Construction of Nijenhuis operators and dendriform trialgebras

Philippe Leroux

Institut de Recherche Mathématique, Université de Rennes I and CNRS UMR 6625
Campus de Beaulieu, 35042 Rennes Cedex, France, pleroux@univ-rennes1.fr

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Abstract: We construct Nijenhuis operators from particular bialgebras called dendriform-Nijenhuis bialgebras. It turns out that such Nijenhuis operators commute with $TD$-operators, kind of Baxter-Rota operators, and therefore closely related dendriform trialgebras. This allows the construction of associative algebras, called dendriform-Nijenhuis algebras made out with nine operations and presenting an exotic combinatorial property. We also show that the augmented free dendriform-Nijenhuis algebra and its commutative version have a structure of connected Hopf algebras. Examples are given.

1 Introduction

Notation: In the sequel $k$ is a field of characteristic zero. Let $(X, \diamond)$ be a $k$-algebra and $(\diamond_i)_{1 \leq i \leq N} : X^\otimes 2 \to X$ be a family of binary operations on $X$. The notation $\diamond \to \sum_i \diamond_i$ will mean $x \diamond y = \sum_i x \diamond_i y$, for all $x, y \in X$. We say that the operation $\diamond$ splits into the $N$ operations $\diamond_1, \ldots, \diamond_N$, or that the operation $\diamond$ is a cluster of $N$ (binary) operations.

Let $(\mathcal{L}, [\cdot, \cdot])$ be a Lie algebra. A Nijenhuis operator $N : \mathcal{L} \to \mathcal{L}$ is a linear map verifying,

$$[N(x), N(y)] + N^2([x, y]) = N([N(x), y] + [x, N(y)]). \quad (1)$$

Solutions of this equation can be constructed by producing operators $\beta : A \to A$ verifying the associative Nijenhuis relation (ANR) on an associative algebra $(A, \mu)$. The associative Nijenhuis relation, i.e.,

$$(ANR) : \quad \beta(x)\beta(y) + \beta^2(xy) = \beta(\beta(x)y + x\beta(y)), \quad (2)$$

for all $x, y \in A$, appears for the first time in [4], see also [6, 5] and the references therein. Such linear maps $\beta$ are then Nijenhuis operators since (2) implies (1) on the Lie algebra $(A, [\cdot, \cdot])$, with
\[ [x, y] = xy - yx, \text{ for all } x, y \in A. \] In the sequel, by Nijenhuis operators, we mean a linear map defined on an associative algebra and verifying (2).

Section 2 prepares the sequel of this work. NS-Algebras are defined and dendriform trialgebras are recalled. We show that Nijenhuis operators on an associative algebra give NS-algebras. The notion of TD-operators is also introduced. Such operators give dendriform trialgebras \cite{15, 6, 10}. In Section 3, the notion of dendriform-Nijenhuis bialgebras \((A, \mu, \Delta)\) is introduced. This notion is the corner-stone of the paper. Indeed, any dendriform-Nijenhuis bialgebra gives two operators \(\beta, \gamma : \text{End}(A) \to \text{End}(A)\), where \(\text{End}(A)\) is the \(k\)-algebra of linear maps from \(A\) to \(A\), which commute one another, i.e., \(\beta \gamma = \gamma \beta\). The first one turns out to be a Nijenhuis operator and the last one, a TD-operator. Section 4 gives examples. Section 5 introduces dendriform-Nijenhuis algebras, which are associative algebras whose associative product splits into nine binary operations linked together by 28 constraints. When the nine operations are gathered in a particular way, dendriform-Nijenhuis algebras become NS-algebras and when gathered in another way, dendriform-Nijenhuis algebras become dendriform trialgebras. Otherwise stated, this means the existence of a commutative diagram between the involved categories,

\[
\begin{array}{c}
\text{Dend. Nijenhuis} \\
\downarrow
\end{array}
\quad
\begin{array}{c}
\text{NS} \\
\downarrow
\end{array}
\quad
\begin{array}{c}
\text{TriDend.} \\
\downarrow
\end{array}
\quad
\begin{array}{c}
\text{As.}
\end{array}
\]

diagram which will be explained in Section 5. Such associative algebras can be easily constructed from dendriform-Nijenhuis bialgebras. Section 6 shows that the augmented free dendriform-Nijenhuis algebra and the augmented free NS-algebra on a \(k\)-vector space \(V\), as well as their commutative versions, have a structure of connected Hopf algebras.

\section{NS-Algebras and dendriform trialgebras}

We present some results relating NS-algebras and dendriform trialgebras to Nijenhuis operators and TD-operators.

\subsection{NS-Algebras}

\textbf{Definition 2.1 [NS-algebra]} A NS-algebra \((A, \prec, \succ, \bullet)\) is a \(k\)-vector space equipped with three binary operations \(\prec, \succ, \bullet : A^\otimes 2 \to A\) verifying,

\[
\begin{align*}
(x \prec y) \prec z :&= x \prec (y \star z), & (x \succ y) \prec z :&= x \succ (y \prec z), & (x \succ y) \succ z :&= x \succ y \succ z, \\
(x \star y) \bullet z + (x \bullet y) \prec z :&= x \succ (y \bullet z) + x \bullet (y \star z),
\end{align*}
\]

where \(\star \mapsto \prec + \succ + \bullet\). The \(k\)-vector space \((A, \star)\) is an associative algebra, i.e., there exits a functor \(F_1 : \text{NS} \to \text{As.}\), where \(\text{NS}\) is the category of NS-algebras and \(\text{As.}\) the category of associative algebras.
Remark: Let \((A, \prec, \succ, \bullet)\) be a NS-algebra. Define three operations \(\prec^{op}, \succ^{op}, \bullet^{op} : A^{\otimes 2} \to A\) as follows,
\[
x \prec^{op} y := y \succ x, \quad x \succ^{op} y := y \prec x, \quad x \bullet^{op} y := y \bullet x, \quad \forall x, y \in A.
\]
Then, \((A, \prec^{op}, \succ^{op}, \bullet^{op})\) is a NS-algebra called the opposite of \((A, \prec, \succ, \bullet)\). NS-Algebras are said to be commutative if they coincide with their opposite, i.e., if \(x \prec y = y \succ x\) and \(x \bullet y := y \bullet x\).

Proposition 2.2 Let \((A, \mu)\) be an associative algebra equipped with a Nijenhuis operator \(\beta : A \to A\). Define three binary operations \(\prec, \succ, \bullet : A^{\otimes 2} \to A\) as follows,
\[
x \prec y := x\beta(y), \quad x \succ y := \beta(x)y, \quad x \bullet y := -\beta(xy),
\]
for all \(x, y \in A\). Then, \(A^\beta := (A, \prec, \succ, \bullet)\) is a NS-algebra.

Proof: Straightforward. \qed

Proposition 2.3 Let \((A, \mu)\) be a unital associative algebra with unit \(i\). Suppose \(\beta : A \to A\) is a Nijenhuis operator. Define three binary operations \(\prec, \succ, \bullet : (A^\beta)^{\otimes 2} \to A^\beta\) as follows,
\[
x \prec y := x\beta(i) \prec y = x \prec \beta(i)y, \quad x \succ y := x \succ \beta(i)y = x\beta(i) \succ y, \quad x \bullet y := x\beta(i) \bullet y = x \bullet \beta(i)y,
\]
for all \(x, y \in A\). Then, \((A, \prec, \succ, \bullet)\) is a NS-algebra.

Proof: Observe that for all \(x \in A\), \(\beta(x)\beta(i) = \beta(x\beta(i))\) and \(\beta(i)\beta(x) = \beta(\beta(i)x)\). Therefore, for all \(x, y \in A\), \(x \prec y := x\beta(i) \prec y = x \prec \beta(i)y = x \prec \beta(i)y\) and similarly for the two other operations. Fix \(x, y, z \in A\). Let us check that,
\[
(x \prec y) \prec z = x \prec (y \bullet z),
\]
where, \(\prec \to \prec \prec + \bullet \prec\). Indeed,
\[
(x \prec y) \prec z = x \prec (y \bullet z) = x \prec \beta(i) (y \bullet \beta(i) z) = x \prec \beta(i) (y \bullet z + \beta(i) z) = x \prec \beta(i) (y \bullet z + \beta(i) z) = x \prec (y \bullet z).
\]

2.2 Dendriform trialgebras

Let us recall some motivations for the introduction of dendriform trialgebras. Motivated by \(K\)-theory, J.-L. Loday first introduced a “non-commutative version” of Lie algebras called Leibniz algebras. Such algebras are described by a bracket \([-, z]\) verifying the Leibniz identity:
\[
[[x, y], z] = [[x, z], y] + [x, [y, z]].
\]
When the bracket is skew-symmetric, the Leibniz identity becomes the Jacobi identity and Leibniz algebras turn out to be Lie algebras. A way to construct such Leibniz algebras is to start with associative dialgebras, i.e., $k$-vector spaces $D$ equipped with two associative products, $\lt$ and $\rt$, and verifying some conditions [14]. The operad $Dias$ associated with associative dialgebras is then Koszul dual to the operad $DiDend$ associated with dendriform dialgebras [14]. A dendriform dialgebra is a $k$-vector space $E$ equipped with two binary operations: $\prec, \succ: E^\otimes 2 \to E$, satisfying the following relations for all $x, y \in E$:

$$(x \prec y) \prec z = x \prec (y \ast z), \quad (x \succ y) \prec z = x \succ (y \prec z), \quad (x \ast y) \prec z = x \succ (y \succ z),$$

where by definition $x \ast y := x \prec y + x \succ y$, for all $x, y \in E$. The dendriform dialgebra $(E, \ast)$ is then an associative algebra such that $\ast \longrightarrow \prec + \succ$. Similarly, to propose a “non-commutative version” of Poisson algebras, J.-L. Loday and M. Ronco [15] introduced the notion of associative trialgebras. It turns out that $Trias$, the operad associated with this type of algebras, is Koszul dual to $TriDend$, the operad associated with dendriform trialgebras.

