TYPES OF FLOW IN GREEN–NAGHDI THERMOELASTICITY THEORY

TIPOS DE FLUXO DA TEORIA TERMOELÁSTICA DE GREEN–NAGHDI

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Abstract: In this paper we will show a review of the Green–Naghdi thermoelasticity theory. Such model have beautiful foundations which contains since the law of heat conduction proposed by Fourier, fundamentals variables such as Helmholtz free energy, entropy, flux of heat and etc. and the constitutive hypothesis. The nature of partial differential equation of parabolic (PDE) type derived from classical thermoelasticity lead us to a mathematical inconsistency well-know as thermal signal speed paradox. We will show how Green–Naghdi model solved those mathematical inconsistency introducing different types of flow and giving us a hyperbolic PDE.

Keywords: Thermoelasticity. Constitutive hypothesis. Thermal signal paradox. Hyperbolic-parabolic EDP. Heat flow.

Resumo: Apresentaremos uma breve revisão da teoria termoelástica de Green–Naghdi. Esse modelo tem seus fundamentos baseados na lei de condução do calor proposta por Fourier, variáveis fundamentais como a energia livre de Helmholtz, entropia, fluxo de calor entre outras e as hipóteses constitutivas. A natureza das equações diferenciais parciais (EDP’s) parabólica derivada da teoria clássica da termoelasticidade nos leva a uma inconsistência matemática chamada de paradoxo de propagação dos sinais térmicos. Mostraremos como o modelo de Green–Naghdi soluciona essa inconsistência matemática introduzindo diferentes tipos de fluxo e nos fornecendo uma EDP hiperbólica.

Palavras-chave: Termoelasticidade. Hipótese constitutiva. Paradoxo dos sinais térmicos. EDP do tipo hiperbólica-parabólica. Fluxo de calor.

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1 INTRODUCTION

Continuum Mechanics is a branch of applied mathematics which study the behavior of a range of materials in deformation state. In this branch of Continuum Mechanics we shall look for two particular theories, elasticity and thermoelasticity. We will present the following content in five sections.

After this introduction, in section 2 we will approach the historical aspects and development of Continuum Mechanics, elasticity and classical and non-classical thermoelasticity, how the Fourier law emerged of heat conduction and how it is related to classical thermoelasticity and also the physical content of the theory.

In section 3 we will show the Green–Naghdi model. The Green–Naghdi model belongs to a non classical thermoelasticity, i.e, hyperbolic PDE. We will present the bird eyes of the model: the types of flow know as type I, type II, type III and the use of thermal displacement and constitutive hypothesis. We will comment in more detail the flow types I and II in sections 3.1 and 3.2.

Once we show the basis of the Green–Naghdi model we will look for linear theory in section 3 and the linear form of constitutive hypothesis lead us to coupled system of hyperbolic PDE containing the temperature-displacement and the temperature equations.

After the presentation of all content of Green–Naghdi theory in section 5 we will discuss our conclusions, the principal differences between the flux type I and II and the linear theory. We also present our current and further works in development in this context.

2 HISTORICAL REVIEW

One of the most important areas of the applied mathematics is the continuum mechanics. The continuum mechanics is responsible to study the action of the force field and velocity in a portion of a continuum material. There are different areas in continuum mechanics such as fluid mechanics, elasticity, thermoelasticity, plasticity and etc. In this work we will look with more attention in elasticity and thermoelasticity theories.

As any other scientific theory, elasticity and thermoelasticity has been passed by a lot of adequations in their existence. The elasticity theory is dated since eighteen century and had been developed in the subsequent centuries (MAUGIN, 2014b; MAUGIN, 2014a). With the
elaboration of the principle of last action, virtual work and the Lagrangian and Hamiltonian formalism the classical mechanics had won a new approach and all of those formalism was used in the deformed body mechanics (CAPECCHI, 2014). From those applications the elasticity theory was born.

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After Robert Hooke show his power of springing bodies in well-know Hooke’s law (HOOKE, 1678) and the extension for isotropic materials by the french mathematician Cauchy (1789-1857) in his postulate (LANDAU; LIFSHITZ, 1970; CHAVEZ, 2013) given by:

$$\sigma = C\epsilon$$  \hspace{1cm} (1)

where $\sigma$ is the Cauchy tensor or stress tensor, $C$ is the elastic constant which define the type of material and $\epsilon$ is the deformation tensor$^3$.

