Commutative Relations
for the Nonlinear Dirac Equation

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Abstract

By constructing the commutative operators chain, we derive conditions for solving the eigenfunctions of Dirac equation and Schrödinger type equation via separation of variables. Detailed calculation shows that, only a few cases can be completely reduced into ordinary differential equation system. So the effective perturbation or approximation methods for the resolution of the spinor equation are necessary, especially for the nonlinear cases.

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1 Introduction

All fermions have spin-\(\frac{1}{2}\), which are naturally described by spinors. Many physicists such as H. Weyl, W. Heisenberg, once proposed using nonlinear spinor equations to establish unified field theory for elementary particles[1, 2]. Some rigorous solutions for the simplest dark nonlinear spinor models were obtained and analyzed[3]-[10]. The nonlinear spinor coupling with self-electromagnetic field was calculated in [11]-[15], regrettably, only roughly approximate results were obtained due to the complexity of the interaction. In contrast, for the linear Schrödinger equation, Pauli equation and Dirac equation coupling with external potential \(V(r)\) and \(\vec{A}(r,\theta)\), there are a lot of refined exact eigen solutions obtained via separation of variables[16]-[27].

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In the resolution of eigenfunctions to the linear Dirac equation, we find that the commutative relations \[ [\hat{H}, \hat{J}_z] = 0, \quad [\hat{H}, \hat{K}] = 0, \quad [\hat{K}, \hat{J}_z] = 0 \] (1.1)
play an important role, where the operators \((\hat{J}_z, \hat{K}, \hat{H})\) stand for angular momentum, spin-orbit coupling and total energy operators respectively. They are good quantum numbers guaranteeing the common eigenfunctions exist. Noticing the definition of operators chain
\[ \hat{J}_z = J_z(\partial_\varphi), \quad \hat{K} = K(\partial_\varphi, \partial_\theta), \quad \hat{H} = H(\partial_\varphi, \partial_\theta, \partial_r), \] (1.2)
which enable us to solve the eigen solutions via separation of variables, and then the original problem reduces to ordinary differential equations. In [15], we derived and simplified the complete nonlinear Dirac equation system, but where it is still a partial differential system depending on coordinate \((r, \theta)\). The primary motivation of this paper is looking for the conditions to reduce the nonlinear Dirac equation into the ordinary differential one, and clarify to how much degree we can expect the method of separating variables to solve the nonlinear problems.

Denote the Minkowski metric by \(\eta_{\mu\nu} = \text{diag}[1, -1, -1, -1]\), Pauli matrices by
\[ \vec{\sigma} = (\sigma^j) = \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}. \] (1.3)
Define 4 × 4 Hermitian matrices as follows
\[ \alpha^\mu = \left\{ \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \right\}, \quad \gamma = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix}. \] (1.4)
In this paper, we adopt the Hermitian matrices (1.4) instead of Dirac matrices \(\gamma^\mu\), because this form is more convenient for calculation.

Our basic problem is to examine a nonlinear spinor field \(\phi\) moving in 4-vector potential \(A^\mu\). The corresponding Lagrangian is generally given by[12, 15]
\[ \mathcal{L} = \phi^+ [\alpha^\mu(h\partial_\mu - eA_\mu) - \mu c\gamma]\phi + F(\hat{\gamma}, \hat{\beta}), \] (1.5)
where \(\mu > 0\) is a constant mass, \(F\) is the nonlinear coupling potential, which is the polynomial of \(\hat{\gamma}\) and \(\hat{\beta}\) defined by
\[ \hat{\gamma} = \phi^+ \gamma \phi, \quad \hat{\beta} = \phi^+ \beta \phi. \] (1.6)
It is well known that \(\hat{\gamma}\) is a true-scalar and \(\hat{\beta}\) a pseudo-scalar. The variation of (1.5) with respect to \(\phi^+\) gives the dynamic equation
\[ h\partial_0 \phi = \hat{H} \phi, \quad \hat{H} \equiv \vec{\alpha} \cdot (-h\nabla - e\vec{A}) + eA_0 + (\mu c - F_\gamma)\gamma - F_\beta \beta, \] (1.7)
where \( F_\gamma = \frac{\partial F}{\partial \gamma} \), \( F_\beta = \frac{\partial F}{\partial \beta} \).

