Reconnection of Non-Abelian Cosmic Strings

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Abstract

Cosmic strings in non-abelian gauge theories naturally gain a spectrum of massless, or light, excitations arising from their embedding in color and flavor space. This opens up the possibility that colliding strings miss each other in the internal space, reducing the probability of reconnection. We study the topology of the non-abelian vortex moduli space to determine the outcome of string collision. Surprisingly we find that the probability of classical reconnection in this system remains unity, with strings passing through each other only for finely tuned initial conditions. We proceed to show how this conclusion can be changed by symmetry breaking effects, or by quantum effects associated to fermionic zero modes, and present examples where the probability of reconnection in a $U(N)$ gauge theory ranges from $1/N$ for low-energy collisions to one at higher energies.

1Permanent address
1 Introduction

The recent revival of interest in cosmic strings is due to developments on both observational and theoretical fronts. On the observational side there is optimism that the next generation of gravitational wave detectors, Advanced LIGO and LISA, will be able to detect the characteristic signature of cosmic string loops as they twist and whip into cusps [1, 2]. Moreover, there exist tantalizing gravitational lensing events, most notably CSL1 and its companion images [3], which point at the existence of a cosmic string with tension $G\mu \sim 4 \times 10^{-7}$. Further spectroscopic analysis should reveal the nature of this system in the near future. Reviews of these developments can be found in [4, 5].

On the theoretical side, the advent of the “warped throat” in realistic type II string compactifications has permitted a resurrection of the (b)old idea [6] that cosmic strings may be superstrings stretched across the sky [7–9]. In this modern guise, the cosmic string network may consist of both fundamental strings, D-strings and wrapped D-branes. Reviews of these recent stringy developments can be found in [5] and [10], while earlier work on cosmic strings from the 1980’s and early 90’s is summarized in [11, 12].
Having admitted the theoretical possibility that cosmic strings may be fundamental strings, the important question becomes: how can we tell? As described in [5, 9], there are two major distinctions between fundamental strings and gauge theoretic solitons in perturbative field theories. The first is the existence of multi-tension string networks consisting of both fundamental strings, and D-strings, and their various bound states. The study of the dynamical scaling properties of such networks is underway [13, 14]. The second distinguishing feature of fundamental strings, and the one we focus on in this paper, is their interaction cross-section. In the abelian-Higgs model it is known that cosmic strings reconnect with unit probability $P = 1$ over a wide range of impact parameters. In contrast, fundamental strings interact with probability $P \sim g_s^2$, where the functional dependence on the angle of incidence and relative velocity of the strings was determined in [15, 16]. Similarly, D-strings may also pass through each other, reconnecting with probability $P < 1$ [16, 17]. A reduced probability for reconnection affects the scaling solution for the string network, resulting in a larger concentration of strings in the sky [18, 19, 2]. Given enough data, it is not implausible that one could extract the probability $P$ from observation. For further work on cosmic superstrings, see [20].

Of course, it may be possible to engineer a gauge theory whose solitonic strings mimic the behavior of fundamental strings. Indeed, since strongly coupled gauge theories are often dual to string theories in warped throats, this must be true on some level. However, restricting attention to weakly coupled gauge theories, we could ask if there exist semi-classical cosmic strings which, like their fundamental cousins, reconnect with probability $P < 1$. A mechanism for achieving this was mentioned by Polchinski in [5]: construct a vortex with extra internal bosonic zero modes. Two vortices could then miss each other in this internal space. It was further noted that symmetry breaking effects would generically give mass to these internal modes, ruining the mechanism except in rather contrived models.

In this paper we present a model which realizes Polchinski’s field theoretic counterexample although, ultimately, in a rather different manner than anticipated. Our model embeds the cosmic string in a non-abelian $U(N)$ gauge theory, so that the string may move in the internal color and flavor space. The formal properties of vortices of this type have been studied extensively of late (see [21–24]) although, until now, not in the context of cosmic strings. Despite the presence of this internal space we find, rather surprisingly, that cosmic strings continue to reconnect with essentially unit probability, passing through each other only for finely tuned initial conditions. This result occurs due to the non-trivial topology in the interior region of the two-vortex moduli space. However, we show that this conclusion is changed when the internal modes gain a mass. Lifting the vortex zero modes naturally leaves behind $N$ different cosmic strings, each of which reconnects only with strings of the same type while passing through other types of strings. The effect of lifting the internal moduli space is therefore to reduce the probability of reconnection at
low-energies to $1/N$. At high energies these effects wash-out, the strings may evolve into each other, and reconnection again occurs with probability one.

In the next section we review the cosmic strings of interest and explain how they gain an internal space of massless modes. In section 3 we study the moduli space of two vortex strings and argue that vortex strings only fail to reconnect for a set of initial conditions of measure zero. In section 4 we examine various further effects in the model, including quantum dynamics on the vortex worldvolume, symmetry breaking masses and fermionic zero modes, and show how these effects reduce the probability of reconnection at low energies. We conclude with the traditional conclusions.

2 Non-Abelian Cosmic Strings

In this paper we study the dynamics of cosmic strings living in a non-abelian $U(N_c)$ gauge theory, coupled to $N_f$ scalars $q_i$, each transforming in the fundamental representation,

$$L = \frac{1}{4e^2} \text{Tr} F_{\mu\nu} F^{\mu\nu} + \sum_{i=1}^{N_f} D_\mu q_i^\dagger D_\mu q_i - \frac{\lambda e^2}{2} \text{Tr} \left( \sum_{i=1}^{N_f} q_i \otimes q_i^\dagger - v^2 \right)^2$$

(2.1)

When $N_c = N_f = 1$, this is simply the abelian-Higgs model while, for $N_c > 1$, it is the simplest non-abelian generalization. The Lagrangian (2.1) enjoys a $SU(N_f)$ flavor symmetry, rotating the scalars. In Section 4 we shall discuss the more realistic situation in which this symmetry is softly broken, but for now let us assume it remains intact. Moreover, we shall restrict attention to the case $N_f = N_c \equiv N^1$. The theory (2.1) has a unique vacuum in which the scalars condense in the pattern

$$q^a_i = v \delta^a_i$$

(2.2)

Here $i = 1, \ldots, N$ is the flavor index, while $a = 1, \ldots N$ is the color index. In what follows we will take our theory to be weakly coupled by requiring the symmetry breaking scale $e v \gg \Lambda$, the scale at which the non-abelian sector confines. The vacuum (2.2) has a mass gap in which the gauge field has mass $m_g^2 \sim e^2 v^2$, while the the scalars have mass $m_q^2 \sim \lambda e^2 v^2$. The symmetry breaking pattern resulting from this condensate puts the theory into what is referred to as the “color-flavor-locked” phase, with

$$U(N_c) \times SU(N_f) \rightarrow SU(N)_{\text{diag}}$$

(2.3)