The principal characteristic of the elasticity theory was the intuitive use of purely mechanical forces (CHAVEZ, 2013). The system studied is composed by one portion of matter and the field forces acting on it, this portion is well defined and its called body $\Omega$. Once we show in equation (1), the stress tensor lead us to a description of the system configuration. There are two possibles configurations, current and initial. The difference between them are in the deformation of the body. We say one body is deformed if the distance between two points that are inside of the body change due to some force acting on it (LANDAU; LIFSHITZ, 1970).

Both quantities in equation (1) $\sigma$ and $\sigma$ by definition are initially considered as infinitesimal. The generalization was made by Piola-Kirchhoff (MAUGIN, 2014b). Those authors also introduced a connection between the current and initial configurations which are effectuated by Piola-Kirchhoff tensor $P_{ij}$.

Searching for a more realistic description of the reality, the french physicist and

$^3$Even we do not explicit the indexes in equation (1) its also common use the Einstein convection to write as follow:

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl}$$  \hspace{1cm} (2)
Duhamel (1797-1862) published an article introducing the thermal effect in the body (DUHAMEL, 1837). The main idea is that one body can change its temperature when there is some dilation occurring, this idea had a strong physical content and since that the new theory was called thermo-mechanic. The change of the body's temperature is caused by the heat theory, which was largely investigated in Europe in the period of industrial revolution post French revolution by a range of authors such as Fourier, Kelvin and Carnot. The most important author for our purpose is Jean Baptiste Joseph Fourier (1768-1830). Fourier studied the nature of the heat conduction in the acclaimed *The analytical theory of heat*.

In this work Fourier proposed with rigor and preciousness the mathematical expression to describe the heat conduction in one material. This expression received his name and its known as Fourier Law of heat conduction which we show below

\[ q = -\kappa \frac{\partial \theta}{\partial x} \]  

(3)

where \( q \) is the heat flux, \( \kappa \) is the thermal conductivity of material and \( \theta \) is the temperature.

It is really important to mention that the legacy of Fourier was not only an achievement in this area but in science in general. Fourier contributed in physics, optics and mechanics, engineering and mainly in mathematics (GRATTAN, 1980; SHOUCAIR, 1989) expanding the functions of sine and cosine series, known as Fourier series.

We already showed in equation (1) that the stress tensor in elasticity theory has a pure mechanical affect. Using the thermodynamic entity, the heat, we introduce a more realistic behavior of studied materials. The first interpretation to thermal effects in elastic materials was introduced by Pierre Duhem (1861-1916), physicist, historian of science and common sense philosophy (LEITE, 2018), published some relevant papers for elasticity theory (DUHEM, 1886; DUHEM, 1893; DUHEM, 1903a; DUHEM, 1903b). Duhem did the connection between thermodynamic and continuum mechanics exploring the notion of virtual work and the D’Alambert formulation of continuum media (LANCZOS, 1986) together with the thermodynamic potentials derived from Gibbs and Helmholtz free energy (GIBBS, 1873; HELMHOLTZ, 2015).

With this interpretation and application of thermodynamic potentials in the elasticity and thermo-mechanic theory emerge a new theory, the classical thermoelasticity. The classical thermoelasticity theory or linear thermoelasticity (TRUESDELL, 1973) basically is a union of elasticity theory together with thermodynamic properties of the system, now named as thermoelastic system. One of the most important theory of classical thermoelasticity is the Maxwell-Cattaneo model (CATTANEO, 1948; GRAFFI, 1980; JOU; LEBON, 2010;
Even with the new formulation of thermoelasticity, the mechanics and thermal effects on materials was not fully explained. In the 20th century the thermoelasticity received a special attention of physicians and mathematicians such as Voigt (VOIGT, 1910), Jeffreys (JEFFREYS, 1930) and Biot (JEFFREYS, 1956). All those authors contributed in a more sophisticated interpretation of the thermoelasticity which include the effect of mechanical and thermal stress in thermal conductivity.

All these theories which include the coupling of thermal and mechanical stress are called non-classical or general thermoelasticity. Other contributions in coupling theory are presented in the literature (WEINER, 1957; NICKELL; SACKMAN, 1968; BOLEY; TOLINS, 1962; NOWACKI, 1965; NOWACKI, 1966a; NOWACKI, 1966b; NOWACKI, 1966c; NOWACKI; IGNACZAK, 1966; LORD; SHULMAN, 1967; MULLER, 1971; GREEN; LAWS, 1972; GREEN; LINDSAY, 1972; GREEN; NAGHDI, 1977; GREEN; NAGHDI, 1978; GREEN; NAGHDI, 1979; GREEN; NAGHDI, 1984; GREEN; NAGHDI, 1991; GREEN; NAGHDI, 1992; GREEN; NAGHDI, 1993). As we showed above, there is a range of classical and non-classical thermoelasticity theories, for our purpose we will focus on Green-Naghdi theory in section 3.

