Regularity lemma’s in a Banach space setting

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Abstract

Szemerédi’s regularity lemma is a fundamental tool in extremal graph theory, theoretical computer science and combinatorial number theory. Lovász and Szegedy [17] gave a Hilbert space interpretation of the lemma and an interpretation in terms of compactness of the space of graph limits. In this paper we prove several compactness results in a Banach space setting, generalising results of Lovász and Szegedy [17] as well as a result of Borgs, Chayes, Cohn and Zhao [5].

1 Introduction

1.1 The regularity lemma

Szemerédi’s regularity lemma [25] is a fundamental tool in extremal graph theory, theoretical computer science and combinatorial number theory. See [16] for a survey. The lemma has many interpretations, variations and extensions. See for example [15, 11, 23, 10, 12, 9, 26, 17, 24, 5].

Very roughly the lemma says something of the form: for each ε > 0 there exists k ∈ N such that the vertex set of any graph can be partitioned into at most k parts, such that for ‘almost’ all pairs of parts the edges between that pair of parts behaves ‘almost’ like a random bipartite graph, where ‘almost’ depends on ε. The weak regularity lemma of Frieze and Kannan [11] weakens the requirements of the partition in the regularity lemma and measures the error of approximation with respect to the cut norm. This has as a consequence that the constant k can be taken to be much smaller. From the perspective of the adjacency matrix of a graph this means that one approximates this matrix with a bounded sum of cut matrices (in particular this gives a low rank approximation) such that their difference is small with respect to the cut norm. This is exactly the point of view we take in this paper: we want to find various types of low rank approximations to matrices and tensors, when measured in a particular norm.

Our work is inspired by the work of Lovász and Szegedy [17] and Borgs, Chayes, Cohn and Zhao [5] relating the compactness of the space of graph limits to Szemerédi’s regularity lemma. We refer to the book by Lovász [19] for more details on graph limits. In [18] Lovász and Szegedy used the weak version of the regularity lemma [11] to assign a limit object to a convergent sequence of dense graphs. This limit object is no longer a graph, but a symmetric measurable function W : [0, 1]2 → [0, 1], called a graphon. In [17] Lovász and

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Szegedy showed that the space of graphons, equipped with the cut metric is compact, interpreting this result as an analytical form of the regularity lemma. Their compactness result implies various kinds of regularity lemma’s varying from weak to very strong. It has recently been extended by Borgs, Chayes, Cohn and Zhao [5] to the space of $\mathbb{R}$-valued functions $W$ with bounded $p$-norm, the $L^p$-graphon space (for any fixed $p > 1$).

1.2 Compactness

We will now describe the compactness of the graphon space, which is denoted by $W$, more precisely (for details concerning definitions we refer to the next section), after which we mention some of the results in the present paper.

Let $W : [0, 1]^2 \to \mathbb{R}$. Consider for $p, q \in [1, \infty)$, $W$ as an operator $W : L^p([0, 1]) \to L^q([0, 1])$. The $p \mapsto q$-operator norm is defined by

$$\|W\|_{p \mapsto q} = \sup_{\|f\|_p = 1} \|Wf\|_q,$$

where $\| \cdot \|_s$ denotes the $s$-norm on the space $L^s([0, 1])$. The norm $\| \cdot \|_{\infty \mapsto 1}$ is equivalent to the cut norm. Call $W \sim W'$ if for each $\varepsilon > 0$ there exists a measure preserving bijection $\tau : [0, 1] \to [0, 1]$ such that $\|W - \tau W'\|_{\infty \mapsto 1} \leq \varepsilon$. Then the result of Lovász and Szegedy [17] can be stated as follows:

the space $(W, \| \cdot \|_{\infty \mapsto 1}) / \sim$ is compact. (1)

The result of Borgs, Chayes, Cohn and Zhao [5] then says that we can replace $W$ with the symmetric functions in $L^p([0, 1]^2)$ of norm 1 for any fixed $p > 1$.

In this paper we will show that in (1) we can also replace the norm $\| \cdot \|_{\infty \mapsto 1}$ by the norm $\| \cdot \|_{q \mapsto q - 1}$, if we replace $W$ by the unit ball of $L^p([0, 1]^2)$, provided that $p > \frac{q}{q - 1}$, cf. Corollary 4.3. In fact, we generalise (1), replacing the space $W$ by a weak-* compact subset of a Banach space $X$, the relation $\sim$ by an equivalence relation obtained from a subgroup of the group of automorphisms of $X$ and the norm $\| \cdot \|_{\infty \mapsto 1}$ by an operator-type norm, cf. Theorem 3.2. From Theorem 3.2 it is then easy to derive the results of Borgs, Chayes, Cohn and Zhao. In Section 4 we will also utilise it to include $q \mapsto \frac{q}{q - 1}$-norms and apply it to higher order tensors. In Section 5 we will apply it to $\ell^p$-spaces.

Our method is based on work of Regts and Schrijver [22]. In [22] the compactness result of Lovász and Szegedy was extended to a general Hilbert space setting, putting emphasis on the possibility of using different norms than the cut norm and the use of groups and moreover using a different method of proof. Consequently, our proof of Theorem 3.2 does not use the martingale convergence theorem. Thus it yields a different proof of the compactness result of Borgs, Chayes, Cohn and Zhao [5]. However, there are some similarities. To prove Theorem 3.2 we need a result from [22], cf. Lemma 3.3 which may be viewed as a weak regularity lemma in a Hilbert space setting, and which generalises weak regularity results from [11, 10].

1.3 Algorithms and applications

Some of the existing versions of the weak and strong regularity lemma’s come with efficient algorithms for finding a low rank approximation (or regularity partition). These algorithms
have been applied to find approximation schemes for various sorts of dense instances of
counting and optimisation problems [11, 10] and to property and parameter testing [1, 6, 7];
see also the book by Lovász [19].

Some of our results can also be put to algorithmic use. In particular, in the ℓ
p

setting, Lemma 5.3 can be used to give approximation algorithms for certain instances of comput-
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ing the matrix \( p \mapsto q \) norm and for finding approximate Nash-equilibria in two player

in games, in a similar spirit as has been done by Barman in \[3\]. In the \( L^p \) setting sampling

algorithms can be applied to find low rank approximations to matrices and tensors yielding

polynomial time approximation algorithms for computing the matrix \( p \mapsto q \) norm. This

work in progress and we will report on it in a forthcoming paper \[21\].

