In this document we characterize the topological phases of the pristine lattice with Bloch Hamiltonian in Eq. 1 of the Main Text, their topological invariants, and their associated Wannier centers. We then discuss the implementation of terms that once added to the Hamiltonian in Eq. 1 of the Main Text, break certain symmetries, possibly transforming the BICs into high-order topological resonances. Finally, we present plots of the energies of the system as a function of system size to conclude that the spread of higher-order topological resonances are not due to finite size effects in the simulations.

A. Irreducible representations of the energy bands of the lattice model

The Hamiltonian in Eq. 1 of the Main Text has $C_{4v}$ symmetry, which is generated by the simultaneous presence of $C_4$ symmetry,

$$\hat{r}_4 h(k_x, k_y) \hat{r}_4^\dagger = h(k_y, -k_x), \quad (S1)$$

and reflection symmetry,

$$\hat{M}_y h(k_x, k_y) \hat{M}_y^\dagger = h(-k_x, k_y), \quad (S2)$$

both of which imply also $C_2$ symmetry as well as reflection symmetries along $y$-denoted $M_{y'}$ and along the two diagonals -denoted $M_{d1}$ and $M_{d2}$. The topology of the crystalline phases of this model can be diagnosed by looking at the representations that the states take at the high-symmetry points (HSPs) of the Brillouin zone. In particular, we are interested in the HSPs $\Gamma$ and $\mathbf{M}$, which are invariant under the full group, $C_{4v}$, as well as the HSPs $\Gamma$, $\mathbf{X}$, and $\mathbf{X'}$, which are invariant only under the little group $C_{2v}$. The representations that each of these bands take at these points is given in Table S1.

The irreducible representations in Table S1 have the character tables detailed in Table S2.

Notice that only the group $C_{4v}$ has a two-dimensional irreducible representation, $E$. This is the representation of the bulk states at zero energy and which coexist with the topological corner BICs.

\begin{table}[h]
\begin{tabular}{cccccc}
\hline
phase & bands & $C_{4v}$ & & $C_{2v}$ & \\
\hline
$|t| < 1$ & $1$ & $B_2$ & $A_1$ & $a_2$ & $b_1$ & $b_2$ \\
& $2,3$ & $E$ & $E$ & $b_1 + b_2$ & $a_1 + a_2$ & \\
& $4$ & $A_1$ & $B_2$ & $a_1$ & $b_2$ & $b_1$ \\
$|t| > 1$ & $1$ & $B_2$ & $B_2$ & $a_2$ & $a_2$ & $a_2$ \\
& $2,3$ & $E$ & $E$ & $b_1 + b_2$ & $b_1 + b_2$ & $b_1 + b_2$ \\
& $4$ & $A_1$ & $A_1$ & $a_1$ & $a_1$ & $a_1$ \\
\hline
\end{tabular}
\caption{Symmetry representations at the high symmetry points of the BZ in both topological ($|t| < 1$) and trivial ($|t| > 1$) phases. Irreducible representations (irreps) at $\Gamma$ and $\mathbf{M}$ are for $C_{4v}$ and irreps at $\mathbf{X}$ and $\mathbf{X'}$ are for $C_{2v}$. Irreps $A_1$, $A_2$, $B_1$, $B_2$, $a_1$, $a_2$, $b_1$, $b_2$ are one dimensional. Irrep $E$ is two-dimensional.}
\end{table}

\begin{table}[h]
\begin{tabular}{cccccc}
\hline
irrep & $C_2$ & $2C_4$ & $2M_v$ & $2M_d$ & \\
$A_1$ & $1$ & $1$ & $1$ & $1$ & \\
$A_2$ & $1$ & $1$ & $-1$ & $-1$ & \\
$B_1$ & $1$ & $-1$ & $1$ & $-1$ & \\
$B_2$ & $1$ & $-1$ & $-1$ & $1$ & \\
$E$ & $2$ & $-2$ & $0$ & $0$ & \\
\hline
\end{tabular}
\caption{Character table for the $C_{4v}$ (left) and $C_{2v}$ (right) groups. The irreducible representations at the HSPs of the Brillouin zone for each energy band is shown in Table S1.}
\end{table}

B. Trivial and topological phases of the model and their Wannier centers

In real space, the topology of the energy bands in the lattice of Fig. 1(a) in the Main Text determines the positions of their Wannier centers [1, 2]. Although the Block Hamiltonian in Eq. 1 in the Main Text has $C_{4v}$ and chiral symmetries, $C_4$ or $C_2$ symmetries alone suffice to fix the positions of the Wannier centers to one of two disconnected maximal Wyckoff positions of the lattice: a ‘trivial’ Wannier center for $|t| > 1$, and a ‘topological’ one, for $|t| < 1$. These two phases are in different atomic limits [3]. The trivial atomic limit ($|t| > 1$) is described by Wannier centers that coincide with the centers of the unit cells [Fig. S1(a)], and the nontrivial atomic limit ($|t| < 1$) has Wannier centers at the corners of the unit cells [Fig. S1(b)]. In particular, when boundaries are open in both directions, the nontrivial atomic limit has a mismatch in the number of Wannier centers relative...
to the number of unit cells [Fig. S1(b)] which results in a fractional density of states at corners [Fig. 1(c), lower panels].