Let coordinate \( x^3 = z \) along the direction of magnetic field \( \vec{B} \), then we locally have\[8, 9, 12, 15\]
\[ \vec{A} = A(r, \theta)(-\sin \varphi, \cos \varphi, 0), \]
which satisfies the Coulomb gauge \( \nabla \cdot \vec{A} = 0 \). In the spherical coordinate system \((r, \theta, \varphi)\), we have
\[ \vec{\sigma} \cdot \nabla = \sigma_r \partial_r + \frac{1}{r} \sigma_\theta \partial_\theta + \frac{1}{r \sin \theta} \sigma_\varphi \partial_\varphi, \]
where \((\sigma_r, \sigma_\theta, \sigma_\varphi)\) is given by
\[ \left\{ \begin{pmatrix} \cos \theta & \sin \theta e^{-\varphi i} \\ \sin \theta e^{\varphi i} & -\cos \theta \end{pmatrix}, \begin{pmatrix} -\sin \theta & \cos \theta e^{-\varphi i} \\ \cos \theta e^{\varphi i} & \sin \theta \end{pmatrix}, \begin{pmatrix} 0 & -ie^{-\varphi i} \\ ie^{\varphi i} & 0 \end{pmatrix} \right\}. \]

Let \( \hat{J} \) be the angular momentum operator for the spinor field
\[ \hat{J} = \vec{r} \times \hat{p} + \frac{1}{2} \hbar \vec{S}, \quad \hat{p} = -\hbar i \nabla, \quad \vec{S} = \text{diag}(\vec{\sigma}, \vec{\sigma}). \]

Then the angular momentum \( \hat{J}_3 = \hat{J}_z \) commutates with the nonlinear Hamilton operator (1.7), and the eigenfunctions of \( \hat{J}_z = -\hbar i \partial_\varphi + \frac{1}{2} \hbar S_3 \) are equivalent to the following form
\[ \phi = (u_1, u_2 e^{\varphi i}, -i v_1, -i v_2 e^{\varphi i})^T \exp \left( \kappa \varphi i - \frac{mc^2}{\hbar t} i \right) \]
with \((\kappa = 0, \pm 1, \pm 2, \cdots)\), where \( u_k, v_k (k = 1, 2) \) are real functions of \((r, \theta)\) but independent on \( \varphi \) and \( t \), the index \( T \) stands for transposed matrix.

Making variable transformation[15]
\[ u = u_1(r, \theta) + u_2(r, \theta) i, \quad v = v_1(r, \theta) + v_2(r, \theta) i, \]
then we have
\[ \begin{cases} \hat{\alpha}_0 = |u|^2 + |v|^2, & \hat{\alpha} = (\bar{u}v - u\bar{v})i, \\ \hat{\gamma} = |u|^2 - |v|^2, & \hat{\beta} = \bar{u}v + u\bar{v}, \end{cases} \]
with \( \hat{\alpha} = \hat{\alpha}(-\sin \varphi, \cos \varphi, 0) \). The eigenfunctions of (1.7) with even parity take the form
\[ u = \sum_{n=-N}^{N} U_n e^{2n\varphi i}, \quad v = \sum_{n=-N}^{N} V_n e^{(-2n+1)\varphi i}, \]
and the eigenfunctions with odd parity take
\[ u = \sum_{n=-N}^{N} U_n e^{(2n+1)\varphi i}, \quad v = \sum_{n=-N}^{N} V_n e^{-2n\varphi i}. \]
where all \((U_n, V_n)\) are real, \(N < \infty\) for some cases, but \(N = +\infty\) for general nonlinear cases. In what follows we only consider \((1.15)\), because for \((1.16)\) we have similar results. By \((1.14)\) and \((1.15)\) we have

\[
\tilde{\alpha}_0 = \sum_{n,m=-N}^{N} (U_n U_m + V_n V_m) \cos(2(n - m)\theta),
\]

\[(1.17)\]

\[
\tilde{\gamma} = \sum_{n,m=-N}^{N} (U_n U_m - V_n V_m) \cos(2(n - m)\theta),
\]

\[(1.18)\]

\[
\tilde{\alpha} = \sum_{n,m=-N}^{N} 2U_n V_m \sin[(2(n + m) - 1)\theta],
\]

\[(1.19)\]

\[
\tilde{\beta} = \sum_{n,m=-N}^{N} 2U_n V_m \cos[(2(n + m) - 1)\theta].
\]

\[(1.20)\]

Obviously, only for the case \(N = 0\), the quantities \(\tilde{\alpha}_0\) and \(\tilde{\gamma}\) are independent of \(\theta\), the following analysis shows only this case can be solved by separation of variables.