The fact that the overall $U(1) \subset U(N_c)$ is broken in the vacuum guarantees the existence of vortex strings characterized by the winding $\int \text{Tr} B = 2\pi k$ for some $k \in \mathbb{Z}$, where the

\footnote{When $N_f < N_c$, the central $U(1)$ remains unbroken and the theory does not admit vortices. For $N_f > N_c$, the resulting cosmic strings are non-abelian generalizations of semi-local vortices [25].}
integral is over the plane transverse to the vortex. The tension of this cosmic string is given by

\[ T = 2\pi v^2 f(\lambda) \]  

(2.4)

where \( f(\lambda) \) is a slowly varying function with \( f(1) = 1 \). The width of the vortex core is given by \( L \sim \max(1/m_y, 1/m_q) \). The parameter \( \lambda \) dictates the behavior of multiple, parallel vortex strings: when \( \lambda > 1 \), parallel vortex strings repel (as in a type II superconductor) while, for \( \lambda < 1 \), parallel strings attract (type I superconductor). In both cases, the force is short ranged, dying off exponentially away from the vortex core. For the critical coupling \( \lambda = 1 \), vortex strings feel no force and multi-soliton solutions exist with parallel vortex strings sitting at arbitrary positions.

Vortex solutions to the theory (2.1) can easily be constructed from solutions to the corresponding abelian theory. Let \( A^*_\mu \) and \( q^\star \) denote gauge and Higgs profiles of the abelian vortex solution. Then we can construct a non-abelian vortex by simply embedding in the upper-left-hand corner thus:

\[
q^a_i = \begin{pmatrix} q^\star & v & \cdots \\ v & \ddots & v \\ v & \cdots & v \end{pmatrix} \quad (A_\mu)^a_b = \begin{pmatrix} A^*_\mu & 0 & \cdots \\ 0 & \ddots & 0 \end{pmatrix}
\]  

(2.5)

This is not the most general embedding. We can act on this configuration with the \( SU(N)_{\text{diag}} \) symmetry preserved by the vacuum to generate new solutions. Dividing out by the stabilizing group, the space of vortex solutions is

\[
\frac{SU(N)_{\text{diag}}}{SU(N-1) \times U(1)} \cong \mathbb{CP}^{N-1}
\]  

(2.6)

The existence of these internal, Goldstone modes, on the string worldsheet means that, at low-energies, the string feels as if it is propagating in a higher dimensional space \( \mathbb{R}^{3,1} \times \mathbb{CP}^{N-1} \). The size (Kähler class) of the internal \( \mathbb{CP}^{N-1} \) space is given by [21]

\[
r = \frac{\tilde{f}(\lambda)}{e^2}
\]  

(2.7)

where \( \tilde{f}(\lambda) \) is, once again, a slowly varying function of \( \lambda \) and it can be shown that \( \tilde{f}(1) = 2\pi \).

A few comments on the literature: vortex zero modes arising through a mechanism of this type were previously studied in [26] although these authors considered unbroken gauge symmetries, a situation which leads to further subtleties. The term “non-abelian strings” is also used to refer to simply-connected gauge groups broken to a discrete subgroup, often
giving rise to several types of cosmic string; see for example [27]. Such strings have rather
different properties from those considered here, such as the existence of string junctions,
and their dynamics shares features with $(p,q)$ string networks [9, 14]. Finally, we make
no attempt to embed our model in a viable GUT, preferring to concentrate instead on
the robust features of our vortex strings. A detailed description of $SO(10)$ GUT strings
can be found, for example, in [28].

The cosmological consequences of these internal modes mimic the behavior of string
moving in higher dimensions. The internal currents carried by the string are akin to
motion in the higher dimensions and, through equipartition of energy, have the effect of
slowing down the motion of the strings in the three dimensions of real space [29]. As we
will explain in Section 4, in our case these internal modes actually gain a small mass from
quantum effects and such currents cease to play a role over large times.

More important for the present discussion is the fact that the internal space (2.6), like
the higher dimensions of string theory, allows vortices to pass without interacting. To see
this, consider two abelian vortices in orthogonal $U(1)$ subgroups, but at different points
$x_1$ and $x_2$ in space,

$$q^a_i = \begin{pmatrix} q^*(x_1) \\ q^*(x_2) \\ \vdots \\ v \end{pmatrix}, \quad (A^*_\mu)^a_b = \begin{pmatrix} A^*_\mu(x_1) \\ A^*_\mu(x_2) \\ \vdots \\ 0 \end{pmatrix} \tag{2.8}$$

In this case, the two strings evolve independently and simply pass through each other: no
reconnection occurs. Of course, if the two vortices instead lie in the same
$U(1)$ subgroup, as in (2.5), then vortices strongly interact and, as we review in the next section, reconnect.
The question is: what happens in the intermediate situations when the two vortices lie in
overlapping $U(1)$ subgroups?

For fundamental strings moving in $d$ compactified dimensions of string theory, one
expects the classical probability of reconnection to be suppressed by the geometric factor
$l_d^d/V$, where $V$ is the volume of the extra dimensions and $l_s = \sqrt{\alpha'}$ is the width of the string
[8]. Together with the inherent probability $P \sim g_s^2$ of fundamental string reconnection
[15, 16]$^2$, this leads to a reduced string cross-section, the net result of which is to increase
the number of cosmic strings seen in the sky [18, 19, 2].

Naively one may imagine that our field theoretic model exhibits similar behavior,
with a critical separation in the internal space distinguishing reconnecting strings from
those that pass through each other. This would then allow one to define a classical


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probability \( P \) of reconnection in this system by coarse graining over the internal space. In the next section we turn to a detailed study of this issue. We shall find that the vortices always reconnect unless they lie in orthogonal subgroups. In some sense, the situation of orthogonal subgroups (2.8) is already the critical separation and the classical probability of reconnection is \( P = 1 \).

3 The Classical Reconnection of Cosmic Strings

In general the non-linear evolution of solitons is a difficult question that requires numerical investigation. However, for the low-energy scattering of cosmic strings we may reliably employ analytical methods in which we restrict attention to the light degrees of freedom describing the positions and internal orientations of the two strings. This method, known as the moduli space approximation [30], has been successfully applied to the abelian Higgs model where it was used to show that vortex strings indeed reconnect [31, 32]. Later numerical simulations revealed that this result is robust, holding for very high energy collisions [33]. These results underpin the statement that gauge theoretic cosmic strings reconnect with probability one. Here we present the moduli space analysis for the non-abelian strings; it is to be hoped that a similar robustness holds for the present result.

3.1 Reconnection of \( U(1) \) Strings

Let us start by recalling how we see reconnection from the moduli space perspective in the case of the abelian Higgs model [31, 32, 17]. One can reduce the dynamics of cosmic strings to that of particles by considering one of two spatial slices shown in Figure 1. The vertical slice cuts the strings to reveal a vortex-anti-vortex pair. After reconnection, this slice no longer intersects the strings, implying the annihilation of this pair. Alternatively, one can slice horizontally to reveal two vortices. Here the smoking gun for reconnection is the right-angle scattering of the vortices at (or near) the interaction point, as shown in Figure 1 (right). Such 90\(^\circ\) degree scattering is a requirement since, as is clear from the figure, the two ends of each string are travelling in opposite directions after the collision. By varying the slicing along the string, one can reconstruct the entire dynamics of the two strings in this manner and show the inevitability of reconnection at low-energies.

