Equivariant asymptotics for Bohr-Sommerfeld
Lagrangian submanifolds

Marco Debernardi (1) and Roberto Paoletti (2) *

1 Introduction

Let $M$ be an $n$-dimensional complex projective manifold, and let $\Omega$ be a
Hodge form on it. Let $(L,h)$ be an Hermitian ample line bundle on $M$,
such that $-2\pi i \Omega$ is the curvature of its unique compatible connection. Let
$L^* \supseteq X \overset{\pi}{\to} M$ be the unit circle bundle in the dual line bundle, and denote
by $\alpha \in \Omega^1(X)$ the normalized connection 1-form on $X$. Thus $\alpha$ is a contact
form on $X$ satisfying $d\alpha = \pi^* (\Omega)$.

The compact Legendrian submanifolds of $X$ play an important role in
the theory of geometric quantization. An immersed Lagrangian submanifold
$\iota : \Lambda \to M$ lifts to an immersed Legendrian submanifold $\tilde{\iota} : \Lambda \to X$
if and only if there exists a non vanishing covariantly constant section of
$\iota^*(L)$. Thus, the Legendrian submanifolds of $X$ determine by projection
distinguished immersed Lagrangian submanifolds of $(M, \Omega)$, so-called Bohr-
Sommerfeld Lagrangian submanifolds. Roughly speaking, from a semiclassical
point of view these (rather than points in the phase space $(M, \Omega)$) are the
geometric counterparts to physical states. Consequently, the quantization of
Bohr-Sommerfeld Lagrangian submanifolds has been an important line of
research in symplectic geometry (see for example [GS3], [BPU], [GT], [BW]
and references therein).

In particular, a systematic procedure for quantizing Bohr-Sommerfeld La-
grangian submanifolds has been developed by Borthwick, Paul and Uribe in
[BPU]. In short, the choice of a half-form $\lambda$ on $\Lambda$ determines a generalized
half-form on $X$ supported on $\Lambda$, essentially the delta-function determined by

*Address. (1): Dipartimento di Matematica F. Casorati, Via Ferrata 1, Università
di Pavia, 27100 Pavia, Italy; (2): Dipartimento di Matematica e Applicazioni, Università
degli Studi di Milano Bicocca, Via R. Cozzi 53, Edificio U5, 20126 Milano, Italy; e-mail:
marco.debernardi@unipv.it, roberto.paoletti@unimib.it
by applying the Szegö kernel to the latter, and taking Fourier components, one then naturally associates to \((\Lambda, \lambda)\) a sequence of holomorphic sections of \(L^\otimes k\), \(u_k \in H^0(M, L^\otimes k)\), for every \(k = 0, 1, 2, \ldots\). The theory of \[BPU\] describes how the local geometry of \((\Lambda, \lambda)\) captures the pointwise asymptotic properties of the sequence \(u_k\).

In this article, we shall suppose given in addition the holomorphic action of a \(g\)-dimensional connected compact Lie group \(G\) on \(M\), Hamiltonian with respect to \(\Omega\). We shall assume that \(0 \in g^*\) is a regular value for the moment map \(\Phi : M \to g^*\); here \(g\) denotes the Lie algebra of \(G\). We shall also suppose that \(L\) is an ample \(G\)-line bundle and that the Hermitian metric \(h\) on \(L\) is \(G\)-invariant.

We recall that, up to topological obstructions, the existence of a linearization amounts to the existence of a moment map \([K\), \([GSI\), §3, and \([GGK\), chapter VI]. More precisely, in the presence of a linearization one recovers a moment map by pairing the connection form on \(X\) with the infinitesimal action of the Lie algebra \(g\). Conversely, to a moment map there is associated an infinitesimal action of \(g\) on \(L\); more precisely, \(\xi \in g\) acts on sections of \(L\) by the operator \(\nabla_{\xi M} + 2\pi i\Phi^\xi\), where \(\nabla\) is the covariant derivative associated to the connection, \(\xi M\) is the vector field on \(M\) generated by \(\xi\), and \(\Phi^\xi = \langle \Phi, \xi \rangle : M \to \mathbb{R}\). The obstruction to extend the infinitesimal action of \(g\) to an action of \(G\) is of topological nature, and the extension certainly exists if \(G\) is simply connected.

In this situation, every space of global holomorphic sections \(H^0(M, L^\otimes k)\) admits a \(G\)-equivariant unitary decomposition over the irreducible representations of \(G\):

\[
H^0(M, L^\otimes k) = \bigoplus_{\varpi} H^0(M, L^\otimes k)_\varpi.
\]

Here, \(\varpi\) runs over the set of highest weights, and thus indexes all finite dimensional irreducible representations \(V_{\varpi}\) of \(G\); for every \(\varpi\), the summand \(H^0(M, L^\otimes k)_\varpi\) is \(G\)-equivariantly isomorphic to a direct sum of finitely many copies of \(V_{\varpi}\).

In particular, if \(u_k \in H^0(M, L^\otimes k)\) is the sequence associated to the pair \((\Lambda, \lambda)\), we have for every \(k = 0, 1, 2, \ldots\) a decomposition \(u_k = \bigoplus_{\varpi} u_{k,\varpi}\), where \(u_{k,\varpi} \in H^0(M, L^\otimes k)_\varpi\). We shall investigate the asymptotic properties of the sequence \(u_{k,\varpi}\), for \(\varpi\) fixed and \(k \to +\infty\). Naturally enough, these are governed by the mutual position of \(\Lambda\) and the zero locus of the moment map, \(\Phi^{-1}(0) \subseteq M\).

For example, when \(G\) is semi-simple and \(\Lambda\) covers a Lagrangian submanifold \(\pi(\Lambda) \subseteq M\), \(\Lambda\) is \(G\)-invariant if and only if \(\pi(\Lambda) \subseteq \Phi^{-1}(0)\) \[GSI\]. If we choose, as we may after averaging, \(\lambda\) itself to be \(G\)-invariant, then so will be each \(u_k\); therefore, \(u_{k,\varpi} = 0\) for every \(\varpi \neq 0\) and \(k \in \mathbb{N}\).
We shall assume instead that $\Lambda$ is transversal to $\Phi^{-1}(0)$; this geometric hypothesis implies a nontrivial decomposition over the irreducibles of $G$.

Incidentally, we remark that any given compact Legendrian submanifold $\Lambda \subseteq X$ may be deformed into one transversal to $\Phi^{-1}(0)$, by a contactomorphism arbitrarily close to the identity. To see this, for some integer $r \geq 1$ let us choose Hamiltonian vector fields $V_1, \ldots, V_r$ on $M$, such that for every $m$ in an open neighbourhood $U$ of $\pi(\Lambda)$ one has $T_mM = \text{span}\{V_1(m), \ldots, V_r(m)\}$. For every $i = 1, \ldots, r$, let $\psi_i : \mathbb{R} \to \text{Diff}(M)$ be the one-parameter group of symplectomorphisms generated by $V_i$.

Theorem [GP] then implies that we can find arbitrarily small $\{t \in \mathbb{R} \mid |t| < \delta\}$, by a contactomorphism, such that $\psi_i(\Lambda) = \psi_i(\Lambda)$ is transversal to $\Phi^{-1}(0)$. For every $i = 1, \ldots, r$, there exist vector fields $\tilde{V}_i$ on $X$ which are $\pi$-related to the $V_i$'s (i.e., the horizontal component of $\tilde{V}_i$ is the horizontal lift of $V_i$ to $X$), for every $i = 1, \ldots, r$, and which generate a one-parameter group of contactomorphisms $\tilde{\psi}_i : \mathbb{R} \to \text{Diff}(X)$ ($[\Omega]$, §4, and $[\text{Ge}]$, Theorem 2.2). Since $\tilde{\psi}_i(t)$ covers $\psi_i(t)$ for every $i$ and $t \in \mathbb{R}$, $\Lambda_\perp =: \tilde{\psi}_1(t_1) \circ \cdots \circ \tilde{\psi}_r(t_r)(\Lambda)$ is a Legendrian submanifold of $X$ transversal to $\Phi^{-1}(0)$.

Here is an explicit example:

**Example 1.1.** Endow $\mathbb{P}^1$ with the Fubini-Study metric, so that $L$ is the hyperplane bundle. Then $X$ is the unit sphere $S^3 \subseteq \mathbb{C}^2$, with projection $\pi : S^3 \to \mathbb{P}^1$ given by the Hopf map. Fix $e^{ia} \in S^1 (-\pi \leq a \leq \pi)$. Let $\iota_a : S^1 \to \mathbb{C} \oplus \mathbb{C}$ be given by $\iota_a(e^{i\theta}) = (\cos(\theta), e^{ia} \sin(\theta))$. Then $\iota(S^1) \subseteq S^3$ is a Legendrian knot for the standard contact structure, and may therefore be viewed as a Bohr-Sommerfeld immersed Lagrangian submanifold of $\mathbb{P}^1$.

Let us consider the Hamiltonian action of $S^1$ on $\mathbb{P}^1$ given by $t \cdot [z_0 : z_1] := [t z_0 : t^{-1} z_1]$, with moment map $\Phi([z_0 : z_1]) = (|z_0|^2 - |z_1|^2)/|z|^2$ (we use $\cdot$ rather than $\circ$ to distinguish the action from the ordinary one given by scalar multiplication). In affine coordinates, $\iota_a(S^1)$ covers the line through the origin of slope $\tan(a)$, and $\Phi^{-1}(0)$ is the unit circle centered at the origin. This example may be generalized in any dimension.

One motivation for studying this problem comes from the following natural question: Let us set

$$M' =: \Phi^{-1}(0) \subseteq M \quad \text{and} \quad M_0 =: M'/G. \quad (1)$$

Thus, $M_0$ is the GIT quotient of $M$ with respect to the action of the complexification $\tilde{G}$ of $G$, and $(L, h, \Omega)$ descend to corresponding orbi-objects...
If we set
\[ X' =: \pi^{-1}(M') \subseteq X \quad \text{and} \quad X_0 =: X'/G, \]
then \( X_0 \) is the circle orbi-bundle of the Hermitian line orbi-bundle \( (L_0^*, h_0) \). Let us momentarily suppose to fix ideas that \( G \) acts freely on \( M' \), so that \( (M_0, \Omega_0) \) is a Kähler manifold, and \( L_0 \) a honest ample line bundle on it. Now if \( \Lambda \subseteq X \) is a (half-weighted) Legendrian submanifold transversal to \( M' \), the intersection \( \Lambda' = \Lambda \cap X' \) determines by projection an immersed (half-weighted) Legendrian submanifold \( \Lambda_0 \rightarrow X_0 \), which we may think of as the reduction of \( \Lambda \). We thus have corresponding half-forms \( u \) on \( X \) and \( u_0 \) on \( X_0 \) in the images of the respective Szegő projectors; taking Fourier components we obtain sequences
\[ u_k \in H^0(M, L^{\otimes k}) \quad \text{and} \quad (u_0)_k \in H^0(M_0, L_0^{\otimes k}). \]
On the other hand, it is well-known that for \( k = 0, 1, 2, \ldots \) there is a natural isomorphism \( H^0(M, L^{\otimes k})^G \cong H^0(M_0, L_0^{\otimes k}) \) [GS1]. One is then led to ask whether, under the latter isomorphism, \( (u_0)_k = u_{k,0} \), at least in some asymptotic sense. More pictorially, does the principle quantization commutes with reduction also hold for the single (transverse, semiclassical) state? To leading order, the relation between the two sequences is governed by the effective potential of the action, defined as the function \( V_{\text{eff}} \) on \( M' \) associating to every \( p \in M' \) the volume of its \( G \)-orbit [BG], and a measure of the mutual position between \( \Lambda \) and the \( G \)-orbit at a given point (another appearance of the effective potential in equivariant asymptotics is described in [P2]). Although \( (u_0)_k \) and \( u_{k,0} \) have the same order of growth, the answer to the question above is negative (see Remark 3.3).

Following Corollary 4.1, we shall also make some remarks regarding the case where \( \Lambda \) is \( G \)-invariant. Some general introductory remarks are in order.

First, while we have followed the general philosophy of Borthwick, Paul and Uribe, we have based our approach on the parametrix for the Szegő kernel constructed by Boutet de Monvel and Sjöstrand in [BS], rather than on the theory of Fourier-Hermite distributions and symplectic spinors as in [BPU]. This follows the approach to equivariant asymptotics already used in [P1], and is inspired by the study of algebro-geometric Szegő kernels by Zelditch in [Z] and its subsequent developments, as in [BSZ], [STZ], [SZ] (in [STZ], in particular, the authors work out scaling asymptotics for toric eigenfunctions). We shall then deal with half-densities, rather than half-forms, on the given Legendrian submanifolds.
We have furthermore made extensive use of the notion of Heisenberg local coordinates introduced in [SZ], for this makes the relation between the local geometry of \( \Lambda \) and the leading term in the asymptotic expansions particularly explicit and simple to express.

Thus, even in the action-free case, our proofs and statements depart somewhat from the corresponding ones in [BPU].

Finally, we have focused on the case of complex projective manifolds. However, given the microlocal description of almost-complex Szegö kernels given in [SZ], our arguments can be extended to the symplectic almost complex category.

Our statements are best expressed by viewing sections of \( L^\otimes k \) as equivariant functions on \( X \). Given that \( \alpha \) and \( \Omega \) endow \( X \) with a \( G \)-invariant volume form, functions and half-densities on \( X \) may be equivariantly and unitarily identified.

Briefly, let \( \mathcal{H}(X) \subseteq C^\infty(X) \) be the Hardy space, and let \( \mathcal{H}(X)_k \) be the \( k \)-th isotype for the \( S^1 \)-action. Then there is a well-known canonical unitary isomorphism
\[
\mathcal{H}(X)_k \cong H^0(M, L^\otimes k),
\]
and we shall use the same symbol for the holomorphic sections and the corresponding equivariant functions. Now, if \( \Lambda \subseteq X \) is a compact Legendrian submanifold, the choice of a smooth half-density \( \lambda \) on it determines a generalized half-density \( \delta_{\Lambda, \lambda} \) on \( X \). Applying the Szegö projector to \( \delta_{\Lambda, \lambda} \), and taking Fourier components, we obtain as before equivariant functions \( u_{k, \varpi} \), for every integer \( k \) and highest weight \( \varpi \). Our key result concerns the asymptotic expansion for an appropriate scaling limit of the sequence \( u_{k, \varpi} \).

More precisely, suppose that \( x \in X \) and \( w \in T_m M \), where \( m = \pi(x) \). We shall often implicitly identify \( T_m M \) with the horizontal tangent space at \( x \) determined by the connection, \( H_x(X/M) \subseteq T_x X \). If \( x \in \Lambda \), the tangent space \( T_x \Lambda \) may be viewed as a Lagrangian subspace of \( T_m M \). Inspired by the results on the scaling limits of Szegö kernels in [BSZ] and [SZ], we shall investigate the asymptotic behavior of \( u_{k, \varpi}(x + w/\sqrt{k}) \), for \( k \to +\infty \) and as \( w \) varies in \( T_m M \). The point \( x + w/\sqrt{k} \) is only well-defined up to the choice of a coordinate system near \( x \), and the ambiguity is \( O(k^{-1}) \); the leading order part of the asymptotic expansion in Theorem 1.1 below is then independent of the choice of local coordinates. For concreteness, we shall at any rate assume that some system of local Heisenberg coordinates has been fixed.

Let us introduce some further pieces of notation. Here orthogonality refers to the standard Euclidean structure on \( \mathbb{C}^n = \mathbb{R}^n \oplus \mathbb{R}^n \).

**Definition 1.1. i):** If \( (h, g) \in S^1 \times G \) and \( x \in X \), let
\[
d_x(h, g) : T_x X \to T_{(h,g)} x X
\]
be the differential of the action.

ii): As above, set $\Lambda' =: \Lambda \cap X'$. Let $H(X/M)_{\Lambda'}$ be the restriction to $\Lambda'$ of the horizontal tangent bundle; thus, $H(X/M)_{\Lambda'}$ has fiber at $x \in \Lambda'$ given by $T_{\pi(x)}M$.

iii): For $m \in M$, let $G_m \subseteq G$ be the stabilizer subgroup of $m$. If $0 \in g^*$ is a regular value of $\Phi$, then $G$ acts locally freely on $M'$; therefore, $G_m$ is a finite subgroup of $G$ for every $m \in M'$.

iv): If $x \in X$ and $m \in M$ let us denote by
\[ g_X(x) = T_x(G \cdot x) \subseteq T_x(X) \quad \text{and} \quad g_M(m) = T_m(G \cdot m) \subseteq T_m(M) \quad (3) \]
the tangent spaces to the respective $G$-orbits. Now given that the $G$-action is horizontal on $X'$, for $x \in X'$ we have a natural identification
\[ g_X(x) \cong g_M(\pi(x)) \subseteq H_x(X/M). \quad (4) \]
It is well-known that if $m \in M'$ then $g_M(m)$ is a $g$-dimensional isotropic subspace of $T_m M$ [GSI].

v): To leading order, we shall see that only the component of $w$ in a certain $n$-dimensional real vector subspace $\bar{N}_\Lambda(x) \subseteq T_{\pi(x)}M$ contributes to $|u_{k,\omega}(x + w/\sqrt{k})|$. More precisely, recall that $T_x\Lambda$ may be viewed as a Lagrangian subspace of $T_{\pi(x)}M$. Now if $\Lambda$ is transversal to $X'$, then
\[ T_x\Lambda \cap g_X(x) = \{0\} \]
for every $x \in \Lambda'$ (Corollary 2.1). Thus, if $x \in \Lambda'$ there is a direct sum decomposition
\[ T_{\pi(x)}M = (T_x\Lambda + g_M(\pi(x)))^\perp \oplus (T_x\Lambda + g_M(\pi(x))) \]
\[ \cong [(T_x\Lambda + g_M(\pi(x)))^\perp \oplus (T_x\Lambda^\perp \cap T_x\Lambda) \oplus [T_x\Lambda' \oplus g_M(\pi(x))]]. \quad (5) \]
If $x \in \Lambda'$, we shall then set
\[ \bar{N}_\Lambda(x) =: (T_x\Lambda + g_M(\pi(x)))^\perp \oplus (T_x\Lambda^\perp \cap T_x\Lambda); \quad (6) \]
\[ \bar{T}_\Lambda'(x) =: T_x\Lambda' + g_M(\pi(x)). \quad (7) \]
Thus, $T_{\pi(x)}M = \bar{N}_\Lambda(x) \oplus \bar{T}_\Lambda'(x)$. We shall denote by $\bar{N}_\Lambda$ and $\bar{T}_\Lambda'$ the rank-$n$ vector sub-bundles of $H(X/M)_{\Lambda'}$ whose fibres at $x \in \Lambda'$ are given by, respectively, (6) and (7).
vi): Suppose again \( x \in \Lambda', m = \pi(x) \). Given \( w \in T_m M \), we shall denote by \( w_j, j = 1, 2, 3, 4 \), the components of \( w \) in the following intrinsic and unique algebraic decomposition:

\[
w = w_a + w_b + w_c + w_d, \tag{8}
\]

where

\[
w_a \in (T_m \Lambda + \mathfrak{g}_M(m))^\perp, \quad w_b \in (T_m \Lambda')^\perp \cap T_m \Lambda, \quad w_c \in T_m \Lambda', \quad w_d \in \mathfrak{g}_M(m).
\]

Thus, \( w' =: w_a + w_b \) and \( w'' =: w_c + w_d \) are the components of \( w \) in \( \tilde{N}_\Lambda(x) \) and \( \tilde{T}_\Lambda'(x) \), respectively.

vii): Let \( \text{dens}_\Lambda \) and \( \text{dens}_{\Lambda}^{(1/2)} \) be the Riemannian density and half-density on \( \Lambda \), respectively. If \( \lambda \) is any smooth half-density on \( \Lambda \), we may write \( \lambda = f_\lambda \text{dens}_{\Lambda}^{(1/2)} \) for a unique smooth function \( f_\lambda \) on \( \Lambda \).

