NONZERO RADIAL SOLUTIONS FOR A CLASS OF ELLIPTIC SYSTEMS WITH NONLOCAL BCS ON ANNULAR DOMAINS

GENNARO INFANTE AND PAOLAMARIA PIETRAMALA

ABSTRACT. We provide new results on the existence, non-existence, localization and multiplicity of nontrivial solutions for systems of Hammerstein integral equations. Some of the criteria involve a comparison with the spectral radii of some associated linear operators. We apply our results to prove the existence of multiple nonzero radial solutions for some systems of elliptic boundary value problems subject to nonlocal boundary conditions. Our approach is topological and relies on the classical fixed point index. We present an example to illustrate our theory.

1. Introduction

In the interesting paper [14], Do Ó, Lorca and Ubilla, motivated by the work of Lee [37] and by their previous paper [13], considered the existence of three positive solutions for the semilinear elliptic system

\begin{align*}
\Delta u + \tilde{f}_1(|x|, u, v) &= 0, \quad |x| \in [R_1, R_0], \\
\Delta v + \tilde{f}_2(|x|, u, v) &= 0, \quad |x| \in [R_1, R_0],
\end{align*}

subject to the non-homogenous boundary conditions (BCs)

\begin{align*}
u|_{\partial B_{R_1}} &= 0 \quad \text{and} \quad u|_{\partial B_{R_0}} = A_1, \\
v|_{\partial B_{R_1}} &= 0 \quad \text{and} \quad v|_{\partial B_{R_0}} = A_2,
\end{align*}

where \( x \in \mathbb{R}^n \), \( 0 < R_1 < R_0 < \infty \), \( A_1, A_2 > 0 \) and \( B_\rho = \{ x \in \mathbb{R}^n : |x| < \rho \} \). The methodology used in [14] is to seek radial solutions of the system (1.1)-(1.2), by means of an auxiliary system of Hammerstein integral equations

\begin{align*}
u(t) &= \int_0^1 k(t, s)\tilde{f}_1(s, u(s), v(s), A_1, A_2) \, ds, \\
v(t) &= \int_0^1 k(t, s)\tilde{f}_2(s, u(s), v(s), A_1, A_2) \, ds,
\end{align*}

2010 Mathematics Subject Classification. Primary 45G15, secondary 34B10, 35B07, 35J57, 47H30.

Key words and phrases. Elliptic system, annular domain, radial solution, multiplicity, non-existence, spectral radius, cone, nontrivial solution, nonlocal boundary conditions, fixed point index.

Partially supported by G.N.A.M.P.A. - INdAM (Italy).
where

\[ k(t, s) = \begin{cases} 
  s(1 - t), & s \leq t, \\
  t(1 - s), & s > t.
\end{cases} \]

The integral equations in (1.3) share the same non-negative kernel and the non-homogeneous terms that occur in (1.2) are incorporated in the nonlinearities \( \hat{f}_1, \hat{f}_2 \) (a similar idea has been fruitfully employed in [15] also in the context of exterior domains). The existence of positive solutions of (1.3) is obtained via the well-known Krasnosel’ski–Guo Theorem on cone compressions and cone expansions (see [19]). The Krasnosel’ski–Guo Theorem and, more in general, topological methods have been used to study the existence of positive solutions for elliptic equations subject to homogeneous BCs on annular domains, see for example the papers by Dunninger and Wang [11, 12], Lan and Lin [35], Lan and Webb [36], Ma [38], Wang [49] and references therein.

The study of nonlocal BCs, in the framework of ODEs, has been initiated by 1908 by Picone [41], who considered multi-point BCs. This topic has been developed by a large number of authors. The motivation for this type of study is driven also by the fact that nonlocal problems occur when modelling several phenomena in engineering, physics and life sciences. For an introduction to nonlocal problems we refer to the reviews by Whyburn [59], Conti [10], Ma [39], Ntouyas [40] and Štikonas [48].

Nonlocal BCs have been studied also in the context of elliptic problems, we mention here the papers by Amster and Maurette [3], Beals [4], Bitsadze and Samarskiï [5], Browder [6], Schechter [45], Skubachevskiï [46, 47], Wang [50], Ye and Ke [63]. In [51] Webb considered the existence of positive radial solutions for the boundary value problem (BVP)

\[
\Delta u + h(|x|)f(u) = 0, \quad |x| \in [R_1, R_0],
\]

\[
u|_{\partial B_{R_0}} = 0 \quad \text{and} \quad (u(R_1\cdot) - \alpha u(R_\eta\cdot))|_{\partial B_1} = 0,
\]

where \( \alpha > 0 \) and \( R_\eta \in (R_1, R_0) \).

Here we develop a theory for the existence of nonzero solutions of systems of Hammerstein integral equations of the type

\[
u(t) = \int_0^1 k_2(t, s)g_2(s)(f_1(s, u(s), v(s))ds,
\]

that is well-suited to prove the existence of nontrivial radial solutions for a class of elliptic systems subject to nonlocal BCs, similar to the ones that occur in (1.4). With this approach the kernels, allowed to change sign, take into account the nonlocalities in the BCs.

The existence of positive solutions of systems of integral equations of the type (1.5) has been widely studied, see for example [1, 8, 9, 11, 12, 17, 18, 20, 21, 33, 34, 35, 30, 61, 62] and references therein. Nonzero solutions of systems of Hammerstein integral equations were
considered in [16]; here we improve the results of [16] in several directions: we allow different
growths in the nonlinearities, discuss non-existence results and provide some criteria that
involve the spectral radii of some suitable associated linear operators.

We illustrate our theory in the special case of a system of nonlinear elliptic BVPs with non-local BCs, that generates two different kernels in the associated system of integral equations, namely

\[ \Delta u + h_1(|x|)f_1(u, v) = 0, \ |x| \in [R_1, R_0], \]
\[ \Delta v + h_2(|x|)f_2(u, v) = 0, \ |x| \in [R_1, R_0], \]
\[ \frac{\partial u}{\partial r} \big|_{\partial B_{R_0}} = 0 \text{ and } (u(R_1 \cdot) - \alpha_1 u(R_0 \cdot))|_{\partial B_1} = 0, \]
\[ \frac{\partial v}{\partial r} \big|_{\partial B_{R_0}} = 0 \text{ and } (v(R_1 \cdot) - \alpha_2 \frac{\partial v}{\partial r}(R_\xi \cdot))|_{\partial B_1} = 0, \]

where \( x \in \mathbb{R}^n, \alpha_1, \alpha_2 \in \mathbb{R}, \) \( 0 < R_1 < R_0 < \infty, \) \( R_\eta, R_\xi \in (R_1, R_0) \) and \( \frac{\partial}{\partial r} \) denotes differentiation in the radial direction \( r = |x|. \)

Here we focus the attention on the existence of solutions that are allowed to change sign, in the spirit of the earlier works [28, 29]. The approach that we use is topological, relies on classical fixed point index theory and we make use of ideas from the papers [16, 26, 27, 29, 35, 36, 51, 55, 58]. In the last Section we present an example that illustrates the applicability of our results.

2. The System of Integral Equations

We begin by stating some assumptions on the terms that occur in the system of Hammerstein integral equations

\[ u(t) = \int_0^1 k_1(t, s)g_1(s)f_1(s, u(s), v(s)) \, ds, \]
\[ v(t) = \int_0^1 k_2(t, s)g_2(s)f_2(s, u(s), v(s)) \, ds, \]

(2.1)

namely:

- For every \( i = 1, 2, \) \( f_i : [0, 1] \times (-\infty, \infty) \times (-\infty, \infty) \to [0, \infty) \) satisfies Carathéodory conditions, that is, \( f_i(\cdot, u, v) \) is measurable for each fixed \( (u, v) \) and \( f_i(t, \cdot, \cdot) \) is continuous for almost every (a.e.) \( t \in [0, 1], \) and for each \( r > 0 \) there exists \( \phi_{i,r} \in L^\infty[0, 1] \) such that

\[ f_i(t, u, v) \leq \phi_{i,r}(t) \text{ for } u, v \in [-r, r] \text{ and a.e. } t \in [0, 1]. \]

- For every \( i = 1, 2, \) \( k_i : [0, 1] \times [0, 1] \to (-\infty, \infty) \) is measurable, and for every \( \tau \in [0, 1] \) we have

\[ \lim_{t \to \tau} |k_i(t, s) - k_i(\tau, s)| = 0 \text{ for a.e. } s \in [0, 1]. \]
For every \( i = 1, 2 \), there exist a subinterval \([a_i, b_i] \subseteq [0, 1] \), a function \( \Phi_i \in L^\infty[0, 1] \), and a constant \( c_i \in (0, 1] \), such that
\[
|k_i(t, s)| \leq \Phi_i(s) \text{ for } t \in [0, 1] \text{ and a.e. } s \in [0, 1],
\]
\[
k_i(t, s) \geq c_i \Phi_i(s) \text{ for } t \in [a_i, b_i] \text{ and a.e. } s \in [0, 1].
\]

For every \( i = 1, 2 \), \( g_i \Phi_i \in L^1[0, 1] \), \( g_i \geq 0 \text{ a.e.} \), and \( \int_0^1 \Phi_i(s)g_i(s) \, ds > 0 \).

We work in the space \( C[0, 1] \times C[0, 1] \) endowed with the norm
\[
\|(u, v)\| := \max\{\|u\|_{\infty}, \|v\|_{\infty}\},
\]
where \( \|w\|_{\infty} := \max\{|w(t)|, t \in [0, 1]\} \).

We recall that a cone \( K \) in a Banach space \( X \) is a closed convex set such that \( \lambda x \in K \) for \( x \in K \) and \( \lambda \geq 0 \) and \( K \cap (-K) = \{0\} \). Take
\[
\tilde{K}_i := \{w \in C[0, 1] : \min_{t \in [a_i, b_i]} w(t) \geq c_i \|w\|_{\infty}\},
\]
and consider the cone \( K \) in \( C[0, 1] \times C[0, 1] \) defined by
\[
K := \{(u, v) \in \tilde{K}_1 \times \tilde{K}_2\}.
\]

For a nontrivial solution of the system (2.1) we mean a solution \((u, v) \in K \) of (2.1) such that \( \|(u, v)\| \neq 0 \). Note that the functions in \( \tilde{K}_i \) are positive on the sub-interval \([a_i, b_i] \) but are allowed to change sign in \([0, 1] \). This type of cone has been introduced by Infante and Webb in [29] and is similar to a cone of non-negative functions first used by Krasnosel’skii, see e.g. [31], and D. Guo, see e.g. [19].

