Quantum Statistics with Classical Particles

Daniel Gottesman

Perimeter Institute for Theoretical Physics, Waterloo, ON N2L 2Y5, Canada

Indistinguishability of particles is normally considered to be an inherently quantum property which cannot be possessed by a classical theory. However, Saunders has argued that this is incorrect, and that classically indistinguishable particles are possible. I make this suggestion concrete by describing a class of microscopic classical theories involving indistinguishable particles hopping stochastically on a graph, and show that it should be possible to experimentally create a physical system realizing a simple model by continuously observing atoms trapped in an optical lattice. The indistinguishable classical particles obey Bose-Einstein statistics, display the associated clustering phenomena, and in appropriate models, can even undergo Bose-Einstein condensation.

Since the discovery of quantum mechanics, much has been made of phenomena such as lasers, Bose-Einstein condensation, superfluidity, and superconductivity. Theoretical and experimental studies of these phenomena have revealed a variety of fascinating and counterintuitive behaviors, consequences of the deviations from classical Maxwell-Boltzman statistics displayed by quantum particles. Indeed, they are associated so strongly with quantum mechanics that they are usually described as "macroscopic quantum phenomena." Indistinguishability, we are told, is a property that can only be possessed by quantum particles and not by classical ones.

The standard folklore of physics says "no." Bach, for instance, has argued that a theory with indistinguishable classical particles is impossible because the trajectories would distinguish those particles. However, Saunders has recently argued that one can imagine classical indistinguishable particles. Rather than having a separate configuration space for each particle, the space of configurations would be a collective one, indicating only the number of particles present at each location. In Saunders’ models, these indistinguishable classical particles still effectively obeyed Maxwell-Boltzman statistics. The difference with quantum particles, he says, is that quantum particles also experience a concentration of measure to a lattice of points (in a system of finite size), whereas classical particles can be anywhere.

However, classical particles which are hopping from point to point on a discrete lattice do have their probability measures concentrated on points. In this paper, I give explicit microscopic stochastic models with indistinguishable classical particles hopping on a discrete lattice. The particles are classical in the sense that at all times, they have definite positions. There are no superpositions and interference cannot occur. These models display Bose-Einstein statistics, unlike Saunders’ continuous models, and can even display Bose-Einstein condensation, normally considered a purely quantum phenomenon. Saunders also did not specify the microscopic dynamics of his particles, leaving open the possibility that a contradiction arises somewhere, whereas my models have a complete microscopic description.

The difference between classical indistinguishable particles in discrete space and continuous space can be easily understood — indistinguishability only has an effect on the equilibrium state of a system when particles have a good chance to be in the same state. For classical particles in a continuous space, the probability is 0 of two particles having the same position and momentum, whereas quantum particles have a finite probability of overlapping due to the uncertainty in position and momentum. On a discrete configuration space, however, classical particles can arrive at the same state with finite probability, and thus can display the distinctive behaviors normally associated with quantum indistinguishability.

Furthermore, the models that I present of classical indistinguishable particles are not just theoretical fantasies; it should be possible to realize some simple instances experimentally. In particular, spin-0 atoms confined to an optical lattice satisfy the Bose-Hubbard model to a good approximation. This produces a system of bosons hopping in a discrete configuration space. These particles are quantum, of course, not classical, but suppose we observe the system at intervals of length ∆t to see how many atoms are present at each lattice site. At each observation, the system collapses to some definite classical configuration. The required measurement nondestructively localizes all particles to individual wells in the optical lattice. This, unfortunately, is beyond current experimental techniques, but may not be for long. Between observations, some particles may hop from one site to another, but if we observe frequently enough, it is extremely unlikely that more than one particle will hop in time ∆t. Interference between different paths is then impossible, and the system becomes, to an excellent approximation, a system of classical bosons.
After I completed the research reported here, I discovered similar work by Kaniadakis and Quarati. Our models are related but distinct. Their models are primarily built in continuous systems with an explicit smearing function in space and velocity to allow the classical particles to overlap (although they do also give one example on a lattice). Another similar approach is the quantum Boltzmann master equation by Jaksch et al., which was used to numerically simulate Bose-Einstein condensation. I work exclusively in discrete systems, allowing my models to be simpler than either of these and to more clearly separate quantum mechanical behavior from indistinguishability; indeed, Kaniadakis and Quarati describe indistinguishability in their models as a residual quantum effect rather than a separate phenomenon, and Jaksch et al. make no comment at all about classicality.

