Strong solutions of SDEs with random and unbounded drifts

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ABSTRACT

In this paper, we are interested in the following forward stochastic differential equation (SDE)

\[ dX_t = b(t, \omega, X_t)dt + \sigma dB_t, \quad 0 \leq t \leq T, \quad X_0 = x \in \mathbb{R}, \]

where the coefficient \( b : [0, T] \times \mathbb{R} \times \Omega \rightarrow \mathbb{R} \) is Borel measurable and of linear growth in the second variable and is adapted. The driving noise \( B_t \) is a \( d \)-dimensional Brownian motion. We obtain the existence and uniqueness of a strong solution in the present situation where the drift is not necessarily deterministic. Let mention that the technique using drift transform based on the solution of the associated backward Kolmogorov equation is not directly applicable here since the system is non-Markovian. The method we rather use is purely probabilistic and relies on Malliavin calculus. As a byproduct, we obtain Malliavin differentiability of the solutions and provide an explicit representation for the Malliavin derivative.

KEYWORDS: Malliavin calculus, random drift, measurable drift, compactness criterion, explicit representation.

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1. Introduction

The first main result of the present paper concerns well-posedness of a class of stochastic differential equations of the form

\[ dX_t = b(t, \omega, X_t) dt + \sigma dB_t, \quad 0 \leq t \leq T, \quad X_0 = x \in \mathbb{R} \]  \quad (1.1)

when the drift coefficient \( b : [0, T] \times \Omega \rightarrow \mathbb{R} \) is merely measurable and of linear growth in the third variable and can be random (i.e. can depend on \( \omega \)), \( \sigma \in \mathbb{R}^d \). The driving noise \( B_t \) is the canonical process \( B_t(\omega) = \omega_t \) on the canonical space \( \Omega := C([0, T], \mathbb{R}^d) \) equipped with the Wiener measure \( P \) and the completion \((\mathcal{F}_t)_{t \in [0, T]}\) of the natural filtration of \( B \).

Since the work of Itô [20], it is well known that the SDE (1.1) admits a unique strong solution when the drift \( b \) is globally Lipschitz continuous and of linear growth. That is, there exists a unique (up to indistinguishably) square integrable process that is \( \mathcal{F} \)-adapted and satisfies (1.1). SDEs are widely applied in stochastic control, in physics, and as a modeling tool, in a number of applied sciences including biology, finance and engineering. Often, the Lipschitz continuity condition is too stringent, as for instance in modeling of switching systems (see e.g. Delong [8] and Heikkilä and Lakshmikantham [19]) or in models of interacting finite (or infinite) particle systems (see e.g. Kondratiev et al. [25] and Albeverio et al. [1]) where the drift \( b \) is typically discontinuous. While existence of weak solutions of (1.1) is a direct consequence of Girsanov theory, the construction of strong solutions is usually a delicate matter. Note however that in the above mentioned applications existence of a solution \( X \) as function of the driving noise (i.e. strong solution) is crucial.

Strong solutions of SDEs with rough coefficients have been extensively studied in the past decades, starting with the seminal works by Zvonkin [40] and Veretennikov [39] and including other important contributions e.g. by Gyöngy and Krylov [17], Gyöngy and Martinez [18] and Krylov and Röckner [26] (see also Fredrizzi and Flandoli [16]). These works eventually build on the analysis of the Kolmogorov partial differential equation associated to the SDE or on pathwise
uniqueness arguments and benefit from the Yamada-Watanabe theorem.

Let us further refer to works by Fang and Zhang [12], Fang et al. [13] and more recently Champagnat and Jabin [5] on uniqueness of SDEs. See also Davie [7] for a path by path uniqueness result. A purely probabilistic approach, initiated by Proske [37] and Meyer-Brandis and Proske [31], and further developed by Menoukeu-Pamen et al. [29] rather uses the Malliavin calculus of variations and white noise analysis for the construction of solutions (see also Banos et al. [2]). This method does not rely on pathwise uniqueness arguments, but rather derives it as a consequence of uniqueness in law and strong existence.

As a common feature in the aforementioned works, the drift coefficient $b$ is assumed bounded and deterministic (i.e. not depending on $\omega$) and sometimes time-independent. This is due to the need to guarantee a Markovian property of the solution, which is paramount for the success of the PDE methods (in finite dimension). Random drift also constitute a clear impediment to the success of the probabilistic method since it crucially uses the property that the Malliavin derivative $DX$ of $X$ solves an ordinary differential equation where $DX$ is the only source of randomness. Regarding the growth of the drift coefficient, let us mention that to the best of our knowledge, the only works considering SDEs with discontinuous and unbounded drifts are the articles by Engelbert and Schmidt [11], Nilssen [33] and Menoukeu-Pamen and Mohammed [28] all considering deterministic coefficients.

In this paper we consider SDEs with coefficients $b$ of the form

$$b(t, \omega, x) = b_1(t, x) + b_2(t, \omega)$$

(1.2)

for some adapted (not necessarily bounded) stochastic process $b_2$, with $b_1$ a Borel function of spatial linear growth. In particular, $b_2$ is possibly path-dependent. When $b_1$ is the gradient of a given function, such SDEs can be seen as dynamics of a diffusion in a random potential see e.g. Kondratiev et al. [25].

SDEs with random coefficients when the drift coefficient does not have the special structure (1.2) have been studied in the literature. For example, the authors in Ocone and Pardoux [35] use the generalise Itô-Ventzell formula for anticipating integrands to study a Stratonovich-type SDE, where the initial condition and drift coefficient are allowed to anticipate the future of driving Brownian motion. They show that the Stratonovich-type SDE with anticipating coefficients has a unique non exploding Malliavin differentiable solution. They assume that the initial condition and drift coefficient are Malliavin smooth and the drift is further sublinear with respect to the spatial coordinate and has derivatives of polynomial growth. Assuming that the drift coefficient satisfies a stochastic Lipschitz condition, Kohatsu-Higa et al. [24] show existence and uniqueness of a class of SDE with random coefficient. They do not prove Malliavin differentiability of the solution in their work.

Our method draws from the Malliavin calculus approach of Proske [37] and Menoukeu-Pamen et al. [29] but avoids the use of white noise analysis. In particular, we prove existence and uniqueness of a strong solution and further derive Malliavin differentiability and non-explosion of the solution. The main difficulties in deriving Malliavin smoothness of the solution to the SDE comes from the fact that we do not require any spatial smoothness of the coefficient. We stress that Malliavin differentiability of solutions is an important additional feature which may have consequences on the study of the stochastic flows of dynamical systems driven by (1.1), and hints at applications to new stochastic transport equations as in the work by Mohammed et al. [32] (see also Flandoli et al. [15] and Fedrizzi and Flandoli [14]).

Let us now give a precise statement of the main results of the paper.

1.1. Probabilistic setting

Let $T \in (0, \infty)$ and $d \in \mathbb{N}$ be fixed and consider a probability space $(\Omega, \mathcal{F}, P)$ equipped with the completed filtration $(\mathcal{F}_t)_{t \in [0,T]}$ of a $d$-dimensional Brownian motion $B$. Throughout the paper, the product $\Omega \times [0, T]$ is endowed with the predictable $\sigma$-algebra. Subsets of $\mathbb{R}^k$, $k \in \mathbb{N}$, are always endowed with the Borel $\sigma$-algebra induced by the Euclidean norm $| \cdot |$. The interval $[0, T]$ is equipped with the Lebesgue measure. Unless otherwise stated, all equalities and inequalities between random variables and processes will be understood in the $P$-almost sure and $P \otimes dt$-almost sure
sense, respectively. For $p \in [1, \infty]$ and $k \in \mathbb{N}$, denote by $S^p(\mathbb{R}^k)$ the space of all adapted continuous processes $X$ with values in $\mathbb{R}^k$ such that $\|X\|_{S^p(\mathbb{R}^k)} := E[(\sup_{t \in [0,T]} |X_t|)^p] < \infty$, and by $H^p(\mathbb{R}^k)$ the space of all predictable processes $Z$ with values in $\mathbb{R}^k$ such that $\|Z\|_{H^p(\mathbb{R}^k)} := E[(\int_0^T |Z_u|^2 \, du)^{p/2}] < \infty$.

### 1.2. Main result

In this section, we present the main result. Let us consider the following conditions

(A1) It holds $b = b_1 + b_2$, where the function $b_1 : [0, T] \times \mathbb{R} \to \mathbb{R}$ is Borel measurable and there is $k_1 \geq 0$ such that for all $x \in \mathbb{R}$, 

$$|b_1(t, x)| \leq k_1(1 + |x|).$$

The function $b_2 : [0, T] \times \Omega \to \mathbb{R}$ is adapted, such that

$$b_2^\exp := E \left[ e^{4b_2^4} |b_2(t, \omega)|^2 \, dt \right].$$

Further assume that $b_2$ is Malliavin differentiable for every $s \in [0, T]$ and its Malliavin derivative $D_t b_2(s, \omega)$ (see the beginning of Section 2 for definition) satisfies

$$b_2^{\text{power}} := \sup_{0 \leq s \leq T} \left( \int_0^T |D_s b_2(t, \omega)|^2 \, dt \right)^{\frac{1}{4}}.$$  \hspace{1cm} (1.3)

(A2) $\sigma \in \mathbb{R}^d$ and $|\sigma|^2 > 0$.

#### Theorem 1.1

Assume that conditions (A1)-(A2) hold. Then there exists a unique global strong solution $X \in S^2(\mathbb{R})$ to the SDE

$$dX_t = b(t, \omega, X_t) \, dt + \sigma \, dB_t, \quad 0 \leq t \leq T, \quad X_0 = x \in \mathbb{R}. \hspace{1cm} (1.5)$$

The proof is given in Section 2. Under the conditions of Theorems 1.1, we show that the unique strong solution of the SDE is Malliavin differentiable see Theorem 3.1. Let us give some examples of drift coefficients satisfying condition (A1).

#### Example 1.2

The example of a random drift term of the form $b_1(t, x) + \varphi(B_t)$, where $\varphi : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ is a Lipschitz continuous functions (in the second variable) seems not to be covered by the existing literature. It is consistent with our assumptions since the Malliavin derivative of $\varphi(t, B_t)$ is bounded, and the exponential moment condition (1.3) is satisfied, at least for $T$ small enough, or for arbitrary $T$ when $\varphi$ is bounded.

A more general example is the path dependent drift case $b(t, \omega, x) := b_1(t, x) + \varphi(B_{0:t})$, where $B_{0:t}$ denotes the path of $B$ up to $t$, and $\varphi : C([0, T], \mathbb{R}^d) \to \mathbb{R}$ is bounded and Lipschitz continuous. It follows e.g. from Cheridito and Nam [6, Proposition 3.2] that $\varphi(B_{0:t})$ has bounded Malliavin derivatives for all $t$.

#### Example 1.3

Let $\alpha \in H^4(\mathbb{R}^d)$ be such that $b_2(t, \omega) := \int_0^t \alpha_s \, dB_s$ satisfies the moment condition (1.3) (which is automatically satisfied when $\alpha$ is deterministic), and $\alpha$ Malliavin differentiable with $(D_t \alpha_t)_t \in H^4$. Then, it follows by Nualart [34, Proposition 1.3.4] that $b_2$ is Malliavin differentiable, and

$$D_s b_2(t) = \alpha_t 1_{\{s \leq t\}} + \int_s^t D_s \alpha_r \, dB_r. \hspace{1cm} (1.6)$$
Thus, by Burkholder-Davis-Gundy inequality, it holds

\[
\sup_{0 \leq s \leq T} E \left[ \left( \int_0^T |D_s b_2(t, \omega)|^2 dt \right)^4 \right] \leq \| \alpha \|_{\mathcal{H}^4(\mathbb{R}^d)} + \sup_{0 \leq s \leq T} \| D_s \alpha \|_{\mathcal{H}^4(\mathbb{R}^d)} < \infty,
\]

which shows that \( \gamma_2^{\text{power}} < \infty \).

In the present paper, we further extensively analyze some properties of the Malliavin derivative of the solution. The remainder of the paper is structured as follows: The next section is mainly dedicated to the proof of Theorem 1.1. As a byproduct of our method, we obtain Malliavin differentiability of the solution. In addition, we derive various results concerning the Malliavin derivative of the solution, including moment estimates and a representation in terms of the space-time local time integral. In the appendix we present a few auxiliary results to make the paper self-contained.

