LETTER TO THE EDITOR

Entanglement and spin squeezing in the two-atom Dicke model

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Abstract. We analyze the relation between the entanglement and spin-squeezing parameter in the two-atom Dicke model and identify the source of the discrepancy recently reported by Banerjee and Zhou et al that one can observe entanglement without spin squeezing. Our calculations demonstrate that there are two criteria for entanglement, one associated with the two-photon coherences that create two-photon entangled states, and the other associated with populations of the collective states. We find that the spin-squeezing parameter correctly predicts entanglement in the two-atom Dicke system only if it is associated with two-photon entangled states, but fails to predict entanglement when it is associated with the entangled symmetric state. This explicitly identifies the source of the discrepancy and explains why the system can be entangled without spin-squeezing. We illustrate these findings in three examples of the interaction of the system with thermal, classical squeezed vacuum and quantum squeezed vacuum fields.

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Entanglement, the most intriguing property of multiparticle systems (or qubits), is one of the key problems in quantum physics and has been the subject of active research in recent years [1]. It describes a multiparticle system which has the astonishing property that the results of a measurement on one particle cannot be specified independently of the results of measurements on the other particles. Therefore, the generation of entanglement between atoms is fundamental not only to demonstrate quantum nonlocality but also would constitute a valuable resource in the fields of quantum information processing, cryptography and quantum computation [2]. In this context, it is not surprising that a tremendous number of theoretical proposals have been made to produce entanglement between separate particles [3]. Several different criteria have been proposed to identify entanglement in two-particle systems, but no definite measure
of entanglement exists for a number of particles larger than two. Entanglement between two particles can be identified by calculating, for example, the Wootters entanglement measure (concurrence) \[4\], or a measure proposed by Peres \[5\] and Horodecki \[6\] given in terms of the negative eigenvalues of the partial transposition of the density matrix of the two-particle system. Recently, Sørensen et al \[7\] have proposed a measure of multiparticle entanglement in terms of the spin-squeezing parameter \[8, 9, 10, 11\]

\[
\xi_{n_i} = \frac{N_a \langle (\Delta S_{n_i})^2 \rangle}{\langle S_{n_i} \rangle^2 + \langle S_{n_k} \rangle^2},
\]

where \(N_a\) is the number of particles, \(n_i, n_j\) and \(n_k\) are three mutually orthogonal unit vectors oriented such that the mean value of one of the spin components, say \(\langle S_{n_k} \rangle\), is different from zero, while the other components \(S_{n_i}\) and \(S_{n_j}\) have zero mean values. The variance \(\langle (\Delta S_{n_i})^2 \rangle\) should be calculated in the plane orthogonal to the mean spin direction. A multiatom system in a coherent state has variances normal to the mean spin direction equal to the standard quantum limit of \(N_a/4\). In this case, \(\xi_{n_i} = \xi_{n_j} = 1\). A system with the variance reduced below the standard quantum limit in one direction normal to the mean spin direction is characterized by \(\xi_{n_i} < 1\), that is spin squeezed in the direction \(n_i\). Sørensen et al \[7\] have shown that multiparticle spin squeezed systems also exhibit entanglement.

However, in recent studies of entanglement in the two-atom Dicke system \[12, 13\] it has been discovered that the spin-squeezing parameter \(\xi_{n_i}\) is not sufficient for predicting entanglement in a multiparticle system. Banerjee \[12\] and Zhou et al \[13\] have shown that the two-atom Dicke system driven by a single mode thermal field, can exhibit an entanglement and at the same time \(\xi_{n_i} > 1\). They have found that in the thermal field the time evolution of the system is represented by a diagonal density matrix

\[
\dot{\rho}(t) = \rho_{gg}(t)|g\rangle\langle g| + \rho_{ee}(t)|e\rangle\langle e| + \rho_{ss}(t)|s\rangle\langle s|,
\]

where

\[
|g\rangle = |g_1\rangle|g_2\rangle,
|e\rangle = |e_1\rangle|e_2\rangle,
|s\rangle = \frac{1}{\sqrt{2}}(|e_1\rangle|g_2\rangle + |g_1\rangle|e_2\rangle),
|a\rangle = \frac{1}{\sqrt{2}}(|e_1\rangle|g_2\rangle - |g_1\rangle|e_2\rangle)
\]

