1 Introduction

In [8], Weinstein introduced the equivariant version of the simplicial de Rham complex. That is a double complex whose components are equivariant differential forms which is called the Cartan model([1]). Weinstein expected that the cohomology group of its total complex is isomorphic to \( H^*(B(G \rtimes H)) \). Here \( B(G \rtimes H) \) is a classifying space of the semi-direct product group. In this paper, we show this conjecture is true.

2 Review of the simplicial de Rham complex

In this section we recall the relation between the simplicial manifold \( NG \) and the classifying space \( BG \). We also recall the notion of the equivariant version of the simplicial de Rham complex.

2.1 The double complex on simplicial manifold

For any Lie group \( G \), we have simplicial manifolds \( NG, NG \) and simplicial \( G \)-bundle \( \gamma : NG \to NG \) as follows:

\[
NG(q) = \underbrace{G \times \cdots \times G}_{q \text{-times}} \ni (h_1, \cdots, h_q) :
\]
face operators $\varepsilon_i : NG(q) \to NG(q - 1)$

$$
\varepsilon_i(h_1, \cdots, h_q) = \begin{cases} 
(h_2, \cdots, h_q) & i = 0 \\
(h_1, \cdots, h_i h_{i+1}, \cdots, h_q) & i = 1, \cdots, q - 1 \\
(h_1, \cdots, h_{q-1}) & i = q
\end{cases} 
$$

The standard definition also involves degeneracy maps but we do not need them here.

For any simplicial manifold $\{X_s\}$, we can associate a topological space $\|X_s\|$ called the fat realization defined as follows.

$$
\|X_s\| \overset{\text{def}}{=} \coprod_n \Delta^n \times X_n / (\varepsilon^i t, x) \sim (t, \varepsilon^i x).
$$

We define $\gamma : \tilde{NG} \to NG$ as $\gamma(g_1, \cdots, g_{q+1}) = (g_1 g_2^{-1}, \cdots, g_q g_{q+1}^{-1})$.

Now we introduce a double complex associated to a simplicial manifold.

**Definition 2.1.** For any simplicial manifold $\{X_s\}$ with face operators $\{\varepsilon_s\}$, we have double complex $\Omega^{p,q}(X) \overset{\text{def}}{=} \Omega^q(X_p)$ with derivatives as follows:

$$
d' = \sum_{i=0}^{p+1} (-1)^i \varepsilon_i^*, \quad d'' := (-1)^p \times \text{the exterior differential on } \Omega^*(X_p).
$$
For $NG$ and $N\bar{G}$ the following holds.

**Theorem 2.1** ([2] [4] [6]). There exist ring isomorphisms

$$H^*(\Omega^*(NG)) \cong H^*(BG), \quad H^*(\Omega^*(N\bar{G})) \cong H^*(EG)$$

Here $\Omega^*(NG)$ and $\Omega^*(N\bar{G})$ means the total complexes.

\[\square\]

### 2.2 Equivariant version

When a Lie group $H$ acts on a manifold $M$, there is the complex of equivariant differential forms $\Omega_H^*(M) := (\Omega^*(M) \otimes S(\mathcal{H}^*))^H$ with suitable differential $d_H$ ([1] [3]). Here $\mathcal{H}$ is the Lie algebra of $H$ and $S(\mathcal{H}^*)$ is the algebra of polynomial functions on $\mathcal{H}$. This is called the Cartan Model. When $M$ is a Lie group $G$, we can define the double complex $\Omega_H^*(NG(*))$ in the same way as in Definition 2.1. This double complex is originally introduced by Weinstein in [8].

### 3 The triple complex on bisimplicial manifold

In this section we construct a triple complex on a bisimplicial manifold.

