BOUND STATES OF THE TWO-DIMENSIONAL
DIRAC EQUATION FOR AN ENERGY-DEPENDENT
HYPERBOLIC SCARF POTENTIAL

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Abstract

We study the two-dimensional massless Dirac equation for a potential that is allowed to depend on the energy and on one of the spatial variables. After determining a modified orthogonality relation and norm for such systems, we present an application involving an energy-dependent version of the hyperbolic Scarf potential. We construct closed-form bound-state solutions of the associated Dirac equation.

Keywords: Dirac equation, energy-dependent potential, hyperbolic Scarf potential

1 Introduction

The two-dimensional massless Dirac equation can be used to model electron transport phenomena in graphene, an atomically thin conducting material that consists of carbon atoms forming a honeycomb lattice structure [21]. This structure gives rise to many unusual properties of graphene, such as its very high electric conductivity, where both electrons and holes serve as charge carriers [3]. Low-energy states of electrons or holes in graphene can be modeled by the two-dimensional massless Dirac equation [9]. In order to confine charge carriers, the Dirac equation must be coupled to a suitable scalar or vector potential. While for the vast majority of such external potentials the Dirac equation will not be solvable, there are some exceptions. Such exceptional cases include coupling to scalar potentials [5] [14] [12] [24], as well as to vector potentials [4] [17] [22]. The purpose of this work is to extend the latter context to the case where the external potential depends on the energy. Quantum systems that feature energy-dependent potentials have been studied in the nonrelativistic case, for a comprehensive theoretical review and introductory examples the reader may refer to [6] and [23]. Energy-dependent potentials appear in a variety of applications, such as hydrodynamics [18], confined quantum systems [19] [27], or multi-nucleon systems [20]. Theoretical applications include the generation of nonrelativistic models with energy-dependent potentials by means of the supersymmetry formalism [26] and through point transformations [7] [25]. In the nonrelativistic context, the presence of
energy-dependent potentials require a modification of the underlying quantum theory, principally affecting orthogonality relation and norm [6]. In this note we show that a similar type of modification is also necessary in the relativistic context if the potential is energy-dependent. There is very few literature on the topic, particular systems were studied for example in [13] [15], [16]. We consider here a particular case of the two-dimensional, massless Dirac equation for an external scalar potential that we assume to depend on the energy and on a single spatial variable. In section 2 we derive a modified orthogonality relation and norm for systems governed by such Dirac equations. Afterwards, we introduce an energy-dependent version of the hyperbolic Scarf potential. The conventional, energy-independent version of this potential was shown to support closed-form zero-energy states [8]. In section 3 we construct bound-state solutions of our Dirac equation for the energy-dependent hyperbolic Scarf potential and give several examples.

2 The relativistic model

We start out by introducing the two-dimensional, massless Dirac equation for an energy-dependent potential. The time-dependent version of this equation that we consider here features a potential that depends only on one of the spatial coordinates. It can be written in atomic units as follows

$$\left[ -i \alpha \cdot \nabla + V(x, i \frac{\partial}{\partial t}) \right] \hat{\Psi}(x,y,t) = i \frac{\partial \hat{\Psi}(x,y,t)}{\partial t}, \quad (x,y,t) \in \mathbb{R}^2 \times (0, \infty),$$

(1)

where \( \alpha = (\sigma_1, \sigma_2) \) has the Pauli spin matrices \( \sigma_1, \sigma_2 \) as components and the potential \( V \) is a continuous function of two variables. We can generate energy-dependence in the potential upon setting

$$\hat{\Psi}(x,y,t) = \exp(-i E t) \Psi(x,y),$$

(2)

introducing the wave number \( k_y \) that describes free motion in the \( y \)-direction, and the real-valued constant \( E \) that will represent the stationary energy of our system. Insertion into (1) gives a stationary Dirac equation of the form

$$\{ -i \alpha \cdot \nabla + [V(x,E) - E] \} \Psi(x,y) = 0, \quad (x,y) \in \mathbb{R}^2.$$

(3)

We observe that the potential now depends on the energy \( E \). Note further that, following the usual convention, we suppress the dependence on the energy \( E \) in the solution \( \Psi \) and its components.