The leading order term of the asymptotic expansion for \( u_{k,\omega}(x + w/\sqrt{k}) \) will depend on both the effective volume of the action at \( \pi(x) \), and a function expressing a pointwise measure of the mutual position between the Legendrian submanifold \( \Lambda \) and the \( G \)-orbit.

Given any \( x \in \Lambda' \), let us choose Heisenberg local coordinates \( (\theta, p, q) \) for \( X \) centered at \( x \). The horizontal tangent space \( H_x(X/M) \) then gets unitarily identified with \( \mathbb{C}^n \), with complex coordinates \( z = p + iq \). Perhaps after applying a unitary transformation in \( z \), we may as well assume that the Lagrangian subspace \( T_x \Lambda \subseteq \mathbb{C}^n \) is defined by \( p = 0 \). Let us choose an orthonormal basis of \( \mathfrak{g}_M(\pi(x)) \) (for the induced metric); the inclusion \( \mathfrak{g}_M(\pi(x)) \subseteq T_{\pi(x)} M \) is then described by a linear map

\[
r \in \mathbb{R}^g \mapsto Rr + i Tr_r, \tag{9}
\]

where \( R \) is an \( n \times g \) real matrix, \( T \) an \( n \times n \) real matrix, and they satisfy

\[
\begin{cases}
\text{rank}(R) = g \\
R^t R + R^t T r Tr = I_g.
\end{cases} \tag{10}
\]

Recalling that \( \mathfrak{g}_M(m) \subseteq T_m M \) is an isotropic subspace when \( m \in M' \), one can see that the complex matrix \( R^t R + i R^t T r Tr \) is symmetric, has positive definite real part, and its determinant is independent of the choices involved. Let \( \Xi_\Lambda : \Lambda' \to \mathbb{C} \) be the smooth function defined by

\[
\Xi_\Lambda(x) =: \frac{\det(R^t R + i R^t T r Tr)^{-1/2}}{V_{\text{eff}}(\pi(x))} \quad (x \in \Lambda'). \tag{11}
\]
The square root of the determinant is determined according to the conventions described in [H], §3.4.

We are now ready to state our main result. Recall that \( n = \dim_C(M), \) \( g = \dim_R(G). \)

**Theorem 1.1.** Suppose that \( 0 \in g^* \) is a regular value of \( \Phi, \) and that the compact Legendrian submanifold \( \Lambda \subseteq X \) is transversal to \( \Phi^{-1}(0). \) Let \( \lambda \) be a smooth half-density on \( \Lambda. \) Fix a highest weight \( \varpi \) for \( G. \) For \( k = 0, 1, 2, \ldots, \) let \( u_{k,\varpi} \) be the component of \( \delta_{\Lambda,\lambda} \) in \( H(X)_{k,\varpi} \subseteq H(X). \) Then:

**i):** if \( x \notin (S^1 \times G) \cdot \Lambda', \) then \( u_{k,\varpi}(x) = O(k^{-\infty}) \) as \( k \to +\infty; \)

**ii):** there exist a positive definite metric \( S \) on \( \tilde{\Lambda} \) and a real quadratic form \( P \) on \( H(X/M)_{\Lambda} \) such that the following holds. If \( x \in (S^1 \times G) \cdot \Lambda', \) let \( (h_j, g_j) \in S^1 \times G, 1 \leq j \leq r_x, \) be the finitely many elements such that \( (h_j, g_j) \cdot x \in \Lambda. \) For every \( j, \) let

\[
x_j =: (h_j, g_j) \cdot x, \quad w_j = [d_x(h_j, g_j)](w) \in H(x)(X/M).
\]

For every \( w \in T_{\pi(x)}M, \) the following asymptotic expansion holds for \( k \to +\infty: \)

\[
u_{k,\varpi}(x + w/\sqrt{k}) \sim k^{(n-g)/2} \frac{\dim(V_\varpi)}{|G_{\pi(x)}|} \frac{1}{\pi^n} \sqrt{\frac{(2\pi)^{(n+g)}}{2^g}} \sum_{j=1}^{r_x} q_0^{(j)}(x, \varpi, k, w) f_\lambda(x_j)
+ \sum_{f \geq 1} k^{(n-g-f)/2} \sum_{j=1}^{r_x} q_f^{(j)}(x, \varpi, k, w),
\]

where \( G_{\pi(x)} \subseteq G \) is the stabilizer of \( \pi(x), \) and for every \( j = 1, \ldots, r_x \) we have

\[
q_0^{(j)}(x, \varpi, k, w) =: h_j^{-k} \chi_\varpi(g_j) \Xi_{\Lambda}(x_j) e^{-S_{x_j}(w', w') - i P_{x_j}(w, w)}.
\]

Here \( \chi_\varpi : G \to \mathbb{C} \) denotes the character of the representation \( V_\varpi. \)

The real quadratic forms \( S \) and \( P \) will be described precisely in the course of the proof (see (73) and (74)); they are determined by \( R \) and \( T. \)

Letting \( w = 0, \) we obtain an asymptotic expansion for \( u_{k,\varpi}(x): \)

\[
u_{k,\varpi}(x) \sim k^{(n-g)/2} \frac{\dim(V_\varpi)}{|G_{\pi(x)}|} \frac{1}{\pi^n} \sqrt{\frac{(2\pi)^{(n+g)}}{2^g}} \sum_{j=1}^{r_x} h_j^{-k} \chi_\varpi(g_j) \Xi_{\Lambda}(x_j) f_\lambda(x_j)
+ L.O.T.. \]
Let us see what the asymptotic expansion of Theorem 1.1 looks like in the action-free case. In this case obviously $\Lambda' = \Lambda$, and $\varpi$ may be disregarded. Here we work in a system of adapted Heisenberg local coordinates (Definition 2.1). As a corollary of (the proof of) Theorem 1.1 we obtain (cfr Theorem 3.12 of [BPU]):

**Corollary 1.1.** Let $\Lambda \subseteq X$ be any compact Legendrian submanifold, $\lambda$ a smooth half-density on $\Lambda$. Let $u_k \in \mathcal{H}(X)_k$ be the components of $\delta_{\Lambda, \lambda}$ in $\mathcal{H}(X)_k$, $k = 0, 1, 2, \ldots$. Then:

i): if $x \not\in S^1 \cdot \Lambda$, then $u_k(x) = O(k^{-\infty})$, as $k \to +\infty$;

ii): if $x \in S^1 \cdot \Lambda$, let $h_1, \ldots, h_{r_x} \in S^1$ be the finitely many elements such that $x_j =: h_j \cdot x \in \Lambda$. Set $m = \pi(x)$, $m_j = \pi(x_j)$. Suppose that $(\theta, p, q)$ is a system of local Heisenberg coordinates for $X$ adapted to $\Lambda$ at $x$. Then for every $w \in T_m M$ the following asymptotic expansion holds as $k \to +\infty$:

$$u_k(x + w/\sqrt{k}) \sim k^{n/2} \left( \frac{2}{\pi} \right)^{n/2} \sum_{j=1}^{r_x} q_0^{(j)}(x, k, w) f_\lambda(x_j)$$

$$+ \sum_{f \geq 1} k^{(n-f)/2} \sum_{j=1}^{r_x} q_f^{(j)}(x, k, w),$$

where for every $j = 1, \ldots, r_x$ we have

$$q_0^{(j)}(x, k, w) =: h_j^{-k} e^{-\|w_j^\perp\|^2 - i \Omega_{m_j}(w_j^\perp, w_j^\parallel)},$$

here $w_j^\perp$ is the component of $w_j = d_x h_j(w)$ perpendicular to $T_{x_j \Lambda}$, and $w_j^\parallel = w_j - w_j^\perp \in T_{x_j \Lambda}$.

In particular, for $w = 0$ we obtain:

$$u_k(x) \sim k^{n/2} \left( \frac{2}{\pi} \right)^{n/2} \sum_{j=1}^{r_x} h_j^{-k} f_\lambda(x_j) + \text{L.O.T.}$$

As a consequence of Corollary 1.1 let us momentarily return to the equivariant setting and take up again the case of an invariant Legendrian submanifold. More precisely, suppose that $G$ is semi-simple and acts freely on $\Phi^{-1}(0)$. Suppose also that $\Lambda \subseteq X$ is a $G$-invariant compact Legendrian submanifold which maps down diffeomorphically under $\pi$ onto a Lagrangian submanifold $\pi(\Lambda) \subseteq M$. Thus, $\pi(\Lambda) \subseteq \Phi^{-1}(0)$, and if we choose a $G$-invariant
smooth half-density \( \lambda = f_\lambda \cdot \text{dens}^{1/2}_\Lambda \) on \( \Lambda \), the corresponding generalized half-density \( \delta_{\Lambda,\lambda} \) is also \( G \)-invariant. As we have mentioned, we then have \( u_{k,\varpi} = 0 \) unless \( \varpi = 0 \), and \( u_{k,0} = u_k \) for every \( k \). Now, \( \Lambda_0 =: \Lambda/G \) is a compact Legendrian submanifold of \( X_0 = X'/G \); let us define (with a slight abuse of language) \( \lambda_0 =: f_\lambda \cdot \text{dens}^{1/2}_{\Lambda_0} \). Then it follows from Corollary 1.1 that, up to the multiplicative factor \( (2/\pi)^{k/2} k^{k/2} \), the asymptotic expansion for the corresponding sequence \( \tilde{u}_k \) has the same leading order term as the asymptotic expansion of \( u_k \).

Let us illustrate the Theorem and the Corollary with some examples.

**Example 1.2.** Let us consider again the setting of Example 1.1. Thus, \( \Lambda \subseteq S^3 \subseteq \mathbb{C}^2 \) is the Legendrian knot given by \( \iota(e^{it}) = (\cos(t), \sin(t)) \). Let us choose the Riemannian half-density on it, so that \( f_\lambda = 1 \).

Let us consider first the action-free case. The Szegő kernel at the level \( k \) is given by

\[
\Pi_k(x, y) = \frac{(k + 1)}{\pi} \langle x, y \rangle^k \quad (x, y \in S^3),
\]

where \( \langle x, y \rangle \) denotes the standard Hermitian product of \( x, y \in \mathbb{C}^2 \) [BSZ]. Since \( \Pi_k \) is self-adjoint with respect to the \( L^2 \)-Hermitian pairing, we have

\[
\langle \Pi_k(\delta_{\Lambda,\lambda}), f \rangle = \langle \delta_{\Lambda,\lambda}, \Pi_k(f) \rangle,
\]

for every \( f \in C^\infty(S^3) \) (we are identifying half-densities with functions by the Riemannian half-density on \( S^3 \)). Thus, setting \( x = (x_0, x_1) \in S^3 \subseteq \mathbb{C}^2 \), we have

\[
u_k(x) = \int_0^{2\pi} \Pi_k(x, (\cos(t), \sin(t))) \, dt = \frac{(k + 1)}{\pi} \int_0^{2\pi} (x_0 \cos(t) + x_1 \sin(t))^k \, dt
\]

\[
= \frac{(k + 1)}{\pi} \int_0^{2\pi} e^{k \log(x_0 \cos(t) + x_1 \sin(t))} \, dt = \frac{(k + 1)}{\pi} \int_0^{2\pi} e^{ikS(t, x)} \, dt,
\]

where \( S(t, x) =: -i \log(x_0 \cos(t) + x_1 \sin(t)) \) (any branch of the logarithm may be used). The latter equalities are meaningless at those values of \( t \) where \( x_0 \cos(t) + x_1 \sin(t) = 0 \); however, the contribution of a neighbourhood of radius \( \epsilon \) of any of these points is \( O(k^{\log(\epsilon)}) \). We shall implicitly introduce cut-off functions vanishing in a small neighbourhood of those points and ignore them in the following.

Clearly, \( \Im(S) \geq 0 \). We have

\[
\frac{\partial S}{\partial t} = -i \frac{-x_0 \sin(t) + x_1 \cos(t)}{x_0 \cos(t) + x_1 \sin(t)}.
\]
Therefore, \( \frac{\partial S}{\partial t}(t_0, x) = 0 \) if and only if there exists \( e^{ih} \in S^1 \) such that \( e^{ih} \cdot x = \nu(t_0) \). Thus, \( \ref{13} \) is rapidly decreasing in \( k \) unless \( x \in S^1 \cdot \Lambda \). Suppose then \( x \in S^1 \cdot \Lambda \), and let \( e^{ih_j} \in S^1 \), where \( h_0, \ldots, h_r \in [0, 2\pi) \), be the elements such that \( e^{ih_j} \cdot x \in \Lambda \). For every \( j = 1, \ldots, r \) there is a unique \( t_j \in [0, 2\pi) \) such that \( e^{ih_j} \cdot x = \nu(t_j) \). Hence the \( t_j \)'s are the only stationary points of \( S \). At every \( h_j \), we have

\[
\frac{\partial^2 S}{\partial t^2} = i,
\]

so that the \( t_j \) are all non-degenerate critical points. We have

\[
x_0 \cos(t_j) + x_1 \sin(t_j) = e^{-ih_j}.
\]

Application of the stationary phase Lemma now yields

\[
u_k(x) \sim \frac{\sqrt{2\pi}}{\pi} \sqrt{k} \cdot \sum_{j=1} e^{-ikh_j} + \text{L.O.T.}
\]

**Example 1.3.** Let us re-examine Example \( \ref{1.2} \) in the presence of the action \( S^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1 \) given by \( t \circ [z_0 : z_1] =: [tz_0 : t^{-1}z_1] \) which we considered in Example \( \ref{1.1} \). For any \( \omega \in \mathbb{Z} \), we now have:

\[
u_{k, \omega}(x) = \frac{(k + 1)}{2\pi^2} \int_0^{2\pi} \int_0^{2\pi} e^{ikS(s, t, x)} e^{i\omega s} dt,
\]

where \( S(s, t, x) := -i \log(x_0 e^{-is} \cos(t) + x_1 e^{is} \sin(t)) \). We have

\[
\frac{\partial S}{\partial t} = \frac{\partial S}{\partial s} = -i \cdot \frac{x_0 e^{-is} \sin(t) + x_1 \cos(t) e^{is}}{x_0 e^{-is} \cos(t) + x_1 e^{is} \sin(t)}.
\]

Thus, \( d_{s,t}S(s_0, t_0, x) = 0 \) if and only if \( e^{ih} \cdot (e^{i\omega_0} \circ x) = \nu(t_0) \) for some \( e^{ih} \in S^1 \) (by \( \ref{16} \)) and \( ||x_0|| = ||x_1|| \) (by pairing \( \ref{16} \) with \( \ref{17} \)). Thus, \( \nu_{k, \omega}(x) \) is rapidly decreasing unless \( x \in (S^1 \times G) \cdot \Lambda' \) (we have \( G = S^1 \) here).

Now suppose \( x \in (S^1 \times G) \cdot \Lambda' \), and let \( (e^{ih_j}, e^{i\omega_j}) \in S^1 \times G \) be the finitely many elements such that \( e^{ih_j} \cdot (e^{i\omega_j} \circ x) \in \Lambda' \), and for every \( j \) let \( t_j \in [0, 2\pi) \) be uniquely determined by the condition that \( e^{ih_j} \cdot (e^{i\omega_j} \circ x) = \nu(t_j) \). The pairs \( (s_j, t_j) \) are the only critical points of \( S(\cdot, \cdot, x) \), and for any \( j = 1, \ldots, r \) the Hessian matrix of \( S \) at \( (s_j, t_j) \) is given by

\[
H(s_j, t_j)(S) = i \begin{pmatrix}
1 & \pm i \\
\pm i & 1
\end{pmatrix}.
\]
Applying the stationary phase Lemma, we now obtain:

\[ u_{k,\omega}(x) \sim \frac{1}{\sqrt{2\pi}} \sum_j e^{i(\omega s_j - kh_j)} + \text{L.O.T.} \]

This agrees with Theorem 1.1. To check this, remark that \(|G_m| = 2\) and \(V_{\text{eff}}(m) = \pi\), for every \(m \in \Phi^{-1}(0)\). The latter equality follows from the fact that every \(G\)-orbit in \(S^3\) has length \(2\pi\), and if an orbit maps to \(\Phi^{-1}(0)\) then it doubly covers its image in \(\mathbb{P}^1\).

As an application, in §4 we shall study the following problem:

**Problem 1.1.** Suppose given two compact Legendrian submanifolds, \(\Lambda, \Sigma \subseteq X\), with specified smooth half-densities \(\lambda\) and \(\sigma\), respectively. Let \(u_{k,\omega}, v_{k,\omega} \in \mathcal{H}(X)_{k,\omega}\) be the components of \(\delta_{\Lambda,\lambda}\) and \(\delta_{\Sigma,\sigma}\), respectively. How can we relate the asymptotic behavior of the Hermitian products \((u_{k,\omega}, v_{k,\omega})\), as \(k \to +\infty\), to the geometry of \(\Lambda, \Sigma\) and \(\Phi^{-1}(0)\)?

