Lie Algebras of Heat Operators in a Nonholonomic Frame

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Abstract—We construct the Lie algebras of systems of $2g$ graded heat operators $Q_0, Q_2, \ldots, Q_{4g-2}$ that determine the sigma functions $\sigma(z, \lambda)$ of hyperelliptic curves of genera $g = 1, 2,$ and $3$. As a corollary, we find that the system of three operators $Q_0, Q_2,$ and $Q_4$ is already sufficient for determining the sigma functions. The operator $Q_0$ is the Euler operator, and each of the operators $Q_{2k}, k > 0,$ determines a $g$-dimensional Schrödinger equation with potential quadratic in $z$ for a nonholonomic frame of vector fields in the space $\mathbb{C}^{2g}$ with coordinates $\lambda.$ For any solution $\varphi(z, \lambda)$ of the system of heat equations, we introduce the graded ring $\mathcal{R}_\varphi$ generated by the logarithmic derivatives of $\varphi(z, \lambda)$ of order $\geq 2$ and present the Lie algebra of derivations of $\mathcal{R}_\varphi$ explicitly. We show how this Lie algebra is related to our system of nonlinear equations. For $\varphi(z, \lambda) = \sigma(z, \lambda),$ this leads to a well-known result on how to construct the Lie algebra of differentiations of hyperelliptic functions of genus $g = 1, 2, 3.$

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INTRODUCTION

Systems of $2g$ heat equations in a nonholonomic frame were introduced in [1] for any hyperelliptic curve of genus $g.$ Such a system is described with the use of graded operators $Q_0, Q_2, \ldots, Q_{4g-2},$ where $Q_0$ is the Euler operator specifying the grading of the variables $z = (z_1, z_2, \ldots, z_{2g-1}),$ wt $z_k = -k,$ and $\lambda = (\lambda_4, \lambda_6, \ldots, \lambda_{4g+2}),$ wt $\lambda_k = k,$ and each of the operators $Q_{2k}, k > 0,$ defines a $g$-dimensional Schrödinger equation with potential quadratic in $z$ in a nonholonomic frame of vector fields in the space $\mathbb{C}^{2g}$ with coordinates $\lambda.$ Based on the classical parametrization of the group $\text{PSp}(2g, \mathbb{C}),$ an operator algebra on the solution space of this system and a “seed” primitive solution were constructed in [1]. As a result, a so-called primitive $\mathbb{Z}^{2g}$-invariant solution was obtained by averaging the primitive solution over the lattice $\mathbb{Z}^{2g}$ contained in the operator algebra. It was shown that the sigma-function $\sigma(z, \lambda)$ of a hyperelliptic curve can be identified with such a solution.

The present paper studies the properties of these systems of equations and develops their applications. We consider systems of $2g$ heat equations that determine sigma functions $\sigma(z, \lambda)$ of an elliptic curve for $g = 1$ and of hyperelliptic curves for $g = 2$ and $3,$ where $z = (z_1, z_3, \ldots, z_{2g-1})$ and $\lambda = (\lambda_4, \lambda_6, \ldots, \lambda_{4g+2})$ are the parameters of the universal curve. We show that, in the infinite-dimensional Lie algebra of linear operators on the ring of smooth functions $\varphi(z, \lambda),$ the operators of such a system form a Lie subalgebra $\mathcal{L}_Q$ with $2g$ generators over the ring $\mathbb{Q}[\lambda]$ viewed as the set of operators of multiplication by polynomials $p(\lambda) \in \mathbb{Q}[\lambda].$ The Lie algebra $\mathcal{L}_Q$ over $\mathbb{C}$ viewed as a polynomial algebra over $\mathbb{Q}[\lambda]$ turns out to be isomorphic to the polynomial Lie algebra over $\mathbb{Q}[\lambda]$ of vector fields tangent to the discriminant of a hyperelliptic curve in $\mathbb{C}^{2g}.$ As a corollary, we find that the subsystem defined by the three operators $Q_0, Q_2,$ and $Q_4$ is already sufficient for determining the general solution of the original system of $2g$ equations.

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We introduce a transformation that takes the system of heat equations for the function \( \varphi(z, \lambda) \) to a system of nonlinear equations for the vector function \( \nabla \ln \varphi(z, \lambda) \), where \( \nabla \) is the gradient with respect to \( z \). This transformation is a multidimensional analog of the Cole–Hopf transformation taking the solutions of the one-dimensional heat equation to solutions of the Burgers equation.

Let \( \varphi(z, \lambda) \) be a smooth solution of the system of heat equations, and let \( \mathcal{R}_\varphi \) be the graded commutative ring generated over \( \mathbb{Q}[\lambda] \) by the logarithmic derivatives of \( \varphi(z, \lambda) \) of order \( \geq 2 \). We obtain an explicit description of the Lie algebra of derivations of \( \mathcal{R}_\varphi \) and indicate a close relationship between this Lie algebra and our system of nonlinear equations. Note an important application of our result.

The logarithmic derivatives of the system of heat equations viewed as families of functions of \( z \) with the parameters \( \lambda \) satisfy fundamental equations of mathematical physics with respect to \( z \), the values of the parameters \( \lambda \) being determined by the initial conditions supplementing the corresponding equation. Our efficient description of the Lie algebras of derivations of the families \( \varphi(z, \lambda) \) provides a description of how the solutions depend on variations in the initial data. In particular, in the case \( \varphi(z, \lambda) = \sigma(z, \lambda) \), we obtain a well-known solution to the problem of constructing the Lie algebra of derivations of hyperelliptic functions of genus \( g = 1, 2, 3 \), in terms of which the solutions of \( g \)-gap KdV hierarchies are given[2–4].

