On the Running Gauge Coupling Constant in the Exact Renormalization Group

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Abstract

In this paper, we investigate the beta-function of the gauge coupling constant \( e \) of the gauged four-fermi theory in the Exact Renormalization Group (ERG) framework. It seems that the presence of the four-fermi interaction strongly affects to the naive RG running of the gauge coupling constant. We show that this strong correction has no physical meaning since the vertex \( \bar{\psi}A\psi \) involves the contribution from a mixing as well as a pure gauge interaction due to the (anomalous) mixing among the photon and the vector composite field. By introducing the auxiliary field for the vector composite field, the situation turns to be rather clear. We adopt the counterterm to cancel the gauge non-invariant correction, and decompose the pure gauge interaction from the contribution of the mixing. We find the beta-function of the gauge coupling constant in the large \( N \) limit.

1 Introduction

The Exact Renormalization Group (ERG) \([1, 2]\) is a powerful tool not only in the statistical physics but also in the particle physics i.e. the dynamical chiral symmetry breaking (D\( \chi \)SB) in the strong coupled gauge theory \([3, 4, 5]\). The ERG enable us to improve the ladder and/or the improved ladder (Higasigima) approximation \([6]\). In the ERG, one can easily incorporate the corrections from the non-ladder diagrams. In Ref. \([4]\), we have applied the ERG method to the chiral critical behavior in QED with the standing (constant) gauge coupling approximation. We have also seen that the naive beta function of gauge coupling constant brings about the non-trivial ultra-violet stable fixed point. It shows the sharp contrast to the Gelmann-Low’s RG beta-function of the gauge coupling constant which is positive semi-definite \( \beta \geq 0 \) and has no ultra-violet (stable) fixed point \([7]\). The additional fixed point in the ERG appears due to the breaking of the Ward-Takahasi identity, i.e. \( Z_1 = Z_2 \). In the gauge invariant calculation, the running of the gauge coupling constant \( e \) is governed only by the photon’s wave function renormalization, \( Z_3 \). However, in the ERG approach, the breaking of the Ward-Takahashi identity \( Z_1 \neq Z_2 \) also contributes to the beta-function of the gauge coupling \( e \) and is enough

\[^{1}\text{A Wilsonian RG }\beta\text{ function and a Gelmann-Low’s one have opposite sign.}\]
large to change the qualitative feature of the continuum limit. Needless to say, we should carefully discuss this result.

The $D\chi_{SB}$ in QCD was also investigated in Ref. [5] partly by using the ERG. In those papers, however, the RG flow of the gauge coupling constant was that in the one-loop perturbation, not in the ERG. If one attempts to solve the $D\chi_{SB}$ only by the ERG, then one will encounter the problem due to the strong correction from the four-fermi interaction, which is inconsistent with the gauge symmetry to the RG beta-function of the gauge coupling constant. It is an obstacle to apply the ERG to solve $D\chi_{SB}$.

The ERG is the continuous version of the block spin transformation and one of the framework to perform the path-integrals. There are three formulation of the ERG, the Wegner-Houghton equation, the Polichinski equation and the evolution equation [1, 2]. They are the functional differential equations for the Wilsonian effective action and/or the Legendre effective action with an infra-red cutoff $\Lambda$. The later is the one particle irreducible part of the Wilsonian effective action. In this paper, we employ the cutoff Legendre effective action $\Gamma_{\Lambda}[\Phi]$ [2] since the Wilsonian effective action strongly depends on the cutoff scheme [8, 9].

The infra-red cutoff is introduced as:

$$S_{\text{cut}}[A_{\mu}, \psi, \bar{\psi}] = \int d^4x \left( \frac{\Lambda^2}{2} Z_3 A_{\mu} C^{-1} (-\partial^2/\Lambda^2) A_{\mu} + Z_2 \bar{\psi} C^{-1} (-\partial^2/\Lambda^2) i\partial \psi \right), \quad (1)$$

where $Z_2$ and $Z_3$ are the wave-function renormalization of the fermion and the photon respectively. The cutoff action $S_{\text{cut}}$ preserves the chiral symmetry; $\psi \rightarrow e^{i\gamma_5} \psi$, and therefore $\Gamma_{\Lambda}[\Phi]$ also respect it. In this paper we do not specify the cutoff functions $C(x), C_{\psi}(x)$.

