A new proof for the Erdős–Ko–Rado theorem for the alternating group

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A subset $S$ of the alternating group on $n$ points is intersecting if for any pair of permutations $\pi, \sigma$ in $S$, there is an element $i \in \{1, \ldots, n\}$ such that $\pi(i) = \sigma(i)$. We prove if $n \geq 5$ and $S$ is intersecting, then $|S| \leq \frac{(n-1)!}{2}$. Also, we prove that provided that $n \geq 5$, then the only sets $S$ that meet this bound are the cosets of the stabilizer of a point of $\{1, \ldots, n\}$. These two results were first proven by Ku and Wong (2007), the proof given in this paper uses an algebraic method that is very different from the original proof.

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1. Introduction

The famous Erdős–Ko–Rado theorem [7] (abbreviated EKR theorem) gives bounds for the sizes of intersecting set systems and characterizes the systems that achieve the bound. Many similar theorems have been proved for other mathematical objects with a relevant concept of “intersection”. See [5,8,13,17,20] for versions of this result for permutations, integer sequences, vector spaces, set partitions and blocks in a $t$-design.

Let $G \leq \text{Sym}(n)$ be a permutation group with the natural action on the set $\{1, \ldots, n\}$. Two permutations $\pi, \sigma \in G$ are said to intersect if $\pi \sigma^{-1}$ has a fixed point. A subset $S \subseteq G$ is, then, called intersecting if any pair of its elements intersect. Clearly, the stabilizer of a point is an intersecting set in $G$ (as is any coset of the stabilizer of a point). We say the group $G$ has the EKR property, if the size of any intersecting subset of $G$ is bounded above by the size of the largest point-stabilizer in $G$. Further, $G$ is said to have the strict EKR property if the only maximum intersecting subsets of $G$ are cosets of the stabilizer of a point. In [5], it was proved that Sym$(n)$ has the strict EKR property. This result caught the attention of several researchers, indeed, the result was proved with vastly different methods in [11,16,23]. Further, researchers have also worked on finding other subgroups of Sym$(n)$ that have the strict EKR property. For example in [15] it is shown that Alt$(n)$ has the strict EKR property, provided that $n \geq 5$.

In [11], the authors prove that the symmetric group has the strict EKR property using an algebraic method which relies strongly on the character theory of the symmetric group. This method establishes an interesting connection between the algebraic properties of the group and properties of a graph (based on the group) which can be used to determine if a version of the EKR theorem holds. A more significant characteristic of this approach is that it introduces a standard way to determine if a permutation group has the strict EKR property. For instance, using this method, in [18] it is proved that the projective general linear group $\text{PGL}(2, q)$ acting on the points of the projective line has the strict EKR property. This motivated the authors of the present paper to provide an alternative proof of the strict EKR property for the alternating group using the method of [11]. In other words, we prove the following.

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**Theorem 1.1.** For \( n \geq 5 \), any intersecting subset of \( \text{Alt}(n) \) has size at most 
\[
\frac{(n - 1)!}{2}.
\]
An intersecting subset of \( \text{Alt}(n) \) achieves this bound if and only if it is a coset of point-stabilizer.

In [15], it is shown that the bound in this theorem actually holds for all \( n \geq 2 \) and the second statement holds provided that \( n \neq 4 \); these facts for \( n < 5 \) are easy to prove. Thus, in this paper, we only consider \( n \geq 5 \).

Throughout the paper we denote \( G_n = \text{Alt}(n) \). The next section provides a brief overview of the method used and explains how this problem can be stated as a question about a graph. In Section 3 a short background of the representation theory of the symmetric group and the alternating group is presented. Section 4 proves some lower bounds on the dimensions of some special representations of \( G_n \) which will be used in Section 5. In Section 5 we give more details about the standard representation of the alternating group and its corresponding module. In Section 6 we give the proof of the main theorem. We conclude with Section 7 in which it is proved that the strict EKR property for the Sym(\( n \)) can be deduced from the fact that \( G_n \) has the strict EKR property.

2. Overview of the method

This section is devoted to explaining the method for the proof of Theorem 1.1. To this goal, we start with recalling the well-known clique–coclique bound; the version we use here was originally proved by Delsarte [6]. Assume \( A = \{A_0, A_1, \ldots, A_d\} \) is an association scheme on \( v \) vertices and let \( \{E_0, E_1, \ldots, E_d\} \) be the corresponding idempotents. (For a detailed discussion about association schemes, the reader may refer to [3] or [4].) For any subset \( S \) of a set \( X \), we denote the characteristic vector of \( S \) in \( X \) by \( v_S \).

**Theorem 2.1 (Clique–Coclique Bound).** Let \( X \) be the union of some of the graphs in an association scheme \( A \) on \( v \) vertices. If \( C \) is a clique and \( S \) is an independent set in \( X \), then
\[
|C||S| \leq v.
\]
If equality holds then
\[
v_C^T E_j v_C v_S^T E_j v_S = 0, \quad \text{for all } j \geq 0.
\]

We refer the reader to [11] for a proof of Theorem 2.1. We will also make use of the following straightforward corollary of this result that was also proved in [11].

**Corollary 2.2.** Let \( X \) be a union of graphs in an association scheme such that the clique–coclique bound holds with equality in \( X \). Assume that \( C \) is a maximum clique and \( S \) is a maximum independent set in \( X \). Then, for \( j > 0 \), at most one of the vectors \( E_j v_C \) and \( E_j v_S \) is not zero.

Let \( G \) be a group and \( D \) a subset of \( G \), which does not include the identity element of \( G \) and is closed under inversion. The Cayley graph of \( G \) with respect to \( D \) is defined to be the graph \( I^*(G; D) \) with vertex set \( G \) in which two vertices \( g, h \) are adjacent if and only if \( gh^{-1} \in D \). If \( D_C \) is the set of all fixed-point-free elements of \( G \), then the graph \( I^*(G; D_C) \) is called the derangement graph of \( G \) and is denoted by \( \Gamma_G \).

If we view \( G \) as a permutation group, two permutations in \( G \) are intersecting if and only if their corresponding vertices are not adjacent in \( \Gamma_G \). Therefore, the problem of classifying the maximum intersecting subsets of \( G \) is equivalent to characterizing the maximum independent sets of vertices in \( \Gamma_G \). Since \( D_C \) is a union of conjugacy classes of \( G \) (namely the derangement conjugacy classes), \( \Gamma_G \) is a union of graphs in the conjugacy class scheme of \( G \). Thus the clique–coclique bound can be applied to \( \Gamma_G \).

Furthermore, the idempotents of the conjugacy class scheme are \( E_\chi \), where \( \chi \) runs through the set of all irreducible characters of \( G \); the entries of \( E_\chi \) are given by
\[
(E_\chi)_{\pi, \sigma} = \frac{\chi(1)}{|G|} \chi(\pi^{-1} \sigma)
\]
(see [2] or [4, Sections 2.2 and 2.7] for a proof of this). The vector space generated by the columns of \( E_\chi \) is called the module corresponding to \( \chi \) or simply the \( \chi \)-module of \( \Gamma_G \). For any character \( \chi \) of \( G \) and any subset \( X \) of \( G \) define
\[
\chi(X) = \sum_{x \in X} \chi(x).
\]
Using Corollary 2.2 and Eq. (1) one observes the following.

**Corollary 2.3.** Assume the clique–coclique bound holds with equality for the graph \( \Gamma_G \) and let \( \chi \) be an irreducible character of \( G \) that is not the trivial character. If there is a clique \( C \) of maximum size in \( \Gamma_G \) with \( \chi(C) \neq 0 \), then
\[
E_\chi v_S = 0
\]
for any maximum independent set \( S \) of \( \Gamma_G \).
In other words, provided that the clique–coclique bound holds with equality, for any module of $\Gamma_C$ (other than the trivial module) the projection of at most one of the vectors $v_C$ and $v_S$ will be non-zero, where $S$ is any maximum independent set and $C$ is any maximum clique.