Under our assumptions, we show that the integral operator
\[
(2.2) \quad T(u, v)(t) := \left( \begin{array}{c}
\int_0^1 k_1(t, s)g_1(s)f_1(s, u(s), v(s)) \, ds \\
\int_0^1 k_2(t, s)g_2(s)f_2(s, u(s), v(s)) \, ds
\end{array} \right) := \left( \begin{array}{c}
T_1(u, v)(t) \\
T_2(u, v)(t)
\end{array} \right),
\]
leaves the cone \( K \) invariant and is compact.

**Lemma 2.1.** The operator (2.2) maps \( K \) into \( K \) and is compact.

**Proof.** Take \((u, v) \in K \) such that \( \|(u, v)\| \leq r \). Then we have, for \( t \in [0, 1] \),
\[
|T_1(u, v)(t)| \leq \int_0^1 \Phi_1(s)g_1(s)f_1(s, u(s), v(s)) \, ds
\]
and therefore
\[
\|T_1(u, v)\|_{\infty} \leq \int_0^1 \Phi_1(s)g_1(s)f_1(s, u(s), v(s)) \, ds.
\]
Then we obtain
\[
\min_{t \in [a_1, b_1]} T_1(u, v)(t) \geq c_1 \int_0^1 \Phi_1(s)g_1(s)f_1(s, u(s), v(s)) \, ds
\]
\[
\geq c_1 \|T_1(u, v)\|_{\infty}.
\]
Hence we have $T_1(u, v) \in K_1$. In a similar manner we proceed for $T_2(u, v)$.
Moreover, the map $T$ is compact since, by routine arguments, the components $T_i$ are compact maps.

The next Lemma summarizes some classical results regarding the fixed point index, for more details see [2, 19]. If $\Omega$ is a open bounded subset of a cone $K$ (in the relative topology) we denote by $\overline{\Omega}$ and $\partial\Omega$ the closure and the boundary relative to $K$. When $\Omega$ is an open bounded subset of $X$ we write $\overline{\Omega}_K = \Omega \cap K$, an open subset of $K$.

**Lemma 2.2.** Let $\Omega$ be an open bounded set with $0 \in \Omega_K$ and $\overline{\Omega}_K \neq K$. Assume that $F : \overline{\Omega}_K \to K$ is a compact map such that $x \neq Fx$ for all $x \in \partial\Omega_K$. Then the fixed point index $i_K(F, \Omega_K)$ has the following properties.

1. If there exists $e \in K \setminus \{0\}$ such that $x \neq Fx + \lambda e$ for all $x \in \partial\Omega_K$ and all $\lambda > 0$, then $i_K(F, \Omega_K) = 0$.
2. If $\mu \varepsilon x \neq Fx$ for all $x \in \partial\Omega_K$ and for every $\mu \geq 1$, then $i_K(F, \Omega_K) = 1$.
3. If $i_K(F, \Omega_K) \neq 0$, then $F$ has a fixed point in $\Omega_K$.
4. Let $\Omega^1$ be open in $X$ with $\overline{\Omega^1} \subset \Omega_K$. If $i_K(F, \Omega_K) = 1$ and $i_K(F, \Omega^1_K) = 0$, then $F$ has a fixed point in $\Omega_K \setminus \overline{\Omega^1}_K$. The same result holds if $i_K(F, \Omega_K) = 0$ and $i_K(F, \Omega^1_K) = 1$.

We use the following (relative) open bounded sets in $K$:

$$K_{\rho_1, \rho_2} = \{(u, v) \in K : \|u\|_\infty < \rho_1 \text{ and } \|v\|_\infty < \rho_2\},$$

and

$$V_{\rho_1, \rho_2} = \{(u, v) \in K : \min_{t \in [a_1, b_1]} u(t) < \rho_1 \text{ and } \min_{t \in [a_2, b_2]} v(t) < \rho_2\}.$$

If $\rho_1 = \rho_2 = \rho$ we write simply $K_\rho$ and $V_\rho$. The set $V_\rho$ (in the context of systems) was introduced by the authors in [23] and is equal to the set called $\Omega^{\rho/c}$ in [16]. $\Omega^{\rho/c}$ is an extension to the case of systems of a set given by Lan [33].

For our index calculations we make use of the following Lemma, similar to Lemma 5 of [16]. The novelty here is the use of different radii, in the spirit of the paper [9]. This choice allows more freedom in the growth of the nonlinearities. The proof of the Lemma is similar to the corresponding one in [16] and is omitted.

**Lemma 2.3.** The sets defined above have the following properties:

- $K_{\rho_1, \rho_2} \subset V_{\rho_1, \rho_2} \subset K_{\rho_1/c_1 \rho_2/c_2}$.
- $(w_1, w_2) \in \partial V_{\rho_1, \rho_2}$ iff $(w_1, w_2) \in K$ and $\min_{t \in [a_i, b_i]} w_i(t) = \rho_i$ for some $i \in \{1, 2\}$ and $\min_{t \in [a_j, b_j]} w_j(t) \leq \rho_j$ for $j \neq i$.
- If $(w_1, w_2) \in \partial V_{\rho_1, \rho_2}$, then for some $i \in \{1, 2\}$ $\rho_i \leq w_i(t) \leq \rho_i/c_i$ for each $t \in [a_i, b_i]$ and for $j \neq i$ we have $0 \leq w_j(t) \leq \rho_j/c_j$ for each $t \in [a_j, b_j]$ and $\|w_j\|_\infty \leq \rho_j/c_j$. 

3. Existence results

We are now able to prove a result concerning the fixed point index on the set $K_{\rho_1, \rho_2}$.

**Lemma 3.1.** Assume that $(I_{\rho_1, \rho_2})$ there exist $\rho_1, \rho_2 > 0$ such that for every $i = 1, 2$

\[ f^{\rho_1, \rho_2}_i < m_i \]

where

\[ f^{\rho_1, \rho_2}_i = \sup \left\{ \frac{f_i(t, u, v)}{\rho_i} : (t, u, v) \in [0, 1] \times [-\rho_1, \rho_1] \times [-\rho_2, \rho_2] \right\} \]

and

\[ \frac{1}{m_i} = \sup_{t \in [0, 1]} \int_0^1 |k_i(t, s)|g_i(s) \, ds. \]

Then $i_K(T, K_{\rho_1, \rho_2}) = 1$.

**Proof.** We show that $\lambda(u, v) \neq T(u, v)$ for every $(u, v) \in \partial K_{\rho_1, \rho_2}$ and for every $\lambda \geq 1$; this ensures that the index is 1 on $K_{\rho_1, \rho_2}$. In fact, if this does not happen, there exist $\lambda \geq 1$ and $(u, v) \in \partial K_{\rho_1, \rho_2}$ such that $\lambda(u, v) = T(u, v)$. Assume, without loss of generality, that $\|u\|_\infty = \rho_1$ and $\|v\|_\infty \leq \rho_2$. Then

\[ \lambda u(t) = \int_0^1 k_1(t, s)g_1(s)f_1(s, u(s), v(s)) \, ds. \]

Taking the absolute value we have

\[ \lambda |u(t)| = \left| \int_0^1 k_1(t, s)g_1(s)f_1(s, u(s), v(s)) \, ds \right|, \]

and then the supremum over $[0, 1]$ gives

\[ \lambda \rho_1 \leq \sup_{t \in [0, 1]} \int_0^1 |k_1(t, s)|g_1(s)f_1(s, u(s), v(s)) \, ds \]

\[ \leq \rho_1 f_1^{\rho_1, \rho_2} \sup_{t \in [0, 1]} \int_0^1 |k_1(t, s)|g_1(s) \, ds = \rho_1 f_1^{\rho_1, \rho_2} \frac{1}{m_1}. \]

Using the hypothesis (3.1) we obtain $\lambda \rho_1 < \rho_1$. This contradicts the fact that $\lambda \geq 1$ and proves the result. \qed

**Remark 3.2.** Take $\omega \in L^1([0, 1] \times [0, 1])$ and denote by

\[ \omega^+(t, s) = \max\{\omega(t, s), 0\}, \quad \omega^-(s) = \max\{-\omega(t, s), 0\}. \]

Then we have

\[ \left| \int_0^1 \omega(t, s) \, ds \right| \leq \max\left\{ \int_0^1 \omega^+(t, s) \, ds, \int_0^1 \omega^-(t, s) \, ds \right\} \leq \int_0^1 |\omega(t, s)| \, ds, \]

since $\omega = \omega^+ - \omega^-$ and $|\omega| = \omega^+ + \omega^-$. 6
Using the inequality above, it is possible to relax the growth assumptions on the nonlinearities $f_i$. This is done by replacing the quantity $\frac{1}{m_i}$ with
\[
\sup_{t \in [0,1]} \left\{ \max \left\{ \int_0^1 k_i^+(t, s)g_i(s) \, ds, \int_0^1 k_i^-(t, s)g_i(s) \, ds \right\} \right\};
\]
this idea has been used, in the case of one equation, in [27].

We give a first Lemma that shows that the index is 0 on a set $V_{\rho_1, \rho_2}$.

**Lemma 3.3.** Assume that

\[(I_{\rho_1, \rho_2}) \text{ there exist } \rho_1, \rho_2 > 0 \text{ such that for every } i = 1, 2\]
\[
(3.2) \quad f_{i, (\rho_1, \rho_2)} > M_i,
\]

where
\[
f_{1, (\rho_1, \rho_2)} = \inf \left\{ \frac{f_1(t, u, v)}{\rho_1} : (t, u, v) \in [a_1, b_1] \times [\rho_1, \rho_1/c_1] \times [-\rho_2/c_2, \rho_2/c_2] \right\},
\]
\[
f_{2, (\rho_1, \rho_2)} = \inf \left\{ \frac{f_2(t, u, v)}{\rho_2} : (t, u, v) \in [a_2, b_2] \times [-\rho_1/c_1, \rho_1/c_1] \times [\rho_2, \rho_2/c_2] \right\},
\]
\[
\frac{1}{M_i} = \inf_{t \in [a_i, b_i]} \int_{a_i}^{b_i} k_i(t, s)g_i(s) \, ds.
\]

Then $i_K(T, V_{\rho_1, \rho_2}) = 0$.

