Determinants of some Hessenberg matrices with generating functions

Abstract: In this paper, we derive some relationships between the determinants of some special lower Hessenberg matrices whose entries are the terms of certain sequences and the generating functions of these sequences. Moreover, our results are generalizations of the earlier results from previous researches. Furthermore, interesting examples of the determinants of some special lower Hessenberg matrices are presented.

Keywords: determinant, Hessenberg matrix, generating function

MSC 2020: 15A15, 15B05, 11C20

1 Introduction

Hessenberg matrices play an important role in both computational and applied mathematics (see [1–5]). For examples, Hessenberg matrix decomposition is the important key of computing the eigenvalue matrix [4] and the rule of the Hessenberg matrix for computing the determinant of general centrosymmetric matrix [5].

A lower Hessenberg matrix \( A_n = (a_{ij}) \) is an \( n \times n \) matrix whose entries above the superdiagonal are all zero but the matrix is not lower triangular, that is, \( a_{ij} = 0 \) for all \( j > i + 1 \),

\[
A_n = \begin{pmatrix}
  a_{00} & a_{01} & 0 & \cdots & 0 \\
  a_{10} & a_{11} & a_{12} & \cdots & 0 \\
  a_{20} & a_{21} & a_{22} & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  a_{n-2,0} & a_{n-2,1} & a_{n-2,2} & \cdots & a_{n-2,n-1} \\
  a_{n-1,0} & a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n-1}
\end{pmatrix}.
\]

Similarly, the \( n \times n \) upper Hessenberg matrix is considered as transpose of the lower Hessenberg matrix \( A_n \). Throughout this paper, we are interested in a lower Hessenberg matrix so in fact our results will be also valid for an upper Hessenberg matrix.

Getu [6] computed determinants of a class of Hessenberg matrices by using a generating function method. The author considered an infinite matrix with 1s in the super diagonal
Then the author showed that if the following equation holds

\[ A(x) = \frac{B(x)}{1 + C(x)} = \sum_{n=0}^{\infty} a_n x^n, \]

then \( a_n = (-1)^n \det D_n \), where \( B(x) = \sum_{n=0}^{\infty} b_n x^n \) and \( C(x) = \sum_{n=0}^{\infty} c_n x^n \) are the generating functions.

Janjic [7] considered a particular case of upper Hessenberg matrices, in which all subdiagonal elements are \(-1\) and showed its relationship with a generalization of the Fibonacci numbers.

Merca [8] showed that determinant of an \( n \times n \) Toeplitz-Hessenberg matrix is expressed as a sum over the integer partitions of \( n \) by using generating function method.

Ramirez [9] derived some relations between the generalized Fibonacci-Narayana sequences, and permanents and determinants of one type of upper Hessenberg matrix.

In 2017, Kilic and Arikan [10] obtained the relationships between determinants of three classes of Hessenberg matrices whose entries are terms of certain sequences, and the generating functions of these sequences.

In this paper, we use the generating function method to determine the relationships between determinants of some special lower Hessenberg matrices whose entries are terms of certain sequences, and generating functions of these sequences. Moreover, we also find interesting examples of determinants of such lower Hesseberg matrices.

2 Main results

**Theorem 2.1.** Let \( \{b_n\}_{n \geq 0} \) be any sequence of complex numbers, and for each \( i = 1, 2, \ldots \), let \( \{c_{ni}\}_{n \geq 0} \) be any sequences of complex numbers where \( c_{0i} \neq 0 \). Suppose that \( B(x) = \sum_{n=0}^{\infty} b_n x^n \) and \( C(x) = \sum_{n=0}^{\infty} c_n x^n \), for all \( i = 1, 2, \ldots, \) are the generating functions for the sequences \( \{b_n\}_{n \geq 0} \) and \( \{c_{ni}\}_{n \geq 0} \), respectively. Then there exists a generating function \( A(x) = \sum_{n=0}^{\infty} a_n x^n \) such that

\[ \sum_{n=0}^{\infty} a_n x^n c_{ni}(x) = B(x). \]

