Free Probability for Pairs of Faces IV: Bi-Free Extremes in the Plane

Dan-Virgil Voiculescu
Department of Mathematics
University of California at Berkeley
Berkeley, CA 94720-3840
E-mail: dvv@math.berkeley.edu

Abstract. We compute the bi-free max-convolution which is the operation on bi-variate distribution functions corresponding to the max-operation with respect to the spectral order on bi-free bi-partite two-faced pairs of hermitian non-commutative random variables. With the corresponding definitions of bi-free max-stable and max-infinitely-divisible laws their determination becomes in this way a classical analysis question.

0. Introduction

The definition and classification of free max-stable laws in [2] had been an unexpected addition to the list of free probability analogues to classical probability items. Here we take the first step in a similar direction in bi-free probability [9]. We show that there is a simple formula for computing the bi-free extremal convolution of probability measures in the plane. This corresponds to computing the distribution of \((a \vee c, b \vee d)\), where \((a, b)\) and \((c, d)\) are two bi-free two-faced pairs of commuting hermitian operators. Like the free extremal convolution on \(\mathbb{R}\) defined in [2], the bi-free extremal convolution in the plane reduces the question of bi-free max-stable laws in the plane to an analysis problem in the classical context which we won’t pursue here.

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The derivation of the formula for the extremal bi-free convolution relies on the partial bi-free $R$-transform we found in [10]. Obviously our result here complements the recent work on operations on bi-free bi-partite hermitian two-faced pairs ([10], [7], [5], [11], [8]).

The present paper has three more sections besides the introduction. Section one contains preliminaries. Section two derives the main technical result, the bi-free extremal convolution for the distributions of hermitian bi-partite pairs in the case the variables are projections. The third section gives the formula for the extremal bi-free convolution in the general case of hermitian variables and the definitions of bi-free max-stable and max-infinitely-divisible laws.

1. Preliminaries

1.1. Throughout the preliminaries $(M, \tau)$ will denote a $W^*$-probability space, that is $M$ is a von Neumann algebra and $\tau$ a normal state. If $A$ is a $C^*$-algebra by $\text{Proj}(A)$ we shall denote the hermitian projections $P = P^* = P^2 \in A$. If $A = N$ a von Neumann algebra and $P, Q \in \text{Proj}(N)$ by $P \vee Q$ and $P \wedge Q$ we denote the least projection $\geq$ both $P$ and $Q$ and, respectively, the largest projection $\leq$ both $P$ and $Q$. If $\mathcal{H}$ is the Hilbert space on which $N$ acts then $P \vee Q$ is the orthogonal projection onto $PH + QH$, while $P \wedge Q$ is the orthogonal projection onto $PH \cap QH$.

1.2. If $M_h = \{m \in M \mid m = m^*\}$, we recall that the spectral order ([1]; see also [2]) on $M_h$ is defined by $a \prec b$ if the spectral projections satisfy

$$E(a; [t, \infty)) \leq E(b; [t, \infty))$$

for all $t \in \mathbb{R}$. Clearly this extends to self-adjoint operator affiliated with $M$, since $E(a; [t, \infty)) \leq E(b; [t, \infty))$ makes sense also under this more general assumption. If $a, b \in M_h$ then $a \vee b$ and $a \wedge b$ are defined by

$$E(a \vee b; (t, \infty)) = E(a; (t, \infty)) \vee E(b; (t, \infty))$$

and

$$E(a \wedge b; [t, \infty)) = E(a; [t, \infty)) \wedge E(b; [t, \infty)).$$

These definitions clearly work also more generally for affiliated self-adjoint operators. In essence the operators $a, b$ are replaced by the right-continuous decreasing family of projections $R \ni t \rightarrow E(a; (t, \infty)) \in \text{Proj}(M)$ and, respectively, by the left-continuous decreasing family of projections $\mathbb{R} \ni t \rightarrow E(a; [t, \infty)) \in \text{Proj}(M).$
1.3. A basic fact underlying free extreme values is the following:

**Lemma 1.1.** If \( P, Q \in \text{Proj}(M) \) are freely independent in \((M, \tau)\), then

\[
\tau(P \lor Q) = \min(\tau(P) + \tau(Q), 1)
\]

and

\[
\tau(P \land Q) = \max(0, \tau(P) + \tau(Q) - 1).
\]

This is a well-known fact. For references see [2] where this is Lemma 2.1 and see [12], [13] for computations. Remark that it is not necessary to require that \( \tau \) be tracial, since its restrictions to the von Neumann algebra generated by two free hermitian operators is always tracial.

