Phase-Entanglement Complementarity with Time-Energy Uncertainty

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We present a unified view of the Berry phase of a quantum system and its entanglement with surroundings. The former reflects the nonseparability between a system and a classical environment as the latter for a quantum environment, and the concept of geometric time-energy uncertainty can be adopted as a signature of the nonseparability. Based on this viewpoint, we study their relationship in the quantum-classical transition of the environment, with the aid of a spin-half particle (qubit) model exposed to a quantum-classical hybrid field. In the quantum-classical transition, the Berry phase has a similar connection with the time-energy uncertainty as the case with only a classical field, whereas the geometric phase for the mixed state of the qubit exhibits a complementary relationship with the entanglement. Namely, for a fixed time-energy uncertainty, the entanglement is gradually replaced by the mixed geometric phase as the quantum field vanishes. And the mixed geometric phase becomes the Berry phase in the classical limit. The same results can be drawn out from a displaced harmonic oscillator model.

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I. INTRODUCTION

A quantum system always need to be isolated from its surroundings, which is referred to an open quantum system. Otherwise, we would have to treat the system and the rest of the universe [1] as a whole. An significant obstacle to this separation is the entanglement [2] between the open system and its environment caused by their quantum nature. The system is exactly described by its reduced density matrix defined as the partial trace of a global state over the Hilbert space of environment. In an ideal case, the decoherence of an open system due to entanglement with its environment is negligible. The effects of the environment represents as parameters, or say classical fields, which are time-dependent in most cases, in the local Hamiltonian of the open system. If the field is slowly altered, the open system will behave like a closed one, namely staying adiabatically in an instantaneous eigenstate of the time-dependent Hamiltonian. The difference is that the open system undergoing a cyclic adiabatic evolution gains a Berry phase [1]. In Berry’s original paper [1], the phase is said to be geometric because it results from the geometric properties of the parameter space of the local Hamiltonian. In other words, it depends on the properties of the external field in evolutionary process, and thus reveals a nonseparable relation between the system and environment.

In the two aforementioned cases, the nonseparability between a system and an environment exhibits either entanglement or Berry phase as the environment is either quantum or classical. This naturally leads to an interesting question: What is their relationship in the quantum-classical transition of the environment? To answer this question, we introduce a model of a spin-half particle (qubit) coupled to an adiabatically rotating quantum-classical hybrid field. And before that, pointed out that the Berry phase is caused by a neglected Hamiltonian and accompanied by a geometric time-energy uncertainty [3–5]. This uncertainty is shown to be a signature of the total nonseparability between the system and the quantum-classical hybrid field. Because of the entanglement between the principal qubit and the field, the original definition of the Berry phase is no longer applicable. We study its two extensions, one of which is presented in this work based on the neglected Hamiltonian and the other is the geometric phase for mixed state [6–8]. In this work, we call the former the Berry phase and the latter the mixed geometric phase. When the classical part of the field vanishes, one can find the Berry phase leads to the definition in the works of vacuum induced Berry phase [9–12] in the Jaynes-Cummings (JC) model [13], where they devise the phase with the aid of a phase shift operator. In the quantum-classical transition of external field and for a fixed time-energy uncertainty, the mixed geometric phase replaces the entanglement gradually, and becomes the Berry phase in the classical limit. This shows a complementary relationship between the mixed geometric phase and the entanglement to reflect the nonseparability.

II. NEGLECTED HAMILTONIAN

Before introducing our model, we first carefully examine the connection between the Berry phase of an open system and its nonseparability with a classical environment. The connection will become apparent if we consider the Berry phase as the adiabatic limit of the Aharonov-Anyandan (AA) phase [14–16]. The latter is an extension of the Berry phase without adiabatic approximation. It has been shown that only for a nonstationary state after a cyclic evolution might the AA phase ap-
where $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli operators of the qubit, $B = B(\sin \theta \cos \omega t, \sin \theta \sin \omega t, \cos \theta)$ is the classical part of the field, and $\mu J$ denotes the quantum part with $J = (J_x, J_y, J_z)$ being the angular momentum operators of a spin-$j$ particle. Experimental realization of the interaction between two spins and observation of the Berry phase for such a system is feasible by current technology [18, 19]. In the frame of quantum optics, the physical meaning of the pure quantum term can be easily understood if we consider the spin operator $J$ as the Schwinger representation [9] of two modes of a quantized optical field. The Hamiltonian $-\mu J \cdot \sigma$ denotes a interacting process between the two modes of optical field and a two level system conserving the total photon number. The whole Hamiltonian (3) could also be regarded as a semiclassical spin star model [20–23]. As shown in Fig. 1, the system qubit interacts with two Heisenberg chains, i.e. two sets of environmental spins. One of the chains has a total spin $j$, and the other is treated as a classical level as an external field $B$.

