SOME EXACT SEQUENCES ASSOCIATED WITH ADJUNCTIONS IN BICATEGORIES. APPLICATIONS.

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Abstract. We prove that the classical result asserting that the relative Picard group of a faithfully flat extension of commutative rings is isomorphic to the first Amitsur cohomology group stills valid in the realm of symmetric monoidal categories. To this end, we built some group exact sequences from an adjunction in a bicategory, which are of independent interest. As a particular byproduct of the evolving theory, we prove a version of Hilbert’s theorem 90 for cocommutatvie coalgebra coextensions (=surjective homomorphisms).

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Introduction

We prove that the classical result asserting that the relative Picard group of a faithfully flat extension of commutative rings is isomorphic to the first Amitsur cohomology group stills valid in the realm of symmetric monoidal categories. To this end, we prove that for any commutative algebra $A = (A, m, e)$ in a symmetric monoidal category $V$ satisfying some technical conditions, there is an exact sequence of groups

\[ 0 \to \text{Aut}_V(I) \xrightarrow{\text{id}} \text{Aut}_V(A) \xrightarrow{\kappa_A} \text{Aut}_{A-\text{cor}}(A \otimes A) \xrightarrow{\alpha_A} \text{Pic}^c(I) \xrightarrow{\text{Pic}^c(e)} \text{Pic}^c(A). \]  

(1)
Details on the group homomorphisms involved are to be found in subsection 4.2. The sequence (1) will be built with the help of an exact sequence of groups associated, under mild conditions, to any adjunction in a bicategory (see Theorem 3.9). The latter generalizes some useful exact sequences associated to a ring extension, used in [15, 21] in the unital case, and in [11] in the realm of ring with local units, to derive adequate versions of Chase-Harrison-Rosenberg’s seven exact sequence [5].

Thus, our results could be of independent interest for extending to wider contexts the aforementioned seven terms sequence. We specialize our general theory to the case of exact sequence (5). Thus, our results could be of independent interest for extending to wider realms of ring with local units, to derive adequate versions of Chase-Harrison-Rosenberg’s seven sequences associated to a ring extension, used in [16, 21] in the unital case, and in [11] in the non-unital case.

Recall that an algebra $A$ over a category $V$ is a monoidal category with underlying ordinary category $V$, tensor product $\otimes$ and monoidal unit $I$. Every comonadic homomorphism of commutative $V$–algebras $\iota : A \to B$, where $V$ is a symmetric monoidal category fulfilling some minimal technical requirements, leads to a homomorphism of the first Amitsur cohomology group $H^1(\iota, \text{Aut}_A^{\mathcal{A}})$ of $\iota$ with coefficients in the functor $\text{Aut}_A^{\mathcal{A}}$ (which is a generalization of the usual units functor, see Lemma 4.10). The problem is easily reduced to prove that, in (1), the kernel of $\kappa_A$ is isomorphic to $H^1(\iota, \text{Aut}_A^{\mathcal{A}})$, whenever $- \otimes A$ is a comonadic functor. Our proof involves some classical results, namely a version of the Bénabou-Roubaud-Beck theorem identifying the category of descent data with an Eilenberg-Moore category (Theorem A.2), and Grothendieck’s isomorphism between the Amitsur first cohomology pointed set and the set of descent data of an effective descent morphism (Proposition A.4). A brief account of the required classical theory is given in the Appendix.

In the final section, we apply our general theory to the bicategory of bimodules. As a particular a version of Hilbert’s theorem 90 for cocommutative coalgebra coextensions (=surjective homomorphisms) (Theorem 4.17) is obtained.

1. Preliminaries

In this section, we list some categorical notions and basic constructions that will be needed. Our basic references on categories are [11, 12, 13].

1.1. Subobjects and quotient objects. Let $a$ be an object of a category $A$. Preorder monomorphisms with range $a$ by setting $j \leq i$ if $j$ is of the form $j = ik$; the equivalence classes for the relation

$$j \leq i \text{ and } i \leq j$$

are called subobjects of $a$. We write $\text{Sub}_A(a)$ for the the class of all subobjects of $a$. We often identify a subobject with a representative monomorphism, and we call the subobject regular etc. if the monomorphism is regular etc.

Dually, one has the collection $\text{Quot}_A(a) = \text{Sub}_{A^{op}}(a)$ of isomorphism classes of epimorphisms with domain $a$ ($A^{op}$ denotes the opposite category of $A$). We shall call an element of $\text{Quot}_A(a)$ a quotient object of $a$. Note that for epimorphisms with domain $a$ we write $j \leq i$ if $j$ is of the form $j = ki$.

1.2. Images and coimages. Recall that a category admits images if any morphism $f$ can be written as $f = ip$ with $i$ monomorphic and $p$ regular epimorphic. The subobject $[i]$ of the codomain of $f$ is called the image of $f$. Dually, a category is said to admit coimages if any morphism $f$ can be written as $f = ip$ with $p$ epimorphic and $i$ regular monomorphic. The quotient object $[p]$ of the domain of $f$ is called the coimage of $f$. We say that a monoidal category admits (co)images if its underlying ordinary category does so.

1.3. Subobjects and quotient objects of (co)algebras. Suppose that $V = (\mathcal{V}, \otimes, I)$ is a fixed monoidal category with underlying ordinary category $\mathcal{V}$, tensor product $\otimes$ and monoidal unit $I$. Recall that an algebra in $V$ (or $V$-algebra) consists of an object $A$ of $V$ endowed with a multiplication $m_A : A \otimes A \to A$ and unit morphism $e_A : I \to A$, subject to the usual associative and identity conditions. These algebras are the objects of a category $\text{Alg}(\mathcal{V})$ with the obvious morphisms.
Dually, one has the notions of $\mathcal{V}$–coalgebra; the corresponding category of $\mathcal{V}$–coalgebras is denoted by $\mathbf{Coalg}(\mathcal{V})$.

Given a $\mathcal{V}$-algebra $A = (A, m_A, e_A)$, we write $\mathcal{J}_A^V(A)$ for the subset of $\text{Sub}_\mathcal{V}(A)$ consisting of those elements $[(J, i_J : J \to A)]$ for which the composite

$$\xi_{i_J} : A \otimes J \xrightarrow{A \otimes i_J} A \otimes A \xrightarrow{m_A} A$$

is an isomorphism. Symmetrically, we let $\mathcal{J}_A^R(A)$ denote the subclass of $\text{Sub}_\mathcal{V}(A)$ consisting of those elements $[(J, i_J : J \to A)]$, for which the composite

$$\xi_{i_J} : J \otimes A \xrightarrow{i_J \otimes A} A \otimes A \xrightarrow{m_A} A$$

is an isomorphism.

Dually, for a $\mathcal{V}$-coalgebra $C = (C, \delta, \epsilon)$, we write $\mathcal{Q}_C^V(C)$ (resp. $\mathcal{Q}_C^R(C)$) for the subset of $\text{Quot}_\mathcal{V}(C)$ consisting of those elements $[(P, \pi_P : C \to P)]$ for which the composite

$$C \xrightarrow{\delta} C \otimes C \xrightarrow{C \otimes \pi_P} C \otimes P$$

(resp.

$$C \xrightarrow{\delta} C \otimes C \xrightarrow{\pi_P \otimes C} P \otimes C$$

is an isomorphism.

1.4. Adjunctions in bicategories. We begin by recalling from [2] that a bicategory $\mathbb{B}$ consists of:

- a class $\text{Ob}(\mathbb{B})$ of objects, or 0-cells;
- a family $\mathbb{B}(A, B)$, for all $A, B \in \text{Ob}(\mathbb{B})$, of hom-categories, whose objects and morphisms are respectively called 1-cells and 2-cells;
- a (horizontal) composition operation, given by a family of functors

$$\mathbb{B}(B, C) \times \mathbb{B}(A, B) \to \mathbb{B}(A, C)$$

whose action on a pair $(g, f) \in \mathbb{B}(B, C) \times \mathbb{B}(A, B)$ is written $g \circ f$;

- identities, given by 1-cells $1_A \in \mathbb{B}(A, A)$, for $A \in \text{Ob}(\mathbb{B})$;

- natural isomorphisms

$$\alpha_{h, g, f} : (h \circ g) \circ f \simeq h \circ (g \circ f), l_f : 1_A \circ f \simeq f \text{ and } r_f : f \circ 1_A \simeq f,$$

subject to two coherence axioms (see [2]).

When the context is clear, we write $[A, B]$ instead of $\mathbb{B}(A, B)$.

We review the concept of adjunction in an arbitrary bicategory along with some of the general theory needed later on.

Fix a bicategory $\mathbb{B}$. An adjunction $(\eta, \varepsilon : f \dashv f^* : B \to A)$ in $\mathbb{B}$ consists of objects $A$ and $B$, 1-cells $f : A \to B$ and $f^* : B \to A$, and 2-cells $\eta : 1_A \to f^* \circ f$, called the unit, and $\varepsilon : f \circ f^* \to 1_B$, called the counit such the following diagrams commute in $[A, B]$ and $[B, A]$, respectively:

$$\begin{array}{ccc}
1_A \circ f & \xrightarrow{\eta \circ f} & f^* \circ f \circ f^* \circ f \\
\xrightarrow{r_f} & & \xrightarrow{\varepsilon \circ f} \\
1_B \circ f & = & 1_B \circ f \\
\end{array}$$

and

$$\begin{array}{ccc}
1_A \circ f^* & \xrightarrow{f \circ \eta^*} & (f \circ f^*) \circ f \\
\xrightarrow{l_f} & & \xrightarrow{f \circ \varepsilon} \\
f^* & = & f^* \circ 1_B.
\end{array}$$
Let \( \eta, \varepsilon : f \dashv f^* : B \to A \) be an adjunction in \( \mathcal{B} \) and let \( X \) be an arbitrary \( 0 \)-cell of \( \mathcal{B} \). Then the functor
\[
[X, f] = f_{\circ -} : [X, A] \to [X, B]
\]
admits as a right adjoint the functor
\[
[X, f^*] = f^*_{\circ -} : [X, B] \to [X, A].
\]
The unit \( \eta^X \) and counit \( \varepsilon^X \) of this adjunction are given by the formulas:
\[
\eta^X_g : g \xrightarrow{\alpha_{f_{\circ -} g}} 1_{A_{\circ -} g} \circ g \xrightarrow{\circ \eta_{f_{\circ -} g}} f^*_{\circ -} f_{\circ -} (f^*_{\circ -} g), \text{ for all } g \in [X, A]
\]
and
\[
\varepsilon^X_{f^* h} : f_{\circ -} (f^*_{\circ -} h) \xrightarrow{\alpha_{f_{\circ -} f_{\circ -} h}} (f_{\circ -} f^*)_{\circ -} h \xrightarrow{\circ \varepsilon_{f_{\circ -} f_{\circ -} h}} 1_B \circ h \xrightarrow{\varepsilon_{1_B}} h, \text{ for all } h \in [X, B].
\]
The situation may be pictured as
\[
\begin{array}{c}
\begin{array}{c}
[X, A] \\
[X, f] = f_{\circ -}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
[X, B] \\
[X, f^*] = f^*_{\circ -}
\end{array}
\end{array}
\]

**Definition.** A 1-cell \( f : A \to B \) in \( \mathcal{B} \) is called invertible if there exist a 1-cell \( g : B \to A \) and isomorphisms \( g \circ f \simeq 1_A \) and \( f \circ g \simeq 1_B \). The 1-cell \( g \) is called a pseudo-inverse of \( f \).

Recall that an adjoint equivalence in \( \mathcal{B} \) is an adjunction in which both the unit and counit are isomorphisms, and that any equivalence is part of an adjoint equivalence.

**Remark 1.5.** If a 1-cell \( h : A \to B \) is invertible, then, for any object \( X \in \mathcal{B} \), both functors \( [X, h] = h_{\circ -} : [X, A] \to [X, A] \) and \( [h, X] = -_{\circ h} : [A, X] \to [A, X] \) are equivalences of categories, and thus they preserve existing limits and colimits. In particular, they preserve monomorphisms and epimorphisms.

The following is an example of bicategory to which some of our general results will be applied.

**Example 1.6. Firm bimodules.** Let \( S \) be a ring, which is not assumed to be unital. A right \( S \)-module \( M \) is said to be firm [23] if the map \( M \otimes_S S \to M \) sending \( m \otimes s \) to \( ms \) is an isomorphism. Thus, the ring \( S \) is said to be firm if the multiplication map \( S \otimes_S S \to S \) is an isomorphism. Firm left modules and firm bimodules are defined analogously. We denote by \( \text{Firm} \) the bicategory whose \( 0 \)-cells are firm rings, the \( 1 \)-cells are firm bimodules and the \( 2 \)-cells are homomorphisms of firm bimodules. The horizontal composition in \( \text{Firm} \) is given by the tensor product of bimodules. Given a homomorphism \( \varphi : R \to S \), where \( R \) and \( S \) are firm rings, we may consider the bimodules \( R_S S \) and \( S_R S \) in the usual way. We say that \( \varphi \) is a homomorphism of firm rings if \( R_S S \) and \( S_R S \) are firm bimodules. In this case, we have \( 1 \)-cells \( S_R S : R \to S \) and \( R_S S : S \to R \), which form an adjunction \( S_R S \dashv R_S S \) in \( \text{Firm} \). Its counit is the multiplication map \( \mu : S \otimes_R S \to S \), while the unit is given by the composite \( \mu : S \xrightarrow{\varepsilon} S \otimes_S S \xrightarrow{\nu} S \otimes_S S \), where \( \nu \) denotes the inverse of the multiplication map \( S \otimes_S S \xrightarrow{\varphi} S \).

**1.7. Mates.** Recall from [17] that for adjunctions \((\eta, \varepsilon : f \dashv g : B \to A)\) and \((\eta', \varepsilon' : f' \dashv g' : B \to A)\) in \( \mathcal{B} \), there is a bijection between 2-cells
\[
\sigma : f \to f' \quad \text{and} \quad \sigma' : g' \to g,
\]
where \( \sigma \) is obtained as the composite
\[
g' \xrightarrow{\eta'_{g'}} 1_{A_{\circ -} g'} \xrightarrow{\eta_{g'}} (g_{\circ -} g')_{\circ -} g' \xrightarrow{\circ \eta_{g'}} g_{\circ -} (g_{\circ -} g') \xrightarrow{\circ \varepsilon'_{g_{\circ -} g'}} g_{\circ -} 1_B \xrightarrow{\varepsilon} g
\]
and \( \sigma' \) is given as the composite
\[
f \xrightarrow{\sigma_{g_{\circ -} g'}} f_{\circ -} (g_{\circ -} g') \xrightarrow{\circ \eta_{g_{\circ -} g'}} f_{\circ -} (g_{\circ -} g') \xrightarrow{\circ \varepsilon_{g_{\circ -} g'}} 1_B f_{\circ -} \xrightarrow{\varepsilon_{1_B}} f'.
\]
In this situation, \( \sigma \) and \( \sigma' \) are called mates under the given adjunctions and this is denoted by \( \sigma \vdash \sigma' \).
Lemma 1.8. If \( \sigma \dashv \varphi \) under adjunctions \((f \dashv g : B \to A)\) and \((f' \dashv g' : B \to A)\). Then \( \sigma \) is an isomorphism iff \( \varphi \) is.

2. Invertible cells associated to an adjunction

Let \( A \) be an object of a bicategory \( \mathcal{B} \). We call a (co)algebra in the monoidal category \([A, A]\) an \( A\)-(co)ring and write \( A\)-ring = \( \text{Alg}([A, A])\) (resp. \( A\)-cor = \( \text{Coalg}([A, A])\)) for the category of \( A\)-(co)rings.

Any 1-cell with a right adjoint generates a ring as well as a coring as follows. If \( \eta_f, \varepsilon_f : f \dashv f^* : B \to A \) is an adjunction in \( \mathcal{B} \), then the triple

\[
\mathcal{S}_f = (f^* \circ f, m_f, \eta_f),
\]

where \( m_f \) is the composite

\[
(r_f \circ f) \cdot ((f^* \circ f) \circ f) \cdot (\alpha_{f^* f^* f})^{-1} : (f^* \circ f) \circ (f^* \circ f) \to f^* \circ f,
\]

is an \( A\)-ring, while the triple

\[
\mathcal{C}_f = (f \circ f^*, \delta_f, \varepsilon_f),
\]

where \( \delta_f \) is the composite

\[
(\alpha_{f^* f^* f}) \cdot (\alpha_{f^* f^* f}^{-1}) : ((f \circ \eta_f) \circ f^*) \cdot (r_f \circ f^*) : f \circ f^* \to (f \circ f^*) \circ (f \circ f^*),
\]

is a \( B\)-coring.

Since \( \mathcal{S}_f \) is an algebra in the monoidal category \([A, A]\), one has the sets \( \mathcal{Y}^r_{[A, A]}(\mathcal{S}_f) \) and \( \mathcal{Y}^l_{[A, A]}(\mathcal{S}_f) \).