**Proof.** Let $e(t) \equiv 1$ for $t \in [0, 1]$. Then $(e, e) \in K$. We prove that

\[(u, v) \neq T(u, v) + \lambda (e, e) \text{ for } (u, v) \in \partial V_{\rho_1, \rho_2} \text{ and } \lambda \geq 0.
\]

In fact, if this does not happen, there exist $(u, v) \in \partial V_{\rho_1, \rho_2}$ and $\lambda \geq 0$ such that $(u, v) = T(u, v) + \lambda (e, e)$. Without loss of generality, we can assume that for all $t \in [a_1, b_1]$ we have

\[\rho_1 \leq u(t) \leq \rho_1/c_1, \quad \min u(t) = \rho_1 \text{ and } -\rho_2/c_2 \leq v(t) \leq \rho_2/c_2.
\]

Then, for $t \in [a_1, b_1]$, we obtain

\[u(t) = \int_0^1 k_1(t, s)g_1(s)f_1(s, u(s), v(s)) \, ds + \lambda e(t),\]

and therefore

\[u(t) \geq \int_{a_1}^{b_1} k_1(t, s)g_1(s)f_1(s, u(s), v(s)) \, ds + \lambda.
\]

Taking the minimum over $[a_1, b_1]$ gives

\[\rho_1 = \min_{t \in [a_1, b_1]} u(t) \geq \rho_1 f_{1, (\rho_1, \rho_2)} \frac{1}{M_1} + \lambda.
\]

Using the hypothesis (3.2) we obtain $\rho_1 > \rho_1 + \lambda$, a contradiction. \qed
In the following Lemma we exploit an idea that was used in [26] and we provide a result of index 0 on \( V_{\rho_1, \rho_2} \) of a different flavour; here we control the growth of just one nonlinearity \( f_i \), at the cost of having to deal with a larger domain. Nonlinearities with different growths were considered, with different approaches, in \([8, 33, 44, 60]\).

**Lemma 3.4.** Assume that
\[(I_{\rho_1, \rho_2})^* \text{ there exist } \rho_1, \rho_2 > 0 \text{ such that for some } i \in \{1, 2\} \text{ we have}
\begin{align*}
(3.3) \quad f_{i, (\rho_1, \rho_2)}^* > M_i,
\end{align*}
where
\[
\begin{align*}
f_{1, (\rho_1, \rho_2)}^* &= \inf \left\{ \frac{f_1(t, u, v)}{\rho_1} : (t, u, v) \in [a_1, b_1] \times [0, \rho_1/c_1] \times [-\rho_2/c_2, \rho_2/c_2] \right\}, \\
f_{2, (\rho_1, \rho_2)}^* &= \inf \left\{ \frac{f_2(t, u, v)}{\rho_2} : (t, u, v) \in [a_2, b_2] \times [-\rho_1/c_1, \rho_1/c_1] \times [0, \rho_2/c_2] \right\}.
\end{align*}
\]
Then \( i_K(T, V_{\rho_1, \rho_2}) = 0 \).

**Proof.** Suppose that the condition \((3.3)\) holds for \( i = 1 \). Let \( (u, v) \in \partial V_{\rho_1, \rho_2} \) and \( \lambda \geq 0 \) such that \( (u, v) = T(u, v) + \lambda(e, e) \). So for all \( t \in [a_1, b_1] \) we have \( \min u(t) \leq \rho_1, 0 \leq u(t) \leq \rho_1/c_1 \) and \(-\rho_2/c_2 \leq v(t) \leq \rho_2/c_2 \) and for \( t \in [a_2, b_2], \min v(t) \leq \rho_2 \). For \( t \in [a_1, b_1], \) as in the proof of Lemma \(3.3\) we have
\[
\begin{align*}
u(t) &\geq \int_{a_1}^{b_1} k_1(t, s) g_1(s) f_1(s, u(s), v(s)) \, ds + \lambda.
\end{align*}
\]
Taking the minimum over \([a_1, b_1]\) gives
\[
\min_{t \in [a_1, b_1]} u(t) \geq \rho_1 f_{1, (\rho_1, \rho_2)}^* \frac{1}{M_1} + \lambda.
\]
Using the hypothesis \((3.3)\) we obtain \( \rho_1 > \rho_1 + \lambda \), a contradiction. \(\square\)

We now state a result regarding the existence of at least one, two or three nontrivial solutions. The proof follows by the properties of fixed point index and is omitted. Note that, by expanding the lists in conditions \((S_5), (S_6)\), it is possible to state results for four or more nontrivial solutions, see for example the paper \([32]\).

**Theorem 3.5.** The system \((2.1)\) has at least one nontrivial solution in \( K \) if one of the following conditions holds.

\[(S_1) \text{ For } i = 1, 2 \text{ there exist } \rho_i, r_i \in (0, \infty) \text{ with } \rho_i/c_i < r_i \text{ such that } (I_{\rho_1, \rho_2}^0) \text{ or } (I_{\rho_1, \rho_2})^*, \]
\[
(I_{r_1, r_2}^0) \text{ hold.}
\]

\[(S_2) \text{ For } i = 1, 2 \text{ there exist } \rho_i, r_i \in (0, \infty) \text{ with } \rho_i < r_i \text{ such that } (I_{\rho_1, \rho_2}^1), (I_{r_1, r_2}^0) \text{ hold.}
\]

The system \((2.1)\) has at least two nontrivial solutions in \( K \) if one of the following conditions holds.
For $i = 1, 2$ there exist $\rho_i, r_i, s_i \in (0, \infty)$ with $\rho_i/c_i < r_i < s_i$ such that $(I_{p_1,p_2}^0)$, 
$[\text{or } (I_{p_1,p_2}^0)^*]$, $(I_{s_1,s_2}^1)$ and $(I_{s_1,s_2}^0)$ hold.

(S4) For $i = 1, 2$ there exist $\rho_i, r_i, s_i \in (0, \infty)$ with $\rho_i < r_i$ and $r_i/c_i < s_i$ such that $(I_{p_1,p_2}^1)$, $(I_{s_1,s_2}^0)$ and $(I_{s_1,s_2}^1)$ hold.

The system (2.1) has at least three nontrivial solutions in $K$ if one of the following conditions holds.

(S5) For $i = 1, 2$ there exist $\rho_i, r_i, s_i, \sigma_i \in (0, \infty)$ with $\rho_i/c_i < r_i < s_i$ and $s_i/c_i < \sigma_i$ such that $(I_{p_1,p_2}^0)$, $(I_{p_1,p_2}^0)^*$, $(I_{s_1,s_2}^0)$, $(I_{s_1,s_2}^0)$ and $(I_{s_1,s_2}^1)$ hold.

(S6) For $i = 1, 2$ there exist $\rho_i, r_i, s_i, \sigma_i \in (0, \infty)$ with $\rho_i < r_i$ and $r_i/c_i < s_i < \sigma_i$ such that $(I_{p_1,p_2}^1)$, $(I_{s_1,s_2}^0)$, $(I_{s_1,s_2}^1)$ and $(I_{s_1,s_2}^1)$ hold.

In the case of $[a_1, b_1] = [a_2, b_2]$ we can relax the assumptions on the nonlinearities $f_i$. In the following two Lemmas we provide a modification of the conditions $(I_{p_1,p_2}^0)$ and $(I_{p_1,p_2}^0)^*$, similar to the one in [16]. An analogous of the Theorem 3.5 holds in this case, we omit the statement of this result.

**Lemma 3.6.** Assume that $[a_1, b_1] = [a_2, b_2] =: [a, b]$ and that $(I_{p_1,p_2}^0)$ there exist $\rho_1, \rho_2 > 0$ such that for every $i = 1, 2$

$$f_{i,(\rho_1,\rho_2)} > M_i,$$

where

$$f_{1,(\rho_1,\rho_2)} = \inf \left\{ \frac{f_1(t,u,v)}{\rho_1} : (t,u,v) \in [a,b] \times [\rho_1, \rho_1/c_1] \times [0, \rho_2/c_2] \right\},$$

$$f_{2,(\rho_1,\rho_2)} = \inf \left\{ \frac{f_2(t,u,v)}{\rho_2} : (t,u,v) \in [a,b] \times [0, \rho_1/c_1] \times [\rho_2, \rho_2/c_2] \right\}.$$

Then $i_K(T, V_{\rho_1,\rho_2}) = 0$.

**Proof.** As in the proof of Lemma 3.3 suppose that there exist $(u,v) \in \partial V_{\rho_1,\rho_2}$ and $\lambda \geq 0$ such that $(u,v) = T(u,v) + \lambda(e,e)$. Without loss of generality, we can assume that for all $t \in [a, b]$ we have

$$\rho_1 \leq u(t) \leq \rho_1/c_1, \quad \min u(t) = \rho_1 \text{ and } 0 \leq v(t) \leq \rho_2/c_2.$$

Then, for $t \in [a, b]$, we obtain

$$u(t) \geq \int_a^b k_1(t,s)g_1(s)f_1(s, u(s), v(s)) \, ds + \lambda.$$

Taking the minimum over $[a, b]$ gives

$$\rho_1 = \min_{t \in [a, b]} u(t) \geq \rho_1 f_{1,(\rho_1,\rho_2)} \frac{1}{M_1} + \lambda.$$

Using the hypothesis (3.3) we obtain $\rho_1 > \rho_1 + \lambda$, a contradiction. \(\square\)
Lemma 3.7. Assume that \([a_1, b_1] = [a_2, b_2] =: [a, b]\) and that 
\((P^0_{\rho_1, \rho_2})^*\) there exist \(\rho_1, \rho_2 > 0\) such that for some \(i \in \{1, 2\}\) we have
\[
(3.5) \quad f_{i, (\rho_1, \rho_2)} > M_i,
\]
where
\[
f_{i, (\rho_1, \rho_2)} = \inf \left\{ \frac{f_i(t, u, v)}{\rho_i} : (t, u, v) \in [a, b] \times [0, \rho_1/c_1] \times [0, \rho_2/c_2] \right\}.
\]
Then \(i_K(T, V_{\rho_1, \rho_2}) = 0\).

Proof. Suppose that the condition \((3.5)\) holds for \(i = 1\). Let \((u, v) \in \partial V_{\rho_1, \rho_2}\) and \(\lambda \geq 0\) such that \((u, v) = T(u, v) + \lambda(e, e)\). So for all \(t \in [a, b]\) we have \(\min u(t) \leq \rho_1, 0 \leq u(t) \leq \rho_1/c_1, 0 \leq v(t) \leq \rho_2/c_2\) and \(\min v(t) \leq \rho_2\). Now, the proof follows as the one of Lemma 3.4. \(\square\)

4. Non-existence results

We now show a non-existence result for problem \((2.1)\).

Theorem 4.1. Assume that one of the following conditions holds.

1. For \(i = 1, 2\),
\[
(4.1) \quad f_i(t, u_1, u_2) < m_i |u_i| \quad \text{for every} \quad t \in [0, 1] \quad \text{and} \quad u_i \neq 0.
\]

2. For \(i = 1, 2\),
\[
(4.2) \quad f_i(t, u_1, u_2) > M_i u_i \quad \text{for every} \quad t \in [a_i, b_i] \quad \text{and} \quad u_i > 0.
\]

3. There exists \(i \in \{1, 2\}\) such that \((4.1)\) is verified for \(f_i\) and for \(j \neq i\) condition \((4.2)\) is verified for \(f_j\).