2. Proofs of Theorem 1.1

2.1. Some notation

In this section, we prove existence and uniqueness of strong solutions for SDEs. Since Malliavin calculus will play an important role in our arguments, we briefly introduce the spaces of Malliavin differentiable random variables and stochastic processes \( \mathcal{D}^{1,p}(\mathbb{R}^d) \) and \( \mathcal{L}^{1,p}(\mathbb{R}^d), \ p \geq 1 \). For a thorough treatment of the theory of Malliavin calculus we refer to Nualart [34]. Let \( \mathcal{M} \) be the class of smooth random variables \( \xi = (\xi^1, \ldots, \xi^d) \) of the form

\[
\xi^i = \varphi^i \left( \int_0^T h^i_1 dW_s, \ldots, \int_0^T h^i_m dW_s \right),
\]

where \( \varphi^i \) is in the space \( \mathcal{C}^{\infty}_{\text{poly}}(\mathbb{R}^n; \mathbb{R}) \) of infinitely continuously differentiable functions whose partial derivatives have polynomial growth, \( h^{1i}, \ldots, h^{ni} \in L^2([0, T]; \mathbb{R}^d) \) and \( n \geq 1 \). For every \( \xi \in \mathcal{M} \) let the operator \( D = (D^1, \ldots, D^d) : \mathcal{M} \to L^2(\Omega \times [0, T]; \mathbb{R}^d) \) be given by

\[
D_t \xi^i := \sum_{j=1}^n \frac{\partial \varphi^i}{\partial x_j} \left( \int_0^T h^i_1 dW_s, \ldots, \int_0^T h^i_m dW_s \right) h^i_j, \quad 0 \leq t \leq T, \ 0 \leq i \leq l,
\]

and the norm \( \| \xi \|_{L^p} := (E[|\xi|^p + \int_0^T |D_t \xi|^p dt])^{1/p} \). As shown in Nualart [34], the operator \( D \) extends to the closure \( \mathcal{D}^{1,p}(\mathbb{R}^d) \) of the set \( \mathcal{M} \) with respect to the norm \( \| \cdot \|_{L^p} \). A random variable \( \xi \) is Malliavin differentiable if \( \xi \in \mathcal{D}^{1,p}(\mathbb{R}^d) \) and we denote by \( D_t \xi \) its Malliavin derivative. Denote by \( \mathcal{L}^{1,p}(\mathbb{R}^d) \) the space of processes \( Y \in \mathcal{H}^2(\mathbb{R}^d) \) such that \( Y_t \in \mathcal{D}^{1,p}(\mathbb{R}^d) \) for all \( t \in [0, T] \), the process \( D_t Y_t \) admits a square integrable progressively measurable version and

\[
\| Y \|_{L^{1,p}(\mathbb{R}^d)} := \| Y \|_{\mathcal{H}^2(\mathbb{R}^d)} + E \left[ \int_0^T |D_t Y_t|^p dt \right] < \infty.
\]

2.2. Proof of Theorem 1.1

In the whole of this section, we assume that conditions (A1) and (A2) are satisfied. The proof of the Theorem 1.1 is given in 5 steps. In the first step, we show that there exists a process \( X^\varepsilon \) satisfying the SDE (1.5) in the weak sense. That is, there is a Brownian motion \( \tilde{B} \) such that \( (X_t^\varepsilon, \tilde{B}_t) \) is a weak solution to the SDE (1.5). Note however that
the solution might not be adapted to the filtration \((\mathcal{F}_t)_{t \in [0,T]}\). Let us mention that if \(X_t\) is adapted to that filtration then \(X_t\) has an explicit representation as a function of \(B_t\) (see for example [27, 30]) and for any other stochastic basis \((\Omega, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t \in [0,T]}, \tilde{P}, \tilde{B})\), the same representation holds with \(\tilde{B}_t\) instead of \(B_t\) and thus \(X_t\) is \((\tilde{\mathcal{F}}_t)_{t \in [0,T]}\)-adapted. The latter indicates that \(X_t\) is a strong solution of (1.5).

In the second step, for \(T\) small, given a sequence \(b_n = b_{1,n} + b_2\) such that \(b_{1,n} : [0, T] \times \mathbb{R} \to \mathbb{R}, n \geq 1\) are smooth coefficients with compact support and converging a.e. to \(b_1\), we show using relative compactness (see Lemma 2.3) that for each \(0 \leq t \leq T\) the sequence of corresponding strong solutions \((X_t^{x,n})_{n \geq 1}\), of the SDEs

\[
dX_t^{x,n} = b_n(t, \omega, X_t^{x,n})dt + \sigma \cdot dB_t, \quad 0 \leq t \leq T, \quad X_0^{x,n} = x \in \mathbb{R}, \quad n \geq 1,
\]

is relatively compact in \(L^2(P; \mathbb{R})\). Let us mention that existence of a unique strong solution to the SDE (2.1) is guaranteed by [35, Theorem 1.1], see also [21].

In step 3, we show that for each \(0 \leq t \leq T\) the above sequence \((X_t^{x,n})_{n \geq 1}\) converges weakly to \(E[X^t|\mathcal{F}_t]\) in the space \(L^2(\Omega, \mathcal{F}_t, P)\). This with step 2 allow to deduce that \((X_t^{x,n})_{n \geq 1}\) converges strongly to \(E[X^t|\mathcal{F}_t]\) in the space \(L^2(\Omega, \mathcal{F}_t, P)\). We also obtain from step 2 that \(E[X^t|\mathcal{F}_t]\) is Malliavin smooth, see Subsection 3.

In step 4, we prove that \(X^t\) is \(\mathcal{F}_t\)-measurable by showing that \(E[X^t|\mathcal{F}_t] = X^t\). The proof is completed by showing uniqueness.

In the last step, we use a pasting argument to show that the result holds for all \(T > 0\). In fact, the linear growth assumption on \(b_1\) and integrability assumption on \(b_2\) ensure by the use of Gronwall lemma that if the solution exists on a small interval then it does not explode. Hence the main task in this step is to show that \(E[|D_xX^t|^2]\) is uniformly in \(n\) for \(0 \leq s \leq t \leq T\).

### 3.2.1. Weak existence.

The following result can be seen as a slight generalization of a result by V.E. Beneš, compare [3, 23]. Therein (and throughout the paper) we denote by \(\mathcal{E}\left(\int q dB\right)\) the Doléan-Dade exponential

\[
\mathcal{E}\left(\int q dB\right) := \exp\left(\int_0^t q_u dB_u - \frac{1}{2} \int_0^t |q_u|^2 du\right).
\]

**Lemma 2.1.** Let \(u\) be given by

\[
u_i = \frac{\sigma_i}{\sigma_1^2 + \ldots + \sigma_d^2} b_i.
\]

Then the process \(Z := \mathcal{E}\left(\int u(r, \omega, \sigma \cdot B_u) dB_u\right)\) is a martingale.

**Proof.** The proof follows from that of [23, Theorem 2.1]. We know that the process \(Z_t\) is a non negative local martingale and thus a supermartingale such that \(E[Z_t] \leq 1\) for any \(0 \leq t \leq T\). Using the same argument as in the proof of [23, Theorem 2.1], one obtains the result by applying Gronwall Lemma provided that \(E\left[\int_0^T |b_2(t, \omega)|^4 dt\right] < \infty\). The later is true by assumption.

The next lemma ensures weak existence, it is a simple adaptation of [22, Proposition 5.3.6].

**Lemma 2.2.** The SDE (1.5) admits a weak solution \(X^t\).

5
Proof. Let \((\Omega, \mathcal{F}, P)\) be a probability space on which a \(d\)-dimensional Brownian motion \(\hat{B}\) is given, and set \(X^x_t := x + \sigma \cdot \hat{B}_t\), \(0 \leq t \leq T\). By (A1), it follows from Lemma 2.1 (see also [3, 4, 23]) that the process \(\mathcal{E} \left( \int u(r, \omega, X^x_t) \mathrm{d}\hat{B}_r \right)\) defines an equivalent probability measure \(Q\) given by

\[
\frac{dQ}{dP} := \mathcal{E} \left( \int u(r, \omega, X^x_t) \mathrm{d}\hat{B}_r \right).
\]

In addition, Girsanov’s theorem asserts that \(B_t = \hat{B}_t - \int_0^t u(r, \omega, X^x_t) \mathrm{d}r\) is a \(Q\)-Brownian motion. Therefore,

\[
X^x_t = x + \int_0^t \sigma \cdot u(s, \omega, X^x_s) \mathrm{d}s + \sigma \cdot B_t \quad \text{Q-a.s., } 0 \leq t \leq T,
\]

showing that \((X^x, B)\) is a weak solution to the SDE (1.5) on the probability space \((\Omega, \mathcal{F}, Q)\).

### 3.2.2. Approximation and compactness

Let \(b_n = b_{1,n} + b_2\) be such that \(b_{1,n} : [0, T] \times \mathbb{R} \to \mathbb{R}, n \geq 1\) are smooth coefficients with compact support and converging a.e. to \(b_1\). Denote by \(X_t^{x,n}\) the unique strong solution to the SDE (2.1) with drift \(b_n\). The following result is key to the compactness argument.

**Lemma 2.3.** If \(T \in (0, \infty)\) is small enough, the strong solution \(X_t^{x,n}\) of the SDE (2.1) satisfies

\[
E \left[ |D_tX_t^{x,n} - D_tX_t^{x',n}|^2 \right] \leq C(k, |x|^2, b_2^{\text{power}}) |t - t'|^\alpha
\]

for all \(0 \leq t' \leq t \leq T\) and some \(\alpha = \alpha(s) > 0\). Moreover,

\[
\sup_{0 \leq t \leq T} E \left[ |D_1X_t^{x,n}|^2 \right] \leq C(k, |x|^2, b_2^{\text{power}}),
\]

where the function \(C(\cdot, \cdot) : [0, \infty)^3 \rightarrow [0, \infty)\) is continuous and increasing in each component, \(b_2^{\text{power}}\) defined in (1.3) and

\[
k = \|	ilde{b}_1\|_{\infty} := \text{ess sup} \left\{ \frac{|b_1(t, z)|}{1 + |z|} : t \in [0, T], z \in \mathbb{R} \right\}.
\]

The combination of Lemma 2.3 and Corollary A.3 yields the following result:

**Corollary 2.4.** For each \(0 \leq t \leq T\), with \(T\) small enough, the sequence \((X_t^{x,n})_{n \geq 1}\), is relatively compact in \(L^2(P; \mathbb{R})\).

**Proof (of Lemma 2.3).** Since the Brownian motions are independent, applying the chain-rule formula for the Malliavin derivatives in the direction of \(i^{th}\) Brownian motion (see e.g. [34]) gives

\[
D^i_1X_t^{x,n} = \sigma_i + \int_0^t D^i_1b_2(u, \omega) \mathrm{d}u + \int_0^t b'_{1,n}(u, X_u^{x,n})D^i_1X_u^{x,n} \mathrm{d}u, \ i = 1, \ldots, d
\]

\(P\text{-a.s. for all } t \leq s \leq T\). Here \(b'_{1,n}(t, x) = \frac{\partial}{\partial x} b_{1,n}(t, x)\) is the spatial derivative of \(b_{1,n}\). Solving (2.5) explicitly gives

\[
D^i_1X_t^{x,n} = \mathcal{E} \int_t^\infty b'_{1,n}(u, X_u^{x,n}) \mathrm{d}u \left( \int_t^\infty D^i_1b_2(u, \omega) \mathrm{d}r - \int_t^\infty b'_{1,n}(r, X_r^{x,n}) \mathrm{d}r + \sigma_i \right).
\]
Let $0 \leq t' \leq t \leq s \leq T$. Using the above representation, we have

$$
D_t^* X^{x,n} - D_t^* X^{x,n} = e^{f_{t'} b_{t',n}(u,X_u^{x,n})du} \left( \int_{t'}^s D_t^* b_2(u,\omega)e^{-f_{x} b_{t,n}(r,X_r^{x,n})dr} du + \sigma_1 \right)
$$

\[ \leq e^{f_{t'} b_{t',n}(u,X_u^{x,n})du} \left( \int_{t'}^s D_t^* b_2(u,\omega)e^{-f_{x} b_{t,n}(r,X_r^{x,n})dr} dr du \right) + \sigma_1 \]

\[ = e^{f_{t'} b_{t',n}(u,X_u^{x,n})du} \left( e^{f_{t'} b_{t',n}(u,X_u^{x,n})du - 1} \right) \]

\[ + \int_{t'}^s D_t^* b_2(u,\omega)e^{-f_{x} b_{t,n}(r,X_r^{x,n})dr} dr du \]

\[ = e^{f_{t'} b_{t',n}(u,X_u^{x,n})du} \left( e^{f_{t'} b_{t',n}(u,X_u^{x,n})du - 1} \right) + \int_{t'}^s D_t^* b_2(u,\omega)e^{-f_{x} b_{t,n}(r,X_r^{x,n})dr} dr du \]

\[ = I_1 + I_2 + I_3. \quad (2.7) \]