**Definition 2.4 [Dendriform trialgebra]** A dendriform trialgebra is a $k$-vector space $T$ equipped with three binary operations: $\prec, \succ, \circ: T^\otimes 2 \to T$, satisfying the following relations for all $x, y \in T$:

$$(x \prec y) \prec z = x \prec (y \circ z), \quad (x \succ y) \prec z = x \prec (y \succ z), \quad (x \circ y) \prec z = x \circ (y \prec z),$$

$$(x \succ y) \circ z = x \circ (y \circ z), \quad (x \prec y) \circ z = x \circ (y \prec z), \quad (x \circ y) \circ z = x \circ (y \circ z),$$

where by definition $x \circ y := x \prec y + x \succ y$, for all $x, y \in T$. The $k$-vector space $(T, \circ)$ is then an associative algebra such that $\circ \longrightarrow \prec + \succ + \circ$. There exists a functor $F_2: TriDend. \to As$.

Observe that these axioms are globally invariant under the transformation $x \prec^{\text{op}} y := y \succ x$, $x \succ^{\text{op}} y := y \prec x$ and $x \circ^{\text{op}} y := y \circ x$. A dendriform trialgebra is said to be commutative if $x \prec y := y \succ x$ and $x \circ y := y \circ x$.

To construct dendriform trialgebras, the use of $t$-Baxter operators, also called Rota-Baxter operators, have been used in [6] and in [10] to generalise [2]. Let $(A, \mu)$ be an associative algebra and $t \in k$. A $t$-Baxter operator is a linear map $\xi: A \to A$ verifying:

$$\xi(x)\xi(y) = \xi(x\xi(y)) + \xi(x)y + txy.$$ 

For $t = 0$, this map is called a Baxter operator. It appears originally in a work of G. Baxter [3] and the importance of such a map was stressed by G.-C. Rota in [17]. We present another way to produce dendriform trialgebras. Let $(A, \mu)$ be a unital associative algebra with unit $i$. The linear map $\gamma: A \to A$ is said to be a TD-operator if,

$$\gamma(x)\gamma(y) = \gamma(\gamma(x)y + x\gamma(y) - x\gamma(i)y),$$

for all $x, y \in A$.

**Proposition 2.5** Let $A$ be a unital algebra with unit $i$. Suppose $\gamma: A \to A$ is a TD-operator. Define three binary operations $\prec_\gamma, \succ_\gamma, \circ_\gamma: A^\otimes 2 \to A$ as follows,

$$x \prec_\gamma y := x\gamma(y), \quad x \succ_\gamma y := \gamma(x)y, \quad x \circ_\gamma y := -x\gamma(i)y, \quad \forall x, y \in A.$$
Then, $A^\gamma := (A, \prec_{\gamma}, \succ_{\gamma}, \circ_{\gamma})$ is a dendriform trialgebra. The operation $\bar{\gamma} : A^{\otimes 2} \to A$ defined by,

$$x\bar{\gamma}y := x\gamma(y) + \gamma(x)y - x\gamma(i)y,$$

is associative.

**Proof:** Straightforward by noticing that $\gamma(i)\gamma(x) = \gamma(x)\gamma(i)$, for all $x \in A$. \qed

### 3 Construction of Nijenhuis operators and $TD$-operators from dendriform-Nijenhuis bialgebras

In [1], Baxter operators are constructed from $\epsilon$-bialgebras. This idea have been used in [2] [10] to produce commuting $t$-Baxter operators. Recall that a $t$-infinitesimal bialgebra (abbreviated $\epsilon(t)$-bialgebra) is a triple $(A, \mu, \Delta)$ where $(A, \mu)$ is an associative algebra and $(A, \Delta)$ is a coassociative coalgebra such that for all $a, b \in A$,

$$\Delta(ab) = a_{(1)} \otimes b_{(2)} + a_{(1)} \otimes b_{(2)} + ta \otimes b.$$

If $t = 0$, a $t$-infinitesimal bialgebra is called an infinitesimal bialgebra or a $\epsilon$-bialgebra. Such bialgebras appeared for the first time in the work of Joni and Rota in [9], see also Aguiar [1], for the case $t = 0$ and Loday [12], for the case $t = -1$.

To produce Nijenhuis operators from bialgebras, we replace the term $ta \otimes b$ in the definition of $t$-infinitesimal bialgebras by the term $-\mu(\Delta(a)) \otimes b$. A physical interpretation of this term can be the following. The two-body system $\mu(a \otimes b) := ab$, for instance a two particule system or a string of letters in informatic, made out from $a$ and $b$ is sounded by a coproduct $\Delta$, representing a physical system. This coproduct “reads sequentially” the system $\mu(a \otimes b)$ giving the systems $\Delta(a)b$ and $a\Delta(b)$. In the case of $t$-infinitesimal bialgebras, $\Delta(ab)$ studies the behavior between the ”sequential reading”, $\Delta(a)b$ and $a\Delta(b)$, and $ta \otimes b$ which can be interpreted as the system $a$ decorrelated with system $b$. In the replacement, $ta \otimes b$ by $\mu(\Delta(a)) \otimes b$, we want to compare the ”sequential reading”, $\Delta(a)b$ and $a\Delta(b)$ to the decorrelated system $\mu(\Delta(a)) \otimes b$ made out with $b$ and the system obtained from the recombinaison of the pieces created by reading the system $a$.

**Definition 3.1 [Dendriform-Nijenhuis bialgebra]** A Dendriform-Nijenhuis bialgebra is a triple $(A, \mu, \Delta)$ where $(A, \mu)$ is an associative algebra and $(A, \Delta)$ is a coassociative coalgebra such that,

$$\Delta(ab) := \Delta(a)b + a\Delta(b) - \mu(\Delta(a)) \otimes b, \ \forall a, b \in A.$$

The $k$-vector space $\text{End}(A)$ of linear endomorphisms of $A$ is viewed as an associative algebra under composition denoted simply by concatenation $TS$, for $T, S \in \text{End}(A)$. Another operation called the convolution product, $*$, defined by $T * S := \mu(T \otimes S)\Delta$, for all $T, S \in \text{End}(A)$ will be used.
Proposition 3.2 Let $(A, \mu, \Delta)$ be a dendriform-Nijenhuis bialgebra. Equip $\text{End}(A)$ with the convolution product $\ast$. Define the operators $\beta, \gamma : \text{End}(A) \to \text{End}(A)$ by $T \mapsto \beta(T) := \text{id} \ast T$ (right shift) and $T \mapsto \gamma(T) := T \ast \text{id}$ (left shift). Then, the right shift $\beta$ is a Nijenhuis operator and the left shift $\gamma$ is a TD-operator. Moreover $\beta \gamma = \gamma \beta$.