Then there is no nontrivial solution of the system \((2.1)\) in \(K\).

Proof. (1) Assume, on the contrary, that there exists \((u, v) \in K\) such that \((u, v) = T(u, v)\) and \((u, v) \neq (0, 0)\). Let, for example, be \(\|u\|_\infty \neq 0\). Then, for \(t \in [0, 1]\),
\[
|u(t)| = \left| \int_0^1 k_1(t, s) g_1(s) f_1(s, u(s), v(s)) ds \right|
\leq \int_0^1 |k_1(t, s)| g_1(s) f_1(s, u(s), v(s)) ds
< m_1 \int_0^1 |k_1(t, s)| g_1(s) |u(s)| ds
\leq m_1 \|u\|_\infty \int_0^1 |k_1(t, s)| g_1(s) ds.
\]
Taking the supremum for \(t \in [0, 1]\), we have
\[
\|u\|_\infty < m_1 \|u\|_\infty \sup_{t \in [0, 1]} \int_0^1 |k_1(t, s)| g_1(s) ds = \|u\|_\infty,
\]
a contradiction.

(2) Assume, on the contrary, that there exists \((u, v) \in K\) such that \((u, v) = T(u, v)\) and \((u, v) \neq (0, 0)\). Let, for example, be \(\|u\|_\infty \neq 0\). Then, for \(t \in [a_1, b_1]\)

\[
u(t) = \int_0^1 k_1(t, s)g_1(s)f_1(s, u(s), v(s))ds \geq \int_{a_1}^{b_1} k_1(t, s)g_1(s)f_1(s, u(s), v(s))ds > M_1 \int_{a_1}^{b_1} k_1(t, s)g_1(s)u(s)ds.
\]

Taking the infimum for \(t \in [a_1, b_1]\), we obtain

\[
\min_{t \in [a_1, b_1]} u(t) > M_1 \inf_{t \in [a_1, b_1]} \int_{a_1}^{b_1} k_1(t, s)g_1(s)u(s)ds.
\]

Take \(\sigma_1 = \min_{t \in [a_1, b_1]} u(t)\). Thus we get

\[
\sigma_1 > M_1 \sigma_1 \inf_{t \in [a_1, b_1]} \int_{a_1}^{b_1} k_1(t, s)g_1(s)ds = \sigma_1,
\]
a contradiction.

(3) Assume, on the contrary, that there exists \((u, v) \in K\) such that \((u, v) = T(u, v)\) and \((u, v) \neq (0, 0)\). Let, for example, be \(\|u\|_\infty \neq 0\). Then the function \(f_1\) satisfies either (4.1) or (4.2) and the proof follows as in the previous cases. \(\square\)

5. Eigenvalue criteria for the existence of nontrivial solutions

In order to state our eigenvalue comparison results, we consider, in a similar way as in [27], the following operators on \(C[0, 1] \times C[0, 1]\)

\[
L(u, v)(t) := \begin{pmatrix}
\int_0^1 |k_1(t, s)|g_1(s)u(s)\, ds \\
\int_0^1 |k_2(t, s)|g_2(s)v(s)\, ds
\end{pmatrix} := \begin{pmatrix}
L_1(u)(t) \\
L_2(v)(t)
\end{pmatrix},
\]

and

\[
L^+(u, v)(t) := \begin{pmatrix}
\int_{a_1}^{b_1} k_1^+(t, s)g_1(s)u(s)\, ds \\
\int_{a_2}^{b_2} k_2^+(t, s)g_2(s)v(s)\, ds
\end{pmatrix} := \begin{pmatrix}
L_1^+(u)(t) \\
L_2^+(v)(t)
\end{pmatrix}.
\]

We denote by \(P\) the cone of positive functions, namely

\[
P := \{w \in C[0, 1] : w(t) \geq 0, t \in [0, 1]\}.
\]

**Theorem 5.1.** The operators \(L\) and \(L^+\) are compact and map \(P \times P\) into \((P \times P) \cap K\).

**Proof.** Note that the operators \(L\) and \(L^+\) map \(P \times P\) into \(P \times P\) (because they have a non-negative integral kernel) and are compact. We now show that they map \(P \times P\) into \((P \times P) \cap K\). Firstly, we do this for the operator \(L\).

We observe that for every \(i = 1, 2\) and for \(t \in [0, 1]\)

\[
|k_i(t, s)| \leq \Phi_i(s),
\]

11
and that, for $t \in [a_i, b_i]$, 
\[ |k_i(t, s)| = k_i(t, s) \geq c_i \Phi_i(t). \]
Thus, with a similar proof as the one in Lemma 2.1 we obtain, for $(u, v) \in P \times P$ and $t \in [0, 1]$, $L(u, v) \in K$. A similar proof works for $L^+$, since for every $i = 1, 2$ and $t \in [0, 1]$, we have
\[ |k^+_i(t, s)| \leq |k_i(t, s)| \leq \Phi_i(s), \]
and, for $t \in [a_i, b_i]$,
\[ k^+_i(t, s) = k_i(t, s) \geq c_i \Phi_i(t). \]
\[ \square \]

We recall that $\lambda$ is an eigenvalue of a linear operator $\Gamma$ with corresponding eigenfunction $\varphi$ if $\varphi \neq 0$ and $\lambda \varphi = \Gamma \varphi$. The reciprocals of nonzero eigenvalues are called characteristic values of $\Gamma$. We will denote the spectral radius of $\Gamma$ by $r(\Gamma) := \lim_{n \to \infty} \|\Gamma^n\|^{\frac{1}{n}}$ and its principal characteristic value (the reciprocal of the spectral radius) by $\mu(\Gamma) = 1/r(\Gamma)$.

The following Theorem is analogous to the ones in [56, 58] and is proven by using the facts that the considered operators leave $P \times P$ invariant, that $P \times P$ is reproducing, combined with the well-known Krein-Rutman Theorem.

**Theorem 5.2.** For $i = 1, 2$, the spectral radius of $L_i$ is nonzero and is an eigenvalue of $L_i$ with an eigenfunction in $P$. A similar result holds for $L^+_i$.

**Remark 5.3.** As a consequence of the two previous theorems, we have that the above mentioned eigenfunction is in $P \cap \tilde{K}_i$.

We consider the following operator on $C[a_1, b_1] \times C[a_2, b_2]$:
\[ \tilde{L}^+(u, v)(t) := \begin{pmatrix} \int_{a_1}^{b_1} k^+_1(t, s)g_1(s)u(s) \, ds \\ \int_{a_2}^{b_2} k^+_2(t, s)g_2(s)v(s) \, ds \end{pmatrix} := \begin{pmatrix} \tilde{L}^+_1(u)(t) \\ \tilde{L}^+_2(v)(t) \end{pmatrix}. \]

In the recent papers [54, 55], Webb developed an elegant theory valid for $u_0$-positive linear operators. It turns out that our operators $\tilde{L}^+_i$ fit within this setting and, in particular, satisfy the assumptions of Theorem 3.4 of [55]. We state here a special case of Theorem 3.4 of [55] that can be used for $\tilde{L}^+_i$.

**Theorem 5.4.** Suppose that there exist $w \in C[a_i, b_i] \setminus \{0\}$, $w \geq 0$ and $\lambda > 0$ such that
\[ \lambda w(t) \geq \tilde{L}^+_i w(t), \text{ for } t \in [a_i, b_i]. \]

Then we have $r(\tilde{L}^+_i) \leq \lambda$.

**Theorem 5.5.** Assume that

$(\Gamma^0_{i+})$ there exist $\varepsilon > 0$ and $\rho_0 > 0$ such that one of the following conditions holds:
Assume, on the contrary, that there exist \((u, v)\). We distinguish two cases. Firstly we discuss the case where \(\lambda_i\) implies that, for \(\lambda \in (0, \rho_0)\),

\[
\begin{align*}
&f_1(t, u, v) \geq (\mu(L_i^+) + \varepsilon)u, \quad \text{for } (t, u, v) \in [a_1, b_1] \times [0, \rho_0] \times [-\rho_0, \rho_0]; \\
&f_2(t, u, v) \geq (\mu(L_i^+) + \varepsilon)v, \quad \text{for } (t, u, v) \in [a_2, b_2] \times [-\rho_0, \rho_0] \times [0, \rho_0].
\end{align*}
\]

Then \(i_K(T, K_\rho) = 0\) for each \(\rho \in (0, \rho_0]\).

**Proof.** Let \(\rho \in (0, \rho_0]\). We show that \((u, v) \neq T(u, v) + \lambda(\varphi_1, \varphi_2)\) for all \((u, v)\) in \(\partial K_\rho\) and \(\lambda \geq 0\), where \(\varphi_i \in K_i \cap P\) is the eigenfunction of \(L_i^+\) with \(\|\varphi_i\|_\infty = 1\) corresponding to the eigenvalue \(\lambda(\varphi_i, \varphi_i)\). This implies that \(i_K(T, K_\rho) = 0\).

Assume, on the contrary, that there exist \((u, v) \in \partial K_\rho\) and \(\lambda \geq 0\) such that \((u, v) = T(u, v) + \lambda(\varphi_1, \varphi_2)\).

We distinguish two cases. Firstly we discuss the case \(\lambda > 0\). Suppose that (5.1) holds. This implies that, for \(t \in [a_1, b_1]\), we have

\[
b(t) = \int_0^1 k_1(t, s)g_1(s)f_1(s, u(s), v(s))ds + \lambda \varphi_1(t)
\]

\[
\geq \int_{a_1}^{b_1} k_1^+(t, s)g_1(s)f_1(s, u(s), v(s))ds + \lambda \varphi_1(t)
\]

\[
\geq (\mu(L_1^+) + \varepsilon) \int_{a_1}^{b_1} k_1^+(t, s)g_1(s)u(s)ds + \lambda \varphi_1(t)
\]

\[
> \mu(L_1^+) \int_{a_1}^{b_1} k_1^+(t, s)g_1(s)u(s)ds + \lambda \varphi_1(t)
\]

\[
= \mu(L_1^+) L_1^+ u(t) + \lambda \varphi_1(t).
\]

Moreover, we have \(u(t) \geq \lambda \varphi_1(t)\) and then \(L_1^+ u(t) \geq \lambda \varphi_1(t) \geq \frac{\lambda}{\mu(L_1^+)} \varphi_1(t)\) in such a way that we obtain

\[
u(t) \geq \mu(L_1^+) L_1^+ u(t) + \lambda \varphi_1(t) \geq 2\lambda \varphi_1(t), \quad \text{for } t \in [a_1, b_1].
\]

By iteration, we deduce that, for \(t \in [a_1, b_1]\), we get

\[
u(t) \geq n\lambda \varphi_1(t) \text{ for every } n \in \mathbb{N},
\]

a contradiction because \(\|u\|_\infty \leq \rho\).

