Type \( \tilde{C} \) Temperley-Lieb algebra quotients and Catalan combinatorics

Sadek Al Harbat, Camilo González and David Plaza

Instituto de Matemática y Física
Universidad de Talca
Talca, Chile

Abstract

We study some algebraic and combinatorial features of two algebras that arise as quotients of Temperley-Lieb algebras of type \( \tilde{C} \), namely, the two-boundary Temperley-Lieb algebra and the symplectic blob algebra. We provide a monomial basis for both algebras. The elements of these bases are parameterized by certain subsets of fully commutative elements. We enumerate these elements according to their affine length.

Keywords: Temperley-Lieb algebras, Fully commutative elements, Catalan triangle.

1. Introduction

The Temperley-Lieb algebra was introduced around fifty years ago from considerations in statistical mechanics [TL71]. Since then it has turned out to be related to many topics of mathematics, including knot theory, algebraic combinatorics, algebraic Lie theory, etc. In 1987, Jones [Jon87] observed that the Temperley-Lieb algebra can be realized as a quotient of the Hecke algebra associated to a Coxeter system of type \( A \). In his thesis, Graham [Gra95] took that observation far beyond its original scope. Concretely, given an arbitrary Coxeter system \((W, S)\), he defined a quotient of the Hecke algebra associated to \((W, S)\) that he called generalized Temperley-Lieb algebra. Furthermore, he showed that for any \((W, S)\) the associated generalized Temperley-Lieb algebra admits a basis indexed by the set of fully commutative (FC for short) elements of \( W \).

Definition 1.1. An element \( w \in W \) is fully commutative if any reduced expression for \( w \) can be obtained from any other by using exclusively braid relations of the form \( st = ts \), \( s, t \in S \).

Full commutativity has been given a proper place by Stembridge in the series of papers [Ste96, Ste97, Ste98]. In particular, and after classifying the Coxeter systems with finitely many FC elements [Ste96], he gave a normal form for FC elements in each of the infinite families of finite Coxeter systems [Ste97]. In this work we focus on type \( B \). Later on the first author gave a normal form for FC elements in the four infinite families of affine Coxeter systems. In this paper we are interested in type \( \tilde{C} \) [AH17].

Let \( K \) be an algebraically closed field, let \( n \) be a positive integer and let \( \delta, \delta_L, \delta_R, \kappa_L, \kappa_R, \kappa \in K^\times \). In this work we study the following \( K \)-algebras.

- The two-boundary Temperley-Lieb algebra with \( n + 1 \) generators: \( 2BTL_n(\delta, \delta_L, \delta_R, \kappa_L, \kappa_R) \).
The symplectic blob algebra with $n + 1$ generators: $SB_n(\delta, \delta_L, \delta_R, \kappa_L, \kappa_R, \kappa)$.

The first algebra was introduced in [DGN09] as a quotient of the $\tilde{C}$-type affine Hecke algebra and was given a diagrammatical presentation. The second algebra was defined in [MGP07] as an algebra of diagrams, then was given by generators and relations in [GMP12] and was studied more deeply later on, see [Ree11] for example.

Given a positive integer $n$ we denote by $W(\tilde{C}_n)$ the affine Coxeter group of type $\tilde{C}_n$ ($W(\tilde{C}_1)$ to be understood as the Coxeter group of type $B_2$). This is, $W(\tilde{C}_n)$ is the group given by the following Coxeter diagram:

$$\begin{array}{ccccccccc}
\sigma_0 & & \sigma_1 & & \sigma_2 & & \cdots & & \sigma_{n-2} & & \sigma_{n-1} & & \sigma_n = t_n
\end{array}$$

Let $\mathcal{T}L\tilde{C}_n(q, Q)$ be the Temperley-Lieb algebra associated to $W(\tilde{C}_n)$. Let $W^c(\tilde{C}_n)$ be the set of FC elements in $W(\tilde{C}_n)$ and $\{b_w \mid w \in W^c(\tilde{C}_n)\}$ the monomial basis of $\mathcal{T}L\tilde{C}_n(q, Q)$ (see Definition 2.4). The starting point of this work is the following observation (already present in [MGP07]):

By specializing the parameters of the two algebras under focus we obtain a sequence of surjective morphisms of $K$-algebras

$$\mathcal{T}L\tilde{C}_n(q, Q) \twoheadrightarrow 2BTL_n(q, Q) \twoheadrightarrow SB_n(q, Q, \kappa),$$

where $2BTL_n(q, Q)$ and $SB_n(q, Q, \kappa)$ denote the specialized algebras (see Section 2 for details). The tower of $\tilde{C}$-type TL algebras defined in [AH17] gives rise to a faithful tower of two boundary TL algebras but not of symplectic blob algebras (see Remark 2.8).

We define two subsets of $W^c(\tilde{C}_n)$, namely:

- The set of positive fully commutative elements, $W^{c+}(\tilde{C}_n)$ (see Definition 3.7).
- The set of blobbed fully commutative elements, $W^c_b(\tilde{C}_n)$ (see Definition 4.1).

The set $W^{c+}(\tilde{C}_n)$ is infinite while $W^c_b(\tilde{C}_n)$ is finite. Furthermore, we have

$$W^c(\tilde{C}_n) \leftrightarrow W^{c+}(\tilde{C}_n) \leftrightarrow W^c_b(\tilde{C}_n).$$

Actually, those three sets index bases of the three algebras above. Indeed, our first main results are the following (where we use the same notation for $b_w$ and its image under the morphisms in (1.2)).

**Theorem A.** The set $\{b_w \mid w \in W^{c+}(\tilde{C}_n)\}$ is a $K$-basis for $2BTL_n(q, Q)$.

**Theorem B.** The set $\{b_w \mid w \in W^c_b(\tilde{C}_n)\}$ is a $K$-basis for $SB_n(q, Q, \kappa)$.

**Definition 1.2.** Let $w \in W^c(\tilde{C}_n)$. We define the affine length of $w$ to be the number of times $t_n$ occurs in a (any) reduced expression of $w$. We denote it by $L(w)$.

Viewing $W(\tilde{C}_n)$ as an “affinization” of $W(B_n)$ allows us to see how the affine length can be a powerful tool in studying the behaviour of FC elements, since we have a finite number of elements of a given affine length and since we know exactly how to get from affine length $k$ to affine length $k+1$. In particular, when $k = \frac{1}{2}(n-1)$, the affine length $L(w)$ is a positive integer, which allows us to define the set $W^c_{\frac{1}{2}}(\tilde{C}_n)$ as the set of elements of affine length $\frac{1}{2}(n-1)$.
Motivated by Theorem A and Theorem B we undertake the task of enumerating the elements of $W^{++c}(\tilde{C}_n)$ and $W^c_b(\tilde{C}_n)$ according to their affine length. Let us make precise the above sentence. Let $n$ and $s$ be integers such that $n \geq 1$ and $s \geq 0$. We define $A^+_n = \{ w \in W^{++c}(\tilde{C}_n) \mid L(w) = s \}$ and $B^+_n = \{ w \in W^c_b(\tilde{C}_n) \mid L(w) = s \}$. Furthermore, we set $a^+_n = |A^+_n|$ and $b^+_n = |B^+_n|$. Then, our goal is to find closed formulas for the numbers $a^+_n$ and $b^+_n$.

There are two main ingredients in our counting. On the one hand, the set $W^{++c}(\tilde{C}_n)$ inherits naturally the normal form from $W^c(\tilde{C}_n)$. Fortunately, the normal form for the elements of $W^{++c}(\tilde{C}_n)$ simplifies drastically. This simplification allows us to introduce a graphical interpretation for any element $w \in W^{++c}(\tilde{C}_n)$, which we call the grid of $w$ and denote by $G(w)$ (see Definition 4.3). Then, we focus on enumerating grids rather than elements. In other words, we transform an algebraic problem into a combinatorial one. Since $W^c_b(\tilde{C}_n) \subset W^{++c}(\tilde{C}_n)$ the same strategy applies for the enumeration of the elements of $W^c_b(\tilde{C}_n)$ as well. On the other hand, we introduce an array of numbers, which we call the Blobbed Catalan triangle. This name is justified since it corresponds to a certain generalization of Forder’s Catalan triangle [For61]. The blobbed Catalan triangle is an infinite matrix formed by non negative integers $\{C_{i,j}\}_{i,j \geq -1}$. The definition and properties of the blobbed Catalan triangle are given in Section 5, where in particular we provide a closed formula for the numbers $C_{i,j}$ in terms of binomial coefficients (see Theorem 5.4). By combining these two ingredients we eventually obtain the following.

Theorem C. Let $n$ and $s$ be integers such that $n \geq 1$ and $s \geq 0$. Then, we have $a^+_n = C_{2n,2s}$.

Theorem D. Let $D^+_n = A^+_n - B^+_n$ and set $d^+_n = |D^+_n| = a^+_n - b^+_n$. Then, there is a closed formula for the numbers $d^+_n$ (see Section 7 for the explicit formulas) which only involves coefficients occurring in the blobbed Catalan triangle. Therefore, we have a closed formula for $b^+_n$. In particular, we obtain $b^+_n = 0$ for $s > n$. Finally, the number

$$p_n := \sum_{s=0}^{n} b^+_s = |W^c_b(\tilde{C}_n)|$$

(1.4)

gives the dimension of $SB_n(q, Q, \kappa)$.

This article is structured as follows. In Section 2 we introduce the algebras under study. In Section 3 we define positive FC elements and prove Theorem A. In Section 4 we define blobbed FC elements and prove Theorem B. In Section 5 we introduce the blobbed Catalan triangle and study the properties of the numbers occurring in it. In Section 6 we prove Theorem C. Finally, in Section 7 we prove Theorem D.

Acknowledgements

Sadek Al Harbat was supported by Fondecyt Postdoctoral grant 3170544. David Plaza was partially supported by FONDECYT project 11160154 and the Inserción en la Academia project pai-conicyt 79150015.
2. Definition of algebras

In this section we define the algebras which we study. Hereinafter, $K$ will denote an algebraically closed field. Let $n$ be a positive integer. Let $W(\tilde{C}_n)$ be the affine Coxeter group of type $\tilde{C}_n$.

**Definition 2.1.** Let $q, Q \in K^\times$. The Hecke algebra of type $\tilde{C}_n$, $H_n(q, Q)$, is the associative unital $K$-algebra with generators $\{g_0, g_1, \ldots, g_{n-1}, g_n\}$ subject to the relations

\[
\begin{align*}
g_i^2 &= (Q - 1)g_i + Q, & & \text{if } i = 0 \text{ or } i = n; \\
g_i^2 &= (q - 1)g_i + q, & & \text{if } 0 < i < n; \\
g_ig_j &= g_jg_i, & & \text{if } |i - j| > 1; \\
g_ig_jg_i &= g_jg_ig_j, & & \text{if } |i - j| = 1 \text{ and } 0 < i, j < n; \\
g_ig_jg_ig_j &= g_jg_ig_jg_i, & & \text{if } \{i, j\} = \{0, 1\} \text{ or } \{i, j\} = \{n - 1, n\}.
\end{align*}
\]

(2.1)

For $x, y$ in a given ring with identity, we define:

\[
\begin{align*}
V(x, y) &= xyx + xy + yx + x + y + 1, \\
Z(x, y) &= xyxy + xyx + yxy + yx + x + y + 1.
\end{align*}
\]

**Definition 2.2.** Let $q, Q \in K^\times$. The Temperley-Lieb algebra of type $\tilde{C}_n$, $TL\tilde{C}_n(q, Q)$, is the associative unital $K$-algebra with generators $\{g_0, g_1, \ldots, g_{n-1}, g_n\}$ subject to the relations as in (2.1) together with

\[
Z(g_0, g_1) = Z(g_{n-1}, g_n) = V(g_i, g_{i+1}) = 0, \text{ for } 0 < i < n.
\]

(2.2)

By setting $U_0 = \frac{1}{\sqrt{q}}(g_0 + 1)$, $U_n = \frac{1}{\sqrt{q}}(g_n + 1)$ and $U_i = \frac{1}{\sqrt{q}}(g_i + 1)$ for $0 < i < n$, we obtain another presentation for $TL\tilde{C}_n(q, Q)$ with generators $\{U_0, U_1, \ldots, U_n\}$ subject to the relations

\[
\begin{align*}
U_0^2 &= \delta U_0, \\
U_n^2 &= \delta U_n; \\
U_i^2 &= \delta U_i, & & \text{if } 0 < i < n; \\
U_iU_j &= U_jU_i, & & \text{if } |i - j| > 1; \\
U_iU_jU_i &= U_i, & & \text{if } |i - j| = 1 \text{ and } 0 < i, j < n; \\
U_iU_jU_iU_j &= k_LU_iU_j, & & \text{if } \{i, j\} = \{0, 1\}; \\
U_iU_jU_iU_j &= k_RU_iU_j, & & \text{if } \{i, j\} = \{n - 1, n\};
\end{align*}
\]

(2.3)

where

\[
\delta_L = \delta_R = \frac{1 + Q}{\sqrt{Q}}, \quad \delta = \frac{1 + q}{\sqrt{q}}, \quad k_L = k_R = \frac{q + Q}{\sqrt{qQ}}.
\]

(2.4)

In order to describe a basis for $TL\tilde{C}_n(q, Q)$ we need to recall the following.

**Definition 2.3.** In a Coxeter system $(W, S)$ of graph $\Gamma$, elements for which one can pass from any reduced expression to any other one only by applying commutation relations are called fully commutative elements. We denote by $W^c(\Gamma)$ the set of fully commutative elements in $W(\Gamma)$.

**Definition 2.4.** Given $w$ in $W^c(\tilde{C}_n)$ and $w = \sigma_{i_1} \ldots \sigma_{i_r}$ any reduced expression of $w$, we set $b_w = U_{i_1} \ldots U_{i_r}$. It is well-known that $b_w$ is well-defined and that the set $\{b_w \mid w \in W^c(\tilde{C}_n)\}$ is a basis for $TL\tilde{C}_n(q, Q)$, which is called the monomial basis.
Remark 2.5. In the literature (for example [Ern12]) we see that $TLC_n(q, Q)$ was defined in the equal parameters case, i.e., when $q = Q$. Sometimes it was defined with a weight function (for example [Gra95]). Now since the $K$-vector space structure of the algebras $TLC_n(q, q)$ and $TLC_n(q, Q)$ is the same, we use the monomial basis for $TLC_n(q, Q)$ exactly in the same way in which it was used in the equal parameters case.

