Integral formulas for a foliated sub-Riemannian manifold

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Abstract
We apply the notion of foliation to a nonholonomic manifold, which was introduced for the geometric interpretation of constrained systems in mechanics. We prove a series of integral formulas for a foliated sub-Riemannian manifold, that is, a Riemannian manifold equipped with a distribution $\mathcal{D}$ and a foliation $\mathcal{F}$ whose tangent bundle is a subbundle of $\mathcal{D}$. Our integral formulas generalize some results for a foliated Riemannian manifold and involve the shape operators of $\mathcal{F}$ with respect to normals in $\mathcal{D}$, the curvature tensor of induced connection on $\mathcal{D}$ and arbitrary functions depending on elementary symmetric functions of eigenvalues of the shape operators. For a special choice of these functions, integral formulas with the Newton transformations of the shape operators of $\mathcal{F}$ are obtained. Application to a foliated sub-Riemannian manifold with restrictions on the curvature and extrinsic geometry of $\mathcal{F}$ and also to codimension-one foliations are given.

Keywords Sub-Riemannian manifold · Foliation · Shape operator · Newton transformation · Mixed scalar curvature

Mathematics Subject Classification 53C12 · 53C17

1 Introduction

The concept of a nonholonomic manifold was introduced for the geometric interpretation of constrained systems in mechanics and thermodynamics. A nonholonomic manifold is a pair $(M, \mathcal{D})$, where $\mathcal{D}$ is a distribution (a subbundle of the tangent bundle) on a smooth manifold $M$. A sub-Riemannian manifold, that is $(M, \mathcal{D})$ equipped with a Riemannian metric $g$ on the distribution $\mathcal{D}$, is the main object of sub-Riemannian geometry, which has several lines of research stemming from optimal control methods, partial differential equations and limits of other geometries, see [3, 7].
Our goal is to apply the notion of foliation (which corresponds to an integrable distribution) to a nonholonomic manifold. In [14] we started to study a sub-Riemannian manifold endowed with a foliation of codimension one, i.e., \((M, D, g)\) equipped with a foliation \(\mathcal{F}\), whose tangent bundle \(T\mathcal{F}\) has codimension one in \(D\). In other words, \(D = T\mathcal{F} \oplus \text{span}(N)\), where \(N\) is a unit vector field orthogonal to the leaves of \(\mathcal{F}\). In [14], we proved a series of integral formulas for a codimension-one foliated sub-Riemannian manifold, which generalize results for a Riemannian manifold in [1, 5].

Integral formulas containing the curvature are useful for solving such problems in differential geometry and analysis, as the existence of foliations with a given geometric property (e.g., being totally geodesic, minimal or totally umbilical), prescribing the generalized mean curvatures of the leaves of a foliation, minimizing functionals defined for tensors on foliated manifolds, e.g., [17]. Integral formulas for foliations are valid for any sub-Riemannian metric and can help in the study of restrictions on the curvature of compact \(G\)-structures, see [10].

Analyzing history of extrinsic geometry of foliations, we see that from the origin it was related to some integral formulas containing the shape operator (or the second fundamental form) of the leaves and its invariants (mean curvature, higher order mean curvatures \(\sigma_r\), etc.) and some expressions corresponding to curvature of \(M\), see surveys [16, 17].

The first known integral formula for a codimension-one foliation of a closed Riemannian manifold by Reeb [12] tells that the integrated mean curvature \(H\) of the leaves is equal to zero:
\[
\int_M H \, \text{dvol}_g = 0.
\]
Its proof is based on the application of the Divergence Theorem to the identity \(\text{div} N = -H\) with \(N\) a unit normal to the leaves. The second formula in the series of total higher mean curvatures \(\sigma_r\)'s of a codimension-one foliation is, see [11],
\[
\int_M (2 \sigma_2 - \text{Ric}_{N, N}) \, \text{dvol}_g = 0, \tag{1}
\]
which is a consequence of the Divergence Theorem applied to \(\nabla_N N + \sigma_1 N\).

In [1], the Newton transformations \(T_r(A)\) of the shape operator \(A : X \mapsto -\nabla_X N\) of the leaves were applied to codimension-one foliations, and a series of integral formulas for \(r \geq 0\) starting with (1) was obtained:
\[
\int_M ((r + 2) \sigma_{r+2} - \text{trace}_\mathcal{F}(T_r(A) \mathcal{R}_N) - g(\text{div}_\mathcal{F} T_r(A), \nabla_N N)) \, \text{dvol}_g = 0. \tag{2}
\]
Here, \(\mathcal{R}_N : Y \mapsto R_{Y, N} N (Y \in T\mathcal{F})\) is the Jacobi operator, \(R_{X, Y} = \nabla_X \nabla Y - \nabla_Y \nabla X - \nabla_{[X, Y]}\) is the Riemannian curvature tensor, and \(\nabla\) is the Levi-Civita connection.

As a consequence of (2), the integrals of \(\sigma_r\) of a codimension-one foliation on a compact manifold \(M^{n+1}(c)\) of constant curvature \(c\) do not depend on the foliation: they depend on \(r, n, c\) and the volume of \((M, g)\) only, see [2, 5]:
\[
\int_M \sigma_r \, \text{dvol}_g = \begin{cases} 
  c^{r/2} \left( \frac{n/2}{r/2} \right) \text{Vol}(M, g), & \text{if } n \text{ and } r \text{ even}, \\
  0, & \text{either } n \text{ or } r \text{ odd}.
\end{cases} \tag{3}
\]
All these formulas were proved by applying the Divergence Theorem to suitable vector fields.

In [13], we have extended (2)–(3) as a series of integral formulas for two complementary orthogonal distributions $\mathcal{D}$ and $\tilde{\mathcal{D}}$ on a foliated Riemannian manifold. This series begins with the following generalization of (1), see [20], which has many applications:

\[
\int_M \left( S_{\text{mix}} + \| h \|^2 + \| h^\perp \|^2 - \| H \|^2 - \| H^\perp \|^2 - \| T \|^2 - \| T^\perp \|^2 \right) \, d\text{vol}_g = 0. \tag{4}
\]

Here, $h, T : \mathcal{D} \times \mathcal{D} \to \tilde{\mathcal{D}}$ and $H = \text{trace}_g h$ are the second fundamental form, the integrability tensor and the mean curvature vector field of $\mathcal{D}$, and similarly for $\tilde{\mathcal{D}}$, $S_{\text{mix}}$ is the mixed scalar curvature—one of the simplest curvature invariants of an almost product structure, e.g., [9, 13, 16, 17, 19, 20]. It is an averaged sum of sectional curvatures of 2-planes that non-trivially intersect with $\mathcal{D}$ and $\tilde{\mathcal{D}}$, and for an adapted orthonormal frame we have

\[
S_{\text{mix}} = \sum_{e_a \in \mathcal{D}} \sum_{e_i \in \tilde{\mathcal{D}}} g(R_{e_i, e_a, e_a, e_i}).
\]

For any compact leaf $L$ of the minimal distribution $\mathcal{D}$ (i.e., $H = 0$) an analog of (4) holds:

\[
\int_L \left( S_{\text{mix}} + \| h \|^2 + \| h^\perp \|^2 - \| T^\perp \|^2 \right) \, d\text{vol}_g|_L = 0. \tag{5}
\]

The following question arises: can the integral formulas of the type (2)–(5) be proved for a foliated sub-Riemannian manifold?

In this article, we answer this question in the affirmative and derive a series of integral formulas (in Theorems 3.2–3.5 and Corollaries 3.2–4.7) for a foliated sub-Riemannian manifold, which generalize (2)–(5) and extend the results in [14], where $\text{codim} \mathcal{F} = 1$. Our integral formulas involve the shape operators of $\mathcal{F}$ with respect to unit normals in $\mathcal{D}$ and some components of the curvature tensor of the induced connection on $\mathcal{D}$. The formulas also include arbitrary functions $f_j (0 \leq j < \dim \mathcal{F})$ depending on elementary symmetric functions of eigenvalues of the shape operators, and for a special choice $f_j = (-1)^j \sigma_{r-j}$ reduce to integral formulas with Newton transformations of the shape operators (and their scalar invariants — the $r$th mean curvatures) of $\mathcal{F}$. We apply our formulas to foliated sub-Riemannian manifolds with restrictions on the curvature and extrinsic geometry of the leaves of $\mathcal{F}$.

2 Preliminaries

2.1 Scalar invariants of the shape operators

Here, we define the shape operator $A_{\xi}$ and its Newton transformations $T_r(A_{\xi})$, the $r$th mean curvatures $\sigma_r(\xi)$ and power sums symmetric functions $\tau_k(\xi)$. Let $\mathcal{D}$ be
an \((n + p)\)-dimensional distribution on a smooth \(m\)-dimensional manifold \(M\), i.e., a subbundle of \(TM\) of rank \(n + p\) (where \(n, p > 0\) and \(n + p < m\)). In other words, to each point \(x \in M\) we assign an \((n + p)\)-dimensional subspace \(\mathcal{D}_x\) of the tangent space \(T_x M\) smoothly depending on \(x\). If \(\mathcal{D}\) is integrable, then it determines a foliation; in this case, the Lie bracket of any two vector fields from the distribution \(\mathcal{D}\) also belongs to \(\mathcal{D}\).