Moreover, the coefficient

\[ a_n = \frac{(-1)^n \det(H_{n+1})}{\prod_{i=1}^{n+1} c_{0i}}, \]

where \( H_{n+1} \) is the \( (n+1) \times (n+1) \) lower Hessenberg matrix defined as follows:

\[
H_{n+1} = \begin{bmatrix}
    b_0 & c_{01} & 0 & 0 & \cdots & 0 \\
    b_1 & c_{11} & c_{02} & 0 & \cdots & 0 \\
    b_2 & c_{21} & c_{12} & c_{03} & \cdots & 0 \\
    \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
    b_{n-1} & c_{n-1,1} & c_{n-2,2} & c_{n-3,3} & \cdots & c_{0,n} \\
    b_n & c_{n,1} & c_{n-1,2} & c_{n-2,3} & \cdots & c_{1,n}
\end{bmatrix}
\]

for all nonnegative integers \( n \).
**Proof.** For each nonnegative integer \( n \), we consider the linear system of equations

\[
\begin{bmatrix}
c_{01} & 0 & 0 & \ldots & 0 & 0 \\
c_{11} & c_{02} & 0 & \ldots & 0 & 0 \\
c_{21} & c_{12} & c_{03} & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
c_{n-1,1} & c_{n-2,2} & c_{n-3,3} & \ldots & c_{0n} & 0 \\
c_{n1} & c_{n-1,2} & c_{n-2,3} & \ldots & c_{1n} & c_{0,n+1}
\end{bmatrix}
\begin{bmatrix}
a_0 \\
a_1 \\
a_2 \\
\vdots \\
a_{n-1} \\
a_n
\end{bmatrix}
= \begin{bmatrix}
b_0 \\
b_1 \\
b_2 \\
\vdots \\
b_{n-1} \\
b_n
\end{bmatrix}.
\tag{2.1}
\]

We notice that the determinant of the first factor of the left-hand side of (2.1) is equal to \( \prod_{i=1}^{n+1} c_{0i} \neq 0 \). Using Cramer’s rule yields a unique solution to (2.1):

\[
a_n = \frac{\begin{vmatrix}
c_{01} & 0 & 0 & \ldots & 0 & b_0 \\
c_{11} & c_{02} & 0 & \ldots & 0 & b_1 \\
c_{21} & c_{12} & c_{03} & \ldots & 0 & b_2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
c_{n-1,1} & c_{n-2,2} & c_{n-3,3} & \ldots & c_{0n} & b_{n-1} \\
c_{n1} & c_{n-1,2} & c_{n-2,3} & \ldots & c_{1n} & b_n
\end{vmatrix}}{\prod_{i=1}^{n+1} c_{0i}}.
\]

Using elementary column operations by interchanging the first column to the \( i \)th column for all \( i = 2, 3, 4, \ldots, n + 1 \), we obtain

\[
\begin{vmatrix}
b_0 & c_{01} & 0 & 0 & \ldots & 0 \\
b_1 & c_{11} & c_{02} & 0 & \ldots & 0 \\
b_2 & c_{21} & c_{12} & c_{03} & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
b_{n-1} & c_{n-1,1} & c_{n-2,2} & c_{n-3,3} & \ldots & c_{0,n} \\
b_n & c_{n1} & c_{n-1,2} & c_{n-2,3} & \ldots & c_{1n}
\end{vmatrix}
= \frac{(-1)^n \det(H_{n+1})}{\prod_{i=1}^{n+1} c_{0i}}.
\]

Now, we consider the infinite linear system of equations:

\[
\begin{bmatrix}
c_{01} & 0 & 0 & \ldots & 0 & 0 & \ldots \\
c_{11} & c_{02} & 0 & \ldots & 0 & 0 & \ldots \\
c_{21} & c_{12} & c_{03} & \ldots & 0 & 0 & \ldots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\
c_{n-1,1} & c_{n-2,2} & c_{n-3,3} & \ldots & c_{0n} & 0 & \ldots \\
c_{n1} & c_{n-1,2} & c_{n-2,3} & \ldots & c_{1n} & c_{1,n+1} & \ldots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots
\end{bmatrix}
\begin{bmatrix}
a_0 \\
a_1 \\
a_2 \\
\vdots \\
a_{n-1} \\
a_n
\end{bmatrix}
= \begin{bmatrix}
b_0 \\
b_1x \\
b_2x^2 \\
\vdots \\
b_{n-1}x^{n-1} \\
b_nx^n
\end{bmatrix}.
\]

Then

\[
c_{01}a_0 = b_0 \\
c_{11}a_0x + c_{02}a_1x = b_1x \\
c_{21}a_0x^2 + c_{12}a_1x^2 + c_{03}a_2x^2 = b_2x^2 \\
\vdots \\
c_{n1}a_0x^n + c_{n-1,2}a_1x^n + c_{n-2,3}a_2x^n + \cdots + c_{1n}a_nx^n = b_nx^n
\]