3 GREEN–NAGHDI MODEL

The Green–Naghdi model was proposed by Paul Mansour Naghdi (1924-1994) and Albert Edward Green (1912-1999) in a serie of papers published between 1973 to 1993 (GREEN; NAGHDI, 1977; GREEN; NAGHDI, 1978; GREEN; NAGHDI, 1979; GREEN; NAGHDI, 1984; GREEN; NAGHDI, 1991; GREEN; NAGHDI, 1992; GREEN; NAGHDI, 1993). As we already mentioned before the bird’s eye of the theory is the flux types, the constitutive assumptions and the thermal displacement. Before we go deeply in the theory we shall show in table 1 the fundamental variables or state variables that we use in its theory.

Once we showed in the table above the state variables we can show three constitutive equations as follow:

$$\rho (\dot{\psi} + \eta \dot{\theta}) + \rho \theta \dot{\xi} + \mathbf{p} \cdot \mathbf{g} + \rho \theta \xi = 0$$  \hspace{1cm} (4)

$$\rho \dot{\eta} = \rho (s + \xi) - div \mathbf{p}$$  \hspace{1cm} (5)

$$\mathbf{q} := \theta \mathbf{p}$$  \hspace{1cm} (6)

$$\sigma = C \epsilon + \alpha \psi (T - T_0), \quad \text{or,} \quad \sigma = \sigma_M + \sigma_T$$  \hspace{1cm} (7)
Table 1: Fundamental Variables

| Variable                          | Symbol |
|----------------------------------|--------|
| thermal displacement             | $\alpha$ |
| Helmholtz free energy            | $\psi$ |
| entropy                          | $\eta$ |
| heat flux                        | $q$    |
| entropy flux                     | $p$    |
| gradient of thermal displacement | $\beta = \frac{\partial \alpha}{\partial X}$ |
| gradient of empirical temperature| $\gamma = \frac{\partial T}{\partial x}$ |
| gradient of absolute temperature | $g = \frac{\partial \theta}{\partial x}$ |
| external rate of heat supply     | $r = \dot{\theta}$ |
| external rate of entropy supply  | $s$    |
| internal rate of entropy supply  | $\xi$  |
| absolute temperature             | $\theta$ |
| empirical temperature            | $T$    |

Equation (4) is called as Helmholtz free energy\(^4\) obtained by the Legendre’s transformation and Maxwell relations, equation (5) is the local entropy balance and the equation (6) is the relation for heat and entropy flux. Equation (7) show us the coupling of thermal and mechanical stress, ($\sigma_M$) and ($\sigma_T$), contribution for general stress in some material as we already mentioned in section 2. Now we will divide the types of flux in two subsections as follow.

3.1 Type I Flux

The type I of heat flux is concerning to a classical thermoelasticity theory. For the deduction and modeling we consider one rigid isotropic conductor material in stationary state\(^5\) in contact with a heat source. Also consider that the state variables $p, \xi, \eta, q, \theta$ and $\psi$ are dependents of empirical temperature $T$ and spacial gradient $\gamma$. In this case the constitutive hypothesis are given by:

$$p = \hat{p}(T, \gamma), \quad \xi = \hat{\xi}(T, \gamma), \quad \eta = \hat{\eta}(T, \gamma), \quad \theta = \hat{\theta}(T, \gamma), \quad \psi = \hat{\psi}(T, \gamma) \quad (8)$$

Deriving with respect to time $t$ the relations for $\theta$ and $\psi$ of equation (8) and replacing into (4) we get:

$$\rho \left( \frac{\partial \hat{\psi}}{\partial T} + \eta \frac{\partial \hat{\theta}}{\partial T} \right) \dot{T} + \rho \left( \frac{\partial \hat{\psi}}{\partial \gamma} + \eta \frac{\partial \hat{\theta}}{\partial \gamma} \right) \dot{\gamma} + \rho \dot{\theta} \dot{\xi} = 0, \quad (9)$$

\(^4\)In the current article we shall adopt $\dot{} = \frac{d}{dt}(*) = \frac{d}{\tau(*)}$ as time derivative for scalar and vector functions.