2.1 Derivation of orthogonality relation and norm

It was shown [6] that in the nonrelativistic scenario the presence of an energy-dependent potential forces a modification of the orthogonality relation and the norm, defined in the usual \( L^2 \)-sense. For this reason, a similar modification is in order to accommodate systems governed by the Dirac equation (3). Our starting point for the construction of a modified orthogonality relation and norm is the continuity equation

$$\frac{\partial P(x,y,t)}{\partial t} = -\nabla J(x,y,t),$$

(4)

where \( P \) and \( J \) denote the relativistic probability density and probability current, respectively, that depend on the two spatial coordinates and on the time \( t \). Let us now assume that \( E_n \) and \( E_m \) are two nonequal energies, for which the stationary Dirac equation (3) admits solutions \( \Psi_n \) and \( \Psi_m \), respectively. The associated solutions of the time-dependent Dirac equation (1) can be
found through (2). In addition, since the potential in our Dirac equation depends only on the x-coordinate, we can separate the y-coordinate off. More precisely, we set for \( j = m \) and \( j = n \)

\[
\hat{\Psi}_j(x, y, t) = \exp(-i E_j t + i k_y y) \Psi_j(x), \tag{5}
\]

Keeping this relation in mind, we will now replace our standard continuity equation (4) by a modified version that satisfies the requirements imposed by an energy-dependent potential. Let us first define the probability density \( P \) and the probability current \( J \). These objects take the same form as in the conventional case, where the potential does not depend on the energy. We have

\[
P(x, y, t) = \hat{\Psi}_m^\dagger(x, y, t) \hat{\Psi}_n(x, y, t) \tag{6}
\]

\[
J(x, y, t) = \hat{\Psi}_m^\dagger(x, y, t) \alpha \hat{\Psi}_n(x, y, t). \tag{7}
\]

Recall that the time-dependency of the expressions on the right sides is governed by (5). We will now show that (6) and (7) satisfy the following modified continuity equation

\[
\frac{\partial P(x, y, t)}{\partial t} + i [V(x, E_n) - V(x, E_m)] \hat{\Psi}_m^\dagger(x, y, t) \hat{\Psi}_n(x, y, t) = -\nabla J(x, y, t), \tag{8}
\]

where the symbol \( \dagger \) denotes the hermitian adjoint. We observe that in contrast to the standard continuity equation (4), the modified version (8) contains an additional term. The presence of this term is similar to the nonrelativistic scenario that was discussed in \([6, 23]\). Let us briefly substitute (6) into the left side of the latter equation. For the sake of legibility we first evaluate the derivative with respect to the time variable. Taking into account the latter definition in combination with (6), the left side of (8) can be integrated as follows

\[
\int t \left\{ \frac{\partial}{\partial s} \hat{\Psi}_m^\dagger(x, y, s) \hat{\Psi}_n(x, y, s) + i [V(x, E_n) - V(x, E_m)] \hat{\Psi}_m^\dagger(x, y, s) \hat{\Psi}_n(x, y, s) \right\} ds = \]

\[
= \left[ 1 - \frac{V(x, E_m) - V(x, E_n)}{E_m - E_n} \right] \hat{\Psi}_m^\dagger(x, y, t) \hat{\Psi}_n(x, y, t). \tag{10}
\]
At this point it is convenient to make use of the relation (5) by substituting it into (10). Similar to the nonrelativistic case [6] [23], this leads to the sought orthogonality relation

\[
\int_{\mathbb{R}} \left[ 1 - \frac{V(x, E_m) - V(x, E_n)}{E_m - E_n} \right] \Psi_m^\dagger(x) \Psi_n(x) \, dx = C \, \delta_{mn},
\]

where \( C \) is a constant. From the orthogonality relation (11) we can now construct the modified norm \( N \) by taking the limit \( m \to n \), resulting in

\[
N(\Psi_n) = \int_{\mathbb{R}} \left[ 1 - \frac{\partial V(x, E_n)}{\partial E_n} \right] \Psi_n^\dagger(x) \, \Psi_n(x) \, dx.
\]

For a solution \( \Psi_n \) of the Dirac equation (3) to represent a bound state, two conditions must be fulfilled: the norm integral (17) must exist and at the same time its integrand must be a nonnegative function. Since the sign of the integral is entirely determined by the factor involving the potential’s derivative, the condition

\[
1 - \frac{\partial V(x, E_n)}{\partial E_n} \geq 0,
\]

must be satisfied for all real numbers \( x \) and energies \( E_n \) associated with the system governed by the Dirac equation (3). Finally let us note that (12) does not constitute a norm in the mathematical sense because it can become negative due to the term containing the energy derivative of the potential.