Let us be concrete. Suppose we have some graph $G$. We can associate to it a simple model of classical bosons by defining the state of the system to be a function $n : G \rightarrow \mathbb{N}$ which gives the number of particles $n_i$ at each site $i$ in the graph. The total number of particles is $N = \sum_i n_i$. These particles can hop from site to site, conserving the total number. For simplicity, we impose artificially the constraint that only one particle in the whole system can hop in a single time step. This means the particles will, strictly speaking, be interacting, but if we work in the limit where hopping is a rare event, then the interaction is negligible.

Suppose $i$ and $j$ are two adjacent sites. Then let us assume there is a probability $p$ to go in one time step from a configuration $(1_i, 0_j)$ (with 1 particle at site $i$ and 0 particles at site $j$) to $(0_i, 1_j)$ (with 0 particles at $i$ and 1 particle at $j$). This probability is independent of $i$ and $j$ (provided they are adjacent) and is the same regardless of the number of particles at all other locations.

However, it is not immediately clear from this what the probability should be of hopping from the configuration $(c_i, d_j)$ to $(c_i-1, d_j+1)$. One constraint that seems sensible is to require that the probability of hopping from configuration $A$ to configuration $B$ is the same as the probability of hopping from configuration $B$ to configuration $A$. The analogous system of distinguishable classical particles hopping on the graph $G$ has this property, since the probability of a particular particle hopping forward is the same as its probability of hopping backwards. This property also corresponds more or less to having a Hermitian Hamiltonian in the analogous quantum system.

If we add the condition that the probability of hopping out of site $i$ is proportional to the number of particles at $i$, we get the probability rule

$$P((c_i, d_j) \rightarrow (c_i-1, d_j+1)) = c(d+1)p,$$

again assuming the lattice sites $i$ and $j$ are adjacent. We should add the additional constraint that $p$ is small, so that the total probability of hopping is always less than one. Then for any configuration $A$, we can calculate the probability of staying at the same configuration as $1 - \sum_B P(A \rightarrow B)$, where the sum is taken over all configurations $B$ which are one hop away from $A$. With $N$ particles, it is sufficient to take $[N^2/4 + (g+1/2)N]p < 1$, where $g$ is the maximum degree of any node in $G$. Systems with distinguishable particles can also obey this transition rule, leading to similar effects (e.g., [2]).

This prescription has the virtue that it agrees with the system produced when we frequently observe particles obeying a Bose-Hubbard model, as with spin-0 atoms in an optical lattice. The standard optical lattice is a cubic lattice, but it is also possible to make other graphs by blocking some sites using a Fermi-Bose mixture [3]. The Bose-Hubbard model [2] has the Hamiltonian

$$H = -J \sum_{\langle i,j \rangle} b_i^\dagger b_j + \sum_i \epsilon_i \hat{n}_i + (U/2) \sum_i \hat{n}_i (\hat{n}_i - 1),$$

where $b_i$ is the annihilation operator at site $i$ and $\hat{n}_i = b_i^\dagger b_i$ is the number operator at site $i$. The basis states of this quantum system are the possible configurations of the above model of classical bosons with the same graph of possible sites. If we observe the system at two times separated by an interval of $\Delta t$, the probability of hopping from a basis configuration $A$ to $B$ is

$$P(A \rightarrow B) = | \langle A | e^{i H \Delta t / \hbar} | B \rangle |^2.$$  

When $\Delta t$ is small, we can approximate

$$e^{i H \Delta t / \hbar} = 1 + i H \Delta t / \hbar + O((\Delta t)^2).$$  

The $\hat{n}_i$ terms do not cause transitions, so for $A \neq B$, we only need to consider the term $-J \sum_{\langle i,j \rangle} b_i^\dagger b_j$ in $H$. If $A$ and $B$ differ on the adjacent sites $i$ and $j$, with $A$ including $(c_i, d_j)$ and $B$ including $((c-1)_i, (d+1)_j)$, then

$$P(A \rightarrow B) = \left( \frac{J \Delta t}{\hbar} \right)^2 | \langle c_i, d_j | b_i^\dagger b_j | (c-1)_i, (d+1)_j \rangle |^2 + O((\Delta t)^3).$$

Recalling that $b_i |c_i\rangle = \sqrt{c_i} |(c-1)_i\rangle$, we find

$$P(A \rightarrow B) = c(d+1)(J \Delta t / \hbar)^2 + O((\Delta t)^3),$$

which agrees with eq. (4) using $p = (J \Delta t / \hbar)^2$. This approximation is valid when $gN J \Delta t / \hbar$, $(N \Delta t / \hbar) \max(c_i)$, and $N^2 U \Delta t / \hbar$ are all much less than 1.

