Algebraic Bethe ansatz for eight vertex model with general open-boundary conditions

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Abstract

By using the intertwiner and face-vertex correspondence relation, we obtain the Bethe ansatz equation of eight vertex model with open boundary conditions in the framework of algebraic Bethe ansatz method. The open boundary condition under consideration is the general solution of the reflection equation for eight vertex model with only one restriction on the free parameters of the right side reflecting boundary matrix. The reflecting boundary matrices used in this paper thus may have off-diagonal elements. Our construction can also be used for the Bethe ansatz of SOS model with reflection boundaries.

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1 Introduction

One of the most important goals of exactly solvable lattice models is to find the eigenvalues and eigenvectors of the transfer matrix of a system, then to obtain the thermodynamic limit of this system.

Bethe examined the completely isotropic case of the $XXX$ model and found the eigenvalues and eigenvectors of its Hamiltonian\cite{1}. After Bethe’s work, Yang and Yang analyzed the anisotropic $XXZ$ model by means of Bethe ansatz\cite{2}. Then, Baxter in his remarkable papers gave a solution for the completely anisotropic $XYZ$ model\cite{3}. He discovered a relation between the quantum $XYZ$ model and eight vertex model which is one of the two dimensional exactly solvable lattice model. Faddeev and Takhtajan simplified Baxter’s formulae and proposed the quantum inverse scattering method or algebraic Bethe ansatz method to solve the six vertex and eight vertex models, whose spin chain equivalent are $XXZ$ spin model and $XYZ$ spin model, respectively\cite{4}. After Yang-Baxter-Faddeev-Takhtajan’s work, a lot of exactly solvable models have been solved by algebraic Bethe ansatz\cite{5,6}, functional Bethe ansatz\cite{7,8}, co-ordinate Bethe ansatz\cite{9}, etc.

Typically, the two-dimensional exactly solvable lattice models are solved by imposing periodic boundary conditions in which the Yang-Baxter equation provides a sufficient condition for the integrability of the models.

$$R_{12}(z_1 - z_2)R_{13}(z_1 - z_3)R_{23}(z_2 - z_3) = R_{23}(z_2 - z_3)R_{13}(z_1 - z_3)R_{12}(z_1 - z_2)$$

(1)

where, the $R$-matrix is the Boltzmann weight for the vertex models in two dimensional statistical mechanics. As usual, $R_{12}(z), R_{13}(z)$ and $R_{23}(z)$ act in $C^n \otimes C^n \otimes C^n$ with $R_{12}(z) = R(z) \otimes 1$, $R_{23}(z) = 1 \otimes R(z)$, etc.

The exactly solvable models with non-periodic boundary conditions have been early studied in ref.[10-14]. Recently, integrable models with open boundary conditions have been attracting a great deal of interests. This was initiated by Cherednik\cite{15} and Sklyanin\cite{16}, they proposed a systematic approach to handle the open boundary condition problems which involves the so called reflection equation (RE).

$$R_{12}(z_1 - z_2)K_1(z_1)R_{21}(z_1 + z_2)K_2(z_2) = K_2(z_2)R_{12}(z_1 + z_2)K_1(z_1)R_{21}(z_1 - z_2)$$

(2)

The open boundary conditions are determined by the boundary reflecting matrix $K$ satisfying the RE.
By using a non-trivial generalization of the quantum inverse scattering method, Sklyanin obtained the Bethe ansatz equation of six vertex model with open boundary conditions by algebraic Bethe ansatz method[16]. The transfer matrix with a particular choice of boundary conditions is quantum group $U_q(sl(2))$ invariant[17,18]. After Sklyanin’s pioneering work, a lot of exactly solvable lattice models with open boundary conditions have been solved. Mezincescu and Nepomechie solved the $A_1^{(1)}$ and $A_2^{(2)}$ vertex models by using the fusion procedure[19]. Foerster and Karowski solved $spl_q(2,1)$ invariant Hamiltonian which contains a non-trivial boundary term by using the nested algebraic Bethe ansatz[20]. Using the same method, de Vega and Gonzalez-Ruiz solved the $A_n$ vertex model, they also analyzed the thermodynamic limit of this system[21]. Yue, Fan and Hou solved the general $SU_q(n|m)$ vertex model[22]. For other progress about open boundary conditions along this direction, see e.g. ref.[23-31].

However, for Baxter’s eight vertex model with open-boundary conditions, little progress has been made. Jimbo et al obtained the difference equation of n point function for semi-infinite $XYZ$ chain [32]. Yu-kui Zhou have recently studied the fused eight vertex model and have found the functional relations for eight vertex model with open boundary conditions[33], in which the reflecting $K$ matrix which satisfy the vertex RE(2) are diagonal.

In this paper, we will use the algebraic Bethe ansatz method to solve the eight vertex model with open boundary conditions. It is known that $K$ matrix is a solution of RE. The general solution of RE for six vertex model was obtained by de Vega[34]. In ref.[35,36], solutions of RE for eight vertex model have been found. The general solution of RE for eight vertex model which has three free parameters was found by two groups[37,38]. In our approach we use the general solution for left and right boundaries. The reflecting boundary $K$ matrices thus may have off diagonal elements. We need only to impose one relation on the free parameters of the right $K$ matrix, if the free parameters of the left $K$ matrix are arbitrarily given.

It is known that in Baxter’s original work, in stead of Yang-Baxter relation, the star-triangular relation plays the key role which can be obtained from Yang-Baxter relation by using the intertwiners. This was later generalized to $Z_n$ Baxter-Belavin model[39,40] to describe the interaction-round-a-face model by Jimbo, Miwa and Okado[41]. This intertwiner method was also used to solve the Bethe ansatz problem for similar cases [42,43]. In order to get algebraic Bethe ansatz equation of eight vertex model with
open-boundary conditions, we need to describe the exchange relations of
the monodromy matrix in the "face language". Thus in our paper, we need
to convert our boundary conditions of vertex model to that of a face model.
Our approach is equivalent to a SOS model with open boundaries satisfying
the face RE. The face RE was first proposed by Behrend, Pearce and Bri
en [44]. In ref.[44], they also find a diagonal solution of face RE for ABF[45]
model. By using the intertwiners, the face RE is derived directly from vertex
RE and the general solution of face RE for eight vertex SOS model are found
by other groups[46,33]. In this paper we actually use a diagonal solution of
face RE at the left boundary and an upper triangular solution at the right
boundary for eight vertex SOS model. This open boundary conditions are
different from the case discussed by Yu-kui Zhou in ref.[33]. Since a diagonal
matrix is a special case of the upper triangular matrix, our approach can
be used to a SOS model with boundaries proposed by Behrend et al. We
can prove that by taking a special case, the Bethe ansatz equation for RSOS
model, ABF model, can also be obtained.

The outline of this paper is as follows. In section 2, we first review
the eight vertex model and reflecting open boundary conditions, the model
under consideration in this paper will also be constructed. In sect.3, by
using the correspondence of face and vertex models, we derive the face RE.
As mentioned above, the face RE will play a key role in the algebraic Bethe
ansatz method for eight vertex model instead of vertex RE. The transfer
matrix with boundary conditions will also be constructed by using the face
weight. In sect.4, we will find the local vaccum for eight vertex model with
a boundary which is the same as the vaccum state given by Baxter[3,7] in
the Bethe ansatz of eight vertex model with periodic boundary condition.
The face boundary matrix derived directly from vertex boundary matrix is
obtained. In sect.5, the Bethe ansatz problem is solved for eight vertex model
with open-boundary conditions. Section 6 contains some discussions and the
further work.

2 Description of the model
2.1 The R matrix

We first start from the R-matrix of the eight vertex model. Denote \( \alpha = (\alpha_1, \alpha_2), \alpha_1, \alpha_2 = 0, 1 \). Let \( g \) and \( h \) be \( 2 \times 2 \) matrices with elements \( g_{ii'} = (-1)^i \delta_{ii'}, h_{ii'} = \delta_{i+1, i'}, i, i' = 0, 1 \).

Define \( 2 \times 2 \) matrices \( I_\alpha = I(\alpha_1, \alpha_2) = h\alpha_1 \otimes g\alpha_2, I_0 = I = \text{identity} \), and define \( I^{(j)}_\alpha = I \otimes I \otimes \cdots \otimes I\alpha \otimes \cdots \otimes I, I\alpha \) is at \( j \)-th space. As usual, \( I^{(j)}_\alpha \) act in \( V = V_1 \otimes \cdots \otimes V_l \), where space \( V \) is consisted of \( l \) two-dimensional spaces.

We then introduce some notations used in this paper. They are

\[
\theta \left[ \begin{array}{c} a \\ b \end{array} \right] (z, \tau) \equiv \sum_{m \in \mathbb{Z}} \exp \left\{ \pi \sqrt{-1} (m + a)[(m + a)\tau + 2(z + b)] \right\}, \quad (3)
\]

\[
\sigma_\alpha(z) \equiv \theta \left[ \begin{array}{c} \frac{1}{2} + \frac{\alpha_1}{2} \\ \frac{1}{2} + \frac{\alpha_2}{2} \end{array} \right] (z, \tau), \quad (4)
\]

\[
h(z) \equiv \sigma_{(0,0)}(z), \quad (5)
\]

\[
\theta^{(i)}(z) \equiv \theta \left[ \begin{array}{c} \frac{1}{2} - \frac{i}{2} \\ \frac{1}{2} \end{array} \right] (z, 2\tau), i = 0, 1, \quad (6)
\]

\[
W_\alpha(z) = \frac{1}{2} \sigma_\alpha(z + \frac{w}{2}). \quad (7)
\]

The R-matrix of eight vertex model takes the form (Fig.1a)

\[
R_{jk}(z) = \sum_\alpha W_\alpha(z) I^{(j)}_\alpha I^{-1}(k) \quad (8)
\]

which satisfy the Yang-Baxter equation (Fig.2a)

\[
R_{ij}(z_i - z_j)R_{ik}(z_i - z_k)R_{jk}(z_j - z_k)
= R_{jk}(z_j - z_k)R_{ik}(z_i - z_k)R_{ij}(z_i - z_j). \quad (9)
\]

It can be proved that the R matrix of eight vertex model satisfy the following unitarity and cross-unitarity conditions,

\[
\text{unitarity} : R_{ij}(z)R_{ji}(-z) = \rho(z) \cdot id, \quad (10)
\]

\[
\text{cross-unitarity} : R_{ij}^{(i)}(z)R_{ji}^{(i)}(-z - 2w) = \rho'(z) \cdot id, \quad (11)
\]
where \( id \) is the identity and \( t_i \) denotes transposition in the \( i \)-th space. \( \rho(z) \) and \( \rho'(z) \) are scalars satisfying

\[
\rho(z) = \rho(-z), \quad (12)
\]
\[
\rho'(z) = \rho'(-z - 2w). \quad (13)
\]

In eqs. (8-11), the indices take the value \( i, j, k = 1, \ldots, l \).

### 2.2 Reflection equation and reflecting boundary conditions

We now deal with an exactly solvable lattice model with reflecting boundary conditions, the R-matrix defined above is the Boltzmann weights for this lattice model. In order to construct the transfer matrix of this system, we must introduce a reflecting boundary matrix \( K(z) \) which is a \( 2 \times 2 \) matrix and satisfy the reflection equation proposed by Cherednik [15] and Sklyanin [16] (Fig.2b).

\[
R_{12}(z_1 - z_2)K_1(z_1)R_{21}(z_1 + z_2)K_2(z_2) = K_2(z_2)R_{12}(z_1 + z_2)K_1(z_1)R_{21}(z_1 - z_2) \quad (14)
\]

As mentioned in the introduction, two groups have obtained independently the general solution \( K(z) \) of this reflection equation for eight vertex model. Here we take the solution \( K(z) \) as [38]

\[
K(z) = \sum_{\alpha} C_{\alpha} \frac{I_{\alpha}}{\sigma_{\alpha}(-z)}, \quad (15)
\]

where \( C_{\alpha} \) are arbitrary parameters. Correspondingly, we have the dual reflection equation which is necessary in the following of this paper (Fig.2c):

\[
R_{12}(z_2 - z_1)\tilde{K}_1(z_1)R_{21}(-z_1 - z_2 - 2w)\tilde{K}_2(z_2) = \tilde{K}_2(z_2)R_{12}(-z_1 - z_2 - 2w)\tilde{K}_1(z_1)R_{21}(z_2 - z_1). \quad (16)
\]

We take the solution of this reflection equation as

\[
\tilde{K}(z) = \sum_{\alpha} \tilde{C}_{\alpha} \frac{I_{\alpha}}{\sigma_{\alpha}(z + w)}, \quad (17)
\]
where $\tilde{C}_\alpha$ are also arbitrary parameters. Usually, we also call the reflecting boundary $K$ and $\tilde{K}$ matrices as right and left boundary matrices, respectively.

