On the Möbius Function and Topology of General Pattern Posets

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Abstract

We introduce a formal definition of a pattern poset which encompasses several previously studied posets in the literature. Using this definition we present some general results on the Möbius function and topology of such pattern posets. We prove our results using a poset fibration based on the embeddings of the poset, where embeddings are representations of occurrences. We show that the Möbius function of these posets is intrinsically linked to the number of embeddings, and in particular to so called normal embeddings. We present results on when topological properties such as Cohen-Macaulayness and shellability are preserved by this fibration. Furthermore, we apply these results to some pattern posets and derive alternative proofs of existing results, such as Björner’s results on subword order. Moreover, we conjecture that shellability is preserved by poset fibrations satisfying certain conditions, which would generalise a result of Quillen’s showing Cohen-Macaulayness is preserved under similar conditions, and we prove a restricted form of this conjecture.

1. Introduction

Pattern occurrence, or more generally the presence of substructures, has been studied on a wide range of combinatorial objects with many different definitions of a pattern, see [Kit11] for an overview of the field. In many of these cases we can use the notion of pattern containment to define a poset on these objects, for example the classical permutation poset. Whilst many such pattern posets have been studied in isolation, there is no general framework for the study of these posets. Yet many of the known results follow a similar theme. By introducing a formal definition of a pattern poset we develop tools for studying these posets, which leads to some general results that helps in understanding why different pattern posets often have a similar structure.

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We say a word $\alpha$ occurs as a $\rho$-pattern in a word $\beta$ if there is a subsequence of $\beta$ satisfying certain conditions $\rho$, and we call this subsequence an occurrence of $\alpha$. We can define a binary relation $\alpha \leq \rho \beta$ if $\alpha$ occurs as a $\rho$-pattern in $\beta$. If $\leq \rho$ satisfies the partial order conditions, then we can define a pattern poset on the set of words in question. A variety of different pattern posets have been studied in the literature, where the main focus is to answer questions on the structure and topology of a pattern poset $P$ and its intervals, that is, the induced subposets $[\alpha, \beta] = \{ \lambda \in P | \alpha \leq \lambda \leq \beta \}$.

The topology of a poset is considered by mapping the poset to a simplicial complex, called the order complex, whose faces are the chains of the poset, that is, the totally ordered subsets. We refer the reader to [Wac07] for an overview of poset topology. A poset is shellable if its maximal chains can be ordered in a certain way. Shellability implies a poset has many nice properties, such as Cohen-Macaulayness. We define a poset $P$ to be Cohen-Macaulay if the order complex of $P$, and of every interval of $P$, is homotopically equivalent to a wedge of top dimensional spheres.

The study of patterns in words has received a lot of attention. Perhaps the simplest type of pattern in a word is that of subword order, that is, $u = u_1 \ldots u_a$ occurs as a pattern in $w = w_1 \ldots w_b$ if there is a subsequence $w_{i_1} \ldots w_{i_a}$ such that $w_{i_j} = u_j$ for all $j = 1, \ldots, a$. Björner [Bjö90] presented a formula for the Möbius function of the poset of words with subword order and showed that this poset is shellable. The poset of words with composition order has the partial order $u \leq w$ if there is a subsequence $w_{i_1} \ldots w_{i_a}$ such that $u_i \leq w_{i_j}$ for all $j = 1, \ldots, a$. A formula for the Möbius function of this poset is given by Sagan and Vatter in [SV06]. Furthermore, the poset of generalised subword order is considered by Sagan and Vatter in [SV06] and McNamara and Sagan in [MS12]. There are many other examples of studies of patterns in words and word representable objects, such as in permutations, set partitions, trees, mesh patterns and more.

Perhaps the most studied pattern posets in recent years is that of permutation patterns, where a permutation is a word on the alphabet of nonnegative integers with no repeated letters. The classical, and most studied, definition of a pattern in a permutation says that $\sigma$ occurs as a pattern in the permutation $\pi$ if there is a subsequence of $\pi$ whose letters have the same relative order of size as the letters of $\sigma$. For example, 213 occurs as a pattern in 35142 in the subsequence 314. The permutation pattern poset has been studied extensively but a complete understanding has proved elusive due to its complex nature. Some formulas for the Möbius function of certain classes of permutations and certain properties of the topology have been given in [SV06, ST10, BJJS11, MS15, Smi14, Smi17, Smi16].

Many of the known results on the Möbius function of pattern posets, including those mentioned above, depend on the number of normal embeddings, defined in various but similar ways. For example, they play an important role in the study of many different classes of intervals of the classical permutation poset. We define an embedding of $\alpha$ in $\beta$ as a sequence of dashes and the letters of $\alpha$, such that the positions of the non-dash letters give an occurrence of $\alpha$.
in $\beta$ and deleting all the dashes results in $\alpha$. For example, $13 - 2 - -$ is an embedding of 132 in 156324. The definition of when an embedding is normal varies, but all follow a similar theme. Perhaps the simplest definition is that of Björner’s for subword order [Bjö90], where the normal condition is that the only positions that can be dashes are the leftmost positions in the maximal consecutive sequences of equal letters.

We introduce a simple definition for normal embeddings which extracts the common theme from those in the literature. Using this definition we prove that the Möbius function of a pattern poset, satisfying certain restrictions, equals the number of normal embeddings, plus an extra term that we describe explicitly. This extra term embodies the variations in the many definitions of normal embeddings. Intriguingly, this extra term often vanishes, or can be shown to be zero, which allows us to compute the Möbius function in polynomial time. Furthermore, our general result can be used to prove many of the existing results on the Möbius function of various pattern posets.

Poset fibrations are instrumental to our results. A fibration of a poset $Q$ consists of another poset $P$, called the total space, and a rank and order preserving surjective map $f : P \rightarrow Q$. Poset fibrations were first studied by Quillen in [Qui78] and a good overview is given in [BWW05]. It was shown by Quillen that Cohen-Macaulayness is maintained across a poset fibration satisfying certain conditions. We conjecture that shellability is also preserved across a poset fibration satisfying similar conditions. We prove a restricted version of this conjecture when the shelling of $P$ can be defined in terms of a linear ordering of $P$.

We introduce a poset fibration on an interval $[\alpha, \beta]$ of a pattern poset. The total space of this fibration is built from the embeddings of $\lambda$ in $\beta$, for all $\lambda \in (\alpha, \beta)$. This total space has a much nicer structure than the original poset, which allows us to compute the Möbius function and topology of the total space. We can then use known results on poset fibrations to get results for the original interval, such as showing an interval $[\alpha, \beta]$ is Cohen-Macaulay if the total space of $[\lambda, \beta]$ is Cohen-Macaulay for all $\lambda \in [\alpha, \beta]$, and we conjecture a similar result for shellability.

It is known that a poset is not shellable if it contains a disconnected subinterval of rank greater than 2. We say an interval is zero split if its set of embeddings can be partitioned into two parts such that no position appears as a dash in embeddings belonging, respectively, to both parts. In [MS15] it is shown that an interval of the classical permutation poset is disconnected if and only if it satisfies a slightly stronger condition than being zero split. We introduce a definition of strongly zero split which generalises this result to pattern posets. This implies that if an interval contains a strongly zero split subinterval of rank greater than 2, then it is not shellable.

In Section 2 we introduce some notation used throughout the paper and in Section 3 we introduce pattern posets. In Section 4 we introduce two poset fibrations on pattern posets. In Section 5 we apply these poset fibrations to prove some results on the Möbius function and topology of pattern posets. In Section 6 we apply the results from Section 5 to the poset of words with
subword order, which provides an alternative proof of Björner’s result on the Möbius function of this poset. We also consider the consecutive permutation pattern poset and provide an alternative proof for the results on the Möbius function given in [EM15]. Finally, in Section 7 we propose some question for future work.

2. Notation and Preliminaries

We begin by introducing some necessary notation on words and posets. For further background on words see [Kit11] and for further background on posets see [Sta12, Chapter 3].

**Definition 2.1.** A word is a sequence of letters from an alphabet $\Sigma$. The length of a word $w$, denoted $|w|$, is the number of letters in the word and we use $w_i$ to denote the letter in position $i$ of $w$. We denote the set of words on the alphabet $\Sigma$ by $\Sigma^*$.  

**Example 2.2.** If $\Sigma = \{0, 1\}$ then 01001 is a word of length 5 on the alphabet $\Sigma$.

Note that we use the convention that the first position is number 1, not 0. We can apply many different restrictions to words to get different combinatorial objects. We are particularly interested in permutations, which can be defined in the following way:

**Definition 2.3.** A permutation is a word with no repeated letters. The reduced form of a permutation $\sigma$ is the permutation $\text{red}(\sigma)$, where if $\sigma_i$ is the $k$’th largest valued letter then $\text{red}(\sigma)_i = k$.

**Example 2.4.** If $\sigma = 264$, then $\text{red}(\sigma) = 132$.

We consider two permutations to be the same if the have the same reduced form. When studying the Möbius function or topology of a poset it is often necessary that there is a unique minimal and unique maximal element, called the bottom and top elements, respectively.

**Definition 2.5.** A poset $P$ is bounded if it has a unique bottom and top element, which we denote $\hat{0}$ and $\hat{1}$, respectively. If $P$ is not bounded we create the bounded poset $\hat{P}$ by adding a bottom and top element. The interior of a bounded poset is obtained by removing the bottom and top elements.

Many of the posets that we look at are infinite, so it makes sense to limit our investigation to smaller subposets.