As well known, a momentum cutoff which cannot be avoided to formulate the ERG, is not consistent with the gauge symmetry. Indeed, Eq. (1) conflicts with the gauge symmetry. Due to the renormalizability problem, the derivatives in $C_{\psi}$ cannot be replaced to the covariant ones $D_\mu = \partial_\mu - ieA_{\mu}$. Thus to compensate the gauge invariance of the total solution, one has to introduce. The gauge non-invariant operators as the counterterms. theory space has to be enlarged to gauge non-invariant dimensions and next it should be restricted to the subspace maintaining the gauge symmetry of the total solution of the ERG. This process is tedious in general and demands more both human efforts and the computer resource.

Recently the several attempts to construct the Wilsonian exact renormalization group consistent with the gauge symmetry are reported [11]. If one can construct it then the above problem is completely avoided. However, their formulations are not accompanied with the non-perturbative approximation method and/or the recipe for extracting the physical information from ‘Wilsonian’ effective action. Hence we must chose either the gauge invariance or the non-perturbative approximation method/the above recipe. Hence for the practical and the non-perturbative analyses, it is necessary to solve the gauge non-invariant counterterms e.g. the photon mass term etc..

The effective action $\Gamma_{\Lambda}[\phi]$ satisfies the certain identity at non-vanishing $\Lambda$ instead of the ordinary Slavnov-Taylor Identity (STI), or the Ward-Takahashi identity. This identity is called the ‘Modified Slavnov-Taylor Identity’ (MSTI) [11]. The MSTI reduces to the STI in the infra-red limit; $\Lambda \rightarrow 0$ and ensures the gauge invariance of the total solutions.
of the ERG. The sub-space consistent with the MSTI can be regard as the theory space of the gauge theory, i.e. the Gauge Invariant Theory Space (GITS). It is in principle also possible to find the counterterms by the fine tuning the initial conditions to make the solution satisfy the STI. The result should coincide with the solution of the MSTI.

2 Exact renormalization group equation

Let us start from the ERG equation for the cutoff Legendre effective action $\Gamma_{\Lambda}[\Phi]$, 
\[ \Lambda \frac{\partial}{\partial \Lambda} \Gamma_{\Lambda}[\Phi] = \text{Str} \left\{ \left( \frac{q^2}{C} \frac{\partial C}{\partial q^2} + 1 \right) \mathbf{1} - \eta \right\} \cdot \left( 1 + C \cdot \frac{\delta}{\delta \Phi^T} \frac{\partial \Phi^T}{\partial \Phi} \right)^{-1}, \] (2)

where we use the condensed notation of the fields $\Phi^T = (A_\mu, \bar{\psi}, \psi^T)$ and of the anomalous dimensions $\eta = \text{diag}(\eta_A, \eta_\psi, \eta_{\bar{\psi}})$. One can easily generalize these to $N$ flavor case. The super trace $\text{Str}$ involves both that of the Lorentz indices and the integral over the space-time coordinates. The matrix $C$ is a following block diagonal matrix,
\[
C^{-1}(q) \equiv \begin{pmatrix}
Z_3 \cdot \Lambda^2 \cdot C^{-1} \cdot \delta_{\mu\nu} & 0 \\
0 & Z_2 \cdot (\Lambda/q)^2 C^{-1} \cdot \delta^T
\end{pmatrix},
\] (3)

where $Z_3$ and $Z_2$ are the wave-function renormalization factors of the photon and of the fermion respectively. Now we choose the cutoff function $C(x)$ as the power like cutoff $C(x) = x^k : k = 1, 2, \cdots [12]$. The anomalous dimensions above are given by,
\[ 2\eta_A = -\Lambda \frac{\partial}{\partial \Lambda} \ln Z_3, \quad 2\eta_{\bar{\psi}} = -\Lambda \frac{\partial}{\partial \Lambda} \ln Z_2. \] (4)

