KPII: Cauchy–Jost function, Darboux transformations and totally nonnegative matrices*

M Boiti1, F Pempinelli1 and A K Pogrebkov2

1 EINSTEIN Consortium, Università del Salento, Lecce, Italy
2 Steklov Mathematical Institute and National Research University Higher School of Economics, Moscow, Russia

E-mail: boiti@le.infn.it, Pempi@le.infn.it and pogreb@mi.ras.ru

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Abstract
Direct definition of the Cauchy–Jost (known also as Cauchy–Baker–Akhiezer) function is given in the case of a pure solitonic solution. Properties of this function are discussed in detail using the Kadomtsev–Petviashvili II equation as an example. This enables formulation of the Darboux transformations in terms of the Cauchy–Jost function and classification of these transformations. Action of Darboux transformations on Grassmanians—i.e. on the space of soliton parameters—is derived and the relation of the Darboux transformations with the property of total nonnegativity of elements of corresponding Grassmanians is discussed.

Keywords: KPII equation, Cauchy–Jost function, Darboux transformation

1. Introduction and notation

In this article we continue the investigation of the Cauchy–Jost function and demonstrate its exceptional role in the study of Inverse problems. This function naturally appeared in the theory of (binary) Bäcklund transformations, see [1, 2]. In [3] this function was considered from an algebro-geometric point of view and it was called there the Cauchy–Baker–Akhiezer kernel. In [4] and [5] this function was studied by means of the \( \tilde{\partial} \)-method for different classes of scattering data. It was proved that by means of this function all equations of the Kadomtsev–Petviashvili hierarchy can be presented in a compact quasilinear form. Then the Jost solutions appear as specific asymptotic values of the Cauchy–Jost function, that motivates our choice of its label. We take the Kadomtsev–Petviashvili II (KPII) equation, [6],

* To the memory of our friend and colleague Peter P Kulish
\( (u_t - 6uu_x + uu_{x_1})_{x_1} = -3uu_{x_2}, \) \hspace{1cm} (1.1)

with Lax operator, \([7, 8],\)

\[ \mathcal{L}(x, \partial_x) = -\partial_x + \partial_x^2 - u(x), \] \hspace{1cm} (1.2)

as an example. In this case the Cauchy–Jost function can be defined as a primitive of the product of the Jost and dual Jost solution:

\[ F(x, \lambda, \mu) = \int_{-\infty}^{\infty} dy \psi(y, \lambda) \varphi(y, \mu) \big|_{y_2 = x_2, y_3 = x_3, \ldots}, \] \hspace{1cm} (1.3)

where \( \varphi(x, \lambda) \) is the Jost solution of the heat equation \( \mathcal{L}(x, \partial_x) \varphi(x, \lambda) = 0 \) and \( \psi(x, \lambda) \) is the Jost solution of its dual equation, \( \lambda \) and \( \mu \) are two complex parameters, \( x = (x_1, x_2, x_3, \ldots) \) (up to some finite number) denotes space and time independent variables. The sign of the bottom limit of the integral in (1.3) must be chosen in order to guarantee its convergence.

It is known, see \([3–5]\) for detail, that the Cauchy–Jost function has a unity residual at \( \mu = \lambda \) and decays when \( \lambda \) or \( \mu \) tend to infinity if the exponential factor \( e^{(\ell(\lambda) - \ell(\mu))x} \) where

\[ \ell(\lambda)x = \sum_{k \geq 1} \lambda^k x_k, \] \hspace{1cm} (1.4)

is removed. In particular in \([4]\) it is shown that, in the case considered there, this function is given in terms of the \( \tau \)-function as

\[ F(x, \lambda, \mu) = \frac{e^{(\ell(\lambda) - \ell(\mu))x}}{\mu - \lambda} \frac{\tau(x - [\lambda^{-1}] + [\mu^{-1}])}{\tau(x)}, \] \hspace{1cm} (1.5)

where dependence on the infinite set of times \( x = (x_1, x_2, x_3, \ldots) \) is assumed and square brackets denote the infinite Miwa vector

\[ [\lambda] = (\lambda, \lambda^2/2, \lambda^3/3, \ldots). \] \hspace{1cm} (1.6)

Instead of (1.3) in \([5]\) the direct definition of the Cauchy–Jost function in the case of a rapidly decaying potential was given by means of the \( \partial \)-problem for it. Jost solutions, potential and other objects result then by proper limiting procedures in terms of this function. In particular, one can use an arbitrary finite number of variables \( x_k \) in (1.4). Then evolutions of the Cauchy–Jost function with respect to these times are given in a closed form and evolutions of the Jost solutions and potential follow consequently. Here we generalize the definition of the Cauchy–Jost function for the pure soliton potentials of the heat equation, investigate its properties and define Darboux transformations in its terms. This enables us to derive the action of these transformations on points of Grassmanians that parametrize soliton potentials.

The article is organized as follows. In section 2 we give some notations and known results for the pure soliton potentials of the heat equation, see e.g. \([9–12]\) for detail. In particular, it is known that soliton potentials are labeled by two numbers \( N_a \) and \( N_b \), so it is natural to speak about \((N_a, N_b)\)-solitons. In section 3 we define the Cauchy–Jost function by means of its analyticity properties with respect to either one of the spectral parameters, \( \lambda \) or \( \mu \), and prove that these properties uniquely define a function \( F(x, \lambda, \mu) \) that in turn defines the Jost solutions and potential of the heat equation in the pure solitonic case. In this sense our approach to the definition of the Cauchy–Jost function is close to that of \([13]\) for the definition of the Jost (Baker–Akhiezer) solutions of the nonstationary Schrödinger equation. In section 5 we introduce discrete transformations of the Cauchy–Jost function by linear equations on it, we show that they are equivalent to the original (bilinear) Darboux transformations of \([1, 2]\) and
specify the latter ones. In accordance to two numbers labeling soliton solutions we single out three types of Darboux transformations: changing \( N_a \rightarrow N_a + 1 \) and preserving \( N_b \), vice versa, changing \( N_b \rightarrow N_b + 1 \) and preserving \( N_a \) and those that do not change any of these numbers, but change other parameters of solitons. We call them \( N_a \)-transformations, \( N_b \)-transformations and 0-transformations, correspondingly, and consider the transformations inverse to the first two. For any of these transformations we derive the action they generate on the corresponding Grassmanian. Discussion of these results and their possible generalizations, as well as concluding remarks are given in section 6. In appendix we prove some linear algebra relations used in the body of the article.

2. Jost solutions and potentials of the KPII equation in the pure solitonic case

We start with some notations used below. Let us have two numbers \( N_a \) and \( N_b \) obeying condition
\[
N_a, N_b \geq 1. \tag{2.1}
\]
Let
\[
N = N_a + N_b, \text{ so that } N \geq 2. \tag{2.2}
\]
We need \( N \) real parameters \( \kappa_1, \kappa_2, \ldots, \kappa_N \) and notation
\[
\ell(\lambda)x = \lambda x_1 + \lambda^2 x_2 + \lambda^3 x_3 + \ldots, \tag{2.3}
\]
\[
\ell_n x = \kappa_n x_1 + \kappa_n^2 x_2 + \kappa_n^3 x_3 + \ldots, \quad n = 1, \ldots, N. \tag{2.4}
\]
where \( \ldots \) denotes higher terms, if necessary, up to some finite number and where \( \lambda \in \mathbb{C} \). We use special diagonal real \( N \times N \) matrices
\[
e^{\ell x} = \text{diag}\{e^{\ell_n x}\}_{n=1}^{N}, \tag{2.5}
\]
\[
r = \text{diag}\{r_n\}_{n=1}^{N}, \quad r_n = \prod_{n' \neq n}^{N} (\kappa_n - \kappa_{n'}). \tag{2.6}
\]

In terms of parameters \( \kappa_j \) we introduce two ‘incomplete’ Vandermonde matrices, i.e. \( N_b \times N \) and \( N \times N_a \) matrices
\[
V = \begin{pmatrix}
1 & \ldots & 1 \\
\kappa_1 & \ldots & \kappa_N \\
\vdots & \ddots & \vdots \\
\kappa_{N_a-1} & \ldots & \kappa_{N_a-1}^N
\end{pmatrix}, \quad V' = \begin{pmatrix}
1 & \ldots & \kappa_{N_a-1}^N \\
\vdots & \ddots & \vdots \\
1 & \ldots & \kappa_{N_a-1}^N
\end{pmatrix}, \tag{2.7}
\]
that obey condition of ‘orthogonality’
\[
V r^{-1} V' = 0. \tag{2.8}
\]
This relation follows from the trivial equality
\[
\sum_{n=1}^{N} \frac{\kappa_n^{m-1}}{r_n} = \delta_{mN'}, \quad m = 1, \ldots, N. \tag{2.9}
\]
that coincides with (2.8) for \( m = 1, \ldots, N - 1 \).
Finally, let $C$ be a $N \times N_b$ real constant matrix. For maximal minors of the rectangular matrices introduced above we use notations of the kind

$$
V(\{n\}) = \det \begin{pmatrix} 1 & \ldots & 1 \\ \kappa_{n_1} & \ldots & \kappa_{n_{N_b}} \\ \vdots & \vdots & \vdots \\ \kappa_{N_1} & \ldots & \kappa_{N_b-1} \end{pmatrix} = \prod_{1 \leq i < j \leq N_b} (\kappa_{n_i} - \kappa_{n_j}), \quad (2.10)
$$

$$
C(\{n\}) = \det \begin{pmatrix} C_{n_1,1} & \ldots & C_{n_1,N_b} \\ \vdots & \vdots & \vdots \\ C_{N_b,1} & \ldots & C_{N_b,N_b} \end{pmatrix}, \quad (2.11)
$$

where $\{n_1, \ldots, n_{N_b}\} \subset \{1, \ldots, N\}$. In these terms we impose on matrix $C$ the following condition.

