $R^+$ be the completion of $\omega_L \otimes_{\mathbb{Z}_p} A^\circ$ for the $\varpi_L, \varpi$-topology for some (any) pseudouniformizer $\varpi_L$ of $L$, and put $R = R^+[[\varpi_L^{-1}, \varpi^{-1}]]$. Then there is a canonical identification

$$Y_L \times_E \text{Spa}(A, A^\circ) \cong \{v \in \text{Spa}(R, R^+) : v(\varpi_L), v(\varpi) < 1\}.$$
Remark 4.3. It is also possible to prove Lemma 4.2 by showing that \( O(1) \) is an ample line bundle on \( X_{L,E} \) as in [21, §6]. We omit further details here.

Definition 4.4. For \( n \) a positive integer, let \( X_{L,E,n} \) be the quotient of \( Y_{L,E} \) by the action of \( \varphi_n^* \). Let \( \pi_n : X_{L,E,n} \to X_{L,E} \) be the natural projection; it is a connected \( n \)-fold étale cover which splits upon base extension from \( E \) to \( F \) (see Remark 3.10).

For \( d = \frac{r}{s} \in \mathbb{Q} \) written in lowest terms (so that \( r, s \in \mathbb{Z} \), \( \gcd(r, s) = 1 \), and \( s > 0 \)), let \( O(d) \) be the vector bundle on \( X_{L,E} \) defined as follows. Start with a trivial line bundle on \( Y_{L,E} \) with a generator \( v \). As in Definition 4.1, promote this to a \( \varphi_n^* \)-equivariant line bundle by specifying that \( \varphi_n^*v = \varphi^{-r}v \); this descends to a line bundle on \( X_{L,E,s} \). Then push forward along \( \pi_s \) to obtain \( O(d) \); for \( d \in \mathbb{Z} \), this agrees with Definition 4.1.

Lemma 4.5. For \( d \in \mathbb{Q} \), the following statements hold.

- (a) If \( d = 0 \), then \( H^0(X_{L,E}, O(d)) = E \).
- (b) If \( d > 0 \), then \( H^0(X_{L,E}, O(d)) \neq 0 \).
- (c) If \( d > d' \), then \( \text{Hom}(O(d), O(d')) = 0 \).
- (d) For any positive integer \( m \), \( O(d)^{\oplus m} \) is isomorphic to a direct sum of copies of \( O(dm) \).

Proof. See for instance [5, §8.2].

Theorem 4.6. Suppose that \( L \) is algebraically closed. Then every vector bundle on \( X_{L,E} \) splits as a direct sum \( \bigoplus_i O(d_i) \) for some \( d_i \in \mathbb{Q} \).

Proof. Modulo the interpretation of vector bundles given in Lemma 4.2, this result is originally due to the author when \( E \) is of characteristic 0 (see [11, Theorem 4.16] in the case \( L = \mathbb{C}_p \) and [12, Theorem 4.5.7] in the general case) and to Hartl–Pink when \( E \) is of characteristic \( p \) [7, Theorem 11.1]. The formulation given here is due to Fargues–Fontaine [5, Théorème 8.2.10], who give an independent proof. See [20, Theorem 3.6.13] for further discussion.

We now use Theorem 4.6 to establish a base case of simple connectivity, following Weinstein [31], Fargues–Fontaine [5], and Scholze [27]. See also [20, Lemma 4.3.10].

Lemma 4.7. Suppose that \( F \) is a completed algebraic closure of \( E \). Then every finite étale cover of \( X_{L,E} \times E \) \( F \) splits.

Proof. In light of the equivalence

\[
\text{F} \text{Et}(X_{L,E} \times E \ F) \cong 2 \lim_{\overrightarrow{E'}} \text{F} \text{Et}(X_{L,E} \times E \ E')
\]

for \( E' \) varying over finite extensions of \( E \) within \( F \) (e.g., apply [21, Proposition 2.6.8] to each term in a finite covering of \( X_{L,E} \) by affinoid subspaces), we may start with a cover \( f : U \to X_{L,E} \times E \) which is the base extension of the cover \( f_0 : U_0 \to X_{L,E} \times E' = X_{L,E'} \) for some finite extension \( E' \) of \( F \). Let \( g_0 : U_0 \to X_{L,E} \times E' \to X_{L,E} \) be the composite projection. Apply Theorem 4.6 to split \( g_0 \circ O_{U_0} \) as a direct sum \( \bigoplus_i O(d_i) \) for some \( d_i \in \mathbb{Q} \).

We claim that in fact \( d_i = 0 \) for all \( i \). To see this, suppose by way of contradiction that \( d_i > 0 \) for some \( i \); since there are only finitely many such \( i \), we may assume without loss of generality that \( d_i = \max_j \{d_j\} \). By Lemma 4.5, \( H^0(X_{L,E}, O(d_i)) \) is nonzero, and any nonzero
element gives rise to a nonzero square-zero element of $H^0(U_0, \mathcal{O}_{U_0})$; however, this yields a contradiction because $X_{L,E}$ is reduced, as then must be $U_0$. Hence $d_i \leq 0$ for all $i$; since $g_0, \mathcal{O}_{U_0}$ is self-dual, we also have $d_i \geq 0$ for all $i$.

Consequently, $d_i = 0$ for all $i$, and so $H^0(U_0, \mathcal{O}_{U_0}) = H^0(X_{L,E}, g_0, \mathcal{O}_{U_0})$ is a finite-dimensional $E$-vector space. This vector space inherits from $U_0$ the structure of an étale $E$-algebra; it follows that the original cover $f$ splits. \hfill $\square$

**Remark 4.8.** One may already deduce from Lemma 4.7 that for $L = \mathbb{C}_p$, $X_{L,E}$ is geometrically simply connected. To do this, one must use the symmetry between $L$ and $F$ coming from Remark 3.11; see Remark 10.3 for further discussion.

