Some Identities on the Twisted $q$-Analogues of Catalan-Daehee Numbers and Polynomials

Dongkyu Lim

Department of Mathematics Education, Andong National University, Andong 36729, Korea; dklim@anu.ac.kr

Abstract: In this paper, the author considers twisted $q$-analogues of Catalan-Daehee numbers and polynomials by using $p$-adic $q$-integral on $\mathbb{Z}_p$. We derive some explicit identities for those twisted numbers and polynomials related to various special numbers and polynomials.

Keywords: $q$-analogue of Catalan-Daehee numbers; $q$-analogue of Catalan-Daehee polynomials; $p$-adic $q$-integral on $\mathbb{Z}_p$; twisted $q$-analogue of Catalan-Daehee numbers; twisted $q$-analogue of Catalan-Daehee polynomials

MSC: 11B68; 11B83; 11S80

1. Introduction

Let $p$ be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ we denote the ring of $p$-adic integers, the field of $p$-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$. The $p$-adic norm $|·|_p$ is normally defined $|p|_p = \frac{1}{p}$. Let $q$ be an indeterminate in $\mathbb{C}_p$ with $|1 − q|_p < p^{−\frac{1}{p^n}}$. The $q$-analogue of $x$ is defined by $[x]_q = \frac{1 − q^x}{1 − q}$.

Note that $\lim_{q \to 1} [x]_q = x$.

Let $f(x)$ be a uniformly differentiable function on $\mathbb{Z}_p$. Then the $p$-adic $q$-integral on $\mathbb{Z}_p$ is defined by [1–3]

$$\int_{\mathbb{Z}_p} f(x)d\mu_q(x) = \lim_{N \to \infty} \sum_{x=0}^{p^N−1} f(x)\mu_q(x + p^N\mathbb{Z}_p)$$

$$= \lim_{N \to \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N−1} f(x)[x]_q^x. \quad (1)$$

From (1), we have

$$q \int_{\mathbb{Z}_p} f(x + 1)d\mu_q(x) = \int_{\mathbb{Z}_p} f(x)d\mu_q(x) + (q − 1)f(0) + \frac{q − 1}{\log q} f'(0), \quad (2)$$

where $f'(0) = \frac{df(x)}{dx}_{|x=0}$.

For $n \in \mathbb{N}$, let $T_p$ be the $p$-adic locally constant space defined by

$$T_p = \bigcup_{n \geq 1} C_{p^n} = \lim_{n \to \infty} C_{p^n},$$

where $C_{p^n} = \{w | w^{p^n} = 1\}$ is the cyclic group of order $p^n$.

For $w \in T_p$, let us take $f(x) = w^xe^{x^2}$. Then, by (1), we get

$$\frac{(q − 1) + \frac{q−1}{\log q}t}{wqe^1 − 1} = \int_{\mathbb{Z}_p} w^xe^{x^2}d\mu_q(x) \quad (3)$$
Thus, by (3), we define the twisted $q$-Bernoulli numbers which are given by the generating function to be

$$
\frac{(q - 1) + \frac{q-1}{\log q} t}{qwe^t - 1} = \sum_{n=0}^{\infty} B_{n,q,w} \frac{t^n}{n!}.
$$

(4)

From (4), we note that

$$qw(B_{q,w} + 1)^n - B_{n,q,w} = \begin{cases} q - 1, & \text{if } n = 0 \\ \frac{q-1}{\log q}, & \text{if } n = 1, \\ 0, & \text{if } n \geq 1, \end{cases}
$$

with the usual convention about replacing $B_{n,q,w}$ by $B_{n,q,w}$.

From (2) and (4), we have

$$
\sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} w^x x^n d\mu_q(x) \frac{t^n}{n!} = \int_{\mathbb{Z}_p} w^x e^t d\mu_q(x)
$$

$$= (q - 1) + \frac{q-1}{\log q} t = \sum_{n=0}^{\infty} B_{n,q,w} \frac{t^n}{n!}.
$$

(5)

Thus, by (5), we get

$$\int_{\mathbb{Z}_p} w^x x^n d\mu_q(x) = B_{n,q,w}, \quad (n \geq 0).$$

(6)

