Infinitely divisible nonnegative matrices, $M$-matrices, and the embedding problem for finite state stationary Markov Chains

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Abstract

This paper explicitly details the relation between $M$-matrices, nonnegative roots of nonnegative matrices, and the embedding problem for finite-state stationary Markov chains. The set of nonsingular nonnegative matrices with arbitrary nonnegative roots is shown to be the closure of the set of matrices with matrix roots in $IM$. The methods presented here employ nothing beyond basic matrix analysis, however it answers a question regarding $M$-matrices posed over 30 years ago and as an application, a new characterization of the set of all embeddable stochastic matrices is obtained as a corollary.

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Introduction

A $Z$-matrix is a matrix that has nonpositive off-diagonal elements. An $M$-matrix is defined as a $Z$-matrix that has a non-negative inverse, or alternatively, as a $Z$-matrix of the form $\alpha I - K$. Here, $K$ is a nonnegative matrix and $\alpha > \rho(K)$, where $\rho$ is the spectral radius of $K$. In fact there are many characterizations of $M$-matrices. A summary of these characterizations can be found in [1] and [2]. The ‘Inverse $M$-matrix problem’ concerns the conditions under which a nonnegative matrix is the inverse of an $M$-matrix. The set of such nonnegative matrices is denoted by $\mathcal{IM}$. Two key surveys of this problem are given by C.Johnson in [3], and more recently C.Johnson and R.Smith in [4]. More than 30 years ago, the question was raised by C.Johnson in [3] asking for which nonnegative matrices $B$, does there exist a sequence of nonnegative matrices $\{K_n\}_{n=1}^{\infty}$ such that

$$\left(K_n\right)^n = B.$$  

Informally put; which nonnegative matrices possess nonnegative matrix roots of arbitrary order. Indeed the question in [3] asked specifically if a nonsingular, non-negative matrix that has arbitrary, nonnegative roots, also has roots which are in $\mathcal{IM}$. We shall see this statement is correct, modulo some further conditions.

The question in [3] is connected with the embedding problem for Markov chains. The latter has been a long standing problem in linear algebra and probability theory since it was first considered by Elfving [5]. The precise formulation of this problem will be given later; however, a connection was made by Kingman in [6], who showed that a Markov chain is embeddable if and only if its stochastic matrix is nonsingular and has arbitrary stochastic matrix roots.

Extensive work was also done on a analogous problem viz the characterization of the class of nonnegative definite matrices having the property that every positive fractional Hadamard power is also nonnegative definite. This was pioneered in [4]. In the context of nonnegative definiteness, such matrices are called infinitely divisible. Following this terminology we make the following definition.

**Definition 1.** A nonnegative matrix $B$ is said to be infinitely divisible if and only if there exists a sequence of nonnegative matrices $\{K_n\}_{n=1}^{\infty}$ such that

$$\left(K_n\right)^n = B.$$  

If in addition $\det(B) > 0$ we say that $B$ is strongly infinitely divisible.

In this paper we develop a theory for these classes of matrices and answer the question in [3]. Although the results will be of interest in other fields, the embedding problem for finite state stationary Markov chains is the primary application intended in this paper.

The paper is organized as follows: the first section is a statement of the main results; the second and third section detail the proofs and framework; and the forth is dedicated to the embedding problem. The reader who is only interested in the results for the embedding
problem may thus proceed directly to section 4; the reader interested primarily in linear algebra may omit section 4 altogether.

1. Main results

The first result is a characterization of the set of strongly infinitely divisible matrices in terms of the exponential map.

**Theorem 1.** An $N \times N$ nonnegative matrix $B$ is strongly infinitely divisible if and only if there exists a $Z$ matrix, $Q$ such that $e^{-Q} = B$.

This result will be used to answer the question in [3].

**Theorem 2.** The set of infinitely divisible nonnegative matrices contains the closure of the set

$$\{ B : B = K^n, \ n \in \mathbb{N}, \ K \in \mathcal{I}M \}.$$  \hspace{1cm} (3)

Furthermore, if $B$ is nonsingular, it is infinitely divisible if and only if it belongs in the closure of this set.

We also prove the following result that relates the strongly infinite divisibility of the matrix to that of its submatrices.

**Theorem 3.** Let $B$ be a strongly infinitely divisible nonnegative matrix. Then following dichotomy holds:

1. If $B$ is irreducible, it is strictly positive

2. If $B$ is reducible, then there exists a permutation matrix $L$ such that $B = LUL^T$ for some infinitely divisible upper block triangular matrix $U$. Furthermore:

   (a) All the matrices on the diagonal blocks of $U$ are strictly positive.

   (b) If $M$ is the number of blocks in $U$, then for every $n \leq M$ the square submatrix, $U^{(n)}$, obtained by deleting the first $n$ blocks from the top rows and left columns, is strongly infinitely divisible.

We prove Theorems 1 and 2 in section 2 and Theorem 3 in the subsequent one, following a number of other algebraic properties. The main results are given above, but there are several other results contained in the following sections that are also of their own interest.
2. Connection to $Z$-matrices and Inverse $M$-matrices

The initial part of this analysis is along very similar lines to the work by Kingman in [6]; however it is not exclusive to stochastic matrices. We begin with a proof of Theorem 1.

Proof of Theorem 1 A proof of the direct implication can be found in [8]. It is presented here for the readers convenience. Suppose that $B = e^{-Q}$ for some $Z$-matrix $Q$. Then an $n$th root of $B$ is $e^{-Q/n}$, which is again the exponential of the negative of a $Z$-matrix. Non negativity follows by taking a sufficiently large $\theta$ so that $-Q/n + \theta I$ is non-negative and then writing

$$e^{-Q/n} = e^{-\theta} e^{-Q/n + \theta I},$$

where $I$ denotes the identity matrix.

Conversely, assume that $B$ is a $N \times N$ strongly infinitely divisible matrix. We first show that the sequence $B^1/n$ contains a subsequence converging to the $N \times N$ identity matrix. Let $M$ be an integer that is divisible for every integer $k$ less than or equal to $N$. Define the sequence $R_n = B^{1/Mn} \geq 0$. The relation $R_n^{Mn} = B$ implies that $R_n$ is bounded and so will have a convergent subsequence, say $R_{n_j}$ with a limit $R$. By the Perron Frobenius theorem, $B$ has a strictly positive eigenvalue $\tilde{\lambda}$ which is the spectral radius of $B$. Elementary considerations tell us that for $R_n$ to be a real, let alone a nonnegative root of $B$, we must take the real root of $\tilde{\lambda}$. By the same reasoning $\tilde{\lambda}^{1/Mn}$ is the spectral radius of $R_n$. Hence the spectral radius of $R$ is 1. A similar argument shows that the determinant of $R$ is on the unit circle, indeed;

$$|\det(R_n)| = |\det(B)|^{1/Mn} \to 1, \quad \text{as } n \to \infty. \quad (5)$$