Proof: Let us show that the right shift $\beta : T \mapsto \text{id} \ast T$ is a Nijenhuis operator. Fix $T, S \in \text{End}(A)$ and $a \in A$. On the one hand,

$$\beta(T) \beta(S)(a) = \mu(id \otimes T) \Delta(a(1)S(a(2))),$$

$$= \mu(id \otimes T)(a(1)(1) \otimes a(2)(1)) S(a(2)(2)) + a(1)S(a(2)(1)) T(S(a(2)(2)) - a(1) a(1)(1) a(1)(2) \otimes S(a(2))),$$

$$= a(1)(1) T(a(1)(2) S(a(2))) + a(1) S(a(2)(1)) T(S(a(2)(2)) - a(1)(1) a(1)(2) T(S(a(2))).$$

On the other hand,

$$\beta(\beta(T) S)(a) = \mu(id \otimes \beta(T) S)(a(1) \otimes a(2)),$$

$$= a(1) S(a(2)(1)) T(S(a(2)(2)))$$

And,

$$\beta(T \beta(S))(a) = \mu(id \otimes T \beta(S))(a(1) \otimes a(2)),$$

$$= a(1) T(a(2)(1) S(a(2)(2))),$$

$$= a(1)(1) T(a(1)(2) S(a(2))),$$

since $\Delta$ is coassociative. Moreover,

$$\beta(\beta(T S))(a) = \mu(id \otimes \beta(T S)(a(1) \otimes a(2)),$$

$$= a(1) \beta(T S)(a(2)),$$

$$= a(1) \mu(id \otimes T S)(a(2)(1) \otimes a(2)(2)),$$

$$= a(1) a(2)(1) T S(a(2)(2)),$$

$$= a(1)(1) a(1)(2) T S(a(2)),$$

since $\Delta$ is coassociative. Similarly, we can show that the left shift $\gamma : T \mapsto T \ast \text{id}$ is a TD-operator. $\square$

A $L$-anti-dipterous algebra $(A, \bowtie, \prec_A)$ is an associative algebra $(A, \bowtie)$ equipped with a right module on itself i.e., $(x \prec_A y) \prec_A z = x \prec_A (y \bowtie z)$ and such that $\bowtie$ and $\prec_A$ are linked by the relation $(x \bowtie y) \prec_A z = x \bowtie (y \bowtie z)$. This notion has been introduced in [11] and comes from a particular notion of bialgebras. See also [16].

Proposition 3.3 Let $(A, \mu, \Delta)$ be a dendriform-Nijenhuis bialgebra. Set $x \bowtie y := \mu(\Delta(x)) y$ and $x \prec_A y := x \mu(\Delta(y))$ defined for all $x, y \in A$. Then, the $k$-vector space $(A, \bowtie, \prec_A)$ is a $L$-anti-dipterous algebra.

Proof: Straightforward by using $\mu(\Delta(xy)) = x \mu(\Delta(y))$, for all $x, y \in A$. $\square$

Let $(A, \mu, \Delta)$ be a dendriform-Nijenhuis bialgebra. We end this subsection by showing that $A$ can admit a left counit and that the $k$-vector space $\text{Der}(A)$ of derivatives from $A$ to $A$ is stable by the right shift $\beta$. 

6
Proposition 3.4 Let \((A, \mu, \Delta)\) be a dendriform-Nijenhuis bialgebra. As a \(k\)-algebra, set \(A := k(S)/\mathcal{R}\), where \(S\) is a non-empty set and \(\mathcal{R}\) is a set of relations. Suppose \(\eta : A \to k\) is a \(\mu\)-homomorphism and \((\eta \otimes id)\Delta := id\) on \(S\). Then, \(\eta\) is a left counit, i.e., \((\eta \otimes id)\Delta := id\) on \(A\).

Proof: Fix \(a, b \in A\). Suppose \(\eta : A \to k\) is a \(\mu\)-homomorphism and \((\eta \otimes id)\Delta(a) := a\) and \((\eta \otimes id)\Delta(b) := b\). Then,
\[
(\eta \otimes id)\Delta(ab) := \eta(a_{(1)})a_{(2)}b + \eta(a)b_2 - \eta(a_{(1)})\eta(a_{(2)})b.
\]
However, \(\eta(a_{(1)})a_{(2)}b := ab\) and \(\eta(a)b_2 := \eta(a)b\) and \(\eta(a_{(1)})\eta(a_{(2)})b := \eta(a_{(1)})a_{(2)}b := \eta(a)b\).
Therefore, \((\eta \otimes id)\Delta(ab) := ab\). \(\square\)

Proposition 3.5 Let \((A, \mu, \Delta)\) be a dendriform-Nijenhuis bialgebra. If \(\partial : A \to k\) is a derivative, i.e., \(\partial(xy) := \partial(x)y + x\partial(y)\), then, so is the linear map \(\beta(\partial) : x \mapsto \mu(id \otimes \partial)\Delta(x)\).

Proof: Straightforward. \(\square\)

4 Examples

Proposition 4.1 Let \(S\) be a set. Suppose \(\Delta : kS \to kS^{\otimes 2}\) is a coassociative coproduct on the free \(k\)-vector space spanned by \(S\). Denote by \(As(S)\) the free associative algebra generated by \(S\) and extend the coproduct \(\Delta\) to \(\Delta_z : As(S) \to As(S)^{\otimes 2}\) as follows,
\[
\Delta_z(s) := \Delta(s), \quad \forall s \in kS,
\]
\[
\Delta_z(ab) := \Delta_z(a)b + a\Delta_z(b) - \mu(\Delta_z(a)) \otimes b,
\]
for all \(a, b \in As(S)\). Then, \((As(S), \Delta_z)\) is a dendriform-Nijenhuis bialgebra.

Proof: Keep notation of Proposition 4.1. The co-operation \(\Delta_z\) is well defined since it does not depend on the writing of a given element \(c \in As(S)\). Indeed, it is straightforward to show that if \(c = ab = a'b' \in As(S)\), with \(a, b, a', b' \in As(S)\), then,
\[
\Delta_z(c) := \Delta_z(a)b + a\Delta_z(b) - \mu(\Delta_z(a)) \otimes b = \Delta_z(a'b') + a'\Delta_z(b') - \mu(\Delta_z(a')) \otimes b'.
\]
Let us show that \(\Delta_z\) is coassociative. Let \(a, b \in As(S)\). Write \(\Delta_z(a) := a_{(1)} \otimes a_{(2)}\) and \(\Delta_z(b) := b_{(1)} \otimes b_{(2)}\). Suppose \((id \otimes \Delta_z)\Delta_z(x) = (\Delta_z \otimes id)\Delta_z(x)\), for \(x = a, b\). By definition, \(\Delta_z(ab) := \Delta_z(a)b + a\Delta_z(b) - \mu(\Delta_z(a)) \otimes b\). On the one hand,
\[
(id \otimes \Delta_z)\Delta_z(ab) := a_{(1)} \otimes \Delta_z(a_{(2)}b) + a_{(1)} \otimes \Delta_z(b_{(2)}) - a_{(1)}a_{(2)} \otimes \Delta_z(b),
\]
\[
:= a_{(1)} \otimes \Delta_z(a_{(2)})b + a_{(1)} \otimes a_{(2)}b_{(1)} \otimes b_{(2)} - a_{(1)}a_{(2)}a_{(2)}(1)a_{(2)}(2) \otimes b
+ ab_{(1)} \otimes \Delta_z(b_{(2)}) - a_{(1)}a_{(2)} \otimes \Delta_z(b).
\]
On the other hand,
\[
(\Delta \otimes \text{id}) \Delta (ab) := \Delta (a(1)) \otimes a(2)b + \Delta (ab(1)) \otimes b(2) - \Delta (a(1)a(2)) \otimes b,
\]
\[
:= \Delta (a(1)) \otimes a(2)b + a(1) \otimes a(2)b(1) \otimes b(2) + a\Delta (b(1)) \otimes b(2) - a(1)a(2) \otimes \Delta (b)
\]
\[
- \Delta (a(1)) \otimes b - a(1)\Delta (a(2)) \otimes b + a(1)a(2) \otimes \Delta (b).
\]

The two equations are equal since \((\text{id} \otimes \Delta) \Delta (x) = (\Delta \otimes \text{id}) \Delta (x)\), for \(x = a, b\). Since \(\Delta \) is supposed to be coassociative on \(kS\) and that \(\text{As}(S)\) is the free associative algebra generated by \(S\), \(\Delta \) is coassociative on the whole \(\text{As}(S)\).

