Three-photon annihilation of the electron-positron pairs

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Abstract

Three-photon annihilation of the electron-positron pairs (= \(e^-, e^+\)-pairs) is considered in the electron rest frame. The energy of the incident positron can be arbitrary. The analytical expression for the cross-section of three-photon annihilation of the \((e^-, e^+)\)-pair has been derived and investigated.

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In this work we discuss three-photon annihilation of the electron-positron pairs (or $(e^-, e^+)$-pairs, for short). Annihilation of the $(e^-, e^+)$-pair is considered in the electron rest frame. The non-relativistic limit for the three-photon annihilation rate $\Gamma_{3\gamma}$ was obtained long ago [1]–[3] (see also [4] and [5]). In our earlier work [6] we considered three-photon annihilation of the electron-positron at arbitrary energies of the colliding particles. However, due to a number of reasons the closed expression for the three-photon annihilation cross-section was not derived in [6]. In this study we want to make the final step in our consideration of the three-photon annihilation and obtain the closed analytical formula for the cross-section of the three-photon annihilation. The analysis of this process is performed in the electron rest frame, while the energy of the incident positron can be arbitrary.

The principal conservation law in the case of three-photon annihilation of the free $(e^-, e^+)$-pair is

$$p_1 + p_2 = k_1 + k_2 + k_3$$  \hspace{1cm} (1)

where $p_1$ and $p_2$ are the 4-vectors of electron and positron momenta, respectively. For these two particles we always have $p_1^2 = m^2$ and $p_2^2 = m^2$, where $m$ is the electron/positron mass (see, e.g., [7]). In the electron rest frame $p_1 = (E_1, 0) = (m, 0)$. In Eq. (1) the three four-vectors $k_1, k_2, k_3$ designate the 4-vectors of photon momenta. Here and below $k_i = (\omega_i, k_i)$. For the real photons one finds $k_i^2 = 0$ (so-called on-shell condition) and $\omega_i^2 = k_i^2$ $(i = 1, 2, 3)$.

According to the rules of QED, the corresponding $S$–matrix element $(S_{fi})$ in the momentum space is

$$S_{fi} = \frac{e^3}{V^3 \sqrt{V}} \sqrt{\frac{m^2}{E_1 E_2}} \sqrt{\frac{(4\pi)^3}{8\omega_1 \omega_2 \omega_3}} (2\pi)^4 \delta^4(p_1 + p_2 - k_1 - k_2 - k_3) M_{fi}$$  \hspace{1cm} (2)

where $e$ is the electron charge, while $V$ is the finite normalization volume, $E_1$ and $E_2$ are the energies of the incident electron and positron, respectively. Also, in this equation $\omega_i$ $(i = 1, 2, 3)$ are the three frequencies/energies of the emitted photons. The notation $M_{fi}$ designates the matrix element which corresponds to the Feynman diagram of the three-photon annihilation (see, e.g., the diagram presented in § 89 of [7]). The explicit form of the matrix element $M_{fi}$ is

$$M_{fi} = \frac{1}{p_1 - k_1 - k_2 - m} \frac{1}{p_1 - k_1 - m} \frac{1}{p_1 - k_1 - k_3 - m} \frac{1}{p_1 - k_1 - m} \epsilon_3$$

$$+ \frac{1}{p_1 - k_1 - k_2 - m} \frac{1}{p_1 - k_1 - m} \frac{1}{p_1 - k_1 - k_3 - m} \frac{1}{p_1 - k_1 - m} \epsilon_1 + \ldots$$  \hspace{1cm} (3)
\[ + \frac{1}{p_1 - k_j - k_l - m} \epsilon_j \frac{1}{p_1 - k_l - m} \epsilon_l + \ldots \] u

where \((i, j, l) = (1, 2, 3)\). The \(v\) and \(u\) are the positron and electron bi-spinors, respectively, while \(k_i\) and \(\epsilon_i\) \((i = 1, 2, 3)\) are the momentum and polarization of the \(i\)-th photon. The total number of terms in the amplitude \(M\) equals six. Each of these six terms in Eq. (3) can be transformed in the following way, e.g., for the first term

\[ M_{321} = \frac{1}{p_1 - k_1 - k_2 - m} \frac{1}{p_1 - k_1 - m} \epsilon_1 + \frac{1}{p_1 - k_1 - k_2 - m} \epsilon_2 \frac{1}{p_1 - k_1 - m} \epsilon_1 \]

\[ (p_1 - k_1 + m) \epsilon_1 u = A_{321} \frac{1}{p_1 - k_1 - k_2 - m} \frac{1}{p_1 - k_1 - m} \epsilon_1 \]

where

\[ A_{321} = \frac{1}{4(p_1 \cdot k_1)(p_2 \cdot k_3)} \] (5)

Here and below, all matrix elements \(M_{ijk}\) and normalization factors \(A_{ijk}\) are designated with the use of three photon indexes which are uniformly related to the corresponding photon lines on Feynman diagram (by reading them from the left to the right).