![Figure 1: The spin star model with two sets of environmental spins. One of the sets is a quantum spin chain with a total spin $j$, and the other is treated at a classical level as an external field $B$.](image-url)

III. QUBIT MODEL

Let us now consider the model of a qubit coupled to a rotating quantum-classical hybrid field with the Hamiltonian

$$H_0 = -\mu J \cdot \sigma - B \cdot \sigma$$

where $\sigma$ is the Pauli matrix, $B$ is the classical part of the field, and $\mu J$ is the quantum part with $J = (J_x, J_y, J_z)$ being the angular momentum operators of a spin-$j$ particle. Experimental realization of the interaction between two spins and observation of the Berry phase for such a system is feasible by current technology [18, 19]. In the frame of quantum optics, the physical meaning of the pure quantum term can be easily understood if we consider the spin operator $J$ as the Schwinger representation [9] of two modes of a quantized optical field. The Hamiltonian $-\mu J \cdot \sigma$ denotes a interacting process between the two modes of optical field and a two level system conserving the total photon number. The whole Hamiltonian (3) could also be regarded as a semiclassical spin star model [20–23]. As shown in Fig. 1, the system qubit interacts with two Heisenberg chains, i.e. two sets of environmental spins. One of the chains has a total spin $j$, and the other is treated as a classical level as an external field $B$.

The Berry phase comes from the fact that even in the adiabatic limit, its effect in a cyclic evolution is finite and nonzero. The uncertainty in energy can be represented as the geometric quantum uncertainty relation [3–5]

$$\Delta E = \frac{\hbar}{2 \pi} \int_0^\tau \Delta E dt,$$

where $\Delta E$ is the energy fluctuation, defined by

$$\Delta E^2 = \langle \psi | H^2 | \psi \rangle - \langle \psi | H | \psi \rangle^2.$$  

It is the distance that the system traverses during its evolution in the projective Hilbert space measured by Fubini-Study metric [3]. In the following section, we will extend the Berry phase to a system in a quantum-classical hybrid field with the aid of the neglected Hamiltonian, and show the connection between the time-energy uncertainty and its entanglement with the field.

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as possible under a fixed time-energy uncertainty.

Furthermore, if we consider the geometric phase for noncyclic evolution [24] and require $|\gamma| = \pi$ corresponding to a reversal of the interference fringes, we can easily obtain a time-energy uncertainty relation in the present model [25]

$$\langle \Delta E \rangle \Delta t \geq \frac{\hbar}{2},$$  

(7)

where $\langle \Delta E \rangle$ is the time-averaged uncertainty in energy during the time $\Delta t$ to gain the geometric phase $|\gamma| = \pi$ and $\hbar$ is the Planck constant. Equality in (7) holds when the classical field is perpendicular to $z$ axis, i.e. $\theta = \pi/2$.