Definition 2.6. Let $\delta, \delta_L, \delta_R, \kappa_L, \kappa_R \in K^\times$. The two-boundary Temperley-Lieb algebra, which we denote by $2BT L_n(\delta, \delta_L, \delta_R, \kappa_L, \kappa_R)$, is the associative unital $K$-algebra with generators $\{U_0, U_1, \ldots, U_n\}$ subject to the relations (2.3) together with

$$U_1 U_0 U_1 = \kappa_L U_1 \quad \text{and} \quad U_{n-1} U_n U_{n-1} = \kappa_R U_{n-1}.$$ (2.5)

By specializing the parameters $\delta, \delta_L, \delta_R, \kappa_L$ and $\kappa_R$ as in (2.4) we obtain a two-parameter version of the two-boundary Temperley-Lieb algebra which we denote by $2BT L_n(q, Q)$. In this setting, we see that $2BT L_n(q, Q)$ is the quotient of $TLC_n(q, Q)$ by the ideal generated by

$$U_1 U_0 U_1 - \kappa_L U_1 \quad \text{and} \quad U_{n-1} U_n U_{n-1} - \kappa_R U_{n-1}.$$ (2.6)

Definition 2.7. Let $\delta, \delta_L, \delta_R, \kappa_L, \kappa_R, \kappa \in K^\times$. The symplectic blob algebra, which we denote by $SB_n(\delta, \delta_L, \delta_R, \kappa_L, \kappa_R, \kappa)$, is the associative unital $K$-algebra with generators $\{U_0, U_1, \ldots, U_n\}$ subject to the relations (2.3) and (2.5), together with

$$IZ = \kappa I \quad \text{and} \quad JIZ = \kappa J,$$ (2.7)

where

$I = \begin{cases} U_1 U_3 \cdots U_{n-1}, & \text{if } n \text{ is even}; \\ U_1 U_3 \cdots U_n, & \text{if } n \text{ is odd} \end{cases}$ and $J = \begin{cases} U_0 U_2 \cdots U_n, & \text{if } n \text{ is even}; \\ U_0 U_2 \cdots U_{n-1}, & \text{if } n \text{ is odd}. \end{cases}$ (2.8)

By specializing the parameters $\delta, \delta_L, \delta_R, \kappa_L$ and $\kappa_R$ as in (2.4) we obtain a three-parameter version of the symplectic blob algebra which we denote by $SB_n(q, Q, \kappa)$. In this setting, we see that $SB_n(q, Q, \kappa)$ is the quotient of $2BT L_n(q, Q)$ by the ideal generated by

$$IZ = \kappa I \quad \text{and} \quad JIZ = \kappa J.$$ (2.9)

With the exception of the Hecke algebra of type $\tilde{C}_n$, all the algebras defined in this section have a diagram calculus given by certain generalizations of classical Temperley-Lieb diagrams (see [MGP07, DGN09, Ern12]). In this paper we do not touch the diagrammatic setting and for this reason we do not recall it here.

Remark 2.8. In [AH17, §6] the first author has defined the morphism $R_n : TLC_n(q, Q) \to TLC_{n+1}(q, Q)$ and has shown the faithfulness of this morphism. We can easily see that the composition with the quotient morphism onto $2BT L_{n+1}(q, Q)$ factors through $2BT L_n(q, Q)$, so $R_n$ gives rise to a morphism of algebras $\tilde{R}_n : 2BT L_n(q, Q) \to 2BT L_{n+1}(q, Q)$. Since the maps $I, J$ defined in [AH17, Definition 5.1] send positive elements -see below- to positive elements (with a slight change concerning the definition of $I$ and $J$ on elements with affine length 1), the faithfulness of $\tilde{R}_n$ follows. The importance of this morphism comes from the fact that it comes from a morphism of $\tilde{C}$-type braid groups, thus the resulting faithful tower of two-boundary T-L algebras encodes most-likely topological data for example. On the other hand the morphism $\tilde{R}_n$ fails to factor through the symplectic blob algebras, so we do not get in this way a tower of symplectic blob algebras, unfortunately.
3. Positive fully commutative elements and the two-boundary Temperley-Lieb algebra

The purpose of this section is to determine a (monomial) basis for $2\mathcal{BTL}_n(q, Q)$. Concretely, we find a subset of the monomial basis of $\mathcal{TL\tilde{C}}_n(q, Q)$ whose elements are mapped under the canonical projection to a basis for $2\mathcal{BTL}_n(q, Q)$. To this end, we begin by recalling the classification of fully commutative elements in type $\tilde{C}_n$.

**Definition 3.1.** Let $w \in W(\tilde{C}_n)$. We define the **affine length** of $w$ to be the number of times $t_n$ occurs in a (any) reduced expression of $w$. We denote it by $L(w)$. In particular, we identify \( \{ w \in W(\tilde{C}_n) \mid L(w) = 0 \} \) with the Coxeter group $W(B_n)$.

Before recalling the classification we write some suitable notations. We define
\[
[i, j] = \sigma_i \sigma_{i+1} \ldots \sigma_j, \quad \text{for } 0 \leq i \leq j < n \quad \text{and} \quad [n, n-1] = 1;
\]
\[
[-i, j] = \sigma_i \sigma_{i-1} \ldots \sigma_1 \sigma_0 \sigma_1 \ldots \sigma_{j-1} \sigma_j, \quad \text{for } 1 \leq i \leq j < n \quad \text{and} \quad [0, -1] = 1.
\] (3.1)

The classification of fully commutative elements of affine length zero in type $\tilde{C}_n$ (or equivalently fully commutative elements in type $B_n$) was given by Stembridge \[Ste97, Theorem 5.1\]. We recall it in the following result.

**Theorem 3.2.** Let $w$ be a fully commutative element in $W^c(B_n)$ different from the identity. Then, $w$ can be written in a unique way as a reduced word of the form
\[
[l_1, g_1][l_2, g_2] \ldots [l_r, g_r]
\] (3.2)

with $n > g_1 > \cdots > g_r \geq 0$ and $|l_t| \leq g_t$ for $1 \leq t \leq r$, such that either

(a) $l_1 > \cdots > l_{r-1} > l_r > 0$;

(b) $l_1 > \cdots > l_s = l_{s+1} = \cdots = l_r = 0$ for some $s \leq r$; or

(c) $l_1 > \cdots > l_{r-1} > -l_r > 0$.

Hereinafter, we call elements of $W^c(B_n)$ satisfying (c) in Theorem 3.2 negative elements. We call positive element any $w \in W^c(B_n)$ which is not negative. We also call elements satisfying (a) in Theorem 3.2 strictly positive elements. Finally, we consider the identity as a strictly positive element.

**Lemma 3.3.** The number of positive elements in $W^c(B_n)$ is \( \binom{2n}{n} \).

**Proof:** By \[Ste97, Proposition 5.9\] we know that
\[
|W^c(B_n)| = (n + 2)C_n - 1,
\] (3.3)

where $C_n = \frac{1}{n+1} \binom{2n}{n}$ denotes the $n$-th Catalan number. Let us denote by $P$, $SP$ and $N$ the set of positive, strictly positive and negative fully commutative elements in $W^c(B_n)$, respectively. We notice that the elements of $SP$ are precisely the elements of $W^c(B_n)$ in which $\sigma_0$ does not appear. We conclude that
\[
|SP| = |W^c(A_{n-1})| = C_n.
\] (3.4)
On the other hand, there is a bijection $N \to SP - \{1\}$, which is obtained by replacing $l_r$ by $-l_r$. Therefore,

$$|N| = C_n - 1. \quad (3.5)$$

Since $W^c(B_n) = P \cup N$, a combination of (3.3) and (3.5) yields the result.

We now explain the classification of elements of $W^c(\tilde{C}_n)$ with positive affine length.

**Theorem 3.4.** [AH17, Theorem 4.7] Let $w \in W^c(\tilde{C}_n)$ with $L(w) \geq 2$. Then $w$ can be written in a unique way as a reduced word of one and only one of the following two forms, for non negative integers $p$ and $k$:

**First type**

$$w = [i, n-1] t_n ([-(n-1), n-1] t_n)^k ([f, n-1])^{-1} \quad (3.6)$$

with $k \geq 1$, $-n < i \leq n$ and $-n < f \leq n$.

**Second type**

$$w = [i_1, n-1] t_n [i_2, n-1] t_n \ldots [i_p, n-1] t_n ([0, n-1] t_n)^k w_r \quad \text{if } p > 0,$$

$$w = ([0, n-1] t_n)^k w_r \quad \text{if } p = 0, \quad (3.7)$$

with $w_r \in W^c(B_n)$ and

- if $k > 0$: $w_r = 1$ or $w_r = [0, r_1][0, r_2] \ldots [0, r_u]$ with $0 \leq r_u < \cdots < r_1 < n$;
- if $p > 0$: $n \geq i_1 > \ldots > i_{p-1} > |i_p| > 0$;
- if $p > 0$ and $i_p < 0$: $k = 0$, $w_r = 1$ and $i_p \neq -(n-1)$;
- if $k = 0$ and $i_p > 0$: $w_r = [l_1, g_1][l_2, g_2] \ldots [l_r, g_r]$ with $|l_1| < i_p$.

The affine length of $w$ of the first (resp. second) type is $k + 1$ (resp. $p + k$) and we have $0 \leq p \leq n$.

Now suppose that $L(w) = 1$, then it has a reduced expression of the form:

$$[i, n-1] t_n v \quad (3.8)$$

where

- if $0 < i \leq n$ then $v = [l_1, g_1][l_2, g_2] \ldots [l_r, g_r]$ such that for $1 \leq j \leq r$ either $l_j = n-j$ or $l_j < i$;
- if $i < 0$ then $v = ([h, n-1])^{-1}$ with $-n < h \leq n$;
- if $i = 0$ then

$$v = \begin{cases} 
([h, n-1])^{-1}, & \text{for } -n < h \leq n, \text{ or} \\
([z, n-1])^{-1}[0, r_1][0, r_2] \ldots [0, r_m], & \text{for } 0 \leq r_m < \ldots r_2 < r_1 < z \leq n. 
\end{cases} \quad (3.9)$$

Conversely, every $w$ of the above form is in $W^c(\tilde{C}_n)$.  

7
We want to extend the notion of positivity to the elements of \( W^c(\tilde{C}_n) \). To this end we need to define the notions of subword and avoidance.

**Definition 3.5.** Let \((W, S)\) be an arbitrary Coxeter system. By a subword of a word \( w = s_1 s_2 \ldots s_r \) \((s_i \in S)\) we mean a word of the form \( s_i s_{i+1} \ldots s_j \), for some \( 1 \leq i \leq j \leq r \).

**Definition 3.6.** Given two elements \( w \) and \( u \) in \( W \) we say that \( w \) contains \( u \) if there exist reduced expressions \( \underline{w} \) of \( w \) and \( \underline{u} \) of \( u \) such that \( \underline{u} \) is a subword of \( \underline{w} \). Otherwise, we say that \( w \) avoids \( u \) or that \( w \) is \( u \)-avoiding.

We are now ready to define the promised extension of the notion of positivity.

**Definition 3.7.** An element \( w \in W^c(\tilde{C}_n) \) is called left-positive (resp. right-positive) if it avoids \( \sigma_1 \sigma_0 \sigma_1 \) (resp. \( \sigma_n-1 \sigma_n \sigma_{n-1} \)). Elements which are both left-positive and right-positive will be called positive. We denote by \( W^{+c}(\tilde{C}_n) \) (resp. \( W^{+c}(\tilde{C}_n) \)) the set of all left-positive (resp. right-positive) elements in \( W^c(\tilde{C}_n) \). Finally, we denote by \( W^{+c+}(\tilde{C}_n) \) the set of all positive elements in \( W^c(\tilde{C}_n) \).

**Remark 3.8.** We notice that the normal form of a given fully commutative element is a reduced expression that allows us to determine if such an element is left-positive (resp. right-positive) or not. More precisely, let \( \underline{w} \) be the normal form of an element \( w \in W^c(\tilde{C}_n) \). Then, \( w \) is left-positive (resp. right-positive) if and only if \( \sigma_1 \sigma_0 \sigma_1 \) (resp. \( \sigma_{n-1} \sigma_n \sigma_{n-1} \)) does not occur as a subword of \( \underline{w} \).

We define \( Lp \) to be the two-sided ideal of \( T L \tilde{C}_n(q, Q) \) generated by \( U_1 U_0 U_1 - \kappa_L U_1 \). We shall, in what follows, investigate the behavior of the elements of the monomial basis of \( T L \tilde{C}_n(q, Q) \) under the canonical surjection \( T L \tilde{C}_n(q, Q) \to T L \tilde{C}_n(q, Q)/Lp \). We keep the same notations for elements of \( T L \tilde{C}_n(q, Q) \) and their images. Our aim is to show that left-positive elements index a basis for \( T L \tilde{C}_n(q, Q)/Lp \). To this end, we study the structure of the ideal \( Lp \).

A precise inspection of the normal forms in Theorem 3.2 and Theorem 3.4 leads to the following classification of non-left-positive elements, where we use the same notation as in the referred theorems.

**Proposition 3.9.** An element \( w \in W^c(\tilde{C}_n) \) is not left-positive if and only if it satisfies one of the following conditions:

1. \( L(w) \geq 2 \) and \( w \) is a first type element. Here \( \sigma_1 \sigma_0 \sigma_1 \) appears at least \( L(w) - 1 \) times, at most \( L(w) + 1 \) times.

2. \( L(w) \geq 2 \) and \( w \) is a second type element with \( k = 0 \), hence \( p \geq 2 \), and either
   
   (a) \( i_p < 0 \) and \( w_r = 1 \), or
   
   (b) \( i_p > 0 \) and \( w_r = [l_1, g_1][l_2, g_2] \ldots [l_r, g_r] \) with \( l_r < 0 \).

   In both cases \( \sigma_1 \sigma_0 \sigma_1 \) appears exactly once.

3. \( L(w) = 1 \) and either

   (a) \( i < 0 \) and \( h \geq 0 \), then \( \sigma_1 \sigma_0 \sigma_1 \) appears once in \( w \),
(b) \( i < 0 \) and \( h < 0 \), then \( \sigma_1 \sigma_0 \sigma_1 \) appears twice in \( w \),

(c) \( i = 0 \), \( v = [h, n - 1]^{-1} \) and \( h < 0 \), then \( \sigma_1 \sigma_0 \sigma_1 \) appears once in \( w \), or

(d) \( i > 0 \) and \( l_r < 0 \), then \( \sigma_1 \sigma_0 \sigma_1 \) appears once in \( w \).

(4) \( L(w) = 0 \) and \( w \) is a negative element in \( W^c(B_n) \). Here \( \sigma_1 \sigma_0 \sigma_1 \) appears exactly once.

