SHRINKING TARGETS FOR COUNTABLE MARKOV MAPS

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Abstract. Let $T$ be an expanding Markov map with a countable number of inverse branches and a repeller $\Lambda$ contained within the unit interval. Given $\alpha \in \mathbb{R}^+$ we consider the set of points $x \in \Lambda$ for which $T^n(x)$ hits a shrinking ball of radius $e^{-n\alpha}$ around $y$ for infinitely many iterates $n$. Let $s(\alpha)$ denote the infimal value of $s$ for which the pressure of the potential $-s\log|T'|$ is below $s\alpha$. Building on previous work of Hill, Velani and Urbaniak we show that for all points $y$ contained within the limit set of the associated iterated function system the Hausdorff dimension of the shrinking target set is given by $s(\alpha)$. Moreover, when $\Lambda = [0, 1]$ the same holds true for all $y \in [0, 1]$. However, given $\beta \in (0, 1)$ we provide an example of an expanding Markov map $T$ with a repeller $\Lambda$ of Hausdorff dimension $\beta$ with a point $y \in \Lambda$ such that for all $\alpha \in \mathbb{R}^+$ the dimension of the shrinking target set is zero.

1. Introduction

Suppose we have a dynamical system $(X, T, \mu)$ consisting of a space $X$ together with a map $T : X \rightarrow X$ and a $T$-invariant ergodic probability measure $\mu$. Let $A$ be a subset of positive $\mu$ measure. Poincaré’s recurrence theorem implies that $\mu$ almost every $x \in X$ will visit $A$ an infinite number of times, i.e. $\bigcap_{m \in \mathbb{N}} \bigcup_{n \geq m} T^{-n}A$ has full $\mu$ measure. This raises the question of what happens when we allow $A$ to shrink with respect to time. How does the size of $\bigcap_{m \in \mathbb{N}} \bigcup_{n \geq m} T^{-n}A(n)$ depend upon the sequence $\{A(n)\}_{n \in \mathbb{N}}$?

We shall consider this question in the setting of hyperbolic maps. Given a Gibbs measure $\mu$, Chernov and Kleinbock have given general conditions according to which $\bigcap_{m \in \mathbb{N}} \bigcup_{n \geq m} T^{-n}A(n)$ will have full $\mu$ measure [CK]. However, when $\sum_{n=0}^{\infty} \mu(A(n))$ is finite it is clear that $\bigcap_{m \in \mathbb{N}} \bigcup_{n \geq m} T^{-n}A(n)$ must be of zero $\mu$ measure. In particular, if $\{A(n)\}_{n \in \mathbb{N}}$ is a sequence of balls which shrink exponentially fast around a point, then $\bigcap_{m \in \mathbb{N}} \bigcup_{n \geq m} T^{-n}A(n)$ must be of zero Lebesgue measure. Thus, in order to understand its geometric complexity we must determine its Hausdorff dimension (see [PT] for an introduction to dimension theory).

In [HV1, HV2] Hill and Velani consider the dimension of the shrinking target set

$$D_y(\alpha) := \bigcap_{m \in \mathbb{N}} \bigcup_{n \geq m} \{x \in X : |T^n(x) - y| < e^{-n\alpha}\}.$$
Let $s(\alpha)$ denote the infimal value of $s$ for which the pressure of the potential $-s \log |T'|$ is below $s \alpha$. In [HV2] it is shown that for an expanding rational maps of the Riemann sphere the dimension of $D_y(\alpha)$ is given by $s(\alpha)$ for all points $y$ contained within the Julia set. Now suppose we have a piecewise continuous map of the unit interval $T$ with repeller $\Lambda$. When $T$ has just finitely many inverse branches, Hill and Velani’s formula for the dimension of $D_y(\alpha)$ is given by $s(\alpha)$ for all points $y$ contained within the Julia set. Now suppose we have a piecewise continuous map of the unit interval $T$ with repeller $\Lambda$. When $T$ has just finitely many inverse branches, Hill and Velani’s formula for the dimension of $D_y(\alpha)$ is given by $s(\alpha)$ for all points $y$ contained within the Julia set. However when $T$ has an infinite number of inverse branches things become more difficult, owing to the unboundedness $|T'|$. In [U] Urbanski showed that for those $y \in \Lambda$ satisfying $\sup \{|(T^n(T^n(y)))|\}_{n \geq 0} < \infty$, the dimension of $D_y(\alpha)$ is equal to $s(\alpha)$. We prove that, even for systems with an infinite number of inverse branches, this formula extends to all points $y \in \Lambda$. Moreover, when $\Lambda = [0, 1]$ we have $\dim H D_y(\alpha) = s(\alpha)$.

2. Statement of results

Before stating our main results we shall introduce some notation and provide some further background.

Definition 2.1 (Expanding Markov Map). Let $V = \{V_i\}_{i \in A}$ be a countable family of disjoint subintervals of the unit interval with non-empty interior. Given $\omega = (\omega_0, \cdots , \omega_{n-1}) \in A^n$ for some $n \in \mathbb{N}$ we let $V_\omega := \cap_{\nu=0}^{n-1} T^{-\nu} V_{\omega_\nu}$. We shall say that $T : \cup_{i \in A} V_i \to [0, 1]$ is an expanding Markov map if $T$ satisfies the following conditions.

1. For each $i \in A$, $T|_{V_i}$ is a $C^1$ map which maps the interior of $V_i$ onto open unit interval $(0, 1)$,

2. There exists $\xi > 1$ and $N \in \mathbb{N}$ such that for all $n \geq N$ and all $x \in \cup_{\omega \in A^n} V_\omega$ we have $|(T^n)'(x)| > \xi^n$,

3. There exists some sequence $\{\rho_n\}_{n \in \mathbb{N}} \subset \mathbb{R}$ with $\lim_{n \to \infty} \rho_n = 0$ such that for all $n \in \mathbb{N}$, $\omega \in A^n$, and all $x,y \in V_\omega$, 

$$e^{-n \rho_n} \leq \frac{|(T^n)'(x)|}{|(T^n)'(y)|} \leq e^{n \rho_n}.$$ 

We shall say that $T$ is a finite branch expanding Markov map if $A$ is a finite set.

The repeller $\Lambda$ of an expanding Markov map is the set of points for which every iterate of $T$ is well-defined, $\Lambda := \cap_{n \in \mathbb{N}} T^{-n}([0, 1])$. We assume throughout that $\#A > 1$. Otherwise $\Lambda$ would either empty or contained within a single point.

Given a point $y \in \Lambda$ in the closure of the repeller and some $\alpha \in \mathbb{R}_+$ we shall be interested in the set of points $x \in \Lambda$ for which $T^n(x)$ hits a shrinking
ball of radius $e^{-n\alpha}$ around $y$ for infinitely many iterates $n$,

$$D_y(\alpha) := \bigcap_{m \in \mathbb{N}} \bigcup_{n \geq m} \{ x \in \Lambda : |T^n(x) - y| < e^{-n\alpha} \}.$$  

More generally, given a function $\varphi : \Lambda \to \mathbb{R}_+$ we let $S_n(\varphi) := \sum_{i=0}^{n-1} \varphi \circ T^i$ and define

$$D_y(\varphi) := \bigcap_{m \in \mathbb{N}} \bigcup_{n \geq m} \{ x \in \Lambda : |T^n(x) - y| < e^{-S_n(\varphi)(x)} \}. \tag{2.2}$$

Sets of the form $D_y(\varphi)$ arise naturally in Diophantine approximation.

**Example 2.1.** Given $\alpha \in \mathbb{R}_+$ we let

$$J(\alpha) := \left\{ x \in [0, 1] : \left| x - \frac{p}{q} \right| < \frac{1}{q^\alpha} \text{ for infinitely many } p, q \in \mathbb{N} \right\}. \tag{2.3}$$

Let $T : [0, 1] \to [0, 1]$ be the Gauss map $x \mapsto \frac{1}{x} - \lfloor \frac{1}{x} \rfloor$ which is an expanding Markov map on the repeller $\Lambda = [0, 1] \setminus \mathbb{Q}$. We define $\psi : \Lambda \to \mathbb{R}$ by $\psi(x) = \log |T'(x)|$ and for each $\alpha > 2$ we let $\psi_\alpha := \left( \frac{\alpha}{2} - 1 \right) \psi$. Then for all $2 < \alpha < \beta < \gamma$ we have,

$$D_0(\psi_\alpha) \subset J(\beta) \subset D_0(\psi_\gamma).$$

In [J, B] Jarðnık and Besicovitch showed that for $\alpha > 2$, $\dim_H(J(\alpha)) = \frac{2}{\alpha}$. By (2.3) this is equivalent to the fact that for all $\alpha > 2$

$$\dim_H D_0(\psi_\alpha) = \frac{2}{\alpha}. \tag{2.4}$$

As we shall see, in sufficiently well behaved settings, the Hausdorff dimension of $D_y(\varphi)$ may be expressed in terms of the thermodynamic pressure.

**Definition 2.2 (Tempered Distortion Property).** Given a real-valued potential $\varphi : \Lambda \to \mathbb{R}$ we define the $n$-th level variation of $\varphi$ by,

$$\text{var}_n(\varphi) := \sup \{ |\varphi(x) - \varphi(y)| : x, y \in V_\omega, \omega \in A^n \}. \tag{2.5}$$

We shall say that a potential $\varphi$ satisfies the tempered distortion condition if $\text{var}_1(\varphi) < \infty$ and $\lim_{n \to \infty} n^{-1} \text{var}_n(S_n(\varphi)) = 0$.

Note that by condition (3) in definition 2.1 the potential $\psi(x) := \log |T'(x)|$ satisfies the tempered distortion condition.

Given a potential $\varphi : \Lambda \to \mathbb{R}$ and a word $\omega \in A^n$ for some $n \in \mathbb{N}$ we define $\varphi(\omega) := \sup \{ \varphi(x) : x \in V_\omega \}$.

**Definition 2.3.** Given a potential $\varphi : \Lambda \to \mathbb{R}$, satisfying the tempered distortion condition, we define the pressure by

$$P(\varphi) := \lim_{n \to \infty} \frac{1}{n} \log \sum_{\omega \in A^n} \exp(S_n(\varphi)(\omega)).$$
This definition of pressure is essentially the same as that given by Mauldin
and Urbański in [MU1, MU2]. We note that the limit always exists, but may
be infinite. Recall that we defined \( \psi(x) \) to be the log-derivative,
\( \psi(x) := \log |T'(x)| \). Given \( \alpha > 0 \) we define \( s(\alpha) \) by,
\[
(2.4) \quad s(\alpha) := \inf \{ s : P(-s\psi) \leq s\alpha \}.
\]
More generally, given a non-negative positive potential \( \phi : \Lambda \rightarrow \mathbb{R} \geq 0 \), satisfying
the tempered distortion condition, we define,
\[
(2.5) \quad s(\phi) := \inf \{ s : P(-s(\psi + \phi)) \leq 0 \}.
\]
The project of trying to determine the Hausdorff dimension of \( D_y(\phi) \)
began with a series of articles due to Hill and Velani [HV1, HV2, HV3].
Whilst Hill and Velani gave the dimension of \( D_y(\phi) \) for an expanding rational
map of the Riemann sphere, the result extends unproblematically to any
expanding Markov map with finitely many inverse branches.

