Nonminimal couplings, gravitational waves, and torsion in Horndeski’s theory

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(Dated: April 11, 2017)

The Horndeski Lagrangian brings together all possible interactions between gravity and a scalar field that yield second-order field equations in four-dimensional spacetime. As originally proposed, it only addresses phenomenology without torsion, which is a non-Riemannian feature of geometry. Since torsion can potentially affect interesting phenomena such as gravitational waves and early Universe inflation, in this paper we allow torsion to exist and propagate within the Horndeski framework. To achieve this goal, we cast the Horndeski Lagrangian in Cartan’s first-order formalism, and introduce wave operators designed to act covariantly on p-form fields that carry Lorentz indices. We find that nonminimal couplings and second-order derivatives of the scalar field in the Lagrangian are indeed generic sources of torsion. Metric perturbations couple to the background torsion and new torsional modes appear. These may be detected via gravitational waves but not through Yang–Mills gauge bosons.

PACS numbers: 04.50.+h
Keywords: Horndeski Theory, Nonvanishing Torsion, Gravitational waves

I. INTRODUCTION

Recently, there has been a surge of interest in Horndeski’s theory [1,3], which is the most general four-dimensional scalar-tensor theory of gravity, without torsion, that has second-order field equations.

Torsion, however, can have dramatic effects in the very early universe [7], which is precisely one regime where scalar fields are thought to be relevant.

In the Einstein–Cartan–Sciama–Kibble (ECSK) theory [2], torsion is generated by fermions and affects only fermions. Its effects are in general very weak, since torsional terms are proportional to $\psi^2$ and hence only important when there is a large fermion density (see sec. 8.4 of Ref. [2] and Ref. [9]). Torsional effects of this magnitude will likely go undetected in any foreseeable particle physics experiment. They may be detectable in cosmological scenarios [10] and in theories that go beyond ECSK in four dimensions (see, e.g., Refs. [11, 21]).

Standard model bosons, on the other hand, do not generate and are not affected by torsion [4].

Most strikingly, torsion is a nonpropagating field. In vacuum, the ECSK theory gives zero torsion, so there can be no torsional modes for gravitational waves.

The kind of nonminimal couplings between gravity and a scalar field that appear in Horndeski’s theory can modify the conclusions drawn from the ECSK theory. Toloza and Zanelli [21] have studied the consequences of including in the action a term that is the product of a scalar field and the Euler 4-form density,

$$L_{TZ} = \phi\epsilon_{abcd} R^{ab} \wedge R^{cd},$$  \hspace{1cm} (1)

where $R^{ab}$ stands for the Lorentz curvature two-form. Here they find a twofold surprise: contrary to expectations, the term in eq. (1) produces nontrivial dynamics (in stark contrast with the uncoupled Euler density) with torsion (which normally requires fermionic fields). This term also appears naturally in several contexts (see Refs. [22, 23]).

In this paper, we study Horndeski’s theory with torsion, i.e., a theory whose relation to the original Horndeski theory is the same as that of the ECSK theory to general relativity.

Our main result can be stated as follows: Nonminimal couplings between gravity and a scalar field generically

$$\frac{1}{2} [A_\mu, A_\nu], \text{not } F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu + \frac{1}{2} [A_\mu, A_\nu],$$ which differs from the former when torsion is present. Both points of view have been studied in the literature. Some examples of the former can be found in Refs. [24, 25]. Some examples of the second point of view, coupling YM bosons and torsion, can be found in Refs. [26, 27].
produce torsion. This nontrivial torsion can be thought of as an effective dark matter, which may in principle be detected via gravitational waves but not through Yang–Mills (YM) gauge bosons.

Since torsion is a non-Riemannian feature of geometry, we find it convenient to work with Cartan’s differential geometry formalism. In section II we setup some useful definitions to deal with the Horndeski Lagrangian in the first-order formalism. We write down the Horndeski Lagrangian in its most general form and deduce its field equations from the variations with respect to the independent fields: the vierbein $e^a$, the spin connection $\omega^{ab}$, and the scalar field $\phi$. An interesting analysis regarding the phenomenology between some nonminimal couplings and torsion components can be found in Ref. [34]. The main difference between the sec. II of current work and Ref. [34] is that in the current article we consider the full Horndeski Lagrangian, and in Ref. [34] they explore the idea of torsion as dark energy in cosmological models.

Before plunging into the study of first-order perturbation theory, in section III we pause for a moment in order to study which torsion-aware wave operators are the most appropriate to act on our fields, which are in general different forms that carry Lorentz indices. In this mostly mathematical section we provide a generalized version of the Weitzenböck identity that relates torsion-aware versions of the Laplace–Beltrami and the Laplace–de Rham operators, which may have an interest of its own.

In section IV we establish the first-order perturbation theory for a theory of gravity, in its first-order formalism guise, and a scalar field, and apply it to the most interesting bits of the Horndeski Lagrangian—namely, those that can lead to gravitational waves. This perturbation theory is nontrivial because, to the best of our knowledge, up until now it has been unclear how to separate the metric from the torsional degrees of freedom in the first-order perturbation of the spin connection on backgrounds with curvature and torsion. For flat backgrounds, linearized gravity in first-order formalism can be found in Ref. [35]. The separation that we achieve in eqs. (7)–(7) is novel and, while it serves as a useful step in establishing our main result—nonminimal couplings lead to nontrivial torsion that “hitches a ride” on gravitational waves—, may also find applications elsewhere.

II. FIRST ORDER FORMALISM FOR HORNDESKI’S THEORY

In this section we analyze the general behavior of the Horndeski Lagrangian without imposing the torsionless condition. The result found in Ref. [21] for the Gauss–Bonnet term coupled to a scalar field [cf. eq. (11)] proves to be a general feature, and nonminimal couplings with a scalar field are shown to be sources of torsion.

A. Preliminaries

Let’s begin with some definitions.

We shall take spacetime to be a four-dimensional smooth manifold $M$ with signature $(-+++)$. Greek indices $\mu, \nu, \ldots = 0, 1, 2, 3$ are used for tensor components in the coordinate basis, while lower-case Latin indices $a, b, \ldots = 0, 1, 2, 3$ are used for the Lorentz (orthonormal) basis. The components of the change-of-basis matrix, $e^a_a$, help us define the one-form vierbein as $e^a = e^a_\mu dx^\mu$. The spacetime metric $g_{\mu\nu}$ can be written as

$$ds^2 = g_{\mu\nu}dx^\mu \otimes dx^\nu = \eta_{ab}e^a \otimes e^b,$$

whence $g_{\mu\nu} = \eta_{ab}e^a_\mu e^b_\nu$. The space of all $p$-forms defined on $M$ is denoted as $\Omega^p(M)$.