Usually, they assume that the sub-Riemannian metric on \(\mathcal{D}\) (the horizontal bundle) is extended to a Riemannian metric on the whole \(M\), also denoted by \(g\). This allows us to define the orthogonal distribution \(\overline{\mathcal{D}}\) (the vertical subbundle) such that \(TM = \mathcal{D} \oplus \overline{\mathcal{D}}\).

**Definition 2.1** A sub-Riemannian manifold \((M, \mathcal{D}, g)\) equipped with a foliation \(\mathcal{F}\) such that the tangent bundle \(T\mathcal{F}\) is a subbundle of \(\mathcal{D}\) will be called a foliated sub-Riemannian manifold.

Let \(\mathcal{F}\) be an \(n\)-dimensional foliation of a sub-Riemannian manifold \((M, \mathcal{D}, g)\). Then \(\mathcal{D} = T\mathcal{F} \oplus N\mathcal{F}\), where \(N\mathcal{F}\) is a \(p\)-dimensional orthogonal complement of \(T\mathcal{F}\) in \(\mathcal{D}\). Set \(N_1\mathcal{F} = \{\xi \in N\mathcal{F} : \|\xi\| = 1\}\). Let \(\mathcal{T}\) denotes the projection from \(TM\) on the vector subbundle \(T\mathcal{F}\). The shape operator \(A_\xi : T\mathcal{F} \rightarrow T\mathcal{F}\) of \(\mathcal{F}\) with respect to the unit normal \(\xi \in N_1\mathcal{F}\) is defined by \(A_\xi : X \mapsto -(\nabla_X \xi)^\top\).

The *elementary symmetric functions* \(\sigma_j(A_\xi)\) of \(A_\xi\) (or, the \(r\)th mean curvatures of \(\mathcal{F}\) in \(\mathcal{D}\)) are coefficients of \(t^r\) in the polynomial equality \(\det(\text{id}_{T\mathcal{F}} + tA_\xi) = \sum_{r=0}^p \sigma_r(A_\xi)t^r\), \(t \in \mathbb{R}\), e.g., [17]. Note that \(\sigma_r(A_\xi) = \sum_{i_1 < \ldots < i_r} \lambda_{i_1}(\xi) \cdots \lambda_{i_r}(\xi)\), where \(\lambda_1(\xi) \leq \cdots \leq \lambda_n(\xi)\) are the eigenvalues of \(A_\xi\). The *power sums* symmetric functions \(\tau_j(A_\xi)\) of \(A_\xi\) are given by \(\tau_j(A_\xi) = \text{trace}(A_\xi^j)\), \(j \in \mathbb{N}\). For short, set \(\sigma_0(\xi) = \sigma_r(A_\xi)\) and \(\tau_r(\xi) = \tau_r(A_\xi)\). For example, \(\sigma_0(\xi) = 1\), \(\sigma_1(\xi) = \text{det } A_\xi\), \(\sigma_1(\xi) = \tau_1(\xi) = \text{trace } A_\xi\) and \(2 \sigma_2(\xi) = \tau_1^2(\xi) - \tau_2(\xi)\). By the Cayley–Hamilton theorem, \(\tau_{n-1+j}(\xi)\) are expressed in terms of sums of lower powers \(\tau_j(\xi)\) \((j < n)\). By the Newton formulas,

\[
\tau_j(\xi) - \tau_{j-1}(\xi)\sigma_1(\xi) + \cdots + (-1)^{j-1}\tau_1(\xi)\sigma_{j-1}(\xi) + (-1)^j j \sigma_j(\xi) = 0 \quad (1 \leq j \leq n),
\]

the power sums \(\tau_j(\xi)\) \((1 \leq j \leq n)\) are polynomials of symmetric functions \(\sigma_i(\xi)\) \((1 \leq j \leq n)\).

**Definition 2.2** The *Newton transformations* \(T_r(A_\xi)\) of the shape operator \(A_\xi\) of a foliated sub-Riemannian manifold \((M, \mathcal{D}, \mathcal{F}, g)\) are defined recursively or explicitly by

\[
T_0(A_\xi) = \text{id}_{T\mathcal{F}}, \quad T_r(A_\xi) = \sigma_r(\xi)\text{id}_{T\mathcal{F}} - A_\xi T_{r-1}(A_\xi) \quad (0 < r \leq n),
\]

\[
T_r(A_\xi) = \sum_{j=0}^r (-1)^j \sigma_{r-j}(\xi) A_\xi^j = \sigma_r(\xi)\text{id}_{T\mathcal{F}} - \sigma_{r-1}(\xi) A_\xi + \cdots + (-1)^r A_\xi^r.
\]
For example, \( T_1(A_\xi) = \sigma_1(\xi) \text{id}_{T^*F} - A_\xi \) and \( T_n(A_\xi) = 0 \). Define a \((1, 1)\)-tensor field

\[
A_\xi := \sum_{j=0}^{n-1} f_j(\tau_1(\xi), \ldots, \tau_n(\xi)) A^j_\xi,
\]

where \( f_j : \mathbb{R}^n \to \mathbb{R} \) \((0 \leq j < n)\) are given functions. We write \( f_j(\tau_1(\xi), \ldots, \tau_n(\xi)) = f_j(\xi) \) shortly. The choice of the RHS for \( A_\xi \) in (9) is natural for the following reasons, [17, Section 2.1.3]:

- the powers \( A^j_\xi \) are the only \((1, 1)\)-tensors, obtained algebraically from \( A_\xi \), while \( \tau_1(\xi), \ldots, \tau_n(\xi) \), or, equivalently, \( \sigma_1(\xi), \ldots, \sigma_n(\xi) \), generate all scalar invariants of \( A_\xi \),
- the Newton transformation \( T_r(A_\xi) \) \((0 \leq r < n)\) of (8) depends on all \( A^j_\xi \) \((0 \leq j \leq r)\).

In this paper, the results obtained for \( A_\xi \) are illustrated by the special case \( A_\xi = T_r(A_\xi) \).

Let \( \nabla^F : TM \times T^*F \to T^*F \) be the induced connection on the vector subbundle \( T^*F \) (and on the leaves of \( F \)). Since the \((1, 1)\)-tensors \( A_\xi \) and \( T_r(A_\xi) \) are self-adjoint, we have

\[
g((\nabla^F_X A_\xi)Y, V) = g((\nabla^F_X A_\xi)V, Y),
\]

\[
g((\nabla^F_X T_r(A_\xi))Y, V) = g((\nabla^F_X T_r(A_\xi))V, Y),
\]

for \( X, Y, V \in T^*F \). The following properties of \( T_r(A) \) are proved similarly as for codimension-one foliations of a Riemannian manifold, see [1].

**Lemma 2.3** For the shape operator \( A_\xi \) of an \( n \)-dimensional foliation \( F \) we have

\[
\text{trace}_F T_r(A_\xi) = (n - r) \sigma_r(\xi),
\]

\[
\text{trace}_F (A_\xi \cdot T_r(A_\xi)) = (r + 1) \sigma_{r+1}(\xi),
\]

\[
\text{trace}_F (A^2 \cdot T_r(A_\xi)) = \sigma_1(\xi) \sigma_{r+1}(\xi) - (r + 2) \sigma_{r+2}(\xi),
\]

\[
\text{trace}_F (T_{r-1}(A_\xi)(\nabla^F_X A_\xi)) = X(\sigma_r(\xi)), \quad X \in T^*F,
\]

\[
k \text{trace}_F (A^{k-1}_\xi \nabla^F_X A_\xi) = X(\tau_k(\xi)), \quad X \in T^*F, \quad k > 0.
\]

**2.2 The induced connection and curvature on \( D \)**

Here, the orthoprojector \( P : TM \to D \) is used to define the induced connection and its curvature tensor associated with a foliated sub-Riemannian manifold, and to prove the Codazzi type equation. The Levi-Civita connection \( \nabla \) on \( (M, g) \) induces a linear connection on \( D \):

\[
\nabla^P_X(PY) = P \nabla_X(Y), \quad X, Y \in \Gamma(TM),
\]
which is compatible with the metric on $\mathcal{D}$:

$$Xg(U, V) = g(\nabla_X^P U, V) + g(U, \nabla_X^P V), \quad U, V \in \mathcal{D}.$$ 

Let $R^P : TM \times TM \to \text{End}(\mathcal{D})$ be the curvature tensor of $\nabla^P$. As for any linear connection, we have the anti-symmetry for the first pair of vectors:

$$R^P_{Y,X} U = - R^P_{X,Y} U.$$ 

Since $\nabla^P$ is compatible with $g$, the following symmetry for the pair of vectors in $\mathcal{D}$ is valid, e.g., [8],

$$g(R^P_{X,Y} U, V) = - g(R^P_{X,Y} V, U). \quad (11)$$

Let $\perp$ be the projection of $TM$ on the vector subbundle $(T\mathcal{F})^\perp$ orthogonal to $\mathcal{F}$ in $TM$. The Codazzi equation for the leaves of a foliation of $(M, g)$ has the form, e.g., [3]:

$$(\nabla_X^f h)(Y, U) = (\nabla_Y h)(X, U) = (R^f_{X,Y} U)^\perp,$$ 

where $h : T\mathcal{F} \times T\mathcal{F} \to (T\mathcal{F})^\perp$ is the second fundamental form of $\mathcal{F}$ in $(M, g)$ defined by

$$h(X, Y) = (\nabla_X Y)^\perp.$$ 

Hence, $g(A_\xi(X), Y) = g(h(X, Y), \xi)$ for all $\xi \in \mathcal{NF}$.