![Diagram showing Wannier center configurations](image)

FIG. S1. Wannier center configuration for (a) the trivial phase, |t| > 1 and (b) the topological phase, |t| < 1 for all bands in the model of Eq. 1 of the Main Text. Gray squares are unit cells. Blue and red circles are the Wannier centers. White circles represent the centers of the unit cells. In (b), Wannier centers in red are in excess relative to those with closed boundaries, inducing filling anomalies that result in fractional density of states at corners.

These two configurations can be diagnosed from symmetry indicator topological invariants, which are derived from the symmetry representations that each of the bands take at the HSPs of the Brillouin zone. Table. S3 compiles the representations for \( C_2 \) and \( C_4 \) symmetries. From these representations, we can calculate the symmetries and filling anomaly topological indices in both topological (\(|t| < 1\)) and trivial (\(|t| > 1\)) phases. Irreducible representations (irreps) at \( \Gamma \) and \( M \) are for \( C_4 \) and irreps at \( X \) and \( X' \) are for \( C_2 \).

### Table S3

| Phase | Bands | \( C_4 \) | \( C_2 \) |
|-------|-------|-----------|-----------|
| \(|t| < 1\) | 1 | \(-1\) | \(+1\) | \(-1\) | \(-1\) |
| \(|t| > 1\) | 2,3 | \(\pm i \pm 1\) | \(\{+1, +1\}\) | \(\{-1, -1\}\) |

When protected by \( C_4 \) symmetry and

\[
P^{(4)} = \frac{1}{2}[X_1^{(2)}](a_1 + a_2)
\]

when protected by \( C_2 \) symmetry [4]. In the topological phase, \( P = (\frac{\pi}{2}, \frac{\pi}{2}) \). With open boundaries, these moments generate an edge-induced filling anomaly [4], an excess in the number of states relative to those with no boundaries. In addition, this configuration has a (nominal) corner-induced filling anomaly [4]: an extra excess or depletion of states caused only in the presence of corners, which can be calculated via the following topological indices,

\[
Q_{\text{corner}}^{(4)} = \frac{1}{4}([X_1] + 2[M_1] + 3[M_2])
\]

when protected by \( C_4 \) symmetry and

\[
Q_{\text{corner}}^{(2)} = \frac{1}{4}([-X_1] - [Y_1] + [M_1])
\]

when protected by \( C_2 \) symmetry [4].
when protected by $C_2$ symmetry.

The filling anomaly causes existence of corner-localized states that constitute topological BICs (if additionally $C_{4v}$ and chiral symmetries are preserved) or topological resonances (if either $C_{4v}$ or chiral are broken). We emphasize that not all lattices with $\mathbf{P} = (\frac{1}{2}, \frac{1}{2})$ have a corner-induced filling anomaly. A case in point is the lattice in Fig. 2(e) in Ref. 4.

The symmetry indicator invariants and their topological indices for polarization and corner filling anomalies for the bands in our model are shown in Table S4 and S5.

C. Constraints on the energy spectrum due to chiral symmetry

Consider the energy eigenstate $|u\rangle$ with energy $\epsilon$, such that

$$h|u\rangle = \epsilon|u\rangle.$$  \hspace{1cm} (S8)

If the Hamiltonian $h$ has chiral symmetry, $\{h, \Pi\} = 0$, then the state $\Pi|u\rangle$ is an energy eigenstate of $h$ with energy $-\epsilon$,

$$h\Pi|u\rangle = -\Pi h|u\rangle = -\epsilon \Pi|u\rangle.$$  \hspace{1cm} (S9)

Thus, the energies in a system with chiral symmetry come in pairs ($\epsilon, -\epsilon$), and their states are related by the chiral operator $\Pi$. From this, it follows that states with $\epsilon = 0$ are either eigenstates of $\Pi$, in which case support only in one sublattice, or come in pairs $(|u\rangle, \Pi|u\rangle)$. In the Hamiltonian of Eq. 1 in the Main Text, examples of the first case are the individual zero energy corner states, while an example of the second case is the subspace of bulk states at zero energy.

D. Implementing the symmetry-breaking perturbations

To generate the results in Fig. 4 of the Main Text, additional hopping terms where added to the Hamiltonian of Eq. 1 in the Main Text. The overall Hamiltonian before introducing losses in the system, Eq. 2 in the Main Text, is

$$h_T(\mathbf{k}) = h(\mathbf{k}) + \Delta_p h_p(\mathbf{k}),$$  \hspace{1cm} (S10)

where $\Delta_p$ is the overall strength of the perturbation and

$$h_p(\mathbf{k}) = T_{x1}\cos k_x + T_{x2}\sin k_x + T_{y1}\cos k_y + T_{y2}\sin k_y$$

$$+ T_1\cos k_x \sin k_y + T_2\sin k_x \cos k_y$$  \hspace{1cm} (S11)

is the Hamiltonian of the additional perturbation, which amounts to hopping terms up to next nearest neighbor

unit cells. The $T$ matrices are all $4 \times 4$ random Hermitian matrices in which each entry has a complex value with a uniform distribution in the range $[0, 1]$. In addition to obeying Hermicity, the $T$ matrices are subject to certain constraints imposed by the symmetries we are interested in preserving. In what follows we detail examples of the constraints on the $T$ matrices used for the preservation of certain symmetries:

