Trouton-Noble paradox revisited

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An apparent paradox is obtained in all previous treatments of the Trouton-Noble experiment; there is a three-dimensional (3D) torque $T$ in an inertial frame $S$ in which a thin parallel-plate capacitor is moving, but there is no 3D torque $T'$ in $S'$, the rest frame of the capacitor. Different explanations are offered for the existence of another 3D torque, which is equal in magnitude but of opposite direction giving that the total 3D torque is zero. In this paper instead of using 3D quantities, e.g., $E$, $B$, $F$, $L$, $T$, .. and their “apparent” transformations we deal with 4D geometric quantities $E$, $B$, $K$, $M$, $N$, ... their Lorentz transformations and equations with them. It is considered in our approach that 4D geometric quantities and not the usual 3D quantities are well-defined both theoretically and experimentally in the 4D spacetime. In analogy with the decomposition of the electromagnetic field $F$ (bivector) into two 1-vectors $E$ and $B$ we introduce decompositions of the torque $N$ and the angular momentum $M$ (bivectors) into 1-vectors $N_s$, $N_t$ and $M_s$, $M_t$ respectively. The torques $N_s$, $N_t$ (the angular momentums $M_s$, $M_t$), taken together, contain the same physical information as the bivector $N$ (the bivector $M$). It is shown that in the frame of “fiducial” observers, in which the observers who measure $N_s$ and $N_t$ are at rest, and in the standard basis, only the spatial components $N_i^s$ and $N_i^t$ remain, which can be associated with components of two 3D torques $T$ and $T_t$. In such treatment with 4D geometric quantities the mentioned paradox does not appear. The presented explanation is in a complete agreement with the principle of relativity and with the Trouton-Noble experiment without the introduction of any additional torque.

1. Introduction

In the experiment [1], see also [2], they looked for the turning motion of a charged parallel plate capacitor suspended at rest in the frame of the earth in order to measure the earth’s motion through the ether. In all previous treatments it is found that in the rest frame of a thin parallel-plate capacitor, the $S'$ frame, there is no three-dimensional (3D) torque $T'$; $T'$ is zero in $S'$ since there is only an electric force between plates. (The 3D vectors will be designated in bold-face.) In the $S$ frame the capacitor moves with uniform velocity $u$ in the positive direction of the $x^1$ - axis. The charges on plates are now in uniform motion producing both an electric field $E$ and a magnetic field $B$. The existence of the magnetic field $B$ in $S$ is responsible for the existence of the 3D magnetic force and this force provides a 3D torque on the charged capacitor. In that way
an apparent paradox, the Trouton-Noble paradox, is obtained and the principle of relativity is violated; there is a 3D torque and so a time rate of change of 3D angular momentum in one inertial frame, but no 3D angular momentum and no 3D torque in another.

Different explanations have been offered for the existence of another 3D torque which is equal in magnitude but of opposite direction giving that the total 3D torque is zero in order to have the agreement with the principle of relativity and experiments. All these previous explanations for the null result of the experiments, see, e.g., [3-7] and references therein, mainly deal with 3D quantities, e.g., $E$, $B$, $F$, $L$, $T$, their “apparent” transformations and equations with them.

However a new explanation of the Trouton-Noble experiment, which is developed in [8], deals from the outset with 4D geometric quantities, their Lorentz transformations (LT) and equations with them. The main point in such geometric approach as in [8] is that the physical meaning, both theoretically and experimentally, is attributed to 4D geometric quantities, and not, as usual, to 3D quantities. The consideration from [8] reveals that there is no need either for the nonelectromagnetic forces and their additional torque, [3-6], or for the angular electromagnetic field momentum and its rate of change, i.e., its additional torque, [7]. Particularly the “resolution” of the Trouton-Noble paradox from [7] is examined in more detail in [8]. There it is shown that, when $r^0 = 0$, where $r^0$ is the temporal component of the lever arm joining the axis of rotation with the point of application of the resultant 3D force, then the 4D torque $N$ (bivector) is zero not only in $S'$, the rest frame of a thin parallel-plate capacitor, but in all other relatively moving inertial frames.

In this paper, sections 3-3.2, the resolution of the Trouton-Noble paradox with the 4D torque $N$ will be given in the case when the temporal component of the lever arm is zero, $r^0 = 0$, in the $S$ frame in which the capacitor is moving. The case $r^0 = 0$ is not investigated in [8], but, in some sense, it corresponds to the case investigated in [7]. Here it will be shown that the torque $N$ as a 4D geometric quantity is the same quantity in both frames $S'$ and $S$, in contrast to the usual approaches, e.g., [7], in which the torque $T'$ in $S'$ is zero, $T' = 0$, while it is different from zero in the $S$ frame, $T \neq 0$.

Furthermore, section 3.1, we shall introduce the decomposition of the 4D torque $N$ into two 1-vectors, the “space-space” torque $N_t$ and the “time-space” torque $N_s$, which is first presented in [9]. In the frame of “fiducial” observers, in which the observers who measure $N_s$ and $N_t$ are at rest, and in the standard basis, only the spatial components $N_s^i$ and $N_t^i$ remain, which can be associated with components of two 3D torques $T_s$ and $T_t$. Note that in all usual approaches, including [7], only the 3D torque $T = r \times F$ is mentioned and considered as the physical one. The 4D torques $N_s$ and $N_t$ will be calculated for both cases, $r^0 = 0$ and $r^0 = 0$. It will be shown in section 3.3 that in the approach with the 4D torques $N_s$ and $N_t$, in the same way as in the approach with the 4D torque $N$ from [8], the principle of relativity is naturally satisfied and there is no paradox.

Some objections to the calculation of $T$ in [7] are raised in section 2, while
the detailed comparison of our calculation of \(N, N_s\) and \(N_t\) and the usual calculation of \(T\) is given in section 4. In section 5 the conclusions are presented.

