Triangular Peg Solitaire Unlimited

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Abstract

Triangular peg solitaire is a well-known one-person game or puzzle. When one peg captures many pegs consecutively, this is called a sweep. We investigate whether the game can end in a dramatic fashion, with one peg sweeping all remaining pegs off the board. For triangular boards of side 6 and 8 (with 21 and 36 holes, respectively) the geometrically longest sweep can occur as the final move in a game. On larger triangular boards, we demonstrate how to construct solutions that finish with arbitrarily long sweeps. We also consider the problem of finding solutions that minimize the total number of moves (where a move is one or more consecutive jumps by the same peg).

1 Introduction

Peg solitaire on a 15-hole triangular board is an old puzzle but it remains popular. In the United States, one can find these puzzles at tables in Cracker Barrel® restaurants. The 15-hole puzzle is also amenable to exhaustive computer search and this is a common programming assignment for computer science classes.

Here we consider peg solitaire on an (equilateral) triangular board with \( n \) holes on each side. This board will be referred to as \( \text{Triangle}(n) \) and can be conveniently presented on an array of hexagons (Figure 1). The \( \text{Triangle}(n) \) board has \( T(n) = n(n + 1)/2 \) holes, where \( T(n) \) is the \( n \)'th triangular number. The notation used to identify the holes in the board is shown in Figure 1—it differs from that given in Beasley [1] and is based on the system for square lattice boards. A nice property of this notation is that the top corner hole is always “a1”.

The rules of the game are simple: start from a board with a peg in every hole but one, called the starting vacancy. Then jump one peg over another into an empty hole, removing the jumped peg from the board. The goal is to choose a sequence of jumps to finish with one peg (the coordinate of this final peg is called the finishing location). This general problem

\footnotetext[1] {Original version at [http://gpj.connectfree.co.uk/gpjr.htm](http://gpj.connectfree.co.uk/gpjr.htm). Converted to \LaTeX{} by the author with some modifications to the text, November 2007.}
of going from a board position with one peg missing to a board position with one peg will be referred to as a **peg solitaire problem**. The special case where the starting vacancy and finishing location are the same is referred to as a **complement problem**. GPJ #28 [6] considered peg solitaire on boards based on a square lattice. Triangular peg solitaire differs in that the holes are based on a triangular lattice, with a maximum of six jumps available into any hole (rather than four). However the theory of the game (in particular the fundamental classes) is very similar, see [1], or [4] for details.

Any solution to a problem on the 15-hole triangular board consists of exactly 13 **jumps** because we start with 14 pegs and finish with 1. However, when the same peg jumps one or more pegs consecutively, we call this one **move**. Given a particular peg solitaire problem, what is the solution with the least number of moves? While this question has historically played an important role in peg solitaire on the standard 33-hole cross-shaped board, it has barely been considered for triangular boards. The following terminology is used in referring to moves involving multiple jumps: when a peg removes $i$ pegs in a single move, we refer to it as a **sweep**, or more specifically, an **$i$-sweep**.

After attempting a peg solitaire problem, many people get the idea to try to solve it backwards from the final peg. What is not so obvious is that this is exactly the same as the original game, where the concepts of “hole” and “peg” are interchanged. In fact the solution to any peg solitaire problem really contains two solutions: the original (“forward” solution) plus this “backward” solution, where the individual jumps are executed in the same direction, but in reverse order, and the starting vacancy and finishing location are swapped. An important observation is that when an $i$-sweep is reversed, we must reverse the individual jumps, and it becomes $i$ single jump moves. In other words, sweeps in forward solutions cannot be sweeps in reversed solutions (and vice versa).

Backward play is hard to comprehend, because our brain does not easily interchange the concepts of “hole” and “peg”. It is easier to understand “backward play” by realizing that it is the same as forward play from the **complement** of the current board position (the complement of a board position is where we replace every hole by a peg, and vice versa). This leads to a theorem used extensively in the remainder of this paper. Suppose we have a board position $B$ and we wonder if it could appear during a solution to a peg solitaire problem.

