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Abstract

Following the ideas of Ore and Li we study $q$-analogues of scalar subresultants and show how these results can be applied to determine the rank of an $F_q$-linear transformation $f$ of $F_{q^n}$. As an application we show how certain minors of the Dickson matrix $D(f)$, associated with $f$, determine the rank of $D(f)$ and hence the rank of $f$.

Keywords: Dickson matrix, subresultant, linearized polynomial

1 Introduction

Let $f(x) = \sum_{i=0}^{k} a_i x^i$ and $g(x) = \sum_{i=0}^{l} b_i x^i$, with $a_kb_l \neq 0$, be two univariate polynomials with coefficients in the field $K$. In elimination theory, the classical resultant of $f$ and $g$ is

$$\text{Res}(f, g) = (-1)^{kl} b_l b_k \prod_{i=1}^{l} f(\xi_i),$$

where $g(x) = b_l \prod_{i=1}^{l} (x - \xi_i)$ with $\xi_1, \xi_2, \ldots, \xi_l \in \overline{K}$ (where $\overline{K}$ denotes the algebraic closure of $K$). For $0 \leq m \leq \min\{k, l\}$ consider the following

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1Note that in many of the cited literature $a_0$ and $b_0$ are used to denote the leading coefficients of $f$ and $g$. 
Result 1.1. The degree of \( \gcd(f, g) \) is \( t \) if and only if \( |R_0(f, g)| = \cdots = |R_{t-1}(f, g)| = 0 \) and \( |R_t(f, g)| \neq 0 \).

The strength of the Result 1.1 is that it provides a way to study the number of common roots of \( f \) and \( g \) only by means of their coefficients.

Now let \( \mathbb{K} \) be a field of characteristic \( p \), and let \( q \) be a power of \( p \). A \( q \)-polynomial over \( \mathbb{K} \) with \( q \)-degree \( m \) is a polynomial of the form \( f(x) = \sum_{i=0}^{m} a_i x^{q^i} \), with \( a_m \neq 0 \) and \( a_0, a_1, \ldots, a_m \in \mathbb{K} \). When \( q = p \) prime, \( q \)-anologue of the classical resultant for \( q \)-polynomials was already mentioned in [14, Chapter 1, Section 7], however, an explicit formula was not given there. An explicit formula can be found for example in [17, page 59].

The subresultant theory was extended to Ore polynomials (cf. [15]) and hence also to the non-commutative ring of \( q \)-polynomials by Li in [11]. Here the non-commutative operation between two \( q \)-polynomials is composition, while addition is defined as usual. Note that this ring is a right-Euclidean domain with respect to the \( q \)-degree, cf. [14]. When \( g = f \circ h \) then we will also say that \( h \) is a symbolic right divisor of \( g \). Note that in the paper of Li the word subresultant is used to what is also known as polynomial subresultant. In the classical theory the \( m \)-th scalar subresultant is the leading coefficient of the \( m \)-th polynomial subresultant. See for example [1, Section 2] for a brief summary, where \( S_m^{(m)} \) corresponds to what we (and
some other authors) call scalar subresultant. For the various notions consult with [9].

Let $K = \mathbb{F}_{q^n}$ and consider $K$ as an $n$-dimensional vector space over $\mathbb{F}_q$. Then there is an isomorphism between the ring of $q$-polynomials

$$\left\{ \sum_{i=0}^{n-1} a_i x^{q^i} : a_0, \ldots, a_{n-1} \in \mathbb{F}_{q^n} \right\}$$

considered modulo $(x^{q^n} - x)$ and the ring of $\mathbb{F}_q$-linear transformations of $\mathbb{F}_{q^n}$. The set of roots of a $q$-polynomial form an $\mathbb{F}_q$-subspace and the dimension of this subspace is the dimension of the kernel of the corresponding $\mathbb{F}_q$-linear transformation. Thus $\deg \gcd(f(x), x^{q^n} - x) = q^{n-k}$, where $k$ is the rank of the $\mathbb{F}_q$-linear transformation of $\mathbb{F}_{q^n}$ defined by $f(x)$. When $n$ is clear from the context, then we will say that $k$ is the rank of $f$.

**Result 1.2** (Ore [14, Theorem 2]). The greatest common symbolic right divisor of two $q$-polynomials is the same as their ordinary greatest common divisor.

It follows that the $q$-subresultant theory can be applied to determine $\gcd(f(x), x^{q^n} - x)$ and hence the rank of $f$. Our contribution to this theory is a direct proof to a $q$-analogue of Result 1.1 providing sufficient and necessary conditions which ensure that $f$ has rank $n - k$ (cf. Theorem 2.1).