2. The choice of the basis in the 4D spacetime and some remarks on the calculation of the 3D torque \(T\) from [7]

The whole investigation will be done in the geometric algebra formalism, see, e.g., [10,11]. Physical quantities will be represented by 4D geometric quantities, multivectors, that are defined without reference frames, i.e., as absolute quantities (AQs) or, when some basis has been introduced, these AQs are represented as 4D coordinate-based geometric quantities (CBGQs) comprising both components and a basis. For simplicity and for easier understanding, only the standard basis \(\{\gamma_\mu; 0, 1, 2, 3\}\) of orthonormal 1-vectors, with timelike vector \(\gamma_0\) in the forward light cone, will be used, but remembering that the approach with 4D geometric quantities holds for any choice of basis.

It is worth noting that the standard basis \(\{\gamma_\mu\}\) corresponds, in fact, to the Einstein’s system of coordinates. In Einstein’s system of coordinates the standard, i.e., Einstein’s synchronization [12] of distant clocks and Cartesian space coordinates \(x^i\) are used in the chosen inertial frame. However different systems of coordinates of an inertial frame are allowed and they are all equivalent in the description of physical phenomena. For example, in [13,14] and in the second and the third paper in [15], two very different, but physically completely equivalent systems of coordinates, Einstein’s system of coordinates and the system of coordinates with a nonstandard synchronization, the everyday (radio) (“\(r^\circ\)”) synchronization, are exposed and exploited throughout the papers.

We shall again, as in [8], examine the situation considered in [7]. Regarding the figures in [7] it has to be remarked that both figures would need to contain the time axes as well. Figure 1 (figure 2) is the projection onto the hypersurface \(t' = 0\) (\(t = 0\)); the distances are simultaneously determined in the \(S'\) (\(S\)) frame. This means that figure 1 corresponds to the choice \(r'^0 = 0\) and figure 2 to the choice \(r^0 = 0\). The LT cannot transform the hypersurface \(t' = 0\) into the hypersurface \(t = 0\) and the distances that are simultaneously determined in the \(S'\) frame cannot be transformed by the LT into the distances simultaneously determined in the \(S\) frame. Such figures could be possible for the Galilean transformations, but for the LT they are meaningless.

The transformations for components of the 3D torque \(T\), equations (1)-(3) in [7], are found, e.g., in Jefimenko’s book [16], equations (8-6.11)-(8-6.13). They are

\[
T_1 = T'_1/\gamma, \quad T_2 = T'_2 + \beta^2 r'_1 F'_3, \quad T_3 = T'_3 - \beta^2 r'_1 F'_2, \quad (1)
\]

where \(\beta = |u|/c, \gamma = (1 - |u|^2/c^2)^{-1/2}\). In [16] these transformations are derived starting with the definition of the 3D torque \(T = r \times F\) and using the LT, but only for the spatial components \(r_i\), i.e., \(r_x, r_y, r_z\), and the transformations for
components of the 3D force $\mathbf{F}$, equations (8.5.1)-(8.5.3) in [16]. Observe that $t = 0$ is chosen to be the time of observation in $S$. Actually what is assumed in that derivation from [16] is not $t = 0$, but that $r^0 = 0$. Then the transformation for the time component is $t^0 = \gamma (r'^0 + \beta r'^1) = 0$, which yields $r^0 = -\beta r'^1$. That relation is used in the derivation of equations (8-6.11)-(8-6.13) in [16], i.e., [11]. Physically $r^0 = 0$ means that the lever arm is simultaneously determined in the $S$ frame in which the capacitor is moving. In the Trouton-Noble experiment the $S$ frame refers to the preferred frame, while the $S'$ frame refers to the Earth. Hence the appropriate choice for the comparison with experiment would be that $r^0 = 0$, i.e., that the lever arm is simultaneously determined in its rest frame, the $S'$ frame. It is worth noting that in [7] Jefimenko deals only with the 3D quantities thus implicitly assuming that the lengths, volumes and angles are well defined in both relatively moving inertial frames, see figures 1 and 2 and, e.g., equation (10) in [7]. Since the relations [11] are obtained for the case $r^0 = 0$, then the mentioned 3D quantities are well-defined only in the $S$ frame, but not in the $S'$ frame. In the 4D spacetime it is not possible to have that, e.g., the lengths, are well defined in both relatively moving inertial frames.

In order to avoid such ambiguities we have calculated in [8] the 4D torque $N$ and showed that there is no Trouton-Noble paradox for the case $r'^0 = 0$, when the 4D geometric quantities are used. From the experimental point of view the other case, $r^0 = 0$, is difficult to realize in measurements. Regardless of that in this paper we shall investigate the case $r^0 = 0$ as well in order to see the fundamental difference between the usual approaches, e.g., [7] and [16], and our approach in which the physical reality is attributed to the 4D geometric quantities.

3. The resolution of the Trouton-Noble paradox representing the torque by the bivector $N$ and by the 1-vectors $N_s$ and $N_t$

The discussion and the results presented in [8] and [9] strongly suggest that the relativistically correct resolution of the Trouton-Noble paradox can be achieved in an unambiguous way by the use of the 4D torque $N$, or $N_s$ and $N_t$, and not with the usual 3D torque $\mathbf{T}$. This will be realized in sections 3.1-3.3.

3.1. The torques $N$, $N_s$ and $N_t$

As shown in [8] the torque $N$, as a 4D AQ, is defined as a bivector $N = r \wedge K$, where $r = x_P - x_O$. $r$ is 1-vector associated with the lever arm, $x_P$ and $x_O$ are the position 1-vectors associated with the spatial point of the axis of rotation and the spatial point of application of the force $K$. $P$ and $O$ are the events whose position 1-vectors are $x_P$ and $x_O$. We shall need to determine $N$ for the Lorentz force $K_L$ which is $K_L = (q/c) \mathbf{F} \cdot u$, where $u$ is the velocity 1-vector and $\mathbf{F}$ is the electromagnetic field $\mathbf{F}(x)$ (bivector). This expression for $K_L$ with $\mathbf{F}$ is used in [8], but in this paper the decomposition of $\mathbf{F}$ into electric and magnetic fields will be employed.
It is proved in the tensor formalism that given an antisymmetric tensor \( F^{ab} \) and a unit time-like four-vector \( n^a \) one can construct two four-vectors, \( E^a = F^{ab}n_b \), and \( B^a = (1/2)\varepsilon^{abcd}F_{cb}n_d \) and, oppositely, that \( F^{ab} \) can be expressed in terms of these two four-vectors \( E^a \), \( B^a \) and \( n^a \) as \( F^{ab} = E^a n_b - E^b n_a + \varepsilon^{abcd}B_{cb}n_d \), see, e.g., [17], section 6, Example 6.1 and [18], or in the covariant form [19], equation (7.58). This decomposition of an antisymmetric tensor is used for the decomposition of the tensor of the electromagnetic field \( F^{ab} \) in [13] [20] [14] and [21]. The same decomposition but in the Clifford algebra formalism is introduced and employed in [22-24] and [9]. The electromagnetic field \( F \) (bivector) is decomposed into 1-vectors of the electric field \( E \) and the magnetic field \( B \) and a unit time-like 1-vector \( v/c \) as