**Definition 3.10.** Given an element \( w \in W^c(\tilde{C}_n) - W^{+c}(\tilde{C}_n) \) we define \( \overline{w} \in W^c(\tilde{C}_n) \) as follows (we use the same notation as in Theorem 3.32 and Theorem 3.34 and consider the same cases as in Proposition 3.13).

1. Suppose that \( L(w) \geq 2 \) and \( w \) is a first type element. We recall that this means that

\[
w = [i, n - 1]t_n([- (n - 1), n - 1] t_n)^k([- f, n - 1])^{-1},
\]

for some integers \( k, i \) and \( f \) such that \( k \geq 1 \), \( -n < i \leq n \) and \( -n < f \leq n \). In this case, we define

\[
\overline{w} = \begin{cases} 
[i, n - 1]t_n([- (n - 1), n - 1] t_n)^k([- f, n - 1])^{-1}, & \text{if } f < 0; \\
\sigma_i t_n [- i, n - 1], & \text{if } k = 1, f = n \text{ and } i = n; \\
[i, n - 1]t_n([- (n - 1), n - 1] t_n)^{k-1}([- f, n - 1])^{-1}, & \text{otherwise.}
\end{cases}
\]

2. Suppose that \( L(w) \geq 2 \) and \( w \) is a second type element with \( k = 0 \) and \( p \geq 2 \).

(a) If \( i_p < 0 \) and \( w_r = 1 \) then \( \overline{w} \) is obtained from the normal form of \( w \) by replacing the block \([i_p, n - 1]\) by the block \([-i_p, n - 1]\).

(b) If \( i_p > 0 \) and \( w_r = [l_1, g_1] [l_2, g_2] \ldots [l_r, g_r] \) with \( l_r < 0 \) then \( \overline{w} \) is obtained from the normal form of \( w \) by replacing the block \([l_r, g_r]\) by the block \([-l_r, g_r]\).

3. Suppose that \( L(w) = 1 \).

(a) If \( i < 0 \) and \( h > 0 \) then \( \overline{w} \) is obtained from the normal form of \( w \) by replacing the block \([i, n - 1]\) by the block \([-i, n - 1]\).

(b) If \( i < 0 \) and \( h < 0 \) then \( \overline{w} \) is obtained from the normal form of \( w \) by replacing the block \([h, n - 1]^{-1}\) by the block \([-h, n - 1]^{-1}\).

(c) If \( i = 0 \), \( v = [h, n - 1]^{-1} \) and \( h < 0 \) then \( \overline{w} \) is obtained from the normal form of \( w \) by replacing the block \([h, n - 1]^{-1}\) by the block \([-h, n - 1]^{-1}\).

(d) If \( i > 0 \) and \( l_r < 0 \) then \( \overline{w} \) is obtained from the normal form of \( w \) by replacing the block \([l_r, g_r]\) by the block \([-l_r, g_r]\).

4. Suppose that \( L(w) = 0 \) and \( w = [l_1, g_1] [l_2, g_2] \ldots [l_r, g_r] \) with \( l_r < 0 \). Then, we define \( \overline{w} = [l_1, g_1] [l_2, g_2] \ldots [-l_r, g_r] \).

It is obvious from the definition that \( l(w) > l(\overline{w}) + 1 \), for any \( w \in W^c(\tilde{C}_n) - W^{+c}(\tilde{C}_n) \). We now extend the bar operator to a linear transformation,

\[
\overline{\cdot} : \text{Span}_K \{ b_x \mid x \in W^c(\tilde{C}_n) - W^{+c}(\tilde{C}_n) \} \rightarrow TL\tilde{C}_n(q, Q),
\]
which is determined in the set \( \{ b_x | x \in W^c(\tilde{C}_n) - W^{+c}(\tilde{C}_n) \} \) by the rule

\[
\overline{b_x} = \begin{cases} 
    b_x, & \text{if } x \text{ is not a first type element;} \\
    b_x, & \text{if } x \text{ is a first type element with } f < 0 \text{ or } k = 1, f = n \text{ and } i = n; \\
    \kappa R b_x, & \text{otherwise.}
\end{cases}
\] (3.10)

**Theorem 3.11.** The set \( X = \{ b_x - \kappa L \overline{b_x} | x \in W^c(\tilde{C}_n) - W^{+c}(\tilde{C}_n) \} \) is a \( K \)-basis of \( Lp \).

**Proof.** It is enough to prove that:
1. The set \( X \) is linearly independent,
2. \( U_1 U_0 U_1 - \kappa L U_1 \in X \),
3. \( X \subset Lp \),
4. The \( K \)-linear space spanned by \( X \) is an ideal of \( T L \tilde{C}_n(q, Q) \).

1. We recall that \( \{ b_x | x \in W^c(\tilde{C}_n) \} \) is the monomial basis of \( T L \tilde{C}_n(q, Q) \). Consider the \( K \)-linear map \( \Phi : T L \tilde{C}_n(q, Q) \rightarrow T L \tilde{C}_n(q, Q) \) determined in the monomial basis by the rule

\[
\Phi(b_x) = \begin{cases} 
    b_x, & \text{if } x \in W^c(\tilde{C}_n) \\
    b_x - \kappa L \overline{b_x}, & \text{if } x \in W^c(\tilde{C}_n) - W^{+c}(\tilde{C}_n).
\end{cases}
\] (3.11)

Since \( l(x) > l(\overline{x}) \), for any non-left-positive element \( x \), we see that \( \Phi \) is a linear map with unipotent triangular matrix for any ordering of the monomial basis compatible with the length. Hence, \( \Phi \) is an isomorphism and

\[
X = \Phi(\{ b_x | x \in W^c(\tilde{C}_n) - W^{+c}(\tilde{C}_n) \})
\] (3.12)

is linearly independent.

2. If \( x = \sigma_1 \sigma_0 \sigma_1 \) then \( \overline{x} = \sigma_1 \). Therefore, \( U_1 U_0 U_1 - \kappa L U_1 = b_x - \kappa L \overline{b_x} \in X \).

3. Let \( \mathcal{L}_p : T L \tilde{C}_n(q, Q) \rightarrow T L \tilde{C}_n(q, Q) / Lp \) be the quotient map. Given \( 1 \leq i \leq j < n \) we have

\[
\mathcal{L}_p(b_{[-i,j]}) = \mathcal{L}_p(U_1 U_{i-1} \cdots U_1 U_0 U_1 \cdots U_j) = \kappa L \mathcal{L}_p(U_1 U_{i-1} \cdots U_2 U_1 U_2 \cdots U_j) = \kappa L \mathcal{L}_p(U_1 U_{i-1} \cdots U_3 U_2 U_3 \cdots U_j) \cdots = \kappa L \mathcal{L}_p(U_1 U_{i+1} \cdots U_j) = \kappa L \mathcal{L}_p(b_{[i,j]})
\] (3.13)

Similarly, we obtain \( \mathcal{L}_p(b_{[-i,j]}^{-1}) = \kappa L \mathcal{L}_p(b_{[i,j]}^{-1}) \). On the other hand, we have

\[
\mathcal{L}_p(b_{\sigma_{n-1} t_{n}^{-1}(n-1) \cdots t_{n}^{-1}(n-1) | t_n}) = \mathcal{L}_p(U_{n-1} U_n U_{n-2} \cdots U_1 U_0 U_1 \cdots U_{n-1} U_n) = \kappa L \mathcal{L}_p(U_{n-1} U_n U_{n-2} \cdots U_2 U_1 U_2 \cdots U_{n-1} U_n) = \kappa L \kappa L \mathcal{L}_p(U_{n-1} U_n U_{n-2} \cdots U_3 U_2 U_3 \cdots U_{n-1} U_n) \cdots = \kappa L \mathcal{L}_p(U_{n-1} U_n U_{n-1} U_n) = \kappa L \kappa L \mathcal{L}_p(U_{n-1} U_n).
\] (3.14)
Similarly, we obtain \( \mathcal{L}_p(b_{|n-1\rangle,n-1\rangle}) = \kappa_L \mathcal{L}_p(U_n U_{n-1} U_n) \). By the way the bar operator was defined and the four formulas above we conclude that \( \mathcal{L}_p(b_x) = \kappa_L \mathcal{L}_p(\overline{b_x}) \), for all \( x \in W^c(\tilde{C}_n) - W^{+c}(\tilde{C}_n) \). Therefore, \( X \subset L_p \).

4. By [3.12], it is enough to show that both \( U_s \Phi(b_x) \) and \( \Phi(b_x) U_s \) belong to the \( K \)-linear space spanned by \( X = \Phi(\{y \mid y \in W^c(\tilde{C}_n) - W^{+c}(\tilde{C}_n)\}) \), for any \( x \in W^c(\tilde{C}_n) - W^{+c}(\tilde{C}_n) \) and any \( 0 \leq s \leq n \). This can be achieved by performing a case-by-case analysis. For the sake of brevity, we only detail the case

\[
x = [i, n-1]t_n([- (n-1), n-1] t_n)^k([f, n-1])^{-1}
\]

with \( 1 \leq k \) and \( 0 \leq f < n \), when the multiplication by \( U_s \) is done on the right. All the other cases are dealt with similarity.

We remark that a first type element is determined by a triple \( (i, k, f) \). In this setting, we set \( b(i, k, f) := b_x \) if \( x \) is as in [3.15]. We also define \( b(i, 0, f) = b_{[i,n-1]t_n(\sigma_{n-1}, \sigma_{n-2}, \ldots, \sigma_f)} \). For instance, with the new notation, the bar operator over monomials indexed by first type elements is given by

\[
\overline{b(i, k, f)} = \begin{cases} b(i, k, -f), & \text{if } f < 0; \\ U_n U_{n-1} U_n, & \text{if } k = 1, f = n \text{ and } i = n; \\ \kappa_R b(i, k - 1, f), & \text{otherwise.} \end{cases}
\]

We split the proof in several cases in accordance with the relation between \( f \) and \( s \).

**Case A** \((s = f)\). We have \( \Phi(b(i, k, f)) U_f = \lambda \Phi(b(i, k, f)) \), where \( \lambda = \delta_L \) and \( \lambda = \delta \) for \( f = 0 \) and \( 0 < f < n \), respectively. In both cases, we are done.

**Case B** \((s = f + 1)\). We have

\[
\Phi(b(i, k, f)) U_{f+1} = \begin{cases} \Phi(b(i, k, -1)) + \kappa_L \Phi(b(i, k, 1)) - \kappa_L \kappa_R \Phi(b(i, k - 1, -1)), \\ \Phi(b(i, k, f + 1)), \\ \kappa_R \Phi(b(i, k, f + 1)), \end{cases}
\]

for \( f = 0, 0 < f < n - 1 \) and \( f = n - 1 \), respectively. Then, \( \Phi(b(i, k, f)) U_{f+1} \) belongs to the \( K \)-linear space spanned by \( X \).

**Case B1** \((s = f - 1)\). Similar to Case B.

**Case C** \((f + 2 \leq s < n)\). A repeated application of the relations in [2.3] together with an inductive argument yield

\[
b(i, k, f) U_s = \kappa_L^{-k} \kappa_R^k b_{[i,n-1]}(U_n U_{n-1} \cdots U_s)(U_{s-2} \cdots U_1 U_0 U_1 \cdots U_f),
\]

for all \( k \geq 1 \). Then, we obtain

\[
\Phi(b(i, k, f)) U_s = b(i, k, f) U_s - \kappa_L \kappa_R b(i, k - 1, f) U_s = 0,
\]

which gives us the result if \( k > 1 \). If \( k = 1 \) we have \( \Phi(b(i, 1, f)) U_s = \kappa_R \Phi(b_y) \), where

\[
y = [i, n-1]t_n(\sigma_{n-1}, \sigma_{n-2}, \ldots, \sigma_s)(\sigma_{s-2} \sigma_{s-3} \cdots \sigma_{f+1})[-f, f].
\]
We remark that $y$ is a non-left-positive element. Therefore, $\Phi(b_y)$ does belong to $X$ and the result follows in this case as well.

**Case C1** ($0 < s \leq f - 2$). Similar to Case C.

**Case D** ($f + 2 \leq s = n$). As in Case C, a repeated application of the relations in (2.3) together with an inductive argument yield

\[
\begin{align*}
b(i, k, f)U_n &= \begin{cases} 
\kappa_L^k \kappa_R^k b_{[i,n]-1}(U_n U_{n-1} \cdots U_1 U_0 \cdots U_f) U_n, & \text{if } f = 0; \\
\kappa_L^{k-1} \kappa_R^k b_{[i,n]-1}(U_n U_{n-1} \cdots U_1 U_0 \cdots U_f) U_n, & \text{if } 0 < f \leq n - 2.
\end{cases}
\tag{3.21}
\end{align*}
\]

for all $k \geq 1$. Then, $\Phi(b(i, k, f))U_n = 0$ which implies the result for $k > 1$. We notice that (3.21) still holds if $k = 0$ and $f = 0$, which gives us the result in the case $k = 1$ and $f = 0$. The remaining case, i.e., $k = 1$ and $0 < f \leq n - 2$, must be split in different cases according to the value of $i$. For $i = n$ we have $\Phi(b(n, 1, f))U_n = \kappa_R \Phi(b_z)$, where

\[
z = t_n \sigma_{n-1} t_n \sigma_{n-2} \sigma_{n-3} \cdots \sigma_1 \sigma_0 \sigma_1 \cdots \sigma_f.
\tag{3.22}
\]

We remark that $z$ is a second type element, that $z$ is non-left positive and that

\[
\overline{z} = t_n \sigma_{n-1} t_n \sigma_{n-2} \sigma_{n-3} \cdots \sigma_f.
\tag{3.23}
\]

All the above gives us the result when $i = n$. We now assume that $i \neq n$. Under this assumption, we notice that there is a generator $U_{n-1}$ on the right of the element $b_{[i,n]-1}$ in (3.21). By moving the rightmost $U_n$ to the left in (3.21) and applying the relation $U_{n-1} U_{n-1} U_n = \kappa_R U_{n-1} U_n$ we obtain

\[
b(i, 1, f)U_n = \begin{cases} 
\kappa_L^2(U_i U_{i-1} \cdots U_1 U_0 U_1 \cdots U_f) U_n, & \text{if } 0 < i < n; \\
\kappa_L \kappa_R^2 (U_0 U_1 \cdots U_f) U_n, & \text{if } i = 0; \\
\kappa_L \kappa_R^2 (U_{[i]} U_{[i]-1} \cdots U_1 U_0 U_1 \cdots U_f) U_n, & \text{if } i < 0.
\end{cases}
\tag{3.24}
\]

Finally, we have

\[
\Phi(b(i, 1, f))U_n = \begin{cases} 
0, & \text{if } i = 0; \\
\kappa_R \Phi(b_u), & \text{otherwise},
\end{cases}
\tag{3.25}
\]

where $u = \sigma_{[i]} \sigma_{[i]-1} \cdots \sigma_1 \sigma_0 \sigma_1 \cdots \sigma_f$ is a non-left-positive element and the result follows.

