Non-gaussian Noise in Quantum Spin Glasses and Interacting Two-level Systems

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We study a general model for non-gaussian 1/f noise based on an infinite range quantum Ising spin system in the paramagnetic state, or equivalently, interacting two-level classical fluctuators. We identify a dilatation interaction term in the dynamics which survives the thermodynamic limit and circumvents the central limit theorem to produce non-gaussian noise even when the equilibrium distribution is that of non-interacting spins. The resulting second spectrum (‘noise of the noise’) itself has a universal 1/f form which we analyze within a dynamical mean field approximation.

PACS numbers: 75.50.Lk, 73.50.Td, 61.20.L-, 61.20.Lc

Noise measurements have long been an important probe of the still poorly understood dynamics of spin glasses [1], structural glasses and two-level systems [2]. Interest in 1/f noise [3] has been further enhanced with recent progress in the development of nano-scale electronic devices such as the single electron transistor [4] and precision charge pumps [5] which are extremely sensitive to offset charge fluctuations [6]. Such noise also plays an important role in determining the coherence time of existing ion-trap realizations of quantum bits [7] and is likely to be even more important in other proposed solid state realizations of qubits such as quantum dots [8], Cooper pair boxes [9], and Josephson junctions [10]. A constant loss tangent in a capacitor leads to a 1/f noise spectrum [11] and loss tangents even as small as $10^{-5}$ would have serious consequences in charge-based quantum bits. Ubiquitous non-equilibrium noise may also be implicated in the coherence time of electrons in wires and films observed via weak localization and via mesoscopic persistent currents [12].

We focus here on non-gaussian fluctuations which are usually connected with critical phenomena [13] or systems having only a few degrees of freedom [14]. We develop a simple model with a large number of degrees of freedom and identify interaction terms that circumvent the central limit theorem and produce non-gaussian fluctuations in the thermodynamic limit even in the paramagnetic state far above any critical point. We analyze the non-gaussian fluctuations using the second spectrum (‘noise of the noise’) [15]. In addition to being of interest for the study of two-level systems, the dynamical model we present is also relevant to models of quantum spin glass behavior [16].

Consider $N$ interacting two-level fluctuators (TLFs) labeled by Ising spin variables $S_i = \pm 1$ whose flipping rates obey

$$\frac{1}{\tau_i(S_i)} = e^{-\left[\tilde{D}_i + h_i S_i + \sqrt{\lambda} \sum_{j \neq i} (\gamma_{ij} S_j S_i)\right]}.$$  (1)

This classical dynamics is equivalent to the imaginary time dynamics of a quantum spin Hamiltonian

$$\mathcal{H} = -\sum_{i=1}^{N} \sigma_i^x + \frac{1}{\tau_i} \sigma_i^z.$$  (2)

with the Ising variables $S$ replaced by Pauli spin matrices $\sigma$. For a spin glass, $h_i$ would typically represent a uniform external magnetic field, i.e. $H_i = 1$. For a structural glass the $h_i$ represent random local site energy differences for the TLFs. $\gamma_{ij}$ is the spin-spin interaction energy and $\tilde{D}_i = \Delta_i + \frac{1}{N} \sum_{j \neq i} [\lambda (\gamma_{ij} + g G_{ij}) S_j]$ is the barrier height which fluctuates around its bare value $\Delta_i$ due to coupling between the TLFs. We take $H_i$, $\Lambda_i$, $G_{ij} = G_{ji}$ and $\Gamma_{ij} = \Gamma_{ji}$ to be random and uniformly distributed on $[-1,1]$, as in the infinite-range Sherrington-Kirkpatrick model [16].

In steady state, detailed balance gives us the single-spin probability ratio

$$\frac{P(S_i = 1)}{P(S_i = -1)} = e^{2 h_i + 2 \sqrt{\lambda} \sum_{j \neq i} \gamma_{ij} S_j}.$$  (3)

Note that this is completely independent of $\tilde{D}_i$ which controls various features of the dynamics but has no effect on the equilibrium statistical mechanics.