\[
F = (1/c)E \wedge v + (IB) \cdot v, \\
E = (1/c)F \cdot v, \\
B = -(1/c^2)I(F \wedge v),
\]

where \( I \) is the unit pseudoscalar and \( v \) is the velocity (1-vector) of a family of observers who measures \( E \) and \( B \) fields. It also holds that \( E \cdot v = B \cdot v = 0 \), which yields that only three components of \( E \) and three components of \( B \) are independent quantities. Observe that \( E \) and \( B \) depend not only on \( F \) but on \( v \) as well.

Using the decomposition of \( F \) into \( E \) and \( B \), the Lorentz force \( K_L \) can be written as \( K_L = (q/c)F \cdot u = (q/c)[(1/c)E \wedge v + (IB) \cdot v] \cdot u \), where \( u \) is the velocity (1-vector) of a charge \( q \), [22-24]. Particularly, from the definition of the Lorentz force \( K_L = (q/c)F \cdot u \) and the relation \( E = (1/c)F \cdot v \) it follows that the Lorentz force ascribed by an observer comoving with a charge, \( u = v \), is purely electric \( K_L = qE \).

The 4D torque \( N = r \wedge K_L \) can be represented as a 4D CBGQ. In \( S' \), the rest frame of the capacitor, it becomes \( N = (1/2)N^\mu \nu s'_\mu r'_\nu \wedge \gamma'_\rho, \) \( N^\mu \nu = r''\nu K''_L^\mu - r''\nu K''_L^\mu \), where the components \( N^\mu \nu \) are determined as \( N^\mu \nu = \gamma''^\rho \cdot (\gamma''^\mu \cdot N) \).

In [8] only the 4D torque \( N \) is considered. Here we shall introduce new 4D torques \( N_s \) and \( N_t \). In fact, they are first introduced in [9]. There the same decomposition as for \( F \), [24], is made for the 4D torque \( N \); it is decomposed into two 1-vectors, the “space-space” torque \( N_s \) and the “time-space” torque \( N_t \), and the unit time-like 1-vector \( v/c \) as

\[
N = (v/c) \cdot (IN_s) + (v/c) \wedge N_t \\
N_s = I(N \wedge v/c), \\
N_t = (v/c) \cdot N,
\]

with the condition

\[
N_s \cdot v = N_t \cdot v = 0;
\]

only three components of \( N_s \) and three components of \( N_t \) are independent since \( N \) is antisymmetric. Here again \( v \) is the velocity (1-vector) of a family of observers who measures \( N_s \) and \( N_t \). Similarly as for \( E \) and \( B \) the 4D torques \( N_s \) and \( N_t \) depend not only on the bivector \( N \) but on \( v \) as well. The relations [3] show that \( N_s \) and \( N_t \) taken together contain the same physical information as the bivector \( N \).
When $N_s$ and $N_t$ are written as CBGQs in the $\{\gamma'_\mu\}$ basis they are

$$N_s = N^\mu_s \gamma'_\mu = (1/2c)\varepsilon^{\alpha\beta\mu\nu}N^\alpha_{\mu} \gamma'_\nu, \quad N_t^i = (1/c)\gamma_T^i \gamma'_\nu. \quad (5)$$

Let us take that the $S'$ frame is the frame of “fiducial” observers, or the $\gamma'_0$ - frame, in which the observers who measure 1-vectors $E$, $B$, $K_L$, $N_s$ and $N_t$ are at rest. Then in $S'$ the velocity $v$ is $v = c\gamma'_0$. In the $\gamma'_0$ - frame and the $\{\gamma'_\mu\}$ basis $v$ has the components $v^\mu = (c, 0, 0, 0)$. Hence in the frame of “fiducial” observers (5) becomes

$$N^0_s = 0, N^i_s = (1/2)\varepsilon^{0jki}N^j_{jk}, \quad N^0_t = 0, \quad N^i_t = 0. \quad (6)$$

It is seen from (6) that $N^0_s = N^0_t = 0$ and only the spatial components remain. $N^i_s$ components are $N^i_s = \gamma^{123} = \gamma^0 K^1_L - \gamma^0 K^2_L, \gamma^0 = N^0_1$ and $N^i_s = N^{12}$. The 1-vector $N_s$ corresponds to the 3D torque $\mathbf{T}$ that is considered in [7] and [16]. On the other hand in [7] and [16], and in all other approaches that deal with the 3D quantities, there is no 3D torque which would correspond to our 1-vector $N_t$. Nevertheless we shall introduce already here another 3D torque $\mathbf{T}_t$, which corresponds to our 4D torque $N_t$. The precise meaning of the mentioned correspondence and of the 3D torque $\mathbf{T}_t$ will be better explained later.

The whole discussion with the torque can be completely repeated for the angular momentum replacing $N$, $N_s$ and $N_t$ by $M$, $M_s$ and $M_t$, see [9]. The angular momentum $M$ as a 4D AQ (bivector) and manifestly Lorentz invariant equation connecting $M$ and $N$ are defined as $M = r \wedge p, \quad N = dM/d\tau$, where $p$ is the proper momentum (1-vector) and $\tau$ is the proper time. The 1-vectors $M_s$ and $M_t$ correspond to $\mathbf{L}$ and $\mathbf{L}_t$ respectively in the usual 3D picture. $\mathbf{L}$ and $\mathbf{L}_t$ are introduced in [25]. The components $L_i$ of the 3D vector $\mathbf{L}$ (which is called the angular momentum) are identified with the “space-space” components of the covariant angular momentum tensor $M^{\mu\nu}$, $M^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu$, and the components $L_{t,i}$ of the 3D vector $\mathbf{L}_t$ (for which a physical interpretation is not given in [25]) are identified with the three “time-space” components of $M^{\mu\nu}$. (We denote Jackson’s $K_L$ with $L_{t,i}$, $\mathbf{K}$ with $\mathbf{L}_t$.)

