A Random Matrix Study of the QCD Sign Problem

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We investigate the severity of the sign problem in a random matrix model for QCD at finite temperature $T$ and baryon chemical potential $\mu$. We obtain analytic expression for the average phase factor – the measure of the severity of the sign problem at arbitrary $T$ and $\mu$. We observe that the sign problem becomes less severe as the temperature is increased. We also find the domain where the sign problem is maximal – the average phase factor is zero, which is related to the pion condensation phase in the QCD with finite isospin chemical potential. We find that, in the matrix model we studied, the critical point is located inside the domain of the maximal sign problem, making the point inaccessible to conventional reweighting techniques. We observe and describe the scaling behavior of the size and shape of the pion condensation near the chiral limit.

I. INTRODUCTION

The phase diagram of QCD at finite temperature and baryon density has been a subject of intense interest during recent years (see, e.g., [1] for review). On the one hand, the experimental heavy-ion collision programs, such as those at CERN SPS and RHIC, as well as planned FAIR facility, demand reliable quantitative understanding of the phase diagram of strongly interacting matter created in those collisions. On the other hand, deriving the relevant predictions from the first principles of QCD is a formidable challenge, because the phenomena associated with phase transitions occur in the domain where the QCD coupling is not small.

One of the features of the QCD phase diagram of particular interest to heavy-ion collision experiments is the QCD critical point. The existence of such a point – an ending point of the first order chiral transition in QCD – was suggested a long time ago [2, 3], and the properties were studied using universality arguments and model calculations more recently [4, 5] (see Ref. [6] for review). The experimental search for the critical point using heavy ion collisions has been proposed in [7]. It is apparent that theoretical knowledge of the location
of the critical point on the phase diagram is important for the success of the experimental search.

The time-tested approach to non-perturbative problems in QCD is the numerical lattice Monte-Carlo calculations. This approach, very powerful at studying QCD thermodynamics at zero baryon density, runs into the sign-problem at nonzero baryon density. The lattice calculations are based on reinterpreting the QCD partition function as a partition function of a classical statistical system, with energy given by the Euclidean action of QCD. This action involves the logarithm of the fermion determinant, which is complex for any nonzero value of the baryon chemical potential $\mu_B$. The Monte Carlo importance sampling technique uses the exponent of the action as the measure of importance and fails because the action is complex.

Several approaches to QCD at finite baryon density are being developed, using various techniques to circumvent or tame the effect of the complexity of the fermion determinant and locate the QCD critical point [8, 9, 10, 11, 12] (see also Refs. [13, 14, 15, 16] for reviews). As the sign problem worsens with increasing $\mu_B$, the hope is that the QCD critical point is located at sufficiently small $\mu_B$, where the sign problem can still be controlled. In many cases it is difficult to judge reliably, either a priori or a posteriori, what the range of validity of the results are, in terms of how large $\mu_B$ can be before the sign problem is out of control. It is therefore necessary to understand better the severity of the sign problem, and its dependence on the variables such as temperature $T$, baryon chemical potential $\mu_B$ and quark mass $m$.

In this paper we use a random matrix model of QCD to assess the severity of the sign problem as a function of $T$, $\mu_B$ and $m$. A similar study at $T = 0$ has been reported in Ref. [17, 18]. Here we shall present analytical\(^1\) results for the random matrix model at nonzero temperature – the regime most relevant for the heavy-ion collision experiments and the lattice studies aimed at discovering the QCD critical point.

As a quantitative measure of the severity of the sign problem we consider the complex phase $e^{i\theta}$ of the fermion determinant $\det \mathbb{D}$, averaged over gauge field configurations of the phase-quenched theory:

$$R \equiv \langle e^{2i\theta} \rangle_{1+1^*} \equiv \left\langle \frac{\det \mathbb{D}}{\det \mathbb{D}^*} \right\rangle_{1+1^*},$$

\(^1\) A numerical study at nonzero $T$ has been reported in Ref. [19].
where $\mathbb{D}$ is the Euclidean space Dirac operator in a given gauge configuration:

$$
\mathbb{D} = \gamma^\mu (\partial_\mu - A_\mu) + m + \mu \gamma_0 ,
$$

and $\mu$ is the quark chemical potential: $\mu = \mu_B/3$.