It proves useful to define an operator $\Sigma_{a_1 \cdots a_q}$ that maps $p$-forms into $(p - q)$-forms,

$$\Sigma_{a_1 \cdots a_q} : \Omega^p(M) \to \Omega^{p-q}(M),$$

and is defined by its action on a $p$-form $\alpha$ as

$$\Sigma^{a_1 \cdots a_q} \alpha = -(-1)^{p(2-q)}* (e^{a_1} \wedge \cdots \wedge e^{a_q} \wedge *\alpha).$$

Here, $*$ stands for the Hodge dual, which maps $p$-forms into $(4 - p)$-forms, $*: \Omega^p(M) \to \Omega^{4-p}(M)$.

When $q = 1$ we find

$$\Sigma^a \alpha = -(* (e^a \wedge *\alpha)).$$

This case is particularly interesting, since $\Sigma_a$ behaves as an exterior derivative: (i) it satisfies Leibniz’s rule,

$$\Sigma_a (\alpha \wedge \beta) = \Sigma_a \alpha \wedge \beta + (-1)^p \alpha \wedge \Sigma_a \beta,$$

and (ii) is nilpotent,

$$\Sigma_a \Sigma^a = 0.$$  

A key difference between $\Sigma_a$ and $d$ is that, while $d$ increases the degree of a differential form by one, $\Sigma_a$ decreases it by the same amount.

In order to write the Horndeski Lagrangian in first-order formalism (and not impose the torsionless condition from the beginning), we will describe the geometry by means of the vierbein one-form $e^a$, the one-form spin connection $\omega^{ab}$, and the scalar field $\phi$. The spin connection and the vierbein represent independent degrees of freedom, and torsion and Lorentz curvature two-forms are given by

$$T^a = De^a = de^a + \omega^a_b \wedge e^b,$$

$$R^{ab} = d\omega^{ab} + \omega^c_a \wedge \omega^{cb}.$$  

2 It is important to notice that the torsionless Horndeski theory has already been studied in the language of differential forms. See Ref. [21] for a treatment of the subject.
A small circle above a quantity will be used to denote the “torsionless version” of that quantity. For instance, the spin connection can always be split as

\[
\omega^{ab} = \hat{\omega}^{ab} + \kappa^{ab},
\]

(10)

where \(\hat{\omega}^{ab}\) stands for the usual torsion-free one-form spin connection derived from the vierbein, and \(\kappa^{ab}\) is the one-form contorsion. In the same way, the Lorentz curvature two-form can be expressed as

\[
R^{ab} = \hat{R}^{ab} + \hat{D}\kappa^{ab} + \kappa^a c \wedge \kappa^b c,
\]

(11)

where \(\hat{R}^{ab}\) is the torsion-independent two-form Riemann curvature, \(\hat{R}^{ab} = d\hat{\omega}^{ab} + \hat{\omega}^a c \wedge \hat{\omega}^b c\), and \(\hat{D}\) stands for the exterior covariant derivative with respect to the torsion-free connection \(\hat{\omega}^{ab}\).

In order to deal with the scalar field \(\phi\) and its derivatives in this first-order formalism context, it proves useful to define the zero-form

\[
Z^a = \Sigma^a d\phi,
\]

(12)

and the one-forms

\[
\pi^a = DZ^a,
\]

(13)

\[
\theta^a = Z^a d\phi.
\]

(14)

Intuitively, one can think of \(Z^a\) as the derivative of \(\phi\) in the direction specified by the \(a\)-index, while \(\pi^a\) and \(\theta^a\) represent \(\partial^2 \phi\) and \((\partial \phi)^2\), respectively.

B. The Horndeski Lagrangian

Using the \(\Sigma^a\) operator and its properties, it is straightforward to work with the Horndeski Lagrangian in the first-order formalism. For instance, the “Fab Four” Lagrangians from Ref. 2 can be rewritten as the four-forms

\[
L_J = \frac{1}{2} V_J (\phi) \epsilon_{abcd}R^{ab} \wedge e^c \wedge \pi^d,
\]

(15)

\[
L_P = \frac{1}{2} V_P (\phi) \epsilon_{abcd}R^{ab} \wedge \theta^c \wedge \pi^d,
\]

(16)

\[
L_G = \frac{1}{2} V_G (\phi) \epsilon_{abcd}R^{ab} \wedge e^c \wedge e^d,
\]

(17)

\[
L_R = \frac{1}{2} V_R (\phi) \epsilon_{abcd}R^{ab} \wedge R^{cd}.
\]

(18)

The Ringo and George cases are straightforward to translate from tensor language into differential forms, while the John and Paul cases prove more interesting. In particular, for Paul it is much more comfortable to work with \(\theta^a\) and \(\pi^a\) instead of the Riemann double-dual.

The same is true for the full Horndeski Lagrangian. In terms of the variables we have defined, the Horndeski Lagrangian four-form reads

\[
L_H (\phi, e, \omega) = \epsilon_{abcd}(2\kappa_1 R^{ab} \wedge e^c \wedge \pi^d +
+ \frac{2}{3} \frac{\partial \kappa_1}{\partial X} \pi^a \wedge \pi^b \wedge \pi^c \wedge \pi^d +
+ 2 \kappa_3 R^{ab} \wedge e^c \wedge \theta^d +
+ \frac{2}{3} \frac{\partial \kappa_3}{\partial X} \theta^a \wedge \pi^b \wedge \pi^c \wedge \pi^d +
+ (F + 2W) R^{ab} \wedge e^c \wedge e^d +
+ \frac{\partial F}{\partial X} \pi^a \wedge \pi^b \wedge e^c \wedge e^d +
+ \frac{\partial W}{\partial X} \theta^a \wedge \pi^b \wedge e^c \wedge e^d +
+ \frac{8}{3} \kappa_5 \pi^a \wedge \pi^b \wedge e^c \wedge e^d +
- \left[ \frac{\partial (F + 2W)}{\partial \phi} - X \kappa_8 \right] \pi^a \wedge e^b \wedge e^c \wedge e^d +
+ \kappa_9 \frac{1}{4!} e^a \wedge e^b \wedge e^c \wedge e^d),
\]

(19)

where the arbitrary functions \((i = 1, 3, 8, 9)\)

\[
k_i = k_i (\phi, X),
\]

(20)

\[
F = F (\phi, X),
\]

(21)

\[
W = W (\phi),
\]

(22)

must satisfy the constraint

\[
C (\phi, X) = \frac{\partial F}{\partial X} - 2 \left( \kappa_3 + 2X \frac{\partial \kappa_3}{\partial X} - \frac{\partial \kappa_1}{\partial \phi} \right) = 0,
\]

(23)

with

\[
X = -\frac{1}{2} Z_a Z^a.
\]

(24)

It is interesting to notice that the Hodge *-operator appears in the Horndeski Lagrangian exclusively through the \(\Sigma^a\) operator. This operator allows us to cast the full Horndeski Lagrangian in an effective Lovelock-like mold 37,38, with the Lorentz one-forms \(\pi^a\) and \(\theta^a\) playing a role similar to that of the vierbein, \(e^a\).