Hence, reconnection of cosmic strings requires both the annihilation of vortex-anti-vortex pairs and the right-angle scattering of two vortices. While the former is expected (at least for suitably slow collisions), to see the latter we must take a closer look at the dynamics of vortices. At critical coupling \( \lambda = 1 \), the static forces between vortices cancel and we may rigorously define the moduli space of solutions to the vortex equations. The
Figure 1: The reconnection cosmic strings. Slicing vertically, one sees a vortex-anti-vortex pair annihilate. Slicing horizontally, one sees two vortices scattering at right angles.

The relative moduli space of two abelian vortices is simply $\mathbb{C}/\mathbb{Z}_2$, where $\mathbb{C}$ is parameterized by $z$, the separation between vortices, and $\mathbb{Z}_2 : z \rightarrow -z$ reflects the fact that the vortices are indistinguishable objects. This $\mathbb{Z}_2$ action means that the single valued coordinate on the moduli space is $z^2$, rather than $z$, an important point in what follows. While the metric on this space is unknown\(^1\), it is known to be smooth [35], looking like the snub-nose cone shown in Figure 2. The motion of two particles at zero impact parameter goes up and over the cone, as shown in the figure, returning down the other side. This motion doesn’t correspond to scattering by 180° (this would be coming back down the same side), but to 90° scattering. This result does not depend on details of the metric on the vortex moduli space, but follows simply from the fact that, near the origin, the space is smooth and the single valued coordinate is $z^2$, rather than $z$.

Before proceeding, we pass some well-known comments on the validity of the moduli space approximation. The energies involved in the collision should be small enough so as not to excite radiation. In the present context, this means that the relative velocity $\dot{z}$ of the vortices should satisfy $TL\dot{z}^2 \ll m_\gamma, m_q$ where $L \sim \max(1/m_\gamma, 1/m_q)$ is the width of the vortex string. For $\lambda \approx 1$ this translates into the requirement that $\dot{z}^2 \ll \epsilon^2$. Similarly, the angle of incidence of the vortices, measured by $z'$, the spatial derivative of the separation along the string, should also satisfy $z'^2 \ll \epsilon^2$. Finally, we should mention

\(^1\)Various properties of the metric on the vortex moduli space have been uncovered in [34].
that the description in terms of particle motion on the moduli space is a little misleading, since a given slice of the string need not follow a geodesic on the moduli space. (For example, waves propagating along the string do not have this property). One should talk instead in terms of the dynamics of the real line $\mathbf{R}$, the spatial extent of the two strings, mapped into the moduli space. The inevitability of reconnection then follows from the single valued nature of $z^2$, together with the free motion of the strings far from the interaction point [17].

3.2 The Moduli Space of Non-Abelian Vortices

We would now like to repeat this analysis for the the non-abelian vortices introduced in Section 2. For the vertical slice shown in Figure 1, the abelian argument carries over. Our vortices have only a single topological protector, $k = \int \text{Tr} \ B / 2\pi$, and a vortex-anti-vortex pair may annihilate regardless of their mutual orientation in the gauge group. One caveat is that a vortex and anti-vortex in orthogonal $U(1)$’s (as in (2.8)) remains a solution, albeit an unstable one. Therefore if we do not allow fluctuations away from this ansatz, the vortex-anti-vortex pair will pass right through each other without annihilating, in accord with the statements in the previous section. This is our first hint that reconnection will occur except for finely tuned initial conditions.

To complete the argument of reconnection, we also need to study when right-angle scattering occurs. For this we require a description of the moduli space of multiple non-abelian vortices in the critically coupled ($\lambda = 1$) theory (2.1). A description of this space arising from modelling the system in terms of string theoretic D-branes was presented in [21]. The moduli space is presented in terms of an algebraic quotient construction, related to the ADHM construction of the instanton moduli space. We now review this construction.
The moduli space of \( k \) vortices in \( U(N) \) gauge theory is a Kähler manifold with real dimension \( 2kN \) which we denote as \( \mathcal{M}_{k,N} \). The construction of [21] presents this space as a \( U(k) \) symplectic quotient construction. We start with a \( k \times k \) complex matrix \( Z \), and a \( k \times N \) complex matrix \( \Psi \), subject to the constraint

\[
[Z, Z^\dagger] + \Psi \Psi^\dagger = r
\]  

where the right-hand-side is proportional to the \( k \times k \) identity matrix, and \( r = 2\pi/e^2 \) as in (2.7). The moduli space \( \mathcal{M}_{k,N} \) is defined as the quotient of the solutions to this constraint, where we divide by the \( U(k) \) action

\[
Z \to UZU^\dagger, \quad \Psi \to U\Psi
\]  

The \( U(k) \) action has no fixed points ensuring that, as in the abelian case, the moduli space of vortices is smooth. Roughly speaking, the eigenvalues of \( Z \) correspond to the positions of \( k \) vortices, while the independent components of \( \Psi \) denote the orientations of these vortices in the internal space. For example, when \( k = 1 \), the scalar \( Z \) decouples and corresponds to the center of mass of the vortex, while \( \Psi \) satisfies \( |\Psi|^2 = r \), modulo the \( U(1) \) gauge action, which reproduces the moduli space \( \mathbb{CP}^{N-1} \) for a single vortex.

The manifold \( \mathcal{M}_{k,N} \) has an \( SU(N)_{\text{diag}} \times U(1)_R \) action. The former results from the symmetry (2.3) acting on the vortex; the latter is the rotational symmetry of the plane. In the construction described above, the action is

\[
SU(N)_{\text{diag}} : \Psi \to \Psi V, \quad U(1)_R : Z \to e^{i\alpha}Z
\]  

The algebraic quotient description of the vortex moduli space presented here arises from studying vortices in a Hanany-Witten set-up [21]. We stress that, despite the D-brane origin of this construction, the resulting moduli space is that of field theoretic vortices; indeed, in [17], this framework was used to elucidate the differences between abelian \((N = 1)\) vortex strings and D-strings moving in vacua. Here we are interested in non-abelian \((N \geq 2)\) strings. To our knowledge, there is no field theoretic derivation that the quotient space \( \mathcal{M}_{k,N} \) coincides with the vortex moduli space and such a proof would be desirable. In the following we shall see that several key features of \( \mathcal{M}_{k,N} \) correctly capture the behavior of vortex strings. Note however that the space \( \mathcal{M}_{k,N} \) naturally inherits a metric from the above construction; this does not coincide with the metric on the moduli space of vortices (interpreted in terms of solitons, it describes co-dimension two objects with long-range polynomial tails). Thankfully, in what follows we will only require topological information about \( \mathcal{M}_{k,N} \).

