Distributive Rings and Some Domains

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Abstract: In this paper, we study many relationships about Distributive ring and other domain such as Dedekind domain and Noetherian domain. We prove if any divisible module over a ring T, then T is Distributive ring. Also we satisfy if T is invariant ring have multiplication ideal this imply T is a Distributive ring. Finally, we study strongly duo ring and related to Distributive ring.

Keywords: Dedekind domain, Distributive ring, Bezout ring, Multiplication module, Strongly duo ring.

1. Introduction

All rings in this paper are commutative with identities [3], all $T$-modules is a unitary modules. Suppose that $T$ is a ring. In [16]: "any module $M$ on $T$ is distributive if $A \cap (B + C) = (A \cap B) + (A \cap C)$, where $A$, $B$ and $C$ are submodules of $M$ and hence any ring $T$ is called Distributive ring if every $T$-module is a distributive ".

From [18]: "T is called Noetherian ring if we have prime ideal $I$ in T, such that $I$ is f-generated". The author [17]: "said semilocal ring T; if it is has finitely many maximal ideals". "The ring T is called Bezout if each f-generated ideal $I \in T$ is principal" [6]. From [5]: "T is called containment-division ring (C.D.R), if for any two ideals $I_1, I_2 \subseteq T$, then $I_1 \subseteq I_2$ iff $I_2$ divides $I_1$". "A ring T is called Artirian if it is satisfies the descending chain condition[4,6]. From" [13]: "T is called P.I.R if any one -sided ideal of T is cyclic". From [17]: "the module M is called divisible if $t \cdot m = m, 0 \neq t \in T, m \in M$". In [12]:" For any $\alpha \in T$, there exist $x \in T$ such that $\alpha^2 x = \alpha$ the element $\alpha$ is called regular". By [1]: "a ring T is called regular if each element in T is regular". Also, in [2]:" $I$ is called maximal ideal if $I \subseteq T$ proper ideal and not contend in any other proper ideal". In [14]:" The module M is called multiplication if for each sub module $K$ of $M K = IM$ for some ideal $I \in T$". Recently, many articles have appeared that support the importance of modules and its relationship to other algebraic concepts [11,3,4,9,15].

2. Main Results
In this section, we study Distributive ring $T$ by depending on some domain. Some new relationships are studied in details, but before that we need to introduce all definitions about the ideas of the topic.

**Definition 2.1.** [10]. "If $T$ is an integral domain then $T$ is called Dedekind domain iff each ideal $I \in T$ is invertible".

**Definition 2.2.** [11]. "Let $I$ be an ideal of $\mathbb{Z}$. Then $I$ is called fractional ideal ($FI$) if every maximal ideal $I_i$ is a $P.I$ over the ring $T_i$ is invertible".

**Remark 2.3.** [11]. "A fractional ideal ($FI$) is called invertible if there exist ($FI^{-1}$) and $II^{-1} = T^n$.

Now we can introduce the following lemma to show the relationship between ($FI$) and Distributive ring.

**Lemma 2.4.** [11]. Let $I$ be an zero fractional ideal ($FI$) of the ring $T$. If $I$ is invertible then $T$ is a Dedekind domain and hence $T$ is Distributive ring.

**Example and Remark (2.5).**

Let $T$ is integral domain then :

1) Every P.I.D is a Dedekind domain.
2) $T$ is P.I.D if and only if any fractional ideal in $T$ is principle.
3) $T$ is unique factorization domain if and only if any fractional ideal in $T$ is principle.
4) Every discreet valuation ring is a Dedekind domain.
5) $T$ is a Dedekind domain if it is C.D.R and Noether in ring.
6) $T$ is a Dedekind domain if and only if any divisible module on $T$ is injective.
7) If $T$ is a Dedekind domain then finite any torsion free submodule is free.

Recall that $T$ is invariant ring if $tT \subseteq Tt$, for each $t \in T$.