### 2.2 Decoupling the Dirac system

Before we can consider applications involving specific energy-dependent potentials, we must solve the Dirac equation (1). Since this equation has two components, it can be written as a system of two equations that must be decoupled. To this end, we represent the solution spinor \( \hat{\Psi} \) in the form (5) and split it up into its two components. We set

\[
\hat{\Psi}(x, y, t) = \frac{1}{2} \exp(-i \, E_n \, t + i \, k_y \, y) \begin{pmatrix} \psi_+(x) \\ \psi_-(x) \end{pmatrix} = \frac{1}{2} \exp(-i \, E_n \, t + i \, k_y \, y) \begin{pmatrix} \psi_1(x) + \psi_2(x) \\ \psi_1(x) - \psi_2(x) \end{pmatrix},
\]

Note that the factor 1/2 was introduced merely to facilitate calculations. Upon substitution of (14) into (1), we obtain the following system of equations [8] for the spinor components \( \psi_1 \) and \( \psi_2 \).

\[
\psi_1''(x) + \left\{ [V(x, E) - E]^2 + i \frac{\partial V(x, E)}{\partial x} - k_y^2 \right\} \psi_1(x) = 0
\]

\[
\psi_2(x) = \frac{1}{k_y} \left\{ \psi_1(x) + i [V(x, E) - E] \psi_1(x) \right\},
\]

where we assume without restriction that \( k_y \neq 0 \). Once the first solution component \( \psi_1 \) has been found from the Schrödinger-type equation (15), the remaining counterpart \( \psi_2 \) is generated by means of (16). These two functions are then substituted into (14) in order to obtain the solution spinor of the stationary Dirac equation. Before we apply the results of this section to a specific model, we rewrite the orthogonality relation (11) and norm (12) in terms of the solutions to the system (15), (16). To this end, we introduce two pairs of functions \( \psi_{1,m}, \psi_{2,m} \) and \( \psi_{1,n}, \psi_{2,n} \).
that are solutions to the latter system for $E = E_m$ and $E = E_n$, respectively. Upon substituting relation (14) into our orthogonality relation (11) and norm (12), we obtain the results

$$\int_{\mathbb{R}} \left[ 1 - \frac{V(x, E_m) - V(x, E_n)}{E_m - E_n} \right] \left\{ \left[ \psi_{1,m}^*(x) + \psi_{2,m}^*(x) \right] \left[ \psi_{1,n}(x) + \psi_{2,n}(x) \right] + \right.$$ 

$$+ \left[ \psi_{1,m}^*(x) - \psi_{2,m}^*(x) \right] \left[ \psi_{1,n}(x) - \psi_{2,n}(x) \right] \right\} \, dx = C \delta_{mn}$$

$$N(\psi_n) = \int_{\mathbb{R}} \left[ 1 - \frac{\partial V(x, E_n)}{\partial E_n} \right] \left[ \left| \psi_{1,n}(x) + \psi_{2,n}(x) \right|^2 + \left| \psi_{1,n}(x) - \psi_{2,n}(x) \right|^2 \right] \, dx. \quad (17)$$

Observe that the dependence on the spatial variable $y$ is gone because the corresponding exponential term from (5) cancels out, leaving a single integral. Note further that we left some irrelevant overall constant factors out.

### 3 Application: hyperbolic Scarf potential

We will now introduce a particular energy-dependent potential, for which our stationary Dirac equation (3) admits bound-state solutions that can be given in closed form. The potential we will focus on reads

$$V(x, E) = -\lambda(E) \sech(x) + \mu(E) \tanh(x) + E, \quad (18)$$

where $\lambda \neq 0$ and $\mu$ are real-valued functions of the energy parameter $E$. We see that the function (18) is an energy-dependent generalization of the hyperbolic Scarf potential. It is known [8] that the conventional, energy-independent version of our potential represents a well for electrons if $\lambda > 0$ and a well for holes if $\lambda < 0$. In what follows we will show that this interpretation can be maintained if the potential is energy-dependent and of the form (18), provided certain constraints are met. Observe that the last term on the right side of (18) will cancel with the same term in our Dirac equation (3). As such, solutions of the latter equation are formally equivalent to zero-energy solutions for the scenario of an energy-independent potential. Let us further remark that the potential (18) has a formal similarity with the potential discussed in [11], as far as the shape of its graph is concerned. However, the latter reference considers the conventional, energy-independent context.