**Lemma 4.10.** The natural map $F \to H^0(X_{L,E} \times_E F, \mathcal{O})$ is an isomorphism.

*Proof.* To check the claim at hand, we may formally reduce to the case where $F$ is the completion of a subfield of countable dimension over $E$. We may then construct a Schauder basis for $F$ over $E$ [1, Proposition 2.7.2/3] to reduce to the assertion that $E \to H^0(X_{L,E}, \mathcal{O})$ is an isomorphism, which is Lemma 4.7(a). \hfill $\square$

**Lemma 4.11.** Let $L'$ be an algebraically closed complete overfield of $L$. Let $F$ be a complete algebraically closed overfield of $E$ and let $F'$ be a complete algebraically closed overfield of $F$. Let $f : U \to X_{L,E} \times_E F$ be a finite étale cover whose base extension to $X_{L',E} \times_E F'$ splits completely. Then $f$ splits completely.

*Proof.* Let $f' : U' \to X_{L',E} \times_E F'$ be the base extension of $f$. Using Remark 4.9, we may equip $H^0(U, \mathcal{O}) = H^0(X_{L,E} \times_E F, f_* \mathcal{O}_U)$ with the structure of a Banach algebra over $F$. Using a Schauder basis argument again, we see that the natural map $H^0(U, \mathcal{O}) \hat{\otimes}_F F' \to H^0(U', \mathcal{O})$ is an isomorphism of Banach algebras. Since $f'$ splits completely, by Lemma 4.10 the ring $H^0(U', \mathcal{O})$ is a finite direct sum of copies of $F$. Consequently, $H^0(U, \mathcal{O})$ is a finite-dimensional reduced $F$-algebra, and hence must itself split completely because $F$ is algebraically closed. This splitting induces the desired splitting of $f$. \hfill $\square$

5. **Abelian covers**

We next give a direct argument to split $\mathbb{Z}/p\mathbb{Z}$-covers.

**Lemma 5.1.** Suppose that $E$ is of characteristic $p$ and that $L \cong \kappa_L((u^L)), F \cong \kappa_F((t^F))$. Assume further that $\Gamma_L, \Gamma_F$ are subgroups of $\mathbb{R}$ and that the norms on $L, F$ are normalized so that

$$|u^i| = p^{-i}, \quad |t^j| = p^{-j} \quad (i \in \Gamma_L, j \in \Gamma_F).$$

Put $k := \kappa_L \otimes_{\mathbb{Q}_p} \kappa_F$ and let $k^{\Gamma_L \times \Gamma_K}$ be the set of functions $\Gamma_L \times \Gamma_F \to k$, with elements written as $k$-valued formal sums over $\Gamma_L \times \Gamma_F$. Consider the map $L \otimes_{\mathbb{Q}_p} F \to k^{\Gamma_L \times \Gamma_K}$ induced by the bilinear map

$$L \times F \to k^{\Gamma_L \times \Gamma_K}, \quad \left( \sum_{i \in \Gamma_L} x^i, \sum_{j \in \Gamma_F} y^j \right) \mapsto \sum_{(i,j) \in \Gamma_L \times \Gamma_F} (x^i \otimes y^j)u^i t^j.$$
Then for any \( r > 0 \), the restriction to \( L \otimes_{\mathbb{F}_q} F \) of the function \( \lambda_r \) on \( k^{\Gamma_L \times \Gamma_K} \) given by

\[
\lambda_r \left( \sum_{(i,j) \in \Gamma_L \times \Gamma_F} x_{i,j} u_i^j \right) = \sup \{ p^{-r} : x_{i,j} \neq 0 \}
\]

computes the tensor product of the given norm on \( F \) and the \( r \)-th power of the given norm on \( L \).

**Proof.** We start by fixing some terminology in order to articulate the argument. By a presentation of an element \( z \in L \otimes_{\mathbb{F}_q} F \), we mean an expression of \( z \) as a finite sum \( \sum_i x_i \otimes y_i \) of simple tensors in \( L \otimes_{\mathbb{F}_q} F \). For a presentation of the form \( \sum_i x_i \otimes y_i \), define the norm of the presentation as \( \max_i \{|x_i|^r | y_i| \} \); by definition, the tensor product seminorm of an element of \( L \otimes_{\mathbb{F}_q} F \) is the infimum of the norms of all presentations of \( z \). For \( z \) admitting a presentation as a simple tensor \( x \otimes y \), the norm of such a presentation equals \( \lambda_r(z) \); it follows formally that \( \lambda_r \) is a lower bound for the tensor product norm. To complete the argument, we will show that every \( z \in L \otimes_{\mathbb{F}_q} F \) admits a presentation with norm \( \lambda_r(z) \); as a bonus, that shows that in this situation, the infimum in the definition of the tensor product norm is always achieved (which need not hold in a more general setting).

Let \( \sum_i x_i \otimes y_i \) be a presentation of \( z \). Let \( L_1 \) (resp. \( F_1 \)) be the \( \mathbb{F}_q \)-vector subspace of \( L \) (resp. \( F \)) spanned by the \( x_i \) (resp. the \( y_i \)). Choose a basis \( e_1, e_2, \ldots \) of \( L_1 \) which is normalized in the following sense:

- the valuations of \( e_1, e_2, \ldots \) form a nondecreasing sequence; and
- if \( e_{i_1}, \ldots, e_{i_2} \) have the same valuation, then their images in the graded ring of \( L \) are linearly independent over \( \mathbb{F}_q \). That is, the images of \( e_i/e_{i_1} \) in \( \kappa_L \) for \( i = i_1, \ldots, i_2 \) are linearly independent over \( \mathbb{F}_q \).

Similarly, choose a normalized basis \( f_1, f_2, \ldots \) of \( F_1 \). Now rewrite the original presentation of \( z \) in the form \( \sum_{i,j} c_{i,j} e_i \otimes f_j \) with \( c_{i,j} \in \mathbb{F}_q \); this presentation has norm equal to \( \lambda_r(z) \). □

**Remark 5.2.** In the notation of Lemma 5.1, the completion of \( L \otimes_{\mathbb{F}_q} F \) injects into \( k^{\Gamma_L \times \Gamma_K} \). It is a subtle problem to describe the image \( R \), and we will not attempt to do so here. One remark we do make is that as a \( k \)-module, \( R \) splits as a direct sum \( \bigoplus_{e_1, e_2 \in \{-, +\}} R_{e_1, e_2} \) in which \( R_{e_1, e_2} \) is the set of elements of \( R \) supported on pairs \((i, j)\) with signs \((e_1, e_2)\); namely, this splitting is obtained by splitting the factors of simple tensors in the form \( x = x_+ + x_0 + x_- \) as per Definition 4.13.