Therefore every eigenvalue of $R$ must be on the unit circle (at this point nonsingularity is essential). By assumption, $R$ is non-negative so by the Perron Frobenius theorem if $R$ is irreducible then every eigenvalue is a root of unity for some $k \leq N$. If $R$ is not irreducible then we may decompose $R$ into the form $PWP^T$ where $P$ is a permutation matrix and $W$ is a block upper triangular matrix in which each diagonal block is an irreducible nonnegative matrix. As the spectrum of $R$ is the union of the spectra of the diagonal blocks in $W$, every eigenvalue is a $k$th root for some integer less then $n \leq N$. Therefore in all cases, $R^{M} = I$. For a review on the Perron Frobenius theorem and irreducible matrices, we direct the reader to [6]. We thus conclude that there is a subsequence $\{n_j\}$ such that

$$\lim_{j \to \infty} B^{1/Mn_j} = I. \quad (6)$$

We can now estimate the decay rate of the diagonal elements of $B^{1/Mn}$ using the inequality

$$|\text{Tr}(B^{1/Mn})| = \left| \sum_{i=1}^{N} \lambda_i^{1/Mn} \right| \leq N \tilde{\lambda}^{1/Mn} = N + O(n^{-1}). \quad (7)$$

Set $k = Mn_j$, and $B^{1/k} = I + A_k$ where $A_k$ is some sequence of matrices with nonnegative off diagonal elements converging to 0 (here and henceforth convergence is in any matrix norm).
From inequality (7) we know that the diagonal elements on the matrix $A_k$ decay as $O(k^{-1})$ as $k \to \infty$. We also have that

$$ (I + A_k)^k = \sum_{m=1}^{k} \binom{k}{m} [A_k]^m = B. \quad (8) $$

Equation (8) along with the fact that all off diagonal elements of $A_k$ are nonnegative and $\binom{k}{m}$ is $O(k^m)$ implies that all elements of $A_k$ decay as $O(k^{-1})$. Indeed, if an off diagonal element were to decay slower, then there would be no negative term in the diagonal to match this slower decay rate and maintain the relationship in (8). We thus have that the sequence $C_k := k(B^{\frac{1}{k}} - I) = kA_k$ (9) is bounded. Hence we find another subsequence of $C_k$, denoted $C_{q_k}$ that is convergent to some limit $C$. We show that $e^C = B$. Let

$$ q_k(B^{\frac{1}{q_k}} - I) = C + \epsilon_{q_k} \quad (10) $$

For some matrices $\epsilon_{q_k}$, that converge to 0. Rearranging, we have for each $q_k$

$$ B = \left( \frac{C}{q_k} + \frac{\epsilon_{q_k}}{q_k} + I \right)^{q_k} \quad (11) $$

and the relation follows from standard estimates and the binomial theorem.

Furthermore, because each $A_{q_k}$ has nonnegative off diagonal elements, $A$ has nonnegative off diagonal elements. Taking $Q = -A$ we complete the proof. □

An interesting implication of Theorem 1 is that the nonnegative roots of infinitely divisible matrices cannot be scattered: they must belong to the same branch of roots. The primary difficulty in dealing with singular infinitely divisible matrices is that nothing similar to Theorem 1 seems to apply. For example, the zero matrix.

In general there is no uniqueness of the $Z$ matrix. In fact, there may be an uncountable family of $Z$ matrices associated an infinitely divisible matrix. In the context of the embedding problem, such an example is provided in [10].

A result central to this paper is Theorem 11 in [3] which states that the primary $n$th root of an $M$-matrix is also an $M$-matrix. This yields in the following (also noted in [3]):

**Theorem 4.** Let $B$ be a nonnegative matrix such that $B = K^n$ for some $K \in IM$ then $B$ is is strongly infinitely divisible.

To prove theorem 2 we also need the following lemma:

**Lemma 1.** The set of infinitely divisible matrices is closed.
Proof. Let \( \{B_n\} \) be a sequence of infinitely divisible matrices converging to \( B \). Then, for any given \( m \in \mathbb{N} \), we can consider the sequence \( B_n^{1/m} \geq 0 \). This sequence contains a convergent subsequence with some limit \( \tilde{B} \). By continuity \( \tilde{B} \) is nonnegative and satisfies \( \tilde{B}^m = B \). \( \square \)

Proof of Theorem 2. Assume there is a sequence \( \{B_n\}_{n=1}^\infty \) converging to \( B \) such that each \( B_n \) is the power of some inverse \( M \)-matrix. This implies each \( B_n \) is infinitely divisible and so by Lemma 1, we have that \( B \) is also infinitely divisible. This completes the first statement of Theorem 2.

Conversely suppose that \( B \) is strongly infinitely divisible and that the off diagonal entries of the associated \( Z \)-matrix, \( Q \), are strictly negative. Then,

\[
B^{-1} = e^Q.
\] (12)

Taking \( n \)th roots for \( n \) sufficiently large we see

\[
B^{-\frac{1}{n}} = I + \frac{Q}{n} + O(n^{-2}).
\] (13)

is an \( Z \)-matrix and specifically because it’s inverse is positive, it is also an \( M \)-matrix. It is clear that such matrices; are dense in the set of infinitely divisible matrices, hence, we may apply Lemma 1 to prove the result. \( \square \)

The above result also indicates that the infinitely divisible matrices, whose associated \( Z \)-matrix have no off-diagonal zeros, always have roots in \( \mathcal{I} \mathcal{M} \). Once one violates this condition, it is easy to construct strongly infinitely divisible matrices that do not have roots in \( \mathcal{I} \mathcal{M} \).

How might powers of inverse \( M \)-matrices be characterized? If \( B \) is indeed the power of an inverse \( M \)-matrix. Then using the series expansion of \( (1 + x)^{-n} \) we see that \( B \) must be of the form

\[
\sum_{k=0}^{\infty} \binom{n+k-1}{k} P^k,
\] (14)

for some nonnegative matrix \( P \) where, in order for the series to converge, \( \rho(P) < 1 \). Conversely if \( B \) it is of the above form for some \( n > 0 \), then it must be power of an inverse \( M \)-matrix.

We will also require the following lemmata.

Lemma 2. The set of matrices with distinct eigenvalues is dense in the set of strongly infinitely divisible nonnegative matrices.

Proof. Consider the set of all \( Z \)-matrices. This set is a convex and satisfies the condition of Corollary 2 in [11], which implies that the set of matrices with distinct eigenvalues is dense in this set. The density of the eigenvalues in the set of infinitely divisible matrices is now simply a consequence of the continuity of the exponential map on matrices. \( \square \)
We also recall the following fact. See [2].

**Lemma 3.** Every $Z$-matrix has a nonnegative eigenvector.

We conclude this section by providing a bound on the eigenvalues of $Z$-matrices generating a infinitely divisible matrix through the exponential map. Thus if one is checking for the existence of said $Z$-matrices for a nonnegative matrix with distinct eigenvalues, one need only check a finite number of them.