The outline for the proof of Theorem 1.1 is the following:
1. Determine the irreducible characters of $G_n$ (Section 3).
2. Show that clique–coclique bound holds with equality for $\Gamma_C$ (Section 5).
3. Find a maximum clique $C$ of $\Gamma_C$ such that $\chi(C) \neq 0$, for any character $\chi$ which is not the standard character (Section 5).
4. Show that the characteristic vector of any maximum independent set of $\Gamma_C$ lies in the direct sum of the trivial and the standard modules (Section 5).
5. Find a basis for the standard module—this basis is made up of characteristic vectors of cosets of point stabilizers (Section 5).
6. Complete the proof by showing that the only linear combination of the basis vectors that gives the characteristic vector for a maximum independent set, is the characteristic vector for the coset of the stabilizer of a point (Section 6).

We conclude this section with the following lemma which will be needed in Section 6. This proof was originally done by Mike Newman [19] (but has not been published elsewhere). The eigenvalues of a graph are the eigenvalues of the adjacency matrix of the graph.

**Proposition 2.4.** Let $X$ be a $k$-regular graph and let $\tau$ be the least eigenvalue of $X$. Assume that there is a collection $C$ of cliques of $X$ of size $w$, such that every edge of $X$ is contained in a fixed number of elements of $C$. Then

$$\tau \geq - \frac{k}{w-1}.$$ 

**Proof.** Assume that every edge of $X$ is contained exactly in $y$ cliques in $C$. Then every vertex of $X$ is contained exactly in \( \frac{k}{w-1} \) cliques. Define a 01-matrix $N$ as follows: the rows of $N$ are indexed by the vertices of $X$ and the columns are indexed by the members of $C$; the entry $N_{x,C}$ is 1 if and only if the vertex $x$ is in the clique $C$. We will, therefore, have

$$NN^T = \frac{y k}{w-1} I + y A(X),$$

where $I$ is the identity matrix and $A(X)$ is the adjacency matrix of $X$. Thus

$$\frac{k}{w-1} I + A(X) = \left( \frac{1}{\sqrt{y}} N \right) \left( \frac{1}{\sqrt{y}} N \right)^T,$$

which implies that the matrix

$$A(X) - \frac{-k}{w-1} I$$

is positive semi-definite and, therefore, if $\tau$ is the least eigenvalue of $X$, then

$$\tau \geq \frac{-k}{w-1}. \quad \Box$$

### 3. Background on the representation theory of the symmetric group

In this section we provide a brief overview of the facts from the representation theory of the symmetric group that will be needed. The reader may refer to any of the books [9,14] and [21] for more details.

Let $G$ be a group and let $V$ be a representation of $G$ with the character $\chi$. For any subgroup $H \leq G$, the restriction of $V$ to $H$, denoted by $V \downarrow^G_H$, is the representation of $H$ with the character $\chi \downarrow^G_H$, where

$$\chi \downarrow^G_H(h) = \chi(h), \quad h \in H.$$  

A weakly decreasing sequence $\lambda = [\lambda_1, \ldots, \lambda_k]$ of positive integers is called a partition of $n$ if $\sum \lambda_i = n$. If $\lambda$ is a partition of $n$ then we write $\lambda \vdash n$. To any partition $\lambda = [\lambda_1, \ldots, \lambda_k] \vdash n$, we associate a Young diagram, which is an array of $n$ boxes having $k$ left-justified rows with row $i$ containing $\lambda_i$ boxes, for $1 \leq i \leq k$. The transpose (or conjugate) of a partition $\lambda \vdash n$, which is denoted by $\lambda^\prime$, is the partition corresponding to the Young diagram which is obtained from that of $\lambda$ by interchanging rows and columns. A partition is symmetric if it is equal to its own conjugate. Fig. 1 displays the Young diagram of the partition $\lambda = [5, 3, 3, 2, 1, 1]$ of 15 and its transpose.

It is well-known that there is a one-to-one correspondence between the partitions of $n$ and the irreducible representations of $\text{Sym}(n)$. Indeed, any irreducible representation of $\text{Sym}(n)$ is of the form $S^\lambda$ where $\lambda$ is a partition of $n$. The spaces $S^\lambda$, which are called the Specht modules, are $\mathbb{C}$-algebras generated by $\lambda$-polytabloids (see [21] for more details).
Let $\lambda$ be a partition of $n$ and let $W$ and $\hat{W}$ be the restrictions of $S^\lambda$ and $\hat{S}^\lambda$ to $G_n$, respectively. Then

(a) if $\lambda$ is not symmetric, then $W$ is an irreducible representation of $G_n$ and is isomorphic to $\hat{W}$; and

(b) if $\lambda$ is symmetric, then $W = W' \oplus W''$, where $W'$ and $W''$ are irreducible but not isomorphic representations of $G_n$.

All the irreducible representations of $G_n$ arise uniquely in this way.

For any conjugacy class $c$ of $G_n$, either $c$ is also a conjugacy class in $\text{Sym}(n)$ or $c \cup c'$ is a conjugacy class in $\text{Sym}(n)$, where $c' = tct^{-1}$, for some $t \not\in G_n$. The second type of conjugacy classes are said to be split. A conjugacy class $c$ of $G_n$ is split if and only if all the cycles in the cycle decomposition of an element of $c$ have odd lengths and no two cycles have the same length.

Suppose $c$ is a conjugacy class of $\text{Sym}(n)$ that is not a conjugacy class in $G_n$. Assume that the decomposition of an element of $c$ contains cycles of odd lengths $q_1 > q_2 > \cdots > q_r$. Then we say $c$ corresponds to the symmetric partition $\lambda = [\lambda_1, \lambda_2, \ldots]$, of $n$ if $q_1 = 2\lambda_1 - 1, q_2 = 2\lambda_2 - 3, q_3 = 2\lambda_3 - 5, \ldots$. This is a correspondence between a split conjugacy classes of $G_n$ and the symmetric partitions of $n$. See Fig. 2. Using this correspondence, we can give equations for the irreducible characters of $G_n$ in terms of the characters of $\text{Sym}(n)$. This result is also proved in Section 5.1 of [9].

**Theorem 3.3.** Let $\lambda$ be a partition of $n$ and let $\chi^\lambda$ be the character of $S^\lambda$. Assume $c$ is a non-split conjugacy class of $G_n$ and $c' \cup c''$ is a pair of split conjugacy classes in $G_n$. Let $\sigma \in c$, $\sigma' \in c'$, $\sigma'' \in c''$ and $\tilde{\sigma} \in c' \cup c''$.

(a) If $\lambda$ is not symmetric, let $\chi_\lambda$ be the character of $W$, then

$$\chi_\lambda(\sigma) = \chi^\lambda(\sigma) \quad \text{and} \quad \chi_\lambda(\sigma') = \chi^\lambda(\sigma') = \chi^\lambda(\tilde{\sigma}).$$

(b) If $\lambda$ is symmetric, let $\chi'_\lambda$ and $\chi''_\lambda$ be the characters of $W'$ and $W''$, respectively, then

$$\chi'_\lambda(\sigma) = \chi''_\lambda(\sigma) = \frac{1}{2} \chi^\lambda(\sigma),$$

and

(i) if $c' \cup c''$ does not correspond to $\lambda$, then

$$\chi'_\lambda(\sigma') = \chi''_\lambda(\sigma') = \chi^\lambda(\sigma') = \chi^\lambda(\sigma'') = \frac{1}{2} \chi^\lambda(\tilde{\sigma}).$$

(ii) if $c' \cup c''$ corresponds to $\lambda$, then

$$\chi'_\lambda(\sigma') = \chi''_\lambda(\sigma') = x \quad \text{and} \quad \chi'_\lambda(\sigma'') = \chi''_\lambda(\sigma'') = y.$$
The values of $x$ and $y$ are
\[
\frac{1}{2} \left[ (−1)^m \pm \sqrt{(−1)^m q_1 \ldots q_r} \right].
\]
where $m = \frac{n^2}{2}$ and the cycle decomposition of an element of $c' \cup c''$ has cycles of odd lengths $q_1, \ldots, q_r$.

