DESCRIPTION OF MIXED MOTIVES

DOOSUNG PARK

Abstract. Assuming the Künneth type standard conjecture, we propose a way to describe objects of mixed motives explicitly. We study their formal properties, and we associate mixed motives to schemes smooth and separated over a field. This serves as a universal cohomology theory. We also discuss ℓ-adic realizations, and we discuss an unconditional construction of 2-motives and their properties.

1. Introduction

1.1. Throughout this paper, fix a perfect field $k$ with an embedding $k \hookrightarrow \overline{k}$ to its algebraic closure.

A conjecture ([11, Definition 2.20]) is that there is an abelian category of mixed motives $\text{MM}$ over $k$ such that for each mixed motive $M$, there is a weight filtration

$$(1.1.1) \quad M = W^r M \to \cdots \to W^{-1} M = 0$$

such that $W^i M / W^{i-1} M$ is a pure motive of weight $i$ for each $i$.

Voevodsky constructed the triangulated category $\text{DM}^{eff}_{gm}(k, \mathbb{Z})$ of geometric effective motives and the triangulated category $\text{DM}_{gm}(k, \mathbb{Z})$ of geometric motives ([18]). If there is a conic over $k$ with no rational points, then $\text{DM}_{gm}(k, \mathbb{Z})$ cannot have a reasonable $t$-structure whose heart is $\text{MM}$ ([18, Proposition 4.3.8]). It is still conjectured that the triangulated category $\text{DM}_{gm}(k, \mathbb{Q})$ of motives with $\mathbb{Q}$-coefficient has a reasonable $t$-structure whose heart is $\text{MM}$. Indeed, in [7] there is a conditional proof of the existence of a reasonable $t$-structure of $\text{DM}_{gm}(k, \mathbb{Q})$ assuming several conjectures including the Hanamura vanishing conjecture given as follows.

(Van$_{\leq n}$) Assume $(C_{\leq n})$. For any $X, Y \in \text{SmProj}_{\leq n}$ and $d, e \geq 0$,

$$\text{Hom}(M_d(X), M_e(Y)[p]) = 0$$

Here, we denote by $\text{SmProj}_{\leq n}$ the category of schemes smooth and projective over $k$ whose dimensions are $\leq n$.

This is one description of the category $\text{MM}$ assuming the existence of a reasonable $t$-structure. How can we describe objects of $\text{MM}$ explicitly? This question is what we want to study in this paper.

For simplicity of notations, we set

$$\text{DM}^{eff}_{gm} := \text{DM}^{eff}_{gm}(k, \mathbb{Q}), \quad \text{DM}_{gm} := \text{DM}_{gm}(k, \mathbb{Q}), \quad L = 1(1)[2].$$

Here, $1$ denotes the object $M(k)$ of $\text{DM}^{eff}_{gm}$.

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1.2. Since MM should be an extension of the category of numerical motives \([8]\), we may need to assume the K"unneth type standard conjecture \([6]\). Even if \(k\) is not algebraically closed field, we can still formulate this conjecture as follows.

\((C_{\leq n})\) For any \(X \in \text{SmProj} \leq n\) and \(d \geq 0\), there is a projector \(p_d : M(X) \to M(X)\) such that the corresponding homomorphism

\[ H^*_\text{ét}(X_{\bar{\kappa}}, \mathbb{Q}_\ell) \to H^*_\text{ét}(X_{\bar{\kappa}}, \mathbb{Q}_\ell) \]

is the composition

\[ H^*_\text{ét}(X_{\bar{\kappa}}, \mathbb{Q}_\ell) \to H^d_{\text{ét}}(X_{\bar{\kappa}}, \mathbb{Q}_\ell) \to H^*_\text{ét}(X_{\bar{\kappa}}, \mathbb{Q}_\ell). \]

Here, \(X_{\bar{\kappa}} := X \times_k \bar{\kappa}\), and the first (resp. second) arrow is the obvious projection (resp. inclusion). We denote by \(M_d(X)\) the image of \(p_d\) in \(\text{DM}_{\text{gm}}\), which exists since \(\text{DM}_{\text{gm}}\) is a pseudo-abelian category.

Then we can provide a description of MM inspired by the weight filtration \([1,1,1]\) as follows.

**Definition 1.3.** Assume \((C_{\leq n})\).

1. We denote by

\[ \text{Gr}_d \text{MM} \leq n \]

the full subcategory of \(\text{DM}_{\text{gm}}\) consisting of elements of the form \(M[-d]\) where \(M\) is a direct summand of \(M_d(X)\) for some \(X \in \text{SmProj} \leq n\). An object of \(\text{Gr}_d \text{MM} \leq n\) is called a pure \(n\)-motive of weight \(d\).

2. Set \(W^d \text{MM} \leq n := 0\) for \(d < 0\). For \(d \geq 0\), we inductively denote by

\[ W^d \text{MM} \leq n \]

the full subcategory of \(\text{DM}_{\text{gm}}\) consisting of objects \(M\) such that there is a distinguished triangle

\[ M' \to M \to M'' \to M'[1] \]

in \(\text{DM}_{\text{gm}}\) with \(M' \in \text{ob} \text{Gr}_d \text{MM} \leq n\) and \(M'' \in \text{ob} W^{d-1} \text{MM} \leq n\). An object of \(W^d \text{MM} \leq n\) is called an \(n\)-motive of weights \(\leq d\).

3. We denote by

\[ \text{MM} \leq n \]

the union of \(W^d \text{MM} \leq n\) for \(d \geq 0\). An object of \(\text{MM} \leq n\) is called an \(n\)-motive.

**Definition 1.4.** Assume \((C_{\leq n})\) for any \(n\).

1. We denote by

\[ W^d \text{MM}_{\text{eff}} \]

the union of \(W^d \text{MM} \leq n\) for \(n \geq 0\). An object of \(W^d \text{MM}_{\text{eff}}\) is called an effective mixed motive of weights \(\leq d\).

2. We denote by

\[ \text{MM}_{\text{eff}} \]

the union of \(\text{MM} \leq n\) for \(n \geq 0\). An object of \(\text{MM}_{\text{eff}}\) is called an effective mixed motive.

3. We denote by

\[ \text{MM} \]

the full subcategory of \(\text{DM}_{\text{gm}}\) consisting of objects of the form \(M(r)\) for \(M \in \text{ob} \text{MM}_{\text{eff}}\) and \(r \in \mathbb{Z}\). An object of \(\text{MM}\) is called a mixed motive.
If one needs to emphasize that our coefficient ring is $\mathbb{Q}$, one may call a mixed motive as a mixed motive with $\mathbb{Q}$-coefficient.

1.5. Then we study formal properties of mixed motives. We first study the weight filtration functor $W^d : \text{MM} \leq n \to W^d \text{MM} \leq n$. For this purpose, we may need to assume the Murre vanishing conjecture ([10, Proposition 5.8]), which is as follows.

(\text{Mur} \leq n) Assume $C \leq n$. For any $X, Y \in \text{SmProj} \leq n$ and $d > e \geq 0$,\n
$$\text{Hom}(M_d(X), M_e(Y)) = 0.$$\n
Then we construct the functor $W^d$ as follows and provide a definition of $n$-motives of weights $\geq d$.

**Theorem 1.6** (Theorem [2.8]). Assume $(C \leq n)$ and $(\text{Mur} \leq n)$. Then the inclusion $W^d \text{MM} \leq n \to \text{MM} \leq n$ admits a left adjoint denoted by $W^d : \text{MM} \leq n \to W^d \text{MM} \leq n$.

**Definition 1.7.** Assume $(C \leq n)$ and $(\text{Mur} \leq n)$. We denote by $W^d \text{MM} \leq n$ the full subcategory of $\text{MM} \leq n$ consisting of objects $M$ such that $W^d - 1 M = 0$. Its object is called an $n$-motive of weights $\geq d$. If $M$ is in the intersection of $W_2 \text{MM} \leq n$ and $W^e \text{MM} \leq n$, we say that $M$ is an $n$-motive of weights in $[d, e]$.

1.8. We also study the duality for mixed motives, which is the generalization of Cartier duality for 1-motives (2.1.5).

**Theorem 1.9** (Theorem [3.2]). Assume $(C \leq n)$ and $(\text{Mur} \leq n)$. Let $M$ be an $n$-motive of weights $\leq d$. Then $\text{Hom}(M, 1(n))$ is an $n$-motive of weights $\geq 2n - d$. Here, $\text{Hom}$ denotes the internal hom of $\text{DM}^{\text{eff}}$.

1.10. An object $M$ of $\text{MM} \leq n$ can be explicitly described as a commutative diagram

$$\begin{array}{cccccc}
Gr_{2n} M & \to & Gr_{2n-1} M & \to & Gr_{2n-2} M & \to & \cdots & \to & Gr_1 M & \to & Gr_0 M \\
\downarrow & & \downarrow & & \downarrow & & \cdots & & \downarrow & & \downarrow \\
M & \to & W^{2n-1} M & \to & W^{2n-2} M & \to & \cdots & \to & W^0 M & \to & 0 \\
\end{array}$$

such that each triangle is a distinguished triangle and $Gr_0 M$ is a pure $n$-motive of weight $d$ for any $d$. Alternatively, $M$ can be described as a commutative diagram

$$\begin{array}{cccccc}
0 & \to & W_{2n} M & \to & \cdots & \to & W_2 M & \to & W_1 M & \to & M \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
Gr_{2n} M & \to & \cdots & \to & Gr_2 M & \to & Gr_1 M & \to & Gr_0 M \\
\end{array}$$

such that each triangle is a distinguished triangle.

In particular, if $n = 1$, then $M$ is precisely a Deligne 1-motive $[Gr_0 M \to W_1 M[1]]$ with a diagram

$$\begin{array}{cccc}
Gr_0 M & \to & 0 \\
\downarrow & & \\
Gr_2 M[1] & \to & W_1 M[1] & \to & Gr_1 M[1] & \to & 0 \\
\end{array}$$

such that
(i) $Gr_0 M$ is a pure 0-motive (a.k.a. lattice),
(ii) $Gr_1 M[1]$ is an abelian variety,
(iii) $Gr_2 M[1]$ is a torus,
(iv) $W_1 M[1]$ is a semi-abelian variety,
(v) the bottom row is an exact sequence.

Thus our description is a generalization of Deligne 1-motives.

1.11. The category $\mathcal{M} M$ should be an abelian category containing the category of numerical motives. We expect that the category of numerical motives is equivalent to the full subcategory of $D M_{g m}$ consisting of finite direct sums $M_1 \oplus \cdots \oplus M_r$

such that each $M_i$ is a pure $n_i$-motive of weight $d_i$ for some $n_i$ and $d_i$. The category of numerical motives is semisimple by [9]. Hence we may need to assume additional conjectures to ensure that the above category is semisimple, which are as follows.

(wVan$_{\leq n}$) Assume (C$_{\leq n}$). For any $X, Y \in SmProj_{\leq n}$ and $e > d \geq 0$,
$$\text{Hom}(M_d(X)[-d], M_e(Y)[-e]) = 0.$$ 

(Semi$_{\leq n}$) Assume (C$_{\leq n}$). For any $X \in SmProj_{\leq n}$ and $d \geq 0$,
$$\text{Hom}(M_d(X), M_d(X))$$

is a finite dimensional semisimple ring.

Then we have the following result.

**Theorem 1.12 (Theorem 4.3).** Assume (C$_{\leq n}$), (Mur$_{\leq n}$), (wVan$_{\leq n}$), and (Semi$_{\leq n}$). Then $MM_{\leq n}$ is an abelian category.

1.13. In [16], the following conjecture is introduced.

(CK$_{\leq n}$) For any $X \in SmProj_{\leq n}$, there are projectors $p_0, \ldots, p_{2n} : M(X) \rightarrow M(X)$ satisfying the condition in (C$_{\leq n}$) such that
$$p_i \circ p_j = 0$$

if $i \neq j$.

If (CK$_{\leq n}$) holds, then for any $X \in SmProj_{\leq n}$, there is a decomposition
$$M(X) = M_0(X) \oplus \cdots \oplus M_{2n}(X)$$
in $D M_{g m}^{\text{eff}}$, which is called the Chow-K"unneth decomposition. This decomposition is not functorial. Set
$$M_{\leq d}(X) := M_0(X) \oplus \cdots \oplus M_d(X)$$
for each $d$. We show that the morphism $M(X) \rightarrow M_{\leq d}(X)$ induced by the decomposition is functorial assuming (CK$_{\leq n}$), (Mur$_{\leq n}$), and the following conjecture.

