INVARIENTS OF SYMPLECTIC AND ORTHOGONAL GROUPS
ACTING ON GL(n, ℂ)-MODULES

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Abstract. Let GL(n) = GL(n, ℂ) denote the complex general linear group and let G ⊂ GL(n) be one of the classical complex subgroups O(n), SO(n), and Sp(2k) (in the case n = 2k). We take a polynomial GL(n)-module W and consider the symmetric algebra S(W). Extending previous results for G = SL(n), we develop a method for determining the Hilbert series H(S(W)^G, t) of the algebra of invariants S(W)^G. Then we give explicit examples for computing H(S(W)^G, t). As a further application, we extend our method to compute also the Hilbert series of the algebras of invariants Λ(S^2V)^G and Λ(Λ^2V)^G, where V = ℂ^n denotes the standard GL(n)-module.

1. Introduction

Let GL(n) = GL(n, ℂ) be the general linear group with its canonical action on the n-dimensional complex vector space V = ℂ^n and let W be a polynomial GL(n)-module. Then W can be written as a direct sum of its irreducible components

W = \bigoplus_\lambda k(\lambda)V^n_\lambda,

where \lambda = (\lambda_1, \ldots, \lambda_n) ∈ ℂ^n, \lambda_1 ≥ \lambda_2 ≥ \cdots ≥ \lambda_n ≥ 0, is a non-negative integer partition and V^n_\lambda is the irreducible GL(n)-module with highest weight \lambda. (In particular, V = V^n_\lambda(1).) We consider the algebra of polynomial functions ℂ[W]. The group GL(n) and its subgroups G act canonically on ℂ[W] by the diagonal action and we can construct the algebra of invariants ℂ[W]^G. It is well known that for any polynomial GL(n)-module W, we have W ≅ W^\lambda when considered as modules over O(n), SO(n), or Sp(2k). Therefore, we may identify ℂ[W]^G with S(W)^G for G being one of O(n), SO(n), or Sp(2k).

We recall the following definition.

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Definition 1.1. Let $A = \bigoplus_{i \geq 0} A^i$ be a finitely generated graded (commutative or non-commutative) algebra over $\mathbb{C}$ such that $A^0 = \mathbb{C}$ or $A^0 = 0$. The Hilbert series of $A$ is the formal power series

$$H(A, t) = \sum_{i \geq 0} (\dim A^i) t^i.$$ 

The Hilbert series $H(A, t)$ is one of the most important numerical invariants of the graded algebra. In particular, it gives information about the lowest degree of the generators of $A$ and the maximal number of generators in each degree.

Both algebras $\mathbb{C}[W]^G$ and $S(W)^G$ have a natural $\mathbb{N}_0$-grading which is inherited, respectively, from the $\mathbb{N}_0$-gradings of $\mathbb{C}[W]$ and $S(W)$. The algebras $\mathbb{C}[W]^G$ and $S(W)^G$ for $G = O(n)$, $SO(n)$, $Sp(2k)$ are isomorphic also as $\mathbb{N}_0$-graded algebras and hence $H(\mathbb{C}[W]^G, t) = H(S(W)^G, t)$.

There are many methods to compute the Hilbert series $H(\mathbb{C}[W]^G, t)$ (see, e.g., [3]). In a series of joint papers of the first named author (see [1] for an account), one more method for computing the Hilbert series $H(S(W)^{SL(n)}, t)$ of the algebra of invariants $S(W)^{SL(n)}$ has been developed. It is based on the method of Elliott [5] from 1903 for finding the non-negative solutions of linear systems of homogeneous Diophantine equations, further developed by MacMahon [13] in his $\Omega$-calculus (or Partition Analysis), and combined with the approach of Berele [2] in the study of cocharacters of algebras with polynomial identities. Our goal in this paper is to extend the latter method and to determine also the Hilbert series of the algebras of invariants $\mathbb{C}[W]^G \cong S(W)^G$ for $G = O(n)$, $G = SO(n)$, or $G = Sp(2k)$. Since $H(\mathbb{C}[W]^G, t) = H(S(W)^G, t)$, in the sequel we shall work in $S(W)$ instead of in $\mathbb{C}[W]$. Our main results are given in Section 4.

In Section 5, using our results from Section 4, we compute the Hilbert series of $\mathbb{C}[W]^G$ for explicit examples of $W$. We point out that in some of the examples the algebra of invariants $\mathbb{C}[W]^G$ is already described in the literature in terms of generators and relations.

In Sections 6 and 7, as a further application of our method, we compute also the Hilbert series of the algebras of invariants $\Lambda(S^2 V)^G$ and $\Lambda(\Lambda^2 V)^G$ for $G = O(n)$, $SO(n)$, $Sp(2k)$, where $\Lambda(W)$ and $\Lambda^2(W)$ denote, respectively, the exterior algebra and the second exterior power of the $GL(n)$-module $W$. Furthermore, for $\Lambda(S^2 V)^{Sp(2k)}$ and $\Lambda(\Lambda^2 V)^{SO(n)}$ we give the degrees of all generators and we point out that these algebras of invariants are isomorphic to exterior algebras.

2. Decomposition of irreducible $GL(2k)$-modules over $Sp(2k)$

In this section $n = 2k$. By $V^{2k}_\lambda$ we denote again the irreducible $GL(2k)$-module with highest weight $\lambda$. Our first goal is to decompose $V^{2k}_\lambda$ as a module over $Sp(2k)$. The irreducible representations of $Sp(2k)$ are indexed by non-negative integer partitions $\mu$ with at most $k$ parts, i.e., $\mu = (\mu_1, \ldots, \mu_k, 0, \ldots, 0)$ (see, e.g., [6, 7]). We denote them by $V^{2k}_{(\mu)}$. For our purposes we need to answer the following question: For which values of $\lambda$ does the one-dimensional trivial $Sp(2k)$-module (i.e., the module $V^{2k}_{(0)}$ with highest weight $\mu = (0, \ldots, 0)$) participate in the decomposition of $V^{2k}_\lambda$ and with what multiplicity? In the above notations we state the following Littlewood-Richardson branching rule.
Proposition 2.1. Let $\lambda$ be a partition in at most $k$ parts. Then

$$V_{\lambda}^{2k} \downarrow \text{Sp}(2k) \cong \bigoplus_{\mu, 2\delta} c_{\mu(2\delta)}^{\lambda} V_{\mu}^{2k},$$

where the sum runs over all partitions $\mu$ and all even partitions $2\delta = (2\delta_1, \ldots, 2\delta_n)$. Here $(2\delta)'$ denotes the transpose partition of $2\delta$ and the coefficients $c_{\mu(2\delta)}^{\lambda}$ are the Littlewood-Richardson coefficients.

Since in the statement of Proposition 2.1 $\lambda$ is a partition in at most $k$ parts, the same holds for the partitions $\mu$. Hence we do obtain the decomposition of $V_{\lambda}^{2k}$ into a direct sum of irreducible $\text{Sp}(2k)$-modules $V_{\mu}^{2k}$. When the partition $\lambda$ has more than $k$ parts, then on the right side of the equation (2) there will appear $\text{Sp}(2k)$-modules $V_{\mu(2\delta)}^{2k}$ for which $\mu$ has more than $k$ parts. The corresponding representations are called inadmissible representations, as they do not define actual representations of $\text{Sp}(2k)$. Then the decomposition of $V_{\lambda}^{2k}$ is obtained from Proposition 2.1 with the use of the modification rules from [11]. These rules state how to replace the inadmissible representations in the branching formula (2) with equivalent admissible ones. For the group $\text{Sp}(2k)$ the modification rule is as follows.

Let $\mu = (p, \mu_2', \ldots, \mu_p')$, i.e., let $\mu$ have $p$ rows with $p > k$. Then the following equivalence formula is derived in [11]:

$$V_{\mu}^{2k} = (-1)^x V_{(\sigma)}^{2k},$$

where the Young diagram of $\sigma$ is obtained from the Young diagram of $\mu$ by the removal of a continuous boundary hook of length $2p-n-2$ starting from the bottom box of the first column of the Young diagram of $\mu$. Here $x$ denotes the depth of the hook, i.e., the number of columns in the hook. We then repeat this process of removal of a continuous boundary hook until we obtain an admissible Young diagram corresponding to a partition $\mu$. We write zero for the multiplicity of $V_{\mu}^{2k}$ in the branching formula if the process stops before we obtain an admissible Young diagram (because $2p-n-2 = 0$) or we obtain a configuration of boxes which is not a Young diagram. The latter happens if for the columns of the configuration corresponding to $\sigma$ the rule $\sigma_1' \geq \sigma_2' \geq \cdots \geq \sigma_p'$ is violated.

Proposition 2.2. No inadmissible representation of $\text{Sp}(2k)$ is equivalent to the trivial one-dimensional $\text{Sp}(2k)$-representation.

Proof. Let $V_{\mu}^{2k}$ be an irreducible $\text{Sp}(2k)$-module corresponding to an inadmissible representation of $\text{Sp}(2k)$. Let us assume that, starting with the Young diagram of the partition $\mu$ and removing continuous boundary hooks, we obtain the admissible Young diagram without boxes corresponding to the partition $(0, \ldots, 0)$. Hence, one step before the end of the process we shall reach a partition $\nu = (\nu_1, 1, \ldots, 1)$ with $p > k$ parts. Since $\nu$ will disappear in the next step, its Young diagram has exactly $2p-2k-2$ boxes, i.e., $\nu_1 + p - 1 = 2p - 2k - 2$ and $\nu_1 = p - 2k - 1 = p - n - 1 < 0$ because $\nu$ is a partition in $p \leq n$ parts. This contradiction shows that $V_{\mu}^{2k}$ cannot be equivalent to the trivial one-dimensional $\text{Sp}(2k)$-module $V_{(0,\ldots,0)}^{2k}$. □

Corollary 2.3. Let $V_{\lambda}^{2k}$ be any irreducible $\text{GL}(2k)$-module. Then the one-dimensional trivial $\text{Sp}(2k)$-module enters the decomposition of $V_{\lambda}^{2k}$ over $\text{Sp}(2k)$ with multiplicity one if and only if $\lambda$ is a partition with even columns, i.e., $\lambda_1 = \lambda_2, \lambda_3 = \lambda_4, \ldots,$
\( \lambda_{2k-1} = \lambda_{2k} \). For all other partitions \( \lambda \) the \( \text{GL}(2k) \)-module \( V_{\lambda}^{2k} \) does not contain a one-dimensional trivial \( \text{Sp}(2k) \)-module.

