Nonseparability and squeezing of continuous polarization variables

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The impact of the operator-valued commutator on nonclassical properties of continuous polarization variables is discussed. The definition of polarization squeezing is clarified to exclude those squeezed states which do not contain any new physics beyond quadrature squeezing. We present a consistent derivation of the general nonseparability criterion for the continuous variables with the operator-valued commutator, and apply it to the polarization variables.

Within the framework of the quantum continuous variables, nonclassical polarization states have recently attracted a particular interest due to their compatibility with the spin variables of atomic systems and due to their simple detection schemes [1–4]. The relevant continuous polarization variables are hermitian Stokes operators (see [2] and references therein):

\[
\begin{align*}
\hat{S}_0 &= \hat{a}_x^\dagger \hat{a}_x + \hat{a}_y^\dagger \hat{a}_y, \\
\hat{S}_1 &= \hat{a}_x^\dagger \hat{a}_x - \hat{a}_y^\dagger \hat{a}_y, \\
\hat{S}_2 &= \hat{a}_y^\dagger \hat{a}_y + \hat{a}_x^\dagger \hat{a}_x, \\
\hat{S}_3 &= i(\hat{a}_y^\dagger \hat{a}_x - \hat{a}_x^\dagger \hat{a}_y)
\end{align*}
\]

(1)

where the \(\hat{a}_x\) and \(\hat{a}_y\) denote the bosonic photon destruction operators associated with the \(x\) and \(y\) orthogonal polarization modes. The Stokes operator \(\hat{S}_0\) commutes with all the others. The operators \(\hat{S}_j, j \neq 0\) obey the commutation relations of the SU(2) Lie algebra:

\[
[\hat{S}_k, \hat{S}_l] = \epsilon_{klm} 2i\hat{S}_m, \quad k,l,m = 1,2,3.
\]

(2)

Simultaneous exact measurements of these Stokes operators are thus impossible in general and their variances are restricted by the uncertainty relations:

\[
V_2 V_3 \geq |\langle \hat{S}_1 \rangle|^2, \quad V_3 V_1 \geq |\langle \hat{S}_2 \rangle|^2, \quad V_1 V_2 \geq |\langle \hat{S}_3 \rangle|^2
\]

(3)

where \(V_j = \langle \hat{S}_j \rangle^2 - \langle \hat{S}_j \rangle^2\) is a shorthand notation for the variance of the quantum Stokes parameter \(\hat{S}_j\). The angle brackets denote expectation values with respect to the state of interest.

Within the last few years, successful generation of polarization squeezed [1,3,5] and polarization entangled [4,6] states has been reported. The respective definitions of polarization squeezing [1,7] and entanglement [2] were formulated. However, the subtleties arising due to the \(q\)-number, i.e. operator-valued, commutator (cf. Eq. (2)) have not been satisfactorily discussed yet and a consistent derivation of criteria for continuous variable polarization entanglement is missing. To provide the answers to these open questions is the aim of this paper.

A state is called polarization squeezed if:

\[
V_k < |\langle \hat{S}_l \rangle| < V_m, \quad k \neq l \neq m = 1,2,3.
\]

(4)

The important difference between quadrature squeezing and polarization squeezing is the discrepancy between coherent and minimum uncertainty states for the latter. A coherent polarization state is defined as a quantum state with both polarization modes having a coherent excitation \(\alpha_x, \alpha_y\): \(\psi_{coh} = |\alpha_x\rangle_x |\alpha_y\rangle_y\). The quantum uncertainty of such a state is equally distributed between the Stokes operators and their variances are all equal to \(V_j = V_{coh} = |\alpha_x|^2 + |\alpha_y|^2 = \langle \hat{n} \rangle\). In analogy to quadrature squeezing, \(V_j < V_{coh}\) seems at first glance to be a natural definition for polarization squeezing. However, due to the SU(2) commutation algebra, a coherent polarization state is not a minimum uncertainty state for all three Stokes operators simultaneously. This was known for atomic states, i.e. for spin coherent states [8] and angular momentum coherent states [9]. The construction of the minimum uncertainty product for the SU(2) algebra and the properties of atomic coherent states were broadly studied around early 70’s [8–11]. Although a polarization state with a sub-shot-noise variance \(V_j < V_{coh}\) is always a non-classical state, it implies nothing more than conventional quadrature or single-mode squeezing observed through the measurement of the Stokes parameter.