We will use the notation of Theorem 3.3 throughout this paper and hence want to emphasize that for representations of $\text{Sym}(n)$ we use $\lambda$ as a superscript and for representations of $\text{Alt}(n)$, the $\lambda$ is a subscript.

The next theorem is known as the Murnaghan–Nakayama Rule; it gives a recursive way to determine the value of a character on a conjugacy class. Before we can state this result, we need to define some terms. The $(i, j)$-block in a Young diagram is the block in the $i$th row (from the top) and the $j$th column (from the left). If a Young diagram contains an $(i, j)$-block but not a $(i + 1, j + 1)$-block then the $(i, j)$-block is part of what is called the boundary of the Young diagram. A skew hook of $\lambda$ is an edge-wise connected (meaning that all blocks are either side by side or one below the other) part of the boundary blocks with the property that removing them leaves a smaller proper Young diagram. The length of a skew hook is the number of blocks it contains.

**Theorem 3.4.** If $\lambda \vdash n$ and $\sigma \in \text{Sym}(n)$ can be written as a product of an $m$-cycle and a disjoint permutation $h \in \text{Sym}(n - m)$, then
\[
\chi^\lambda(\sigma) = \sum_\mu (-1)^{r(\mu)} \chi^{\mu}(h),
\]
where the sum is over all partitions $\mu$ of $n - m$ that are obtained from $\lambda$ by removing a skew hook of length $m$, and $r(\mu)$ is one less than the number of rows of the removed skew hook.

For a proof of this theorem, the reader may refer to [21]. We will state, without proof, two simple applications of this rule.

**Corollary 3.5.** Let $\sigma$ be an $n$-cycle in $\text{Sym}(n)$, then
\[
\chi^\lambda(\sigma) = \begin{cases} (-1)^{n - \lambda_1}, & \text{if } \lambda = [\lambda_1, 1^{n - \lambda_1}]; \\ 0, & \text{otherwise}. \end{cases}
\]

Partitions $\lambda \vdash n$ of the form $\lambda = [\lambda_1, 1^{n - \lambda_1}]$ are called hooks, similarly partitions of the form $[\lambda_1, 2, 1^{n - \lambda_1 - 2}]$, for $\lambda_1 > 1$ are called near hooks.

**Corollary 3.6.** Let $\sigma$ be the product of two disjoint $n/2$-cycles in $\text{Sym}(n)$, then
\[
\chi^\lambda(\sigma) \in \{0, \pm1, \pm2\}.
\]

In the following section we will define a new type of partition and show that these new partitions, along with near-hooks, are the only partitions for which $\chi^\lambda(\sigma)$ in the above corollary could be equal to $−2$.

**4. Two-layer hooks**

Assume $\lambda = [\lambda_1, \ldots, \lambda_k]$ is a partition of $n$ such that $k \geq 3, \lambda_2 + \lambda_3 \geq 5, \lambda_3 \leq 2$ and $\lambda_1 - \lambda_2 = \hat{\lambda}_1 - \hat{\lambda}_2 > 0$. Then we say $\lambda$ is a two-layer hook. In fact, a two-layer hook is a partition whose Young diagram is obtained by “appropriately gluing” two hooks of lengths greater than 1. See Fig. 3 for some examples of two-layer hooks. Note that if $\lambda \vdash n$ is a two-layer hook, then $\hat{\lambda}$ is also a two-layer hook and $n$ must be at least 8. Note also that a near hook is not a two-layer hook.

**Lemma 4.1.** Let $\lambda$ be a partition of $n$ and let $\sigma$ be a permutation in $\text{Sym}(n)$ that is the product of two disjoint $n/2$-cycles. If $\chi^\lambda(\sigma) = −2$, then $\lambda$ is either a two-layer hook or a symmetric near hook.

**Proof.** According to the Murnaghan–Nakayama Rule and Corollary 3.5, $\lambda$ should have two skew-hooks of length $n/2$ and deleting each of them should leave a hook of length $n/2$. If we denote $\lambda = [\lambda_1, \ldots, \lambda_k]$, then this obviously implies that $k > 1$.

If $k = 2$, then $\lambda$ must be the partition $[\frac{n}{2}, \frac{n}{2}]$ (since if $\lambda = [\lambda_1, \lambda_2]$, where $\lambda_1 > \lambda_2$, then $\lambda$ will not have two skew-hooks of length $n/2$); in this case we can calculate the character value at $\sigma$ to be 2. Thus $k \geq 3$.  

![Fig. 3. Two-layer hooks.](image-url)
If \( \lambda_3 > 2 \), then the partition \( \lambda' \) obtained from \( \lambda \) by deleting any skew-hook will have \( \lambda'_3 \geq 2 \) which implies that \( \lambda' \) is not a hook. Thus \( \lambda_3 \leq 2 \).

Let \( \lambda_1 - \lambda_2 = s \) and \( \hat{\lambda}_1 - \hat{\lambda}_2 = t \). Assume \( \mu \) and \( \nu \) are the two skew hooks of \( \lambda \) of length \( n/2 \). Since they have length \( n/2 \), we may assume that \( \mu \) contains the last box of the first row and \( \nu \) contains the last box of the first column. The lengths of \( \mu \) and \( \nu \) being both equal to \( n/2 \) implies that

\[
(s + 1) + (\lambda_2 - 1) + (\hat{\lambda}_2 - 1) - 1 = (t + 1) + (\hat{\lambda}_2 - 1) + (\lambda_2 - 1) - 1,
\]

which yields \( s = t \).

If \( s = t = 0 \) then \( \lambda = [\lambda_1, \lambda_1, 2, \ldots, 2] \). If we denote the number of rows in \( \lambda \) by \( k \), then according to the Murnaghan–Nakayama Rule

\[
\chi^\lambda(\sigma) = (1) r(\mu)(1)r(\lambda) \mu + (1) r(\nu)(1)r(\lambda) \nu
= (1) r(\mu)(1)^k + (1) r(\lambda) (1)^k - 1
= 2.
\]

Finally, note that if \( \lambda_2 + \hat{\lambda}_2 < 5 \), then either \( \lambda_2 + \hat{\lambda}_2 = 2 \) or 4. In the former case, \( \lambda \) is a hook and obviously it cannot have two skew-hooks of length \( n/2 \). In the latter case, \( \lambda \) must be a near hook. If it is not symmetric then it cannot have two skew-hooks.

These imply that if \( \lambda \) is neither symmetric near hook nor a two layer hook, then \( \chi^\lambda(\sigma) \neq -2 \); this completes the proof. \( \square \)

The following lemma provides a lower bound on the dimension of a symmetric near hook.

