Singular supersymmetric $\sigma$-models

T.S. Nyawelo$^a$, F. Riccioni$^b$, J.W. van Holten$^c$

NIKHEF
PO Box 41882
1009 DB Amsterdam
The Netherlands

S. Groot Nibbelink$^d$
(CITA National Fellow)

Univ. of Victoria, Dept. of Physics and Astronomy
PO Box 3055 STN CSC
Victoria BC, V8W 3P6
Canada

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Abstract
Supersymmetric non–linear $\sigma$–models are described by a field dependent Kähler metric determining the kinetic terms. In general it is not guaranteed that this metric is always invertible. Our aim is to investigate the symmetry structure of supersymmetric models in four dimensional space-time in which metric singularities occur. For this purpose we study a simple anomaly-free extension of the supersymmetric $CP^1$ model from a classical point of view. We show that the metric singularities can be regularized by the addition of a soft supersymmetry-breaking mass parameter.

$^a$ e-mail: tinosn@nikhef.nl
$^b$ e-mail: fabio@nikhef.nl
$^c$ e-mail: v.holten@nikhef.nl
$^d$ e-mail: grootnib@uvic.ca
1. Introduction

The scalar fields of supersymmetric $\sigma$–models in four dimensions take values in Kähler manifolds [1, 2]. In some supersymmetric field theories the Kähler metric develops a zero mode; then the model becomes singular in the sense that some of the kinetic terms vanish in the vacuum state, and correspondingly some couplings diverge. The central issue we want to investigate in this paper is how to treat supersymmetric field theories in which these types of complications occur. This analysis is in particular relevant for supersymmetric non-linear $\sigma$–model building based on homogeneous Kählerian cosets $G/H$ [3].

Supersymmetric pure $\sigma$–models on cosets, including among others the grassmannian models on $SU(n+m)/[SU(n) \times SU(m) \times U(1)]$ and the models on manifolds $SO(2n)/U(n)$, are known to be anomalous [4, 5, 6], since they incorporate chiral fermions in non-trivial representations of the holonomy group. These anomalies can be removed by coupling additional chiral superfields carrying specific line-bundle representations of the group $G$ [7, 8]. In a study of anomaly-free extended supersymmetric $\sigma$–models on coset manifolds of the type $SO(2n)/U(n)$, it was found that the resulting field metric can develop singularities in the form of zero-modes, and therefore may not be positive definite [3]. Indeed, upon gauging all or part of the isometry group the $D$-term potential sometimes forces the scalar fields to take vacuum expectation values at the geometric singularities of field space.

In order to gain an understanding of this situation, in this paper we study an anomaly-free extension of the $d = 4$ supersymmetric $CP^1$-model, where the scalar fields take values in $SU(2)/U(1)$, and some of the isometries are gauged. In addition to the chiral multiplet parametrizing the coset manifold, anomaly cancellation requires the inclusion of other chiral multiplets. The simplest choice corresponds to a single supermultiplet with the scalar component defining a section of a complex line bundle. We couple these matter multiplets to a gauge multiplet, focusing in particular on the case where the full $SU(2)$ isometry group is gauged. If one considers the most general Kähler potential corresponding to this geometry, one realizes that depending on the parameters, the resulting metric can vanish for particular values of the scalars. Moreover, in many cases the potential drives the scalars to a vacuum value exactly at these singular points. At this singularity, some of the four-fermi couplings explode, while the mass terms for the fermions stay in general finite.

Singularities can occur in two places: either the kinetic term of the scalar parametrizing $CP^1$ or the kinetic term of the scalars parametrizing the section of the line bundle can vanish in the vacuum. When the two singularities occur at the same point, the vacuum preserves both supersymmetry and the whole $SU(2)$ gauge symmetry.

The paper is organized as follows. In section 2 we describe the isometry structure of the scalar manifold, and we show how a generic choice of the Kähler
potential leads to geometrical singularities. In Section 3 we describe the gauged version of the anomaly-free $CP^1$-model. In section 4 we classify the possible vacua, discussing general consequences for the gauge symmetries and particle spectra. Section 5 discusses a modification of the model containing a soft supersymmetry breaking mass term which preserves the full non-linear $SU(2)$. The mass term acts as a regulator, as it displaces the vacuum away from the singular point. The particle spectrum in this regulated model is computed and shown to be sensitive to the behaviour of the Kähler potential in the limit of small regulator mass. In section 6 we present some examples, showing that the various types of behaviour of the spectra in the limit of small regulator mass can all be realized in actual models. In section 7 we summarize our results. In the two appendices we present our notation and we give the complete expressions for the lagrangeans both off-shell and in the unitary gauge.

2. The model

We consider a simple supersymmetric $\sigma$-model, in which the vanishing of kinetic terms occurs in a restricted subset of the parameter domain. The model is based on the supersymmetric $CP^1$-model, where the scalar fields take values in $SU(2)/U(1)$. As the pure supersymmetric $CP^1$ model in four dimensions is anomalous, we include another chiral multiplet, transforming as a contravariant vector on the $CP^1$ manifold. The complete field content of the model is therefore specified by a complex scalar superfield $\Phi = (z, \psi_L, H)$ and a second complex scalar superfield $A = (a, \varphi_L, B)$. These superfields define representations of the isometry group $SU(2)$; on the scalar fields they take the infinitesimal form

$$\delta z = \epsilon + i\theta z + \bar{\epsilon}z^2, \quad \delta a = -i\theta a - 2\bar{\epsilon}za.$$ (1)

Here $\theta$ is the parameter of $U(1)$ phase transformations, and $(\epsilon, \bar{\epsilon})$ are the complex parameters of the broken off-diagonal $SU(2)$ transformations. We take the fields $z$ and $a$ to be dimensionless; dimensionful fields are obtained by introducing a parameter $f$ with the dimension of inverse mass (in natural units in which $c = h = 1$), and making the replacements

$$z \to fz, \quad a \to fa,$$ (2)

and similarly for other fields to be introduced. The infinitesimal transformations of such other field components (chiral spinors, auxiliary fields) are found by requiring the isometries to commute with supersymmetry [9]. Observe, that the opposite linear $U(1)$ transformations of the multiplets are precisely as required for cancellation of the isometry anomalies.

