ON A CONJECTURE OF HUANG–LIAN–YAU–YU

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Abstract. We verify a formula on the solution rank of the tautological system arising from ample complete intersections in a projective homogeneous space of a semisimple group conjectured by Huang–Lian–Yau–Yu [1]. As an application, we prove the existence of the rank one point for such a system, where mirror symmetry is expected.

Introduction

In this note we verify a formula conjectured by Huang–Lian–Yau–Yu [1]. This is a formula on the solution rank of a tautological system associated with “ample complete intersections” in a projective homogeneous space of a semisimple group.

1. Conventions.

(1) We work with complex algebraic varieties.

(2) The cohomology groups will be taken with respect to sheaves of complex vector spaces, or complexes of sheaves of complex vector spaces.

(3) The notation $H^m(A, B)$ is potentially confusing: it can mean the relative cohomology of $A$ with respect to $B$, or the sheaf cohomology of $A$ with coefficients in $B$. In the situations below, the reader should distinguish the use by recalling the previously set up notation.

2. Let $X$ be an $n$-dimensional, smooth, projective variety with an action of a connected algebraic group $G$. Let $L_1, \ldots, L_r$ be ample invertible sheaves on $X$. For a section $b_i \in \Gamma(X, L_i)$, define $Y_{b_i}$ to be the vanishing scheme of $b_i$. Define $V = \prod_{i=1}^r \Gamma(X, L_i)$.

When $r = 1$, and $L_1 = \omega_X^{-1}$, a $\mathcal{D}_V$-module, the tautological system, was introduced by Lian–Song–Yau [3] in order to study the periods of Calabi–Yau hypersurfaces in $X$. We shall not need the precise definition of tautological systems. The upshot is that among its solutions are “period integrals” $\int_{\Omega}$, where $\Omega$ is some canonically defined differential [4]. We also mention that tautological systems specialize to the Gelfand–Kapranov–Zelevinsky $A$-hypergeometric systems when $G$ is a torus.

For general $r$, set $\mathbb{P} = \text{Proj}(\text{Sym}^*(L_1 \oplus \cdots \oplus L_r))$. Assume further that $L_i$ are $G$-equivariant for all $i$ and $L_1 \otimes \cdots \otimes L_r = \omega_X^r$. Then $\mathbb{P}$ admits an action of $G \times \mathbb{G}_m^{r-1}$, and the sheaf $\mathcal{O}_{\mathbb{P}}(r) = \omega_{\mathbb{P}}^r$. Hence, one can define a tautological system for $\mathbb{P}$. This system is what we refer to as the tautological system of complete intersections.

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When $X$ is a projective homogeneous variety of a semisimple group $G$, Huang, Lian, Yau, and Yu were able to give a cohomological interpretation (see (3.2) below) of the solution space of this tautological system. They subsequently formulated the following conjecture, which we shall verify in this note.

3. Theorem (Conjecture of Huang–Lian–Yau–Yu). Let notation be as above. Assume that $X$ is a projective homogeneous variety of a semisimple group $G$. The solution space of the tautological system $\tau$ at a point $b = (b_1, \ldots, b_r) \in V$ is identified with the homology group

$$H_n(X - \bigcup_{i=1}^r Y_{b_i}).$$

The proof we present is somehow unrelated to the theory of tautological systems, but uses the cohomological interpretation of Huang–Lian–Yau–Yu to verify their conjecture. Huang–Lian–Yau–Yu proved that the said solution space can be identified with

$$H_{n+r-1}(U_b, U_b \cap D),$$

where $U_b$ is the complement of the hypersurface in $\mathbb{P}$ determined by $b$, and $D$ the union of fiber-wise coordinate axes. Our work is to prove (3.1) and (3.2) are naturally isomorphic. The proof of the isomorphism between these groups uses some common methods in singularity theory and eventually reduces to a simple problem in combinatorics.

In the last section, we give an application of the main theorem. We prove the existence of a certain special points, called “rank one points” in the moduli space of Calabi–Yau intersections in a Grassmannian. These are the points where the solution rank of the Picard–Fuchs equations equals one. That is, all, but one, period integrals of the canonical differential $\Omega$ (of the Calabi–Yau complete intersections) become singular. These points are the candidates of the so-called “large radius limits” in the story of mirror symmetry.

For the ease of exposition we shall be using cohomology instead of homology. Everything about homology can be deduced by taking duality.