The eigenvalues of \( Z \) are dimensionless and correspond to the positions of the vortices multiplied by the mass scale \( v \).
3.3 Reconnection of $U(2)$ Strings

We first discuss the case of $k = 2$ vortices in the $N_c = N_f = 2$ gauge theory. Both $Z$ and $\Psi$ are $2 \times 2$ matrices (although for different reasons), and each suffers a $U(2)$ action (3.2). We project out the trivial center of mass motion of the system by requiring $\text{Tr} Z = 0$ and, following [36], use the $U(2)$ action to place $Z$ in upper-triangular form. We write

$$Z = \begin{pmatrix} z & \omega \\ 0 & -z \end{pmatrix}, \quad \Psi = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$$

(3.4)

This does not fix all gauge degrees of freedom, but leaves a surviving $U(1)_1 \times U(1)_2 \times Z_2 \subset U(2)$ gauge symmetry acting as:

$$U(1)_1 : U = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & 0 \end{pmatrix}, \quad U(1)_2 : U = \begin{pmatrix} 0 & 0 \\ 0 & e^{i\phi} \end{pmatrix}$$

(3.5)

under which $a_i$ transforms with charge $(1, 0)$, $b_i$ with charge $(0, 1)$ and $\omega$ with charge $(1, -1)$. The coordinate $z$ is neutral. Meanwhile, the $Z_2$ action is

$$Z_2 : U = \frac{-1}{\sqrt{1 + |\zeta|^2}} \begin{pmatrix} -1 & \bar{\zeta} \\ \zeta & 1 \end{pmatrix}$$

(3.6)

with the parameter $\zeta = 2z/\omega$. Finally, the constraints arising from (3.1) read

$$\sum_{i=1}^{2} |a_i|^2 = r - |\omega|^2, \quad \sum_{i=1}^{2} |b_i|^2 = r + |\omega|^2, \quad a_1 \bar{b}_1 + a_2 \bar{b}_2 = 2\bar{z}\omega$$

(3.7)

Counting degrees of freedom, we have 6 complex parameters in $Z$ and $\Psi$ subject to two real constraints and one complex constraint (3.7), together with the two $U(1)$ actions (3.5). This leaves us with a smooth moduli space $\tilde{M}_{2,2}$ of 3 complex dimensions. (Recall that we have factored out the center of mass degree of freedom so the full moduli space is $M_{2,2} \cong \mathbb{C} \times \tilde{M}_{2,2}$). The rest of this section is devoted to studying this space.

The Asymptotic Regime

To get a feel for the physical interpretation of the various parameters, it is instructive to examine the regime of far separated vortices with $z \gg r$. We have $|\omega| \sim 1/|z|$ and the constraints (3.7), combined with the $U(1)^2$ action (3.5), restrict $a_i$ and $b_i$ to lie in independent $\mathbb{CP}^1$’s, up to $1/|z|^2$ corrections. In this limit the $Z_2$ action reads

$$Z_2 : \begin{cases} z \leftrightarrow -z \\ a_i \leftrightarrow b_i \end{cases}$$

(3.8)
interchanging the position and orientation of the two vortices. Thus, asymptotically, the moduli space is simply

$$\tilde{\mathcal{M}}_{2,2} \to \frac{\mathbb{C} \times \mathbb{CP}^1 \times \mathbb{CP}^1}{\mathbb{Z}_2}$$

(3.9)

Two Submanifolds

To continue our exploration of this space, it will prove useful to seek out a couple of special submanifolds. These correspond to the two extreme cases described in Section 2 in which we understand that reconnection does/does not occur.

The first such submanifold corresponds to the situation (2.5) where the vortices lie in the same $U(1)$ subgroup. As we mentioned in Section 2, such vortices must always scatter at right-angles. We can impose this condition through the requirement that the two orientation vectors lie parallel: $a_i \sim b_i$. We will refer to this submanifold as $\mathcal{M}|_{U(1)} \subset \tilde{\mathcal{M}}_{2,2}$. By an $SU(2)_F$ action, we can choose a representative point, say $a_2 = b_2 = 0$. Then the constraints read

$$|a_1|^2 = r - |\omega|^2, \quad |b_1|^2 = r + |\omega|^2, \quad a_1 \bar{b}_1 = 2 \bar{z} \omega$$

(3.10)

This system was previously studied in [36]. On this submanifold, $a_i$ and $b_i$ are both even under the $\mathbb{Z}_2$ action (3.6), while $\omega$ and $z$ are odd: $(\omega, z) \to -(\omega, z)$. The calculation of [36] shows that this manifold is asymptotic to $\mathbb{C}/\mathbb{Z}_2$, with a smooth metric at the origin, as depicted in Figure 2. Acting now with the $SU(2)_F$ action sweeps out a $\mathbb{CP}^1$ at each point, leaving us with a space which is topologically

$$\mathcal{M}|_{U(1)} \cong \mathbb{C}/\mathbb{Z}_2 \times \mathbb{CP}^1$$

(3.11)

Note that as $z \to 0$, the $\mathbb{CP}^1$ does not vanish. In this limit the equations (3.10) are solved by $|\omega|^2 = r$ and $a_i = 0$, while $b_i$ parameterize a $\mathbb{CP}^1$ with Kähler class $2r$.

Let us now turn to the submanifold describing vortices in orthogonal $U(1)$ subgroups as in (2.8). The vortices should now pass through each other without interacting. This submanifold is defined by the requirement $\omega = 0$, and we will refer to it as $\mathcal{M}|_{\omega=0}$. The first two equations in (3.7) tell us that $a_i$ and $b_i$ each define a point on $\mathbb{CP}^1$, while the third equation, which reads $a_i \bar{b}_i = 0$, requires these points to be antipodal. Again, acting with the $SU(2)_F$ symmetry then sweeps out a $\mathbb{CP}^1$. We still have to divide out by the

\footnote{Topologically $\mathbb{C}/\mathbb{Z}_2 \cong \mathbb{C}$. We keep the former to emphasize that this description also captures the asymptotic metric on the space.}

\footnote{This situation is similar to the theory describing two D-strings, in which $r = 0$ and there is no $\Psi$ field, forcing $\omega = 0$ [17].}

\footnote{Generically the orbits of the $SU(2)_F$ action on $\tilde{\mathcal{M}}_{2,2}$ are three dimensional. They degenerate to two dimensional orbits on $\mathcal{M}|_{U(1)}$ and $\mathcal{M}|_{\omega=0}$.}
The $\mathbb{Z}_2$ gauge action which this time acts as in (3.8), exchanging $z \leftrightarrow -z$ and, at the same time, mapping antipodal points on $\mathbb{C}P^1$. We therefore have,

$$\mathcal{M}|_{\omega=0} \cong \frac{\mathbb{C} \times \mathbb{C}P^1}{\mathbb{Z}_2}$$

But what happens at the origin? If we set $\omega = z = 0$ then $\Psi$ feels the full force of the restored $U(2)$ gauge symmetry, resulting in a unique solution to the constraints (3.1). This means that when the vortices live in orthogonal $U(1)$’s, as in (2.8), the internal space collapses as they approach each other! $\mathcal{M}|_{\omega=0}$ can be thought of as the cone over $(S^1 \times \mathbb{C}P^1)/\mathbb{Z}_2$ (which can alternately be described as the non-trivial $S^2$ bundle over $S^1$, or as the connect sum $\mathbb{R}P^3 \# \mathbb{R}P^3$). Note that the submanifold $\mathcal{M}|_{\omega=0}$ is singular at $z = 0$. This is an artifact of restricting attention to this subspace; the full manifold $\mathcal{M}_{2,2}$ should be smooth at the point $\omega = z = 0$.

