On the numerical solution of fractional Riccati differential equations

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Abstract
In this paper, a method has been proposed to solve Fractional differential equations of the form $(D^\alpha)(y(t)) = f(t, y(t)), 0 < \alpha \leq 1$, where $D^\alpha$ denotes the Caputo fractional derivative by applying the New iterative method proposed by Daftardar-Gejji and Jafari on the implicit version of fractional trapezoidal formula. We discussed the error analysis of the proposed method. We solved fractional Riccati differential equation to prove the efficiency of the proposed method.

Keywords
Fractional order differential equations, Caputo fractional derivative, New iterative method, Numerical Method.

AMS Subject Classification
34A08, 65L20.

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1. Introduction
The study of fractional differential equations is an interesting area of research because of its applications in various fields such as Biology, Physics, Fluid dynamics and Engineering. There are various analytical methods for solving fractional differential equations. But there are equations which do not have the exact analytic solution. Hence numerical techniques were attracted by many researchers. Several numerical methods have been developed for solving Fractional differential equation such as Finite Difference method [1], wavelets method [2, 3], Adomian Decomposition method [4], Homotopy Analysis method [5], Fractional Euler method [6] and Adams method [7].

The New iterative method (NIM) introduced by Daftardar-Gejji and Jafari [8] is one of the most powerful iterative technique for solving a broad class of non linear equations such as Partial differential equations, Fractional differential equations and boundary value problems etc. The NIM has been further explored by many researchers [9–13].

Through this paper we introduced a numerical method by applying the new iterative method introduced by Daftardar-Gejji and jafari on the implicit version of fractional trapezoidal formula [14] for solving fractional differential equations of the form [7],

$$(D^\alpha)(y(t)) = f(t, y(t)), \quad t \in [0, T], T > 0, 0 < \alpha \leq 1$$

with initial conditions,

$$y^{(k)}(0) = y^k_0, \quad k = 0, 1, 2, ..., n - 1,$$

where $n = \lceil \alpha \rceil$ is the first integer not less than $\alpha$ and $D^\alpha$ denotes the Caputo type fractional derivative.

The paper is structured as follows. In preliminaries we gave some basic definitions and results. New iterative method is discussed in section 3, the numerical method is introduced in section 4 and error analysis of the numerical method is discussed in section 5. The fractional Riccati differential equation is solved to show the efficiency of the numerical method through section 6.
2. Preliminaries

Through this section we give some basic definitions from Fractional calculus.

Definition 2.1. [16] The Riemann-Liouville fractional integral of order \( \alpha > 0 \) of a function \( f : (0, \infty) \rightarrow \mathbb{R} \) is defined by,

\[
(i^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt.
\]

Definition 2.2. [16] The Caputo fractional derivative of order \( \alpha > 0 \) of a function \( f : (0, \infty) \rightarrow \mathbb{R} \) is defined by,

\[
(D^\alpha f)(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{f^{(n)}(t)}{(x-t)^{\alpha+n-1}} dt
\]

where \( n = [\alpha] + 1 \), \([\alpha]\) is the integer part of \( \alpha \).

It is known that the IVP (1.1) is equivalent to the Volterra integral equation [15],

\[
y(t) = y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s, y(s))}{(t-s)^{\alpha-1}} ds, \quad t \leq T.
\]

If the function \( f \) is continuous and satisfy the Lipschitz condition with respect to the second variable with Lipschitz constant \( L \), then the IVP (1.1) has a unique solution on some \([0, T]\).

Theorem 2.3. [7] If \( y \in C^2[0, T] \), then there is a constant \( c_T^\alpha \) depending only on \( \alpha \) such that,

\[
\left| \int_0^{t_{k+1}} (t_{k+1}-t)^{\alpha-1} y(t)dt \right| \leq c_T^\alpha \|y''\| \Delta t_{k+1} ^{\alpha}.
\]

2.1 Fractional Riccati differential equation

The Riccati differential equation named after the Italian nobleman Count Jacopo Francesco Riccati(1676-1754). Here we consider the general fractional Riccati differential equation of the form,

\[
(D^\alpha)(y(t)) = p(t) + q(t)y(t) + r(t)y^2(t), \quad t \in [0, T], T > 0
\]

with initial conditions

\[
y^{(k)}(0) = y^{(k)}_0, \quad k = 0, 1, 2, ..., n - 1, 0 < \alpha \leq 1,
\]

where \( p(t) \), \( q(t) \) and \( r(t) \) are given functions and \( D^\alpha \) denotes the Caputo derivative.