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Some general facts about cohomology

4. This paragraph discusses how to compute relative cohomology. The notation we use shall be in agreement of the notation employed later in the note. Let $E$ be a smooth complex manifold. Let $D = \bigcup_{i \in I} D_i$ be a simple normal crossing divisor on $E$. Let $i : D \to E$ be the inclusion map. For $J \subset I$, define $D_J = \bigcap_{j \in J} D_j$. Define $D^{(m)} = \prod_{#J=m} D_J$. Let $i_m : D^{(m)} \to E$ be the natural map. Then there is
we conclude that we have an exact sequence
\[ i_\ast C_D \to [i_{1\ast} C_{D(1)} \to i_{2\ast} C_{D(2)} \to \cdots] \]

Let \( j : U \to E \) be the open embedding of the complement of \( D \). Then there is an exact sequence
\[ 0 \to j_! C_U \to C_E \to i_\ast C_D \to 0. \]

It follows from (4.1) that we have an exact sequence
\[ j_! C_U \to C_E \to i_{1\ast} C_{D(1)} \to i_{2\ast} C_{D(2)} \to \cdots. \]

Let \( u : F \to E \) be a closed embedding of a complex submanifold of \( E \). Let \( v : U \cap F \to F \) be the inclusion, and let \( F_i = D^{(i)} \times_E F = \bigsqcup_{j \in J_i} F \cap D_j \). As an abuse of notation, we shall denote the induced map \( F_m \to F \) also by \( i_m \). By the exactness of \( u^{-1} \) we get an exact sequence
\[ u^{-1} j_! C_U \to C_F \to u^{-1} i_{1\ast} C_{D(1)} \to \cdots. \]

By proper base change, we know
\[ u^{-1} j_! C_U = v_! C_{U \cap F}. \]

Since \( i_m \) is proper, \( i_{m \ast} = i_{m!} \). By proper base change we also have \( u^{-1} i_{m \ast} = i_{m \ast} \).

we conclude that we have an exact sequence
\[ v_! C_{U \cap F} \to C_F \to i_{1 \ast} C_{F_1} \to \cdots. \]

By definition, \( H^m(F, F \cap D) = H^m(F, v_! C_{U \cap F}) \). Thus we deduce that
\[ H^m(F, F \cap D) = H^m(F, C_F \to i_{1 \ast} C_{F_1} \to \cdots). \]

5. Let \( B \) be a smooth complex algebraic variety of pure dimension \( n \). Let \( L_1, \ldots, L_r \) be invertible sheaves on \( B \). Let
\[ \mathbb{P} = \text{Proj}(\text{Sym}^* (L_1 \oplus \cdots \oplus L_r)) \]

be the projective space bundle associated with the direct sum of the \( L_i \)'s. Let \( E \) be the geometric vector bundle associated with \( L_1^\vee \oplus \cdots \oplus L_r^\vee \). Then \( E - \zeta(B) \), where \( \zeta : B \to E \) is the zero section, is a \( \mathcal{G}_m \)-torsor on \( \mathbb{P} \).

Since \( E \) is a direct sum of line bundles, it makes sense to talk about whether the \( i \)th factor of a point on \( E \) is zero. Let \( D_i \) be the divisor of \( E \) consisting of these points. The projection \( D_i \to B \) is the geometric vector bundle associated with the direct sum \( \bigoplus_{k \neq i} L_k^\vee \).

More generally, for a subset \( J \subset \{1, 2, \ldots, r\} \), we define \( D_J = \bigcap_{j \in J} D_j \). The projection \( D_J \to B \) is thus the geometric vector bundle associated with the direct sum \( L_J = \bigoplus_{k \notin J} L_k \).

6. Let the notation be as in (5). Let \( b = (b_1, \ldots, b_r) \in H^0(B, \bigoplus_{i=1}^r L_i) \) be a nonzero section.

(1) The section \( b \) gives rise to a section of the invertible sheaf \( \mathcal{O}_\mathbb{P}(1) \) on \( \mathbb{P} \). The nonvanishing locus of this section is denoted by \( U_b \subset \mathbb{P} \). Then \( U_b \) is an algebraic variety of pure dimension \( n + r - 1 \).