We also recall from [2] the definitions of extremal free convolutions for probability measures on \( \mathbb{R} \). If \( \mu \) is a probability measures on \( \mathbb{R} \), its distribution function is \( F(t) = \mu((-\infty, t]) \). If \( \mu \) and \( \nu \) are probability measure on \( \mathbb{R} \) with distribution functions \( F(t) \) and \( G(t) \), then \( \mu \lor \nu \) and \( \mu \land \nu \) are defined via their distribution functions \( H(t) = \max(0, F(t) + G(t) - 1) \) and, respectively, \( K(t) = \min(F(t) + G(t), 1) \). If \( \mu \) and \( \nu \) are the distributions of \( a, b \in M_h \) with respect to \( \tau \), then \( \mu \lor \nu \) and \( \mu \land \nu \) are the distributions of \( a \lor b \) and \( a \land b \). It is also convenient to have corresponding operations on distribution functions of probability measures on \( \mathbb{R} \). If \( F, G \) are two such distribution functions, then \( F \lor G = (F + G - 1)_+ \) and \( F \land G = \min(F + G, 1) \).

1.4. We conclude the section of preliminaries by recalling some basics about the free \( R \)-transform and the partial bi-free \( R \)-transform.

If \( a \in (\mathcal{A}, \varphi) \) is a non-commutative random-variable in a Banach-algebra probability space, let \( G_a(z) = \varphi((z1 - a)^{-1}) \) be the Green-function, or Cauchy-transform of the distribution of \( a \), which is a holomorphic function in a neighborhood of \( \infty \in \mathbb{C} \cup \{\infty\} \) on the Riemann sphere. Then \( K_a(z) \) defined in a neighborhood of 0 \( \in \mathbb{C} \) and taking values in \( \mathbb{C} \) is the inverse of \( G_a \), that is \( K_a(0) = \infty \) and \( G_a(K_a(z)) = z \). Further \( R_a(z) = K_a(z) - z^{-1} \) is defined in a neighborhood of 0. If \( a \) and \( b \) are free, then \( R_{a+b}(z) = R_a(z) + R_b(z) \) in a neighborhood of 0.

If \( (a, b) \) is a two-faced bi-free pair in \((\mathcal{A}, \varphi)\) we consider

\[
G_{a,b}(z, w) = \varphi((z1 - a)^{-1}(w1 - b)^{-1})
\]
defined in a neighborhood of \((\infty, \infty) \in (\mathbb{C} \cup \{\infty\})^2\). The reduced partial bi-free \(R\)-transform of \((a, b)\) is
\[
\tilde{R}_{a,b}(z, w) = 1 - \frac{zw}{G_{a,b}(K_a(z), K_b(w))}
\]
defined in a neighborhood of \((0, 0) \in \mathbb{C}^2\). If \((a, b)\) and \((c, d)\) are bi-free in \((\mathcal{A}, \varphi)\), then
\[
\tilde{R}_{a+c,b+d}(z, w) = \tilde{R}_{a,b}(z, w) + \tilde{R}_{c,d}(z, w)
\]
in a neighborhood of \((0, 0) \in \mathbb{C}^2\) (see [10]).

If \((a, b)\) are commuting hermitian operators in a \(C^*\)-probability space \((\mathcal{A}, \varphi)\), then the joint distribution of \((a, b)\) is given by a probability measure on \(\mathbb{R}^2\) with compact support \(\mu_{a,b}\) and
\[
G_{a,b}(z, w) = \int \int (z - x)^{-1}(w - y)^{-1}d\mu_{a,b}(x, y).
\]
The measures \(\mu_{a,b}, \mu_{c,d}\) and \(\mu_{a+c,b+d}\) when \((a, b)\) and \((c, d)\) are bi-free in \((\mathcal{A}, \varphi)\) are related by additive bi-free convolution
\[
\mu_{a,b} \boxplus \boxplus \mu_{c,d} = \mu_{a+c,b+d}.
\]

2. Two-faced pairs of commuting projections

In this section we shall compute the bi-free extremal convolution in the case of the distributions of commuting projections. This is the bi-free generalization of the free probability result in Lemma 1.1. We begin with a sequence of lemmas.