**Condition 2.1.** For every $n$, $1 \leq n \leq N_b$, there exists a subset $\{n_1, \ldots, n_{N_b-1}\}$ of $\{1, \ldots, N\}$ such that

$$
C(n, n, \ldots, n_{N_b-1}) \neq 0. \quad (2.12)
$$

Let $C'$ be a constant nonzero $N_a \times N$ matrix that is ‘orthogonal’ to the matrix $C$ in the sense that in analogy to (2.8) it obeys

$$
C' rC = 0, \quad (2.13)
$$

where zero in the rhs is the $N_a \times N_b$ matrix. In appendix it is shown that thanks to (2.1) matrix $C'$ exists and obeys condition above. It is also shown there that matrices $C$ and $C'$ are complementary in the sense that for any $N$-vector $w$ there exist such $N_b$-vector $w_b$ and $N_a$-vector $w_a$ that

$$
w = C w_b + r C'^T w_a, \quad (2.14)
$$

where $T$ denotes transposition.

In these terms the $\tau$-functions of the $(N_a, N_b)$-soliton solution of the KPII equation are given by determinants

$$
\tau(x) = \det \left( V e^{\ell x} C \right), \quad \tau'(x) = \det \left( C' e^{-\ell x} V' \right), \quad (2.15)
$$

that read as

$$
\tau(x) = \sum_{1 \leq n_1 < \ldots < n_{N_a} \leq N} C(\{n\}) V(\{n\}) \prod_{j=1}^{N_b} e^{\ell_{n_j} x}, \quad (2.16)
$$

$$
\tau'(x) = \sum_{1 \leq n_1 < \ldots < n_{N_b} \leq N} C'(\{n\}) V'(\{n\}) \prod_{j=1}^{N_a} e^{-\ell_{n_j} x}
$$

by means of the Binet–Cauchy formula. The $(N_a, N_b)$-soliton solution is given by the standard equalities

$$
u(x) = -2 \partial_x^2 \log \tau(x) \equiv -2 \partial_x^2 \log \tau'(x), \quad (2.17)
$$

where both expressions coincide as these $\tau$-functions are proportional (see appendix):
\( \tau(x) = \text{const} \cdot \left( \prod_{n=1}^{N} e^{\ell_n x} \right) \tau'(x) \).

Relation (2.16) also clarifies the meaning of condition 2.1: if there exists such \( n_0 \) that for any set \( n_1 < \ldots < n_{N_b-1} \) the maximal minor \( C(n_0, n_1, \ldots, n_{N_b-1}) = 0 \) then \( \tau(x) \) is independent of \( \tau_{n_0} \). Thus in this case we have either \((N_a - 1, N_b)_-\), or \((N_a, N_b-1)_-\)-soliton solution given by \( \mathcal{N} - 1 \) parameters \( \tau_x \). Thanks to (2.17) the same is valid for the matrix \( C' \).

Relation (2.18) demonstrates the well known fact that the potential \( u(x) \) in the pure soliton case is invariant under transformations

\[
C \to Cc, \quad C' \to c'C',
\]

where \( c \) and \( c' \) are arbitrary nondegenerate matrices of size \( N_b \times N_b \) and \( Na \times Na \) correspondingly. Thus soliton solutions of the \( \text{KPII} \) equation are parametrized by points of a real Grassmanian \( \text{Gr}_{N_a,N_b} \), or \( \text{Gr}_{N_a,N_a} \).

In what follows we also need the \( N \)th order polynomial

\[
R(\lambda) = \prod_{n=1}^{N} (\lambda - \tau_n),
\]

so that by (2.6):

\[
r_n = \left. \frac{dR(\lambda)}{d\lambda} \right|_{\lambda = \tau_n} \quad \text{and} \quad \sum_{n=1}^{N} \frac{\tau_n^j}{(\lambda - \tau_n)r_n} = \frac{\lambda^j}{R(\lambda)}, \quad j = 0, \ldots, N - 1.
\]

Let \( \tau_m = (-1)^m \sum_{1 \leq i_1 < \ldots < i_m \leq N} \tau_{i_1} \ldots \tau_{i_m} \), and \( s_0 = 1 \), i.e. let them denote the symmetric polynomials of \( -\tau_1, \ldots, -\tau_N \). Then we can also write \( R(\lambda) = \sum_{m=0}^{N} \lambda^{N-m}s_m \). Taking into account that any \( \tau_n \) is a root of \( R(\lambda) \) we get

\[
\frac{R(\lambda)}{\lambda - \tau_n} = \sum_{j=0}^{N-1-j} R_{N-1-j}(\lambda) \tau_n^j,
\]

where

\[
R_j(\lambda) = \sum_{i=0}^{j} \lambda^{N-i} s_i \equiv \left( \frac{R(\lambda)}{\lambda^{N-j}} \right)_+.
\]

It is easy to see that \( R(\lambda) = \lambda^{N-j} R_j(\lambda) = \sum_{i=j+1}^{N} \lambda^{N-i} s_i \), i.e. it equals a polynomial of order \( N - j - 1 \).

3. Cauchy–Jost function

Let us consider the case where the number \( k \) of times being switched on is not less than 3, so we write \( x = (x_1, \ldots, x_k) \) and, see (2.3), \( \ell(\lambda)x = \sum_{j=1}^{k} \lambda^j x_j \). Then using the above notation we set the following definition.

\textbf{Definition 3.1.} We say that a function \( F(x, \lambda, \mu) \), where \( \lambda \) and \( \mu \) are in \( \mathbb{C} \), is the Cauchy–Jost function of a \((N_a, N_b)\)-soliton solution of the \( \text{KPII} \) equation if the function

\[
f(x, \lambda, \mu) = e^{(\ell(\lambda) - \ell(\mu))x} F(x, \lambda, \mu)
\]

(3.1)
is such that the product \((\lambda - \mu)f(x, \lambda, \mu)\) is a polynomial of order \(N_b\) with respect to \(\mu\), if
\[
\text{res}_{\mu=\lambda} f(x, \lambda, \mu) = R(\lambda);
\]  
(3.2)
and if for some \(\mathcal{N} \times N_b\) (see (2.2)) constant matrix \(C\) obeying condition 2.1 we have that
\[
\sum_{m=1}^{\mathcal{N}} F(x, \lambda, \kappa_m) C_{mk} = 0, \quad k = 1, \ldots, N_b.
\]  
(3.3)

Then we prove that

**Theorem 3.1.** Function \((\lambda - \mu)f(x, \lambda, \mu)\) is an entire function of \(\lambda\) for any \(x\) such that \(\tau'(x) \neq 0\) (see (2.15)) and, for any \(\mu \in \mathbb{C}\), product \((\lambda - \mu)f(x, \lambda, \mu)\) is a polynomial of order \(N_a\) with respect to \(\lambda\). Values of the function \(F(x, \lambda, \mu)\) at points \(\lambda = \kappa_m\) obey
\[
\sum_{l=1}^{\mathcal{N}} C'_{jl} F(x, \kappa_l, \mu) = 0, \quad j = 1, \ldots, N_a,
\]  
(3.4)
where \(C'\) is matrix defined in (2.13). Function \(f(x, \lambda, \mu)\) has representation
\[
f(x, \lambda, \mu) = \frac{R(\mu)}{\mu - \lambda} - \sum_{j=1}^{N_a} \lambda^{j-1} \sum_{m=1}^{\mathcal{N}} \left( (C' e^{-\ell x} V')^{-1} (C' e^{-\ell x}) \right)_{jm} \frac{R(\mu)}{\mu - \kappa_m}\]
(3.5)
\[
= \frac{R(\lambda)}{\lambda - \mu} + \sum_{n=1}^{\mathcal{N}} \frac{R(\lambda)}{\lambda - \kappa_n} \sum_{j=1}^{N_a} (e^{\ell x} C V^e \ell x)^{-1} n_{j} \mu^{j-1}.
\]  
(3.6)

**Proof.** Taking into account that by the stated condition product \((\mu - \lambda)f(x, \lambda, \mu)\) is a polynomial with respect to \(\mu\) we can write
\[
(\mu - \lambda)f(x, \lambda, \mu) = \sum_{m=1}^{\mathcal{N}} f(x, \lambda, \kappa_m) \frac{(\kappa_m - \lambda)R(\mu)}{(\mu - \kappa_m)r_m},
\]  
(3.7)
where we used (2.21) and (2.22). Then by (3.2) we get
\[
\sum_{m=1}^{\mathcal{N}} f(x, \lambda, \kappa_m) r_m = -1.
\]  
(3.8)