Now we consider the case \(\lambda = 0\). We have, for \(t \in [a_1, b_1]\),

\[
u(t) = \int_0^1 k_1(t, s)g_1(s)f_1(s, u(s), v(s))ds
\]

\[
\geq \int_{a_1}^{b_1} k_1^+(t, s)g_1(s)f_1(s, u(s), v(s))ds \geq (\mu(L_1^+) + \varepsilon) L_1^+ u(t).
\]

Since \(L_1^+ \varphi_1(t) = r(L_1^+) \varphi_1(t)\) for \(t \in [0, 1]\), we have, for \(t \in [a_1, b_1]\),

\[
L_1^+ \varphi_1(t) = L_1^+ \varphi_1(t) = r(L_1^+) \varphi_1(t),
\]

\[
\frac{\lambda}{\mu(L_1^+)} \varphi_1(t) = r(L_1^+) \varphi_1(t),
\]

\[
\frac{\lambda}{\mu(L_1^+)} \varphi_1(t) = \frac{\lambda \varphi_1(t)}{\mu(L_1^+) + \varepsilon} L_1^+ u(t).
\]

\[
\Rightarrow \mu(L_1^+) \varphi_1(t) \geq \lambda \varphi_1(t) \Rightarrow \mu(L_1^+) \geq \lambda.
\]
and we obtain \( r(L^+_1) \geq r(L^+_1) \). On the other hand, we have, for \( t \in [a_1, b_1] \),
\[
u(t) \geq (\mu(L^+_1) + \varepsilon)L^+_1 u(t) = (\mu(L^+_1) + \varepsilon)\bar{L}^+_1 u(t).
\]
where \( u(t) > 0 \). Thus, utilizing Theorem 5.5, we have \( r(\bar{L}^+_1) \leq \frac{1}{\mu(L^+_1) + \varepsilon} \) and therefore \( r(L^+_1) \leq \frac{1}{\mu(L^+_1) + \varepsilon} \) and thus \( \mu(L^+_1) + \varepsilon \leq \mu(L^+_1) \), a contradiction.

\( \square \)

**Remark 5.6.** Note that condition \((5.1)\) holds, for example, if
\[
mu(L^+_1) < \liminf_{u \to a^+} \inf_{t \in [a_1, b_1]} \frac{f_1(t, u, v)}{u}, \text{ uniformly w.r.t. } v \in \mathbb{R}.
\]
A similar type of condition has been used in \([8]\).

**Theorem 5.7.** Assume that
\((I^0_\infty)\) there exists \( R_1 > 0 \) such that the following conditions hold:
\[
(5.2) \quad f_1(t, u, v) \geq (\mu(L^+_1) + \varepsilon)u, \text{ for } (t, u, v) \in [a_1, b_1] \times [cR_1, +\infty) \times \mathbb{R};
\]
\[
f_2(t, u, v) \geq (\mu(L^+_2) + \varepsilon)v, \text{ for } (t, u, v) \in [a_2, b_2] \times \mathbb{R} \times [cR_1, +\infty).
\]
Then \( i_K(T, K_R) = 0 \) for each \( R \geq R_1 \).

**Proof.** Let \( R \geq R_1 \). We show that \((u, v) \neq T(u, v) + \lambda(\varphi_1, \varphi_2)\) for all \((u, v)\) in \( \partial K_R \) and \( \lambda \geq 0 \), where \( \varphi_i \in \tilde{K}_1 \cap P \) is the eigenfunction of \( L^+_i \) with \( \|\varphi_i\|_\infty = 1 \) corresponding to the eigenvalue \( 1/\mu(L^+_i) \). This implies that \( i_K(T, K_R) = 0 \).

Assume, on the contrary, that there exist \((u, v)\) in \( \partial K_R \) and \( \lambda \geq 0 \) such that \((u, v) = T(u, v) + \lambda(\varphi_1, \varphi_2)\).

Suppose that \( \|u\|_\infty = R \) and \( \|v\|_\infty \leq R \). We have \( u(t) \geq c\|u\|_\infty = cR \geq cR_1 \) for \( t \in [a_1, b_1] \), thus condition \((5.2)\) holds. Hence, we have \( f(t, u(t), v(t)) \geq (\mu(L^+_1) + \varepsilon)u(t) \) for \( t \in [a_1, b_1] \).

This implies, proceeding as in the proof of Theorem 5.5, for the case \( \lambda > 0 \), that for \( t \in [a_1, b_1] \)
\[
u(t) \geq (\mu(L^+_1) + \varepsilon)L^+_1 u(t) + \lambda \varphi_1(t) \geq 2\lambda \varphi_1(t).
\]

Then \( u(t) \geq n\lambda \varphi_1(t) \) for every \( n \in \mathbb{N} \), a contradiction because \( \|u\|_\infty = R \).

The proof in the case \( \lambda = 0 \) is treated as in the proof of Theorem 5.5. \( \square \)

**Theorem 5.8.** Assume that
\((I^1_0)\) there exist \( \varepsilon > 0 \) and \( \rho_0 > 0 \) such that the following conditions hold:
\[
f_1(t, u, v) \leq (\mu(L_1) - \varepsilon)|u|, \text{ for all } (t, u, v) \in [0, 1] \times [-\rho_0, \rho_0] \times [-\rho_0, \rho_0];
\]
\[
f_2(t, u, v) \leq (\mu(L_2) - \varepsilon)|v|, \text{ for all } (t, u, v) \in [0, 1] \times [-\rho_0, \rho_0] \times [-\rho_0, \rho_0].
\]
Then \( i_K(T, K_\rho) = 1 \) for each \( \rho \in (0, \rho_0] \).
Proof. Let \( \rho \in (0, \rho_0] \). We prove that \( T(u, v) \neq \lambda(u, v) \) for \( (u, v) \in \partial K_\rho \) and \( \lambda \geq 1 \), which implies \( i_K(T, K_\rho) = 1 \). In fact, if we assume otherwise, then there exists \( (u, v) \in \partial K_\rho \) and \( \lambda \geq 1 \) such that \( \lambda(u, v) = T(u, v) \). Therefore,

\[
|u(t)| \leq \lambda|u(t)| = |T_1(u, v)(t)| = \left| \int_0^1 k_1(t, s)g_1(s)f_1(s, u(s), v(s))ds \right|
\]

\[
\leq \int_0^1 |k_1(t, s)|g_1(s)f_1(s, u(s), v(s))ds \leq (\mu(L_1) - \varepsilon) \int_0^1 |k_1(t, s)|g_1(s)|u(s)|ds
\]

\[
= (\mu(L_1) - \varepsilon)L_1|u|(t).
\]

Thus, we have that, for \( t \in [0, 1], \)

\[
|u(t)| \leq (\mu(L_1) - \varepsilon)L_1[(\mu(L_1) - \varepsilon)L_1|u|(t)]
\]

\[
= (\mu(L_1) - \varepsilon)^2L_1^2|u|(t) \leq \cdots \leq (\mu(L_1) - \varepsilon)^nL_1^n|u|(t),
\]

thus, taking the norms, \( 1 \leq (\mu(L_1) - \varepsilon)^n\|L_1^n\| \), and then

\[
1 \leq (\mu(L_1) - \varepsilon)\lim_{n \to \infty} \|L_1^n\|^\frac{1}{n} = \frac{\mu(L_1) - \varepsilon}{\mu(L_1)} < 1,
\]

a contradiction. \( \square \)

**Theorem 5.9.** Assume that

\((1)_{\infty}\) there exist \( \varepsilon > 0 \) and \( R_1 > 0 \) such that the following conditions hold:

\[
f_1(t, u, v) \leq (\mu(L_1) - \varepsilon)|u|, \text{ for } |u| \geq R_1, |v| \geq R_1, \text{ and a.e. } t \in [0, 1];
\]

\[
f_2(t, u, v) \leq (\mu(L_2) - \varepsilon)|v|, \text{ for } |u| \geq R_1, |v| \geq R_1, \text{ and a.e. } t \in [0, 1].
\]

Then there exists \( R_0 > 0 \) such that \( i_K(T, K_R) = 1 \) for each \( R > R_0 \).

*Proof.* Since the functions \( f_i \) satisfy Carathéodory condition, there exists \( \phi_{i, R_1} \in L^\infty[0, 1] \) such that

\[
f_i(t, u, v) \leq \phi_{i, R_1}(t) \text{ for } u, v \in [-R_1, R_1] \text{ and a.e. } t \in [0, 1].
\]

Hence, we have

\[
f_1(t, u, v) \leq (\mu(L_1) - \varepsilon)|u| + \phi_{1, R_1}(t) \text{ for all } u, v \in \mathbb{R} \text{ and a.e. } t \in [0, 1],
\]

and

\[
f_2(t, u, v) \leq (\mu(L_2) - \varepsilon)|v| + \phi_{2, R_1}(t) \text{ for all } u, v \in \mathbb{R} \text{ and a.e. } t \in [0, 1].
\]

Denote by \( \text{Id} \) the identity operator. Since for \( i = 1, 2 \) the operators \((\mu(L_i) - \varepsilon)L_i\) have spectral radius less than one, we have that the operators \((\text{Id} - (\mu(L_i) - \varepsilon)L_i)^{-1}\) exist and are bounded. Moreover, from the Neumann series expression,

\[
(\text{Id} - (\mu(L_i) - \varepsilon)L_i)^{-1} = \sum_{k=0}^{\infty}((\mu(L_i) - \varepsilon)L_i)^k
\]

we obtain that \((\text{Id} - (\mu(L_i) - \varepsilon)L_i)^{-1}\) map \( P \) into \( P \), since the operators \( L_i \) have this property.
Take for $i = 1, 2$

$$C_i := \int_0^1 \Phi_i(s)g_i(s)\phi_{i,R_i}(s)ds,$$

and

$$R_0 := \max\{\|\text{Id} - (\mu(L_i) - \varepsilon)L_i)^{-1}C_i\|_\infty, \ i = 1, 2\} \in \mathbb{R}.$$ 

Now we prove that for each $R > R_0$, $T(u, v) \neq \lambda(u, v)$ for all $(u, v) \in \partial K_R$ and $\lambda \geq 1$, which implies $i_K(T, K_R) = 1$. Otherwise there exist $(u, v) \in \partial K_R$ and $\lambda \geq 1$ such that $\lambda(u, v) = T(u, v)$. Suppose that $\|u\|_\infty = R$ and $\|v\|_\infty \leq R$.

