OPTIMAL INVESTMENT AND DIVIDEND PAYMENT STRATEGIES WITH DEBT MANAGEMENT AND REINSURANCE

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ABSTRACT. This paper derives the optimal debt ratio, investment and dividend payment strategies for an insurance company. The surplus process is jointly determined by the reinsurance strategies, debt levels, investment portfolios and unanticipated shocks. The objective is to maximize the total expected discounted utility of dividend payments in finite-time period subject to three control variables. The utility functions are chosen as the logarithmic and power utility functions. Using dynamic programming principle, the value function is the solution of a second-order nonlinear Hamilton-Jacobi-Bellman equation. The explicit solution of the value function is derived and the corresponding optimal debt ratio, investment and dividend payment strategies are obtained. In addition, the investment borrowing constraint, dividend payment constraint and impacts of reinsurance policies are considered and their impacts on the optimal strategies are analyzed. Further, to incorporating the interest rate risk, the problem is studied under a stochastic interest rate model.

1. Introduction. The dividend payment schemes are vital for publicly traded companies. Not only does it represent an important signal about a firm’s future growth opportunities and profitability, but also it may influence the investment and financing decisions of firms and the wealth of the shareholders. On the one hand, if...
dividends are too high, firms’ retained profits may become low, thereby weakening their financial stability or undermining growth opportunities. On the other hand, companies’ stocks may become less popular due to low dividends. In insurance companies, because of the nature of their products, insurers tend to accumulate relatively large amounts of cash, cash equivalents, and investments in order to pay future claims and avoid financial ruin. The payment of dividends to shareholders may reduce an insurer’s ability to survive adverse investment and underwriting experience. The study of optimal dividend payment and liability management of an insurance company has become a high priority task.

Managing the surplus and designing dividend payment policies have been investigated by many researchers under various risk models since the dividend payment problem was proposed by [7]. There have been increasing efforts on using advanced methods of stochastic control to study the optimal dividend policy, see [3], [9], [10], [13], [18], [19], [11] and [23]. The optimal dividend problem is usually phrased as the management’s problem of determining the optimal timing and dividend payments in the presence of bankruptcy risk, and the objective is to maximize the expected value of the accumulated discounted utility of the dividend payments until ruin. A survey of some recent progress and open problems in identifying optimal dividend payment strategies is concluded in [1].

Recently, debt has been widely used as a popular tool to increase the leverage of financial status for both individual investors and publicly traded companies. When the market is highly leveraged, the undertaken liabilities become very risky for the risk takers, and the financial system also becomes vulnerable since the small changes in asset values will significantly influence the financial institutions’ surplus status. The insurance companies, providing protections for the whole financial system, are also vulnerable when they overtake liabilities, and no one could survive in the turmoil. The failure of AIG has taught us a lesson and leads us to address the key issues in liability management of insurance companies and tackle the public concerns on how to avoid the risks of either total collapse of our financial system and economy or inject trillions of taxpayer dollars into the financial system as US did in 2008 financial crisis (see [17]). The determination of optimal liability level has its short- and long-run effect on the insurance company’s performance. More importantly, it is a crucial issue to the vulnerabilities of the whole financial system. Following the formulation in [12], the liability ratio, defined as the ratio of liability and surplus ($L_t/X_t$), measures the leverage level of the insurance companies. As the actual liability ratio exceeds the optimal liability ratio, the probability of default rises.

To protect insurance companies against the impact of claim volatilities, reinsurance is a standard tool with the goal of reducing and eliminating risks. The primary insurance carrier pays the reinsurance company a certain part of the premiums. In return, the reinsurance company is obliged to share the risk of large claims. Proportional reinsurance is one of such reinsurance policies. Using such an approach, the reinsurance company covers a fixed percentage of losses and the premium of reinsurance is determined. The most common nonproportional reinsurance policy is the so-called excess-of-loss reinsurance, within which the cedent (primary insurance carrier) will pay all of the claims up to a pre-given level of amount (termed retention level). Related work in both reinsurance schemes under various framework can be found in [2], [15], [14], [6], [22], [5] and references therein.
Further, we allow the investment of surplus in a continuous-time financial market and the management of the dividend payment policy. Investment strategies with and without constraints are both considered in our model. The borrowing constraint, which is consistent with the golden rule “never borrow money to do risky investment”, guarantees that the insurers cannot put too much money in risky assets for the sake of risk management. Therefore two rules for the investment strategies are imposed: (i) The weight of the risky asset should be no more than one; (ii) short-selling risky asset is prohibited. The proportion of capital invested in the risky asset is denoted as a control variable taking values in \([0, 1]\). With the investment borrowing constraint, designing the optimal investment strategies become much more complicated. \([4]\) considered the optimal investment policy and dividend payment strategy in an insurance company with independent and identically distributed claims. See also \([21]\) and \([20]\) for related work.

In this study, we choose logarithm and power utility functions to describe the risk aversion level for different types of insurers to find the optimal capital requirement or leverage that balances risk against expected growth and return. Optimal debt ratio will be determined. Meanwhile, the investment and dividend strategies under both utility criteria, that is, the maximization of the total expected discounted utility of dividend payments in finite-time period, are developed. The stochastic control problem considered in this formulation contains three control variables: liability ratio, investment portfolio ratio and dividend payment rate. By dynamic programming principle, the value function obeys a second-order nonlinear partial differential equation (PDE) generally. The value function and the corresponding optimal strategies are obtained analytically.

In addition, we consider the investment borrowing constraint and dividend payment constraint and their impacts on the optimal strategies. Under both constraints, the optimal investment and dividend payment strategies are significantly different from those without constraints. Moreover, the impact of reinsurance strategies on the debt management and dividend payment policies are studied. Further, to consider the interest rate risk in the long term, a stochastic interest rate model is proposed. The Hamilton-Jacobi-Bellman (HJB) equation becomes more generalized but more complicated. Due to the nonlinearity, explicit solutions are generally not able to be obtained for this type of PDE. \([8]\) introduced a subsolution–supersolution method to obtain existence of classical solutions of the HJB equation. In our formulation, the stochastic control problem can be solved analytically.

The rest of the paper is organized as follows. A general formulation of asset value, debt, surplus, risky and riskless assets, insurance liabilities, claim rates, dividend strategies, and assumptions are presented in Section 2. Section 3 deals with optimal debt ratio, investment and dividend payment strategies with both logarithm and power utilities with no constraints. Dividend payment constraint is considered in Section 4. The investment borrowing constraint is presented and studied in Section 5. Section 6 deals with ruin probabilities under optimal strategies. The impact of reinsurance strategies under optimal strategies are studied in Section 7. Section 8 deals with a stochastic interest rate model. The value functions and optimal strategies under both utility functions are derived. Finally, concluding remarks are provided in Section 9.

2. Formulation. For a large insurer, the change of surplus process \(X(t)\) is described as the difference between the change of asset value \(A(t)\) and claims related
to liabilities \( L(t) \). In addition, when the insurer incurs a liability at time \( t \), he receives a premium for the amount insured. The collected premium will increase assets and surplus at time \( t \). Denote by \( \alpha \) the premium rate, which represents the cost of protection per dollar of insurance liabilities. The asset value increasing from the insurance sales during the time period \( [t, t + dt] \) is denoted as \( \alpha L(t)dt \).

We further consider the tool of proportional reinsurance to reduce the primary insurer’s risk. Within proportional scheme, the reinsurance company covers a fixed percentage of losses. Let \( \lambda \in (0, 1) \) be an exogenous retention level for the reinsurance policy. During the time period \( [t, t + dt] \), the primary insurer pays \( h(\lambda)L(t)dt \) as the reinsurance premium, where \( h(\lambda) \) be reinsurance charge rate (the cost of reinsurance protection per dollar of reinsured liabilities). Then only \( \lambda L(t)dt \) will be covered by the primary insurance company. Further, from the point of view of insurance company, it is naturally that the reinsurance cost should be less than the premium collected per dollar of the liability. That is, it is intuitively that \( \alpha > h(\lambda) \).

At this premium rate \( \alpha \) and reinsurance retention level \( \lambda \), there is an elastic demand for insurance contract and the insurer decides how much insurance \( L(t) \) to offer at that premium rate and reinsurance retention level. One natural control variable of the insurance company is its liability, the insurance policies sold. Let \( \pi(t) = L(t)/X(t) \) be the debt ratio of the insurance company. Then, the leverage, which is described as the ratio between asset values and surplus, can be written as \( A(t)/X(t) = 1 + \pi(t) \). To avoid the insurance liabilities being too large, the insurers will decide the optimal liabilities to manage the sale of insurance policies.

We consider the financial market with a riskless asset \( B(t) \) and a risky asset \( S(t) \) with prices satisfying

\[
\begin{align*}
\frac{dB(t)}{B(t)} &= r dt, \\
\frac{dS(t)}{S(t)} &= \mu(t) dt + \sigma(t) dW_1(t),
\end{align*}
\] (1)

where \( r \) is the interest rate, \( \mu(t) \) is the varying drift of the risky asset, \( \sigma(t) \) is the corresponding volatility and \( W_1(t) \) is a standard Brownian motion.

We assume that the asset value \( A(t) \) is invested in the financial market. The investment behavior of the insurer is modelled as a portfolio process \( k(t) \), where proportional surplus \( k(t) \) was invested in the risky asset \( S(t) \). Then

\[
\frac{dA(t)}{A(t)} = (r + \mu(t) - \mu(t)) dt + k(t) \sigma(t) dW_1(t),
\] (2)

Hence, in view of (2), the surplus process in the absence of claims and dividend payment can be denoted by \( \bar{X}(t) \) and follows

\[
d\bar{X}(t) = [(\alpha - h(\lambda))L(t) + A(t)][(r + \mu(t) - \mu(t)) dt + k(t) \sigma(t) dW_1(t)].
\] (3)

**Remark 1.** In view of (3), we assume that the net premium can be invested in the financial market immediately after receipt. It is consistent with the insurance contract’s premium payment scheme. Since both the gross premium received from the written policy and the reinsurance premium payout occur in advance for each coverage period \( [t, t + dt] \), we can invest the net premium \( (\alpha - h(\lambda))L(t) \) in the same financial market with asset value \( A(t) \) at time \( t \).

