Soliton-impurity interaction in two Ablowitz-Ladik chains with different coupling

R S Kamburova and M T Primatarowa
Georgi Nadjakov Institute of Solid State Physics, Bulgarian Academy of Sciences, 1784 Sofia, Bulgaria
E-mail: krad@issp.bas.bg

Abstract. The interaction of solitons with point defects in a system of coupled Ablowitz-Ladik (AL) chains is studied numerically. The system is a discrete analog of coupled nonlinear Schrödinger equations. Two types of interchain coupling are investigated: one which admits reduction of the system to the standard integrable AL model (dispersive coupling) and one which couples opposite sites of the chains and does not admit reduction to the AL model (nondispersive coupling). The action of the two coupling types is additive and they can compensate each other in some cases. We have obtained that the single-peak bound soliton-defect solution (attractive impurity) is stable against perturbations, while the double-peak bound soliton-defect solution (repulsive impurity) is unstable and can be easily destroyed. Linear point defects do not influence the period of energy transfer and it is close to the period for the homogeneous case.

1. Introduction
The study of nonlinear waves is receiving much attention due to the potential application within different branches of physics from nonlinear optics to Bose-Einstein condensate. The interplay between discrete diffraction and nonlinearity leads to the formation of discrete solitons. They promise an efficient way to control switching of optical signals in a system of coupled waveguides. So waveguide-based devices have received considerable attention in literature and this field has been extensively explored theoretically and experimentally [1,2]. A discrete coupler involving two waveguides which exchange power as a result of weak overlap of their evanescent fields is the basic realization of a waveguide switch. The rate of power swapped back and forth between waveguides depends on the strength of the coupling, the degree of similarity of the waveguides and the initial pulse energy [3-7]. Waveguide arrays are particularly interesting because of their possible applications in signal processing [8-12]. All considerations regard discrete soliton switching mainly in homogenous waveguides with constant or smoothly varying coupling between them. However in practical applications the properties of inhomogeneous waveguides are more interesting and rather inevitable for switching [13-17].

Widely investigated are the standard discrete nonlinear Schrödinger (NLS) equation, as well as the completely integrable discrete Ablowitz-Ladik (AL) equation [18-20]. Although the two equations have the same linear properties and yield the same NLS equation in the continuum limit, their nonlinear properties are different. This leads to differences in the dynamics of narrow solitons (bright or dark) for the two models. Soliton solutions in two coupled discrete nonlinear chains were found and their stability was investigated in [21-24].
In the present paper we study the interaction of solitons with impurities in two Ablowitz-Ladik chains with a complicated coupling that includes linear, nonlinear and dispersive interactions between the chains.

2. The model

We shall consider two parallel chains of particles described by the following system of coupled Ablowitz-Ladik equations:

\[
\begin{align*}
    i \frac{\partial \alpha_n}{\partial t} &= M(\alpha_{n+1} + \alpha_{n-1})(1 + \gamma |\alpha_n|^2) + [d_1(\beta_{n+1} + \beta_{n-1}) + 2d_2\beta_n](1 + \gamma |\alpha_n|^2) + \varepsilon \delta_{n,0}\alpha_n \\
    i \frac{\partial \beta_n}{\partial t} &= M(\beta_{n+1} + \beta_{n-1})(1 + \gamma |\beta_n|^2) + [d_1(\alpha_{n+1} + \alpha_{n-1}) + 2d_2\alpha_n](1 + \gamma |\beta_n|^2) + \varepsilon \delta_{n,0}\beta_n
\end{align*}
\]  

(1)

\(\alpha_n(t) [\beta_n(t)]\) is the amplitude of an excitation at site \(n\) of the first (second) chain, interacting with an impurity of the strength \(\varepsilon\) localized at the point \(n_0\). \(M\) is the coupling interaction between neighboring particles in one and the same chain. The two chains are coupled to each other through the real parameters \(d_1\) and \(d_2\). The parameter \(d_2\) governs the interchain coupling between opposite sites (nondispersive), while \(d_1\) is a coupling of the AL type and allows reduction of the system to the AL model which is completely integrable. Both interactions include linear and nonlinear terms. The parameter \(\gamma\) determines the type of the soliton solution (bright for \(\gamma > 0\) and dark for \(\gamma < 0\)) of the AL equation. In what follows we consider only bright solitons and set \(\gamma = 1\) due to the scaling property of the AL system.