In the usual picture with 3D quantities one can connect $\mathbf{T}$ and $\mathbf{T}_t$ with the above mentioned $\mathbf{L}$ and $\mathbf{L}_t$ by the relations $\mathbf{T} = d\mathbf{L}/dt$ and $\mathbf{T}_t = d\mathbf{L}_t/dt$.

As shown in [8] all “space-space” components $N^{ij}$ and the “time-space” components $N^{0i}$ are zero, $N^{0\mu\nu} = 0$, in $S'$ and in the case when $r^{00} = 0$. Accordingly, in $S'$, the whole bivector $N$ is zero, $N = 0$, when $r^{00} = 0$. As already mentioned the case $r^{00} = 0$ is the appropriate choice for the comparison with experiment since the lever arm is simultaneously determined in $S'$ in which the capacitor is at rest. The essential difference between our geometric approach and the usual covariant picture is the presence of the basis. The existence of the basis causes that every 4D CBGQ is invariant under the passive LT; the components transform by the LT and the basis by the inverse LT leaving the whole 4D CBGQ unchanged. This means that a CBGQ represents the same physical quantity for relatively moving 4D observers. For the torque $N = \ldots$
\[(1/2)N^\mu\nu\gamma'_\mu \wedge \gamma'_\nu,\] this yields that the components transform by the LT as
\[
N^{23} = N'^{23}, \quad N^{31} = \gamma(N'^{31} - \beta N'^{303}), \quad N^{12} = \gamma(N'^{12} + \beta N'^{02}), \quad N^{10} = N'^{10}, \quad N^{02} = \gamma(N'^{02} + \beta N'^{12}), \quad N^{03} = \gamma(N'^{03} + \beta N'^{13}), \quad (7)
\]
whereas the basis \(\gamma'_\mu \wedge \gamma'_\nu,\) transform by the inverse LT giving that the whole 4D torque \(N\) is unchanged
\[
N = (1/2)N^\mu\nu\gamma'_\mu \wedge \gamma'_\nu = (1/2)N^\mu\nu\gamma_\mu \wedge \gamma_\nu. \quad (8)
\]
From (8) it is concluded in [8] that the whole 4D torque \(N\) is zero not only in the rest frame of the capacitor but in all other relatively moving inertial frames of reference. Thus it is proved in [8] that for \(r^{00} = 0\) the principle of relativity is naturally satisfied and there is no Trouton-Noble paradox.

It is immediately seen from the definitions of \(N_s\) and \(N_t, (3),\) that they are also zero in all relatively moving inertial frames of reference when \(r^{00} = 0,\)
\[
r^{00} = 0 \Rightarrow N_s = N_t = 0. \quad (9)
\]
Again we conclude that there is no Trouton-Noble paradox.

3.2. The 4D torque \(N\) when \(r^{00} = 0\)

Let us now consider the case when \(r^{00} \neq 0,\) but \(r^{00} = 0\) \((r^{00} = -\beta r^{11}).\) This case is not investigated in [8]. As already mentioned the case \(r^{00} = 0\) corresponds to that one explored in [7] section 2 and [16]. In the \(S'\) frame, which is again chosen to be the frame of “fiducial” observers, the velocity \(u\) of the capacitor is \(u = c\gamma'_0,\) or \(u = u^\mu\gamma'_\mu,\) where \(u^\mu = (c, 0, 0, 0).\) The components \(r^{\mu\nu}\) are \(r^{\mu\nu} = (r^{00}, r^{11}, r^{12}, 0)\)
where \(r^{11} = r'_x = -a' \sin \Theta', r^{12} = r'_y = a' \cos \Theta', r^{22} = r'_z = 0,\) see figure 1 in [7]. Hence in \(S'\) \(v = u = c\gamma'_0.\) As already mentioned in that case the Lorentz force is purely electric, \(K_L = qE,\) where \(q\) is the total charge residing on the positive plate. Then \(K'_L = (0, qE^{13}, qE^{23}, 0)\) or \(K'_L = (0, F'_x, F'_y, 0),\) where \(F'_x\) and \(F'_y\) are the components of the 3D force \(F\) that are the same as in [7], but written with primed quantities; \(F'_x = C r'_x, F'_y = C r'_y, F'_z = 0\) and \(C = -\sigma^2 A'/2\epsilon_0 a'.\)
This yields that the components \(N^{\mu\nu}\) are
\[
N'^{12} = N^{013} = N^{023} = 0, \quad N'^{01} = r^{00}K'^{11} - r^{11}K'^{00} = r^{00}F'_x, \quad N'^{02} = r^{00}F'_y, \quad N'^{03} = 0. \quad (10)
\]
where the result that \(N'^{12} = r^{11}K'^{11} - r^{22}K'^{11} = r'_x F'_y - r'_y F'_x = 0\) is obtained inserting the explicit expressions for \(r'_x, r'_y,\) and \(F'_x, F'_y\) into \(N'^{12}.\) Hence in \(S'\) the whole torque \(N\) is given as
\[
r^{00} = 0, \quad N = (1/2)N^\mu\nu\gamma'_\mu \wedge \gamma'_\nu = N^{01} \gamma'_0 \wedge \gamma'_1 + N^{02} \gamma'_0 \wedge \gamma'_2, \quad N^{01} = -\beta r'_x F'_x, \quad N^{02} = -\beta r'_x F'_y. \quad (11)
\]
In the $S$ frame $N = (1/2)N^{\mu\nu} \gamma_\mu \wedge \gamma_\nu$ and the components $N^{\mu\nu}$ can be determined writing $N^{\mu\nu} = r^\mu K^\nu_L - r^\nu K^\mu_L$ and using the LT for $r^\mu$ and $K^\mu_L$, or using directly the LT for $N^{\mu\nu}$, which are given by (7). Taking into account that in $S'$ all $N^{ij} = 0$ we find that the “space-space” components $N^{ij}$ are $N^{12} = \gamma \beta N^{02}$ and $N^{13} = N^{23} = 0$. From the above relations we see that the “time-space” components $N^{0i}$ are the following, $N^{01} \neq 0$, $N^{02} \neq 0$ and $N^{03} = 0$, which yields that the whole $N$ in $S$