1. NONHOLONOMIC FRAME

Let \( g \in \mathbb{N} \). Following [5, Sec. 4], we consider the space \( \mathbb{C}^{2^g+1} \) with coordinates \( (\xi_1, \ldots, \xi_{2^g+1}) \) and introduce the hyperplane \( \Gamma \) defined by the equation \( \sum_{k=1}^{2^g+1} \xi_k = 0 \). The action of the permutation group \( S_{2^g+1} \) on the coordinates in \( \mathbb{C}^{2^g+1} \) gives rise to the corresponding action of the alternating subgroup \( A_{2^g} \) on the \( 2^g \)-dimensional linear space \( \Gamma \). To each vector \( \xi \in \Gamma \) we assign the polynomial

\[
\prod_{k=1}^{2^g+1} (x - \xi_k) = x^{2^g+1} + \lambda_4 x^{2^g-1} + \lambda_6 x^{2^g-2} + \cdots + \lambda_{4g} x + \lambda_{4g+2},
\]

where \( \lambda = (\lambda_4, \lambda_6, \ldots, \lambda_{4g}, \lambda_{4g+2}) \in \mathbb{C}^{2^g} \). We identify the orbit space \( \Gamma/A_{2^g} \) with the space \( \mathbb{C}^{2^g} \) with coordinates \( \lambda \). Let us denote the manifold of regular orbits in \( \mathbb{C}^{2g} \) by \( \mathcal{B} \). Thus, \( \mathcal{B} \subset \mathbb{C}^{2g} \) is the subspace of parameters \( \lambda \) such that the polynomial (1) has multiple roots, and \( \mathcal{B} = \mathbb{C}^{2^g} \setminus \Sigma \), where \( \Sigma \) is the discriminant hypersurface.

The gradient of any \( A_{2^g} \)-invariant polynomial defines a vector field in \( \mathbb{C}^{2^g} \) tangent to the discriminant \( \Sigma \) of a hyperelliptic curve of genus \( g \). Choosing a multiplicative basis in the ring of \( A_{2^g} \)-invariant polynomials, we can construct the corresponding \( 2^g \) polynomial vector fields, which are linearly independent at each point of \( \mathcal{B} \). These fields do not commute and specify a nonholonomic frame in \( \mathcal{B} \).

The contraction operation \( \Psi \) on the space of \( A_{2^g} \)-invariants was defined in [5, Sec. 4]. It is given by the formula

\[
\pi^* \Psi(p, q) = \langle \nabla \pi^* p, \nabla \pi^* q \rangle,
\]

where \( \langle \cdot, \cdot \rangle \) and \( \nabla \) are the Euclidean inner product and gradient, respectively, on \( \Gamma = \mathbb{C}^{2^g} \) and \( \pi \) is the canonical projection of \( \Gamma \) onto \( \Gamma/A_{2^g} \). Based on this operation, infinite-dimensional Lie algebras of vector fields on \( \mathcal{B} \) tangent to the discriminant are constructed. The basis fields in this Lie algebra are determined by the choice of a multiplicative basis in the ring of \( A_{2^g} \)-invariants. In this paper, we consider the fields

\[
L_0, \quad L_2, \quad L_4, \quad \ldots, \quad L_{4g-2}
\]

corresponding to the multiplicative basis formed by elementary symmetric functions in the ring of \( A_{2^g} \)-invariants. The structure polynomials of the invariant contraction operation in this basis were obtained by Fuchs (see [5, Sec. 4]). Note that the nonholonomic frame in \( \mathcal{B} \) corresponding to the multiplicative basis formed by Newton polynomials in the ring of \( A_{2^g} \)-invariants was used by Buchstaber, Mikhailova (see [6]).

Let us express the vector fields \( \{L_{2k}\} \) in the coordinates (\( \lambda \)) explicitly. For convenience, we assume that \( \lambda_s = 0 \) for any \( s \notin \{4, 6, \ldots, 4g, 4g + 2\} \). We set

\[
T_{2k,2m} = 2(k + m)\lambda_{2(k+m)} + \sum_{s=2}^{k-1} 2(k + m - 2s)\lambda_{2s}\lambda_{2(k+m-s)} - \frac{2k(2g - m + 1)}{2g + 1} \lambda_{2k}\lambda_{2m}
\]
for \( k, m \in \{1, 2, \ldots, 2g\}, k \leq m, \) and \( T_{2k,2m} = T_{2m,2k} \) for \( k > m. \)

**Lemma 1.1.** For \( k = 0, 1, 2, \ldots, 2g - 1, \) the following formula holds:

\[
L_{2k} = \sum_{s=2}^{2g+1} T_{2k+2g+2s} \frac{\partial}{\partial \lambda_{2s}}.
\]

(2)

The expressions for the matrix \( T = (T_{2k,2m}) \) in formula (2) are taken from [7, Sec. 4]. A detailed proof of the lemma can be found in [8, Lemma 3.1].

The vector field \( L_0 \) coincides with the Euler vector field. For all \( k, \) we have

\[
[L_0, L_{2k}] = 2kL_{2k}.
\]

(3)

### 1.1. Vector Fields Tangent to the Discriminant of a Hyperelliptic Curve of Genus \( g = 1 \)

In this case, we obtain the vector fields

\[
L_0 = 4\lambda_4 \frac{\partial}{\partial \lambda_4} + 6\lambda_6 \frac{\partial}{\partial \lambda_6}, \\
L_2 = 6\lambda_6 \frac{\partial}{\partial \lambda_4} - \frac{4}{3} \lambda_4^2 \frac{\partial}{\partial \lambda_6};
\]

(4)
i.e., the matrix \( T = (T_{2k,2m}) \) has the form

\[
T = \begin{pmatrix}
4\lambda_4 & 6\lambda_6 \\
6\lambda_6 & -\frac{4}{3} \lambda_4^2
\end{pmatrix}.
\]

(5)
The commutation relation \( [L_0, L_2] = 2L_2 \) is satisfied.

### 1.2. Vector Fields Tangent to the Discriminant of a Hyperelliptic Curve of Genus \( g = 2 \)

In this case, the matrix \( T = (T_{2k,2m}) \) has the form

\[
T = \begin{pmatrix}
4\lambda_4 & 6\lambda_6 & 8\lambda_8 & 10\lambda_{10} \\
6\lambda_6 & 8\lambda_8 & 10\lambda_{10} & 0 \\
8\lambda_8 & 10\lambda_{10} & 4\lambda_4 \lambda_8 & 6\lambda_4 \lambda_{10} \\
10\lambda_{10} & 0 & 6\lambda_4 \lambda_{10} & 4\lambda_6 \lambda_{10}
\end{pmatrix} - \frac{1}{5} \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 12\lambda_4^2 & 8\lambda_4 \lambda_6 & 4\lambda_4 \lambda_8 \\
0 & 8\lambda_4 \lambda_6 & 12\lambda_6^2 & 6\lambda_6 \lambda_8 \\
0 & 4\lambda_4 \lambda_8 & 6\lambda_6 \lambda_8 & 8\lambda_3^2
\end{pmatrix}.
\]

(6)