For $|t|_p < p^{-\frac{1}{\log q}}$, the twisted $(\lambda, q)$-Daehee polynomials are defined by generating function to be (cf. [4])

$$
\sum_{n=0}^{\infty} D_{n,q,w}(x|\lambda) \frac{t^n}{n!} = \frac{2(q - 1) + \lambda \frac{q-1}{\log q} \log(1 + t)}{wq^2(1 + t)^\lambda - 1} (1 + t)^{\lambda x}.
$$

(7)

When $x = 0$, $D_{n,q,w}(\lambda) = D_{n,q,w}(0|\lambda)$ are called the twisted $(\lambda, q)$-Daehee numbers. In particular,

$$D_{0,q,w}(1) = \frac{2(q - 1)}{wq^2 - 1}.
$$

The twisted Catalan-Daehee numbers are defined by [5]

$$\frac{1}{2} \log(1 - 4t) \frac{t^n}{\sqrt{1 - 4t} - 1} = \sum_{n=0}^{\infty} d_{n,w} t^n.
$$

(8)

If we take $w = 1$ in the twisted Catalan-Daehee numbers, $d_n = d_{n,1}$, are the Catalan-Daehee numbers in [6–8].

We note that

$$\sqrt{1 + t} = \sum_{m=0}^{\infty} (-1)^{m-1} \left(\begin{array}{c} 2m \\ m \end{array}\right) \left(\frac{1}{4}\right)^m \left(\frac{1}{2m - 1}\right) t^m.
$$

(9)

By replacing $t$ by $-4t$ in (9), we get

$$\sqrt{1 - 4t} = 1 - 2 \sum_{m=0}^{\infty} \left(\begin{array}{c} 2m \\ m \end{array}\right) \frac{1}{m + 1} t^{m+1} = 1 - 2 \sum_{m=0}^{\infty} C_m t^{m+1},
$$

(10)

where $C_m$ is the Catalan number.
From (8) and (10), Dolgy et al. showed a relation between the Catalan-Daehee numbers and the Catalan numbers in [6]:

\[ d_n = \begin{cases} 
\frac{4^n}{n+1} - \frac{1}{n-m} \sum_{m=0}^{n-1} C_m, & \text{if } n \geq 1 \\
1, & \text{if } n = 0 
\end{cases} \]

Catalan-Daehee numbers and polynomials were introduced in [7] and considered the family of linear differential equations arising from the generating function of those numbers in order to derive some explicit identities involving Catalan-Daehee numbers and Catalan numbers. In [8], several properties and identities associated with Catalan-Daehee numbers and polynomials were derived by utilizing umbral calculus techniques. Dolgy et al. gave some new identities for those numbers and polynomials derived from \( p \)-adic Volkenborn integrals on \( \mathbb{Z}_p \) in [6]. Recently, Ma et al. introduced and studied \( q \)-analogues of the Catalan-Daehee numbers and polynomials with the help of \( p \)-adic \( q \)-integral on \( \mathbb{Z}_p \) in [9]. The aim of this paper is to introduce \( q \)-analogues of the twisted Catalan-Daehee numbers and polynomials by using \( p \)-adic \( q \)-integer on \( \mathbb{Z}_p \), and derive some explicit identities for those twisted numbers and polynomials related to various special numbers and polynomials.

2. The Twisted \( Q \)-Analogues of Catalan-Daehee Numbers

For \( t \in \mathbb{C}_p \) with \( |t|_p < p^{-\frac{1}{r+1}} \) and for \( \omega \in T_p \), we have

\[ \int_{\mathbb{Z}_p} w^x (1 - 4t)^{\frac{x}{2}} d\mu_q(x) = \frac{q - 1 + \log_q^2 \frac{1}{2} \log(1 - 4t)}{q \omega \sqrt{1 - 4t - 1}}. \]  \hspace{1cm} (11)

In the view of (11), we define the twisted \( q \)-analogue of Catalan-Daehee numbers which are given by the generating function to be

\[ \frac{q - 1 + \log_q^2 \frac{1}{2} \log(1 - 4t)}{q \omega \sqrt{1 - 4t - 1}} = \sum_{n=0}^{\infty} d_{n,q,w} t^n. \]  \hspace{1cm} (12)

Note that \( \lim_{q \to 1} d_{n,q,w} = d_{n,w} \) \( (n \geq 0) \), which is the twisted Catalan-Daehee numbers in [5].

