BOTT PERIODICITY, SUBMANIFOLDS, AND VECTOR BUNDLES

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Abstract. We sketch a geometric proof of the classical theorem of Atiyah, Bott, and Shapiro [3] which relates Clifford modules to vector bundles over spheres. Every module of the Clifford algebra $Cl_k$ defines a particular vector bundle over $S^{k+1}$, a generalized Hopf bundle, and the theorem asserts that this correspondence between $Cl_k$-modules and stable vector bundles over $S^{k+1}$ is an isomorphism modulo $Cl_{k+1}$-modules. We prove this theorem directly, based on explicit deformations as in Milnor’s book on Morse theory [8], and without referring to the Bott periodicity theorem as in [3].

Introduction

Topology and Geometry are related in various ways. Often topological properties of a specific space are obtained by assembling its local curvature invariants, like in the Gauss-Bonnet theorem. Bott’s periodicity theorem is different: A detailed investigation of certain totally geodesic submanifolds in specific symmetric spaces leads to fundamental insight not just for these spaces but for whole areas of mathematics. This geometric approach was used originally by Bott [4, 5] and Milnor in his book on Morse theory [8] where the stable homotopy of the classical groups was computed. Later Bott’s periodicity theorem was re-interpreted as a theorem on K-theory [2, 3, 1], but the proofs were different and less geometric. However we feel that the original approach of Bott and Milnor can prove also the K-theoretic versions of the periodicity theorem. As an example we discuss Theorem (11.5.) from the fundamental paper [3] by Atiyah, Bott and Shapiro, which relates Clifford modules to vector bundles over spheres. The argument in [3] uses explicit computations of the right and left hand sides of the stated isomorphism, and depends on the Bott periodicity theorem for the orthogonal groups. Instead we prove bijectivity of the relevant
comparison map directly. In consequence the Bott periodicity theorem for the orthogonal groups is now implied by its algebraic counterpart in the representation theory of Clifford algebras [3]. This gives a positive response to the remark in [3, page 4]: “It is to be hoped that Theorem (11.5) can be give a more natural and less computational proof”, cf. also [7, page 69]. We will concentrate on the real case which is more interesting and less well known than the complex theory. Much of the necessary geometry was explained to us by Peter Quast [12].

1. Poles and Centrioles

We start with the geometry. A symmetric space is a Riemannian manifold $P$ with an isometric point reflection $s_p$ (called symmetry) at any point $p \in P$, that is $s_p \in \hat{G}$ = isometry group of $P$ with $s_p(\exp_p(v)) = \exp_p(-v)$ for all $v \in T_pP$. The map $s : p \mapsto s_p : P \to \hat{G}$ is called Cartan map; it is a covering onto its image $s(P) \subset \hat{G}$ which is also symmetric.\footnote{$s(P) \subset \hat{G}$ is a connected component of the set $\{g \in \hat{G} : g^{-1} = g\}$. When we choose a symmetric metric on $\hat{G}$ such that $g \mapsto g^{-1}$ is an isometry, $s(P)$ is a reflective submanifold and hence totally geodesic, thus symmetric.}

The composition of any two symmetries, $\tau = s_q s_p$ is called a transvection. It translates the geodesic $\gamma$ connecting $p = \gamma(0)$ to $q = \gamma(r)$ by $2r$ and acts by parallel translation along $\gamma$, see next figure. The subgroup of $\hat{G}$ generated by all transvections (acting transitively on $P$) will be called $G$.

Two points $o, p \in P$ will be called poles if $s_p = s_o$. The notion was coined for the north and south pole of a round sphere, but there are many other spaces with poles; e.g. $P = SO_{2n}$ with $o = I$ and $p = -I$, or the Grassmannian $P = \mathbb{G}_n(\mathbb{R}^{2n})$ with $o = \mathbb{R}^n$ and $p = (\mathbb{R}^n)^\perp$. A geodesic $\gamma$ connecting $o = \gamma(0)$ to $p = \gamma(1)$ is reflected into itself at $o$ and $p$ and hence it is closed with period 2.

Now we consider the midpoint set $M$ between poles $o$ and $p$,

$$M = \{m = \gamma\left(\frac{1}{2}\right) : \gamma \text{ geodesic in } P \text{ with } \gamma(0) = o, \ \gamma(1) = p\}.$$ For the sphere $P = S^n$ with north pole $o$, this set would be the equator, see figure below.
Theorem 1. \[11\] \(M\) is the fixed set of an isometric involution \(r\) on \(P\).

Proof. In the example of the sphere \(P = S^n\), the equator \(M\) is the fixed set of \(-s_o = -I \circ s_o\). Here, \(-I\) is the deck transformation\(^2\) of the covering \(S^n \to \mathbb{RP}^n = S^n/\{\pm I\}\). In the general case we consider the covering \(P \to s(P)\). Since \(s(P)\) is again symmetric, we have \(s(P) = P/\Delta\) for some discrete freely acting group \(\Delta \subset \hat{G}\) normalized by all symmetries and centralized by all transvections.\(^3\) Since \(s_o = s_p\), the points \(o\) and \(p\) are identified in \(s(P)\). Thus there is a unique \(\delta \in \Delta\) with \(\delta(o) = p\). This will be the analogue of \(-I\) in the case \(P = S^n\). We will show that \(\delta\) has order 2 and preserves any geodesic \(\gamma\) with \(\gamma(0) = o\) and \(\gamma(1) = p\). In fact, let \(\tau\) be the transvection along \(\gamma\) from \(o\) to \(p\). Then \(\tau^2(o) = o\) and therefore

\[
\delta(p) = \delta(\tau(o)) = \tau(\delta(o)) = \tau(p) = o.
\]

Thus \(\delta^2\) fixes \(o\) which shows \(\delta^2 = \text{id}\) since \(\Delta\) acts freely. Hence \(\{I, \delta\} \subset \Delta\) is a subgroup and \(\bar{P} = P/\{\text{id}, \delta\}\) a symmetric space. Under the projection \(\pi : P \to \bar{P}\), the geodesic \(\gamma\) is mapped onto a closed geodesic doubly covered by \(\gamma\), thus \(\delta\) preserves \(\gamma\) and shifts its parameter by 1, and \(\gamma\) has period 2.

