Symbolic Dynamics of the Diamagnetic Kepler Problem Without Involving Bounces

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Without involving bounce events, a Poincaré section associated with the axes is found to give a map on the annulus for the diamagnetic Kepler problem. Symbolic dynamics is then established based on the lift of the annulus map. The correspondence between the coding derived from this axis Poincaré section is compared with the coding based on bounces. Symmetry is used to reduce the symbolic dynamics. By means of symbolic dynamics the admissibility of periodic orbits is analyzed, and the symmetry of orbits discussed.

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I. INTRODUCTION

The diamagnetic Kepler problem (DKP), or a hydrogen atom moving in a uniform magnetic field, is a simple physical system for studying the correspondence relation between classical chaos and quantum behavior \cite{1}. In the semiclassical quantization of a classically chaotic system, quantum properties are related to periodic orbits or closed orbits \cite{2,3}. An effective method to locate orbits, and then encode and classify them is essential to the study. For the \(n\)-disk scattering system \cite{5} and stadium billiard \cite{6,7}, symbolic dynamics has been established by taking a Poincaré section on the boundary of reflection. The symbolic encoding of the orbits and symbolic dynamics developed for the four-disk system have been extended to the smooth Hamiltonian system for the DKP \cite{8,9}. However, bounces are not always well defined for a soft potential. For the hyperbolic potential system, symbolic dynamics is established by introducing an appropriate Poincaré section without referring to bounce events \cite{11}. Here we use this method to construct symbolic dynamics for the DKP. The paper is organized as follows. In Sec. II, we describe the method to construct a Poincaré section with the Birkhoff coordinates. In Sec. III we discuss the symbolic encoding of orbits and ordering rules of symbolic sequences. We examine the correspondence between this coding and the coding based on bounces for the touching four-disk billiard in Sec. IV. In Sec. V, we classify orbits according to their codes and symmetries. Finally, in Sec. VI, we give a summary and conclusion.

II. THE POINCARÉ SECTION

The classical dynamics of a hydrogen atom in a uniform magnetic field \(B\) along the \(z\)-axis is described by the Hamiltonian:

\[
H = \frac{p_z^2}{2m_e} - \frac{e^2}{r} + \omega z + \frac{1}{2} m_e \omega^2 (x^2 + y^2)
\]  

(1)

where \(l_z\) is the \(z\) component of the angular momentum, and \(\omega = \frac{eB}{2m_e}\) the cyclotron frequency. In cylindrical coordinates \((\rho,z,\phi)\) and atomic units, at the angular momentum \(l_z = 0\), the Hamiltonian can be rewritten as

\[
H = \frac{\rho^2}{2} + \frac{\rho_z^2}{2} - \frac{\rho^2}{8} - \frac{1}{\sqrt{\rho^2 + z^2}}
\]

(2)

which is an even function of the coordinates and momenta. Its symmetry group consists of the identity \(e\) and the reflection \(\sigma_\rho\). Introducing the semi-parabolic coordinates

\[
\mu^2 = \sqrt{\rho^2 + z^2} - z,
\]

\[
\nu^2 = \sqrt{\rho^2 + z^2} + z,
\]

(3)

\[
d\tau = \frac{1}{2} dt / \sqrt{\rho^2 + z^2},
\]

(4)
we may convert the Hamiltonian to

\[ h = \frac{p_\mu^2}{2} + \frac{p_\nu^2}{2} - \epsilon(\mu^2 + \nu^2) + \frac{1}{8}\mu^2\nu^2(\mu^2 + \nu^2) \equiv 2, \]

where \( \epsilon = E(2\omega)^{-2/3} \) is the scaled energy and \( h \) is the fixed “pseudo energy”. In Fig. 1, we show an orbit at \( \epsilon = 0 \). Irregular orbits at \( \epsilon = 0 \) displayed in the configuration space \((z, \rho)\) or in the Poincaré section \((\nu, p_\nu)\) show the chaotic behavior of the dynamics.

The transformed Hamiltonian \( h \) has a higher symmetry than \( H \). Its symmetry group consists of the identity \( e \), two reflections \( \sigma_\mu, \sigma_\nu \) across the \( \mu, \nu \) axes, two diagonal reflections \( \sigma_{13}, \sigma_{24} \), and three rotations \( C_4, C_2 \) and \( C_4 \) by \( \pi/2, \pi \) and \( 3\pi/2 \) around the center, respectively.

Our Poincaré section is chosen as follows. Imagine that the \( \mu \) and \( \nu \) axes are both of a finite width and length. A counterclockwise contour is taken along the perimeter of the area forming by the two crossing imaginary rectangles. The Poincaré section is then obtained by recording the position and the tangent component of the momentum along the contour, i.e., the Birkhoff canonical coordinates \([3]\) at intersecting points with the contour where an orbit enter the inside of the contour. The length of the contour is infinite. It is more convenient to transform the contour to one with a finite length. For example, in the first quadrant, the transformations \( s = -\mu/(1 + \mu) \) along the positive \( \mu \) axis and \( s = \nu/(1 + \nu) \) along the positive \( \nu \) axis convert the segment of the original contour in the first quadrant into an interval of length 2 parametrized with \( s \in [-1, 1] \). For our purpose here, the variable corresponding to the momentum may be take as \( v = -p_\mu/p \) at the positive \( \mu \) axis, and \( v = p_\nu/p \) at the positive \( \nu \) axis, where \( p = \sqrt{p_\mu^2 + p_\nu^2} \). In this way we may parametrize the whole contour with \( s \in [-1, 7] \), and define corresponding \( v \). The rotational symmetry under \( C_4, C_2 \) and \( C_4 \) in the original configurational space becomes the translational symmetry of shifting \( s \) by a multiple of 2 in the \( s-v \) plane. The dynamics on the Poincaré surface is then represented by a map on the annulus \( s \in [-1, 7] \) and \( v \in [-1, 1] \).