The classical boson model is a Markov process, and can thus be described via a transition matrix $M$ giving the probability to hop between configurations. Because the probability of hopping forward is the same as the probability of hopping backwards, the transition matrix is symmetric, and is thus doubly stochastic (each row and each column of $M$ sums to 1). For an arbitrary initial state, a Markov process asymptotes to an equilibrium state, an eigenvector of $M$ with eigenvalue +1.
For a doubly stochastic matrix, the uniform mixture over all configurations is a possible equilibrium state. If the graph $G$ is connected, the Markov chain is irreducible, and if $p$ is small, there is a non-zero probability of staying in the same configuration, so the Markov chain is aperiodic. When both of these are true, the uniform mixture will be the only equilibrium state $\Gamma$.

The equilibrium state of the Markov process therefore corresponds to thermodynamic equilibrium. For instance, suppose we consider an extremely simple case with two lattice sites and two particles. For classical distinguishable particles, there are four configurations, two with one particle on each site, and two with both particles on the same site. The equilibrium state thus has probability 1/2 of having both particles in the same location. For the classical boson model, there are only three configurations, the same two with both particles on the same site, but only one with one particle on each site. The equilibrium state thus has probability 2/3 of having both particles in the same location, displaying the usual clustering effect associated with Bose-Einstein statistics. Naturally, this clustering becomes more prevalent in larger systems with a high density of particles.

This simple model of classical bosons illustrates the basic point — that classical particles can be indistinguishable — but cannot display the more exciting properties associated with indistinguishable quantum particles. If we want to make classical analogues of macroscopic quantum phenomena, we will need models where the particles have additional properties. As a first step, let us introduce energy into the model. Since we wish energy to be conserved at each transition, the new model will have two types of particles, which I will call “atoms” and “photons.” This enables us to have both interesting dynamics and conservation of energy by shifting energy back and forth between the atoms and the photons.

Atoms will hop on a graph, just as in the simple model. However, now each site $i$ in the graph will have an integral value of energy $E_i$ associated with it. We assume that adjacent sites of the graph have energy values that are within ±1 of each other. (The energies of adjacent sites may be the same.) The number of atoms is conserved in each transition. We will assume the atoms are classical bosons, so the possible atom configurations are described by the number of atoms at each site.

Photons have a graph of their own, and may hop on it also as in the previous model. The photons will be indistinguishable classical particles too. When photons hop between different sites on the photon graph, the number of photons is conserved, but photons can be created or destroyed when an atom makes a transition. In particular, we associate a photon site $k$ with each edge $\langle i, j \rangle$ of the atom graph for which $|E_i - E_j| = 1$. When an atom makes a transition along the edge $\langle i, j \rangle$, a photon must be created or destroyed at the site $k$ in order to conserve energy. Each photon is considered to have unit energy.

| Transition type | Probability |
|-----------------|-------------|
| $i \rightarrow j, E_i = E_j$ | $n_i(n_j + 1)p$ |
| $i \rightarrow j, E_i = E_j + 1$, create $\gamma$ at $k$ | $n_i(n_j + 1)(m_k + 1)p$ |
| $i \rightarrow j, E_i = E_j - 1$, destroy $\gamma$ at $k$ | $n_i(n_j + 1)m_kp$ |
| $i \rightarrow j, |E_i - E_j| > 1$ | 0 |
| $k \rightarrow l$ | $m_k(m_l + 1)q$ |

TABLE I: Transition probabilities in the classical boson energy model. Sites $i$ and $j$ are adjacent atom sites with energies $E_i$ and $E_j$. Sites $k$ and $l$ are adjacent photon ($\gamma$) sites, and if appropriate, photon site $k$ is associated to the edge $\langle i, j \rangle$. There are $n_i$ and $n_j$ atoms at sites $i$ and $j$ respectively, and $m_k$ and $m_l$ photons at sites $k$ and $l$.