We further consider the future claims, which are against insurer’s liabilities incurred earlier. The future claims are the required payments to the insured policy
holders. Surplus declines by the amount of future claims. Denoted by $R(t)$ the future claims up to time $t$. Then we assume that the claims are proportional to the amount of insurance liabilities $L(t)$. Hence, the accumulated claims up to time $T$ is denoted as

$$R(T) = \int_0^T c(t)L(t)dt,$$

where $c(t)$ can be considered as a claim rate against liabilities.

We are now working on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$, where $\mathcal{F}_t$ is the $\sigma$-algebra generated by $\{W_1(s) : 0 \leq s \leq t\}$ and $\{\mathcal{F}_t\}$ is the filtration satisfying the usual conditions. A dividend strategy $D(\cdot)$ is an $\mathcal{F}_t$-adapted process $\{D(t) : t \geq 0\}$ corresponding to the accumulated amount of dividends paid up to time $t$ such that $D(t)$ is a nonnegative and nondecreasing stochastic process that is right continuous and has left limits with $D(0^-) = 0$. In this paper, we consider the optimal dividend strategy where the dividend payments are proportional to the surplus with a dividend payment rate $z(t)$. As a result, we write $D(t)$ as

$$dD(t) = z(t)X(t)dt,$$

where $z(t)$ is an $\mathcal{F}_t$-adapted process. Thus, taking into account the impact of reinsurance, the insurer’s surplus process in the presence of claims and dividend payments is given by

$$dX(t) = d\tilde{X}(t) - \lambda dR(t) - dD(t).$$

Together with the initial condition, (4) follows

$$\begin{cases}
    dX(t) = \{[(\alpha - h(\lambda))L(t) + A(t)][r + (\mu(t) - r)k(t)] - \lambda c(t)L(t) - z(t)X(t)\}dt \\
    + [\alpha - h(\lambda))L(t) + A(t)]k(t)\sigma(t)dW_1(t), \\
    X(0) = x > 0.
\end{cases}$$

(5)

For the portfolio weight $k(t)$, we assume that $\forall T \in (0, \infty)$,

$$\mathbb{E}\int_0^T k^2(t)dt < \infty.$$

Recall that $\pi(t)$ represents the debt ratio, the decision maker manages the underwriting process and determines the optimal liability ratio. Thus, $\pi(t)$ is a control variable. Denote $\Gamma = [0, M]$, $0 < M < \infty$. We are considering the optimal controls in a compact set. Assume that $\pi \in \Gamma$. (5) can be written as

$$\begin{cases}
    \frac{dX(t)}{X(t)} = \{k(t)(\mu(t) - r)[\pi(t)(\alpha - h(\lambda) + 1) + 1] + r[\pi(t)(\alpha - h(\lambda) + 1) + 1] \\
    - \pi(t)\lambda c(t) - z(t)\}dt + [\pi(t)(\alpha - h(\lambda) + 1) + 1]k(t)\sigma(t)dW_1(t), \\
    X(0) = x.
\end{cases}$$

(6)

For dividend payment rate $z(t)$, we assume $z(t)$ is non-negative and subject to an upper bound. A strategy $u(\cdot) = \{(k(t), \pi(t), z(t)) : t \geq 0\}$ being progressively measurable with respect to $\{W_1(s) : 0 \leq s \leq t\}$ is called an admissible strategy. Denote the collection of all admissible strategies or admissible controls by $\mathcal{A}$. Then the admissible strategy set $\mathcal{A}$ can be defined as

$$\mathcal{A} = \left\{u(t) = (k(t), \pi(t), z(t)) \in \mathbb{R} \times \Gamma \times \mathbb{R} : \mathbb{E}\int_0^T k^2(t)dt < \infty; 0 \leq z(t) \leq N < \infty, \text{for some constant } N > 0\right\}.$$
The representative financial institute is risk averse and the objective is to maximize the expectation of the discounted value of the utility of dividend in finite-time horizon. Denote by $\rho > 0$ the discount factor. For an arbitrary admissible triplet $u = (k, \pi, z)$, the performance function is the expected discounted dividend until $T$, and is given by

$$J(x, t, u(\cdot)) = \mathbb{E}_{x,t} \left[ \int_t^T e^{-\rho s} U(z(s)X(s))ds + e^{-\rho T} U(X(T)) \right],$$

where $\mathbb{E}_{x,t}$ denotes the expectation conditioned on $X(t) = x$, $U$ denotes the utility function to the dividend payment and terminal surplus.

We are interested in finding the optimal dividend payment rate, investment strategy and debt ratio to maximize the performance function $J(x, t, u(\cdot))$. Define $V(x, t)$ as the optimal value of the corresponding problem. That is,

$$V(x, t) = \sup_{u(\cdot) \in A} J(x, t, u(\cdot)). \quad (7)$$

To solve a stochastic control problem, one usually uses a dynamic programming approach. This in turn requires considering the generator (an operator) of the controlled process involved and using it to derive a partial differential equation, known as HJB equation, satisfied by the value function. The solution of the HJB equation then yields the optimal control and optimal value function. Assuming the existence of optimal control, for an arbitrary $V(\cdot, \cdot) \in C^2(\mathbb{R} \times [0, \infty))$, define an operator $L^u$ by

$$L^u V(x, t) = \frac{1}{2} V_{xx}(x, t) [\pi(\alpha - h(\lambda) + 1) + 1]^2 k^2 \sigma^2 x^2 \left. + V_x(x, t) \left( k(\mu - r)[\pi(\alpha - h(\lambda) + 1) + 1] + r[\pi(\alpha - h(\lambda) + 1) + 1] - \pi \lambda c - z \right) \right],$$

where $V_x$ and $V_{xx}$ denote the first-order and the second-order partial derivatives with respect to $x$, respectively. Let $V_t$ denote the first-order partial derivatives with respect to $t$. Formally, the value function (7) satisfies the Hamilton-Jacobi-Bellman equation

$$\max_u \{ V_t(x, t) + L^u V(x, t) + e^{-\rho t} U(zx) \} = 0. \quad (8)$$

The proof of (8) is presented in Appendix A.

Using $k$, $\pi$ and $z$ to represent the controls, (8) can be rewritten as

$$0 = \max_{k, \pi} \left\{ V_x(x, t) \left\{ k(\mu - r)[\pi(\alpha - h(\lambda) + 1) + 1] \right. \right.$$

$$+ \frac{1}{2} V_{xx}(x, t) x^2 [\pi(\alpha - h(\lambda) + 1) + 1]^2 k^2 \sigma^2$$

$$+ V_x(x, t) \left. \left\{ r[\pi(\alpha - h(\lambda) + 1) + 1] - \pi \lambda c \right\} \right\} + \max_z \left[ -zx V_z(x, t) + e^{-\rho t} U(zx) \right] + V_t(x, t) \right\} \quad (9)$$

with terminal condition

$$V(x, T) = e^{-\rho T} U(x). \quad (10)$$

3. Optimal controls and value function.

3.1. Logarithmic utility. Assume the utility function $U(x) = \ln x$. We construct a solution of (9) with the form

$$V(x, t) = A(t) \ln x + B(t). \quad (11)$$
With appropriate values of \( A(t) \) and \( B(t) \), (11) will be verified to be the solution of (9) later. To determine \( A(t) \) and \( B(t) \), we plug (11) into (9). Then we have

\[
0 = \max_{k, \pi} \left\{ k(\mu - r)(\alpha - h(\lambda) + 1) + 1 - \frac{1}{2}[\pi(\alpha - h(\lambda) + 1) + 1]2k^2\sigma^2 \right. \\
+ \left. \left\{ \rho[\pi(\alpha - h(\lambda) + 1) + 1 - \pi\lambda c] \right\} A(t) + \left[ A(t) + e^{-\rho t} \right] \ln x \right. \\
+ \left. \max_{z}[-zA(t) + e^{-\rho t} \ln z] + B(t). \right. \tag{12}
\]

For simplicity, let

\[ \Psi = \pi(\alpha - h(\lambda) + 1) + 1. \]

Note that \( \Psi \geq 1 \). Then (12) can be written as

\[
0 = \max_{k, \pi} \left\{ k(\mu - r)\Psi - \frac{1}{2}\Psi^2k^2\sigma^2 + \Psi\left( r - \frac{\lambda c}{\alpha - h(\lambda) + 1} \right) \right\} A(t) \\
+ \left[ A(t) + e^{-\rho t} \right] \ln x + \max_{z}[-zA(t) + e^{-\rho t} \ln z] + B(t) + \frac{\lambda c A(t)}{\alpha - h(\lambda) + 1}. \tag{13}
\]

To proceed,

\[
0 = \max_{k, \pi} \left\{ -\frac{1}{2}\Psi^2\sigma^2 \left( k - \frac{\mu - r}{\Psi\sigma^2} \right) \right. \\
+ \left. \left( \rho - \frac{\lambda c}{\alpha - h(\lambda) + 1} \right) \right\} A(t) \\
+ \left[ A(t) + e^{-\rho t} \right] \ln x + \max_{z}[-zA(t) + e^{-\rho t} \ln z] + B(t) \\
+ \left. A(t) \left( \frac{\lambda c}{\alpha - h(\lambda) + 1} + \frac{(\mu - r)^2}{2\sigma^2} \right). \right. \tag{14}
\]

In view of (14), for fixed \( \pi \), the portfolio weight \( k^* \) is obtained as

\[ k^* = \frac{\mu - r}{\Psi\sigma^2}, \]

and the optimal dividend payment rate follows

\[ z^* = \frac{e^{-\rho t}}{A(t)}. \]

In addition, if

\[ r > \frac{\lambda c}{\alpha - h(\lambda) + 1}, \]

the maximal value can be achieved when \( \pi^* = M \) since \( \alpha > h(\lambda) \). If

\[ r < \frac{\lambda c}{\alpha - h(\lambda) + 1}, \]

the maximal value of can be achieved when \( \pi^* = 0 \). If

\[ r = \frac{\lambda c}{\alpha - h(\lambda) + 1}, \]

\( \pi^* \) can be any number in \( \Gamma \).