From the inequality (5.3), we have, for $t \in [0, 1]$,

$$|u(t)| \leq \lambda|u(t)| = |T_1(u, v)(t)| = \left|\int_0^1 k_1(t, s)g_1(s)f_1(s, u(s), v(s))ds\right| \leq \int_0^1 |k_1(t, s)|g_1(s)f_1(s, u(s), v(s))ds \leq (\mu(L_1) - \varepsilon)\int_0^1 |k_1(t, s)|g_1(s)|u(s)|ds + C_i = (\mu(L_i) - \varepsilon)L_i|u(t)| + C_1,$$

which implies

$$(\text{Id} - (\mu(L_1) - \varepsilon)L_1)|u(t)| \leq C_1.$$ 

Since $(\text{Id} - (\mu(L_1) - \varepsilon)L)^{-1}$ is non-negative, we have

$$|u(t)| \leq (\text{Id} - (\mu(L_1) - \varepsilon)L_1)^{-1}C_1 \leq R_0.$$

Therefore, we have $\|u\|_\infty \leq R_0 < R$, a contradiction. \qed

The index results in Sections 2 and 5 can be combined in order to establish results on existence of multiple nontrivial solutions for the system (2.1), we refer to [35] for similar statements.

6. AN AUXILIARY SYSTEM OF ODEs

We now present some results regarding the following system of ODEs

$$u''(t) + g_1(t)f_1(t, u(t), v(t)) = 0, \quad \text{a.e. on } [0, 1],$$

$$v''(t) + g_2(t)f_2(t, u(t), v(t)) = 0, \quad \text{a.e. on } [0, 1],$$

with the BCs

$$u'(0) = 0, \quad \alpha_1 u(\eta) = u(1), \quad 0 < \eta < 1, $$

$$v'(0) = 0, \quad v(1) = \alpha_2 v'(\xi), \quad 0 < \xi < 1.$$ 

Here we focus on the case $\alpha_1 < 0, \ 0 < \alpha_2 < 1 - \xi$, that leads to the case of solutions that are positive on some sub-intervals of $[0, 1]$ and are allowed to change sign elsewhere.
To the system (6.1)-(6.2) we associate the system of Hammerstein integral equations

\[
\begin{align*}
    u(t) &= \int_0^1 k_1(t, s)g_1(s)f_1(s, u(s), v(s))\,ds, \\
    v(t) &= \int_0^1 k_2(t, s)g_2(s)f_2(s, u(s), v(s))\,ds,
\end{align*}
\]

where the Green’s functions are given by

\[
\begin{align*}
k_1(t, s) &= \frac{1}{1-\alpha_1}(1-s) - \begin{cases}
    \frac{\alpha_1}{1-\alpha_1}(\eta - s), & s \leq \eta, \\
    0, & s > \eta
\end{cases} - \begin{cases}
    t - s, & s \leq t, \\
    0, & s > t
\end{cases}, \\
k_2(t, s) &= (1-s) - \begin{cases}
    \alpha_2, & s \leq \xi, \\
    0, & s > \xi
\end{cases} - \begin{cases}
    t - s, & s \leq t, \\
    0, & s > t
\end{cases}.
\end{align*}
\]

The Green’s function \(k_1\) has been studied in [29], where it was shown that we may take

\[
\Phi_1(s) = 1 - s,
\]

arbitrary \([a_1, b_1] \subset [0, \eta]\) and \(c_1 = (1-\eta)/(1-\alpha_1)\).

Regarding \(k_2\), this has been studied in [22]; we may take

\[
\Phi_2(s) = 1 - s,
\]

arbitrary \([a_2, b_2] \subset [0, \xi]\) and \(c_2 = 1 - \alpha_2 - \xi\).

The results of the previous Sections, for example Theorem 3.5, can be applied to the system (6.3).

6.1. Optimal intervals. We now assume that \(g_1 = g_2 \equiv 1\) and we seek the ‘optimal’ \([a_i, b_i]\) such that

\[
M_i(a_i, b_i) = \left(\inf_{t \in [a_i, b_i]} \int_{a_i}^{b_i} k_i(t, s)\,ds\right)^{-1}
\]

is a minimum. This type of problem has been tackled in the past in the case of second and higher order BVPs in [7, 24, 25, 42, 52, 53, 57].

Since in \([0, 1] \times [0, 1]\) the kernel \(k_1\) is non-positive only for

\[
\frac{1 - \alpha_1 \eta}{1 - \alpha_1} \leq t \leq 1 \text{ and } 0 \leq s \leq \frac{1 - \alpha_1}{-\alpha_1} t + \frac{1}{\alpha_1},
\]

by direct calculation, we have

\[
\int_0^1 |k_1(t, s)|\,ds = \begin{cases}
    -\frac{t^2}{2} + \frac{1 - \alpha_1}{1 - \alpha_1} \left(\frac{\eta^2}{2} - \alpha_1 \eta^2 + \frac{1}{2}\right) - \frac{\eta^2}{2} =: \vartheta_1(t), & 0 \leq t \leq \frac{1 - \alpha_1 \eta}{1 - \alpha_1}, \\
    -\alpha_2 + 2 - \frac{t^2}{2} + \frac{1 - \alpha_1 - \alpha_2 \eta^2 + 2}{2 \alpha_1 (1 - \alpha_1)} =: \vartheta_2(t), & \frac{1 - \alpha_1 \eta}{1 - \alpha_1} \leq t \leq 1,
\end{cases}
\]

\[
\begin{align*}
    -\alpha_2 + 2 - \frac{t^2}{2} + \frac{1 - \alpha_1 - \alpha_2 \eta^2 + 2}{2 \alpha_1 (1 - \alpha_1)} =: \vartheta_2(t), & \frac{1 - \alpha_1 \eta}{1 - \alpha_1} \leq t \leq 1,
\end{align*}
\]

\[
\int_0^1 |k_2(t, s)|\,ds = \begin{cases}
    -\alpha_2 + 2 - \frac{t^2}{2} + \frac{1 - \alpha_1 - \alpha_2 \eta^2 + 2}{2 \alpha_1 (1 - \alpha_1)} =: \vartheta_2(t), & \frac{1 - \alpha_1 \eta}{1 - \alpha_1} \leq t \leq 1,
\end{cases}
\]
and therefore we obtain

\[ 1/m_1 = \sup_{t \in [0,1]} \int_0^1 |k_1(t, s)| \, ds \]

\[ = \begin{cases} \frac{1}{1 - \alpha_1} \left( \frac{\eta^2}{2} - \alpha_1 \eta^2 + \frac{1}{2} \right) - \frac{\eta^2}{2} = \vartheta_1(0), & \text{if } -2\alpha_1 \eta^2 + \alpha_1 + 1 \geq 0, \\ \frac{-\alpha_1 + 2}{\alpha_1} + \frac{-\alpha_1 - \alpha_1^2 \eta^2 + 2}{-2\alpha_1} = \vartheta_2(1), & \text{if } -2\alpha_1 \eta^2 + \alpha_1 + 1 \leq 0. \end{cases} \]

Firstly we note that \(\frac{1 - \alpha_1 \eta}{1 - \alpha_1} \geq \eta\). For arbitrary \(0 \leq a < b \leq \eta\), the kernel \(k_1\) is a positive, non-increasing function of \(t\). Thus we have

\[ 1/M_1(a, b) = \min_{t \in [a,b]} \int_a^b k_1(t, s) \, ds = \int_a^b k_1(b, s) \, ds. \]

Note that \(\inf_{0 \leq a < b} M_1(a, b) = M_1(0, b)\) and we get

\[ 1/M_1(0, b) = \int_0^b k_1(b, s) \, ds = \left( \frac{1 - \alpha_1 \eta}{1 - \alpha_1} - b \right) b \]

Now we have

\[ \max_{\theta < b \leq \eta} \left\{ \left( \frac{1 - \alpha_1 \eta}{1 - \alpha_1} - b \right) b \right\} = \begin{cases} \frac{(1 - \alpha_1 \eta)^2}{4(1 - \alpha_1)^2}, & \text{if } \frac{1 - \alpha_1 \eta}{2(1 - \alpha_1)} < \eta, \\ \frac{\eta(1 - \eta)}{1 - \alpha_1}, & \text{if } \frac{1 - \alpha_1 \eta}{2(1 - \alpha_1)} \geq \eta. \end{cases} \]

Therefore we may take as optimal interval

\[ [a_1, b_1] = \begin{cases} [0, \frac{1 - \alpha_1 \eta}{2(1 - \alpha_1)}], & \text{if } \frac{1 - \alpha_1 \eta}{2(1 - \alpha_1)} < \eta, \\ [0, \eta], & \text{if } \frac{1 - \alpha_1 \eta}{2(1 - \alpha_1)} \geq \eta. \end{cases} \]

The kernel \(k_2\) in \([0, 1] \times [0, 1]\) is non-positive only for

\[ 1 - \alpha_2 \leq t \leq 1 \text{ and } 0 \leq s \leq \xi; \]

by direct calculation, we have

\[ \int_0^1 |k_2(t, s)| \, ds = \begin{cases} -t^2/2 - \alpha_2 \xi + 1/2 =: \theta_1(t), & 0 \leq t \leq 1 - \alpha_2, \\ -t^2/2 + 2 \xi t - 2 \xi + \alpha_2 \xi + 1/2 =: \theta_2(t), & 1 - \alpha_2 \leq t \leq 1, \end{cases} \]

and therefore we obtain

\[ 1/m_2 = \sup_{t \in [0,1]} \int_0^1 |k_2(t, s)| \, ds = \max\{\theta_1(0), \theta_2(1)\} = \theta_1(0) = -\alpha_2 \xi + 1/2. \]

For arbitrary \(0 \leq a < b \leq \xi\), the kernel \(k_2\) is a positive, non-increasing function of \(t\). Thus we have

\[ 1/M_2(a, b) = \min_{t \in [a,b]} \int_a^b k_2(t, s) \, ds = \int_a^b k_2(b, s) \, ds. \]
Note that $\inf_{0 \leq a < b} M_2(a, b) = M_2(0, b)$ and we get

$$1/M_2(0, b) = \int_0^b k_2(b, s) \, ds = \left(\frac{1 - \alpha_1 \eta}{1 - \alpha_1} - b\right)b$$

Now we have

$$\max_{0 < b \leq \xi} \{(1 - \alpha_2)b \leq b^2\} = \begin{cases} \frac{(1 - \alpha_2)^2}{4}, & \text{if } 1 - \alpha_2 < \xi, \\ (1 - \alpha_2)\xi - \xi^2, & \text{if } 1 - \alpha_2 \geq \xi. \end{cases}$$

Therefore we may take as optimal interval

$$[a_2, b_2] = \begin{cases} [0, \frac{1 - \alpha_2}{2}], & \text{if } 1 - \alpha_2 < \xi, \\ [0, \xi], & \text{if } 1 - \alpha_2 \geq \xi. \end{cases}$$

7. Radial solutions of systems of elliptic PDEs

We now turn back our attention to the systems of BVPs

$$\Delta u + h_1(|x|) f_1(u, v) = 0, \quad |x| \in [R_1, R_0],$$

$$\Delta v + h_2(|x|) f_2(u, v) = 0, \quad |x| \in [R_1, R_0],$$

(7.1)

$$\frac{\partial u}{\partial r}|_{\partial B_{R_0}} = 0 \quad \text{and} \quad (u(R_1 \cdot) - \alpha_1 u(R_0 \cdot))|_{\partial B_1} = 0,$$

$$\frac{\partial v}{\partial r}|_{\partial B_{R_0}} = 0 \quad \text{and} \quad (v(R_1 \cdot) - \alpha_2 \frac{\partial v}{\partial r}(R_0 \cdot))|_{\partial B_1} = 0,$$

where $x \in \mathbb{R}^n$, $\alpha_1 < 0$, $0 < \alpha_2 < 1$, $0 < R_1 < R_0 < \infty$, $R_0, R_1 \in (R_1, R_0)$.