**Lemma 5.3.** Every étale \( \mathbb{Z}/p\mathbb{Z} \)-cover of \( X_{L,E} \times E F \) splits.

**Proof.** By Remark 3.8 we may assume that \( E \) is of characteristic \( p \). By Lemma 4.11, we may check the claim after enlarging both \( L \) and \( F \); we may thus also assume that both fields are spherically complete. We may then identify \( L \) and \( F \) with \( \kappa_L((u^t L)) \) and \( \kappa_F((t^{Fr})) \), respectively; moreover, the groups \( \Gamma_L, \Gamma_F \) must be divisible and the fields \( \kappa_L, \kappa_F \) must be algebraically closed.

Define the rings \( R := \bigcap_i B^i_{L,E} \otimes_{\mathbb{F}_q} F = \mathcal{O}(Y_{L,E} \times E F) \), \( k := \kappa_L \otimes_{\mathbb{F}_q} \kappa_F \). Recall that since we are in characteristic \( p \), by Remark 3.8 we may interpret \( B^i_{L,E} \otimes_{\mathbb{F}_q} F \) as a completion of \( L \otimes_{\mathbb{F}_q} F \); as per Lemma 5.1 and Remark 5.2, we may write any element \( x \in R \) as a formal sum \( \sum_{(i,j) \in \Gamma_L \times \Gamma_F} x_{i,j} u_i^j \) with \( x_{i,j} \in k \), and use this representation to compute \( \lambda_r \) for all \( r > 0 \). We may further decompose \( x \) canonically as \( \sum_{e_1, e_2 \in \{-, +\}} x_{e_1, e_2} \) as in Remark 5.2.
Recall the general Artin–Schreier construction: on any space of characteristic $p$ we have an exact sequence of étale sheaves

$$0 \to \mathbb{F}_p \to \mathcal{O} \xrightarrow{\varphi^{-1}} \mathcal{O} \to 0.$$ 

On the quasi-Stein space $Y_{L,E} \times_E F$, this yields isomorphisms

$$H^i((Y_{L,E} \times_E F)_{et}, \mathbb{Z}/p\mathbb{Z}) \cong H^i(\varphi, R) \quad (i = 0, 1).$$

Quotienting by $\varphi_L$ and considering the Hochschild–Serre spectral sequence yields a commutative diagram

$$
\begin{array}{ccc}
0 & \xrightarrow{\quad} & H^1(\varphi, k^p) \\
\downarrow & & \downarrow \\
0 & \xrightarrow{\quad} & H^1(\varphi, R^e) \\
\end{array}
\begin{array}{ccc}
\xrightarrow{\quad} & H^1((\Spec(k)/\Phi)_{et}, \mathbb{Z}/p\mathbb{Z}) & \xrightarrow{\quad} & H^1((X_{L,E} \times_E F)_{et}, \mathbb{Z}/p\mathbb{Z}) & \xrightarrow{\quad} & H^1(\varphi, R)^{\varphi_L}.
\end{array}
$$

with exact rows. By Drinfeld’s lemma for schemes (Theorem 2.3), the terms in the top row vanish. From formal sums, we see that the left vertical arrow is an isomorphism. Consequently, to finish we must check that $H^1(\varphi, R)^{\varphi_L} = 0$.

Before proceeding, we record a key observation. Again by the Artin–Schreier construction and the Hochschild–Serre spectral sequence, we have an exact sequence

$$H^1((\Spec(L \otimes_{\mathbb{F}_q} F)_{et}, \mathbb{Z}/p\mathbb{Z}) \to H^1(\varphi, L \otimes_{\mathbb{F}_q} F)^{\varphi_L} \to H^2(\varphi, (L \otimes_{\mathbb{F}_q} F)^{\varphi_L}) = 0.$$ 

By Theorem 2.3 again, the first term in this sequence vanishes; we thus deduce that

(5.3.1) $H^1(\varphi, L \otimes_{\mathbb{F}_q} F)^{\varphi_L} = 0$.

Note that the action of $\varphi - 1$ on $R$ preserves the decomposition $R \cong \bigoplus_{e_1, e_2} R_{e_1, e_2}$. Consequently, it suffices to check that $H^1(\varphi, R_{e_1, e_2})^{\varphi_L} = 0$ for all $e_1, e_2$.

First, suppose that $(e_1, e_2) = (0, +)$. Any class in $H^1(\varphi, R_{0, +})$ may be represented by an element $y_{0,+} \in R_{0, +}$ which is a convergent sum of products supported on $\{0\} \times [1, \infty)$; namely, all but finitely many terms already have the right support, and each remaining term can be replaced by its image under a suitable power of Frobenius to fix its support. However, we can now replace $y_{0,+}$ with

$$y_{0,+} + z^p - z, \quad z = \sum_{m=0}^{\infty} y_{0,+}^p,$$

to see that $y_{0,+}$ represents the zero class in $H^1(\varphi, R_{0, +})$. We deduce that $H^1(\varphi, R_{0, +}) = 0$ even before taking $\varphi_L$-invariants. Similar considerations apply in the cases $(e_1, e_2) = (+, 0), (+, +)$.

Next, suppose that $(e_1, e_2) = (0, -)$. Any class in $H^1(\varphi, R_{0, +})^{\varphi_L}$ may be represented by an element of $R_{0, +}$ which is a finite sum of products supported on $\{0\} \times (-\infty, 0)$; in particular, this yields an element of $L \otimes_{\mathbb{F}_q} F$ whose image in $H^1(\varphi, L \otimes_{\mathbb{F}_q} F)$ is again $\varphi_L$-invariant. By (5.3.1), this image vanishes, from which we deduce that $H^1(\varphi, R_{0, -})^{\varphi_L} = 0$. Similar considerations apply in the cases $(e_1, e_2) = (-, 0), (-, -), (0, 0)$.

Finally, suppose that $(e_1, e_2) = (+, -)$. Any class in $H^1(\varphi, R_{+, -})^{\varphi_L}$ may be represented by an element $y_{+, -}$ of $R_{+, -}$ which is a convergent sum of products supported on $[1, \infty) \times (-\infty, 0)$. 