(wVan'$_{\leq n}$) Assume (C$_{\leq n}$). For any $X, Y \in SmProj_{\leq n}$ and $d > e \geq 0$,
$$\text{Hom}(M_d(X), M_e(Y)[-1]) = 0.$$ 

1.14. One of the motivations for mixed motives is to construct a sort of universal cohomology for schemes not necessarily projective over $k$. Assuming (C$_{\leq n}$), we construct the $\ell$-adic realization functor
$$R_\ell : MM_{\leq n} \rightarrow \text{Rep}_{\text{Gal}(\overline{k}/k)}(\mathbb{Q}_\ell).$$

Here, $\text{Rep}_{\text{Gal}(\overline{k}/k)}(\mathbb{Q}_\ell)$ denotes the category of $\mathbb{Q}_\ell$-representations of $\text{Gal}(\overline{k}/k)$. 

Let $U$ be a scheme smooth and separated over $k$ whose dimension is $\leq n$. To study the $\ell$-adic cohomology using $\text{MM}_{\leq n}$, we need to associate $n$-motives
\[(1.14.1) \quad M_0(U), \ldots, M_{2n}(U), M_0^c(U), \ldots, M_{2n}^c(U).\]

Let us introduce a weaker version of $(\text{Semi}_{\leq n})$, which is as follows.

$(\text{Semi}'_{\leq n})$ Assume $(\text{Semi}_{\leq n-1})$. For any $d$ and morphism $f : M \to N$ in $\text{MM}_{\leq n}$ such that
\begin{itemize}
  \item[(i)] $(d < 2n-2$ and $M$ is an $(n-1)$-motive of weights $\leq d$) or $(d = 2n-2$ and $M$ is a pure $(n-1)$-motive of weight $d$),
  \item[(ii)] $N$ is a pure $n$-motive of weight $d$.
\end{itemize}
the kernel, cokernel, and image of $f$ exist, and the cokernel of $f$ is a pure $n$-motive of weight $d$.

We also use the following conjectures.

$(\text{Res}_{\leq n})$ Resolution of singularities holds for any integral scheme separated over $k$ whose dimension is $\leq n$.

Assuming several conjectures, we can construct the motives $(1.14.1)$ and show
\begin{itemize}
  \item[1.15] $(\text{Theorem 1.15})$ Assume $(\text{CK}_{\leq n})$, $(\text{Mur}_{\leq n})$, $(\text{wVan}'_{\leq n})$, $(\text{Res}_{\leq n})$, and $(\text{Semi}'_{\leq n})$. Then for any integral scheme $U$ smooth over $k$ whose dimension is $\leq n$,
\[
R_\ell(M_d(U)[-d]) \cong H^d_\ell(U_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell), \quad R_\ell(M_d^c(U)[-d]) \cong H^d_\ell(U_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell).
\]
\end{itemize}

1.16. If $n = 2$, then $(\text{CK}_{\leq 2})$ is known $(15)$. Thus we have an unconditional construction of $\text{MM}_{\leq 2}$. Moreover, $(\text{Mur}_{\leq 2})$ is known $(12\text{ Theorem 7.3.10})$, $(\text{Res}_{\leq 2})$ is known, and we prove $(\text{Semi}'_{\leq 2})$ and $(\text{wVan}'_{\leq 2})$. Thus we get the following result

$\text{Theorem 1.17}$ $(\text{Theorem 1.17})$. When $n = 2$, the conclusions of Theorems 1.16 and 1.15 hold.

1.18. Thus we have an unconditional construction of $\text{MM}_{\leq 2}$ and its several properties. However, since $(\text{wVan}_{\leq 2})$ and $(\text{Semi}_{\leq 2})$ are open, in Theorem 1.12, we have only a conditional proof that $\text{MM}_{\leq 2}$ is an abelian category.

In [1], Ayoub introduced 2-motives using the 2-motivic $\ell$-structure. The advantage of his definition is that the category of Ayoub 2-motives is abelian. However, it requires at least some vanishing conjectures to show that $M_2(X)[-2]$, $M_3(X)[-3]$, and $M_4(X)[-4]$ are Ayoub 2-motives where $X$ is a surface smooth and projective over $k$. Our conjecture is that the category of our 2-motives is a full subcategory of the category of Ayoub 2-motives.

For general $n$, even though the conclusion of Theorem 1.15 depends on several conjectures, there is a chance that for a specific $U$, the motives $(1.14.1)$ are easy to construct. Then the argument in Theorem 1.15 can be used to such a case.

1.19. Organizations. In Section 2, we define the weight filtration, and we study its functoriality. In Section 3, we study the dual $\text{Hom}(M, 1(n))$ of an $n$-motive $M$. 

\[
H^d_\ell(U_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) := H^{2n-d}_{\ell, c}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)(n), \quad H^d_\ell(U_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) := H^{2n-d}_{\ell, c}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)(n).
\]
In Section 4, we provide a conditional proof that $MM_{\leq n}$ is an abelian category. In Section 5, we construct motives $M_0(U), \ldots, M_{2m}(U)$ and $M_0^c(U), \ldots, M_{2m}^c(U)$ in $MM_{\leq n}$ for any scheme $U$ smooth and separated over $k$ whose dimension is $\leq n$. In Section 6, we define the $\ell$-adic realization functor, and we compare $M_d(U)[-d]$ with the $\ell$-adic cohomology theory. In Section 7, we discuss 2-motives.

2. Weight filtration

Lemma 2.1. Let $\mathcal{T}$ be a triangulated category, and let

$$
\begin{array}{c}
M' \xrightarrow{u} M \xrightarrow{v} M'' \xrightarrow{w} M'[1] \\
\downarrow f \\
N' \xrightarrow{u'} N \xrightarrow{v'} N'' \xrightarrow{w'} N'[1]
\end{array}
$$

(2.1.1)

be a diagram in $\mathcal{T}$ such that each row is a distinguished triangle. Assume that

$\text{Hom}(M', N'') = 0$, $\text{Hom}(M', N''[-1]) = 0$.

Then we have the following results.

(1) There is a unique morphism $f' : M' \to N'$ making the above diagram commutative.

(2) There is a unique morphism $f'' : M'' \to N''$ making the above diagram commutative.

(3) There is a unique pair of morphisms $(f' : M' \to N', f'' : M'' \to N'')$ making the above diagram into a morphism of distinguished triangles in $DM_{\text{eff}}^c$.

Proof. (1) Consider the exact sequence

$$
\text{Hom}(M', N'')[-1] \xrightarrow{\text{Hom}(M', N')} \text{Hom}(M', N) \xrightarrow{\text{Hom}(M', N'')} \text{Hom}(M', N''),
$$

of abelian groups. Since $\text{Hom}(M', N'') = 0$ and $\text{Hom}(M', N''[-1]) = 0$, the second arrow is an isomorphism. Thus there is a unique morphism $f' : M' \to N'$ such that $u'f' = fu$.

(2) Consider the exact sequence

$$
\text{Hom}(M'[-1], N'') \xrightarrow{\text{Hom}(M', N'')} \text{Hom}(M, N'') \xrightarrow{\text{Hom}(M', N'')} \text{Hom}(M', N'')
$$

of abelian groups. Since $\text{Hom}(M', N'') = 0$ and $\text{Hom}(M', N''[-1]) = 0$, the second arrow is an isomorphism. Thus there is a unique morphism $f'' : M'' \to N''$ such that $f''u = v'f$.

(3) Choose $f'$ as in (1). Since each row in (2.1.1) is a distinguished triangle, there is a morphism $f'' : M'' \to N''$ making (2.1.1) into a morphism of distinguished triangles in $DM_{\text{eff}}^c$. The uniqueness of $f''$ follows from (2).

Lemma 2.2. Under the notations and hypotheses of Lemma 2.1, if $f$ is an isomorphism, then $f'$ and $f''$ are isomorphisms.

Proof. By Lemma 2.1, we also have a morphism

$$
\begin{array}{c}
N' \xrightarrow{u'} N \xrightarrow{v'} N'' \xrightarrow{w'} N'[1] \\
\downarrow g' \downarrow f_{[-1]} \downarrow g'' \downarrow f'[1] \\
M' \xrightarrow{u} M \xrightarrow{v} M'' \xrightarrow{w} M'[1]
\end{array}
$$
of distinguished triangles in $\mathcal{T}$. Consider the diagram

$$
\begin{array}{ccc}
M' & \xrightarrow{u} & M \\
\downarrow{id} & & \downarrow{id} \\
M' & \xrightarrow{u} & M
\end{array}
$$

in $\mathcal{T}$. Then both

$$(g' f' : M' \to M', g'' f'' : M'' \to M''), \ (\text{id} : M' \to M', \text{id} : M'' \to M'')$$

make the above diagram into a morphism of distinguished triangles in $\mathcal{T}$. Thus by Lemma 2.1, $g' f' = \text{id}$ and $g'' f'' = \text{id}$. By the same argument, $f' g' = \text{id}$ and $f'' g'' = \text{id}$. Thus $f'$ and $f''$ are isomorphisms. □

**Lemma 2.3.** Let $\mathcal{T}$ be a triangulated category, and let

$$
\begin{array}{ccc}
M' & \xrightarrow{u} & M \\
\downarrow{f'} & & \downarrow{f''} \\
N' & \xrightarrow{v} & N
\end{array}
$$

be a commutative diagram in $\mathcal{T}$. If

$$\text{Hom}(M', N'') = 0, \quad \text{Hom}(M', N''[-1]) = 0, \quad \text{Hom}(M'', N') = 0,$$

then the above diagram can be uniquely extended to a morphism

$$
\begin{array}{ccc}
M' & \xrightarrow{u} & M \\
\downarrow{f'} & & \downarrow{f''} \\
N' & \xrightarrow{v} & N
\end{array}
$$

of distinguished triangles in $\mathcal{T}$.

**Proof.** Note that such an $f$ exists since the rows are distinguished triangles. Hence it remains to show that $f$ is unique. Let $f, f_0 : M \to N$ be morphisms making (2.3.1) still commutative. Then

$$v'(f - f_0) = v'f - v'f_0 = f''v - f''v = 0,$$

so $f - f_0 = u'g$ for some morphism $g : M \to N'$ in $\mathcal{T}$. Thus

$$u'g = (f - f_0)u = fu - f_0u = u'f' - u'f' = 0.$$

Consider the commutative diagram

$$
\begin{array}{ccc}
M' & \xrightarrow{u} & M \\
\downarrow{g u} & & \downarrow{g u[1]} \\
N' & \xrightarrow{v'} & N
\end{array}
$$

in $\mathcal{T}$. By Lemma 2.1, $g u = 0$. Then $g = hv$ for some morphism $h : M'' \to N'$ in $\mathcal{T}$. Since $\text{Hom}(M'', N') = 0$, $h$ should be 0. Thus $f = f_0$. □

**Proposition 2.4.** Assume $(C_{\leq n})$. For any $X, Y \in \text{SmProj}_{\leq n}$, $d, e \geq 0$, and $p > 0$,

$$\text{Hom}(M_d(X), M_e(Y)[p]) = 0.$$
Proof. We may assume that $Y$ is connected. Since $M_d(X)$ (resp. $M_e(Y)$) is a direct summand of $M(X)$ (resp. $M(Y)$), it suffices to show that

$$\text{Hom}(M(X), M(Y)[p]) = 0.$$ 

Then it suffices to show that

$$\text{Hom}(M(X \times Y), 1(r)[2r + p]) = 0$$

where $r$ is the dimension of $Y$ by [14, Theorem 16.24] (see [13] to remove the assumption of resolution of singularity in our case). This follows from [14, Vanishing Theorem 19.3]. □

Proposition 2.5. Assume $(C \leq n)$ and $(\text{Mur} \leq n)$. Let $M$ be a pure $n$-motive of weight $d$, and let $N$ be a pure $n$-motive of weight $e$. If $d > e$, then

$$\text{Hom}(M, N) = 0, \quad \text{Hom}(M, N[-1]) = 0.$$ 

Proof. Choose varieties $X$ and $Y$ projective over $k$ with dimensions $\leq n$ such that $M$ (resp. $N$) is a direct summand of $M_d(X)[d]$ (resp. $M_e(Y)[e]$). Then it suffices to show that

$$\text{Hom}(M_d(X), M_e(Y)[d - e]) = 0, \quad \text{Hom}(M_d(X), M_e(Y)[d - e - 1]) = 0.$$ 

The first one follows from Proposition 2.4. The second one follows from Proposition 2.4 if $d > e + 1$ and from $(\text{Mur} \leq n)$ if $d = e + 1$. □

2.6. For each $n$-motive $M$, by definition, we can choose morphisms

$$M = W^{2n}M \to W^{2n-1}M \to \cdots \to W^{-1}M = 0$$

in $\text{DM}^{\text{eff}}_g$ such that a cone of $W^d M \to W^{d-1}M$ is a pure $n$-motive of weight $d$.