**Definition 2.6.** An interval of a poset $P$ is the induced subposet $[\sigma, \pi] := \{\lambda \in P | \sigma \leq \lambda \leq \pi\}$. We denote the interior of an interval by $(\sigma, \pi)$ and the half open intervals $[\sigma, \pi) = [\sigma, \pi] \setminus \{\pi\}$ and $(\sigma, \pi] = [\sigma, \pi] \setminus \{\sigma\}$.

We also recall some general poset terminology that is used throughout.
Definition 2.7. Let $P$ be a bounded poset. A chain $c$, of length $|c| = \ell$, in $P$ is a totally ordered subset of elements $c_1 < c_2 < \cdots < c_\ell$. The rank of an element $\alpha \in P$, denoted $rk_P(\alpha)$ or $rk(\alpha)$ when the poset is clear, is the length of the longest chain from $\hat{0}$ to $\alpha$, minus 1. The rank of $P$ is given by $rk(P) = rk(\hat{1})$. A poset is pure if all the maximal chains have the same length.

The join of any two elements $\alpha$ and $\beta$, denoted $\alpha \vee \beta$, of a poset is the smallest element that lies above both $\alpha$ and $\beta$. An element $\alpha$ is covered by $\beta$, which we denote by $\alpha \prec \beta$, if $\alpha < \beta$ and there is no $\kappa$ such that $\alpha < \kappa < \beta$.

In this paper all posets are assumed to be pure. Two questions often asked of any poset are “What is the Möbius function?” and “Is it shellable?”. A poset is shellable if the maximal chains can be ordered in a “nice” way, see Section 5.4 for a formal definition. Shellability has many interesting consequences for the structure and topology of a poset. The Möbius function is defined as follows:

Definition 2.8. The Möbius function on a poset $P$ is defined recursively, where for any elements $a, b \in P$ we have $\mu(a, a) = 1$, $\mu(a, b) = 0$ if $a \nleq b$ and if $a < b$ then:

$$\mu(a, b) = - \sum_{c \in [a, b)} \mu(a, c).$$

The Möbius function of bounded poset $P$ is $\mu(P) = \mu(\hat{0}, \hat{1})$ and the Möbius number of a poset $P$ is $\hat{\mu}(P) := \mu(P)$.

We are also interested in looking at the structure of the poset. For example, it is known that a poset is not shellable if it has any disconnected subintervals of rank greater than 2, where disconnected is defined by:

Definition 2.9. A bounded poset is disconnected if the interior can be split into two disjoint sets, which we call components, such that elements from separate components are incomparable.

See Figure 2.1 for an example of a disconnected bounded pure poset. In order to study these properties of a poset we use poset fibrations, which were first introduced by Quillen in [Qui78] and have many nice properties, see [BWW05] for a good overview.

Definition 2.10. A poset fibration is a rank and order preserving surjective map between posets.

3. Pattern Posets

The idea of pattern occurrence has been used to consider a variety of different combinatorial objects. The notion of permutation pattern occurrence has received the most attention, but many others have also been considered. Pattern occurrence can be used to define a partial order and thus a poset. In this section we give a general definition of such a poset. First we need to formally define pattern occurrence.
Figure 2.1: The Hasse diagram of a disconnected bounded pure poset $P$ with rank $\text{rk}(P) = 4$ and Möbius function $\mu(P) = 1$, where each element $\alpha$ is labelled by the value of $\mu(\hat{0}, \alpha)$.

**Definition 3.1.** Given two words $\sigma, \pi$ on some alphabet $\Sigma$, we say that the subsequence $\pi_{a_1} \ldots \pi_{a_k}$ is a $(\rho, \kappa)$-occurrence of $\sigma$ in $\pi$ if $\pi_{a_1} \ldots \pi_{a_k} \sim_{\rho} \sigma$, where $\sim_{\rho}$ is some binary relation we call a pattern relation and $a_1, \ldots, a_k$ satisfies the position conditions $\kappa$. We call $(\rho, \kappa)$ a pattern pair.

**Example 3.2.** Some examples of pattern relations are:

- Two words are related by $\sim_\omega$ if they are equal, which gives subword order, see [Bjö90].

- Two words are related by $\sim_\delta$ if their letters appear in the same relative order of size, which gives the classical permutation patterns, see [MS15].

- Given a poset $Q$, two words $\sigma$ and $\pi$ are related by $\sigma \sim_{\omega(Q)} \pi$ if $\sigma_i \leq_{Q} \pi_i$ for all $i$, which gives generalised subword order, see [MS12].

Some examples of position conditions are:

- If $\kappa_\emptyset = \emptyset$, then there are no conditions on the positions of the subsequence.

- If $\kappa_c = \{\forall i : a_i + 1 = a_{i+1}\}$, we require that the subsequence occurs consecutively, such as in the consecutive permutation poset [BFS11, EM15].

- Given an infinite binary string $A$, if $\kappa_A = \{a_i + 1 = a_{i+1} | A_i = 1\}$ we get vincular pattern posets, which were introduced in [BF14].

We can use our notion of pattern relations to define a poset as follows:

**Definition 3.3.** Consider a pattern pair $(\rho, \kappa)$. Given $\sigma, \pi \in \Sigma^*$ we define a binary relation by $\sigma \leq_{\rho, \kappa} \pi$ if there is a $(\rho, \kappa)$-occurrence of $\sigma$ in $\pi$. If $\leq_{\rho, \kappa}$ is reflexive, antisymmetric and transitive, then we define a pattern poset $P(B, \rho, \kappa)$ as the poset with elements $B \subseteq \Sigma^*$ and partial order $\leq_{\rho, \kappa}$.

**Example 3.4.** Consider the pattern relations and position conditions defined in Example 3.2 and let $S$ and $A$ be the set of all permutations and set of all words on the positive integers, respectively.
• The poset $P(S, \delta, \kappa_\emptyset)$ is the classical permutation pattern poset, see [MS15].

• The poset $P(S, \delta, \kappa_c)$ is the consecutive permutation pattern poset, see [BFS11, EM15].

• The poset $P(A, \omega, \kappa_\emptyset)$ is the subword order poset, see [Bjö90].

• If $N$ is the chain of natural numbers, then $P(A, \omega(N), \kappa_\emptyset)$ is the composition poset, see [SV06].

• If $B \subseteq A$ is the set of Dyck words, then $P(B, \omega, \kappa_\emptyset)$ is the Dyck pattern poset, see [BBF+14].

When it is clear what poset we are considering we drop the subscript and use the notation $\leq$.

**Definition 3.5.** We say a pattern poset $P$ is closed if for every $\pi \in P$ and for every position $1 \leq i \leq |\pi|$ there is a unique operation that changes only $\pi_i$ to create an element $\alpha \in P$ with $\alpha \leq \pi$, and we call this operation decreasing position $i$. We say a pattern poset $P$ is an equivalence pattern poset if for every $\pi \in P$ every subword of $\pi$ is an occurrence of at most one element of $P$.

Here we give a list of known pattern posets and whether they are closed or equivalence posets:

**Lemma 3.6.**

• The subword order poset is a closed equivalence pattern poset.

• The classical permutation poset is a closed equivalence pattern poset.

• The consecutive permutation poset is a non-closed equivalence pattern poset.

• The Dyck pattern poset is a non-closed equivalence pattern poset.

• Consider the poset of words with generalised subword order $P_G$, where $G$ is the underlying poset. The pattern poset $P_G$ is closed if and only if $G$ is a rooted forest, and $P_G$ is an equivalence poset if and only if every element of $G$ has rank at most 1.

**Proof.** The equivalence results are straightforward to verify so we just check the closed condition. The poset of words with subword order and the classical permutation poset are closed, where the decreasing operation is deleting the letter. The consecutive pattern poset is non-closed because the decreasing operation is also deleting a letter but can only be applied to the first and last letters. The Dyck pattern poset is non-closed because we need to delete exactly two letters to get an element of rank one less. In the poset of words with generalised subword order the decreasing operation is decreasing the value of a letter according to the underlying poset, or deleting the letter if it is an atom of the underlying poset. So the decreasing operation is unique if and only if the underlying poset is a rooted forest. \qed
Note that in a closed equivalence pattern poset every subword of every element is an occurrence of exactly one element of the pattern poset. We use the word equivalence because the equivalence condition implies the pattern relation is an equivalence relation.

4. Poset Fibration

In this section we introduce a poset fibration which we use to study pattern posets in Section 5. Poset fibrations where first studied by Quillen in [Qui78] and have many nice properties. We can define a poset fibration using the embeddings of a pattern poset, where an embedding is defined as follows:

**Definition 4.1.** Consider a pattern poset $P$ and two elements $\alpha \leq \beta$ of $P$. An embedding of $\alpha$ in $\beta$ is a sequence of length $|\beta|$, consisting of the letters of $\alpha$ and dashes, such that the non-dash letters are exactly the positions of an occurrence of $\alpha$ in $\beta$ and removing the dashes results in $\alpha$.

Let $E^{\alpha,\beta}$ be the set of embeddings of $\alpha$ in $\beta$. Given an embedding $\eta \in E^{\alpha,\beta}$, we call the positions of the dashed letters in $\eta$ the empty positions and let the zero set of $\eta$, denoted $Z(\eta)$, be the set of empty positions in $\eta$.

**Example 4.2.** In the classical permutation poset $-23-1$ is an embedding of $231$ in $156243$. In the composition poset $121$ is an embedding of $121$ in $13211$.