1.4 Organisation

In the next section we will discuss some preliminaries and set up some notation. In Section

3 we will state and prove Theorem 3.2, the aforementioned generalisation of \( (1) \). We will

also deduce some consequences from it. In Section 4 we will apply this theorem to

\( L^p \) spaces and in Section 5 to \( ℓ^p \) spaces.

2 Preliminaries and notation

In this section we will give some preliminaries on Lebesgue spaces and set up some nota-
tion. We refer to \[8\] for functional analytic background and to \[13\] for measure theoretic

background.

For a measure space \((X, Ω, μ)\) and \( p \in [1, ∞) \) we denote by \( L^p(X) \) the linear space of

complex or real-valued function \( f : X \to \mathbb{C} \) (or \( \mathbb{R} \)) with bounded \( p \)-norm, which is defined

as

\[
\|f\|_p := \left( \int |f|^p dμ \right)^{1/p} \quad \text{for } p < ∞, \text{ and}
\]

\[
\|f\|_∞ := \inf_{t \geq 0}\{|f(x)| \leq t | \text{ for } μ-\text{almost all } x}\).
\]

For our results it often does not matter whether we use real or complex-valued functions.
So we generally do not distinguish between the complex and real-valued cases. Moreover,
we often omit the reference to \( Ω \) and \( μ \). In case \( X = [0, 1]^l \) for some \( l \in \mathbb{N} \) we will always
equip it with the Borel (or Lebesgue) sigma algebra and with the Lebesgue measure \( λ \).

For a set \( S \), and \( p \in [1, ∞) \), \( ℓ^p(S) \) is just equal to \( L^p(S) \) with \( Ω \) the power set of \( S \) and \( μ \)
the counting measure.

For a normed space \((Y, \| \cdot \|)\) we denote its closed unit ball by \( B(Y, \| \cdot \|) \), which is defined as \( \{y \in Y | \|y\| \leq 1\} \). Often we just write \( B(Y) \). Let \((X, μ)\) be a probability space,
i.e. \( μ(X) = 1 \). An important property of the space \( L^p(X) \), that we will often use, is the
nesting of the closed unit balls: for any \( 1 < p < q < ∞ \) we have

\[
B(L^∞(X)) \subset B(L^q(X)) \subset B(L^p(X)) \subset B(L^1(X)).
\]

For the closed unit balls in \( L^p(S) \) the opposite inclusions hold:

\[
B(ℓ^1(S)) \subset B(ℓ^p(S)) \subset B(ℓ^q(S)) \subset B(ℓ^∞(S)).
\]
If a group $G$ acts on a space $X$ this induces an action on the functions from $X$ to $\mathbb{C}$ (or $\mathbb{R}$) via $gf(x) := f(g^{-1}x)$ for $g \in G$, a function $f$ and $x \in X$. Moreover, $G$ has a natural action on $X'$ for any $l \in \mathbb{N}$. In particular, if $(X, \mu)$ is a measure space and a group $G$ acts on $X$ such that $\mu(gA) = \mu(A)$ for all measurable sets $A$ and $g \in G$ (then we call elements of $G$ measure preserving bijections), then $G$ acts on $L^p(X')$ for any $p$ and $l$ and preserves the $p$-norm. If a group acts on a space $X$ and $S \subseteq X$ is $G$-stable and weak-* compact set. If $G \subseteq X$ is $G$-stable and weak-* compact set. If $G \subseteq X$ is $G$-stable and weak-* compact set. If $G \subseteq X$ is $G$-stable and weak-* compact set. If $G \subseteq X$ is $G$-stable and weak-* compact set. If $G \subseteq X$ is $G$-stable and weak-* compact set.

3 Compact orbit spaces in Banach spaces

In this section we will state and proof our main results concerning compact orbit spaces in Banach spaces and discuss some consequences, which may be viewed as regularity lemma’s in a Banach space setting.

3.1 The main theorems

Before we can state our results, we need some definitions. Let $X = (X, \| \cdot \|)$ be a normed space and let $R$ be a bounded subset of $X^*$, the dual space of $X$. We define a seminorm $\| \cdot \|_R$ and pseudo metric $d_R$ on $X$ by

$$\|x\|_R := \sup_{r \in R} |r(x)| \quad d_R(x, y) := \|x - y\|_R$$

for $x, y \in X$.

For a metric space $(X, d)$ let $\text{Aut}(X)$ denote the group of invertible maps $g : X \to X$ that preserve the metric. Let $G$ be a subgroup of $\text{Aut}(X)$. Define a pseudo metric $d/G$ on $X$ by

$$(d/G)(x, y) := \inf_{g \in G} d(x, gy)$$

for $x, y \in X$. Note that since $(d/G)(x, y)$ is just equal to the distance between the $G$-orbits of $x$ and $y$, this implies that $d/G$ is indeed a pseudo metric. For our purposes it is sometimes more convenient to work with $(X, d/G)$ than with $X/G$, but note that $(X, d/G)$ is compact if and only if $X/G$ is compact. Recall that a (pseudo) metric space is called totally bounded of for each $\varepsilon > 0$ it can be covered with finitely many balls of radius $\varepsilon$.

We can now state our first result about compactness of orbit spaces in Banach spaces, which we will prove in Section 3.2.

**Theorem 3.1.** Let $(X, \| \cdot \|)$ be a Banach space and let $G$ be a subgroup of $\text{Aut}(X)$. Let $R \subseteq B(X^*)$, be $G$-stable, and let $W \subseteq X$ be a $G$-stable and weak-* compact set. If $(W, d_R/G)$ is totally bounded, then $(W, d_R/G)$ is compact.

Showing that $(W, d_R/G)$ is totally bounded may not be very simple. However, if $W$ is somehow ‘close’ to a Hilbert space (as will be made precise below), then totally boundedness of $(W, d_R/G)$ can be deduced from totally boundedness of the space of sums of elements from $R$ modulo $G$.