The dimensionless $CP^1$ Kähler potential

$$K_\sigma(\bar{z}, z) = \ln(1 + \bar{z}z)$$ (3)
is invariant under the isometry transformations (1) up to the real part of a holomorphic function:

\[ \delta K_\sigma(\bar{z}, z) = F(z) + \bar{F}(\bar{z}), \quad F(z; \theta, \bar{\epsilon}) = \frac{i}{2} \theta + \bar{\epsilon}z. \]  

(4)

The transformations of the scalar \( a \) can therefore be rewritten as

\[ \delta a = -2F(z; \theta, \bar{\epsilon})a. \]  

(5)

It follows, that the dimensionless real scalar

\[ X = \bar{a}a e^{2K_\sigma(\bar{z}, z)} = \bar{a}a (1 + \bar{z}z)^2, \]  

(6)

is an invariant under the full set of isometries. With this observation in mind, we take as the starting point for our supersymmetric model a Kähler potential

\[ K(\Phi, \bar{\Phi}; A, \bar{A}) = K_\sigma(\Phi, \bar{\Phi}) + K_m(\Phi, \bar{\Phi}; A, \bar{A}) = \ln(1 + \Phi\bar{\Phi}) + K_m(\Omega), \]  

(7)

with \( K_m(\Omega) \) a real function of the real superfield

\[ \Omega = \bar{A}A e^{2K_\sigma(\bar{z}, z)}, \]  

(8)

of which the real scalar quantity \( X \) is the lowest component. The kinetic terms of scalars and chiral spinors in the action are then given by

\[ S = \int d^4 x \int d^2 \theta d^2 \bar{\theta} K(\Phi, \bar{\Phi}; A, \bar{A}) = -\int d^4 x G_{IL} \left( \partial \bar{Z}^L \cdot \partial Z^I + \psi^I_L \bar{\psi}^L \right) + ... \]  

(9)

Here \( Z^I = (z, a) \) and \( \psi^I_L = (\psi_L, \varphi_L) \), \( I = (z, a) \), denote the scalar and spinor components of the respective superfields \( \Phi^I = (\Phi, A) \), and the dots represent four-fermion interactions. The full component action and its derivation are presented in appendices A and B. The Kähler metric in field space is obtained from the Kähler potential:

\[ G_{IL} = \frac{\partial^2 K}{\partial Z^I \partial \bar{Z}^L} = \begin{pmatrix} 2M(X) + 4\bar{z}zXM'(X) & 2a\bar{z}(1 + \bar{z}z)M'(X) \\ 2a\bar{z}(1 + \bar{z}z)M'(X) & (1 + \bar{z}z)^2M'(X) \end{pmatrix}, \]  

(10)

where we have introduced the \( SU(2) \)-invariant function \( M(X) \) defined in terms of \( K_m(X) \) as

\[ M(X) = \frac{1}{2} + XK'_m(X). \]  

(11)

The primes in the equations denote derivatives w.r.t. \( X \). The determinant of the metric is

\[ \det G = 2M'(X)M(X). \]  

(12)
Positive definite kinetic terms are obtained if both

\[ M(X) > 0, \quad M'(X) > 0. \]  

(13)

We then have a standard non-linear field theory, which is well-behaved below a cut-off \( \Lambda^2 \sim \mathcal{O}(1/f^2) \), the parameter determining the characteristic scale of the Kähler manifold.

In contrast, if one of the two factors is negative: \( \det G < 0 \), the theory contains ghosts and is inconsistent. If one of the two factors vanishes, we have critical case and we must resort to some regularization in order to investigate if the model still allows for some physically interesting interpretation. Observe however, that with the given field content it is not possible to construct an \( SU(2) \)-invariant superpotential \( W(\Phi^I) \). With such a flat potential the vacuum expectation values of the scalar fields are undetermined, but it is natural to suppose they are to be fixed in the region where the model is well behaved according to the criterion of eq. (13).

It is important to stress that the vanishing of the kinetic terms for the scalar fields corresponds to the divergence of some four-fermi couplings in the lagrangian, once one solves the equations for the auxiliary fields (see Appendix A for the details). The four–fermion interactions take the general form

\[
L_{4\text{fermi}} = K_{z\bar{z}z\bar{z}} \bar{\psi}_R \psi_L \bar{\psi}_L \psi_R + K_{\bar{a}a\bar{a}a} \bar{\varphi}_R \varphi_L \bar{\varphi}_L \varphi_R \\
+ \{ K_{z\bar{z}a\bar{a}} \bar{\psi}_R \psi_L \bar{\varphi}_L \varphi_R + \text{perm.} \},
\]

(14)

with the curvature components given by

\[
K_{z\bar{z}z\bar{z}} = -4M + 8M'X ,
\]

\[
K_{\bar{a}a\bar{a}a} = M'' + XM''' - XM''^2/M' ,
\]

(15)

\[
K_{z\bar{z}a\bar{a}} = 2M' + 2XM'' - 2XM''^2/M' .
\]

The last term in eq. (14) is short-hand for four combinations with two \( \psi \)'s and two \( \varphi \)'s which contribute to the four-fermion terms. From the expressions (15) it follows, that subsets of the four-fermion terms diverge at the kinetic singularities.

3. The gauged \( CP^1 \)-model

The critical case is of importance when the isometry group of the model is gauged. As well-known, the gauging of the supersymmetric model involves several steps:
- modification of the kinetic terms by introducing gauge-covariant derivatives;
- addition of a \( D \)-term potential;
- addition of Yukawa couplings for the fermions;
- introduction of kinetic terms for the gauge superfields.
A simplification occurs however, because in the model with broken local $SU(2)$ the Goldstone bosons are absorbed completely in the longitudinal components of the massive charged vector bosons, and we can analyze the model in the unitary gauge $\bar{z} = z = 0$, with $X = \bar{a}a$. In this gauge the metric (10) is automatically diagonal:

$$G_{IL} = \begin{pmatrix} 2M(X) & 0 \\ 0 & M'(X) \end{pmatrix}. \quad (16)$$

The expressions for the gauge-covariant derivatives of the complex scalar fields read

$$D_\mu z = \partial_\mu z - igA_\mu z - \frac{g}{\sqrt{2}} (W_\mu^+ + W^-_\mu z^2) \simeq -\frac{g}{\sqrt{2}} W_\mu^+; \quad (17)$$

$$D_\mu a = \partial_\mu a + igA_\mu a + \sqrt{2}gW^-_\mu za \simeq \partial_\mu a + igA_\mu a,$$

whilst the covariant derivatives of the fermions become

$$D_\mu \psi_L = \partial_\mu \psi_L - igA_\mu \psi_L - \sqrt{2}gW^-_\mu z\psi_L \simeq \partial_\mu \psi_L - igA_\mu \psi_L$$

$$D_\mu \varphi_L = \partial_\mu \varphi_L + igA_\mu \varphi_L + \sqrt{2}gW^-_\mu (z\varphi_L + a\psi_L)$$

$$\simeq \partial_\mu \varphi_L + igA_\mu \varphi_L + \sqrt{2}gW^-_\mu a\psi_L. \quad (18)$$

The last expression on each line is the one in the unitary gauge. We have introduced the notation $W^\pm_\mu$ for the charged gauge fields corresponding to the broken $SU(2)$ transformations parametrized by $(\epsilon, \bar{\epsilon})$; $A_\mu$ is the gauge field of the $U(1)$ transformations. Note, that for the unitary gauge to be valid, we must assume that the charged vector bosons are massive, i.e. $\langle M(X) \rangle > 0$. In the critical case $\langle M(X) \rangle = 0$ the charged vector bosons become massless again, and the choice of the unitary gauge is not allowed.