Proposition 2.7. Assume $(C \leq n)$ and $(\text{Mur} \leq n)$. Let $M$ be a pure $n$-motive of weight $d$, and let $N$ be an $n$-motive of weights $\leq e$. If $d > e$, then

$$\text{Hom}(M, N) = 0, \quad \text{Hom}(M, N[-1]) = 0.$$ 

Proof. Choose morphisms $W^{2n}N \to \cdots \to W^{-1}N$ as above. By Proposition 2.5, the homomorphisms

$$\text{Hom}(M, W^d N) \to \text{Hom}(M, W^{d-1}N),$$

$$\text{Hom}(M, W^d N[-1]) \to \text{Hom}(M, W^{d-1}N[-1])$$

are injective for each $d$ since a cocone of $W^d N \to W^{d-1}N$ is a pure $n$-motive of weight $d$. Thus

$$\text{Hom}(M, N) = \text{Hom}(M, W^{2n}N) \to \text{Hom}(M, W^{-1}N) = 0,$$

$$\text{Hom}(M, N[-1]) = \text{Hom}(M, W^{2n}N[-1]) \to \text{Hom}(M, W^{-1}N[-1]) = 0$$

are injective, so we are done. □

Theorem 2.8. Assume $(C \leq n)$ and $(\text{Mur} \leq n)$. The inclusion functor

$$W^d \text{MM} \leq n \to \text{MM} \leq n$$

admits a left adjoint denoted by $W^d : \text{MM} \leq n \to W^d \text{MM} \leq n$. 

Proof. We may assume that $Y$ is connected. Since $M_d(X)$ (resp. $M_e(Y)$) is a direct summand of $M(X)$ (resp. $M(Y)$), it suffices to show that
2.9. We will complete the proof of Theorem 2.8 in 2.11. For each $n$-motive, set $W^dM$ as in 2.6. We will construct a functor structure on $W^d$. Let $f : M \to N$ be a morphism of $n$-motives. We will show that there is a unique set of morphisms
\[ \{ W^{2n-1}f : W^{2n-1}M \to W^{2n-1}N, \ldots, W^0f : W^0M \to W^0N \} \]
in $\text{DM}_{gm}^{\text{eff}}$ such that the induced diagram
\[
\begin{array}{c}
M = W^{2n}M \xrightarrow{f} W^{2n-1}M \xrightarrow{} \cdots \xrightarrow{} W^dM \\
\downarrow{W^{2n-1}} \quad \quad \quad \downarrow{W^d} \\
N = W^{2n}N \xrightarrow{} W^{2n-1}N \xrightarrow{} \cdots \xrightarrow{} W^dN
\end{array}
\]
of $n$-motives commutes.

For sufficiently large $d$, $W^dM \cong M$ and $W^dN \cong N$, so the above claim holds for such a $d$. Let us use an induction on $d$. Assume that the above claim holds for $d$. Let $Gr_dM$ (resp. $Gr_dN$) be a cocone of $W^dM \to W^{d-1}M$ (resp. $W^dN \to W^{d-1}N$). Consider the induced diagram
\[
\begin{array}{c}
Gr_dM \xrightarrow{} W^dM \xrightarrow{} W^{d-1}M \xrightarrow{} Gr_dM[1] \\
\downarrow{W^d} \quad \quad \quad \downarrow{W^{d-1}} \\
Gr_dN \xrightarrow{} W^dN \xrightarrow{} W^{d-1}N \xrightarrow{} Gr_dN[1]
\end{array}
\]
of $n$-motives. By Proposition 2.7
\[ \text{Hom}(Gr_dM, W^{d-1}N) = 0, \quad \text{Hom}(Gr_dM, W^{d-1}N[-1]) = 0. \]
Thus by Lemma 2.1, the above diagram can be uniquely extended to a morphism of distinguished triangles:
\[
\begin{array}{c}
Gr_dM \xrightarrow{} W^dM \xrightarrow{} W^{d-1}M \xrightarrow{} Gr_dM[1] \\
\downarrow{W^d} \quad \quad \quad \downarrow{W^{d-1}} \\
Gr_dN \xrightarrow{} W^dN \xrightarrow{} W^{d-1}N \xrightarrow{} Gr_dN[1]
\end{array}
\]
Thus the above claim holds for $d - 1$. This completes the induction process. Now the functoriality of $W^d$ follows from the uniqueness.

**Proposition 2.10.** Assume $(C_{\leq n})$ and $(\text{Mur}_{\leq n})$. Let $M$ be an $n$-motive, and let $N$ be an $n$-motive of weights $\leq e$. Suppose that $M$ admits a sequence of morphisms
\[ M = W^{2n}M \to \cdots \to W^{d-1}M = 0 \]
such that for each $r \geq d$, a cocone of $W^rM \to W^{r-1}M$ is a pure $n$-motive of weight $r$. If $d > e$, then
\[ \text{Hom}(M, N) = 0, \quad \text{Hom}(M, N[-1]) = 0. \]

**Proof.** By Proposition 2.7 the homomorphisms
\[ \text{Hom}(W^rM, N) \to \text{Hom}(W^{r-1}M, N) \]
\[ \text{Hom}(W^rM, N[-1]) \to \text{Hom}(W^{r-1}M, N[-1]) \]
are injective since $r \geq d > e$. Thus
\[ \text{Hom}(M, N) = \text{Hom}(W^{2n}M, N) \to \text{Hom}(W^{d-1}M, N) = 0, \]
\[ \text{Hom}(M, N[-1]) = \text{Hom}(W^{2n}M, N[-1]) \to \text{Hom}(W^{d-1}M, N[-1]) = 0 \]
are injective, so we are done. \[\square\]

2.11. **Proof of Theorem 2.8.** It remains to show that $W^d$ is the left adjoint of the inclusion functor. For any $n$-motive $L$ of weights $\leq d$, it suffices to show that the induced homomorphism

$$\text{Hom}(W^d M, L) \to \text{Hom}(M, L)$$

is an isomorphism. By Proposition 2.10 it suffices to show that a cocone $N$ of $M \to W^d M$ admits a sequence of morphisms

$$(2.11.1) \quad N = W^{2d} N \to \cdots \to W^d N = 0$$

such that for each $e \geq d$, a cocone of $W^{e+1} N \to W^e N$ is a pure $n$-motive of weight $r$.

For $e \geq d$, let $W^e N$ be a cocone of $W^e M \to W^d M$. By the octahedral axiom, there is a commutative diagram

$$\begin{array}{cccccc}
Gr_e M & \xrightarrow{id} & Gr_e M & \to & 0 & \to Gr_e M[1] \\
\downarrow & & \downarrow & & \downarrow & \\
W^{e+1} N & \to & W^{e+1} M & \to & W^d M & \to W^{e+1} N[1] \\
\downarrow & & \downarrow & & \downarrow & \\
W^e N & \to & W^e M & \to & W^d M & \to W^e N[1] \\
\downarrow & & \downarrow & & \downarrow & \\
Gr_e M[1] & \xrightarrow{id} & Gr_e M[1] & \to & 0 & \to Gr_e M[2]
\end{array}$$

in $\text{DM}^{eff}_{gm}$ such that each row and column is a distinguished triangle. Thus we obtain (2.11.1), and a cocone of each $W^{e+1} N \to W^e N$ is isomorphic to $Gr_e M$, which is a pure $n$-motive of weight $e$. \[\square\]

Now see Definition 1.7 for the definition of $W^d_{MM \leq n}$, $n$-motives of weights $\geq d$, and $n$-motives of weights in $[d, e]$.

**Proposition 2.12.** Assume $(C_{\leq n})$ and $(\operatorname{Mur}_{\leq n})$. Then the inclusion functor

$$W_d : \text{MM}_{\leq n} \to \text{MM}_{\leq n}$$

admits a right adjoint denoted by

$$W_d : \text{MM}_{\leq n} \to W_d \text{MM}_{\leq n}.$$  

**Proof.** For any $n$-motive $M$, choose a cone of $M \to W^{d-1} M$, and let us denote it by $W_d M$. We will show that $M \to W_d M$ has a functor structure. Let $f : M \to N$ be a morphism of $n$-motives. Consider the induced diagram

$$\begin{array}{cccccc}
W_d M & \to & M & \to & W^{d-1} M & \to & W_d M[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
W_d N & \to & N & \to & W^{d-1} N & \to & W_d N[1]
\end{array}$$

of $n$-motives. By Proposition 2.10

$$\text{Hom}(W_d M, W^{d-1} N) = 0, \quad \text{Hom}(W_d M, W^{d-1} N[-1]) = 0.$$
Thus by Lemma 2.1 the above diagram can be uniquely extended to a morphism of distinguished triangles:

\[
\begin{array}{ccc}
W_dM & \longrightarrow & M \\
\downarrow w_d & & \downarrow f \\
W_dN & \longrightarrow & N
\end{array}
\]

(2.12.1) \[
\begin{array}{ccc}
W_dM & \longrightarrow & W^{d-1}M \\
\downarrow w_d & & \downarrow w_d^{-1}f \\
W_dN & \longrightarrow & W^{d-1}N
\end{array}
\]

The functoriality of \(W_d\) follows from the uniqueness.

It remains to show that \(W_d\) is the right adjoint of the inclusion functor. For any \(n\)-motive \(L\) of weights \(\geq d\), it suffices to show that the induced homomorphism

\[
\text{Hom}(L, W_d M) \rightarrow \text{Hom}(L, M)
\]

is an isomorphism. This follows from Proposition 2.10 since a cone of \(W_d M \rightarrow M\) is isomorphic to \(W_d^{-1} M \), which is an \(n\)-motive of weight \(\leq d-1\). \(\Box\)

**Proposition 2.13.** Assume \((C_{\leq n})\) and \((\text{Mur}_{\leq n})\). Consider the functors \(W_d : \text{MM}_{\leq n} \rightarrow W_d \text{MM}_{\leq n}\) and \(W^d : \text{MM}_{\leq n} \rightarrow W^d \text{MM}_{\leq n}\). For any \(d\), there is a natural transformation \(\partial : W^{d-1} \rightarrow W_d[1]\) such that

\[
W_d \rightarrow \text{id} \rightarrow W^{d-1} \partial \rightarrow W_d[1]
\]

is a distinguished triangle of functors.

**Proof.** For each \(n\)-motive \(M\), choose a morphism \(\partial : W^{d-1} M \rightarrow W_d M[1]\) of \(n\)-motives such that

\[
W_d M \rightarrow M \rightarrow W^{d-1} M \rightarrow W_d M[1]
\]

is a distinguished triangle. It remains to show that for any morphism \(f : M \rightarrow N\) of \(n\)-motives, the diagram

\[
\begin{array}{ccc}
W^{d-1}M & \longrightarrow & W_d M[1] \\
\downarrow w^{d-1}f & & \downarrow w_d f[1] \\
W^{d-1}N & \longrightarrow & W_d N[1]
\end{array}
\]

of \(n\)-motives commutes. This follows from the commutativity of (2.12.1). \(\Box\)

**Proposition 2.14.** Assume \((C_{\leq n})\) and \((\text{Mur}_{\leq n})\). For any \(d \geq e\), there is a natural isomorphism

\[
W^d W_e \cong W_e W^d
\]

of functors.

**Proof.** Consider the commutative diagram

\[
\begin{array}{ccc}
W_{d+1} M \\
\downarrow \\
W_e M & \longrightarrow & M \\
\downarrow & & \downarrow \text{id} \\
W_e W^d M & \longrightarrow & W^d M \\
\downarrow & & \downarrow \\
W_{d+1} M[1]
\end{array}
\]
in $\text{DM}^{\text{eff}}_{gm}$ where the rows and the second column are distinguished triangles in Proposition 2.13. By the octahedral axiom, this can be completed to a commutative diagram

$$
\begin{array}{cccccc}
W_{d+1}M & \overset{id}{\longrightarrow} & W_{d+1}M & \longrightarrow & 0 & \longrightarrow W_{d+1}M[1] \\
\downarrow & & \downarrow & & \downarrow & \\
W_eM & \longrightarrow & M & \longrightarrow & W^{e-1}M & \longrightarrow W_eM[1] \\
\downarrow & & \downarrow & & \downarrow & \\
W_eW^dM & \longrightarrow & W^dM & \longrightarrow & W^{e-1}M & \longrightarrow W_eW^dM[1] \\
\downarrow & & \downarrow & & \downarrow & \\
W_{d+1}M[1] & \overset{id}{\longrightarrow} & W_{d+1}M[1] & \longrightarrow & 0 & \longrightarrow W_{d+1}M[2]
\end{array}
$$

(2.14.1)

in $\text{DM}^{\text{eff}}_{gm}$ where each row and column is a distinguished triangle. Consider the diagram

$$
\begin{array}{cccccc}
W_{d+1}M & \longrightarrow & W_eM & \longrightarrow & W^dW_eM & \longrightarrow W_{d+1}M[1] \\
\downarrow & & \downarrow & & \downarrow & \\
W_{d+1}M & \longrightarrow & W_eM & \longrightarrow & W^dW_eM & \longrightarrow W_{d+1}M[1]
\end{array}
$$

in $\text{DM}^{\text{eff}}_{gm}$ where the first row is the first column in (2.14.1), and the second row is induced by the distinguished triangle in Proposition 2.13. By Lemmas 2.1 and 2.2, there is a unique isomorphism $W^dW_eM \cong W^dW_eM$ making the above diagram commutative. Now the functoriality follows from the uniqueness, so this gives a natural isomorphism $W^dW_e \cong W^dW_e$ of functors. □