Consider the formal expansion \( \sum_{i,j} y_{i,j}u^it^j \) of \( y_{i,-} \); since the resulting class in \( H^1(\varphi, R_{+, -}) \) is \( \varphi_L \)-invariant, for every \( i, j \) and every nonnegative integer \( m \) we have
\[
\sum_{n \in \mathbb{Z}} y_{ip^n,jp^n}^{p^{-n}} = \sum_{n \in \mathbb{Z}} \varphi_{K_i}(y_{ip^n,jp^n+m}^{p^{-n}}).
\]
For \( m \) sufficiently large (depending on \( i, j \)), the sum on the right is identically zero. From this, it follows that the formal sum \( \sum_{n=0}^{\infty} y_{i,-}^{p^n} \) converges to an element of \( R_{+, -} \), and hence that the class of \( y_{i,-} \) in \( H^1(\varphi, R_{+, -}) \) vanishes. Similar considerations apply in the case \((e_1, e_2) = (-, +)\).

\[\square\]

6. Inputs from \( p \)-adic differential equations

We next bring in some relevant input from the theory of \( p \)-adic differential equations.

**Hypothesis 6.1.** Throughout \([6]\), assume that \( E \) is of characteristic 0. Let \( K_0 \) be the completion of \( F(T) \) for the Gauss norm. (This field often called the field of analytic elements over \( F \) in the variable \( T \); this terminology is due to Krasner \([23, 24]\).) Let \( K \) be a finite tamely ramified extension of \( K_0 \).

**Definition 6.2.** The derivation \( \frac{d}{dT} \) on \( F(T) \) is submetric for the Gauss norm, so it extends continuously to a derivation on \( K \) with operator norm 1. By a differential module over \( K \), we will mean a finite-dimensional \( K \)-vector space \( V \) equipped with a derivation \( D \) satisfying the Leibniz rule with respect to \( \frac{d}{dT} \). The example to keep in mind is a finite étale \( K \)-algebra equipped with the unique \( K \)-linear derivation extending \( \frac{d}{dT} \).

Let \( V \) be a differential module over \( K \) of constant rank \( n \). We define the subsidiary radii of \( V \) as in \([6] \) Definition 9.8.1; this is a multisubset of \( (0, 1] \) of cardinality \( n \), and is invariant under base extension \([6] \) Proposition 10.6.6. Geometrically, the subsidiary radii may be interpreted as the radii of convergence of local horizontal sections of \( V \) in a generic unit disc \([6] \) Theorem 11.9.2].

Let \( e^{-f_1(V)}, \ldots, e^{-f_n(V)} \) be the subsidiary radii of \( V \) listed in ascending order (with multiplicity), and define
\[
F_i(V) := f_1(V) + \cdots + f_i(V) \quad (i = 1, \ldots, n).
\]

**Remark 6.3.** Suppose that \( K \) is also a finite tamely ramified extension of the completion of \( K_0(T) \) for the Gauss norm. Then any differential module \( V \) with respect to \( \frac{d}{dT} \) is also a differential module with respect to \( \frac{d}{dT} \), and the two resulting definitions of subsidiary radii coincide. This follows from the geometric interpretation in terms of convergence in a generic disc.

**Remark 6.4.** One can generally predict properties of the subsidiary radii of a differential module \( V \) by modeling them by the inverse norms of the eigenvalues of some linear transformation \( T_V \) on some \( n \)-dimensional vector space over \( K \), subject to the functoriality properties
\[
T_{V_1 \oplus V_2} = T_{V_1} \oplus T_{V_2}, \quad T_{V_1 \otimes V_2} = T_{V_1} \otimes 1_{V_2} + 1_{V_1} \otimes T_{V_2}.
\]

**Lemma 6.5** (Christol–Dwork). Let \( V \) be a differential module over \( K \) of rank \( n \). Let \( v \in V \) be a cyclic vector, i.e., an element such that \( v, D(v), \ldots, D^{n-1}(v) \) form a basis of \( K \). (Such elements always exist; see for example \([16] \) Theorem 5.4.2.) Write \( D^n(v) = a_0 + a_1v + \ldots + a_{n-1}v^{n-1} \).
\[\cdots + a_{n-1}D^{n-1}(v) \text{ with } a_0, \ldots, a_{n-1} \in K. \] Form the multiset consisting of \(p^{-1/(p-1)}|\lambda|^{-1}\) as \(\lambda\) varies over the roots of the polynomial \(T^n - a_{n-1}T^{n-1} - \cdots - a_0\) in an algebraic closure of \(K\). Then this multiset coincides with the subsidiary radii of \(K\) in its values less than \(p^{-1/(p-1)}\).

Proof. The original reference is \([3\, \text{Théorème 1.5}]\); see also \([16\, \text{Theorem 6.5.3}]\). \(\square\)

**Lemma 6.6.** Let \(V\) be a differential module over \(K\) of rank \(n\). Let \(V'\) be the restriction of \(V\) along the unique continuous \(F\)-linear homomorphism \(K \to K\) taking \(T\) to \(T^p\). View \(V'\) as a differential module over \(K\) of rank \(n\) using the derivation \(D' = pT^{p-1}D\) (this is called the Frobenius pushforward of \(V\)). Then the subsidiary radii of \(V'\) coincide with the multiset consisting of

\[\begin{cases} \{p\rho\} \cup \{(p^{-p/(p-1)}) \times (p-1)\} & \rho > p^{-1/(p-1)} \\ \{p^{-1}\rho\} \times p & \rho \leq p^{-1/(p-1)} \end{cases}\]

for \(\rho\) running over the subsidiary radii of \(V\).

Proof. See \([16\, \text{Theorem 10.5.1}]\). \(\square\)

**Lemma 6.7.** Let \(V\) be a differential module over \(K\) of rank \(n\). Suppose that the subsidiary radii of \(V\) are all equal to some value \(\rho < 1\). Then at least \(n\) of the subsidiary radii of \(V' \otimes V\) are strictly greater than \(\rho\).

