SL(2, R) ⊗ SU(2)/R² STRING MODEL IN CURVED SPACETIME
AND EXACT CONFORMAL RESULTS †

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ABSTRACT

Pursuing further the recent methods in the algebraic Hamiltonian approach to gauged WZW models, we apply them to the bosonic SL(2, R)_{-k'} ⊗ SU(2)_k / (R ⊗ R̃) model recently investigated by Nappi and Witten. We find the global space and compute the conformally exact metric and dilaton fields to all orders in the 1/k expansion. The semiclassical limit k', k → ∞ of our exact results agree with the lowest order perturbation computation which was done in the Lagrangian formalism. We also discuss the supersymmetric type-II and heterotic versions of this model and verify the non-renormalization of e^Φ √−G.

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The non-compact coset (or equivalently non-compact gauged WZW model) approach to string theory in curved spacetime \[ \text{1} \text{ 2} \] has been under investigation during the past year, following the two dimensional black hole interpretation of Witten \[ \text{3} \]. Many of the later papers \[ \text{4-13} \] have used the gauged WZW Lagrangian approach in a unitary gauge that yields an effective sigma model with a classical background metric, antisymmetric tensor and one loop dilaton. In the patch of the spacetime that is given by a particular unitary gauge, the interpretation of the geometry ranged from black hole singularities to cosmology for various cosets.

In recent papers we showed how to improve on the perturbative Lagrangian results by using algebraic Hamiltonian techniques to compute globally valid \[ \text{14} \] and conformally exact \[ \text{15} \text{ 16} \] geometrical quantities such as the metric and dilaton (and, in principle, other fields) in gauged WZW models. We have applied the method to bosonic, heterotic and type-II supersymmetric 4D string models that use the non-compact cosets \( SO(3,2)/SO(3,1), \text{IR} \times SO(2,2)/SO(2,1) \) and \( \text{IR}^2 \times SO(2,1)/\text{IR} \).

There are just two remaining cosets that are relevant to strings on four dimensional curved spacetime backgrounds. These are \( SL(2,\text{IR}) \times SL(2,\text{IR})/\text{IR}^2 \) and \( SL(2,\text{IR}) \times SU(2)/\text{IR}^2 \) (for notational convenience we do not distinguish between \( \text{IR} \) and \( U(1) \)). The latter coset was investigated recently by Nappi and Witten \[ \text{17} \] in the conformally perturbative semi-classical limit. In a particular patch of the geometry they showed that the effective sigma model metric describes an expanding and recollapsing universe. This coset was also examined before \[ \text{10} \text{ 12} \]. In the present note we reinvestigate this coset with the methods of \[ \text{14} \text{ 15} \] with the purpose of finding the global space, including all dual patches, interpreting the geometry and obtaining the conformally exact metric and dilaton fields. We will discuss the bosonic as well as the heterotic and type-II supersymmetric versions of the string model. The geometry for the \( SL(2,\text{IR}) \times SL(2,\text{IR})/\text{IR}^2 \) coset is obtainable just by an analytic continuation from the one investigated presently, and therefore we will not comment about it anymore in this paper.

It is convenient to parametrize the group element \( g \) of \( G \) as follows

\[
g = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}, \quad g_1 \in SL(2,\text{IR}), \quad g_2 \in SU(2),
\]

