Hilbert and Poincare problems for semi-linear equations in domains with rectifiable boundaries

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Abstract
Recall that the research of boundary-value problems with arbitrary measurable data is due to the famous dissertation of Luzin where he has studied the Dirichlet problem for harmonic functions in the unit disk.

In the last paper [26], it was studied Hilbert, Poincare and Neumann boundary-value problems with arbitrary measurable data for generalized analytic functions and generalized harmonic functions with applications to the relevant problems of mathematical physics.

The present paper is devoted to the study of the boundary-value problems with arbitrary measurable boundary data in domains with rectifiable boundaries for the corresponding semi-linear equations with suitable nonlinear sources.

For this purpose, here it is constructed completely continuous operators generating nonclassical solutions of the Hilbert and Poincare boundary-value problems with arbitrary measurable data for the Vekua type equations and the Poisson equations, respectively.

On this base, it is first proved the existence of solutions of the Hilbert boundary-value problem with arbitrary measurable data in any domains with rectifiable boundaries for the nonlinear equations of the Vekua type.

It is necessary to note that our approach is based on the geometric interpretation of boundary values as angular (along nontangential paths) limits in comparison with the classical variational approach in PDE.

The latter makes it is also possible to obtain the theorem on the existence of nonclassical solutions of the Poincare boundary-value problem on the directional derivatives and, in particular, of the Neumann problem with arbitrary measurable data to the Poisson equations with nonlinear sources in Jordan domains with rectifiable boundaries.

As consequences, then it is given a series of applications of these results to some problems of mathematical physics describing such phenomena as diffusion with physical and chemical absorption, plasma states and stationary burning.

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1 Introduction

Recall that the boundary-value problems for analytical functions and their generalizations go to the famous Riemann dissertation (1851) and to the following works of Hilbert (1904, 1912, 1924) and Poincare (1910).

The research of boundary-value problems with arbitrary measurable data is due to the known dissertation of Luzin where he has studied the corresponding Dirichlet problem for harmonic functions in the unit disk \( \mathbb{D} := \{ z \in \mathbb{C} : |z| < 1 \} \).

In this connection, recall that the following deep result of Luzin was one of the main theorems of his dissertation, see e.g. his paper [14], dissertation [15], p. 35, and its reprint [16], p. 78, adopted to the segment \([0, 2\pi]\).

**Theorem A.** For any measurable function \( \varphi : [0, 2\pi] \to \mathbb{R} \), there is a continuous function \( \Phi : [0, 2\pi] \to \mathbb{R} \) such that \( \Phi' = \varphi \) a.e. on \([0, 2\pi]\).

Just on the basis of Theorem A, Luzin proved the next significant result of his dissertation, see e.g. [16], p. 87.

**Theorem B.** Let \( \varphi : \mathbb{R} \to \mathbb{R} \) be a \( 2\pi \)-periodic measurable function. Then there is a harmonic function \( u \) in \( \mathbb{D} \) such that \( u(z) \to \varphi(\vartheta) \) for a.e. \( \vartheta \in \mathbb{R} \) as \( z \to e^{i\vartheta} \) along any nontangential path.

Note that the Luzin dissertation was later on published only in Russian as book [16] with comments of his pupils Bari and Men’shov already after his death. A part of its results was also printed in Italian [17]. However, Theorem A was published in English in the Saks book [27] as Theorem VII(2.3). Hence Frederick Gehring in [7] has rediscovered Theorem B and his proof on the basis of Theorem A in fact coincided with the original proof of Luzin.

Corollary 5.1 in [22] has strengthened Theorem B as the next, see also [24].

**Theorem C.** For each measurable function \( \varphi : \partial \mathbb{D} \to \mathbb{R} \), the space of all harmonic functions \( u : \mathbb{D} \to \mathbb{R} \) with the angular (along nontangential paths) limits \( \varphi(\zeta) \) for a.e. \( \zeta \in \partial \mathbb{D} \) has the infinite dimension.
The latter was key to establish the following result on the existence of nonclassical solutions to the Hilbert boundary-value problem for analytic functions in [22], Theorems 3.1 and Remark 5.2.

**Theorem D.** Let $D$ be a Jordan domain in $\mathbb{C}$ with a rectifiable boundary and let $\lambda : \partial D \to \mathbb{C}$, $|\lambda(\zeta)| \equiv 1$, and $\varphi : \partial D \to \mathbb{R}$ be measurable functions. Then the space of analytic functions $f : D \to \mathbb{C}$ with the angular limits

$$\lim_{z \to \zeta} \Re \{\overline{\lambda(\zeta)} \cdot f(z)\} = \varphi(\zeta) \quad \text{for a.e. } \zeta \in \partial D$$

(1.1)

has the infinite dimension.

In turn, on the base of the latter result, it was derived the corresponding theorems on the existence of nonclassical solutions to the Poincare boundary-value problem on the directional derivatives and, in particular, to the Neumann problem for harmonic functions in [23], Theorems 3, 4 and 5.

**Theorem E.** Let $D$ be a Jordan domain in $\mathbb{C}$ with a rectifiable boundary, $\nu : \partial D \to \mathbb{C}$, $|\nu(\zeta)| \equiv 1$, and $\varphi : \partial D \to \mathbb{R}$ be measurable functions. Then the space of harmonic functions $u : D \to \mathbb{R}$ with the angular limits

$$\lim_{z \to \zeta} \frac{\partial u}{\partial \nu}(z) = \varphi(\zeta) \quad \text{for a.e. } \zeta \in \partial D$$

(1.2)

has the infinite dimension.

Here we apply the standard designation for the directional derivative

$$\frac{\partial u}{\partial \nu}(z) := \lim_{t \to 0} \frac{u(z + t \cdot \nu) - u(z)}{t}.$$  

(1.3)

**Theorem F.** Let $D$ be a Jordan domain in $\mathbb{C}$ with a rectifiable boundary, $\varphi : \partial D \to \mathbb{R}$ be a measurable function and let $n = n(\zeta)$ denote the unit interior normal to $\partial D$ at a point $\zeta$. Then the space of harmonic functions $u : D \to \mathbb{R}$ such that, for a.e. $\zeta \in \partial D$, there exist

1) a finite limit along the normal $n(\zeta)$

$$u(\zeta) := \lim_{z \to \zeta} u(z),$$

(1.4)
2) the normal derivative

\[ \frac{\partial u}{\partial n} (\zeta) := \lim_{t \to 0^+} \frac{u(\zeta + t \cdot n) - u(\zeta)}{t} = \varphi(\zeta), \] (1.5)

3) the angular limit (along nontangential paths)

\[ \lim_{z \to \zeta} \frac{\partial u}{\partial n} (z) = \frac{\partial u}{\partial n} (\zeta), \] (1.6)

has the infinite dimension.

In the last paper [26], it was studied Hilbert, Poincare and Neumann boundary-value problems with arbitrary measurable data for the so-called generalized analytic functions and generalized harmonic functions with sources and given applications to relevant problems of mathematical physics.

In this connection, let us recall that the monograph [29] was devoted to the generalized analytic functions, i.e., continuous complex valued functions \( h(z) \) of the complex variable \( z = x + iy \) in \( W^{1,1}_{1,\text{loc}} \) satisfying equations of the form

\[ \partial \bar{z} h + ah + bh = c, \quad \partial \bar{z} := \frac{1}{2} \left( \partial_x + i \cdot \partial_y \right), \] (1.7)

where it was assumed that the complex valued functions \( a, b \) and \( c \) belong to the class \( L^p \) with some \( p > 2 \) in the corresponding domain \( D \subseteq \mathbb{C} \).

The paper [26] contained Theorem 1 on the existence of nonclassical solutions of the Hilbert boundary-value problem with arbitrary measurable boundary data for generalized analytic functions with sources \( g \), when \( a \equiv 0 \equiv b \),

\[ \partial \bar{z} h(z) = g(z) \] (1.8)

with the real valued functions \( g \) in the class \( L^p, p > 2 \).

Moreover, the paper [26] included Theorem 6 (Corollary 6) on the existence of continuous solutions in \( W^{2,p}_{\text{loc}} \) to the Poincare (Neumann) boundary-value problem with arbitrary measurable boundary data for generalized harmonic functions with sources \( G \) in \( L^p, p > 2 \), satisfying the Poisson equations

\[ \Delta U(z) = G(z). \] (1.9)
The present paper is devoted to the study of the boundary-value problems with arbitrary measurable boundary data in domains with rectifiable boundaries for the corresponding semi-linear equations with suitable nonlinear sources.

Namely, the first part of the paper is devoted to the Hilbert boundary-value problem with arbitrary measurable boundary data in Jordan domains $D$ with rectifiable boundaries for the nonlinear Vekua type equations of the form

$$\partial_z f(z) = h(z) \cdot q(f(z)) \quad \text{a.e. in } D, \quad (1.10)$$

where $h : D \to \mathbb{C}$ is a function in the class $L^p(D)$ for $p > 2$ and $q : \mathbb{C} \to \mathbb{C}$ is a continuous function with

$$\lim_{w \to \infty} \frac{q(w)}{w} = 0. \quad (1.11)$$

The second part of the paper is devoted to the Poincare (and Neumann) boundary-value problem with arbitrary measurable boundary data in Jordan domains $D$ with rectifiable boundaries for the nonlinear Poisson equations

$$\triangle U(z) = H(z) \cdot Q(U(z)) \quad \text{a.e. in } D, \quad (1.12)$$

where $H : D \to \mathbb{R}$ is a function in the class $L^p(D)$ for $p > 2$ and $Q : \mathbb{R} \to \mathbb{R}$ is a continuous function with

$$\lim_{t \to \infty} \frac{Q(t)}{t} = 0. \quad (1.13)$$

For this purpose, it is first established the existence of completely continuous operators generating nonclassical solutions of the Hilbert and Poincare boundary-value problems with arbitrary measurable data for the equations of the Vekua and Poisson types (1.8) and (1.9), respectively.