**Theorem 1** (Hill, Velani). Let \( T \) be a finite branch expanding Markov map
with repeller \( \Lambda \) and let \( \phi : \Lambda \rightarrow \mathbb{R} \) a non-negative potential which satisfies the
tempered distortion condition. Then, for all \( y \in \Lambda \) we have \( \dim_H D_y(\phi) = s(\phi) \).

Given the neat connection between Diophantine approximation and shrinking
target sets for the Gauss map it is natural to try to generalise Theorem 1
to the setting of expanding Markov maps with an infinite number of inverse
branches. However, for such maps things can become much more delicate.

Note that we always have \( \Lambda_0 \subseteq \Lambda \subseteq \overline{\Lambda} \). Indeed, when \( T \) is a finite branch
Markov map \( \Lambda_0 = \Lambda = \overline{\Lambda} \), up to a countable set. However, for Markov maps
with infinitely many inverse branches both of these containments may be
strict.

In [U] Urbański proves the following extension of Theorem 1 to points
\( y \in \Lambda_0 \) for an infinite branch expanding Markov map.

**Theorem 2** (Urbański). Let \( T \) be an expanding Markov map with repeller
\( \Lambda \) and let \( \phi : \Lambda \rightarrow \mathbb{R} \) a non-negative potential which satisfies the
tempered distortion condition. Then, for every \( y \in \Lambda_0 \) we have \( \dim_H D_y(\phi) = s(\phi) \).

In terms of dimension \( \Lambda_0 \) is a large set, with \( \dim_H \Lambda_0 = \dim_H \Lambda \) [MU1].
However, it follows from Bowen’s equation combined with the strict monoto-
nicity of the pressure function for finite iterated function systems (see
[F2 Chapter 5]) that for any \( T \) ergodic measure with \( \dim_H \mu = \dim_H \Lambda \),
\( \mu(\Lambda_0) = 0 \). For example, when \( T \) is the Gauss map and \( \mathcal{G} \) the Gauss measure,
which is ergodic and equivalent to Lebesgue measure \( \mathcal{L} \), then \( \Lambda_0 \) is the set of
badly approximable numbers with \( \dim_H \Lambda_0 = 1 \) and \( \mathcal{L}(\Lambda_0) = \mathcal{G}(\Lambda_0) = 0 \).

Our main theorem extends the above result to all \( y \in \Lambda \).

**Theorem 3.** Let \( T \) be an expanding Markov map with repeller \( \Lambda \) and let
\( \phi : \Lambda \rightarrow \mathbb{R} \) be a non-negative potential which satisfies the tempered distortion
condition. Then, for every \( y \in \Lambda \) we have \( \dim_H D_y(\phi) = s(\phi) \).
Note that in Example 2.1 \(0 \notin \Lambda = \mathbb{R} \setminus \mathbb{Q}\), so it is clear that for certain maps \(\dim_H D_y(\varphi) = s(\varphi)\) holds for \(y \in \overline{\Lambda} \setminus \Lambda\). The following theorem shows that this holds whenever \(\Lambda\) is dense in the unit interval.

**Theorem 4.** Let \(T\) be an expanding Markov map with a repeller \(\Lambda\) satisfying \(\Lambda = [0, 1]\) and let \(\varphi : \Lambda \to \mathbb{R}\) a non-negative potential which satisfies the tempered distortion condition. Then, for every \(y \in [0, 1]\) we have \(\dim_H D_y(\varphi) = s(\varphi)\).

Returning to Example 2.1 we let \(T\) denote the Gauss map and \(\psi_\alpha := (\frac{\alpha}{2} - 1) \psi\) and let \(\alpha > 2\). By the Jarník Besicovitch theorem [J, B] we have \(\dim_H D_0(\psi_\alpha) = \frac{2}{\alpha}\). It follows from Theorem 2 [U] that \(\dim_H D_y(\psi_\alpha) = \frac{2}{\alpha}\) also holds for all badly approximable numbers \(y\). By Theorem 4 we see that \(\dim_H D_y(\psi_\alpha) = \frac{2}{\alpha}\) for all \(y \in [0, 1]\).

We remark that Bing Li, BaoWei Wang, Jun Wu, Jian Xu have independently obtained a proof of Theorem 4 in the special case in which \(T\) is the Gauss map, as well some interesting results concerning targets which shrink at a super-exponential rate [BBJJ]. However, the methods used in [BBJJ] rely upon certain properties of continued fractions which do not hold in full generality.

Now suppose that \(\Lambda \neq [0, 1]\) and \(y \in \overline{\Lambda} \setminus \Lambda\). It might seem reasonable to conjecture that again \(\dim_H D_y(\varphi) = s(\varphi)\). However this is not always the case and, as the following theorem demonstrates, this conjecture fails in rather a dramatic way.

Given \(\Phi : \mathbb{N} \to \mathbb{R}^+\) we define,

\[
S_y(\Phi) := \bigcap_{m \in \mathbb{N}} \bigcup_{n \geq m} \{x \in X : d(T^n(x), y) < \Phi(n)\}.
\]

**Theorem 5.** Let \(\Phi : \mathbb{N} \to \mathbb{R}_{>0}\) be any strictly decreasing function satisfying \(\lim_{n \to \infty} \Phi(n) = 0\). Then, for each \(\beta \in (0, 1)\) there exists an expanding Markov map \(T\) with a repeller \(\Lambda\) with \(\dim_H \Lambda = \beta\) together with a point \(y \in \overline{\Lambda}\) satisfying \(\dim_H S_y(\Phi) = 0\).

Thus, even for \(\Phi\) which approaches zero at a subexponential rate we can have \(\dim_H S_y(\Phi) = 0\). We remark that \(s(\alpha)\) is always strictly positive.

We begin in Section 4 we prove the upper bound in Theorems 3 and 4 simultaneously with an elementary covering argument. In Section 5 we introduce and prove a technical proposition which implies the lower bounds in both Theorems 3 and 4. In Section 6 we prove Theorem 5. We conclude in Section 7 with some remarks.

3. Infinite iterated function systems

In order to make the proof more transparent we shall employ the language of iterated function systems.

Let \(T : \cup_{i \in \mathcal{A}} V_i \to [0, 1]\) be a countable Markov map. We associate an iterated function system \(\{\phi_i\}_{i \in \mathcal{A}}\) corresponding to \(T\) in the following way.
Proof. Suppose $P$ as $\varphi$ must take $P(\Lambda)$ repeller $\xi$ there need not be any such $\varphi$. Given $\tau = (\tau_1, \ldots, \tau_n) \in \mathcal{A}^n$ for some $n \in \mathbb{N}$ we let $\phi_\tau := \phi_{\tau_1} \circ \cdots \circ \phi_{\tau_n}$. Sets of the form $\phi_\tau([0,1])$ are referred to as cylinder sets.

Take $\omega \in \mathcal{A}^\mathbb{N}$. Note that by definition 2.1 (2) we have $\text{diam}(\phi_\omega([0,1])) \leq \xi^{-n}$ for all $n \geq N$. Thus, we may define,

$$\pi(\omega) := \bigcap_{n \in \mathbb{N}} \phi_\omega([0,1]).$$

This defines a continuous map $\pi : \Sigma \to [0,1]$.

Since the intervals $\{V_i\}_{i \in \mathcal{A}}$ have disjoint interiors the iterated function system $\{\phi_i\}_{i \in \mathcal{A}}$ satisfies the open set condition (see [F1, Section 9.2]) and $\pi(\Sigma) \setminus \Lambda$ is countable. By definition 2.1 (1) we have $T \circ \pi(\omega) = \pi \circ \sigma(\omega)$ for all $\omega \in \pi^{-1}(\Lambda)$. Thus, $T : \Lambda \to \Lambda$ and $\sigma : \Sigma \to \Sigma$ are conjugate up to a countable set.

In Definition 2.3 we have used a slightly modified version of the definition given in [MU2, (2.1)]. Nevertheless, the following theorems may be proved in essentially the same way as the proofs given in [MU2].

**Theorem 6** (Mauldin, Urbański). Given a countable Markov map $T$ with repeller $\Lambda$ we have $\text{dim}_H \Lambda = \inf \{s : P(-s\psi) \leq 0\}$.

When $T$ has finitely many branches there is a unique $s(\Lambda)$ such that $P(-s(\Lambda)\psi) = 0$ and $\text{dim}_H \Lambda = s(\Lambda)$. However, Mauldin and Urbański have shown that when $T$ has countably many inverse branches we can have $P(-t\psi) < 0$ for all $t \geq \inf \{s : P(-s\psi) \leq 0\}$ and consequently there is no such $s(\Lambda)$ (see [MU1, Example 5.3]). Similar examples show that in general there need not be any $s$ satisfying $P(-s(\psi + \varphi)) = 0$ and consequently we must take $s(\varphi) := \inf \{s : P(-s(\psi + \varphi)) \leq 0\}$ in Theorems 3 and 4.

The pressure $P$ has the following finite approximation property.

**Theorem 7** (Mauldin, Urbański). Let $T$ be a countable Markov map and $\varphi : \Lambda \to \mathbb{R}$ a potential satisfying the tempered distortion condition. Then $P(\varphi) = \sup \{P_F(\varphi) : F \subseteq \mathcal{A} \text{ is a finite set}\}$.

**Corollary 1.** Let $\varphi : \Lambda \to \mathbb{R}$ be a non-negative potential satisfying the tempered distortion condition. Then $P(-s(\varphi)(\psi + \varphi)) \leq 0$.

**Proof.** Suppose $P(-s(\varphi)(\psi + \varphi)) > 0$. Then, by Theorem 7 $P_F(-s(\varphi)(\psi + \varphi)) > 0$ for some finite set $F \subseteq \mathcal{A}$. However $\psi + \varphi$ is bounded on $\mathcal{A}$ as $\text{var}_1(\psi), \text{var}_1(\varphi) < \infty$, and hence $s \mapsto P_F(-s(\varphi)(\psi + \varphi))$ is continuous. Thus, there exists $t > s(\varphi)$ for which $P(-t(\psi + \varphi)) > 0 \geq P_F(-t(\psi + \varphi)) > 0$.

Since $\psi + \varphi \geq 0$, $s \mapsto P(-s(\psi + \varphi))$ is non-increasing and hence, $t \leq \inf \{s : P(-s(\psi + \varphi)) \leq 0\}$. Since $s(\varphi) < t$ this is a contradiction. \qed
Corollary 2. Let $T$ be a countable Markov map. Then for all potentials $\varphi : \Lambda \to \mathbb{R}$, satisfying the tempered distortion condition, $s(\varphi) > 0$.

Proof. Since $\psi + \varphi \geq 0$ and $\mathcal{A} \geq 2$ it follows from Definition 2.3 that $P(-s(\psi + \varphi)) \geq \log 2 > 0$ for all $s \leq 0$. If, however, $s(\varphi) \leq 0$ then by Corollary 1 there exists some $s \leq 0$ with $P(-s(\psi + \varphi)) \leq 0$, which is a contradiction.

\[ \square \]

4. Proof of the upper bound in Theorems 3 and 4

In this section we use a standard covering argument to prove a uniform upper bound on the dimension of $D_y(\varphi)$, which entails the upper bounds in Theorems 3 and 4.

