On the Success Probability of Three Detectors for the Box-Constrained Integer Linear Model

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Abstract—This paper is concerned with detecting an integer parameter vector inside a box from a linear model that is corrupted with a noise vector following the Gaussian distribution. One of the commonly used detectors is the maximum likelihood detector, which is obtained by solving a box-constrained integer least squares problem that is NP-hard. Two other popular detectors are the box-constrained rounding and Babai detectors due to their high efficiency of implementation. In this paper, we first present formulas for the success probabilities (the probabilities of correct detection) of these three detectors for two different situations: the integer parameter vector is deterministic and is uniformly distributed over the constraint box. Then, we give two simple examples to respectively show that the success probability of the box-constrained rounding detector can be larger than that of the box-constrained Babai detector and the latter can be larger than the success probability of the maximum likelihood detector when the parameter vector is deterministic, and prove that the success probability of the box-constrained rounding detector is always not larger than that of the box-constrained Babai detector when the parameter vector is uniformly distributed over the constraint box. Some relations between the results for the box constrained and ordinary cases are presented, and two bounds on the success probability of the maximum likelihood detector, which can easily be computed, are developed. Finally, simulation results are provided to illustrate our main theoretical findings.

Index Terms—Linear model, box-constrained integer least squares detector, box-constrained rounding detector, box-constrained Babai detector, success probability.

I. INTRODUCTION

Suppose that we have the following box-constrained linear model:

\[ y = Ax + v, \quad v \sim \mathcal{N}(0, \sigma^2 I), \]

\[ x \in B \equiv \{ x \in \mathbb{Z}^n : \ell \leq x \leq u, \quad \ell, u \in \mathbb{Z}^n, \ell < u \}, \]

where \( y \in \mathbb{R}^m \) is an observation vector, \( A \in \mathbb{R}^{m \times n} \) with \( m \geq n \) is a deterministic full column rank model matrix, \( \hat{x} \in \mathbb{R}^n \) is an integer parameter vector and \( v \in \mathbb{R}^m \) is a noise vector following the Gaussian distribution \( \mathcal{N}(0, \sigma^2 I) \) with given \( \sigma \).

This paper studies the detection of \( \hat{x} \), which can be deterministic or random in the box \( B \), from (1). This problem arises from many applications. For example, in some wireless communications systems (see e.g., [1]–[3]), \( \hat{x} \) is a random vector which is uniformly distributed over the box \( B \); in transportation science (see, e.g., [4]) and image processing (see, e.g., [5]), \( \hat{x} \) is a deterministic vector in a finite box \( B \); in power electronics (see, e.g., [6]) and Global Position System (GPS) (see, e.g., [7]), \( \hat{x} \) is a deterministic vector in an infinite box, i.e., \( B = \mathbb{Z}^n \).

One of the most commonly used methods to detect \( \hat{x} \) is to solve the following Box-constrained Integer Least Squares (BILS) problem:

\[ \min_{x \in B} \| y - Ax \|_2^2, \quad (3) \]

whose solution, denoted by \( x^{\text{BILS}} \), is the maximum likelihood estimator of \( \hat{x} \) due to the fact that \( v \sim \mathcal{N}(0, \sigma^2 I) \).

If \( B = \mathbb{Z}^n \) in (2), then (1) is referred to as an ordinary linear model, and the maximum likelihood estimator of \( \hat{x} \), denoted by \( x^{\text{OLS}} \), is the solution of the Ordinary Integer Least Squares (OILS) problem:

\[ \min_{x \in \mathbb{Z}^n} \| y - Ax \|_2^2. \quad (4) \]

One of the widely used approaches to solving (3) or (4) is sphere decoding, which consists of two steps: reduction and discrete search. Reduction is the process of using some lattice reduction strategies to preprocess (3) or (4). Discrete search is the process of finding the solution of the preprocessed problem with certain search algorithm. The most popular reduction method for preprocessing the OILS problem (4) is the Lenstra-Lenstra-Lovász (LLL) reduction [8] [9]. It consists of two kinds of strategies: size reductions and column permutations, which reduce the magnitudes of the off-diagonal entries and reorder the columns of the triangular factor of the QR factorization of \( A \), respectively. For the BILS problem (3), the size reductions make the constraint box too complicated to be handled, thus instead of using the LLL reduction, some column reordering strategies are frequently utilized to preprocess (3).

The commonly used column reordering strategies includes Vertical-Bell Laboratories Layered Space Time (V-BLAST) [10] and Sorted QR Decomposition (SQRD) [11] which use the information of \( A \) only, and those developed in [12], [13], which use not only the information of \( A \), but also the information of \( y \) and \( B \). The most widely used discrete search strategy for (3) is the Schnorr-Euchner search algorithm [14], which is an improvement of the Fincke-Pohst search algorithm.

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By taking the box-constraint into account, the Schnorr-Euchner search algorithm has been modified to solve (3), for more details, see, e.g., 1 and 12.

Although the reduction step can usually reduce the computational cost of solving (3) or (4) (see, e.g., 11, 12, 16, 17), it has been respectively shown in 13 and 19 (a simplified proof can be found in 20) that (3) and (4) are NP-hard. Therefore, in practical applications, especially for real-time applications, an efficient and effective suboptimal algorithm is often used to detect \( \hat{x} \) instead of exactly solving (3) or (4) to get the optimal solution.

For the OILS problem, the ordinary rounding detector \( x^{\text{OR}} \) and the Babai detector \( x^{\text{BR}} \), which are respectively obtained by the Babai rounding off and nearest plane algorithms 21, are frequently used as suboptimal detectors for \( \hat{x} \). (In the ordinary case, actually it is more appropriate to use the term “estimator” than “detector”. However, as we mainly focus on the box constrained case in this paper and “detector” is the standard term used in communications in this case. For simplicity, we will use “detector” for both cases.) By taking the box constraint (2) into account, one can easily modify the algorithms for \( x^{\text{OR}} \) and \( x^{\text{BR}} \) to get box-constrained rounding detector \( x^{\text{BR}} \) and box-constrained Babai detector \( x^{\text{BB}} \) for \( \hat{x} \) satisfying both 1 and 2. In communications, \( x^{\text{BB}} \) is referred to as zero-forcing detector, while \( x^{\text{BR}} \) is called the nulling and canceling detector 10 (also called zero-forcing decision-feedback equalization detector 11).

One of the most popular measures to characterize how good a detector is its success probability, which is the probability of the detector being equal to \( \hat{x} \), see, e.g., 12, 22–24. There are some other measures to characterize the performance of a detector, such as the bit error rate (see, e.g., 25).

For the estimation of \( \hat{x} \) in the ordinary linear model (1), the formulas of the success probability \( P^{\text{OR}} \) of the rounding detector \( x^{\text{OR}} \) and the success probability \( P^{\text{OP}} \) of the Babai detector \( x^{\text{BR}} \) have been given in 23 and 17, respectively. Equivalent formulas of \( P^{\text{OR}} \) and \( P^{\text{OP}} \) were given earlier in 17, which considers the OILS problem in a different format in GPS. It is shown in 7 that \( P^{\text{OR}} \leq P^{\text{OP}} \). The success probability \( P^{\text{OL}} \) of the solution \( x^{\text{OL}} \) of (3) was given in 26, which considers the OILS problem in a different format in GPS. Furthermore, it has been shown in 26 that \( x^{\text{OL}} \) has the highest success probability among a class of the so-called admissible estimators, which include \( x^{\text{OR}} \) and \( x^{\text{BR}} \).

For the detection of \( \hat{x} \) satisfying both 1 and 3, it is also generally believed that the success probability \( P^{\text{BR}} \) of the rounding detector \( x^{\text{BR}} \) is not larger than the success probability \( P^{\text{BB}} \) of the Babai detector \( x^{\text{BB}} \), and \( P^{\text{BR}} \) is not larger than the success probability \( P^{\text{PL}} \) of the solution \( x^{\text{BIL}} \) of 3.

Since detecting \( \hat{x} \), which can be deterministic or random in the box \( B \), from 1 and 3 arises from many applications, and success probability is one of the often used measures of the goodness of a detector, this paper develops formulas for the success probabilities of the rounding, Babai and maximal likelihood detectors and investigate their relationships. Specifically, the contributions of this paper are summarized as follows (part of this work has been presented in a conference paper 27).

