Reasoning with Justifiable Exceptions in Contextual Hierarchies (Appendix)

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Abstract. This paper is an appendix to the paper “Reasoning with Justifiable Exceptions in Contextual Hierarchies” by Bozzato, Serafini and Eiter, 2018 \cite{bozzato2018reasoning}. It provides further details on the language, the complexity results and the datalog translation introduced in the main paper.

1 SROIQ syntax and semantics

Table 1 presents the syntax and semantics of SROIQ operators and axioms. In the table, \(A\) is any atomic concept, \(C\) and \(D\) are any concepts, \(P\) and \(R\) are any atomic roles (and for \(\ast\) simple in the context of a knowledge base \(\mathcal{K}\)), \(S\) and \(Q\) are any (possibly complex) roles, \(a\) and \(b\) are any individual constants, and \(n\) stands for any positive integer.

2 Reasoning and complexity: more details

In what follows, we assume the setting of \cite{bozzato2018reasoning} for the complexity analysis.

**Proposition 1.** Deciding whether a CAS-interpretation \(\mathcal{I}_{\text{CAS}}\) of a sCKR \(\mathcal{R}\) is a CKR-model is coNP-complete.

Informally, \(\mathcal{I}_{\text{CAS}}\) can be refuted if it is not a justified CAS-model of \(\mathcal{R}\), which can be checked in polynomial time using the techniques in \cite{bozzato2018reasoning}, or some preferred model \(\mathcal{I}_{\text{CAS}}\) exists; the latter can be guessed and checked in polynomial time. The coNP-hardness is shown, already under data complexity, by a reduction from a restricted version of UNSAT. We shall discuss in the context of c-entailment under data complexity below.

**Theorem 1.** Given a ranked sCKR \(\mathcal{R}\), a context name \(c\) and an axiom \(\alpha\), deciding whether \(\mathcal{R} \models c : \alpha\) is \(\Delta^p_2\)-complete for profile-based preference.

**Proof (Sketch).** For profile-based comparison, we can compute the lexicographic maximum profile \(p^*\) of a CKR-model by extending a partial profile \((l_1, l_2, \ldots, l_n)\), \(i = n, n-1, \ldots, 0\) using an NP oracle in polynomial time; asking for each possible value \(v\) whether \(l_i = v\) is possible. We then can check with the NP oracle whether every justified CAS-model \(\mathcal{I}_{\text{CAS}}\) having this profile fulfills \(\mathcal{I}_{\text{CAS}} \not\models c : \alpha\).
The $\Delta^p_2$-hardness is shown by a reduction from deciding the last bit $x_n$ of the lexicographic maximum satisfying assignment of a SAT instance $E = \bigwedge_{i=1}^m \gamma_i$ over propositional atoms $X = \{x_1, \ldots, x_n\}$.

Without loss of generality, $E$ is a 3SAT instance (with duplicate literals allowed) and each clause $\gamma_i$ in $E$ is either positive or negative.

Then we construct $\mathcal{R}$ as follows. Let $V_i$, $i = 1, \ldots, n$ and $F, T, A$ be concepts, $P_1, P_2, P_3, N_1, N_2, N_3$ be roles, and $x_1, \ldots, x_n$, $c_1, \ldots, c_m$ be individual constants. We use totally ordered contexts $c_0 < c_1 < \cdots < c_{n+1}$. The knowledge bases of the contexts contain the following axioms

- the knowledge base of $c_{i+1}$ contains the defeasible axioms $D(V_i \subseteq F)$ for all $i = 1, \ldots, n$
- the knowledge base of $c_i$, $i = 1, \ldots, n$ contains the defeasible axiom $D(V_i \subseteq T)$
- the knowledge base of $c_0$ that contains the inclusion axioms: $T \sqcap F \subseteq \bot$, $T \subseteq A$, $F \subseteq A$, $\bigcap_{j=1}^3 \exists P_j(T \sqcap A) \subseteq \bot$, and $\bigcap_{j=1}^3 \exists P_j(F \sqcap A) \subseteq \bot$, and the assertions $V_i(x_i), h = 1, \ldots, n$ and $P_j(c_i, x_i)$ for $i = 1, \ldots, m$ and $j = 1, 2, 3$ such that the clause $\gamma_i$ is of form $x_i \vee x_i \vee x_i$
- $N_j(c_i, x_i)$ for $i = 1, \ldots, m$ and $j = 1, 2, 3$ such that the clause $\gamma_i$ is of form $\neg x_i \vee \neg x_i \vee x_i$

Intuitively, we must at context $c_0$ make for each $x_i$ an exception to either $V_i \subseteq F$ or $V_i \subseteq T$; the respective single minimal clashing set is $\{V_i(x_i), \neg F(x_i)\}$ resp. $\{V_i(x_i), \neg T(x_i)\}$.

One can show that the justified CAS-models $\mathcal{X}_{\text{CAS}}$ of the CKR correspond 1-1 to the satisfying assignments $\sigma$ of $E$. Furthermore, under profile-based preference, keeping...
$V_i \subseteq T$ is preferred over keeping $V_i \subseteq F$, and thus by the context ordering the lexicographic maximum $\sigma^*$ that satisfies $E$ will be reflected in every non-preferred model. Consequently, $\mathcal{R} \models \alpha_0 : T(x_0)$ holds iff $\sigma(x_0) = \text{true}$.

**Theorem 2.** Deciding where an sCKR $\mathcal{R}$ entails a Boolean CQ $\gamma$ is \(\Pi_2^0\)-complete for profile-based preference.

**Proof (Sketch).** Similarly as for $\mathcal{C}$-entailment, a CKR-model $\mathcal{Y}_{\text{CAS}}$ that does not entail $\gamma$ can be guessed and checked with the help of an NP oracle (ask whether no CKR-model $\mathcal{Y}_{\text{CAS}}$ of $\mathcal{R}$ exists that is preferred to $\mathcal{Y}_{\text{CAS}}$ and whether $\gamma$ is entailed in $\mathcal{Y}_{\text{CAS}}$); note that the profiles of interpretations are easy to calculate. The $\Pi_2^0$-hardness is inherited from ordinary CKR.

### 2.1 Data complexity

Concerning the data complexity, i.e., the CKR $\mathcal{R}$ is fixed and only the assertions in the knowledge modules vary,

**Proposition 2.** Deciding whether a given CAS-interpretation $\mathcal{Y}_{\text{CAS}}$ of a sCKR $\mathcal{R}$ is a CKR-model is coNP-complete under data complexity.

**Proof (Sketch).** The membership is inherited from the general case. The hardness part follows from the particular reduction of deciding ODD SAT to $\mathcal{C}$-entailment under data complexity, which amounts for particular inputs to a reduction from a variant of UN-SAT, and will be discussed in this context.

**Theorem 3.** Deciding whether $\mathcal{R} \models \mathcal{C} : \alpha$ is $\Delta_2^p[O(\log n)]$-complete for profile-based preference, under data complexity.

**Proof (Sketch).** The membership in $\Delta_2^p[O(\log n)]$ is established by exploiting that $\Delta_2^p[O(\log n)] = \text{P}^{\text{NP}[\text{fl}]}$ (cf. [3]): we can compute, with parallel NP oracle queries, in a constant number of rounds the optimal profile $P^* = (t'_1, \ldots, t'_n)$ of any clashing assumption $\chi$ of a CKR-model, as $n$ is constant: in each round, we extend the partial profile $((t''_1, \ldots, t''_{i-1})$ with $t'_i$, asking for each possible value $v$ whether $t'_j = v$ is possible. In a last round, we can then decide with a single oracle call $\mathcal{R} \models \mathcal{C} : \alpha$ based on $P^*$.