(2) Let \( D_i \subset \mathbb{P} \) be the image of \( D_i \).
(3) The section \( b \) gives rise to a function \( \varphi_b : E \to \mathbb{A}^1 \) by fiber-by-fiber pairing:

\[
\varphi(b)(\ell_1, \ldots, \ell_r) = \sum_{i=1}^{r} (\ell_i, b_i(x)).
\]

Let \( F_b = \varphi_b^{-1}(1) \). We mention in passing that \( F_b \) is an analogue of the Milnor fiber of a quasi-homogeneous singularity.

(4) Let \( E_i \) be the closed subvariety of \( F_b \) defined by the vanishing of \( \ell_i \). Thus the \( E_i \)'s are the intersection of \( F_b \) with a collection of divisors on \( E \) with normal crossings.

(5) Let \( F_1 = \prod E_i, \quad F_2 = \prod E_i \cap E_j, \quad \text{and so on.} \)

(6) Let \( Y_i \) be the hypersurface in \( B \) defined by the vanishing locus of \( b_i \). Let \( U_i \) be the complement of \( Y_i \). Let \( U_b = \bigcup_{i=1}^{r} U_i \)

**Lemma 7.** The projection \( F_b \to \mathbb{P} \) induces an isomorphism \( F_b \cong U_b \) of algebraic varieties. Hence the pair \((F_b, \bigcup E_i)\) is isomorphic to the pair \((U_b, U_b \cap \mathbb{D})\).

*Proof.* The problem being local, we can assume that the \( L_i \)'s are trivial bundle on \( B \). Then the \( b_i \)'s are given by a collection of regular functions on \( B \). In this case, \( F_b \) is given by

\[
\{(x, \ell_1, \ldots, \ell_r) \in B \times \mathbb{A}^r : \sum b_i(x)\ell_i = 1\}
\]

whereas

\[
U_b = \{(x, [u_1, \ldots, u_r]) \in B \times \mathbb{P}^{r-1} : \sum b_i(x)u_i \neq 0\}.
\]

It is easy to see then that the natural map \( F_b \to U_b \) sending \((x, \ell_1, \ldots, \ell_r)\) to \((x, [\ell_1, \ldots, \ell_r])\) is well-defined and is an isomorphism of algebraic varieties. \( \square \)

The lemma above shows that the validity of Theorem \ref{thm:main} will follow from the following proposition.

**Proposition 8.** There is an isomorphism

\[
H^m(F_b, \bigcup E_i) \cong H^{m-r+1}(U_1 \cap \cdots \cap U_r).
\]

The first step of the proof the proposition needs the explicit form of Cayley’s trick.

**Lemma 9.** The projection \( F_b \to U_b \) is an \( \mathbb{A}^{r-1} \)-bundle. In particular, \( F_b \) and \( U_b \) have the same homotopy type.

*Proof.* The problem being local, we can assume the \( L_i \)'s are trivial. Then \( F_b \) is defined by \( \sum b_i(x)\ell_i = 1 \). If \( U_k \) is the open subset of \( X \) consisting of \( x \in B \) such that \( b_k(x) \neq 0 \). Then \( U_k \subset U_b \) and we can define an open embedding

\[
\varphi_k : U_k \times \mathbb{A}^{r-1} \to F_b
\]

by

\[
(x, \ell_1, \ldots, \ell_{k-1}, \ell_k, \ldots, \ell_r) \mapsto \left(x, \ell_1, \ldots, \ell_{k-1}, \frac{1 - \sum_{i \neq k} b_i(x)\ell_i}{b_k(x)}, \ell_{k+1}, \ldots, \ell_r\right).
\]

Clearly the union of the images of \( \varphi_k \) equals \( F_b \). This completes the proof. \( \square \)

Now we turn to the proof of Proposition \ref{prop:8}.
The proof

10. In the sequel, in order to avoid using the notion of “complexes in the derived category”, we shall insist on working with the abelian category of complex of sheaves. Cohomology groups are taken only in the last step. We shall also now employ the notation of §4, with $F = F_b$ (the other notation are compatible with the situation of §4). The groups $H^\bullet(F_b, \bigcup E_i)$ are the hypercohomology groups of the complex

$$C_F \rightarrow i_1^*C_F_1 \rightarrow i_2^*C_F_2 \rightarrow \cdots.$$ 

Let $\pi : F_b \rightarrow U_b$ be the natural projection, which is a smooth, affine morphism that is a torsor of a vector bundle. Then the above hypercohomology can be computed on $U_b$ by taking direct images:

$$R\pi_*C_F \rightarrow R\pi_*i_1^*C_F_1 \rightarrow R\pi_*i_2^*C_F_2 \rightarrow \cdots. \tag{10.1}$$

Again, we recapitulate that the items

$$R\pi_*i_k^*C_F_k \tag{10.2}$$

are to be thought as complexes of sheaves, and the above displayed equation should be thought as a double complex, or a complex in the abelian category of complexes of sheaves. Precisely, we fix an injective resolution $I^\bullet$ of $C_{E_i_1 \cap \cdots \cap E_i_s}$ and the item (10.2) is regarded as the complex $\pi_*I^\bullet$.