**Lemma 2.1.** Let \((\mathcal{A}, \varphi)\) be a \(C^*\)-probability space and let \(P = P^* = P^2 \in \mathcal{A}\) and \(\varphi(P) = p\). Then we have
\[
G_P(z) = \frac{p}{z - 1} + \frac{1 - p}{z} = \frac{z + p - 1}{z(z - 1)}
\]
and
\[
zK_P(K_P - 1) = K_P + p - 1.
\]

**Lemma 2.2.** Let \((\mathcal{A}, \varphi)\) be a \(C^*\)-probability space and let \((P, Q)\) be a two-faced pair in \(\mathcal{A}\), so that \(P = P^* = Q^*, P = P^2, Q = Q^2, [P, Q] = 0, \varphi(P) = p, \varphi(Q) = q, \varphi(PQ) = r\). Then we have:
\[
G_{P,Q}(z, w) = \frac{(z + p - 1)(w + q - 1) + (r - pq)}{zw(z - 1)(w - 1)}.
\]
Lemma 2.3. Under the same assumptions as in Lemma 2.2, we have:
\[ G_{P,Q}(K_P, K_Q) = \frac{zw((K_P + p - 1)(K_Q + q - 1) + (r - pq))}{(K_P + p - 1)(K_Q + q - 1)}. \]

Lemma 2.4. Under the same assumptions as in Lemma 2.2, we have:
\[ \mathcal{R}_{P,Q}(z, w) = \frac{r - pq}{(K_P(z) + p - 1)(K_Q(w) + q - 1) + r - pq}. \]

The proofs of the preceding four lemmata are straightforward computations and will be omitted.

Lemma 2.5. Let \( \mu \) be a probability measure on \([0, 2]^2 \subset \mathbb{R}^2\) and let
\[ G(z, w) = \iint (z - x)^{-1} (w - y)^{-1} d\mu(x, y). \]
Let further \( z_n \in (2, \infty) \), \( w_n \in (2, \infty) \) be such that \( z_n \to 2 \), \( w_n \to 2 \) as \( n \to \infty \). Then we have:
\[ \lim_{n \to \infty} (z_n - 2)(w_n - 2)G(z_n, w_n) = \mu(\{(2, 2)\}). \]

Proof. Let \( F_n(x, y) = (z_n - 2)(z_n - x)^{-1}(w_n - 2)(w_n - y)^{-1} \) where \((x, y) \in [0, 2]^2\). Then \( |F_n| \leq 1 \) for \((x, y) \in [0, 2]^2\) and \( F_n \) converges pointwise to the indicator function of \( \{(2, 2)\} \). By Lebesgue’s dominated convergence theorem we have
\[ \lim_{n \to \infty} \iint F_n(x, y)d\mu(x, y) = \mu(\{(2, 2)\}) \]
which is what we wanted to prove.

Lemma 2.6. In a \( C^*\)-probability space \((A, \varphi)\) let \((P, Q)\) and \((P', Q')\) be bi-free two-faced pairs so that \( P = P^* = P^2 \), \( Q = Q^* = Q^2 \), \( P' = P'^* = P'^2 \), \( Q' = Q'^* = Q'^2 \), \( [P, Q] = [P', Q'] = [P, Q'] = [P', Q] = 0 \) and \( \varphi(P) = p \), \( \varphi(Q) = q \), \( \varphi(PQ) = r \), \( \varphi(P') = p' \), \( \varphi(Q') = q' \), \( \varphi(P'Q') = r' \) and let \( \delta = r - pq \), \( \delta' = r' - p'q' \). Then for \((z, w)\) in some neighborhood of \((0, 0) \in \mathbb{C}^2\) we have
\[ G_{P+P', Q+Q'}(K_{P+P'}(z), K_{Q+Q'}(w)) = \]
\[ = zw(1 - (1 + \delta^{-1}(K_P(z) + p - 1)(K_Q(w) + q - 1))^{-1} - (1 + \delta'^{-1}(K_P(z) + p' - 1)(K_Q(w) + q' - 1))^{-1})^{-1}. \]
In case \( \delta = 0 \) we set here \( (1 + \delta^{-1}(K_P(z) + p - 1)(K_Q(w) + q - 1))^{-1} = 0 \) and we adopt also a similar rule if \( \delta' = 0 \).
Proof. We have