3. New Iterative Method [8]

In this section we described the new iterative method proposed by Daftardar-Gejji and Jafari(DJM) useful for solving a general functional equations of the form

\[
y = f + L(y) + N(y),
\]

where \( f \) is a given function \( L \) and \( N \) are linear and nonlinear operators respectively. It is assumed to be that the DJM solution of equation (1.1) has the form

\[
y = \sum_{i=0}^{\infty} y_i
\]

The linear operator \( L \) can be written as,

\[
L(\sum_{i=0}^{\infty} y_i) = \sum_{i=0}^{\infty} L(y_i)
\]

The non linear operator \( N \) in Equation (3.1) is decomposed in DJM as follows,

\[
N(\sum_{i=0}^{\infty} y_i) = N(y_0) + \sum_{i=0}^{\infty} (N(\sum_{j=0}^{i} y_j) - N(\sum_{j=0}^{i-1} y_j))
\]

From Equations (3.2), (3.3) and (3.4), equation (3.1) is equivalent to

\[
\sum_{i=0}^{\infty} y_j = f + \sum_{i=0}^{\infty} L(y_i) + \sum_{i=0}^{\infty} \left( N(\sum_{j=0}^{i} y_j) - N(\sum_{j=0}^{i-1} y_j) \right)
\]

From equation (3.5), the DJM series terms are generated as follows,

\[
y_0 = f
\]

\[
y_1 = L(y_0) + N(y_0)
\]

\[
y_{m+1} = L(y_m) + N(y_0 + y_1 + y_2 + ... + y_m) - N(y_0 + y_1 + y_2 + ... + y_{m-1})
\]

\[
m = 1, 2, 3, ...
\]

The k-term approximate solution is given by,

\[
y = \sum_{i=0}^{k-1} y_i
\]

For suitable integer \( k \).

4. Numerical Method

In this section we shall derive a new numerical method for the numerical solution of the IVP (1.1). We apply NIM technique on the implicit version of fractional trapezoidal formula to derive the method.

To obtain the numerical solution of IVP (1.1), we take partition of the interval \([0, T]\) as \(0 = t_0 < t_1 < t_2 < ... < t_n = T\). These points are called mesh points. A sufficient small spacing between the points is given by, \( h = t_{k+1} - t_k, k = 0, 1, 2, ..., n - 1 \), which is called the step length. We also have

\[
t_k = t_0 + kh, \quad j = 0, 1, 2, ..., n.
\]
If $y_k$ is an approximation to $y(t_k)$, the implicit version of fractional trapezoidal formula is given by,

\[
y_{k+1} = \sum_{j=0}^{n-1} \frac{t_{k+1}^j}{j!} y_0 + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^{k} a_{j,k+1} f(t_j, y_j) + \frac{h^\alpha}{\Gamma(\alpha+2)} a_{k+1,k+1} f(t_{k+1}, y_{k+1})
\]

where,

\[
a_{j,k+1} = \begin{cases} 
  k^{\alpha+1} - (k-\alpha)(k+1)^\alpha & : j = 0 \\
  (k-j+1)^{\alpha+1} - (k-j)^{\alpha+1} & : 1 \leq j \leq k \\
  1 & : j = k+1
\end{cases}
\]

Equation (4.1) is of the form (3.1), where,

\[
y = y_{k+1} \\
f = \sum_{j=0}^{n-1} \frac{t_{k+1}^j}{j!} y_0 + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^{k} a_{j,k+1} f(t_j, y_j) \\
N(y) = \frac{h^\alpha}{\Gamma(\alpha+2)} f(t_{k+1}, y_{k+1})
\]

