Sequence of families of lattice polarized $K3$ surfaces, modular forms and degrees of complex reflection groups

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August 9, 2024

Abstract

We introduce a sequence of families of lattice polarized $K3$ surfaces. This sequence is closely related to complex reflection groups of exceptional type. Namely, we obtain modular forms coming from the inverse correspondences of the period mappings attached to our sequence. We study a non-trivial relation between our modular forms and invariants of complex reflection groups. Especially, we consider a family concerned with the Shephard-Todd group No.34 based on arithmetic properties of lattices and algebro-geometric properties of the period mappings.

Introduction

In the 20th century, Brieskorn founded an interesting theory which connects finite real reflection groups, Klein singularities and families of rational surfaces (see [Br]). For example, according to this theory, a family of rational surfaces defined by the equation

$$z^2 = y^3 + (\alpha_2 x^3 + \alpha_8 x^2 + \alpha_{14} x + \alpha_{20})y + (x^5 + \alpha_{12} x^3 + \alpha_{18} x^2 + \alpha_{24} x + \alpha_{30})$$

(0.1)

is characterized by the real reflection group $W(E_8)$. Namely, the theory enables us to interpret the parameters $\alpha_2, \alpha_8, \alpha_{12}, \alpha_{14}, \alpha_{18}, \alpha_{20}, \alpha_{24}$ and $\alpha_{30}$ as the invariants of $W(E_8)$ (see [Sl] Chapter IV or [H] Chapter 5). Many researchers have been attracted by this theory and they have tried to generalize it. Indeed, Arnold suggested that it is an interesting problem to obtain an analogous theory for finite complex reflection groups (see [A] p.20). There are many works for that problem based on various ideas, viewpoints and techniques.

On the other hand, the author has studied modular forms derived from periods of $K3$ surfaces and realized a potential of those modular forms to be applied to studies for complex reflection groups. In this paper, we introduce a sequence of families of $K3$ surfaces whose period mappings are closely related to complex reflection groups.

Let $U$ be the even hyperbolic lattice of rank 2. Let $A_m$ or $E_m$ be the root lattices of rank $m$. Then, the $K3$ lattice $L_{K3}$ is given by $U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$. Here, if a lattice $\Lambda$ with the intersection matrix $(c_{ij})$ is given, the lattice given by $(nc_{ij})$ is denoted by $\Lambda(n)$. The 2-homology group of a $K3$ surface $S$ is isometric to $L_{K3}$. The Néron-Severi lattice, which is denoted by $NS(S)$, is a sublattice of $H_2(S, \mathbb{Z})$ of signature $(1, \rho - 1)$, if $S$ is an algebraic $K3$ surface. The transcendental lattice $Tr(S)$ is the orthogonal complement of $NS(S)$ in $H_2(S, \mathbb{Z})$. Then, $Tr(S)$ is an even lattice of signature $(2, 20 - \rho)$.

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In this paper, we introduce a sequence of even lattices

$$A_3 \subset A_2 \subset A_1 \subset A_0 = A,$$

where

$$\begin{align*}
A_0 &= U \oplus U \oplus A_2(-1) \oplus A_1(-1), \\
A_1 &= U \oplus U \oplus A_2(-1), \\
A_2 &= U \oplus U \oplus A_1(-1), \\
A_3 &= U \oplus \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix},
\end{align*}$$

(0.2)

Keywords: $K3$ surfaces ; Modular forms ; Complex reflection groups ; Compactifications defined by arrangements.

Mathematics Subject Classification 2020: Primary 14J28 ; Secondary 11F11, 20F55, 32S22.
Set $M_j = A_j^*$ in $L_{K^3}$. Then, $A_j$ is of type $(2, 5 - j)$ and $M_j$ is of type $(1, 14 + j)$. We introduce a sequence of analytic sets

$$A_3 \subset A_2 \subset A_1 \subset A_0 = \mathbb{A}$$

with $\dim(A_j) = 7 - j$ (for detail, see Lemma 2.1 and 4.1). Then, we obtain an explicit family

$$\varpi_j : \mathfrak{F}_j \rightarrow A_j \quad (j \in \{0, 1, 2, 3\})$$

of $M_j$-polarized $K^3$ surfaces. This roughly means that the Néron-Severi lattice of a generic member of $\mathfrak{F}_j$ is $M_j$. These families $\mathfrak{F}_j$ constitute a sequence of families of $K^3$ surfaces indicated in the following diagram:

$$\begin{array}{ccccccc}
\mathfrak{F}_3 & \overset{i_3}{\longrightarrow} & \mathfrak{F}_2 & \overset{i_2}{\longrightarrow} & \mathfrak{F}_1 & \overset{i_1}{\longrightarrow} & \mathfrak{F}_0 \\
\downarrow \varpi_3 & & \downarrow \varpi_2 & & \downarrow \varpi_1 & & \downarrow \varpi_0 \\
A_3 & \overset{i_3}{\longrightarrow} & A_2 & \overset{i_2}{\longrightarrow} & A_1 & \overset{i_1}{\longrightarrow} & A_0
\end{array}$$

Here, $i_j$ and $\tilde{i}_j$ are natural inclusion. The period domain for $\mathfrak{F}_j$ is given by a connected component $D_j$ of

$$D_{M_j} = \{ [\xi] \in \mathbb{P}(A_j \otimes \mathbb{C}) \mid \xi A_j \xi = 0, \xi A_j \tilde{\xi} > 0 \},$$

which is $(5 - j)$-dimensional.

The main theme of the present paper is to show that there is an interesting and non-trivial relation between the sequence (0.4) and finite complex reflection groups of exceptional type.

The first purpose of this paper is to study the period mapping of the family $\mathfrak{F}_0$. The inverse of the period mapping gives a pair of meromorphic modular forms on $D_0$ (see Definition 5.1). We will obtain a system of generators of the ring of these modular forms by applying techniques for periods of $K^3$ surfaces (Theorem 5.1 and 5.2). In short, we study a family of $K^3$ surfaces defined by the equation

$$z^2 = y^3 + (a_0 x^5 + a_4 x^4 + a_8 x^3) y + (a_2 x^7 + a_6 x^6 + a_{10} x^5 + a_{14} x^4)$$

and we show that the parameters $a_2, a_4, a_6, a_8, a_{10}$ and $a_{14}$ with positive weight induce a system of generators of the ring of modular forms.

Our modular forms are highly expected to have a closed relation with the complex reflection group No.34 in the list of Shephard-Todd [ST] (see also [LT] Appendix D), because three times the weights of the modular forms (namely, $6, 12, 18, 24, 30$ and $42$) are equal to the degrees of the group. This group has the maximal rank among finite complex reflection groups of exceptional type. This expectation is based on not only the above mentioned apparent similarity between the weights and the degrees, but also the following fact. There are exact descriptions of the period mappings for the subfamilies $\mathfrak{F}_1, \mathfrak{F}_2, \mathfrak{F}_3$ via invariants of complex reflection groups. Precisely, the period mappings for $\mathfrak{F}_1, \mathfrak{F}_2, \mathfrak{F}_3$ (resp.) derive Hermitian (Siegel, Hilbert, resp.) modular forms with explicit expressions via the invariants of the complex reflection group No.33 (No.31, No.23, resp.) of rank $r_j$ and a system of appropriate theta functions (for detail, see Section 4.3; see also Remark 5.2). Let $\left( w^{(j)}_1, \ldots, w^{(j)}_r \right)$ be the weights of the modular forms as in Table 1. Then, the degrees of the complex reflection groups are given by $\left( \kappa_j w^{(j)}_1, \ldots, \kappa_j w^{(j)}_r \right)$. Here, the weights of the theta functions account for the integer $\kappa_j$.

**Table 1: Modular forms coming from $K^3$ surfaces and complex reflection groups**

| $j$ | Modular forms | Weights | $K^3$ surfaces | Reflection groups | $r_j$ | $\kappa_j$ |
|-----|--------------|---------|----------------|--------------------|------|---------|
| 0   | see Definition 5.1 | 2, 4, 6, 8, 10, 14 | This paper | No.34 | 6 | 3 |
| 1   | Hermitian | 4, 6, 10, 12, 18 | NS2 | No.33 | 5 | 1 |
| 2   | Siegel | 4, 6, 10, 12 | CD | No.31 | 4 | 2 |
| 3   | Hilbert | 2, 6, 10 | Na1 | No.23 | 3 | 1 |

In these works for the families $\mathfrak{F}_j$ ($j \in \{1, 2, 3\}$), the Satake-Baily-Borel compactifications for bounded symmetric domains play big roles. However, in the case of $\mathfrak{F}_0$, the Satake-Baily-Borel compactifications are inadequate, because we need to consider modular forms defined on a complement of an arrangement
of hyperplanes (see Section 3 and 5). So, instead of the Satake-Baily-Borel compactifications, we will consider the Looijenga compactifications constructed in [L2]. The Looijenga compactifications are coming from arithmetic arrangements of hyperplanes. We can regard them as interpolations between the Satake-Baily-Borel compactifications and toroidal compactifications. Their properties are essential for our construction of modular forms.

By the way, there exists a double covering of every member of $\mathfrak{F}_0$, which is a $K3$ surface also. We obtain the family

$$\varpi_j : G_j \rightarrow \mathfrak{A}_j \quad (j \in \{0, 1, 2, 3\})$$

whose members are such double coverings and we have the following diagram:

$$\begin{array}{ccc}
G_j & \xrightarrow{\varphi_j} & \mathfrak{F}_j \\
\downarrow{\varpi_j} & & \downarrow{\varpi_j} \\
\mathfrak{A}_j & & 
\end{array}$$

(0.7)

Here, $\varphi_j$ is a correspondence given by the double covering.

The second purpose of this paper is to determine the transcendental lattice for $G_0$. The family $G_0$ has interesting features. For example, it naturally contains the famous family of Kummer surfaces coming from principally polarized Abelian surfaces. Moreover, $G_0$ is a natural extension of the family studied by Matsumoto-Sasaki-Yoshida [MSY], whose periods are solutions of the hypergeometric equation of type $(3, 6)$. In spite of interesting properties of $G_j \ (j \in \{0, 1, 2, 3\})$, it is not straightforward to determine the lattices for them. For example, if $j \in \{1, 2, 3\}$, the lattices for $G_j$ were determined via precise arguments or heavy calculations (for detail, see Section 6). In the present paper, we will determine the transcendental lattice for $G_0$, based on the result of $G_1$ and arithmetic properties of even lattices (Theorem 6.1). As a result, the transcendental lattices $B_j$ for $G_j$ are given as follows.

$$\begin{align*}
B_0 &= U(2) \oplus U(2) \oplus \begin{pmatrix} -2 & 0 & 1 \\ 0 & -2 & 1 \\ 1 & 1 & -4 \end{pmatrix}, \\
B_1 &= A_1(2) = U(2) \oplus U(2) \oplus A_2(-2), \\
B_2 &= A_2(2) = U(2) \oplus U(2) \oplus A_1(-2), \\
B_3 &= A_3(2) = U(2) \oplus \begin{pmatrix} 4 & 2 \\ 2 & -4 \end{pmatrix}.
\end{align*}$$

(0.8)

Especially, we note that $B_0$ is not just $A_0(2)$.

It is an interesting problem to describe our meromorphic modular forms via the invariants of the group No.34 and explicit special functions (like theta functions). Furthermore, it may be quite meaningful to understand why the period mappings for the sequence (0.4) are related with complex reflection groups. While the methods in this paper and preceding papers [CD], [Na1] and [NS2] are just based on algebro-geometric properties of $K3$ surfaces and arithmetic properties of modular forms, the author does not know the fundamental reason why the complex reflection groups work effectively as in Table 1. The author expects that there exists an unrevealed principle underlying the relation between our sequence of the families and complex reflection groups. There are indications which support this expectation. For example, Sekiguchi [Se] studies Arnold’s problem based on methods of Frobenius potentials and he obtains a family of rational surfaces. Although his standpoint and methods are widely different from ours, a direct calculation shows that our $K3$ surface of (0.6) is related to Sekiguchi’s rational surface (see Remark 2.2). The author is hoping that a new principle will rationalize our families of $K3$ surfaces from the viewpoint of complex reflection groups in near future, as Brieskorn’s theory enables us to explain the essence of the family of the rational surfaces of (0.1).