A generalization of the coset \( \text{IR} \times SO(2,2)_{-k}/SO(2,1)_{-k} \) was given in \[ 2 \] as \( \text{IR} \times SL(2,\text{IR})_{-k_1} \times SL(2,\text{IR})_{-k_2}/SL(2,\text{IR})_{-k_1-k_2} \). The exact results for any \( k_1, k_2 \) may be deduced from \[ 13 \] with a little effort. Another model based on \( \text{IR} \times SL(2,\text{IR})_{k_1} \) may be viewed as the limit \( k_2 \to \infty \) of the generalized case, and therefore need not be considered separately.
with
\[ g_1 = \begin{pmatrix} a & u \\ -v & b \end{pmatrix}, \quad ab + uv = 1 \] (2)
and
\[ g_2 = e^{i\gamma_2} e^{i\sigma_3} e^{i\beta_2}, \] (3)
where \(\sigma_i, \; i = 1, 2, 3\) are the standard \(2 \times 2\) Pauli matrices. We will be interested below in the generators of left (right) transformations \(\delta g = \epsilon L g (\delta g = g \epsilon R)\). For \(SL(2, \mathbb{R})\) they are defined by \(J_i g = -t_i^{(1)} g (\bar{J}_i g = g t_i^{(1)})\) and for \(SU(2)\) one has \(I_i g = -t_i^{(2)} g (\bar{I}_i g = g t_i^{(2)})\). The Pauli matrix representations of the \(SL(2, \mathbb{R})\) and \(SU(2)\) generators are
\[
\begin{align*}
t_1^{(1)} &= \begin{pmatrix} i \sigma_2 & 0 \\ 0 & 0 \end{pmatrix}, & t_2^{(1)} &= \begin{pmatrix} \sigma_2 & 0 \\ 0 & 0 \end{pmatrix}, & t_3^{(1)} &= \begin{pmatrix} i \sigma_3 & 0 \\ 0 & 0 \end{pmatrix}, & t_i^{(2)} &= \begin{pmatrix} 0 & 0 \\ 0 & \sigma_i \end{pmatrix}.
\end{align*}
\] (4)

We gauge an abelian subgroup \(H\) isomorphic to \(\mathbb{R} \otimes \mathbb{R}\) generated by the following infinitesimal transformations of \(g_1, \; g_2\)
\[
\begin{align*}
\delta g_1 &= \epsilon \frac{\sigma_3}{2} g_1 + (\epsilon \cos \alpha + \epsilon \sin \alpha) g_1 \frac{\sigma_3}{2} \\
\delta g_2 &= i \sqrt{\frac{k'}{k}} (\epsilon \frac{\sigma_2}{2} g_2 + (-\epsilon \sin \alpha + \epsilon \cos \alpha) g_2 \frac{\sigma_2}{2}) .
\end{align*}
\] (5)
The relative coefficients in various terms have been fixed to insure gauge invariance \(\bar{\epsilon}\). The generators of \(H\) may be identified by writing (5) in the form \(\delta g = i(\epsilon Q g + \bar{\epsilon} \bar{Q} g)\),
\[
\begin{align*}
Q &= J_3 - \sin \alpha \; \bar{J}_3 + \sqrt{\frac{k'}{k}} \cos \alpha \; \bar{I}_2 \\
\bar{Q} &= -\sqrt{\frac{k'}{k}} I_2 - \cos \alpha \; \bar{J}_3 - \sqrt{\frac{k'}{k}} \sin \alpha \; \bar{I}_2 .
\end{align*}
\] (6)
Then it is easy to check that the condition for gauge invariance, or anomaly cancellation \(\bar{\epsilon}\) in the gauged WZW model is satisfied. Namely,

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\(2\) There is a more general set of coefficients that are consistent with gauge invariance, as follows: In the transformation of \(g_1\) replace \(\cos \alpha\) with \(\epsilon_1 \cos \alpha\) and in the transformation of \(g_2\) replace \(\cos \alpha\) by \(\epsilon_2 \cos \alpha\) and \(\sin \alpha\) by \(\epsilon_1 \epsilon_2 \sin \alpha\). The \(\epsilon_i\) are independently chosen as \(\pm\). Such sign switches generate non-trivial discrete duality transformations of the type discussed in \(\bar{\epsilon}\). These are the analog of the vector/axial (or \(R \to 1/R\)) duality in the 2D black hole model. In the discussion that follows we will mention the effect of these signs on the geometry.
\[ Tr(Q_L^a Q_L^b \begin{pmatrix} -k' & 0 \\ 0 & k \end{pmatrix}) = Tr(Q_R^a Q_R^b \begin{pmatrix} -k' & 0 \\ 0 & k \end{pmatrix}) , \] (7)