The cross-section of the three-photon annihilation is

\[ d\sigma = \int \frac{V^2}{TV} \left| S_{fi} \right|^2 V \frac{d^3 k_1}{(2\pi)^3} V \frac{d^3 k_2}{(2\pi)^3} V \frac{d^3 k_3}{(2\pi)^3} \]

\[ = \frac{e^6}{2\omega_1^2 \omega_2^2 2\omega_3^2} \int \delta^4(p_1 + p_2 - k_1 - k_2 - k_3) |M_{fi}|^2 \frac{d^3 k_1}{2\omega_1} \frac{d^3 k_2}{2\omega_2} \frac{d^3 k_3}{2\omega_3} \]

where in our present case \(v = \frac{p_2}{E_2}\) is the velocity of incoming positron, while \(T\) designates the finite (but very large) time. To finish these calculations one needs to obtain the closed analytical formulas for the \(M_{fi}\) matrix element. In general, such calculations is not an easy task. However, to simplify these calculations and final expressions for the amplitudes \(M_{321}\) and \(M_{ijk}\) one can introduce a few additional conditions on the photon polarization 4-vectors \(\epsilon_i\) \((i = 1, 2, 3)\). Below we shall assume that the following conditions are obeyed for the three polarization 4-vectors of photons \(\epsilon_i\) \[ \epsilon_i \cdot k_i = 0 \quad , \quad \epsilon_i \cdot \epsilon_i = -1 \quad , \quad \epsilon_i \cdot p_1 = 0 \] (7)

where \(i = 1, 2, 3\). In this work \(p_1 = (m, 0)\) and we can choose \(\epsilon_i = (0, \varepsilon_i)\), where \(\varepsilon_i\) are the three unit-norm 3D-vectors.

Now, by applying the conditions, Eq. (7), and using relation \(ab = 2(a \cdot b) - ba\), where \(a\) and \(b\) are the two arbitrary 4-vectors, one finds

\[ (p_1 - k_1 + m)\epsilon_1 u = -k_1 \epsilon_1 u + \epsilon_1 (-p_1 + m) u = -k_1 \epsilon_1 u \]

(8)
since \((p_1 - m)u = 0\), where \(u\) is the electron bi-spinor. Analogously, since \(0 = \psi(p_2 + m)\), where \(\psi\) is the positron bispinor, we can simplify computations of Eq. (4). Finally, for the \(M_{321}\) matrix element one finds

\[
M_{321} = -A_{321} \cdot \psi[\epsilon_3 k_3 \epsilon_2 k_1 \epsilon_1]u + 2A_{321}(\epsilon_3 \cdot p_2)\psi[\epsilon_2 k_1 \epsilon_1]u
\]

\[
= A_{321} \cdot \psi[2(\epsilon_3 \cdot p_2)\epsilon_2 k_1 \epsilon_1 - \epsilon_3 k_3 \epsilon_2 k_1 \epsilon_1]u
\]

In the case of the \((i jl)\) diagram the analogous expression for the \(M_{ijl}\) matrix element takes the form

\[
M_{ijl} = A_{ijl} \cdot \psi[2(\epsilon_i \cdot p_2)\epsilon_j k_l \epsilon_l - \epsilon_l k_i \epsilon_j k_l \epsilon_l]u
\]

where \(i \neq j \neq l = (1,2,3)\) and

\[
A_{ijl} = \frac{1}{4(p_1 \cdot k_i)(p_2 \cdot k_l)}
\]

The conjugate amplitude \(M_{ijl}^{*}\) is

\[
M_{ijl}^{*} = A_{ijl} \cdot \psi[2(\epsilon_i \cdot p_2)\epsilon_l k_i \epsilon_j - \epsilon_l k_i \epsilon_j k_i \epsilon_l]v
\]