**Definition 2.15.** Assume $(C \leq n)$ and $(\text{Mur} \leq n)$. For each $d$, set

$$Gr_d := W^dW_e.$$

**Proposition 2.16.** Assume $(C \leq n)$, $(\text{Mur} \leq n)$, and $(\text{wVan} \leq n)$. Let $M$ be an $n$-motive of weights $\geq d$, and let $N$ be an $n$-motive of weights $\leq e$. If $d > e$, then

$$\text{Hom}(N, M) = 0.$$ 

**Proof.** Consider the distinguished triangle

$$Gr_eN \rightarrow N \rightarrow W^{e-1}N \rightarrow W_eN[1]$$

in $\text{DM}^{\text{eff}}_{gm}$. To show that $\text{Hom}(N, M) = 0$, it suffices to show that

$$\text{Hom}(Gr_eN, M) = 0, \quad \text{Hom}(W^{e-1}N, M) = 0.$$ 

Repeating this process, we reduce to the case when $N$ is a pure $n$-motive of weight $e$. By the same argument, we reduce to the case when $M$ is a pure $n$-motive of weight $d$. Then $\text{Hom}(N, M) = 0$ by $(\text{wVan} \leq n)$. □
3. Duality

**Proposition 3.1.** Assume \((C_{\leq n})\) and \((\text{Mur}_{\leq n})\), and let \(M\) be an object of \(\text{DM}_{\text{gm}}^{\text{eff}}\). If there is a distinguished triangle

\[
M' \xrightarrow{f} M \xrightarrow{g} M'' \xrightarrow{h} M'[1]
\]

in \(\text{DM}_{\text{gm}}^{\text{eff}}\) such that \(M'\) (resp. \(M''\)) is an \(n\)-motive of weights in \([b, c]\) (resp. \([a, b-1]\)), then \(M\) is an \(n\)-motive of weights in \([a, c]\).

**Proof.** Let us use an induction on \(c\). If \(c = b - 1\), then we are done since \(M' = 0\). Assume that \(c \geq b\). We have the distinguished triangle

\[
W^cM' \xrightarrow{u} M' \xrightarrow{v} W^{c-1}M' \xrightarrow{\partial} W^cM'[1]
\]

by Proposition 2.13. Then by the octahedral axiom, there is a commutative diagram

\[
\begin{array}{cccccccc}
W^cM' & \xrightarrow{id} & W^cM' & \rightarrow & 0 & \rightarrow & W^cM'[1] \\
\downarrow{u} & & \downarrow{f} & & \downarrow{g} & & \downarrow{h} & \downarrow{u[1]} \\
M' & \xrightarrow{f} & M & \rightarrow & M'' & \xrightarrow{h} & M'[1] \\
\downarrow{v} & & \downarrow{id} & & \downarrow{id} & & \downarrow{v[1]} \\
W^{c-1}M & \rightarrow & N & \rightarrow & M'' & \rightarrow & W^{c-1}M'[1] \\
\downarrow{\partial} & & \downarrow{\partial} & & \downarrow{\partial[1]} & & \downarrow{\partial[1]} \\
W^cM'[1] & \xrightarrow{id} & W^cM'[1] & \rightarrow & 0 & \rightarrow & W^cM'[1] \\
\end{array}
\]

in \(\text{DM}_{\text{gm}}^{\text{eff}}\) such that each row and column is a distinguished triangle. Consider the third row. By induction on \(c\), \(N\) is an \(n\)-motive of weights in \([a, c-1]\). Then consider the second column. By definition, \(M\) is an \(n\)-motive of weights \(\leq c\). We also have that \(W^aM = 0\) since \(W^aW^cM = 0\) and \(W^aN = 0\). Thus \(M\) is an \(n\)-motive of weights in \([a, c]\). \(\square\)

**Theorem 3.2.** Assume \((C_{\leq n})\) and \((\text{Mur}_{\leq n})\). Let \(M\) be an \(n\)-motive of weights \(\leq d\). Then \(\text{Hom}(M, 1(n))\) is an \(n\)-motive of weights \(\geq 2n - d\).

**Proof.** We may assume that \(M\) is an \(n\)-motive of weights \(\leq d\). Let us use an induction on \(d\). If \(d < 0\), then we are done since \(M = 0\). If \(d \geq 0\), we have the distinguished triangle

\[
W_dM \rightarrow M \rightarrow W^{d-1}M \rightarrow W_dM[1]
\]

in \(\text{DM}_{\text{gm}}^{\text{eff}}\). Then we get the distinguished triangle

\[
\text{Hom}(W_dM[1], 1(n)) \rightarrow \text{Hom}(W^{d-1}M, 1(n)) \rightarrow \text{Hom}(M, 1(n)) \rightarrow \text{Hom}(W_dM, 1(n))
\]

in \(\text{DM}_{\text{gm}}^{\text{eff}}\). By induction, \(\text{Hom}(W^{d-1}M, 1(n))\) is an \(n\)-motive of weights \(\geq 2n-d+1\). Since \(W_dM\) is a pure motive of weight \(d\), \(\text{Hom}(W_dM, 1(n))\) is a pure motive of weight \(2n - d\). Thus by Proposition 2.11 \(\text{Hom}(M, L^n)\) is an \(n\)-motive of weights \(\geq 2n - d\). \(\square\)
4. ABELIAN CATEGORY

Lemma 4.1. Assume \((C_{\leq n})\) and \((\text{Mur}_{\leq n})\). Let \(M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3\) be morphisms in \(\text{DM}^{\text{eff}}_{gm}\) such that \(M_1\) and \(M_2\) are \(n\)-motives. If \(\text{Gr}_d f\) has a retraction for each \(d\), and if \(gf = 0\), then \(g = 0\).

Proof. We may assume that \(M_1\) and \(M_2\) are \(n\)-motives for some \(n\). For \(d \geq 0\), consider the induced commutative diagram

\[
\begin{array}{ccc}
\text{Gr}_d M_1 & \xrightarrow{\text{Gr}_d f} & \text{Gr}_d M_2 \\
\downarrow & & \downarrow \\
M_1 & \xrightarrow{f} & M_2 \\
\downarrow & & \downarrow \\
W^{d-1} M_1 & \xrightarrow{W^{d-1} f} & W^{d-1} M_2
\end{array}
\]

in \(\text{DM}^{\text{eff}}_{gm}\). The composition \(\text{Gr}_d M_1 \to M_3\) is 0 since \(gf = 0\). Thus the composition \(\text{Gr}_d M_2 \to M_3\) is 0 since \(\text{Gr}_d f\) has a retraction. Then \(g\) factors through \(W^{d-1} M_2 = 0\), so \(g = 0\). □

Proposition 4.2. If \((\text{Semi}_{\leq n})\) holds, then \(\text{Gr}_d \text{MM}_{\leq n}\) is a semisimple abelian category.

Proof. Follows from [8, Lemma 2]. □

Theorem 4.3. Assume \((C_{\leq n})\), \((\text{Mur}_{\leq n})\), \((\text{Semi}_{\leq n})\), \((\text{wVan}_{\leq n})\), and \((\text{Semi}_{\leq n})\). Then the category \(\text{MM}_{\leq n}\) is an abelian category.

Proof. Let \(f : M_1 \to M_2\) be a morphism of \(n\)-motives, and assume that \(W^d M_1 \cong M_1\) and \(W^d M_2 \cong M_2\). Note that for any morphism of pure \(n\)-motives of the same weight, kernel, cokernel, and image exist by Proposition 4.2. Moreover, any kernel has a retraction, and any cokernel has a section.

(I) Construction of kernels. Consider the induced commutative diagram

\[
\begin{array}{ccc}
\ker W^{d-1} f & \xrightarrow{h} & W^{d-1} M_1 \\
\downarrow & & \downarrow \text{g}_1 \\
\ker \text{Gr}_d f[1] & \xrightarrow{h'} & \text{Gr}_d M_3[1]
\end{array}
\]

in \(\text{DM}^{\text{eff}}_{gm}\), where the upper row is given by induction on \(d\). Since \(h'\) is a morphism of pure \(n\)-motives of the same weight, \(h'\) has a retraction denoted by \(u'\). The composition \(v := u'g_1h\) makes the above diagram still commutative. Let \(\ker f\) be a cocone of \(v\). Then we get a morphism \(\ker f \to M_1\) from the above diagram.

We will show that \(\ker f\) is the kernel of \(f\). Consider a morphism \(f' : M_3 \to M_1\) such that \(ff' = 0\). By induction on \(d\), there is a unique morphism \(w : W^{d-1} M_3 \to \ker W^{d-1} f\) in \(\text{DM}^{\text{eff}}_{gm}\) such that \(hw = W^{d-1} f'\). There is also a unique morphism \(w' : \text{Gr}_d M_3[1] \to \ker \text{Gr}_d f[1]\) such that \(h'w' = \text{Gr}_d f'[1]\) by Proposition 4.2. Then
we have the induced commutative diagram
\[
\begin{array}{ccccccc}
W^{d-1}M_1 & \xrightarrow{a'} & \text{im} W^{d-1} & \xrightarrow{b} & W^{d-1}M_2 \\
\downarrow g_1 & & \downarrow c & & \downarrow g_2 \\
Gr_dM_1 & \xrightarrow{c'} & \text{im} Gr_f[1] & \xrightarrow{b'} & Gr_dM_2[1]
\end{array}
\]

in $\text{DM}_{\text{gm}}^{\text{eff}}$. The morphism $v : \ker W^{d-1}f \to \ker Gr_d f[1]$ makes the above diagram still commutative since
\[
vw = u'g_1hw = u'g_1(hw) = u'(h'w')g_1 = u'h'w'g_1 = w'g_1.
\]
Consider the induced commutative diagram
\[
\begin{array}{ccccccc}
Gr_dM_3 & \xrightarrow{w'[-1]} & M_3 & \xrightarrow{w} & W^{d-1}M_3 & \xrightarrow{w'} & Gr_dM_3[1] \\
\downarrow \text{ker} Gr_d f & & \downarrow \text{ker} f & & \downarrow \text{ker} W^{d-1}f & & \downarrow \text{ker} Gr_d f[1]
\end{array}
\]
in $\text{DM}_{\text{gm}}^{\text{eff}}$. Then
\[
\text{Hom}(Gr_dM_3, \ker W^{d-1}f) = 0, \quad \text{Hom}(Gr_dM_3, \ker W^{d-1}f[-1]) = 0,
\]
by Propositions 2.5 and 2.16. Thus there is a unique morphism $M_3 \to \ker f$ in $\text{DM}_{\text{gm}}^{\text{eff}}$, making the above diagram commutative. Now the uniqueness of kernel follows from Lemma 2.3 and the fact that $w$ and $w'$ are unique.

(II) Construction of cokernels. Let $f : M_1 \to M_2$ be a morphism of $n$-motives. The construction of its cokernel is similar to that of kernel, but we need to use $W_{d-1}$ instead of $W^{d-1}$. We do not repeat the proof here.

(III) Final step of the proof. It remains to show that the cokernel of the kernel agrees with the kernel of the cokernel. By induction on $d$, we suppose that this holds for $W^{d-1}f$. Consider the induced commutative diagram
\[
\begin{array}{ccccccc}
W^{d-1}M_1 & \xrightarrow{a} & \text{im} W^{d-1} & \xrightarrow{b} & W^{d-1}M_2 \\
\downarrow g_1 & & \downarrow c & & \downarrow g_2 \\
Gr_dM_1 & \xrightarrow{c'} & \text{im} Gr_f[1] & \xrightarrow{b'} & Gr_dM_2[1]
\end{array}
\]
in $\text{DM}_{\text{gm}}^{\text{eff}}$, where $c$ (resp. $c'$) is obtained from the cokernel of the kernel (resp. the kernel of the cokernel) of $f$. We only need to show that $c = c'$.

By Proposition 4.2, there are a retraction $a''$ of $a'$ and a section $b''$ of $b'$. Then
\[
ca = a'g_1 = b''b'a'g_1 = b''(b'a')g_1 = b''g_2(ba) = b''g_2ba = b''b'c'a = c'a.
\]
Thus $c = c'$ by Lemma 4.1.

\begin{corollary}
Assume $(C_{\leq n})$, $(\text{Mur}_{\leq n})$, $(\text{Semi}_{\leq n})$, and $(\text{wVan}_{\leq n})$ for any $n$. Then the category $\text{MM}$ is an abelian category.
\end{corollary}

\begin{proof}
For each $a \geq 0$, let $\mathcal{A}_a$ be the full subcategory of $\text{MM}$ consisting of objects of the form $M(r)$ for $M \in \text{ob} \text{MM}^{\text{eff}}$ and $r \geq -a$. By 19, the functor
\[
\text{DM}_{\text{gm}}^{\text{eff}} \to \text{DM}_{\text{gm}}^{\text{eff}}
\]
given by \( F \mapsto F(a) \) is fully faithful. Thus \( \mathcal{A}_a \simeq \text{MM}^{\text{ff}} \), which is an abelian category by Theorem 4.3. Then \( \text{MM} \) is an abelian category since it is the union of \( \mathcal{A}_a \) for \( a \geq 0 \). □

**Proposition 4.5.** Assume \((C_{\leq n}), (\text{Mur}_{\leq n}), (\text{Semi}_{\leq n}), \) and \((\text{wVan}_{\leq n})\). Let \( f : M \to N \) be a morphism in \( W^d \text{MM}_{\leq n} \).