1) We present formulas for the success probabilities \( P^{\text{BR}} \) and \( P^{\text{BB}} \) of the box-constrained rounding detector \( x^{\text{BR}} \) in Theorems 11 and 2 corresponding to the case that \( \hat{x} \) is a deterministic parameter vector and the case that \( \hat{x} \) is uniformly distributed over \( B \) (an assumption often made for multi-input multi-output (MIMO) applications, see, e.g., 28), respectively. We also give a formula for the success probability \( P^{\text{BB}} \) of the box-constrained Babai detector \( x^{\text{BB}} \) for the case that \( \hat{x} \) is deterministic in Theorem 3 and develop formulas for the success probabilities \( P^{\text{BR}} \) and \( P^{\text{PL}} \) of the BILS detector \( x^{\text{BIL}} \) (i.e., the solution to (3)), in Theorems 5 and 7, respectively, to the case that \( \hat{x} \) is deterministic and \( \hat{x} \) is uniformly distributed over \( B \), respectively.

2) Since it is difficult to compute \( P^{\text{BR}} \), \( P^{\text{BB}} \) and \( P^{\text{PL}} \), we present some good bounds on them in Corollaries 11, respectively, and these bounds can easily be computed. For the same reason, we present good upper bounds on \( P^{\text{BR}} \) and \( P^{\text{PL}} \) in Theorems 6 and 8, respectively.

3) We give relations among various success probabilities – the major contribution of this paper. We compare the success probabilities of the same type of detectors in different circumstances in Corollaries 11, and Theorem 3. We also compare the success probabilities of the three types of detectors in the same circumstances. Specifically, Example 1 shows that \( P^{\text{BR}} \) can be larger than \( P^{\text{BB}} \), while Theorem 9 rigorously shows that \( P^{\text{BR}} \leq P^{\text{BB}} \), and Example 2 shows that \( P^{\text{PL}} \) can be larger than \( P^{\text{BB}} \), although it is true that \( P^{\text{PL}} \leq P^{\text{BB}} \) (this result can be obtained from, e.g., 29, p18).

The practical significance of our contribution is as follows. Firstly, as can be seen from Sec. 11-A, the complexities of computing \( x^{\text{BR}} \) and \( x^{\text{BB}} \) are the same and are dominated by the QR factorization of \( A \), thus, the proved inequality \( P^{\text{BR}} \leq P^{\text{BB}} \) indicates that in practical applications, one should usually use \( x^{\text{BR}} \) instead of \( x^{\text{BB}} \) to detect \( \hat{x} \) if \( \hat{x} \) is uniformly distributed over the constraint box \( B \). Secondly, since \( P^{\text{BR}} \leq P^{\text{PL}} \), if \( P^{\text{BR}} \), which can be efficiently computed (see Theorem 3), is close to 1, then one can just use \( x^{\text{BR}} \) to detect \( \hat{x} \), and hence there is no need to spend extra time to obtain \( x^{\text{PL}} \). Thirdly, from simulations in Sec. 11-V, we can see that the upper bound on \( P^{\text{BR}} \) given in Theorem 8 is close to \( P^{\text{BR}} \). Thus the former can be used as an approximation to the latter, which is difficult to compute. Fourthly, it is known that \( x^{\text{BB}} \) has the highest success probability over all the detectors (see, e.g., 29, P.18) when \( \hat{x} \) is uniformly distributed over the constraint box \( B \). Thus, if the upper bound on \( P^{\text{PL}} \) is much smaller than 1, then there is no detector which can detect \( \hat{x} \) with high probability and one should try to improve the physical setting.

The rest of the paper is organized as follows. We present formulas for \( P^{\text{BR}} \), \( P^{\text{BB}} \), \( P^{\text{PL}} \), \( P^{\text{BB}} \), \( P^{\text{PL}} \), and \( P^{\text{BIL}} \) in Section 11. In Section 11-III, we study the relationships among them. Simulation tests to illustrate our main results are provided in Section 11-V. Finally, this paper is summarized in Section 11-V.

Notation. We use \( e_i \) to denote the \( i \)-th column of the identity matrix \( I \). For \( x \in \mathbb{R}^n \), we use \( |x| \) to denote its nearest integer vector, i.e., each entry of \( x \) is rounded to its nearest integer (if there is a tie, the smaller integer is chosen).
For a vector $x$, $x_{i:j}$ denotes the subvector of $x$ formed by entries $i, i+1, \ldots, j$. For a matrix $A$, $A_{i:j; i:j}$ denotes the submatrix of $A$ formed by rows and columns $i, i+1, \ldots, j$. For a box $B = \{x \in \mathbb{Z}^n : \ell \leq x \leq u, \ell, u \in \mathbb{Z}^n\}$, sometimes we also write it as $B = \prod_{i=1}^n B_i$ with $B_i = \{x_i \in \mathbb{Z} : \ell_i \leq x_i \leq u_i, \ell_i, u_i \in \mathbb{Z}\}$. For a random vector $v$ following the normal distribution with mean $\bar{v}$ and covariance matrix $\Sigma$, we write $v \sim N(\bar{v}, \Sigma)$.

For the sake of reading convenience, we provide a list of success probability symbols and the corresponding detectors in Table I. Note that this paper is mainly concerned with the quantities in the second part of this table, although the quantities in the first part are also involved in some results. For the ordinary case, $\hat{x}$ is a fixed unknown integer vector, so the quantities in the first part of the table do not need to have the subscript $D$ or $R$.

Table I

| Symbol | Detector |
|--------|----------|
| $P_{DOR}$ | ordinary rounding |
| $P_{DBR}$ | ordinary Babai |
| $P_{DOIL}$ | solution for ordinary Integer Least Squares (ILS) |
| $P_{DBR}^D$ | deterministic box-constrained rounding |
| $P_{DBR}^R$ | random box-constrained rounding |
| $P_{DBB}^D$ | deterministic box-constrained Babai |
| $P_{DBB}^R$ | deterministic box-constrained Babai |
| $P_{DOIL}^D$ | deterministic box-constrained ILS |
| $P_{DOIL}^R$ | solution for random box-constrained ILS |

II. Success probabilities of some detectors

In this section, we derive formulas for $P_{DOR}^D$, $P_{DBR}^R$, $P_{DBR}^D$, $P_{DOIL}^D$, and $P_{DOIL}^R$ (see Table I). Note that the formula for $P_{DOR}^R$ has been derived in [24, Th.1].

A. Definitions of $x_{DBR}^D$ and $x_{DBB}^R$

In this subsection, we introduce the box-constrained rounding detector $x_{DBR}^D$ and the box-constrained Babai detector $x_{DBB}^R$.

Let $A$ in (1) have the following QR factorization

$$A = QR,$$

where $Q \in \mathbb{R}^{n \times n}$ has orthonormal columns (i.e., $Q^T Q = I$) and $R \in \mathbb{R}^{n \times n}$ is nonsingular upper triangular with positive diagonal entries. Define $\hat{y} = Q^T y$ and $\hat{v} = Q^T v$. Then, left multiplying both sides of (1) by $Q^T$ yields

$$\hat{y} = R\hat{x} + \hat{v}, \quad \hat{v} \sim N(0, \sigma^2 I)$$

and the BILS problem (3) is equivalent to

$$\min_{x \in B} \|y - R x\|_2^2.$$  \hspace{1cm} (7)

Later on we mainly work on the transformed model (6) and the transformed BILS problem (7).

Let

$$d = R^{-1} \hat{y},$$

which is the real solution to (7) or (3) with the box $B$ replaced by $\mathbb{R}^n$. The box-constrained rounding detector $x_{DBR}^D$ of $\hat{x}$ in (6) is computed as follows (see, e.g., [30]):

$$x_{DBR}^D = \begin{cases} \ell_i, & \text{if } |d_i| < \ell_i, \\ [d_i], & \text{if } \ell_i \leq |d_i| \leq u_i, \quad i = 1, \ldots, n. \end{cases}$$

The box-constrained Babai detector $x_{DBB}^R$ is computed in the following way (see, e.g., [30]):

$$c_i = (\bar{y}_i - \sum_{j=1}^n r_{ij} \hat{x}_{DBB}^j)/r_{ii}, \quad \text{with} \quad \sum_{j=n+1}^n r_{nj} \hat{x}_{DBB}^j = 0,$$

$$x_{DBB}^D = \begin{cases} \ell_i, & \text{if } |c_i| < \ell_i, \\ [c_i], & \text{if } \ell_i \leq |c_i| \leq u_i, \quad i = n, \ldots, 1. \\ u_i, & \text{if } |c_i| > u_i. \end{cases}$$

B. Success probability of the box-constrained rounding detector

In this subsection, we develop formulas for $D_{DBR}^D$, $D_{DOIL}^D$, $D_{DBR}^R$, and $D_{DOIL}^R$, which are the success probabilities of $x_{DBR}^D$ for deterministic $\hat{x}$ and for random $\hat{x}$ which is uniformly distributed over $B$, respectively. We also give a lower bound on them. We first present a formula for $D_{DBR}^D$.