\(^5\)For all flux types in this work we will consider rigid isotropic conductors materials in stationary state and we will resume its name calling simply stationary solid body.
where the equation (9) should be verified for any choice of \( \dot{T}, \dot{\gamma} \) and \( \frac{\partial \gamma}{\partial \chi} \). Regarding the nature of constitutive hypothesis we have:

\[
\psi = \hat{\psi}(\theta), \quad \eta = -\frac{\partial \hat{\psi}}{\partial \theta}, \quad \hat{\p}(\theta, g) \cdot g + \rho \theta \hat{\xi}(\theta, g) = 0. \quad (10)
\]

Now, considering:

\[
\rho \theta \dot{\eta} = \rho r - \text{div} q, \quad (11)
\]

where \( q = \hat{q}(\theta, g) = \theta \hat{\p}(\theta, g) \).

Assuming the relations for Helmholtz free energy \( \psi \), entropy \( \eta \) and classical Fourier heat equation \( q \) on the form

\[
\hat{\psi} = c(\theta - \theta \ln \theta), \quad \hat{\eta} = c \ln \theta, \quad \hat{q} = -k \nabla \theta, \quad (13)
\]

where \( c \) e \( k \) are constants named as specific heat and thermal conductivity and \( \nabla \equiv \frac{\partial}{\partial X} \) is the gradient operator. If we replace the equations (9) in the equations (6) and (7), after simplifying we get the

\[
\rho c \dot{\theta} = \rho r + k \nabla^2 \theta = \rho r + k \Delta \theta, \quad (14)
\]

where \( \Delta = \nabla^2 = \frac{\partial}{\partial X} \cdot \frac{\partial}{\partial X} \) is the laplacian.

For some materials such as He, the parabolic PDE described in equation (14) in low temperature \( (T \leq 4 K) \) lead us to a paradox called speed propagation of thermal signal. This paradox was expected by Landau (1941) and experimentally validated by Peshkov (1944), Wang, Wagner and Donnelly (1987)\(^7\). Several authors proposed different approaches to solve this paradox, in our context we will see in next section how Green–Naghdi change the parabolic PDE to hyperbolic PDE and how it extinguish the paradox.

### 3.2 Type II Flux

For the development of type II flux we will consider the thermal displacement \( \alpha \) showed in table 1. If \( \alpha = \alpha(x, t) \) is an scalar and \( \beta \) its the respective gradient given by following expressions:

\[
\alpha = \int_{t_0}^{t} T(x, t)dt + \alpha_0, \quad \beta = \frac{\partial \alpha}{\partial x} \quad (15)
\]

where \( t_0 \) is the initial time and \( \alpha_0 \) is the thermal displacement in \( t = t_0 \).

\(^6\)Note all these identities are tensorial identities. If the reader is interesting in a more mathematical description search for (CHAVEZ, 2013).

\(^7\)A beautiful description and evolution of law physics and the discovery of superfluid helium is found in (VINEN, 2004).
Now, consider one stationary solid body in contact with a heat source and the state variables \( p, \xi, \eta, q, \theta \) and \( \psi \) dependents of \( T, \alpha, \) and \( \beta. \) The constitutive hypothesis are given by:

\[
p = \hat{p}(T, \alpha, \beta), \quad \xi = \hat{\xi}(T, \alpha, \beta), \quad \eta = \hat{\eta}(T, \alpha, \beta), \quad \theta = \hat{\theta}(T, \alpha, \beta), \quad \psi = \hat{\psi}(T, \alpha, \beta)
\]  

(16)

Deriving the relations for \( \theta \) and \( \psi \) and replacing into (4) we have:

\[
\rho \left( \frac{\partial \hat{\psi}}{\partial T} + \eta \frac{\partial \hat{\theta}}{\partial T} \right) \dot{T} + \rho \left( \frac{\partial \hat{\psi}}{\partial \alpha} + \eta \frac{\partial \hat{\theta}}{\partial \alpha} \right) T + \rho \left( \frac{\partial \hat{\psi}}{\partial \beta} + \eta \frac{\partial \hat{\theta}}{\partial \beta} \right) \gamma +
\]

\[
p \left( \frac{\partial \hat{\theta}}{\partial T} + \beta \frac{\partial \hat{\theta}}{\partial \alpha} + \frac{\partial \beta}{\partial x} \frac{\partial \hat{\theta}}{\partial \beta} \right) + \rho \theta \hat{\xi} = 0,
\]