Set

\[ u_{i,n} = \frac{\sigma_i}{\sigma_1^2 + \cdots + \sigma_d^2} b_n. \quad (2.8) \]

Then using Girsanov transform and Hölder inequality, we have

\[ E[I_2^2] = \sigma^2 E \left[ e^{f_{t'} b_{t',n}(u,X_u^{x,n})du} \left( e^{f_{t'} b_{t',n}(u,X_u^{x,n})du - 1} \right)^2 \right] \]

\[ = \sigma^2 E \left[ \left( \int_0^T u_n(r,\omega, x + \sigma \cdot B_r)dB_r \right) e^{f_{t'} b_{t',n}(u,x+\sigma B_u)du} \left( e^{f_{t'} b_{t',n}(u,x+\sigma B_u)du - 1} \right)^2 \right] \]

\[ \leq \sigma^2 E \left[ e^{2 \sum_{i=1}^d \int_0^T u_{i,n}(r,\omega, x + \sigma \cdot B_r)dB_r - \sum_{i=1}^d \int_0^T u_{i,n}^2(r,\omega, x + \sigma \cdot B_r)dr} e^{4 \sum_{i=1}^d \int_0^T u_{i,n}(r,\omega, x + \sigma \cdot B_r)du} \right]^{\frac{1}{2}} \]

\[ \times E \left[ \left( e^{f_{t'} b_{t',n}(u,x+\sigma B_u)du - 1} \right)^4 \right]^{\frac{1}{2}} \]

\[ \leq E \left[ e^{4 \sum_{i=1}^d \int_0^T u_{i,n}(r,\omega, x + \sigma \cdot B_r)dB_r - 8 \sum_{i=1}^d \int_0^T u_{i,n}^2(r,\omega, x + \sigma \cdot B_r)dr} \right]^{\frac{1}{2}} E \left[ e^{12 \sum_{i=1}^d \int_0^T u_{i,n}^2(r,\omega, x + \sigma \cdot B_r)dr} \right]^{\frac{1}{2}} \]

\[ \times E \left[ e^{16 \sum_{i=1}^d \int_0^T u_{i,n}(x+\sigma B_u)du} \right]^{\frac{1}{2}} E \left[ \left( e^{f_{t'} b_{t',n}(u,x+\sigma B_u)du - 1} \right)^4 \right]^{\frac{1}{2}} \sigma^2. \quad (2.9) \]

If follows from the Girsanov theorem applied to the martingale $4 \sum_{i=1}^d \int_0^T u_{i,n}(r,\omega, x + \sigma \cdot B_r)dB_r$ that the first term in (2.9) is equal to one. Next, we wish to use conditions on $b_n$ and thus $u_{i,n}$ to show that the second term is finite for $T$ small enough. Using Hölder inequality, we have

\[ E \left[ e^{12 \sum_{i=1}^d \int_0^T u_{i,n}^2(r,\omega, x + \sigma \cdot B_r)dr} \right] \leq \prod_{i=1}^d E \left[ e^{12d \int_0^T u_{i,n}^2(r,\omega, x + \sigma \cdot B_r)dr} \right]^{\frac{1}{2}} \leq \prod_{i=1}^d E \left[ e^{12d \int_0^T \frac{\sigma_i^2}{\sigma_1^2 + \cdots + \sigma_d^2} b_n^2(r,\omega, x + \sigma \cdot B_r)dr} \right]^{\frac{1}{2}}. \]

Let us focus on each component of the above product. Using the condition on $b_n$, Hölder inequality successively and
the independence of the Brownian motion, we get
\[
E \left[ e^{12d \int_0^T \frac{\sigma^2}{(\sigma_1^2 + \cdots + \sigma_d^2)^2} k^2(r,\omega,x+\sigma B_t) \, dr} \right] \leq E \left[ e^{24c_{d,\sigma} \int_0^T (k^2(1+x+\sigma B_t)|^2 + |b_2(t,\omega)|^2) \, dt} \right] \\
\leq E \left[ e^{48c_{d,\sigma} \int_0^T k^2(1+|x|+\sigma B_t)|^2 \, dt} \right] \frac{k}{2} \times E \left[ e^{48c_{d,\sigma} \int_0^T |b_2(t,\omega)|^2 \, dt} \right] \frac{k}{2} \\
\leq C e^{48c_{d,\sigma} k^2 T (1+|x|)^2} E \left[ e^{48c_{d,\sigma} k^2 T |\sigma B_t|^2 \, dt} \right] \frac{k}{2} \\
\leq C e^{48c_{d,\sigma} k^2 T (1+|x|)^2} E \left[ e^{48c_{d,\sigma} k^2 T \sum_{i=1}^d \sup_{0 \leq t \leq T} |\sigma_i B_t|^2} \right] \frac{k}{2} \\
\leq C e^{48c_{d,\sigma} k^2 T (1+|x|)^2} \prod_{i=1}^d E \left[ e^{48c_{d,\sigma} k^2 T \sup_{0 \leq t \leq T} |\sigma_i B_t|^2} \right] \frac{k}{2}. \tag{2.10}
\]

In the above, \(c_{d,\sigma} := \frac{d \sigma^2}{(\sigma_1^2 + \cdots + \sigma_d^2)^2} \). Now, using exponential expansion and the Doob maximal inequality, we have
\[
E \left[ e^{48c_{d,\sigma} k^2 T \sup_{0 \leq t \leq T} |\sigma_i B_t|^2} \right] = 1 + \sum_{p=1}^\infty \frac{(48c_{d,\sigma} k^2 T)^p}{p!} E \left[ \sup_{0 \leq t \leq T} |B_t|^2 \right] \\
\leq 1 + \sum_{p=1}^\infty \frac{(48k^2 dT)^p}{p!} \left( \frac{2p}{2p-1} \right)^2 \frac{(2p)!}{2p!} T^p. \tag{2.11}
\]

The inequality comes from the fact that \(\frac{d \sigma^2}{(\sigma_1^2 + \cdots + \sigma_d^2)^2} \leq d \). Next, applying the ratio test to the series \(\sum_{p} a_p\) with
\[
a_p := \frac{(48k^2 dT)^p}{p!} \left( \frac{2p}{2p-1} \right)^2 \frac{(2p)!}{2p!} T^p \text{ for } p \geq 1, \]
one can easily show that the series converges for example for
\[
T = T_1 \leq \frac{1}{4 \sqrt{3} d k^2 T}. \tag{2.12}
\]

Hence the second term in (2.9) is finite for small \(T\).

Now, using power and exponential expansion, and dominated convergence theorem in the last term, we get by linearity of the expectation
\[
E \left[ \left( e^{I^i_{\nu_1} b_{i,n}(u,x+\sigma B_u) \, du} - 1 \right)^4 \right] = E \left[ e^{I^i_{\nu_1} 4b_{i,n}(u,x+\sigma B_u) \, du} - 4 e^{I^i_{\nu_1} 3b_{i,n}(u,x+\sigma B_u) \, du} + 6 e^{I^i_{\nu_1} 2b_{i,n}(u,x+\sigma B_u) \, du} - 4 e^{I^i_{\nu_1} b_{i,n}(u,x+\sigma B_u) \, du} + 1 \right] \\
= E \left[ \sum_{q=1}^{\infty} \frac{\left( \int_{-\nu_1}^{\nu_1} 4b_{i,n}(u,x+\sigma B_u) \, du \right)^q}{q!} - 4 \sum_{q=1}^{\infty} \frac{\left( \int_{-\nu_1}^{\nu_1} 3b_{i,n}(u,x+\sigma B_u) \, du \right)^q}{q!} \\
+ 6 \sum_{q=1}^{\infty} \frac{\left( \int_{-\nu_1}^{\nu_1} 2b_{i,n}(u,x+\sigma B_u) \, du \right)^q}{q!} - 4 \sum_{q=1}^{\infty} \frac{\left( \int_{-\nu_1}^{\nu_1} b_{i,n}(u,x+\sigma B_u) \, du \right)^q}{q!} \right] \\
\leq \sum_{q=1}^{\infty} E \left[ \frac{\left( \int_{-\nu_1}^{\nu_1} 4b_{i,n}(u,x+\sigma B_u) \, du \right)^q}{q!} \right] + 4 \sum_{q=1}^{\infty} E \left[ \frac{\left( \int_{-\nu_1}^{\nu_1} 3b_{i,n}(u,x+\sigma B_u) \, du \right)^q}{q!} \right] + 6 \sum_{q=1}^{\infty} E \left[ \frac{\left( \int_{-\nu_1}^{\nu_1} 2b_{i,n}(u,x+\sigma B_u) \, du \right)^q}{q!} \right] - 4 \sum_{q=1}^{\infty} E \left[ \frac{\left( \int_{-\nu_1}^{\nu_1} b_{i,n}(u,x+\sigma B_u) \, du \right)^q}{q!} \right].
\]
that each term in (3) is bounded by $C(T, \|b\|_{\infty}, |x|)|t - t'|$, where $C(||b||_{\infty}, |x|)$ is a continuous function depending on $\|b\|_{\infty}, x, \|\sigma\|^2$ and $T$. More specifically, one has:

$$\sum_{q=1}^{\infty} E \left[ \left| \int_{t'}^{t} b_{1,n}(u, x + \sigma \cdot B_u) du \right|^q \right] \leq C \sum_{q=1}^{\infty} E \left[ \left( \int_{t'}^{t} b_{1,n}(u, x + \sigma \cdot B_u) du \right)^{2q} \right]^{1/2} \leq \sum_{q=1}^{\infty} \frac{C^{2q}(1 + |x|^q)(t - t_0)^{2q}}{q!} \left( \sum_{q=1}^{\infty} \frac{1}{4q} \right)^{1/2} (t - t')^{1/2} \leq C \frac{\exp\{C\sigma T(1 + |x|^2)\}|t - t'|^{1/2}}{T}.$$ (2.15)

Similarly, it can be proved that $E \left[ e^{16 \int_{t'}^{t} b_{1,n}(u, x + \sigma \cdot B_u) du} \right]$ is bounded by $C \frac{\exp\{C\sigma T(1 + |x|^2)\}|t - s|^{1/2}}{T}$. Therefore there exists a constant $C$ depending on $\sigma$ such that

$$E[F_1^2] \leq C \frac{\exp\{C\sigma T(1 + k^2)(1 + |x|^2)\}|t - t'|^{1/2}}{T^{1/2}}.$$ (2.16)

Repeated application of the Hölder inequality yields

$$E[F_2^2] = E \left[ \left( \int_{t'}^{t} D_x b_2(u, \omega) e^{-\int_{t'}^{u} b_{1,n}(r, X_r^{\omega, n}) dr} du \right)^2 \right] \leq E \left[ \left( \int_{t'}^{t} D_x b_2(u, \omega) \right)^2 \left( \int_{t'}^{t} e^{-\int_{t'}^{u} b_{1,n}(r, X_r^{\omega, n}) dr} du \right)^{1/2} \right] (t - t')^{1/2}.$$
\[
\begin{align*}
&\leq E\left[\left(\int_{t'}^t \left(D_{t'}^b b_2(u, \omega)\right)^2 \, du\right)^{\frac{1}{2}}\right] E\left[\int_{t'}^t e^{-\int_s^t 4b'_{i,n}(r,X_{r}^\omega) \, dr} \, du\right]^{\frac{1}{2}}\frac{1}{(t-t')^{1/2}} \\
&\leq C|t-t'|^{1/2} E\left[\left(\int_{0}^T \left(D_{t'}^b b_2(u, \omega)\right)^2 \, du\right)^{2}\right]^{\frac{1}{2}} E\left[\int_{0}^T e^{-\int_s^t 4b'_{i,n}(r,X_{r}^\omega) \, dr} \, du\right]^{\frac{1}{2}} \\
&\leq C(t_2^{\text{power}})^{1/2}|t-t'|^{1/2} E\left[\int_{0}^T e^{-\int_s^t 4b'_{i,n}(r,X_{r}^\omega) \, dr} \, du\right]^{\frac{1}{2}}. \quad (2.16)
\end{align*}
\]

Again, using Girsanov transform, similar reasoning as before gives

\[
E[I_2^s] \leq C(t_2^{\text{power}})^{1/2} \frac{C}{T} \exp\{C_\sigma T(1+k^2)(1+|x|)^2\}|t-t'|^{\frac{1}{2}}
\]

for \(0 \leq t' \leq t < T\), with \(T\) small enough.