The average in Eq. (1), denoted by $\langle \ldots \rangle_{1+1^*}$, is taken over the gauge field configuration ensemble with the phase of the determinant removed (quenched), making the measure of path integration manifestly positive: $e^{-S_{\text{YM}}} \lvert \det \mathbb{D} \rvert^2$. This phase-quenched theory can be viewed as a theory with 1 quark and 1 conjugate quark or, due to $(\det \mathbb{D}(\mu))^* = \det \mathbb{D}(-\mu)$, two quarks with opposite chemical potentials, i.e., QCD at finite isospin chemical potential $\mu_I = 2\mu$ (see, e.g., [20, 21]).

The average phase factor $R$ can be recast as the ratio of two partition functions [17, 19]:

$$
R \equiv \langle e^{2i\theta} \rangle_{1+1^*} = \frac{\langle (\det \mathbb{D})^2 \rangle_0}{\langle |\det \mathbb{D}|^2 \rangle_0} = \frac{Z_{1+1}}{Z_{1+1^*}},
$$

where $\langle \ldots \rangle_0$ denotes average over gauge configurations in a theory without quarks (with pure Yang-Mills measure $e^{-S_{\text{YM}}}$).

II. RANDOM MATRIX THEORY (RMT) AND QCD PARTITION FUNCTION

A. The random matrix model

The Chiral Random Matrix Theory [22] approximates the QCD partition function by an integral over random matrix ensemble. We introduce temperature as first done in Ref. [23] and chemical potential as in Ref. [24]. The resulting random matrix model has been used in Ref. [5] to study the QCD phase diagram. The partition function in the model is given by:

$$
Z_{N_i} = \int \mathcal{D}X \ e^{-N \text{tr} XX^\dagger} \det^{N_i} \mathbb{D} = \langle \det^{N_i} \mathbb{D} \rangle_X ,
$$

where $\mathbb{D}$ is the $2N \times 2N$ matrix approximating the Dirac operator:

$$
\mathbb{D} = \begin{pmatrix} m & iX + C \\ iX^\dagger + C & m \end{pmatrix},
$$

with

$$
C = \mu \mathbb{1}_N + iT \begin{pmatrix} \mathbb{1}_{N/2} & 0 \\ 0 & -\mathbb{1}_{N/2} \end{pmatrix}.
$$
$X$ is an $N \times N$ complex random matrix; $\mathds{1}_N$ is the $N \times N$ identity matrix. The deterministic matrix $C$ defined by Eq. (6) accounts for the effect of the chemical potential $\mu$ and of the 2 smallest Matsubara frequencies: $+\pi T$ and $-\pi T$. For simplicity, we absorb the coefficient $\pi$ into $T$. The integration in Eq. (4) is over the real and imaginary components of the matrix $X$: $\mathcal{D}X = \prod_{i,j=1}^{N} dX_{ij} dX_{ij}^*$. The Dirac determinant can be written as a Grassmann integral:

$$\det_{\mathcal{N}f} \mathds{1}_n = \int \mathcal{D}\psi^f_R \mathcal{D}\psi^f_L \exp \left[ \sum_{f} \left( \begin{array}{c} \psi^f_{R*} \\ \psi^f_{L*} \end{array} \right)^T \left( \begin{array}{cc} m & iX + C \\ iX^\dagger + C & m \end{array} \right) \left( \begin{array}{c} \psi^f_{R} \\ \psi^f_{L} \end{array} \right) \right],$$

where the Grassmann integration is over the spinors: $\mathcal{D}\psi^f = \prod_{i=1}^{N_f} d\psi_i^f d\psi_i^{f*}$. Following the logic of the Hubbard-Stratonovich transformation, 4-fermion interaction can be rewritten using fermion bilinears with the help of a new auxiliary $N_f \times N_f$ complex matrix $A$ (flavor matrix). Performing the Grassmann integration one then obtains

$$Z_{N_f} = \int DA e^{-N \operatorname{tr} AA^\dagger} \det^N_{\mathcal{F}} \left( \begin{array}{cc} A + m & \mu + iT \\ \mu + iT & A^\dagger + m \end{array} \right) \det^N_{\mathcal{F}} [T \rightarrow -T].$$