Quantum-Classical Hybrid Field. – We will now derive the results for the full Hamiltonian (3). One can diagonalize the Hamiltonian in the subspace of $\{|U(t)|m\rangle \uparrow \rangle, |U(t)|m+1\rangle \downarrow \rangle$, where $|m\rangle$ is the eigenvector of $J_z$ with $J_z|m\rangle = m|m\rangle$ and $U(t) = \exp[-i\omega t(-j + J_z + \sigma_z/2)] \exp[-i\theta(J_y + \sigma_y/2)]$. In the unitary transformation, a term $-j\omega t$ is introduced to eliminate the influence of the $\pi$ phase brought by a odd $j$ [26]. The instantaneous eigenstates of $H_0$ corresponding to eigenvalues $\varepsilon_m$ are

$$|\psi_+^m\rangle = U(t) \left( \cos \frac{\alpha_m m}{2} |m\rangle \uparrow \rangle + \sin \frac{\alpha_m m}{2} |m+1\rangle \downarrow \rangle \right),$$

$$|\psi_-^m\rangle = U(t) \left( \sin \frac{\alpha_m m}{2} |m\rangle \uparrow \rangle - \cos \frac{\alpha_m m}{2} |m+1\rangle \downarrow \rangle \right),$$  

(8)

with $\tan \alpha_m = \mu \sqrt{(j+1) - m(m+1)}/[B + \mu(m+1/2)]$. Here, the values of index $m$ range from $-(j+1)$ to $j$. For $m = j$ or $-(j+1)$, only two states $|\psi_+^j\rangle = U(t)|j\rangle \uparrow \rangle$ and $|\psi_-^{-(j+1)}\rangle = -U(t)|-j\rangle \downarrow \rangle$ are physically possible. They are direct products of two spin-coherent states aligned or against the classical field, and the states of spin-$j$ are the closest to the classical states [27]. The concurrence [28] of states (8) can be written as

$$C = \sin \alpha_m$$  

(9)

which is the degree of entanglement between the system qubit and its external field.

In the case of the classical field discussed above, the dynamics of the field $B$ is not described by the Hamiltonian (3), but is considered as a time-dependent variable. A quantum field cannot be treated in this way anymore, as its state can be influenced by the interaction with the principal qubit. We have to take into account a Hamiltonian driving the quantum field to rotate with the same frequency as the classical field. The most natural choice of the Hamiltonian is $H_j = (-j + J_z)\omega$. Staying in the instantaneous eigenstates (8) requires the whole system to satisfy the Schrödinger equation

$$i|\dot{\psi}_m^\pm\rangle = (H_0 + H_j + H_\Delta)|\psi_m^\pm\rangle,$$  

(10)

with $|\psi_m^\pm\rangle = c_m^\pm |\psi_m^\pm\rangle$. Consequently, the neglected Hamiltonian can be found as $H_\Delta = \dot{U}(t)U(t^\dagger(t) - H_j =$

$$\pi(1 - \cos \theta) \cos \alpha_m$$  

(11)

where we omit the $\pm$ sign for clarity. The second term in (11) shows a correction in Berry phase aroused by the quantum part of the field. We call this term quantum field induced Berry phase (QBP), as it is caused by quantum fluctuation and entanglement. In addition, when the classical field vanishes, one can choose $\theta = 0$ due to the symmetry of the Hamiltonian (3) and find the QBP has the same physical meaning as the vacuum induced Berry phase [9–12]. A remarkable quantum nature of QBP can be understood by its sharp contrast to the classical one. Namely, the Berry phase is robust against the perturbations by classical environment, particularly is invariant under a changing of the strength of the field. But it is sensitive to the quantum perturbation even though the polarization direction of the spin-$j$ particle in the eigenstates remains parallel to the classical field. From the form of QBP in (11) one can surmise it reveal to a solid angle in a space corresponding to the quantum fluctuation of spin-$j$. The QBP reaches its maximum for the most entangled states with $m = 0$ or $-1/2$ and vanishes for the two separable eigenstates $|\psi_j^+\rangle$ and $|\psi_-^{(j+1)}\rangle$. And in the former case the states give rise to the maximum dispersion for square $J^2$ of the spin angular momentum, and the latter two states lead to the minimum [27].