### III. Symbolic Dynamics Based on Lifting

For the map on an annulus describing the dynamics on the Poincaré section, it is useful to consider its lift. In the lifted space on which the lift map is defined the area \( s \in [-1, 7] \) and \( v \in [-1, 1] \) consists of a fundamental domain (FD), which is uniformly divided into 8 strips according to \( s \). The image of the FD in the lifted space is partly shown in Fig. 2(a), where the 8 strips of the FD are marked with the numbers 0 to 7, and four strips right of the FD marked with +0 to +3. In the figure we draw only the images of the strips 1 and 2. Two zones marked with 1 and 1′ of the strip 1 are mapped to the zone 1 of strip +3 and the zone 1′ of strip +2, respectively. The notation representing the image of strip 2 is analogous. The rotational symmetry of the Hamiltonian corresponds to the translational symmetry in the lifted space. For example, the image of strip 0 is in the strips 6 and 7, and can be obtained from the image of strip 2 (in the strips +0 and +1) by shifting to the left by 2 strips. Similarly, we show part of the preimage of the FD in Fig. 2(b). Since on the annulus forward and backward maps rotate in opposite directions, strips are on the right of their preimages, in contrast with images.

For the full FD, the rotation number of orbits is between 0 and 2. Taking the rotational symmetry into account, we may consider only the reduced domain (RD) consisting of strips 1 and 2. When the RD is regarded as the annulus, from Fig. 1 the integral part of the rotation numbers is 3 and 4 for zone 2, which is the joint zone of 1′ and 2′, but 4 and 5 for zone 1. As shown in Fig. 3(a), the RD may be correspondingly divided into three regions according to the rotation number. The two lines of demarcation are marked by \( B_0 \) and \( B_2 \). In the figure we have also drawn some stable and unstable manifolds. (The stable manifolds have a segment roughly parallel to lines \( B_0 \) and \( B_2 \), and the unstable manifolds are a mirror image of the stable manifolds with respect to the line \( v = 0 \).) In the RD, tangencies between stable and unstable manifolds can be easily seen. Among them, the prominent ones are in the top-left and bottom-right regions of the three. At a given point the stable and unstable directions, i.e., tangent directions of manifolds, can be determined with the procedure suggested by Greene [14], and then tangent points found. Two lines connecting such tangent points, which are marked by \( C_0 \) and \( C_2 \), give a further partition of the RD. The final partition of the RD into five regions is shown in Fig. 3(a), where these regions are marked by \( L_0, R_0, R_1, R_2 \), and \( L_2 \). In the figure the two boundaries of the RD are marked by \( D_0 \) and \( D_2 \). This partition is the partition according to the preimage. Its preimage gives the partition according to the present, as shown in Fig. 3(b), where the five regions are marked with \( L_0, R_0, R_1, R_2, \) and \( L_2 \). By means of this partition, or equivalently the partition of Fig. 3(a) according to the preimage, we may code an orbit with a doubly infinite sequence

\[ \cdots s_{-1} \bullet s_0 s_1 \cdots, \]
where • indicates the present. We may refer to this coding as the axis coding, and the coding based on bounces in the literature the bounce coding.

The ordering is essential to establishing symbolic dynamics. The natural order is well defined in the lifted space. The 5-piece preimage of the RD arranged from left to right in the lifted space consist of \( L_0 •, R_0 •, R_1 •, R_2 • \) and \( L_2 • \), as can be seen from Fig. 3(b). Thus, according to the natural order we have

\[
L_0 • < R_0 • < R_1 • < R_2 • < L_2 • .
\]

From the preimage of the RD shown in Fig. 3(b) we see an opposite arrangement of the preimage pieces in the lifted space. A more precise description of ordering is given by the ordering of unstable manifolds on a transversal stable foliation. It is numerically verified that under the backward map the ordering is reversed in the regions \( R_0 •, R_1 • \) and \( R_2 • \), but preserved in \( L_0 • \) and \( L_2 • \). We may define the parity of a finite string by the oddness of the total number of the letters \( R_0, R_1 \) and \( R_2 \) contained in the string. Any odd leading string in backward sequences will then reverse the ordering \( 3 \) of backward sequences.

Similarly, from the partition shown in Fig. 3(b) according to the present, we have the ordering

\[
•L_0 < •R_0 < •R_1 < •R_2 < •L_2 .
\]

An odd leading string also reverses this ordering for forward sequences.