1 Arithmetic arrangement of hyperplanes and Looijenga compactification

Looijenga constructed compactifications for bounded symmetric domains of type $IV$ derived from arithmetic arrangements of hyperplanes. First, we will survey his result. For detail, see [L2].
Let $V$ be an $(n+2)$-dimensional vector space over $\mathbb{C}$ with a non-degenerated symmetric bilinear form $\varphi$. Now, we suppose that $(V, \varphi)$ has been defined over $\mathbb{Q}$ and $\varphi$ is of signature $(2, n)$ for the $\mathbb{Q}$-structure. Then, the set $\{[v] \in V \mid \varphi(v, v) = 0, \varphi(v, v) > 0\}$ has two connected components $\mathcal{D}$ and $\mathcal{D}_-$. We take $\mathcal{D}$ from $\{\mathcal{D}, \mathcal{D}_-\}$. This is a bounded symmetric domain of type $IV$. For a linear subspace $L$ of $V$, we set $\mathcal{D}_L = \mathcal{D} \cap \mathcal{F}(L)$. The orthogonal group $O(\varphi)$ is an algebraic group over $\mathbb{Q}$. Let $\Gamma$ be an arithmetic subgroup of $O(\varphi)$. We set $\mathcal{X} = \mathcal{D}/\Gamma$.

If a hyperplane $H$ of $V$ is defined over $\mathbb{Q}$ and of signature $(2, n-1)$, it gives a hypersurface $\mathcal{D}_H \neq \phi$. Suppose $\mathcal{H}$ is a $\Gamma$-invariant arrangement of hyperplanes satisfying this property. Such an arrangement is said to be arithmetic if it is given by a finite union of $\Gamma$-orbits.

Set
\[ \mathcal{D}^0 = \mathcal{D} - \bigcup_{H \in \mathcal{H}} \mathcal{D}_H. \]

Looijenga [L2] constructs a natural compactification of $X^\circ = \mathcal{D}^0/\Gamma$. Namely, the Looijenga compactification is given by
\[ \widehat{X}^\circ_L := \mathcal{D}^0/\Gamma, \]
where
\[ \mathcal{D}^o_L = \mathcal{D}^0 \sqcup \bigsqcup_{L \in \mathcal{P}(\mathcal{H}|_\varphi)} \pi_L(\mathcal{D}^0) \sqcup \bigsqcup_{\sigma \in \Sigma(\mathcal{H})} \pi_\sigma(\mathcal{D}^0). \]

The disjoint union of (1.2) admits an appropriate topology and $\mathcal{D}^0$ is an open and dense set in $\mathcal{D}^0_L$. We remark that the Looijenga compactification $\widehat{X}^\circ_L$ coincides with the Satake-Baily-Borel compactification $\widehat{X}^{SBB}$ when $\mathcal{H} = \phi$. In the following, we will see the meaning of (1.1) and (1.2).

Letting $L$ be a subspace of $V$, we have a natural projection $\pi_L : \mathbb{P}(V) - \mathbb{P}(L) \to \mathbb{P}(V/L)$. Let $\mathcal{P}(\mathcal{H}|_\varphi)$ be a set of subspaces $L$ of $V$ such that there exists $z \in L$ with $\varphi(z, \bar{z}) > 0$. We note that $L \in \mathcal{P}(\mathcal{H}|_\varphi)$ if and only if $\mathcal{D}_L \neq \phi$. Also, we remark that $\mathcal{P}(\mathcal{H}|_\varphi)$ is a partially ordered set (see [L1] Section 2 and 3).

For a $\mathbb{Q}$-isotropic line $I$ in $V$, $\varphi$ defines a bilinear form on the $n$-dimensional space $I^\perp/I$ of signature $(1, n-1)$. Taking the choice of $\mathcal{D}$ from $\{\mathcal{D}, \mathcal{D}_-\}$ into account, we have an $n$-dimensional cone $C_I$ in $I^\perp/I$. Let $C_{I^+}(\subset I^\perp/I)$ be the convex hull of $(I^\perp/I) \cap \mathcal{C}_I$. If a member $H \in \mathcal{H}$ contains $I$, then it naturally determines a hyperplane $H_{I^+}/I$ of $I^\perp/I$ of signature $(1, n-2)$. The hyperplanes $H_{I^+}/I$ with $H_{I^+}/I \cap C_{I^+} \neq \phi$ give a decomposition $\Sigma(\mathcal{H})_I$ of the cone $C_{I^+}$ into locally rational cones.

We set $\Sigma(\mathcal{H}) = \bigcup_I \Sigma(\mathcal{H})_I$. For a cone $\sigma \in \Sigma(\mathcal{H})_I(\subset \Sigma(\mathcal{H}))$, we have the $\Sigma$-support space $V_\sigma(\subset I^\perp/I)$. Namely, $V_\sigma$ contains $I$ and corresponds to the $\mathbb{C}$-span of the cone $\sigma(\subset I^\perp/I)$. Hence, $V_\sigma$ is given by the intersection of $I^\perp/I$ and the members $H$ of $\mathcal{H}$ such that $H \supset I$. We put $\pi_\sigma = \pi_{V_\sigma}$.

Set
\[ \overline{\mathcal{D}}^{\Sigma(\mathcal{H})} = \prod_{\sigma \in \Sigma(\mathcal{H})} \pi_\sigma(\mathcal{D}). \]

Then, $X^{\Sigma(\mathcal{H})} = \overline{\mathcal{D}}^{\Sigma(\mathcal{H})}/\Gamma$ is a normal analytic space.

We have a blowing up $\widehat{X}^\circ \to X^{\Sigma(\mathcal{H})}$, which is coming from the connected components of intersections of members $H \in \mathcal{H}$. The Looijenga compactification $\widehat{X}^\circ_L$ of (1.1) is equal to a blowing down $\widehat{X}^\circ \to \widehat{X}^\circ_L$.

**Theorem 1.1.** ([L2] Corollary 7.5) Suppose that every $\pi_\sigma(\mathcal{D})$ in (1.3) is not $(n-1)$-dimensional. Then, the algebra
\[ \bigoplus_{k \in \mathbb{Z}} H^0(\mathcal{D}^0, \mathcal{O}(L^k))^{\Gamma}, \]
where $\mathcal{L}$ is the natural automorphic bundle over $\mathcal{D}$, is finitely generated with positive degree generators. Its $\text{Proj}$ gives the Looijenga compactification $\widehat{X}^\circ_L$ of (1.1). The boundary $\widehat{X}^\circ - X^\circ$ is the strict transform of the boundary $X^{\Sigma(\mathcal{H})} - X$. Especially, \[ \text{codim} \left( \widehat{X}^\circ_L - X^\circ \right) \geq 2 \] holds.
Let $\mathcal{L}$ be an ample line bundle on $X^s$ such that $\mathcal{L}|_{X^s} = \mathcal{L}|_{X^s}$. It is shown in [2] that every meromorphic $\Gamma$-invariant automorphic form whose poles are contained in $\mathcal{H}$ is corresponding to a meromorphic section $s$ of $\mathcal{L}$ such that $s|_{X^s}$ is holomorphic.

1.1 Lattice $A$ and arrangement $\mathcal{H}$

We set
\[ A = A_0 = U \oplus U \oplus A_2(-1) \oplus A_1(-1). \] (1.4)
For this lattice $A$, we set
\[ \Gamma = \hat{O}(A) \cap O^+(A). \] (1.5)
Here, $\hat{O}(A)$ is the stable orthogonal group: $\hat{O}(A) = \text{Ker} \ (O(A) \to \text{Aut}(A^\vee/A))$, where $A^\vee = \text{Hom}(A, \mathbb{Z})$. Also, $O^+(A)$ is the subgroup of $O(A)$ which preserves the connected component $\mathcal{D}$. The lattice $A$ satisfies the Kneser conditions in the sense of Gritsenko-Hulek-Sankaran [GHS]. Therefore, we have the following result.

**Proposition 1.1.** Let $\Delta(A)$ be the set of vectors $v \in A$ such that $(v \cdot v) = -2$. The group $\Gamma$ of (1.5) is generated by reflections $\sigma_\delta : z \mapsto z + (z \cdot \delta)\delta$ for $\delta \in \Delta(A)$ and it holds $\text{Char}(\Gamma) = \{\text{id}, \text{det}\}$.

Here, we note that the intersection number of elements $v_1$ and $v_2$ of lattices are often denoted by $(v_1 \cdot v_2)$ in this paper.

**Lemma 1.1.** The group $\Gamma$ is isomorphic to the projective orthogonal group $PO^+(A)$.

**Proof.** Let $\{\alpha_1, \alpha_2, \alpha_3\}$ be a basis of $A_2(-1) \oplus A_1(-1)$ with $(\alpha_j \cdot \alpha_j) = -2 \ (j \in \{1, 2, 3\})$, $(\alpha_1 \cdot \alpha_2) = 1$ and $(\alpha_k \cdot \alpha_3) = 0 \ (k \in \{1, 2\})$. Then, $y_1 = \frac{3}{2} \alpha_1 + \frac{1}{2} \alpha_2$, $y_2 = \frac{5}{2} \alpha_1 + \frac{1}{2} \alpha_2$ and $y_3 = \frac{1}{2} \alpha_3$ generate the discriminant group $A^\vee/A$. It follows that $-id_{O^+(A)} \notin \Gamma$ and $O^+(A)/\Gamma \cong \mathbb{Z}/2\mathbb{Z}$. Hence, the assertion follows.

We consider the case $V = A \oplus \mathbb{C}$. Let $e_j, f_j \ (j \in \{1, 2\})$ be elements of $U^{\oplus 2}$ satisfying $(e_j \cdot e_k) = (f_j \cdot f_k) = 0$ and $(e_j \cdot f_k) = \delta_{j,k}$. Let $\{\alpha_1, \alpha_2, \alpha_3\}$ be the basis of $A_2(-1) \oplus A_1(-1)$ as in Lemma 1.1 A vector $v \in V$ is given by the form
\[ v = \xi_1 e_1 + \xi_2 e_1 + \xi_3 e_2 + \xi_4 f_2 + \xi_5 \alpha_1 + \xi_6 \alpha_2 + \xi_7 \alpha_3. \] (1.6)
Let us consider a hyperplane $H_0 = \{\xi_7 = 0\}$ in $V$. For $\Gamma$ of (1.5), the $\Gamma$-orbits of $H_0$ give an arithmetic arrangement $\mathcal{H}$.

**Lemma 1.2.** The above arrangement $\mathcal{H}$ of hyperplanes satisfies the condition of Theorem 1.1

**Proof.** We will prove that all members of $\mathcal{H}$ contain a non-zero common subspace of the negative-definite vector space $\langle \alpha_1, \alpha_2, \alpha_3 \rangle_\mathbb{C}$. Then, it is guaranteed that our arrangement $\mathcal{H}$ satisfies the condition of Theorem 1.1 as in the argument of [1] Section 6.

The hyperplane $H_0 = \{\xi_7 = 0\}$ of $V$ is generated by the basis $\{e_1, f_1, e_2, f_2, \alpha_1, \alpha_2\}$. Using the notation in the proof of Lemma 1.1 $A^\vee/A$ is generated by $y_1, y_2$ and $y_3$. We note that every $\gamma \in \Gamma$ fixes $y_j \in A^\vee/A$. This implies that we can take a basis of the subspace $\gamma H_0$ such that this basis is an extension of $\langle \alpha_1, \alpha_2 \rangle$. Therefore, every member of $\mathcal{H}$ contains the 2-dimensional subspace $\langle \alpha_1, \alpha_2 \rangle_\mathbb{C}$ of $\langle \alpha_1, \alpha_2, \alpha_3 \rangle_\mathbb{C}$. \]

**Remark 1.1.** The condition of Theorem 1.1 is not always satisfied. For example, under the notation \[ (1.6) \], let us take a hyperplane $H'_0 = \{\xi_5 - \xi_6 = 0\}$. We can see that the arrangement $\mathcal{H}'$ of $\gamma H'_0$ for $\gamma \in \Gamma$ does not satisfy the condition of Theorem 1.1. This condition is closely related to whether an arrangement gives the zero of a modular form or not. In fact, we can see that $\mathcal{H}$ is not the zero set of a modular form, whereas $\mathcal{H}'$ does.

For our symmetric space $\mathcal{D} = D_6$, which is a connected component of $\mathcal{D}_{\mathcal{M}_6}$ of [0.5], set
\[ D^o = D - \bigcup_{H \in \mathcal{H}} D_H. \] (1.7)
Due to Theorem 1.1 and Lemma 1.2, we have the following result for our arrangement $\mathcal{H}$. 