Proof. This follows from the existence of a refined spectral decomposition of \(V\) over a suitable tamely ramified extension of \(K\); see \([16\, \text{Theorem 10.6.2, Theorem 10.6.7}]\). \(\square\)

**Hypothesis 6.8.** For the remainder of \([6]\) let \(D\) be a one-dimensional affinoid space over \(F\) (viewed as an adic space). Let \(E\) be a vector bundle over \(D\) of rank \(n\) equipped with an \(F\)-linear connection.

**Definition 6.9.** Let \(x \in D\) be a point of type 2 or 5 in the sense of Definition \([1, \text{Definition 1.4}]\). Then the residue field \(H(x)\) may be viewed as a finite tamely ramified extension of \(K_0\) in some fashion. We may thus define the quantities \(f_i(E, x), F_i(E, x)\) for \(i = 1, \ldots, n\) by viewing the fiber \(E_x\) as a differential module over \(H(x)\). By Remark \([6.3]\) this definition does not depend on auxiliary choices.

Now let \(x \in D\) be a point of type 3 or 4. Then after replacing \(F\) with a suitable extension field, we may lift \(x\) to a point of type 2 and apply the previous paragraph to define \(f_i(E, x), F_i(E, x)\).

**Remark 6.10.** The previous definition can again be interpreted in terms of the convergence of local horizontal sections. See \([19]\) for an overview of the results one obtains in this manner.

**Lemma 6.11.** Let \(U\) be an open disc in \(D\) bounded by \(x\). If \(f_1(E, x) = 0\), then the restriction of \(E\) to \(U\) has trivial connection.

Proof. This is an instance of the Dwork transfer principle \([16\, \text{Theorem 9.6.1}]\). \(\square\)

**Lemma 6.12.** Let \(U\) be an open annulus in \(D\) and let \(T\) be a coordinate on \(U\). For each positive integer \(m\), let \(U_m\) be the \(m\)-fold cover of \(U\) with coordinate \(T^{1/m}\). Suppose that there exists a finite étale cover \(\pi : D' \to D\) such that the pullback of \(E\) to \(D'\) has trivial connection. If \(f_1(E, x) = 0\) for each \(x\) in the skeleton of \(U\), then there exists a positive integer \(m \leq \deg(\pi)!\) not divisible by \(p\) such that the pullback of \(E\) to \(U_m\) has trivial connection.
**Proof.** The condition that $f_1(\mathcal{E}, x) = 0$ for each $x$ in the skeleton of $U$ means, in classical language, that $\mathcal{E}$ satisfies the Robba condition on $U$. We may then apply [16, Theorem 13.5.5] to construct an exponent $A \in \mathbb{Z}_p^n$ for $\mathcal{E}$ on $U$ in the sense of the Christol–Mebkhout theory of $p$-adic exponents [16, Theorem 13]. We claim that $A \in \mathbb{Q}^n$; this may be checked after shrinking $U$, so we may reduce to the case where $\pi^{-1}(U)$ admits a connected component $U'$ which is itself an annulus. Let $d$ be the degree of $U' \to U$; then $dA$ is an exponent of the pullback of $\mathcal{E}$ to $U'$, as is the zero vector because this connection is trivial. By [16, Theorem 13.5.6], $dA \in \mathbb{Z}^n$, proving the claim.

Let $m$ be the lowest common denominator of $A$. By [16, Theorem 13.6.1], the pullback of $\mathcal{E}$ to $U_m$ is unipotent. However, this pullback is also semisimple, as this may be checked after pulling back along $\pi$; consequently, the pullback of $\mathcal{E}$ to $U_m$ is trivial. $\square$

7. Relative connections

We now extend the discussion of $p$-adic connections to a relative setting, with the key case being when the base field is replaced by a Fargues–Fontaine curve.

**Hypothesis 7.1.** Throughout [17] let $A$ be a uniform Huber ring over $\mathbb{Q}_p$.

**Definition 7.2.** Let $\mathcal{M}(A)$ denote the Gel’fand spectrum of $A$ in the sense of Berkovich, i.e., the space of all bounded multiplicative seminorms on $A$, equipped with the evaluation topology; it may be identified with the maximal Hausdorff quotient of the adic spectrum $\text{Spa}(A, A^\circ)$. Since $A$ is assumed to be uniform, the supremum over $\mathcal{M}(A)$ is a norm on $A$ which induces the correct topology on $A$. For $\alpha \in \mathcal{M}(A)$, let $\ker(\alpha) := \alpha^{-1}(0)$ be the inverse image of 0 under $\alpha$, and let $\mathcal{H}(\alpha) := A/\ker(\alpha)$ denote the (completed) residue field of $\alpha$.

For $\alpha \in \mathcal{M}(A)$, we say that a rational localization $A \to B$ (in the sense of Huber rings) encircles $\alpha$ if the map $\mathcal{M}(B) \to \mathcal{M}(A)$ identifies $\mathcal{M}(B)$ with a neighborhood of $\alpha$ in $\mathcal{M}(A)$.

Such neighborhoods form a neighborhood basis of $\alpha$ in $\mathcal{M}(A)$.

**Definition 7.3.** Equip the ring $A(T)$ with the Gauss extension of the norm on $A$. For each $\alpha \in \mathcal{M}(A)$, let $\tilde{\alpha} \in \mathcal{M}(A(T))$ be its Gauss extension. Let $S$ be the multiplicative subset of $s \in A(T)$ for which $\tilde{\alpha}(s) \neq 0$ for all $\alpha \in \mathcal{M}(A)$ (equivalently, the coefficients of $s$ have no common zero in $\mathcal{M}(A)$). We may then form a new Huber ring $R_A$ by completing the algebraic localization $A(T)_S$ for the supremum of the seminorms induced by $\tilde{\alpha}$ for each $\alpha \in \mathcal{M}(A)$. By construction, $R_A$ is again a uniform Huber ring; in the case where $A = K$ is a nonarchimedean field with norm $\alpha$, the ring $R_K$ is simply the completion of $K(T)$ with respect to the multiplicative norm $\tilde{\alpha}$. Note that any homomorphism $A \to B$ of uniform Huber rings induces a homomorphism $R_A \to R_B$.

