Dynamic self-triggered control for nonlinear systems with delays *

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Abstract: Self-triggered control (STC) is a resource efficient approach to determine sampling instants for Networked Control Systems (NCS). Recently, a dynamic STC strategy based on hybrid Lyapunov functions for nonlinear NCS has been proposed in Hertneck and Allgöwer (2021b), however with limitation to NCS without transmission delays. In this paper, we extend this strategy for nonlinear NCS with transmission delays. The capability to handle systems with delays makes it possible to use the resulting dynamic STC mechanism in many practical scenarios where instant transmissions without delays cannot be guaranteed. The proposed dynamic STC mechanism guarantees stability despite bounded transmission delays. The effectiveness of the mechanism is illustrated with a numerical example and compared to state-of-the-art literature.

Keywords: Event-triggered and self-triggered control, Control under communication constraints

1. INTRODUCTION

Event-triggered control (ETC) and self-triggered control (STC) are resource efficient approaches to determine sampling instants for Networked Control Systems (NCS) (cf. Heemels et al. (2012)). Whilst sampling instants for ETC are determined by a state-dependent trigger rule that is monitored continuously, for STC at each sampling instant the next sampling instant is determined using available state information. It has been shown in Mazo et al. (2009); Anta and Tabuada (2010) that STC can reduce the network load for NCS in contrast to the classical periodic sampling significantly.

Whilst STC for linear systems is well studied (see, e.g., Heemels et al. (2012); Brunner et al. (2019) and the references therein), fewer results are available for nonlinear systems. In Anta and Tabuada (2010); Delimpaltadakis and Mazo (2020, 2021), the state space is divided into isochronous manifolds with the same sampling interval. In Benedetto et al. (2013); Tiberi and Johansson (2013); Theodosis and Dimarogonas (2018), Lipschitz continuity properties are used to determine sampling instants such that the decrease of a Lyapunov function can be guaranteed. Recently, in Hertneck and Allgöwer (2021b); Hertneck and Allgöwer (2021c), hybrid Lyapunov functions and a dynamic variable that captures the past system behavior are used to determine sampling instants.

With the exception of Benedetto et al. (2013); Theodosis and Dimarogonas (2018), the aforementioned approaches for nonlinear systems neglect delays between sampling of the states and the application of the respective feedback. However, neglecting such delays is typically not realistic for NCS, where delays may, e.g., arise from bandwidth limitations and from congestion of packets in the network due to a high network load.

In this paper, we present a dynamic STC mechanism based on hybrid Lyapunov functions for nonlinear NCS with transmission delays. The mechanism is based on Hertneck and Allgöwer (2021b). However, due to the delays, significant modifications are required. In contrast to the delay-free case, the sampling induced error is not reset to zero at sampling instants. As a result, more complex hybrid Lyapunov functions are required, that we adapt to our setup from the framework of Heemels et al. (2010). In contrast to the approaches from the literature (Benedetto et al. (2013); Theodosis and Dimarogonas (2018)), the proposed approach can guarantee asymptotic stability of the origin instead of only convergence to a set around the origin. Moreover, we demonstrate with a numerical example from Theodosis and Dimarogonas (2018), that the proposed dynamic STC mechanism can lead to significantly less sampling instants than the mechanism from Theodosis and Dimarogonas (2018).

The remainder of this paper is organized as follows. In Section 2, we present the considered setup. Hybrid Lyapunov functions for systems with delays are discussed in Section 3. The details of the proposed dynamic STC mechanism are presented and stability guarantees are derived in Section 4. In Section 5, a numerical example is given. The paper is concluded in Section 6.
Notation and definitions

The real numbers are denoted by \( \mathbb{R} \) and the nonnegative real numbers by \( \mathbb{R}_+ \). The natural numbers are denoted by \( \mathbb{N} \), and we define \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \). Moreover, we define the even natural numbers including 0 by \( \mathbb{N}_e := \{0, 2, 4, 6, \ldots\} \).

A continuous function \( \alpha : \mathbb{R}_+ \to \mathbb{R}_+ \) is a class \( K \) function if it is strictly increasing and \( \alpha(0) = 0 \). It is a class \( K_{\infty} \) function if it is of class \( K \) and it is unbounded. A continuous function \( \beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \) is a class \( KL \) function, if \( \beta(\cdot, t) \) is of class \( K \) for each \( t \geq 0 \) and \( \beta(q, \cdot) \) is nonincreasing and satisfies \( \lim_{q \to \infty} \beta(q, t) = 0 \) for each \( q \geq 0 \).

A function \( \beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \) is a class \( KCL \) function if for each \( r \geq 0 \), \( \beta(\cdot, \cdot, r) \) and \( \beta(\cdot, r, \cdot) \) belong to class \( KL \).

We use (Carnevale et al., 2007, Definitions 1-3), that are originally taken from Goebel and Teel (2006), to characterize a hybrid model of the considered NCS and corresponding hybrid time domains, trajectories and solutions. Moreover, we adapt the definitions of maximal solutions and \( t \)-completeness from Goebel and Teel (2006).

2. SETUP

In this section, we describe the setup of the paper and give a precise problem statement. The plant is given by

\[
\dot{x} = f_p(x, \hat{u}),
\]

where \( x(t) \in \mathbb{R}^{n_x} \) is the plant state with initial condition \( x(0) = x_0 \) and \( \hat{u}(t) \in \mathbb{R}^{n_u} \) is the last input that has been received by the plant. The input is generated by the static state-feedback controller \( u = g_c(x) \). The function \( f_p \) is assumed to be continuous and \( g_c \) is assumed to be continuously differentiable.