**Lemma 4.2.** If a symmetric partition \( \lambda \) of \( n \geq 8 \) is a near hook, then \( \chi(1) > 2n - 2 \).

**Proof.** Since \( \lambda \) is a symmetric near hook we know that \( \lambda = [n/2, 2, 1^{n-2}] \) and we can calculate the hook lengths directly

\[
\text{hl}(\lambda) = (n - 1) \left( \frac{n}{2} \right)^2 \left( \frac{n}{2} - 2 \right)!
\leq (n - 1) \frac{n^2}{4} (n - 4)! = \frac{n(n - 1)}{2(n - 2) 2} \frac{n(n - 2)!}{2}
< \frac{n(n - 2)!}{2}
\]

since \( n \geq 8 \). Putting this bound into the hook-length formula (Lemma 3.1) gives the lemma. \( \square \)

Next we prove that the same lower bound holds for the dimension of a two-layer hook.

**Lemma 4.3.** If a partition \( \lambda \) is a two-layer hook, then \( \chi^\lambda(1) > 2n - 2 \).

**Proof.** Let \( \lambda = [\lambda_1, \lambda_2, \ldots, \lambda_k] \). According to the hook-length formula, it suffices to show that \( \text{hl}(\lambda) < n(n - 2)!/2 \). We proceed by induction on \( n \). It is easy to see the lemma is true for \( n = 8 \). Let \( n \geq 10 \) and without loss of generality assume that \( \lambda_1 > \hat{\lambda}_1 \). This implies \( \lambda_2 \geq 3 \). We compute

\[
\text{hl}(\lambda) = (\lambda_1 + \hat{\lambda}_1 - 1)(\lambda_1 + \hat{\lambda}_2 - 2)(\lambda_2 + \hat{\lambda}_1 - 2)(\lambda_2 + \hat{\lambda}_2 - 3) \cdot \frac{(\lambda_1 - 1)!}{s + 1} \frac{(! \hat{\lambda}_1!}{s + 1} (\lambda_2 - 2)! (\hat{\lambda}_2 - 2)!
\]

where \( s \) is as in Lemma 4.1. One can re-write this as

\[
\text{hl}(\lambda) = \frac{\lambda_1 + \hat{\lambda}_1 - 1}{\lambda_1 + \lambda_2 - 2} \frac{\lambda_1 + \hat{\lambda}_2 - 2}{\lambda_1 + \lambda_2 - 3} \frac{\lambda_2 + \hat{\lambda}_1 - 2}{\lambda_2 + \lambda_1 - 3} \frac{\lambda_2 + \hat{\lambda}_2 - 3}{\lambda_2 + \lambda_2 - 4} \cdot (\lambda_1 - 1)(\lambda_2 - 2)
\]

\[
= \left( \lambda_1 + \hat{\lambda}_1 - 1 \right) \frac{\lambda_1 + \hat{\lambda}_2 - 2}{\lambda_1 + \lambda_2 - 2} \frac{\lambda_2 + \hat{\lambda}_1 - 2}{\lambda_2 + \lambda_1 - 2} \frac{\lambda_2 + \hat{\lambda}_2 - 3}{\lambda_2 + \lambda_2 - 3} \cdot (\lambda_1 - 1) (\lambda_2 - 2) \text{hl}(\hat{\lambda}),
\]

where \( \hat{\lambda} = [\lambda_1 - 1, \lambda_2 - 1, \lambda_3, \ldots, \lambda_k] \) is the partition whose Young diagram is obtained from that of \( \lambda \) by removing the last boxes of the first and the second rows. We can simplify (2) as

\[
\text{hl}(\lambda) = \left( 1 + \frac{1}{\lambda_1 + \hat{\lambda}_1 - 2} \right) \left( 1 + \frac{1}{\lambda_1 + \hat{\lambda}_2 - 2} \right) \left( 1 + \frac{1}{\lambda_2 + \hat{\lambda}_1 - 3} \right) \left( 1 + \frac{1}{\lambda_2 + \hat{\lambda}_2 - 4} \right) (\lambda_1 - 1)(\lambda_2 - 2) \cdot \text{hl}(\hat{\lambda}).
\]
We now observe the following facts:

1. \( \lambda_1 + \hat{\lambda}_1 - 2 > n/2 \); thus
   \[
   1 + \frac{1}{\lambda_1 + \hat{\lambda}_1 - 2} < \frac{n + 2}{n}.
   \]

2. By Lemma 4.1 and the definition of a two-layer hook, \( \lambda_1 - \lambda_2 = \hat{\lambda}_1 - \hat{\lambda}_2 \); hence \( \lambda_1 + \hat{\lambda}_1 = \lambda_2 + \hat{\lambda}_1 \). On the other hand
   \[
   \lambda_1 + \hat{\lambda}_2 + \lambda_2 + \hat{\lambda}_1 - 5 = n - 1,
   \]
   hence
   \[
   1 + \frac{1}{\lambda_1 + \hat{\lambda}_2 - 3} = \frac{n}{n - 2}, \quad 1 + \frac{1}{\lambda_2 + \hat{\lambda}_1 - 3} = \frac{n}{n - 2}.
   \]

3. Since \( \lambda_2 + \hat{\lambda}_2 \geq 5 \), we have
   \[
   1 + \frac{1}{\lambda_2 + \hat{\lambda}_2 - 4} < 2.
   \]

4. Since \( \lambda_1 + \lambda_2 \leq n - 1 \), we have
   \[
   (\lambda_1 - 1)(\lambda_2 - 2) \leq \frac{(n - 4)^2}{4}.
   \]

The partition \( \tilde{\lambda} \) is either a near hook or a two-layer hook. In the first case, because \( \lambda \) is a two-layer hook, we have
   \[
   \tilde{\lambda}_1 - 2 = \tilde{\lambda}_1 - \tilde{\lambda}_2 = \lambda_1 - \lambda_2 = \hat{\lambda}_1 - \hat{\lambda}_2 = \hat{\lambda}_1 - 2 = \tilde{\lambda}_1 - 2,
   \]
that is, the sizes of the first row and the first column of \( \lambda \) are equal which imply that \( \tilde{\lambda} \) is symmetric and, thus, according to Lemma 4.2,
   \[
   \text{hl}(\tilde{\lambda}) < \frac{(n - 2)(n - 4)!}{2}.
   \]

If \( \hat{\lambda} \) is a two layer hook, then the same bound holds by the induction hypothesis. Therefore
   \[
   \text{hl}(\hat{\lambda}) < 2 \cdot \frac{n + 2}{n} \cdot \frac{n}{n - 2} \cdot \frac{n}{n - 2} \cdot \frac{(n - 4)^2}{4} \cdot \frac{(n - 2)(n - 4)!}{2}
   \]
   \[
   = \frac{n(n + 2)(n - 4)^2(n - 4)!}{4(n - 2)}
   \]
   \[
   = \frac{(n + 2)(n - 4)^2}{2(n - 2)^2(n - 3)} \cdot \frac{n(n - 2)!}{2}
   \]
   \[
   < \frac{n(n - 2)!}{2}. \quad \Box
   \]

5. The standard module

The representation of Sym\((n)\) corresponding to \([n]\) is called the trivial representation, the character for this representation is equal to 1 for every permutation. If \( \lambda = [n - 1, 1] \), then the irreducible representation \( S^\lambda \) of the Sym\((n)\) is called the standard representation. For \( n \geq 5 \), \( \lambda = [n - 1, 1] \) is not symmetric; hence the restriction \( V \) of \( S^\lambda \) to \( G_n \) is also irreducible. We also call this representation the standard representation of \( G_n \) and \( V \) is the standard module of \( G_n \). The value of the character of the standard representation on a permutation \( \sigma \) is the number of elements of \([1, \ldots, n]\) fixed by \( \sigma \) minus 1.