Traditionally zeroes are used in embeddings instead of dashes, but we use dashes as this allows us to consider words that contain zeroes. We can define a poset of embeddings in the following way:

**Definition 4.3.** Consider any interval $[\sigma, \pi]$ in a pattern poset $P$. Let $A(\sigma, \pi)$ be the poset of all embeddings of $\lambda$ in $\pi$, for all $\lambda \in (\sigma, \pi)$. The partial order of $A(\sigma, \pi)$ is given by $\eta \leq \phi$ if $Z(\eta) \supseteq Z(\phi)$ and $\alpha \leq \beta$, where $\alpha$ and $\beta$ are the words obtained by removing the empty positions from $\eta$ and $\phi$, respectively. Let $A^*(\sigma, \pi)$ be the poset of all embeddings of $\lambda$ in $\pi$, for all $\lambda \in (\sigma, \pi)$. Let $f^P_{\pi} : A(\sigma, \pi) \rightarrow (\sigma, \pi)$ be the map that maps all embeddings of $\lambda$ in $\pi$ to $\lambda$.

**Example 4.4.** Consider the permutation pattern poset with $\pi = 243516$, then $-213-- \leq -213-4$ however $-2 -- \not\leq -213-4$.

So we have a poset fibration where $f^P_{\pi}$ is the projection map and $A(\sigma, \pi)$ is the total space. Where it is clear we often use the notation $f$, dropping the $P$ and $\pi$. In Section 5 we use this poset fibration to prove some results on pattern posets.

4.1. Another Poset Fibration for Closed Pattern Posets

The notion of normal embeddings plays an important role in many of the existing results on the Möbius function of pattern posets. For example, normal embeddings appear in results on the poset of words with subword order [Bjö90, Bjö93], the poset of words with composition order [SV06], the poset of words...
with generalised subword order \([SV06, MS12]\) and the classical permutation poset \([BJJS11, Smi14, Smi17, Smi16]\). We generalise these notions of normal embeddings and present a simple definition for a normal embedding for any closed pattern poset. This definition is different from some definitions in the literature but extracts the common aspect of all of them, and the variation is then accounted for in our formula for the Möbius function in Section 5. First we introduce adjacencies, sometimes called runs, which play an important role in defining a normal embedding.

**Definition 4.5.** Consider a closed pattern poset \(P := P(B, \rho, \kappa)\). An adjacency in an element \(\sigma \in P\) is a maximal sequence of consecutive positions such that decreasing any letter of the adjacency yields the same element relative to \(\sim_{\rho, \kappa}\). An adjacency of length 1 is trivial, the tail of a non-trivial adjacency is all but the first letter of the adjacency and trivial adjacencies have no tails.

**Example 4.6.** Consider the classical permutation poset. The permutation \(\pi = 2341657\) has adjacencies 234, 1, 65, 7 and the tails are 34 and 5. In the composition poset the only non-trivial adjacencies are sequences of 1’s.

Given any embedding \(\eta\) of \(\sigma\) in \(\pi\) in a closed pattern poset we can apply the inverse of the decreasing operation, which we call the *increasing operation*, to any letter of \(\eta\) to get an embedding of some element \(\beta\) in \(\pi\), with \(\sigma \prec \beta\). We say a position of \(\eta\) is full if we cannot apply the increasing operation to it. Moreover, we say a position is fillable if applying the increasing operation once to that position results in the position being full.

For example, a position is full in a closed equivalence pattern poset if and only if it is non-empty and fillable if only if it is empty. A position is full in the composition poset if \(\eta_i = \pi_i\) and fillable if \(\eta_i = \pi_i - 1\) or \(\eta_i = -1\) and \(\pi_i = 1\).

Using our definition of adjacency we can define a normal embedding, which appears frequently in the results on pattern posets:

**Definition 4.7.** Consider a closed pattern poset \(P := P(B, \rho, \kappa)\) and two elements \(\sigma, \pi \in P\). An embedding \(\eta\) of \(\sigma\) in \(\pi\) is normal if all the positions that are in a tail of any adjacency in \(\pi\) are full in \(\eta\) and all other positions are fillable. Let \(\text{NE}(\sigma, \pi)\) denote the number of normal embeddings of \(\sigma\) in \(\pi\).

An embedding \(\eta\) is representative if for every adjacency of \(\pi\) the corresponding letters in \(\eta\) have all the empty letters to the left, the full letters to the right and at most one non-full non-empty letter positioned between them. Let \(\hat{E}^{\sigma, \pi}\) be the set of representative embeddings of \(\sigma\) in \(\pi\).

**Example 4.8.** Consider the classical permutation poset. The embeddings of 213 in 231645 are:

\[
2 - 13 - - - 213 - - - 2 - 1 - 3 - - 21 - 3 - - 2 - 1 - - 3 - - 21 - 3
\]

The representative embeddings are \(-213 - - -\) and \(-21 - - 3\) and the only normal embedding is \(-21 - - 3\).
Note that in a closed equivalence pattern poset an embedding is representative if there is no empty position to the right of a non-empty position in the same adjacency, and normal if all positions in the tail of an adjacency are non-empty.

The definition of normal in Definition 4.7 is equivalent to the definition of normal in the poset of words with subword order, see [Bjo90], and the classical permutation poset, see [Smi17]. However, the definition is not equivalent to the definitions given for the poset of words with composition order [SV06] or generalised subword order [MS12]. In these cases our definition of normal embeddings gives a subset of the normal embeddings according to the previous definitions. We account for these differences in the formulas we present in Section 5.1.

Using the notion of representative embeddings we can define another poset fibration, with a smaller total space than the fibration defined in the previous subsection. This allows us to simplify many of the results presented in Section 5.

**Definition 4.9.** Consider an interval $[\sigma, \pi]$ of a closed pattern poset. Let $R(\sigma, \pi)$ and $R^*(\sigma, \pi)$ be the posets of all the representative embeddings of $\lambda$ in $\pi$, for all $\lambda \in (\sigma, \pi)$ and $\lambda \in [\sigma, \pi)$, respectively.

**Remark 4.10.** Consider an interval $[\sigma, \pi]$ of a closed pattern poset and any $\eta \in R^*(\sigma, \pi)$. To obtain an element that covers $\eta$ in $\hat{R}^*(\sigma, \pi)$ we must increase the rightmost non-full position of an adjacency, as increasing any other position would result in a non-representative embedding. So let $\hat{\eta}_i$ be the number of increasing operations required until every letter of the $i$'th adjacency is full and let $[0, \hat{\eta}_1] \times \cdots \times [0, \hat{\eta}_t]$ be the chain of integers from 0 to $\hat{\eta}_i$. The interval $[\eta, 1]$ in $\hat{R}^*(\sigma, \pi)$ is isomorphic to the product of chains $[0, \hat{\eta}_1] \times \cdots \times [0, \hat{\eta}_t]$.

The poset of representative embeddings $R(\sigma, \pi)$ is a subposet of the poset of all embedding $A(\sigma, \pi)$. So we can define a poset fibration onto $[\sigma, \pi]$ by restricting $f_P^\pi$ to $R(\sigma, \pi)$, which has the total space $R(\sigma, \pi)$ and the projection map $f_P^\pi|_{R(\sigma, \pi)}$. When the context is clear we simply use $f$ to denote the projection map.

5. Results on Pattern Poset

5.1. The Möbius Function of Intervals of a Pattern Poset

In this subsection we focus on the Möbius function of pattern posets. The following result, which is the dual of Corollary 3.2 in [Wal81], proves very useful:

**Proposition 5.1.** Given a poset fibration $f : P \to Q$:

$$\hat{\mu}(Q) = \hat{\mu}(P) + \sum_{q \in Q} \hat{\mu}(Q_{<q})\hat{\mu}(f^{-1}(Q_{\geq q})).$$

Applying Proposition 5.1 to the poset fibrations given in Section 4 gives the following results:
Theorem 5.2. If $[\sigma, \pi]$ is an interval of a pattern poset, then:

$$
\mu(\sigma, \pi) = \hat{\mu}(A(\sigma, \pi)) + \sum_{\lambda \in [\sigma, \pi]} \mu(\sigma, \lambda) \hat{\mu}(A^*(\lambda, \pi)) \tag{5.1}
$$

$$
= \sum_{\eta \in E^{\sigma, \pi}} \mu(\eta, \hat{1}) + \sum_{\lambda \in [\sigma, \pi]} \hat{\mu}(\sigma, \lambda) \mu(A^*(\lambda, \pi)). \tag{5.2}
$$

Proof. Applying Proposition 5.1 to the poset fibration given in Definition 4.3 gives Equation (5.1). The posets $A(\sigma, \pi)$ and $A^*(\sigma, \pi)$ can be considered as the union of the intervals $(\eta, \hat{1})$ and $[\eta, \hat{1})$, respectively, for all $\eta \in E^{\sigma, \pi}$. Applying an inclusion-exclusion argument for the Möbius function we get:

$$
\hat{\mu}(A(\sigma, \pi)) = \sum_{\eta \in E^{\sigma, \pi}} \mu(\eta, \hat{1}) + \sum_{S \subset E^{\sigma, \pi}} (-1)^{|S|} \hat{\mu}\left(\bigcap_{\eta \in S}(\eta, \hat{1})\right), \tag{5.3}
$$

$$
\hat{\mu}(A^*(\sigma, \pi)) = \sum_{\eta \in E^{\sigma, \pi}} \hat{\mu}([\eta, \hat{1})] + \sum_{S \subset E^{\sigma, \pi}} (-1)^{|S|} \hat{\mu}\left(\bigcap_{\eta \in S}(\eta, \hat{1})\right). \tag{5.4}
$$

Note that in Equation (5.4) we use the intersections of $(\eta, \hat{1})$ instead of $[\eta, \hat{1})$, because these are equivalent as $\eta$ will never be in the intersections. Moreover, $[\eta, \hat{1})$ has the unique bottom element $\eta$ and thus the Möbius number equals zero. Therefore, the second term on the right hand side of Equation (5.4) equals zero. Therefore, the second term on the right hand side of Equation (5.3) is equal to $\hat{\mu}(A^*(\sigma, \pi))$, so we have:

$$
\hat{\mu}(A(\sigma, \pi)) = \sum_{\eta \in E^{\sigma, \pi}} \mu(\eta, \hat{1}) + \hat{\mu}(A^*(\sigma, \pi)). \tag{5.5}
$$

Combining Equations (5.1) and (5.5) gives Equation (5.2).