Recall from [12, Remark 4.2] that for any monomorphic 2-cell \( i_h : h \to f^* \circ f \),

\[
\xi_{ih} = (f^* \circ \xi_{ih}) \cdot \alpha_{f^* f}, f, h
\]

and

\[
\xi_{ih}^* = (\xi_{ih}^* \circ f) \cdot \alpha_{h, f^* f}, f,
\]

where \( \xi_{ih} \) and \( \xi_{ih}^* \) are the composites

\[
\xi_{ih} : f \circ h \xrightarrow{\eta_h \circ h} f \circ (f^* \circ f) \xrightarrow{\alpha_{f^* f}^{-1} \circ f} (f \circ f^*) \circ f \xrightarrow{\varepsilon_f \circ f} 1_B \circ f \xrightarrow{1_f} f,
\]

and

\[
\xi_{ih}^* : h \circ f^* \xrightarrow{i_h \circ f^*} (f^* \circ f) \circ f^* \xrightarrow{\alpha_{f^* f} \circ f} f^* \circ (f \circ f^*) \xrightarrow{f \circ \varepsilon_f} f^* \circ 1_B \xrightarrow{r_f} f^*,
\]

respectively.

We write \( \mathcal{Y}_{f}^{A,l} \) (resp. \( \mathcal{Y}_{f}^{A,r} \)) for the subset of \( \mathcal{Y}^r_{[A, A]}(\mathcal{S}_f) \) (resp. \( \mathcal{Y}^l_{[A, A]}(\mathcal{S}_f) \)) determined by those subobjects \([h, i_h]\) with \( h \) invertible.

Proposition 2.1. Let \( \eta_f, \varepsilon_f : f \dashv f^* : B \to A \) be an adjunction in \( \mathcal{B} \) such that \( \eta_f \) is monomorphic in \([A, A]\) and \( h : A \to A \) be an invertible 1-cell. If there is an isomorphism \( \sigma : f \circ h \to f \) in \([A, B]\), then \([h, i_h]\) \( \in \mathcal{Y}_{f}^{A,l} \), where \( i_h \) is the composite \( h \xrightarrow{\eta_h \circ h} f^* \circ f \circ h \xrightarrow{f \circ \varepsilon_f} f^* \).

Proof. Suppose that \( h \) is invertible and that there is an isomorphism \( \sigma : f \circ h \to f \) in \([A, B]\). Since \( \eta_f \) is assumed to be monomorphic in \([A, A]\), it follows from Remark 1.5 that \( h \xrightarrow{\eta_h \circ h} f^* \circ f \circ h \) is monomorphic in \([A, A]\). Then, since \( \sigma \) is an isomorphism, \( i_h \) must be a monomorphism too. Now, since the functor \( f^* \circ \varepsilon \) : \([A, B] \to [A, A]\) is right adjoint for the functor \( f \circ \varepsilon \) : \([A, A] \to [A, B]\) with \( \eta_f \circ \varepsilon \) as unit and \( \varepsilon_f \circ \varepsilon \) as counit, it follows that \( \sigma = (\varepsilon_f \circ f) \cdot (f \circ \varepsilon_f) = \xi_{ih} \). Therefore, \( \xi_{ih}^* = f^* \circ \xi_{ih} = f^* \circ \sigma \) is an isomorphism too and hence \([h, i_h]\) \( \in \mathcal{Y}_{f}^{A,r} \).

\( \square \)

Proposition 2.2. In the situation of Proposition 2.1, suppose that \( h^* \) is a pseudo-inverse of \( h \). Then there is a monomorphic 2-cell \( i_{h^*} : h^* \to f^* \circ f \) such that \([h^*, i_{h^*}] \) \( \in \mathcal{Y}_{f}^{A,r} \).

\[ \footnote{For simplicity of exposition we sometimes treat \( \mathcal{B} \) as a 2-category which is justified by the coherence theorem (see [19]) asserting that every bicategory is biequivalent to a 2-category. Consequently, we sometimes omit brackets in the horizontal compositions and suppress the associativity constraints \( \alpha \) and the unitality constraints \( \ell \) and \( r \).} \]
Proof. Composing the adjunction \( f \dashv f^* \) with \( h \dashv h^* \) yields an adjunction \( f o h \dashv h^* o f^* \). Since \( f o h \simeq f \) and since adjoints are unique up to unique isomorphism, one has an isomorphism \( \tau : h^* o f^* \simeq f^* \). Now, if we take \( i_h \) to be the composite \( i_h : h^* \xrightarrow{h^* o \eta_f} h^* o f^* o f \xrightarrow{\eta_f} f^* o f, \) then the result is proved in exactly the same way as Proposition \( 2.1 \) but this time using the adjunction \(-\circ f^* \dashv -\circ f : \mathcal{B}, A \rightarrow [A, A].\)

\[ \square \]

Proposition 2.3. Let \( \eta, \varepsilon : f \dashv f^* : \mathcal{B} \rightarrow A \) be an adjunction and \((\eta_h, \varepsilon_h : h \dashv h^* : A \rightarrow A)\) be an adjoint equivalence in \( \mathcal{B} \). Then for any 2-cell \( i_h : h \rightarrow f^* o f, \) the following are equivalent:

(i) \( \xi_{ih} : f o h \rightarrow f \) is an isomorphism;
(ii) \( \xi^*_{ih} : (f^* o f) o h \rightarrow f^* o f \) is an isomorphism;
(iii) \( \xi^*_{ih} : h o f^* \rightarrow f^* \) is an isomorphism;
(iv) \( \xi^*_{ih} : h o (f^* o f) \rightarrow f^* o f \) is an isomorphism.

Moreover, \( i_h \) is monomorphism in \([A, A]\) provided any (and hence all) of the above conditions hold.

Proof. Since (i) is equivalent to (ii) and (iii) is equivalent to (iv) by [12] Remark 4.2 and its dual, we have only to show that (i) and (iii) are equivalent.

Note first that composing the adjunction \((\eta_f, \varepsilon_f : f \dashv f^*)\) with \((\eta_h, \varepsilon_h : h \dashv h^*)\) yields an adjunction \((\eta, \varepsilon : f o h \dashv h^* o f^*),\) where \( \eta \) and \( \varepsilon \) are the composites

\[ 1_A \overset{\eta_h}{\rightarrow} h^* o h \overset{h^* o \eta_f}{\rightarrow} h^* o f^* o f o h \]

and

\[ f o h o h^* o f^* \overset{f o \eta_f}{\rightarrow} f o f^* \overset{\varepsilon_f}{\rightarrow} 1_B, \]

respectively.

Consider now the composite

\[ \overline{\xi}_{ih} : f^* \xrightarrow{\eta_f o f} h^* o f^* o f o h \overset{\eta_f o f^* o \xi_{ih} o f^*}{\rightarrow} h^* o f^* o f o f^* \overset{h^* o f^* o \varepsilon_f}{\rightarrow} h^* o f^*, \]

which is the mate of \( \xi_{ih} \) under the adjunctions \((f^* o h \dashv h^* o f^*)\) and \((f \dashv f^*).\) A straightforward calculation, using the expression for \( \eta \) and \( \varepsilon, \) shows that \( \overline{\xi}_{ih} \) is the composite

\[ f^* \overset{\eta_f o f^*}{\rightarrow} h^* o f^* \overset{h^* o \xi_{ih} o f^*}{\rightarrow} h^* o f^* o f o f^* \overset{h^* o f^* o \varepsilon_f}{\rightarrow} h^* o f^*, \]

and therefore

\[ \overline{\xi}_{ih} = (h^* o \xi^*_{ih}) \cdot (\eta_h o f^*), \]

implying – since both \( h^* \) and \( \eta_h \) are invertible 1-cells – that \( \xi_{ih} \) is an isomorphism if \( \xi^*_{ih} \) is. In the light of Lemma [13], one now concludes that (i) and (iii) are equivalent.

Finally, each of the conditions (i)-(iv) implies that \( \xi_{ih} \) is an isomorphism, and then \( i_h \) is a monomorphism by Proposition \( 2.1.\) This completes the proof.

\[ \square \]

Proposition 2.4. \( \mathcal{J}^A_{f, f} = \mathcal{J}^A_{f, r}. \)

Proof. By symmetry, it suffices to prove the inclusion \( \mathcal{J}^A_{f, f} \subseteq \mathcal{J}^A_{f, r}. \) To this end consider an arbitrary element \([h, i_h] \in \mathcal{J}^A_{f, f}. \) Since \( h \) is an invertible 1-cell, we need only show that \([h, i_h] \in \mathcal{J}^A_{f, f} \) and \( \mathcal{J}_{f, r} \) is an isomorphism, and then \( i_h : f o h \rightarrow h \) is also an isomorphism by Remark [12] Remark 4.2. Applying now Proposition \( 2.3 \) gives that both \( \xi_{ih} : h o f^* \rightarrow f^* \) and \( \xi^*_{ih} : h o (f^* o f) \rightarrow f^* o f \) are isomorphisms. Thus, \([h, i_h] \in \mathcal{J}^A_{f, f} \) and hence \([h, i_h] \in \mathcal{J}^A_{f, r}.\)

\[ \square \]

Definition 2.5. We write \( \mathcal{J}^A_f \) to denote either \( \mathcal{J}^A_{f, f} \) or \( \mathcal{J}^A_{f, r}. \)
3. Exact sequences associated with adjunctions in bicategories. Applications.

Fix an adjunction \( \eta_f, \varepsilon_f : f \dashv f^* : B \to A \) in \( \mathcal{B} \). In this section we suppose, with the exception of Subsection 3.1, that \( \mathcal{B} \) is a bicategory such that each hom-category admits finite limits and images, and that the 2-cell \( \eta_f : 1_A \to f^* \circ f \) is a monomorphism in \( [A, A] \). In this case, \( \text{Sub}_{[A, A]}(f^* \circ f) \) has a monoid structure formed as in [12 Proposition 3.2], and throughout this paper, when considering \( \text{Sub}_{[A, A]}(f^* \circ f) \) as a monoid, we always mean this monoid structure.

3.1. Automorphisms and invertible subobjects. One can easily verify that the assignment taking a 2-cell \( s : 1_A \to 1_A \) to the composite

\[
f^* \xrightarrow{f^* \cdot \varepsilon_f} f^* \circ f \xrightarrow{s \circ f} f^* \xrightarrow{\eta_f} f^*,
\]

yields a monoid morphism

\[
\varpi : [A, A](1_A, 1_A) \to [B, A](f^*, f^*),
\]

which gives, by restriction, a homomorphism of groups

\[
\varpi_0 : \text{Aut}_{[A, A]}(1_A) \to \text{Aut}_{[B, A]}(f^*).
\]

between the groups of automorphisms of the objects \( 1_A \) and \( f^* \), respectively.

Proposition 3.1. The map \( \varpi_0 \) is a monomorphism of groups.

Proof. If \( s \in \text{Aut}_{[A, A]}(1_A) \) is such that \( \varpi_0(s) = 1_{f^*} \), then \( 1_{f^*} = f^* \cdot (s \circ f) \cdot f^* \cdot \varepsilon_f \cdot \eta_f \) and hence \( \lambda_f = 1_{f^*} \cdot (s \circ f) \cdot f^* \cdot \varepsilon_f \cdot \eta_f \). But since \( \lambda_f = f^* \cdot (1_A \circ f^*) \) and since \( f^* \) is invertible, it follows that \( 1_{f^*} \circ f^* = s \circ f^* \) and hence \( 1_{f^*} \circ f^* = s \circ f \). Direct calculation then shows that \( \eta_f \cdot 1_{f^*} = \eta_f \cdot s \). Now, since \( \eta_f \) is assumed to be monomorphic, the map

\[
[A, A](1_A, \eta_f) : [A, A](1_A, 1_A) \to [A, A][1_A, f^* \circ f]
\]

is injective, implying that \( 1_{f^*} = s \). Thus, \( \varpi_0 \) is a monomorphism of groups. □

For any \( \lambda \in \text{Aut}_{[B, A]}(f^*) \), form the pullback

\[
\begin{array}{ccc}
1_A & \xrightarrow{\eta_f} & f^* \\
\downarrow{f_\lambda} & & \downarrow{\lambda f} \\
L_A & \xrightarrow{\varpi_0} & f^* \circ f
\end{array}
\]

(5)

Since, by hypothesis, \( \eta_f \) is a monomorphism in \( [A, A] \), so too is \( i_\lambda \), and thus \( (f_\lambda, i_\lambda) \) represents an element of \( \text{Sub}_{[A, A]}(f^* \circ f) \), implying – since pullbacks are unique up to isomorphism – that the assignment \( \lambda \mapsto [(f_\lambda, i_\lambda)] \) yields a map \( D_f : \text{Aut}_{[B, A]}(f^*) \to \text{Sub}_{[A, A]}(f^* \circ f) \).

Consider now the diagram

\[
\begin{array}{cccc}
(f^* \circ f) \circ (f^* \circ f) & \xrightarrow{\alpha_f^{-1}} & (f^* \circ f) \circ (f^* \circ f) & \xrightarrow{\alpha_f} & (f^* \circ f) \circ (f^* \circ f) \\
\downarrow{(\lambda f) \circ (f^* \circ f)} & & \downarrow{(\lambda f) \circ (f^* \circ f)} & & \downarrow{(\lambda f) \circ (f^* \circ f)} \\
(f^* \circ f) \circ (f^* \circ f) & \xrightarrow{\alpha_f^{-1}} & (f^* \circ f) \circ (f^* \circ f) & \xrightarrow{\alpha_f} & (f^* \circ f) \circ (f^* \circ f)
\end{array}
\]

(1)

(2)

(3)

(4)

\[
\begin{array}{ccc}
\end{array}
\]

in which rectangles (1) and (2) commute by naturality of \( \alpha \), rectangle (3) commutes by naturality of composition, while rectangle (4) commutes by naturality of \( r \). Thus the outer rectangle of the diagram is also commutative, and using now that

\[
m_f = (r f^*) \cdot ((f^* \circ \varepsilon_f) \circ f^*) \cdot (\alpha_f \cdot f \cdot f) \cdot (\lambda f)^{-1},
\]

indeed, we need this assumption only for the hom-category \([A, A]\).
we get
\[(\lambda \circ f) \cdot m_f = m_f \cdot ((\lambda \circ f) \circ (f^* \circ f))\] (6)

It then follows from (5) that
\[(\lambda \circ f) \cdot m_f \cdot (i_\lambda \circ (f^* \circ f)) = m_f \cdot ((\lambda \circ f) \circ (f^* \circ f)) \cdot (i_\lambda \circ (f^* \circ f))\]
\[= m_f \cdot (\eta_f \circ (f^* \circ f)) \cdot (p_\lambda \circ (f^* \circ f))\]
\[= l_{f^* \circ f} \cdot (p_\lambda \circ (f^* \circ f))\] (7)

Since the morphisms \(\lambda, l_{f^* \circ f}\) and \(p_\lambda\) all are isomorphisms, one concludes that the composite 
\(m_f \cdot (i_\lambda \circ (f^* \circ f))\) is also an isomorphism and hence we have:

**Proposition 3.2.** Under the hypotheses above, \(D_f(\lambda) \in \mathcal{J}^r_{[A,A]}(S_f)\) for all \(\lambda \in \textbf{Aut}_{[B,A]}(f^*)\).

We shall need the following easy lemma:

**Lemma 3.3.** In an arbitrary category, a commutative diagram \(gf = yx\) with \(g\) isomorphism is a pullback iff \(x\) is an isomorphism.

**Proposition 3.4.** The map \(D_f : \textbf{Aut}_{[B,A]}(f^*) \to \textbf{Sub}_{[A,A]}(f^* \circ f)\) is a homomorphism of monoids.

**Proof.** Quite obviously, the diagram

\[
\begin{array}{c}
1_A \xrightarrow{\eta_f} f^* \circ f \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
1_A \xrightarrow{f^* \circ f} \quad f^* \circ f \\
\end{array}
\]

is a pullback, showing that \(D_f(1_f^*) = [1_A, \eta_f] = 1\).

Next, for any two elements \(\lambda, \lambda' \in \textbf{Aut}_{[B,A]}(f^*)\), consider the diagram

in which Diagrams (I) and (II) commute by (5), Diagram (III) commutes by (4), Diagrams (IV), (V) commute by naturality of composition, Diagram (VII) commutes by naturality of \(l\), and Diagram (VI) commutes since \(\eta_f : 1_A \to f^* \circ f\) is the unit for the multiplication \(m_f\). Thus the
outer diagram, which by naturality of $l$ can be rewritten as

\[
\begin{array}{c}
\xymatrix{ f_x \circ f_y \ar[r]^{i_x \ast i_y} \ar[d]_{p_x \circ p_y} & (f_x \ast f_y) \circ (f_x \ast f_y) \ar[r]^{m_f} & f_x \ast f_y \\
I_A \ast I_A \ar[r]_{\eta_f} & I_A \ast I_A \ar[r]^{(\lambda \lambda') \ast f} & f_x \ast f_y,}
\end{array}
\]

commutes, and since all the 2-cells $l_{i_x}$, $\lambda$, $\lambda'$, $p_x$ and $p_y$ (and hence also $l_{i_x} \cdot (p_x \circ p_y)$) and $(\lambda \lambda') \ast f$ are isomorphisms, it follows from Lemma 3.3 that the diagram is a pullback. Then, in particular, the composite $m_f \cdot (i_x \ast i_y)$ is a monomorphism, and thus

\[D_f(\lambda \lambda') = [(f_x \circ f_y, m_f \cdot (i_x \circ i_y))].\]

Moreover,

\[[(f_x, i_x)] \cdot [(f_y, i_y)] = [(f_x \circ f_y, m_f \cdot (i_x \circ i_y))]
\]

in $\text{Sub}_{[A, A]}(f \ast f)$ by [12, Remark 3.3]. Thus

\[D_f(\lambda \lambda') = [(f_x, i_x)] \cdot [(f_y, i_y)] = D_f(\lambda) \cdot D_f(\lambda'),\]

and hence $D_f$ is a homomorphism of monoids.

\[\square\]

Remark 3.5. Putting $\lambda' = \lambda^{-1}$ in the proof of Proposition 3.4 gives that for any $\lambda \in \text{Aut}_{[B, A]}(f^*)$, the 1-cell $f_\lambda$ defined in (5) is invertible.

