Chapter 1

MULTIPARTITE GREENBERGER-HORNE-ZEILINGER PARADOXES FOR CONTINUOUS VARIABLES

Serge Massar and Stefano Pironio
Service de Physique Théorique, CP 225,
Université Libre de Bruxelles, 1050 Brussels, Belgium

Abstract We show how to construct Greenberger-Horn-Zeilinger type paradoxes for continuous variable systems. We give two examples corresponding to 3 party and 5 party paradoxes. The paradoxes are revealed by carrying out position and momentum measurements. The structure of the quantum states which lead to these paradoxes is discussed.

Keywords: Continuous variables, non-locality, Greenberger-Horne-Zeilinger paradox

When studying continuous variables systems, described by conjugate variables with commutation relation \([x, p] = i\), it is natural to inquire how non-locality can be revealed in those systems. Experimentally the operations that are easy to carry out on such systems involve linear optics, squeezing and homodyne detection. Using these operations, the states that can be prepared are Gaussian states and the measurements that can be performed are measurements of quadratures. But Gaussian states possess a Wigner function which is positive everywhere and so provide a trivial local-hidden variable model for measurement of \(x\) or \(p\).

To exhibit non-locality in these systems, it is thus necessary to drop some of the requirements imposed by current day experimental techniques. For instance one can invoke more challenging measurements such as photon counting measurements or consider more general states that will necessitate higher order non-linear couplings to be produced. Using these two approaches it has recently been possible to extend from discrete variables to continuous variables systems the usual non-locality tests: Bell inequalities \([1, 2, 3]\), Hardy’s non-locality proof \([4]\) and the Greenberger-Horne-Zeilinger paradox \([5, 6, 7]\).
Greenberger-Horne-Zeilinger (GHZ) paradoxes [8] as formulated by Mermin [9] are particularly elegant and simple ways of demonstrating the non-locality of quantum systems since the argument can be carried out at the level of operators only. The existence of a generalization of the original GHZ paradox for qubits to continuous variables was first pointed out by Clifton [5] and was studied in more details in [6] and [7]. The paradox presented in [7] involve measurements of the parity of the number of photons, while in [5] and [6], it is associated with position and momentum variables. It is this last case that we will consider here. We shall summarize the results of [6] and show that the multipartite multidimensional GHZ paradoxes introduced in [10] can easily be generalized to the case of continuous variables by exploiting the non-commutative geometry of the phase space. This idea is closely related to the technique used to embed finite-dimensional quantum error correcting code in the infinite-dimensional Hilbert space of continuous variables systems [11].

Let us introduce the dimensionless variables

$$\tilde{x} = \frac{x}{\sqrt{\pi}L} \quad \text{and} \quad \tilde{p} = \frac{p L}{\sqrt{\pi}},$$

(1.1)

where $L$ is an arbitrary length scale. Consider the translation operators in phase space

$$X^\alpha = \exp(i\alpha \tilde{x}) \quad \text{and} \quad Y^\beta = \exp(i\beta \tilde{p}).$$

(1.2)

These unitary operators obey the commutation relation

$$X^\alpha Y^\beta = e^{i\alpha \beta / 2\pi} Y^\beta X^\alpha,$$

(1.3)

which follows from $[\tilde{x}, \tilde{p}] = i/\pi$ and the identity $e^A e^B = e^{[A,B]} e^B e^A$ (valid if $A$ and $B$ commute with their commutator). The continuous variable GHZ paradoxes will be built out of these operators.

Let us first consider the case of three spatially separated parties, A, B, C, each of which possess one part of an entangled system described by the canonical variables $x_A, p_A, x_B, p_B, x_C$ and $p_C$. Consider the operators $X_{j}^{\pm\pi}$ and $Y_{j}^{\pm\pi}$ acting on the space of party $j$ ($j = A, B, C$). Since $\alpha \beta = \pm \pi^2$, it follows from (1.3), that these operators obey the commutations relations $X_{j}^{\pm\pi} Y_{j}^{\pm\pi} = -Y_{j}^{\pm\pi} X_{j}^{\pm\pi}$. Using these operators let us construct the following four GHZ operators:

$$
\begin{align*}
V_1 &= X_A^\pi \quad X_B^\pi \quad X_C^\pi \\
V_2 &= X_A^{-\pi} \quad Y_B^{-\pi} \quad Y_C^\pi \\
V_3 &= Y_A^\pi \quad X_B^{-\pi} \quad Y_C^{-\pi} \\
V_4 &= Y_A^{-\pi} \quad Y_B^\pi \quad X_C^{-\pi}
\end{align*}
$$

(1.4)

These four operators give rise to a GHZ paradox as we now show. First note that the following two properties hold:
1. $V_1, V_2, V_3, V_4$ all commute. Thus they can be simultaneously diagonalized (in fact there exists a complete set of common eigenvectors).