The expression for the \(|M|^2\) value is reduced to the sum of the six matrix element \(M_{ijl}^{*} M_{321}\), where \((i, j, l) = (1, 2, 3)\). The analytical formula for the \(M_{ijl}^{*} M_{321}\) matrix element averaged over the initial electron and positron states can be written in the form

\[
M_{ijl}^{*} M_{321} = A_{ijl} A_{321} \cdot B_{ijl} = \frac{A_{ijl} A_{321}}{16m^2}(B_1 - m^2 B_2)
\]

where

\[
B_1 = 4(\epsilon_i \cdot p_2)(\epsilon_3 \cdot p_2)B_{1a} - 2(\epsilon_i \cdot p_2)B_{1b} - 2(\epsilon_3 \cdot p_2)B_{1c} + B_{1d}
\]

and

\[
B_2 = 4(\epsilon_i \cdot p_2)(\epsilon_3 \cdot p_2)B_{2a} - 2(\epsilon_i \cdot p_2)B_{2b} - 2(\epsilon_3 \cdot p_2)B_{2c} + B_{2d}
\]

where the explicit expressions for the eight traces \(B_{1k}\) and \(B_{2k} (k = a, b, c, d)\) are

\[
B_{1a} = Tr[p_2 \epsilon_i k_l \epsilon_j p_1 \epsilon_2 k_1 \epsilon_1] , \quad B_{2a} = Tr[\epsilon_i k_l \epsilon_j \epsilon_2 k_1 \epsilon_1]
\]

(16)

\[
B_{1b} = Tr[p_2 \epsilon_i k_l \epsilon_j p_1 \epsilon_3 k_2 \epsilon_1] , \quad B_{2b} = Tr[\epsilon_i k_l \epsilon_j \epsilon_3 k_2 \epsilon_1]
\]

(17)

\[
B_{1c} = Tr[p_2 \epsilon_i k_l \epsilon_j k_i \epsilon_1 p_2 k_1 \epsilon_1] , \quad B_{2c} = Tr[\epsilon_i k_l \epsilon_j \epsilon_2 k_1 \epsilon_1]
\]

(18)

\[
B_{1d} = Tr[p_2 \epsilon_i k_l \epsilon_j k_i \epsilon_1 p_1 \epsilon_3 k_2 \epsilon_1] , \quad B_{2d} = Tr[\epsilon_i k_l \epsilon_j \epsilon_3 k_2 \epsilon_1]
\]

(19)
Thus, the problem of three-photon annihilation of the electron-positron pair at arbitrary energies of the colliding particles is reduced to the analytical computation of these eight traces. The explicit formulas for all individual traces, Eq. (16) - Eq. (19), as well as for $B_{ijl}$ can be obtained directly from the author (some restrictions may apply).

The formulas given above correspond to the case when the polarizations of all photons (i.e. $\epsilon_i, i = 1, 2, 3$) are known. If this is not the case, then in the formulas presented above one needs to compute the sums over polarizations of all final photons. Below, we perform the polarization summation using the standard replacement $\sum_{\lambda=0,3} \epsilon^{(\lambda)}_\mu \epsilon^{(\lambda)}_\nu = -g_{\mu\nu}$. However, the last condition in Eq. (7) ($\epsilon \cdot p_1 = 0$) which can contradict such a replacement. Following the procedure developed in [8], one finds that there is no contradiction in the electron rest frame. This means that the standard replacement $\sum_{\lambda=0,3} \epsilon^{(\lambda)}_\mu \epsilon^{(\lambda)}_\nu = -g_{\mu\nu}$ can be used in this study to perform the polarization summation.

After computing all traces summed over photon polarizations, the explicit expression for the $M_{ijl}^* M_{321}$ value is written in the form

$$M_{ijl}^* M_{321} = \frac{A_{ijl} \cdot A_{321}}{16m^2} D_{ijl}$$  \hspace{1cm} (20)

for $(i, j, l) = (1, 2, 3)$ and $D_{ijl}$ denotes the resulting traces computed for each of these cases.