On the other side, polynomial in (3.7) is of order \(\mathcal{N} - 1\), while by the stated condition it must be of the order \(N_b < \mathcal{N}\), see (2.2). Thus by (2.23) (with \(\lambda\) substituted by \(\mu\)) we have that
\[
\sum_{m=1}^{\mathcal{N}} f(x, \lambda, \kappa_m) \frac{(\kappa_m - \lambda)\kappa_m^j}{r_m} = 0, \quad 0 \leq j \leq N_a - 2,
\]
where it is assumed that \(N_a \geq 2\). Thanks to (3.8) this means that
\[
\sum_{m=1}^{\mathcal{N}} f(x, \lambda, \kappa_m) \frac{\kappa_m^j}{r_m} = -\lambda^j, \quad 0 \leq j \leq N_a - 1,
\]
that is valid also for $N_a = 1$. By means of notation (2.7) the latter equality can be written as
\[
\sum_{m=1}^N f(x, \lambda, \varkappa_m)(r^{-1}\mathcal{V}')_{mn} = -\lambda^{n-1}, \quad 1 \leq n \leq N_a.
\] (3.9)

Next, equation (3.3) in terms of function $f$ means that
\[
\sum_{m=1}^N f(x, \lambda, \varkappa_m)(e^{\xi}C)_{mn} = 0, \quad n = 1, \ldots, N_b,
\]
where (2.5) was used. Thus due to (2.14) there exists such $N_a$-vector $\tilde{f}_j(x, \lambda)$ that
\[
f(x, \lambda, \varkappa_m) = \sum_{j=1}^{N_a} \tilde{f}_j(x, \lambda) (C' e^{-\xi})^{-1}_{jm}.
\]

Inserting this equality in the previous one we have
\[
f(x, \lambda, \varkappa_m) = -\sum_{n=1}^{N_a} \lambda^{n-1} (e^{\xi}C' e^{-\xi})^{-1}_{nm},
\] (3.10)

Because of (3.8), equation (3.7) can be written in the form
\[
f(x, \lambda, \mu) = \frac{R(\mu)}{\lambda - \mu} \sum_{m=1}^N f(x, \lambda, \varkappa_m) \frac{R(\mu)}{r_m} + \sum_{m=1}^N f(x, \lambda, \varkappa_m) \frac{R(\mu)}{(\mu - \varkappa_m)r_m}.
\]

that gives (3.5) due to (3.8) and (3.10).

This proves that $(\lambda - \mu)F(x, \lambda, \mu)$ is an entire function of $\mu$ for any $x$ (det $\tau'(x) \neq 0$) and any $\lambda \in \mathbb{C}$. Moreover, (3.5) shows that product $(\lambda - \mu)f(x, \lambda, \mu)$ is a polynomial of order $N_a$ and
\[
\text{res}_{\lambda=\mu} f(x, \lambda, \mu) = -R(\mu).
\]

In order to prove (3.4) we notice that by (3.5) and (2.7)
\[
f(x, \varkappa_l, \mu) = \frac{R(\mu)}{\mu - \varkappa_l} - \sum_{m=1}^N (\mathcal{V}'(C' e^{-\xi}\mathcal{V}')^{-1}C' e^{-\xi})_{lm} \frac{R(\mu)}{\mu - \varkappa_m}.
\] (3.11)

Then direct summation gives $\sum_{l=1}^N C'_{\mu} e^{-\xi} f(x, \varkappa_l, \mu) = 0$, that thanks to (3.1) is nothing but (3.4). We have also an analog of equation (3.9) that sounds as
\[
\sum_{l=1}^N (\mathcal{V}'^{-1})_m f(x, \varkappa_l, \mu) = \mu^{m-1}, \quad 1 \leq m \leq N_b.
\] (3.12)
This follows if in the first term of (3.11) we use (2.22), while the second term cancels out thanks to (2.8).

We see that the properties of the function \( f(x, \lambda, \mu) \) are symmetric with respect to the variables \( \lambda \) and \( \mu \). Thus it is natural to expect that besides (3.5) it also has representation (3.6). Indeed, it is clear that this representation gives correctly the polynomial structure of \((\mu - \lambda)f(x, \lambda, \mu)\). Thus validity of (3.6) is equivalent to the condition that values

\[
f(x, \lambda, x_m) = -\frac{R(\lambda)}{\lambda - x_m} + R(\lambda) \sum_{i=1}^{N} \frac{1}{\lambda - x_i} \left(e^{\xi_i} C(\mathcal{V} e^{\xi_i} C)^{-1} \mathcal{V}\right)_{jm}
\]

(3.13)

obey (3.3) that follows by the direct summation and equation (3.1). Notice that the above relation is symmetric to (3.11). Let us mention, that (3.13) gives (3.9) by the arguments used to derive (3.12).

**Corollary 3.1.** One can exchange \( \lambda \) and \( \mu \), matrices \( C \) and \( C' \), relations (3.3) and (3.4), etc in definition 3.1 and in theorem 3.1.

**Remark 3.1.** Relations (3.10), (3.11) and (3.13) show that values \( f(x, x_m, \mu) \) and \( f(x, x_m, \lambda) \) are regular functions of \( \lambda \) and \( \mu \) thanks to (2.21) in spite of the pole behavior of \( f(x, \lambda, \mu) \) at \( \lambda = \mu \). Nevertheless, the pole behavior of \( f(x, \lambda, \mu) \) has essential consequence for the properties of these values. Thanks to (3.10) and (2.7) we have that \( f(x, \lambda, x_m)|_{\lambda=x_0} = -\left(V'(C'e^{-\xi}V')^{-1}C'e^{-\xi}\right) \). On the other side by (3.11) and (2.21) \( f(x, x_0, \mu)|_{\mu=x_0} = r_m \delta_{jm} - \left(V'(C'e^{-\xi}V')^{-1}C'e^{-\xi}\right) \), so that

\[
F(x, x_0, \mu)|_{\mu=x_0} = r_m \delta_{jm} + F(x, \lambda, x_m)|_{\lambda=x_0},
\]

(3.14)

where (3.1) was used.

4. Properties of the Cauchy–Jost function

Relations (3.5) and (3.6) can be written in the form

\[
\tau'(x) f(x, \lambda, \mu) = \frac{R(\mu) \tau'(x)}{\mu - \lambda} - \sum_{m=1}^{N} \sum_{i,j=1}^{N_\mu} (-1)^{j+i} \lambda^{j-1} \det(C_{j} e^{-\xi_j} C_{j})_{jm} \frac{e^{-\xi_j} R(\mu)}{\mu - x_m},
\]

\[
\tau(x) f(x, \lambda, \mu) = \frac{R(\lambda) \tau(x)}{\mu - \lambda} + \sum_{n=1}^{N} \frac{e^{\xi_n} R(\lambda)}{\lambda - x_n} \sum_{i,j=1}^{N_\mu} (-1)^{j+i} C_{m} \det(\mathcal{V} e^{\xi_n} C_{j})_{jm} \mu^{-1},
\]

that follow from (2.15) and where notation \( \mathcal{V}_j \) denotes matrix with removed \( j \)th row and \( j \)th column. Let also \( C_i \) and \( V_j \) denote matrices \( C' \) and \( V \) with removed \( i \)th row and \( C_i \) and \( V_j \) denote matrices \( C \) and \( V \) with removed \( i \)th column. Now by using the Binet–Cauchy formula like in (2.16), (2.17) we get
\[
\tau'(x)f(x, \lambda, \mu) = \frac{R(\mu)\tau'(x)}{\mu - \lambda} - \sum_{m=1}^{N} \frac{e^{-\ell_{mx}}R(\mu)}{\mu - \varsigma_m} \sum_{i,j=1}^{N} (-1)^{i+j}\lambda^{i-1} C_{im}^r
\]
\[
\times \sum_{1 \leq n_1 < \ldots < n_{N_\lambda} \leq N} C_i'(n_1, \ldots, n_{N_\lambda-1}) V_j'(n_1, \ldots, n_{N_\lambda-1}) \prod_{k=1}^{N_\lambda-1} e^{-\ell_{mk}},
\]
\[
\tau(x)f(x, \lambda, \mu) = \frac{R(\lambda)\tau(x)}{\mu - \lambda} + \sum_{n=1}^{N} \frac{e^{\ell_{nx}}R(\lambda)}{\lambda - \varsigma_n} \sum_{i,j=1}^{N} (-1)^{i+j}\mu^{i-1} C_{nm} \mu^{j-1}
\]
\[
\times \sum_{1 \leq n_1 < \ldots < n_{N_\mu} \leq N} C_i(n_1, \ldots, n_{N_\mu-1}) V_j(n_1, \ldots, n_{N_\mu-1}) \prod_{k=1}^{N_\mu-1} e^{\ell_{nk}},
\]