Proposition 3.6. The monoid structure on $\text{Sub}_{[A, A]}(f \ast f)$ restricts to a group structure on $\mathcal{I}^A_f$.

Moreover, $D_f$ induces a group homomorphism

\[\overline{D}_f : \text{Aut}_{[B, A]}(f^*) \to \mathcal{I}^A_f.\]

Proof. The 2-cell $\eta_f$ is monomorphic by assumption. Since, quite obviously, $1_A$ is invertible, and $[(1_A, \eta_f)] \in \mathcal{I}^A_{[A, A]}(S_f)$, it follows that $[(1_A, \eta_f)] \in \mathcal{I}^A_f$.

Next, if $[(h, i_h)], [(g, i_g)] \in \mathcal{I}^A_f$, then clearly $h \circ g$ is invertible. Observe that $\xi_{i_h} : S_f \circ g \to S_f$ is an isomorphism as $[(g, i_g)] \in \mathcal{I}^A_f \subseteq \mathcal{I}^A_{[A, A]}(S_f)$. Since $i_h$ is a monomorphism, we get from Remark 3.3 the 2-cell $i_h \circ g : h \circ g \to S_f$ is a monomorphism. On the other hand, $m_f \cdot (i_h \circ i_g) = \xi_{i_h} \cdot (i_h \circ i_g)$, and it follows that the 2-cell $i_h \circ g := m_f \cdot (i_h \circ i_g)$ is monomorphic. Thus, by [12, Remark 3.3], $[(h, i_h)] \cdot [(g, i_g)] = [(h \circ g, i_{h \circ g})]$ in $\text{Sub}_{[A, A]}(f \ast f)$. Moreover, $[(h \circ g, i_{h \circ g})]$ lies in $\mathcal{I}^A_{[A, A]}(S_f)$ by exactly the same argument as in the proof of [12, Proposition 3.5]. Thus $[(h \circ g, i_{h \circ g})] \in \mathcal{I}^A_f$, and hence $\mathcal{I}^A_f$ inherits the structure of a monoid from $\text{Sub}_{[A, A]}(f \ast f)$. In view of Proposition 3.1, it is easy to see that if $[(h, i_h)] \in \mathcal{I}^A_f$, then its two-sided inverse is $[(h^*, i_h^*)]$, where $h^*$ the pseudo-inverse of $h$. Therefore, $\mathcal{I}^A_f$ is in fact a group.

In light of Proposition 2.4 and Remark 3.5 it follows from Proposition 2.3 that $D_f(\lambda) \in \mathcal{I}^A_f$, for any $\lambda \in \text{Aut}_{[B, A]}(f^*)$. Proposition 3.4 guarantees then that $D_f(\lambda)$ induces a homomorphism of groups $\overline{D}_f : \text{Aut}_{[B, A]}(f^*) \to \mathcal{I}^A_f$.

\[\square\]

Theorem 3.7. The following sequence of groups

\[1 \to \text{Aut}_{[A, A]}(1_A) \xrightarrow{\overline{\varpi}_0} \text{Aut}_{[B, A]}(f^*) \xrightarrow{\overline{D}_f} \mathcal{I}^A_f\]

is exact.

Proof. To say that the sequence is exact at $\text{Aut}_{[A, A]}(1_A)$ is to say that $\overline{\varpi}_0$ is injective, which is indeed the case by Proposition 3.1.
To prove exactness at $\text{Aut}_{[B, A]}(f^*)$, we have to show that $\text{Ker}(D_f) = \text{Im}(\varpi_0)$. For any $s \in \text{Aut}_{[A, A]}(1_A)$, the diagram

\[
\begin{array}{ccc}
\eta_f & \xrightarrow{\eta_f} & f^* \circ f \\
\downarrow (l_\alpha)^{-1} & & \downarrow (l_{f^*})^{-1} \\
1_A \circ 1_A & \xrightarrow{(1_A \circ f^*) \circ f} & (1_A \circ f^*) \circ f \\
\downarrow s \circ 1_A & & \downarrow (s \circ f^*) \circ f \\
1_A \circ 1_A & \xrightarrow{(1_A \circ f^*) \circ f} & (1_A \circ f^*) \circ f \\
\downarrow l_\alpha & & \downarrow l_{f^*} \\
1_A & \xrightarrow{\eta_f} & f^* \circ f
\end{array}
\]  

(8)

is commutative, as can be seen easily using the naturality of $l$ and of $\alpha$ and the fact that

\[
(1_A \circ u) \circ v \xrightarrow{\alpha_{1_A \circ u \circ v}} 1_{A \circ (u \circ v)} \xrightarrow{l_{u \circ v}} l_u \circ v \xrightarrow{\eta_{u \circ v}} 1_u \circ v
\]

is a commutative diagram for all 1-cells $u, v : A \to A$ (e.g. [15 Proposition 1.1]). Since $s = l_\alpha \cdot (s \circ 1_A) \cdot (l_\alpha)^{-1}$ by naturality of $l$ and $\varpi_0(s) = l_{f^*} \circ (s \circ f^*) \circ (l_{f^*})^{-1}$, it follows that (8) may be rewritten in the form

\[
\begin{array}{ccc}
\eta_f & \xrightarrow{\eta_f} & f^* \circ f \\
\downarrow s & & \downarrow \varpi_0(s) \circ f \\
1_A & \xrightarrow{\eta_f} & f^* \circ f
\end{array}
\]

Since both $s$ and $\varpi_0(s)$ are invertible 2-cells, it follows from Lemma [3,3] that the diagram above is a pullback, implying that $D_f(\varpi_0(s)) = [\eta_f] = 1$ in $\mathcal{F}^A$. Since $s \in \text{Aut}_{[A, A]}(1_A)$ was arbitrary, $\text{Im}(\varpi_0) \subseteq \text{Ker}(D_f)$.

Next, if $\lambda \in \text{Aut}_{[B, A]}(f^*)$ is such that $D_f(\lambda) = 1$, then there is an automorphism $s : 1_A \to 1_A$ such that the diagram

\[
\begin{array}{ccc}
1_A & \xrightarrow{\eta_f} & f^* \circ f \\
\downarrow s & & \downarrow \lambda \circ f \\
1_A & \xrightarrow{\eta_f} & f^* \circ f
\end{array}
\]

is a pullback, implying that in the diagram

\[
\begin{array}{ccccc}
f \xrightarrow{r_f^{-1}} & f \circ 1_A & \xrightarrow{f \circ \eta_f} & f \circ (f^* \circ f) & \xrightarrow{f \circ (\lambda \circ f)} & f \circ (f^* \circ f) \xrightarrow{\alpha_{f, f^* \circ f}^{-1}} (f \circ f^*) \circ f \\
\downarrow f \circ s & & \downarrow f \circ (f \circ \eta_f) & & \downarrow f \circ (f \circ \eta_f) & & \downarrow (f \circ f^*) \circ f \\
\end{array}
\]

the triangle commutes, while the trapezoid commutes by [2]. It then follows that the mate of $\lambda$ under the adjunction $f \dashv f^*$, which is the composite

\[l_f \cdot (\varepsilon_f \circ f) \cdot \alpha_{f, f^* \circ f}^{-1} \cdot (f \circ \lambda \circ f) \cdot (f \circ f^*) \circ f \cdot r_f^{-1},\]
is in fact equal to the composite \( r_f \cdot (f \circ s) \cdot r_f^{-1} \). Direct inspection using the fact that the diagram
\[
\begin{array}{ccc}
(f \circ 1_A) \circ f^* & \xrightarrow{\alpha_{f, 1_A} \cdot f^*} & f \circ (1_A \circ f^*) \\
\downarrow r_f \circ f^* & & \downarrow f \circ f^* \\
\downarrow f \circ f^* & & \downarrow f \circ f^*
\end{array}
\]
commutes, shows that the mate of the last composite under the adjunction \( f \dashv f^* \) is just \( \omega_0(s) = l_f \cdot (s \circ f^*) \cdot (l_f)^{-1} \). This proves that \( \omega_0(s) = \lambda \). Thus \( \text{Ker}(\Lambda_f) \subseteq \text{Im}(\omega_0) \), and hence \( \text{Ker}(\Lambda_f) = \text{Im}(\omega_0) \). \( \square \)

3.2. An exact sequence involving the Picard group. For any object \( A \) of \( \mathcal{B} \), define the Picard Group of \( A \), denoted \( \text{Pic}(A) \), to be the collection of isomorphism-classes \([h] \) of invertible 1-cells \( h : A \to A \) with product and inverses defined by
\[
[h] \cdot [g] = [h \circ g] \quad \text{and} \quad [h]^{-1} = [h^*],
\]
where \( h^* \) is a pseudo-inverse of \( h \). As easily seen, \( \text{Pic}(A) \) is a well-defined group with identity element \([1_A] \).

**Proposition 3.8.** The assignment that takes \([h, i_h] \in \mathcal{J}_f^A \) to \([h] \) defines a group homomorphism
\[
\Omega_f : \mathcal{J}_f^A \to \text{Pic}(A).
\]

**Proof.** For any \([((h, i_h), [h]) \in \mathcal{J}_f^A \), \([h] \in \text{Pic}(A) \) by the very definition of \( \mathcal{J}_f^A \). The product \([([h, i_h]), ([h', i_{h'}], [h', i_{h'}}) \in \mathcal{J}_f^A \) is the pair \(([h \circ h'], i_{h \circ h'}) \), where \( i_{h \circ h'} \) is the composite
\[
i_{h \circ h'} : h \circ h' \xrightarrow{h \circ i_{h'}} \varphi \circ (f^* \circ f) \xrightarrow{m_f} f^* \circ f
\]
(see the proof of Proposition 5.6). Therefore, \( \Omega_f \) preserves the product, and hence is a group homomorphism. \( \square \)

**Theorem 3.9.** The sequence of groups
\[
1 \to \text{Aut}_{[B,A]}(1_A) \xrightarrow{\omega_f} \text{Aut}_{[B,A]}(f^*) \xrightarrow{\Lambda_f} \mathcal{J}_f^A \xrightarrow{\Omega_f} \text{Pic}(A)
\]
is exact.

**Proof.** By Theorem 3.4 it suffices to show that the sequence is exact at \( \mathcal{J}_f^A \). So, suppose \([([h, i_h]), [h, i_h]) \in \mathcal{J}_f^A \) is such that \( \Omega_f([([h, i_h]), [h, i_h])] = [h] = [1_A] \). Then there exists an isomorphism \( \tau : h \to 1_A \) in \([A, A]\). Define \( \lambda \) to be the composite
\[
f^* \xrightarrow{\xi_{i_h}^{-1} \cdot f^*} h \circ f^* \xrightarrow{\tau \circ f^*} 1_A \circ f^* \xrightarrow{l_f \circ f^*} f^*.
\]
It is clear that \( \lambda \in \text{Aut}_{[B,A]}(f^*) \). We claim that \( \Lambda_f(\lambda) = [([h, i_h]) \). Indeed, we know that the diagram
\[
\begin{array}{ccc}
(h \circ f^*) \circ f & \xrightarrow{\alpha_{l_f, f^*} \cdot f^*} & h \circ (f^* \circ f) \\
\downarrow \circ f \circ f^* & & \downarrow \circ f \circ f^* \\
(1_A \circ f^*) \circ f & \xrightarrow{\alpha_{l_f, f^*} \cdot f^*} & 1_A \circ (f^* \circ f)
\end{array}
\]
commutes by naturality of \( \alpha \), and \( l_f \circ f^* \cdot \alpha_{l_f, f^*} \cdot f = l_f \circ f \) by one of the two coherence axioms (see [15 Proposition 1.1]). Since \( \xi_{i_h}^{-1} = \alpha_{l_f, f^*} \cdot ((\xi_{i_h}^{-1} \circ f)(\xi_{i_h}^{-1}) \circ f) \) by the dual of [12 Remark 4.2], the 2-cell \( \lambda \circ f \) can be rewritten as follows
\[
f^* \circ f \xrightarrow{\xi_{i_h}^{-1} \cdot f^*} h \circ (f^* \circ f) \xrightarrow{\tau \circ (f^* \circ f)} 1_A \circ (f^* \circ f) \xrightarrow{l_f \circ f^*} f^* \circ f.
\]
In the following diagram
\[
\begin{array}{cccccc}
h & \xrightarrow{(r_h)^{-1}} & h \circ 1_A & \xrightarrow{\tau \circ 1_A} & 1_A \circ 1_A & \xrightarrow{\lambda_h = r_h} & 1_A, \\
\downarrow{i_h} & & \downarrow{h \circ \eta_f} & \downarrow{\lambda \circ 1_A} & \downarrow{\lambda_h \circ \eta_f} & \downarrow{\eta_f} & \\
f^* \circ f & \xrightarrow{(\xi_{ih})^{-1}} & h \circ (f^* \circ f) & \xrightarrow{\tau \circ (f^* \circ f)} & 1_A \circ (f^* \circ f) & \xrightarrow{f^* \circ f} & f^* \circ f
\end{array}
\]
Square (2) commutes by naturality of composition, while Square (3) commutes by naturality of \( \lambda \).

We claim that Square (1) is also commutative. Indeed, using that \( \text{Aut}(S) \) where \( S \) injective, then we can apply Theorem 3.9 to the adjunction
\[
\exists \text{ a map } \pi : \text{Pic} \to \text{Inv}
\]
then clearly the 2-cell
\[
\begin{array}{c}
\lambda \circ f \\
\downarrow{\eta_f} \\
f^* \circ f
\end{array}
\]
commutes, and since the composite \( r_h \cdot (\tau \circ 1_A) \cdot (r_h)^{-1} \) is an isomorphism, it follows from Lemma 3.3 that the diagram is a pullback. Hence \( \overline{\Omega}_f(\lambda) = \{[h, i_h]\} \), and thus \( \text{Ker}(\Omega_f) \subseteq \text{Im}(\overline{\Omega}_f) \).

Now, if \( i_h : h \to f^* \circ f \) is such that there are an automorphism \( \lambda \in \text{Aut}_{[A, B]}(f^*) \) and a pullback
\[
\begin{array}{c}
h \\
\downarrow{i_h} \\
f^* \circ f
\end{array}
\]
then clearly the 2-cell \( p_h : h \to 1_A \) is an isomorphism and thus \( \Omega_f([h, i_h]) = [h] = [1_A] \). Thus \( \text{Im}(\overline{\Omega}_f) \subseteq \text{Ker}(\Omega_f) \) and hence \( \text{Ker}(\Omega_f) = \text{Im}(\overline{\Omega}_f) \). This completes the proof.

**Example 3.10.** Let \( \varphi : R \to S \) be a homomorphism of firm rings as in Example 13. If \( \varphi \) is injective, then we can apply Theorem 5.9 to the adjunction \( \exists \text{ S}_R \to R \text{ S}_S \) in \( \text{Firm} \), and we get the exact sequence of groups
\[
1 \longrightarrow \text{Aut}(R \text{ RR}) \longrightarrow \text{Aut}(S \text{ SR}) \longrightarrow \text{Inv}_R(S) \longrightarrow \text{Pic}(R),
\] (10)
where \( \text{Aut}(R \text{ RR}) \) (resp. \( \text{Aut}(S \text{ SR}) \)) denote the group of \((R, R)\)-bimodule (resp. \((S, R)\)-bimodule) automorphisms of \( R \) (resp. \( S \)). \( \text{Inv}_R(S) \) is the group of invertible \( R \)-submodules of \( S \), and \( \text{Pic}(R) \) is the Picard group of the ring \( R \). The exact sequence (10) was obtained, as a generalization of the unital case [21], in [10] Proposition 1.4 for any extension of rings with the same set of local units. Every such an extension is clearly an injective homomorphism of firm rings.

Given an arbitrary category \( C \), we write \( \pi_0(C) \) for the collection of the isomorphism classes of objects of \( C \). For any \( C \in C, |C| \) denotes the class of \( C \). Clearly, for any functor \( S : C \to D \), the assignment \( |C| \to |S(C)| \) yields a map \( \pi_0(S) : \pi_0(C) \to \pi_0(D) \).
Quite obviously, the assignment \([h] \mapsto [f \circ h]\) yields a map \(\text{Pic}(A) \xrightarrow{[f \circ -]} \pi_0([A, B])\), where \(\pi_0([A, B])\) denotes the pointed set of the isomorphism classes \([g]\) of 2-cells \(g : A \to B\) with a distinguished class \([f]\). Since \(f \circ A \simeq f\), \([f \circ -]\) is morphism of pointed sets.

**Theorem 3.11.** The following sequence of pointed sets

\[
\mathcal{F}^f_f \xrightarrow{\Omega_f} \text{Pic}(A) \xrightarrow{[f \circ -]} \pi_0([A, B])
\]

is exact.