Eq. (19) gives the full Horndeski Lagrangian in Carter’s first-order formalism. The Horndeski Theorem 1 states that, when torsion vanishes, this is the most general scalar-tensor Lagrangian that gives rise to second-order equations for the metric. When torsion is allowed to exist, however, Horndeski’s theorem is no longer valid. Indeed, it is quite easy to come up with new terms, explicitly involving torsion, that don’t spoil the second-order nature of the field equations. For the sake of simplicity, in this article we will concern ourselves solely with the Horndeski Lagrangian as shown in eq. (19). The generalization of the Horndeski theorem for the case of nonvanishing torsion, i.e., the answer to the question “What is the most general Lagrangian that leads to second-order field equations for the metric on a spacetime with torsion?” remains an open problem and will be considered elsewhere.
C. Field Equations

In order to derive the field equations in the first-order paradigm, we treat \( \omega^{ab}, e^a, \) and \( \phi \) as independent degrees of freedom. \(^3\)

Explicitly performing the variation with respect to the spin connection yields the three-form equation \( \mathcal{E}_{ab} = 0, \) where

\[
\mathcal{E}_{ab} = -\epsilon_{abcd} T^c \wedge \left[ \kappa_1 \pi^d + \kappa_3 \theta^d + (F + 2W) e^d \right] + \\
+ \epsilon_{abcd} e^c \wedge \left[ d\kappa_1 \wedge \pi^d + \kappa_1 R^d e^c \right] + d\kappa_3 \wedge \theta^d + \\
- \kappa_3 d\phi \wedge \pi^d + \frac{1}{2} d(F + 2W) \wedge e^d \right] + \\
- \frac{1}{2} (Z_a \epsilon b c e d - Z_b \epsilon c a d e) \left[ \kappa_1 R^c d + \\
+ \pi^c \wedge \left( \frac{\partial \kappa_1}{\partial X} \pi^d + 2 \frac{\partial \kappa_3}{\partial X} \theta^d + \frac{\partial F}{\partial X} e^d \right) \right] + \\
+ \frac{1}{2} \left( \kappa_8 \theta^c - \left[ \frac{\partial}{\partial \phi} (2F + 2W) - X \kappa_8 \right] e^c \right) \wedge e^d \right] \wedge e^c \right)
\]

\[
(25)
\]

The field equations obtained from variation with respect to the vierbein and the scalar field, on the other hand, read

\[
\mathcal{E}_a = E_a + \Sigma^b \left( S_b + T_b + U_b \right) Z_a = 0,
\]

\[
\mathcal{E} = E + \Sigma^b \left( S_b + T_b + U_b \right) = 0,
\]

where

\[
E_d = \epsilon_{abcd} \left[ 2 \kappa_1 R^{ab} \wedge \pi^c + \frac{2}{3} \frac{\partial \kappa_1}{\partial X} \pi^a \wedge \pi^b \wedge \pi^c + \\
+ 2 \kappa_3 R^a e^b + \frac{2}{3} \frac{\partial \kappa_3}{\partial X} \theta^a \wedge \pi^b \wedge \pi^c + \\
+ 2 (F + 2W) R^{ab} \wedge e^c + 2 \frac{\partial F}{\partial X} \pi^a \wedge \pi^b \wedge e^c + \\
+ 2 \kappa_8 \theta^a \wedge \pi^b \wedge e^c + \frac{1}{2} \kappa_8 e^a \wedge e^b \wedge e^c + \\
- 3 \left[ \frac{\partial}{\partial \phi} (2F + 2W) - X \kappa_8 \right] \pi^a \wedge e^b \wedge e^c \right],
\]

\[
(28)
\]

\[
E = \epsilon_{abcd} \left[ 2 \left( \frac{\partial \kappa_1}{\partial \phi} - \kappa_3 \right) R^{ab} \wedge e^c \wedge \pi^d + \\
+ 2 \left( \frac{1}{3} \frac{\partial^2 \kappa_1}{\partial \phi^2} - \frac{\partial \kappa_3}{\partial X} \right) \pi^a \wedge \pi^b \wedge \pi^c \wedge e^d + \\
+ 2 \frac{\partial \kappa_3}{\partial \phi} \theta^a \wedge \pi^b \wedge \pi^c \wedge e^d + \\
+ 2 \frac{\partial^2 \kappa_3}{\partial \phi \partial X} \theta^a \wedge \pi^b \wedge \pi^c \wedge e^d + \\
+ \frac{1}{2} \left( \frac{\partial F}{\partial \phi} + 2 \frac{\partial W}{\partial \phi} \right) R^{ab} \wedge e^c \wedge e^d + \\
+ \frac{1}{2} \left( \frac{\partial^2 F}{\partial \phi \partial X} - \kappa_8 \right) \pi^a \wedge \pi^b \wedge \pi^c \wedge e^d + \\
+ \frac{1}{2} \frac{\partial \kappa_8}{\partial \phi} \theta^a \wedge \pi^b \wedge e^c \wedge e^d + \\
- \left[ \frac{\partial^2 (F + 2W)}{\partial \phi^2} - X \frac{\partial \kappa_8}{\partial \phi} \right] \pi^a \wedge e^b \wedge e^c \wedge e^d + \\
+ \frac{1}{4!} \frac{\partial \kappa_8}{\partial \phi} e^a \wedge e^b \wedge e^c \wedge e^d \right],
\]

\[
(29)
\]

\[
Z = \left[ 2 \kappa_3 \wedge R^{ab} + 2 \frac{\partial \kappa_3}{\partial X} \wedge \pi^a \wedge \pi^b + \\
+ d\kappa_8 \wedge \pi^a \wedge e^b + \\
+ d \pi^a \wedge \left( 4 \frac{\partial \kappa_3}{\partial X} \pi^b + \kappa_8 e^b \right) \right] \wedge e^c Z^d + \\
+ 2 \epsilon_{abcd} \left[ \kappa_3 R^c d + \frac{\partial \kappa_3}{\partial X} \pi^a \wedge \pi^b + \\
+ \kappa_8 \pi^a \wedge e^b \right] \wedge T^c Z^d,
\]

\[
(30)
\]

and the common 4-form variables \( S_a, T_a \) and \( U_a \) are given by

\[
S_d = \epsilon_{abcd} \left[ D \pi^a \wedge e^b \wedge \left( 2 \frac{\partial \kappa_1}{\partial X} \pi^c + 2 \frac{\partial \kappa_3}{\partial X} \theta^c + \frac{\partial F}{\partial X} e^c \right) + \\
+ \pi^a \wedge e^b \wedge dX \wedge \left( \frac{\partial^2 \kappa_1}{\partial X^2} \pi^c + 2 \frac{\partial^2 \kappa_3}{\partial X^2} \theta^c + \frac{\partial^2 F}{\partial X^2} e^c \right) + \\
+ \frac{1}{2} \pi^a \wedge e^b \wedge dX \wedge \left( \theta^c \frac{\partial \kappa_3}{\partial X} \pi^d - e^c \frac{\partial F}{\partial X} \frac{\partial \kappa_8}{\partial \phi} \right) \right],
\]