The plant state \( x(t_k) \) is sampled at sampling instants \( t_k \), \( k \in \mathbb{N}_e \), that are determined by a sampling mechanism to be specified later. At each sampling instant, the input \( u(t_k) \) is computed based on the sampled state \( x(t_k) \) and sent over the network. The values arrive at the actuator after a transmission delay of \( \tau_k \) time units, resulting in an update of \( \hat{u} \) at the corresponding times \( t_{k+1} = t_k + \tau_k \). We assume that the maximum delay is bounded as \( \tau_k \leq \tau_{\text{mad}} \) for all \( k \in \mathbb{N}_e \) and some known \( \tau_{\text{mad}} > 0 \). Between update times, the values of \( \hat{u} \) are kept constant, which resembles a zero-order hold (ZOH) scenario. We assume for simplicity, that the plant is sampled at \( t_0 = 0 \) and that initially \( \hat{u}(0) = g_c(0) \) for each \( t \in [0, \tau_0] \). For analysis purposes, we denote by \( \hat{x} \) the state that is corresponding to the current value of \( \hat{u} \), i.e., \( \hat{x}(t) = x(t_k) \), \( t \in [t_k + \tau_k, t_{k+2} + \tau_{k+2}] \) and \( \hat{x}(t) = 0 \) for \( t \in [0, \tau_0] \), resembling again a ZOH scenario.

The sampling induced delay is denoted by \( e := \hat{x} - x \).

Similar as in Hertneck and Allgöwer (2021b), we consider in this paper a dynamic STC mechanism that determines at each sampling instant \( t_k \) the next sampling instant \( t_{k+1} \) using current states of the plant and an additional dynamic variable \( \eta(t) \in \mathbb{R}^{n_\eta} \) that encodes the past system behavior.

The dynamic STC mechanism can thus be described by \( t_{k+2} = \Gamma(t_k, \eta(t_k), \eta(t_{k+1})), \) where \( \Gamma : \mathbb{R}_+ \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_\eta} \to [\tau_{\min}, \infty) \). The sampling mechanism will be designed such that \( t_{\min} \geq \tau_{\text{mad}} \), resembling the so-called small delay case.

The dynamic variable is updated at sampling instants based on its current value and on current state information and remains constant between sampling instants. Its update can thus be described as \( \eta(t_{k+2}) = S(\eta(t_k), x(t_k), e(t_k)) \) for some \( S \), where \( S : \mathbb{R}^{n_x} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_\eta} \to \mathbb{R}^{n_\eta} \).

We model the NCS as a hybrid system \( H_{\text{STC}} \). For that, we introduce some auxiliary variables. We use the timer \( \tau(t) \in \mathbb{R}_+ \) to keep track of the time since the last sampling instant, the variable \( \tau_{\text{max}}(t) \in \mathbb{R}_+ \) that encodes the next sampling interval, a variable \( s(t) \in \mathbb{R}^{n_s} \) to store the value of \( x(t_k) \) until the next input update and a boolean variable \( \ell(t) \in \{0, 1\} \) to keep track of whether the next event for the NCS is a sampling instant or an update of the input. In particular, \( \ell = 0 \) represents the situation that the next event will be related to a sampling instant and \( \ell = 1 \) represents that the next event will be an update of \( \hat{u} \). Using \( \xi := [x^\top, e^\top, s^\top, \eta^\top, \tau, \tau_{\text{max}}, \ell]^\top \), \( f(x, e) := f_p(x, k(x + e)) \) and \( g(x, e) := -f(x, e) \), we obtain the hybrid system \( H_{\text{STC}} \)

\[
\begin{align*}
\dot{\xi} &= F(\xi) \\
\xi^+ &= G(\xi) \quad \in D
\end{align*}
\]

with \( F(\xi) := (f(x, e), g(x, e), 0, 0, 0, 0, 0, 0)^\top \),

\[
G(\xi) := \begin{cases}
(x^\top, e^\top, -S(\eta, x, e)^\top, 0, 1) & \ell = 0 \\
(x^\top, (s + e)^\top, -(s + e)^\top, 1) & \ell = 1
\end{cases}
\]

where \( C := \{0 \leq \ell \leq 1 \} \) is the set of allowed values for \( \ell \). Here the choice of \( s^* \) is made as in Heemels et al. (2010) to simplify analysis later.

The jumps of \( H_{\text{STC}} \) represent sampling events and update events in an alternating fashion. This justifies the choice of \( k = 2j \in \mathbb{N}_e \), where \( j \) describes the jumps of the hybrid system, to characterize sampling instants. We further describe the hybrid time before the sampling or update event at time \( t_j \) by \( r_j = (t_j, j-1) \) and the hybrid time directly after the event by \( r^+_j = (t_j, j) \).

In this paper, our goal is to design functions \( \Gamma \) and \( S \), such that asymptotic stability of the origin of \( H_{\text{STC}} \) is guaranteed for a region of attraction \( \mathcal{R} \) according to the following definition.

**Definition 1.** For the hybrid system \( H_{\text{STC}} \), the set \( \{x, e, s, \eta, \tau, \tau_{\text{max}}, \ell : x \in \mathcal{R} \} \) is asymptotically stable with region of attraction \( \mathcal{R} \subseteq \mathbb{R}^{n_x} \), if there exists \( \beta \in \mathcal{K}L \) such that all corresponding maximal solutions \( x(t, 0) \in \mathcal{R} \) are \( t \)-complete and satisfy for all \( (t, j) \in \text{dom} \xi \)

\[
\begin{bmatrix}
\|x(t, j)\| \\
\|e(t, j)\| \\
\|s(t, j)\| \\
\|
eta(t, j)\|
\end{bmatrix} \leq \beta
\begin{bmatrix}
\|x(0, 0)\| \\
\|e(0, 0)\| \\
\|s(0, 0)\| \\
\|\eta(0, 0)\|
\end{bmatrix} \\
\|t, j\|
\end{bmatrix}.
\]

3. HYBRID LYAPUNOV FUNCTIONS FOR SYSTEMS WITH DELAYS

In this section, we present how a bound on a hybrid Lyapunov function similar to the one used in Hertneck and Allgöwer (2021b) can be obtained for the setup with
delays that is considered in this paper. The bound will be useful to determine sampling instants. We adapt the following condition from [Heemels et al., 2010, Condition IV.1] to our setup.