From (7) and (12), we have

\[ \sum_{n=0}^{\infty} d_{n,q,w} t^n = \frac{1}{2} \left( \frac{2(q - 1 + \log_q^2 \frac{1}{2} \log(1 - 4t))}{w^2 q^2 (1 - 4t - 1)} \right) (q \omega \sqrt{1 - 4t + 1}) \]

\[ = \frac{1}{2} \left( \sum_{l=0}^{\infty} 4^l D_{1,q,w^2}(1) \left( \frac{(-1)^l}{l!} \right) \right) \left( 1 + qw - 2qw \sum_{m=0}^{\infty} C_m t^{m+1} \right) \]

\[ = \frac{1 + qw}{q^2 q^2 - 1} \sum_{n=0}^{\infty} (-4)^n D_{n,q,w}(1) t^n - qw \sum_{n=1}^{\infty} \left( \sum_{m=0}^{n-1} \frac{(-4)^{n-m-1}}{(n-m-1)!} D_{n-m-1,q,w^2}(1) C_m \right) t^n \]  \hspace{1cm} (13)

\[ = \frac{q^2 - 1}{q^2 q^2 - 1} + \sum_{n=0}^{\infty} \left( \frac{q}{n!} - 2 \right) (-4)^n D_{n,q,w}(1) t^n - qw \sum_{n=1}^{\infty} \left( \sum_{m=0}^{n-1} \frac{(-4)^{n-m-1}}{(n-m-1)!} D_{n-m-1,q,w^2}(1) C_m \right) t^n \]

Therefore, by comparing the coefficients on the both sides of (13), we obtain the following theorem.
Theorem 1. For \( n \geq 0 \) and \( w \in T_p \), we have
\[
 d_{n,q,w} = \begin{cases} 
 \frac{q^2 - 1}{wq^2 - 1} - \frac{n}{(n-m)q^2 - 1}, & \text{if } n = 0, \\
 1 + qw \left( -4 \right)^n D_{n,q,w} (1) - qw \sum_{m=0}^{n-1} \frac{(-4)^{n-m-1}}{(n-m-1)!} 2^{2n-2m-1} D_{n-m-1,q,w} (1) C_m, & \text{if } n \geq 1.
\end{cases}
\]
Specially, \( w = 1 \) and \( q \to 1 \), we have

Corollary 1 (Theorem 1, [6]). For \( n \geq 0 \), we have
\[
 d_n = \begin{cases} 
 (-4)^n D_n (1) - \sum_{m=0}^{n-1} \frac{(-4)^{n-m-1}}{(n-m-1)!} 2^{2n-2m-1} D_{n-m-1} (1) C_m, & \text{if } n = 0, \\
 1, & \text{if } n \geq 1.
\end{cases}
\]

Now, from (6) and (12), we observe that
\[
\sum_{n=0}^{\infty} d_{n,q,w} t^n = \frac{q - 1 + \frac{q^n - 1}{\log q} \log(1 - 4t)}{qw \sqrt{1 - 4t} - 1} = \int_{Z_p} w^x (1 - 4t)^{\frac{x}{2}} d\mu_q (x)
\]
\[
= \sum_{m=0}^{\infty} \left( \frac{1}{2} \right)^{m} \frac{1}{m!} (\log(1 - 4t))^m \int_{Z_p} w^x x^m d\mu_q (x)
\]
\[
= \sum_{m=0}^{\infty} \left( \frac{1}{2} \right)^{m} B_{m,q,w} \sum_{n=m}^{\infty} S_1 (n, m) \frac{1}{n!} (-4t)^n
\]
\[
= \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} 2^{2n-m} (-1)^m B_{m,q,w} S_1 (n, m) \right) \frac{t^n}{n!},
\]
where \( S_1 (n, m), (n, m \geq 0) \) is the Stirling number of the first kind which is defined by [1–20]
\[
(x)_n = \sum_{l=0}^{n} S_1 (n, l) x^l, \quad (n \geq 0).
\]
Here, \( (x)_0 = 1, (x)_n = x(x - 1) \cdots (x - n + 1), (n \geq 1) \).

Therefore, by (14), we obtain the following theorem.