\(^2\)A deck transformation of \(\pi : P \to \bar{P}\) is an isometry \(\delta\) of \(P\) with \(\pi \circ \delta = \pi\).

\(^3\)Consider a symmetric space \(P\) and a covering \(\pi : P \to P/\Delta\) for some discrete freely acting group \(\Delta\) of isometries on \(P\). Then \(P/\Delta\) is again symmetric if and only if each symmetry \(s_p\) of \(P\) maps \(\Delta\)-orbits onto \(\Delta\)-orbits. Thus for each \(\delta \in \Delta\) we have \(s_p(\delta x) = \delta s_p(x)\) for all \(x \in P\), and \(\delta \in \Delta\) is independent of \(x\), by discreteness. Thus \(s_p\delta = \delta s_p\), in particular \(s_p\delta s_p = \delta \in \Delta\). For any other symmetry \(s_q\) we have the same equation \(s_q\delta = \delta s_q\) with the same \(\delta \in \Delta\), again by discreteness. Thus \(\delta^{-1}s_p s_q\delta = s_p \delta^{-1}\delta s_q = s_p s_q\), and \(\delta\) commutes with the transvection \(s_p s_q\) (see also [14, Thm. 8.3.11]).
We put $r = s_o \delta$. This is an involution since $s_o$ and $\delta$ commute: $\delta' = s_o \delta s_o \in \Delta$ sends $o$ to $p$ like $\delta$, thus $\delta' = \delta$. Then $r$ fixes the midpoint $m = \gamma(\frac{1}{2})$ of any geodesic $\gamma$ from $o$ to $p$ since $s_o(\delta(\gamma(\frac{1}{2}))) = s_o(\gamma(\frac{3}{2})) = s_o(\gamma(-\frac{1}{2})) = \gamma(\frac{1}{2})$. Thus $M \subset \text{Fix}(r)$.

Vice versa, assume that $m \in P$ is a fixed point of $r$. Thus $s_o m = \delta m$. Join $o$ to $m$ by a geodesic $\gamma$ with $\gamma(0) = o$ and $\gamma(\frac{1}{2}) = m$. Then $\gamma(-\frac{1}{2}) = s_o(m) = \delta(m) = \delta(\gamma(\frac{1}{2}))$, and the projection $\pi : P \to \bar{P} = P/\{\text{id}, \delta\}$ maps $\gamma : [-\frac{1}{2}, \frac{1}{2}] \to P$ onto a geodesic loop $\bar{\gamma} = \pi \circ \gamma$, that is a closed geodesic of period 1 (since $\bar{P}$ is symmetric). Thus $\gamma$ extends to a closed geodesic of period 2 doubly covering $\bar{\gamma}$, and $\delta$ shifts the parameter of $\gamma$ by 1. Therefore $\gamma(1) = \delta(o) = p$. Hence $m$ is the midpoint of $\gamma|_{[0,1]}$ from $o$ to $p$. Thus $M \supset \text{Fix}(r)$.

Connected components of the midpoint set $M$ are called centrioles [6]. Connected components of the fixed set of an isometry are totally geodesic (otherwise shortest geodesic segments in the ambient space with end points in the fixed set were not unique, see figure below); if the isometry is an involution, its fixed components are called reflective.

Most interesting are connected components containing midpoints of geodesics with minimal length between $o$ and $p$ ("minimal centrioles").

Each such midpoint $m = \gamma(\frac{1}{2})$ determines its geodesic $\gamma$ uniquely: if there were two geodesics of equal length from $o$ to $p$ through $m$, they could be made shorter by cutting the corner.

There exist chains of minimal centrioles (centrioles in centrioles),

$$P \supset P_1 \supset P_2 \supset \ldots$$

Peter Quast [12, 13] classified all such chains with at least 3 steps starting with a compact simple Lie group $P = G$. Up to group coverings, the result is as follows. The chains 1, 2, 3 occur in Milnor [8].

| No. | $G$ | $P_1$ | $P_2$ | $P_3$ | $P_4$ | restr. |
|-----|-----|-------|-------|-------|-------|-------|
|     | $(S)O_{4n}$ | $SO_{4n}/U_{4n}$ | $U_{2n}/Sp_n$ | $G_p(H^n)$ | $Sp_p$ | $p = \frac{7}{2}$ |
| 2   | $(S)U_{2n}$ | $G_n(C^{2n})$ | $U_n$ | $G_p(C^n)$ | $U_p$ | $p = \frac{5}{2}$ |
| 3   | $Sp_n$ | $Sp_n/U_n$ | $U_n/SO_n$ | $G_p(R^n)$ | $SO_p$ | $p = \frac{5}{2}$ |
| 4   | $Spin_n$ | $Q_n$ | $(S^1 \times S^{n-3})/\pm$ | $S^{n-4}$ | $S^{n-5}$ | $n \geq 5$ |
| 5   | $E_7$ | $E_7/(S^1 E_6)$ | $S^1 E_6/F_4$ | $\mathbb{O} \mathbb{P}^2$ | $-$ |     |
By $G_p(K^n)$ we denote the Grassmannian of $p$-dimensional subspaces in $K^n$ for $K \in \{ \mathbb{R}, \mathbb{C}, \mathbb{H} \}$. Further, $Q_n$ denotes the complex quadric in $\mathbb{CP}^{n+1}$ which is isomorphic to the real Grassmannian $G_2^+(\mathbb{R}^{n+2})$ of oriented 2-planes, and $O\mathbb{P}^2$ is the octonionic projective plane $F_4/Spin_9$.