Consider in $\mathbb{R}^n$, $n \geq 2$, the equation

(7.2) \quad $$\Delta w + h(|x|) f(w) = 0,$$ for a.e. $|x| \in [R_1, R_0]$.

with the BCs

$$\frac{\partial w}{\partial r}|_{\partial B_{R_0}} = 0 \quad \text{and} \quad (w(R_1 \cdot) - \alpha_1 w(R_0 \cdot))|_{\partial B_1} = 0,$$

or

$$\frac{\partial w}{\partial r}|_{\partial B_{R_0}} = 0 \quad \text{and} \quad (w(R_1 \cdot) - \alpha_2 \frac{\partial w}{\partial r}(R_0 \cdot))|_{\partial B_1} = 0.$$

In order to establish the existence of radial solutions $w = w(r)$, $r = |x|$, we proceed as in \[33, 35, 36\] and we rewrite (7.2) in the form

(7.3) \quad $$w''(r) + \frac{n - 1}{r} w'(r) + h(r) f(w(r)) = 0 \quad \text{a.e. on } [R_1, R_0].$$

Set $w(t) = w(r(t))$ where, for $n \geq 3$,

$$r(t) = (\gamma + (\beta - \gamma)t)^{-1/(n-2)}, \quad \text{for } t \in [0, 1],$$

with $\gamma = R_0^{-(n-2)}$ and $\beta = R_1^{-(n-2)}$, and for $n = 2$,

$$r(t) = R_0^{1-t} R_1^t, \quad \text{for } t \in [0, 1].$$
Take for \( n \geq 3 \)

\[
\phi(t) = \frac{(\beta - \gamma)}{(n - 2)}(\gamma + (\beta - \gamma)t)^{-2(n-1)/(n-2)},
\]

and for \( n = 2 \)

\[
\phi(t) = \left( R_0(1 - t) \log \frac{R_0}{R_1} \right)^2.
\]

Then the equation (7.3) becomes

\[
w''(t) + \phi(t) h(r(t)) f(w(t)) = 0, \quad \text{a.e. on } [0, 1],
\]

subject to the BCs

\[
w'(0) = 0, \quad \alpha w(\eta) = w(1), \quad 0 < \eta < 1,
\]

or

\[
w'(0) = 0, \quad \alpha w'(\xi) = w(1), \quad 0 < \xi < 1.
\]

Thus, to the system (7.1) we can associate the system of Hammerstein integral equations

\[
\begin{align*}
u(t) &= \int_0^1 k_1(t, s) g_1(s) f_1(u(s), v(s)) \, ds, \\
v(t) &= \int_0^1 k_2(t, s) g_2(s) f_2(u(s), v(s)) \, ds,
\end{align*}
\]

(7.4)

where \( k_1 \) is as in (6.4), \( k_2 \) is as in (6.5) and

\[g_i(t) := \phi(t) h_i(r(t)).\]

The results of the previous Sections can be applied to the system (7.4), yielding results for the system (7.1), we refer to [35, 36] for the results that may be stated.

We illustrate in the following example that all the constants that occur in the Theorem 3.5 can be computed.

**Example 7.1.** Consider in \( \mathbb{R}^2 \), the system of BVPs

\[
\begin{align*}
\Delta u + f_1(u, v) &= 0, \quad |x| \in [1, e], \\
\Delta v + f_2(u, v) &= 0, \quad |x| \in [1, e],
\end{align*}
\]

(7.5)

\[
\begin{align*}
\left. \frac{\partial u}{\partial r} \right|_{\partial B_1} &= 0 \quad \text{and} \quad \left. (u(\cdot) + u(\sqrt{2})) \right|_{\partial B_1} = 0, \\
\left. \frac{\partial v}{\partial r} \right|_{\partial B_1} &= 0 \quad \text{and} \quad \left. \left( v(\cdot) - \frac{1}{4} \frac{\partial v}{\partial r} (\sqrt{e}^{\cdot}) \right) \right|_{\partial B_1} = 0.
\end{align*}
\]

To the system (7.3) we associate the system of second order ODEs

\[
\begin{align*}
u''(t) + e^2(1 - t)^2 f_1(u(t), v(t)) &= 0, \quad t \in [0, 1], \\
v''(t) + e^2(1 - t)^2 f_2(u(t), v(t)) &= 0, \quad t \in [0, 1], \\
u'(0) &= 0, \quad u(1/2) + u(1) = 0, \\
\frac{dv}{dt}(0) &= 0, \quad v'(1/4) = 4v(1).
\end{align*}
\]
Now we have

\[
\frac{1}{m_1} = \sup_{t \in [0, 1]} \int_0^1 |k_1(t, s)|g_1(s) \, ds
\]

\[
= \max \left\{ \sup_{t \in [0, 1/2]} \left\{ -\frac{e^2}{384} \left(-128 t^3 + 32 t^4 + 192 t^2 - 65\right) \right\}, \right. \\
\left. \sup_{t \in [1/2, 3/4]} \left\{ -\frac{e^2}{384} \left(160 t^4 + 864 t^2 - 608 t^3 + 19 - 400 t\right) \right\}, \right. \\
\left. \sup_{t \in [3/4, 1]} \left\{ \frac{1}{2} t + \frac{5}{4} e^2 t^4 + 15/2 t^2 e^2 - 5 t^3 e^2 + \frac{467}{384} e^2 - \frac{119}{24} t e^2 - \frac{3}{8} \right\} \right\}, \]

and

\[
\frac{1}{m_2} = \sup_{t \in [0, 1]} \int_0^1 |k_2(t, s)|g_2(s) \, ds
\]

\[
= \max \left\{ \sup_{t \in [0, 1/4]} \left\{ -\frac{e^2}{768} \left(-256 t^3 + 64 t^4 + 384 t^2 - 155\right) \right\}, \right. \\
\left. \sup_{t \in [1/4, 3/4]} \left\{ -\frac{e^2}{768} \left(-256 t^3 + 64 t^4 + 384 t^2 - 155\right) \right\}, \right. \\
\left. \sup_{t \in [3/4, 1]} \left\{ -\frac{e^2}{768} \left(67 - 296 t - 256 t^3 + 64 t^4 + 384 t^2\right) \right\} \right\}. \]

We fix \([a_1, b_1] = [a_2, b_2] = [0, 1/4] \], obtaining

\[
\frac{1}{M_1} = \inf_{t \in [0, 1/4]} \int_0^{1/4} k_1(t, s)g_1(s) \, ds = \inf_{t \in [0, 1/4]} \left\{ -\frac{e^2}{3072} \left(-377 + 256 t^4 - 1024 t^3 + 1536 t^2\right) \right\},
\]

and

\[
\frac{1}{M_2} = \inf_{t \in [0, 1/4]} \int_0^{1/4} k_2(t, s)g_2(s) \, ds = \inf_{t \in [0, 1/4]} \left\{ -\frac{e^2}{3072} \left(-377 + 256 t^4 - 1024 t^3 + 1536 t^2\right) \right\}.
\]

By direct computation, we get

\[
c_1 = \frac{1}{4}; \quad m_1 = \frac{384}{65e^2}; \quad M_1 = \frac{384}{37e^2}; \quad c_2 = \frac{1}{2}; \quad m_2 = \frac{768}{155e^2}; \quad M_2 = \frac{384}{37e^2}.
\]

Let us now consider

\[
f_1(u, v) = \frac{1}{4}(|u|^3 + |v|^3 + 1), \quad f_2(u, v) = \frac{1}{3}(|u|^{3/2} + v^2).
\]
Then, with the choice of $\rho_1 = 1/6$, $\rho_2 = 1/3$, $r_1 = r_2 = 1$, $s_1 = 3$ and $s_2 = 5$, we obtain
\[
\inf \left\{ f_1(u,v) : (u,v) \in [0,4\rho_1] \times [-2\rho_2,2\rho_2] \right\} = f_1(0,0) > M_1 \rho_1,
\]
\[
\sup \left\{ f_1(u,v) : (u,v) \in [-r_1,r_1] \times [-r_2,r_2] \right\} = f_1(1,1) < m_1 r_1,
\]
\[
\sup \left\{ f_2(u,v) : (u,v) \in [-r_1,r_1] \times [-r_2,r_2] \right\} = f_2(1,1) < m_2 r_2,
\]
\[
\inf \left\{ f_1(u,v) : (u,v) \in [s_1,4s_1] \times [-2s_2,2s_2] \right\} = f_1(s_1,0) > M_1 s_1,
\]
\[
\inf \left\{ f_2(u,v) : (u,v) \in [-4s_1,4s_1] \times [s_2,2s_2] \right\} = f_2(0,s_2) > M_2 s_2.
\]
Thus the conditions $(L^0_{\rho_1,\rho_2})^*$, $(L^1_{r_1,r_2})$ and $(L^0_{s_1,s_2})$ are satisfied; therefore the system (7.5) has at least two nontrivial solutions.

**References**

[1] R. P. Agarwal, D. O'Regan and P. J. Y. Wong, *Constant-sign solutions of systems of integral equations*, Springer, Cham, 2013.

[2] H. Amann, Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, *SIAM. Rev.*, **18** (1976), 620–709.

[3] P. Amster and M. Maurette, An elliptic singular system with nonlocal boundary conditions, *Nonlinear Anal.*, **75** (2012), 5815–5823.

[4] R. Beals, Nonlocal elliptic boundary value problems, *Bull. Amer. Math. Soc.*, **70** (1964), 693–696.