**Theorem 2.6.** Let $I$ be ($FI$) in $T$. If $I$ is a f-generated and left principal then $T$ is a Distributive ring.

**Proof:** The set $\chi = I(T : I)$ sub-set of $T$. Let $I$ be a f.g.ideal. Therefor $\chi_i = I_i(T : I_i)$. On the other hand, $I_i \neq 0$ is a (P.I). This means $\chi_i = T_i$. Hence $T = I(I$ is invertible ideal $)$. But $I$ is ($FI$). Then $T$ is a Dedekind domain and thus it is Distributive ring.

In the next theorem, we try to obtain that $T$ is a Distributive ring in case an ideal $I$ is a projective module over $T$.

**Theorem 2.7.** Let $I$ be an-ideal of the ring $T$. If $I$ satisfy the following conditions
1- $I$ is ($FI$).
2- $I$ is a projective module. Then $T$ is a Distributive-ring.
Proof: Let $I$ be $(FI)$ and let $I$ is a projective module. So this ideals to the following: $\exists J$ as a module $\exists I \oplus J = T$. Let $M : T \to I$ be a projective mapping with $M(\alpha x) = \alpha x$. Also we take the mapping $\psi : I \to T$ with $\chi$ th factor. Hence $\forall a = \sum \psi(x)ax; x \in I$ and $\psi(x) = 0 \forall x$. Let $0 \neq b \in I$ and $x \in T$. So $qbx = 0$. $I \in \sum Tq, \psi(x) = 0$. Hence $0 \neq a \in I$. Suppose that $a = \frac{a_1}{b_1}$ and $b = \frac{a_2}{b_2}$; $a_1, a_2, b_1, b_2 \in T$. Hence $a_1, a_2 \in I$ and $a_2b_2\psi(b) = \psi(a_1, a_2) = b_1a_2\psi(a)$. Therefor $aqx = \psi(a) \in T$. Then $IJ \subseteq T$. But $b = M\psi(b)bx$ and hence $1 = bxq$. Thus $IJ = T$. (I invertible ). Hence $T$ is a Dedekind domain and from (lemma 2.4); $T$ is a Distributive ring.

Remark 2.8. If $I$ is a (P.I) and $(FI)$ of the ring $T$, then it is invertible fractional ideal and hence $T$ is a Distributive ring. Therefore need to deal with (P.I.D) in the next results.

Definition 2.9. [9]. "Let $T$ be a domain. Then $T$ is called Dedekind finite if and only if every regular element of $T$ is a unit ".

(*) Any ring $T$ is called quoring if all regular elements of $T$ is invertible. Hence every Dedekind finite domain is a Distributive finite ring.

Theorem 2.10. Let $E$ be an $T$-module. If $T$ is divisible $T$-module, then $T$ is a Distributive finite ring.

Proof: Clear. Since $T$ is divisible $T$-module, then $\chi T = T$. Hence for every regular element $\chi$ is a unit. Then $T$ is a Dedekind finite domain. Thus $T$ is Distributive finite ring.

Corollary 2.11. Every quoring $T^n$ such that $n$ is finite is Distributive finite ring.

Corollary 2.12. Every Artinian ring $T$ with zero nilradical is a Distributive finite ring.

Theorem 2.13. [12]. A ring $T$ is Dedekind finite iff it is Artinian and contains nilradical.

Corollary 2.1. Let $M$ be a right multiplication module over invariant ring $T$ with commutative multiplication ideals then $M$ is finite generated if it is finite dimensional.

Definition 2.15. [10]. An integral-domain $T$, Then $T$ is called generalized domain $G(D)$ iff every $I \in F(D), I = (I^{-1})^{-1}$ is invertible.

Theorem 2.16. Let $T$ be an integral domain and let $M$ be finitely generated free $T$-module of rank one. Then $T$ is Dedekind domain and Distributive, if there exist $m_i$ set of elements $m$ of $M$, and $f_i$ $T$-maps such that $f_i : M \to T$, and $\sum f_i(m)m_i = m$. finite for all $m \in M$.