**Case D1** ($0 = s \leq f - 2$). Similar to Case D.

**Corollary 3.12.** The set $\{b_w \mid w \in W^+(\widehat{C}_n)\}$ is a $K$-basis of $TL\widehat{C}_n(q, Q)/Lp$.

**Proof.** We use the same notation for $b_w$ and its image in the quotient. On the one hand, the proof of the first claim in Theorem 3.11 reveals that $X \cup \{b_w \mid w \in W^+(\widehat{C}_n)\}$ is a $K$-basis for $TL\widehat{C}_n(q, Q)$. On the other hand, Theorem 3.11 shows that $X$ is a $K$-basis of $Lp$. The result follows. □

We define $Rp$ to be the ideal of $TL\widehat{C}_n(q, Q)/Lp$ generated by $U_{n-1}U_n U_{n-1} - \kappa_R U_{n-1}$. We now want to mimic the previous strategy in order to determine a basis for the ideal $Rp$. The first step in this case is to give a classification of the left-positive elements that are not right-positive. A moment’s thought reveals that first type elements are not left-positive. Another moment’s thought
In this setting, we have 0 \leq i < n, and one of the following.

(a) 0 < i < n and \( v \) is of the form (3.2) with \( l_i = n - 1 \) and \( l_j = n - j \) or 0 \leq l_j < i for 2 \leq j \leq r.

(b) \( i = 0 \) and \( v = ([h, n - 1])^{-1} \) with 0 \leq h < n.

(c) \( i = 0 \) and \( v = ([z, n - 1])^{-1} [0, r_1][0, r_2] \cdots [0, r_m] \) with 0 \leq r_m < \ldots < r_2 < r_1 < z < n.

Furthermore, in the three cases \( \sigma_{n-1} \sigma_n \sigma_{n-1} \) occurs exactly once.

\textbf{Definition 3.14.} For \( w \in W^c(\tilde{C}_n) - W^{c+}(\tilde{C}_n) \) we define \( \tilde{w} \in W^{c+}(\tilde{C}_n) \) by (we use the same notation and cases considered in Proposition 3.13)

(a)
\[
\tilde{w} = \begin{cases} 
[i, l_a][l_{a+1}, g_{a+1}] \cdots [l_r, g_r], & \text{if } i \leq l_a; \\
(l_{a+1}, i)^{-1}[l_{a+1}, g_{a+1}] \cdots [l_r, g_r], & \text{if } i > l_a,
\end{cases}
\]
where \( \alpha = \max \{1 \leq j \leq r \mid l_j = n - j \} \).

(b)
\[
\tilde{w} = \begin{cases} 
[0, h], & \text{if } h > 0; \\
\sigma_0 \sigma_1 \sigma_0, & \text{if } h = 0.
\end{cases}
\]

(c)
\[
\tilde{w} = [0, z][0, r_1] \cdots [0, r_m].
\]

Finally, we define a \( K \)-linear map \( \tilde{\Phi} : \text{Span}_K \{ b_w \mid w \in W^{c+}(\tilde{C}_n) - W^{c+}(\tilde{C}_n) \} \rightarrow TL\tilde{C}_n(q, Q)/Lp \) determined by \( b_w = b_{\tilde{w}} \).

\textbf{Theorem 3.15.} The set \( \{ b_w \mid w \in W^{c+}(\tilde{C}_n) \} \) is a \( K \)-basis of \( (TL\tilde{C}_n(q, Q)/Lp)/Rp \). Therefore, the set of positive fully commutative elements indexes a monomial basis for 2\( BTL_n(q, Q) \).

\textbf{Proof.} Consider the \( K \)-linear map \( \Phi' : TL\tilde{C}_n(q, Q)/Lp \rightarrow TL\tilde{C}_n(q, Q)/Lp \) determined by
\[
\Phi'(b_w) = \begin{cases} 
b_w, & \text{if } w \in W^{c+}(\tilde{C}_n); \\
b_w - \kappa R b_w, & \text{if } w \in W^{c+}(\tilde{C}_n) - W^{c+}(\tilde{C}_n).
\end{cases}
\]

Since \( l(w) > l(\tilde{w}) \) we see that \( \Phi' \) is an isomorphism of \( K \)-vector spaces. By the same arguments as the ones used in Theorem 3.11 we obtain that the set \( \{ \Phi'(b_w) \mid w \in W^{c+}(\tilde{C}_n) - W^{c+}(\tilde{C}_n) \} \) is a \( K \)-basis for \( Rp \). Therefore, \( \{ b_w \mid w \in W^{c+}(\tilde{C}_n) \} \) is a \( K \)-basis of \( (TL\tilde{C}_n(q, Q)/Lp)/Rp \). Finally, we notice that
\[
(TL\tilde{C}_n(q, Q)/Lp)/Rp \simeq 2BTL_n(q, Q).
\]
\[\square\]
We conclude this section by providing a classification for positive fully commutative elements.

If we look back at Theorem 3.4, we notice that the first type elements cannot be positive, while we
can give some conditions on the normal form of second type elements and elements of affine length
one, so that they would be positive. We sum up in the notations of Theorem 3.4. Let \( w \in W^c(\tilde{C}_n) \)
be a positive element. If \( L(w) \geq 2 \), it has the following form, for non negative integers \( p \) and \( k \):
\[
\begin{align*}
w &= [i_1, n-1] t_n [i_2, n-1] t_n \cdots [i_p, n-1] t_n ([0, n-1] t_n)^k w_r \quad \text{if } p > 0, \\
w &= ([0, n-1] t_n)^k w_r \quad \text{if } p = 0,
\end{align*}
\] (3.31)
with \( w_r \in W^c(B_n) \) and

\[ \begin{align*}
\bullet & \text{ if } k > 0: w_r = 1 \text{ or } w_r = [0, r_1][0, r_2] \cdots [0, r_u] \text{ with } 0 \leq r_u < \cdots < r_1 < n; \\
\bullet & \text{ if } p > 0: n \geq i_1 > \ldots > i_{p-1} > i_p > 0; \\
\bullet & \text{ if } k = 0: w_r \text{ is a positive element in } W^c(B_n) \text{ with } l_1 < i_p.
\end{align*} \]

Now suppose that \( L(w) = 1 \), then \( w \) has a reduced expression of the form \( [i, n-1] t_n v \) where

\[ \begin{align*}
\bullet & \text{ if } 0 < i \leq n \text{ then } v \text{ is a positive element in } W^c(B_n) \text{ with } l_1 < i; \\
\bullet & \text{ if } i = 0 \text{ then } v = 1 \text{ or } v \text{ is equal to } [0, r_1][0, r_2] \cdots [0, r_m] \text{ for } 0 \leq r_m < \ldots < r_2 < r_1 < n.
\end{align*} \]

In order to provide a uniform description of the positive fully commutative elements we need a small modification of the notation introduced in (3.1). To this end, we define
\[
\langle i, j \rangle = \sigma_i \sigma_{i+1} \ldots \sigma_j
\] (3.32)
for \( 0 \leq i \leq j \leq n \). With this notation at hand, we can reformulate the above description of positive
fully commutative elements as follows.

**Proposition 3.16.** Let \( n > 1 \) be an integer. Let \( w \) be a positive element in \( W^c(\tilde{C}_n) \) other than the
unit. Then there exists a positive integer \( k \) such that
\[
w = \langle l_1, r_1 \rangle \langle l_2, r_2 \rangle \cdots \langle l_k, r_k \rangle,
\] (3.33)
for some integers \( l_i \) and \( r_i \) such that

1. \( n \geq l_1 \geq l_2 \geq \ldots \geq l_k \geq 0; \)
2. \( n \geq r_1 \geq r_2 \geq \ldots \geq r_k \geq 0; \)
3. \( l_i \leq r_i; \)
4. If \( l_{i+1} = l_i \) then \( l_i = 0; \)
5. If \( r_{i+1} = r_i \) then \( r_i = n.\)

Conversely, every \( w \) of the form (3.33) is in \( W^{++}(\tilde{C}_n) \).

We call the elements \( \langle l_s, r_s \rangle, 1 \leq s \leq k \), the **rigid blocks** of \( w. \)
4. Blobbed fully commutative elements and the symplectic blob algebra.

In the previous section we found a monomial basis for the two-boundary Temperley-Lieb algebra which was indexed by positive fully commutative elements. In this section we find a monomial basis for the symplectic blob algebra which is indexed by blobbed fully commutative elements. Let us begin by defining these elements.

**Definition 4.1.** An element \(w \in W^+\) is called \(I\)-blobbed (resp. \(J\)-blobbed) if it avoids \(IJI\) (resp. \(JIJ\)), where
\[
I = \left\{ \sigma_1\sigma_3 \cdots \sigma_n \quad \text{if } n \text{ is odd; } \sigma_1\sigma_3 \cdots \sigma_{n-1} \quad \text{if } n \text{ is even,} \right. \\
J = \left\{ \sigma_0\sigma_2 \cdots \sigma_{n-1} \quad \text{if } n \text{ is odd; } \sigma_0\sigma_2 \cdots \sigma_n \quad \text{if } n \text{ is even.} \right. \\
\]
(4.1)

Elements which are both \(I\)-blobbed and \(J\)-blobbed will be called blobbed. We denote by \(I^{\tilde{C}}(\tilde{C})\) the set of all \(I\)-blobbed (resp. \(J\)-blobbed) elements in \(W^+\). Finally, we denote by \(W^{\tilde{C}}(\tilde{C})\) the set of all blobbed elements in \(W^+\).

Our goal in this section is to demonstrate the following result.

**Theorem 4.2.** The set \(\{bw \mid w \in W^{\tilde{C}}(\tilde{C})\}\) is a \(K\)-basis for the symplectic blob algebra \(SB_n(q, Q, \kappa)\).

We utilize the same strategy as the one used in the previous section. For this reason and for the sake of brevity we only provide a sketch of the proof of Theorem 4.2. The rest of this section is devoted to this aim. The first step in order to reach our goal is to provide a normal form for positive elements in which we can see the occurrences of \(IJI\) and \(JIJ\). We stress that the normal form for positive elements given in Proposition 3.16 is not well suited for this purpose. It is convenient to introduce a graphical interpretation for the elements of \(W^{\tilde{C}}(\tilde{C})\).

**Definition 4.3.** Let \(w = \langle l_1, r_1 \rangle \langle l_2, r_2 \rangle \cdots \langle l_k, r_k \rangle \in W^{\tilde{C}}(\tilde{C})\). The grid of the element \(w\) is the set
\[
G(w) := \{(i, j) \in \mathbb{Z}^2 \mid 1 \leq i \leq k; l_i \leq j \leq r_i \}, \\
\]
(4.2)
up to horizontal translation.

**Example 4.4.** Let \(w = \langle 7, 8 \rangle \langle 4, 8 \rangle \langle 3, 7 \rangle \langle 1, 4 \rangle \langle 0, 1 \rangle \langle 0, 0 \rangle \in W^{\tilde{C}}(\tilde{C})\). The grid of \(w\) can be described graphically as

\[
\text{[Graphical representation of the grid]} \\
\]
(4.3)

From now on we do not distinguish between a grid and its graphical interpretation. This geometric presentation gives rise to another normal form of positive fully commutative elements, in which our point of viewing a fully commutative element is to determine the reduced expression of maximal blocks of commuting generators. Indeed, moving a line with slope \(-2\) from left to right through the grid, we see that each intersection of the grid with such a line is made of commuting
generators. We call for short oblique of \( G(w) \) (or of \( w \)) such an intersection. We identify a point \((i, j)\) with the generator \( \sigma_j \). This allows us to identify an oblique with an element of \( W(\mathring{C}_n) \) by considering the product of the generators involved in the oblique. We recall that the generators involved in a given oblique commute so it is not necessary to specify an order for the aforementioned product. Let us illustrate the above with an example.

**Example 4.5.** Let \( w = \langle 7, 8 \rangle \langle 4, 8 \rangle \langle 3, 7 \rangle \langle 1, 4 \rangle \langle 0, 1 \rangle \langle 0, 0 \rangle \in W^{+c}(\mathring{C}_8). \)

![Diagram](image)

In this case we have six obliques. If we label them from left to right as \( O_1, \ldots, O_6 \) then the corresponding elements in \( W(\mathring{C}_n) \) are

\[
O_1 = \sigma_1; \quad O_2 = O_4 = \sigma_1 \sigma_3 \sigma_5 \sigma_7 = I; \quad O_3 = \sigma_0 \sigma_2 \sigma_4 \sigma_6 \sigma_8 = J; \quad O_5 = \sigma_0 \sigma_4 \sigma_6 \sigma_8; \quad O_6 = \sigma_7. \tag{4.5}
\]

**Proposition 4.6.** Let \( w \in W^{+c}(\mathring{C}_n) \). Then, \( w \) is the product of the obliques of \( G(w) \) taken from left to right. Furthermore, if we write the generators involved in each oblique in increasing order then we obtain a well-defined reduced expression for \( w \), which we call the oblique form of \( w \).

**Proof.** The result follows by induction on the number of obliques of \( G(w) \) once we notice that any generator involved in the leftmost oblique can be pushed to the left of the expression (3.33) for \( w \) and that the element obtained from \( w \) by eliminating its leftmost oblique is positive. \( \square \)

**Example 4.7.** If \( w \) is as in Example 4.5 then its oblique form is given by

\[
\sigma_4 \sigma_1 \sigma_3 \sigma_5 \sigma_7 \sigma_0 \sigma_2 \sigma_4 \sigma_6 \sigma_8 \sigma_1 \sigma_3 \sigma_5 \sigma_7 \sigma_0 \sigma_4 \sigma_6 \sigma_8 \sigma_7. \tag{4.6}
\]

Now let \((W, S)\) be a Coxeter system and let \( w \) be a FC element in \( W \). By Matsumoto’s theorem and the definition of full commutativity, we know that the number of occurrences of a generator \( s \in S \) in \( w \) is well-defined. Moreover, suppose that \( s \) occurs \( k \) times in \( w \). Given a reduced expression \( w \) of \( w \), let us order the occurrences of \( s \) in \( w \) from left to right, for example \( s^i \) for \( 1 \leq i \leq k \). Again by Matsumoto’s theorem and full commutativity, the relative positions of \( s^1, s^2, \ldots, s^k \) are independent of the reduced expression. In other terms we can mark the occurrences of any simple reflexion in some reduced expression of \( w \). We choose to mark by coloring in what follows.