**Proof.** By Propositions 2.1 and 2.2 and the modification rules stated between them, it is sufficient to calculate in (2) the Littlewood-Richardson coefficient \( c_{\mu(2\delta)'}^{\lambda} \) for the partition \( \mu = (0, \ldots, 0) \) and to show that

\[
    c_{\mu(2\delta)'}^{\lambda} = \begin{cases} 
        1, & \text{if } \lambda = (2\delta)' \\
        0, & \text{otherwise.}
    \end{cases}
\]

In order to calculate \( c_{\mu(2\delta)'}^{\lambda} \), we start with the diagram of \( (2\delta)' \) and add to it the boxes of the diagram of \( \mu \) to obtain the diagram of \( \lambda \) filling in the boxes from \( \mu \) with integers following the Littlewood-Richardson rule (see, e.g., [12]). Then \( c_{\mu(2\delta)'}^{\lambda} \) is equal to the possible ways to do these fillings in. Since the diagram of \( \mu \) has no boxes, the only diagram we obtain, and exactly once, is the diagram of \( \lambda \) when \( \lambda_1 = \lambda_2, \lambda_3 = \lambda_4, \ldots, \lambda_{2k-1} = \lambda_{2k} \).

3. Decomposition of irreducible \( \text{GL}(n) \)-modules over \( \text{O}(n) \) and \( \text{SO}(n) \)

In this section we answer the following question: When do the trivial one-dimensional \( \text{O}(n) \)- and \( \text{SO}(n) \)-modules enter the decomposition of \( V_{\lambda}^{\omega} \)? We use a similar approach as in Section 2. We start with the description of the structure of \( V_{\lambda}^{\omega} \) as an \( \text{O}(n) \)-module. The irreducible representations of \( \text{O}(n) \) are indexed by partitions \( \mu \) with \( \mu_1' + \mu_2' \leq n \), i.e., the sum of the lengths of the first two columns of the Young diagram of \( \mu \) should be at most \( n \) (see, e.g., [6, 7]). We denote the corresponding \( \text{O}(n) \)-modules by \( V_{[\mu]}^{\omega} \). With these notations the following Littlewood-Richardson rule holds.

**Proposition 3.1.** [8, 11] Let \( \lambda \) be a partition with at most \( n/2 \) parts. Then

\[
    V_{\lambda}^{\omega} \downarrow \text{O}(n) \cong \bigoplus_{\mu, 2\delta} c_{\mu(2\delta)}^{\lambda} V_{[\mu]}^{\omega},
\]

where the sum runs over all partitions \( \mu \) and all even partitions \( 2\delta = (2\delta_1, \ldots, 2\delta_n) \).

When \( \lambda \) has more than \( n/2 \) parts the above branching formula does not hold and we use again the modification rules from [11]. A module \( V_{[\mu]}^{\omega} \) of \( \text{O}(2k) \) or \( \text{O}(2k+1) \) and its corresponding representation are called inadmissible if \( \mu = (p, \mu_2', \ldots, \mu_q')' \) and \( p > k \), i.e., if the first column in the Young diagram of \( \mu \) has more than \( k \) boxes. For inadmissible \( \text{O}(n) \)-representations the following equivalence formula holds (see [11]):

\[
    V_{[\mu]}^{\omega} = (-1)^x \varepsilon V_{[\sigma]}^{\omega},
\]

where the Young diagram of \( \sigma \) is obtained from the Young diagram of \( \mu \) by the removal of a continuous boundary hook of length \( 2p - n \) starting in the first column of \( \mu \). Here again \( x \) denotes the depth of the hook and \( \varepsilon \) is the determinant of the matrix of the particular group element from \( \text{O}(n) \) acting on \( V_{\lambda}^{\omega} \). As before, we repeat the process of removal of a continuous boundary hook until we obtain an admissible Young diagram or until we obtain a configuration of boxes which is not a Young diagram. In the latter case we write zero in the branching formula. Using this modification rule and repeating the arguments in the proof of Proposition 2.2 we obtain the following statement.
Proposition 3.2. No inadmissible representation of $O(n)$ is equivalent to the trivial one-dimensional $O(n)$-representation.

Then, as in Section 2, we obtain the description of the $GL(n)$-modules $V^n_\lambda$ which contain the trivial $O(n)$-module $V^n_{(0,\ldots,0)}$.

Corollary 3.3. Let $V^n_\lambda$ be any irreducible $GL(n)$-module. Then the one-dimensional trivial $O(n)$-module enters the decomposition of $V^n_\lambda$ over $O(n)$ with multiplicity one if and only if $\lambda$ is an even partition. For all other partitions $\lambda$ the $GL(n)$-module $V^n_\lambda$ does not contain the $O(n)$-module $V^n_{(0,\ldots,0)}$.

When we consider the subgroup $SO(n)$, then $\varepsilon = 1$ for any group element in the equivalence formula (3). Furthermore, all irreducible $O(n)$-modules $V^n_{\mu}$ remain irreducible when restricted to $SO(n)$, except for the case $n = 2k$ and $\mu = (\mu_1,\ldots,\mu_k,0,\ldots,0)$ with $\mu_k \neq 0$. Such representations split into two irreducible $SO(n)$-representations. Using these considerations we make the following observation.

Proposition 3.4. The only inadmissible $SO(n)$-module which is equivalent to the one-dimensional trivial $SO(n)$-module is $V^n_{n}$ with $\mu = (1,1,\ldots,1)$.

Corollary 3.5. Let $V^n_\lambda$ be any irreducible $GL(n)$-module. Then the one-dimensional trivial $SO(n)$-module enters the decomposition of $V^n_\lambda$ over $SO(n)$ with multiplicity one if and only if $\lambda$ is either an even or an odd partition.

Proof. In view of Propositions 3.1 and 3.4 we only need to evaluate the Littlewood-Richardson coefficient $c^{\lambda}_{\mu(2\delta)}$ for $\mu = (0,\ldots,0)$ and $\mu = (1,1,\ldots,1)$. For $\mu = (0,\ldots,0)$ this is trivial.

When $\mu = (1,1,\ldots,1)$ we use the following Pieri rule (see, e.g., [6]): $c^{\lambda}_{\mu(2\delta)} = 1$ if and only if we can obtain $\lambda$ from $2\delta$ by adding one box to each row. In all other cases $c^{\lambda}_{\mu(2\delta)} = 0$. In other words, the only possibility for $\lambda$ is $\lambda = (2\delta_1 + 1,\ldots,2\delta_n + 1)$. Thus the statement follows. \qed

4. Determining the Hilbert series

The goal of this section is to determine the Hilbert series $H(\mathbb{C}[W]^G,t)$ for $G = O(n), SO(n)$, and $Sp(2k)$ by using Hilbert series of multigraded algebras. We recall that if

$$A = \bigoplus_{\mu \in \mathbb{N}_0^n} A(\mu)$$

is an algebra with an $\mathbb{N}_0^n$-grading, then the Hilbert series of $A$ with respect to this grading is defined by

$$H(A,x_1,\ldots,x_n) = \sum_{\mu = (\mu_1,\ldots,\mu_n) \in \mathbb{N}_0^n} \dim A(\mu)x_1^{\mu_1}\cdots x_n^{\mu_n}.$$ 

One example of an algebra with an $\mathbb{N}_0^n$-grading is the $GL(n)$-module $V^n_\lambda$ together with its weight space decomposition. The Hilbert series of $V^n_\lambda$ with respect to this grading has the form

$$H(V^n_\lambda,x_1,\ldots,x_n) = S_\lambda(x_1,\ldots,x_n),$$
where $S_{\lambda}(x_1, \ldots, x_n)$ is the Schur polynomial corresponding to the partition $\lambda$. Consequently, any polynomial $GL(n)$-module $W$ has an $\mathbb{N}_0^n$-grading and a corresponding Hilbert series which is again expressed via Schur polynomials.

Let $W$ be any polynomial $GL(n)$-module. The symmetric algebra $S(W)$ has the following decomposition into irreducible $GL(n)$-modules

\[(4) \quad S(W) = \bigoplus_{l \geq 0} S^l W = \bigoplus_{l \geq 0} \bigoplus_{\lambda} m_l(\lambda) V_{\lambda}^n,\]

where the second sum runs over all partitions $\lambda \in \mathbb{N}_0^n$. Thus $S(W)$ possesses a natural $\mathbb{N}_0$-grading coming from the decomposition into homogeneous components and a natural $\mathbb{N}_0^n$-grading coming from the weight space decomposition of each $V_{\lambda}^n$. As in [1], we consider the following Hilbert series of $S(W)$, which takes into account both gradings

\[H(S(W); X, t) = \sum_{l \geq 0} H(S^l W, x_1, \ldots, x_n)t^l = \sum_{l \geq 0} \left( \sum_{\lambda} m_l(\lambda) S_{\lambda}(X) \right)t^l,\]

where $X = (x_1, \ldots, x_n)$ and $S_{\lambda}(X)$ denotes again the Schur polynomial corresponding to the partition $\lambda$. Furthermore, as in [1] again, we introduce the following two multiplicity series of $H(S(W); X, t)$:

\[M(H(S(W))); X, t) = \sum_{l \geq 0} \left( \sum_{\lambda} m_l(\lambda) X^\lambda \right)t^l,\]

\[M'(H(S(W)); v_1, \ldots, v_n, t) = \sum_{l \geq 0} \left( \sum_{\lambda} m_l(\lambda) v_1^{\lambda_1} v_2^{\lambda_2} \cdots v_n^{\lambda_n} \right)t^l.\]

The second multiplicity series is obtained from the first one using the change of variables