If only if $\Gamma$ is symmetric, we can write the full equilibrium probability distribution in terms of a Boltzmann factor derived from a Hamiltonian

$$P[S] = \frac{1}{Z} e^{-\frac{1}{\lambda} \sum_i h_i S_i + \frac{1}{\lambda} \sum_{i < j} \Gamma_{ij} S_i S_j},$$  (4)

where $Z$ is the partition function. Notice that the matrix $G$ need not be symmetric. The approach to equilibrium might depend on this, but the equilibrium does not.

For large enough $\gamma$ (low enough temperature) the system undergoes a glass transition into a frozen state. Near this transition the dynamics may also become non-gaussian, but we restrict our attention here to the high temperature paramagnetic regime where $\gamma$ is small. Our central result is that even if $\gamma = 0$ so that the equilibrium distribution is that of non-interacting spins, the second spectrum indicates non-gaussian dynamics for any $\lambda \neq 0$. 
The term $\sum_{j \neq i} \lambda \Lambda_{ij} S_j$ is a ‘dilatation’ term. It causes all the barriers to collectively and simultaneously increase or decrease in a correlated manner. In contrast to this, interactions through the $g G_{ij}$ and $\gamma \Gamma_{ij}$ terms cause barrier fluctuations which are statistically independent on different sites (at least for small $g$ and $\gamma$ so that the system is above the critical point). As a consequence of the central limit theorem, one expects that only the dilatation term survives to produce non-gaussian fluctuations in the thermodynamic limit of this infinite-range model.

A possible physical realization of such a dilatation term is the elastic interaction between TLFs. In addition to the dipole term one could have an expansion of the lattice coupled to the impurity positions [19]. The motion of a defect then changes the hydrostatic pressure by an amount proportional to $\frac{1}{t}$ throughout the volume $V$ [20]. Since the barriers would tend to go up uniformly with increasing hydrostatic pressure, a long-range dilatation term with an amplitude $\sim N^{-1}$ would be present [19]. Such a term thus could be important in small grains. We have chosen for simplicity to scale the interaction with $N^{-\frac{1}{2}}$ in order to obtain a sensible thermodynamic limit in the present model.

To ensure that the power spectrum of the model decays as $1/t$, we take the bare barriers to have a flat distribution on a fixed interval $\Delta_i \in [\Delta_{\text{min}}, \Delta_{\text{max}}]$. We carry out simulations using Eq. (4). To average, each “measurement” was repeated 1600 times, where $\Lambda_i, \Lambda_j, G_{ij}$ and $\Gamma_{ij}$ are assigned new values each time. Using $\delta t$ as a discretizing time interval, the noise signal is defined as $V(t \delta t) = N^{-1/2} \sum_{i=1}^{N} S_i(t \delta t)$. Henceforth, we set $\delta t = 1$.

In this letter, $t, t_2$ and $\nu, \nu_2$ are dimensionless integer times and frequencies, respectively. The normalization $N^{-1/2}$ ensures that $\langle V^2(t) \rangle = 1$, independent of $N$.

To study the time-time correlation of the noise, we use the power spectrum defined as $\langle W(\nu) \rangle = \langle M^{-1/2} \sum_{t=1}^{M} e^{2\pi i \nu t/M} V(t) \rangle^2$, where $M$ is the length of the time sequence. We see in Fig. 1 that the power spectrum is insensitive to weak interactions.

To study non-gaussian properties of the noise, we use the second spectrum introduced by Restle et al. [16,17]. Given a time sequence of length $M$, we split it into $M_2$ shorter time sequences of length $M_1$, each, and define the second spectrum as

$$\langle W^{(2)}(\nu) \rangle = \left\langle \left| \frac{1}{\sqrt{M_2}} \sum_{t_2=1}^{M_2} e^{2\pi i \nu t_2/M_2} \sum_{\nu=\nu_{\text{min}}}^{\nu_{\text{max}}} W_{t_2}(\nu) \right|^2 \right\rangle,$$

$$W_{t_2}(\nu) = \left| \frac{1}{\sqrt{M_1}} \sum_{t=t_2M_1}^{(t_2+1)M_1} e^{2\pi i \nu t/M_1} V(t) \right|^2. \quad (5)$$