Theorem 2. For \( n \geq 0 \) and \( w \in T_p \), we have
\[
(-4)^n d_{n,q,w} = \frac{1}{n!} \sum_{m=0}^{n} 2^{2n-m} B_{m,q,w} S_1 (n, m).
\]

By binomial expansion, we get
\[
\int_{Z_p} w^x (1 - 4t)^{\frac{x}{2}} d\mu_q (x) = \sum_{n=0}^{\infty} (-4)^n \int_{Z_p} w^x \left( \frac{x}{2} \right)^m d\mu_q (x) \frac{t^n}{n!}.
\]
(15)

From (12) and (15), we obtain the following corollary.

Corollary 2. For \( n \geq 0 \) and \( w \in T_p \), we have
\[
\int_{Z_p} w^x \left( \frac{x}{2} \right)^m d\mu_q (x) = (-1)^n 2^{2n} d_{n,q,w} = \frac{1}{n!} \sum_{m=0}^{n} \left( \frac{1}{2} \right)^{m} B_{m,q,w} S_1 (n, m).
\]

For the case \( w = 1 \) and \( q \to 1 \), we have the following.
Corollary 3 (Theorem 2, [6]). For \( n \geq 0 \), we have

\[
(-1)^n d_n = \frac{1}{n!} \sum_{m=0}^{n} 2^{2n-m} B_m S_1(n, m).
\]

The twisted \( q \)-analogue of \( \lambda \)-Dahee polynomials are given by the \( p \)-adic \( q \)-integral on \( \mathbb{Z}_p \) to be

\[
\int_{\mathbb{Z}_p} w^n (1 + t)^{\lambda y + z} d\mu_q(y) = \frac{(q - 1) + \lambda \frac{q - 1}{\log q} \log(1 + t)}{qw(1 + t)^\lambda - 1} (1 + t)^x
\]

(16)

\[
= \sum_{n=0}^{\infty} \tilde{D}_{n,q,w}(x|\lambda) \frac{t^n}{n!}.
\]

When \( x = 0 \), \( \tilde{D}_{n,q,w}(\lambda) = \tilde{D}_{n,q,w}(0|\lambda) \) \((n \geq 0)\) are called the twisted \( q \)-analogue of \( \lambda \)-Dahee numbers. Note that

\[
\tilde{D}_{0,q,w}(\lambda) = \frac{q - 1}{qw - 1}.
\]

From (16), we note that

\[
\sum_{n=0}^{\infty} (-1)^n 4^n \tilde{D}_{n,q,w}(\lambda) \left(\frac{1}{2}\right) = \frac{q - 1 + \frac{q - 1}{\log q} \log(1 - 4t)}{qw(1 - 4t)^{\frac{1}{2}} - 1}
\]

(17)

\[
= \sum_{n=0}^{\infty} d_{n,q,w} t^n.
\]

Thus, by (17), we get

\[
d_{n,q,w} = (-1)^n \frac{4^n}{n!} \tilde{D}_{n,q,w}(\lambda) \left(\frac{1}{2}\right), \quad (n \geq 0).
\]

Let us take \( t = \frac{1}{4}(1 - e^{2t}) \) in (12). Then we have

\[
\sum_{k=0}^{\infty} d_{k,q,w}(\lambda) \left(\frac{1}{4}\right) (1 - e^{2t})^k = \frac{q - 1 + \frac{q - 1}{\log q} t}{qw e^t - 1} = \int_{\mathbb{Z}_p} w^x e^{xt} d\mu_q(x)
\]

(18)

\[
= \sum_{n=0}^{\infty} B_{n,q,w} \frac{t^n}{n!}.
\]

On the other hand,

\[
\sum_{k=0}^{\infty} d_{k,q,w}(\lambda) (-1)^k \left(\frac{1}{4}\right) (e^{2t} - 1)^k = \sum_{k=0}^{\infty} (-1)^k k! d_{k,q,w}(\lambda) \left(\frac{1}{4}\right) \frac{1}{k!} (e^{2t} - 1)^k
\]

(19)

\[
= \sum_{k=0}^{\infty} (-1)^k k! d_{k,q,w} 2^{-2k} \sum_{n=k}^{\infty} S_2(n, k) 2^n \frac{t^n}{n!}
\]

where \( S_2(n, k) \) \((n, k \geq 0)\) is the Stirling number of the second kind which is defined by

\[
x^n = \sum_{l=0}^{n} S_2(n, l)(x)_l, \quad (n \geq 0).
\]