$$N^0 = 0, \quad N = (1/2)N^{\mu\nu} \gamma_\mu \wedge \gamma_\nu = N^{01} \gamma_0 \wedge \gamma_1 + N^{02} \gamma_0 \wedge \gamma_2 + N^{12} \gamma_1 \wedge \gamma_2, \quad N^{01} = -\gamma r^r F'_r, \quad N^{02} = -\gamma \beta r^r F'_r, \quad N^{12} = -\gamma \beta^2 r^r F'_r. \quad (12)$$

Consequently, when $r^0 = 0$, we find that in the $S$ frame, in which the capacitor is moving, the components $N^{\mu\nu}$ that are different from zero are not only the “time-space” components $N^{01}$ and $N^{02}$, but also the “space-space” component $N^{12}$.

Of course, due to the invariance of any CBGQ under the passive LT, $N$ from (12) is equal to $N$ from (11).

$$r^0 = 0, \quad N^{12} = N^{12}. \quad (13)$$

This means that the principle of relativity is again naturally satisfied and there is no Trouton-Noble paradox for the whole 4D torque $N$.

It is interesting to explore the connection between our result for the whole 4D torque $N$ (12) and the usual result for the 3D torque $T$ from [7]. From [7] we know that $N^{12} = \gamma N^{12} + \gamma \beta N^{02} = \gamma (r^1 K^2_L - r^2 K^1_L) + \beta (r^0 K^2_L - r^2 K^0_L)$. Since $S'$ is the frame of “fiducial” observers and at the same time it is the rest frame of the capacitor, $v = u = c\gamma_0$, one can write $N^{12}$ in terms of the components of the 3D force $F$, which yields $N^{12} = \gamma (r^r F'_r - r^r F'_r) + \gamma \beta (-\beta r^r F'_r)$ Then all “space-space” components $N^{ij}$ can be written in terms of the components of the 3D torque $T$ as

$$N^{12} = \gamma (T^r_z - \beta^2 r^r F'_r), \quad N^{31} = \gamma (T^r_y + \beta^2 r^r F'_y), \quad N^{23} = T^r_z, \quad (14)$$

When it is taken that all $N^{ij} = 0$ and that $F'_r = 0$ then $N^{23} = N^{31} = 0$ and only $N^{12}$ remains, $N^{12} = \gamma (-\beta^2 r^r F'_r)$, as in (12). This component $N^{12}$ corresponds to the component $T_z$, equation (4) in [7]. However, it is worth noting that in all usual approaches, including [7], only three components of the 3D torque $T$ are considered to be physical. In our approach physically important, measurable, quantity is the whole 4D torque $N$. When that $N$ is written as a CBGQ then it contains sixteen components $N^{\mu\nu}$ (six of them are independent) together with the bivector basis $\gamma_\mu \wedge \gamma_\nu$ and equation (13) holds.

3.3. The 4D torques $N_s$ and $N_t$ when $r^0 = 0$

Let us now calculate the torques $N_s$ and $N_t$ in the case when $r^0 = 0$. Again we assume that $S'$ is the frame of “fiducial” observers, i.e., $v = u = c\gamma_0$. Then,
as seen from (11), all “space-space” components \( N^{rj} \) are zero. Using (5), or (13), one finds that in our case all \( N^0_\mu = 0 \) in \( S' \). According to that the whole \( N_s \) is zero, \( N_s = N^\mu_\mu' = 0, \) when \( r^0 = 0. \) From (14) it follows that \( N^0_t = 0, \) \( N^r_i = N^{r0}_i, \) where \( N^{r0}_i = r^0 K^r_i \). Hence we have

\[
\begin{align*}
  r^0 &= 0, \quad N_s = N^\mu_\mu' = 0, \quad N_t = N^\mu_\mu', \\
  N^0_t &= 0, \quad N^r_i = -\beta r^j i F^j_s, \quad N^{r2}_i = -\beta r^j i F^j_s, \quad N^{r3}_i = 0. 
\end{align*}
\tag{15}
\]

Next we determine \( N_s \) and \( N_t \) in the \( S \) frame. Note that, e.g., \( N^\mu_\mu' \) transforms under the LT as every 1-vector transforms, which means that components \( N^\mu_\mu' \) (of the “space-space” torque \( N_s \)) transform to the components \( N^\mu_\mu' \) (of the same torque \( N_s \) in the \( S \) frame); there is no mixing with the components of the “time-space” torque \( N_t \)

\[
N^0_s = \gamma (N^0_s + \beta N^{12}_s), \quad N^1_s = \gamma (N^{r1}_s + \beta N^0_s), \quad N^{2,3}_s = N^{2,3}_s. \tag{16}
\]

The unit 1-vectors \( \gamma'_\mu \) transform by the inverse LT. Only with such transformations, e.g., the CBGQs \( N^\mu_\mu' \) and \( N^\mu_\mu' \) are the same quantity \( N_s \) in \( S' \) and \( S \) frames, \( N_s = N^\mu_\mu' = N^\mu_\mu' \). The same is fulfilled for \( N^\mu_\mu' \). Thus it holds that in the \( S \) frame too \( N^0_s = 0 \), i.e., the torque \( N_s \) is zero in all relatively moving inertial frames of reference, \( N_s = 0 \), when \( r^0 = 0. \) In order to find the explicit expression for the torque \( N_t \) as a CBGQ in the \( S \) frame we can simply perform the LT of the 1-vector \( N_t = N^\mu_\mu' \). Then both torques \( N_s \) and \( N_t \) as CBGQs in the \( S \) frame can be written as

\[
\begin{align*}
  r^0 &= 0, \quad N_s = N^\mu_\mu' = 0, \quad N_t = N^\mu_\mu', \\
  N^0_t &= \gamma N^{r1}_t, \quad N^1_t = \gamma N^{r1}_t, \quad N^2_t = N^{r2}_t, \quad N^3_t = N^{r3}_t = 0, \tag{17}
\end{align*}
\]

where the components \( N^r_i \) are given by (15). Obviously it again holds that \( N_s \) and \( N_t \) from (15) are equal to \( N_s \) and \( N_t \) from (17)

\[
\begin{align*}
  r^0 &= 0, \quad N_{s,t} = N^\mu_\mu', \quad N^0_{s,t} = N^\mu_\mu', \tag{18}
\end{align*}
\]