Proof: Assume that $f(m) = \sum f_i(m)m_i = \sum r_i e_i$. Since $M$ is finitely generated then $M$ is projective of rank one, and $M$ is invertible fractional ideal of $T$. Also $T$ is Dedekind domain, And all
fractional ideal are principle. Now, let \( I, J, H \), three principle fractional ideal, then \( I, J, H \), are principle. Let \( I = \sum a_f(m),\ J = \sum b_f(m),\ H = \sum c_f(m) \).

Then
\[
I \cap (J + H) = (\sum a_f(m)) \cap ((\sum b_f(m)) + (\sum c_f(m)))
= (\sum a_f(m)) \cap (b \sum f_i(m) + c \sum f_i(m))
= (a \sum f_i(m)) \cap ((b + c) \sum f_i(m))
= (a \cap (b + c))(\sum f_i(m))
= (a \cap b) + (a \cap c)(\sum f_i(m))
= (a \cap b)(\sum f_i(m)) + (a \cap c)(\sum f_i(m))
= (\sum f_i(m)) \cap b(\sum f_i(m)) + a(\sum f_i(m)) \cap c(\sum f_i(m))
= (\sum f_i(m) \cap \sum b f_i(m)) + (\sum a f_i(m) \cap \sum c f_i(m))
= (I \cap J) + (\cap H)
\]

Hence \( T \) is Distributive ring.

**Theorem 2.17.** Let \( T \) be a left invariant ring with commutative multiplication of ideals. Then \( T \) is left Distributive Dedekind ring and a \( T \)-module \( M = tT \) where \( T = (t_i)r + (t_j)r \) and \( M = \bigoplus_{i=1}^{n} t_iT, j \neq i \).

**Proof:** Since \( T \) is left invariant ring with commutative multiplication of ideals, then every left ideal of \( T \) is a multiplication and \( T \) finite-dimensional ring. Therefor \( M \) is left finite-dimensional module. Thus \( M \) is a finitely generated. So \( M \) is projective and krull dimension. Hence \( T \) is Artinian, therefor \( T \) is left Notherin ring. Hence \( T \) is Distributive Dedekind ring. Let \( t = t_1 + t_2 + \cdots + t_n \in M, i = 1, 2, \ldots, n \). Since \((t_i)r \) is an ideal of left invariant ring \( T \), and by assumption that \( T = (t_i)r + (t_j)r \), then there exist an element \( m_i \in (t_i)r \) and \((1 - m_i) \in \cap_{j \neq i} (t_i)r \), such that \( t(1 - m_i) = t_i \), then \( t_iT \subseteq tT \) for every \( i \). Hence \( M = tT \).

**Theorem 2.18.** Let \( T \) be an integral domain and \( M \) be \( T \)-module. If \( T \) is regular multiplication, then the following are equivalent.

1) Every ideal \( I \) contained all regular element of \( T \), the ring \( T/I \) is finite direct sum of (P.I.R).
2) \( I = MI \).
3) \( T \) is Distributive Dedekind domain.
4) \( T \) is Dedekind finite.
Proof : (1) → (2) Suppose that $I$ an ideal of $T$. Then $I$ are regular ideal (since $T$ is regular multiplication and $I$ has only regular element ). Let $I = g_1^{r_1}g_2^{r_2}...g_n^{r_n}$, such that $g_1,g_2,...,g_n$ are distinct maximal ideal of $T$, and $r_1,r_2,...,r_n$ are positive integer. Then $T/I \cong \oplus_{i=1}^{r_n}T/g_i^{r_i}$, where $I$ is regular. Then $g_i^{r_i}$ is regular ideals of $T$. So any ideal $I$ of $T$ containing $g_i^{r_i}$ is a multiplication $T$-module. Thus $I = M$.  