In order to deal with the systems with open boundary conditions, let us define two forms of standard "row-to-row" monodromy matrices $S_1(z_1)$ and $T_1(z_1)$ which act in the space $V = V_1 \otimes V_2 \otimes \cdots \otimes V_l$ by

\[
S_1(z_1) = R_{l1}(u_l + z_1)R_{l-1,1}(u_{l-1} + z_1) \cdots R_{31}(u_3 + z_1), \\
T_1(z_1) = R_{13}(z_1 - u_3)R_{14}(z_1 - u_4) \cdots R_{1l}(z_1 - u_l),
\]

(18)

where $(u_l, u_{l-1}, \cdots u_3) \equiv \{u_i\}$ are arbitrary parameters. $S_2(z_2)$ and $T_2(z_2)$ can also be similarly defined.

Considering $\{u_i\}$ are the same for $S_1(z_1)$ and $S_2(z_2)$, and noticing that two $R$ matrices acting on four different spaces commute with each other, we find

\[
R_{21}(z_1 - z_2)S_1(z_1)S_2(z_2) = R_{21}(z_1 - z_2)R_{l1}(u_l + z_1)R_{l2}(u_l + z_2) \cdots.
\]

(19)

Using Yang-Baxter equation repeatedly, we have

\[
R_{21}(z_1 - z_2)S_1(z_1)S_2(z_2) = S_2(z_2)S_1(z_1)R_{21}(z_1 - z_2).
\]

(20)

Similarly, we can also obtain

\[
T_1(z_1)R_{12}(z_1 + z_2)S_2(z_2) = S_2(z_2)R_{12}(z_1 + z_2)T_1(z_1),
\]

(21)

\[
T_2(z_2)T_1(z_1)R_{12}(z_1 - z_2) = R_{12}(z_1 - z_2)T_1(z_1)T_2(z_2).
\]

(22)

For the periodic boundary condition cases which are studied extensively before, the transfer matrix is defined as the trace of the standard "row-to-row" monodromy matrix. But for the open boundary conditions cases, instead of the standard "row-to-row" monodromy matrix, we should define the "double-row" monodromy matrices which take the form:

\[
k_1(z_1) = T_1(z_1)K_1(z_1)S_1(z_1), \\
k_2(z_2) = T_2(z_2)K_2(z_2)S_2(z_2).
\]

(23)
Using the relations listed above, we can prove $k_i(z_i)$ satisfy the reflection equation

$$R_{12}(z_1 - z_2)k_1(z_1)R_{21}(z_1 + z_2)k_2(z_2) = k_2(z_2)R_{12}(z_1 + z_2)k_1(z_1)R_{21}(z_1 - z_2).$$  \hspace{1cm} (24)$$

As used usually in the framework of the quantum inverse scattering method, $k_i(z_i)$ are $2 \times 2$ matrix with elements defined as operators acting in the space $V' = V_3 \otimes V_4 \otimes \cdots \otimes V_l$ which is the so called quantum space, the spaces $V_1$ and $V_2$ are the auxiliary spaces. Eqs.(14) and (24) shows that $k(z)$ is the co-module of $K(z)$.

### 2.3 The transfer matrix

Now, let’s formulate the transfer matrix with open boundary conditions.

$$t(z_i) = Tr_{V_i} \left\{ \tilde{K}_i(z_i) k_i(z_i) \right\} = \sum_{kl} \tilde{K}(z_i) k_{ik}(z_i),$$  \hspace{1cm} (25)$$

with $i = 1, 2$. Since the transfer matrices are defined as the trace over the auxiliary spaces $V_i, i = 1, 2$, they should be independent of $V_1$ and $V_2$, and are represented as operators acting in the quantum space $V_3 \otimes \cdots \otimes V_l$. With the help of the unitarity, cross-unitarity relations of R matrix, Yang-Baxter relation, reflection equation and its dual reflection equation, we can prove that the transfer matrices with different spectrum commute with each other[47],

$$t(z_1)t(z_2) = t(z_2)t(z_1).$$  \hspace{1cm} (26)$$

This ensures the integrability of this system.

The aim of this paper is to find the eigenvalues and eigenvectors of the transfer matrix which defines the Hamiltonian of the system under consideration. We will use the algebraic Bethe ansatz method to solve this problem. The transfer matrix is defined as a linear function of the elements of the "double-row" monodromy matrix. So, it is necessary to find the proper linear combinations of the elements of the "double-row" monodromy matrix whose commutation relations are suitable for algebraic Bethe ansatz. Besides this,
we also need to find an "vaccum" state which is independent of the spectrum $z$. It is well known that this "vaccum" state can be obtained easily for six vertex model with periodic- or open-boundary conditions. For eight vertex model, it is not a trivial problem. We will study the commutation relations and the "vaccum" state problems in the following sections.

3 Commutation relations

It is known that for six vertex model and other trigonometric vertex models we can obtain the necessary commutation relations directly from the reflection equation in which $k(z)$ is the "double-row" monodromy matrix. But for the eight vertex model whose R matrix has eight non-zero elements, we can not obtain such relations directly from the reflection equation. We have to use the vertex-face correspondence to solve this problem. That means we should properly combine the elements of $k(z)$ so that we can find simple commutation relations which can be dealt with by algebraic Bethe ansatz method.

3.1 Face-vertex correspondence

We first define a two element column vectors $\phi_{m,\mu}(z)$, $\mu = 0, 1, m \in Z$, whose $k$-th element is [7,41,49] (Fig.1c)

$$ \phi_k^{m,\mu} = \theta^{(k)}(z + (-1)^{\mu}wa + w\beta), $$

where $a = m + \gamma, \gamma, \beta \in C, k = 0, 1$. We call $m$ the face weight, $\mu$ the face index, which take values $0, 1$. $\phi$ is usually called the three-spin operator. It can be proved that we can find row vectors $\tilde{\phi}, \tilde{\phi}$ satisfying the following conditions for generic $w, \beta, \gamma$.

$$ \tilde{\phi}_{m+\tilde{\mu},\mu}(z)\phi_{m+\nu,\nu}(z) = \delta_{\mu\nu}, $$

$$ \tilde{\phi}_{m,\mu}(z)\phi_{m,\nu}(z) = \delta_{\mu\nu}, $$

where

$$ \tilde{\mu} \equiv (-1)^{\mu}, \tilde{\nu} \equiv (-1)^{\nu}. $$
The above relations can also be written in other forms
\[
\sum_{\nu=0}^{1} \phi_{m+\check{\nu},\nu}(z) \tilde{\phi}_{m+\check{\nu},\nu}(z) = I,
\]
\[
\sum_{\mu=0}^{1} \phi_{m,\mu}(z) \bar{\phi}_{m,\mu}(z) = I.
\] (29)

As usual, \( I \) is the 2 \( \times \) 2 unit matrix. (Fig.3)

We define the face Boltzmann weights for the interaction-round-a-face model (IRF) as follows[7,41,45,49]:
\[
W(m|z)_{\mu\nu} = \frac{h(z+w)}{h(w)}
\]
\[
W(m|z)_{\mu\nu} = \frac{h(w(m+\gamma) - (-1)^{\mu} z)}{h(w(m+\gamma))}, \mu \neq \nu,
\]
\[
W(m|z)_{\mu\nu} = \frac{h(z)h(w(m+\gamma) - (-1)^{\mu} w)}{h(w)h(w(m+\gamma))}, \mu \neq \nu,
\] (30)

where the face indices \( \mu, \nu \) take the values 0, 1. The other face Boltzmann weights are defined as zeroes, so we can see explicitly that for a given face weight \( m \), we only have six non-zero face Boltzmann weights at all. Traditionally, the face Boltzmann weights for eight vertex SOS model are denoted as \( W_z \left[ \begin{array}{cc} a & b \\ c & d \end{array} \right] \). Its relation with the notations used in this paper is (Fig.1b):
\[
W(m|z)_{\mu\nu} = W_z \left[ \begin{array}{cc} m + \check{\mu} & m + \check{\nu} \\ m + \check{\mu} & m \end{array} \right]
\] (31)

The reasons that we use notations (30) are that a lot of zero face Boltzmann weights will not appear in our calculation. It is also convenient to compare the results of eight vertex model case with the six vertex model case.

The face Boltzmann weights of IRF model defined above have a relation with the R-matrix of eight vertex model which is usually called the face-vertex correspondence[3,7,41].

\[
R_{12}(z_1 - z_2)\phi_{m+\check{\nu},\nu}(z_1)\phi_{m+\check{\nu},\nu}(z_2)
= \sum_{\mu',\nu'} W(m|z_1 - z_2)_{\mu',\nu'} \phi_{m+\check{\mu}',\nu'}(z_2)\phi_{m+\check{\mu}',\nu'}(z_1)
\] (32)
where \( \phi^{(i)} \) denote that it act in \( i \)-th space \((i = 1, 2)\).

With the help of the properties of \( \phi, \tilde{\phi}, \bar{\phi} \) \((28,29)\), we can derive the following relations from the above face-vertex correspondence relation.

\[
\begin{align*}
\tilde{\phi}^{(1)}_{m+\mu,\mu}(z_1)R_{12}(z_1-z_2)\phi^{(2)}_{m+\nu,\nu}(z_2) &= \sum_{\mu',\nu'} W(m|z_1-z_2)\tilde{\phi}^{(1)}_{m+\mu+\nu,\mu'}(z_1)\phi^{(2)}_{m+\nu',\nu'}(z_2), \quad (33) \\
\bar{\phi}^{(2)}_{m+\mu,\mu}(z_2)R_{12}(z_1-z_2)\phi^{(1)}_{m+\nu,\nu}(z_1) &= \sum_{\mu',\nu'} W(m|z_1-z_2)\bar{\phi}^{(2)}_{m+\mu,\nu'}(z_2)\tilde{\phi}^{(1)}_{m+\nu',\mu'}(z_1), \quad (34) \\
\bar{\phi}^{(2)}_{m,\mu}(z_2)R_{12}(z_1-z_2)\phi^{(1)}_{m,\mu}(z_1) &= \sum_{\nu',\mu'} W(m-\mu-\nu'|z_1-z_2)\mu'\nu'\bar{\phi}^{(1)}_{m-\mu-\nu',\mu'}(z_1)\phi^{(2)}_{m-\nu',\nu'}(z_2), \quad (35) \\
\bar{\phi}^{(2)}_{m+\mu+\nu,\nu}(z_2)\bar{\phi}^{(1)}_{m+\mu,\mu}(z_1)R_{12}(z_1-z_2) &= \sum_{\mu',\nu'} W(m|z_1-z_2)\mu'\nu'\bar{\phi}^{(1)}_{m+\mu+\nu',\mu'}(z_1)\phi^{(2)}_{m+\nu',\nu'}(z_2). \quad (36)
\end{align*}
\]

All of these relations obtained above have described the correspondence between face and vertex models. Usually, we call \( \phi \) the intertwiner of face-vertex correspondence (Fig.4).

### 3.2 Commutation relations for elements of the face boundary reflecting k matrix

As mentioned above, in order to obtain a comparatively simple commutation relations which can be dealt with by algebraic Bethe ansatz method, we should change the ”vertex” reflection equations to ”face” reflection equations, since where the new ”R-matrix” - face Boltzmann weights have only six non-zero elements instead of eight non-zero elements in ”vertex” case. For this purpose, by using the three-spin operator \( \phi \), we change the vertex boundary reflecting matrix to face boundary reflecting matrix, and find the commutation relations between the elements of the ”face” type monodromy matrix, which are useful for the quantum inverse scattering method.