**Proof.** Since \(\mathcal{F}^f_f \subseteq \mathcal{F}^{[A, A]}([S_f])\), it is clear that \(([f \circ -] : \Omega_f)([(h, i_h)]) = [f]\) for all \([(h, i_h)] \in \mathcal{F}^f_f\). So it remains to show that if \([g] \in \text{Pic}(A)\) is such that

\([f \circ -]([g]) = [f]\),

then there exists \([(h, i_h)] \in \mathcal{F}^f_f\) with \([g] = \Omega_f([(h, i_h)]) = [h]\). Since \([f \circ g] = [f]\), there is an isomorphism \(\sigma : f \circ g \to f\) in \([A, B]\). It then follows from Proposition 2.11 that \([(g, i_g)] \in \mathcal{F}^f_f\), where \(i_g\) is the composite \(g \xrightarrow{\eta_g \circ q} f \circ f \circ g \xrightarrow{f \circ \alpha} f \circ f\). Then clearly \(\Omega_f([(g, i_g)]) = [g]\). \(\square\)

### 3.3. Comonadicity

Recall from [12, p. 172] that there is a map \(\Gamma_f : \mathcal{F}^{[A, A]}([S_f]) \to \text{End}_{B-cor}(\mathcal{E}_f)\) that takes \([(h, i_h)] \in \mathcal{F}^{[A, A]}([S_f])\) to the composite

\[
(f \circ f) \xrightarrow{\xi_{\mathcal{E}_f}} (f \circ h) \circ f \xrightarrow{\alpha^f_{f, h, f}} f \circ (h \circ f) \xrightarrow{\xi_{\mathcal{E}_f}} f \circ f.
\]

**Proposition 3.12.** Suppose that the functor \([A, f] = f \circ - : [A, A] \to [A, B]\) is comonadic. Then \(\Gamma_f\) restricts to an isomorphism of groups

\[
\Gamma_f : \mathcal{F}^{f_f} \to \text{Aut}_{B-cor}(\mathcal{E}_f).
\]

**Proof.** The functor \([A, f]\) is precomonadic if and only if the unit of the adjunction \([A, f] \dashv [A, f^*]\) is a componentwise monomorphism. So \(\eta_f : 1 \to f^* \circ f\) is right pure in the monoidal category \([A, A]\) (meaning that \(\eta_f o h : 1 o h \to (f^* \circ f) o h\) is monomorphic for all 1-cells \(h : A \to A\)), provided the functor \([A, f]\) is (pre)comonadic. Consequently, according to [12, Proposition 4.4], \(\mathcal{F}^{[A, A]}([S_f])\) inherits the structure of a monoid from \(\text{Sub}_{[A, A]}([f^* \circ f])\). Moreover, the map \(\Gamma_f : \mathcal{F}^{[A, A]}([S_f]) \to \text{End}_{B-cor}(\mathcal{E}_f)\) is an isomorphism of monoids by [12, Theorem 4.9]. If \([(h, i_h)] \in \mathcal{F}^f_f\), then \(\xi_{\mathcal{E}_f}\) is an isomorphism and hence is so \(\Gamma_f([(h, i_h)])\). Thus, \(\Gamma_f\) restricts to a monomorphism \(\Gamma_f : \mathcal{F}^{f_f} \to \text{Aut}_{B-cor}(\mathcal{E}_f)\) of groups. To show that \(\Gamma_f\) is surjective, note first that if \([(h, i_h)] \in \mathcal{F}^{[A, A]}([S_f])\) is such that \(\Gamma_f([(h, i_h)]) \in \text{Aut}_{B-cor}(\mathcal{E}_f)\), then \([(h, i_h)] \in \mathcal{F}^{f_f}([S_f])\). Indeed, if the composite

\[
\Gamma_f([(h, i_h)]) : f \circ f \xrightarrow{\xi_{\mathcal{E}_f}} (f \circ h) \circ f \xrightarrow{\alpha^f_{f, h, f}} f \circ (h \circ f) \xrightarrow{\xi_{\mathcal{E}_f}} f \circ f
\]

is an isomorphism, then \(f \circ \xi_{\mathcal{E}_f}\) is also an isomorphism. But by hypothesis the functor \([A, f] = f \circ -\) is comonadic, and in particular conservative. Hence \(\xi_{\mathcal{E}_f}\) is an isomorphism too. Thus \([(h, i_h)] \in \mathcal{F}^{f_f}([S_f])\). Consider now any \(\alpha \in \text{Aut}_{B-cor}(\mathcal{E}_f)\). Then, since \(\Gamma_f^{-1}\) is a morphism of monoids, one has the following equalities in \(\mathcal{F}^{[A, A]}([S_f])\):

\[
\Gamma_f^{-1}(\alpha) \cdot \Gamma_f^{-1}(\alpha^{-1}) = \Gamma_f^{-1}(1_{\mathcal{E}_f}) = [(1_A, \eta_f)].
\]

Similarly, \(\Gamma_f^{-1}(\alpha^{-1})\cdot \Gamma_f^{-1}(\alpha) = [(1_A, \eta_f)]\). If now \(\Gamma_f^{-1}(\alpha) = [(h, i_h)]\) and \(\Gamma_f^{-1}(\alpha^{-1}) = [(h', i_{h'})]\), then since by [12, Proposition 4.4], the product \([(h, i_h)] : [(h', i_{h'})] \in \mathcal{F}^{f_f}([S_f])\) is the pair \([(h \circ h'), (i_{h \circ h'})]\), where \(i_{h \circ h'}\) is the composite

\[
i_{h \circ h'} : h \circ h' \xrightarrow{i_h \circ i_{h'}} (f^* \circ f) o (f^* \circ f) \xrightarrow{m_{f^*}} f^* \circ f,
\]

it follows that \(h \circ h' \simeq 1_A\) and \(h' \circ h \simeq 1_A\). Hence \([h] \in \text{Pic}(A)\). Since \(\Gamma_f(\Gamma_f^{-1}(\alpha)) = \alpha\) is an isomorphism, \(\Gamma_f^{-1}(\alpha) \in \mathcal{F}^{f_f}([S_f])\), as we have shown above. Thus, \(\Gamma_f^{-1}(\alpha) \in \mathcal{F}^f_f\), and hence \(\Gamma_f\) is surjective. This completes the proof. \(\square\)
Remark 3.13. We have proved in passing that, when the functor
\[ [A, f] = f \circ - : [A, A] \to [A, B] \]
is comonadic, then
\[ \mathcal{F}_f^A = \mathcal{F}_{[A, A]}(\mathcal{S}_f) \cap \mathcal{F}_{[A, A]}(\mathcal{S}_f). \]

As a corollary, we get:

**Proposition 3.14.** Whenever the functor
\[ [A, f] = f \circ - : [A, A] \to [A, B] \]
is comonadic, we have an equality of groups
\[ \mathcal{F}_f^A = (\mathcal{F}_{[A, A]}(\mathcal{S}_f))^\times, \]
where \((-)^\times\) is the functor taking a monoid to its group of invertible elements.

**Remark 3.15.** In [12] Section 5 some sufficient conditions for the comonadicity of the functor
\[ [A, f] = f \circ - : [A, A] \to [A, B] \]
are investigated. Concretely, if \([A, f]\) preserves equalizers and \(\eta_f\)
is right regular \(A\)-pure (see [12] Definition 5.1), then \([A, f]\) is comonadic. This generalizes the “faithfully flat” classical situation. The functor \([A, f]\) also becomes comonadic if \(f\) is a separable 1–cell (that is, if \(\eta_f\) is a split monomorphism in the category \([A, A]\)) (see [12] Proposition 5.5).

3.4. **Duality.** Let \(\mathcal{B}\) be a bicategory whose hom-categories admit finite colimits and coimages and let \(\eta_f, \varepsilon_f : f \rightarrow f^* : B \to A\) be an adjunction in \(\mathcal{B}\) such that \(\varepsilon_f : f \circ f^* \to 1_B\) is epimorphic in \([B, B]\). Let \(\mathcal{C}_f\) be the corresponding \(B\)-co-ring. Write \(Q_f^{B, l}(\text{resp. } Q_f^{B, r})\) for the subset of \(Q_{[B, B]}^{f}(\mathcal{C}_f)(\text{resp. } Q_{[B, B]}^{f}(\mathcal{C}_f))\) determined the elements \([h, i_h]\) with \(h \in \text{Pic}(B)\). Then \(Q_f^{B, l} = Q_f^{B, r}\) and we write \(Q_f^{B}\) to denote either \(Q_f^{B, l}\) or \(Q_f^{B, r}\).

Recall that for any bicategory \(\mathcal{B}\), \(\mathcal{B}^{co}\) is a bicategory obtained from \(\mathcal{B}\) by reversing 2-cells, i.e., \(\mathcal{B}^{co}(A, B) = \mathcal{B}(B, A)^{op}\). Applying now Theorems 3.9 and 3.11 to the bicategory \(\mathcal{B}^{co}\) gives:

**Theorem 3.16.** We have an exact sequence of groups
\[ 1 \to \text{Aut}_{[B, B]}(1_B) \xrightarrow{\sim} \text{Aut}_{[A, B]}(f) \xrightarrow{\mathbf{T}_f} Q_f^B \xrightarrow{\Omega_f} \text{Pic}(B), \]
and an exact sequence of pointed sets
\[ Q_f^B \xrightarrow{\Omega_f} \text{Pic}(B) \xrightarrow{[f \circ -]} \pi_0([B, A]). \]
Here,
\[ \sim (1_B \xrightarrow{\sim} 1_B) = f \xrightarrow{(b)^{-1}} 1_B \circ f \xrightarrow{\eta_f} 1_B \circ f \xrightarrow{\varepsilon_f} f, \]
\[ [f \circ -] = ([P, \pi_P]), \text{ where } f \circ f^* \xrightarrow{\varepsilon_f} 1_B \text{ is a pushout, and} \]
\[ \sigma_P \xrightarrow{\sigma_P f} f \circ f^* \xrightarrow{\pi_P} P. \]

When the functor \([B, f^*]\) is monadic, we have, by [12] Theorem 4.11, that the map
\[ \Gamma_{f^*} : Q_{[B, B]}^f(\mathcal{C}_f) \to \text{End}_{\mathbb{A}^{-\text{ring}}}(\mathcal{S}_f), \]
given by
\[ (p, \pi_P) \mapsto (f^* \circ f \xrightarrow{f^* \circ \pi_P} f^* \circ (\pi_P f) \xrightarrow{\alpha_f^{-1}} f \circ (\pi_P f) \circ f \xrightarrow{\xi_f^{-1}} f \circ f) \]
is an isomorphism of monoids.

Now, the dual version of Proposition 3.12 yields

**Proposition 3.17.** Suppose that the functor \([B, f^*] = f^* \circ - : [B, B] \to [B, A]\) is monadic. Then \(\Gamma_{f^*}\) restricts to an isomorphism of groups
\[ \Gamma_{f^*} : Q_f^B \to \text{Aut}_{\mathbb{A}^{-\text{ring}}}(\mathcal{S}_f). \]
Example 3.18. Let \( \varphi : R \to S \) be a homomorphism of firm rings as in Example 1.6. Now, the adjunction \( s_S R \dashv \rho S_\circ \) in \( \text{Firm} \) leads to the functor \( S \otimes - : \text{Firm}(S, S) \to \text{Firm}(S, R) \) which is monadic according to Beck’s Theorem. Moreover, the isomorphism \( S \otimes S \cong S \) becomes an isomorphism of \( R \)-rings, so that, by Proposition 3.17, we get an isomorphism of groups \( Q^S_S \cong \text{Aut}_{R-\text{ring}}(S) \). We thus get from Theorem 3.16 an exact sequence of groups

\[
1 \longrightarrow \text{Aut}(S S_R) \longrightarrow \text{Aut}(S S) \longrightarrow \text{Aut}_{R-\text{ring}}(S) \longrightarrow \text{Pic}(S),
\]

which generalizes [10, Proposition 2.3].

4. Applications

In this section, we apply the results from Section 3 to an adjoint pair in a bicategory of bimodules. This bicategory is built over an abstract monoidal category subject to some requirements, which, of course, are fulfilled by the category of abelian groups, recovering the usual bicategory of bimodules. With this tool at hand, we treat the case of a homomorphism of commutative algebras. In particular, the group isomorphism involving first Amitsur cohomology and the Picard group is proved.

4.1. The bicategory of bimodules. Suppose that \( \mathcal{V} = (\mathcal{V}, \otimes, I) \) is a monoidal category such that the category \( \mathcal{V} \) admits reflexive coequalizers, and that the latter are preserved, as in the biclosed case, for instance, by the functors \( M \otimes -, -(\otimes M) : \mathcal{V} \to \mathcal{V}, \) for all \( M \in \mathcal{V} \). We will briefly recall basic notions and results about (commutative) monoids and modules over them in monoidal categories; all can be found in [18].

For simplicity of exposition we treat \( \otimes \) as strictly associative and \( I \) as a strict unit, which is justified by Mac Lane’s coherence theorem [18].

Recall that, for a \( \mathcal{V} \)-algebra \( A = (A, m_A, e_A) \), a left \( A \)-module is a pair \( (M, \rho_M) \), where \( M \) is an object of \( \mathcal{V} \) and \( \rho_M : A \otimes M \to M \) is a morphism in \( \mathcal{V} \), called the action (or the \( A \)-action) on \( M \), such that \( \rho_M(m_A \otimes M) = \rho_M(A \otimes M) \) and \( \rho_M(e_A \otimes M) = 1 \).

The left \( A \)-modules are the objects of a category \( \mathcal{V}^A \). A morphism of left \( A \)-modules is a morphism in \( \mathcal{V} \) of the underlying \( \mathcal{V} \)-objects that commutes with the actions of \( A \). In a similar manner, one defines the category \( \mathcal{V}_A \) of right \( A \)-modules.

If \( A \) and \( B \) are algebras in \( \mathcal{V} \), then an \( (A, B) \)-bimodule \( M \) in \( \mathcal{V} \) is an object of \( \mathcal{V} \) with commuting left \( A \)-module and right \( B \)-module structures. The category of \( (A, B) \)-bimodules is denoted \( \mathcal{V}^A_B \).

If \( (M, \rho_M) \in \mathcal{V}_A \) and \( (N, \rho_N) \in \mathcal{V}_B \), then the tensor product of \( (M, \rho_M) \) and \( (N, \rho_N) \) over \( A \) is the object part of the following (reflexive) coequalizer

\[
\xymatrix{ M \otimes A \otimes N \ar[r]^{\rho_M \otimes N} & M \otimes N \ar[r]^{\rho_M \otimes N} & M \otimes A N. }
\]

Moreover, if \( M \in \mathcal{V}^A_B \) and \( N \in \mathcal{V}^C_B \), then \( M \otimes A N \in \mathcal{V}^C_B \). It then follows, in particular, that for a fixed \( \mathcal{V} \)-algebra \( A \), the category \( \mathcal{V}^A_B \) of \( (A, A) \)-bimodules in \( \mathcal{V} \) is (non-symmetric) monoidal category with tensor product of two \( (A, A) \)-bimodules being their tensor product over \( A \) and the unit for this tensor product being the \( (A, A) \)-bimodule \( A \).

This allows us (see, for example, [3]) to construct the bicategory \( \text{Bim}(\mathcal{V}) \) in which:

- Objects are \( \mathcal{V} \)-algebras,
- \( \text{Bim}(\mathcal{V})(A, B) = \mathcal{V}^A_B \);
- 2-cells are bimodule morphisms.

Although the 1-cells in a bicategory are usually denoted using the arrow symbols, we sometimes, as here, find it convenient to write \( A \leadsto B \) instead of \( A \to B \). Thus, \( M : A \leadsto B \) means that \( M \) is a \( (B, A) \)-bimodule.

The horizontal composite \( N M \) of \( M : A \leadsto B \) and \( N : B \leadsto C \) is the \( (C, A) \)-bimodule \( N \otimes B M \), while the vertical composition of two 2-cells is the ordinary composition of bimodule morphisms.

We write \( I \) for the trivial \( \mathcal{V} \)-algebra \( (I, r_I = l_I : I \otimes I \to I, 1_I : I \to I) \). Then, for any \( \mathcal{V} \)-algebra \( A \), the category \( \text{Bim}(\mathcal{V})(I, A) \) is (isomorphic to) the category of left \( A \)-modules \( \mathcal{V} \), while
the category \( \text{Bim}(\mathcal{V})(A, I) \) is (isomorphic to) the category of right \( A \)-modules \( \mathcal{V}_A \). Moreover, if \( M : A \hookrightarrow B \) is an \((B, A)\)-bimodule, then the diagrams

\[
\begin{array}{ccc}
\text{Bim}(\mathcal{V})(I, A) & \xrightarrow{M_{\text{L}}} & \text{Bim}(\mathcal{V})(I, B) \\
\mathcal{V}_A & \xrightarrow{M \otimes A} & \mathcal{V}_B
\end{array}
\]

and

\[
\begin{array}{ccc}
\text{Bim}(\mathcal{V})(B, I) & \xrightarrow{\circ M} & \text{Bim}(\mathcal{V})(A, I) \\
\mathcal{V}_B & \xrightarrow{- \otimes B M} & \mathcal{V}_A
\end{array}
\]

where the vertical morphisms are the isomorphisms, are both commutative.