$v_1 = x_1, v_2 = x_1 x_2, \ldots, v_n = x_1 \cdots x_n.$

The following theorem is the main tool to calculate the Hilbert series of the algebras of invariants $\mathbb{C}[W]^G$ for $G = Sp(2k), O(n), SO(n)$, which is done in the next sections:

**Theorem 4.1.** Let $W$ be as above.

(i) The Hilbert series of the algebra of invariants $\mathbb{C}[W]^{Sp(2k)}$ (where $n = 2k$) is given by

\[H(\mathbb{C}[W]^{Sp(2k)}, t) = M'(H(S(W)); 0, 1, 0, 1, \ldots, 0, 1, t).\]

(ii) The Hilbert series of the algebra of invariants $\mathbb{C}[W]^{O(n)}$ is

\[H(\mathbb{C}[W]^{O(n)}, t) = M_n(t),\]

where $M_n$ is defined iteratively in the following way:

\[M_1(x_2, \ldots, x_n, t) = \frac{1}{2} \left( M(H(S(W)); -1, x_2, \ldots, x_n, t) + M(H(S(W)); 1, x_2, \ldots, x_n, t) \right),\]

\[M_2(x_3, \ldots, x_n, t) = \frac{1}{2} \left( M_1(-1, x_3, \ldots, x_n, t) + M_1(1, x_3, \ldots, x_n, t) \right)\]

\[\ldots \ldots \]

\[M_n(t) = \frac{1}{2} \left( M_{n-1}(-1, t) + M_{n-1}(1, t) \right).\]

(iii) The Hilbert series of the algebra of invariants $\mathbb{C}[W]^{SO(n)}$ is

\[H(\mathbb{C}[W]^{SO(n)}, t) = M'_n(t),\]
Thus, using Corollary 2.3, we obtain

\[ M'_1(v_2, \ldots, v_n, t) = \frac{1}{2}(M'(H(S(W)); -1, v_2, \ldots, v_n, t) + M'(H(S(W)); 1, v_2, \ldots, v_n, t)), \]

\[ M'_2(v_3, \ldots, v_n, t) = \frac{1}{2}(M'_1(-1, v_3, \ldots, v_n, t) + M'_1(1, v_3, \ldots, v_n, t)) \]

\[ \ldots \ldots \]

\[ M'_{n-1}(v_n, t) = \frac{1}{2}(M'_{n-2}(-1, v_n, t) + M'_1(1, v_n, t)), \]

\[ M'_n(t) = M'_{n-1}(1, t). \]

**Proof.** Since the graded algebras \( \mathbb{C}[W]^G \) and \( S(W)^G \) are isomorphic and \( H(\mathbb{C}[W]^G, t) = H(S(W)^G, t) \) when \( G = \text{Sp}(2k), O(n), \text{SO}(n) \), we shall work in \( S(W)^G \) instead of \( \mathbb{C}[W]^G \).

(i) Equation (1) implies

\[ S(W)^{\text{Sp}(2k)} = \bigoplus_{\ell \geq 0} \bigoplus_{\lambda} m_\ell(\lambda)(V_\lambda^{2\ell})^{\text{Sp}(2k)}. \]

Then, using Corollary 2.3 we obtain

\[ S(W)^{\text{Sp}(2k)} = \bigoplus_{\ell \geq 0} \bigoplus_{\lambda_1 = \lambda_2, \ldots, \lambda_{2k-1} = \lambda_{2k}} m_\ell(\lambda)(V_\lambda^{2\ell})^{\text{Sp}(2k)}. \]

Thus,

\[ H(\mathbb{C}[W]^{\text{Sp}(2k)}, t) = H(S(W)^{\text{Sp}(2k)}, t) = \sum_{\ell \geq 0} \left( \sum_{\lambda_1 = \lambda_2, \ldots, \lambda_{2k-1} = \lambda_{2k}} m_\ell(\lambda) \right) t^\ell. \]

Moreover, if we evaluate the multiplicity series \( M'(H(S(W)); v_1, \ldots, v_k, t) \) at the point \((v_1, v_2, \ldots, v_{2k-1}, v_{2k}) = (0, 1, \ldots, 0, 1)\) we obtain

\[ M'(H(S(W)); 0, 1, \ldots, 0, 1, t) = \sum_{\ell \geq 0} \left( \sum_{\lambda_1 = \lambda_2, \ldots, \lambda_{2k-1} = \lambda_{2k}} m_\ell(\lambda) \right) t^\ell = \]

\[ H(S(W)^{\text{Sp}(2k)}, t) = H(\mathbb{C}[W]^{\text{Sp}(2k)}, t). \]

(ii) Similarly, for \( \mathbb{C}[W]^{O(n)} \) we obtain, using Corollary 2.3

\[ \mathbb{C}[W]^{O(n)} \cong S(W)^{O(n)} = \bigoplus_{\ell \geq 0} \bigoplus_{\lambda \text{ an even partition}} m_\ell(\lambda)(V_\lambda^{2\ell})^{O(n)}. \]

This implies

\[ H(\mathbb{C}[W]^{O(n)}, t) = H(S(W)^{O(n)}, t) = \sum_{\ell \geq 0} \left( \sum_{\lambda \text{ an even partition}} m_\ell(\lambda) \right) t^\ell = M_n(t), \]

where \( M_n \) is defined as in the statement of Theorem 4.1 (ii).

(iii) For \( \mathbb{C}[W]^{SO(n)} \) we obtain, using Corollary 3.3

\[ H(\mathbb{C}[W]^{SO(n)}, t) = H(S(W)^{SO(n)}, t) = \sum_{\ell \geq 0} \left( \sum_{\lambda \text{ an even or odd partition}} m_\ell(\lambda) \right) t^\ell = M'_n(t), \]
where $M'_n$ is defined as in the statement of Theorem 4.1 (iii).

5. Examples

In this section we use our results from Section 4 to compute the Hilbert series $H(\mathbb{C}[W]^{\text{Sp}(2k)}, t)$, $H(\mathbb{C}[W]^{O(n)}, t)$, and $H(\mathbb{C}[W]^{SO(n)}, t)$ for several explicit $GL(n)$-modules $W$. Using the results we obtain, we can determine in each case the minimal degree of the generators of the respective algebras of invariants and in particular the number of generators of lowest degree. However, in general, we cannot give complete information about how many generators there are and in what degrees and we cannot determine if there are relations between them. Nevertheless, the knowledge of the Hilbert series simplifies the solution of the problem to find a presentation of the algebra in terms of generators and defining relations (see, e.g., [3]). We point out when, to our knowledge, the respective example is already known in the literature, and in particular when the algebra $\mathbb{C}[W]^G$ is a polynomial algebra.

In the cases we consider below the input data is the decomposition (1) of the $GL(n)$-module $W$ into a sum of irreducible submodules. Then the Hilbert series of $W$ is

$$H(W, x_1, \ldots, x_n) = \sum_{\lambda} k(\lambda) S_{\lambda}(x_1, \ldots, x_n) = \sum_{\mu=(\mu_1, \ldots, \mu_n) \in \mathbb{N}_0^n} a_{\mu} x_1^{\mu_1} \cdots x_n^{\mu_n}$$

and the Hilbert series of $S(W)$ as a multigraded algebra is

$$H(S(W), x_1, \ldots, x_n) = \prod_{\mu=(\mu_1, \ldots, \mu_n) \in \mathbb{N}_0^n} \frac{1}{(1 - x_1^{\mu_1} \cdots x_n^{\mu_n})^{a_{\mu}}}.$$

In the first three examples we know in advance the decomposition of $S(W)$ into irreducible $GL(n)$-modules, described in (1). Hence, we can easily determine the multiplicity series $M$ and $M'$. In all three cases the algebra of invariants $\mathbb{C}[W]^G$, for $G = SO(n)$ or $\text{Sp}(2k)$, is known to be a polynomial algebra and the degrees of its generators are given in [10] for irreducible $W$ and in [14] for reducible $W$. It follows that $\mathbb{C}[W]^{O(n)}$ is also a polynomial algebra in these three cases and the expressions for the respective Hilbert series which we find below are enough to describe all generators of $\mathbb{C}[W]^{O(n)}$.

Example 5.1. Let $V = \mathbb{C}^n$ denote the standard $GL(n)$-module and let $W = S^2V$ be the second symmetric power of $V$. In other words, $W = V^{\lambda}_\Lambda$ with $\lambda = (2, 0, \ldots, 0)$. The decomposition (1) is known in this case (see, e.g., [7]) and is given by

$$S(S^2V) = \bigoplus_{\ell \geq 0} \bigoplus_{|\lambda|=2\ell} V^{\lambda}_{\Lambda_{\text{even}}}.$$

Thus the multiplicity series $M(H(S(S^2V)); x_1, \ldots, x_n, t)$ and $M'(H(S(S^2V)); v_1, \ldots, v_n, t)$ are, respectively, equal to (see also [11]):

$$M(H(S(S^2V)); x_1, \ldots, x_n, t) = \prod_{i=1}^{n} \frac{1}{1 - (x_1 \cdots x_i)^2 t^i},$$

$$M'(H(S(S^2V)); v_1, \ldots, v_n, t) = \prod_{i=1}^{n} \frac{1}{1 - v_i^2 t^i}.$$
Using Theorem 4.1, we obtain

\[ H(\mathbb{C}[S^2V]^{Sp(2k)}, t) = \prod_{i=1}^{k} \frac{1}{1 - t^{2i}}, \text{ where } n = 2k, \]

(5)

\[ H(\mathbb{C}[S^2V]^{O(n)}, t) = H(\mathbb{C}[S^2V]^{SO(n)}, t) = \prod_{i=1}^{n} \frac{1}{1 - t^i}. \]

In [10] it is shown that the algebra of invariants \( \mathbb{C}[S^2V]^{Sp(2k)} \) is a polynomial algebra with \( k \) generators in degrees respectively 2, 4, \ldots, 2k.