(2) → (3) Since $I = M$ then I is multiplication module. Thus $T$ is multiplication domain. Hence $T$ is Distributive Dedekind domain.

(3) → (4) Since $T$ is Dedekind domain and regular multiplication, then every regular ideal in $T$ is invertible. But every regular element is invertible. Therefor $T$ is a quoring ring. Hence $T$ is Dedekind finite.

(4) → (1) Since $T$ is regular multiplication and Dedekind finite, then every regular ideal I of T is invertible multiplication $T$-module. So $T/I$ is multiplication ring and (P.I.R). Hence $T/I$ is finite direct sum of (P.I.R).

Remark 2.29. If $I$ is regular prime ideal in the regular multiplication ring $T$, then $I$ is maximal ideal. And every multiplication ring is regular multiplication but the convers is not true. Also $M$ is regular $T$-module if any $x \in M, A_{nnT}(x) = 0$.

Definition 2.20. [2]. "A ring $T$ is called $P$-Noetheran if it is $(\frac{T}{I})$ -Noetherian such that $P$ is prime ideal of $T$".

The next theorem explain the relationship between Noetherian ring $T$ and Distributive ring.

Theorem 2.21. Let $T$ be a P-Noetherian ring. Then for every maximal ideal $p$ of $T$, $T$ is a Distributive ring.

Proof: For an ideal of $T$, if $P$ is a maximal of $T$, there exists $S_p \subseteq I_p$ and $F_p$ is a f-generated sub ideal of $I$ such that $S_p I \subseteq F_p$. $S_p$ generated unit ideal. Then it is true for some finie subset $\{S_{p_1},...,S_{p_n}\}$ of them. Hence $I = (S_{p_1},...,S_{p_n})I \subseteq F_{p_1} + ... + F_{p_n} \subseteq I$. Therefor $I = F_{p_1} + ... + F_{p_n}$ is a f-generated ideal of $T$. Then $T$ is a Noetherian ring thus $T$ is a Distributive ring.

(\*) Consider the proper ideal of $T$ is a proper multiply of prime ideals and denoted by $(\otimes P)$. Therefor any ring $T$ is called $(\otimes P)$-ring if $T$ satisfy (\*) and hence $T$ is $(\otimes P)$-ring. So it is a Noetherian ring and finitely it is Distributive ring.
In the next result we introduce a clear relationship between $P$-ring and Distributive ring.

**Corollary 2.22.** For each maximal ideal $I$ of $T$, if $T$ is a $P$-ring, then it is a Distributive ring.

**Proof:** Assume that $T$ is a ring such that each ideal is maximal. So $T$ is a Noetherian ring if $T$ is a $P$-ring. Thus $T$ is a Distributive ring.

**Corollary 2.23** Every locally Noetherian ring is a Distributive ring.

**Proposition 2.24.** Let $T$ be a ring. If $T$ is an Artinian ring, then it is a Distributive ring.

Before prove proposition 2.24, we need to introduce the following lemma.

**Lemma 2.25.** Let $T$ be a commutative ring. Then $H = \bigcap P_i = \bigcap P_i$ is a nilpotent ideal such that $H^k = (0), k \geq 1$.

**Proof:** We denote $P_i$ to the finitely distinct primes ideal of $T$. Therefor all powers of $H$ is decrease and hence we find the smallest $k$ such that $HH^k = H^k$. Assume that $H^k = (0)$, and $0 \neq I$ is maximal. So $IH^k \neq (0)$. Clear $I = (x)$ is a principal. Therefor $xHH^k = xH^k \neq (0)$ and hence $(x) = xH$. $x = xy$, $y \in H$. Then $(1 - y)x = 0$. $(1 - y)$ is a unite because we have $(1 - y) \subseteq$ maximal ideal. C! So $H^k = (0)$.

Now, we return to prove proposition 2.24.

