Gauge Fields and Unparticles

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Abstract

We show that a rigorous path integral method of introducing gauge fields in the UnParticle lagrangian leads to somewhat different and more complicated vertexes than those currently used.
INTRODUCTION

The idea of a scalar field that represents a particle of indefinite mass, introduced by Georgi [1] [2], was extended to a gauged field by Terning et al [3]. The unparticle action was taken in the nonlocal form:

\[ I = \int d^4x d^4y \psi^\dagger(x) K(x - y) \psi(y) \]  

Where \( K \) denotes the inverse of the Unparticle’s propagator. To include a gauge field \( A \), an additional term is included:

\[ U(x, y, \gamma) = P \left[ \exp \left( -ig \int_{x,\gamma}^y A_\mu(w) dw^\mu \right) \right] \]

Here \( P \) denotes a path ordering, and \( \gamma \) denotes a path from \( x \) to \( y \). Then

\[ I = \int d^4x d^4y \psi^\dagger(x) K(x - y) U(x, y, \gamma) \psi(y) \]

Terning et al [3] do not specify the path \( \gamma \), but they make the assumption that it is always such that

\[ \frac{\partial}{\partial y^\mu} U(x, y, \gamma) = -igU(x, y, \gamma) A_\mu(y) \]

This is a very old idea that goes back to Mandlestam [4], but it can not be quite correct. It requires that for all \( x \) and \( y \), the path that goes from \( x \) to \( y + dy \) must have gone first from \( x \) to \( y \). However, the paths between all point pairs must be exactly defined before the integral in Eq. (3) can be calculated. Whatever the definition of the path from \( x \) to \( y + dy \), it cannot be expected to have gone through \( y \) just because someone wishes to compute the derivative.

In the following, we investigate the consequences of defining the path integral between any two points \( x \) and \( y \) as the straight line from \( x \) to \( y \). We show that the resulting vertexes satisfy the Ward-Takahashi identities [5], but they do lead to vertexes that are rather more complicated than those found in Ref. [3]. In a later work, we will show that a Terning-type vertex can be obtained by a different method of introducing gauge fields into the Unparticle action.
STRAIGHT LINE PATH

Choosing the path as the straight line from \( x \) to \( y \) leads to

\[
U(x, y) = P \left[ \exp \left( -ig \int_0^1 A_\mu(w(\lambda)) \, dw^\mu(\lambda) \right) \right]
\]  

(5)

where

\[
w^\mu(\lambda) = (1 - \lambda) x^\mu + \lambda y^\mu
\]  

(6)

The UP-gauge-UP vertex is defined by

\[
ig \Gamma^\mu(y, x, z) = -\frac{\delta^3 I}{\delta A_\mu(x) \delta \bar{\psi}(y) \delta \psi(z)} \bigg|_{A=0}
\]  

(7)

Fourier transforming:

\[
ig \Gamma^\mu(p, q, p + q) (2\pi)^4 \delta(p' - p - q) = \int d^4xd^4yd^4ze^{i(p'z - pq)} \cdot ig \Gamma^\mu(y, x, z)
\]  

(8)

Using

\[
\frac{\delta}{\delta A_\mu(w)} U(x, y) \bigg|_{A=0} = -ig \int_0^1 d\lambda \delta(w - (1 - \lambda) x - \lambda y)(y^\mu - x^\mu)
\]  

(9)

and with \( S(k) \) denoting the Unparticle propagator in momentum space,

\[
K(x - y) = \int \frac{d^4k}{(2\pi)^4} S^{-1}(k) e^{ik(x-y)}
\]  

(10)

we get

\[
\Gamma^\mu(p, q, p + q) = -i \int_0^1 d\lambda \frac{\partial}{\partial k^\mu} S^{-1}(k) \bigg|_{k=-(p+\lambda q)}
\]  

(11)

We show that this satisfies the Ward-Takahashi identity. We consider first the scalar UP case, where the propagator depends on \( k \) through \( s = k^2 = p^2 + 2(p \cdot q)\lambda + q^2\lambda^2 \). Then

\[
\Gamma^\mu = 2i \int_0^1 d\lambda (p^\mu + \lambda q^\mu) \frac{dS^{-1}}{ds}
\]  

(12)

and with \( \frac{ds}{d\lambda} = 2(p \cdot q + \lambda q^2) \) we get

\[
q^\mu \Gamma_\mu = i \int_0^1 d\lambda \frac{ds}{d\lambda} \frac{dS^{-1}}{ds} = i \left[ S^{-1}(p + q) - S^{-1}(p) \right]
\]  

(13)
the WT relation. If now the UP is a fermion, then

\[ S^{-1} = \gamma^\mu k_\mu g(s) \]  

(14)

and

\[ \Gamma^\mu = -i \int_0^1 d\lambda \left[ \gamma_\mu g + 2\gamma^\alpha (p_\alpha + \lambda q_\alpha) (p^\mu + \lambda q^\mu) \frac{dg}{ds} \right] \]  

(15)

then

\[ q^\mu \Gamma_\mu = -i \int_0^1 d\lambda \gamma^\alpha \left[ q_\alpha g + (p_\alpha + \lambda q_\alpha) \frac{dg}{d\lambda} \right] \]

\[ = i\gamma^\alpha \left[ (p_\alpha + q_\alpha) g \left( (p + q)^2 \right) - p_\alpha g \left( p^2 \right) \right] \]

\[ = i \left[ S^{-1} (p + q) - S^{-1} (p) \right] \]

the WT identity.