In the action-free case, and in the setting of Fourier-Hermite distributions and symplectic spinors, this was carried out in [BPU]. Broadly speaking, we shall see that:

- if \((S^1 \times G) \cdot \Lambda \cap \Sigma \cap (\Phi \circ \pi)^{-1}(0) = \emptyset\), then \((u_{k,\omega}, v_{k,\omega}) = O(k^{-\infty})\) as \(k \to +\infty\);

- if, more generally, the map \(S^1 \times G \times \Lambda \to X\) given by the action is transversal to \(\Sigma' = \Sigma \cap (\Phi \circ \pi)^{-1}(0)\), then there is an asymptotic expansion

  \[ (u_{k,\omega}, v_{k,\omega}) \sim k^{-g/2} \rho_0 + \sum_{f \geq 1} k^{-(g+f)/2} \rho_f, \]

  where the leading term \(\rho_0\) is described explicitly, and is a sum of terms corresponding to each \((h_j, g_j) \in S^1 \times G\) such that \((h_j, g_j) \cdot \Lambda \cap \Sigma' \neq \emptyset\);

- a similar asymptotic expansion holds when \(S^1 \times G \times \Lambda \to X\) meets \(\Sigma'\) nicely; the order of the leading term depends on the dimension of the inverse image of \(\Sigma'\) in \(S^1 \times G \times \Lambda\), and the leading coefficient is determined by certain integrals on this inverse image.

The present work covers part of the PhD thesis of the first author at the University of Pavia.

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2 Preliminaries

In this section we shall collect a number of preliminary technical results, and begin a more precise description of the microlocal background of the quantization scheme outlined in the introduction. In §2.3 we shall prove statement i) of Theorem 1.1.

2.1 The distribution defined by a half-density on a Legendrian submanifold.

The given complex structure $J$ on $M$ and the unique compatible connection form $\alpha$ on $X$ determine a Riemannian metric and a volume form $\text{vol}_X := \alpha \wedge (d\alpha)^n$. The generator $\frac{\partial}{\partial \theta}$ of the $S^1$-action on $X$ spans the vertical tangent bundle $V(X/M) := \ker(d\pi) \subseteq TX$:

$$V(X/M) = \text{span} \left\{ \frac{\partial}{\partial \theta} \right\},$$

and $TX = V(X/M) \oplus H(X/M) \cong V(X/M) \oplus \pi^*(TM)$.

Now let $\Lambda \subseteq X$ be a compact Legendrian submanifold, endowed with the induced Riemannian metric. Suppose $x \in \Lambda$. In view of the above isomorphism, any basis $b$ of $T_x\Lambda$ can be naturally extended to a basis $\tilde{b} = (\frac{\partial}{\partial \theta}, b, J_{\pi(x)}b)$ of $T_{\pi(x)}X$, where $J_{\pi(x)}$ denotes the complex structure at $\pi(x) \in M$. By construction, $\tilde{b}$ is orthonormal if so is $b$. Thus the map $b \mapsto \tilde{b}$ yields an embedding $Bs(\Lambda)_{\text{ort}} \hookrightarrow Bs(X)_{\text{ort}}$, with obvious equivariance properties; here $Bs(\Lambda)_{\text{ort}}$ is the principal $O(n)$-bundle of orthonormal frames of $\Lambda$, and $Bs(X)_{\text{ort}}$ is the principal $O(2n + 1)$-bundle of orthonormal frames of $X$.

Given this, half-densities on $X$ restrict to half-densities on $\Lambda$; conversely, half-densities on $\Lambda$ extend to half-densities for $X$ defined on $\Lambda$.

On the upshot, any choice of a smooth half-density $\lambda$ on $\Lambda$ determines a generalized half-density on $X$ supported on $\Lambda$, $\delta_{\Lambda, \lambda}$, as follows: If $\beta$ is a smooth half-density on $X$, let $\beta_{\Lambda}$ denote the induced half-density on $\Lambda$; thus the product $\beta_{\Lambda} \otimes \lambda$ is a density on $\Lambda$. Then

$$\delta_{\Lambda, \lambda}(\beta) := \int_{\Lambda} \beta_{\Lambda} \otimes \lambda.$$

Suppose $\beta = b \text{dens}_{X}^{(1/2)}$, $\lambda = f_{\lambda} \text{dens}_{\Lambda}^{(1/2)}$ with $b \in C^\infty(X)$, $f_{\lambda} \in C^\infty(\Lambda)$ (here $\text{dens}_{X}^{(1/2)}$, $\text{dens}_{\Lambda}^{(1/2)}$ denote the Riemannian half-densities on $X$ and $\Lambda$, respectively). Then $\beta_{\Lambda} = b|_{\Lambda} \text{dens}_{\Lambda}^{(1/2)}$, and

$$\delta_{\Lambda, \lambda}(\beta) := \int_{\Lambda} (b|_{\Lambda} \cdot f_{\lambda}) \text{dens}_{\Lambda}. \quad (19)$$
2.2 Adapted local coordinates and the microlocal structure of $\delta_{\Lambda,\sigma}$.

Heisenberg coordinates for circle bundles are discussed in [SZ]. The context of [SZ] is the symplectic and almost complex category; we recall that the construction of local Heisenberg coordinates at a given $x_0 \in X$ involves the choice of preferred local coordinates and a preferred frame for $L$ at $m_0 = \pi(x_0) \in M$. Although it isn’t strictly necessary, in the present complex projective setting the preferred local coordinates and frames involved may as well be assumed holomorphic.

In short, suppose that $(z_1, \ldots, z_n)$ is a system of preferred local holomorphic coordinates for $M$ at $m_0$, so that the Hermitian metric satisfies $(g - i\Omega)|_{m_0} = \sum_{j=1}^n dz_j \otimes d\overline{z}_j$. Let $e_L$ be a preferred local holomorphic frame for $L$ at $p_0$, with dual frame $e^*_L$, such that $e^*_L(z_0) = x_0$. The associated system of local Heisenberg coordinates for $X$ centered at $x_0$, $\rho : U \subseteq (-\pi, \pi) \times \mathbb{C}^n \rightarrow V \subseteq X$, is

$$
\rho(\theta, z) = e^{i\theta} a(z)^{-1/2} e^*_L(z);
$$

here $a =: \|e^*_L\|^2 = \|e_L\|^{-2}$.

Write $z = (z_1, \ldots, z_n) = p + iq$, where $p, q \in \mathbb{R}^n$, and set $pdq =: \sum_j p_j dq_j$, $qdp =: \sum_j q_j dp_j$. By [SZ], §1.2, the connection form $\alpha$ has the local representation

$$
\alpha = d\theta + pdq - qdp + \beta(p, q),
$$

with $\beta = O(\|z\|^2)$.

**Definition 2.1.** Suppose that $\Lambda \subseteq X$ is a compact Legendrian submanifold, and that $x_0 \in \Lambda$. A system of Heisenberg local coordinates $(\theta, p, q)$ centered at $x_0$ is called adapted to $\Lambda$ at $x_0$ if $\Lambda$ is tangent to the submanifold $\{\theta = 0, p = 0\}$ at $x_0$.

Any system of Heisenberg local coordinates at $x_0$ may be turned into one adapted to $\Lambda$ at $x_0$ simply by applying a suitable unitary transformation in the $z$ coordinates.

Suppose the Heisenberg local coordinates $(\theta, p, q)$ are adapted to $\Lambda$ at $x_0$. Then $\Lambda$ is locally defined by $\theta = f(q)$, $p = h(q)$, where $(f, h) : V \rightarrow \mathbb{R} \times \mathbb{C}^n$ vanishes to second order at $x_0$. Thus, $F(\theta, q) = \theta - f(q)$ and $H(p, q) = p - h(q)$ are local defining functions for $\Lambda$ on $V$. Actually,

**Lemma 2.1.** $f$ vanishes to third order at the origin.

**Proof.** By assumption, the restriction of $\alpha$ to $\Lambda$ vanishes identically; therefore, $df = -h dq + q dh - \beta(h, q)$, which vanishes to second order at $q = 0$. 

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The $q$’s may be naturally viewed as local coordinates on $\Lambda$. Let $D_\Lambda(q)$ be the local coordinate expression for the Riemannian half-density $\operatorname{dens}\text{ }^{(1/2)}\Lambda$ on $\Lambda$. In view of $\text{(19)}$, we conclude:

**Lemma 2.2.** Suppose $x_0 \in \Lambda$, and choose adapted Heisenberg local coordinates at $x_0$, defined in an open neighborhood $V \ni x_0$. Up to a smoothing contribution, the restriction of $\delta_{\Lambda,\Lambda}$ to $C^\infty_c(V)$ is a Fourier integral

$$
\frac{1}{(2\pi)^{n+1}} \int \int_{\mathbb{R} \times \mathbb{R}^n} e^{i(\tau F + \eta H)} f_\lambda(q) D_\Lambda(q) d\tau d\eta. \tag{21}
$$

Here $(\tau, \eta) \in \mathbb{R} \times \mathbb{R}^n$, $\eta \cdot H = \sum k \eta_k H_k$, where $H(p, q) = p - h(q)$, and $D_\Lambda$ is the local coordinate expression for the Riemannian density of $\Lambda$ (the $q$’s restrict to a system of local coordinates on $\Lambda$). By our choices, $D_\Lambda(0) = 1$. The factor $(2\pi)^{-(n+1)}$ in front of $(21)$ comes from the fact that $\operatorname{codim}(\Lambda, X) = n + 1$.

Let $\{V_j\}$ be an open cover of $X$ such that whenever $V_j \cap \Lambda \neq \emptyset$ there exist Heisenberg local coordinates on $V_j$ adapted to $\Lambda$ at some $x_j \in V_j \cap \Lambda$. Let $\sum_j \rho_j = 1$ be a partition of unity subordinate to the open cover $\{V_j\}$. Then $\delta_{\Lambda,\Lambda} = \sum_j \delta_{\Lambda,\Lambda}(j)$, where each $\delta_{\Lambda,\Lambda}(j) =: \rho_j \delta_\Lambda$ is either a smoothing operator or a Fourier integral as $(21)$.

The construction of a system of Heisenberg local coordinates adapted to $\Lambda$ at some $x \in \Lambda$ may be varied smoothly with $x$. More precisely, let $B_{2n+1}(0, \varepsilon) \subseteq \mathbb{R}^{2n+1} \cong \mathbb{R} \times \mathbb{C}^n$ be the open ball of radius $\varepsilon$ centered at the origin and having radius $\varepsilon > 0$. Then:

**Lemma 2.3.** Fix $y \in \Lambda$. Then there exist

i): open neighborhoods $y \in U \subseteq \Lambda$ and $y \in V \subseteq X$, and

ii): a smooth map $\kappa : U \times B_{2n+1}(0, \varepsilon) \to V$,

such that the following holds: For every $x \in U$, the restricted map $\kappa_x = \kappa(x, \cdot) : B_{2n+1}(0, \varepsilon) \to V$ is a Heisenberg local chart adapted to $\Lambda$ at $x$.

Let $(\theta(x), p(x), q(x))$ be the local coordinates associated to $\kappa_x$ ($x \in U$). It is then clear that we may also find smoothly varying local defining functions $(F_x, H_x) : V \to \mathbb{R} \times \mathbb{C}^n$, of the form $F_x = \theta(x) - f_x(q(x))$, $H_x = p(x) - h_x(q(x))$.

**Lemma 2.4.** Suppose $x_0 \in X$ and fix Heisenberg local coordinates $(\theta, z)$ centered at $x_0$. Let $m_0 =: \pi(x_0) \in M$ and for some $\delta > 0$ consider a smooth path $\gamma : (-\delta, \delta) \to M$ satisfying $\gamma(0) = m_0$. Let $\tilde{\gamma} : (-\delta, \delta) \to X$ be the unique horizontal lift of $\gamma$ to $X$ satisfying $\tilde{\gamma}(0) = x_0$. Then the local Heisenberg coordinates of $\tilde{\gamma}$ are of the form $(\theta(t), z(t))$, where $z(t) \in \mathbb{C}^n$ are the (holomorphic) preferred local coordinates of $\gamma(t)$, and $\theta(t) \in (-\pi, \pi)$ vanishes to third order at $t = 0$. 

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Proof. If \( z(t) \) are the preferred coordinates of \( \gamma(t) \), then clearly the Heisenberg coordinates of \( \tilde{\gamma}(t) \) have the form \( (\theta(t), z(t)) \) for some smooth real function \( \theta(t) \) vanishing at the origin. Now let \( z'(0) = \rho_0 + iq_0 \) be the tangent vector of \( \gamma \) at \( t = 0 \) (expressed in local coordinates). Thus, \( z(t) = (tp_0 + P(t)) + i(tq_0 + Q(t)) \), where \( P(t) \) and \( Q(t) \) vanish to second order at \( t = 0 \). Since \( \tilde{\gamma} \) is horizontal, the pull-back \( \tilde{\gamma}^*(\alpha) \in \Omega^1(-\delta,\delta) \) vanishes identically. Now in view of (20) and the horizontality of \( \tilde{\gamma} \),

\[
0 = \tilde{\gamma}^*(\alpha) = \left[ \theta'(t) + (tp_0 + P(t))(q_0 + Q'(t)) - (tq_0 + Q(t))(p_0 + P'(t)) \right] dt + O(t^2).
\]

It follows that \( \theta'(t) = O(t^2) \), since the first order terms cancel out.

2.3 The equivariant setting.

Recall that the connection form \( \alpha \) and \( \pi^*(\Omega) \) naturally endow \( X \) with a \( G \)-invariant volume form. This yields a unitary and equivariant identification of functions and half-densities, which with some abuse of language will be implicit in the following discussion.

Let \( L^2(X) \) be the Hilbert space of square-integrable half-densities on \( X \). By the theory of [BS], the Schwartz kernel of the Szegö projector \( \Pi_X : L^2(X) \to \mathcal{H}(X) \subseteq L^2(X) \) is microlocally a Fourier integral

\[
\Pi_X(x,y) = \int_0^{+\infty} e^{it\psi(x,y)} \zeta(x,y,t) \, dt.
\]  

(22)

The phase \( \psi \) is the restriction to \( X \times X \) of a smooth function on \( L^* \times L^* \) defined in the neighbourhood of \( (x_0, x_0) \), satisfying \( \Im(\psi) \geq 0 \). The Taylor series of \( \psi \) along the diagonal \( L^* \subseteq L^* \times L^* \) is:

\[
\psi(x + h, x + k) \sim i \sum_{i,j} \frac{\partial^{i+j} \rho}{\partial z^i \partial \bar{z}^j} r(x) \, h^i \, \bar{k}^j,
\]

where \( \rho = 1 - \|\cdot\|^2 \) is the defining function for \( X \subseteq L^* \). The amplitude \( \zeta(x,y,t) \in S^n(X \times X \times \mathbb{R}^+) \) is a classical symbol of the form

\[
\zeta(x,y,t) \sim \sum_{k=0}^{\infty} t^{n-k} \zeta_k(x,y).
\]  

(23)

A complete discussion of the almost analytic geometry involved, together with a description of the leading term, is in [BS], [Z], [SZ]. It follows in particular that the wave front of \( \Pi_X \) is the closed isotropic cone

\[
\Sigma = \{(x, r\alpha_x, x, -r\alpha_x) : x \in X, r > 0 \} \subseteq (T^* X \setminus \{0\}) \times (T^* X \setminus \{0\}).
\]
let \( L \) direct sum of copies of \( V \) given by \((g) \) induces a Hamiltonian action (for the canonical symplectic structur e) \( \tilde{P} \) moment map. Then \( L \) Lagrangian submanifold \( P_{\text{GS2}} \). Thus, \( \chi \) is given by \( \mu \) of (24). To this end, let us remark that the action \( D \) on an \( S \) complex phase, whose wave front satisfies:

\[ G \] In the first equality, we have made use of the \( \Phi \) on \( S \) of \( \psi \) the character of the representation \( \psi \) [Di].

We need to recall some basic facts concerning the the microlocal structure of (24). To this end, let us remark that the action \( \mu : G \times X \to X \) naturally induces a Hamiltonian action (for the canonical symplectic structur e) \( \tilde{\mu} : G \times (T^*X \setminus \{0\}) \to T^*X \setminus \{0\} \). Let \( \Psi : T^*X \setminus \{0\} \to g^* \) be the associated moment map. Then \( P_{\psi} \) is a Fourier integral operator, associated to the Lagrangian submanifold

\[ \Lambda_0 =: \{ (\nu_1, \nu_2) : \Psi(\nu_1) = 0, \nu_2 = \tilde{\mu}(g, \nu_1) \} \subseteq (T^*X \setminus \{0\}) \times (T^*X \setminus \{0\}) \]  

[GS2]. Thus, \( P_{\psi} \) obviously extends to a bounded operator \( P_{\psi} : \mathcal{D}'(X) \to \mathcal{D}'(X) \), and the composition \( P_{\psi} \circ \Pi_X \) is a Fourier integral operator with complex phase, whose wave front satisfies:

\[
\text{WF}(P_{\psi} \circ \Pi_X) = \{ (x, r\alpha_x, y, -r\alpha_y) : r > 0, \Psi(x, r\alpha_x) = 0, y = \mu(g, x) \}
\]
\[
= \{ (x, r\alpha_x, y, -r\alpha_y) : r > 0, \Phi(x) = 0, y = \mu(g, x) \}. \]  

In the first equality, we have made use of the \( G \)-invariance of \( \alpha \), and in the second we have used the equality \( \Psi(x, r\alpha_x) = r\Psi(x, \alpha_x) = r\Phi(x) \) [GS1].

On the upshot,

\[
\text{WF}(P_{\psi} \circ \Pi_X(\delta_{A,\lambda})) \subseteq \{ (x, r\alpha_x) : x \in G \cdot A, \Phi(x) = 0, r > 0 \}. \]  

Therefore, if \( x \not\in (S^1 \times G) \cdot A' \), then \( P_{\psi} \circ \Pi_X(u) \) is smooth on an \( S^1 \)-invariant neighborhood of \( x \). Given that \( u_{k,\psi} \) is the \( k \)-th Fourier component of \( P_{\psi} \circ \Pi_X(u) \), we obtain:

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Proposition 2.1. If \( x \notin (S^1 \times G) \cdot \Lambda' \), then \( u_{k,\infty}(x) = O(k^{-\infty}) \) as \( k \to +\infty \).