First we shall consider the homogeneous case \((\varepsilon = 0)\). We briefly outline the influence of the two different couplings on the soliton properties [24]. Equation (1) can be derived from the Hamiltonian

\[
H = \sum_n [M(\alpha_n^* \alpha_{n-1} + \alpha_n^* \alpha_{n+1} + \beta_n^* \beta_{n-1} + \beta_n^* \beta_{n+1}) + d_1(\alpha_n^* \beta_{n-1} + \alpha_{n-1}^* \beta_n + \beta_n^* \beta_{n+1} + \beta_{n+1}^* \beta_n) + 2d_2(\alpha_n^* \beta_n + \alpha_n^* \beta_{n+1})]
\]

(2)

using the deformed Poisson brackets [19,20]

\[
\begin{align*}
    \{\alpha_n, \alpha_m^*\} &= i(1 + |\alpha_n|^2)\delta_{n,m}, & \{\beta_n, \beta_m^*\} &= i(1 + |\beta_n|^2)\delta_{n,m} \\
    \{\alpha_n, \alpha_m\} &= \{\alpha_n^*, \alpha_m^*\} = 0, & \{\beta_n, \beta_m\} &= \{\beta_n^*, \beta_m^*\} = 0
\end{align*}
\]

(3)

and the equations of motion

\[
\frac{\partial \alpha_n}{\partial t} = \{H, \alpha_n\}, \quad \frac{\partial \beta_n}{\partial t} = \{H, \beta_n\}.
\]

(4)

The system (1) is nonintegrable but has two integrals of motion, the Hamiltonian \(H\) and the total number of particles \(N\)

\[
N = \sum_n [\ln(1 + |\alpha_n|^2) + \ln(1 + |\beta_n|^2)].
\]

(5)

For \(d_2 = 0\) and the symmetric reduction \(\alpha_n(t) \equiv \beta_n(t)\) the system (1) turns in an AL equation with the well known bright soliton solution:

\[
\alpha_n(t) = \beta_n(t) = \sinh \frac{1}{L} \text{sech} \frac{n - vt}{L} e^{i(kn - \omega t)}
\]

\[
v = -2(M + d_1)L \sinh \frac{1}{L} \sin k, \quad \omega = 2(M + d_1) \cosh \frac{1}{L} \cos k
\]

(6)
The parameters $k$ (wavenumber) and $L$ (width) determine the velocity $v$ and frequency $\omega$ of the soliton. In this case the conserved quantities have the form

$$H = 8(M + d_1) \sinh \frac{1}{L} \cos k, \quad N = 4/L$$

(7)

and it holds $\omega = \partial H/\partial N$.

Figure 1. Switching of the bound soliton-defect solution for $M = -1$, $L = 2$, $\varepsilon = -0.32$ (attraction) and (a): $|d_1| = 0.628$, $d_2 = 0$; (b): $d_1 = 0$, $|d_2| = 0.628$. The time is in units $1/|M|$.

3. Soliton-impurity interaction

Now we consider the inhomogeneous static case $\varepsilon \neq 0$, $k = v = 0$. For $d_1 = d_2 = 0$ the system (1) decomposes into two standard AL equations which possess the following bound soliton-defect solution:

$$\alpha_n(t) = \beta_n(t) = \sinh \frac{1}{L} \sech \left( \frac{|n - n_0|}{L} + \Delta \right) e^{-i\omega t},$$

$$\omega = 2M \cosh \frac{1}{L}, \quad \tanh \Delta = \varepsilon / \left( 2M \sinh \frac{1}{L} \right).$$

(8)

For $\Delta > 0$ the function $|\alpha_n(t)|$ has a single maximum at $n = n_0$ (single pick solution). For $\Delta < 0$ the function $|\alpha_n(t)|$ has two maxima at $n = n_0 \pm \Delta L$ (double pick solution).