Next, we write all the dimensionful quantities in terms of the infra-red cutoff $\Lambda$, i.e. $\Phi = \Lambda^d \hat{\Phi}$, $p = \Lambda \hat{p}$ and $\mathcal{L}_\Lambda = \Lambda^d \hat{\mathcal{L}}_t$ where $d_\phi$, $t = \ln \Lambda_0/\Lambda$ and $\hat{\mathcal{L}}_t$ are the canonical dimension of the field; $d_\phi = (d - 2)/2$, the cutoff scale factor and the action density respectively. We also write the dimensionless cutoff Legendre effective action $\hat{\Gamma}_t[\hat{\phi}] = \int d^d x \hat{\mathcal{L}}_t(\hat{\phi})$. Then, we have,
\[
\Lambda \frac{\partial}{\partial \Lambda} \Gamma_{\Lambda} = -\Lambda^d \left( \frac{\partial}{\partial t} + d_\phi \Delta_\phi + \Delta_\theta - d \right) \hat{\Gamma}_t,
\] (5)

where $\Delta_\phi$ and $\Delta_\theta$ count the degree of the field and that of the derivative $\partial_\mu$ respectively. Since the cutoff function $C$ preserves the chiral symmetry, the effective action $\hat{\Gamma}_t[\hat{\phi}]$ also respects it.

The initial boundary condition of the ERG flow equation is given by
\[ \Gamma_{\Lambda=\infty}[\Phi] = S_{\text{bare}}[\Phi], \] (6)

and at $\Lambda = 0$ the cutoff Legendre effective action $\Gamma_{\Lambda}[\Phi]$ coincide with the ordinary effective action; $\Gamma_{\Lambda=0}[\Phi] = \Gamma[\Phi]$. 

3
3 Structure of RG beta-functions in large N limit

Now let us consider $N$-flavor massless QED with the four-fermi operators. The fermionic field above is understood as $N$ Dirac fields, i.e. $\psi \rightarrow \psi_i \ (i = 1, \cdots, N)$. After rewriting the dimensionful parameters by the unit $\Lambda$, we write the initial effective action $\Gamma$ as,

$$\Gamma_0[\Phi] = \int d^d x \left\{ \frac{1}{4} Z_3 F_{\mu \nu}^2 + \frac{1}{2\alpha} (\partial_\mu A_\mu)^2 + \frac{1}{2} m^2 A_\mu^2 + Z_2 \bar{\psi} i\gamma^\mu \psi + e \bar{\psi} A_\mu \psi - \frac{1}{2} G_\mu (\bar{\psi} \gamma_\mu \psi)^2 \right\}, \quad (7)$$

where $m$ is the photon mass counterterm to cancel the gauge non-invariant correction to the photon mass. The gauge invariance requires the additional renormalization condition $m^2 = 0$ at $\Lambda = 0$ or the Modified Slavnov-Taylor Identity (MSTI) \([\square]\) at the finite cutoff $\Lambda \neq 0$.

Let us consider the large $N$ limit, i.e.

$$e^2 \rightarrow e^2/N, \quad G_\nu \rightarrow G_\nu/N, \quad N \rightarrow \infty. \quad (8)$$

In this limit, RG flow of the four-fermi operator closes into the functional space $\{G^{\mu \nu}(P)\}$ as,

$$- \frac{1}{2N} \int \frac{d^4 P}{(2\pi)^4} (\bar{\psi} \gamma_\mu \psi)(P)G^{\mu \nu}(P)(\bar{\psi} \gamma_\mu \psi)(-P), \quad (9)$$

where $(\bar{\psi} \gamma_\mu \psi)(P)$ represents a Fourier transform of the local composite operator $\bar{\psi}(x)\gamma_\mu \psi(x)$. If we start from the action \([\square]\), the multi-fermi operators should take the form,

$$\frac{1}{2N^{n-1}} \int \prod_{i=1}^n \left( \frac{d^4 P_i}{(2\pi)^4} (\bar{\psi} \gamma_\mu \psi)(P_i) \right) G^{\mu_1 \cdots \mu_n}(P_1, \cdots, P_{n-1})(2\pi)^4\delta^4(\sum_i P_i), \quad (10)$$

and another multi-fermi operator like $\bar{\psi} i\gamma^\mu (\bar{\psi} \gamma_\mu \psi)^2$ cannot appear. The RG beta-function for the momentum dependent four-fermi operator can be read,

$$\frac{\partial}{\partial t} G^{\mu \nu}(P) = -2G^{\mu \nu}(P) - 2P^2 \frac{\partial}{\partial P^2} G^{\mu \nu}(P) + G^{\mu \rho}(P)I_{\rho \sigma}(P)G^{\sigma \nu}(P), \quad (11)$$

where $I_{\rho \sigma}(P)$ is a cutoff scheme dependent function given by,

$$I_{\mu \nu}(P) = 4(k + 1 - \eta_\psi) \int \frac{d^4 q}{(2\pi)^4} C(q)S^2(q)C(P - q)S(P - q) tr[\gamma_\mu (P - q) \gamma_\nu q], \quad (12)$$

and called threshold function. In the large $N$ limit we have $\eta_\psi = 0$. The function $S(q)$ corresponding to fermion’s propagator is given by

$$S(q) = 1/(1 + q^2C(q^2)). \quad (13)$$

We can decompose $G^{\mu \nu}(P)$ and $I_{\rho \sigma}(P)$ into two parts, the transverse part and the longitudinal one, i.e.