It is interesting to establish the relation of polarization squeezing to two-mode squeezing, i.e. quadrature entanglement. For two-mode squeezing, the nonclassical correlations are created between two spatially separated modes. For polarization squeezing, quantum correlations are created between two orthogonal polarization modes. However, by the appropriate choice of variables and basis, the correlations within a two-mode system can be redistributed so that polarization squeezing is transformed into two-mode squeezing and vice versa. This effect was already observed in the experiments: Polarization squeezing and quadrature entanglement were observed in the same nonlinear system of cold 4-level atoms, depending of the choice of the mode basis [13]. Furthermore, the two schemes to generate continuous variable (CV) polarization entanglement have proven to be equivalent: superimposing two polarization squeezed beams on a beam splitter [2,6] or overlapping two quadrature entangled beams with an orthogonally polarized coherent beam each [4]. As an example of a basis transformation which translates both two-mode effects into each other let us view quadrature entanglement in terms of new variables having the mathematical form of Stokes operators:

\[
\begin{align*}
\hat{s}_0 &= \hat{a}_A^\dagger \hat{a}_A + \hat{a}_B^\dagger \hat{a}_B, \\
\hat{s}_1 &= \hat{a}_A^\dagger \hat{a}_A - \hat{a}_B^\dagger \hat{a}_B, \\
\hat{s}_2 &= \hat{a}_A^\dagger \hat{a}_B + \hat{a}_B^\dagger \hat{a}_A, \\
\hat{s}_3 &= i(\hat{a}_B^\dagger \hat{a}_A - \hat{a}_A^\dagger \hat{a}_B)
\end{align*}
\]

(5)
where $A$ and $B$ are the two output spatially separated beams, the quadrature entangled beams. To be specific, suppose that the quadrature entanglement emerges in the interference of two amplitude-squeezed beams with equal squeezing $V^+$ and coherent amplitude $\alpha$ on a beam splitter [12], where the input amplitude squeezing is quantified by the variances $V^+ < 1 < V^-$ of the quadrature operators $\hat{X}_{A,B} = \hat{a}_{A,B}^\dagger + \hat{a}_{A,B}$ and $\hat{X}_{A,B} = i(\hat{a}_{A,B}^\dagger - \hat{a}_{A,B})$. The variances of the new “Stokes” operators $\hat{s}_i$ of Eq. (5) for the noncommuting pair $\hat{s}_1, \hat{s}_3$ are equal to

$$v_1 = 2\alpha^2 V^- > |\langle \hat{s}_2 \rangle|, \quad v_3 = 2\alpha^2 V^+ < |\langle \hat{s}_2 \rangle|.$$  \hfill (6)

Thus quadrature entanglement with anti-correlated amplitudes and correlated phases exhibits squeezing in $\hat{s}_3$.

Along with polarization squeezing, CV polarization entanglement [2] has proven to be a useful tool in quantum communication. There is no unique criterion to quantify CV entanglement in general, in particular for mixed states. For the generalization and comparison of different sum and product entanglement criteria for CVs with a c-number commutator see, e.g., Ref. [16]. The formulation of the EPR criterion [14] and of the nonseparability criterion [15] for the Stokes operators was presented in Ref. [2] and further elaborated in Ref. [4] on the basis of the generalized Heisenberg relation. However, a consistent derivation of the nonseparability condition was not given. In the following, the general expression for the nonseparability criterion using CVs with an operator-valued commutator is derived and the use of the generalized Heisenberg uncertainty relation is justified and emphasized.