#### 3.1 General solution of the governing equation

Our starting point is the observation that the Dirac equation (3) for our potential (18) is exactly-solvable. The general solution provided in [8] persists under the generalization regarding the energy-dependent parameters. Since the explicit form of the solution spinor (14) is too long to be shown here, we focus on the function $\psi_1$ that is determined by the Schrödinger-type equation (15). This equation reads after incorporation of (18)

$$\psi_1''(x) + \left\{ -k_y^2 + \mu(E)^2 + \sech^2(x) \left[ \lambda(E)^2 + i \mu(E) - \mu(E)^2 \right] + \sech(x) \tanh(x) \left[ i \lambda(E) - \right.$$ 

$$- 2 \lambda(E) \mu(E) \right\} \psi_1(x) = 0. \quad (19)$$
The general solution of this equation for $\psi_1$ is given by

$$
\psi_1(x) = c_1 \left[ 1 - i \sinh(x) \right]^{-\frac{1}{2}} \left[ 1 + i \sinh(x) \right]^{\frac{5}{4} - \frac{3}{2} + \frac{i}{2} + \frac{1}{4}} 2F_1 \left[ a, b, c, \frac{1}{2} - \frac{i}{2} \sinh(x) \right] + \\
+ c_2 \left[ 1 - i \sinh(x) \right]^{-\frac{1}{2}} \left[ 1 + i \sinh(x) \right]^{\frac{5}{4} - \frac{3}{2} + \frac{i}{2} + \frac{1}{4}} 2F_1 \left[ 1 - a, 1 - b, 2 - c, \frac{1}{2} - \frac{i}{2} \sinh(x) \right].
$$

(20)

Here, $c_1, c_2$ are arbitrary constants and $2F_1$ stands for the hypergeometric function $[\Pi]$. Furthermore, the following abbreviations are in use

$$
a = \frac{1}{2} - \frac{1}{4} \sqrt{[-1 + 2 \lambda(E) - 2 i \mu(E)]^2 - \frac{1}{4} \sqrt{[1 + 2 \lambda(E) + 2 i \mu(E)]^2 + k_y^2 - \mu(E)^2}}
$$

$$
b = \frac{1}{2} - \frac{1}{4} \sqrt{[-1 + 2 \lambda(E) - 2 i \mu(E)]^2 - \frac{1}{4} \sqrt{[1 + 2 \lambda(E) + 2 i \mu(E)]^2 - k_y^2 - \mu(E)^2}}
$$

$$
c = 1 - \frac{1}{2} \sqrt{[1 + 2 \lambda(E) + 2 i \mu(E)]^2}.
$$

(21)

These expressions can be simplified further once the sign of the radicands is known. We will discuss this in detail further below. Observe that in (21) we did not include an argument to indicate the dependency of $a$, $b$ and $c$ on the energy $E$. We note that the function $\psi_2$ in (14) can now be obtained from (20) through the relation (16), determining the general solution of our Dirac equation (3) for the potential (18).

### 3.2 Construction of bound states

We will now impose additional conditions on (20) in order to extract bound-state solutions and their corresponding energies. For such solutions, the norm integral (17) must exist and the sign condition (13) must be fulfilled. We will investigate these two aspects separately.

**Existence of the norm integral.** Let us first ensure that the norm (17) exists in the present case. To this end, we observe that our solution (20) becomes in general unbounded at the infinities due to the hypergeometric functions it contains. We modify the latter solution by setting $c_1 = 1$ and $c_2 = 0$, removing its second term on the right side. Next, we recall that the hypergeometric function degenerates to a polynomial if its first argument equals a nonpositive integer. Taking into account that this argument is given by $a$ and defined in (21), we obtain the constraint

$$
\frac{1}{2} - \frac{1}{4} \sqrt{[1 + 2 \lambda(E) + 2 i \mu(E)]^2} - \frac{1}{4} \sqrt{[-1 + 2 \lambda(E) - 2 i \mu(E)]^2 + k_y^2 - \mu(E)^2} = -n,
$$

(22)

for a nonnegative integer $n$. Since the first two complex roots on the left side can take two values each, we can simplify (22) by distinguishing four possible cases, depending on the sign of the two roots. These cases are

$$
\frac{1}{2} - \lambda(E) + \sqrt{k_y^2 - \mu(E)^2} = -n
$$

(23)

$$
\frac{1}{2} + \lambda(E) + \sqrt{k_y^2 - \mu(E)^2} = -n
$$

(24)

$$
1 + i \mu(E) + \sqrt{k_y^2 - \mu(E)^2} = -n
$$

(25)