**Example 4.2 [Dendriform-Nijenhuis bialgebras from duplications]** Keep notation of Proposition 4.1. Define the coproduct \(\Delta : kS \rightarrow kS^\otimes 2\) as follows:
\[\Delta (s) := s \otimes s.\]

Then, \((\text{As}(S), \Delta)\) is a dendriform-Nijenhuis bialgebra. For instance,
\[\Delta (s_1s_2) = s_1 \otimes s_1s_2 + s_1s_2 \otimes s_2 - s_1^2 \otimes s_2, \forall s_1, s_2 \in S.\]

## 5 Dendriform-Nijenhuis algebras

**Definition 5.1 [Dendriform-Nijenhuis algebra]** A Dendriform-Nijenhuis algebra is a \(k\)-vector space \(DN\) equipped with nine operations \(\rhd, \rhd, \uparrow, \downarrow, \prec, \succ, \cdot : DN^\otimes 2 \rightarrow DN\) verifying 28 relations. To ease notation, \(7\) sums operations are introduced:
\[
\begin{align*}
    x \lhd y & := x \lhd y + x \rhd y + x \prec y, \\
    x \rhd y & := x \rhd y + x \lhd y + x \succ y, \\
    x \cdot y & := x \cdot y + x \downarrow y + x \uparrow y, \\
    x \and y & := x \and y + x \rhd y + x \downarrow y, \\
    x \lor y & := x \rhd y + x \lhd y + x \downarrow y, \\
    x \prec y & := x \prec y + x \succ y + x \cdot y,
\end{align*}
\]
and
\[
x \pdr y := x \pdr y + x \rhd y + x \lhd y + x \pdr y + x \prec y + x \succ y + x \cdot y.
\]
That is,
\[
x \pdr y := x \lhd y + x \rhd y + x \cdot y := x \and y + x \lor y + x \pdr y,
\]
for all \(x, y, z \in A\). The 28 relations are presented in two matrices. The first one is a \(7 \times 3\)-matrix denoted by \(M_1^{ij}(i=1,\ldots,7; j=1,\ldots,3)\). The second one is a \(7 \times 1\)-matrix denoted by \(M_1^{i}(i=1,\ldots,7)\).
Then, its vertical structure

Keep notation of Definition 5.1. Let

There exists a sym-

vertical structure

The

DN,

v := \( y \uparrow x \), \( y \triangleright x \), \( y \triangleright x \). A dendriform-Nijenhuis algebra is said to be commutative when it coincides with its opposite. For any \( x, y \in T \), observe that \( x \ast y := y \ast x \). Observe that the matrix of relations \( M^1 \) has 3 centers of symmetry. The first one, \( M^1_{22} \), corresponds to the first bloc of three rows, the second one, \( M^1_{52} \), to the second bloc of three rows. The last one is \( M^1_{7} \). There are also 3 centers of symmetry for the matrix of relations \( M^2 \). The first one, \( M^2_{3} \), corresponds to the first bloc of three rows, the second one, \( M^2_{5} \), to the second bloc of three rows. The last one is \( M^2_{7} \).

Theorem 5.2 Keep notation of Definition 5.1. Let \( DN \) be a dendriform-Nijenhuis algebra. Then, its vertical structure \( DN_v := (DN, \preceq, \triangleright, \ast) \) is a NS-algebra and its horizontal structure \( DN_h := (DN, \wedge, \lor, \ast) \) is a dendriform trialgebra.
Proof: Let $DN$ be a dendriform-Nijenhuis algebra. The vertical structure $DN_v := (DN, \lhd, \rhd, \bullet)$ is a $NS$-algebra. Indeed for all $x, y, z \in DN$,

$$
\sum_{i=1,2,3,7} M^1_{i1} - \sum_{i=4,5,6} M^1_{i1} \iff (x \lhd y) \lhd z = x \lhd (y \rhd z),
$$

$$
\sum_{i=1,2,3,7} M^1_{i2} - \sum_{i=4,5,6} M^1_{i2} \iff (x \rhd y) \rhd z = x \rhd (y \lhd z),
$$

$$
\sum_{i=1,2,3,7} M^1_{i3} - \sum_{i=4,5,6} M^1_{i3} \iff (x \rhd y) \rhd z = x \rhd (y \rhd z),
$$

$$
\sum_{i=1,2,3,7} M^2_i - \sum_{i=4,5,6} M^2_i \iff (x \rhd y) \rhd z + (x \bullet y) \rhd z = x \rhd (y \bullet z) + x \bullet (y \rhd z).
$$

The horizontal structure $DN_h := (DN, \lhd, \rhd, \bullet)$ is a dendriform trialgebra. Indeed for all $x, y, z \in DN$,

$$
\sum_{j=1,2,3} M^1_{ij} - M^2_{i1} \iff (x \wedge y) \wedge z = x \wedge (y \rhd z),
$$

$$
\sum_{j=1,2,3} M^1_{ij} - M^2_{i1} \iff (x \vee y) \vee z = x \vee (y \wedge z),
$$

$$
\sum_{j=1,2,3} M^1_{ij} - M^2_{i1} \iff (x \rhd y) \vee z = x \rhd (y \vee z),
$$

$$
\sum_{j=1,2,3} M^1_{ij} - M^2_{i1} \iff (x \rhd y) \rhd z + (x \bullet y) \rhd z = x \rhd (y \bullet z) + x \bullet (y \rhd z).
$$

□

Remark: In a categorical point of view, Theorem 5.2 gives two functors $F_v$ and $F_h$ represented in the following diagram:

$$
\begin{array}{ccc}
\text{Dend. Nijenhuis} & \xrightarrow{F_v} & \text{NS} \\
F_h \downarrow & f \downarrow & F_1 \\
\text{TriDend.} & \xrightarrow{F_2} & \text{As.}
\end{array}
$$

This diagram commutes, i.e., $F_2 F_h = f = F_1 F_v$.

**Definition 5.3** Let $(\mathcal{N}, \prec, \succ, \bullet)$ be a $NS$-algebra. Set $\mathcal{N}(2) := k\{ \prec, \succ, \bullet \}$. A $TD$-operator $\gamma$ on $\mathcal{N}$ is a linear map $\gamma : \mathcal{N} \rightarrow \mathcal{N}$ such that,
1. There exist $i \in \mathcal{N}$ and two linear maps,

$$\ast_1 : \mathcal{N} \otimes k(\gamma(i)) \to \mathcal{N}, \quad \text{and} \quad \ast_2 : k(\gamma(i)) \otimes \mathcal{N} \to \mathcal{N},$$

$$x \otimes \lambda \gamma(i) \mapsto \lambda x \ast_1 \gamma(i), \quad \text{and} \quad \lambda \gamma(i) \otimes x \mapsto \lambda \gamma(i) \ast_2 x,$$

such that $x \ast \gamma(i) \ast_2 y = x \ast_1 \gamma(i) \circ y : = -x \circ y$, for all $\circ \in \mathbb{N}(2)$ and $x, y \in \mathcal{N}$,

2. In addition,

$$\gamma(x) \circ \gamma(y) = \gamma(x) \circ y + x \circ \gamma(y) + x \circ y,$$

for all $x, y \in \mathcal{N}$ and $\gamma(x) \ast_1 \gamma(i) = \gamma(i) \ast_2 \gamma(x)$.

3. For all $\circ \in \mathbb{N}(2)$, $x, y \in \mathcal{N}$, $\gamma(i) \ast_2 \gamma(x) \circ y = \gamma(i) \ast_2 (\gamma(x) \circ y)$ and $x \circ \gamma(y) \ast_1 \gamma(i) = (x \circ \gamma(y)) \ast_1 \gamma(i)$.

**Proposition 5.4** Let $(A, \mu)$ be a unital associative algebra with unit $i$. Suppose there exists a Nijenhuis operator $\beta : (A, \mu) \to (A, \mu)$ which commutes with a $TD$-operator $\gamma : (A, \mu) \to (A, \mu)$. Suppose $\gamma(i) = \beta(i)$. Then, $\gamma$ is a $TD$-operator on the $NS$-algebra $A^\beta$.

**Proof:** By proposition 2.2, $A^\beta$ is a $NS$-algebra. By applying Proposition 2.3, items 1 and 3 of Definition 5.3 hold since $\gamma(i) = \beta(i)$ and $\gamma(i) \gamma(x) = \gamma(x) \gamma(i)$, for all $x \in A$. Fix $x, y \in A$.

$$\gamma(x) \prec_\beta \gamma(y) = \gamma(x) \beta(\gamma(y)) = \gamma(x) \gamma(\beta(y)),$$

$$\gamma(x) \beta(\gamma(y)) = \gamma(x) \gamma(\beta(y)) - x \gamma(i) \beta(y),$$

$$\gamma(x) \prec_\beta y + x \prec_\beta \gamma(y) + x \prec_\beta y.$$

Checking the three other equations is straightforward. \hfill \Box

**Remark:** Observe that $\gamma(x) \star_\beta \gamma(y) = \gamma(\gamma(x) \star_\beta y + x \star_\beta \gamma(y))$ and thus $x \star \gamma(y) = \gamma(x) \star_\beta y + x \star_\beta \gamma(y) + x \bar{\star} \beta y$ is an associative product, or that $\gamma : (A, \star) \to (A, \star_\beta)$ is a morphism of associative algebras.

**Proposition 5.5** Let $(\mathcal{N}, \prec, \succ, \ast)$ be a $NS$-algebra, with $\ast \longrightarrow \prec + \succ + \ast$ and $\gamma$ be a $TD$-operator on $\mathcal{N}$. Denote by $i \in \mathcal{N}$ the element which verifies items 1, 2 and 3 of Definition 5.3.