A bisimplicial manifold is a sequence of manifolds with horizontal and vertical face and degeneracy maps which commute with each other. A bisimplicial map is a sequence of maps commuting with horizontal and vertical face and degeneracy maps. Let $H$ be a subgroup of $G$. We define a bisimplicial manifold $NG(*) \ltimes NH(*)$ as follows;

$$NG(p) \ltimes NH(q) := \underbrace{G \times \ldots \times G}_{p\text{-times}} \times \underbrace{H \times \ldots \times H}_{q\text{-times}}$$

Horizontal face operators $\varepsilon_i^G : NG(p) \ltimes NH(q) \to NG(p - 1) \ltimes NH(q)$ are the same as the face operators of $NG(p)$. Vertical face operators $\varepsilon_i^H : NG(p) \ltimes NH(q) \to NG(p) \ltimes NH(q - 1)$ are

$$\varepsilon_i^H(\vec{g}, h_1, \ldots, h_q) = \begin{cases} (\vec{g}, h_2, \ldots, h_q) & i = 0 \\ (\vec{g}, h_1, \ldots, h_i, h_{i+1}, \ldots, h_q) & i = 1, \ldots, q - 1 \\ (h_q \vec{g}^{-1} h_q^{-1}, h_1, \ldots, h_{q-1}) & i = q \end{cases}$$
Here $\vec{g} = (g_1, \ldots, g_p)$.

We define the bisimplicial map $\gamma_\times : N\tilde{G}(p) \times N\tilde{H}(q) \to NG(p) \times NH(q)$ as $\gamma_\times(\vec{g}, h_1, \ldots, h_{q+1}) = (h_{q+1}\gamma(\vec{g})h_{q+1}^{-1}, \gamma(h_1, \ldots, h_{q+1}))$. Now we fix the semi-direct product operator $\cdot_\times$ of $G \rtimes H$ as $(g, h) \cdot_\times (g', h') := (ghg'\cdot_1 h^{-1}, hh')$, then $G \rtimes H$ acts $N\tilde{G}(p) \times N\tilde{H}(q)$ by right as $(\vec{g}, h) \cdot (g, h) = (h^{-1}\vec{g}gh, hh)$. Since $\gamma_\times((\vec{g}, h)) = \gamma_\times((\vec{g}, h))$, one can see that $\gamma_\times$ is a principal $(G \times H)$-bundle. 

We can also check that $EG \times EH$ acts $NG(p) \times NH(q)$ by right as $(\vec{g}, h) \cdot (g, h) = (h^{-1}\vec{g}gh, hh)$. Since $\gamma_\times((\vec{g}, h)) = \gamma_\times((\vec{g}, h))$, one can see that $\gamma_\times$ is a principal $(G \times H)$-bundle since $(G \times H)$ is an absolute neighborhood retract (see for example [4] P.73). Hence $NG(* \times NH(*))$ is a model of $B(G \times H)$.

**Definition 3.1.** For the bisimplicial manifold $NG(*) \times NH(*)$, we have a triple complex as follows:

$$\Omega^{p,q,r}(NG(*) \times NH(*)) \overset{\text{def}}{=} \Omega^r(NG(p) \times NH(q))$$

Derivatives are:

$$d' = \sum_{i=0}^{p+1} (-1)^i (\varepsilon_i^G)^*, \quad d'' = \sum_{i=0}^{q+1} (-1)^i (\varepsilon_i^H)^* \times (-1)^p$$

$$d''' = (-1)^{p+q} \times \text{the exterior differential on } \Omega^r(NG(p) \times NH(q)).$$

Let $C^*(X)$ denote the set of singular cochains of a topological space $X$. We can also define the triple complex $C^{p,q,r}(NG(*) \times NH(*))$ in the same way. Applying the de Rham theorem and the lemma below twice, we can see that the total complex $\Omega^r(NG \times NH)$ of the triple complex in the Definition 3.1 is quasi-isomorphic to the total complex of $C^{p,q,r}(NG(*) \times NH(*))$.

**Lemma 3.1** ([4], lemma 1.19). Let $K^p_{1,q}$ and $K^p_{2,q}$ be 1.quadrant double complexes, i.e. $K^p_{1,q} = K^p_{2,q} = 0$ if either $p < 0$ or $q < 0$. Suppose $f : K^*_1 \to K^*_2$ is a homomorphism of double complexes and suppose $f^{p,q} : H^p(K^*_1, d_1^*) \to H^p(K^*_2, d_2^*)$ is an isomorphism. Then also $f^* : H^*(K_1, d_1) \to H^*(K_2, d_2)$ is an isomorphism.

**Remark 3.1.** Let $C_*(X)$ denote the set of singular chains of a topological space $X$. We can also define the triple complex $C_{p,q,r}(NG(*) \times NH(*)) := C_r(NG(p) \times NH(q))$ of the singular chains in the same way.
4 Main theorem

Theorem 4.1. If $H$ is compact, there exists an isomorphism

$$H(\Omega^*_H(NG)) \cong H(\Omega^*(NG \rtimes NH)) \cong H^*(B(G \rtimes H))$$

Here $\Omega^*_H(NG)$ means the total complex in subsection 2.2.