It is important to remark that the decomposition of \( N \) into \( N_s \) and \( N_t \), equations (13) and (14), will be fulfilled for both representations (15) and (17). All this shows that the principle of relativity is again naturally satisfied and there is no Trouton-Noble paradox when the theory is formulated with 4D torques \( N_s \) and \( N_t \).

4. Comparison with the 3D torques \( T \) and \( T_t \)

The relations (15) indicate that in the frame of “fiducial” observers, here the \( S' \) frame, and in the \( \{ \gamma'_\mu \} \) basis one can make the identification

\[
\begin{align*}
  v^\mu &= (c, 0, 0, 0)^t : T'_{s,t} = N^r_i = (1/2)c^0jki N^r_j k v^0_0 = (1/2)c^0jki N^r_j k, \\
  T'_{s,t} &= N^{r0}_s = (1/c) N^{r0}_s v^0_0 = N^{r0}_s. \tag{19}
\end{align*}
\]
Observe that such identification is fulfilled only for the components. The 3D vector \( T' \) is \( T' = T'_i \hat{v} + T'_j \hat{j} + T'_k \hat{k} \), as a geometric quantity in the 3D space, is constructed multiplying the unit 3D vectors \( \hat{v}, \hat{j}, \hat{k} \). In contrast to it the 4D vector \( N_s \) in the \( \{ \gamma'_\mu \} \) basis is a geometric quantity in the 4D spacetime. It consists from the components \( N'_s' \) and the basis \( \{ \gamma'_\mu \} \). In this case, when \( S' \) is the frame of “fiducial” observers \( N_s \) is given as \( N_s = 0\gamma'_0 + N'_s\gamma'_i \). The same holds for \( T'_i \) and \( N_t \) in the \( \{ \gamma'_\mu \} \) basis. The bases in the 3D space and the 4D spacetime are completely different.

In the usual approaches in which components of the “physical” quantities \( T, L, B \) and \( E \) are derived from components of the corresponding covariant quantities \( N'^\mu, M'^\mu \) and \( F'^\mu \), see [25] and for \( F'^\mu \) [26] section 11.9, the components \( T'_i \) and \( T'_{t,i} \) are not defined by (19) than they are identified with the “space-space” and “time-space” components respectively of the torque four-tensor \( N'^\mu \). Thus, in the \( S' \) frame, they are determined as

\[
T'_i = (1/2)\varepsilon_{i\alpha\beta\gamma} N'^{\alpha\beta\gamma}, \quad T'_{t,i} = N'^{0\alpha}_{\alpha i},
\tag{20}
\]

This is completely analogous to the way in which the components of 3D vectors \( B' \) and \( E' \) are identified with the “space-space” and the “time-space” components respectively of the covariant expression for the electromagnetic field \( F'^\mu \); \( B'_i = (1/2c)\varepsilon_{i\alpha\beta\gamma} F'^{\alpha\beta\gamma}, \quad E'_i = F'^{0\alpha}_{\alpha i} \) [26] section 11.9. Obviously the components \( T'_i \) correspond to \( -B'_i \) and \( T'_{t,i} \) to \( -E'_i \). In these equations we use the notation in which components of \( T', B' \) and \( E' \) are written with lowered (generic) subscripts, since they are not the spatial components of 4D quantities. This refers to the third-rank antisymmetric \( \varepsilon \) tensor too. The super- and subscripts are used only on components of 4D quantities. Geometric quantities in the 3D space, \( B' \) and \( E' \) are also formed by the multiplication of the components \( B'_i \) and \( E'_i \) with the unit 3D vectors \( \hat{v}, \hat{j}, \hat{k} \). However it is important to note once again that the covariant quantities \( N'^\mu \) and \( F'^\mu \) (and \( M'^\mu \)) are only components (numbers) that are (implicitly) determined in Einstein’s system of coordinates. Components are frame-dependent numbers (frame-dependent because the basis refers to a specific frame). Components tell only part of the story, while the basis contains the rest of the information about the considered physical quantity. These facts are completely overlooked in all usual covariant approaches and in the above identifications (20) and those for \( B_i, E_i, L_i \) and \( L_{t,i} \).

After this digression let us go back to the identification (19). It is visible from (19) and (15) that in the case when \( \nu^0 = 0 \) all components \( T'_i = 0 \) and the components \( T'_{t,i} \) are the same as \( N'^{\alpha}_{\alpha i} \) from (15). Such result for \( T'_i \) agrees with that one from [7], but in [7] there are no components \( T'_{t,i} \), i.e., the 3D vector \( T'_i \).