![Figure 1: The general triangular board with hole coordinates.](image-url)
**Forward/Backward Theorem:** Board position $\mathcal{B}$ can appear during a solution to a peg solitaire problem if and only if both

1. $\mathcal{B}$ can be reduced to a single peg (the finishing location) using peg solitaire jumps.
2. The complement of $\mathcal{B}$ can be reduced to a single peg (the starting location) using peg solitaire jumps.

A proof of this theorem just boils down to the equivalence of playing the game forward or backward. It may seem obvious, but is key to understanding how to reach complex sweep positions.

A practical problem that arises is finding a triangular board to play on. Fifteen hole Triangle(5) boards are common, but cannot be easily extended to larger triangular boards. The best board I have found is a Chinese Checkers set, which allows one to play on boards as large as Triangle(13). To help follow the arguments below, I recommend playing out the solutions on a Chinese Checkers set. It is particularly helpful to see solutions played forward and backward.

## 2 Maximal sweeps on odd-side triangular boards

Triangular boards support the longest sweeps of any peg solitaire board. This is because from any board location the total number of possible jumps is even. If the board size $n$ is odd, there exist sweeps that jump into or over every single location on the board. Such sweeps are maximal in the sense that they are the longest sweep geometrically possible on the board. The figure below shows examples of the first four maximal sweeps.

![Maximal Sweeps on Triangle](image)

Figure 2: Maximal Sweeps on Triangle($n$), where $n = 3, 5, 7$ and 9. These sweeps are shown starting and ending at a1, but can begin and end at other board locations.

The length of this sweep is $3T((n-1)/2) = 3(n^2 - 1)/8$. Related geometrical tours on a triangular lattice have appeared in GPJ #20 \[5\] under “Trapezoidal Tours”. As these sweep patterns become larger and larger, how can we be sure a peg can traverse the entire pattern? If we consider the pattern of the sweep as a graph, one of the most elementary theorems in graph theory, due to Euler, ensures that such a traversal is always possible when there are at most two nodes of odd degree. Since maximal sweeps have all nodes of even degree, they can always be traversed, and the starting and ending board locations must be the same. This theorem also guarantees that the more complex sweep patterns seen later can be traversed.
Note that the sweep length divided by the board size approaches $3/4$ as the board size increases. If this sweep is the final move to a solitaire game it removes nearly $3/4$ of the pegs that we started with in one move! But can these maximal sweeps be realized during a solitaire game? If we take the complement of the sweep pattern, not a single jump is possible. Thus, by the forward-backward theorem, maximal sweep patterns cannot appear in solitaire games on odd size boards.

### 3 Maximal sweeps on even-side triangular boards

The same maximal sweep on an odd triangular board is also maximal on the even board one size larger, because the added row cannot be reached to extend the sweep. However, the added row can make it possible to reach the sweep position during a peg solitaire game. On the \textbf{Triangle}(6) and \textbf{Triangle}(8) boards this can be worked out by hand, Figure 3 shows this process on \textbf{Triangle}(6):

![Figure 3](image)

Figure 3: Constructing a solitaire solution that finishes with the 9-sweep to a1. (a) Backward: playing from the complement of the sweep pattern to c5. (b) Forward: The completed solution from the c5 vacancy ending with a 9-sweep to a1.

The forward solution in Figure 3b has only 9 moves. In the final section of this paper, we prove that it’s \textit{impossible} to solve this problem in fewer than 9 moves. Thus, besides containing a maximal length sweep, the solution of Figure 3b solves the problem in the minimum number of moves. This solution was discovered before 1975 by Harry O. Davis [3].

There are three problems on \textbf{Triangle}(6) that can contain a 9-sweep:

1. Vacate c5, play to finish at a1 with the last move a 9-sweep (Figure 3).
2. Vacate c5, play to finish at a4 with the last move a 9-sweep.
3. Vacate c5, play to finish at a4 with the second to the last move a 9-sweep.

The reader is invited to solve problems 2 and 3. As in Figure 3, the trick is to figure out what the final sweep must look like, and then solve backwards (playing forward from the complement of the sweep position). Then exactly reverse the jumps in your “backward” solution and you will reproduce the sweep position.

On Triangle(8), the 18-sweep can also be reached. There are three problems on this board that can contain the 18-sweep:

1. Vacate c5, play to finish at a1 with the last move an 18-sweep (Figure 4).
2. Vacate b6 or e6, play to finish at c8 with the last move an 18-sweep.
3. Vacate c5, play to finish at b6 with the second to the last move an 18-sweep.