As for \(I_1\), once more repeated use of Cauchy-Schwartz inequality gives the existence of a constant \(C\) that may change from one line to the other such that

\[
\begin{align*}
E[I_2^s] &= E\left[e^{-2 \int_{t}^{t'} b'_{i,n}(r,X_{r}^\omega) \, dr} e^{-2 \int_{t}^{t'} b'_{i,n}(r,X_{r}^\omega) \, dr} \left(\int_{t}^{t'} \left(D_{t'}^b b_2(u, \omega) - D_{t}^b b_2(u, \omega)\right) e^{-\int_{u}^{t'} b'_{i,n}(r,X_{r}^\omega) \, dr} \, du\right)^2\right] \\
&\leq E\left[e^{-2 \int_{t}^{t'} b'_{i,n}(r,X_{r}^\omega) \, dr}\right]^{\frac{1}{4}} E\left[e^{-2 \int_{t}^{t'} b'_{i,n}(r,X_{r}^\omega) \, dr}\right]^{\frac{1}{4}} \\
&\times E\left[\left(\int_{t}^{t'} \left(D_{t'}^b b_2(u, \omega) - D_{t}^b b_2(u, \omega)\right) e^{-\int_{u}^{t'} b'_{i,n}(r,X_{r}^\omega) \, dr} \, du\right)^2\right]^{\frac{1}{4}} E\left[\left(\int_{t}^{t'} \left(D_{t'}^b b_2(u, \omega) - D_{t}^b b_2(u, \omega)\right) e^{-\int_{u}^{t'} b'_{i,n}(r,X_{r}^\omega) \, dr} \, du\right)^2\right]^{\frac{1}{4}} \\
&\leq CE\left[e^{-2 \int_{t}^{t'} b'_{i,n}(r,X_{r}^\omega) \, dr}\right]^{\frac{1}{4}} E\left[e^{-2 \int_{t}^{t'} b'_{i,n}(r,X_{r}^\omega) \, dr}\right]^{\frac{1}{4}} E\left[\left(\int_{t}^{t'} b'_{i,n}(r,X_{r}^\omega) \, dr\right)^2\right]^{\frac{1}{4}} E\left[\left(\int_{t}^{t'} b'_{i,n}(r,X_{r}^\omega) \, dr\right)^2\right]^{\frac{1}{4}} \\
&\times \left\{E\left[\left(\int_{t}^{t'} \left(D_{t'}^b b_2(u, \omega)\right)^2 \, du\right)^{\frac{1}{2}}\right]^{\frac{1}{4}} + E\left[\left(\int_{t}^{t'} \left(D_{t}^b b_2(u, \omega)\right)^2 \, du\right)^{\frac{1}{2}}\right]^{\frac{1}{4}}\right\} \\
&\leq CE\left[e^{-2 \int_{t}^{t'} b'_{i,n}(r,X_{r}^\omega) \, dr}\right]^{\frac{1}{4}} E\left[e^{-2 \int_{t}^{t'} b'_{i,n}(r,X_{r}^\omega) \, dr}\right]^{\frac{1}{4}} E\left[\left(\int_{t}^{t'} b'_{i,n}(r,X_{r}^\omega) \, dr\right)^2\right]^{\frac{1}{4}} E\left[\left(\int_{t}^{t'} b'_{i,n}(r,X_{r}^\omega) \, dr\right)^2\right]^{\frac{1}{4}} \\
&\times \left\{E\left[\left(\int_{t}^{t'} \left(D_{t'}^b b_2(u, \omega)\right)^2 \, du\right)^{\frac{1}{2}}\right]^{\frac{1}{4}} + E\left[\left(\int_{t}^{t'} \left(D_{t}^b b_2(u, \omega)\right)^2 \, du\right)^{\frac{1}{2}}\right]^{\frac{1}{4}}\right\}. \quad (2.17)
\end{align*}
\]

Once again, using Girsanov theorem and the linear growth condition on the drift \(b_1\), one can show that the expectations \(E\left[\int_{t}^{t'} e^{-\int_{u}^{t'} 8b'_{i,n}(r,X_{r}^\omega) \, dr} \, du\right]\) and \(E\left[e^{-8 \int_{t}^{t'} b'_{i,n}(r,X_{r}^\omega) \, dr}\right]\) are bounded by \(\frac{C}{T} \exp\{C_\sigma T(1+k^2)(1+|x|)^2\}\) and that of \(E\left[e^{-8 \int_{t}^{t'} b'_{i,n}(r,X_{r}^\omega) \, dr}\right]\) is bounded by \(\frac{C}{T} \exp\{C_\sigma T(1+k^2)(1+|x|)^2\}|t-t'|^{1/2}\). Moreover, the assumption on \(b_2\)
insures that the two last integral terms on the right hand side of (2.18) are bounded by $C$. Therefore,

$$E[b_2^2] \leq C(b_2^{\text{power}})^{1/2} \frac{C}{T} \exp\{C_n T (1 + k^2 (1 + |x|^2)) |t - t'|^{\frac{1}{2}} \} \tag{2.19}$$

for $0 \leq t' \leq t \leq T_1$ with $T_1$ small enough.

Combining (2.9)-(2.19), there exists a function $C = C(k, |x|^2, b_2^{\text{power}}) > 0$ depending on $k, b_2$ and $|x|^2$ such that

$$E \left[ |D_t X_t^{x,n} - D_{t'} X_t^{x,n}|^2 \right] \leq C(k, |x|^2, b_2^{\text{power}})|t - t'|^{\alpha}$$

for $0 \leq t' \leq t \leq T$ with $T$ small enough and $\alpha = 1/2$. Thus the first part of the Lemma is shown. Taking $t' > s$ above, $D_{t'} X_t^{x,n} = 0$, which implies

$$\sup_{0 \leq t \leq T} E \left[ |D_t X_t^{x,n}|^2 \right] \leq C(k, |x|^2, b_2^{\text{power}}).$$

This proves the lemma. \hfill $\square$

### 3.2.3. Weak convergence to the weak solution.

In this step, we show that for each $0 \leq t \leq T$ the above sequence $(X_t^{x,n})_{n \geq 1}$ converges weakly to $E \left[ X_t^{x} \mid F_t \right]$ in the space $L^2(\Omega; P; F_t)$.

**Lemma 2.5.** Assume $b_2^{\text{up}} < \infty$ and $\Omega$ is the canonical space. Choose the sequence $b_{1,n} : [0, T] \times \mathbb{R} \to \mathbb{R}$, $n \geq 1$ as before, and let $(X_t^{x,n})_{n \geq 1}$ be the corresponding strong solutions to the SDE (2.1). Then for each $0 \leq t \leq T$, and each function $h : \mathbb{R} \to \mathbb{R}$ of polynomial growth, the sequence $(h(X_t^{x,n}))_{n \geq 1}$ is uniformly bounded in $L^2(\Omega; P; F_t)$ and converges weakly to $E \left[ h(X_t^{x}) \mid F_t \right]$ in this space.

**Proof.** Let us first show that $(h(X_t^{x,n}))_{n \geq 1}$ is bounded in $L^2(\Omega; P; F_t)$. In fact, using Girsanov transform, Hölder inequality and the fact that $(1 + |z|^p)e^{-|z|^2/2t}$ can be bounded by $C_p e^{-\frac{|z|^2}{4t}}$, where $C_p$ is a constant depending on $p$, we have

$$\sup_n E \left[ |h(X_t^{x,n})|^2 \right] \leq E \left[ e^{2 \sum_{i=1}^d \int_0^t \varphi_i(u, r, x + \sigma, B_u)dB_u - 2 \sum_{i=1}^d \int_0^t \varphi_i(u, x + \sigma, B_u)dt} \right]^{\frac{1}{2}}$$

$$\times E \left[ e^{2 \sum_{i=1}^d \int_0^t \varphi_i(u, x + \sigma, B_u)dt} \right]^{\frac{1}{2}} E \left[ |h(x + \sigma \cdot B_t)|^4 \right]^{\frac{1}{4}}$$

$$\leq CE \left[ |h(x + \sigma \cdot B_t)|^4 \right]^{\frac{1}{4}}$$

$$= C \left( \frac{1}{4 \pi t \alpha} \int_{\mathbb{R}} |h(x + z)|^4 e^{-|z|^2/4t} \frac{dz}{|z|^2} \right)^{\frac{1}{4}}$$

$$\leq \frac{C}{(2 \pi t \alpha)^{\frac{1}{4}}} \left( |x|^4 \int_{\mathbb{R}} e^{-\frac{|z|^2}{4t \alpha}} dz + \int_{\mathbb{R}} e^{-\frac{|z|^2}{4t \alpha}} dz \right)^{\frac{1}{4}} < \infty. \tag{2.20}$$

The constant $C_p$ above depends on $|b_2|, \|b_1\|, \|\sigma\|^2$ and $|x|$. The weak convergence of $(h(X_t^{x,n}))_{n \geq 1}$ to $E \left[ h(X_t^{x}) \mid F_t \right]$ in $L^2(\Omega; P; F_t)$, first notice that the space

$$\left\{ \mathcal{E} \left( \int_0^T \varphi(u)dB_u \right) : \varphi \in C_0^1([0, T], \mathbb{R}^d) \right\}$$

is a dense subspace of $L^2(\Omega, P)$. Here $C_0^1([0, T], \mathbb{R}^d)$ is the space of bounded continuous differentiable functions on $[0, T]$ and with values in $\mathbb{R}^d$ and $\varphi$ is the derivative of $\varphi$. Hence, it is enough to show that

$$\left( h(X_t^{x,n}) \mathcal{E} \left( \int_0^T \varphi(u)dB_u \right) \right)_{n \geq 1}$$
converges to \( E[h(X^x_T)]F_t \) in expectation. Since \( \Omega \) is a Wiener space, we know from the Cameron-Martin theorem, see e.g. [38], that for every \( h \) measurable, 

\[
E \left[ h(X^x_T)E \left( \int_0^T \phi_r dB_r \right) \right] = \int \Omega h(X^x_T(\omega + \varphi))dP(\omega) . \tag{2.21}
\]

Let \( \varphi \in C^1_b([0, T], \mathbb{R}^d) \). For every \( n \), the process \( \tilde{X}^{x,n} \) given by \( \tilde{X}^{x,n}_t(\omega) := X^{x,n}_t(\omega + \varphi) \) solves the SDE

\[
d\tilde{X}^{x,n}_t = \left(b_1(t, \tilde{X}^{x,n}_t) + \tilde{b}_2(\omega) + \sigma \varphi \right)dt + \sigma dB_t
\]

where \( \tilde{b}_2(\omega) := b_2(\omega + \varphi) \). To see this, let \( H \in L^2(\Omega, P) \) and apply (2.21) and the fact that \( X^{x,n} \) solves the SDE (2.1) to get

\[
E[\tilde{X}^{x,n}_tH] = E \left[ X^{x,n}_tH(\omega - \varphi)E \left( \int_0^T \phi(u)dB_u \right) \right]
\]

\[
= E \left[ \left( x + \int_0^t b_1(u, X^{x,n}_u) + b_2(u)d\omega + \sigma B_t \right)H(\omega - \varphi)E \left( \int_0^T \phi dB \right) \right]
\]

\[
= E \left[ \left( x + \int_0^t b_1(u, X_t^{x,n}(\omega + \varphi)) + b_2(\omega + \varphi)d\omega + \sigma B_t(\omega + \varphi) \right)H \right]
\]

\[
= E \left[ \left( x + \int_0^t b_1(u, \tilde{X}^{x,n}_u(\omega)) + \tilde{b}_2(\omega) + \sigma \varphi d\omega + \sigma B_t(\omega) \right)H \right],
\]

where the last equality follows by the fact that \( B_t(\omega + \varphi) = B_t(\omega) + \varphi \), since \( B \) is the canonical process. This proves the claim. Since \( X^x \) satisfies the SDE (without being adapted to the filtration \( \{F_t\} \)), with respect to a probability measure \( Q \) which is equivalent to \( P \), see the proof of Lemma 2.2, the above arguments show that \( \tilde{X}^{x}(\omega) := X^{x}(\omega + \varphi) \) satisfies

\[
d\tilde{X}^{x}_t = \left(b_1(t, \tilde{X}^{x}_t) + \tilde{b}_2(\omega) + \sigma \varphi \right)dt + \sigma dB_t \quad \text{P-a.s.} \tag{2.23}
\]