Next we discuss the $D$-term potential. First we recall, that the isometries (1) can be obtained locally as gradients of a set of real Killing potentials, defined by

$$M(\theta, \epsilon, \bar{\epsilon}) = \frac{\theta(1 - \bar{\epsilon}z) + 2i(\epsilon\bar{z} - \bar{\epsilon}z)}{1 + \bar{z}z} M(X). \quad (19)$$

Indeed, the variations (1) are given by

$$\delta Z^I = -iG^{IL} \frac{\partial M}{\partial Z^L}. \quad (20)$$

Now the auxiliary $D$-fields couple to these Killing potentials, and after elimination of the $D$-fields the potential for the model with fully gauged $SU(2)$ becomes

$$V_D = \frac{g^2}{2} \frac{\partial M}{\partial \epsilon} \frac{\partial M}{\partial \bar{\epsilon}} + \frac{g^2}{2} \left( \frac{\partial M}{\partial \theta} \right)^2 = \frac{g^2}{2} M^2(X). \quad (21)$$
Finally, we also have to introduce kinetic terms for the gauge fields. They are of the canonical form

\[ \mathcal{L}_{\text{gauge}} = -\frac{1}{2} F^+(W) \cdot F^-(W) - \frac{1}{4} F^2(A). \]  

(22)

In this expression the quadratic terms for the auxiliary \( D \)-fields have been left out, their elimination giving rise to the scalar potential (21). For the full action, including the fermionic terms required by supersymmetry, we again refer to appendix B, eqs. (83) and (88).

4. Analysis of the particle spectrum

We begin our analysis of the particle content of the model by studying the bosonic part of the \( CP^1 \)-model, which up to the kinetic terms for the gauge bosons is described by the action

\[ \mathcal{L}_B = -g^2 M(X) W^+ \cdot W^- - M'(X) |Da|^2 - \frac{g^2}{2} M^2(X). \]  

(23)

The vacuum expectation value of the scalar field \( a \) is derived by minimizing the potential, which leads to the condition

\[ M(X) M'(X) a = 0. \]  

(24)

It is clear that a priori there may be three ways to solve this equation:
(a) \( a = 0 \) is always a solution.
(b) There may be a value \( a = a_0 \) such that \( M'_0 = M'(X_0) = 0 \); if the potential reaches its minimum here, then the model is critical in the sense discussed above, and we have to be careful in the analysis of the physical realization of the theory.
(c) The solution \( M_0 = M(X_0) = 0 \) is also logically allowed; it implies that the charged vector bosons become massless. It may happen that at the same time \( X_0 M'(X_0) = 0 \); then all \( SU(2) \) gauge bosons become massless and the full gauge symmetry is restored. In that case the complex scalars \( (z, \bar{z}) \) are no longer Goldstone bosons, and the unitary gauge can not be used to eliminate them. We also observe, that if the solution \( M_0 = 0 \) exists, it is necessarily the absolute minimum of the potential: \( V_0 = 0 \), and supersymmetry is apparently restored as well. Of course, the standard way to describe a situation in which the gauge symmetry is restored, is to reformulate the physics in terms of linear representations of the gauge symmetry.

Assuming therefore that supersymmetry is broken and \( M_0 > 0 \), it depends on the precise form of \( M(X) \) which solution is the true minimum of the potential; actually, it can happen that both conditions hold simultaneously. Quite generally, we can derive the linearized field equation for fluctuations around the vacuum, by making the expansion

\[ a = a_0 + \sqrt{Z_a} \delta a, \]  

(25)
where the normalization factor $Z_a$ is still to be determined. With $a_0$ a solution of eq. (24), the quadratic part of the action for the linearized field fluctuations becomes

$$
\mathcal{L}_{\text{lin}} = -g^2 M_0 W^+ \cdot W^- - g^2 X_0 M'_0 A_\mu^2 - Z_a M'_0 |\partial_\mu \delta a|^2 - \frac{g^2}{2} M_0^2
$$

(26)

$$
- g^2 Z_a M_0 M'_0 |\delta a|^2 - \frac{g^2 Z_a}{2} \left( M_0 M''_0 + M'_0^2 \right) (a_0 \delta a + a_0 \delta \bar{a})^2 + \ldots
$$

First we consider the solution (a): $a_0 = 0$ with $M'(0) > 0$. Taking $Z_a = 1/M'(0)$ we get a canonically normalized model for the fluctuating field, satisfying in the linearized limit a Klein-Gordon equation

$$
(-\square + m_a^2) \delta a = 0,
$$

(27)

with the mass given by

$$
m_a^2 = g^2 M(0) = \frac{g^2}{2}.
$$

(28)

This conclusion can be extended to the case $M'(0) = 0$, although $Z_a$ diverges in that case, because mass and kinetic terms are then rescaled by the same infinite normalization factor, and the mass remains finite. In fact, from eq. (14) it turns out that also the four-fermion terms stay finite in this case.

In contrast, in case (b) with $X_0 = |a_0|^2 > 0$ and $M'_0 = 0$, some contributions to the mass terms of the scalars $\delta a$ in eq. (26) generally diverge. This indicates that one of the degrees of freedom does not describe a propagating particle: it decouples from the spectrum of states. In all cases, when $X_0 M'_0 = 0$, the mass of the $U(1)$ gauge boson vanishes.

Returning to the possibility (a) with unbroken $U(1)$ gauge symmetry, the mass spectrum of the bosons is well-defined and can be read off from the action (26); they are summarized in table 1 below:

| mass     | $m_a^2$ | $m_A^2$ | $m_W^2$ |
|----------|---------|---------|---------|
| value    | $g^2 M(0) = g^2/2$ | 0       | $g^2 M(0) = g^2/2$ |

Table 1: Boson mass spectrum for $X_0 = 0$

Next we turn to the spectrum of fermions. With $U(1)$ not broken, the fermions must fall into charged states. A positively charged Dirac fermion is formed by the quasi-Goldstone fermion $\psi_L$ and the gaugino $\lambda^+_R$:

$$
\Psi = \sqrt{2M_0} \psi_L + \lambda^+_R,
$$

(29)

whereas the remaining fermions $\bar{\chi}_L$, $\lambda^-_L$ and $\lambda_R$ remain massless. The mass of the Dirac fermion (29) is $m_\Psi^2 = 2g^2 M(0) = g^2$. A straightforward calculation shows,
that in the scenario with manifest non-broken $U(1)$ the standard supertrace formula for the mass spectrum is satisfied:

$$\text{Str} \, m^2 = \sum_j (-1)^{2j} (2J + 1) m_J^2 = 0. \quad (30)$$

5. Softly broken supersymmetry

As eq. (24) with $X_0 > 0$ implies that $M_0 M_0' = 0$, the phase with spontaneously broken $U(1)$ symmetry is always critical. The analysis of the theory is then complicated by the appearance of infinities at the classical level. However, a completely finite theory is obtained by adding an $SU(2)$-invariant soft supersymmetry breaking scalar mass term $\Delta V(X) = -\mu^2 X$ to the potential. In the following we take the point of view that the critical model is the limit of this regulated theory when the soft mass term is taken to vanish.