**Proof:** From lemma 2.25; $H^k = (0)$ We have maximal ideals $i_1, i_2, \ldots, A, i_r$, $(0) = i_1 i_2 \ldots i_r$. Take $T \triangleright i_1 \triangleright i_1 i_2 \triangleright \cdots \triangleright i_1 i_2 \ldots i_r = (0)$. Since $T$ is an Artinian ring, so $i^{th}$ quotient in $(\ast)$ is Artinian. Also over the filed $\frac{T}{i_j}$ these quotients are Noetherian module over the ring $T$. So $T$ is a Noetherian ring. Thus $T$ is a Distributive ring.

**Example 2.26.** The ring $\mathbb{Z}$ is a Distributive ring because it is a Noetherain ring.

**Remark 2.27.** Any homomorphic image of a Distributive ring is Distributive.

**Definition 2.28.** A ring $T$ is called Co-Noetherian if the inj-hull of simple module over T is Artinian.

**Proposition 2.29.** Let $T$ be a ring. If

1) $T$ is a Co-Noetherian ring.
2) $T$ is a semilocal commutative ring. Then $T$ is a Distributive ring.
Proof: Take \( C \) be injective hull such that \( C = C \left( \frac{T}{J} \right) = \bigoplus_{i=1}^{n} C(K_i) \ni K_i \) are simple module over \( T \) and \( J \) is the Jacobson radical of \( T \) and let \( H = I_0 \subseteq I_1 \subseteq \ldots \), is a scenting chain ideals of \( T \). Suppose that \( B_i = \text{ann}(I_i) \). So \( C \ni I_1 \ni I_2 \ni \ldots \). Also, there exists \( d \) integer such that \( B_d = B_{d+s}, s \geq 1 \). Then \( \text{Hom}(I_{d+s}/I_d, C) = B_d/B_{d+s} = 0 \). Since \( C \) is a cogenerator, then \( I_{d+s} = I_d, d \geq 1 \). Hence \( T \) is a Noetherian ring and thus \( T \) is a Distributive ring.

Example 2.30. \( \mathbb{R}[x_1, \ldots, x_n] \) is a right Distributive ring, because it is a Noetherian ring and hence it is an Artinian ring.

(\( \clubsuit \)) The module \( M \) is distributive if and only if the lattic of the submodule \( L(M) \) is distributive.

Recall that \( T \) is strongly regular ring if for each \( t \in T \) theare \( y \in T \) satisfy \( t^2y = t \). The lattice of each f-generated submodule of f-generated \( T \)-module \( M \) is \( L(M) \), and \( L(T_T) \) is the lattice of all principal right ideals of \( T \).

Corollary 2.31. If \( T \) be arbitrary ring and \( M \) be f-generated regular \( T \)-module, then \( L(M) \) is distributive if and only if the ring \( S = \text{End}(M) \) is strongly regular.

Theorem 2.32. If \( T \) be any ring and \( M \) a f-generated \( T \)-module with \( \text{ann}(m) = 0 \) for each \( m \in M \). If:

1) Every f-generated regular submodule is multiplication \( T \)-module.

2) \( S = \text{End}(M) \) is strongly regular ring. Then \( T \) is Dedekind-distributive.

Proof: Let \( m \in M ; \text{ann}(m) = 0 \). Therefor \( m \) is regular element in \( M \). Since \( M \) is f- generated and has regular element, then \( M \) is regular \( T \)-module. By assumption that every f-generated regular submodule is multiplication \( T \)-module. Then each regular ideals are multiplication \( T \)-module. \( T \) is regular multiplication ring. Since \( \text{End}(M) \) is strongly regular ring, then from theorem(2.32) and by (\( \clubsuit \)), every f-generated submodule in \( M \) is distributive. Thus \( M \) is distributive and by the fact that every multiplication ring is Dedekind domain. Thus \( T \) is Dedekind distributive ring.