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Proposition 1.2.
\[ \text{codim} \left ( \frac{D^\circ}{T} - \frac{D^\circ}{T} \right ) \geq 2. \]

2 Family $\mathfrak{F}_0$ of K3 surfaces with Picard number 15

In the present paper, we will consider the Looijenga compactification coming from the arithmetic arrangement $\mathcal{H}$ in Section 1.1. In order to obtain an explicit model of the compactification, we will introduce a family of elliptic K3 surfaces whose transcendental lattice is $\mathbf{A}$ in (1.4) (see Theorem 2.1). Periods for K3 surfaces are very important in our argument. We remark that the period mapping for a family of K3 surfaces is essentially related to the arithmetic property of the transcendental lattice for a generic member of the family.

For $a = (a_0, a_2, a_4, a_6, a_8, a_{10}, a_{14}) \in \mathbb{C}^7 - \{0\} =: \mathcal{C}_a$, we consider the hypersurface $S_a$ defined by an equation
\[ S_a : z^2 = y^3 + (a_0 x^5 + a_4 x^4 w + a_6 x^3 w^2 + a_8 x^2 w^3 + a_{10} x w^4 + a_{14}) \]
of weight 30 in the weighted projective space $\text{Proj}(\mathbb{C}[x, y, z, w]) = \mathbb{P}(4, 10, 15, 1)$. We have a natural action of $\mathbb{C}^*$ on $\mathbb{P}(4, 10, 15, 1)$ given by $(x, y, z, w) \mapsto (x, y, z, \lambda^{-1} w)$ and that on $\mathcal{C}_a$ given by $a \mapsto \lambda \cdot a = (\lambda^6 a_k) = (a_0, \lambda^2 a_2, \lambda^4 a_4, \lambda^6 a_6, \lambda^8 a_8, \lambda^{10} a_{10}, \lambda^{14} a_{14})$.

By applying [NSI] Proposition 3.1, we have the following result.

**Lemma 2.1.** Let $\mathfrak{A}_0 = \mathfrak{A}$ be the set of parameters $a \in \mathcal{C}_a$ such that $S_a$ is a K3 surface. Then, $\mathfrak{A}_0$ is a subset of $\{a \in \mathcal{C}_a \mid a_0 \neq 0\} \cup \{a \in \mathcal{C}_a \mid a_2 \neq 0\}$ such that
\[ \mathcal{C}_a - \mathfrak{A}_0 = C' \cup C'' \]
where
\[ \left\{ \begin{array}{l}
C' = \{a \in \mathcal{C}_a \mid a_0 \neq 0, a_{10} = a_{12} = a_{18} = 0\} \subset \{a_0 \neq 0\}, \\
C'' = \{a \in \mathcal{C}_a \mid a_2 \neq 0, a_0 = a_{10} = a_{12} = a_{18} = 0\} \subset \{a_2 \neq 0\}.
\end{array} \right. \]

**Remark 2.1.** The surface $S_a$ degenerates to a rational surface if $a_0 = a_2 = 0$.

We have a family
\[ \varpi_0 : \mathfrak{F}_0 = \{S_a \text{ of } (2.1) \mid a \in \mathfrak{A}_0\} \to \mathfrak{A}_0 \]
of elliptic K3 surfaces.

2.1 Singular fibres

The Weierstrass equation (2.1) defines an elliptic surface $\pi_a : S_a \to \mathbb{P}^1(\mathbb{C})$. For a generic point $a \in \mathfrak{A}_0$, we have singular fibres for $\pi_a$ of Kodaira type
\[ III^* + IV^* + 7I_1, \]
as illustrated in Figure 1. Now, $\pi_a^{-1}(\infty)$ ($\pi_a^{-1}(0)$, resp.) is a singular fibre of Kodaira type $III^*$ ($IV^*$, resp.). Each gives an $E_7$-singularity and an $E_6$-singularity, respectively.

Set $x_0 = \frac{z}{w}$ and
\[ g_2'(x_0, a) = a_0^5 + a_4 x_0^4 + a_8 x_0^3, \quad g_3'(x_0, a) = a_2 x_0^7 + a_6 x_0^6 + a_{10} x_0^5 + a_{14} x_0^4. \]
Let $r(a)$ be the resultant of $g_2'(x_0, a)$ and $g_3'(x_0, a)$ in $x_0$. Also, let $R(x_0, a)$ be a polynomial in $x_0$ coming from the discriminant of the right hand side of (2.1) in $y$. So, the discriminant of $\frac{1}{x_0} R(x_0, a)$ in $x_0$ is given by $r(a)^3 d_{44}(a)$, where $d_{84}(a)$ can be calculated as a polynomial in $a$ of weight 84.

The following lemma can be proved by arguments of elliptic surfaces as in [NSI] Section 3, [HU] Section 6 and [Na3] Section 1. In particular, we can determine the Kodaira type of each singular fibre by [NSI] Remark 3.5.

**Lemma 2.2.** Except for the generic case corresponding to the fibres of (2.2), the types of the singular fibres for the elliptic surface $\pi_a : S_a \to \mathbb{P}^1(\mathbb{C})$ ($a \in \mathfrak{A}_0$) are given by the following.
If $a \in \mathfrak{A}_0$ satisfies $r(a) = 0$, there is a new singular fibre of Kodaira type $II$ on the elliptic surface $S_a$. Such a singular fibre does not acquire any new singularities.

If $a \in \mathfrak{A}_0$ satisfies $d_{84}(a) = 0$, two of singular fibres of Kodaira type $I_1$ in (2.2) collapse into a singular fibre of type $I_2$:

$$III^* + IV^* + I_2 + 5I_1$$

In this case, a new $A_1$-singularity appears on the $K3$ surface $S_a$.

If $a_0 = 0$, there are the singular fibres of Kodaira type

$$II^* + IV^* + 6I_1$$
on $S_a$. In this situation, the $E_7$-singularity of (2.2) turns into an $E_8$-singularity.

If $a_{14} = 0$, there are the singular fibres of Kodaira type

$$III^* + III^* + 6I_1$$
on $S_a$. The $E_6$-singularity of (2.2) turns into an $E_7$-singularity.

2.2 Local period mapping

Set

$$\tilde{\mathfrak{A}} = \mathfrak{A} - \{a \in \mathfrak{C}_a : a_0a_{14}d_{84}(a) = 0\}.$$  \hspace{1cm} (2.3)

Let $F$ be a general fibre for the elliptic surface $\pi_a$ and $O$ be the zero section. Let $C_0, \ldots, C_7$ ($D_0, \ldots, D_6$, resp.) be nodal curves in the singular fibre of type $III^*$ ($IV^*$, resp.) indicated Figure 1. For $a \in \tilde{\mathfrak{A}}$, the lattice generated by

$$F, O, C_1, \ldots, C_7, D_1, \ldots, D_6$$  \hspace{1cm} (2.4)

is a sublattice of $\text{NS}(S_a)$ whose intersection matrix is

$$\mathbb{M} = \mathbb{M}_0 = U \oplus E_7(-1) \oplus E_6(-1).$$  \hspace{1cm} (2.5)

Let $L_{K3}$ be the $K3$ lattice : $L_{K3} = II_{3,19}$. Applying \cite{N2} Theorem 1.14.4 (see also \cite{Mo} Theorem 2.8), we can see that $\mathbb{M}$ has a unique primitive embedding into $L_{K3}$ up to isometry. The orthogonal complement of $\mathbb{M}$ with respect to the unimodular lattice $L_{K3}$ is given by $\mathbb{A}$ of (1.4). We have an isometry $\psi: H_2(S_a, \mathbb{Z}) \rightarrow L_{K3}$ such that

$$\psi(F) = \gamma_8, \quad \psi(O) = \gamma_9, \quad \psi(C_j) = \gamma_{9+j}, \quad \psi(D_k) = \gamma_{16+k} \quad (j \in \{1, \ldots, 7\}, k \in \{1, \ldots, 6\}).$$

Then, the sublattice $\langle \gamma_8, \ldots, \gamma_{22} \rangle_{\mathbb{Z}}$ in $L_{K3}$ is isometric to the lattice $\mathbb{M}$ of (2.5). This is a primitive sublattice, because $|\text{det}(\mathbb{M})| = 6$ is square-free. Hence, we can take $\gamma_1, \ldots, \gamma_7 \in L_{K3}$ such that $\{\gamma_1, \ldots, \gamma_7, \gamma_{8}, \ldots, \gamma_{22}\}$ gives a basis of $L_{K3}$. Let $\{\delta_1, \ldots, \delta_{22}\}$ be the dual basis of $\{\gamma_1, \ldots, \gamma_{22}\}$ with respect to the unimodular lattice $L_{K3}$. Then, the intersection matrix of the sublattice $\langle \delta_1, \ldots, \delta_7 \rangle_{\mathbb{Z}}$ is equal to the intersection matrix of $\mathbb{A}$.

Figure 1: Singular fibres for $\pi_a : S_a \rightarrow \mathbb{P}^1(\mathbb{C})$
Proposition 2.1. (The canonical form for \( a_0 \neq 0 \)) If \( a \in A \cap \{a_0 \neq 0\} \), (2.1) is transformed to the Weierstrass equation

\[
S(u) : z^2 = y^3 + (x^5 + u_4x^4w^4 + u_8x^3w^8)y + (u_2x^7w^2 + u_6x^6w^6 + u_{10}x^5w^{10} + u_{14}x^4w^{14}).
\]  

(2.6)

Proof. By putting

\[
x \mapsto \frac{x}{a_0}, \quad y \mapsto \frac{y}{a_6^2}, \quad z \mapsto \frac{z}{a_6^3},
\]


and

\[
u_2 = \frac{a_2}{a_0}, \quad \nu_4 = a_4, \quad \nu_6 = a_6, \quad \nu_8 = a_0a_8, \quad \nu_{10} = a_0a_{10}, \quad \nu_{14} = a_0^2a_{14},
\]

we obtain (2.6). \( \square \)

Here, we put \( u = (u_k) = (u_2, u_4, u_6, u_8, u_{10}, u_{14}) \in \mathbb{C}^6 - \{0\} =: C_u. \) Let \( U^* \) be a subset of \( C_u \) of codimension 3 such that

\[
C_u - U^* = \{u \in C_u \mid \nu_{10} = \nu_{12} = \nu_{18} = 0\}.
\]

(2.7)

For \( \lambda \in \mathbb{C}^* \) and \( u = (u_k) \in U, \) set \( \lambda \cdot u = (\lambda^k u_k). \) This action induces an isomorphism \( \lambda : S(u) \to S(\lambda \cdot u). \)

Letting \([u] = (u_2 : u_4 : u_6 : u_8 : u_{10} : u_{14}) \in \mathbb{P}(2, 4, 6, 8, 10, 14)\) be the point which is corresponding to \( u \in C_u, \) we set \( U = \{[u] \in \mathbb{P}(2, 4, 6, 8, 10, 14) \mid u \in U^*\}. \) The above action of \( C^* \) on \( U^* \), we naturally defines the family

\[
\{S([u]) \mid [u] \in U\} \to U.
\]

(2.8)

Definition 2.1. Let \( \pi_1 : S_1 \to \mathbb{P}^1(\mathbb{C}) \) and \( \pi_2 : S_2 \to \mathbb{P}^1(\mathbb{C}) \) be two elliptic surfaces. Suppose that there exist a biholomorphic mapping \( f : S_1 \to S_2 \) and \( \varphi \in \text{Aut}(\mathbb{P}^1(\mathbb{C})) \) with \( \varphi \circ \pi_1 = \pi_2 \circ f. \) Then, these two elliptic surfaces are said to be isomorphic as elliptic surfaces.