The $\Delta_2^p[O(\log n)]$-hardness is shown by a reduction from deciding whether among given 3SAT instances $E_1, \ldots, E_l$, $l \geq 1$ on disjoint atoms, where duplicate literals in clauses are allowed, and an odd number of $E_k$ is satisfied by some assignment that does not set all atoms in $E_k$ to false. The $\Delta_2^p[O(\log n)]$-completeness of this problem, which we refer to as ODD SAT follows from [5]. Without loss of generality, we may assume that $E_2$ is only satisfiable if $E_{l-1}$ is, that $l$ is even, that all $E_k$ have the same number of variables, that the clauses in them are monotone, and that each satisfying assignment of $E_k = E_k(x'_1, \ldots, x'_n)$ sets either all atoms to false or otherwise $x'_i$ to true.

Then we construct $\mathcal{R}$ similar as for $\text{P}^{\text{NP}}$-hardness follows. Let $F, T, A, V, Y, O$ be concepts, $P_1, P_2, P_3, N_1, N_2, N_3, C, R$ be roles, and $\alpha$ and $x_1^k, \ldots, x_n^k, e_1^k, \ldots, e_m^k$ be individual constants for the variables and clauses in $E_i$, respectively. We use totally ordered contexts $c_0 < c_1 < c_2$. The knowledge bases of the contexts contain the following axioms.
– the knowledge base of $c_2$ contains the defeasible axioms $D(V \subseteq F)$.
– the knowledge base of $c_1$ contains the defeasible axiom $D(V \subseteq T)$
– the knowledge base of $c_0$ that contains the inclusion axioms:

$$T \cap F \subseteq \bot, \quad T \subseteq A, \quad F \subseteq A, \quad \bigcap_{j=1}^{3} \exists N_j, (T \cap A) \subseteq \bot, \quad \bigcap_{j=1}^{3} \exists P_j, (F \cap A) \subseteq \bot,$$

and the assertions

$$V(x_i^j), \text{ for all } i \text{ and } j.$$

$$P_j(c_i^j, x_i^j) \text{ for } i = 1, \ldots, m \text{ and } j = 1, 2, 3 \text{ such that the clause } \chi_i^j \text{ of } E_k \text{ is of form }$$

$$x_{i_1}^k \lor x_{i_2}^k \lor x_{i_3}^k,$$

$$N_j(c_i^j, x_i^j) \text{ for } i = 1, \ldots, m \text{ and } j = 1, 2, 3 \text{ such that the clause } \gamma_i^j \text{ is of form }$$

$$\neg x_{i_1}^k \lor \neg x_{i_2}^k \lor \neg x_{i_3}^k,$$

$$C(x_{1}^{k+1}, x_{k+2}^l), R(a, x_1^{k+1}) \text{ for } k = 0, 2, \ldots, l-2.$$

Intuitively, we must at context $c_0$ make for each $x_i^j$ an exception to either $V \subseteq F$ or $V \subseteq T$; the respective single minimal clashing set is $[V(x_i^j), \neg F(x_i^j)]$ resp. $[V(x_i^j), \neg T(x_i^j)]$.

One can show that the (preference-less) CKR-\_CAS of the CKR correspond 1-1 to the combinations of satisfying assignments $\sigma_1, \ldots, \sigma_l$ of $E_1, \ldots, E_l$, respectively. Furthermore, under profile-based preference, keeping $V \subseteq T$ is preferred over keeping $V \subseteq F$, and thus by the context ordering for each $E^d$ an assignment $\sigma_d$ that sets $x_i^j$ to true.

In case an odd number of $x_i^j$ is set to true, for some $x_{1}^{k+1}$ from the assertion $C(x_1^{k+1}, x_{k+2})$ and $T \cap \exists C.F \subseteq Y$, one can derive $Y(x_{1}^{k+1})$, and then from $R(a, x_1^{k+1})$ and the axiom $O \subseteq \exists R.Y$ that $O(a)$ holds. On the other hand, $O(a)$ cannot be derived if an even number of $x_i^j$ is set to true.

Consequently, $\mathcal{K} \models c_0: O(a)$ holds iff the instance of ODD SAT is a yes-instance.

This shows $\Delta^0_2[O(\log n)]$-hardness.

We remark that the reduction in the proof establishes coNP-hardness of model checking under data complexity: if we consider $l = 2$ and an $E_2$ that is satisfied only if all atoms are set to false, then for the clashing assumption $\chi$ consisting of $\langle V \subseteq T, x_i^j \rangle$ for all atoms $x_i^j$ gives rise to a (canonical) CKR-model $\mathcal{K}_{CAS}$ of $\mathcal{K}$ that can be constructed in polynomial time, and moreover $\mathcal{K}_{CAS}$ is preferred iff $E_2$ is unsatisfiable; this shows coNP-hardness (a simpler, direct construction for $E_2$ is clearly possible).

That $CQ$ entailment remains $\Pi^P_2$-complete under data complexity is a simple consequence that membership in $\Pi^P_2$ holds for the general case, and that the inherited $\Pi^P_2$-hardness of $CQ$-answering for ordinary CKR knowledge bases (without context hierarchies) holds under data complexity.

### 2.2 Complexity of c-Entailment under local preference

As for local preference at a context $c$, let for any context $c'$ above $c$ denote $X(c,c')$ the set of all clashing assumptions $(\sigma,e)$ for defeasible axioms at $c'$ made at $c$ in some CKR-model of $\mathcal{K}$. 
Call a context \( c'\) a connector for \( c \), if it directly covers \( c \) and for every \( c'' \) and \( c''' \), if and \( c'' \) covers \( c''' \), then \( c''' \) covers \( c' \) (i.e., every path in the covers-graph from a node \( c'' \) above \( c' \) to \( c \) must pass through \( c' \)).

Consider the following property of the local preference >:

\begin{enumerate}
\item[(CP)] If \( c' \) is a connector for \( c \) and (i) \( X_2(c') \subseteq \chi_1(c) \), (ii) \( X_2(c') \not\subseteq \chi_2(c) \), and (iii) \( \chi_1(c'') = \chi_2(c''') \), for each \( c'' > c \) such that \( c''' \neq c'' \), then \( \chi_2(c) > \chi_1(c) \).
\end{enumerate}

That is, the worst possible overriding at a connector for \( c \) is always less preferred, if the clashing assumptions agree on the contexts that are not above \( c' \) or reachable from some such context. This condition seems to be plausible for local preference.

Let the global preference on CAS-models \( \Sigma_{\text{CAS}}^1 = \langle \Sigma, \chi_1 \rangle \), \( \Sigma_{\text{CAS}}^2 = \langle \Sigma, \chi_2 \rangle \) induced by a local preference \( \chi_1(c) > \chi_2(c) \) on clashing assumptions of contexts \( c \) be as follows: \( \Sigma_{\text{CAS}}^1 \) is preferred to \( \Sigma_{\text{CAS}}^2 \), if there exists some \( c \in N \) s.t. \( \chi_1(c) > \chi_2(c) \) and for no context \( c' \neq c \in N \) it holds that \( \chi_1(c') < \chi_2(c') \).

**Theorem 4.** Suppose \( \mathcal{R} \) is a SCKR with global preference induced by a local preference > that is polynomial-time decidable and satisfies (CP). Then e-entailment \( \mathcal{R} \models c : \alpha \) is \( \Pi_2^{\text{P}} \)-complete. Furthermore, the \( \Pi_2^{\text{P}} \)-hardness even holds for ranked hierarchies with three levels.