A chain is extendible beyond $P_k$ if and only if $P_k$ contains poles again. E.g. among the Grassmannians $P_3 = G_p(K^n)$ only those of half dimensional subspaces ($p = \frac{n}{2}$) enjoy this property: Then $(E, E^\perp)$ is a pair of poles for any $E \in G_{n/2}(K^n)$, and the corresponding midpoint set is the group $O_{n/2}, U_{n/2}, Sp_{n/2}$ since its elements are the graphs of orthogonal $K$-linear maps $E \to E^\perp$, see figure below.

![Diagram](image)

2. Centrioles with topological meaning

Points in minimal centrioles are in 1:1 correspondence to minimal geodesics between the corresponding poles $o$ and $p$. Thus minimal centrioles sometimes can be viewed as low-dimensional approximations of the full path space $\Lambda$, the space of all $H^1$-curves $\lambda : [0, 1] \to P$ with $\lambda(0) = o$ and $\lambda(1) = p$. This is due to the Morse theory for the energy function $E$ on $\Lambda$ where $E(\lambda) = \int_0^1 |\lambda'(t)|^2 dt$. We may decrease the energy of any path $\lambda$ by applying the gradient flow of $-E$ (left figure).

![Diagram](image)

Most elements of $\Lambda$ will be flowed to the minima of $E$ which are the shortest geodesics between $o$ and $p$. The only exceptions are the domains of attraction (“unstable manifolds”) for the other critical points,

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4$H^1$ means that $\lambda$ has a derivative almost everywhere which is square integrable.
Replacing any path $\lambda$ by a geodesic polygon with $N$ vertices, we may replace $\Lambda$ by a finite dimensional manifold, cf. [8].
the non-minimal geodesics between \( o \) and \( p \). The codimension of the unstable manifold is the index of the critical point, the maximal dimension of any subspace where the second derivative of \( E \) (taken at the critical point) is negative. If \( \beta \) denotes the smallest index of all non-minimal critical points, any continuous map \( f : X \to \Lambda \) from a connected cell complex \( X \) of dimension \(< \beta \) can be moved away from these unstable manifolds and flowed into a connected component of the minimum set, that is into some centriole \( P_1 \). Thus \( f \) is homotopic to a map \( \tilde{f} : X \to P_1 \).

But this works only if all non-minimal geodesics from \( o \) to \( p \) have high index \((\geq \beta)\). Which symmetric spaces \( P \) have this property? An easy example is the sphere, \( P = S^n \). A nonminimal geodesic \( \gamma \) between poles \( o \) and \( p \) covers a great circle at least one and a half times and can be shortened within any 2-sphere in which it lies (right figure above). There are \( n - 1 \) such 2-spheres perpendicular to each other since the tangent vector \( \gamma'(0) = e_1 \) is contained in \( n - 1 \) perpendicular planes \( \text{Span}(e_1, e_i) \) with \( i \geq 2 \) in the tangent space. Thus the index is \( \geq n - 1 \), in fact \( \geq 2(n - 1) \) since any such geodesic contains at least 2 conjugate points where it can be shortened by cutting the corner, see figure.

For the classical groups we can argue similarly. E.g. in \( SO_{2n} \), a shortest geodesic from \( I \) to \( -I \) is a product of \( n \) half turns, planar rotations by the angle \( \pi \) in \( n \) perpendicular 2-planes in \( \mathbb{R}^{2n} \). A non-minimal geodesic must make an additional full turn and thus a \( 3\pi \)-rotation in at least one of these planes, say in the \( x_1x_2 \)-plane. This rotation belongs to the rotation group \( SO_3 \subset SO_{2n} \) in the \( x_1x_2x_k \)-space for any \( k \in \{3, \ldots, 2n\} \). Using \( SO_3 = S^3/\pm \), we lift the \( 3\pi \)-rotation to \( S^3 \) and obtain a \( 3/4 \) great circle which can be shortened. There are \( 2n - 2 \) coordinates \( x_k \) and therefore \( 2n - 2 \) independent contracting directions, hence the index of a nonminimal geodesic in \( SO_{2n} \) is \( \geq 2n - 2 \) (compare [8, Lemma 24.2]). The index of the spaces \( P_k \) can be bounded from below in a similar way, see next section for the chain of \( SO_n \). This implies the homotopy version of the periodicity theorem:

**Theorem 2.** When \( n \) is even and sufficiently large, we have for \( G = SO_{4n}, SU_{2n}, Sp_n \) (notations of table 2):

\[
\pi_k(G) = \pi_{k-1}(P_1) = \pi_{k-2}(P_2) = \pi_{k-3}(P_3) = \pi_{k-4}(P_4).
\]

Together with table 2 this implies the following periodicities:

\[
\pi_{k+2}(SU_n) = \pi_k(SU_{n/2}),
\]
\[
\begin{align*}
\pi_{k+4}(SO_n) &= \pi_k(Sp_{n/2}), \\
\pi_{k+4}(Sp_n) &= \pi_k(SO_{n/2}).
\end{align*}
\]

3. Clifford modules

For compact matrix groups \( G \) containing \(-I\), there is a linear algebra interpretation for the iterated midpoint sets \( M_j \) and their components \( P_j \). A geodesic \( \gamma \) in \( G \) with \( \gamma(0) = I \) is a one-parameter subgroup, and when \( \gamma(1) = -I \), then \( \gamma(\frac{1}{2}) = J \) is a complex structure, \( J^2 = -I \). Thus the midpoint set \( M_1 \) is the set of complex structures in \( G \). When the connected component \( P_1 \) of \( M_1 \) contains antipodal points \( J_1 \) and \(-J_1 \), there is a next midpoint set \( M_2 \subset P_1 \). It consists of points \( J_1\gamma(\frac{1}{2}) \) where \( \gamma \) is a one-parameter subgroup in \( G \) with \( \gamma(1) = -I \) such that \( J_1\gamma(t) \) is a complex structure for all \( t \),

\[ J_1\gamma J_1\gamma = -I. \]

(\#)

In particular the midpoint \( J = \gamma(\frac{1}{2}) \) anticommutes with \( J_1 \) (since \( J_1J_1J = -I \iff J_1J = -J_J1 \)), and when \( \gamma \) is minimal, this condition is sufficient for (\#): then both \( J_1\gamma J_1 \) and \(-\gamma^{-1} \) are shortest geodesics from \(-I \) to \( I \) with midpoint \( J \), so they must agree. By in-duction hypothesis, we have anticommuting complex structures \( J_n \in G \) with \( J_i \in P_i \) for \( i < k \), and \( P_k \) is a connected component of the set

(3)

\[ M_k = \{ J \in G : J^2 = -I, \ J J_i = -J_i J \text{ for } i < k \} \]

of complex structures \( J \in G \) which anticommute with \( J_1, \ldots, J_{k-1} \). To finish the induction step we choose some \( J_k \in P_k \).