Therefore, by (18) and (19), we obtain the following theorem.
Theorem 3. For $n \geq 0$, we have

$$B_{n,q,w} = \sum_{k=0}^{n} (-1)^k S_2(n,k) 2^{n-2k} d_{k,q,w}.$$ 

Now, we observe that

$$\int_{\mathbb{R}} w^q (1 - 4t)^{2y} d\mu_q(y) = \frac{(q - 1) + \frac{q-1}{\log q} \log(1 - 4t)}{wq \sqrt{1 - 4t} - 1} (1 - 4t)^{\frac{x}{2}}.$$ 

We define the twisted Catalan-Daehee polynomials which are given by the generating function to be

$$\frac{q - 1 + \frac{q-1}{\log q} \log(1 - 4t)}{wq \sqrt{1 - 4t} - 1}(1 - 4t)^{\frac{x}{2}} = \sum_{n=0}^{\infty} d_{n,q,w}(x) t^n. \quad (20)$$

When $x = 0$, $d_{n,q,w} = d_{n,q,w}(0) \ (n \geq 0)$ are the twisted Catalan-Daehee numbers in (12).

Note that

$$\begin{align*}
(1 - 4t)^{\frac{x}{2}} &= \sum_{l=0}^{\infty} \frac{(x/2)^l}{l!} (\log(1 - 4t))^l = \sum_{l=0}^{\infty} \frac{x^l}{2^l} \sum_{m=1}^{\infty} S_1(m,l)(-4)^m \frac{m^l}{m!} \\
&= \sum_{m=0}^{\infty} \sum_{l=0}^{m} S_1(m,l) \frac{(-4)^m}{m!} \left(\frac{x}{2}\right)^l \frac{m^l}{m!}.
\end{align*} \quad (21)$$

Thus, by (12), (20) and (21), we get

$$\sum_{n=0}^{\infty} d_{n,q,w}(x) t^n = \frac{q - 1 + \frac{q-1}{\log q} \log(1 - 4t)}{wq \sqrt{1 - 4t} - 1} (1 - 4t)^{\frac{x}{2}}$$

$$= \left( \sum_{k=0}^{\infty} d_{k,q,w} \frac{k^l}{k!} \right) \sum_{m=0}^{\infty} \sum_{l=0}^{m} S_1(m,l) \frac{(-4)^m}{m!} \left(\frac{x}{2}\right)^l \frac{m^l}{m!} d_{n-m,q,w}.$$  

(22)

By comparing the coefficients on both sides (22), we obtain the following theorem.

Theorem 4. For $n \geq 0$, we have

$$d_{n,q,w}(x) = \sum_{m=0}^{n} \sum_{l=0}^{m} S_1(m,l) (-1)^m \frac{2^{2m-l}}{m!} d_{n-m,q,w} x^l.$$ 

For the case $w = 1$ and $q \to 1$, we have the following.

Corollary (Theorem 5, [6]). For $n \geq 0$, we have

$$d_n(x) = \sum_{l=0}^{n} \left( \sum_{m=1}^{n} (-1)^m \frac{2^{2m-l}}{m!} S_1(m,l) d_{n-m,q,w} \right) x^l.$$ 

3. Conclusions

To summarize, we introduced twisted $q$-analogues of Catalan-Daehee numbers and polynomials and obtained several explicit expressions and identities related to them. We expressed the twisted $q$-analogues of Catalan-Daehee numbers in terms of the twisted...
(λ, q)-Daehee numbers, and of the twisted q-Bernoulli numbers and Stirling numbers of the first kind in Theorems 1 and 2. We also derived an identity involving the twisted q-Bernoulli numbers, twisted q-analogues of Catalan-Daehee numbers and Stirling numbers of the second kind in Theorem 3. In addition, we obtain an explicit expression for the twisted q-analogues of Catalan-Daehee polynomials which involve the twisted q-analogues of Catalan-Daehee numbers and Stirling numbers of the first kind in Theorem 4.

In recent years, many special numbers and polynomials have been studied by employing various methods, including: generating functions, p-adic analysis, combinatorial methods, umbral calculus, differential equations, probability theory and analytic number theory. We are now interested in continuing our research into the application of ‘twisted’ and ‘q-analogue’ versions of certain interesting special polynomials and numbers in the fields of physics, science, and engineering as well as mathematics.