Let us now dwell on the geometry of \( \Lambda' \). To this end, let us first recall the following basic fact from [GS1]:

Lemma 2.5. For every \( m \in M' \), \( T_m M' \) is the symplectic annihilator \( g_M(m)^0 \) of the isotropic subspace \( g_M(m) \). In particular, \( T_m M' \) is a co-isotropic subspace of \( T_m M \).

We deduce:

Corollary 2.1. Suppose that \( \Lambda \) is transversal to \( X' \), and \( x \in \Lambda' \). Then
\[
T_x \Lambda \cap T_x (G \cdot x) = 0.
\]

Proof. Let \( m =: \pi(x) \in M' \). Since both \( T_x \Lambda \) and \( T_x (G \cdot x) = g_X(x) \) are horizontal subspaces of \( T_x X \), it is equivalent to prove that \( T_x \Lambda \cap g_M(m) = 0 \), where in the latter equality \( T_x \Lambda \) is identified with the Lagrangian subspace \( d_x \pi(T_x \Lambda) \subseteq T_m M \). Passing to symplectic annihilators, we have
\[
(T_x \Lambda \cap g_M(m))^0 = T_x \Lambda^0 + g_M(m)^0 = T_x \Lambda + T_m M' = T_m M,
\]
by Lemma 2.5 and the transversality assumption.

By horizontality, this means:

Corollary 2.2. Let \( \Lambda \subseteq X \) be a Legendrian submanifold transversal to \( X' \). If \( x \in \Lambda' \), we have
\[
T_x \Lambda \cap (V_x(X/M) \oplus g_X(x)) = \{0\}.
\]

Since the action of \( S^1 \times G \) on \( X' \) is (locally) free, we conclude:

Corollary 2.3. Suppose that \( \Lambda \subseteq X \) is a compact Legendrian submanifold transversal to \( X' \), and that \( x \in (S^1 \times G) \cdot \Lambda' \). There are then only finitely many \((h_j, g_j) \in S^1 \times G\) such that \((h_j, g_j) \cdot x \in \Lambda\).

Now suppose \( x \in \Lambda' \), and let us choose Heisenberg local coordinates \((\theta, z)\) adapted to \( \Lambda \) centered at \( x \), defined on some open neighborhood \( V \ni x \). Let \( F(\theta, q) = \theta - f(q) \) and \( H(p, q) =: p - h \) be defining functions for \( \Lambda \cap V \), as in [2.2]. By construction, the locus \( \theta = 0 \) is tangent to the horizontal tangent bundle at the origin, and \( d_0 f = 0 \). Given that the \( G \)-action on \( X \) is horizontal at any \( x \in X' \), it follows that \( d_x F(\xi(x)) = 0 \). In view of Corollary 2.1, we obtain:

Corollary 2.4. There exist an open neighbourhood \( E \) of \( 0 \in \mathfrak{g} \) and \( c > 0 \) such that \( \|H(\mu_{\exp_\xi}(x))\| \geq c\|\xi\| \) for every \( \xi \in E \).
2.4 A unitary invariant of pairs of Lagrangian subspaces

The invariant introduced in this section will be used in §4. Let \((V, \Omega_V, J)\) be a unitary vector space; that is, \(V\) is a \(2r\)-dimensional real vector space, \(\Omega_V\) a linear symplectic structure on \(V\), and \(J \in \mathrm{GL}(V)\) is a complex structure compatible with \(\Omega_V\). Leaving \(\Omega_V\) and \(J\) understood, let \(U(V)\) denote its unitary group, and let \(\mathrm{Gr}_{\mathrm{Lag}}(V)\) be the its Lagrangian Grassmanian (the manifold parametrizing Lagrangian vector subspaces of \(V\)). Given \(L, L' \in \mathrm{Gr}_{\mathrm{Lag}}(V)\), let \(U(V)_{L, L'} \subseteq U(V)\) be the subset of unitary transformations mapping \(L\) onto \(L'\).

**Definition 2.2.** For every \(L\), we let \(\iota_J(L, L) = 1\). If \(c = \dim(L \cap L') < n\), suppose that \(\psi \in U(V)_{L, L'}\) satisfies \(\psi(L \cap L') \subseteq L \cap L'\). Let \(\mathcal{B}\) be an orthogonal real basis of \(L\) whose first \(c\) vectors lie in \(L \cap L'\). The matrix of \(\psi\) in the basis \(\mathcal{B}\), viewed as an orthonormal complex basis of \(V\), has a block diagonal form, whose first \(c \times c\) block is a real orthogonal matrix and whose second \((r - c) \times (r - c)\) block is a unitary matrix \(A + iB \in U(r - c)\), where \(A, B \in M_{r - c}(\mathbb{R})\) and \(B\) is non-singular. Then \(\iota_J(L, L') = |\det(B)|\).

For example, if \(V = \mathbb{C}^2\) with its standard unitary structure, and \(L, L' \subseteq \mathbb{C}^2\) are two distinct lines through the origin, then \(\iota_J(L, L') = |\sin(\vartheta)|\), where \(\vartheta\) is the angle between \(L\) and \(L'\).

For every \(c = 0, \ldots, n\), let

\[
D_c = \{(L, L') \in \mathrm{Gr}_{\mathrm{Lag}}(V) \times \mathrm{Gr}_{\mathrm{Lag}}(V) : \dim(L \cap L') = c\}.
\]

We leave it to the reader to check the following:

**Lemma 2.6.** \(\iota_J: \mathrm{Gr}_{\mathrm{Lag}}(V) \times \mathrm{Gr}_{\mathrm{Lag}}(V) \to \mathbb{R}^*\) is well-defined, \(U(r)\)-invariant (with respect to the action \(R \cdot (L, L') = (RL, RL')\)), and symmetric. It is continuous on \(D_c\), for every \(c = 0, \ldots, n\).

3 Proof of Theorem 1.1

As before, let \(u =: \Pi_X(\delta_{\Lambda, \lambda}) \in \mathcal{H}(X)\), and denote by \(u_{k, \varpi} \in \mathcal{H}_{k, \varpi}(X)\) its \(S^1 \times G\)-equivariant components.

Suppose \(x \in (S^1 \times G) \cdot \Lambda'. \) Choose local Heisenberg coordinates \((\theta, z) = (\theta, p, q)\) centered at \(x\), defined on an open neighbourhood \(V \ni x\) \((z = p + iq\) and \(p, q \in \mathbb{R}^n)\). We shall denote by \(x + w\) the point in \(V\) having Heisenberg local coordinates \((0, w)\) \((w \in \mathbb{C}^n)\).
Given \( w \in \mathbb{C}^n \), let us consider the asymptotics of \( u_{k,\varpi}(x + w/\sqrt{k}) \) for \( \varpi \) fixed and \( k \to +\infty \). We have:

\[
\begin{align*}
 u_{k,\varpi}(x + w/\sqrt{k}) &= \frac{\dim(V_\varpi)}{(2\pi)^{n+2}} \int_G \int_{\pi} u \left( \mu_{g^{-1}} \circ r_{e^{i\theta}}(x + w/\sqrt{k}) \right) \\
 & \quad \times \chi_\varpi(g^{-1}) e^{-ik\theta} \, dg \, d\theta.
\end{align*}
\] (28)

Here \( \mu \) and \( r \) denote the \( G \)- and \( S^1 \)-actions on \( X \) (we shall occasionally also use a dot to denote group action on a given point).

Let \( \{(e^{i\theta_j}, g_j)\}, 1 \leq j \leq N_m \), be the finitely many elements of \( S^1 \times G \) such that \( x_j =: (e^{i\theta_j}, g_j) \cdot x \in \Lambda \) (Corollary 2.3); \( N_m \) depends only on \( m = \pi(x) \in M \). Since the action of \( G \) on \( X' \) is locally free, but not necessarily free, it may happen that \( x_j = x_{j'} \) for \( j \neq j' \).

We shall now show that, perhaps after disregarding a rapidly decaying contribution, the integration over \( S^1 \times G \) may be localized near the \( (e^{i\theta_j}, g_j)'s \). Using standard basic facts from the theory of wave fronts \([Du]\), \([H]\), and recalling that we are identifying functions and densities by means of the Riemannian volume forms, one can prove the following:

**Lemma 3.1.** For \( y \in X \), define the smooth map \( \Upsilon_y : S^1 \times G \to X \) by \( \Upsilon_y(h, g) = \mu_{g^{-1}} \circ r_\varpi(y) \) \( (h \in S^1, \, g \in G) \). Then:

i): \( \Upsilon_y \) is an immersion, for every \( y \in X' \);

ii): the pull-back \( \Upsilon_y^*(u) \) is a well-defined generalized half-density on \( S^1 \times G \);

iii): the singular support of \( \Upsilon_y^*(u) \) satisfies

\[
\text{SS} \left( \Upsilon_y^*(u) \right) \subseteq \{(h, g) \in S^1 \times G : \mu_{g^{-1}} \circ r_h(y) \in \Lambda' \}.
\]

Now suppose \( \epsilon > 0 \) is suitably small; similarly, choose a suitably small open neighborhood \( E \) of the unit \( e \in G \). For every \( j = 1, \ldots, N_m \), let

\[
D_j =: \{e^{i\theta} : |\vartheta - \vartheta_j| < \epsilon\}, \quad \text{and} \quad E_j =: g_{j^{-1}}^{-1} \cdot E \subseteq G.
\]

Thus, \( T_j =: D_j \times E_j \subseteq S^1 \times G \) is an open neighborhood of \( (e^{i\vartheta_j}, g_j^{-1}) \). Let \( T_0 \subseteq S^1 \times G \) be an open subset such that \( (e^{i\vartheta_j}, g_j^{-1}) \notin \mathcal{T}_0 \), for every \( j \), and such that \( S^1 \times G = \bigcup_{j=0}^{N_m} T_j \). Let \( \gamma_j(h, g) = 1 \) be a partition of unity subordinate to the open cover \( \mathcal{T} = \{T_j\}_{j=0}^{N_m} \) of \( S^1 \times G \). We have, with \( dh = (2\pi)^{-1} d\vartheta \\)

\[
\begin{align*}
 u_{k,\varpi}(x + w/\sqrt{k}) &= \frac{\dim(V_\varpi)}{(2\pi)^{n+1}} \sum_{j=0}^{N_m} \int_{T_j} \gamma_j(h, g) u \left( \mu_{g^{-1}} \circ r_h(x + w/\sqrt{k}) \right) \\
 & \quad \times \chi_\varpi(g^{-1}) h^{-k} \, dg \, dh = \sum_j u_{k,\varpi}(x + w/\sqrt{k})_j,
\end{align*}
\] (29)
where \( u_{k,\varpi}(x + w/\sqrt{k}) \) is defined to be the \( j \)-th summand in (29).

**Lemma 3.2.** \( u_{k,\varpi}(x + w/\sqrt{k}) = O(k^{-\infty}) \) as \( k \to +\infty \).

**Proof.** Perhaps after restricting the open neighborhood \( V \) of \( x \), we may assume that

\[
\text{dist}_X (\mu_{g^{-1}} \circ r_h(y), \Lambda') > \epsilon_1
\]

for some given sufficiently small \( \epsilon_1 > 0 \) and every \( (h, g) \in T_0, y \in V \). Therefore, as \( y \in V \) varies, the generalized functions \( \psi_0(h, g) \mathcal{Y}^*_y(u) \in \mathcal{D}'(S^1 \times G) \) are smooth and have bounded derivatives. Taking Fourier components, we deduce that there exist \( C_N > 0, N = 1, 2, \ldots \), such that \( |u_{k,\varpi}(y)| < C_N k^{-N} \) for every \( y \in V \). Since \( x + w/\sqrt{k} \in V \) for \( k \gg 0 \), the statement follows.

Next we shall focus on the asymptotics of each \( u_{k,\varpi}(x + w/\sqrt{k}) \), \( 1 \leq j \leq N_p \). Recalling (28), we have:

\[
u_{k,\varpi}(x + w/\sqrt{k}) = \frac{\dim(V_{\varpi})}{(2\pi)^{n+1}} \int_{T_j} \int_X \tilde{\Pi} \left( \mu_{g^{-1}} \circ r_h(x + w/\sqrt{k}), y \right) \delta_{\lambda,\lambda}(y) \\
\gamma_j(h, g) \chi_{\varpi}(g^{-1}) h^{-k} dy dh.
\]

For every \( j \), set \( V_j =: (e^{i\theta_j}, g_j) \cdot V \). We may assume that if \( (h, g) \in T_j \) and \( y \not\in V_j \), then

\[
\text{dist}_X (\mu_{g^{-1}} \circ r_h(x + w/\sqrt{k}), y) > \epsilon_3
\]

for some \( \epsilon_3 > 0 \) and every \( k \gg 0 \). Given that the Szegö kernel is smoothing away from the diagonal, it follows from (31) that

\[
u_{k,\varpi}(x + w/\sqrt{k}) \sim \frac{\dim(V_{\varpi})}{(2\pi)^{n+1}} \int_{T_j} \int_{V_j} \tilde{\Pi} \left( \mu_{g^{-1}} \circ r_h(x + w/\sqrt{k}), y \right) \delta_{\lambda,\lambda}(y) \\
\gamma_j(h, g) \varrho_j(y) \chi_{\varpi}(g^{-1}) h^{-k} dg dh dy,
\]

for an appropriate compactly supported bump function \( \varrho_j \) on \( V_j \), identically equal to one near \( x_j \). Here the symbol \( \sim \) means that the difference between the left and right hand side is \( O(k^{-\infty}) \).

For every \( j \), the local Heisenberg coordinates on \( V \) determine by translation local Heisenberg coordinates on \( V_j \) centered at \( x_j \). We may then compose with an appropriate \( A_j \in U(n) \) in the \( z \)-variable, so as to obtain a system of local Heisenberg coordinates \( (\theta^{(j)}, z^{(j)}) \) adapted to \( \Lambda \) at \( x_j \), in the sense of 

With these coordinates understood, \( \mu_{g^{-1}} \circ r_{e^{i\vartheta}}(x + w/\sqrt{k}) = x_j + w_j/\sqrt{k}, \) where \( w_j = A_j(w) \); furthermore,

\[
\mu_{(g^{-1}g_0) \circ r_{e^{i(\vartheta + \vartheta_0)}}}(x + w/\sqrt{k}) = \mu_{g^{-1}} \circ r_{e^{i\vartheta}}(x_j + w_j/\sqrt{k})
\]

(\( g \in E, |\vartheta| < \epsilon \)).

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To simplify, when focusing on one $j$ at a time, we shall write $(\theta, p, q)$ for $(\varphi^{(j)}, q^{(j)}, p^{(j)})$. Thus, $F_j(q) = \theta - f_j(q)$ and $H_j(p, q) = p - h_j$ will denote local defining functions for $\Lambda$ in $V_j$.

Thus,

$$F_j(q) = \theta - f_j(q), \quad H_j(p, q) = p - h_j.$$ 

We may assume that the Szegő kernel can be represented on each $V_j \times V_j$ by a Fourier integral as in (22), and that $\delta_{\Lambda, \lambda}$ is represented on each $V_j$ by a Fourier integral as in (21); we shall thus apply (22) to (21), write the $k$-th Fourier component of the result as an oscillatory integral, and study the asymptotics of the latter by the Lemma of stationary phase [H].

Let us fix an orthonormal basis of $g$, and identify the latter with $\mathbb{R}^g$. We may assume that the exponential map, $\exp_G : g \to G$, induces a diffeomorphism $E' = \exp^{-1}_G(E) \to E$. Thus, the linear coordinates on $E'$ become local coordinates on $E$.

**Lemma 3.3.** Let $(s_j, S_j) : E' \subseteq g \to \mathbb{R} \times \mathbb{C}^n$ be defined by the condition that $\mu e^{-\xi}(x_j)$ has adapted Heisenberg local coordinates $(s_j(\xi), S_j(\xi))$. Then $S_j$ is an embedding and $s_j$ vanishes to third order at $0 \in g$.

**Proof.** The first statement holds because the $G$-action on $\Phi^{-1}(0) \subseteq M$ is locally free, and the second follows from Lemma 2.4 since $G$ acts horizontally on $(\Phi \circ \pi)^{-1}(0) \subseteq X$.

By construction of Heisenberg local coordinates, we have:

**Lemma 3.4.** Suppose $y \in V_j$ has adapted Heisenberg local coordinates $(\theta, z)$. Let $\text{dist}_M$ be the geodesic distance function on $M$. Then (perhaps after restricting $V_j$ and $E'$):

$$\frac{1}{2}||S_j(\xi) - z|| \leq \text{dist}_M(\mu e^{-\xi}(\pi(x_j)), \pi(y)) \leq 2||S_j(\xi) - z||.$$

Let us write $\psi_j$ and $\zeta_j$ for the phase and amplitude in (22) on $V_j \times V_j$, and define

$$\Psi_j(\tau, \eta, t, g, \vartheta, y) =: t\psi_j(\mu g^{-1} \circ r e^{i\vartheta}(x_j + w_j/\sqrt{k}), y)$$
$$+ \tau F_j(y) + \eta \cdot H_j(y) - \vartheta.$$

(34)
Clearly, $\Im(\Psi_j) = t \Im(\psi_j) \geq 0$.

Performing the change of variables $t \mapsto kt$, $\eta \mapsto k\eta$, $\tau \mapsto k\tau$, we obtain:

$$u_{k,\infty}(x + w/\sqrt{k})_j \sim k^{n+2} \dim(V_{\infty}) \frac{\dim(V_{\infty})}{(2\pi)^{n+2}} \int_{\R} e^{-ikt\phi} \int_{\R^n} e^{-i\phi\eta} \int_{\R^n} e^{-i\eta\tau} x_j \chi_{\infty}(g^{-1} g_j) \cdot \gamma_j(e^{i(\phi_j + \phi)}, g_j^{-1} g) \zeta_j(\mu_{g^{-1}} \circ r_{\phi}(x_j + w_j/\sqrt{k}), y, kt) \cdot \varepsilon_j(y) f_{\lambda}(q) D_{\lambda}(q) d\tau d\eta d\phi dy.$$  \(35\)

**Remark 3.1.** Arguing as in the proof of Theorems 2.3.1 and 2.2.2 of [Du], an oscillatory integral like \(35\) can be evaluated asymptotically by implicitly introducing a cut-off in the norm of \((t, \tau, \eta)\), vanishing for large values of argument (this justifies the integration by parts in the proof of Lemma 3.5).