Recall that the real Clifford algebra \( Cl_k \) is the associative real algebra with 1 which is generated by \( \mathbb{R}^k \) with the relations \( vw + wv = -2\langle v, w \rangle \). Equivalently, an orthonormal basis \( e_1, \ldots, e_k \) of \( \mathbb{R}^k \subset Cl_k \) satisfies

\[ e_i e_j + e_j e_i = -2\delta_{ij}. \]

A representation of \( Cl_k \) is an algebra homomorphism from \( Cl_k \) into some matrix algebra \( \mathbb{K}^{n \times n} \) with \( \mathbb{K} \in \{ \mathbb{R}, \mathbb{C}, \mathbb{H} \} \); the space \( \mathbb{K}^n \) on which the matrices operate is called Clifford module \( S \). A representation maps the vectors \( e_i \) onto matrices \( J_i \) with the same relations \( J_i^2 = -I \) and \( J_i J_j = -J_j J_i \) for \( i \neq j \). Thus a \( Cl_k \) module is nothing but a Clifford system, a family of \( k \) are anticommuting complex structures, and the midpoint set \( M_{k+1} \subset P_k \) between \( J_k \) and \(-J_k \) can be viewed as the set of extensions of a given \( Cl_k \)-module (defined by \( J_1, \ldots, J_k \)) to a \( Cl_{k+1} \)-module.

The algebraic theory of the Clifford representations is rather easy (cf. [7]). They are direct sums of irreducible representations, and in the real case there is just one irreducible \( Cl_k \)-module \( S_k \) (up to isomorphisms)
when $k \not\equiv 3 \mod 4$, while there are two with equal dimensions when $k \equiv 3 \mod 4$. For $k = 0, \ldots, 8$ we have

**Theorem 3.**

\begin{equation}
S_k = \mathbb{R} \circ C \circ \mathbb{H} \circ \mathbb{H}^2 \circ \mathbb{C}^4 \circ \mathbb{O} \circ \mathbb{O}^2
\end{equation}

and further we have the periodicity theorem for Clifford modules,

\begin{equation}
S_{k+8} = S_k \otimes S_8.
\end{equation}

For $k = 3$ and $k = 7$, the two different module structures are given by left and right multiplications of $\mathbb{R}^k = K' := \mathbb{K} \oplus \mathbb{R} \cdot 1$ on $S_k = K$ for $K = \mathbb{H}, \mathbb{O}$.

4. INDEX OF NONMINIMAL GEODESICS

From (3) we have gained a uniform description for all iterated centrioles $P_k$ of $G$ in terms of Clifford systems. This can be used for a calculation of the lower bound $\beta$ for the index of nonminimal geodesics in all $P_k$, cf. [8].

**Theorem 4.** Let $SO_n = G \supset P_1 \supset P_2 \supset \cdots \supset P_k \supset \cdots$ be the chain \cite{1} of iterated centrioles where $n$ is divisible by a high power of 2. Then for each $k$ there is some $\beta$ depending on $n, k$ such that the index of nonminimal geodesics from $J_k$ to $-J_k$, and $\beta \to \infty$ for $n \to \infty$.

**Proof.** Let $\tilde{\gamma} = J_k \gamma : [0, 1] \to P_k$ be a non-minimal geodesic from $J_k$ to $-J_k$. Then $\gamma(t) = e^{\pi t A}$ for some $A \in \mathfrak{so}_n$. Since $\tilde{\gamma}(t)$ anticommutes with $J_i$ for all $i < k$, it follows that $\gamma(t)$ and $A$ commute with $J_i$. Further, from $\tilde{\gamma}(t)^2 = -I$ we obtain $J_k e^{\pi t A} J_k^{-1} = e^{-\pi t A}$ and therefore $A$ anticommutes with $J_k$. Thus we have computed the tangent space of $P_k$ at $J_k$:

\begin{equation}
T_{J_k} P_k = \{ J_k A : A \in \mathfrak{so}_n, \ A J_k = -J_k A, \ A J_i = J_i A \text{ for } i < k \}.
\end{equation}

Since $\gamma(1) = -I$, the (complex) eigenvalues of $A$ have the form $a i$ with $i = \sqrt{-1}$ and $a$ an odd integer.

To relate these eigenvalues to the index we argue similar as in [8, p. 144-147]. We split $\mathbb{R}^n$ into a sum of subspaces $V_j$ being invariant under the linear maps $A, J_1, \ldots, J_k$ and being minimal with respect to this property. All $J_i, i < k$, preserve the (complex) eigenspaces $E_a$ of $A$, corresponding to the nonzero eigenvalue $ai$, while $J_k$ interchanges $E_a$ and $E_{-a}$. Thus by minimality of $V_j$, there is just one pair $\pm a$ such that $V_j$ is the real part of $E_a + E_{-a}$. Therefore $J' := A/a$ is an additional
complex structure on $V_j$ commuting with $J_i$ ($i < k$) and anticommuting with $J_k$, and $J_{k+1} := J_k J'$ is a complex structure which anticommutes with all $J_1, \ldots, J_k$. Hence $V_j$ is an irreducible $Cl_k$-module. Moreover, $A = a_j J'$ on $V_j$ for some nonzero integer $a_j$ while $A = 0$ on $V_0$. By choice of the sign of $J'|V_j$ we may assume that all $a_j > 0$. Hence $a_j \in \{1, 3, 5, \ldots\}$.