**Lemma 4.8.** Let \((W, S)\) be a Coxeter system of graph \( \Gamma \), let \( s, t, u \in S \) be such that the subgraph of \( \Gamma \) determined by \( s, t, u \) is connected and \( s \) commutes with \( u \), suppose \( s, t, u \in \text{Supp}(w) \) for \( w \in W^c \) then the following holds: \( t \) occurs between \( s \) and \( u \) in a reduced expression of \( w \) if and only if \( t \) occurs between \( s \) and \( u \) in any reduced expression of \( w \).
Proof: The result follows directly from Matsumoto’s theorem and the definition of full commutativity.

Proposition 4.9. Let \( w \in W^{+c}(\tilde{C}_n) \), then \( w \) contains \( I \) (resp. \( J \), resp. \( JIJ \), resp. \( JIJ \)) if and only if \( G(w) \) contains \( G(I) \) (resp. \( G(J) \), resp. \( G(JIJ) \), resp. \( G(JIJ) \)).

Proof. The “if” part follows immediately from Proposition 4.6 once we remember that \( G(w) \) was defined up to horizontal translation.

For proving the “only if” part we assume that \( n \) is odd (the case \( n \) even is treated similarly). Suppose that \( w \) contains \( I \), this gives a reduced expression \( \overline{w} \) of \( w \) in which we can see \( I = \sigma_n \sigma_{n-2} \cdots \sigma_3 \sigma_1 \).

By marking the letters involved in \( I \) we obtain

\[
\overline{w} = u \sigma_n \sigma_{n-2} \cdots \sigma_3 \sigma_1 v, \tag{4.7}
\]

where \( l(w) = l(u) + l(v) + (n + 1)/2 \). We consider the normal form of \( w \) given by Proposition 3.16.

In this form, the rigid block of some marked \( s \in S \) is uniquely determined by \( s \), it is to be denoted by \([s]\). In the notation of Proposition 3.16 we say that the rigid block \( (l_s, r_s) \) is directly on the left (resp. right) of \( l \) if \( t = s + 1 \) (resp. \( t = s - 1 \)).

Now \([\sigma_n]\) and \([\sigma_{n-2}]\) are consecutive. Otherwise, any rigid block between those two blocks must contain \( \sigma_{n-1} \), which contradicts - by Lemma 4.8 - the fact that in \( \overline{w} \) the generator \( \sigma_{n-1} \) does not occur between \( \sigma_n \) and \( \sigma_{n-2} \). Moreover, \([\sigma_n]\) is on the left of \([\sigma_{n-2}]\), otherwise \( \sigma_{n-1} \) occurs between them by the normal form of Proposition 3.16. We sum up: \([\sigma_n]\) is directly on the left of \([\sigma_{n-2}]\). We apply the same argument to \([\sigma_{n-2}]\) and \([\sigma_{n-4}]\) and so on, arriving to \( \sigma_3 \) and \( \sigma_1 \). That means: the \( \frac{n+1}{2} \) rigid blocks \([\sigma_n][\sigma_{n-2}] \cdots [\sigma_3][\sigma_1]\) are consecutive in this order. Consequently, \( G(w) \) contains \( G(I) \).

The same argument applies replacing \( I \) by \( J \).

Now suppose that \( w \) contains \( JIJ \). Then, there exists a reduced expression \( \overline{w} \) of \( w \) in which we can see \( JIJ \). By marking the letters involved in \( JIJ \) we obtain

\[
\overline{w} = u \sigma_n \cdots \sigma_3 \sigma_1 \sigma_0 \sigma_n \cdots \sigma_3 \sigma_1 v, \tag{4.8}
\]

where \( l(w) = l(u) + l(v) + 3(n + 1)/2 \). Now consider the rigid block \([\sigma_n]\), it is directly on the left of \( [\sigma_n] \) due to the conditions in Proposition 3.16. Moreover, it is directly on the left of \([\sigma_{n-2}]\) by the argument above and finally it is directly on the left of \([\sigma_{n-1}]\), because the position of \( \sigma_{n-1} \) between \( \sigma_n \) and \( \sigma_{n-1} \) is independent of the reduced expression. We sum up: \([\sigma_n] = [\sigma_{n-1}] = [\sigma_{n-2}] \) and this rigid block is directly on the right of \([\sigma_n]\). By applying the very same argument \((n - 3)/2 \) times we obtain that \([\sigma_i] = [\sigma_{i-1}] = [\sigma_{i-2}] \) and that this block is directly on the right of \([\sigma_i]\) for \( i \) odd, \( 3 \leq i \leq n \). Finally, by considering the relative positions of \( \sigma_1 \), \( \sigma_1 \) and \( \sigma_0 \) we conclude that \([\sigma_1] = [\sigma_0]\) and that this block is directly on the right of \([\sigma_1]\). All the above implies that \( G(w) \) contains \( G(JIJ) \).

The same argument applies replacing \( JIJ \) by \( JIJ \).

Remark 4.10. Proposition 4.9 states a strong property that does not hold in general. For instance let \( w = \sigma_1 \sigma_2 \sigma_3 \sigma_0 \sigma_1 \sigma_2 \) and \( w' = \sigma_2 \sigma_1 \). Then, \( G(w') \subset G(w) \) as is illustrated in (4.9). However, \( w \) does not contain \( w' \).

\[
G(w) \longleftarrow \bullet \rightarrow G(w') \tag{4.9}
\]
Lemma 4.11. Let \( w \in W^{+c}(\hat{\mathcal{C}}_n) \). Let \( O_1, \ldots, O_m \) be the obliques of \( G(w) \) labelled from left to right. If \( O_i = O_{i+2j} = I \) for some positive integers \( i \) and \( j \) then

\[
O_{i+1} = O_{i+3} = \ldots = O_{i+2j-1} = J \quad \text{and} \quad O_{i+2} = O_{i+4} = \ldots = O_{i+2j-2} = I.
\]

(4.10)

Similarly, if \( O_i = O_{i+2j} = J \) for some positive integers \( i \) and \( j \) then

\[
O_{i+1} = O_{i+3} = \ldots = O_{i+2j-1} = I \quad \text{and} \quad O_i = O_{i+3} = \ldots = O_{i+2j-1} = J.
\]

(4.11)

Proof. We only prove (4.10). Equation (4.11) is treated similarly. If we have two \( G(I) \)'s in a grid of a positive element then, in order to satisfy the conditions in Proposition 3.10 we are forced to locate all the points “in between” of the two occurrences of \( G(I) \), as illustrated in (4.12) when \( n = 14 \) and \( j = 7 \). The result follows.

\[
\begin{array}{c}
\includegraphics[width=0.8\textwidth]{diagram.png}
\end{array}
\]

(4.12)

Corollary 4.12. Let \( w \in W^{+c}(\hat{\mathcal{C}}_n) \). Then, the oblique form of \( w \) is given by

\[
O_1 \ldots O_r(IJ)^k IO'_1 \ldots O'_s,
\]

(4.13)

for some integer \( k \geq 1 \) and some obliques \( O_i \) and \( O'_i \) such that \( O_i \neq I \), \( O'_i \neq J \), \( O'_i \neq I \) for all \( i \) and \( O_i \neq J \) for \( i \neq r \).

Proof: The result follows by a direct application of Proposition 4.6 and Lemma 4.11.

We are now ready to reapply the strategy used in the previous section in order to prove Theorem 4.2. We recall from Section 2 that \( SB_h(q, Q, k) \) is the quotient of \( 2BTL_n(q, Q) \) by the ideal generated by the elements \( IJI - \kappa I \) and \( JIJ - \kappa J \). We define \( IJ \) to be the ideal of \( 2BTL_n(q, Q) \) generated by \( IJI - \kappa I \). We are going to find a \( K \)-basis for this ideal. To this end we define a map \( \overline{\cdot} : W^{+c}(\hat{\mathcal{C}}_n) \to \hat{\mathcal{W}}^c(\hat{\mathcal{C}}_n) \) given by

\[
\bar{w} = O_1 \ldots O_r(IJ)^{k-1} IO'_1 \ldots O'_s,
\]

(4.14)

where \( w \) is as in (4.13). We define a \( K \)-linear map

\[
\overline{\cdot} : \text{Span}_K \{ b_w \mid w \in W^{+c}(\hat{\mathcal{C}}_n) - \hat{\mathcal{W}}^c(\hat{\mathcal{C}}_n) \} \to 2BTL_n(q, Q)
\]

(4.15)
determined by \( \overline{\bar{w}} = b_{\bar{w}} \). By using the same methods as the ones used in the proof of Theorem 3.11 we can see that the set

\[
Y := \{ b_w - \kappa \bar{w} \mid w \in W^{+c}(\hat{\mathcal{C}}_n) - \hat{\mathcal{W}}^c(\hat{\mathcal{C}}_n) \}
\]

(4.16)
is a $K$-basis for the ideal $Ib$. Furthermore, one can see that the set
\[ Y \cup \{ b_w \mid w \in \mathcal{W}_b^c(\tilde{C}_n) \} \]
is a $K$-basis for $2BT_L(n,q)$ and, therefore, $\{ b_w \mid w \in \mathcal{W}_b^c(\tilde{C}_n) \}$ is a $K$-basis for the quotient $2BT_L(n,q)/Ib$.

Let $Jb$ be the ideal of $2BT_L(n,q)/Ib$ generated by the element $JIJ - \kappa J$. We notice that the oblique form for an element $I$-blobbed which is not $J$-blobbed reduces to
\[ O_1 \ldots O_r JIJO_1' \ldots O_s', \]
for some obliques $O_i$ and $O_i'$ different from $I$ and $J$. We define a map
\[ \tilde{\eta}: \mathcal{W}_b^c(\tilde{C}_n) -\mathcal{W}_b^c(\tilde{C}_n) \rightarrow 2BT_L(n,q)/Ib \]
given by
\[ \tilde{w} = O_1 \ldots O_r JIJO_1' \ldots O_s', \]
if $w$ is as in (4.18). Then, we extend this map to a $K$-linear map
\[ \tilde{\eta}: \text{Span}_K \{ b_w \mid w \in \mathcal{W}_b^c(\tilde{C}_n) - \mathcal{W}_b^c(\tilde{C}_n) \} \rightarrow 2BT_L(n,q)/Ib \]
determined by $\tilde{b}_w = b_{\tilde{w}}$. As before, we can conclude that the set
\[ Z := \{ b_w - \kappa b_w \mid w \in \mathcal{W}_b^c(\tilde{C}_n) - \mathcal{W}_b^c(\tilde{C}_n) \} \]
is a $K$-basis for $Jb$ and that the set $Z \cup \{ b_w \mid w \in \mathcal{W}_b^c(\tilde{C}_n) \}$ is a $K$-basis of $2BT_L(n,q)/Ib$. Therefore, the set $\{ b_w \mid w \in \mathcal{W}_b^c(\tilde{C}_n) \}$ is a $K$-basis of
\[ (2BT_L(n,q)/Ib)/Jb \cong SB_n(q,Q,\kappa). \]

This finishes the sketch of proof of Theorem 4.2.

**Remark 4.13.** The grid presentation used in this work can be extended to express any element in $\mathcal{W}_b^c(\tilde{C}_n)$. More generally, all fully commutative elements of the three finite and four affine infinite families of Coxeter groups have a grid presentation coming from their normal forms. So we can use the oblique point of view to obtain a normal form of maximal blocks of commuting generators.

### 5. Catalan combinatorics

In this section we introduce and study a variation of Forder’s Catalan triangle \[\text{[For61]}\], which we call the blobbed Catalan triangle. It might be unclear for the reader why we would care about such a triangle. However, we point out that all the information needed to perform the enumeration of positive and blobbed fully commutative elements in the forthcoming sections is encoded in this triangle. Let us start by recalling the construction of Forder’s Catalan triangle\[\text{[Sha76]}\]. It is worth mentioning that in the literature there is a confusion between Forder’s triangle and Shapiro’s triangle\[\text{[Sha76]}\]. It is common to find authors who work with Forder’s triangle but who cite Shapiro’s work. But they are different triangles!
Definition 5.1. The Catalan triangle is the infinite matrix $(c_{i,j})_{-1\leq i,j}$ defined recursively as follows:

1. $c_{-1,-1} = 1$;
2. $c_{-1,j} = 0$, for all $j \geq 0$;
3. $c_{i,-1} = 0$, for all $i \geq 0$;
4. $c_{i,j} = c_{i-1,j-1} + c_{i-1,j+1}$, for all $i,j \geq 0$.

The left-hand side of (5.1) shows a Catalan triangle truncated at row 10 and at column 10. We have omitted the $(-1)$-th row and the $(-1)$-th column, as well as all zeroes. The numbers appearing in the Catalan triangle have the following combinatorial interpretation. Consider the infinite directed graph obtained from the Catalan triangle by replacing positive numbers by vertices and with arrows as illustrated in the right-hand side of (5.1). Then, for a given vertex $V$ the number of directed paths from the highest vertex to $V$ is equal to the number replaced by $V$. In particular, the numbers occurring in the 0-th column coincide with the Catalan numbers, since in this case the relevant paths are Dick paths. In formulas, we have $C_n = c_{2n,0}$.

Definition 5.2. The blobbed Catalan triangle is the infinite matrix $(C_{i,j})_{-1\leq i,j}$ defined recursively as follows:

1. For $i = -1$ or $i = 0$ and for all $j$ we have
   \[
   C_{i,j} = \begin{cases} 
   1, & \text{if } i + j \equiv 0 \mod 2; \\
   0, & \text{otherwise}.
   \end{cases}
   \]
2. $C_{i,-1} = 0$, for all $i \geq 1$;
3. $C_{i,j} = C_{i-1,j-1} + C_{i-1,j+1}$, for all $i \geq 1$ and all $j \geq 0$.

The blobbed Catalan triangle truncated at column 8 and at row 8 is shown in (5.2). We have omitted the entire $(-1)$-th column, as well as all zeroes.
The blobbed Catalan triangle is not a triangle but a rectangle! The choice of the name triangle instead of rectangle is justified by the fact that our primary (but not exclusive) interest is in the numbers that appear under the main diagonal, since the numbers located above this diagonal are only powers of 2. Like for the numbers occurring in the classical Catalan triangle, the numbers located under the main diagonal in the blobbed Catalan triangle have a combinatorial interpretation as follows. First, we erase all the zeroes and all the numbers appearing above the main diagonal, as illustrated in the left-hand side of (5.3). Then, as before, we construct a directed graph by replacing numbers by vertices. But this time, we add two edges between each pair of consecutive vertices located in the “hypotenuse” of the triangle, as is shown in the right-hand side of (5.3). Then, for a given vertex $V$, the number of directed paths from the highest vertex to $V$ is the number replaced by $V$.

\begin{align*}
1 \\
2 & 2 \\
6 & 14 & 16 \\
20 & 50 & 62 & 64 \\
70 & 182 & 238 & 254 & 256
\end{align*}

\( (5.3) \)

**Lemma 5.3.** The numbers $C_{i,j}$ satisfy:

1. $C_{j,0} = C_{j-1,1}$, for all $j \geq 0$.
2. $C_{i,j} = \sum_{k=0}^{j} C_{i-1-k,j+1-k}$, for all $i, j \geq 1$.
3. $C_{i,j} = \sum_{k=0}^{j} C_{i-1-k,j+1+k}$, for all $i, j \geq 1$.