We henceforth suppose in addition that the category \( \mathcal{V} \) admits, besides reflexive coequalizers, all finite limits and image factorizations. Then, for any two \( \mathcal{V} \)-algebras \( A \) and \( B \), the category \( \mathcal{V}_A \), being the Eilenberg-Moore category for the monad \( A \otimes - \otimes B \): \( \mathcal{V} \to \mathcal{V} \), also admits finite limits (see, for example, [4]) and image factorizations (see [1]). So we are in a position to apply Theorems 3.9 and 3.11 to obtain the following result.

**Theorem 4.1.** Let \( \mathcal{V} = (\mathcal{V}, \otimes, I) \) be a monoidal category with \( \mathcal{V} \) admitting finite limits, image factorizations and reflexive coequalizers. Assume that the latter are preserved by the tensor product, and let \( A, B \) be two \( \mathcal{V} \)-algebras. Then for any adjunction \( \eta, \varepsilon : M : A \Rightarrow B \) in \( \text{Bim}(\mathcal{V}) \) with monomorphic \( \eta_M : A \to M^* \otimes B M = M^* \circ M \), the following sequence of groups

\[
1 \to \text{Aut}_{\mathcal{V}_A}(A) \xrightarrow{i_0} \text{Aut}_{\mathcal{V}_B}(M^*) \xrightarrow{\Omega_M} \Omega_M \xrightarrow{[M \otimes - \rightarrow \text{Pic}(A)]}
\]

is exact. Moreover, the following sequence of pointed sets

\[
\mathcal{V}_A \xrightarrow{M_{\text{L}}} \text{Pic}(A) \xrightarrow{[M \otimes - \rightarrow \pi_0(\mathcal{V}_A)]}
\]

is exact.

**Example 4.2.** Each morphism of \( \mathcal{V} \)-algebras \( \iota : A \to B \) leads to two bimodules \( B_{\iota} : A \hookrightarrow B \) and \( B_{\iota}^* : B \hookrightarrow A \) which are both equal to \( B \) as objects of \( \mathcal{V} \) but with the bimodule structures defined by

\[
(B \otimes B \xrightarrow{m_B} B, B \otimes A \xrightarrow{B \otimes B M} B \otimes B \xrightarrow{m_B} B)
\]

and

\[
(A \otimes B \xrightarrow{\iota \otimes B} B \otimes B \xrightarrow{m_B} B, B \otimes B \xrightarrow{m_B} B).
\]

In fact \( B_{\iota}^* \) is right adjoint for \( B_{\iota} \) in \( \text{Bim}(\mathcal{V}) \) with

\[
A \xrightarrow{\iota} B \simeq B \otimes_B B = B^* \circ B_{\iota}
\]

as unit and

\[
B_{\iota} \circ B_{\iota}^* = B \otimes_B B \xrightarrow{\overline{m_B}} B
\]

as counit. Here \( B \otimes_B B \xrightarrow{\overline{m_B}} B \) is the unique morphism making the triangle

\[
\begin{array}{ccc}
B \otimes B & \xrightarrow{\overline{m_B}} & B \\
\downarrow m_B \downarrow & \downarrow & \downarrow \overline{m_B} \\
B & \xrightarrow{\overline{m_B}} & B
\end{array}
\]

commute.

It therefore follows that every morphism \( \iota : A \to B \) of \( \mathcal{V} \)-algebras gives rise to two functors:
• the forgetful functor \( \iota_* = B^\prime \circ - : \mathcal{B}^\prime \to \mathcal{V} \), where for any left \( \mathcal{B} \)-module \((M, \varrho_M)\), \( \iota_*(M, \varrho_M) \) is a left \( \mathcal{A} \)-module via the action

\[
A \otimes M \overset{\varrho_M}{\longrightarrow} B \otimes M \overset{\iota_*}{\longrightarrow} M;
\]

• the change-of-base functor \( \iota^* = B \circ - : \mathcal{V} \to \mathcal{B}^\prime \), where for any (left) \( \mathcal{A} \)-module \((N, \rho_N)\), \( \iota^*(N, \rho_N) = B \otimes_A N \) and \( B \otimes_A N \) is a (left) \( \mathcal{B} \)-module via the action

\[
B \otimes B \otimes_A N \overset{m_B \otimes_A N}{\longrightarrow} B \otimes A N.
\]

Since \( B^\prime \) is right adjoint to \( B \), in \( \text{Bim}(\mathcal{V}) \), it follows that the forgetful functor \( \iota_* \) is right adjoint to the change-of-base functor \( \iota^* \).

If we specialize Theorem 4.1 to the adjunction \( B \dashv B^\prime \) in \( \text{Bim}(\mathcal{V}) \), we obtain the following exact sequence of groups:

\[
1 \to \mathbf{Aut}_{\mathcal{A}^\mathcal{V}_\pi}(A) \overset{\pi_0}{\longrightarrow} \mathbf{Aut}_{\mathcal{A}^\mathcal{V}_{\pi_B}}(B^\prime) \overset{\mathcal{I}_{B^\prime}}{\longrightarrow} \mathcal{I}_{B} \overset{\mathcal{J}_{B}}{\longrightarrow} \mathbf{Pic}(A)
\]

and the following exact sequence of pointed sets:

\[
\mathcal{J}_{B} \overset{\Omega_M}{\longrightarrow} \mathbf{Pic}(A) \overset{[B, \otimes_A -]}{\longrightarrow} \pi_0(\mathcal{B}^\prime_A)
\]

Let now assume that \( \mathcal{V} \) is a symmetric monoidal category with symmetry \( \tau \). Recall that a \( \mathcal{V} \)-algebra is called commutative if the multiplication map is unchanged when composed with the symmetry.

Given a morphism \( \iota : A \to B \) of commutative \( \mathcal{V} \)-algebras, consider the associated adjunction \( B \dashv B^\prime \) in \( \text{Bim}(\mathcal{V}) \). Write \( S_i \) for \( \mathcal{S}_i \). Then \( S_i = B^\prime \otimes B_i \simeq B \), where the left and right actions of \( A \) on \( B \) are given by the compositions

\[
\rho_l : A \otimes B \overset{\iota^*}{\longrightarrow} B \otimes B \overset{m_B}{\longrightarrow} B \quad \text{and} \quad \rho_r : B \otimes A \overset{\iota^*}{\longrightarrow} B \otimes B \overset{m_B}{\longrightarrow} B
\]

respectively. Since \( \iota \) is a morphism of commutative \( \mathcal{V} \)-algebras, these actions coincide (in the sense that \( \rho_r = \rho_l \cdot \tau_{B,A} \)), and one concludes that \( \text{Sub}_{\mathcal{A}^\mathcal{V}_\pi}(S_i) = \text{Sub}_{\mathcal{A}^\mathcal{V}_\pi}(S_i) = \text{Sub}_{\mathcal{V}}(S_i) \). Therefore \( \mathcal{I}_{A^\mathcal{V}_\pi}(S_i) = \mathcal{I}_{\mathcal{V}}(S_i) \).

Similarly, write \( \xi_i \) for \( \mathcal{S}_i \). Then \( \xi_i \) is the \( \mathcal{B} \)-bimodule \((B \otimes_A B, m_B \otimes_A B, B \otimes_A m_B)\) equipped with the coproduct

\[
B \otimes_A \mathcal{B} \otimes_B B : B \otimes_A B \to (B \otimes_A B) \otimes_B (B \otimes_A B) \simeq B \otimes_A B \otimes_A B
\]

and counit \( m_B : B \otimes_A B \to B \).

The unit \( e \) of \( A \) can be seen as a morphism of commutative \( \mathcal{V} \)-algebras \( I \to A \). If \( e \) is a monomorphism, using that \( \mathcal{V}_A = \mathcal{V}_A \) and that \( \mathcal{V}_I = \mathcal{V} \), we get from (11) and (12) the following exact sequences of groups

\[
1 \to \mathbf{Aut}_{\mathcal{V}}(I) \overset{\pi_0}{\longrightarrow} \mathbf{Aut}_{\mathcal{V}_\pi}(A) \overset{\mathcal{I}_A}{\longrightarrow} \mathcal{I}_{A} \overset{\mathcal{J}_{A}}{\longrightarrow} \mathbf{Pic}(I)
\]

and of pointed sets

\[
\mathcal{J}_{A} \overset{\Omega_A}{\longrightarrow} \mathbf{Pic}(I) \overset{[A, \otimes -]}{\longrightarrow} \pi_0(\mathcal{V}).
\]

It is easy to see that \( \mathcal{J}_{\mathcal{V}}(S_i) = \mathcal{J}_{\mathcal{V}}(A) \) and that \( \mathcal{J}_{\mathcal{V}}(S_i) = \mathcal{J}_{\mathcal{V}}(A) \).

**Proposition 4.3.** Let \( A \) be a commutative \( \mathcal{V} \)-algebra with monomorphic unit \( e : I \to A \). Then \( \mathcal{J}_{\mathcal{V}}(A) \) is a commutative monoid, while \( \mathcal{J}_{A} \) is an abelian group.

**Proof.** Since \( \mathcal{V} \) is symmetric, the monoid structure on \( \mathcal{J}_{\mathcal{V}}(A) \) is easily seen to be commutative. This implies – since by Proposition 3.6 the monoid structure on \( \mathcal{J}_{\mathcal{V}}(A) \) restricts to the group structure on \( \mathcal{J}_{A} \) – that the group \( \mathcal{J}_{A} \) is abelian. \( \square \)

**Lemma 4.4.** For any commutative \( \mathcal{V} \)-algebra \( A \), \( \mathcal{J}_{\mathcal{V}}(A) = \mathcal{J}_{\mathcal{V}}(A) \).


Proof. For any subject $i_\mathcal{V} : J \to A$ of $A$, consider the diagram

$$A \otimes J \xrightarrow{\tau_{A,J}} A \otimes A \xrightarrow{\tau_{A,A}} A \otimes A$$

in which the rectangle commutes by naturality of $\tau$. Since $A$ is commutative, $m_A : \tau_{A,A} = m_A$, and hence $\xi_{ij} \cdot \tau_{A,A,J} = m_A \cdot (\tau_{A,A,J}) = \xi_{ij}$. Thus $\xi_{ij} \cdot \tau_{A,A,J} = \xi_{ij}^\rho$ and hence $\xi_{ij}^\rho$ is an isomorphism (i.e. $[(i_j, J)] \in \mathcal{I}_\mathcal{V}^V(A)$) if $\xi_{ij}$ is so (i.e. $[(i_j, J)] \in \mathcal{I}_\mathcal{V}^V(A)$). Therefore, $\mathcal{I}_\mathcal{V}^V(A) = \mathcal{I}_\mathcal{V}^V(A)$.

**Proposition 4.5.** Let $A$ be a commutative $\mathcal{V}$-algebra such that the functor $A \otimes - : \mathcal{V} \to \mathcal{V}$ is comonadic. Then

$$\text{End}_{A,\mathcal{V}}(e_c) = \text{Aut}_{A,\mathcal{V}}(e_c).$$

**Proof.** Since the functor $A \otimes - : \mathcal{V} \to \mathcal{V}$ is assumed to be comonadic, the map

$$\Gamma_A : \mathcal{I}_\mathcal{V}^V(A) = \mathcal{I}_\mathcal{V}^V(S_c) \to \text{End}_{A,\mathcal{V}}(e_c)$$

is an isomorphism of monoids by [12, Theorem 4.9]. But since

- the monoid isomorphism $\Gamma_A : \mathcal{I}_\mathcal{V}^V(A) \to \text{End}_{A,\mathcal{V}}(e_c)$ restricts to an isomorphism $\overline{\Gamma}_A : \mathcal{I}_\mathcal{V}^V(A) \to \text{End}_{A,\mathcal{V}}(e_c)$ of groups by Proposition 4.12 and
- $\mathcal{I}_\mathcal{V}^V(A) = \mathcal{I}_\mathcal{V}^V(A) \cap \mathcal{I}_\mathcal{V}^V(A) = \mathcal{I}_\mathcal{V}^V(A)$ by Remark 4.12 and by Lemma 4.4.

it follows that $\Gamma_A = \overline{\Gamma}_A$, and hence $\text{Aut}_{A,\mathcal{V}}(S_c) = \text{End}_{A,\mathcal{V}}(e_c)$. □

**Remark 4.6.** Since for any commutative $\mathcal{V}$-algebra $A$, $\mathcal{V}$ is a symmetric monoidal category with tensor product $- \otimes_A -$ and monoidal unit $(A, m_A)$, and since the monoid of endomorphisms of the monoidal unit of any monoidal category is commutative (e.g., (21, 1.3.3.1)), it follows that $\text{End}_{A,\mathcal{V}}(A, m_A)$ is a commutative monoid, and $\text{Aut}_{A,\mathcal{V}}(A)$ is an abelian group. Since $A = A^{op}$ for any commutative $\mathcal{V}$-algebra $A$, it follows that $\mathcal{V}A = A \otimes \mathcal{V}$, and since $A \otimes A$ is again a commutative $\mathcal{V}$-algebra, we get that $\text{End}_{A,\mathcal{V}}(A \otimes A)$ is a commutative monoid. Then the inclusions

$$\text{Aut}_{A,\mathcal{V}}(e_c) \subseteq \text{End}_{A,\mathcal{V}}(e_c) \subseteq \text{End}_{A,\mathcal{V}}(A \otimes A)$$

imply that $\text{End}_{A,\mathcal{V}}(e_c)$ is a commutative monoid, and that $\text{Aut}_{A,\mathcal{V}}(e_c)$ is an abelian group.

4.2. Amitsur cohomology and Picard group. We still assume that $\mathcal{V}$ is symmetric with symmetry $\tau$, and also that $\mathcal{V}$ admits reflexive coequalizers that are preserved by the tensor product, and all finite limits and image factorizations.

For a commutative algebra $A = (A, m_e)\in \mathcal{V}$, write $\text{Pic}^e(A)$ for the subgroup of $\text{Pic}(A)$ consisting of all classes of invertible $(A, A)$-bimodules $(M, \rho_i : A \otimes M \to M, \rho_r : M \otimes A \to M)$ such that $\rho_r = \tau_{A,A,M} \cdot \rho_l$. Then $\text{Pic}^e(A)$ is easily seen to be an abelian group. Moreover, given a morphism $\iota : A \to B$ of commutative $\mathcal{V}$-algebras,

$$\text{Pic}^\iota(e) : \text{Pic}^e(A) \to \text{Pic}^e(B)$$

defined by $\text{Pic}^\iota(e)([P]) = [B \otimes A P]$, is a homomorphism of abelian groups.

It is clear that $\text{Pic}^e(I) = \text{Pic}(I)$. It is also clear that $[A \otimes -]$ factors through $\text{Pic}^e(e) : \text{Pic}^e(I) \to \text{Pic}^e(A)$, i.e. the following diagram

$$\begin{array}{ccc}
\text{Pic}^e(I) \ar{[A \otimes -]} \ar{\pi_0(A)} & & \text{Pic}^e(A) \\
\text{Pic}^e(e) \ar{\downarrow} & & \\
\text{Pic}^e(A) & & \end{array}$$

where the unlabeled morphism is the canonical embedding, is commutative. It then follows – since (14) is an exact sequence of pointed sets – that

$$
\begin{array}{c}
\mathcal{J}_A^I \xrightarrow{\Omega_A} \text{Pic}^c(I) \xrightarrow{\text{Pic}^c(e)} \text{Pic}^c(A)
\end{array}
$$

is an exact sequence of abelian groups, provided \( e : I \to A \) is monomorphic.

**Theorem 4.7.** If \( A \) is such that the functor \( A \otimes - : \mathcal{V} \to \mathcal{A} \mathcal{V} \) is comonadic, then there exists an exact sequence of abelian groups

$$
\begin{array}{c}
0 \rightarrow \text{Aut}_\mathcal{V}(I) \xrightarrow{\lambda_0} \text{Aut}_{\mathcal{V}A}(A) \xrightarrow{\kappa_A} \text{End}_{A-\text{cor}}(C_e) \xrightarrow{o_A} \text{Pic}^c(A).
\end{array}
$$

**Proof.** By Proposition 3.12, the isomorphism \( \Gamma_A : \mathcal{J}_A^I(A) \to \text{End}_{A-\text{cor}}(C_e) \) of monoids restricts to an isomorphism

$$
\Gamma_A : \mathcal{J}_A^I \to \text{End}_{A-\text{cor}}(C_e).
$$

Write \( \kappa_A \) for \( \Gamma_A \) and write \( o_A \) for \( \Omega_A(\Gamma_A)^{-1} \). Then by combining (13) with (14), one obtains the exact sequence (16).

As an immediate consequence we deduce:

**Proposition 4.8.** Suppose that \( A = (A, m, e) \) is a commutative \( \mathcal{V} \)-algebra such that that the functor \( A \otimes - : \mathcal{V} \to \mathcal{A} \mathcal{V} \) is comonadic. Then one has an isomorphism of groups

$$
\text{Coker}(\kappa_A) \cong \text{Ker}(\text{Pic}^c(e)),
$$

where \( \text{Coker}(\kappa_A) \) is the cokernel of the homomorphism \( \kappa_A \).