$$
-\lambda(E) + \sqrt{k_y^2 - \mu(E)^2} = -n.
$$

(26)
Next, let us show that the last two cases can be discarded. To this end, we first assume that the root in (25) and (26) is real-valued. This implies \( \mu = 0 \), such that the energy \( E \) completely disappears from the condition. As a consequence, no stationary energies can be determined. If we assume that the root in (25) is imaginary, then (25) results in \( n = -1 \), which is not a valid assignment due to the restriction that \( n \) must be a nonnegative integer. Finally, if the root in (26) takes imaginary values, we obtain \( n = 0 \) and \(|\mu(E)| = k_y / \sqrt{2}\). While this is in principle acceptable, we will see below that bound states of our system can only be constructed if \( \mu \) is a constant. As such, the energy \( E \) will once more disappear from our condition (26). For these reasons we only retain the conditions (23) and (24) that were obtained by assuming that the two first complex roots on the left side of (22) take the same sign. Now, our condition (22) can be rewritten using (23) and (24) as follows

\[
\frac{1}{2} - \epsilon \lambda(E_n) + \sqrt{k_y^2 - \mu(E_n)^2} = -n \quad \text{and} \quad |k_y| \geq |\mu(E_n)|, \tag{27}
\]

where \( n \) is a nonnegative integer and the new parameter \( \epsilon \) can take either the value positive one or negative one. Since our potential coefficients \( \lambda \) and \( \mu \) depend on the stationary energy, we cannot determine an explicit formula for those energies from (27), unless more information about the coefficients is known. Before we continue, a general remark on the role of the parameter \( k_y \) is in order. We observe that a condition is placed on \( k_y \) in order for (27) to deliver real-valued energies. While this condition on \( k_y \) is relatively mild, there are cases where stronger constraints are imposed. A typical example of such a case is the work [3], where bound-state solutions of the Dirac equations are sought at zero energy. It is found that bound states can be constructed provided \( k_y \) is constrained to attain certain values. This type of constraint does not appear in the present case because we do not set the energy to a fixed value. Let us for now assume that (27) is satisfied. The stationary energies defined in the latter condition belong to bound-state type solutions of (19), given by the functions

\[
\psi_{1,n}(x) = [1 - i \sinh(x)]^{-\frac{1}{2}} \frac{1}{2} \left[ 1 + 2i(\lambda(E) + 2\mu(E)) \right] [1 + i \sinh(x)]^{-\frac{1}{2}} \left[ 1 - 2i(\lambda(E) - \mu(E)) \right] \\
\times \ {}_2F_1 \left[ -n, -n - 2, 1 - \frac{1}{2}, \frac{1}{2} - \frac{1}{2} \sinh(x) \right], \tag{28}
\]

where irrelevant overall factors were omitted. Since the first argument of the hypergeometric function in (28) is a nonpositive integer, we can express it as follows

\[
\psi_{1,n}(x) = [1 - i \sinh(x)]^{-\frac{1}{2}} \left[ 1 + 2i(\lambda(E) + 2\mu(E)) \right] [1 + i \sinh(x)]^{-\frac{1}{2}} \left[ 1 - 2i(\lambda(E) - \mu(E)) \right] \\
\times P_n \left( -\frac{1}{2} \sqrt{1 + 2i(\lambda(E) + 2\mu(E))} \right) \left[ 1 + i \sinh(x) \right]^{-\frac{1}{2} - \frac{1}{2} \lambda(E) - \mu(E)} \left[ 1 + i \sinh(x) \right]^{-\frac{1}{2} + \lambda(E) - \mu(E)}, \tag{29}
\]

Here, the symbol \( P_n \) stands for a Jacobi polynomial of degree \( n \). Before we continue, let us point out that the functions (29) do not lead to bound states of our Dirac equation (3) unless the parameters satisfy certain conditions. In particular, existence and positiveness of our norm (17) is not guaranteed in general. In order to find out more about this, let us now analyze the asymptotic behavior of the solution (29) at the infinities. For the sake of simplicity we first assume that \( n = 0 \), such that the Jacobi polynomial becomes equal to one and only two factors on the right side of (29) remain. The asymptotics of these factors for large values of \(|x|\) is determined by the real parts of their exponents. In particular, at least one of the exponents must have negative real part. In case one of the exponents has positive real part, it must be less than the absolute value of its counterpart. For \( n = 0 \) and \( \epsilon = 1 \), (29) simplifies to

\[
\psi_1(x) = (1 - i \sinh(x))^{-\frac{1}{2} - \frac{1}{2} \lambda(E) + \frac{1}{2} \mu(E)} [1 + i \sinh(x)]^{-\frac{1}{2} + \frac{1}{2} \lambda(E) + \frac{1}{2} \mu(E)}. \tag{30}
\]
We see that the exponents satisfy our requirements if the condition $\lambda(E) > \frac{1}{2}$ is satisfied. Let us now evaluate (29) for $n = 0$ and $\epsilon = -1$. This gives

\[
\psi_1(x) = [1 - i \sinh(x)]^{\frac{1+\lambda(E)}{2}} \frac{\mu(E)}{2} [1 + i \sinh(x)]^{\frac{\lambda(E) - \mu(E)}{2}}.
\]