**Definition 7.4.** By a differential module over $R_A$, we will mean a finite projective $R_A$-module $M$ equipped with a derivation $D$ satisfying the Leibniz rule with respect to $\frac{d}{dt}$. We define the subsidiary radii of $M$ at $\alpha \in \mathcal{M}(A)$ by base extension from $R_A$ to $R_{\mathcal{H}(\alpha)}$ and application of Definition [6.2]. For $\alpha \in \mathcal{M}(A)$ let $e^{-f_1(M, \alpha)}, \ldots, e^{-f_n(M, \alpha)}$ be the subsidiary radii of $M$ at $\alpha$ listed in ascending order (with multiplicity), and define

$$F_i(M, \alpha) := f_1(M, \alpha) + \cdots + f_i(M, \alpha) \quad (i = 1, \ldots, n).$$
Lemma 7.5. Let $I$ be a closed subinterval of $(0, \infty)$. Let $M$ be a differential module over $R_A$ of constant rank $n$ for $A = B_{L,E}^I \otimes_{W(\mathcal{O}_E)} \mathcal{O}_F$. Then for $i = 1, \ldots, n$, the function 

$$t \mapsto F_i(M, \lambda_t)$$

on $I$ is convex in $t$.

Proof. It suffices to check that for any $\epsilon > 0$, the function 

$$t \mapsto \sum_{j=1}^{i} \max\{f_j(M, \lambda_t), \epsilon\}$$

is convex. It further suffices to check this claim locally around some interior point $t \in I$.

We first check the claim for $\epsilon = \frac{1}{p-1} \log p$. Apply the cyclic vector theorem to choose a cyclic vector $v$ for $M \otimes_{R_A} R_{B_{L,E}^I \otimes_{W(\mathcal{O}_E)} \mathcal{O}_F}$. By Corollary 3.5, $v$ is also a cyclic vector for $M \otimes_{R_A} R_{B_{L,E}^I \otimes_{W(\mathcal{O}_E)} \mathcal{O}_F}$ for some closed subinterval $J$ of $I$ containing $t$ in its interior. The claim then follows at once from Lemma 6.5 and the convexity of $t \mapsto \log \lambda_t(x)$ (see Definition 3.3).

To conclude, it suffices to check the claim for $\epsilon = \frac{p^m - p^{m-1}}{p-1} \log p$ for each nonnegative integer $m$. For this, we apply the previous argument to treat the base case $m = 0$ and Lemma 6.6 to handle the induction step. \qed

Definition 7.6. Let $X$ be an adic space over some nonarchimedean field $K$ over $\mathbb{Q}_p$. Let $D$ be a subset of the analytic affine line over $\mathbb{Q}_p$ containing the Gauss point (the boundary of the closed unit disc). Let $\mathcal{E}$ be a vector bundle of constant rank $n$ on $X \times_D E$ equipped with an $\mathcal{O}_X$-linear connection. By base extension to Definition 6.2, we may define the functions $f_i(\mathcal{E}, x)$ and $F_i(\mathcal{E}, x)$ for $x \in X$ and $i \in \{1, \ldots, n\}$.

Hypothesis 7.7. For the remainder of §7, let $D$ be a rational subspace of the analytic affine line over $F$ containing the Gauss point, and let $\mathcal{E}$ be a vector bundle of constant rank $n$ on $X_{L,E} \times_E D$ equipped with an $\mathcal{O}_{(X_{L,E} \times_E F)}$-linear connection.

Lemma 7.8. For $i = 1, \ldots, n$, the function $x \mapsto f_i(\mathcal{E}, x)$ on $X_{L,E} \times_E F$ is constant. We will hereafter denote by $f_i(\mathcal{E})$ this constant value.

Proof. We proceed by strong induction on $i$. We start by using Remark 3.10 to make an identification of topological spaces

$$\{\lambda_t : t > 0\} \times_E F \cong (0, +\infty) \times \widehat{\mathbb{Z}}.$$ 

We then note that 

$$t \mapsto \sup\{F_i(\mathcal{E}, x) : x \in \{t\} \times \widehat{\mathbb{Z}}\}$$

is both convex (by Lemma 7.3) and invariant under $t \mapsto tq$; it must therefore be constant. Denote by $c$ the constant value.

Our next goal is to check that $F_i(\mathcal{E}, x)$ is in fact constant on $(0, +\infty) \times \widehat{\mathbb{Z}}$. This is guaranteed in case $c$ equals the known constant value of $F_{i-1}(\mathcal{E}, x)$, as this is a lower bound on $F_i(\mathcal{E}, x)$; we may thus assume that the difference 

$$c' = \sup\{f_i(\mathcal{E}, x) : x \in \{t\} \times \widehat{\mathbb{Z}}\}$$

is positive (and again independent of $t$).
By combining Lemma 6.6 and Lemma 7.5 as in the proof of Lemma 7.5 (and using that \( c' > 0 \)), we see that for any fixed \( t \), the set of \( y \in \hat{\mathbb{Z}} \) for which \( x = (t, y) \) maximizes \( F_i(\mathcal{E}, x) \) is nonempty and open in \( \hat{\mathbb{Z}} \). For any such \( y \), the function \( t \mapsto F_i(\mathcal{E}, (t, y)) \) on \((0, \infty)\) is convex (again by Lemma 7.5), has \( c \) as an upper bound everywhere, and achieves this bound at one point. It therefore is identically equal to \( c \).

Returning to Remark 3.10 and applying the previous paragraph, we now see that for any fixed \( t \), the set of \( y \in \hat{\mathbb{Z}} \) for which \( x = (t, y) \) maximizes \( F_i(\mathcal{E}, x) \) is also invariant under the translation action of \( \mathbb{Z} \subset \hat{\mathbb{Z}} \). Since it is nonempty and dense, it must cover \( \hat{\mathbb{Z}} \); this completes the proof that \( F_i(\mathcal{E}, x) = c \) for all \( t \) and all \( x \in \{ \lambda_0 \} \times_E F \).