In this section we prove that the characteristic vector of any maximum intersecting subset of \( G_n \) is in the direct sum of the trivial module and the standard module of \( G \). To do this, we will show that the clique–coclque bound holds with equality and, moreover, for each \( \lambda \) which is neither \([n]\) nor \([n - 1, 1]\), there is a clique \( C \) of maximum size with \( E_{\lambda V} \neq 0 \). From this, using Corollary 2.2, we conclude that \( E_{\lambda V} = 0 \) for any maximum independent set, unless \( E_{\lambda} \) is the projection to either the trivial module or the standard module. According to Corollary 2.3, it is sufficient to show that for each irreducible representation \( \chi \) of \( G_n \) there is a maximum clique \( C \) in \( T_{G_n} \) with \( \chi(C) \neq 0 \). This is very similar to what was done in [11, Section 5]. To do this, we will consider two cases, first when \( n \) is odd and second when it is even.

5.1. Case 1: \( n \) is odd:

Assume \( n \geq 5 \) to be odd. Theorem 1.1 of [1] proves that there is a decomposition of the arcs of the complete digraph \( K_n^+ \) on \( n \) vertices into \( n - 1 \) directed cycles of length \( n \). Each of these cycles corresponds to an \( n \)-cycle in \( G_n \). Since no two such decompositions share an arc in \( K_n^+ \), no two of the corresponding permutations intersect. Let \( C \) be the set of these
permutations together with the identity element of \( G_n \). Then \( C \) is a clique in \( I_{G_n} \) of size \( n \). The set of all \( n \)-cycles from \( \text{Sym}(n) \) form a pair of split conjugacy classes \( c_0' \) and \( c_0'' \) in \( G_n \). Thus all the non-identity elements of \( C \) lie in \( c_0' \cup c_0'' \).

**Lemma 5.1.** Let \( n \geq 5 \) be odd. Then for any irreducible character \( \chi \) of \( G_n \), other than the standard character, we have \( \chi(C) \neq 0 \), where \( C \) is the set defined above.

**Proof.** Theorem 3.3 gives all the irreducible representations of \( G_n \). First consider the case where \( \chi \) is the character of the restriction of the representation \( S^k \), where \( \lambda \) is not symmetric, to \( G_n \). Let \( \chi' \) be the character of \( S^k \) then \( \chi = \chi' \). According to Theorem 3.3, \( \chi' \) has the same values on \( c_0' \) and \( c_0'' \) and this value is equal to the value of \( \chi' \) on \( c_0' \cup c_0'' \). We compute

\[
\chi'(C) = \sum_{x \in C} \chi'(x) = \chi'(1) + (n - 1)\chi'(\sigma),
\]

where \( \sigma \) is a cyclic permutation of length \( n \). Using the corollary of the Murnaghan–Nakayama Rule (Corollary 3.5), we have \( \chi'(\sigma) \in \{0, \pm 1\} \). Therefore, if \( \chi'(C) = 0 \), since \( \chi'(1) > 0 \), it must be that \( \chi'(\sigma) = -1 \) and then \( \chi'(1) = n - 1 \). The representations corresponding to the partition \([n - 1, 1]\) and its transpose, \([2, 1, \ldots, 1]\), are the only representations of \( \text{Sym}(n) \) of dimension \( n - 1 \) and according to Theorem 3.2, their restrictions to \( G_n \) are both isomorphic to the standard representation of \( G_n \).

Next assume that \( \chi \) is the character of one of the two irreducible representations \( W' \) or \( W'' \), where \( W = W' \oplus W'' \) is the restriction of \( S^k \) to \( G_n \); in this case \( \lambda \) must be symmetric. Thus, using the notation of Theorem 3.3, \( \chi = \chi' \) (the case when \( \chi = \chi'' \) is identical, so we omit it). If \( \lambda \) is not the hook \([n + 1]/2, 1, \ldots, 1]\), then according to Theorem 3.3, we have

\[
\chi'(C) = \sum_{x \in C} \chi'(x) = \frac{1}{2} \chi'(1) + (n - 1)\frac{1}{2} \chi'(\sigma).
\]

Thus, as in the previous case, if \( \chi'(C) = 0 \), then we must have \( \chi'(1) = n - 1 \) which is a contradiction.

The final case that we need to consider is when \( \chi \) is the character of one of the two irreducible representations whose sum is the representation formed by restricting \( S^k \) to \( G_n \) where \( \lambda = [(n + 1)/2, 1, \ldots, 1] \). Again we assume that \( \chi = \chi' \) (since the case for \( \chi = \chi'' \) is identical) and using Theorem 3.3, we have

\[
\chi'(C) = \sum_{x \in C} \chi'(x) = \chi'(1) + \sum_{x \in C \cap c_0'} \chi'(x) + \sum_{x \in C \cap c_0''} \chi'(x)
\]

\[
= \frac{1}{2} \chi'(1) + r' \frac{1}{2} \left( -1 \frac{n-1}{2} + \sqrt{-1 \frac{n+1}{2}} n \right) + r'' \frac{1}{2} \left( -1 \frac{n-1}{2} - \sqrt{-1 \frac{n+1}{2}} n \right),
\]

where \( r' = |C \cap c_0'| \) and \( r'' = |C \cap c_0''| \). Note that \( r' + r'' = n - 1 \). Hence, if \( \chi'(C) = 0 \), then we must have

\[
- \chi'(1) = r' \left( -1 \frac{n-1}{2} + \sqrt{-1 \frac{n+1}{2}} n \right) + r'' \left( -1 \frac{n-1}{2} - \sqrt{-1 \frac{n+1}{2}} n \right).
\]

(3)

Note that

\[
\chi'(1) = \frac{2^{n-1}(n - 2)!!}{(n - 1)!!},
\]

where \( a!! = a(a - 2)(a - 4) \cdots 2 \) if \( a \) is even integer and \( a!! = a(a - 2)(a - 4) \cdots 1 \), if \( a \) is odd. Consider the following two cases. If \( 4 \nmid n - 1 \), then (3) implies that

\[
- \frac{2^{n-2}(n - 2)!!}{(n - 1)!!} = -(n - 1) + \sqrt{-n(r' - r')}.
\]

(4)

It follows, then, that \( r' = r'' \) and so

\[
\frac{2^{n-1}(n - 2)!!}{(n - 1)!!} = n - 1,
\]

since this only holds for \( n = 3 \), this is a contradiction.

On the other hand, if \( 4 \mid n - 1 \), then (3) implies that

\[
- \frac{2^{n-1}(n - 2)!!}{(n - 1)!!} = r' \left( 1 + \sqrt{n} \right) + r'' \left( 1 - \sqrt{n} \right);
\]

(5)
that is,
\[
\frac{2^{n-1}(n-2)!!}{(n-1)!!} = -(n - r'' - 1) (\sqrt{n} + 1) + r'' (\sqrt{n} - 1)
\leq (n - 1) (\sqrt{n} - 1) \leq n^{\frac{5}{2}}.
\]

Note that
\[
\frac{2^{n-1}(n-2)!!}{(n-1)!!} = \frac{2^{n-1} n!!}{n (n-1)!!} > \frac{2^{n-1}}{n};
\]

thus (6) yields
\[
2^{n-1} < n^{\frac{5}{2}}.
\]

It is easily seen that this inequality fails for all \( n \geq 9 \). Finally, note that (4) and (5) lead us to contradictions if \( n = 5 \) and if \( n = 7 \), respectively. This completes the proof of the lemma. \( \Box \)

Next we consider when \( n \) is even

5.2. Case 2: \( n \) is even

In this part, we assume \( n \geq 6 \) to be even. According to Theorem 1.1 in [1], the arcs of the complete digraph \( K_n^* \) can be decomposed to \( n - 1 \) pairs of vertex-disjoint directed cycles of length \( n/2 \). Each of these pairs corresponds to a permutation in \( G_n \), which is a product of two cyclic permutations of length \( n/2 \). Let \( C \) be the set of these permutations together with the identity element of \( G_n \). Then, similar to the previous part, \( C \) is a clique in \( \Gamma_{G_n} \). Note that the non-identity elements of \( C \) lie in a non-split conjugacy class \( c \) of \( G_n \). Now we prove the equivalent of Lemma 5.1 for even \( n \), using this set \( C \).