(17)

The relations above should be true for all choice of \( \dot{T}, \frac{\partial \beta}{\partial x} \) and \( \gamma. \) Then we get:

\[
\frac{\partial \hat{\theta}}{\partial \beta} = \theta, \quad \frac{\partial \hat{\psi}}{\partial T} + \eta \frac{\partial \hat{\theta}}{\partial T} = 0, \quad \frac{\partial \hat{\psi}}{\partial \alpha} + \rho \frac{\partial \hat{\theta}}{\partial \beta} = 0,
\]  

(18)

\[
\rho \left( \frac{\partial \hat{\psi}}{\partial \alpha} + \hat{\eta} \frac{\partial \hat{\theta}}{\partial \alpha} \right) T + \hat{p} \beta \frac{\partial \hat{\theta}}{\partial \alpha} + \rho \theta \hat{\xi} = 0.
\]  

(19)

Now if we suppose that \( \psi \) and \( \theta \) of the form:

\[
\psi = \hat{\psi}(\theta) = c(\theta - \theta \ln \theta) + \frac{1}{2} k \beta \cdot \beta, \quad \theta = a + bT
\]  

(20)

for \( a \) and \( b \) real and positive constants.

Using the equations (18) and (20) we have the following relations:

\[
\hat{p} = -\frac{\rho k \beta}{b}, \quad \xi = 0, \quad \hat{\eta} = c \ln \theta
\]

(21)

From the equations (21) and entropy balance (5) we have the hyperbolic PDE:

\[
c b \alpha = r + \frac{ka}{b} \Delta \alpha \quad \text{or}, \quad c b \dot{T} = \dot{r} + \frac{ka}{b} \Delta T.
\]

(22)

In the special case of equation (22) if we consider \( \dot{r} = 0, \) we return to the classical wave equation, in which the propagation speed is give by \( v = \sqrt{\frac{ak}{cb}}. \)
4 LINEAR THEORY OF TYPE II FLUX

In some cases is more convenient to use the linear form of the theory. In this section we will develop the foundations of Green–Naghdi linear theory of type II flux. Consider the following constitutive hypothesis:

\[ \psi = \hat{\psi}(T, \gamma, F, X), \quad \theta = \hat{\theta}(T, \gamma, F, X), \quad T = \hat{T}(T, \gamma, F, X), \quad p = \hat{p}(T, \gamma, F, X), \quad \eta = \hat{\eta}(T, \gamma, F, X), \quad \xi = \hat{\xi}(T, \gamma, F, X). \]  

(23)

For a linear body we assume

\[ |\nabla u|, |\nabla \dot{u}|, |\theta - \theta_0|, |\dot{\theta}|, |\nabla \theta| \leq \delta \]

where \( \delta \) is a small number while \( \theta_0 > 0 \) is the reference temperature and being constant such that \( S(0, \theta_0) = 0, \eta(0, \theta_0) = 0 \). In linear body \( S = T = \sigma \) is the Cauchy tensor or first Piola-Kirchhoff tensor.

Furthermore for linear theory of type II we will regard the relations for Helmholtz free energy, gradient of thermal displacement and infinitesimal deformation \( E = \frac{1}{2}[\nabla u + (\nabla u)']^2 \) as follow:

\[ \rho_0 \psi = \frac{1}{2} \lambda (tr E)^2 + \mu tr E^2 - \frac{c \theta^2}{2 \theta_0} - \frac{E \beta^*}{3(1 - 2 \nu)} \theta tr E + \frac{k^*}{2 \theta_0} \beta \cdot \beta, \]  

(24)

where \( k^* \) is a constant, \( \rho_0 \) is the mass density in the reference configuration, \( \lambda \) and \( \mu \) are the Lamé coefficient’s, \( E \) is the Young modulus, \( \nu \) is the Poisson rate, \( c \) is the specific heat and \( \beta^* \) is the volumetric expansion coefficient\(^9\). It is easy to show that \( c \geq 0 \), directly from second thermodynamic law and no further restrictions as long as \( \xi = 0 \) in our case, for more details of those restrictions look for (BARGMANN; FAVATA; GUIDUGLI, 2014).