Based on the ordering rules, metric representations for both forward and backward sequences may be introduced to construct the symbolic plane. Every forward sequence \( s_0 s_1 \ldots s_n \ldots \) may correspond to a number \( \alpha \), represented in base 5, between 0 and 1 as follows. We define the correspondence of symbol \( \alpha \) to number \( \mu_i \in \{0,1,2,3,4\} \) as \( L_0 \rightarrow 0, R_0 \rightarrow 1, R_1 \rightarrow 2, R_2 \rightarrow 3 \) and \( L_2 \rightarrow 4 \) if the leading string \( s_0 s_1 \ldots s_i \ldots \) is even, and as \( \{L_0, R_0, R_1, R_2, L_2\} \rightarrow \{4,3,2,1,0\} \) otherwise. Finally, we define

\[
\alpha = \sum_{i=0}^{\infty} \mu_i 5^{-(i+1)}.
\]

In this way forward sequences are ordered according to their \( \alpha \)-values. Similarly, we assign to a backward sequence \( \ldots s_{-m} \ldots s_{-2} s_{-1} \) the number \( \beta \) defined by

\[
\beta = \sum_{j=1}^{\infty} \nu_j 5^{-j},
\]

where \( \nu_j \in \{0,1,2,3,4\} \) is determined by \( s_{-j} \) and the oddness of the leading string \( s_{-j+1} \ldots s_{-2} s_{-1} \). In the symbolic plane every orbit point corresponds to a point \((\alpha, \beta)\) in the unit square, where \( \alpha \) and \( \beta \) are associated with the forward and backward sequences of the orbit point, respectively. We show the symbolic plane for \( \epsilon = 0 \) in Fig. 3(a), where approximately 68000 points of several real orbits in the RD are drawn. The corresponding pruning front for the partition lines \( \bullet C_0, \bullet C_2 \) and the boundary lines \( \bullet D_0 \) and \( \bullet D_2 \) is shown in Fig. 3(b).

So far we have not considered the reflectional symmetry. In the lifted space this symmetry corresponds to the invariance of the RD under a \( \pi \)-rotation around the center (or the reflection with respect to the center). From Fig. 3(a) it is seen that the image of zone 1 of strip 1 is in strip +3, hence still in 1 after wrapping. On the contrary, the image 1' of strip 1 is in strip 2. However, a \( \pi \)-rotation can put the image back into strip 1. So, using the reflectional symmetry, we may focus only on strip 1, which may be regarded as the minimal domain (MD). In this way the 5-letter symbolic dynamics is reduced to a 3-letter one, and the \( \pi \)-rotation changes the parity of \( R_1 \). More specifically speaking, we have the following ordering for the minimal domain

\[
\bullet L_0 < \bullet R_0 < \bullet R_1, \quad L_0 • < R_0 • < R_1 •,
\]

and only \( R_0 \) in a leading string reverses the ordering. The symbolic plane of the MD is shown in Fig. 3 where 34,000 points of several real orbits are drawn together with the pruning fronts. The primary pruning front of \( \bullet D_0 \) is roughly diagonal, while that of \( \bullet C_0 \) is almost vertical. The latter encloses a forbidden zone near the top in the symbolic plane, which is responsible for many interesting bifurcations. In the 5-letter code for the RD, the second and fourth quadrants of the symbolic plane are forbidden. The symbolic plane for the 3-letter code of the MD looks more compact.

Let us discuss the pruning front in some detail. A point \( Q \bullet D_0 P \) on the partition line \( \bullet D_0 \) discards a rectangle in the symbolic plane as a forbidden zone, the top-right corner of which is the point \( Q \bullet D_0 P \), as shown in Fig. 3. When representing the point in the symbolic plane, we have replaced \( D_0 \) by its right limit, i.e., a point with \( s = 0_+ \).
Its allowed zone is the rectangle, the lower-left corner of which is the same point. Similarly, a point \( Q \cdot C_0P \) discards a rectangle as its forbidden zone whose two lower corners are \( Q \cdot L_0P \) and \( Q \cdot R_0P \). The point also determines two allowed zones. The above mentioned two corners form the upper-right corner of the left allowed zone and the upper-left corner of the right one. We have numerically found the following 5 points on the partition lines:

\[
P_1: \ldots \cdot R_1L_0R_1L_0R_1L_0R_1^2R_2^2R_3^2 \cdot C_0R_1R_0R_1R_0R_1 \cdot \ldots,
\]

\[
P_2: \ldots \cdot R_1L_0R_1R_0L_0L_0R_1^2 \cdot C_0R_1^3R_0^2R_0L_0R_1 \cdot \ldots,
\]

\[
P_3: \ldots \cdot R_1^5L_0R_0^2R_2^2L_0L_0R_1 \cdot C_0R_2^2R_3^2R_4L_0 \cdot \ldots,
\]

\[
P_4: \ldots \cdot R_1L_0R_1L_0R_1L_0R_1L_0R_1^2 \cdot L_0L_0L_0L_0L_0 \cdot \ldots,
\]

\[
P_5: \ldots \cdot R_1L_0R_1L_0L_0R_1^2L_0L_1L_0L_1L_0 \cdot \ldots,
\]