Theorem 1. Let $\hat{x}$ in (1) or (6) be a deterministic integer parameter vector that satisfies (2), then

$$D_{DBR}^D = \frac{\det(R)}{(\sqrt{2\pi\sigma})^n} \int_{I_n} \cdots \int_{I_1} \exp\left(-\frac{\|R\xi\|_2^2}{2\sigma^2}\right) \, d\xi_1 \cdots d\xi_n,$$

where

$$I_i := I_i(\hat{x}) = \begin{cases} (-\infty, 1/2], & \text{if } \hat{x}_i = \ell_i \\ (-\frac{1}{2}, \frac{1}{2}], & \text{if } \ell_i < \hat{x}_i < u_i, \quad i = 1, \ldots, n. \\ (-\frac{1}{2}, \infty), & \text{if } \hat{x}_i = u_i \end{cases}$$

Proof: Since $\hat{x}$ is deterministic and $\hat{v} \sim N(0, \sigma^2 I)$, by (6) and (3), we have

$$d = R^{-1} \hat{v} \sim N(0, \sigma^2 (R^T R)^{-1}).$$

Then, by (9) and (12), we can conclude that

$$x_{DBR}^D = \hat{x} \iff e_i^T R^{-1} \hat{v} \in I_i, \quad i = 1, \ldots, n.$$  \hspace{1cm} (14)

Therefore, (11) holds.

From (11) and (12), we see that $D_{DBR}^D$ depends on the position of $\hat{x}$ in the box $B$, thus we also write $D_{DBR}^D$ as $D_{DBR}^D(\hat{x})$. To compute $D_{DBR}^D$, we need to know the positions of $\hat{x}_i$ on $[\ell_i, u_i]$ for $i = 1, \ldots, n$. In practice this information is unknown. However, it is easy to observe from (11) that $D_{DBR}^D$ has a lower bound which does not rely on this information.

Corollary 1. Under the conditions of Theorem 1, we have

$$D_{DBR}^D \geq \frac{\det(R)}{(\sqrt{2\pi\sigma})^n} \int_{-1/2}^{1/2} \cdots \int_{-1/2}^{1/2} \exp\left(-\frac{\|R\xi\|_2^2}{2\sigma^2}\right) \, d\xi_1 \cdots d\xi_n$$

$$= P_{DOR}^D,$$
where the lower bound is reached if and only if \( \ell_i < \hat{x}_i < u_i \)
for \( i = 1, \ldots, n \).

The lower bound on \( P_{BR} \) in Corollary 1 is equal to the success probability of the ordinary rounding detector \( P_{OR} \), which can be found in [23] Th. 1. It is easy to understand this. In fact, the ordinary case can be regarded as a special case of the box-constrained case with \( \ell_i = -\infty \) and \( u_i = \infty \), thus, \( \ell_i < \hat{x}_i < u_i \) for \( i = 1, \ldots, n \).

The following theorem gives a formula for \( P_{BR} \).

**Theorem 2.** Suppose that \( \hat{x} \) in (1) is a random integer parameter vector that is uniformly distributed over \( \mathcal{B} \), and \( \hat{x} \) and \( v \) are independent, then

\[
P_{BR} = \frac{\det(R)}{(2\pi\sigma)^n} \prod_{i=1}^{n} \left( \alpha_i \int_{-\infty}^{\ell_i} dx_i + \beta_i \int_{1/2}^{\ell_i} dx_i \right)
\]

\[
\times \cdots \left( \alpha_1 \int_{-\infty}^{\ell_1} dx_1 + \beta_1 \int_{1/2}^{\ell_1} dx_1 \right) \exp \left( -\frac{\|R\|_2^2}{2\sigma^2} \right)
\]

\[
= \frac{\det(R)}{(2\pi\sigma)^n} \prod_{i=1}^{n} \omega_i \int_{\tilde{I}(\omega_i)}^{\infty} \cdots \int_{\tilde{I}(\omega_1)}^{\infty} \exp \left( -\frac{\|R\|_2^2}{2\sigma^2} \right) dx_1 \cdots dx_n,
\]

where for \( 1 \leq i \leq n \),

\[
\alpha_i = \frac{1}{u_i - \ell_i + 1}, \quad \beta_i = \frac{u_i - \ell_i}{u_i - \ell_i + 1}.
\]

\[
\tilde{I}(\omega_i) = \left\{ \begin{array}{ll}
(-\infty, +\infty) & \text{if } \omega_i = \alpha_i \\
[-1/2, 1/2] & \text{if } \omega_i = \beta_i
\end{array} \right.
\]

**Proof:** See Appendix A.

Although the formula for \( P_{BR} \) given in Theorem 2 can be used for computation, it may be too expensive when the dimension \( n \) is a little large, hence, Theorem 2 is of little practical use.

By Theorem 2 we get the following corollary.

**Corollary 2.** Under the conditions of Theorem 2 we have

\[
P_{BR} \geq P_{OR}.
\]

and

\[
\lim_{i \to -\infty \text{ or } u_i \to \infty} P_{BR} = P_{OR}.
\]

**Proof:** From (15), by applying Corollary 1 one can obtain (18). From (15) and Corollary 1 it is easy to see that (19) holds.

This corollary shows the relation between \( P_{BR} \) and \( P_{OR} \). The latter is a strict lower bound on the former for a finite box \( \mathcal{B} \). But the two quantities will be close when \( \mathcal{B} \) is big enough.

### C. Success probability of the box-constrained Babai detector

In this subsection, we develop a formula for \( P_{BB} \). Since \( P_{BB} \) depends on the position of \( \hat{x} \) in the box \( \mathcal{B} \), we also give a lower bound and an upper bound on \( P_{BB} \). The following theorem presents a formula for \( P_{BB} \).

**Theorem 3.** Let \( \hat{x} \) in (1) be a deterministic integer parameter vector that satisfies (2), then

\[
P_{BB} = \prod_{i=1}^{n} \omega_i(r_{ii}),
\]

where

\[
\omega_i(r_{ii}) = \begin{cases}
\frac{1}{2} \left[ 1 + \phi_\sigma(r_{ii}) \right], & \text{if } \hat{x}_i = \ell_i \text{ or } \hat{x}_i = u_i \\
\phi_\sigma(r_{ii}), & \text{if } \ell_i < \hat{x}_i < u_i
\end{cases}
\]

with

\[
\phi_\sigma(\zeta) = \frac{\zeta}{\sqrt{2\pi}\sigma} \exp \left( -\frac{\frac{\zeta^2}{2}}{2\sigma^2} \right)
\]

\[
= \frac{\sqrt{\pi}}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\zeta} \exp \left( -\frac{\frac{\zeta^2}{2}}{2\sigma^2} \right) d\zeta.
\]

**Proof:** See Appendix A.

Although the formula for \( P_{BB} \) given in Theorem 3 can be used for computation, it may be too expensive when the dimension \( n \) is a little large, hence, Theorem 3 is of little practical use.

By Theorem 3 we get the following corollary.

**Corollary 3.** Under the conditions of Theorem 3 we have

\[
P_{BR} \geq P_{BB}.
\]

and

\[
\lim_{i \to -\infty \text{ or } u_i \to \infty} P_{BR} = P_{BB}.
\]

**Proof:** From (15), by applying Corollary 1 one can obtain (18). From (15) and Corollary 1 it is easy to see that (19) holds.

This corollary shows that \( P_{BR} = P_{BB} \).
D. Success probability of the BILS detector

In this subsection, we give formulas for $P_{D}^{\text{OL}}$ and $P_{R}^{\text{OL}}$. Since both $P_{D}^{\text{DL}}$ and $P_{R}^{\text{DL}}$ are complicated to be computed, we also give an upper bound on each of them. We first consider the deterministic situation.