The canonical form (2.6) naturally gives an elliptic surface \( \pi_{[u]} : S([u]) \to \mathbb{P}^1(\mathbb{C}). \)

Lemma 2.3. Two elliptic surfaces \( \pi_{[u_1]} : S([u_1]) \to \mathbb{P}^1(\mathbb{C}) \) and \( \pi_{[u_2]} : S([u_2]) \to \mathbb{P}^1(\mathbb{C}) \) are isomorphic as elliptic surfaces if and only if \([u_1] = [u_2] \in \mathbb{P}(2, 4, 6, 8, 10, 14). \)

Proof. We can prove it by an argument which is similar to the proof of [Na3] Lemma 1.1. \( \square \)

Let us take a generic point \( a \in \tilde{A} \) of (2.3). Since \( \tilde{A} \subset A \cap \{a_0 \neq 0\}, \) we obtain the corresponding surface \( S([u]) \) for a parameter \([u] \in \mathbb{P}(2, 4, 6, 8, 10, 14) = \text{Proj}(\mathbb{C}[u_2, u_4, u_6, u_8, u_{10}, u_{14}]), \) which is given by the canonical form (2.6). We set

\[
P(\tilde{A}) = \{[u] \in \mathbb{P}(2, 4, 6, 8, 10, 14) \mid \text{there exists } a \in \tilde{A} \text{ such that } S_a \text{ is identified with } S([u]) \text{ of (2.6)}\}.
\]

By an argument which is similar to [Na3] p.41, using Lemma 2.3, also, we can obtain a local period mapping defined on sufficiently small neighborhood around \([u]\) in \( P(\tilde{A}). \) By gluing the local period mappings, we obtain the period mapping

\[
\Phi_1 : P(\tilde{A}) \to D
\]

given by

\[
[u] \mapsto \left( \int_{\psi_{[u]}^{-1}(\gamma_1)} \omega_{[u]} : \cdots : \int_{\psi_{[u]}^{-1}(\gamma_r)} \omega_{[u]} \right),
\]

where \( \omega_{[u]} \) is a unique holomorphic 2-form on \( S([u]) \) up to a constant factor and

\[
\psi_{[u]} : H_2(S([u]), \mathbb{Z}) \to L_{K3}
\]

is an appropriate isometry, called \( S \)-marking. We note that the period mapping (2.9) is a multivalued analytic mapping.

We call the pair \( (S([u]), \psi_{[u]}) \) an \( S \)-marked \( K3 \) surface. By applying Torelli's theorem to the above local period mappings, we can show the following theorem as in the proof of [Na3] Theorem 1.1 and Corollary 1.1.

Theorem 2.1. For a generic point \( a \in \tilde{A} \) of (2.3), the Picard number of \( S_a \) is 15. The intersection matrix of the \( \text{Néron-Severi lattice NS}(S_a) \) (the transcendental lattice \( \text{Tr}(S_a), \) resp.) is equal to \( M \) of (2.3) (\( A \) of (4.4), resp.).
2.3 Double covering \( K_a \) of \( S_a \)

The \( K3 \) surface \( S_a \) of (2.1) is transformed to

\[
Z^2 = Y^3 + \left( a_4 + a_6X + \frac{a_8}{X} \right) Y + \left( a_6 + a_2X + \frac{a_{10}}{X^2} + \frac{a_{14}}{X^2} \right)
\]

(2.10)

by the birational transformation

\[
x \mapsto X, \quad y \mapsto X^2Y, \quad z \mapsto X^3Z.
\]

We have a double covering

\[
K_a : Z^2 = Y^3 + \left( a_4 + a_6U^2 + \frac{a_8}{U^2} \right) Y + \left( a_6 + a_2U^2 + \frac{a_{10}}{U^2} + \frac{a_{14}}{U^4} \right)
\]

(2.11)

of (2.10). There is a Nikulin involution on \( K_a \) given by

\[
\iota_{K_a} : (U, Y, Z) \mapsto (-U, Y, -Z).
\]

This means that it satisfies \( \iota_{K_a}^* \omega_K = \omega_K \), where \( \omega_K \) is a unique holomorphic 2-form up to a constant factor.

We have a family \( V_0 : \mathfrak{S}_0 \to \mathfrak{A}_0 \) of \( K3 \) surfaces, where \( \mathfrak{S}_0 = \{ K_a \text{ of (2.11)} | a \in \mathfrak{A}_0 \} \). In fact, this surface \( K_a \) can be regarded as a natural generalization of the Kummer surface for a principally polarized Abelian surface. Therefore, we call a member of \( \mathfrak{S}_0 \) a Kummer-like surface in this paper. More precisely, see Section 4. We remark that the double covering \( \varphi_0 \) in (0.7) is given by this \( \iota_{K_a} \).

Remark 2.2. The surface \( K_a \) of (2.11) has another involution

\[
j_{K_a} : (U, Y, Z) \mapsto (-U, Y, Z).
\]

This is not a Nikulin involution. The minimal resolution of the quotient surface \( K/\langle j_{K_a} \rangle \) is given by the equation

\[
\Sigma : z'^2 = y'^3 + \left( a_0x'^3 + a_4x'^2 + a_8x' \right)y' + \left( a_2x'^4 + a_6x'^3 + a_{10}x'^2 + a_{14}x' \right).
\]

This is a rational surface. The surface \( \Sigma \) is very similar to a surface appearing in Sekiguchi’s recent work \[5\], in which he studies algebraic Frobenius potentials, deformation of singularities and Arnold’s problem. Namely, he obtains the equation in the form

\[
z'^2 = f_{E_7(1)} := y'^3 + (x'^3 + t_2x'^2 + t_4x')y + (t_1x'^4 + t_3x'^3 + t_5x'^2 + t_7x) + s_3y'^2
\]

of a rational surfaces. Putting \( s_3 = 0 \), we can see an apparent correspondence to our \( \Sigma \).

The author is expecting that there is a non-trivial and unrevealed theory connecting our work of the moduli of \( K3 \) surfaces and Sekiguchi’s result of Frobenius potentials.

3 Moduli space of M-polarized \( K3 \) surfaces

In this section, letting \( M \) be the lattice of (2.5) of signature (1,14), we consider the moduli space of \( M \)-polarized \( K3 \) surfaces. An \( M \)-polarized \( K3 \) surface is a pair \((S, j)\) of a \( K3 \) surface \( S \) and a primitive embedding \( j : M \hookrightarrow \text{NS}(S) \). Two \( M \)-polarized \( K3 \) surfaces \((S_1, j_1)\) and \((S_2, j_2)\) are said to be isomorphic if there exists an isomorphism \( f : S_1 \to S_2 \) of \( K3 \) surfaces such that \( j_2 = f_\ast \circ j_1 \). In this paper, \( \text{NS}(S) \) is often regarded as a sublattice of the homology group \( H_2(S, \mathbb{Z}) \). We note that \( \text{NS}(S) \) is identified with the sublattice \( H^2(S, \mathbb{Z}) \cap H^{1,1}(S, \mathbb{R}) \) of the cohomology group \( H^2(S, \mathbb{Z}) \) by the Poincaré duality. This is denoted by the same notation in the discussion below.

Let \( V(S)^+ \) be the connected component of \( V(S) = \left\{ x \in H_2^{1,1}(S) \mid (x \cdot x) > 0 \right\} \) which contains the class of a Kähler form on \( S \). Let \( \Delta(S)^+ \) be the subset of effective classes of \( \Delta(S) = \left\{ \delta \in \text{NS}(S) \mid (\delta \cdot \delta) = -2 \right\} \). Set \( C(S)^+ \) of \( C(S) \), which is defined by the condition \((x \cdot x) > 0 \), is called the Kähler cone. We set \( \text{NS}(S)^+ = C(S)^+ \cap H^2(S, \mathbb{Z}) \) and \( \text{NS}(S)^{++} = C(S)^+ \cap H^2(S, \mathbb{Z}) \).

Due to Theorem 2.1, we can take a point \( \tilde{a} \in \tilde{A} \) and an \( S \)-marking \( \psi_0 : H_2(S_0, \mathbb{Z}) \to L_{K3} \) such that \( \psi_0^{-1}(M) = \text{NS}(S_0) \), where \( S_0 = S_0 \) is called a reference surface. Letting \( \Delta(M) = \left\{ \delta \in M \mid (\delta \cdot \delta) = -2 \right\} \),
we set \( \Delta(M)^+ = \{ \delta \in \Delta(M) \mid \psi^{-1}_0(\delta) \in \text{NS}(S_0) \text{ gives an effective class} \} \). The set \( V(M) = \{ y \in \mathbb{M}_R \mid (y \cdot y) > 0 \} \) has two connected components. We suppose the component \( V(M)^+ \) contains \( \psi_0(x) \) for \( x \in V(S_0)^+ \). Set \( C(M)^+ = \{ y \in V(M)^+ \mid (y \cdot \delta) > 0, \text{ for all } \delta \in \Delta(M)^+ \} \). An \( M \)-polarized \( K \) surface \((S,j)\) is called a pseudo-ample \( M \)-polarized \( K \) surface if \( j(C(M)^+) \cap \text{NS}(S) \) is effective.

For a \( K \) surface \( S \), let \( \psi : H_2(S, \mathbb{Z}) \to L_{K3} \) be an isometry of lattices with \( \psi^{-1}(M) \subset \text{NS}(S) \). We call the pair \((S,\psi)\) of such \( S \) and \( \psi \) is called a marked \( K \) surface. If \((S,\psi^{-1}|_M)\) is a pseudo-ample \( M \)-polarized \( K \) surface, then \((S,\psi)\) is called a pseudo-ample marked \( M \)-polarized \( K \) surface. For two pseudo-ample marked \( M \)-polarized \( K \) surfaces \((S_1,\psi_1)\) and \((S_2,\psi_2)\), we suppose \((S_1,\psi_1^{-1}|_M)\) and \((S_2,\psi_2^{-1}|_M)\) are isomorphic as \( M \)-polarized \( K \) surfaces. Then, \((S_1,\psi_1)\) and \((S_2,\psi_2)\) are said to be isomorphic as pseudo-ample \( M \)-polarized \( K \) surfaces. Also, if there is an isomorphism \( f : S_1 \to S_2 \) such that \( \psi_1 = \psi_2 \circ f_* \), we say \((S_1,\psi_1)\) and \((S_2,\psi_2)\) are isomorphic as pseudo-ample marked \( M \)-polarized \( K \) surfaces.

By gluing local moduli spaces of marked \( M \)-polarized \( K \) surfaces, we have the fine moduli space \( \mathcal{M}_M \) of marked \( M \)-polarized \( K \) surfaces. Then, we have the period mapping

\[
\text{per} : \mathcal{M}_M \to \mathcal{D}_M,
\]

where \( \mathcal{D}_M \) is given in (0.5).

Let \( \mathcal{M}_M^a(\subset \mathcal{M}_M) \) be the set of isomorphism classes of pseudo-ample marked \( M \)-polarized \( K \) surfaces. By restricting (3.1) to \( \mathcal{M}_M^a \), we have a surjective mapping

\[
\text{per} : \mathcal{M}_M^a \to \mathcal{D}_M.
\]

The group \( \Gamma(M) = \{ \sigma \in \text{O}(L_{K3}) \mid \sigma(m) = m \text{ for all } m \in M \} \) acts on \( \mathcal{M}_M \) by \((S,\psi) \mapsto (S,\psi \circ \sigma)\). Then, \( \mathcal{M}_M^a/\Gamma(M) \) gives the set of isomorphism classes of pseudo-ample \( M \)-polarized \( K \) surfaces.

**Theorem 3.1.** (Dolgachev \([D]\), Section 3) The period mapping (3.2) induces the bijection

\[
\mathcal{M}_M^a/\Gamma(M) \simeq \mathcal{D}_M/\hat{O}(A) = \mathcal{D}/\Gamma,
\]

where \( \Gamma \) is given in (1.5). Especially, \( \mathcal{D}/\Gamma \) gives the set of isomorphism classes of pseudo-ample \( M \)-polarized \( K \) surfaces.

Let us take a reference surface \( S_0 = S_\tilde{a} \) for \( \tilde{a} \in \tilde{A} \) with the divisors (2.4) and the \( S \)-marking \( \psi_0 : H_2(S_0, \mathbb{Z}) \to L_{K3} \) such that \( \text{NS}(S_0) = \psi^{-1}_0(M) \). For a pseudo-ample marked \( M \)-polarized \( K \) surface \((S,\psi)\), as in the proof of \([Na3]\) Theorem 2.3, we can show that there is an isometry \( \psi : H_2(S, \mathbb{Z}) \to L_{K3} \) satisfying the following conditions:

(i) \( \psi^{-1}(M) \subset \text{NS}(S) \),

(ii) \( \psi^{-1} \circ \psi_0(F), \psi^{-1} \circ \psi_0(O), \psi^{-1} \circ \psi_0(C_j) \) (\( j \in \{1, \ldots, 7\} \)), \( \psi^{-1} \circ \psi_0(D_k) \) (\( k \in \{1, \ldots, 6\} \)) are effective divisors,

(iii) \( \psi^{-1} \circ \psi_0(F) \) is a nef divisor.