\[ G_{P+P',Q+Q'}(K_{P+P'}(z), K_{Q+Q'}(w)) = zw(1 - \tilde{R}_{P+P',Q+Q'}(z, w))^{-1} \]

\[ = zw(1 - \tilde{R}_{P,Q}(z, w)) \]

\[ - \tilde{R}_{P',Q'}(z, w)^{-1}. \]

Note that if \( \delta = 0 \), \( P \) and \( Q \) are classically independent, so that \( \tilde{R}_{P,Q}(z, w) = 0 \) and similarly \( \tilde{R}_{P',Q'}(z, w) = 0 \) if \( \delta' = 0 \). On the other hand, Lemma 2.4 gives that \( \tilde{R}_{P,Q}(z, w) = (1 + \delta^{-1}(K_P(z) + p - 1)(K_Q(w) + q - 1))^{-1} \) if \( \delta \neq 0 \) and a similar fact for \( \tilde{R}_{P',Q'}(z, w). \)

Lemma 2.7. Under the same assumptions as in Lemma 2.6 we have for \( (z, w) \) in some neighborhood of \( (0, 0) \in \mathbb{C}^2 \) that

\[ (K_{P+P'}(z) - 2)(K_{Q+Q'}(w) - 2)G_{P+P',Q+Q'}(K_{P+P'}(z), K_{Q+Q'}(w)) \]

\[ = (K_P(z) + K_{P'}(z) - z^{-1} - 2)(K_Q(w) + K_{Q'}(w) - w^{-1} - 2) \]

\[ zw(1 - (1 + \delta^{-1}(K_P(z) + p - 1)(K_Q(w) + q - 1))^{-1} \]

\[ - (1 + \delta'^{-1}(K_{P'}(z) + p' - 1)(K_{Q'}(w) + q' - 1))^{-1}, \]

this being an equality of holomorphic functions.

Proof. This follows from the preceding lemma after multiplication with

\[ (K_{P+P'}(z) - 2)(K_{Q+Q'}(w) - 2) = \]

\[ (K_P(z) + K_{P'}(z) - z^{-1} - 2)(K_Q(w) + K_{Q'}(w) - w^{-1} - 2). \]

Since the conclusion of Lemma 2.6 was actually an equality of germs of holomorphic functions near \( (0, 0) \in \mathbb{C}^2 \) it might seem that there may be a problem with infinities when \( z \) or \( w \) is 0. It is easily seen looking at the right-hand side that this is not the case since \( (K_P(z) + K_{P'}(z) - z^{-1} - 2)z \) is holomorphic in a neighborhood of \( z = 0 \) and \( (K_Q(w) + K_{Q'}(w) - w^{-1} - 2)w \) is holomorphic in a neighborhood of \( w = 0 \), while

\[ (1 - (1 + \delta^{-1}(K_P(z) + p - 1)(K_Q(w) + q - 1))^{-1} \]

\[ - (1 + \delta'^{-1}(K_{P'}(z) + p' - 1)(K_{Q'}(w) + q' - 1))^{-1} \]

is holomorphic in a neighborhood of \( (0, 0) \in \mathbb{C}^2 \) and \( \neq 0. \)

Lemma 2.8. Assuming \( p > 0, q > 0, p' > 0, q' > 0 \), the equality which is the conclusion of Lemma 2.7 for \( (z, w) \) in a neighborhood of \( (0, 0) \in \mathbb{C}^2 \) extends analytically to \( (z, w) \) in a neighborhood of \( [0, \infty)^2 \subset \mathbb{C}^2 \).
**Proof.** Let \( t_0 \) and \( s_0 \) be the least upper bounds of the supports of the probability measures \( \mu_{P+P'} \) and \( \mu_{Q+Q'} \) on \( \mathbb{R} \). Then \( G_{P+P'}(z) \) on \( (t_0, \infty] \) and \( G_{Q+Q'}(w) \) on \( (s_0, \infty] \) are strictly decreasing taking the values \([0, \infty)\) so that \( K_{P+P'}(z) \) and \( K_{Q+Q'}(w) \) have analytic continuations along \([0, \infty)\) to a neighborhood of \([0, \infty)\) (the functions are viewed as taking values in the Riemann sphere \( \mathbb{C} \cup \{\infty\}\)). This implies the analytic extension of