**Theorem 5.** Let $B = e^{-Q}$ be a strongly infinitely divisible nonnegative matrix. Let $v$ be a nonnegative eigenvector of $Q$ with associated eigenvalue $\lambda$. Then every eigenvalue $\lambda$ of $Q$ satisfies

$$|\text{Im}(\lambda)| \leq |\tilde{\lambda}(n - 1) - \log(\det(B))|. \quad (15)$$

**Proof.** Assume that $B = e^{-Q}$ for some $Z$ matrix $Q$, with entries $Q_{ij}$. Then

$$(\tilde{\lambda} + Q_{ii})v_i = -\sum_{j \neq i} Q_{ij}v_j \geq 0. \quad (16)$$

In particular we have that $-Q_{ii} \leq \tilde{\lambda}$, which provides an upper bound on the diagonal elements. Combining this with

$$\prod e^{-Q_{ij}} = \det(B), \quad (17)$$

we can deduce that $-Q_{ii} \geq -\tilde{\lambda}n + \log(\det(B))$, and thus we conclude that the matrix

$$-Q + (\tilde{\lambda}(n - 1) - \log(\det(B)))I \quad (18)$$

is nonnegative with nonnegative eigenvalue $\tilde{\lambda}(n - 1) - \log(\det(B))$. By the Perron-Frobenius theorem, this is its spectral radius; therefore,

$$|\text{Im}(\lambda)| \leq |\tilde{\lambda}(n - 1) - \log(\det(B))|. \quad (19)$$

The above results and proofs are similar in spirit to those presented in [12] and we shall see that when dealing with a stochastic matrix, the bound simplifies considerably.
3. Algebraic properties.

Given the known invariant zero patterns for \( M \)-matrices, it is natural to ask similar questions regarding infinitely divisible matrices. The following result is one such immediate observation:

**Lemma 4.** If, for a strongly infinitely divisible matrix \( B \), \( B_{ij} = 0 \) for \( i \neq j \), then on any \( Z \) matrix \( Q \) such that \( B = e^{-Q} \), we have \((Q^n)_{ij} = 0 \) for every \( n \in \mathbb{N} \).

**Proof.** Let \( B = e^{-Q} \), where \( Q \) is a \( Z \) matrix and write \( Q = \theta I + (Q - \theta I) \), where \( \theta \) is chosen sufficiently large so that \((Q - \theta I)\) has negative diagonal elements. Then
\[
B = e^{-Q} = e^{(\theta I - Q) - \theta I} = e^{-\theta} e^{\theta I - Q}.
\]

Now \( e^{-\theta} > 0 \), so for \( B_{ij} \) to be zero, we must have that
\[
(e^{\theta I - Q})_{ij} = 0.
\]
But this a positive power series of nonnegative matrices; therefore, for \( B_{ij} \) to be 0
\[
[(\theta I - Q)^n]_{ij} = 0.
\]
for every \( n \). In particular (22) implies that \((Q^n)_{ij} = 0 \) for each \( n \) (as the term we are considering is off diagonal).

Lemma 4 yields the following corollaries:

**Corollary 1.** For a strongly infinitely divisible nonnegative matrix \( B \), \( B_{ij} = 0 \) \((i \neq j)\) implies that \((B^t)_{ij} = (e^{-tQ})_{ij} = 0 \) for all \( t \in \mathbb{R} \).

Along similar methods, one can prove other similar statements, for instance taking \( \theta \) sufficiently large in (22) we can deduce.

**Corollary 2.** A strongly infinitely divisible matrix has strictly positive diagonal elements.

In order to ‘test’ if a matrix is infinitely divisible, it is of interest to know what operations infinitely divisible nonnegative matrices are closed under. More generally, how we might alter an infinitely divisible matrix such that it remains infinitely divisible? Infinitely divisible matrices are not closed under addition. Even in the \( 2 \times 2 \) case consider the example where
\[
A = \begin{pmatrix} 2 & 6 \\ 3 & 5 \end{pmatrix},
\]
then \( A \) and \( A^T \) are both infinitely divisible but \( \det(A + A^T) < 0 \). This example also shows that the set of infinitely divisible matrices is not convex.
We can also show that the product of two infinitely divisible stochastic matrices need not be infinitely divisible. As a specific counterexample, we show the product of two embeddable stochastic matrices need not be embeddable. Consider the two intensity matrices
\[
Z_1 = \begin{pmatrix} -2 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix},
\]
(24)
\[
Z_2 = \begin{pmatrix} -\frac{1}{2} & \frac{1}{12} & \frac{5}{12} \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{pmatrix}.
\]
(25)

Then the associated stochastic matrices are:
\[
E_1 := \exp(Z_1) \approx \begin{pmatrix} 0.135 & 0.233 & 0.632 \\ 0 & 0.368 & 0.632 \\ 0 & 0 & 1 \end{pmatrix},
\]
(26)
and
\[
E_2 = \exp(Z_2) \approx \begin{pmatrix} 0.607 & 0.018 & 0.375 \\ 0 & 0.050 & 0.950 \\ 0 & 0 & 1 \end{pmatrix}.
\]
(27)

The principle branch of the logarithm of \( E_2E_1 \) yields a matrix that negative off diagonal entries. Furthermore, this matrix has distinct positive eigenvalues, so that the only possible intensity matrix that can generate \( E_2E_1 \) is the principal branch of the logarithm. Therefore the matrix \( E_2E_1 \) is not embeddable.

Curiously though, \( E_1 \) and \( E_2 \) are elements of \( \mathcal{IM} \) and hence infinitely divisible. Furthermore it is easily shown that \( E_1E_2 \in \mathcal{IM} \), so that \( E_1E_2 \) is embeddable. In summary,

Let \( A, B \) be two infinitely divisible matrices. Then:

- \( AB \) need not be infinitely divisible.
- The product of \( M \)-matrices need not be the power of an \( M \)-matrix.
- If \( AB \) is infinitely divisible, then \( BA \) need not be.
- If \( AB \) is the power of an \( M \)-matrix, \( BA \) need not be.

There is however an important class of embeddable matrices for which the product of them is again embeddable.

**Theorem 6.** Let \( A \) and \( B \) be two commuting infinitely divisible matrices. Then \( AB = BA \) is infinitely divisible.
Proof. We can without loss of generality suppose that \(A, B\) have distinct eigenvalues, and then use Lemma \[\text{1}\] In this case, any matrix function of \(A\) or \(B\) is primary, and therefore is a polynomial of \(A\) or \(B\) respectively (see \[\text{13}\] for more details regarding matrix functions). Hence, for any given \(m \in \mathbb{N}\) we have that \(A^m\) and \(B^m\) commute. Therefore \((AB)^{\frac{1}{m}} = A^{\frac{1}{m}}B^{\frac{1}{m}} \geq 0\). \hfill \Box

It is shown in \[\text{3}\] that inverse \(M\)-matrices are closed under multiplication and addition by strictly positive diagonal matrices. It is natural to inquire if this property extends to every infinitely divisible matrix. In other words, if a nonnegative matrix \(B\) is infinitely divisible if and only if \(BD\) and \(B + D\) are infinitely divisible for every strictly positive diagonal matrix \(D\). It turns out that infinitely divisible matrices are not closed under positive diagonal multiplication. For example, consider the infinitely divisible matrix

\[
\begin{pmatrix}
\frac{2}{5} & \frac{2}{5} & \frac{1}{5} \\
0 & \frac{1}{2} & \frac{1}{2} \\
0 & 0 & 1
\end{pmatrix}.
\]

(28)

We can multiply this by a diagonal matrix to get,

\[
\begin{pmatrix}
\frac{2}{5} & \frac{2}{5} & \frac{1}{5} \\
0 & \frac{1}{2} & \frac{1}{2} \\
0 & 0 & \frac{1}{2}
\end{pmatrix}.
\]

(29)

Computing the principal logarithm of matrix (29) we see it is not infinitely divisible. We can however establish the following weaker results.

Lemma 5. Let \(B\) be a strongly infinitely divisible matrix and \(L\) a strictly positive monomial matrix. Then \(L^{-1}BL\) is strongly infinitely divisible. This implies \(LB\) is strongly infinitely divisible if and only if \(BL\) is strongly infinitely divisible.