1. If \( M \) is a pure \( n \)-motive of weight \( d \), then \( \ker f \) is a pure \( n \)-motive of weight \( d \).
2. If \( N \) is a pure \( n \)-motive of weight \( d \), then \( \text{cok} f \) is a pure \( n \)-motive of weight \( d \).

**Proof.**

(1) In the proof of Theorem 4.3, we have a distinguished triangle

\[
\ker \text{Gr}_d f \to \ker f \to \ker W^{d-1} f \to \ker \text{Gr}_d f[1]
\]

in \( \text{DM}_{\text{eff}} \). Since \( \ker W^{d-1} f = 0 \) by assumption, \( \ker \text{Gr}_d f \cong \ker f \). This is a pure \( n \)-motive of weight \( d \).

(2) Its proof is dual to that of (1). □

5. MOTIVES ASSOCIATED WITH SCHEMES

**Definition 5.1.** Assume \((C_{\leq n})\). Let \( C_{\leq n} \) denote the category defined as follows.

(i) An object \( F \) is a sequence \( M_{\leq 2n}(F) \to M_{\leq 2n-1}(F) \to \cdots \to M_{\leq -(d)} = 0 \) in \( \text{DM}_{\text{eff}}^{\text{gm}} \) such that for each \( d \), \( M_d(F)[-d] \) is an \( n \)-motive of weights \( \leq d \) for each \( d \). Here, let \( M_d(F) \) be a cocone of the morphism \( M_{\leq d}(F) \to M_{\leq d-1}(F) \) in \( \text{DM}_{\text{eff}}^{\text{gm}} \).

(ii) A morphism \( F \to G \) is a commutative diagram

\[
\begin{array}{cccccc}
M_{\leq 2n}(F) & \to & M_{\leq 2n-1}(F) & \to & \cdots & \to M_{\leq 0}(F) \\
\downarrow f_{\leq 2n} & & \downarrow f_{\leq 2n-1} & & & \downarrow f_{\leq 0} \\
M_{\leq 2n}(G) & \to & M_{\leq 2n-1}(G) & \to & \cdots & \to M_{\leq 0}(G)
\end{array}
\]

(iii) The composition of morphisms is given by composing the commutative diagrams.

We have the functor \( \pi : C_{\leq n} \to \text{DM}_{\text{eff}}^{\text{gm}} \) given by

\( \pi(F) = M_{\leq 2n}(F) \).

We often omit \( \pi \) for brevity. For each \( d \), we denote by \( M_{\leq d} C_{\leq n} \) (resp. \( M_{\geq d} C_{\leq n} \)) the full subcategory of \( C_{\leq n} \) consisting of objects \( F \) such that \( M_{\leq e}(F) = 0 \) for \( e > d \) (resp. \( e < d \)).

**Lemma 5.2.** Assume \((C_{\leq n}), (\text{Mur}_{\leq n}), \) and \((\text{wVan}_{\leq n})\). Let \( F \) be an object of \( M_{\geq d} C_{\leq n} \), and \( G \) be objects of \( M_{\leq e} C_{\leq n} \). If \( d > e \), then

\[
\text{Hom}(F,G) = 0, \quad \text{Hom}(F,G[-1]) = 0.
\]

**Proof.** Let us use an induction on \( d \). If \( e = -1 \), then we are done since \( \pi(F) = 0 \). Hence assume that \( e \geq 0 \). Consider the distinguished triangle

\[
M_e(G) \to M_{\leq e}(G) \to M_{\leq e-1}(G) \to M_e(G)[1]
\]
in $\text{DM}^{\text{eff}}_{\text{gm}}$. By induction,

$$\text{Hom}(F, M_{\leq e-1}(G)) = 0, \quad \text{Hom}(F, M_{\leq e-1}(G)[-1]) = 0.$$ 

Thus we reduce to the case when $G = M_e(G)$. Similarly, we reduce to the case when $F = M_d(F)$.

Then $\text{Hom}(F, G) = 0$ follows from (Mur$_{\leq n}$), and $\text{Hom}(F, G[-1]) = 0$ follows from (wVan$_{\leq n}$).

**Proposition 5.3.** Assume $(C_{\leq n})$, (Mur$_{\leq n}$), and (wVan$_{\leq n}$). Then the functor $\pi : C_{\leq n} \to \text{DM}^{\text{eff}}_{\text{gm}}$ is fully faithful.

**Proof.** Let $f : \pi(F) \to \pi(G)$ be a morphism in $\text{DM}^{\text{eff}}_{\text{gm}}$. Set $f_{\leq 2n} := f$, and consider the commutative diagram

$$
\begin{array}{cccc}
M_{2n}(F) & \longrightarrow & M_{\leq 2n}(F) & \longrightarrow & M_{\leq 2n-1}(F) & \longrightarrow & M_{2n}(F)[1] \\
\downarrow f_{\leq 2n} & & & & & & \\
M_{2n}(G) & \longrightarrow & M_{\leq 2n}(G) & \longrightarrow & M_{\leq 2n-1}(G) & \longrightarrow & M_{2n-1}(G)[1]
\end{array}
$$

in $\text{DM}^{\text{eff}}_{\text{gm}}$. By 5.2

$$\text{Hom}(M_{2n}(F), M_{\leq 2n-1}(G)) = 0, \quad \text{Hom}(M_{2n}(F), M_{\leq 2n-1}(G)[-1]) = 0.$$ 

Thus by Lemma 2.7 there is a unique morphism $f_{\leq 2n-1} : M_{\leq 2n-1}(F) \to M_{\leq 2n-1}(G)$ in $\text{DM}^{\text{eff}}_{\text{gm}}$ making the above diagram commutative. Repeating this process, we can construct uniquely a morphism $g : F \to G$ in $C_n$ such that $\pi(g) = f$.

**Proposition 5.4.** Assume $(C_{\leq n})$, (Mur$_{\leq n}$), and (wVan$_{\leq n}$).

1. For any $d$, the inclusion functor $M_{\leq d}C_{\leq n} \to C_{\leq n}$ admits a left adjoint denoted by $M_{\leq d}$.
2. For any $d$, the inclusion functor $M_{\geq d}C_{\leq n} \to C_{\leq n}$ admits a right adjoint denoted by $M_{\geq d}$.
3. For any $d$, there is a natural transformation $\delta : M_{\leq d-1} \to M_{\leq d}[1]$ such that

$$M_{\geq d} \to \text{id} \to M_{\leq d-1} \xrightarrow{\delta} M_{\geq d}[1]$$

is a distinguished triangle of functors.
4. For any $d > e$, there is a natural isomorphism $M_{\leq d}M_{\geq e} \sim M_{\geq e}M_{\leq d}$ of functors.
5. For any object $F$ of $M_{\leq d}C_{\leq n}$, $\text{Hom}(F; L^n)$ is an object of $M_{\geq 2n-d}C_{\geq n}$.

**Proof.** The proofs of (1), (2), (3), and (4) are parallel to those of Theorem 2.8 Propositions 2.12, 2.13, and 2.14 respectively if we use Lemma 5.2 instead of Proposition 2.7. The proof of (5) is parallel to that of Theorem 3.2 if we use Lemma 5.2 and Theorem 3.2 instead of Proposition 2.7 and the fact that $\text{Hom}(M, 1(n))$ is a pure $n$-motive of weights $2n - d$ for any pure $n$-motive $M$ of weight $d$.

**Definition 5.5.** Assume $(C_{\leq n})$, (Mur$_{\leq n}$), and (wVan$_{\leq n}$). For each $d$, set

$$M_d := M_{\geq d}M_{\leq d}.$$
**Proposition 5.6.** Assume \((C \leq n, \text{ Mur} \leq n, \text{ and wVan}' \leq n)\). Let \(f : F \to G\) be a morphism in \(C \leq n\). Then for each \(d\), there is a unique morphism \(f_{\leq d} : M_{\leq d}(F) \to M_{\leq d}(G)\) such that the diagram

\[
\begin{array}{c}
F \xrightarrow{f} M_{\leq d}(F) \\
\downarrow f \downarrow \downarrow f_{\leq d} \\
G \xrightarrow{f_{\leq d}} M_{\leq d}(G)
\end{array}
\]

in \(\text{DM}^{\text{eff}}_{gm}\) commutes.

**Proof.** By Proposition \([5.4]\) we have the diagram

\[
\begin{array}{c}
M_{\geq d+1}(F) \xrightarrow{f} F \xrightarrow{f} M_{\leq d}(F) \xrightarrow{f} M_{\geq d+1}(F)[1] \\
\downarrow \downarrow \downarrow \downarrow \\
M_{\geq d+1}(G) \xrightarrow{f} G \xrightarrow{f} M_{\leq d}(G) \xrightarrow{f} M_{\geq d+1}(G)[1]
\end{array}
\]

in \(\text{DM}^{\text{eff}}_{gm}\) where the rows are distinguished triangles. By Lemma \([5.2]\)

\[
\text{Hom}(M_{\geq d+1}(F), M_{\leq d}(G)) = 0, \quad \text{Hom}(M_{\geq d+1}(F), M_{\leq d}(G)[−1]) = 0.
\]

Thus by Lemma \([2.1]\) there is a unique morphism \(f_{\leq d} : M_{\leq d}(F) \to M_{\leq d}(G)\) making the above diagram commutative. \(\square\)

**5.7.** Assume \((C \leq n, \text{ Mur} \leq n, \text{ and wVan}' \leq n)\). Let

\[
F \xrightarrow{f} G \xrightarrow{g} H \xrightarrow{h} F[1]
\]

be a distinguished triangle in \(\text{DM}^{\text{eff}}_{gm}\). Assume that \(F\) and \(G\) be objects of \(C \leq n\). Then by Proposition \([5.3]\) \(f\) can be considered as a morphism in \(C \leq n\). Thus we have the commutative diagram

\[
\begin{array}{c}
F = M_{\leq 2n}(F) \xrightarrow{a_{2n}} M_{\leq 2n-1}(F) \xrightarrow{a_{2n-1}} \cdots \xrightarrow{a_1} M_{\leq 0}(F) \\
\downarrow f_{\leq 2n} \downarrow \downarrow f_{\leq 2n-1} \downarrow \downarrow f_{\leq 0} \\
G = M_{\leq 2n}(G) \xrightarrow{b_{2n}} M_{\leq 2n-1}(G) \xrightarrow{b_{2n-1}} \cdots \xrightarrow{b_1} M_{\leq 0}(G)
\end{array}
\]

in \(\text{DM}^{\text{eff}}_{gm}\).

Choose distinguished triangles

\[
M_d(F) \xrightarrow{p_d} M_{\leq d}(F) \xrightarrow{a_d} M_{\leq d-1}(F) \xrightarrow{p_d} M_d(F)[1],
\]

\[
M_d(G) \xrightarrow{q_d} M_{\leq d}(G) \xrightarrow{b_d} M_{\leq d-1}(G) \xrightarrow{q_d} M_d(G)[1]
\]

in \(\text{DM}^{\text{eff}}_{gm}\). By Propositions \([2.10, 5.2]\) and \([2.1]\) there is a unique morphism \(f_d : M_d(F) \to M_d(G)\) in \(\text{DM}^{\text{eff}}_{gm}\) such that the diagram

\[
\begin{array}{c}
M_d(F) \xrightarrow{p_d} M_{\leq d}(F) \xrightarrow{a_d} M_{\leq d-1}(F) \xrightarrow{p_d} M_d(F)[1] \\
\downarrow f_d \downarrow \downarrow f_{\leq d} \downarrow \downarrow f_{d}[1] \\
M_d(G) \xrightarrow{q_d} M_{\leq d}(G) \xrightarrow{b_d} M_{\leq d-1}(G) \xrightarrow{q_d} M_d(G)[1]
\end{array}
\]
in DM\textsuperscript{eff}\textsubscript{gm} commutes. Assume that \( f_d[-d] \) has a kernel, cokernel, and image in MM\( \leq \alpha \) for any \( d \). By the octahedral axiom, for some objects \( U_d \) and \( V_d \) of DM\textsuperscript{eff}\textsubscript{gm}, there are commutative diagrams

\[
\begin{array}{ccccccccc}
\text{im } f_d[-1] & \rightarrow & 0 & \rightarrow & \text{im } f_d & \rightarrow & \text{id} & \rightarrow & \text{im } f_d \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\ker f_d & \rightarrow & M_{\leq d}(F) & \rightarrow & U_d & \rightarrow & \ker f_d[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
M_d(F) & \rightarrow & M_{\leq d}(F) & \rightarrow & M_{\leq d-1}(F) & \rightarrow & M_d(F)[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\text{im } f_d & \rightarrow & 0 & \rightarrow & \text{im } f_d[1] & \rightarrow & \text{id} & \rightarrow & \text{im } f_d[1] \\
\end{array}
\]

\[
\begin{array}{ccccccccc}
\text{cok } f_d[-1] & \rightarrow & 0 & \rightarrow & \text{cok } f_d & \rightarrow & \text{id} & \rightarrow & \text{cok } f_d \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\text{im } f_d & \rightarrow & M_{\leq d}(G) & \rightarrow & V_d & \rightarrow & \text{im } f_d[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
M_d(G) & \rightarrow & M_{\leq d}(G) & \rightarrow & M_{\leq d-1}(G) & \rightarrow & M_d(G)[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\text{cok } f_d & \rightarrow & 0 & \rightarrow & \text{cok } f_d[1] & \rightarrow & \text{id} & \rightarrow & \text{cok } f_d[1] \\
\end{array}
\]

in DM\textsuperscript{eff}\textsubscript{gm} where each row and column is a distinguished triangle. Here, we set

\[
\ker f_d := (\ker(f_d[-d]))[d], \quad \text{im } f_d := (\text{im}(f_d[-d]))[d],
\]

\[
\text{cok } f_d := (\text{cok}(f_d[-d]))[d]
\]

for brevity.