### 2 Operators and Commutative Relations

In spherical coordinate system, by straightforward calculation, we have the following explicit operator relations\[15, 16\]

\[
\partial_x = \sin \theta \cos \varphi \partial_r + \frac{1}{r} \cos \theta \cos \varphi \partial_\theta - \frac{1}{r \sin \theta} \sin \varphi \partial_\varphi,
\]

\[(2.1)\]

\[
\partial_y = \sin \theta \sin \varphi \partial_r + \frac{1}{r} \cos \theta \sin \varphi \partial_\theta + \frac{1}{r \sin \theta} \cos \varphi \partial_\varphi,
\]

\[(2.2)\]

\[
\partial_z = \cos \theta \partial_r - \frac{1}{r \sin \theta} \partial_\varphi,
\]

\[(2.3)\]

For orbit angular momentum operator \(\hat{L} = \vec{r} \times \hat{p}\), we have

\[
\hat{L}_x = \hbar i (\sin \varphi \partial_\theta + \cot \theta \cos \varphi \partial_\varphi),
\]

\[(2.4)\]

\[
\hat{L}_y = \hbar i (-\cos \varphi \partial_\theta + \cot \theta \sin \varphi \partial_\varphi),
\]

\[(2.5)\]

\[
\hat{L}_z = -\hbar i \partial_\varphi,
\]

\[(2.6)\]

\[
\hat{L}^2 = -\hbar^2 (\partial_\theta^2 + \cot \theta \partial_\theta + \frac{1}{\sin \theta^2} \partial_\varphi^2).
\]

\[(2.7)\]
For the nonlinear system (1.7), detailed calculation shows that

\[
\frac{i}{\hbar}[\hat{H}, \hat{J}_x] = \sin \varphi \partial_\theta (eV - F_\gamma \gamma - F_\beta \beta) + e \left( \cot \theta \cos^2 \varphi A + \sin^2 \varphi \partial_\theta A \right) \alpha^1 + e \sin \varphi \cos \varphi (\cot \theta A - \partial_\theta A) \alpha^2 - eA \cos \varphi \alpha^3, \tag{2.8}
\]

\[
\frac{i}{\hbar}[\hat{H}, \hat{J}_y] = -\cos \varphi \partial_\theta (eV - F_\gamma \gamma - F_\beta \beta) + e \sin \varphi \cos \varphi (\cot \theta A - \partial_\theta A) \alpha^1 + e(\cot \theta \sin^2 \varphi A + \cos^2 \varphi \partial_\theta A) \alpha^2 - eA \sin \varphi \alpha^3, \tag{2.9}
\]

\[
\frac{i}{\hbar}[\hat{H}, \hat{J}_z] = 0. \tag{2.10}
\]

(2.10) shows $\hat{J}_z$ is still a good quantum number for the nonlinear Dirac equation. By (2.8)-(2.10) and relation $[H, \hat{J}^2] = [H, \hat{J}] \cdot \hat{J} + \hat{J} \cdot [H, \hat{J}]$, we have

\[
[\hat{H}, \hat{J}^2] = \hbar^2 \left( L_\gamma + L_\beta - eL_V - eL_A (\sin \varphi \alpha^1 - \cos \varphi \alpha^2) + D_V + eD_A \right), \tag{2.11}
\]

in which

\[
L_\gamma = -\partial_\theta^2 F_\gamma - \cot \theta \partial_\theta F_\gamma - 2\partial_\theta F_\gamma \partial_\theta, \tag{2.12}
\]