For a subset $Y$ of a linear space $X$ and $k \in \mathbb{N}$ we define

$$k \cdot Y := \{y_1 + \ldots + y_k \mid y_i \in Y\}.$$
Note that when \( Y \) is convex, \( k \cdot Y \) is just equal to \( kY \). Let \( X \) be a normed space and let \( W \subseteq X \). We call \( W \) Hilbert-small, if there exists a Hilbert space \( H \) with \( B(H) \subseteq B(X) \) and a function \( c : (0, \infty) \to \mathbb{N} \) such that \( W \subseteq c(\varepsilon)B(H) + \varepsilon B(X) \) for each \( \varepsilon > 0 \). Note that this condition implies that \( X^* \subseteq X \) as \( H \subseteq X \) implies that \( X^* \subseteq H^* = H \). We have the following theorem:

**Theorem 3.2.** Let \((X, \| \cdot \|)\) be a Banach space and let \( G \) be a subgroup of \( \text{Aut}(X) \). Let \( R \subseteq B(X^*) \), be \( G \)-stable and let \( W \subseteq X \) be a \( G \)-stable and weak-* compact set, which is Hilbert-small. If \((k \cdot R) / G \) is totally bounded with respect to \( \| \cdot \| \) for each \( k \in \mathbb{N} \), then \((W, d_R / G)\) is compact.

Observe that when \( X \) is a Hilbert space, Theorem 3.2 reduces to \([22\text{ Theorem 2.1}]\).

Before we give a proof of Theorem 3.2, let us remark that the compactness of the \( L^p \)-graphon space proved by Borgs, Chayes, Cohn and Zhao \([5]\) follows almost immediately from it. Indeed, take \( W = B(L^p([0, 1]^2)) \subseteq X = L^1([0, 1]^2) \) (for any \( p \in (1, \infty) \)) and \( H = L^2([0, 1]^2) \). Then \( W \) is Hilbert-small by \([5]\), or see Lemma 4.4. Taking \( R = \{X_{A \times B} \mid A, B \subseteq [0, 1] \text{ measurable}\} \), makes \( \| \cdot \| \) into the cut norm. As group \( G \) we take the group of measure preserving bijections \( \phi : [0, 1] \to [0, 1] \). Then \( d_R / G \) is equal to \( \delta_{\square} \), the cut metric. Since any measurable set \( A \) can be mapped onto any interval of length \( \lambda(A) \) by a measure preserving bijection, cf. \([20]\), it follows that \( k \cdot R / G \) is compact (see \([22]\)). So, as \( B(L^p([0, 1]^2)) \) is weak-* compact by the Banach-Alaoglu theorem, Theorem 3.2 implies that \((B(L^p([0, 1]^2)), d_R / G)\) is compact.

With little additional effort we can derive something similar as Theorem C.7 in \([5]\): Given a function \( \kappa : (0, \infty) \to (0, \infty) \), let \( \kappa_{\kappa} := \cap_{\kappa > 0} \kappa(\varepsilon) B(L^\infty([0, 1]^2)) + \varepsilon B(L^1([0, 1]^2)) \). Then by Theorem 3.2 \((W_{\kappa}, d_R / G)\) is compact. To see this note that \( W_{\kappa} \) is clearly Hilbert-small. So we just need to check that it is weak-* compact. Since \( W_{\kappa} \) is bounded (with respect to \( \| \cdot \|_1 \) we may assume that is contained in \( B(L^1([0, 1]^2)) \subseteq B(L^\infty([0, 1]^2))^* \). Since the latter space is weak-* compact, it suffices to show that if \( C \subseteq W_{\kappa} \), then its weak-* closure is again contained in \( W_{\kappa} \). Let \( f \) be any element of the weak-* closure of \( C \) and let \( \varepsilon > 0 \). Define \( U := \{x \mid f(x) > \kappa(\varepsilon)\} \) and \( V := \{x \mid f(x) < -\kappa(\varepsilon)\} \) and let \( \delta > 0 \). Then there exists \( g \in C \) such that \( \int_U (f - g)d\lambda \leq \delta \) and \( -\int_V (f - g)d\lambda \leq \delta \). This implies

\[
\int_{U \cup V}(f - g)d\lambda = \int_U f d\lambda - \int_V f d\lambda \leq \int_U g d\lambda - \int_V g d\lambda + 2\delta \\
\leq \int_U |g| d\lambda + 2\delta \leq \lambda(U \cup V)\kappa(\varepsilon) + \varepsilon + 2\delta.
\]

Since \((2)\) holds for any \( \delta > 0 \), this shows that \( f \in W_{\kappa} \).

### 3.2 Proofs of Theorem 3.1 and 3.2

The proofs of Theorems 3.1 and 3.2 follow the same pattern as the proof of Theorem 2.1 in \([22]\). Let us start with the proof of Theorem 3.1.

**Proof of Theorem 3.1** Since \((W, d_R / G)\) is totally bounded it suffices to show that \((W, d_R / G)\) is complete. Let \( a_1, a_2, \ldots \) be a Cauchy sequence in \((W, d_R / G)\). We may assume that \( (d_R / G)(a_i, a_{i+1}) < 2^{-i} \) for each \( i \). Next find iteratively \( g \in G \) such that \( d_R(a_i, ga_{i+1}) < 2^{-i} \) and replace \( a_{i+1} \) by \( ga_{i+1} \). Then \( d_R(a_i, a_j) < 2^{-i+1} \) for all \( i \) and \( j \geq i \) and hence this sequence is Cauchy with respect to \( d_R \). We will show that this sequence is also convergent. Let for \( n \in \mathbb{N} \), \( A_n \) denote the weak-* closure of the set \( \{a_n, a_{n+1}, \ldots\} \). Since the \( a_i \in W \) and the
latter space is weak-* compact by assumption, we have that $A := \cap_n A_n$ is not empty. Let $a \in A$. We will show that $a_n$ converges to $a$ with respect to $d_R$. Let $\varepsilon > 0$. Choose $N$ such that for all $n, m > N$, $d_R(a_n, a_m) \leq \varepsilon$. We will show that $d_R(a_n, a) < \varepsilon$ for all $n > N$. To this end fix $r \in R$. Then there exists $m \geq N$ such that $|(r, a_m - a)| \leq \varepsilon$. Then for any $n > N$ we have

$$|(r, a - a_n)| \leq |(r, a - a_m)| + |(r, a_m - a_n)| \leq 2\varepsilon,$$

showing that $d_R(a_n, a) \leq 2\varepsilon$. This shows that $(a_n)$ converges to $a$ with respect to $d_R / G$, proving the theorem. \hfill $\square$

To prove Theorem 3.2, we need the following lemma from [22], which may be viewed as a weak regularity lemma in a Hilbert space setting. In particular, choosing appropriate $R$, we recover results from [11] and [10] respectively.