For \( \mathbb{C}[S^2V]^{SO(n)} \) we use, in the notations of Section 3, that

\[ S^2V \downarrow SO(n) \cong V_{(2,0, \ldots, 0)}^{n} \oplus V_{(0,0, \ldots, 0)}^{n}. \]

Hence, it follows from [10] that \( \mathbb{C}[S^2V]^{SO(n)} \) is a polynomial algebra too with \( n \) generators in degrees respectively 1, 2, \ldots, \( n \). Therefore, \( \mathbb{C}[S^2V]^{O(n)} \) is also a polynomial algebra and the formula (5) implies that \( \mathbb{C}[S^2V]^{O(n)} \) and \( \mathbb{C}[S^2V]^{SO(n)} \) have the same sets of generators.

**Example 5.2.** Next, let us take \( W = \Lambda^2V \), the second exterior power of \( V \). Then \( W = V_{\lambda}^\circ \) with \( \lambda = (1, 1, 0, \ldots, 0) \) and it is known that

\[ S(\Lambda^2V) = \bigoplus_{\lambda \geq 0} \bigoplus_{\lambda} V_{\lambda}^n, \]

where the second sum runs over all partitions \( \lambda \) with \( |\lambda| = 2l \) and such that \( \lambda_{2i-1} = \lambda_{2i} \) for \( i = 1, \ldots, [n/2] \). When \( n \) is odd we also have \( \lambda_n = 0 \) (see, e.g., [7]). Then for the multiplicity series \( M \) and \( M' \) one obtains (see also [10])

\[ M(H(S(\Lambda^2V))); x_1, \ldots, x_n, t) = \prod_{i=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{1}{1 - (x_1 \cdots x_{2i})t^i}, \]

\[ M'(H(S(\Lambda^2V))); v_1, \ldots, v_n, t) = \prod_{i=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{1}{1 - v_{2i}t^i}. \]

Thus,

\[ H(\mathbb{C}[\Lambda^2V]^{Sp(2k)}, t) = \prod_{i=1}^{k} \frac{1}{1 - t^{2i}}, \]

\[ H(\mathbb{C}[\Lambda^2V]^{O(n)}, t) = \prod_{i=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{1}{1 - t^i}, \]

\[ H(\mathbb{C}[\Lambda^2V]^{SO(n)}, t) = \begin{cases} \prod_{i=1}^{k} \frac{1}{1 - t^{2i}}, & \text{if } n = 2k, \\ \frac{1}{1 - t^k} \prod_{i=1}^{k-1} \frac{1}{1 - t^{2i}}, & \text{if } n = 2k + 1. \end{cases} \]

It is known (see, e.g., [10]) that the algebra of invariants \( \mathbb{C}[\Lambda^2V]^G \) for \( G = SO(n) \) is a polynomial algebra with the following generators: if \( n = 2k + 1 \) there are \( k \) generators in degrees respectively 2, 4, \ldots, 2k; if \( n = 2k \) there are \( k-1 \) generators in degrees respectively 2, 4, \ldots, 2(k-1) and one generator in degree \( k \). Therefore, the
algebra $\mathbb{C}[A^2V]^G$ for $G = O(n)$ is also a polynomial algebra and the above formula implies that there are $\lfloor \frac{n}{2} \rfloor$ generators in degrees $2i$ for $i = 1, \ldots, \lfloor \frac{n}{2} \rfloor$.

For $\mathbb{C}[A^2V]^{Sp(2k)}$ we use, in the notations of Section 7 we find that

$$\Lambda^2V \downarrow Sp(2k) \cong V_n^{(1,1,0,\ldots,0)} \oplus V_n^{(0,0,\ldots,0)}.$$ 

Therefore, it follows from (10) that $\mathbb{C}[A^2V]^{Sp(2k)}$ is also a polynomial algebra with $k$ generators in degrees respectively 1, 2, \ldots, $k$.

**Example 5.3.** Let $W = V \oplus \Lambda^2V$. The decomposition (11) of $S(W)$ can be found in the following way (see also [1] and, in the language of symmetric functions, [12, the second edition, page 76, Example 4])

$$S(V \oplus \Lambda^2V) = S(V) \otimes S(\Lambda^2V) = \bigoplus_{\lambda} V^n_{\lambda},$$

where the last sum is over all partitions $\lambda \in \mathbb{N}_0^n$. Hence, for the multiplicity series we obtain (see also [1])

$$M'(H(S(W)); v_1, \ldots, v_n, t) = \prod_{2 \leq n} \frac{1}{(1 - v_{2i-1}t^i)(1 - v_{2i}t^i)}, \text{ for } n = 2k,$$

$$M'(H(S(W)); v_1, \ldots, v_n, t) = \prod_{2 < n} \frac{1}{(1 - v_{2i-1}t^i)(1 - v_{2i}t^i)}, \text{ for } n = 2k+1.$$

Therefore, when $n = 2k$ we obtain

$$H(\mathbb{C}[W]^{Sp(2k)}, t) = \prod_{i \leq k} \frac{1}{1 - t^i},$$

$$H(\mathbb{C}[W]^{O(2k)}, t) = \prod_{i \leq k} \frac{1}{(1 - t^i)^2},$$

$$H(\mathbb{C}[W]^{SO(2k)}, t) = \frac{1}{(1 - t^{2k})(1 - t^k)} \prod_{i \leq k-1} \frac{1}{(1 - t^i)^2}. $$

For $n = 2k + 1$ we have

$$H(\mathbb{C}[W]^{O(2k+1)}, t) = \frac{1}{1 - t^{n+1}} \prod_{i \leq k} \frac{1}{(1 - t^i)^2},$$

$$H(\mathbb{C}[W]^{SO(2k+1)}, t) = \frac{1}{1 - t^{k+1}} \prod_{i \leq k} \frac{1}{(1 - t^i)^2}. $$

It is shown in [13] that $\mathbb{C}[W]^{SO(2k)}$ and $\mathbb{C}[W]^{SO(2k+1)}$ are polynomial algebras and, moreover, that up to adding trivial summands $W$ is a maximal representation with this property. (Schwarz calls such representations maximally coregular). The algebra $\mathbb{C}[W]^{SO(2k)}$ has $2k$ generators, two in each of the degrees $2, 4, \ldots, 2(k-1)$ and two more generators – one in degree $k$ and one in degree $2k$. The algebra $\mathbb{C}[W]^{SO(2k+1)}$ has $2k + 1$ generators – two in each of the degrees $2, 4, \ldots, 2k$ and one generator in degree $k + 1$. Therefore, we can conclude that the algebras $\mathbb{C}[W]^{O(2k)}$ and $\mathbb{C}[W]^{O(2k+1)}$ are also polynomial algebras and we can determine the degrees of the generators from the above expressions of the respective Hilbert series. For $\mathbb{C}[W]^{O(2k)}$ we obtain that there are $2k$ generators, two in each of the
degrees $2, 4, \ldots, 2k$ and for $\mathbb{C}[W]^{O(2k+1)}$ there are $2k + 1$ generators – two in each of the degrees $2, 4, \ldots, 2k$ and one generator in degree $2k + 2$.

For the polynomiality of $\mathbb{C}[W]^{\text{Sp}(2k)}$ we may use [4] again, or we may show it directly as follows. Corollary 2.3 and the equation (6) imply that $\mathbb{C}[W]^{\text{Sp}(2k)} = \mathbb{C}[\Lambda^2 V]^{\text{Sp}(2k)}$. Therefore, $\mathbb{C}[W]^{\text{Sp}(2k)}$ is a polynomial algebra with $k$ generators in degrees $1, 2, \ldots, k$.

Even if the decomposition (4) is not known, for fixed and not very big values of $n$ we can still determine the Hilbert series of $M$. In order to do this we use an algorithm described in [1] for determining the multiplicity series $M$ and $M'$. Below we give several explicit examples.

**Example 5.4.** In this example we take $n = 2$ and $W = S^3 V$. Then by [1]

$$M(H(S(S^3 V)), x_1, x_2, t) = \frac{1 - x_1^3 x_2 t + x_1^4 x_2^2 t^2}{(1 - x_1^3 t)(1 - x_2 x_1^2 t)(1 - x_1^6 x_2^3 t^4)},$$

$$M'(H(S(S^3 V)), v_1, v_2, t) = \frac{1 - v_1 v_2 t + v_1^4 v_2^2 t^2}{(1 - v_1 t)(1 - v_2 v_1 t)(1 - v_2 v_1 t)(1 - v_2 t^4)}.$$

Therefore,

$$H(\mathbb{C}[S^3 V]^{\text{Sp}(2)}, t) = \frac{1}{1 - t^4},$$

$$H(\mathbb{C}[S^3 V]^{O(2)}, t) = \frac{1}{(1 - t^3)^2(1 - t^4)},$$

$$H(\mathbb{C}[S^3 V]^{\text{SO}(2)}, t) = \frac{1 + t^4}{(1 - t^2)^2(1 - t^4)}.$$

The above results imply that the algebra $\mathbb{C}[S^3 V]^{\text{Sp}(2)}$ has one generator which is of degree $4$; the algebra $\mathbb{C}[S^3 V]^{O(2)}$ has exactly two generators in degree $2$ and at least one generator in degree $4$; and the algebra $\mathbb{C}[S^3 V]^{\text{SO}(2)}$ has two generators in degree $2$ and at least two generators in degree $4$.

Since $\text{Sp}(2) = \text{SL}(2)$, it is of course well-known that the algebra $\mathbb{C}[S^3 V]^{\text{Sp}(2)}$ is a polynomial algebra in one variable generated by the discriminant of cubic polynomials, i.e., there is a natural choice of the generator in this case.