Proof. At first we recall Getzler’s result in [5]. When a Lie group $H$ acts on a manifold $M$ by left, there are simplicial manifolds $\{M \rtimes NH^{(q)}\}$ with face operators:

$$\varepsilon_i(u, h_1, \ldots, h_q) = \begin{cases} (u, h_2, \ldots, h_q) & i = 0 \\ (u, h_1, \ldots, h_i h_{i+1}, \ldots, h_q) & i = 1, \ldots, q - 1 \\ (h_q u, h_1, \ldots, h_{q-1}) & i = q \end{cases}$$

Theorem 4.2 ([5]). If $H$ is compact, there is a cochain map between the total complex of the double complex $\Omega^*(M \rtimes NH(\ast))$ and $(\Omega^*_H(M), d_H)$ which induces an isomorphism in cohomology.

As a corollary of this theorem, we obtain the following statement.

Corollary 4.1. For any fixed $p$, the total complex of the double complex $\Omega^*(NG(p) \rtimes NH(\ast))$ is quasi-isomorphic to $(\Omega^*_H(G^p), (-1)^p d_H)$

Hence using the Lemma 3.1, we can see that $H^*(\Omega^*_H(NG))$ is isomorphic to $H^*(\Omega^*(NG \rtimes NH))$.

Now we prove the existence of another isomorphism. Let $S_r(X)$ denote the set of singular simplexes of a topological space $X$. For the triple simplicial set $S_r(NG(p) \rtimes NH(q))$, we have the fat realization

$$\coprod_{r,p,q \geq 0} \Delta^p \times \Delta^q \times \Delta^r \times S_r(NG(p) \rtimes NH(q))/\sim.$$

with suitable identifications. This is a CW complex and the set of $n$-cells are in one-to-one correspondence with $\coprod_{r+p+q=n} S_r(NG(p) \rtimes NH(q))$. Its homology group coincides with the homology group of the total complex of the triple complex $C_{p,q,r}(NG \rtimes NH)$.

So we need to show the cohomology group of this CW complex is isomorphic to $H^*(\|NG \rtimes NH\|)$. We recall that the map $\rho : \Delta^r \times S_r(X) \to X$ which
is defined as $\rho(t, \sigma_r) := \sigma_r(t)$ induces an isomorphism $H_\ast(\coprod_r \Delta^r \times S_r(X)/\sim) \cong H_\ast(X)$ (see for instance [4] P.82). Hence for any fixed $p, q$, the following map $\rho_{p,q}$ which is same as $\rho$ induces an isomorphism in homology.

$$\rho_{p,q} : \coprod_r \Delta^r \times S_r(NG(p) \rtimes NH(q))/\sim \rightarrow NG(p) \rtimes NH(q).$$

By applying the lemma below, we see for any fixed $p$, $\| \rho_{p,\ast} \| : \coprod_q \Delta^q \times (\coprod_r \Delta^r \times S_r(NG(p) \rtimes NH(q))/\sim)/\sim \rightarrow \coprod_q \Delta^q \times NG(p) \rtimes NH(q)/\sim$ induces an isomorphism in homology.

**Lemma 4.1** ([4], Lemma 5.16). Let $f : \{X_\ast\} \rightarrow \{X'_\ast\}$ be a simplicial map of simplicial spaces such that $f_p : X_p \rightarrow X'_p$ induces an isomorphism in homology with coefficients in a ring $\lambda$ for all $p$. Then $\| f\| :\| X_\ast \| \rightarrow\| X'_\ast \|$ also induces an isomorphism in homology and cohomology with coefficients in $\lambda$.

Hence again applying this lemma we can see

$$\| \rho_{\ast,\ast} \| : \coprod_p \Delta^p \times (\coprod_q \Delta^q \times (\coprod_r \Delta^r \times S_r(NG(p) \rtimes NH(q))/\sim))/\sim \rightarrow \coprod \Delta^p \times (\coprod_q \Delta^q \times NG(p) \rtimes NH(q)/\sim)/\sim,$$

induces an isomorphism in cohomology. This completes the proof of Theorem 4.1. □

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