Since (12) holds for all \( x \), then

\[ A(t) + e^{-\rho t} = 0. \]

We have

\[ A(t) = \frac{1}{\rho}e^{-\rho t} + \beta_1, \]

where \( \beta_1 \) is a constant. Then,

\[ V(x, t) = \left( \frac{1}{\rho}e^{-\rho t} + \beta_1 \right) \ln x + B(t). \]
Plugging $k$, $\pi$ and $z$ with $k^*$, $\pi^*$ and $z^*$. Let

$$\Psi^* = \pi^*(\alpha - h(\lambda) + 1) + 1.$$  

We obtain

$$\frac{(\mu - r)^2}{2\sigma^2} A(t) - e^{-\rho t}(\ln A(t) + 1 + \rho t) + A(t)\{r\Psi^* - \pi^*\lambda c\} + B_t(t) = 0.$$  

Hence, we have

$$B(t) = -\int_0^t \left\{ \frac{(\mu - r)^2}{2\sigma^2} A(s) - e^{-\rho s}(\ln A(s) + 1 + \rho s) + A(s)\{r\Psi^* - \pi^*\lambda c\} \right\} ds + \beta_2, \quad (15)$$

where $\beta_2$ is a constant.

Referring to the terminal condition in (10), we have $B(T) = 0$, and

$$\frac{1}{\rho} e^{-\rho T} + \beta_1 = e^{-\rho T}.$$  

Then,

$$\beta_1 = \frac{\rho - 1}{\rho} e^{-\rho T},$$

and

$$A(t) = \frac{1}{\rho} e^{-\rho t} + \frac{\rho - 1}{\rho} e^{-\rho T}. \quad (16)$$

On the other hand, in view of (15), we have

$$\beta_2 = \int_0^T \left\{ \frac{(\mu - r)^2}{2\sigma^2} A(s) - e^{-\rho s}(\ln A(s) + 1 + \rho s) + A(s)\{r\Psi^* - \pi^*\lambda c\} \right\} ds.$$  

Then, in view of (15),

$$B(t) = \int_t^T \left\{ \frac{(\mu - r)^2}{2\sigma^2} A(s) - e^{-\rho s}(\ln A(s) + 1 + \rho s) + A(s)\{r\Psi^* - \pi^*\lambda c\} \right\} ds. \quad (17)$$

Hence, combining (11), (16) and (17), $V(x,t)$ is obtained explicitly. The optimal strategy $u^*(t)$ follows

$$\pi^* = \begin{cases} 
M, & \text{if } r > \frac{\lambda c}{a - h(\lambda) + 1}, \\
0, & \text{if } r < \frac{\lambda c}{a - h(\lambda) + 1}, \\
\text{any } \pi \in \Gamma, & \text{if } r = \frac{\lambda c}{a - h(\lambda) + 1}, 
\end{cases}$$

$$k^* = \frac{\mu - r}{\rho},$$

$$z^* = \frac{1}{\rho} e^{-\rho(T-t)}.$$  

(18)

3.2. Power utility. Now we consider power utility function, i.e.,

$$U(x) = \frac{x^\gamma}{\gamma}, \quad x > 0,$$

where $\gamma \in (0,1)$.

Recall that the operator follows

$$\mathcal{L}^n V(x,t) = \frac{1}{2} V_{xx}(x,t)\Psi^2 k^2 \sigma^2 x^2$$

$$+ V_x(x,t) \left\{ k(\mu - r)\Psi + r\Psi - \frac{\Psi - 1}{\alpha - h(\lambda) + 1} \lambda c - z \right\} x,$$
where
\[ \Psi = \pi(\alpha - h(\lambda) + 1) + 1 \geq 1, \]
and
\[ \pi = \frac{\Psi - 1}{\alpha - h(\lambda) + 1}. \]

The HJB equation under the power utility function follows
\[ \max_{u} \{ V_t(x,t) + \mathcal{L}u V(x,t) + e^{-\rho t} \frac{(zx)^\gamma}{\gamma} \} = 0. \]

It follows
\[ 0 = V_t(x,t) + \max_{k, \pi} \left\{ V_x(x,t) x k(\mu - r) \Psi + \frac{1}{2} V_{xx}(x,t) x^2 \Psi^2 k^2 \sigma^2 \right. \]
\[ + V_z(x,t) \left( r \Psi - \frac{\Psi - 1}{\alpha - h(\lambda) + 1} \lambda c \right) \right\} + \max_{z} \left[ -z x V_z(x,t) + e^{-\rho t} \frac{(zx)^\gamma}{\gamma} \right], \]
with boundary condition
\[ V(x,T) = e^{-\rho T} \frac{x^\gamma}{\gamma}. \]

Consider the solution with the form
\[ V(x,t) = p(t) \frac{x^\gamma}{\gamma}. \]

Then
\[ 0 = p_t(t) \frac{x^\gamma}{\gamma} + \max_{k, \pi} \left\{ p(t) x^{\gamma - 1} x k(\mu - r) \Psi + \frac{1}{2} (\gamma - 1) p(t) x^{\gamma - 2} x^2 \Psi^2 k^2 \sigma^2 \right. \]
\[ + p(t) x^{\gamma - 1} x \left( r \Psi - \frac{\Psi - 1}{\alpha - h(\lambda) + 1} \lambda c \right) \right\} + \max_{z} \left[ -z x p(t) x^{\gamma - 1} + e^{-\rho t} \frac{(zx)^\gamma}{\gamma} \right], \]
i.e.,
\[ 0 = \max_{k, \pi} \left\{ -\frac{1}{2} (1 - \gamma) \Psi^2 k^2 \sigma^2 + k(\mu - r) \Psi + r \Psi - \frac{\Psi - 1}{\alpha - h(\lambda) + 1} \lambda c \right\} \gamma p(t) \]
\[ + \max_{z} \left[ -z \gamma p(t) + e^{-\rho t} z^\gamma \right] + p_t(t). \]

Then we have
\[ 0 = \max_{k, \pi} \left\{ -\frac{1}{2} (1 - \gamma) \Psi^2 \sigma^2 \left( k - \frac{\mu - r}{(1 - \gamma) \sigma^2 \Psi} \right)^2 \right. \]
\[ + \left( r - \frac{\lambda c}{\alpha - h(\lambda) + 1} \right) \Psi + \frac{1}{2} \frac{(\mu - r)^2}{(1 - \gamma) \sigma^2} + \frac{\lambda c}{\alpha - h(\lambda) + 1} \right\} \gamma p(t) \]
\[ + \max_{z} \left[ -z \gamma p(t) + e^{-\rho t} z^\gamma \right] + p_t(t). \quad (19) \]

Thus,
\[ k^* = \frac{\mu - r}{(1 - \gamma) \sigma^2 \Psi}, \quad (20) \]
\[ z^* = \left( p(t) e^{\rho t} \right)^{\frac{1}{1-\gamma}}. \quad (21) \]

In addition, if \( r > \frac{\lambda c}{\alpha - h(\lambda) + 1} \), the maximal value can be achieved when \( \pi^* = M \) as \( \alpha > h(\lambda) \); if \( r < \frac{\lambda c}{\alpha - h(\lambda) + 1} \), then the maximal value can be achieved when \( \pi^* = 0 \); if \( r = \frac{\lambda c}{\alpha - h(\lambda) + 1} \), \( \pi^* \) can be any number in \( \Gamma \).
Substituting (20) and (21) into (19), we have

\[
0 = p(t) + \gamma \left[ \left( r - \frac{\lambda c}{\alpha - h(\lambda) + 1} \right) \Psi^* + \frac{1}{2} \left( \frac{(\mu - r)^2}{(1 - \gamma)\sigma^2} + \frac{\lambda c}{\alpha - h(\lambda) + 1} \right) p(t) \right. \\
+ (1 - \gamma) \left( e^{\rho t} \right)^{\frac{1}{1 - \gamma}} p(t)^{\frac{1}{1 - \gamma}}.
\]

Let

\[
f(t) = p(t)^{\frac{1}{1 - \gamma}}.
\]

That is,

\[
p(t) = f(t)^{1 - \gamma}.
\]

Then

\[
0 = \gamma \left[ \left( r - \frac{\lambda c}{\alpha - h(\lambda) + 1} \right) \Psi^* + \frac{1}{2} \left( \frac{(\mu - r)^2}{(1 - \gamma)\sigma^2} + \frac{\lambda c}{\alpha - h(\lambda) + 1} \right) f(t)^{1 - \gamma} \\
+ (1 - \gamma) \left( e^{\rho t} \right)^{\frac{1}{1 - \gamma}} f(t)^{-\gamma} + (1 - \gamma) f_t(t) f(t)^{-\gamma}.
\]

Thus, we have

\[
0 = f_t(t) + \gamma \left[ \left( r - \frac{\lambda c}{\alpha - h(\lambda) + 1} \right) \Psi^* + \frac{1}{2} \left( \frac{(\mu - r)^2}{(1 - \gamma)\sigma^2} + \frac{\lambda c}{\alpha - h(\lambda) + 1} \right) f(t) \\
+ \left( e^{\rho t} \right)^{\frac{1}{1 - \gamma}},
\]

where

\[
\Psi^* = \pi^*(\alpha - h(\lambda) + 1) + 1.
\]

Since \( p(T) = e^{-\rho T} \), we have \( f(T) = e^{-\frac{T}{1 - \gamma}} \). Then

\[
f(t) = f(T) e^{\kappa(T - t)} + \int_t^T e^{\kappa(s - t)} e^{-\frac{s}{1 - \gamma}} ds \\
= e^{-\frac{T}{1 - \gamma}} e^{\kappa(T - t)} + \int_t^T e^{\kappa(s - t)} e^{-\frac{s}{1 - \gamma}} ds \\
= \left[ 1 + \left( \kappa - \frac{\rho}{1 - \gamma} \right)^{-1} \right] e^{\kappa(T - t) - \left( \kappa - \frac{\rho}{1 - \gamma} \right)^{-1} e^{-\frac{\rho}{1 - \gamma} t}},
\]

where

\[
\kappa = \frac{\gamma}{1 - \gamma} \left[ \left( r - \frac{\lambda c}{\alpha - h(\lambda) + 1} \right) \Psi^* + \frac{1}{2} \left( \frac{(\mu - r)^2}{(1 - \gamma)\sigma^2} + \frac{\lambda c}{\alpha - h(\lambda) + 1} \right). \right.
\]