2. The product $V_1 V_2 V_3 V_4 = -I_{ABC}$ equals minus the identity operator.

These properties are easily proven using the commutations relations $X_j^\pm Y_j^\pm = -Y_j^\pm X_j^\pm$. Any common eigenstate of $V_1, V_2, V_3, V_4$ will give rise to a GHZ paradox. Indeed suppose that the parties measure the hermitian operators $x_j$ or $p_j$, $j = A, B, C$ on this common eigenstate. The result of the measurement associates a complex number of unit norm to either the $X_j$ or $Y_j$ unitary operators. If one of the combinations of operators that occurs in eq. (1.4) is measured, a value can be assigned to one of the operators $V_1, V_2, V_3, V_4$. Quantum mechanics imposes that this value is equal to the corresponding eigenvalue. Moreover - due to property 2 - the product of the eigenvalues is $-1$.

But this is in contradiction with local hidden variables theories. Indeed in a local hidden theory one must assign, prior to the measurement, a complex number of unit norm to all the operators $X_j$ and $Y_j$. Then taking the product of the four c-numbers assigned simultaneously to $V_1, V_2, V_3, V_4$ yields $+1$ instead of $-1$.

Remark that all other tests of non-locality for continuous variable systems [1, 2, 3, 4, 7] use measurements with a discrete spectrum (such as the parity photon number) or involving only a discrete set of outcome (such as the probability that $x > 0$ or $x < 0$). In our version of the GHZ paradox for continuous variables this discrete character doesn’t seem to appear at first sight. However it turn out that is is also the case thought in a subtle way because eq. (1.4) can be viewed as an infinite set of 2 dimensional paradoxes (see [6] for more details).

In [10], GHZ paradoxes for many parties and multi-dimensional systems where constructed. These paradoxes where build using $d$-dimensional unitary operators with commutation relations:

$$XY = e^{2\pi i/d} YX$$

which is a generalization of the anticommutation relation of spin operators for two-dimensional systems. Using $X^\alpha$ and $Y^\beta$ and choosing the coefficients $\alpha$ and $\beta$ such that $\alpha \beta = 2\pi^2/d$ with $d$ integer, this commutation relation can be realized in a continuous variable systems and so all the paradoxes presented in [10] can be rephrased with minor modifications in the context of infinite-dimensional Hilbert space.

Let us for instance generalise to continuous variables the paradox for 5 parties each having a 4 dimensional systems described in [10]. We now consider the operators $X^{\pm q}, Y^{q}$ and $Y^{-3q}$ where $q = \pi/\sqrt{2}$. They obey the commutation relation $X^{\pm q} Y^{q} = e^{\pm \pi/2} Y^{q} X^{\pm q}$ and $X^{\pm q} Y^{-3q} = e^{\pm \pi/2} Y^{-3q} X^{\pm q}$. 

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Consider now the six unitary operators

\[
\begin{align*}
W_1 &= X_A^q X_B^q X_C^q X_D^q X_E^q, \\
W_2 &= X_A^{-q} Y_B^{-3q} Y_C^{-q} Y_D^{-q} Y_E^{-q}, \\
W_3 &= Y_A^q X_B^{-q} Y_C^{-3q} Y_D^{-q} Y_E^{-q}, \\
W_4 &= Y_A^q Y_B^{-q} X_C^{-q} Y_D^{-3q} Y_E^{-q}, \\
W_5 &= Y_A^q Y_B^{-q} Y_C^{-q} X_D^{-q} Y_E^{-3q}, \\
W_6 &= Y_A^{-3q} Y_B^q Y_C^{-q} Y_D^{-q} X_E^{-q}
\end{align*}
\]

One easily shows that these six unitary operators commute and that their product is minus the identity operator. Furthermore if one assigns a classical value to \(x_j\) and to \(p_j\) for \(j = A, B, C, D, E\), then the product of the operators takes the value +1. Hence, using the same argument as in the three party case, we have a contradiction.