Finally, the third part includes a series of applications of the results on the Poincare and Neumann boundary-value problems to some nonlinear equations of mathematical physics modeling, for instance, such phenomena as physical and chemical absorption with diffusion, plasma states, stationary burning etc.

Here we use the geometric interpretation of boundary values as angular (along nontangential paths) limits that is a traditional tool in the geometric function theory, see e.g. monographs [6], [11], [16], [20] and [21].
2 Definitions and preliminary remarks

First of all, recall that a **completely continuous** mapping from a metric space $M_1$ into a metric space $M_2$ is defined as a continuous mapping on $M_1$ which takes bounded subsets of $M_1$ into relatively compact ones of $M_2$, i.e. with compact closures in $M_2$. When a continuous mapping takes $M_1$ into a relatively compact subset of $M_1$, it is nowadays said to be **compact** on $M_1$.

Note that the notion of completely continuous (compact) operators is due essentially, in the special space of bilinear forms in $l_2$, to Hilbert who requires the operator to map weakly convergent sequences into strongly convergent sequences that, in reflexive spaces, is equivalent to Definition VI.5.1 for the Banach spaces in [5] which is due to F. Riesz, see also the comments of Section VI.12 in [5]. The latter just coincides with the above definition in the special case.

Recall more some definitions and the fundamental result of the celebrated paper [13]. Leray and Schauder extend as follows the Brouwer degree to compact perturbations of the identity $I$ in a Banach space $B$, i.e. a complete normed linear space. Namely, given an open bounded set $\Omega \subset B$, a compact mapping $F : B \to B$ and $z \notin \Phi(\partial \Omega)$, $\Phi := I - F$, the **(Leray–Schauder) topological degree** $\deg[\Phi, \Omega, z]$ of $\Phi$ in $\Omega$ over $z$ is constructed from the Brouwer degree by approximating the mapping $F$ over $\Omega$ by mappings $F_\varepsilon$ with range in a finite-dimensional subspace $B_\varepsilon$ (containing $z$) of $B$. It is showing that the Brouwer degrees $\deg[\Phi_\varepsilon, \Omega_\varepsilon, z]$ of $\Phi_\varepsilon := I_\varepsilon - F_\varepsilon$, $I_\varepsilon := I|_{B_\varepsilon}$, in $\Omega_\varepsilon := \Omega \cap B_\varepsilon$ over $z$ stabilize for sufficiently small positive $\varepsilon$ to a common value defining $\deg[\Phi, \Omega, z]$ of $\Phi$ in $\Omega$ over $z$.

This topological degree “algebraically counts” the number of fixed points of $F(\cdot) - z$ in $\Omega$ and conserves the basic properties of the Brouwer degree as additivity and homotopy invariance. Now, let $a$ be an isolated fixed point of $F$. Then the **local (Leray–Schauder) index** of $a$ is defined by $\text{ind}[\Phi, a] := \deg[\Phi, B(a, r), 0]$ for small enough $r > 0$. $\text{ind}[\Phi, 0]$ is called by **index** of $F$. In particular, if $F \equiv 0$, correspondingly, $\Phi \equiv I$, then the index of $F$ is equal to 1.
Let us formulate the main result in [13], Theorem 1, see also the survey [18].

**Proposition 1.** Let $B$ be a Banach space, and let $F(\cdot, \tau) : B \to B$ be a family of operators with $\tau \in [0, 1]$. Suppose that the following hypotheses hold:

(H1) $F(\cdot, \tau)$ is completely continuous on $B$ for each $\tau \in [0, 1]$ and uniformly continuous with respect to the parameter $\tau \in [0, 1]$ on each bounded set in $B$;

(H2) the operator $F := F(\cdot, 0)$ has finite collection of fixed points whose total index is not equal to zero;

(H3) the collection of all fixed points of the operators $F(\cdot, \tau)$, $\tau \in [0, 1]$, is bounded in $B$.

Then the collection of all fixed points of the family of operators $F(\cdot, \tau)$ contains a continuum along which $\tau$ takes all values in $[0, 1]$.

Let us go back to the discussion of the results of Luzin in Introduction.

**Remark 1.** Applying the Cantor ladder type functions, namely, continuous nondecreasing functions $C : [0, 2\pi] \to \mathbb{R}$ with $C(0) = 0$, $C(2\pi) = 1$ and $C'(t) = 0$ for a.e. $t \in [0, 2\pi]$, see e.g. Section 8.15 in [8], we may assume in Theorem A that $\Phi(0) = 0 = \Phi(2\pi)$. On the same base, using uniform continuity of the function $\Phi$ on $[0, 2\pi]$ and applying sequentially fragmentations of the segment to arbitrarily small parts, we may assume in Theorem A that $|\Phi(t)| < \varepsilon$ for every prescribed $\varepsilon > 0$ and, in particular, that $|\Phi(t)| < 1$ for all $t \in [0, 2\pi]$. Thus, in view of arbitrariness of $\varepsilon > 0$, there is the infinite collection of such $\Phi$ for each $\varphi$. Furthermore, applying series of pair of (nondecreasing and nonincreasing) functions of the Cantor ladder type on the segments $[2^{-(k+1)}\pi, 2^{-k}\pi]$, $k = 1, 2, \ldots$ it is easy to see that the space of such functions $\Phi$ has the infinite dimension.

By the proof of Theorem B, see [15], [16] or [7], $u(z) = \frac{\partial}{\partial \vartheta} U(z)$, where

$$U(re^{i\vartheta}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 - 2r \cos(\vartheta - t) + r^2} \Phi(e^{it}) \, dt ,$$

(2.1)
i.e., for a function \( \Phi \) from Theorem A, \( u \) can be calculated in the explicit form

\[
u(re^{i\vartheta}) = -\frac{r}{\pi} \int_0^{2\pi} \frac{(1-r^2) \sin(\vartheta-t)}{(1-2r\cos(\vartheta-t)+r^2)^2} \Phi(e^{it}) \, dt.
\] (2.2)

**Remark 2.** Later on, it was shown by Theorems 3 in [25] that the Luzin harmonic functions \( u(z) \) can be represented as the Poisson–Stieltjes integrals

\[
\mathcal{U}_\Phi(z) = \frac{1}{2\pi} \int_{-\pi}^\pi P_r(\vartheta-t) \, d\Phi(e^{it}) \quad \forall \, z = re^{i\vartheta}, \, r \in (0,1), \, \vartheta \in [-\pi,\pi],
\] (2.3)

where \( P_r(\Theta) = (1-r^2)/(1-2r\cos\Theta+r^2), r < 1, \Theta \in \mathbb{R} \), is the Poisson kernel.

The corresponding analytic functions in \( \mathbb{D} \) with the real parts \( u(z) \) can be represented as the corresponding Schwartz–Stieltjes integrals

\[
\mathcal{S}_\Phi(z) = \frac{1}{2\pi} \int_{\partial\mathbb{D}} \frac{\zeta+z}{\zeta-z} \, d\Phi(\zeta), \quad z \in \mathbb{D},
\] (2.4)

because of the Poisson kernel is the real part of the (analytic in the variable \( z \)) Schwartz kernel \((\zeta+z)/(\zeta-z)\). Integrating (2.4) by parts, see Lemma 1 and Remark 1 in [25], we obtain also the more convenient form of the representation

\[
\mathcal{S}_\Phi(z) = \frac{z}{\pi} \int_{\partial\mathbb{D}} \frac{\Phi(\zeta)}{(\zeta-z)^2} \, d\zeta, \quad z \in \mathbb{D}.
\] (2.5)

### 3 On completely continuous Hilbert operators

Recall that in paper [26], we considered generalized analytic functions \( f \) with sources \( g \in L^p, p > 2 \), in the class \( W^{1,1}_{\text{loc}} \) that satisfy the equation

\[
\frac{\partial f}{\partial \bar{z}} = g, \quad \frac{\partial}{\partial \bar{z}} := \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right), \quad z = x + iy,
\] (3.1)

and studied for them the Hilbert boundary-value problem in Jordan domains with rectifiable boundaries under arbitrary boundary data that are measurable over the natural parameter.

In particular, Theorem 1 in [26] stated that, for arbitrary measurable functions \( \lambda : \partial\mathbb{D} \to \mathbb{C}, \, |\lambda(\zeta)| \equiv 1 \), and \( \varphi : \partial\mathbb{D} \to \mathbb{R} \), there exist generalized analytic
functions \( f : D \to \mathbb{C} \) with any source \( g : D \to \mathbb{R} \) in the class \( L^p(\mathbb{D}) \), \( p > 2 \), that have the angular limits

\[
\lim_{z \to \zeta} \Re \left\{ \overline{\lambda(\zeta)} \cdot f(z) \right\} = \varphi(\zeta) \quad \text{a.e. on } \partial \mathbb{D} \, .
\]  

(3.2)

Furthermore, the space of such functions \( f \) has the infinite dimension.

Thus, the Hilbert boundary-value problem always has many solutions in the given sense for each such coefficient \( \lambda \), boundary date \( \varphi \) and source \( g \). Of course, axiom of choice by Zermelo makes it possible to choose one of such correspondence named further as a Hilbert operator but the latter with such a random choice can be completely discontinuous. Later on, to apply the approach of Leray-Schauder for extending Theorem 1 in [26] to the generalized analytic functions, satisfying nonlinear equations of the Vekua type, we need just the complete continuity of such correspondence.