We first change the matrix \( \bar{K}(z) \) defined in eqs.(16,17) to face boundary reflecting matrix \( \bar{K}(m|z)_{\mu}^{\nu} \), using the unitarity properties of the intertwiner
(29), we find

\[ \tilde{K}(z) = \sum_{\mu} \left\{ \phi_{m,\mu}(-z) \tilde{\phi}_{m,\mu}(-z) \tilde{K}(z) \right\} \times \sum_{\nu} \left[ \phi_{m-\hat{\mu}+\hat{\nu},\nu}(z) \tilde{\phi}_{m-\hat{\mu}+\hat{\nu},\nu}(z) \right] \]

\[ = \sum_{\mu\nu} \phi_{m,\mu}(-z) \tilde{\phi}_{m-\hat{\mu}+\hat{\nu},\nu}(z) \tilde{K}(m|z)_{\mu}^\nu, \]

where

\[ \tilde{K}(m|z)_{\mu}^\nu \equiv \phi_{m,\mu}(-z) \tilde{K}(z) \phi_{m-\hat{\mu}+\hat{\nu},\nu}(z) \]  

(37)

Thus, the transfer matrix of eight vertex model with open-boundary conditions can be rewritten as

\[ t(z) = Tr \left( \tilde{K}(z)k(z) \right) \]

\[ = Tr \left\{ \sum_{\mu\nu} \phi_{m,\mu}(-z) \tilde{\phi}_{m-\hat{\mu}+\hat{\nu},\nu}(z) \tilde{K}(m|z)_{\mu}^\nu k(z) \right\} \]

\[ = \sum_{\mu\nu} \left[ \phi_{m-\hat{\mu}+\hat{\nu},\nu}(z) k(z) \phi_{m,\mu}(-z) \right] \tilde{K}(m|z)_{\mu}^\nu \]

\[ \equiv \sum_{\mu\nu} k(m|z)_{\mu}^\nu \tilde{K}(m|z)_{\mu}^\nu, \]  

(38)

which is true for arbitrary \( m \). Here we also introduce the definition of face boundary reflecting \( k \) matrix as:

\[ k(m|z)_{\mu}^\nu = \phi_{m-\hat{\mu}+\hat{\nu},\nu}(z) k(z) \phi_{m,\mu}(-z). \]

(39)

We call \( m \) and \( m - \hat{\mu} + \hat{\nu} \) the initial and final weight of \( k(m|z)_{\mu}^\nu \), respectively. Thus, we have written out the transfer matrix by using the face form of the model.

Next, we will derive the face reflection equation directly from the vertex reflection equation by using the intertwiner. Multiply both sides of eq.(14) from left by \( \tilde{\phi}_{m+\hat{\mu}_0,\nu_0}(z_1) \phi_{m+\hat{\mu}_0+\hat{\nu}_0,\nu_0}(z_2) \), from right by \( \phi_{m+\hat{\mu}_3,\nu_3}(-z_1) \phi_{m+\hat{\mu}_3+\hat{\nu}_3,\nu_3}(-z_2) \), notice the properties such as \( \tilde{\phi}^{(2)} \) commutes with \( \tilde{\phi}^{(1)} \) and
$k_1(z_1)$, use the face-vertex correspondence relations. We can get the face reflection equation (see Appendix A and Fig.5,6) [44,46,33],

$$k(m + \hat{\mu}_2 + \hat{\nu}_1|z_1)^{\mu_2}_{\mu_1} k(m + \hat{\mu}_3 + \hat{\nu}_3|z_2)^{\nu_3}_{\nu_2}$$

$$W(m|z_1 + z_2)^{\nu_3}_{\nu_2} W(m|z_1 - z_2)^{\mu_2}_{\mu_1}$$

$$= k(m + \hat{\mu}_1 + \hat{\nu}_2|z_2)^{\nu_2}_{\nu_0} k(m + \hat{\mu}_3 + \hat{\nu}_3|z_1)^{\mu_3}_{\mu_1}$$

$$W(m|z_1 + z_2)^{\mu_3}_{\mu_0} W(m|z_1 - z_2)^{\mu_2}_{\nu_2}. \quad (40)$$

Here and below summation over repeated indices are assumed. One can find that this equation is true for arbitrary $\mu_0, \nu_0, \mu_3, \nu_3$. We should point out here that this face reflection equation is different from the one proposed by Behrend et al in ref.[44], if the cross-inversion relation of face Boltzmann weights [41] is applied, the two equations are equivalent.

Now, we will let the indices in the above relation take special values so that we can find the necessary commutation relations. When $\mu_0 = \nu_0 = 0, \mu_3 = \nu_3 = 1$, we get

$$k(m|z_1)^1_0 k(m - 2|z_2)^1_0 W(m|z_1 + z_2)^{01}_{01} W(m|z_1 - z_2)^{00}_{00}$$

$$= k(m|z_2)^1_0 k(m - 2|z_1)^1_0 W(m|z_1 + z_2)^{01}_{01} W(m|z_1 - z_2)^{11}_{11}. \quad (41)$$

From the definition of face Boltzmann weights, we know that $W(m|z)^{00}_{00} = W(m|z)^{11}_{11} = \frac{b(x+w)}{\mu(w)}$, so we find

$$k(m|z_1)^1_0 k(m - 2|z_2)^1_0 = k(m|z_2)^1_0 k(m - 2|z_1)^1_0. \quad (42)$$

This means that the position of $z_1$ and $z_2$ can be exchanged in this form.

Let $\mu_0 = \nu_0 = \mu_3 = \nu_3 = 0$, we obtain

$$k(m + 2|z_2)^0_0 k(m|z_1)^1_0 W(m|z_1 + z_2)^{00}_{00} W(m|z_1 - z_2)^{01}_{01}$$

$$= k(m|z_1)^0_1 k(m|z_2)^0_0 W(m|z_1 + z_2)^{01}_{00} W(m|z_1 - z_2)^{00}_{00}$$

$$- k(m|z_2)^0_0 k(m|z_1)^0_0 W(m|z_1 + z_2)^{01}_{00} W(m|z_1 - z_2)^{00}_{01}$$

$$- k(m|z_2)^0_0 k(m|z_1)^1_0 W(m|z_1 + z_2)^{10}_{00} W(m|z_1 - z_2)^{00}_{01}. \quad (43)$$

Denote

$$A(m|z) \equiv k(m|z)^0_0,$$

$$D(m|z) \equiv k(m|z)^1_1,$$

$$B(m|z) \equiv k(m|z)^1_0,$$

$$C(m|z) \equiv k(m|z)^0_1. \quad (44)$$
So, the matrix \( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \) is the boundary \( k \) matrix in the "face" form.

Relations (42,43) give the commutation relations of \( BB \) and \( AB \), respectively. In order to use the algebraic Bethe ansatz method, we must also get the commutation relation of \( DB \).

Let \( \mu_0 = \mu_3 = \nu_3 = 1, \nu_0 = 0 \), exchange \( z_1 \) and \( z_2 \), we get one relation, exchange \( z_1 \) and \( z_2 \) in equation (43), we find another relation, combine the two relations and with the help of equation (43), we can find the commutation relation of \( DB \):

\[
\begin{align*}
k(m + 2|z_2)W(m + 2|z_1 + z_2)W(m + 2|z_1 - z_2)W(m + 2|z_1 + z_2)W(m + 2|z_1 - z_2)W(m + 2|z_1 + z_2)
\end{align*}
\]

\[
\begin{align*}
&= k(m|z_2)W(m + 2|z_1 + z_2)W(m + 2|z_1 - z_2)W(m + 2|z_1 + z_2)W(m + 2|z_1 - z_2)W(m + 2|z_1 + z_2)
\end{align*}
\]

\[
\begin{align*}
&= k(m|z_2)W(m + 2|z_1 + z_2)W(m + 2|z_1 - z_2)W(m + 2|z_1 + z_2)W(m + 2|z_1 - z_2)W(m + 2|z_1 + z_2)
\end{align*}
\]

\[
\begin{align*}
&= k(m|z_2)W(m + 2|z_1 + z_2)W(m + 2|z_1 - z_2)W(m + 2|z_1 + z_2)W(m + 2|z_1 - z_2)W(m + 2|z_1 + z_2)
\end{align*}
\]

In the right hand side of this equation, behind \( k(m|z_2) \), we find not only term \( k(m|z_1) \) but also term \( k(m|z_1) \); this will cause trouble in the proceeding of the algebraic Bethe ansatz method, especially in the case where the thermodynamic limit of this system is taken. In order to solve this problem, we should reformulate \( A \) and \( D \) as \( A \) and \( D \) so that when we commute \( D \) with \( B \), only term \( D \) exists behind \( B(z_2) \) which is the notation of \( k(m|z_2) \).

It is not easy to find \( D \) by direct calculation, but the work of Sklyanin[16] gives us a hint to formulate the term \( D \). Sklyanin pointed out in his paper that the term \( D \) in six vertex model case is one of the elements of the inverse matrix of the monodromy matrix. Now we will study the inverse of face form monodromy matrix for the eight vertex model case.
For simplicity, abusing the face weights $m$ etc, we can write the face reflection equation as
\[ W_{12}(z_1 - z_2)k_1(z_1)W_{21}(z_1 + z_2)k_2(z_2) = k_2(z_2)W_{12}(z_1 + z_2)k_1(z_1)W_{21}(z_1 - z_2). \quad (46) \]
It is just the same as the vertex reflection equation. If the inverse of $k(z)$ exists, we have
\[ k_2^{-1}(z_2)W_{12}(z_1 - z_2)k_1(z_1)W_{21}(z_1 + z_2) = W_{12}(z_1 + z_2)k_1(z_1)W_{21}(z_1 - z_2)k_2^{-1}(z_2). \quad (47) \]
That means that $k_2^{-1}(-z_2)$ and $k_2(z_2)$ has similar exchange relation with $k_1(z_1)$ in formalism. We have known that the commutation relation of $k(m + 2|z_2)_0^0$ with $k(m|z_1)_0^0$ is comparatively simple which is also useful for algebraic Bethe ansatz. From the above equation, we assume that the commutation relation of $k^{-1}(m + 2|z_2)_0^0$ with $k(m|z_1)_0^0$ should have the similar properties in formalism. We know that $k_2^{-1}(z_2)_0^0$ is a linear combination of $A$ and $D$ in six vertex model. We hope that this is also true for the case of eight vertex model. Fortunately, we have obtained the expected result.

Define
\[ Q(m|z - w)_{\nu_3}^{\mu_2} = k(m + \hat{\mu}_3 + \hat{\nu}_2|z - w)_{\mu_1}^{\mu_2}W(m|2z - w)_{\nu_1}^{\nu_2}W(m|w - w)_{01}^{\mu_1\nu_1}, \]
\[ Q'(m|z - w)_{\nu_2}^{\mu_1} = k(m|z - w)_{\mu_1}^{\mu_2}W(m|2z - w)_{\nu_1}^{\nu_2}W(m|w - w)_{01}^{\mu_1\nu_2}, \quad (48) \]
where, as usual, summation over repeated indices is assumed and $0 = 1, \bar{1} = 0$. We can show (Appendix B)
\[ Q(m|z - w)_{\nu_3}^{\mu_2}k(m + \hat{\nu} + \hat{\nu}'|z)_{\nu'}^{\nu''} = \rho(m, \nu|z)\delta_{\nu\nu''}, \]
\[ k(m + \hat{\mu}' + \hat{\mu}'|z)_{\mu}^{\mu'}Q'(m|z - w)_{\mu'}^{\mu''} = \rho'(m, \mu|z)\delta_{\mu\mu''}, \quad (49) \]
where $\rho(m, \nu|z)$, and $\rho'(m, \mu|z)$ are scalars of the “quantum space”. We can rescale $Q$ and $Q'$ such that $\rho, \rho' \rightarrow 1$. But this is not necessary in the following derivation. Multiply (40) by $Q(m + \hat{\nu} + \hat{\mu}_0|z_2 - w)_{\nu_0}^{\nu_0}$ from left and by $Q'(m + \hat{\mu}_3 + \hat{\nu}'|z_2 - w)_{\nu_3}^{\nu_3}$ from right. Summation over $\nu_0, \nu_3$ gives
\[ Q(m + \hat{\nu} + \hat{\mu}_0|z_2 - w)_{\nu_0}^{\nu_0}k(m + \hat{\mu}_3 + \hat{\nu}_2|z_1)_{\mu_1}^{\mu_2}W(m|z_1 + z_2)_{\nu_1}^{\nu_2}\]
\[ \times W(m|z_1 - z_2)_{\nu_1}^{\nu_2}\delta_{\nu_1\nu_2}k(m + \hat{\mu}_3 + \hat{\nu}_3|z_2)_{\mu_1}^{\mu_2}Q'(m + \hat{\mu}_3 + \hat{\nu}'|z_2 - w)_{\nu_3}^{\nu_3}\]
\[ W(m|z_1 + z_2)_{\mu_1}^{\mu_2}W(m|z_1 - z_2)_{\nu_1}^{\nu_2}\]
\[ = \rho(m + \hat{\nu} + \hat{\mu}_0, \nu_2|z_2)\delta_{\nu_1\nu_2}k(m + \hat{\mu}_3 + \hat{\nu}_3|z_1)_{\mu_1}^{\mu_2}Q'(m + \hat{\mu}_3 + \hat{\nu}'|z_2 - w)_{\nu_3}^{\nu_3}\]
\[ W(m|z_1 + z_2)_{\mu_1}^{\mu_2}W(m|z_1 - z_2)_{\nu_1}^{\nu_2}. \quad (50) \]
In the derivation of LHS, we use the fact that $W(m|z_1 + z_2)^{\mu_1 \mu_2}_{\nu_1 \nu_2} \neq 0$ only if $\hat{v}_1 + \hat{\mu}_2 = \hat{v}_2 + \hat{\mu}_3$. The equation becomes