Recall that the \( T \)- module \( M \) is strongly duo module if for every \( K \) submodule of \( M \) then \( T\sigma(K, M) = K \). Also \( T \) is right strongly duo ring if \( T \) is right strongly duo \( T \)-module.

(\( \clubsuit \)) If \( T \) is right strongly duo ring, then it is right Artinian if it is right Neotherian ring.
Theorem 2.33. If $T$ be an integral domain and $M$ be $T$-module, then $T$ is Dedekind Distributive ring if for any $m_1, m_2 \in M$ there exists $t_1, t_2, t_3, t_4 \in T$ such that $t_1 + t_2 = 1$ and $m_1 t_1 = m_2 t_3$, $m_2 t_2 = m_1 t_4$, and $T = (m_1: m_2 T) + (m_2; m_1 T)$.

Proof: Let $K_1, K_2, K_3$ be three submodule of $M$, and let $k \in (K_1 \cap (K_2 + K_3))$, to show that $M$ is distributive module, let $k = m_1 + m_2$, $m_1 \in K_2$ and $m_2 \in K_3$. By assumption we have $t_1 + t_2 = 1$, $m_1 t_1 \in m_2 T$, and $m_2 t_2 \in m_1 T$, then $kt_2 = (m_1 + m_2)t_2 = m_1 t_2 + m_2 t_2 \in kT \cap m_1 T \subset K_1 \cap K_2$, and $kt_1 = m_1 t_1 + m_2 t_1 \in kT \cap m_2 T \subset K_1 \cap K_3$, then $kt_1 + kt_2 = k(t_1 + t_2) = k \in (K_1 \cap K_2) + (K_1 \cap K_3)$. Thus $M$ is distributive $T$-module, therefore $T$ is distributive, hence it is Dedekind Distributive ring.

Remarks 2.34.

1) If $T$ is a right quasi-invariant ring and $M$ is a right Bezout $T$-module. Then $T$ is Distributive ring.
2) If $T$ is a Bezout ring and $T$ is right quasi-invariant, then $T$ is left Distributive ring, and if $T$ is left quasi–invariant then $T$ is right distributive ring.
3) If $T$ is right Bezout ring and $T / J(T)$ is finite direct product of division ring, then $T$ is right Distributive semilocal ring.
4) Every right Bezout domain is Dedekind domain. (since Bezout domain is P.I.D.
5) If $T$ is divisible right duo ring, then it is right strongly duo.

Proposition 2.35. Let $T$ is a right duo right Neotherian ring and $T_T$ is divisible. Then $T$ is Dedekind Distributive ring.

Proof: Since $T$ is right duo and $T_T$ is divisible from Remark 3.34 (5); $T$ is a right strongly duo. And by (**) $T$ is a right Artinian, hence $T$ is Dedekind Distributive ring.

Theorem 2.36. If $T$ is a ring and $M$ be multiplication $T$-module with any $a \in M$, ann($a$) = 0. Then $T$ is a Distributive ring if $S = \text{End}(M)$ is strongly regular ring.

Proof: Let $M$ be a $T$-module and $S = \text{End}(M)$ is strongly regular ring. Since $M$ is multiplication than it is $f$-generated and every $K$ submodule of $M$ is $f$-generated, and we have ann($a$) = 0 for all $a \in M$ then $M$ is regular $T$-module. Since $S = \text{End}(M)$ is strongly regular ring then the lattice of $f$-generated $K$ of $M$ is distributive, by (**) we get; $M$ is distributive and hence $T$ is a Distributive ring.

3.Conclusions

The main results in this paper showed the relationship Distributive ring and other domains such as the Dedekind domain and the Noetherian domain as a theory (2.6,2.7). Finally we introduced a good result about divisible module, Distributive ring such that we satisfy if T is invariant ring have multiplication ideal this imply T is a Distributive ring and we study strongly duo ring and related to Distributive ring.

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