**Theorem 5.** Let $\hat{x}$ in (1) be a deterministic integer parameter vector that satisfies (2) and $x^{\text{DL}}$ be any solution to the BILS problem (7), then the success probability $P_{D}^{\text{DL}}$ of $x^{\text{DL}}$ satisfies

$$P_{D}^{\text{DL}} = \frac{1}{(2\pi\sigma)^{n}} \int_{S^{\text{DL}}} \exp \left( -\frac{1}{2\sigma^{2}} \| \xi \|_{2}^{2} \right) d\xi,$$

where

$$S^{\text{DL}} = \{ \xi \ | \ 2(\hat{x} - \hat{x})^{T} R^{T} \xi \leq \| R(\hat{x} - \hat{x}) \|_{2}^{2} \ \text{for all} \ \hat{x} \in B \}.$$

**Proof:** See Appendix [B].

From (26) we see that $P_{D}^{\text{DL}}$ depends on $\hat{x}$, thus we also write $P_{D}^{\text{DL}}(\hat{x})$ and $P_{D}^{\text{DL}}(\hat{x})$. Like $P_{D}^{\text{DL}}(\hat{x})$ (see (11) and (20)), one cannot use (25) to compute $P_{D}^{\text{DL}}(\hat{x})$, since we do not know $\hat{x}$. Furthermore, the set $S^{\text{DL}}$ in (26) is complicated and it is difficult to calculate $P_{D}^{\text{DL}}(\hat{x})$ from the computational perspective, even if $\hat{x}$ is known.

By Theorem 5 we can obtain the following corollary.

**Corollary 4.** The success probability $P_{OL}^{\text{OL}}$ of the solution $x^{\text{OL}}$ to the OILS problem, and $P_{D}^{\text{DL}}$ of the solution $x^{\text{DL}}$ to the BILS problem (7) satisfy

$$P_{OL}^{\text{OL}} \leq P_{D}^{\text{DL}}.$$

**Proof.** By using the same method as that used for deriving $P_{D}^{\text{DL}}$ in Theorem 5, one can easily obtain

$$P_{OL}^{\text{OL}} = \frac{1}{(2\pi\sigma)^{n}} \int_{S^{\text{OL}}} \exp \left( -\frac{1}{2\sigma^{2}} \| \xi \|_{2}^{2} \right) d\xi,$$

where

$$S^{\text{OL}} = \{ \xi \ | \ 2(\hat{x} - \hat{x})^{T} R^{T} \xi \leq \| R(\hat{x} - \hat{x}) \|_{2}^{2} \ \text{for all} \ \hat{x} \in B \}.$$

Note that $S^{\text{OL}}$ is the same as $S^{\text{DL}}$ in (26), except that the constraint set $B$ is replaced by $\mathbb{Z}^{n}$. Thus, $S^{\text{OL}} \subseteq S^{\text{DL}}$. Then, comparing the expressions for $P_{OL}^{\text{OL}}$ here and $P_{D}^{\text{DL}}$ in (25), we obtain $P_{OL}^{\text{OL}} \leq P_{D}^{\text{DL}}$. □

In the following, we give some upper bounds on $P_{D}^{\text{DL}}(\hat{x})$, one of them can be calculated easily without using any information of $\hat{x}$.

**Theorem 6.** Let $\hat{x}$ in (1) be a deterministic integer parameter vector that satisfies (2) and $x^{\text{OL}}$ be any solution to the BILS problem (7). Let $\phi_{\sigma}(\cdot)$ be defined in (22). For any $x$ such that $\hat{x} \in B$ and $x \neq \hat{x}$,

$$P_{D}^{\text{DL}} \leq \frac{1}{2} \left[ 1 + \phi_{\sigma}(\| R(\hat{x} - \hat{x}) \|_{2}) \right].$$

In particular,

$$P_{D}^{\text{DL}} \leq \frac{1}{2} \left[ 1 + \phi_{\sigma}(\min_{1 \leq i \leq n} \| R_{1, i} \|_{2}) \right].$$

**Proof:** See Appendix [C].

Note that the upper bound given by (28) is independent of the box $B$, thus it also holds when $u_{i} - \ell_{i}, 1 \leq i \leq n$, tends to infinity. Hence, the right-hand side of (28) is also an upper bound on $P_{OL}^{\text{OL}}$. Furthermore, from Theorem 5 we can see that $P_{D}^{\text{DL}}$ becomes larger as the box gets smaller for fixed $A$, so the upper bound (28) becomes sharper as the box gets smaller.

In the rest of this section, we consider the case that $\hat{x}$ is uniformly distributed over $B$. By using the technique for deriving (34) in Appendix [A], we can easily obtain the following result.

**Theorem 7.** Suppose that $\hat{x}$ in (1) is a random integer parameter vector that is uniformly distributed over $B$, and $x$ and $\sigma$ are independent. Let $x^{\text{DL}}$ be any solution to the BILS problem (7), then the success probability $P_{D}^{\text{DL}}$ of $x^{\text{DL}}$ satisfies

$$P_{D}^{\text{DL}} = \prod_{i=1}^{n} \left( \frac{1}{u_{i} - \ell_{i} + 1} \right) \sum_{\hat{x} \in B} \phi_{\sigma}(\| R_{1, i} \|_{2}^{2}),$$

where $P_{D}^{\text{DL}}(\hat{x})$ denotes the success probability of the BILS estimator when $\hat{x} = \hat{x}$.

Although in theory it is possible to obtain $P_{D}^{\text{DL}}$, it is challenging to compute it numerically for high dimension $n$ and big box $B$. But the formula (29) is useful in analysis.

By Theorem 7 and Corollary 4 we can easily obtain the following result.

**Corollary 5.** The success probability $P_{OL}^{\text{OL}}$ of the solution $x^{\text{OL}}$ to the OILS problem, and $P_{D}^{\text{DL}}$ of the solution $x^{\text{DL}}$ to the BILS problem (7) satisfy

$$P_{OL}^{\text{OL}} \leq P_{D}^{\text{DL}}.$$

By (28) and (29), one can easily see that the right-hand side of (28) is also an upper bound on $P_{D}^{\text{DL}}$. But we can get a sharper upper bound. Before presenting the upper bound, we need to introduce the following lemma.

**Lemma 1.** Suppose that $U \in \mathbb{R}^{n \times n}$ is an upper triangular matrix with positive diagonal entries, and $a_{i} > 0$ (where either $a_{i} < \infty$ or $a_{i} = \infty$) for $1 \leq i \leq n$. Then

$$\int_{-a_{n}}^{a_{n}} \cdots \int_{-a_{1}}^{a_{1}} \exp \left( -\| U \xi \|_{2}^{2} \right) d\xi_{1} \cdots d\xi_{n} \leq \prod_{i=1}^{n} \int_{-a_{i}}^{a_{i}} \exp \left( -u_{i}^{2} \right) dt.$$

**Proof:** See Appendix [D].

Here we make a remark. If $a_{i} = 1/2$ for $i = 1, \ldots, n$ in (30), an inequality equivalent to (30) was derived and used to show that the success probability of ordinary rounding detectors cannot be larger than the success probability of ordinary Babai detectors in (7).

The following theorem gives an upper bound on $P_{D}^{\text{DL}}$, which can easily be computed.

**Theorem 8.** Suppose that $\hat{x}$ in (1) is a random integer parameter vector that is uniformly distributed over $B$, and $\hat{x}$ and $v$ are independent. Let $x^{\text{DL}}$ be any solution to the BILS problem (7), then the success probability $P_{D}^{\text{DL}}$ of $x^{\text{DL}}$ satisfies

$$P_{D}^{\text{DL}} \leq \prod_{i=1}^{n} \left( \frac{1}{u_{i} - \ell_{i} + 1} + \frac{u_{i} - \ell_{i} + 1}{u_{i} - \ell_{i} + 1} \phi_{\sigma}(\| R_{1, i} \|_{2}^{2}) \right) := \mu_{D}^{\text{DL}}.$$
Proof: See Appendix E

Here we give a remark about a practical use of the upper bound \( \mu_{BL} \). If \( \mu_{BL} \) is much smaller than 1, then one may give up detection without bothering to find a detector. Later in Section III-B we will give another remark about \( \mu_{BL} \) when we compare \( P_{BR} \) and \( P_{BILS}^{\text{R}} \). In Section IV we will give some numerical examples to show how tight the upper bound is.