Consider the diagram

\[
\begin{array}{ccc}
\ker f_d & \rightarrow & M_{\leq d}(F) \\
\uparrow & & \uparrow \quad f_{\leq d} \\
M_{\leq d}(G) & \rightarrow & \ker f_d[1] \\
\end{array}
\]

in DM\textsuperscript{eff}\textsubscript{gm}. Since \( f_{\leq d} u_d = 0 \), there is a morphism \( \eta : U_d \rightarrow M_{\leq d}(G) \) in DM\textsuperscript{eff}\textsubscript{gm} making the above diagram commutative. Then by the octahedral axiom, for some
object $M_{\leq d}(H)$ of $\text{DM}_{gm}^{\text{eff}}$, there is a commutative diagram

$$
\begin{array}{ccc}
\text{im} f_d & \xrightarrow{id} & \text{im} f_d \\
\alpha_d & \downarrow & \downarrow \\
U_d & \xrightarrow{\eta} & M_{\leq d}(G) \\
\alpha'_d & \downarrow & \downarrow \\
M_{\leq d-1}(F) & \xrightarrow{\varphi} & V_d \\
\alpha''_d & \downarrow & \downarrow \\
\text{im} f_d[1] & \xrightarrow{id} & \text{im} f_d[1]
\end{array}
\xrightarrow{\text{id}}
\begin{array}{ccc}
0 & \xrightarrow{id} & 0 \\
\downarrow & \downarrow & \downarrow \\
M_{\leq d-1}(F) & \xrightarrow{\varphi'} & M_{\leq d}(H) \\
\downarrow & \downarrow & \downarrow \\
M_{\leq d-1}(F)[1] & \xrightarrow{id} & M_{\leq d-1}(F)[1]
\end{array}
\xrightarrow{\text{id}}
\begin{array}{ccc}
\text{cok} f_d & \xrightarrow{id} & \text{cok} f_d \\
\downarrow & \downarrow & \downarrow \\
M_{\leq d-1}(F) & \xrightarrow{\varphi''} & M_{\leq d-1}(F)[1] \\
\downarrow & \downarrow & \downarrow \\
0 & \xrightarrow{id} & 0
\end{array}
$$

in $\text{DM}_{gm}^{\text{eff}}$ such that each row and column is a distinguished triangle. Now choose a distinguished triangle

$$
M_{\leq d-1}(F) \xrightarrow{f_{\leq d-1}} M_{\leq d-1}(G) \xrightarrow{\theta_d} W_d \xrightarrow{\theta'_d} M_{\leq d-1}(F)[1]
$$

in $\text{DM}_{gm}^{\text{eff}}$. By the octahedral axiom, there are commutative diagrams

$$
\begin{array}{ccc}
\ker f_d & \xrightarrow{id} & \ker f_d \\
\downarrow & \downarrow & \downarrow \\
M_{\leq d}(F) & \xrightarrow{f_d} & M_{\leq d}(G) \\
\downarrow & \downarrow & \downarrow \\
\ker f_d[1] & \xrightarrow{id} & \ker f_d[1]
\end{array}
\xrightarrow{\text{id}}
\begin{array}{ccc}
0 & \xrightarrow{id} & 0 \\
\downarrow & \downarrow & \downarrow \\
M_{\leq d-1}(F) & \xrightarrow{\theta'_d} & W_d \\
\downarrow & \downarrow & \downarrow \\
0 & \xrightarrow{id} & 0
\end{array}
$$

in $\text{DM}_{gm}^{\text{eff}}$ such that each row and column is a distinguished triangle.
Now by the octahedral axiom, we can choose an object $M_d(H)$ and a commutative diagram

\[
\begin{array}{ccccccc}
\ker f_{d-1} & \rightarrow & 0 & \rightarrow & \ker f_{d-1}[1] & \rightarrow & \ker f_{d-1}[1] \\
-\mu_d'' & \downarrow & & & \gamma_d & \downarrow & \mu_d''[1] \\
\cok f_d & \rightarrow & M_{\leq d}(H) & \rightarrow & W_d & \rightarrow & \cok f_d[1] \\
\mu_d & \downarrow & \id & & \gamma_d' & \downarrow & \mu_d[1] \\
M_d(H) & \rightarrow & M_{\leq d}(H) & \rightarrow & M_{\leq d-1}(H) & \rightarrow & M_d(H)[1] \\
\mu_d' & \downarrow & \gamma_d'' & \downarrow & \mu_d'[1] \\
\ker f_{d-1}[1] & \rightarrow & 0 & \rightarrow & \ker f_{d-1}[2] & \rightarrow & \ker f_{d-1}[2] 
\end{array}
\]

in $\text{DM}^{\text{eff}}_{gm}$, where each row and column is a distinguished triangle. Then we get the distinguished triangle

\[(5.7.1) \ \cok f_d[-d] \xrightarrow{\mu_d[-d]} M_d(H)[-d] \xrightarrow{\nu_d[-d]} \ker f_{d-1}[-d+1] \xrightarrow{\mu_d''[-d]} \cok f_{d-1}[-d+1]
\]
in $\text{DM}^{\text{eff}}_{gm}$.

**Proposition 5.8.** Assume $(C_{\leq n})$, $(\text{Mur}_{\leq n})$, and $(\text{wVan}_{\leq n})$. Let $F \xrightarrow{f} G \xrightarrow{g} H \rightarrow F[1]$ be a distinguished triangle in $\text{DM}^{\text{eff}}_{gm}$ such that $F$ and $G$ are objects of $C_{\leq n}$. Assume that

(i) for each $d$, the morphism $f_d[-d]$ in 5.7 has a kernel, cokernel, and image in $W^d\text{MM}_{\leq n}$,

(ii) for each $d$, the cokernel of $f_d[-d]$ is a pure $n$-motive of weight $d$.

Then $H$ is an object of $C_{\leq n}$.

**Proof.** In 5.7 we have constructed morphisms

\[H = M_{\leq 2n}(H) \rightarrow M_{\leq 2n-1}(H) \rightarrow \cdots \rightarrow M_{\leq -1}(H) = 0\]
in $\text{DM}^{\text{eff}}_{gm}$ such that for each $d$, $M_d(H)[-d]$ admits the distinguished triangle $(5.7.1)$. By assumption, $\cok f_d[-d]$ is a pure $n$-motive of weight $d$. Then $M_d(H)[-d]$ is an $n$-motive of weights $\leq d$ since $\ker f_{d-1}[-d+1]$ is an $n$-motive of weights $\leq d - 1$. Thus $H$ is an object of $C_{\leq n}$.

**Definition 5.9.** Let $\mathcal{T}$ be a triangulated category. We say that a finite sequence

\[0 \rightarrow F_1 \xrightarrow{u_1} \cdots \xrightarrow{u_{r-1}} F_r \rightarrow 0\]

in $\mathcal{T}$ with $r \geq 3$ is **exact** if there are distinguished triangles

\[F_1 \xrightarrow{v_2} F_2 \xrightarrow{w_2} C_2 \rightarrow F_1[1],\]
\[C_2 \xrightarrow{v_3} F_2 \xrightarrow{w_3} C_3 \rightarrow C_2[1],\]
\[\cdots,\]
\[C_{r-2} \xrightarrow{v_r} F_{r-2} \xrightarrow{w_r} C_{r-1} \rightarrow C_{r-2}[1],\]
\[C_{r-1} \xrightarrow{v_r} F_{r-1} \xrightarrow{w_r} C_r \rightarrow C_{r-1}[1]\]
in $\mathcal{T}$ such that $u_i = v_i w_i$ for $i = 2, \ldots, r - 2$. 
Proposition 5.10. Under the notations and hypotheses of \([\text{3}, \text{Theorems 16.1.3, 16.1.4}]\) the finite sequence
\[
0 \longrightarrow M_{2n+1}(H)[-2n-1] \xrightarrow{h_{2n+1}[-2n-1]} M_{2n}(F)[-2n] \xrightarrow{f_{2n}[-2n]} M_{2n}(G)[-2n] \xrightarrow{g_{2n}[-2n]} M_{2n}(H)[-2n] \\
\ldots \xrightarrow{h_0} M_0(F) \xrightarrow{f_0} M_0(G) \longrightarrow M_0(H) \longrightarrow 0
\]
is exact in the sense of Definition \([\text{5.9}]\).

Proof. This follows from the distinguished triangles
\[
\ker f_d[-d] \to M_d(F)[-d] \to \im f_d[-d] \to \ker f_d[-d+1], \\
\im f_d[-d] \to M_d(G)[-d] \to \cok f_d[-d] \to \im f_d[-d], \\
\cok f_d[-d] \mu_d[-d] M_d(H)[-d] \xrightarrow{\mu_d'[-d]} \ker f_{d-1}[-d+1] \mu_d''[-d+1] \cok f_d[-d+1]
\]
in \(\text{DM}_{\text{eff}}^\text{gm}\). \(\square\)

Proposition 5.11. Assume \((\text{CK}_{\leq n})\). Then for any \(X \in \text{SmProj}_{\leq n}\), \(M(X)\) is an object of \(\mathcal{C}_{\leq n}\).

Proof. We just need to set \(M_{\leq d}(X) := M_0(X) \oplus \cdots \oplus M_d(X)\), which is possible because of \((\text{CK}_{\leq n})\). \(\square\)

Definition 5.12. Let \(Z\) be a scheme of finite type over \(k\), and let \(\mathcal{Z} = \{Z_1, \ldots, Z_r\}\) be a closed cover of \(Z\). Let \(\text{C}(\text{PSh}^{tr})\) denote the category of complexes of presheaves with transfers on the category of schemes smooth over \(k\). Recall the object \(M(X) := C^{\ast}Z_{tr}(X)\) of \(\text{C}(\text{PSh}^{tr})\) in \([13]\). Consider the Čech double complex
\[
(5.12.1) \quad \bigoplus_{|I|=r} M(Z_I) \to \cdots \to \bigoplus_{|I|=1} M(Z_I)
\]
in \(\text{C}(\text{PSh}^{tr})\) where \(Z_I := Z_{i_1} \times_Z \cdots \times_Z Z_{i_r}\) if \(I = \{i_1, \ldots, i_r\}\). We denote by \(M(\mathcal{Z})\) its total complex.

Lemma 5.13. Under the above notations and hypotheses, the induced morphism
\[
M(\mathcal{Z}) \to M(\mathcal{Z})
\]
in \(\text{DM}_{\text{eff}}^\text{gm}\) is an isomorphism.