So, let us construct a completely continuous Hilbert operator generating generalized analytic functions with sources \( g : D \to \mathbb{C} \) in the class \( L^p \), \( p > 2 \), and the boundary condition (3.2) for prescribed measurable functions \( \lambda \) and \( \varphi \).

For this purpose, let us first consider the known linear singular operator

\[
T_g(z) := \frac{1}{\pi} \int_{\mathbb{C}} g(w) \frac{d m(w)}{z - w} ,
\]  

(3.3)

where we assume that \( g \) is extended by zero outside of \( \mathbb{D} \).

**Remark 3.** By Theorem 1.14 in [29] the function \( T_g \) has the generalized derivative by Sobolev \( \partial T_g / \partial \bar{z} = g \) if \( g \in L^1(\mathbb{D}) \). Moreover, by Theorem 1.36 in [29] the function \( T_g \in W^{1,p}_{\text{loc}} \) if \( g \in L^p(\mathbb{D}) \), \( p > 1 \).

Furthermore, if \( g \in L^p(\mathbb{D}) \), \( p > 2 \), then by Theorem 1.19 in [29]

\[
|T_g(z)| \leq M_1 \| g \|_p \quad \forall z \in \mathbb{C} ,
\]  

(3.4)

\[
|T_g(z_1) - T_g(z_2)| \leq M_2 \| g \|_p |z_1 - z_2|^\alpha \quad \forall z_1, z_2 \in \mathbb{C} ,
\]  

(3.5)

where the constants \( M_1 \) and \( M_2 \) depend only on \( p > 2 \), and \( \alpha = (p-2)/p \). Thus, the linear operator \( T_g \) is completely continuous on compact sets in \( \mathbb{C} \) and, in particular, on \( \mathbb{D} \) by Arzela-Ascoli theorem, see e.g. Theorem IV.6.7 in [3].
Next, since $T_g$ is continuous, we have the measurable boundary function

$$
\varphi_g(\zeta) := \lim_{z \to \zeta} \Re \left\{ \overline{\lambda(\zeta)} \cdot T_g(z) \right\} = \Re \left\{ \overline{\lambda(\zeta)} \cdot T_g(\zeta) \right\}, \quad \forall \zeta \in \partial \mathbb{D} .
$$

Thus, the generalized analytic functions $f$ with the source $g$ satisfying the Hilbert condition (3.2) can be get as the sums $f = T_g + C$ with analytic functions $C$ satisfying, in the sense of angular limits, the Hilbert boundary condition

$$
\lim_{z \to \zeta} \Re \left\{ \overline{\lambda(\zeta)} \cdot C(z) \right\} = \psi(\zeta) := \varphi(\zeta) - \varphi_g(\zeta) \quad \text{a.e. on } \partial \mathbb{D} .
$$

In turn, by the construction of Theorem 2.1 in [22], such analytic functions $C$ can be obtained as the products of 2 analytic functions $A$ and $B$. The first

$$
A(z) = e^{ia(z)} , \quad a(z) := \frac{1}{2 \pi i} \int_{\partial \mathbb{D}} \frac{\alpha_\lambda(\zeta)}{z - \zeta} d\zeta, \quad z \in \mathbb{D} ,
$$

with $\alpha_\lambda(\zeta) := \arg \lambda(\zeta)$, where $\arg \omega$ is the principal value of the argument of $\omega \in \mathbb{C}, |\omega| = 1$, i.e., the number $\alpha \in (-\pi, \pi]$ such that $\omega = e^{i\alpha}$; the second one

$$
B(z) = S_\Phi(z) = \frac{z}{\pi} \int_{\partial \mathbb{D}} \frac{\Psi(\zeta)}{(\zeta - z)^2} d\zeta, \quad z \in \mathbb{D} ,
$$

see Remark 2, where $\Psi$ is an antiderivative of the function $\psi e^{\beta}$ in Theorem A, $\beta(\zeta)$ is the angular limit of $\text{Im } a(z)$ as $z \to \zeta \in \partial \mathbb{D}$ a.e., see e.g. Corollary 4 in [25], $\text{Re } a(z) \to \alpha_\lambda(\zeta)$ as $z \to \zeta \in \partial \mathbb{D}$ a.e., see e.g. Corollary IX.1.1 in [9].

Thus, analytic functions $C$ can be represented in more convenient form

$$
C(z) = A(z) \cdot \left[ S_\Phi(z) - S_{\Phi_g}(z) \right] ,
$$

where $\Phi$ and $\Phi_g$ are antiderivatives of $\varphi e^{\beta}$ and $\varphi_g e^{\beta}$ as functions of $\theta \in [0, 2\pi]$ in Theorem A, respectively. Note that the analytic functions $A$ and $S_\Phi$ do not depend on the sources $g$ at all. Let us choose the function $\Phi_g$ in a suitable way.

From this point on, we demand that all sources $g$ have compact supports in the unit disk and belong to a disk $D_\rho := \{ z \in \mathbb{C} : |z| \leq \rho \}$ with a radius $\rho \in (0, 1)$. Then the function $T_g(z)$, $z \in \mathbb{C}$, is analytic in a neighborhood of the unit circle $\partial \mathbb{D}$ and, in particular, $T_g(\zeta)$ is differentiable in $\theta \in \mathbb{R}, \zeta = e^{i\theta}$.
Moreover, we have that, for all $\zeta = e^{i\vartheta}$, $\vartheta \in [0, 2\pi],$

$$\{T_g\}_\vartheta(\zeta) = i\zeta T'_g(\zeta) = \frac{\zeta}{\pi i} \int_{\mathbb{D}_\rho} g(w) \frac{dm(w)}{(\zeta - w)^2} \quad \forall \zeta \in \partial \mathbb{D}. \quad (3.11)$$

Let us denote by $\Lambda$ an antiderivative for the function $\lambda e^{i\beta}$ as a function of $\vartheta \in [0, 2\pi]$ in Theorem A, see also Remark 1.

Then the following function $\Phi_g$ is an antiderivative for the function $\varphi_g e^{i\beta}$:

$$\Phi_g(\zeta) := \text{Re} \left\{ \Lambda(\zeta)T_g(\zeta) - \int_0^\vartheta \Lambda(\xi)\{T_g\}_\vartheta(\xi)\,d\vartheta + S(\vartheta) \right\}, \quad (3.12)$$

where $S : [0, 2\pi] \to \mathbb{C}$ is either zero or a singular function of the form

$$S(\vartheta) := C(\vartheta) \int_0^{2\pi} \Lambda(\xi)\{T_g\}_\vartheta(\xi)\,d\vartheta, \quad \zeta = e^{i\vartheta}, \xi = e^{i\theta}, \vartheta, \theta \in [0, 2\pi], \quad (3.13)$$

with a singular function $C : [0, 2\pi] \to [0, 1]$ of the Cantor ladder type, see Section 8.15 in [8], i.e., $C$ is continuous, nondecreasing, $C(0) = 0$, $C(2\pi) = 1$ and $C' = 0$ a.e. on $[0, 2\pi]$.

Let us show that the Hilbert operator $\mathcal{H}_g^*$ generated by the sums $T_g + C$ under the given choice of $\Phi$ and $\Phi_g$ in (3.10) is completely continuous on compact sets in $\mathbb{D}$. Recall that the analytic functions $A$ and $S_{\Phi}$ in the representation (3.10) of $C$ do not depend on the sources $g$. Hence by Remark 3, it remains to show that the linear operator $S_{\Phi_g}$ is completely continuous.

Indeed, by the construction of $\Phi_g$ in (3.12) and relations (3.3) and (3.11)

$$|\Phi_g(\zeta)| \leq \frac{1}{\pi} \frac{\|g\|_1}{1 - \rho} + 2 \cdot \frac{\|g\|_1}{(1 - \rho)^2} \leq c_\rho \cdot \|g\|_1 \leq C_\rho \cdot \|g\|_p \quad \forall \zeta \in \partial \mathbb{D} \quad (3.14)$$

with $c_\rho = 3/(1 - \rho)^2$ and $C_\rho = 3\pi/(1 - \rho)^2$, respectively. Hence, by (2.5)

$$|S_{\Phi_g}(z)| \leq C_{\rho,r} \cdot \|g\|_p, \quad \forall z \in \mathbb{D}_r, \ r \in (0, 1), \quad (3.15)$$

$$|S_{\Phi_g}(z_1) - S_{\Phi_g}(z_2)| \leq C_{\rho,r} \cdot \|g\|_p \cdot |z_1 - z_2|, \quad \forall z_1, z_2 \in \mathbb{D}_r, \ r \in (0, 1), \quad (3.16)$$

where the constants $C_{\rho,r}$ and $C_{\rho,r}^*$ depend only on the radii $\rho$ and $r \in (0, 1)$.

Thus, the operator $S_{\Phi_g}$ is completely continuous on compact sets in $\mathbb{D}$ again
by the Arzela-Ascoli theorem. Combining it with Remark 3, we obtain the following conclusion.

**Lemma 1.** Let \( \lambda : \partial D \to \mathbb{C}, \ |\lambda(\zeta)| \equiv 1 \), and \( \varphi : \partial D \to \mathbb{R} \) be measurable. Then there is a Hilbert operator \( \mathcal{H}^*_g \) over \( g : D \to \mathbb{C} \) in \( L^p(D), \ p > 2 \), with compact supports in \( D \), generating generalized analytic functions \( f : D \to \mathbb{C} \) with the sources \( g \) and the angular limits

\[
\lim_{z \to \zeta} \text{Re} \left\{ \lambda(\zeta) \cdot f(z) \right\} = \varphi(\zeta) \quad \text{a.e. on } \partial D, \tag{3.17}
\]

whose restriction to sources \( g \) with \( \text{supp} g \subseteq D_\rho \) is completely continuous over \( D_\rho \) for each \( \rho \) and \( r \in (0, 1) \).