We shall specialize to $N_f = 2$ quark flavors. The integral in Eq. (8) is performed over $2 \times N_f \times N_f = 8$ variables which are real and imaginary parts of the elements of the complex $N_f \times N_f = 2 \times 2$ flavor matrix $A$. We shall define potential $\Omega_{1+1}(A)$ as

$$Z_{1+1} \equiv \int DA e^{-N\Omega_{1+1}(A)},$$

i.e.,

$$\Omega_{1+1}(A) = \operatorname{Tr} \left[ AA^\dagger - \frac{1}{2} \ln \{(A + m)(A^\dagger + m) - (\mu + iT)^2\} \times [T \rightarrow -T]\right].$$
B. Phase quenched partition function

The phase-quenched partition function is given by

$$Z_{1+1^*} = \int \mathcal{D}X e^{-N \text{tr} XX^\dagger} \det \mathbb{D} \det \mathbb{D}^* = \langle |\det \mathbb{D}|^2 \rangle_X. \quad (11)$$

Following the steps outlined above, the phase quenched partition function $Z_{1+1^*}$ can be written as

$$Z_{1+1^*} = \int \mathcal{D}A e^{(-N \text{tr} AA^\dagger)} \det \frac{N}{2} \left( \begin{array}{cc} A + m & \mu \tau_3 + iT \\ \mu \tau_3 + iT & A^\dagger + m \end{array} \right) \det \frac{N}{2} [T \to -T] \quad (12)$$

$$\equiv \int \mathcal{D}A e^{-N \Omega_{1+1^*}(A)}. \quad (13)$$

Here $\tau_3$ is the Pauli matrix, and we defined another potential $\Omega_{1+1^*}(A) — a$ function of a $2 \times 2$ complex flavor matrix $A$. For a generic matrix $A$, $\Omega_{1+1^*}(A) \neq \Omega_{1+1}(A)$, due to the presence of the Pauli matrix in Eq. (12).

C. Thermodynamic limit $N \to \infty$ and the solution of the model

In the thermodynamic limit $N \to \infty$, the random matrix partition function $Z_{1+1}$ can be calculated analytically using the saddle-point approximation by minimizing $\Omega_{1+1}(A)$ with respect to $A$. The minimum is given by a multiple of the unit matrix: $A = a \mathbb{1}$ with $a – \text{real}$.

For the $1 + 1^*$ theory the minimum of $\Omega_{1+1^*}(A)$ is also given by a multiple of the unit matrix, except for a region on the phase diagram where another, deeper minimum is given by a non-diagonal matrix $A$ [24]. This breaks the U(1) ($\tau_3$ isospin) symmetry of the theory in this region and is associated with the pion condensation in QCD.

Outside of the pion condensation region, i.e., when the minimum of $\Omega_{1+1^*}$ is given by a multiple of the unit matrix, the minimum values of the two potentials coincide. Indeed, for any real $a$, $\Omega_{1+1}(a \mathbb{1}) = \Omega_{1+1^*}(a \mathbb{1}) \equiv \Omega(a)$, where we defined

$$\Omega(a) = 2a^2 - \ln \{|(a + m)^2 - (iT + \mu)^2|[a + m]^2 - (iT - \mu)^2\}. \quad (14)$$

Then the saddle-point value of $A = a \mathbb{1}$ for both $Z_{1+1}$ and $Z_{1+1^*}$ is determined by minimizing the potential $\Omega(a)$ with respect to $a$:

$$a - \frac{(a + m)[(a + m)^2 - \mu^2 + T^2]}{[(a + m)^2 - \mu^2 + T^2]^2 + 4\mu^2T^2} = 0. \quad (15)$$
III. AVERAGE PHASE FACTOR IN PHASE QUENCHED THEORY

A. General result: arbitrary $m$, $\mu$ and $T$

Since the leading exponential behavior of the partition functions $Z_{1+1}$ and $Z_{1+1}^*$ is the same (see previous section), it will cancel in the ratio $R$. Therefore, we have to take into account the preexponential factors, which are determined by the second order derivatives of the potential function $\Omega_{1+1}(A)$ and $\Omega_{1+1}^*(A)$ with respect to all elements of flavor matrix $A$:

$$Z_Q \xrightarrow{N \to \infty} \left( \frac{2\pi}{N} \right)^4 \left( \det \Omega''_Q \right)^{-\frac{1}{2}} e^{-N\Omega_Q(A)} \bigg|_{A=A_{sp}} ,$$

where $Q$ indicates the quark content of the theory, $1 + 1$ or $1 + 1^*$, and

$$\det \Omega''_Q \equiv \det \left( \frac{\partial^2 \Omega_Q}{\partial A_\alpha \partial A_\beta} \right).$$