Thus,

\[
p(t) = f(t)^{1 - \gamma} \\
= \left\{ \left[ 1 + \left( \kappa - \frac{\rho}{1 - \gamma} \right)^{-1} \right] e^{\kappa(T - t) - \left( \kappa - \frac{\rho}{1 - \gamma} \right)^{-1} e^{-\frac{\rho}{1 - \gamma} t} \right\}^{1 - \gamma}.
\]

(22)
In addition, the optimal strategy \( u^* \) is given by

\[
\pi^* = \begin{cases} 
M, & \text{if } r > \frac{\lambda c}{\alpha - h(\lambda) + 1}, \\
0, & \text{if } r < \frac{\lambda c}{\alpha - h(\lambda) + 1}, \\
\text{any } \pi \in \Gamma, & \text{if } r = \frac{\lambda c}{\alpha - h(\lambda) + 1},
\end{cases}
\]

\[
k^* = \mu - r \left(1 - \gamma \right) \sigma^2 \left[\pi^*(\alpha - h(\lambda) + 1) + 1\right],
\]

\[
z^* = p(t) \frac{1}{1 - e^{-\rho t}} e^{\frac{\rho}{\rho - 1} t}.
\]

4. **Dividend payment constraint.** Generally, the accumulated amount of dividends paid up to time \( t \) denoted by \( D(t) \) is a nonnegative and nondecreasing stochastic process that is right continuous and has left limits with \( D(0^-) = 0 \). In our work, we consider the optimal dividend strategy where the dividend payments are proportional to the surplus with a dividend payment rate \( z(t) \). That is, the dividend payment in infinitesimal period is \( z(t)X(t)dt \). \( z(t) \) can be considered as the ratio between the dividend payment and current surplus. Practically, it is unlikely that the company will result in a deficit in the surplus due to the dividend payment process. Hence, it is intuitively that we should set a constraint on the dividend payment rate such that \( z(t) < 1 \). Together with the nonnegativity of the dividend payment, we will set a constraint for the dividend payment strategy that \( 0 < z(t) < 1 \) in this section.

Let us consider log utility. Define

\[
Y(z) = -zA(t) + e^{-\rho t} \ln z.
\]

We have the first and second order derivatives

\[
Y_z(z) = -A(t) + e^{-\rho t} \frac{1}{z},
\]

and

\[
Y_{zz}(z) = -e^{-\rho t} \frac{1}{z^2}.
\]

**Theorem 4.1.** The optimal portfolio weight \( z^* \) follows

- If \( \rho < 1 \), the optimal dividend payment rate

\[
z^* = \frac{\rho}{1 + (\rho - 1)e^{-\rho(T-t)}}
\]

in (18) is achieved.

- If \( \rho > 1 \), the optimal dividend payment rate \( z^* = 1 \).

**Proof.** In view of (23), \( Y(z) \) is a concave function. In addition, \( Y(0) \to -\infty \), and \( Y_z(z) \) is monotonically decreasing. If \( \rho < 1 \), we have

\[
A(t) - e^{-\rho t} = e^{-\rho t} \frac{1 - \rho}{\rho} (1 - e^{-\rho(T-t)}) > 0.
\]

Then the optimal dividend payment rate

\[
z^* = \frac{\rho}{1 + (\rho - 1)e^{-\rho(T-t)}}
\]

can be achieved. Otherwise, when \( \rho > 1 \), due to the monotonicity of \( Y_z \), the optimal dividend payment rate is achieved at \( z^* = 1 \). \( \Box \)
Remark 2. For the case of power utility function, a similar result cannot be obtained. Note that $p(t)$ in (22) has a much more complex structure, and depends on $\pi^*$. This is different from the case with log utility where $A(t)$ is independent of the choice of $\pi^*$. Thus a simple comparison between $z^*$ and its boundaries of dividend payment constraints cannot be achieved without details of the parameter values. However, it is shown that the dividend payment constraints has significant impact on the optimal dividend payment rate.

5. Investment borrowing constraint. In the real markets, investors cannot put too much money in risky assets for the sake of risk management. For example, there is a golden rule “never borrow money to do risky investment”. That is, there is a natural constraint on the portfolio so that the total weight of the risky assets should be no more than 1. On the other hand, for a small investor, short selling is fairly risky and may cost more than normal transaction, and as a result, only few investors do that. In fact, in some real market, short selling is prohibited.

5.1. Logarithmic utility. In this section, we deal with portfolio selection problems with these borrowing constraints in the logarithmic utility case. That is, we will focus on the situations where the optimal portfolio weight $k^* \in [0,1]$.

Let

$$H(k, \Psi) = k(\mu - r)\Psi - \frac{1}{2}\Psi^2 k^2 \sigma^2 + \Psi \left(r - \frac{\lambda c}{\alpha - h(\lambda) + 1}\right) .$$

Recall that $k \in [0,1]$ and $\pi \in [0,M]$. We will discuss the optimal controls in the following cases.

If

$$r > \frac{\lambda c}{\alpha - h(\lambda) + 1},$$

$\Psi$ can be maximized as $M(\alpha - h(\lambda) + 1) + 1$. Hence, considering the quadratic term, we have the optimal strategies

$$\pi^* = M, \quad k^* = \frac{\mu - r}{\sigma^2(M(\alpha - h(\lambda) + 1) + 1)} .$$

(24)

If

$$r < \frac{\lambda c}{\alpha - h(\lambda) + 1},$$

to maximize the third term of the right side of $H(k, \Psi)$, $\Psi$ should be minimized as 1. Then, to guarantee that the optimal portfolio ratio is in the range of $[0,1]$, we need to further consider the range of $r$. We further assume that $r > \mu - \sigma^2$. If

$$\mu - \sigma^2 < r < \frac{\lambda c}{\alpha - h(\lambda) + 1},$$

then $\frac{\mu - r}{\sigma^2} < 1$. Hence we obtain the optimal strategies

$$\pi^* = 0, \quad k^* = \frac{\mu - r}{\sigma^2} .$$

Note that we don’t need to worry about the case when $\mu - \sigma^2 > \frac{\lambda c}{\alpha - h(\lambda) + 1}$. In this case, if

$$\frac{\lambda c}{\alpha - h(\lambda) + 1} < \mu - \sigma^2 < r,$$
we have the same optimal strategies in (24). In addition, if
\[ \frac{\lambda c}{\alpha - h(\lambda) + 1} < r < \mu - \sigma^2, \]
we also have the same optimal strategies in (24).

Now the only cases we need to consider are the ones such that \( r < \mu - \sigma^2 \). If \( r < \mu - \sigma^2 < \lambda c \alpha - h(\lambda) + 1 \), we have
\[ \mu - \sigma^2 - \frac{\lambda c}{\alpha - h(\lambda) + 1} < 0. \]

Further,
\[ H_k(k, \Psi) = -\Psi^2 \sigma^2 k + \Psi(\mu - r), \]
\[ H_\Psi(k, \Psi) = -\Psi \sigma^2 k^2 + k(\mu - r) + r - \frac{\lambda c}{\alpha - h(\lambda) + 1}. \]

Then
\[ H_k(k, \Psi) - H_\Psi(k, \Psi) = (\Psi - k)(\mu - r - \sigma^2 k \Psi) + \frac{\lambda c}{\alpha - h(\lambda) + 1} - r. \]

Hence, when \( k \Psi \leq 1 \),
\[ H_k(k, \Psi) - H_\Psi(k, \Psi) = (\Psi - k)(\mu - r - \sigma^2 k \Psi) + \frac{\lambda c}{\alpha - h(\lambda) + 1} - r \]
\[ \geq (\Psi - k)(\mu - r - \sigma^2) + \frac{\lambda c}{\alpha - h(\lambda) + 1} - r \]
\[ \geq 0. \]

In addition, if \( k \Psi \geq 1 \),
\[ H_\Psi(1, \Psi) \leq -\sigma^2 + \mu - r + r - \frac{\lambda c}{\alpha - h(\lambda) + 1} \]
\[ = \mu - \sigma^2 - \frac{\lambda c}{\alpha - h(\lambda) + 1} < 0. \]

Then, \( H(k, \Psi) \) is maximized if \( k = 1 \) and \( \Psi = 1 \). Thus, the optimal strategies follow
\[ \pi^* = 0, \]
\[ k^* = 1. \]

If \( r < \frac{\lambda c}{\alpha - h(\lambda) + 1} < \mu - \sigma^2 \), we have
\[ \mu - \sigma^2 - \frac{\lambda c}{\alpha - h(\lambda) + 1} > 0. \]

Note that (25) still holds. On the other hand,
\[ H_\Psi(1, 1) = -\sigma^2 + \mu - r + r - \frac{\lambda c}{\alpha - h(\lambda) + 1} \]
\[ = \mu - \sigma^2 - \frac{\lambda c}{\alpha - h(\lambda) + 1} \]
\[ > 0. \]
In addition,
\[ H_{\Psi}(1, \Psi) = 0 \Rightarrow \Psi = \frac{1}{\sigma^2} \left( \mu - \frac{\lambda c}{\alpha - h(\lambda) + 1} \right). \]
Since \( \frac{1}{\sigma^2} \left( \mu - \frac{\lambda c}{\alpha - h(\lambda) + 1} \right) > 1 \), the optimal strategies follows
\[ \pi^* = \frac{1}{\sigma^2(\alpha - h(\lambda) + 1)} \left( \mu - \frac{\lambda c}{\alpha - h(\lambda) + 1} \right) - \frac{1}{\alpha - h(\lambda) + 1}, \]

\[ k^* = 1. \]

We conclude the results of the optimal strategies in the following.

