TETRAHEDRON EQUATIONS, BOUNDARY STATES AND HIDDEN STRUCTURE OF $U_q(D_n^{(1)})$

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Abstract. Simple periodic $3d \to 2d$ compactification of the tetrahedron equations gives the Yang-Baxter equations for various evaluation representations of $U_q(sl_n)$. In this paper we construct an example of fixed non-periodic $3d$ boundary conditions producing a set of Yang-Baxter equations for $U_q(D_n^{(1)})$. These boundary conditions resemble a fusion in hidden direction.

The tetrahedron equation can be viewed as a local condition providing existence of an infinite series of Yang-Baxter equations. In the applications to quantum groups the method of tetrahedron equation is a powerful tool for generation of $R$-matrices and $L$-operators for various “higher spin” evaluation representations. This has been demonstrated in [1] for $U_q(\hat{sl}_n)$ and in [4] for super-algebras $U_q(\hat{gl}_{n|m})$.

The main principle producing the cyclic $\hat{sl}_n$ structure is the trace in hidden “third” direction. In this paper we introduce another boundary condition, a certain boundary states still providing the existence of effective Yang-Baxter equation and integrability.

We shall start with a short remained of a (super-)tetrahedron equation and $\hat{sl}_n$ compactification in their elementary form. The simplest known tetrahedron equation in the tensor product of six spaces $B_1 \otimes F_2 \otimes \cdots \otimes F_5 \otimes B_6$ is

$$R_{B_1F_2F_3}R_{B_1F_4F_5}R_{F_2F_4B_6}R_{F_3F_5B_6} = R_{F_3F_5B_6}R_{F_2F_4B_6}R_{B_1F_4F_5}R_{B_1F_2F_3},$$

where $F_i = \{|0\rangle, |1\rangle\}_i$ is a representation space of Fermi oscillator

$$f^+|0\rangle = |1\rangle, \quad f^-|1\rangle = |0\rangle.$$

Odd operators $f^\pm$ in different components $i$ of their tensor product anti-commute and $(f^\pm_i)^2 = 0$. It is convenient to introduce projectors

$$M_i = f^+_i f^-_i, \quad M^0_i = f^-_i f^+_i, \quad [f^+_i, f^-_i]_+ = M^0_i + M_i = 1.$$

Operator $M^0_i$ is the projector to vacuum, $M_i$ is the occupation number and $M^0_i M = 0$.

Space $B_i$ stands for representation space of $i$-th copy of $q$-oscillator,

$$b^+ b^- = 1 - q^{2N}, \quad b^- b^+ = 1 - q^{2N+2}, \quad q^N b^\pm = b^\pm q^{N\pm1}.$$
In this paper we imply the unitary Fock space representation, \((b^-)^\dagger = b^+\), defined by

\[ N|n\rangle = |n\rangle n, \quad b^-|0\rangle = 0, \quad |n\rangle = \frac{b^+}{\sqrt{q^2;q^2}}|0\rangle, \quad n \geq 0, \]

where \((x; q^2)_n = (1 - x)(1 - q^2x) \cdots (1 - q^{2n-2}x)\). In terms of creation, annihilation and occupation number operators the R-matrices in \((1)\) are given \([4]\) by

\[ R_{B_1 F_2 F_3} = M_2^0 M_3^0 - q^{N_1 + 1} M_2^0 M_3^0 + q^{N_1} M_2^0 M_3 - M_2 M_3 + b_1 f_3^+ f_3^f - b_2 f_2^+ f_2^f \]

and

\[ R_{F_1 F_2 B_3} = M_1^0 M_2^0 + M_1 M_2^0 q^{N_1 + 1} - M_1^0 M_2^0 q^{N_1} - M_2 M_3 + f_1 f_2^+ b_3^f - f_2 f_2^f b_3^f \]

Both operators \(R\) are unitary roots of unity. The constant tetrahedron equation \((1)\) can be verified in the operator language straightforwardly.