Proof. Let \(B = e^{-Q}\) for some \(Z\) matrix \(Q\). Clearly \(L^{-1}BL = L^{-1}e^{-QL} = e^{-L^{-1}QL}\). Since \(Q\) is a \(Z\) matrix, \(L^{-1}QL\) is a \(Z\) matrix because \(L^{-1}\) is non-negative and diagonal elements are only mapped to diagonal elements. Hence, \(L^{-1}BL\) must be infinitely divisible.

Now assume that \(LB\) is strongly infinitely divisible, then so is \(L^{-1}LBL = BL\). A similar argument applies if we assume that \(BL\) is strongly infinitely divisible. \hfill \Box

With regards to the embedding problem, the case of interest is when \(L\) is a permutation matrix. The above Lemma implies that if \(P\) is an embeddable stochastic matrix, then so is \(LPLL^{-1} = LLP^T\). We can now prove Theorem \[\text{3}\]

Proof of Theorem. The matrix strongly infinitely divisible matrix \(B\) is either irreducible or reducible, if it is irreducible, then it is known that there exists an \(m \in \mathbb{N}\) such that \(B^m\) is strictly positive. However this violates the invariance of zero patterns of Corollary \[\text{4}\] unless \(B\) is strictly positive. This yields the first statement of the dichotomy
If the matrix $B$ is reducible then the decomposition $B = LUL^T$ as stated in Theorem \(3 \text{ii}\) is a well known fact, see [9]. Furthermore, in this decomposition, the diagonal block matrices must be irreducible and thus by the same argument as in the preceding paragraph, this diagonal block matrices must be strictly positive.

We now show that the submatrices $U^{(n)}$, as defined in the statement of Theorem 3, must be infinitely divisible. Without loss of generality assume $U$ is upper block triangular and that the eigenvalues are distinct. By Lemma 5, $U$ is infinitely divisible. Consider the associated $Z$-matrix $\bar{Q}$ and the submatrix of $\bar{Q}$, denoted $\bar{Q}^{(n)}$, obtained by deleting the first $n$ blocks from the top rows and left columns. Because $\bar{Q}$ must be a polynomial of $U$, $\bar{Q}^{(n)}$ depends only on the entries in $U^{(n)}$. Therefore we conclude that that $\bar{Q}^{(M)}$ is a $Z$ matrix and $e^{-\bar{Q}^{(n)}} = U^{(n)}$.

In light Theorem 3, whenever dealing with strongly infinitely divisible nonnegative matrices, we may without loss of generality assume that it is strictly positive or upper block triangular.

4. The embedding problem for finite state stationary Markov chains

The embedding problem for Markov chains has been a long standing problem in linear algebra and probability theory since it was first considered by Elfving [5]. It raises the question if a given discrete finite state Markov chain can be interpreted as having arisen from a continuous stationary Markov chain that has been observed at discrete intervals. Such Markov chain is called embeddable. This problem has found applications in a diverse number of fields, such as sociology [10], credit ratings [12] and biology [14]. A Markov chain with stochastic Matrix $P$ is embeddable if and only if there exists an intensity matrix $R$ such that

$$P = e^{R}$$

The reader is directed to Singer and Spilerman [10] for the definition of an intensity matrix and a wide variety of of examples illustrating the depth of this problem. Kingman [6] showed that a Markov chain was embeddable if and only if it was nonsingular and had stochastic matrix roots of arbitrary order. We thus recognize the embeddable matrices as a special case of strongly infinitely divisible matrices.

For convenience and clarity we note what our key results entail for the embedding problem for stochastic matrices. Before this, however, there are a few things to verify. The following was proved recently by EB Davies [15] and can be proved in a similar way to Lemma 2.

**Theorem 7.** (EB Davies, 2010) The set of matrices with distinct eigenvalues dense in the set of embeddable stochastic matrices.

In light of the above theorem, we now realize the implication of Lemma 2 on the embedding problem to deduce what was and can show what was proved by Kingman [6] without
the additional assumption that the roots are also stochastic. [1]

**Theorem 8.** Assume a stochastic matrix $P$ is strongly infinitely divisible, then $P$ is embeddable.

**Proof.** It suffices to consider a stochastic matrix $P$ with distinct eigenvalues. By Theorem [1], if $P$ has nonnegative roots for all $n$. Then there is a $Z$ matrix $Q$ such that

$$P = e^{-Q}. \quad (31)$$

Let $u$ be the vector of length $N$, all of whose entries are 1. Since $P$ has distinct eigenvalues, $Q = \log(P)$, is a polynomial of $P$. It follows that $u$ is an eigenvector of $Q$, with eigenvalue 0. However because $Q$ is a $Z$ matrix, this implies that $-Q$ must in fact be a intensity matrix and hence $P$ is embeddable. \[\Box\]

It is useful to note that, in the case of stochastic matrices, these $M$-matrices must be of a specific form. If a stochastic matrix $P$ is the inverse of an $M$ matrix then we have that $P^{-1} = sI - K$ for some $s > \rho(K)$ and $K$ is nonnegative. However, we know that $u$, the vector consisting of ones as defined above, must be an eigenvector for $K$. Let $\lambda = \rho(K)$, so that $\lambda = s - 1$. Defining $H = \frac{1}{\lambda}K$, where $H$ is now stochastic, we have

$$P^{-1} = s \left( I - \frac{s - 1}{s} H \right). \quad (32)$$

Thus when an embeddable stochastic matrix $P$ is the power of an inverse $M$ matrix, it must be of the form

$$P = (1 - \epsilon)^m (I - \epsilon H)^{-m}. \quad (33)$$

Where $\epsilon = \frac{s - 1}{s}$. We can thus classify the set of embeddable stochastic matrices: those stochastic matrices which can be infinitesimally perturbed to be in the form $(1 - \epsilon)^m (I - \epsilon H)^{-m}$. More formally, we state the following result.

**Corollary 3.** A stochastic matrix is embeddable if and only if it is nonsingular and in the closure of the set:

$$\{ P : P = (1 - \epsilon)^m (I - \epsilon H)^{-m}, \ 0 \leq \epsilon < 1, \ m \in \mathbb{N}, \ H \text{ stochastic} \}. \quad (34)$$

Bounds on the eigenvalues for the intensity matrices of embeddable stochastic matrices have been developed in some length. Notably, Runnenbergs’ condition [17] which states that the eigenvalues of a values of an $n \times n$ intensity matrix must be an element of the set

\[1\] A stochastic matrix may have nonnegative nonstochastic roots, an example is given in [10]
\[
\{ z : \pi \left( \frac{1}{2} \pm \frac{1}{n} \right) \leq \text{arg}(z) \leq \left( \frac{3}{2} \pm \frac{1}{n} \right) \}.
\] (35)

The utility of this result however, diminishes rapidly in higher dimensions. More in the spirit of this analysis is the related bound proven in [12]:

\[
|\text{Im}(\log(\lambda))| \leq -\log(\det(P)).
\] (36)

If we apply our bound derived at the end of Section 1 to the case of stochastic matrices, we arrive at (36).