Recall that the Dickson matrix associated with $f(x) = \sum_{i=0}^{n-1} a_i x^{q^i} \in \mathbb{F}_{q^n}[x]$ is

$$D(f) := \begin{pmatrix}
a_0 & a_1 & \cdots & a_{n-1} \\
a_0^{q} & a_1^{q} & \cdots & a_{n-2}^{q} \\
\vdots & \vdots & \ddots & \vdots \\
a_0^{q^{n-1}} & a_1^{q^{n-1}} & \cdots & a_0^{q^{n-1}}
\end{pmatrix}.$$  

It is well-known that the rank of $f$ equals the rank of $D(f)$, see for example [18, Proposition 4.4] or [13, Proposition 5]. In some recent constructions of maximum scattered subspaces and MRD-codes it was crucial to the determine the rank of certain Dickson matrices (cf. [6, Section 7] and [7, Section 5]). In these papers this was done by considering certain minors of such matrices and excluding the possibility that their determinants vanish at the same time. On the other hand, in [4, Section 3] Dickson matrices were used to prove non-existence results of certain MRD-codes. This was done by proving that, for a certain choice of the parameters, all $6 \times 6$ submatrices of a $9 \times 9$ Dickson matrix have zero determinant. As an application of Theorem
2.1 we show that it is enough to investigate the nullity of the determinant of at most $k + 1$ well-defined minors to decide whether $f$ has rank $n - k$. This result can significantly simplify the above mentioned arguments.

To state here the main result of this paper we introduce the notion $D_m(f)$ to denote the $(n - m) \times (n - m)$ matrix obtained from $D(f)$ after removing its first $m$ columns and last $m$ rows. Our main result is the following.

**Theorem 1.3.** $\dim_{\mu}(\ker f) = \mu$ if and only if

$$|D_0(f)| = |D_1(f)| = \ldots = |D_{\mu-1}(f)| = 0$$

and $|D_\mu(f)| \neq 0$.

Results in a similar direction have been obtained recently in [5] where for each $q$-polynomial $f$ of $q$-degree $k$, $k$ conditions were given, in terms of the coefficients of $f$, which are satisfied if and only if $f$ has rank $n - k$ (there is a hidden $(k + 1)$-th condition here as well, namely the assumption that the coefficient of $x^{q^k}$ in $f$ is non-zero). Independently, in [16] it was proved that the rank of $f$ is $n - m$ if and only if a certain $k \times k$ matrix has rank $k - m$. If $m - k$, then this result gives back the main result of [5].

## 2 Scalar $q$-subresultants

Consider $f(x) = \sum_{i=0}^k a_i x^{q^i}$ and $g(x) = \sum_{l=0}^l b_i x^{q^l}$, two $q$-polynomials with coefficients in $\mathbb{F}_q$ such that $a_kb_l \neq 0$. Put

$$q^\ell = \deg \gcd(f, g).$$

By Result 1.2, $\mu$ also equals the $q$-degree of the symbolic greatest common right divisor of $f$ and $g$.

For $m \leq \min\{k, l\}$ we define the $(k + l - 2m) \times (k + l - 2m)$ matrix $R_{m,q}(f, g)$ as follows:

$$
\begin{pmatrix}
  a_k^{q^1} m_{1} & a_k^{q^2} m_{1} & \cdots & a_k^{q^{k-m_l+1}} m_{1} & \cdots & a_k^{q^{k-m_{l+2}}} m_{1} & a_k^{q^{k-m_{l+2}}} m_{1} \\
  0 & a_k^{q^1} m_{2} & \cdots & a_k^{q^{k-m_{l+2}}} m_{2} & \cdots & a_k^{q^{k-m_{l+2}}} m_{2} & a_k^{q^{k-m_{l+2}}} m_{2} \\
  \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
  0 & \ldots & 0 & a_k & \ldots & a_k & a_k \\
  b_l^{q^k-m_{l-1}} & b_l^{q^{k-m_{l-1}}} & \cdots & b_l^{q^{k-m_{l+2}}} & \cdots & b_l^{q^{k-m_{l+2}}} & b_l^{q^{k-m_{l+2}}} \\
  0 & b_l^{q^{k-m_{l-2}}} & \cdots & b_l^{q^{k-m_{l+2}}} & \cdots & b_l^{q^{k-m_{l+2}}} & b_l^{q^{k-m_{l+2}}} \\
  \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
  0 & \ldots & 0 & b_l & \ldots & b_l & b_l \\
\end{pmatrix}
$$


Note that $R_{m+1,q}(f,g)$ is obtained from $R_{m,q}(f,g)$ by removing its first and last columns, and its first and $(l-m+1)$-th rows.

We state here the $q$-analogue of Result 1.1.

**Theorem 2.1.** The $q$-degree of $\gcd(f,g)$ is $\mu$ if and only if $|R_{0,q}(f,g)| = \cdots = |R_{\mu-1,q}(f,g)| = 0$ and $|R_{\mu,q}(f,g)| \neq 0$.