where \( Q_L = (Q_L, \tilde{Q}_L) \) and \( Q_R = (Q_R, \tilde{Q}_R) \) are the matrix representations of \( Q, \tilde{Q} \) applied on the left and right of \( g \), as follows from (4)(5). The reason for the \( k', k \) insertions is understood by writing the action in terms of traces over \( g \) rather than \( g_1, g_2 \) individually.

The central charge of the model is

\[ c = \frac{3k'}{k' - 2} + \frac{3k}{k + 2} - 2 . \] (8)

This model may be taken in only four dimensions by requiring the appropriate central charge, as suggested in [19], or one can tensor this model with another conformal field theory representing some internal space and demand that the total central charge be \( c_{tot} = 26 \) (or \( c_{tot} = 15 \) when supersymmetric). If, as in [17], one requires for the curved spacetime model the same central charge as that of flat Minkowski space, namely \( c = 4 \), then (8) gives the condition \( k' = k + 4 \). This particular choice of the central charge has the feature that both \( k' \) and \( k \) can continously reach infinite values and therefore the 1–loop results obtained in [17] correspond to a conformal model with the correct central charge. Any other choice of \( c \) requires finite \( k' \) or \( k \) even if one of them is infinite, and therefore higher loop corrections are of substantial importance.

With the Hamiltonian method (see [15] for more details) the metric and dilaton for the coset \( G/H=SL(2, \mathbb{R})_{-k'} \otimes SU(2)_k / (\mathbb{R} \otimes \bar{\mathbb{R}}) \) are determined by comparing the two sides of the following equation

\[ (L_0 + \bar{L}_0) T = -\frac{1}{e^{\Phi} \sqrt{-G}} \partial_\mu (G^{\mu\nu} e^\Phi \sqrt{-G} \partial_\nu T) , \quad \mu, \nu = 0, 1, 2, 3 , \] (9)

where \( L_0, \bar{L}_0 \), the zero modes of the stress tensors for the left and right movers respectively, are represented as second order differential operators on group space, and \( T \) is a tachyon level state which is taken to be a gauge singlet. The appropriate algebraic expression for \( L_0 \), which contains the exact dependence on \( k \) and \( k' \), is

\[ L_0 = \frac{\Delta_{SL(2, \mathbb{R})}}{-k' + 2} + \frac{\Delta_{SU(2)}}{k + 2} - \frac{Q_L^2}{k'} - \frac{\tilde{Q}_L^2}{k'} , \] (10)

where, as seen from (1), \( (Q_L, \tilde{Q}_L) = (J_3, -\sqrt{k'/k} I_2) \) are the zero modes of the left-moving subgroup currents whose central extensions are \( (k', k') \). The Casimir operators defined in terms of hermitian generators are
\[ \Delta_{SL(2, \mathbb{R})} = -J_3^2 + \frac{1}{2}(J_+ J_- + J_- J_+) , \quad \Delta_{SU(2)} = I_1^2 + I_2^2 + I_3^2 , \]  

(11)

where \( J_\pm = J_2 \mp J_1 \). The corresponding expressions for the right movers are

\[ \bar{L}_0 = \frac{\Delta_{SL(2, \mathbb{R})}}{-k' + 2} + \frac{\Delta_{SU(2)}}{k + 2} - \frac{Q_R^2}{k'} - \frac{\tilde{Q}_R^2}{k'} , \]  