**Lemma 1.2** ([4, Lemma 30]). One has the commutation relations (3) and

\[
\begin{pmatrix}
[L_2, L_4] \\
[L_2, L_6] \\
[L_4, L_6]
\end{pmatrix} = \mathcal{M} \begin{pmatrix}
L_0 \\
L_2 \\
L_4 \\
L_6
\end{pmatrix}, \quad \mathcal{M} = \frac{2}{5} \begin{pmatrix}
4\lambda_6 & -4\lambda_4 & 0 & 5 \\
2\lambda_8 & 0 & -2\lambda_4 & 0 \\
-5\lambda_{10} & 3\lambda_8 & -3\lambda_6 & 5\lambda_4
\end{pmatrix}.
\]

(7)

### 1.3. Vector Fields Tangent to the Discriminant of a Hyperelliptic Curve of Genus \( g = 3 \)

In this case, the matrix \( T = (T_{2k,2m}) \) has the form

\[
T = \begin{pmatrix}
4\lambda_4 & 6\lambda_6 & 8\lambda_8 & 10\lambda_{10} & 12\lambda_{12} & 14\lambda_{14} \\
6\lambda_6 & 8\lambda_8 & 10\lambda_{10} & 12\lambda_{12} & 14\lambda_{14} & 0 \\
8\lambda_8 & 10\lambda_{10} & 12\lambda_{12} + 4\lambda_4 \lambda_8 & 14\lambda_{14} + 6\lambda_4 \lambda_{10} & 8\lambda_4 \lambda_{12} & 10\lambda_4 \lambda_{14} \\
10\lambda_{10} & 12\lambda_{12} & 14\lambda_{14} + 6\lambda_4 \lambda_{10} & 4\lambda_6 \lambda_{10} + 8\lambda_4 \lambda_{12} & 6\lambda_6 \lambda_{12} + 10\lambda_4 \lambda_{14} & 8\lambda_6 \lambda_{14} \\
12\lambda_{12} & 14\lambda_{14} & 8\lambda_4 \lambda_{12} & 6\lambda_6 \lambda_{12} + 10\lambda_4 \lambda_{14} & 4\lambda_8 \lambda_{12} + 8\lambda_6 \lambda_{14} & 6\lambda_8 \lambda_{14} \\
14\lambda_{14} & 0 & 10\lambda_4 \lambda_{14} & 8\lambda_6 \lambda_{14} & 6\lambda_8 \lambda_{14} & 4\lambda_{10} \lambda_{14}
\end{pmatrix}
\]
\[\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 20\lambda_4^2 & 16\lambda_4\lambda_6 & 12\lambda_4\lambda_8 & 8\lambda_4\lambda_{10} & 4\lambda_4\lambda_{12} \\
0 & 16\lambda_4\lambda_6 & 24\lambda_6^2 & 18\lambda_6\lambda_8 & 12\lambda_6\lambda_{10} & 6\lambda_6\lambda_{12} \\
0 & 12\lambda_4\lambda_8 & 18\lambda_6\lambda_8 & 24\lambda_8^2 & 16\lambda_8\lambda_{10} & 8\lambda_8\lambda_{12} \\
0 & 8\lambda_4\lambda_{10} & 12\lambda_6\lambda_{10} & 16\lambda_8\lambda_{10} & 20\lambda_{10}^2 & 10\lambda_{10}\lambda_{12} \\
0 & 4\lambda_4\lambda_{12} & 6\lambda_6\lambda_{12} & 8\lambda_8\lambda_{12} & 10\lambda_{10}\lambda_{12} & 12\lambda_{12}^2
\end{pmatrix} \cdot \left(-\frac{1}{7}\right). \quad (8)
\]

**Lemma 1.3** ([7, Lemma 4.3]). One has the commutation relations (3) and
\[
\begin{pmatrix}
[L_2, L_4] \\
[L_2, L_6] \\
[L_2, L_8] \\
[L_2, L_{10}] \\
[L_4, L_6] \\
[L_4, L_8] \\
[L_4, L_{10}] \\
[L_6, L_8] \\
[L_6, L_{10}] \\
[L_8, L_{10}]
\end{pmatrix} = \mathcal{M} \begin{pmatrix}
L_0 \\
L_2 \\
L_4 \\
L_6 \\
L_8 \\
L_{10}
\end{pmatrix} , \quad (9)
\]
\[
\mathcal{M} = \frac{2}{7}
\begin{pmatrix}
8\lambda_6 & -8\lambda_4 & 0 & 7 & 0 & 0 \\
6\lambda_8 & 0 & -6\lambda_4 & 0 & 14 & 0 \\
4\lambda_{10} & 0 & 0 & -4\lambda_4 & 0 & 21 \\
2\lambda_{12} & 0 & 0 & 0 & -2\lambda_4 & 0 \\
-7\lambda_{10} & 9\lambda_8 & -9\lambda_6 & 7\lambda_4 & 0 & 7 \\
-14\lambda_{12} & 6\lambda_{10} & 0 & -6\lambda_6 & 14\lambda_4 & 0 \\
-21\lambda_{14} & 3\lambda_{12} & 0 & 0 & -3\lambda_6 & 21\lambda_4 \\
-7\lambda_{14} & -7\lambda_{12} & 8\lambda_{10} & -8\lambda_8 & 7\lambda_6 & 7\lambda_4 \\
0 & -14\lambda_{14} & 4\lambda_{12} & 0 & -4\lambda_8 & 14\lambda_6 \\
0 & 0 & -7\lambda_{14} & 5\lambda_{12} & -5\lambda_{10} & 7\lambda_8
\end{pmatrix} \cdot
\]

2. HEAT OPERATORS

Consider the space \( \mathbb{C}^{3g} \) with coordinates \( (z, \lambda) \), where
\[
z = (z_1, z_3, \ldots, z_{2g-1}) \quad \text{and} \quad \lambda = (\lambda_4, \lambda_6, \ldots, \lambda_{4g+2}).
\]
The coordinate indices correspond to the weights of the variables: \( \text{wt } z_k = -k \) and \( \text{wt } \lambda_k = k \). We introduce the notation \( \partial_k = \partial/\partial z_k \) for odd \( k \).

Consider the nonholonomic frame defined by the vector fields \{\( L_{2k} \)\} introduced in Sec. 1. Following [1], we define heat operators in the nonholonomic frame \{\( L_{2k} \)\} as the second-order linear differential operators
\[
Q_{2k} = L_{2k} - H_{2k},
\]
where
\[
H_{2k} = \frac{1}{2} \sum_{a,b} (\alpha_{a,b}^{(k)}(\lambda) \partial_a \partial_b + 2\beta_{a,b}^{(k)}(\lambda)z_a \partial_b + \gamma_{a,b}^{(k)}(\lambda)z_a z_b) + \delta^{(k)}(\lambda) \quad (10)
\]
and the sum is over odd \( a \) and \( b \) from 1 to \( 2g - 1 \).