We prove this result directly by following the proof of the classical Result 1.1. Theorem 2.1 will easily follow from Proposition 2.3.

**Proposition 2.2.** Recall $q^\mu = \deg \gcd(f,g)$ and let $m \leq \mu$. Let $c(x) = \sum_{i=0}^{k-m} c_i x^i$ and $d(x) = \sum_{i=0}^{l-m} d_i x^i$ be $q$-polynomials over $\mathbb{F}_q$ with $c_{k-m} = a_{q^\mu}^d$, $d_{l-m} = b_{q^\mu}^d$, and their other coefficients are considered as unknowns. Then the set of solutions for these coefficients such that

$$d \circ f - c \circ g = 0$$

form a $(\mu - m)$-dimensional affine $\mathbb{F}_q$-space.

**Proof.** First assume that $f$ and $g$ have only simple roots.

Let $r$ be the greatest common monic symbolic right divisor of $f$ and $g$ and suppose that (2) holds for some $c$ and $d$. Then $f = f_1 \circ r$ and $g = g_1 \circ r$ and (2) yields $d \circ f_1 = c \circ g_1$, thus $d$ is zero on $f_1(\ker g_1)$ (in this proof the kernel is always taken over $\mathbb{F}_q$) and $c$ is zero on $g_1(\ker f_1)$. Since the greatest common symbolic right divisor of $f_1$ and $g_1$ is the identity map, it follows that $\gcd(f_1,g_1) = x$ and hence $\ker f_1 \cap \ker g_1 = \{0\}$. Thus $\dim_q f_1(\ker g_1) = \dim_q \ker g_1 = l - \mu$ and similarly $\dim_q g_1(\ker f_1) = k - \mu$. It follows that the unique $q$-polynomial $d_1$ of $q$-degree $l - \mu$ and with leading coefficient $b_{q^\mu}^d$ which vanishes on $f_1(\ker g_1)$ is a divisor of $d$. By Result 1.2 $\gcd(d,d_1) = d_1$ is also a symbolic right divisor of $d$, i.e. $d = d_2 \circ d_1$, for some monic $d_2$ with $q$-degree $(\mu - m)$. Similarly, the unique $q$-polynomial $c_1$ of $q$-degree $k - \mu$ and with leading coefficient $a_{q^\mu}^c$ which vanishes on $g_1(\ker f_1)$ is a symbolic right divisor of $c$, i.e. $c = c_2 \circ c_1$, for some monic $c_2$ with $q$-degree $(\mu - m)$.

Note that

$$d_1 \circ f_1 - c_1 \circ g_1$$

has $q$-degree $k + l - 2\mu - 1$ (the coefficient of $x^{q^\mu+(-2\mu)}$ vanishes because of the assumptions on the leading coefficients of $c$ and $d$) and it vanishes on $\ker f_1 \oplus \ker g_1$. Thus it is the zero polynomial.

Then

$$c_2 \circ c_1 \circ g_1 = c \circ g_1 = d \circ f_1 = d_2 \circ d_1 \circ f_1 = d_2 \circ c_1 \circ g_1$$
and hence \( c_2 = d_2 \). On the other hand, if \( c_2 = d_2 \), then we clearly have a solution since \((2)\) becomes \( d_2 \circ (d_1 \circ f_1 - c_1 \circ g_1) \circ r \) with the zero polynomial in the middle.

Since we can choose the first \((\mu - m)\) coefficients of \( d_2(x) - \sum_{i=0}^{\mu-m} \hat{d}_i x^i \) arbitrarily, the assertion follows. More precisely, if \( d_1(x) = \sum_{j=0}^{\mu-m} \hat{d}_j x^j \) with \( \hat{d}_{\mu-m} = b_1^{k-m} \) and with coefficients out of range defined as 0, then \( d(x) \) is of the form

\[
\sum_{i=0}^{k-m} \sum_{j=0}^{i} \hat{d}_{i-j} \hat{q}^i \cdot x^j,
\]

with \( \hat{d}_k \in \mathbb{F}_q \) for \( 0 \leq k \leq \mu - m - 1 \), \( \hat{d}_{\mu-m} = 1 \) and \( \hat{d}_l = 0 \) for \( l > \mu - m \). These polynomials form a \((\mu - m)\)-dimensional affine \( \mathbb{F}_q \)-space and as we have seen, any such \( d(x) \) uniquely defines a \( c(x) \) for which \((2)\) holds.