These problems are more difficult than those on Triangle(6), but are still quite reasonable to work out by hand. Figure 4 shows the solution to the first problem, played forward. This solution was discovered, as usual, by attempting to play backward from the complement of the sweep position. This solution contains 15-moves, but it is possible to solve this problem in 14 moves (without the 18-sweep), so this solution does not have the minimum number of moves.

On Triangle(10), a computational search for a peg solitaire solution containing a maximal 30-sweep has come up empty (although a solution was found which ends with a 29-sweep). Triangle(12) does not appear to have a maximal 45-sweep solution either. I haven’t checked all possible configurations of this 45-sweep, but my program has shown the most obvious candidates can’t be reached from a single vacancy start. It appears that the Triangle(8) board is the largest triangular board for which a solution to a peg solitaire problem can contain a maximal sweep.
4 Long sweeps on arbitrarily large boards

Although maximal sweeps appear not to be attainable in peg solitaire problems on large triangular boards, it turns out sweeps of only slightly reduced length are possible. A very special solution was discovered by hand on the 78-hole Triangle(12) board. This solution finishes with a 42-sweep when run forward, and is shown in Figure 5.

This solution is remarkable because it can be extended to even larger triangular boards. We can extend the final sweep to cover the bottom of Triangle(14) and the complement can still be reduced to a single peg. We do this by keeping moves along the bottom row the same, while extending other moves vertically. In particular moves 1 & 2 end at the same board locations but begin from the bottom row of the board. The U-shaped moves 9 & 10 become vertically elongated, but have the same starting and ending board locations. Finally additional moves need to be added after move 11 to reduce the remaining pattern of pegs to a single survivor (which does not end up at the same location as in Figure 5b). We can continue stepping the board size up by 2 and still reduce the complement to a single peg up to Triangle(20). However if we try this on Triangle(22), we find we can no longer reduce the remaining pattern to a single peg.
As the board is extended, the middle of the lower portion of the board becomes similar to the original problem on Triangle(12). This key insight suggests that we may be able to perform an inductive step, and extend the board indefinitely. This is in fact possible, and will be described below. In effect we can construct solutions to peg solitaire problems on triangular boards of arbitrarily large size. Not only that, these solutions finish with arbitrarily long sweep moves!

Figure 6: Fitting the inductive components together to make a long finishing sweep on Triangle(24).

To complete the inductive step, we add another solution for Triangle(12) underneath the first one to obtain a solution on the 300-hole Triangle(24) board. Figure 6 shows the overall geometry of the solution; we use “component A” (Figure 5) in the upper part of the board and “component B” (Figure 7) in the lower part of the board. In the reversed solution, the remaining (white) areas of the board are cleared by extensions of the moves to clear A, and the final move snakes down vertically through both A and B to the bottom of the board.

The final sweep pattern will be close to the maximal sweep, but component A contains a “defect” in this sweep pattern, and so must component B. Figuring out exactly where to place this defect in component B is critical to making everything work out. At first I tried putting the defect symmetrically over the y-axis of the board, but this never seems to work out. What did work was to put it slightly off center.

While component A can be considered a solution itself on Triangle(12), component B is inherently tied to the board above it, for there must be moves which link the two boards. Figure 7 shows how to solve component B, which looks very similar to component A. Note, however, that the U-shaped moves actually go in the opposite direction from before. The entire solution is constructed to enable the final move to pass vertically down the board.

The final sweep pattern always starts at a1 and finishes at a3. The initial vacancy, or finishing location for the reversed solution is always directly under a3 on the lowest row that has a hole in vertical alignment with a3 (on Triangle(24), this starting vacancy is at k23). The entire solution on Triangle(24), when executed in the forward direction, has a final move
Figure 7: Inductive Component B. Backward: showing how to reduce the complement of the sweep pattern to one peg. The top two holes (in green) are part of the board above.

which is a 191-sweep (Figure 8):

Figure 8: Forward: the finishing 191-sweep on Triangle(24). This sweep pattern is the maximal 198-sweep with two “defects”.