**Condition 1.** Consider some sets \( \mathcal{X} \subseteq \mathbb{R}^{n_x} \) and \( \mathcal{E} \subseteq \mathbb{R}^{n_e} \). There exist a function \( \tilde{W} : \{0,1\} \times \mathcal{E} \times \mathcal{E} \to \mathbb{R}_{\geq 0} \) with \( \tilde{W}(\ell,\ldots) \) locally Lipschitz for all \( \ell \in \{0,1\} \), a locally Lipschitz function \( \tilde{V} : \mathcal{X} \to \mathbb{R}_{\geq 0} \), \( \mathcal{K}_\infty \) functions \( \tilde{\beta}_1, \tilde{\beta}_2, \tilde{\beta}_W \) and \( \tilde{\beta}_V \), continuous functions \( H_1 : \mathcal{X} \times \mathcal{E} \to \mathbb{R}_{\geq 0} \), constants \( L_i \in \mathbb{R}, \gamma_i > 0 \) for \( i \in \{0,1\}, \epsilon \in \mathbb{R} \) and \( \lambda \in [0,1] \) such that
\[
\tilde{W}(1, \epsilon, e, s) \leq \lambda \tilde{W}(0, e, s) \quad (3a)
\]
and for all \( \epsilon \in \mathcal{E}, s \in \mathcal{E} \) and \( \ell \in \{0,1\} \),
\[
\begin{aligned}
\frac{\partial \tilde{W}(\ell, e, s)}{\partial e}, g(x,e) &\leq L_{\ell} \tilde{W}(\ell, e, s) + H_{\ell}(x,e) \quad (4)
\\
\nabla \tilde{V}(x), f(x,e) &\leq -\tilde{V}(x) - H_1^2(x,e) + \gamma_1^2 \tilde{W}(\ell, e, s) \quad (5)
\end{aligned}
\]
holds for all \( x \in \mathcal{X}, s \in \mathcal{E}, \ell \in \{0,1\} \) and almost all \( e \in \mathcal{E}, s \in \mathcal{E} \), \( \ell \in \{0,1\} \),
\[
\begin{aligned}
\|g(x,e)\| \leq L\|e\| + H(x,e) \quad (6)
\\
\nabla \tilde{V}(x), f(x,e) &\leq -\epsilon \tilde{V}(x) - H_1^2(x,e) + \gamma_1^2 \|e\|^2 \quad (7)
\end{aligned}
\]
In particular, we obtain from [Heemels et al., 2010, Theorem V.3] for \( W(e) = \|e\| \) and any \( \lambda \in (0,1) \) that if Assumption 1 holds, then Condition 1 holds with \( \tilde{V}(x) = V(x), \tilde{W}(\ell, e, s) = \max \{\|e\|, \|e+s\|\} \), \( \ell = 0 \), \( \max \{\lambda \|e\|, \|e+s\|\} \), \( \ell = 1 \), \( H_0(x,e) = H_1(x,e) = H(x,e), L_0 = L, L_1 = L, L_0 = \frac{L}{\gamma_0}, L_1 = \gamma_1 = \frac{L}{\gamma_1} \). Assumption 1 can typically be verified for various different choices of \( \epsilon, \gamma \) and \( L \) for the same function \( \tilde{V}(x) \), leading therefore also to various different parameters \( \epsilon, \gamma_0, \gamma_1, L_0, L_1 \) in Condition 1.

Using Condition 1, we next derive a bound on a hybrid Lyapunov function for the time between two sampling instants. This bound will subsequently be used by the dynamic STC mechanism to determine sampling instants. For the bound, we use the following definition.

**Definition 2.** Consider the differential equations
\[
\begin{aligned}
\dot{\phi}_0 &= -2 \left( L_0 + \frac{\epsilon}{\gamma_0} \right) \phi_0 - \gamma_0 (\phi_0^2 + 1) \quad (8a) \\
\dot{\phi}_1 &= -2 \left( L_1 + \frac{\epsilon}{\gamma_1} \right) \phi_1 - \gamma_1 (\phi_1^2 + 1) \quad (8b)
\end{aligned}
\]
and some \( c_{\ell} \in \mathbb{R}_{>0} \). If
\[
\gamma_1 \phi_1(\tau) \geq \gamma_0 \phi_0(\tau) > 0 \quad \text{for all } 0 \leq \tau \leq \tau_{\text{mad}}
\]
holds, then we define the function \( T_{\text{max}}(\gamma_0, \gamma_1, L_0, L_1, \phi_0(0), \phi_1(0), \lambda, c_{\ell}, \tau_{\text{mad}}) \) as the maximum time \( T_{\text{max}} \geq \tau_{\text{mad}} \) such that
\[
\gamma_0 \phi_0(\tau) \geq \lambda^2 c_{\ell} \quad \text{for all } 0 \leq \tau \leq T_{\text{max}}
\]
holds. Otherwise, we set
\[
T_{\text{max}}(\gamma_0, \gamma_1, L_0, L_1, \phi_0(0), \phi_1(0), \lambda, c_{\ell}, \tau_{\text{mad}}) = 0.
\]

**Proposition 1.** Consider any maximal solution \( \xi \) to \( \mathcal{H}_{\text{STC}} \) at time \( t_k = (t_k, k) \) for some \( k \in \mathbb{N} \) and let Condition 1 hold for \( \gamma_0, \gamma_1, L_0, L_1 \) and \( \epsilon \) on \( \mathcal{X} := \{x | \tilde{V}(x) < c_{\ell}\} \) and \( \mathcal{E} := \{x - x | x \in \mathcal{X}, x \in \mathcal{X}\} \) for some \( c_{\ell} \in \mathbb{R}_{>0} \). Suppose \( t_{k+2} - t_k \leq T_{\text{max}}(\gamma_0, \gamma_1, L_0, L_1, \phi_0(0), \phi_1(0), \lambda, c_{\ell}, \tau_{\text{mad}}) \) for some \( c_{\ell} > 0 \). Consider the hybrid Lyapunov function
\[
U(\xi) := \tilde{V}(x) + \gamma_1 \phi_1(\tau) \tilde{W}(\ell, e, s)
\]
and the maximum possible time between sampling instants and the maximum possible value for \( \tau_{\text{mad}} \) increase as \( \epsilon \) increases and decrease as \( \epsilon \) decreases.

The dynamic STC mechanism will exploit different parameter combinations for Condition 1 and their effect on the respective functions \( U \) and \( T_{\text{max}} \) by searching at each sampling interval a parameter combination for which a certain bound on the system state can be guaranteed for a preferably large sampling interval based on (12). To handle the transition between the different parameter combinations, we will later use the constant \( c_{\ell} \).