**Proof (Sketch).** The membership in \( \Pi_2^{\text{P}} \) follows by a guess and check argument, as we can guess a CKR-model \( \Sigma_{\text{CAS}} \) of \( \mathcal{R} \) such that (i) \( \Sigma_{\text{CAS}} \not\models c : \alpha \) and (ii) no CKR-model \( \Sigma'_{\text{CAS}} \) of \( \mathcal{R} \) exists such that \( \Sigma'_{\text{CAS}} > \Sigma_{\text{CAS}} \). As local model checking in absence of preferences is polynomial, and local preference is polynomial decidable, \( \mathcal{R} \models c : \alpha \) is decidable in non-deterministic polynomial time with an NP oracle, and thus in \( \Sigma_2^{\text{P}} \), which implies the result.

The \( \Pi_2^{\text{P}} \)-hardness of e-entailment under the given assertion can be shown by a reduction from evaluating a QBF of the form \( \forall X \exists Y \not\models \mu (X, Y) \), where w.l.o.g. \( E = \bigwedge_{i=1}^3 Y_i \) is a monotone 3CNF (each clause is either positive or negative and has size 3, with duplicate literals allowed), and \( Y = X' Y' \), where \( E \) contains the clauses \( x_i \lor x'_i \lor \neg x_j \lor \neg x'_j \) (i.e., \( x_i \equiv \neg x'_j \)) for each \( x_i \in X \), and \( \mu \) is a particular assignment to \( Y \) such that \( \forall X \exists Y (X, \mu(Y)) \) evaluates to true.

We construct a CKR \( \mathcal{R} \) as follows. We use contexts \( c_0 \) and \( c_p \), \( c_{\gamma'} \), for all \( p \in X \cup X' \) and \( c_{\gamma}, c_{\gamma''} \). The context ordering is:

\begin{itemize}
\item \( c_0 < c_p < c_{\gamma'} \), for all \( p \in X \cup X' \),
\item \( c_0 < c_{\gamma'} < c_{\gamma''} \).
\end{itemize}

Let \( V_i, i = 1, \ldots, 2|X| + |Y| \) and \( F, T, A \) be concepts, \( P_1, P_2, P_3, N_1, N_2, N_3 \) be roles, and \( p \in X \cup X' \), \( y_j \in Y \) and \( c_1, \ldots, c_m \) be individual constants.

The knowledge bases of the contexts contain the following axioms:

\begin{itemize}
\item the knowledge base of \( c_p \) (resp., \( c_{\gamma'} \)) contains the defeasible axiom \( D(V_i \sqsubseteq T) \) (resp.,
\item \( D(V_i \sqsubseteq p) \)) if \( p = x_i \) (resp., if \( p = x'_i \) and \( i = 2j \)),
\item the knowledge base of \( c_{\gamma} \) the defeasible axioms \( D(V_{2|x|+j} \sqsubseteq T) \) if \( \mu(y_j) = \text{true} \), and
\item \( D(V_{2|x|+j} \sqsubseteq F) \) otherwise, for all \( j = 1, \ldots, |Y| \)
\end{itemize}
– the knowledge base of $c_{\gamma'}$ the defeasible axioms $D(V_{2\gamma+j} \subseteq F)$ if $\mu(y_j) = \text{true}$, and $D(V_{2\gamma+j} \subseteq T)$ otherwise, for all $j = 1, \ldots, |\gamma'|$
– the knowledge base of $c_0$ that contains the inclusion axioms:
\[
T \cap F \subseteq \bot, \quad T \subseteq A, \quad F \subseteq A, \quad \bigcap_{j=1}^{3} \exists N_j(T \cap A) \subseteq \bot, \quad \text{and} \quad \bigcap_{j=1}^{3} \exists P_j(F \cap A) \subseteq \bot,
\]
and the assertions
\[
V_h(x_h), \quad V_{\alpha+h}(x_h'), \quad \forall h = 1, \ldots, n, \text{and} \quad V_{2\gamma+j}(y_j) \quad j = 1, \ldots, |\gamma'|.
\]
– $P_j(c_i, y_i)$ for $i = 1, \ldots, m$ and $j = 1, 2, 3$ such that the clause $\gamma_j$ is of form $v_{i_1} \lor v_{i_2} \lor v_{i_3}$,
– $N_j(c_i, y_i)$ for $i = 1, \ldots, m$ and $j = 1, 2, 3$ such that the clause $\gamma_j$ is of form $\neg v_{i_1} \lor \neg v_{i_2} \lor \neg v_{i_3}$.

Informally, either $V_p \subseteq T$ or $V_p \subseteq F$ is overridden in each CKR-model for each $p = x_i$, resp. $p = x'_i$, which correspond to truth assignments to $X$ and $X'$; as $x_i \leftrightarrow \neg x'_i$. Justified CAS-models are only comparable for preference if the correspond to the same assignment. On the other hand, by the assumption of $\mu$ for $E$, we have some CKR-model $\mathcal{J}_{\text{CAS}}$ in which all overriding of $V_j \subseteq T$ or $V_j \subseteq F$ for atoms $y_j$ happens for the axioms at $c_{\gamma'}$. That is, the clashing assumption of $\mathcal{J}_{\text{CAS}}$ includes the set $\mathcal{X}_\gamma(c')$ of clashing assumptions defined above for $c = c_0$ and $c' = c_{\gamma'}$.

We may assume that if for a given assignment $\sigma$ to $X \cup X'$ some other assignment $\mu'$ to $Y'$ exists that makes $E(\sigma(X \cup X'), Y')$ true, then some fixed variable $y_j$ must in $\mu'$ have, regardless of $\sigma$, a different value than in $\mu$.

Note that $c_{\gamma}$ is a connector of $c$ (as is $c_0$ for every atom $p$). Under the assumption that the local preference $\prec$ satisfies the property (CP), it follows that the corresponding CKR-model $\mathcal{J}^{c}_{\text{CAS}}$ will then be preferred to the model $\mathcal{J}_{\text{CAS}}$ for $\sigma, \mu$.

Consequently, $\mathcal{R} \models c_0 : T(y_j)$ respectively $\mathcal{R} \models c_0 : F(y_j)$ holds iff the formula $\forall X \forall Y \neq \mu E(X, Y)$ evaluates to true. This shows $\Pi^c_2$-hardness of $c$-entailment under a global preference induced by any polynomial-time decidable local preference $\chi_2(c) > \chi_1(c)$ that satisfies (CP). A particular such preference is e.g. profile-based based preference a the local level.

We note that the contexts $c_{\gamma'}$ and $c_{\gamma''}$ can be replaced by copies $c_{\gamma'}$ and $c_{\gamma''}$, for all $\gamma' \in \gamma''$: each $c_{\gamma'}$ is a connector. Thus, the $\Pi^c_2$-hardness carries over to the case of a ranked hierarchy with three levels. In case of two levels, no context-sensitive overriding is possible and the setting is subsumed by the one of ordinary CKR, for which $c$-entailment is coNP-complete.

3 Translation rule set tables

Rule sets for the proposed translation are shown in the tables in following pages. SROIQ-RL input and deduction rules are presented in Table 2. Table 3 lists global and local translations and output rules. Table 4 shows input rules for defeasible axioms. Overriding rules are shown in Table 5; defeasible inheritance rules are reported in Table 6 and test rules are shown in Table 7. Finally, the newly introduced rules and constraints for overriding level preference are shown in Table 8.
Table 2. SROIQ-RL input and deduction rules

**SROIQ-RL input translation \( I_a(S, c) \)**

| Rule | Description |
|------|-------------|
| (irl-nom) | \( a \in \text{NI} \Rightarrow \{ \text{nom}(a, c) \} \) |
| (irl-clsl) | \( A \in \text{NC} \Rightarrow \{ \text{cls}(A, c) \} \) |
| (irl-rol) | \( R \in \text{NR} \Rightarrow \{ \text{rol}(R, c) \} \) |
| (irl-inst1) | \( A(a) \Rightarrow \{ \text{insta}(a, A, c, \text{main}) \} \) |
| (irl-inst2) | \( A(a) \Rightarrow \{ \text{insta}(a, A, c) \} \) |
| (irl-triple) | \( R(a, b) \Rightarrow \{ \text{triple}(a, R, b, c, \text{main}) \} \) |
| (irl-ntriple) | \( a \neq b \Rightarrow \{ \text{ntriple}(a, R, b, c) \} \) |
| (irl-cls) | \( A \subseteq B \Rightarrow \{ \text{subClass}(A, B, c) \} \) |
| (irl-top) | \( \top(a) \Rightarrow \{ \text{insta}(a, \text{top}, c) \} \) |
| (irl-bot) | \( \bot(a) \Rightarrow \{ \text{insta}(a, \text{bot}, c) \} \) |