**Theorem 5.1.** The optimal portfolio weight \( k^* \) and \( \pi^* \) follows

- If \( r > \frac{\lambda c}{\alpha - h(\lambda) + 1} \), the optimal portfolio strategies follow
  \[ \pi^* = M, \]
  \[ k^* = \frac{\mu - r}{\sigma^2(M(\alpha - h(\lambda) + 1) + 1)}. \]

- If \( \mu - \sigma^2 < r < \frac{\lambda c}{\alpha - h(\lambda) + 1} \), the optimal portfolio strategies follow
  \[ \pi^* = 0, \]
  \[ k^* = \frac{\mu - r}{\sigma^2}. \]

- If \( r < \mu - \sigma^2 < \frac{\lambda c}{\alpha - h(\lambda) + 1} \), the optimal portfolio strategies follow
  \[ \pi^* = 0, \]
  \[ k^* = 1. \]

- If \( r < \frac{\lambda c}{\alpha - h(\lambda) + 1} < \mu - \sigma^2 \), the optimal portfolio strategies follow
  \[ \pi^* = \frac{1}{\sigma^2(\alpha - h(\lambda) + 1)} \left( \mu - \frac{\lambda c}{\alpha - h(\lambda) + 1} \right) - \frac{1}{\alpha - h(\lambda) + 1}, \]
  \[ k^* = 1. \]

- If \( r = \frac{\lambda c}{\alpha - h(\lambda) + 1} < \mu - \sigma^2 \), the optimal portfolio strategies follow
  \[ \pi^* = \text{any } \pi \text{ in } \Gamma, \]
  \[ k^* = \frac{\mu - r}{\sigma^2(\pi^*(\alpha - h(\lambda) + 1) + 1)}. \]

**Proof.** The details are provided above. \( \square \)

5.2. **Power utility.** In this section, we deal with portfolio selection problems with these borrowing constraints in the power utility case. That is, we will focus on the situations where the optimal portfolio weight \( k^* \in [0, 1] \).

Analogously, let
\[ \tilde{H}(k, \Psi) = -\frac{1}{2} (1 - \gamma) \Psi^2 k^2 \sigma^2 + k(\mu - r) \Psi + \Psi \left( \frac{\lambda c}{\alpha - h(\lambda) + 1} \right). \]

Then,
\[ \tilde{H}(k, \Psi) = -\frac{1}{2} (1 - \gamma) \Psi^2 \sigma^2 \left( k - \frac{\mu - r}{(1 - \gamma) \sigma^2 \Psi} \right)^2 \]
\[ + \left( \frac{\lambda c}{\alpha - h(\lambda) + 1} \right) \Psi + \frac{1}{2} (\mu - r)^2. \]
Recall that $k \in [0, 1]$ and $\pi \in [0, M]$. We will discuss the optimal controls in the following cases.

If $$r > \frac{\lambda c}{\alpha - h(\lambda) + 1},$$
$\Psi$ can be maximized as $M(\alpha - h(\lambda) + 1) + 1$. Hence, considering the quadratic term, we have the optimal strategies

$$\pi^* = M,$$
$$k^* = \frac{\mu - r}{\sigma^2(1 - \gamma)(M(\alpha - h(\lambda) + 1) + 1)}. \tag{26}$$

Note that $M$ is a large number, thus $k^* \in [0, 1]$.

If $$r < \frac{\lambda c}{\alpha - h(\lambda) + 1},$$
to maximize the third term of the right side of $\tilde{H}(k, \Psi)$, $\Psi$ should be minimized as 1. Then, to guarantee that the optimal portfolio ratio is in the range of $[0, 1]$, we need further consider the range of $r$. We further assume that $r > \mu - (1 - \gamma)\sigma^2$. If $$\mu - (1 - \gamma)\sigma^2 < r < \frac{\lambda c}{\alpha - h(\lambda) + 1},$$
then $\frac{\mu - r}{(1 - \gamma)\sigma^2} < 1$. Hence we obtain the optimal strategies

$$\pi^* = 0,$$
$$k^* = \frac{\mu - r}{(1 - \gamma)\sigma^2}.$$

Similarly, we don’t need to worry about the case when $\mu - (1 - \gamma)\sigma^2 > \frac{\lambda c}{\alpha - h(\lambda) + 1}$.

In this case, if $$\frac{\lambda c}{\alpha - h(\lambda) + 1} < \mu - (1 - \gamma)\sigma^2 < r,$$
we have the same optimal strategies in (26). In addition, if $$\frac{\lambda c}{\alpha - h(\lambda) + 1} < r < \mu - (1 - \gamma)\sigma^2,$$
the same optimal strategies are obtained in (26).

Now the only cases we need consider are the ones such that $r < \mu - (1 - \gamma)\sigma^2$. If $$r < \mu - (1 - \gamma)\sigma^2 < \frac{\lambda c}{\alpha - h(\lambda) + 1},$$
we have

$$\mu - (1 - \gamma)\sigma^2 - \frac{\lambda c}{\alpha - h(\lambda) + 1} < 0.$$

Further,

$$\tilde{H}_k(k, \Psi) = -(1 - \gamma)\Psi^2\sigma^2 k + \Psi(\mu - r),$$
$$\tilde{H}_\Psi(k, \Psi) = -(1 - \gamma)\Psi\sigma^2 k^2 + k(\mu - r) + r - \frac{\lambda c}{\alpha - h(\lambda) + 1}.$$

Then

$$\tilde{H}_k(k, \Psi) - \tilde{H}_\Psi(k, \Psi) = (\Psi - k)(\mu - r - (1 - \gamma)\sigma^2 k\Psi) + \frac{\lambda c}{\alpha - h(\lambda) + 1} - r.$$
Hence, if $k\Psi \leq 1$,
\[
\tilde{H}_k(k, \Psi) - \tilde{H}_\Psi(k, \Psi) = (\Psi - k)(\mu - r - (1 - \gamma)\sigma^2k\Psi) + \frac{\lambda c}{\alpha - h(\lambda) + 1} - r.
\]
\[
\geq (\Psi - k)(\mu - r - (1 - \gamma)\sigma^2) + \frac{\lambda c}{\alpha - h(\lambda) + 1} - r
\]
\[
\geq 0.
\]
(27)

In addition, if $k\Psi \geq 1$,
\[
\tilde{H}_\Psi(1, \Psi) \leq -(1 - \gamma)\sigma^2 + \mu - r + r - \frac{\lambda c}{\alpha - h(\lambda) + 1}
\]
\[
= \mu - (1 - \gamma)\sigma^2 - \frac{\lambda c}{\alpha - h(\lambda) + 1}
\]
\[
< 0.
\]

Then, $\tilde{H}(k, \Psi)$ is maximized if $k = 1$ and $\Psi = 1$. Thus, the optimal strategies are
\[
\pi^* = 0,
\]
\[
k^* = 1.
\]

If
\[
r < \frac{\lambda c}{\alpha - h(\lambda) + 1} < \mu - (1 - \gamma)\sigma^2,
\]
we have
\[
\mu - (1 - \gamma)\sigma^2 - \frac{\lambda c}{\alpha - h(\lambda) + 1} > 0.
\]

Note that (27) still holds. On the other hand,
\[
\tilde{H}_\Psi(1, 1) = -(1 - \gamma)\sigma^2 + \mu - r + r - \frac{\lambda c}{\alpha - h(\lambda) + 1}
\]
\[
= \mu - (1 - \gamma)\sigma^2 - \frac{\lambda c}{\alpha - h(\lambda) + 1}
\]
\[
> 0.
\]

In addition,
\[
\tilde{H}_\Psi(1, \Psi) = 0 \Rightarrow \Psi = \frac{1}{(1 - \gamma)\sigma^2} \left( \mu - \frac{\lambda c}{\alpha - h(\lambda) + 1} \right).
\]

Since $\frac{1}{(1 - \gamma)\sigma^2} \left( \mu - \frac{\lambda c}{\alpha - h(\lambda) + 1} \right) > 1$, the optimal strategies are
\[
\pi^* = \frac{1}{(1 - \gamma)\sigma^2(\alpha - h(\lambda) + 1)} \left( \mu - \frac{\lambda c}{\alpha - h(\lambda) + 1} \right) - \frac{1}{\alpha - h(\lambda) + 1},
\]
\[
k^* = 1.
\]

We conclude the results of the optimal strategies in the following.

**Theorem 5.2.** The optimal portfolio weight $k^*$ and $\pi^*$ follows

- If $r > \frac{\lambda c}{\alpha - h(\lambda) + 1}$, the optimal portfolio strategies follow
  \[
  \pi^* = M,
  \]
  \[
  k^* = \frac{\mu - r}{(1 - \gamma)\sigma^2(M(\alpha - h(\lambda) + 1) + 1)}.
  \]
If $\mu - (1 - \gamma)\sigma^2 < r < \frac{\lambda c}{\alpha - h(\lambda) + 1}$, the optimal portfolio strategies follow

$$\begin{align*}
    \pi^* &= 0, \\
    k^* &= \frac{\mu - r}{(1 - \gamma)\sigma^2}. \\
\end{align*}$$

If $r < \mu - (1 - \gamma)\sigma^2 < \frac{\lambda c}{\alpha - h(\lambda) + 1}$, the optimal portfolio strategies follow

$$\begin{align*}
    \pi^* &= 0, \\
    k^* &= 1. \\
\end{align*}$$

If $r < \frac{\lambda c}{\alpha - h(\lambda) + 1} < \mu - (1 - \gamma)\sigma^2$, the optimal portfolio strategies follow

$$\begin{align*}
    \pi^* &= \frac{1}{(1 - \gamma)\sigma^2(\alpha - h(\lambda) + 1)} \left( \mu - \frac{\lambda c}{\alpha - h(\lambda) + 1} - \frac{1}{\alpha - h(\lambda) + 1} \right), \\
    k^* &= 1. \\
\end{align*}$$

If $r = \frac{\lambda c}{\alpha - h(\lambda) + 1} < \mu - (1 - \gamma)\sigma^2$, the optimal portfolio strategies follow

$$\begin{align*}
    \pi^* &= \text{any } \pi \text{ in } \Gamma, \\
    k^* &= \frac{\mu - r}{\sigma^2(\pi^*(\alpha - h(\lambda) + 1) + 1)}. \\
\end{align*}$$

**Proof.** The details are provided above. 

### 6. Ruin probability

We will study the impact of the optimal portfolio weight and dividend payment strategies on the ruin probability. Denote by $\tau = \inf\{t \geq 0 : X(t) < 0\}$ the time of financial ruin. We will consider the probability of ruin as follows.