**Lemma 2.4** The following Codazzi-type equation is true:

$$(\nabla_X^f A_\xi) Y - (\nabla_Y^f A_\xi) X = - R^f_{X,Y} \xi, \quad X, Y \in T\mathcal{F}, \quad \xi \in \mathcal{NF}. \quad (13)$$

**Proof** From (12), for all vectors $X, Y, U \in T\mathcal{F}$ we get

$$g((\nabla_X^f A_\xi) Y - (\nabla_Y^f A_\xi) X, U) + g(h(X, U), \nabla_Y \xi) - g(h(Y, U), \nabla_X \xi) = - g(R^f_{X,Y} \xi, U). \quad (14)$$

Applying the orthoprojector from $TM$ on the vector subbundle $(T\mathcal{F})^\perp$, we find

$$g(R^f_{X,Y} \xi, U) = g(R^f_{X,Y} \xi, U)$$

$$+ g(\nabla_X((\nabla_Y \xi)^\perp) - \nabla_Y((\nabla_X \xi)^\perp) - \nabla_{[X,Y]}^f \xi, U). \quad (15)$$

Using the equalities $[X, Y]^\perp = 0$ (since $T\mathcal{F}$ is integrable), (15) and

$$g(h(X, U), \nabla_Y \xi) = g(\nabla_Y \xi, (\nabla_X U)^\perp) = - g(\nabla_X((\nabla_Y \xi)^\perp), U),$$

$$g(h(Y, U), \nabla_X \xi) = g(\nabla_X \xi, (\nabla_Y U)^\perp) = - g(\nabla_Y((\nabla_X \xi)^\perp), U),$$

in (14) completes the proof. \qed

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2.3 The $\mathcal{F}$-divergence of $(1, 1)$-tensors on $\mathbb{D}$

An orthonormal frame $\{e_i, e_a\}_{i \leq n, a \leq p}$ on $\mathbb{D} = T\mathcal{F} \oplus N\mathcal{F}$ built either of single vectors or of local vector fields is said to be adapted, whenever $e_i \in T\mathcal{F}$ and $e_a \in N\mathcal{F}$. Following [1, 13], define the $\mathcal{F}$-divergence of $(1, 1)$-tensors $A^k_\xi$ and $T_\xi(A^k_\xi)$ by

$$\text{div}_\mathcal{F} A^k_\xi = \sum_{i=1}^n (\nabla^\mathcal{F}_e_i A^k_\xi) e_i, \quad \text{div}_\mathcal{F} T_\xi(A^k_\xi) = \sum_{i=1}^n (\nabla^\mathcal{F}_e_i T_\xi(A^k_\xi)) e_i. \quad (16)$$

Note that $\text{div}_\mathcal{F} T_0(A^k_\xi) = \text{div}_\mathcal{F} \text{id}_{T\mathcal{F}} = \nabla^\mathcal{F}_{e_i} (\text{id}_{T\mathcal{F}} e_i) - \text{id}_{T\mathcal{F}} (\nabla^\mathcal{F}_{e_i} e_i) = 0$.

For any $X \in \mathbb{D}$ and $\xi \in N_1\mathcal{F}$, define a linear operator $R^P_{X, \xi} : T\mathcal{F} \to T\mathcal{F}$ by

$$R^P_{X, \xi} : V \to (R^P_{V, X} \xi)^\top, \quad V \in T\mathcal{F}.$$ 

The following result generalizes [1, Lemma 2.2], see also [13, Lemmas 2.4 and 2.5].

**Proposition 2.5** (a) The following formula is valid for any $\xi \in N_1\mathcal{F}$ and $X \in T\mathcal{F}$:

$$g(\text{div}_\mathcal{F} A^k_\xi, X) = \sum_{k<n} \left( (A^k_\xi X)(f^k(\xi)) \right. \right.$$  

$$+ f^k(\xi) \sum_{j=1}^k \left( \frac{1}{k - j + 1} (A^{j-1}_\xi X)(\tau_{k-j+1}(\xi)) \right)$$

$$\left. + \text{trace}_\mathcal{F} (A^{k-j}_\xi R^P_{(-A_j)X, \xi}) \right) \quad (17)$$

(b) For $f_j = (-1)^j \sigma_{r-j}$ and $0 < r < n$, (17) gives the following:

$$g(\text{div}_\mathcal{F} T_\xi(A^k_\xi), X) = \sum_{j=1}^r \text{trace}_\mathcal{F}(T_{r-j}(A^k_\xi) R^P_{(-A_j)X, \xi}). \quad (18)$$

**Proof** (a) We will calculate at a point $x \in M$. One may assume $\nabla_{e_i} \xi \mid_x \in (T\mathcal{F})_x$ for all $i$. Decomposing $A^k_\xi = A_\xi A^{k-1}_\xi$ for $k \geq 1$ and using (16), we get at a point $x$,

$$\text{div}_\mathcal{F} A^k_\xi = A_\xi \text{div}_\mathcal{F} A^{k-1}_\xi + \sum_{i=1}^n (\nabla^\mathcal{F}_{e_i} A^k_\xi) A^{k-1}_\xi e_i. \quad (19)$$

Since (19) is tensorial, it is valid for any point of $M$. Using (13), we find for $X \in (T\mathcal{F})_x$,

$$\sum_{i=1}^n g((\nabla^\mathcal{F}_{e_i} A^k_\xi) A^{k-1}_\xi e_i, X) = \sum_{i=1}^n g(A^{k-1}_\xi e_i, (\nabla^\mathcal{F}_{e_i} A^k_\xi) X)$$

$$= \sum_{i=1}^n g(A^{k-1}_\xi e_i, (\nabla^\mathcal{F}_{X} A^k_\xi) e_i + R^P_{X, e_i} \xi)$$

$$= \text{trace}_\mathcal{F}(A^{k-1}_\xi \nabla^\mathcal{F}_{X} A^k_\xi) + \text{trace}_\mathcal{F}(A^{k-1}_\xi R^P_{X, \xi}).$$
For $X \in (T^\mathcal{F})_\xi$ and for $k \geq 1$, (19) gives us
\[
g(\text{div}_\mathcal{F} A^k_\xi, X) = -g(A_\xi \text{div}_\mathcal{F} A^{k-1}_\xi, X) + \text{trace}_\mathcal{F}(A^{k-1}_\xi \nabla^\mathcal{F}_X A_\xi) + \text{trace}_\mathcal{F}(A^{k-1}_\xi R^P_{X,\xi}).
\]

The above and the last identity of Lemma 2.3, yield the inductive formula
\[
g(\text{div}_\mathcal{F} A^k_\xi, X) = g(\text{div}_\mathcal{F} A^{k-1}_\xi, -A_\xi X) + \frac{1}{k} X(\tau_k(\xi)) \]
\[
+ \text{trace}_\mathcal{F}(A^{k-1}_\xi R^P_{(A_\xi)-1,X,\xi}).
\]

By induction, from (20) we obtain the following for $k \geq 1$:
\[
g(\text{div}_\mathcal{F} A^k_\xi, X) = \sum_{j=1}^k \left( \frac{1}{k-j+1}(A^{j-1}_\xi X)(\tau_{k-j+1}(\xi)) \right.
\]
\[
+ \text{trace}_\mathcal{F}(A^{k-j}_\xi R^P_{(A_\xi)-j+1,X,\xi}) \right).
\]

For $k = 1$, (21) reads as $g(\text{div}_\mathcal{F} A_\xi, X) = X(\tau_1(\xi)) + \text{trace}_\mathcal{F} R^P_{X,\xi}.$ For $(1, 1)$-tensor $A_\xi$, see (9), we get
\[
g(\text{div}_\mathcal{F} A_\xi, X) = \sum_{k<n} \left(g(\nabla^\mathcal{F}_X f_k(\xi), A^k_\xi X) + f_k(\xi)g(\text{div}_\mathcal{F} A^k_\xi, X) \right).
\]

Thus, (17) follows from (21).