Note also, that if \( \epsilon > 0 \), \( c_{\ell} = \gamma_1 \phi_1(0) > 0 \) and \( T_{\text{max}}(\gamma_0, \gamma_1, L_0, L_1, \phi_0(0), \phi_1(0), \lambda, c_{\ell}, \tau_{\text{mad}}) \geq \tau_{\text{mad}} \), Proposition 1 can be used to obtain a stability guarantee for \( \mathcal{H}_{\text{STC}} \) for periodic sampling with sampling interval \( T_{\text{max}}(\gamma_0, \gamma_1, L_0, L_1, \phi_0(0), \phi_1(0), \lambda, c_{\ell}, \tau_{\text{mad}}) \).

Footnote 2: This is essentially the same Assumption as [Hertneck and Allgöwer, 2021b, Assumption 1], for which a thorough discussion can be found in [Hertneck and Allgöwer, 2021b].

Footnote 3: Note that \( \mathcal{E} = \{x - x | x \in \mathcal{X}, x \in \mathcal{X}\} \) is the Minkowski sum of \( \mathcal{X} \) and \( -\mathcal{X} \).

**Proof:** See Appendix A.1

Proposition 1 delivers a bound on the function \( U(\xi) \). Note that the bound as well as the function \( U(\xi) \) depend on the actual parameters in Condition 1. Particularly, if \( \epsilon > 0 \), then the bound is decreasing and if \( \epsilon < 0 \), the bound is increasing as time increases. Different values of \( \epsilon \) in Condition 1 also lead to different values for \( \gamma \), \( L_0 \) and \( L_1 \), which influence \( T_{\text{max}}(\gamma_0, \gamma_1, L_0, L_1, \phi_0(0), \phi_1(0), \lambda, c_{\ell}, \tau_{\text{mad}}) \). In general, the maximum possible time between sampling instants and the maximum possible value for \( \tau_{\text{mad}} \) increase as \( \epsilon \) increases and decrease as \( \epsilon \) decreases.
4. DYNAMIC STC FOR SYSTEMS WITH DELAYS

In this section, we present the details of the dynamic STC mechanism for nonlinear NCSs with delays. Note that the general approach for the dynamic STC mechanism is adapted from Hertneck and Allgöwer (2021b), where systems without delays were considered. 

4.1 General idea for dynamic STC for systems with delays

We assume subsequently, that there are \( n_p \) different parameter sets \( \epsilon_p, \gamma_{0,p}, \gamma_{1,p}, L_0,p, L_1,p, \phi_0,p(0), \phi_1,p(0), p \in \{1, \ldots, n_p\} \), for which Condition 1 holds for \( \mathcal{X} \) and \( \mathcal{E} \) for the same functions \( \mathcal{V} \) and \( \mathcal{W} \) and the same \( \lambda \in (0,1) \). For simplicity, we use subsequently the abbreviation \( \mathcal{P}_p = (\epsilon_p, \gamma_{0,p}, \gamma_{1,p}, L_0,p, L_1,p, \phi_0,p(0), \phi_1,p(0)) \).

From Proposition 1, we obtain for each parameter set a bound on a different function \( U \) with different differential equations in Definition 2 for \( \phi_1 \) and \( \phi_2 \). We thus define \( \phi_{0,p} \) and \( \phi_{1,p} \) as the solutions to (8) and \( U_p \) as the respective function \( U \) for parameter set \( p \), i.e., \( U_p(\xi) = \mathcal{V}(x) + \gamma_{0,p}(0) \phi_{0,p}(\tau) \mathcal{W}(e, \epsilon, s) \). Further we make the following assumption on one of the parameter sets to which we assign the index 1.

Assumption 2. It holds that \( e_1 > 0 \). Moreover, for \( \mathcal{P}_1 \), it holds that \( T_{\text{max}}(\mathcal{P}_1, \lambda, \gamma_{1,1}, \mathcal{I}_1, 0), \tau_{\text{mad}} \geq \tau_{\text{mad}} \).

Assumption 2 ensures that there is at least one parameter set for which the corresponding function \( U_1 \) can be used as a hybrid Lyapunov function for periodic sampling with sampling interval larger or equal to \( \tau_{\text{mad}} \). It will be used as a back-up by the dynamic STC mechanism to derive stability guarantees.

We will further use the function \( U_1 \) as reference for the dynamic STC mechanism. In particular, the dynamic STC mechanism will search at sampling instant \( r_k \) for \( p \in \{1, \ldots, n_p\} \) for which Proposition 1 ensures using the respective parameter set that for a chosen \( m \in \mathbb{R} > 0 \)

\[
U_1(\xi(r_{k+2}^-)) \leq e^{-e_1(t_{k+2}^- - t_k)} \sum_{i=0}^{m-1} U_1(\xi(r_{k-2}^-)) \tag{14}
\]

holds for a preferably large value of \( r_{k-2} \). Notice that this is an adaptation of the mechanism from Hertneck and Allgöwer (2021b) to our setup with delays. Different to Hertneck and Allgöwer (2021b), we use the function \( U_1 \) instead of \( V \), which is required since the sampling induced error is not reset to 0 at sampling instants due to the delays.

The dynamic STC mechanism uses the dynamic variable to store past values of \( U_1 \) in order to evaluate the right-hand side of (14). Note that \( U_1(\xi(r_{k}^+)) = \mathcal{V}(x(r_k)) + \gamma_{1,1} \phi_{1,1}(0) \mathcal{W}(1, e(r_k), -e(r_k)), i.e., the value of \( U_1(\xi(r_{k}^+)) \) directly after a sampling event can be determined using only the values of \( x(r_k) \) and \( e(r_k) \) at the sampling event.