Now consider the case when \( f \) and \( g \) may have multiple roots. Let \( f = x^{q^{k_1}} \circ \hat{f} \) and \( g = x^{q^{l_1}} \circ \hat{g} \) where \( \hat{f} \) and \( \hat{g} \) have only simple roots. W.l.o.g. assume \( l_1 \leq k_1 \). We want to find the dimension of the solutions of

\[
d \circ x^{q^{k_1}} \circ \hat{f} = c \circ x^{q^{l_1}} \circ \hat{g},
\]

under the given assumptions on the degrees and leading coefficients of \( c \) and \( d \). Clearly, the multiplicities of the roots of the left hand side and the right hand side have to coincide and hence \( c = \hat{c} \circ x^{q^{l_1-k_1}} \). Let \( \hat{d} \) and \( \hat{c} \) denote the \( q \)-polynomials whose coefficients are the \( q^{-k_1} \)-th roots of the coefficients of \( d \) and \( c \), respectively. Then the solutions of the previous system correspond to the solutions of

\[
x^{q^{k_1}} \circ \hat{d} \circ \hat{f} = x^{q^{k_1}} \circ \hat{c} \circ \hat{g}
\]

and hence to those of

\[
\hat{d} \circ \hat{f} = \hat{c} \circ \hat{g},
\]

where the \( q \)-degree of \( \hat{d} \) is \((l - l_1) - (m - l_1)\) and the \( q \)-degree of \( \hat{c} \) is \((k - k_1) - (m - l_1)\). The roots of the \( q \)-polynomials \( \hat{f} \) and \( \hat{g} \) are simple, thus we can apply the first part of this proof for these polynomials. The leading coefficients of \( \hat{d} \) and \( \hat{c} \) are \( b_1^{k-m-k_1} \) and \( a_1^{q^{l_1-k_1}} \), respectively; the leading coefficients of \( \hat{f} \) and \( \hat{g} \) are \( a_1^{q^{k_1}} \) and \( b_1^{q^{l_1-k_1}} \), respectively. Since \( b_1^{k-m-k_1} = b_1^{q^{l_1-k_1}} \) and \( a_1^{q^{l_1-k_1}} = a_1^{q^{k_1}} \), the conditions on the leading coefficients also hold. Note that the \( q \)-degree of \( \gcd(\hat{f}, \hat{g}) \) is \( \mu - l_1 \). Then the dimension of the solutions of this system is \((\mu - l_1) - (m - l_1) = \mu - m\).
Proposition 2.3. Suppose \( m \leq \mu \). Then the nullity of the matrix \( R_{m,q}(f,g) \) is \( \mu - m \).

Proof. Let \( f, g, c, d \) be defined as before, then

\[
d \circ f - c \circ g = \sum_{i=0}^{m-1} d_i \sum_{j=0}^{k} a_j^q x^{q^{i+1}} - \sum_{i=0}^{k-m} c_i \sum_{j=0}^{l} b_j^q x^{q^{i+1}} = \\
\sum_{i=0}^{k+l-m} \left( \sum_{j=0}^{i} d_i j a_j^{q^{i-j}} - c_i j b_j^{q^{i-j}} \right) x^{q^i}.
\]

The \( q \)-degree of \( r := \gcd(f,g) \) is \( \mu \geq m \) and \( r \mid d \circ f - c \circ g \), thus \( d \) and \( c \) form a solution to \( d \circ f - c \circ g = 0 \) if and only if the \( q \)-degree of \( d \circ f - c \circ g \) is less than \( m \). In another words, we only have to concentrate on the coefficients of terms with \( q \)-degree \( i \in \{m, m+1, \ldots, k + l - m\} \) in \( d \circ f - c \circ g \).

Note that the coefficient of \( q^k \) is \( d_k - c_k \) (coefficients out of range are considered to be 0), which is 0 because of our assumptions on \( c \) and \( d \). Now let

\[
v = (d_{l-m-1}, d_{l-m-2}, \ldots, d_0, -c_{k-m-1}, -c_{k-m-2}, \ldots, -c_0)
\]

and

\[
b = (b_l^{k-m} a_{l-1}^{q^{l-m}} - a_k^{q^{l-m}} b_l^{q^k}, \ldots, b_l^{k-m} a_{2m-l}^{q^{l-m}} - a_k^{q^{l-m}} b_{2m-l}^{q^k}).
\]

We claim that

\[
v R_{m,q}(f,g) = -b
\]

holds if and only if

\[
\sum_{j=0}^{i} d_i j a_j^{q^{i-j}} - c_i j b_j^{q^{i-j}} = 0
\]

for all \( m \leq i \leq k + l - m - 1 \). To see this we show that the \((k+l-2m-t)\)-th coordinates in the vectors at the left and right hand side of (3) coincide if and only if (4) holds with \( i = m + t \). Indeed, in

\[
\sum_{j=0}^{m+t} d_{m+t-j} a_j^{m+t-j} - c_{m+t-j} b_j^{m+t-j}
\]
Corollary 2.4. \(d_m + t_j \neq 0\) only if \(j \in \{m + t, m + t - 1, \ldots, 2m + t - l\}\) and \(c_{m + t_j} \neq 0\) only if \(j \in \{m + t, m + t - 1, \ldots, 2m + t - k\}\). Thus, after changing indices in the summation, (5) equals

\[
\sum_{j=0}^{l-m} d_{m-j} b^{q-m-j} = \sum_{j=0}^{k-m} c_{k-m-j} b^{q-m-j}.
\]