(12)

where, as seen from (11), the right moving current \( Q_R = (-J_3 \sin \alpha + I_2 \sqrt{k'/k} \cos \alpha) \) has central extension \( k' \sin^2 \alpha + k(\sqrt{k'/k} \cos \alpha)^2 = k' \), and similarly \( \tilde{Q}_R = (-\tilde{J}_3 \cos \alpha - \tilde{I}_2 \sqrt{k'/k} \sin \alpha) \) has central extension \( k' \cos^2 \alpha + k(\sqrt{k'/k} \sin \alpha)^2 = k' \). The formula for \( \bar{L}_0 \) can be simplified by noticing that

\[ -\frac{Q_R^2}{k'} - \frac{\tilde{Q}_R^2}{k'} = -\frac{J_3^2}{k'} - \frac{\tilde{J}_3^2}{k} . \]  

(13)

Thus, despite the complicated intermediate steps, \( \bar{L}_0 \) and \( L_0 \) end up having the same structure in terms of the respective left and right moving currents. Note also that the group Casimir operators are always the same for the left and and right movers, i.e. \( \Delta_{SL(2, \mathbb{R})} = \Delta_{SL(2, \mathbb{R})} \) and \( \Delta_{SU(2)} = \Delta_{SU(2)} \).

Now consider the differential operator form of the left and right generators whose action on the group element reproduces the matrix representation of (11). We take the hermitian \( SL(2, \mathbb{R}) \) generators in the form given in [16] with an extra factor of \( i \)

\[
J_3 = \frac{i}{2} (v \partial_v - a \partial_a - u \partial_u) , \quad \bar{J}_3 = \frac{i}{2} (a \partial_a - u \partial_u + v \partial_v)
\]

\[
J_+ = i (b \partial_u - v \partial_a) , \quad \bar{J}_+ = -ia \partial_a
\]

\[
J_- = ia \partial_v , \quad \bar{J}_- = i(u \partial_a - b \partial_v)
\]  

(14)

In the above expressions the parameters \( a, u, \) and \( v \) were taken as the independent ones, whereas \( b = (1 - uv)/a \). For the \( SU(2) \) currents we have the following expressions

\[
I_1 = \frac{i}{2} \left( \frac{\cos 2\gamma}{\sin 2s} \partial_\beta - \sin 2\gamma \partial_s - \cos 2\gamma \cot 2s \partial_\gamma \right)
\]

\[
I_2 = \frac{i}{2} \partial_\gamma
\]

\[
I_3 = \frac{i}{2} \left( \frac{\sin 2\gamma}{\sin 2s} \partial_\beta + \cos 2\gamma \partial_s - \sin 2\gamma \cot 2s \partial_\gamma \right)
\]  

(15)

and
\[ I_1 = \frac{i}{2} \left( -\cos 2\beta \cot 2s \partial_\beta - \sin 2\beta \partial_s + \frac{\cos 2\beta}{\sin 2s} \partial_\gamma \right) \]
\[ I_2 = -\frac{i}{2} \partial_\beta \]
\[ I_3 = \frac{i}{2} \left( \sin 2\beta \cot 2s \partial_\beta - \cos 2\beta \partial_s - \frac{\sin 2\beta}{\sin 2s} \partial_\gamma \right) . \]

The \( SL(2, \mathbb{R}) \) and \( SU(2) \) Lie algebras are indeed obeyed by the right and left generators separately and moreover any left generator commutes with any right generator.

We are now prepared to consider the differential equation obeyed by a gauge singlet state \( T \) at the tachyon level. A priori \( T(g_1, g_2) \) is a function of the 6 group parameters \( u, v, a, \beta, \gamma, s \). However, since it is assumed to be a gauge singlet, it can be a function of only gauge singlet combinations of these parameters, of which there are four, i.e. the four dimensional spacetime. Then it automatically obeys the gauge invariance conditions

\[ QT = 0 , \quad \bar{Q}T = 0 . \]

From this one can also prove immediately that the condition \( (L_0 - \bar{L}_0)T = 0 \) for closed bosonic strings is satisfied.