**Proof:** All the statements follow immediately from Definition 5.2. \(\square\)

The following result provides a closed formula for the numbers occurring below the main diagonal of the blobbed Catalan triangle.

**Theorem 5.4.** Let $i$ and $j$ be integers with the same parity such that $0 \leq j \leq i$. Then, we have

$$C_{i,j} = \sum_{k=\frac{j}{2}}^{i} \binom{i}{k}.$$

(5.4)

**Proof:** We proceed by induction on $i$. If $i = 0$ or $i = 1$ then the result is clear. We suppose that $i > 1$ and that (5.4) holds for $i - 1$. The proof splits naturally into two cases in accordance with the parity of $i$. We only treat the case $i$ even since the case $i$ odd is handled similarly. For the rest of the proof we assume $i$ is even. Let $j$ be an even integer with $0 \leq j \leq i$. For $j = 0$ our induction hypothesis yields

$$C_{i,0} = C_{i-1,1} = \sum_{k=\frac{j}{2}}^{i} \binom{i-1}{k} = \binom{i-1}{\frac{j}{2}} + \binom{i-1}{\frac{i}{2}} = \binom{i}{\frac{i}{2}},$$

(5.5)

as we wanted to show. For $j = i$ we have $C_{i,i} = 2^i$ and (5.4) holds in this case as well.

We now assume $0 < j < i$. By Definition 5.2 we have $C_{i,j} = C_{i-1,j-1} + C_{i-1,j+1}$, then our induction
hypothesis shows that
\[ C_{i,j} = \sum_{k=\frac{i-1}{2}}^{\frac{i+j-1}{2}} \binom{i-1}{k} + \sum_{k=\frac{i-j}{2}}^{\frac{i-1}{2}} \binom{i-1}{k}. \] (5.6)

By collecting in consecutive pairs the terms appearing in the sums in (5.6) and by applying the well-known Pascal’s identities we can rewrite these sums as
\[ \sum_{k=\frac{i-j}{2}}^{\frac{i+j-1}{2}} \binom{i-1}{k} = \sum_{k=1}^{\frac{i}{2}} \binom{i}{\frac{i}{2}+2k-1} \] and
\[ \sum_{k=\frac{i-j}{2}}^{\frac{i}{2}} \binom{i-1}{k} = \sum_{k=0}^{\frac{i}{2}} \binom{i}{\frac{i}{2}+2k}. \] (5.7)

Finally, by combining (5.6) and (5.7) we obtain
\[ C_{i,j} = \sum_{k=0}^{j} \binom{\frac{i}{2}+k}{\frac{i}{2}} = \frac{1}{2^{\frac{i}{2}(i+j)}} \sum_{k=0}^{\frac{i}{2}(i-j)} \binom{i}{k}. \] (5.8)

Corollary 5.5. For all \( i \geq 1 \) we have \( C_{2i,0} = \binom{2i}{i} \).

Proof: This is just the case \( j = 0 \) of Theorem 5.4.

We conclude this section by showing how the results obtained in this section allow to recover the main result in [LO18]. Concretely, we prove that any binomial coefficient can be written as a 2-power weighted sum of numbers occurring in the Catalan triangle. This result is not needed in the sequel. We begin by considering central binomial coefficients.

Lemma 5.6. For all \( i \geq 1 \) we have
\[ \binom{2i}{i} = \sum_{k=1}^{i} 2^k C_{2i-k-1,k-1}. \] (5.9)

Proof: We recall that \( C_{2i,0} \) counts the number of paths from the vertex \((0,0)\) to the vertex \((2i, 0)\) in the blobbed Catalan triangle. These paths can be classified into \( i \) disjoint sets according to the number of steps on the hypotenuse. For \( 1 \leq k \leq i \), let \( P_k \) be the set of paths with \( k \) steps on the hypotenuse. Let \( p \) be a path in \( P_k \). The steps on the hypotenuse can be performed in \( 2^k \) different ways. Once \( p \) completed the \( k \) steps on the hypotenuse, it is forced to make one step in southwest direction till it reaches the vertex \((k+1, k-1)\). Finally, the number of ways in which \( p \) can be completed to reach the vertex \((2i, 0)\) is given by \( C_{2i-k-1,k-1} \). To see this just reverse the arrows in the classical Catalan triangle and count the number of paths from \((2i, 0)\) to \((k+1, k-1)\). Summing up, we have
\[ C_{2i,0} = \sum_{k=1}^{i} 2^k C_{2i-k-1,k-1}. \] (5.10)

The result is now a consequence of Corollary 5.5.
We now treat the case \( s \). Thus, the result in this case follows by combining Lemma 3.3 and Corollary 5.5.

\[ W \quad \text{denotes the cardinality of the set of positive fully commutative elements of affine length zero in } \mathcal{C} \]

**Enumeration of positive fully commutative elements.**

We proceed by induction on \( n \). Theorem 6.1. (and unique) result in this section is the following.

**Theorem 6.1.** Let \( n \) and \( s \) be integers such that \( n \geq 1 \) and \( s \geq 0 \). Then, \( a_n^s = C_{2n, 2s} \).

**Proof:**  We proceed by induction on \( s \). We begin by treating the case \( s = 0 \). We recall that \( a_n^0 \) denotes the cardinality of the set of positive fully commutative elements of affine length zero in \( W^c(\mathcal{C}_n) \). The above set coincides with the set of positive fully commutative elements in \( W^c(B_n) \). Thus, the result in this case follows by combining Lemma 5.3 and Corollary 5.5.

We now treat the case \( s = 1 \). We notice that the set \( A_{n+1}^0 \) splits into two disjoint subsets. Namely, the subset formed by the elements that contain \( \sigma_n \) and the subset formed by the elements that do not contain \( \sigma_n \). The set of elements in \( A_{n+1}^0 \) that contain \( \sigma_n \) can be identified with \( A_n^1 \). On the other hand, the set of elements in \( A_{n+1}^0 \) that do not contain \( \sigma_n \) can be identified with \( A_n^0 \). Then, we have \( a_{n+1}^0 = a_n^1 + a_n^0 \). By combining Lemma 5.3 and the result for \( s = 0 \) we conclude

\[ a_n^1 = a_{n+1}^0 - a_n^0 = C_{2n+2, 0} - C_{2n, 0} = C_{2n+1, 1} - C_{2n, 0} = C_{2n, 2}, \] (6.1)

which gives us the result for \( s = 1 \).

We now assume that \( s \geq 2 \) and that the result holds for integers less than \( s \). The set \( A_{n+1}^{s-1} \)
we have the elements of $W_n$ and Type III elements with elements of $D_n$ define decomposes into four disjoint subsets according to the behavior of the grids of its elements in the $n$-th row. We illustrate such a decomposition in (6.2) for $s = 6$. 

$$
\begin{array}{cccc}
 n + 1 & \text{Type I} & \text{Type II} & \text{Type III} \\
 n & \bullet \bullet \bullet \bullet \bullet \circ & \bullet \bullet \bullet \bullet \bullet & \bullet \bullet \bullet \bullet \bullet \circ \\
\end{array}
$$

(6.2)

By disregarding the $(n+1)$-th row, we can identify Type I elements with elements of $A_n^{s-2}$, Type II and Type III elements with elements of $A_n^{s-1}$ and Type IV elements with elements of $A_n^s$. Therefore, we have

$$a_{n+1}^{s-1} = a_n^{s-2} + a_n^{s-1} + a_n^{s-1} + a_n^s.\quad (6.3)$$

Then, property (3) in Definition 5.2 and our induction hypothesis yield

$$a_n^s = C_{2(n+1),2(s-1)} - C_{2n,2(s-2)} - 2C_{2n,2(s-1)}$$

$$= C_{2n+1,2s-3} + C_{2n+1,2s-1} - C_{2n,2(s-2)} - 2C_{2n,2(s-1)}$$

$$= C_{2n,2(s-2)} + C_{2n,2(s-1)} + C_{2n,2(s-2)} + C_{2n,2s} - C_{2n,2(s-2)} - 2C_{2n,2(s-1)}$$

$$= C_{2n,2s}.\quad (6.4)$$

\[\square\]

7. Enumeration of blobbed fully commutative elements.

The goal of this section is to count the elements of $W_b^c(\tilde{C}_n)$. We set $w_n^I = IJI$ and $w_n^J = JIJ$. We recall that the elements of $W_b^c(\tilde{C}_n)$ are the positive fully commutative elements that avoid $w_n^I$ and $w_n^J$. We notice that for $n \geq 2$ we have

$$w_n^I = \begin{cases} 
\langle n-1, n \rangle \langle n-3, n-1 \rangle \langle n-5, n-3 \rangle \cdots \langle 1, 3 \rangle \langle 0, 1 \rangle, & \text{if } n \text{ is even}; \\
\langle n, n \rangle \langle n-2, n \rangle \langle n-4, n-2 \rangle \cdots \langle 1, 3 \rangle \langle 0, 1 \rangle, & \text{if } n \text{ is odd},
\end{cases} \quad (7.1)$$

and

$$w_n^J = \begin{cases} 
\langle n, n \rangle \langle n-2, n \rangle \langle n-4, n-2 \rangle \cdots \langle 0, 2 \rangle \langle 0, 0 \rangle, & \text{if } n \text{ is even}; \\
\langle n-1, n \rangle \langle n-3, n-1 \rangle \langle n-5, n-3 \rangle \cdots \langle 0, 2 \rangle \langle 0, 0 \rangle, & \text{if } n \text{ is odd}.
\end{cases} \quad (7.2)$$

If $n = 1$ we have $w_1^I = \langle 1, 1 \rangle \langle 0, 1 \rangle$ and $w_1^J = \langle 0, 1 \rangle \langle 0, 0 \rangle$. As in the previous section, we enumerate the elements of $W_b^c(\tilde{C}_n)$ according with their affine length. Given integers $n \geq 1$ and $s \geq 0$ we define

$$B_n^s = \{ w \in W_b^c(\tilde{C}_n) | L(w) = s \} \quad (7.3)$$

and set $b_n^s = \# B_n^s$. We also define $D_n^s = A_n^s - B_n^s$ and set $d_n^s = |D_n^s| = a_n^s - b_n^s$. We compute the numbers $\{a_n^s\}$ are known by Theorem 6.1 the knowledge of $d_n^s$ is equivalent to the knowledge of $b_n^s$. We stress that the elements of $D_n^s$ are the positive FC elements in $W^c(\tilde{C}_n)$ of affine length $s$ that contain $w_n^I$ or $w_n^J$. By Proposition 4.9 we can see that an element $w \in D_n^s$ if and only if $G(w)$ has $s$ black dots in the $n$-th row and contains $G(w_n^I)$ or $G(w_n^J)$. Sometimes we abuse notation and think of $D_n^s$ as the set formed by the grids of its elements. We begin our counting by treating the cases $s = 0$ and $s > n$.

**Theorem 7.1.** Let $n$ be a positive integer. Then, $d_n^0 = 0$ and $d_n^s = a_n^s$, if $s > n$. Consequently, $b_n^0 = a_n^0$ and $b_n^s = 0$ if $s > n$.

24
Proof: Let $w \in A^*_n$. If $s = 0$ then $w$ does not contain the generator $\sigma_n$. Thus $w$ cannot contain $w^I_n$ or $w^J_n$ since the generator $\sigma_n$ occurs in both $w^I_n$ and $w^J_n$. We conclude that $d^I_n = 0$. We now assume that $s > n$. In this case $G((n, n)(n-1, n)\ldots(0, n))$ is contained in $G(w)$ and this forces to $G(w)$ to contain $G(w^I_n)$, as illustrated in (7.4) for $(n, s) = (6, 7)$ and $(n, s) = (7, 8)$, respectively. Therefore, $w \in D^*_n$ and $d^*_n = a^*_n$. □

Theorem 7.1 confirms the fact that $W^*_n(C_n)$ is finite. We continue our counting with the case $s = 1$. We first introduce some suitable notations.

Definition 7.2. We define $I^-_{n,r}$ to be the subset of $D^I_n$ formed by all the grids that are obtained from $G(w^I_n)$ by adding black dots exclusively on its left and with $r$ dots added in its leftmost non-empty column. Similarly, we define $I^+_{n,r}$ to be the subset of $D^I_n$ formed by all the grids that are obtained from $G(w^I_n)$ by adding black dots exclusively on its right and with $r$ dots added in its highest non-empty diagonal. We also define $J^-_{n,r}$ and $J^+_{n,r}$ by considering $w^J_n$ rather than $w^I_n$. Furthermore, we set $i^-_{n,r} = |I^-_{n,r}|$, $i^+_{n,r} = |I^+_{n,r}|$, $j^-_{n,r} = |J^-_{n,r}|$ and $j^+_{n,r} = |J^+_{n,r}|$. Finally, we define

\[ i^-_n := \sum_{r=0}^{n-1} i^-_{n,r} \quad \text{and} \quad i^+_n := \sum_{r=0}^{n-1} i^+_{n,r} \quad (7.5) \]

\[ j^-_n := \sum_{r=0}^{n-1} j^-_{n,r} \quad \text{and} \quad j^+_n := \sum_{r=0}^{n-1} j^+_{n,r}. \quad (7.6) \]

Lemma 7.3. Let $n$ and $r$ be integers such that $0 \leq r < n$. We have $i^-_{n, r} = i^+_{n, r}$ and $j^-_{n, r} = j^+_{n, r}$.