How can we understand the result that the internal space collapses at the origin of $\mathcal{M}|_{\omega=0}$ from the perspective of the soliton solutions? In fact, it is rather simple to see. The solutions of the form (2.8) generically transform non-trivially under the $SU(2)_{\text{diag}}$ vacuum symmetry, sweeping out the $\mathbb{C}P^1$ internal space. However, as $x_1 \to x_2$ (corresponding to $z \to 0$), the relevant part of $A_\mu$ and $q$ approaches the unit matrix, and the $SU(N)_{\text{diag}}$ symmetry no longer acts. Two coincident vortices in orthogonal $U(1)$ sectors have no internal space! This is one of the important points that allows for reconnection to generically occur in this model.

Reconnection and the Origin of Moduli Space

Having determined the topology of these two submanifolds, let us now examine whether reconnection takes place on each. We start with $\mathcal{M}|_{U(1)}$. Here the argument proceeds as for the abelian case: the submanifold $\mathbb{C}/\mathbb{Z}_2$ is a smooth cone, as shown in Figure 2, with $z^2$ the single valued coordinate at the tip of the cone. This ensures that any trajectory hitting the tip of the cone results in right angle scattering. The $\mathbb{Z}_2$ does not act on the internal space $\mathbb{C}P^1$ and it plays no role in the discussion of reconnection.

What about the space $\mathcal{M}|_{\omega=0}$, describing orthogonal vortices? Here the issue is somewhat clouded by the singularity at the center of the space. However, consider first the resolved space where the $\mathbb{C}P^1$ does not degenerate at the origin. Since the $\mathbb{Z}_2$ action has no fixed points, such a manifold is smooth. In contrast to the previous case, a trajectory through the origin at $z = 0$ now corresponds to vortices passing straight through each other; there is no right angle scattering. The reason for this is that the $\mathbb{Z}_2$ gauge symmetry does not act only on $\mathbb{C}$ but also on the internal space; it exchanges both the positions and the identities of the particles. This means that, near the origin, $z$ is the single valued coordinate rather than $z^2$ and right-angle scattering does not occur. The
true motion in $\mathcal{M}_{|\omega=0}$, in which the $\mathbb{C}P^1$ degenerates, can be thought of as the limiting case of this discussion. The need to take this degenerative limit is necessary since, as we shall presently see, $\omega = 0$ corresponds to the only case of no reconnection; if the moduli space $\mathcal{M}_{|\omega=0}$ were not described by this singular limit then, by continuity, vortices in the neighborhood of $\omega = 0$ should also pass through each other.

Let us now show that, as promised, vortices of arbitrary orientation always scatter at right angles unless $\omega = 0$. As we have seen above, the key to this lies in the $\mathbb{Z}_2$ action (3.6). In particular, we are interested in this action as the vortices approach each other and $z \to 0$. Then we see that, provided $\zeta = 2z/\omega \to 0$, the $\mathbb{Z}_2$ action on $\Psi$ and $\omega$ can be absorbed in the $U(1)^2$ gauge transformations, leaving only $z \to -z$. In other words, for all $\omega \neq 0$, the single valued coordinate near the origin is $z^2$ rather than $z$. Using the general arguments described above, this implies right-angle scattering and reconnection of cosmic strings.

To complete the argument, we need to make sure that restricting to $z = 0$ where the vortices coincide defines a complete submanifold. Let us denote this as $\mathcal{M}_{|z=0}$. It may be defined in a coordinate independent manner as the fixed locus of the $U(1)_R$ action (since the resulting action on $\omega$ is gauge equivalent to a $SU(2)_F$ rotation). When $z = 0$, the $\mathbb{Z}_2$ action (3.6) can be absorbed into the axial combination of the $U(1)_1 \times U(1)_2$ gauge symmetry (3.5) and $\mathcal{M}_{|z=0}$ can be thought of as the resolution of the $\mathbb{Z}_2$ fixed point.

What is $\mathcal{M}_{|z=0}$? Upon setting $z = 0$, the constraints (3.7) read

$$|a_1|^2 + |a_2|^2 + |\omega|^2 = r, \quad |b_1|^2 + |b_2|^2 - |\omega|^2 = r, \quad a_i \bar{b}_i = 0 \quad (3.13)$$

where we must still quotient by the $U(1)^2$ action (3.5). The first constraint, together with the $U(1)_1$ action, defines a copy of $\mathbb{C}P^2$ (although it doesn’t inherit the round Fubini-Study metric). For a generic point $p \in \mathbb{C}P^2$, the second and third constraints in (3.13) determine $b_i$ uniquely up to a phase, which is gauged away by $U(1)_2$. If this were true globally, we would have

$$\mathcal{M}_{|z=0} \cong \mathbb{C}P^2 \quad (3.14)$$

However, there are two exceptional points. Firstly, when $a_i = 0$ and $|\omega|^2 = r$, the $b_i$’s parameterize a $\mathbb{C}P^1$ rather than a point. We have $\mathcal{M}_{|U(1)} \cap \mathcal{M}_{|z=0} \cong \mathbb{C}P^1$. Secondly, when $\omega = 0$, the full $U(2)$ gauge symmetry is restored and the $a_i$’s parameterize a point rather than a $\mathbb{C}P^1$; we have $\mathcal{M}_{|\omega=0} \cap \mathcal{M}_{|z=0} \cong \{0\}$. In fact, it is a rather cute fact that after making these two adjustments, (3.14) remains correct! To see this, we may think of $\mathcal{M}_{|z=0}$ as a fiber over the interval $|\omega| \in [0, \sqrt{r}]$. The phase of $\omega$ shrinks to zero at each end due to the action of the gauge symmetry. In the middle of the interval, $a_i$ and $b_i$, modulo the constraints (3.13) and the $U(1)^2$ gauge action, define a $\mathbb{C}P^1$. The phase of $\omega$ is fibered over this to yield a $S^3$ (to see this, note that rotating the phase of $\omega$ is
gauge equivalent to an $SU(2)_F$ flavor transformation). We therefore have a description of $\mathcal{M}|_{z=0}$ in terms of an $S^3$ fibration over the interval, degenerating to a point at one end and to $\mathbb{C}P^1$ at the other. This is precisely $\mathbb{C}P^2$.