**Lemma 3.3 ([22]).** Let $H$ be any Hilbert space. Let $R \subseteq B(H)$ be closed under multiplication by elements of $[-1, 1]$. Then for any $k \in \mathbb{N}$,

$$B(H) \subseteq k \cdot R + \frac{1}{\sqrt{k}} B(H, \| \cdot \|_R).$$

For convenience of the reader we will give a proof.

**Proof.** Let us denote the inner product on $H$ by $\langle \cdot, \cdot \rangle$. Let $a \in B(H)$. Write $a_0 = a$. If $\|a_i\|_R > \frac{1}{\sqrt{k}}$ for some $i \geq 0$, then there exists $r \in R$ such that $|\langle a_i, r \rangle| > \frac{1}{\sqrt{k}}$. Set $a_{i+1} = a_i - \langle a_i, r \rangle r$. Note that by induction we have that $a_{i+1} - a_0 \in i \cdot R \subseteq k \cdot R$, as $|\langle a_i, r \rangle| \leq 1$ by Cauchy-Schwarz. Then

$$\|a_{i+1}\|^2 = \|a_i\|^2 - 2\langle a_i, r \rangle^2 + \langle a_i, r \rangle^2 \|r\|^2 = \|a_i\|^2 - \langle a_i, r \rangle^2 (2 - \|r\|^2) < \|a_i\|^2 - 1/k,$$

by definition of $r$ and the fact that $R \subseteq B(H)$. By induction we have that $\|a_{i+1}\|^2 < 1 - i/k$. This shows that for some $i \leq k$ we must have $\|a_i\|_R \leq \frac{1}{\sqrt{k}}$ and hence finishes the proof. \hfill $\square$

We can now prove Theorem 3.2.

**Proof of Theorem 3.2.** By Theorem 3.1 it suffices to show that $(W, d_R / G)$ is totally bounded. To this end choose $\varepsilon > 0$. Set $k := \lceil c^2 \varepsilon^{-2} \rceil$. Since $W$ is Hilbert-close, there exists a Hilbert space $H$ and a constant $c > 0$ such that $W \subseteq cB(H) + \varepsilon B(X)$. Set $k := \lceil c^2 \varepsilon^{-2} \rceil$. As $k \cdot R / G$ is totally bounded (in $X$), there exists a finite set $F \subseteq k \cdot R$ such that $k \cdot R \subseteq GF + \frac{\varepsilon}{c} B(X)$. Now note that, since $H^* = H$ and duality reverses inclusions, we have $B(X^*) \subseteq B(H) \subseteq B(X) \subseteq B(X, \| \cdot \|_R)$. By Lemma 3.3 we have

$$W \subseteq cB(H) + \varepsilon B(X) \subseteq c(k \cdot R + e^{-1/2} B(H, \| \cdot \|_R)) + \varepsilon B(X) \subseteq$$

$$c(GF + \frac{\varepsilon}{c} B(X)) + \varepsilon B(H, \| \cdot \|_R) + \varepsilon B(X, \| \cdot \|_R) \subseteq GcF + 3\varepsilon B(X, \| \cdot \|_R).$$

This shows that $(W, d_R / G)$ is totally bounded and finishes the proof. \hfill $\square$
3.3 Weak and strong regularity

The following is implicit in the proof of Theorem 3.2 and may be viewed as an extension of the weak regularity lemma from the Hilbert space setting to a Banach space setting.

**Lemma 3.4.** Let \((X, \| \cdot \|)\) be a Banach space, let \(W \subseteq B(X)\) and let \(R \subseteq B(X^*)\). Suppose that \(W\) is Hilbert-small for some Hilbert space \(H\), with function \(c : (0,1) \rightarrow \mathbb{N}\). Then for each \(\varepsilon > 0\) there exists \(k \leq \lceil c(\varepsilon)/\varepsilon \rceil^2\) such that for each \(w \in W\) there exists \(x_1, \ldots, x_l \in R\), with \(l \leq k\), such that \(\|w - c\sum_{i=1}^l x_i\|_R \leq 2\varepsilon\).

In [17], Lovász and Szegedy applied the compactness of the graphon space, cf. [1], to derive an approximation results for graphons, cf. [17, Lemma 5.2]. This result implies several types of regularity lemma’s varying from the weak regularity lemma [11], to the derive an approximation results for graphons, cf. [17, Lemma 5.2]. This result implies several types of regularity lemma’s varying from the weak regularity lemma [11], to the original lemma [25], to a ‘super strong’ variant [11]. See [19] for more details. We can derive something similar in our Banach space setting:

**Lemma 3.5.** Let \((X, \| \cdot \|)\) be a Banach space, let \(G \subseteq \text{Aut}(X)\), let \(R \subseteq X^*\) and let \(W \subseteq X\) be G-stable and suppose that \((W, d_R / G)\) is compact. Let for \(k \in \mathbb{N}\), \(Y_k \subseteq W\) be G-stable such that \(Y := \bigcup_{k \in \mathbb{N}} Y_k\) is dense in \(W\) (w.r.t. \(\| \cdot \|\)). Let \(h : (0,\infty) \times \mathbb{N} \rightarrow \mathbb{N}\) be any function. Then for any \(\varepsilon > 0\) there exists \(n \in \mathbb{N}\) such that for any \(w \in W\) there exists \(w' \in W\) and \(y \in Y_n\), with \(m \leq n\), such that

\[
\|w - w'\|_R \leq h(\varepsilon, m) \quad \text{and} \quad \|w' - y\| \leq \varepsilon.
\]

The proof goes along the same lines as the proof of Lemma 5.2 in [17].

**Proof.** We may assume that \(h\) is monotonically decreasing in its second variable. Let \(\varepsilon > 0\). Since \(Y\) is dense in \(W\), for each \(w \in W\), there exists \(n \in \mathbb{N}\) and \(y \in Y_n\) such that \(\|w - y\| \leq \varepsilon\). Let \(f(w)\) denote the smallest \(n\) such that there exists \(y \in Y_n\) with \(\|w - y\| \leq \varepsilon\). For \(w \in W\) let \(O(w)\) denote the open ball defined as

\[
O(w) := \{w' \in W \mid (d_R / G)(w, w') < h(\varepsilon, f(w))\}.
\]

By Theorem 5.1 it follows that there exists \(w_1, \ldots, w_l\) such that the union of the \(O(w_i)\) contains \(W\). This means that for any \(w \in W\) there exists \(i \in \{1, \ldots, l\}\) and \(y \in Y_{m_i}\), with \(m_i \leq h(\varepsilon, f(w_i))\), such that \(\|w_i - y\| \leq \varepsilon\) and \((d_R / G)(w, w_i) < h(\varepsilon, f(w_i))\). Choosing \(n := \max_{i=1,\ldots,l} f(w_i)\) and a suitable \(g \in G\) applied to both \(w_i\) and \(y\) we arrive at the desired conclusion.

