Mass-Radius diagram for compact stars

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Abstract. The compact stars represent the final stage in the evolution of ordinary stars, they are formed when a star ceases its nuclear fuel, in this point the process that sustain its stability will stop. After this, the internal pressure can no longer stand the gravitational force and the star colapses [2]. In this work we investigate the structure of these stars which are described by the equations of Tolman-Openheimer-Volkof (TOV) [1]. These equations show us how the pressure varies with the mass and radius of the star. We consider the TOV equations for both relativistic and non-relativistic cases. In the case of compact stars (white dwarfs and neutron stars) the internal pressure that balances the gravitational pressure is essentially the pressure coming from the degeneracy of fermions. To have solved the TOV equations we need an equation of state that shows how this internal pressure is related to the energy density or mass density. Instead of using polytropic equations of state we have solved the equations numerically using the exact relativistic energy equation for the model of fermion gas at zero temperature. We obtain results for the mass-radius relation for white dwarfs and we compared with the results obtained using the polytropic equations of state. In addition we discussed a good fit for the mass-radius relation.

1. Introduction
The compact stars are the final stage of the evolution of an ordinary star, and they also constitute a laboratory of tests in general relativity [2][4]. We can study the effects of general relativity on the compact stars when it is taken into account [1]. The compact stars are formed of matter in high densities, in the case of the white dwarfs they are constituted of degenerate electron gas or in the case of neutron star that are formed by neutron degenerate gas [2]. The mass of a white dwarf is typically 0.5 to 1 M⊙ (M⊙ represents the solar mass) and they have a radii of the order of 10000 km, however the mass of the neutron stars is about 1.4 to 3 M⊙ and the radii is typically 10 km [5].

2. Differential Equations
We always assume in our accounts that the configuration of the mass of the star is static and spherically symmetric, in this case the pressure and the density of the star are functions of only the radial coordinate r [1]. In the compact star there are two forces acting on the star, one is the gravitational force and the other one came from the degeneracy pressure in which is the force that essentially balances the gravitational one. For the structure of a newtonian star we have
the following equations [2]:

\[ \frac{dp}{dr} = -\frac{Gm(r)\rho(r)}{r^2} \tag{1a} \]

\[ \frac{dm}{dr} = 4\pi r^2 \rho(r), \tag{1b} \]

where \( P \) denotes pressure, \( G \) is the gravitational constant, and \( \rho \) density, the mass \( m(r) \) represents the mass inside of the radius \( r \).

Note that while the \( dm/dr \) is positive, \( dp/dr \) must always be negative. We solve the equations numerically with the initial conditions \( m(0) = 0 \) and \( p(0) = p_0 \) in the region at the center of the star, with increasing radius the mass also increases while the pressure decreases occasionally reaching zero in the surface of the star [2]. However for an isotropic, general relativistic, static, ideal fluid sphere in hydrostatic equilibrium we have the Tolman-Oppenheimer-Volkoff (TOV) equations that describes the structure of the star [1]

\[ \frac{dp}{dr} = -\frac{Gm(r)\epsilon(r)}{c^2 r^2} \left[ 1 + \frac{p(r)}{\epsilon(r)} \right] \left[ 1 + 4\pi r^3 \rho(r) \right] \left[ 1 - \frac{2Gm(r)}{c^2 r} \right]^{-1} \tag{2a} \]

\[ \frac{dm}{dr} = \frac{4\pi r^2 \epsilon(r)}{c^2}, \tag{2b} \]

the \( \epsilon(r) \) represents the energy density, in case of the newtonian equations we can also substitute

\[ \epsilon = \rho c^2, \tag{3} \]

where \( c \) is the speed of light.

The new three terms in the equilibrium equation (2a) are corrections terms. The general relativistic effects become important when the star is sufficiently compact, i.e., when the factor \( 2Gm(r)/c^2r \) approaches unity [4] and when the pressure is considerable compared to energy density.

### 3. Equation of State EoS

To solve numerically the coupled differential equations in both cases, general relativistic and non-relativistic, we must have an equation of state (EoS). A first approach for the EoS is the polytrope approximation, at zero temperature we have [3]

\[ P = K \rho^{1+\frac{2}{n}}, \tag{4} \]

where \( K \) is a constant, and \( n \) is the politropic index in which defines if the electrons in the gas are relativistic or non-relativistic. The value of the constant \( K \) depends on the politropic index.