In order to find the global geometry we must construct the 4 gauge invariant combinations of group parameters that are identified with the 4D spacetime. These are

\[ s , \quad r = uv , \quad \rho = \beta + \gamma - \frac{1}{2} \sqrt{\frac{k'}{k}} \frac{\cos \alpha}{1 + \sin \alpha} \ln \frac{a}{b} , \quad \lambda = \gamma - \beta + \frac{1}{2} \sqrt{\frac{k'}{k}} \frac{\cos \alpha}{1 - \sin \alpha} \ln \frac{u}{v} . \]

where \( b \) is to be expressed in terms of \( a, u, v \) as above. One can check that, these are indeed invariants by extracting the infinitesimal transformation of each parameter from eq.(5).

When the Virasoro operators \( L_0, \bar{L}_0 \) are now applied on a function of the 4 global string variables \( X^\mu = (r, s, \rho, \lambda) \), i.e. \( T(X) \), they may be rewritten as second order differential operators in 4 dimensional spacetime rather than the 6 dimensional group space, by using the chain rule. Then we are in a position to compare to the right hand side of (9) and extract the inverse metric \( G^{\mu\nu}(X) \) from the double derivative terms and the dilaton \( \Phi(X) \) from the single derivative terms. We emphasize that the exact dependence on \( k, k' \) is included in these expressions. Finally, by taking the inverse of \( G^{\mu\nu} \) we derive the line element

\[ dS^2 = 2(k' - 2) \left( G_{rr} dr^2 + G_{ss} ds^2 + G_{\rho\rho} d\rho^2 + G_{\lambda\lambda} d\lambda^2 \right) , \quad (19) \]
with
\[
G_{rr} = -\frac{1}{4r(1-r)} \\
G_{ss} = \frac{k + 2}{k' - 2} \\
G_{\rho\rho} = \frac{1 + \sin \alpha}{1 - \sin \alpha} \frac{1 - r}{B_s + \frac{k'}{k}r} \\
G_{\lambda\lambda} = \frac{1 - \sin \alpha}{1 + \sin \alpha} \frac{r}{rA_s + \frac{k'}{k}(1-r)},
\]
where \(A_s\), and \(B_s\) are functions defined as follows
\[
A_s = 2 \frac{k + 1}{k} + \frac{1 - \sin \alpha}{1 + \sin \alpha} \frac{k'}{k} - 2 \left(\cot^2 s - \frac{2}{k}\right) \\
B_s = 2 \frac{k + 1}{k} + \frac{1 - \sin \alpha}{1 + \sin \alpha} \frac{k'}{k} + 2 \left(\tan^2 s - \frac{2}{k}\right).
\]
The expression for the dilaton is
\[
\Phi = \frac{1}{2} \ln \left[\sin^2 2s (rA_s + \frac{k'}{k}(1-r)) \left((1-r)B_s + \frac{k'}{k}r\right)\right] + \Phi_0.
\]
Let us now take the large \(k, k'\) limit and compare our global semi-classical geometry to the one discussed in [17]. In the global space the semi-classical line element and dilaton take the form
\[
dS^2 = 2k' \frac{dr^2}{4r(r-1)} + 2k \left(ds^2 + \frac{d\rho^2}{\tan^2 s - \frac{r}{r-1} \tan^2 \left(\frac{\pi}{4} - \frac{\alpha}{2}\right)} + \frac{d\lambda^2}{\cot^2 s - \frac{r-1}{r} \cot^2 \left(\frac{\pi}{4} - \frac{\alpha}{2}\right)}\right), \\
\Phi = \ln[r \tan^2 \left(\frac{\pi}{4} - \frac{\alpha}{2}\right) \cos^2 s - (r - 1) \sin^2 s] + \Phi'_0.
\]
The scalar curvature for the semi-classical geometry is
\[
R = \frac{7}{4} \frac{4r(r-1)(\cos 2s - \sin \alpha)^2 - \frac{1}{k} \sin^2 2s(1-2r + \sin \alpha)^2}{(\sin^2 s (1 + \sin \alpha) + r (\cos 2s - \sin \alpha))^2} \\
+ \frac{5}{4} \frac{1}{k}(1-2r)(\sin \alpha - \cos 2s) + \frac{1}{k} \cos 2s (1-2r + \sin \alpha)}{\sin^2 s (1 + \sin \alpha) + r (\cos 2s - \sin \alpha)} + 3\left(\frac{1}{k} - \frac{1}{k'}\right).
\]
\(^3\) Note that the combination \(e^\Phi \sqrt{-G}\) is \(k, k'\) independent. Either the exact or the semi-classical metric and dilaton give the same result. The non-renormalization of this quantity has been noticed for all coset models [15][16].
From the definition (18) of the global variables, and the ranges of the group parameters from which they are constructed, we deduce that $s$ is periodic and taken in the range $0 < s < \pi$, while $r, \rho, \lambda$ take values from $-\infty$ to $+\infty$. In the range $0 < r < 1$ there are no singularities in the metric, dilaton or scalar curvature. At generic values of $s$ there are no curvature singularities at $r = 0, 1$ even though the metric appears singular; this is simply a coordinate singularity which is eliminated by a redefinition of variables. However, at $r = 0$ and $s = 0, \pi$ the dilaton and scalar curvature blow up. Similarly, at $r = 1$ and $s = \pi/2$ these quantities are singular. These points are shown by heavy dots on the figure. In addition, as may be seen from the metric, for $r > 1$ or $r < 0$ there are curvature and dilaton singularities at arbitrary values of $r$ when