The quantity $W_{t_2}(\nu)$ is a single shot “power spectrum” of $V(t)$ in the time window $t_2$. Physically, the second spectrum is a spectrum of the power spectrum’s wandering within a chosen frequency band, $[\nu_{\text{min}}, \nu_{\text{max}}]$. It is a four-point correlation function which probes the noise of the noise [16,17]. For the non-interacting (gaussian) case the second spectrum becomes flat [18] and takes the value $\sum_{\nu_{\text{min}}}^{\nu_{\text{max}}} \langle W(\nu) \rangle^2$. We see in Fig. 2 that for $\lambda = 0$ the second spectrum decreases for increasing $N$, merging into its non-interacting (gaussian) counter part for large $N$. However, for $\lambda = 0.5$ the second spectrum is independent of $N$ for large $N$. Thus, the dilatation term in Eq. (4) is the only term that survives the central limit theorem.

**FIG. 1** Power spectrum as a function of frequency for a non-interacting ($\lambda = g = h = \gamma = 0$) and two interacting ($\lambda = g = h = \gamma = 0.25, 0.5$) cases. Other parameters are: $\Delta_{\text{max}} = 14, \Delta_{\text{min}} = 1, M = 2^{19}$ and $N = 100$. $\langle W(\nu) \rangle$ is found to be independent of $N$. Thin dotted line is a plot of the analytical expression for the non-interacting case, Eq. (4). Typical barrier fluctuations for the ($\lambda = g = h = \gamma = 0.5$) case are $\sim 10\%$ and $\sim 100\%$ of $\Delta_{\text{max}}$ and $\Delta_{\text{min}}$, respectively, indicating weak or moderate interactions.

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**FIG. 2** Second spectrum as a function of frequency for $\lambda = 0$ and 0.5 with $N = 100, 200, 400, 800, 1200$. Other parameters are: $g = h = \gamma = 0.5, M_1 = M_2 = 2^{19}, \Delta_{\text{max}} = 15, \Delta_{\text{min}} = 0.5, \nu_{\text{min}} = 125$ and $\nu_{\text{max}} = 175$. Data for the $\lambda = 0$ case is shifted down one decade for clarity. Dots (○) show the second spectrum for the non-interacting (gaussian) case.
All the other terms produce only gaussian fluctuations in the thermodynamic limit.

We now calculate the power spectrum and second spectrum analytically. First, we set $g = h = \gamma = 0$ since, as shown above, these terms do not contribute to non-gaussian properties of the model and the power spectrum is insensitive to weak interactions. Second, we set $\Delta_i = (\Delta^{\text{max}} - \Delta^{\text{min}})i/N + \Delta^{\text{min}}$. Third, we replace $A_i \in [-1,1]$ by $A_i = 1$. Mathematically, it is always possible to make a variable (‘gauge’) transformation $S'_i = \text{sign}(A_i)S_i$ such that all $A_i$ are positive in the new basis. Simulations have shown that the results for these two cases are qualitatively the same, with the second spectrum of the $A_i = 1$ case somewhat larger as expected.

We now use the Suzuki-Trotter decomposition \[21\] and recast the quantum model, Eq. \[4\], into a classical model with the partition function

$$Z = \left\{ \prod_{i=1}^{N} \prod_{t=1}^{M} \sum_{S_i(t)} \sum_{j} \delta\left[ \tilde{V}(t) - \frac{1}{\sqrt{N}} \sum_{j} S_j(t) \right] \right\} \times \left( \frac{1}{N} \sum_{i,t} \left[ K_i(t)S_i(t)S_i(t+1) + \frac{1}{2} \ln \left[ \sinh[2\epsilon_i(t)] \right] \right] \right).$$

