On automorphisms of finite $p$-groups

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It is proved in [J. Group Theory, 10 (2007), 859-866] that if $G$ is a finite $p$-group such that $(G, Z(G))$ is a Camina pair, then $|G|$ divides $|\text{Aut}(G)|$. We give a very short and elementary proof of this result.

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1 Introduction Let $G$ be a finite non-abelian $p$-group. The problem “Does the order, if it is greater than $p^2$, of a finite non-cyclic $p$-group divide the order of its automorphism group?” is a well-known problem [6, Problem 12.77] in finite group theory. Gaschütz [4] proved that any finite $p$-group of order at least $p^2$ admits a non-inner automorphism of order a power of $p$. It follows that the problem has an affirmative answer for finite $p$-groups with center of order $p$. This immediately answers the problem positively for finite $p$-groups of maximal class. Otto [7] also gave an independent proof of this result.

Fouladi et al. [3] gave a supportive answer to the problem for finite $p$-groups of co-class 2. For more details on this problem, one can see the introduction in the paper of Yadav [8]. In [8, Theorem A], Yadav proved that if $G$ is a finite $p$-group such that $(G, Z(G))$ is a Camina pair, then $|G|$ divides $|\text{Aut}(G)|$. He also proved the important result [8, Corollary 4.4] that the group of all class-preserving outer automorphisms is non-trivial for finite $p$-groups $G$ with $(G, Z(G))$ a Camina pair.

In this paper, we give different and very short proofs of these results of Yadav using elementary arguments.

Let $G$ be a finite $p$-group. Then $(G, Z(G))$ is called a Camina pair if $xZ(G) \subseteq x^G$ for all $x \in G - Z(G)$, where $x^G$ denotes the conjugacy class of $x$ in $G$. In particular, if $(G, G')$ is a Camina pair, then $G$ is called a Camina $p$-group.

2 Proofs We shall need the following lemma which is a simple modification of a lemma of Alperin [12, Lemma 3].

Lemma 2.1. Let $G$ be any group and $B$ be a central subgroup of $G$ contained in a normal subgroup $A$ of $G$. Then the group $\text{Aut}_A^B(G)$ of all automorphisms of $G$ that induce the identity on both $A$ and $G/B$ is isomorphic onto $\text{Hom}(G/A, B)$.

Theorem 2.2. Let $G$ be a finite $p$-group such that $(G, Z(G))$ is a Camina pair. Then $|G|$ divides $|\text{Aut}(G)|$.

Proof. Observe that $Z(G) \leq G' \leq \Phi(G)$ and, therefore, $Z(G) \leq Z(M)$ for every maximal subgroup $M$ of $G$. Suppose that $Z(G) < Z(M_1)$ for some maximal subgroup $M_1$ of $G$. Let $G = M_1 \langle g_1 \rangle$, where $g_1 \in G - M_1$ and $g_1^2 \in M_1$. Let $g \in Z(M_1) - Z(G)$. Then

$$|Z(G)| \leq ||g, G|| = ||g, M_1 \langle g_1 \rangle|| = ||g, \langle g_1 \rangle|| \leq p$$
implies that \(|Z(G)| = p\). The result therefore follows by Gaschütz [3]. We therefore suppose that \(Z(G) = Z(M)\) for every maximal subgroup \(M\) of \(G\). We prove that \(C_G(M) \leq M\). Assume that there exists \(g_0 \in C_G(M_0) - M_0\) for some maximal subgroup \(M_0\) of \(G\). Then \(G = M_0(g_0)\) and thus \(g_0 \in Z(G)\), because \(g_0\) commutes with \(M_0\). This is a contradiction because \(Z(G) \leq \Phi(G)\). Therefore \(C_G(M) \leq M\) for every maximal subgroup \(M\) of \(G\). Consider the group \(\text{Aut}^{Z(G)}_M(G)\) which is isomorphic to \(\text{Hom}(G/M, Z(G))\) by Lemma 2.1. It follows that \(\text{Aut}^{Z(G)}_M(G)\) is non-trivial. Let \(\alpha \in \text{Aut}^{Z(G)}_M(G) \cap (\text{Inn}(G))\). Then \(\alpha\) is an inner automorphism induced by some \(g \in C_G(M) = Z(M)\). Since \(Z(G) = Z(M)\), \(\alpha\) is trivial. It follows that

\[ |(\text{Aut}^{Z(G)}_M(G))((\text{Inn}(G))| = |(\text{Aut}^{Z(G)}_M(G))((\text{Inn}(G))| = |Z(G)||G/Z(G)| = |G|, \]

because \(Z(G)\) is elementary abelian by Theorem 2.2 of [3]. This completes the proof. 

**Corollary 2.3.** Let \(G\) be a finite Camina \(p\)-group. Then \(|G|\) divides \(|\text{Aut}(G)|\).

**Proof.** It is a well known result [2] that nilpotence class of \(G\) is at most 3. Also, it follows from [5, Lemma 2.1, Theorem 5.2, Corollary 5.3] that \((G, Z(G))\) is a Camina pair. The result therefore follows from Theorem 2.2.

An automorphism \(\alpha\) of \(G\) is called a class-preserving automorphism of \(G\) if \(\alpha(x) \in x^G\) for each \(x \in G\). The group of all class-preserving automorphisms of \(G\) is denoted by \(\text{Aut}_c(G)\). An automorphism \(\beta\) of \(G\) is called a central automorphism if \(x^{-1}\beta(x) \in Z(G)\) for each \(x \in G\). It is easy to see that if \((G, Z(G))\) is a Camina pair, then the group of all central automorphisms fixing \(Z(G)\) element-wise is contained in \(\text{Aut}_c(G)\).

**Remark 2.4.** It follows from the proof of Theorem 2.2 that if \(G\) is a finite \(p\)-group such that \((G, Z(G))\) is a Camina pair and \(|Z(G)| \geq p^2\), then

\[ |\text{Aut}_c(G)| \geq |(\text{Aut}^{Z(G)}_M(G))((\text{Inn}(G))| = |G|. \]

Thus, in particular, we obtain the following result of Yadav [8].

**Corollary 2.5 ( [8 Corollary 4.4]).** Let \(G\) be a finite \(p\)-group such that \((G, Z(G))\) is a Camina pair and \(|Z(G)| \geq p^2\). Then \(\text{Aut}_c(G)/\text{Inn}(G)\) is non-trivial.

The following example shows that Remark 2.4 is not true if \(|Z(G)| = p\).

**Example 2.6.** Consider a finite \(p\)-group \(G\) of nilpotence class 2 such that \((G, Z(G))\) is a Camina pair and \(|Z(G)| = p\). Since \(\text{cl}(G) = 2\), \(\exp(G/Z(G)) = \exp(G')\) and hence \(G' = Z(G) = \Phi(G)\). Let \(|G| = p^n\), where \(n \geq 3\), and let \(\{x_1, x_2, \ldots, x_{n-1}\}\) be the minimal generating set of \(G\). Then

\[ |\text{Aut}_c(G)| \leq \prod_{i=1}^{n-1} |x_i^G| = p^{n-1} = |G/Z(G)|. \]

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