**Example 5.5.** Let again $n = 2$ and $W = S^4 V$. Then in [1] it is shown that

$$M(H(S(S^4 V)), x_1, x_2, t) = \frac{1 - x_1^3 x_2 t + x_1^4 x_2^2 t^2}{(1 - x_1^3 t)(1 - x_2 x_1^2 t)(1 - x_1^6 x_2^3 t^4)(1 - x_1^6 x_2^3 t^4)},$$

$$M'(H(S(S^4 V)), v_1, v_2, t) = \frac{1 - v_1 v_2 t + v_1^4 v_2^2 t^2}{(1 - v_1 t)(1 - v_2 v_1 t)(1 - v_2 v_1 t)(1 - v_2 v_1 t)(1 - v_2 t^3)}.$$

Therefore,

$$H(\mathbb{C}[S^4 V]^{\text{Sp}(2)}, t) = \frac{1}{(1 - t^2)(1 - t^3)};$$

$$H(\mathbb{C}[S^4 V]^{O(2)}, t) = \frac{1}{(1 - t)(1 - t^2)^2(1 - t^3)};$$

$$H(\mathbb{C}[S^4 V]^{\text{SO}(2)}, t) = \frac{1 + t^3}{(1 - t)(1 - t^2)^2(1 - t^3)}.$$

Thus, the algebra $\mathbb{C}[S^4 V]^{\text{Sp}(2)}$ has one generator in degrees $2$ and $3$; the algebra $\mathbb{C}[S^4 V]^{O(2)}$ has one generator in degree $1$, at least two generators in degree $2$, and
at least one generator in degree 3; and the algebra \( \mathbb{C}[S^4V]^{SO(2)} \) has one generator in degree 1, two generators in degree 2, and at least two generators in degree 3.

It follows from [24], by using again the isomorphism \( \text{Sp}(2) = \text{SL}(2) \), that \( \mathbb{C}[S^4V]^{\text{Sp}(2)} \) is a polynomial algebra with two generators in degrees 2 and 3, respectively.

**Example 5.6.** Let \( n = 3 \) and \( W = S^3V \). Then using the formulas for the multiplicity series given in [1] we obtain that

\[
H(\mathbb{C}[S^3V]^{O(3)}, t) = \frac{(1 + t)^4(1 + t^8)(1 + t^2 + t^4 + 3t^6 + 5t^8 + 3t^{10} + t^{12} + t^{14} + t^{16})}{(1 - t^2)(1 - t^4)^2(1 - t^{10})}.
\]

For the Hilbert series of the algebra \( \mathbb{C}[S^3V]^{SO(3)} \) we obtain using again [1]

\[
H(\mathbb{C}[S^3V]^{SO(3)}, t) = \frac{t^{14} + t^{13} - 2t^{11} + t^9 + 5t^8 + 5t^7 + 5t^6 + t^5 - 2t^3 + t + 1}{(1 - t^3)^2(1 - t^5)(1 - t^2)^2(1 - t^4)(1 + t)}.
\]

6. **The Algebra of Invariants \( \Lambda(S^2V)^G \) for \( G = O(n), SO(n), \text{Sp}(2k) \)**

As a further application of our results and methods developed in Sections 2, 3, and 4 in this and the next section we determine the Hilbert series \( H(\Lambda(S^2V)^G, t) \) and \( H(\Lambda(S^2V)^G, t) \) for \( G = O(n), SO(n), \text{Sp}(2k) \). Here again \( V = \mathbb{C}^n \) denotes the standard representation of \( \text{GL}(n) \). Since the modules \( \Lambda(S^2V) \) and \( \Lambda(S^2V) \) are finite dimensional, the Hilbert series of the respective algebras of invariants are polynomials.

For convenience, for the rest of the section we denote \( W = \Lambda(S^2V) \). The decomposition of \( W \) into irreducible \( \text{GL}(n) \)-modules can be derived using, e.g., the formulas in [12] the second edition, page 79, Example 9 (b)]. We obtain

\[
W = \bigoplus_{\lambda} V_\lambda^n,
\]

where the sum runs over all partitions \( \lambda = (\alpha_1 + 1, \ldots, \alpha_p + 1|\alpha_1, \ldots, \alpha_p) \) in the Frobenius notation with \( \alpha_1 \leq n - 1 \). If we take into account also the \( \mathbb{N}_0 \)-grading of \( W \), given by the decomposition into irreducible components, we obtain

\[
W = \bigoplus_{i=0}^{n(n+1)/2} \Lambda^i(S^2V) = \bigoplus_{i=0}^{n(n+1)/2} \bigoplus_{|\lambda|=2i} V_\lambda^n,
\]

where again the sum is over all partitions \( \lambda = (\alpha_1 + 1, \ldots, \alpha_p + 1|\alpha_1, \ldots, \alpha_p) \) in the Frobenius notation with \( \alpha_1 \leq n - 1 \). Notice that the condition \( \alpha_1 \leq n - 1 \) implies \( i \leq n(n + 1)/2 \).

Using our results from Sections 2, 3, and 4 we obtain the following expressions.
For $u$ then we set $\alpha$ from Section 4. Notice that since $|H|$ in the definition of $\alpha$ with partitions of the form $\lambda$ and define the polynomial $\lambda|H|\alpha$, we obtain after some computations

$$H(W^{SO(n)}, t) = \sum_{i \geq 0} \left( \sum_{\lambda = (\alpha_1 + 1, \ldots, \alpha_p + 1 | \alpha_1, \ldots, \alpha_p)} \frac{1}{1} t^i \right),$$

$$H(W^{Sp(2k)}, t) = \sum_{i \geq 0} \left( \sum_{\lambda = (\alpha_1 + 1, \ldots, \alpha_p + 1 | \alpha_1, \ldots, \alpha_p)} \frac{1}{1} t^i \right),$$

where $\lambda'$ denotes the transpose partition to $\lambda$.

We recall that if we have a partition $\lambda$ of the form $\lambda = (\alpha_1 + 1, \ldots, \alpha_p + 1 | \alpha_1, \ldots, \alpha_p)$ then necessarily $\alpha_1 > \alpha_2 > \ldots > \alpha_p \geq 0$. Therefore partitions of the type $\lambda = (\alpha_1 + 1, \ldots, \alpha_p + 1 | \alpha_1, \ldots, \alpha_p)$ are in one-to-one correspondence with partitions of the form $\alpha = (\alpha_1, \ldots, \alpha_p)$ with $p$ distinct parts. Moreover, $|\lambda| = 2|\alpha| + 2p$.

To determine the above three Hilbert series we fix one $p$, set $X = \{x_1, \ldots, x_p\}$ and define the polynomial

$$H_p(X, t) = \sum_{i \geq 0} \left( \sum_{\alpha = (\alpha_1, \ldots, \alpha_p)} x_1^{\alpha_1 - (p-1)} x_2^{\alpha_2 - (p-2)} \ldots x_p^{\alpha_p} \right) t^i.$$  

The polynomial $H_p(X, t)$ is in some sense an analogue of the multiplicity series from Section 4. Notice that since $\alpha$ is a partition with distinct parts, all exponents in the definition of $H_p(X, t)$ are non-negative integers.

We rewrite the above polynomial in the form

$$H_p(X, t) = \sum_{i \geq 0} x_1^{-(p-1)} x_2^{-(p-2)} \ldots x_p^{-1} t^i \left( \sum_{\alpha = (\alpha_1, \ldots, \alpha_p)} x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_p^{\alpha_p} \right) t^i.$$  

Then we set $u_1 = x_1 t, u_2 = x_1 x_2 t^2, \ldots, u_p = x_1 \ldots x_p t^p$ and obtain

$$H_p(X, t) = \frac{t^{p+1}/2}{u_1 \cdots u_p-1} \sum_{i \geq 0} \sum_{\alpha = (\alpha_1, \ldots, \alpha_p)} u_1^{\alpha_1-2} u_2^{\alpha_2-3} \ldots u_p^{\alpha_p-1} u_p^{\alpha_p}.$$  

Now we notice that the polynomial $H_p(X, t)$ is the $(n-p)$-th partial sum of the power series

$$H_p^{\text{inf}}(X, t) = \frac{t^{p+1}/2}{u_1 \cdots u_p-1} \sum_{i \geq 0} \sum_{\alpha = (\alpha_1, \ldots, \alpha_p)} u_1^{\alpha_1-2} u_2^{\alpha_2-3} \ldots u_p^{\alpha_p-1} u_p^{\alpha_p}.$$  

For $H_p^{\text{inf}}(X, t)$ we obtain after some computations

$$H_p^{\text{inf}}(X, t) = \frac{t^{p+1}/2}{u_k^{p+1}} \prod_{k=1}^{p} \frac{1}{1 - u_k}.$$
Using the change of variables \( v_1 = x_1, v_2 = x_1 x_2, \ldots, v_p = x_1 \cdots x_p \) we have

\[
H_p(X, t) = H'_p(V, t) = \sum_{i \geq 0} \left( \sum_{\alpha = (\alpha_1, \ldots, \alpha_p)} v_1^{\alpha_1 - \alpha_2 - 1} v_2^{\alpha_2 - \alpha_3 - 1} \cdots v_{p-1}^{\alpha_{p-1} - \alpha_p - 1} v_p^{\alpha_p} \right) t^i
\]

and

\[
H_p^{\text{inf}}(X, t) = (H'_p)^{\text{inf}}(V, t) = \sum_{i \geq 0} \left( \sum_{\alpha = (\alpha_1, \ldots, \alpha_p)} v_1^{\alpha_1 - \alpha_2 - 1} v_2^{\alpha_2 - \alpha_3 - 1} \cdots v_{p-1}^{\alpha_{p-1} - \alpha_p - 1} v_p^{\alpha_p} \right) t^i.
\]

Hence,

\[
(H_p)^{\text{inf}}(V, t) = \frac{t^{\frac{p(p+1)}{2}}}{\prod_{k=1}^{p} 1 - v_k t^k}.
\]

As we mentioned above, the polynomial \( H_p(X, t) \) consists of all terms from \( H_p^{\text{inf}}(X, t) \) of the form \( t^{\frac{p(p+1)}{2}} u_1^{\alpha_1} \cdots u_p^{\alpha_p} \) such that \( a_1 + \cdots + a_p \leq n - p \). Therefore, using (10), we obtain that the polynomial \( H'_p(V, t) \) consists of all terms from \( (H'_p)^{\text{inf}}(V, t) \) of the form \( t^{\frac{p(p+1)}{2}} v_1^{a_1} \cdots v_p^{a_p} t^{a_1 + 2 a_2 + \cdots + p a_p} \) with \( a_1 + \cdots + a_p \leq n - p \).