III. RELATIONSHIPS AMONG \( P_{BR} \), \( P_{BB} \) AND \( P_{BILS}^{\text{R}} \)

In this section, we investigate the relationships among \( P_{BR} \), \( P_{BB} \) and \( P_{BILS}^{\text{R}} \). Specifically, on the one hand, we give a simple example to show that \( P_{BR} > P_{BB} \) and then rigorously show that \( P_{BB} > P_{BR} \). On the other hand, since it is well-known that \( P_{BR} \leq P_{BILS}^{\text{R}} \) (see [29] P.18), we give a simple example to show that \( P_{BB} > P_{BR} \) may hold.

A. Relationship between \( P_{BB} \) and \( P_{BR} \)

It has been shown in [7] eq. (20)] that the success probability of the ordinary rounding detector cannot be larger than that of the ordinary Babai detector, i.e., \( P_{POD} \leq P_{BOD} \). For the box-constrained case, if the deterministic \( \hat{x} \) satisfies \( \ell_i < x_i < u_i \) for \( 1 \leq i \leq n \), then by Corollaries 1 and 3 \( P_{D}^{\text{BR}} = P_{D}^{\text{OR}} \) and \( P_{D}^{\text{BB}} = P_{D}^{\text{OB}} \) which imply that \( P_{D}^{\text{BB}} \leq P_{D}^{\text{BR}} \). When \( x_i = \ell_i \) or \( x_i = u_i \) for some \( i \), our simulations indicate that in general the experiment success probability of \( x_{BB} \) is smaller than that of \( x_{BR} \). However, the following example shows that in this case it is possible that \( P_{D}^{\text{BB}} > P_{D}^{\text{BR}} \).

Example 1. Suppose that in (6) \( \sigma = 1 \), \( R = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} \), \( \hat{x} = \ell_1 \) and \( \hat{x}_2 = \ell_2 \). Then, by Theorems 1 and 2, we have

\[
P_{D}^{\text{BR}} = \frac{2}{2\pi} \int_{-\infty}^{1/2} \int_{-\infty}^{1/2} \exp(-\frac{1}{2}||x||^2)dx_1dx_2 = 0.6192, \\
P_{D}^{\text{BB}} = \frac{1}{4}(1 + \phi_1(1))(1 + \phi_1(2)) = 0.5818.
\]

Thus, \( P_{D}^{\text{BB}} > P_{D}^{\text{BR}} \).

Unlike the deterministic situation, when \( \hat{x} \) is uniformly distributed over \( B \), we will show in Theorem 9 below that \( P_{R} \leq P_{BR} \).

Theorem 9. Suppose that \( \hat{x} \) in (1) is a random integer parameter vector that is uniformly distributed over \( B \), and \( \hat{x} \) and \( v \) are independent, then

\[
P_{R}^{\text{BR}} \leq P_{R}.
\]  (31)

Proof: See Appendix E

B. Relationships between \( P_{BB} \) and \( P_{BILS}^{\text{R}} \)

We mentioned before that \( x_{BL} \) is optimal in terms of the success probability if \( \hat{x} \) is uniformly distributed over the constraint box \( \beta \). Thus we have

\[
P_{R}^{\text{BB}} \leq P_{R}.
\]  (32)

In Theorem 8 when \( R \) is diagonal, the upper bound \( \mu_{BL} \) on \( P_{BR} \) in (30) becomes \( P_{BR} \) (see (24)). Thus, in this case, \( P_{BB} \) holds with equality. It is easy to understand this as \( x_{BB} = x_{BL} \) in this case. If \( R \) is nearly diagonal, the upper bound \( \mu_{BL} \) will be close to the lower bound \( P_{BR} \), thus it must be tight. In practice, if we find \( \mu_{BL} \) is close to \( P_{BR} \), then we do not need to solve the BILS problem to find \( x_{BB} \) and we can just use \( x_{BB} \) as the detector.

Although the inequality (32) holds, Example 2 below shows that the success probability \( P_{D}^{\text{BR}} \) of the box-constrained Babai detector \( x_{BB} \) can be larger than the success probability \( P_{D}^{\text{BB}} \) of the BILS detector \( x_{BL} \) if \( \hat{x} \) is a deterministic parameter vector.

Example 2. Suppose that in (6), \( \sigma = 1 \), \( R = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix} \), \( B = [1, 2] \times [1, 2] \) and \( \hat{x} = [2, 2]^T \).

Since \( \hat{x} = [2, 2]^T \), by (20), we have

\[
P_{D}^{\text{BB}} = \frac{1}{4}[1 + \phi_2(2)]^2 = 0.7079.
\]

By (20) and some simple calculations, we can see that \( \xi \in S \) if and only if \( \xi \) satisfies all of the following inequalities:

\[
\begin{align*}
\xi_2 &\geq -\frac{1}{2}\xi_1 - \frac{5}{4} \quad (\text{take } x = [1, 1]^T), \\
\xi_1 &\geq -1 \quad (\text{take } x = [1, 2]^T), \\
\xi_2 &\geq \frac{1}{2}\xi_1 - \frac{5}{4} \quad (\text{take } x = [2, 1]^T).
\end{align*}
\]

Thus,

\[
S_{BB} = \left\{ \xi \mid \xi_2 \geq -\frac{1}{2}\xi_1 - \frac{5}{4}, \xi_1 \geq 0 \right\} \cup \left\{ \xi \mid \xi_2 \geq -\frac{1}{2}\xi_1 - \frac{5}{4}, -1 \leq \xi_1 \leq 0 \right\}.
\]

Then, by (23), we obtain that

\[
P_{D}^{\text{BB}} = \frac{1}{2\pi} \int_{S_{BB}} \exp(-\frac{1}{2}(\xi_1^2 + \xi_2^2))d\xi = 0.6845.
\]

Thus, \( P_{D}^{\text{BB}} > P_{D}^{\text{BR}} \).

Example 2 shows that \( P_{D}^{\text{BB}} > P_{D}^{\text{BL}} \) may hold when the true parameter vector \( \hat{x} \) is on the boundary of the constraint box. But if \( \hat{x} \) is inside the box, the following theorem shows that \( P_{D}^{\text{BB}} \leq P_{D}^{\text{BL}} \) always holds.

Theorem 10. Suppose that \( \hat{x} \) in (1) is a deterministic integer parameter vector which satisfies \( \ell_i < \hat{x}_i < u_i \) for \( i = 1, \ldots, n \), then

\[
P_{D}^{\text{BR}} \leq P_{D}^{\text{BL}}.
\]  (33)

Proof: Since \( \ell_i < \hat{x}_i < u_i \) for \( i = 1, \ldots, n \), By Corollary 3 \( P_{D}^{\text{BB}} = P_{D}^{\text{OB}} \) (recall \( P_{D}^{\text{OB}} \) is the success probability of the ordinary Babai detector). Let \( P_{D}^{\text{POL}} \) denote the success probability of the solution \( x_{OILS} \) to the OILS problem (3), then by (26), we have \( P_{D}^{\text{OB}} \leq P_{D}^{\text{POL}} \). Then, by Corollary 4 one can see that (33) holds.

IV. SIMULATION RESULTS

In this section, we do numerical tests to illustrate our theoretical findings. As \( \hat{x} \) is typically assumed to be uniformly distributed over \( B \) in communications, we consider this case only in this section. The formulas for \( P_{BR} \), \( P_{BB} \) and \( P_{BILS}^{\text{R}} \) have been derived in Section III and their relationships have been established in Section III. We would like to compare...
them numerically. However, the cost of computing $P_{BB}^{\text{BB}}$ or $P_{BB}^{\text{BB}}$ is extremely high when $n$ is large, so we compare the experimental and theoretical success probability of $P_{BB}^{\text{BB}}$ only. Note that the experimental success probability of a detector is the number of correct detection divided by the total number of tests, and the theoretical success probability of $P_{BB}^{\text{BB}}$ is obtained by (29).

Since the column permutation strategy V-BlAST is commonly used in practical applications to improve the decoding performance of $x^{\text{BB}}$, we also compute the success probability of $x^{\text{BB}}$ after V-BlAST is applied in computing the QR factorization of $A$ to see its effect. This $x^{\text{BB}}$ is referred to as the V-BlAST aided $x^{\text{BB}}$. Note that $x^{\text{BB}}$ and $x^{\text{BB}}$ are not changed by column permutations.