THE VERTEX INTEGRAL

It is generally assumed \[1\], \[2\] that the Fourier transform of the inverse propagator goes as a power of the invariant momentum squared:

\[ S^{-1}(k) \doteq (k^2)^\nu \]  

(16)

where \( \nu = 2 - d_u \), \( d_u \) being the unparticle dimension. We therefore need to find the integral

\[ f_\nu (p, p') = \int_0^1 s^\nu d\lambda \]  

(17)

This can be written as

\[ f_\nu = \frac{1}{2\sqrt{q^2}} \int_{s_0}^{s_1} ds \frac{s^\nu}{\sqrt{s - A}} \]  

(18)

where \( s_0 = p^2 \), \( s_1 = p'^2 \) and

\[ A = p^2 - \frac{(p \cdot q)^2}{q^2} \]  

(19)

If \( \nu \) is not a half integer, the integral in Eq. (18) can be done as an infinite series, giving

\[ f_\nu (p, p') = g_\nu \left( p'^2, A \right) - g_\nu \left( p^2, A \right) \]  

(20)
where
\[ g_\nu (s, A) = \frac{s^{\nu + \frac{1}{2}}}{2\sqrt{q^2}} \sum_{k=0}^{\infty} \frac{(2k - 1)!!}{2^k k! (\nu - k + \frac{1}{2})} \left( \frac{A}{s} \right)^k \]  

(21)

THE SCALAR VERTEX

In this section we show another way of calculating the vertex integral and derive the vertex for a scalar unparticle. The vertex is given in terms of the vertex integral as

\[ \Gamma^\mu = 2i \left. \frac{\partial}{\partial p^\mu} f_\nu \right|_q \]  

(22)

where the scalar integral is

\[ f_\nu = \int_0^1 d\lambda (s(\lambda))^\nu \]  

(23)

with \( s = (p + \lambda q)^2 \), this can be expanded in a Taylor series in \( \lambda \) and then integrated to give

\[ f_\nu = A^\nu \sum_{k=0}^{\infty} \frac{\Gamma (\nu + 1)}{k! (2k + 1) \Gamma (\nu - k + 1)} \left( \frac{q^2}{A} \right)^k \left[ (1 + B)^{2k+1} - B^{2k+1} \right] \]  

(24)

where

\[ A = p^2 - \frac{(p \cdot q)^2}{q^2} = \frac{p^2 p'^2 - (p \cdot p')^2}{(p' - p)^2} \]  

and

\[ 1 + B = \frac{p' \cdot q}{q^2} \]
\[ B = \frac{p \cdot q}{q^2} \]

This can be expressed in terms of the hypergeometric functions

\[ _2F_1 (\alpha, \beta, \gamma; z) = 1 + \frac{\alpha \beta}{\gamma} z + \frac{\alpha (\alpha + 1) \beta (\beta + 1)}{\gamma (\gamma + 1) 2!} z^2 + \ldots \]  

(25)

as

\[ f_\nu = A^\nu \left[ (1 + B) _2F_1 \left( \frac{1}{2}, -\nu, \frac{3}{2}, -\frac{q^2}{A} (1 + B)^2 \right) - B _2F_1 \left( \frac{1}{2}, -\nu, \frac{3}{2}, -\frac{q^2}{A} B^2 \right) \right] \]  

(26)

Defining

\[ Q_\nu (z) = _2F_1 \left( \frac{1}{2}, -\nu, \frac{3}{2}, -\frac{q^2}{A} z^2 \right) \]  

(27)
and using
\[ \frac{\partial B}{\partial p_\mu} \bigg|_q = \frac{q_\mu}{q^2} \]
\[ \frac{\partial A}{\partial p_\mu} \bigg|_q = 2 \left( p^\mu - \frac{(p \cdot q)}{q^2} q^\mu \right) \]
\[ = \frac{2}{q^2} [p^\mu (p' \cdot q) - p'^\mu (p \cdot q)] \]
\[ C_\nu = (1 + B) Q_\nu (1 + B) - B Q_\nu (B) \]
we get
\[ \Gamma^\mu = 2i \left\{ \frac{q_\mu}{q^2} \left[ p'^{2\nu} - p^{2\nu} \right] + \frac{2\nu A^{\nu-1}}{q^2} \left[ p^\mu (p' \cdot q) - p'^\mu (p \cdot q) \right] C_{\nu-1} \right\} \]
This is considerably more complicated than the result found in Ref. [3].
When \( \nu = 1 \), this reduces to
\[ \Gamma^\mu = 2i (p'^\mu + p^\mu) \]
The expected result for a scalar particle.

CONCLUSIONS

We find that a rigorous application of the path integral method of introducing gauge fields into the unparticle Lagrangian leads to vertexes that are considerably more complicated than those found by Terning et al. [3] We will show in a later work that there is another method of combining gauge fields and unparticles does lead to the Terning result.

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