\[
L_\beta = -\partial_\theta^2 F_\beta - \cot \theta \partial_\theta F_\beta - 2\partial_\theta F_\beta \partial_\theta, \tag{2.13}
\]

\[
L_V = -\partial_\theta^2 V - \cot \theta \partial_\theta V - 2\partial_\theta V \partial_\theta, \tag{2.14}
\]

\[
L_A = -\partial_\theta^2 A - \cot \theta \partial_\theta A - 2\partial_\theta A \partial_\theta + \cot^2 \theta A, \tag{2.15}
\]

\[
D_V = -i(e\partial_\theta V - \partial_\theta F_\gamma \gamma) (\sin \varphi S_1 - \cos \varphi S_2), \tag{2.16}
\]

\[
D_A = (\partial_\theta A + \cot \theta A) \gamma \beta + 2A \cot \theta \left( \cot \theta (\cos \varphi \alpha^1 + \sin \varphi \alpha^2) - \alpha^3 \right) \partial \varphi. \tag{2.17}
\]

Define the spin-orbit coupling operator by

\[
\hat{K} \equiv \gamma (\hat{L} \cdot \hat{S} + \hbar) = \hbar \gamma - \hbar i (K_\theta \partial_\theta + K_\varphi \partial_\varphi), \tag{2.18}
\]

where $K_\theta$ and $K_\varphi$ defined by

\[
K_\theta = \text{diag} \begin{bmatrix} 0 & -ie^{-\varphi i} \\ ie^{\varphi i} & 0 \end{bmatrix}, \tag{2.19}
\]

\[
K_\varphi = \text{diag} \begin{bmatrix} 1 & -\cot \theta e^{-\varphi i} \\ -\cot \theta e^{\varphi i} & 1 \end{bmatrix}, \tag{2.20}
\]

then we have

\[
\hat{K}^2 = \hat{J}^2 + \frac{1}{4}\hbar^2. \tag{2.21}
\]

Therefore, $\hat{K}$ is the first order form of $\hat{J}^2$. For (1.7), we can check

\[
[\hat{H}, \hat{K}] = K_V + K_A + K_\beta, \tag{2.22}
\]
In which

\[ K_V = -\hbar i (c \partial_\theta V - \partial_\theta F_\gamma)\gamma (\sin \varphi S_1 - \cos \varphi S_2), \]  
(2.23)

\[ K_A = -\hbar e \left( A\gamma (\sin \varphi \alpha - \cos \varphi \alpha^2) - (\partial_\theta A + \cot \theta A + 2A\partial_\theta)\beta \right), \]  
(2.24)

\[ K_\beta = 2\hbar \gamma \beta F_\beta + \hbar (\partial_\theta F_\beta + 2F_\beta \partial_\theta) (\sin \varphi \alpha - \cos \varphi \alpha^2) + 
2\hbar F_\beta \left( \cot \theta (\cos \varphi \alpha + \sin \varphi \alpha^2) - \alpha^3 \right) \partial_\varphi. \]  
(2.25)

In some sense, (2.22)-(2.25) reflect the influence of parameters on the integrability.

3 Conditions for Separation of Variables

For the nonlinear Hamiltonian (1.7), the commutative relation (2.10) shows \( \hat{J}_z \) is still a good quantum number, but (2.22) shows \( \hat{K} \) is not. In what follows, we look for a new operator \( \hat{T} \) and the conditions, such that the systems (1.7) can be solved by separation of variables. This is equivalent to the existence of operator \( \hat{T} = T(\partial_\theta, \partial_\varphi) \)
or \( \hat{T}' = T'(\partial_r, \partial_\varphi) \), which satisfies commutative relations

\[ [\hat{J}_z, \hat{T}] = 0, \quad [\hat{H}, \hat{T}] = 0. \]  
(3.1)

We only consider the following first order operator

\[ \hat{T} = T_0 - \hbar i (T_\theta \partial_\theta + T_\varphi \partial_\varphi), \]  
(3.2)

where \( (T_0, T_\theta, T_\varphi) \) are all Hermitian matrices, and their components are smooth functions of \( (r, \theta, \varphi) \). The higher order operators can be similarly constructed, but they seem not to be functionally independent of (3.2), and the highest order derivatives should matches that in \( \hat{H} \). For Dirac equation, it is enough to look for the linear operators similar to (3.2). But for the Schrödinger and Pauli equations, we should consider the second order operator similar to \( \hat{L}^2 \) as shown below. Different from \( \hat{p} \) and \( \hat{J} \), the operators constructed from the procedure may have not manifest physical meanings.