Applying NIM on Equation (4.1), we obtain the 4-term solution as,

\[
y = y_0 + y_1 + y_2 + y_3 \\
= y_0 + N(y_0) + N(y_0 + y_1) - N(y_0) + \\
N(y_0 + y_1 + y_2) - N(y_0 + y_1) \\
= y_0 + N(y_0 + y_1 + y_2) \\
= y_0 + N(y_0 + N(y_0) + N(y_0 + y_1) - N(y_0)) \\
= y_0 + N(y_0 + N(y_0 + N(y_0))) \\
= y_0 + N(y_0 + \frac{h^\alpha}{\Gamma(\alpha+2)} f(t_{k+1}, \sum_{j=0}^{n-1} \frac{t_{k+1}^j}{j!} y_0) + \\
\frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^{k} a_{j,k+1} f(t_j, y_j) + \frac{h^\alpha}{\Gamma(\alpha+2)} f(t_{k+1}, \sum_{j=0}^{n-1} \frac{t_{k+1}^j}{j!} y_0) + \\
\frac{h^\alpha}{\Gamma(\alpha+2)} a_{k+1,k+1} f(t_{k+1}, y_{k+1}))
\]

then the above equation become,

\[
y_{k+1} = \sum_{j=0}^{n-1} \frac{t_{k+1}^j}{j!} y_0 + \frac{h^\alpha}{\Gamma(\alpha+2)} f(t_{k+1}, \sum_{j=0}^{n-1} \frac{t_{k+1}^j}{j!} y_0) + \\
\frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^{k} a_{j,k+1} f(t_j, y_j) + \frac{h^\alpha}{\Gamma(\alpha+2)} a_{k+1,k+1} f(t_{k+1}, y_{k+1})
\]

For $j = 0, 1, 2, \ldots$

\[
y_{k+1} = y_0 + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^{k} a_{j,k+1} f(t_j, y_j) + \frac{h^\alpha}{\Gamma(\alpha+2)} a_{k+1,k+1} f(t_{k+1}, y_{k+1})
\]

Then we prove that the result is true for $j = k+1$.

5. Error Analysis of the Numerical Method

Theorem 5.1. Suppose the solution $y(t)$ of the IVP (1.1) satisfies the condition

\[
\left| \int_0^{t_{k+1}} (t_{k+1} - t)^{\alpha-1} (D^\alpha y)(t) \, dt \right| = C_{t_{k+1}} h^\delta \leq C \gamma h^\delta \quad \text{for some } C, \gamma \geq 0 \quad \text{and } f \text{ satisfies Lipschitz condition in the second variable with Lipschitz constant } L.
\]

Then for some suitably chosen $T > 0$, we have, max $|y(t_j) - y_j| = O(h^\delta)$, where, $N = \lfloor \frac{T}{h} \rfloor$

Proof. We will show that for sufficiently small h,

\[
|y(t_j) - y_j| \leq ch^\delta
\]

for $j = 0, 1, 2, \ldots N$, where c is a suitable constant.

The proof will be based on mathematical induction. Clearly the hypothesis is true for $j = 0$.

Now assume that the induction hypothesis is true for $j = 1, 2, \ldots k$. Then we prove that the result is true for $j = k+1$. 

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Now,\[
\frac{1}{\Gamma\alpha} \int_0^{k+1} (t_{k+1} - t)^{\alpha - 1} f(t, y(t)) dt - \frac{h^\alpha}{\alpha(\alpha + 1)} \sum_{j=0}^k a_{j,k+1} f(t_j, y_j) \leq \frac{1}{\Gamma\alpha} \int_0^{k+1} (t_{k+1} - t)^{\alpha - 1} f(t, y(t)) dt - \frac{h^\alpha}{\alpha(\alpha + 1)} \sum_{j=0}^k a_{j,k+1} f(t_j, y_j) + \frac{h^\alpha}{\alpha + 2} \sum_{j=0}^k a_{j,k+1} f(t_j, y_j)\]
\[+ \frac{h^\alpha}{\alpha + 2} \sum_{j=0}^k [f(t_j, y_j) - f(t_{j+1}, y_{j+1})] \leq \frac{C \gamma \lambda^\delta}{\Gamma\alpha} + \frac{Lk^\delta \sum_{j=0}^k a_{j,k+1}}{\Gamma\alpha + 2} \leq \frac{CT^\gamma h^\delta}{\Gamma\alpha} + \frac{Lk^\delta a_{i+\delta}}{\Gamma\alpha + 1} (5.1)\]