The standard derivation [10,17] of the uncertainty relation for operators $\hat{A}$ and $\hat{B}$ uses the Schwarz inequality in the form

$$V_AV_B \geq |\langle \Delta \hat{A}\Delta \hat{B} \rangle|^2,$$  \hfill (7)

where $\Delta \hat{A} = \hat{A} - \langle \hat{A} \rangle$, $\Delta \hat{B} = \hat{B} - \langle \hat{B} \rangle$ and the quantities on the left of (7) are the variances of the relevant operators (cf. Eq. 3). The basic uncertainty relation can be written in a variety of forms, for example [10]:

$$V_AV_B \geq \frac{1}{4} |\langle \{\Delta \hat{A}, \Delta \hat{B} \} \rangle|^2 + \frac{1}{4} |\langle [\Delta \hat{A}, \Delta \hat{B}] \rangle|^2,$$  \hfill (8)

where the anticommutator and commutator of the two operators are defined by

$$\{\Delta \hat{A}, \Delta \hat{B} \} = \hat{A}\hat{B} + \hat{B}\hat{A} - 2\langle \hat{A} \rangle\langle \hat{B} \rangle,$$

$$[\Delta \hat{A}, \Delta \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} = [\hat{A}, \hat{B}].$$  \hfill (9)

For noncommuting operators, both contributions on the right of (8) are in general positive nonzero quantities and the strongest inequality is obtained when both are retained. However, valid but weaker inequalities result when one or other of the contributions is neglected. Thus, removal of the anticommutator term leads to the frequently-used form [17]

$$V_AV_B \geq \frac{1}{4} |\langle [\Delta \hat{A}, \Delta \hat{B}] \rangle|^2.$$  \hfill (10)

This provides the strongest available inequality when $\langle [\Delta \hat{A}, \Delta \hat{B}] \rangle$ vanishes. The commutator $[\Delta \hat{A}, \Delta \hat{B}]$ is often a c-number as, for example, with the position and momentum operators. In this case the variance product has a universal state-independent lower limit,

$$V_AV_B \geq \frac{1}{4} |\langle \Delta \hat{A}, \Delta \hat{B} \rangle|^2.$$  \hfill (11)

However, the commutator is sometimes an operator quantity, or q-number, and the anticommutator $\{\Delta \hat{A}, \Delta \hat{B} \}$ is usually a q-number. Both contributions on the right of (8) then depend on the state of the system and there is no reason to remove any of them. The inequality in (10) remains valid but the full form in (8) provides a stronger inequality with a higher minimum value of the variance product. There is no universal uncertainty relation in such cases, as in the examples of the angular momentum operators and of the Stokes-parameter operators considered here.

The derivation of the nonseparability criterion for CV position $x$ and momentum $p$ having a c-number commutator [15] considers an overall system composed of two subsystems, $c$ and $d$, described by operators

$$\hat{A} = |a|\hat{x}_c + \frac{1}{a}\hat{x}_d, \quad \hat{B} = |a|\hat{p}_c - \frac{1}{a}\hat{p}_d,$$  \hfill (12)

$$[\hat{x}_i, \hat{p}_j] = i\delta_{ij} \quad (i, j = c, d), \quad [\hat{A}, \hat{B}] = i \left(a^2 - \frac{1}{a^2}\right).$$  \hfill (13)

The restrictions on the sum of the two variances are direct consequences of the uncertainty relation: With the use of (7) and the Cauchy inequality $V_A^2 + V_B^2 \geq 2V_AV_B$ it follows that

$$V_A + V_B \geq 2|\langle \Delta A\Delta B \rangle|.$$  \hfill (14)

Thus, with the Heisenberg uncertainty relation taken in the form (11), all states must satisfy

$$V_AV_B \geq \frac{1}{4} \left(a^2 - \frac{1}{a^2}\right)^2 \quad \text{and} \quad V_A + V_B \geq \left|a^2 - \frac{1}{a^2}\right|.$$  \hfill (15)

It is shown in [15] that separable states of the two subsystems must satisfy the stronger inequality

$$V_A + V_B \geq a^2 + \frac{1}{a^2},$$  \hfill (16)

Nonseparable or entangled states thus exist in the region defined by

$$\left|a^2 - \frac{1}{a^2}\right| \leq V_A + V_B < a^2 + \frac{1}{a^2},$$  \hfill (17)
where the lower limit on the left comes from the development of the Heisenberg uncertainty relation in (15) and the upper limit on the right comes from the nonseparability criterion in [15] in its sufficient form.