\[
zw(K_P(z) + K_{P'}(z) - z^{-1} - 2)(K_Q(w) + K_{Q'}(w) - w^{-1} - 2)
\]

as a holomorphic function, taking values in \( \mathbb{C} \), to a neighborhood of \([0, \infty)^2 \subset \mathbb{C}^2\). On the other hand, similarly, since \( p > 0, p' > 0, q > 0, q' > 0 \) the functions \( K_P(z), K_{P'}(z), K_Q(w), K_{Q'}(w) \) have analytic continuations to a neighborhood of \([0, \infty)\) taking values for \( z, w \in [0, \infty) \) in \((1, \infty]\). If \( z, w \in [0, \infty), (1 + \delta^{-1}(K_P(z) + p - 1)(K_Q(w) + q - 1))^{-1} \) is well-defined when \( \delta \neq 0 \) since \((K_P(z) + p - 1)(K_Q(w) + q - 1) > pq > 0 \) while either \( \delta^{-1} > 0 \) or \( 0 \) or \( \delta^{-1} \geq p^{-1}q^{-1} \) so that the quantity to be inverted is \( > 0 \). Similar reasoning takes care of the term involving \( \delta' \).

For the analytic extension of

\[
G_{P+P',Q+Q'}(K_{P+P'}(z), K_{Q+Q'}(w))
\]

it suffices to remark that \( G_{P+P',Q+Q'}(z, w) \) is analytic in a neighborhood of \((t_0, \infty] \times (s_0, \infty] \subset (\mathbb{C} \cup \{\infty\})^2\) and \( K_{P+P'}(z), K_{Q+Q'}(w) \) on \([0, \infty)\) take values in \((t_0, \infty]\) and \((s_0, \infty]\), respectively. That

\[
(1 - (1 + \delta^{-1}(K_P(z) + p - 1)(K_Q(w) + q - 1))^{-1}
- (1 + \delta'^{-1}(K_{P'}(z) + p' - 1)(K_{Q'}(w) + q' - 1))^{-1})
\]

is \( \neq 0 \) for \((z, w) \in [0, \infty)^2\) follows from the fact the left-hand side is finite and

\[
zw(K_P(z) + K_{P'}(z) - z^{-1} - 2)(K_Q(w) + K_{Q'}(w) - w^{-1} - 2)
\]

\[
= zw(K_{P+P'}(z) - 2)(K_{Q+Q'}(w) - 2) \neq 0.
\]

\[\Box\]

**Lemma 2.9.** If \((\mathcal{A}, \varphi)\) is a \(W^*\)-probability space and \( P = P^* = P^2 \in \mathcal{A}, P' = P'^* = P'^2 \in \mathcal{A} \) then \( P \wedge P' = E(P + P', \{2\}) \).

The preceding lemma is a well-known fact.
Lemma 2.10. Let \((\mathcal{A}, \varphi)\) be a \(W^*-\)probability space. Then under the assumptions of Lemma 2.6, if \(p + p' - 1 > 0\) and \(q + q' - 1 > 0\) we have
\[
\varphi((P \land P')(Q \land Q')) = (p + p' - 1)(q + q' - 1)(1 - (1 + \delta^{-1}pq)^{-1} - (1 + \delta'^{-1}p'q')^{-1})^{-1}.
\]

Here in case \(\delta = 0\) we set \((1 + \delta^{-1}pq)^{-1} = 0\) and adopt a similar rule if \(\delta' = 0\). In case \(p + p' - 1 \leq 0\) or \(q - q' - 1 \leq 0\) we have \(\varphi((P \land P')(Q \land Q')) = 0\).