**SROIQ-RL deduction rules \( P_f \)**

| Rule | Description |
|------|-------------|
| (irl-subcnj) | \( A_1 \land A_2 \subseteq B \Rightarrow \{ \text{subConj}(A_1, A_2, B, c) \} \) |
| (irl-supforall) | \( \exists R \subseteq \text{S} \Rightarrow \{ \text{supForall}(R, S, c) \} \) |
| (irl-sat) | \( \exists R \subseteq \text{S} \Rightarrow \{ \text{supEx}(R, S, c) \} \) |
| (irl-inv2) | \( \exists R \subseteq \text{S} \Rightarrow \{ \text{inv2}(R, S, c) \} \) |
| (irl-unsat) | \( \exists R \subseteq \text{S} \Rightarrow \{ \text{unsat}(R, S, c) \} \) |
| (irl-irr) | \( \exists R \subseteq \text{S} \Rightarrow \{ \text{irr}(R, S, c) \} \) |
Table 3. Global, local and output rules

Global input rules $I_{gl}(C)$
- (igl-covers) $c_1 < c_2 \rightarrow \{\text{prec}(c_1, c_2)\}$
- (igl-level) $l(c_i) = n \rightarrow \{\text{level}(c_i, n + 1)\}$

Local input rules $I_{loc}(Km, c)$
- (ilc-subevalat) $\text{eval}(A, c_1) \subseteq B \rightarrow \{\text{subEval}(A, c_1, B, c)\}$
- (ilc-subevalr) $\text{eval}(R, c_1) \subseteq T \rightarrow \{\text{subEval}(R, c_1, T, c)\}$

Local deduction rules $P_{bc}$
- (plc-subevalat) $\text{instd}(x, b, c, t) \leftarrow \text{subEval}(a, c_1, b, c), \text{instd}(x, a, c_1, t)$.
- (plc-subevalr) $\text{tripled}(x, y, c, t) \leftarrow \text{subEval}(r, c_1, s, c), \text{tripled}(x, r, y, c_1, t)$.
- (plc-subevalatp) $\text{instd}(x, b, c, t) \leftarrow \text{subEval}(a, c_1, h, c_2), \text{instd}(x, a, c_1, t), \text{prec}(c, c_2)$.
- (plc-subevalrp) $\text{tripled}(x, s, c, t) \leftarrow \text{subEval}(r, c_1, s, c_2), \text{tripled}(x, r, y, c_1, t), \text{prec}(c, c_2)$.

Output translation $O(a, c)$
- (o-concept) $A(a) \rightarrow \{\text{instd}(a, A, c, \text{main})\}$
- (o-role) $R(a, b) \leftrightarrow \{\text{tripled}(a, R, b, c, \text{main})\}$

Table 4. Input rules $I_{cl}(S, c)$ for defeasible axioms

| Rule          | Expression |
|---------------|------------|
| (id-inst)     | $D(A(a)) \rightarrow \{\text{def_insta}(A, a, c)\}$ |
| (id-triple)   | $D(R(a, b)) \rightarrow \{\text{def_triple}(R, a, b, c)\}$ |
| (id-ninst)    | $D(\neg A(a)) \rightarrow \{\text{def_ninsta}(A, a, c)\}$ |
| (id-ntriple)  | $D(\neg R(a, b)) \rightarrow \{\text{def_ntriple}(R, a, b, c)\}$ |
| (id-subc)     | $D(A \subseteq B) \rightarrow \{\text{def_subclass}(A, B, c)\}$ |
| (id-subcnj)   | $D(A_1 \sqcap A_2 \subseteq B) \rightarrow \{\text{def_subcnj}(A_1, A_2, B, c)\}$ |
| (id-subex)    | $D(\exists R.A \subseteq B) \rightarrow \{\text{def_subex}(R, A, B, c)\}$ |
| (id-supex)    | $D(A \subseteq \exists R.\{a\}) \rightarrow \{\text{def_supex}(A, R, a, c)\}$ |
| (id-forall)   | $D(A \subseteq \forall R.B) \rightarrow \{\text{def_supforall}(A, R, B, c)\}$ |
| (id-leqone)   | $D(A \subseteq \leq 1R.T) \rightarrow \{\text{def_supleqone}(A, R, c)\}$ |
| (id-subr)     | $D(R \subseteq S) \rightarrow \{\text{def_subr}(R, S, c)\}$ |
| (id-subrc)    | $D(R \cap S \subseteq T) \rightarrow \{\text{def_subrc}(A_1, A_2, B, c)\}$ |
| (id-dis)      | $D(\text{Dis}(R, S)) \rightarrow \{\text{def_dis}(R, S, c)\}$ |
| (id-inv)      | $D(\text{Inv}(R, S)) \rightarrow \{\text{def_inv}(R, S, c)\}$ |
| (id-irr)      | $D(\text{Ir}(R)) \rightarrow \{\text{def_irr}(R, c)\}$ |
Table 5. Deduction rules $P_D$ for defeasible axioms: overriding rules