We also know that the diagonal elements in any intensity matrix are always nonpositive and the rows sum to 0. Hence using Gershgorin’s disc theorem, the imaginary part of any eigenvalue of an intensity matrix is nonpositive; therefore

\[
0 \geq \text{Im} \log(\lambda) \geq \log(\det(P))
\] (37)

Inequality (37) dramatically simplifies the procedure for determining whether a stochastic matrix is embeddable. For practical purposes we can usually, without lost of generality, restrict ourselves to the case of distinct eigenvalues, as this may always be obtained after a infinitesimal perturbation by Lemma 2. In this case one needs only to check branches of the logarithm with imaginary part in the domain above. For example, if \( P \) is a 5 \( \times \) 5 matrix, and the determinant of \( P \) is small, say 0.00001, we need to check only 16 cases.

Theorem 3 also has a probabilistic interpretation: this result implies that there are in fact only two types of finite state, stationary, continuous Markov chains. One type corresponds to a process whereby, from any state, it may, with positive probability, reach any other state in any given time interval. This type corresponds to Theorem 3 (i). The other type is when there is a hierarchy of systems described by some sequence of square upper triangular block matrices \( P^n, n \leq M \), each modeling a continuous Markov chain in its own right. The practical application of this result is that one can determine if a stochastic matrix is embeddable by checking if the stochastic matrices defined by submatrices \( P^n \) are embeddable. I.e our result introduces a new necessary condition.

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Erratum to “Infinitely divisible nonnegative matrices, $M$-matrices, and the embedding problem for finite state stationary Markov Chains.”

Alexander Van-Brunt

November 29, 2022

Abstract

This erratum corrects the proof of Theorem 1 in [1].

Erratum to the proof of Theorem 1

We recall the definition of an infinitely divisible nonnegative matrix as given in [1].

Definition 1. A nonnegative $N \times N$ matrix $B$ is infinitely divisible if there exists a sequence of nonnegative matrices $\{K_n\}_{n=1}^{\infty}$ such that for every $n \in \{1, 2, 3, \ldots\}$

$$(K_n)^n = B. \quad (1)$$

If, in addition, $\det(B) > 0$ then $B$ is strongly infinitely divisible.

The matrices $K_n$ are not assumed to be primary matrix functions of $B$.

The main result of [1] is the following theorem.

Theorem 1. $B$ is strongly infinitely divisible if and only if there exists a $Z$-matrix, $Q$, such that

$$B = e^{-Q}. \quad (2)$$

One error in [1] is that it is claimed the relation $K_n \geq 0$ and $(K_n)^n = B$ imply that the sequence $K_n$ is uniformly bounded. However this is false. For example, if we consider, $N = 2, B = 0$ then unbounded sequence of nonnegative matrices.

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1This example was communicated by Professor Roger Horn.
\[ K_n = \begin{bmatrix} 0 & n \\ 0 & 0 \end{bmatrix}. \]  

satisfy \( (K_n)^n = B \) for every \( n \in \{1, 2, 3, \ldots \} \). This example shows that in order to deduce that \( K_n \) is uniformly bounded, \( B \) cannot be permitted to be singular without some additional hypothesis.

The direct implication of Theorem 1 is proved in [2], page 260. We will henceforth concern ourselves only with the reverse implication. Throughout this article we consider \( N \times N \) matrices for a fixed integer \( N \geq 1 \). \( M \) denotes an integer that is divisible by every integer less than or equal to \( N \).

Let us begin with the following Lemma, which succinctly illustrates key ingredient for the proof of Theorem 1.

**Lemma 1.** Suppose that \( B \) is a nonnegative, nonsingular \( N \times N \) matrix and that there exists a sequence of positive integers \( \{n_k\}_{k=1}^\infty \) with \( n_k \to \infty \) as \( k \to \infty \) and nonnegative matrices \( R_{nk} \) such that, for each \( k \),

\[(R_{nk})^{n_k} = B \]  

and

\[ \|R_{nk} - I\| \leq C n_k^{-1} \]  

where \( I \) denotes the \( N \times N \) identity matrix and \( \| \cdot \| \) is the maximum row sum norm. Then there exists a \( Z \)-matrix, \( Q \) such that

\[ B = e^{-Q}. \]

**Proof.** By assumption the sequence

\[ Q_n := n_k [R_{nk} - I] \]  

is uniformly bounded. So passing to a convergent subsequence, also denoted \( Q_{nk} \), we have that \( -Q_{nk} \) converges to a \( Z \) matrix, denoted as \( -Q \). Rearranging (4) and taking the limit we deduce

\[ B = \lim_{k \to \infty} [I - n_k^{-1} Q_{nk}]^{n_k} = e^{-Q}. \]

Thus we see the proof of Theorem 1 will consist of showing two steps. First, to show that the strong infinite divisibility of a nonnegative matrix \( B \) implies the existence of a subsequence of nonnegative matrices \( R_{nk} \) satisfying the identity (4) which converge to the identity. Secondly that the rate of convergence of \( R_{nk} \) to the identity matrix is at least \( O(n_k^{-1}) \), as per equation (5).
We will first prove these two steps in the irreducible case. Then we will use the analysis of the irreducible case to prove the general case. The proof of the irreducible case is broken into a series of smaller Lemmas. The first of these Lemmas is as follows.

**Lemma 2.** Suppose that $B$ is an irreducible, nonnegative, nonsingular $N \times N$ matrix and that there exists a sequence of positive integers $\{n_k\}_{k=1}^{\infty}$ with $n_k \to \infty$ as $k \to \infty$ and nonnegative matrices $K_{n_k}$ such that

$$(K_{n_k})^{n_k} = B \quad (9)$$

Then

1. $K_{n_k}$ is uniformly bounded
2. If $R$ is the limit of any convergent subsequence of $K_{n_k}$ then $R^M = I$.

**Proof.** Let $\lambda = \rho(B)$ be the spectral radius of $B$ and $v = [v_i]$ be the unique nonnegative eigenvector such that $Bv = \lambda v$ and $v_1 + v_2 + \cdots + v_N = 1$. The entries of $v$, (the Perron eigenvector of $B$), are strictly positive. We claim that $v$ is also an eigenvector of $K_{n_k}$, for every $k$, with eigenvalue $\lambda^{1/n_k}$. The identity (9) ensures that each $K_{n_k}$ commutes with $B$ and hence

$$BK_{n_k}v = K_{n_k}Bv = \lambda K_{n_k}v. \quad (10)$$

Since $K_{n_k}$ is nonnegative and nonzero, the vector $K_{n_k}v$ is nonnegative and nonzero, so (10) ensures that, $K_{n_k}v$ is a nonnegative eigenvector of $B$. By uniqueness of the Perron eigenvector, $K_{n_k}v$ is a positive scalar multiple of $v$. Thus $K_{n_k}v = \mu_{n_k}v$ for some $\mu_{n_k} > 0$ and so $v$ is an eigenvector of $K_{n_k}$. The identity (9) ensures that $\mu_{n_k} = \lambda^{1/n_k}$.

Let $C = \max_i v_i$ and $c = \min_i v_i > 0$. Then, for each $i \in \{1, 2, \ldots, N\}$,

$$c \sum_j (K_{n_k})_{ij} \leq \sum_j (K_{n_k})_{ij}v_j = \lambda^{1/n_k}v_i \leq \lambda^{1/n_k}C \quad (11)$$

for every $k$. Thus the row sums of $K_{n_k}$ are uniformly bounded and, as $K_{n_k}$ is nonnegative, it follows that the entries of $K_{n_k}$ are uniformly bounded.