Taking the standard relations of the kind
\[
\sum_{i=1}^{N_\lambda} (-1)^{i} C_{mi} C_i(n_1, \ldots, n_{N_\lambda-1}) = -C(m, n_1, \ldots, n_{N_\lambda-1}),
\]
\[
\sum_{j=1}^{N_\mu} (-1)^{j}\lambda^{j-1} V_j'(n_1, \ldots, n_{N_\mu-1}) = -\prod_{j=1}^{N_\mu} (\varsigma_{n_j} - \lambda) V_j'(n_1, \ldots, n_{N_\mu-1})
\]
into account we rewrite the above equalities in the form
\[
\tau'(x)f(x, \lambda, \mu) = \frac{R(\mu)\tau'(x)}{\mu - \lambda} - R(\mu) \sum_{m=1}^{N} \frac{e^{-\ell_{mx}}}{\mu - \varsigma_m} \sum_{i,j=1}^{N} (-1)^{i+j}\lambda^{i-1} C_{im}^r
\]
\[
\times \sum_{1 \leq n_1 < \ldots < n_{N_\lambda} \leq N} C_i'(n_1, \ldots, n_{N_\lambda-1}) V_j'(n_1, \ldots, n_{N_\lambda-1}) \prod_{k=1}^{N_\lambda-1} e^{-\ell_{mk}} (\varsigma_{n_k} - \lambda),
\]
\[
\tau(x)f(x, \lambda, \mu) = \frac{R(\lambda)\tau(x)}{\mu - \lambda} + R(\lambda) \sum_{n=1}^{N} \frac{e^{\ell_{nx}}}{\lambda - \varsigma_n} \sum_{i,j=1}^{N} (-1)^{i+j}\mu^{i-1} C_{nm} \mu^{j-1}
\]
\[
\times \sum_{1 \leq n_1 < \ldots < n_{N_\mu} \leq N} C_i(n_1, \ldots, n_{N_\mu-1}) V_j(n_1, \ldots, n_{N_\mu-1}) \prod_{k=1}^{N_\mu-1} e^{\ell_{nk}} (\varsigma_{n_k} - \mu).
\]

We introduce two new functions, \(\tau(x, \lambda, \mu)\) and \(\tau'(x, \lambda, \mu)\) in such a way that
\[
\frac{f(x, \lambda, \mu)}{\mu - \lambda} = \frac{\tau(x, \lambda, \mu)}{\mu - \lambda} = \frac{\tau'(x, \lambda, \mu)}{\mu - \lambda} \tau'(x).
\]
so that due to the above
\[
\frac{\tau'(x, \lambda, \mu)}{R(\mu)} = \tau'(x) - (\mu - \lambda) \sum_{m=1}^{N} e^{\ell_{x,m}} \prod_{k=1}^{N} e^{-\ell_{x,m}(x_{n_k} - \lambda)} \\
\times \sum_{1 \leq n_1 < \ldots < n_{N-1} \leq N} C'(m, n_1, \ldots, n_{N-1}) V'(n_1, \ldots, n_{N-1}) \prod_{k=1}^{N-1} e^{-\ell_{x,m}(x_{n_k} - \lambda)}.
\]

\[
\frac{\tau(x, \lambda, \mu)}{R(\lambda)} = \tau(x) + (\mu - \lambda) \sum_{n=1}^{N} e^{\ell_{x,n}} \prod_{k=1}^{N} e^{\ell_{x,n}(x_{n_k} - \mu)} \\
\times \sum_{1 \leq n_1 < \ldots < n_{N-1} \leq N} C(n, n_1, \ldots, n_{N-1}) V(n_1, \ldots, n_{N-1}) \prod_{k=1}^{N-1} e^{\ell_{x,n}(x_{n_k} - \mu)}.
\]

Thanks to (4.1) and definition 3.1 the ratio \(\frac{\tau'(x, \lambda, \mu)}{R(\mu)}\) decays as \(\mu^{-N}\) when \(\mu \to \infty\), and thanks to the theorem 3.1 ratio \(\frac{\tau(x, \lambda, \mu)}{R(\lambda)}\) decays as \(\lambda^{-N}\) when \(\lambda \to \infty\). Thus

\[
\tau(x, \lambda, \mu) = \sum_{1 \leq n_1 < \ldots < n_{N-1} \leq N} C([n_1]) V([n_1]) \prod_{j=1}^{N} e^{\ell_{x,j}(x_{n_j} - \mu)} \prod_{j=1}^{N} (\lambda - x_{n_j}),
\]

(4.2)

\[
\tau'(x, \lambda, \mu) = \sum_{1 \leq n_1 < \ldots < n_{N-1} \leq N} C'(([n_1]) V'([n_1]) \prod_{j=1}^{N} e^{\ell_{x,j}(x_{n_j} - \lambda)} \prod_{j=1}^{N} (\mu - x_{n_j}),
\]

(4.3)

as residuals of the rhs’s of these equalities coincide with residuals of the previous ones thanks to (2.10) and (2.21). Notice that these relations follow from expressions for \(\tau(x)\) and \(\tau'(x)\) in (2.16) and (2.17) under substitution \(e^{\ell_{x,n}} \to e^{\ell_{x,n} \frac{x_n - x_{n_k}}{x_n - \lambda}}\), that up to some unessential constants is nothing but a double Miwa shift in (1.5). Finally we get

\[
\tau(x, \lambda, \mu) = \sum_{1 \leq n_1 < \ldots < n_{N-1} \leq N} C([n_1]) V([n_1]) \prod_{n} e^{\ell_{x,n}(\mu - x_{n_0})} \prod_{\{n\}} (\lambda - x_{n_0}),
\]

(4.4)

\[
\tau'(x, \lambda, \mu) = \sum_{1 \leq n_1 < \ldots < n_{N-1} \leq N} C'(([n_1]) V'([n_1]) \prod_{\{n\}} e^{\ell_{x,n}(\lambda - x_{n_0})} \prod_{\{n\}} (\mu - x_{n_0}),
\]

(4.5)

where \([n]\) denotes set of indexes involved in summation and \([\bar{n}]\) denotes the set complementary to \([n]\) in \(\{1, \ldots, N\}\).

In virtue of (2.21) and (2.16), (2.17), relations (4.4) and (4.5) give

\[
\tau(x, \lambda, \lambda) = R(\lambda) \tau(x), \quad \tau'(x, \lambda, \lambda) = R(\lambda) \tau'(x).
\]

(4.6)

Jost solution \(\varphi(x, \mu)\) of the heat equation, Jost solution \(\psi(x, \lambda)\) of the dual heat equation and the soliton solution \(u(x)\) are given as

\[
\varphi(x, \mu) = \lim_{\lambda \to \infty} \lambda^{-N_{\mu} + 1} f(x, \lambda, \mu) e^{\ell(x)};
\]

(4.7)

\[
\psi(x, \lambda) = \lim_{\mu \to \infty} \mu^{-N_{\mu} + 1} f(x, \lambda, \mu) e^{-\ell(x)};
\]

(4.8)
\[ u(x) = -2 \lim_{\lambda, \mu \to \infty} \lambda^{-N_\mu + 1} \mu^{-N_\mu + 1} f_{\mu}(x, \lambda, \mu), \quad \text{(4.9)} \]

so that thanks to (4.1)-(4.5):

\[ \varphi(x, \mu) = \frac{e^{-\mu \lambda x}}{\tau(x)} \sum_{1 \leq n_1 < \ldots < n_N \leq N} C(\{n_i\}) V(\{n_i\}) \prod \mathcal{e}^{\lambda \gamma n}\lambda - x_n, \quad \text{(4.10)} \]

\[ \psi(x, \lambda) = \frac{e^{-\ell \lambda x}}{\tau(x)} \sum_{1 \leq n_1 < \ldots < n_N \leq N} C'(\{n_i\}) V'(\{n_i\}) \prod \mathcal{e}^{-\mu \gamma n}(\lambda - x_n), \quad \text{(4.11)} \]

while for the potential \( u(x) \) we rederive (2.18). It is worth to mention that in the pure solitonic case the Jost solutions themselves can be defined by means of the obvious analogs of definition 3.1. In order to avoid singularities with respect to the spectral parameter we normalized the Jost and dual Jost solutions in a way that \( \varphi(x, \mu)e^{\ell (\mu x)} \) is a polynomial of order \( N_\mu \) with respect to \( \mu \) and \( \psi(x, \lambda)e^{\ell (\lambda x)} \) is a polynomial of order \( N_\lambda \) with respect to \( \lambda \).