It follows from (24) and for our purpose that

\[ p = -\frac{k^*}{\theta_0} \beta. \]  

(25)

\(^8\)This relation is easily put in tensor notation using the idea of a symmetric tensor:

\[ X_{ij} = \frac{1}{2}[X_{ij} + X_{ji}]. \]

\(^9\)For more details of all those constants and coefficients look for (CHAVEZ, 2013).
\[
T = \sigma = \lambda (tr \, E) \, \mathbf{I} + 2 \mu \, E - \frac{E \, \beta^* \, \theta}{3(1 - 2\nu)} \, \mathbf{I},
\]
\[
\rho_0 \, \eta = c \frac{\theta}{\theta_0} + \frac{E \, \beta^*}{3(1 - 2\nu)} \, tr \, E
\] (27)

Now we will deduce first of all the temperature-displacement equilibrium equation for the three dimensional space \( \mathbb{R}^3 \). Consider one body defined in the set \( \Omega = \mathbb{B} \times (0, T) \), where \( \mathbb{B} \subset \mathbb{R}^3 \). For this, consider yet that displacement vector is given by \( u = u(X, t) = (u, v, w) \)'\(^{10}\), for \( u = u(x, y, z, t), v = v(x, y, z, t), w = w(x, y, z, t) \) with \( u : \mathbb{B} \times (0, T) \to \mathbb{R} \) and \( X = (x, y, z) \). We also consider that the temperature is given by \( \theta = \theta(x, y, z, t)' \).

Following the reference (RACKE; JIANG, 2000), we started the linear elasticity in which the elasticity modulus for a isotropic material \( \Omega \in \mathbb{R}^3 \) is a symmetric fourth order tensor given by:

\[
S = S(x) = C_{ijkl} = C_{ijkl}(x, 0, 0) =
\begin{bmatrix}
2\mu + \lambda & 0 & 0 & 0 & 0 & 0 \\
0 & 2\mu + \lambda & 0 & 0 & 0 & 0 \\
0 & 0 & 2\mu + \lambda & 0 & 0 & 0 \\
0 & 0 & 0 & \mu & 0 & 0 \\
0 & 0 & 0 & 0 & \mu & 0 \\
0 & 0 & 0 & 0 & 0 & \mu
\end{bmatrix}
\]

Now defining the generalized gradient for the displacement \( u \)

\[
\mathcal{D} =
\begin{bmatrix}
\partial_1 & 0 & 0 \\
0 & \partial_2 & 0 \\
0 & 0 & \partial_3 \\
\partial_3 & \partial_2 & 0 \\
\partial_3 & 0 & \partial_1 \\
\partial_2 & \partial_1 & 0
\end{bmatrix}
\]

where \( \partial_i \) represents the operation partial differential in relation to the variable \( x_i \).

The idea is use the conservation of linear momentum \( div \, T + \rho \, b = \rho \, \ddot{u} \). In terms of elastic tensor and generalized gradient we rewrite the conservation of linear momentum as

\[
\mathcal{D}' \, S \, \mathcal{D} \, u + \mathcal{D}' \, \Gamma \, \theta + \rho \, b = \rho \, \ddot{u}
\]

\(^{10}\)The symbol ‘\(^{'}\) denote the transposition operator of vector field \( u \).
where \( \rho = \rho(x) \) is a density symmetric matrix \( \Gamma = \Gamma(x) \) is a vector with coefficients defining the stress-temperature tensor and in our case we consider that \( \Gamma = \frac{E\beta^*}{\gamma(1-2\nu)} \).

We assume all functions are smooth, at least classes \( C^2 \) in space and \( C^1 \) in time. Applying \( u \) in \( D \) we get:

\[
D \mathbf{u} = \begin{bmatrix}
\partial_1 & 0 & 0 & 0 & 0 \\
0 & \partial_2 & 0 & 0 & 0 \\
0 & 0 & \partial_3 & 0 & 0 \\
0 & \partial_3 & 0 & \partial_1 & 0 \\
\partial_2 & \partial_1 & 0 & 0 & 0 \\
\end{bmatrix} \begin{bmatrix}
u \\
v \\
w \\
\end{bmatrix} = \begin{bmatrix}
\partial_u \\
\partial_v \\
\partial_w \\
\end{bmatrix}
\]