Let us now split the integration in $dg \, dy$ as follows. Let $\text{dist}_G$ denote the Riemannian distance function on $G$, and for $k = 1, 2, \ldots$ define open sets $A_{jk}, B_{jk} \subseteq E \times V_j$ by

$$A_{jk} =: \{(g, y) \in E \times V_j : \gamma(g, y) > k^{-1/3}\},$$

$$B_{jk} =: \{(g, y) \in E \times V_j : \gamma(g, y) < 2k^{-1/3}\},$$  \(36\)

where

$$\gamma(g, y) =: \max\{\text{dist}_G(g, e), \text{dist}_M(\mu_{g^{-1}}(\pi(x)), \pi(y))\}.  \(37\)$$

Let $a_{jk} + b_{jk} = 1$ be a partition of unity on $E \times V_j$ subordinate to the open cover $E \times V_j = A_{jk} \cup B_{jk}$. By construction, $a_{jk}$ and $b_{jk}$ may be chosen $S^1$-invariant. In local coordinates, we may actually assume that

$$a_{jk}(\xi, z) = a_j(\sqrt{k} \xi, \sqrt{k} z), \quad b_{jk}(\xi, z) = b_j(\sqrt{k} \xi, \sqrt{k} z),$$  \(38\)

for fixed functions $a_j, b_j$. Then

$$u_{k,\infty}(x + w/\sqrt{k})_j \sim u_{k,\infty}(x + w/\sqrt{k})_{ja} + u_k(x + w/\sqrt{k})_{jb},$$

where $u_{k,\infty}(x + w/\sqrt{k})_{ja}$ is obtained by multiplying the integrand in \(35\) by $a_{jk}(y)$ - and integration is thus over $A_{jk}$ - and similarly for $u_{k,\infty}(x + w/\sqrt{k})_{jb}$.

**Lemma 3.5.** $u_{k,\infty}(x + w/\sqrt{k})_{ja} = O(k^{-N}), N = 1, 2, \ldots$.

**Proof.** In view of Lemma 3.5, for every $(e^{i\phi}, \exp G(\xi)) \in S^1 \times E$, we have

$$\text{dist}_G(\mu_{e^{-\epsilon}} \circ r_{\phi}(x_j + w_j/\sqrt{k}), y) \geq \text{dist}_M \left( \mu_{e^{-\epsilon}} \left( \pi(x_j + w_j/\sqrt{k}) \right), \pi(y) \right) \geq \frac{1}{2} \|z - S_j(\xi)\| + O(k^{-1/2}).$$  \(39\)
Denote by $\text{dist}_X$ the geodesic distance function on $X$. Fix an open neighborhood $R \ni x_j$ with compact closure, contained in the chosen chart adapted to $\Lambda$. Let

$$C := \max_{x' \in R} \{ \| d_{x'} H_j \| \}.$$  

If $x', x'' \in R$ have local coordinates $(\theta', z')$, $(\theta'', z'')$ then

$$\| H_j(x') - H_j(x'') \| \leq C \| z' - z'' \|. \quad (40)$$

Here $\| \cdot \|$ is the standard norm in $\mathbb{C}^n$.

Choose $c \in (0, C)$ satisfying the conclusions of Corollary 2.4.

Now let us set

$$A_{jk}^{(1)} =: \left\{ (g, y) \in A_{jk} : \text{dist}_M (\mu_{g^{-1}}(\pi(x)), \pi(y)) > \frac{c}{10 C} \gamma(g, y) \right\}, \quad (41)$$

$$A_{jk}^{(2)} =: \left\{ (g, y) \in A_{jk} : \text{dist}_M (\mu_{g^{-1}}(\pi(x)), \pi(y)) < \frac{c}{5 C} \gamma(g, y) \right\}. \quad (42)$$

Let $\tau_1 + \tau_2 = 1$ be a partition of unity of $A_{jk}$ subordinate to the open cover $A_{jk}^{(1)} \cup A_{jk}^{(2)} = A_{jk}$. We may assume that $\tau_1$ and $\tau_2$ are fixed functions of $(g, y)$, independent of $k$.

Suppose first that $(g, y) \in A_{jk}^{(1)}$. By Lemma 3.4 and the hypothesis $\gamma(g, y) > k^{-1/3}$, this implies $\| z - S_j(\xi) \| > c k^{-1/3} / (20 C)$. Given this, (39) implies

$$\text{dist}_X (\mu_{e^{-t}} \circ r_\vartheta(x_j + w_j / \sqrt{k}), y) \geq \frac{1}{3} \| z - S_j(\xi) \| \quad (43)$$

for $k \gg 0$. In view of Corollary 1.3 of [BS], we deduce with $g = \exp_G(\xi)$:

$$\left| d_t \Psi_j (\mu_{g^{-1}} \circ r_\vartheta(x_j + w_j / \sqrt{k}), y) \right| = \left| \psi_j (\mu_{g^{-1}} \circ r_\vartheta(x_j + w_j / \sqrt{k}), y) \right| \geq C \| z - S_j(\xi) \|^2, \quad (44)$$

with $C_1 > 0$ an appropriate constant.

The differential operator $L^{(1)} =: \psi^{-1} \frac{\partial}{\partial t}$ is thus well-defined and smooth on $A_{jk}^{(1)}$, is positively homogeneous of degree $-1$ in $t$, and satisfies

$$L^{(1)} \Psi_j = 1, \quad \| L^{(1)} (1, y, g) \| \leq C_2 / \| z - S_j(\xi) \|^2.$$
If on the other hand \((g, y) \in A_{jk}^{(2)}\) and \(k \gg 0\), then necessarily \(\text{dist}_G(g, e) = \gamma(g, y) > k^{-1/3}\). If \(g = \exp_G(\xi)\), we also deduce

\[
\|S_j(\xi) - z\| \leq 2 \text{dist}_M(\mu_{g^{-1}}(\pi(x)), \pi(y)) < \frac{2c}{5C}\gamma(\exp_G(\xi), y)
\]

\[
= \frac{2c}{5C}\text{dist}_G(g, e) \leq \frac{4c}{5C}\|\xi\|.
\]

Thus, if \(\xi \in g, g = \exp_G(\xi) \in R\), and \(c > 0\) is as in Corollary 2.3 then

\[
\|d_\eta \Psi_j\| = \|H_j(y)\| = \|H_j(y) - H_j(\mu_g(x)) + H_j(\mu_g(x))\| \\
\geq \|H_j(\mu_g(x))\| - \|H_j(\mu_g(x)) - H_j(y)\| \\
\geq c\|\xi\| - C\|S_j(\xi) - z\| \\
\geq \left(c - \frac{4c}{5C}\right)\|\xi\| \geq C_3\|\xi\| \geq C_3\|\xi\|^2, \quad (45)
\]

since we may assume \(\|\xi\| < 1/2\) if \(g = \exp_G(\xi) \in R\). Arguing as above, one can then produce a linear first order differential operator \(L^{(2)}\) on \(A_{jk}^{(2)}\), positively homogeneous of degree \(-1\) in \(\eta\) and with no zero order term, such that \(L^{(2)}\Psi_j = 1\), whence \(L^{(2)}(e^{i_k\Psi_j}) = ike^{i_k\Psi_j}\), and \(\|L^{(2)}\| \leq C_4\|\xi\|^2\).

Then \(L =: \tau_1L^{(1)} + \tau_2L^{(2)}\) is a first order linear partial differential operator on \(A_j\), positively homogeneous of degree \(-1\) in \((t, \eta)\), having no zero order term, and satisfying \(L\Psi_j = 1\). Hence \(L(e^{i_k\Psi_j}) = ike^{i_k\Psi_j}\), and \(\|L\| \leq C'/(\|z, \xi\|^2)\) for some \(C' > 0\). Let \(L^T\) be the transpose operator (norms and transposes are in the given local coordinates); then for every \(s = 1, 2, \ldots\) there exists a constant \(C_s > 0\) such that

\[
\|L^T\|^s \leq C_s\|z, \xi\|^{2s}. \quad (46)
\]

In \(L^T\) and its powers only \(t\)- and \(\eta\)-derivatives occur, and the coefficients are functions of \((g, y)\).

Now let us set \(S = \mathbb{R} \times \mathbb{R}^n \times (0, +\infty) \times (-2\epsilon, 2\epsilon)\), and \(dX =: dt\,dy\,dt\,d\vartheta\). Let us write \(e^{i_k\Psi_j} F_j\) for the integrand in the expression for \(u_{k, w}(x + w/\sqrt{k})_{ja}\); the latter is obtained from (45) by inserting the additional factor \(a_{jk}\). Then (Remark 3.1)

\[
u_{k, w}(x + w/\sqrt{k})_{ja} \sim k^{n+2-s} \frac{\dim(V_w)}{(2\pi)^{n+2}i^s} e^{-ik\vartheta} \int_S \int_{A_{jk}} e^{i_k\Psi_j} (L^T)^s(F_j) dX \, dg \, dy. \quad (47)
\]

Introducing radial coordinates in the \((z, g)\)-variables and invoking (46),

\[
\left|u_{k, w}(x + w/\sqrt{k})_{ja}\right| \leq D_s k^{n+2-s} \int_{k^{-1/3}}^\infty r^{2n+g-1-2s} dr \\
= D'_s k^{-(n-g-s)/3},
\]

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where $D_s, D'_s$ are appropriate positive constants. This completes the proof of Lemma 3.5.

We shall now determine the asymptotics of $u_{k,\omega}(x + w/\sqrt{k})_{jb}$. To this end, let us perform the following change of integration variables:

$$\theta' = \theta, \quad p' = p - h_j(q), \quad q' = q. \quad (48)$$

At the origin, the Jacobian of this transformation is the identity. In the new coordinates, the phase function (34) becomes

$$\Psi_j(\tau, \eta, t, g, \vartheta, \theta, \rho, \theta', \rho', q') = t\psi_j \left( \mu_{g^{-1}} \circ r_\vartheta \left( x_j + \frac{w_j}{\sqrt{k}} + R \left( \frac{w_j}{\sqrt{k}} \right) \right), (\theta', p' + h_j(q'), q') + \tau (\theta' - f_j(q')) + \eta \cdot \rho' - \vartheta, \right) \quad (49)$$

where $R : C^n \to C^n$ vanishes to second order at the origin. Thus, $R_k = R \left( \frac{w_j}{\sqrt{k}} \right) = O(k^{-1})$ as $k \to +\infty$. In the following, we shall work in the new coordinates and omit the primes for notational simplicity.

We shall next rescale our coordinates by a factor $k^{-1/2}$, as follows. First, let us rescale the local coordinates on $G$ in the neighbourhood $E \ni e$, by writing $\xi = \nu/\sqrt{k}$; thus on $E$ we have $g = \exp_G(\xi) = \exp_G(\nu/\sqrt{k})$. Let us also rescale in the same manner the new coordinates (48) centered at $x_j$ in the horizontal direction; more precisely, let us write $(\theta, p, q) = (\theta, \rho/\sqrt{k}, s/\sqrt{k})$. Here $\nu \in \mathbb{R}^g$ (given our choice of an orthonormal basis of $g$) and $r, s \in \mathbb{R}^n$. In Heisenberg coordinates, $(\theta, \rho/\sqrt{k}, s/\sqrt{k})$ corresponds to $(\theta, (r + is)/\sqrt{k} + h_j(s)/k)$.

By the definition (36) of $B_{jk}$, integration in $d\nu dr ds$ takes place over a ball of radius $O(k^{1/6})$ in $\mathbb{R}^g \times \mathbb{C}^n$.

Let us express (49) in rescaled coordinates, and define

$$\Psi_{jk}(\tau, \eta, t, \nu, \vartheta, \rho, r, s) = t\psi_j \left( \mu_{\nu^{-1}} \circ r_\vartheta \left( x_j + \frac{w_j}{\sqrt{k}} + R \left( \frac{w_j}{\sqrt{k}} \right) \right), \rho, r, s \right). \quad (50)$$

By Lemma 2.1, $f_j$ vanishes to third order at the origin. Thus $f_j(s/\sqrt{k}) = k^{-3/2} f_{jk}(s)$ for a smooth function $f_{jk}$ vanishing to third order at the origin. We obtain

$$\Psi_{jk} = t\psi_j \left( \mu_{\nu^{-1}} \circ r_\vartheta \left( x_j + \frac{w_j}{\sqrt{k}} + R_k \right), (\theta, k^{-1/2} r + k^{-1} h_j(s), k^{-1/2} s) \right) + \tau \theta + k^{-1/2} \eta \cdot r - \vartheta + k^{-3/2} f_{jk}(s). \quad (51)$$

As usual, we may identify $w_j$ with a tangent vector in $T_{m_j} M$, where $m_j =: \pi(x_j)$; clearly, $m_j = g_j \cdot m$. Let $\nu_M$ be the vector field on $M$ generated by $\nu \in g$. 

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Lemma 3.6. The adapted Heisenberg coordinates of $\mu_{e^{-\nu/\sqrt{k}}} \circ r_\theta(x_j + w_j/\sqrt{k} + R_k)$ are

$$\left( \vartheta - \frac{2}{k} \Omega_{m_j}(\nu_M(m_j), w_j) + Q \left( \frac{w_j}{\sqrt{k}}, \frac{\nu}{\sqrt{k}} \right), \frac{1}{\sqrt{k}} (w_j - \nu_M(m_j)) + T \left( \frac{w_j}{\sqrt{k}}, \frac{\nu}{\sqrt{k}} \right) \right),$$

where $Q, T : \mathbb{C}^n \times \mathbb{R}^k \to \mathbb{C}^n$ vanish at the origin to third and second order, respectively.

Thus, $Q_k = Q \left( \frac{w_j}{\sqrt{k}}, \frac{\nu}{\sqrt{k}} \right) = O(k^{-3/2})$ and $T_k = T \left( \frac{w_j}{\sqrt{k}}, \frac{\nu}{\sqrt{k}} \right) = O(k^{-1})$ as $k \to +\infty$ for fixed $\nu$ as $k \to +\infty$.

Proof. Clearly, the preferred holomorphic coordinates of $\mu_{e^{-\nu/\sqrt{k}}} (m_j + w_j/\sqrt{k} + R_k)$ are $k^{-1/2}(w_j - \nu_M(m_j)) + T \left( \frac{w_j}{\sqrt{k}}, \frac{\nu}{\sqrt{k}} \right)$, for some $\mathbb{C}^n$-valued function $T$ vanishing to second order at the origin. By construction, the Heisenberg coordinates of $\mu_{e^{-\nu/\sqrt{k}}} \circ r_\theta(x_j + w_j/\sqrt{k} + R_k)$ then have the form

$$\left( \theta(1/\sqrt{k}), \frac{1}{\sqrt{k}} (w_j - \nu_M(m_j)) + T \left( \frac{w_j}{\sqrt{k}}, \frac{\nu}{\sqrt{k}} \right) \right),$$

for some smooth function $\theta : (-\delta, \delta) \to \mathbb{R}$.

To determine the latter, let us momentarily set $\vartheta = 0$ and consider the path, defined for sufficiently small $s, t \in \mathbb{R}$,

$$\gamma_s : t \in (-\delta, \delta) \mapsto \mu_{e^{-\nu}} (x_j + s w_j + R(s w_j)),$$

where $R$ is as in (49); thus, $R(s w_j)$ is a smooth function $(-\delta, \delta) \to \mathbb{C}^n$ vanishing to second order at $s = 0$. Let us write $w_j = p_w + iq_w$, $\nu_M(p_j) = p_\nu + iq_\nu$. The preferred coordinates of $\pi(\gamma_s(t))$ are given by $(s p_w - t p_\nu) + i(s q_w - t q_\nu) + Q(s w, t \nu)$, where $Q : \mathbb{C}^n \times \mathbb{R}^k \to \mathbb{C}^n$ vanishes to second order at the origin. Thus, the Heisenberg coordinates of $\gamma_s(t)$ have the form

$$\left( \theta(s, t), (s p_w - t p_\nu) + i(s q_w - t q_\nu) + Q(s w, t \nu) \right),$$

for some real-valued smooth function $\theta(s, t)$.

Claim 3.1. $\theta(s, t) = (s t) \cdot d_0 + \theta_1(s w, t \nu)$, where $d_0$ depends only on $w$ and $\nu$, while $\theta_1$ vanishes to third order at $s = t = 0$.

Proof. By construction, $\theta(s, 0)$ vanishes identically. Therefore, $\theta(s, t) = t \theta_1(s, t)$, for a smooth function $\theta_1$. Given that $G$ acts horizontally on $X'$, Lemma 3.3 implies that $\theta(0, t)$ vanishes to third order at $t = 0$. Thus, $\theta_1(s, t) = at^2 + ts b(t) + s d(s, t)$. The claim follows by writing $d(s, t) = d_0 + d_1(s, t)$, where $d_1(0, 0) = 0$. 

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Next we shall determine $d_0$ by use of (20). The pull-back $\gamma^*_s(\alpha) \in \Omega^1(-\delta, \delta)$ is given by

$$
\gamma^*_s(\alpha)(t) = \{sd_0 + [(sq_w - tp_{\nu})p_{\nu} - (sp_w - tp_{\nu})q_{\nu}] \} dt + G_1(s, t) dt
$$

where $G_1(s, t)$ vanishes to second order at $s = t = 0$. On the other hand, if $\nu_X$ is the vector field on $X$ generated by $\nu \in \mathfrak{g}$, then

$$
\alpha(\nu_X) = \Phi^\nu = :<\Phi, \nu >
$$

(equation (5.1) of [GS1]). Since $\Phi^{-1}(0)$ is $G$-invariant, $\frac{\partial \Phi^\nu}{\partial t} |_{(0, t)} = 0$ for every $t$. Thus,

$$
\Phi(\gamma_s(t)) = s d_{m_j} \Phi(w_j) + G_2(s, t),
$$

where $G_2$ vanishes to second order at the origin. Hence,

$$
\gamma^*_s(\alpha)(t) = -\langle \Phi(\gamma_s(t)), \nu \rangle dt = - s \left[d_{m_j} \Phi(w_j), \nu \right] dt + G_3(s, t) dt
$$

where $G_3$ vanishes to second order, and $\Phi^\nu = :\langle \Phi, \nu \rangle$ is the Hamiltonian function associated to $\nu$. Thus $G_1 = G_3$, and $d_0 = -2 \Omega_{m_j} (\nu_M(m_j), w_j)$. Lemma 3.6 now follows in the case $\vartheta = 0$ by letting $s = t = k^{-1/2}$; in general we need only notice that $r_{\vartheta}$ corresponds in local Heisenberg coordinates to a translation by $\vartheta$.