Theorem 5.3. If $[\sigma, \pi]$ is an interval of a closed equivalence pattern poset, then:

$$
\mu(\sigma, \pi) = (-1)^{|\pi| - |\sigma|} E(\sigma, \pi) + \sum_{\lambda \in [\sigma, \pi]} \mu(\sigma, \lambda) \hat{\mu}(A^*(\lambda, \pi)). \tag{5.6}
$$

Proof. In a closed equivalence pattern poset the interval $[\eta, \hat{1})$, for any $\eta \in E^{\sigma, \pi}$, is a boolean lattice of rank $|\pi| - |\sigma|$, so has Möbius number $(-1)^{|\pi| - |\sigma|}$. To see this note in a closed equivalence pattern poset a letter in $\eta$ is fillable if and only if it is empty. So the interval $[\eta, \hat{1})$ contains all elements obtained by filling empty positions of $\eta$, of which there are $|\pi| - |\sigma|$, and so it is a boolean lattice of rank $|\pi| - |\sigma|$. Therefore, the result follows from Theorem 5.2.

Theorem 5.4. If $[\sigma, \pi]$ is an interval of a closed pattern poset, then:

$$
\mu(\sigma, \pi) = (-1)^{|\pi| - |\sigma|} NE(\sigma, \pi) + \sum_{\lambda \in [\sigma, \pi]} \mu(\sigma, \lambda) \hat{\mu}(R^*(\lambda, \pi)). \tag{5.7}
$$
Proof. If we consider the posets $R(\sigma, \pi)$ and $R^*(\sigma, \pi)$ and apply an analogous argument to that used in the proof of Theorem 5.2 to derive Equation (5.2), then we get the following equation:

$$
\hat{\mu}(R^*(\sigma, \pi)) = \sum_{\eta \in \hat{E}^{\sigma, \pi}} \hat{\mu}(\eta, \hat{1}) + \sum_{S \subseteq \hat{E}^{\sigma, \pi}, |S| > 1} (-1)^{|S|} \hat{\mu} \left( \bigcap_{\eta \in S} (\eta, \hat{1}) \right). \quad (5.8)
$$

By Remark 4.10 we know that $[\eta, \hat{1}]$ is the Cartesian product of chains, for any $\eta \in \hat{E}^{\sigma, \pi}$. Moreover, these chains all have length at most 1 if and only if $\eta$ is normal. If $\eta$ is normal then there are $|\pi| - |\sigma|$ chains of length 1, the rest having length 0. The Mőbius function of a chain is 1 if the chain has length 0, $-1$ if the chain has length 1 and 0 otherwise. Furthermore, the Mőbius function of the Cartesian product of posets is the product of the Mőbius functions. Therefore, $\mu(\eta, \hat{1})$ equals $(-1)^{|\pi| - |\sigma|}$ if $\eta$ is normal and 0 otherwise. So the first term on the right hand side of Equation (5.8) equals $(-1)^{|\pi| - |\sigma|} \text{NE}(\sigma, \pi)$, which completes the proof.

5.2. Disconnected Intervals of a Pattern Poset

In this subsection we study the property of disconnectedness in pattern posets. Proposition 5.3 of [MS15] gives a characterisation of when an interval of the classical permutation poset is disconnected, based on whether the set of embeddings can be split in a certain way, and we generalise this result to closed equivalence pattern posets. First note that in a closed equivalence pattern poset an embedding is uniquely determined by its zero set.

Definition 5.5. An interval $[\sigma, \pi]$ of a pattern poset, with $\text{rk}(\sigma, \pi) \geq 2$, is zero split (resp. rep-zero split) if the embedding set (resp. representative embedding set) can be split into two disjoint non-empty sets $E_1$ and $E_2$ such that $Z(E_1) \cap Z(E_2) = \emptyset$, where $Z(E_i)$ is the union of the zero sets of the elements of $E_i$. We call $E_1$ and $E_2$ a zero split partition of the embedding set. We say an interval of rank $k \leq 1$ is never zero split.

We say that an interval $[\sigma, \pi]$, with $\text{rk}(\sigma, \pi) \geq 2$, is strongly zero split if there exists a zero split partition $E_1$ and $E_2$ of $\hat{E}^{\sigma, \pi}$ which satisfies the following condition: For all $\eta_1 \in E_1$ and $\eta_2 \in E_2$ there does not exist a pair $z_1 \in Z(\eta_1)$ and $z_2 \in Z(\eta_2)$ such that the embeddings in $\pi$ with zero sets $Z(\eta_1) \setminus \{z_1\}$ and $Z(\eta_2) \setminus \{z_2\}$ are embeddings of the same element $\lambda$ in $\pi$.

Example 5.6. Consider the interval $[41253, 41627385]$ of the classical permutation poset. The embeddings are $\eta_1 = 41 - 253 - -$ and $\eta_2 = --41 - 253$. If we partition the embeddings into the sets $E_1 = \{\eta_1\}$ and $E_2 = \{\eta_2\}$, then this is a zero split partition but not a strongly zero split partition. To see this is not a strongly zero split partition, note that $Z(\eta_1) = \{3, 7, 8\}$ and $Z(\eta_2) = \{1, 2, 5\}$. The embeddings with zero sets $Z(\eta_1) \setminus \{3\}$ and $Z(\eta_2) \setminus \{5\}$ are 415263 and 415263, respectively, which are both embeddings of 415263. Therefore, the condition for a strongly zero split partition is violated.
Remark 5.7. Any partition of the embeddings of a rank 1 interval is a zero split partition, however we assume rank 1 intervals to be non-zero split. The reason for this is that we are interested in zero split partitions because they imply disconnectivity, however a rank 1 interval has an empty interior so cannot be disconnected. Therefore, for consistency it makes sense to assume all rank 1 intervals are not zero split.

Next we give some properties of being zero split in relation to the posets defined in Section 4. In an interval \([\sigma, \pi]\) of a closed equivalence pattern poset the join of any two embeddings \(\alpha, \beta \in A^*(\sigma, \pi)\) is given by the embedding with the zero set \(Z(\alpha) \cap Z(\beta)\). We can use this to show that an interval \([\sigma, \pi]\) being zero split is intrinsically related to the connectedness of the posets \(A(\sigma, \pi), A^*(\sigma, \pi)\) and \([\sigma, \pi]\). Note that given any embedding \(\eta \in E^{\sigma, \pi}\) there is a unique representative embedding \(rp(\eta) \in \hat{E}^{\sigma, \pi}\) obtained by moving all empty position to the left and full positions to the right in each adjacency.

Lemma 5.8. Consider an interval \([\sigma, \pi]\) of a closed equivalence pattern poset, with \(rk(\sigma, \pi) \geq 2\), then the following conditions are equivalent:

1. \([\sigma, \pi]\) is zero split,
2. \(A^*(\sigma, \pi)\) is disconnected,
3. \([\sigma, \pi]\) is rep-zero split,
4. \(R^*(\sigma, \pi)\) is disconnected,

Furthermore, if \(rk(\sigma, \pi) \geq 3\), then the above conditions are equivalent to:

5. \(A(\sigma, \pi)\) is disconnected.
6. \(R(\sigma, \pi)\) is disconnected.

Proof. Case (1) \(\implies\) (2). Suppose that \([\sigma, \pi]\) is zero split with the partition \(E_1\) and \(E_2\) of \(E^{\sigma, \pi}\). Let \(P_1\) and \(P_2\) be the elements of \(A^*(\sigma, \pi)\) that contain an element of \(E_1\) and \(E_2\), respectively. Note that any two atoms \(\eta_1 \in E_1\) and \(\eta_2 \in E_2\) have \(Z(\eta_1) \cap Z(\eta_2) = \emptyset\), so their join is \(\hat{1}\). Therefore, \(P_1\) and \(P_2\) are disconnected components of \(A^*(\sigma, \pi)\).

Case (2) \(\implies\) (1). Suppose \(A^*(\sigma, \pi)\) is disconnected with components \(P_1\) and \(P_2\), which have atoms \(E_1\) and \(E_2\), respectively. The join of any elements \(\eta_1 \in E_1\) and \(\eta_2 \in E_2\) equals \(\hat{1}\) which implies that the intersection of their zero set is empty. Moreover, because this is true for any pair, it implies \(E_1\) and \(E_2\) form a zero split partition of \(E^{\sigma, \pi}\).

Case (1) \(\implies\) (3): Suppose \([\sigma, \pi]\) is zero split with the zero split partition \(E_1\) and \(E_2\). Let \(r(E_i)\) be obtained by removing the non representative embeddings from \(E_i\). We claim the sets \(r(E_1)\) and \(r(E_2)\) form a rep-zero split partition of the set of representative embeddings. We know that the intersection of zero sets is empty because \(Z(r(E_1)) \cap Z(r(E_2)) \subseteq Z(E_1) \cap Z(E_2) = \emptyset\). However, we must check that \(r(E_1)\) and \(r(E_2)\) are non-empty. Without loss of generality
suppose \( r(E_1) \) is empty, then \( E_1 \) contains no representative embeddings. So for any \( \eta \in E_1 \), we have \( \text{rp}(\eta) \in E_2 \). This is only possible if there is exactly one empty position in \( \eta \), which contradicts \( |\pi| - |\sigma| \geq 2 \). Therefore, \( r(E_1) \) must be non-empty, so \( r(E_1) \) and \( r(E_2) \) is a valid rep-zero split partition of \( \hat{E}^{\sigma, \pi} \).