Define next the “monodromy” of \(R\)-matrices as the ordered product

\[ R_{\Delta_n(B_1 F_2), F_3} = R_{B_1, F_2, F_3} R_{B_2, F_2, F_3} \cdots R_{B_n, F_2, F_3} \rightleftharpoons \prod_{j=1}^{n} R_{B_{1,j}, F_{2,j}, F_3} . \]

Here the convenient “co-product” notation stands for a tensor power of corresponding spaces,

\[ \Delta_n(B_1) = \bigotimes_{j=1}^{n} B_{1,j}, \quad \Delta_n(F_2) = \bigotimes_{j=1}^{n} F_{2,j} . \]

The repeated use of \((1)\) provides

\[ R_{\Delta_n(B_1 F_2), F_3} R_{\Delta_n(B_1 F_4), F_5} R_{\Delta_n(F_2 F_4), B_6} R_{\Delta_n(F_2 F_4), B_6} = R_{F_3, F_5, B_6} R_{\Delta_n(F_2 F_4), B_6} R_{\Delta_n(B_1 F_4), F_5} R_{\Delta_n(B_1 F_4), F_5} . \]

Note the conservation laws:

\[ v^{-M_1}u^{-M_5} \left( \frac{u}{v} \right)^N_6 R_{F_3, F_5, B_6} = R_{F_3, F_5, B_6} v^{-M_1}u^{-M_5} \left( \frac{u}{v} \right)^N_6 . \]

Multiplying \((10)\) by the \(u, v\)-term in \(F_3 \otimes F_5 \otimes B_6\) and by \(R_{F_3, F_5, B_6}^{-1}\), and making then the traces over \(F_3 \otimes F_5 \otimes B_6\), we come to the Yang-Baxter equation

\[ L_{\Delta_n(B_1 F_2)}(v)L_{\Delta_n(B_1 F_4)}(u)R_{\Delta_n(F_2 F_4)}(u/v) = R_{\Delta_n(F_2 F_4)}(u/v)L_{\Delta_n(B_1 F_4)}(u)L_{\Delta_n(B_1 F_2)}(v) , \]

where

\[ L_{\Delta_n(B_1 F_2)}(v) = \text{Str} \left( v^{-M_3} R_{\Delta_n(B_1 F_2), F_3} \right), \quad R_{\Delta_n(F_2 F_4)}(w) = \text{Tr} \left( w^{N_6} R_{\Delta_n(F_2 F_4), B_6} \right) . \]

This is the case of \(q(\varphi_n)\). Two-dimensional \(R\)-matrices \((13)\) have the centers

\[ J_i = \sum_{j=1}^{n} M_{i,j} \] for fermions and \[ J_1 = \sum_{j=1}^{n} N_{1,j} \] for bosons.
Irreducible components of $R$-matrices and $L$-operators \([13]\) correspond to fixed values of $J_i$. In particular, $\Delta_n(F)$ is the sum of all antisymmetric tensor representations of $sl_n$,

\[
\text{dim } \Delta_n(F) = 2^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!}.
\]

The Dirac spinor representation of $D_n$ has the same dimension $2^n$, it is the direct sum of two irreducible Weyl spinors with dimensions $2^{n-1}$. It is evident intuitively, the structure of $D_n$ will appear if the total occupation number $J$ of $\Delta_n(F)$ is not a center of $L$-operators and $R$-matrices, but all operators preserve just the parity of $J$. Also, since the dimension of vector representation of $D_n$ is $2n$, we need to double the number of bosons.

Consider now two copies of \([11]\) and further of \([11]\) glued in the “second” direction. This consideration keeps the desired space $\Delta_n(F)$ and doubles the number of bosons. The repeated use of \([11]\) provides

\[
R_{\Delta(B_1)F_2F_3} R_{\Delta(B_1)F_4F_3} R_{F_2F_3B_6} R_{\Delta'(F_3F_5)B_6}
= R_{\Delta'(F_3F_5)B_6} R_{F_2F_3B_6} R_{\Delta(B_1)F_1\Delta(F_3)} R_{\Delta(B_1)F_2\Delta(F_3)},
\]

where

\[
R_{\Delta(B_1)F_2F_3} = R_{B_1F_2F_3} R_{B_1'F_2'F_3'} \quad \text{and} \quad R_{\Delta'(F_3F_5)B_6} = R_{F_3F_5B_6} R_{F_3'F_5'B_6}.
\]