**Lemma 5.2.** Let \( n \geq 6 \) be even. Then for any irreducible character \( \chi \) of \( G_n \), which is not the standard character, we have \( \chi(C) \neq 0 \), where \( C \) is as defined above.

**Proof.** First consider the case \( \chi = \chi_\lambda \), where \( \lambda \) is not symmetric. Using the notation of Theorem 3.3, we have
\[
\chi_\lambda(C) = \sum_{x \in C} \chi_\lambda(x) = \chi^\lambda(1) + (n - 1) \chi^\lambda(\sigma),
\]

where \( \sigma \) is a product of two disjoint cyclic permutations of length \( n/2 \). Now, suppose \( \chi_\lambda(C) = 0 \). Then
\[
- \chi^\lambda(1) = (n - 1) \chi^\lambda(\sigma).
\]

Using the Murnagahan–Nakayama Rule, we have \( \chi^\lambda(\sigma) \in \{0, \pm 1, \pm 2\} \) (see Corollary 3.6). If \( \chi^\lambda(\sigma) = 0 \) or \( \pm 2 \), then Eq. (7) yields a contradiction with the fact that \( \chi^\lambda(1) \) is strictly positive. Also if \( \chi^\lambda(\sigma) = -1 \), then we must have \( \chi^\lambda(1) = n - 1 \) which contradicts the fact that the standard representation and its conjugate are the only irreducible representations of \( \text{Sym}(n) \) of dimension \( n - 1 \). Hence, suppose \( \chi^\lambda(\sigma) = -2 \). Then \( \chi^\lambda(1) = 2n - 2 \). But according to Lemma 4.1, \( \lambda \) must be a two-layer hook or a symmetric near hook. Then by Lemmas 4.2 and 4.3, the dimension of \( \chi \) is strictly greater than \( 2n - 2 \).

Next consider the case where \( \chi \) is the character of one of the two irreducible representations in the restriction of the representation \( S^1 \) to \( G_n \), where \( \lambda \) is symmetric; so \( \chi = \chi^\lambda_1 \) or \( \chi^\lambda_2 \). We will show that \( \chi^\lambda_1(C) \neq 0 \); the proof that \( \chi^\lambda_2(C) \neq 0 \) is similar. We have
\[
\chi^\lambda_1(C) = \sum_{x \in C} \chi^\lambda_1(x) = \frac{1}{2} \chi^\lambda(1) + (n - 1) \frac{1}{2} \chi^\lambda(\sigma),
\]

where \( \sigma \) is a product of two disjoint \( n/2 \)-cycles. If \( \chi^\lambda(C) = 0 \), then with the same argument as above, we get a contradiction. \( \Box \)

We now prove the main theorem of this section.

**Proposition 5.3.** Let \( S \) be an intersecting subset of \( G_n \) of size \( (n - 1)!/2 \) and let \( v_S \) be the characteristic vector of \( S \). Then the vector \( v_S - \frac{1}{n} \mathbf{1} \) is in the standard module of \( G_n \).

**Proof.** Let \( S_{1,1} \) be the point stabilizer for \( 1 \) in \( G_n \); so \( S_{1,1} \) is an independent set of size \( \frac{(n-1)!}{2} \) in \( \Gamma_{G_n} \). Then the cliques defined in Lemmas 5.1 and 5.2, together with \( S_{1,1} \) prove that the clique–coclique bound holds with equality for \( \Gamma_{G_n} \). Given any irreducible character \( \chi \) of \( G_n \), except the standard character and the trivial character, according to Lemmas 5.1 and 5.2, there is a maximum clique \( C \), such that \( \chi(C) \neq 0 \). Hence, according to Corollary 2.3, we have \( E_{\chi} v_S = 0 \), for any maximum independent set \( S \). This implies that if \( \chi \) is neither the trivial nor the standard character, then \( E_{\chi} v_S = 0 \). It is not hard to see that the vector \( v_S - \frac{1}{n} \mathbf{1} \) is orthogonal to \( \mathbf{1} \); hence it cannot lie in the trivial module of \( G_n \). Therefore, for any maximum independent set \( S \), the vector \( v_S - \frac{1}{n} \mathbf{1} \) belongs to the standard module of \( G_n \). \( \Box \)
In the remainder of this section, we will provide a basis for the standard module of $G_n$. For any pair $i, j \in \{1, \ldots, n\}$, define $S_{i,j}$ to be the set of all permutations $\pi \in S_n$ such that $\pi(i) = j$. Note that $S_{i,j}$ are cosets of point stabilizers in $G_n$ under the natural action of $G_n$ on $\{1, \ldots, n\}$ and that they are maximum intersecting sets in $G_n$. Define $v_{i,j}$ to be the characteristic vector of $S_{i,j}$, for all $i, j \in \{1, \ldots, n\}$.

**Lemma 5.4.** The set

$$B := \left\{ v_{i,j} - \frac{1}{n} \mathbf{1} \mid i, j \in [n-1] \right\}$$

is a basis for the standard module $V$ of $G_n$.

**Proof.** According to Proposition 5.3, we have $B \subset V$ and since the dimension of $V$ is equal to $|B| = (n-1)^2$, it suffices to show that $B$ is linearly independent. Note, also, that since $\mathbf{1}$ is not in the span of $v_{i,j}$ for $i, j \in [n-1]$, it is enough to prove that the set $\{v_{i,j} \mid i, j \in [n-1]\}$ is independent.

Define a matrix $H$ to have the vectors $v_{i,j}$, with $i, j \in [n-1]$, as its columns. Then the rows of $H$ are indexed by the elements of $G_n$ and the columns are indexed by the ordered pairs $(i,j)$, where $i,j \in [n-1]$; we will also assume that the ordered pairs are listed in lexicographic order. It is easy to see that

$$H^T H = \frac{(n-1)!}{2} I_{(n-1)^2} + \frac{(n-2)!}{2} (A(K_{n-1}) \otimes A(K_{n-1})),$$

where $I_{(n-1)^2}$ is the identity matrix of size $(n-1)^2$ and $A(K_{n-1})$ is the adjacency matrix of the complete graph $K_{n-1}$. The distinct eigenvalues of $A(K_{n-1})$ are $-1$ and $n-2$; thus the eigenvalues of $A(K_{n-1}) \otimes A(K_{n-1})$ are $-(n-2), 1, (n-2)^2$. This implies that the least eigenvalue of $H^T H$ is

$$\frac{(n-1)!}{2} - \frac{(n-2)(n-2)!}{2} > 0.$$

This proves that $H^T H$ is non-singular and hence full rank. This, in turn, proves that $\{v_{i,j} \mid i, j \in [n-1]\}$ is linearly independent. \(\square\)

6. **Proof of the main theorem**

Define the $|G_n| \times n^2$ matrix $A$ to be the matrix whose columns are the characteristic vectors $v_{i,j}$ of the sets $S_{i,j}$, for all $i, j \in \{1, \ldots, n\}$. We will use $\mathbf{1}_n$ to denote the all ones vector of length $n$ and $\mathbf{0}_n$, the all zeros vector of length $n$; if the length is clear from context, the subscript will be omitted. Then since $A$ has constant row-sums, the vector $\mathbf{1}_n$ is in the column space of $A$; thus in the light of Lemma 5.4, we observe the following.