There is a slight difference between the paradox (1.6) and the 4-dimensional paradox described in [10]. The origin of this difference is that in a \(d\)-dimensional Hilbert space, if unitary operators \(X, Y\) obey \(XY = e^{i\pi/d} YX\), then \(X^d = Y^d = I\) (up to a phase which we set to 1). In the 4-dimensional case, this implies that \(X^3 = X^\dagger\) and \(Y^3 = Y^\dagger\). In the continuous case these relations no longer hold and the GHZ operators \(W_i\)'s must be slightly modified, i.e. the operator \(X^{-q} = X^{q\dagger}\) and \(Y^{-3q} = Y^{3q\dagger}\) have to be explicitly introduced in order for the product of the \(W_i\)'s to give minus the identity. Note that the same remark apply for the previous paradox (1.4) where in the discrete 2-dimensional version \(X^\dagger = X\) and \(Y^\dagger = Y\).

As we mentioned earlier the GHZ state are not Gaussian states. A detailed analysis of the common eigenstates of \(V_1, V_2, V_3, V_4\) is given in [6]. Let us give an example of such an eigenstate. Define the following coherent superpositions of infinitely squeezed states:

\[
\begin{align*}
|\uparrow\rangle &= \frac{1}{\sqrt{2}} \sum_{k=-\infty}^{\infty} \left( |\tilde{x} = 2k\rangle + i|\tilde{x} = 2k + 1\rangle \right), \\
|\downarrow\rangle &= \frac{1}{\sqrt{2}} \sum_{k=-\infty}^{\infty} \left( |\tilde{x} = 2k\rangle - i|\tilde{x} = 2k + 1\rangle \right),
\end{align*}
\]

where \(|\tilde{x}\rangle = |x = \sqrt{\pi} L\tilde{x}\rangle\). Then a common eigenstate of the four GHZ operators \(V_1, V_2, V_3, V_4\) is

\[
(|\uparrow\rangle_A |\uparrow\rangle_B |\uparrow\rangle_C - |\downarrow\rangle_A |\downarrow\rangle_B |\downarrow\rangle_C \big) / \sqrt{2}.
\]

However as shown in [6], for any choice of the eigenvalues of the operators \(V_k\), there is an infinite family of eigenvectors, i.e. the eigenspace is infinitely degenerate.
In summary we have shown the existence of multidimensional and multipartite GHZ paradoxes for continuous variable systems. These paradoxes involve measurements of position and momentum variables only, but the states which are measured are complex and difficult to construct experimentally.

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References

[1] K. Banaszek and K. Wódkiewicz, Phys. Rev. A 58, 4345 (1998).
[2] A. Kuzmich, I. A. Walmsley and L. Mandel, Phys. Rev. Lett. 85, 1349 (2000).
[3] Z. Chen, J. Pan, G. Hou and Y. Zhang, Phys. Rev. Lett. 88 040406 (2002).
[4] B. Yurke, M. Hillery and D. Stoler, Phys. Rev. A 60, 3444 (1999).
[5] R. Clifton, Phys. Lett. A 271, 1 (2000).
[6] S. Massar and S. Pironio, Phys. Rev. A 64 062108 (2001).
[7] Z. Chen and Y. Zhang, Phys. Rev. A 65 044102 (2001).
[8] D. M. Greenberger, M. Horne, A. Zeilinger, in *Bell’s Theorem, Quantum Theory, and Conceptions of the Universe*, M. Kafatos, ed., Kluwer, Dordrecht, The Netherlands (1989), p. 69.
[9] N. D. Mermin, Phys. Rev. Lett. 65, 3373 (1990) and Phys. Today, 43(6), 9 (1990).
[10] N. Cerf, S. Massar, S. Pironio, *Greenberger-Horne-Zeilinger paradoxes for many qudits*, quant-ph/0107031.
[11] D. Gottesman, A. Kitaev and J. Preskill, Phys. Rev. A 64 012310 (2001).