For all $x, y \in \mathcal{N}$, define nine operations as follows,

$$\gamma(x) \prec y = \gamma(x) \succ y, \quad \gamma(x) \succ y = \gamma(x) \prec y, \quad x \prec \gamma y = \gamma(x) \prec y,$$

$$x \succ \gamma y = x \ast \gamma(y), \quad x \downarrow \gamma y = \gamma(x) \circ y,$$

and,

$$x \bar{\prec} \gamma y = -x \ast \gamma(i) y, \quad x \bar{\succ} \gamma y = -x \circ \gamma(i) y, \quad x \ast \gamma y = -x \circ \gamma(i) y.$$
for all $x, y \in A$. Define also sums operations as follows,

\[
\begin{align*}
x \cdot_\gamma y & := x \ldownarrow_\gamma y + x \nearrow_\gamma y + x \searrow_\gamma y := \gamma(x) > y + x > \gamma(y) + x \searrow_\gamma y, \\
x \langle_\gamma y & := x \leftarrow_\gamma y + x \rightarrow_\gamma y + x \leftarrow_\gamma y := \gamma(x) < y + x < \gamma(y) + x \leftarrow_\gamma y, \\
x \vee_\gamma y & := x \downarrow_\gamma y + x \uarrow_\gamma y + x \downarrow_\gamma y := x \downarrow_\gamma y + x \leftarrow_\gamma y,
\end{align*}
\]

Checking the 26 other axioms does not present any difficulties.

**Proof:** Let us check the relation $M^1_{11} := (x \langle_\gamma y) \langle_\gamma z = x \langle_\gamma (y \star_\gamma z)$ of Definition 5.1.

\[
\begin{align*}
(x \langle_\gamma y) \langle_\gamma z & = (x < \gamma(y)) < \gamma(z); \\
& = x < (\gamma(y) \cdot_\gamma \gamma(z)); \\
& = x < \gamma(y \star_\gamma z); \\
& = x \langle_\gamma (y \star_\gamma z).
\end{align*}
\]

Similarly, let us check the relation $M^1_{51} := (x \langle_\gamma y) \searrow_\gamma z = x \searrow_\gamma (y \vee_\gamma z)$.

\[
\begin{align*}
(x \langle_\gamma y) \searrow_\gamma z & = (x < \gamma(y)) < \gamma(i) \cdot_2 z, \\
& = x < (\gamma(y) \cdot_1 \gamma(i) \cdot_2 z); \\
& = x < (\gamma(y) \cdot_1 \gamma(i) \cdot z); \\
& = x < (\gamma(i) \cdot_2 \gamma(y) \cdot z); \\
& = x < \gamma(i) \cdot_2 (\gamma(y) \cdot z); \\
& = x \searrow_\gamma (y \vee_\gamma z),
\end{align*}
\]

Checking the 26 other axioms does not present any difficulties. \qed

## 6 Transpose of a dendriform-Nijenhuis algebra

**Definition 6.1 [Transpose of a dendriform-Nijenhuis algebra]** A dendriform-Nijenhuis algebra $DN_1 := (DN, \backslash_1, \langle_1, \nearrow_1, \downarrow_1, \searrow_1, \bullet_1)$ is said to be the *transpose* of a dendriform-Nijenhuis algebra $DN_2 := (DN, \backslash_2, \langle_2, \nearrow_2, \downarrow_2, \searrow_2, \bullet_2)$ if for all $x, y \in DN$,

\[
\begin{align*}
x \backslash_1 y & = x \backslash_2 y, \\
x \langle_1 y & = x \langle_2 y, \\
x \nearrow_1 y & = x \nearrow_2 y, \\
x \downarrow_1 y & = x \downarrow_2 y, \\
x \searrow_1 y & = x \searrow_2 y, \\
x \bullet_1 y & = x \bullet_2 y.
\end{align*}
\]
Definition 6.2 A Nijenhuis operator on a dendriform trialgebra \((TD, \prec, \succ, \circ)\) is a linear map \(\beta : TD \to TD\) such that for all \(x, y \in TD\) and \(\circ \in \{\prec, \succ, \circ\},\)
\[
\beta(x) \circ \beta(y) = \beta(\beta(x) \circ y + x \circ \beta(y) - \beta(x \circ y)).
\]

Remark: If \(\ast\) denotes the associative operation of a dendriform trialgebra, then \(\beta(x) \ast \beta(y) = \beta(\beta(x) \ast y + x \ast \beta(y) - \beta(x \ast y))\). Therefore, the map \(\beta\) is a Nijenhuis operator on the associative algebra \((TD, \ast)\) and a morphism of associative algebra \((A, \star) \to (A, \ast)\), where the associative operation \(\star\) is defined by \(x \star y := \beta(x) \ast y + x \ast \beta(y) - \beta(x \ast y)\), for all \(x, y \in TD\).

Proposition 6.3 Let \(A\) be a unital associative algebra. Suppose \(\beta : A \to A\) is a Nijenhuis operator which commutes with a \(TD\)-operator \(\gamma : A \to A\). Then, \(\beta\) is a Nijenhuis operator on the dendriform trialgebra \(A^\gamma\).

Proof: By Proposition \[\text{(2)}\] \(A^\gamma\) is a dendriform trialgebra. Fix \(x, y \in A\). For instance,
\[
\beta(x) \prec_{\gamma} \beta(y) := \beta(x) \gamma(\beta(y)),
\]
\[
= \beta(x) \beta(\gamma(y)),
\]
\[
= \beta(\beta(x) \gamma(y) + x \beta(\gamma(y)) - \beta(x \gamma(y))),
\]
\[
= \beta(\beta(x) \gamma(y) + x \gamma(\beta(y)) - \beta(x \gamma(y))),
\]
\[
= \beta(\beta(x) \prec_{\gamma} y + x \prec_{\gamma} \beta(y) - \beta(x \prec_{\gamma} y)).
\]

\(\Box\)

Proposition 6.4 Let \((TD, \prec, \succ, \circ)\) be a dendriform trialgebra and \(\beta\) be a Nijenhuis operator. For all \(x, y \in TD\), define nine operations as follows,
\[
x \setminus_{\beta} y = \beta(x) \succ y, \ x \setminus y = x \prec \beta(y), \ x \int_{\beta} y = x \succ \beta(y), \ x \setminus_{\beta} y = x \prec \beta(y),
\]
\[
x \setminus_{\beta} y = x \circ \beta(y), \ x \setminus_{\beta} y = x \circ \beta(y),
\]
\[
\text{and,} \quad x \setminus_{\beta} y := -\beta(x \prec y), \ x \setminus_{\beta} y := -\beta(x \succ y), \ x \setminus_{\beta} y := -\beta(x \circ y), \ \forall \ x, y \in A.
\]

The sum operations are defined as in Definition \[\text{(5)}\] Then, for all \(x, y, z \in TD\), (the label \(\beta\) and being omitted,
\[
(x \setminus y) \setminus z = x \setminus (y \setminus z), \ (x \setminus y) \setminus z = x \setminus (y \setminus z), \ (x \setminus y) \setminus z = x \setminus (y \setminus z),
\]
\[
(x \setminus y) \setminus z = x \setminus (y \setminus z), \ (x \setminus y) \setminus z = x \setminus (y \setminus z), \ (x \setminus y) \setminus z = x \setminus (y \setminus z),
\]
\[
(x \setminus y) \setminus z = x \setminus (y \setminus z), \ (x \setminus y) \setminus z = x \setminus (y \setminus z), \ (x \setminus y) \setminus z = x \setminus (y \setminus z),
\]
\[
(x \setminus y) \setminus z = x \setminus (y \setminus z), \ (x \setminus y) \setminus z = x \setminus (y \setminus z), \ (x \setminus y) \setminus z = x \setminus (y \setminus z).
\]
\[(x \triangleleft y) \bowtie z + (x \triangledown y) \triangleright z = x \triangleright (y \bowtie z) + x \bowtie (y \triangledown z),\]
\[(x \triangledown y) \triangleright z + (x \triangledown y) \triangleright z = x \triangleright (y \bowtie z) + x \bowtie (y \triangledown z),\]
\[(x \circledast y) \bowtie z + (x \circledast y) \triangleright z = x \triangleright (y \bowtie z) + x \bowtie (y \triangledown z),\]
\[(x \triangledown y) \bowtie z + (x \triangledown y) \triangleright z = x \triangleright (y \bowtie z) + x \bowtie (y \triangledown z),\]
\[(x \triangledown y) \bowtie z + (x \triangledown y) \triangleright z = x \triangleright (y \bowtie z) + x \bowtie (y \triangledown z),\]
\[(x \circledast y) \bowtie z + (x \circledast y) \triangleright z = x \triangleright (y \bowtie z) + x \bowtie (y \triangledown z).\]