Proof. Let us use an induction on \(r\). If \(r = 1\), then this is obvious. If \(r = 2\), then this follows from \([19\text{ Theorems 16.1.3, 16.1.4}]\). For \(r > 2\), consider the closed covers
\[
\mathcal{W} := \{Z_1, \ldots, Z_{r-1}\}, \\
\mathcal{W}' := \{Z_1, \ldots, Z_{r-1}, Z_r\}
\]
of \(W := Z_1 \cup \cdots \cup Z_{r-1}\) and \(W' := Z_1, r \cup \cdots \cup Z_{r-1}, r\) respectively. Here, set \(Z_{i,r} := Z_i \times Z_{Z_r}\). Then \(M(\mathcal{W})\) and \(M(\mathcal{W}')\) are the total complexes of the double complexes
\[
\bigoplus_{|I|=r, r \notin I} M(Z_I) \to \cdots \to \bigoplus_{|I|=1, m \notin I} M(Z_I) \\
\bigoplus_{|I|=r, r \in I} M(Z_I) \to \cdots \to \bigoplus_{|I|=2, m \in I} M(Z_I)
\]
in \(\text{C}(\text{PSh}^{tr})\) respectively. Combining with \((5.12.1)\), we have the induced distinguished triangle
\[
M(\mathcal{W}') \to M(\mathcal{W}) \oplus M(Z_1) \to M(\mathcal{Z}) \to M(\mathcal{W})[1]
\]
in $\text{DM}^{\text{eff}}_{gm}$. Now consider the induced commutative diagram

$$
\begin{array}{cccccc}
M(\mathcal{W}) & \longrightarrow & M(\mathcal{W}) \oplus M(Z_1) & \longrightarrow & M(Z) & \longrightarrow & M(\mathcal{W})[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
M(W) & \longrightarrow & M(W) \oplus M(Z_1) & \longrightarrow & M(Z) & \longrightarrow & M(W)[1]
\end{array}
$$

in $\text{DM}^{\text{eff}}_{gm}$. The claim holds for $r = 2$, so the second row is a distinguished triangle. The first, second, and fourth vertical arrows are isomorphisms by induction on $r$. Thus the third vertical arrow is an isomorphism. □

**Lemma 5.14.** Assume $(\text{Res}_{\leq n})$. Let $U$ be an integral scheme smooth and separated over $k$. Then

$$\text{Hom}(M^c(U), L^n) \cong M(U).$$

**Proof.** Let $\mathcal{T}$ be the full subcategory of $\text{DM}^{\text{eff}}_{gm}$ generated by objects of the form $M(X)[n]$ where $n \in \mathbb{Z}$ and $X$ is a scheme smooth and projective over $k$ whose dimension is $\leq n$. By [18, Theorem 4.3.2], the functor

$$\mathcal{T} \to \mathcal{T}$$

given by $F \mapsto \text{Hom}(\text{Hom}(F, L^n), L^n)$ is an equivalence. In the proof of [18, Theorem 4.3.7], we have that

$$\text{Hom}(M(U), L^n) \cong M^c(U).$$

Thus the conclusion follows from these. □

**5.15.** Assume $(C_{\leq n})$, $(\text{Mur}_{\leq n})$, $(\text{wVan}'_{\leq n-1})$, $(\text{Semi}_{\leq n-1})$, and $(\text{Res}_{\leq n})$, and let $U$ be an integral scheme smooth and separated over $k$ whose dimension is $\leq n$. By $(\text{Res}_{\leq n})$, we can choose a closed immersion $i : Z \to X$ and a projective smooth morphism $X \to k$ of schemes such that

(i) the complement of $i$ is $U$,

(ii) $Z = Z_1 \cup \cdots \cup Z_r$ is a divisor with strict normal crossings.

For $1 \leq i \leq r$, let $K_i$ be the total complex of the double complex

$$\bigoplus_{|I|=i} M(Z_I) \to \cdots \to \bigoplus_{|I|=1} M(Z_I)$$

in $\text{C}(\text{PSh}^{tr})$ where $Z_I := Z_{i_1} \times z \cdots \times z Z_{i_r}$ if $I = \{i_1, \ldots, i_r\}$. Then we have the distinguished triangle

$$K_{i+1} \to \bigoplus_{|I|=i+1} M(Z_I) \to K_i \to K_{i+1}[1]$$

for each $1 \leq i \leq r-1$. By Proposition 5.11, $\bigoplus_{|I|=i} M(Z_I)$ is an object of $C_{\leq n-2}$ if $i > 1$ and of $C_{\leq n-1}$ if $i = 1$. In particular, $K_r$ is an object of $C_{\leq n-2}$ if $r > 1$ and of $C_{\leq n-1}$ if $r = 1$. By Theorem 4.3 and Proposition 4.8, we can apply Proposition 5.8 to the above distinguished triangle. Thus each $K_i$ is an object of $C_{\leq n-2}$ if $i > 0$ and of $C_{\leq n-1}$ if $i = 0$. In particular, $M(Z) = K_0$ is an object of $C_{\leq n-1}$. In Proposition 5.10 we have the sequence

$$M_{2n-2}(K_1) \to \bigoplus_{|I|=1} M_{2n-2}(Z_I) \to M_{2n-2}(K_0) \to M_{2n-3}(K_1).$$

Since $K_1$ is an object of $C_{\leq n-2}$, $M_{2n-2}(K_1) = M_{2n-3}(K_1) = 0$. By Proposition 5.10

$$\bigoplus_{|I|=1} M_{2n-2}(Z_I) \cong M_{2n-2}(K_0).$$

Thus $M_{2n-2}(Z)$ is a pure $(n-1)$-motive of weight $2n-2$.

Let us consider the following conjecture, which is a weaker version of $(\text{Semi}_{\leq n})$. 
(Semi\(_{\leq n}\)) Assume (Semi\(_{\leq n-1}\)). For any \(d\) and morphism \(f: M \to N\) in \(\mathcal{M}_{\leq n}\) such that

(i) \((d < 2n - 2\) and \(M\) is an \((n-1)\)-motive of weights \(\leq d\)) or \((d = 2n - 2\) and \(M\) is a pure \((n-1)\)-motive of weight \(d\)),

(ii) \(N\) is a pure \(n\)-motive of weight \(d\),

the kernel, cokernel, and image of \(f\) exist, and the cokernel of \(f\) is a pure \(n\)-motive of weight \(d\).

Assume (Semi\(_{\leq n}\)). Then we can apply Proposition 5.8 to the distinguished triangle

\[
M(Z) \to M(X) \to M^c(U) \to M(Z)[1]
\]

in \(\text{DM}^{eff}_g\) (14 Theorem 16.15], so \(M^c(U)\) is an object of \(\mathcal{C}_{\leq n}\). Since \(\text{Hom}(M^c(U), L^n) \cong M(U)\) by Lemma 5.14, \(M(U)\) is an object of \(\mathcal{C}_{\leq n}\) by Proposition 5.4. Thus we get the following result.

**Theorem 5.16.** Assume (CK\(_{\leq n}\)), (Mur\(_{\leq n}\)), (wVan\(_{\leq n}\)), (Res\(_{\leq n}\)), and (Semi\(_{\leq n}\)). Then for any integral scheme \(U\) smooth over \(k\) whose dimension is \(\leq n\), \(M(U)\) and \(M^c(U)\) are objects of \(\mathcal{C}_{\leq n}\).

### 6. Realizations

**6.1.** Let

\[
R_\ell: \text{DM}^{eff}_g \to D_c^b(k_{\text{ét}}, \mathbb{Q}_\ell)
\]

denote the \(\ell\)-adic realization functor ([14 Remark 7.2.25]), which commutes with the Grothendieck six operations ([4 Definition A.1.10]). Here, \(D_c^b(k_{\text{ét}}, \mathbb{Q}_\ell)\) denotes the bounded derived category of constructible étale \(\mathbb{Q}_\ell\)-sheaves.

**Lemma 6.2.** Let \(X\) be a purely \(n\)-dimensional scheme smooth over \(k\). Then there are canonical isomorphisms

\[
R_\ell(M(X)) \cong R\Gamma_{\text{ét},c}(X_{\overline{k}}, \mathbb{Q}_\ell)(n)[2n], \quad R_\ell(M^c(X)) \cong R\Gamma_{\text{ét}}(X_{\overline{k}}, \mathbb{Q}_\ell)(n)[2n].
\]

**Proof.** Let \(p: X \to \text{Spec} \ k\) denote the structure morphism. By the Poincaré duality ([4 Definition A.1.10(3)]), there is a canonical isomorphism

\[
M(X) \cong pp^*1(n)[2n].
\]

Thus

\[
R_\ell(M(X)) \cong pp^*\mathbb{Q}_\ell(n)[2n] \cong R\Gamma_{\text{ét},c}(X_{\overline{k}}, \mathbb{Q}_\ell)(n)[2n].
\]

By [14 Theorem 16.24] (see [19] to remove the assumption of resolution of singularities), for any scheme \(T\) smooth over \(k\) and \(i \in \mathbb{Z}\) there is a canonical isomorphism

\[
\text{Hom}(M(T)[i], M^c(X)) \cong \text{Hom}(M(T \times X)[i], 1(n)[2n]).
\]

On the other hand, by the Grothendieck six operations formalism there is a canonical isomorphism

\[
\text{Hom}(M(T)[i], pp^*M(X)) \cong \text{Hom}(M(T \times X), 1).
\]

Thus there is a canonical isomorphism

\[
M^c(X) \cong pp^*1(n)[2n].
\]

Then

\[
R_\ell(M^c(X)) \cong pp^*\mathbb{Q}_\ell(n)[2n] \cong R\Gamma_{\text{ét}}(X_{\overline{k}}, \mathbb{Q}_\ell)(n)[2n].
\]

\[\Box\]
6.3. Let $X$ be a purely $n$-dimensional scheme smooth and projective over $k$. By [3 0.3], there is an isomorphism

$$R\Gamma_{\text{et}}(X_{\overline{k}}, \mathbb{Q}_\ell) \cong \bigoplus_{d=0}^{2n} H^{d}_{\text{et}}(X_{\overline{k}}, \mathbb{Q}_\ell)[-d].$$

Thus with the étale homology notation

$$H^{d}_{\text{et}}(X_{\overline{k}}, \mathbb{Q}_\ell) \cong H^{2n-d}_{\text{et}}(X_{\overline{k}}, \mathbb{Q}) \cong H^{2n-d}_{\text{et}}(X_{\overline{k}}, \mathbb{Q})(n)$$

we have an isomorphism

$$(6.3.1) \quad R_\ell(M(X)) \cong \bigoplus_{d=0}^{2n} H^{d}_{\text{et}}(X_{\overline{k}}, \mathbb{Q}_\ell)[d].$$

6.4. Assume $(C \subseteq n)$. For $X \in \text{SmProj}_{\leq n}$, the projector $p_d : M(X) \to M(X)$ in $\mathbb{L}_{\mathbb{Z}}$ corresponds to the projector

$$\bigoplus_{d=0}^{2n} H^{d}_{\text{et}}(X_{\overline{k}}, \mathbb{Q}_\ell)[d] \to H^{d}_{\text{et}}(X_{\overline{k}}, \mathbb{Q}_\ell)[d] \to \bigoplus_{d=0}^{2n} H^{d}_{\text{et}}(X_{\overline{k}}, \mathbb{Q}_\ell)[d].$$

Thus we have an isomorphism

$$R_\ell(M_d(X)[−d]) \cong H^{d}_{\text{et}}(X_{\overline{k}}, \mathbb{Q}_\ell).$$

Consider the usual $t$-structure on the derived category $D^b_c(k_{\text{et}}, \mathbb{Q}_\ell)$ whose heart is $\text{Rep}_{\text{Gal}(\overline{k}/k)}(\mathbb{Q}_\ell)$. Then $R_\ell(M_d(X)[−d])$ is in $\text{Rep}_{\text{Gal}(\overline{k}/k)}(\mathbb{Q}_\ell)$.

For each $d$, let $\tau_{\leq d}$ and $\tau_{\geq d}$ denote the homological truncation functors of $D^b_c(k_{\text{et}}, \mathbb{Q}_\ell)$. By the following proposition, we obtain a functor

$$R_\ell : \text{MM}_{\leq n} \to \text{Rep}_{\text{Gal}(\overline{k}/k)}(\mathbb{Q}_\ell).$$

**Proposition 6.5.** Assume $(C \subseteq n)$. For any $n$-motive $M$, $R_\ell(M)$ is in $\text{Rep}_{\text{Gal}(\overline{k}/k)}(\mathbb{Q}_\ell)$.

**Proof.** We may assume that $M$ is an $n$-motive of weights $\leq d$. If $d = −1$, we are done since $M = 0$. Hence assume that $d \geq 0$. Consider the distinguished triangle

$$G_{\text{et}}M \to M \to W^{d-1}M \to G_{\text{et}}M[1]$$

in $\text{DM}^{\text{eff}}_{gm}$. By induction, $R_\ell(W^{d-1}M)$ is in the heart of $D^b_c(k_{\text{et}}, \mathbb{Q}_\ell)$, and by 6.3 $R_\ell(G_{\text{et}}M)$ is in the heart. Thus $R_\ell(M)$ is also in the heart. \qed

**Proposition 6.6.** (1) For any object $F$ of $M_{\leq d}C_{\leq n}$, $R_\ell(F)$ is in $\tau_{\geq 0}\tau_{\leq d}D^b_c(k_{\text{et}}, \mathbb{Q}_\ell)$.

(2) For any object $F$ of $M_{\geq d}C_{\leq n}$, $R_\ell(F)$ is in $\tau_{\leq 2n}\tau_{\geq d}D^b_c(k_{\text{et}}, \mathbb{Q}_\ell)$.