$$Q(m + \hat{v} + \hat{\mu}_0|z_2 - w)^{\mu_0}_{\nu_0} k(m + \hat{\mu}_3 + \hat{v}'|z_1)^{\mu_3}_{\nu_3}$$

$$W(m|z_1 + z_2)^{\nu_3}_{\mu_3} W(m|z_1 - z_2)^{\mu_1 \nu_1}_{\mu_0 \nu_0}$$

$$= k(m + \hat{\mu}_3 + \hat{v}_3|z_1)^{\mu_3}_{\nu_3} Q'(m + \hat{\mu}_3 + \hat{v}'|z_2 - w)^{\nu'}_{\nu_3}$$

$$W(m|z_1 + z_2)^{\nu_3}_{\mu_3} W(m|z_1 - z_2)^{\mu_1 \nu_1}_{\mu_0 \nu_0}$$

$$\times \rho(m + \hat{v} + \hat{\mu}_0, \nu|z_2)$$

$$\rho'(m + \hat{\mu}_3 + \hat{v}', \nu'|z_2)$$  \hspace{1cm} (51)

which is similar to (40). Put $\mu_0 = \nu = \nu' = 0$, $\mu_3 = 1$, and notice $m + \hat{0} + \hat{0} = m + 2, m + 0 + \hat{1} = m$. We then have

$$Q(m + 2|z_2 - w)^{\mu_0}_{\nu_0} k(m|z_1)^{\mu_1}_{\nu_1} W(m|z_1 + z_2)^{\mu_1}_{\nu_1} W(m|z_1 - z_2)^{\nu_0}_{\mu_0}$$

$$+ Q(m + 2|z_2 - w)^{\mu_0}_{\nu_0} k(m|z_1)^{\mu_1}_{\nu_1} W(m|z_1 + z_2)^{\mu_1}_{\nu_1} W(m|z_1 - z_2)^{\nu_0}_{\mu_0}$$

$$+ Q(m + 2|z_2 - w)^{\mu_0}_{\nu_0} k(m|z_1)^{\mu_1}_{\nu_1} W(m|z_1 + z_2)^{\mu_1}_{\nu_1} W(m|z_1 - z_2)^{\nu_0}_{\mu_0}$$

$$= k(m|z_1)^{\mu_1}_{\nu_1} Q'(m|z_2 - w)^{\mu_0}_{\nu_0} W(m|z_1 + z_2)^{\mu_0}_{\nu_0} W(m|z_1 - z_2)^{\nu_0}_{\mu_0} \frac{\rho(m + 2, 0|z_2)}{\rho'(m, 0|z_2)}$$  \hspace{1cm} (52)

Using the same derivation as that in Appendix B, we can calculate

$$\frac{\rho(m + 2, 0|z_2)}{\rho'(m, 0|z_2)} = \frac{(-1)h(wa)h(w(a + 1))}{h(w(a - 1))h(w(a + 2))}$$  \hspace{1cm} (53)

where $a \equiv m + \gamma$ and $\gamma$ is a parameter in defining $\phi$. From (48) and considering the explicit form of $W$, we can write $Q, Q'$ by components of $k(z)$,

$$Q(m|z - w)^{\mu_1}_{\nu_1} = k(m - 2|z - w)^{\mu_1}_{\nu_1} W(m|2z - w)^{\mu_1}_{\nu_1} W(m|w)^{\mu_1}_{\nu_1}$$  \hspace{1cm} (54)

which is proportional to $B(m - 2|z - w)$, and

$$Q(m|z - w)^{\mu_0}_{\nu_0} = - k(m|z - w)^{\mu_0}_{\nu_0} W(m|2z - w)^{\mu_0}_{\nu_0} W(m|w)^{\mu_1}_{\nu_1}$$

$$- Q'(m|z - w)^{\mu_0}_{\nu_0} W(m|2z - w)^{\mu_1}_{\nu_1} W(m|w)^{\mu_0}_{\nu_0}$$

$$\equiv - \tilde{D}(m|z - w) W(m|w)^{\mu_1}_{\nu_1}$$  \hspace{1cm} (55)
where

\[
\tilde{D}(m|z) \equiv -k(m|z)_0^0 W(m|2z+w)_0^0 + k(m|z)_1^1 W(m|2z+w)_1^0
\]

\[
= \frac{-Q(m|z)}{W(m|w)_0^0}
\]

(56)

is indeed a linear combination of \(A(m|z) \equiv k(m|z)_0^0\) and \(D(m|z) \equiv k(m|z)_1^1\). Substitute (53–55) into (52) and write \(k(m|z_2-w)_1^1\) in the equation in terms of \(\tilde{D}(m|z_2-w)\) and \(A(m|z_2-w)\). We then change the parameter \(z_2-w\) to \(z_2\) giving the expected equation

\[
\tilde{D}(m+2|z_2)B(m|z_1)
\]

\[
= B(m|z_1)\tilde{D}(m|z_2)\frac{h(z_1-z_2-w)h(z_1+z_2+2w)}{h(z_1+z_2+w)h(z_1-z_2)}
\]

\[
+ B(m|z_2)A(m|z_1)\frac{h(2z_1)h(z_1+z_2+w(a+2))h(2z_2+2w)}{h(2z_1+w)h(z_1+z_2+w)h(w(a+1))}
\]

\[
+ B(m|z_2)\tilde{D}(m|z_1)\frac{h(z_2-z_1+w(a+1))h(2z_2+2w)h(w)}{h(2z_1+w)h(z_1-z_2)h(w(a+1))}.
\]

(57)

where \(h(z) \equiv \sigma_0(z)\). In the derivation we have used a formula of \(\theta\) function [7]

\[
h(u+x)h(u-x)h(v+y)h(v-y) - h(u+y)h(u-y)h(v+x)h(v-x) = h(u+v)h(u-v)h(x+y)h(x-y).
\]

(58)

We need also to change \(D\) in the exchange relation of \(A\) and \(B\) by linear combination of \(\tilde{D}\) and \(A\). Thus (43) is rewritten as

\[
A(m+2|z_2)B(m|z_1)
\]

\[
= B(m|z_1)A(m|z_2)\frac{h(z_1+z_2)h(z_1-z_2+w)}{h(z_1+z_2+w)h(z_1-z_2)}
\]

\[
- B(m|z_2)A(m|z_1)\frac{h(2z_1)h(z_1-z_2+w(a+1))h(w)}{h(2z_1+w)h(z_1-z_2)h(w(a+1))}
\]

\[
- B(m|z_2)\tilde{D}(m|z_1)\frac{h(-z_1-z_2+wa)h(w)^2}{h(z_1+z_2+w)h(2z_1+w)h(w(a+1))}.
\]

(59)

We have also used (58) in the derivation. With the permutation relations of \(AB\), \(\tilde{D}B\) and \(BB\) (57,59,42), we can obtain the algebraic Bethe ansatz equations provided
1. The transfer matrix $t(z)$ is a linear combination of $A$ and $\tilde{D}$.

2. There is a "vaccum state" of the quantum space which is an eigenstate of $A$ and $\tilde{D}$ but not an eigenstate of $B$.

We will study these problems in section 4.

4 Vacuum state and boundary conditions

4.1 Vacuum state

The algebraic Bethe ansatz requires to construct a state of the "quantum" space, which is an eigenstate of operators $A$ and $\tilde{D}$ with all spectrum parameter $z$. This state is called a vacuum state, which is also an eigenstate of $\tilde{D}$. Before introducing the vacuum state, let us make some preperations. We first express $S$ and $T$ (see Eq.(18)) by the "face language", change their auxiliary indices to face indices, and express $k(m|z)_{\mu'}^\nu$ by such expressions of $S$ and $T$. From (23,39), the operator $k(z)$ with "face" indices can be written as

$$k(m|z)_{\mu'}^\nu = \tilde{\phi}_{m+\mu'-\nu',\mu}(z)T(z)K(z)S(z)\phi_{m,\nu'}(-z).$$

(60)

From (28,29) we have

$$K(z) = \sum_{\mu\nu} \{ \phi_{m_0+\mu-\nu,\mu}(z)\tilde{\phi}_{m_0+\mu-\nu,\mu}(z)K(z)\phi_{m_0,\nu}(-z)\tilde{\phi}_{m_0,\nu}(-z) \}. \quad (61)$$

Combining these two equations gives

$$k(m|z)_{\mu'}^\nu = \sum_{\mu\nu} K(m_0|z)_{\mu'}^\nu T(m-\nu', m_0-\nu|z)_{\mu'\mu} S(m, m_0|z)_{\nu'\nu}, \quad (62)$$

where we define

$$K(m_0|z)_{\mu'}^\nu \equiv \tilde{\phi}_{m_0+\mu-\nu,\mu}(z)K(z)\phi_{m_0,\nu}(-z),$$

$$T(m-\nu', m_0-\nu|z)_{\mu'\mu} \equiv \tilde{\phi}_{m+\mu'-\nu',\mu}(z)T(z)\phi_{m_0+\mu-\nu,\mu}(z),$$

$$S(m, m_0|z)_{\nu'\nu} \equiv \tilde{\phi}_{m_0,\nu}(-z)S(z)\phi_{m,\nu'}(-z).$$

(63)

(64)

(65)

Equation (62) is true for all $m$ and $m_0$.

Next, assume that we can properly choose the parameters of the right reflecting matrix such that for given $m_0$ and all $z$, $K(m_0|z)_{11}^0 = 0$. This
is possible. We will study this problem in the second part of this section. Actually, this requirement constitute the only restriction on the boundary matrices in our approach.

Then, multiply \( \tilde{\phi}(1)_{m,1}(z) \tilde{\phi}(2)_{m_0+1,0}(-z) \) from left and multiply \( \phi^{(1)}_{m_0+1,0}(z) \phi^{(2)}_{m,1}(-z) \) from right to equation (21). Using face-vertex correspondence relations 32-36) we can prove the following exchange relation of \( S \) and \( T \) with face indices,

\[
W(m + 1|2z) \frac{1}{11} S(m - 1, m_0 + 1|z) T(m, m_0|z)_{10} \\
+ W(m + 1|2z) \frac{01}{10} S(m + 1, m_0 + 1|z)_{00} T(m, m_0|z)_{00} \\
= W(m_0 + 1|2z) \frac{00}{00} T(m + 1, m_0 + 1|z)_{11} S(m, m_0|z)_{11} \\
+ W(m_0 - 1|2z) \frac{00}{00} T(m + 1, m_0 - 1|z)_{10} S(m m_0|z)_{10},
\]

which will be useful in our derivation. For later convenience we rewrite \( S, T \) as

\[
S(z) = R_{00}(u + z) R_{1-1,0}(u-1 + z) \cdots R_{10}(u + z), \\
T(z) = R_{01}(z - u_1) R_{02}(z - u_2) \cdots R_{0l}(z - u_l).
\]

They are acting on the space \( V_0 \otimes V_1 \otimes \cdots \otimes V_l \), where \( V_0 \) is the auxiliary space. The final preparation is the following observation. In Eqs.(32-36), the summation over face indices has at most two terms in eight vertex model. In the following cases, due to the non zero condition of \( W(m|z)_{\mu \nu} \), there is actually only one term in RHS of the equations,

1. Eq.(32) \( \text{when } \mu = \nu \)
2. Eq.(33) \( \text{when } \mu \neq \nu \) \hspace{1cm} (68)
3. Eq.(35) \( \text{when } \mu \neq \nu \).