**Lemma 6.1.** The characteristic vector of any maximum intersecting subset of $G_n$ lies in the column space of $A$.

We denote by $A_{i,j}$ the column of $A$ indexed by the pair $(i,j)$, for any $i,j \in \{1, \ldots, n\}$. Define the matrix $\overline{A}$ to be the matrix obtained from $A$ by deleting all the columns $A_{i,n}$ and $A_{n,j}$ for any $i,j \in [n-1]$. Note that $\overline{A}$ is also obtained from $H$ by adding the column $A_{n,n}$. With a similar method as in the proof of [18, Proposition 10], we prove the following.

**Lemma 6.2.** The characteristic vector of any maximum intersecting subset of $G_n$ lies in the column space of $\overline{A}$.

**Proof.** According to Lemma 6.1, it is enough to show that the two matrices $A$ and $\overline{A}$ have the same column space. Obviously, the column space of $\overline{A}$ is a subspace of the column space of $A$; thus we only need to show that the vectors $A_{i,n}$ and $A_{n,j}$ are in the column space of $\overline{A}$, for any $i,j \in [n-1]$. Since $G_n$ is two transitive, it suffices to show this for $A_{1,n}$. Define the vectors $v$ and $w$ as follows:

$$v := \sum_{i \neq 1,n} \sum_{j \neq n} A_{i,j} \quad \text{and} \quad w := (n-3) \sum_{j \neq n} A_{1,j} + A_{n,n}.$$

The vectors $v$ and $w$ are in the column space of $\overline{A}$. It is easy to see that for any $\pi \in G_n$,

$$v_\pi = \begin{cases} n-2, & \text{if } \pi(1) = n \\ n-2, & \text{if } \pi(n) = n \\ n-3, & \text{otherwise} \end{cases} \quad \text{and} \quad w_\pi = \begin{cases} 0, & \text{if } \pi(1) = n \\ n-2, & \text{if } \pi(n) = n \\ n-3, & \text{otherwise} \end{cases}$$

Thus

$$(v - w)_\pi = \begin{cases} n-2, & \text{if } \pi(1) = n \\ 0, & \text{if } \pi(n) = n \\ 0, & \text{otherwise} \end{cases}$$

which means that $(n-2)A_{1,n} = v - w$. This completes the proof. \(\square\)
If the columns of $\tilde{A}$ are arranged so that the first $n$ columns correspond to the pairs $(i, i)$, for $i \in \{1, \ldots, n\}$, and the rows are arranged so that the first row corresponds to the identity element, and the next $|D_{G_n}|$ rows correspond to the elements of $D_{G_n}$ (recall that these are the derangements of $G_n$), then $\tilde{A}$ has the following block structure:

\[
\begin{bmatrix}
1 & 0 \\
0 & M \\
B & C
\end{bmatrix}
\]

Note that the rows and columns of $M$ are indexed by the elements of $D_{G_n}$ and the pairs $(i, j)$ with $i, j \in \{n - 1\}$ and $i \neq j$, respectively; thus $M$ is a $|D_{G_n}| \times (n - 1)(n - 2)$ matrix. The block $0$ represents the all-zeros matrix of size $|D_{G_n}| \times n$. We will next prove that $M$ is full rank.

**Proposition 6.3.** For all $n \geq 5$, rank of $M$ is $(n - 1)(n - 2)$.

**Proof.** First assume $n$ is odd. Consider the submatrix $M_1$ of $M$ that is comprised of all the rows in $M$ that are indexed by cyclic permutations of length $n$. Set $T = M_1^T M_1$; it suffices to show that $T$ is non-singular. Consider all types of entries of $T$. If $i, j, k, l$ are in $\{n - 1\}$, then the following are all possible cases for the pairs $(i, j)$ and $(k, l)$.

- $i = k$ and $j = l$; in this case $T_{(i,j),(k,l)} = (n - 2)!$; because the number of all $n$-cycles mapping $i$ to $j$ is $(n - 2)!$.
- $i = l$ and $j = k$; in this case $T_{(i,j),(k,l)} = 0$; because the only case in which an $n$-cycle can swap $i$ and $j$ is $n = 2$.
- $i = k$ and $j \neq l$; in this case $T_{(i,j),(k,l)} = 0$; because there is no permutation mapping $i$ to two different numbers.
- $i \neq k$ and $j = l$; again $T_{(i,j),(k,l)} = 0$.
- $i \neq l$ and $j = k$; in this case $T_{(i,j),(k,l)} = (n - 3)!$; because the number of all $n$-cycles mapping $i$ to $j$ and $j$ to $i$ is $(n - 3)!$.
- $i = l$ and $j \neq k$; in this case $T_{(i,j),(k,l)} = (n - 3)!$; with a similar reasoning as above.
- $\{i, j\} \cap \{k, l\} = \emptyset$; in this case $T_{(i,j),(k,l)} = (n - 3)!$; because the number of $n$-cycles mapping $i$ to $j$ and $k$ to $l$ is $\binom{n - 3}{2} (n - 4)! = (n - 3)!$.

Therefore, one can write $T$ as

\[
T = (n - 2)! I + (n - 3)! A(X),
\]

where $I$ is the identity matrix of size $(n - 1)(n - 2)$ and $A(X)$ is the adjacency matrix of the graph $X$ defined as follows: the vertices of $X$ are all the ordered pairs $(i, j)$ where $i, j \in \{n - 1\}$ and $i \neq j$; the vertices $(i, j)$ and $(k, l)$ are adjacent in $X$ if and only if either $\{i, j\} \cap \{k, l\} = \emptyset$, or if $i = l$ and $j \neq k$, or if $i \neq k$ and $j = l$. Note that $X$ is a regular graph of valency $(n - 2)(n - 3)$.

Our next result, **Lemma 6.4**, will show that the least eigenvalue of $X$ is greater than or equal to $-(n - 3)$; thus using (8), the least eigenvalue of $T$ is at least

\[
(n - 2)! - (n - 3)! (n - 3)! > 0;
\]

therefore $T$ is non-singular and the proof is complete for the case $n$ is odd.

Now assume $n$ to be even. Consider the subset of $D_{G_n}$ which consists of all the permutations of $G_n$ whose cycle decomposition includes two cycles of length $n/2$ and let $M_2$ be the submatrix of $M$ whose rows are indexed by these permutations. Define $U = M_2^T M_2$. With a similar approach as for the previous case, one can write $U$ as

\[
U = \frac{2(n - 2)!}{n} I + \frac{2(n - 3)!}{n} A(X).
\]

According to **Lemma 6.4**, the least eigenvalue of $U$ is at least

\[
\frac{2(n - 2)!}{n} - \frac{2(n - 3)!}{n} (n - 3) = \frac{2(n - 3)!}{n} > 0;
\]

therefore $U$ is non-singular and the proof is complete. \qed

**Lemma 6.4.** Let $n > 3$ and $X$ be the graph defined in **Proposition 6.3**. The least eigenvalue of $X$ is at least $-(n - 3)$.

**Proof.** First note that any cyclic permutation $\alpha = (i_1, i_2, \ldots, i_{n - 1})$ of $\{1, 2, \ldots, n - 1\}$ corresponds to a unique clique of size $n - 1$ in $X$; namely the clique $C_{\alpha}$ induced by the vertices $\{(i_1, i_2), (i_2, i_3), \ldots, (i_{n - 2}, i_{n - 1}), (i_{n - 1}, i_1)\}$. We claim that any edge of $X$ is contained in exactly $(n - 4)!$ cliques of form $C_{\alpha}$.