Suppose that along a subsequence of $n_k$ (which we also denote by $n_k$), $K_{n_k}$ converges to a limit, denoted by $R$. The positive vector $v$ is an eigenvector of $R$ with eigenvalue 1, since

$$Rv = \lim_{k \to \infty} K_{n_k}v = \lim_{k \to \infty} \lambda^{1/n_k}v = v. \quad (12)$$

Let $\{\lambda_i\}_{i=1}^{N}$ denote the eigenvalues of $R$ counting multiplicities. We now argue that $|\lambda_i| = 1$ for every $i$. To start with, we note that

$$|\det(K_{n_k})| = |\det(B)|^{1/n_k}. \quad (13)$$

therefore $|\det(R)| = 1$. Because $|\lambda_i| \leq \lim_{k \to \infty} \rho(B)^{1/n_k} = 1$ and
\[ 1 = |\det(R)| = \prod_{i}^{N} |\lambda_i| \]  
(14)

it follows that \(|\lambda_i| = 1\) for each \(i\).

We now employ Theorem 8.3.5 in [3], which ensures that every eigenvalue is semisimple and hence \(R\) is diagonalizable.\(^2\)

The irreducible normal form for a nonnegative matrix, ensures that there exists a permutation matrix \(P\) such that \(PRP^T\) is an upper triangular block matrix with irreducible diagonal blocks (see [3], page 532). As the spectrum of \(R\) is the union of the spectra of the diagonal blocks, we can employ Corollary 8.4.6 in [3] to deduce that each eigenvalue of \(R\) is a \(m\)th root of unity for some \(m \leq N\). Hence \(R^M = I\) follows as \(R\) is diagonalizible. \(\square\)

As an immediate corollary we have the following.

**Corollary 1.** Let \(B\) be an \(N \times N\) irreducible strongly infinitely divisible nonnegative matrix. Then there exists a sequence of positive integers \(\{n_k\}_{k=0}^{\infty}, n_k \to \infty\) as \(k \to \infty\) and nonnegative matrices \(R_{n_k}\) such that

\[
(R_{n_k})^{n_k} = B
\]  
(15)

and

\[
\lim_{k \to \infty} R_{n_k} = I
\]  
(16)

**Proof.** Consider the sequence \(R_n := (K_{Mn})^M\), then each \(R_n\) is nonnegative and \((R_n)^n = (K_{Mn})^{Mn} = B\). As \(B\) is strongly infinitely divisible, it is nonsingular and so we may apply Lemma [2] to deduce that \(K_{Mn}\) is uniformly bounded. Suppose that we pass to a convergent subsequence \(K_{Mn_k}\) with limit \(R\). Then by Lemma [2]

\[
I = R^M = \lim_{k \to \infty} (K_{Mn_k})^M = \lim_{k \to \infty} R_{n_k}
\]  
(17)

This is the first of the two steps for the irreducible case. We now turn to show that this sequence of nonnegative matrices, \(R_{n_k}\), satisfies the convergence estimate [3]. The starting point is to study the rate of convergence of the trace.

**Lemma 3.** Let \(B\) be an \(N \times N\) matrix. Suppose there exists a sequence of positive integers \(\{n_k\}_{k=1}^{\infty}\) such that \(n_k \to \infty\) as \(k \to \infty\) and a sequence \(R_{n_k}\) of matrices with \((R_{n_k})^{n_k} = B\) and

\[
\lim_{k \to \infty} R_{n_k} = I.
\]  
(18)

Then there exists positive constants \(C_1, C_2\) independent of \(k\) such that

\(^2\)More precisely see the exercise on page 531 immediately after the statement of the Theorem.
\[ |\text{tr}(R_{nk} - I)| \leq C_1 n_k^{-1} + C_2(\|R_{nk} - I\|^2), \quad (19) \]

where \( \| \cdot \| \) denotes the maximum row sum norm.

**Remark 1.** This Lemma does not require \( B \) or \( R_{nk} \) to be irreducible or nonnegative.

**Proof.** Our key tool to prove this lemma is Jacobi’s formula, which states that for a differentiable map \( A : \mathbb{R} \to \mathbb{R}^{n \times n} \)

\[
\frac{d}{dt} \det A(t) = \text{tr} \left( \text{adj}(A(t)) \frac{dA(t)}{dt} \right) \quad (20)
\]

We may assume that \( n_k \) is a subsequence such that \( R_{nk} \neq I \) for every \( n_k \). If there exists an integer \( L \geq 0 \) such that \( R_{nk} = I \) for all \( n_k \geq L \), then there is nothing to prove. Therefore without loss of generality we may assume that, for any given \( k \), the map

\[
A_{nk}(t) := I + t \frac{R_{nk} - I}{\|R_{nk} - I\|} \quad (21)
\]

is well defined. Note that at \( t = \|R_{nk} - I\| \) we have \( A_{nk}(t) = R_{nk} \). We may use Taylor’s reminder theorem around \( t = 0 \) to deduce that for some \( \gamma_{nk}(t) \)

\[
\det A_{nk}(t) = 1 + \text{tr} \left( (\text{adj}(A(0)) \frac{dA(t)}{dt} \big|_0 \right) t + \gamma_{nk}(t)t^2 \quad (22)
\]

\[
= 1 + \text{tr} \left( \frac{R_{nk} - I}{\|R_{nk} - I\|} \right) t + \gamma_{nk}(t)t^2. \quad (23)
\]

Setting \( t = \|R_{nk} - I\| \) this implies

\[
\det R_{nk} = 1 + \text{tr}(R_{nk} - I) + \gamma_{nk}\|R_{nk} - I\|^2. \quad (24)
\]

Furthermore at \( t = \|R_{nk} - I\| \) the constants \( \gamma_{nk} \) depend only on \( R_{nk} \).

Let us now analyse the left hand side of (24). If we consider Taylor’s remainder theorem for the function \( f(t) = x^t \), for \( 0 < t < 1 \), and a fixed \( x > 0 \), we have that

\[
f(t) = 1 + \xi t \quad (25)
\]

for \( \xi = \xi(t) \) such that \( |\xi| \leq C_1 \) for a \( C_1 > 0 \) depending only on \( x \). Then if we apply this with \( x = \det(B) \) and \( t = n_k^{-1} \) we have

\[
\det(R_{nk}) = \det(B)^{1/n_k} = 1 + \xi_{nk}^{-1} \quad (26)
\]

for some constants \( \xi_{nk} \) where \( |\xi_{nk}| \leq C_1 \) for an \( C_1 > 0 \) depending only on \( \det(B) \). Combining equations (24) and (26)

\[
\xi_{nk} n_k^{-1} = \text{tr}(R_{nk} - I) + \gamma_{nk}\|R_{nk} - I\|^2. \quad (27)
\]
The final step is to show that the reminder constants $\gamma_{nk}$ can be chosen such that $|\gamma_{nk}| \leq C_2$ for a constant $C_2 > 0$ independent of $k$. Let $\{\lambda_{i,nk}\}_{i=1}^N$ denote the eigenvalues of $R_{nk}$ counting multiplicities. We may express $\det A_{nk}(t)$ as
\[ \det A_{nk}(t) = \prod_{i=1}^N \left( 1 + t \frac{\lambda_{i,nk}}{\|R_{nk} - I\|} \right) \] (28)

It is a standard result that $\rho(A) \leq \|A\|$ for any matrix $A$. We refer the reader to [3], page 347, for the statement and proof, valid for any matrix norm. From this fact it follows
\[ \left| \frac{\lambda_{i,nk} - 1}{\|R_{nk} - I\|} \right| \leq 1 \] (29)
and so the expression (28) is well defined for each $k$. Then, $\det A_{nk}(t)$, being a polynomial in $t$ with uniformly bounded coefficients, has uniformly bounded derivatives. The uniform boundedness of $\gamma_{nk}$ follows.