In other words,

\[
H'_p(V, t) = \sum_{a_1 + \cdots + a_p \leq n - p} v_1^{a_1} \cdots v_p^{a_p} t^{\frac{p(p+1)}{2} + a_1 + 2 a_2 + \cdots + p a_p}.
\]

Next, we come back to determining the Hilbert series \( H(W^O(n), t) \). We have the following proposition.

**Proposition 6.1.**

\[
H(W^O(n), t) = 1 + t + \sum_{p=2}^{n} \begin{cases} 
\frac{t^{\frac{p(p+1)}{2}}}{2} & \sum_{a_1 + \cdots + a_p \leq n - p} t^{2 a_1 + 4 a_2 + \cdots + p a_p} \\
& a_1, \ldots, a_{p-2} \text{ even, } a_{p-1} \text{ odd} \\
\frac{t^{\frac{p(p+1)}{2}}}{2} & \sum_{a_1 + \cdots + a_p \leq n - p} t^{2a_1 + 4 a_2 + \cdots + (p-1)a_{p-1}} \\
& a_1, \ldots, a_{p-1} \text{ even} \end{cases}
\]

**Proof.** A partition \( \lambda = (\alpha_1 + 1, \ldots, \alpha_p + 1 | \alpha_1, \ldots, \alpha_p) \) is even if and only if either the following three conditions hold

- \( p \) is even;
- \( \alpha_1, \alpha_3, \ldots, \alpha_{p-1} \) are even;
- \( \alpha_1 - \alpha_2 = 1, \alpha_3 - \alpha_4 = 1, \ldots, \alpha_{p-1} - \alpha_p = 1. \)

or

- \( p \) is odd;
- \( \alpha_1, \alpha_3, \ldots, \alpha_{p-1} \) are even and \( \alpha_p = 0; \)
- \( \alpha_1 - \alpha_2 = 1, \alpha_3 - \alpha_4 = 1, \ldots, \alpha_{p-2} - \alpha_{p-1} = 1. \)
Therefore (7) implies

\[
H(W^{O(n)}, t) = \sum_{i \geq 0} \left( \sum_{p=0}^{n} \sum_{\lambda=(\alpha_1, \ldots, \alpha_p) \atop \alpha_1 \leq n-1, |\lambda|=\alpha, \alpha \text{ an even partition}} \frac{1}{(1-t^i)^{\alpha_1+\cdots+\alpha_p+1}} \right) t^i = \sum_{i \geq 0} \left( \sum_{p=0}^{n} \sum_{\alpha=1-p} \frac{1}{(1-t^i)^{\alpha}} \right) t^i,
\]

where the last sum runs over partitions \(\alpha = (\alpha_1, \ldots, \alpha_p)\) with distinct parts and such that the above conditions hold.

We rewrite the above as

\[
\sum_{i \geq 0} \left( \sum_{p=0}^{n} \sum_{\alpha=1-p} \frac{1}{(1-t^i)^{\alpha}} \right) t^i = \sum_{p=0}^{n} \sum_{\alpha=1-p} \frac{1}{(1-t^i)^{\alpha}} t^i.
\]

Next, we fix one even and non-zero \(p\) and consider the polynomial \(H'_p(V, t)\). We notice that the monomial \(v_1^{\alpha_1-\alpha_2-1}v_2^{\alpha_2-\alpha_3-1} \ldots v_p^{\alpha_p-1-\alpha_{p-1}-1} v_{p+1}^{\alpha_{p+1}}\) evaluated at the point \((0, v_2, 0, v_4, \ldots, 0, v_p)\) is non-zero if and only if \(\alpha_1 - \alpha_2 = 1, \alpha_3 - \alpha_4 = 1, \ldots, \alpha_{p-1} - \alpha_p = 1\). Therefore we set

\[
M_p(v_2, v_4, \ldots, v_p, t) = H'_p(0, v_2, 0, v_4, \ldots, 0, v_p, t)
\]

\[
= \sum_{i \geq 0} \left( \sum_{\alpha=(\alpha_1, \ldots, \alpha_p) \atop \alpha_1 \leq n-1, |\alpha|=i-p} v_2^{\alpha_1-\alpha_2-2}v_4^{\alpha_3-\alpha_4-2} \ldots v_{p-2}^{\alpha_{p-3} - \alpha_{p-1}-2} v_p^{\alpha_{p-1}-1} \right) t^i.
\]

Next, \(\alpha_1, \alpha_3, \ldots, \alpha_{p-1}\) are even numbers if and only if all exponents in the above expression except the last one are even numbers and \(\alpha_{p-1} - 1\) is odd. Therefore we can define iteratively

\[
M_p^{(1)}(v_4, \ldots, v_p, t) = \frac{1}{2} (M_p(1, v_4, \ldots, v_p, t) + M_p(-1, v_4, \ldots, v_p, t))
\]

\[
\ldots
\]

\[
M_p^{(p-1)}(v_p, t) = \frac{1}{2} (M_p^{(p-2)}(1, v_p, t) + M_p^{(p-2)}(-1, v_p, t)).
\]

Finally, we define

\[
M_p^{(p/2)}(t) = \frac{1}{2} (M_p^{(p/2-1)}(1, t) - M_p^{(p/2-1)}(-1, t)).
\]

The next step is to consider the case when \(p\) is an odd number and \(p > 1\). Then the monomial \(v_1^{\alpha_1-\alpha_2-1}v_2^{\alpha_2-\alpha_3-1} \ldots v_{p-1}^{\alpha_{p-1} - \alpha_p - 1} v_p^{\alpha_p}\) evaluated at the point \((0, v_2, 0, v_4, \ldots, 0, v_{p-1}, 0)\) is non-zero if and only if \(\alpha_1 - \alpha_2 = 1, \alpha_3 - \alpha_4 = 1, \ldots,
\[ \alpha_{p-2} - \alpha_{p-1} = 1 \] and \( \alpha_p = 0 \). Therefore we set

\[ N_p(v_2, v_4, \ldots, v_{p-1}, t) = H'_p(0, v_2, 0, v_4, \ldots, 0, v_{p-1}, 0, t) \]

\[
= \sum_{i \geq 0} \left( \sum_{\alpha=(\alpha_1, \ldots, \alpha_p) \atop \alpha_1 \leq \alpha_2 = 1, \ldots, \alpha_{p-2} - \alpha_{p-1} = 1, \alpha_p = 0} \frac{1}{\alpha!} v^{\alpha_1-2} v_4^{\alpha_3-2} \cdots v_{p-1}^{\alpha_p-2} \right) t^i.
\]

We notice again that \( \alpha_1, \alpha_3, \ldots, \alpha_p \) are even numbers if and only if all exponents in the above expression are even numbers. Hence, similarly to the previous case we can define iteratively

\[ N_p^{(1)}(v_4, \ldots, v_{p-1}, t) = \frac{1}{2} (N_p(1, v_4, \ldots, v_{p-1}, t) + N_p(-1, v_4, \ldots, v_{p-1}, t)) \]

\[ \ldots \]

\[ N_p^{(p-2)}(t) = \frac{1}{2} (N_p^{(p-1)}(1, t) + N_p^{(p-1)}(-1, t)). \]

Finally, we consider the cases \( p = 0 \) and \( p = 1 \). For \( p = 0 \) we set \( H'_0(V, t) = 1 \). For \( p = 1 \) we have

\[ H'_1(v_1, t) = \sum_{\alpha_1 = 0}^{n-1} v_1^\alpha t^{\alpha+1}. \]

Since the partition \( \lambda = (\alpha_1 + 1 | \alpha_1) \) is even if and only if \( \alpha_1 = 0 \), we evaluate \( H'_1(v_1, t) \) at the point \( (0, t) \) and obtain \( H'_1(0, t) = t \).

The statement of the proposition follows now from (11) and the observation that

\[ H(W^{O(n)}, t) = H'_0(V, t) + H'_1(0, t) + \sum_{p=2}^{n} \frac{M_p^{(p/2)}(t)}{p-\text{even}} + \sum_{p=3}^{n} N_p^{(p-1)}(t). \]

We determine the Hilbert series \( H(W^{Sp(2k)}, t) \) and \( H(W^{SO(n)}, t) \) in a similar way by using respectively (9) and (8). To determine \( H(W^{Sp(2k)}, t) \) we notice that for a partition \( \lambda = (\alpha_1 + 1, \ldots, \alpha_p + 1 | \alpha_1, \ldots, \alpha_p) \) the transpose \( \lambda' \) is even if and only if

- \( p \) is even;
- \( \alpha_1, \alpha_3, \ldots, \alpha_{p-1} \) are odd numbers;
- \( \alpha_1 - \alpha_2 = 1, \alpha_3 - \alpha_4 = 1, \ldots, \alpha_{p-1} - \alpha_p = 1 \).

Therefore, by fixing \( p \) even and evaluating the series \( H'_p(V, t) \) at well-chosen points we obtain:

**Proposition 6.2.** Let \( n = 2k \). Then

\[ H(W^{Sp(2k)}, t) = 1 + \sum_{p=2}^{n} \frac{t^{p(p+1)}}{p(p-\text{even})} \left( \sum_{\alpha_1 + \cdots + \alpha_n \leq n-p \atop \alpha_1 \text{-even}} t^{2\alpha_1 + 4\alpha_2 + \cdots + p\alpha_n} \right). \]

In the next section we determine the degrees of all generators of \( \Lambda(S^2V)^{Sp(2k)} \) and show that it is isomorphic to an exterior algebra (see Corollary 7.3).
It remains to determine the Hilbert series \( H(W^{SO(n)}, t) \). First we notice that a partition \( \lambda = (\alpha_1 + 1, \ldots, \alpha_p + 1 | \alpha_1, \ldots, \alpha_p) \) is odd if and only if either the following four conditions hold

- \( n \) is even;
- \( p \) is odd;
- \( \alpha_1, \alpha_3, \ldots, \alpha_{p-2}, \alpha_p \) are odd numbers and \( \alpha_1 = n - 1 \);
- \( \alpha_2 - \alpha_3 = 1, \alpha_4 - \alpha_5 = 1, \ldots, \alpha_{p-1} - \alpha_p = 1 \).

or

- \( n \) is even;
- \( p \) is even;
- \( \alpha_1, \alpha_3, \ldots, \alpha_{p-1} \) are odd numbers and \( \alpha_1 = n - 1 \);
- \( \alpha_2 - \alpha_3 = 1, \alpha_4 - \alpha_5 = 1, \ldots, \alpha_{p-2} - \alpha_{p-1} = 1 \) and \( \alpha_p = 0 \).