We can thus choose \( n_m = m - 1 \) and define

\[
S(\eta, x, e) := (\eta_2 \ldots \eta_{m-1} \mathcal{V}(x) + \gamma_{1,1} \phi_{1,1}(0) \mathcal{W}(1, e, -e))^T \tag{15}
\]

where \( \eta \) denotes the \( i \)-th element of \( \eta \). For this choice of \( S(\eta, x, e) \), it holds at time \( r_k \) for \( k \geq 2m \) that

\[
\sum_{i=0}^{m-1} U_1(\xi(r_{k-2}^-)) = \mathcal{V}(x(r_k)) + \gamma_{1,1} \phi_{1,1}(0) \mathcal{W}(1, e(r_k), -e(r_k)) + \sum_{i=1}^{m-1} \eta_i(r_k). \tag{16}
\]

Note that if we chose \( c_U = \gamma_{1,1} \phi_{1,1}(0), \) then (13) implies that

\[
U_1(\xi(r_{k+2})^+) \leq e^{-e_1(t_{k+2}^- - t_k)} U_p(\xi(r_k^-)), \tag{17}
\]

i.e., choosing \( c_U \) according to (16) makes it possible to verify (14) based on (13) from Proposition 1 for any \( p \in \{1, \ldots, n_p\} \). To use Proposition 1, the dynamic STC mechanism requires further to ensure that (11) holds, which is equivalent to the conditions \( U_p(\xi(r_k))^\prime \leq c_X \) and

\[
e^{-e_1(t_{k+2}^- - t_k)} U_p(\xi(r_k^-))^\prime \leq c_X, \tag{18}
\]

where \( K(x(r_k), e(r_k), \eta(r_k), c_X) \) and \( K(x(r_k), e(r_k), \eta(r_k), c_X) \) depend on the initial conditions for \( \eta \), which can typically be chosen by the user to tune the initial behavior of the STC mechanism and which does not influence stability guarantees.

4.2 Implementation of the dynamic STC mechanism and stability result

Recall that the idea for the dynamic STC mechanism is to maximize \( \tau_{\text{max}}(r_k^-) = t_{k+2}^- - t_k \) such that (18) and thus (14) hold. We will use a similar approach as in Hertneck and Allgöwer (2021c) to realize this. For any parameter set \( p \in \{1, \ldots, n_p\} \), we note that if

\[
e^{-e_1 \tau_{\text{max}}(r_k^-)} U_p(\xi(r_k^-))^\prime \leq e^{-e_1 \tau_{\text{max}}(r_k^-)} K(x(r_k), e(r_k), \eta(r_k), c_X) \tag{19}
\]

and

\[
\tau_{\text{max}}(r_k^-) \leq T_{\text{max}}(\mathcal{P}_p, \lambda, c_U, \tau_{\text{mad}}) \tag{20}
\]

hold with \( c_U = \gamma_{1,1} \phi_{1,1}(0) \), then it follows from Proposition 1 due to (13) that (14) holds for \( r_{k+2} \). The next step is thus to maximize \( \tau_{\text{max}}(r_{k+1}) \) such that (19) and (20)
hold at least for one \( p \in \{2, \ldots, n_p\} \). Note that (19) can be rewritten as
\[
(-\epsilon_p + \epsilon_1) \tau_max(r_k^+) \leq \left( \frac{K(x(r_k), e(r_k), \eta(r_k), \epsilon_p)}{U_p(\xi(r_k^+))} \right).
\]
Suppose that \( K(x(r_k), e(r_k), \eta(r_k), \epsilon_p) \geq U_p(\xi(r_k^+)) \). \( H \) \( K \) \( \tau_max(r_k^+) \) such that (19) and (20) hold for a given \( p \in \{2, \ldots, n_p\} \) is straightforward. If \( -\epsilon_p + \epsilon_1 > 0 \), then we obtain
\[
\tau_max(r_k^+) = \min \left\{ \log \left( \frac{K(x(r_k), e(r_k), \eta(r_k), \epsilon_p)}{U_p(\xi(r_k^+))} \right) - \log(U_p(\xi(r_k^+))), -\epsilon_p + \epsilon_1 \right\},
\]
\[
T_max(P_p, \lambda, \epsilon_p, \tau_{rad}) \}
\]
Otherwise, i.e., if \( -\epsilon_p + \epsilon_1 \leq 0 \), we can directly use the maximum value \( \tau_max(r_k^+) = T_max(P_p, \lambda, \epsilon_p, \tau_{rad}) \).

The case that \( K(x(r_k), e(r_k), \eta(r_k), \epsilon_p) < U_p(\xi(r_k^+)) \) is typically not relevant and therefore omitted \(^4\) by the dynamic STC mechanism in this paper. In this case, the respective parameter set will be discarded.

Using the above discussion, it is possible for any \( p \in \{2, \ldots, n_p\} \) to search efficiently for a preferably large value for \( \tau_max(r_k^+) \) such that (14) holds. The dynamic STC mechanism can thus simply probe iteratively for all \( p \in \{2, \ldots, n_p\} \) the maximum value for \( \tau_max(r_k^+) \) for which it can be guaranteed that (14) holds. The maximum value is then used to determine the next sampling instant.

There may also be the situation, that for no \( p \in \{2, \ldots, n_p\} \) a guarantee that (14) holds can be obtained based on Proposition 1. In this case, the dynamic STC mechanism can choose \( \tau_max(r_k^+) = T_max(P_1, \lambda, \epsilon_p, \tau_{rad}) \) with \( \epsilon_p = \tau_{rad}(1,0) \) as a fall-back strategy. Then Proposition 1 implies due to Assumption 2 that
\[
U_1(\gamma(r_k^+)) \leq e^{-\epsilon_1 \tau_max(r_k^+)},
\]
i.e., a certain decrease of \( U_1 \) is guaranteed in this case which will be useful to obtain stability guarantees.

The overall procedure to determine \( \tau_max(r_k^+) \) is summarized in Algorithm 1, which is an adaption of (Hertneck and Allgöwer, 2021c, Algorithm 2) to the setup and notation of this paper.

We therefore omit a detailed explanation of Algorithm 1 and instead refer to (Hertneck and Allgöwer, 2021c, Section III B) for a thorough discussion of the Algorithm. We note that Algorithm 1 ensures either that (14) holds or that (21) holds and that it guarantees that \( \tau^+ \geq \tau_{min} = T_max(P_1, \lambda, \epsilon_p, \tau_{rad}) \geq \tau_{rad}(1,0) \). We can now state the following result.