We now rework the derivation of [15] for the basic operator commutation relations more general than those given in (12), (13):

\[
\hat{A} = \hat{A}_c + \hat{A}_d, \quad \hat{B} = \hat{B}_c - \hat{B}_d, \quad \{\hat{A}_c, \hat{B}_d\} = [\hat{A}_c, \hat{B}_d] = 0,
\]

\[
\{\hat{A}, \hat{B}\} = [\hat{A}_c, \hat{B}_c] - [\hat{A}_d, \hat{B}_d].
\]

Here the nonzero commutators may themselves be operators. The uncertainty relations (15) are generalized to

\[
V_AV_B \geq \left\langle \Delta \hat{A}_c\Delta \hat{B}_c\right\rangle - \left\langle \Delta \hat{A}_d\Delta \hat{B}_d\right\rangle^2,
\]

\[
V_A + V_B \geq 2 \left| \left\langle \Delta \hat{A}_c\Delta \hat{B}_c\right\rangle - \left\langle \Delta \hat{A}_d\Delta \hat{B}_d\right\rangle \right|. \tag{21}
\]

Note that these relations reduce to those in (7) and (14) when there is only a single system, c or d. By substitution of (18) into (12, 14), the Eq. (3, 4) in [15] can be reworked for the pair of variables with the q-number commutator giving the sufficient nonseparability criterion. The main difference to the derivation of [15] is the replacement of the universal limit in (17) by the state-dependent contribution containing the mean value of the operator-valued commutator (19) and the retainment of the state-dependent anticommutator contribution. Nonseparable or entangled states must then satisfy the condition

\[
2 \left| \left\langle \Delta \hat{A}_c\Delta \hat{B}_c\right\rangle - \left\langle \Delta \hat{A}_d\Delta \hat{B}_d\right\rangle \right| \leq V_A + V_B < 2 \left| \left\langle \Delta \hat{A}_c\Delta \hat{B}_c\right\rangle + 2 \left\langle \Delta \hat{A}_d\Delta \hat{B}_d\right\rangle \right|, \tag{22}
\]

where the lower limit on the left comes from the development of the Heisenberg uncertainty relation in (21) and the upper limit on the right comes from the generalization of the sufficient nonseparability criterion (see Appendix). A derivation of the nonseparability criterion in its necessary and sufficient in the case of the q-number commutator still remains a challenge. The sufficient general product criterion was derived in Ref. [16], where the standard form of the Heisenberg uncertainty relation was used to derive an upper limit for the product of two variances.

The expectation values that occur in the limits of Eq. (22) can be calculated either from the forms shown in (22) or from the square root of the form shown in (8).

With this latter form, the examples given below emphasize that the contributions of both the anticommutator and the commutator must be retained to obtain the most reliable and accurate limits.

**Example 1: Korolkova et al.** The experiment proposed in [2] uses bright light beams labelled a and b, each with x and y polarization components, all of which have identical coherent amplitudes denoted by \(\alpha\). Their polarization squeezing properties are conveniently expressed in terms of the variances of the quadrature operators defined by

\[
\hat{X}_{ax}^+ = \hat{a}_x^+ + \hat{a}_x, \quad \hat{X}_{ax}^- = i(\hat{a}_x^+ - \hat{a}_x)
\]

with \([X_{ax}^+, X_{ax}^-] = 2i\) and variances denoted \(V_{ax}^+, V_{ax}^-\) (similarly for \(ay, bx, and by\)). Consider the example with

\[
\hat{A} = \hat{S}_{1c} + \hat{S}_{1d}, \quad \hat{B} = \hat{S}_{3c} - \hat{S}_{3d},
\]

where the subsystems c and d refer to light beams produced by interference of beams a and b, as described in [2]. These have mean values of the Stokes parameters given by

\[
\langle \hat{S}_{1c} \rangle = \langle \hat{S}_{1d} \rangle = \langle \hat{S}_{3c} \rangle = \langle \hat{S}_{3d} \rangle = 0,
\]

\[
\langle \hat{S}_{2c} \rangle = \langle \hat{S}_{2d} \rangle = 2\alpha^2. \tag{25}
\]