Throughout the proof we shall let $\rho_n$ denote

$$\rho_n := \max \{ \var_n(A_n(\psi)), \var_n(A_n(\varphi)) \}.$$ 

Since both $\psi$ and $\varphi$ satisfy the tempered distortion condition, $\lim_{n \to \infty} \rho_n = 0$.

Proposition 4.1. For every $y \in [0, 1]$ we have $\dim_H D_y(\varphi) \leq s(\varphi)$.

Proof. For each $n \in \mathbb{N}$ and $\omega \in \mathcal{A}^n$ we define,

$$V^{\varphi,n}_\omega := \left\{ x \in V_\omega : |T^n(x) - y| < e^{-\inf_{z \in V_\omega} S_n(\varphi)(z)} \right\}.$$ 

Clearly every $x \in D_y(\varphi)$ is in $V^{\varphi,n}_\omega$ for infinitely many $n \in \mathbb{N}$ and $\omega \in \mathcal{A}^n$. Moreover, by the mean value theorem we have,

$$\text{diam}(V^{\varphi,n}_\omega) \leq e^{-\inf_{z \in V_\omega} S_n(\phi)(z) - \inf_{z \in V_\omega} S_n(\varphi)(z)} \leq e^{-\sup_{z \in V_\omega} S_n(\psi)(z) - \inf_{z \in V_\omega} S_n(\varphi)(z)} \leq e^{\sup_{z \in V_\omega} S_n(-s(\phi + \varphi))(z) + 2n\rho_n} \leq e^{S_n(-s(\phi + \varphi))(\omega) + 2n\rho_n}.$$ 

Choose $s > s(\varphi)$, so there exists some $t < s$ with $P(-t(\phi + \varphi)) \leq 0$. By condition (2) in definition 2.1 together with $\varphi \geq 0$ we have $S_n(\phi + \varphi) \geq n \log \xi$ for all sufficiently large $n$ and hence $P(-s(\phi + \varphi)) < 0$. Take $\epsilon > 0$ with $\epsilon < -P(-s(\phi + \varphi))$. Since $\lim_{n \to \infty} \rho_n = 0$ there exists some $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have,

$$\sum_{\omega \in \mathcal{A}^n} \{ \exp(S_n(-s(\phi + \varphi))(\omega)) \} < e^{-n\epsilon - 2n\rho_n}.$$ 

Now choose some $\delta > 0$. Since $\rho_n \to 0$ and $S_n(\phi + \varphi) \geq n \log \xi$ for all sufficiently large $n$, it follows from (4.2) that we may choose $n_1 \geq n_0$ so that for all $n \geq n_1$ diam$(V^{\varphi,n}_\omega) < \delta$. Moreover, $\bigcup_{n \geq n_1} \{ V^{\varphi,n}_\omega \}_{\omega \in \mathcal{A}^n}$ forms a countable cover of $D_y(\varphi)$. Applying (4.2) together with (4.3) we see that for
all $n_1 \geq n_0$, 
\[ \sum_{n \geq n_1} \sum_{\omega \in \mathcal{A}^n} \text{diam}(V_{\omega}^n)^s \leq \sum_{n \geq n_1} \sum_{\omega \in \mathcal{A}^n} e^{\sup_{z \in V_{\omega}} S_n(-s(\varphi + \phi))(z) + 2n s p_n} \]
\[ \leq \sum_{n \geq n_1} e^{-n \epsilon} \leq \sum_{n \geq n_0} e^{-n \epsilon} < \infty. \]

Thus, $\mathcal{H}^s_\delta(D_y(\varphi)) \leq \sum_{n \geq n_0} e^{-n \epsilon}$ for all $\delta > 0$ and hence $\mathcal{H}^s(D_y(\varphi)) \leq \sum_{n \geq n_0} e^{-n \epsilon} < \infty$. Thus, $\dim \mathcal{H}(D_y(\varphi)) \leq s$ and since this holds for all $s > s(\varphi)$ we have $\dim \mathcal{H}(D_y(\varphi)) \leq s(\varphi)$.

5. Proof of the lower bound in Theorems 3 and 4

In order to prove the lower bound to Theorems 3 and 4 we shall introduce the positive upper cylinder density condition. The condition essentially says that there is a sequence of arbitrarily small balls, surrounding a point $y \in [0, 1]$, such that each ball contains a collection of disjoint cylinder sets whose total length is comparable to the diameter of the ball. As we shall see, given any countable Markov map $T$ with repeller $\Lambda$ this condition is satisfied for all $y \in \Lambda$, and if $\overline{\Lambda} = \Lambda$, this condition is satisfied for all $y \in [0, 1]$. The substance of the proof lies in showing that for any point $y \in [0, 1]$, for which the positive upper cylinder density condition is satisfied, we have $\dim \mathcal{H}(D_y(\varphi)) \geq s(\varphi)$.

**Definition 5.1** (Positive upper cylinder density). Suppose we have an expanding Markov map with a corresponding iterated function system $\{\phi_i\}_{i \in \mathcal{A}}$. Given $y \in \overline{\Lambda}$, $n \in \mathbb{N}$ and $r > 0$ we define,
\[ C(y, n, r) := \{\phi_{\tau}([0, 1]) : \tau \in \mathcal{A}^n, \phi_{\tau}([0, 1]) \subset B(y, r)\}. \]
We shall say that the iterated function system $\{\phi_i\}_{i \in \mathcal{A}}$ has positive upper cylinder density at $y$ if there is a family of natural numbers $(\lambda_r)_{r \in \mathbb{R}^+}$ with $\lim_{r \to 0} \lambda_r = \infty$ and $\limsup_{r \to 0} \lambda_r^{-1} \log r < 0$, for which
\[ \limsup_{r \to 0} r^{-1} \sum_{A \in C(y, \lambda_r, r)} \text{diam}(A) > 0. \]

**Proposition 5.1.** Let $T$ be an expanding Markov map with associated iterated function system $\{\phi_i\}_{i \in \mathcal{A}}$. Suppose that $\{\phi_i\}_{i \in \mathcal{A}}$ has positive upper cylinder density at $y \in \overline{\Lambda}$. Then for each non-negative potential $\varphi : \Lambda \to \mathbb{R}$ which satisfies the tempered distortion condition we have $\dim \mathcal{H}(D_y(\varphi)) \geq s(\varphi)$.

Combining Proposition 5.1 with Lemmas 5.1 and 5.2 completes the proof of the lower bound in Theorems 3 and 4 respectively.

**Lemma 5.1.** Let $T$ be an expanding Markov map. Then the corresponding iterated function system $\{\phi_i\}_{i \in \mathcal{A}}$ has positive upper cylinder density at every $y \in \Lambda$. 
Proof. Suppose that \( y \in \Lambda \). Then there exists some \( \omega \in \Sigma \) such that \( y \in \phi_{\omega|n}([0,1]) \) for all \( n \in \mathbb{N} \). We shall define \( (\lambda_r)_{r \in \mathbb{R}_+} \) by
\[
\lambda_r := \min \{ n \in \mathbb{N} : 2 \text{diam} (\phi_{\omega|n}([0,1])) \leq r \}.
\]
Clearly \( \lim_{r \to 0} \lambda_r = \infty \). Moreover,
\[
r < 2 \text{diam} (\phi_{\omega|\lambda_r-1}([0,1])) \leq 2 \zeta^{-\lambda_r+1},
\]
so \( \limsup_{r \to \infty} \lambda_r^{-1} \log r \leq - \log \xi < 0 \).

Given any \( n \in \mathbb{N} \) choose \( r_n := 2 \text{diam} (\phi_{\omega|n}([0,1])) \). Clearly \( \lambda_{r_n} = n \) and \( \phi_{\omega|n}([0,1]) \in C(y,n,r_n) \). Hence,
\[
\limsup_{r \to 0} r^{-1} \sum_{A \in C(y,\lambda_r,r)} \text{diam}(A) \geq \frac{1}{2}.
\]
\[\square\]

Lemma 5.2. Suppose \( T \) is an expanding Markov map with \( \overline{\Lambda} = [0,1] \). Then the corresponding iterated function system \( \{ \phi_i \}_{i \in \mathcal{A}} \) has positive upper cylinder density at every \( y \in [0,1] \).

Proof. Suppose \( T \) satisfies \( \overline{\Lambda} = [0,1] \). Then for any \( n \in \mathbb{N} \) we have
\[
[0,1] \subseteq \overline{\Lambda} \subseteq \bigcup_{\omega \in \mathcal{A}^n} \phi_{\omega}(\Lambda) \subseteq \bigcup_{\omega \in \mathcal{A}^n} \phi_{\omega}([0,1]).
\]

We define \( (\lambda_r)_{r \in \mathbb{R}_+} \) by
\[
\lambda_r := \left\lceil \frac{- \log r + \log 2}{\log \xi} \right\rceil.
\]
Clearly \( \lim_{r \to 0} \lambda_r = \infty \) and \( \limsup_{r \to 0} \lambda_r^{-1} \log r = - \log \xi < 0 \).

Suppose \( y \in [0, \frac{1}{2}] \). Given any \( r < \frac{1}{2} \) and any \( \omega \in \mathcal{A}^{\lambda_r} \) we have
\[
\text{diam} (\phi_{\omega}([0,1])) \leq \xi^{-\lambda_r} < r/2.
\]
Now \( C(y,n,r) \) contains all but the right most member of
\[
\mathcal{I} := \{ \phi_{\omega}([0,1]) : \phi_{\omega}([0,1]) \cap [y, y+r) \neq \emptyset \},
\]
if such a member exists. By (5.1) \( \sum_{A \in \mathcal{I}} \text{diam}(A) \geq r \), so by (5.2) we have,
\[
\sum_{A \in C(y,\lambda_r,r)} \text{diam}(A) \geq r/2.
\]
By symmetry (5.3) also holds for \( y \in [\frac{1}{2}, 1] \).