Theorem 1. Assume there are \( n_p \) different parameter sets \( P_p, p \in \{1, \ldots, n_p\} \), for which Condition 1 holds for the same function \( \mathcal{V}(x), \mathcal{X} = \{x | \mathcal{V}(x) < c_X\} \) and \( \mathcal{E} = \{x \in \mathcal{X}, x \in \mathcal{X}'\} \) for some \( c_X \in \mathbb{R}_{>0} \). Let Assumption 2 hold. Consider \( H_{STC} \) with \( S(y, x) \) and \( \Gamma(x, \eta) \) defined according to (15) and by Algorithm 1. Then the set \( \{x, e, \eta, \tau, \tau_{max}\} : x = 0, e = 0, s = 0, \eta = 0\) is asymptotically stable with region of attraction \( \mathcal{E} \).

\(^4\) Note that a similar case study as in (Hertneck and Allgöwer, 2021b, Section IV B) can be used to address this case.

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**Algorithm 1** Computation of \( \tau_{max} \) for the proposed STC mechanism.

1: \( K \leftarrow K(x(r_k), e(r_k), \eta(r_k), \epsilon_p) \)
2: \( c_U = \gamma_{1,1}\phi_{1,0}(0) \)
3: \( h \leftarrow T_{max}(P_1, \lambda, \epsilon, \tau_{rad}) \)
4: for each \( p \in \{2, \ldots, n_p\} \) do
5: \( U_p = \mathcal{V}(x(r_k)) + \gamma_{1,1}\phi_{1,0}(0) \mathcal{V}(1, e(r_k), -e(r_k)) \)
6: \( h_{max} = T_{max}(P_p, \lambda, c_U, \tau_{rad}) \)
7: if \( K \geq h_{max} \) then
8: if \( -\epsilon_p + \epsilon_1 > 0 \) then
9: \( h_{p} \leftarrow \min \left\{ h_{max}, \frac{\log(K) - \log(U_p)}{-\epsilon_p + \epsilon_1} \right\} \)
10: else
11: \( h_{p} \leftarrow h_{max} \)
12: end if
13: else
14: \( h_{p} \leftarrow 0 \)
15: end if
16: if \( h_{p} > h \) then
17: \( h \leftarrow h_{p} \)
18: end if
19: end for
20: \( \mathcal{F}(x(r_k), e(r_k), \eta(r_k)) \leftarrow h \)

\( \mathcal{R} := \{x \in \mathbb{R}^{n+1} | \mathcal{V}(x) + \gamma_{1,1}\phi_{1,0}(0) \mathcal{V}(1, x) < c_X\} \)

and for any complete solution \( \xi, t_{k+2} - t_k \geq t_{min} = T_{max}(P_1, \lambda, \gamma_{1,1}\phi_{1,0}(0), \tau_{rad}) \) holds for all \( k \in \mathbb{N} \).

**Proof:** See Appendix A.2.

**Remark 1.** The region of attraction \( \mathcal{R} \) as defined in Theorem 1 is smaller than the set \( \mathcal{X} \). This is not surprising as the system state can leave the set \( \mathcal{R} \) before \( u \) is updated for the first time. In fact, under the assumption that \( e(0, 0) = 0 \), the region of attraction could be extended to \( \mathcal{X} \), however assuming \( e(0, 0) = 0 \) is in general rather restrictive.

**Remark 2.** The assumption that \( e(0, 0) = -x(0, 0) \) can be relaxed in Theorem 1. In particular, the theorem holds also for all initial conditions with \( U_1(\gamma(r_k^+)) \leq c_X \) and \( \hat{x}(0, 0) \in \mathcal{X} \). However, then the region of attraction has a more complex shape, which would complicate to determine the required size of \( \mathcal{X} \) and \( \mathcal{E} \).

---

5. EXAMPLE

In this section, we illustrate the proposed dynamic STC mechanism with a numerical example from the literature. The system that we consider is the same as in (Theodosis and Dimarogonas, 2018, Example 2). The dynamics of the plant are
\[
\dot{x} = -x \sin^2(x^2) + \dot{u} \cos(x^2)
\]
and we consider the state feedback \( u = -x \cos(x^2) \). As in (Theodosis and Dimarogonas, 2018, Example 2), we consider \( \tau_{rad} = 0.00004 \). Using \( \cos((x + e)^2) = \cos(x^2) - a_1(x^2 + 2xe) + e \cos((x + e)^2) = a_2e \) for varying parameters \( a_1, a_2 \in [-1, 1] \), we note that
\[
\dot{u} = -x \cos(x^2) = (x + e) \cos((x + e)^2) = -x \cos(x^2) - e(a_1(xe + 2x^2) + a_2e)
\]
leading to \( f(x, e) = -x - e \cos(x^2) \), where \( a_3(x, e) = a_2 + a_1(ex + 2x^2) \) and to \( g(x, e) = -f(x, e) \). We consider the
In the interval $[0\text{s}, 10\text{s}]$, we have experienced a total number of 117 sampling instants. Note that this is only 5.47% of the number of sampling instants that were reported for the approach from Theodosis and Dimarogonas (2018). In the simulation, the sampling interval eventually converges to a value of about 0.095 s which means an improvement by factor 7 in comparison to periodic sampling with $t_{\text{min}}$.

6. CONCLUSION

In this paper, we have presented a dynamic STC mechanism for nonlinear NCS with transmission delays. From a technical point of view, the main difference to the delay free case is that more involved hybrid Lyapunov functions are required, leading further to more parameters that influence the behavior of the dynamic STC mechanism. The effectiveness of the approach was demonstrated with a numerical example. By taking transmission delays into account, it becomes possible to apply dynamic STC based on hybrid Lyapunov functions to realistic NCS setups, where delays are not negligible. Future work therefore includes to investigate the interplay between dynamic STC and realistic network setups.