Thus we have (Fig.7)

\[
R_{12}(z - z_2) \phi^{(1)}_{m + 2,0}(z_1) \phi^{(2)}_{m + 1,0}(z_2) \\
= W(\ |z - z_2\ |_{00} \phi^{(2)}_{m + 2,0}(z_2) \phi^{(1)}_{m + 1,0}(z_1)),
\]

\[
\tilde{\phi}^{(1)}_{m - 1,1}(z_1) R_{12}(z - z_2) \phi^{(2)}_{m + 1,0}(z_2) \\
= W(m|z_1 - z_2) \phi^{(1)}_{m + 1,0}(z_1) \phi^{(2)}_{m,0}(z_2),
\]

\[
(69) \hspace{1cm} (70)
\]
\[
\tilde{\phi}_{m,1}^{(2)}(z_2) R_{12}(z_1 - z_2) \phi_{m,0}^{(1)}(z_1)
= W(m|z_1 - z_2)_{01}^{01} \phi_{m+1,0}^{(1)}(z_1) \phi_{m-1,1}^{(2)}(z_2),
\]

(71)

which will be repeatedly used in the proof of vacuum state.

The above are preparations for introducing the vacuum state. Now, define the vacuum state as

\[
|0 > ^m_{m_0} \equiv \phi_{m_0,0}^{(0)}(u_1) \phi_{m_0-1,0}^{(l-1)}(u_{l-1}) \cdots \phi_{m_0-(l-2),0}^{(2)}(u_2) \phi_{m_0-(l-1),0}^{(1)}(u_1)
\]

(72)

where \( m = m_0 - l \). This is precisely the same vacuum state introduced by Baxter in the original work of Bethe ansatz for eight vertex model with periodic boundary conditions [3]. For the vacuum state defined in (72) we can show that \( S(m,m_0|z)_{00} \) and \( T(m,m_0|z)_{11} \) change \( |0 > ^m_{m_0+1} \) to \( |0 > ^{m-1}_{m_0-1} \), while \( S(m,m_0|z)_{11} \) and \( T(m,m_0|z)_{00} \) change \( |0 > ^m_{m_0} \) to \( |0 > ^m_{m_0+1} \) (with some coefficients). The operators \( S(m,m_0|z)_{01} \) and \( T(m,m_0|z)_{10} \) change \( |0 > ^m_{m_0} \) to zero. Following is the proof.

We have

\[
S(m,m_0|z)_{00} = \phi_{m_0,0}^{(0)}(-z) S(z) \phi_{m,0}^{(0)}(-z)
= \phi_{m_0,0}^{(0)}(-z) R_{l0}(u_l + z) R_{l-1,0}(u_{l-1} + z) \cdots R_{1,0}(u_1 + z) \phi_{m_0,0}^{(0)}(-z_1).
\]

(73)

Since vectors and operators belonging to different spaces may change their positions in an equation, we obtain

\[
S(m,m_0|z)_{00} |0 > ^m_{m_0} = \phi_{m_0,0}^{(0)}(-z) R_{l0}(u_l + z) \phi_{m_0,0}^{(0)}(u_l) R_{l-1,0}(u_{l-1} + z) \phi_{m_0-1,0}^{(l-1)}(u_{l-1})
\]

\[ \cdots R_{20}(u_2 + z) \phi_{m_0-(l-2),0}^{(2)}(u_2) R_{10}(u_1 + z) \phi_{m_0-(l-1),0}^{(1)}(u_1) \phi_{m_0,0}^{(0)}(-z) \].

(74)

By using (69), we move \( \phi_{m_0,0}^{(0)}(-z) \) towards left step by step. At each step we eliminate an \( R \) matrix getting a \( W \) factor, and change the face weight of \( \phi_{m_0,0}^{(0)}(-z) \) and that of \( \phi_{m,0}^{(i)} \) encountered. Thus we have

\[
S(m,m_0|z)_{00} |0 > ^m_{m_0} = \cdots R_{20}(u_2 + z) \phi_{m_0+2,0}^{(2)}(u_2) \phi_{m_0,0}^{(0)}(u_1)
\]

20
\[ \times W( |u_1 + z|_0^0) \]

\[ = \cdots \]

\[ = \phi_{m_0,0}(-z)\phi_{m_0,0}(-z)\phi_{m_0-1,0}(u_1)\cdots \phi_{1,0}(u_1) \]

\[ \times \prod_{i=1}^l W( |u_i + z|_0^0). \]  

(75)

Due to the orthogonal relation (28), this becomes

\[ S(m, m_0|z)_00|0 > m_{m_0} = \prod_{i=1}^l W( |u_i + z|_0^0)|0 > m_{m_0-1} \equiv s_{00}(z)|0 > m_{m_0-1}. \]  

(76)

Similarly, one can show

\[ S(m, m_0|z)_01|0 > m_{m_0} = \phi_{m_0,1}(-z)\phi_{m_0,0}(-z)\cdots \]

\[ = 0. \]  

(77)

For the action of \( S_{11} \), from (65,67) and (72), we write

\[ S(m, m_0|z)_{11}|0 > m_{m_0} \]

\[ = \phi_{m_0,1}(-z)R_{l_0}(u_l + z)\phi_{m_0,0}(u_l)R_{l-1,0}(u_{l-1} + z)\phi_{m_0-1,0}(u_{l-1}) \]

\[ \cdots R_{1,0}(u_1 + z)\phi_{m+1,0}(u_1)\phi_{m_1,1}(-z). \]  

(78)

Using (71) we move \( \phi_{m_0,1}(-z) \) towards right step by step. At each step we eliminate an \( R \) matrix getting a \( W \) factor, and change the face weight of \( \phi_{m_0,1}(-z) \) and \( \phi_{m_0,0} \) encountered. We then have (Fig.8)

\[ S(m, m_0|z)_{11}|0 > m_{m_0} = \prod_{i=1}^l W(m_0 - (i - 1)|u_i + z|_{01}^1) = \phi_{m_0-l,1}(-z) \]

\[ \times \phi_{m_0-l,0}(-z)\phi_{m_0-l+1,0}(u_l)\cdots \phi_{m+2,0}(u_1) \]

\[ \times \prod_{i=1}^l W(m_0 - (i - 1)|u_i + z|_{01}^1)|0 > m_{m_0+1} \equiv s_{11}(m|z)|0 > m_{m_0+1}. \]  

(79)
here \( m_0 - l = m \). Similar derivation can be performed for \( T \), giving

\[
T(m, m_0|z)_{00}|0 \geq_{m_0}^{m} = \prod_{i=1}^{l} W( |z - u_i^{00}|)^{00}_{00}|0 \geq_{m_0+1}^{m+1} \\
\equiv t_{00}(z)|0 \geq_{m_0+1}^{m+1},
\]

\[
T(m, m_0|z)_{10}|0 \geq_{m_0}^{m} = 0, \quad (80)
\]

\[
T(m, m_0|z)_{11}|0 \geq_{m_0}^{m} = \prod_{i=1}^{l} W(m + (i - 1)|z - u_i^{10}|)^{10}_{10}|0 \geq_{m_0-1}^{m-1} \\
\equiv t_{11}(m|z)|0 \geq_{m_0-1}^{m-1}, \quad (81)
\]

which completes our proof.

We now assume that in (62), \( K(m_0|z)_{1}^{0} = 0 \), and consider \( \hat{\mu} \equiv (-1)^{m} \), obtaining

\[
k(m|z)_{0}^{0} = K(m_0|z)_{0}^{0}T(m - 1, m_0 - 1|z)_{00}S(m, m_0|z)_{00} \\
+ K(m_0|z)_{0}^{0}T(m - 1, m_0 + 1|z)_{00}S(m, m_0|z)_{01} \\
+ K(m_0|z)_{1}^{1}T(m - 1, m_0 + 1|z)_{01}S(m, m_0|z)_{01}, \quad (82)
\]

\[
k(m|z)_{1}^{1} = K(m_0|z)_{0}^{0}T(m + 1, m_0 - 1|z)_{10}S(m, m_0|z)_{10} \\
+ K(m_0|z)_{0}^{0}T(m + 1, m_0 + 1|z)_{10}S(m, m_0|z)_{11} \\
+ K(m_0|z)_{1}^{1}T(m + 1, m_0 + 1|z)_{11}S(m, m_0|z)_{11}. \quad (83)
\]

By (66), \( k(m|z)_{1}^{1} \) can be written as

\[
k(m|z)_{1}^{1} = \frac{K(m_0|z)_{0}^{0}}{W(m_0 - 1|2z)^{00}_{00}} \left\{ W(m + 1|2z)^{11}_{11}S(m - 1, m_0 + 1|z)_{10}T(m, m_0|z)_{10} \\
+ W(m + 1|2z)^{01}_{10}S(m + 1, m_0 + 1|z)_{00}T(m, m_0|z)_{00} \right\} \\
+ K(m_0|z)_{1}^{1}T(m + 1, m_0 + 1|z)_{11}S(m, m_0|z)_{11} \\
+ \left\{ K(m_0|z)_{1}^{1} \right\} - \left\{ \frac{K(m_0|z)_{0}^{0}W(m_0 + 1|2z)^{01}_{10}}{W(m_0 - 1|2z)^{00}_{00}} \right\} \\
\times T(m + 1, m_0 + 1|z)_{11}S(m, m_0|z)_{11}. \quad (84)
\]

Acting on the vacuum state \( |0 \geq_{m_0}^{m} \), and by equations (76,79-81), these two operators yield

\[
k(m|z)_{0}^{0}|0 \geq_{m_0}^{m} = K(m_0|z)_{0}^{0}T(m - 1, m_0 - 1|z)_{00}s_{00}(z)|0 \geq_{m_0-1}^{m-1} + 0 + 0 \\
= K(m_0|z)_{0}^{0}t_{00}(z)s_{00}(z)|0 \geq_{m_0}^{m} \\
\equiv \tau(m|z)_{0}^{0}|0 \geq_{m_0}^{m}, \quad (85)
\]

22
and

$$k(m|z)_{1}\big|0 >_{m-0} = \frac{K(m_0|z_0)W(m + 1|z_0)^{01}_{10} - W(m_0 - 1|z_0)^{00}_{10} S_{00}(z) t_{00}(z)|0 >_{m-0}}{W(m_0 - 1|z_0)^{00}_{10}}\times s_{11}(m + 1|z_0) s_{11}(m|z_0)|0 >_{m-0}$$

$$\equiv \tau(m|z)_{1}\big|0 >_{m-0}.$$ (86)

From equations (85,86) we see that when $K(m_0|z)_{1}^{0} = 0$, the vacuum state $|0 >_{m-0}$ is indeed an eigenstate of $A = k(m|z)_{0}$ and $D = k(m|z)_{1}$. Thus it is also an eigenstate of $D$. The eigenvalues depend on $m$, $\{u_i\}$, $l$ and $z$, the spectrum of $A$ and $D$.