$$G^{\mu \nu}(P) = G_T(P) \left( g^{\mu \nu} - \frac{P^\mu P^\nu}{P^2} \right) + G_L(P) \frac{P^\mu P^\nu}{P^2}, \quad (14)$$

$$I_{\mu \nu}(P) = I_T(P) \left( g_{\mu \nu} - \frac{P_\mu P_\nu}{P^2} \right) + I_L(P) \frac{P_\mu P_\nu}{P^2} \quad (15)$$
Then the RG flow equations of two parts of the four-fermi operator decouple each other and we will find,

$$\frac{\partial}{\partial t} G_{T,L}(P) = -2G_{T,L}(P) - 2P^2 \frac{\partial}{\partial P^2} G_{T,L}(P) + I_{T,L}(P) \cdot [G_{T,L}(P)]^2.$$  \hspace{1cm} (16)

The higher operators do not contribute to the RG flow of $G_{T,L}(P)$ by a lack of the six-fermi operator in the large $N$ limit.

Next, let us consider the gauge sector. In large $N$ limit, $\bar{\psi} A \psi$ interaction has the form

$$\int \frac{d^4P}{(2\pi)^4} A_\mu(P) \left[ \Gamma_T(P) \left( g^{\mu\nu} - \frac{P^\mu P^\nu}{P^2} \right) + \Gamma_L(P) \frac{P^\mu P^\nu}{P^2} \right] (\bar{\psi} \gamma_\nu \psi)(-P).$$  \hspace{1cm} (17)

The photon two point function is also decomposed into two parts, the transverse part and the longitudinal part,

$$\frac{1}{2} \int \frac{d^4P}{(2\pi)^4} A_\mu(P) \left[ \Pi_T(P) \left( g^{\mu\nu} - \frac{P^\mu P^\nu}{P^2} \right) + \Pi_L(P) \frac{P^\mu P^\nu}{P^2} \right] A_\nu(-P).$$  \hspace{1cm} (18)

The above functions $\Pi_{T,L}(P)$ and $\Gamma_{T,L}(P)$ have the following expansions,

$$\begin{align*}
\begin{cases} 
\Pi_T(P) = m^2 + Z_3 P^2 + \cdots, \\
\Pi_L(P) = m^2 + \alpha^{-1} P^2 + \cdots, \\
\Gamma_T(P) = e/\sqrt{N} + \cdots, \\
\Gamma_L(P) = e/\sqrt{N} + \cdots.
\end{cases}
\end{align*}$$  \hspace{1cm} (19)

Here, the longitudinal parts $\Pi_L(P)$ and $\Gamma_L(P) - e_0/\sqrt{N}$ break the gauge symmetry, where $e_0$ is a bare gauge coupling. Hence, they should vanish at $\Lambda = 0$. The quasi-locality of the threshold functions and of the bare action requires $\Pi_T(0) = \Pi_L(0)$ and $\Gamma_T(0) = \Gamma_L(0)$. Therefore the transverse part of the vacuum polarization at $P^2 = 0$ should vanish at $\Lambda = 0$, i.e. $\Pi_T(P = 0) = m^2 = 0$. The RG flow equations for $\Pi_{T,L}(P)$ and $\Gamma_{T,L}(P)$ can be found,

$$\frac{\partial}{\partial t} \Pi_{T,L}(P) = \left( 2 - 2P^2 \frac{\partial}{\partial P^2} \right) \Pi_{T,L}(P) + I_{T,L}(P) \cdot [\Gamma_{T,L}(P)]^2,$$  \hspace{1cm} (20)

$$\frac{\partial}{\partial t} \Gamma_{T,L}(P) = -2P^2 \frac{\partial}{\partial P^2} \Gamma_{T,L}(P) + I_{T,L}(P) \cdot \Gamma_{T,L}(P) \cdot G_{T,L}(P),$$  \hspace{1cm} (21)

where for convenience, we deal with the vertices of the bare photon $A_\mu$ instead of that of the renormalized photon $\hat{A}_\mu = \sqrt{Z_3} A_\mu$.

