Enhanced quantization is an improved program for overcoming difficulties which may arise during an ordinary canonical quantization procedure. We review here how this program applies for a particle on circle.

Keywords: Quantization, coherent states, periodic coordinates.

1. Introduction

Conventional canonical quantization works very well for many systems but it has also notorious drawbacks. Enhanced Quantization\(^1\) is an improved quantization procedure which has matured through years.\(^2\) It is only recently that the name of “Enhanced Quantization” (EQ) has been found. This procedure yields a new interpretation of the very process of quantization that encompasses the usual canonical formalism and offers additional features as well.

EQ yields a new sense to and improves major points which have been discussed for a long time in the ordinary quantization version. For instance,\(^3\) a) the invariance of the theory under canonical transformations is ensured under EQ; b) EQ may remove singularities in classical solutions (Hydrogen atom and quantum toy models\(^3\) and other simple cosmological models\(^4\)); c) the triviality of certain quantum field models is now traded for a non trivial behavior (pseudo-free theories) and d) EQ preserves the metric positivity in quantum gravity kinematics.

Let us review now the main ingredients of the EQ program.\(^3\)
(i) From pairs \( P \) and \( Q \) of self-adjoint operators (a stronger condition than Hermiticity), one generates unitary operators acting on a fiducial state \( |\eta\rangle \). This provides a set of coherent states
\[
|p, q\rangle = e^{-\frac{i}{\hbar} qP} e^{i p Q} |\eta\rangle
\]
spanning the Hilbert space \( \mathcal{H} \).

(ii) Referring to the action principle, arbitrary variations of \( |\psi(t)\rangle \) for a microscopic system are not accessible to a macroscopic observer who can only change the velocity or position of the microscopic system. Hence, the only state she/he could make are those represented by \( |p(t), q(t)\rangle \). Thus, restricting the quantum action functional
\[
A_Q = \int_0^T \langle \psi(t)| \left[ i\hbar \partial_t - \mathcal{H} \right] |\psi(t)\rangle \, dt
\]
to only coherent states yields
\[
A_{Q(\mathcal{H})} = \int_0^T \langle p(t), q(t)| \left[ i\hbar \partial_t - \mathcal{H} \right] |p(t), q(t)\rangle \, dt
= \int_0^T \left[ p(t) \dot{q}(t) - H_c(p(t), q(t)) \right] \, dt
\]
The restricted action (3) may be called enhanced classical action since it is of a classical form but, because still \( \hbar > 0 \), it includes certain quantum modifications. The usual classical action is given by
\[
A_C = \int_0^T \left[ p(t) \dot{q}(t) - H_c(p(t), q(t)) \right] \, dt,
\]
where \( H_c(p(t), q(t)) = \lim_{\hbar \to 0} H(p(t), q(t)) \).

According to this point of view, enhanced classical theory forms a subset of quantum theory, and they both co-exist just like they do in the real world where \( \hbar > 0 \).

In this paper, we review the EQ program for a classical particle moving on a circle of finite radius. The particle motion on the circle has been studied in different ways but none of previous contributions addresses the issue of the relationship between classical and quantum actions which is our main concern here. The present study leads to yet another set of coherent states which serves to unite the classical and quantum theories for such a system. At the quantum level, we find that the non trivial topology induces a possibly shifted momentum easily reabsorbed by a canonical transformation.
2. Enhanced quantization on the circle

Consider a particle on the circle $S^1$ parametrized by $\theta \in [-\pi, \pi)$. At the quantum level, position and momentum operators $Q$ and $P$, respectively, obey the commutation relation:

$$[Q, P] = i\hbar. \quad (5)$$

The spectrum of the operator $Q$ is bounded in $[-\pi, \pi)$ modulo $2\pi$. The EQ program starts with two self-adjoint operators $P$ and $Q$. Let us investigate the self-adjoint possibilities of the operator $P$.

**Self-adjoint extension of $P$.** Let us review,\(^\text{14}\) the properties of $P = -i\hbar \partial_\theta$ acting on $L^2([-\pi, \pi), d\theta)$.

Call $D(P)$ the domain of $P$. Consider the inner product for any two functions $\psi, \varphi, \psi', \varphi' \in L^2([-\pi, \pi), d\theta)$. If we adopt the boundary conditions $\varphi(\pm \pi) = 0$ and make no restriction on $\psi$, then $D(P) = \{ \varphi; \varphi, \varphi' \in L^2([-\pi, \pi), d\theta); \varphi(\pi) = \varphi(-\pi) = 0 \}$ and $\langle \psi, P\varphi \rangle = \langle P\psi, \varphi \rangle$ holds on $D(P)$. Hence $P$ is symmetric, i.e. $P^\dagger = P$ on $D(P)$. However, the domain of $P^\dagger$ is larger, $D(P^\dagger) = \{ \varphi; \varphi, \varphi' \in L^2([-\pi, \pi), d\theta) \} \supset D(P)$.