### 4.2 Boundary conditions

The algebraic Bethe ansatz requires a vacuum state, which needs the right boundary to satisfy

$$K(m_0|z)_{1}^{0} = 0.$$ (87)

Also, the transfer matrix $t(z)$ must be a linear combination of $A$ and $D$, which impose the left boundary to satisfy

$$K(m_0|z)_{1}^{0} = 0$$ (88)

and

$$K(m_0|z)_{1}^{0} = 0,$$ (89)

for $m_0$ which we will specify in the next section. We will see that $m_0$ is constrained with $m$, $m_0$ of vacuum state by

$$m + l\hat{0} = m + l = m_0,$$

$$m + (-\hat{1} + \hat{0})l' = m + 2l' = m_0.$$ (90)

In equation (90), $l$ is the column number of the lattice and $l'$ is a positive integer, which will be the number of $B$ operators in constructing the eigenstates of $t(z)$ (see (105)).
From the general solution of RE (15,17),

\[ K(z) = \sum_{\alpha} C_{\alpha} \frac{I_{\alpha}}{\sigma_{\alpha}(-z)}, \]
\[ \tilde{K}(z) = \sum_{\alpha} \tilde{C}_{\alpha} \frac{I_{\alpha}}{\sigma_{\alpha}(z+w)}, \]

and the definitions

\[ K(m|z)_{\mu} = \tilde{\phi}_{m0+\tilde{\mu},\mu}(z)K(z)\phi_{m0,\nu}(-z), \]
\[ \tilde{K}(m|z)_{\mu} = \tilde{\phi}_{m,\mu}(-z)\tilde{K}(z)\phi_{m-\tilde{\mu},\nu}(z), \]

we can derive the following results (Appendix C),

\[ K(m|z)_{1}^0 = (-1)^2 \sum_{\alpha} C_{\alpha}(-1)^{a_{\alpha}}\sigma_{\alpha}(wa + w\beta - \frac{1}{2}) \frac{h(z + w\beta + w - \frac{1}{2})h(wa - w)}{h(-z + w\beta - \frac{1}{2})h(wa)}, \]
\[ \tilde{K}(m|z)_{1}^0 = \sum_{\alpha} \tilde{C}_{\alpha}(-1)^{a_{\alpha}}\sigma_{\alpha}(-w(a - 1) + w\beta - \frac{1}{2}) \frac{h(-z + w\beta - \frac{1}{2})h(wa)}{h(z + w\beta + w - \frac{1}{2})h(wa)}, \]
\[ \tilde{K}(m|z)_{0}^1 = (-1)^2 \sum_{\alpha} \tilde{C}_{\alpha}(-1)^{a_{\alpha}}\sigma_{\alpha}(w(a + 1) + w\beta - \frac{1}{2}) \frac{h(z + w\beta + w - \frac{1}{2})h(wa)}{h(-z + w\beta - \frac{1}{2})h(wa)}, \]

where \( a \equiv m + \gamma \). We can easily see from (92-94) that those conditions (87-89) are actually independent of \( z \), and depend only on \( \{C_{\alpha}\}, \{\tilde{C}_{\alpha}\} \) and parameters \( \beta, \gamma, w, \tau \), etc.. For any given generic \( \{\tilde{C}_{\alpha}\} \), we may solve the equation

\[ \sum_{\alpha} \tilde{C}_{\alpha}(-1)^{a_{\alpha}}\sigma_{\alpha}(2\eta) = 0. \]  

The LHS of (95) is a doubly quasi-periodic holomorphic function of \( \eta \). From the quasi-periodicity we see that [7,40,41] it has four zeros in \( \Lambda_{\tau} : \eta \to \eta + 1, \eta \to \eta + \tau \). Assume \( \eta_1, \eta_2 \) are two different zeros, we obtain \( \alpha \) and \( \beta \) by solving

\[ -w(a - 1) + w\beta - \frac{1}{2} = 2\eta_1, \]
\[ w(a + 1) + w\beta - \frac{1}{2} = 2\eta_2. \]
Then from \( a = m^0 + \gamma \) we obtain \( \gamma \) according to a given \( m^0 \). Using such \( \beta, \gamma \) from generic \( \{ C_\alpha \} \), one may construct \( \phi, \bar{\phi} \) and \( \tilde{\phi} \) with which the equations (88) and (89) are satisfied. Since \( \beta \) and \( \gamma \) are completely determined, equation (87) is a constraint for \( \{ C_\alpha \} \) at the right boundary. This is the only restriction we must impose for our approach. There are 3 free parameters at the right boundary for general solution (15), (an overall scalar of \( K \) is not important). With the constraint (87), there are still 2 free parameters left. If we further require

\[
K(m^0|z)_0 = 0, \tag{97}
\]

then only one free parameter at right boundary survives. We can show

\[
\frac{K(m^0|z)_1}{K(m^0|z)_0} = h(\xi - z)h(aw + \xi + z)
\]

where \( \xi \) is the right boundary free parameter introduced in [44], and similarly for the left boundary. In this case the left and right boundary are equivalent to that of SOS model (except a symmetric factor) introduced by Behrend, Pearce and Brien in [44] where they derive the solutions directly from the face reflection equation. This implies that our approach can be used for the Bethe ansatz of SOS model with such boundary conditions.

5 Bethe ansatz

We see that for a given \( m^0 \), properly choosing \( \beta \) and \( \gamma \), we can ensure

\[
\tilde{K}(m^0|z)_0 = \tilde{K}(m^0|z)_1 = 0 \tag{98}
\]

at the left boundary. We need the parameters \( \{ C_\alpha \} \) at the right boundary to satisfy

\[
\sum_\alpha C_\alpha(-1)^{a_2} = (w + w\beta - \frac{1}{2}) = 0 \tag{99}
\]

for an integer \( m_0 \), where \( a \equiv m_0 + \gamma \), to ensure \( K(m^0|z)_1 = 0 \). Assume that \( 2l' = m^0 - (m_0 - l) \) is a positive even integer, where \( l \) is the number of column of the lattice. We can proceed the standard algebraic Bethe ansatz [3,4,7,16] as following.
We have the transfer matrix from Eq. (38),

\[ t(z) = \tilde{K}(m^0|z)_0^0k(m^0|z)_0^0 + \tilde{K}(m^0|z)_1^1k(m^0|z)_1^1. \] (100)

Due to (44) and (56), \( t(z) \) can be rewritten as

\[ t(z) = \mu_0(z)A(m^0|z) + \mu_1(z)\tilde{D}(m^0|z) \] (101)

We also have the exchange relations of \( A, \tilde{D} \) with \( B \), Eq. (57) and (59). They can be compactly written as

\[
\begin{align*}
A(m|u)B(m - 2|v) &= a_{00}(m, u, v)B(m - 2|v)A(m - 2|u) \\
&\quad + b_{00}(m, u, v)B(m - 2|u)A(m - 2|v) \\
&\quad + b_{01}(m, u, v)B(m - 2|u)\tilde{D}(m - 2|v),
\end{align*}
\] (102)

\[
\begin{align*}
\tilde{D}(m|u)B(m - 2|v) &= a_{11}(m, u, v)B(m - 2|v)\tilde{D}(m - 2|u) \\
&\quad + b_{10}(m, u, v)B(m - 2|u)A(m - 2|v) \\
&\quad + b_{11}(m, u, v)B(m - 2|u)\tilde{D}(m - 2|v).
\end{align*}
\] (103)

Besides, we have (42), the exchange relation of \( B \)'s,

\[ B(m|u)B(m - 2|v) = B(m|v)B(m - 2|u). \] (104)

Consider a vector of the quantum space

\[ \Phi = B(m^0 - 2|z_1)B(m^0 - 4|z_2)\cdots B(m^0 - 2l'|z_{l'})|0 >_{m_o}, \] (105)

where the number of \( B \) is \( l' \), \( m^0 - 2l' = m \). Due to (104), it is symmetric for \( z_i \)'s. We will show that for properly chosen \( z_1, \ldots, z_l \) which satisfy the so called Bethe ansatz equations, \( \Phi \) is an eigenvector (eigenstate) of \( t(z) \).

We have

\[
\begin{align*}
t(z)\Phi &= \left\{ \mu_0(z)A(m^0|z) + \mu_1(z)\tilde{D}(m^0|z) \right\} \Phi \\
&= \left\{ \mu_0(z)A(m^0|z) + \mu_1(z)\tilde{D}(m^0|z) \right\} \\
&\quad \times B(m^0 - 2|z_1)\cdots B(m|z_{l'})|0 >_{m_o} \\
&= \left\{ B(m^0 - 2|z_1) \left[ \mu_0(z)a_{00}'A(m^0 - 2|z) + \mu_1(z)a_{11}'\tilde{D}(m^0 - 2|z) \right] \right\}
\end{align*}
\]
\begin{align*}
+ B(m^0 - 2|z) \left[ \mu_0(z)b'_{00}A(m^0 - 2|z_1) + \mu_0(z)b'_{01} \tilde{D}(m^0 - 2|z_1) \right] \\
+ \mu_1(z)b'_{10}A(m^0 - 2|z_1) + \mu_1(z)b'_{11} \tilde{D}(m^0 - 2|z_1) \right] \\
\times B(m^0 - 4|z_2) \cdots |0 \rangle_m

= \left\{ B(m^0 - 2|z_1) \left[ (\mu a')_0 A(m^0 - 2|z) + (\mu a')_1 \tilde{D}(m^0 - 2|z) \right] \\
+ B(m^0 - 2|z) \left[ (\mu b')_0 A(m^0 - 2|z_1) + (\mu b')_1 \tilde{D}(m^0 - 2|z_1) \right] \right\} \\
\times B(m^0 - 4|z_2) \cdots |0 \rangle_m

= \left\{ B(m^0 - 2|z_1) B(m^0 - 4|z_2) \left[ (\mu a'a'')_0 A(m^0 - 4|z) \\
+ (\mu a'a'')_1 \tilde{D}(m^0 - 4|z) \right] \\
+ B(m^0 - 2|z) B(m^0 - 4|z_2) \left[ (\mu b'a'')_0 A(m^0 - 4|z_1) \\
+ (\mu b'a'')_1 \tilde{D}(m^0 - 4|z_1) \right] \right\} \\
\times B(m^0 - 6|z_3) \cdots |0 \rangle_m,
\end{align*}

where $a', a'', a''_1, b', b''$ are $2 \times 2$ matrices. The matrices $a', a'', a''_1$ are diagonal. The elements of these matrices are determined by $m^0, z$ and $z_i$ via Eqs. (102,103). The notations of these matrices are

\begin{align*}
a(m^0, z, z_1) & \equiv a', \\
a(m^0 - 2, z, z_2) & \equiv a'', \\
a(m^0 - 2, z_1, z_2) & \equiv a''_1, \\
b(m^0, z, z_1) & \equiv b', \\
b(m^0 - 2, z, z_2) & \equiv b''
\end{align*}

for short. The notations $(\mu \cdots)_i$ represents the $i$-th component of the product of the row vector $\mu$ with matrix \ldots. In the following derivation, the notations are similar. Repeatedly using Eqs. (102,103) to move $A$ and $\tilde{D}$ to the right of all $B$’s, we obtain

\begin{align*}
t(z)\Phi & = B(m^0 - 2|z_1) \cdots B(m^0 - 2l'|z_l) \\
& \times \left[ (\mu a'a'' \cdots a^{(l')}|_0 A(m^0 - 2l'|z) + (\mu a'a'' \cdots a^{(l')}|_1 \tilde{D}(m^0 - 2l'|z)) \right] |0 \rangle_m
\end{align*}

27
Therefore, noticing $m_0 - 2l' = m$, we can write $t(z)\Phi$ as

$$
t(z) = \left[ (\mu a'' \cdots a^{(r')}_0) \lambda_0(z) + (\mu a' \cdots a^{(r')}_1) \lambda_1(z) \right] \times B(m^0 - 2|z_1) \cdots B(m|z_{r'}) |0 >^m_{m_0} + \cdots + \left[ (\mu b a'' \cdots a^{(r')}_0) \lambda_0(z_1) + (\mu b a' \cdots a^{(r')}_1) \lambda_1(z_1) \right] \times B(m^0 - 2|z) B(m^0 - 4|z_2) \cdots B(m|z_{r'}) |0 >^m_{m_0} + \cdots + \left[ (\cdots) \lambda_0(z_2) + (\cdots) \lambda_1(z_2) \right] B(m^0 - 2|z) B(m^0 - 4|z_1) \times B(m^0 - 6|z_3) \cdots |0 >^m_{m_0} + \cdots
$$

Therefore, noticing $m_0 - 2l' = m$, we can write $t(z)\Phi$ as

$$
t(z) = \left[ (\mu a'' \cdots a^{(r')}_0) \lambda_0(z) + (\mu a' \cdots a^{(r')}_1) \lambda_1(z) \right] \times B(m^0 - 2|z_1) \cdots B(m|z_{r'}) |0 >^m_{m_0} + \cdots + \left[ (\mu b a'' \cdots a^{(r')}_0) \lambda_0(z_1) + (\mu b a' \cdots a^{(r')}_1) \lambda_1(z_1) \right] \times B(m^0 - 2|z) B(m^0 - 4|z_2) \cdots B(m|z_{r'}) |0 >^m_{m_0} + \cdots + \left[ (\cdots) \lambda_0(z_2) + (\cdots) \lambda_1(z_2) \right] B(m^0 - 2|z) B(m^0 - 4|z_1) \times B(m^0 - 6|z_3) \cdots |0 >^m_{m_0} + \cdots
$$