Otherwise stated, \((TD, \triangleright \triangleright, \triangleright \triangleright, \circledast \triangleright, \circledast \triangleright, \bowtie \triangleright, \bowtie \triangleright, \circledast \triangleright, \circledast \triangleright)\) is a dendriform-Nijenhuis algebra where \(\triangleright \triangleright\) plays the rôle of \(\triangleright \triangleright\), \(\circledast \triangleright\) for \(\circledast \triangleright\), \(\circledast \triangleright\) for \(\circledast \triangleright\), \(\triangledown \triangleright\) for \(\triangledown \triangleright\), \(\bowtie \triangleright\) for \(\bowtie \triangleright\), \(\triangledown \triangleright\) for \(\triangledown \triangleright\), \(\triangledown \triangleright\) for \(\triangledown \triangleright\), \(\bowtie \triangleright\) for \(\bowtie \triangleright\), \(\bowtie \triangleright\) for \(\bowtie \triangleright\), \(\circledast \triangleright\) for \(\circledast \triangleright\), \(\circledast \triangleright\) for \(\circledast \triangleright\), \(\circledast \triangleright\) for \(\circledast \triangleright\), and \(\circledast \triangleright\) for \(\circledast \triangleright\).
7 Dendriform-Nijenhuis algebras from dendriform-Nijenhuis bialgebras

Theorem 7.1 Let \((A, \mu, \Delta)\) be a dendriform-Nijenhuis bialgebra and consider \(\text{End}(A)\) as an associative algebra under composition, equipped with the convolution product \(*\). Then, there exist two dendriform-Nijenhuis algebra structures on \(\text{End}(A)\), the one being the transpose of the other.

Proof: By Proposition 3.2, the right shift \(\beta\) is a Nijenhuis operator which commutes with the left shift \(\gamma\) which is a \(TD\)-operator. If \(\text{id} : A \rightarrow A\) denotes the identity map, then observe that \(\beta(\text{id}) := \text{id} * \text{id} := \gamma(\text{id})\). Use Proposition 6.5 to conclude. \(\square\)

Remark: Let \(T, S \in \text{End}(A)\). By applying Proposition 5.5 the nine operations are given by,

\[
T \searrow S \ := \ \beta \gamma(T)S := (\text{id} * T * \text{id})S, \\
T \nwarrow S \ := \ \gamma(T) \beta(S) := T(\text{id} * S * \text{id}), \\
T \nearrow S \ := \ \beta(T) \gamma(S) := (id * T)(S * id), \\
T \swarrow S \ := \ \gamma(T) \beta(S) := (T * id)(id * S), \\
T \downarrow S \ := \ -\beta(T \gamma(S)) := -\text{id} * (T(S * id)), \\
T \downarrow S \ := \ -\beta(T \gamma(S)) := -\text{id} * ((T * id)S), \\
T \searrow S \ := \ -T \gamma(id) \beta(S) := -T(id * id)(id * S), \\
T \searrow S \ := \ -\beta(T \gamma(id))S := -(id * T)(id * id)S, \\
T \bullet S \ := \ \beta(T \gamma(id))S := id * (T(id * id)S).
\]

The horizontal structure is a dendriform trialgebra given by,

\[
T \wedge S \ := \ T \nearrow S + T \nwarrow S + T \uparrow S := (id * T)(S * id) + T(id * S * id) - \text{id} * (T(S * id)), \\
T \vee S \ := \ T \nwarrow S + T \nearrow S + T \downarrow S := (id * T * id)S + (T * id)(id * S) - \text{id} * ((T * id)S), \\
T \star S \ := \ T \searrow S + T \searrow S + T \bullet S := -T(id * id)(id * S) - (id * T)(id * id)S + \text{id} * (T(id * id)S).
\]

The vertical structure is a NS-algebra given by,

\[
T \triangleright S \ := \ T \nearrow S + T \nwarrow S + T \searrow S := (id * T)(S * id) + (id * T * id)S - (id * T)(id * id)S, \\
T \triangleleft S \ := \ T \nwarrow S + T \nearrow S + T \searrow S := T(id * S * id) + (T * id)(id * S) - T(id * id)(id * S), \\
T \circ S \ := \ T \downarrow S + T \downarrow S + T \bullet S := -\text{id} * (T(S * id)) - \text{id} * ((T * id)S) + \text{id} * (T(id * id)S).
\]

The associative operation \(\star\) is the sum of the nine operations, i.e.,

\[
T \star S \ := \ (id * T)(S * id) + (id * T * id)S - (id * T)(id * id)S \\
+ T(id * S * id) + (T * id)(id * S) - T(id * id)(id * S) \\
- \text{id} * (T(S * id)) - \text{id} * ((T * id)S) + \text{id} * (T(id * id)S).
\]
Let us recall what an operad is, see [12, 7, 8] for instance.

Let $P$ be a type of algebras, for instance the $NS$-algebras, and $P(V)$ be the free $P$-algebra on the $k$-vector space $V$. Suppose $P(V) := \bigoplus_{n\geq 1} P(n) \otimes S_n V^\otimes n$, where $P(n)$ are right $S_n$-modules. Consider $P$ as an endofunctor on the category of $k$-vector spaces. The structure of the free $P$-algebra of $P(V)$ induces a natural transformation $\pi : P\circ P \to P$ as well as $u : Id \to P$ verifying usual associativity and unitarity axioms. An algebraic operad is then a triple $(P, \pi, u)$. A $P$-algebra is then a $k$-vector space $V$ together with a linear map $\pi_A : P(A) \to A$ such that $\pi_A \circ \pi(A) = \pi_A \circ P(\pi_A)$ and $\pi_A \circ u(A) = Id_A$. The $k$-vector space $P(n)$ is the space of $n$-ary operations for $P$-algebras. We will always suppose there is, up to homotheties, a unique 1-ary operation, the identity, i.e., $P(1) := k\text{Id}$ and that all possible operations are generated by composition from $P(2)$. The operad is said to be binary. It is said to be quadratic if all the relations between operations are consequences of relations described exclusively with the help of monomials with two operations. An operad is said to be non-symmetric if, in the relations, the variables $x, y, z$ appear in the same order. The $k$-vector space $P(n)$ can be written as $P(n) := P'(n) \otimes k[S_n]$, where $P'(n)$ is also a $k$-vector space and $S_n$ the symmetric group on $n$ elements. In this case, the free $P$-algebra is entirely induced by the free $P$-algebra on one generator $P(k) := \bigoplus_{n\geq 1} P'(n)$. The generating function of the operad $P$ is given by:

$$f^P(x) := \sum (-1)^n \frac{\dim P(n)}{n!} x^n := \sum (-1)^n \dim P'(n)x^n.$$  

Below, we will indicate the sequence $(\dim P'(n))_{n\geq 1}$.

### 8.1 On the free $NS$-algebra

Let $V$ be a $k$-vector space. The free $NS$-algebra $\mathcal{N}(V)$ on $V$ is by definition, a $NS$-algebra equipped with a map $i : V \mapsto \mathcal{N}(V)$ which satisfies the following universal property: for any linear map $f : V \to A$, where $A$ is a $NS$-algebra, there exists a unique $NS$-algebra morphism $\tilde{f} : \mathcal{N}(V) \to A$ such that $\tilde{f} \circ i = f$. The same definition holds for the free dendriform-Nijenhuis algebra.

Since the three operations of a $NS$-algebra have no symmetry and since compatibility axioms involve only monomials where $x$, $y$ and $z$ stay in the same order, the free $NS$-algebra is of the form:

$$\mathcal{N}(V) := \bigoplus_{n\geq 1} \mathcal{N}_n \otimes V^\otimes n.$$  

In particular, the free $NS$-algebra on one generator $x$ is $\mathcal{N}(k) := \bigoplus_{n\geq 1} \mathcal{N}_n$, where $\mathcal{N}_1 := kx$, $\mathcal{N}_2 := k(x \cdot x) \oplus k(x \cdot x) \oplus k(x \cdot x)$. The space of three variables made out of three operations is of dimension $2 \times 3^2 = 18$. As we have 4 relations, the space $\mathcal{N}_3$ has a dimension equal to $18 - 4 = 14$. Therefore, the sequence associated with the dimensions of $(\mathcal{N}_n)_{n\in\mathbb{N}}$ starts with $1, 3, 14\ldots$ Finding the free $NS$-algebra on one generator is an open problem. However, we will show that the augmented free $NS$-algebra over a $k$-vector space $V$ has a connected Hopf
algebra structure. Before, some preparations are needed. To be as self-contained as possible, we introduce some notation to expose a theorem due to Loday [12].