The key observation is the existence of a family of eigenvectors of operator $R_{\Delta'(F_3F_5)B_6}$:

\[
R_{\Delta'(F_3F_5)B_6} \langle \psi_{\Delta(F_3)}(v) \psi_{\Delta(F_5)}(u) \psi_{B_6}(u/v) \rangle = \langle \psi_{\Delta(F_3)}(v) \psi_{\Delta(F_5)}(u) \psi_{B_6}(u/v) \rangle,
\]

where

\[
\Delta(F) = F' \otimes F, \quad \langle \psi_{\Delta(F)}(v) \rangle = (1 + v^{-1}f^{++}f^{+})|0\rangle,
\]

and in the unitary basis \([3]\)

\[
\langle 2k + 1| \psi_B(w) \rangle = 0, \quad \langle 2k| \psi_B(w) \rangle = w^k \sqrt{\frac{(q^{4k+4}; q^4)_{\infty}}{(q^{4k+2}; q^4)_{\infty}}}.
\]

The normalization of $\psi_B$ is given by

\[
\langle \overline{\psi_B}(w) | (b^\pm)^{2m} | \psi_B(w) \rangle = w^m \frac{(q^{2+4m}; q^4)_{\infty}}{(w^2; q^4)_{\infty}}.
\]

Considering now a length-$n$ chain of \([16]\) in the “third” direction and applying vectors $\psi_{\Delta(F_3)}(u)$, $\psi_{\Delta(F_5)}(v)$ and $\psi_B(u/v)$, we come to the Yang-Baxter equation

\[
L_{\Delta_n(\Delta(B_1)F_2)}(v) L_{\Delta_n(\Delta(B_1)F_4)}(u) R_{\Delta_n(F_2F_4)}(u/v)
= R_{\Delta_n(F_2F_4)}(u/v) L_{\Delta_n(\Delta(B_1)F_4)}(u) L_{\Delta_n(\Delta(B_1)F_2)}(v),
\]

without trace construction:

\[
L_{\Delta_n(\Delta(B_1)F_2)}(v) = \langle \overline{\psi_{\Delta(F_3)}}(v) R_{\Delta_n(\Delta(B_1)F_2), \Delta(F_3)} | \psi_{\Delta(F_3)}(v) \rangle.
\]
and
\begin{equation}
R_{\Delta_n(F_2 F_4)}(w) = \langle \overline{\psi}_{B_6}(w) | R_{\Delta_n(F_2 F_4), B_6} | \psi_{B_6}(w) \rangle.
\end{equation}

Matrix elements of $R_{\Delta_n(F_2 F_4)}(w)$ can be calculated with the help of (21) and similar identities. The invariants of $L$-operator (23) and $R$-matrix (24) are:

The parity of $J_2 = \sum M_{2,j}$, similar parity of $J_4$ and

\begin{equation}
J_1 = \sum_{j=1}^{n} (N_{1,j} - N'_{1,j}).
\end{equation}

A choice of different spectral parameters in bra- and ket-vectors in (23,24) is equivalent to the choice of equal spectral parameters by means of a gauge transformation.

The structure of $D_n$ representation ring can be verified explicitly by a direct calculation of matrix elements of $R$-matrix (24) for small $n$ and check of factor powers of $\det(\lambda - R)$.

As to $2n$-bosons space, irreducible components of $\Delta_n(\Delta(B_1))$ are in general infinite dimensional. However, a choice of Fock and anti-Fock space representations, $\text{Spectrum}(N_{1,j}) = 0, 1, 2, \ldots$ and $\text{Spectrum}(N'_{1,j}) = -1, -2, -3, \ldots$, makes $\Delta_n(\Delta(B_1))$ a direct sum of symmetric tensors of $O(2n)$.

The main result of this paper is a step forward to a classification of integrable boundary conditions in three-dimensional models. At least two scenarios are hitherto known: quasi-periodic boundary condition (13) and the boundary states condition (23,24). These conditions can be imposed for a layer-to-layer transfer matrix in different directions independently. In both scenarios the spectral parameters of effective two-dimensional models reside the boundary. Also, the boundary admits twists making the quantum groups classification inapplicable [3]. It worth noting one more possible scenario of integrable boundary conditions: yet unknown 3d reflection operators satisfying the tetrahedron reflection equations [2].

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**References**

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