Now comes the fundamental difference between the approaches with 3D quantities and our approach with 4D geometric quantities. \( N_s \) and \( N_t \) as CBGQs in the \( S' \) frame are determined by equation (5). When \( S' \) is the frame of “fiducial” observers then (5) becomes (6) and in that case the identification (19) is possible. In order to find \( N_s \) and \( N_t \) as CBGQs in the \( S' \) frame, in which the capacitor is moving, we have to transform by the LT all quantities, which determine \( N_s \) and \( N_t \) in (5) or (6), i.e., \( N'^{\mu\nu}, v'_\mu \) and \( \gamma'_\mu \), from \( S' \) to \( S \). This
procedure yields the LT (16) for the components $N^\mu_i$ and similarly for $N^\mu_j$. It is important to note that “fiducial” observers are moving in $S$. Therefore the components $v^\mu$ of their velocity in $S$, which are obtained by the LT from $v^\mu = (c, 0, 0, 0)$, are $v^\mu = (\gamma c, \gamma \beta c, 0, 0)$. Of course, for the whole CBGQ $v$ it holds that $v = v^\mu \gamma_\mu = v^\mu \gamma_\mu$. Hence, in order to find the components $N^\mu_k$ and $N^\mu_j$ in $S$ it is not enough to transform only $N^{\alpha \beta}$ but the components $v^\mu$ as well. In $S$, as seen from (16), the 4D torque $N_\alpha$ contains not only the spatial components $N^\alpha_0$ but the temporal component $N^\alpha_0 = \gamma \beta N^\alpha_0$ as well. Also it follows from (16) that the components $N^\mu_i$ transform again to the components $N^\mu_k$ of the same 1-vector $N_\alpha$ in $N^\alpha_0$ from (16) and the unit 1-vectors $\gamma_\mu$ and the same for $N_i$. Hence, in order to find the components $N^\mu_i$ and the 1-vector $\gamma_\mu$ it is obtained that $N^\mu_k = N^\mu_0 \gamma_\mu = N^\mu_0 \gamma_\mu$, and the same for $N_0$.

On the other hand, in order to get the transformations for the components $T_i$, (11), one has to suppose that only the components $N^{\alpha \beta}$ and not $v^\mu$ are transformed from $S'$ to $S$. It can be interpreted as that the components $v^\mu = (c, 0, 0, 0)$ from $S'$ are again transformed to the same $v^\mu = (c, 0, 0, 0)$ in $S$. Hence it is supposed that the same identification as (19), or (20), is valid not only in the $S'$ frame, the frame of “fiducial” observers, but also in the relatively moving $S$ frame. This yields the following transformations

$$
T_1 = T'_1, \quad T_2 = \gamma (T'_2 - \beta T'_{1,3}), \quad T_3 = \gamma (T'_3 + \beta T'_{1,3}),
$$

$$
T_{1,1} = T'_{1,1}, \quad T_{1,2} = \gamma (T'_{1,2} + \beta T'_3), \quad T_{1,3} = \gamma (T'_{1,3} - \beta T'_2). \quad (21)
$$

The transformations for $T_i$ from (21) are completely equivalent to the transformations (11). Note that the transformations (21) are the same as the transformations for the components $N^{\mu \nu}$ (7). This is obvious since the identifications (20) say that the components of the 3D $T'$ and $T'_i$ are the components of the 4D torque $N$ and according to that they transform like the components of $N^{\mu \nu}$.

Furthermore it is visible from (21) that the components $T_i$ in $S$ are expressed by the mixture of components $T'_k$ of the 3D vector $T'$ and the components $T'_{i,k}$ of another 3D vector $T'_i$ from $S'$. This is the reason that the components of the usual 3D torque $T$ will not vanish in the $S$ frame even if they vanish in the $S'$ frame, i.e., that there is the Trouton-Noble paradox in the usual approaches to special relativity.

Let us now see what is with the transformations of bases. In [7] both $T'$ and $T$, as geometric quantities in the 3D space, are constructed multiplying the components $T'_i$ and $T_i$ given by equations (11), by the unit 3D vectors $i', j', k'$ and $i$, $j$, $k$ respectively. The components $T_i$ are determined by the transformations (21), or (11), but there is no transformation which transforms the unit 3D vectors $i'$, $j'$, $k'$ into the unit 3D vectors $i$, $j$, $k$. Hence it is not true that the 3D vector $T = T_i i + T_j j + T_k k$ is obtained by the LT from the 3D vector $T' = T'_i i' + T'_j j' + T'_k k'$. Accordingly $T$ and $T'$ determined from (21), or (11), are not the same quantity for relatively moving inertial observers, $T \neq T'$. Moreover, as already said, the torque $T'_i$ is not mentioned in [7]. Completely different situations is with 4D geometric quantities, the torques $N$, $N_\alpha$ and
$N_t$, which are Lorentz invariant quantities, independent of the chosen reference frame and the system of coordinates in it. All this together shows that although the identification (19) is possible in the frame of the “fiducial” observers, here the $S'$ frame, it is not possible in any other relatively moving frame, here the $S$ frame.

Comparing (21) with the usual transformations for $E_k$ and $B_k$, which are

\begin{align*}
B_1 &= B'_1, \quad B_2 = \gamma(B'_2 - \beta E'_3/c), \quad B_3 = \gamma(B'_3 + \beta E'_2/c), \\
E_1 &= E'_1, \quad E_2 = \gamma(E'_2 + \beta c B'_3), \quad E_3 = \gamma(E'_3 - \beta c B'_2),
\end{align*}

(22)
equation (11.148) in [26], we see that the transformations for $T_i$ and $T_{t,i}$ are the same as (22). In all previous treatments of special relativity the transformations for the components $B_k$ and $E_k$, (22), are considered to be the LT of the 3D electric and magnetic fields. The same opinion exists in connection with the transformations for $T_i$, (21), and the transformations for $L_i$, equation (11) in [25]. However in [23, 24] (Clifford algebra formalism) and [21] (tensor formalism with tensors as 4D geometric quantities) it is proved in different manners that the transformations for the 3D vectors $E$, $B$, (22) and equations (11.149) in [25], are not the LT but the “apparent” transformations (AT) (for the name see [27]), which do not refer to the same 4D quantity. (Previously I often called these usual transformations for $E$ and $B$ as the standard transformations.)

From the analogy between the transformations for the components $B_k$ and $E_k$, (22), on the one hand and (21) on the other hand we can conclude that the transformations (21) for $T_i$ and $T_{t,i}$, i.e., (1), are also the AT. The same holds for the transformations for $L_i$ equation (11) in [25], as explained in detail in [9].