Case (3) \( \Rightarrow \) (1): Suppose \([\sigma, \pi]\) is rep-zero split, with rep-zero split partition \( E_1 \) and \( E_2 \). Let \( B_i = \{ \eta \in E^{\sigma, \pi} | \text{rp}(\eta) \in E_i \} \), for \( i = 1, 2 \). Suppose there exists a pair \( \eta \in B_1 \) and \( \phi \in B_2 \) such that \( Z(\eta) \cap Z(\phi) \neq \emptyset \). This implies \( Z(\text{rp}(\eta)) \cap Z(\text{rp}(\phi)) \neq \emptyset \). However, because \( \text{rp}(\eta) \in E_1 \) and \( \text{rp}(\phi) \in E_2 \), this contradicts \( E_1 \) and \( E_2 \) being a valid rep-zero split partition. Therefore, \( Z(B_1) \cap Z(B_2) = \emptyset \) so \( B_1 \) and \( B_2 \) form a zero split partition.

Case (3) \( \iff \) (4): This follows by arguments analogous to (1) \( \Rightarrow \) (2) and (2) \( \Rightarrow \) (1).

Cases (2) \( \Rightarrow \) (5) and (4) \( \Rightarrow \) (6): These follow trivially because \( A(\sigma, \pi) \) (resp. \( R(\sigma, \pi) \)) is obtained from \( A^*(\sigma, \pi) \) (resp. \( R^*(\sigma, \pi) \)) by removing the atoms.

Case (5) \( \Rightarrow \) (2): If \( A(\sigma, \pi) \) is disconnected then \( A^*(\sigma, \pi) \) is connected only if there is an embedding \( \eta \in E^{\sigma, \pi} \) contained in elements from both components of \( A(\sigma, \pi) \). However, as the interval \([\eta, 1]\) is a Boolean lattice, with rank greater than 2, it cannot be disconnected. Therefore, no such embedding exists so \( A^*(\sigma, \pi) \) is disconnected.

Case (6) \( \Rightarrow \) (4): This follows by a similar argument to that used in the case (5) \( \Rightarrow \) (2), where the only alteration is that \([\eta, 1]\) is a product of chains, so again it cannot be disconnected when it has rank greater than 2.

By Lemma 5.8, when considering closed equivalence pattern posets we can consider either zero splitness or rep-zero splitness. For simplicity we drop the rep prefix and simply refer to zero splitness, which can be checked by looking at either the embedding set or representative embedding set. We now use Lemma 5.8 to consider the disconnectivity of \([\sigma, \pi]\):

**Proposition 5.9.** Consider an interval \([\sigma, \pi]\) of a closed equivalence pattern poset, where \( \text{rk}(\sigma, \pi) \geq 3 \). The interval \([\sigma, \pi]\) is disconnected if and only if \([\sigma, \pi]\) is strongly zero split.

**Proof.** Suppose that \([\sigma, \pi]\) is strongly zero split with the partition \( E_1 \) and \( E_2 \) of \( E^{\sigma, \pi} \). Then \( A(\sigma, \pi) \) is disconnected by Lemma 5.8. So the only way that \([\sigma, \pi]\) is not disconnected is if there are two embeddings \( \kappa_1 \) and \( \kappa_2 \) in separate components of \( A(\sigma, \pi) \) such that \( f(\kappa_1) = f(\kappa_2) \), where \( f \) is the poset fibration map. First note that if \( f(\kappa_1) = f(\kappa_2) \), then for any \( \phi_1 \leq \kappa_1 \) there exists a \( \phi_2 \leq \kappa_2 \) such that \( f(\phi_1) = f(\phi_2) \). Therefore, we need only consider the case that \( \kappa_1 \) and \( \kappa_2 \) are atoms.

So suppose \( \kappa_1 \) and \( \kappa_2 \) are atoms with zero sets \( Z(\kappa_i) = Z(\eta_i) \setminus \{z_i\} \), where \( \eta_i \in E_i \) and \( z_i \in Z(\eta_i) \), for \( i = 1, 2 \). However, this implies that the embeddings with zero sets \( Z(\eta_1) \setminus \{z_1\} \) and \( Z(\eta_2) \setminus \{z_2\} \) are embeddings of the same element \( f(\kappa_1) \) in \( \pi \), which is exactly the forbidden situation in the definition of strongly zero split. Therefore, we cannot have elements from separate
components mapping to the same element, so \([\sigma, \pi]\) is disconnected. The other
direction follows by applying a similar argument in reverse.

The proof of Proposition 5.9 allows us to see what the disconnected
components of \([\sigma, \pi]\) look like when \([\sigma, \pi]\) is strongly zero split.

**Corollary 5.10.** If \([\sigma, \pi]\) is an interval of a closed equivalence pattern poset,
with \(\text{rk}(\sigma, \pi) \geq 3\), which is strongly zero split by the partition \(E_1\) and \(E_2\),
then \([\sigma, \pi]\) is disconnected with components

\[ P_i = \{ \lambda \in (\sigma, \pi) \mid \lambda = \pi \setminus S \text{ for some } S \subset Z(E_i) \}, \quad \text{for } i = 1, 2, \]

where \(\pi \setminus S\) is obtained from \(\pi\) by removing the letters \(\pi_i\), for all \(i \in S\).

Applying Proposition 5.9 and Corollary 5.10 to the classical permutation
poset implies Proposition 5.3 of [MS15]. Also note that it may be possible
to derive results similar to those in this section for non-closed or non-equivalence
pattern posets, which we leave as an open problem.

5.3. Cohen-Macaulayness Preserved by a Poset Fibration

A poset is **Cohen-Macaulay** if the order complex of every interval of the
poset is homotopically equivalent to a wedge of top dimensional spheres. Given a
poset \(P\) and an element \(p \in P\) define the induced subposet \(P_\leq p = \{ q \in P \mid q < p \}\)
and similarly define \(P_\geq p, P_<_p\) and \(P_>_p\). It was first shown in [Qui78] that the
Cohen-Macaulay property is preserved across a poset fibration \(f : P \to Q\) if the
sets \(f^{-1}(Q_{\geq q})\), known as the fibres, satisfy certain conditions. The following is a
variation of these results and is the dual pure form of Theorem 5.2 of [BWW05]:

**Proposition 5.11.** Let \(P\) and \(Q\) be pure posets and let \(f : P \to Q\) be a
poset fibration. Assume that for all \(q \in Q\) there is some \(p_q \in P\) such that
\(f^{-1}(Q_{<q}) = P_{<p_q}\) and \(f^{-1}(Q_{\geq q})\) is Cohen-Macaulay. If \(P\) is Cohen-Macaulay,
then \(Q\) is Cohen-Macaulay.

We can alter the lower ideal condition of Proposition 5.11 to get the following
result:

**Proposition 5.12.** Let \(P\) and \(Q\) be pure posets and let \(f : P \to Q\) be a
poset fibration. Assume that for all \(q \in Q\) there is some \(p_q \in P\) such that
\(Q_{<q} = f(P_{<p_q})\) and \(f^{-1}(Q_{\geq q})\) is Cohen-Macaulay. If \(P\) is Cohen-Macaulay,
then \(Q\) is Cohen-Macaulay.

**Proof.** Consider the posets \(Q^i = Q_{\leq i} \cup P_{>i}\) and maps \(f_i : Q^{i-1} \to Q^i\) where:

\[
\alpha \leq_{Q^i} \beta \iff \begin{cases} 
\alpha \leq_Q \beta, & \text{and } \text{rk}(\alpha), \text{rk}(\beta) \leq i \\
\alpha \leq_P \beta, & \text{and } \text{rk}(\alpha), \text{rk}(\beta) > i \\
\alpha \leq_Q f(\beta), & \text{and } \text{rk}(\alpha) \leq i, \text{rk}(\beta) > i 
\end{cases},
\]

\[f_i(q) = \begin{cases} 
q, & \text{if } \text{rk}(q) \neq i \\
f(q), & \text{if } \text{rk}(q) = i 
\end{cases}.
\]
Note that $Q^0 = P$ and $Q^{rk(Q)} = Q$, so $Q^0$ is Cohen-Macaulay by our assumption and we proceed by an inductive argument. Assume $Q^{i-1}$ is Cohen-Macaulay, for some $i > 0$, and consider $Q^i$. We apply Proposition 5.11 to $f_i$. Given any $q \in Q^i$, if $rk(q) \neq i$ then $f_i^{-1}(Q^i_{\geq q})$ is $[q, \hat{1}]$ in $Q^{i-1}$, which is an interval of a Cohen-Macaulay poset and so is Cohen-Macaulay. If $rk(q) = i$ then $f_i^{-1}(Q^i_{\geq q}) = f^{-1}(Q_{\geq q})$ which we assumed to be Cohen-Macaulay. Furthermore, if $rk(q) \neq i$, then $f_i^{-1}(Q^i_{< q}) = f^{-1}(P_{< q})$, which we assumed to be Cohen-Macaulay. So the conditions of Proposition 5.11 are satisfied for $f_i$ which implies $Q^i$ is Cohen-Macaulay. Therefore, by induction $Q$ is Cohen-Macaulay.