According condition \([\hat{J}_z, \hat{T}] = 0\), we can solve

\[ T_0 = \begin{pmatrix}
P_{11} & P_{12} \cot \theta e^{-i\phi} & P_{13} & P_{14} \cot \theta e^{-i\phi} \\
P_{21} \cot \theta e^{i\phi} & P_{22} & P_{23} \cot \theta e^{i\phi} & P_{24} \\
P_{31} & P_{32} \cot \theta e^{-i\phi} & P_{33} & P_{34} \cot \theta e^{-i\phi} \\
P_{41} \cot \theta e^{i\phi} & P_{42} & P_{43} \cot \theta e^{i\phi} & P_{44}
\end{pmatrix}, \]  
(3.3)
\[ T_{\theta} = \begin{pmatrix} M_{11} & M_{12}e^{-i\phi} & M_{13} & M_{14}e^{-i\phi} \\ M_{21}e^{i\phi} & M_{22} & M_{23}e^{i\phi} & M_{24} \\ M_{31} & M_{32}e^{-i\phi} & M_{33} & M_{34}e^{-i\phi} \\ M_{41}e^{i\phi} & M_{42} & M_{43}e^{i\phi} & M_{44} \end{pmatrix}, \] 

\[(3.4)\]

\[ T_{\varphi} = \begin{pmatrix} N_{11} & N_{12} \cot \theta e^{-i\phi} & N_{13} & N_{14} \cot \theta e^{-i\phi} \\ N_{21} \cot \theta e^{i\phi} & N_{22} & N_{23} \cot \theta e^{i\phi} & N_{24} \\ N_{31} & N_{32} \cot \theta e^{-i\phi} & N_{33} & N_{34} \cot \theta e^{-i\phi} \\ N_{41} \cot \theta e^{i\phi} & N_{42} & N_{43} \cot \theta e^{i\phi} & N_{44} \end{pmatrix}, \] 

\[(3.5)\]

where \((P_{kl}, M_{kl}, N_{kl})\) are Hermitian matrices, and their components are functions of \((r, \theta)\), the factor \(\cot \theta\) is introduced for convenience of the following resolution.

The relation \([\hat{H}, \hat{T}]\) can be expressed as

\[
[\hat{H}, \hat{T}] = H_{\theta\theta} \partial_{\theta}^2 + H_{\theta r} \partial_{\theta} + (H_{\theta \varphi} + H_{\theta \varphi} \partial_{\varphi}) \partial_{\theta} + (H_{r} + H_{r\varphi} \partial_{\varphi}) \partial_{r} \\
+ (H_{0} + H_{\varphi} \partial_{\varphi} + H_{\varphi \varphi} \partial_{\varphi}^2),
\]

\[(3.6)\]

where the coefficient matrices can be obtained by straightforward calculation, and they can be easily generated by computer. \((H_{\theta\theta}, H_{\theta r}, H_{\theta \varphi}, H_{r \varphi})\) are only functions of \((P_{kl}, M_{kl}, N_{kl})\), but the others depend on their first order derivatives. The operator \(\partial_{\varphi}\) should be replaced by matrix \(D_{\varphi}\) due to solution (1.12),

\[
\partial_{\varphi} \leftrightarrow D_{\varphi} = i \text{ diag}(\kappa, \kappa + 1, \kappa, \kappa + 1),
\]

\[(3.7)\]

\[
\partial_{\varphi}^2 \leftrightarrow D_{\varphi}^2 = -\text{diag} (\kappa^2, (\kappa + 1)^2, \kappa^2, (\kappa + 1)^2).
\]

\[(3.8)\]