Following the notation of [BSZ], let us set

$$
K_2(u, v) = : u \cdot \overline{v} - \frac{1}{2} (\|u\|^2 + \|v\|^2)
$$

$$
i \Im (u \cdot \overline{v}) - \frac{1}{2} \|u - v\|^2 \quad (u, v \in \mathbb{C}^n).
$$

In view of the asymptotic expansion for the phase discussed in the proof of the scaling limit of the Szegö kernel in [BSZ] and [SZ], Lemma 3.6 implies
that $\Psi_{jk}$ in (51) has an asymptotic $k$-expansion of the form

$$
\Psi_{jk} \sim it \left[ 1 - e^{i(\vartheta - \frac{2}{k} \Omega_{m_j}(\nu \mu (m_j), w_j) - \theta)} \right] - \vartheta + \tau \theta + \frac{1}{\sqrt{k}} \eta \cdot r
$$

$$
- \frac{it}{k} e^{i(\vartheta - \theta)} K_2(w_j - \nu M(m_j), r + is) + O(k^{-3/2})
$$

$$
= it \left[ 1 - e^{i(\vartheta - \theta)} \right] - \vartheta + \tau \theta + \frac{1}{\sqrt{k}} \eta \cdot r
$$

$$
- \frac{t}{k} e^{i(\vartheta - \theta)} \left[ 2 \Omega_{m_j}(\nu M(m_j), w_j) + i K_2(w_j - \nu M(m_j), r + is) \right]
$$

$$
+ t e^{i(\vartheta - \theta)} P \left( \frac{1}{\sqrt{k}} (r + is), \frac{w}{\sqrt{k}} \right),
$$

where $P : \mathbb{C}^n \times \mathbb{R}^6 \rightarrow \mathbb{C}$ vanishes to third order at the origin. Hence, $P_k =: P \left( \frac{1}{\sqrt{k}} (r + is), \frac{w}{\sqrt{k}} \right) = O(k^{-3/2})$ for fixed $r, s \in \mathbb{R}^n$ as $k \rightarrow +\infty$.

On the upshot, we have $u_{k,\omega}(x + w/\sqrt{k})_j \sim u_{k,\omega}(x + w/\sqrt{k})_{jb}$ and

$$
u_{k,\omega}(x + w/\sqrt{k}) \rho = \dim(V_{\omega}) k^{2-g/2} e^{-ik\theta l} \int_{\mathbb{R}^8} \int_{\mathbb{R}^n} F_{jk}(\nu, s) d\nu ds;
$$

here, performing the coordinate change $\eta \rightarrow \sqrt{k} \eta$, we have set

$$
F_{jk}(\nu, s) =: k^{n/2} \int_0^{+\infty} e^{ikS} A dr d\theta d\tau d\eta d\vartheta dt;
$$

where the complex phase

$$
S(\vartheta, \tau, \eta, \nu, \nu, \rho) =: it \left[ 1 - e^{i(\vartheta - \theta)} \right] - \vartheta + \tau \theta + \eta \cdot r
$$

satisfies $\Im(S) \geq 0$, and the amplitude $A$ is

$$
A =: e^{-it e^{i(\vartheta - \theta)}} \left[ 2 \Omega_{m_j}(\nu \mu (m_j), w_j) + i K_2(w_j - \nu M(m_j), r + is) \right]
$$

$$
e^{ikt e^{i(\vartheta - \theta)}} P \left( \frac{1}{\sqrt{k}} (r + is), \frac{w}{\sqrt{k}} \right) g_j(y) f_\lambda(q) D_\Lambda(q)
$$

$$
+ b_j(k^{-1/6} (\nu, r + is)) \zeta_j(\mu_{g^{-1}} \circ r_\theta(x_j + w_j/\sqrt{k}), y, kt)
$$

$$
+ \gamma_j(e^{i(\vartheta - \theta)}, g_j^{-1} e^{\nu/\sqrt{k}}) \chi_{\omega}(e^{-\nu/\sqrt{k}} g_j) H_G(\nu/\sqrt{k});
$$

in (53), the integration from 0 to $+\infty$ refers to $dt$. In (57), $H_G$ denotes the Haar density on $G$, expressed in the local coordinates given by the exponential chart; thus, $H_G(0) = 1$. The factor $b_j(k^{-1/6} (\nu, r + is))$ comes from setting $\xi = \nu/\sqrt{k}$ and $z = (r + is)/\sqrt{k}$ in $b_{jk}$ given by (58).

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As above, let us write \( w_j = p_w + iq_w \), with \( p_w, q_w \in \mathbb{R}^n \). The term
\[
2 \Omega_{m_j} (v_M(m_j), w_j) + i K_z (w_j - v_M(m_j), r + i s)
\]
appearing in the exponent in the first factor of \([57]\), may be rewritten:
\[
2 (p_v q_w - q_w p_v) - (q_w - q_v) \cdot r + (p_w - p_v) \cdot s
- \frac{i}{2} (||p_w - p_v - r||^2 + ||q_w - q_v - s||^2).
\]

**Lemma 3.7.** i): As \( k \to +\infty \), there is an asymptotic expansion, uniform on compact subsets of \( \mathbb{R}^g \times \mathbb{R}^n \),
\[
F_{jk}(\nu, s) \sim k^{n/2-2} Z_{j0}(\nu, s) + \sum_{f \geq 1} k^{(n-f)/2-2} Z_{jf}(\nu, s).
\]

The coefficient of the leading term is:
\[
Z_{j0}(\nu, s) = \frac{(2\pi)^{n+2}}{\pi^n} \chi_{\omega}(g_j) f_\lambda(x_j)
\times e^{-\frac{1}{2}||p_w-p_v||^2} e^{-i(p_w-p_v)s-2i(p_v q_w-q_v p_w)-\frac{i}{2}||q_w-q_v-s||^2}.
\]

ii): there exist positive constants \( c > 0 \) and \( C_f > 0 \) for every \( f = 1, 2, \ldots \) such that \( |Z_{jf}(\nu, s)| < C_f e^{-c(||s||^2+||\nu||^2)} \), for every \( (\nu, s) \in \mathbb{R}^g \times \mathbb{R}^n \).

iii): For every \( \ell = 0, 1, 2, \ldots \) there exists \( D_\ell > 0 \) such that
\[
|F_{jk}(\nu, s) - \sum_{f=0}^{\ell} k^{(n-f)/2-2} Z_{jf}(\nu, s)| \leq D_\ell k^{(n-\ell-1)/2-2} e^{-c(||s||^2+||\nu||^2)}
\]
for every \( (\nu, s) \in \mathbb{R}^g \times \mathbb{R}^n \).

**Proof.** i): On any fixed compact subset of \( \mathbb{R}^g \times \mathbb{R}^n \), we have \( H_G(\nu/\sqrt{k}) = 1 + O(k^{-1/2}) \), \( \mu_{e^{i/\sqrt{k}}} = \text{id} + O(k^{-1/2}) \), \( \chi_x(e^{-\nu/\sqrt{k}} g_j) = \chi_x(g_j) + O(k^{-1/2}) \) as \( k \to +\infty \). Incorporating the terms \( O(k^{-1/2}) \) into the amplitude, \([55]\) may be interpreted as an oscillatory integral, with complex phase \([56]\), and whose amplitude may be developed in descending powers of \( k^{-1/2} \).

Since \( r = \frac{\partial S}{\partial q} \), the asymptotic contribution to \( F_{jk}(s) \) from the region \( ||r|| \geq 1 \), say, is \( O(k^{-\infty}) \). Therefore, we may assume that both \( r \) and \( s \) are bounded in norm, and so \( b_j (k^{-1/6}(\nu, v)) = 1 \) if \( k \gg 0 \). The proof of the following is left to the reader:

**Claim 3.2.** The phase \( S \) has only one stationary point \( (\vartheta_0, t_0, \theta_0, \tau_0, \eta_0, r_0) \), given by \( t_0 = \tau_0 = 1 \), \( \vartheta_0 = \theta_0 = 0 \), \( r_0 = \eta_0 = 0 \). The Hessian of \( S \) at this

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stationary point is

\[
\begin{bmatrix}
1 & -i & -1 & 0 & 0 & 0 \\
-i & 0 & i & 0 & 0 & 0 \\
-1 & i & 1 & -i & 0 & 0 \\
0 & 0 & -i & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -i & I_n \\
0 & 0 & 0 & 0 & 0 & -iI_n
\end{bmatrix}.
\]

Now (i) follows in view of (23) and (58) by the complex stationary phase Lemma (Theorem 7.7.5 of [H]).

ii): By our choice of adapted Heisenberg local coordinates for \( \Lambda \) centered at \( x_j \), and by Corollary 2.4 \( \nu \mapsto p_\nu \) is an injective \( \mathbb{R} \)-linear map \( g \to \mathbb{R}^n \); therefore, so is the affine map \( A_j : g \oplus \mathbb{R}^n \to \mathbb{C}^n \) given by

\[ A_j(\nu, s) = (p_w - p_\nu) + i(q_w - q_\nu - s). \]

Hence there exist \( c, d > 0 \) such that

\[ \|p_w - p_\nu\|^2 + \|q_w - q_\nu - s\|^2 \geq c (\|\nu\|^2 + \|s\|^2) - d. \]

(59)

Now ii) follows in view of (58) and i).

iii): In view of the cut-off \( b_j(k^{-1/6} (\nu, r + is)) \), we may suppose \( \|\nu, s\| \leq k^{1/6} \) (and \( |r| \leq 1 \), as above). Let us make the coordinate change \( s' = s/k^{1/6} \), \( \nu' = \nu/k^{1/6} \), so that \( \nu' \) and \( s' \) may be assumed to be bounded. Then (59) may still be interpreted as an oscillatory integral, with complex phase \( S \), and whose amplitude is \( S_1/6 \). We may then apply the stationary phase Lemma as in i), ii) and plug back in \( s = k^{1/6} s' \) and \( \nu = k^{1/6} \nu' \) in the result.

This completes the proof of Lemma 3.7.

In view of (54), Lemma 3.7 implies the asymptotic expansion

\[ u_{k,\varpi}(x + w/\sqrt{k})_{j_b} = k^{(n-g)/2} \varrho_0(k, \varpi, w)^{(j)} + \sum_{f \geq 1} k^{(n-g-f)/2} \varrho_f(k, \varpi, w, x)^{(j)}, \]

(60)

where

\[ \varrho_f(k, \varpi, w, x)^{(j)} = \frac{\dim(V_\varpi)}{(2\pi)^{n+2}} e^{-ik\theta_j} \int_{\mathbb{R}^g} \int_{\mathbb{R}^n} Z_{jf}(\nu, s) d\nu ds. \]

(61)

In particular, the coefficient of the leading term is

\[ \varrho_0(k, \varpi, w)^{(j)} = \frac{\dim(V_\varpi)}{\pi^n} e^{-ik\theta_j} \chi_\varpi(g_j) f_\lambda(x_j) \]

\[ \int_{\mathbb{R}^g} \int_{\mathbb{R}^n} e^{-\frac{1}{2} \|p_w - p_\nu\|^2} e^{-i(p_w - p_\nu) s - \frac{1}{2} \|q_w - q_\nu - s\|^2} d\nu ds. \]

(62)
To integrate in $ds$, let us make the change of variables $s \rightsquigarrow s + q_\nu - q_w$, and recall that the function $e^{-\|x\|^2/2}$ on $\mathbb{R}^n$ equals its own Fourier transform. We obtain:

$$g_0(k, \omega, w)^{(j)} = \frac{\dim(V_\omega)}{\pi^n} (2\pi)^{\frac{n}{2}} e^{-i k \omega} \chi_\omega(g_j) f_\lambda(x_j) \int_{\mathbb{R}^n} e^{-\|p_\omega-p_\nu\|^2} e^{-i(p_\omega-p_\nu)(q_\omega-q_\nu) + 2\{p_\nu q_\omega - q_\nu p_\omega\}} d\nu. \quad (63)$$

Let us now decompose $w_j$ as in (8), and to simplify our notation let us write $w_\nu$ for $(w_j)_\nu, \ldots (j$ being fixed in our argument). Let us write $w_\alpha, \ldots$ in local coordinates as column vectors $(p_\alpha q_\alpha)^t, \ldots \in \mathbb{R}^{2n}$. More precisely, we have:

**Claim 3.3.** i): $w_\alpha = \begin{pmatrix} p_\alpha \\ 0 \end{pmatrix}$ where $p_\alpha \cdot p_\nu = 0 \forall \nu \in \mathfrak{g}$;

ii): $w_b = \begin{pmatrix} 0 \\ q_b \end{pmatrix}$ and $w_c = \begin{pmatrix} 0 \\ q_c \end{pmatrix}$ for appropriate $q_b, q_c \in \mathbb{R}^n$, and $p_\nu \cdot q_c = 0, \forall \nu \in \mathfrak{g}$;

iii): $w_d = \begin{pmatrix} p_{\nu w} \\ q_{\nu w} \end{pmatrix}$, for a unique $\nu_w \in \mathfrak{g}$.

**Proof.** For the second statement of ii), recall that $w_c \in T_{m_j} \mathcal{N} \subseteq T_{m_j} \mathcal{M}$, and the latter is the symplectic annihilator of $\mathfrak{g}_M(m_j)$. Everything else is immediate.

**Lemma 3.8.** Let $p_{w_j}, q_{w_j} \in \mathbb{R}^n$ be as in Claim 3.3 Then for every $\nu \in \mathfrak{g}$ one has

$$p_\nu \cdot q_w - q_\nu \cdot p_w = p_\nu \cdot q_b - q_\nu \cdot p_a. \quad (64)$$

**Proof.** The left hand side of (64) is the symplectic pairing $\Omega_{m_j}(\nu_M(m_j), w_j)$. We have $p_w = p_\alpha + p_{\nu w}, q_w = q_b + q_c + q_{\nu w}$. However, recalling that $\mathfrak{g}_M(m_j) \subseteq T_{m_j} \mathcal{M}$ is an isotropic subspace, we have $\Omega_{m_j}(\nu_M(m_j), \nu_{\omega}(m_j)) = 0$ for every $\nu \in \mathfrak{g}$; therefore, $p_{\nu w}$ and $q_{\nu w}$ may be ignored. The statement then follows from ii) of Claim 3.3.

Let us make the change of variables $\beta = \nu - \nu_w$ in (63). The real part of the exponent in (63) then is $-\|p_\alpha\|^2 - \|p_\beta\|^2$, while the imaginary part may be written as

$$- (p_\alpha - p_\beta) \cdot [(q_b + q_c) - q_\beta] - 2 [(p_\beta + p_{\nu w}) \cdot q_b - (q_\beta + q_{\nu w}) \cdot p_a] = \left[ \Omega_{m_j}(w_j - w_d, w_a) + 2 \Omega_{m_j}(w_j, w_d) \right] - (p_\beta \cdot q_b - 3 q_\beta \cdot p_a) - p_\beta \cdot q_\beta. \quad (65)$$
Let us set \( C(w_j) = \Omega_{m}(w_j - w_d, w_a) + 2 \Omega_{m}(w_j, w_d). \) We may then rewrite the right hand side of (63) as
\[
\rho_{0}(k, \varpi, w^{(j)}) = \dim(V_{\varpi}) (2\pi)^{\frac{n}{2}} e^{-ik\varpi} \chi_{\varpi}(g_j) f_{\lambda}(x_j) e^{-\|w_a\|^2 + iC(w_j)} \int_{\mathbb{R}^g} e^{-\|p_{\beta}\|^2 + ip_{\beta} \cdot q_{\beta}} e^{-i(p_{\beta} q_{\beta} - 3q_{\beta} a_{\beta})} d\beta.
\] (66)

In (66), the integral is over \( g \), identified with \( \mathbb{R}^g \) by means of an orthonormal basis for the Haar metric. Thus, the Lebesgue measure \( d\beta \) corresponds to the Haar measure at the identity \( e \in G \). To make our statement more intrinsic, we shall now rewrite the latter integral as an integral over \( g_{M}(m_j) \subseteq T_{m_j} M \), with the induced metric.

**Lemma 3.9.** Fix \( t \in M' \). Suppose that \( \mathcal{B} \) is an orthonormal basis of \( g \) for the Haar metric, and that \( \mathcal{B}_t \) is an orthonormal basis of \( g_{M}(t) \) for the induced metric from \( T_{m_j} M \). Identify \( g \) with \( g_{M}(t) \) by the linear isomorphism \( \xi \mapsto \xi \mathcal{M}(t) \). Let \( A = M_{\mathcal{B}}^{\mathcal{B}}(\text{id}_g) \) be the matrix of the base change. Then
\[
|\det(A)| = \frac{1}{V_{\text{eff}}(t) |G_t|},
\]
where \( V_{\text{eff}}(t) \) is the effective potential at \( t \), and \( G_t \subseteq G \) is the stabilizer subgroup of \( t \).

**Remark 3.2.** Since \( m_j = \mu_{g_j}(m) \), we have \( V_{\text{eff}}(m_j) = V_{\text{eff}}(m) \) and \( |G_{m_j}| = |G_m| \).