where the last two points are on \( \bullet D_0 \). The first three points correspond to lower corners of the top three rectangles with dotted sides in Fig. 6, while the other two to the upper-right corners of the two bottom rectangles. It can be verified that the point \((R_0R_1^2R_0^3R_1L_0R_1^2)\infty \cdot (R_0^3R_1^2R_0^3R_1L_0R_1^2)\infty \) is in the forbidden zone of \( P_2 \), so the periodic sequence \((R_0R_1^2R_0^3R_1L_0R_1^2)\infty \) is nonadmissible. One can verify the admissibility of the sequence \((R_1L_0R_1R_0L_0^2R_0R_1R_0R_1^3)\infty \). In fact, its 0, 1, 2, 4, 5, 7, 9th shifts are in the allowed zone of \( P_1 \), while its 3, 8th shifts in that of \( P_3 \). At the same time, its 0, 1, 3, 5, 6, 8th shifts are in the allowed zone of \( P_4 \), while its 2, 4, 7, 9th shifts in that of \( P_3 \). In this way we may examine the admissibility of periodic sequences from the known coding of partition lines. The above orbit \((R_1L_0R_1R_0R_0^2R_0R_1)\infty \), written in the 5-letter code, is \((R_1R_2R_1R_0R_1^2R_0R_1L_2R_1)\infty \). Symmetry of sequences by means of the 5-letter code will be discussed in Sec. V.

**IV. CORRESPONDENCE BETWEEN THE AXIS CODES AND THE BOUNCE CODES**

In the literature the symbolic description based on bounces is used. Bounces are well defined for billiards. We may explore the relation between the axis codes and the bounce codes by examining a billiard. For simplicity, let us consider the touching four-disk billiard. As shown in Fig. 3(a), the boundary of the billiard is formed by the four disks which are labeled with a, b, c, and d, respectively. Taking the boundary as a Poincaré section, we have the corresponding Birkhoff coordinates similar to those of the stadium billiard, which we denote by \((r, u)\) to distinguish from the above \((s, v)\), and hence the associated map on the annulus. We normalize the boundary length as unity, and measure the length from the touching point of disks a and d. As shown in Fig. 3(b), the fundamental domain of the annulus consists of four strips corresponding to the four disks. The image of the strip of disk a, which has three pieces in three other strips, determines the partition of the reduced domain consisting of a single strip. The three regions of the RD are labeled with \( \bullet 0 \), \( \bullet 1 \), and \( \bullet 2 \) in the figure. By the argument based on lifting, we have the ordering

\[ \bullet 0 < \bullet 1 < \bullet 2. \tag{11} \]

Since every bounce on a disk reverses the ordering of orbits, each symbol has an odd parity. That is, for forward symbolic sequences, a string of an odd length is odd, while that of an even length is even. We have similar ordering rules for backward symbolic sequences. This is our version of the symbolic dynamics described in Ref. 3.

When the strip of \( r \in [0, 0.25] \) is divided at \( r = 0.125 \), and an eighth of the FD is considered, the image of the line \( r = 0.125 \) has three pieces respectively in strips b, c and d. The partition of a half strip, which is determined by putting all the pieces of its image together back into the half strip, will involve symbols more than three.

To explore the correspondence between the axis codes and the bounce codes, we examine how an orbit starting from a point on an axis hits different disks. This gives the relation between the phase spaces \((s, v)\) associated with the axis contour and \((r, u)\) associated with the billiard boundary. Consider the orbit AC grazing against disk d in Fig. 3(a). It corresponds to a point of \( s \in (-1, 0) \), or equivalently a point of \( s \in (1, 2) \) when the RD is considered. Starting from point A on the x-axis, orbits right of this grazing orbit hit disk d before intersecting an axis again, while orbits on its left hit the y-axis directly before having a bounce on disk c. The orbit ABO, which passes through the origin, separates orbits crossing the y-axis from those crossing the x-axis. The orbits of the type AC correspond to line \( \bullet C_2 \) in the RD of the s-v phase space since the line demarcates the order preserving region without a bounce from the order reversing region with a bounce. Similarly, the orbits of type ABO correspond to line \( \bullet B_2 \). Thus, bisecting the region \( \bullet R_1 \) into \( \bullet R_+ \) and \( \bullet R_- \) with line \( s = 1 \), we may re-label the right half of the RD as in Fig. 3(b). Comparing this figure with Fig. 3(b), we see that \( \bullet R_+ \) and \( \bullet R_- \) belong to \( \bullet 0 \), which associates with orbits leaving from disk a and bouncing on disk d. The region \( \bullet L_2 \) has no direct correspondence with any bounce. Orbits from the region, after
missing a bounce, will hit disk c before intersecting again an axis. We may re-label the region as (●1). Consequently, we have the correspondence:

\[ R_+ \cdot R_2 \rightarrow \bullet 0; \quad L_2 X \rightarrow \bullet 1, \]

where \( X \) stands for any symbol.

The other half of the RD in \( s-v \) space may be analyzed in a similar way. The re-labeling of regions is also shown in Fig. 9(b), which indicates the correspondence:

\[ R_-, R_0 \rightarrow \bullet 2; \quad L_0 X \rightarrow \bullet 1. \]

Putting Fig. 9(a) on the top of 3(b), we see that the allowed symbolic pair are

\[ L_2 R_+, L_2 R_2;\quad R_2 R_+, R_2 R_2, R_2 L_2; \quad R_+ R_-, R_2 R_0, R_+ L_0; \quad R_0 R_-, R_0 R_0, R_0 L_0; \quad L_0 R_-, L_0 R_0. \]

Two rules can then be extracted:

1. \( L_2 \) and \( L_0 \) are always followed by a symbol of odd parity.
2. \( R_+ \) follows either \( L_2 \), or \( R_2 \), or \( R_- \), while \( R_- \) follows either \( L_0 \), or \( R_0 \), or \( R_+ \).