\[
(31)
\]

\[
T_d = 2 \epsilon_{abcd} \left[ \kappa_1 R^{ab} + \frac{\partial \kappa_1}{\partial X} \pi^a \wedge \pi^b + 2 \frac{\partial \kappa_3}{\partial X} \pi^a \wedge \theta^b + \\
+ \frac{2}{3} \frac{\partial F}{\partial X} \pi^a \wedge e^b + \frac{1}{2} \kappa_8 e^a \wedge \theta^b + \\
- \frac{3}{2} \left( \frac{\partial}{\partial \phi} (2F + 2W) - X \kappa_8 \right) \pi^a \wedge e^b \wedge \pi^c \wedge \theta^d \right]
\]

\[
(32)
\]

\[
U_c = \epsilon_{abcd} \left[ -R^{ab} \wedge e^c \wedge \left( C_{ed}^c + \frac{\partial \kappa_1}{\partial X} \delta^d e \theta^e \right) Z \pi^f + \\
- \pi^a \wedge \pi^b \wedge e^c \wedge \left( C_{ed}^c + \frac{2 \partial^2 \kappa_1}{3 \partial X^2} \pi^d \right) Z \pi^f + \\
+ \pi^a \wedge e^b \wedge e^c \wedge M^d e \wedge e^a \wedge e^b \wedge e^c \wedge K^d e \right].
\]

\[
(33)
\]

\(^3\) Note that \( Z^a \) depends on \( e^a \) and the derivatives of \( \phi \) through the \( \Sigma^a \) operator. This dependence must be taken into account when performing the variations with respect to \( e^a \) and \( \phi \).
In eq. $33$, $C^a_b$, $\bar{C}^a_b$, $K^a_b$, and $M^a_b$ are one-forms defined as

\[
C^a_b = 2d\phi \left[ \frac{\partial \kappa_3}{\partial X} Z^a Z_b - \left( \kappa_3 - \frac{\partial \kappa_1}{\partial \phi} \right) \delta^a_b \right] + e^a Z_b \frac{\partial F}{\partial X}, \tag{34}
\]

\[
\bar{C}^a_b = 2d\phi \left[ \frac{\partial \kappa_3}{\partial X^2} Z^a Z_b - \left( 3 \frac{\partial \kappa_1}{\partial X} + \frac{\partial^2 \kappa_1}{\partial \phi \partial X} \right) \delta^a_b \right] + e^a Z_b \frac{\partial^2 F}{\partial X^2}, \tag{35}
\]

\[
K^a_b = \left[ \frac{\partial^2 F}{\partial \phi^2} (F + 2W) - X \frac{\partial \kappa_8}{\partial \phi} \right] d\phi \delta^a_b + \frac{1}{4!} e^a Z_b \frac{\partial \kappa_9}{\partial X}, \tag{36}
\]

\[
M^a_b = \left( 2 \left[ \kappa_8 - \frac{\partial^2 F}{\partial \phi \partial X} \right] \delta^a_b - \frac{\partial \kappa_8}{\partial X} Z^a Z_b \right) d\phi + e^a Z_b \frac{\partial F}{\partial \phi} - X \kappa_8 . \tag{37}
\]

The one-forms $C^a_b$ and $\bar{C}^a_b$ satisfy the properties

\[
\Sigma^b C^a_a = Z^a C, \tag{38}
\]

\[
\Sigma^b \bar{C}^a_a = Z^a \frac{\partial C}{\partial X}, \tag{39}
\]

where $C (\phi, X) = 0$ is the Horndeski constraint $20$. Here it is important to observe that in the terms $Z$ and $T_a$ torsion appears explicitly as a result of nonminimal couplings. Torsional degrees of freedom are also contained inside the Lorentz curvature through the torsionless one-form, as shown in eq. $11$.

As the quickest glance at eqs. $29$-$31$ will show, the full Horndeski theory is extremely complicated. Actually, it may be more accurate to think of it as a family of theories, each one defined by a choice of the arbitrary functions $\kappa_i$, $F$, and $W$. There are, however, several general observations to be made.

First, using the properties of the $\Sigma^a$ operator it is straightforward to derive the field equations obtained from the independent variations of $e^a$ and $\omega^{ab}$, without imposing the torsionless condition. To achieve the same feat in the standard Palatini tensor formalism would have been impractical, to say the least.

Second, the field equations directly show that torsion arises, in general, from every nonminimal coupling with the scalar field, and from the terms depending on $\pi^a = D \Sigma^a \phi$. For instance, see the term $T_a$ in eqs. $29$ and $31$, and the first term of eq. $29$.

In order to recover the standard torsionless dynamics, one cannot simply impose $T^a = 0$ on the equations of motion. This happens because, in the general setting, the dynamics of $\phi$ and the torsion become fully intertwined, generically leading to $T^a \sim \partial \phi$. Therefore, imposing $T^a = 0$ in these cases will lead to $\phi = \text{const.}$, freezing the dynamics of the scalar field. The important point here is that the standard torsionless case corresponds to a constraint on the more general Cartan geometry framework. This problem seems to have been known for a long time; see, e.g., sec. 1.7.1 of Ref. $32$. The solution for it can be be written in a very practical way in terms of the $\Sigma^a$ operator. First, we have to include the torsionless condition via a two-form Lagrangian multiplier with a Lorentz index, $\Lambda_a$,

\[
L_H \rightarrow \bar{L}_H = L_H + \Lambda_a \wedge T^a, \tag{40}
\]

whence we get the new equations of motion

\[
\bar{\mathcal{E}}_a = \mathcal{E}_a - D \Lambda_a = 0, \tag{41}
\]

\[
\bar{\mathcal{E}} = \mathcal{E} = 0, \tag{42}
\]

\[
\bar{\mathcal{E}}^{ab} = \mathcal{E}^{ab} - \frac{1}{2} \left( \Lambda^a \wedge e^b - \Lambda^b \wedge e^a \right) = 0, \tag{43}
\]

\[
T^a = 0. \tag{44}
\]

Using the $\Sigma^a$ operator, it is possible to solve $\bar{\mathcal{E}}^{ab} = 0$ for $\Lambda_a$. We find

\[
\Lambda^a = 2 \Sigma_b \mathcal{E}^{ab} + \frac{1}{2} e^a \wedge \Sigma_b \mathcal{E}^{bc}. \tag{45}
\]

Therefore, the standard field equations for the torsionless Horndeski theory are recovered in this setting as

\[
\mathcal{E}^a - 2 D \Sigma_b \mathcal{E}^{ab} + \frac{1}{2} e^a \wedge d \Sigma_b \mathcal{E}^{bc} \bigg|_{T^a = 0} = 0, \tag{46}
\]

\[
\mathcal{E} \bigg|_{T^a = 0} = 0. \tag{47}
\]

This behavior is in stark contrast with the standard Einstein–Cartan case with minimally coupled fields. In this case, $T^a = 0$ is an equation of motion in vacuum, and therefore it is unnecessary to use a Lagrangian multiplier. In fact, in this case only fermionic fields can be a source of non-propagating torsion (see, e.g., sec. 8.4 of Ref. $3$).