Now, put

\[
\tilde{u}_{i,n} := \frac{\sigma_i}{\sigma_1 + \cdots + \sigma_d}(b_{1,n} + \tilde{b}_2) \quad \text{and} \quad \tilde{u}_i = \frac{\sigma_i}{\sigma_1 + \cdots + \sigma_d}(b_1 + \tilde{b}_2). \tag{2.24}
\]

It follows by Girsanov theorem that

\[
E \left[ h(X^{x,n}_t)E \left( \int_0^T \phi_r dB_r \right) - E \left[ h(X^x_T)|F_t \right]E \left( \int_0^T \phi_r dB_r \right) \right] = E \left[ \left(h(X^{x,n}_t) - h(X^x_T)\right)E \left( \int_0^T \phi_r dB_r \right) \right]
\]

\[
= E \left[ h(x + \sigma \cdot B_t) \left( \mathbb{E} \left( \int_0^T \tilde{u}_n(r, \omega, x + \sigma \cdot B_r) + \tilde{\varphi}_r dB_r \right) - \mathbb{E} \left( \int_0^T \tilde{u}(r, \omega, x + \sigma \cdot B_r) + \varphi_r dB_r \right) \right) \right]. \tag{2.25}
\]
Using the fact that $|e^a - e^b| \leq |e^a + e^b||a - b|$, the Hölder inequality and Burkholder-Davis-Gundy inequality, we get

$$
E\left[h(X_t^{x,n})\mathcal{E}\left(\int_0^T \varphi_r dB_r\right)\right] - E\left[h(X_t^x)\mathcal{E}\left(\int_0^T \varphi_r dB_r\right)\right]
\leq C E\left[\left(\int_0^T \left(\tilde{u}_n(r, x, x + B_r) + \varphi_r\right) dB_r\right)^\frac{5}{4}\right]^\frac{4}{5}
\times \left\{ 1 + \mathcal{E}\left(\int_0^T \left(\tilde{u}_n(r, x + B_r) - \tilde{u}(r, x + B_r)\right) dB_r\right)^4 \right\}^{\frac{1}{4}}
+ \mathcal{E}\left(\int_0^T \left(\tilde{u}_n(r, x, x + B_r) + \tilde{u}(r, x + B_r)\right) dB_r\right)^4
\leq I_1 \times I_{2,n} \times (I_{3,n} + I_{4,n}).
$$

That $I_1$ is finite was proved in the computations leading to (2.20). Observe that

$$
\mathcal{E}\left(\int_0^T \left(\tilde{u}(r, x, x + B_r) + \varphi_r\right) dB_r\right) = \mathcal{E}\left(\int_0^T u_n(r, x + B_r) dB_r\right) \mathcal{E}\left(\int_0^T \tilde{b}_1 dB_r\right) \mathcal{E}\left(\int_0^T \varphi_r dB_r\right)
\times \exp\left(\int_0^T \varphi_r u_n(r, x + B_r) + \varphi_r \tilde{b}_2(r) + \tilde{b}_2(r) u_n(r, x + B_r) dr\right).
$$

Thus, $I_{2,n}$ is bounded by similar argument as in the proof of Lemma 2.3 since $\tilde{\phi}$ is bounded. Using the dominated convergence theorem, we get that $I_{3,n}$ and $I_{4,n}$ converge to 0 as $n$ goes to infinity. \hfill \Box

The following result is a corollary of the compactness result given by Lemma 2.3 and Corollary 2.4.

**Proposition 2.6.** For any fixed $t \in [0, T]$, with $T$ small and $x \in \mathbb{R}$, the sequence $(X_t^{n,x})_{n \geq 1}$ of strong solutions to the SDE (2.1) converges strongly in $L^2(\Omega, P; \mathbb{R})$ to $E\left[X_t^x|\mathcal{F}_t\right]$.

**Proof.** Observe that by the compactness criteria for each $t$, there exists a subsequence $(X_t^{x,n_k})_{k \geq 1}$ that converges strongly in $L^2(\Omega, P)$. From the previous lemma, we get by setting $h(x) = x, x \in \mathbb{R}$ that $(X_t^{x,n_k})_{n \geq 1}$ converges weakly to $E\left[X_t^x|\mathcal{F}_t\right]$ in $L^2(\Omega, P)$ and therefore by the uniqueness of the limit there exists a subsequence $n_k$ such that $(X_t^{x,n_k})_{n \geq 1}$ converges strongly to $E\left[X_t^x|\mathcal{F}_t\right]$ in $L^2(\Omega, P)$. The convergence then holds for the entire sequence by uniqueness of limit. Indeed, by contradiction, suppose that for some $t, i$, there exist $\epsilon > 0$ and a subsequence $n_i, i \geq 0$ such that

$$
||X_t^{x,n_j} - E\left[X_t^x|\mathcal{F}_t\right]||_{L^2(\Omega, P)} \geq \epsilon.
$$

We also know by the compactness criteria that there exists a further subsequence of $n_m, m \geq 0$ of $n_i, i \geq 0$ such that

$$
X_t^{x,n_m} \text{ converges to } \bar{X}_t \text{ in } L^2(\Omega, P) \text{ as } m \text{ goes to } \infty.
$$

Nevertheless, $(X_t^{x,n_k})_{n \geq 1}$ converges weakly to $E\left[X_t^x|\mathcal{F}_t\right]$ in $L^2(\Omega, P)$, and hence by the uniqueness of the limit, we have

$$
\bar{X}_t = E\left[X_t^x|\mathcal{F}_t\right].
$$

13
Since
\[ \|X_t^{x,n,m} - E[X_t^x]|F_t]\|_{L^2(\Omega, P)} \geq \epsilon, \]
this is a contradiction. \qed

### 3.2.4. Adaptedness of the weak solution and uniqueness.

Finally, we show that the weak solution \( X_t^x \) is \((F_t)_{t \in [0,T]}\)-adapted and unique.

**Theorem 2.7.** The weak solution \( X_t^x \) to the SDE (1.5) is \( F_t \)-measurable.

**Proof.** Let us first show that \( X_t^x \) is \( F_t \)-measurable. Let \( h \) be a globally Lipschitz continuous function. By Proposition 2.6, there exists a subsequence \( n_k, k \geq 0 \), such that \( h(X_t^{x,n_k}) \) converges to \( h(E[X_t^x]|F_t]) \) P-a.s. as \( k \) goes to infinity. Moreover, we know that \( h(X_t^{x,n_k}) \) converges to \( E[h(X_t^x)|F_t] \) weakly in \( L^2(\Omega, P) \) as \( k \) goes to infinity. We get from the uniqueness of the limit that
\[ h(E[X_t^x]|F_t]) = E[h(X_t^x)|F_t] \quad \text{P-a.s.} \]
Since the above holds for any arbitrary globally Lipschitz continuous function, it follows that \( X_t^x \) is \( F_t \)-measurable. \Box

**Proposition 2.8.** The SDE (1.5) satisfies the pathwise uniqueness property.

**Proof.** Let \( X_t^x \) and \( \tilde{X}_t^x \) be two solutions to the SDE (1.5). For \( \varphi \in C_b^1([0,T], \mathbb{R}^d) \), we have
\[ E[X_t^x \mathcal{E} \left( \int_0^T \varphi(u)dB_u \right)] = \int_\Omega X_t^x(\omega + \varphi) dP(\omega), \quad (2.27) \]
where, as shown in the course of the proof of Lemma 2.5, \( X_t^x(\omega + \varphi) \) satisfies the SDE (2.22). Similarly \( \tilde{X}_t^x(\omega + \varphi) \) satisfies the same SDE. Thus, since the drift is of linear growth, it follows that \( (X_t^x(\omega + \varphi), B) \) and \( (\tilde{X}_t^x(\omega + \varphi), B) \) have the same distribution. In fact, using that the distributions \( P^x \) and \( \tilde{P}^x \) of \( X^x \) and \( \tilde{X}^x \), respectively are equal to \( P \) (see the construction in [22, Proposition 3.6]) it follows that by assumptions on \( b \) and the linear growth of \( b_1 \) that
\[ \int_0^T |b_1(t, X^x(t)) + (\sigma \dot{\varphi}_t + b_2(t, \omega + \varphi))|^2 dt < \infty \quad \text{P-a.s.} \]
The same holds if \( X^x \) is replaced by \( \tilde{X}^x \) and \( P^x \) by \( \tilde{P}^x \). Thus, a simple adaptation of the proof of [22, Proposition 3.10] shows that \( (X_t^x(\omega + \varphi), B) \) and \( (\tilde{X}_t^x(\omega + \varphi), B) \) have the same distribution. Hence, for all \( t, \varphi \), we have
\[ E\left[ X_t^x \mathcal{E} \left( \int_0^T \dot{\varphi}_u dB_u \right) \right] = E\left[ \tilde{X}_t^x \mathcal{E} \left( \int_0^T \dot{\varphi}_u dB_u \right) \right] \]
from which pathwise uniqueness follows. \Box

### 3.2.5. Global existence.

Since the small time \( T_1 \) for which the solution exists does not depend on the initial condition (see (2.12)) one can use a standard pasting argument to show that the solution exists for all time \( T > 0 \). In addition using the linear growth condition on \( b_1 \) and the integrability condition on \( b_2 \), it follows from Gronwall lemma that the unique solution does not explode.

The proof of Theorem 1.1 is now complete.

### 3. Malliavin differentiability

#### 3.1. Differentiability of the strong solution

In this subsection, we show that the unique strong solution of the SDE (1.5) constructed in the previous subsection is Malliavin differentiable and we derive a representation formula of the Malliavin derivative.
Theorem 3.1. Assume that \( (A1)-(A2) \) hold. Let \( X \) be the unique strong solution to the SDE (1.5). It holds \( X_t \in D^{1,2}(\mathbb{R}) \) for all \( t \in [0, T] \).

The proof of this result for small time interval follows directly from Lemma 2.3. In order to control the Malliavin derivative of the process on arbitrary time intervals, we need the following result whose proof is similar to that of [28, Lemma 2.6].

**Lemma 3.2.** Let \( t_0 \in [0, 1] \) and let \( \eta : \Omega \to \mathbb{R} \) be a \( \mathcal{F}_{t_0} \)-measurable random variable independent of the \( P \)-augmented filtration generated by the Brownian motion \( B \). Let \( b : [0, 1] \times \Omega \times \mathbb{R} \to \mathbb{R} \) such that \( b = b_1 + b_2 \) satisfies \((A1)\), with \( b_1 \) a smooth coefficient with compact support satisfying a global linear growth condition. Denote by \( X_{t_0}^{\eta} \) the unique strong solution (if it exists) to the SDE (1.5) starting at \( \eta \) and with drift coefficient \( b \). Then we can find a positive number \( \delta_0 \) independent of \( t_0 \) and \( \eta \) (but may depend on \( \|b_1\|_{\infty} \)) such that

\[
E \exp[\delta_0 \sup_{t_0 \leq t \leq 1} |X_t^{t_0, \eta}|^2] \leq C_1 E \exp\{C_2 \delta_0 \eta |^2\},
\]

where \( C_1, C_2 \) are positive constants independent of \( \eta \), but may depend on \( k_1 \) and \( b_2 \). Moreover, \( C_1 \) may depend on \( \delta_0 \).

Furthermore, if the right hand side of (3.1) is finite then the above expectation is finite.