Then \([\hat{H}, \hat{T}] = 0\) is equivalent to the following equations

\[
H_{\theta\theta} = H_{r\theta} = 0, \quad (3.9)
\]

\[
H_{\theta} + H_{\theta \varphi} D_{\varphi} = 0, \quad (3.10)
\]

\[
H_{r} + H_{r \varphi} D_{\varphi} = 0, \quad (3.11)
\]

\[
H_{0} + H_{\varphi} D_{\varphi} + H_{\varphi \varphi} D_{\varphi}^2 = 0. \quad (3.12)
\]

To solve equations (3.9)-(3.12), we should solve the simple and algebraic relations at first, then the complex and differential equations, otherwise the calculation will become terribly complicated. The differential equations including in (3.9)-(3.12) are linear, and one component usually satisfies two independent equations, which lead to constant solutions. So it is easy to resolution.
By (3.9), we can solve
\[
\begin{pmatrix}
W_1 & -iW_2e^{-i\phi} & W_3 & W_4e^{-i\phi} \\
iW_2e^{i\phi} & W_1 & -W_4e^{i\phi} & W_3 \\
W_3 & -W_4e^{-i\phi} & W_1 & iW_2e^{-i\phi} \\
W_4e^{i\phi} & W_3 & -iW_2e^{i\phi} & W_1
\end{pmatrix},
\]
(3.13)
in which \(W_k = W_k(r, \theta, \kappa), \quad (k = 1, 2, 3, 4)\). Again by (3.10), we find that \(W_1 = W_3 = W_4 = 0\), and \(W_2 \neq 0\) is a constant. Obviously \(\hat{T}\) can be only determined by a constant factor and constant translation, so we choose \(W_2 = 1\) to fix the solution. Then in equivalent sense we have
\[
T_\theta = K_\theta.
\] (3.14)

For the other equations in (3.10)-(3.12), detailed calculation shows that the solutions for \(P_{kl}\) and \(N_{kl}\) are highly underdetermined. Expressing \(P_{kl}\) by \(N_{kl}\), we can get the following results via long but simple calculation
\[
\begin{align*}
V &= V(r), \\
A &= r^{-1}U(\theta), \\
\partial_\theta F_\gamma &= 0, \\
F_\beta &= 0,
\end{align*}
\] (3.15)
\[
T_0 - hiT_\varphi D_\varphi = (K_0 - hiK_\varphi D_\varphi) + eU\text{diag}(\sigma_\theta, -\sigma_\theta) + \lambda_0,
\] (3.16)
where \(\lambda_0\) is a constant. If we set \(\hat{T} = \hat{K}\) when \(U = 0\), we get \(\lambda_0 = 0\). Substituting \(D_\varphi \rightarrow \partial_\varphi\) into (3.16), in equivalent sense, we finally get
\[
\begin{align*}
T_0 &= K_0 + eU\text{diag}(\sigma_\theta, -\sigma_\theta), \\
T_\theta &= K_\theta, \\
T_\varphi &= K_\varphi.
\end{align*}
\] (3.17)

In the spherical coordinate system, the above procedure is invertible, so equations (3.15) actually form the sufficient and necessary conditions of the separation of variables to spinor equation with axisymmetry. Noticing the relation (3.17), conditions (3.15) can also be verified by the commutative relations (2.22)-(2.25).

\{\(V = A = F_\beta = \partial_\theta F_\gamma = 0\)\} is just the simplest nonlinear case discussed in [3]-[9].
\{\(V = V(r), A = F_\beta = 0, F_\gamma = G(r)\)\} are the cases discussed in [16] and [17]-[26].
\{\(V = \frac{Ze}{r}, A = \frac{e}{r\sin\theta}(a - b \cos \theta), F_\beta = F_\gamma = 0\)\} is the case calculated in [27].

For the general case of (3.15), although the separation of variables is valid for the Dirac equation (1.7), but the solutions to the reduced ordinary differential equations usually can not be expressed by elementary functions in finite form. So some auxiliary numerical computation is still necessary.