### III. Wave Operators, Torsion, and the Weitzenböck Identity

Our goal in this section is to define a wave operator that can act on differential forms that carry Lorentz indices, such as the vierbein, $e^a$. We need this operator because our treatment of gravitational waves relies on perturbations of the vierbein and the spin connection, $\omega^{ab}$, which are the natural independent degrees of freedom for a spacetime with torsion.

Let $\Phi$ be a scalar (i.e., without Lorentz indices) $p$-form,

\[
\Phi = \frac{1}{p!} \Phi_{\mu_1 \cdots \mu_p} dx^\mu_1 \wedge \cdots \wedge dx^\mu_p. \tag{48}
\]

There are at least two wave operators that can conceivably act on $\Phi$. The Laplace–de Rham operator,

\[
\Box_d R = d^\dagger d + dd^\dagger, \tag{49}
\]

is defined as the anticommutator of the exterior derivative and the exterior coderivative, $d^\dagger = *d*$ [for dimensions other than four or signatures other than $(-++++)$].
TABLE I. Many different derivatives are defined in this section. This table collects all definitions and some of their most important properties.

| Symbol | Definition | Change in form degree | Key property |
|--------|------------|-----------------------|--------------|
| d      | dxμ∂a     | +1                    | ds² = 0      |
| D      | d + ω     | +1                    |              |
| D̃     | d + ‾ω     | +1                    |              |
| d†     | *d*       | −1                    |              |
| D†     | *D*       | −1                    |              |
| D‡     | *D‡       | −1                    |              |
| Σ_a   | − *(e_a ∧ *) | −1                     | Σ_aΣ_a = 0   |
| D‡     | −Σ_aDΣ_a  | −1                    |              |
| D‡     | −Σ_aDΣ_a  | −1                    | D‡ = D†     |
| Σ_a   | Σ_aD + DΣ_a | 0                     |              |
| Σ_a   | Σ_aD + DΣ_a | 0                     | D_a = e_aμ∇μ |

While unconventional, writing the Weitzenböck identity as in eq. (50) proves to be useful for our purposes and is equivalent to more common approaches. In words, the Weitzenböck identity states that the difference between the standard Laplace–Beltrami operator, which is nontrivial when the geometry has nonvanishing torsion. As a first step, one may be inclined to define the de Rham Lorentz-covariant codervative as D† = *D*, in perfect analogy with d† = *d*. We find, however, that a more useful definition is

\[ D† = -Σ_aDΣ_a. \]  

This is equivalent to the first definition when torsion is zero,

\[ *D* = -Σ_aDΣ_a, \]  

but not in general. What makes definition (55) useful is that the wave operator built from it satisfies a generalized version of the Weitzenböck identity (50).

Let us define the generalized Laplace–de Rham operator as

\[ \Box_{\text{dR}} = DD† + D†D. \]  

It is possible to prove that \( \Box_{\text{dR}} \) satisfies the following generalized Weitzenböck identity:

\[
\begin{align*}
\Box_{\text{dR}} & \Phi^{a_1 \cdots a_m} = \Box_B \Phi^{a_1 \cdots a_m} + \Sigma_c D^2 \Sigma^c \Phi^{a_1 \cdots a_m} \\
& = \Box_B \Phi^{a_1 \cdots a_m} + \Sigma_c (R^c_b \Sigma^b \Phi^{a_1 \cdots a_m} + R^{a_1} b \Sigma^c \Phi^{a_2 \cdots a_m} + \cdots + R^{a_1 a_2} \Sigma^c \Phi^{a_3 \cdots a_m} - b) .
\end{align*}
\]  

In eq. (58) we have introduced the generalized Laplace–Beltrami operator

\[ \Box_B = -D^a D_a , \]  

where

\[ D_a = \Sigma_a D + DΣ_a . \]  

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| Symbol | Definition | Change in form degree | Key property |
|--------|------------|-----------------------|--------------|
| d      | dxμ∂a     | +1                    | ds² = 0      |
| D      | d + ω     | +1                    |              |
| D̃     | d + ‾ω     | +1                    |              |
| d†     | *d*       | −1                    |              |
| D†     | *D*       | −1                    |              |
| D‡     | *D‡       | −1                    |              |
| Σ_a   | − *(e_a ∧ *) | −1                     | Σ_aΣ_a = 0   |
| D‡     | −Σ_aDΣ_a  | −1                    |              |
| D‡     | −Σ_aDΣ_a  | −1                    | D‡ = D†     |
| Σ_a   | Σ_aD + DΣ_a | 0                     |              |
| Σ_a   | Σ_aD + DΣ_a | 0                     | D_a = e_aμ∇μ |

the definition of d† must be modified with a judiciously chosen sign]. This operator satisfies the Weitzenböck identity

\[ \Box_{\text{dR}} \Phi = \Box_B \Phi + \Sigma_a \left( R^c_b \wedge \Sigma^b \Phi \right) , \]  

where \( \Box_B = -\nabla^\mu \nabla_\mu \) is the usual Laplace–Beltrami operator built from the torsion-free covariant derivative \( \nabla_\mu \). While unconventional, writing the Weitzenböck identity as in eq. (50) proves to be useful for our purposes and is equivalent to more common approaches. In words, the Weitzenböck identity states that the difference between the two wave operators acting on \( \Phi \) is related to the curvature of the manifold and does not involve derivatives of \( \Phi \).

By definition, the Laplace–Beltrami operator carries no information about torsion. Since it is the Riemann curvature two-form, \( R^{ab} \), that appears in the second term on the right-hand side of eq. (50), this means that eq. (50) has no information at all about torsion, even when torsion is present in spacetime. This is consistent with the fact that the Laplace–de Rham operator is defined without any reference to torsion.