So, \( S \cdot D \mathbf{u} \) is given by

\[
\begin{bmatrix}
2\mu + \lambda & 0 & 0 & 0 & 0 \\
0 & 2\mu + \lambda & 0 & 0 & 0 \\
0 & 0 & 2\mu + \lambda & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix} \begin{bmatrix}
\partial_u \\
\partial_v \\
\partial_w \\
\end{bmatrix} + \begin{bmatrix}
\partial_u \\
\partial_v \\
\partial_w \\
\end{bmatrix}
\]

\[
(2\mu + \lambda) \partial_u + \lambda \partial_v + \lambda \partial_w = \mu \begin{bmatrix}
\partial_u \\
\partial_v \\
\partial_w \\
\end{bmatrix}
\]

So we have

\[
D' S \cdot D \mathbf{u} = \begin{bmatrix}
\partial_1 & 0 & 0 & 0 & \partial_2 \\
0 & \partial_2 & 0 & \partial_3 & 0 \\
0 & 0 & \partial_3 & 0 & \partial_1 \\
0 & 0 & \partial_3 & \partial_2 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix} \begin{bmatrix}
\partial_u \\
\partial_v \\
\partial_w \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
(2\mu + \lambda) \partial_u + \lambda \partial_v + \lambda \partial_w = \mu \begin{bmatrix}
\partial_u \\
\partial_v \\
\partial_w \\
\end{bmatrix}
\end{bmatrix}
\]

Then

\[
D' S \cdot D \mathbf{u} = \begin{bmatrix}
\mu \Delta u + (\lambda + \mu) \nabla \cdot \mathbf{u} \\
\mu \Delta v + (\lambda + \mu) \nabla \cdot \mathbf{u} \\
\mu \Delta w + (\lambda + \mu) \nabla \cdot \mathbf{u} \\
\end{bmatrix}
\]
or rewriting the above expression we have:

\[ D' S D u = \mu \Delta u + (\lambda + \mu) \nabla \text{div} u. \]

On the other hand,

\[ D' \Gamma \theta = \Gamma D' \theta = \Gamma \nabla \theta = \frac{E \beta^*}{3(1 - 2\nu)} \nabla \theta. \]

Therefore we can conclude the linear momentum conservation is given by

\[ D' S D u + D' \Gamma \theta + \rho b = \mu \Delta u + (\lambda + \mu) \nabla \text{div} u + \frac{E \beta^*}{3(1 - 2\nu)} \nabla \theta + \rho b = \rho \ddot{u}. \]

Following the same assumptions is easy to show that for the multidimensional space \( \mathbb{R}^n \) we have the stress-temperature relation given by:

\[ (\lambda + \mu) \nabla \text{div} u + \nabla^2 u - \frac{E \beta^*}{3(1 - 2\nu)} \nabla \theta + \rho b = \rho \ddot{u} \]

Now we will introduce the temperature equation. Consider the internal energy is given by the following equation:

\[ \rho_0 e = \rho_0(\psi + \eta \theta) = \frac{1}{2} \lambda (tr \ E)^2 + \mu tr \ E^2 + \frac{c \theta^2}{2 \theta_0} + \frac{k^*}{2 \theta_0} \beta \cdot \beta. \]

Deriving with respect to time the equation (29) and using (25), organizing we get the expression:

\[ \frac{c \dot{\theta}}{\rho_0 \theta_0} + \frac{E \beta^*}{3 \rho_0(1 - 2\nu)} tr \dot{E} = s + \xi + \frac{k^*}{\theta_0 \rho} \text{div} \beta. \]

and now, assuming \( \xi = 0 \) we simplify the above equation to

\[ \frac{c \dot{\theta}}{\rho_0 \theta_0} + \frac{E \beta^*}{3 \rho_0(1 - 2\nu)} tr \dot{E} = s + \frac{k^*}{\theta_0 \rho} \text{div} \beta. \]

Multiplying both sides of (30) by \( \rho_0 \theta_0 \) and assuming \( \rho = \rho_0 \) and taking into consideration that \( r = \theta_0 s \) we have:

\[ c \dot{\theta} + \frac{E \theta_0 \beta^*}{3 (1 - 2\nu)} tr \dot{E} = \rho r + k^* \text{div} \beta. \]
Taking the time derivative of (31) we obtain:

\[ c \ddot{\theta} + \frac{E \theta_0 \beta^*}{3(1 - 2\nu)} \mathrm{tr} \ddot{E} = \rho \dot{r} + k^* \text{div} \dot{\beta}. \quad (32) \]