(b) By inductive definition (7), we obtain the following for $r > 0$:
\[
div_\mathcal{F} T_r(A_\xi) = \nabla^\mathcal{F}_r(\xi) - A_\xi \text{div}_\mathcal{F} T_{r-1}(A_\xi) - \sum_{i=1}^n (\nabla^\mathcal{F}_{e_i} A_\xi)(T_{r-1}(A_\xi) e_i).
\]

Note that the $(1, 1)$-tensor $\nabla^\mathcal{F}_{e_i} A_\xi$ is self-adjoint. Then, using (10), the third identity of Lemma 2.3 and Codazzi-type equation (13), we get the following:
\[
g(\text{div}_\mathcal{F} T_r(A_\xi), X) = g(\text{div}_\mathcal{F} T_{r-1}(A_\xi), (-A_\xi)X) + \text{trace}_\mathcal{F}(T_{r-1}(A_\xi)R^P_{X,\xi}).
\]

By induction, from (22) we obtain (18). \hfill \Box

**Example 2.6** Let the distribution $T^\mathcal{F}$ be $P$-curvature invariant, that is
\[
R^P_{X,Y} V \in T^\mathcal{F}, \quad X, Y, V \in T^\mathcal{F}.
\]

Then, by (11), equation (18) implies that $\text{div}_\mathcal{F} T_r(A_\xi) = 0$ for all $r$ and $\xi$. Note that (23) is satisfied, if $\nabla_X Y \in \Gamma(T^\mathcal{F})$ for all $X, Y \in \Gamma(T^\mathcal{F}).$ The following equality (for some $c \in \mathbb{R}$) is sufficient for (23):

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\[
R^P_{X,Y} Z = c \left( g(Y, Z)X - g(X, Z)Y \right), \quad X, Y, Z \in \mathcal{D}.
\]  
(24)

**Remark 2.7** Let \( \mathcal{F} \) be a foliation of a Riemannian manifold \((M, g)\) and \( \mathcal{D} = TM \); thus, \( R^P = R \). Using the symmetry \( g(R_{X,Y} U, V) = g(R_{U,V} X, Y) \) of the curvature tensor, we simplify (22) to the following form, see [13, Proposition 2.9] and [1, Lemma 2.2]:

\[
\text{div}_{\mathcal{F}} T_r(A_\xi) = - A_\xi \text{div}_{\mathcal{F}} T_{r-1}(A_\xi) + \sum_{i=1}^{n} \left( R(\xi, T_{r-1}(A_\xi) e_i) e_i \right)^\top.
\]

**2.4 The \( \mathcal{F} \)-divergence of vector fields on \( \mathcal{D} \)**

For a local vector field \( \xi \) in \( N_1 \mathcal{F} \), define the vector field \( Z(\xi) = (\nabla_{\xi} \xi)^\top \). The value of \( Z(\xi) \) at any point \( x \in M \) does not depend on the extension of \( \xi \in (N_1 \mathcal{F})_x \) as a local vector field in \( N_1 \mathcal{F} \). The \( \mathcal{F} \)-divergence of a vector field \( Y \) on \( M \) is defined by

\[
\text{div}_{\mathcal{F}} Y = \sum_{i \leq n} g(\nabla_{e_i} Y, e_i),
\]

where \( \{e_i\} \) is a local orthonormal frame of \( T\mathcal{F} \).

For a vector field \( Y \) on a compact Riemannian manifold \((M, g)\) without boundary, or on a compact leaf \( L \) of \( \mathcal{F} \), the Divergence Theorem reads as

\[
\int_M (\text{div} \, Y) \, d\text{vol}_g = 0, \quad \int_L (\text{div}_{\mathcal{F}} Y) \, d\text{vol}_g |_L = 0.
\]  
(25)

Let \( d\omega^1_x \) be the volume form on the fiber \((N_1 \mathcal{F})_x \) (a unit sphere) with the induced metric.

Applying Lemma 2.10 given below, and (17), we generalize [13, Proposition 2.8].

**Proposition 2.8** The \( \mathcal{F} \)-divergence of a vector field given by

\[
Y(x) = \int_{(N_1 \mathcal{F})_x} A_\xi Z(\xi) \, d\omega^1_x
\]

at any point \( x \in M \) is

\[
(\text{div}_{\mathcal{F}} Y)(x) = \int_{(N_1 \mathcal{F})_x} \left( g(\text{div}_{\mathcal{F}} A_\xi, Z(\xi)) + \text{trace}_{\mathcal{F}}(A_\xi R^P_{\xi,\xi}) \\
+ \sum_{k<n} \left( f_k(\xi) \tau_{k+2}(\xi) - \frac{f_k(\xi)}{k+1} \xi(\tau_{k+1}(\xi)) \right) \\
+ \sum_{a \leq p} g(A_\xi (\nabla_{\xi} e_a)^\top, \nabla_{\xi} e_a) \right) \, d\omega^1_x,
\]  
(26)
where the underlined term is given in (17) with \( X = Z(\xi) \). In particular, the \( \mathcal{F} \)-divergence of a vector field given by \( Y(x) = \int_{(N_1, \mathcal{F})} T_r(A_x)Z(\xi) \, d\omega_x^\perp \) at any point \( x \in M \) is

\[
(\text{div}_\mathcal{F} Y)(x) = \int_{(N_1, \mathcal{F})} \left( g(\text{div}_\mathcal{F} T_r(A_x), Z(\xi)) \right. \\
- \frac{\xi(\sigma_{r+1}(\xi))}{(r+2)\sigma_{r+2}(\xi)} \\
+ \sigma_1(\xi)\sigma_{r+1}(\xi) + \text{tr}_\mathcal{F}(T_r(A_x)\mathcal{R}^P_{\xi, \xi}) \right. \\
+ \left. \sum_{a \leq p} g(T_r(A_x)(\nabla_e\xi)^\top, \nabla_\xi e_a) \right) \, d\omega_x^\perp. 
\]

where the underlined term is given by (18) with \( X = Z(\xi) \).

**Proof** To prove (26), assume that \( \nabla_X \xi \in (T \mathcal{F})_x \) for any \( X \in T_x M \), and calculate

\[
(\text{div}_\mathcal{F} Y)(x) = \sum_{i \leq n} \int_{(N_1, \mathcal{F})} g(\nabla_{e_i}(A_x Z(\xi)), e_i) \, d\omega_x^\perp \\
= \int_{(N_1, \mathcal{F})} \left( g(\text{div}_\mathcal{F} A_x, Z(\xi)) + \sum_{i \leq n} g(\nabla_{e_i} Z(\xi), A_x e_i) \right) \, d\omega_x^\perp. 
\]

By Lemma 2.10 in what follows, we find the integrand \( \sum_{i \leq n} g(\nabla_{e_i} Z(\xi), A_x e_i) \) in (28),

\[
\sum_{k < n} f_k(\xi) \tau_{k+2}(\xi) + \text{tr}_\mathcal{F}(A_x \mathcal{R}^P_{\xi, \xi}) \\
- \text{tr}_\mathcal{F}(A_x (\nabla^\mathcal{F}_\xi A_x)) + \sum_{a \leq p} g(A_x (\nabla_e\xi)^\top, \nabla_\xi e_a). 
\]

We transform the third term in (29), using (9) and the last identity of Lemma 2.3, as

\[
\text{tr}_\mathcal{F}(A_x \nabla^\mathcal{F}_\xi A_x) = \sum_{k < n} \frac{f_k(\xi)}{k+1} \xi(\tau_{k+1}(\xi)).
\]

Next, find

\[
\sum_{i=1}^n g(\nabla_{e_i}^\mathcal{F} Z(\xi), A_x e_i) = \text{tr}_\mathcal{F}(A_x \mathcal{R}^P_{\xi, \xi}) \\
+ \sum_{k < n} \left( f_k(\xi)\tau_{k+2}(\xi) - \frac{f_k(\xi)}{k+1} \xi(\tau_{k+1}(\xi)) \right) \\
+ \sum_{a \leq p} g(A_x (\nabla_e\xi)^\top, \nabla_\xi e_a). 
\]
Since the result (26) is tensorial, it is valid for any \( x \in M \). The proof of (27) is similar.

\[ \square \]

Remark 2.9 The integrals over \((N_1 \mathcal{F})_x\) when \( x \in M \) can be found explicitly. To show this, set \( I_{\lambda_1, \ldots, \lambda_p} = \int_{\|y\|=1} y^\lambda \, d\omega_{p-1} \), where \( \lambda = (\lambda_1, \ldots, \lambda_p) \), \( y = (y_1, \ldots, y_p) \) and \( p > 1 \). Then,

\[
I_{\lambda_1, \ldots, \lambda_p} = \frac{2}{\Gamma(p/2 + 1/2) \sum_{a \leq p} \lambda_a} \prod_{a \leq p} \frac{1}{2} (1 + (-1)^{\lambda_a}) \Gamma\left(\frac{1 + \lambda_a}{2}\right),
\]

see e.g., [13], where \( y^\lambda = \prod_{a \leq p} y_a^{\lambda_a} \), and \( \Gamma \) is the Gamma function. For example,

\[
I_{2\lambda_1, 0, \ldots, 0} = 2 \pi \frac{p-1}{\Gamma(p/2 + 1/2)} \frac{\Gamma(1/2 + \lambda_1)}{\Gamma(p/2 + \lambda_1)} , \quad I_{0, \ldots, 0} = \frac{2 \pi p/2}{\Gamma(p/2)} = \text{vol}(S^{p-1}(1)).
\]

The following lemma generalizes [1, Lemma 3.1], see also [13, Lemma 2.7].