The indices $\alpha$ and $\beta$ run through eight values labeling eight independent components of the complex $2 \times 2$ matrix $A$: $A_\alpha$ and $A_\beta = (A_{11}, A_{11}^*, A_{12}, A_{12}^*, A_{21}, A_{22}^*, A_{22}, A_{22}^*)$. Evaluating determinants in Eq. (17) at the saddle point $A = A_{sp}$ we find, using notations given below in Eq. (20):

$$\det \Omega''_{1+1} = [x^2 - y^2]^4 \quad \text{and} \quad \det \Omega''_{1+1}^* = [x^2 - y^2]^2 [u^2 - v^2]^2.$$  

Since $\Omega_{1+1}(A_{sp}) = \Omega_{1+1}^*(A_{sp})$ outside of the region of pion condensation in $1 + 1^*$ theory, the exponential factors in Eq. (16) cancel in the ratio $R$ and the average phase factor is given by

$$R \equiv \langle e^{2i\theta} \rangle_{1+1} = Z_{1+1} / Z_{1+1}^* = \left[ \frac{\det \Omega''_{1+1}}{\det \Omega''_{1+1}^*} \right]^{-\frac{1}{2}} \left. = \frac{u^2 - v^2}{x^2 - y^2} \right|_{A=A_{sp}} ,$$

where we define $x, y, u, v$ as follows:

$$x = 1 - \frac{T^2 - \mu^2}{W} - \frac{8T^2 \mu^2 (a + m)^2}{W^2} ;$$

$$y = \frac{(a + m)^2}{W} \left( 1 - \frac{8T^2 \mu^2}{W} \right) ;$$

$$u = 1 - \frac{T^2 + \mu^2}{W} ;$$

$$v = \frac{(a + m)^2}{W} ;$$

$$W = [(a + m)^2 + T^2 - \mu^2]^2 + 4\mu^2T^2 .$$
FIG. 1: The contours of the average phase factor $R$ for $m = 0.07$. The first order phase transition line in $1+1$ theory and the critical point are also shown (see discussion in Section III F).

where $a$ is a solution of the saddle-point equation (15), and the global minimum of $\Omega(a)$ in Eq. (14).

Using Eqs. (19) and (20) we can now obtain the average phase factor $R$ for any values of $T$, $\mu$ and $m$. In general, this has to be done numerically, but in certain limiting cases, discussed below, explicit analytical results can be derived as well. As an illustration, the average phase factor contours on the $T\mu$ plane for $m = 0.07$ are plotted in Figure 1.

B. Arbitrary $T$ and $\mu$ in the chiral limit $m = 0$

In the chiral limit $m = 0$ the solution of the saddle-point equation (15) can be found explicitly. In the high-temperature phase it is simply $a = 0$. Then, from (19) and (20) the average phase factor $R$ can be written explicitly:

$$R \equiv \langle e^{2i\theta} \rangle_{1+1}^* = \frac{[T^2 + \mu^2]^2 - (T^2 - \mu^2)]^2}{[(T^2 + \mu^2)^2 - (T^2 - \mu^2)^2]} \quad (m = 0).$$

From (21), the contour where the average phase factor vanishes is determined by: $T^2 + \mu^2 = 1$.

The contour plot of $R(T, \mu)$ at $m = 0$ is shown in Figure 2.
FIG. 2: The contours of the average phase factor $R$ in the chiral limit $m = 0$. The chiral symmetry transition line and tri-critical point (TCP) are also shown.

C. Small $m$ and $\mu$

At small $m$ and $\mu$ the solution to the fifth-order polynomial equation for the saddle point $a$ can be found in the form of an expansion, which at zero temperature is given by

$$a \approx 1 - \frac{m^2}{2} + \frac{\mu^2}{2} \quad (T = 0, \quad m \sim \mu^2 \ll 1),$$

and thus

$$R \approx 1 - \frac{2\mu^2}{m} = 1 - \frac{4\mu^2}{m^2} \quad (T = 0, \quad m \sim \mu^2 \ll 1),$$

where we used the fact that the $R = 0$ value is achieved at the boundary of the pion condensation phase in $1 + 1^*$ theory, and that, in QCD, the pion condensation occurs at $\mu_u = -\mu_d = m_\pi/2$ to define the pion mass: $m_\pi^2 = m/2$ in the units employed in the random matrix model. This result is in agreement with the earlier $T = 0$ calculation [17, 18].