Such an isometry \( \psi \) is called the \( P \)-marking in \([Na3]\). By an argument which is similar to the proof of \([Na3]\) Lemma 2.1, we can prove the following lemma.

**Lemma 3.1.** For any pseudo-ample marked \( M \)-polarized \( K \) surface \((S,\psi)\), there exists \( a \in \tilde{A} \) such that \((S,\psi)\) is given by the elliptic \( K \) surface \( \pi_\alpha : S_a \to \mathbb{P}^1(\mathbb{C}) \) given by the Weierstrass equation (2.1). Especially, the divisor \( \psi^{-1} \circ \psi_0(F) \), which is effective and nef, gives a general fibre for \( \pi_\alpha \).

Let us take two pseudo-ample marked \( M \)-polarized \( K \) surfaces \((S_a,\pi_\alpha)\) and \((S_{a'},\pi_{a'})\) in the sense of Lemma 3.1. If they are isomorphic, then the types of the singular fibres for \( \pi_a \) coincide with those for \( \pi_{a'} \). According to Lemma 2.2, we have the following facts:

- for \( a \in \{ a_0 \neq 0 \} \), then \( \pi_a^{-1}(\infty) \) is of Kodaira type \( III^* \),
- for \( a \in \{ a_0 = 0 \} \), then \( \pi_a^{-1}(\infty) \) is of Kodaira type \( II^* \).

Hence, \( S_a \) for \( a \in \{ a_0 \neq 0 \} \) is not isomorphic to \( S_{a'} \) for \( a' \in \{ a_0 \neq 0 \} \) as pseudo-ample \( M \)-polarized \( K \) surfaces. If \( a_0 \neq 0 \), we have a canonical form given by (2.6). On the other hand, if \( a_0 = 0 \), we have the following result.
Proposition 3.1. (The canonical form for \(a_0 = 0\)) If \(a \in \mathfrak{A} \cap \{a_0 = 0\}\), (2.1) is transformed to the Weierstrass equation

\[
S_1(t) : z^2 = y^3 + (t_4 x^4 w^4 + t_10 x^3 w^{10}) y + (x^7 + t_6 x^6 w^6 + t_{12} x^5 w^{12} + t_{18} x^4 w^{18}).
\]

Proof. By putting

\[
x \mapsto \frac{x}{a_2}, \quad y \mapsto \frac{y}{a_2^2}, \quad z \mapsto \frac{z}{a_2^2},
\]

and

\[
t_4 = a_4, \quad t_6 = a_6, \quad t_{10} = a_2 a_8, \quad t_{12} = a_2 a_{10}, \quad t_{18} = a_2^2 a_{14},
\]

we obtain (3.3).

Now, we naturally obtain \(S_1([t]) \) for \([t] \in \mathbb{P}(4, 6, 10, 12, 18) = \text{Proj}(\mathbb{C}[t_4, t_6, t_{10}, t_{12}, t_{18}])\). Let \(\mathcal{T}\) be a subset of \(\mathbb{P}(4, 6, 10, 12, 18)\) such that \(\mathbb{P}(4, 6, 10, 12, 18) - \mathcal{T} = \{t_{10} = t_{12} = t_{18} = 0\}\).

The above argument guarantees that the set of isomorphism classes of pseudo-ample \(M\)-polarized \(K3\) surfaces is given by the disjoint union \(U \cup T\). Therefore, we have the following result.

Theorem 3.2. The period mapping in Theorem 3.1 has an explicit form

\[
\Phi : U \sqcup T \simeq \mathcal{D}/\Gamma,
\]

where \(\Gamma\) is given in (1.5). Especially, the injection \(\mathcal{P}(\mathfrak{A}) \hookrightarrow \mathcal{D}/\Gamma\), which is induced by the visualized period mapping \(\Phi_1\) of (2.9), is extended to (3.4).

Let \(\mathcal{H}\) be the arithmetic arrangement of hyperplanes defined in Section 1.1. By virtue of Lemma 2.2, \(\mathcal{H}\) corresponds to pseudo-ample \(M\)-polarized \(K3\) surfaces given by the canonical form (3.3). So, from Theorem 3.2, the restriction of the period mapping \(\Phi\) of (3.4) gives an isomorphism

\[
\Phi|_{\mathcal{T}} : \mathcal{T} \simeq \left(\bigcup_{\mathcal{H} \in \mathcal{H}} \mathcal{D}_{\mathcal{H}}\right)/\Gamma.
\]

We remark that (3.5) coincides with the period mapping of Corollary 2.1. The detailed results of (3.5) will be summarized in Section 4.

For \(\mathcal{D}\) of (1.7), we have an isomorphism

\[
\Phi|_{\mathcal{U}} : \mathcal{U} \simeq \mathcal{D}/\Gamma.
\]

Since \(\mathcal{P}(\mathfrak{A}) \subset \mathcal{U}\), (3.6) is an extension of \(\mathcal{P}(\mathfrak{A}) \hookrightarrow \mathcal{D}/\Gamma\), which is derived from \(\Phi_1\) of (2.9). By abuse of notation, this \(\Phi|_{\mathcal{U}}\) will be denoted by \(\Phi\) in Section 5.

4 Sequence of families of \(K3\) surfaces and complex reflection groups

Let \(\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3\) be subvarieties of \(\mathfrak{A}_0\) explicitly given by

\[
\begin{align*}
\mathfrak{A}_1 &= \{a \in \mathfrak{A}_0 \mid a_0 = 0\}, \\
\mathfrak{A}_2 &= \{a \in \mathfrak{A}_0 \mid a_0 = a_{14} = 0\}, \\
\mathfrak{A}_3 &= \{a \in \mathfrak{A}_0 \mid a_0 = a_{14} = \mathcal{M}(a) = 0\}.
\end{align*}
\]

Here,

\[
\mathcal{M}(a) := \left(\frac{a_2 a_{10} + \frac{a_3}{27} \frac{a_6^2}{4}}{a_2^2}\right)^2 + \frac{1}{27} a_4 a_6 + 6 a_2 a_8)^2 = 0
\]

coincides with the modular equation of the Humbert surface for the minimal discriminant which is studied in Theorem 5.4 via an appropriate transformation.

We have the subfamilies

\[
\varpi_j : \mathfrak{F}_j \to \mathfrak{A}_j \quad (j \in \{1, 2, 3\})
\]

of \(\varpi : \mathfrak{F}_0 = \{S_a \mid a \in \mathfrak{A}_0\} \to \mathfrak{A}_0\). They are indicated in the diagram (1.4). Also, we need another subvariety \(\mathfrak{A}_1' = \{a \in \mathfrak{A}_0 \mid a_{14} = 0\}\) and another subfamily \(\varpi'_1 : \mathfrak{F}_1' \to \mathfrak{A}_1'\) such that \(\mathfrak{A}_1 \cap \mathfrak{A}_1' = \mathfrak{A}_2\).
4.1 Transcendental lattices for subfamilies of $\mathfrak{F}_0$

The above mentioned subfamilies of $\mathfrak{F}_0$ are precisely studied in [CD], [Na1], [CMS] and [Na3]. For each case, it is important to determine the transcendental lattice in order to study the moduli space of the corresponding lattice polarized $K3$ surfaces. By surveying the results of those papers, we have the following result.

**Proposition 4.1.** The intersection matrices of the transcendental lattices of a generic member of the subfamilies $\mathfrak{F}_1, \mathfrak{F}'_1, \mathfrak{F}_2, \mathfrak{F}_3$ of $\mathfrak{F}_0$ are given in Table 2.

| Parameters | Transcendental lattices | $K3$ surfaces |
|------------|--------------------------|---------------|
| $\mathfrak{A}_1$ | $A_1 = U \oplus U \oplus A_2(-1)$ | Na3 |
| $\mathfrak{A}'_1$ | $A'_1 = U \oplus U \oplus A_1(-1)^{2\oplus 2}$ | CMS |
| $\mathfrak{A}_2$ | $A_2 = U \oplus U \oplus A_1(-1)$ | CD |
| $\mathfrak{A}_3$ | $A_3 = U \oplus \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$ | Na1 |

The lattices in (0.2) are based on Proposition 4.1 and Theorem 2.1.

4.2 Transcendental lattices for subfamilies of $\mathcal{G}_0$ of Kummer-like surfaces

In Section 2.3, we have the family $\mathfrak{G}_0 : \mathcal{G}_0 = \{K_a \mid a \in \mathfrak{A}_0\} \rightarrow \mathfrak{A}_0$ of $K3$ surfaces. This family contains interesting and important families of algebraic $K3$ surfaces. For example, the subfamily $\mathcal{G}_2$ over $\mathfrak{A}_2$ coincides with the well-known family of Kummer surfaces derived from principally polarized Abelian surfaces. Also, the subfamily $\mathfrak{G}'_1$ over $\mathfrak{A}'_1$ is the family of $K3$ surfaces given by the double covering of $\mathbb{P}^2(\mathbb{C})$ branched along six lines, studied by Matsumoto-Sasaki-Yoshida [MSY].

**Proposition 4.2.** The intersection matrices of the transcendental lattices of a generic member of the subfamilies $\mathfrak{G}_1, \mathfrak{G}'_1, \mathfrak{G}_2, \mathfrak{G}_3$ of $\mathfrak{G}_0$ are given in Table 3.

| Parameters | Transcendental lattices | $K3$ surfaces |
|------------|--------------------------|---------------|
| $\mathfrak{A}_1$ | $B_1 = U(2) \oplus U(2) \oplus A_2(-2)$ | NS3 |
| $\mathfrak{A}'_1$ | $B'_1 = U(2) \oplus U(2) \oplus A_1(-1)^{2\oplus 2}$ | MSY |
| $\mathfrak{A}_2$ | $B_2 = U(2) \oplus U(2) \oplus A_1(-2)$ | Kummer surface |
| $\mathfrak{A}_3$ | $B_3 = U(2) \oplus \begin{pmatrix} 4 & 2 \\ 2 & -4 \end{pmatrix}$ | Na2 |

By the way, we are able to determine the transcendental lattices in Table 2 for the subfamilies $\mathfrak{F}_0$ by a relatively simple way. However, the proof of Proposition 4.2 is much more complicated. We need a delicate argument for each case. For detail, see the beginning of Section 6.

Proposition 4.2 is necessary for the proof of Theorem 6.1, which is the main theorem of Section 6. The lattices in (0.8) are based on Proposition 4.2 and Theorem 6.1.

4.3 Relation between modular forms and invariants of complex reflection groups via theta functions

For the subfamilies $\mathfrak{F}_1, \mathfrak{F}_2$ and $\mathfrak{F}_3$, there is a non-trivial relationship between the period mappings for them and a complex reflection group of rank $r_j = 6 - j$.

In each family of $K3$ surfaces in Table 1, we have the period mapping

$$\Phi_j : P_j \simeq \mathcal{D}_j/\Gamma_j$$  (4.2)
where $\mathcal{P}_j$ is a Zariski open set in the weighted projective space whose weights are given in Table 1, $D_j$ is a $(5 - j)$-dimensional symmetric domain and $\Gamma_j$ is a subgroup of the orthogonal group of the lattice $A_j$ in Table 2. We note that $D_j$ can be obtained as a restriction of $\mathcal{D}_4$. 

Finite complex reflection groups are listed by Shephard-Todd ST (see also LT). Note that real reflection groups are contained in this list.

A complex reflection group of rank $r_j$ acts on the polynomial ring $\mathbb{C}[X_1, \ldots, X_{r_j}]$. We can find generators of the ring of invariants for this action. Let $(w_1^{(j)}, \ldots, w_{r_j}^{(j)}) \ (j \in \{1, 2, 3\})$ be the set of weights given in Table 1. There is a system $\{g_{w_1}, g_{w_2}(X_1, \ldots, X_{r_j}), \ldots, g_{w_{r_j}}(X_1, \ldots, X_{r_j})\}$ of generators of the ring of invariants. Here, $w_j \in \mathbb{Z}$ is the integer in Table 1 and $g_{w_j}$ is a polynomial of degree $\kappa_j w_j^{(j)}$. For example, if $j = 3$, the invariants for the group No.23 are famous Klein’s icosahedral invariants introduced in [K]. In the preceding papers in Table 4, we have the explicit theta expressions of the inverse of $\Phi_j$ of (4.2) via appropriate systems of theta functions.