In summary, as two vortices collide their relative orientations define a point $p \in \mathbb{C}P^2$. The vortices undergo $90^\circ$ scattering (and, hence, strings undergo reconnection) unless $p$ coincides with the special point $\omega = 0$ on $\mathbb{C}P^2$. Thus the reconnection probability $P$ is unity. Note that we did not assume geodesic motion on the moduli space, and the argument for reconnection goes through even for strings carrying different currents in the internal space. Although this may seem surprising, similar results were observed numerically for Witten’s superconducting strings, resulting in excess charge build up at the interaction point [37].

The behavior of the vortices passing through the special point $\omega = 0$ presumably depends on the quantity $\zeta = 2z/\omega$ as $z \to 0$. For $\zeta \to \infty$, we have seen that the vortices pass through each other unscathed. We suspect that for other values of $\zeta$, the vortices undergo scattering an angle less than $90^\circ$.

### 3.4 Reconnection of $U(N)$ Strings

We now turn to the case of vortices in $U(N)$ gauge theories. The details are similar to the $U(2)$ case so we shall be brief. The matrix $Z$ remains $2 \times 2$, while $\Psi$ is now a $2 \times N$ matrix. Once again we may employ the auxiliary $U(2)$ gauge action to place $Z$ in upper triangular form:

$$Z = \begin{pmatrix} z & \omega \\ 0 & -z \end{pmatrix}, \quad \Psi = \begin{pmatrix} a_1 & \ldots & a_N \\ b_1 & \ldots & b_N \end{pmatrix}$$

(a choice which is preserved by the remnant $U(1)_1 \times U(1)_2 \times \mathbb{Z}_2$ action of equations (3.5) and (3.6). The same analysis of the previous section shows that asymptotically,

$$\tilde{\mathcal{M}}_{2,N} \to \frac{\mathbb{C} \times \mathbb{C}P^{N-1} \times \mathbb{C}P^{N-1}}{\mathbb{Z}_2}$$

As we have seen, the question of reconnection boils down to the $\mathbb{Z}_2$ action which, since it is unchanged, ensures that $z^2$ is the single-valued coordinate as $z \to 0$ provided $\zeta \to 0$. This time the manifold $\mathcal{M}|_{z=0}$ has complex dimension $2N - 2$, and is defined by the quotient construction,

$$\sum_{i=1}^N |a_i|^2 + |\omega|^2 = r \quad , \quad \sum_{i=1}^N |b_i|^2 - |\omega|^2 = r \quad , \quad \sum_{i=1}^N a_i\bar{b}_i = 0$$

One can view this space as a fibration over the interval$^1$ $|\omega|^2 \in [0, r]$. To ensure that the space $\mathcal{M}|_{z=0}$ is smooth, one must check that spheres degenerate at either end of the

$^1$We’re grateful to James Sparks for discussions and explanations regarding these issues.
interval, rather than a more complicated space. For $|\omega|^2 \neq 0, r$, we may use the $U(1)_2$ action to set the phase of $\omega$ to a constant. Then the constraints (3.17) define the Stiefel manifold $V(2, N) \cong U(N)/U(N - 2)$ of orthonormal two-frames in $\mathbb{C}^N$. Dividing by the remaining $U(1)_1$ action, the fiber over a generic point is $V(2, N)/U(1)_1$.

At the two end points of the interval some submanifold of this fiber degenerates. When $|\omega|^2 = r$, so $a_i = 0$, the constraints (3.17), together with the $U(1)_2$ action, ensure that the fiber shrinks to $\mathbb{C}P^{N-1} \cong U(N)/U(N - 1) \times U(1)$. This means that the degenerating cycle at $|\omega|^2 = r$ is

$$[U(N - 2) \times U(1)] / [U(N - 1) \times U(1)] \cong S^{2N-3}$$ (3.18)

Meanwhile, at the other end of the interval, when $\omega = 0$, the full $U(2)$ gauge action is restored. When combined with the constraints (3.17), this gives rise to the Grassmanian of complex two-planes in $\mathbb{C}^N$, which can be described as $G(2, N) \cong U(N)/U(N - 2) \times U(2)$. At this end the degenerating cycle is

$$[U(N - 2) \times U(2)] / [U(N - 2) \times U(1)] \cong S^3$$ (3.19)

We therefore find that the cycles that degenerate at either end of the interval are indeed spheres, and the space $\mathcal{M}|_{z=0}$ is smooth.

In summary, the submanifold $\mathcal{M}|_{z=0}$ is smooth and reconnection occurs unless the vortices collide over the complex codimension 2 submanifold $\mathcal{M}|_{z=0} \cap \mathcal{M}|_{\omega=0} \cong G(2, N)$, therefore $P = 1$.

### 4 Symmetry Breaking and Quantum Effects

So far we have discussed the situation in which the $SU(N_f)$ flavor symmetry of the model is unbroken. Since global symmetries are unlikely to be exact in Nature, in this section we discuss various mechanisms by which the flavor symmetry is broken and/or the internal modes on the string are lifted.

#### 4.1 Symmetry Breaking and Monopole Pair Creation

We start by introducing explicit symmetry breaking terms into the Lagrangian. We will present two examples in which the moduli space of vortices gets lifted, leaving behind $N$ different cosmic strings, each carrying magnetic flux in a different part of the gauge group.

The simplest symmetry breaking term is a mass for the scalar fields $q_i$. After a suitable
unitary transformation, we have the potential,

$$V = \frac{\lambda e^2}{2} \text{Tr} \left( \sum_i q_i \otimes q_i^\dagger - v^2 \right)^2 + \sum_i m_i^2 q_i^\dagger q_i \quad (4.1)$$

For $m_i^2 \ll \lambda e^2 v^2$, the theory still lies in the Higgs phase, with vacuum expectation value

$$q_i^a = \sqrt{v^2 - m_i^2} \frac{\delta_i^a}{\lambda e^2} \equiv \mu_i \delta_i^a \quad (\text{no sum over } i) \quad (4.2)$$

The symmetry breaking pattern now becomes,

$$U(N_c) \times SU(N_f) \rightarrow SU(N) \text{diag} \rightarrow U(1)^{N-1}_{\text{diag}} \quad (4.3)$$

where the first, spontaneous, breaking occurs at the scale $e^2 v^2$, while the second, explicit breaking, occurs at the scales $m_i$. With the $SU(N)_{\text{diag}}$ broken, we can no longer sweep out a moduli space of vortex solutions as in (2.6) and the internal $\mathbb{C}P^{N-1}$ space is lifted. What remains are $N_c$ distinct vortex solutions in which the non-abelian field strength has non-vanishing component within only one of the diagonal $U(1) \subset U(N_c)$. Let $B$ denote the adjoint valued magnetic field in the direction of the strings. Then we may embed an abelian vortex in the $i^{th}$ $U(1)$ subgroup of $U(N_c)$ with

$$B \sim \text{diag}(0, \ldots, 1, \ldots, 0) \quad (4.4)$$

Such a vortex has tension $T_i \sim \mu_i^2$. Since each of these vortices is supported by the same topological quantum number, $\int \text{Tr} B$, only one of these strings is globally stable; the others may all decay into the string with the lowest tension. We shall discuss one such mechanism for this decay shortly.