The various required quantities defined above are readily calculated from expressions given in [2]. Thus,

\[
V_A = V_B = \alpha^2 \left( V_{ax}^+ + V_{ay}^+ + V_{bx}^+ + V_{by}^+ \right) \tag{26}
\]

\[
\left\langle \Delta \hat{S}_{1c}\Delta \hat{S}_{3c} \right\rangle = \frac{1}{4} \alpha^2 \left( V_{ax}^+ - V_{ax}^- + V_{ay}^+ - V_{ay}^- - V_{bx}^+ - V_{bx}^- + V_{by}^+ + V_{by}^- \right) - 2\alpha^2 = - \left\langle \Delta \hat{S}_{1d}\Delta \hat{S}_{3d} \right\rangle \tag{27}
\]

so that

\[
\left| \left\langle \Delta \hat{S}_{1c}\Delta \hat{S}_{3c} \right\rangle \right| = \left| \left\langle \Delta \hat{S}_{1d}\Delta \hat{S}_{3d} \right\rangle \right| = \left\{ \frac{1}{16} \alpha^4 \left( V_{ax}^+ - V_{ax}^- + V_{ay}^+ - V_{ay}^- - V_{bx}^+ - V_{bx}^- + V_{by}^+ + V_{by}^- \right)^2 + 4\alpha^4 \right\}^{1/2}. \tag{28}
\]

This form shows the provenance of the two contributions to the variance product (8), with the first term in the square root coming from the anticommutator and the second from the commutator, equal to \(2i\langle S_2 \rangle\) in this example. In the simple case where the four modes making up the polarization-squeezed beams all have equal quadrature squeezing \(V^+\), \(V_A = V_B = 4\alpha^2V^+\), only the contribution of the commutator survives in (28) to give

\[
\left| \left\langle \Delta \hat{S}_{1c}\Delta \hat{S}_{3c} \right\rangle \right| = \left| \left\langle \Delta \hat{S}_{1d}\Delta \hat{S}_{3d} \right\rangle \right| = 2\alpha^2. \tag{29}
\]

The range (22) for nonseparability therefore becomes

\[
0 \leq V^+ < 1. \tag{30}
\]

Note that the proper general form of the upper limit in the entanglement criterion in (22) was not given in [2] but the correct form of the specific result (30) was nevertheless derived there.

**Example 2: Bowen et al.** The experiment performed in [4] also uses bright light beam subsystems labelled c and d (x and y in notations of [4]), each of which has \(H\) and \(V\)
polarization components, with coherent amplitudes $\alpha_H$ for both the $H$ components and $\alpha_V$ for both the $V$ components. Both $H$ components have quadrature variances $V_H^+ \alpha_H$ and $V_H^- \alpha_H$, with a similar notation for the common $V$ variances. The $H$ light beams are produced by interference of primary quadrature-squeezed beams, then combined with much more intense $V$ beams to form the $c$ and $d$ subsystems, in the manner described in [4]. The Stokes parameter operators have the same general properties as outlined above but with a generalization to allow for a phase difference $\theta$ between the $H$ and $V$ components. For the performed experiment with $\theta = \pi/2$, the Stokes operators are related to those used in [2], which correspond to $\theta = 0$, by

$$\hat{S}_1(\pi/2) = \hat{S}_1(0), \quad \hat{S}_2(\pi/2) = -\hat{S}_3(0), \quad \hat{S}_3(\pi/2) = \hat{S}_2(0). \quad (31)$$

The expectation values of the $\theta = \pi/2$ operators are

$$\langle \hat{S}_1 \rangle = \alpha_H^2 - \alpha_V^2, \quad \langle \hat{S}_2 \rangle = 0, \quad \langle \hat{S}_3 \rangle = 2\alpha_H \alpha_V, \quad (32)$$

in agreement with (25) when $\alpha_H = \alpha_V = \alpha$ and the conversion (31) is used.

The quantities needed for evaluation of the entanglement criteria, again for the $\theta = \pi/2$ operators, are [4]

$$V \left( \hat{S}_{1c} \pm \hat{S}_{1d} \right) = 2\alpha_H^2 V_H^+, \quad V \left( \hat{S}_{2c} \pm \hat{S}_{2d} \right) = 2\alpha_V^2 V_H^-, \quad V \left( \hat{S}_{3c} \pm \hat{S}_{3d} \right) = 2\alpha_H^2 V_H^+, \quad (33)$$

$$\left| \langle \Delta \hat{S}_1 \Delta \hat{S}_2 \rangle \right| = 2\alpha_H \alpha_V \left| V_H^+ - V_H^- \right|, \quad (34)$$

with the same results for subsystems $c$ and $d$. With use of the form of variance product given in (8), the first two expressions in (34) result from the commutator, as is evident from comparison with (2) and (32), and the third from the anticommutator.