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2 The ERG flow equation involves at most the second functional derivative, no higher derivative with respect to the field. Therefore the eight-fermi operator does not contribute to the RG flow of the four-fermi operators.

3 By a virtue of the chiral symmetry, the Pauli term $F^{\mu\nu} \bar{\psi} \sigma_{\mu\nu} \psi$ does not appear, therefore the anomalous magnetic moment of the ‘electron’ vanishes.

4 In another words, $I_{T,L}(P)$ are analytical at $P = 0$. 

5
Now, let us consider the solutions of the RG flow equations \((14)\). It is more convenient to introduce the inverse of \(G_{T,L}(P)\) i.e. \(M_{T,L}(P) \equiv [G_{T,L}(P)]^{-1}\). Multiplying \(-[M_{T,L}(P)]^2\) to both side of Eq. \((14)\), we find linear partial differential equations,

\[
\frac{\partial}{\partial t} M_{T,L}(P) = 2M_{T,L}(P) - 2P^2 \frac{\partial}{\partial P^2} M_{T,L}(P) - I_{T,L}(P).
\]

The initial boundary condition is given by \(M_{T,L}(P) = 1/G_V\) at \(t = 0\). One may easily find

\[
\tilde{M}_{T,L}(\tilde{P}; \Lambda(t)) = \frac{1}{G_v} - \int_0^t dt' e^{-2t'} I_{T,L}(e^{2t'} \tilde{P}^2),
\]

where \(\tilde{M}_{T,L} = e^{-2t}M_{T,L}\) and \(\tilde{P} = e^{-t}P\) are the inverse of the dimensionful four fermi vertex and the dimensionful momenta respectively. Since the last term of Eq. \((23)\) corresponds to the fermionic bubble diagram, the inverse of \(\tilde{M}_{T,L}\) gives the famous chain sums.

Especially for \(\tilde{P} = 0\), we find,

\[
G_{R,T,L} = G_V/(1 - I(P = 0)\Lambda_0^2 G_V).
\]

If we take \(G_v \to 1/I_{T,L}(P = 0)\Lambda_0^2\), then \(G_{R,T,L}\) blows up to infinity or equivalently \(\tilde{M}_{T,L}(\tilde{P} = 0; \Lambda = 0)\) vanishes. Consequently, the four-fermi vertex acquires a massless pole in a vector channel\[^5\]. In the strong coupling region \(G_v > 1/I_{T,L}(P = 0)\Lambda_0^2 > 0\), the true vacua breaks the Lorentz symmetry i.e. \(\langle \bar{\psi}\gamma_{\mu}\psi \rangle \neq 0\), since \(\tilde{M}_{T,L}(0; \Lambda(t))\) correspond to the mass terms of the transverse and longitudinal modes of the vector composite field and they turn to a negative value in the strong coupling region.

In our RG equation, when the vector four-fermi operator acquires a pole structure above, then \(\Pi_{T,L}(P)\) should have a same pole structure due to the last term of Eq. \((21)\). The longitudinal part of it conflicts to \(\sqrt{N} \Gamma_{T,L}(P) = e, \quad \sqrt{N} \Gamma_{T,L}(P) = e\), \(N^{-1}\Pi_T(P) = Z_3 P^2 + m^2, \quad N^{-1}\Pi_T(P) = m^2\), \(\Pi_{T,L}(P) = \Pi_L(0)\) and \(\Gamma_{T,L}(0) = \Gamma_{L}(0)\), the constant part of transversal part \(\Pi_T(P = 0)\) should also vanish at \(\Lambda = 0\).

### 4 Solution in derivative expansion

In this section, we would like to explore the solutions in the derivative expansion. Let us restrict the sub-theory space to the following one.