Here, $\tilde{V}(t)$ is an auxiliary field used to decouple the spins. Furthermore, $\epsilon_i(t) = \frac{1}{2} e^{-\Delta} - \lambda \tilde{V}(t)$ and $K_i(t) = \frac{1}{2} \ln \left( \coth[\epsilon_i(t)] \right)$, where $\beta = 1/k_B T$ is the inverse temperature. In Eq. \[5\], we have replaced $\delta[\tilde{V}(t) - N^{-1/2} \sum_{j \neq i} S_j(t)]$ by $\delta[\tilde{V}(t) - N^{-1/2} \sum_{j=1}^{N} S_j(t)]$ and, thereby, introduced an error of order $O(N^{-1/2})$ which vanishes in the $N \to \infty$ limit.

Using Eq. \[4\], the power spectrum for the non-interacting case may in the large $N$ and large $M$ limit be shown to take the well known form

$$\langle W(\nu; \lambda = 0) \rangle \sim \text{atan}(\frac{\epsilon^{\Delta^{\text{min}}}}{\nu M}) - \text{atan}(\frac{-\epsilon^{\Delta^{\text{max}}}}{\nu M}).$$

From Eq. \[4\], we see that for intermediate frequencies $\langle W(\nu; \lambda = 0) \rangle \sim 1/\nu$, independent of the temperature. Henceforth, we set $\beta/M = 1$ so that the analytical and simulation results are directly comparable.

We plot Eq. \[4\] in Fig. \[1\] using the same parameters as in the simulations. There is nice agreement between the simulation and analytical results. At the highest frequencies, $\nu > 10^6$, results from the simulations are higher compared to the analytical result. This is expected since in the simulations the time sequences are discrete, while in the analytical calculations a continuum description is used to integrate out the sums. It is precisely at these highest frequencies that the difference between the two methods becomes important.

We now apply the dynamical mean-field approximation (DMFA) \[8\] by replacing the formal averaging over $\tilde{V}$ in Eq. \[3\] with the requirement that all (multi-point) correlation functions of $\tilde{V}(t)$ are self-consistently identical to the corresponding correlations functions of $V(t)$. For example, $\langle \tilde{V}(t)\tilde{V}(t') \rangle_{\tilde{V}} = \langle V(t)V(t') \rangle_{S}$. Here, $\langle \rangle_{S}$ and $\langle \rangle_{\tilde{V}}$ are averages over the spin configurations and the auxiliary field, respectively. As for other infinite-range spin models \[15\], this approximation becomes exact in the $N \to \infty$ limit.

For a classical 1D Ising model $\langle S(t)S(t')S(t''t''') \rangle = \langle S(t)S(t') \rangle \langle S(t'') \rangle \langle S(t''') \rangle$, given that the overlap between the time intervals $[t, t']$ and $[t'', t''']$ is zero. Thus, the term $\langle W_{t_1}(\nu) \ W_{t_2}(\nu') \rangle_{S}$ in Eq. \[3\] may be replaced by $\langle W_{t_1}(\nu) \ W_{t_2}(\nu') \rangle_{S} + O(M_2^{-1/2})$, where the error $O(M_2^{-1/2})$ vanishes in the $M_2 \to \infty$ limit. Using Eq. \[3\] and the DMFA we obtain

$$\langle W_{t_2}(\nu; \lambda) \rangle_{S} = \frac{1}{M_1^{(t_1+1)*M_2}} \sum_{t',t''=t_2M_1} \exp \left[ -2e^{-\Delta} \sum_{t=t'}^{t''} e^{-\lambda \tilde{V}(t)} \right].$$