Now we can finally deduce the rate of convergence in the irreducible case.

**Lemma 4.** Let $B$ be an $N \times N$ irreducible nonnegative matrix. Suppose there exists a sequence of positive integers $\{n_k\}_{k=1}^\infty$ such that $n_k \to \infty$ as $k \to \infty$ and a sequence of nonnegative matrices $R_{nk}$ such that, for each $k$,
\[ (R_{nk})^{n_k} = B \] (30)
and
\[ \lim_{k \to \infty} R_{nk} = I. \] (31)

Then there is a constant $C > 0$ such that for all $k$
\[ \|R_{nk} - I\| \leq C n_k^{-1} \] (32)
where $\|\cdot\|$ is the maximum row sum matrix norm.

**Proof.** Let $\lambda = \rho(B)$ be the spectral radius of $B$ and $v = [v_i]$ be the unique nonnegative eigenvector such that $Bv = \lambda v$ and $v_1 + v_2 + \cdots + v_N = 1$. The entries of $v$, (the Perron eigenvector of $B$), are strictly positive. In a similar derivation to (11) we can conclude
\[ \sum_{j=1}^N ((R_{nk})_{ij} - \delta_{ij})v_j = (\lambda^{1/n_k} - 1)v_i \] (33)
where $\delta_{ij}$ is the Kronecker delta function. If we sum up over $i$, we may write this as
\[ \text{tr}(R_{nk} - I) + \sum_{\{i \neq j\}} (R_{nk})_{ij}v_j = (\lambda^{1/n_k} - 1)\|v\|_{\ell_1} \] (34)
where $\|\cdot\|_{\ell_1}$ denotes the $\ell_1$ vector norm in $\mathbb{R}^N$. By a Taylor series expansion of $$(\lambda^{1/n_k} - 1)\|v\|_{\ell_1}$$ we deduce that $|(\lambda^{1/n_k} - 1)\|v\|_{\ell_1}| \leq C_3 n_k^{-1}$ for some constant $C_3 > 0$ depending only on $\lambda$. This gives us an estimate on the RHS of (34). Furthermore Lemma 3 implies that

Now we can finally deduce the rate of convergence in the irreducible case.

**Lemma 4.** Let $B$ be an $N \times N$ irreducible nonnegative matrix. Suppose there exists a sequence of positive integers $\{n_k\}_{k=1}^\infty$ such that $n_k \to \infty$ as $k \to \infty$ and a sequence of nonnegative matrices $R_{nk}$ such that, for each $k$,
\[
\left| \text{tr}(R_{nk} - I) \right| \leq C_1 n_k^{-1} + C_2 \| K_{nk} - I \|^2. \tag{35}
\]

for positive constants \( C_1, C_2 > 0 \) uniform in \( k \). Then using these estimates in equation (34) we deduce

\[
c \left| \sum_{i \neq j} (R_{nk})_{ij} \right| \leq C_3 n_k^{-1} + C_1 n_k^{-1} + C_2 \| R_{nk} - I \|^2 \tag{36}
\]

where \( c = \min_i v_i > 0 \). As \( R_{nk} \) is nonnegative then this implies that for each distinct pair \( i, j \) with \( i \neq j \)

\[
(R_{nk})_{ij} \leq \hat{C}_1 n_k^{-1} + \hat{C}_2 \| R_{nk} - I \|^2. \tag{37}
\]

for \( \hat{C}_1 = c^{-1}(C_1 + C_2) \) and \( \hat{C}_2 = c^{-1}C_3 \). We may then in turn substitute this result into (33) to deduce that for each \( 0 \leq i \leq N \).

\[
|R_{nk} - I| \leq N \hat{C}_3 n_k^{-1} + N \hat{C}_1 \| R_{nk} - I \|^2 + NC_2 n_k^{-1} \tag{38}
\]

Thus we have arrived at the conclusion that for all \( 0 \leq i, j \leq N \)

\[
|R_{nk} - I| - \delta_{ij} | \leq N \hat{C}_3 n_k^{-1} + N \hat{C}_1 \| R_{nk} - I \|^2 + NC_2 n_k^{-1}. \tag{39}
\]

This in turns readily implies that

\[
\| R_{nk} - I \| \leq N^2 \hat{C}_3 n_k^{-1} + N^2 \hat{C}_1 \| R_{nk} - I \|^2 + N^2 C_2 n_k^{-1}. \tag{40}
\]

and so, as \( \| R_{nk} - I \|^2 \leq \frac{1}{2} \| R_{nk} - I \| \), for \( k \) sufficiently large, we can then conclude

\[
\| R_{nk} - I \| \leq C_4 n_k^{-1} \tag{41}
\]

for some constant \( C_4 > 0 \).

This completes our analysis of the irreducible case. In particular Corollary \[\Box\] combined with Lemma \[\Box\] implies that if we have an irreducible, infinitely divisible matrix \( B \), then automatically there will exist a subsequence \( R_{nk} \) which satifies the hypothesis of Lemma \[\Box\].

We can now begin to analyse the general case. For \( K_{nk} \) as in \[\Box\], the fundamental insight will be provided by the relation \( (PK_{nk}P^T)^{nk} = P(K_{nk})^{nk}P^T = PBP^T \), for any permutation matrix \( P \). We will without loss of generality assume that \( B \) is in irreducible normal form henceforth.

We now show that if \( B \) is in irreducible normal form, there will exist nonnegative roots of \( B \) that which have the same zero block pattern.

**Lemma 5.** Assume that \( B \) is a strongly infinitely divisible matrix in irreducible normal form. Then for each \( n \) there exists a matrix \( R_n \) such that

\[
(R_n)^n = B \tag{42}
\]

where each \( R_n \) is nonnegative and has the same zero block pattern as \( B \).
Proof. As in the proof of Corollary 1 we may consider the sequence \( R_n := (K_{Mn})^{Mn} \). We claim that for each \( i \), \( (R_n)_{ii} > 0 \). To show this first observe that as \( K_{Mn} \) must be non-singular. In addition, as it is nonnegative we have for some \( \alpha > 0 \) and a permutation matrix \( P \) that
\[
K_{Mn} \geq \alpha P
\]
(43)
Which implies that
\[
R_n = (K_{Mn})^M \geq (\alpha P)^M = \alpha^M I.
\]
(44)
Hence \( (R_n)_{ii} > \alpha^M \) for each \( i \) and some \( \alpha > 0 \).

Next we observe that \( B_{ij} = 0 \) only if \( (R_n)_{ij} = 0 \). This follows from noting that
\[
(R_n)_{ij}^2 = \sum_{m=1}^{n} (R_n)_{im}(R_n)_{mj} \geq (R_n)_{ii}(R_n)_{ij} \geq \alpha^M (R_n)_{ij}
\]
(45)
and then proceeding by induction. The fact that we have \( B_{ij} = 0 \) only if \( (R_n)_{ij} = 0 \) implies that \( R_n \) can be written as the same zero-block structure as \( B \). In particular, if a block is zero in \( B \), then we have shown that the same block must be zero in \( R_n \).