By summing both sides of the aforementioned equalities, we obtain

\[
a_0C_0(x) + a_1xC_1(x) + \cdots + a_nx^nC_{n+1}(x) + \cdots = B(x),
\]
as required. \qed
Example 2.1. For any integers \( m \geq 1 \) and \( n \geq 0 \), the determinant of \((n + 1) \times (n + 1)\) matrix

\[
\begin{pmatrix}
1 & m & 0 & 0 & \ldots & 0 \\
1 & m + 1 & m & 0 & \ldots & 0 \\
1 & m + 2 & m + 2 & m & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & m + n - 1 & m + 2(n - 2) & m + 3(n - 3) & \ldots & m \\
1 & m + n & m + 2(n - 1) & m + 3(n - 2) & \ldots & m + n
\end{pmatrix}
\]

is equal to 1 if \( n = 0, 1 \), and \((n - m)((n - 1) - m)\cdots(2 - m)\) if \( n \geq 2 \).

Applying Theorem 2.1 by setting \( H_{n+1} \) is the required determinant for all \( n = 0, 1, 2, \ldots \), \( B(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \), and \( C(x) = \sum_{n=0}^{\infty} (m + kn)x^n = \frac{m + (k - m)x}{(1-x)^2} \), for all \( k = 1, 2, 3, \ldots \), there exists a generating function

\[
A(x) = \sum_{n=0}^{\infty} a_n x^n
\]
such that

\[
\sum_{n=0}^{\infty} a_n x^n \frac{m + (n + 1 - m)x}{(1-x)^2} = \frac{1}{1-x}
\]

for all integers \( n \geq 2 \). Since \( \prod_{i=1}^{n+1} c_{i0} = m^{n+1} \), and \( a_n = \frac{(-1)^n(\det(H_{n+1}))}{\prod_{i=1}^{n+1} c_{i0}} \) for all \( n = 0, 1, 2, \ldots \), we obtain that

\[
\det(H_1) = 1, \quad \det(H_2) = 1, \quad \text{and} \quad \det(H_{n+1}) = (n - m)((n - 1) - m)\cdots(2 - m)
\]

for all integers \( n \geq 2 \).

If we set \( C_k(x) = C(x) \) for all integers \( k \geq 1 \) in Theorem 2.1, then we obtain the following Hessenberg determinant \( H_{n+1} \), which is a generalization of [6, Proposition 1].

Corollary 2.1.1. Assume that \( B(x) = \sum_{n=0}^{\infty} b_n x^n \), and \( C(x) = \sum_{n=0}^{\infty} c_n x^n \); \( c_0 \neq 0 \) are the generating functions for the sequences of complex numbers \( \{b_n\}_{n \geq 0} \) and \( \{c_n\}_{n \geq 0} \), respectively. Then there exists a generating function

\[
A(x) = \frac{B(x)}{C(x)}
\]

Furthermore, the coefficient

\[
a_n = \frac{(-1)^n \det(H_{n+1})}{c_{n+1}^{n+1}},
\]

where the \((n + 1) \times (n + 1)\) lower Hessenberg matrix \( H_{n+1} \) is defined as follows:

\[
H_{n+1} = \begin{bmatrix}
b_0 & c_0 & 0 & 0 & \ldots & 0 \\
b_1 & c_1 & c_0 & 0 & \ldots & 0 \\
b_2 & c_2 & c_1 & c_0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
b_{n-1} & c_{n-1} & c_{n-2} & c_{n-3} & \ldots & c_0 \\
b_n & c_n & c_{n-1} & c_{n-2} & \ldots & c_1
\end{bmatrix}
\]

for all nonnegative integers \( n \), and also if \( D(x) \) is the generating function for the sequences \( \{\det(H_{n+1})\}_{n \geq 0} \), then \( D(x) = c_0 A(-c_0 x) \).
Example 2.2. For any sequence of complex numbers \( \{b_n\}_{n \geq 0} \), we have the determinant of \((n + 1) \times (n + 1)\) matrix

\[
\begin{vmatrix}
  b_0 & 1 & 0 & \cdots & 0 \\
  b_1 & 1 & 1 & \cdots & 0 \\
  b_2 & 1 & 1 & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  b_{n-1} & 1 & 1 & \cdots & 1 \\
  b_n & 1 & 1 & \cdots & 1 \\
\end{vmatrix} = \begin{cases} 
  b_0 & \text{if } n = 0, \\
  (-1)^n(b_n - b_{n-1}) & \text{if } n \geq 1.
\end{cases}
\]

Applying the aforementioned corollary by letting \( C(x) = \sum_{n=0}^{\infty} b_n x^n = \frac{1}{1-x} \), and \( B(x) = \sum_{n=0}^{\infty} b_n x^n \) directly gives the required determinant.