With the addition of the regulator mass term, the full potential becomes

$$V(X) = V_D(X) + \Delta V(X) = \frac{g^2}{2} M^2(X) - \mu^2 X. \quad (31)$$

As a result the minimum of the potential is shifted to the position where

$$g^2 M'M a = \mu^2 a. \quad (32)$$

Then either $a_0 = 0$ and $U(1)$ is not broken, as discussed previously; or $U(1)$ is broken, $X_0 = |a_0|^2 > 0$ and

$$g^2 M_0'M_0 = \mu^2. \quad (33)$$

Hence the soft supersymmetry breaking term shifts the vacuum of the model away from the critical point. Taking the broken $U(1)$ invariance into account, we parametrize the complex scalar field $a$ as

$$a = \left( \sqrt{X_0} + \sqrt{Z_h h} \right) e^{i Z_\theta / 2}, \quad (34)$$

where again $Z_h$ and $Z_\theta$ are normalization constants to be fixed such that we obtain canonically normalized kinetic terms. The bosonic terms in the action then become in the unitary gauge

$$\mathcal{L}_{\text{bos}} = -g^2 M_0 W^+ \cdot W^- - \frac{Z_h M_0'}{2} (\partial_\mu h)^2 - \frac{Z_\theta X_0 M_0'}{2} \left( \partial_\mu \theta + g \sqrt{\frac{2}{Z_\theta}} A_\mu \right)^2 - \frac{g^2}{2} M_0^2 - \mu^2 X_0 - \frac{g^2 X_0}{M_0'} \left( M_0 M_0'' + M_0' \right) h^2 + ...$$

$$= -m_W^2 W^+ \cdot W^- - \frac{m_A^2}{2} A_\mu^2 - V_0 - \frac{1}{2} (\partial_\mu h)^2 - \frac{m_h^2}{2} h^2 + ..., \quad (35)$$
where we have only written out the quadratic terms which determine the linearized field equations for the fluctuating part of the fields. To get the final result, we have taken
\[ Z_h = \frac{1}{M_0}, \quad Z_\theta = \frac{1}{X_0 M_0'}, \tag{36} \]
and
\[ m_W^2 = g^2 M_0, \quad m_A^2 = 2g^2 X_0 M_0', \quad m_h^2 = \frac{2g^2 X_0}{M_0'} \left(M_0 M_0'' + M_0'^2\right), \tag{37} \]
Also we have redefined the abelian vector field to absorb the Goldstone mode in the usual way:
\[ A_\mu \to \tilde{A}_\mu = A_\mu + \frac{1}{g} \sqrt{\frac{Z_\theta}{2}} \partial_\mu \theta. \tag{38} \]
This is equivalent to the choice of unitary gauge for the broken \( U(1) \) symmetry.

Finally, as \( M_0 \) and \( M_0' \) are related by (33), we can eliminate \( M_0 ', M_0 '' \) and \( M_0 ''' \) and the \( U(1) \) breaking parameter \( X_0 \) in favor of the physical parameters \( m_W^2, m_A^2, m_h^2 \) and the soft supersymmetry breaking parameter \( \mu^2 \):
\[ g^2 M_0 = m_W^2, \quad M_0' = \frac{\mu^2}{m_W^2}, \quad X_0 = \frac{m_A^2 m_W^2}{2g^2 \mu^2}, \quad M_0'' = \frac{g^2 \mu^4}{m_A^2 m_W^6} \left(m_h^2 - m_A^2\right). \tag{39} \]
For values \( \mu^2 > 0 \) the model obviously describes massive charged and neutral vector bosons plus a massive real Higgs scalar \( h \).

In the limit \( \mu^2 \to 0 \) we can now distinguish various possible scenarios:

a. If there is a number \( n > 0 \) such that for small \( \mu \) values \( X_0 \sim \mu^{2n} \), then the \( U(1) \) symmetry is restored when the regulator mass vanishes; it also follows, that \( m_A^2 m_W^2 \sim \mu^{2(n+1)} \). As \( X_0 \to 0 \) implies that \( M_0 \to 1/2 \), we find in this limit that \( m_W^2 = g^2/2 \) is finite and non-zero. Therefore \( M'(0) \sim \mu^2 \), and \( m_A^2 \sim \mu^{2(n+1)} \), which vanishes in the limit \( \mu^2 \to 0 \) as expected when \( U(1) \) is restored. The last relation (39) finally implies, that
\[ m_h^2 \sim \mu^{2(n-1)} M''(0). \tag{40} \]
For finite \( M''(0) < \infty \), the scalar mass then remains finite for \( n = 1 \), and vanishes for \( n > 1 \). For \( n < 1 \) the scalar mass diverges. If \( M''(0) \) itself vanishes as \( \mu^p \), \( p > 0 \), these constraints can be further relaxed. The upshot is, that if \( X_0 \) vanishes at least as \( \mu^2 \), then in the limit \( \mu^2 \to 0 \) we reobtain the results of section 4.

b. If in the supersymmetric limit of vanishing \( \mu^2 \) the vacuum expectation value \( X_0 > 0 \), then the \( U(1) \) symmetry remains broken. However, in the standard scenario with \( m_W^2 > 0 \) the \( U(1) \) gauge boson is massless in the limit \( \mu^2 \to 0 \). This apparent contradiction is resolved by looking at the kinetic term for the Goldstone field: it turns out that in the limit its effective charge
\[ g_{eff} = g \sqrt{\frac{2}{Z_\theta}} = m_A \to 0. \]
Therefore in this case there is a decoupling: the \( U(1) \) symmetry broken by \( X_0 \) is a global one, while the \( U(1) \) gauge symmetry remains unbroken. Finally the \( h \)-scalar mass becomes

\[
m_h^2 \sim \frac{M''_0}{\mu^2}.
\] (41)

Thus the scalar mass diverges, unless \( M''_0 \sim \mu^p \) with \( p \geq 2 \).

In contrast, if there is a number \( k > 0 \) such that for small \( \mu \) values \( M_0 \sim \mu^{2k} \), then \( m_W^2 \to 0 \), and \( m_A^2 \sim \mu^{2(1-k)} \). In this case one can not trust the limit \( \mu^2 \to 0 \) to describe the critical \( CP^1 \)-model, as the restauration of the \( SU(2) \) symmetry is expected to be accompanied by the reappearance of light bosons \((z, \bar{z})\), and it is no longer allowed to use the unitary gauge.

Nevertheless for finite \( \mu^2 \) the regularized model is well-defined and the mass spectrum can be computed. First of all, \( m_A^2 \) becomes large for small \( \mu \) when \( k > 1 \); it is finite and \( \mu \)-independent for \( k = 1 \), and it vanishes for \( k < 1 \). Therefore the limit is well-behaved if \( k \leq 1 \). In that case the Higgs mass behaves as

\[
m_h^2 - m_A^2 \sim \mu^{2(2k-1)} M''_0.
\] (42)

For finite \( M''_0 \) this implies that the masses remain finite if \( 1/2 \leq k \leq 1 \). In particular, for \( k = 1 \) we have \( m_h^2 = m_A^2 \), both non-zero and finite. For \( k = 1/2 \) \( m_h^2 \) can be finite non-zero whilst \( m_A^2 = 0 \).