Because of (102,103) and (104) we see that $t(z)\Phi$ must be a linear combination of

$$
B(m^0 - 2|z_1) \cdots B(m|z_{r'}) |0 >^m_{m_0} \equiv \Psi_0 = \Phi
$$
$$
B(m^0 - 2|z) B(m^0 - 4|z_2) \cdots |0 >^m_{m_0} \equiv \Psi_1
$$

$$
\vdots
$$
$$
B(m^0 - 2|z) B(m^0 - 4|z_1) \cdots B(\cdots|z_{i-1}) B(\cdots|z_{i+1}) \cdots |0 >^m_{m_0} \equiv \Psi_i
$$

$$
\vdots
$$
$$
B(m^0 - 2|z) B(m^0 - 4|z_1) \cdots B(m|z_{r'-1}) |0 >^m_{m_0} \equiv \Psi_{r'}(111)
$$
This is because we can always change the order of $B$’s such that $z_i$’s inside $B$’s are arranged according to the order of $i$. So we have

$$t(z)\Phi \equiv C_0^1\Psi_0 + C_1^1\Psi_1 + C_2^1\Psi_2 + \cdots + C_{l'}^1\Psi_{l'}$$  \hspace{1cm} (112)$$

The problem now is that although the forms of $C_0^1$ and $C_1^1$ are simple and clear, $C_i^1$ for $i \geq 2$ are represented by a complicated summation. However, using the fact that $\Phi$ is a symmetric function of $z_i$’s, we can greatly simplify the calculation. Let us exchange $z_i$ and $z_1$ in $\Phi$. This does not change $\Phi$. Then we can use the above standard procedure to have

$$t(z)\Phi = C_0^i\Psi + C_1^i\Psi_1 + \cdots + C_i^i\Psi_i + C_{i+1}^i\Psi_{i+1} + \cdots$$  \hspace{1cm} (113)$$

where $C_0^i$ and $C_1^i$ can be obtained by exchanging $z_i$ and $z_1$ in $C_0^1$ and $C_1^1$. Assume $\Psi_0, \Psi_1, \ldots, \Psi_{l'}$ are linearly independent vectors. Then each coefficient for the linear decomposition of $t(z)\Phi$ by $\{\Psi_i\}$ is unique. Thus we have $C_i^1 = C_i^i$. Put all $C_i^1 = 0$ for $i \neq 0$ in (112) we have

$$t(z)\Phi = C_0^1\Phi \equiv \tau(z)\Phi.$$  \hspace{1cm} (114)$$

It is, $\Phi$ is an eigenstate of the transfer matrix $t(z)$ with eigenvalue

$$\tau(z) = \mu(z)a'a'' \cdots a^{(l')}\lambda(z)$$  \hspace{1cm} (115)$$

The spectrum parameters $z_i$ are determined by the $l'$ conditions $C_i^1 = 0, i = 1, \ldots, l'$. The first condition is

$$C_1^1 = \mu(z)b'a'' \cdots a_{1}^{(l')}\lambda(z_1) = 0.$$  \hspace{1cm} (116)$$

Other $l' - 1$ conditions can be obtained by exchanging $z_1$ and $z_i$ in (116). These are the Bethe ansatz equations. Using the explicit form of (101-103) (i.e. Eqs.(56), (100), (57) and (59)), we can prove that these equations are actually independent of the spectrum parameter $z$. This implies that $\Phi$ is an eigenstate of all transfer matrices with arbitrary spectrum.

To end this section, we present here some results. The left boundary matrix is diagonal, we can show that its diagonal elements can be explicitly written as:

$$\tilde{K}(m^0|z)_0^0 = h((a^0 - 1)w)h(\xi - z - w)h((a^0 + 1)w + \xi + z)F(z),$$

$$\tilde{K}(m^0|z)_1^1 = h((a^0 + 1)w)h(z + \xi + w)h((a^0 - 1)w + \xi - z)F(z),$$
where $\tilde{\xi}$ is the left boundary free parameter. We notice that this solution is identified with the solution given in ref. [44]. While $F(z)$ is a function of $z$ depending on the scale of the left boundary matrix, which is not essential. So we get

$$
\begin{align*}
\mu_0(z) &= h(2z + 2w)h(\tilde{\xi} - z)h(z + \tilde{\xi} + a^0w)F(z), \\
\mu_1(z) &= h(w)h(z + \tilde{\xi} + w)h(\tilde{\xi} - z + (a^0 - 1)w)F(z).
\end{align*}
$$

The eigenvalue of the transfer matrix of eight vertex model with open boundary conditions is

$$
\tau(z) = \mu_0(z)\lambda_0(z) \prod_{i=1}^{l'} \frac{h(z_i + z)h(z_i - z + w)}{h(z_i + z + w)h(z_i - z)}
+ \mu_1(z)\lambda_1(z) \prod_{i=1}^{l'} \frac{h(z_i - z - w)h(z_i + z + 2w)}{h(z_i + z + w)h(z_i - z)}. \quad (117)
$$

Here $\{z_i\}$ should satisfy the Bethe ansatz equations:

$$
\begin{align*}
\frac{\lambda_0(z_i)}{\lambda_1(z_i)} \prod_{j=1, j\neq i}^{l'} \frac{h(z_i + z_j)h(z_i - z_j + w)}{h(z_i - z_j - w)h(z_i + z_j + 2w)} &= \\
&= \frac{h(w)h(\tilde{\xi} - z_i + w(a^0 - 1))h(\tilde{\xi} + z_i + w)}{h(2z_i)h(\tilde{\xi} + z_i + wa^0)h(\tilde{\xi} - z_i)} \quad (118)
\end{align*}
$$

for $i = 1, \ldots, l'$. From (85, 86) and the definitions of $A, \tilde{D}$, we have

$$
\begin{align*}
\lambda_0(z) &= K(m_0|z) \prod_{i=1}^{l} \left[ \frac{h(z + u_i + w)h(z - u_i + w)}{[h(w)]^2} \right], \\
\lambda_1(z) &= \left[ K(m_0|z) - K(m_0|z) \frac{h(2z + w(a_0 + 1))h(w)}{h(w(a_0 + 1))h(2z + w)} \right] \frac{h(2z + w)h(w(a_0 + 1))}{h(w)h(wa_0)} \\
&\times \prod_{i=1}^{l} \left[ \frac{h(z + u_i)h(z - u_i)}{[h(w)]^2} \right],
\end{align*}
$$

where $a^0 \equiv m^0 + \gamma = a + 2l'$, $a_0 \equiv m_0 + \gamma = a + l$, $a \equiv m + \gamma$. 

30
6 Discussions

We can obtain our trigonometric limit as the following. The intertwiner (or three spin operator) defined in (27) is

$$\phi_{m,\mu}^k(z) = \theta \left[ \frac{1}{2} - \frac{k}{2} \right] (z + (-1)^\mu w (m + \gamma) + w \beta, 2\tau)$$  \hspace{1cm} (119)

Define $\gamma' = \gamma + \frac{\tau}{2w}, \beta' = \beta + \frac{\tau}{2w}$ and note $\mu = 0, 1$. Equation (27) reads

$$\phi_{m,\mu}^k(z) = \theta \left[ \frac{1}{2} - \frac{k}{2} \right] (z + (-1)^\mu w (m + \gamma') + w \beta' + (\mu - 1)\tau, 2\tau)$$

$$= \xi(\mu) \theta \left[ \frac{\mu-k}{2} \right] (z + (-1)^\mu w (m + \gamma') + w \beta', 2\tau), \hspace{1cm} (120)$$

where

$$\xi(\mu) = e^{-2\pi i \left( \frac{\mu-1}{4} \right) \left( \frac{\mu-1}{2} \tau + z + (-1)^\mu w (m + \gamma') + w \beta' + \frac{1}{2} \right)}$$  \hspace{1cm} (121)

is independent of $k$. We then rescale $\phi$ to $\phi' = \xi^{-1}\phi$. At the same time, we must perform a gauge transformation which change $W$ to $W'$ to ensure the face-vertex correspondence. When $\tau \to i\infty, \phi_{m,\mu}^k(z) \to \delta_{\mu k}$. $W'$ goes to a trigonometric R matrix, which is different with the R matrix in ref.[16] only by a constant factor. The boundary condition $K(m|z)_1^0 = 0, \tilde{K}(m|z)_1^0 = \tilde{K}(m|z)_0^1 = 0$ (if we add $K(m|z)_0^1 = 0$) also approach that of ref.[16]. Thus we can show that the trigonometric limit of our model is that of ref.[16]. It is a six vertex model with integrable reflection boundaries. In such model, the number of $B$'s ($l'$) in the Bethe ansatz state $\Phi$ is arbitrary. It is reasonable that the eigenstate of transfer matrix with maximum absolute value of eigenvalue (ETMM) is a Bethe ansatz state. Each such six vertex model can be attained as a limit of a sequence of eight vertex model with reflection boundaries. In this limit procedure, the Bethe ansatz equations, vacuum states and operators $A, B, C, D$ are all approaching that of six vertex model. Thus we can reasonably assume in this sequence of eight vertex models there is a sequence of Bethe ansatz states which approach the ETMM of the six vertex model. Thus it is quite possible that these Bethe ansatz states are ETMM of the eight vertex models, especially when they are very close to the limit six vertex model since the eigenvalues of the transfer matrix $t(z)$.
are discrete[7]. The fact whether a Bethe ansatz state is with the maximum absolute value of eigenvalue, should not depend on continuous parameters if there is no phase transition in the procedure, also since eigenvalues are discrete. For the above six vertex model, when $l$ is given (the column number is given), the true discrete variable is $l'$. The above sequence of Bethe ansatz states should have same $l'$. Thus, it is reasonable that in our approach, when the left and right boundary condition determine a proper $l'$, the Bethe ansatz state has a eigenvalue of maximum absolute value, which is the most important state in thermodynamics.

We know that Johnson, Krinsky and McCoy calculated the energy of excitations of the $XYZ$ model after Baxter obtains the Bethe ansatz of eight vertex model with periodic boundary conditions[48]. Now, using the results presented in this paper, we may also calculate the energy of the excitations of the $XYZ$ model with boundaries. By analyzing the eigenvalues of the transfer matrix, one may also get the boundary free energy, thermodynamic limit and finite size corrections. Other physical phenomena are also worth of studying such as surface critical exponents and scaling, the central charges in conformal field theory etc.. It is well known that eight vertex model is equivalent to SOS model, if one impose some restrictions on SOS model, we can obtain the restricted SOS model (ABF model). So, if we impose some restrictions on the Bethe ansatz of eight vertex model with boundaries, we should find the Bethe ansatz for the ABF model with boundary conditions. All of these work worth be studied in the future.

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Appendix A  Quadratic relation of the components of the face type $k(z)$

The left hand side of equation (24) is $R_{12}(z_1 - z_2)k_1(z_1)R_{21}(z_1 + z_2)k_2(z_2)$. We multiply it from left by $\tilde{\phi}^{(1)}_{m+\tilde{\mu}_0,\mu_0}(z_1)\tilde{\phi}^{(2)}_{m+\tilde{\mu}_0+\nu_0,\nu_0}(z_2)$ and multiply it from right by $\phi^{(1)}_{m+\tilde{\mu}_3,\mu_3}(-z_1)\phi^{(2)}_{m+\tilde{\mu}_3+\nu_3,\nu_3}(-z_2)$ obtaining

$$LHS = \tilde{\phi}^{(1)}_{m+\tilde{\mu}_0,\mu_0}(z_1)\tilde{\phi}^{(2)}_{m+\tilde{\mu}_0+\nu_0,\nu_0}(z_2)R_{12}(z_1 - z_2)\cdots.\quad(122)$$

Using (34) we can eliminate $R_{12}$ to have

$$LHS = \begin{pmatrix} W(m|z_1 - z_2)_{\mu_0\nu_0} \phi^{(1)}_{m+\tilde{\mu}_1,\nu_1}(z_1)\phi^{(2)}_{m+\tilde{\mu}_1+\nu_1,\mu_1}(z_2) & R_{21}(z_1 + z_2)k_2(z_2)\phi^{(1)}_{m+\tilde{\mu}_3,\mu_3}(-z_1)\phi^{(2)}_{m+\tilde{\mu}_3+\nu_3,\nu_3}(-z_2) \end{pmatrix}.\quad(123)$$

Move $\tilde{\phi}^{(2)}$ over $k_1(z_1)$ and move $\phi^{(1)}$ over $k_2(z_2)$, so that they are at the left and right side of $R_{21}$ respectively. LHS becomes

$$LHS = \begin{pmatrix} \cdots \phi^{(2)}_{m+\tilde{\mu}_1,\nu_1}(z_2)R_{21}(z_1 + z_2)\phi^{(1)}_{m+\tilde{\mu}_3,\mu_3}(-z_1) & \cdots \end{pmatrix} = \begin{pmatrix} \cdots W(m|z_1 + z_2)_{\nu_2\mu_2}\phi^{(1)}_{m+\tilde{\mu}_2+\nu_2,\mu_2}(z_2) & \phi^{(2)}_{m+\tilde{\mu}_3+\nu_3,\nu_3}(-z_1) \end{pmatrix} \cdot \cdots.\quad(124)$$

Since the Boltzmann weight $W$ is non zero only if $\tilde{\nu}_2 + \tilde{\mu}_3 = \nu_1 + \mu_2$, we have

$$LHS = \begin{pmatrix} \cdots \phi^{(1)}_{m+\tilde{\mu}_3+\nu_3,\nu_3}(-z_1)W(m|z_1 + z_2)_{\nu_2\mu_2}\phi^{(2)}_{m+\tilde{\mu}_2+\nu_2,\mu_2}(z_2) & \cdots \end{pmatrix}.\quad(125)$$

Now $k_1(z_1)$ and $k_2(z_2)$ all have their “own” $\tilde{\phi}$ and $\phi$ at left and right side. By definition one conclude

$$LHS = k(m+\tilde{\mu}_2+\nu_1|z_1)_{\mu_2}k(m+\tilde{\mu}_3+\nu_3|z_2)_{\mu_3}$$

$$\times W(m|z_1 - z_2)_{\mu_0\nu_0}W(m|z_1 + z_2)_{\nu_1\mu_1}.\quad(126)$$

The derivation of RHS is similar.