Choose two of these irreducible modules, say $V_j$ and $V_h$. By (4), there is a module isomorphism $V_j \to V_h$ as $Cl_{k+1}$-modules when $k + 1 \equiv 3 \pmod{4}$ (Case 1) and as $Cl_k$-modules when $k + 1 \equiv 3 \pmod{4}$ (Case 2). This remains true when we alter the $Cl_{k+1}$-module structure of $V_h$ in Case 1 by changing the sign of $J_{k+1}$ (and thus that of $J'$) on $V_h$. With this identification we have $V_j + V_h = V_j \otimes \mathbb{R}^2$ and $B = I \otimes \left( j^{-1} \right)$ (with $B = 0$ on the other submodules) commutes with all $J_j, j \leq k$, and the same is true for $e^{\mu B}$. Putting $A_u = e^{uB}Ae^{-uB}$, we have $J_k A_u \in T_h P_k$ by (6).

**Case 1: $k + 1 \equiv 3 \pmod{4}$:** We have modified our identification of $V_j$ and $V_h$ by changing the sign of $J_{k+1}$ on $V_h$. Thus on $V_j + V_h = V_j \otimes \mathbb{R}^2$ we have $A = J' \otimes D$ where $D = \text{diag}(a_j, -a_h) = cI + D'$ with $D' = \text{diag}(b, -b)$ for $b = \frac{1}{2}(a_j + a_h)$ and $c = \frac{1}{2}(a_j - a_h)$. Let us consider the family of geodesics $J_k \gamma_u$ from $J_k$ to $-J_k$ in $P_k$ with $\gamma_u(t) = e^{tA_u} = e^{uB} \gamma(t) e^{-uB}$. The point $\gamma(t) = e^{\pi t e^{\pi t D'}}$ is fixed under conjugation with the rotation matrix $e^{uB} = \left( \begin{array}{cc} \cos u & -\sin u \\ \sin u & \cos u \end{array} \right)$ precisely when $e^{\pi t D'} = \text{diag}(e^{\pi tb}, e^{-\pi tb})$ is a multiple of the identity matrix which happens for $t = 1/b$. If one of the eigenvalues $a$ of $A$ is $> 1$, say $a_h \geq 3$, then $b = \frac{1}{2}(a_j + a_h) \geq 2$ and $1/b \in (0, 1)$. All $\gamma_u$ are geodesics connecting $I$ to $-I$ on $[0, 1]$. By “cutting the corner” it follows that $\gamma$ can no longer be locally shortest beyond $t = 1/b$, see figure. If there is at least one eigenvalue $a_h > 1$, we have $r - 1$ index pairs $(j, h)$, hence the index of non-minimal geodesics is at least $r - 1$.

**Case 2: $k + 1 \equiv 3 \pmod{4}$:** In this case, the product $J_0 := J_1 J_2 \cdots J_{k-1}$ is a complex structure\(^6\) which commutes with $A$ and anticommutes with $J_k$ (since $k - 1$ is odd). Thus $A$ can be viewed as a complex matrix, using $J_0$ as the multiplicaton by $i$. Let $E_a \subset V_j$ be the eigenspace of $J_0$ with $J_0 E_a = a E_a$.

\(^6\)Putting $S_n = (J_1 \cdots J_n)^2$ we have

\[
S_n = J_1 J_n J_1 \cdots J_n = (-1)^{n-1} S_{n-1} J_n^2 = (-1)^n S_{n-1},
\]

thus $S_n = (-1)^s J^s$ with $s = n + (n - 1) + \cdots + 1 = \frac{1}{2}n(n + 1)$. When $n = k - 1 \equiv 1 \pmod{4}$, we have $s \equiv 1 \mod{4}$, hence $S_n = -I$. 
A corresponding to the eigenvalue \(ia\) where \(a\) is any odd integer. Then \(E_a\) is invariant under the \(J_i\), \(i < k\), which commute with \(A\), but is it also invariant under \(J_k\) which anticommutes with \(A\) and with \(i = J_0\) (since \(k - 1\) is odd). By minimality we have \(V_j = E_a\), hence \(A = aJ_o\).

As before, we consider the linear map \(B = (I - I)\) on \(V_j + V_h = V_j \otimes \mathbb{R}^2\) and the family of geodesics \(\gamma_u(t) = e^{tA}A = e^{uB}\gamma(t)e^{-uB}\). This time, \(A = J' \otimes D\) where \(D = \text{diag}(a_j, a_h) = cI + D'\) with \(c = \frac{1}{2}(a_j + a_h)\) and \(D' = \text{diag}(b, -b)\) with \(b = \frac{1}{2}(a_j - a_h)\). Thus the element \(\gamma(t) = e^{tD}e^{\pi tD'}\) is fixed under conjugation with the rotation matrix \(e^{uB} = \left( \begin{array}{cc} \cos u & -\sin u \\ \sin u & \cos u \end{array} \right)\) precisely when \(e^{\pi tD'} = \text{diag}(e^{\pi t}, e^{-\pi t})\) is a multiple of the identity matrix which happens for \(t = 1/b\). If \(b > 1\), we obtain an energy-decreasing deformation by cutting the corner, see figure above. We need to show that there are enough pairs \((j, h)\) with \(b > 1\) when \(\gamma\) is non-minimal.