For the Schrödinger equation
\[
hi\partial_0 \phi = \hat{H} \phi, \quad \hat{H} = -\frac{\hbar^2}{2m} \left(\partial_r^2 + \frac{2}{r} \partial_r + \frac{1}{\hbar^2 r^2} \hat{L}^2\right) + V(r, \theta),
\] (3.18)

the calculation is simple. Obviously, for $\hat{L}_z = -\hbar \hat{\phi}$, we have
\[ [\hat{H}, \partial_\phi] = 0, \quad \partial_\phi \leftrightarrow \kappa \hat{i}. \quad (3.19) \]

We look for the following second order operator
\[ \hat{\mathcal{Y}} = Y_0 + Y_\theta \partial_\theta + \hat{L}^2. \quad (3.20) \]

By $[\hat{\mathcal{Y}}, \partial_\phi] = 0$, we get
\[ \partial_\phi Y_0 = \partial_\phi Y_\theta = 0. \quad (3.21) \]

By $[\hat{H}, \hat{\mathcal{Y}}] = 0$, we find
\[ V = W(r) + \frac{1}{r^2} U(\theta), \quad (3.22) \]
\[ Y_0 = \lambda_0 - 2mU(\theta), \quad Y_\theta = 0, \quad (3.23) \]

where $U(\theta)$ and $W(r)$ are arbitrary functions. (3.22) is the condition to solve the eigenfunctions of (3.18) via separating variables. Different from the Dirac equation, (3.22) shows $V$ can vary with $\theta$.

## 4 Some Remarks

From the above calculation, we get the following insights and experiences:

**\textbf{(R1)}** The above procedure has general significance for separation of variables. The method is also suitable for higher dimensional cases. For Hamiltonian
\[ \hat{H}_n = H_n(\partial_1, \partial_2, \cdots, \partial_n), \quad (4.1) \]

the separation of variables is equivalent to the existence of the following Hermitian operators chain
\[ \hat{H}_1 = H_1(\partial_1), \quad \hat{H}_2 = H_2(\partial_1, \partial_2), \quad \cdots, \quad \hat{H}_n = H_n(\partial_1, \partial_2, \cdots, \partial_n), \quad (4.2) \]

which form a Abelian Lie algebra
\[ [\hat{H}_j, \hat{H}_k] = 0, \quad (j, k = 1, \cdots, n). \quad (4.3) \]

The operators chain $\{\hat{H}_k\}$ reflects the intrinsic symmetry of the system, which can be obtained via simple calculation. In general, there may not be a complete chain similarly to the case of (1.7), but the incomplete chain is still helpful to simplify the dynamic
equation as done in [15]. In addition, the existence of operators chain depends on the coordinate system.

(R2) The commutative relations (4.3) are conditional relations, that is, it is enough for them to hold only for the eigenfunctions, rather than for the whole function space of the solutions. The solution to the dark spinors is an example[3]-[9]. So we can use the information of solutions for solving the operators chain.

(R3) For the spinor with self-electromagnetic interaction, we have \( \partial_\theta V \neq 0 \) and \( rA \neq eU(\theta) \). Then according to (3.15), we can not construct an operator \( \hat{T} \) commuting with \( \hat{H} \). This means (1.7) has not complete operators chain in the spherical coordinate system, so the nonlinear Dirac equation can not be generally reduced to ordinary differential equations. Consequently, the equation derived in [16] can not be further simplified. Some physical parameters such as \( g/2 = 1.001159652\cdots \) may only be computed by perturbation.

(R4) (2.22) provides a measurement for integrability in some sense. By (2.25), we find \( K_\beta \) influences many terms, so \( \tilde{\beta} \) is a very bad parameter for the existence of eigen solutions. The dynamical reason for the Pauli’s exclusion principle is one of the most fundamental problems unsettled. After all, the reason should be something rooting in the complete Dirac equation, so the suspicion that \( \tilde{\beta} \) is related to this phenomena might be reasonable.

(R5) The above method is related to the concept ‘superintegrability’, which was introduced at the classical level by Wojciechowski[28] and at the quantum level by Kuznetsov[29]. Superintegrability also deals with the symmetry and integrability of a Hamiltonian system, and many elaborate models were solved. Besides (4.2), the super-integrability demands extra commutative operators. There are extensive literatures, e.g. [28]-[40] and references therein. However, most of the papers studied classical Hamiltonian system and Schrödinger type equations, and few works dealt with spinor equations. The above treatment seems to be more straightforward and simpler, the information of the solutions and symmetries of the system is automatically used, and the resolution seems unnecessarily to introduce extra commutative relations to (4.2).

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