Since the norm \(\| \cdot \|\) on \(X\) satisfies \(\|x\| \leq \|x\|_R\) for each \(x \in X\), taking \(h(\varepsilon, m) = \varepsilon\) for all \(m\), Lemma 3.5 implies several types of weak regularity lemmas by taking different choices of \(Y_k\). In particular, taking \(Y_k := k \cdot R\), it implies Lemma 3.4 without the explicit bound on \(n\) though. More importantly, it allows to use different types of approximations not based on sums of elements from \(R\). We will elaborate a bit more on this in the next two sections, where we will apply the results obtained here to the spaces \(l^p(\mathbb{N}^d)\) and \(L^p([0,1]^d)\).

4 Compact orbit spaces in \(L^p(([0,1]^d))\)

In this section we will apply Theorems 3.1 and 3.2 to the Banach space \(X = L^p([0,1]^d)\) for some fixed \(s \in [1,\infty)\) and \(l \in \mathbb{N}\). Let for any \(q \in [1,\infty] \),

\[
R_{l}^q := \{r_1 \otimes \ldots \otimes r_l \mid r_i \in B(L^q([0,1]))\} \subseteq B(L^q([0,1]^d))
\]
Let $S_{[0,1]}$ be the group of measure preserving bijections $\tau : [0,1] \to [0,1]$ (i.e., $\tau^{-1}$ is also measure preserving). We will derive the following result from Theorems 3.1 and 3.2 generalising the compactness result of Borgs, Chayes, Cohn and Zhao [5] mentioned earlier.

**Theorem 4.1.** Let $p \in (1,\infty]$ and let $q$ be such that $q > \frac{p}{p-1}$. Fix $l \in \mathbb{N}$. Then the space $(B(L^p([0,1]^2)), d_{R^l})/S_{[0,1]}$ is compact.

We will prove Theorem 4.1 in Section 4.1 below. Let us first make some remarks and state some consequences.

Borg, Chayes, Cohn and Zhao [5], who proved Theorem 4.1 for $l = 2$ and $q = \infty$, already noted that the same result cannot be true when $p = 1$. We note here that in case $p = 2$ we really need that $q > 2$ since $B(L^2([0,1]^2), d_{R^2})/S_{[0,1]}$ is not totally bounded. To see this take $f_i$ to be the function which is constant $i$ on the rectangle $[0,1/i] \times [0,1/i]$. Since for rank 2 functions $f$ we have $\|f\|_{R^2} \leq \|f\|_2 \leq 2\|f\|_{R^2}$, we may as well replace $d_{R^2}$ by $d_2$. Then for any $\pi \in S_{[0,1]}$ we have for $i = 2^k < j = 2^l$,

$$\int_{[0,1]^2} |f_i - \pi f_i|^2 \geq \int_{[0,1]^2} |f_i - f_j|^2 \geq \frac{(2^k - 2^l)^2}{2^l} \geq 1 - 2^{k-l} \geq 1/2,$$

implying that we need to take $q > 2$ when $p = 2$. This example of course generalises to any $l > 2$ and $p, q$ such that $q = p^s$.

Janson [14] showed, using the Riesz-Thorin theorem, that in case $p = \infty$ (and $l = 2$), Theorem 4.1 remains true if one replaces the $\| \cdot \|_{R^l}$ norm by the $p \to q_0$ operator norm for any $p_0, q_0 > 1$. See [14] Lemma E.6. We can also use the Riesz-Thorin theorem in our setting for $l = 2$. (We need to work over the complex numbers though. This does not cause any problems for real-valued functions, as the complex and real operator norms are within a constant factor of each other.) Let us take $p > 1$ and $q > \frac{p}{p-1} = p^r$. Taking $f \in B(L^p([0,1]^2))$, we then have $\|f\|_{p \to p} \leq 1$. Define for $\theta \in (0,1)$ $p_\theta$ and $q_\theta$ by

$$\frac{1}{p_\theta} := \frac{1 - \theta}{p} + \frac{\theta}{q} \quad \text{and} \quad \frac{1}{q_\theta} := \frac{1 - \theta}{p} + \frac{\theta}{q^s}. \quad (3)$$

Then by the Riesz-Thorin theorem (cf. [4] Theorem 1.1.1) $\|f\|_{p_\theta \to q_\theta} \leq \|f\|_{p \to q}^\theta = \|f\|_{R^l}^\theta$.

By Theorem 4.1 this implies the following:

**Corollary 4.2.** Let $p, q \in (1,\infty]$ such that $q > \frac{p}{p-1}$. Let $\theta \in (0,1)$ and let $p_\theta$ and $q_\theta$ be defined as in (3). Then $(B(L^p([0,1]^2))), \| \cdot \|_{p_\theta \to q_\theta})/S_{[0,1]}$ is compact.

By Lemma 3.5 the following is a direct corollary to Theorem 4.1 and the previous corollary.

**Corollary 4.3.** Let $p, q \in (1,\infty]$ such that $q > \frac{p}{p-1}$. Fix $l \in \mathbb{N}$. Let for $k \in \mathbb{N}, F_k \subset B(L^p([0,1]^l))$ be $S_{[0,1]}$-stable such that $F := \cup_{k \in \mathbb{N}} F_k$ is a dense subset of $B(L^p([0,1]^l))$. Let $h : (0,\infty) \times \mathbb{N} \to \mathbb{N}$ be any function. Then for any $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that for any $g \in B(L^p([0,1]^l))$ there exists $g' \in B(L^p([0,1]^l))$ and $f \in F_n$, with $m \leq n$, such that

$$\|g - g'\|_p \leq h(\varepsilon, m) \quad \text{and} \quad \|g' - f\|_{R^l} \leq \varepsilon. \quad (4)$$

In case $l = 2$ and $p_\theta$ and $q_\theta$ are defined by (3), the $R^l_\theta$ norm may be replaced by $\| \cdot \|_{p_\theta \to q_\theta}$ in (4).
There are various choices for the sets $F_k$ in the corollary above. For example one can take $F_k$ to be those functions that are sums of at most $k$ rectangles (i.e., a rectangle is a function of the form $c \chi_{A_1 \times \cdots \times A_t}$ with $A_i$ measurable and $c$ a constant.) A special case is to take $F_k$ to be those sums of rectangles that are coming from a partition of $[0,1]$ into at most $k$ sets. Another choice would be to take $F_k$ to be those functions that take only $k$ different values (i.e. step functions with $k$ steps).