[5] A. V. Bitsadze and A. A. Samarski˘ı, Some elementary generalizations of linear elliptic boundary value problems (Russian), *Anal. Dokl. Akad. Nauk SSSR*, **185** (1969), 739–740.

[6] F. Browder, Non-local elliptic boundary value problems, *Amer. J. Math.*, **86** (1964), 735–750.

[7] A. Cabada, G. Infante and F. A. F. Tojo, Nonzero solutions of perturbed Hammerstein integral equations with deviated arguments and applications, *arXiv:1306.6560 [math.CA]*, (2013).

[8] X. Cheng and Z. Zhang, Existence of positive solutions to systems of nonlinear integral or differential equations, *Topol. Methods Nonlinear Anal.*, **34** (2009), 267–277.

[9] X. Cheng and C. Zhong, Existence of positive solutions for a second-order ordinary differential system, *J. Math. Anal Appl.*, **312** (2005), 14–23.

[10] R. Conti, Recent trends in the theory of boundary value problems for ordinary differential equations. *Boll. Un. Mat. Ital.*, **22** (1967), 135–178.

[11] D. R. Dunninger and H. Wang, Existence and multiplicity of positive solutions for elliptic systems, *Nonlinear Anal.*, **29** (1997), 1051–1060.

[12] D. R. Dunninger and H. Wang, Multiplicity of positive radial solutions for an elliptic system on an annulus, *Nonlinear Anal.*, **42** (2000), 803–811.

[13] J. M. do Ó, S. Lorca and P. Ubilla, Local superlinearity for elliptic systems involving parameters, *J. Differential Equations*, **211** (2005), 1–19.

[14] J. M. do Ó, S. Lorca and P. Ubilla, Three positive solutions for a class of elliptic systems in annular domains, *Proc. Edinb. Math. Soc.* (2), **48** (2005), 365–373.

[15] J. M. do Ó, J. Sánchez, S. Lorca and P. Ubilla, Positive solutions for a class of multiparameter ordinary elliptic systems, *J. Math. Anal. Appl.*, **332** (2007), 1249–1266.
[16] D. Franco, G. Infante and D. O’Regan, Nontrivial solutions in abstract cones for Hammerstein integral systems, *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.*, 14 (2007), 837–850.

[17] C. S. Goodrich, Nonlocal systems of BVPs with asymptotically superlinear boundary conditions, *Comment. Math. Univ. Carolin.*, 53 (2012), 79–97.

[18] C. S. Goodrich, Nonlocal systems of BVPs with asymptotically sublinear boundary conditions, *Appl. Anal. Discrete Math.*, 6 (2012), 174–193.

[19] D. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, Boston, 1988.

[20] J. Henderson and R. Luca, Existence and multiplicity for positive solutions of a system of higher-order multi-point boundary value problems, *NoDEA Nonlinear Differential Equations Appl.*, 20 (2013), 1035–1054.

[21] J. Henderson and R. Luca, Positive solutions for systems of second-order integral boundary value problems, *Electron. J. Qual. Theory Differ. Equ.*, 70 (2013), 21 pp.

[22] G. Infante, Eigenvalues of some non-local boundary-value problems, *Proc. Edinb. Math. Soc.*, 46 (2003), 75–86.

[23] G. Infante and P. Pietramala, Eigenvalues and non-negative solutions of a system with nonlocal BCs, *Nonlinear Stud.*, 16 (2009), 187–196.

[24] G. Infante and P. Pietramala, A cantilever equation with nonlinear boundary conditions *Electron. J. Qual. Theory Differ. Equ.*, Spec. Ed. I, No. 15 (2009), 1–14.

[25] G. Infante and P. Pietramala, Perturbed Hammerstein integral inclusions with solutions that change sign, *Comment. Math. Univ. Carolin.*, 50 (2009), 591–605.

[26] G. Infante and P. Pietramala, Existence and multiplicity of non-negative solutions for systems of perturbed Hammerstein integral equations, *Nonlinear Anal.*, 71 (2009), 1301–1310.

[27] G. Infante, P. Pietramala and F. A. F. Tojo, Nontrivial solutions of local and nonlocal Neumann boundary value problems, arXiv:1404.1390 [math.CA], (2014).

[28] G. Infante and J. R. L. Webb, Nonzero solutions of Hammerstein integral equations with discontinuous kernels, *J. Math. Anal. Appl.*, 272 (2002), 30–42.

[29] G. Infante and J. R. L. Webb, Three point boundary value problems with solutions that change sign, *J. Integral Equations Appl.*, 15 (2003), 37–57.

[30] G. L. Karakostas, Existence of solutions for an n-dimensional operator equation and applications to BVPs, *Electron. J. Differential Equations*, 71 (2014), 17 pp.

[31] M. A. Krasnosel’skiĭ and P. P. Zabreĭko, *Geometrical methods of nonlinear analysis*, Springer-Verlag, Berlin, (1984).

[32] K. Q. Lan, Multiple positive solutions of Hammerstein integral equations with singularities, *Diff. Eqns and Dynam. Syst.*, 8 (2000), 175–195.

[33] K. Q. Lan, Multiple positive solutions of semilinear differential equations with singularities, *J. London Math. Soc.*, 63 (2001), 690–704.

[34] K. Q. Lan and W. Lin, Multiple positive solutions of systems of Hammerstein integral equations with applications to fractional differential equations, *J. Lond. Math. Soc.*, 83 (2011), 449–469.

[35] K. Q. Lan and W. Lin, Positive solutions of systems of singular Hammerstein integral equations with applications to semilinear elliptic equations in annuli, *Nonlinear Anal.*, 74 (2011), 7184–7197.

[36] K. Q. Lan and J.R.L. Webb, Positive solutions of semilinear differential equations with singularities, *J. Differential Equations*, 148 (1998), 407–421.

[37] Y.-K. Lee, Multiplicity of positive radial solutions for multiparameter semilinear elliptic systems on an annulus, *J. Differential Equations*, 174 (2001), 420–441.
[38] R. Ma, Existence of positive radial solutions for elliptic systems, *J. Math. Anal. Appl.*, **201** (1996), 375–386.

[39] R. Ma, A survey on nonlocal boundary value problems, *Appl. Math. E-Notes*, **7** (2001), 257–279.

[40] S. K. Ntouyas, Nonlocal initial and boundary value problems: a survey, *Handbook of differential equations: ordinary differential equations. Vol. II*, 461–557, Elsevier B. V., Amsterdam, 2005.

[41] M. Picone, Su un problema al contorno nelle equazioni differenziali lineari ordinarie del secondo ordine, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, **10** (1908), 1–95.

[42] P. Pietramala, A note on a beam equation with nonlinear boundary conditions, *Bound. Value Probl.*, (2011), Art. ID 376782, 14 pp.

[43] R. Precup, Componentwise compression-expansion conditions for systems of nonlinear operator equations and applications, *Mathematical models in engineering, biology and medicine*, 284–293, AIP Conf. Proc., 1124, Amer. Inst. Phys., Melville, NY, 2009.

[44] R. Precup, Existence, localization and multiplicity results for positive radial solutions of semilinear elliptic systems, *J. Math. Anal. Appl.*, **352** (2009), 48–56.

[45] M. Schechter, Nonlocal elliptic boundary value problems, *Ann. Scuola Norm. Sup. Pisa*, **20** (1966), 421–441.

[46] A. L. Skubachevskii, Nonclassical boundary value problems. I, *J. Math. Sci. (N. Y.)*, **155** (2008), 199–334.

[47] A. L. Skubachevskii, Nonclassical boundary value problems. II, *J. Math. Sci. (N. Y.)*, **166** (2010), 377–561.

[48] A. Štikonas, A survey on stationary problems, Green’s functions and spectrum of Sturm-Liouville problem with nonlocal boundary conditions, *Nonlinear Anal. Model. Control*, **19** (2014), 301–334.

[49] H. Wang, On the existence of positive solutions for semilinear elliptic equations in the annulus, *J. Differential Equations*, **109** (1994), 1–7.

[50] Y. Wang, Solutions to nonlinear elliptic equations with a nonlocal boundary condition, *Electron. J. Differential Equations*, **05** (2002), 16 pp.

[51] J. R. L. Webb, Positive solutions of some three point boundary value problems via fixed point index theory, *Nonlinear Anal.*, **47** (2001), 4319–4332.

[52] J. R. L. Webb, Multiple positive solutions of some nonlinear heat flow problems, *Discrete Contin. Dyn. Syst.*, suppl. (2005), 895–903.

[53] J. R. L. Webb, Optimal constants in a nonlocal boundary value problem, *Nonlinear Anal.*, **63** (2005), 672–685.

[54] J. R. L. Webb, Solutions of nonlinear equations in cones and positive linear operators, *J. Lond. Math. Soc.*, **82** (2010), 420–436.

[55] J. R. L. Webb, A class of positive linear operators and applications to nonlinear boundary value problems, *Topol. Methods Nonlinear Anal.*, **39** (2012), 221–242.

[56] J.R.L. Webb and G. Infante, Positive solutions of nonlocal boundary value problems: a unified approach, *J. London Math. Soc.*, **74** (2006), 673–693.

[57] J. R. L. Webb and G. Infante, Nonlocal boundary value problems of arbitrary order, *J. London Math. Soc.*, **79** (2009), 238–258.

[58] J. R. L. Webb and K. Q. Lan, Eigenvalue criteria for existence of multiple positive solutions of nonlinear boundary value problems of local and nonlocal type, *Topol. Methods Nonlinear Anal.*, **27** (2006), 91–115.

[59] W. M. Whyburn, Differential equations with general boundary conditions, *Bull. Amer. Math. Soc.*, **48** (1942), 692–704.
[60] Z. Yang, Positive solutions to a system of second-order nonlocal boundary value problems, *Nonlinear Anal.*, **62** (2005), 1251–1265.

[61] Z. Yang, Positive solutions for a system of nonlinear Hammerstein integral equations and applications, *Appl. Math. and Comput.*, **218** (2012), 11138–11150.

[62] Z. Yang and Z. Zhang, Positive solutions for a system of nonlinear singular Hammerstein integral equations via nonnegative matrices and applications, *Positivity*, **16** (2012), 783–800.

[63] H. Ye and Y. Ke, A $p$-Laplace equation with nonlocal boundary condition in a perforated-like domain, *Bull. Belg. Math. Soc. Simon Stevin*, **20** (2013), 895–908.

**Gennaro Infante**, Dipartimento di Matematica e Informatica, Università della Calabria, 87036 Arcavacata di Rende, Cosenza, Italy

*E-mail address: gennaro.infante@unical.it*

**Paolamaria Pietramala**, Dipartimento di Matematica e Informatica, Università della Calabria, 87036 Arcavacata di Rende, Cosenza, Italy

*E-mail address: pietramala@unical.it*