For further analysis of the physical meaning of QBP and the whole Berry phase in (11) we calculate the geometric time-uncertainty relation (1) during a period of rotation $\gamma = 2\pi/\omega$. Substituting the expressions of $|\psi_m^\pm\rangle$ and $H$ into (1) leads to

$$S^2 = 2\pi \sqrt{1 - \cos^2 \theta \cos^2 \alpha_m}.$$  

(12)

Similar to QBP, it is increased by the entanglement between the qubit and field. It is interesting to note that if one defines an angle $\cos \theta_m = \cos \theta \cos \alpha_m$, there also exists a relationship of solid angle and perimeter between $\gamma$ and $S^3$ as the case of the classical field. That is, the classical part and quantum part of the external field parallel to each other behave as an effective classical field in another direction. This deflection caused by the fluctuation of the quantum field is most visible when $\theta = 0$, where only the quantum field contributes to $S^3$ and $\gamma$, and vanishes for $\theta = \pi/2$ with the classical field making the most contribution. Moreover, for a fixed value of geometric phase $\gamma = \pi$, the inequality (7) still holds for the qubit system with a quantum-classical hybrid field. And the entanglement between the principal qubit and its external field reduces the value of $\langle \Delta E \rangle \Delta t$ for a given
In the above discussion about QBP, we treat the system qubit and the quantum field as a composite system when we consider their evolution. We actually remove terms irrelevant to the qubit in the total phase of the whole system and obtain the Berry phase (11). That is, the Berry phase (11) depends on the evolution of the quantum field entangled with the qubit. To study the phase determined only by the geometry of the path of the qubit, we derive the time-dependent mixed state

$$\rho = \sin^2 \frac{\alpha_m}{2} |\psi^+\rangle\langle\psi^+| + \cos^2 \frac{\alpha_m}{2} |\psi^-\rangle\langle\psi^-|,$$

by tracing out the quantum field in the eigenstates (8), where $|\psi^\pm\rangle$ is defined in (4). Here, we only give the case for the pure state $|\psi_m\rangle$, the mixed geometric phase for $|\psi_m^\pm\rangle$ can be easily obtained by changing $\alpha_m$ into $\pi + \alpha_m$. We can calculate the mixed geometric phase by using the definition with a kinematic description [6–8] and get

$$\gamma^{\text{mix}} = \arg \left( \sin^2 \frac{\alpha_m}{2} e^{i\gamma_+} + \cos^2 \frac{\alpha_m}{2} e^{i\gamma_-} \right) = \arctan \left( \cos \alpha_m \tan \gamma_- \right),$$

where $\gamma_\pm$ are the Berry phases (5) resulting from the classical field. The phase $\gamma^{\text{mix}}$ is manifestly gauge invariant and can be experimentally tested in interferometry.

Let us give a further discussion about the relationship among these quantities related to the nonseparability between the system and the external field. For a fixed value of $\theta$, $\gamma^{\text{mix}} \leq \gamma^q$, and the entanglement widens their gaps. They are equal to the Berry phase $\gamma_-$ for a separable eigenstate. It is important to note that the time-energy uncertainty leads to an upper bound of the entanglement $C \leq S^q/(2\pi)$. The equality holds when the mixed geometric phase $\gamma^{\text{mix}}$ vanishes. These reveal a complementary relationship between the mixed geometric phase $\gamma^{\text{mix}}$ and the entanglement to reflect the nonseparability between the system and the external field, while $S^q$ and $\gamma^q$ can be considered to be their sum. For a fixed amount of $S^q$, the entanglement is gradually replaced by the geometric phase $\gamma^{\text{mix}}$ in the quantum-classical transition of external field, and becomes the Berry phase when the quantum field vanishes. In a figurative sense, Berry phase is semiclassical entanglement between a quantum system and a classical environment.

### IV. HARMONIC OSCILLATOR

We also study a displaced harmonic oscillator which is another canonical example of the Berry phase [29] to verify our above discussions. Here we add a qubit as the part of the system environment, which is described in terms of the Pauli operators $\sigma_\pm = (\sigma_x \pm i\sigma_y)/2$. Suppose the qubit-oscillator interaction is characterized by a JC term [13], we get