We split $\tilde{V}$ into fast and slow components, $\tilde{V} = \tilde{V}_f + \tilde{V}_s$, and then assume that $\tilde{V}_s$ is constant over the time interval $[t', t'']$ whereas $\sum_{t'=t}^{t''} e^{-\lambda \tilde{V}_f(t)}$ is replaced by its second cumulant average. The second cumulant depends only on the first spectrum, which from Fig. \[1\] we know is universal and independent of the all couplings including $\lambda$. In Eq. \[8\] we may therefore make the approximation $\sum_{t'=t}^{t''} e^{-\lambda \tilde{V}_f(t)} \simeq [t' - t''] e^{-\lambda \tilde{V}_f(t_2)+\xi}$ where $\xi$ is an unimportant constant which will not contribute to the second spectrum computed to order $\lambda^2$ below. Applying the approximation to Eq. \[8\] we find

$$\langle W_{t_2}(\nu; \lambda) - W(\nu;0) \rangle \simeq \frac{e^{\Delta^{\text{min}}} (1/\nu) e^{-\Delta^{\text{max}} - \Delta^{\text{min}}}}{e^{\Delta^{\text{max}} - \Delta^{\text{min}}}}.$$

within the frequency band $1 \ll \nu \ll M_2^{-1} e^{-\Delta^{\text{min}} - \lambda}$. Here, $\nu \in \left[ 0, \frac{1}{M_2} \right]$. Eq. \[8\] shows that the shape of $\langle W_{t_2}(\nu; \lambda) \rangle_{S}$ is not modified by the interactions between the fast and the slow fluctuators. Because the barrier distribution is flat, there is no net effect on the first spectrum, but the mean square wandering produces the second spectrum.

Expanding Eq. \[4\] to second order in $\lambda$ and ensemble averaging over the fluctuations of $\tilde{V}_f(t_2)$ we see that the non-gaussian contribution to the second spectrum is, up to a simple normalization factor, determined by the slow part of the first spectrum:
\[
\begin{align*}
\langle \Delta W^{(2)}(\nu_2; \lambda) \rangle_{\nu_2, \lambda} & \sim \langle W^{(2)}(\nu_2; \lambda) - W^{(2)}(\nu_2; 0) \rangle_{\nu_2, \lambda} \\
& \approx \left[ \lambda e^{-\Delta_{\min} \left( \frac{\nu_\text{max} - \nu_\text{min}}{\Delta_{\max} - \Delta_{\min}} \right)^2} \right]^2 W(\nu_2; \lambda = 0) \approx \frac{A_0}{\nu_2}, \quad (10)
\end{align*}
\]

where \( A_0 = \frac{[\lambda e^{-\Delta_{\min} \left( \frac{\nu_\text{max} - \nu_\text{min}}{\Delta_{\max} - \Delta_{\min}} \right)^2}]^2}{\nu_2} \). From Eq. (10) and the universality of the first spectrum shown in Fig. [3] we see that, apart from the prefactor, the shape of the non-gaussian part of second spectrum is independent of all interaction parameters as well as the size and position of the frequency window used to probe it, provided we restrict the frequency band to be inside \( \nu \ll \frac{\lambda e^{-\Delta_{\min} - \lambda}}{\Delta_{\max} - \Delta_{\min}} \). In Fig. [3] we plot simulation results for four different sets of parameters. We see that the simulation results collapse onto the line predicted by the analytical result. The small deviations may be explained by the fact that \( \lambda \) is not infinitesimal, by practical limitations on the available dynamical range which cause the inequality below Eq. (3) to be imperfectly satisfied, and by the unavoidable poor sampling statistics for the slowest fluctuators.

In the present model, the second spectrum has been found to decay close to \( 1/f \); that is, the \( 1/f \) noise itself has \( 1/f \) noise. This is the same result seen experimentally in semiconductors [22,33], discontinuous metal films [23], spin glasses [1] and carbon resistors [17]. Our model is an infinite-range interaction model in which we have chosen the simplest possible dynamics that circumvents the central limit theorem to yield non-gaussian noise even in the thermodynamic limit of the weakly correlated paramagnetic phase. To the extent that the Ising variables we use could represent entire droplets of spins, our model could also potentially be relevant to the strongly correlated regime of spin glasses [23].

We thank M. Weissman, J. Sethna, M. Kardar, J. Carini and N. Ma, for useful discussions. This work is supported by the Norwegian Research Council 13527/410 and by NSF DMR-0087133 and the ITP at UCSB under NSF PHY-9907949.

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