There is a more general EoS without the polytrope approximation that is given by [2]

\[ P = \frac{\epsilon_0}{24} \left[ (2x + 3 - 3x)(1 + x^2)^{1/2} + 3 \sinh^{-1}(x) \right], \tag{5} \]

where \( x \) is a dimensionless variable given by \( x = P_f/m_e c \), \( P_f \) is the Fermi moment, \( m_e \) is the electron mass and \( \epsilon_0 \) is a constant which carries the dimension of pressure. In terms of relative parameter \( x \) the density is given by

\[ \rho = 9.738 \times 10^3 \mu x^3 \text{ g/cm}^3, \tag{6} \]

\( \mu \) is the ratio between the mass number and atomic number. If we assume that the white dwarf is made of \(^4\text{He}\), \(^{12}\text{C}\) or \(^{16}\text{O}\), hence \( \mu = 2 \).

In order to compare the equation (4) with (5) we plot these functions in figure (1).
We can see that the politropic approaches have a density regime in which they work as an approximation of the general EoS. For low densities the non-relativistic politropic EoS will work, but for densities between $10^5 - 10^7$ g/cm$^3$ neither non-relativistic nor relativistic EoS correspond to the general solution for the EoS. So for the mass-radius relation it’s important that we use the general EoS.

4. Mass-radius relation
Using the set of equations (1) or (2) we can calculate the mass-radius relation and to verify the effects of the general relativity on the model of white dwarfs stars.

![Figure 2. Mass-radius diagram for the general relativistic and newtonian ODE.](image_url)
In the figure (2) we have solved the set of equations (1) and (2) using the general EoS (5). In “Newtonian with SR corrections” we solved the equations (1) and use energy density instead of mass density according to equation (3) and sum the contributions of the energy density of the electrons.

We can determine the maximum mass for the models of white dwarfs and realize that the difference between the models are small.

| Model                        | Maximum mass          |
|------------------------------|-----------------------|
| Newton                       | $1.4559 \times M_\odot$ |
| Newton with SR corrections   | $1.4358 \times M_\odot$ |
| TOV                          | $1.4154 \times M_\odot$ |

In figure (3) we calculated the mass-radius relation for the TOV equations (2) using the general EoS (5) and the non-relativistic EoS (equation (4) for $n = 3/2$).

We see that the non-relativistic EoS works for the mass-radius relation for low values of mass and large radius until approximately values of 15000 km.

In addition we try to find good fits for the mass-radius relation as a possible analytical correlation between the mass and radius of a white dwarf star. We fit the two curves in figure (3) as plotted in following graphics.
In figure (4) we fit the mass-radius relation generated with TOV and non-relativistic EoS using,

\[ \frac{M}{M_{\odot}} = 2.08 \times 10^{-6} \left( \frac{R}{R_{\odot}} \right)^{-3}. \]  

Figure 4. Non-relativistic mass-radius diagram fit.

We see that the non-relativistic EoS in general relativistic calculations doesn’t modifies the well known non-relativistic Newtonian result obtained by Chandrasekhar in [6]. We also fit the general relativistic results in order to obtain an analytical expression for mass-radius relation in that case. In figure (5) we fit the mass-radius relation generated with TOV and general EoS

Figure 5. General relativistic mass-radius diagram fit.
using:

\[
\frac{M}{M_\odot} = \frac{aR + b}{e^{cR^2} + d}
\]  

(8)

where \( a, b, c \) and \( d \) are constants given by:

\[
\begin{align*}
a &= 2.325 \times 10^{-5} \text{ km}^{-1} \\
b &= 0.4617 \\
c &= 7.277 \times 10^{-9} \text{ km}^{-2} \\
d &= -0.644
\end{align*}
\]  

(9)

The fit (8) can be useful for white dwarfs with radius until values approximately 17000 km.

5. Acknowledgments

I would like to thank CAPES (Coordenação de Aperfeiçoamento de Pessoal de Nível Superior) for the financial support.

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