Letting \( r \to 0 \) proves the lemma. \( \square \)

Before going into details we shall give a brief outline of the proof of Proposition 5.1. We begin by taking \( s < s(\varphi) \) and extracting a certain finite set of words \( B \) such that \( P_B(-s(\varphi) > 0 \). In addition, we take a Bernoulli measure \( \mu \) supported on \( B^\mathbb{N} \) with \( h(\mu) = t \int (\varphi + \varphi) d\mu \) for some \( t > s \). We then construct a tree structure, iteratively, in the following way. Let \( \Gamma_{q-1} \) be the finite collection of words in the tree at stage \( q - 1 \) and
\( \gamma_{q-1} \) denote the length of those words. At stage \( q \) we take \( \alpha_q \) so large that \( \alpha_q^{-1} \max \{ S_{\gamma_{q-1}}(\psi)(\omega), S_{\gamma_{q-1}}(\varphi)(\omega) : \omega \in \Gamma_q \} \) is negligible. We then take a ball of radius \( B(y, r_q) \) so that \( r_q < \exp(-\alpha_q \int \varphi d\mu) \) and \( B(y, r_q) \) contains a collection of disjoint cylinder sets whose total width is comparable to \( r_q \), corresponding to a finite collection of words \( \mathcal{R}_q \) of length \( \lambda_q \). This is made possible by the upper cylinder density condition. We then choose \( \beta_q \) so that \( \exp(-\beta_q \int \varphi d\mu) \) is greater than, but comparable with, \( r_q \). \( \Gamma_q \) consists of all continuations of \( \Gamma_{q-1} \) of length \( \gamma_q := \beta_q + \lambda_q \) so that \( \beta_q |\omega| \in \mathcal{R}_q \) and \( \omega_\nu \) is chosen freely from \( \mathcal{B} \) for all \( \gamma_{q-1} < \nu \leq \beta_q \). Having constructed our tree we shall define \( S \) to be a certain subset of its limit points for which \( \omega |\beta_q \) behaves “typically” with respect to \( \mu \) for each \( q \). Given \( \omega \in \mathcal{S} \) we have \( S_{\beta_q}(\varphi)(\pi(\omega)) = \beta_q \int \varphi d\mu < -\log r_q \) so \( \beta_q |\omega| \gamma_q \in \mathcal{R}_q \) implies \( |T^{\beta_q}(\pi(\omega)) - y| < \exp(-S_{\beta_q}(\varphi)(\pi(\omega))) \). Hence \( \pi(S) \subset \mathcal{D}_\varphi(\varphi) \). At each stage \( \beta_q \), \( S \) consists of approximately \( \beta_q h(\mu) \) intervals of diameter approximately \( \exp(-\beta_q \int \varphi d\mu) \). Moreover, for all \( \omega \in S \), \( \beta_q |\omega| \gamma_q \in \mathcal{R}_q \). The total diameter of cylinders corresponding to words from \( \mathcal{R}_q \) is about \( r_q \approx \exp(-\beta_q \int \varphi d\mu) \), and so at stage \( \gamma_q \) \( S \) consists of approximately \( \beta_q h(\mu) \) intervals of diameter roughly \( \exp(-\beta_q \int (\psi + \varphi) d\mu) \), giving an optimal covering exponent of \( t > s \). The fact that \( \beta_q \geq \alpha_q \) will be shown to imply that we cannot obtain a cover which is more efficient, and as such \( \dim_H \pi(S) \geq t \).

**Proof of Proposition 5.1.** Choose \( s < s(\varphi) \) so that \( P(-s(\psi + \varphi)) > 0 \). Without loss of generality we may assume that \( s > 0 \). Now take \( \epsilon \in (0, P(-s(\psi + \varphi))) \). Since \( \lim_{n \to \infty} \rho_n = 0 \), it follows from the definition of pressure that for all sufficiently large \( n \) we have,

\[
\sum_{\omega \in \mathcal{A}^n} \exp(S_n(-s(\psi + \varphi))(\omega)) > e^{\epsilon n + 2n \epsilon s \rho_n}.
\]

Consequently, for all sufficiently large \( n \) we have,

\[
\sum_{\tau \in \mathcal{A}^n} e^{-s(S_n(\psi)(\tau) + S_n(\phi)(\tau))} > e^{\epsilon n}.
\]

By choosing some large \( k \) we obtain,

\[
\sum_{\tau \in \mathcal{A}^k} e^{-s(S_k(\psi)(\tau) + S_k(\phi)(\tau))} > 6.
\]

Thus, there exists some finite subset \( \mathcal{F} \subset \mathcal{A}^k \) with

\[
\sum_{\tau \in \mathcal{F}} e^{-s(S_k(\psi)(\tau) + S_k(\phi)(\tau))} > 6.
\]

Note that \( s > 0 \) and for each \( \tau \in \mathcal{F} \), \( S_k(\psi)(\tau) > 0 \) and \( S_k(\varphi)(\tau) > 0 \), so \( e^{-s(S_k(\psi)(\tau) + S_k(\phi)(\tau))} \in (0, 1) \) for every \( \tau \in \mathcal{F} \).

The finite set \( \mathcal{F} \) inherits an order \( \prec_s \) from the order on \([0, 1] \) in a natural way by \( \tau_1 \prec_s \tau_2 \) if and only if \( \sup \phi_{\tau_1}([0, 1]) \leq \inf \phi_{\tau_2}([0, 1]) \). Partition \( \mathcal{F} \) into two disjoint sets \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) so that if \( \tau \in \mathcal{F}_1 \) then its successor under
follows from the fact that $B$ is in $\mathcal{F}_2$ and if $\tau \in \mathcal{F}_2$ then its successor under $<_s$ is in $\mathcal{F}_1$. Clearly we may choose one $m \in \{1, 2\}$ so that

\begin{equation}
\sum_{\tau \in \mathcal{F}_m} e^{-s(S_k(\psi)(\tau)+S_k(\phi)(\tau))} \geq \frac{1}{2} \sum_{\tau \in \mathcal{F}} e^{-s(S_k(\psi)(\tau)+S_k(\phi)(\tau))} > 3.
\end{equation}

Since $s > 0$, $S_k(\psi)(\tau) > 0$ and $S_k(\phi)(\tau) \geq 0$, $e^{-s(S_k(\psi)(\tau)+S_k(\phi)(\tau))} < 1$ for every $\tau \in \mathcal{F}$. Thus we may remove both the smallest and the largest element from $\mathcal{F}_m$, under the order $<_s$, to obtain a set $\mathcal{B} \subset \mathcal{F}_m$ satisfying

\begin{equation}
\sum_{\tau \in \mathcal{B}} e^{-s(S_k(\psi)(\tau)+S_k(\phi)(\tau))} > 1.
\end{equation}

Let $c := \max \{ S_k(\psi)(\tau) + S_k(\phi)(\tau) : \tau \in \mathcal{F} \} > 0$. Given any $\omega_1, \omega_2 \in A^n$ and $\tau_1, \tau_2 \in \mathcal{B}$ with either $\omega_1 \neq \omega_2$ or $\tau_1 \neq \tau_2$, or both, we have,

\begin{equation}
|x - y| \geq \max \left\{ e^{-S_n(\psi)(\omega_1)-c}, e^{-S_n(\psi)(\omega_2)-c} \right\}
\end{equation}

for all $x \in (\phi_{\omega_1} \circ \phi_{\tau_1})([0, 1])$ and $y \in (\phi_{\omega_1} \circ \phi_{\tau_2})([0, 1])$. When $\omega_1 \neq \omega_2$ this follows from the fact that $\mathcal{B}$ contains neither the maximal nor the minimal element of $\mathcal{F}$ under $<_s$. When $\omega_1 = \omega_2$ but $\tau_1 \neq \tau_2$ this follows from the fact that since $\tau_1, \tau_2 \in \mathcal{B} \subset \mathcal{F}_m$, $\tau_1$ cannot be the successor of $\tau_2$ and $\tau_2$ cannot be the successor of $\tau_1$.

Since $\mathcal{B}$ is finite and for each $\omega \in \Sigma S_k(\psi)(\omega) \geq k \log \xi$ and $S_k(\psi)(\omega) \geq 0$, we may take $t \in (s, 1)$ satisfying

\begin{equation}
\sum_{\tau \in \mathcal{B}} e^{-t(S_k(\psi)(\tau)+S_k(\phi)(\tau))} = 1.
\end{equation}

We define a $k$-th level Bernoulli measure $\mu$ on $\mathcal{B}^\mathbb{N}$ by defining $p(\tau)$ for $\tau \in A^k$ by $p(\tau) := e^{-t(S_k(\psi)(\tau)+S_k(\phi)(\tau))}$ and setting $\mu([\tau_1, \cdots, \tau_n]) = p_{\tau_1} \cdots p_{\tau_n}$ for each $(\tau_1, \cdots, \tau_n) \in \mathcal{B}^n$. We define,

\begin{align*}
\mathbb{E}(S_k(\psi)) &:= \sum_{\tau \in \mathcal{B}} p(\tau) S_k(\psi)(\tau) \\
\mathbb{E}(S_k(\phi)) &:= \sum_{\tau \in \mathcal{B}} p(\tau) S_k(\phi)(\tau).
\end{align*}

Choose a decreasing sequence $\{\delta_q\}_{q \in \mathbb{N}} \subset \mathbb{R}_{>0}$ so that $\prod_{q \in \mathbb{N}} (1 - \delta_q) > 0$. Take $q \in \mathbb{N}$. By Kolmogorov’s strong law of large numbers combined with Egorov’s theorem there exists set $S_q \subset \mathcal{B}^\mathbb{N}$ with $\mu(S_q) > 1 - \delta_q$ and $N_q(q) \in \mathbb{N}$ such that for all $\omega = (\omega_\nu)_{\nu \in \mathbb{N}} \in S_q$ with $\omega_\nu \in \mathcal{B}$ for each $\nu \in \mathbb{N}$ and all
\[ n \geq N(q) \] we have,

\begin{align}
(5.12) \quad & \frac{1}{n} \sum_{\nu=1}^{n} S_k(\psi)(\omega_{\nu}) < \mathbb{E}(S_k(\psi)) + \frac{1}{q} \\
(5.13) \quad & \frac{1}{n} \sum_{\nu=1}^{n} S_k(\varphi)(\omega_{\nu}) < \mathbb{E}(S_k(\varphi)) + \frac{1}{q} \\
(5.14) \quad & \frac{1}{n} \sum_{\nu=1}^{n} \log p_{\omega_{\nu}} < \sum_{\tau \in \mathcal{B}} p(\tau) \log p(\tau) + \frac{1}{q} \\
& \quad \quad \quad = -t \left( \mathbb{E}(S_k(\psi)) + \mathbb{E}(S_k(\varphi)) \right) + \frac{1}{q} \\
& \quad \quad \quad < -t \left( \frac{1}{n} \sum_{\nu=1}^{n} S_k(\psi)(\omega_{\nu}) + \mathbb{E}(S_k(\varphi)) \right) + \frac{2}{q} \\
& \quad \quad \quad \leq -t \left( \frac{1}{n} S_{nk}(\psi)(\omega_{\nu})_{\nu=1}^{n} + \mathbb{E}(S_k(\varphi)) \right) + \frac{2}{q}.
\end{align}

Clearly we may assume that \((N(q))_{q \in \mathbb{N}}\) is increasing and \(N(1) \geq 2\).

Now fix

\[ \zeta \in \left( 0, \limsup_{r \to 0} r^{-1} \sum_{A \in C(y, \lambda, r)} \text{diam}(A) \right), \]

\[ d \in \left( \limsup_{r \to 0} \lambda r^{-1} \log r, 0 \right). \]

We shall now give an inductive construction consisting of a quadruple of rapidly increasing sequences of natural numbers \((\alpha_q)_{q \in \mathbb{N} \cup \{0\}}, (\beta_q)_{q \in \mathbb{N} \cup \{0\}}, (\gamma_q)_{q \in \mathbb{N} \cup \{0\}}, (\lambda_q)_{q \in \mathbb{N} \cup \{0\}}\), a sequence of positive real numbers \((r_q)_{q \in \mathbb{N} \cup \{0\}}\) and a pair of sequences of finite sets of words \((\mathcal{R}_q)_{q \in \mathbb{N} \cup \{0\}}\) and \((\Gamma_q)_{q \in \mathbb{N} \cup \{0\}}\). First set \(\alpha_0 = \beta_0 = \gamma_0 = 0\), \(\lambda_0 = 1\) and \(\Lambda_0 = \Gamma_0 = \emptyset\). For each \(q \in \mathbb{N}\) we define

\[ \alpha_q := 10kq^2 \gamma_{q-1} N(q) N(q+1) \left[ \log \zeta^{-1} c(3 + 2\rho_{\lambda_{q-1}}) \max \left\{ S_{\gamma_{q-1}}(\psi)(\tau) + S_{\gamma_{q-1}}(\varphi)(\tau) : \tau \in \Gamma_{q-1} \right\} \right]. \]

Note that since \(\Gamma_{q-1}\) is finite \(\alpha_q\) is well defined.