Note that,\[
\frac{h^\alpha}{\alpha(\alpha + 1)} \sum_{j=0}^k a_{j,k+1} \leq \frac{h^\alpha}{\alpha(\alpha + 1)} \sum_{j=0}^{k+1} a_{j,k+1} = \int_0^{k+1} (t_{k+1} - t)^{\alpha - 1} dt = \frac{1}{\alpha k^\alpha + 1} \leq \frac{1}{\alpha^\alpha}\]

Now,\[
\frac{1}{\Gamma\alpha} \int_0^{k+1} (t_{k+1} - t)^{\alpha - 1} f(t, y(t)) dt - \frac{h^\alpha}{\alpha(\alpha + 1)} \sum_{j=0}^k a_{j,k+1} f(t_j, y_j) \leq \frac{1}{\Gamma\alpha} \int_0^{k+1} (t_{k+1} - t)^{\alpha - 1} f(t, y(t)) dt - \frac{h^\alpha}{\alpha(\alpha + 1)} \sum_{j=0}^k a_{j,k+1} f(t_j, y_j) + \frac{h^\alpha}{\alpha + 2} \sum_{j=0}^k a_{j,k+1} f(t_j, y_j)\]
\[+ \frac{h^\alpha}{\alpha + 2} [f(t_{k+1}, y_{k+1}) - f(t_{k+1}, \sum_{j=0}^{n-1} \frac{t_{k+1} f(t_j, y_j)}{j!}) + \frac{h^\alpha}{\alpha + 2} \sum_{j=0}^k a_{j,k+1} f(t_j, y_j)]\]
\[\leq \frac{C \gamma \lambda^\delta}{\Gamma\alpha} + \frac{Lk^\delta a_{i+\delta}}{\Gamma\alpha + 1} + \frac{h^\alpha L}{\Gamma\alpha + 2} + \frac{CT^\gamma h^\delta}{\Gamma\alpha} + \frac{Lk^\delta a_{i+\delta}}{\Gamma\alpha + 1} + \frac{h^\alpha L}{\Gamma\alpha + 2} (5.4)\]

by using (5.2) and Lipschitz condition of \(f\). Now,\[
[y(t_{k+1}) - y_{k+1}] = \frac{1}{\Gamma\alpha} \int_0^{k+1} (t_{k+1} - t)^{\alpha - 1} f(t, y(t)) dt - \frac{h^\alpha}{\alpha + 2} \sum_{j=0}^k a_{j,k+1} f(t_j, y_j) - \frac{h^\alpha}{\alpha + 2} f(t_{k+1})\]
\[+ \sum_{j=0}^{n-1} \frac{t_{k+1} f(t_j, y_j)}{j!} + \frac{h^\alpha}{\alpha + 2} \sum_{j=0}^k a_{j,k+1} f(t_j, y_j) + \frac{h^\alpha L}{\Gamma\alpha + 2} (5.3)\]

\(\sum_{j=0}^{n-1} \frac{t_{k+1} f(t_j, y_j)}{j!} + \frac{h^\alpha}{\alpha + 2} \sum_{j=0}^k a_{j,k+1} f(t_j, y_j) + \frac{h^\alpha L}{\Gamma\alpha + 2} (5.3)\)
According to the proposed method, the numerical solution to equation (6.1) in the interval [0, 1] with \( h = 0.1 \) is graphically illustrated in Figure (1) and tabulated in Table (1).