In the tests, for each fixed $n$, constraint box $B = [0, u]^n$ and signal-to-noise ratio (SNR), we generated 100 $A$’s with $a_{ij}, 1 \leq i, j \leq n$, independently and identically following the standard Gaussian distribution $N(0, 1)$. Then, for each generated $A$, we randomly generated 100 $x$’s $\in Z^n$ that follow the uniform distribution over $B$, and 100 $y$’s $\in R^n$ that follow the Gaussian distribution $N(0, \sigma^2 I)$, where $\sigma$ is found from the following equation (see [32, Appendix C]):

$$SNR = 10 \log_{10} \frac{u(u + 2)}{12\sigma^2},$$

and then computed the corresponding vector $y$ based on the linear model (1). For each instance, we computed $x^{\text{BB}}$ by (9), $x^{\text{BB}}$ by (10), the V-BlAST aided $x^{\text{BB}}$ by (11), and $x^{\text{BL}}$ by using the sphere decoding method in [12]. Finally, we computed their experimental success probabilities, which are denoted by “Rounding”, “Babai”, “Babai-VBLAST-E” and “BILS”, respectively. We also computed the average of the upper bound $\mu^{\text{BL}}$ (after V-BlAST is applied in computing the QR factorization of $A$) on $P_{BB}^{\text{BB}}$ given in Theorem 8 to be denoted by “BILS-UB”, and computed the average of the theoretical success probability of the V-BlAST aided Babai point via (24), to be denoted by “Babai-VBLAST-T”.

Figures 1 and 2 display the test results for SNR=4 : 4 : 32 dB, $n = 20$ with $B = [0, 1]^n$ and $B = [0, 7]^n$, respectively. Figures 3 and 4 show the test results for $n = 5 : 5 : 40$, SNR=15 dB with $B = [0, 1]^n$ and $B = [0, 7]^n$, respectively.

From Figures 14 one can see that the experimental success probability of $x^{\text{BB}}$ is less than that of $x^{\text{BB}}$, which is less than that of the V-BlAST aided $x^{\text{BB}}$, and $x^{\text{BL}}$ has the highest success probability. These observations are consistent with the inequality $P_{BB}^{\text{BB}} \leq P_{BB}^{\text{BB}}$ (see 31), the fact that V-BlAST can improve the success probability of $x^{\text{BB}}$ (more details on this can be found in [24]), and the inequality $P_{BB}^{\text{BB}} \leq P_{BB}^{\text{BB}}$ (see [32]). Those figures also show that “Babai-VBLAST-E” and “Babai-VBLAST-T” are almost the same, which means the theoretical $P_{BB}^{\text{BB}}$ matches very well with the experimental $P_{BB}^{\text{BB}}$.

Figures 12 also show that all the (experimental) success probabilities of $x^{\text{BB}}, x^{\text{BB}}$ (and the V-BlAST aided $x^{\text{BB}}$), and $x^{\text{BL}}$ increase as SNR increases, and decrease as the box size increases. Figures 54 show that the success probabilities of all the detectors decrease when the box size increases for fixed SNR and $n$. These can easily be explained by Theorems 24 and 7 respectively.
In the following, we list observations from Figures 3-4 for fixed SNR and constraint box as $n$ increases, and give some explanations:

- The success probability of the Babai detector $x_{BB}$ does not change much when $n$ increases. Since the entries of the tested $A$ independently and identically follow the standard Gaussian distribution $N(0,1)$, by (33) p99, the entries of the R-factor $R$ of the QR factorization (see (5)) are independent, and $r_{ij}^2$, $1 \leq i \leq n$, follow the Chi-square distribution with degree $n-i+1$ (so the mean of $r_{ij}^2$ is $n-i+1$), and $r_{ij}$, $1 \leq i \neq j \leq n$, follow the standard Gaussian distribution. Suppose that $n$ increases to (say) $n'$ and we denote the new R-factor by $R'$.

- Roughly speaking, the $n'$ - $n$ leading diagonal entries of $R'$ are large and the rest are more or less the same as the diagonal elements of $R$. From (24), we see that after $n$ increases to $n'$, the product in the formula of $P^R_{BR}$ for $R'$ has $n'$ - $n$ more factors, corresponding to the $n'$ - $n$ leading diagonal entries of $R'$. Our numerical test indicated that each of these $n'$ - $n$ factors is close to 1. The rest $n$ factors in the product are more or less the same as those in the product corresponding to $R$. This explains why $P^R_{BR}$ does not change much when $n$ increases.

- The success probability of the V-BLAST aided Babai detector $x_{BB}$ increases. Generally speaking, applying V-BLAST will increase the smallest diagonal entries of $R$ and decrease largest ones (note that det($R$) is unchanged), i.e., the gap between the largest one and the smallest one decreases. This leads to the increase of $P^R_{BR}$; see (24) for more details. For the sake of convenience we denote the new R-factor after applying V-BLAST to $R$ by $R_{v}$ and the new R-factor after applying V-BLAST to $R'$ (defined in the preceding item for dimension $n'$) by $R'_{v}$. From the preceding item, the $n$ diagonal entries of $R_{v}$ are more or less the same as those of $R$, but $R'$ has $n'$ - $n$ extra large entries. Roughly speaking, the first $n$ diagonal entries of $R'_{v}$ are larger than the $n$ diagonal entries of $R_{v}$. The rest $n'$ - $n$ diagonal entries of $R'_{v}$ are large. From (24) we see the formula for $P^R_{BR}(R'_{v})$ involves a product of $n'$ factors, in which the $n'$ factors are larger than the $n$ factors in the product involved in the formula for $P^R_{BR}(R_{v})$ and the rest $n'$ - $n$ factors are close to 1 for the tested cases. Therefore, $P^R_{BR}(R'_{v})$ is larger than $P^R_{BR}(R_{v})$.

- The upper bound $\mu_{BL}$ on $P^R_{BR}$ increases when $n$ increases. From the above explanation, when $n$ increases to $n'$, denote the new R-factor by $R'$, then the $n'$ - $n$ extra terms in (50) are very close to 1. Furthermore, since $R'$ has more rows than $R$ and their nondiagonal entries follow the same distribution, $\|Re_i\|^2/r_{ij}$ for $1 \leq j \leq n$ are usually larger than $\|Re_i\|^2/r_{ij}$ for $1 \leq i \leq n$, so from (50), we can see that $\mu_{BL}$ increases with $n$.

To clearly see how good the upper bound given in Theorem 8 on the success probability $P^R_{BR}$ of $x_{BB}$, we display the ratio of “BILS-UB” to “BILS” versus SNR = 4 : 4 : 32 dB with $n = 20$, $B = [0,1]^n$ and $B = [0,7]^n$ in Table (11) and display the ratio of “BILS-UB” to “BILS” versus $n = 5 : 5 : 40$ with SNR=15 dB and $B = [0,1]^n$ and $B = [0,7]^n$ in Table (11). From these two tables, we can see that “BILS-UB” is close to “BILS” for high SNR or small box, so the upper bound $\mu_{BL}$ given in Theorem 8 on the success probability $P^R_{BR}$ is sharp for high SNR or small box.

V. Conclusion

In this paper, we have investigated the success probabilities of the box-constrained rounding detectors $x_{BR}$, the box-constrained Babai detectors $x_{BB}$ and the BILS detectors $x_{BL}$ for detecting an integer parameter vector $\hat{x} \in B$ in the linear model (1), and studied their relationships for two cases: $\hat{x}$ is deterministic and $\hat{x}$ is uniformly distributed over $B$. We first developed formulas for the success probabilities $P^D_{BR}$ and $P^R_{BR}$ for $x_{BR}$, the success probability $P^D_{BB}$ for $x_{BB}$, and the success probabilities $P^D_{BL}$ and $P^R_{BL}$ for $x_{BL}$. Since it is time consuming to compute $P^D_{BL}$ and $P^R_{BL}$, we also developed upper bounds, which can easily be calculated, on them. Then, we gave two examples to show that both $P^D_{BR} > P^D_{BB}$ and $P^D_{BR} > P^D_{BL}$ are possible, and rigorously proved that $P^D_{BR} < P^D_{BL}$ always holds.

In MIMO applications, often the entries of $A$ are assumed to independently and identically follow the standard Gaussian distribution (see, e.g., (34)). A closed-form expression of $P^R_{BR}$ has been developed in (35) and (32) for this class of random matrices $A$. Although we do not have a formula for $P^D_{BR}$ or $P^D_{BL}$ for random $A$, we can see that $P^R_{BR} \leq P^R_{BB} \leq P^R_{BL}$ still hold if the entries of $A$, $v$ and $\hat{x}$ are independent random variables, since it holds for any realization of $A$ which implies it also holds for random matrix $A$.