**Proof.** We have the following almost sure equality

\[
X_t^{t_0, \eta} = \eta + \int_{t_0}^t \left( b_1(u, X_u^{t_0, \eta}) + b_2(u, \omega) \right) du + \sigma \cdot (B_t - B_{t_0}), \quad t_0 \leq t \leq 1.
\]

Successive application of Hölder’s inequality to (3.2) yields

\[
|X_t^{t_0, \eta}|^2 \leq 4|\eta|^2 + 4 \left( \int_{t_0}^t |b(u, X_u^{t_0, \eta})|^2 du \right)^2 + 4 \left( \int_{t_0}^t |b_2(u, \omega)|^2 du \right)^2 + 4|\sigma|^2 |B_t - B_{t_0}|^2
\]

\[
\leq 4|\eta|^2 + 4 \left( \int_{t_0}^t k_1 (1 + |X_u^{t_0, \eta}|) du \right)^2 + 4 \left( \int_{t_0}^t |b_2(u, \omega)|^2 du \right)^2 + 4|\sigma|^2 |B_t - B_{t_0}|^2
\]

\[
\leq 4|\eta|^2 + 8k_1^2 (t - t_0) \left( \int_{t_0}^t (1 + |X_u^{t_0, \eta}|^2) du + 4(t - t_0) \int_{t_0}^t |b_2(u, \omega)|^2 du + 4|\sigma|^2 |B_t - B_{t_0}|^2 \right)
\]

\[
\leq 4|\eta|^2 + 8k_1^2 (t - t_0)^2 + 8k_1^2 (t - t_0) \int_{t_0}^t |X_u^{t_0, \eta}|^2 du + 4(t - t_0) \int_{t_0}^t |b_2(u, \omega)|^2 du + 4|\sigma|^2 |B_t - B_{t_0}|^2, \quad \text{a.s.}
\]

\[
(3.3)
\]

Take the supremum on both sides of (3.3) and multiply by \( \delta_0 \) to get

\[
\delta_0 \sup_{t_0 \leq t \leq 1} |X_t^{t_0, \eta}|^2 \leq 4\delta_0|\eta|^2 + 8k_1^2 \delta_0 + 8k_1^2 \int_{t_0}^1 \delta_0 \sup_{0 \leq u \leq s} |X_u^{t_0, \eta}|^2 ds
\]

\[
+ 4(t - t_0) \delta_0 \int_{t_0}^1 |b_2(u, \omega)|^2 du + 4|\sigma|^2 \delta_0 \sup_{t_0 \leq t \leq 1} |B_t - B_{t_0}|^2, \quad \text{a.s.}
\]

\[
(3.4)
\]
Applying Gronwall’s lemma to (3.4), we have
\[
\delta_0 \sup_{t_0 \leq t \leq 1} |X_t^{\tau,\omega}|^2 \leq \left\{4\delta_0|\eta|^2 + 8k_1^2\delta_0 + 4\delta_0|\sigma|^2 \sup_{t_0 \leq u \leq 1} |B_t - B_{t_0}|^2 + 4\delta_0 \int_{t_0}^1 |b_2(u, \omega)|^2 \, du \right\} e^{8k_1^2}, \text{ a.s.}
\]
Now, set \( C_2 := 4e^{8k_1^2} \). Then taking exponential on both sides of (8), we have
\[
\exp \left\{ \delta_0 \sup_{t_0 \leq t \leq 1} |X_t^{\tau,\omega}|^2 \right\} \leq \exp \left\{ 2C_2\delta_0 k_2^2 \right\} \times \exp \left\{ \delta_0 C_2 |\eta|^2 \right\} \times \exp \left\{ C_2\delta_0 |\sigma|^2 \sup_{t_0 \leq u \leq 1} |B_u - B_{t_0}|^2 \right\} \times \exp \left\{ C_2\delta_0 \int_{t_0}^1 |b_2(u, \omega)|^2 \, du \right\}, \text{ P-a.s.} \quad (3.5)
\]
Taking expectations in the above and using once more Hölder inequality, we get
\[
E \exp \left\{ \delta_0 \sup_{t_0 \leq t \leq 1} |X_t^{\tau,\omega}|^2 \right\} \leq \exp \left\{ 2C_2\delta_0 k_2^2 \right\} \cdot E \left[ \exp \left\{ 3C_2\delta_0 |\eta|^2 \right\} \right] \times \exp \left\{ 3C_2\delta_0 |\sigma|^2 \sup_{t_0 \leq u \leq t} |B_u - B_{t_0}|^2 \right\} \frac{1}{2} \times E \left[ \exp \left\{ 3C_2\delta_0 \int_{t_0}^1 |b_2(u, \omega)|^2 \, du \right\} \right]^\frac{1}{2} \quad (3.6)
\]
The result follows provided that we find \( \delta_0 \) independent of \( \eta \) and \( t_0 \) such that
\[
E \left[ \exp \left\{ 3C_2\delta_0 |\sigma|^2 \sup_{t_0 \leq u \leq 1} |B_u - B_{t_0}|^2 \right\} \right] < \infty. \quad (3.7)
\]
(3.7) is obtained from the use of the exponential series expansion of the left hand side followed by Doob’s maximal inequality. The ratio test gives convergence of the series for instance if \( \delta_0 < \min \left( \frac{1}{12C_2 |\sigma|^2}, \frac{1}{C_2} \right) \) (see for example [28, Lemma 2.6] for details). From this (3.6) yields
\[
E \exp \left\{ \delta_0 \sup_{t_0 \leq t \leq 1} |X_t^{\tau,\omega}|^2 \right\} \leq C_1 E e^{C_2\delta_0 |\eta|^2}. \quad (3.8)
\]
Note that \( C_1, C_2 \) and \( \delta_0 \) are independent of \( \eta \) and \( t_0 \) (but may depend on \( \|\tilde{b}_1\|_\infty \) and \( |\sigma|^2 \)). Thus (3.1) is valid for the above choice of \( \delta_0 \).

Next, using Lemma 3.2, we prove that under the conditions of Theorem 1.1 the Malliavin derivative is bounded in the \( L^2(\Omega, P) \) norm.

**Proof (of Theorem 3.1).** First recall that by the second part of Lemma 2.3, the sequence of strong solutions of the SDE (2.1) satisfies
\[
\sup_{0 \leq t \leq T} \sup_{n \geq 1} E \left[ |D_t X_s^{\tau,\omega}|^2 \right] \leq C(\|\tilde{b}_1\|_\infty, |x|, b_2^{\text{power}}).
\]
Since the Malliavin derivative is a closable operator (see e.g. [34, Exercise 1.2.3]), it follows from Proposition 2.6 and Theorem 2.7 that \( X_t^\tau \) is Malliavin smooth.

It remains to prove integrability of the derivative. This is done by induction. Choose \( \delta_0 \) as in Lemma 3.2 and define
\[
\tau := \frac{\delta_0}{64d\sqrt{2k^2}} \leq T_1,
\]
16
let $s_i = i\tau$ and $x_i := X_{s_i}^{x,n}, i \geq 1$. It follows form the previous argument that the result is valid for $0 \leq t \leq s_1$.

Assume that there exists a Malliavin differentiable solution $\{X_t, 0 \leq t \leq s_m\}$. Set $t$ such that $s_m \leq t < s_{m+1}$. Let $(X_t^{x,n})_{n \geq 1}$ be the approximating sequence defined by (2.1) it satisfies the a.s.

relation

$$X_t^{x,m,n} = X_{s_m}^n + \int_{s_m}^t b_n(u, \omega, X_u^{x,m,n})du + \sigma (B_2 - B_{s_m}), \quad s_m \leq t \leq s_{m+1}.$$  

Using the chain-rule for the Malliavin derivatives, we have component wise

$$D^i_s X^{x,m,n} = \begin{cases} 
D^i_s X_{s_m}^n + \int_{s_m}^t \left( b_{i,1,n}(u, X_u^{x,m,n})D^i_s X_u^{x,m,n} + D^i_s b_2(t, \omega) \right)du & \text{if } s \leq s_m \\
\sigma_i + \int_{s_m}^t b_{i,1,n}(u, X_u^{x,m,n})D^i_s X_u^{x,m,n} + D^i_s b_2(t, \omega)du & \text{if } s > s_m \end{cases}$$  

(3.9)

P.a.s., for all $0 \leq s \leq t$ and solving explicitly gives

$$D^i_s X^{x,m,n} = \begin{cases} 
eq \left( f_{s_m}^t b_{i,1,n}(u, X_u^{x,m,n})du \left( \int_{s_m}^t D^i_s b_2(u, \omega)e^{-f_{s_m}^u b_{i,1,n}(r, X_r^{x,m,n})dr}du + D^i_s X_{s_m}^n \right) & \text{if } s \leq s_m \\
\sigma_i + \int_{s_m}^t b_{i,1,n}(u, X_u^{x,m,n})du \left( \int_{s_m}^t D^i_s b_2(u, \omega)e^{-f_{s_m}^u b_{i,1,n}(r, X_r^{x,m,n})dr}du + \sigma_i \right) & \text{if } s > s_m \end{cases}$$  

(3.10)

P.a.s., for all $0 \leq s \leq t$. Let us first focus on $D^i_s X^{x,m,n}$, when $s \leq s_m$. Using Hölder inequality, we have

$$E \left[ |D^i_s X^m|^2 \right] \leq 2E \left[ \left( \int_{s_m}^t b_{i,1,n}(u, X_u^{x,m,n})du \right)^2 \right]$$

+ $2E \left[ \left( \int_{s_m}^t b_{i,1,n}(u, X_u^{x,m,n})du \right)^2 \right]$

$$= 2E \left[ \left( \int_{s_m}^t b_{i,1,n}(u, X_u^{x,m,n})du \right)^2 \right]$$

+ $2E \left[ \left( \int_{s_m}^t b_{i,1,n}(u, X_u^{x,m,n})du \right)^2 \right]$

$$= I_1 + I_2.$$

Let us now consider the conditional expectation part in $I_2$. As before, using Girsanov theorem and Hölder inequality, we have

$$E \left[ \int_{s_m}^t b_{i,1,n}(u, X_u^{x,m,n})du \right] \left| \mathcal{F}_{s_m} \right|$$

$$= E \left[ \mathcal{E} \left( \int_{s_m}^t b_{i,1,n}(u, X_u^{x,m,n} + \sigma \cdot (B_u - B_{s_m}))du \right) \right] \left| \mathcal{F}_{s_m} \right|$$

$$\leq E \left[ \left( 2 \sum_{i=1}^m \int_{s_m}^t u_{i,n}(r, \omega, X_{s_m}^n + \sigma \cdot (B_r - B_{s_m}))dr \left| \mathcal{F}_{s_m} \right| \right)^\frac{1}{2} \right]$$

$$\times E \left[ \left( \left( \sum_{i=1}^m \int_{s_m}^t u_{i,n}^2(r, \omega, X_{s_m}^n + \sigma \cdot (B_r - B_{s_m}))dr \left| \mathcal{F}_{s_m} \right| \right)^\frac{1}{2} \right]$$

$$\leq C e^{\frac{3m^2}{2}} \tau^{1/2} (1 + |X_{s_m}^n|^2).$$  

(3.11)
Next let us consider the last term. We can show as in [28, Proposition 4.10 and (4.28)] that there exists a constant $C > 0$ such that

$$E \left[ e^{s I^1_{s_m}} \right] \leq e^{\tau C k_1 |X^n_m|}. \quad (3.13)$$

Combining (3.11)-(3.13) and using Hölder inequality, we get

$$I_2 \leq C E \left[ e^{\frac{4}{4}r(1+|X^n_m|^2)} e^{\tau C k_1 |X^n_m||D^n_{s_m} X^n_s|^2} \right]$$

$$\leq C E \left[ e^{\frac{1}{4}2k_1 (1+|X^n_m|^2)} e^{\tau C k_1 |X^n_m|} \right] e^{\tau C k_1 |X^n_m|^2} \right] \leq C e^{C_{2\delta_0} |x|^2}. \quad (3.14)$$

Let us notice that one can show as in the proof of Lemma 2.3 that there exists $C > 0$ such that $E[|D^n_{s_m} X^n_m|^p] \leq C$ for $p \geq 1$ and by induction hypothesis it follows that $E[|D^n_{s_m} X^n_m|^p] \leq C$ for $p = 4$. The choice of $\tau$ combined with Lemma 3.2 ensures that one can find $C > 0$ such that $E[e^{12k_1 r(1+|X^n_m|^2)}] \leq C e^{C_{2\delta_0} |x|^2}$. Furthermore, using successive approximation, one can show that $E[e^{4 C k_1 |X^n_m|}] \leq C_1 e^{C_2 k_1 |x|}$, where $C_1$ depends on $b_2$ and $C_2$ is a positive constant.

This can be shown as in the proof of Lemma 3.2 using the Gronwall lemma and the probability distribution of the Brownian motion.