An example of the usefulness of this construction is provided by classical electromagnetism on a curved spacetime. Let \( A \) be the electromagnetic potential one-form and \( F = dA \) its associated field strength two-form. Maxwell equations in vacuum can be written as

\[ D†F = d†dA = 0 . \]  

Choosing the Lorenz gauge, d†A = 0, we can use eq. (50) to find

\[ \Box_B A + \Sigma_a R^a b \Sigma^b A = 0 , \]  

or, in standard tensor language,

\[ -\nabla^\lambda \nabla_\lambda A_\mu + \hat{R}_{\mu \nu} A^\nu = 0 , \]  

where \( \hat{R}_{\mu \nu} \) is the standard torsionless Ricci tensor. It is interesting to notice that this result holds even when the background geometry has nonvanishing torsion. The electromagnetic field only interacts with the torsionless sector of the geometry. The same happens with all YM gauge bosons: they only can interact with the torsionless sector of the geometry.\(^4\)

Extending the de Rham definition of the wave operator for the case of a p-form with \( m \) free Lorentz indices, such as

\[ \Psi^{a_1 \cdots a_m} = \frac{1}{p!} \Psi^{a_1 \cdots a_m} \rho_1 \cdots \rho_p dx^{\rho_1} \wedge \cdots \wedge dx^{\rho_p} , \]  

is nontrivial when the geometry has nonvanishing torsion. As a first step, one may be inclined to define the de Rham Lorentz-covariant codervative as D† = *D*, in perfect analogy with d† = *d*. We find, however, that a more useful definition is

\[ D† = -\Sigma_a DΣ_a . \]  

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4 See footnote on page.
In the torsionless case, the operator $\mathcal{D}_a = \Sigma_a \mathcal{D} + \mathcal{D} \Sigma_a$ can be shown to satisfy $\mathcal{D}_a = e_a^\mu \nabla_\mu$, meaning that it matches the usual torsionless covariant derivative $\nabla = \partial + \Gamma$, and the standard Weitzenböck identity (50) is recovered.

Some useful properties satisfied by $\mathcal{D}_a$ are:

$$\mathcal{D}_a (\alpha \wedge \beta) = \mathcal{D}_a \alpha \wedge \beta + \alpha \wedge \mathcal{D}_a \beta,$$

(61)

$$[\Sigma_a, \mathcal{D}_b] = - (\Sigma_a \mathcal{T}_c) \Sigma_c,$$

(62)

$$[\mathcal{D}_a, \mathcal{D}_b] = D^2 \Sigma_a + \Sigma_a D^2 + \Sigma_b D^2 \Sigma_a - \Sigma_b D^2 \Sigma_a + - (\Sigma_a \mathcal{T}_c) \Sigma_c - (\Sigma_b \mathcal{T}_c) \mathcal{D}_c,$$

(63)

where $\alpha$ is a $p$-form and $\beta$ is a $q$-form. In particular, eq. (61) implies that $\mathcal{D}_a$ obeys Leibniz’s rule without any correcting signs.

From the above discussion, it seems clear that in order to have waves interacting with torsion, it is necessary for the field to have free Lorentz indices. This is precisely the case of gravitational waves in the Horndeski case, as we shall see in the next section.

IV. GRAVITATIONAL WAVES AND TORSIONAL MODES

Let us consider a background geometry described by $e^a$, $\bar{\omega}^{ab}$ and $\bar{\phi}$. Linear perturbations around this background are described by:

$$e^a \to e^a = e^a + \frac{1}{2} h^a,$$

(64)

$$\bar{\omega}^{ab} \to \omega^{ab} = \bar{\omega}^{ab} + u^{ab},$$

(65)

$$\bar{\phi} \to \phi = \bar{\phi} + \varphi,$$

(66)

where we have the 1-forms $h^a = h^a e^b$ and $u^{ab} = u^{ab} e^c$, and the 0-form $\varphi$. In terms of the metric, to first-order in the perturbations we have:

$$g = \eta_{ab} e^a \otimes e^b$$

$$= \eta_{ab} e^a \otimes e^b + \frac{1}{2} (h_{ab} + h_{ba}) e^a \otimes e^b,$$

$$= (\bar{g}_{\mu\nu} + h^+_\mu) dx^\mu \otimes dx^\nu,$$

where

$$h^\pm_{ab} = \frac{1}{2} (h_{ab} \pm h_{ba})$$

are just the symmetric and anti-symmetric part of $h_{ab}$. The standard theory of gravitational waves is formulated just in terms of $h^+_{ab}$. We could keep this anti-symmetric part $h^-_{ab}$, but at the end it is possible to prove that it corresponds to an infinitesimal local Lorentz transformation. Since the Horndeski Lagrangian eq. (19) is locally Lorentz invariant, it is possible to gauge away that piece and to keep only the symmetric part. Therefore, from now on and for the sake of simplicity we will just suppose that $h_{ab}$ is symmetric $h_{ab} = h_{ba}$.

In standard general relativity the perturbation in the geometry is described only in terms of $h_{\mu\nu}$, since the perturbation in the connection depends on the $h_{\mu\nu}$ through the torsionless condition. But when considering nonvanishing torsion, the vierbein and the spin connection correspond to independent degrees of freedom. Therefore, the perturbation 1-forms $h^a = h^a e^b$ and $u^{ab} = u^{ab} e^c$ are independent too. In fact, it is always possible to split the linear perturbation 1-form $u^{ab}$ in two pieces, one carrying all the dependency in $h^a$ and one completely independent (and associated to linear perturbations in contorsion).

In order to see this, let us write down the 2-form torsion as $T^a = De^a$. Then its linear perturbations under eqs. (64-66) are given by:

$$\bar{T}^a \to T^a = T^a + \frac{1}{2} D h^a + u^a_b \wedge \bar{e}^b;$$

$$= T^a + \frac{1}{2} D h^a + \frac{1}{2} \bar{e}^a_b \wedge h^b + u^a_b \wedge \bar{e}^b.$$

(67)

with

$$D h^a = d h^a + \bar{\omega}^a_b \wedge h^b,$$

where $\bar{\omega}_b^a$ corresponds to the torsionless piece of the background spin connection. But torsion may be written also in terms of the contorsion 1-form $\kappa^{ab}$ as $T^a = \kappa^{a} \wedge \bar{e}^b$, and therefore we have to linear order on perturbations:

$$\bar{T}^a \to T^a = T^a + \frac{1}{2} \bar{\kappa}^a_b \wedge h^b + q^a_b \wedge \bar{e}^b,$$

(68)

where $q^{ab}$ corresponds to perturbation in the contorsion $\bar{\kappa}^{ab} \to \kappa^{ab} = \bar{\kappa}^{ab} + q^{ab}$. The relations eqs. (67) and (68) may seem contradictory, since one of them includes derivatives of $h^a$ and the other doesn’t. There is no contradiction though; it is straightforward to see that $u^{ab}$ must be of the form

$$u^{ab} = \bar{u}^{ab} + q^{ab},$$

where

$$\frac{1}{2} D h^a + \bar{u}^a_b \wedge \bar{e}^b = 0.$$

(69)

In order to avoid an algebraic nightmare mixing both covariant derivatives $D$ and $\bar{D}$, it is convenient to define new perturbation variables as:

$$U_{ab} = \bar{u}_{ab} - \frac{1}{2} \left( \bar{\Sigma}_a \bar{\kappa}_{bc} \otimes h^c - \bar{\Sigma}_b \bar{\kappa}_{ac} \otimes h^c \right),$$