The first rule explains that \( L_0 X \) or \( L_2 X \) has the same parity as the bounce code 1. The second rule can be used to refine \( R_1 \) into \( R_+ \) and \( R_- \), and then further to convert them into bounce codes. For example, an orbit shown in Fig. 9(a) has the axis codes \((R_2 R_1 R_0 R_1)\). The codes are first refined as \((R_2 R_1 R_0 R_1)\), and then converted to \((0^3 2^3)\). Another orbit shown in Fig. 9(b) has the codes \((R_2 R_1 L_0 R_0 R_1)\). This sequence is converted as follows:

\[ (R_2 R_1 L_0 R_0 R_1) \rightarrow (R_2 R_1 R_- R_0 R_0) \rightarrow (0^3 2^3) \rightarrow (0^3 2^3) \]

The ordering \( \sigma \) can be refined as

\[ •L_0 < •R_0 < •R_- < •R_+ < •R_2 < •L_2, \]

which, under a cyclic transformation, becomes

\[ •R_+ < •R_2 < •L_2 < •L_0 < •R_0 < •R_- \]

This ordering is then converted to

\[ •0 < •1 < •2, \]

which is just the ordering of the bounce codes. A detailed analysis of the relation between axis codes and bounce codes will be presented elsewhere.

V. SYMBOLIC SEQUENCES AND SYMMETRY OF ORBITS

The configurational space of the original dynamics is the half \( \rho-\nu \) plane, which converts to the first quadrant of the \( \mu-\nu \) plane. The rotational symmetry in the \( \mu-\nu \) plane is the invariance under the transformation from \((\mu, \nu, p_\mu, p_\nu)\) to \((-\nu, \mu, -p_\nu, p_\mu)\) and \((\nu, -\mu, p_\nu, -p_\mu)\) (denoted by \( \rho, \pi \) and \( \bar{\rho} \)). This symmetry has nothing to do with the original dynamics. Confining the dynamics to the RD, we remove the symmetry.

As for the reflectional symmetry, we may focus only on \( \sigma_{\rho} \) : \((\mu, -\nu, p_\mu, -p_\nu)\) since we can composite \( \sigma_{\nu} = \pi \circ \sigma_{\rho} \), \( \sigma_{13} = \sigma_{\mu} \circ \bar{\rho} \) and \( \sigma_{24} = \sigma_{\mu} \circ \rho \). We shall denote \( \sigma_{\mu} \) simply by \( \sigma \). While \( \rho, \pi \) and \( \bar{\rho} \) have no effect in the \( s-v \) space, \( \sigma \) leads to \((s, v) \rightarrow (2 - s, -v)\), which, written in symbols, is \( L_0 \leftrightarrow L_2, R_0 \leftrightarrow R_2 \) and \( R_1 \leftrightarrow R_1 \). To discuss the reflectional symmetry, the 5-letter code must be used. Thus, if an orbit is symmetric under reflection, its symbolic sequence is invariant up to a shift under the symbolic transformation of \( \sigma \). For example, sequence \((R_1 R_2 R_1 R_0)\) of period 6 is symmetric under \( \sigma \), while \((R_2 R_1)\) is not. This two orbits are shown in Fig. 9(a).

The Hamiltonian dynamics has the time-reversal symmetry: \((\mu, \nu, p_\mu, p_\nu; \tau) \rightarrow (\mu, \nu, p_\mu, p_\nu; -\tau) = (\mu, \nu, -p_\mu, -p_\nu; \tau)\). This transformation, denoted by \( T \), leads to \((s, v) \rightarrow (2-s, v)\) in the \( s-v \) space. Written in symbols, this means \( L_0 \leftrightarrow L_2, R_0 \leftrightarrow R_2 \), \( R_1 \leftrightarrow R_1 \), \( R_2 \leftrightarrow R_0 \) and \( L_2 \leftrightarrow L_0 \). Thus, the effect of \( \sigma \) on a sequence is to reverse the sequence and apply \( \sigma \) to it. For example, transformation \( T \) converts sequence \((R_2 R_1 R_2 L_2)\) to \((R_2 R_1 R_2 L_2)\). We may combine \( T \) and \( \sigma \) to define the transformation \( T_{\sigma} \) of reversal by \( \sigma \). In the
3-letter symbolic dynamics we identify $L_2$ with $L_0$, and $R_2$ with $R_0$, so $T$ just reverses a sequence. Transformation $T_\sigma$ converts the above just mentioned orbit $(R_1^3 L_0 R_0 R_1 R_2^2)_{\infty}$ to $(R_2^2 R_1 R_0 L_0 R_1^3)_{\infty}$. The two orbits $(R_1^3 L_0 R_0 R_1 R_2^2)_{\infty}$ and $(R_2^2 R_1 R_0 L_0 R_1^3)_{\infty}$ are shown in Fig. 9(b). (In the configuration space orbits $(R_1^3 L_0 R_0 R_1 R_2^2)_{\infty}$ and $(R_2^2 R_1 R_2 L_2 R_1^3)_{\infty}$ are not distinguishable.)