In order to get the evolutions of the Cauchy–Jost function with respect to times \( x_1, x_2, \ldots \), we differentiate (3.5):

\[ f_{\mu}(x, \lambda, \mu) = -R(\mu) \sum_{j=1}^{N_\mu} \lambda^{j-1} \sum_{m=1}^{N_\mu} \left\{ \sum_{n=1}^{N_\mu} \left( C' e^{-\ell \gamma n} e^{-\ell \gamma m} \right)_{n, m} \right\} \frac{1}{\mu - x_m}, \quad \text{(4.12)} \]

where (2.7) was used. By means of (3.10) and (3.5) this can be written as

\[ f_{\mu}(x, \lambda, \mu) = -\sum_{n=1}^{N_\mu} f(x, \lambda, x_n) \frac{x_n^k}{R_m} f(x, x_n, \mu). \quad \text{(4.12)} \]

Taking analyticity properties of \( f(x, \lambda, \mu) \) into account, we have

\[ f_{\mu}(x, \lambda, \mu) = f(x, \lambda, \mu)(\lambda^{k} - \mu^{k}) - \frac{1}{2\pi i} \oint_{\gamma} \frac{d\nu \nu^k}{R(\nu)} f(x, \lambda, \nu) f(x, \nu, \mu), \]

where we used (2.22) and where contour \( \gamma \) encircles all \( x_1, \ldots, x_N \), \( \lambda \) and \( \mu \). Because of (2.3) and (3.1) this gives

\[ F_{\mu}(x, \lambda, \mu) = -\frac{1}{2\pi i} \oint_{\gamma} \frac{d\nu \nu^k}{R(\nu)} F(x, \lambda, \nu) F(x, \nu, \mu). \quad \text{(4.13)} \]

In particular, thanks to (2.2), (2.21) and (4.7), (4.8), we derive (1.3):

\[ F_{\mu}(x, \lambda, \mu) = \psi(x, \lambda) \varphi(x, \mu). \quad \text{(4.14)} \]

5. Darboux transformations

5.1. Direct transformations

We start with the \((N_\mu, N_\lambda)\)-soliton potential and denote by tilde all Darboux transformed objects. Taking into account that the soliton solutions are parametrized by two discrete parameters, \( N_\mu \) and \( N_\lambda \), we can single out three types of Darboux transformations: A-transformation,
Theorem 5.1. Darboux transformed Cauchy–Jost function \( \tilde{F}(x, \lambda, \mu) \) is given as solution of the following equations:

**A-transformation:**

\[
\frac{\tilde{F}(x, \lambda, \mu)}{R(\lambda)} = \frac{F(x, \lambda, \mu)}{R(\lambda)} - \frac{F(x, \lambda, \kappa_{N+1}) \tilde{F}(x, \kappa_{N+1}, \mu)}{\tilde{r}_{N+1}},
\]

and auxiliary condition:

\[
\sum_{i=1}^{N+1} a_i \tilde{F}(x, \kappa_i, \mu) = 0,
\]

where \( a_1, \ldots, a_{N+1} \) are arbitrary real parameters, \( a_{N+1} \neq 0 \);

**B-transformation:**

\[
\frac{\tilde{F}(x, \lambda, \mu)}{R(\mu)} = \frac{F(x, \lambda, \mu)}{R(\mu)} + \frac{\tilde{F}(x, \lambda, \kappa_{N+1}) F(x, \kappa_{N+1}, \mu)}{\tilde{r}_{N+1}},
\]

and auxiliary condition:

\[
\sum_{n=1}^{N+1} \tilde{F}(x, \lambda, \kappa_n) \frac{b_n}{r_n} = 0,
\]

where \( b_1, \ldots, b_{N+1} \) are arbitrary real parameters, \( b_{N+1} \neq 0 \); and

**0-transformation:**

\[
\tilde{F}(x, \lambda, \mu) = F(x, \lambda, \mu) - \sum_{i=1}^{N} \tilde{F}(x, \lambda, \kappa_i) b_i \sum_{j=1}^{N} a_j F(x, \kappa_j, \mu),
\]

where \( a_1, \ldots, a_{N} \) and \( b_1, \ldots, b_{N} \) are arbitrary real parameters such that

\[
\text{arb} \equiv \sum_{n=1}^{N} a_n r_n b_n \neq -1.
\]

Remark 5.1. Cauchy–Jost functions constructed by these means can have singularities with respect to \( x \)-variables. We discuss this problem below.
Proof for the $\Lambda$-transformation. Let $\tilde{f}$ be defined by $\tilde{F}$ as in (3.1). By (5.1) we can rewrite (5.3) as

$$\tilde{f}(x, \lambda, \mu) = (\lambda - x_{N+1}) \left\{ f(x, \lambda, \mu) - \frac{f(x, \lambda, x_{N+1})}{R(x_{N+1})} \tilde{f}(x, x_{N+1}, \mu) \right\}, \quad (5.9)$$

that proves (3.2) for the transformed $\tilde{f}$ and proves $\tilde{f}$ to be a polynomial of order $Na + 1$ with respect to $\lambda$, if it exists. Reduction of (5.3) gives

$$\tilde{F}(x, x_{l}, \mu) = \frac{F(x, x_{l}, \mu)}{r_{l}} - \frac{F(x, x_{l}, x_{N+1})}{r_{l}} \tilde{F}(x, x_{N+1}, \mu) \quad (5.10)$$

for any $l = 1, \ldots, N$. But in virtue of (3.2) the value $\tilde{f}(x, x_{N+1}, \mu)$ is not defined by this equality as for $\lambda = x_{N+1}$ it gives an identity. This remark motivates the necessity of condition (5.4) with $a_{N+1}$ to be different from zero.

By means of (3.4) we get from (5.10) that

$$\sum_{l=1}^{N} C'_{jl} r_{l} \tilde{F}(x, x_{l}, \mu) = 0, \quad j = 1, \ldots, N_a. \quad (5.11)$$

Matrix $\tilde{C}'$ corresponding to the transformed case must be of size $(Na + 1) \times (N + 1)$, so we use (5.4) to sum up (5.10) from 1 to $N$, that gives

$$\tilde{F}(x, x_{N+1}, \mu) = \frac{\sum_{l=1}^{N} a_{l} F(x, x_{l}, \mu)}{\sum_{l=1}^{N} a_{l} F(x, x_{l}, x_{N+1}) - a_{N+1}}. \quad (5.12)$$

Inserting this in the rhs of (5.3) we get

$$\tilde{F}(x, \lambda, \mu) = (\lambda - x_{N+1})$$

$$\times \left\{ F(x, \lambda, \mu) - \frac{F(x, \lambda, x_{N+1}) \sum_{l=1}^{N} a_{l} F(x, x_{l}, \mu)}{\sum_{l=1}^{N} a_{l} F(x, x_{l}, x_{N+1}) - a_{N+1}} \right\}, \quad (5.13)$$

that specializes transformations introduced in [1, 2], if one differentiates both sides by $x_{1}$ and factorizes the result in the product of $\lambda$- and $\mu$-dependent functions due to (4.14).

In order to complete the proof we have to show that there exists a matrix $\tilde{C}'$ that obeys tilde version of relation (3.4). This matrix is given, in fact, by equality (5.11) and auxiliary condition (5.4). Expression for the matrix $\tilde{C}'$ obeying tilde analog of (2.13), $\tilde{C}'\tilde{r} = 0$, follows by (5.13) and (3.3). Thus

$$\tilde{C} = \left( \begin{array}{cc} C & a_{N+1} \\ -a_{N+1} & C' \end{array} \right), \quad \tilde{C}'_{\tilde{r}} = \left( \begin{array}{cc} C' r & 0 \\ a & a_{N+1} \end{array} \right), \quad (5.14)$$

where we introduced row
\[ a = (a_1, \ldots, a_N), \]  

and where 0 denotes zero \( N \)-row.

**Proof for the \( \beta \)-transformation is close to the above.** By (3.1) and (5.1) we rewrite (5.5) as

\[
\tilde{f}(x, \lambda, \mu) = (\mu - x_{N+1}^b) \left\{ f(x, \lambda, \mu) + \frac{\tilde{f}(x, \lambda, x_{N+1})}{R(x_{N+1})} f(x, x_{N+1}, \mu) \right\},
\]

that proves (3.2) for the transformed \( \tilde{f} \) and proves it to be a polynomial of order \( N_b + 1 \) with respect to \( \mu \). Value \( \tilde{f}(x, \lambda, x_{N+1}) \) is not defined by this equality, so we use (5.6) under condition \( b_{N+1}^b \neq 0 \). Taking into account that for \( m = 1, \ldots, N \) reduction of (5.5) gives

\[
\tilde{F}(x, \lambda, x_m) = \frac{F(x, \lambda, x_m) b_m}{r_m} - \frac{\sum_{m=1}^{N} F(x, \lambda, x_m) b_m}{r_m} + b_{N+1}^b r_m, \quad (5.17)
\]

we get thanks to (5.6)

\[
\tilde{F}(x, \lambda, x_{N+1}) = - \frac{\sum_{m=1}^{N} F(x, \lambda, x_m) b_m}{r_m} + b_{N+1}^b r_m, \quad (5.18)
\]

so that finally

\[
\tilde{F}(x, \lambda, \mu) = (\mu - x_{N+1}^b) \times \left\{ F(x, \lambda, \mu) - \frac{F(x, x_{N+1}^b, \mu) \sum_{m=1}^{N} F(x, \lambda, x_m) b_m}{r_m} \right\}. \quad (5.19)
\]

This equality being differentiated by \( x_1 \) and factorized with respect to \( \lambda \)- and \( \mu \)-dependence by (4.14) leads to the standard form of the binary Darboux transformation, [1, 2].

