On open flat sets in spaces with bipolar comparison

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Abstract

We show that if a Riemannian manifold satisfies (3,3)-bipolar comparisons and has an open flat subset then it is flat. The same holds for a version of MTW where the perpendicularity is dropped.

In particular we get that the (3,3)-bipolar comparison is strictly stronger than the Alexandrov comparison.

1 Introduction

We say that a metric space $X$ satisfies the $(k,l)$-bipolar comparison if for any $a_0, a_1, \ldots, a_k; b_0, b_1, \ldots, b_l \in X$ there are points $\hat{a}_0, \hat{a}_1, \ldots, \hat{a}_n, \hat{b}_0, \hat{b}_1, \ldots, \hat{b}_n$ in the Hilbert space $\mathbb{H}$ such that

$$|\hat{a}_0 - \hat{b}_0|_\mathbb{H} = |a_0 - b_0|_X, \quad |\hat{a}_i - \hat{a}_0|_\mathbb{H} = |a_i - a_0|_X, \quad |\hat{b}_i - \hat{b}_0|_\mathbb{H} = |b_i - b_0|_X$$

for any $i, j$ and

$$|\hat{x} - \hat{y}|_\mathbb{H} \geq |x - y|_X$$

for any $x, y \in \{a_0, a_1, \ldots, a_k, b_0, b_1, \ldots, b_l\}$.

This definition was introduced in [5]. The class of compact length metric spaces satisfying $(k,0)$-bipolar comparison with $k \geq 2$ coincide with the class of Alexandrov spaces with nonnegative curvature, (for $k = 2$ it is just one of the equivalent definitions, for arbitrary $k$ see [1], [3]). In general $(k,l)$-bipolar comparisons (with $k$ or $l \geq 2$) for length metric spaces are stronger conditions than nonnegative curvature condition and they describe some new interesting classes of spaces. In particular, we prove in [5] that for Riemannian manifolds $(4,1)$-bipolar comparison is equivalent to the conditions related to the continuity of optimal transport. Also in [5] we together with coauthors describe classes of Riemannian manifolds satisfying $(k,l)$-bipolar comparisons for almost all $k, l$ excepting $(2,3)$ and $(3,3)$-bipolar comparisons. In particular it was not known if $(3,3)$-bipolar comparison differs from Alexandrov’s comparison. In this note the affirmative answer is obtained as a corollary of some rigidity result for spaces with $(3,3)$-bipolar comparison. To formulate exact statements we need some definitions and notations.

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Let $M$ be a Riemannian manifold and $p \in M$. The subset of tangent vectors $v \in T_p$ such that there is a minimizing geodesic $[p,q]$ in the direction of $v$ with length $|v|$ will be denoted as $\operatorname{TIL}_p$. The interior of $\operatorname{TIL}_p$ is denoted by $\operatorname{IL}_p$; it is called tangent injectivity locus at $p$. If at $\operatorname{TIL}_p$ is convex for any $p \in M$, then $M$ is called CTIL.

Riemannian manifold $M$ satisfies MTW if the following holds. For any point $p \in M$, any $W \in \operatorname{TIL}_p$ and tangent vectors $X, Y \in T_p$, such that $X \perp Y$ we have

\[ \frac{\partial^4}{\partial^2s \partial^2t} |\exp_p(s \cdot X) - \exp_p(W + t \cdot Y)|^2_M \leq 0 \]

at $t = s = 0$.

This definition was introduced by Xi-Nan Ma, Neil Trudinger and Xu-Jia Wang in [7], Cedric Villani studied a synthetic version of this definition ([9]). If the same inequality holds without the assumption $X \perp Y$ Riemannian manifold $M$ satisfies MTW$^\perp$ [2].

MTW and CTIL are necessary condition for TCP (transport continuity property). In [2], Alessio Figalli, Ludovic Rifford and Cédric Villani showed that a strict version of CTIL and MTW provide a sufficient condition for TCP. A compact Riemannian manifold $M$ is called TCP if for any two regular measures with density functions bounded away from zero and infinity the generalized solution of Monge–Ampère equation provided by optimal transport is a genuine (continuous) solution.

Let us denote by $\mathcal{M}(k,l)$ the class of smooth complete Riemannian manifolds satisfying $(k,l)$–bipolar comparison and by $\mathcal{M}_{\geq 0}$ the class of complete Riemannian manifolds with nonnegative sectional curvature.