Lemma 2.10 Let \( \{e_i, e_a\} \) be an adapted orthonormal frame of \( D = T\mathcal{F} \oplus N\mathcal{F} \) such that

- \( \nabla_X^D e_i = 0 \) (i \leq n) and \( g(\nabla_X e_a, e_b) = 0 \) \( (a, b \leq p) \) for any vector \( X \in T_x M \),
- \( \nabla^P_Y e_i = 0 \) (i \leq n) for any vector \( Y \in \mathcal{D}_x \)

at a point \( x \in M \). Then for any unit vector \( \xi \in (N_1 \mathcal{F})_x \) we have

\[
g(\nabla^D_X \xi, e_j) = g(A^2_\xi e_i, e_j) + g(R^P_{e_i, \xi} \xi, e_j) + g((\nabla^D_\xi A_\xi)e_i, e_j) + \sum_{a \leq p} g(\nabla^D_\xi e_a, e_i) g(\nabla e_a \xi, e_j).
\]

\[
\nabla^D_X \xi = \nabla^D_X \xi + g(\nabla^D_X \xi, e_j) e_j + g(\nabla^D_X \xi, P \nabla^D_\xi e_j) + g(\nabla^D_\xi e_j, \nabla^D_X \xi) + g(\nabla^D_\xi e_j, P \nabla^D_\xi e_j).
\]

Proof Taking covariant derivative of \( g(\nabla^D_X \xi, e_j) = -g(\xi, \nabla_X e_j) \) with respect to \( e_i \), we find

\[
-g(\nabla^D_X \xi, \nabla_X e_j) = g(\nabla_X^D \xi, e_j) + g(\nabla_X^D \xi, P \nabla_X^D e_j) + g(\nabla_X^D \xi, P \nabla_X^D e_j).
\]

For a foliation \( \mathcal{F} \), we obtain

\[
g(A^2_\xi e_i, e_j) = \nabla^D_\xi g(\xi, \nabla_X e_j) = g(\nabla^D_\xi \xi, \nabla_X e_j) + g(\xi, \nabla^D_\xi P \nabla_X e_j).
\]

Therefore, we calculate at the point \( x \in M \):

\[
g(A^2_\xi e_i, e_j) + g(R^P_{e_i, \xi} \xi, e_j) - g((\nabla^D_\xi A_\xi)e_i, e_j) = g(A^2_\xi e_i, e_j) + g(R^P_{e_i, \xi} \xi, e_j) - g((\nabla^D_\xi A_\xi)e_i, e_j) = g(A^2_\xi e_i, e_j) - g(\nabla^D_\xi \xi, \nabla_X e_j) - g(\nabla_X^D \xi, e_j) + g(\nabla_X^D \xi, e_j).
\]

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Using (30) and the following equalities at \( x \in M \):

\[
P\nabla_{e_i} \xi = \sum_{j \leq n} g(\nabla_{e_i} \xi, e_j) e_j, \quad P\nabla_{\xi} e_i = \sum_{a \leq p} g(\nabla_{\xi} e_i, e_a) e_a,
\]

\[
g(A_{\xi}^2 e_i, e_j) = - \sum_{k \leq n} g(\nabla_{e_i} \xi, e_k) g(\nabla_{e_k} e_j, \xi),
\]

we simplify the last line in (31) as

\[
g(\nabla_{e_i} A(\xi), e_j) - \sum_{k=1}^{n} g(\nabla_{\xi} e_k, e_i) g(\nabla_{e_k} e_j, \xi).
\]

From the above, the claim follows. \( \square \)

### 3 Main results

Here, we prove integral formulas for a foliated sub-Riemannian manifold \((M, D, \mathcal{F}, g)\).

The idea is to find the divergence of a suitable vector field and apply the Divergence Theorem and Lemma 3.1 in what follows to \((M, g)\) or to a compact leaf of a foliation with the induced metric. We will integrate over the normal sphere bundle \( N_1 \mathcal{F} \subset N\mathcal{F} \) of \( \mathcal{F} \) with the induced metric.

**Lemma 3.1** ([4, Note 7.1.1.1]) *For any function \( f : N_1 \mathcal{F} \to \mathbb{R} \), we have the Fubini formula,*

\[
\int_{N_1 \mathcal{F}} f(\xi) \, d\omega_{\perp} = \int_{M} \left( \int_{(N_1 \mathcal{F})_x} f(\xi) \, d\omega_{\perp}^x \right) \, dvol_g,
\]

where \( d\omega_{\perp} \) is the volume form on \( N_1 \mathcal{F} \) and \( dvol_g \) is the volume form of \((M, g)\). *Differentiation along \( M \) commutes with the integration along the fibers \((N_1 \mathcal{F})_x\).*

The next theorem generalizes [13, Theorem 3.1].

**Theorem 3.2** *Let \((M, D, \mathcal{F}, g)\) be a foliated sub-Riemannian manifold with \( D = T\mathcal{F} \oplus N\mathcal{F} \). Then for any compact leaf \( L \) of \( \mathcal{F} \) we get the following integral formula:*

\[
\int_{N_1 \mathcal{F}|_L} \left( g(\text{div}_\mathcal{F} A_{\xi}, Z(\xi)) + \sum_{k<n} \left( f_k(\xi) \tau_{k+2}(\xi) - \frac{f_k(\xi)}{k+1} \xi(\tau_{k+1}(\xi)) \right) \right. \\
\left. + \text{trace}_\mathcal{F}(A_{\xi} R_{\xi,e_a}^{P}) + \sum_{a \leq p} g(\nabla_{e_a}^T \nabla_{e_a} \xi, \nabla_{e_a} e_a) \right) \, d\omega_{\perp}^L = 0,
\]

where \( d\omega_{\perp}^L \) is the volume form on \( N_1 \mathcal{F}|_L \), and the underlined term is given by (17) with \( X = Z(\xi) \). *For the special case \( f_j = (-1)^j \sigma_{r-j}(\xi) \), (32) gives the following*
integral formula:

$$\int_{\mathcal{N}_1:F|L} \left( g(\text{div}_{\mathcal{F}} T_{r}(A_{\xi}), Z(\xi)) - \xi(\sigma_{r+1}(\xi)) ight. \
- (r + 2) \sigma_{r+2}(\xi) + \sigma_1(\xi) \sigma_{r+1}(\xi) \
+ \text{trace}_{\mathcal{F}}(T_{r}(A_{\xi})R_{\xi,\xi}) \
+ \sum_{a \leq p} g(T_{r}(A_{\xi})(\nabla e_{a},\xi)^{\top}, \nabla_{\xi} e_{a}) \right) \, d\omega_{L} = 0, \quad (33)$$

where the underlined term is given by (18) with $X = Z(\xi)$.

**Proof** This follows from Proposition 2.8, Lemma 3.1 and the Divergence Theorem (25)(b). \qed

For any vector field $Y$ tangent to $\mathcal{F}$, we have

$$\text{div} \, Y = \text{div}_{\mathcal{F}} \, Y - g(Y, H^\perp) - g(Y, \widetilde{H}), \quad (34)$$

where $H^\perp$ is the mean curvature vector of $\mathcal{N}\mathcal{F}$, and $\widetilde{H}$ is the mean curvature vector field of $\mathcal{D}$. Recall that a distribution $\mathcal{D}$ is minimal if $\widetilde{H} = 0$. There are topological restrictions for the existence of a Riemannian metric on closed manifold, for which a given distribution is minimal, see [18]. Using Proposition 2.8 and Lemma 3.1, we get the following integral formulas for a closed foliated sub-Riemannian manifold, which generalize [13, Theorem 3.3].

**Theorem 3.3** Let $(M, D, \mathcal{F}, g)$ be a closed foliated sub-Riemannian manifold with $D = T\mathcal{F} \oplus \mathcal{N}\mathcal{F}$ and a minimal orthogonal distribution $\mathcal{D}$. Then the following integral formula is valid:

$$\int_{\mathcal{N}_1:F} \left( g(\text{div}_{\mathcal{F}} A_{\xi}, Z(\xi)) ight. \
+ \sum_{k < n} \left( f_k(\xi) \tau_{k+2}(\xi) + \frac{\tau_{k+1}(\xi)}{k + 1} (\xi(f_k(\xi)) - f_k(\xi)\tau_1(\xi)) \right) \
+ \text{trace}_{\mathcal{F}}(A_{\xi}R_{\xi,\xi}^P) - g(A_{\xi} Z(\xi), H^\perp) \
+ \sum_{a \leq p} g(A_{\xi}(\nabla e_{a},\xi)^{\top}, \nabla_{\xi} e_{a}) \right) \, d\omega_{L}^\perp = 0, \quad (35)$$

where the underlined term is given by (17) with $X = Z(\xi)$. 

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For the special case $f_j = (-1)^j \sigma_{r-j}(\xi)$, (35) gives

$$
\int_{N^1F} \left( g(\text{div}_{\xi} Tr(A\xi), Z(\xi)) - (r + 2) \sigma_{r+2}(\xi) - g(Tr(A\xi)Z(\xi), H^\perp) \\
+ \text{trace}_{\xi} (Tr(A\xi) R^P_{\xi, \xi}) \\
+ \sum_{a \leq p} g(Tr(A\xi)(\nabla e_a\xi)^\top, \nabla\xi e_a) \right) d\omega^\perp = 0,
$$

(36)

where the underlined term is given by (18) with $X = Z(\xi)$.