| Rule       | Description                                                                 |
|------------|-----------------------------------------------------------------------------|
| (ovr-inst) | $\text{ovr}$(insta, x, y, c1, c) $\leftarrow\text{def}_\text{insta}(x, y, c1), \text{prec}(c, c1), \text{not test}_\text{failsinst}(x, y, c)$. |
| (ovr-triple) | $\text{ovr}$(triplea, x, y, z, c1, c) $\leftarrow\text{def}_\text{triplea}(x, y, z, c1), \text{prec}(c, c1), \text{not test}_\text{failsinrel}(x, y, z, c)$. |
| (ovr-ninst) | $\text{ovr}$(ninsta, x, y, c1, c) $\leftarrow\text{def}_\text{ninsta}(x, y, c1), \text{prec}(c, c1), \text{instd}(x, z, c, \text{main})$. |
| (ovr-triple) | $\text{ovr}$(triplea, x, y, z, c1, c) $\leftarrow\text{def}_\text{triplea}(x, y, z, c1), \text{prec}(c, c1), \text{tripled}(x, y, z, c, \text{main})$. |
| (ovr-subc) | $\text{ovr}$(subClass, x, y, z, c1, c) $\leftarrow\text{def}_\text{subClass}(x, y, z, c1), \text{prec}(c, c1), \text{instd}(x, y, c, \text{main}), \text{not test}_\text{failsinrel}(x, z, c)$. |
| (ovr-cnj) | $\text{ovr}$(subConj, x, y, z, c1, c) $\leftarrow\text{def}_\text{subConj}(x, y, z, c1), \text{prec}(c, c1), \text{instd}(x, y, c, \text{main}), \text{not test}_\text{failsinrel}(x, z, c)$. |
| (ovr-subex) | $\text{ovr}$(subEx, x, y, z, c1, c) $\leftarrow\text{def}_\text{subEx}(x, y, z, c1), \text{prec}(c, c1), \text{tripled}(x, y, w, c, \text{main}), \text{not test}_\text{failsinrel}(x, z, c)$. |
| (ovr-sup) | $\text{ovr}$(supEx, x, y, z, c1, c) $\leftarrow\text{def}_\text{supEx}(x, y, z, c1), \text{prec}(c, c1), \text{not test}_\text{failsinrel}(x, y, w, c)$. |
| (ovr-forall) | $\text{ovr}$(supForall, x, y, z, r, w, c1, c) $\leftarrow\text{def}_\text{supForall}(x, y, z, r, w, c1), \text{prec}(c, c1), \text{instd}(x, z, c, \text{main}), \text{tripled}(x, y, r, c, \text{main}), \text{not test}_\text{failsinrel}(y, w, c)$. |
| (ovr-lem) | $\text{ovr}$(supLemma, x, r, s, z, c1, c) $\leftarrow\text{def}_\text{supLemma}(x, r, s, z, c1), \text{prec}(c, c1), \text{instd}(x, z, c, \text{main}), \text{tripled}(x, r, s, c, \text{main}), \text{tripled}(x, r, s, c, \text{main}), \text{not test}_\text{failsinrel}(x, z, c)$. |
| (ovr-subr) | $\text{ovr}$(subRole, x, y, z, r, c1, c) $\leftarrow\text{def}_\text{subRole}(x, y, z, r, c1), \text{prec}(c, c1), \text{tripled}(x, y, z, c, \text{main}), \text{not test}_\text{failsinrel}(x, y, c)$. |
| (ovr-subc) | $\text{ovr}$(subChains, x, y, z, r, t, c1, c) $\leftarrow\text{def}_\text{subChains}(x, y, z, r, t, c1), \text{prec}(c, c1), \text{tripled}(x, y, z, c, \text{main}), \text{tripled}(y, z, t, c, \text{main}), \text{not test}_\text{failsinrel}(x, z, c)$. |
| (ovr-dis) | $\text{ovr}$(dis, x, y, z, c1, c) $\leftarrow\text{def}_\text{dis}(x, y, z, c1), \text{prec}(c, c1), \text{tripled}(x, y, z, c, \text{main})$. |
| (ovr-inv1) | $\text{ovr}$(inv, x, y, z, c1, c) $\leftarrow\text{def}_\text{inv}(x, y, z, c1), \text{prec}(c, c1), \text{tripled}(x, y, z, c, \text{main}), \text{not test}_\text{failsinrel}(x, y, z, c)$. |
| (ovr-inv2) | $\text{ovr}$(inv, x, y, z, c1, c) $\leftarrow\text{def}_\text{inv}(x, y, z, c1), \text{prec}(c, c1), \text{tripled}(y, x, z, c, \text{main}), \text{not test}_\text{failsinrel}(x, y, z, c)$. |
Table 6. Deduction rules $P_{D}$ for defeasible axioms: inheritance rules

| Rule Type | Description |
|-----------|-------------|
| (prop-inst) | $\text{inst}(x, z, c, t) \leftarrow \text{inst}(x, z, c, t), \text{prec}(c, c_j), \text{not ovr}(\text{insta}, x, z, c_j, c)$. |
| (prop-triple) | $\text{triple}(x, r, y, c, t) \leftarrow \text{triple}(x, r, y, c, t), \text{prec}(c, c_j), \text{not ovr}(\text{triplea}, x, r, y, c, t)$. |
| (prop-ninst) | $\text{unsat}(t) \leftarrow \text{ninst}(x, z, c, t), \text{inst}(x, z, c, t)\text{, not ovr}(\text{ninsta}, x, z, c_j, c)$. |
| (prop-triple) | $\text{unsat}(t) \leftarrow \text{ntriple}(x, r, y, c, t), \text{triple}(x, r, y, c, t$, |
| (prop-subc) | $\text{inst}(x, z, c, t) \leftarrow \text{subClass}(y, z, c, t), \text{inst}(x, y, c, t)\text{, not ovr}(\text{subClass}, x, y, z, c_j, c)$. |
| (prop-cnj) | $\text{inst}(x, z, c, t) \leftarrow \text{conj}(x, y, z, c, t), \text{inst}(x, y, c, t)\text{, not ovr}(\text{conj}, x, y, z, c_j, c)$. |
| (prop-subcx) | $\text{inst}(x, z, c, t) \leftarrow \text{subEx}(v, y, z, c, t), \text{triple}(x, v, x', c, t), \text{inst}(x', y, c, t), \text{prec}(c, c_j), \text{not ovr}(\text{subEx}, x, v, y, z, c_j, c)$. |
| (prop-supcx) | $\text{triple}(x, r, y, x', c, t) \leftarrow \text{subEx}(r, t, x', c, t), \text{inst}(x, y, c, t), \text{prec}(c, c_j), \text{not ovr}(\text{supEx}, x, r, y, x', c, t)$. |
| (prop-forall) | $\text{inst}(x, y, c, t) \leftarrow \text{supForall}(x, y, c, t), \text{inst}(x, z, c, t)\text{, not ovr}(\text{supForall}, x, y, z, c_j, c)$. |
| (prop-leqone) | $\text{unsat}(t) \leftarrow \text{supLeqOne}(x, y, z, c, t), \text{inst}(x, z, c, t)\text{, not ovr}(\text{supLeqOne}, x, y, z, r, c_j, c)$. |
| (prop-subr) | $\text{triple}(x, w, x', c, t) \leftarrow \text{subRole}(v, w, c, t), \text{triple}(x, v, x', c, t), \text{prec}(c, c_j), \text{not ovr}(\text{subRole}, x, v, w, c, t)$. |
| (prop-subc) | $\text{triple}(x, w, z, c, t) \leftarrow \text{subChain}(u, v, w, c, t), \text{triple}(x, u, y, c, t)\text{, not ovr}(\text{subChain}, x, u, v, w, c, t)$. |
| (prop-dis) | $\text{unsat}(t) \leftarrow \text{dis}(u, v, c, t), \text{triple}(x, u, y, c, t)\text{, not ovr}(\text{dis}, x, y, u, v, c, t)$. |
| (prop-inv1) | $\text{triple}(y, v, x, c, t) \leftarrow \text{inv}(u, v, c, t), \text{triple}(x, u, y, c, t), \text{prec}(c, c_j), \text{not ovr}(\text{inv}, x, y, u, v, c, t)$. |
| (prop-inv2) | $\text{triple}(x, u, y, c, t) \leftarrow \text{inv}(u, v, c, t), \text{triple}(y, v, x, c, t), \text{prec}(c, c_j), \text{not ovr}(\text{inv}, x, y, u, v, c, t)$. |
| (prop-irr) | $\text{unsat}(t) \leftarrow \text{irr}(u, c, t), \text{triple}(x, u, x, c, t), \text{prec}(c, c_j), \text{not ovr}(\text{irr}, x, u, c, t)$. |
Table 7. Deduction rules $P_2$ for defeasible axioms: test rules