It is simple enough to alter our model to arrange for all $N_c$ strings to have the same tension. We introduce a new, adjoint valued, scalar field $\phi$, with canonical kinetic term, and consider the potential,

$$V = \frac{\lambda e^2}{2} \text{Tr} \left( \sum_i q_i \otimes q_i^\dagger - v^2 \right)^2 + \sum_i q_i^\dagger (\phi - m_i)^2 q_i \quad (4.5)$$

This is a variant on the potentials that appear in $\mathcal{N} = 2$ SQCD. Unlike the potential (4.1), symmetry breaking in the pattern (4.3) now occurs regardless of the relative values of $m_i$ and (non-zero) $v^2$. The unique, gapped, vacuum is given by

$$q_i^a = v^2 \delta_i^a \quad , \quad \phi = \text{diag}(m_1, \ldots, m_N) \quad (4.6)$$

In this theory we again have $N_c$ different vortices, with magnetic flux (4.4), but now with equal tension (2.4). (Note that if we also include an explicit mass $M$ for the adjoint
scalar $\phi$ then symmetry breaking only occurs for suitably small $M$, but the tensions of the vortices remain equal).

Both deformations (4.1) and (4.5) result in $N_c$ different types of vortices, each embedded in a different, orthogonal, $U(1)$ subgroup of $U(N_c)$. This ensures that two strings colliding with energies $E \ll \Delta m_i$ fall into one of the two categories discussed in Section 2: either the strings are of the same type (i.e. inhabit the same $U(1)$) and they reconnect; or they are of different types, and pass through each other. Unlike the situation where the strings enjoyed an internal moduli space, there is no need for fine tuning to make the strings miss each other: the potential does the job for us. Therefore at energies smaller than the mass splittings $\Delta m_i$, the classical probability for reconnection is $1/N$. At energies much larger than this, the masses are negligible and the probability increases to unity (at least whenever the moduli space approximation of the previous section is valid).

Confined Monopoles

In fact, quantum effects give rise to a finite probability for reconnection even for distinct strings. This can occur if one string turns into another through the creation of a confined monopole. Here we give an estimate of the magnitude of this effect. The presence of confined monopoles, acting like beads on the cosmic string, may have other interesting cosmological consequences as explored in [38–40].

Strings living in different $U(1)$ subgroups are supported by the same topological invariant $\int \text{Tr} B$, suggesting that they may transmute into each other. The change of the string, from one type to another, occurs by a kink on the string worldsheet which, from the four-dimensional perspective, has the interpretation of a confined magnetic monopole. These monopoles were described in [41] and further explored in [23, 42]. Similar objects were previously discovered in $\mathbb{Z}_2$ strings in [43]. The mass of the kink on the worldsheet is

$$M_{\text{kink}} \sim r \Delta m_i \sim \frac{2\pi \langle \phi \rangle}{e^2} \sim M_{\text{monopole}}$$

which has the same parametric dependence as the mass of the unconfined magnetic monopole. (In the supersymmetric context the equality $M_{\text{kink}} = M_{\text{monopole}}$ is exact).

Reconnection for different abelian strings requires the quantum pair creation of a monopole-anti-monopole on the string as shown in Figure 3. One can estimate the probability for reconnection to occur by treating the worldsheet dynamics as a $d = 1 + 1$ quantum field theory. For simplicity let us model the reconnection of two almost static strings at incident angle $\theta$ by the shape shown in Figure 3. Then reconnection reduces
the energy of the configuration by

$$\Delta V = -4T a \tan^2(\theta/2) + 2M_{\text{monopole}}$$

(4.8)

The reconnected region is specified to be $-a < x < a$ where $x$ is the worldsheet spatial coordinate. This is the same potential arising in electron-positron pair creation in a constant electric field in $d = 1 + 1$, for which the electric field times the positron charge is now given by $2T \tan^2(\theta/2)$. The famous result by Schwinger [44], evaluating the bounce action of a circular loop in Euclidean space, gives the decay width as

$$\Gamma \sim \exp \left( -\frac{\pi M_{\text{monopole}}^2}{2T \tan^2(\theta/2)} \right) \sim \exp \left( -\frac{\pi^2 (\Delta m)^2}{e^4 v^2 \tan^2(\theta/2)} \right)$$

(4.9)

This computation ignores the relative velocity of the strings, and is valid only for almost parallel strings for which the exponent is large (and negative). It would be interesting to better quantify the role of these confined monopoles for other impact parameters.

4.2 Quantum Effects

Until now, much of our discussion has been purely classical. Indeed, we have chosen the four-dimensional symmetry breaking scale $e^2 v^2$ to be suitably high so the theory is weakly coupled. Nevertheless, the theory on the vortex string is necessarily strongly coupled at low-energies: it is the two-dimensional $\mathbb{CP}^{N-1}$ sigma model.
For now let us set the masses $m_i$ discussed in the previous section to zero, ensuring that the $SU(N)_{\text{diag}}$ symmetry is exact. The resulting low-energy quantum dynamics on $\mathbb{CP}^{N-1}$ is well understood. The Mermin-Wagner-Coleman theorem guarantees that the ground state wavefunction spreads over $\mathbb{CP}^{N-1}$, resulting in a unique vacuum state for the string. More quantitatively [45], the size of the vortex moduli space evolves under RG flow, resulting in dynamical transmutation and a mass gap for the internal modes on the vortex string. The one-loop beta function leads to the strong coupling scale

$$\Lambda_{\mathbb{CP}^{N-1}} = \mu \exp\left(-\frac{2\pi r(\mu)}{N_c}\right)$$

where, from equation (2.7), we have $r \sim 2\pi/e^2$ at the symmetry breaking scale $\mu = ev$. Thus the currents discussed previously, which classically may travel along the worldsheet at the speed of light, become massive and do not persist. One can show using the large $N$ expansion that the theory confines and all dynamical degrees of freedom are singlets of $SU(N)_{\text{diag}}$ [45].

In terms of reconnection, the quantum effects do little to change the story: at low energies $E \ll \Lambda_{\mathbb{CP}^{N-1}}$, the strings lie in a unique ground state and reconnect with unit probability. At higher energies, $\Lambda_{\mathbb{CP}^{N-1}} < E < ev$, asymptotic freedom of the sigma model ensures that the classical analysis of the previous section is valid and, once again, the strings reconnect. At energies $E \gg ev$, numerical simulations are required to determine the issue. Introducing masses $m_i$ as in (4.1) or (4.5) leads to a weakly coupled theory when $\Delta m_i \gg \Lambda_{\mathbb{CP}^{N-1}}$ and the results of the previous subsection hold only in this regime.