The possibilities for entanglement with the three pairings of the Stokes parameters are readily evaluated by substitution of the expressions from (33) and (34) in (22). With the vertically-polarized input beams assumed to be much brighter than the horizontally-polarized beams ($\alpha_V \gg \alpha_H$), the entanglement criterion is difficult to satisfy for the pairs of Stokes operators $\hat{S}_1, \hat{S}_2$ and $\hat{S}_3, \hat{S}_1$. However, the entanglement criterion for the $\hat{S}_2, \hat{S}_3$ pair is the same as that for quadrature entanglement and the corresponding polarization entanglement was demonstrated experimentally [4].

To conclude, the usual uncertainty relation in the form of (11) is generalized to the equivalent forms in (7) and (8) when the commutator of the operators of interest is itself an operator, not a $c$-number, and when the anticommutator of the operators is nonzero [10,17]. The corresponding contributions to the minimum variance product on the right of (8) are then both state-dependent, in contrast to the state-independent form of (11), and there is no general reason to neglect either of them. The generalization of the uncertainty relation also affects the range of values for the variance sum in the usual nonseparability or entanglement criterion [15] reproduced in (17), which is converted to the more general form in (22). The retention of the anticommutator contribution in (8) has the effect of increasing the upper limit for entanglement on the right of (22). The operator representations of the Stokes polarization parameters provide examples of operator-valued commutation relations, where the more general theory is needed for the description of recently proposed [2] or performed [4,6] experiments.

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APPENDIX A: DERIVATION OF THE UPPER LIMIT IN EQ. (22)

Consider the variables \( \hat{A} = \hat{A}_c + \hat{A}_d \) and \( \hat{B} = \hat{B}_c - \hat{B}_d \) (18) of two subsystems \( c \) and \( d \). They obey the commutator relations \([\hat{A}_c, \hat{B}_d] = [\hat{B}_c, \hat{A}_d] = 0; [\hat{A}_c, \hat{B}_c] \neq 0 \) and \([\hat{A}_d, \hat{B}_d] \neq 0 \) where the nonzero commutators can be some nontrivial operators. The variances of \( \hat{A}, \hat{B} \) are defined as \( V_z = \langle (\Delta \hat{Z})^2 \rangle_\rho = \langle \hat{Z}^2 \rangle_\rho - \langle \hat{Z} \rangle_\rho^2 \) with \( Z = A, B \) and \( \Delta \hat{Z} = \hat{Z} - \langle \hat{Z} \rangle \).

**Theorem.** For any separable state \( \rho_{\text{sep}} \) the following inequality holds:

\[
V_A + V_B \geq 2 \left| \langle \Delta \hat{A}_c \Delta \hat{B}_c \rangle \right| + 2 \left| \langle \Delta \hat{A}_d \Delta \hat{B}_d \rangle \right| \quad (A1)
\]

**Proof.** A separable quantum state \( \rho_{\text{sep}} \) can be written as a convex decomposition

\[
\rho_{\text{sep}} = \sum_j p_j \rho_{j c} \otimes \rho_{j d}. \quad (A2)
\]

Using this decomposition we can directly compute the sum of the variances \( V_A + V_B \). The averaging in the expressions below is performed over the product density matrix \( \rho_{\text{sep}} = \sum_j p_j \rho_{j c} \otimes \rho_{j d} \). We obtain:

\[
V_A + V_B = \sum_j p_j \left( \langle \hat{A}^2 \rangle_j + \langle \hat{B}^2 \rangle_j \right) - \left( \sum_j p_j \langle \hat{A} \rangle_j \right)^2 - \left( \sum_j p_j \langle \hat{B} \rangle_j \right)^2
\]

\[
= \sum_j p_j \left( \langle \hat{A}_c^2 \rangle_j + \langle \hat{A}_d^2 \rangle_j + \langle \hat{B}_c^2 \rangle_j + \langle \hat{B}_d^2 \rangle_j \right) - \left( \sum_j p_j \langle \hat{A}_c \rangle_j \right)^2 - \left( \sum_j p_j \langle \hat{B}_c \rangle_j \right)^2
\]