Then, we fix \( p \) and evaluate the series \( H_p(V, t) \) from (11) at well-chosen points to obtain:

**Proposition 6.3.**

(i) Let \( n = 2k + 1 \). Then

\[
H(W^{SO(n)}, t) = H(W^{O(n)}, t).
\]

(ii) Let \( n = 2k \). Then

\[
H(W^{SO(n)}, t) = 1 + \sum_{p=2}^{n} t^{\frac{p(p+1)}{2}} \left( \sum_{a_1 + \cdots + a_p \leq n-p \atop a_1, \ldots, a_{p-2} \text{ even}, a_{p-1} \text{ odd}} t^{2a_1 + 4a_2 + \cdots + pa_p} \right)
+ \sum_{p=3}^{n} t^{\frac{p(p+1)}{2}} \left( \sum_{a_1 + \cdots + a_p \leq n-p \atop a_1 \text{ even}} t^{2a_1 + 4a_2 + \cdots + (p-1)a_{p-1}} \right)
+ \sum_{p=1}^{n} t^{\frac{p(p+1)}{2}} \left( \sum_{a_1 + \cdots + a_p = n-p \atop a_1, \ldots, a_{p-1} \text{ even}, a_p \text{ odd}} t^{a_1 + 3a_2 + \cdots + pa_p} \right)
+ \sum_{p=2}^{n} t^{\frac{p(p+1)}{2}} \left( \sum_{a_1 + \cdots + a_p = n-p \atop a_1 \text{ even}} t^{a_1 + 3a_2 + \cdots + (p-1)a_p} \right).
\]

**Corollary 6.4.** Let \( n = 2k + 1 \). Then

\[
\Lambda(S^2V)^{O(2k+1)} = \Lambda(S^2V)^{SO(2k+1)}.
\]

**Remark 6.5.** One immediate observation from the results in this section is that the lowest degree of the invariants in \( \Lambda(S^2V)^{Sp(2k)} \) is 3, whereas the lowest degree of the invariants in \( \Lambda(S^2V)^{O(n)} \) and \( \Lambda(S^2V)^{SO(n)} \) is 1. In all three cases there is only one invariant in lowest degree. As we already mentioned, in the next section we discuss in more detail the degrees of invariants of \( \Lambda(S^2V)^{Sp(2k)} \).
7. The algebra of invariants $\Lambda(\Lambda^2 V)^G$ for $G = O(n), SO(n), Sp(2k)$

In this section we set $W = \Lambda(\Lambda^2 V)$. The approach for computing the Hilbert series $H(\Lambda(\Lambda^2 V)^G, t)$ is very similar to the one in the previous section and we shall sketch the proofs only. One remark here is that the algebra of invariants $\Lambda(\Lambda^2 V)^{O(n)}$ is already described in terms of generators and defining relations in [9]. Moreover, the results in [9] show that $\Lambda(\Lambda^2 V)^{O(n)}$ and $\Lambda(\Lambda^2 V)^{SO(n)}$ are isomorphic to exterior algebras.

The decomposition of $W$ into irreducible $GL(n)$-modules can again be determined using, e.g., the formulas from [12] pages 78-79, Example 9 (a). The exact formula is

$$W = \bigoplus_{\lambda} V_{\lambda}^n = \bigoplus_{i=0}^{n(n-1)/2} \bigoplus_{|\lambda|=2i} V_{\lambda}^n,$$

where the sum runs over all partitions $\lambda = (\alpha_1 - 1, \ldots, \alpha_p - 1 | \alpha_1, \ldots, \alpha_p)$ in the Frobenius notation and $\alpha_1 \leq n - 1$. Then for the Hilbert series of the respective algebras of invariants we obtain the following expressions:

$$H(W^{O(n)}, t) = \sum_{i \geq 0} \left( \sum_{\lambda = (\alpha_1 - 1, \ldots, \alpha_p - 1 | \alpha_1, \ldots, \alpha_p)}^{\lambda \leq n - 1, |\lambda|=2i} 1 \right) t^i;$$

$$H(W^{SO(n)}, t) = \sum_{i \geq 0} \left( \sum_{\lambda = (\alpha_1 - 1, \ldots, \alpha_p - 1 | \alpha_1, \ldots, \alpha_p)}^{\lambda \leq n - 1, |\lambda|=2i} 1 \right) t^i;$$

$$H(W^{Sp(2k)}, t) = \sum_{i \geq 0} \left( \sum_{\lambda = (\alpha_1 - 1, \ldots, \alpha_p - 1 | \alpha_1, \ldots, \alpha_p)}^{\lambda \leq n - 1, |\lambda|=2i} \lambda' \text{ an even partition} \right) t^i,$$

where $\lambda'$ denotes the transpose partition to $\lambda$.

We notice that partitions of the type $\lambda = (\alpha_1 - 1, \ldots, \alpha_p - 1 | \alpha_1, \ldots, \alpha_p)$ are in one-to-one correspondence with partitions of the form $\alpha = (\alpha_1, \ldots, \alpha_p)$ with $p$ distinct positive parts. Moreover $|\lambda| = 2|\alpha|$.

As in the previous section we fix $p$, set $X = \{x_1, \ldots, x_p\}$ and define the following analogue of the multiplicity series

$$H_p(X, t) = \sum_{i \geq 0} \left( \sum_{\alpha = (\alpha_1, \ldots, \alpha_p) \geq 0}^{\alpha_1 \leq n - 1, |\alpha|=i} x_1^{\alpha_1-p} x_2^{\alpha_2-(p-1)} \cdots x_p^{\alpha_p-1} \right) t^i.$$

We make the same transformations as in the previous section.

\[ H_p(X, t) = \sum_{i \geq 0} x_{-i} x_2^{-p-1} \cdots x_2^{-1} \left( \sum_{\alpha = (a_1, \ldots, a_p) > 0, a_1 \leq n-1} x_1^{a_1} x_2^{a_2} \cdots x_p^{a_p} \right) t^i. \]

Then we set again \( u_1 = x_1 t, u_2 = x_1 x_2 t^2, \ldots, u_p = x_1 \cdots x_p t^p \) and obtain

\[ H_p(X, t) = \frac{t^{p(p+1)/2}}{u_1 \cdots u_p} \sum_{i \geq 0} \sum_{\alpha = (a_1, \ldots, a_p) > 0, a_1 \leq n-1} u_1^{a_1-1} u_2^{a_2-1} \cdots u_p^{a_p-1} u_i. \]

Now we notice that the polynomial \( H_p(X, t) \) is the \((n-p-1)\)-st partial sum of the power series

\[ H_p^{\text{inf}}(X, t) = \frac{t^{p(p+1)/2}}{u_1 \cdots u_p} \sum_{i \geq 0} \sum_{\alpha = (a_1, \ldots, a_p) > 0, a_1 \leq n-1} u_1^{a_1-1} u_2^{a_2-1} \cdots u_p^{a_p-1} v_i. \]

We derive after some computations that

\[ H_p^{\text{inf}}(X, t) = t^\frac{p(p+1)}{2} \prod_{k=1}^p \frac{1}{1 - u_k}. \]

Using the change of variables \( v_1 = x_1, v_2 = x_1 x_2, \ldots, v_p = x_1 \cdots x_p \) we obtain

\[ H_p(X, t) = H_p'(V, t) = \sum_{i \geq 0} \left( \sum_{\alpha = (a_1, \ldots, a_p) > 0, a_1 \leq n-1} v_1^{a_1-1} v_2^{a_2-1} \cdots v_p^{a_p-1} v_i \right) t^i \]

and

\[ H_p^{\text{inf}}(X, t) = (H_p')^{\text{inf}}(V, t) = \sum_{i \geq 0} \left( \sum_{\alpha = (a_1, \ldots, a_p) > 0} v_1^{a_1-1} v_2^{a_2-1} \cdots v_p^{a_p-1} v_i \right) t^i. \]

Therefore,

\[ (H_p')^{\text{inf}}(V, t) = t^\frac{p(p+1)}{2} \prod_{k=1}^p \frac{1}{1 - v_k}. \]

The polynomial \( H_p(X, t) \) consists of all terms from \( H_p^{\text{inf}}(X, t) \) of the form \( t^\frac{p(p+1)}{2} u_1^{a_1} \cdots u_p^{a_p} \) such that \( a_1 + \cdots + a_p \leq n - p - 1 \). Therefore, the polynomial \( H_p'(V, t) \) consists of all terms from \( (H_p')^{\text{inf}}(V, t) \) of the form \( t^\frac{p(p+1)}{2} v_1^{a_1} \cdots v_p^{a_p} t^{a_1 + 2a_2 + \cdots + pa_p} \) with \( a_1 + \cdots + a_p \leq n - p - 1 \). In other words,

\[ (H_p')^{\text{inf}}(V, t) = \sum_{a_1 + \cdots + a_p \leq n-p-1} v_1^{a_1} \cdots v_p^{a_p} t^\frac{p(p+1)}{2} + a_1 + 2a_2 + \cdots + pa_p. \]

The following propositions now hold.
Proposition 7.1.

\[ H(W^{O(n)},t) = 1 + \sum_{p=2}^{n-1} \sum_{\frac{p(p+1)}{2}, \text{ p-even}} t^{\sum \alpha_i} \sum_{\frac{\sum a_i}{2} \leq n-p-1, a_1, \ldots, a_p \text{ even}} t^{2\alpha_1+4\alpha_2+\ldots+p\alpha_p} \].