Let $k^*$, $\pi^*$ and $z^*$ be the optimal portfolio and dividend payment strategies obtained previous sections, respectively. Let

$$\bar{Q}(t) = k^*(t)(\mu(t) - r)\Psi^* + r\Psi^* - \pi^*\lambda c(t) - z^*(t),$$

and

$$\bar{Q}(t) = \Psi^*k^*(t)\sigma(t).$$

By choosing the optimal strategies, (6) can be rewritten as

$$\begin{align*}
    \frac{dX(t)}{X(t)} &= \bar{Q}(t)dt + \bar{Q}(t)dW(t), \\
    X(0) &= x. \\
\end{align*}$$

Hence,

$$X(t) = x \exp \left\{ \int_0^t (\bar{Q}(s) - \frac{1}{2}\bar{Q}^2(s))ds + \bar{Q}(s)dW_1(s) \right\}.$$

Note that $X(t)$ follows a geometric Brownian motion process when optimal controls are adopted. Then $X(t) > 0$ and $\mathbb{P}(\tau < \infty) = 0$ since $x > 0$. It shows that financial ruin can be completely avoided if optimal investment and dividend payment strategies are executed. This is because the debt ratio sets a implicit constraint on the written liability size such that the written liability $L(t)$ should depend on the surplus status $X(t)$.

In the classical Cramér-Lundberg model with the optimal barrier dividend payment strategy, when total discounted dividend payment is maximized, the company will almost surely be financial ruined. In both cases of utility functions, when liability size is well managed with debt ratio, the financial ruin can be immunized from abrupt asset value shocks, even though the total discounted dividend payment is
maximized. This is one of the advantages of this model. Intuitively, when surplus is low, the company should choose to write much less new policies to monitor the risk of financial ruin.

7. Impact of reinsurance policy.

7.1. Logarithmic utility. In this subsection, we will analyze the impact of reinsurance strategies on the optimal investment strategies in the logarithmic utility case. The impact will be discussed in four scenarios.

Scenario One. When \( r > \frac{\lambda c}{\alpha - h(\lambda) + 1} \), the optimal investment strategies follow

\[
k^* = \frac{\mu - r}{\sigma^2(M(\alpha - h(\lambda) + 1) + 1)}.
\]

In addition, increasing the reinsurance cost \( h(\lambda) \) will decrease the portfolio weight in the risky asset. The retention level \( \lambda \) is assumed to be positive. The reinsurance cost \( h(\lambda) \) is negatively correlated to the retention level \( \lambda \). Thus \( h(\lambda) < 0 \). By simple calculation, it is not hard to find that

\[
k^* \lambda = \frac{Mh(\lambda)(\mu - r)}{(M(\alpha - h(\lambda) + 1) + 1)^2\sigma^2}.
\]

Since \( k^* \lambda \) is negative, it is shown that the insurance company should put less capital in the risky asset if the company chooses to reduce the reinsurance cost by increasing the retention level. This is consistent with the intuition. When the retention level \( \lambda \) is higher, the insurance companies will be exposed to more investment risks. Decreasing the liability will reduce the risks and help the company avoid financial ruin.

Scenario Two. When \( r = \frac{\lambda c}{\alpha - h(\lambda) + 1} \), the decision maker has much freedom to determine the optimal debt ratio and investment strategy as long as \( k^* \Psi^* = \frac{\mu - r}{\sigma^2} \). We call this riskfree rate as an ideal rate. In this case, (13) can be rewritten as

\[
0 = \max_{k, \pi} \left\{ k(\mu - r)\Psi - \frac{1}{2}\Psi^2k^2\sigma^2 \right\} A(t) + [A_t(t) + e^{-\rho t} \ln x + \max_z[-zA(t) + e^{-\rho t} \ln z] + B_t(t) + \frac{A(t)}{\alpha - h(\lambda) + 1}].
\]

Hence,

\[
0 = [A_t(t) + e^{-\rho t}] \ln x + \max_z[-zA(t) + e^{-\rho t} \ln z] + B_t(t)
\]

\[
+ A(t)\left( \frac{1}{\alpha - h(\lambda) + 1} + \frac{(\mu - r)^2}{2\sigma^2} \right).
\]

The HJB equation is largely simplified in this case.

Scenario Three. In this scenario we will assume that there is no borrowing constraint. When \( r < \frac{\lambda c}{\alpha - h(\lambda) + 1} \), the optimal investment strategies \( k^* = \frac{\mu - r}{\sigma^2} \). In this case, the weight of the risky asset in the portfolio is constant. Considering the return from the risk-free asset is lower than the benchmark, the investment strategy is not surprising.

Scenario Four. In this scenario we take into account the borrowing constraint and its impact. In view of the results in Theorem 5.1, the impact of borrowing constraint
only effects when \( r < \min \left\{ \frac{\lambda c}{\alpha-h(\lambda)+1}, \mu - \sigma^2 \right\} \). In this case, the decision maker will put all his stakes into the risky asset. Plugging it to (6), we have

\[
\begin{aligned}
\frac{dX(t)}{X(t)} &= \{\pi(t)[\mu(t)(\alpha - h(\lambda) + 1) - \lambda c(t)] + \mu(t) - z(t)\} dt \\
&\quad + [\pi(\alpha - h(\lambda) + 1) + 1] \sigma(t) dW_1(t), \\
X(0) &= x.
\end{aligned}
\]

In this scenario, because of the borrowing constraint, we set our investment strategy as \( k^* = 1 \). To maximize the objective function, we can choose the optimal liability size and use the debt ratio as the control parameter. Hence, we can write a new HJB equation by using the dynamic programming principle as follows

\[
0 = \max \left\{ V_x(x,t) x \pi(\alpha - h(\lambda) + 1) - \lambda c \right\} \\
+ \frac{1}{2} V_{xx}(x,t) x^2 \left[ \pi(\alpha - h(\lambda) + 1) + 1 \right]^2 \sigma^2 \\
+ \max \left\{ -z x V_x(x,t) + e^{-\rho t} U_1(z x) \right\} + V_x(x,t) x \mu + V_t(x,t).
\]

By using the similar techniques in Section 3.1, we assume that the form of the solution to (28) as

\[
V(x,t) = \mathcal{A}(t) \ln x + \mathcal{B}(t).
\]

To determine \( \mathcal{A}(t) \) and \( \mathcal{B}(t) \), we plug (29) into (28). Then we have

\[
0 = \max \left\{ \pi(\mu - \sigma^2)(\alpha - h(\lambda) + 1) - \lambda c - \frac{1}{2} \left[ \pi(\alpha - h(\lambda) + 1) \right]^2 \sigma^2 \right\} \mathcal{A}(t) \\
+ \mathcal{A}_t(t) + e^{-\rho t} \ln x + \max \left\{ -z \mathcal{A}(t) + e^{-\rho t} \ln z \right\} + \mathcal{A}(t)(\mu - \frac{1}{2} \sigma^2) + \mathcal{B}_t(t).
\]

Then,

\[
\pi^* = \frac{(\mu - \sigma^2)(\alpha - h(\lambda) + 1) - \lambda c}{(\alpha - h(\lambda) + 1)^2 \sigma^2}.
\]

(30) is meaningful only if

\[
\mu > \sigma^2 + \frac{\lambda c}{\alpha - h(\lambda) + 1}.
\]

Otherwise, a negative debt ratio can’t provide much information.

### 7.2. Power utility.

In this subsection, we consider the case with power utility. Similarly, the impact will be discussed in four scenarios.

**Scenario One.** When \( r > \frac{\lambda c}{\alpha-h(\lambda)+1} \), the optimal investment strategies follow

\[
k^* = \frac{\mu - r}{(1 - \gamma) \sigma^2 (\alpha - h(\lambda) + 1) + 1}.
\]

The impact of the reinsurance policy in the power utility case is similar to that of logarithmic case. Note that the reinsurance cost \( h(\lambda) \) is negatively correlated to the retention level \( \lambda \). Thus \( h_1(\lambda) < 0 \). Further, the insurance company should put less capital in the risky asset if the company chooses to reduce the reinsurance cost by increasing the retention level. This is consistent with the intuition. When retention level \( \lambda \) is higher, the insurance companies will be exposed to more risks. Decreasing the liability will reduce the risks and help the company avoid financial ruin. The amount of the risks exposed to the risky assets in the financial market then is exposed to the risk aversion level \( \gamma \).
\textbf{Scenario Two.} When \( r = \frac{\lambda c}{\alpha - h(\lambda) + 1} \), the decision maker has much freedom to determine the optimal debt ratio and investment strategy as long as \( k^* = \left( \frac{\mu - r}{1 - \gamma \sigma^2} \right) \). We call this riskfree rate as an ideal rate. In this case, (19) can be rewritten as

\[
0 = \left\{ \frac{1}{2} (\mu - r)^2 + \frac{\lambda c}{\alpha - h(\lambda) + 1} \right\} \gamma p(t) + \max_z \left[ -z \gamma p(t) + e^{-\rho t} z \gamma \right] + p_t(t).
\]

Hence, the HJB equation is largely simplified in this case.

\textbf{Scenario Three.} In this scenario we will assume that there is no borrowing constraint. When \( r < \frac{\lambda c}{\alpha - h(\lambda) + 1} \), the optimal investment strategies \( k^* = \left( \frac{\mu - r}{1 - \gamma \sigma^2} \right) \). In this case, the weight of the risky asset in the portfolio is constant for given level of risk aversion. Considering the return from the riskfree asset is sufficiently low, the investment strategy is reasonable.