Proof: We just prove $i^-_{n, r} = i^+_{n, r}$, the other equality is treated similarly. It is enough to exhibit a bijection between $I^-_{n, r}$ and $I^+_{n, r}$. Such a bijection is given by a rotation of the region on the right of $G(w^I_n)$ in $45^\circ$ clockwise followed by a reflection through a vertical edge. We depicted such transformations in (7.7) for $n = 8$. We notice that under this bijection the points located on the highest diagonal are mapped to points in the leftmost column. □

Lemma 7.3 allows us to define

\[ i_{n, r} := i^-_{n, r} = i^+_{n, r} \quad \text{and} \quad j_{n, r} := j^-_{n, r} = j^+_{n, r}. \quad (7.8) \]

Similarly, we define

\[ i_n := i^-_n = i^+_n \quad \text{and} \quad j_n := j^-_n = j^+_n. \quad (7.9) \]

\[ * \text{Here “diagonal” means a line with slope } -1. \]
Lemma 7.4. Let $n$ be a positive integer. We have

$$d_{n}^{1} = \begin{cases} (i_{n})^{2}, & \text{if } n \text{ is even;} \\ (j_{n})^{2}, & \text{if } n \text{ is odd.} \end{cases}$$

Proof: We assume that $n$ is even. We need to count the number of elements in $D_{n}^{1}$, that is, the number of positive fully commutative elements of affine length one that contain $w_{n}^{l}$ or $w_{n}^{r}$. Since $\sigma_{n}$ occurs twice in $w_{n}^{l}$ when $n$ is even, we conclude that $w_{n}^{l}$ cannot be contained in an element of affine length one. For this reason we only care about $w_{n}^{l}$. In (7.11) we have depicted $G(w_{n}^{l})$, for $n = 2$, $n = 4$, $n = 6$ and $n = 8$, where we have added white dots to indicate the positions where we can add black dots keeping the affine length one. Since any addition of black dots in the left is independent of any addition of black dots in the right, and vice-versa, we conclude that

$$d_{n}^{1} = i_{n}^{r} \cdot i_{n}^{r} = i_{n}^{2}. \quad (7.10)$$

The result for $n$ odd is treated similarly.

$$i_{n} = \begin{cases} C_{n,0}, & \text{if } n \text{ is even;} \\ 0, & \text{if } n \text{ is odd;} \end{cases} \quad \text{and} \quad j_{n} = \begin{cases} 0, & \text{if } n \text{ is even;} \\ \frac{1}{2}C_{n,1}, & \text{if } n \text{ is odd.} \end{cases} \quad (7.13)$$

Lemma 7.5. Let $n$ and $r$ be integers such that $0 \leq r < n$. Then, we have

$$i_{n,r} = \begin{cases} C_{n-2-r,r}, & \text{if } n \text{ is even;} \\ 0, & \text{if } n \text{ is odd;} \end{cases} \quad \text{and} \quad j_{n,r} = \begin{cases} 0, & \text{if } n \text{ is even;} \\ \frac{1}{2}C_{n-1-r,r}, & \text{if } n \text{ is odd.} \end{cases} \quad (7.12)$$

Consequently, $i_{n} = C_{n,0}$, if $n$ is even; and $j_{n} = \frac{1}{2}C_{n,1}$, if $n$ is odd.

Proof: We only prove the formulas for $i_{n,r}$ and $i_{n}$. We recall that $i_{n,r} = i_{n,r}^{r} = i_{n,r}^{r}$. Thus it is enough to show that $i_{n,r}^{r}$ matches with the value given in (7.12). If $n$ is odd then $w_{n}^{l}$ has already two occurrences of the generator $t_{n}$. Therefore, $I_{n,r}^{r} = \emptyset$ since $I_{n,r}^{r}$ is by definition a subset of $D_{n}^{1}$. We conclude that $i_{n,r} = i_{n,r}^{r} = 0$. We now assume that $n$ is even and proceed by induction. It can be easily checked that

$$i_{2,0}^{r} = 1 = C_{0,0} \quad \text{and} \quad i_{2,1}^{r} = 1 = C_{-1,1}, \quad (7.14)$$

which provides the base of our induction for $n = 2$. We now assume that (7.12) holds for $n$ and we will prove it for $n + 2$. The key point here is that the $G(w_{n+2}^{l})$ and its left region “contains” $G(w_{n}^{l})$ and its left region, as illustrated in (7.13) for $n = 6$. Therefore, any element of $I_{n-2}^{r}$ is given by a number of black dots added in the leftmost non-empty column of $G(w_{n+2}^{l})$ and some element of $I_{n}^{r}$.

(7.15)
If we add 0 or 1 black dots in the leftmost non-empty column of \( G(w^I_{n+2}) \) then we can consider any element of \( J^+_{n+2} \) in order to construct an element of \( I^+_{n+2} \). Thus, our induction hypothesis and Lemma 5.3 imply
\[
i^+_{n+2,0} = i^+_{n+2,1} = \sum_{k=0}^{n-1} i^+_{n,k} = \sum_{k=0}^{n-1} C_{n-2-k,k} = C_{n-1,1} = C_{n,0}. \tag{7.16}
\]
We now assume that \( 2 \leq r \leq n \). If we add \( r \) black dots in the leftmost non-empty column of \( G(w^I_{n+2}) \) then we are forced to add at least \( r - 1 \) black dots in its second column, otherwise the configuration obtained would not be a grid of a positive fully commutative element. Therefore,
\[
i^+_{n+2,r} = \sum_{k=0}^{n-r} i^+_{n,k} = \sum_{k=0}^{n-r} C_{n-2-k,k} = \sum_{k=0}^{n-r} C_{(n-r)-1-k,r-1+k} = C_{n+2-2-r,r}. \tag{7.17}
\]
Furthermore, we note that \( i^+_{n+2,n+1} = 1 = C_{1,n+1} \). This complete the proof of (7.12). Finally, (7.13) is now a consequence of Lemma 5.3 and (7.12).

**Theorem 7.6.** Let \( n \) be a positive integer. We have
\[
d^I_n = \begin{cases} 
(C_{n,0})^2 & \text{if } n \text{ is even} \\
\frac{1}{2}C_{n,1}^2 & \text{if } n \text{ is odd}
\end{cases}
\]

**Proof:** The result follows by a direct application of Lemma 7.4 and Lemma 7.5.

We recall that our goal in this section is to compute the numbers \( \{d^I_n\} \). So far we have already computed these numbers for \( s = 0 \), \( s = 1 \) and \( s > n \). To deal with the case \( 2 \leq s \leq n \) we need a bit more of notation.

**Definition 7.7.** Let \( n \) be a positive integer and \( t \) be a non-negative integer. We define the set \( \mathcal{I}^{\leftarrow,t}_{n} \) to be the set of grids obtained from \( G(w^I_{n}) \) by adding black dots exclusively on its left and exactly \( t \) black dots on the \( n \)-th row. We define \( \mathcal{I}^{\rightarrow,t}_{n} \) in a similar way but this time we consider the region located on the right of \( G(w^I_{n}) \). We also define \( \mathcal{J}^{\leftarrow,t}_{n} \) and \( \mathcal{J}^{\rightarrow,t}_{n} \) by considering \( w^I_{n} \) rather than \( w^I_{n} \). Finally, we define \( i^{\leftarrow,t}_{n} \), \( i^{\rightarrow,t}_{n} \), \( j^{\leftarrow,t}_{n} \) and \( j^{\rightarrow,t}_{n} \) to be the cardinalities of the sets \( \mathcal{I}^{\leftarrow,t}_{n} \), \( \mathcal{I}^{\rightarrow,t}_{n} \), \( \mathcal{J}^{\leftarrow,t}_{n} \) and \( \mathcal{J}^{\rightarrow,t}_{n} \), respectively.

**Lemma 7.8.** Let \( n \) be a positive integer and \( t \) be a non-negative integer. We have
\[
i^{\leftarrow,t}_{n} = i^{\rightarrow,t}_{n} \quad \text{and} \quad j^{\leftarrow,t}_{n} = j^{\rightarrow,t}_{n}. \tag{7.18}
\]

**Proof:** We only prove \( i^{\leftarrow,t}_{n} = i^{\rightarrow,t}_{n} \). The other equality is treated similarly. In order to see that \( i^{\leftarrow,t}_{n} = i^{\rightarrow,t}_{n} \) it is enough to exhibit a bijection between \( \mathcal{I}^{\rightarrow,t}_{n} \) and \( \mathcal{I}^{\leftarrow,t}_{n} \). Such a bijection is described graphically as follows. Let \( G \in \mathcal{I}^{\rightarrow,t}_{n} \). We draw lines starting from the 0-th row with slope \( -1 \) connecting the points in \( G \) that are not involved in \( G(w^I_{n}) \). Then, we rotate these lines around the point located in the 0-th row in \( 45^\circ \) clockwise and locate the rotated points at the same level as they were before the rotation. After doing this we apply a reflection through a vertical line. Finally, we locate \( G(w^I_{n}) \) on the right of the resultant configuration. We illustrate in (7.19) the bijection for \( n = 10 \) and \( t = 3 \). Some remarks are in order. Blue dots have no special meaning. They must be thought simply as black dots. The difference between black and blue dots is that the black ones are forced to appear in any element of \( \mathcal{I}^{\leftarrow,3}_{10} \) and \( \mathcal{I}^{\rightarrow,3}_{10} \), whereas the blue ones are just a choice we did
to illustrate the bijection. Finally, we notice that under this bijection the points on the $n$-th row are mapped to points on the $n$-th row.

\[ (7.19) \]

**Remark 7.9.** A practical consequence of Lemma 7.8 is that the direction of the arrows in the symbols $i_n^{\leftarrow t}$, $i_n^{\rightarrow t}$, $j_n^{\leftarrow t}$ and $j_n^{\rightarrow t}$ is irrelevant. For this reason we can relax the notation by dropping the arrows. Concretely, we define

\[ i_n^t := i_n^{\leftarrow t} = i_n^{\rightarrow t} \quad \text{and} \quad j_n^t := j_n^{\leftarrow t} = j_n^{\rightarrow t}. \quad (7.20) \]

**Lemma 7.10.** Let $n$ and $s$ be integers with $2 \leq s \leq n$. Then,

\[ d_n^s = \begin{cases} 
\sum_{k=0}^{s-1} j_{n}^k i_{n}^{s-1-k} + \sum_{k=0}^{s-2} j_{n}^k j_{n}^{s-2-k} - 2 \sum_{k=0}^{s-2} j_{n}^k i_{n}^{s-2-k}, & \text{if } n \text{ is even;} \\
\sum_{k=0}^{s-1} j_{n}^k j_{n}^{s-1-k} + \sum_{k=0}^{s-2} j_{n}^k i_{n}^{s-2-k} - 2 \sum_{k=0}^{s-2} j_{n}^k i_{n}^{s-2-k}, & \text{if } n \text{ is odd.} 
\end{cases} \quad (7.21) \]

**Proof:** For the sake of brevity we only prove the result for $n$ even. The case $n$ odd is handled similarly and is left to the reader. We recall that $d_n^s$ is the cardinality of $D_n^s$ and that $D_n^s$ is the set of positive fully commutative elements of affine length $s$ that contain $G(w_n^J)$ or $G(w_n^I)$. The main obstacle to carry out the counting of such elements is that there are elements in $D_n^s$ that contain $G(w_n^J)$ or $G(w_n^I)$ more than once. In order to overcome this drawback we need a couple of definitions:

1. First, we label the $s$ black dots appearing in the $n$-th row of any element of $D_n^s$ from left to right as $P_1, P_2, \ldots, P_s$.
2. Given an element $x \in D_n^s$ and an integer $1 \leq u \leq s$ we say that $w_n^I$ appears in $x$ at position $u$ if there is an occurrence of $G(w_n^I)$ in $x$ in which $P_u$ is involved.
3. Given an element $x \in D_n^s$ and an integer $1 \leq u < s$ we say that $G(w_n^I)$ appears in $x$ at position $u$ if there is an occurrence of $G(w_n^I)$ in $x$ in which $P_u$ and $P_{u+1}$ are involved.
4. Finally, we define the sets

\[ D_n^s(I, u) = \{ x \in D_n^s | G(w_n^I) \text{ appears in } x \text{ at position } u \} \quad (7.22) \]

\[ D_n^s(J, u) = \{ x \in D_n^s | G(w_n^J) \text{ appears in } x \text{ at position } u \} \quad (7.23) \]

We split the proof into two cases according the parity of $s$.  

28
Case A. We suppose that $s$ is odd and set $s := (s + 1)/2$. We can decompose the set $D_n^s$ as a disjoint union of the sets

\[
\begin{align*}
D_n^s(I, s - 1) \backslash D_n^s(I, s) & \quad D_n^s(J, s) \backslash D_n^s(I, s) \\
D_n^s(I, s - 1) \backslash D_n^s(J, s) & \quad D_n^s(I, s + 1) \backslash D_n^s(J, s) \\
D_n^s(J, s - 2) \backslash D_n^s(I, s - 1) & \quad D_n^s(J, s + 1) \backslash D_n^s(I, s + 1) \\
D_n^s(I, s - 2) \backslash D_n^s(J, s - 2) & \quad D_n^s(I, s + 2) \backslash D_n^s(J, s + 1)
\end{align*}
\]

(7.24)

With this decomposition at hand, we have reduced the proof of the theorem to compute the cardinality of each one of the sets occurring in (7.24). We begin by determining the cardinality of $D_n^s(I, s)$. By definition, each element of $D_n^s(I, s)$ has a $G(w_n^j)$ at position $s$. Furthermore, to the left and to the right of this occurrence of $G(w_n^j)$ there are $s - 1$ black dots in the $n$-th row, as illustrated for $s = 7$ in (7.25).

\[
\begin{array}{c}
P_1 P_2 P_3 P_4 P_5 P_6 P_7 \\
\ldots \ldots \ldots \ldots \ldots \ldots \\
\end{array}
\]

\[
w_n^j
\]

(7.25)

By using the fact that any addition of black dots on the left is independent of any addition of black dots on the right and vice-versa, we conclude that

\[
|D_n^s(I, s)| = i_n^{s-1} \cdot i_n^{s-1}.
\]

(7.26)

We now compute $|D_n^s(J, s - u) \backslash D_n^s(I, s - u + 1)|$, for $1 \leq u < s$. By definition, each element of $D_n^s(J, s - u)$ has a $G(w_n^j)$ at position $s - u$. Furthermore, to the left of this occurrence of $G(w_n^j)$ there are $s - u - 1$ black dots located in the $n$-th row. On the other hand, to the right of the aforementioned occurrence of $G(w_n^j)$ there are $s + u - 2$ black dots located in the $n$-th row. Therefore, we have

\[
|D_n^s(J, s - u)| = j_n^{s-u-1} \cdot j_n^{s+u-2}.
\]

(7.27)

To determine the cardinality of $D_n^s(J, s - u) \backslash D_n^s(I, s - u + 1)$, it only remains to know the cardinality of $D_n^s(J, s - u) \cap D_n^s(I, s - u + 1)$. By definition, an element of $D_n^s(J, s - u) \cap D_n^s(I, s - u + 1)$ has a $G(w_n^j)$ at position $s - u$ and a $G(w_n^j)$ at position $s - u + 1$. These elements have $s - u - 1$ black dots in the $n$-th row to the left of the aforementioned occurrence of $G(w_n^j)$ and $s + u - 2$ black dots in the $n$-th row to the right of the aforementioned occurrence of $G(w_n^j)$. The above is illustrated for
$s = 13$ (and therefore $s = 7$) and $u = 4$ in (7.28), where we have depicted the relevant occurrences of $G(w_n^I)$ and $G(w_n^J)$ in black and red, respectively.