For $j = 1, 2$, we have simple expressions of the above mentioned results. There exists a system $\{\vartheta_1^{(j)}(Z_j), \ldots, \vartheta_{r_j}^{(j)}(Z_j)\}$ of theta functions $D_j \ni Z_j \mapsto \vartheta_{r_j}^{(j)}(Z_j) \in \mathbb{C}$ of weight $1/\kappa_j$ such that

$$D_j \ni Z_j \mapsto \left(g_{w_1}^{(j)} \left(\vartheta_1^{(j)}(Z_j), \ldots, \vartheta_{r_j}^{(j)}(Z_j)\right), \ldots, g_{w_{r_j}}^{(j)} \left(\vartheta_1^{(j)}(Z_j), \ldots, \vartheta_{r_j}^{(j)}(Z_j)\right)\right) \in \mathcal{P}_j$$

(4.3)

gives a ratio of modular forms on $D_j$ with respect to $\Gamma_j$ and it coincides with the inverse of the period mapping $\Phi_j$ of (4.2),

- If $j = 1$, the invariants for the group No.33 are given in [Mu] and the explicit form (4.3) is established in [NS2] Theorem 5.1, using the theta functions of [DK]. In this case, (4.3) is given by a ratio of Hermitian modular forms for the unitary group $U(2, 2)$ concerned with the imaginary quadratic field of the simplest discriminant.

- If $j = 2$, using the invariants for the group No.31 and the theta functions given in [R] Section 4, one can obtain the expression (4.3) by combining the results [CD] Theorem 3.5 and [R] Section 4 (see also [NS2] Section 5.1). In this case, (4.3) is given by a ratio of well-known Siegel modular forms of degree 2.

Also, refer to Remark 5.2

5 Meromorphic modular forms

Let $D^*$ be the connected component of $\{\xi \in A \otimes \mathbb{C} \mid (\xi \cdot \xi) = 0, (\xi \cdot \bar{\xi}) > 0\}$ which projects to $D$. For $D^o$ of [1.7], let $(D^o)^*$ be a subset of $D^*$ which projects to $D^o$.

Based on the fact stated in Theorem 1.1 below, we will use the following terminology.

**Definition 5.1.** A holomorphic function $f : (D^o)^* \rightarrow \mathbb{C}$ given by $Z \mapsto f(Z)$ is called a meromorphic modular form of weight $k \in \mathbb{Z}$ and character $\chi \in \text{Char}(\Gamma)$ with poles in $\mathcal{H}$, if $f$ satisfies

(i) $f(\lambda Z) = \lambda^{-k} f(Z)$ \ (for all $\lambda \in \mathbb{C}^*$),

(ii) $f(\gamma Z) = \chi(\gamma) f(Z)$ \ (for all $\gamma \in \Gamma$).

The vector space of the meromorphic modular forms of weight $k \in \mathbb{Z}$ and $\chi \in \text{Char}(\Gamma)$ with poles in $\mathcal{H}$ is denoted by $A^o_k(\Gamma, \chi)$. Then, the ring of the meromorphic modular forms is given by

$$A^o(\Gamma) = \bigoplus_{k \in \mathbb{Z}} \bigoplus_{\chi \in \text{Char}(\Gamma)} A^o_k(\Gamma, \chi).$$

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In this section, we will construct the generator of this ring. Recalling Proposition 1.1, we will consider the cases of $\chi = \text{id}$ and $\chi = \det$. The structure of the ring $A^n(\Gamma)$ is determined by Theorem 5.1 and 5.2.

By the way, in [HU] (see also [Na3]), period mappings for lattice polarized $K3$ surfaces and the canonical orbibundles on the Satake-Baily-Borel compactifications of symmetric spaces are effective to construct holomorphic modular forms. However, the Satake-Baily-Borel compactifications are useless for our purpose, because we want to obtain not holomorphic modular forms but meromorphic modular forms. Accordingly, we will use the Looijenga compactification for $D^0$ of (1.7) and $\Gamma$ of (1.5), instead of the Satake-Baily-Borel compactification.

Since Proposition 1.2 and (2.7) hold, by Hartogs’s extension theorem, the period mapping $\Phi$ of (3.6) is extended to the isomorphism

$$\tilde{\Phi} : \tilde{U} \simeq D^0 / \Gamma$$

(5.1)
between the weighted projective space $\tilde{U} = \mathbb{P}(2, 4, 6, 8, 10, 14)$ and the Looijenga compactification. Our construction of meromorphic modular forms is based on this period mapping.

5.1 Meromorphic modular forms of character id

There exists a unique holomorphic 2-form $\omega_u$ on $S(u)$ of (2.6) up to a constant factor. This is explicitly given by $\frac{dz_0 \wedge dz_0}{z_0^4}$, where $x_0 = \frac{z}{w^2}$, $y_0 = \frac{w}{w^2}$, $z_0 = \frac{z}{w^4}$. The action of $\lambda \in \mathbb{C}^*$ given by $S(u) \to S(\lambda \cdot u)$, which defines the family (2.8), induces the relation

$$\lambda^w \omega_{\lambda \cdot u} = \lambda^{-1} \omega_u.$$  

(5.2)

**Theorem 5.1.** The ring $A^n(\Gamma, \text{id})$ of meromorphic modular forms of character id is isomorphic to the polynomial ring $\mathbb{C}[u_2, u_4, u_6, u_8, u_{10}, u_{14}]$. Here, a polynomial of weight $k$ defines a modular form of weight $k$.

**Proof.** We have a principal $\mathbb{C}^*$-bundle $\text{pr} : (D^0)^* \to D^0$. The quotient space $Q = D^0 / \Gamma$ is identified with a Zariski open set $U$ of the weighted projective space $\tilde{U} = \mathbb{P}(2, 4, 6, 8, 10, 14)$ via the period mapping. Since $\text{pr}$ is equivalent under the action of $\Gamma = O^+(A)$, we have a principal $\mathbb{C}^*$-bundle $\text{pr} : (D^0)^*/\Gamma \to Q$. Let $O_Q(1)$ be the line bundle over $Q$ associated with $\text{pr}$ and set $O_Q(k) = O_Q(1)^{\otimes k}$. Recalling the definition of the associated bundle, we can regard a section of $O_Q(k)$ as a holomorphic function $(D^0)^* \ni Z \mapsto s(Z) \in \mathbb{C}$ satisfying

$$s(\lambda Z) = \lambda^{-k} s(Z), \quad s(\gamma Z) = s(Z),$$  

(5.3)

where $\lambda \in \mathbb{C}^*$ and $\gamma \in \Gamma$. From (2.7), $\tilde{U} - U$ is an analytic subset such that codim $(\tilde{U} - U) \geq 2$. So, via Hartogs’s phenomenon, the inclusion $u_U : U \hookrightarrow \tilde{U}$ induces the isomorphism

$$u_U^* : \text{Pic}(\tilde{U}) \simeq \text{Pic}(U).$$  

(5.4)

Now, we have Pic$(\tilde{U}) \simeq \mathbb{Z}$ and

$$\bigoplus_{k \in \mathbb{Z}} H^0(\tilde{U}, O_{\tilde{U}}(k)) = \mathbb{C}[u_2, u_4, u_6, u_8, u_{10}, u_{14}],$$  

(5.5)

because $\tilde{U} = \text{Proj}(\mathbb{C}[u_2, u_4, u_6, u_8, u_{10}, u_{14}])$ is a weighted projective space. From (5.4) and (5.5), we have

$$\bigoplus_{\mathcal{L} \in \text{Pic}(Q)} H^0(Q, O_Q(\mathcal{L})) \simeq \mathbb{C}[u_2, u_4, u_6, u_8, u_{10}, u_{14}].$$  

(5.6)

Due to (5.2), the period mapping gives the following diagram:

$$u = (u_2, u_4, u_6, u_8, u_{10}, u_{14}) \xrightarrow{\text{period mapping}} Z \xrightarrow{\text{C}^*-\text{action}} \lambda Z \xrightarrow{\text{C}^*-\text{action}}$$

$$\lambda^{-1} \cdot u = (\lambda^{-2} u_2, \lambda^{-4} u_4, \lambda^{-6} u_6, \lambda^{-8} u_8, \lambda^{-10} u_{10}, \lambda^{-14} u_{14})$$

From Definition 5.1, together with (5.3) and (5.6), we have the assertion. \qed
The integer $\kappa_0 = 3$ in Table 1 is coming from this theorem. Namely, $(2\kappa_0, 4\kappa_0, 6\kappa_0, 8\kappa_0, 10\kappa_0, 14\kappa_0) = (6, 12, 18, 24, 30, 42)$ is equal to the degrees of the group $\text{No.34}$ (see [LT] Appendix D).

5.2 Meromorphic modular forms of character $\det$

We will study the orbifold

$$\mathcal{O} = [(D^o)^*/(\Gamma \times \mathbb{C})]$$

in order to construct modular forms for $\det$.

First, let us observe the action of $\Gamma$ on $D^o$ precisely. The action of $\Gamma$ on $D^o$ is effective. We set

$$\mathcal{H}_{D^o} = \bigcup_{g \in \Gamma} \{ [Z] \in D^o | g([Z]) = [Z] \}.$$

Also, letting $\Gamma _{|Z}$ be the stabilize subgroup with respect to $[Z] \in D^o$, we set

$$\mathcal{G}_{D^o} = \{ [Z] \in D^o | \Gamma _{|Z} \text{ is neither \{id}_\Gamma \text{ nor \{id}_\Gamma , \sigma_d \} for } \delta \in \Delta(A) \}.$$

According to Proposition 1.1, $\mathcal{H}_{D^o}$ is a countable union of reflection hypersurfaces and $\mathcal{G}_{D^o}$ is a countable union of analytic subsets of codimension at least 2.

Let $\mathcal{H}_Q$ and $\mathcal{G}_Q$ be the images of $\mathcal{H}_{D^o}$ and $\mathcal{G}_{D^o}$ by the projection $D^o \to Q = D^o/\Gamma$, respectively. From Proposition 1.1 it follows that

$$\Gamma ' : \{ \gamma \in \Gamma | \gamma \text{ is given by a product of reflections of even numbers } \} = \{ \gamma \in \Gamma | \det(\gamma) = 1 \}. \quad (5.8)$$

We set $Q_1 = D^o/\Gamma '$. The action of $\Gamma ' \text{ on } D^o - \mathcal{G}_{D^o}$ is free. Let us naturally define $\mathcal{H}_{Q_1}$ and $\mathcal{G}_{Q_1}$, respectively. Recall that the period mapping $\Phi$ gives an identification $\mathcal{U} \simeq Q$ (see (3.6)). Set $\mathcal{H}_U = \Phi ^{-1}(\mathcal{H}_Q)$ and $\mathcal{G}_U = \Phi ^{-1}(\mathcal{G}_Q)$. Then, $\mathcal{H}_U$ gives a divisor on the weighted projective space $\tilde{U}$ and there exists a weighted homogeneous polynomial $\Delta_U(u) \in \mathbb{C}[u_2, u_4, u_6, u_8, u_{10}, u_{14}]$ such that

$$\mathcal{H}_U = \left\{ [u] \in \tilde{U} = \mathbb{P}(2, 4, 6, 8, 10, 14) | \Delta_U(u) = 0 \right\}. \quad (5.9)$$

We have the double covering $U_1$ of $U - \mathcal{G}_U$ branched along $\mathcal{H}_U - \mathcal{G}_U$:

$$U_1 = \left\{ ([u], s) \in (U - \mathcal{G}_U) \times \mathbb{C} | s^2 = \Delta_U(u) \right\}. \quad (5.10)$$

We can obtain the lift $\Phi_{Q_1} : U_1 \to Q_1 - \mathcal{G}_{Q_1}$ of $\Phi_{|D^o - \mathcal{G}_{D^o}}$ so that $\Phi_{Q_1}$ is equivalent under the action of $\Gamma '/\Gamma ' \simeq \mathbb{Z}/2\mathbb{Z}$. Also, $\Phi_{Q_1}$ is lifted to $\Phi_{D^o} : U_{D^o} \to D^o - \mathcal{G}_{D^o}$, which is equivalent under the action of $\Gamma$. We can consider the pull-back $U_{(D^o)^*} \to U_{D^o}$ of the principal bundle $U^* \to U$ by the composition $U_{D^o} \to U_1 \to U - \mathcal{G}_U \leftrightarrow U$. Then, we have the lifted period mapping

$$\Phi_{(D^o)^*} : U_{(D^o)^*} \simeq (D^o)^* - \mathcal{G}_{(D^o)^*},$$

where $\mathcal{G}_{(D^o)}$ is the preimage of $\mathcal{G}_{D^o}$ under the projection.