It was mentioned above, that

$\mathcal{M}_{\geq 0} = \mathcal{M}_{(k,0)}$

for $k \geq 2$ and it is obvious from definition, that

$\mathcal{M}_{(k',l')} \subset \mathcal{M}_{(k,l)}$

if $k' \geq k$ and $l' \geq l$. It is proven in [5] that

$\mathcal{M}_{\geq 0} = \mathcal{M}_{(2,2)} = \mathcal{M}_{(3,1)}$

and

$\mathcal{M}_{(4,1)} = \mathcal{M}_{(k,l)}$

for $k \geq 4$ and $l \geq 1$. The most interesting fact proven in [5] is that

$\mathcal{M}_{(4,1)} = \mathcal{M}_{\operatorname{CTIL}} \cap \mathcal{M}_{\operatorname{MTW}^\perp}$,

where $\mathcal{M}_{\operatorname{CTIL}}$, $\mathcal{M}_{\operatorname{MTW}^\perp}$ are classes of smooth Riemannian manifolds satisfying CTIL and MTW$^\perp$ correspondingly. In particular this implies that $\mathcal{M}_{(4,1)} \neq \mathcal{M}_{\geq 0}$.
In this paper we prove the following two results.

1.1. **Theorem.** Let $M$ be a complete Riemannian manifold that satisfies $(3,3)$-bipolar comparison and contains a nonempty open flat subset. Then $M$ is flat.

1.2. **Theorem.** Let $M$ be a complete Riemannian manifold that satisfies $MTW \neq \perp$ and contains a nonempty open flat subset. Then $M$ is flat.

1.3. **Corollary.** We have that $\mathcal{M}_{(3,3)} \neq \mathcal{M}_{\geq 0}$.

Theorem 1.1 follows from Proposition 2.2 and Theorem 1.2 follows from Proposition 2.3, proved in the next section.

As a related result we would like to mention a rigidity result for manifolds with nonnegative sectional curvature with flat open subsets by Dmitrii Panov and Anton Petrunin [8].

2 Proofs

For points $a, b, c$ in a manifold we denote by $\angle[a_b_c]$ the angle at $a$ of the triangle $[abc]$.

2.1. **Key lemma.** Let $M$ be a complete Riemannian manifold that satisfies $(3,3)$-bipolar comparison. Assume that for the points $x, p, q, x_q$ in $M$ there is a triangle $[\tilde{p} \tilde{q} \tilde{x}]$ in the Euclidean plane $\mathbb{E}^2$ such that

$$|x_p - p|_M = |\tilde{x} - \tilde{p}|_{\mathbb{E}^2}, \quad |p - q|_M = |\tilde{p} - \tilde{q}|_{\mathbb{E}^2}, \quad |q - x_q|_M = |\tilde{q} - \tilde{x}|_{\mathbb{E}^2}$$

and moreover a neighborhood $N \subset \mathbb{E}^2$ of the base $[\tilde{p} \tilde{q}]$ admits a globally isometric embedding $\iota$ into $M$ such that $\iota([\tilde{p} \tilde{x}] \cap N) \subset [px_p]$ and $\iota([\tilde{q} \tilde{x}] \cap N) \subset [qx_q]$. Then $x_p = x_q$ and the triangle $[px_q]$ can be filled by a flat geodesic triangle.

![Diagram](image)

**Proof.** Set $p_- = p$ and $q_- = q$.

Choose points $p_0, p_+ \in [p_-, x_p] \cap (N)$ so that the points $p_- = p_0, p_+ = x_p$ appear in the same order on $[p_-, x_p]$. Analogously, choose points $q_0, q_+ \in [q_-, x_q] \cap (N)$ so that the points $q_- = q_0, q_+ = x_q$ appear in the same order on $[q_-, x_q]$. Denote by $\tilde{p}_-, \tilde{p}_+, \tilde{q}_-, \tilde{q}_0, \tilde{q}_+$ the corresponding points on the sides of triangle $[\tilde{p} \tilde{q} \tilde{x}]$; so $\tilde{p}_- = \tilde{p}$ and $\tilde{q}_- = \tilde{q}$.

Applying the comparison to $a_0 = p_0, a_1 = p_- = a_2 = p_+, a_3 = x_p; \quad b_0 = q_0, b_1 = q_-, b_2 = q_+, b_3 = x_q$, we get a model configuration $\tilde{p}_0, \tilde{p}_-, \tilde{x}_p, q_0, \tilde{q}_-, \tilde{q}_+, x_q$ in the Hilbert space $\mathbb{H}$.
Note that from the comparison it follows that the quadruple $\hat{p}_-, \hat{p}_0, \hat{p}_+, \hat{x}_p$ lies on one line and the same holds for the quadruple $\hat{q}_-, \hat{q}_0, \hat{q}_+, \hat{x}_q$.

Since

$$|\hat{p}_0 - \hat{q}_+|_{\|} \geq |\hat{p}_0 - \hat{q}_+|_{M}, \quad |\hat{p}_0 - \hat{q}_0|_{\|} = |\hat{p}_0 - \hat{q}_0|_{M}, \quad |\hat{q}_0 - \hat{q}_+|_{\|} = |\hat{q}_0 - \hat{q}_+|_{M},$$

we have $\angle[\hat{q}_0 \hat{p}_0] \geq \angle[\hat{q}_0 \hat{p}_0]$. The same way we get that $\angle[\hat{q}_0 \hat{p}_0] \geq \angle[\hat{q}_0 \hat{p}_0]$. Since the sum of adjacent angles is $\pi$, these two inequalities imply that

$$\angle[\hat{q}_0 \hat{p}_0] = \angle[\hat{q}_0 \hat{p}_0].$$

The same way we get that

$$\angle[\hat{p}_0 \hat{p}_0] = \angle[\hat{p}_0 \hat{p}_0].$$

From the angle equalities, we get that

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$$|\hat{p}_- - \hat{q}_+|_{\|} \leq |\hat{p}_- - \hat{q}_+|_{M}$$

and the equality holds if the points $\hat{p}_-, \hat{q}_+$ lie in one plane and on the opposite sides from the line $\hat{p}_0 \hat{q}_0$. By (3,3)-bipolar comparison the equality in 1 indeed holds.