Example 2.3. For any integer \( n \geq 0 \), the determinant of \((n + 1) \times (n + 1)\) matrix

\[
\begin{vmatrix}
  2 & i & 0 & \cdots & 0 \\
  -i & 1 & i & \cdots & 0 \\
  0 & i & 1 & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & \cdots & i \\
  0 & 0 & 0 & \cdots & 1 \\
\end{vmatrix} = L_n,
\]

where \( \{L_n\}_{n \geq 0} \) is the sequence of the Lucas numbers.

Applying Corollary 2.1.1 by letting \( C(x) = 1 + x + ix^2 \), and \( B(x) = 2 - ix \), there exists a generating function \( A(x) = \sum_{n=0}^{\infty} a_n x^n \) such that the generating function for the sequences \( \{H_{n+1}\}_{n \geq 0} \):

\[
D(x) = iA(-ix) = \frac{2 + i(-ix)}{i + (-ix) + i(-ix)^2} = \frac{2 - x}{1 - x - x^2},
\]

which is the generating function for the sequence of the Lucas numbers.

Example 2.4. For any integer \( n \geq 0 \), we have the determinant of \((n + 1) \times (n + 1)\) matrix

\[
\begin{vmatrix}
  1 & 1 & 0 & \cdots & 0 \\
  -1 & 1 & 1 & \cdots & 0 \\
  1 & 0 & 1 & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  (-1)^n & 0 & 0 & \cdots & 1 \\
  (-1)^{n+1} & 0 & 0 & \cdots & 1 \\
\end{vmatrix} = n + 1.
\]

Applying Corollary 2.1.1 by letting \( C(x) = 1 + x \), and \( B(x) = \sum_{n=0}^{\infty} (-1)^n x^n \) directly gives the required determinant.

Corollary 2.1.2. Suppose that \( B(x) = \sum_{n=0}^{\infty} b_n x^n \), \( C(x) = \sum_{n=0}^{\infty} c_n x^n \); \( c_0 \neq 0 \), and \( D(x) = \sum_{n=0}^{\infty} d_n x^n \); \( d_0 \neq 0 \) are the generating functions for sequences \( \{b_n\}_{n \geq 0} \), \( \{c_n\}_{n \geq 0} \), and \( \{d_n\}_{n \geq 0} \), respectively. Then there exists a generating function \( A(x) = \sum_{n=0}^{\infty} a_n x^n \) such that

\[
A(x) = \frac{B(x) + a_0(D(x) - C(x))}{D(x)}.
\]

In addition, the coefficient

\[
a_n = \frac{(-1)^n \det(H_{n+1})}{c_0 d_0^n},
\]
where the \((n + 1) \times (n + 1)\) matrix \(H_{n+1}\) is defined as follows:

\[
H_{n+1} = \begin{bmatrix}
    b_0 & c_0 & 0 & 0 & \ldots & 0 \\
    b_1 & c_1 & d_0 & 0 & \ldots & 0 \\
    b_2 & c_2 & d_1 & d_0 & \ldots & 0 \\
    \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
    b_{n-1} & c_{n-2} & d_{n-2} & d_{n-3} & \ldots & d_0 \\
    b_n & c_n & d_{n-1} & d_{n-2} & \ldots & d_1 \\
\end{bmatrix}
\]

for all nonnegative integers \(n\).

The proof of this corollary is analogous to the proof Theorem 2.1. Note that the matrix \(H_{n+1}\) in this corollary is included in [10, Theorem 2.12].