Recall that a bialgebra \((H, \mu, \Delta, \eta, \kappa)\) is a unital associative algebra \((H, \mu, \eta)\) together with co-unital coassociative coalgebra \((H, \Delta, \kappa)\). Moreover, it is required that the coproduct \(\Delta\) and the counit \(\kappa\) are morphisms of unital algebras. A bialgebra is connected if there exists a filtration \((F_rH)\), such that \(H = \bigcup_r F_rH\), where \(F_0H := k1_H\) and for all \(r\),

\[
F_rH := \{x \in H; \Delta(x) - 1_H \otimes x - x \otimes 1_H \in F_{r-1}H \otimes F_rH \}.
\]

Such a bialgebra admits an antipode. Consequently, connected bialgebras are connected Hopf algebras.

Let \(P\) be a binary quadratic operad. By a unit action [12], we mean the choice of two linear applications:

\[ v : P(2) \to P(1), \quad \varepsilon : P(2) \to P(1), \]

giving sense, when possible, to \(x \circ 1\) and \(1 \circ x\), for all operations \(\circ \in P(2)\) and for all \(x\) in the \(P\)-algebra \(A\), i.e., \(x \circ 1 = v(\circ)(x)\) and \(1 \circ x = \varepsilon(\circ)(x)\). If \(P(2)\) contains an associative operation, say \(\ast\), then we require that \(x \ast 1 := x := 1 \ast x\), i.e., \(v(\ast) := Id := \varepsilon(\ast)\). We say that the unit action, or the couple \((v, \varepsilon)\) is compatible with the relations of the \(P\)-algebra \(A\) if they still hold on \(A_+ := k1 \circ A\) as far as the terms as defined. Let \(A, B\) be two \(P\)-algebras such that \(P(2)\) contains an associative operation \(\ast\). Using the couple \((v, \varepsilon)\), we extend binary operations \(\circ \in P(2)\) to the \(k\)-vector space \(A \otimes k1 \oplus k1 \otimes B \oplus A \otimes B\) by requiring:

\[
\begin{align*}
(a \otimes b) \circ (a' \otimes b') &:= (a \ast a') \otimes (b \circ b') \quad \text{if} \quad b \circ b' \neq 1 \otimes 1, \quad (1) \\
(a \otimes 1) \circ (a' \otimes 1) &:= (a \circ a') \otimes 1, \quad \text{otherwise.} \quad (2)
\end{align*}
\]

The unit action or the couple \((v, \varepsilon)\) is said to be coherent with the relations of \(P\) if \(A \otimes k1 \oplus k1 \otimes B \oplus A \otimes B\), equipped with these operations is still a \(P\)-algebra. Observe that a necessary condition for having coherence is compatibility.

One of the main interest of these two concepts is the construction of a connected Hopf algebra on the augmented free \(P\)-algebra.

**Theorem 8.1 (Loday [12])** Let \(P\) be a binary quadratic operad. Suppose there exists an associative operation in \(P(2)\). Then, any unit action coherent with the relations of \(P\) equips the augmented free \(P\)-algebra \(P(V)_+\) on a \(k\)-vector space \(V\) with a coassociative coproduct \(\Delta : P(V)_+ \to P(V)_+ \otimes P(V)_+,\) which is a \(P\)-algebra morphism. Moreover, \(P(V)_+\) is a connected Hopf algebra.

**Proof:** See [12] for the proof. However, we reproduce it to make things clearer. Let \(V\) be a \(k\)-vector space and \(P(V)\) be the free \(P\)-algebra on \(V\). Since the unit action is coherent, \(P(V)_+ \otimes P(V)_+\) is a \(P\)-algebra. Consider the linear map \(\delta : V \to P(V)_+ \otimes P(V)_+,\) given by \(v \mapsto 1 \otimes v + v \otimes 1\). Since \(P(V)\) is the free \(P\)-algebra on \(V\), there exists a unique extension of \(\delta\) to a morphism of augmented \(P\)-algebra \(\Delta : P(V)_+ \to P(V)_+ \otimes P(V)_+.\) Now, \(\Delta\) is coassociative since the morphisms \((\Delta \otimes id)\Delta\) and \((id \otimes \Delta)\Delta\) extend the linear map \(V \to P(V)^{\otimes 3}_+\) which maps
v to $1 \otimes 1 \otimes v + 1 \otimes v \otimes 1 + v \otimes 1 \otimes 1$. By unicity of the extension, the coproduct $\Delta$ is coassociative. The bialgebra we have just obtained is connected. Indeed, by definition, the free $P$-algebra $P(V)$ can be written as $P(V) := \bigoplus_{n \geq 1} P(V)_n$, where $P(V)_n$ is the $k$-vector space of products of $n$ elements of $V$. Moreover, we have $\Delta(x \circ y) := 1 \otimes (x \circ y) + (x \circ y) \otimes 1 + x \otimes (1 \circ y) + y \otimes (x \circ 1)$, for all $x, y \in P(V)$ and $\circ \in P(2)$. The filtration of $P(V)_+$ is then $FrP(V)_+ = k.1 \oplus \bigoplus_{1 \leq n \leq r} P(V)_n$. Therefore, $P(V)_+ := \bigcup_r FrP(V)_+$ and $P(V)_+$ is a connected bialgebra. □

We will use this theorem to show that there exists a connected Hopf algebra structure on the augmented free $NS$-algebra as well as on the augmented free commutative $NS$-algebra.

**Proposition 8.2** Let $\mathcal{N}(V)$ be the free $NS$-algebra on a $k$-vector space $V$. Extend the binary operations $\prec, \succ$ and $\bullet$ to $\mathcal{N}(V)_+$ as follows:

$$x \succ 1 := 0, \; 1 \succ x := x, \; 1 \prec x := 0, \; x \prec 1 := x, \; x \bullet 1 := 0 := 1 \bullet x,$$

So that, $x \star 1 = x = 1 \star x$ for all $x \in \mathcal{N}(V)$. This choice is coherent.

**Remark:** We cannot extend the operations $\succ$ and $\prec$ to $k$, i.e., $1 \succ 1$ and $1 \prec 1$ are not defined.

**Proof:** Keep notation introduced in that Section. Firstly, let us show that this choice is compatible. Let $x, y, z \in \mathcal{N}(V)_+$. We have to show for instance that the relation $(x \prec y) \prec z = x \prec (y \star z)$ holds in $\mathcal{N}(V)_+$. Indeed, for $x = 1$ we get $0 = 0$. For $y = 1$ we get $x \prec z = x \prec z$ and for $z = 1$ we get $x \prec y = x \prec y$. The same checking can be done for the 3 other equations. The augmented $NS$-algebra $\mathcal{N}(V)_+$ is then a $NS$-algebra.

Secondly, let us show that this choice is coherent. Let $x_1, x_2, x_3, y_1, y_2, y_3 \in \mathcal{N}(V)_+$. We have to show that, for instance:

$$(x_1 \otimes y_1) \prec (x_2 \otimes y_2) \prec (x_3 \otimes y_3) = (x_1 \otimes y_1) \prec ((x_2 \otimes y_2) \star (x_3 \otimes y_3)).$$

Indeed, if there exists a unique $y_i = 1$, the other belonging to $\mathcal{N}(V)$, then, by definition we get:

$$(x_1 \star x_2 \star x_3) \otimes ((y_1 \prec y_2) \prec y_3) = (x_1 \star x_2 \star x_3) \otimes (y_1 \prec (y_2 \star y_3)),$$

which always holds since our choice of the unit action is compatible. If $y_1 = y_2 = y_3 = 1$, then the equality holds since our choice is compatible. If $y_2 = 1$, $y_1 = 1$ and $y_3 \in \mathcal{N}(V)$, we get: $0 = 0$, similarly if $y_1 = 1$, $y_3 = 1$ and $y_2 \in \mathcal{N}(V)$. If $y_1 \in \mathcal{N}(V)$, $y_2 = 1$ and $y_3 = 1$, the two hand sides are equal to $(x_1 \star x_2 \star x_3) \otimes y_1$. Therefore, this equation holds in $\mathcal{N}(V) \otimes 1.k \oplus k.1 \otimes \mathcal{N}(V) \oplus \mathcal{N}(V) \otimes \mathcal{N}(V)$. Checking the same thing with the 3 other relations shows that our choice of the unit action is coherent. □

**Corollary 8.3** There exists a connected Hopf algebra structure on the augmented free $NS$-algebra as well as on the augmented free commutative $NS$-algebra.