It follows that configuration $\hat{p}_0, \hat{p}_- \hat{p}_+, \hat{x}_p, \hat{q}_0, \hat{q}_- \hat{q}_+, \hat{x}_q$ is isometric to the configuration $\hat{p}_0, \hat{p}_- \hat{p}_+, \hat{x}_q, \hat{q}_0, \hat{q}_- \hat{q}_+$; in particular, $\hat{x}_q = \hat{x}_p$.

By (3,3)-bipolar comparison $|x_p - x_q|_{M} \leq |\hat{x}_q - \hat{x}_p|_{\|}$; therefore $x_p = x_q$; so we can set further $x = x_p = x_q$.

Note that we also proved that the angles at $p$ and $q$ in the triangle $[pqx]$ coincide with their model angles; that is,

$$\angle[p \hat{q}] = \angle[p \hat{q}], \quad \angle[q \hat{p}] = \angle[q \hat{p}].$$

By the lemma on flat slices (see for example [4]), there is a global isometric embedding $\iota'$ of the solid model triangle $[\hat{p}\hat{q}\hat{x}]$ to $M$ which sends $[\hat{p}\hat{q}]$ to $[pq]$ and $[\hat{p}\hat{x}]$ to $[px]$. Note that $\iota'$ has to coincide with $\iota$ on $N$. It follows that $\iota'$ maps $[\hat{q}\hat{x}]$ to $[q\hat{x}]$, which finishes the proof.

Theorem 1.1 and Theorem 1.2 follow from the propositions below.

2.2. Proposition. Let $M$ be a complete Riemannian manifold that satisfies (3,3)-bipolar comparison. Then any point $p \in M$ admits a neighborhood $U \ni p$ such that if $U$ contains a nonempty open flat subset, then $U$ is flat.

Proof. Given a point $p$ consider a convex neighborhood $U \ni p$ such that injectivity radius at any point of $U$ exceeds the diameter of $U$; in particular any two points $p, q \in U$ are connected by unique minimizing geodesic $[pq]$ which lies in $U$. Denote by $F$ an open flat subset in $U$; we can assume that $F$ is convex.

Note that by the key lemma we have the following:

Claim. For any $x \in U$ and any $p, q \in F$ the triangle $[pqx]$ admits a geodesic isometric filling by a flat triangle.
Indeed, set \( x_p = x \). Consider a plane triangle \([\tilde{p}\tilde{q}\tilde{x}]\) that has the same angle at \( \tilde{p} \) and the same adjacent sides as the triangle \([pqx]\). Since \( F \) is flat and convex there is a flat open geodesic surface \( \Sigma \) containing \([pq]\) and a part of \([px]\) near \( p \). Choose a direction at \( q \) that runs in \( \Sigma \) at the angle \( \angle[\tilde{q}\tilde{x}\tilde{p}] \) to \([qp]\). Consider the geodesic in this direction of the length \( |\tilde{q}\tilde{x}| \). Since diameter of \( U \) exceeds the injectivity radius at \( q \), this geodesic is minimizing. It remains to apply the key lemma.

From the claim, it follows that the sectional curvature \( \sigma_x(X, Y) \) vanishes for any point \( x \in U \) and any two velocity vectors \( X, Y \in T_x \) of minimizing geodesics from \( x \) to \( F \). Since the set of such sectional directions is open, curvature vanish at \( x \); hence the result. \( \square \)

2.3. Proposition. Let \( M \) be a complete Riemannian manifold that satisfies MTW\(^K\). Then any point \( p \in M \) admits a neighborhood \( U \ni p \) such that if \( U \) contains a nonempty open flat subset, then \( U \) is flat.

Proof. For a given \( p \in M \) let us take a neighborhood \( U \ni p \) as in the proof of the previous proposition. The same proof as (Thm 1.2 [5]) shows that \( U \) satisfies (4,1)-bipolar comparison (CTIL condition is not necessary, because we stay away from cut-locus). Again, same proof as (the Thm 1.2 [5]) shows that inside this neighborhood (4,1)-bipolar comparison is equivalent to (4,4)-bipolar comparison. Further note that (4,4)-bipolar comparison implies (3,3)-bipolar comparison. Now we can follow the same lines as in the proof of Proposition 2.2, because (3,3)-bipolar comparison is used only locally in the proof. \( \square \)

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