Now, using our general result, we can extend the result of Ref. [17, 18] to nonzero temperatures. In this case, the saddle point is given by:

$$a \approx \sqrt{1 - T^2} - \frac{m}{2} \left( \frac{1 - 2T^2}{1 - T^2} \right) + \frac{\mu^2}{2} \left( \frac{1 - 4T^2}{\sqrt{1 - T^2}} \right) \quad (T < 1, \quad m \sim \mu^2 \ll 1),$$

and

$$R \approx 1 - \sqrt{1 - T^2} \left( \frac{2\mu^2}{m} \right) = 1 - \sqrt{1 - T^2} \left( \frac{4\mu^2}{m^2} \right) \quad (T < 1, \quad m \sim \mu^2 \ll 1).$$
which generalizes Eq. (23). Equation (25) shows that the sign problem diminishes at higher temperatures, which can be seen also on the contour plot in Figure 1.

D. $R = 0$ contour

The contour where $R$ vanishes is of particular interest to our study – the sign problem reaches its maximum there, i.e., the fluctuations of the phase completely wash out the magnitude.

We can obtain an explicit equation for the $R = 0$ contour by setting $u = v$ in equations (19), (20):

$$u = v \implies [(a + m)^2 + T^2 - \mu^2]m = 2\mu^2a.$$  \hspace{1cm} (26)

Solving the quadratic equation (26) for $a$ and substituting the solution into the saddle-point equation (15) one finds for the $R = 0$ contour:

$$T^2 = 1 - \mu^2 + \frac{m^2}{\mu^2 - m^2} - \frac{m^2}{4(\mu^2 - m^2)^2},$$  \hspace{1cm} (27)

in agreement with an earlier result [25].

It is interesting to consider the limiting behavior (shape) of this contour as $m \to 0$. As can be seen from Figures 1 and 2 the contour develops a singularity (a kink) at $\mu = 0$, $T = 1$ in this limit. Near this kink the shape and location of the contour scale with $m$, i.e., the contour at different values of $m$ can be obtained by rescaling

$$t \to \lambda^{2/3}t \quad \text{and} \quad \mu^2 \to \lambda^{2/3}\mu^2 \quad \text{as} \quad m \to \lambda m,$$  \hspace{1cm} (28)

where we introduced $t \equiv T^2 - 1$. This can be seen upon expanding Eq. (27) in $\mu$, $t$ and $m$ in the regime $t : \mu^2 : m^{2/3}$ fixed as $m \to 0$:

$$t = -\mu^2 - \frac{m^2}{4\mu^4} + O(m^{4/3}).$$  \hspace{1cm} (29)

For example, the point where the $R = 0$ contour reaches maximum temperature slides to $T \to 1$ and $\mu \to 0$ as

$$T^2_* = 1 - \frac{3}{24^{1/3}} m^{2/3} + O(m^{4/3}) \quad \text{and} \quad \mu^2_* = \frac{1}{24^{1/3}} m^{2/3} + O(m^{4/3}).$$  \hspace{1cm} (30)

This result may be useful for analysis of QCD simulations using the improved reweighting [8] or techniques which treat the sign problem by separating the phase from the absolute
value of the determinant. The reweighting methods break down at $R = 0$, and the problem is to distinguish the signatures of the critical point, which are similar to those of the breakdown of the reweighting \[26, 27\]. By doing simulations at different values of $m$ and comparing to scaling \[28\] one can determine whether the observed signatures are those of the genuine critical point of $1 + 1$ theory or those of the breakdown of the reweighting method.