**Proof.** (1) Let us use an induction on $d$. If $d = 0$, then we are done by Proposition 6.5. Thus assume that $d > 0$. Consider the distinguished triangle

$$R_\ell(M_d(F)) \to R_\ell(F) \to R_\ell(M_{\leq d-1}(F)) \to R_\ell(M_d(F)[1])$$

in $D^b_c(k_{\text{et}}, \mathbb{Q}_\ell)$. By Proposition 6.5, $R_\ell(M_d(F)[−d])$ is in $\tau_{\geq 0}\tau_{\leq d}D^b_c(k_{\text{et}}, \mathbb{Q}_\ell)$. Thus $R_\ell(M_d(F))$ is in $\tau_{\geq d}\tau_{\leq d}D^b_c(k_{\text{et}}, \mathbb{Q}_\ell)$. Since $R_\ell(M_{\leq d-1}(F))$ is in $\tau_{\leq 0}\tau_{\leq d-1}D^b_c(k_{\text{et}}, \mathbb{Q}_\ell)$ by induction on $d$, $R_\ell(F)$ is in $\tau_{\leq 0}\tau_{\leq d-1}D^b_c(k_{\text{et}}, \mathbb{Q}_\ell)$.

(2) Let us use an induction on $d$. If $d = 2n$, then we are done by Proposition 6.5. Thus assume that $d < 2n$. Consider the distinguished triangle

$$R_\ell(M_{d+1}(F)) \to R_\ell(F) \to R_\ell(M_d(F)) \to R_\ell(M_{\geq d+1}(F)[1])$$
in $D^b_e(k_{\text{et}},Q_\ell)$. As above, $R_\ell(M_d(F)[-d])$ is in $\tau_{\geq d}\tau_{\leq d}D^b_e(k_{\text{et}},Q_\ell)$. Since $R_\ell(M_{d+1}(F))$ is in $\tau_{d+1}\leq 2nD^b_e(k_{\text{et}},Q_\ell)$ by induction on $d$, $R_\ell(F)$ is in $\tau_{\geq d}\tau_{\leq 2n}D^b_e(k_{\text{et}},Q_\ell)$. □

**Proposition 6.7.** Assume $(C_{\leq n})$, $(\text{Mur}_{\leq n})$, and $(\text{wVan}_{\leq n-1})$, Let $F$ be an object of $C_{\leq n}$. Then there is a unique isomorphism

$$R_\ell(M_{\leq d}(F)) \to \tau_{\leq d}R_\ell(F)$$

making the diagram

$$R_\ell(M_{d+1}(F)) \to R_\ell(F) \to R_\ell(M_{\leq d}(F)) \to R_\ell(M_{d+1}(F))$$

$$\tau_{d+1}R_\ell(F) \to R_\ell(F) \to \tau_{\leq d}R_\ell(F) \to \tau_{d+1}R_\ell(F)$$

in $DM^{gm}_{\text{eff}}$ commutative.

**Proof.** By Proposition 6.6, $R_\ell(M_{d+1}(F))$ is in $\tau_{\geq d+1}D^b_e(k_{\text{et}},Q_\ell)$. Thus the conclusion follows from Propositions 2.1 and 2.2. □

**Proposition 6.8.** Assume $(C_{\leq n})$, $(\text{Mur}_{\leq n})$, and $(\text{wVan}_{\leq n-1})$, Let $F$ be an object of $C_{\leq n}$. Then there is a unique isomorphism

$$R_\ell(M_d(F)) \to \tau_{\geq d}\tau_{\leq d}R_\ell(F)$$

making the diagram

$$R_\ell(M_d(F)) \to R_\ell(M_{\leq d}(F)) \to R_\ell(M_{d-1}(F)) \to R_\ell(M_d(F))[1]$$

$$\tau_{\geq d}\tau_{\leq d}R_\ell(F) \to \tau_{\leq d}R_\ell(F) \to \tau_{d-1}R_\ell(F) \to \tau_{\geq d}\tau_{\leq d}R_\ell(F)$$

in $DM^{gm}_{\text{eff}}$ commutative. Here, the vertical arrow is defined in Proposition 6.7.

**Proof.** By Proposition 6.6, $R_\ell(M_d(F))$ is in $\tau_{\geq d+1}D^b_e(k_{\text{et}},Q_\ell)$. Thus the conclusion follows from Propositions 2.1 and 2.2. □

**Theorem 6.9.** Assume $(\text{CK}_{\leq n})$, $(\text{Mur}_{\leq n})$, $(\text{wVan}_{\leq n})$, $(\text{Res}_{\leq n})$, and $(\text{Semi}^\prime_{\leq n})$. Then for any integral scheme $U$ smooth over $k$ whose dimension is $\leq n$,

$$R_\ell(M_d(U)[-d]) \cong H^d_{\text{et}}(U_{\overline{k}},Q_\ell), \quad R_\ell(M^e_d(U)[-d]) \cong H^{d,e}_{\text{et}}(U_{\overline{k}},Q_\ell).$$

**Proof.** We may assume that $U$ has pure $n$-dimension. Here, recall that we use the étale homology notation

$$H^d_{\text{et}}(U_{\overline{k}},Q_\ell) := H^{2n-d}_{\text{et}}(U_{\overline{k}},Q_\ell)(n), \quad H^{d,e}_{\text{et}}(U_{\overline{k}},Q_\ell) := H^{2n-d}_{\text{et}}(U_{\overline{k}},Q_\ell)(n).$$

By Propositions 5.10 and 6.8, it suffices to show that

$$\tau_{\geq d}\tau_{\leq d}R_\ell(M(U)) \cong H^d_{\text{et}}(U_{\overline{k}},Q_\ell), \quad \tau_{\geq d}\tau_{\leq d}R_\ell(M^e(U)) \cong H^{d,e}_{\text{et}}(U_{\overline{k}},Q_\ell).$$

These follow from Lemma 6.2. □
7. 2-Motives

Proposition 7.1. The condition (Semi_{≤1}) holds.

Proof. Let $X$ be a connected curve smooth and projective over $k$ with $k' := \Gamma(X, \mathcal{O}_X)$. Then $M(X) \cong M_0(X) \oplus M_1(X) \oplus M_2(X)$ with

$$M_0(X) \cong M(k), \quad M_1(X) \cong \text{Pic}^0(X), \quad M_2(X) \cong M(k') \otimes \mathbb{L}. $$

The rational equivalence and numerical equivalence are equal for codimension 0. Thus by [8], Hom($M_0(X), M_0(X)$) is a semisimple ring. By [19],

$$\text{Hom}(M_2(X), M_2(X)) \cong \text{Hom}(M_0(X), M_0(X)).$$

Thus Hom($M_2(X), M_2(X)$) is a semisimple ring. The category of abelian varieties up to isogeny is a semisimple category, and Hom($M_1(X), M_1(X)$) is isomorphic to the group of homomorphisms Pic$^0(X) \to \text{Pic}^0(X)$ of abelian varieties up to isogeny by [17, Proposition 4.5]. These mean that Hom($M_1(X), M_1(X)$) is a semisimple ring.

□

Proposition 7.2. The condition (Van_{≤1}) holds.

Proof. Since (Mur_{≤2}) holds by [12, Theorem 7.3.10], it remains to show that for any $X, Y \in \text{SmProj}_{≤1}$, $p < 0$, and $2 \geq d, e \geq 0$ with $d > e + p - 1$,

$$\text{Hom}(M_d(X), M_e(Y)[p]) = 0.$$ 

We may assume that $X$ and $Y$ are connected. If $e = 0$, then it suffices to show that

$$\text{Hom}(M(X), 1[p]) = 0.$$

This holds by [14, Vanishing Theorem 19.3]. If $e = 1$, then it suffices to show that $\text{Hom}(M(X), M(Y)[p]) = 0$. Hence it suffices to show that the induced homomorphism

$$\text{Hom}(M(X), M(Y)[-1]) \to \text{Hom}(M(X), M_0(Y)[-1])$$

of abelian groups is an isomorphism. By [14, Theorems 16.24, 19.1], it suffices to show that the induced homomorphism

$$H^1(X \times Y, \mathbb{Z}(1)) \to H^1(X \times \text{Spec} k'', \mathbb{Z}(1))$$

of motivic cohomology groups is an isomorphism where $k'' := \Gamma(Y, \mathcal{O}_Y)$. Since $X$ and $Y$ are projective over $k$,

$$H^1(X \times Y, \mathbb{Z}(1)) \cong \Gamma(X \times Y, \mathcal{O}_{X \times Y}^*) \cong k^* \times k^{''*},$$

$$H^1(X \times \text{Spec} k'', \mathbb{Z}(1)) \cong \Gamma(X \times \text{Spec} k'', \mathcal{O}^*_{X \times \text{Spec} k''}) \cong k^* \times k^{''*},$$

by [14, Corollary 4.2] where $k' := \Gamma(X, \mathcal{O}_X)$. This proves the claim when $e = 1$.

If $e = 2$ and $p \leq -2$, then we are done by [14, Vanishing Theorem 19.3]. If $e = 2$ and $p = -1$, then it suffices to show that the induced homomorphism

$$\text{Hom}(M(X), 1(1)[1]) \to \text{Hom}(M_0(X), 1(1)[1])$$

is an isomorphism. By [14, Theorem 16.24], it suffices to show that the induced homomorphism

$$H^1(X, \mathbb{Z}(1)) \to H^1(k', \mathbb{Z}(1))$$

of motivic cohomology groups is an isomorphism where $k' := \Gamma(X, \mathcal{O}_X)$. Since $X$ is projective over $k$,

$$H^1(X, \mathbb{Z}(1)) \cong k'^*, \quad H^1(k', \mathbb{Z}(1)) \cong k'^*$$
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by [14] Corollary 4.2. This proves the claim when $e = 2$ and $p = -1$. □

**Proposition 7.3.** The condition $(\text{Semi'}_{\leq 2})$ holds.

**Proof.** Note that $(\text{Semi}_{\leq 1})$ holds by Proposition 7.1. Let $f : M \to N$ be a morphism in $\text{MM}_{\leq 2}$ such that

(i) $(d < 2$ and $M$ is an 1-motive of weights $\leq d)$ or $(d = 2$ and $M$ is a pure 1-motive of weight $d)$,

(ii) $N$ is a pure 2-motive of weight $d$.

If $d = 0$ or $d = 1$, then $N$ is a pure $d$-motive of weight $d$ by [15]. Since $(\text{Mur}_{\leq 2})$ (resp. $(\text{Semi}_{\leq 1})$, resp. $(\text{Van}_{\leq 1})$) holds by [12, Theorem 7.3.10] (resp. Proposition 7.1, resp. Proposition 7.2), the kernel, cokernel, and image of $f$ exist by Theorem 4.3. The cokernel of $f$ is a pure $d$-motive of weight $d$ by Proposition 4.5. Thus the remaining case is when $d = 2$.

Hence assume that $d = 2$. Then since $M$ is a pure 1-motive of weight 2, $M$ is a finite direct sum of $1(1)$. By [12, Lemma 7.4.1], there is a decomposition $N \cong N' \oplus N''$ in $\text{DM}_{gm}^{\text{eff}}$ for some pure 2-motives $N'$ and $N''$ of weight 2 such that

$$\text{Hom}(1(1), N') = \text{Hom}(N'', 1(1)) = 0, \quad N' \cong 1(1)^{\oplus r}$$

for some $r \geq 0$. Consider the induced morphism $f' : M \to N'$. From this, we see that

$$\ker f \cong \ker f', \quad \text{cok} f \cong \text{cok} f' \oplus N'', \quad \text{im} f \cong \text{im} f'.$$

Thus the kernel, cokernel, and image of $f$ exist, and the cokernel of $f$ is a pure 2-motive of weight 2. □

**Proposition 7.4.** The condition $(\text{wVan'}_{\leq 2})$ holds.

**Proof.** We need to show that for any $X, Y \in \text{SmProj}_{\leq 2}$ and $4 \geq d > e \geq 0$,

$$\text{Hom}(M_d(X), M_e(Y)[-1]) = 0.$$

Since $\text{Hom}(M_r(S), \mathcal{L}^2) \cong M_{4-r}(S)$ by [14], this is equivalent to showing that

$$\text{Hom}(M_{4-e}(Y), M_{4-d}(X)[-1]) = 0.$$

If $e \geq 2$, then $4 - d \leq 1$. Thus we reduce to the case when $e \leq 1$. Assuming this, the conclusion follows from Proposition 7.2. □

7.5. We have verified $(\text{wVan'}_{\leq 2})$ and $(\text{Semi'}_{\leq 2})$. Thus by [14,16] we get the following result.

**Theorem 7.6.** When $n = 2$, the conclusions of Theorems 1.6, 1.9, and 1.15 hold.

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**Institut für Mathematik, Universität Zürich, Winterthurerstr. 190, 8057 Zürich, Switzerland**

*E-mail address*: doosung.park@math.uzh.ch