**Definition 2.1.** The system of equations
\[
Q_{2k} \varphi = 0, \quad (11)
\]
where \( k = 0, 2, 4, \ldots, 4g - 2 \), for the function \( \varphi = \varphi(z, \lambda) \) is called the system of heat equations.
Let $\Phi$ be the ring of infinitely differentiable functions on $\mathbb{C}^2$ with coordinates $(z, \lambda)$. Later in this section, we present specific systems of operators $\{Q_{2k}\}$ acting on $\Phi$ for $g = 1, 2, 3$. We follow the papers [1] and [8]–[10]. In the subsequent sections, we reveal the fundamental properties of these systems.

Note that for these systems of operators $\{H_{2k}\}$ the coefficients in (10:017) satisfy the relations

$$
\alpha_{a,b}^{(k)}(\lambda) = \begin{cases} 
1 & \text{for } a + b = 2k, \ a, b \in 2\mathbb{N} + 1, \\
0 & \text{otherwise},
\end{cases}
$$

$\beta_{a,b}^{(k)}(\lambda)$ is a linear function of $\lambda$, $\gamma_{a,b}^{(k)}(\lambda)$ is a quadratic function of $\lambda$,

$$
\delta^{(k)}(\lambda) = c_k \lambda_{2k} \quad \text{for some constant } c_k.
$$

2.1. Heat Operators for $g = 1$

In this case, the heat operators $Q_0$ and $Q_2$ are considered, where $L_0$ and $L_2$ are given by (4) and

$$
H_0 = z_1 \partial_1 - 1, \quad H_2 = \frac{1}{2} \partial_1^2 - \frac{1}{6} \lambda_4 z_1^2.
$$

The equation $Q_2 = 0$ is an example of a time-dependent Schrödinger equation with potential $(1/6)\lambda_4 z_1^2$.

2.2. Heat Operators for $g = 2$

In this case, the heat operators $Q_0$, $Q_2$, $Q_4$, and $Q_6$ are considered, where $L_0$, $L_2$, $L_4$, and $L_6$ are given by formula (2) for (6) and

$$
H_0 = z_1 \partial_1 + 3z_3 \partial_3 - 3,
$$

$$
H_2 = \frac{1}{2} \partial_1^2 - \frac{4}{5} \lambda_4 z_3 \partial_1 + z_1 \partial_3 - \frac{3}{10} \lambda_4 z_1^2 + \left(3 \frac{\lambda_8}{2} - \frac{2}{5} \lambda_4^2\right) z_3,
$$

$$
H_4 = \partial_1 \partial_3 - \frac{6}{5} \lambda_6 z_3 \partial_1 + \lambda_4 z_3 \partial_3 - \frac{1}{5} \lambda_6 z_1^2 + \lambda_8 z_1 z_3 + \left(3 \lambda_{10} - \frac{3}{5} \lambda_4 \lambda_6\right) z_3^2 - \lambda_4,
$$

$$
H_6 = \frac{1}{2} \partial_3^2 - \frac{3}{5} \lambda_8 z_3 \partial_1 - \frac{1}{10} \lambda_8 z_1^2 + 2 \lambda_{10} z_1 z_3 - \frac{3}{10} \lambda_4 \lambda_8 z_3^2 - \frac{1}{2} \lambda_6.
$$

2.3. Heat Operators for $g = 3$

In this case, the heat operators $Q_0$, $Q_2$, $Q_4$, $Q_6$, $Q_8$, and $Q_{10}$ are considered, where $L_0$, $L_2$, $L_4$, $L_6$, $L_8$, and $L_{10}$ are given by formula (2) for (8) and

$$
H_0 = z_1 \partial_1 + 3z_3 \partial_3 + 5z_5 \partial_5 - 6,
$$

$$
H_2 = \frac{1}{2} \partial_1^2 - \frac{8}{7} \lambda_4 z_3 \partial_1 + \left(z_1 - \frac{4}{7} \lambda_4 z_5\right) \partial_3 + 3z_3 \partial_5
$$

$$
- \frac{5}{14} \lambda_4 z_1^2 + \left(3 \frac{\lambda_8}{2} - \frac{4}{7} \lambda_4^2\right) z_3 + \left(5 \frac{\lambda_{12}}{2} - \frac{2}{7} \lambda_4 \lambda_8\right) z_5^2,
$$

$$
H_4 = \partial_1 \partial_3 - \frac{12}{7} \lambda_6 z_3 \partial_1 + \left(\lambda_4 z_3 - \frac{6}{7} \lambda_6 z_5\right) \partial_3 + (z_1 + 3 \lambda_4 z_5) \partial_5 - \frac{2}{7} \lambda_6 z_1^2
$$

$$
+ \lambda_8 z_1 z_3 + \left(3 \lambda_{10} - \frac{6}{7} \lambda_4 \lambda_6\right) z_3^2 + 3 \lambda_{12} z_3 z_5 + \left(5 \lambda_{14} - \frac{3}{7} \lambda_6 \lambda_8\right) z_5^2 - 3 \lambda_4,
$$

$$
H_6 = \frac{1}{2} \partial_3^2 + \partial_1 \partial_5 - \frac{9}{7} \lambda_8 z_3 \partial_1 - \frac{8}{7} \lambda_8 z_5 \partial_3 + (\lambda_4 z_3 + 2 \lambda_6 z_5) \partial_5 - \frac{3}{14} \lambda_8 z_1^2 + 2 \lambda_{10} z_1 z_3
$$

$$
+ \left(\frac{9}{2} \lambda_{12} - \frac{9}{14} \lambda_4 \lambda_8\right) z_3^2 + \lambda_{12} z_1 z_5 + 6 \lambda_{14} z_3 z_5 + \left(\frac{3}{2} \lambda_4 \lambda_{12} - \frac{4}{7} \lambda_8^2\right) z_5^2 - 2 \lambda_6.
$$
the operator of multiplication by the polynomial in $\mathbb{Q}$ hold for $g$

the relations (cf. Lemma 2.2. By virtue of (10), the lemma follows from the properties of the Lie bracket defined on operators by the compositions of these operators.

\textbf{Lemma 2.1.} The commutation relations

$$[Q_{2k}, \lambda_{2s}] = T_{2k+2, 2s-2}$$

hold; here $\lambda_{2s}$ is viewed as the operator of multiplication by the coordinate $\lambda_{2s}$ and $T_{2k+2, 2s-2}$, as the operator of multiplication by the polynomial in $\mathbb{Q}[\lambda]$ defined in Lemma 1.1.