To be precise, in QCD, the scaling behavior of the phase transition boundary in the phase quenched, $1 + 1^*$ theory (i.e., pion condensation at finite isospin chemical potential) should be similar to \[28\], but with critical exponent $1/(\beta \delta)$ replacing the mean-field exponent $2/3$. The value $1/(\beta \delta) \approx 0.54$ is the ratio of critical scaling dimensions of the energy-like and ordering-field-like operators in the $O(4)$ universality class of the QCD phase transition (for two flavors). The scaling shape of the pion condensation boundary, given by Eq. (29) in the random matrix model, in QCD will also be correspondingly different.

E. $R = 0$ domain and the sign problem

One can understand the underlying reason that the sign problem becomes severe in the $R = 0$ region by looking at the distribution of the zeros of the det $D$ as a function of $\mu$. These zeros can be also viewed as eigenvalues of the random matrix $P$, defined as\(^2\)

$$D = (\mu \mathbb{I}_{2N} - P) \gamma_0, \quad \text{where} \quad \gamma_0 = \begin{pmatrix} 0 & \mathbb{1}_N \\ \mathbb{1}_N & 0 \end{pmatrix}.$$

The locations of the zeros are random, fluctuating together with matrix $X$ in the ensemble in Eq. (4). For $N \to \infty$ the density of zeros develops finite region of support with a sharp boundary, which one can see already quite clearly for a finite, but large, matrix on Figure 3\(^3\).

The fluctuations of the phase of det $D$ become large when $\mu$ enters the domain of the zeros. More explicitly, one can write

$$\arg \det D = \sum_i \arg (\mu - \lambda_i), \quad (32)$$

where the sum is over all eigenvalues $\lambda_i$ of the matrix $P$. When $\mu$ is away from the domain of support of the eigenvalue density, the fluctuations in the eigenvalue positions do not affect

\(^2\) This is the analog of the propagator matrix introduced in \[28\].

\(^3\) At $T = 0$ these distributions have been studied in Ref. \[29\].
 FIG. 3: The distribution of zeros of $\text{det} \mathbb{D}$ in the complex $\mu$ plane for a random matrix model with finite $N = 1000$ at $T = 0.5$ and $m = 0.07$. The filled circles indicate solutions of the $R = 0$ contour equation (27), bounding the region of the maximal sign problem.

the phase of the determinant significantly. On the other hand, when $\mu$ is in the domain filled with eigenvalues, there are eigenvalues which come very close to $\mu$ (as close as $1/\sqrt{N}$) and even small fluctuations in the eigenvalue positions translate into large fluctuations of the phase, causing severe sign problem.

To confirm this picture, one can compare the solutions of the equation (27) with the distribution of the eigenvalues at the same values of $T$ and $m$, as it is shown in Figure 3. The comparison clearly shows that $R = 0$ (the sign problem is most severe) for the values of $\mu$ which fall inside the domain of eigenvalues.

F. QCD sign problem vs RMT

When drawing conclusions from our random matrix study for QCD one should bear in mind the following. In QCD, in thermodynamic $V \to \infty$ limit, the partition functions $Z_{1+1}$ and $Z_{1+1}^*$ are already different at the exponential level. Indeed, expressing the partition
function via the pressure, $Z = \exp(VP(T, \mu))$, one finds that

$$ R = \frac{Z_{1+1}}{Z_{1+1}^*} = \exp[V(P_{1+1} - P_{1+1}^*)]. \tag{33} $$

As an illustration, consider sufficiently low temperatures ($T \ll m_\pi$). With exponential precision, the $\mu$-dependence of pressure is given by the masses of the lightest particles with nonzero charge to which the chemical potential couples, i.e., $P_{1+1}(T, \mu) - P_{1+1}(T, 0) \sim \mu^2 e^{-m_N/(3T)}$ and $P_{1+1}^*(T, \mu) - P_{1+1}^*(T, 0) \sim \mu^2 e^{-m_\pi/(2T)}$, as long as $T \ll m_\pi$. Since $P_{1+1}(T, 0) = P_{1+1}^*(T, 0)$, and neglecting $e^{-m_N/(3T)}$ compared to $e^{-m_\pi/(2T)}$ we can write (with double exponential precision):

$$ R(T, \mu) = \frac{Z_{1+1}(T, \mu)}{Z_{1+1}^*(T, \mu)} \sim \exp[-V \mu^2 e^{-m_\pi/(2T)}], \tag{34} $$

which means for $T \ll m_\pi$ the phase factor $R$ is exponentially small for large $V$. This is not surprising, since the “warm” gas of pions is very much different from the gas of baryons at the same temperature.