\[
NG_T(P) = G_2 P^2 + G_V, \quad NG_L(P) = G_V, \quad \sqrt{N} \Gamma_T(P) = \Gamma_2 P^2 + e, \quad \sqrt{N} \Gamma_L(P) = e, \quad N^{-1}\Pi_T(P) = Z_3 P^2 + m^2, \quad N^{-1}\Pi_T(P) = m^2,
\]

where the \(O(\partial^0)\) parts of the longitudinal and the transversal vertices should coincide due to quasi-locality of the threshold functions and the bare action. Inserting these and the \(^5\)Nota bene, only a massless composite particle whose binding energy is zero can be stable since the fermi fields remain massless due to the chiral symmetry.
truncated threshold functions; \( I_T(P) = -P^2/6\pi^2 + 4\Omega, \ I_L(P) = 4\Omega \), we have

\[
\begin{align*}
\frac{\partial}{\partial t} e &= 4\Omega eG, \tag{28} \\
\frac{\partial}{\partial t} G_v &= -2G_v + 4\Omega G_v^2, \tag{29} \\
\frac{\partial}{\partial t} G_2 &= -4G_2 + 8\Omega G_v G_2 - \frac{1}{6\pi^2} G_v^2, \tag{30} \\
\frac{\partial}{\partial t} \Gamma_2 &= -2\Gamma_2 + 4\Omega (eG_2 + \Gamma_2 G_v) - \frac{1}{6\pi^2} eG_v, \tag{31} \\
\frac{\partial}{\partial t} Z_3 &= \frac{1}{6\pi^2} e^2 - 8\Omega e\Gamma_2. \tag{32}
\end{align*}
\]

Note that, in the large \( N \) limit the RG flow equations (28)-(32) forms a closed system in the total theory space, and therefore they describe exact results.

The solution of Eqs. (28)-(32) can be found as follows. First, integrating Eq. (28), we have

\[
G_v = \frac{G_v(0)}{2\Omega G_v(0) + e^2 (1 - 2\Omega G_v(0))}, \tag{33}
\]

where \( G_v(0) \) is a bare four-fermi coupling constant. For another coupling constants we have

\[
\begin{align*}
e &= \frac{e_0}{1 - 2\Omega G_v(0)} (1 - 2\Omega G_v), \tag{34} \\
G_2 &= -\frac{t}{6\pi^2} G_v^2, \tag{35} \\
\Gamma_2 &= \frac{eG_2}{G_v}, \tag{36} \\
Z_3 &= 1 + \frac{t}{6\pi^2} e^2, \tag{37}
\end{align*}
\]

where we choose the initial boundary condition as; \( G^{(2)}(0) = \Gamma^{(2)}(0) = 0, \ e(0) = e_0 \) and \( Z_3(0) = 1 \).

Using these results, we can realized that the renormalized gauge coupling constant \( \hat{e}^2 \equiv e^2/Z_3 \) satisfies

\[
\frac{\partial}{\partial t} \hat{e}^2 = -\hat{e}^4 + 8\Omega \left(1 - \frac{t}{6\pi^2} \hat{e}^2\right) \hat{e}^2 G_v. \tag{38}
\]

The ‘beta-function’ of \( \hat{e}^2 \) has the positive region and the sign of the ‘beta-function’ changes at

\[
\hat{e}^2 = 8\Omega (6\pi^2 - e^2 t) G_v. \tag{39}
\]

The last term of Eq. (38) still breaks th WT identity and makes a dominant effect to the running of the ‘naive’ gauge coupling constant in both the ultra-violet and the infra-red regions.

\footnote{The approximation corresponding to that of Ref. \[4\] will be found by setting \( G_2 = \Gamma_2 = 0 \).}

\footnote{Here the RG flow of the renormalized gauge coupling constant \( \hat{e} \) depends on the cutoff scale parameter \( t \) explicitly, since the shadow of the RG flow on the \( \hat{e} - G_v \) plane does not draw the unique flow on the \( \hat{e} - G_v \) plane although the ERG flow on the full theory space; \{\( \hat{e}, G_v, G_2, \Gamma_2 \)\} does.}
5 Introducing the Auxiliary field

Let us introduce the auxiliary field \( V_\mu \) and give the counterterm for the mixing among \( A_\mu \) and \( V_\mu \) by the following Gaussian integral,

\[
\mathcal{N} = \int DV_\nu \exp \left\{ - \int d^4x \frac{1}{2} M_V^2 \left( V_\mu + \frac{1}{\sqrt{N} M_V^2} \bar{\psi}_i \gamma_\mu \psi_i + \theta A_\mu \right)^2 \right\},
\]

where \( \theta \) is a certain constant determined by the gauge invariance. In the language of CJT effective action [13], it is equivalent to introduce the composite source \( \Sigma_\mu \) for the vector composite field as: \( \Sigma_\mu \left( N^{-1/2} M_V^2 \bar{\psi}_i \gamma_\mu \psi_i + \theta A_\mu \right) \) except a physically non-important source quadratic term. The effective Lagrangian density \( L \) at the ultra-violet cutoff becomes,