Appendix A

Proof of Theorem 2

Proof: Note that (16) is just the expansion of (15). Thus we need only to prove (34) and (15).

Since $\hat{x}$ is uniformly distributed over $B$, for any $\hat{x} \in B$,

$$\Pr(\hat{x} = \hat{x}) = \frac{1}{\prod_{i=1}^{n}(u_i - \ell_i + 1)}.$$
Thus,

$$\Pr(x^{\text{BR}} = \tilde{x}) = \sum_{\forall \hat{u} \in \mathcal{B}} \Pr(x^{\text{BR}} = \tilde{x} | \hat{u} = \tilde{x}) \Pr(\hat{u} = \tilde{x})$$

$$= \sum_{\forall \hat{u} \in \mathcal{B}} \Pr(x^{\text{BR}} = \tilde{x}) \Pr(\hat{u} = \tilde{x})$$

$$= \sum_{\forall \hat{u} \in \mathcal{B}} P^{\text{BR}}_D(\tilde{x}) \Pr(\hat{u} = \tilde{x})$$

$$= \frac{1}{\prod_{i=1}^{n}(u_i - \ell_i + 1)} \sum_{\forall \hat{u} \in \mathcal{B}} P^{\text{BR}}_D(\tilde{x}). \ (34)$$

For any $\tilde{x} \in \mathcal{B}$, we denote

$$\tilde{x} = \left[\frac{\tilde{x}}{\tilde{x}_n}\right], \quad \mathcal{B} = \{ \tilde{x} \in \mathbb{Z}^{n-1} : \ell_{1:n-1} \leq \tilde{x} \leq u_{1:n-1} \},$$

where $\ell$ and $u$ are defined in (3). To simplify notation, we also denote

$$\gamma_i = u_i - \ell_i + 1, \quad f(\xi) = \exp\left(-\frac{\|R\xi\|^2}{2\sigma^2}\right).$$

Then, by Theorem $1$ and (34), we see that to show (35) we only need to show

$$\frac{1}{\prod_{i=1}^{n} \gamma_i} \sum_{\forall \hat{u} \in \mathcal{B}} \int_{I_1(\tilde{x})} \cdots \int_{I_n(\tilde{x})} f(\xi) d\xi_1 \cdots d\xi_n$$

$$= \left(\alpha_n \int_{-\infty}^{\infty} d\xi_n + \beta_n d\xi_n \right)^{1/2} \cdots$$

$$\times \left(\alpha_1 \int_{-\infty}^{\infty} d\xi_1 + \beta_1 d\xi_1 \right)^{1/2} f(\xi). \ (35)$$

Then, we can easily see that

$$\mathcal{S}_2 \setminus \mathcal{S}_3 \subseteq \mathcal{S}_1 \subseteq \mathcal{S}_2. \quad (39)$$

Thus,

$$\Pr(\tilde{\nu} \in \mathcal{S}_2) - \Pr(\tilde{\nu} \in \mathcal{S}_3) \leq P^{\text{BR}}_{\text{UL}} \leq \Pr(\tilde{\nu} \in \mathcal{S}_2). \quad (40)$$

Appendix B

Proof of Theorem 5

Proof: Define sets $\mathcal{S}_1, \mathcal{S}_2$ and $\mathcal{S}_3$ as

$$\mathcal{S}_1 = \{ \tilde{\nu} | \tilde{\nu} \sim \mathcal{N}(0, \sigma^2 I), \ y = R\tilde{x} + \tilde{v}, \ x^{\text{UL}} = \tilde{x} \}, \quad (36)$$

$$\mathcal{S}_2 = \{ \tilde{\nu} | \tilde{\nu} \sim \mathcal{N}(0, \sigma^2 I), \ y = R\tilde{x} + \tilde{v}, \ ||\tilde{v}\|^2 \leq ||y - R\tilde{x}\|^2 \text{ for } \forall \tilde{x} \in \mathcal{B} \}, \quad (37)$$

$$\mathcal{S}_3 = \{ \tilde{\nu} | \tilde{\nu} \sim \mathcal{N}(0, \sigma^2 I), \ y = R\tilde{x} + \tilde{v}, \ ||\tilde{v}\|^2 = ||y - R\tilde{x}\|^2 \text{ for some } \tilde{x} \neq \tilde{x}, \tilde{x} \in \mathcal{B} \}. \quad (38)$$

By the fact that $\tilde{x}_n$ can be any integer in $[l_n, u_n]$ and (12),

$$\Pr(\tilde{\nu} \in \mathcal{S}_3) = 0. \quad (41)$$
Let \( \tilde{v} \in S_2 \), then there exists at least one \( \tilde{x} \in B \) such that \( \tilde{x} \neq \hat{x} \) and \( \| \tilde{v} \|^2_2 = \| \tilde{y} - R\tilde{x} \|^2_2 \). Note that by (6), we have \( \tilde{v} = \tilde{y} - R\hat{x} \). Then, with (6), we have
\[
\| \tilde{v} \|^2_2 = \| \tilde{y} - R\hat{x} \|^2_2 = \| (\tilde{y} - R\hat{x}) - R(\hat{x} - \hat{x}) \|^2_2 = \| \tilde{v} \|^2_2 + \| R(\hat{x} - \hat{x}) \|^2_2 - 2(\hat{x} - \hat{x})^T R^T \tilde{v}.
\]
Thus
\[
2(\hat{x} - \hat{x})^T R^T \tilde{v} = \| R(\hat{x} - \hat{x}) \|^2_2
\]
which indicates that \( \tilde{v} \) lies on an \((n-1)\)-dimensional plane. Since \( \tilde{v} \) is an \( n \)-dimensional Gaussian random variable, (41) holds.

By (40)-(41), we can see that
\[
P_{\text{DL}}^{\text{UL}} = \Pr(\tilde{v} \in S_2).
\]

For \( \tilde{v} \in S_2 \) and \( x \in B \),
\[
\| \tilde{y} - R\hat{x} \|^2_2 = \| (\tilde{y} - R\hat{x}) - R(\hat{x} - \hat{x}) \|^2_2 = \| \tilde{v} \|^2_2 + \| R(\hat{x} - \hat{x}) \|^2_2 - 2(\hat{x} - \hat{x})^T R^T \tilde{v}.
\]
which implies that \( \| \tilde{v} \|^2_2 \leq \| \tilde{y} - R\hat{x} \|^2_2 \) if and only if
\[
2(\hat{x} - \hat{x})^T R^T \tilde{v} \leq \| R(\hat{x} - \hat{x}) \|^2_2.
\]
Then, from (42) and the fact that \( \tilde{v} \sim \mathcal{N}(0, \sigma^2 I) \), we can conclude that (27) holds.

**APPENDIX D**

**PROOF OF LEMMA 1**

*Proof:* An inequality which is equivalent to (36) for \( a_i = 1/2 \) is given in (7). As a reviewer pointed out, when all \( a_i \) are finite, we could prove (36) by applying that inequality via a change of variables. For the reader's convenience, we give a proof without referring to (7).