Turning to the fermion sector, the quadratic part of the lagrangean is

\[
L_{\text{ferm}} = -2M_0 \bar{\psi}_L \bar{\psi}_L - M_0 \bar{\varphi}_L \bar{\varphi}_L - \bar{\lambda}_L \bar{\varphi}_L \lambda_L - \bar{\lambda}_L \bar{\varphi}_L \lambda_L - \bar{\lambda}_L \bar{\varphi}_L \lambda_L - \bar{\lambda}_L \bar{\varphi}_L \lambda_L
\]

\[
+4gM_0 \left( \bar{\lambda}_R \psi_L + \bar{\psi}_L \lambda_R \right) + 2ig\sqrt{2}X_0 M'_0 \left( \bar{\lambda}_R \varphi_L - \varphi_L \lambda_R \right).
\] (43)

This is diagonalized by defining the Dirac spinors

\[
\Psi = \sqrt{2M_0} \psi_L + \lambda_R, \quad \Phi = \sqrt{M'_0} \varphi_L - i\lambda_R.
\] (44)

In terms of these fields, the expression (43) becomes

\[
L_{\text{ferm}} = -\bar{\Psi} \bar{\hat{\psi}} \Psi - \bar{\Phi} \bar{\hat{\varphi}} \Phi - \bar{\lambda}_L \bar{\hat{\varphi}} \lambda_L - 2g\sqrt{2M_0} \bar{\Psi} \Psi + 2g\sqrt{2X_0 M'_0} \bar{\Phi} \Phi.
\] (45)

It follows, that we have the fermion mass spectrum as given in table 2:

| mass | \( m_{\bar{\psi}}^2 \) | \( m_{\bar{\Phi}}^2 \) | \( m_{\bar{\lambda}}^2 \) |
|------|-----------------|-----------------|-----------------|
| value| \( 2g^2M_0 \)   | \( 2g^2X_0M'_0 \)| \( 0 \)          |

Table 2: Fermion mass spectrum in the presence of soft supersymmetry breaking.

Combining the boson and fermion mass spectra we obtain a supertrace formula including soft supersymmetry breaking:

\[
\text{Str} m^2 = -\frac{2g^2M_0}{M'_0} (M_0' - X_0 M'') = -2m_W^2 - m_A^2 + m_h^2.
\] (46)
In particular, if in the limit $\mu^2 \to 0$ the $U(1)$ symmetry remains broken: $X_0 > 0$, and if in this limit the model is well-behaved, there are numbers $\omega^2 > 0$ and $1/2 \leq k \leq 1$ such that for small $\mu^2$ to first approximation

$$m_W^2 = \omega^2 \mu^{2k}, \quad m_A^2 = \frac{2g^2 X_0}{\omega^2} \mu^{2(1-k)},$$

$$m_h^2 = \frac{2g^2 X_0}{\omega^2} \mu^{2(1-k)} \left(1 + \frac{\omega^6}{g^2 \mu^{6k-4}} M_0''\right).$$

Then

$$\text{Str } m^2 = -2\omega^2 \mu^{2k} + 2\omega^4 X_0 M_0'' \mu^{2(2k-1)} \mu^2 \to 0 \begin{cases} 0, & \text{if } 1/2 < k \leq 1; \\ 2\omega^4 X_0 M_0'', & \text{if } k = 1/2. \end{cases}$$

In fact, the supertrace vanishes even for $k > 1$, but that is because the difference $m_h^2 - m_A^2$ then vanishes, even though both masses diverge individually.

The results (46) and (48) can be compared with standard results for the supertrace formula [11, 12, 13]. This provides an excellent check on our results, as in the general form the supertrace of the mass matrix is computed in a gauge-independent way. Specifically, from the general lagrangean (83) and (86) we obtain

$$\text{Str } m^2 = \text{tr} \left( m_0^2 - 2m_{1/2}^2 + 3m_1^2 \right)$$

with the traces of the mass matrices for the various spins given by

$$\text{tr } m_1^2 = 2g^2 (M_0 + X_0 M_0'),$$

for the vector bosons;

$$\text{tr } m_{1/2}^2 = 4g^2 (M_0 + X_0 M_0'),$$

for the fermions; and finally

$$\text{tr } m_0^2 = 2G^J V_{IJ} = \frac{2(M_0 + X_0 M_0')}{M_0 M_0'} V_0' + \frac{2X_0 V_0''}{M_0}. $$

To obtain this result, we have used the general expression for $G^J$ in eq.(75) in appendix A. Now observing that by definition of the vacuum state $X_0 V_0' = 0$, and that

$$\frac{2X_0 V_0''}{M_0} = \frac{2g^2 X_0}{M_0} \left( M_0 M_0'' + M_0'^2 \right),$$

the final expression for the supertrace of the mass matrix takes the form (46):

$$\text{Str } m^2 = \frac{2g^2 M_0}{M_0'} (X_0 M_0'' - M_0').$$
The regulator mass $\mu^2$ does not appear explicitly in this expression, because it does not contribute to $V''$. Observe, that the result (54) even holds for $X_0 = 0$, due to the equality $g^2 M_0 = \mu^2/M'_0$. Finally observe, that in this derivation we have not used the unitary gauge at all.

6. Examples
In this section we provide examples of models with the properties conjectured in sections 4 and 5.

1. Let

$$K_m(X) = \kappa_1 X + \frac{\kappa_2}{2} X^2$$

Then

$$M(X) = \frac{1}{2} + \kappa_1 X + \kappa_2 X^2, \quad M'(X) = \kappa_1 + 2\kappa_2 X, \quad M''(X) = 2\kappa_2.$$ (56)

1.a If $\kappa_1 > 0$ and $\kappa_2 \geq 0$, then $M(X)$ and $M'(X)$ have no zeros for $X \geq 0$, and the potential reaches its minimum for $X = 0$; it follows that the $U(1)$ symmetry is preserved, whilst supersymmetry is broken as described in section 4, eqs. (27) and (28).

1.b If $\kappa_1 > 0$ and $\kappa_2 < 0$, then $M(X)$ possesses a zero for

$$X_0 = \frac{1}{\kappa_1 \left(\sqrt{1 + \frac{2|\kappa_2|}{\kappa_1}} - 1\right)}.$$ (57)

However, if we include the soft breaking term (31), we find that at the minimum $M < 0$ and $M' < 0$, and the latter condition remains true in the limit $\mu^2 \to 0$; hence the model contains tachyons. We do not consider this case further.