We then choose \(r_q > 0\) so that,

\begin{align}
(5.15) \quad & - \log r_q > k^{-1} (\alpha_q - \gamma_{q-1}) \left( \mathbb{E}(S_k(\varphi)) + \frac{1}{q} \right) + \gamma_{q-1} c + q, \\
& \quad \quad \quad + \sum_{A \in C(y, \lambda r_q, r_q)} \text{diam}(A) > \zeta r_q
\end{align}

and also

\[ \lambda r^{-1} \log r < d. \]
Let $\lambda_q := \lambda_{r_q}$. We may choose $\mathcal{R}_q$ to be a finite set of words $\tau \in \mathcal{A}^{\lambda_q}$ so that for each $\tau \in \mathcal{R}_q \phi_{\tau}([0, 1]) \subset B(y, r_q)$ and

$$\sum_{\tau \in \mathcal{R}_q} \text{diam} (\phi_{\tau}([0, 1])) > \zeta r_q.$$

Let $\beta_q$ be the largest integer satisfying $k|(|\beta_q - \gamma_{q-1}|$ and

$$- \log r_q > k^{-1}(\beta_q - \gamma_{q-1}) \left( \mathbb{E}(S_k(\nu)) + \frac{1}{q} \right) + \gamma_{q-1}c + q.$$  \hfill (5.16)

We let $\gamma_q := \beta_q + \lambda_q$. We define $\Gamma_q$ by,

$$\Gamma_q := \left\{ \omega \in \mathcal{A}^{\gamma_q} : \omega|_{\gamma_{q-1}} \in \Gamma_{q-1}, \gamma_{q-1} \omega|_{\beta_q} \in \mathcal{B}^{k^{-1}(\beta_q - \gamma_{q-1})}, \beta_q \omega|_{\gamma_q} \in \mathcal{R}_q \right\}.$$

Note that since $\mathcal{B}$, $\Gamma_{q-1}$ and $\mathcal{R}_q$ are finite, so is $\Gamma_q$.

We inductively define a sequence of measures $\mathcal{W}_q$ supported on $\Gamma_q$.

For each $\omega \in \mathcal{A}^{\omega}$ and $\tau \in \mathcal{R}_q$ we let

$$q (\omega, \tau) := \frac{\text{diam} (\phi_{\omega} \circ \phi_{\tau}([0, 1]))}{\sum_{\tau \in \mathcal{R}_q} \text{diam} (\phi_{\omega} \circ \phi_{\tau}([0, 1]))}.$$

Now by the definition of $\Gamma_q$, each $\omega^q \in \Gamma_q$ is of the form $\omega^q = (\omega^{q-1}, \kappa_1^q, \cdots, \kappa_{k-1}(\beta_q - \gamma_{q-1}), \tau_q)$ where $\omega^{q-1} \in \Gamma_{q-1}$, $\kappa_1^q \in \mathcal{B}$ for $\nu = 1, \cdots, k^{-1}(\beta_q - \gamma_{q-1})$ and $\tau_q \in \mathcal{R}_q$. We set,

$$\mathcal{W}_q(\omega^q) = \mathcal{W}_{q-1}(\omega^{q-1}) \left( \prod_{\nu=1}^{k^{-1}(\beta_q - \gamma_{q-1})} p(\kappa_\nu) \right) q \left( (\omega^{q-1}, \kappa_1^q, \cdots, \kappa_{k-1}(\beta_q - \gamma_{q-1}), \tau_q) \right).$$

Define $\Gamma := \{ \omega \in \Sigma : \omega|_{\gamma_q} \in \Gamma_q \text{ for all } q \in \mathbb{N} \}$ and extend the sequence $(\mathcal{W}_q)_{q \in \mathbb{N}}$ to a measure $\mathcal{W}$ on $\Gamma$ in the natural way.

We let $S \subseteq \Gamma$ denote the subset,

$$S := \{ \omega \in \Gamma : [\gamma_{q-1}|\omega|_{\beta_q}] \cap S_q \neq \emptyset \text{ for all } q \in \mathbb{N} \}. $$ \hfill (5.17)

**Lemma 5.3.** For all $\omega \in S$ and $n \in \mathbb{N}$ we have $\pi(\omega) \in \phi_{\omega|_{n}}((0, 1))$.

**Proof.** Suppose for a contradiction that $\omega \in S$ and for some $N \in \mathbb{N}$ $\pi(\omega) \notin \phi_{\omega|_{N}}((0, 1))$. Then for all $n \geq N$ we have $\pi(\omega) \in \phi_{\omega|_{n}}((0, 1)) = \partial \phi_{\omega|_{n}}((0, 1))$.

However, given $N \in \mathbb{N}$ we may choose $q$ with $\gamma_q > N$. Then $\omega_{\gamma_q+1} \in \mathcal{B}$ by the construction of $S$. Consequently $\phi_{\gamma_q+1}([0, 1])$ is in neither the left most, nor the right most interval amongst,

$$\{ \phi_{\omega|_{\kappa(\ell)}} \circ \phi_{\tau}([0, 1]) : \tau \in \mathcal{F} \}.$$

Hence, $\pi(\omega) \notin \partial \phi_{\omega|_{\gamma_q}}((0, 1)).$ \hfill $\Box$

**Lemma 5.4.** $\pi(S) \subseteq \mathcal{D}_y(\nu)$. 
Proof. Take $\omega \in S$. By Lemma 5.3 we have $\pi(\omega) \in \phi_{\omega | n}((0, 1)) \subseteq V_{\omega | n}$ and hence $S_n(\varphi)(\omega) \leq S_n(\varphi)(\omega)$ for all $n \in \mathbb{N}$ and in particular for each $q \in \mathbb{N}$,

$$S_{\beta_q}(\varphi)(\omega) \leq S_{\beta_q}(\varphi)(\omega) \leq S_{\beta_q - \gamma_q - 1}(\varphi)(\gamma_q - 1|\omega) + c\gamma_q - 1$$

$$\leq \sum_{\nu = 1}^{k - 1(\beta_q - \gamma_q - 1)} S_k(\varphi)(\gamma_q - 1 + (\nu - 1)k|\omega) + nk + c\gamma_q - 1.$$ By (5.13) combined with the fact that $[\gamma_q - 1|\omega] \cap S_q \neq \emptyset$,

$$S_{\beta_q}(\varphi)(\omega) \leq k - 1(\beta_q - \gamma_q - 1) \left( E(S_k(\varphi)) + \frac{1}{q} \right) + c\gamma_q - 1.$$ Thus, by the definition of $r_q$ we have, $r_q < e^{-S_{\beta_q}(\varphi)(\omega)}$.

$$T^{\beta_q}(\pi(\omega)) = \pi(\sigma^{\beta_q}(\omega)) \in \phi_{\beta_q|\omega}|_{\gamma_q}([0, 1])$$

Since $\omega \in S \subseteq \Gamma$, $\beta_q|\omega|\gamma_q \in R_q$ and hence

$$T^{\beta_q}(\pi(\omega)) \in \phi_{\beta_q|\omega}|_{\gamma_q}([0, 1]) \subseteq B(y, r_q) \subseteq B(y, e^{-S_{\beta_q}(\varphi)(\omega)})$$ Since this holds for all $q \in \mathbb{N}$, $\pi(\omega) \in \mathcal{F}_y(\varphi)$. 

\begin{lemma}
Suppose $\omega \in S$. Given $q \in \mathbb{N}$ and $\gamma_q - 1 < n \leq \beta_q$ we have,

$$- \log W_q(\omega|n) \geq t \left( S_n(\psi)(\omega|n) + k - 1(n - \gamma_q - 1)E(S_k(\varphi)) \right)$$

$$- 3\gamma_q - 1 \left( \frac{n}{q - 1} - 2\gamma_q - 1\rho_{\lambda_q - 1} - \frac{2n}{q} - N(q)c, \right.$$}

$$- \log W_q(\omega|\gamma_q) \geq t S_\gamma(\psi)(\omega|\gamma_q) - \frac{3\gamma_q}{q} - 2\gamma_q \rho_{\lambda_q}.$$ \end{lemma}

Proof. We prove the lemma by induction. The lemma is trivial for $q = 0$. Now suppose that

$$- \log W_{q - 1}(\omega|\gamma_q) \geq t S_{q - 1}(\psi)(\omega|\gamma_q - 1) - \frac{3\gamma_q - 1}{q - 1} - 2\gamma_q - 1\rho_{\lambda_q - 1}.$$ Take $\gamma_q - 1 < n \leq \beta_q$ consider $\ell(n) := \lfloor k - 1(n - \gamma_q - 1) \rfloor$. If $\ell(n) < N(q)$ then clearly

$$S_n(\psi)(\omega|n) \leq S_{q - 1}(\psi)(\omega|\gamma_q - 1) + S_{n - \gamma_q}(\psi)(\gamma_q - 1|\omega|n)$$

$$\leq S_{\gamma_q}(\psi)(\omega|\gamma_q - 1) + N(q)c,$$

$$k - 1(n - \gamma_q - 1)E(S_k(\varphi)) \leq N(q)c$$

Since $t < 1$ and $N(q - 1) \leq N(q)$ it follows from the inductive hypothesis together with the definition of $W_q$ that,

$$- \log W_q([\omega|n]) \geq - \log W_{q - 1}([\omega|\gamma_q - 1])$$

$$\geq t \left( S_n(\psi)(\omega|n) + k - 1(n - \gamma_q - 1)E(S_k(\varphi)) \right)$$

$$- \frac{3\gamma_q - 1}{q - 1} - 2\gamma_q - 1\rho_{\lambda_q - 1} - 2N(q)c.$$
On the other hand, if $\ell(n) \geq N(q)$ then by equation (5.14) together with $[\gamma_q-1|\omega|\beta_q] \cap S_q \neq \emptyset$ we have

\[
\sum_{\nu=k^{-1}\gamma_q}^{k^{-1}\gamma_q+\ell(n)-1} \log p(\omega_{kq+1}, \ldots, \omega_{kq+k}) < -t \left( S_{k\ell(n)}(\psi)(\gamma_q-1|\omega|\gamma_q-1 + k\ell(n)) + \ell(n)E(S_k(\varphi)) \right) + \frac{2n}{q} \\
< -t \left( S_{n-\gamma_q-1}(\psi)(\omega|n-\gamma_q-1) + k^{-1}(n-\gamma_q-1)E(S_k(\varphi)) \right) + 2c + \frac{2n}{q}.
\]