Taking a more carefully look for the last term of right side of equation (32). Remember that \( \alpha = \alpha(X, t) \) and for the definition of thermal displacement considering that \( \dot{\alpha} = T = \theta \) follows that:

\[ \beta = \frac{\partial \alpha}{\partial X} \Rightarrow \dot{\beta} = \frac{\partial \dot{\alpha}}{\partial X} = \frac{\partial \theta}{\partial X} = \nabla \theta. \]

then

\[ \text{div} \dot{\beta} = \text{div} (\nabla \theta) = \nabla^2 \theta = \Delta \theta. \]

Replacing the last identity in equation (32) we obtain

\[ c \ddot{\theta} + \frac{E \theta_0 \beta^*}{3(1 - 2\nu)} \text{tr} \ddot{E} = \rho \dot{r} + k^* \nabla^2 \theta. \]

It is easy to show that \( \text{tr} \ddot{E} = \dddot{\text{div} u} \) then finally we have the hyperbolic PDE for the temperature \( \theta \) as follow:

\[ c \ddot{\theta} + \frac{E \theta_0 \beta^*}{3(1 - 2\nu)} \dddot{\text{div} u} = \rho \dot{r} + k^* \nabla^2 \theta. \quad (33) \]

In summary follows from equations (28) and (33) we have the coupled system of PDE:

\[ (\lambda + \mu) \nabla \text{div} u + \mu \nabla^2 u - \frac{E \beta^*}{3(1 - 2\nu)} \nabla \theta + \rho \mathbf{b} = \rho \ddot{u}, \quad (34) \]

\[ c \ddot{\theta} + \frac{E \beta^*}{3(1 - 2\nu)} \dddot{\text{div} u} = \rho \dot{r} + k^* \nabla^2 \theta, \quad (35) \]

where \( \text{div} \) is the divergence operator with respect to \( X \).

Note that without heat source \( r \) and body force \( b \), considering the restrictions \( c > 0, k^* \geq 0, \lambda + 2\mu \geq 0 \), the system of PDE (34)-(35) lead us to harmonic propagation of waves without damping.
5 CONCLUSIONS

We started showing in the section 2 the development of continuum mechanics, elasticity, the inclusion of the thermal stress in the material and how a change of temperature affect the mechanical stress and vice and versa ultimately, how it culminated in the thermoelasticity theory. We also commented the physical interpretation given by Fourier, Biot and Voigt. Still in this section we showed the classical and non-classical thermoelasticity and how the Green–Naghdi theory was proposed.

In section 3 we gave details of Green–Naghdi thermoelasticity theory showing the fundamental variables such as $\psi$, $\eta$, $q$, and $p$, etc and constitutive hypothesis (4), (5), (6) and (7). We gave a more attention to the equation (7) because it have the information of the coupling between thermal and mechanical stress for the general stress in thermoelastic material.

We also derived in sections 3.1 and 3.2 the two types of flow of the model. In those sections we mentioned how the parabolic PDE obtained in (14) derived from type I flux lead us to a mathematical inconsistency according to experimental data. To solve this paradox we showed the Green-Naghdi type II flux and from the nature of constitutive hypothesis and assumptions lead us to the hyperbolic PDE in equation (22). The modification for parabolic to hyperbolic PDE solves the mathematical inconsistency and also show to us the essence of classical and non-classical thermoelasticity, the changes of the type of the PDE.

In non-classical theory as a special case of equation (22) we also showed we can return to the classical wave equation with the propagation velocity $v = \sqrt{\frac{ak}{ct^2}}$. Its well know such hyperbolic PDE have exactly solutions, and remove the mathematical inconsistency and turn the theory accord to experimental data. Its well know such hyperbolic PDE have exact solutions, remove the mathematical inconsistency and transform the theory according to experimental data. It is important to note that the difference in the constitutive hypothesis have a critical importance in the both models. Type I flux was a good approach but type II flux is a more realistic model due to the physical content. We can classify the types of flow in the scale of physical content to I to III even we do not show the type III flux.

In section 4 we showed the linear theory of type II flux. The linear theory although seems more complicated due to regarded much information about the system and we have a system of coupled PDE (34)-(35) to solve, the theory is convenient due to have different methods for solve the linear problems but non-linear is complex and not always have a solution.

Other point extremely important to mention is the solution of equations (14), the general form of (22) and the PDE coupled system (34)-(35) is not an easy task. There is a lot of current works in the area solving those equations using numerical methods and our next step is try to
use finite element to solve PDE equations of Green–Naghdi theory for linear and non-linear theory.

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