1.c If $\kappa_1 < 0$ and $\kappa_2 > 0$ with $\kappa_1^2 < 2\kappa_2$, then $M(X)$ has no zeros; however, $M'(X) = 0$ at

$$X_0 = \frac{|\kappa_1|}{2\kappa_2} < 1 \quad \Rightarrow \quad M_0 = \frac{1}{2} \left(1 - \frac{\kappa_1^2}{2\kappa_2}\right).$$ (58)

This is the absolute minimum of the $D$-term potential. If we now include the soft supersymmetry breaking term with small $\mu^2$, we have to first approximation

$$X_1 = X_0 + \Delta X, \quad M_1 = M_0, \quad M'_1 = 2\kappa_2 \Delta X,$$ (59)

with

$$\Delta X = \frac{\mu^2}{2\kappa_2 g^2 M_0} = \frac{2\mu^2}{g^2 (2\kappa_2 - \kappa_1^2)}.$$ (60)

In this case the mass spectrum of bosons reads to first approximation

$$m_W^2 = \frac{g^2}{2} \left(1 - \frac{\kappa_1^2}{2\kappa_2}\right), \quad m_A^2 = \frac{4\mu^2 |\kappa_1|}{2\kappa_2 - \kappa_1^2}, \quad m_h^2 = \frac{g^4 |\kappa_1|}{2\mu^2} \left(1 - \frac{\kappa_1^2}{2\kappa_2}\right)^2.$$ (61)
For the fermions we obtain the masses
\[ m^2_\Psi = 2m^2_W = g^2 \left(1 - \frac{\kappa^2}{2\kappa_2}\right), \quad m^2_\Phi = m^2_A = \frac{2\mu^2}{2\kappa_2 - \kappa_1^2}, \quad m_{\lambda^-} = 0. \tag{62} \]

1.d If \( \kappa_1 < 0 \) and \( \kappa_2 > 0 \) with \( \kappa_1^2 > 2\kappa_2 \), then \( M(X) \) has two zeros, at
\[ X_\pm = \frac{1}{|\kappa_1| \left(1 \pm \sqrt{1 - \frac{2\kappa_2}{\kappa_1^2}}\right)}. \tag{63} \]

Again, including the soft breaking term the model contains tachyons at \( X_+ \). At \( X_- \) it is well-behaved, with
\[ M_-= 0, \quad M'_- = |\kappa_1| \sqrt{1 - \frac{2\kappa_2}{\kappa_1^2}}. \tag{64} \]

The physical minimum of the potential now occurs at
\[ X_1 = X_- + \Delta X, \tag{65} \]
where to first approximation in \( \mu^2 \)
\[ \Delta X = \frac{\mu^2}{g^2 M'^2_-}, \quad M_1 = M'_- \Delta X, \quad M'_1 = M'_-. \tag{66} \]

It follows, that the bosonic mass spectrum in this approximation reads
\[ m^2_W = \frac{\mu^2}{M'_-}, \quad m^2_h \approx m^2_A = 2g^2 X_- M'_- = 2g^2 \sqrt{1 - \frac{2\kappa_2}{\kappa_1^2}} \frac{\sqrt{1 - \frac{2\kappa_2}{\kappa_1^2}}}{1 - \sqrt{1 - \frac{2\kappa_2}{\kappa_1^2}}}. \tag{67} \]

This corresponds to the results (47) with \( k = 1 \) and \( \omega^2 = 1/M'_- \). The fermionic mass spectrum becomes
\[ m^2_\Psi = 2m^2_W = \frac{2\mu^2}{M'_-}, \quad m^2_\Phi = m^2_A = 2g^2 X_- M'_-, \quad m^2_{\lambda^-} = 0. \tag{68} \]

2. The above analysis is typical for models which do not have simultaneous zeros of \( M(X) \) and its derivative \( M'(X) \). If such simultaneous zeros exist, as in the above model with \( 2\kappa_2 = \kappa_1^2 \), the analysis is changed. As a generic example, consider the model
\[ M(X) = \frac{1}{2} (\kappa X - 1)^n, \quad M'(X) = \frac{n\kappa}{2} (\kappa X - 1)^{n-1}. \tag{69} \]

When \( n \) is a positive integer, an \( n \)-fold zero of \( M(X) \) occurs at \( X = 1/\kappa \); it is also an \( (n-1) \)-fold zero of \( M'(X) \). The minimum of the potential, including the soft breaking term, is at
\[ \frac{n\kappa g^2}{4} (\kappa X - 1)^{2n-1} = \mu^2 \quad \Rightarrow \quad (\kappa X - 1) = \left(\frac{4\mu^2}{n\kappa g^2}\right)^{\frac{1}{2n-1}}. \tag{70} \]
It follows that at the minimum to lowest order in $\mu^2$:

$$\kappa X_1 = 1 + \left( \frac{4\mu^2}{n\kappa g^2} \right)^\frac{1}{2n-1}, \quad M_1 = \frac{1}{2} \left( \frac{4\mu^2}{n\kappa g^2} \right)^\frac{n}{2n-1}, \quad M'_1 = \frac{n\kappa}{2} \left( \frac{4\mu^2}{n\kappa g^2} \right)^\frac{n-1}{2n-1}. \quad (71)$$

Then the spectrum of boson masses becomes

$$m^2_W = \frac{g^2}{2} \left( \frac{4\mu^2}{n\kappa g^2} \right)^\frac{2}{2n-1}, \quad m^2_A = ng^2 \left( \frac{4\mu^2}{n\kappa g^2} \right)^\frac{n-2}{2n-1}, \quad (72)$$

with the Higgs mass to lowest order in $\mu^2$:

$$m^2_h = \frac{3}{2} m^2_A, \quad n = 2,$$

$$m^2_h = m^2_A, \quad n > 2. \quad (73)$$

In all expressions we have kept only the terms of leading order in $\mu^2$ for small $\mu^2$. The fermion mass spectrum for these models reads

$$m^2_\Psi = g^2 \left( \frac{4\mu^2}{n\kappa g^2} \right)^\frac{n}{2n-1}, \quad m^2_\Phi = ng^2 \left( \frac{4\mu^2}{n\kappa g^2} \right)^\frac{n-2}{2n-1}, \quad m^2_{\lambda^-} = 0. \quad (74)$$

We observe, that in the limit $\mu^2 \to 0$ and for $n$ a positive integer, all masses vanish, even though $X_0 = 1/\kappa$ remains finite and non-zero. For $n = 1$ the masses $m^2_A = m^2_\Phi$ are finite non-zero, whilst for $1/2 < n < 1$ they diverge.

### 7. Discussion

In this paper we have investigated non-linear $\sigma$-models with singular metrics, such that the kinetic terms of some fields vanish. Such models are for example supplied by gauged supersymmetric extensions of well-known coset-models.

We have shown by general arguments and by regularization based upon the addition of soft supersymmetry breaking terms to the potential, that different types of behavior are possible. For example, the subset of linear gauge symmetries can be realized manifestly, or in a spontaneously broken mode; this is reflected in the mass of the corresponding vector bosons.