Moreover, by the definition of $W_q$ we have,

\[
-\log W_q([\omega|n]) \geq -\log W_{q-1}([\omega|\gamma_q-1]) - \sum_{\nu=0}^{\ell(n)-1} \log p(\omega_{kq+1}, \ldots, \omega_{kq+k}) \\
\geq t \left( S_{\gamma_q-1}(\psi)(\omega|\gamma_q-1) + S_{n-\gamma_q-1}(\psi)(\omega|n-\gamma_q-1) + k^{-1}(n-\gamma_q-1)E(S_k(\varphi)) \right) \\
- \frac{3\gamma_q-1}{q-1} - 2\gamma_q-1\rho_{\gamma_q-1} - 2c - \frac{2n}{q} \\
\geq t \left( S_{n}(\psi)(\omega|n) + k^{-1}(n-\gamma_q-1)E(S_k(\varphi)) \right) \\
- \frac{3\gamma_q-1}{q-1} - 2\gamma_q-1\rho_{\gamma_q-1} - N(q)c - \frac{2n}{q}.
\]

In particular we have

\[
-\log W_q([\omega|\beta_q]) \geq t \left( S_{\beta_q}(\psi)(\omega|\beta_q) + k^{-1}(\beta_q - \gamma_q-1)E(S_k(\varphi)) \right) \\
- \frac{3\gamma_q-1}{q-1} - 2\gamma_q-1\rho_{\gamma_q-1} - N(q)c - \frac{2\beta_q}{q}.
\]

Note that,

\[
-\log W_q([\omega|\gamma_q]) = -\log W_q([\omega|\beta_q]) - \log q(\omega|\beta_q, \beta_q|\omega|\gamma_q) \\
= -\log W_q([\omega|\beta_q]) - \log \left( \frac{\text{diam} \left( \phi_{\omega|\gamma_q}([0,1]) \right)}{\sum_{\tau \in \mathcal{R}_q} \text{diam} \left( \phi_{\omega|\beta_q \circ \phi_r([0,1])} \right)} \right) \\
\geq -\log W_q([\omega|\beta_q]) - t \log \left( \frac{\text{diam} \left( \phi_{\omega|\gamma_q}([0,1]) \right)}{\sum_{\tau \in \mathcal{R}_q} \text{diam} \left( \phi_{\omega|\beta_q \circ \phi_r([0,1])} \right)} \right).
\]

Clearly,

\[
-\log \text{diam} \left( \phi_{\omega|\gamma_q}([0,1]) \right) \geq S_{\gamma_q}(\psi)(\omega|\gamma_q) - \gamma_q\rho_{\gamma_q}
\]
Moreover,
\[
\sum_{\tau \in \mathcal{R}_q} \text{diam} (\phi_{\omega|\beta_q \circ r}([0,1])) \geq \sum_{\tau \in \mathcal{R}_q} \exp (-S_{\gamma_q} (\psi)(\omega|\beta_q, \tau))
\]
\[
\geq e^{-S_{\beta_q}(\psi)(\omega|\beta_q)} \sum_{\tau \in \mathcal{R}_q} e^{-S_{\lambda_q}(\psi)(\tau)}
\]
\[
\geq e^{-S_{\beta_q}(\psi)(\omega|\beta_q) - \lambda_q \rho \lambda_q} \sum_{\tau \in \mathcal{R}_q} \text{diam} (\phi_{\tau}([0,1]))
\]
\[
\geq e^{-S_{\beta_q}(\psi)(\omega|\beta_q) - \lambda_q \rho \lambda_q} \zeta r.
\]

Note that from the definition of $\beta_q$ and $c$ we have,
\[
- \log r \leq k - 1 (\beta_q - \gamma_q - 1) E(S_k(\varphi)) + c(\gamma_q - 1) + q
\]

Combining these inequalities we see that,
\[
- \log \mathcal{W}_q ([\omega|\gamma_q]) \geq t S_{\gamma_q} (\psi)(\omega|\gamma_q) - \gamma_q \rho \gamma_q - N(q)c - \frac{2\beta_q}{q}
\]
\[
- \frac{3\gamma_q - 1}{q - 1} - 2\gamma_q - 1 \rho \lambda_{q-1} - \lambda_q \rho \lambda_q - c(\gamma_q - 1) - q + \log \zeta
\]
\[
\geq t S_{\gamma_q} (\psi)(\omega|\gamma_q) - \frac{3\gamma_q}{q} - 2\gamma_q \rho \lambda_q,
\]
since $\gamma_q \geq \beta_q \geq \alpha_q$ and by the definition of $\alpha_q$,
\[
\alpha_q > q \left( \frac{3\gamma_q - 1}{q - 1} + 2\gamma_q - 1 \rho \lambda_{q-1} + c(\gamma_q - 1) + q - \log \zeta \right).
\]

We define a Borel measure $\mu$ by $\mu(A) := \mathcal{W}(S \cap \pi^{-1}(A))$ for Borel sets $A \subseteq [0,1]$.

**Lemma 5.6.** $\mu([0,1]) > 0$.

**Proof.** This follows immediately from the fact that
\[
\mathcal{W}(S) \geq \prod_{q \in \mathbb{N}} (1 - \delta_q) > 0.
\]

**Lemma 5.7.** For all $\omega \in S$ we have
\[
\liminf_{r \to 0} \frac{\log \mu(B(\pi(\omega), r))}{\log r} \geq t.
\]

**Proof.** For the proof of Lemma 5.7 we shall require some additional notation. Given a pair of functions $f$ and $g$, depending on $q \in \mathbb{N}$ and $r \in (0,1)$, we shall write,
\[
f(q,r) \geq g(q,r) - \eta(q,r),
\]
to denote that for each \( \epsilon > 0 \) there exists an \( N \in \mathbb{N} \) and a \( \delta > 0 \) such that given any \((q, r) \in \mathbb{N} \times (0, 1)\) with \( q > N \) and \( r < \delta \) we have

\[
(5.19) \quad f(q, r) \geq g(q, r) - \epsilon.
\]

Note that by (5.15) \( r_q < e^{-q} \) for all \( q \in \mathbb{N} \) and by Definition 5.1 this implies that \( \lim_{q \to \infty} \lambda_q = \lim_{q \to \infty} \lambda_{r_q} = \infty \) and hence \( \lim_{q \to \infty} \rho_{\lambda_q} = 0 \). Thus for any function \( g : \mathbb{N} \times (0, 1) \to \mathbb{R} \),

\[
g(q, r) - \rho_{\lambda_q} \geq g(q, r) - \eta(q, r).
\]

Similarly, it follows from the definition of \( \beta_q \) that

\[
g(q, r) - cN(q)N(q + 1)\beta_q^{-1} \geq g(q, r) - \eta(q, r).
\]

Firstly we show that for any \( x = \pi(\omega) \) with \( \omega \in S \) \( B(x, r) \) and \( r > 0 \) for which there exists \( q \in \mathbb{N} \) and \( l \in \mathbb{N} \) with \( \gamma_{q-1} \leq l < \beta_q \) such that

\[
B(x, r) \cap \pi(S) \subseteq \phi_{\omega|l}([0, 1]) \text{ but } B(x, r) \cap \pi(S) \nsubseteq \phi_{\omega|l+1}([0, 1])
\]

satisfies

\[
(5.20) \quad \frac{\log \mu(B(x, r))}{\log r} \geq t - \eta(q, r).
\]

Indeed, as \( B(x, r) \cap \pi(S) \subseteq \phi_{\omega|l}([0, 1]) \) it follows from Lemma 5.5 that,

\[
- \log \mu(B(x, r)) \geq - \log W([\omega|l])
= - \log W_q([\omega|l])
\geq tS_l(\psi)(\omega|l) - \frac{3\gamma_{q-1}}{q-1} - 2\gamma_{q-1}\rho_{\lambda_{q-1}} - \frac{2l}{q} - N(q)c
= - \log W_q([\omega|l])
\geq tS_l(\psi)(\omega|l) - \frac{6l}{q-1} - 2l\rho_{\lambda_{q-1}},
\]

since \( l \geq \gamma_{q-1} > qN(q)c \). Since \( S_l(\psi)(\omega|l) \geq l \log \xi \) this implies

\[
\frac{\log \mu(B(x, r))}{S_l(\psi)(\omega|l)} \geq t - \log \xi^{-1} \left( \frac{6}{q-1} + 2\rho_{\lambda_{q-1}} \right).
\]

However, \( B(x, r) \cap \pi(S) \nsubseteq \phi_{\omega|l+1}([0, 1]) \) and hence \( B(x, r) \cap \pi(S) \nsubseteq \phi_{\omega|l+1}(\kappa(l))([0, 1]) \) where \( \kappa(l) := k[k^{-1}(l + 1)] \). It follows that \( B(x, r) \cap \pi(S) \) intersects \( \phi_{x|\kappa(l)}([0, 1]) \), for some \( \tau \in S \), as well as \( \phi_{\omega|\kappa(l)}([0, 1]) \). Since \( \kappa(l) \leq \beta_q \) and \( \omega, \tau \in S \), \( (\kappa(l) - k)|\omega|, (\kappa(l) - k)|\tau| \kappa(l) \in \mathcal{B} \). Thus, by (5.10),

\[
r \geq \frac{1}{2} e^{-S_n(\psi)(\omega|\kappa(l) - k) - c}
\geq e^{-S_n(\psi)(\omega|l) - c - \log 2}.
\]

Thus,

\[
\frac{\log \mu(B(x, r))}{\log r} \geq \left( 1 + \frac{c + \log 2}{\log r} \right) \left( t - \log \xi^{-1} \left( \frac{6}{q-1} + 2\rho_{\lambda_{q-1}} \right) \right)
\]
which implies the first claim (5.20).

Secondly, we show that given \( \omega \in S, x \in [0, 1] \) and \( r > 0 \) for which \( B(x, r) \cap \pi(S) \subseteq \phi_{\omega|\beta_q}[0, 1] \) and yet \( B(x, r) \cap \pi(S) \not\subseteq \phi_{\omega|\beta_q} \circ \phi_T([0, 1]) \) for any \( \tau \in R_q \) we have,

\[
\frac{\log \mu(B(x, r))}{\log r} \geq t - \eta(q, r).
\]

From the proof of Lemma 5.5 we have,

\[
-\log \mathcal{W}_q([\omega|\beta_q]) \quad \geq \quad t \left( S_{\beta_q}(\psi)(\omega|\beta_q) + k^{-1}(\beta_q - \gamma_{q-1})E(S_k(\varphi)) \right) - \frac{3\gamma_{q-1}}{q-1} - 2\gamma_{q-1}\rho_{\lambda_{q-1}} - N(q)c - \frac{2\beta_q}{q} \\
- \log r_q \quad \leq \quad k^{-1}(\beta_q - \gamma_{q-1})E(S_k(\varphi)) + c(\gamma_{q-1} + 1) + q \\
\sum_{\tau \in R_q} \text{diam} \left( \phi_{\omega|\beta_q \circ \tau}([0, 1]) \right) \quad \geq \quad e^{-S_{\beta_q}(\psi)(\omega|\beta_q) - \lambda q \rho_{\lambda} \xi r_q}.
\]

Suppose \( r > r_q \). Then by the first two inequalities together with the fact that \( B(x, r) \subseteq \phi_{\omega|\beta_q}[0, 1] \) we have

\[
-\log \mu(B(x, r)) \quad \geq \quad -\log \mathcal{W}_q([\omega|\beta_q]) \\
\geq \quad -t \log r - \left( \frac{3\gamma_{q-1}}{q-1} + 2\gamma_{q-1}\rho_{\lambda_{q-1}} + N(q)c + \frac{2\beta_q}{q} + c(\gamma_{q-1} + 1) + q \right).
\]