### 4.1 Proof of Theorem 4.1

To prove Theorem 4.1 we need some preliminary results.

The next lemma implies that $L^p([0,1])$ is Hilbert-close for $H = L^2([0,1])$ for any $p > 1$.

**Lemma 4.4.** Let $(X, \mu)$ be any probability space. Let $p > p' \geq 1$ and $\epsilon > 0$. Then there exists a constant $C$ such that $B(L^p(X)) \subseteq CB(L^{p'}(X)) + \epsilon B(L^{p'}(X))$.

**Proof.** Define for convenience $B^s := B(L^s(X))$, for any $s \in [1,\infty)$. Fix any $K \geq 1$ and let $f \in B^p$. Let $A := \{ x \in [0,1] \mid |f(x)| > K \}$. Then $f = f_1 + f_2$ with $f_1 = f_{X[0,1]\setminus A}$ and $f_2 = f_{X_A}$. Clearly, $f_1 \in KB^\infty$. Next we consider $\|f_2\|_{p'}$:

$$\|f_2\|_{p'}^p = \int |f_2|^p d\lambda \leq \int |f|^p |K|^{-p} d\lambda \leq K^{-p} \int |f|^p d\lambda \leq K^{-p} \int |\eta|^p d\lambda.$$

Hence choosing $C$ in such a way that $C^1 = p'/p' = \epsilon$ we obtain the desired result. \qed

**Lemma 4.5.** Let $p > s \geq 1$ and $\epsilon > 0$. Then for any $k \in \mathbb{N}$ there exists a constant $C$ such that $R_k^p \subseteq CR_k^s + \epsilon R_k^s$.

**Proof.** The proof is by induction on $k$. The case $k = 1$ follows by Lemma 4.4. Now suppose $k > 1$. Choose any $p' \in (s,p)$ and let $\delta > 0$. By induction, there exists a constant $C_1$ such that $R_{k-1}^p \subseteq C_1 R_{k-1}^s + \delta R_{k-1}^s$ and $R_k^p \subseteq C_1 R_k^s + \delta R_k^s$. Let us define for $s,t \in [1,\infty]$,

$$R_{k-1}^t := \{ f_1 \otimes f_2 \mid f_1 \in R_{k-1}^s, f_2 \in R_1^t \}.$$

Then

$$R_k^p \subseteq C_1^2 (R_k^s + \delta R_k^s + C_1 \delta (R_{k-1,1}^s + R_{k-1,1}^s)) + \delta^2 R_k^p.$$  \hspace{1cm} (5)

Next choose $\eta > 0$. Then by induction there is a constant $C_2$ such that $R_1^p \subseteq C_2 R_1^s + \eta R_1^s$ and $R_{k-1}^p \subseteq C_2 R_{k-1}^s + \eta R_{k-1}^s$. Plugging this into (5) we obtain that

$$R_k^p \subseteq CR_k^s + C_1 \eta \delta (R_{k-1,1}^s + R_{k-1,1}^s) + \delta^2 R_k^p,$$

where $C := C_1^2 + 2\delta C_1 C_2$. Finally, using that $R_l^s \subseteq R_l^s$ for any $t \geq s$ and $l \in \mathbb{N}$, and first choosing $\delta$ such that $\delta^2 \leq 1/3\epsilon$ and then $\eta$ such that $C_1 \delta \eta \leq 1/3\epsilon$, we obtain the desired inclusion. \qed

**Proof of Theorem 4.1.** Let us denote $R_k^s$ by $R$, $R_1^s$ by $R'$ and $S_{[0,1]}$ by $G$. Define $X = L^{q'}([0,1])$, where $q' = \frac{1}{q}$. The proof proceeds in a number of steps.

First of all note that $B(L^{\infty}([0,1]),d_R/G)$ is compact by the argument given below Theorem 3.2 (This is essentially the compactness of the graphon space.) We will next show:

$$B(L^\infty([0,1]),d_R/G)$$ is compact. \hspace{1cm} (6)
To do so let \((f_n)\) be a sequence in \(B(L^\infty([0,1]^I))\). By compactness of \(B(L^\infty([0,1]^I), d_R/G)\) it has a convergent subsequence (which we may assume to be \((f_n)\) itself) that converges to some \(f \in B(L^\infty([0,1]^I))\) with respect to \(d_R/G\). So it suffices to show that \((f_n)\) converges to \(f\) with respect to \(d_R/G\). Let \(\epsilon > 0\). Let \(C\) be the constant from Lemma 4.5 such that \(R_1^j \subseteq R_1^j \subset CR^n_\infty\). Fix \(N \in \mathbb{N}\) such that \((d_R/G)(f_n-f) < \epsilon/C\) for all \(n \geq N\). In other words, for each \(n \geq N\) there exists \(g_n \in S_{[0,1]}\) such that \(\|g_nf_n - f\|_R < \epsilon/C\). Then, writing \(f_n' = g_nf_n\),

\[
\|f_n' - f\|_{R_1} = \sup_{r \in R_1^j} r(f_n' - f) \leq \sup_{r_1 \in CR^n_\infty} r_1(f_n' - f) + \sup_{r_2 \in eR_1^j} r_2(f_n' - f)
\leq C\|f_n' - f\|_{R_1^n} + \epsilon\|f_n' - f\|_R \leq \epsilon + 2\epsilon = 3\epsilon,
\]

showing that \(f_n\) converges to \(f\) with respect to \(d_R/G\). This proves 5. We will now show:

\[
(B(L^p([0,1]^I)), d_R/G) \text{ is totally bounded in } X. \tag{7}
\]

To do so, let \(K\) be the constant from Lemma 4.4 such that \(B(L^p([0,1]^I)) \subseteq KB(L^\infty([0,1]^I)) + \epsilon B(L^\infty([0,1]^I))\) and note that \(B(L^q([0,1]^I)) = B(X) \subseteq B(X, \|\cdot\|_R)\). Write \(\epsilon' = \epsilon/K\). By compactness of \((B(L^\infty([0,1]^I)), d_R/G)\), there exists a finite set \(F \subseteq B(L^\infty([0,1]^I))\) such that \(B(L^\infty([0,1]^I)) \subseteq GF + \epsilon'B(L^\infty([0,1]^I), \|\cdot\|_R)\). Then

\[
B(L^p([0,1]^I)) \subseteq KGF + Ke'B(L^\infty([0,1]^I), \|\cdot\|_R) + \epsilon B(X, \|\cdot\|_R) \\
\subseteq KGF + 2\epsilon B(X, \|\cdot\|_R),
\]

proving (7).