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Due to (11), we can conclude that $x(t,j) \leq c_X$, then it follows that $x(t,j) \in \mathcal{X}$. Further since $x(t,k+1) = \tilde{x}(t,k)$ for $t_k \leq t \leq t_{k+1}$ and $x(t,k+2) = \tilde{x}(t,k)$ for $t_{k+1} \leq t \leq t_{k+2}$, we can conclude that for $r_k \leq (t,j) \leq r_{k+2}$, $x(t,j) \in \mathcal{X}$ further implies $e(t,j) = \hat{x}(t,j) - x(t,j) \in \mathcal{E}$. Moreover, we note that $s(r_k^+ + 1) = e(r_k^+)$ and $s(r_{k+1}^+) = -e(r_{k+1}^+)$, i.e. $e(t,j) \in \mathcal{E}$ further implies $s(t,j) \in \mathcal{E}$ for $r_k^+ \leq (t,j) \leq r_{k+2}$.

Next, using (4) and (5) from Condition 1, and (8), we obtain for $x \in \mathcal{X}$, $e \in \mathcal{E}$, and $\tau \in [0,t_{k+2} - t_k]$

$$
\frac{d}{dt}U(x(t,k+1)) \leq -eU(x(t,k+1))
$$

(4.1) and due to the comparison Lemma (cf. Khalil, 2002, p. 102)

$$
U(x(t,k+1)) \leq e^{-c(t-t_k)}U(x(t_k^+))
$$

(4.2) Due to (11), this implies $U(x(t,k+1)) < c_X$ and thus $x(t,k+1) \in \mathcal{X}$ and $(t,k+1) \in \mathcal{E}$. We can now use this information to conclude that

$$
U(x(t,k+1)) \leq e^{-c(t-t_k)}U(x(t_k^+))
$$

(4.3) holds for $t \in [t_k,t_{k+1}]$. Recall that $e(r_k^+ + 1) = s(r_k^+ + 1) + e(r_k^+)$, $s(r_k^+ + 1) = -s(r_k^+ + 1) + e(r_k^+)$ and $\tau(r_k^+ + 1) = \tau(r_k^+ + 1) \leq \tau_{\text{mad}}$. We can thus conclude from (3b) and (9)

$$
U(x(r_k^+ + 1)) = \tilde{V}(x(r_k^+ + 1)) + \gamma_0\phi_{0}(\tau(r_k^+ + 1))\tilde{W}(0,e(r_k^+),s(r_k^+ + 1))
$$

(4.4) and (A.4)

Using again (A.1), the comparison Lemma and the iterative argumentation as previously, we obtain that

$$
U(t,k+2) \leq e^{-c(t-t_k)}U(x(r_k^+))
$$

(5.5) holds for $t \in [t_{k+1},t_{k+2}]$ and thus (12) holds for $r_k^+ \leq (t,j) \leq r_{k+2}$. Next, we note that $e(r_{k+2}^+ + 1) = e(r_{k+2})$ and $s(r_{k+2}^+) = -e(r_{k+2})$. Using (3a) and (10), we thus obtain from (12) for $(t,j) = r_{k+2}$ that (13) holds.

Finally note that (A.5) implies that $\tilde{V}(x(r_{k+1}^+)) \leq c_X$ and thus that $x(r_{k+1}^+) \in \mathcal{X}$. Hence $\hat{x}(r_{k+2}) = x(r_{k+1}^+) \in \mathcal{X}$. \hfill $\blacksquare$

A.2 Proof of Theorem 1

Recall that sampling instances are described by the jumps of $\mathcal{H}_{\text{STC}}$ that occur between the hybrid times $r_k = (t_k,k-1)$ and $r_k^+ = (t_k,k)$, $k \in \mathbb{N}_c$ and the corresponding update occurs between hybrid times $r_{k+1} = (t_{k+1},k)$ and $r_{k+1}^+ = (t_{k+1},k+1)$. The first sampling instance is at $r_0^+ = (0,0)$ (modeled in the definition of $\mathcal{H}_{\text{STC}}$ by the initial condition restriction $e(0,0) = -x(0,0)$, $s(0,0) = x(0,0)$, $\eta(0,0) \in \mathbb{R}^n_+$, $\tau(0,0) = 0$, $\tau_{\text{mad}}(0,0) = \tau(x(0,0),e(0,0),\eta(0,0))$ and $\ell(0,0) = 1$). Now consider an arbitrary sampling instance $r_k^+ \in \mathbb{N}_c$ with $\tau_{\text{mad}}(r_k^+) = \Gamma(x(r_k),e(r_k),\eta(r_k))$, where $\Gamma(x(r_k),e(r_k),\eta(r_k))$ is defined by Algorithm 1. Obviously, $h \geq \tau_{\text{mad}}(P_1,\lambda,\varepsilon_U,\tau_{\text{mad}})$ and $h \leq \max \tau_{\text{mad}} \cdot \tau_{\text{max}}(P_{\eta,\lambda,U},\tau_{\text{mad}}) =: \tau_{\text{min}}$ with $c_U = \gamma_1\phi_{1,0}(0)$ in Algorithm 1. Due to Assumption 2, it holds that $\tau_{\text{min}} \geq \tau_{\text{mad}}$ and thus we can conclude that $\tau_{\text{max}} \geq \tau_{\text{mad}} - \delta \leq \tau_{\text{mad}}$. Suppose that $U_1(x(r_k^+)) < c_X$ and $\hat{x}(r_k^+) \in \mathcal{X}$. If Algorithm 1 outputs $\Gamma(x(r_k),e(r_k),\eta(r_k)) = \tau_{\text{min}}$, then

$$
\left\{1,e^{-c(t-t_k)}\right\}U_1(x(r_k^+)) < c_X
$$

holds since $\ell_1 > 0$ and $U_1(x(r_k^+)) \leq c_X$, i.e., (11) holds for the respective parameter. Thus it follows from Proposition 1 for $P_1$ with $c_U = \gamma_1\phi_{1,0}(0)$ that (21) holds and that $x(r_{k+2}) \in \mathcal{X}$ in this case.