\textbf{Scenario Four.} In this scenario we take into account the borrowing constraint and its impact. In view of the results in Theorem 5.2, the impact of borrowing constraint only effects when \( r < \min \left\{ \frac{\lambda c}{\alpha - h(\lambda) + 1}, \mu - (1 - \gamma) \sigma^2 \right\} \). In this case, the decision maker will put all his stakes into the risky asset. Plugging it to (6), we have

\[
\frac{dX(t)}{X(t)} = \left\{ \pi(t) \{ \mu(t) (\alpha - h(\lambda) + 1) - \lambda c(t) \} + \mu(t) - z(t) \right\} dt + \left[ \pi(\alpha - h(\lambda) + 1) + 1 \right] \sigma(t) dW(t),
\]

\[
X(0) = x.
\]

In this scenario, due to the borrowing constraint, we set our investment strategy as \( k^* = 1 \). To maximize the objective function, we can choose the optimal liability size and use the debt ratio as the control parameter. Hence, we can write a new HJB equation by using the dynamic programming principle as follows

\[
0 = \max \left\{ \pi V_x(x, t) x \pi [\mu(\alpha - h(\lambda) + 1) - \lambda c] + \frac{1}{2} V_{xx}(x, t) x^2 \pi (\alpha - h(\lambda) + 1)^2 \sigma^2 \right\} + \max_z \left[ -zx V_x(x, t) + e^{-\rho t} U_1(zx) \right] + V_x(x, t)x\mu + V_t(x, t).
\]

By using the similar techniques in Section 3.2, we assume that the solution to (31) is of the form:

\[
V(x, t) = p(t) \frac{x^\gamma}{\gamma}.
\]

To determine \( p(t) \), we plug (32) into (31). Then we have

\[
0 = \max \left\{ \frac{1}{2} (1 - \gamma) \Psi^2 \sigma^2 + \mu \Psi - \frac{\Psi - 1}{\alpha - h(\lambda) + 1} \lambda c \right\} \gamma p(t) + \max_z \left[ -z \gamma p(t) + e^{-\rho t} z \gamma \right] + p_t(t).
\]

Then,

\[
\pi^* = \frac{(\mu - (1 - \gamma) \sigma^2)(\alpha - h(\lambda) + 1) - \lambda c}{(\alpha - h(\lambda) + 1)^2 (1 - \gamma) \sigma^2}.
\]

(33) is meaningful only if

\[
\mu > (1 - \gamma) \sigma^2 + \frac{\lambda c}{\alpha - h(\lambda) + 1}.
\]

Otherwise, a negative debt ratio is inconsistent with our intuition.
8. **Stochastic interest rate.** In this section, we will assume that the bond’s return rate \( r(t) \) is a random variable. Assume the interest rate \( r(t) \) follows the Vasicek process. That is, we assume
\[
\begin{align*}
    dr(t) &= \theta(\bar{r} - r(t))dt + \bar{\sigma}dW_2(t), \\
    r(0) &= r,
\end{align*}
\]
where \( \bar{r} \) represents the expectation of long-term interest rate, and \( \theta > 0 \) represents the speed of reversion to \( \bar{r} \). Assume that \( \bar{r} > 0 \). \( \bar{\sigma}dW_2(t) \) represents the random shocks of interest rate due to the financial market and other economic performance.

Furthermore, to guarantee the existence of classical solutions to the corresponding HJB equations, we need additional assumptions on the drift of the risky asset \( S(t) \). Specifically, we assume that \( \mu(t) \) satisfies the polynomial growth condition, i.e.,
\[
\forall \mu(t) \in C(\mathbb{R}), \quad \mu(t) \leq \theta_1(1 + |t|^{\phi_1}),
\]
where \( \theta_1 \) and \( \phi_1 \) are positive constants.

8.1. **Logarithmic utility.** By incorporating the stochastic interest rate process, the value function will be denoted as \( V(x, r, t) \). By using the dynamic programming principle, (9) can be written as
\[
0 = \max_{k, \pi} \left[ V_x(x, r, t)x\{k(\mu - r)[\pi C(t)(\alpha - h(\lambda) + 1) + 1]\right.
\]
\[
+ \frac{1}{2}V_{xx}(x, r, t)x^2[\pi(\alpha - h(\lambda) + 1) + 1]^2k^2\sigma^2
\]
\[
+ V_x(x, r, t)x\{\pi(\alpha - h(\lambda) + 1) + 1 - \pi\lambda c\} \right]
\]
\[
+ \max_z[-zx V_x(x, r, t) + e^{-\rho t}U_1(zx)] + V_t(x, r, t)
\]
\[
+ \frac{1}{2}\sigma^2V_{rr}(x, r, t) + \theta(\bar{r} - r)V_r.
\]
with terminal condition
\[
V(x, r, T) = e^{-\rho T}\ln x.
\]
We construct a solution of (35) with the form
\[
V(x, r, t) = e^{-\rho t}(C(t)\ln x + Y(r, t)),
\]
To determine \( C(t) \) and \( Y(r, t) \), we plug (37) into (35). Recall that \( \Psi = \pi(\alpha - h(\lambda) + 1) + 1 \). Then we have
\[
e^{-\rho t}\left\{ \max_{k, \pi} C(t)[-\frac{1}{2}\sigma^2k^2\Psi^2 + k(\mu - r)\Psi + r\Psi - \pi\lambda c] + (1 - C(t)\rho + D_t(t))\ln x
\right.
\]
\[
+ \max_z[-C(t)z + \ln z] + Y_t(r, t) + \frac{1}{2}\sigma^2Y_{rr}(r, t) + \theta(\bar{r} - r)Y_r(r, t) - \rho Y(r, t) \right\} = 0.
\]
(38)

Since (35) holds for all \( x \), we have
\[
1 - C(t)\rho + D_t(t).
\]
Then,
\[
C(t) = me^{-\rho t} + \frac{1}{\rho},
\]
where \( m \) is a constant to be determined. Thus,
\[
V(x, r, t) = e^{-\rho t}\left[ \left( me^{-\rho t} + \frac{1}{\rho} \right)\ln x + Y(r, t) \right].
\]
Considering the boundary condition (36), \( m \) follows
\[
m = e^{\rho T} \left( 1 - \frac{1}{\rho} \right).
\]

Let
\[
\tilde{C}(t) = \left( 1 - \frac{1}{\rho} \right) e^{\rho (T-t)} + \frac{1}{\rho}.
\]

Hence, the value function follows
\[
V(x, r, t) = e^{-\rho t} \left\{ \tilde{C}(t) \ln x + Y(r, t) \right\}.
\]

To proceed, we will determine the form of \( Y(r, t) \). In view of (38), the optimal portfolio \( k^* \) is obtained as
\[
k^* = \frac{\mu - r}{\sigma^2 \Psi},
\]
and the optimal dividend payment rate follows
\[
z^* = \frac{1}{\tilde{C}(t)}.
\]

Substituting the optimal controls (39) and (40) into (35), it yields that
\[
\max_{\pi} e^{-\rho t} \left\{ \max_z \left[ -\tilde{C}(t) z + \ln z \right] + Y_t(r, t) + \frac{1}{2} \sigma^2 Y_{rr}(r, t) + \theta (\bar{r} - r) Y_r(r, t) - \rho Y(r, t) + \tilde{C}(t) \left[ \frac{(\mu - r)^2}{2\sigma^2} + r \Psi - \pi \lambda c \right] + \ln \tilde{C}(t) - 1 \right\} = 0.
\]

(41) yields
\[
Y_t(r, t) + \frac{1}{2} \sigma^2 Y_{rr}(r, t) + \theta (\bar{r} - r) Y_r(r, t) - \rho Y(r, t) + G(r, t) = 0,
\]
where
\[
G(r, t) = \tilde{C}(t) \left[ \frac{(\mu - r)^2}{2\sigma^2} + \Psi^* \left( r - \frac{\lambda c}{\alpha - h(\lambda) + 1} \right) \right] + \ln \tilde{C}(t) - 1.
\]

\( \Psi^* \) is the optimal value. Similar to (18), the optimal debt ratio follows
\[
\pi^* = \begin{cases} 
M & \text{if } r > \frac{\lambda c}{\alpha - h(\lambda) + 1}, \\
0 & \text{if } r < \frac{\lambda c}{\alpha - h(\lambda) + 1}, \\
\text{any } \pi \in \Gamma & \text{if } r = \frac{\lambda c}{\alpha - h(\lambda) + 1}.
\end{cases}
\]

On the other hand, the boundary condition (36) yields
\[
Y(r, T) = 0.
\]

By (34) and using the subsolution–supersolution method in [16] (Section 7.2), there exists a classical solution \( \tilde{V}(r, t) \) to (42) satisfying the corresponding boundary condition (44). Hence,
\[
\tilde{V}(x, r, t) = e^{-\rho t} \left( \tilde{C}(t) \ln x + \tilde{Y}(r, t) \right)
\]
is a classical solution of (41).
Further, combining (39), (40) and (43), the optimal strategy $u^*(t)$ follows

$$
\pi^* = \begin{cases} 
M, & \text{if } r > \frac{\lambda c}{\alpha - h(\lambda) + 1}, \\
0, & \text{if } r < \frac{\lambda c}{\alpha - h(\lambda) + 1}, \\
\text{any } \pi \in \Gamma, & \text{if } r = \frac{\lambda c}{\alpha - h(\lambda) + 1}.
\end{cases}
$$

$$
k^* = \frac{\mu - r}{(\pi^* - h(\lambda) + 1) + 1}\sigma^2
$$

$$
z^* = \frac{1}{C(t)}
$$

8.2. **Power utility.** In this case, HJB equation becomes

$$
0 = \max_{k, \pi} \{ V_x(x, r, t) \left[ k(\mu - r) (\pi - h(\lambda) + 1) + 1 \right] \\
+ \frac{1}{2} V_{xx}(x, r, t) x^2 [\pi(\alpha - h(\lambda) + 1) + 1]^2 k^2 \sigma^2 \\
+ V_r(x, r, t) x [r (\pi(\alpha - h(\lambda) + 1) + 1) - \lambda \sigma_p] \\
+ \max \left[ -z_x V_x(x, r, t) + e^{-\rho t}\frac{(zx)^\gamma}{\gamma} \right] \\
+ V_t(x, r, t) + \frac{1}{2} \sigma^2 V_{rr}(x, r, t) + \theta(r - r)V_r, \tag{45}
\}
$$

with terminal condition

$$
V(x, r, T) = e^{-\rho T}\frac{x^\gamma}{\gamma}.
$$

We construct a solution of (45) with the form

$$
V(x, r, t) = e^{-\rho t}\frac{p(r, t)}{\gamma} \frac{x^\gamma}{\gamma}. \tag{46}
$$