We will need the following description of \( \kappa_A \):

**Proposition 4.9.** In the circumstances above, \( \kappa_A(\lambda) = (A \otimes \lambda^{-1}) \cdot (\lambda \otimes A) \) for every \( \lambda \in \text{Aut}_{\mathcal{V}A}(A) \).

**Proof.** Recall first that for any \( [(J, i_J)] \in \mathcal{J}_A^I \), \( \Gamma_A([(J, i_J)]) \) is the composite

$$
A \otimes A \xrightarrow{\xi^I_{J}} A \otimes J \otimes A \xrightarrow{A \otimes \xi^I_J} A \otimes A.
$$

Now, take any \( \lambda \in \text{Aut}_{\mathcal{V}A}(A) \) and form the pullback

\[
\begin{array}{ccc}
A & \xrightarrow{i_\lambda} & A \\
\downarrow{p_\lambda} & & \downarrow{\lambda} \\
I & \xrightarrow{\epsilon} & A
\end{array}
\]

Then \( \overline{\theta}_A(\lambda) = [(A_\lambda, i_\lambda)] \). Moreover, \( m \cdot (i_\lambda \otimes A) = \lambda^{-1} \cdot (p_\lambda \otimes A) \) by (17). Thus

$$
\xi^I_{i_\lambda} = \lambda^{-1} \cdot (p_\lambda \otimes A).
$$

Then, since \( \xi^I_{i_\lambda} = \xi^I_{i_\lambda} \cdot \tau_{A, A_\lambda} \), it follows that

$$
\xi^I_{i_\lambda} = \lambda^{-1} \cdot (A \otimes p_\lambda).
$$
Considering now the diagram

\[
\begin{array}{c}
\overset{(\xi'_\alpha)^{-1}\otimes A}{A\otimes A} \xrightarrow{A\otimes \xi^{-1}\otimes A} A\otimes J\otimes A \\
\xrightarrow{A\otimes \xi\otimes A} A\otimes J\otimes A \\
\xrightarrow{A\otimes \xi\otimes A} A\otimes J\otimes A \\
\xrightarrow{A\otimes \xi^{-1}} A\otimes A
\end{array}
\]

in which the rectangle commutes by naturality of composition, and using (17) and (18), one concludes that \(\kappa_A(\lambda) = (A\otimes \lambda^{-1}) \cdot (\lambda \otimes A)\). \(\square\)

Our next objective is to prove that the group \(\text{Ker} (\text{Pic}^c(e))\) is isomorphic to a suitable Amitsur cohomology group \(H^1(e, \text{Aut}^{\mathfrak{Alg}})\). In order to describe this cohomology group, and to prove the existence of the aforementioned isomorphism, we need to use some classical results which, for the convenience of the reader, are recalled in the Appendix. Let us first describe the functor \(\text{Aut}^{\mathfrak{Alg}}\), which is a particular case of the given at the beginning of the Appendix.

Let \(\mathcal{E}\) be the opposite of the category of commutative \(V\)-algebras, \(\text{CAlg}(V)\). It is well-known (e.g., [12, Corollary C.1.1.9]) that, under our assumptions on \(V\), \(\mathcal{E}\) has pullbacks and they are constructed as tensor products. It is routine to check that the assignments

\[\text{Alg} : \mathcal{E}^{op} \rightarrow \text{CAT}.
\]

Since for any morphism \(\iota : B \rightarrow A\) in \(\mathcal{E}\) (i.e., a morphism \(\iota : A \rightarrow B\) of commutative \(V\)-algebras), the functor \(\iota^* : \text{Alg} \rightarrow \text{Alg}\) admits as a right adjoint the forgetful functor \(\iota_* : \text{Alg} \rightarrow \text{Alg}\) (see Example [12]), give rise to an \(\mathcal{E}\)-indexed category (see the Appendix)

\[
\text{Alg} : \mathcal{E}^{op} \rightarrow \text{CAT}.
\]

and this is certainly the case, since the tensor product in \(V\) preserves reflexive coequalizers by our assumption on \(V\). Thus, \(\text{Alg}\) admits products.

**Lemma 4.10.** The functor \(\text{Aut}^{\mathfrak{Alg}} : (\mathcal{E} \downarrow A)^{op} \rightarrow \text{Group}\) is described on objects as

\[
\text{Aut}^{\mathfrak{Alg}}(A \downarrow B) = \text{Aut}_{\mathfrak{Alg}}(B)
\]

and on morphisms as

\[
\text{Aut}^{\mathfrak{Alg}}(f : (B', \iota') \rightarrow (B, \iota))(\sigma) = m_{B'} \cdot (B' \otimes f) \cdot (B' \otimes \sigma) \cdot (B' \otimes e_B)
\]

for all \(\sigma \in \text{Aut}_{\mathfrak{Alg}}(B)\).

**Proof.** Fix a commutative \(V\)-algebra \(A\), and a left \(A\)-module \(M_0\). We have the functor

\[
\text{Aut}_{M_0}^{\mathfrak{Alg}} : (\mathcal{E} \downarrow A)^{op} \rightarrow \text{Group}
\]
sending each object \( \iota : B \to A \) of \( \mathcal{E} \downarrow A \) (i.e. a morphism \( \iota : A \to B \) of commutative \( V \)-algebras) to the group

\[
\text{Aut}_{\mathcal{M}_0}^{\mathcal{V}}(\iota) = \text{Aut}_{\mathcal{V}}(\iota^*(M_0)).
\]

Note that since \( \iota^*(M_0) = (B \otimes_A M_0, m_{B \otimes_A M_0}) \), \( \text{Aut}_{\mathcal{M}_0}^{\mathcal{V}}(\iota) = \text{Aut}_{\mathcal{V}}(B \otimes_A M_0) \), where \( B \otimes_A M_0 \) is a left \( B \)-module via \( m_{B \otimes_A M_0} : B \otimes B \otimes_A M_0 \to B \otimes_A M_0 \).

Let us describe explicitly the action of \( \text{Aut}_{\mathcal{M}_0}^{\mathcal{V}} \) on morphisms. If \( f : (B', \iota') \to (B, \iota) \) is a morphism in \( (\mathcal{E} \downarrow A)^{\text{op}} \) making the triangle

\[
\begin{array}{ccc}
A & \xrightarrow{\iota} & B \\
\downarrow f & & \downarrow \iota' \\
B' & \to & B'
\end{array}
\]

commute, then

\[
\text{Aut}_{\mathcal{M}_0}^{\mathcal{V}}(f) : \text{Aut}_{\mathcal{M}_0}^{\mathcal{V}}(\iota) \to \text{Aut}_{\mathcal{M}_0}^{\mathcal{V}}(\iota')
\]
takes \( \sigma \in \text{Aut}_{\mathcal{M}_0}^{\mathcal{V}}(\iota) \) to the composite

\[
B' \otimes_A M_0 \simeq B' \otimes_B (B \otimes_A M_0) \xrightarrow{B' \otimes_B \sigma} B' \otimes_B (B \otimes_A M_0) \simeq B' \otimes_A M_0.
\]

Since the following split coequalizer diagram

\[
\begin{array}{ccc}
B' \otimes_B B \otimes_A M_0 & \xrightarrow{\sigma} & B' \otimes_B B \otimes_A M_0 \\
\downarrow a & \downarrow & \downarrow a \\
B' \otimes_B B \otimes_A M_0 & \xrightarrow{\sigma} & B' \otimes_B B \otimes_A M_0 \\
\end{array}
\]

where \( a \) is the composite \( (m_B \otimes_A M_0) \cdot (B' \otimes f \otimes_A M_0) \), is the defining coequalizer for \( B' \otimes_B (B \otimes_A M_0) \), it follows that \( \text{Aut}_{\mathcal{M}_0}^{\mathcal{V}}(f)(\sigma) \) is the unique (iso)morphism \( B' \otimes_A M_0 \to B' \otimes_A M_0 \) making the diagram

\[
\begin{array}{ccc}
B' \otimes_B B \otimes_A M_0 & \xrightarrow{\sigma} & B' \otimes_B B \otimes_A M_0 \\
\downarrow & \downarrow & \downarrow \\
B' \otimes_B B \otimes_A M_0 & \xrightarrow{\sigma} & B' \otimes_B B \otimes_A M_0 \\
\end{array}
\]

commute. But since \( (m_B \otimes_A M_0) \cdot (B' \otimes f \otimes_A M_0) \cdot (B' \otimes e_B \otimes_A M_0) = 1 \), it follows that

\[
\text{Aut}_{\mathcal{M}_0}^{\mathcal{V}}(f)(\sigma) = (m_B \otimes_A M_0) \cdot (B' \otimes f \otimes_A M_0) \cdot (B' \otimes \sigma) \cdot (B' \otimes e_B \otimes_A M_0).
\]

For us of interest is the case where \( M_0 = A \) with the left regular action of \( A \) on \( A \). Since \( \text{Aut}_{\mathcal{M}_0}^{\mathcal{V}}(A \hookrightarrow B) = \text{Aut}_{\mathcal{V}}(B \otimes_A A) \) and since the defining coequalizer diagram for \( B \otimes_A A \) is the following split one

\[
\begin{array}{ccc}
B \otimes_B B \otimes e_A & \xrightarrow{\iota} & B \otimes_B B \otimes e_A \\
\downarrow & \downarrow & \downarrow \\
B \otimes_B B \otimes e_A & \xrightarrow{\iota} & B \otimes_B B \otimes e_A \\
\end{array}
\]

it follows that the group \( \text{Aut}_{\mathcal{M}_0}^{\mathcal{V}}(\iota) \) is canonically isomorphic to \( \text{Aut}_{\mathcal{V}}(B) \).

Note that since for each morphism \( \iota : A \to B \) of commutative \( V \)-algebras, \( \text{Aut}_{\mathcal{M}_0}^{\mathcal{V}}(\iota) \) is an abelian group by Remark \( \ref{remark:abelian-groups} \) it follows that the functor \( \text{Aut}_{\mathcal{M}_0}^{\mathcal{V}} \) takes values in the category of abelian groups.
Now consider the augmented simplicial object in $\mathcal{E} = \mathsf{CAlg}(\mathcal{V})^{op}$

$$(A/I)_* : I \xrightarrow{e} A \xrightarrow{i_0} A \otimes A \xrightarrow{i_0} A \otimes A \otimes A \xrightarrow{i_2} A \otimes A \otimes A \otimes A \ldots,$$  

associated to the morphism $e : I \to A$, which is a particular case of [25] in the Appendix. By applying the functor $\mathsf{Aut}_{\mathcal{V}}^{31g}$ to $(A/I)_*$, and computing cohomology, we get the first Amitsur cohomology group $\mathcal{H}^1(e, \mathsf{Aut}_{\mathcal{V}}^{31g})$. The reader is referred to the Appendix for details.

**Theorem 4.11.** Suppose that $A = (A, m, e)$ is a $\mathcal{V}$-algebra such that the functor $A \otimes - : \mathcal{V} \to \mathcal{A}^\mathcal{V}$ is comonadic. Then there is a natural isomorphism $\mathcal{H}^1(e, \mathsf{Aut}_{\mathcal{V}}^{31g}) \simeq \ker(\mathsf{Pic}^e(e))$.

**Proof.** According to Proposition [13], it is enough to show that there is a natural isomorphism $\mathsf{Coker}(e_A) \simeq \mathcal{H}^1(e, \mathsf{Aut}_{\mathcal{V}}^{31g})$.

Writing $G_*$ for the comonad on $\mathcal{A}^\mathcal{V}$ generated by the adjunction

$\xymatrix{ \mathcal{V} \ar@<0.5ex>[rr]^{e_* = A \otimes -} \ar@/_0.5ex/[rr]_{e_* = U} & & \mathcal{A}^\mathcal{V} \ar@<0.5ex>[ll] \ar@/_0.5ex/[ll]}

$\xymatrix{ \mathcal{V} \ar@<0.5ex>[rr]^{e_* = A \otimes -} \ar@/_0.5ex/[rr]_{e_* = U} & & \mathcal{A}^\mathcal{V} \ar@<0.5ex>[ll] \ar@/_0.5ex/[ll]}

and write $G_\mathsf{-Coalg}(A, m_A)$ the set of all $G_\mathsf{-coalg}$-structures on $(A, m_A) \in \mathcal{A}^\mathcal{V}$. We know from [12] Proposition 4.5] that $G_\mathsf{-Coalg}(A, m_A) = \mathsf{End}_{A \mathsf{-cor}}(e_\mathcal{E})$ and that $\mathsf{End}_{A \mathsf{-cor}}(e_\mathcal{E}) = \mathsf{Aut}_{A \mathsf{-cor}}(e_\mathcal{E})$ by Corollary [13]. On the other hand, $\mathsf{Des}_{31g}(A, m_A) = Z^1(e, \mathsf{Aut}_{\mathcal{V}}^{31g})$ by Proposition [13] and $\mathsf{Des}_{31g}(A, m_A) = G_\mathsf{-Coalg}(A, m_A)$ by Theorem [13]. It follows that $Z^1(e, \mathsf{Aut}_{\mathcal{V}}^{31g}) = \mathsf{Aut}_{A \mathsf{-cor}}(e_\mathcal{E})$.

Applying the functor $\mathsf{Aut}_{\mathcal{V}}^{31g}$ to (21), and using the fact that for any commutative $\mathcal{V}$-algebra $S$, $\mathsf{Aut}_{\mathcal{V}}^{31g}(S) = \mathsf{Aut}_{\mathcal{A}^\mathcal{V}}(S)$ is an abelian group by Remark [13], we get the following simplicial abelian group

$$(A/I, \mathsf{Aut}_{\mathcal{V}}^{31g})_* : \mathcal{V}(I) \xrightarrow{\mathsf{Aut}_{\mathcal{V}}^{31g}(e)} \mathsf{Aut}_{\mathcal{V}}(A) \xrightarrow{\mathsf{Aut}_{\mathcal{V}}^{31g}((x_{0})_i)} \mathsf{Aut}_{\mathcal{A}^\mathcal{V}}(A \otimes A) \ldots$$

and the corresponding complex $\mathcal{C}(A/I, \mathsf{Aut}_{\mathcal{V}}^{31g})$ of abelian groups

$$0 \to \mathsf{Aut}_\mathcal{V}(I) \xrightarrow{\mathsf{Aut}_{\mathcal{V}}^{31g}(e)} \mathsf{Aut}_\mathcal{V}(A) \xrightarrow{\Delta_{n+1}} \mathsf{Aut}_{\mathcal{A}^\mathcal{V}}(A \otimes A) \xrightarrow{\Delta_n} \ldots$$

where

$$\Delta_n = \prod_{i=0}^{n} \mathsf{Aut}_{\mathcal{V}}^{31g}(i_{n})^{(-1)^n}, \quad n \geq 1.$$ 

Since $i_0 = A \otimes e, i_1 = e \otimes A$ and since the multiplication in the tensor product $\mathcal{V}$-algebra $A \otimes A$ is given by the composite $(m_A \otimes m_A) \cdot (A \otimes A_{A, A} \otimes A)$, it follows from (20) that

$$\mathsf{Aut}_{\mathcal{V}}^{31g}(i_0)(\lambda) = (m_A \otimes m_A) \cdot (A \otimes A_{A, A} \otimes A) \cdot (A \otimes A \otimes e) \cdot (A \otimes e)$$

and

$$\mathsf{Aut}_{\mathcal{V}}^{31g}(i_1)(\lambda) = (m_A \otimes m_A) \cdot (A \otimes A_{A, A} \otimes A) \cdot (A \otimes e \otimes A) \cdot (A \otimes A \otimes e)$$

for all $\lambda \in \mathsf{Aut}_{\mathcal{A}^\mathcal{V}}(A, m_A)$. 