Any \(J \in P_k\) defines a \(\mathbb{C}\)-linear map \(J_kJ\) since \(J_kJ\) commutes with \(J_i\) and hence with \(J_o\). Thus a path \(\lambda : I \to P_k\) from \(J_k\) to \(-J_k\) defines a family of \(\mathbb{C}\)-linear maps, and its complex determinant \(\det(J_k\lambda)\) is a path in \(S^1\) starting and ending at \(\det(\pm I) = 1\) (recall that the dimension \(n\) is even). This loop in \(S^1\) has a mapping degree which is apparently invariant under homotopy; it decomposes the path space \(\Lambda P_k\) into infinitely many connected components. If \(\lambda = \text{a geodesic, } \lambda(t) = J_ke^{\pi tA}\), then \(\det J_k^{-1}\lambda(t) = e^{\pi t\text{trace }A}\), hence its mapping degree is \(\frac{1}{2}\) \(\text{trace }A/i\). Since \(\text{trace }A/i = m \sum_j a_j\), we may fix \(c := \sum_j a_j\) (which means fixing the connected component of \(\Lambda P_k\)) and we may assume that \(\lvert c\rvert\) is much smaller than \(r\) (the number of submodules \(V_j\)). Let \(p\) denote the sum of the positive \(a_j\) and \(-q\) the sum of the negative \(a_j\). Then \(p + q \geq r\) since all \(\lvert a_j\rvert \geq 0\), and \(p - q = c\) which means roughly \(p \approx q \approx r/2\). Assume for the moment \(c = 0\). If there is some eigenvalue \(a_h\) with \(\lvert a_h\rvert > 1\), say \(a_h = -3\), there are many positive \(a_j\) with \(a_j - a_h \geq 4\), more precisely \(\sum_{a_j > 0}(a_j - a_h) \geq 4 \cdot r/2 = 2r\), and this is a lower bound for the index. In the general case this result has to be corrected by the comparably small number \(c\). In contrast, if all \(a_j = \pm 1\), the geodesic \(\gamma\) consists of simultaneous half turns in \(n/2\) perpendicular planes; these are shortest geodesics from \(I\) to \(-I\) in \(SO_n\).\(^7\)

\(^7\)Any one-parameter subgroup \(\gamma\) in \(SO_n\) is a family of planar rotations in perpendicular planes. When \(\gamma(1) = -I\), all rotation angles are odd multiples of \(\pi\). The squared length of \(\gamma\) is the sum of the squared rotation angles. Thus the length is minimal if all rotation angles are just \(\pm \pi\).
5. Vector bundles over spheres

Clifford representations have a direct connection to vector bundles over spheres and hence to K-theory. Every vector bundle $E \to S^{k+1}$ is trivial over each of the two closed hemispheres $D_+, D_- \subset S^{k+1}$, but along the equator $S^k = D_+ \cap D_-$ the fibres over $\partial D_+$ and $\partial D_-$ are identified by some map $\phi : S^k \to O_n$ called \textit{clutching map}.

Homotopic clutching maps define equivalent vector bundles. Thus vector bundles over $S^{k+1}$ are classified by the homotopy group $\pi_k(O_n)$. When we allow adding of trivial bundles (stabilization), $n$ may be arbitrarily high. Let $V_k$ be the set of vector bundles over $S^{k+1}$ up to equivalence and adding of trivial bundles ("stable vector bundles"). Then

$$V_k = \lim_{n \to \infty} \pi_k(O_n).$$

Hence we could apply Theorem 2 in order to classify stable vector bundles over spheres. However, a separate argument based on the same ideas but also using Clifford modules will give more information.

A $\text{Cl}_k$ module $S = \mathbb{R}^n$ or the corresponding Clifford system $J_1, \ldots, J_k \in O_n$ defines a peculiar map $\phi = \phi_S : S^k \to O_n$ which is \textit{linear}, that is a restriction of a linear map $\phi : \mathbb{R}^{k+1} \to \mathbb{R}^{n \times n}$, where we put

$$\phi_S(e_{k+1}) = I, \quad \phi_S(e_i) = J_i \text{ for } i \leq k.$$

The bundles defined by such clutching maps $\phi_S$ are called \textit{generalized Hopf bundles}. In the cases $k = 1, 3, 7$, these are the classical complex, quaternionic, and octonionic Hopf bundles over $S^{k+1}$.

In fact, $\text{Cl}_k$-modules are in 1:1 correspondence to linear maps $\phi : S^k \to O_n$ with the identity matrix in the image. To see this, let $\phi$ be such map and $W = \phi(\mathbb{R}^{k+1})$ its image. Then $S_W := \phi(S^k) \subset O_n$. Thus $\phi$ is an isometry for the inner product $\langle A, B \rangle = \frac{1}{n} \text{trace}(A^TB)$ on $\mathbb{R}^{n \times n}$ since $\phi(S^k) \subset O_n$ and $O_n$ lies in the unit sphere of $\mathbb{R}^{n \times n}$. 

For all $A, B \in S_W$ we have $(A + B) \in \mathbb{R} \cdot O_n$. On the other hand, 
$(A + B)^T(A + B) = 2I + A^TB + B^TA$, thus $A^TB + B^TA = tI$ for some $t \in \mathbb{R}$. From the inner product with $I$ we obtain $t = 2\langle A, B \rangle$. Inserting $A = I$ and $B \perp I$ yields $B + B^T = 0$, and for any $A, B \perp I$ we obtain $AB + BA = -2\langle A, B \rangle I$. Thus $\phi |\mathbb{R}^k$ defines a $Cl_k$-representation on $\mathbb{R}^n$.

Atiyah, Bott and Shpiro [3] reduced the theory of vector bundles over spheres to the simple algebraic structure of Clifford modules by showing that all vector bundles over spheres are generalized Hopf bundles plus trivial bundles, see Theorem 5 below. We sketch a different proof of this theorem using the original ideas of Bott and Milnor. We will homotopically deform the clutching map $\phi : S^k \rightarrow G = SO_n$ of the given bundle $E \rightarrow S^{k+1}$ step by step into a linear map. Since adding of trivial bundles is allowed, we may assume that the rank $n$ of $E$ is divisible by a high power of 2.