Let $J$ be a subset of $\{1, 2, \ldots, r\}$. Let $\overline{J}$ be the complement of $J$ in $\{1, 2, \ldots, r\}$. Let $U_J = \bigcup_{k \in J} U_k$. Then we know from Lemma 9 that $E_i_1 \cap \cdots \cap E_i_s \rightarrow U_{\overline{\{i_1, \ldots, i_s\}}}$ is an affine space bundle, hence the direct image of the fixed injective resolution of $C_{E_i_1 \cap \cdots \cap E_i_s}$ becomes an injective resolution of $C_{U_{\overline{\{i_1, \ldots, i_s\}}}}$. It follows that the complex (10.1) is of the form

$$C_{U_b} \rightarrow \bigoplus_{#J=1} Rj_{1J}^*C_{U_{\overline{\{i_1, \ldots, i_s\}}}} \rightarrow \bigoplus_{#J=2} Rj_{1J}^*C_{U_{\overline{\{i_1, \ldots, i_s\}}}} \rightarrow \cdots \rightarrow Rj_{11}^*C_{U_1} \oplus \cdots \oplus Rj_{rs}^*C_{U_r}. \tag{10.3}$$

where $j_J$ is the inclusion of $U_J$ into $U_b$.

We have reduced our problem to a topological one. Thus we will work in the category of locally compact, Hausdorff, topological spaces. The situation is that we are given an open covering $U_b = U_1 \cup \cdots \cup U_r$ of topological spaces, and let $\iota : U_1 \cap \cdots \cap U_r \rightarrow U_b$ be the open immersion of the deepest intersection. Proving Proposition 8 reduces to proving the (double) complex (10.3) and the complex $R\iota_*C_{U_1 \cap \cdots \cap U_r} \cap [1-r]$ have the same hypercohomology.

Notation. For the later combinatorial manipulation, from now on we use $(i_1, \ldots, i_m)$ to denote the complex $Rj_{i_1, \ldots, i_m}C_{U_{i_1, \ldots, i_m}}$.

More generally, let

$$(i_1^{(1)} \cap \cdots \cap i_k^{(1)}, i_1^{(2)} \cap \cdots \cap i_k^{(2)}, \ldots, i_m^{(1)} \cap \cdots \cap i_m^{(m)})$$

be the direct image, to $U_b$, of the constant sheaf on the open subset

$$(U_{i_1^{(1)}} \cap \cdots \cap U_{i_k^{(1)}}) \cup \cdots \cup (U_{i_1^{(m)}} \cap \cdots \cap U_{i_m^{(m)}}).$$
Moreover, we shall use the usual addition to denote direct sum. Therefore, our final task is to prove the following exercise in topology.

**Proposition 11.** The (double) complex

\[(1, \ldots, r) \to \sum_i (i) \to \sum_{i<j} (i, j) \to \cdots \to \sum_i (i) \]

admits a chain map to \((1 \cap \cdots \cap r) - (r - 1)]\) that is a quasi-isomorphism.

The proof is by induction. We first deal with the cases when \(r = 2\) and \(r = 3\), which are also bases of the inductive proof.

**Example 11.1.** Assume \(r = 2\). Then \(U_b = U_1 \cup U_2\). The complex (10.1) is

\[(1, 2) \to (1) + (2)\]

But this is clearly quasi-isomorphic to the complex \((1 \cap 2)[-1]\), by the Mayer–Vietories principle.