Appendix B  Left and Right inverse matrices of $k(z)$

When $n = 2$, we have

$$R_{12}(z) = R_{21}(z) = R_{12}(z)^{t_1t_2}.\quad(127)$$
When \( z = -w \), we have
\[
P_-(12)R_{12}(-w) = R_{12}(-w)P_-(12) = R_{12}(-w),
\]
where \( P_-(12) \) is the anti-symmetric operator of space \( V_1 \otimes V_2 \) satisfying \( P_-(12)^2 = P_-(12) \). On the other hand the \( R \) matrix have the property
\[
I_\alpha^{(i)}I_\alpha^{(j)}R_{ij}(z)[I_\alpha^{(i)}]^{-1}[I_\alpha^{(j)}]^{-1} = R_{ij}(z).
\]
Thus
\[
I_\alpha^{(3)}P_-(12)R_{32}(z_2)R_{31}(z_1)P_-(12)[I_\alpha^{(3)}]^{-1} = P_-(12)I_\alpha^{(3)}R_{32}(z_2)R_{31}(z_1)I_\alpha^{(3)}P_-(12) = P_-(12)[I_\alpha^{(2)}]^{-1}R_{32}(z_2)R_{31}(z_1)I_\alpha^{(1)}P_-(12).
\]
It is not difficult to show
\[
I_\alpha^{(1)}I_\alpha^{(2)}P_-(12) = (-1)^{\alpha_1+\alpha_2}P_-(12) = P_-(12)[I_\alpha^{(1)}]^{-1}[I_\alpha^{(2)}]^{-1}.
\]
Thus
\[
I_\alpha^{(3)}P_-(12)R_{32}(z_2)R_{31}(z_1)P_-(12)[I_\alpha^{(3)}]^{-1} = P_-(12)R_{32}(z_2)R_{31}(z_1)P_-(12) \equiv U.
\]
The operator \( U \) acting on \( V_1 \otimes V_2 \otimes V_3 \) is invariant under the similar transformation by \( I_\alpha^{(3)} \), which implies that \( U \) is equivalent to a unit operator on the \( V_3 \). From (127) we see that this is also true for
\[
U' = P_-(12)R_{23}(z_2)R_{13}(z_1)P_-(12).
\]
We then consider the RE (14) for the case \( z_1 = z - w, z_2 = z \), and have
\[
G = R_{12}(-w)k_1(z - w)R_{21}(2z - w)k_2(z)
\]
\[
= R_{12}(-w)T_1(z - w)K_1(z - w)S_1(z - w)R_{21}(2z - w)T_2(z)K_2(z)S_2(z).
\]
Due to (21),
\[
G = R_{12}(-w)T_1(z - w)K_1(z - w)T_2(z)R_{21}(2z - w)S_1(z - w)K_2(z)S_2(z)
\]
\[
= R_{12}(-w)T_1(z - w)T_2(z)K_1(z - w)R_{21}(2z - w)K_2(z)S_1(z - w)S_2(z).
\]
Rewrite $R_{12}(-w)$ as $P_- (12) R_{12}(-w)$, and move $R_{12}(-w)$ towards the right. Each time when it goes over a pair of $R_{12} R_{2i}$ in $T_1 T_2$ by YBE (9), we rewrite $R_{12}(-w)$ as $P_- (12) R_{12}(-w)$ and leave $P_-$. Then we obtain

$$ G = [P_- (12) R_{23} R_{13} R_{24} R_{14} P_- (12) \cdots P_- (12) R_{2i} R_{1i} P_- (12)]$$

$$ R_{12}(-w) K_1(z-w) R_{21}(2z-w) K_2(z) S_1(z-w) S_2(z)$$

$$ \equiv \ [M] \times \cdots . \quad (134)$$

Move $R_{12}(-w)$ to the left side of $S_1$ by RE. One has

$$ G = [M] K_2(z) R_{12}(2z-w) K_1(z-w) R_{21}(-w) S_1(z-w) S_2(z). \quad (135)$$

Similarly, we move $R_{21}(-w)$ step by step to the right side of $S_1 S_2$ obtaining

$$ G = [M] \cdots [N] R_{21}(-w)$$

$$ [N] = [P_- (12) R_{12} R_{21} P_- (12) P_- (12) R_{l-1,2} R_{l-1,1} P_- (12) \cdots P_- (12) R_{32} R_{31} P_- (12)]. \quad (136)$$

From the property of $U$ and $U'$, we see that $G$ is proportional to an identity operator in the "quantum" space $V' = V_3 \otimes \cdots \otimes V_l$. Using the similar derivation as in Appendix A, we multiply $\tilde{\phi}^{(1)} \tilde{\phi}^{(2)}$ from left and multiply $\phi^{(1)} \phi^{(2)}$ from right of $G$, and conclude that when $z_1 = z - w, z_2 = z$, both LHS and RHS of (40) are proportional to identity operator in the quantum space $V'$. Properly choosing indices and noticing $W(m|w) \phi^\mu \phi^\nu = \cdots \delta_{\mu \nu} \delta_{\mu' \nu'}$, we have (49).

From the above derivation, we see also that $G$ is anti-symmetric to the classical (auxiliary) indices of space $V_1 \otimes V_2$ and is independent of $m$. Thus

$$ \rho'(m, \mu|z) = \tilde{\phi}^{\mu}_{m+\mu,\mu}(z-w) \tilde{\phi}^{\mu}_{m+\mu,\mu}(z) G_{ij}^{\mu \nu}$$

$$ \times \phi^{\mu}_{m+1,1}(-z+w) \phi^{\nu}_{m+1,0}(-z)$$

$$ = \left\{ \tilde{\phi}^{1}_{1}(z-w) \tilde{\phi}^{0}_{0}(z) - \tilde{\phi}^{0}_{0}(z-w) \tilde{\phi}^{1}_{0}(z) \right\}$$

$$ \left\{ \phi^{1}_{1}(-z+w) \phi^{0}_{0}(-z) - \phi^{0}_{0}(-z+w) \phi^{1}_{0}(-z) \right\} G_{10}^{10}. \quad (137)$$

Similarly, $\rho(m, \nu|z)$ can also be expressed as $G_{10}^{10}$ multiplied by a factor which depends only on $\phi, \tilde{\phi}$. Therefore the ratio $\frac{\rho'(m+2|z)}{\rho'(m|z)}$ is independent of $G_{10}^{10}$. It is completely determined by $\phi, \tilde{\phi}$. Direct calculation gives (53).
Appendix C Derivation of the boundary condition

From definition we have

\[ \phi_{m,\mu}^k(z) = \theta \left[ \frac{1}{2} \frac{z}{2} \frac{1}{2} \right] \left( z + (-1)^{\mu}w a + w \beta, 2 \tau \right) \]

\[ \equiv \theta^{(k)}(z + w((-1)^{\mu}a + \beta)) \]

\[ \equiv \theta^{(k)}(z + \chi), \quad (138) \]

where \( a \equiv m + \gamma \). When \( \alpha_1, \alpha_2 \) are integers, by the expression of \( \theta \) function, we have

\[ \theta^{(k)}(z + \chi + \alpha_1 \tau + \alpha_2) \]

\[ = e^{-2\pi i \left( \frac{\alpha_1}{2} \right) \left( \frac{\alpha_1}{2} \tau + z + \chi + \frac{1}{2} \right) + 2\pi i \left( \frac{1}{2} - \frac{k}{2} \right) \alpha_2} \theta^{(k-\alpha_1)}(z + \chi), \quad (139) \]

giving

\[ h^{\alpha_1} g^{\alpha_2} \phi_{m,\mu}(z) = (-1)^{\alpha_2} e^{2\pi i \left( \frac{\alpha_1}{2} \right) \left( \frac{\alpha_1}{2} \tau + z + \chi + \frac{1}{2} + \alpha_2 \right)} \phi_{m,\mu}(z + \alpha_1 \tau + \alpha_2). \quad (140) \]

On the other hand, from the property of zeros of doubly quasi-periodic holomorphic function, one can show[49]

\[ \text{Det} \begin{bmatrix} \theta^{(0)}(z_1) & \theta^{(0)}(z_2) \\ \theta^{(1)}(z_1) & \theta^{(1)}(z_2) \end{bmatrix} = C \times h(z_1 + z_2 - \frac{1}{2}) h(z_1 - z_2), \quad (141) \]

where \( C \) is independent of \( z_1 \) and \( z_2 \). If we write

\[ A \equiv \begin{bmatrix} \phi_{m+0,0}^0(z) & \phi_{m+0,1}^0(z) \\ \phi_{m+1,0}^1(z) & \phi_{m+1,1}^1(z) \end{bmatrix} \]

\[ (142) \]

then \( \tilde{\phi}_{m+\mu,\mu}^k(z) \) are elements of its inverse matrix. Thus for any column vector \( \Psi \), the quantity

\[ B \equiv \sum_k \tilde{\phi}_{m+1,1}^k(z) \Psi^k \]

can be written as

\[ B = \frac{1}{\text{Det} A} \text{Det} \begin{bmatrix} \phi_{m+0,0}^0(z) & \Psi^0 \\ \phi_{m+0,0}^1(z) & \Psi^1 \end{bmatrix}. \quad (143) \]
Combining (140) and (143) gives
\[
B'_\alpha \equiv \bar{\phi}_{m-2,1}(z)h^{\alpha_1}g^{\alpha_2}\phi_{m,0}(-z)
= \left( \frac{1}{\text{Det} A} \right) \text{Det} \begin{bmatrix}
\theta^{(0)}(z + wa + w\beta) & \theta^{(0)}(-z + wa + w\beta + \alpha_1\tau + \alpha_2) \\
\theta^{(1)}(z + wa + w\beta) & \theta^{(1)}(-z + wa + w\beta + \alpha_1\tau + \alpha_2)
\end{bmatrix}
\times e^{2\pi i (\frac{\alpha_1}{2}) (z + wa + w\beta + \frac{1}{2} + \alpha_2)} \times (-1)^{\alpha_2}.
\] (144)

Two determines can be obtained from (141), thus
\[
B'_\alpha = -\left[ \sigma_\alpha(wa + w\beta - \frac{1}{2})\sigma_\alpha(-z)(-1)^{\alpha_2} \right] \left[ h(z + w\beta + w - \frac{1}{2})h(wa - w) \right]^{-1}
\] (145)

Substituting the definition of \( K(m|z)_1^0 \) and the expression of \( K(z)(15,63) \), we have
\[
K(m|z)_1 = (-1)^2 \frac{\sum_\alpha C_\alpha(-1)^{\alpha_2}\sigma_\alpha(wa + w\beta - \frac{1}{2})}{h(z + w\beta + w - \frac{1}{2})h(wa - w)}.
\] (146)

Other elements of \( K(m|z) \) can be similarly obtained. They are.

\[ K(m|z)_0 = a \to -a \text{ in RHS of the above equation}. \]
\[ K(m|z)_1 = a \to -a \text{ in RHS of equation (147)}. \]
\[ \tilde{K}(m|z)_0 = a \to -a \text{ in RHS of equation (148)}. \]
\[ \tilde{K}(m|z)_1 = a \to -a \text{ in RHS of equation (149)}. \]
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