(31)

This time we see that our requirements on imply $\lambda(E) < -\frac{1}{2}$. If we now drop the assumption of vanishing $n$, we must consider the effect that the Jacobi polynomial in (29) has on our conditions for $\lambda$. To this end, we observe that the Jacobi polynomial depends on the variable $x$ merely through the term $1 - i \sinh(x)$. As such, the degree of the polynomial adds $n$ to the exponent of the first factor on the right side of (30) and (31), respectively. As a direct consequence, our constraint on $\lambda$ becomes for $\epsilon = 1$

\[
\lambda(E) > n + \frac{1}{2}.
\]

(32)

This generalizes a condition found in [8] to the energy-dependent potential (18). If (32) is satisfied, bound-state solutions of the Dirac equation (3) describe the behavior of electrons. Next, if we choose $\epsilon = -1$, we arrive at the condition

\[
\lambda(E) < -n - \frac{1}{2},
\]

(33)

Similar to (32), this is a generalization of a constraint valid for the energy-independent version of (18). If (33) holds, then bound-state solutions of the Dirac equation (3) model the behavior of holes [8]. Hence, if either condition (32) or (33) are satisfied, then the corresponding solution in (29) vanishes at both infinities. Furthermore, the derivative of (29) with respect to $x$ shows the same behavior, implying that the remaining component (16) forming our solution spinor (14) vanishes at both infinities. For the sake of brevity we omit to show this rigorously, as it would require a similar series of considerations as done above for the function (29). It now follows that the density $|\psi_+|^2 + |\psi_-|^2$ also vanishes at the infinities, establishing existence of the norm integral in (17).

**Sign of the norm integral.** It now remains to study the sign of the norm (17) in order to ensure that condition (13) is fulfilled. The expression on the left side of this condition reads after substitution of our potential (18)

\[
1 - \frac{\partial V(x, E)}{\partial E} = \lambda'(E) \text{sech}(x) - \mu'(E) \tanh(x).
\]

(34)

Note that for a fixed value of $E$, this expression stays bounded on the whole real line, such that it cannot affect existence of the integral (17). In order to satisfy condition (13), the right side of (34) must be nonnegative for all $x$ and all admissible values of $E$. This is only possible if the coefficient of the hyperbolic secant is positive and if the hyperbolic tangent term is not present. In other words, we must impose the simultaneous conditions

\[
\lambda'(E) > 0 \quad \text{and} \quad \mu(E) = \text{constant},
\]

(35)

for all values of the energy $E$. In summary, if one of the conditions (32) or (33) is met and if in addition the sign condition (35) is fulfilled, then the functions (29) generate bound-state solutions of the Dirac equation (3) with energy-dependent potential (18) by means of (16) and (14).
The case $\mu = 0$. Before we conclude this section, let us briefly comment on a particular case of our potential (18) that arises if the parameter $\mu$ vanishes. The resulting potential, consisting of a single secant term, satisfies the sign condition (35) and as such allows for the construction of bound-state solutions to our Dirac equation (3). The secant potential is of importance especially in applications of graphene, as it was shown to match the graphene top-gate structure [10] [12]. In the latter references, bound-state solutions of the Dirac equation for an energy-independent secant potential were obtained, within a quasi-exactly solvable setting and at zero energy, respectively. Let us add that zero-energy solutions of the Dirac equation have also been found for different types of potentials, see for example [5]. In the present case of energy-dependence in the potential, bound-state solutions and their associated stationary energies can be obtained directly from (29) and (27), respectively, by setting $\mu = 0$. We will comment on this case below when discussing examples. For small values of $\mu$, the hyperbolic Scarf potential (18) is a deformation of the secant potential, which is an even function. Due to this property, the effective complex potential in the Schrödinger-type equation (19) features $\mathcal{PT}$-symmetry [2).