Consider the edge $\{a, b, c, d\}$. If $a \neq d$ and $b = c$ or $a = d$ and $b \neq c$, then there are $(n - 4)!$ cyclic permutations of form $(a, b, d, c, \ldots, \ldots)$ and this edge is in exactly $(n - 4)!$ of the cliques. If $\{a, b\} \cap \{c, d\} = \emptyset$ then there are again $(n - 4)!$ cyclic permutations of form $(a, b, c, \ldots, d, \ldots)$ (as there are $(n - 4)!$ ways to assign a position for the pair $c, d$, and then there are $(n - 5)!$ ways to arrange other elements of $\{1, \ldots, n - 1\}$ in the remaining spots).

Thus the claim is proved. If $\tau$ denotes the least eigenvalue of $X$, then we can apply **Proposition 2.4** to $X$ to get that

\[
\tau \geq \frac{-k}{w - 1} = -\frac{(n - 2)(n - 3)}{n - 2} = -(n - 3). \quad \square
\]
Now we are ready to prove the main theorem.

**Proof** (Theorem 1.1). Let $S$ be an intersecting set of permutations in $G_n$. Then $S$ is an independent set in the graph $\Gamma_{G_n}$. In Sections 5.1 and 5.2 cliques of size $n$ in $\Gamma_{G_n}$ are given. By the clique–coclique bound no independent set is larger than $\frac{(n-1)^{n-1}}{2}$; thus the bound in Theorem 1.1 holds.

Further suppose that $S$ is of maximum size (namely $\frac{(n-1)^{n-1}}{2}$) and let $v_S$ be the characteristic vector of $S$. To complete the proof of this theorem, it is enough to show that $S = S_{ij}$, for some $i, j \in \{1, \ldots, n\}$.

Without loss of generality, we may assume that $S$ includes the identity element. By Lemma 6.2, $v_S$ is in the column space of $A$, thus

$$
\begin{bmatrix}
1_n \\
0 \\
B \\
C
\end{bmatrix}
\begin{bmatrix}
v^T \\
w^T
\end{bmatrix}
= v_S
$$

for some vectors $v$ and $w$. Since the identity is in $S$, no elements from $D_{G_n}$ are in $S$, and the characteristic vector of $S$ has the form

$$v_S = \begin{bmatrix}
1 \\
0 \\
t^T
\end{bmatrix}
$$

for some vector $t$. Thus we have $1_n \cdot v = 1$, $Mw = 0_{|G_n|}$, and $Bv^T + Cw^T = t^T$. According to Proposition 6.3, $M$ is full rank; therefore, $w$ is the zero vector and so $Bv^T = t^T$.

Furthermore, for any $x \in \{1, \ldots, n\}$, there is a permutation $g_x \in G_n$ which has only $x$ as its fixed point. Then by a proper permutation of the rows of $B$, one can write

$$B = \begin{bmatrix}
1_n \\
B'
\end{bmatrix} \quad \text{and} \quad Bv^T = \begin{bmatrix}
v^T \\
B'v^T
\end{bmatrix}.$$ 

Since $Bv^T$ is equal to the $01$-vector $t$, the vector $v$ must also be a $01$-vector. But, on the other hand, $1_n \cdot v = 1$, thus we conclude that exactly one of the entries of $v$ is equal to 1. This means that $v_S$ is the characteristic vector of the stabilizer of a point. □

7. Concluding remarks

An interesting result of Theorem 1.1 is that it implies that the symmetric group also has the strict EKR property. To show this we will state two results that were first pointed out by Pablo Spiga [22]; the proof we give of the first result is due to Chris Godsil [10].

**Theorem 7.1.** Let $G$ be a transitive subgroup of $\text{Sym}(n)$ and let $H$ be a transitive subgroup of $G$. If $H$ has the EKR property, then $G$ has the EKR property.

**Proof.** The group $H$ has the EKR property and is transitive, so the size of the maximum coclique is $|H|/n$. Further, the graph $\Gamma_H$ is vertex transitive so its fractional chromatic number is $n$ (see [12, Chapter 7] for details about the fractional chromatic number of a graph).

The embedding $\Gamma_H \to \Gamma_G$ is a homomorphism, so the fractional chromatic number of $\Gamma_G$ is at least the fractional chromatic number of $\Gamma_H$. The graph $\Gamma_G$ is also vertex transitive, so

$$n \leq \frac{|G|}{\alpha(\Gamma_G)}$$

where $\alpha(\Gamma_G)$ is the size of a maximum independent set. Thus $\alpha(\Gamma_G) \leq \frac{|G|}{n}$, and since $G$ is transitive the stabilizer of a point achieves this bound. □

If the groups $G$ and $H$ in the above theorem are 2-transitive then we can say something about when the strict EKR property holds for $G$.

**Theorem 7.2.** Let $G$ be a 2-transitive subgroup of $\text{Sym}(n)$ and let $H$ be a 2-transitive subgroup of $G$. If $H$ has the strict EKR property then $G$ has the strict EKR property.

**Proof.** Since $H$ has the strict EKR property, it also has the EKR property and by the previous result $G$ also has the EKR property. Assume that $S$ is a coclique in $\Gamma_G$ of size $|G|/n$ that contains the identity; we will prove that $S$ is the stabilizer of a point.

Let $\{x_1 = \text{id}, \ldots, x_{|G/H|}\}$ be a left transversal of $H$ in $G$ and set $S_i = S \cap x_iH$. Then for each $i$ the set $x_i^{-1}S_i$ is an independent set in $\Gamma_H$ with size $|H|/n$. Since $H$ has the strict EKR property each $x_i^{-1}S_i$ is the coset of a stabilizer of a point.

Since $x_1 = \text{id}$, the identity is in $S_1$ which means that $S_1$ is the stabilizer of a point and we can assume that $S_1 = H_{x_\alpha}$ for some $\alpha \in \{1, \ldots, n\}$. We need to show that every permutation in $S$ also fixes the point $\alpha$. Assume that there is a $\pi \in S$ that...
does not fix $\alpha$. Since $S$ is intersecting, for every $\sigma \in S_1$ the permutation $\sigma \pi^{-1}$ fixes some element (but not $\alpha$ and not $\pi(\alpha)$), from this it follows that

$$H_\alpha \pi^{-1} \subseteq \bigcup_{\beta \neq \pi(\alpha), \beta \neq \sigma \pi(\alpha)} G_\beta = \bigcup_{\beta \neq \pi(\alpha)} (G_\beta \cap H_\alpha \pi^{-1}).$$

Assume that $\sigma \pi^{-1} \in G_\beta \cap H_\alpha \pi^{-1}$, then $\beta^\sigma \pi^{-1} = \beta$ and $\alpha^\sigma = \alpha$. The permutation $\sigma \pi^{-1}$ must map $(\alpha, \beta)$ to $(\alpha^\pi^{-1}, \beta)$. Since the group $H$ is 2-transitive there are exactly $|H|/n(n-1)$ such permutations and we have that

$$|G_\beta \cap H_\alpha \pi^{-1}| = \frac{|H|}{n(n-1)}.$$

From this we have that the size of $H_\alpha \pi^{-1}$ is

$$\sum_{\beta \neq \pi(\alpha)} \frac{|H|}{n(n-1)} = (n-2) \frac{|H|}{n(n-1)},$$

but since this is strictly less than $\frac{|H|}{n}$, this is a contradiction. □

This theorem along with Theorem 1.1 provides an alternative proof of the following theorem which was initially proved in [5].

**Corollary 7.3.** For any $n \geq 2$, the group $\text{Sym}(n)$ has the strict EKR property.

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