\[
+ 2 \left( \sum_j p_j \langle \hat{A}_c \rangle_j \langle \hat{A}_d \rangle_j - \sum_j p_j \langle \hat{B}_c \rangle_j \langle \hat{B}_d \rangle_j \right) - \left( \sum_j p_j \langle \hat{A} \rangle_j \right)^2 - \left( \sum_j p_j \langle \hat{B} \rangle_j \right)^2 =
\]

\[
\sum_j p_j \left( \langle (\Delta \hat{A}_c)^2 \rangle_j + \langle (\Delta \hat{A}_d)^2 \rangle_j + \langle (\Delta \hat{B}_c)^2 \rangle_j + \langle (\Delta \hat{B}_d)^2 \rangle_j \right)
\]

\[
\sum_j p_j \left( \langle \hat{A}_j^2 \rangle + \langle \hat{B}_j^2 \rangle \right) - \left( \sum_j p_j \langle \hat{A} \rangle_j \right)^2 - \left( \sum_j p_j \langle \hat{B} \rangle_j \right)^2. \quad (A3)
\]

Let us estimate the limits for the last two lines in (A3). We use the Schwarz inequality in the form (7) and \( V_A^2 + V_B^2 \geq 2V_AV_B \) and get:

\[
\sum_j p_j \left( \langle (\Delta \hat{A}_c)^2 \rangle_j + \langle (\Delta \hat{B}_c)^2 \rangle_j + \langle (\Delta \hat{A}_d)^2 \rangle_j + \langle (\Delta \hat{B}_d)^2 \rangle_j \right)
\]

\[
\geq 2 \left| \langle \Delta \hat{A}_c \Delta \hat{B}_c \rangle \right| + 2 \left| \langle \Delta \hat{A}_d \Delta \hat{B}_d \rangle \right|.
\]

Note that the application of the Schwarz inequality (7) corresponds to the use of the generalized uncertainty relation: Eq. (7) is readily re-expressed in the form Eq. (8) and the anti-commutator term is retained. Furthermore, it can be easily shown [15] using the Cauchy-Schwarz inequality \( \sum p_j \langle \hat{A}_j \rangle \langle \hat{B}_j \rangle \leq \sum p_j |\langle \hat{A}_j \rangle| \) that the lower bound for the last line in (A3) is zero,

\[
\sum_j p_j \left( \langle \hat{A}_j^2 \rangle + \langle \hat{B}_j^2 \rangle \right) - \left( \sum_j p_j \langle \hat{A} \rangle_j \right)^2 - \left( \sum_j p_j \langle \hat{B} \rangle_j \right)^2 \geq 0.
\]

Hence, for any separable state (A2) the inequality (A1) holds, which proves our statement.

It follows from Eq. (A1) and uncertainty relations (7), (8) that the nonseparable or entangled states have to satisfy Eq. (22):

\[
2 \left| \langle \Delta \hat{A}_c \Delta \hat{B}_c \rangle - \langle \Delta \hat{A}_d \Delta \hat{B}_d \rangle \right| \leq V_A + V_B < 2 \left| \langle \Delta \hat{A}_c \Delta \hat{B}_c \rangle \right| + 2 \left| \langle \Delta \hat{A}_d \Delta \hat{B}_d \rangle \right|. \quad (A4)
\]

In contrast to Eqs. (5-7) [18] in Ref. [16] the lower limit in (A1) and hence the upper limit in Eqs. (22), (A4) does not depend on the particular form of the convex decomposition in (A2). However, the lower bound for \( V_A + V_B \) (A1) and the limits in the nonseparability criterion Eq. (22), (A4) do depend on the quantum state under consideration. There is no universal separability limit for the sum or product of the two variances \( V_A, V_B \) in the case of the operator-valued commutators \([\hat{A}_c, \hat{B}_c] \) and \([\hat{A}_d, \hat{B}_d] \). Nevertheless, the inequalities of Eqs. (A1), (22), (A4) provide a sensible operational sufficient criterion for nonseparability which can be readily verified in an experiment (see examples in the paper).

[17] E. Merzbacher, Quantum Mechanics, 3rd edn (Wiley, New York, 1998) pp. 217-220.

[18] Please note that Eq. (7) in Ref. [16] contains a misprint but appears correctly in the corresponding manuscript quant-ph/0210155 on the E-print arXive.