Imposing the boundary condition $\varphi(\pi) = e^{2\pi i \alpha} \varphi(-\pi)$, for a given $\alpha \in [0,1)$, enlarges the domain of $P$ and reduces the domain of $P^\dagger$ so that

$$\tilde{D}(P_\alpha) = \{ \varphi; \varphi, \varphi' \in L^2([-\pi, \pi), d\theta); \varphi(\pi) = e^{2\pi i \alpha} \varphi(-\pi) \} = \tilde{D}(P^\dagger_\alpha). \quad (6)$$

**Coherent states.** One denotes $P$ by $P_\alpha$, where $\alpha \in [0,1)$ labels the different inequivalent representations of the momentum operator. We work in units such that $Q$ is dimensionless and so the dimension of $P$ is that of $\hbar$. Define eigenvectors $|\theta\rangle$ for the operator $Q$, obeying $\langle \theta | P | \theta' \rangle = \delta_{S^1}(\theta - \theta')$, where $\delta_{S^1}$ is periodic on $S^1$, as well as eigenvectors $|n, \alpha\rangle$ of $P_\alpha$, satisfying $\langle n, \alpha | m, \alpha \rangle = \delta_{n,m}$, such that

$$Q |\theta\rangle = \theta |\theta\rangle, \quad P_\alpha |n, \alpha\rangle = p_{n,\alpha} |n, \alpha\rangle. \quad (7)$$

One has $\langle \theta | P_\alpha |n, \alpha\rangle = (-i\hbar) \partial_\theta \langle \theta | n, \alpha \rangle = p_{n,\alpha} \langle \theta | n, \alpha \rangle$. The spectrum of $P_\alpha$ on the circle is such that $p_{n,\alpha} = \hbar(n+\alpha), (n, \alpha) \in \mathbb{Z} \times [0,1)$ and corresponds to the normalized wave functions

$$\langle \theta | n, \alpha \rangle = \frac{1}{\sqrt{2\pi}} e^{i(n+\alpha)\theta}. \quad (8)$$

The unitary operators $e^{-\frac{i}{\hbar} q P_\alpha}$ and $e^{-\frac{i}{\hbar} p Q}$, where $(q, p) \in S^1 \times \mathbb{R}$ allow us to define a set of states

$$|p, q\rangle = e^{-\frac{i}{\hbar} q P_\alpha} e^{-\frac{i}{\hbar} p Q} |\eta_\alpha\rangle, \quad (9)$$

where $\eta_\alpha = \{ n, \alpha \}$. The set $\{ |p, q\rangle \}$ is orthonormal and forms a complete basis for $L^2([-\pi, \pi), d\theta)$.
where $|\eta_\alpha\rangle$ is called the fiducial state. One proves that the states (9) are normalized $\langle p,q|p,q \rangle = \langle \eta_\alpha|\eta_\alpha \rangle = 1$, and they satisfy a resolution of unity:

$$\int_{\mathbb{R} \times S^1} |p,q\rangle \langle p,q| \frac{dpdq}{2\pi\hbar} = I_B .$$

(10)

The set of states $\{|p,q\rangle\}$ forms an overcomplete family of normalized states which can be called coherent states.

We can now discuss the dynamics associated with such states by introducing a general quantum Hamiltonian of the form $H(P,e^{iQ},e^{-iQ})$. Consider the restricted quantum action associated with $|\psi(t)\rangle \rightarrow |p(t),q(t)\rangle$ which leads to

$$A_Q(R) = \int_0^T \langle p(t),q(t)|\left[i\hbar \partial_t - H\right]|p(t),q(t)\rangle \, dt .$$

(11)

A class of fiducial vectors $|\eta_\alpha\rangle$ is chosen in the domain of both $Q$ and $P_\alpha$ such that these states satisfy

$$\langle \eta_\alpha|Q|\eta_\alpha \rangle = 0 \quad \text{and} \quad \langle \eta_\alpha|P_\alpha|\eta_\alpha \rangle = \hbar \alpha .$$

(12)

Hence $|\eta_\alpha\rangle$ obeying the condition (12) can be considered analogs of the “physically centered” fiducial vectors $|\eta\rangle$ for the canonical case.