**Proof** Using $\sum_{a \leq p} (\nabla e_a e_a)^\top = PH^\perp$, (34) with $Y(x) = \int_{(N^1F)_x} A\xi Z(\xi) d\omega^\perp_x$ at any point $x \in M$ and our assumption $\tilde{H} = 0$, we get

$$
(\text{div} Y)(x) = (\text{div}_{\xi} Y)(x) + \sum_{a \leq p} \int_{(N^1F)_x} g(\nabla e_a(A\xi Z(\xi)), e_a) d\omega^\perp_x \\
= (\text{div}_{\xi} Y)(x) - \int_{(N^1F)_x} g(A\xi Z(\xi), H^\perp) d\omega^\perp_x.
$$

(37)

Assuming $\nabla_X\xi \perp N^1F$ for any $X \in T_xM$ and $\xi \in (N^1F)_x$ at some point $x \in M$, we get

$$
\text{div} \xi = \text{div}_{\xi} \xi = -\tau_1(\xi).
$$

Using this equality, we also find

$$
\text{div} \left( \frac{f_k(\xi)}{k+1} \tau_{k+1}(\xi) \right) = \frac{f_k(\xi)}{k+1} \text{div}(\tau_{k+1}(\xi)) + \frac{1}{k+1} \tau_{k+1}(\xi) \text{div} f_k(\xi) \\
= \frac{f_k(\xi)}{k+1} \left( \tau_{k+1}(\xi) \text{div} \xi + \xi(\tau_{k+1}(\xi)) \right) + \frac{1}{k+1} \tau_{k+1}(\xi) \text{div} f_k(\xi) \\
= \frac{f_k(\xi)}{k+1} \xi(\tau_{k+1}(\xi)) + \frac{1}{k+1} \tau_{k+1}(\xi) (f_k(\xi) - f_k(\xi) \tau_1(\xi)).
$$

Then at the point $x \in M$ we obtain the equality for tensors

$$
\text{div} \int_{(N^1F)_x} \left( A\xi Z(\xi) + \sum_{k<n} \frac{f_k(\xi) \tau_{k+1}(\xi)}{k+1} \xi \right) d\omega^\perp_x \\
= \int_{(N^1F)_x} \left( g(\text{div}_{\xi} A\xi, Z(\xi)) \\
+ \sum_{k<n} \left( f_k(\xi) \tau_{k+2}(\xi) + \frac{1}{k+1} \tau_{k+1}(\xi) (f_k(\xi) - f_k(\xi) \tau_1(\xi)) \right) \\
+ \text{trace}_{\xi} (A\xi R^P_{\xi, \xi}) - g(A\xi Z(\xi), H^\perp) \right)
$$
\[ + \sum_{a \leq p} g(A_\xi (\nabla_{e_a} \xi)^\top, \nabla_\xi e_a) \right) d \omega_x^\perp, \]

see (17) for the underlined term. Applying the Divergence Theorem (25)(a) yields (35).

For the Newton transformations of \( A_\xi \), similarly to (37), we have

\[ (\text{div } Y)(x) = (\text{div}_\mathcal{F} Y)(x) - \int_{(N_1\mathcal{F})_x} g(T_r(A_\xi) Z(\xi), H^\perp)) d \omega_x^\perp, \]

where \( Y(x) = \int_{(N_1\mathcal{F})_x} T_r(A_\xi) Z(\xi) d \omega_x^\perp \) at any point \( x \in M \). Observe that

\[ \text{div}(\sigma_{r+1}(\xi) \xi) = -\sigma_1(\xi) \sigma_{r+1}(\xi) + \xi(\sigma_{r+1}(\xi)) \]

for all \( \xi \in N_1\mathcal{F} \). Then at any point \( x \in M \) we obtain

\[ \text{div} \int_{(N_1\mathcal{F})_x} \left( T_r(A_\xi) Z(\xi) + \sigma_{r+1}(\xi) \xi \right) d \omega_x^\perp \]

\[ = \int_{(N_1\mathcal{F})_x} \left( g(\text{div}_\mathcal{F} T_r(A_\xi), Z(\xi)) - (r + 2) \sigma_{r+2}(\xi) - g(T_r(A_\xi) Z(\xi), H^\perp) \right) \]

\[ + \text{trace}_\mathcal{F}(T_r(A_\xi) \mathcal{R}_P^{\xi}) + \sum_{a \leq p} g(T_r(A_\xi)(\nabla_{e_a} \xi)^\top, \nabla_\xi e_a) \right) d \omega_x^\perp, \]

see (27) for the underlined term. Using the Divergence Theorem (25)(a) yields (36).

\[ \Box \]

**Example 3.4** We shall look at first members of series (35)–(36). For \( A_\xi = A_\xi^2 \), by (35) we get

\[ \int_{N_1\mathcal{F}} \left( g(\text{div}_\mathcal{F} A_\xi^2, Z(\xi)) + \tau_4(\xi) - \frac{1}{3} \tau_3(\xi) \tau_1(\xi) + \text{trace}_\mathcal{F}(A_\xi^2 \mathcal{R}_P^{\xi}) \right. \]

\[ - g(A_\xi^2 Z(\xi), H^\perp) + \sum_{a \leq p} g(A_\xi^2(\nabla_{e_a} \xi)^\top, \nabla_\xi e_a) \right) d \omega_x^\perp = 0. \]

(38)

For \( r = 2 \), (36) gives

\[ \int_{N_1\mathcal{F}} \left( 4 \sigma_4(\xi) + g(T_2(A_\xi) Z(\xi), H^\perp) \right. \]

\[ - \text{trace}_\mathcal{F}(T_2(A_\xi) \mathcal{R}_P^{\xi} + T_1(A_\xi) \mathcal{R}_P^{\xi} - \mathcal{R}_P^{\xi} Z(\xi), \xi) \]

\[ - \sum_{a \leq p} g(T_2(A_\xi)(\nabla_{e_a} \xi)^\top, \nabla_\xi e_a) \right) d \omega_x^\perp = 0. \]

(39)

Let \((M, \mathcal{D}, \mathcal{F}, g)\) be a foliated sub-Riemannian manifold with \( \mathcal{D} = T\mathcal{F} \oplus N\mathcal{F} \), then
• $\mathcal{N}\mathcal{F}$ will be called $P$-auto-parallel, if $(\nabla_X Y)^\top = 0$ for all $X, Y \in \mathcal{N}\mathcal{F},$
• $\mathcal{F}$ will be called $P$-minimal, if $\sigma_1(\xi) = 0$ for all $\xi \in \mathcal{N}_1\mathcal{F},$
• $\mathcal{F}$ will be called $P$-totally umbilical, if

$$A_\xi = (\sigma_1(\xi)/n) \text{id}_{T\mathcal{F}}, \quad \xi \in \mathcal{N}_1\mathcal{F}. \quad (40)$$

Obviously, auto-parallel, minimal and totally umbilical distributions are $P$-auto-parallel, $P$-harmonic and $P$-totally umbilical, respectively, but the opposite is not true.

The mixed scalar $P$-curvature $S^P_{\text{mix}}$ is defined (similarly to $S_{\text{mix}}$) as an averaged sum of sectional $P$-curvatures of 2-planes that non-trivially intersect with each of the distributions $T\mathcal{F}$ and $\mathcal{N}\mathcal{F}$; for an adapted orthonormal frame $e_a, e_i$, we have

$$S^P_{\text{mix}} = \sum_{e_a \in T\mathcal{F}} \sum_{e_i \in T\mathcal{F}^\perp} g(R^P_{e_i, e_a} e_a, e_i).$$

Let $h^\perp, T^\perp : \mathcal{N}\mathcal{F} \times \mathcal{N}\mathcal{F} \to (\mathcal{N}\mathcal{F})^\perp$ and $H^\perp = \text{trace}_g h^\perp$ be the second fundamental form, the integrability tensor and the mean curvature vector field of $\mathcal{N}\mathcal{F}$, and similarly for $T\mathcal{F}$; in our case of a foliation, we have $T = 0$.

The following integral formulas generalize (4) and (5).

**Theorem 3.5** Let $(M, \mathcal{D}, \mathcal{F}, g)$ be a foliated sub-Riemannian manifold with $\mathcal{D} = T\mathcal{F} \oplus \mathcal{N}\mathcal{F}$.