The explicit formulas for $D_{ijl}$ are

$$D_{123} = 32m^2[2(k_1 \cdot k_3)^2 + (k_1 \cdot p_2)(k_3 \cdot p_1) + (k_1 \cdot p_1)(k_3 \cdot p_2) + (k_1 \cdot k_3)(2m^2 - p_1 \cdot p_2)] \hspace{1cm} (21)$$

$$D_{132} = 32m^2[(k_1 \cdot k_3)(k_2 \cdot p_1) - (k_1 \cdot p_2)(4k_2 \cdot k_3 + k_2 \cdot p_1 - 2k_2 \cdot p_2) + (k_1 \cdot p_1)(-k_2 \cdot k_3 \hspace{1cm} (22)$$

$$+ k_2 \cdot p_2) + (k_1 \cdot k_2)(k_3 \cdot p_1) + (k_1 \cdot k_2)(p_1 \cdot p_2)]$$

$$D_{213} = 32m^2[ -(k_1 \cdot k_2)(k_3 \cdot p_1) + (k_1 \cdot p_2)(k_3 \cdot p_1) + (k_1 \cdot p_1)(k_2 \cdot k_3 - k_3 \cdot p_2) \hspace{1cm} (23)$$

$$- 4(k_1 \cdot k_2)(k_3 \cdot p_1) + 2(k_1 \cdot p_2)(k_3 \cdot p_1) + (k_1 \cdot k_3)(k_2 \cdot p_1 + p_1 \cdot p_2)]$$

$$D_{231} = 64(k_1 \cdot p_2)[(k_1 \cdot p_1)(-m^2 - 2k_2 \cdot k_3 + k_2 \cdot p_2 + k_3 \cdot p_2) - (k_1 \cdot k_2 + k_1 \cdot k_3)(p_1 \cdot p_2) + (k_1 \cdot p_2)(k_2 \cdot p_1 + k_3 \cdot p_1 + 2p_1 \cdot p_2)] \hspace{1cm} (24)$$

$$D_{312} = 64(k_1 \cdot k_2)[m^4 + 2k_3 \cdot p_1(m^2 + k_3 \cdot p_2) + 2m^2(p_1 \cdot p_2) - (k_3 \cdot p_2)(m^2 + 4p_1 \cdot p_2)] \hspace{1cm} (25)$$

$$D_{321} = -128(k_1 \cdot p_2)[(k_1 \cdot p_2)(k_3 \cdot p_1) + (k_1 \cdot p_1)(m^2 - k_3 \cdot p_2) - (k_1 \cdot k_3)(k_3 \cdot p_1 + p_1 \cdot p_2)] \hspace{1cm} (26)$$

In actual computations the sum $(M_{123} + M_{132} + M_{213} + M_{231} + M_{312} + M_{321}) \cdot M_{321}$ must be multiplied by an additional factor (= degeneracy factor) $S = \frac{1}{3!} = \frac{1}{6}$. This factor is needed,
since the final state in the case of three-photon annihilation contains three identical photons. This means that the $|M_{fi}|^2$ term from Eq. (1) is represented in the form

$$|M_{fi}|^2 = \frac{1}{6}(M_{123} + M_{132} + M_{213} + M_{231} + M_{312} + M_{321}) \cdot M_{321}$$  \hfill (27)$$

Now, we want to obtain the explicit formula for the cross-section of the three-photon annihilation. First, consider the computation of the following auxiliary integral

$$I = \int \frac{d^3k_1}{2\omega_1} \frac{d^3k_2}{2\omega_2} \frac{d^3k_3}{2\omega_3} \delta^4(p_1 + p_2 - k_1 - k_2 - k_3) \cdot f(k_1, k_2, k_3)$$  \hfill (28)$$

where $f(x, y, z)$ is an arbitrary, in principle, function of three variables $x, y$ and $z$. The 3D-integral over the $k_3$ is reduced to the following four-dimensional integral with the use of the relation (see, e.g., [7])

$$\frac{d^3k_3}{2\omega_3} = \int_{-\infty}^{+\infty} d^4k_3 \delta(k_3 \cdot k_3 - 0) \Theta[(k_3)_0]$$  \hfill (29)$$

where $\Theta(x) = 1$, if $x \geq 0$ and zero otherwise. The notation $(b)_0$ means 0-component of the four-vector $b$. With the use of this formula the expression, Eq. (28), takes the form

$$dI = \int \frac{d^3k_1}{2\omega_1} \frac{d^3k_2}{2\omega_2} \delta[(p_1 + p_2 - k_1 - k_2)^2] \Theta(E_1 + E_2 - \omega_1 - \omega_2) \cdot f(k_1, k_2, p_1 + p_2 - k_1 - k_2)$$