In contrast, in the random matrix model, as we have seen, the exponential of the volume (i.e., the matrix size $N$) cancels in the ratio $R$.

However, in QCD, for $T \sim T_c$ and higher, the difference between the pressures, $P_{1+1}(T, \mu) - P_{1+1}^*(T, \mu)$, although remaining of order $V \mu^2$, becomes smaller. Lattice studies indicate remarkable similarity of the partition functions $Z_{1+1}$ and $Z_{1+1}^*$, manifested, e.g., in the similar slopes $dT_c/d\mu^2$ of the pseudo-critical lines, and it has been also argued that the difference between the slopes is suppressed in the large $N_c$ limit of QCD [31]. It may be also added that in QCD at asymptotically large $T \gg T_c$ the difference vanishes due to the asymptotic freedom – quark flavors decouple from each other, and the pressure is independent of the relative sign of the quark chemical potentials. Thus the random matrix model results might be a useful guide to the sign problem in QCD at least in the range of temperatures near $T_c$, which is of much experimental and theoretical interest.

We wish to stress again that for the comparison of QCD sign problem to the RMT to be meaningful the lattice 4-volume $V$ should be finite. More quantitatively, the system

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4 For simplicity, we denote by $V$ the 4-dimensional Euclidean volume of the finite-temperature system: $V \equiv V_{\text{space}}/T$.

5 Interestingly, using the condition $R < 1$ one could derive the mass inequality: $m_\pi/2 < m_N/3$ [30].

6 Of course, as $T \to 0$, the ratio $R \to 1$, which is the reflection of the fact that for $T = 0$ there is no dependence on any chemical potential (for $\mu < m_\pi/2$), neither baryon nor isospin.
should be within the so-called epsilon-regime \cite{32,33}, where the RMT description of the QCD becomes exact. In QCD, as the volume is increased, the role of the exponential factor eventually becomes dominant and $R$ vanishes exponentially. The crossover between the epsilon-regime and the thermodynamic limit can be studied, e.g., along the lines of Ref. \cite{34}.

With the preceding discussion in mind, let us take the point of view \cite{26} that the sign problem becomes intractable (even on a finite volume), when we enter the domain of pion condensation in the phase-quenched theory, i.e., $R \equiv \langle e^{2i\theta} \rangle = 0$ domain of the RMT. In this context, it is interesting to see where, relative to this domain, is the critical point of the $1+1$ theory. In the 2-flavor random matrix model we studied, the critical point, as well as the whole first order transition line, is always inside the $R = 0$ domain, as Figures \cite{1} and \cite{2} demonstrate.

IV. SUMMARY AND CONCLUSIONS

We investigated the strength of the QCD sign problem and its dependence on temperature $T$ and quark mass $m$ using the random matrix model. We observed that the sign problem diminishes at higher temperatures, which is a welcomed property, anticipated, e.g., in \cite{20}, and relied upon in improved reweighting techniques, e.g., Ref. \cite{8}.

We also observed that the strength of the sign problem is related to the position of the pion-condensation region in the phase-quenched theory, equivalent to the theory where the baryon chemical potential is replaced by the isospin chemical potential (of the same absolute value per quark), as already discussed in Ref. \cite{26}. In particular, this underscores the importance of understanding the phase diagram of QCD at finite isospin chemical potential \cite{20,21,35,36,37}.

We observed and generalized to QCD the scaling behavior of the shape of the pion condensation region in the $T\mu$ plane near $T = T_c$ as $m \to 0$, where it develops a singularity (kink). This allows us to understand how the phase diagram of QCD with finite isospin chemical potential evolves towards the chiral limit. One of the practical applications of this result is to guard the reweighting techniques against a possible breakdown of the reweighting method by studying the dependence of the breakdown point on the quark mass.

In the random matrix model we studied, the critical point of the $1+1$ flavor theory falls
within the $R = 0$ domain of the maximal sign problem (see Figures [1] and [2]). This means that reweighting methods cannot access the critical point in such a theory. However, it is possible, that a 3-flavor theory, like QCD with a strange quark, where the critical point lies at smaller values of $\mu$ (tunable by the strange quark mass), that the critical point is outside the pion condensation domain. What happens in this case is an interesting question which we hope to address in future work.

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