(6)

Since \(d_{l-m} = b^{q-m}_k\) and \(c_{k-m} = a^{q-m}_k\), the \((k + l - 2m - t)\)-th coordinates on the left and right hand side of (3) coincide if and only if

\[
\sum_{j=0}^{l-m-1} d_{l-m-j} b^{q-m-j} = \sum_{j=0}^{k-m-1} c_{k-m-j} b^{q-m-j}.
\]

and this happens if and only if (6) equals zero.

Thus the dimension of the kernel of the \(\mathbb{F}_q\)-linear transformation of \(\mathbb{F}_q^{k+l-2m}\) defined by \(x \mapsto xR_{n,q}(f, g)\) is the same as the dimension of the set of solutions of (2) and this finishes the proof.

Corollary 2.4. Let \(f\) be a \(q\)-polynomial over \(\mathbb{F}_{q^n}\) and put \(g(x) = x^{q^n} - x\). Then \(\dim_q(\ker f) = \mu\) if and only if

\[
|R_{0,q}(f, g)| = |R_{1,q}(f, g)| = \ldots = |R_{\mu-1,q}(f, g)| = 0
\]

and \(|R_{\mu,q}(f, g)| \neq 0\).

As an illustration, the \((n+k) \times (n+k)\) matrix \(R_{0,q}(f, g)\) in the particular case when \(g(x) = x^{q^n} - x\) and \(f(x) = \sum_{i=0}^k a_i x^{q^i}\) has the following form:

\[
\begin{pmatrix}
    a^{q^n-1}_k & a^{q^n-1}_{k-1} & \ldots & a^{q^n-1}_0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
    0 & a^{q^n-2}_k & \ldots & a^{q^n-2}_1 & a^{q^n-2}_0 & \ldots & 0 & 0 & 0 & 0 \\
    \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
    0 & 0 & \ldots & 0 & 0 & \ldots & a_k & a_{k-1} & \ldots & a_0 \\
    1 & 0 & \ldots & 0 & 0 & \ldots & 0 & -1 & 0 & \ldots \\
    0 & 1 & \ldots & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots \\
    \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
    0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & 0 & -1 \\
\end{pmatrix}.
\]
The matrix $R_{m,q}(f, g)$ can be obtained from $R_{0,q}(f, g)$ by removing its first and last $m$ columns and its first $m$ rows together with the $(n+1)$-th, $(n+2)$-th, $\ldots$, $(n+m)$-th rows.

Let $f(x) = \sum_{i=0}^{n-1} a_i x^i$ and $g(x) = x^{q^m} - x$. If we substitute $a_k = 0$ in $R_{m,q}(f, g)$, then its determinant equals either $|R_{m,q}(f, g)|$ or $-R_{m,q}(f, g)$. This argument can be iterated and hence one can use Corollary 2.4 even if the $q$-degree of $f$ is not known, by considering the $2(n-1-2m) \times (2n-1-2m)$ $m$-th scalar $q$-subresultants of $\sum_{i=0}^{n-1} a_i x^{q^i}$ and $g(x)$.

3 A connection with Dickson matrices

In this section we prove Theorem 1.3 but before that we need some preparation.

Result 3.1 (Schur’s determinant identity, [3]). Consider the square matrix

$$M := \begin{pmatrix} X & Y \\ Z & W \end{pmatrix},$$

where $W$ is also square and invertible. Then $\det(M) = \det(W) \det(X - Y W^{-1} Z)$.

Corollary 3.2. Consider the square matrices

$$M := \begin{pmatrix} A & B & C \\ I_l & O & -I_l \end{pmatrix},$$

$$N := \begin{pmatrix} B & A + C \end{pmatrix},$$

where $A$ and $C$ are $k \times l$ matrices, $B$ is $k \times (k - l)$, $I_l$ denotes the $l \times l$ identity matrix and $O$ is the $l \times (k - l)$ zero matrix. Then $\det(M) = (-1)^{(k-l+1)} \det(N)$.

Proof. Result 3.1 with $X = \begin{pmatrix} A & B \end{pmatrix}$, $Y = C$, $Z = \begin{pmatrix} I_l & O \end{pmatrix}$ and $W = -I_l$ gives

$$\det(M) = \det(-I_l) \det(\begin{pmatrix} A & B + C \end{pmatrix} \begin{pmatrix} I_l & O \end{pmatrix}) = (-1)^l \det(A + C \quad B).$$

The result follows since $N$ can be obtained from $(A + C \quad B)$ by $l(k-l)$ column changes.