We prove the lemma by changing variables in the integral. Let
\[
T = \begin{bmatrix}
1 & -\frac{1}{u_{11}} U_{1.2:n} \\
0 & I_{n-1}
\end{bmatrix},
\]
then
\[
UT = \begin{bmatrix}
u_{11} \\
0 \quad U_{2:n,2:n}\end{bmatrix}.
\]
Define \( \xi = T\eta \), i.e.,
\[
\xi_1 = \eta_1 - \frac{1}{u_{11}} U_{1.2:n} \eta_{2:n}, \quad \xi_{2:n} = \eta_{2:n}.
\]

First we consider the case that \( a_1 < \infty \). Since \( \xi_1 \in [-a_1, a_1] \),
\[
\eta_1 \in \{ \eta \mid \eta_1 \leq \frac{1}{u_{11}} U_{1.2:n} \eta_{2:n}, \quad a_1 + \frac{1}{u_{11}} U_{1.2:n} \eta_{2:n} \}
\]
Then we have
\[
\int_{u_{11}}^{a_1} \cdots \int_{u_{11}}^{a_1} \exp(-\|U\xi\|^2) \, d\xi_1 \cdots d\xi_n = \int_{u_{11}}^{a_1} \cdots \int_{u_{11}}^{a_1} \exp(-u_{11}^2 \eta_1^2 - \|U_{2:n,2:n} \eta_{2:n}\|^2) \, d\eta_1 \cdots d\eta_n
\]
\[
\int_{u_{11}}^{a_1} \cdots \int_{u_{11}}^{a_1} \exp(-u_{11}^2 \eta_1^2) \, d\eta_1 \times \int_{u_{11}}^{a_1} \cdots \int_{u_{11}}^{a_1} \exp(-\|U_{2:n,2:n} \eta_{2:n}\|^2) \, d\eta_2 \cdots d\eta_n.
\]

According to (41), eq. (68), we have
\[
\int_{u_{11}}^{a_1} \cdots \int_{u_{11}}^{a_1} \exp(-u_{11}^2 \eta_1^2) \, d\eta_1 \leq \int_{-a_1}^{a_1} \exp(-u_{11}^2 \xi_1^2) \, d\xi_1,
\]
which actually can easily be observed from the graph of the density function of the normally distributed random variable with \( 0 \) mean. Therefore,
\[
\int_{-a_1}^{a_1} \cdots \int_{-a_1}^{a_1} \exp(-\|U\xi\|^2) \, d\eta_1 \cdots d\eta_n
\]
\[
\leq \int_{-a_1}^{a_1} \cdots \int_{-a_1}^{a_1} \exp(-u_{11}^2 \xi_1^2) \, d\xi_1
\]
\[
\times \int_{u_{11}}^{a_1} \cdots \int_{u_{11}}^{a_1} \exp(-\|U_{2:n,2:n} \eta_{2:n}\|^2) \, d\eta_2 \cdots d\eta_n.
\]

Thus, by (43), one can easily show that
\[
\int_{-a_1}^{a_1} \cdots \int_{-a_1}^{a_1} \exp(-\|U\xi\|^2) \, d\eta_1 \cdots d\eta_n
\]
\[
\leq \int_{-a_1}^{a_1} \dots \int_{-a_1}^{a_1} \exp(-u_{11}^2 \xi_1^2) \, d\xi_1 \int_{-a_1}^{a_1} \exp(-u_{11}^2 \xi_2^2) \, d\xi_2 \\
\times \int_{-a_1}^{a_1} \cdots \int_{-a_1}^{a_1} \exp(-\|U_{3:n,3:n} \eta_{3:n}\|^2) \, d\eta_3 \cdots d\eta_n
\]
\[
\leq \cdots \leq \prod_{i=1}^{n} \int_{-a_1}^{a_1} \exp(-u_{11}^2 \xi_i^2) \, d\xi_i.
\]
Hence, (30) holds for finite $a_i$, $1 \leq i \leq n$.

In the following, we show that (30) holds if some or all $a_i$ are infinity. To show this, by the above analysis, it suffices to show that (46) still holds if $a_1 = \infty$. Note that

$$\int_{-\infty}^{a_1} \cdots \int_{-\infty}^{\infty} \exp \left( -\frac{1}{2\sigma^2} \|R_i \eta \|_2^2 \right) \, d\eta_1 \cdots d\eta_n$$

where the second equality is due to the fact that the improper integral is convergent. Then taking limit on both sides of (46) as $\alpha \to \infty$ leads to the desired inequality.

**APPENDIX E**

**Proof of Theorem 3**

**Proof:** To prove (30), we first prove the following inequality

$$P_{D}^{\text{BL}}(\hat{x}) \leq \frac{\det(R^{-T})}{(\sqrt{2\pi\sigma})^n} \times \int_{\tilde{I}_i(\hat{x})} \cdots \int_{\tilde{I}_n(\hat{x})} \exp \left( -\frac{1}{2\sigma^2} \|R^{-T} \eta \|_2^2 \right) \, d\eta_1 \cdots d\eta_n, \quad (47)$$

where

$$\tilde{I}_i(\hat{x}) := \tilde{I}_1(\hat{x}) \times \tilde{I}_2(\hat{x}) \times \cdots \tilde{I}_n(\hat{x}) \quad (48)$$

with

$$\tilde{I}_i(\hat{x}) := \begin{cases} (-\infty, \|R_i \eta \|_2^2/2], & \hat{x}_i = \ell_i \\ \|R_i \eta \|_2^2/2, & \ell_i < \hat{x}_i < u_i \\ (-\infty, \|R_i \eta \|_2^2/2, \infty), & \hat{x}_i = u_i \end{cases} \quad (49)$$

for $1 \leq i \leq n$.

By Theorem 3 and setting $\eta = R^T \xi$, we have

$$P_{D}^{\text{BL}}(\hat{x}) \leq \frac{\det(R^{-T})}{(\sqrt{2\pi\sigma})^n} \times \int_{\tilde{S}(\hat{x})} \exp \left( -\frac{1}{2\sigma^2} \|R^{-T} \eta \|_2^2 \right) d\eta_1 \cdots d\eta_n, \quad (50)$$

where

$$\tilde{S}(\hat{x}) = \{ \eta \mid (x - \hat{x})^T \eta \leq \|R(x - \hat{x})\|_2^2/2 \, \text{for} \, \forall x \in \mathcal{B} \}. \quad (51)$$

We take some special $x \in \mathcal{B}$ so that we will get a set, which is included in $\tilde{S}(\hat{x})$, but is more structured so that we can derive an upper bound on $P_{D}^{\text{BL}}(\hat{x})$, which can be easily computed. For $1 \leq i \leq n$, define

$$x^{(i)} = \begin{cases} \hat{x} + e_i, & \hat{x}_i = \ell_i \\ \hat{x} + e_i, & \ell_i < \hat{x}_i < u_i \\ \hat{x} - e_i, & \hat{x}_i = u_i \end{cases} \quad (i)$$

then $x^{(i)} \in \mathcal{B}$. For $1 \leq i \leq n$, take $x = x^{(i)}$, then the inequality in (51) just becomes $\eta_i \in \tilde{I}_i(\hat{x})$, where $\tilde{I}_i(\hat{x})$ is defined in (49). Then, by (48) and (51), $\tilde{S}(\hat{x}) \subseteq \tilde{I}(\hat{x})$. Therefore, (47) holds.

In the following, we use Theorem 7 and (47) to prove (30).

To simplify notation, denote

$$\gamma_i = u_i - \ell_i + 1, \quad h(\eta) = \exp \left( -\frac{\|R_i \eta \|_2^2}{2\sigma^2} \right).$$

Then, by Theorem 7 and (47), we have

$$P_{R_1}^{\text{BL}} \leq \frac{1}{\prod_{i=1}^n \gamma_i (\sqrt{2\pi\sigma})^n} \sum_{\forall \bar{x} \in \bar{I}(\hat{x})} \prod_{i=1}^n h(\eta) \, d\eta_1 \cdots d\eta_n$$

where the second equality is from (22) and the integral transformation.

**APPENDIX F**

**Proof of Theorem 9**

**Proof:** Applying Lemma 1 by taking $U = R/(\sqrt{2\sigma})$ to each term in the sum in (10), which is the expanded version of (15), and then combining all the terms into the same form as (15), we obtain

$$P_{R}^{\text{BR}} \leq \frac{\det(R)}{(\sqrt{2\pi\sigma})^n} \prod_{i=1}^n \left( \frac{1}{u_i - \ell_i + 1} \int_{-\infty}^{\infty} \exp \left( -\frac{r_{ii}^2}{2\sigma^2} \right) dt \right) + \frac{u_i - \ell_i}{u_i - \ell_i + 1} \int_{-1/2}^{1/2} \exp \left( -\frac{r_{ii}^2}{2\sigma^2} \right) dt. \quad (52)$$

Then using the facts that

$$\frac{r_{ii}}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} \exp \left( -\frac{r_{ii}^2}{2\sigma^2} \right) dt = 1$$

and (see (22))

$$\frac{r_{ii}}{\sqrt{2\pi\sigma}} \int_{-1/2}^{1/2} \exp \left( -\frac{r_{ii}^2}{2\sigma^2} \right) dt = \phi_0(r_{ii})$$

we have

$$P_{R}^{\text{BR}} \leq \prod_{i=1}^n \left[ \frac{1}{u_i - \ell_i + 1} + \frac{u_i - \ell_i}{u_i - \ell_i + 1} \phi_0(r_{ii}) \right]$$

where the right-hand side is just $P_{R}^{\text{BR}}$ by Theorem 4.