(a) If $M$ is a closed manifold and orthogonal distribution $\tilde{\mathcal{D}}$ is minimal, then we get

$$\int_M (S^P_{\text{mix}} + \|P \circ h\|^2 + \|P \circ h^\perp\|^2 - \|P H\|^2 - \|P H^\perp\|^2 - \|P \circ T^\perp\|^2) \, d\text{vol}_g = 0. \quad (41)$$

(b) If $\mathcal{F}$ is $P$-minimal, then for any compact leaf $L$ of $\mathcal{F}$ we get the following:

$$\int_L (S^P_{\text{mix}} + \|P \circ h\|^2 + \|P \circ h^\perp\|^2 - \|P \circ T^\perp\|^2) \, d\text{vol}_g|_L = 0. \quad (42)$$

**Proof** (a) From (35) with $A_\xi = \text{id}_{T\mathcal{F}}$ (or, from (36) with $r = 0$), we get

$$\int_{\mathcal{N}_1\mathcal{F}} \left( \tau_1^2(\xi) - \tau_2(\xi) - \text{trace}_{\mathcal{F}} R^P_{\xi, \xi} \right. $$

$$\left. + g(\mathcal{Z}(\xi), H^\perp) - \sum_{a \leq p} g((\nabla_{e_a} \xi)^\top, \nabla_{\xi} e_a) \right) \, d\omega^\perp = 0. \quad (43)$$

Note that $\tau_1^2(\xi) - \tau_2(\xi) = 2 \sigma_2(\xi)$. Let $\xi = \sum_{a \leq p} y_a e_a$, where $y_a \in \mathbb{R}$, be any unit vector field in $\mathcal{N}\mathcal{F}$. For any 2-homogeneous function $f(\xi, \xi) = \frac{1}{2} \sum_{a \leq p} y_a^2$.
\[ \sum_{a,b \leq p} f(e_a, e_b) y_a y_b, \] we have

\[
\int_{(N_1 \mathcal{F})_x} f(\xi, \xi) \, d\omega^\perp_x = \tilde{T}_2 \sum_{a \leq p} f(e_a, e_a),
\]

where \( \tilde{T}_2 = \int_{(N_1 \mathcal{F})_x} y_a^2 \, d\omega^\perp_x = 2\pi \frac{p^{-1}}{\Gamma(p/2 + 1)} \Gamma(3/2). \)

see Remark 2.9. Applying this to the terms of (43), we find

\[
\int_{(N_1 \mathcal{F})_x} (N_1 F) x f(\xi, \xi) \, d\omega^\perp_x = \tilde{T}_2 \sum_{a \leq p} f(e_a, e_a),
\]

where \( \tilde{T}_2 = \int_{(N_1 \mathcal{F})_x} y_a^2 \, d\omega^\perp_x = 2\pi \frac{p^{-1}}{\Gamma(p/2 + 1)} \Gamma(3/2). \)

Using (44), we reduce (43) to the equality (41).

(b) We simplify (32) for \( A_\xi = \text{id}_{T \mathcal{F}} \) (the same gives (33) for \( r = 0 \)) as

\[
\int_{(N_1 \mathcal{F})_x} (\tau_2(\xi) - \tau_1^2(\xi)) \, d\omega^\perp_x = \tilde{T}_2 (\| P \circ h \|^2 - \| PH \|^2),
\]

\[
\int_{(N_1 \mathcal{F})_x} g(Z(\xi), H^\perp) \, d\omega^\perp_x = \tilde{T}_2 \| PH^\perp \|^2,
\]

\[
\int_{(N_1 \mathcal{F})_x} \text{trace} \mathcal{F} R^p_{\xi, \xi} \, d\omega^\perp_x = \tilde{T}_2 \sum_{a \leq p} \text{trace} \mathcal{F} R^p_{e_a, e_a} = \tilde{T}_2 S_{\text{mix}}^p,
\]

\[
\int_{(N_1 \mathcal{F})_x} \sum_{a \leq p} g((\nabla e_a \xi)^\top, \nabla \xi e_a) \, d\omega^\perp_x = \tilde{T}_2 (\| P \circ h \|^2 - \| P \circ T^\perp \|^2).
\]

Using (44), we reduce (43) to the equality (41).

Example 3.6 If \( N \mathcal{F} \) is \( P \)-auto-parallel, then \( (\nabla e_a \xi)^\top = (\nabla \xi e_a)^\top = 0 \) and \( Z(\xi) = 0, \) hence (38) and (39) shorten to the following integral formulas:

\[
\int_{N_1 \mathcal{F}} \left( \tau_2(\xi) - \xi(\tau_1(\xi)) + \text{trace} \mathcal{F} R^p_{\xi, \xi} + \sum_{a \leq p} g((\nabla e_a \xi)^\top, \nabla \xi e_a) \right) \, d\omega^\perp_x = 0.
\]

Using (44) and conditions \( \tau_1(\xi) = 0 \) and \( P \circ T^\perp = 0, \) we reduce (45) to the equality (42).

Example 3.6 If \( N \mathcal{F} \) is \( P \)-auto-parallel, then \( (\nabla e_a \xi)^\top = (\nabla \xi e_a)^\top = 0 \) and \( Z(\xi) = 0, \) hence (38) and (39) shorten to the following integral formulas:

\[
\int_{N_1 \mathcal{F}} \left( \tau_4(\xi) - \frac{1}{3} \tau_3(\xi) \tau_1(\xi) + \text{trace} \mathcal{F} (A^2_{\xi} R^p_{\xi, \xi}) \right) \, d\omega^\perp_x = 0,
\]

\[
\int_{N_1 \mathcal{F}} \left( 4 \sigma_4(\xi) - \text{trace} \mathcal{F} (T_2(2 \xi) R^p_{\xi, \xi}) \right) \, d\omega^\perp_x = 0.
\]

Similarly to (43), one can transform double integrals (46) to the integrals over \( M. \)
4 Applications of main results

Here, we apply results of Sect. 3 to a foliated sub-Riemannian manifold with restrictions on the curvature and extrinsic geometry, and also to foliations of codimension one.

**Corollary 4.1** (of Theorem 3.5) Let \((M, \mathcal{D}, \mathcal{F}, g)\) be a foliated sub-Riemannian manifold with \(\mathcal{D} = T\mathcal{F} \oplus N\mathcal{F}\) and \(P \circ T^\perp = 0\). If \(S^p_{\text{mix}} > 0\), then \(\mathcal{F}\) has no compact \(P\)-minimal leaves.

**Proof** By conditions, \(P \circ T^\perp = 0\) is valid. Thus, (42) reduces to the integral formula
\[
\int_L (S^P_{\text{mix}} + \|P \circ h\|^2 + \|P \circ h^\perp\|^2) \, d\operatorname{vol}_g |_L = 0,
\]
and the claim follows. \(\Box\)

**Corollary 4.2** A closed sub-Riemannian manifold \((M, \mathcal{D}, g)\) with a minimal distribution \(\mathcal{D}\) does not admit
(i) \(P\)-minimal foliations \(\mathcal{F}\), \(T\mathcal{F} \subseteq \mathcal{D}\), with \(P\)-auto-parallel \(N\mathcal{F}\) and \(S^P_{\text{mix}} > 0\).
(ii) \(P\)-totally umbilical foliations \(\mathcal{F}\), \(T\mathcal{F} \subseteq \mathcal{D}\), with \(P\)-auto-parallel \(N\mathcal{F}\) and \(S^P_{\text{mix}} < 0\).

**Proof** This follows directly from (41). For (ii) we use \(\|P \circ h\|^2 - \|PH\|^2 = -\frac{n-1}{n} \|PH\|^2\). \(\Box\)

**Corollary 4.3** (of Theorem 3.3) Let \((M, \mathcal{D}, \mathcal{F}, g)\) be a closed foliated sub-Riemannian manifold with \(\mathcal{D} = T\mathcal{F} \oplus N\mathcal{F}\). If \(\mathcal{D}\) is minimal and \(N\mathcal{F}\) is \(P\)-auto-parallel, then
\[
\int_{N_1\mathcal{F}} \left( \sigma_{k+2}(\xi) - \frac{1}{k+1} \, \sigma_{k+1}(\xi) \right) \, d\omega^\perp = 0, \quad (47)
\]
\[
\int_{N_1\mathcal{F}} \left( (r+2) \, \sigma_{r+2}(\xi) - \text{trace}_\mathcal{F}(T\mathcal{F}(A^\xi_\xi) \, R^P_{\xi,\xi}) \right) \, d\omega^\perp = 0. \quad (48)
\]

**Proof** This follows directly from (36) with \(A^\xi_\xi = A^k(\xi)\) and from (35). \(\Box\)

The total \(k\)-th mean curvatures of \(\mathcal{F}\) are defined by
\[
\sigma_{\mathcal{D},k}(\mathcal{F}) = \int_{N_1\mathcal{F}} \sigma_k(\xi) \, d\omega^\perp, \quad \tau_{\mathcal{D},k}(\mathcal{F}) = \int_{N_1\mathcal{F}} \tau_k(\xi) \, d\omega^\perp.
\]

Note that \(\sigma_{2s+1}(\mathcal{F}) = \tau_{2s+1}(\mathcal{F}) = 0\) and \(\sigma_0(\mathcal{F}) = \tau_0(\mathcal{F}) = n \frac{2\pi^{p/2}}{\Gamma(p/2)} \operatorname{Vol}(M, g)\), see Remark 2.9.

The next corollary of Theorem 3.2 generalizes (3), see also [5, Theorem 1.1] and [13, Corollary 4.3] on \(\mathcal{D} = TM\) (and [14, Corollary 1] and [1, Section 4.1] for codimension-one \(\mathcal{F}\)).