In order to complete the proof we have to show that there exist matrices \( \tilde{C} \) and \( \tilde{C}' \) that obey tilde versions of relations (3.3) and (3.4). By means of (3.3) we get from (5.17) that

\[
\sum_{m=1}^{N} \tilde{F}(x, \lambda, x_m) r_m C_{mj}^b = 0, \quad j = 1, \ldots, N_b, \quad (5.20)
\]

that together with (5.6) shows that these matrices equal

\[
\tilde{C} = \begin{pmatrix} r^c & b^c \\ 0 & b_{N+1}^c \end{pmatrix}, \quad \tilde{C}' = \begin{pmatrix} c^c, -c^b \\ b_{N+1}^c \end{pmatrix}, \quad (5.21)
\]

where we introduced column

\[
b = (b_1, \ldots, b_N)^T, \quad (5.22)
\]

and where 0 denotes zero \( N \)-row.
Proof for the 0-type transformation. Let us denote
\[ F(x, \xi, \zeta_m) = F(x, \xi, \mu)|_{\mu=\zeta_m}, \]
so that by (3.14)
\[ F(x, \lambda, \zeta_m)|_{\lambda=\xi_0} = F(x, \xi, \zeta_m) - r_0 \delta_{lm}. \]

Then setting \( \mu = \zeta_m \) in (5.7), multiplying it by \( a_m \) and summing up we get
\[ \sum_{m=1}^{N} \tilde{F}(x, \lambda, \zeta_m) b_m = \frac{\sum_{i=1}^{N} F(x, \lambda, \zeta_i) b_i}{\sum_{i=1}^{N} a_j F(x, \xi_j, \zeta_i) b_i + 1}. \]
so that by (5.7) we derive
\[ \tilde{F}(x, \lambda, \mu) = F(x, \lambda, \mu) \]
\[ = \frac{\sum_{i=1}^{N} F(x, \lambda, \zeta_i) b_i \sum_{j=1}^{N} a_j F(x, \xi_j, \mu) b_i}{\sum_{i=1}^{N} a_j F(x, \xi_j, \zeta_i) b_i + 1}. \]
Properties of \( \tilde{F}(x, \lambda, \mu) \) with respect to \( \lambda \) and \( \mu \) are determined by the properties of \( F(x, \lambda, \mu) \) and obviously coincide with them. By reductions of (5.26) and thanks to (5.23) and (5.24) we get
\[ \tilde{F}(x, \xi_0, \mu) = F(x, \xi_0, \mu) \]
\[ = \frac{\sum_{i=1}^{N} (F(x, \xi_0, \zeta_i) - r_0 \delta_{i0}) b_i \sum_{j=1}^{N} a_j F(x, \xi_j, \mu) b_i}{\sum_{i=1}^{N} a_j F(x, \xi_j, \zeta_i) b_i + 1}, \]
\[ \tilde{F}(x, \xi_0, \mu)|_{\mu=\zeta_m} = F(x, \xi_0, \zeta_m) \]
\[ = \frac{\sum_{i=1}^{N} (F(x, \xi_0, \zeta_i) - r_0 \delta_{i0}) b_i \sum_{j=1}^{N} a_j F(x, \xi_j, \zeta_m) b_i}{\sum_{i=1}^{N} a_j F(x, \xi_j, \zeta_i) b_i + 1}, \]
\[ \tilde{F}(x, \lambda, \zeta_m) = F(x, \lambda, \zeta_m) \]
\[ = \frac{\sum_{i=1}^{N} F(x, \lambda, \xi_0) b_i \sum_{j=1}^{N} a_j F(x, \xi_j, \zeta_m) b_i}{\sum_{i=1}^{N} a_j F(x, \xi_j, \zeta_i) b_i + 1}, \]
\[ \tilde{F}(x, \lambda, \zeta_m)|_{\lambda=\xi_0} = F(x, \lambda, \zeta_m) - r_0 \delta_{jm} \]
\[ = \frac{\sum_{i=1}^{N} (F(x, \lambda, \zeta_i) - r_0 \delta_{i0}) b_i \sum_{j=1}^{N} a_j F(x, \xi_j, \zeta_m) b_i}{\sum_{i=1}^{N} a_j F(x, \xi_j, \zeta_i) b_i + 1}. \]

Relations (5.28) and (5.30) prove that \( \tilde{F}(x, \lambda, \mu) \) obeys
\[ \tilde{F}(x, \xi_0, \mu)|_{\mu=\zeta_m} = \tilde{F}(x, \lambda, \zeta_m)|_{\lambda=\xi_0} + m \delta_{jm}, \]
i.e. the tilde analog of relation (3.14) for \( F(x, \lambda, \mu) \) with the same matrix \( r \). Next, by (5.27)
where notation (5.8) was used. On the other side, applying $C'$ from the left to (5.27), we get by (3.4)

$$
\sum_{j=1}^{N} C'_{jl} \tilde{F}(x, \kappa_l, \mu) = \sum_{j=1}^{N} C'_{jl} r_j b_j = \sum_{i,j=1}^{N} a_j F(x, \kappa_j, \mu) \sum_{i,j=1}^{N} a_i F(x, \kappa_i, \mu) b_i + 1
$$

that, thanks to condition (5.8) and (5.31), proves the tilde version of relation (3.4), where

$$
\tilde{C}' = C'(E_{N \times N} - rb \otimes a) \frac{1}{1 + (arb)}
$$

and where we used notations (5.15) and (5.22) for row $a$ and column $b$.

Next, we apply matrix $C$ to (5.29) from the right. Notice that due to (5.23) and (5.24) equality (3.3) gives

$$
\sum_{m=1}^{N} F(x, \kappa_l, \kappa_m) C_{mj} = r_j C_{lj}.
$$

Then like above we derive that

$$
\tilde{C} = (E_{N \times N} + b \otimes ar) C.
$$

Thus function $\tilde{F}(x, \lambda, \mu)$ given by (5.7) obeys all properties of the Cauchy–Jost function in definition 3.1. Taking into account that

$$
(E_{N \times N} - \frac{rb \otimes a}{1 + (arb)}) r(E_{N \times N} + b \otimes ar) = r,
$$

we see that new matrices $\tilde{C}$ and $\tilde{C}'$ obey relation (2.13): $\tilde{C}' r \tilde{C} = 0$.

Thanks to (2.14) we can write column $b$ and row $a$ as

$$
b = C v' + r C^T v, \quad a = w' C + w C^T r^{-1},
$$

so that thanks to (2.13) $C' rb = C' r^2 C^T v, ar C = w C^T C$ and $arb = w' C' r^2 C^T v + w C^T C v'$. This means that without loss of generality one can set $w' = v' = 0$. Then $arb = 0$ and we get

$$
\tilde{C} = (E_{N \times N} + r C^T v \otimes w C^T) C,
$$

$$
\tilde{C}' r = C' r (E_{N \times N} - r C^T v \otimes w C^T)
$$

that means that 0-transformation is parametrized by $N_j$-column $v$ and $N_a$-row $w$.

5.2. Inverse transformations

Let now deal with the Cauchy–Jost function $\tilde{F}$ parametrized by $\tilde{N}_a = N_a + 1, \tilde{N}_b = N_b$, parameters $\kappa_1, \ldots, \kappa_{N+1}$ and by matrix $C$ (or $C'$) as (see (3.3), (3.4))....
\[
\sum_{m=1}^{N+1} \tilde{F}(x, \lambda, x_m) \tilde{C}_{mi} = 0, \quad i = 1, \ldots, N_b, \\
\sum_{j=1}^{N+1} \tilde{C}_{ji} \tilde{F}(x, \mu, \lambda) = 0, \quad j = 1, \ldots, N_a + 1.
\] (5.37)

Then the same relation (5.3) supplies us with the transformation inverse to the considered above \(A\)-transformation, i.e., gives a function \(\tilde{F}(x, \lambda, \mu)\) with parameters \(N_a\) and \(N_b\) with \(x_{N+1}\) omitted. First, in analogy to (5.23) we denote

\[\tilde{\tilde{F}}(x, \lambda, x_{N+1}, \mu)\]

so that thanks to remark 3.1 and the tilde-version of (3.14) we have

\[\tilde{\tilde{F}}(x, \lambda, x_{N+1})|_{\mu=x_{N+1}} = \tilde{F}(x) - \tilde{r}_{N+1}.
\] (5.39)

Then equality (5.3) at \(\mu = x_{N+1}\) reduces to

\[\frac{F(x, \lambda, x_{N+1})}{R(\lambda)} = \frac{\tilde{r}_{N+1} \tilde{\tilde{F}}(x, \lambda, x_{N+1})}{R(\lambda) \tilde{F}(x) - r_{N+1}},
\] (5.40)

so that (5.3) gives explicitly the transformed function \(F(x, \lambda, \mu)\):

\[F(x, \lambda, \mu) = \frac{1}{\lambda - x_{N+1}} \left[ \frac{\tilde{F}(x, \lambda, \mu)}{\tilde{F}(x) - r_{N+1}} \right].
\] (5.41)