$$= \frac{1}{4} \int \omega_1 d\omega_1 d\Omega_1 d\Omega_2 \int_0^a \omega_2 d\omega_2 f(k_1, k_2, p_1 + p_2 - k_1 - k_2) \delta[(p_1 + p_2 - k_1 - k_2)^2]$$  \hfill (30)$$

$$= \frac{1}{8} \int \omega_1 d\omega_1 d\Omega_1 d\Omega_2 \int_0^a \omega_2 d\omega_2 f(k_1, k_2, p_1 + p_2 - k_1 - k_2) \delta[m^2 + E_1 E_2 - p_1 \cdot p_2]$$

$$- (E_1 + E_2)(\omega_1 + \omega_2) + \omega_1 \omega_2 - k_1 \cdot k_2 + (k_1 + k_2) \cdot (p_1 + p_2)]$$

where $a = E_1 + E_2 - \omega_1 (\geq 0)$. The total number of variables in this (final) formula is five ($5 = 9 - 4$) as expected.

In our present case $p_1 = (m, 0)$, i.e. $E_1 = m$ and such a substitution simplifies all following formulas. In particular, for the auxiliary integral $I$ one finds

$$dI = \frac{1}{8} \omega_1 d\omega_1 d\Omega_1 d\Omega_2 \cdot F \cdot \frac{m^2 + mE_2 - (m + E_2)\omega_1 + |p_2| \omega_1 \cos \Theta_1}{(m + E_2 + |p_2| \cos \Theta_2 + \omega_1 - \omega_1 \cos \Psi_{12})^2}$$  \hfill (31)$$

$$= \frac{1}{8} F \cdot \frac{m^2 + mE_2 - (m + E_2)\omega_1 + |p_2| \omega_1 \cos \Theta_1}{(m + E_2 + |p_2| \cos \Theta_2 + \omega_1 - \omega_1 \cos \Psi_{12})^2} \cdot \omega_1 d\omega_1 \sin \Theta_1 d\phi_1 \sin \Theta_2 d\phi_2$$

where $|p_2| = \sqrt{E_2^2 - m^2}$ and notation $F$ stands for the function $f$ from Eq. (30) which also contains all mentioned substitutions for the $k_3$ and $\omega_2$ variables. The notations $\Theta_1$ and $\Theta_2$
mean the two angles between the positron and first and second photon, respectively. The notation \( \Psi_{12} \) designates the angle between the two photons (photons 1 and 2). Computation of the \( \cos \Psi_{12} \) value is performed with the use of addition theorem for spherical harmonics (see, e.g., [9])

\[
\cos \Psi_{12} = \cos \Theta_1 \cos \Theta_2 + \sin \Theta_1 \sin \Theta_2 \cos (\phi_1 - \phi_2)
\]

(32)

\[
= \cos \Theta_1 \cos \Theta_2 + \sin \Theta_1 \sin \Theta_2 \cos \phi_2 - \sin \Theta_1 \sin \Theta_2 \sin \phi_1 \sin \phi_2
\]

The following integration over four angular variables does not present any difficulty.

Now, by using the explicit formulas Eq.(5), Eq.(6) and Eq.(21) - Eq.(26) one can finish these calculations and obtain the final formulas for the cross-section of the three-photon annihilation. The complete formula which contains all terms is extremely complicated. In fact, its derivation has been made with the use a symbolic algebra system, such as Maple [10]. Here we want to illustrate our calculations only for one term in \( |M|^2 \). Calculations of other five terms are almost identical. Let us present the final formula for the differential cross-section of the three-photon annihilation. In the electron rest frame one finds from Eq.(6) and Eq.(31)

\[
\frac{d\sigma}{d\omega_1 d\Theta_1 d\Theta_2} = \frac{e^6}{(2\pi)^5} \frac{m\omega_1 (4\pi)^3}{E_2} \left| M \right|^2 \frac{m^2 + mE_2 - (m + E_2)\omega_1 + |p_2| \omega_1 \cos \Theta_1}{(m + E_2 + |p_2| \cos \Theta_2 + \omega_1 - \omega_1 \cos \Psi_{12})^2}
\]

(33)

where \( |M|^2 = \frac{1}{6}(M_{123}^* + M_{132}^* + M_{213}^* + M_{231}^* + M_{312}^* + M_{321}^*) \cdot M_{321} \). For simplicity, consider only the first term \( M_{123}^* \cdot M_{321} \) (analysis of other terms is very similar). For the \( M_{123}^* \cdot M_{321} \) term one finds from formulas presented above

\[
M_{123}^* M_{321} = A_{123} \cdot A_{321} D_{123} = \frac{8Q}{R}
\]