Let us introduce the abbreviation

$$R_m(f) := R_{m,q}(f, g),$$

where $g(x) = x^{q^m} - x$ and $f(x) = \sum_{i=0}^{n-1} a_i x^{q^i}$ for some $a_i \in \mathbb{F}_{q^n}$.
Lemma 3.3. \( |D_m(f)| = |R_m(f)|. \)

Proof. Note that \( D_{n-1}(f) = R_{n-1}(f) = (a_{n-1}), \) so we may assume \( m < n - 1. \) Let \( T_k \) denote the \( k \times k \) anti-diagonal matrix whose non-zero entries equal to one and let \( I_k \) denote the \( k \times k \) identity matrix. By \( O \) we will always denote a zero matrix whose dimension will be clear from the context. We distinguish two cases.

If \( m \geq (n - 1)/2, \) then \( 2n - 1 - 2m \leq n \) and hence \( R_m(f) \) has the form:

\[
\begin{pmatrix}
A & B \\
I_{n-1-m} & O
\end{pmatrix},
\]

where \( B = T_{n-m}D_m(f)T_{n-m}. \) We have

\[
\det\left(\begin{pmatrix}
A & B \\
I_{n-1-m} & O
\end{pmatrix}\right) = (-1)^{(n-m-1)(n-m)} \det\left(\begin{pmatrix}
B & A \\
O & I_{n-1-m}
\end{pmatrix}\right),
\]

and hence by Result 3.1

\[
|R_m(f)| = |B| = |D_m(f)|.
\]

If \( m < (n - 1)/2, \) then first consider the last \( m \) rows of \( R_m(f): \) for \( k = 0, 1, \ldots, m - 1 \) the \((2n - 2m - 1 - k)-\text{th}\) row of \( R_m(f) \) contains only one non-zero entry, namely, a 1 at position \( n - 1 - m - k. \) Then it is easy to see by row expansion applied to the last \( m \) rows that:

\[
(-1)^{(n-1)m} |R_m(f)| = \det\left(\begin{pmatrix}
A & B & C \\
I_{n-2m-1} & O & -I_{n-2m-1}
\end{pmatrix}\right),
\]

where \( A \) and \( C \) are \((n-m) \times (n-2m-1)\) matrices and

\[
(B \ A + C) = T_{n-m}D_m(f)T_{n-m}.
\]

According to Corollary 3.2,

\[
(-1)^{(n-1)m} |R_m(f)| = (-1)^{(n-2m-1)(m+2)} T_{n-m}D_m(f)T_{n-m},
\]

which proves the assertion.

\( \square \)

Lemma 3.3 immediately yields Theorem 1.3.

For some \( s \) with \( \gcd(s, n) = 1 \) put \( \sigma := q^s. \) The set of \( \sigma \)-polynomials over \( \mathbb{F}_{q^n} \) is isomorphic to the skew-polynomial ring \( \mathbb{F}_{q^n}[t, \sigma] \) where \( t\sigma = \sigma^q t \) for
all $\alpha \in \mathbb{F}_{q^n}$. Analogies for some of the results of Section 2 should hold in these non-commutative polynomial rings as well. Next we show a generalization of Theorem 1.3 for $\sigma$-polynomials.

Consider the $\sigma$-polynomial $f(x) = \sum_{i=0}^{n-1} a_i x^{q^i} \in \mathbb{F}_{q^n}[x]$, which is also a $q$-polynomial. As before, by $\ker f$ we will denote $\gcd(f(x), x^{q^n} - x)$ and similarly to $D(f)$ we define

$$D_\sigma(f) := \begin{pmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ a_{q-1} & a_0 & \cdots & a_{q-2} \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \cdots & a_0 \end{pmatrix}.$$  

We will denote by $D_{m,\sigma}(f)$ the $(n - m) \times (n - m)$ matrix obtained from $D_\sigma(f)$ after removing its first $m$ columns and last $m$ rows. Because of the applications it might be useful to have conditions on other minors of $D_\sigma(f)$. In the next corollary we show some results also in this direction.

**Corollary 3.4.** If $f(x) = \sum_{i=0}^{n-1} a_i x^{q^i} \in \mathbb{F}_{q^n}[x]$ with $\gcd(s, n) = 1$, then $\dim_q(\ker f) = \mu$ if and only if

$$|D_0,\sigma(f)| = |D_1,\sigma(f)| = \cdots = |D_{\mu-1},\sigma(f)| = 0 \quad (8)$$

and $|D_{\mu,\sigma}(f)| \neq 0$.