$$s = \theta(\alpha, r), \quad \text{or} \quad \pi - \theta(\alpha, r)$$

(25)

where $\tan^2 \theta(\alpha, r) = \frac{r}{r-1} \tan^2 \left( \frac{\pi}{4} - \frac{\alpha}{2} \right)$. The singularities are shown as lines in the global $(r, s)$ space in Fig.1. These lines approach asymptotically $s = s_0 = \theta(\alpha, \infty)$. On this figure we have also indicated the signature in the basis $(r, s, \rho, \lambda)$. We see that, as in other examples we studied there is a single time coordinate, but the role of time switches from one coordinate to another in different patches. Geodesics can cross the dotted boundary lines at $r = 0, 1, \infty$ and $s = 0, \pi$ (periodic), thus connecting all the regions into each other in principle.

The geodesics can be explicitly solved with the method described in [14], but we will not discuss this point here in detail. However, let us mention what happens when a geodesic touches one of the singularity lines. A generic geodesic (time-like or light-like) seems to come tangentially to the singularity and then bounce from it. But actually what it does is move into another “world” that is a copy of the original one and glued to it at the singularity lines. It is described by the same invariant coordinates $(r, s, \rho, \lambda)$, but has a different discrete gauge invariant label. This corresponds to multicovers of the group space. This phenomenon occurs in all previous models as well, even for the 2D black hole, although it has not been discussed explicitly. We do not have the space to explain this phenomenon here, but simply say that it can be reconstructed from the explicit geodesics for the entire group manifold (as in [14]) and cannot be seen by concentrating on only the 4 global invariants (or working in specific gauges). The point is that there are additional discrete gauge invariants that label the different “worlds”.

As in other non-compact coset models, there are interesting discrete duality transformations [6][7][14] in the present model. The dual models and patches are generated by the
If these signs are inserted in the expressions for \( Q, \tilde{Q} \) in (1) and in the invariants (18) then instead of (20) and (22) the dual metric and dilaton get generated. The result is the following: in (20) and (22) replace \((s, \rho, \lambda)\) by \((\frac{\pi}{2} - s, \tilde{\lambda}, \tilde{\rho})\), where \((\tilde{\lambda}, \tilde{\rho})\) is a new point in the \((\rho, \lambda)\) space. Another way to view this result is to leave the \((s, \rho, \lambda)\) unchanged, but instead interchange \(r\) with \((r - 1)\) and also change the parameter \(\alpha\) to \(-\alpha\). In the second point of view the global \((r, s)\) space will be described by a figure obtained from Fig.1 by a left-right reflection at \(r = \frac{1}{2}\) and by changing \(s_0\) to \(\frac{\pi}{2} - s_0\). So the dual regions can be obtained by comparing these two figures.