We can use Proposition 5.12 to consider the Cohen-Macaulay property on pattern poset. First we note that the lower ideal condition of Proposition 5.12 is always satisfied for pattern posets.

**Lemma 5.13.** Let $[\sigma, \pi]$ be an interval of a pattern poset along with the poset fibration $f : A(\sigma, \pi) \to [\sigma, \pi]$, we have $[\sigma, \lambda] = f(A(\sigma, \pi)_{< \ell})$, for every $\ell \in f^{-1}(\lambda)$.

**Proof.** First note that $f^{-1}(\lambda) = E^\lambda : \pi$, so we need to show that given any element $\kappa \in [\sigma, \lambda]$ and $\ell \in E^\lambda : \pi$ there is an embedding of $\kappa$ in $\pi$ that is contained in $\ell$. Let $\phi$ be an embedding of $\kappa$ in $\lambda$ and create an embedding $\psi$ by replacing the non-empty positions of $\ell$ with $\phi$. So $\psi$ is an embedding of $\kappa$ in $\pi$ and clearly $\kappa \leq \ell$. This completes the proof.

**Corollary 5.14.** Let $[\sigma, \pi]$ be an interval of a closed pattern poset along with the poset fibration $f : R(\sigma, \pi) \to [\sigma, \pi]$, we have $[\sigma, \lambda] = f(R(\sigma, \pi)_{< \ell})$, for every $\ell \in f^{-1}(\lambda)$.

So applying Proposition 5.12 and Lemma 5.13 implies the following result:

**Theorem 5.15.** Let $[\sigma, \pi]$ be an interval of a pure pattern poset such that $A^*(\lambda, \pi)$ is Cohen-Macaulay for all $\lambda \in (\sigma, \pi)$. If $A(\sigma, \pi)$ is Cohen-Macaulay, then so is $[\sigma, \pi]$.

**Corollary 5.16.** Let $[\sigma, \pi]$ be an interval of a closed pattern poset such that $R^*(\lambda, \pi)$ is Cohen-Macaulay for all $\lambda \in (\sigma, \pi)$. If $R(\sigma, \pi)$ is Cohen-Macaulay, then so is $[\sigma, \pi]$.

A Cohen-Macaulay poset cannot contain a disconnected subposet of rank greater than 2. Therefore, Proposition 5.9 implies the following result:

**Corollary 5.17.** If $[\sigma, \pi]$ is an interval of a closed equivalence pattern poset and contains a strongly zero split subinterval of rank greater than 2, then $[\sigma, \pi]$ is not Cohen-Macaulay, thus not shellable.
5.4. Shellability Preserved by a Poset Fibration

We conjecture that Propositions 5.11 and 5.12, and also Theorem 5.1(i) of [BWW05], can be generalised to show that shellability is also preserved across a poset fibration. First we give the formal definition of CL-shellability.

**Definition 5.18.** Given a bounded poset $P$ a rooted interval is an interval $[\alpha, \beta]$ of $P$ and a chain $c$ from $\hat{0}$ to $\alpha$, and is denoted $[\alpha, \beta]_c$. We similarly define a rooted element $\alpha_c$. A chain-edge labelling of a poset assigns an integer $\lambda(a \lessdot b, c)$ to each edge $a \lessdot b$ and chain $c$ from $\hat{0}$ to $a$, such that if two maximal chains coincide along their bottom $d$ edges, then their labels also coincide along these edges.

A chain is increasing in a chain-edge labelling, if the labels read from bottom to top are strictly increasing and a chain is decreasing if the labels are weakly decreasing. A chain-edge labelling of $P$ is a CL-labelling if every rooted interval $[\alpha, \beta]_c$ of $P$ has a unique increasing maximal chain and this chain lexicographically precedes all other chains.

Given any chain $c$ let $c_i$ denote the element with rank $i$ in $c$, let $c_{<i}$ denote the chain of all elements of rank less than $i$ in $c$ and let $c \cdot a$ denote the chain $c$ concatenated with the element or chain $a$. Let $at^P(\alpha)$ denote the set of atoms of $[\alpha, 1]$ in $P$. It was shown in [BW83] that a poset is CL-shellable if and only if it admits a recursive atom ordering, which is defined as follows:

**Definition 5.19.** A bounded poset $P$ is said to admit a recursive atom ordering (RAO) if there is an ordering $a_1, \ldots, a_t$ of $at^P(\alpha)$ for every rooted interval $[\alpha, 1]_c$ that satisfies:

(R1) If $r = rk(\alpha) \geq 1$, then the elements of $\Omega^P_c(\alpha)$ must appear first in the ordering, where $\Omega^P_c(\alpha)$ contains the elements of $at^P(\alpha)$ which cover an element ordered before $\alpha$ in the ordering of the atoms of $[c_{r-1}, 1]_{c<r}$.

(R2) For all $i < j$ if $a_i, a_j < y$, then there is a $k < j$ and an atom $z \in at^P(a_j)$ such that $y \geq z > a_k$.

We drop the subscripts and superscripts from $\Omega^P_c(\alpha)$ and $at^P(\alpha)$ when the context is clear.

**Remark 5.20.** An RAO induces a CL-labelling on the poset in the following way. Consider a rooted element $b_c$ and an ordering $\beta_1, \ldots, \beta_k$ of $at(b)$. Let $i$ be the largest integer such that $\beta_i \in \Omega_c(b)$ and let $r = rk(b) - 1$, and define the labelling

$$
\lambda(b \lessdot \beta_j, c) = \begin{cases} 
\lambda(c_r \lessdot b, c_{\leq r}) - (j - i) - 1, & \text{if } j \leq i, \\
\lambda(c_r \lessdot b, c_{\leq r}) + j - i, & \text{if } j > i.
\end{cases}
$$

So the elements of $\Omega_c(b)$ have labels less than the previous label in the chain and all other elements have labels greater than the previous label in the chain. This labelling is shown to be a CL-labelling in Theorem 3.2 in [BW83].
So a chain is increasing if at every step $c_i < c_{i+1}$ we have that $c_{i+1}$ does not cover any element ordered before $c_i$ in the ordering of $\text{at}(c_{i-1})$. And similarly a chain is decreasing if at every step $c_{i+1}$ does cover an element ordered before $c_i$. Note that the Möbius function of a shellable poset equals the number of decreasing chains, with the sign given by the rank.

We refer the reader to [Wac07] for further information on these definitions. We now present the main conjecture in this section that generalises Propositions 5.11 and 5.12 above and Theorem 5.1(i) of [BWW05]:

**Conjecture 5.21.** Let $P$ and $Q$ be pure posets, $f : P \rightarrow Q$ be a poset fibration and assume that $f^{-1}(Q \geq q)$ is shellable, for all $q \in Q$.

(a) Suppose there is some $p_q \in P$ such that $f^{-1}(Q < q) = P < p_q$. If $P$ is shellable, then $Q$ is shellable.

(b) Suppose there is some $p_q \in P$ such that $Q < q = f(P < p_q)$. If $P$ is shellable, then $Q$ is shellable.

(c) If $Q$ is shellable, then $P$ is shellable.

Note that in Conjecture 5.21 we consider the fibres $f^{-1}(Q \geq q)$ rather than $f^{-1}(Q > q)$, however if $f^{-1}(Q \geq q)$ is shellable then $f^{-1}(Q > q)$ is shellable by [Bjö80, Theorem 4.1], because the latter is obtained by deleting all atoms from the former. Therefore, we can replace $f^{-1}(Q > q)$ with $f^{-1}(Q \geq q)$ in Conjecture 5.21 if required. If Conjecture 5.21(b) is true, then by Lemma 5.13 the following results would follow immediately:

**Conjecture 5.22.** Consider an interval $[\sigma, \pi]$ of a pure pattern poset $P$.

(a) If $A(\lambda, \pi)$ is shellable for all $\lambda \in [\sigma, \pi)$, then $[\sigma, \pi]$ is shellable.

(b) If $P$ is a closed pattern poset and $R(\lambda, \pi)$ is shellable for all $\lambda \in [\sigma, \pi)$, then $[\sigma, \pi]$ is shellable.

The main difficulty in proving the first two parts of Conjecture 5.21 lies in the fact that the labellings of the posets and the fibres do not necessarily coincide. If we restrict to a case where these labellings do coincide then we are able to prove the result.

A linear order $\prec$ of $P$ is an ordering of all elements of $P$. We say that a linear order $\prec$ induces an RAO on $P$ if the ordering of $\text{at}(\alpha)$ by $\prec$ satisfies the RAO conditions for all rooted intervals $[\alpha, 1]_c$ of $P$. Given a poset fibration $f : P \rightarrow Q$ define $r^*_f(q)$ as the earliest element of $f^{-1}(q)$ in the linear ordering $\prec$ and define $\prec_f$ as the linear order on $Q$ where $a \prec_f b$ if and only if $r^*_f(a) \prec r^*_f(b)$. Moreover, given a chain $c$ in $Q$ let $r^*_f(c)$ be the chain $r^*_f(c_1) \prec r^*_f(c_2) \prec \cdots$. We drop the subscripts and superscripts from $r^*_f(q)$ when the context is unambiguous.

A linear order can be used to induce an RAO on our posets which ensures that the shellings of the posets and fibres coincide. This leads to the following result which is a restricted version of Conjecture 5.21(a).
Proposition 5.23. Let $P$ and $Q$ be pure posets, $f : P \to Q$ be a poset fibration and $\prec$ a linear ordering of $P$. Suppose that $f^{-1}(Q_{<q}) = P_{<r(q)}$ and $\prec$ induces an RAO on $f^{-1}(Q_{>q})$, for all $q \in Q$. If $\prec$ induces an RAO on $P$, then $\prec_f$ induces an RAO on $Q$, thus $Q$ is shellable.