Putting (46) into (45), it yields that

$$
0 = \max_{k, \pi} \{ e^{-\rho t}p(r, t) x^{\gamma-1} \left[ k(\mu - r) (\pi - h(\lambda) + 1) + 1 \right] \\
+ \frac{1}{2} (\gamma - 1)e^{-\rho t}p(r, t) x^{\gamma-2} x^2 [\pi(\alpha - h(\lambda) + 1) + 1]^2 k^2 \sigma^2 \\
+ e^{-\rho t}p(r, t) x^{\gamma-1} x [r (\pi(\alpha - h(\lambda) + 1) + 1) - \lambda \sigma_p] \\
+ \max \left[ -z_x e^{-\rho t}p(r, t) x^{\gamma-1} + e^{-\rho t}\frac{(zx)^\gamma}{\gamma} \right] \\
+ \left( -\rho e^{-\rho t}p(r, t) \frac{x^\gamma}{\gamma} + e^{-\rho t}p_t(r, t) \frac{x^\gamma}{\gamma} \right) + \frac{1}{2} \sigma^2 e^{-\rho t}p_{rr}(r, t) \frac{x^\gamma}{\gamma} \\
+ \theta(\bar{r} - r)e^{-\rho t}p_r(r, t) \frac{x^\gamma}{\gamma},
\}
$$

i.e.,

$$
0 = \max_{k, \pi} \left\{ -\frac{1}{2} (1 - \gamma) \Psi^2 \sigma^2 \left( k - \frac{\mu - r}{(1 - \gamma) \sigma^2 \Psi} \right)^2 \\
+ \left( r - \frac{\lambda c}{\alpha - h(\lambda) + 1} \right) \Psi + \frac{1}{2} \frac{(\mu - r)^2}{(1 - \gamma) \sigma^2} + \frac{\lambda c}{\alpha - h(\lambda) + 1} \right\} \gamma p(r, t) \\
+ \max \left[ -z \gamma p(r, t) + z^\gamma \right] \\
- \rho p(r, t) + p_t(r, t) + \frac{1}{2} \gamma^2 p_{rr}(r, t) + \theta(\bar{r} - r)p_r(r, t). \tag{47}
\}
Thus,

\[ k^* = \frac{\mu - r}{(1 - \gamma)\sigma^2 \Psi}, \] (48)

\[ z^* = p(r, t) \frac{1}{\gamma}. \] (49)

In addition, if \( r > \frac{\lambda c}{\alpha - h(\lambda) + 1} \), the maximal value can be achieved when \( \pi^* = M \) as \( \alpha > h(\lambda) \); if \( r < \frac{\lambda c}{\alpha - h(\lambda) + 1} \), then the maximal value can be achieved when \( \pi^* = 0 \); if \( r = \frac{\lambda c}{\alpha - h(\lambda) + 1} \), \( \pi^* \) can be any number in \( \Gamma \).

Substituting (48) and (49) into (47), we have

\[ 0 = \left[ \left( r - \frac{\lambda c}{\alpha - h(\lambda) + 1} \right) \Psi^* + \frac{1}{2} (\mu - r)^2 + \frac{\lambda c}{\alpha - h(\lambda) + 1} \right] \gamma p(r, t) \]

\[ + (1 - \gamma) (p(r, t))^{\gamma} - \rho p(r, t) + p_t(r, t) + \frac{1}{2} \bar{\sigma}^2 p_{rr}(r, t) + \theta (\bar{r} - r) p_r(r, t), \] (50)

with

\[ p(r, T) = 1. \] (51)

Analogously to the case of logarithmic utility function, by using the method of upper and lower solutions in Pao (1993) (Section 7.2), we can find a classical solution to (50)-(51).

9. **Concluding remarks.** Debt management has been an important tool to monitor the the leverage of financial status for both individual investors and publicly traded companies. How to chose a proper debt level is vital for financial institutions as well as the whole financial system. We investigated the optimal debt ratio, investment and dividend payment strategies for an insurer whose objective is to maximize the total expected discounted utility of dividend payment in maximize period. Considering logarithmic and power utility functions, we obtained the explicit solution of the value function and the corresponding optimal debt ratio, investment and dividend payment strategies. In addition, we analyzed the impact of borrowing constraints, dividend payment constraint and reinsurance policies to the optimal strategies. Finally, we extended the results to a model with Vasicek stochastic interest rate process.

In this paper, we worked under a diffusion model. Since the surplus of insurance companies declines by the claims that occur randomly, it might be better to study the same problem under compound Poisson model and jump-diffusion models. Further, a dividend process is not necessarily absolutely continuous. Insurance companies may distribute dividends at unrestricted payment rate at random discrete times. In such a scenario, the surplus level changes drastically on a dividend payday. Thus, abrupt or discontinuous changes occur due to a “singular” type dividend distribution policy. The “singular” type of dividend policy will make the associated HJB equation much more complicated. However, in general, we cannot expect closed-form solutions under these models, and have to resort to numerical method to solve corresponding HJB equations. We shall report these results in our future research.

**Appendix A. Proof of Eq. (8).** Fix \((x, t) \in [0, T) \times \mathbb{R}, \) and \( u \in \mathcal{A}. \) Let \( X(\cdot) \) be the trajectory associated with the control \( \bar{u}(\cdot) \doteq (\bar{k}(\cdot), \bar{\pi}(\cdot), \bar{z}(\cdot)), \) and \( t' \in (t, T]. \) By using the Itô lemma, we have
Hence, combining (58) and (63), we conclude that (8) holds.

\[ V(X(t'), t') - V(X(t), t) = \int_t^{t'} f(X(s), s, \bar{u}(s)) ds + \int_t^{t'} g(X, s, \bar{u}) dW_t(s), \quad (52) \]

where

\[ f(x, s, \bar{u}) = V_x(x, s) + \frac{1}{2} V_{xx}(x, s)[\bar{\pi}(x - h(\lambda) + 1)]^2 k^2 \sigma^2 x^2 + V_z(x, s)[k(\mu - r)[\bar{\pi}(x - h(\lambda) + 1) + 1] + r[\bar{\pi}(x - h(\lambda) + 1) + 1 - \pi \lambda c - \bar{\pi}], \quad (53) \]

and

\[ g(x, s, \bar{u}) = V_x(x, s)[k(\mu - r)[\pi(x - h(\lambda) + 1) + 1] + r[\pi(x - h(\lambda) + 1) + 1 - \pi \lambda c - \bar{\pi}) x. \]

Dividing \( t' - t \) and taking expectations on both sides of (52), we have

\[ \frac{EV(X(t'), t') - V(X(t), t)}{t' - t} = \frac{1}{t' - t} \mathbb{E} \int_t^{t'} f(X(s), s, \bar{u}(s)) ds. \quad (54) \]

Using the dynamic programming principle, we have

\[ \frac{V(X(t), t) - EV(X(t'), t')}{t' - t} \geq \frac{1}{t' - t} \mathbb{E} \int_t^{t'} e^{-\rho s} U(\bar{z}(s)X(s)) ds. \quad (55) \]

Combining (54) and (55), we have

\[ \frac{1}{t' - t} \mathbb{E} \int_t^{t'} [f(X(s), s, \bar{u}(s)) + e^{-\rho s} U(\bar{z}(s)X(s))] ds \leq 0. \quad (56) \]

Letting \( t' \downarrow t \), (56) yields

\[ f(x, t, \bar{u}) + e^{-\rho t} U(\bar{z}x) \leq 0 \quad (57) \]

for all \( \bar{u} \in \mathcal{A} \). Hence,

\[ \max_u \{V(x, t) + \mathcal{L}^u V(x, t) + e^{-\rho t} U(\bar{z}x)\} \leq 0. \quad (58) \]

On the other hand, for any \( \varepsilon > 0 \) and any \( t' \in (t, T] \), there exists a control \( \bar{u}_\varepsilon(\cdot) \equiv (\bar{k}_\varepsilon(\cdot), \bar{\pi}_\varepsilon(\cdot), \bar{z}_\varepsilon(\cdot)) \) such that

\[ V(x, t) - \varepsilon (t' - t) \leq \mathbb{E} \left[ \int_t^{t'} e^{-\rho s} U(\bar{z}_\varepsilon(s)X(s)) ds + V(X(t'), t') \right]. \quad (59) \]

Then, together with (55), rearrange (59), we have

\[ \varepsilon \geq -\frac{1}{t' - t} \mathbb{E} \int_t^{t'} [f(X(s), s, \bar{u}_\varepsilon(s)) + e^{-\rho s} U(\bar{z}_\varepsilon(s)X(s))] ds. \quad (60) \]

Hence, we have

\[ \varepsilon \geq -\frac{1}{t' - t} \mathbb{E} \int_t^{t'} [V_t(x, t) + \mathcal{L} \bar{u}_\varepsilon V(x, t) + e^{-\rho t} U(\bar{z}_\varepsilon x)] ds. \quad (61) \]

As \( t' \downarrow t \), we have

\[ -\varepsilon \leq V_t(x, t) + \mathcal{L} \bar{u}_\varepsilon V(x, t) + e^{-\rho t} U(\bar{z}_\varepsilon x) \quad (62) \]

for any \( \varepsilon > 0 \). Thus,

\[ -\varepsilon \leq \max_u \{V_t(x, t) + \mathcal{L}^u V(x, t) + e^{-\rho t} U(zx)\}. \quad (63) \]

Hence, combining (58) and (63), we conclude that (8) holds. \( \square \)
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