**Proof.** As recorded in Lemma 1.1, we have
\[
\varphi(P \land P') = (p + p' - 1)_+, \quad \varphi(Q \land Q') = (q + q' - 1)_+
\]
after observing that \(P\) and \(P'\) being free, the restriction of \(\varphi\) to the algebra generated by \(P\) and \(P'\) is a trace and a similar fact for \(Q\) and \(Q'\). Clearly, if \(\varphi(P \land P') = 0\) or \(\varphi(Q \land Q') = 0\), we must have also \(\varphi((P \land P')(Q \land Q')) = 0\) since \([P \land P', Q \land Q'] = 0\) and \(0 \leq (P \land P')(Q \land Q') \leq P \land P'.\) Thus we are left with proving the lemma when \(p + p' - 1 > 0\) and \(q + q' - 1 > 0\).

Thus \(\mu_{P+P'}\{\{2\}\} = p + p' - 1 > 0\), \(\mu_{Q+Q'}\{\{2\}\} = q + q' - 1 > 0\) and \(\text{supp } \mu_{P+P'} \subset [0, 2]\), \(\text{supp } \mu_{Q+Q'} \subset [0, 2]\). By considerations along the lines in the proof of Lemma 2.8, if \(t_n \in (0, \infty), t_n \uparrow \infty\) then \(K_P(t_n) \downarrow 1\), \(K_{P'}(t_n) \downarrow 1\), \(K_Q(t_n) \downarrow 1\), \(K_{Q'}(t_n) \downarrow 1\), \(K_{P+P'}(t_n) \downarrow 2\), \(K_{Q+Q'}(t_n) \downarrow 2\). Taking \(z = w = t_n\) in the equality in Lemma 2.7 extended according to Lemma 2.8, we get that the limit of the left-hand side in view of Lemma 2.5 is
\[
\mu_{P+P',Q+Q'}\{\{(2, 2)\}\} = \varphi(E(P + P', \{2\})E(Q + Q', \{2\})) = \varphi((P \land P')(Q \land Q')).
\]

On the other hand,
\[
(K_P(t_n) + K_{P'}(t_n) - t_n^{-1} - 2)t_n = (K_P(t_n) - 1)t_n + (K_{P'}(t_n) - 1)t_n - 1
\]
and this converges as \(n \to \infty\), by the simpler analogue of Lemma 2.5 for Cauchy transforms in one variable, to \(p + p' - 1\). Similarly \((K_Q(t_n) + K_{Q'}(t_n) - t_n^{-1} - 2)t_n\) converges to \(q + q' - 1\). On the other hand
\[
(1 - (1 + \delta^{-1})(K_P(t_n) + p - 1)(K_Q(t_n) + q - 1))^{-1}
\]
\[
- (1 + \delta'^{-1})(K_{P'}(t_n) + p' - 1)(K_{Q'}(t_n) + q' - 1))^{-1}
\]
converges to \((1 - (1 + \delta^{-1}pq)^{-1} - (1 + \delta'^{-1}p'q'))\) no matter whether \(\delta\) and \(\delta'\) are \(\neq 0\) or \(= 0\). \(\Box\)
The last lemma after some simple algebraic work on the formulae will give the final result of the computations in this section, which we record as a theorem.

**Theorem 2.1.** Let \((\mathcal{A}, \varphi)\) be a \(W^*\)-probability space and let \(P = P^* = P^2 \in \mathcal{A}, Q = Q^* = Q^2 \in \mathcal{A}, P' = P'^* = P'^2 \in \mathcal{A}, Q' = Q'^* = Q'^2 \in \mathcal{A}\) be such that \([P, Q] = 0, [P', Q] = [P, Q'] = 0, [P', Q'] = 0\) and \((P, Q)\) and \((P', Q')\) are bi-free in \((\mathcal{A}, \varphi)\). Then we have \(\varphi(P \wedge P') = (\varphi(P) + \varphi(P') - 1)_+, \varphi(Q \wedge Q') = (\varphi(Q) + \varphi(Q') - 1)_+\) and if \(\varphi(P \wedge P') > 0, \varphi(Q \wedge Q') > 0, \varphi(PQ) > 0, \varphi(P'Q') > 0\) then we have

\[
\frac{\varphi(P \wedge P') \varphi(Q \wedge Q')}{\varphi((P \wedge P')(Q \wedge Q'))} = \frac{\varphi(P) \varphi(Q)}{\varphi(PQ)} + \frac{\varphi(P') \varphi(Q')}{\varphi(P'Q')} - 1.
\]

If any of the numbers \(\varphi(P \wedge P'), \varphi(Q \wedge Q'), \varphi(PQ), \varphi(P'Q')\) is 0, then \(\varphi((P \wedge P')(Q \wedge Q')) = 0\).