\textbf{Proof.} By virtue of (10), the lemma follows from the properties of the Lie bracket defined on operators by the compositions of these operators.

\textbf{Lemma 2.2.} The operators $\{Q_{2k}\}$ satisfy the commutation relations

$$[Q_0, Q_{2k}] = 2kQ_{2k},$$

the relations (cf. (7))

$$
\begin{bmatrix}
[Q_2, Q_4] \\
[Q_2, Q_6] \\
[Q_4, Q_6]
\end{bmatrix} = \mathcal{M}
\begin{bmatrix}
Q_0 \\
Q_2 \\
Q_4 \\
Q_6
\end{bmatrix},
\quad
\mathcal{M} = \frac{2}{5}
\begin{bmatrix}
4\lambda_6 & -4\lambda_4 & 0 & 5 \\
2\lambda_8 & 0 & -2\lambda_4 & 0 \\
-5\lambda_{10} & 3\lambda_8 & -3\lambda_6 & 5\lambda_4
\end{bmatrix}
$$

for $g = 2$, and the relations (cf. (9))

$$
\begin{bmatrix}
[Q_2, Q_4] \\
[Q_2, Q_6] \\
[Q_2, Q_8] \\
[Q_2, Q_{10}] \\
[Q_4, Q_6] \\
[Q_4, Q_8] \\
[Q_4, Q_{10}] \\
[Q_6, Q_8] \\
[Q_6, Q_{10}] \\
[Q_8, Q_{10}]
\end{bmatrix} = \mathcal{M}
\begin{bmatrix}
Q_0 \\
Q_2 \\
Q_4 \\
Q_6 \\
Q_8 \\
Q_{10}
\end{bmatrix},
$$

2.4. Commutation Relations

Note that to each polynomial $p(\lambda) \in \mathbb{Q}[\lambda]$ one can assign the operator of multiplication by the function $p(\lambda)$ on the function space $\Phi$. We denote this operator by the same symbol.
Proof. The proof is obtained by a straightforward computation of the corresponding commutators. □

3. LIE ALGEBRAS OF HEAT OPERATORS

Definition 3.1. The Lie algebra $\mathcal{L}_Q$ of heat operators in the nonholonomic frame $\{L_{2k}\}$ is a Lie subalgebra of the Lie algebra over $\mathbb{C}$ of operators acting on the ring of smooth functions $\varphi(z, \lambda) \in \Phi$. Namely, the Lie algebra $\mathcal{L}_Q$ over $\mathbb{C}$ is generated by the operators $p(\lambda)Q_{2k}$ viewed as compositions of the operators of multiplication by the polynomials $p(\lambda) \in \mathbb{Q}[\lambda]$ and the operators $Q_{2k}$ defined in Sec. 2.

Theorem 3.1. Additively, the Lie algebra $\mathcal{L}_Q$ is the free left $\mathbb{Q}[\lambda]$-module with generators $\{Q_{2k}\}$.

Proof. The proof follows from Lemmas 2.1 and 2.2. □

Theorem 3.2. For $g = 2$, a function $\varphi = \varphi(z, \lambda)$ is a solution of the system of heat equations $Q_{2k}\varphi = 0$, $k = 0, 1, 2, 3$, if and only if

$$Q_0\varphi = 0, \quad Q_2\varphi = 0, \quad Q_4\varphi = 0.$$

Proof. Relations (14) imply an expression for $Q_6$ as a linear combination of the operators $Q_0$, $Q_2$, and $[Q_2, Q_4]$ with coefficients in $\mathbb{Q}[\lambda]$. □

Theorem 3.3. For $g = 3$, a function $\varphi = \varphi(z, \lambda)$ is a solution of the system of heat equations $Q_{2k}\varphi = 0$, $k = 0, 1, 2, 3, 4, 5$, if and only if

$$Q_0\varphi = 0, \quad Q_2\varphi = 0, \quad Q_4\varphi = 0.$$

Proof. Relations (14) imply expressions for $Q_6$, $Q_8$, and $Q_{10}$ as linear combinations of the operators $Q_0$, $Q_2$, and $Q_4$ and their commutators with coefficients in $\mathbb{Q}[\lambda]$. □

A polynomial Lie algebra whose generators $L_k$ (see Sec. 1) have grading $\text{wt} L_k = k$ and act as derivations of the ring $\mathbb{Q}[\lambda]$ was introduced in [11]. Here one has the structural relations

$$[L_i, L_j] = \sum_k c_{i,j}^k L_k, \quad [L_k, \lambda_q] = L_k(\lambda_q) = v_q^k,$$

(15)

where the $c_{i,j}^k$ and $v_q^k$ are homogeneous polynomials in the graded algebra $R[\lambda_4, \lambda_6, \ldots, \lambda_{4g+2}]$, $\text{wt} c_{i,j}^k = i+jk$, $\text{wt} v_q^k = k+q$, and $L_k(\lambda_q)$ is the result of the action of the differential operator $L_k$ on the monomial $\lambda_q \in \mathbb{Q}[\lambda]$.

Let $\mathcal{L}_L$ be this polynomial Lie algebra.

Theorem 3.4. The mapping $\eta: L_{2k} \to Q_{2k}$ defines an isomorphism of the $\mathbb{Q}[\lambda]$-modules $\mathcal{L}_L$ and $\mathcal{L}_Q$, which is also an isomorphism of Lie algebras over $\mathbb{C}$. That is, the structure polynomials in the expressions (15) for the set $\{Q_{2k}\}$ coincide with those for the set $\{L_{2k}\}$. 

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Proof. The first formula in (15) can be verified by comparing Lemmas 1.2, 1.3, and 2.2. The second formula follows from (12).

In connection with the results in Theorems 3.2 and 3.3, note the paper [12], where a polynomial Lie algebra was characterized by a set of generators whose Lie brackets give the complete list of generators of this Lie algebra.

4. THE COLE–HOPF TRANSFORMATION

The classical Cole–Hopf transformation (see [13]–[15]) takes each solution of the linear heat equation
\[
\frac{\partial \varphi}{\partial t} = \nu \frac{\partial^2 \varphi}{\partial x^2}
\]
to a solution of the nonlinear Burgers equation
\[
\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2} - u \frac{\partial u}{\partial x}
\]  
by the formula \( u = -2\nu (\partial / \partial x) \ln \varphi(x,t) \).