**Proof.** Suppose \( \mathcal{B} = \{v_1, \ldots, v_g\} \), \( \mathcal{B}_t = \{w_1, \ldots, w_g\} \) so that \( w_j = \sum_{i=1}^{g} a_{ij} v_i \), where \( A = [a_{ij}] \). Hence
\[
w_1 \wedge \cdots \wedge w_g = \det(A) v_1 \wedge \cdots \wedge v_g.
\] (67)

Let \( \text{dens}_G \) be the Haar density on \( G \), so that \( \int_G \text{dens}_G = 1 \); hence \( \text{dens}_G(v_1 \wedge \cdots \wedge v_g) = 1 \). Let \( \text{dens}_t \) be the pull-back to \( G \) of the invariant metric density on the orbit \( G \cdot t \cong G/G_t \) under the degree - \( |G_t| \) covering map \( g \mapsto g \cdot t \). By invariance,
\[
\text{dens}_t = V_{\text{eff}}(t) \cdot |G_t| \cdot \text{dens}_G.
\] (68)

By construction,
\[
1 = \text{dens}_t(w_1 \wedge \cdots \wedge w_g) = |\det(A)| \cdot V_{\text{eff}}(t) \cdot |G_t| \cdot \text{vol}_G(v_1 \wedge \cdots \wedge v_g)
= |\det(A)| \cdot V_{\text{eff}}(t) \cdot |G_t|.
\]
Now let $\beta$ and $u$ denote the linear coordinates on $\mathfrak{g}$ associated to the basis $\mathcal{B}$ and $\mathcal{B}_q$, respectively; thus, $\beta = Au$. By the Lemma,

$$d\beta = (V_{\text{eff}}(q) | G_q)^{-1} du. \quad (69)$$

We have already exploited the following consequence of Corollary 2.1: working in adapted Heisenberg local coordinates, the projection of $\mathfrak{g}_M(m_j) \subseteq T_{m_j} M \cong \mathbb{C}^n \cong \mathbb{R}^n \times \mathbb{R}^n$ onto $\mathbb{R}^n \times \{0\}$,

$$p_{\text{real}} : \nu_{M}(m_j) \mapsto p_\nu, \quad (\nu \in \mathfrak{g})$$

is injective. Hence there exists a linear map $T : \mathbb{R}^n \to \mathbb{R}^n$ such that $q_\nu = T(p_\nu)$ for every $\nu \in \mathfrak{g}$ (if we so wish, we may determine $T$ uniquely by imposing that it vanishes on the Euclidean orthocomplement of $p_{\text{real}}(\mathfrak{g}_M(p_j)) \subseteq \mathbb{R}^n$). We shall think of $T$ as an $n \times n$ real matrix.

On the upshot, identifying $\mathfrak{g} \cong \mathbb{R}^g$ by $\mathcal{B}_{m_j}$, and $T_{m_j} M \cong \mathbb{C}^n$ by the given choice of adapted Heisenberg local coordinates, the inclusion $\mathfrak{g} \cong \mathfrak{g}_M(m_j) \hookrightarrow T_{m_j} M$ may be written

$$\iota(u) = Ru + iT Ru \quad (u \in \mathbb{R}^g), \quad (70)$$

for a certain $n \times g$ real matrix $R$ of maximal rank $g$. Since $\mathcal{B}_{m_j}$ is orthonormal for the induced metric, we have $RR^t + R^tTR = I_g$.

**Lemma 3.10.** $R^tTR$ is a $g \times g$ symmetric matrix.

**Proof.** Since $m_j \in M'$, $\mathfrak{g}_M(m_j) \subseteq T_{m_j} M$ is an isotropic subspace. Thus, for every $\xi, \nu \in \mathfrak{g}$ we have

$$\Omega_{m_j}(\xi_M(m_j), \nu_M(m_j)) = \Omega_0 \left( \begin{pmatrix} p_\xi \\ q_\xi \end{pmatrix}, \begin{pmatrix} p_\nu \\ q_\nu \end{pmatrix} \right) = \Omega_0 \left( \begin{pmatrix} R\xi \\ TR \xi \end{pmatrix}, \begin{pmatrix} R\nu \\ TR \nu \end{pmatrix} \right) = \xi^t(R^tTR - R^tTR)\nu,$$

where $\Omega_0$ denotes the standard symplectic structure on $\mathbb{R}^n \times \mathbb{R}^n$.

Now (66) may be rewritten as

$$g_0(k, \varpi, w)^{(j)} = \frac{\dim(V_{\varpi})}{V_{\text{eff}}(q) | G_q|} (2\pi)^{\frac{n+1}{2}} e^{-ik\theta} \chi_\varpi(g_j) f_\lambda(x_j) e^{-\|w_a\|^2 + iC(w_j)} \int_{\mathbb{R}^g} e^{-u^t(R^tR + iR^tTR)u} e^{-iu^tR^t(q_b - 3T^tp_a)} du. \quad (71)$$
Up to a scalar factor \((2\pi)^{\frac{n}{2}}\), the integral in (71) is the evaluation at \(R^t(q_b - 3T^t p_a)\) of the Fourier transform of the function \(e^{-\frac{1}{2}(u,Au)}\) on \(\mathbb{R}^g\), where \(A = 2(R^tR + i R^tTR)\) is a \(g \times g\) complex symmetric matrix with positive definite real part. By Theorem 7.6.1 of [H], we have

\[
\varrho_0(k, \varpi, w)^{(j)} = \frac{\dim(V_{\varpi})}{V_{\text{eff}}(q)} \left[ \frac{1}{|G_q|} \sqrt{\frac{(2\pi)^{\frac{n+g}{2}}}{2^g}} \right] e^{-ik\varpi} \chi_{\varpi}(g_j) f_\lambda(x_j) \det \left( R^tR + i R^tTR \right)^{-1/2} \exp(-Q(w_j)) \tag{72}
\]

where

\[
Q(w_j) = \|p_a\|^2 - iC(w_j) + \frac{1}{4} \left( R^t(q_b - 3T^t p_a), (R^tR + i R^tTR)^{-1} R^t(q_b - 3T^t p_a) \right)
= S(w'_j) + iP(w_j), \tag{73}
\]

where \(S\) and \(P\) denote real valued quadratic forms. Here \((,\,\,)\) is the standard Euclidean scalar product on \(\mathbb{R}^g\). Thus, if \((R^tR + i R^tTR)^{-1} = F + iG\), where \(F\) and \(G\) are \(g\times g\) real symmetric matrices,

\[
S(w'_j) =: \|p_a\|^2 + \frac{1}{4} \left( R^t(q_b - 3T^t p_a), FR^t(q_b - 3T^t p_a) \right). \tag{74}
\]

Recall that \(w'_j = w_a + w_b\) in the decomposition described in Definition 1.1 and Claim 3.3.

**Lemma 3.11.** \(S(w'_j) \geq 0\), and equality holds only if \(w'_j = 0\).

**Proof.** Since \(F\) is positive definite by construction, both summands in (74) are \(\geq 0\). Suppose \(S(w'_j) = 0\). Then both summands vanish, whence \(p_a = 0\) and \(R^t q_b = 0\). Thus we are reduced to proving:

**Lemma 3.12.** If \(R^t q_b = 0\), then \(q_b = 0\).

**Proof.** By construction, the range of \(R\) is

\[
p_{\text{real}}(\mathfrak{g}_M(m_j)) = \{ p_\nu : \nu \in \mathfrak{g} \} \subseteq \mathbb{R}^n.
\]

Recall that the symplectic annihilator of \(\mathfrak{g}_M(m_j)\) is given by \(\mathfrak{g}_M(m_j)^0 = T_{m_j} M'\). Hence, in view of the identification \(T_{m_j} M \cong \mathbb{R}^n \oplus \mathbb{R}^n\) (and viewing as usual \(T_{x_j} \Lambda\) as a subspace of \(T_{m_j} M\),

\[
\ker(R^t) = \left\{ q \in \mathbb{R}^n : (0, q) = 0 \forall \nu \in \mathfrak{g} \right\} = \left\{ q \in \mathbb{R}^n : \left( \begin{array}{c} 0 \\ q \end{array} \right) \in \mathfrak{g}_M(m_j)^0 \right\} = \left\{ q \in \mathbb{R}^n : \left( \begin{array}{c} 0 \\ q \end{array} \right) \in T_{x_j} \Lambda' \right\}.
\]

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By definition, \( w_b \in (T_{x_j} \Lambda')^\perp \). Thus if \( R'tq_b = 0 \) then

\[
\begin{pmatrix}
0 \\
q_b
\end{pmatrix} \in T_{x_j} \Lambda' \cap (T_{x_j} \Lambda')^\perp = \{0\}.
\]

This completes the proof of Theorem 1.1.

**Remark 3.3.** Let us now consider the question described in the introduction, i.e. whether quantization commutes with reduction for a transverse compact Legendrian submanifold \( \Lambda \subseteq X \). For the sake of brevity, we shall make fairly brutal simplifying assumptions, and leave it to the interested reader to work out more general cases.

Let us assume that \( G \) acts freely on \( M' \), and - to fix ideas - that \( \Lambda \) meets every \( S^1 \times G \)-orbit in \( X' \) at most once. Thus, \( \Lambda' =: \Lambda \cap X' \) is an \( (n - g) \)-dimensional isotropic submanifold, which projects diffeomorphically onto a compact Legendrian submanifold \( \Lambda_0 \subseteq X_0 =: X'/G \) (\( \Lambda \) and \( \Lambda_0 \) actually map down diffeomorphically onto Lagrangian submanifolds in \( M \) and \( M_0 \), respectively). Also, let us choose as a half-density on \( \Lambda \) the Riemannian half-density \( \text{dens}^{(1/2)}(\Lambda) \), so that \( f_{\Lambda'} = 1 \).

Let us fix \( x \in \Lambda' \), and let \( x_0 \in X_0 \) be its image in \( X_0 \). To simplify, let us also assume that \( \Lambda \) is perpendicular to the \( G \)-orbit \( G \cdot x \) at \( x \), so that -referring to (70) - we have \( T = 0 \) and \( R^t R = I_g \). In view of Theorem 1.1, we then have:

\[
u_{k,0}(x) \sim \frac{\pi^{-n}}{V_{\text{eff}}(\pi(x))} \sqrt{\frac{(2\pi)^{n+g}}{2^g} k^{(n-g)/2} + \sum_{f \geq 1} \theta_f k^{(n-g-f)/2}}.
\]

Now there are two natural ways to induce a half-density on \( \Lambda' \cong \Lambda_0 \): One is to choose the Riemannian half-density, \( \Lambda' = \text{dens}^{(1/2)}_{\Lambda_0} \), so that \( f_{\Lambda'} = 1 \). The other is to divide \( \text{dens}^{(1/2)}_{\Lambda} \) by the half-density on \( g^* \cong g \) associated to the Haar metric. Let \( \Lambda'' \) be the half-density obtained in this manner. By arguments similar to those in Lemma 3.9 one can see that \( f_{\Lambda''}(x_0) = V_{\text{eff}}(\pi(x))^{-1/2} \).

If \( u_k' \in H^0(M_0, L_0^\otimes k) \cong \mathcal{H}_k(X_0) \) is the sequence associated to \( \Lambda' \), then by Corollary 1.1 we have:

\[
u'_k(x) \sim \frac{(2\pi)^{g}}{\pi^n} k^{(n-g)/2} + \sum_{f \geq 1} \theta_f k^{(n-f)/2}.
\]

If on the other hand \( u_k'' \in H^0(M_0, L_0^\otimes k) \cong \mathcal{H}_k(X_0) \) is the sequence associated to \( \Lambda'' \), the leading order term gets multiplied by \( V_{\text{eff}}(\pi(x))^{-1/2} \).
4 The Hermitian products.

Let us now assume that $\Lambda, \Sigma \subseteq X$ are two compact Legendrian submanifolds, and that $\lambda$ and $\sigma$ are given smooth half-densities on $\Lambda$ and $\Sigma$, respectively. Let $u := \Pi_X(\delta_{\Lambda,\lambda})$, $v := \Pi_X(\delta_{\Sigma,\sigma})$. Let as usual $\varpi$ be a fixed highest weight of $G$. We shall study in this section the asymptotics of the Hermitian products $(u_k,\varpi, v_k,\varpi)_{L^2(X)}$ as $k \to +\infty$.

In the action-free case, we shall reproduce expansions similar to those in [BPU], except for some differences due to the fact that we are dealing with half-densities rather than half-forms.

To this end, let us first of all recall that we are unitarily and equivariantly identifying functions and half-densities on $X$. Furthermore, the self-duality pairing $<,>$ and the $L^2$-unitary product $(,)_{L^2}$ of smooth half-densities $\tau = f \cdot \text{dens}_X$ and $v = g \cdot \text{dens}_X$ are related by $\int_X f \cdot \overline{g} \text{dens}_X = (\tau, v)_{L^2} = \langle \tau, v \rangle$.

Suppose then that $u_t$, $t > 0$, is a family of smooth half-densities on $X$ such that $u_t \to u$ as $t \to 0$, in the topology of the space of all generalized half-densities whose wave front is conormal to $\Lambda$. In view of the self-adjointness of the orthogonal projector $\Pi_{k,\varpi}$ on $\mathcal{H}(X)_{k,\varpi}$, we obtain:

\[
(u_k,\varpi, v_k,\varpi)_{L^2(X)} = \lim_{t \to 0} \left( \Pi_{k,\varpi}(u_t), v_k,\varpi \right)_{L^2(X)} = \lim_{t \to 0} (u_t, v_k,\varpi)_{L^2(X)}
\]

\[
= \lim_{t \to 0} \left< u_t, \overline{v_k,\varpi} \right> = \left< u, \overline{v_k,\varpi} \right>
\]

\[
= \int_{\Lambda} f_{\lambda} \cdot \overline{v_k,\varpi} \text{dens}_{\Lambda}, \tag{75}
\]

where $\text{dens}_{\Lambda}$ is the Riemannian density on $\Lambda$.

4.1 The transverse case.

Consider the smooth map given by group action restricted to $\Lambda$,

$$
\Upsilon : (h, g, x) \in S^1 \times G \times \Lambda \mapsto (h, g) \cdot x \in X,
$$

To fix ideas, suppose first that $\Upsilon$ is transversal to $\Sigma' = \Sigma \cap \left( (\Phi \circ \pi)^{-1}(0) \right)$. In this case, $\Upsilon^{-1}(\Sigma')$ is a finite set:

$$
\Upsilon^{-1}(\Sigma') = \{ \tilde{y}_1, \ldots, \tilde{y}_r \},
$$

where $\tilde{y}_j = (h_j, g_j, y_j)$ for some $h_j \in S^1$, $g_j \in G$ and $y_j \in \Lambda'$. Hence $\tilde{\gamma}_j := \Upsilon(\tilde{y}_j) = (h_j, g_j) \cdot \tilde{y}_j \in \Sigma'$ for every $j$. 37
Now let $U_j \subseteq \Lambda$ be some arbitrarily small neighbourhood of $y_j$. Since $v_{k,\omega} = O(k^{-\infty})$ away from $\Sigma'$, in view of (75) we have

$$
(u_{k,\omega}, v_{k,\omega})_{L^2(X)} \sim \sum_{j=1}^{r} \int_{U_j} f_{\lambda} \cdot \overline{v_{k,\omega}} \, \text{dens}_{\Lambda}.
$$

Let us fix Heisenberg local coordinates $(p, q, \theta)$ for $X$ centered at $\hat{y}_j$ and adapted to $\Sigma$, defined on an open neighbourhood $V_j \ni \hat{y}_j$. Thus, $\Sigma \cap V_j \subseteq V_j$ is defined by conditions $\theta = f(q)$ and $p = h(q)$, as described in §2.2. We may arrange, given our assumptions, that $T_{\hat{y}_j} \Sigma' = \text{span} \left\{ \frac{\partial}{\partial q_1} \bigg|_{\hat{y}_j}, \ldots, \frac{\partial}{\partial q_{n-g}} \bigg|_{\hat{y}_j} \right\}$. 

The following is left to the reader:

**Lemma 4.1.** Given (77), we have

$$
\mathfrak{g}_X(\hat{y}_j) = \text{span} \left\{ \frac{\partial}{\partial p_{n-g+1}} \bigg|_{\hat{y}_j} + t_{n-g+1}, \ldots, \frac{\partial}{\partial p_n} \bigg|_{\hat{y}_j} + t_n \right\},
$$

for appropriate $t_{n-g+1}, \ldots, t_n \in T_{\hat{y}_j} \Sigma'$.

Let us now consider the Legendrian submanifold

$$
\hat{y}_j \in \Lambda_j := \Upsilon \left( \{(h_j, g_j)\} \times \Lambda \right) \subseteq X,
$$

obtained by 'translating' $\Lambda$ by the action of $(h_j, g_j) \in S^1 \times G$. Given (77) and Lemma 4.1, the present transversality assumption implies:

**Lemma 4.2.** In the above situation, $(p_1, \ldots, p_{n-g})$ restrict to local coordinates on $\Lambda'_j$ centered at $\hat{y}_j$, and $(p_1, \ldots, p_{n-g}, q_{n-g+1}, \ldots, q_n)$ restrict to local coordinates on $\Lambda_j$ centered at $\hat{y}_j$.

Therefore $(p_1, \ldots, p_{n-g}, q_{n-g+1}, \ldots, q_n)$ may be viewed in a natural manner as local coordinates on $\Lambda$ centered at $y_j$, defined on some open neighbourhood $U_j \subseteq \Lambda$. In order to apply Theorem 1.1 we need to relate these coordinates on $\Lambda$ to the local Heisenberg coordinates on $X$. Given $x = (x_1, \ldots, x_n)$, to simplify our notation let us write $x = (x', x'')$, where $x' = (x_1, \ldots, x_{n-g})$, $x'' = (x_{n-g+1}, \ldots, x_n)$. The following is left to the reader:

We have:
Lemma 4.3. There exists an \( \mathbb{R} \)-linear map
\[
A_j : \mathbb{R}^n \to \mathbb{C}^n \cong \{0\} \oplus \mathbb{C}^n \subseteq \mathbb{R} \oplus \mathbb{C}^n
\]
such that if \( y \in U_j \subseteq \Lambda \) has local coordinates \( \frac{1}{\sqrt{k}} (p', q'') \) on \( \Lambda \), then it has local Heisenberg coordinates \( \frac{1}{\sqrt{k}} A_j (p', q'') + O(k^{-1}) \).