At this point, it now suffices to check that the function \( \alpha \mapsto F_i(\mathcal{E}, \alpha) \) on \( B_{L,E}^{[t]} \mathcal{E}_{W(\mathbb{Q})}_E \mathcal{O}_F \) is constant. As in the proof of Lemma 7.5, it suffices to check that for any \( \epsilon > 0 \), the function

\[
\alpha \mapsto \sum_{j=1}^{i} \max \{ f_j(\mathcal{E}, \alpha), \epsilon \}
\]

is constant; again using Lemma 6.6 we may further reduce to the case \( \epsilon = \frac{1}{p-1} \log p \). To check this case, let \( M \) be the base extension of \( \mathcal{E} \) to this ring, let \( v \) be a cyclic vector of \( M \), and set notation as in Lemma 6.5. Then the coefficients \( a_0, \ldots, a_{n-1} \) belong to \( B_{L,E}^{[t]} \mathcal{E}_{W(\mathbb{Q})}_E \mathcal{O}_F \) for some interval \( J \) containing \( t \) in its interior; moreover, by the previous arguments, for each coefficient \( a_i \) which contributes to the Newton polygon for \( \alpha = \lambda_i \), its image in \( \text{Gr}(B_{L,E}^{[t]} \mathcal{E}_{W(\mathbb{Q})}_E \mathcal{O}_F) \) belongs to \( \text{Gr} F \). From this, we read off the claim. \( \square \)

Lemma 7.9. There exists a unique decomposition \( \mathcal{E} = \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_k \) of vector bundles with connection (with nonzero summands) with the following properties.

(i) For \( i = 1, \ldots, k \), we have \( f_1(\mathcal{E}_i) = \cdots = f_{\text{rank}(\mathcal{E}_i)}(\mathcal{E}_i) \).

(ii) We have \( f_1(\mathcal{E}_1) > \cdots > f_1(\mathcal{E}_k) \).

Proof. We may assume that \( n > 0 \). Let \( j \) be the largest integer such that \( f_j(\mathcal{E}) = f_1(\mathcal{E}) \); it suffices to split \( \mathcal{E} \) as a direct sum \( \mathcal{E}_1 \oplus \mathcal{E}_2 \) with \( \text{rank}(\mathcal{E}_1) = j \) such that \( f_i(\mathcal{E}_1) = f_1(\mathcal{E}) \) for all \( i \) and \( f_i(\mathcal{E}_2) < f_1(\mathcal{E}) \) for all \( i \). For this, we may assume that \( j < n \), and use Frobenius pushforwards (as in Lemma 6.6, but see more specifically the proof of [16 Theorem 12.2.2]) to reduce to the case where \( f_1(\mathcal{E}) > p^{-1}(p-1) \). In this case, set notation as in the proof of Lemma 7.5 we may then deduce the claim by applying a suitable version of Hensel’s lemma, such as [16 Theorem 2.2.2], to the polynomial coming from the cyclic vector. \( \square \)

8. Elimination of One Parameter

Hypothesis 8.1. Throughout §8 suppose that every finite étale cover of \( X_{L,E} \times_E F \) splits completely. Let \( D \) be a one-dimensional smooth affinoid algebra over \( F \). Let \( x \in D \) be a point of type 2, 3, 4 in the sense of Definition 1.4; in the type 2 case, assume further that \( D \) is a strict neighborhood of \( x \).

Let \( \rho : \pi_1^{\text{prof}}(X_{L,E} \times_E D) \to \text{GL}(V) \) be a discrete representation on a finite-dimensional \( F \)-vector space \( V \). We will make various statements that hold not for \( \rho \) itself, but its restriction to \( \pi_1^{\text{prof}}(X_{L,E} \times D') \) for some connected étale neighborhood \( D' \) of \( x \) in \( D \) (which in the type 2 case must again be a strict neighborhood of some preimage of \( x \)). To indicate this restriction, we will say that such statements hold after replacing \( D \).
Definition 8.2. Let \( f : U \to X_{L,E} \times_E D \) be a finite étale cover such that the restriction of \( \rho \) to \( \pi_1^{\text{prof}}(U) \) splits completely. We then obtain a diagonal action of \( \pi_1^{\text{prof}}(X_{L,E} \times_E D, \overline{\pi}) \) on \( f_\ast \mathcal{O}_U \otimes_F V \); let \( \mathcal{E}_\rho \) be the fixed submodule for this action, viewed as a vector bundle on \( X_{L,E} \times_E D \).

For \( n = \dim(V) \), for \( y \in D \) of type 2 or 5 with residual genus 0, we may define quantities \( f_1(\mathcal{E}_\rho, y), \ldots, f_n(\mathcal{E}_\rho, y) \) using Lemma 7.8 for \( y \) of type 3 or 4, we may define these quantities by enlarging \( F \) to lift \( y \) to a point of type 2 or 5 (which will necessarily have residual genus 0) and then applying Lemma 7.8 after suitable rescaling. (It is possible to extend the construction to points of type 2 or 5 of positive genus, but this is not crucial here.)

Lemma 8.3. Suppose that \( f_i(\mathcal{E}_\rho, y) = 0 \) for all \( i \) and \( y \). Then \( \rho \) becomes trivial after replacing \( D \).

Proof. We first verify that for each geometric point \( z \) of \( X_{L,E} \times_E F \), after replacing \( D \), the restriction of \( \rho \) to \( \pi_1^{\text{prof}}(z \times_F D) \) becomes trivial.

- If \( x \) is of type 3, then after replacing \( D \), we may assume that \( D \) is an annulus. By Lemma 6.12, the restriction of \( \rho \) to \( \pi_1^{\text{prof}}(z \times_F D) \) becomes trivial after replacing \( D \).
- If \( x \) is of type 4, then after replacing \( D \), we may assume that \( D \) is a disc. By Lemma 6.11, the restriction of \( \rho \) to \( \pi_1^{\text{prof}}(z \times_F D) \) is trivial.
- If \( x \) is of type 2, then for all but finitely many specializations \( x' \) of \( x \), we may find an open disc \( V \) in \( D \) bounded by \( x' \); by Lemma 6.11, the restriction of \( \rho \) to \( \pi_1^{\text{prof}}(z \times_F V) \) is trivial. For each of the other remaining specializations \( x' \), we may find an open annulus \( V \) in \( D \) bounded by \( x' \) at one end; by Lemma 6.12, the restriction of \( \rho \) to \( \pi_1^{\text{prof}}(z \times_F V) \) becomes trivial after replacing \( D \). Combining these results, we see that after replacing \( D \), the restriction of \( \rho \) to \( \pi_1^{\text{prof}}(z \times_F D) \) factors through \( \pi_1^{\text{prof}}(C_\ell) \) where \( C \) is the residual curve of \( D \) at \( x \) and \( \ell \) is the residue field of \( \mathcal{H}(z) \). Since \( C \) is proper by hypothesis, we may apply [20, Corollary 4.19] to deduce that \( \pi_1^{\text{prof}}(C_\ell) \to \pi_1^{\text{prof}}(C) \) is a homeomorphism; consequently, after replacing \( D \), the restriction of \( \rho \) to \( \pi_1^{\text{prof}}(z \times_F D) \) becomes trivial.