**Funding:** The work of D. Lim was partially supported by the National Research Foundation of Korea (NRF) grant funded by the Korean government (MSIT) NRF-2021R1C1C1010902.

**Data Availability Statement:** Not applicable.

**Acknowledgments:** The author would like to thank the referees for their comments and suggestions which improved the original manuscript in its present form.

**Conflicts of Interest:** The author declares no conflict of interest.

**References**

1. Kim, T. On a q-analogue of the p-adic log gamma functions and related integrals. *J. Number Theory* **1999**, *76*, 320–329. [CrossRef]
2. Kim, T. q-Volkenborn integration. *Russ. J. Math. Phys.* **2002**, *9*, 288–299.
3. Araci, S.; Acikgoz, M.; Kilicman A. Extended p-adic q-invariant integral on Z_p associated with applications of umbral calculus. *Adv. Differ. Equ.* **2013**, *2013*, 96. [CrossRef]
4. Park, J.-W. On the q-Daehee polynomials with q-parameter. *J. Comput. Anal. Appl.* **2016**, *20*, 11–20.
5. Lim, D. Some explicit expressions for twisted Catalan-Daehee numbers. *Symmetry* **2022**, in press.
6. Dolgy, D.V.; Jang, G.-W.; Kim, D.S.; Kim, T. Explicit expressions for Catalan-Daehee numbers. *Proc. Jangjeon Math. Soc.* **2017**, *20*, 1–9.
7. Kim, T.; Kim, D.S. Differential equations associated with Catalan-Daehee numbers and their applications. *Rev. Real Acad. Cienc. Exactas Fisicas Nat. Ser. A Mat.* **2016**, *111*, 1071–1081. [CrossRef]
8. Kim, T.; Kim, D.S. Some identities of Catalan-Daehee numbers arising from umbral calculus. *Appl. Comput. Math.* **2017**, *16*, 177–189.
9. Ma, Y.; Kim, T.; Kim, D.S.; Lee, H. A study on q-analogues of Catalan-Daehee numbers and polynomials. *arXiv* **2021**, arXiv:2105.12013v1.
10. Kim, D.S.; Kim, T. Catalan-Daehee numbers and polynomials. *Appl. Math. Sci.* **2013**, *7*, 5969–5976. [CrossRef]
11. Kim, D.S.; Kim, T. A new approach to Catalan numbers using differential equations. *Russ. J. Math. Phys.* **2017**, *24*, 465–475. [CrossRef]
12. Kim, D.S.; Kim, T. Triple symmetric identities for w-Catalan polynomials. *J. Korean Math. Soc.* **2017**, *54*, 1243–1264.
13. Kim, T. An analogue of Bernoulli numbers and their applications. *Rep. Fac. Sci. Engrg. Saga Univ. Math.* **1994**, *22*, 21–26.
14. Kim, T. A note on Catalan numbers associated with p-adic integral on Z_p. *Proc. Jangjeon Math. Soc.* **2016**, *19*, 493–501.
15. Kim, T.; Kim, D.S.; Seo, J.-J. Symmetric identities for an analogue of Catalan polynomials. *Proc. Jangjeon Math. Soc.* **2016**, *19*, 515–521.
16. Kim, T.; Kim, D.S.; Seo, J.-J.; Kwon, H.-I. Differential equations associated with λ-Changhee polynomials. *J. Nonlinear Sci. Appl.* **2016**, *9*, 3098–3111. [CrossRef]
17. Ozden, H.; Cangul, I.N.; Simsek, Y. Remarks on q-Bernoulli numbers associated with Daehee numbers. *Adv. Stud. Contemp. Math. (Kyungshang)* **2009**, *19*, 41–48.
18. Sharma, S.K.; Khan, W.A.; Araci, S.; Ahmed, S.S. New type of degenerate Daehee polynomials of the second kind. *Adv. Differ. Equ.* **2020**, *2020*, 1–14. [CrossRef]
19. Simsek, Y. Analysis of the p-adic q-Volkenborn integrals: An approach to generalized Apostol-type special numbers and polynomials and their applications. *Cogent Math.* **2016**, *3*, 1269393. [CrossRef]
20. Simsek, Y. Apostol type Daehee numbers and polynomials. *Adv. Stud. Contemp. Math.* **2016**, *26*, 555–566.