We thus have a variety of possible types of transitions in this model, with transition probabilities given in table I. As with the previous model, we only allow a single particle to make a hopping transition in each time step, plus the possibility that a single photon is also created or destroyed. To determine the transition probabilities, it seems reasonable to assume the probability of absorbing a photon increases proportionally to the number of photons available to be absorbed. From this, it follows that the requirement that the Markov chain be doubly stochastic implies that the probability to emit a photon into a site with $m_k$ photons must be proportional to $m_k + 1$. We allow different probabilities $p$ and $p'$ for an atom hopping to a site with the same or different energy, and a separate probability $q$ for photons hopping.

Assuming the atom and photon graphs are connected, and that $p$, $p'$, and $q$ are sufficiently small, there is again a unique equilibrium state, which is a uniform mixture over all configurations. There are two conserved quantities, the total energy $E$ and the total number of atoms $N_A$. If there are $n_i$ atoms at site $i$, then the total number of photons must be $n_P = E - \sum n_i E_i$. Clearly, the allowed configurations must satisfy $\sum n_i E_i \leq E$ as well as $\sum n_i = N_A$. Assume there are a total of $V$ photon sites.

We can thus write down a formula for $\langle n_i \rangle$, the expected number of atoms at site $i$. There are $\binom{n+V-1}{n_P}$ total possible configurations for $n_P$ photons, so the total number of configurations of all kinds is

$$T = \sum_{\{n_i\}} \binom{n_P + V - 1}{n_P},$$

(7)

where the sum is taken over atom configurations $\{n_i\}$ satisfying the constraints on total energy and total atom number. We thus have

$$\langle n_j \rangle = \frac{1}{T} \sum_{\{n_i\}} n_j \binom{n_P + V - 1}{n_P}.$$

(8)

To proceed further, we wish to treat the photon system as a thermal bath, so we work in the limit where $E$ and $V$
are both very large, but $E/V$ is constant. Then we expect most of the energy of the system to be in the photons, so the constraint $E_A = \sum_i n_i E_i \le E$ becomes unimportant, and we can neglect terms of order $E_A/(E + V)$. Letting $x = n_P/(n_P + V - 1)$ and $x_0 = E/(E + V)$, we have

$$\ln \left( \frac{n_P + V - 1}{n_P} \right) \approx (n_P + V) h(x) \approx (E + V) h(x_0) + E_A \ln x_0,$$

where we have used Stirling’s formula to approximate the binomial coefficient, and $h(x) = -x \ln x - (1-x) \ln(1-x)$.

By identifying $\beta = -\ln x_0$, we thus produce the standard canonical ensemble for the atoms. Indeed, the procedure above simply mimics the usual argument moving from the microcanonical ensemble to the canonical ensemble. Since the sum over configurations only considers the number of atoms at each site, the atoms behave as bosons, not as Maxwell-Boltzman particles.

We can thus immediately apply the standard results about thermodynamics of bosons. For instance, appropriate systems will display Bose-Einstein condensation; we need only find an assignment of energies to a graph which produces the correct number of states of a given energy. For instance, suppose we take the positive octant of a square lattice in 3 dimensions, and let the energy of site $(x, y, z)$ be $x + y + z (x, y, z \ge 0)$. We then replicate the density of states for a 3-dimensional harmonic trap with appropriate trap frequency, and in this system, bosons can undergo Bose-Einstein condensation [11].

The exact nature of the thermal bath is presumably not important, and it could consist of regular quantum photons or a different species of atoms. While the model discussed above would be difficult to replicate experimentally, given the variety of possibilities, it might be possible to find some system which could experimentally realize classical Bose-Einstein condensation.

Many further extensions are possible. One could consider interacting particles and could add more particle types and more properties such as charge to try to replicate other macroscopic quantum phenomena such as superconductivity. One could add velocities to make bosonic versions of lattice gas automata [12]. However, deterministic classical boson models face a special challenge. In a deterministic model with distinguishable particles, two particles with the same properties that start with the same state must retain the same state throughout their evolution; if they were to separate, there would be no way to determine which of the two particles would go in which direction. We might worry that this would preclude any interesting effects of bosonic particle statistics, but luckily models with indistinguishable particles do not face this problem: It is perfectly possible for two indistinguishable particles with the same state to head off in different directions without violating any symmetry.