For any given \( z \), it formally follows that after replacing \( D \), there exists some neighborhood \( U \) of \( z \) in \( X_{L,E} \times_E F \) such that the restriction of \( \rho \) to \( \pi_1^{\text{prof}}(U \times_F D) \) factors through \( \pi_1^{\text{prof}}(U) \). By compactness, we may choose \( D \) uniformly over some finite set of geometric points \( z \) for which the neighborhoods \( U \) form a covering of \( X_{L,E} \); we then deduce that (after replacing \( D \)) the restriction of \( \rho \) to \( \pi_1^{\text{prof}}(X_{L,E} \times_E D) \) factors through \( \pi_1^{\text{prof}}(X_{L,E} \times_E F) \). As we are working under Hypothesis 8.1, the latter group is trivial; this proves the claim.

Lemma 8.4. Under no additional hypotheses, \( \rho \) becomes trivial after replacing \( D \).

Proof. Assume by way of contradiction that the conclusion fails for some \( \rho \). We may assume that \( \rho \) is irreducible with nontrivial simple image, and remains so after replacing \( D \) arbitrarily.

Suppose first that \( \lim_{y \to x} f_1(\mathcal{E}_\rho, y) = 0 \). By [16, Theorem 11.3.2], each function \( f_i(\mathcal{E}_\rho, y) \) restricts to a linear function on the skeleton of some neighborhood of \( x \) in \( D \). By Lemma 7.3 and Lemma 8.3, no more than one of the functions \( f_i(\mathcal{E}_\rho' \otimes \mathcal{E}_\rho, y) = f_i(\mathcal{E}_\rho' \otimes \mathcal{E}_\rho, y) \) can be identically zero; however, by Lemma 6.7, this is only possible if \( \dim(\rho) = 1 \).

Suppose next that \( \lim_{y \to x} f_1(\mathcal{E}_\rho, y) > 0 \). Again, by Lemma 7.3 and Lemma 8.3, no more than one of the limits \( \lim_{y \to x} f_i(\mathcal{E}_\rho' \otimes \mathcal{E}_\rho, y) \) can be zero, but again by Lemma 6.7, this is only possible if \( \dim(\rho) = 1 \).
In both cases, the image of $\rho$ is a cyclic group of prime order. If this order is prime to $p$, we obtain a contradiction against Lemma [8.3]; if the order is $p$, we obtain a contradiction against Lemma [5.3].

9. Simple connectivity

We now put everything together to prove Theorem 0.1.

**Theorem 9.1.** Every finite étale cover of $X_{L,E} \times_E F$ splits completely. In particular, Theorem 0.1 holds.

**Proof.** By Remark 3.8, we may assume that $E$ is of characteristic 0. By Lemma 4.7, the claim holds in case $F$ is a completed algebraic closure of $E$. By transfinite induction, it suffices to prove that if the claim holds for $F$, then it holds for the completed algebraic closure of $F'$ where $F'$ is the completion of $F(T)$ for some multiplicative norm.

Note that $F'$ determines a point of the adic affine line over $F$ of type 2, 3, or 4. We may thus apply Lemma 8.4 to conclude.

10. Drinfeld’s lemma for diamonds

To conclude, we reinterpret Theorem 0.1 in Scholze’s language of diamonds [27, 28] and give the formulation of Drinfeld’s lemma for diamonds described in [20, §4.3], which see for more details.

**Definition 10.1.** We write $X^\diamond$ for the diamond associated to the adic space $X$ over $E$. For $A$ a Huber ring over $E$, we write $\text{Spd}(A)$ as shorthand for $\text{Spa}(A, A^\diamond)$.

**Remark 10.2.** For any Huber ring $A$ over $E$, we have an identification

$$(Y_{L,E} \times_E \text{Spa}(A, A^\diamond))^\diamond \cong \text{Spd}(L) \times \text{Spd}(A)$$

where the product on the right is the absolute categorical product. Note that if $A$ is a perfectoid $E$-algebra, then $\text{Spd}(A) = \text{Spd}(A^\per)$ as diamonds; the extra data of the choice of $A$ as an untilt of $A^\per$ is equivalent to the specification of a structure morphism $\text{Spd}(A^\per) \to \text{Spd}(E)$.

**Remark 10.3.** Continuing with the previous remark, suppose that $F$ is a perfectoid field of characteristic $p$ containing $E$. From Remark 3.11 we have an identification

$$\text{Ft}((X_L \times_E F)^\diamond) \cong \text{Ft}(\text{Spd}(L) \times \text{Spd}(F))/\phi_L).$$

On the other hand, as in Definition 2.2 for $X := \text{Spd}(L) \times \text{Spd}(F)$ we may identify the right side with

$$\text{Ft}(X/\Phi) \cong \text{Ft}(X/(\phi_L, \phi_E)) \times_{\text{Ft}(X/\phi_X)} \text{Ft}(X),$$

in which $L$ and $F$ play symmetric roles. This interpretation leads naturally to Drinfeld’s lemma for analytic spaces; we will curtail the discussion here, but see [20, §4.3] for more details.

**Theorem 10.4.** Let $X_1, \ldots, X_n$ be connected qcqs diamonds. Then for any geometric point $\overline{x}$ of $X := X_1 \times \cdots \times X_n$, the map

$$\pi_1^{\text{prof}}(X/\Phi, \overline{x}) \to \prod_{i=1}^n \pi_1^{\text{prof}}(X_i, \overline{x})$$

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is an isomorphism of profinite groups.

Proof. The case where \( n = 2, X_1 = \text{Spd}(L), X_2 = \text{Spd}(F) \) is Theorem 0.1. For the reduction of the general case to this result, see [20, Theorem 4.3.14].

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