### 4.3 Fermionic Zero Modes

The low-energy quantum dynamics of the string can be dramatically changed by the inclusion of fermionic zero modes [45]. We may add Weyl fermions $\xi$ and $\psi$ to the bulk theory with Yukawa couplings of the schematic form,

$$L_{\text{Yukawa}} = \bar{\psi} \xi q$$

Any such coupling will lead to chiral fermionic zero modes $\chi$ propagating on the vortex string. The exact nature of these zero mode depends on the properties of $\xi$ and $\psi$ and, as we now discuss, different color and flavor representations for $\xi$ and $\chi$ will lead to very different low-energy physics for the cosmic strings. Here we sketch two examples. More details will be given in a future publication.

First consider the example in which $\psi_i$ transforms, like $q_i$, in the fundamental $N_c$ of the gauge group, as well as the fundamental $N_f$ of the flavor group (recall that $N_c = N_f = N$), while $\xi$ is a singlet under both. Then the Yukawa coupling (4.11) can be shown to give
rise to a single chiral zero mode on the worldsheet with kinetic term,

\[ L_{\text{zeromode}} = i \bar{\chi} \not{\partial} \chi \]  

(4.12)

Such zero modes do not couple to \( \mathbb{C}P^{N-1} \) modes on the string and do not affect the low-energy dynamics. (Note that the four-dimensional anomaly can be cancelled by the addition of further fermions transforming in the conjugate representation which may give rise to further fermionic zero modes but do not qualitatively change the low-energy string dynamics).

A more interesting example comes if we consider \( \psi \) to transform, once again, in the \((N_c, N_f)\) representation of \( U(N_c) \times SU(N_f) \), while \( \xi \) transforms in the adjoint representation of \( U(N_c) \) (and is a singlet under \( SU(N_f) \)). In this case index theorems ensure the existence of \( N \) zero modes \( \chi_i \) on the worldsheet. However, crucially, they now couple to the strongly interacting \( \mathbb{C}P^{N-1} \) sector of the theory. Let \( \pi_i, i = 1, \ldots, N \) define homogeneous coordinates on \( \mathbb{C}P^{N-1} \), such that \( \sum_{i=1}^{N} |\pi_i|^2 = r \), with \( \mathbb{C}P^{N-1} \) obtained after identifying \( \pi_i \equiv e^{i\alpha_i} \pi_i \). Then the fermionic zero modes on the worldsheet can be shown to couple to a bosonic \( U(1) \) current,

\[ L_{\text{current}} = i \bar{\chi}_i \left( \pi_j \not{\partial} \pi_j \right) \chi_i \]  

(4.13)

Once again, the gauge anomaly can be cancelled by the addition of conjugate fermions in four dimensions. In fact, such action guarantees that the \( d = 1 + 1 \) theory on the string is non-chiral, cancelling a related sigma-model anomaly on the worldsheet [46].

So what is the consequence of the interaction (4.13)? The crucial point, as explained in [45], is worldsheet chiral symmetry breaking. Classically, the \( U(1) \) chiral symmetry acts as \( \chi_i \rightarrow e^{i\beta \gamma^5} \chi_i \) while, quantum mechanically, only a \( \mathbb{Z}_2 \) is non-anomalous. The strong coupling dynamics on the worldsheet induces a condensate for the zero modes, \( \langle \chi \chi \rangle \sim \Lambda_{\mathbb{C}P^{N-1}} \), breaking the discrete chiral symmetry yet further: \( \mathbb{Z}_2 \rightarrow \mathbb{Z}_2. \) We find a situation similar to that of Section 4.1, in which the moduli space of vacua is lifted, now at a scale \( \Lambda_{\mathbb{C}P^{N-1}} \), resulting \( N \) different ground states. Recent studies of the low-energy dynamics of these vortex strings in both supersymmetric theories identify these ground states with the \( N_c \) vortex strings lying in orthogonal, diagonal \( U(1) \subset U(N_c) \) subgroups [23]. Once again we have a situation in which the strings reconnect with probability \( P = 1/N \) at energy scales \( E \ll \Lambda_{\mathbb{C}P^{N-1}} \), and with unit probability at higher energies where the sigma-model becomes asymptotically free and the classical moduli space approximation holds.
5 Summary and Conclusions

We have studied the low-energy dynamics of cosmic strings embedded in a $U(N_c)$ gauge theory with $N_f = N_c \equiv N$ scalars transforming in the fundamental representation. The cosmic strings in this theory obtain an internal $\mathbb{CP}^{N-1}$ space in which they move. We presented a number of deformations which lift this internal space at a scale $M$, leaving behind $N_c$ types of vortex string, each embedded in a different diagonal $U(1) \subset U(N_c)$. Strings of the same type reconnect, while strings of different types do not interact. In this manner, the classical probability of two cosmic strings reconnecting at energies $E \ll M$ is $P = 1/N$.

A reconnection probability of $P = 1/N$ may also be achieved by simply considering $N$ decoupled abelian-Higgs models. Our strings are distinguished from this trivial case by two effects. Firstly, even at energies $E \ll M$, the quantum creation of confined magnetic monopoles may lift the probability above $P = 1/N$. We have only been able to compute this effect in the limit of small velocity and small angle of incidence where it is negligible. Away from this regime it may be the dominant contribution to reconnection and a better understanding of this process is desired. The second effect occurs at energy scales $E \gg M$ where the classical probability for reconnection increases to $P = 1$, at least when the moduli space approximation is valid. We find it interesting that semi-classical magnetic strings in non-abelian gauge theories remain strongly coupled at large $N_c$, distinguishing them from their non-perturbative electric counterparts in QCD-like theories, which are expected to interact with coupling $\sim 1/N_c^2$.

For energies beyond the moduli space approximation we have been unable to determine the probability of reconnection analytically, although experience with the abelian-Higgs model suggests that it may remain unity up to very high energies. It would, of course, be interesting to develop numerical simulations to extract the functional dependence of the probability over the full ranges of impact velocities and incidence angles.

Finally, an important, outstanding problem is to determine how the scaling of the string network is affected by the presence of a threshold scale $M$, and the associated confined monopoles, acting like beads on string, which are created after reconnection. One would expect monopoles of to also be created by the Kibble mechanism during formation of the initial string network. It seems plausible that a suitably chosen $M$ may skew the velocity distribution of the strings, giving rise to a larger concentration of low-energy strings. This would distinguish our non-abelian cosmic strings from others such as abelian strings ($P = 1$), strongly coupled QCD-like strings ($P \sim 1/N_c^2$), weakly coupled fundamental strings ($P \sim g_s^2$) and D-strings/wrapped D-branes ($P < 1$). One can only hope that cosmic strings are one day observed, presenting us with the challenge of deciding between these different possibilities.
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