**Example 11.2.** Assume that \(r = 3\). Then the complex (10.1) is

\[(1, 2, 3) \to (1, 2) + (1, 3) + (2, 3) \to (1) + (2) + (3)\]

Consider

\[
\begin{array}{ccc}
(2, 3) & \longrightarrow & (2, 3) \\
\downarrow & & \downarrow \\
(1, 2, 3) & \longrightarrow & (1, 2) + (1, 3) + (2, 3) \longrightarrow (1) + (2) + (3)
\end{array}
\]

where the left vertical arrow is given by the inclusion and the right vertical one is given by the Mayer–Vietories of \(U_2 \cup U_3\) with respect to the covering \(U_2\) and \(U_3\). Taking the quotient complex yields a complex quasi-isomorphic to (11.3), which is

\[(1, 2, 3) \to (1, 2) + (1, 3) \to (1) + (2 \cap 3)\]

As a second step, we consider the item \((1, 2, 3)\) on the left. Form the diagram

\[
\begin{array}{ccc}
(1, 2, 3) & \longrightarrow & (1, 2, 3) \\
\downarrow & & \downarrow \\
(1, 2, 3) & \longrightarrow & (1, 2) + (1, 3) \longrightarrow (1) + (2 \cap 3)
\end{array}
\]

The right vertical arrow being induced by the Mayer–Vietories of \(U_1 \cup U_2 \cup U_3\) with respect to the covering \(U_1 \cup U_2\) and \(U_1 \cup U_3\). Thus taking the quotient yields a quasi-isomorphic complex which is

\[0 \to (1, 2 \cap 3) \to (1) + (2 \cap 3)\]

By the Mayer–Vietories for the space \(U_1 \cup (U_2 \cap U_3)\), this complex equals \((1 \cap 2 \cap 3)[-2]\). We win.

**Proof of Proposition 11.** The pattern can already be seen in the above example when \(r = 3\): we modify, from right to left, extra terms. Starting with the complex

\[(1, 2, \ldots, r) \to \sum_{i=1}^r (1, \ldots, \hat{i}, \ldots, r) \to \cdots \to \sum_{1 \leq i < j \leq r} (i, j) \to \sum_{i=1}^r (i),\]
we first modify the right-most item by considering
\[
\begin{array}{c}
(r - 1, r) \\
\downarrow \quad \downarrow \\
\cdots \quad \sum_{1 \leq i < j \leq r} (i, j) \\
\quad \quad \quad \sum_{i=1}^r (i).
\end{array}
\]

Using the Mayer–Vietories for \( U_{r-1} \cap U_r \) we see the quotient complex, which is quasi-isomorphic to the started one, is of the form
\[
C_1 : (1, \ldots, r) \to \cdots \to \sum_{1 \leq i < j \leq r-2} (i, j) + \sum_{i=1}^{r-2} [(i, r-1) + (i, r)] \to \sum_{i=1}^{r-2} (i) + (r-1 \cap r).
\]

Now look at the third from the last term in \( C_1 \); namely \( \sum_{1 \leq i < j < k \leq r} (i, j, k) \), and let \( Q_1 = \sum_{1 \leq i < j \leq r-2} (i, r-1, r) \) which contains those terms where both indices \( r-1 \) and \( r \) appear. We form the diagram
\[
\begin{array}{c}
Q_1 \\
\downarrow \\
\cdots \to \sum_{1 \leq i < j < k \leq r} (i, j, k) \\
\quad \quad \quad \sum_{1 \leq i < j \leq r-2} (i, j) + \sum_{i=1}^{r-2} [(i, r-1) + (i, r)] \\
\quad \quad \quad \to \cdots.
\end{array}
\]

Each summand of \( Q_1 \), say \( (i, r, r-1) \), fits into an exact sequence
\[
(i, r, r-1) \to (i, r) + (i, r-1) \to (i, r \cap r-1).
\]

Using these sequences we can replace the complex \( C_1 \) by its quasi-isomorphic quotient \( C_2 \), and eliminate all the terms appearing in \( Q_1 \). The resulting complex is
\[
C_2 : (1, \ldots, r) \to \cdots \to \sum_{1 \leq i < j < k \leq r-2} (i, j, k) + \sum_{1 \leq i < j \leq r-2} [(i, j, r-1) + (i, j, r)] \\
\quad \quad \to \sum_{1 \leq i < j \leq r-2} (i, j) + \sum_{1 \leq i \leq r-2} (i, r-1 \cap r) \to \sum_{i=1}^{r-2} (i) + (r-1 \cap r).
\]