Note also that \( B(x, r) \subseteq \phi_{\omega|\beta_q}[0, 1] \) implies \( -\log r > \beta_q \log \xi > \gamma_{q-1} \log \xi \) and hence,

\[
\frac{\log \mu(B(x, r))}{\log r} \quad \geq \quad t - \log \xi^{-1} \left( \frac{3}{q-1} + 2\rho_{\lambda_{q-1}} + \frac{N(q)c + c(\gamma_{q-1} + 1) + q}{\beta_q} + \frac{2}{q} \right) \geq t - \eta(q, r).
\]

Now suppose that \( r \leq r_q \) and let \( T \) denote the following collection,

\[
T := \left\{ \tau \in R_q : \frac{\text{diam} \left( \phi_{\omega|\beta_q} \circ \phi_T([0, 1]) \cap B(x, r) \right)}{\text{diam} \left( \phi_{\omega|\beta_q} \circ \phi_T([0, 1]) \right)} > \frac{1}{2} \right\}.
\]

We also define \( B_T(x, r) \subseteq B(x, r) \) by,

\[
B_T(x, r) := \bigcup_{\tau \in T} \phi_{\omega|\beta_q} \circ \phi_T([0, 1])
\]

From the definition of \( \mu \) and \( \mathcal{W} \) we see that for each \( \tau \in R_q \) we have,

\[
\mu(\phi_{\omega|\beta_q} \circ \phi_T([0, 1])) \quad \leq \quad \mathcal{W}_q([\omega|\beta_q], \tau) \\
\leq \quad \mathcal{W}_q([\omega|\beta_q]) \cdot \frac{\text{diam} \left( \phi_{\omega|\beta_q} \circ \phi_T([0, 1]) \right)}{\sum_{\tau \in R_q} \text{diam} \left( \phi_{\omega|\beta_q} \circ \phi_T([0, 1]) \right)}.
\]
Hence, as \( t < 1 \),
\[
\mu(B_T(x, r)) \leq \sum_{\tau \in \mathcal{T}} \mu(\phi_{\omega|\beta_q} \circ \phi_{\tau}([0, 1]))
\]
\[
\leq W_q ([\omega|\beta_q]) \frac{\sum_{\tau \in \mathcal{T}} \text{diam} (\phi_{\omega|\beta_q} \circ \phi_{\tau}([0, 1]))}{\sum_{\tau \in \mathcal{R}_q} \text{diam} (\phi_{\omega|\beta_q} \circ \phi_{\tau}([0, 1]))}
\]
\[
\leq W_q ([\omega|\beta_q]) \left( \frac{\sum_{\tau \in \mathcal{T}} \text{diam} (\phi_{\omega|\beta_q} \circ \phi_{\tau}([0, 1]))}{\sum_{\tau \in \mathcal{R}_q} \text{diam} (\phi_{\omega|\beta_q} \circ \phi_{\tau}([0, 1]))} \right)^t
\]
\[
\leq 2W_q ([\omega|\beta_q]) \left( \sum_{\tau \in \mathcal{R}_q} \text{diam} (\phi_{\omega|\beta_q} \circ \phi_{\tau}([0, 1])) \right)^{-t} r^t.
\]
Piecing the previous inequalities together with the observations from the proof of Lemma 5.5 we obtain
\[
- \log \mu (B_T(x, r))
\]
\[
\geq -t \log r - \left( \frac{3q-1}{q-1} + 2\gamma_{q-1}\rho_{\lambda_q-1} + N(q)c + \frac{2\beta_q}{q} + q - \log \zeta - \log 2 \right).
\]
Now \( \lambda_q < d \log r_q \leq d \log r \), where \( d < 0 \) is the constant as appears in the positive upper cylinder density condition. Hence,
\[
(5.22) \quad \frac{\log \mu (B_T(x, r))}{\log r}
\]
\[
\geq t - \log \xi^{-1} \left( \frac{3}{q-1} + 2\rho_{\lambda_q-1} + \frac{N(q)c + q - \log \zeta + \log 2}{\beta_q} + \frac{2}{q} \right) + d\rho_{\lambda_q}
\]
\[
\geq t - \eta(q, r).
\]
Consider the set \( \mathcal{C} := \{ \tau \in \mathcal{R}_q : \phi_{\omega|\beta_q} \circ \phi_{\tau}([0, 1]) \cap B(x, r) \neq \emptyset, \tau \notin \mathcal{T} \} \).
It is clear that \( \mathcal{C} \) contains at most two elements, with \( \phi_{\omega|\beta_q} \circ \phi_{\tau}([0, 1]) \) containing either \( \inf \) \( B(x, r) \) or \( \sup \) \( B(x, r) \). We shall show that for \( \tau \in \mathcal{C} \) we have,
\[
(5.23) \quad \frac{\log \mu (\phi_{\omega|\beta_q} \circ \phi_{\tau}([0, 1]) \cap B(x, r))}{\log r} \geq t - \eta(q, r).
\]
Take \( \tau \in \mathcal{C} \) and assume that \( \sup \) \( B(x, r) \) \( \phi_{\omega|\beta_q} \circ \phi_{\tau}([0, 1]) \) ie. \( \phi_{\omega|\beta_q} \circ \phi_{\tau}([0, 1]) \) intersects the right hand boundary of \( B(x, r) \). Since \( \tau \notin \mathcal{T} \) we have
\[
\text{diam} (\phi_{\omega|\beta_q} \circ \phi_{\tau}([0, 1]) \cap B(x, r)) < \frac{1}{2} \text{diam} (\phi_{\omega|\beta_q} \circ \phi_{\tau}([0, 1])).
\]
Choose \( \tilde{\omega} \in S \) such that \( \pi (\tilde{\omega}) \) is on the right hand side of \( \phi_{\omega|\beta_q} \circ \phi_{\tau}([0, 1]) \cap B(x, r) \cap \pi (S) \).
Define \( \tilde{r} := |\pi (\tilde{\omega}) - \inf (\phi_{\omega|\beta_q} \circ \phi_{\tau})([0, 1])|, \) and consider \( B (\pi (\tilde{\omega}), \tilde{r}) \). Since \( \pi (\tilde{\omega}) \) is on the right hand side of \( (\phi_{\omega|\beta_q} \circ \phi_{\tau})([0, 1]) \cap B(x, r) \cap \pi (S) \) and
\[
\text{diam} (\phi_{\omega|\beta_q} \circ \phi_{\tau}([0, 1]) \cap B(x, r)) < \frac{1}{2} \text{diam} (\phi_{\omega|\beta_q} \circ \phi_{\tau}([0, 1])),
\]
we have
\[(\phi_{\omega}|_{\beta_q} \circ \phi_{r})([0, 1]) \cap B(x, r) \cap \pi(S) \subseteq B(\pi(\tilde{\omega}), \tilde{r}) \subseteq (\phi_{\omega}|_{\beta_q} \circ \phi_{r})([0, 1])\]
and \(\tilde{\omega}|_{\gamma_q} = (\omega|_{\beta_q}, \tau).

We consider two cases. First suppose that \(B(\pi(\tilde{\omega}), \tilde{r}) \subseteq \phi_{\omega}|_{\beta_{q+1}}([0, 1])\). It follows from Lemma 5.5 that,
\[-\log \mu (B(\pi(\tilde{\omega}), \tilde{r})) \geq -\log \mathcal{W}_{q+1} ([\tilde{\omega}|_{\beta_{q+1}}]) \geq t \left(S_{\beta_{q+1}}(\psi)(\omega|_{\beta_{q+1}}) + k^{-1}(\beta_{q+1} - \gamma_{q-1}) \exp(S_k(\varphi))\right) - \frac{3\gamma_q}{q} - 2\gamma_q \rho \lambda - \frac{2\beta_{q+1}}{q + 1} - N(q + 1)c \geq t \beta_{q+1} \log \xi - \left(k \log \xi + cN(q + 1) + \frac{5\beta_{q+1} + 2\beta_{q+1}\rho \lambda}{q}\right).

Hence,
\[-\log \mu \left((\phi_{\omega}|_{\beta_q} \circ \phi_{r})([0, 1]) \cap B(x, r)\right)_{\beta_{q+1}} \geq t \log \xi^{-1} \left(\frac{k \log \xi + cN(q + 1)}{\beta_{q+1}} + \frac{5}{q} + 2\rho \lambda\right).

Since \(B(x, r) \cap \pi(S) \not\subset (\phi_{\omega}|_{\beta_q} \circ \phi_{r})([0, 1])\) for any \(r' \in R_q\), it follows from (5.10) that
\[-\log r \leq -\max \left\{S_{\gamma_q}(\psi)(r') : r' \in \Gamma_q\right\} - c \leq \alpha_{q+1} \log \xi < \beta_{q+1} \log \xi.

Thus,
\[\frac{\log \mu \left((\phi_{\omega}|_{\beta_q} \circ \phi_{r})([0, 1]) \cap B(x, r)\right)}{\log r} \geq t - \log \xi^{-1} \left(\frac{k \log \xi + cN(q + 1)}{\beta_{q+1}} + \frac{5}{q} + 2\rho \lambda\right) \geq t - \eta(q, r).

Now suppose that \(B(\pi(\tilde{\omega}), \tilde{r}) \subset \phi_{\omega}|_{\beta_{q+1}}([0, 1])\). Then we may apply (5.20) to obtain
\[\frac{\log \mu (B(\pi(\tilde{\omega}), \tilde{r}))}{\log \tilde{r}} \geq t - \eta(q + 1, \tilde{r}).
\]

Clearly \(\tilde{r} < 2r\) and so \(\lim_{r \to \infty} \frac{\log \tilde{r}}{\log r} \geq 1\) and hence,
\[\frac{\log \mu \left((\phi_{\omega}|_{\beta_q} \circ \phi_{r})([0, 1]) \cap B(x, r)\right)}{\log r} \geq t - \eta(q, r).
\]

By symmetry the same holds if \(\phi_{\omega}|_{\beta_q} \circ \phi_{r}([0, 1])\) intersects the left hand boundary of \(B(x, r)\). This proves the claim (5.23).

Recall that,
\[B(x, r) \cap \pi(S) \subseteq B_\tau(x, r) \cup \left(\bigcup_{\tau' \in C} (\phi_{\omega}|_{\beta_q} \circ \phi_{r})([0, 1]) \cap B(x, r)\right).
\]
Noting that \( \#C \leq 2 \) we obtain,
\[
\mu(B(x,r)) \leq \mu(B_T(x,r)) + \sum_{\tau \in C} \mu((\phi_{\omega|\beta_q} \circ \phi_r)([0,1]) \cap B(x,r)) \\
\leq 3 \max \{ \mu(B_T(x,r)) \} \cup \{ \mu((\phi_{\omega|\beta_q} \circ \phi_r)([0,1]) \cap B(x,r)) : \tau \in C \}.
\]
By combining with (5.22) and (5.23),
\[
\frac{\log \mu(B(x,r)) - \log 3}{\log r} \geq t - \eta(q,r),
\]
which implies (5.21).

To complete the proof of the Lemma we fix \( \omega \in S \), let \( x = \pi(\omega) \) and consider a ball \( B(\pi(\omega),r) \) of radius \( r > 0 \). Now choose \( q(r) \in N \) so that \( B(x,r) \cap \pi(S) \subseteq \phi_{\omega|\beta_q(r)^{-1}}([0,1]) \) but \( B(x,r) \cap \pi(S) \nsubseteq \phi_{\omega|\beta_q(r)}([0,1]) \).