As \(B(L^p([0,1]^I))\) is weak-* compact by the Banach-Alaoglu theorem, this finishes the proof by Theorem 3.1.

\[\square\]

5 Compact orbit spaces in \(\ell^p(\mathbb{N}^l)\)

In this section we will apply Theorem 5.1 to the Banach space \(X = \ell^p(\mathbb{N}^l)\) for \(p \in (1, \infty]\). Let \(S_N\) be the group of invertible maps \(\tau : \mathbb{N} \to \mathbb{N}\). We will derive the following result:

**Theorem 5.1.** Let \(q \in [1, \infty)\) and let \(p > q\). Fix \(l \in \mathbb{N}\). Then the space \(B(\ell^q(\mathbb{N}^l)), d_p)/S_N\) is compact.

We will prove this result in Section 5.1 below. Let us first note that we really need \(q < p\). Lemma 5.3 below is not true for \(p = q\) and is a direct corollary of Theorem 5.1. It seems that one cannot only take \(q = p\) when both of them are equal to 2, but then one has to use a different (weaker) metric and the orthogonal group instead of \(S_N\), cf. [22].

By Lemma 5.5 the following is a direct corollary to Theorem 5.1.

**Corollary 5.2.** Let \(q \in [1, \infty)\) and let \(p > q\). Fix \(l \in \mathbb{N}\). Let for \(k \in \mathbb{N}\), \(X_k \subset B(\ell^q(\mathbb{N}^l))\) be \(S_N\)-stable such that \(X := \cup_{k \in \mathbb{N}} X_k\) is a dense subset of \(B(\ell^q(\mathbb{N}^l))\). Let \(h : (0, \infty) \times \mathbb{N} \to \mathbb{N}\) be any function. Then for any \(\epsilon > 0\) there exists \(n \in \mathbb{N}\) such that for any \(y \in B(\ell^q(\mathbb{N}^l))\) there exists \(y' \in B(\ell^q(\mathbb{N}^l))\) and \(x \in X_m\), with \(m \leq n\), such that

\[
\|y - y'\|_p \leq h(\epsilon, m) \quad \text{and} \quad \|y' - x\|_q \leq \epsilon.
\]
We can take \( X_k := k \cdot B(\ell^q(N)) \) in the corollary above. Then the weak version of the corollary says that for each \( \varepsilon > 0 \) there is \( k \) such that each \( y \in B(\ell^q(N^i)) \) can be approximated by a rank \( k \) tensor in the \( p \)-norm. Unfortunately, Corollary 5.2 does not give any explicit bounds on \( k \). It is not to be expected that the bounds on \( k \) will be much better than the bound given by Lemma 5.3 below (at least for \( p = \infty \)). Indeed, for \( p = \infty \), Alon, Lee, Schraibman and Vempala [2] showed that for any \( n \times n \) Hadamard matrix \( M \) and \( \varepsilon \in (0, 1) \) one has that for any matrix \( M' \), such that \( \| M - M' \|_\infty \leq \varepsilon \), the rank of \( M' \) is at least \( (1 - \varepsilon^2)n \).

Since all entries of a Hadamard matrix are 1 or \(-1\), we need to take \( q = \Omega(\log n) \), to make sure that \( \| M \|_q \) is bounded. The bound given by Lemma 5.3 on \( k \) is then of order \( \varepsilon^{-\log n} \), which is polynomial in \( n \).

### 5.1 Proof of Theorem 5.1

For any \( p \in [0, 1, \infty] \), a subset \( K \subset S \) and \( x \in \ell^p(S) \) we define \( x_K \in \ell^p(S) \) by

\[
x_K := \begin{cases} x(i) & \text{if } i \in K, \\ 0 & \text{otherwise.} \end{cases}
\]

We need the following lemma for the proof of Theorem 5.1.

**Lemma 5.3.** Let \( S \) be any set. Let \( q \in [1, \infty) \) and let \( p > q \). For any \( \varepsilon > 0 \) there exists \( k = k(q, p, \varepsilon) \in \mathbb{N} \) such that for any \( x \in B(\ell^q(S)) \) there exists a set \( K \subset S \) of size at most \( k \) such that \( \| x_K - x \|_p \leq \varepsilon \). In case \( p = \infty \) we can take \( k = e^{-\eta} \).

**Proof.** Let \( C \subseteq S \) be a countable set of points, which we identify with \( \mathbb{N} \), such that \( x(i) = 0 \) for \( i \notin C \). We may assume that \( x \) satisfies \( |x(1)| \geq |x(2)| \geq \ldots \). Since \( \| x \|_q \leq 1 \), it follows that \( |x(i)|^q \leq 1/i \) for all \( i \). Let \( K = \{1, \ldots, k\} \) for some \( k \in \mathbb{N} \) to be fixed later. Then for \( p < \infty \),

\[
\| x_K - x \|_p^p = \sum_{i \in K} |x(i)|^p \leq \sum_{i \geq k} (1/i)^p/q.
\]

It is a well known fact that the series \( \sum_{i=1}^{\infty} (1/i)^s \) converges for every \( s > 1 \). This implies that for \( k \) large enough, (8) will be bounded by \( \varepsilon^p \), which finishes the first part of the proof.

In case \( p = \infty \), we have

\[
\| x_K - x \|_\infty = \sup_{i\geq k} |x(i)| \leq (1/i)^{1/q}.
\]

So taking \( k \) such that \( (1/k)^{1/q} \leq \varepsilon \), we are done. \( \Box \)

Note that Lemma 5.3 is not true if one replaces \( q \) by \( p \). One can simply take \( x(i) = n^{-p} \) for \( i = 1, \ldots, n^p \) and \( x(i) = 0 \) for \( i > n^p \). Then for any constant \( k \) not depending on \( n \) and any set \( K \subset \mathbb{N} \) of size \( k \) we have \( \| x_K - x \|_p^p \leq (n^p - k)n^{-p} = 1 - kn^{-p} = 1 - o(1) \).