(34)

where

\[
Q = 2[k_1 \cdot (p_1 + p_2 - k_1 - k_2)]^2 + (k_1 \cdot p_2)[p_1 \cdot (p_1 + p_2 - k_1 - k_2)] + (k_1 \cdot p_1)[p_2 \cdot (p_1
\]

(35)

\[
+ p_2 - k_1 - k_2)] + (2m^2 - p_1 \cdot p_2)(|k_1 \cdot (p_1 + p_2 - k_1 - k_2)]
\]

\[
= 2\omega_1^2 [m + E_2 - |p_2| \cos \Theta_1 - \omega_2 + \omega_2 \cos \Psi_{12}]^2 m \omega_1 (m + E_2 - \omega_1 - \omega_2)(E_2 - |p_2| \cos \Theta_2)
\]

\[
+ m \omega_1 [m^2 + E_2 m + |p_2| \omega_1 \cos \Theta_1 - |p_2| \omega_2 \cos \Theta_2 - E_2 (\omega_1 + \omega_2)] + m \omega_1 (2m - E_2)(m + E_2
\]

\[
- |p_2| \omega_2 \cos \Theta_1 + \omega_2 \cos \Psi_{12})
\]
and

\[ R = [p_1 \cdot (p_1 + p_2 - k_1 - k_2)](p_2 \cdot k_1)[p_2 \cdot (p_1 + p_2 - k_1 - k_2)] \tag{36} \]

\[ = m^2 E_2 \omega_1^2 (m + E_2 - \omega_1 - \omega_2)(m^2 + mE_2 - E_2 \omega_1 - E_2 \omega_2 + |p_2| \omega_1 \cos \Theta_1 + |p_2| \omega_2 \cos \Theta_2) \]

where the function \( \omega_2 \) is

\[ \omega_2 = \frac{m^2 + mE_2 - (m + E_2) \omega_1 + |p_2| \omega_1 \cos \Theta_1}{m + E_2 + |p_2| \cos \Theta_2 + \omega_1 - \omega_1 \cos \Psi_{12}} \tag{37} \]

and \(|p_2| = \sqrt{E_2^2 - m^2}\). Note that all these formulas contain only five variables already mentioned above \( \omega_1, \cos \Theta_1, \cos \Theta_2 \) and \( \cos \Psi_{12} \) (i.e. \( \phi_1 \) and \( \phi_2 \)). The same conclusion is true about the five other terms in the formula for the \(|M|^2\) factor in Eq. (39). Formally, after the integration over these five variables \((\omega_1, \cos \Theta_1, \cos \Theta_2, \phi_1 \) and \( \phi_2 \)) the final expression (total cross-section of the three-photon annihilation of the \((e^-, e^+)-\)pair) is the function of only one actual variable \( \gamma_p = \frac{E_p}{m} \), where \( \gamma_p \) is the Lorentz gamma-factor of the incoming (= fast) positron. The well known Dirac’s formula for the total cross-section of the two-photon annihilation of the \((e^-, e^+)-\)pair also depends upon the positron gamma-factor \( \gamma_p \) only, if it is written in the electron rest frame [1].

The formulas given above allow one to describe and determine the angular and energy distribution of the photons emitted during three-photon annihilation of the electron-positron pair (= \((e^-, e^+)-\)pair). Our results are obtained in the electron rest frame. The energy of the incident positron \( E_2 (\equiv E_p) \) can be arbitrary. The analytical expression has finally been derived for the differential cross-section of three-photon annihilation of the electron-positron pair. It is shown that such a cross-section is an explicit function of the five variables \((\omega_1, \cos \Theta_1, \cos \Theta_2, \phi_1 \) and \( \phi_2 \)) which describe three-photon annihilation of the \((e^-, e^+)-\)pair. In fact, our formula for the differential cross-section of three-photon annihilation of the electron-positron pair can be simplified even further, since it contains only four actual variables \((\omega_1, \cos \Theta_1, \cos \Theta_2 \) and \( \phi_1 - \phi_2 \)). In our next study we want to consider the four-photon annihilation of the \((e^-, e^+)-\)pair.