(70)

$$V_{ab} = q_{ab} + \frac{1}{2} \left( \bar{\Sigma}_a \bar{\kappa}_{bc} \otimes h^c - \bar{\Sigma}_b \bar{\kappa}_{ac} \otimes h^c \right),$$

(71)

5 It is very important to remember that in order to study cases of astrophysical interest, it is necessary to go at least to second-order in the perturbations of curvature. In the current article we are not interested in modelling a particular phenomena, but just studying how gravitational waves could interact with torsion at first-order. Detailed calculations to second-order for particular astrophysical situations will be presented elsewhere.
and therefore $u^{ab} = \dot{w}^{ab} + q^{ab} = U^{ab} + \Sigma^{ab}$. After some algebra, and using the fact that contorsion and torsion are related trough

$$\kappa_{ab} = \frac{1}{2} (\Sigma_a T_b - \Sigma_b T_a + e^c \Sigma_{ab} T_c) ,$$

it is possible to prove that eq. (65) becomes

$$\frac{1}{2} \bar{D} \dot{h}^a + U^{ab} \wedge \dot{e}^b + \frac{1}{2} \bar{\Sigma}^a (h_b \wedge \bar{T}^b) = 0 , \tag{72}$$

and eq. (68) becomes the linear order torsion perturbation equation

$$\bar{T}^a \rightarrow T^a = \bar{T}^a + \Sigma^a_b \wedge \dot{e}^b - \frac{1}{2} \bar{\Sigma}^a (h_b \wedge \bar{T}^b) . \tag{73}$$

From eq. (72), and after some algebra, it is possible to get a closed expression for $U^{ab}$,

$$U^{ab} = \frac{1}{2} \left( \Sigma^a \bar{D} h^b - \Sigma^b \bar{D} h^a \right) . \tag{74}$$

Since $\bar{\omega}^{ab} \rightarrow \omega^{ab} = \bar{\omega}^{ab} + U^{ab} + \Sigma^{ab}$, we have to linear order for the Lorentz curvature

$$\bar{R}^{ab} \rightarrow R^{ab} = \bar{R}^{ab} + \bar{D} (U^{ab} + \Sigma^{ab}) . \tag{75}$$

One can show that the scalar field ‘curvature’ $Z^a = \Sigma^a d \phi$ is perturbed to linear order under $\phi \rightarrow \phi = \tilde{\phi} + \varphi$ and $\bar{e}^a \rightarrow e^a = \bar{e}^a + \frac{1}{2} h^a$ as

$$\bar{Z}^a \rightarrow Z^a = \bar{Z}^a + \Sigma^a d \varphi - \frac{1}{2} h^a \bar{Z}^a . \tag{76}$$

As we have seen in section [II] nonminimal couplings and second-order derivatives terms in the Horndeski lagrangian are sources of torsion. In this general case, torsion propagates through the ‘contorsional mode’ $\Sigma^{ab}$ and the background torsion $\bar{T}^a$ interacts with the metric modes $h^a$. However, our intuition may lead us to the idea that the Einstein–Hilbert term can give rise just to the wave equation and interactions of $h^a$ with the background curvature, as in the standard torsionless case. That is not the case. As we shall see, even the Einstein–Hilbert term gives rise to both, metrical modes interacting with the background torsion and propagating torsional modes.

In order to see this, let us consider a lagrangian in the Horndeski family of the form

$$\mathcal{L}^{(4)} (e, \omega, \phi) = \mathcal{L}^{(4)}_{\text{EH}} + \text{other terms},$$

where this other terms are the ones giving rise to torsion through nonminimal couplings and/or second-order derivatives of $\phi$. The Einstein–Hilbert 4-form term is given by

$$\mathcal{L}_{\text{EH}}^{(4)} (e, \omega) = \frac{1}{4 \kappa_4} \epsilon_{abed} R^{ab} \wedge e^c \wedge e^d ,$$

and therefore the equations of motion take the form

$$\delta_e \mathcal{L}^{(4)} (e, \omega, \phi) = \delta_e \mathcal{L}^{(4)}_{\text{EH}} + \delta_e \phi [\text{other terms}] = 0 , \tag{77}$$

$$\delta_\omega \mathcal{L}^{(4)} (e, \omega, \phi) = \delta_\omega \mathcal{L}^{(4)}_{\text{EH}} + \phi \omega [\text{other terms}] = 0 , \tag{78}$$

$$\delta_\phi \mathcal{L}^{(4)} (e, \omega, \phi) = \delta_\phi \phi [\text{other terms}] = 0 , \tag{79}$$

with

$$\delta_e \mathcal{L}^{(4)}_{\text{EH}} (e, \omega) = \frac{1}{2 \kappa_4} \epsilon_{abcd} R^{ab} \wedge e^c \wedge \delta e^d ,$$

$$\delta_\omega \mathcal{L}^{(4)}_{\text{EH}} (e, \omega) = \frac{1}{2 \kappa_4} \epsilon_{abcd} \delta \omega^{ab} \wedge \mathcal{T}^c \wedge \delta e^d.$$
the ‘divergence’ $\mathcal{D}_\alpha h^\alpha = \left( \Sigma_a \mathcal{D} + \tilde{\Sigma}_a \right) h^\alpha$ transforms as $\mathcal{D}_\alpha h^\alpha \rightarrow \mathcal{D}_\alpha h^\alpha$ where $\mathcal{D}_\alpha h^\alpha$ is given by

$$
\mathcal{D}_\alpha h^\alpha = \left[ -\overline{D}_a \overline{D} \zeta_b + \mathcal{D}_\alpha \Sigma_b h^\alpha - \tilde{\Sigma}_{ab} \overline{R}_c \zeta^c + -\tilde{\Sigma}^a \overline{\mathcal{D}}_a \left( \mathcal{D}_c \zeta_b + \overline{D}_b \zeta_c - \eta_{cb} \overline{D}_p \zeta^p \right) + \tilde{\Sigma}_{ab} \overline{\mathcal{D}}_a \Sigma_c h^\alpha \right] e^b. 
$$

(83)

where $\overline{D}_a = \tilde{\Sigma}_a \overline{D} + \overline{\Sigma}_a$. It means we can always choose the ‘Lorenz gauge’

$$
\mathcal{D}_\alpha h^\alpha = 0
$$

with a vector field $\zeta$ such that the right-hand side of eq. (83) vanishes.