We declare $N = e_{k+1}$ to be the “north pole” of $S^k$. First we deform $\phi$ such that $\phi(N) = I$ and $\phi(-N) = -I$. Thus $\phi$ maps each meridian from $N$ to $-N$ in $S^k$ onto some path from $I$ to $-I$ in $G$, an element of $\Lambda G$. The meridians $\mu_v$ are parametrized by $v \in S^{k-1}$ where $S^{k-1}$ is the equator of $S^k$. Therefore $\phi$ can be considered as a map $\phi : S^{k-1} \rightarrow \Lambda G$. Using the negative gradient flow for the energy function $E$ on the path space $\Lambda G$ as in section 2 we may shorten all $\phi(\mu_v)$ simultaneously to minimal geodesics from $I$ to $-I$ and obtain a map $\tilde{\phi} : S^{k-1} \rightarrow \Lambda_o G$ where $\Lambda_o G$ is the set of shortest geodesics from $I$ to $-I$, the minimum set of $E$ on $\Lambda G$. Let $m(\gamma) = \gamma(\frac{1}{2})$ be the midpoint of any geodesic $\gamma : [0, 1] \rightarrow G$. Thus we obtain a map $\phi_1 = m \circ \tilde{\phi} : S^{k-1} \rightarrow P_1$, and we may replace $\phi$ by the geodesic suspension over $\phi_1$ from $I$ and $-I$.

We repeat this step replacing $G$ by $P_1$ and $\phi$ by $\phi_1$. Again we choose a “north pole” $N_1 = e_k \in S^{k-1}$ and deform $\phi_1$ such that $\phi_1(\pm N_1) = \pm J_1$ for some $J_1 \in P_1$. Now we deform the curves $\phi_1(\mu_1)$ for all meridians $\mu_1 \subset S^{k-1}$ to shortest geodesics, whose midpoints define a map $\phi_2 : S^{k-2} \rightarrow P_1$, and then we replace $\phi_1$ by a geodesic suspension from $\pm J_1$ over $\phi_2$. This step is repeated $(k - 1)$-times until we reach a map.
$\phi_{k-1} : S^1 \to P_{k-1}$. This loop can be shortened to a geodesic loop $\tilde{\gamma} = J_{k-1} \gamma : [0,1] \to P_{k-1}$ (which is a closed geodesic since $P_{k-1}$ is symmetric) starting and ending at $J_{k-1}$, such that $\tilde{\gamma}$ and $\gamma$ are shortest in their homotopy class.

We have $\gamma(t) = e^{2\pi i A}$ for some $A \in T_{J_{k-1}} P_{k-1}$. Since $\gamma$ is closed, the (complex) eigenvalues of $A$ have the form $ai$ with $a \in \mathbb{Z}$ and $i = \sqrt{-1}$. To compute these eigenvalues we argue as in section 4. We split $\mathbb{R}^n$ into $V_0 = \ker A$ and a sum of subspaces $V_j$ which are invariant under the linear maps $A, J_1, \ldots, J_{k-1}$ and minimal with respect to this property. As before, $A = aJ'$ for some nonnegative integer $a$, and $J_k = J_{k-1} J'$ is a complex structure anticommuting with $J_1, \ldots, J_{k-1}$. Hence $V_j$ is an irreducible $Cl_k$-module with dimension $m_k$, see (4), (5).

Since the clutching map of the given vector bundle $E \to S^{k+1}$ (after the deformation) is determined by $\gamma, J_1, \ldots, J_{k-1}$ which leave all $V_j, j \geq 0$, invariant, the vector bundle splits accordingly as $E = E_0 \oplus \sum_{j>0} E_j$ where $E_0$ is trivial.

We claim that the minimality of $\gamma$ implies $a_j = 1$ for all $j$ and hence $A = J_k$. In fact, the geodesic variation $\gamma_u$ of section 4 shows that $|a_j - a_h| < 2$ for all $j, h$, otherwise we could shorten $\gamma$ by cutting the corner.

Now suppose that, say, $a_1 \geq 2$. We may assume that $V_0 = \ker A$ contains another copy $\tilde{V}_1$ of $V_1$ as a $Cl_{k-1}$-module: if not, we extend $E_0$ by the trivial bundle $S^{k+1} \times \tilde{V}_1$. Thus we have eigenvalues 0 and $a_1$ on $\tilde{V}_1 \oplus V_1$ with difference $\geq 2$, in contradiction to the minimality of the geodesic.

We have shown $E = E_0 \oplus E_1$ where $E_0$ is trivial and $E_1$ is a generalized Hopf bundle for the Clifford system $J_1, \ldots, J_k$ on $\sum_{j>0} V_j$.

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The clutching map $\phi : S^k \to SO_n$ splits into components $\phi_j : S^k \to SO(V_j)$. The domain $S^k$ is the union of totally geodesic spherical $(k-1)$-discs $D_v, v \in S^1$, centered at $v$ and perpendicular to $S^1$. All $D_v$ have a common boundary $S^{k-2}$. Since $\phi_0|_{D_v}$ is constant in $v$, it is contractable along $D_v$ to a constant map.
Let $\mathcal{M}_k$ the set of equivalence classes of $Cl_k$-modules, modulo trivial $Cl_k$-representations. We have studied the map

$$\hat{\alpha} : \mathcal{M}_k \to \mathcal{V}_k$$

which assigns to each $S \in \mathcal{M}_k$ the corresponding generalized Hopf bundle over $S^{k+1}$. It is additive with respect to direct sums. We have just proved that $\hat{\alpha}$ is onto. But it is not 1:1. In fact, every $Cl_{k+1}$-module is also a $Cl_k$-module since $Cl_k \subset Cl_{k+1}$. This defines a restriction map $\rho : \mathcal{M}_{k+1} \to \mathcal{M}_k$. Any $Cl_k$-module $S$ which is really a $Cl_{k+1}$-module gives rise to a contractible clutching map $\phi_S : S^k \to SO_n$ and hence to a trivial vector bundle since $\phi_S$ can be extended to $S^{k+1}$ and thus contracted over one of the half spheres $D_+, D_- \subset S^{k+1}$. Thus $\hat{\alpha}$ sends $\rho(M_{k+1})$ into trivial bundles and hence it descends to an additive map

$$\alpha : A_k := \mathcal{M}_k / \rho(M_{k+1}) \to \mathcal{V}_k.$$ 

We claim that this map is injective: if a stable bundle $\hat{\alpha}(S)$ is trivial for some $Cl_k$-module $S$, then $S$ is (the restriction of) a $Cl_{k+1}$-module.