We shall investigate the evolution of an AL bound soliton which at the initial time is launched in one of the chains

$$\alpha_n(0) = \sinh \frac{1}{L} \sech \left( \frac{|n - n_0|}{L} + \Delta \right), \quad \beta_n(0) = 0$$

(9)
solving numerically the system (1). The simulations are carried out for 1000 sites of each chain, \( n_0 = 500 \) and periodic boundary conditions. The width of the solitons is \( L = 2 \), which underline the discreteness of the system. For wide solitons we can use the continuum approximation and (1) turns in a system of coupled NLS equations. \( M = -1 \).

In the homogeneous linear case \( (\varepsilon = 0, \gamma = 0) \) the excitation will transfer from one chain to the other and back with a period \( t_0 = \pi/|d| \), where \( d \) is the total linear coupling.

For our complicated homogeneous model \( (\varepsilon = 0, \gamma = 1) \), which is not linear we observe that an energy exchange between the two chains take place with nearly the same period

\[
t_0 = \frac{\pi}{2|d_1 + d_2|} \tag{10}
\]

and the energy exchange rate depends on the strength of the coupling. For small values of the coupling constants the soliton is only partially transferred. When the coupling increases the transferred rate grows. This behavior is due to the nonlinear coupling terms. We obtained that a soliton can transfer (perfect soliton switching) when the simple condition

\[
4|d_1 + d_2|L^2 \gg 1 \tag{11}
\]

is fulfilled.

We have observed the differences between the two types of interchain coupling. For large values of the coupling constants the soliton switching is preserve for the coupling between opposite sites of the chains \( d_2 \). For the dispersive coupling \( d_1 \) an energy exchange with the same period \( t_0 \) takes place but the soliton formation is destroyed. Obviously to achieve soliton

\[
\begin{align*}
|\alpha_n|^2 & \\
|\beta_n|^2 & \\
|\alpha_n|^2 & \\
|\beta_n|^2 & \\
\end{align*}
\]

Figure 2. Switching of the bound soliton-defect solution with \( \varepsilon = 0.32 \) (repulsion) for (a): \( |d_1| = 0.628, d_2 = 0 \); (b): \( d_1 = 0, |d_2| = 0.628 \).
switching $|d_1| < |M|$ has to be fulfilled. We have obtained that for wide solitons ($L \gg 1$) the dispersive character of the coupling $d_1$ vanish and the solitons in the two chains are stable [24].

Figure 1 shows the influence of the attractive impurity over the soliton transfer when the coupling between the two chains is of type $d_1$ [figure 1(a)] or $d_2$ [figure 1(b)]. The process does not depend on the sign of the coupling. We have obtained that the single-peak bound soliton-defect solution is stable against perturbations. The period of the energy transfer is 2.5 for both cases and is in a good agreement with the value calculated from (10).

Figures 2 and 3 show the evolution of a double pick bound soliton-defect solution (repulsive impurity) which is unstable and can be easily destroyed. It begins to oscillate for small defects (figure 2) or splits into two solitons which propagate with opposite velocities (figure 3). Figure 4 demonstrates the interplay between the two types of coupling. Their action is additive. When the constants $d_1$ and $d_2$ are of the same sign [figure 4(a)] the soliton switching is similar to that for one coupling’s type but with the greater strength. Evidence for this behavior is the magnitude of the transfer period and the oscillations period. When the two couplings are of different signs the resulting coupling decreases and can be set to nearly zero as shown on figure 4(b). We observe that the soliton $\alpha_n$ remains in the first chain where it was initially excited and in the second chain $\beta_n \approx 0$.

![Figure 3](image)

**Figure 3.** Splitting of the bound soliton-defect solution $|\alpha_n|$ for a repulsive defect with $\varepsilon = 0.64$ [(a), (b)]; $\varepsilon = 1.03$ [(a'), (b')]. $|d_1| = 0.628$, $d_2 = 0$ [(a), (a')] and $d_1 = 0$, $|d_2| = 0.628$ [(b), (b')]. The evolution of the $|\beta_n|$ function is similar.

4. Conclusion
We have studied the interaction of discrete solitons with impurities in a system of coupled Ablowitz-Ladik chains. Two types of interchain coupling are introduced and compared. The coupling of the AL type admits reduction of the system to an AL model. Perfect soliton switching
Figure 4. Time evolution of the bound soliton-defect solution for a repulsive defect with \( \varepsilon = 0.32 \) and (a): \( d_1 = d_2 = 0.314 \); (b): \( d_1 = -d_2 = 0.314 \).

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