Proof. To show this we check that both conditions of an RAO are satisfied given any rooted interval $[a, \hat{i}]$, in $Q$. First we check Condition (R1). Consider $a_i \in \Omega^Q_{\alpha}$ and $a_j \not\in \Omega^Q_{\alpha}$ and let $\lambda$ be the element ordered before $\alpha$ that is covered by $a_i$. By the condition $f^{-1}(Q_{<a_i}) = P_{<r(a_i)}$ we know that $r(\lambda) \prec r(a_i)$ and it is straightforward to see $r(\lambda) < r(\alpha)$, therefore $r(a_i) \in \Omega^P_{r(\alpha)}(r(\alpha))$. Moreover, by a similar argument, $r(a_j) \not\in \Omega^P_{r(\alpha)}(r(\alpha))$. Therefore, $r(a_i) \prec r(a_j)$ so $a_i \prec_f a_j$ and thus Condition (R1) is satisfied.

Now we check Condition (R2). Consider two elements $a_i \prec_f a_j$ in $a\Omega^Q_{\alpha}$ and some $y > a_i, a_j$. So, $r(a_i)$ and $r(a_j)$ are atoms of $f^{-1}(Q_{>a})$ and by the condition $f^{-1}(Q_{<\lambda}) = P_{<r(\lambda)}$ we know that $r(\lambda) > r(a_i), r(a_j)$. Moreover, because $\prec$ induces an RAO on $f^{-1}(Q_{>a})$, there exists an atom $\hat{a}$ of $f^{-1}(Q_{>a})$ and element $z \in Q$ with $\hat{a} \prec r(a_j)$ and $\hat{a} \prec z \leq y$. Therefore, $f(\hat{a}) \prec_f a_j$ and $f(\hat{a}) \prec f(z) \leq y$, so Condition (R2) is satisfied. 

We also get a restricted case of Conjecture 5.21(b), whose proof we omit, as it follows by an argument analogous to that used in the proof of Proposition 5.12.

Proposition 5.24. Let $P$ and $Q$ be pure posets, $f : P \to Q$ a poset fibration and $\prec$ a linear ordering of $P$. Suppose that $Q_{<q} = f(P_{<r(q)})$ and $\prec$ induces an RAO in $f^{-1}(Q_{>q})$, for all $q \in Q$. If $\prec$ induces an RAO in $P$, then $\prec_f$ induces an RAO in $Q$, thus $Q$ is shellable.

Proposition 5.24 and Lemma 5.13 imply the following restricted version of Conjecture 5.22:

Theorem 5.25. Consider an interval $[\sigma, \pi]$ of a pure pattern poset $P$.

(a) If $A(\sigma, \pi)$ has a linear order which induces an RAO on $A(\sigma, \pi)$ and $A^*(\lambda, \pi)$ for all $\lambda \in (\sigma, \pi)$, then $[\sigma, \pi]$ is shellable.

(b) Suppose $P$ is a closed pattern poset. If $R(\sigma, \pi)$ has a linear order which induces an RAO on $R(\sigma, \pi)$ and $R^*(\lambda, \pi)$ for all $\lambda \in (\sigma, \pi)$, then $[\sigma, \pi]$ is shellable.

6. Applications

In this section we use the results from Section 5 to examine the poset of words with subword order and the consecutive permutation poset. First we introduce a lemma that proves useful. To show that $R^*(\sigma, \pi)$ and $R(\sigma, \pi)$ are shellable using a recursive atom ordering does not require that we check every rooted interval $[\lambda, \hat{i}]$. In fact it suffices to prove there is an ordering of the atoms of $R^*(\sigma, \pi)$ which satisfies Condition (R2).
Lemma 6.1. Consider an interval $[\sigma, \pi]$ of a closed pattern poset $P$. If there is an ordering $\prec$ of $E^{\sigma, \pi}$ which satisfies Condition (R2), then $R^*(\sigma, \pi)$ and $R(\sigma, \pi)$ are shellable. Moreover, if $P$ is also an equivalence pattern poset, then $\hat{\mu}(R^*(\sigma, \pi)) = (-1)^{|\pi| - |\sigma| - 1}|V(\sigma, \pi)|$, where $V(\sigma, \pi)$ is the set of embeddings $\eta \in \hat{E}^{\sigma, \pi}$ such that $at(\eta) = \Omega(\eta)$.

Proof. Given any $\alpha \in R^*(\sigma, \pi)$ we refer to the filling position of each element $\beta \in at(\alpha)$ as the position of the letter increased to get from $\alpha$ to $\beta$. First we show that $R^*(\sigma, \pi)$ is shellable. By Remark 4.10 we know that $[\alpha, \hat{1}]$ is isomorphic to a product of chains. Therefore, every pair in $at(\alpha)$ are covered by their join, so any ordering of the atoms satisfies Condition (R2). Define an RAO on $R^*(\sigma, \pi)$ in the following way. Consider any rooted element $\phi \in R^*(\sigma, \pi)$. If $rk(\phi) = 0$, then order $at(\phi)$ according to $\prec$. If $rk(\phi) = 1$ order the elements of $\Omega(\phi)$ in increasing order of the filling position and then the remaining elements in any order. If $rk(\phi) > 1$, then order $at(\phi)$ by the order of the filling positions induced by the ordering of $at(c_1)$. It is straightforward to see that this ordering satisfies Conditions (R1) and (R2), so we have an RAO of $R^*(\sigma, \pi)$, so it is shellable. Furthermore, $R(\sigma, \pi)$ is obtained by removing the atoms of $R^*(\sigma, \pi)$, and so $R(\sigma, \pi)$ is shellable by [BW83, Theorem 8.1].

Now we show the M"obius function result. Using the RAO we defined on $R^*(\sigma, \pi)$ a chain $c$ is decreasing if the empty positions of $c_1$ are filled in the reverse order of $at(c_1)$ and $c_2$ is in $\Omega(c_1)$, which implies every atom of $at(c_1)$ must be in $\Omega(c_1)$. So the number of decreasing chains is $|V(\sigma, \pi)|$.

Corollary 6.2. Consider an interval $[\sigma, \pi]$ of a closed equivalence pattern poset. If there is an ordering of $E^{\sigma, \pi}$ which satisfies Condition (R2), then $A^*(\sigma, \pi)$ and $A(\sigma, \pi)$ are shellable.

Proof. In a closed equivalence pattern poset $[\eta, \hat{1}]$ is isomorphic to a boolean lattice, for all $\eta \in E^{\sigma, \pi}$. Therefore, the proof follows by the same argument used to prove shellability in Lemma 6.1. □

6.1. Poset of Words With Subword Order

It was shown in [Bjö90] that any interval $[u, w]$ of the poset of words with subword order is shellable, thus Cohen-Macaulay, and the M"obius function equals the number of normal embeddings with sign given by the rank. In this section we give an alternative proof of Cohen-Macaulayness and the M"obius function result on this poset. Moreover, if Conjecture 5.21(a) is true then our proof also implies shellability.

Note that the poset of words with subword order is a closed equivalence pattern poset and that the definition of normal embedding given by Björner is equivalent to Definition 4.7 when applied to this poset. First we introduce some notation regarding an embedding $\eta \in R^*(u, w)$. Given a pair of positions $i$ and $j$ in $\eta$ that are empty and non-empty, respectively, then moving $i$ to $j$ means setting the position $i$ as empty and the position $j$ as non-empty.
Proposition 6.3. Consider an interval \([u, w]\) of the poset of words with subword order. The interval \([u, w]\) is Cohen-Macaulay and

\[\mu(u, w) = (-1)^{|w| - |u|} NE(u, w).\]

Proof. The poset of words with subword order is a closed equivalence pattern poset. So first we show that \(R(u, w)\) and \(R^*(u, w)\) are shellable. By Lemma 6.1 it suffices to order the elements of \(E^u.w\) in a way satisfying Condition (R2).

Define the position word of an embedding as the nonempty positions listed in increasing order and the order \(\prec\) on \(E^u.w\) as the lexicographic order on the position words. To show that \(\prec\) satisfies Condition (R2), consider any two embeddings \(\eta_i, \eta_j \in E^u.w\), with \(\eta_i \prec \eta_j\), and some \(y > \eta_i, \eta_j\). Let \(a\) (resp. \(b\)) be the leftmost non-empty position of \(\eta_i\) (resp. \(\eta_j\)) that is empty in \(\eta_j\) (resp. \(\eta_i\)). The \(a\)'th letter of \(\eta_i\) and \(b\)'th letter of \(\eta_j\) correspond to the same letter in \(u\). Therefore, moving \(b\) to \(a\) in \(\eta_j\) gives a valid embedding \(\eta_k\) with \(\eta_k \preceq \eta_j\). Moreover, let \(z \in at(\eta_j)\) be the embedding obtained by filling \(a\) in \(\eta_j\), then \(\eta_k < z \leq y\), so Condition (R2) is satisfied. So \([u, w]\) is Cohen-Macaulay by Corollary 5.16.