are the collective states of the two-atom system \[14\], and \(|g_i\rangle, |e_i\rangle\) are the ground and excited states of the \(i\)th atom, respectively. In the Dicke system the antisymmetric state \(|a\rangle\) is completely decoupled from the remaining states, and then the simple three-state representation of the two-atom Dicke system can be applied with the ground product state \(|g\rangle\), the excited product state \(|e\rangle\) and the maximally entangled symmetric state \(|s\rangle\). Since the density matrix of the system is diagonal and the symmetric state \(|s\rangle\) is a maximally entangled state, an entanglement can be produced in the Dicke system by a suitable population of the state \(|s\rangle\). This is exactly the situation considered by Banerjee \[12\] and Zhou et al \[13\].
In this letter, we clarify the discrepancy between entanglement and the spin-squeezing parameter. The parameter $\xi_{n_i}$ has been proposed as a simple and robust method to identify entanglement of a large number of atoms, so we believe that a detailed analysis of the discrepancy is of general interest. We show that in the two-atom Dicke model, the parameter $\xi_{n_i}$ correctly predicts entanglement only if the system is in the two-photon entangled states which are linear superpositions of the collective ground state $|g\rangle$ and the upper state $|e\rangle$, but fails to predict entanglement if the system is in the entangled symmetric state $|s\rangle$.

In order to show this more quantitatively, we start from the definition of the parameter $\xi_{n_i}$, which we can write in terms of the density matrix elements of the system as

$$\xi_{n_i} = 2\langle(\Delta S_{n_i})^2\rangle = 1 + \rho_{ss} - 2|\rho_{eg}| \cos \theta,$$

where $\theta$ is the angle between $n_i$ and the direction of maximum squeezing. In the derivation of (4), we have used the Kitagawa and Ueda’s definition of $\xi_{n_i}$ in which the variance $\langle(\Delta S_{n_i})^2\rangle$, calculated in the $n_i$ direction, is compared to the maximum spin $\langle S_{n_k}\rangle = N_a/2$ in the normal $n_k$ direction. For simplicity, we have assumed that the mean spin direction coincides with the $z$ axis and calculated the variance in the $n_i$ direction which coincides with the $x$ axis. This is not an essential feature if the system is driven by a thermal or squeezed vacuum field, since in this case the mean values $\langle S_x\rangle$ and $\langle S_y\rangle$ are zero for all values of the parameters involved [3]. In a more general case of a coherently driven atoms, where $\langle S_x\rangle$ and $\langle S_y\rangle$ are different from zero, one can adjust the angle $\theta$ such that the maximum squeezing will coincide with the direction of the rotated nonzero spin components.

We see from (4) that the parameter $\xi_{n_i}$ depends on the population $\rho_{ss}$ of the entangled symmetric state and the two-photon coherence $\rho_{eg}$. Hence, spin squeezing will be produced in the direction $\theta$ when $|\rho_{eg}| > \rho_{ss}/2$. Note that the spin-squeezing parameter involves the two-photon coherences with no dependence on one-photon coherences. This indicates that the spin squeezing can only be generated by two-photon processes. Thus, the spin squeezing is inherent multi-atom effect arising from the collective evolution of the Dicke system.

We now determine general conditions for entanglement in the two-atom Dicke model using the Peres-Horodecki measure of entanglement given by the quantity [5, 6]

$$E = \max\left(0, -2 \sum_i \mu_{i-}\right),$$

where the sum is taken over the negative eigenvalues $\mu_{i-}$ of the partial transposition of the density matrix $\hat{\rho}$ of the system. The value $E = 1$ corresponds to maximum entanglement between the atoms whilst $E = 0$ describes completely separated atoms.