Choosing this gauge and using properties eqs. (62) and (63), it is possible to show that eq. (80) in terms of $h^a$ corresponds to

$$
\mathcal{D} \hat{h}^a + \Sigma_{ab} \left( \hat{R}^a_{\ b} \hat{h}^b + \hat{\Theta}^a_{\ b} \hat{h}^b + \hat{\Theta}^a_{\ b} \hat{h}^b \right) + - A_d + B_d + \frac{1}{2} \epsilon_{d} \left( C - \Sigma_{c} \left[ A^c + B^c \right] \right) + + \epsilon_{abc} \left( \hat{R}^a_{\ b} \hat{h}^c + 2 \hat{\Sigma}^a c \hat{h} + \hat{\Theta}^a_{\ b} \hat{h}^c \right) + \cdots = 0
$$

(84)

where ‘$\cdots$’ stands for other Horndeski terms contributions. $\mathcal{D} \hat{h}^a$ is given by the Generalized Weitzenböck identity eq. (65).

$$
\mathcal{D} \hat{h}^a = - \overline{D}_a \overline{D}_a \hat{h}^a + \Sigma_{ab} \left( \hat{R}^a_{\ b} \hat{h}^b - \hat{R}^a_{\ b} \hat{h}^b \right),
$$

and $A_d$, $B_d$ and $C$ are the torsional terms

$$
A_m = (\tilde{\Sigma}_b \mathcal{T}_a) \mathcal{D}^a \hat{h}^b + \hat{h}^a \mathcal{D} \Sigma_{b} \mathcal{T}_a,
$$

$$
B_m = (\Sigma_b \mathcal{T}_a) \Sigma^a \left[ \mathcal{D} \hat{h}^a + \mathcal{D} \hat{h}^a - \mathcal{D} \hat{h}^a \right] + + \frac{1}{2} \left[ \mathcal{D} \mathcal{T}_m \hat{h}^a + \Sigma^a \left( \mathcal{D} \hat{h}^a \mathcal{T}_a - \mathcal{T}_a \mathcal{D} \hat{h}^a \right) \right],
$$

$$
C = \mathcal{D} \hat{h}^a \Sigma_{bc} \mathcal{T}_a + \left( \Sigma_b \mathcal{T}_a \right) \Sigma^a \mathcal{D} \hat{h}^a.
$$

The equation for the propagation of linear perturbations is given by replacing eq. (84) in eq. (80). Doing so, we observe that in the context of nonvanishing torsion:

1. The metric wave $\tilde{h}_{ab}$ couples to the background torsion besides to the background curvature.

2. The metric wave $\tilde{h}_{ab}$ couples to an independent propagating torsion wave mode $\gamma^{ab}$.

3. Some of the coupling between $\tilde{h}_{ab}$ and the background torsion is through the trace $\tilde{h}$. All this dependence has been ‘packed’ in the Lorentz-vector 1-form $B_m$, but the important point is that the ‘traceless’ variable $\tilde{h}_{ab}$ no longer lead to equations without the trace $\tilde{h}$.

V. CONCLUSIONS

When YM bosons are described by connections on fiber bundles, their field strength is given by $F = dA + \frac{1}{2}[A, A]$, regardless of the curvature and torsion of the spacetime (basis) manifold. The YM Lagrangian, $-\frac{1}{4} (F \wedge F)$, only has information about the connection $A$ and the background spacetime metric $g_{\mu\nu}$ needed to construct the Hodge $\ast$-operator. Therefore, YM bosons will be sensitive to the spacetime Riemann (metric) curvature but oblivious to torsion. Of all Standard Model fields, torsion only interacts, albeit very weakly, with fermions in the ECSK theory. Since it is always possible to “pack” torsional terms in an effective stress-energy tensor, it may seem tempting to consider torsion as a dark matter candidate (see, e.g., Ref. [41]).

Adopting geometry as a solution to the dark matter problem is an idea with a rich history (see, e.g., Ref. [41]). There are, however, at least two potential weak points worth considering:

- In the pure ECSK theory, torsion does not propagate in vacuum and fermions are its only (very weak) source. Therefore, in order to consider the idea seriously it is necessary to look for more general theories in $d = 4$ and new torsion sources.

- The same “darkness” of torsion (i.e., its lack of interaction with YM fields) that makes the idea attractive also makes it hard to falsify in any foreseeable accelerator physics experiment. Therefore it seems appropriate to find a torsion-sensitive phenomenon outside of the Standard Model in order to test the idea of torsion as dark matter.

In this paper we have explored solutions to both of these issues. Regarding the first point, in sec. IV we take Horndeski’s theory and allow it to develop nonzero torsion by casting it in Cartan’s first-order formalism. The main result of this exercise is that every nonminimal coupling of the geometry with $\phi$ and every term in the Lagrangian with second derivatives of $\phi$ are generic sources of torsion. This was to be expected in the light of previous work, such as sec. 1.7.1 of Ref. [39] on the Brans–Dicke theory and Ref. [21] on nonminimal coupling with the Gauss–Bonnet term. The main novelty of sec. II is the development of new mathematical techniques based on the properties of the $\Sigma^a$ operator, making it possible to work with the full Horndeski Lagrangian in first-order formalism and without imposing the torsionless condition.

In sec. IV we explored the idea of using gravitational waves as a probe for torsion, and in sec. III we introduced the necessary mathematical tools to address this problem. In particular, in sec. III we developed a generalization of the Laplace–de Rham operator, $\square_{\text{deR}} = d^D d + d d^D$, to a new operator $\square_{\text{deR}} = D^D \mathcal{D} + D^D D^D$ which acts covariantly on $p$-forms with Lorentz indices, where $D^D = -\Sigma_a D^a$. In sec. IV we showed that any Horndeski La-
interations with the background torsion. It is clear that the Horndeski theorem breaks down in this case of nonvanishing torsion: there are many new torsional terms which can be added to the Lagrangian which give rise only to second-order field equations. What is the most general Lagrangian for this case remains as an open problem.

- We have shown in sec. [14] that gravitational waves interact with the background torsion, and a new torsional mode appears. However, the phenomenology of this interaction still remains to be modeled. Even further, in any realistic astrophysical scenario it is necessary to go up to second-order in perturbations (see, e.g., Ref. [42]).

- It is not yet clear which, if any, of the Horndeski family members generate suitable dark matter profiles. With sufficiently precise observations, one may hope to use this information to select the most appropriate Lagrangians, or at least rule out some of them. The same is true regarding GW propagation. Some ideas have been proposed about this point in Ref. [43], but only for the torsionless case.

- The cosmological implications of Horndeski's theory have been studied only on particular cases (see, e.g., Ref. [21]).

**ACKNOWLEDGMENTS**

We are grateful to Antonella Cid, José M. Izquierdo, Patricio Mella, Julio Oliva, Patricio Salgado, and Jorge Zanelli for many enlightening conversations. This research was partially funded by Fondecyt grants 1130653, 1150719 (FI), and 3160437 (OV), and by Conicyt scholarships 72160340 (FC-T), 21160784 (JB), 21161574 (PM), and 21161099 (DN) from the Government of Chile.
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