But since in the diagram

- the top right triangle commutes since $\tau$ is symmetry;
- the left and the bottom right triangles commute since $e$ is the unit of $A$, and
- the rectangle commutes since $\lambda$ is an automorphism of the left $A$-module $(A, m)$,

it follows that $\text{Aut}^\text{alg}_i(i_0) = \lambda \otimes A$ and $\text{Aut}^\text{alg}_i(i_1) = A \otimes \lambda$
and since $\Delta_l = \text{Aut}^\text{alg}_i(i_0) \cdot (\text{Aut}^\text{alg}_i(i_1))^{-1} = (\text{Aut}^\text{alg}_i(i_1))^{-1} \cdot \text{Aut}^\text{alg}_i(i_0)$, it follows that $\Delta_l = (A \otimes \lambda^{-1}) \cdot (\lambda \otimes A)$. Hence one has commutativity in

$$\text{Aut}_{\text{alg}}(A) \xrightarrow{\kappa} \text{Aut}_{A \otimes \text{cor}}(\mathbb{C}_e)$$

$$\text{Aut}_{\text{alg}}(A) \xrightarrow{\lambda} 2^1(e) \otimes \text{Aut}_{A \otimes \mathcal{V}}(A \otimes A),$$

implying that $\text{Coker}(\kappa_A) \simeq H^1(e, \text{Aut}^\text{alg}_i(i))$. This completes the proof of the theorem. \hfill $\Box$

It is well-known (e.g., [13]) that for any commutative $\mathcal{V}$-algebra $A$, one has

$$\mathcal{E} \downarrow A = (A \downarrow \mathcal{C} \mathcal{A} \mathcal{G}(\mathcal{V}))^{pp}.$$  
Moreover, the co-slice category $A \downarrow \mathcal{C} \mathcal{A} \mathcal{G}(\mathcal{V})$ is isomorphic to the category $\mathcal{C} \mathcal{A} \mathcal{G}(\mathcal{A} \mathcal{V})$. In other words, to give a commutative monoid $B$ in the symmetric monoidal category $\mathcal{A} \mathcal{V}$ is to give a morphism $i : A \to B$ of commutative monoids in $\mathcal{V}$. The latter morphism serves as the unit morphism of the $\mathcal{A} \mathcal{V}$-monoid $B$. Write $(i)$ for the corresponding commutative monoid in the symmetric monoidal category $\mathcal{A} \mathcal{V}$. Then a (left) $(i)$-module in $\mathcal{A} \mathcal{V}$ consists of a (left) $A$-module
structure $A \otimes M \to M$ together with a morphism $\rho : B \otimes_A M \to M$ in $A^V$. A straightforward calculation shows that the composite
$$B \otimes M \xrightarrow{\rho \otimes M} B \otimes_A M \xrightarrow{\rho} M$$
makes $M$ into a $B$-module. In other direction, if $\varrho : B \otimes M \to M$ is a $B$-structure on $M$, then the pair $(M, A \otimes M \xrightarrow{\rho \otimes M} B \otimes M \xrightarrow{\varrho} M)$ is a left $A$-module and $\varrho = \varrho' \cdot \rho_B : M \to M$ for a unique $\varrho' : B \otimes_AM \to M$. Then $(M, \varrho')$ is a left $(\imath \cdot )$-module in $\mathbf{P}^V$. It is easily checked that the above constructions are inverse to each other, and hence give an isomorphism $(\imath \cdot (\mathbf{P}^V)) \simeq \mathbf{P}^V$ of categories. This allows us to identify the change-of-base functor $\imath^* = B \otimes_A - : A^V \to B^V$ with the functor $(\imath) \otimes_A - : A^V \to (\imath \cdot (A^V))$.

One then constructs an $E \downarrow A$-indexed category
$$\mathfrak{Alg}/A : (E \downarrow A)^{op} \to \mathbf{CAT}$$
as follows: If $(\imath : A \to B)$ is an object of $(E \downarrow A)^{op}$, then $\mathfrak{Alg}/A(\imath) = \mathbf{P}^V$ and if
$$A \xleftarrow{\imath} \xrightarrow{f} B \xleftarrow{\imath} B'$$
is a morphism in $(E \downarrow A)^{op}$, then $f^*$ is the functor $B' \otimes_R - : \mathbf{P}^V \to \mathbf{P}^V$. This $E \downarrow A$-indexed category satisfies the Beck-Chevalley condition (see the Appendix) and applying Theorem 4.11 gives:

**Theorem 4.12.** Suppose that $\imath : A \to B$ is a morphism of commutative $\mathcal{V}$-algebras such that the change-of-base functor $B \otimes A - : A^\mathcal{V} \to B^\mathcal{V}$ is comonadic. Then there is a natural isomorphism
$$H^1(\imath, \mathfrak{Alg}^\mathcal{V}/A) \simeq \ker(\text{Pic}^\mathcal{V}(\imath)).$$

Let $R \subseteq A$ be an extension of commutative rings. If $\imath : R \to A$ denotes the inclusion map, then $H^1(\imath, \mathfrak{Alg}_{R}^\mathcal{V}/A)$ is just the first Amitsur cohomology group $H^1(A/R, U)$, where $U$ denotes the “units” functor. When $\imath$ is a faithfully flat extension of commutative rings, then the change-of-base functor $A \otimes_R - : \mathcal{V} \to \mathcal{V}$ is comonadic (see, for example, [20]). Moreover, specializing Theorem 4.12 to this case gives the following well-known result (see, for example, [6], Corollary 4.6):

**Corollary 4.13.** Let $A$ be a faithfully flat commutative $R$-algebra and let $\imath : R \to A$ be the inclusion map. Then there is a natural isomorphism
$$H^1(A/R, U) \to \ker(\text{Pic}^\mathcal{V}(\imath)).$$

Theorem 4.12 also implies, in view of Remark 3.15:

**Corollary 4.14.** Let $A$ be a separable commutative $R$-algebra and let $\imath : R \to A$ be the inclusion map. Then there is a natural isomorphism
$$H^1(A/R, U) \to \ker(\text{Pic}^\mathcal{V}(\imath)).$$

4.3. **Bicomodules.** As observed in Subsection 3.4 there are dual versions of the exact sequences built in Subsections 3.1, 3.2, and 3.3 from an adjunction in a bicategory. One reason of recording explicitly them is to have statements tailored to concrete situations, as Examples 3.10 and 3.12 illustrate. Next, with the same motivation, we will consider the bicategory of bicomodules, and we will record some exact sequences derived from an adjunction in this setting. We close with some applications.

Suppose that $\mathcal{V} = (\mathcal{V}, \otimes, I)$ is a monoidal category with equalizers such that all the functors $X \otimes - : \mathcal{V} \to \mathcal{V}$ as well as $- \otimes X : \mathcal{V} \to \mathcal{V}$ for $X \in \mathcal{V}$, preserve equalizers. Coalgebras and (left, right, bi-) comodules in $\mathcal{V}$ can be defined as algebras and left (right, bi-) modules in the opposite monoidal category $(\mathcal{V}^{op}, \otimes, I)$. The resulting categories are denoted by $\mathbf{Coalg}(\mathcal{V})$, $\mathcal{C}^\mathcal{V}$, $\mathcal{C}^{\mathcal{V}^{op}}$, $\mathcal{C}^{\mathcal{V}^D}$, $\mathcal{C}$ and $\mathcal{D}$ being coalgebras in $\mathcal{V}$. 
Let $C, D, E$ be $V$-coalgebras. Dualizing the tensor product of bimodules, the cotensor product $X \square_C Y$ of a $(D, C)$-bicomodule $(X, \theta^l, \theta^r)$ and a $(C, E)$-bicomodule $(Y, \theta^l, \theta^r)$ over $C$ is defined to be the equalizer of the pair of morphisms

$$X \square_C Y \xrightarrow{\kappa_X Y} X \otimes Y \xrightarrow{\theta^r \otimes 1 \otimes \theta^l} X \otimes C \otimes Y.$$ 

Note that $X \square_C Y$ is a $(D, E)$-bicomodule.

Recall (for example, from [4]) and coimage factorizations (see, [1]).

Suppose in addition that $V$ admits, besides equalizers, finite colimits and coimage factorizations. In this case, for any two $V$-coalgebras $C$ and $C'$, the category $C^V C' = \text{Bicom}(V)(C, C')$, being the category of Eilenberg-Moore algebras for the comonad $C \otimes - \otimes C'$, also admits coequalizers (see, for example, [3]) and coimage factorizations (see, [1]).

Applying Theorem 4.15 gives:

**Theorem 4.15.** Suppose that $V$ admits finite colimits, coimage factorizations and equalizers that are preserved by the tensor product. Let $C$ and $D$ be $V$-coalgebras and $\Lambda : D \rightarrow C$ be a 1-cell admitting a right adjoint $\Lambda' : C \rightarrow D$ with unit $\eta_D : 1_D \rightarrow \Lambda' \circ \Lambda = \Lambda \square_D \Lambda'$. Suppose $\varepsilon_\Lambda : \Lambda \circ \Lambda = \Lambda' \square_D \Lambda'$ and counit $\varepsilon_A : A \circ A = A' \square_D A'\rightarrow \varepsilon_C$. If $\varepsilon_\Lambda$ is epimorphic, then

$$1 \rightarrow \text{Aut}_{C^V V}(C) \xrightarrow{\omega_0} \text{Aut}_{C^V V}(\Lambda) \xrightarrow{\pi} \text{Pic}(C)$$

is an exact sequence of groups, while

$$\text{Pic}(C) \xrightarrow{\pi} \text{Pic}(C) \rightarrow \pi_0(C^V D)$$

is an exact sequence of pointed sets.

Each morphism $\phi : D \rightarrow C$ of $V$-coalgebras determines a bicomodule $D_\phi : D \rightarrow C$ defined to be $D$ together with the coactions

$$(D \xrightarrow{\delta_D} D \otimes D, D \xrightarrow{\delta_D} D \otimes D \xrightarrow{D \otimes \phi} D \otimes C)$$

and a bicomodule $D^\phi : C \rightarrow D$ defined to be $C$ together with the coactions

$$(D \xrightarrow{\delta_D} D \otimes D, D \otimes D \xrightarrow{\phi \otimes D} C \otimes D, D \xrightarrow{\delta_D} D \otimes D).$$

$D^\phi$ is right adjoint to $D_\phi$ in $\text{Bicom}(V)$ with

$$D_\phi \circ D^\phi = D \square_D D \simeq D \xrightarrow{\phi} C$$

as counit and

$$D \xrightarrow{\varepsilon_D} D \square_D D = D^\phi \circ D_\phi$$

as unit. Here $D \xrightarrow{\varepsilon_D} D \square_D D$ is the unique morphism making the triangle

$$D \xrightarrow{\delta_D} D \otimes D \xrightarrow{\varepsilon_{D, D}} D \square_D D$$
commute.

It follows that a morphism \( \phi : D \to C \) of \( \mathcal{V} \)-coalgebras gives rise to two functors

\[
\phi^* = - \square_C D^\phi : \mathcal{V}^C \to \mathcal{V}^D
\]

\((Y, \theta_Y) \in \mathcal{V}^C \mapsto (Y \square_C D, Y \square_C \delta_D)\),

known as the change-of-cobase functor; and

\[
\phi_* = - \square_D D^\phi : \mathcal{V}^D \to \mathcal{V}^C,
\]

where for any right \( D \)-comodule \((X, \theta_X)\), \(\phi_* (X, \theta_X) = X\) is a right \( C \)-comodule via the coaction

\[
X \xrightarrow{\theta_X} X \otimes D \xrightarrow{X \otimes \phi} X \otimes C;
\]

Since \( D^\phi \) is right adjoint to \( D^\phi \) in \( \text{Bicom}(\mathcal{V}) \), it follows that \( \phi^* \) is left adjoint to \( \phi_* \).

Assume further that \( \mathcal{V} \) is symmetric with symmetry \( \tau \). It is well-known that the category of cocommutative \( \mathcal{V} \)-coalgebras, \( \mathcal{CCoalg}(\mathcal{V}) \), has pullbacks and they are constructed as cotensor products: For any two morphism \( \phi : D \to C \) and \( \phi' : D' \to C \) in \( \mathcal{CCoalg}(\mathcal{V}) \), the diagram

\[
\begin{array}{ccc}
D' \square_C D & \xrightarrow{p_D} & D \\
\downarrow{p_{D'}} & & \downarrow{\phi} \\
D' & \xrightarrow{\phi'} & C,
\end{array}
\]

where \( p_D = \varepsilon_D \square_C D \) and \( p_{D'} = D' \square_C \varepsilon_D \), is a pullback in \( \mathcal{CCoalg}(\mathcal{V}) \).

Define a \( \mathcal{CCoalg}(\mathcal{V}) \)-indexed category

\( \mathcal{Ecoalg} : (\mathcal{CCoalg}(\mathcal{V}))^{op} \to \mathbf{CAT} \)

by setting

\[
\mathcal{Ecoalg}(C) = \mathcal{V}^C \quad \text{and} \quad \mathcal{Ecoalg}(D \xrightarrow{\phi} C) = \mathcal{V}^C \xrightarrow{\phi^*} \mathcal{V}^D,
\]

where \( \phi^* = - \square_C D : \mathcal{V}^C \to \mathcal{V}^D \) is the change-of-cobase functor. Since for any morphism \( \phi : D \to C \) in \( \mathcal{CCoalg}(\mathcal{V}) \), the functor \( \phi^* = - \square_C D \) admits as a left adjoint the forgetful functor \( \mathcal{V}^D \xrightarrow{\phi_* = - \square_D D} \mathcal{V}^C \), the \( \mathcal{CCoalg}(\mathcal{V}) \)-indexed category \( \mathcal{Ecoalg} \) admits coproducts if it satisfies the Beck-Chevalley condition, i.e., for any morphism \( \phi' : D' \to C \) in \( \mathcal{CCoalg}(\mathcal{V}) \), the diagram

\[
\begin{array}{ccc}
\mathcal{V}^D' & \xrightarrow{(p_D')^*} & \mathcal{V}^D \square_C D \\
\downarrow{(\phi')^*} & & \downarrow{(p_D)^*} \\
\mathcal{V}^C & \xrightarrow{\phi^*} & \mathcal{V}^D
\end{array}
\]

commutes up to canonical isomorphism. It is easily to check that this condition is equivalent to saying that for any \( X \in \mathcal{V}^D \), one has an isomorphism

\[
X \square_D (D' \square_C D) \simeq X \square_C D,
\]

and this is certainly the case, since the tensor product in \( \mathcal{V} \) preserves reflexive equalizers by our assumption on \( \mathcal{V} \). Thus, \( \mathcal{Ecoalg} \) admits coproducts.

As in the case of algebras, given a cocommutative \( \mathcal{V} \)-coalgebra \( C \), we get the functor

\( \text{Aut}_{\mathcal{C}Coalg} : (\mathcal{CCoalg} \downarrow C)^{op} \to \mathbf{Group} \)

sending each object \( \phi : D \to C \) of \( (\mathcal{CCoalg} \downarrow C)^{op} \) (i.e. a morphism \( \phi : D \to C \) of cocommutative \( \mathcal{V} \)-coalgebras) to the group

\[
\text{Aut}_{\mathcal{C}Coalg} \phi (D \xrightarrow{\phi} C) = \text{Aut}_{\mathcal{V}^D} (D).
\]
On morphisms it is defined in the following way: Given a morphism $f : (D, \phi) \to (D', \phi')$ in $(\mathcal{C}_{\text{Coalg}} \downarrow C)^{\text{op}}$ (i.e., a commutative diagram

$$
\begin{array}{ccc}
\phi' & \xrightarrow{f} & D' \\
\downarrow & & \downarrow \\
\phi & \xrightarrow{} & D
\end{array}
$$

in $\mathcal{C}_{\text{Coalg}} \downarrow C$, then

$$
\mathbf{Aut}_{\text{coalg}}^{\mathcal{C}}(f : (D, \phi) \to (D', \phi'))(\sigma) = (\varepsilon_D \otimes D') \cdot (\sigma \otimes D') \cdot (\phi \otimes D') \cdot \delta_D,
$$

(23)

for all $\sigma \in \mathbf{Aut}_{\mathcal{V}}(D)$.

Note that, since for each morphism $\phi : D \to C$ of cocommutative $\mathcal{V}$-coalgebras, $\mathbf{Aut}_{\text{coalg}}^{\mathcal{C}}(\phi)$ is an abelian group, it follows that the functor $\mathbf{Aut}_{\text{coalg}}^{\mathcal{C}}$ also takes values in the category of abelian groups.

Given a morphism $\phi : D \to C$ of cocommutative $\mathcal{V}$-coalgebras, consider the associated augmented simplicial object

$$(D/C)_* : \cdots \to D^3 \xrightarrow{\partial_0} D^2 \xrightarrow{\partial_1} D^1 \xrightarrow{\partial_2} D^0 \xrightarrow{\phi} C,$$

where

- $D^0 = D$
- $D^n = D \square_C D \cdots \square_C D$ for all $n \geq 1$ (n+1)-times
- $\partial_i = D'(\square_C D)^{n+1-i} : D^{n+1} \to D^n$ for all $0 \leq i \leq n$
- $s_j = D'(\square_C D)^{(n+j)} : D^{(n-1)} \to D^{(n)}$ for all $0 \leq j \leq n$.

Applying the functor $\mathbf{Aut}_{\text{coalg}}^{\mathcal{C}} : (\mathcal{C}_{\text{Coalg}} \downarrow C)^{\text{op}} \to \text{Group}$ to $(D/C)_*$, we obtain the following augmented cosimplicial group

$$(D/C, \mathbf{Aut}_{\text{coalg}}^{\mathcal{C}})_* : \mathbf{Aut}_{\mathcal{V}C}(C) \xrightarrow{\mathbf{Aut}_{\text{coalg}}^{\mathcal{C}}(\phi)} \mathbf{Aut}_{\mathcal{V}D}(D) \xrightarrow{\mathbf{Aut}_{\text{coalg}}^{\mathcal{C}}(\partial_0)} \mathbf{Aut}_{\mathcal{V}D \square_C D}(D \square_C D) \xrightarrow{\mathbf{Aut}_{\text{coalg}}^{\mathcal{C}}(\partial_1)} \cdots$$

Since for any $n \geq 0$, $D^n$ is a cocommutative $\mathcal{V}$-coalgebra, all the categories $\mathcal{V}D^n$ are symmetric monoidal with monoidal unit $D^n$, it follows that $(D/C, \mathbf{Aut}_{\text{coalg}}^{\mathcal{C}})_*$ is in fact an augmented abelian cosimplicial group.