In some cases the singularities imply the vanishing of all vector boson masses, which one expects to be accompanied by the reappearance of light bosons in the physical spectrum. However, this is difficult to show while staying in the original framework, as in particular it invalidates the use of the unitary gauge. Of course, one could abandon the present approach and return to ordinary Yang-Mills theories with matter in linear representations; however, a more sophisticated formulation of the present models is possible and may shed light on this issue [14].
Our treatment of the $\sigma$-model is based firmly on the classical action, although some features of our model were motivated by quantum aspects like the absence of holonomy and gauge anomalies. Higher order quantum corrections [10] may change the behavior of the models by renormalization of the Kähler potential.

Another extension of interest would be to study what happens if one only gauges the linear stability group, i.e. $U(1)$ in the $CP^1$-model. This changes the $D$-terms and allows for the introduction of a Fayet-Iliopoulos term. Therefore it is likely that such models offer less problems to obtain physically reasonable spectra of masses. In the present paper we have not analyzed this modification.

Finally, we have not studied in detail the issue of the appearance of tachyons in certain parameter ranges. The standard lore is that in this range the model is inconsistent.

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Appendix A

In this appendix we collect some results for the Kähler metric and its derivatives.

The metric is given in (10):

\[
G_{I\bar{J}} = \frac{\partial^2 K}{\partial Z^I \partial \bar{Z}^J} = \begin{pmatrix}
G_{z\bar{z}} & G_{z\bar{a}} \\
G_{a\bar{z}} & G_{a\bar{a}}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\frac{1+2(1+2\bar{z}z)X K'_m + 4\bar{z}zX^2 K''_m}{(1+\bar{z}z)^2} & 2\bar{z}a(1+\bar{z}z)\left(K'_m + X K''_m\right) \\
2\bar{a}z(1+\bar{z}z)\left(K'_m + X K''_m\right) & \left(1+\bar{z}z\right)^2\left(K'_m + X K''_m\right)
\end{pmatrix}
\]

with the inverse

\[
G^{I\bar{J}} = \begin{pmatrix}
\frac{(1+\bar{z}z)^2}{1+2X K'_m} & -\frac{2\bar{z}a(1+\bar{z}z)}{1+2X K'_m} \\
-\frac{2\bar{a}z(1+\bar{z}z)}{(1+\bar{z}z)^2} & \frac{1}{K'_m + X K''_m} + \frac{4\bar{z}X}{1+2X K'_m}
\end{pmatrix}
\]  

(75)

We define the differential operators

\[
\partial = dZ^I \frac{\partial}{\partial Z^I} = dz \frac{\partial}{\partial z} + da \frac{\partial}{\partial a}, \quad \bar{\partial} = \bar{d}Z^\bar{I} \frac{\partial}{\partial \bar{Z}^\bar{I}} = d\bar{z} \frac{\partial}{\partial \bar{z}} + d\bar{a} \frac{\partial}{\partial \bar{a}}
\]

such that \(d = \partial + \bar{\partial}\). Their action on the variable \(X\) is given by

\[
\partial X = \frac{2\bar{z}dz}{1+\bar{z}z}X + \bar{a}da(1+\bar{z}z)^2, \quad \bar{\partial} X = \frac{2\bar{z}d\bar{z}}{1+\bar{z}z}X + ad\bar{a}(1+\bar{z}z)^2.
\]

(77)

Applying these differential operators to the metric (10) we get

\[
\partial G_{z\bar{z}} = -\frac{2\bar{z}dz}{(1+\bar{z}z)^3}\left[1 - 2(1+\bar{z}z)X K'_m - (4+10\bar{z}z)X^2 K''_m - 4\bar{z}zX^3 K'''_m\right]
\]

\[
+2\bar{a}da\left[(1+\bar{z}z)K'_m + (1+6\bar{z}z)X K''_m + 2\bar{z}zX^2 K'''_m\right]
\]

\[
\partial G_{z\bar{a}} = 2\bar{a}dz\left[K'_m + 5X K''_m + 2X^2 K'''_m\right]
\]

\[
+2\bar{z}da(1+\bar{z}z)\left[K'_m + 3X K''_m + X^2 K'''_m\right]
\]

\[
\partial G_{a\bar{z}} = 2\bar{a}dz\left[(1+2\bar{z}z)K'_m + (1+6\bar{z}z)X K''_m + 2\bar{z}zX^2 K'''_m\right]
\]

\[
+2\bar{z}a^2da(1+\bar{z}z)^3\left[2K''_m + X K'''_m\right]
\]

\[
\partial G_{a\bar{a}} = 2\bar{z}dz(1+\bar{z}z)\left[K'_m + 3X K''_m + X^2 K'''_m\right]
\]

\[
+\bar{a}da(1+\bar{z}z)^4\left[2K''_m + X K'''_m\right].
\]

(78)
and their complex conjugates. Next we compute the mixed second derivative of the metric components:

$$\bar{\partial}\partial G_{IJ}L = d\bar{z}dz G_{II}z\bar{z} + d\bar{a}da G_{I\bar{a}\bar{a}} + d\bar{z}da G_{I\bar{a}a\bar{a}} + d\bar{a}da G_{I\bar{a}a\bar{a}}. \quad (79)$$

This gives the following results:

$$G_{zz,zz} = \frac{1}{(1 + \bar{z}z)^4} \left[ -2 + 4\bar{z}z + 4(1 + \bar{z}z)^2 X K'_m + (8 + 64\bar{z}z + 68(\bar{z}z)^2) X^2 K''_m 
+ 16\bar{z}z(2 + 5\bar{z}z) X^3 K'''_m + 16(\bar{z}z)^2 X^4 K''''_m \right]$$

$$G_{zz,a\bar{a}} = G_{a\bar{a},zz} = [G_{zz,\bar{a}a}]^* = \langle G_{z\bar{a},z\bar{a}} \rangle^*$$

$$= \frac{4\bar{a}z}{(1 + \bar{z}z)} \left[ (1 + \bar{z}z)K'_m + (5 + 11\bar{z}z) X K''_m + (2 + 11\bar{z}z) X^2 K''_m 
+ 2\bar{z}zX^3 K''''_m \right]$$

$$G_{z\bar{a},\bar{a}a} = G_{z\bar{a},\bar{a}a} = G_{a\bar{a},z\bar{a}} = G_{a\bar{a},z\bar{a}}$$

$$= 2(1 + 2\bar{z}z) K'_m + (3 + 14\bar{z}z) X K''_m + (2 + 12\bar{z}z) X^2 K'''_m + 4\bar{z}zX^3 K''''_m$$

$$G_{\bar{a}a,\bar{a}a} = G_{\bar{a}a,\bar{a}a} = [G_{a\bar{a},\bar{a}a}]^* = [G_{a\bar{a},z\bar{a}}]^*$$

$$= 2\bar{z}a(1 + \bar{z}z)^3 \left[ 4K''_m + 5X K''_m + X^2 K'''_m \right]$$

$$G_{a\bar{a},a\bar{a}} = (1 + \bar{z}z)^4 \left[ 2K''_m + 4X K''_m + X^2 K'''_m \right]. \quad (80)$$