**Remark 4.** Note that the nonlinear operator \( \mathcal{H}^*_g \) constructed above is not bounded except the trivial case \( \Phi \equiv 0 \) because then \( \mathcal{H}^*_\Phi = \mathcal{A} \cdot \mathcal{S} \Phi \neq 0 \). However, the restriction of the operator \( \mathcal{H}^*_g \) to \( D_\rho \) under each \( r \in (0, 1) \) is bounded at infinity in the sense that \( \max_{z \in D_\rho} |\mathcal{H}^*_g(z)| \leq M \cdot \|g\|_p \) for some \( M > 0 \) and all \( g \) with large enough \( \|g\|_p \). Note also that by Remark 1 we are able always to choose \( \Phi \) for any \( \varphi \), including \( \varphi \equiv 0 \), which is not identically 0 in the unit disk \( D \).

4 On Hilbert problem for semi-linear equations

In this section we study the solvability of the Hilbert boundary-value problem for nonlinear equations of the Vekua type \( \partial z f(z) = h(z)q(f(z)) \) in the unit disk \( D \). The Leray–Schauder approach described in Section 2 allows us to reduce the problem to the study of the corresponding linear equation from our last paper [26] on the basis of Lemma 1 in the previous section on completely continuous Hilbert operator \( \mathcal{H}^*_g \) and Remark 4 on its boundedness at infinity.

In the proof of the next theorem, the initial operator \( F(\cdot) := F(\cdot, 0) \equiv 0 \). Hence \( F \) has the only one fixed point (at the origin) and its index is equal to 1 and, thus, hypothesis (H2) in Proposition 1 will be automatically satisfied.
**Theorem 1.** Let $\lambda : \partial D \to \mathbb{C}$, $|\lambda(\zeta)| \equiv 1$, and $\varphi : \partial D \to \mathbb{R}$ be measurable. Suppose that $h : \mathbb{D} \to \mathbb{C}$ is in $L^p(D)$ for $p > 2$ with compact support in $D$ and $q : \mathbb{C} \to \mathbb{C}$ is a continuous function with $\lim_{w \to \infty} q(w) = 0$.

Then there is $f : \mathbb{D} \to \mathbb{C}$ in the class $W^{1,p}_{\text{loc}} \cap C^\alpha_{\text{loc}}(D)$ with $\alpha = (p - 2)/p$,

\[ \partial_{\bar{z}} f(z) = h(z) \cdot q(f(z)) \quad \text{a.e. in } \mathbb{D}, \tag{4.2} \]

with the angular limits

\[ \lim_{z \to \zeta} \Re \left\{ \lambda(\zeta) \cdot f(z) \right\} = \varphi(\zeta) \quad \text{a.e. on } \partial \mathbb{D}. \tag{4.3} \]

**Proof.** If $\|h\|_p = 0$ or $\|q\|_C = 0$, then any analytic function from Theorem 2.1 in [26] gives the desired solution of (4.2). Thus, we may assume that $\|h\|_p \neq 0$ and $\|q\|_C \neq 0$. Set $q_*(t) = \max_{|w| \leq t} |q(w)|$, $t \in \mathbb{R}^+ := [0, \infty)$. Then the function $q_* : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and nondecreasing and, moreover, by (4.1)

\[ \lim_{t \to \infty} \frac{q_*(t)}{t} = 0. \tag{4.4} \]

By Lemma 1 and Remark 4 we obtain the family of operators $F(g; \tau) : L^p_h(\mathbb{D}) \to L^p_h(\mathbb{D})$, where $L^p_h(\mathbb{D})$ consists of functions $g \in L^p(\mathbb{D})$ with supports in the support of $h$,

\[ F(g; \tau) : = \tau h \cdot q(H_g^*) \quad \forall \tau \in [0, 1] \tag{4.5} \]

which satisfies all groups of hypothesis H1-H3 of Theorem 1 in [13], see Proposition 1. Indeed:

H1). First of all, by Lemma 1 the function $F(g; \tau) \in L^p_h(\mathbb{D})$ for all $\tau \in [0, 1]$ and $g \in L^p_h(\mathbb{C})$ because the function $q(H_g^*)$ is continuous and, furthermore, the operators $F(\cdot ; \tau)$ are completely continuous for each $\tau \in [0, 1]$ and even uniformly continuous with respect to the parameter $\tau \in [0, 1]$.

H2). The index of the operator $F(g; 0)$ is obviously equal to 1.
H3). Let us assume that solutions of the equations \( g = F(g; \tau) \) is not bounded in \( L^p_h(\mathbb{D}) \), i.e., there is a sequence of functions \( g_n \in L^p_h(\mathbb{D}) \) with \( \|g_n\|_p \to \infty \) as \( n \to \infty \) such that \( g_n = F(g_n; \tau_n) \) for some \( \tau_n \in [0, 1], n = 1, 2, \ldots \).

However, then by Remark 4 we have that, for some constant \( M > 0 \),
\[
\|g_n\|_p \leq \|h\|_p q_* (M \|g_n\|_p)
\]
and, consequently,
\[
\frac{q_* (M \|g_n\|_p)}{M \|g_n\|_p} \geq \frac{1}{M \|h\|_p} > 0 \tag{4.6}
\]
for all large enough \( n \). The latter is impossible by condition (4.4). The obtained contradiction disproves the above assumption.

Thus, by Theorem 1 in [13] there is a function \( g \in L^p_h(\mathbb{D}) \) with \( F(g; 1) = g \), and by Lemma 1 the function \( f := \mathcal{H}^*_g \) gives the desired solution of (4.2).

In particular, choosing \( \lambda \equiv 1 \) in Theorem 1 we obtain the following consequence on the Dirichlet problem for the nonlinear equations of the Vekua type.

**Corollary 1.** Let \( \varphi : \partial \mathbb{D} \to \mathbb{R} \) be a measurable function, \( h : \mathbb{D} \to \mathbb{C} \) be a function in the class \( L^p(\mathbb{D}) \) for \( p > 2 \) with compact support in \( \mathbb{D} \) and let \( q : \mathbb{C} \to \mathbb{C} \) be a continuous function with condition (4.1).

Then there is a function \( f : \mathbb{D} \to \mathbb{C} \) in the class \( W^{1, p}_{\text{loc}} \cap C^\alpha_{\text{loc}}(\mathbb{D}) \) with \( \alpha = (p - 2)/p \), satisfying equation (4.2) a.e. with the angular limits
\[
\lim_{z \to \zeta} \Re f(z) = \varphi(\zeta) \quad \text{a.e. on } \partial \mathbb{D}. \tag{4.7}
\]

**Remark 5.** Moreover, by the proof of Theorem 1 \( f \) is a generalized analytic function with a source \( g \in L^p(\mathbb{D}), \ f = \mathcal{H}^*_g, \) where \( \mathcal{H}^*_g \) is the Hilbert operator described in the last section (with the simplest analytic function \( A \equiv 1 \) in the case of Corollary 1), Lemma 1, and the support of \( g \) is in the support of \( h \) and the upper bound of \( \|g\|_p \) depends only on \( \|h\|_p \) and on the function \( q \).

In addition, the source \( g : \mathbb{D} \to \mathbb{C} \) is a fixed point of the nonlinear operator \( \Omega_g := h \cdot q(\mathcal{H}^*_g) : L^p_h(\mathbb{D}) \to L^p_h(\mathbb{D}) \), where \( L^p_h(\mathbb{D}) \) consists of functions \( g \) in \( L^p(\mathbb{D}) \) with supports in the support of \( h \).
5 On the Hilbert problem in rectifiable domains

**Theorem 2.** Let $D$ be a Jordan domain in $\mathbb{C}$ with a rectifiable boundary, $\lambda : \partial D \to \mathbb{C}$, $|\lambda(\zeta)| \equiv 1$, and $\varphi : \partial D \to \mathbb{R}$ be measurable over natural parameter. Suppose that $h : D \to \mathbb{C}$ is in $L^p(D)$ for $p > 2$ with compact support in $D$ and $q : \mathbb{C} \to \mathbb{C}$ is a continuous function with

$$\lim_{w \to \infty} \frac{q(w)}{w} = 0.$$  

(5.1)

Then there is $f : D \to \mathbb{C}$ in the class $W^{1,p}_{\text{loc}} \cap C^\alpha_{\text{loc}}(D)$ with $\alpha = (p - 2)/p$,

$$\partial_x f(\xi) = h(\xi) \cdot q(f(\xi)) \quad \text{a.e. in } D,$$

(5.2)

and the angular limits

$$\lim_{\zeta \to \omega} \text{Re} \left\{ \lambda(\omega) \cdot f(\xi) \right\} = \varphi(\omega) \quad \text{a.e. on } \partial D.$$  

(5.3)

**Proof.** Let $c$ be a conformal mapping of $D$ onto $\mathbb{D}$ that exists by the Riemann mapping theorem, see e.g. Theorem II.2.1 in [9]. Now, by the Carathéodory theorem, see e.g. Theorem II.3.4 in [9], $c$ is extended to a homeomorphism $\tilde{c}$ of $\overline{D}$ onto $\overline{\mathbb{D}}$. Set $c_* = \tilde{c}|_{\partial D}$. If $\partial D$ is rectifiable, then by the theorem of F. and M. Riesz length $c_*^{-1}(E) = 0$ whenever $E \subset \partial \overline{D}$ with $|E| = 0$, see e.g. Theorem II.C.1 and Theorems II.D.2 in [11]. Conversely, by the Lavrentiev theorem $|c_*(E)| = 0$ whenever $E \subset \partial D$ and length $E = 0$, see [12], see also the point III.1.5 in [21].