As seen from sections 3.1 - 3.3, the 4D torques $N$, $N_s$ and $N_t$ are determined taking that $S'$, the rest frame of the capacitor, is the frame of “fiducial” observers. In section 3 in [7], under the title “The Trouton-Noble paradox as an electrodynamic paradox,” Jefimenko calculated the 3D torque $T$ in the $S$ frame using direct nonrelativistic electromagnetic calculations and not the transformations for the components $T_i$. The whole calculation is exclusively made with the 3D quantities $E$, $B$, $F$ and $T$. In our approach the case corresponding to section 3 from [7], but with the 4D geometric quantities, would be that the $S$ frame is the frame of “fiducial” observers and that $r^0 = 0$. This will not be investigated here since the analogous case with Jackson’s paradox is examined in detail in [9], where it is shown that there is no paradox in that case as well and that the principle of relativity is naturally satisfied when the 4D geometric quantities are used.

In section 4 in [7] an additional 3D torque is introduced, which comes from the rate of change of the 3D angular electromagnetic field momentum, and which balances the rate of change of the 3D angular mechanical momentum of the moving capacitor. It is argued in [7] that such procedure resolves the paradox with the 3D torque. In our approach with 4D geometric quantities there is no paradox and consequently there is no need for any additional torque.

However it is worth noting that even in the case when the electromagnetic
field is concerned one has to deal with 4D AQs and not with the usual 3D quantities. Recently, [8], I have developed an axiomatic geometric formulation of electromagnetism with the bivector $F$ as the primary quantity for the whole electromagnetism. There, [8], it is shown that the most important quantity for the momentum and energy of the electromagnetic field is the observer independent stress-energy vector $T(n)$ (1-vector), which is a vector-valued linear function on the tangent space at each spacetime point $x$ describing the flow of energy-momentum through a hypersurface with normal $n = n(x)$. From that $T(n)$ some new observer independent expressions (AQs) are obtained for the energy density $U$ contained in an electromagnetic field, for the Poynting vector $S$, for the momentum density $g$ and for the angular-momentum density $M$. They are all written in terms of $F$. The most general expressions for $T(n)$, $U$, $S$, $g$ and $M$, but with 1-vectors $E$ and $B$ and not with $F$, can be simply obtained inserting the decomposition of $F$, [2], into the relations with $F$ from [8]. We consider that the 4D geometric quantities $T(n)$, $U$, $S$, $g$ and $M$ have to be used in all investigations of the electromagnetic field instead of the usual expressions with the 3D $E$ and $B$.

5. Conclusions

The main conclusion that can be drawn from the whole consideration is that the relativistically correct description of physical phenomena without any paradoxes can be achieved in the consistent way with 4D geometric quantities as physical quantities in the 4D spacetime. On the other hand, as seen from all previous treatments of the Trouton-Noble paradox and from our extensive discussion, the use of 3D quantities and their AT necessarily leads to different ambiguities and inconsistencies.

Regarding the measurements of the 4D geometric quantities we give the following remark. In order to check the validity of the relation [8] (and the similar relations for $N_s$ and $N_t$), and of the physical law $N = dM/d\tau$, the experimentalists have to measure all six independent components of $N^\mu\nu$ (or equivalently of $N_s^\mu$ and $N_t^\mu$), and also of $M^\mu\nu$ ($M_s^\mu$ and $M_t^\mu$), in both frames $S'$ and $S$. Obviously it is more complicated than in the usual approaches with the 3D quantities, in which the experimentalists measure only three components of the 3D torque $T$ and three components of the 3D angular momentum $L$ in both frames $S'$ and $S$. However, the comparisons with different experiments that are presented in [15] and [23, 24, 8] and in this paper explicitly show that in the 4D spacetime the relativistically correct results refer to the 4D geometric quantities and not, as generally accepted, to the 3D quantities.

References

[1] Trouton F T and Noble H R 1903 Philos. Trans. R. Soc. London Ser. A 202 165
[2] Hayden H C 1994 *Rev. Sci. Instrum.* **65** 788
[3] von Laue M 1911 *Phys. Zeits.* **12** 1008
[4] Pauli W 1958 *Theory of Relativity* (New York: Pergamon)
[5] Singal A K 1993 *Am. J. Phys.* **61** 428
[6] Teukolsky S A 1996 *Am. J. Phys.* **64** 1104
[7] Jefimenko O D 1999 *J. Phys. A: Math. Gen.* **32** 3755
[8] Ivezić T 2005 *Found. Phys. Lett.* **18** 401
[9] Ivezić T 2006 *physics/0602105*, to be published in 2006 *Found. Phys.* **36** (10)
[10] Hestenes D 1966 *Space-Time Algebra* (New York: Gordon and Breach); 1999 *New Foundations for Classical Mechanics* 2nd edn (Dordrecht: Kluwer); 2003 *Am. J Phys.* **71** 691
[11] Doran C and Lasenby A 2003 *Geometric algebra for physicists* (Cambridge: Cambridge University Press)
[12] Einstein A 1905 *Ann. Physik.* **17** 891 tr. by Perrett W and Jeffery G B 1952 in *The Principle of Relativity* (New York: Dover)
[13] Ivezić T 1999 *Found. Phys. Lett.* **12** 507
[14] Ivezić T 2001 *Found. Phys.* **31** 1139
[15] Ivezić T 2002 *Found. Phys. Lett.* **15** 27; 2001 *physics/0103026*; 2001 *physics/0101091*
[16] Jefimenko O D 1997 *Retardation and Relativity* (Star City, WV: Electret Scientific)
[17] Ludvigsen M 1999 *General Relativity, A Geometric Approach* (Cambridge: Cambridge University Press)
[18] Sonego S and Abramowicz M A 1998 *J. Math. Phys.* **39** 3158
[19] Møller C 1972 *The Theory of Relativity* 2nd edn (Oxford: Clarendon Press)
[20] Ivezić T 1999 *Found. Phys. Lett.* **12** 105
[21] Ivezić T 2003 *Found. Phys.* **33** 1339
[22] Ivezić T 2002 *hep-th/0207250v2*
[23] Ivezić T 2005 *Found. Phys. Lett.* **18** 301
[24] Ivezić T 2005 *Found. Phys.* **35** 1585
[25] Jackson J D 2004 *Am. J. Phys.* **72** 1484
[26] Jackson J D 1977 *Classical Electrodynamics* 2nd edn (New York: Wiley)
[27] Rohrlich F 1966 *Nuovo Cimento B* **45** 76