A. The non-relativistic limit of the differential cross-section

Let us obtain the explicit formula for the differential cross-section \( d\sigma_{3\gamma} \) at non-relativistic energies of the colliding particles (or positron in our case). First, note that in the non-
relativistic limit one finds for the $\omega_2$ frequency

$$\omega_2 = \frac{m^2 + mE_2 - (m + E_2)\omega_1}{(m + E_2 + \omega_1 - \omega_1 \cos \Psi_{12})^2} = \frac{(m + E_2)(m - \omega_1)}{m + E_2 + \omega_1 - \omega_1 \cos \Psi_{12}}$$  \hspace{1cm} (38)$$

In other words, in the non-relativistic limit the $\omega_2$ frequency does not depend any photon-positron angles $\Theta_1, \phi_1$ and/or $\Theta_2, \phi_2$. However, it explicitly depends upon the inter-electron angle $\cos \Psi_{12}$.

The final formula for the differential cross-section of the three-photon annihilation is written in the form (in the electron rest frame)

$$\frac{d\sigma_{3\gamma}}{d\omega_{1}d\Omega_{1}d\Omega_{2}} = \frac{2e^6}{\pi^2 m^2 \gamma_p |\mathbf{v}|} |M| \frac{m^2 + mE_2 - (m + E_2)\omega_1 + |\mathbf{p}_2| \omega_1 \cos \Theta_1}{(m + E_2 + |\mathbf{p}_2| \cos \Theta_2 + \omega_1 - \omega_1 \cos \Psi_{12})^2}$$

$$= \frac{2e^6}{6 \cdot 8\pi^2 m^2 \gamma_p |\mathbf{v}|} |(M_{123} + mM_{132} + M_{213} + M_{231} + M_{312} + M_{321}) \cdot M_{321}| \frac{d\omega_2}{d(\cos \Psi_{12})}$$  \hspace{1cm} (39)$$

where $|M|^2 = (M_{123} + mM_{132} + M_{213} + M_{231} + M_{312} + M_{321}) \cdot M_{321}$ (the factor $\frac{1}{6}$ has been moved in front of Eq. (39)). Now, consider the non-relativistic limit of the $d\sigma_{3\gamma}$ cross-section. In the non-relativistic limit we can integrate over all angular variables, but one ($\Psi_{12}$). This gives us an additional factor $8\pi^2$ and the formula, Eq. (39) takes the form

$$d\sigma_{3\gamma} = \frac{2e^6}{6m^2 |\mathbf{v}|} |M|^2 \frac{d\omega_1}{d(\cos \Psi_{12})} \frac{d\omega_2}{d(\cos \Psi_{12})} d(\cos \Psi_{12}) = \frac{2e^6}{6m^2 |\mathbf{v}|} |M|^2 d\omega_1 d\omega_2$$  \hspace{1cm} (40)$$

where the expression $\omega_3 = 2m - \omega_1 - \omega_2$ must be used everywhere in the $|M|^2$ factor. The formula Eq. (40) essentially coincides with the expression (89.14) from [7]. Such a coincidence will be exact, if we can extract an additional factor $4$ from $|M|^2$.

The final step in computations of the non-relativistic limit for the cross-section $d\sigma_{3\gamma}$ is to obtain the explicit formula for the $|M|^2 = (M_{123} + mM_{132} + M_{213} + M_{231} + M_{312} + M_{321}) \cdot M_{321}$. From the non-relativistic relations $k_i + k_j = 2m - k_l$ one finds that $k_i \cdot k_j = (2m - k_l)^2$, where $(i,j,l) = (1,2,3)$. Now, we can find the analytical expression for the $m^2(M_{123} + mM_{132} + M_{213} + M_{231} + M_{312} + M_{321}) \cdot M_{321}$. For instance, the formula for the first term (i.e. for the $m^2M_{123} \cdot M_{321}$ term) is

$$m^2M_{123} \cdot M_{321} = \frac{1}{256m^4 \omega_1^2 \omega_3^2} \cdot 64m^4 \left[ 4(m - \omega_2)^2 + \omega_1 \omega_3 + m(m - \omega_2) \right]$$

$$= \frac{4(m - \omega_2)^2 + \omega_1 \omega_3 + m(m - \omega_2)}{4\omega_1^2 \omega_3^2}$$  \hspace{1cm} (41)$$

The derivation of explicit formulas for other six matrix elements $m^2M_{ijl} \cdot M_{321}$ is also straightforward. In turn, the known formulas for these matrix elements allows one to obtain the
non-relativistic limit of the differential cross-section of the three-photon annihilation of the
$$(e^-, e^+)$$–pair.

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