**Corollary 4.4** Let \((M, \mathcal{D}, \mathcal{F}, g)\) be a closed foliated sub-Riemannian manifold with \(\mathcal{D} = T\mathcal{F} \oplus N\mathcal{F}\) and minimal distribution \(\mathcal{D}\) such that \(N\mathcal{F}\) is \(P\)-auto-parallel and (24) holds. Then \(\sigma_{\mathcal{D},r}(\mathcal{F})\) depends on \(r, n, p, c\) and the volume of \((M, g)\) only, i.e., the following integral formula is valid:
\[
\sigma_{\mathcal{D},r}(\mathcal{F}) = \begin{cases} 
\frac{2\pi^{p/2}}{\Gamma(p/2)} \left( \frac{n}{2} \right)^{r/2} c^{r/2} \operatorname{Vol}(M, g), & \text{if } n \text{ and } r \text{ even}, \\
0, & \text{if } n \text{ or } r \text{ odd}.
\end{cases} \quad (49)
\]
Moreover, if \( \mathcal{F} \) is \( P \)-minimal, then

\[
\tau_{D,k}(\mathcal{F}) = \begin{cases} 
\frac{2\pi^{p/2}}{\Gamma(p/2)} (-c)^{k/2} \text{Vol}(M, g), & n \text{ and } k \text{ even}, \\
0, & \text{either } n \text{ or } k \text{ odd}.
\end{cases}
\] (50)

**Proof** By (24), \( \mathcal{R}^P_{\xi,\xi} = c \text{id}_\mathcal{F} \). By definition of \( \tau_k(\xi) \) and the first identity of Lemma 2.3,

\[
\text{trace}_\mathcal{F}(A^k_\xi \mathcal{R}^P_{\xi,\xi}) = c \tau_k(\xi), \quad \text{trace}_\mathcal{F}(T_r(A^k_\xi \mathcal{R}^P_{\xi,\xi})) = c (n - r) \sigma_r(\xi).
\]

By (47) and conditions, we get \( \tau_{D,k+2}(\mathcal{F}) = -c \tau_{D,k}(\mathcal{F}) \). By (48), we get \( \sigma_{D,r+2}(\mathcal{F}) = \frac{c(n-r)}{r+2} \sigma_{D,r}(\mathcal{F}) \). Then (by induction), from the above we obtain (49) and (50). \( \square \)

Similarly to (49)–(50) formulas are true when \((M, D, g)\) is a \( P \)-Einstein manifold, i.e.,

\[
\text{trace}_\mathcal{F} \mathcal{R}^P_{X,\xi} = C g(X, \xi), \quad X \in D, \quad \xi \in N_1 \mathcal{F}
\] (51)

for some \( C \in \mathbb{R} \), and a foliation \( \mathcal{F} \) is \( P \)-totally umbilical, see (40). Note that for \( D = TM \), condition (51) is satisfied for Einstein manifolds \((M, g)\).

The following corollary of (36) generalizes result in [13, Example 4.4] (see also [14, Corollary 2] and [1, Section 4.2] for codimension-one foliations).

**Corollary 4.5** Let \((M, D, \mathcal{F}, g)\) be a closed foliated sub-Riemannian manifold with a minimal distribution \( \tilde{D} \), \( D = TM \oplus N\mathcal{F} \) and the following conditions: (40), (51), and \( N\mathcal{F} \) is \( P \)-auto-parallel. Then \( \sigma_{D,r}(\mathcal{F}) \) depends on \( r, n, p, C \) and the volume of \((M, g)\) only, i.e., the following integral formula is valid:

\[
\sigma_{D,r}(\mathcal{F}) = \begin{cases} 
\frac{(C/n)^{r/2}}{r/2} \left( \frac{n/2}{r/2} \right) \text{Vol}(M, g), & n, r \text{ even}, \\
0, & r \text{ odd}.
\end{cases}
\] (52)

**Proof** By (40), we get \( T_r(A_\xi) = \frac{n-r}{n} \sigma_r(\xi) \text{id}_\mathcal{F} \). By (51), we also get

\[
\text{trace}_\mathcal{F} \mathcal{R}^P_{Z(\xi),\xi} = 0, \quad \text{trace}_\mathcal{F} \mathcal{R}^P_{\xi,\xi} = C,
\]

where \( C \geq 0 \) by Corollary 4.2. Thus, (36) becomes

\[
\sigma_{D,r+2}(\mathcal{F}) = (C/n) \frac{n-r}{r+2} \sigma_{D,r}(\mathcal{F}).
\]

Using induction similarly to Corollary 4.4, yields (52). \( \square \)
Finally, we will briefly discuss the case of a transversally orientable foliation $\mathcal{F}$ of codimension one in $\mathcal{D}$ with a unit normal $N$, i.e., $\mathcal{D} = T\mathcal{F} \oplus \text{span}(N)$, on a sub-Riemannian manifold with a minimal distribution $\tilde{\mathcal{D}}$, see [14]. Put

$$Z = \nabla^P_N N, \quad R^P_X = R^P_{X,N}, \quad A = A_N, \quad \sigma_{r+2} = \sigma_{r+2}(N), \quad \tau_k = \tau_k(N).$$

The leafwise divergence of $T_r(A)$ ($r > 0$) satisfies, compare with (18),

$$g(\text{div}_\mathcal{F} T_r(A), X) = \sum_{j \leq r} \text{trace}_{\mathcal{F}}(T_{r-j}(A) R^P_{(-A)j-1X})$$

for any vector field $X$ tangent to $\mathcal{F}$. This yields the following corollaries of Theorems 3.2–3.3.

**Corollary 4.6** For any compact leaf $L$ of a codimension-one foliated sub-Riemannian manifold, (33) reads as

$$\int_L \left( \sum_{1 \leq j \leq r} \text{trace}_{\mathcal{F}}(T_{r-j}(A) R^P_{(-A)j-1Z}) - N(\sigma_{r+1}) + \sigma_1 \sigma_{r+1} 
- (r + 2) \sigma_{r+2} + \text{trace}_{\mathcal{F}}(T_r(A) R^P_N) + g(T_r(A) Z, Z) \right) \, \text{dvol}_g|_L = 0.$$

**Corollary 4.7** For a closed sub-Riemannian manifold $(M, \mathcal{D}, g)$ with a minimal distribution $\tilde{\mathcal{D}}$ and $\mathcal{D} = T\mathcal{F} \oplus \text{span}(N)$, by (36) we get

$$\int_M \left( \sum_{1 \leq j \leq r} \text{trace}_{\mathcal{F}}(T_{r-j}(A) R^P_{(-A)j-1Z}) 
- (r + 2) \sigma_{r+2} + \text{trace}_{\mathcal{F}}(T_r(A) R^P_N) \right) \, \text{dvol}_g = 0.$$

**Example 4.8** (a) For $r = 0$, by Corollary 4.6, we obtain the integral formula

$$\int_L (\tau_2 - N(\tau_1) + \text{Ric}^P_{N,N} + g(Z, Z)) \, \text{dvol}_g|_L = 0,$$

where $\text{Ric}^P_{N,N}$ is the *Ricci P-curvature* in the $N$-direction:

$$\text{Ric}^P_{N,N} = \text{trace}_{\mathcal{F}} R^P_N = \sum_{i \leq n} g(R^P_{e_i,N}, N, e_i).$$

Thus, a codimension-one foliation $\mathcal{F}$ with $\tau_1 = \text{const}$ and $\text{Ric}^P_{N,N} > 0$ has no compact leaves.
(b) For \( r = 0 \), by Corollary 4.7, we obtain the following generalization of (1):

\[
\int_M (2 \sigma_2 - \text{Ric}^p_{N,N}) \, d\text{vol}_g = 0.
\]

Let \( \text{dim} \mathcal{F} = 1 \), then \( \sigma_2 = 0 \) and the Gaussian \( P \)-curvature of \( \mathcal{D} \) is a function on \( M \) defined by \( K^P = g(R^p_{X,N} N, X) \), where \( X \) is a unit (local) vector tangent to \( \mathcal{F} \). In this case, (53) means that the integral Gaussian \( P \)-curvature vanishes:

\[
\int_M K^P \, d\text{vol}_g = 0.
\]

5 Conclusion

Integral formulas (32), (33), (35), (36), and, in particular, (41), can be used for better understanding the geometry of (sub-)Riemannian manifolds equipped with foliations. We delegate to the future the derivation of integral formulas

- for the case of \( \mathcal{D} = \mathcal{D}_1 \oplus \mathcal{D}_2 \) — the sum of two smooth distributions, see [13] for \( \mathcal{D} = T M \); in other words, we take a non-integrable distribution \( \mathcal{D}_1 \) instead of a foliation \( \mathcal{F} \) and \( \mathcal{D}_2 \) instead of \( N \mathcal{F} \). (This appears, for example, when \( \mathcal{D} \subset T M \) is a hyperdistribution, whose shape operator has an eigenvalue of constant multiplicity, and \( \mathcal{D}_1 \) is its eigen-distribution).
- for foliations defined outside of a “singularity set” \( \Sigma \), e.g., a finite union of pairwise disjoint closed submanifolds of codimension greater than 2 of a closed manifold \( M \), under additional assumption of convergence of certain integrals. (Then, instead of the Divergence theorem, we apply the following, see [9, Lemma 2]: if \( X \) is a vector field on \( M \setminus \Sigma \) such that \( \int_M \| X \|^2 \, d\text{vol}_g < \infty \), then \( \int_M \text{div} X \, d\text{vol}_g = 0 \).)
- for a geometrical structure with a \( G \)-connection on a sub-Riemannian manifold, e.g., for holomorphic foliations of complex sub-Riemannian manifolds, see [10, 19] for \( \mathcal{D} = T M \).
- for a sub-Riemannian manifold equipped with several orthogonal foliations (in \( \mathcal{D} \)) and certainly defined \( S^p_{\text{mix}} \). (Mathematicians have shown a specific interest in manifolds equipped with several distributions, e.g., webs of foliations and multiply warped products, e.g., [15]).

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