A direct evaluation using (12) yields

$$\langle p(t),q(t)|\left[i\hbar \partial_t\right]|p(t),q(t)\rangle = (\hbar \alpha + p) \dot{q} .$$

(13)

Moreover, one proves that

$$H_\alpha(p(t),q(t)) = \langle \eta_\alpha|H(P_\alpha + p,e^{iQ},e^{-iQ+q})|\eta_\alpha \rangle .$$

(14)

Equations (13) and (14) imply that the restricted quantum action is of the form

$$A_Q(R) = \int_0^T \left\{\left[p \dot{q} - H_\alpha(p(t),q(t))\right] \right\} dt .$$

(15)

One notices that the part $\tilde{A}_C = \int_0^T \left[p \dot{q} - H_\alpha(p(t),q(t))\right] dt$, as in the ordinary situation\textsuperscript{1} can be related to a classical action $A_C$ up to $\hbar$ corrections using

$$H_\alpha(p,q) = H_{c,\alpha}(p,q) + O(\hbar;p,q) .$$

(16)

$H_{c,\alpha}(p,q)$ is viewed as the usual classical Hamiltonian. Note that the quantum parameter $\alpha$ induces a surface term $\hbar \alpha \dot{q}$ in $A_Q(R)$ which makes no influence on the enhanced classical equations of motion whatsoever. Thus, one has

$$A_Q(R) = A_C + O(\hbar) .$$

(17)
Let us apply the above formalism to the particular instance of a particle governed by the following (periodic) dynamics
\begin{equation}
\mathcal{H}(P_\alpha, e^{iQ}, e^{-iQ}) = P_\alpha^2 + V(e^{iQ}, e^{-iQ}),
\end{equation}
\begin{equation}
V(e^{iQ}, e^{-iQ}) = a_0 + \sum_{n=1}^m \left[ a_n \cos nQ + b_n \sin nQ \right],
\end{equation}
in mass units such that $1/2\mu = 1$ and with $m$ a positive integer. As a fiducial vector, we consider
\begin{equation}
\eta_\alpha(\theta) := \langle \theta | \eta_\alpha \rangle = Ne^{(r/\hbar)(\cos \theta - 1) + i\alpha \theta},
\end{equation}
where $r/\hbar > 0$, $I_0(z)$ stands for a modified Bessel function, and $N$ is a normalization fixed by $\langle \eta_\alpha | \eta_\alpha \rangle = 1$. Among its properties, $|\eta_\alpha(\theta)|$ is even, periodic and fulfills (12). For large $r/\hbar \gg 1$ and for $|\theta| \leq \pi$, one makes the approximation
\begin{equation}
|\eta_\alpha(\theta)|^2 = N^2 e^{\frac{2\pi}{\hbar} \sqrt{\cos \theta - 1}} \lesssim KN^2 e^{-r/\hbar},
\end{equation}
for a large constant $K$ implying that $|\theta| \lesssim \sqrt{\hbar/r}$ is small. Therefore, $\eta_\alpha(\theta)$ acts as a large $\theta$-value cut-off. Calculating the diagonal coherent state matrix elements of $\mathcal{H}$, and denoting $\alpha' = \hbar \alpha$, we come to the restricted quantum action given by
\begin{equation}
A_{Q(R)} = \int_0^T \left\{ (p + \alpha') \dot{q} - \left[ (p + \alpha')^2 + V(e^{iQ}, e^{-iQ}) \right] + O(\hbar) \right\} dt.
\end{equation}
Therefore, up to constants and a canonical shift in momentum ($p \rightarrow p - \alpha'$),
\begin{equation}
A_{Q(R)} = \int_0^T \left\{ pq - \left[ p^2 + V(e^{iQ}, e^{-iQ}) \right] + O(\hbar) \right\} dt
= A_C + O(\hbar).
\end{equation}
Evaluating (21), some expectation values (for instance $\langle \eta_\alpha | (P_\alpha + p)^2 | \eta_\alpha \rangle$) contain terms $\alpha'$ of order $O(\hbar)$. We have preferred to remove these terms in a unified way by making a canonical shift of momenta at the last stage (22). Another precision, the quantity $\langle \eta_\alpha | V(e^{i(Q+q)}, e^{-i(Q+q)}) | \eta_\alpha \rangle$ contains, strictly speaking, at the first order of approximation, terms of order $O(\hbar/r)$. These are subtleties without any consequence for the result.
Acknowledgements

The organizers of the XXIXth International Colloquium on Group-Theoretical Methods in Physics, Nankai, China, are warmly thank for their welcome and hospitality. Discussions with John R. Klauder are gratefully acknowledged. Research at Perimeter Institute is supported by the Government of Canada through Industry Canada and by the Province of Ontario through the Ministry of Research and Innovation.

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