Proof. A partition \( \lambda = (\alpha_1 - 1, \ldots, \alpha_p - 1|\alpha_1, \ldots, \alpha_p) \) is even if and only if

- \( p \) is even;
- \( \alpha_1, \alpha_3, \ldots, \alpha_{p-1} \) are even;
- \( \alpha_1 - \alpha_2 = 1, \alpha_3 - \alpha_4 = 1, \ldots, \alpha_{p-1} - \alpha_p = 1 \).

Therefore, fixing \( p \) even and evaluating the polynomial \( H'_p(V,t) \) in (12) at well-chosen points we obtain the desired result. \( \square \)

As we mentioned before, the algebra of invariants \( \Lambda(\Lambda^2 V)^{O(n)} \) is already described in terms of generators and defining relations in [9]. More precisely, when \( n = 2k \), there are \( k-1 \) generators in degrees respectively 3, 7, \ldots, \( 4k-5 \), and when \( n = 2k+1 \), there are \( k \) generators in degrees respectively 3, 7, \ldots, \( 4k-1 \). There are no other relations among the described generators besides anticommutativity. Thus, it is shown in [9] that \( \Lambda(\Lambda^2 V)^{O(n)} \) is isomorphic to an exterior algebra.

Next, we notice that Proposition 6.2 and Proposition 7.1 imply the following corollary.

Corollary 7.2.

\[ H(\Lambda(S^2 V)^{Sp(2k)},t) = H(\Lambda(\Lambda^2 V)^{O(2k+1)},t). \]

Corollary 7.3. The algebra of invariants \( \Lambda(S^2 V)^{Sp(2k)} \) is isomorphic to an exterior algebra with \( k \) generators in degrees 3, 7, \ldots, \( 4k-1 \).

Proof. For any reductive group \( G \), it follows from cohomology theory of Lie algebras that \( \Lambda(g^*)^0 \) is isomorphic to an exterior algebra (see e.g., [9]). In particular, this shows that \( \Lambda(S^2 V)^{Sp(2k)} \) is isomorphic to an exterior algebra. Furthermore, it is a well-known fact that if \( A \) is an exterior algebra generated by \( f_1, \ldots, f_k \), such that \( f_1, \ldots, f_k \) have no other relations among them except anticommutativity, then

\[ H(A,t) = 1 + \sum_{i=1}^{k} \sum_{1 \leq i_1 < \ldots < i_{l} \leq k} t^{\sum_{j=1}^{l} \deg f_{i_j}}. \]

Therefore, the statement follows from Corollary 7.2 and the results for \( \Lambda(\Lambda^2 V)^{O(2k+1)} \) proved in [9] and recalled above. \( \square \)

It remains to consider the algebras \( \Lambda(\Lambda^2 V)^{Sp(2k)} \) and \( \Lambda(\Lambda^2 V)^{SO(n)} \).
Proposition 7.4. Let $n = 2k$. Then

$$H(W^{Sp(2k)}, t) = 1 + t + \sum_{p=2}^{n-1} t^{\frac{p(p+1)}{2}} \left( \sum_{a_1 + \cdots + a_p \leq n-p-1 \atop a_1, \ldots, a_p - \text{even}, a_p - \text{odd}} t^{2a_1 + 4a_2 + \cdots + pa_p} \right)$$

$$+ \sum_{p=3}^{n-1} t^{\frac{p(p+1)}{2}} \left( \sum_{a_1 + \cdots + a_p \leq n-p-1 \atop a_1, \ldots, a_p - \text{even}} t^{2a_1 + 4a_2 + \cdots + (p-1)a_{p-1}} \right).$$

Proof. For a partition $\lambda = (\alpha_1 - 1, \ldots, \alpha_p - 1 | \alpha_1, \ldots, \alpha_p)$ the transpose $\lambda'$ is even if and only if either the following three conditions hold

- $p$ is even;
- $\alpha_1, \alpha_3, \ldots, \alpha_{p-1}$ are odd numbers;
- $\alpha_1 - \alpha_2 = 1, \alpha_3 - \alpha_4 = 1, \ldots, \alpha_{p-1} - \alpha_p = 1$;

or

- $p$ is odd;
- $\alpha_1, \alpha_3, \ldots, \alpha_p$ are odd numbers and $\alpha_p = 1$;
- $\alpha_1 - \alpha_2 = 1, \alpha_3 - \alpha_4 = 1, \ldots, \alpha_{p-2} - \alpha_{p-1} = 1$.

Therefore, by fixing $p$ and evaluating the series $H'_t(V, t)$ from (12) at well-chosen points we obtain the statement of the proposition. \hfill \qed

Corollary 7.5. $H(\Lambda(\Lambda^2 V)^{Sp(2k)}, t) = H(\Lambda(S^2 V)^{O(2k-1)}, t) = H(\Lambda(S^2 V)^{SO(2k-1)}, t)$.

Remark 7.6. Proposition 7.4 implies that the lowest degree of an invariant in $\Lambda(\Lambda^2 V)^{Sp(2k)}$ is 1 and that there is only one invariant in lowest degree.

Proposition 7.7. (i) Let $n = 2k + 1$. Then

$$H(W^{SO(n)}, t) = H(W^{O(n)}, t) = 1 + \sum_{p=2}^{n-1} t^{\frac{p(p+1)}{2}} \left( \sum_{a_1 + \cdots + a_p \leq n-p-1 \atop a_1, \ldots, a_p - \text{even}} t^{2a_1 + 4a_2 + \cdots + pa_p} \right).$$

(ii) Let $n = 2k$. Then

$$H(W^{SO(n)}, t) = 1 + \sum_{p=2}^{n-1} t^{\frac{p(p+1)}{2}} \left( \sum_{a_1 + \cdots + a_p \leq n-p-1 \atop a_1, \ldots, a_p - \text{even}} t^{2a_1 + 4a_2 + \cdots + pa_p} \right)$$

$$+ \sum_{p=1}^{n-1} t^{\frac{p(p+1)}{2}} \left( \sum_{a_1 + \cdots + a_p \leq n-p-1 \atop a_1, \ldots, a_{p-1} - \text{even}} t^{a_1 + 3a_2 + \cdots + pa_{p-1} + 1} \right).$$
Proof. A partition \( \lambda = (\alpha_1 - 1, \ldots, \alpha_p - 1|\alpha_1, \ldots, \alpha_p) \) is odd if and only if the following conditions hold

- \( n \) is even;
- \( p \) is odd;
- \( \alpha_1, \alpha_3, \ldots, \alpha_{p-2}, \alpha_p \) are odd numbers and \( \alpha_1 = n - 1 \);
- \( \alpha_2 - \alpha_3 = 1, \alpha_4 - \alpha_5 = 1, \ldots, \alpha_{p-1} - \alpha_p = 1 \).

Then, we fix \( p \) odd and evaluate the series \( H_p'(V, t) \) from (12) at well-chosen points to obtain the result. \( \square \)

By using again the argument that \( \Lambda(g^\ast)^\theta \) is isomorphic to an exterior algebra, we notice that \( \Lambda(\Lambda^2 V)^{SO(n)} \) is an exterior algebra for all \( n \) (this was already pointed out in [9]). Therefore, Proposition 7.7 implies the following two corollaries.

**Corollary 7.8.** Let \( n = 2k + 1 \). Then

\[
\Lambda(\Lambda^2 V)^{SO(2k+1)} = \Lambda(\Lambda^2 V)^{O(2k+1)}.
\]

**Corollary 7.9.** Let \( n = 2k \). Then \( \Lambda(\Lambda^2 V)^{SO(2k)} \) is an exterior algebra with \( k - 1 \) generators in degrees respectively 3, 7, \ldots, 4k - 5 and one more generator in degree 2k - 1. There are no other relations among the described generators besides anticommutativity.

Proof. To prove the statement we need to show that

\[
H(\Lambda(\Lambda^2 V)^{SO(2k)}, t) = 1 + \sum_{l=1}^{k-1} \sum_{1 \leq i_1 < \ldots < i_l \leq k-1} t^{\sum_{j=1}^l \deg f_{i_j}} + t^{2k-1}(1 + \sum_{l=1}^{k-1} \sum_{1 \leq i_1 < \ldots < i_l \leq k-1} t^{\sum_{j=1}^l \deg f_{i_j}}),
\]

where \( f_1, \ldots, f_{k-1} \) are elements in degrees respectively 3, 7, \ldots, 4k - 5. Proposition 7.1 and the results from [9] imply that

\[
1 + \sum_{p=2}^{n-1} \sum_{p \text{ even}} t^{\frac{p(p+1)}{2}} \left( \sum_{a_1 + \cdots + a_p = n-p-1 \atop a_1, \ldots, a_p \text{ even}} t^{2a_1 + 4a_2 + \cdots + pa_p} \right) = 1 + \sum_{l=1}^{k-1} \sum_{1 \leq i_1 < \ldots < i_l \leq k-1} t^{\sum_{j=1}^l \deg f_{i_j}}.
\]

Therefore, we only need to transform the expression

\[
\sum_{p=1}^{n-1} \sum_{p \text{ odd}} t^{\frac{p(p+1)}{2}} \left( \sum_{a_1 + \cdots + a_p = n-p-1 \atop a_i \text{ even}} t^{a_1 + 3a_2 + \cdots + pa_p + \frac{p+1}{2}} \right).
\]
It is not difficult to see that for $n = 2k$
\[
\sum_{p=1 \atop p=\text{even}}^{n-1} t^{\frac{p(p+1)}{2}} \left( \sum_{a_1+\cdots+a_{\frac{p}{2}}=n-p-1 \atop a_1-\text{even}} t^{a_1+3a_2+\cdots+pa_{\frac{p}{2}}+1} \right)
\]
\[
= t^{2k-1} + t^{2k-1} \left( \sum_{p=2 \atop p=\text{even}}^{n-1} t^{\frac{p(p+1)}{2}} \left( \sum_{a_1+\cdots+a_{\frac{p}{2}}=n-p-1 \atop a_1,\ldots,a_{\frac{p}{2}}-\text{even}} t^{2a_1+4a_2+\cdots+pa_{\frac{p}{2}}} \right) \right),
\]
Thus the statement follows. 

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