In previous studies of the present coset [10] [12] [17] the choice of unitary gauges forced the authors to concentrate on some of the patches in our figure. In specific gauges it is not possible to recognize the full global space. To see how this happens, let us consider one of the gauges in [17], namely \(a = b = \cos \psi\) and \(u = v = \sin \psi\), so that the four independent parameters are \(r = uv = \sin^2 \psi, \rho = \beta + \gamma, \gamma = \beta - \gamma\) and \(s\). Then our metric and dilaton in (23) become identical to those displayed by Nappi and Witten [17] when \(k = k'\). We see that in this gauge we only recover the patch labelled II in Fig.1, and moreover \(\rho, \lambda\) appear to be periodic, although in the full global space this is not true. The second gauge in [17] parametrized by \(x\) is also insufficient to reveal the structure of Fig.1. Similar comments apply to the gauges studied in [10] [12].

We now move on to a brief discussion of the type-II supersymmetric and heterotic versions of the string theory based on the present coset. The algebraic structure of the supersymmetric coset and a description of the action has been given elsewhere [19]. We wish to compute the conformally exact metric and dilaton following the general discussion in [17]. The main point is that the presence of the supersymmetric fermionic partners leads to a modification of the gauge currents in such a way that they acquire new central charges \(k + 2, k' - 2\) that are shifted by the Coxeter numbers of the corresponding groups. This has an important impact on the \(k, k'\) dependence of the Virasoro operators \(L_0, \bar{L}_0\) as well as on the geometry that follows from them through the Dalambertian.

We first explain how the shifting in the level occurs since there are some new features not encountered before. The complications are due to the fact that there are two different central extensions and the gauge currents are constructed by mixing currents that have these different extensions. For definiteness we consider the left movers which are assumed

\[
\begin{align*}
\epsilon_1, \epsilon_2 \text{ dependence of these quantities are given by } &
\rho = \beta + \epsilon_1 \epsilon_2 \gamma - \frac{\pi}{2} \sqrt{\frac{k'}{k} \frac{\cos \alpha}{1 + \sin \alpha}} \ln \frac{u}{v} \text{ and } \\
\lambda = \epsilon_1 \epsilon_2 \gamma - \beta + \frac{\epsilon_2}{2} \sqrt{\frac{k'}{k} \frac{\cos \alpha}{1 - \sin \alpha}} \ln \frac{u}{v}.
\end{align*}
\]
to be supersymmetric. If the right movers are also supersymmetrized (as in the type-II superstring) the same discussion will apply to them. We introduce left-moving fermions classified in $G/H$. This means $SL(2, \mathbb{R})/\mathbb{R}$ fermions $\psi_1, \psi_2$ and $SU(2)/U(1)$ fermions $\chi_1, \chi_3$. The left-moving part of gauge currents are modified by the presence of the fermions as follows \[20\]

$$J_3 \rightarrow J'_3 = J_3 + \psi_+ \psi_- , \quad I_2 \rightarrow I'_2 = I_2 + \chi_+ \chi_- ,$$

(26)

where $\psi_\pm = \psi_2 \pm \psi_1$ and $\chi_\pm = \chi_3 \pm i\chi_1$. Next we construct the gauge currents for the heterotic string

$$Q_{het} = J'_3 - \sin \alpha J_3 + \sqrt{\frac{k'}{k}} \cos \beta I_2$$

$$\tilde{Q}_{het} = -\sqrt{\frac{k'-2}{k+2}} I'_2 - \cos \alpha J_3 - \sqrt{\frac{k'}{k}} \sin \alpha I_2 .$$