In order to find matrices \(C\) and \(C'\) obeying (3.3) and (3.4) notice that thanks to the first equalities in (5.37) and (5.38) we have by (5.41)

\[\sum_{m=1}^{N} F(x, \lambda, x_m) \tilde{C}_{mi} = 0, \quad i = 1, \ldots, N_b.
\] (5.42)

for all \(i = 1, \ldots, N_b\). Thus \(C\) equals matrix \(\tilde{C}\) with removed last row. Next, let us consider values \(F(x, \lambda, \mu)\) for \(l = 1, \ldots, N\) as given by (5.41). To sum up them we write the second equality in (5.37) in the form \(\sum_{i=1}^{N} \tilde{C}_{ji} \tilde{F}(x, \lambda, x_i) = -\tilde{C}_{j,N+1} \tilde{F}(x, x_{N+1}, \mu)\), so that \(\sum_{i=1}^{N} \tilde{C}_{ji} \tilde{F}(x, \lambda, x_{N+1}) = -\tilde{C}_{j,N+1} \tilde{F}(x)\) thanks to (5.38). Then

\[\sum_{j=1}^{N} \tilde{C}_{ji} \tilde{F}(x, \lambda, \mu) = -\tilde{r}_{N+1} \tilde{C}_{j,N+1} \tilde{F}(x) - r_{N+1},
\] (5.43)

where \(j = 1, \ldots, N_a + 1\). Thus in order to get zero in the rhs for \(j = 1, \ldots, N\), we have first to reduce matrix \(C'\) by means of (2.20) to the form where the last column has all zeros with exception to the last element. Then matrix \(C'\) is given by omitting the last column and last row of matrix \(C'\) and by multiplying the diagonal matrix \(\text{diag}\{r_i/\tilde{r}_i\}\) from the right.

Derivation of the \(B^{-1}\)-transformation is performed in the same way. We assume that \(\tilde{F}(x, \lambda, \mu)\) with parameters \(x_1, \ldots, x_{N+1}\) \((\tilde{N}_a = N_a, \tilde{N}_b = N_b + 1)\) and matrix \(\tilde{C}\) (or \(\tilde{C}'\)) is given and we use (5.5) to define \((\tilde{N}_a, \tilde{N}_b)\)-soliton function \(\tilde{F}(x, \lambda, \mu)\) with parameters \(x_1, \ldots, x_{N}\). Setting in (5.5) \(\lambda = x_{N+1}\) we get due to (5.38)

\[\frac{F(x, x_{N+1}, \mu)}{R(\mu)} = \frac{\tilde{F}(x, x_{N+1}, \mu) / \tilde{r}_{N+1}}{R(\lambda) / \tilde{F}(x)},
\] (5.44)
so that under this substitution (5.5) reads as

\[
\frac{F(x, \lambda, \mu)}{R(\mu)} = \frac{1}{R(\mu)} \left[ \tilde{F}(x, \lambda, \mu) - \frac{\tilde{F}(x, \lambda, \kappa_{N+1}) \tilde{F}(x, \kappa_{N+1}, \mu)}{F(x)} \right],
\]

(5.45)

We see that the pole at \( \mu = \kappa_{N+1} \) in the rhs (see (5.1)), thanks to (5.38), is compensated by a zero of numerator at this point, that proves that \( \kappa_{N+1} \) is excluded from parameters of the function \( F \). Now for \( l, m = 1, \ldots, N \)

\[
\frac{F(x, \lambda, \mu)}{R(\mu)} = \frac{1}{R(\mu)} \left[ \tilde{F}(x, \lambda, \kappa_l, \mu) - \frac{\tilde{F}(x, \lambda, \kappa_{N+1}) \tilde{F}(x, \kappa_{N+1}, \mu)}{F(x)} \right],
\]

(5.46)

\[
\frac{F(x, \lambda, \kappa_m)}{r_m} = \frac{1}{r_m} \left[ \tilde{F}(x, \lambda, \mu) - \frac{\tilde{F}(x, \lambda, \kappa_{N+1}) \tilde{F}(x, \kappa_{N+1}, \mu)}{F(x)} \right],
\]

(5.47)

where (2.22) and its tilde version were used. Thus, thanks to the tilde version of (3.4) we have that

\[
\sum_{i=1}^{N} \tilde{C}_{i}^j F(x, \lambda, \mu) = 0, \quad j = 1, \ldots, N_a,
\]

(5.48)

that means that

\[
C' = \| \tilde{C}_{i}^j \|_{i=1}^{N} = 0,
\]

(5.49)

i.e. the transformed matrix \( C' \) equals matrix \( \tilde{C}' \) with the last column omitted. Next, by (5.47) and tilde version of (3.3)

\[
\sum_{m=1}^{N} F(x, \lambda, \kappa_m) \tilde{r}_m \tilde{C}_{mj} = - \frac{\tilde{r}_{N+1} \tilde{F}(x, \lambda, \kappa_{N+1}) \tilde{C}_{N+1,j}}{F(x)},
\]

(5.50)

where \( j = 1, \ldots, N_b + 1 \). By means of (2.20) we reduce matrix \( \tilde{C} \) to the form where the last row equals \((0, \ldots, 0, 1)\). Then the transformed matrix \( C \) obeying (3.3) is given by means of relation

\[
rC = \| (\tilde{rC})_g \|_{i=1}^{N_b},
\]

(5.51)

where we used notation (2.6). Thus the transformed matrix \( rC \) equals to the reduced as above matrix \( \tilde{C} \) with the last column and last row being omitted. It is easy to see that matrices \( C \) and \( C' \) obey (2.13) because matrices \( \tilde{C} \) and \( \tilde{C}' \) obey the tilde version of this equality.

6. Discussion

In [5] we used the KPII equation to demonstrate that the Cauchy–Jost function is the very natural tool for formulation and investigation of the Inverse problem. Here this approach is developed by including the case of soliton solutions. In definition 3.1 the Cauchy–Green function was defined directly by its analyticity properties with respect to the spectral parameters. We proved that the standard relation (1.3), specific for the KPII case, results from this definition. We presented properties of the Cauchy–Jost function in detail and gave linear equations defining Darboux transformations in its terms. We proved that these transformations are naturally decomposed in three classes: transformations of \( A, B \) and \( 0 \) types, given by (5.3), (5.5) and
(5.7) with auxiliary conditions (5.4), (5.6) and (5.8) correspondingly. This classification is
generic and relations (5.3), (5.5) and (5.7) define both, direct and inverse
transformations.

In a sense the Cauchy–Jost function is more informative than the potential \( u(x) \). In order
to demonstrate this let us denote as \( u_{N_a N_b}(x) \), \( F_{N_a N_b}(x, \lambda, \mu) \), etc, objects corresponding to
\((N_a, N_b)\)-soliton case. Notice that, in contrast to \( \tau \)-function formulation where condition (2.1)
was necessary, relations (3.5) and (3.6) allow, under the standard assumption that \( \sum_{\ell=0}^\infty = 0 \),
the case where either \( N_a \), or \( N_b \), or both are equal to zero. In particular by (3.1) we get

\[
F_{0,0}(x, \lambda, \mu) = \frac{e^{(\ell(\mu) - \ell(\lambda))x}}{\mu - \lambda},
\]

(6.1)
because \( R(\lambda) \equiv 1 \) (see (2.21)) in the case of absence of solitons. Of course, thanks to (4.9),
\( u_{0,0}(x) \equiv 0 \). In the case of arbitrary positive \( N_a \) and \( N_b \) we get by (3.6)

\[
F_{N_a,0}(x, \lambda, \mu) = \frac{R_{N_a,0}(\lambda)e^{(\ell(\mu) - \ell(\lambda))x}}{\mu - \lambda},
\]

(6.2)
where \( R_{N_a,0}(\lambda) \) is given by (2.21) for \( N = N_a \). The same (6.2) follows from (3.5), where \( C' \)
can be chosen as diagonal (unity) \( N_a \times N_a \) matrix and the same result we get by \( N_a \) consecutive
\( A \)-transformations (5.3), (5.4) starting from (6.1). Matrix \( C \) obeying (2.13) in this case equals
to zero, as well as potential \( u_{N_a,0}(x) \) due to (3.1) and (4.9). While both potentials \( u_{0,0} \) and \( u_{N_a,0} \)
are equal to zero, their Cauchy–Jost functions are different. Thus the \( B \)-transformation of
\( F_{0,0}(x, \lambda, \mu) \) gives \( F_{0,1}(x, \lambda, \mu) \) and potential \( u_{0,1}(x) \) again equals zero identically, but the same
transformation applied to \( F_{N_a,0}(x, \lambda, \mu) \) by (5.5) and (5.6) leads to the Cauchy–Jost function
\( F_{N_a,1}(x, \lambda, \mu) \) that corresponds to generic \((N_a, 1)\)-soliton solution \( u_{N_a,1}(x) \) with matrices \( C_{N_a,1} \)
and \( C'_{N_a,1} \) given by (5.21).