Lemma 5.1. The lifted period mapping $\Phi_{(D^o)^*}$ induces an isomorphism $[\Phi_{(D^o)^*}] : [U_{(D^o)^*}/(\mathbb{C}^* \times \Gamma)] \simeq [(D^o)^* - \mathcal{G}_{(D^o)^*})/(\mathbb{C}^* \times \Gamma)].$

Proof. The proof is similar to [Na3] Section 3.3. See the diagram [5.11] also.

\[ U_{(D^o)^*} \text{ C^*-bundle } \xrightarrow{C^*-bundle} U_{D^o} \text{ C^*-bundle } \xrightarrow{C^*-bundle} (D^o)^* - \mathcal{G}_{(D^o)^*}, \]
Proposition 5.1. The Picard group Pic(Ω) of the orbifold Ω is isomorphic to $\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})$.

Proof. Set $\mathcal{V}_1 = \{([u], s) \in \tilde{U} \times \mathbb{C} \mid s^2 = \Delta_U(u)\}$. Then, $\mathcal{U}_1$ of \[5.10\] satisfies $\mathcal{U}_1 \subset \mathcal{V}_1$ and $\text{codim}(\mathcal{V}_1 - \mathcal{U}_1) \geq 2$, from (2.7). Recall that $\Phi$ in \[5.11\] is extended to the identification \[5.1\]. This $\Phi$ is lifted to $\tilde{\Phi}_Q_1 : \mathcal{V}_1 \simeq \tilde{Q}_1$, which is equivalent under the $(\mathbb{Z}/2\mathbb{Z})$-action. Here, $\tilde{Q}_1$ is a double covering of $\mathcal{D}^o/\Gamma^L$.

We consider the orbifold $\mathcal{V}_1 = [\mathcal{V}_1/(\mathbb{Z}/2\mathbb{Z})]$ with the structure morphism $p_{\mathcal{V}_1} : \mathcal{V}_1 \to \tilde{U}$. Now, the Picard group $\text{Pic}(\mathcal{V}_1)$ is generated by $\mathcal{O}_{\mathcal{V}_1}(1) := p_{\mathcal{V}_1}^* \mathcal{O}_\mathcal{U}(1)$ and the generator $g$ of $\mathbb{Z}/2\mathbb{Z}$. We remark that this generator $g$ is corresponding to $\text{det} \in \text{Char}(\Gamma)$ (see Proposition \[1.1\] and \[5.8\]). The divisor $\{\Delta_U(u) = 0\}$ is corresponding to the reflection hypersurfaces for our lattice $\mathcal{A}$ via $\Phi$ in \[5.11\]. Therefore, any element of $\text{Pic}(\mathcal{V}_1)$ is given by $\mathcal{O}_{\mathcal{V}_1}(1)^{\otimes k} \otimes g^l$ for $k \in \mathbb{Z}$ and $l \in \mathbb{Z}/2\mathbb{Z}$. By considering $\tilde{\Phi}_Q_1 : \mathcal{V}_1 \simeq [\tilde{Q}_1/(\mathbb{Z}/2\mathbb{Z})]$, we obtain $\text{Pic} \left(\left[\tilde{Q}_1/(\mathbb{Z}/2\mathbb{Z})\right]\right) \simeq \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})$.

Recall that the action of $\Gamma'$ on $\mathcal{D}^o - \mathcal{S}_{\mathcal{D}}$ is free. From the fact that $\text{codim} \left(\tilde{Q}_1 - Q_1\right) \geq 2$, we have

$$\text{Pic}(\Omega) = \text{Pic}(\{(\mathcal{D}^o)^*/(\mathbb{C} \times \Gamma)\}) = \text{Pic}(\{Q_1/(\mathbb{Z}/2\mathbb{Z})\}) \simeq \text{Pic} \left(\left[\tilde{Q}_1/(\mathbb{Z}/2\mathbb{Z})\right]\right) \simeq \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z}).$$

When we consider holomorphic sections of line bundles, analytic subsets of codimension at least 2 do not affect the results due to Hartogs’s phenomenon. So, in this section, we shall omit such analytic sets. Namely, from now on, we will often omit such sets (like $\mathcal{S}_{\mathcal{U}}$ or $\mathcal{S}_Q$).

Proposition 5.2. The weight of $\Delta_U(u)$ is equal to 98.

Proof. Since $\tilde{U}$ is a Zariski open set in $\tilde{U} = \mathbb{P}(2,4,6,8,10,14)$, the canonical bundle $\Omega_{\tilde{U}_1}$ is calculated as

$$\Omega_{\mathcal{U}_1} \simeq \mathcal{O}_{\mathcal{U}}(−2 − 4 − 6 − 8 − 10 − 14) = \mathcal{O}_{\mathcal{U}}(−44).$$

Let $p_1$ be the double covering $\mathcal{U}_1 \to \mathcal{U}$ branched along $\mathcal{S}_{\mathcal{U}_1}$. We obtain the isomorphism $\Omega_{\mathcal{U}_1} \simeq p_1^* \Omega_{\mathcal{U}} \otimes \mathcal{O}_{\mathcal{U}_1}(\mathcal{S}_{\mathcal{U}_1})$, by considering the holomorphic differential forms. So, we have

$$\Omega_{\mathcal{U}_1/(\mathbb{Z}/2\mathbb{Z})} \simeq [p_1]^* \Omega_{\mathcal{U}} \otimes \mathcal{O}_{\mathcal{U}_1/(\mathbb{Z}/2\mathbb{Z})}(\mathcal{S}_{\mathcal{U}_1}) \simeq [p_1]^* \Omega_{\mathcal{U}}(−44) \otimes \mathcal{O}_{\mathcal{U}_1/(\mathbb{Z}/2\mathbb{Z})}(\mathcal{S}_{\mathcal{U}_1}).$$

From the proof of Proposition \[5.1\] the orbifold $\tilde{\Omega}$ is equivalent to $[\mathcal{U}_1/(\mathbb{Z}/2\mathbb{Z})]$. So, $\text{Pic}(\{\mathcal{U}_1/(\mathbb{Z}/2\mathbb{Z})\})$ is isomorphic to $\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})$. Let $d$ be the weight of $\Delta_U(u)$. Then, we have $[p_1]^* \Omega_{\mathcal{U}}(d) \simeq \mathcal{O}_{\mathcal{U}_1/(\mathbb{Z}/2\mathbb{Z})}(2\mathcal{S}_{\mathcal{U}_1})$. This implies that the direct summand $\mathbb{Z}/2\mathbb{Z}$ of $\text{Pic}(\{\mathcal{U}_1/(\mathbb{Z}/2\mathbb{Z})\})$ is generated by $[p_1]^* \Omega_{\mathcal{U}}(−d/2) \otimes \mathcal{O}_{\mathcal{U}_1/(\mathbb{Z}/2\mathbb{Z})}(\mathcal{S}_{\mathcal{U}_1})$. Here, we do not need to worry about whether $−d/2$ is an integer, for all weights of $\mathcal{U}$ are even numbers.

Since $\mathcal{D}^o$ is a Zariski open set in a quadratic hypersurface in the projective space $\mathbb{P}^6(\mathbb{C})$, we apply the adjunction formula to $\mathcal{D}^o$ and obtain $\Omega_{\mathcal{D}^o} \simeq \mathcal{O}_{\mathcal{D}^o}(7 − 2) = \mathcal{O}_{\mathcal{D}^o}(5)$, where the weight is concordant with the $\mathbb{C}^*$-action indicated in (5.7). This implies that the canonical orbibundle $\Omega_{\mathcal{O}}$ is isomorphic to $\mathcal{O}_{\mathcal{O}}(5) \otimes \text{det}$. By summing up the above properties, we have

$$\mathcal{O}_{\mathcal{O}}(5) \otimes \text{det} \simeq \Omega_{\mathcal{O}} \simeq \Omega_{\mathcal{U}_1/(\mathbb{Z}/2\mathbb{Z})}$$

$$\simeq [p_1]^* \Omega_{\mathcal{U}}(−44) \otimes \mathcal{O}_{\mathcal{U}_1/(\mathbb{Z}/2\mathbb{Z})}(\mathcal{S}_{\mathcal{U}_1})$$

$$\simeq [p_1]^* \Omega_{\mathcal{U}} \left(−44 + \frac{d}{2}\right) \otimes \left([p_1]^* \Omega_{\mathcal{U}} \left(−\frac{d}{2}\right) \otimes \mathcal{O}_{\mathcal{U}_1/(\mathbb{Z}/2\mathbb{Z})}(\mathcal{S}_{\mathcal{U}_1})\right)$$

and we obtain $d = 98$. □

Theorem 5.2. (1) There exist holomorphic functions $s_7$ of weight 7 and $s_{42}$ of weight 42 on $(\mathcal{D}^o)^*$ such that

$$s_7^2 = u_{14}, \quad s_{42}^2 = d_{s_{44}}(u),$$

where $d_{s_{44}}(u)$ is the polynomial studied in Lemma \[2.2\]. (2) The holomorphic function $s_{49} = s_7 s_{42}$ on $(\mathcal{D}^o)^*$ gives a modular form of weight 49 and character $\text{det}$. It holds $\mathcal{A}^o(\Gamma, \text{det}) = s_{49} \mathcal{A}^o(\Gamma, 1\text{d})$. 

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Proof. (1) By the argument in this section, the divisor \( \{u \in U \mid \Delta_U(u) = 0\} \) corresponds to the union of reflection hyperplanes of \( \{H_\delta \mid \delta \in \Delta(A)\} \) - \( \mathcal{H} \). Here, \( H_\delta \) is a reflection hyperplane defined by \( \delta \in \Delta(A) \) and \( \mathcal{H} \) is the arrangement of \( \gamma H_0 (\gamma \in \Gamma, H_0 = \{\xi_0 = 0\}) \) as in Section 1.1. Recalling the observation of degenerations of our lattice polarized K3 surfaces in Lemma 2.2 and the meaning of the canonical form \( (\ref{eq:2.6}) \), we have

\[
\{ [u] \in U \mid d_{s_4}(u) = 0 \} \cup \{ [u] \in U \mid u_{14} = 0 \} \subset \{ [u] \in U \mid \Delta_U(u) = 0 \}.
\] (5.12)

The inclusion \( (\ref{eq:5.12}) \) means that \( u_{14} d_{s_4}(u) \) divides \( \Delta_U(u) \) in \( \mathbb{C}[u_2, u_4, u_6, u_8, u_{10}, u_{14}] \). On the other hand, Proposition 5.2 says that \( \Delta_U(u) \) is of weight 98. Thus, we have the irreducible decomposition

\[
\Delta_U(u) = \text{const} \cdot u_{14} d_{s_4}(u).
\] (5.13)

Since the double covering \( p_1 : U_1 \to U \) in the diagram \( (\ref{eq:5.11}) \) is branched along the divisor \( \{ [u] \in U \mid \Delta_U(u) = 0 \} \), \( (\ref{eq:5.13}) \) implies that there is a holomorphic function \( s_7 \) (s_{42}, resp.) on \( (D^o)^* \) satisfying \( s_7^2 = u_{14} (s_{24}^2 = d_{s_4}(u), \text{resp.}) \).

(2) Recall that \( \text{det} \in \text{Char}(\Gamma) \) is coming from the action of \( \Gamma/\Gamma' \simeq \mathbb{Z}/2\mathbb{Z} \) which defines the double covering \( p_1 : U_1 \to U \). By the Definition \( (\ref{eq:5.1}) \) and the meaning of \( (\ref{eq:5.10}) \), every modular form of character \( \text{det} \) vanishes on the preimage of \( \mathbb{R}_D \) by the canonical projection \( (D^o)^* \to D^o \). Since this \( p_1 \) is branched along the divisor \( \{ \Delta_U(u) = 0 \} \), \( (\ref{eq:5.12}) \) and \( (\ref{eq:5.13}) \) show that every modular form of character \( \text{det} \) is given by a product of \( s_40 := s_7 s_{42} \) and a modular form of character id.

\[\square\]

Remark 5.1. The Weierstrass equation of \( (\ref{eq:2.1}) \) is essential for our purpose.