Index the rows and columns of $D_\sigma(f)$ from 0 to $n - 1$. For $0 \leq m \leq \dim_q(\ker f)$ if $J, K \subseteq \{0, 1, \ldots, n - 1\}$ are two sets of $m$ consecutive integers modulo $n$ then let $M_{J,K}(f)$ denote the $(n - m) \times (n - m)$ matrix obtained from $D_\sigma(f)$ after removing its rows and columns with indices in $J$ and $K$, respectively. Then

$$|M_{J,K}(f)| = 0 \iff |D_{m,\sigma}(f)| = 0.$$  

**Proof.** Consider $f$ as a $q$-polynomial with $\dim_q(\ker f) = \mu$. This happens if and only if $D(f)$ has rank $\mu$. Recall that rows and columns of $D(f)$ are indexed from 0 to $n - 1$ and let $P$ denote the permutation matrix for which the $i$-th row of $PA$ is the $si$-th row of $A$ (considered modulo $n$). Then $PAP^{-1} = D(f)$ and hence the rank of $D_\sigma(f)$ is the same as the rank of $D(f)$ (cf. also [8, Remark 2.3]). Note that $D_\sigma(f)$ is the Dickson matrix of a $\sigma$-polynomial considered as an $\mathbb{F}_{q^n}$-linear transformation of $\mathbb{F}_{q^n}$ with kernel a $\mu$-dimensional $\mathbb{F}_{q^n}$-subspace of $\mathbb{F}_{q^n}$. By Theorem 1.3 this happens if and only if the conditions on $|D_{m,\sigma}(f)|$ holds for $0 \leq m \leq \mu$.

For the second part take $0 \leq m \leq \dim_q(\ker f)$. Note that for any $\sigma$-polynomial $g(x) = \sum_{i=0}^{n-1} b_i x^{q^i} \in \mathbb{F}_{q^n}[x]$ and for any non-negative integer $t$ the rank of $g(x)$ is the same as
1. the rank of $g(x)^q$ considered modulo $x^n - x$,

2. the rank of $\hat{g}(x) := \sum_{i=0}^{n-1} b_i x^i$ (since $D_\sigma(g)^T = D_\sigma(\hat{g})$, where by $T$ we denote matrix transposition).

Suppose $J = \{j, j + 1, \ldots, j + m - 1\}$ and $K = \{k, k + 1, \ldots, k + m - 1\}$ considered modulo $n$. Then $f_1(x) := f(x)^{q^{n-k-m}}$ modulo $x^n - x$ has the same rank as $f(x)$ and $|M_{\sigma, K}(f_1)| = |M_{\sigma, K}(f)^{q^{n-k-m}}$ where $K' = \{n - m, m + 1, \ldots, n - 1\}$. Then $f_1(x)$ has the same rank as $f_1(x)\text{ and } |M_{\sigma, K}(f_1)| = |M_{\sigma, K}(f_1)|^{q^{n-k-m}}$ where $J' = \{0, 1, \ldots, m - 1\}$. By definition $M_{\sigma, K}(f_2) = D_{m, \sigma}(f_2)$, and hence

$$|D_{m, \sigma}(f_2)| = 0 \iff |M_{\sigma, K}(f_1)| = 0 \iff |M_{\sigma, K}(f_1)| = 0 \iff |M_{\sigma, K}(f)| = 0.$$ 

Recall $0 \leq m \leq \dim_q(\ker f)$. Since $f_2$ and $f$ has the same rank, it follows from the first part of the assertion that $D_{m, \sigma}(f_2) = 0 \iff D_{m, \sigma}(f) = 0$ and this finishes the proof.

### 3.1 Applications

A $q$-polynomial $f(x) \in \mathbb{F}_{q^n}$ is called scattered if $\{f(x)/x : x \in \mathbb{F}_{q^n}\setminus\{0\}\}$ (the set of directions determined by the graph of $f$) has maximum size, that is $(q^n - 1)/(q - 1)$. Put $U_f = \{x, f(x) : x \in \mathbb{F}_{q^n}\}$, which is an $n$-dimensional $\mathbb{F}_q$-subspace of $\mathbb{F}_{q^n}^2$. The linear set of $\text{PG}(1, q^n)$ defined by $f$ is the set of projective points $L_f := \{(x, f(x)) : x \in \mathbb{F}_{q^n}\setminus\{0\}\}$. The weight of a point $\langle(a, b)\rangle_{\mathbb{F}_{q^n}} \in \text{PG}(1, q^n)$ w.r.t. the $\mathbb{F}_q$-subspace $U_f$ is $\dim_q(\langle(a, b)\rangle_{\mathbb{F}_{q^n}} \cap U_f)$. The polynomial $f$ is scattered if and only if the points of $L_f$ have weight 1. In this case $L_f$ and $U_f$ are called maximum scattered. This happens if and only if the $\mathbb{F}_q$-linear transformations of $\mathbb{F}_{q^n}$ in the $\mathbb{F}_{q^n}$-subspace $M := \langle x, f(x)\rangle_{\mathbb{F}_{q^n}}$ have rank at least $n - 1$. Equivalently, $M$ is equivalent to an $\mathbb{F}_{q^n}$-linear maximum rank distance (MRD for short) code of $\mathbb{F}_{q^n}^n$ with minimum distance $n - 1$. For more details about these objects and the relations among them we refer to [16, Section 13.3.6] and the references therein.