\[
L = \frac{1}{4} \bar{Z}_3 F_{\mu\nu}^2 + \frac{1}{2} M_A^2 A^2 + \frac{\bar{e}}{\sqrt{N}} \bar{\psi}_i A \psi_i + \frac{1}{2} M_V^2 V^2 + M_{\text{mixing}}^2 A \cdot V + \frac{1}{\sqrt{N}} \bar{\psi}_i V \psi_i + \cdots,
\]

where we introduce new variables, \( \bar{Z}_3, M_A^2 \) and \( \bar{e} \) to distinguish from those in the previous section. Now we hold the ambiguity of the initial values of \( M_V^2 \) and \( y \) as \( G_V = 1/M_V^2 \) and \( y = 1 \) at the ultra-violet cutoff \( \Lambda_0 \).

For the leading order in the derivative expansion, we have the RG equations;

\[
\frac{\partial}{\partial t} \bar{e} = \frac{\partial}{\partial t} y = 0,
\]

\[
\frac{\partial}{\partial t} M_A^2 = 2 M_A^2 - 4 \bar{e}^2 \Omega,
\]

\[
\frac{\partial}{\partial t} M_V^2 = 2 M_V^2 - 4 \Omega,
\]

\[
\frac{\partial}{\partial t} M_{\text{mixing}}^2 = 2 M_{\text{mixing}}^2 - 4 \bar{e} \Omega.
\]

Here, the RG flow equations of \( O(\partial^0) \) coupling constants are not affected by the higher derivative couplings. By the condition \( G_V = 1/M_V^2 \), the four-fermi coupling is not generated, since once \( G_V \) vanishes then the beta-function of \( G_V \) is also vanish for each scale.\(^{8}\)

Hence at all scale we have \( G_V = 0 \) and the beta-function of \( e \) does not have the anomalous region. Therefore the physical running of the gauge coupling is governed only by the wave-function renormalization of the gauge field.

To see the RG running of the physical gauge coupling, we must calculate the wave-function renormalizations. Let us write the kinetic terms of the gauge field and composite vector field as:

\[
L \sim \frac{1}{4} \bar{Z}_3 F_{\mu\nu} F_{\mu\nu} + \frac{1}{2} Z_M F_{\mu\nu} G_{\mu\nu} + \frac{1}{4} Z_V G_{\mu\nu} G_{\mu\nu} + \cdots,
\]

where \( G_{\mu\nu} \) is a field strength of the vector composite field \( V_\mu \), i.e. \( G_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu \).

The gauge invariant kinetic term mixing \( Z_M \) does not affect the RG flow of the gauge coupling and the large-\( N \) limit. \(^{8}\)The RG flow of \( G_V \) is also give by Eq. (29).
coupling since the rotation diagonalizing the kinetic terms, should take a form by the
gauge transformation low of the gauge field,
\[
\begin{pmatrix}
A_\mu \\
V_\mu
\end{pmatrix}
= \begin{pmatrix}
1 & \delta_1 \\
0 & \delta_2
\end{pmatrix}
\begin{pmatrix}
\tilde{A}_\mu \\
\tilde{V}_\mu
\end{pmatrix},
\]
(47)
which preserves the Z-factor of the gauge field. We find the RG equation for \( Z_3 \) as,
\[
\frac{\partial}{\partial t} \tilde{Z}_3 = \frac{\bar{e}^2}{6\pi^2},
\]
(48)
and the solution \( \tilde{Z}_3 = t\bar{e}^2/6\pi^2 \).

Next, the counterterm \( M_{mixing}^2(\bar{e}) \) should be constrained by
\[
\left( \frac{\partial}{\partial \bar{e}} \right)\left( \frac{\partial}{\partial \bar{M}_{mixing}} \right) M_{mixing}^2(\bar{e}) = \frac{\partial}{\partial t} M_{mixing}^2,
\]
(49)
with a boundary condition \( M_{mixing}^2 \to 0 \) as \( e_R \equiv \bar{e}/\sqrt{\bar{Z}_3} \to 0 \). One can easily get \( M_{mixing}^2 = 2\bar{e}\Omega \). In the same manner, we also find \( \tilde{M}_A^2 = 2\bar{e}^2\Omega \) for the photon mass counterterm.