By stacking additional copies of component B under the diagram of Figure 7, we can extend this process indefinitely. The net result is that on Triangle(12i), we can construct a solution to a solitaire problem that finishes with a sweep of length $54i^2 - 13i + 1$ (this is $4i - 1$ shorter than the maximal sweep length). The board itself has $72i^2 + 6i$ holes, so asymptotically this finishing sweep removes $3/4$ of the pegs on the board. The forward solution consists of $18i^2 + 19i - 3$ jumps, followed by the final sweep move. By reordering a few jumps, we can reduce the total number of moves in the forward solution slightly to $18i^2 + 14i - 3$. Table 1
gives statistics on these solutions as the board size increases.

| i | Board   | Board size (# holes) | Final sweep length (# jumps) | % pegs removed by final sweep | Forward solution length (# moves) |
|---|---------|----------------------|-----------------------------|------------------------------|----------------------------------|
| 1 | Triangle (12) | 78                   | 42                          | 55.3%                        | 29                               |
| 2 | Triangle (24) | 300                  | 191                         | 64.1%                        | 97                               |
| 3 | Triangle (36) | 666                  | 448                         | 67.5%                        | 201                              |
| 10| Triangle (120) | 7,260               | 5,271                       | 72.6%                        | 1,937                            |

Table 1: Statistics on long sweep solutions on Triangle$(12^i)$.

Although this construction technique has given us solutions on boards with sides a multiple of 12, it is not hard to extend it to Triangle$(n)$ for any even $n \geq 12$. The way to do this is using the same technique we used to extend component $A$ on Triangle$(12)$ to Triangle$(14)$, Triangle$(16)$, etc.

5 Short solutions

Given a board and a (solvable) peg solitaire problem, what is the least number of moves that can solve it? While this question has been important in the history of the standard 33-hole cross-shaped board, it has not received much attention on triangular boards. Informally, this question can be rephrased: “What is the solution which involves touching the smallest number of pegs?”

In 1966, Harry O. Davis studied short solutions on the Triangle$(5)$ board analytically [2]. He was able to find minimal solutions “for all starting locations” and prove that his solutions were the shortest possible. In particular, he found a 10 move solution to the a1-complement, and proved that the problem could not be solved in less than 10 moves.

I have now completed exhaustive computer calculations on all peg solitaire problems on boards up to Triangle$(7)$, and all complement problems on Triangle$(8)$. Table 2 summarizes my results, for more information with examples and complete lists of shortest solutions see [4].

In Table 2 we count only distinct peg solitaire problems that cannot be reduced to another problem by means of rotation or reflection of the board. For example, on the Triangle$(5)$ board, Table 2 indicates that there are 17 distinct peg solitaire problems, but only 12 are solvable. Of these 12 solvable problems, two can be done in a minimum of 9 moves, six in 10 moves and four in 11 moves. Surprisingly, over half the problems on the Triangle$(6)$ board can be solved in 9 moves, so on average, it’s possible to solve problems on this board in fewer moves than for Triangle$(5)$!
### Table 2: Summary of minimal solution lengths (found by computational search) on triangular boards of side 5–8 († - complement problems only).

| Board   | Total holes | Peg solitaire problems total | Peg solitaire problems solvable | # with minimal solution length |
|---------|-------------|------------------------------|---------------------------------|-------------------------------|
| Triangle(5) | 15          | 17                           | 12                              | 2 6 4 - - - - - - - - - - - - |
| Triangle(6) | 21          | 29                           | 29                              | 16 11 2 - - - - - - - - - - - - |
| Triangle(7) | 28          | 27                           | 27                              | - - - 19 8 - - - - - - - - - - - - |
| Triangle(8) | 36          | 80                           | 80                              | - - - - - - - 1† 5† 2† - - - - - - - - - - - - |

There are 80 different problems on Triangle(8), I have only run the 8 complement problems plus a few non-complement problems. Among the complement problems, only one can be accomplished in 13 moves, the a7-complement (Figure 9) [due to board symmetry, this problem is equivalent to the complement problem at 5 other board locations: a2, b2, b8, g7 or g8]. The computer run also determined that this 13-move solution is unique in the sense that any other 13-move solution to the a7-complement must contain the exact same set of jumps, although not necessarily in the same order. I have found several other 13-move solutions to non-complement problems on the Triangle(8) board.