3.3 Applications

Even though we were able to construct the general form of bound-state solutions through (29), we can only give an implicit equation (27) for the associated stationary energies. This changes once more information is known about the parameter functions $\lambda$ and $\mu$. Therefore, we will now choose particular cases of those functions and apply the results of the previous sections. It will turn out that, depending on the parameter values, the resulting stationary energies form infinite sequences that can be unbounded or have an accumulation point.

3.3.1 Linear energy-dependence

In our first example let us employ the following settings

$$
\lambda(E) = \alpha E \quad \mu(E) = \beta,
$$

(36)

where $\alpha > 0$ and $\beta$ are real-valued constants. We observe that these settings are compatible with the requirement (35), a necessary condition for the construction of bound states. Furthermore, the constant $\beta$ is allowed to vanish, in which case (18) turns into the hyperbolic secant potential. The remaining conditions for norm existence will be verified further below. Upon substitution of (36) into the potential (18), we obtain

$$
V(x,E) = -\alpha E \text{sech}(x) + \beta \tanh(x) + E.
$$

(37)

This potential depends linearly on the energy in its first and third term. Next, let us determine the stationary energies of the system governed by the Dirac equation (3) and the potential (37). These energies can be found from equation (27), where we must first provide a value for $\epsilon$. This value depends on which of the two conditions (32), (33) we want to satisfy. We choose the first of these conditions, implying that $\epsilon = 1$. Upon substitution of this value in combination with (36) into (27), we obtain the following condition

$$
\frac{1}{2} - \alpha E + \sqrt{k_y^2 - \beta^2} = -n, \quad |k_y| \geq \beta.
$$

Solving for $E$ will give an explicit formula for our stationary energies. In order to indicate this, we amend $E$ by an index $n$, arriving at

$$
E_n = \frac{1}{\alpha} \left(n + \frac{1}{2} + \sqrt{k_y^2 - \beta^2}\right), \quad |k_y| \geq \beta.
$$

(38)
Figure 1: The stationary energies (38) for \(0 \leq n \leq 20\). Parameter settings are \(\alpha = 1\), \(k_y = 2\) and \(\beta = 3/2\).

These values are positive and increase linearly with \(n\). Figure 1 shows the lowest stationary energies for a particular setting of our parameters in (36). Next, we must check for which values of the parameters \(\alpha\), \(\beta\) these energies comply with the existence condition (32) of the norm. Note that due to our parameter choice \(\epsilon = 1\), we do not consider the second condition (33). Taking into account the definition of \(\lambda\) in (36), we substitute (38) into (32). This gives

\[
\lambda(E_n) > n + 1/2,
\]

so that after simplification we obtain

\[
\sqrt{k_y^2 - \beta^2} > 0. \quad (39)
\]

This constraint is satisfied since it coincides with our requirement in (38). Since condition (35) is already satisfied, it follows that the stationary energies (38) are associated with solutions that are normalizable in the sense (17). There are infinitely many such bound-state solutions because the constraint (39) is fulfilled for all values of \(n\). Since the closed form of the bound-state solutions is very long, we restrict ourselves to showing only the function \(\psi_1\). To this end, we insert the settings (36) and (38) into (29), arriving at the result

\[
\psi_{1,n}(x) = [1 - i \sinh(x)]^{1/2} \left[1 + \frac{1}{2} \left(1 + n + \sqrt{k_y^2 - \beta^2 + 4i\beta}\right) \sinh(x)\right]^{1/2} \times
\]

\[
P_n\left(-\frac{1}{2}\sqrt{(2 + 2n + 2\sqrt{k_y^2 - \beta^2} + 2i\beta)^2 - 4\left(-2n - 2\sqrt{k_y^2 - \beta^2} + 2i\beta\right)}\right) \sinh(x). \quad (40)
\]

Note that we included the parameter \(n\) as an index in order to indicate the bound-state character. In order to construct the solution to the Dirac equation (3), we calculate \(\psi_{2,n}\) by means of (16). After that, we can calculate the norm of these bound-state solutions by means of (17). Since we know that the norm integral exists, it remains to ensure that it gives a nonnegative result. To this end, we recall that the sign of the norm is determined by expression (34). In the present case, this expression is obtained by substituting (36) and evaluating the derivatives, giving

\[
1 - \frac{\partial V(x, E_n)}{\partial E_n} = \alpha \text{sech}(x). \quad (41)
\]