| (test-inst)                      | test(nlit(x, y, c)) ← def_insta(x, y, c), prec(c, c). |
| (const-inst)                    | testfails(nlit(x, y, c), over(nlit, x, y, c, c)). |
| (test-triple)                   | test(nrel(x, r, y, c)) ← def_triple(x, r, c), prec(c, c). |
| (const-triple)                  | testfails(nrel(x, r, y, c), over(triplea, x, r, y, c, c)). |
| (test-sube)                     | test(nlit(x, z, c)) ← def_subclass(y, z, c), instd(x, y, c, main), prec(c, c). |
| (const-sube)                    | testfails(nlit(x, z, c), over(subClass, x, y, z, c, c)). |
| (test-sube)                     | test(nlit(x, z, c)) ← def_subconj(y, y, z, c), instd(x, y1, c, main), instd(x, y2, c, main), prec(c, c). |
| (const-sube)                    | testfails(nlit(x, z, c), over(subConj, x, y1, y2, z, c1, c)). |
| (test-subex)                    | test(nlit(x, z, c)) ← def_subex(x, y, z, c), tripled(x, r, w, c, main), instd(w, y, c, main), prec(c, c). |
| (const-subex)                   | testfails(nlit(x, z, c), over(subEx, x, r, y, z, c1, c)). |
| (test-supex)                    | test(nrel(x, r, w, c)) ← def_supex(y, r, w, c), instd(x, y, c, main), prec(c, c). |
| (const-supex)                   | testfails(nrel(x, r, w, c), over(supEx, x, r, y, w, c, c)). |
| (test-supforall)                | test(nlit(y, w, c)) ← def_supforall(x, r, w, c), instd(x, z, c, main), tripled(x, r, y, c, main), prec(c, c). |
| (const-supforall)               | testfails(nlit(y, w, c), over(supforall, x, y, z, r, w, c, c)). |
| (test-sub)                      | test(nrel(x, x, y, c)) ← def_sub(x, x, c1), tripled(x, r, w, c, main), prec(c, c1). |
| (const-sub)                     | testfails(nrel(x, x, y, c), over(subRole, x, r, x, c1, c)). |
| (test-subc)                     | test(nrel(x, r, y, c, c)) ← def_subc(x, r, c1), tripled(x, r, y, c, main), prec(c, c1). |
| (const-subc)                    | testfails(nrel(x, r, y, c, c), over(subChain, x, y, z, r, x, c1, c)). |
| (test-inv1)                     | test(nrel(x, x, y, c)) ← def_inv(x, y, c), tripled(x, r, y, c, main), prec(c, c). |
| (test-inv2)                     | test(nrel(y, r, x, c)) ← def_inv(x, y, c), tripled(x, y, c, main), prec(c, c). |
| (const-inv1)                    | testfails(nrel(x, x, y, c), over(inv, x, y, r, x, c1, c)). |
| (const-inv2)                    | testfails(nrel(y, r, x, c), over(inv, x, y, r, c1, c)). |
| (test-fail1)                    | testfails(nlit(x, z, c)) ← instd(x, z, c, nlit(x, z, c)), not sat(nlit(x, z, c)). |
| (test-fail2)                    | testfails(nrel(x, r, y, c)) ← tripled(x, r, y, c, nrel(x, r, y, c)), not sat(nrel(x, r, y, c)). |
| (test-add1)                     | instd(x, z, c, nlit(x, z, c)) ← test(nlit(x, z, c)). |
| (test-add2)                     | tripled(x, r, y, c, nrel(x, r, y, c)) ← nrel(x, r, y, c). |
| (test-copy1)                    | instd(x1, y, c, t) ← instd(x1, y, c, main), test(t). |
| (test-copy2)                    | tripled(x1, r, y, c, t) ← tripled(x1, r, y, c, main), test(t). |
Table 8. Deduction rules $P_D$ for defeasible axioms: preference rules

| Rule | Expression |
|------|------------|
| $(pref\text{-inst})$ | $\text{ovrLevel\_Inst}(x, y, c, n) \leftarrow \text{ovr\_Inst}(x, y, c, n)$, $[1 : n]$ |
| $(wconst\text{-inst})$ | $\text{ovrLevel\_Inst}(x, y, c, n) \leftarrow \text{ovr\_Inst}(x, y, c, n)$, $[1 : n]$ |
| $(pref\text{-triple})$ | $\text{ovrLevel\_Triple}(x, y, c, n) \leftarrow \text{ovr\_Triple}(x, y, c, n)$, $[1 : n]$ |
| $(wconst\text{-triple})$ | $\text{ovrLevel\_Triple}(x, y, c, n) \leftarrow \text{ovr\_Triple}(x, y, c, n)$, $[1 : n]$ |
| $(pref\text{-ntriple})$ | $\text{ovrLevel\_nTriple}(x, y, c, n) \leftarrow \text{ovr\_nTriple}(x, y, c, n)$, $[1 : n]$ |
| $(wconst\text{-ntriple})$ | $\text{ovrLevel\_nTriple}(x, y, c, n) \leftarrow \text{ovr\_nTriple}(x, y, c, n)$, $[1 : n]$ |
| $(pref\text{-sub})$ | $\text{ovrLevel\_SubClass}(x, y, z, c, n) \leftarrow \text{ovr\_SubClass}(x, y, z, c, n)$, $[1 : n]$ |
| $(wconst\text{-sub})$ | $\text{ovrLevel\_SubClass}(x, y, z, c, n) \leftarrow \text{ovr\_SubClass}(x, y, z, c, n)$, $[1 : n]$ |
| $(pref\text{-conj})$ | $\text{ovrLevel\_Conj}(x, y, c, n) \leftarrow \text{ovr\_Conj}(x, y, c, n)$, $[1 : n]$ |
| $(wconst\text{-conj})$ | $\text{ovrLevel\_Conj}(x, y, c, n) \leftarrow \text{ovr\_Conj}(x, y, c, n)$, $[1 : n]$ |
| $(pref\text{-sub}), (wconst\text{-sub})$ | $\text{ovrLevel\_SubEx}(x, y, z, c, n) \leftarrow \text{ovr\_SubEx}(x, y, z, c, n)$, $[1 : n]$ |
| $(pref\text{-sup})$, $(wconst\text{-sup})$ | $\text{ovrLevel\_SubEx}(x, y, z, c, n) \leftarrow \text{ovr\_SubEx}(x, y, z, c, n)$, $[1 : n]$ |
| $(pref\text{-forall})$ | $\text{ovrLevel\_SubAll}(x, y, z, c, n) \leftarrow \text{ovr\_SubAll}(x, y, z, c, n)$, $[1 : n]$ |
| $(wconst\text{-forall})$ | $\text{ovrLevel\_SubAll}(x, y, z, c, n) \leftarrow \text{ovr\_SubAll}(x, y, z, c, n)$, $[1 : n]$ |
| $(pref\text{-leqone})$, $(wconst\text{-leqone})$ | $\text{ovrLevel\_leqOne}(x, y, z, c, n) \leftarrow \text{ovr\_leqOne}(x, y, z, c, n)$, $[1 : n]$ |
| $(pref\text{-sub}), (wconst\text{-sub})$ | $\text{ovrLevel\_SubRole}(x, y, z, c, n) \leftarrow \text{ovr\_SubRole}(x, y, z, c, n)$, $[1 : n]$ |
| $(pref\text{-sub}), (wconst\text{-sub})$ | $\text{ovrLevel\_SubRole}(x, y, z, c, n) \leftarrow \text{ovr\_SubRole}(x, y, z, c, n)$, $[1 : n]$ |
| $(pref\text{-dis})$, $(wconst\text{-dis})$ | $\text{ovrLevel\_Dis}(x, y, z, c, n) \leftarrow \text{ovr\_Dis}(x, y, z, c, n)$, $[1 : n]$ |
| $(pref\text{-inv})$, $(wconst\text{-inv})$ | $\text{ovrLevel\_Inv}(x, y, z, c, n) \leftarrow \text{ovr\_Inv}(x, y, z, c, n)$, $[1 : n]$ |
| $(pref\text{-irr})$, $(wconst\text{-irr})$ | $\text{ovrLevel\_Irr}(x, y, z, c, n) \leftarrow \text{ovr\_Irr}(x, y, z, c, n)$, $[1 : n]$ |
4 Translation correctness: more details

Given a CAS-interpretation $\hat{\mathcal{I}}_{\text{CAS}} = (\mathcal{I}, \chi)$, (similarly to the CKR case in [1]) we can build from its components a corresponding Herbrand interpretation $I(\mathcal{I}_{\text{CAS}})$ of the program $PK(\mathcal{R})$ as the smallest set of literals containing:

- all facts of $PK(\mathcal{R})$;
- $\text{insta}(a, A, c, \text{main})$, if $I(c) \models A(a)$;
- $\text{triplied}(a, R, b, c, \text{main})$, if $I(c) \models R(a, b)$;
- each ovr-literal from $OVR(\mathcal{I}_{\text{CAS}})$;
- each literal $l$ with environment $l \neq \text{main}$, if $\text{test}(t) \in I(\mathcal{I}_{\text{CAS}})$ and $l$ is in the head of a rule $r \in \text{grnd}(PK(\mathcal{R}))$ with $\text{Body}(r) \subseteq I(\mathcal{I}_{\text{CAS}})$;
- $\text{test}(t)$, if $\text{test}\_\text{fails}(t)$ appears in the body of an overriding rule $r$ in $\text{grnd}(PK(\mathcal{R}))$ and the head of $r$ is an ovr literal in $OVR(\mathcal{I}_{\text{CAS}})$;
- $\text{unsat}(t) \in I(\mathcal{I}_{\text{CAS}})$, if adding the literal corresponding to $t$ to the local interpretation of its context $c$ violates some axiom of the local knowledge $K_c$;
- $\text{test}\_\text{fails}(t)$, if $\text{unsat}(t) \notin I(\mathcal{I}_{\text{CAS}})$.
- $\text{ovrlevel}(p(e), n)$, if the corresponding ovr-literal appears in $OVR(\mathcal{I}_{\text{CAS}})$ with $a$ in context $c$ and $\text{level}(c, n) \in PK(\mathcal{R})$.

Note that $\text{unsat}(\text{main})$ is not included in $I(\mathcal{I}_{\text{CAS}})$.

Lemma 1. Let $\mathcal{R}$ be a sCKR in $\SigmaROIQ$-RLD normal form, then:

(i). for every (named) justified clashing assumption $\chi$, the interpretation $S = I(\tilde{\mathcal{I}}(\chi))$ is an answer set of $PK(\mathcal{R})$;
(ii). every answer set $S$ of $PK(\mathcal{R})$ is of the form $S = I(\tilde{\mathcal{I}}(\chi))$ with $\chi$ a (named) justified clashing assumption for $\mathcal{R}$.

Proof (Sketch). Intuitively, as we are interested in computing the correspondence with (not necessarily optimal) answer sets of $PK(\mathcal{R})$ (namely, of the rules part of the program, not including weak constraints), the newly added weak constraints rules in $P_D$ do not influence the construction of such answer sets and the result can be proved along the lines of Lemma 6 in [1].

Let us consider $S = I(\tilde{\mathcal{I}}(\chi))$ defined above and the reduct $G_S(PK(\mathcal{R}))$ of $PK(\mathcal{R})$ with respect $S$. Note that the NAF literals in $PK(\mathcal{R})$ considered in computing such reduct involve instances of ovr, test\_fails and unsat. We can then proceed to prove the lemma by showing that the answer sets of $PK(\mathcal{R})$ coincide with the sets $S = I(\tilde{\mathcal{I}}(\chi))$ where $\chi$ is a justified clashing assumption of $\mathcal{R}$.

(i). Assuming that $\chi$ is a justified clashing assumption, we show that $S = I(\tilde{\mathcal{I}}(\chi))$ is an answer set of $PK(\mathcal{R})$. We first that $S \models G_S(PK(\mathcal{R}))$, that is for every rule instance $r \in G_S(PK(\mathcal{R}))$ it holds that $S \models r$. We can prove this by examining the possible rule forms that occur in $G_S(PK(\mathcal{R}))$. Here we show some representative cases (see also [1]):

- (prl\_insta): then $\text{insta}(a, A, c, t) \in I(\tilde{\mathcal{I}}(\chi))$ and, by definition of the translation, $A(a) \in K_c$ (as $t$ can only be main). This implies that $I(c) \models A(a)$ and thus $\text{insta}(a, A, c, \text{main})$ is added to $I(\tilde{\mathcal{I}}(\chi))$. 

– (prl-subc): then \( \{ \text{subClass}(A, B, c), \text{instd}(a, A, c, t) \} \subseteq I(\tilde{\chi}(\varphi)) \). By definition of the translation we have \( A \subseteq B \in K_c \). For the construction of \( I(\tilde{\chi}(\varphi)) \), if \( t = \text{main} \) then \( I(c) \models A(a) \). This implies that \( I(c) \models B(a) \) and \( \text{instd}(a, B, c, t) \) is added to \( I(\tilde{\chi}(\varphi)) \). Otherwise, if \( t \neq \text{main} \) then \( \text{instd}(a, B, c, t) \) is directly added to \( I(\tilde{\chi}(\varphi)) \) by its construction.

– (pic-evalat): then \( \{ \text{subEval}(A, c_1, B, c), \text{instd}(a, A, c_1, t) \} \subseteq I(\tilde{\chi}(\varphi)) \). Thus we have that \( \text{eval}(A, c_1) \subseteq B \in K_c \). For the construction of \( I(\tilde{\chi}(\varphi)) \), if \( t = \text{main} \) then \( I(c_1) \models A(a) \); This implies that \( I(c) \models B(a) \) and \( \text{instd}(a, B, c, t) \) is added to \( I(\tilde{\chi}(\varphi)) \). Otherwise, if \( t \neq \text{main} \) then \( \text{instd}(a, B, c, t) \) is directly added to \( I(\tilde{\chi}(\varphi)) \) by its construction.

– (pref-subc): then \( \{ \text{level}(c_1, n), \text{ovr}(\text{subClass}, a, A, B, c_1, c) \} \subseteq I(\tilde{\chi}(\varphi)) \). That is, \( \text{ovr}(\text{subClass}, a, A, B, c_1, c) \) appears in \( \text{ovr}(\tilde{\chi}(\varphi)) \): \( \text{ovrlevel}_{\text{subClass}}(a, A, B, c, n) \) is then added to \( I(\tilde{\chi}(\varphi)) \) by its construction.

Minimality of \( S = I(\tilde{\chi}(\varphi)) \) w.r.t. the (positive) deduction rules of \( G_\varphi(\text{PK}(\mathbb{R})) \) can then be motivated as in the original proof in [1]: thus, \( I(\tilde{\chi}(\varphi)) \) is an answer set of \( \text{PK}(\mathbb{R}) \).

(ii). Let \( S \) be an answer set of \( \text{PK}(\mathbb{R}) \). We show that there is some justified clashing assumption \( \chi \) for \( \mathcal{K} \) such that \( S = I(\tilde{\chi}(\varphi)) \) holds.

Note that as \( S \) is an answer set for the CKR program, all literals on \( \text{ovr} \) and \( \text{testfails} \) in \( S \) are derivable from the reduct \( G_\varphi(\text{PK}(\mathbb{R})) \). By the definition of \( I(\tilde{\chi}(\varphi)) \) we can easily build a model \( \mathfrak{3}_S = \langle \mathfrak{3}_S, \chi_S \rangle \) from the answer set \( S \) as follows: for every \( c \in \mathbb{N} \), we build the local interpretation \( I_S(c) = \langle A_c, \chi_S \rangle \) as follows:

\[- A_c = \{ d | d \in \text{NI} \};
- A_c^I = a, \text{ for every } a \in \text{NI};
- R_c^I = \{ (d, d') \in A_c \times A_c | S \models \text{triple}(d, R, d', c, \text{main}) \} \text{ for } R \in \text{NR};\]

Finally, \( N_S(c) = \{ (a, e) | I_\varphi(a, c') = p, \text{ovr}(p(e), c) \in S \} \). We have to show that \( \mathfrak{3}_S \) meets the definition of a least justified CAS-model for \( \mathcal{K} \), that is:

(i) for every \( a \in K_c \) (strict axiom), and \( c' \leq c \), \( I_S(c') \models a \);
(ii) for every \( D(a) \in K_c \) and \( c' < c \), if \( d \notin \{ e | (a, e) \in \chi(c') \} \), then \( I_S(c') \models \phi_d(d) \).