The transformations $\sigma$, $T$ and $T_\sigma$ may be used to generate admissible sequences from a known one. When performing a transformation on a sequence results in the same sequence up to a shift, the sequence is symmetric. In Table I, we give the non-repeating strings of 15 admissible periodic sequences, the initial values $(s, v)$ and the symmetry of their corresponding orbits.

Orbits which pass through the origin play a special role in the semiclassical theory of closed orbits. Let us analyze their symmetry in some detail. Right at the origin the momentum $v$ is ill-defined. However, for an orbit passing through the origin we may consider its left and right limits since both of them are well-defined. It can be seen that around the origin we have codes $BD\tau$ and $BD\bar{\tau}$ for the two limits, where $B$ stands for $B_0$ or $B_2$. In the RD $D_0$ should have been identified with $D_2$ since the dynamics is on the annulus. Here by $D_0$ and $D_2$ we mean $s = 0_+$ and $2_-$, respectively. It can be further verified that the first $D_0(D_2)$ and second $D_2(D_0)$ are related as $(s, v) \leftrightarrow (2 - s, v)$, which is just the transformation $T$. This means that, after identifying $D_0$ with $D_2$, the sequence of an orbit passing through the origin must be $T$-invariant. For example, the orbit shown in Fig. 9(c) has the sequences $(B_2 D_2^2 R_0 B_0 D_2 R_2)_{\infty}$ and $(B_0 D_2^2 R_2 D_2^2 B_0)_{\infty}$ as its right and left limits. Written more precisely, the two limiting sequences are $(R_2 L_2 R_1 R_0 L_0 R_2)_{\infty}$ and $(R_2 R_1 L_0 R_0 R_1 L_2 R_2)_{\infty}$, respectively. The $T$ transformation of the former is $(R_0 R_1 L_2 R_2 R_2 R_2 L_0 R_0)_{\infty}$, which turns out to be just the 4-th shift of the latter. (A further observation discovers that the codes $BD\tau\cdots$ can be equally well written as $BDB\cdots$ since the third point is on the common segment of $\bullet D$ and $\bullet B$, where $D$, similar to $B$, stands for $D_0$ or $D_2$.)

Many short closed orbits have been found in Ref. [2]. Non-repeating strings or their doubled strings of the periodic sequences corresponding to these orbits are listed in Table II, where sequences and initial values $v$ are for the right limit orbits starting at $s = 0_+$. Sequences for left limit orbits may be obtained by exchanging $D_0$ with $D_2$. While the labeling of Ref. [2] for orbits are given in the table, we arrange them according to the order of sequences and symmetry, and shall refer this numbering to orbits. More precisely speaking, the two sequences which relate to each other by $\sigma$ are always listed successively, and ordered according to one of the two without referring to the order of the other. The orbit shown in Fig. 9(c) is orbit 27.

From the table it is clearly seen that symbolic coding provides a way to classify orbits. For example, according to the coding orbits 1, 3, 5, 15, 25 and 2, 4, 10, 20 form a big family of the type $BD^2 R_1^3 BD^2 R_1^3$. When $n$ is odd the non-repeating string of an orbit reduces to half. We have doubled it in the table to make it look more like that for an even $n$. There are two more orbits (29 and 30) for which doubled strings are given. We can see many pairs of orbits which should be created in a bifurcation related to $\bullet C$. A pair of orbits 29 and 30 are given in Fig. 9(d) to show that they have the same topology. Other examples are orbits 6 and 8, orbits 11 and 13, and orbits 28 and 30. In the table 8 orbits are obtained by the transformation $\sigma$, which is indicated in the last column of the table. The initial values of $v$ for two such orbits related by $\sigma$ are different only in their sign, so only one $v$ is given. The sequences listed for these orbits then correspond to a left limit.

VI. CONCLUSION

In the above we have given a natural way to construct symbolic dynamics for the the diamagnetic Kepler problem. Our choice of the Poincaré section avoids any ambiguity in identifying bounces. A lift of the phase space helps us to understand the dynamics. Tangencies between stable and unstable manifold foliations play an important role in the construction of symbolic dynamics. Based on the symbolic coding we have analyzed the rotational and reflectional symmetry, and the time reversal symmetry as well. Symbolic coding provides us a convenient way to classify orbits. For a billiard limit we have established the relation between our coding and the coding based on bounces.