Finally, a combination of (7.27) and (7.29) yields

$$|D_n^s(I, s - u) \cap D_n^s(I, s - u + 1)| = j_n^{s-u-1}i_n^{s+u-2}. \quad (7.29)$$

We obtain

$$|D_n^s(J, s - u) \cap D_n^s(I, s - u + 1)| = j_n^{s-u-1}(j_n^{s+u-2} - i_n^{s+u-2}). \quad (7.30)$$

Similarly, we obtain

$$|D_n^s(I, s - u) \cap D_n^s(J, s - u)| = i_n^{s-u-1}(i_n^{s+u-1} - j_n^{s+u-2});$$

$$|D_n^s(J, s + u) \cap D_n^s(I, s + u)| = j_n^{s-u-2}(j_n^{s+u-1} - i_n^{s+u-1});$$

$$|D_n^s(I, s + u) \cap D_n^s(J, s + u - 1)| = i_n^{s-u-1}(i_n^{s+u-1} - j_n^{s+u-2}); \quad (7.31)$$

for $1 \leq u < s$, $0 \leq u < s - 1$ and $1 \leq u < s$, respectively. We now combine (7.24), (7.26), (7.30) and (7.31) to obtain

$$d_n^s = \sum_{u=1}^{s-1} j_n^{s-u-1}(j_n^{s+u-2} - i_n^{s+u-2}) + \sum_{u=1}^{s-1} i_n^{s-u-1}(i_n^{s+u-1} - j_n^{s+u-2}) + \sum_{u=0}^{s-2} i_n^{s-u-2}(j_n^{s+u-1} - i_n^{s+u-1}). \quad (7.32)$$

In order to match the formula in (7.32) with the one in (7.21) we collect the terms formed by products of $j$’s, products of $i$’s and the products of $i$’s and $j$’s. For instance, the products of $j$’s in (7.32) are

$$\sum_{k=0}^{s-2} j_n^k j_n^{2s-3-k} + \sum_{k=s-1}^{2s-3} j_n^k j_n^{2s-3-k} = \sum_{k=0}^{2s-3} j_n^k j_n^{2s-3-k}. \quad (7.33)$$

Finally, going back to the normal $s$, we conclude that the products of $j$’s in (7.32) is given by

$$\sum_{k=0}^{s-2} j_n^k j_n^{s-2-k}, \quad (7.34)$$
which is the same that appears in \((7.21)\). The other terms are treated similarly. This finishes the proof of the theorem in this case.

**Case B.** We suppose that \(s\) is even and set \(s := s/2\). In this case we can decompose \(D_n^s\) as a disjoint union of the sets

\[
egin{align*}
D_n^s(I, s) &= D_n^s(I, s) \setminus D_n^s(J, s) \\
D_n^s(I, s - 1) \setminus D_n^s(I, s) &= D_n^s(I, s) \setminus D_n^s(I, s + 1) \\
D_n^s(I, s - 1) \setminus D_n^s(J, s - 1) &= D_n^s(I, s + 2) \setminus D_n^s(J, s) \\
D_n^s(I, s - 2) \setminus D_n^s(I, s - 1) &= D_n^s(I, s + 2) \setminus D_n^s(I, s + 1) \\
\vdots & \ \vdots \\
D_n^s(I, 2) \setminus D_n^s(J, 2) &= D_n^s(I, s - 1) \setminus D_n^s(J, s - 2) \\
D_n^s(I, 1) \setminus D_n^s(I, 2) &= D_n^s(I, s - 1) \setminus D_n^s(I, s - 1) \\
D_n^s(I, 1) \setminus D_n^s(J, 1) &= D_n^s(I, s) \setminus D_n^s(J, s - 1).
\end{align*}
\]

(7.35)

The rest of the argument carries over for this case. \(\square\)

**Lemma 7.11.** Let \(n\) be a positive integer and \(t\) be a non-negative integer. We have

\[
i_n^t = \begin{cases} 
C_{n, 2t}, & \text{if } n \text{ is even;} \\
C_{n, 2t + 1}, & \text{if } n \text{ is odd.}
\end{cases}
\]

(7.36)

\[
j_n^t = \begin{cases} 
(1/2)C_{n + 1, 2t + 1}, & \text{if } n \text{ is even;} \\
(1/2)C_{n + 1, 2t}, & \text{if } n \text{ is odd.}
\end{cases}
\]

(7.37)

**Proof:** We only prove (7.36). Equation (7.37) is treated similarly. We split the proof into two cases in accordance with the parity of \(n\). We recall from Remark 7.9 that \(i_n^t = i_n^{t^{-1}} = i_n^{-1} = i_n^t\). Thus it is enough to show that \(i_n^{t^{-1}} = i_n^{-1}\) matches with the value given in (7.36).

**Case A:** \(n\) is even. If \(t = 0\) we have \(I_n^{t^{-1}} = I_n^{-1}\), then Lemma 7.5 implies

\[
i_n^{t^{-1}} = i_n^{-1} = i_n = C_{n, 0},
\]

(7.38)
as we wanted to show. We now assume that \(t > 0\). The key point here is that any element of \(I_n^{t^{-1}}\) can be seen as an element of \(I_{n + 2t}\) with at least \(2t - 1\) black dots added in the leftmost column. The above is achieved by adding black dots to \(G(w_n^t)\) in order to obtain \(G(w_{n + 2t}^t)\), as illustrated in (7.39) for \(n = 10\) and \(t = 3\), where we have drawn in blue the points needed to pass from \(G(w_{10}^t)\) to \(G(w_{16}^t)\). We have also depicted the black dots that are forced to appear. In particular, there are \(5 = 2t - 1\) black dots forced to appear in the leftmost non-empty column.

\[
(7.39)
\]
By combining Lemma 5.3 and Lemma 7.5 we obtain

\[
i_n^{t→t} = \sum_{k=2t-1}^{n+2t-1} i_n^{t,k} = \sum_{k=0}^{n} C_{n+2t-2-(k+2t-1),k+2t-1} = \sum_{k=0}^{n} C_{n-1-k,2t-1+k} = C_{n,2t}.
\]

**Case B:** \( n \) is odd. We claim that

\[
i_n^{t→t} = i_{n-1}^{t→t} + i_{n-1}^{t→t+1}.
\]  \( (7.40) \)

We notice that if \( (7.40) \) holds, then property (3) in Definition 5.2 and an application of the theorem for the even case yield

\[
i_n^{t→t} = i_{n-1}^{t→t} + i_{n-1}^{t→t+1} = C_{n-1,2t} + C_{n-1,2t+2} = C_{n,2t+1},
\]  \( (7.41) \)

which is what we want to show. So that to finish the proof of the theorem we only need to check \( (7.40) \). To see why the above formula is correct, we notice that the re is a bijection between \( I_n^{t→t} \) and the disjoint union of \( I_{n-1}^{t→t} \) and \( I_{n-1}^{t→t+1} \). Such a bijection is given as follows. First, we draw \( G(w_n^t) \). Then, we draw \( t \) black dots in the \( n \)-th row to the left of \( G(w_n^t) \). The occurrence of these \( t \) dots forces the occurrence of \( t \) black dots in the \((n-1)\)-th row. Beside these black dots, we can still add another black dot in the intersection of the \((n-1)\)-th row with the leftmost non-empty column. We refer to such a dot as the special dot. The special dot is depicted in blue in \( (7.42) \) for \( n = 11 \) and \( t = 3 \). We conclude that the elements of \( I_n^{t→t} \) split into classes according whether we use the special dot or not. Let \( G \) be an element of \( I_n^{t→t} \). If the special dot appears in \( G \) then by erasing the whole \( n \)-th row we obtain an element of \( I_{n-1}^{t→t+1} \). If the special dot does not appear in \( G \) then by erasing the whole \( n \)-th row we obtain an element of \( I_{n-1}^{t→t} \). This gives the promised bijection and completes the proof of \( (7.40) \).

\[
\begin{align*}
n & \quad \quad \quad \\
n & \quad \quad \quad \\
\end{align*}
\]

(7.42)

**Theorem 7.12.** Let \( n \) and \( s \) be integers with \( 2 \leq s \leq n \). Then,

\[
d_n^s = \begin{cases} 
\sum_{k=0}^{s-2} \left[ C_{n,2k}(C_{n,2(s-1-k)} - C_{n+1,2(s-k)-3}) + \frac{1}{4} C_{n+1,2k+1}C_{n+1,2(s-k)-3} \right] + C_{n,2(s-1)}C_{n,0}, \\
\sum_{k=0}^{s-1} C_{n,2(s-k)-3}(C_{n,2k+1} - C_{n+1,2k}) + \frac{1}{4} \sum_{k=0}^{s-1} C_{n+1,2k}C_{n+1,2(s-1-k)}, 
\end{cases}
\]

for \( n \) even and odd, respectively.

**Proof:** The result follows by a direct application of Lemma 7.10 and Lemma 7.11. Table 1 and Table 2 show some values of \( d_n^s \) and \( b_n^s \), respectively. There, rows correspond to \( n \) and columns to \( s \). We recall that \( b_n^s = a_n^s - d_n^s \) and that the value of \( a_n^s \) is given by Theorem 5.4 and Theorem 6.1. We collect the information obtained in this section in a polynomial, \( P_n(v) \in \mathbb{N}[v] \), defined by \( P_n(v) = \sum_{s=0}^{n} b_n^s v^s \). Of course, \( p_n := P_n(1) = |W_n^s(C_n)| \). Theorem 4.2 tells us that \( \dim SB_n(q,Q,\kappa) = p_n \). The first nine terms of the sequence \( (p_n) \) are 5, 19, 84, 335, 1428, 5748, 24104, 97287, 404148.
Table 1: Values of \( d_n \)

| \( n \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|-------|---|---|---|---|---|---|---|---|---|---|
| 1     | 1 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 |
| 2     | 0 | 4 | 16 | 16 | 16 | 16 | 16 | 16 | 16 | 16 |
| 3     | 0 | 36 | 184 | 1024 | 64 | 64 | 64 | 64 | 64 | 64 |
| 4     | 0 | 36 | 184 | 1024 | 64 | 64 | 64 | 64 | 64 | 64 |
| 5     | 0 | 100 | 500 | 1000 | 256 | 1024 | 1024 | 1024 | 1024 | 1024 |
| 6     | 0 | 400 | 1825 | 3160 | 64 | 4096 | 4096 | 4096 | 4096 | 4096 |
| 7     | 0 | 1225 | 6370 | 11711 | 14868 | 16384 | 16384 | 16384 | 16384 | 16384 |
| 8     | 0 | 4900 | 23716 | 44100 | 57428 | 65536 | 65536 | 65536 | 65536 | 65536 |
| 9     | 0 | 15876 | 84672 | 164304 | 221004 | 262144 | 262144 | 262144 | 262144 | 262144 |

Table 2: Values of \( b_n \)

| \( b_n \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---------|---|---|---|---|---|---|---|---|---|---|
| 1       | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2       | 2 | 6 | 10 | 3 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3       | 3 | 20 | 41 | 20 | 3 | 0 | 0 | 0 | 0 | 0 |
| 4       | 4 | 70 | 146 | 90 | 26 | 3 | 0 | 0 | 0 | 0 |
| 5       | 5 | 252 | 572 | 412 | 157 | 32 | 3 | 0 | 0 | 0 |
| 6       | 6 | 924 | 2108 | 1673 | 778 | 224 | 38 | 3 | 0 | 0 |
| 7       | 7 | 3432 | 8213 | 7072 | 3733 | 1304 | 303 | 44 | 3 | 0 |
| 8       | 8 | 12870 | 30850 | 28050 | 16402 | 6714 | 1954 | 394 | 50 | 3 |
| 9       | 9 | 48620 | 120260 | 115112 | 72608 | 33044 | 11156 | 2792 | 497 | 56 |

References

[AH17] S. Al Harbat. On the fully commutative elements of type \( \tilde{C} \) and faithfulness of related towers. *Journal of Algebraic Combinatorics*, 45(3):803–824, 2017.

[DGN09] J. De Gier and A. Nichols. The two-boundary Temperley–Lieb algebra. *Journal of Algebra*, 321(4):1132–1167, 2009.

[Ern12] D. C. Ernst. Diagram calculus for a type affine C Temperley–Lieb algebra, I. *Journal of Pure and Applied Algebra*, 216(11):2467–2488, 2012.

[For61] H. G. Forder. Some problems in combinatorics. *The Mathematical Gazette*, 45(353):199–201, 1961.

[GMP12] R. Green, P. Martin, and A. Parker. A presentation for the symplectic blob algebra. *Journal of Algebra and Its Applications*, 11(03):1250060, 2012.

[Gra95] J. J. Graham. *Modular representations of Hecke algebras and related algebras*. PhD thesis, University of Sydney, 1995.

[Jon87] V. F. Jones. Hecke algebra representations of braid groups and link polynomials. *Annals of mathematics*, 126(2):335–388, 1987.
[LO18] K.-H. Lee and S.-J. Oh. Catalan triangle numbers and binomial coefficients. *Contemporary Mathematics*, 713:165–185, 2018.

[MGP07] P. Martin, R. Green, and A. Parker. Towers of recollement and bases for diagram algebras: planar diagrams and a little beyond. *Journal of Algebra*, 316(1):392–452, 2007.

[Ree11] A. Reeves. Tilting modules for the symplectic blob algebra. *arXiv preprint arXiv:1111.0146v2*, 2011.

[Sha76] L. W. Shapiro. A Catalan triangle. *Discrete Mathematics*, 14(1):83–90, 1976.

[Ste96] J. R. Stembridge. On the fully commutative elements of Coxeter groups. *Journal of Algebraic Combinatorics*, 5(4):353–385, 1996.

[Ste97] J. Stembridge. Some combinatorial aspects of reduced words in finite Coxeter groups. *Transactions of the American Mathematical Society*, 349(4):1285–1332, 1997.

[Ste98] J. R. Stembridge. The enumeration of fully commutative elements of Coxeter groups. *Journal of Algebraic Combinatorics*, 7(3):291–320, 1998.

[TL71] H. N. Temperley and E. H. Lieb. Relations between the ‘percolation’ and ‘colouring’ problem and other graph-theoretical problems associated with regular planar lattices: some exact results for the ‘percolation’ problem. *Proc. R. Soc. Lond. A*, 322(1549):251–280, 1971.