Hence $c_*$ and $c_*^{-1}$ transform measurable sets into measurable sets. Indeed, every measurable set is the union of a sigma-compact set and a set of measure zero, see e.g. Theorem III(6.6) in [27], and continuous mappings transform compact sets into compact sets. Thus, functions $\lambda : \partial D \to \mathbb{C}$, $|\lambda| \equiv 1$, and $\varphi : \partial D \to \mathbb{R}$ are measurable with respect to the natural parameter on $\partial D$ if and only if the functions $\tilde{\lambda} = \lambda \circ c_*^{-1} : \partial \mathbb{D} \to \mathbb{C}$ and $\tilde{\varphi} = \varphi \circ c_*^{-1} : \partial \mathbb{D} \to \mathbb{R}$ are so.

Now, set $\tilde{h} = h \circ C \cdot \overline{C'}$, where $C$ is the inverse conformal mapping to $c$, $C := c^{-1} : \mathbb{D} \to D$. Then it is clear by the hypotheses of Theorem 2 that $\tilde{h}$
has compact support in \( \mathbb{D} \) and belongs to the class \( L^p(\mathbb{D}) \). Consequently, by Theorem 1 there is \( \hat{f} : \mathbb{D} \to \mathbb{C} \) in the class \( W^{1,p}_\text{loc} \cap C^\alpha_\text{loc}(\mathbb{D}) \) with \( \alpha = (p-2)/p \),

\[
\partial_z \hat{f}(z) = \hat{h}(z) \cdot q(\hat{f}(z)) \quad \text{a.e. in } \mathbb{D},
\]

and the angular limits

\[
\lim_{z \to \zeta} \text{Re} \left\{ \overline{\lambda(\zeta)} \cdot \hat{f}(z) \right\} = \hat{\phi}(\zeta) \quad \text{a.e. on } \partial \mathbb{D}. \tag{5.5}
\]

Moreover, by Remark 5 \( \hat{f} \) is a generalized analytic function with a source \( \hat{g} \in L^p(\mathbb{D}) \), \( \hat{f} = \mathcal{H}_{\hat{g}}^* \), where \( \mathcal{H}_{\hat{g}}^* \) is the Hilbert operator described in Section 3, Lemma 1, and associated with \( \lambda \) and \( \hat{\phi} \), and the support of \( \hat{g} \) is in the support of \( \hat{h} \) and the upper bound of \( \|\hat{g}\|_p \) depends only on \( \|\hat{h}\|_p \) and on the function \( q \).

In addition, \( \hat{g} : \mathbb{D} \to \mathbb{C} \) is a fixed point of the nonlinear operator \( \hat{\Omega}_{\hat{g}} : L^p(\mathbb{D}) \to L^p(\mathbb{D}) \), where \( L^p(\mathbb{D}) \) consists of functions \( g^* \) in \( L^p(\mathbb{D}) \) with supports in the support of \( \hat{h} \).

Next, setting \( f = \hat{f} \circ c \), by simple calculations, see e.g. Section 1.C in \([1]\), we obtain that \( \frac{\partial f}{\partial \xi} = \frac{\partial \hat{f}}{\partial \xi} \circ c \cdot c' \) and, consequently, the function \( f : D \to \mathbb{C} \) is in the class \( W^{1,p}_\text{loc} \cap C^\alpha_\text{loc}(D) \) with \( \alpha = (p-2)/p \) and satisfies equation (5.2). Moreover, \( f \) is a generalized analytic function with the source \( g = \hat{g} \circ c \) in the class \( L^p(D) \), \( f(\xi) = \mathcal{H}_{\hat{g}}^*(c(\xi)) \), and the support of \( g \) is in the support of \( h \) and the upper bound of \( \|g\|_p \) depends only on \( \|h\|_p \), the function \( q \) and the domain \( D \).

It remains to show that \( f \) has the angular limits as \( \xi \to \omega \in \partial D \) and satisfies the boundary condition (5.3) a.e. on \( \partial D \). Indeed, by the Lindelöf theorem, see e.g. Theorem II.C.2 in \([1]\), if \( \partial D \) has a tangent at a point \( \omega \), then \( \arg [c_*(\omega) - c(\xi)] - \arg [\omega - \xi] \to \text{const} \) as \( \xi \to \omega \). In other words, the images under the conformal mapping \( c \) of sectors in \( D \) with a vertex at \( \omega \in \partial D \) is asymptotically the same as sectors in \( \mathbb{D} \) with a vertex at \( \zeta = c_*(\omega) \in \partial \mathbb{D} \). Consequently, nontangential paths in \( D \) are transformed under \( c \) into nontangential paths in \( \mathbb{D} \) and inversely a.e. on \( \partial D \) and \( \partial \mathbb{D} \), respectively, because the rectifiable boundary \( \partial D \) has a tangent a.e. and \( c_* \) and \( c_*^{-1} \) keep sets of the length zero.
In particular, choosing $\lambda \equiv 1$ in Theorem 2 we obtain the following consequence on the Dirichlet problem for the nonlinear equations of the Vekua type.

**Corollary 2.** Let $D$ be a Jordan domain with a rectifiable boundary, $\varphi : \partial D \to \mathbb{R}$ be measurable, $h : D \to \mathbb{C}$ be in $L^p(D)$, $p > 2$, with compact support in $D$, and let $q : \mathbb{C} \to \mathbb{C}$ be a continuous function with condition (5.1).

Then there is $f : D \to \mathbb{C}$ in the class $W^{1,p}_{\text{loc}} \cap C^{\alpha}_{\text{loc}}(D)$ with $\alpha = (p - 2)/p$, satisfying equation (5.2), and the angular limits

$$\lim_{\xi \to \omega} \Re f(\xi) = \varphi(\omega) \quad \text{a.e. on } \partial D. \quad (5.6)$$

**Remark 6.** Moreover, by the proof of Theorem 2 $f$ is a generalized analytic function with a source $g \in L^p(D)$ whose support is in the support of $h$ and the upper bound of $\|g\|_p$ depends only on $\|h\|_p$, the function $q$ and the domain $D$.

In addition, $g = \tilde{g} \circ c$ and $f = H^* \circ c$, where $c$ is a conformal mapping of $D$ onto $\mathbb{D}$, $\tilde{g} : \mathbb{D} \to \mathbb{C}$ is a fixed point of the nonlinear operator $\tilde{\Omega}_{g_*} := \tilde{h} \cdot q(\mathcal{H}^*_{g_*}) : L^p(\mathbb{D}) \to L^p(\mathbb{D})$, where $L^p(\mathbb{D})$ consists of functions $g_*$ in $L^p(\mathbb{D})$ with supports in the support of $\tilde{h} := h \circ C \cdot \overline{c}$, $C = c^{-1}$, $\mathcal{H}^*_{g_*}$ is the Hilbert operator described in Section 3 and associated with $\lambda = \lambda \circ c_*^{-1}$ and $\tilde{\varphi} = \varphi \circ c_*^{-1}$. Here $c_* : \partial D \to \partial \mathbb{D}$ is the homeomorphic boundary correspondence under the mapping $c$.

6 On completely continuous Poincaré operators

In Section 7 of [26], we considered the Poincaré boundary-value problem on the directional derivatives and, in particular, the Neumann problem with arbitrary measurable boundary data over the natural parameter for the Poisson equations

$$\Delta U(z) = G(z) \quad (6.1)$$

with real valued functions $G$ of classes $L^p(D)$ with $p > 2$ in Jordan’s domains $D$ in $\mathbb{C}$ with rectifiable boundaries.

Recall that a continuous solution $U$ of (6.1) in the class $W^{2,p}_{\text{loc}}$ was called in [26] a **generalized harmonic function with the source $G$** and that by
the Sobolev embedding theorem such a solution belongs to the class $C^1$, see Theorem I.10.2 in [28].

As usual, here $\frac{\partial u}{\partial \nu}(\xi)$ denotes the derivative of $u$ at the point $\xi \in D$ in the direction $\nu \in \mathbb{C}$, $|\nu| = 1$, i.e.,

$$\frac{\partial u}{\partial \nu}(\xi) := \lim_{t \to 0} \frac{u(\xi + t \cdot \nu) - u(\xi)}{t} .$$

The Neumann boundary value problem is a special case of the Poisson problem on the directional derivatives with the unit interior normal $n = n(\omega)$ to $\partial D$ at the point $\omega$ as $\nu(\omega)$, see Corollary 4 further.

By Theorem 6 in [26], for each measurable functions $\nu : \partial D \to \mathbb{C}$, $|\nu(\zeta)| \equiv 1$ and $\varphi : \partial D \to \mathbb{R}$, $G : \mathbb{D} \to \mathbb{R}$ in $L^p(\mathbb{D})$, $p > 2$, there is a generalized harmonic function $U : \mathbb{D} \to \mathbb{R}$ with the source $G$ that have the angular limits

$$\lim_{z \to \zeta} \frac{\partial U}{\partial \nu}(z) = \varphi(\zeta) \quad \text{a.e. on } \partial D .$$

Furthermore, the space of such functions $U$ has the infinite dimension.

As it follows from constructions in the proofs of Theorem 1 and 6 in [26], see especially (2.6) there, one of such functions $U$ can be presented as a sum of the logarithmic (Newtonian) potential $N_G$ of the source $G$,

$$N_G(z) := \frac{1}{2\pi} \int_{\mathbb{C}} \ln|z - w| G(w) \, dm(w) ,$$

where $dm(w)$ corresponds to the Lebesgue measure in the plane, i.e., the area, and the harmonic function

$$\gamma(z) := \text{Re} \int_{0}^{z} \left\{ \mathcal{H}_{G/2}(\xi) - T_{G/2}(\xi) \right\} \, d\xi ,$$

where $\mathcal{H}_{\lambda}$ is the Hilbert operator described in Section 3 but with $\lambda = \bar{\nu}$ and where we assumed that $G \in L^p(\mathbb{D})$, $p > 2$, with compact support in $\mathbb{D}$.