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Table I. Some orbits which do not pass through the origin.

| No. | $s$      | $v$         | Sequence                  | Period | Symmetry | Bound Codes |
|-----|----------|-------------|---------------------------|--------|----------|-------------|
| 1   | 0.502590559992 | 0.000000000377 | $R_0R_1R_2^3R_1R_0$     | 6      | $\sigma,T,T_\sigma$ | $2^{0}0^{2}$  |
| 2   | 0.525466094731  | 0.083479979410 | $R_2^3L_0R_1^2R_0$     | 6      | $T_\sigma$ | $2^{2}102$ |
| 3   | 0.597726644479  | 0.000000000105 | $R_0$                   | 4      | $T_\sigma$ | $2^4$       |
| 4   | 0.62776845085   | 0.000000000692 | $R_2^3R_1^2$           | 6      | $T_\sigma$ | $2^{0}2^{2}$ |
| 5   | 0.634615582851  | 0.34368565696  | $R_1L_2R_1R_0$         | 4      | $T_\sigma$ | 212         |
| 6   | 0.634764050232  | 0.000000004571 | $R_2^3R_1^2R_0^2$      | 8      | $T_\sigma$ | $2^{0}202^{2}$ |
| 7   | 0.637812891717  | 0.466976827386 | $R_2R_1R_0R_0$         | 4      | $\sigma,T,T_\sigma$ | $20^{2}2$ |
| 8   | 0.644118560693  | 0.000000282982 | $R_2^3R_1^2$           | 16     | $T_\sigma$ | $2^{3}(02)^{1}0^{2}$ |
| 9   | 0.645908548293  | 0.86051268032  | $R_2^3R_1^2L_2R_2^3L_0R_0L_0$ | 16 | $1(02)^{3}0^{2}1201$ |
| 10  | 0.653504728307  | 0.022526445642 | $R_0L_0R_2^3R_0^2$     | 26     | $21(02)^{3}1^{2}$ |
| 11  | 0.653578148187  | 0.014059036967 | $R_0L_0R_2^6R_0^2$     | 30     | $21(02)^{1}1^{2}$ |
| 12  | 0.671107671805  | 0.000000002635 | $R_0L_0R_2^3L_0R_0$    | 8      | $T_\sigma$ | 210201      |
| 13  | 0.685136083296  | 0.257524880819 | $R_1R_2R_1R_0^2L_0R_2^3R_0^2$ | 16 | $20^{2}2^{0}1(20)^{3}2^{2}$ |
| 14  | 0.714849526728  | 0.061364237283 | $R_2R_1R_2R_1R_0^2R_1L_2R_1$ | 10     | $20^{2}2^{0}2^{1}$ |
| 15  | 0.744712436526  | 0.064776613634 | $R_2^3L_0R_0R_1R_2^3R_1$ | 8      | $20120^{3}$ |
Table II. Sequences of orbits passing through the origin. By $O_i$ we mean the $i$-th orbit according to the numbering in the first column.