**Example 2.5.** For any integer \(n \geq 0\), we have

\[
\begin{vmatrix}
    b_0 & 1 & 0 & 0 & \ldots & 0 \\
    b_1 & 1 & 1 & 0 & \ldots & 0 \\
    b_2 & 1 & 2 & 1 & \ldots & 0 \\
    \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
    b_{n-1} & 1 & n-1 & n-2 & \ldots & 1 \\
    b_n & 1 & n & n-1 & \ldots & 2 \\
\end{vmatrix}
= \begin{cases}
    b_0 & \text{if } n = 0, \\
    b_0 - b_1 & \text{if } n = 1, \\
    (-1)^n(b_{n-2} - 2b_{n-1} + b_n) & \text{if } n \geq 2.
\end{cases}
\]

Applying Corollary 2.1.2 by letting \(B(x) = \sum_{n=0}^{\infty} b_n x^n\), \(C(x) = \sum_{n=0}^{\infty} c_n x^n\) and \(D(x) = \sum_{n=0}^{\infty} (n+1) x^n = \frac{1}{(1-x)^2}\) directly yields the required determinant.

The following Theorem is a slight generalization of [10, Theorem 2.16].

**Theorem 2.2.** Let \(\{b_n\}_{n \geq 0}\) be any sequence of complex numbers, and let \(\{c_n\}_{n \geq 0}\) be any sequences of complex numbers, where \(c_0 \neq 0\). Suppose that \(B(x) = \sum_{n=0}^{\infty} b_n x^n\), and \(C(x) = \sum_{n=0}^{\infty} c_n x^n\) are the generating functions for the sequences \(\{b_n\}_{n \geq 0}\), and \(\{c_n\}_{n \geq 0}\), respectively. Then there exists a generating function \(A(x) = \sum_{n=0}^{\infty} a_n x^n\) such that

\[
C(x)A(x) + c_0 x A'(x) = B(x).
\]

Furthermore, the coefficient

\[
a_n = \frac{(-1)^n \det(H_{n+1})}{(n + 1)! c_{n+1}},
\]

where the \((n + 1) \times (n + 1)\) matrix \(H_{n+1}\) is defined as follows:

\[
H_{n+1} = \begin{bmatrix}
    b_0 & c_0 & 0 & 0 & \ldots & 0 \\
    b_1 & c_1 & 2c_0 & 0 & \ldots & 0 \\
    b_2 & c_2 & c_1 & 3c_0 & \ldots & 0 \\
    \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
    b_{n-1} & c_{n-1} & c_{n-2} & c_{n-3} & \ldots & n c_0 \\
    b_n & c_n & c_{n-1} & c_{n-2} & \ldots & c_1 \\
\end{bmatrix}
\]

for all nonnegative integers \(n\).

**Proof.** For each nonnegative integer \(n\), we consider the linear system of equations:
We notice that the determinant of the first factor on the left-hand side of this system is equal to 

\[(n + 1)!c_0^{n+1} \neq 0.\]

It follows that there exists a unique solution to the system (2.2):

\[
\begin{vmatrix}
    c_0 & 0 & 0 & \ldots & 0 & b_0 \\
    c_1 & 2c_0 & 0 & \ldots & 0 & b_1 \\
    c_2 & c_1 & 3c_0 & \ldots & 0 & b_2 \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    c_{n-1} & c_{n-2} & c_{n-3} & \ldots & nc_0 & b_{n-1} \\
    c_n & c_{n-1} & c_{n-2} & \ldots & c_1 & b_n \\
\end{vmatrix} =
\begin{vmatrix}
    a_0 \\
    a_1 \\
    a_2 \\
    \vdots \\
    a_{n-1} \\
    a_n \\
\end{vmatrix} =
\begin{vmatrix}
    b_0 \\
    b_1 \\
    b_2 \\
    \vdots \\
    b_{n-1} \\
    b_n \\
\end{vmatrix}.
\]

By using elementary column operations by interchanging the first column to the ith column for all

\[i = 2, 3, 4, \ldots, n + 1,\]

we obtain

\[
(-1)^n a_n = \frac{\det(H_{n+1})}{(n + 1)!c_0^{n+1}}.
\]

Now, let \(A(x) = \sum_{n=0}^{\infty} a_n x^n\) be the generating functions of such sequence \(\{a_n\}_{n \geq 0}\). Then

\[
C(x)A(x) + c_0x A'(x) = \left(\sum_{n=0}^{\infty} c_0 x^n\right)\left(\sum_{n=0}^{\infty} a_n x^n\right) + c_0 x \left(\sum_{n=0}^{\infty} a_n x^n\right)
\]

\[
= \sum_{n=0}^{\infty} \left(\sum_{i+j=n} c_0 a_j\right) x^n + \sum_{n=0}^{\infty} c_0 n a_n x^n
\]

\[
= \sum_{n=0}^{\infty} \left(\sum_{i+j=n} c_0 a_j + c_0 n a_n\right) x^n
\]

\[
= \sum_{n=0}^{\infty} b_n x^n = B(x),
\]

as required. \(\square\)