Since the generation of the spin squeezing is independent of the one-photon coherences, we will look into conditions for entanglement which are determined by the population of the collective states and the two-photon coherences. Note, that in the Dicke model, $\rho_{aa} = 0$. In this case, the density matrix of the system in the basis
\{ |e_1, e_2⟩, |e_1, g_2⟩, |g_1, e_2⟩, |g_1, g_2⟩ \} can be written as
\[ \hat{\rho} = \begin{pmatrix} \rho_{ee} & 0 & 0 & \rho_{eg} \\ 0 & \frac{1}{2} \rho_{ss} & \frac{1}{2} \rho_{ss} & 0 \\ 0 & \frac{1}{2} \rho_{ss} & \frac{1}{2} \rho_{ss} & 0 \\ \rho_{ge} & 0 & 0 & \rho_{gg} \end{pmatrix}. \] (6)

Following the Peres-Horodecki criterion for entanglement, we find that the eigenvalues of the partial transposition of \( \hat{\rho} \) are
\[ \mu_{1\pm} = \frac{1}{2} \rho_{ss} \pm |\rho_{eg}|, \]
\[ \mu_{2\pm} = \frac{1}{2} \left\{ (\rho_{ee} + \rho_{gg}) \pm \left[ (\rho_{ee} - \rho_{gg})^2 + \rho_{ss}^2 \right]^{\frac{1}{2}} \right\}. \] (7)

It is obvious that \( \mu_{1+} \) and \( \mu_{2+} \) are always positive. The eigenvalues \( \mu_{1-} \) and \( \mu_{2-} \) become negative if and only if
\[ |\rho_{eg}| > \frac{1}{2} \rho_{ss} , \] (8)
or
\[ \rho_{ss} > 2 \sqrt{\rho_{ee} \rho_{gg}} . \] (9)

We are now in a position to understand quantitatively the discrepancy between entanglement and the spin squeezing parameter. It is seen that there are two criteria for entanglement in the two-atom Dicke model. The first criterion, Equation (8), is associated with the two-photon coherence and population of the symmetric state. The second criterion, Equation (9), is associated only with the populations of the collective states. It is evident that the criterion (8) overlaps with the criterion for spin squeezing, see Equation (4). Therefore, in the absence of the two-photon coherences, the two-atom system can still be entangled, in accordance with the criterion (9), but cannot exhibit spin-squeezing, which is associated with the criterion (8). This explicitly identifies the source of the discrepancy found by Banerjee [12] and Zhou et al [13] and explains why the two-atom Dicke system can be entangled without spin-squeezing.

In the situations where the criterion (8) is satisfied, there are entangled states generated which can be found by the diagonalization of the density matrix (6). We find that the diagonalization leads to eigenstates
\[ |\Psi_+⟩ = \left[ (\Pi_+ - \rho_{ee}) |g⟩ + \rho_{eg} |e⟩ \right] / \left[ (\Pi_+ - \rho_{ee})^2 + |\rho_{eg}|^2 \right]^{\frac{1}{2}}, \]
\[ |\Psi_-⟩ = \left[ \rho_{ge} |g⟩ + (\Pi_- - \rho_{gg}) |e⟩ \right] / \left[ (\Pi_- - \rho_{gg})^2 + |\rho_{eg}|^2 \right]^{\frac{1}{2}}, \]
\[ |\Psi_s⟩ = |s⟩, \]
\[ |\Psi_a⟩ = |a⟩, \] (10)

with the diagonal probabilities
\[ \Pi_+ = \frac{1}{2} (\rho_{gg} + \rho_{ee}) + \frac{1}{2} \left[ (\rho_{gg} - \rho_{ee})^2 + 4 |\rho_{eg}|^2 \right]^{\frac{1}{2}}, \]
\[ \Pi_- = \frac{1}{2} (\rho_{gg} + \rho_{ee}) - \frac{1}{2} \left[ (\rho_{gg} - \rho_{ee})^2 + 4 |\rho_{eg}|^2 \right]^{\frac{1}{2}}, \]
It is evident from (10), that in the presence of the two-photon coherence, the system evolves into entangled states which are linear superpositions of the collective ground state $|g\rangle$ and the upper state $|e\rangle$. The entangled symmetric state remains unchanged in the presence of two-photon processes. Thus, spin squeezing and entanglement created by the two-photon coherences are both associated with the two-photon entangled states $|\Psi_\pm\rangle$.

As an example to illustrate our findings, consider the two-atom Dicke system driven by a broadband squeezed vacuum field. In the steady-state, nonzero matrix elements are [3, 15]

$$
\rho_{ee} = \frac{N^2 (2N + 1) - (2N - 1)|M|^2}{(2N + 1)(3N^2 + 3N + 1 - 3|M|^2)} ,
$$

$$
\rho_{ss} = \frac{N (N + 1) - |M|^2}{3N^2 + 3N + 1 - 3|M|^2} ,
$$

$$
|\rho_{eg}| = \frac{|M|}{(2N + 1)(3N^2 + 3N + 1 - 3|M|^2)} ,
$$

(12)

where $N$ is the intensity of the squeezed field and $M$ is the two-photon correlation function [16].