Well known problem of the description of soliton solutions of the KPII equation is the problem of their regularity on the \((x_1, x_2)\)-plane for any \( x_3 \), that thanks to (2.18) is equivalent to the absence of zeros of the \( \tau \)-function, see [1]. In order to formulate condition of regularity here we assume, without loss of generality, that

\[
x_1 < x_2 < \ldots < x_{N_a},
\]

(6.3)
so that by (2.10) all maximal minors \( \mathcal{V}(\{n_i\}), n_1 < n_2 < \ldots < n_{N_a} \), in (2.16) are positive.
Then for the positivity of \( \tau \)-function in (2.16) it is enough to impose condition that all maximal
minors of the matrix \( C \) are nonnegative:

\[
C(\{n_i\}) \geq 0, \quad n_1 < n_2 < \ldots < n_{N_a},
\]

(6.4)
i.e. that matrix \( C \) is totally nonnegative (TNN). Necessity of this condition was proved in [14].
In terms of the Cauchy–Jost function regularity of the potential \( u(x) \) is equivalent to regularity of
this function itself thanks to relations (4.1). It is easy to see that Darboux transformations do
not guarantee that function \( \tilde{F}(x, \lambda, \mu) \) given by any direct transformation is regular. By (5.14)
for the \( A \)-transformation (or (5.21) for the \( B \)-transformation) it is clear that the TNN property
of matrix \( \tilde{C} \) (correspondingly, \( C' \)) implies with necessity this property for matrix \( C \) (or \( C' \)).
Thus regularity of the initial potential \( u(x) \) is necessary condition for regularity of the Darboux
transformed potential \( \tilde{u}(x) \).

Vice versa, for the inverse \( A \)-transformation we get that TNN property of matrix \( C \) is
preserved under reduction procedure given after (5.42). The same for the inverse \( B \)-transformation
for matrix \( C' \) as follows from (5.49). Summarizing the above considerations we prove that an
arbitrary regular \((N_a, N_b)\)-soliton solution can be reached from the zero one by successive
direct Darboux transformations that preserve regularity property at every step.
The results of direct transformations, relations (5.13) and (5.19) together with properties of the Cauchy–Jost function given in section 3, can be used as a tool to control absence of singularities in the transformed objects. Let us consider the example of A-transformation, and by the above assume regularity of the original potential $u(x)$ and, thus, of the original Cauchy–Jost function $F(x, \lambda, \mu)$. We see that the regularity of the transformed function $\tilde{F}(x, \lambda, \mu)$ is equivalent to the absence of zeros in the denominator in the rhs of (5.13). Thus by (4.1), (4.2) and notations (2.4), (2.21) and (3.1) we have that

$$\sum_{l=1}^{N} a_l F(x, \kappa_l, \kappa_{N+1}) - a_{N+1}$$

$$= \frac{1}{\tau(x)} \sum_{1 \leq n_1 < \ldots < n_N \leq N} C(\{n_i\}) V(\{n_i\}) \prod_{j=1}^{N} e^{\ell_j \kappa_{j+1} - \kappa_j} \times \frac{a_n e^{(\ell_{N+1} - \ell_n) x}}{\kappa_{N+1} - \kappa_n} \prod_{i=1}^{N} \frac{e^{\ell_j \kappa_{j+1} - \kappa_j} - a_{N+1}}{\kappa_{N+1} - \kappa_n}.$$  (6.5)

By assumption, $\tau(x)$ is positive, that means that the whole factor in the second line is positive as well (we use condition 2.1 here). Now, thanks to (6.3) the sign of the product in the last line equals $(-1)^{N_{N+1}}$ and in a generic situation one cannot make the sum in the third line to be sign determined by the choice of parameters $a_k$. But let us assume that, say, only the last parameter in the set $\{a_1, \ldots, a_{N+1}\}$ is different from zero, $a_n = a d_n$. Then $n_{N+1} = N$ due to condition on the indexes of summation in (6.5), so that

$$\sum_{l=1}^{N} a_l F(x, \kappa_l, \kappa_{N+1}) - a_{N+1}$$

$$= \frac{a d_{N+1}}{\tau(x)} \sum_{1 \leq n_1 < \ldots < n_N \leq N-1} C(\{n_i, N\}) V(\{n_i, N\}) \prod_{j=1}^{N-1} e^{\ell_j \kappa_{j+1} - \kappa_j} \times \frac{e^{(\ell_{N+1} - \ell_n) x}}{\kappa_{N+1} - \kappa_n} - a_{N+1}.$$  (6.6)

Thus choice $a_{N+1} a < 0$ guarantees that all terms in the rhs have the same sign, and thus the resulting soliton solution is regular. Correspondingly, by [14] this means that matrix $\tilde{C}$ given in (5.14) is totally nonnegative. The problem of how generic are TNN matrices constructed in this way deserves further investigation.

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**Appendix. Proof of relations (2.14) and (2.19)**

By definition the $\tilde{N} \times N$ real matrix $C$ obeys condition 2.1, thus it can be written in the form

$$C = p \left( \begin{array}{c} l_n \\ d \end{array} \right) c,$$  (A.1)
where \( p \) is a permutation matrix, \( I_N \) is the unity matrix, \( I_N \) is the unity \( N \times N \)-matrix, \( d \) is an \( N \times N \)-matrix and \( N \times N \)-matrix \( c \) is nondegenerate. Then relation (2.13) gives that there exists such \( N \times N \)-matrix \( \pi \) that

\[
C' r = \pi' (-d, I_N) p^{-1}.
\]

(A.2)

Thanks to this any maximal minor of the matrix \( C'r \) is proportional to \( \det \pi' \), thus by the stated condition matrix \( \pi' \) is nonsingular. Then \( C^T C = c^T (I_N + d^T d) c \) and \( C' \pi' C = c' (I_N + dd^T) c^T \), that proves that both these products are positive (and thus invertible) matrices. Let us introduce \( \mathcal{N} \times \mathcal{N} \) matrices

\[
\pi = C (C^T C)^{-1} C^T, \quad \pi' = r C^T (C' \pi' C^T)^{-1} C' r.
\]

(A.3)

Thanks to this definition and (2.13) these matrices are orthogonal projectors:

\[
\pi^2 = \pi, \quad \pi'^2 = \pi', \quad \pi \pi' = 0 = \pi' \pi,
\]

(A.4)

and because of the above definitions we have for them

\[
p^{-1} \pi p = \begin{pmatrix} (I_N + d^T d)^{-1}, & (I_N + d^T d)^{-1} d^T \\ d (I_N + d^T d)^{-1}, & d (I_N + d^T d)^{-1} d^T \end{pmatrix},
\]

(A.5)

\[
p^{-1} \pi' p = \begin{pmatrix} d^T (I_N + dd^T)^{-1} d, & -d^T (I_N + dd^T)^{-1} d \\ -(I_N + dd^T)^{-1} d, & (I_N + dd^T)^{-1} d \end{pmatrix}.
\]

(A.6)

Using now obvious relations of the kind

\[
(I_N + d^T d)^{-1} + d^T (I_N + dd^T)^{-1} d = (I_N + d^T d)^{-1} [I_N + (I_N + d^T d) d^T (I_N + dd^T)^{-1} d] = (I_N + d^T d)^{-1} [I_N + d^T (I_N + dd^T)^{-1} d] = I_N,
\]

we prove by (A.5) and (A.6) that projectors \( \pi \) and \( \pi' \) are complementary in the sense that

\[
\pi + \pi' = E_{\mathcal{N}},
\]

(A.7)

where \( E_{\mathcal{N}} \) is the \( \mathcal{N} \times \mathcal{N} \) unitary matrix. This gives equality (2.14), where

\[
w_b = (C^T C)^{-1} C^T w \text{ and } w_a = (C' \pi' C^T)^{-1} C' r w.
\]

In order to prove equation (2.19) we write the Vandermonde matrix \( \mathcal{V} \) (see (2.7)) in the form \( \mathcal{V} = \tau(u, I_N) p^{-1} \), where \( p \) is the same permutation matrix as in (A.1), \( \tau \) and \( u \) are some \( N \times N \) and \( N \times N \) matrices defined by this equality. Then by (2.8)

\[
\tau^{-1} \tau'(u) \equiv r^{-1} \tau'(u) \equiv r^{-1} \tau(u) \tau'(u)
\]

where \( \tau' \) is a \( N \times N \) matrix. By definition both matrices \( \tau \) and \( \tau' \) are nonsingular. Let us introduce two diagonal matrices \( e_{N}^{e} \) and \( e_{N}^{e} \) defined by the equality (see (2.5))

\[
p^{-1} e^{e} p = \begin{pmatrix} e_{N}^{e} & 0 \\ 0 & e_{N}^{e} \end{pmatrix}.
\]

Then using (A.1) we can write the first equality in (2.15) in the form \( \tau(x) = \det (\tau(ue_{N}^{e} + e_{N}^{e} d) c) \). Finally we get
\[
\tau(x) = \det v \det c \det e_{N_x}^{\pm} \det \left( u + e_{N_x}^{\pm}de_{N_y}^{\pm} \right),
\]
\[
\tau'(x) = (-1)^{N_x} \det v' \det c' \det e_{N_x}^{\pm} \det \left( u + e_{N_x}^{\pm}de_{N_y}^{\pm} \right),
\]
where the second relation is derived in analogy. These relations prove (2.19).

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