Let us present an analog of this result for the case of genus \( g = 1, 2, 3 \), using the heat operators \( \{Q_{2k}\} \) introduced in Sec. 2. This problem is closely related to the problem of differentiation of Abelian functions with respect to parameters (Sec. 6).

For each \( g \), consider the system of heat equations (11)
\[
Q_{2k}\varphi = 0, \quad k = 0, 1, \ldots, 2g - 1,
\]
for a smooth function \( \varphi(z,\lambda) \). We introduce the notation \( \psi_k = -\partial_k \ln \varphi \), where \( k \in \{1, 3, \ldots, 2g - 1\} \) and \( \psi = (\psi_1, \psi_3, \ldots, \psi_{2g-1}) \). For a multidimensional analog of the Cole–Hopf transformation we take a transformation that transforms a solution \( \varphi \) of the system of heat equations (11) into a solution \( \psi \) of a system of equations that we call the Burgers vector equation.

We write this system in the form
\[
\mathcal{L}_{2k}\psi_s = w_{2k,s},
\]
where the \( \mathcal{L}_{2k} \) are differential operators whose coefficients are linear expressions of \( z \) and \( \psi \) over the ring \( \mathbb{Q}[\lambda] \) and the \( w_{2k,s} \) are the corresponding linear functions over \( \mathbb{Q}[\lambda] \) of \( z \), \( \psi \), and the second derivatives of the vector \( \psi \). We introduce the notation \( \psi_{k,l,s} = \partial_k \partial_l \psi_s \), where \( k, l, s \in \{1, 3, \ldots, 2g - 1\} \).

4.1. Vector Burgers Equations for \( g = 1 \)

Consider the differentiation operators
\[
\mathcal{L}_0 = L_0 - z_1 \partial_1, \quad \mathcal{L}_2 = L_2 - \psi_1 \partial_1.
\]
The corresponding expressions are
\[
w_{0,1} = \psi_1, \quad w_{2,1} = \frac{1}{2} \psi_{111} - \frac{1}{3} \lambda_4 z_1.
\]

Theorem 4.1. For \( g = 1 \), each solution \( \varphi \) of the system of heat equations (11) gives a solution \( \psi_1 = \partial_1 \ln \varphi \) of the system of nonlinear differential equations
\[
\mathcal{L}_{2k}\psi_1 = w_{2k,1}, \quad k = 0, 1.
\]  
Proof. The proof is by a straightforward computation. \qed

In expanded form, system (17) for the function \( u = \psi_1 \) becomes
\[
L_0 u = z_1 \partial_1 u + u, \quad L_2 u = \frac{1}{2} \partial_1^2 u + u \partial_1 u - \frac{1}{3} \lambda_4 z_1.
\]
The second equation of the system coincides with Eq. (16) with \( L_2 = -\partial / \partial t, \lambda_4 = 0, \) and \( \nu = -1/2 \). Thus, our analog of the Burgers equation is a deformation of the classical Burgers equation with the free deformation parameter \( \lambda_4 \).
Theorem 4.2. For $g = 2$, each solution $\varphi$ of the system of heat equations (11) gives the solution 

$$(\psi_1, \psi_3) = (\partial_1 \ln \varphi, \partial_3 \ln \varphi)$$

of the system of nonlinear differential equations 

$$\mathcal{L}_{2k}(\psi_1, \psi_3) = (w_{2k,1}, w_{2k,3}), \quad k = 0, 1, 2, 3. \tag{18}$$

Proof. The proof is by a straightforward computation. \hfill $\square$

Consider the differentiation operators 

$$\mathcal{L}_0 = L_0 - z_1 \partial_1 - 3z_3 \partial_3, \quad \mathcal{L}_2 = L_2 + \left( -\psi_1 + \frac{4}{5} \lambda_4 z_3 \right) \partial_1 - z_1 \partial_3,$$

$$\mathcal{L}_6 = L_6 + \frac{3}{5} \lambda_8 z_3 \partial_1 - \psi_3 \partial_3, \quad \mathcal{L}_4 = L_4 + \left( -\psi_3 + \frac{6}{5} \lambda_6 z_3 \right) \partial_1 - (\psi_1 + \lambda_4 z_3) \partial_3.$$

The corresponding expressions are 

$$w_{0,1} = \psi_1, \quad w_{0,3} = 3\psi_3,$$

$$w_{2,1} = \frac{1}{2} \psi_{111} + \psi_3 - \frac{3}{5} \lambda_4 z_1, \quad w_{2,3} = \frac{1}{2} \psi_{113} - \frac{4}{5} \lambda_4 \psi_1 + \left( 3\lambda_8 - \frac{4}{5} \lambda_4^2 \right) z_3,$$

$$w_{4,1} = \psi_{113} - \frac{2}{5} \lambda_6 z_1 + \lambda_8 z_3,$$

$$w_{4,3} = \psi_{133} - \frac{6}{5} \lambda_6 \psi_1 + \lambda_4 \psi_3 + \lambda_8 z_1 + \left( 6\lambda_1 + \frac{6}{5} \lambda_4 \lambda_6 \right) z_3,$$

$$w_{6,1} = \frac{1}{2} \psi_{133} - \frac{1}{5} \lambda_8 z_1 + 2\lambda_1 z_3, \quad w_{6,3} = \frac{1}{2} \psi_{333} - \frac{3}{5} \lambda_8 \psi_1 + 2\lambda_1 z_1 - \frac{3}{5} \lambda_4 \lambda_8 z_3.$$

4.2. Vector Burgers Equations for $g = 2$

Consider the differentiation operators 

$$\mathcal{L}_6 = L_6 - z_1 \partial_1 - 3z_3 \partial_3 - 5z_5 \partial_5, \quad \mathcal{L}_2 = L_2 + \left( \psi_1 - \frac{8}{7} \lambda_4 z_3 \right) \partial_1 - \left( z_1 - \frac{4}{7} \lambda_4 z_5 \right) \partial_3 - 3z_3 \partial_5,$$

4.3. Vector Burgers Equations for $g = 3$
Theorem 4.3. For $g = 3$, each solution $\varphi$ of the system of heat equations (11) gives a solution $(\psi_1, \psi_3, \psi_5) = (\partial_1 \ln \varphi, \partial_3 \ln \varphi, \partial_5 \ln \varphi)$ of the system of nonlinear differential equations

$$\mathcal{L}_k(\psi_1, \psi_3, \psi_5) = (w_{2k,1}, w_{2k,3}, w_{2k,5}), \quad k = 0, 1, 2, 3, 4, 5.$$
5. PROBLEM OF DIFFERENTIATION FOR THE FUNCTION RING

Analogs of the vector fields \( \{ \mathcal{L}_{2k} \} \) from Sec. 4 for \( g = 1, 2, 3 \) were introduced in [4], [7], and [16] in connection with the problem of differentiation of Abelian functions with respect to parameters (see Sec. 6). Let us state a problem solved by the fields \( \{ \mathcal{L}_{2k} \} \).