The case $s > s_m$ is similar (and easier) since the term $D^n_{s_m} X^n_m$ is not involved, it is replaced by the constant $\sigma$.

Since $b_2 < \infty$, using the Hölder inequality, the term $I_1$ can also be bounded using similar arguments as above. Thus the Malliavin derivative of the approximating sequence $(X^n_m)_{n \geq 1}$ has a uniform bound on $[0,T]$ which does not depend on $n$. Therefore, $D_x X_t \in L^2(\Omega, P; \mathbb{R}^d)$ for $0 \leq s \leq t \leq T$.

Remark 3.3. Let us observe that if $b_1$ is globally bounded then it follows from the condition on $b_2$ and Girsanov theorem that Lemma 2.3 holds for all $T$. Therefore, we do not need the above argument and the Malliavin differentiability of the solution directly follows from the compactness argument.

3.2. Representation and moment bounds for the Malliavin derivative

In this subsection we give an explicit representation of the Malliavin derivative $DX^x$ of the solution $X^x$ of the SDE (1.5). Such representation can be very useful to derive results concerning $DX^x$. See e.g. Theorem 3.7 for an application. The representation we obtain will be given in terms of the time-space local time studied in details in [9].

In order to define the local time-space integral with respect to $L^{X^x}(t, z)$, we first start by introducing the space $(\mathcal{H}^x, \|\|)$ of functions $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ with the norm

$$\|f\|_x := 2 \left( \int_0^1 \int_\mathbb{R} f^2(s, z) \exp \left( \frac{|z-x|^2}{2 s} \right) \frac{dx \, dz}{\sqrt{2\pi s}} \right)^{\frac{1}{2}} + \int_0^1 \int_\mathbb{R} |f(s, x)| \exp \left( - \frac{|z-x|^2}{2 s} \right) \frac{dx \, dz}{s \sqrt{2\pi s}}$$

See for example [9]. Endowed with this normed, $(\mathcal{H}^x, \|\|)$ is a Banach space. It follows from [2, Lemma 2.7 and Definition 2.8] that the local time-space integral of $f \in \mathcal{H}^x$ with respect to $L^{X^x}(t, z)$ is well defined and we have

$$\int_0^T f(s, z) L^{X^x}(ds, dz) = \int_0^T f(s, z) I_{[0,t]}(s) L^{X^x}(ds, dz). \quad (3.15)$$

Let us point that as already observed in [2, Remark 2.9], functions $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ of spacial linear growth uniformly in $t$ belong to $\mathcal{H}^x$ and thus the above local space-time integral exists for $x \in \mathbb{R}$. 

18
We will also need the following representation which will play a key role in our argument (see for example [2, Lemma 2.11]). Let \( f \in \mathcal{H} \) be Lipschitz continuous in space. Then for all \( t \in [0, T] \), it holds
\[
\int_0^t f'(s, X_x^z) \, ds = - \int_0^t f(s, z)L^{X^z}(ds, dz). \tag{3.16}
\]
Moreover, the local time-space integral of \( f \in \mathcal{H}_0 \) admits the decomposition (see the proof of Theorem 3.1 in [9])
\[
\int_0^t \int \mathbb{R} f(s, z) L^{B^z}(ds, dz) = \int_0^t f(s, B^z_s) \, dB_s + \int_{T-t}^T f(T-s, \tilde{B}^z_s) \, d\tilde{W}_s - \int_{T-t}^T f(T-s, \tilde{B}^z_s) \frac{\tilde{B}_s}{T-s} \, ds,
\]
for every \( 0 \leq t \leq T \), a.s., \( B^z \) is the Brownian motion started at \( x \) and \( \tilde{B} \) is the time-reversed Brownian motion, that is
\[
\tilde{B}_t \coloneqq B_{T-t}, \quad 0 \leq t \leq T. \tag{3.17}
\]
Further, the process \( \tilde{W}_t, \) \( 0 \leq t \leq T \), is an independent Brownian motion with respect to the filtration \( \mathcal{F}_t^{\tilde{B}} \) generated by \( \tilde{B}_t \) and satisfies:
\[
\tilde{W}_t = \tilde{B}_t - B_T + \int_t^T \frac{\tilde{B}_s}{T-s} \, ds. \tag{3.18}
\]

We are now ready to give an explicit representation of the Malliavin derivative of the unique strong solution to the SDE (1.5) in terms of a local time integral.

**Theorem 3.4.** Assume that the conditions of Theorem 1.1 are satisfied. Assume in addition the following:

\[
\sup_{0 \leq s \leq T} E \left[ \left( \int_0^T |D_s b_2(t, \omega + \varphi)|^2 \, dt \right)^{2 \epsilon} \right] < \infty \tag{3.19}
\]

for every \( \varphi \in C^1_b([0, T], \mathbb{R}^d) \). For every \( 0 \leq s \leq t \leq T \), the \( i \)-th component of the Malliavin derivative of the unique strong solution to the SDE (1.5) admits the following representation:
\[
D^i_t X^z_t = e^t \int_0^t \int_0^T \mathbb{R} D^i_s b_2(u, \omega) e^{-t \int_0^u f_k \, dr} b_1(r, z) L^{X^z}(dr, dz) \, du + \sigma_t, \quad i = 1, \ldots, d. \tag{3.20}
\]

\( L^{X^z}(ds, dz) \) is the integration with respect to the time-space local time of \( X^z \).

Before proving the above theorem we will need some auxiliary results. The next one generalises [2, Lemma A.2] to the case where the integrand is of spatial linear growth.

**Remark 3.5.** Let us notice that using Cameron-martin-Girsanov theorem, one can show that the bound (3.19) holds if for \( \epsilon > 0 \) small, we have
\[
\sup_{0 \leq s \leq T} E \left[ \int_0^T |D_s b_2(t, \omega)|^{4+\epsilon} \, dt \right] < \infty. \tag{3.21}
\]
Lemma 3.6. Let \( f : [0, T] \times \mathbb{R} \to \mathbb{R} \) be of spacial linear growth uniformly in \( t \). Then for every \( t \in [0, T] \), \( k \in \mathbb{R} \) and subset \( K \subset \mathbb{R} \), it holds
\[
\sup_{x \in K} E \left[ \exp \left( k \int_0^t \int f(s, z)L^B^x(ds, dz) \right) \right] < \infty,
\]
provided that \( T \) is small enough. In the above, \( L^B^x(ds, dz) \) is the integration with respect to the time-space local time of \( B^x \).

Proof. It follows from (3.2) and the Hölder inequality that
\[
E \left[ \exp \left( k \int_0^t \int f(s, z)L^B^x(ds, dz) \right) \right] \leq \left( E \left[ \exp \left( 2k \int_0^t f(s, B^x_s)dB^x_s \right) \right] \right)^{\frac{1}{2}} \times \left( E \left[ \exp \left( 4k \int_{T-t}^T f(T-s, B^x_{T-s})dW^x_s \right) \right] \right)^{\frac{1}{4}} \times \left( E \left[ \exp \left( -4k \int_{T-t}^T f(T-s, B^x_{T-s}) \frac{B^x_{T-s}}{T-s}ds \right) \right] \right)^{\frac{1}{8}} = I_1 \times I_2 \times I_3.
\]

Let us consider \( I_1 \). Using Hölder inequality, we have
\[
E \left[ \exp \left( 2k \int_0^t f(s, B^x_s)dB^x_s \right) \right] \leq E \left[ \exp \left( 8k^2 \int_0^t f^2(s, B^x_s)ds \right) \right]^{\frac{1}{8}}.
\]
The Girsanov theorem applied to the martingale \( 2k \int_0^t f(s, B^x_s)dB^x_s \) yields that the first term in (2.9) is equal to one. Similar arguments as in the proof of Lemma 2.3 (i.e. using power series expansion of the exponential function) enables to conclude that the above the second term is finite for small \( T \).

Next we wish to study the boundedness \( I_3 \). It was already shown in [2, Lemma A.2] that
\[
E \left[ \exp \left( k \int_0^T \frac{|B^x_{T-s}|}{T-s}ds \right) \right] < \infty.
\]
Hence to show the boundedness of \( I_3 \), it suffices to show that
\[
E \left[ \exp \left( k \int_0^T \frac{|B^x_{T-s}|^2}{T-s}ds \right) \right] < \infty
\]
for \( T \) small enough. Indeed, using exponential expansion, and the Hölder inequality, we have
\[
E \left[ \exp \left( k \int_0^T \frac{|B^x_{T-s}|^2}{T-s}ds \right) \right] = \sum_{n=1}^{\infty} \frac{1}{n!} E \left[ \left( k \int_0^T \frac{|B^x_{T-s}|^2}{T-s}ds \right)^n \right] \leq \sum_{n=1}^{\infty} \frac{k^n}{n!} \int_0^T \frac{E|B^x_{T-s}|^{2n}}{(T-s)^n}ds \times T^{n-1} = \sum_{n=1}^{\infty} \frac{k^n}{n! \cdot 2^{n(n!)}} T \times T^{n-1} = \sum_{n=1}^{\infty} (Tn)^n \frac{(2n)!}{2^{n(n!)}}.
\]
Using once more the ratio test, one deduces that the above sum is finite for small \( T \). Combining arguments for the bounds of \( I_1 \) and \( I_3 \) enables to conclude that \( I_2 \) is bounded as well. \( \square \)
Proof (of Theorem 3.4). Let $b_{1,n}$ be a sequence of smooth drifts approximating $b_1$. Then, using (2.6) and (3.16), we have

\[
D_t^X u_{z,n} = e^{-\int_t^s f_u b_{1,n}(u,z)L^{X_{z,u}}(du, dz)} \left( \int_t^s D_t^b b_2(u, \omega)e^{\int_t^s f_u b_{1,n}(r,z)L^{X_{z,u}}(dr, dz)} du + \sigma_t \right). \tag{3.24}
\]

It follows from Corollary 2.4 that $(X_{z,u}^{X^n})_{n \geq 1}$ is relatively compact in $L^2(\Omega, P)$ and $(\|D_s X_t^{X^n}\|_{L^2(P \otimes dt)})_{n \geq 1}$ is uniformly bounded in $n$. Hence by [34, Lemma 1.2.3], $(D_s X_t^{X^n})_{n \geq 1}$ converges weakly to $D_s X_t^X$ in $L^2(P; \mathbb{R})$. Thus, in order to conclude, we need to show that

\[
\left\{ e^{-\int_t^s f_u b_{1,n}(u,z)L^{X_{z,u}}(du, dz)} \left( \int_t^s D_t^b b_2(u, \omega)e^{\int_t^s f_u b_{1,n}(r,z)L^{X_{z,u}}(dr, dz)} du + \sigma_t \right) \mathbb{E} \left( \int_0^T \phi_t dB_t \right) \right\}_{n}
\]

converges to

\[
e^{-\int_t^s f_u b_{1,n}(u,z)L^{X_{z,u}}(du, dz)} \left( \int_t^s D_t^b b_2(u, \omega)e^{\int_t^s f_u b_{1,n}(r,z)L^{X_{z,u}}(dr, dz)} du + \sigma_t \right) \mathbb{E} \left( \int_0^T \phi_t dB_t \right)
\]

in expectation for every $\phi \in C_b^1([0, T], \mathbb{R}^d)$. We will only show that

\[
\left\{ e^{-\int_t^s f_u b_{1,n}(u,z)L^{X_{z,u}}(du, dz)} \int_t^s D_t^b b_2(u, \omega)e^{\int_t^s f_u b_{1,n}(r,z)L^{X_{z,u}}(dr, dz)} du \mathbb{E} \left( \int_0^T \phi_t dB_t \right) \right\}_{n}
\]

converges to $e^{-\int_t^s f_u b_{1,n}(u,z)L^{X_{z,u}}(du, dz)} \int_t^s D_t^b b_2(u, \omega)e^{\int_t^s f_u b_{1,n}(r,z)L^{X_{z,u}}(dr, dz)} du \mathbb{E} \left( \int_0^T \phi_t dB_t \right)$ in expectation. Using Girsanov theorem and the Cameron-Martin theorem, we have

\[
L = \mathbb{E} \left[ \left( \int_0^T \phi_t dB_t \right) \left\{ e^{-\int_t^s f_u b_{1,n}(u,z)L^{X_{z,u}}(du, dz)} \int_t^s D_t^b b_2(u, \omega)e^{\int_t^s f_u b_{1,n}(r,z)L^{X_{z,u}}(dr, dz)} du \right\} \right]
\]