For the interaction of the system with a thermal field, $M = 0$, and then using (12) it is straightforward to prove that both criteria (8) and (9) are not satisfied for all values of $N$. Moreover, the inequality $\xi_{ns} > 1$ always holds indicating that both entanglement and spin-squeezing are not present in the steady-state two-atom Dicke system driven by a thermal field.

The situation is different when the system is driven by a classical squeezed field with the maximal two-photon correlations $M = N$. In this case the inequality $\rho_{ss} < 2\sqrt{\rho_{ee}\rho_{gg}}$ always holds in contradiction to [9]. However, we find that the inequality [8] can be satisfied as $|\rho_{eg}| \neq 0$. According to (11) and (8), the criterion for both entanglement and spin squeezing can be determined by positive values of a parameter

$$
2|\rho_{eg}| - \rho_{ss} = \frac{N (1 - 2N)}{(2N + 1)(3N + 1)} .
$$

(13)

Equation (13) shows that the system driven by the classical squeezed field will exhibit entanglement and spin-squeezing when $N < 1/2$. We illustrate these features in Figure (1a), where we plot the entanglement measure $E$ and the squeezing parameter $\xi_{ns}$ as a function of the intensity $N$. The figure clearly demonstrates that with the condition [8], the squeezing parameter correctly predicts entanglement induced by the two-photon coherences. We should note here that the steady-state of the system driven by a classical squeezed field is a mixed state. Thus, the squeezing parameter correctly predicts entanglement in a mixed state if the entanglement is generated by the two-photon coherences.
When the system is driven by a quantum squeezed field with perfect correlations $|M|^2 = N(N + 1)$, the populations of the diagonal states (11) are profoundly affected by the presence of the strong two-photon correlations $M$ such that the steady-state of the system is a pure state $|\Psi_+\rangle$ [15, 17]. Since $\rho_{ss} = 0$, the criterion (9) is not satisfied, and therefore entanglement is determined solely by the criterion (8) which is always satisfied as $|\rho_{eg}| > 0$. Since the inequality $|\rho_{eg}| > 0$ always holds, the system exhibits entanglement and spin-squeezing for all $N$. This feature is seen in Figure 1(b), where we show the entanglement measure $E$ and squeezing parameter $\xi_{nx}$ for the quantum squeezed field. We see that the entanglement and spin squeezing are present for all $N$. The entanglement and spin squeezing increase with increasing $N$ and attain their maximal values, $E = 1$ and $\xi_{nx} = 0$, for large $N$.

As we have mentioned above, the steady-state of the system driven by the quantum squeezed field is a pure state. We find from (10) that the pure state is the entangled state given by [15, 17]

$$|\Psi_+\rangle = \frac{1}{\sqrt{2N+1}} \left[ \sqrt{N+1}|g\rangle + \sqrt{N}|e\rangle \right].$$

(14)

The pure state is a non-maximally entangled state, and reduces to a maximally entangled state for $N \gg 1$.

In summary, we have clarified the discrepancy between the entanglement and spin-squeezing parameter recently reported by Banerjee [12] and Zhou et al [13]. We have found that there are two criteria for entanglement in the two-atom Dicke system, one associated with the two-photon coherences and population of the symmetric state, and the other associated with populations of all the collective states. We have shown that the criterion for spin squeezing overlaps with only one of the two criteria for entanglement, that involving the two-photon coherences. Therefore, if entanglement is produced by the other criterion, one obtains entanglement without spin squeezing. Thus, our calculations

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**Figure 1.** Entanglement measure $E$ (solid line) and the squeezing parameter $\xi_{nx}$ (dashed line) as a function of the intensity $N$ for (a) classical squeezed field with $|M| = N$, and (b) quantum squeezed field with $|M| = \sqrt{N(N + 1)}$. 
demonstrate that the spin-squeezing parameter correctly predicts entanglement in the
two-atom Dicke system if the entanglement is created by the two-photon coherences.
Moreover, the current study provides a clear physical picture of different processes which
can create entanglement in the two-atom Dicke system.

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