Let \( y \left( \frac{1}{\sqrt{k}} (p', q'') \right) \) denote the point in \( U_j \) having local coordinates \( \frac{1}{\sqrt{k}} (p', q'') \). By Theorem 1.1 and Lemma 4.3, passing to rescaled coordinates on \( U_j \) we may then write the \( j \)-th summand in (76) as:
\[
\int_{U_j} f_y \cdot \overline{v_{k, \omega}} \text{dens}_\Lambda = k^{-n/2} \int_{\mathbb{R}^n} f_y \left( (k^{-1/2} (p', q'')) \right) \overline{v_{k, \omega}} \left( k^{-1/2} A_j (p', q'') + O(k^{-1}) \right) D_\Lambda \left( k^{-1/2} (p', q'') \right) \, dp' \, dq''
\]

Inserting the asymptotic expansion of Theorem 1.1 in (79), we conclude

Proposition 4.1. If \( \Upsilon : S^1 \times G \times \Lambda \to X \) is transversal to \( \Sigma' \), the \( j \)-th summand in (83) is
\[
\int_{U_j} f_y \cdot \overline{v_{k, \omega}} \text{dens}_\Lambda \sim k^{-g/2} \rho_0^{(j)} + \sum_{f \geq 1} k^{-(g+f)/2} \rho_f^{(j)},
\]
where
\[
\rho_0^{(j)} = \frac{\dim(V_{\omega})}{|G_{\pi(y_j)}|} \frac{1}{\pi^n} \sqrt{\frac{(2\pi)^{n+g}}{2^g}} h_j^k \chi_{\omega}(g_j^{-1}) \overline{\chi_{\Lambda}(y_j)} f_{\lambda}(y_j) f_{\sigma}(y_j)
\cdot \int_{\mathbb{R}^n} e^{-s \gamma_j \left( A_j(p', q''), A_j(p', q'') \right)} - iT_{\gamma_j} \left( A_j(p', q''), A_j(p', q'') \right) \, dp' \, dq''.
\]

In the action-free case, the present transversality assumption means that \( S^1 \times \Lambda \to X \) is transverse to \( \Sigma \). For every \( j = 1, \ldots, r \), \( T_{\pi(y_j)} \Lambda_j \subseteq T_{\pi(y_j)} M \) is a Lagrangian subspace transversal to \( T_{\pi(y_j)} \Sigma \). Thus, in the given Heisenberg local coordinates adapted to \( \Sigma \) at \( \hat{y}_j \), we have
\[
T_{\pi(y_j)} \Lambda_j = \{ (p, Z_j p) : p \in \mathbb{R}^n \} \subseteq T_{\pi(y_j)} M \cong \mathbb{R}^n \oplus \mathbb{R}^n,
\]
where \( Z_j \) is a symmetric matrix. Therefore, the \( p \)'s restrict to a system of local coordinates on \( \Lambda_j \) (whence on \( \Lambda \)), and \( A_j(p) = p + iZ_j p \).

Let \( \gamma_{\pi(y_j)} : \text{Gr}_{\text{lag}}(T_{\pi(y_j)} M) \times \text{Gr}_{\text{lag}}(T_{\pi(y_j)} M) \to \mathbb{R} \) be the invariant introduced in section 2.3, let us write \( J_j = J_{\pi(y_j)} \). Applying the asymptotic expansion of Corollary 1.1

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Corollary 4.1. Suppose that the two projections \( \Lambda \rightarrow M \) and \( \Sigma \rightarrow M \) are transversal. Let \( \Upsilon : S^1 \times \Lambda \rightarrow \Sigma \) be the map induced by the action, and suppose \( \Upsilon^{-1}(\Sigma) = \{ \bar{y}_1, \ldots, \bar{y}_r \} \), where \( \bar{y}_j = (h_j, y_j) \). Set \( \bar{y}_j =: h_j \cdot y_j \) and \( \Lambda_j =: h_j \cdot \Lambda \) for every \( j \). Then

\[
(u_k, v_k) \sim \rho_0 + \sum_{f \geq 1} k^{-f/2} \rho_f,
\]

where

\[
\rho_0 = \frac{(2\pi)^{\frac{n}{2}}}{\pi^n} \sum_{j=1}^{r} h_j^k \frac{f_{\lambda}(y_j)}{f_{\sigma}(y_j)} \nu_j (T_{\bar{y}_j} \Lambda_j, T_{\bar{y}_j} \Sigma)^{-1} \int_{\mathbb{R}^n} e^{-\|p\|^2 + ip^\dagger Z_j p} dp.
\]

4.2 The clean case.

Now we shall make the following more general hypothesis:

i): \( \Lambda \) and \( \Sigma \) are both transversal to \( X' \); let us set \( \Lambda' =: \Lambda \cap X', \Sigma' =: \Sigma \cap X' \).

ii): the smooth map given by group action restricted to \( \Lambda \),

\[
\Upsilon : (h, g, x) \in S^1 \times G \times \Lambda \mapsto (h, g) \cdot x \in X,
\]

meets \( \Sigma' \) nicely; by this, we mean that every connected component of \( \Upsilon^{-1}(\Sigma') \) is a manifold, and that for every \( \varsigma = (h, g, x) \in \Upsilon^{-1}(\Sigma') \) we have

\[
T_{\varsigma} (\Upsilon^{-1}(\Sigma')) = (d_{\varsigma} \Upsilon)^{-1} (T_{\Upsilon(\varsigma)} \Sigma').
\]

iii): there exist integers \( r_{\Sigma}, r_{\Lambda} \geq 1 \) such that for every \( x \in \Sigma' \) and \( y \in \Lambda' \) one has

\[
|\Sigma \cap (G \cdot x)| = r_{\Sigma} \quad \text{and} \quad |\Lambda \cap (G \cdot y)| = r_{\Lambda}.
\]

iv): \( G \) acts freely on \( M' \).

Definition 4.1. Let us set \( \hat{\Upsilon} =: \Upsilon^{-1}(\Sigma') \subseteq S^1 \times G \times \Lambda \). Let \( \pi_{\Lambda} : S^1 \times G \times \Lambda \rightarrow \Lambda \) be the projection onto the third summand, and let us set \( Y =: \pi_{\Lambda}(\hat{Y}) \subseteq \Lambda' \).

Lemma 4.4. Let \( \Lambda' = \Lambda \cap X' \). Then there exists an open neighbourhood \( V \subseteq \Lambda \) of \( \Lambda' \) such that \( \Upsilon \) is immersive on \( S^1 \times G \times V \).

Proof. This follows from the horizontality of \( \Lambda \) and of the \( G \)-action on \( X' \), and from Corollary 2.1.

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Proposition 4.2. Suppose that the hypothesis i), ii) and iii) above are satisfied. Let $\tilde{\mathcal{Y}_1}, \ldots, \tilde{\mathcal{Y}_r} \subseteq S^1 \times G \times \Lambda'$ be the connected components of $\tilde{\mathcal{Y}}$, and let $\mathcal{Y}_j =: \pi_\Lambda(\tilde{\mathcal{Y}_j}) \subseteq \Lambda'$. Then:

i): for every $j = 1, \ldots, r$, there exists $h_j \in S^1$ such that 
\[ \tilde{\mathcal{Y}_j} \subseteq \{h_j\} \times G \times \Lambda'; \]

ii): every $\mathcal{Y}_j$ is a submanifold, and the induced map $\pi_j : \tilde{\mathcal{Y}_j} \to \mathcal{Y}_j$ is an unramified covering;

iii): the $\mathcal{Y}_j$’s, with possible repetitions, are the connected components of $\mathcal{Y}$.

Proof. i): Suppose that $(h, g, x) \in \tilde{\mathcal{Y}_j}$ for some $j$, and consider $(a, v, w) \in T_{(h, g, x)} \tilde{\mathcal{Y}_j}$. Since $\Upsilon(\mathcal{Y}_j) \subseteq \Sigma$ and $\Sigma$ is Legendrian, we conclude that
\[ 0 = \alpha_{\Upsilon(h, g, x)} (d_{(h, g, x)} \Upsilon(a, v, w)) = \alpha_{\Upsilon(h, g, x)} \left( a \frac{\partial}{\partial \theta} + d_{(h, g, x)} \Upsilon(0, v, w) \right) \]
\[ = a + \alpha_{\Upsilon(h, g, x)} (d_{(h, g, x)} \Upsilon(0, v, w)) = a. \]

The latter equality follows from the horizontality of $\Lambda$ and of the $G$-action on $X'$. Since $\tilde{\mathcal{Y}_j}$ is connected, the statement follows.

ii) and iii): Let $\tilde{\pi}_j : \tilde{\mathcal{Y}_j} \to \Lambda'$ be the projection. If $\tilde{\pi}_j$ is not an immersion, by part i) there exists $(h, g, x) \in \tilde{\mathcal{Y}_j}$ and a tangent vector of the form $(0, v, 0) \in T_{(h, g, x)} \tilde{\mathcal{Y}_j}$, for some $0 \neq v \in T_g G$. By Lemma 4.3
\[ 0 \neq d_{(h, g, x)} \Upsilon((0, v, 0)) \in \left[ T_{\Upsilon(h, g, x)} \Sigma \right] \cap \left[ T_{\Upsilon(h, g, x)} (G \cdot \Upsilon(h, g, x)) \right], \]
against Corollary 2.1.

Suppose now that $h \in \{h_1, \ldots, h_r\} \subseteq S^1$. Let 
\[ \mathcal{Y}(h) =: \bigcup_{h_j = h} \tilde{\mathcal{Y}_j}. \]

Suppose that $y \in \pi_\Lambda(\mathcal{Y}(h))$; there are as many inverse images of $y$ in $\mathcal{Y}(h)$ as there are group elements $g \in G$ such that $(h, g) \cdot y = g \cdot (h \cdot y) \in \Sigma$; in other words,
\[ |\mathcal{Y}(h) \cap \pi_\Lambda^{-1}(y)| = |\Sigma \cap (G \cdot h \cdot y)| = d_\Sigma. \]
(82)

On the other hand, in an immersion with compact domain the number of points in a inverse image can only jump up. Therefore, given $\mathcal{Y}(h)$, the cardinality of a fibre has to constant for each map $\tilde{\mathcal{Y}_j} \to \mathcal{Y}_j$, for every connected component $\tilde{\mathcal{Y}_j}$ of $\mathcal{Y}(h)$. If on the other $f : X \to Y$ is an immersion, and
\[ |f^{-1}(y)| \text{ is constant for every } y \in f(X), \text{ then } f(X) \text{ is a manifold and the} \]
\[ \text{induced map } X \to f(X) \text{ is an unramified covering.} \]

We note in passing that by the same argument, and given the symmetry of our hypothesis on \( \Lambda \) and \( \Sigma \), we also have:

**Proposition 4.3.** For every \( j = 1, \ldots, r \), let \( \Sigma_j = \Upsilon(\tilde{Y}_j) \). Then the \( \Sigma_j \)'s are disjoint manifolds, and are the connected components of \( \Upsilon(\tilde{Y}) \). The induced map \( \tilde{Y} \to \Upsilon(\tilde{Y}) \) is an unramified covering.

**Definition 4.2.** For every \( j = 1, \ldots, r \), set \( c_j =: n - \dim(Y_j) \) and let \( d_j \) be the degree of the unramified cover \( \pi_j : \tilde{Y}_j \to Y_j \). Thus, for every \( y \in Y_j \) there exist distinct \( s_{ij}(y), \ldots, s_{d_j,j}(y) \in G \) such that \( (h_j, s_{ij}(y), y) \in \tilde{Y}_j \), and therefore \( (h_j, s_{ij}(y)) \cdot y \in \Sigma', i = 1, \ldots, d_j \). Locally on \( Y_j \) near \( y \) we may think of \( s_{ij} \)'s as \( G \)-valued smooth maps. The \( s_{ij} \)'s are not globally well-defined as smooth maps \( Y_j \to G \); nonetheless, collectively they do define a smooth map from \( Y_j \) to the appropriate symmetrized product of \( G \).

Now let \( U_j \subseteq \Lambda \) be some arbitrarily small tubular neighbourhood of the submanifold \( Y_j \subseteq \Lambda \). Since \( v_{k,\infty} = O(k^{-\infty}) \) away from \( \Sigma' \), in view of (75) we have

\[
(u_{k,\infty}, v_{k,\infty})_{L^2(X)} \sim \sum_{j=1}^{r} \int_{U_j} f_{\lambda} \cdot v_{k,\infty} \text{ dens}_\Lambda. \quad (83)
\]

**Remark 4.1.** Since the \( Y_j \)'s are not necessarily all distinct, (83) is not literally true. However, to avoid making our exposition too heavy, we shall be slightly vague on this; we shall thus act as the \( Y_j \)'s were all disjoint. In the following computations, each summand in (83) will split as the sum of various other contributions, and we shall not sum the same contribution twice.

Suppose \( 1 \leq j \leq r \). For every \( y \in Y_j \), we may find an open neighbourhood \( y \in S \subseteq Y_j \) which is uniformly covered by \( \tilde{\pi}_j \), meaning that \( \tilde{\pi}_j^{-1}(S) = \bigcup_{i=1}^{d_j} \tilde{S}_i \subseteq \tilde{Y}_j \), a disjoint union where each \( \tilde{S}_i \) projects diffeomorphically onto \( S \) under \( \tilde{\pi}_j \), and \( (h_j, s_{ij}(y), y) \in \tilde{S}_i \).

Perhaps after restricting \( S \), by Lemma 4.4 we may further assume that for each \( i \) the map induced by \( \Upsilon \), \( (h_j, s_{ij}(y), y) \mapsto (h_j, s_{ij}(y)) \cdot y \) is a diffeomorphism onto its image,

\[
\tilde{S}_i =: \Upsilon(\tilde{S}_i) \subseteq \Sigma. \quad (84)
\]

We may then find a finite open cover \( \{S_{ja}\}_{a \in A} \) of \( Y_j \) with the following properties:
i): each $S_{ja}$ is the domain of a coordinate chart, say
\[ R_{ja} = (r_1, \ldots, r_{n-c_j}) : S_{ja} \to B_{n-c_j}(0, \epsilon) \subseteq \mathbb{R}^{n-c_j}, \]
for some $\epsilon > 0$;
ii): each $S_{ja}$ is uniformly covered by $\tilde{\pi}_j$, and $\tilde{\pi}_j^{-1}(S_{ja}) = \bigcup_{i=1}^{d_j} \tilde{S}_{ija}$ is a disjoint union, where for each $i, a$
\[ \tilde{S}_{ija} = \{(h_{ij}, s_{ij}(y), y) : y \in S_{ja}\} \subseteq \tilde{Y}_j; \quad (85) \]
iii): $\Upsilon$ induces a diffeomorphism $\tilde{S}_{ija} \sim \hat{S}_{ija} =: \Upsilon(\tilde{S}_{ija}) \subseteq \Sigma$ for every $i, a$;
iv): for every $i, a$ there exist an open neighbourhood $T_{ija} \subseteq X$ of $\hat{S}_{ija}$, and a smooth map $\kappa = \kappa_{ija} : \hat{S}_{ija} \times T_{ija} \to B_{2n+1}(0, \epsilon)$, such that for every $y \in \hat{S}_{ija}$ the partial function $\kappa_{y} : T_{ija} \to B_{2n+1}(0, \epsilon)$ is a Heisenberg chart adapted to $\Sigma$ at $y$ (Lemma 2.3).

Now recall that for every $j$ we have fixed a tubular neighbourhood $U_j \subseteq \Lambda$ of $Y_j$; let $p_j : U_j \to Y_j$ be the projection, and set $U_{ja} =: p_j^{-1}(S_{ja}) \subseteq U_j$. Thus, $\{U_{ja}\}_{a \in A}$ is a finite open cover of $U_j$. By introducing a partition of unity $\sum_a \varphi_{ja} = 1$, we may decompose the $j$-th summand in (83) as
\[ \int_{U_j} f_{\lambda} \cdot \overline{v_{k,\omega}} \text{dens}_{\Lambda} = \sum_a \int_{U_{ja}} \varphi_{ja} f_{\lambda} \cdot \overline{v_{k,\omega}} \text{dens}_{\Lambda}. \quad (86) \]

We are thus reduced to considering the asymptotics of each summand in (86). Given (85) we may apply a relative version of the argument in §4.1; rescaling will now be in the coordinates in $U_{ja}$ which are transversal to $Y_j$.

We now leave it to the reader to verify that, using the local coordinates $R_{ja} = (r_1, \ldots, r_{n-c_j})$ on $S_{ja}$, one obtains an asymptotic expansion
\[ \int_{U_{ja}} \varphi_{ja} f_{\lambda} \cdot \overline{v_{k,\omega}} \text{dens}_{\Lambda} \sim k^{(n-g-c_j)/2} \int_{\mathbb{R}^{n-c_j}} \rho_0^{(ja)}(r) \, dr \]
\[ + \sum_{f \geq 1} k^{(n-g-c_j-f)/2} \rho_f^{(ja)} \]

where, in view of the asymptotic expansion of Theorem 1.1
\[ \rho_0^{(ja)}(r) = \frac{\dim(V_{\omega}) \, 1}{V_{\text{eff}}[\pi(y(r))] \, \pi^n \sqrt{\frac{(2\pi)^n + g}{2g} \sum_l h_j[k] \chi_{\omega}[s_{lj}(y(r))]} \, e^{-S_{e}(z)-iP_{e}(z)} \, dz, \]

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for quadratic forms $S_r, P_r$ on $\mathbb{R}^{c_j}$, with $S_r$ positive definite. In the action-free case, this becomes:

$$
\int_{U_{ja}} \varphi_{ja} f_\lambda \cdot \overline{v_k} \text{d} \sigma \sim k^{(n-c_j)/2} \int_{\mathbb{R}^{n-c_j}} \rho_0^{(ja)}(r) \, dr
+ \sum_{f \geq 1} k^{(n-c_j-f)/2} \rho_f^{(ja)}
$$

where

$$
\rho_0^{(ja)}(r) = \frac{\dim(V_\varphi)}{V_{\text{eff}}[\pi(y(r))]} \frac{(2\pi)^{\frac{n}{2}}}{\pi^n} h_j^k \varphi_{ja}(y(r)) f_\lambda(y(r)) f_\sigma(\overline{h_j \cdot y(r)} T_{ja}(y(r)),
$$

$$
T_{ja}(y(r)) =: \mathcal{U}_{h_j,y(r)} \left( T_{h_j,y(r)}, A_j, T_{h_j,y(r)} \Sigma \right)^{-1} \int_{\mathbb{R}^{c_j}} e^{-\|p\|^2 + ip' Z_{h_j,y(r)} p} \, dr',
$$

$Z_r$ being an appropriate $c_j \times c_j$ symmetric matrix.

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