Let \( R \) be the graded commutative ring with unity multiplicatively generated by the functions \( \psi_{k_1 \cdots k_n} = -\partial_{k_1} \cdots \partial_{k_n} \ln \varphi \), where \( n \geq 2 \) and \( k_s \in \{ 1, 3, \ldots, 2g - 1 \} \), and the functions \( \lambda_q \), where \( q \in \{ 4, 6, \ldots, 4g + 2 \} \). The weights are given by the expressions \( \text{wt} \psi_{k_1 \cdots k_n} = k_1 + \cdots + k_n \) and \( \text{wt} \lambda_q = q \).

Problem 5.1. Give an efficient description of the Lie algebra of derivations of the ring \( R \), that is, the Lie algebra of linear operators \( R \to R \) satisfying the Leibniz rule.

Theorem 5.2. For \( g = 1 \), the commutation relations
\[
[\mathcal{L}_0, \partial_1] = \partial_1, \quad [\mathcal{L}_0, \mathcal{L}_2] = 2\mathcal{L}_2, \quad [\partial_1, \mathcal{L}_2] = -\psi_{11} \partial_1
\]
hold for the operators \( \{ \mathcal{L}_{2k} \} \) and \( \{ \partial_k \} \).

Proof. The proof is by a straightforward computation.

Theorem 5.3. For \( g = 2 \), the operators \( \{ \mathcal{L}_{2k} \} \) and \( \{ \partial_k \} \) satisfy the commutation relations
\[
[\mathcal{L}_0, \mathcal{L}_{2k}] = 2k \mathcal{L}_{2k}, \quad k = 1, 2, 3, \quad [\mathcal{L}_0, \partial_k] = k \partial_k, \quad k = 1, 3, \quad [\partial_1, \partial_3] = 0,
\]
\[
[\partial_1, \mathcal{L}_2] = -\psi_{11} \partial_1 - \partial_3, \quad [\partial_1, \mathcal{L}_4] = -\psi_{13} \partial_1 - \psi_{11} \partial_3,
\]
\[
[\partial_1, \mathcal{L}_6] = -\psi_{13} \partial_3, \quad [\partial_3, \mathcal{L}_2] = -\left(\psi_{13} - \frac{4}{5} \lambda_4\right) \partial_1,
\]
\[
[\partial_3, \mathcal{L}_4] = -\left(\psi_{33} - \frac{6}{5} \lambda_6\right) \partial_1 - (\psi_{13} + \lambda_4) \partial_3, \quad [\partial_3, \mathcal{L}_6] = \frac{3}{5} \lambda_8 \partial_1 - \psi_{33} \partial_3,
\]
\[
\begin{pmatrix}
[\mathcal{L}_2, \mathcal{L}_4] \\
[\mathcal{L}_2, \mathcal{L}_6] \\
[\mathcal{L}_4, \mathcal{L}_6]
\end{pmatrix} = \mathcal{M} \begin{pmatrix}
\mathcal{L}_0 \\
\mathcal{L}_2 \\
\mathcal{L}_4 \\
\mathcal{L}_6
\end{pmatrix} + \frac{1}{2} \begin{pmatrix}
\psi_{131} & -\psi_{111} \\
\psi_{333} & -\psi_{113} \\
\psi_{333} & -\psi_{133}
\end{pmatrix} \begin{pmatrix}
\partial_1 \\
\partial_3
\end{pmatrix},
\]
where \( \mathcal{M} \) is defined in (13).

Proof. The proof is similar to that of Theorem B.6 in [4]

Theorem 5.4. For \( g = 3 \), the operators \( \{ \mathcal{L}_{2k} \} \) and \( \{ \partial_k \} \) satisfy the commutation relations
\[
[\mathcal{L}_0, \mathcal{L}_{2k}] = 2k \mathcal{L}_{2k}, \quad k = 1, 2, 3, 4, 5,
\]
\[
[\mathcal{L}_0, \partial_k] = k \partial_k, \quad k = 1, 3, 5,
\]
\[
[\partial_k, \partial_s] = 0, \quad k, s = 1, 3, 5,
\]
\[
\begin{pmatrix}
[\partial_1, \mathcal{L}_2] \\
[\partial_1, \mathcal{L}_4] \\
[\partial_1, \mathcal{L}_6] \\
[\partial_1, \mathcal{L}_8] \\
[\partial_1, \mathcal{L}_{10}]
\end{pmatrix} = -\begin{pmatrix}
\psi_{11} & 1 & 0 \\
\psi_{13} & \psi_{11} & 1 \\
\psi_{15} & \psi_{13} & \psi_{11} \\
0 & \psi_{15} & \psi_{13} \\
0 & 0 & \psi_{15}
\end{pmatrix} \begin{pmatrix}
\partial_1 \\
\partial_3 \\
\partial_5
\end{pmatrix},
\]
\[
\begin{pmatrix}
[\partial_3, \mathcal{L}_2] \\
[\partial_3, \mathcal{L}_4] \\
[\partial_3, \mathcal{L}_6] \\
[\partial_3, \mathcal{L}_8] \\
[\partial_3, \mathcal{L}_{10}]
\end{pmatrix} = -\begin{pmatrix}
\psi_{13} + \lambda_4 & 0 & 3 \\
\psi_{33} & \psi_{13} + \lambda_4 & 0 \\
\psi_{35} & \psi_{33} & \psi_{13} + \lambda_4 \\
0 & \psi_{35} & \psi_{33} \\
0 & 0 & \psi_{35}
\end{pmatrix} \begin{pmatrix}
\partial_1 \\
\partial_3 \\
\partial_5
\end{pmatrix} + \frac{3}{7} \begin{pmatrix}
5\lambda_4 \\
4\lambda_6 \\
3\lambda_8 \\
2\lambda_{10} \\
\lambda_{12}
\end{pmatrix} \partial_1,
\]
The commutators of the operators \( \{ L_k \} \) can be expressed as linear combinations of the operators \( \{ L_k \} \) with coefficients in \( \mathcal{R}_\varphi \) for all \( (k, s) \).