**Corollary 3.5.** Consider the $q$-polynomial $f(x) = \sum_{i=0}^{n-1} a_i x^i \in \mathbb{F}_{q^n}[x]$ and with $y$ as a variable consider the matrix

$$H := \begin{pmatrix}
    y & a_1 & \cdots & a_{n-1} \\
    a_{n-1}^q & y^q & \cdots & a_{n-2}^q \\
    \vdots & \vdots & \ddots & \vdots \\
    a_1^{q^{n-1}} & a_2^{q^{n-1}} & \cdots & y^{q^{n-1}}
\end{pmatrix}.$$
The determinant of the \((n-m) \times (n-m)\) matrix obtained from \(H\) after removing its first \(m\) columns and last \(m\) rows is a polynomial \(H_m(y) \in \mathbb{F}_{q^n}[y]\). Then the following holds:

1. The roots of \(H_0(y)\) are in \(\mathbb{F}_{q^n}\),

2. the number of points of weight \(\mu\) of \(L_f\) w.r.t. \(U_f\) is the same as the number of common roots of \(H_0(y), H_1(y), \ldots, H_{\mu-1}(y)\) which are not roots of \(H_\mu(y)\),

3. in particular \(f(x)\) is scattered if and only if \(H_0(y)\) and \(H_1(y)\) have no common roots.

Proof. Let \(y_0\) be a root of \(H_0(y)\). Note that Lemma 3.3 does not require the coefficients of \(f\) to be in \(\mathbb{F}_{q^n}\), thus also for \(y_0 \in \mathbb{F}_q\) we have \(0 = H_0(y_0) = |R_0, q(y_0x + \sum_{i=1}^{n-1} a_i x^{q^i}, x^{q^n} - x)|\) and hence by Theorem 2.1 there exists \(x_0 \in \mathbb{F}_{q^n}\backslash\{0\}\) such that \(y_0 = -\sum_{i=1}^{n-1} a_i x_0^{q^i-1}\). Here the right-hand side is in \(\mathbb{F}_{q^n}\) and hence \(y_0 \in \mathbb{F}_{q^n}\).

By Theorem 1.3 \(H_0(y_0) = H_1(y_0) = \ldots = H_{\mu-1}(y_0) = 0\) and \(H_\mu(y_0) \neq 0\) hold if and only if the \(q\)-polynomial \((y_0 - a_0)x + f(x) \in \mathbb{F}_{q^n}[x]\) has nullity \(\mu\), equivalently, the point \((1, a_0 - y_0)_{\mathbb{F}_{q^n}}\) has weight \(\mu\).

The last part follows from the fact that \(f\) is scattered if and only if \(L_f\) does not have points of weight larger than 1. 

In [2] Part 3. of Corollary 3.5 is used to derive sufficient and necessary conditions for \(f(x) = bx^q + x^{q^i} \in \mathbb{F}_{q^n}[x]\) to be a scattered polynomial and to prove [6, Conjecture 7.5] regarding the number of scattered polynomials of this form.

In [4] the authors study MRD-codes with maximum idealisers, or equivalently, the problem of finding sets of distinct integers \(\{t_0, t_1, \ldots, t_k\}\) such that every \(\mathbb{F}_q\)-linear transformation of \(\mathbb{F}_{q^n}\) in the \(\mathbb{F}_{q^n}\)-subspace \(\langle x^{q^0}, x^{q^1}, \ldots, x^{q^k}\rangle_{\mathbb{F}_{q^n}}\) has rank at least \(n - k\). In [4, Corollary 3.6] it is stated that in \(M := \langle x, x^q, x^{q^2}, x^{q^3}\rangle_{\mathbb{F}_{q^n}}\) one can find an \(\mathbb{F}_q\)-linear transformation of \(\mathbb{F}_{q^n}\) with rank at most 5 and hence the set of integers \(\{0, 1, 2, 4\}\) does not satisfy the above mentioned condition. In [4] this was proved by calculating sixteen \(6 \times 6\) submatrices of \(D(f)\), where \(f(x) = -x + (1 + c q) x^q + cx^{q^2} - x^{q^3}\) and \(c \in \mathbb{F}_{q^n}\) satisfies certain conditions, and by proving that each of them has zero determinant. According to Theorem 1.3 the same result follows also by calculating only \([D_0(f)], [D_1(f)], [D_2(f)], [D_3(f)]\) and by proving that all of them are zero.
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