Despite the fact that Lemma 5.3 is really simple, it can be utilised for algorithmic purposes, since we can go over all bounded size subsets of \( \{1, \ldots, n\} \) in time polynomial in \( n \). This is used in [21] to give approximation algorithms for computing matrix \( p \mapsto q \) norms.

Now we can give a proof of Theorem 5.1.

**Proof of Theorem 5.1.** Write \( W = B(\ell^q(N^i)) \). By the Banach-Alaoglu Theorem \( W \) is weak-* compact. So by Theorem 5.1 it suffices to show that \( (W, d_p)/S_N \) is totally bounded. To this end choose \( \varepsilon > 0 \). Let \( k = k(q, p, \varepsilon) \) be the constant supplied by Lemma 5.3. Denote
by \([n]\) the set of positive integers \(\{1, \ldots, n\}\) for any \(n \in \mathbb{N}\). Let \(F\) be a finite set of points in \(B(\ell^p([k]^l]))\) such that for each \(y \in B(\ell^p([k]^l]))\) there exists \(x \in F\) with \(\|x - y\|_p \leq \varepsilon/2\). By Lemma 5.3 for each \(w \in W\) there exists a set \(K \subset \mathbb{N}^l\) of size at most \(k\) such that \(\|w_K - w\|_p \leq \varepsilon/2\). Then there exists \(\sigma \in S_N\) such that \(\sigma(K) \subset [k]^l\), as the \(l\)-tuples in \(K\) can contain at most \(k^l\) distinct numbers. This implies that \(d_p(S_N F, W) \leq \varepsilon\) and finishes the proof.

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References

[1] N. Alon, E. Fischer, M. Krivelevich and M. Szegedy: Efficient testing of large graphs, Combinatorica 20 (2000), 451–476.

[2] N. Alon, T. Lee, A. Shraibman and S. Vempala: The approximate rank of a matrix and its algorithmic applications, in: Proceedings of the forty-fifth annual ACM symposium on Theory of computing, pp. 675–684. ACM, 2013.

[3] S. Barman: An approximate version of Carathéodory’s theorem with applications to approximating Nash equilibria and dense bipartite subgraphs, arXiv preprint arXiv:1406.2296 (2014).

[4] J. Bergh and J. Löfström: Interpolation Spaces An Introduction, Grundlheren der mathematischen Wissenschaften 223, Springer-Verlag Berlin Heidelberg New York, 1976.

[5] C. Borgs, J.T. Chayes, H. Cohn and Y. Zhao: An \(L^p\) theory of sparse graph convergence I: limits, sparse random graph models, and power law distributions, arXiv preprint arXiv:1401.2906 (2014).

[6] C. Borgs, J.T. Chayes, L. Lovász, V.T. Sós, B. Szegedy and K. Vesztergombi: Graph limits and parameter testing, in: Proceedings of the thirty-eighth annual ACM symposium on Theory of computing, pp. 261–270. ACM, 2006.

[7] C. Borgs, J.T. Chayes, L. Lovász, V.T. Sós and K. Vesztergombi: Convergent sequences of dense graphs I: Subgraph frequencies, metric properties and testing, Advances in Mathematics 219 (2008), 1801–1851.

[8] J.B. Conway: A Course in Functional Analysis–Second Edition, Graduate Texts in Mathematics, 96, Springer Science & Business Media, New York, 1990.

[9] J.N. Cooper: A permutation regularity lemma, The Electronic Journal of Combinatorics 13 (2006), R22.

[10] W. Fernandez de la Vega, M. Karpinski, R. Kannan and S. Vempala: Tensor decomposition and approximation schemes for constraint satisfaction problems, in Proceedings of the thirty-seventh annual ACM symposium on Theory of computing, pp. 747-754, ACM, 2005.
[11] A. Frieze and R. Kannan: Quick approximation to matrices and applications, *Combinatorica* 19 (1999), 175-220.

[12] B. Green: A Szemerédi-type regularity lemma in abelian groups, with applications, *Geometric and Functional Analysis* 15 (2005), 340–376.

[13] P.R. Halmos: *Measure theory*, Graduate Texts in Mathematics, 18, Springer, New York, 2014.

[14] S. Janson: Graphons, cut norm and distance, couplings and rearrangements, *NYJM Monographs* Vol 4, 2013.

[15] Y. Kohayakawa: Szemerédi’s regularity lemma for sparse graphs, in: *Foundations of computational mathematics*, pp. 216–230, Springer Berlin Heidelberg, 1997.

[16] J. Komlós and M. Simonovits: Szemerédi’s Regularity Lemma and its applications in graph theory, *Combinatorics, Paul Erdos is Eighty*, (D. Miklos et. al, eds.), Bolyai Society Mathematical Studies, 2, 295–352, 1996.

[17] L. Lovász and B. Szegedy: Szemerédi’s Lemma for the analyst, *Geometric and Functional Analysis* 17 (2007), 252–270.

[18] L. Lovász and B, Szegedy: Limits of dense graph sequences, *Journal of Combinatorial Theory, Series B* 96 (2006), 933–957.

[19] L. Lovász: *Large Networks and Graph Limits*, American Mathematical Society, Providence, Rhode Island, 2012.

[20] T. Nishiura: Measure-Preserving Maps of $\mathbb{R}^n$, *Real Analysis Exchange* 24 (1999), 837–842.

[21] G. Regts: Approximation algorithms for matrix norms from regularity lemma’s in Banach spaces, in preparation.

[22] G. Regts and A. Schrijver: Compact orbit spaces in Hilbert spaces and limits of edge-colouring models, to appear in *European Journal of Combinatorics*, arXiv preprint, arXiv:1210.2204 (2012).

[23] V. Rödl and Jozef Skokan: Regularity Lemma for k-uniform hypergraphs, *Random Structures and Algorithms* 25 (2004), 1-42.

[24] A. Scott: Szemerédi’s regularity lemma for matrices and sparse graphs, *Combinatorics, Probability and Computing* 20 (2011), 455–466.

[25] E. Szemerédi: Regular partitions of graphs, in: Problèmes combinatoires et théorie des graphes, (Proceedings Colloque International C.N.R.S., Paris-Orsay, 1976) [Colloques Internationaux du C.N.R.S. No 260], Editions du C.N.R.S., Paris, 1978, pp. 399–401.

[26] T.C. Tao: Szemerédi’s regularity lemma revisited, *Contributions Discrete Math.* 1 (2006), 8–28.