Denoting by $\mathcal{P}_G^*$ the given correspondence between such sources $G$ and the generalized harmonic functions with the sources $G$ and the Poincare boundary condition (6.3), we see that $\mathcal{P}_G^*$ is a completely continuous operator over each
disk $|z| < r < 1$ because the operators $H'_{G/2}$ and $T_{G/2}$ are so and, in addition, the indefinite integral as well as the operator of taking Re are bounded and linear. Thus, by Lemma 1 and Remark 4 we come to the following statements.

**Lemma 2.** Let $\nu : \partial \mathbb{D} \to \mathbb{C}$, $|\nu(\zeta)| \equiv 1$, and $\varphi : \partial \mathbb{D} \to \mathbb{R}$ be measurable functions. Then there is a Poincare operator $P^*_G$ over the sources $G : \mathbb{D} \to \mathbb{R}$ in $L^p(\mathbb{D})$, $p > 2$, with compact supports in $\mathbb{D}$, generating generalized harmonic functions $U : \mathbb{D} \to \mathbb{R}$ with the sources $G$ and the angular limits (6.3), whose restriction to sources $G$ with $\text{supp } G \subseteq \mathbb{D}_\rho$ is completely continuous over $\mathbb{D}_r$ for each $\rho$ and $r \in (0, 1)$.

**Remark 7.** Moreover, we may assume that the restriction of the operator $P^*_G$ to $\mathbb{D}_r$ under each $r \in (0, 1)$ is bounded at infinity in the sense that $\max_{z \in \mathbb{D}_r} |P^*_G(z)| \leq M \cdot \|G\|_p$ for some $M > 0$ and all $G$ with large enough $\|G\|_p$.

## 7 On Poincare problem for semi-linear equations

In this section we study the solvability of the Poincare boundary-value problem for semi-linear Poisson equations of the form $\Delta U(z) = H(z) \cdot Q(U(z))$ in the unit disk $\mathbb{D}$. Again the Leray–Schauder approach allows us to reduce the problem to the study of the linear Poisson equation from our last paper [26] on the basis of Lemma 2 on completely continuous Poincare operator $P^*_G$ and Remark 7 on its boundedness at infinity from the previous section.

Note that hypothesis (H2) in Section 2 will be automatically satisfied in the proof of the next theorem because the initial operator $F(\cdot) := F(\cdot, 0) \equiv 0$ and hence $F$ has the only one fixed point (at the origin) and its index is equal to 1.

**Theorem 3.** Let $\nu : \partial \mathbb{D} \to \mathbb{C}$, $|\nu(\zeta)| \equiv 1$, and $\varphi : \partial \mathbb{D} \to \mathbb{R}$ be measurable functions. Suppose that $H : \mathbb{D} \to \mathbb{R}$ is a function in the class $L^p(\mathbb{D})$ for $p > 2$ with compact support in $\mathbb{D}$ and $Q : \mathbb{R} \to \mathbb{R}$ is a continuous function with

$$\lim_{t \to \infty} \frac{Q(t)}{t} = 0.$$  (7.1)
Then there is a function \( U : \mathbb{D} \rightarrow \mathbb{R} \) in \( W^{2,p}_{\text{loc}}(\mathbb{D}) \cap C^{1,\alpha}_{\text{loc}}(\mathbb{D}) \) with \( \alpha = (p-2)/p \),

\[
\triangle U(z) = H(z) \cdot Q(U(z)) \quad \text{a.e. in } \mathbb{D}, \tag{7.2}
\]

and the angular limits

\[
\lim_{z \to \zeta} \frac{\partial U}{\partial \nu}(z) = \varphi(\zeta) \quad \text{a.e. on } \partial \mathbb{D}. \tag{7.3}
\]

**Proof.** If \( \|H\|_p = 0 \) or \( \|Q\|_C = 0 \), then any harmonic function from Theorem 3 in [24] gives the desired solution of (7.2). Thus, we may assume that \( \|H\|_p \neq 0 \) and \( \|Q\|_C \neq 0 \). Set \( Q_*(t) = \max_{|\tau| \leq t} |Q(\tau)|, \; t \in \mathbb{R}^+ := [0, \infty) \). Then the function \( Q_* : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) is continuous and nondecreasing and, moreover, by (7.1)

\[
\lim_{t \to \infty} Q_*(t) t^{-1} = 0. \tag{7.4}
\]

By Lemma 2 and Remark 7 we obtain the family of operators \( F(G; \tau) : L^p_H(\mathbb{D}) \rightarrow L^p_H(\mathbb{D}) \), where \( L^p_H(\mathbb{D}) \) consists of functions \( G \in L^p(\mathbb{D}) \) with supports in the support of \( H \),

\[
F(G; \tau) := \tau H \cdot Q(P^*_G) \quad \forall \; \tau \in [0,1] \tag{7.5}
\]

which satisfies all groups of hypothesis H1-H3 of Theorem 1 in [3], see Proposition 1. Indeed:

H1). First of all, by Lemma 2 the function \( F(G; \tau) \in L^p_H(\mathbb{D}) \) for all \( \tau \in [0,1] \) and \( G \in L^p_H(\mathbb{C}) \) because the function \( Q(P^*_G) \) is continuous and, furthermore, the operators \( F(\cdot; \tau) \) are completely continuous for each \( \tau \in [0,1] \) and even uniformly continuous with respect to the parameter \( \tau \in [0,1] \).

H2). The index of the operator \( F(\cdot; 0) \) is obviously equal to 1.

H3). Let us assume that solutions of the equations \( G = F(G; \tau) \) is not bounded in \( L^p_H(\mathbb{D}) \), i.e., there is a sequence of functions \( G_n \in L^p_H(\mathbb{D}) \) with \( \|G_n\|_p \rightarrow \infty \) as \( n \rightarrow \infty \) such that \( G_n = F(G_n; \tau_n) \) for some \( \tau_n \in [0,1], \; n = 1, 2, \ldots \). However, then by Remark 7 we have that, for some constant \( M > 0 \),

\[
\|G_n\|_p \leq \|H\|_p \cdot Q_*(M\|G_n\|_p)
\]
and, consequently, 
\[
\frac{Q_* (M \|G_n\|_p)}{M \|G_n\|_p} \geq \frac{1}{M \|H\|_p} > 0 \quad (7.6)
\]
for all large enough \(n\). The latter is impossible by condition (7.4). The obtained contradiction disproves the above assumption.

Thus, by Theorem 1 in [13] there is a function \(G \in L^p_H(D)\) with \(F(G; 1) = G\), and by Lemma 2 the function \(U := P_G^*\) gives the desired solution of (7.2).

**Remark 8.** Moreover, by the proof of Theorem 3 the function \(U\) is a generalized analytic function with a source \(G \in L^p_H(D)\), \(U = P_G^*\), where \(P_G^*\) is the Poincare operator described in the last section, Lemma 2, and the support of \(G\) is in the support of \(H\) and the upper bound of \(\|G\|_p\) depends only on \(\|H\|_p\) and on the function \(Q\).

In addition, the source \(G : D \to C\) is a fixed point of the nonlinear operator \(\Omega_G := h \cdot Q(P_G^*) : L^p_H(D) \to L^p_H(D)\), where \(L^p_H(D)\) consists of functions \(G\) in \(L^p(D)\) with supports in the support of \(H\).

We are able to say more in Theorem 3 for the case of \(\text{Re} \ n(\zeta)\nu(\zeta) > 0\), where \(n(\zeta)\) is the inner normal to \(\partial D\) at the point \(\zeta\). Indeed, the latter magnitude is a scalar product of \(n = n(\zeta)\) and \(\nu = \nu(\zeta)\) interpreted as vectors in \(\mathbb{R}^2\) and it has the geometric sense of projection of the vector \(\nu\) into \(n\). In view of (7.3), since the limit \(\varphi(\zeta)\) is finite, there is a finite limit \(U(\zeta)\) of \(U(z)\) as \(z \to \zeta\) in \(D\) along the straight line passing through the point \(\zeta\) and being parallel to the vector \(\nu\) because along this line
\[
U(z) = U(z_0) - \int_0^1 \frac{\partial U}{\partial \nu} (z_0 + \tau (z - z_0)) \, d\tau . \quad (7.7)
\]
Thus, at each point with condition (7.3), there is the directional derivative
\[
\frac{\partial U}{\partial \nu} (\zeta) := \lim_{t \to 0} \frac{U(\zeta + t \cdot \nu) - U(\zeta)}{t} = \varphi(\zeta) . \quad (7.8)
\]
In particular, in the case of the Neumann problem, \(\text{Re} \ n(\zeta)\nu(\zeta) \equiv 1 > 0\), where \(n = n(\zeta)\) denotes the unit inner normal to \(\partial D\) at the point \(\zeta\), and we have by Theorem 3 and Remark 8 the following significant result.
Corollary 3. Let $\varphi : \partial D \to \mathbb{R}$ be measurable, $H : D \to \mathbb{R}$ be in $L^p(D)$, $p > 2$, with compact support in $D$ and let $Q : \mathbb{R} \to \mathbb{R}$ be a continuous function with condition (7.1).