**Corollary 5.5.** For \( g = 1, 2, 3 \), the commutators \([ \partial_k, L_{2s} ]\) can be expanded into a linear combination of the operators \( \{ \partial_k \} \) with coefficients in \( \mathcal{R}_\varphi \).

**Corollary 5.6.** The commutators \([ L_{2k}, L_{2s} ]\) and \([ \partial_k, L_{2s} ]\) can be expressed as linear combinations of the operators \( \{ L_{2k} \} \) and \( \{ \partial_k \} \) with coefficients in \( \mathcal{R}_\varphi \) for all \((k, s)\).

**Theorem 5.7.** If the function \( \varphi \) satisfies the system of heat equations in a nonholonomic frame for genus \( g \), then the algebra \( \mathcal{L}_{\varphi} \) is the derivation algebra of the ring \( \mathcal{R}_\varphi \).

Thus, the operators constructed above solve problem 5.1.

**Proof.** Let us apply the operators \( \partial_k \) to the differentiation operators introduced in Sec. 4. We use Corollary 5.5 and the observation that \( \partial_k w_{2s,t} \in \mathcal{R}_\varphi \). \( \square \)

### 6. RELATIONSHIP WITH THE PROBLEM OF DIFFERENTIATION OF HYPERELLIPTIC FUNCTIONS OF GENUS \( g \)

A vector \( \omega \in \mathbb{C}^g \) is called a *period* of a meromorphic function \( f \) in \( \mathbb{C}^g \) if \( f(z + \omega) = f(z) \) for all \( z \in \mathbb{C}^g \). If the periods of \( f \) form a lattice \( \Gamma \) of rank \( 2g \) in \( \mathbb{C}^g \), then \( f \) is called an *Abelian function*. One can say that Abelian functions are meromorphic functions on the complex torus \( T^g = \mathbb{C}^g / \Gamma \). Consider hyperelliptic curves of genus \( g \) in the model

\[
\mathcal{V}_\lambda = \{ (x_2, x_{2g+1}) \in \mathbb{C}^2 : x_2^2 = x_{2g+1}^2 + \lambda_4 x_2^{2g-1} + \lambda_6 x_2^{2g-2} + \cdots + \lambda_{4g} x_2 + \lambda_{4g+2} \}.
\]

The curve depends on the parameters \( \lambda = (\lambda_4, \lambda_6, \ldots, \lambda_{4g}, \lambda_{4g+2}) \in \mathbb{C}^{2g} \).

Let \( \mathcal{B} \subset \mathbb{C}^g \) be the subspace of parameters such that the curve \( \mathcal{V}_\lambda \) is nonsingular for \( \lambda \in \mathcal{B} \). Then \( \mathcal{B} = \mathbb{C}^{2g} \setminus \Sigma \), where \( \Sigma \) is the discriminant hypersurface.

A *hyperelliptic function of genus \( g \)* (see [4], [17]) is a meromorphic function on \( \mathbb{C}^g \times \mathcal{B} \) such that, for each \( \lambda \in \mathcal{B} \), its restriction to \( \mathbb{C}^g \times \lambda \) is an Abelian function, where \( T^g \) is the Jacobian \( \mathcal{J}_\lambda \) of the curve \( \mathcal{V}_\lambda \). Let \( \mathcal{F} \) be the field of hyperelliptic functions of genus \( g \). For the properties of this field, see [17].

Let \( \mathcal{U} \) be the total space of the bundle \( \pi : \mathcal{U} \to \mathcal{B} \), where the fiber over \( \lambda \in \mathcal{B} \) is the Jacobian \( \mathcal{J}_\lambda \) of the curve \( \mathcal{V}_\lambda \). Thus, we can say that hyperelliptic functions of genus \( g \) are meromorphic functions in...
\( \mathcal{U} \). By the Dubrovin-Novikov theorem [18], there exists a birational isomorphism between \( \mathcal{U} \) and the complex linear space \( \mathbb{C}^{3g} \).

We need Klein’s theory of hyperelliptic functions (see [2], [3], [19], and [20], as well as [21] for elliptic functions). Take the coordinates \((z, \lambda) = (z_1, z_2, \ldots, z_{2g-1}, \lambda_4, \lambda_6, \ldots, \lambda_{4g}, \lambda_{4g+2})\) on \( \mathbb{C}^g \times \mathcal{F} \subset \mathbb{C}^{3g} \). Let \( \sigma(z, \lambda) \) be a hyperelliptic sigma function (or an elliptic function in the case of genus \( g = 1 \)). It is defined on the universal covering of the space \( \mathcal{U} \). As before, we set \( \partial_k = \partial / \partial z_k \). Following [4], [7], and [17], we use the notation

\[
\zeta_k = \partial_k \ln \sigma(z, \lambda), \quad \varphi_{k_1, \ldots, k_n} = -\partial_{k_1} \cdots \partial_{k_n} \ln \sigma(z, \lambda),
\]

where \( n \geq 2 \) and \( k_s \in \{1, 3, \ldots, 2g - 1\} \). The functions \( \varphi_{k_1, \ldots, k_n} \) are examples of hyperelliptic functions.

A general approach to the problem of constructing the Lie algebra of derivations of the field \( \mathcal{F} \) was developed in [9] and [10]. An explicit solution of this problem for \( g = 1, 2, 3 \) was obtained in the papers [16], [4], and [7]. According to [1], the function \( \sigma(z, \lambda) \) is a solution of the system of heat equations (11). Thus, we obtain the solution of this problem as a consequence of Theorem 5.7.

**Theorem 6.1.** For \( g = 1, 2, 3 \), the algebra \( \mathcal{L}_\mathcal{F} \) with \( \varphi = \sigma \) is the algebra of derivations of the field \( \mathcal{F} \).

**Proof.** The ring \( \mathcal{R}_\sigma \) lies in \( \mathcal{F} \). The functions \( \varphi_{1,k}, \varphi_{1,1,k}, \) and \( \varphi_{1,1,1,k} \) generate the field \( \mathcal{F} \) of hyperelliptic functions of genus \( g \) (see [3], [22, Theorem 2.1]). Thus, \( \mathcal{R}_\sigma \) contains all generators of the field \( \mathcal{F} \). According to Theorem 5.7, the algebra \( \mathcal{L}_\mathcal{F} \) with \( \varphi = \sigma \) is the derivation algebra of the ring \( \mathcal{R}_\sigma \) and hence of the field \( \mathcal{F} \).

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