**Proof of the claim.** Let $S$ be a $Cl_k$-module and $\phi = \phi_S : S^k \to G$ the corresponding clutching map (that is $\phi(e_{k+1}) = I$, $\phi(e_i) = J_i$). We assume that $\phi$ is contractible, that is it extends to $\hat{\phi} : D^{k+1} \to G$, possibly after adding to $S$ an element in $\rho(M_{k+1})$. The closed disk $D^{k+1}$ will be considered as the northern hemisphere $D^{k+1}_+ \subset S^{k+1}$. Repeating the argument above for the surjectivity, we consider the meridians $\mu_v$ between $N = e_{k+1} \in S^k$ and $-N$, but this time there are much more such meridians, not only those in $S^k$ but also those through the hemisphere $D^{k+1}_+$. They are labeled by $v \in D^k_+ := D^{k+1}_+ \cap N^+$. Applying the negative energy gradient flow we deform the curves $\phi(\mu_v)$ to minimal geodesics without changing those in $\phi(S^k)$ which are already minimal. Then we obtain the midpoint map $\hat{\phi}_1 : D^k_+ \to P_1$ with $\phi_1(v) = m(\hat{\phi}(\mu_v))$ which extends the given midpoint map $\phi_1$ of $\phi$. This step is repeated $k$ times until we reach $\hat{\phi}_k : D^1_+ \to P_k$ which is a path from $J_k$ to $-J_k$ in $P_k$. This path can be shortened to a minimal geodesic.

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Footnote: In fact, both $\mathcal{V}_k$ and $A_k$ are abelian groups with respect to direct sums, not just semigroups, and $\alpha$ is a group homomorphism. Using the tensor product, $\mathcal{V} = \sum_k \mathcal{V}_k$ and $A = \sum_k A_k$ become rings and $\alpha$ a ring homomorphism, see [3].
in $P_k$ whose midpoint is a complex structure $J_{k+1}$ anticommuting with $J_1, \ldots, J_k$. Thus we have shown that our $\text{Cl}_k$-module $S$ is extendible to a $\text{Cl}_{k+1}$-module, that is $S \in \rho(\mathcal{M}_{k+1})$. This finishes the proof of the injectivity.

**Theorem 5.** [3] Every vector bundle over $S^k$ splits stably into a trivial bundle and a generalized Hopf bundle. More precisely, the map $\alpha : \mathcal{A}_k = \mathcal{M}_k / \rho(\mathcal{M}_{k+1}) \rightarrow \mathcal{V}_k$ sending the equivalence class of a $\text{Cl}_k$-module $S$ onto its generalized Hopf bundle is an isomorphism.

From (4) one easily obtains the groups $\mathcal{A}_k$ since the modules $S_k$ in (4) are the (one or two) generators of $\mathcal{M}_k$. If $S_k = \rho(S_{k+1})$, then $\mathcal{A}_k = 0$. This happens for $k = 2, 4, 5, 6$. For $k = 0, 1$ we have

$$\rho(S_{k+1}) = S_k \oplus S_k = 2S_k,$$

hence $\mathcal{A}_0 = \mathcal{A}_1 = \mathbb{Z}_2$. For $k = 3, 7$ there are two generators for $\mathcal{M}_k$, say $S_k$ and $S_k'$, and $\rho(S_{k+1}) = S_k \oplus S_k'$, thus $\mathcal{A}_3 = \mathcal{A}_7 = \mathbb{Z}$. Hence

$$\begin{array}{cccccccc}
 k & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
 \mathcal{A}_k & \mathbb{Z}_2 & \mathbb{Z}_2 & 0 & \mathbb{Z} & 0 & 0 & 0 & \mathbb{Z}
\end{array}$$

and because of the periodicity (5) we have $\mathcal{A}_{k+8} = \mathcal{A}_k$.

Consequently, the list (9) for $\mathcal{A}_k$ is the same as that for $\mathcal{V}_k$ and for $\pi_k(\text{O}_n)$, $n$ large (see (7). Thus we have also computed the stable homotopy of $\text{O}_n$.

We have seen that the following objects are closely related and obey the same periodicity theorem:

- Iterated centrioles of $\text{O}_n$,
- stable homotopy groups of $\text{O}_n$,
- Clifford modules,
- stable vector bundles over spheres.

**References**

[1] M. Atiyah: K-Theory. Benjamin 1967

[2] M. Atiyah, R. Bott, *On the periodicity theorem for complex vector bundles*, Acta Math. 112 (1964), 229 - 247

[3] M. Atiyah, R. Bott, A. Shapiro, *Clifford modules*, Topology 3 (1964), 3-38

[4] R. Bott, *The stable homotopy of the classical groups*, Proc. Nat. Acad. Sci. U.S.A. 43 (1957), 933–935.

[5] R. Bott, *The stable homotopy of the classical groups*, Ann. Math. 70 (1959), 313–337.

[6] Chen, B.-Y., Nagano, T., *Totally geodesic submanifolds of symmetric spaces II*, Duke Math. J. 45 (1978), 405 - 425

[7] H.B. Lawson, M.-L. Michelson: Spin Geometry, Princeton 1989

[8] J. Milnor: Morse Theory, Princeton 1963
[9] Mitchell, S. A., *The Bott filtration of a loop group*, in: Springer Lect. Notes Math. 1298 (1987), 215 - 226
[10] Mitchell, S. A.: *Quillens theorem on buildings and the loops on a symmetric space*, Enseign. Math. (II)34 (1988), 123 - 166
[11] Nagano, T., *The involutions of compact symmetric spaces*, Tokyo J. Math. 11 (1988), 57 - 79
[12] P. Quast: *Complex structures and chains of symmetric spaces*, Habilitationsschrift Augsburg 2010 (available from the author)
[13] P. Quast: *Centrioles in Symmetric Spaces*, Nagoya Math. J. 211 (2013), 51 - 77
[14] Wolf, J.A.: *Spaces of constant curvature*, 5th edition, Publish or Perish, Wilmington (Delaware) 1984

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