Then one can find generalized harmonic functions $U : D \to \mathbb{R}$ with a source $G \in L^p(D)$ satisfying equation (7.2) such that a.e. on $\partial D$ there exist:

1) the finite limit along the normal $n(\zeta)$

$$U(\zeta) := \lim_{z \to \zeta} U(z) ,$$

2) the normal derivative

$$\frac{\partial U}{\partial n}(\zeta) := \lim_{t \to 0} \frac{U(\zeta + t \cdot n(\zeta)) - U(\zeta)}{t} = \varphi(\zeta) ,$$

3) the angular limit

$$\lim_{z \to \zeta} \frac{\partial U}{\partial n}(z) = \frac{\partial U}{\partial n}(\zeta) .$$

8 On Poincare problem in Jordan domains with rectifiable boundaries

Theorem 4. Let $D$ be a Jordan domain with a rectifiable boundary and let $\nu : \partial D \to \mathbb{C}, |\nu| \equiv 1$, and $\varphi : \partial D \to \mathbb{R}$ be measurable over the natural parameter.

Suppose that $H : D \to \mathbb{R}$ is in $L^p(D)$ for $p > 2$ with compact support in $D$ and $Q : \mathbb{R} \to \mathbb{R}$ is a continuous function with

$$\lim_{t \to \infty} \frac{Q(t)}{t} = 0 . \quad (8.1)$$

Then there is a function $U : D \to \mathbb{R}$ in $W^{2,p}_{loc}(D) \cap C^{1,\alpha}_{loc}(D)$ with $\alpha = (p-2)/p,$

$$\Delta U(\xi) = H(\xi) \cdot Q(U(\xi)) \quad \text{a.e. in } D , \quad (8.2)$$

and the angular limits

$$\lim_{\xi \to \omega} \frac{\partial U}{\partial \nu}(\xi) = \varphi(\omega) \quad \text{a.e. on } \partial D . \quad (8.3)$$
Proof. Arguing similarly to the first and second items in the proof of Theorem 2, we see that \( \tilde{\nu} := \nu \circ c_*^{-1} \) and \( \tilde{\varphi} := \varphi \circ c_*^{-1} \) are measurable over natural parameters, where \( c_* := \tilde{c}|_{\partial D} : \partial D \to \partial \mathbb{D} \) is the restriction to the boundary of the homeomorphic extension \( \tilde{c} \) of \( c \) to \( \mathbb{D} \) onto \( \mathbb{D} \).

Now, set \( \tilde{H} = |C'|^2 \cdot H \circ C \), where \( C := c^{-1} : \mathbb{D} \to D \). Then it is clear by the hypotheses of Theorem 4 that \( \tilde{H} \) has compact support in \( D \) and belongs to the class \( L^p(D) \). Consequently, by Theorem 3 there is \( \tilde{U} : D \to \mathbb{R} \) in \( W^{1,p}_{\text{loc}}(D) \cap C^{1,\alpha}_{\text{loc}}(D) \) with \( \alpha = (p-2)/p \),

\[
\triangle \tilde{U}(z) = \tilde{H}(z) \cdot Q(\tilde{U}(z)) \quad \text{a.e. in } D \tag{8.4}
\]

and the angular limits

\[
\lim_{z \to \zeta} \frac{\partial \tilde{U}}{\partial \nu}(z) = \tilde{\varphi}(\zeta) \quad \text{a.e. on } \partial D . \tag{8.5}
\]

Moreover, by Remark 8 the function \( \tilde{U} \) is a generalized harmonic function with a source \( \tilde{G} \in L^p(D), \tilde{U} = \mathcal{P}_{\tilde{G}}^* \) where \( \mathcal{P}_{\tilde{G}}^* \) is the Poincare operator described in Section 6, Lemma 2, and associated with \( \tilde{\nu} \) and \( \tilde{\varphi} \), and the support of \( \tilde{G} \) is in the support of \( \tilde{H} \) and the upper bound of \( \|	ilde{G}\|_p \) depends only on \( \|	ilde{H}\|_p \) and on the function \( Q \).

In addition, \( \tilde{G} : \mathbb{D} \to \mathbb{C} \) is a fixed point of the nonlinear operator \( \tilde{\Omega}_{G_*} := \tilde{H} \cdot Q(\mathcal{P}_{\tilde{G}_*}^*) : L^p(B) \to L^p(B) \), where \( L^p(B) \) consists of functions \( G_* \) in \( L^p(D) \) with supports in the support of \( \tilde{H} \).

Next, setting \( U = \tilde{U} \circ c \), by simple calculations, see e.g. Section 1.C in [1], we obtain that \( \triangle U = |c'|^2 \cdot \triangle \tilde{U} \circ c \) and, consequently, the function \( U : D \to \mathbb{C} \) is in the class \( W^{1,p}_{\text{loc}}(D) \cap C^{1,\alpha}_{\text{loc}}(D) \) with \( \alpha = (p-2)/p \) that satisfies equation (8.2), \( U \) is a generalized harmonic function with a source \( G \in L^p(D) \) and, moreover, \( U(\xi) = \mathcal{P}_G^*(c(\xi)) \), where \( \mathcal{P}_G^* \) is the Poincare operator from Section 6, \( G = \tilde{G} \circ c \), and the support of \( G \) is in the support of \( H \) and the upper bound of \( \|G\|_p \) depends only on \( \|H\|_p \), the function \( Q \) and the domain \( D \).

Finally, arguing similarly to the last item in the proof of Theorem 2, we show that (8.5) implies (8.3). \( \square \)
Remark 9. Moreover, by the proof of Theorem 4 the function $U$ is a generalized harmonic function with a source $G$ in the class $L^p(D)$ whose support is in the support of $H$ and the upper bound of $\|G\|_p$ depends only on $\|H\|_p$, the function $Q$ and the domain $D$.

In addition, $G = \tilde{G} \circ c$ and $U = \mathcal{P}^*_G \circ c$, where $c$ is a conformal mapping of $D$ onto $\mathbb{D}$, $\tilde{G} : \mathbb{D} \to \mathbb{C}$ is a fixed point of the nonlinear operator $\tilde{\Omega}_G := \tilde{H} \cdot Q(\mathcal{P}^*_G) : L^p_{\tilde{H}}(\mathbb{D}) \to L^p_{\tilde{H}}(\mathbb{D})$, where $L^p_{\tilde{H}}(\mathbb{D})$ consists of functions $G_*$ in $L^p(\mathbb{D})$ with supports in the support of $\tilde{H} := H \circ c \cdot \overline{c'}$, $C = c^{-1}$, $\mathcal{P}^*_G$ is the Poincare operator described in Section 6 and associated with $\tilde{\nu} = \nu \circ c_*^{-1}$ and $\tilde{\varphi} = \varphi \circ c_*^{-1}$. Here $c_* : \partial D \to \partial \mathbb{D}$ is the homeomorphic boundary correspondence under the mapping $c$.

We are able to say more in Theorem 4 for the case of $\Re n(\zeta) \nu(\zeta) > 0$, where $n(\zeta)$ is the inner normal to $\partial D$ at the point $\zeta$. Indeed, the latter magnitude is a scalar product of $n = n(\zeta)$ and $\nu = \nu(\zeta)$ interpreted as vectors in $\mathbb{R}^2$ and it has the geometric sense of projection of the vector $\nu$ into $n$. In view of (8.3), since the limit $\varphi(\zeta)$ is finite, there is a finite limit $U(\zeta)$ of $U(z)$ as $z \to \zeta$ in $D$ along the straight line passing through the point $\zeta$ and being parallel to the vector $\nu$ because along this line

$$U(z) = U(z_0) - \int_0^1 \frac{\partial U}{\partial \nu}(z_0 + \tau(z - z_0)) \, d\tau. \quad (8.6)$$

Thus, at each point with condition (8.3), there is the directional derivative

$$\frac{\partial U}{\partial \nu}(\zeta) := \lim_{t \to 0} \frac{U(\zeta + t \cdot \nu) - U(\zeta)}{t} = \varphi(\zeta). \quad (8.7)$$

In particular, in the case of the Neumann problem, $\Re n(\zeta) \nu(\zeta) \equiv 1 > 0$, where $n = n(\zeta)$ denotes the unit interior normal to $\partial D$ at the point $\zeta$, and we have by Theorem 4 and Remark 9 the following significant result.

**Corollary 4.** Let $D$ be a Jordan domain in $\mathbb{C}$ with a rectifiable boundary and $\varphi : \partial D \to \mathbb{R}$ be measurable over the natural parameter.
Suppose that \( H : D \to \mathbb{R} \) is in \( L^p(D) \), \( p > 2 \), with compact support in \( D \). Then one can find a generalized harmonic function \( U : D \to \mathbb{R} \) with a source \( G \in L^p(D) \) satisfying equation (8.2) such that a.e. on \( \partial D \) there exist:

1) the finite limit along \( n(\zeta) \)

\[
U(\zeta) := \lim_{z \to \zeta} U(z),
\]

2) the derivative

\[
\frac{\partial U}{\partial n}(\zeta) := \lim_{t \to 0} \frac{U(\zeta + t \cdot n(\zeta)) - U(\zeta)}{t} = \varphi(\zeta),
\]

3) the angular limit

\[
\lim_{z \to \zeta} \frac{\partial U}{\partial n}(z) = \frac{\partial U}{\partial n}(\zeta),
\]

where \( n = n(\zeta) \) is the unit inner normal at points \( \zeta \in \partial D \).

9 The Poincare problem in physical applications

Theorem 4 on the Poincare boundary-value problem with arbitrary measurable boundary data over the natural parameter in Jordan domains with rectifiable boundaries can be applied to mathematical models of physical and chemical absorption with diffusion, plasma states, stationary burning etc.