The relations between the original coupling constants i.e. before introducing the auxiliary field \( \Gamma_t[A_\mu, \bar{\psi}, \psi] \) and those after introducing the auxiliary field \( \Gamma_t[A_\mu, \bar{\psi}, \psi, V_\mu] \) can be easily found. First for the gauge coupling constant \( e \) we have,
\[
e = \bar{e} - \left[ \frac{M_{mixing}^2}{M_V^2} \right] = \bar{e}(1 - 2\Omega G_V). \]
(50)
The mappings among another parameters e.g. the wave-function renormalizations also can be found. For each scale \( t \), the auxiliary field \( V_\mu \) can be integrated out since the loop corrections of \( V_\mu \) are dropped in the large \( N \) limit. Taking account of the tree diagrams we have the relations;
\[
\begin{align*}
Z_3 &= \tilde{Z}_3 + \left[ \frac{M_{mixing}^4}{M_V^4} Z_V - 2 \frac{M_{mixing}^2}{M_V^2} Z_M \right], \\
m^2 &= \tilde{M}_A^2 - \left[ \frac{M_{mixing}^4}{M_V^4} \right], \\
G_2 &= -\frac{Z_V}{M_V^2}, \\
\Gamma_2 &= \left[ \frac{M_{mixing}^2 Z_V}{M_V^2} \right] - \frac{Z_M}{M_V}. \end{align*}
\]
(51)-(54)
One can find the solutions (34)-(37) by using \( \tilde{Z}_3/\bar{e}^2 = Z_M/\bar{e} = Z_V = t/6\pi^2 \), \( G_V = 1/M_V^2 \), \( M_{mixing}^2 = 2\bar{e}\Omega \) and \( e_0 = \bar{e}(1 - 2\Omega G_V(0)) \). The terms in the square bracket are the contributions from the gauge non-invariant corrections, since these proposal to \( M_{mixing}^2 \).

\footnote{It is similar to the coupling reduction \cite{14}.}
In the language of the ordinary perturbation theory, $M^2_{\text{mixing}}$ corresponds to a quadratically divergent one-loop contribution like a photon’s mass correction. As well-known, such a correction breaks the WT identity. The gauge invariant vertices also suffer from the such correction through quadratically diverging renormalization parts, i.e. for example, the terms in the square bracket in Eqs. (51), (54) and (54). The anomalous running of $e$ in Eq. (38) is essentially a result form these corrections. By introducing the auxiliary field $V_\mu$, we can decompose these contributions from the gauge invariant contributions i.e. $\bar{e}, \bar{Z}_3$ etc..

Hence the strong correction of the RG flow of the gauge coupling constant from the $eG_V$ term is physically meaningless. It is due to the fact that the $A_\mu \bar{\psi} \gamma_\mu \psi$ vertex is not a purely gauge interaction but including a mixing among the gauge field and the vector composite field and that the wave-function renormalization factor $Z_3$ also suffers from a mixing.

In the large $N$ limit, we introduced the auxiliary field for the vector composite field $V_\mu \sim \bar{\psi} \gamma_\mu \psi$. Then we could easily distinguish the pure gauge interaction from the mixing among the gauge field and the vector composite field. After resolving a mixing, we found the correct RG running of the gauge coupling constant $e$.

The MSTI leads to the relation corresponding to Eq. (50) but another relations, for example Eq. (51), since they are the gauge invariant vertices. The MSTI tells the informations for the counterterms compensating the gauge invariance but the recipes to distinguish the contribution from the gauge non-invariant sub-diagrams from the purely gauge invariant corrections.

For a finite $N$, we cannot easily solve a mixing therefore the analyses will be more complicated. An essential difference from the case of the large $N$ limit is the propagation of the auxiliary fields. The $eG_V$ term is composed of a mixing among the vector composite field and a propagation of the vector (scalar) composite field $\bar{\psi} \gamma_\mu \psi$ $(\bar{\psi} \psi)$. The later one should also contribute to the wave-function renormalization factor $(Z_2)$ of the fermion, and cancel to each other. To see this, we must introduce an auxiliary field for the scalar composite field too.

For the non-abelian case like QCD, there are no physical particle which has a same quantum number with gluons since the colored particles should be confined. Hence the vertex $A_\mu^a \bar{\psi} T^a \gamma_\mu \psi$ does not have a pole structure in a strict sense. However the gauge non-invariant renormalization parts will affect the Wilsonian RG flow of the gauge coupling constant.

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