(27)

and the gauge currents for the type-II string

$$Q_{II} = J'_3 - \sin \alpha J_3 + \sqrt{\frac{k'-2}{k+2}} I'_2$$

$$\tilde{Q}_{II} = -\sqrt{\frac{k'-2}{k+2}} I'_2 - \cos \alpha J_3 - \sqrt{\frac{k'-2}{k+2}} \sin \alpha I'_2 .$$

(28)

Note the shifts in the $k, k'$ that are quite different from each other and from the bosonic case in \[3\]. These shifts are required in the quantum theory in order to maintain the gauge invariance of the stress tensor in each case. This can be verified by demanding that the operator products of the stress tensor and the gauge currents vanish. The stress tensor for the left movers is

$$T = T_G + T_{\psi \chi} - \frac{(Q_L)^2}{k' - 2} - \frac{\tilde{Q}_L^2}{k' - 2} ,$$

(29)

where $(Q_L, \tilde{Q}_L)$ are the left-moving pieces of the modified gauge currents. Note that their central extensions have also shifted relative to the bosonic case. Here $T_G$ is the standard Sugawara stress tensor for the group $SL(2, \mathbb{R})_{-k'} \otimes SU(2)_k$ and $T_{\psi \chi}$ is the stress tensor for the free fermions $(\psi_1, \psi_2, \chi_1, \chi_3)$. The stress tensor for the right movers will be like the bosonic case if we consider the heterotic string, and it will be like above if we consider the type-II string.
Now that we have the exact $k, k'$ dependence of the operators we can compute the transformations they generate on the group parameters. The transformation law will now be sensitive to the shifts in $k, k'$ and therefore the construction of the invariants will change accordingly. The previous $r, s$ invariants remain as in (18) but $\rho, \lambda$ get modified. In the type-II case we have

$$\rho = \beta + \gamma - \frac{1}{2} \sqrt{\frac{k' - 2}{k + 2}} \cos \alpha \ln \frac{a}{b}, \quad \lambda = \gamma - \beta + \frac{1}{2} \sqrt{\frac{k' - 2}{k + 2}} \cos \alpha \ln \frac{u}{v}. \quad (30)$$

while for the heterotic case we obtain

$$\rho = \beta + \sqrt{\frac{k'(k + 2)}{k(k' - 2)}} \gamma - \frac{1}{2} \sqrt{\frac{k'}{k}} \cos \alpha \ln \frac{a}{b}, \quad \lambda = \sqrt{\frac{k'(k + 2)}{k(k' - 2)}} \gamma - \beta + \frac{1}{2} \sqrt{\frac{k'}{k}} \cos \alpha \ln \frac{u}{v}. \quad (31)$$

Using these expressions we can now repeat the procedure of (9) to compute the metric and dilaton. For the type-II case the result is quite simple: the exact expressions coincide with the semi-classical expressions in (23) except for shifting $k'$ to $(k' - 2)$ and $k$ to $(k + 2)$. However, for the heterotic string the result is considerably complicated and will not be given here.

In this paper we did not discuss the antisymmetric tensor, or torsion field, that is present for this coset model. In the algebraic approach it can be computed by considering states that satisfy the closed string conditions $L_0 = \bar{L}_0$ without demanding that $Q = \bar{Q} = 0$ on the state. In the coset approach such states must be included in the spectrum. What will then happen is that the Dalambertian will have to include a “spin connection” in addition to the metric. The torsion field can then be extracted by comparing the derivative form of $L_0 + \bar{L}_0$ and the Dalambertian form. We have not applied this method in detail to the current coset model.

In summary, we have computed the conformally exact metric and dilaton and gave the global space for the curved spacetime string model based on the coset $SL(2, \mathbb{R}) \times SU(2)/\mathbb{R}^2$. We have also discussed the supersymmetric type-II and heterotic versions of the model. As seen, the finite $k, k'$ corrections are substantial not only numerically but also structurally.
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