**6. Numerical Examples**

**Example 1.** Consider the fractional Riccati equation

\[
D^\alpha y(t) = \frac{1}{2} y(t) (1 - y(t)); \quad t > 0, \quad 0 < \alpha \leq 1 \quad (6.1)
\]

\[
y(0) = \frac{1}{2}.
\]

At \( \alpha = 1 \) equation (6.1) has exact solution

\[
y(t) = \frac{e^t}{1 + e^t}
\]

\[
\frac{h^\alpha}{\Gamma(\alpha + 1)} + \sum_{j=0}^{n-1} \left( \frac{h^\alpha}{\Gamma(\alpha + 1)} \right)^j = \frac{C T h^\delta}{\Gamma(\alpha + 1)} + \frac{L k T h^{\alpha + \delta}}{\Gamma(\alpha + 1)} + \frac{h^\alpha L}{\Gamma(\alpha + 1)} + \frac{h^{\alpha + \delta}}{\Gamma(\alpha + 1)}
\]

\[
\leq C T h^\delta + L k T h^{\alpha + \delta} + h^\alpha L + h^{\alpha + \delta} \leq \varepsilon
\]

where \( c \) is a positive constant.

**Example 2.** Consider the fractional Riccati equation

\[
D^\alpha y(t) = 1 - y(t)^2; \quad t > 0, \quad 0 < \alpha \leq 1 \quad (6.2)
\]

\[
y(0) = 0.
\]

At \( \alpha = 1 \) equation (6.2) has exact solution

\[
y(t) = \frac{e^{2t} - 1}{e^{2t} + 1}
\]

\[
\begin{array}{c|c|c|c}
| t | \text{Exact (} \alpha = 1 \text{)} | \text{Approx (} \alpha = 1 \text{)} | \text{Approx (} \alpha = 0.5 \text{)} \\
|---|---|---|
| 0 | 0.512497 | 0.512496 | 0.544369 |
| 0.1 | 0.524979 | 0.524977 | 0.562421 |
| 0.2 | 0.537429 | 0.537426 | 0.576057 |
| 0.3 | 0.549833 | 0.549829 | 0.587376 |
| 0.4 | 0.562177 | 0.562171 | 0.597199 |
| 0.5 | 0.574443 | 0.574435 | 0.605946 |
| 0.6 | 0.586617 | 0.586609 | 0.613872 |
| 0.7 | 0.598688 | 0.598678 | 0.621142 |
| 0.8 | 0.610639 | 0.610629 | 0.627873 |
| 0.9 | 0.622459 | 0.622448 | 0.634148 |
| 1 | 0.634148 | 0.634148 | 0.634148 |
\end{array}
\]

**Figure 1.** Comparison between the exact and the numerical solution of Example 1 for different values of \( \alpha \).
The exact and the approximate solutions of Example 2 for various values $\alpha$.

**Table 2.**

| $\alpha$ | Exact ($\alpha = 1$) | Approx ($\alpha = 0.75$) | Approx ($\alpha = 0.5$) | Approx ($\alpha = 0.25$) |
|----------|-----------------------|--------------------------|-------------------------|--------------------------|
| 0        | 0.099068              | 0.099250                 | 0.104742                 | 0.109322                 |
| 0.1      | 0.197173              | 0.197946                 | 0.204075                 | 0.209614                 |
| 0.2      | 0.291113              | 0.290896                 | 0.300739                 | 0.309629                 |
| 0.3      | 0.385130              | 0.378087                 | 0.398042                 | 0.410981                 |
| 0.4      | 0.480127              | 0.473151                 | 0.504969                 | 0.530706                 |
| 0.5      | 0.575104              | 0.563856                 | 0.606713                 | 0.647245                 |
| 0.6      | 0.665104              | 0.650307                 | 0.696157                 | 0.740985                 |
| 0.7      | 0.753098              | 0.738021                 | 0.784027                 | 0.833029                 |
| 0.8      | 0.838194              | 0.821399                 | 0.869381                 | 0.920041                 |
| 0.9      | 0.920984              | 0.902134                 | 0.951990                 | 1.010981                 |

Figure 2. Comparison between the exact and the numerical solution of Example 2 for the value of $\alpha = 1$.

**7. Conclusion**

In this paper we introduced a numerical method based on the New iterative method which is applied in implicit version of fractional Trapezoidal formula to solve fractional Riccati type differential equations. Also We discussed the convergence through Error analysis. The efficiency of this method is shown through two numerical examples.

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