We can now compute the components of the Kähler connection and curvature; in the unitary gauge $z = \bar{z} = 0$ the non-trivial ones read explicitly

$$\Gamma^K_{IJ} = G^{KK}G_{IK}K_{IJ} \rightarrow$$

$$\Gamma^z = 2\bar{a} \frac{K'_m + X K''_m}{1 + 2X K'_m}, \quad \Gamma^a = \bar{a} \frac{2K''_m + X K'''_m}{K'_m + X K''_m}. \quad (81)$$
and their complex conjugates, for the connection; and for the curvature:

\[ K_{I J L L} = G_{I J L L} - G^{K} K_{I K} J G_{K L L} \rightarrow \]

\[ K_{\bar{z} \bar{z} \bar{z}} = -2(1 - 2X K'_{m} - 4X^2 K''_{m}) ; \]

\[ K_{\bar{z} \bar{a} \bar{z}} = K_{\bar{a} \bar{z} \bar{z}} = K_{\bar{z} \bar{a} \bar{a}} \]

\[ = \frac{2}{1+2X K'_{m}} \left( K'_{m} + 3X K''_{m} + X^2 K'''_{m} + 2X^2 K'_{m} K''_{m} + 2X^3 (K'_{m} K'''_{m} - K''_{m} K''_{m}) \right) ; \]

\[ K_{a \bar{a} a \bar{a}} = \frac{1}{K'_{m} + X K'_{m}} \left( 2K'_{m} K''_{m} + 2X (K'_{m} K'''_{m} - K''_{m} K''_{m}) + X^2 K'_{m} K''_{m} \right) \]

\[ + X^3 (K''_{m} K'''_{m} - K'''_{m}^2) \] . \hspace{1cm} (82)

**Appendix B**

In the singular limit of supersymmetric σ-models, the elimination of auxiliary fields is a delicate procedure. Therefore in this appendix we present the off-shell lagrangean for our models (7). The Lagrange density for the gauged supersymmetric CP\( ^1 \) model plus the chiral matter multiplet is given on the next page. The Lagrange density of the gauge multiplet is:

\[ \mathcal{L}_{\text{gauge}} = -\frac{1}{2} F^{+}(W) \cdot F^{-}(W) - \frac{1}{4} F^{2}(A) \]

\[ -\overline{\lambda} \rightarrow \Phi \lambda \rightarrow \overline{\lambda} \rightarrow \Phi \lambda \rightarrow -\overline{\lambda} \rightarrow \Phi \lambda \rightarrow D^{+} D^{-} + \frac{1}{2} D^{2} + \xi D . \] \hspace{1cm} (83)

Here \( \xi \) is the Fayet-Iliopoulos parameter, which can only be included in the model with gauged linear \( U(1) \) (and hence \( W^{\pm} = 0 \)); furthermore the decomposition of the vector multiplet \( V = (W_{\mu}, \lambda, D) \) is defined by

\[ V = V^{i} \tau_{i}, \quad V^{\pm} = \frac{1}{\sqrt{2}} (V^{1} \pm iV^{2}) , \quad V = V^{3} . \] \hspace{1cm} (84)

In particular: \( W^{\pm} = (W^{1} \pm iW^{2})/\sqrt{2} , A = W^{3} , \) and

\[ F^{\pm}_{\mu \nu}(W) = (\partial_{\nu} \mp ig A_{\nu}) W^{\pm}_{\nu} - (\partial_{\nu} \mp ig A_{\nu}) W^{\mp}_{\mu} , \]

\[ F_{\mu \nu}(A) = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} - ig W^{+}_{\mu} W^{\pm}_{\nu} + ig W^{+}_{\nu} W^{\pm}_{\mu} . \] \hspace{1cm} (85)

Elimination of the auxiliary \( D \)-fields from (83) with \( \xi = 0 \), and using the Lagrange density below, then leads to the potential \( V_{D} \), eq. (21).
In addition to the potential $V_D$ generated by the $D$-terms, we observe that the
equations for the auxiliary fields \((H, B)\) and their complex conjugates become

\[
G_{\bar{z}z} H + G_{a\bar{z}} B = G_{\bar{z}z,a} \bar{\psi}_R \psi_L + 2G_{\bar{z}z,a} \bar{\varphi}_R \psi_L,
\]

\[
G_{z\bar{a}} H + G_{a\bar{a}} B = G_{z\bar{a},a} \bar{\psi}_R \psi_L + 2G_{z\bar{a},a} \bar{\varphi}_R \psi_L,
\]

and their conjugates.

In the unitary gauge \(z = \bar{z} = 0\) this becomes

\[
\mathcal{L}_{\sigma+m} = -2M(X) \left( \frac{g^2}{2} \bar{W}^- \cdot W^+ + \bar{\psi}_L \bar{\psi} \right) - HH
\]

\[
- M'(X) \left( D\bar{a} \cdot Da + \bar{\varphi}_L \bar{\varphi}_L - \bar{B}B \right) - gM(X)D
\]

\[
- 2 \left( \bar{a} \bar{D}_\mu a \right) \left[ M'(X) \bar{\psi}_L \gamma^\mu \psi_L + M''(X) \bar{\varphi}_L \gamma^\mu \varphi_L \right]
\]

\[
+ \sqrt{2} gM'(X) \left[ -aW_\mu^- \bar{\varphi}_L \gamma^\mu \psi_L + \bar{a}W_\mu^+ \bar{\psi}_L \gamma^\mu \varphi_L \right]
\]

\[
- M''(X) \left( \bar{a} \bar{B} \bar{\varphi}_R \varphi_L + aB \varphi_L \varphi_R \right)
\]

\[
- 4M'(X) \left( \bar{a}H \bar{\varphi}_R \psi_L + aH \bar{\varphi}_L \psi_R \right)
\]

\[
- 4 \left( M(X) - 2XM'(X) \right) \bar{\psi}_R \psi_L \bar{\psi}_L \psi_R
\]

\[
+ \left( M''(X) + XM'''(X) \right) \bar{\varphi}_R \varphi_L \bar{\varphi}_L \varphi_R
\]

\[
+ 8 \left( M'(X) + XM''(X) \right) \bar{\varphi}_R \psi_L \bar{\psi}_L \varphi_R
\]

\[
+ 2\sqrt{2} g \left[ \sqrt{2} M(X) \left( \bar{\lambda}_R \psi_L + \bar{\psi}_L \lambda_R^+ \right) \right]
\]

\[
+ iM'(X) \left( \bar{a} \lambda_R \varphi_L - a \varphi_L \lambda_R \right) \]

In case the superpotential vanishes the dependence on the charged vector fields \(W^\pm\) and their superpartners \(\lambda^\pm\) and auxiliary fields \(D^\pm\) disappears, and the model is indistinguishable of that with gauged \(U(1)\) only.

In the unitary gauge the equations for the auxiliary fields become

\[
M(X) H = 2\bar{a}M'(X) \bar{\varphi}_R \psi_L,
\]

\[
M'(X) B = \bar{a}M''(X) \bar{\varphi}_R \varphi_L.
\]
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