Next we eliminate the term
\[
Q_2 = \sum_{1 \leq i < j \leq r-2} (i, j, r-1, r)
\]
in the forth from the last term in \( C_2 \). We then achieve the complex
\[
C_3 : (1, \ldots, r) \to \cdots \to \sum_{1 \leq i < j < k \leq r-2} (i, j, k, l) + \sum_{1 \leq i < j < k \leq r-2} [(i, j, k, r-1) + (i, j, k, r)] \\
\quad \quad \to \sum_{1 \leq i < j < k \leq r-2} (i, j, k) + \sum_{1 \leq i < j \leq r-2} (i, j, r-1 \cap r) \\
\quad \quad \to \sum_{1 \leq i < j \leq r-2} (i, j) + \sum_{1 \leq i \leq r-2} (i, r-1 \cap r) \to \sum_{i=1}^{r-2} (i) + (r-1 \cap r).
\]
Inductively, we can finally achieve the complex

\[ C_{r-1} : 0 \to (1, \ldots, r-2, r-1 \cap r) \]

\[ \to \cdot \cdot \cdot \to \sum_{1 \leq i < j < k \leq r-2} (i, j, k) + \sum_{1 \leq i < j \leq r-2} (i, j, r-1 \cap r) \]

\[ \to \sum_{1 \leq i < j \leq r-2} (i, j) + \sum_{1 \leq i \leq r-2} (i, r-1 \cap r) \]

By induction on \( r \), the complex \( C_{r-1} \) is the complex associated with the covering \( U_1, \ldots, U_{r-2}, U_{r-1} \cap U_r \) shifted by 1, which by induction hypothesis computes

\[ U_1 \cap \cdots \cap U_r \left[ -(r-2) - 1 \right] = U_1 \cap \cdots \cap U_r \left[ -(r-1) \right] \]

as desired. \( \square \)

**An application**

12. Let \( X = G/P \) be a projective homogeneous space of a semisimple algebraic group. Let \( L_1, \ldots, L_r \) be a collection of \( G \)-equivariant invertible sheaves on \( X \) such that \( L_1 \otimes \cdots \otimes L_r = \omega_X^\vee \). Then a complete intersection in \( X \) with respect to \( L_1, \ldots, L_r \) is a Calabi–Yau variety.

Let \( V = \prod H^0(X, L_i) \). As we have mentioned, there is a \( D_V \) module \( \tau \), the tautological system, whose local solution space is identified with a homology group, as in the main theorem. We say a point \( b \in V \) is a rank one point, if the space of formal power series solution of \( \tau \) at \( b \) is 1-dimensional. These points are important in the classical story of “mirror symmetry”. As observed by Huang–Lian–Zhu [2], and independently by S. Bloch, the cohomological description of a tautological system can be used in the search of rank one points.

**Proposition 13.** Let the notation be as in 12. Assume further \( X = \text{Grass}(d, N) \) is a Grassmannian acted by \( G = \text{SL}_N \). Then \( \tau \) admits a rank one point.

**Proof.** Write \( L_i = \mathcal{O}_X(d_i) \), where \( \mathcal{O}_X(1) \) is the generator of the Picard group of \( X \) (which defines the Plücker embedding of \( X \)). It is well-known that \( \omega_X = \mathcal{O}_X(-N) \). Consider, in terms of the Plücker coordinate, the hypersurface \( \Pi \) defined by

\[ x_{1,2, \ldots, d} \cdot x_{2,3, \ldots, d+1} \cdots x_{N,1, \ldots, d-1} = 0. \]

Then we have \( \mathcal{O}_X(-\Pi) = \omega_X \). Since \( \bigotimes L_i = \omega_X^\vee \), we have \( \sum d_i = N \). Define

\[ b_1^{(0)} = x_{1,2, \ldots, d} \cdots x_{d_1, d_1+1, \ldots, d_1+d} \in H^0(X, \mathcal{O}_X(d_1)), \]

\[ \vdots \]

\[ b_r^{(0)} = x_{N-d_r+1, \ldots, N} \cdots x_{N,1, \ldots, d-1} \in H^0(X, \mathcal{O}_X(d_r)). \]
We get a point $b^{(0)} = (b_1^{(0)}, \ldots, b_r^{(0)}) \in V$. Let $Y_i^{(0)}$ be the vanishing locus of $b_i^{(0)}$. Then $\Pi$ equals the union of all $Y_i^{(0)}$. By [2, Proposition 8.6], $\dim H_{\dim X}(X - \Pi) = 1$. This, together with the main theorem, implies that, up to scaling, $\tau$ admits a unique nonzero formal solution around $b^{(0)} \in V$. Thus $b^{(0)}$ is a rank one point in the parameter space $V$ of Calabi–Yau complete intersections in $X = \text{Grass}(d, N)$. □

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