| No. | label | $\nu$ | Sequence | Period | Symmetry | Bound Codes |
|-----|-------|-------|----------|--------|----------|-------------|
| 1   | 30    | 0.9740712322977 | $D_0R_1^3B_2D_2^2R_1^3B_0^4D_0$ | 22 | 21(02)$^301(20)^4$ |
| 2   | 24    | 0.9728988884484 | $D_0R_1^3B_2D_2^2R_1^3B_0D_0$ | 20 | 21(02)$^410(20)^3$ |
| 3   | 19    | 0.97034076638   | $D_0R_1^3B_2D_2^3R_1^3B_0D_0$ | 18 | 21(02)$^501(20)^3$ |
| 4   | 13    | 0.96706416031   | $D_0R_1^3B_2D_2^3R_1^3B_0^4D_0$ | 16 | 21(02)$^510(20)^2$ |
| 5   | 9     | 0.962773942237  | $D_0R_1^3B_2D_2^3R_1^3B_0D_0$ | 16 | 21(02)$^501(20)^2$ |
| 6   | 29    | 0.959653990845  | $D_0R_1^3B_2D_2^3R_1^3B_0D_0$ | 16 | 21(02)$^50212^2020$ |
| 7   | 14    | 0.95869766319   | $D_0R_1^3B_2D_2^3R_1^3B_0D_0$ | 16 | $\sigma \circ O_6$ $2^210^2(20)^21(20)^2$ |
| 8   | 28    | 0.950042745974  | $D_0R_1^3B_2D_2^3L_0^2L_0^3R_1^3B_0D_0$ | 16 | 21(02)$^511020$ |
| 9   | 15    | 0.956722093284  | $D_0R_1^3B_2D_2^3L_0^2L_0^3R_1^3B_0D_0$ | 12 | 21(02)$^51020$ |
| 10  | 8     | 0.952489366877  | $D_0R_1^3B_2D_2^3L_0^2L_0^3R_1^3B_0D_0$ | 14 | 21(02)$^5210^220$ |
| 11  | 27    | 0.952489366877  | $D_0R_1^3B_2D_2^3L_0^2L_0^3R_1^3B_0D_0$ | 14 | 21(02)$^5210^220$ |
| 12  | 23    | 0.952489366877  | $D_0R_1^3B_2D_2^3L_0^2L_0^3R_1^3B_0D_0$ | 14 | $\sigma \circ O_{11}$ $12(20)^21(20)^2$ |
| 13  | 20    | 0.950042745974  | $D_0R_1^3B_2D_2^3L_0^2L_0^3R_1^3B_0D_0$ | 14 | 21020$^120$ |
| 14  | 22    | 0.950042745974  | $D_0R_1^3B_2D_2^3L_0^2L_0^3R_1^3B_0D_0$ | 14 | $\sigma \circ O_{13}$ $1^2020120201$ |
| 15  | 5     | 0.947457897321  | $D_0R_1^3B_2D_2^3L_0^2L_0^3R_1^3B_0D_0$ | 10 | 21020$^120$ |
| 16  | 17    | 0.940875954726  | $D_0R_1^3B_2D_2^3L_0^2L_0^3R_1^3B_0D_0$ | 12 | 21020$^120^2$ |
| 17  | 18    | 0.93500646896   | $D_0R_1^3B_2D_2^3L_0^2L_0^3R_1^3B_0D_0$ | 12 | $\sigma \circ O_{16}$ $2^210^220$ |
| 18  | 12    | 0.93500646896   | $D_0R_1^3B_2D_2^3L_0^2L_0^3R_1^3B_0D_0$ | 12 | 21020$^120$ |
| 19  | 16    | 0.93500646896   | $D_0R_1^3B_2D_2^3L_0^2L_0^3R_1^3B_0D_0$ | 12 | $\sigma \circ O_{18}$ $201^2201$ |
| 20  | 3     | 0.931024837844  | $D_0R_1^3B_2D_2^3L_0^2L_0^3R_1^3B_0D_0$ | 8 | 21020$^120$ |
| 21  | 11    | 0.918172430278  | $D_0R_1^3B_2D_2^3L_0^2L_0^3R_1^3B_0D_0$ | 10 | 21020$^120$ |
| 22  | 10    | 0.891672484472  | $D_0R_1^3B_2D_2^3L_0^2L_0^3R_1^3B_0D_0$ | 10 | $\sigma \circ O_{21}$ $12^20120^2$ |
| 23  | 7     | 0.89732522729    | $D_0R_1^3L_0B_0D_2^2L_2R_1^3B_0D_0$ | 10 | 2101$^1$ |
| 24  | 6     | 0.891672484472  | $D_0R_1^3L_0B_0D_2^2L_2R_1^3B_0D_0$ | 10 | $\sigma \circ O_{23}$ $1^212^2$ |
| 25  | 2     | 0.891672484472  | $D_0R_1^3L_0B_0D_2^2L_2R_1^3B_0D_0$ | 6 | 2101$^1$ |
| 26  | 21    | 0.864842403049   | $D_0R_1^3L_0B_0D_2^2L_2R_1^3B_0D_0$ | 10 | 2102$^2$ |
| 27  | 4     | 0.84966377360    | $D_0R_2B_2D_2^2R_2B_0D_0$ | 8 | 210$^2$ |
| 28  | 26    | 0.844140836919   | $D_0R_2B_2D_2^2R_2B_0B_0D_0$ | 20 | $1^3(20)^22^210^2(20)^2$ |
| 29  | 25    | 0.803121053916   | $D_0R_2B_2D_2^2R_2B_0B_0D_0$ | 20 | $210(02)^3210(02)^3$ |
| 30  | 31    | 0.790624022578   | $D_0R_2B_2D_2^2L_2L_2R_1^3L_0B_0D_0$ | 20 | $1^3(20)^21^3(20)^2$ |
| 31  | 1     | 0.707106780832   | $D_0D_2^2D_0D_0$ | 4 | $1^2$ |
FIG. 1. Boundary of the transformed potential for the diamagnetic Kepler problem at $\epsilon = 0$, and a typical orbit.

FIG. 2. The image (a) and preimage (b) of the fundamental domain.

FIG. 3. The partition of the reduced domain according to preimage (a) and to present (b).

FIG. 4. (a) Symbolic plane of the reduced domain. Approximately 68000 real orbit points are shown. (b) Primary pruning front. Points on $\bullet D_0$ and $\bullet D_2$ are marked with diamonds, and points on $\bullet C_0$ and $\bullet C_2$ with crosses.

FIG. 5. Symbolic plane of the minimal plane. Approximately 34000 real orbit points are shown together with the primary pruning front. Points on $\bullet D_0$ are marked with diamonds, and points on $\bullet C_0$ with crosses.

FIG. 6. Admissible and forbidden orbits are determined by five points on two partition lines.

FIG. 7. The touching four-disk billiard (a) and its phase space (b). Here the billiard boundary is taken as a Poincaré section. The Birkhoff coordinates are denoted by $(r, u)$.

FIG. 8. The grazing orbit and center-passing orbit (a) determine the correspondence between $(s, v)$ space and the bounce-based $(r, u)$ space (b).

FIG. 9. Examples of orbits: (a) $\sigma$-symmetric $\left(R_1 R_2^2 R_1 R_0^3\right)_\infty$ (solid) and asymmetric $\left(R_1^2 R_0^4\right)_\infty$ (dashed); (b) an orbit $\left(R_1^4 L_0 R_0 R_1 R_2^2\right)_\infty$ (solid) and its $T_\sigma$-transformation image $\left(R_2^2 R_1 R_2 L_2 R_1^4\right)_\infty$ (dashed); (c) an orbit passing through the origin (orbit 27 in Table II); (d) orbits $D_0 R_2 R_1^2 L_0 B_0 D_0^3 L_2 R_1^2 R_0 B_0 D_0$ (solid) and $D_0 R_2 R_1^2 L_0 B_2 D_2^3 R_2 R_1^2 R_0 B_0 D_0$ (dashed) with the same topology.