**Proof.** The formulae for \(\varphi(P \wedge P'), \varphi(Q \wedge Q')\) are not new (see Lemma 1.1) and it is obvious that if any of \(\varphi(P \wedge P'), \varphi(Q \wedge Q'), \varphi(PQ) \varphi(P'Q')\) is 0, then so is \(\varphi((P \wedge P')(Q \wedge Q'))\). Thus using the notation \(\varphi(P) = p, \varphi(P') = p', \varphi(Q) = q, \varphi(Q') = q', \varphi(PQ) = r, \varphi(P'Q') = r'\) which we used in the lemmata, we may assume \(p + p' > 1, q + q' > 1, r > 0, r' > 0\). Turning to the result of Lemma 2.10, remark that if \(\delta = r - pq \neq 0\) then

\[
(1 + \delta^{-1}pq)^{-1} = \delta + p q = (r - pq) r^{-1} = 1 - pqr^{-1}.
\]

If \(\delta = 0\), then \(r = pq\) and \(1 - pqr^{-1} = 0\) which is in agreement with the rule that \((1 + \delta^{-1}pq)^{-1}\) is 0 if \(\delta = 0\). A similar remark about \(\delta'\). Hence the right-hand side of the formula in Lemma 2.10 is

\[
\varphi(P \wedge P') \varphi(Q \wedge Q') (pqr^{-1} + p'q'r'^{-1} - 1)^{-1}
\]

so that the formula gives

\[
\frac{\varphi(P \wedge P') \varphi(Q \wedge Q')}{\varphi((P \wedge P')(Q \wedge Q'))} = pqr^{-1} + p'q'r'^{-1} - 1
\]

which is in agreement with the previous remark.

\[
\varphi(P) \varphi(Q) \varphi(P') \varphi(Q') - 1.
\]

\(\square\)

3. **Bi-free max-convolution in the plane**

In this section \((\mathcal{A}, \varphi)\) will be a von Neumann algebra with a normal state \(\varphi\). If \((a, b)\) is a bi-partite hermitian two-faced pair in \((\mathcal{A}, \varphi)\), that is a pair of commuting hermitian operators \(a, b \in \mathcal{A}\), let \(E(a, b; \omega)\) denote its spectral measure where \(\omega \subset \mathbb{R}^2\) is a Borel set and let
\( \mu_{a,b}(\omega) = \varphi(E(a, b; \omega)) \) be the probability measure on \( \mathbb{R}^2 \) which is the distribution of \( (a, b) \). Let further \( F_{a,b}(s, t) = \mu_{a,b}((\mathbb{R} \times \mathbb{R}) \setminus (-\infty, 0] \times (-\infty, 0]) \) be the distribution function of the measure \( \mu_{a,b} \). We recall that such functions \( F(s, t) \) are such that \( s_1 \leq s_2, t_1 \leq t_2 \Rightarrow F(s_1, t_1) \leq F(s_2, t_2) \), \( s_n \downarrow s_0, t_n \downarrow t_0 \Rightarrow F(s_n, t_n) \downarrow F(s_0, t_0) \) as \( n \to \infty \) and \( s_1 \leq s_2, t_1 \leq t_2 \Rightarrow F(s_2, t_2) - F(s_1, t_2) - F(s_2, t_1) + F(s_1, t_1) \geq 0 \). Moreover, since this is the distribution function of a probability measure with compact support, we have \( 0 \leq F(s, t) \leq 1 \) and there is \( C > 0 \) so that \( F(s, t) = 0 \) if \( \min(s, t) \leq -C \) and \( F(s, t) = 1 \) if \( \min(s, t) \geq C \). If we want to deal with probability measures without a condition of compact support, we will require that \( F \) be defined on \( [−\infty, ∞]^2 \) and satisfy \( F(s, t) = 0 \) if \( \min(s, t) = −\infty \) and \( \lim_{n↑+∞} F(n, n) = 1 \).