Since both the constant \(\alpha\) and the hyperbolic secant function are positive, the norm (17) of our bound-state solutions generated by (40) will also be positive. Now that we have found the functions \(\psi_{1,n}\) and \(\psi_{2,n}\), we can determine the components \(\psi_+\) and \(\psi_-\) of the Dirac spinor (14) through addition and subtraction, respectively. Figure 2 shows these components for a particular parameter setting and the first values of \(n\). Note that we normalized the functions shown in the figure, such that \(|\psi_+|^2 = |\psi_-|^2 = 1|\).
3.3.2 Inverse-power energy-dependence

We will now employ a new set of parameter values for our energy-dependent potential (18). Even though we are using the same form of the potential, it will turn out that in this example the sequence of stationary energies does not increase linearly, but converges to zero. We make the following parameter definitions

\[ \lambda(E) = -\frac{\alpha}{E}, \quad \mu(E) = \beta, \]  

(42)

where the \( \alpha > 0 \) and \( \beta \) are real constants. These settings comply with the condition (13) that ensures the integrand in the norm (17) to be a nonnegative function. Let us add that \( \beta \) can be zero, such that this example includes the case of a hyperbolic secant potential. Next, upon substitution of the parameters (42) into our potential (18) we obtain

\[ V(x,E) = \frac{\alpha}{E} \text{sech}(x) + \beta \tanh(x) + E, \]  

(43)

In contrast to its counterpart (37) from the previous example, this potential has inverse energy dependence in its first term. We will now construct the stationary energies supported by the Dirac equation (3) with potential (43). To this end, we must solve equation (27) with respect to \( E \), where we again choose the parameter value \( \epsilon = 1 \). After incorporation of the settings (42) we get the condition

\[ \frac{1}{2} + \frac{\alpha}{E} + \sqrt{k_y^2 - \beta^2} = -n, \quad |k_y| \geq \beta. \]

We now obtain our stationary energies by solving for \( E \). Upon renaming \( E = E_n \) we arrive at

\[ E_n = -\frac{2 \alpha}{2n + 1 + 2 \sqrt{k_y^2 - \beta^2}}, \quad |k_y| \geq \beta. \]  

(44)

These energy values are negative and increase monotonically with \( n \), as shown in figure 3. Next we need to find out how many stationary energies are provided by (44), let us verify that our parameter \( \lambda \) satisfies the condition (33), guaranteeing existence of the norm (17). We substitute the settings (42) and (44) into (33), arriving at

\[ \sqrt{k_y^2 - \beta^2} > 0. \]
Since we are assuming that $|k_y| \geq \beta$, this condition is fulfilled for all values of our parameters. Therefore we have an infinite numbers of stationary energies \((44)\) that accumulate at $E = 0$.

In addition, these energies belong to solutions that are normalizable in the sense of our norm \((17)\). These solutions can be constructed from the function $\psi_1$ that is defined in \((29)\). Upon substitution of our current parameter setting \((42)\) and the stationary energies \((44)\) we obtain its explicit form

$$\psi_{1,n}(x) = [1 - i \sinh(x)]^{-\frac{1}{4}} [1 + 2n + 2\sqrt{k_y^2 - \beta^2} - 2i\beta] [1 + i \sinh(x)]^{\frac{1}{4}} [1 + 2n - 2\sqrt{k_y^2 - \beta^2} + 2i\beta] \times$$

$$\times P_n \left( \pm \frac{1}{2} \sqrt{2 + 2n + 2\sqrt{k_y^2 - \beta^2} + 2i\beta}, -\frac{1}{2} \sqrt{2 + 2n - 2\sqrt{k_y^2 - \beta^2} + 2i\beta} \right) [i \sinh(x)]. \quad (45)$$

where we included an index $n$ to emphasize the bound-state character. After calculating the function $\psi_{2,n}$ from \((45)\), we can determine the solution spinor \((14)\) of our Dirac equation \((3)\) by calculating the functions $\psi_+$ and $\psi_-$. Since the explicit expressions of these functions are too long to be shown here, we restrict ourselves to present their graphs for a particular parameter setting, see figure 4. Recall that we do not need to verify normalizability according to \((17)\), as this is guaranteed by \((33)\) and the fact that $\mu$ is independent of the energy.
4 Concluding remarks

In this work we have demonstrated how to construct bound-state solutions of the massless Dirac equation for an energy-dependent potential. Our approach of decoupling the Dirac equation relies on the potential depending on only one of the spatial variables. If this condition is fulfilled, bound states for energy-dependent potentials different from (18) can be constructed, provided the Schrödinger-type equation (15) renders exactly-solvable. In addition, the sign condition (13) for the modified norm must be verified. In most cases, this condition will either give restrictions on the parameters of the potential or dictate that the problem’s domain must be restricted.

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