The first circle of such applications is relevant to reaction-diffusion problems. Problems of this type are discussed in [4], p. 4, and, in detail, in [2]. A nonlinear system is obtained for the density \( U \) and the temperature \( T \) of the reactant. Upon eliminating \( T \) the system can be reduced to equations of the type (8.2),

\[
\Delta U = \sigma \cdot Q(U)
\]

(9.1)

with \( \sigma > 0 \) and, for isothermal reactions, \( Q(U) = U^\beta \) where \( \beta > 0 \) that is called the order of the reaction. It turns out that the density of the reactant \( U \) may be zero in a subdomain called a dead core. A particularization of results in Chapter 1 of [4] shows that a dead core may exist just if and only if \( \beta \in (0, 1) \) and \( \sigma \) is
large enough, see also the corresponding examples in [10]. In this connection, the following statement may be of independent interest.

**Corollary 5.** Let $D$ be a Jordan domain in $\mathbb{C}$ with a rectifiable boundary, $\nu : \partial D \to \mathbb{C}, |\nu| \equiv 1$, and $\varphi : \partial D \to \mathbb{R}$ be measurable over the natural parameter.

Suppose that $H : D \to \mathbb{R}$ is a function in the class $L^p(D)$ for $p > 2$ with compact support in $D$.

Then there is a solution $U : D \to \mathbb{R}$ in the class $W^{2,p}_{loc}(D) \cap C^{1,\alpha}_{loc}(D)$ with $\alpha = (p - 2)/p$ of the semi-linear Poisson equation

$$\triangle U(\xi) = H(\xi) \cdot U^\beta(\xi), \quad 0 < \beta < 1, \quad a.e. \text{ in } D \quad (9.2)$$

satisfying the Poincare boundary condition on directional derivatives

$$\lim_{\xi \to \omega} \frac{\partial U}{\partial \nu}(\xi) = \varphi(\omega) \quad a.e. \text{ on } \partial D \quad (9.3)$$

in the sense of the angular limits.

Note also that certain mathematical models of a thermal evolution of a heated plasma lead to nonlinear equations of the type $[9,11]$. Indeed, it is known that some of them have the form $\triangle \psi(u) = f(u)$ with $\psi'(0) = \infty$ and $\psi'(u) > 0$ if $u \neq 0$ as, for instance, $\psi(u) = |u|^{q-1}u$ under $0 < q < 1$, see e.g. [4]. With the replacement of the function $U = \psi(u) = |u|^q \cdot \text{sign } u$, we have that $u = |U|^Q \cdot \text{sign } U$, $Q = 1/q$, and, with the choice $f(u) = |u|^q \cdot \text{sign } u$, we come to the equation $\triangle U = |U|^q \cdot \text{sign } U = \psi(U)$.

**Corollary 6.** Let $D$ be a Jordan domain in $\mathbb{C}$ with a rectifiable boundary, $\nu : \partial D \to \mathbb{C}, |\nu| \equiv 1$ and $\varphi : \partial D \to \mathbb{R}$ be measurable over the natural parameter.

Suppose also that $H : D \to \mathbb{R}$ is a function in the class $L^p(D)$ for $p > 2$ with compact support in $D$.

Then there is a solution $U : D \to \mathbb{R}$ in the class $W^{2,p}_{loc}(D) \cap C^{1,\alpha}_{loc}(D)$ with $\alpha = (p - 2)/p$ of the semi-linear Poisson equation

$$\triangle U(\xi) = H(\xi) \cdot |U(\xi)|^{\beta-1}U(\xi), \quad 0 < \beta < 1, \quad a.e. \text{ in } D \quad (9.4)$$
satisfying the Poincare boundary condition on directional derivatives (9.3).

Finally, we recall that in the combustion theory, see e.g. [3], [19] and the references therein, the following model equation
\[ \frac{\partial u(z,t)}{\partial t} = \frac{1}{\delta} \cdot \Delta u + e^u, \quad t \geq 0, \quad z \in D, \] (9.5)
takes a special place. Here \( u \geq 0 \) is the temperature of the medium and \( \delta \) is a certain positive parameter. We restrict ourselves here by the stationary case, although our approach makes it possible to study the parabolic equation (9.5), see [10]. Namely, the corresponding equation of the type (8.2) is appeared here after the replacement of the function \( u \) by \(-u\) with the function \( Q(u) = e^{-u} \) that is bounded at all.

**Corollary 7.** Let \( D \) be a Jordan domain in \( \mathbb{C} \) with a rectifiable boundary, \( \nu : \partial D \to \mathbb{C}, |\nu| \equiv 1 \), and \( \varphi : \partial D \to \mathbb{R} \) be measurable over the natural parameter.

Suppose also that \( H : D \to \mathbb{R} \) is a function in the class \( L^p(D) \) for \( p > 2 \) with compact support in \( D \).

Then there is a solution \( U : D \to \mathbb{R} \) in the class \( W^{2,p}_{\text{loc}}(D) \cap C^{1,\alpha}_{\text{loc}}(D) \) with \( \alpha = (p - 2)/p \) of the semi-linear Poisson equation
\[ \Delta U(\xi) = H(\xi) \cdot e^{U(\xi)} \quad \text{a.e. in } D \] (9.6)
satisfying the Poincare boundary condition on directional derivatives (9.3).

**Remark 10.** In the above corollaries \( U \) is a generalized harmonic function with a source \( G \in L^p(D) \) whose support is in the support of \( H \) and the upper bound of \( \|G\|_p \) depends only on \( \|H\|_p \), the function \( Q \) and the domain \( D \).

In addition, \( G = \tilde{G} \circ c \) and \( U = \mathcal{P}_G^* \circ c \), where \( c \) is a conformal mapping of \( D \) onto \( \mathbb{D} \), \( \tilde{G} : \mathbb{D} \to \mathbb{C} \) is a fixed point of the nonlinear operator \( \tilde{\Omega}_{G_\ast} := \tilde{H} \cdot Q(\mathcal{P}_{G_\ast}^*) : L^p_\tilde{H}(\mathbb{D}) \to L^p_\tilde{H}(\mathbb{D}) \), where \( L^p_\tilde{H}(\mathbb{D}) \) consists of functions \( G_\ast \) in \( L^p(\mathbb{D}) \) with supports in the support of \( \tilde{H} := H \circ C \cdot C^{-1}, C = c^{-1}, \mathcal{P}_G^* \) is the Poincare operator described in Section 6 and associated with \( \tilde{\nu} = \nu \circ c^{-1} \) and \( \tilde{\varphi} = \varphi \circ c^{-1} \).

Here \( G_\ast : \partial D \to \partial \mathbb{D} \) is the homeomorphic boundary correspondence under the mapping \( c \).
10 Neumann problem in physical applications

In turn, Corollary 4 can be applied to the study of the physical phenomena discussed by us in the last section. In the connection, the particular cases of the function $Q(t)$ of the forms $t^\beta$ and $|t|^{\beta-1}t$ with $\beta \in (0, 1)$ and $e^t$ will be useful.

**Corollary 8.** Let $D$ be a Jordan domain in $\mathbb{C}$ with a rectifiable boundary and $\varphi : \partial D \to \mathbb{R}$ be measurable over the natural parameter. Suppose that $H : D \to \mathbb{R}$ is a function in the class $L^p(D)$, $p > 2$, with compact support in $D$.

Then one can find a generalized harmonic function $U : D \to \mathbb{R}$ with a source $G \in L^p(D)$ that is a solution $U : D \to \mathbb{R}$ in the class $W^{2,p}_{\text{loc}}(D) \cap C^{1,\alpha}_{\text{loc}}(D)$ with $\alpha = (p - 2)/p$ of the semi-linear Poisson equation

$$
\triangle U(\xi) = H(\xi) \cdot U^\beta(\xi), \quad 0 < \beta < 1, \quad \text{a.e. in } D \quad (10.1)
$$

such that a.e. on $\partial D$ there exist:

1) the finite limit along the $n(\zeta)$

$$
U(\zeta) := \lim_{z \to \zeta} U(z),
$$

2) the derivative

$$
\frac{\partial U}{\partial n}(\zeta) := \lim_{t \to 0} \frac{U(\zeta + t \cdot n(\zeta)) - U(\zeta)}{t} = \varphi(\zeta),
$$

3) the angular limit

$$
\lim_{z \to \zeta} \frac{\partial U}{\partial n}(z) = \frac{\partial U}{\partial n}(\zeta),
$$

where $n = n(\zeta)$ is the unit inner normal at points $\zeta \in \partial D$.

**Corollary 9.** Under hypotheses of Corollary 8, there is a solution $U$ in the class $W^{2,p}_{\text{loc}}(D) \cap C^{1,\alpha}_{\text{loc}}(D)$ with $\alpha = (p - 2)/p$ of the semi-linear Poisson equation

$$
\triangle U(\xi) = H(\xi) \cdot |U(\xi)|^{\beta-1}U(\xi), \quad 0 < \beta < 1, \quad \text{a.e. in } D \quad (10.2)
$$

such that a.e. on $\partial D$ all the conclusion 1)-3) of Corollary 8 hold, i.e., $U$ is a generalized solution of the Neumann problem for (10.2) in the given sense.
**Corollary 10.** Under hypotheses of Corollary 8, there is a solution $U$ in the class $W^{2,p}_{\text{loc}}(D) \cap C^{1,\alpha}_{\text{loc}}(D)$ with $\alpha = (p - 2)/p$ of the semi-linear Poisson equation

$$\triangle U(\xi) = H(\xi) \cdot e^{U(\xi)} \quad \text{a.e. in } D \quad (10.3)$$

such that a.e. on $\partial D$ all the conclusion 1)-3) of Corollary 8 hold, i.e., $U$ is a generalized solution of the Neumann problem for (10.3) in the given sense.

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