Condition (i) should be proved in the local case where \( c' = c \) and in the "propagating" case where \( c' < c \). The second case can be shown as a special case of (ii), where overridding to strict axiom is never applicable. Thus, considering \( c' = c \), we verify the condition by showing that, for every \( K_c \), we have \( I(c) \models K_m \). This can be shown by cases considering the form of all of the axioms \( \beta \in \mathcal{L}_c, \mathbb{N} \) that can occur in \( K_c \). For example (the other cases are similar):

- Let \( \beta = A(a) \in K_c \), then, by rule (prl-instd), \( S \models \text{instd}(a, A, c, \text{main}) \). This directly implies that \( a^I \models A^I(c) \).
- Let \( \beta = A \subseteq B \in K_c \), then \( S \models \text{subClass}(A, B, c) \). If \( d \in A^I(c) \), then by definition \( S \models \text{instd}(d, A, c, \text{main}) \): by rule (prl-subc) we obtain that \( S \models \text{instd}(d, B, c, \text{main}) \) and thus \( d \in B^I(c) \).
- Let \( \beta = \text{eval}(A, c_1) \subseteq B \in K_c \), then \( S \models \text{subEval}(A, c_1, B, c) \). If \( d \in A^I(c_1) \), then by definition \( S \models \text{instd}(d, A, c_1, \text{main}) \) and \( S \models \text{instd}(c', C, \text{main}) \). By rule (pic-evalat) we obtain that \( S \models \text{instd}(d, B, c, \text{main}) \): hence, by definition \( d \in B^I(c) \).
To prove condition (ii), let us assume that $D(\beta) \in K'_i$ with $c < c'$. We can proceed by cases on the possible forms of $\beta$ as in the original proof in [1], by considering the propagation along the coverage relation. For example:

- Let $\beta = A(a)$. Then, by definition of the translation, we have that $S \models \text{insta}(a, A, c', \text{main})$. Suppose that $(A(x), a) \notin \chi_S(c)$. Then by definition, $\text{ovr}(\text{insta}, a, A, c', c) \notin \text{OVR}(\hat{\chi}(\epsilon))$. Note that we have $S \models \text{prec}(c, c')$ by construction. By the definition of the reduction, the corresponding instantiation of rule (prop-inst) has not been removed from $G_S(PK(\hat{\epsilon}))$; this implies that $S \models \text{insta}(a, A, c, \text{main})$. By definition, this means that $a^{\text{inst}}(a) \in A^{\text{inst}}$.

- Let $\beta = A \subseteq B$. Then, by definition of the translation, we have that $S \models \text{subClass}(A, B, c')$. As above, we also have $S \models \text{prec}(c, c')$. Let us suppose that $b^{\text{inst}}(c) \in A^{\text{inst}}$: then $S \models \text{insta}(b, A, c, \text{main})$. Suppose that $(A \subseteq B, b) \notin \chi_S(c)$; by definition, $\text{ovr}(\text{subClass}, b, A, c', c) \notin \text{OVR}(\hat{\chi}(\epsilon))$. By the definition of the reduction, the corresponding instantiation of rule (prop-subc) has not been removed from $G_S(PK(\hat{\epsilon}))$: this implies that $S \models \text{insta}(b, B, c, \text{main})$. Thus, by definition, this means that $b^{\text{inst}}(c) \in B^{\text{inst}}$.

We have shown that $\mathcal{S}$ is a CAS-model of $\mathcal{R}$: using the same reasoning in the original proof in [1] we can also prove the $\mathcal{S}$ corresponds to the least model and that $\chi_S$ is justified, thus proving the result.

**Lemma 2.** Let $\mathcal{R}$ be a sCKR in SROIQ-RLD normal form with ranked context hierarchy. Then, $\mathcal{S}$ is a CKR model of $\mathcal{R}$ if and only if there exists a (named) justified clashing assumption $\chi$ s.t. $I(\hat{\mathcal{S}}(\chi))$ is an optimal answer set of $PK(\mathcal{R})$.

**Proof (Sketch).** To prove the result, we have to show that, $\hat{\mathcal{S}}$ is a CKR model iff:

- (i) there exists a (named) justified clashing assumption $\chi$ s.t. $I(\hat{\mathcal{S}}(\chi))$ is an answer set of $PK(\mathcal{R})$.
- (ii) $I(\hat{\mathcal{S}}(\chi))$ is an optimal answer set of $PK(\mathcal{R})$.

Condition (i) is directly derived from Lemma [1] and the definition of CKR model in Definition 10.

To prove (ii), we have to show the correspondence of the lexicographic order on global profiles $p(\chi)$ with the order induced by objective function $H^{PK(\mathcal{R})}(S)$ on answer sets. That is, $I(\hat{\mathcal{S}}(\chi))$ is optimal iff there does not exist a justified $\chi'$ s.t. $p(\chi') < p(\chi)$.

First of all, we note that weak constraints are only associated to instances of overridings (i.e. ovr atoms): thus the optimization of the answer sets is only dependent on minimization of aspects related to such atoms (which, on the other hand, are related to the clashing assumptions in $\chi$).

Suppose that $\chi$ is preferred, that is there does not exist a justified $\chi'$ s.t. $p(\chi') < p(\chi)$. Thus, for every such $\chi'$ we have $p(\chi') > p(\chi)$. By the definition of lexicographic order on profiles, this means that if $p(\chi) = (l_0, \ldots, l_0)$ and $p(\chi') = (l'_0, \ldots, l'_0)$ some $j \in \{0, \ldots, n\}$ exists such that $l_n = l'_n, \ldots, l_{n-1} = l'_{n-1}, l_{n-1} = l'_{n-1}, \ldots, l_j = l'_j$. This means that there exist at least an "additional" $(\alpha, e) \in \chi'(c)$ for a context $c$ such that $(\alpha) = j$. That is, either all elements in $\chi$ have level smaller than $j$ or $\chi'$ has more elements at the level $j$. Considering then the interpretation $S' = I(\hat{\mathcal{S}}(\chi'))$, can show that it necessarily
has an higher cost with respect $S = I(\hat{\chi}(c))$. Since $(a, e) \in \chi'(c)$, by construction of $S'$ we have that the corresponding $ovr(p(e)) \in S'$ and $ovrlevel(p(e), j) \in S'$: this causes the instantiation of the weak constraint rule in relative to $ovrlevel(p(e), j)$, which adds a weak constraint violation to $S'$ at level $j$ and with cost 1. Considering the definition of the optimization function $H^{PK(3)}$ from [4]:

- if the violation in $S'$ is at a level bigger than all of the violations in $S$, the level function $f_{PK(3)}(j)$ in the definition of $H^{PK(3)}$ is assured to add an higher cost than all of the lower levels $f_{PK(3)}(i)$;
- if the violation in $S'$ is at the same level of the (higher) violation in $S$, then the additional cost 1 of the violation assures that level cost of $j$ in $S'$ is bigger than in $S$.

Thus, we have that in both case $H^{PK(3)}(S') > H^{PK(3)}(S)$. This shows the optimality of $I(\hat{\chi}(\chi))$.

The other direction can be shown similarly: supposing that $S = I(\hat{\chi}(\chi))$ is optimal, then for all other $S' = I(\hat{\chi}(\chi'))$ we have $H^{PK(3)}(S') > H^{PK(3)}(S)$. Thus, by the definition of the function, we have that there exists at least a violation on a $ovr(p(e))$ with higher level or higher level cost at a level $j$. Considering the corresponding clashing assumption sets, we can analogously map back to the definition of lexicographic ordering on profiles, obtaining that $p(\chi') > p(\chi)$. Thus, $\chi$ is preferred and we proved the result.
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