Write $\mathbf{Pic}^\mathcal{C}(C)$ for the subgroup of $\mathbf{Pic}(C)$ consisting of all classes of invertible $(C, C)$-bicomodules $(X, \vartheta_X : X \to C \otimes X, \vartheta_r : \to X \otimes C)$ such that $\rho_r = \tau_C \cdot \rho_l$. Then $\mathbf{Pic}^\mathcal{C}(C)$ is an abelian group. Moreover, given a morphism $\phi : D \to C$ of cocommutative $\mathcal{V}$-coalgebras, the map

$$\mathbf{Pic}^\mathcal{C}(\phi) : \mathbf{Pic}^\mathcal{C}(C) \to \mathbf{Pic}^\mathcal{C}(D)$$

defined by $\mathbf{Pic}^\mathcal{C}(\phi)([P]) = [P \square_C D]$, is a homomorphism of abelian groups.

Now with the complex of abelian groups

$$0 \to \mathbf{Aut}_{\mathcal{V}C}(C) \xrightarrow{\mathbf{Aut}_{\text{coalg}}^{\mathcal{C}}(\phi)} \mathbf{Aut}_{\mathcal{V}D}(D) \xrightarrow{\Delta_1} \mathbf{Aut}_{\mathcal{V}D \square_C D}(D \square_C D) \xrightarrow{\Delta_2} \cdots$$

$$\Delta_n = \prod_{i=0}^n \mathbf{Aut}_{\text{coalg}}^{\mathcal{C}}(\partial_n)^{(-1)^n}, \quad n \geq 1,$$

corresponding to the augmented abelian cosimplicial group $(D/C, \mathbf{Aut}_{\text{coalg}}^{\mathcal{C}})_*$, by arguments similar to those used in the of Theorems, we derive the following result.
Theorem 4.16. Suppose that \( \phi : D \to C \) is a morphism of cocommutative \( \mathcal{V} \)-coalgebras such that the change-of-cobase functor \( \phi^* = -\square_C D : \mathcal{V}^C \to \mathcal{V}^D \) is monadic. Then there is a natural isomorphism

\[
\mathcal{H}^1(\phi, \text{Aut}^{\text{C coalg}}_C) \simeq \text{Ker}(\text{Pic}^C(\phi)).
\]

Specializing Theorem 4.16 to the case where \( \mathcal{V} = \text{Vect}_k \) is the category of vector spaces over a field \( k \) and using that for any cocommutative \( k \)-coalgebra \( C \), \( \text{Pic}^C(C) = 0 \) (see, for example, [8, Proposition 3.2.14] or [7, Proposition 4.1]), we get the following version of Hilbert’s theorem 90 for cocommutative coalgebras:

Theorem 4.17. Suppose that \( \phi : D \to C \) is a morphism of cocommutative \( \mathcal{V} \)-coalgebras such that the change-of-cobase functor \( -\square_C D : \text{Vect}^C_k \to \text{Vect}^D_k \) is monadic. Then

\[
\mathcal{H}^1(\phi, \text{Aut}^{\text{C coalg}}_C) = 0.
\]

Appendix A. Some classical results and constructions

Let \( \mathcal{A} \) be a category with pullbacks. An \( \mathcal{A} \)-indexed category \( \mathcal{X} \) is a pseudo-functor \( \mathcal{A}^{op} \to \text{CAT} \), where \( \text{CAT} \) denotes the 2-category of locally small (but possibly large) categories, explicitly given by the data of a family of categories \( \mathcal{X}(a) \), indexed by the objects of \( \mathcal{A} \), with change of base functors

\[
i^* : \mathcal{X}(a) \to \mathcal{X}(b)
\]

for each morphism \( \iota : b \to a \) of \( \mathcal{A} \) and with additional structure expressing the idea of a pseudo-functor (see [13], [22]).

A simple but important example of an \( \mathcal{A} \)-indexed category is the so-called basic \( \mathcal{A} \)-indexed category \( \mathcal{A}^{\downarrow} : \mathcal{A}^{op} \to \text{CAT} \) that to any object \( a \in \mathcal{A} \) associates the slice category \( \mathcal{A}^{\downarrow} a \), and to a morphism \( \iota : b \to a \) the functor \( \iota^* : \mathcal{A}^{\downarrow} a \to \mathcal{A}^{\downarrow} b \) given by pulling back along \( \iota \).

Fix and object \( a \) of \( \mathcal{A} \), and \( \mathcal{X} : \mathcal{A}^{op} \to \text{CAT} \) an \( \mathcal{A} \)-indexed category. For each object \( x \in \mathcal{X}(a) \), let us define a functor

\[
\text{Aut}^\mathcal{X}_a : (\mathcal{A}^{\downarrow} a)^{op} \to \text{Group}
\]

sending each object \( b \xrightarrow{\kappa} a \) of \( \mathcal{A}^{\downarrow} a \) to the group

\[
\text{Aut}^\mathcal{X}_a(\kappa) \overset{\text{def}}{=} \text{Aut}_{\mathcal{X}(b)}(i^*(\kappa)). \tag{24}
\]

A.1. Descent. Consider the augmented simplicial complex

\[
(b/a)_* : \quad \cdots \to (b/a)_2 \overset{\partial_0}{\to} (b/a)_1 \overset{\partial_0}{\to} (b/a)_0 \overset{i}{\to} a
\]

associated to any morphism \( \iota : b \to a \) in \( \mathcal{A} \), where

- \( (b/a)_0 = b \)
- \( (b/a)_n = b \times_a b \times_a \cdots \times_a b \) for all \( n \geq 1 \)
- \( \partial_i = < p_1, p_2, \ldots, p_{i-1}, p_{i+1}, \ldots, p_{n+1} : a_n \to a_{n-1} \) for all \( 0 \leq i \leq n \)
- \( s_j = b \times_a b \times_a \cdots \times_a b \times_a b \times_a b \times_a \cdots \times_a b : (b/a)_{n-1} \to (b/a)_n \) for all \( 0 \leq j \leq (n+1)-\text{times} \)
- \( \Delta_{b/a} : b \to b \times_a b \)

Here \( p_i : b \times_a b \times_a \cdots \times_a b \to b \) is the projection to the \( i \)-th factor, while \( \Delta_{b/a} \) is the diagonal morphism \( b \to b \times_a b \).

Let us recall from [13] the definition of the category \( \text{Des}_\mathcal{X}(\iota) \) of \( \mathcal{X} \)-descent data relative to \( \iota \). Its objects are pairs \((x, \vartheta)\), with \( x \) an object of \( \mathcal{X}(b) \) and \( \vartheta : \partial_1^\iota(x) \simeq \partial_0^\iota(x) \) an isomorphism in
\( \mathcal{X}(b \times_a b) \) such that \( s_0^i(\partial) = 1 \) and the diagram

\[
\begin{array}{ccc}
\partial_2^\ast \partial_1^\ast (x) & \xrightarrow{\partial_2^\ast (\vartheta)} & \partial_2^\ast \partial_0^\ast (x) \\
\downarrow & & \downarrow \\
\partial_1^\ast \partial_0^\ast (x) & \xrightarrow{\partial_1^\ast (\vartheta)} & \partial_1^\ast \partial_0^\ast (x)
\end{array}
\]

commutes in \( \mathcal{X}(b \times_a b) \). Here the labeled isomorphisms are the canonical ones of the \( \mathcal{A} \)-indexed category \( \mathcal{X} \) coming from the simplicial identities

\[ \partial_i \partial_j = \partial_{j-1} \partial_i \quad (i < j). \]

A morphism \( f : (x, \vartheta) \to (y, \theta) \) in \( \text{Des}_\mathcal{X}(i) \) is a morphism \( f : x \to y \) in \( \mathcal{X}(b) \) which commutes with the descent data \( \vartheta \) and \( \theta \) in the sense that the diagram

\[
\begin{array}{ccc}
\partial_1^\ast (x) & \xrightarrow{\vartheta} & \partial_0^\ast (x) \\
\partial_1^\ast (f) & & \partial_0^\ast (f) \\
\partial_1^\ast (y) & \xrightarrow{\theta} & \partial_0^\ast (y)
\end{array}
\]

commutes in \( \mathcal{X}(b \times_a b) \).

If \( z \) is an object of \( \mathcal{X}(a) \), then \( \iota^\ast (z) \) comes equipped with a canonical descent datum given by the composite

\[ \partial_1^\ast (\iota^\ast (z)) \simeq (i \partial_1^\ast) (z) = (i \partial_0^\ast) (z) \simeq \partial_0^\ast (\iota^\ast (z)) \]

of canonical isomorphisms. In other words, the functor \( \iota^\ast \) factors as

\[
\begin{array}{ccc}
\mathcal{X}(a) & \xrightarrow{K_i} & \text{Des}_\mathcal{X}(i) \\
\downarrow \iota^\ast & & \downarrow U \\
\mathcal{X}(b) & & \end{array}
\]

where \( U \) is the evident forgetful functor, and \( K_i \) sends \( z \in \mathcal{X}(a) \) to \( \iota^\ast (z) \) equipped with the canonical descent datum.

**Definition A.1.** \( \iota \) is called an \( \mathcal{X} \)-descent morphism if \( K_i \) is full and faithful, and an effective \( \mathcal{X} \)-descent morphism if \( K_i \) is an equivalence of categories.

Let \( x \in \mathcal{X}(b) \). Write \( \text{Des}_\mathcal{X}(x) \) for the set of all descent data on \( x \). Two descent data \( (x, \vartheta) \) and \( (x, \vartheta') \) on \( x \) are called equivalent if they are isomorphic objects in the category \( \text{Des}_\mathcal{X}(i) \). The set of equivalence classes of descent data on \( x \) is denoted by \( \text{Des}_\mathcal{X}(x) \). If \( x = \iota^\ast (y) \) for some \( y \in \mathcal{X}(a) \), then \( \text{Des}_\mathcal{X}(\iota^\ast (y)) \) is a pointed set with the class of canonical descent datum as a distinguished element.

**Definition A.2.** An \( \mathcal{A} \)-indexed category \( \mathcal{X} \) has products (resp. coproducts) if for each morphism \( \iota : b \to a \) in \( \mathcal{A} \), the change of base functor \( \iota^\ast : \mathcal{X}(a) \to \mathcal{X}(b) \) admits a right (resp. left) adjoint \( \Pi_\iota : \mathcal{X}(b) \to \mathcal{X}(a) \) (resp. \( \Sigma_\iota : \mathcal{X}(b) \to \mathcal{X}(a) \)) and the Beck-Chevalley condition is satisfied, i.e., for every pullback diagram

\[
\begin{array}{ccc}
c & \xrightarrow{q} & b \\
p \downarrow & & \downarrow \\
c' \xleftarrow{p} & & \iota \\
\downarrow & & \downarrow \\
a & \xleftarrow{\iota'} & a,
\end{array}
\]

...
the following diagram

\[ \begin{array}{ccc}
\mathcal{X}(b) & \xrightarrow{q^*} & \mathcal{X}(c) \\
\Pi & \downarrow & \Pi \\
\mathcal{X}(a) & \xrightarrow{\iota^*} & \mathcal{X}(b) \\
\end{array} \]

(resp. \[ \begin{array}{ccc}
\mathcal{X}(b') & \xrightarrow{p^*} & \mathcal{X}(c) \\
\Sigma & \downarrow & \Sigma \\
\mathcal{X}(a) & \xrightarrow{\iota} & \mathcal{X}(b) \\
\end{array} \] )

commutes up to canonical isomorphism.

We shall need the following version of the Bénabou-Roubaud-Beck theorem (cf. [14, Proposition B1.5.5]):

**Theorem A.3.** For an \( \mathcal{A} \)-indexed category \( \mathcal{X} : \mathcal{A}^{op} \to \text{CAT} \) having products (resp. coproducts) and for an arbitrary morphism \( \iota : b \to a \) in \( \mathcal{A} \), the category \( \text{Des}_\mathcal{A}(\iota) \) of descent data with respect to \( \iota \) is isomorphic to the Eilenberg-Moore category of coalgebras (resp. algebras) for the comonad (resp. monad) \( \mathcal{G}_\iota \) (resp. \( \mathcal{T}_\iota \)) on \( \mathcal{X}(b) \) generated by the adjoint pair \( \iota^* \dashv \Pi : \mathcal{X}(b) \to \mathcal{X}(a) \) (resp. \( \Sigma \dashv \iota^* : \mathcal{X}(a) \to \mathcal{X}(b) \)). Moreover, modulo this equivalence, the functor \( K_\iota : \mathcal{X}(a) \to \text{Des}_\mathcal{X}(\iota) \) corresponds to the comparison functor \( \mathcal{X}(a) \to (\mathcal{X}(b))^{\mathcal{G}_\iota} \) (resp. \( \mathcal{X}(a) \to (\mathcal{X}(b))^{\mathcal{T}_\iota} \)). Thus, \( \iota \) is an effective \( \mathcal{X} \)-descent morphism if and only if the functor \( \iota^* : \mathcal{X}(a) \to \mathcal{X}(b) \) is monadic (resp. co-monadic).

**A.2. Amitsur cohomology.** Let \( F \) be a functor on the category \( (\mathcal{A} \downarrow a)^{op} \) with values in the category of groups. Applying \( F \) to the augmented simplicial object \( [25] \), one gets a coaugmented cosimplicial group

\[
\begin{array}{ccc}
(b/a, F)_* : & F(a) & \xrightarrow{F(a)} F(b) \\
& F(\partial_0) & \xrightarrow{F(\partial_0)} F(b \times_a b) \\
& F(\partial_1) & \xrightarrow{F(\partial_1)} F(b \times_a b) \\
& F(\partial_2) & \xrightarrow{F(\partial_2)} F(b \times_a b) \\
& \vdots & \vdots \\
& F(\partial_n) & \xrightarrow{F(\partial_n)} F(b \times_a b) \\
\end{array}
\]

with cofaces \( F(\partial_i) \) and codegeneracies \( F(s_i) \), and hence one has the non-abelian 0-cohomology group \( \mathcal{H}^0((b/a, F)_*) \), and the non-abelian 1-cohomology pointed set \( \mathcal{H}^1((b/a, F)_*) \). More precisely, \( \mathcal{H}^0((b/a, F)_*) \) is the equalizer of the pair \( (F(\partial_0), F(\partial_1)) \).

On the other hand, a 1-cocycle is an element \( x \in F(b \times_a b) \) such that

\[
F(\partial_1)(x) = F(\partial_2)(x) \cdot F(\partial_0)(x)
\]

in \( F(b \times_a b \times_a b) \). Write \( Z^1((b/a, F)_*) \) for the set of 1-cocycles. This set is pointed with point the unit element of \( F(b \times_a b \times_a b) \). Two 1-cocycles \( x \) and \( x' \) are equivalent if

\[
x' = F(\partial_1)(y) \cdot x \cdot F(\partial_0)(y)^{-1}
\]

for some element \( y \in F(b \times_a b) \). This is an equivalence relation on \( Z^1((b/a, F)_*) \) and \( \mathcal{H}^1((b/a, F)_*) \) is defined as the factor-set of equivalence classes of 1-cocycles, and it is a pointed set.

We call \( \mathcal{H}^0((b/a, F)_*) \) (resp. \( \mathcal{H}^1((b/a, F)_*) \)) the zeroth Amitsur cohomology group of \( \iota : a \to b \) with values in \( F \) (resp. the first Amitsur cohomology pointed set of \( \iota : a \to b \) with values in \( F \)) and denote it by \( \mathcal{H}^0(\iota, F) \) (resp. \( \mathcal{H}^1(\iota, F) \)).

If for each \( n \geq 1 \), \( F(b \times_a b \times_a \cdots \times_a b) \) is an abelian group (which is indeed the case if the functor \( F \) factors through the category of abelian groups), it is possible to define higher cohomology groups as follows. Let

\[
C(\iota, F) : 0 \to F(a) \xrightarrow{F(\iota)} F(b) \xrightarrow{\Delta_2} F(b \times_a b) \xrightarrow{\Delta_3} \cdots
\]

\[
\Delta_n = \prod_{i=0}^{n} (F(\partial_n))^{(-1)^n}, \quad n \geq 1
\]
be the complex of abelian groups associated to the abelian cosimplicial group \((F/\iota)^*\). The cohomology groups of this complex are called the Amitsur cohomology groups of \(\iota : a \to b\) with values in \(F\) and are denoted by \(\mathcal{H}^*(\iota,F)\).

The following result of Grothendieck is to be found in \[\text{[13]}\].

**Proposition A.4.** Let \(X : A^{\text{op}} \to \text{CAT}\) be an \(A\)-indexed category, \(\iota : b \to a\) a morphism in \(A\) and \(x \in X(a)\). Then the assignment that takes \(\alpha \in \text{Aut}_{X(b \times_a b)}(\partial^*(x))\), where \(\partial\) denotes the common value of \(\iota \cdot \partial_0\) and \(\iota \cdot \partial_1\), to the composite

\[
\partial_1^R(\iota^*(x)) \simeq (\iota \cdot \partial_1)^R(x) = \partial^R(x) \xrightarrow{\alpha} \partial^R(x) \simeq (\iota \cdot \partial_0)^R(x) = \partial_0^R(\iota^*(x))
\]

yields an isomorphism

\[
\Upsilon^{\iota,x} : Z^1(\iota, \text{Aut}_{X}^X) \simeq \text{Des}_X(\iota^*(x))
\]

of the pointed sets. When \(\iota\) is an effective \(X\)-descent morphism, \(\Upsilon^{\iota,x}\) induces an isomorphism

\[
\tilde{\Upsilon}^{\iota,x} : H^1(\iota, \text{Aut}_{X}^X) \simeq \text{Des}_X(\iota)
\]

of pointed sets.

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