If Algorithm 1 outputs $h = \ell_{\delta} > \tau_{\text{min}}$ for some $p \in \{2,\ldots,\ell_p\}$, then we know from the algorithm that (19) and (20) hold in this case for the respective parameter $p$ and $c_U = \gamma_1\phi_{1,0}(0)$. Moreover, we know that $U(x(r_k^+)) \leq K(x(r_k),e(r_k),\eta(r_k),c_X)$, as $p$ would otherwise have been skipped. Recall that since $K(x(r_k),e(r_k),\eta(r_k),c_X) \leq c_X$ and $\ell_1 > 0$, (19) implies that $e^{-c(t-t_k)}U_{p_1}(x(r_k^+)) \leq c_X$, holds, and hence (11) holds for the respective parameter set. We can hence conclude in this case from Proposition 1 that

$$
U_1(x(r_k^+)) \leq \min\left\{c_1e^{-\epsilon_1},\frac{U_1(x(r_k^+))}{m_{\ell_{\delta}+1}} + \sum_{k=1}^{m_{\ell_{\delta}+1}} \eta(r_k)\right\}
$$

(6.6) holds and that $\hat{x}(r_{k+2}) \in \mathcal{X}$.

Since either (21) or (A.6) hold, we can thus infer that

$$
U_1(x(r_k^+)) \leq e^{-c\epsilon_1}\max\left\{U_1(x(r_k^+),\eta_1(r_k),\ldots,\eta_{m-1}(r_k))\right\}
$$

(7.7)
and that $U_1(\xi(r_{k+2}^+)) \leq c_X$. Observe that $U_1(\xi(r_{k+2}^+)) = \hat{V}(x(0, 0)) + \gamma_1 \phi_{1, 0}(0) \dot{W}(1, x(0, 0), 0) \leq c_X$ due to the definition of $\hat{R}$. Hence, we obtain by induction for all $k \in \mathbb{N}_0$ that $U_1(\xi(r_k^+)) \leq c_X$ and that $\xi(r_k^+) \in \mathcal{X}$. Together with the fact that $\tau_{\min} \leq t_{k+2} - t_k \leq \tau_{\max}$ for all $k \in \mathbb{N}_c$ this further implies that $\xi$ is $t$-complete.

Note that (A.7) is similar as equation (26) in the preprint of Hertneck and Allgöwer (2021b) with $U_1(\xi(r_k))$ instead of $V(x(r_k))$. Using similar steps as in the proof of (Hertneck and Allgöwer, 2021b, Theorem 1), we thus obtain the bounds

$$U_1(\xi(r_k^+)) \leq \hat{\beta}_1 \left( \left\| x(r_0) \right\|, t_k, k \right) \tag{A.8}$$

and

$$\left\| \eta(r_k^+) \right\| \leq (m - 1) \hat{\beta}_1 \left( \left\| x(r_0) \right\|, t_k, k \right) \tag{A.9}$$

for a class $\mathcal{K}$ function $\hat{\beta}_1$ that are valid at sampling instants. To show asymptotic stability it thus remains to derive a similar bound between sampling instants, which works again analogous as in the proof of (Hertneck and Allgöwer, 2021b, Theorem 1). In particular, we first note that

$$\left\| \eta(t, j) \right\| = \left\| \eta(r_k^+) \right\| \tag{A.10}$$

holds for $r_k^+ \leq (t, j) \leq r_{k+2}$. For $x(t, j), e(t, j)$ and $s(t, j)$, we can use again Proposition 1 to derive a bound. Recall that the conditions of Proposition 1 hold for all $k \in \mathbb{N}_c$ at least for one $p \in \{1, \ldots, n_p\}$, for which $U_p(\xi(r_k^+)) \leq K(x(r_k), e(r_k), s(r_k), c_X)$ and $t_{k+2} - t_k \leq \tau_{\max}(P_p, \lambda, c_U, \tau_{\max})$. Thus, we obtain from the proposition that

$$U_p(\xi(t, j)) \leq e^{\max\{c_p, 1\} \tau_{\max}} U_p(\xi(r_k^+)) \leq e^{\max\{c_p, 1\} \tau_{\max}} K(x(r_k), e(r_k), s(r_k), c_X). \tag{A.11}$$

We note that

$$K(x(r_k), e(r_k), s(r_k), c_X) \leq \frac{1}{m} \left( U_1(r_k^+) + \sum_{i=1}^{m-1} \eta_i(r_k) \right) \leq m \hat{\beta}_1 \left( \left\| x(r_0) \right\|, t_k, k \right). \tag{A.12}$$

Further, it holds due to Definition 2 that $\gamma_{1, p} \phi_{1, 0}(\tau) \geq c_U$ for $\tau \in [0, \tau_{\max}]$ and $\gamma_{0, p} \phi_{0, p}(\tau) \geq c_U$ for $\tau \in [0, \tau_{\max}(P_p, \lambda, c_U, \tau_{\max})]$. Using also the bounds on $\dot{V}$ and $\dot{W}$ from Condition 1, we can conclude that there exist a function $\tilde{\beta}_{\mathcal{P}, p} \in \mathcal{K}_\infty$, such that

$$U_p(\xi(t, j)) \leq \tilde{\beta}_{\mathcal{P}, p} \left( \left\| x(t, j) \right\|, e(t, j), s(t, j) \right) \leq \beta_{\mathcal{P}, \max}(\xi(t, j)) \leq \max\left\{ \tilde{\beta}_{\mathcal{P}, 1}, \ldots, \tilde{\beta}_{\mathcal{P}, n_p} \right\} \tag{A.13}$$

holds for $r_k^+ \leq (t, j) \leq r_{k+2}$ where $\tilde{\beta}_p \in \mathcal{K}$. Combining now (A.9), (A.10) and (A.13), we obtain for $r_k^+ \leq (t, j) \leq r_{k+2}$ that

$$\left\| x(t, j) \right\| \leq m \hat{\beta}_1 \left( \left\| x(r_0) \right\|, t_k, k \right) \tag{A.14}$$

holds. Using a time shift, we can thus conclude that

$$\left\| x(t, j) \right\| \leq \beta \left( \left\| x(0, 0) \right\|, e(0, 0), s(0, 0) \right), (t, j) \tag{A.15}$$

holds for

$$\beta(\cdot, t, j) \leq \hat{\beta}_2(\cdot, \max\{t - \tau_{\max}, 0\}, \max\{j - 1, 0\}) + (m - 1) \hat{\beta}_1(\cdot, \max\{t - \tau_{\max}, 0\}, \max\{j - 1, 0\}) \tag{A.16}$$

that

5 Note that this proof is given in the preprint https://arxiv.org/abs/2109.06657.