If \( (a, b) \) and \( (a', b') \) are bi-free bi-partite hermitian pairs, it’s always possible to find a realization in a von Neumann algebra \( (\mathcal{A}, \varphi) \) of the joint distribution so that \( [a, b'] = [a', b] = 0 \). Note further that the joint distribution of \( (a \vee a', b \vee b') \) does not depend on the realization, but only on the distributions \( \mu_{a,b}, \mu_{a', b'} \) since

\[
E(a \vee a', b \vee b'; (−∞, s] \times (−∞, t])
= E(a \vee a'; (−∞, s])E(b \vee b'; (−∞, t])
= (E(a; (−∞, s]) \wedge E(a'; (−∞, s]))(E(b; (−∞, t]) \wedge E(b'; (−∞, t]))
\]

and \( \varphi(E(a \vee a', b \vee b'; (−∞, s] \times (−∞, t])) \) can be computed using the results in Section 2 from the distributions of the bi-free two-faced pairs \( E(a; (−∞, s]), E(b; (−∞, t])) \) and \( E(a'; (−∞, s]), E(b'; (−∞, t])) \).

**Definition 3.1.** If \( F \) and \( G \) are distribution functions of probability measures with compact support on \( \mathbb{R}^2 \) we define their bi-free max-convolution or alternatively also called bi-free upper extremal convolution \( H = F \vee G \) to be such that if \( F_j, G_j, H_j \) \( (j = 1, 2) \) are their distribution function marginals we have \( H_j = F_j \vee G_j \) \( (j = 1, 2) \) and

\[
\frac{H_1(s)H_2(t)}{H(s, t)} = \frac{F_1(s)F_2(t)}{F(s, t)} + \frac{G_1(s)G_2(t)}{G(s, t)} - 1
\]

if \( F(s, t) > 0, G(s, t) > 0, H_1(s) > 0, H_2(t) > 0 \) and \( H(s, t) = 0 \) otherwise.

That the above gives a well-defined distribution function of a probability measure with compact support is a consequence of the discussion preceding the definition and of Theorem 2.1. Note also that to see that the distribution function of \( a \wedge a' \) is given by \( F_a \vee F_{a'} \) it is not necessary to assume \( a \) and \( a' \) are in a tracial \( W^* \)-probability space, since the restriction of \( \varphi \) to the weak closure of the \( * \)-algebra generated
by \( \{I, a, a'\} \) will be a tracial normal state. In essence, \( \vee \vee \) gives the distribution of \((a \vee a', b \vee b')\) in the realizations of the joint distributions of \((a, a')\) and \((b, b')\) where the commutations \( [a, b] = [a', b] = 0 \) hold.

The further remark is that actually \( a, b, a', b' \) only appear here via their spectral scales \( E(a; (-\infty, t]) \) etc. and thus the operations extend to affiliated unbounded self-adjoint operators and distributions of any probability measures on \( \mathbb{R}^2 \).

Having defined \( \vee \vee \) we can now define bi-free max-stable and bi-free max-infinitely divisible laws on \( \mathbb{R}^2 \).

**Definition 3.2.** A distribution function \( F \) of a probability measure on \( \mathbb{R}^2 \) is bi-freely max-stable if there are \( a_n, b_n, c_n, d_n \in \mathbb{R}, a_n > 0, c_n > 0 \) so that

\[
(F \vee \vee F \vee \vee F \vee \vee \ldots F)(a_n x + b_n, c_n y + d_n) \to F(x, y)
\]

as \( n \to \infty \).

**Definition 3.3.** A distribution function \( F \) of a probability measure on \( \mathbb{R}^2 \) is bi-freely max-infinitely-divisible if for each \( n \in \mathbb{N} \) there is a distribution function \( F_n \) so that

\[
F_n \vee \vee F_n \vee \vee \ldots F_n = F.
\]

**Remark 3.1.** The definitions in this section show that, in the simplest bi-free case of bi-partite hermitian two-faced pairs that have distributions given by probability measures on \( \mathbb{R}^2 \), the basic extreme value questions about bi-free max-stable and bi-free max-infinitely are transformed by the operation \( \vee \vee \) into “classical” questions. Clearly these questions are more difficult than univariate free extreme value questions ([2], [3]). It is a natural question whether like in [2], where free max-stable laws were related to classical “peaks over thresholds”, the “classical” questions to which bi-free extremes laws lead in this simplest case are also related to some classical extremes questions ([4], [6]).

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