Now either \( B(x,r) \cap \pi(S) \nsubseteq \phi_{\omega|\beta_q(r)}([0,1]) \), in which case we apply (5.20) or \( B(x,r) \cap \pi(S) \nsubseteq \phi_{\omega|\beta_q(r)}([0,1]) \) in which case we apply (5.21). In both cases we obtain,
\[
(5.26) \quad \frac{\log \mu(B(x,r))}{\log r} \geq t - \eta(q,r).
\]
By (5.24) whenever \( q(r) \leq Q \) we have
\[
r \geq \exp \left(-\max \{ S_{\gamma_Q}(\psi)(\tau') : \tau' \in \Gamma_Q \} - c \right) > 0.
\]
Hence, \( \lim_{r \to 0} q(r) = \infty \). Therefore, by (5.26) we have
\[
(5.27) \quad \liminf_{r \to 0} \frac{\log \mu(B(\pi(\omega),r))}{\log r} \geq t.
\]
□

To complete the proof of Proposition 5.1 we recall the following standard Lemma.

**Lemma 5.8.** Let \( \nu \) be a finite Borel measure on some metric space \( X \). Suppose we have \( J \subseteq X \) with \( \nu(J) > 0 \) such that for all \( x \in J \)
\[
\liminf_{r \to 0} \frac{\log \nu(B(x,r))}{\log r} \geq d.
\]
Then \( \dim_H J \geq d \).

**Proof.** See [P2] Proposition 2.2] . □

Thus by Lemmas 5.7 and 5.6 we have
\[
\dim_H \pi(S) \geq t > s.
\]
Hence, by Lemma 5.4 the Hausdorff dimension of \( D_{\psi}(\varphi) \) is at least \( s \). Since this for all \( s < s(\varphi) \), we have
\[
\dim_H D_{\psi}(\varphi) \geq s(\varphi).
\]
□
6. Proof of Theorem 5

Proof of Theorem 5. We begin by defining a sequence \((r_n)_{n \in \mathbb{N}}\) by

\[
(6.1) \quad r_n := \min \left\{ \left( \frac{2 + \sum_{q \in \mathbb{N}} e^{-q/n}}{e^{-2n^2} \cdot e^{-\frac{1}{2} (\Phi(n) - \Phi(n+1))}} \right)^{-n^2}, \frac{1}{2} \right\}
\]

Note that since \(\Phi\) is strictly decreasing each \(r_n > 0\). Now take \(n_0 > 2\) so that \(\Phi(n_0) < \left(1 - 2^{1-\beta^{-1}}\right)\) and \(\sum_{n \geq n_0} e^{-\beta n} < 1\). For each \(n \geq n_0\) we choose some closed interval \(V_n \subset (\Phi_{n+1}, \Phi_n)\) of length \(r_n\), which is always possible, since \(r_n < \Phi(n) - \Phi(n+1)\). Note that since each \(r_n < e^{-n}\) we have \(\sum_{n \geq n_0} r_n^\beta \leq \sum_{n \geq n_0} e^{-\beta n} < 1\). Hence, \(r_1 = r_2 := 2^{-\beta^{-1}} \left(1 - \sum_{n \geq n_0} r_n^\beta\right)^{\beta^{-1}} > 0\). Note also that \(1 - \Phi(n_0) > 2^{1-\beta^{-1}} > 2r_1\).

Thus, we may choose two disjoint closed intervals \(V_1, V_2\) of width \(r_1 = r_2\) contained within \((\Phi(n_0), 1)\).

We now let \(A := \{n \in \mathbb{N} : n \geq n_0\} \cup \{1, 2\}\). Define \(T : \bigcup_{n \in A} V_n \to [0, 1]\) to be the unique expanding Markov map which maps each of the intervals \(\{V_n\}_{n \in A}\) onto \([0, 1]\) in an affine and orientation preserving way. First note that,

\[
(6.2) \quad \sum_{n \in A} \text{diam}(V_n)^\beta = r_1^\beta + r_2^\beta + \sum_{n \geq n_0} r_n^\beta = 1.
\]

Thus, \(\dim_{\mathcal{H}} \Lambda = \beta\) by Moran’s formula.

Take \(n \geq n_0\) and consider \(S_0^{(n)}(\Phi) := \{x \in \Lambda : |T^n(x)| < \Phi(n)\}\). Since \(T\) is orientation preserving it follows from the construction of \(T\) that we can cover \(S_n(\Phi)\) with sets of the form \(V_\omega = \bigcap_{j=0}^{n-1} T^{-j} V_\omega\) where \(\omega \in C_n := \{\omega \in A^{n+1} : \omega_{n+1} \geq n\}\). Since \(T\) is piecewise linear we have \(\text{diam} V_\omega = \prod_{j=1}^{n+1} r_{\omega_j}\) for each \(\omega \in A^{n+1}\). It follows that for any \(m > n_0\) we may cover \(S_0(\Phi)\) with the family \(\bigcup_{n \geq m} \{V_\omega : \omega \in C_n\}\).
Now take $\epsilon > 0$. For all $n > \epsilon^{-1}$ we have,

$$\sum_{\omega \in C_n} (\text{diam} V_{\omega})^\epsilon \leq \sum_{\omega \in C_n} (r_{\omega_1} \cdots r_{\omega_n})^\epsilon$$

$$= \left( \sum_{n \in A} r_n^\epsilon \right)^n \cdot \sum_{q \geq n} r_q^\epsilon$$

$$\leq \left( 2 + \sum_{q \in \mathbb{N}} e^{-sq} \right)^n \cdot \sum_{k \geq n} \left( 2 + \sum_{q \in \mathbb{N}} e^{-q/k} \right)^{-k^2} \cdot e^{-2k^2}$$

$$\leq \left( 2 + \sum_{q \in \mathbb{N}} e^{-sq} \right)^n \cdot \left( 2 + \sum_{q \in \mathbb{N}} e^{-q/n} \right)^{-n^2 \epsilon} \cdot \sum_{k \geq n} e^{-2kn\epsilon}$$

$$\leq \left( 2 + \sum_{q \in \mathbb{N}} e^{-sq} \right)^n \cdot \left( 2 + \sum_{q \in \mathbb{N}} e^{-q/n} \right)^{-n^2} \cdot \sum_{k \geq n} e^{-k}$$

$$\leq e^{-n^2} \sum_{k \in \mathbb{N}} e^{-k}.$$

Thus, for all $m > \epsilon^{-1}$ we have,

$$\sum_{n \geq m} \sum_{\omega \in C_n} (\text{diam} V_{\omega})^\epsilon \leq \sum_{n \geq m} e^{-n^2} \sum_{k \in \mathbb{N}} e^{-k} \leq \left( \sum_{k \in \mathbb{N}} e^{-k} \right)^2 < \infty.$$

Since $\lim_{m \to \infty} \sup \{ \text{diam} V_{\omega} : \omega \in C_m \} = 0$ it follows that $\dim_H S_0(\Phi) < \epsilon$. As this holds for all $\epsilon > 0$ we have $\dim_H S_0(\Phi) = 0$. □

We note that by Corollary 2 $s(\alpha) > 0$ for all $\alpha \in \mathbb{R}_{>0}$.

7. Remarks

Both Theorems 3 and 4 may be extended in a number of ways with some minor alterations of the proof.

Given $\Phi : \mathbb{N} \times \Lambda \to (0, 1)$ we define

$$S_y(\Phi) := \bigcap_{m \in \mathbb{N}} \bigcup_{n \geq m} \{ x \in \Lambda : |T^n(x) - y| < \Phi(n, x) \}.$$

Theorems 3 and 4 both deal with the case where $\Phi$ is multiplicative, ie. $\Phi(n + m, x) = \Phi(n, T^m(x)) \cdot \Phi(m, x)$, for all $n, m \in \mathbb{N} \cup \{0\}$ and $x \in \Lambda$.

Indeed, when $\Phi$ is multiplicative, we may take $\varphi : x \mapsto -\log \Phi(0, x)$ so that $\Phi(n, x) = \exp(-S_n(\varphi)(x))$ and $S_y(\Phi) = D_y(\varphi)$.

We say that $\Phi$ is almost multiplicative if there exists some constant $C > 1$ such that,

$$C^{-1} < \frac{\Phi(n, T^m(x)) \cdot \Phi(m, x)}{\Phi(n + m, x)} < C,$$
for all $n, m \in \mathbb{N}$ and $x \in \Lambda$. Examples include the norms of certain matrix products (see [FL, IY]). Given $\omega \in \mathcal{A}^n$ we let $\Phi(\omega) := \sup \{ \Phi(n, x) : x \in V_\omega \}$. Following Feng and Lau [FL] one may define a pressure function, $P(s, \Phi) \to \mathbb{R}$ by

$$P(s, \Phi) := \lim_{n \to \infty} \frac{1}{n} \sum_{\omega \in \mathcal{A}^n} (\Phi(\omega) \cdot ||\psi'_\omega||_\infty)^s,$$

and let $s(\Phi) := \inf \{ s : P(s, \Phi) \leq 0 \}$. Technical modifications to the proof of Theorems 3 and 4 show that whenever $T$ is a countable Markov map and $\Phi$ is almost multiplicative, $\dim_{\mathcal{H}} S_y(\Phi) = s(\Phi)$ for all $y \in \Lambda$, and if $\overline{\Lambda} = [0, 1]$ then $\dim_{\mathcal{H}} S_y(\Phi) = s(\Phi)$ for all $y \in \overline{\Lambda}$.

Instead of considering the sets $D_y(\varphi)$ we can consider sets of the form,

$$L_y(\varphi) := \left\{ x \in \Lambda : \limsup_{n \to \infty} \frac{\log d(T^n(x), y)}{S_n(\varphi)(x)} = -1 \right\}.$$

When $T$ is a countable Markov map we have $\dim_{\mathcal{H}} L_y(\varphi) = \dim_{\mathcal{H}} D_y(\varphi) = s(\varphi)$ for all $y \in \Lambda$ and when $T$ is a countable Markov map satisfying $\overline{\Lambda} = [0, 1]$ we have $\dim_{\mathcal{H}} L_y(\varphi) = \dim_{\mathcal{H}} D_y(\varphi) = s(\varphi)$ for all $y \in [0, 1]$. To prove the upper bound we note that $L_y(\varphi) \subset \dim_{\mathcal{H}} D_y((1 - \delta)\varphi)$ for all $\delta \in (0, 1)$ and $\lim_{\delta \to 0} \dim_{\mathcal{H}} D_y((1 - \delta)\varphi) = \lim_{\delta \to 0} s((1 - \delta)\varphi) = s(\varphi)$. To prove the lower bound requires a technical adaptation of the proof of Proposition 5.1 removing those points $x$ for which $T^n(x)$ moves too close to $y$.

One can also consider what happens when we replace assumption (1) in Definition 2.1 with the weaker assumption that $T$ is modelled by a subshift of finite type. If the corresponding matrix is finitely primitive (see [MU2, Section 2.1]) then one may adapt the proofs of Theorems 3 and 4 with only minor modifications. However, to determine the dimension of $D_y(\varphi)$ for an arbitrary countable subshift of finite type would require further innovation.

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