**Proof:** The first claim comes from the fact that our choice is coherent and from Theorem 8.1. For the second remark, observe that our choice is in agreement with the symmetry relations defining a commutative $NS$-algebra since for instance $x \prec^{op} 1 := 1 \succ x := x$ and $1 \succ^{op} x := x \prec 1 := x$, for all $x \in \mathcal{N}(V)$. □
8.2 On the free dendriform-Nijenhuis algebra

The same claims hold for the free dendriform-Nijenhuis algebra. The associated operad is binary, quadratic and non-symmetric. The free dendriform-Nijenhuis algebra on a $k$-vector space $V$ is of the form:

$$\mathcal{DN}(V) := \bigoplus_{n\geq 1} \mathcal{DN}_n \otimes V^\otimes n.$$ 

In particular, on one generator $x$,

$$\mathcal{DN}(k) := \bigoplus_{n\geq 1} \mathcal{N}_n,$$

where $\mathcal{N}_1 := kx$, $\mathcal{N}_2 := k(x \uparrow x) \oplus k(x \downarrow x) \oplus k(x \searrow x) \oplus k(x \nearrow x) \oplus k(x \swarrow x) \oplus k(x \nwarrow x) \oplus k(x \searrow x) \oplus k(x \nearrow x) \oplus k(x \bullet x)$. The space of three variables made out of nine operations is of dimension $2 \times 9^2 = 162$. As we have 28 relations, the space $\mathcal{N}_3$ has a dimension equal to $162 - 28 = 134$. Therefore, the sequence associated with the dimensions of $(\mathcal{DN}_n)_{n\in\mathbb{N}}$ starts with 1, 9, 134… Finding the free dendriform-Nijenhuis algebra on one generator is an open problem.

However, there exists a connected Hopf algebra structure on the augmented free dendriform-Nijenhuis algebra as well as on the augmented free commutative dendriform-Nijenhuis algebra.

**Proposition 8.4** Let $\mathcal{DN}(V)$ be the free dendriform-Nijenhuis algebra on a $k$-vector space $V$. Extend the binary operations $\searrow$ and $\nearrow$ to $\mathcal{DN}(V)_+$ as follows:

$$x \searrow 1 := x, \quad 1 \searrow x := 0, \quad 1 \nearrow x := x, \quad x \nearrow 1 := 0, \quad \forall x \in \mathcal{DN}(V).$$

In addition, for any other operation, $\diamond \in \{\swarrow, \nwarrow, \uparrow, \downarrow, \searrow, \nearrow, \bullet\}$ choose:

$$x \diamond 1 := 0 = 1 \diamond x, \quad \forall x \in \mathcal{DN}(V).$$

Moreover, this choice is coherent.

**Remark:** We cannot extend the operations $\searrow$ and $\searrow$ to $k$, i.e., $1 \searrow 1$ and $1 \searrow 1$ are not defined.

**Proof:** Keep notation introduced in that Section. Firstly, let us show that this choice is compatible. Let $x, y, z \in \mathcal{DN}(V)_+$. We have to show for instance that the relation $(x \searrow y) \searrow z = x \searrow (y \diamond z)$ holds in $\mathcal{DN}(V)_+$. Indeed, for $x = 1$ we get $0 = 0$. For $y = 1$ we get $x \searrow z = x \searrow z$ and for $z = 1$ we get $x \searrow y = x \searrow y$. We do the same thing with the 27 others and quickly found that the augmented dendriform-Nijenhuis algebra $\mathcal{DN}(V)_+$ is still a dendriform-Nijenhuis algebra.

Secondly, let us show that this choice is coherent. Let $x_1, x_2, x_3, y_1, y_2, y_3 \in \mathcal{DN}(V)_+$. We have to show that, for instance:

$$(Eq. M_1^{11}) \quad ((x_1 \otimes y_1) \searrow (x_2 \otimes y_2)) \searrow (x_3 \otimes y_3) = (x_1 \otimes y_1) \searrow ((x_2 \otimes y_2) \diamond (x_3 \otimes y_3)).$$
Indeed, if there exists a unique $y_1 = 1$, the other belonging to $\mathcal{DN}(V)$, then, by definition we get:
\[
(x_1 \bar{x}_2 \bar{x}_3) \otimes (y_1 \kern 0.15cm \kern 0.15cm \kern 0.15cm \downarrow y_2 \kern 0.15cm \kern 0.15cm \kern 0.15cm \downarrow y_3 = (x_1 \bar{x}_2 \bar{x}_3) \otimes y_1 \kern 0.15cm \kern 0.15cm \kern 0.15cm \downarrow (y_2 \bar{y}_3),
\]
which always holds since our choice of the unit action is compatible. Similarly for $y_1 = y_2 = y_3 = 1$. If $y_1 = 1 = y_2$ and $y_3 \in \mathcal{DN}(V)$, we get: $0 = 0$, similarly if $y_1 = 1 = y_3$ and $y_2 \in \mathcal{DN}(V)$. If $y_1 \in \mathcal{DN}(V)$ and $y_2 = 1 = y_3$, the two hand sides of $(\text{Eq. } M_{11}^1)$ are equal to $(x_1 \bar{x}_2 \bar{x}_3) \otimes y_1$. Therefore, $(\text{Eq. } M_{11}^1)$ holds in $\mathcal{DN}(V) \otimes 1.k \oplus k.1 \otimes \mathcal{DN}(V) \oplus \mathcal{DN}(V) \otimes \mathcal{DN}(V)$. Checking the same thing with the 27 other relations shows that our choice of the unit action is coherent. □

**Corollary 8.5** There exists a connected Hopf algebra structure on the augmented free dendriform-Nijenhuis algebra as well as on the augmented free commutative dendriform-Nijenhuis algebra.

**Proof:** The first claim comes from the fact that our choice is coherent and from Theorem 8.1. For the second remark, observe that our choice is in agreement with the symmetry relations defining a commutative dendriform-Nijenhuis algebra since for instance $x \kern 0.15cm \downarrow \kern 0.15cm \kern 0.15cm \kern 0.15cm \downarrow \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15cm \kern 0.15 cm

9 Conclusion

There exists another way to produce Nijenhuis operators by defining another type of bialgebras. Instead of defining the coproduct $\Delta$ by $\Delta(ab) := \Delta(a)b + a\Delta(b) - \mu(\Delta(a)) \otimes b$ on an associative algebra $(A, \mu)$, the following definition can be chosen: $\Delta(ab) := \Delta(a)b + a\Delta(b) - a \otimes \mu(\Delta(b))$. The generalisation of what was written is straightforward by observing that the right shift $\beta$ becomes a $TD$-operator whereas the left shift $\gamma$ becomes a Nijenhuis operator.

References

[1] M. Aguiar. Infinitesimal bialgebras, pre-lie and dendriform algebras. to appear in “Hopf algebras: Proceedings from an International Conference held at DePaul University”; arXiv:math.QA/0211074.

[2] M. Aguiar and J.-L. Loday. Quadri-algebras. arXiv:math.QA/0309171.

[3] G. Baxter. An analytic problem whose solution follows from a simple algebraic identity. Pacific J. Math., 10:731–742, 1960.

[4] J. Cariñena, J. Grabowski, and G. Marmo. Quantum bi-hamiltonian systems”. Internat. J. Modern Phys. A, 15(30):4797–4810, 2000.

[5] K. Ebrahimi-Fard. On the associative Nijenhuis relation. eprint, arXiv:math-ph/0302062.

[6] K. Ebrahimi-Fard. Loday-type algebras and the Rota-Baxter relation. Letters in Mathematical Physics, 61(2):139–147, 2002.
[7] B. Fresse. Koszul duality of operads and homology of partition posets. *Preprint 2002.*

[8] V. Ginzburg and M. Kapranov. Koszul duality for operads. *Duke Math. J.*, 76(1):203–272, 1994.

[9] S.A. Joni and G.-C. Rota. Coalgebras and bialgebras in combinatorics. *Stud. Appl. Math.*, 61:93–139, 1979.

[10] Ph. Leroux. Ennea-algebras. *eprint, arXiv:math.QA/0309213, (version 2).*

[11] Ph. Leroux. On some remarkable operads constructed from Baxter operators. *In preparation.*

[12] J.-L. Loday. Scindement d’associativité et algèbres de Hopf.

[13] J.-L. Loday. Une version non commutative des algèbres de Lie: Les algèbres de Leibniz. *L’Enseignement Math.*, 39:269–293, 1993.

[14] J.-L. Loday. Dialgebras. *in Dialgebras and related operads, Lecture Notes in Math.*, 1763:7–66, 2001.

[15] J.-L. Loday and M. Ronco. Trialgebras and families of polytopes. *eprint arXiv:math.QA/0205043.*

[16] J.-L. Loday and M. Ronco. Algèbres de Hopf colibres. *C.R.Acad.Sci Paris*, 337(Ser. I):153–158, 2003.

[17] G.-C. Rota. Baxter operators, an introduction. *in Gian-Carlo Rota on Combinatorics: Introductory papers and commentaries (J.P.S. Kung, Ed.) Birkhauser, Boston*, pages 504–512, 1995.