One could also try to create models of classical fermions. A straightforward way of doing this is to simply impose a constraint that no site can contain more than one of the particles. This prescription immediately reproduces the statistical mechanics of fermions using the techniques described above for bosons, and as before, could be produced experimentally by frequently observing quantum fermions moving in an optical lattice. However, it is somewhat unsatisfying from a philosophical perspective. The constraint causes the models to behave like fermions even if the particles in the model are actually distinguishable. It is unclear if there is a true distinction between distinguishable and indistinguishable classical fermions, but one place to look might be in models where two or more particles can hop at the same time (as in [4]). Then, for instance, if two particles switch locations, this produces a different configuration when the particles are distinguishable, but not if they are indistinguishable. Unfortunately, I do not see how such models could be realized experimentally, as a double hop in a single time step opens the possibility for interference between observations of the quantum systems.

In summary, I have presented some models of classical bosons, demonstrating an error in the standard folklore that indistinguishability is an inherently quantum property. Indeed, we see that indistinguishability and quantum behavior are separate phenomena; each can exist without the other. In practice, small objects like atoms tend to be both quantum and indistinguishable. Larger objects have more accessible internal degrees of freedom, so tend to lose indistinguishability. Large objects also tend to lose quantum coherence, which is perhaps why indistinguishability and quantum behavior have been considered in the past to be so closely associated.

In this paper I have only examined the equilibrium behavior of the classical models to show that they can reproduce the equilibrium behavior of quantum indistinguishable systems. Of course, it is also possible to study the non-equilibrium behavior of classical boson or classical fermion models, for instance to examine transport properties of the models. Indeed, it is in the realm of dynamics that we can expect to see a difference between the quantum and classical models, as interference can play a role in the quantum systems but not in the classical ones.

The classical boson models offer a new perspective for understanding macroscopic quantum phenomena. They may even provide an arena to make improved concrete predictions about such phenomena: Classical systems are much easier to simulate and analyze than quantum systems, so if classical boson (or fermion) models can be created which replicate the major properties of interesting systems such as high-$T_c$ superconductors, they would provide a very useful technique for understanding those systems. Indeed, since collective phenomena can persist for systems large enough to experience significant decoherence, it might even be that classical boson or fermion models are more accurate than existing quantum models.
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* E-mail: dgottesman@perimeterinstitute.ca

[1] S. Saunders, “On the explanation for quantum statistics,” Studies in the History and Philosophy of Modern Physics, forthcoming; “Are quantum particles objects,” Analysis, forthcoming.

[2] A. Bach, *Indistinguishable classical particles*, Springer (New York, 1997).

[3] G. Kaniadakis and P. Quarati, “Classical model of bosons and fermions,” Phys. Rev. E 49, 5103–5110 (1994).

[4] G. Kaniadakis and P. Quarati, “Classical bosons and fermions in the contemporary transition description,” Physica 218, 183–190 (1995).

[5] D. Jaksch, C. W. Gardiner, P. Zoller, “Quantum kinetic theory II: Simulation of the quantum Boltzmann master equation,” Phys. Rev. A 56, 575-586 (1997), quant-ph/9701008.

[6] D. Jaksch, C. Bruder, J. I. Cirac, C. W. Gardiner, and P. Zoller, “Cold bosonic atoms in optical lattices,” Phys. Rev. Lett. 81, 3108 (1998), cond-mat/9805329.

[7] G. Bianconi and A.-L. Barabási, “Bose-Einstein condensation in complex networks,” Phys. Rev. Lett. 86, 5632-5635 (2001), cond-mat/0011224.

[8] V. Ahufinger, L. Sanchez-Palencia, A. Kantian, A. Sanpera, and M. Lewenstein, “Disordered ultracold atomic gases in optical lattices: A case study of Fermi-Bose mixtures,” cond-mat/0508042.

[9] M. P. A. Fisher, P. B. Weichman, G. Grinstein, and D. S. Fisher, “Boson localization and the superfluid-insulator transition,” Phys. Rev. B 40, 546 (1989).

[10] W. Feller, *An Introduction to Probability Theory and Its Applications*, vol. I, 3rd edition, John Wiley and Sons (New York, 1968).

[11] S. R. de Groot, G. J. Hooyman, and C. A. ten Seldam, “On the Bose-Einstein condensation,” Proc. Roy. Soc. London, series A 203, 266-286 (1950).

[12] D. H. Rothman, S. Zaleski, “Lattice gas models of phase separation: interfaces, phase transitions, and multiphase flow,” Rev. Mod. Physics 66, 1417–1479 (1994).