**Example 2.6.** For any integer \(n \geq 0\), we have the determinant of \((n + 1) \times (n + 1)\) matrix

\[
\text{det}(H_{n+1}) := \begin{vmatrix}
1 & 2 & 0 & 0 & \ldots & 0 \\
\frac{1}{2 \cdot 1!} & 2 & 0 & 0 & \ldots & 0 \\
\frac{1}{2^2 \cdot 2!} & 0 & -1 & 6 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\frac{1}{2^{n-1} \cdot (n-1)!} & 0 & 0 & 0 & \ldots & 2n \\
\frac{1}{2^n \cdot n!} & 0 & 0 & 0 & \ldots & -1 \\
\end{vmatrix} = (-1)^n(n + 1).
\]
Let \( B(x) = \sum_{n=0}^{\infty} \frac{1}{2^n n!} x^n = e^x \) and \( C(x) = 2 - x \). By Theorem 2.2, there exists a generating function \( A(x) = \sum_{n=0}^{\infty} a_n x^n \) satisfying \( C(x)A(x) + c_0x A'(x) = B(x) \). Then

\[
(2 - x)A(x) + 2xA'(x) = e^x
\]

\[
A'(x) + \left( \frac{1}{x} - \frac{1}{2} \right) A(x) = \frac{e^x}{2x},
\]

which is a linear differential equation of order 1 having the integrating factor \( I(x) = xe^{\frac{x}{2}} \). Hence, the general solution to this linear differential equation is expressed as follows:

\[
xe^{\frac{x}{2}} A(x) = \int \frac{1}{2} \, dx = \frac{x}{2} + c,
\]

where \( c \) is an arbitrary constant. For \( x = 0 \), we have \( c = 0 \), and hence,

\[
A(x) = \frac{1}{2} e^{\frac{x}{2}} = \sum_{n=0}^{\infty} \frac{1}{2^n n!} x^n.
\]

That is, \( a_n = \frac{1}{2^n n!} \) for all integers \( n \geq 0 \). It follows that

\[
\frac{1}{2^n n!} = a_n = (-1)^n \frac{\det(H_{n+1})}{2^n (n+1)!}, \quad \text{or} \quad \det(H_{n+1}) = (-1)^n (n+1),
\]

for all integers \( n \geq 0 \).

**Acknowledgement:** This research was supported by Faculty of Science (International SciKU Branding, ISB), Kasetsart University, Thailand.

**Conflict of interest:** Authors state no conflict of interest.

**Data availability statement:** The datasets generated during and/or analysed during the current study are available from the corresponding author on reasonable request.

**References**

[1] Y. H. Chen and C. Y. Yu, *A new algorithm for computing the inverse and the determinant of a Hessenberg matrix*, Appl. Math. Comput. 218 (2011), 4433–4436.

[2] M. Elouafi and A. D. Aiat Hadji, *A new recursive algorithm for inverting Hessenberg matrices*, Appl. Math. Comput. 214 (2009), 497–499.

[3] G. H. Golub and C. F. Van Loan, *Matrix Computations*, 3rd ed., Johns Hopkins University Press, Baltimore and London, 1996, 341–352.

[4] J. Maroulas, *Factorization of Hessenberg matrices*, Linear Algebra Appl. 506 (2016), 226–243.

[5] D. Zhao and H. Li, *On the computation of inverse and determinant of a kind of special matrices*, Appl. Math. Comput. 250 (2015), C, 721–726.

[6] S. Getu, *Evaluating determinants via generating function*, Math. Magazine 64 (1991), no. 1, 45–53.

[7] M. Janjic, *Hessenberg matrices and integer sequences*, J. Integer Seq. 13 (2010), Article 10.7.8.

[8] M. Merca, *A note on the determinant of a Toeplitz-Hessenberg matrix*, Special Matrices 1 (2013), 10–16.

[9] J. L. Ramirez, *Hessenberg matrices and the generalized Fibonacci-Narayana sequence*, Filomat 29 (2015), no. 7, 1557–1563.

[10] E. Kılıç and T. Arıkan, *Evaluation of Hessenberg determinants via generating function approach*, Filomat 31 (2017), no. 15, 4945–4962.