1. For chiral symmetry

Under chiral symmetry, $\Pi h(\mathbf{k})\Pi = -h(\mathbf{k})$, all $T$ matrices must obey

$$\{T, \Pi\} = 0.$$  \hspace{1cm} (S12)

2. For $C_4$ symmetry

Let us first focus on the nearest neighbor $T$ matrices. To first satisfy $C_2$ symmetry, $\hat{r}_2 h(k_x, k_y)\hat{r}_2^\dagger = h(-k_x, -k_y)$, we require

$$[T_{x1}, \hat{r}_2] = 0, \quad \{T_{x2}, \hat{r}_2\} = 0.$$  \hspace{1cm} (S13)

Now, to satisfy $C_4$ symmetry, $\hat{r}_4 h(k_x, k_y)\hat{r}_4^\dagger = h(k_y, -k_x)$, we additionally require

$$T_{y1} = \hat{r}_4 T_{x1}\hat{r}_4^\dagger, \quad T_{y2} = -\hat{r}_4 T_{x2}\hat{r}_4^\dagger.$$  \hspace{1cm} (S14)

The two next nearest neighbor $T$ matrices are odd under $C_2$ symmetry. Take first $T_1$ to obey

$$\{T_1, \hat{r}_2\} = 0,$$  \hspace{1cm} (S15)

and then determine $T_2$ via the constraint due to $C_4$ symmetry,

$$T_2 = -\hat{r}_4 T_2\hat{r}_4^\dagger.$$  \hspace{1cm} (S16)

3. For reflection symmetry

Under reflection symmetry along $x$, $\hat{M}_x h(k_x, k_y)\hat{M}_x^\dagger = h(-k_x, k_y)$, four $T$ matrices are even under $M_x$ and two are odd,

$$[T_{x1}, \hat{M}_x] = 0, \quad [T_{y1}, \hat{M}_x] = 0,$$

$$[T_{y2}, \hat{M}_x] = 0, \quad [T_1, \hat{M}_x] = 0,$$

$$\{T_{x2}, \hat{M}_x\} = 0, \quad \{T_2, \hat{M}_x\} = 0.$$  \hspace{1cm} (S17)

If more than one symmetry is to be preserved, the constraints due to each of them have to be met simultaneously. Once the $T$ matrices are chosen, an inverse Fourier transform allows to implement the hopping terms in real
space. For example, the nearest neighbor perturbations along $x$ lead to

$$T_{x1} \cos k_x + T_{x2} \sin k_x \rightarrow \sum_{x,y} \sum_{\alpha,\beta=1}^4 c_{(x,y),\alpha}^\dagger \frac{1}{2} \left( \frac{T_{x1} - iT_{x2}}{2} \right)_{\alpha,\beta} c_{(x+1,y)\beta} + h.c., \quad (S18)$$

where the sum over $x$ and $y$ run over the coordinate of unit cells in the entire lattice.

I. Scaling of energies with lattice size

Bound states exponentially penetrate into the bulk of the lattice. Resonances, on the other hand, will have both a ‘corner’ component, which will exponentially penetrate into the lattice, and a ‘bulk’ component, which will not. In our scheme for detection of BICs which introduces loss to the bulk but not the corners, the difference in the penetration between BICs and resonances will be manifested by zero imaginary energies for BICs and non-zero imaginary energies for resonances. To rule out the possibility that finite size effects interfere in this differentiation, in this section, we show how the values of the imaginary components of energy vary as the lattice size $n$ increases. This is shown in Fig. S2. In (a), we show the imaginary energies for a lattice with the original Hamiltonian, Eq. 1 of the Main Text, with added perturbations that preserve both $C_{4v}$ and chiral symmetries (implemented as described in Section D). Under these symmetries, corner BICs are protected and thus their imaginary energies are zero (red dots). Bulk states (blue dots), on the other hand, will acquire non-zero imaginary energies due to the losses in the bulk. In (b), perturbations are added to the original lattice, Eq. 1 of the Main Text, which break reflection symmetries. This breaks the mechanism protection of BICs, which the cease to exist as they hybridize with bulk states to form resonances. This is manifested in the fact that the imaginary energies are not zero anymore (purple dots). Notice that the values converge for lattice sizes as low as $n = 16$.

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FIG. S2. Imaginary components of energy as a function of lattice size $n$ for a fixed value of system size $n_s = 4$. (a) Lattice preserving both $C_{4v}$ and chiral symmetries. (b) A lattice that breaks $C_{4v}$ down to only $C_4$ while keeping chiral symmetry. Only in (a) there are BICs (red dots, which are 4-fold degenerate). In (b), as the lattice size increases, the imaginary components of the energy of resonances converge to a non-zero imaginary value (purple dots). The insets are zoomed in versions of (a) and (b) around Im(E)=0.