Next we consider the Möbius function result using Lemma 6.1. Consider any embedding \(\eta\) and the embedding \(\phi\) obtained by filling the rightmost empty position \(i\) of \(\eta\), then \(\phi \notin \Omega(\eta)\). To see this note that if \(\phi \in \Omega(\eta)\) then there is an element \(\psi \prec \eta\) with \(\psi < \phi\), where \(\psi\) is obtained from \(\eta\) by moving a letter \(j > i\) to \(i\). However, if \(j\) is in the same adjacency as \(i\), then \(\psi\) would not be representative, so there must be a letter with a different value between \(i\) and \(j\). However, this means we are moving \(j\) across a letter with a different value, so the order of the letters is different which implies \(\psi\) is not an embedding of \(u\). Therefore, \(\phi \notin \Omega(\eta)\), so \(\Omega(\eta) \neq at(\eta)\), for any \(\eta \in E^u.w\), so by Lemma 6.1 \(\mu(R^*(u, w)) = 0\). Moreover, as this is true for any interval \([u, w]\) the Möbius function result follows from Theorem 5.4. \(\square\)

6.2. Consecutive Permutation Poset

The Möbius function and topology of the consecutive permutation poset has been studied in [BFS11, SW12, EM15], and a formula for the Möbius function has been developed. To state this formula we first introduce some notation. A permutation is monotone if it is of the form \(12 \ldots n\) or \(n \ldots 21\). A permutation \(\sigma\) is a prefix of \(\pi\) if \(\pi_1 \ldots \pi_{|\sigma|}\) is an occurrence of \(\sigma\), similarly define a suffix, and a \(\sigma\) is bifix of \(\pi\) if it is both a prefix and suffix of \(\pi\). The exterior of \(\pi\), denoted \(x(\pi)\), is the longest bifix of \(\pi\) and the interior of \(\pi\), denoted \(i(\pi)\), is \(\pi_2 \ldots \pi_{|x(\pi)|-1}\).

Theorem 6.4. [EM15, Theorem 2.1] The Möbius function of any interval \([\sigma, \pi]\) of the consecutive pattern poset is:

\[\mu(\sigma, \pi) = \begin{cases} 
\mu(\sigma, x(\pi)), & \text{if } |\pi| - |\sigma| > 2 \text{ and } \sigma \leq x(\pi) \not\leq i(\pi) \\
1, & \text{if } |\pi| - |\sigma| = 2, \pi \text{ is non-monotone and } \sigma \in \{i(\pi), x(\pi)\} \\
(-1)^{|\pi| - |\sigma|}, & \text{if } |\pi| - |\sigma| < 2 \\
0, & \text{otherwise}
\end{cases}\]
The consecutive pattern poset is a non-closed equivalence pattern poset, so we can use Theorem 5.2 to provide an alternative proof of Theorem 6.4. So to compute \( \mu(\sigma, \pi) \) we need to know \( \mu(\eta, \hat{1}) \), for each \( \eta \in E^{\sigma, \pi} \), and \( \hat{\mu}(A^*(\lambda, \pi)) \), for all \( \lambda \in [\sigma, \pi] \). Note that given any embedding \( \eta \) of \( \sigma \) in \( \pi \) there are at most two positions that can be filled in \( \eta \), the positions immediately left and right of the occurrence. So any element \( \phi \geq \eta \) is obtained by a sequence of left/right fillings. We say an embedding \( \eta \) is a prefix embedding if the non-empty positions are the initial \( k \) positions, and similarly define a suffix embedding.

**Lemma 6.5.** Given any interval \([\sigma, \pi]\) of the consecutive permutation poset and embedding \( \eta \in E^{\sigma, \pi} \), we have

\[
\mu(\eta, \hat{1}) = \begin{cases} 
0, & \text{if } |\pi| - |\sigma| > 2, \\
0, & \text{if } |\pi| - |\sigma| = 2 \text{ and } \eta \text{ is a prefix or suffix embedding}, \\
(-1)^{|\pi| - |\sigma|}, & \text{otherwise}.
\end{cases}
\]

**Proof.** If \(|\pi| - |\sigma| < 2\) the result is trivial, so suppose \(|\pi| - |\sigma| \geq 2\). There are at most three elements of rank 2 in \([\eta, \hat{1}]\), obtained from \( \eta \) in the following way: \( \alpha_1 \) obtained by two left fillings, \( \alpha_2 \) obtained by two right fillings and \( \alpha_3 \) obtained by a left and right filling. It is straightforward to see that \( \mu(\eta, \alpha_1) = \mu(\eta, \alpha_2) = 0 \) and \( \mu(\eta, \alpha_3) = -1 \). Note that if \( \alpha_1 \) or \( \alpha_2 \) equal \( \hat{1} \) then \( \eta \) is a prefix or suffix embedding, so the case \(|\pi| - |\sigma| = 2\) is complete. If \(|\pi| - |\sigma| > 2\), consider any element \( \phi \) with rank greater than 2. If \( \phi \) does not contain \( \alpha_3 \), then it must be obtained by only filling left or only filling right positions, so \([\eta, \phi]\) is a chain and thus \( \mu(\eta, \phi) = 0 \). If \( \phi \) contains \( \alpha_3 \), then by a simple inductive argument it can be seen that \( \mu(\eta, \phi) = 0 \), because \( \alpha_3 \) contains all the elements \( \kappa \) with \( \mu(\eta, \kappa) \neq 0 \).

This completes the proof. \( \square \)

**Lemma 6.6.** Given any interval \([\sigma, \pi]\) of the consecutive permutation poset, we have

\[
\hat{\mu}(A^*(\sigma, \pi)) = \begin{cases} 
1, & \text{if } \sigma = x(\pi) \text{ and } \sigma \not\subseteq i(\pi), \\
0, & \text{otherwise}.
\end{cases}
\]

**Proof.** Given two embeddings \( \eta_1 \) and \( \eta_2 \), let \( i \) and \( j \) be the leftmost and rightmost positions, respectively, that are non-empty in \( \eta_1 \) or \( \eta_2 \). Then the join of \( \eta_1 \) and \( \eta_2 \) is obtained by setting positions \( i \) through \( j \) as non-empty and all other positions as empty. So if \( \sigma \) is not a bifix of \( \pi \) then the join of the atoms of \( A^*(\sigma, \pi) \) is less than \( \hat{1} \), so \( \hat{\mu}(A^*(\sigma, \pi)) = 0 \).

Next note that \( \pi \) cannot contain a non-exterior bifix not contained in the interior. To see this let \( \alpha \) be such a bifix, then \( \alpha \) is also a bifix of \( x(\pi) \), which means that \( \alpha \) occurs as the suffix of the prefix occurrence of \( x(\pi) \), which is in the interior, giving a contradiction.

Suppose \( \sigma \) is a bifix and let \( \eta_1 \) and \( \eta_2 \) be the prefix and suffix embeddings, respectively. The join of any set of \( k \geq 0 \) embeddings, that doesn’t contain both \( \eta_1 \) and \( \eta_2 \), contributes \((-1)^k\) to the \( \hat{\mu}(A^*(\sigma, \pi)) \) and every other element
contributes 0, by the Crosscut Theorem, see [Sta12, Corollary 3.9.4]. Therefore,
\[
\hat{\mu}(A^*(\sigma, \pi)) = -\sum_{S \subseteq E^{\sigma, \pi} \setminus \{\eta_1, \eta_2\}} (-1)^{|S|} + (-1)^{|S|+1} + (-1)^{|S|+1},
\]
(6.1)
where we get three terms from considering the sets $S$, $S \cup \{\eta_1\}$ and $S \cup \{\eta_2\}$. Equation (6.1) equals 1 if $E^{\sigma, \pi} = \{\eta_1, \eta_2\}$ and 0 otherwise, which completes the proof.

Combining Lemmas 6.5 and 6.6 and Theorem 5.2 provides an alternative proof of Theorem 6.4. Moreover, Theorem 4.3 of [EM15] states that an interval of the consecutive permutation poset is shellable if and only if it has no disconnected subintervals of rank greater than 2. It is straightforward to define a shelling on $A^*(\sigma, \pi)$ if $[\sigma, \pi]$ has no disconnected subintervals. So Theorem 5.15 can be used to provide an alternative proof of the Cohen-Macaulayness of these posets. Moreover, if Conjecture 5.22 is true we can provide an alternative proof of shellability.

7. Future work

We have introduced a general definition of a pattern poset and given some results that apply to these posets. In Section 6 we applied these results to two previously studied posets and showed that our results can provide alternative proofs for existing results on these posets. There are many other pattern posets, some of which have been previously studied and many of which have not, and applying the results and techniques we have presented here could be very helpful in the study of these posets. For example, very little is known of the Dyck path poset introduced in [BBF14], which seems to have many nice properties, such as the sign of the Möbius function being alternating. Can we apply some of the results we have introduced to learn more about this poset?

One particular pattern poset for which there are many open problems is the classical permutation poset. The results from Section 5.1 imply the main results of [Smi17], and the results from Section 5.2 imply some of the results in [MS15]. Whether we can apply the results from Sections 5.3 and 5.4 to determine the topology of intervals of the classical permutation poset is still open? We conjecture that if an interval $[\sigma, \pi]$ of the classical permutation pattern poset does not contain any zero split subintervals, then $[\sigma, \pi]$ is shellable. If we can find an ordering of the embeddings of such intervals which satisfies Condition (R2), then Lemma 6.1 would prove that these intervals are Cohen-Macaulay and allow us to compute the Möbius function of these intervals. Moreover, if Conjecture 5.22 is true, such an ordering would prove these intervals are shellable.

Our definition of normal is not equivalent to those given for the poset of words with composition order or for generalised subword order. Therefore, the results on the Möbius function of these posets given in [SV06] and [MS12] do not follow immediately from Theorem 5.4. However, it seems reasonable to hope that with some work one could apply Theorem 5.4 to provide an alternative
proof of these results. Moreover, such an alternative proof might provide further insight into the structure of these posets.

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