$$H_0 = \nu b^\dagger b + g(\sigma_- b^\dagger + \sigma_+ b),$$

where $b^\dagger = a^\dagger - \beta^*$, $b = a - \beta$ are created displacement and annihilation operators with the frequency $\nu$, $g$ is the coupling constant. We take $\beta = |\beta|e^{-i\omega t}$ and $\beta^* = |\beta|e^{i\omega t}$ to be slowly rotating parameters. One can find its instantaneous eigenstates in the subspace $\{\mathcal{V}(t)|n\rangle\uparrow, \mathcal{V}(t)|n+1\rangle\downarrow\}$, where $\mathcal{V}(t) = \exp[\beta b^\dagger - \beta^* b] \exp[-i\omega t(\sigma_+ + 1)/2]$, and $|n\rangle$ satisfies $a^\dagger a|n\rangle = n|n\rangle$. The instantaneous eigenstates with eigenvalues $\varepsilon_n^\pm$ are derived as

$$|\psi_n^\pm\rangle = \mathcal{V}(t) \left( \cos \frac{\alpha_n}{2}|n\rangle\uparrow - \sin \frac{\alpha_n}{2}|n+1\rangle\downarrow \right),$$

$$|\psi_n^\mp\rangle = \mathcal{V}(t) \left( \sin \frac{\alpha_n}{2}|n\rangle\uparrow + \cos \frac{\alpha_n}{2}|n+1\rangle\downarrow \right),$$

where $\tan \alpha_n = g\sqrt{n+1}/\nu$. The concurrence of states $|\psi_n^\pm\rangle$ is

$$C_h = \sin \alpha_n,$$

which is zero for the state $|0\rangle\downarrow$, corresponding to the eigenstate $|\psi_n^-\rangle$ in (16) with $n = -1$. We also choose a Hamiltonian to drive the rotation of the qubit with the frequency of $\beta$, which is $H_q = \omega(\sigma_+ + 1)/2$. The neglected Hamiltonian in this case is $H_\Delta = \omega a^\dagger a$. Follow the same steps of the qubit case in the above section, we obtain the time-energy uncertainty

$$S_h^q = 2\pi \sqrt{4(2n+1)|\beta|^2 + \sin^2 \alpha_n},$$

the Berry phase

$$\gamma_h^q = 2\pi |\beta|^2 \pm \pi(1 - \cos \alpha_n),$$

and the mixed geometric phase

$$\gamma_h^{\text{mix}} = 2\pi |\beta|^2.$$

Here, both the phases are defined modulo $2\pi$. Obviously there exist an entanglement induced term in each of the expression of the time-energy uncertainty and the Berry phase. And the entanglement and the mixed geometric phase are complementary for a fixed value of $S_h^q$.

### V. SUMMARY

In this work, we present a viewpoint of the Berry phase that it reflects the nonseparability between an open system and its classical environment just like the entanglement between a system and a quantum environment. This unified view of the two concepts inspires us to explore their properties and connection in the quantum-classical transition of the environment. The viewpoint is supported by the fact that the Berry phase can be considered as the effect of a neglected Hamiltonian which
affects the system but has no effect on the eigenvalues and eigenstates in the adiabatic limit. This understanding can be obtained by checking the relation between the Berry phase and the AA phase, and provides an approach to extend the Berry phase to the system in a quantum-classical hybrid environment. In addition, the geometric time-energy uncertainty as an accumulating result of the neglected Hamiltonian in a cyclic evolution is found to be a signature of the nonseparability not only for a classical environment but also for a quantum-classical hybrid one.

Based on foregoing considerations we investigate a qubit under an adiabatically rotating quantum-classical hybrid field. The entanglement between the principal qubit and the field introduces a correction to the time-energy uncertainty and a corresponding effect on the Berry phase. For a fixed time-energy uncertainty, the entanglement is gradually replaced by the mixed geometric phase, determined solely by the geometry of the system, in the quantum-classical transition of external field. That is, the geometric phase and entanglement have complementary relationship to reflect the nonseparability between the system and the external field. We also make similar calculations for a model of displaced harmonic oscillator to verify these conclusions, in which a qubit acts as the quantum part of the field.

We have been very careful to introduce a quantum part of the environment in our models without giving up the conditions of adiabaticity and cyclicity in Berry’s original definition of geometric phase. These models allow us to treat the geometric phases and quantum entanglement uniformly. We believe that, by removing the restrictions in this work, one may uncover more general connections among geometric aspects of quantum mechanics and different quantum correlations [30].

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