Let us denote \( B_{ij} \) as the \( i \)th, \( j \)th block of \( B \) and \( K_{n_kij} \) the \( i \)th, \( j \)th block of \( K_{n_k} \) which we now understand to have the same structure. A crucial observation is that for each \( i \) and \( n_k \) we have \( B_{ii} = (K_{n_kii})^{n_k} \). As a consequence, if \( B \) is infinitely divisible then so are the diagonal blocks \( B_{ii} \). Furthermore as each \( B_{ii} \) is irreducible, it follows that each \( K_{n_kii} \) is irreducible (the powers of reducible matrices can never create irreducible matrix). This allows us to apply our analysis on these blocks \( B_{ii} \) and sequence of matrices \( K_{n_kii} \) via Lemmas 2 - 4 to deduce properties of convergence to the appropriate sized identity matrix. This serves as the platform to study the reducible case.

Let us begin with the following Lemma which is a precursor to study the rate of convergence in the general case.

**Lemma 6.** Let \( B \) be an \( N \times N \) irreducible nonnegative matrix. Suppose there exists a sequence of positive integers \( \{n_k\}_{k=1}^{\infty} \) such that \( n_k \to \infty \) as \( k \to \infty \) and a sequence of nonnegative matrices \( R_{n_k} \) such that, for each \( k \),
\[
(R_{n_k})^{n_k} = B
\]
(46)
and
\[
\lim_{k \to \infty} R_{n_k} = I
\]
(47)
Then there exists an \( \alpha > 0 \) such that
\[
\alpha^{1/n_k} R_{n_k} - I \geq 0
\]
(48)
for all \( n_k \) sufficiently large.
Proof. By Lemma 4 we deduce that
\[ R_{n_k} = I + \epsilon_{n_k} \]  
where \( \epsilon_{n_k} \) is an error term, which for some \( C_1 > 0 \) satisfies \( \| \epsilon_{n_k} \| \leq C_1 n_k^{-1} \) and \( \epsilon_{n_k} \) has nonnegative off diagonal entries. Furthermore \( C_1 \) can be chosen independently of \( k \).

Consider then, via a Taylor series expansion of \( \alpha^{1/n_k} \),
\[ \alpha^{1/n_k} R_{n_k} - I = (1 + \ln(\alpha)\alpha^{1/n_k} n_k^{-1} + \xi_{\alpha} n_k^{-2})(I + \epsilon_{n_k}) - I \]  
for constants \( \xi_{\alpha} \) satisfying \( |\xi_{\alpha}| \leq C_2 \) for a constant \( C_2 \) depending only on \( \alpha \).

Taking \( \alpha \) so large such that \( \ln(\alpha) > C_1 \) implies that
\[ \alpha^{1/n_k} R_{n_k} - I = \ln(\alpha)\alpha^{1/n_k} n_k^{-1} I + \epsilon_{n_k} + \mathcal{O}(n_k^{-2}) \geq 0 \]  
for any \( i \) and \( n_k \) sufficiently large. Therefore
\[ \alpha^{1/n_k} R_{n_k} - I \geq 0 \]  
for \( n_k \) sufficiently large. \( \square \)

We can now prove that a strongly infinitely divisible nonnegative matrix always satisfies the hypothesis of Lemma 1, and consequently prove Theorem 1.

**Lemma 7.** Suppose that \( B \) is an \( N \times N \) strongly infinitely divisible nonnegative matrix. Then there exists a sequence of positive integers \( \{n_k\}_{k=1}^{\infty} \) with \( n_k \to \infty \) as \( k \to \infty \) and nonnegative matrices \( R_{n_k} \) such that
\[ (R_{n_k})^{n_k} = B \]  
\[ \|R_{n_k} - I\| \leq C n_k^{-1} \]  

**Proof.** Let \( K_n \) be a sequence on nonnegative matrices satisfying \( (K_n)^n = B \) for each \( n \in \{1, 2, 3..., \} \). Without loss of generality assume that \( B \) is in irreducible normal form. Set \( B_{ij} \) as the \( i \)th, \( j \)th block of \( B \). Similarly set \( K_{nij} \) the \( i \)th, \( j \)th block of \( K_n \).

Step 1. Firstly we show that \( K_n \) is uniformly bounded. By definition, for every \( n, \)
\[ B_{ij} = \sum_{m_1, m_2, ..., m_{n-1}} K_{nm_1m_2...m_{n-1}nm} K_{nmnj} \]  
By nonnegativity, if we restrict the summation to \( m_1 = m_2 = ... = i \), then we have the inequality
\[ B_{ij} \geq (K_{nii})^{n-1} K_{nij} = (K_{nii})^{-1} B_{ii} K_{nij}. \]  
Rearranging we have that
\[ K_{nii} B_{ij} \geq B_{ii} K_{nij}. \]
By Lemma 2, $K_{nii}$ is uniformly bounded. We claim that because $B_{ii}$ is nonsingular then this implies that $K_{nij}$ must be uniformly bounded. This is apparent by using an inequality similar to (43):

\[ B_{ii} \geq \beta P_{ii} \]

for some $\beta > 0$ and an appropriately sized permutation matrix $P_{ii}$. From the inequality (57) we can then deduce

\[ K_{nii}B_{ij} \geq \beta P_{ii}K_{nij}. \]

Multiplying both sides by $(P_{ii})^{-1}$, which is a nonnegative matrix, we then deduce

\[ (P_{ii})^{-1}K_{nii}B_{ij} \geq \beta K_{nij}. \]

Therefore $K_n$ is uniformly bounded.

Step 2. Consider the sequence $K_{Mn}$. By step one, $K_{Mn}$ is uniformly bounded and so we can pass to a convergent subsequence, denoted $K_{Mn_k}$. Because each $B_{ii}$ is irreducible and $(K_{Mn_k})^{Mn_k} = B_{ii}$, we can employ Lemma 2 on each $R_{ii}$ to deduce $(R_{ii})^M = I_{ii}$

Step 3. For each $k$ set $R_{nk} = (K_{Mn_k})^M$. As each $R_{nk} := (K_{Mn_k})^M$ converges to $I_{ii}$, and $B_{ii}$ is irreducible, we can apply Lemma 6 to each $R_{nk}$ to deduce that for some $\alpha > 0$

\[ \epsilon_{nk} := \alpha^{1/nk}R_{nk} - I \geq 0 \]

for all $k$ sufficiently large. By nonnegativity of $\epsilon_{nk}$

\[ \alpha B = (I + \epsilon_{nk})^{nk} \geq I + nk\epsilon_{nk}. \]

Or in other words

\[ \frac{1}{nk}(\alpha B - I) \geq \alpha^{1/nk}R_{nk} - I \geq 0. \]

Finally noting that $\alpha^{1/nk} = 1 + O(n_k^{-1})$ we deduce, for some constant $C$ depending only on $B$,

\[ Cn_k^{-1} \geq \|\alpha^{1/nk}R_{nk} - I\| \geq \|R_{nk} - I\| + O(n_k^{-1}) \]

which completes the proof.

\[ \square \]

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