We have another expression of elliptic K3 surfaces with singular fibres of type \( (\ref{eq:2.2}) \) given by the equation

\[
z_7^2 = y_1^3 + (b_1 x_1^5 w_1 + b_4 x_1^4 w_1^4 + b_7 x_1^3 w_1^7)y_1 + (b_0 x_1^8 + b_3 x_1^7 w_1^3 + b_6 x_1^6 w_1^6 + b_9 x_1^5 w_1^9)
\] (5.14)
of weight 24, where \( (x : y : z : w) \in \mathbb{P}(3, 8, 12, 1) \). However, the expression \( (\ref{eq:5.14}) \) is not appropriate to construct modular forms. Although \( (\ref{eq:5.14}) \) can be birationally transformed to the Weierstrass form \( (\ref{eq:2.1}) \), it is impossible to construct correct modular forms from the parameters in \( (\ref{eq:5.14}) \). It seems that we can obtain modular forms of weight 1, 3, 4, 6, 7, 9 on \( (D^o)^* \) from \( (\ref{eq:5.14}) \), which is a complement of an arrangement corresponding to the condition \( b_0 = 0 \). However, we can see that this arrangement does not satisfy the condition of Theorem 1.1. So, the expression \( (\ref{eq:5.14}) \) induces an erroneous use of the theory of the Looijenga compactifications.

For example, it seems that we can obtain a modular form of weight 7 coming from the parameter \( b_7 \). We can see that the zero set of this modular form coincides with the arrangement \( \mathcal{H} \) in Lemma 1.2. However, as stated in [L2] (see also [L3] Section 6), if an arrangement gives the zero set of a modular form, then this does not satisfy the condition of Theorem 1.1. This contradicts to Lemma 1.2.

Thus, our expression of \( (\ref{eq:2.1}) \) is suitable for the theory of the Looijenga compactifications and effective to construct modular forms.

Remark 5.2. Theorem 5.2 supports the relation between our sequence of the families and complex reflection groups. The weight 42 of \( s_{42} \) is coming from the discriminant of the right hand side of \( (\ref{eq:2.6}) \). We note that 126 = 42\( \kappa_0 \), where \( \kappa_0 = 3 \) as in Table 1, is equal to the number of reflections of order 2 for the group No.34 (see [LT] Appendix D).

Such a phenomenon occurs in each case \( \mathfrak{F}_j \) (\( j \in \{1, 2, 3\} \)). In the case for \( j = 1 \) \( (2, 3, \text{resp.}) \), there is a holomorphic function of weight 45 \( (30, 15, \text{resp.}) \) coming from the discriminant of the elliptic K3 surfaces of \( \mathfrak{Na}^3, (\mathfrak{CD}), \mathfrak{Na}^3, \text{resp.} \). Then, \( 45 = 45\kappa_1 = 30\kappa_2 = 15 = 15\kappa_3, \text{resp.} \) is equal to the number of reflections of order 2 for the group No.33 (No.31, No.23, resp.). Each of these holomorphic functions gives a factor of a modular form of a non-trivial character. Also, there are explicit expressions of them in terms of the theta functions in Table 4.

6 Transcendental lattice for family \( \mathfrak{G}_0 \) of Kummer-like surfaces with Picard number 15

In Section 2, we determined the lattice structure for the family \( \mathfrak{F}_0 \) via a natural consideration of singular fibres for elliptic surfaces. Also, in fact, the lattices for the subfamilies \( \mathfrak{F}_1, \mathfrak{F}_2, \mathfrak{F}_3 \) in Proposition 4.1 can be determined in a similar way.
However, as for the families $\mathcal{S}_j$ ($j \in \{1, 2, 3\}$) and $\mathcal{S}_j'$ of the Kummer-like surfaces, it is much harder to determine their lattices correctly. For example,

- The family $\mathcal{S}_2$ is the family of the Kummer surfaces for principally polarized Abelian surfaces. For a precise study for the lattice structure of the Kummer surfaces, Nikulin [NH] introduces a particular lattice which is called the Kummer lattice. Also, Morrison [Mo] studies an interesting viewpoint called the Shioda-Inose structure for $K3$ surfaces whose Picard numbers are greater than 17.

- In order to determine the transcendental lattice for the family $\mathcal{S}_1'$ of Kummer-like surfaces with Picard number 16, Matsumoto-Sasaki-Yoshida [MSY] (see also [Y]) study hypergeometric integrals of type $(3, 6)$ and calculate intersection numbers of the chambers coming from the configuration of six lines on $\mathbb{P}^2(\mathbb{C})$ by applying a delicate technique of twisted homologies.

- For the transcendental lattice for the family $\mathcal{S}_1$ of Kummer-like surfaces with Picard number 16, Shiga and the author [NS3] have a geometric construction of 2-cycles on a generic member of $\mathcal{S}_1'$ taking into account the fact that a generic member of $\mathcal{S}_1$ is a double covering of that of $\mathcal{S}_1$. This construction is also based on hard calculations of local monodromies for elliptic surfaces.

In this section, we will determine the transcendental lattice for the family $\mathcal{S}_0$. It is a non-trivial problem to determine it. If there were a double covering $S_a \to K_a$ for generic members of $\mathcal{S}_0$ and $\mathcal{S}_1$, we could calculate the lattice for $\mathcal{S}_0$ from the lattice for $\mathcal{S}_1$ in Theorem 2.1 via a technique of Nikulin [NB] Section 2. However, in practice, we can prove that there is no such a double covering for generic $a \in \mathcal{S}_0$. This proof can be given in a similar way to the proof of [NS3] Theorem 6.3.

Now, let us remark the fact that our family $\mathcal{S}_0$ naturally contains the families $\mathcal{S}_1$ and $\mathcal{S}_1'$. We will determine the transcendental lattice for $\mathcal{S}_0$ based on this fact. The main result in this section is indebted to heavy calculations for $\mathcal{S}_1'$ in [MSY] and those for $\mathcal{S}_1$ in [NS3].

The following result for even lattices is necessary for our proof.

**Lemma 6.1.** ([Mo], Corollary 2.10) Suppose $12 \leq \rho \leq 20$. Let $T$ be an even lattice of signature $(2, 20-\rho)$. Then, the primitive embedding $T \hookrightarrow L_{K3}$ is unique up to isometry.

The following theorem is the main result of this section.

**Theorem 6.1.** The transcendental lattice of a generic member of $\mathcal{S}_0$ is given by the intersection matrix

$$U(2) \oplus U(2) \oplus \begin{pmatrix} -2 & 0 & 1 \\ 0 & -2 & 1 \\ 1 & 1 & -4 \end{pmatrix}.$$

**Proof.** We identify the $K3$ lattice $L_{K3} = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$ with the 2-homology group of $K3$ surfaces.

Let $e_j, f_j$ ($j \in \{1, 2, 3\}$) be elements of $U^{\oplus 3}$ satisfying $(e_j \cdot e_k) = (f_j \cdot f_k) = 0$ and $(e_j \cdot f_k) = \delta_{j,k}$. Let $p_j, q_j$ ($j \in \{1, \ldots, 8\}$) be elements of $E_8(-1)^{\oplus 2}$ with the intersection numbers defined by the Dynkin diagram in Figure 2. Also, put $\nu_j = p_j + q_j$.

![Figure 2: Basis of $E_8(-1)^{\oplus 2}$](image)

By reference to [Ma], we put

$$\begin{align*}
\lambda_1 &= -\nu_5 + \nu_7 + 2(e_1 + f_1), & \mu_1 &= -\nu_4, \\
\lambda_2 &= \nu_7 + \nu_8 + 2(e_1 + e_2 + e_3 + f_3), & \mu_2 &= \nu_6.
\end{align*}$$

(6.1)
Then, \{\lambda_1, \mu_1, \lambda_2, \mu_2\} gives a basis of the lattice \(U(2)^{\oplus 2}\). This basis defines a primitive embedding \(U(2)^{\oplus 2} \to \mathbb{L}_{K3}\).

Let us recall Proposition 4.2. By extending the basis (6.1), we have a primitive embedding of the transcendental lattice \(B_1 = U(2)^{\oplus 2} \oplus A_2(-2)\), for the family \(\mathfrak{G}_1\), into \(L_{K3}\) given by

\[
B_1 = \langle \lambda_1, \mu_1, \lambda_2, \mu_2, \nu_1, \nu_2 \rangle_{\mathbb{Z}},
\]

(6.2)

where the transcendental lattice \(B_2 = U(2)^{\oplus 2} \oplus A_1(-1)^{\oplus 2}\) for the subfamily \(\mathfrak{G}_2\) is a primitive sublattice of \(L_{K3}\) explicitly given by

\[
B_2 = \langle \lambda_1, \mu_1, \lambda_2, \mu_2, \nu_1 \rangle_{\mathbb{Z}}.
\]

(6.3)

According to Lemma 6.1, this embedding is unique up to isometry. So, we can fix the embedding given by (6.2) and (6.3) without loss of generality.

We remark that the family \(\mathfrak{G}_1\) also contains \(\mathfrak{G}_2\) as a subfamily. Hence, its transcendental lattice \(B_1' = U(2)^{\oplus 2} \oplus A_1(-1)^{\oplus 2}\) should be an extension of the lattice \(B_2\) of (6.3). So, we have the following explicit basis:

\[
B_1' = \langle \lambda_1, \mu_1, \lambda_2, \mu_2, p_1, q_1 \rangle_{\mathbb{Z}}.
\]

(6.4)

We note that the expression (6.4) is guaranteed by the fact that the lattice \(B_2\) of (6.3) is invariant under the involution given by interchanging two \(A_1(-1)\) summands of the lattice \(B_1'\) (see [MSY]; see also [Y] Chapter IX).

Our family \(\mathfrak{G}_0\) contains \(\mathfrak{G}_1\) and \(\mathfrak{G}_1'\). So, the transcendental lattice for a generic member of \(\mathfrak{G}_0\) is a primitive lattice of \(L_{K3}\), of rank 7 and given by the basis which is an extension of (6.2) and (6.4). Such a lattice is given by the explicit basis \(\langle \lambda_1, \mu_1, \lambda_2, \mu_2, p_1, q_1, \nu_2 \rangle_{\mathbb{Z}}\) whose intersection matrix is \(U(2)^{\oplus 2} \oplus \begin{pmatrix} -2 & 0 & 1 \\ 0 & -2 & 1 \\ 1 & 1 & -4 \end{pmatrix}\).

Since we have a double covering \(K_a \to S_a\) for generic members of \(\mathfrak{G}_0\) and \(\mathfrak{G}_0\), we can testify the correctness of Theorem 6.1. Namely, when \(\text{Tr}(K_a)\) is given, we can calculate the intersection matrix of \(\text{Tr}(S_a)\). Let \(\Lambda_0 = \left(\text{Tr}(K_a) \otimes \mathbb{Q}\right) \cap \left(U^{\oplus 3} + \frac{1}{2} E_8(-2)\right)\), where \(E_8(-2)\) is the lattice generated by \(\nu_1, \ldots, \nu_8\), \(\text{Tr}(S_a)\) is isometric to the lattice \(\Lambda_0(2)\) (see [N2] Section 2). In our case, \(\Lambda_0(2)\) is given by the direct sum of \(U^{\oplus 2}\) and \(\langle \nu_1/2, q_1, \nu_2/2 \rangle_{\mathbb{Z}}(2)\). This is isometric to

\[
U^{\oplus 2} \oplus \begin{pmatrix} -1 & -1 & 1/2 \\ -1 & -2 & 1/2 \\ 1/2 & 1/2 & -1 \end{pmatrix}(2) \simeq U^{\oplus 2} \oplus \begin{pmatrix} -2 & -2 & 1 \\ -2 & -4 & 1 \\ 1 & 1 & -2 \end{pmatrix} \simeq U^{\oplus 2} \oplus \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix} = A.
\]

This is concordant with Theorem 2.1.

**Acknowledgment**

The author would like to thank Professor Manabu Oura and Professor Jiro Sekiguchi for valuable suggestions from the viewpoint of complex reflection groups. He also appreciates the reviewer’s helpful comments to improve the manuscript. This work is supported by JSPS Grant-in-Aid for Scientific Research (22K03226, 18K13383), JST FOREST Program (JPMJFR2235) and MEXT LEADER.

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