Figure 9: The minimal 13-move solution to the a7-complement on Triangle(8). Note: more than one move is sometimes shown between board snapshots.

### 6 Bounds on the length of the shortest solution

It is quite difficult to determine the shortest solution on boards larger than Triangle(8). However, we can obtain a lower bound on the length of the shortest solution by dividing the board into “Merson Regions” (named after Robin Merson who first used this concept in 1962 [1 p. 203]). The shape of a region is chosen such that when it is completely filled with pegs, there is no way to remove a peg in the region without a move that originates in the region. On the edge of the board the regions can be corners or two consecutive holes, but in
the interior we must use hexagons 2 holes on a side (7 holes total). Figure 10 shows three triangular boards divided into Merson Regions.

Figure 10: Dividing Triangle(6), Triangle(8) and Triangle(10) into “Merson Regions” (R is the number of regions).

Any region that starts out full must have at least one move starting from inside it. Since the starting position has every hole filled by a peg except one, all regions start full except possibly the region that contains the starting location. If we start in a corner, then this region starts out empty but is filled by the first move, hence there still must be a move out of this corner region. We can summarize these results as follows: If R is the number of Merson Regions, then

1. If the starting vacancy is a corner, or is not in any region (white board locations in Figure 10), then any solution to the peg solitaire problem (no matter where it finishes) has at least R moves.
2. If the starting vacancy is in a region (but not a corner), then any solution to the peg solitaire problem (no matter where it finishes) has at least R − 1 moves.

For example, on Triangle(6), this proves that any solution to “Vacate c5, finish at a1” will take at least 9 moves. On Triangle(8), this analysis indicates that the a7-complement must take at least 12 moves. It can be seen that the 13-move solution in Figure 9 contains one “extra” move above this minimum, in that there are two moves out of the central (blue) hexagon region. Therefore, 13 moves is not proved the shortest possible by this method, although exhaustive computer search indicates no 12-move solution exists.

As the triangular board becomes very large, the number of interior hexagons eventually dominates the number of regions, because this is the only region count growing quadratically. We can then “tile” the board with these hexagons without leaving any gaps (except near the edge of the board). So no solution can be shorter than the number of holes in the board divided by 7. No matter how large the triangular board, we cannot hope to find a solution which does better than a 7-sweep averaged over all moves. Of course, averaging anywhere near this would be quite remarkable—note that the 13-move solution in Figure 9 removes an average of only $34/13 \approx 2.6$ pegs per move.

Using the long sweep solution in the last section, we can obtain an upper bound on the length of the shortest solution of any problem on Triangle(12i). This upper bound is a solution of length $18i^2 + 14i - 3$ on a board of size $72i^2 + 6i$. As i increases, this shows that there exist
solutions that average (asymptotically) a 4-sweep for every move. Therefore, combining the result of the last paragraph, we see that for large triangular boards, the shortest solution must average between 4 and 7 pegs captured per move. Of course, finding any such solution will be very difficult.

References

[1] J. Beasley, *The Ins and Outs of Peg Solitaire*, Oxford Univ. Press, Oxford, New York, 1992.

[2] Letter to Martin Gardner, 16 June 1966, now preserved in the Bodleian Library. Information communicated by J. D. Beasley.

[3] M. Gardner, Penny Puzzles, in *Mathematical Carnival*, 12–26, Alfred A. Knopf, Inc., 1975 (reprint of an original article which appeared in *Scientific American*, February 1966).

[4] [http://www.geocities.com/gibell.geo/pegsolitaire/](http://www.geocities.com/gibell.geo/pegsolitaire/) (should this web address change in the future I suggest a search on keywords: “triangular peg solitaire George Bell”)

[5] G. Jelliss, The Games and Puzzles Journal, Issue 20, February 2001 [http://www.gpj.connectfree.co.uk/gpjb.htm](http://www.gpj.connectfree.co.uk/gpjb.htm)

[6] J. Beasley (editor), The Games and Puzzles Journal, Issue 28, September 2003 [http://www.gpj.connectfree.co.uk/gpjj.htm](http://www.gpj.connectfree.co.uk/gpjj.htm) [arXiv:0811.0851 [math.CO]]