\[
- e^{-\int_t^s f_u b_{1,n}(u,z)L^{X_{z,u}}(du, dz)} \int_t^s D_t^b b_2(u, \omega)e^{\int_t^s f_u b_{1,n}(r,z)L^{X_{z,u}}(dr, dz)} du \mathbb{E} \left( \int_0^T \phi_t dB_t \right)
\]

\[
= \mathbb{E} \left[ \left( \int_0^T \phi_t dB_t \right) \left\{ e^{-\int_t^s f_u b_{1,n}(u,z)L^{X_{z,u}}(du, dz)} \int_t^s D_t^b b_2(u, \omega)e^{\int_t^s f_u b_{1,n}(r,z)L^{X_{z,u}}(dr, dz)} du \right\} \right]
\]

\[
- e^{-\int_t^s f_u b_{1,n}(u,z)L^{X_{z,u}}(du, dz)} \int_t^s D_t^b b_2(u, \omega)e^{\int_t^s f_u b_{1,n}(r,z)L^{X_{z,u}}(dr, dz)} du \mathbb{E} \left( \int_0^T \phi_t dB_t \right)
\]

\[
+ e^{-\int_t^s f_u b_{1,n}(u,z)L^{X_{z,u}}(du, dz)} \int_t^s D_t^b b_2(u, \omega)e^{\int_t^s f_u b_{1,n}(r,z)L^{X_{z,u}}(dr, dz)} du \mathbb{E} \left( \int_0^T \phi_t dB_t \right)
\]

\[
- e^{-\int_t^s f_u b_{1,n}(u,z)L^{X_{z,u}}(du, dz)} \int_t^s D_t^b b_2(u, \omega)e^{\int_t^s f_u b_{1,n}(r,z)L^{X_{z,u}}(dr, dz)} du \mathbb{E} \left( \int_0^T \phi_t dB_t \right)
\]
\[ E \left[ \int_0^T \psi_t d\mathcal{B}_t \right] \left\{ e^{-f_r^t f_b b_{1,n}(u,z)L^{X,n}}(du,dz) \right\} \]

\[ \times \int_t^s D_1^t b_2(u, \omega) \left( e^{f_r^t f_b b_{1,n}(r,z)L^{X,n}}(dr,dz) - e^{f_r^s f_b b_{1,n}(r,z)L^{X,n}}(dr,dz) \right) \, du \]

\[ + \left( e^{-f_r^t f_b b_{1,n}(u,z)L^{X,n}}(du,dz) - e^{-f_r^s f_b b_{1,n}(u,z)L^{X,n}}(du,dz) \right) \int_t^s D_1^t b_2(u, \omega) e^{f_r^s f_b b_{1,n}(r,z)L^{X,n}}(dr,dz) \, du \]

\[ \leq I_{1,n} + I_{2,n}. \]  

(3.25)

Let us concentrate on \( I_{1,n} \). Repeated use of Hölder inequality, Girsanov transform, the bound on \( D_1^t b_2(u, \omega + \phi) \) and the fact that \( |e^x - 1| \leq |x|(e^x + 1) \) gives

\[ I_{1,n} \leq E \left[ e^{-2f_r^t f_b b_{1,n}(u,z)L^{X,n}}(du,dz) \right]^{1/2} \times E \left[ \left( \int_t^s \left( (D_1^t b_2(u, \omega + \phi))^2 \, du \right) \right)^{2/3} \right]^{1/4} \]

\[ \leq C E \left[ \int_t^s \left\{ \int_0^T \tilde{u}_n(r, \omega, x + \sigma \cdot B_t) + \tilde{\phi}_r \right\} dB_t \right] e^{-2f_r^t f_b b_{1,n}(r,z)L^{X,n}}(dr,dz) \]

\[ \times \left( \int_t^s E \left[ \left( e^{f_r^t f_b b_{1,n}(r,z)L^{X,n}}(dr,dz) - e^{f_r^s f_b b_{1,n}(r,z)L^{X,n}}(dr,dz) \right)^4 \right] \, du \right)^{1/4} \]

\[ \leq C E \left[ \int_0^T \left\{ \int_0^T \tilde{u}_n(r, \omega, x + \sigma \cdot B_t) + \tilde{\phi}_r \right\} dB_t \right]^{2/3} 1/4 E \left[ e^{-4f_r^t f_b b_{1,n}(r,z)L^{X,n}}(dr,dz) \right]^{1/4} \]

\[ \times \left( \int_t^s E \left[ \left( e^{f_r^t f_b b_{1,n}(r,z)L^{X,n}}(dr,dz) - e^{f_r^s f_b b_{1,n}(r,z)L^{X,n}}(dr,dz) \right)^{1/2} \right]^{1/2} \right)^{1/4} \]

\[ \times E \left[ \left( e^{f_r^t f_b b_{1,n}(r,z)L^{X,n}}(dr,dz) - e^{f_r^s f_b b_{1,n}(r,z)L^{X,n}}(dr,dz) \right)^{15} \right]^{1/2} \]
by Girsanov transform and Cauchy-Schwartz inequality, we have

\[ \mathbb{E}\left( \int_0^T \left\{ \tilde{u}_n(r, \omega, x + \sigma \cdot B_t) + \tilde{\varphi}_t \right\} dB_t \right)^2 \leq C \mathbb{E}\left( \int_0^T \left\{ \tilde{u}_n(r, \omega, x + \sigma \cdot B_t) + \tilde{\varphi}_t \right\} dB_t \right)^{1/4} \mathbb{E}\left[ \left. e^{-4f_t} f_{b,n(r,z)L^1\sigma\|B_t^z}\right| \delta \right]^{1/4}, \]

In the above, \( B_t^z := \sum_{i=1}^n \omega_i B_t^i \) is a standard Brownian motion. Using Cauchy-Schwartz inequality, the Novikov’s condition on \( b_2 \) and Beneš Theorem, the first term is finite for small time \( T \). Using Lemma 3.6 and [10, Proposition 2.1.1] enables to conclude that the second term is bounded. Using once more Cauchy-Schwartz inequality, Girsanov transform and Lemma 3.6, one deduces that the fourth and fifth terms are bounded for small time \( T \). Let us now focus on the third term. By Girsanov transform and Cauchy-Schwartz inequality, we have

\[ \mathbb{E}\left[ \int_0^T \left\{ \tilde{u}_n(r, \omega, x + \sigma \cdot B_t) + \tilde{\varphi}_t \right\} dB_t \right] \]

\( \leq C \mathbb{E}\left[ \int_0^T \left\{ \tilde{u}_n(r, \omega, x + \sigma \cdot B_t) + \tilde{\varphi}_t \right\} dB_t \right] \]
are satisfied. Then, the Malliavin derivative of the unique strong solution to the SDE (1.5) satisfies:

\[
E \left[ |D_t X^n_s - D_{t'} X^n_s|^2 \right] \leq C(\|b_1\|_{\infty}, |x|, b^{power}_n) |t - t'|^\alpha
\]

for 0 \leq t' \leq t \leq T_1, \alpha = \alpha(s) > 0 and

\[
\sup_{0 \leq t \leq T_1} E \left[ |D_t X^n_s|^2 \right] \leq C(\|\tilde{b}_1\|_{\infty}, |x|, b^{power}_n),
\]

where C : \([0, \infty)^3 \to [0, \infty)\) is continuous and increasing in each component and can differ from line to line, T_1 is small, and \(\|\tilde{b}_1\|_{\infty}\) defined in (2.4).

**Proof.** Let \(b_{1,n}\) be a sequence of smooth functions with compact support converging a.e. to \(b_1\) and such that \(b_{1,n}\) satisfy a uniform global linear growth; that is \(\|b_{1,n}\|_{\infty} \leq \|\tilde{b}_1\|_{\infty}\), see (2.4) for definition. Let \(X^n\) be the solution of the SDE (2.1) with drift \(b_n := b_{1,n} + b_2\). Then by Lemma 2.3, for every \(n \in \mathbb{N}\) it holds

\[
E \left[ |D_t X^n_s - D_{t'} X^n_s|^2 \right] \leq C(\|\tilde{b}_1\|_{\infty}, |x|, b^{power}_n) |t - t'|^\alpha
\]

for 0 \leq t' \leq t \leq T_1, \alpha = \alpha(s) > 0 and

\[
\sup_{0 \leq t \leq T_1} E \left[ |D_t X^n_s|^2 \right] \leq C(\|\tilde{b}_1\|_{\infty}, |x|, b^{power}_n),
\]

where the r.h.s. do not depend on \(n\). In particular, \(D_t X^n_s\) is bounded in \(L^2(\Omega, P)\). Thus, it follows from Corollary 2.4 and [34, Lemma 1.2.3] that (up to a subsequence) \((D_t X^n_s)\) converges to \(D_t X_s\) in the weak topology of \(L^2(\Omega, P)\). Since the function \(L^2 \ni A \mapsto E[|A|^2]\) is convex and lower semicontinuous, it is weakly lower semicontinuous. Thus, taking limits in \(n\) on both sides above gives the results. \(\square\)
A. Compactness criteria

The suggested construction of the strong solution for the SDE (1.5) is based on the subsequent relative compactness criteria from Malliavin calculus (see [36].)

**Theorem A.1.** Let \( \{ (\Omega, \mathcal{A}, P); H \} \) be a Gaussian probability space, that is \( (\Omega, \mathcal{A}, P) \) is a probability space and \( H \) a separable closed subspace of Gaussian random variables in \( L^2(\Omega) \), which generate the \( \sigma \)-field \( \mathcal{A} \). Denote by \( D \) the derivative operator acting on elementary smooth random variables in the sense that

\[
D(f(h_1, \ldots, h_n)) = \sum_{i=1}^n \partial_i f(h_1, \ldots, h_n)h_i, \quad h_i \in H, f \in C^\infty_0(\mathbb{R}^n).
\]

Further let \( D_{1,2} \) be the closure of the family of elementary smooth random variables with respect to the norm

\[
\|F\|_{1,2} := \|F\|_{L^2(\Omega)} + \|DF\|_{L^2(\Omega; H)}.
\]

Assume that \( C \) is a self-adjoint compact operator on \( H \) with dense image. Then for any \( c > 0 \) the set

\[
G = \{ G \in D_{1,2} : \|G\|_{L^2(\Omega)} + \|C^{-1}DG\|_{L^2(\Omega; H)} \leq c \}
\]

is relatively compact in \( L^2(\Omega) \).

The relative compactness criteria in this paper required the following result (see [36, Lemma 1]).

**Lemma A.2.** Let \( v_s, s \geq 0 \) be the Haar basis of \( L^2([0, 1]) \). For any \( 0 < \alpha < 1/2 \) define the operator \( A_\alpha \) on \( L^2([0, 1]) \) by

\[
A_\alpha v_s = 2^{k\alpha}v_s, \quad \text{if} \quad s = 2^k + j
\]

for \( k \geq 0, 0 \leq j \leq 2^k \) and

\[
A_\alpha 1 = 1.
\]

Then for all \( \beta \) with \( \alpha < \beta < (1/2) \), there exists a constant \( c_1 \) such that

\[
\|A_\alpha f\| \leq c_1 \left( \|f\|_{L^2([0, 1])} + \left( \frac{1}{0} \int_{0}^{1} \left( \frac{\|f(t) - f(t')\|^2}{|t - t'|^{1+2\beta}} \right)^{1/2} \right) dt \right)^{1/2}.
\]

The next compactness criteria comes from Theorem A.1 and Lemma A.2.

**Corollary A.3.** Let \( \{X_n\}_{n \geq 1} \subseteq D_{1,2} \) be a sequence of \( \mathcal{F}_t \)-measurable random variables such that there exist constants \( \alpha > 0 \) and \( C > 0 \) with

\[
\sup_n E \left[ |D_tX_n - D_{t'}X_n|^2 \right] \leq C|t - t'|^\alpha
\]

for \( 0 \leq t' \leq t \leq 1 \), and

\[
\sup_{n} \sup_{0 \leq t \leq 1} E \left[ |D_tX_n|^2 \right] \leq C.
\]

Then the sequence \( \{X_n\}_{n \geq 1} \) is relatively compact in \( L^2(\Omega) \).
B. An auxiliary result

The following key result generalises [7, Proposition 2.2.] to the case of functions with spatial polynomial growth.

**Proposition B.1.** Let \( B \) be a \( d \)-dimensional Brownian motion starting from the origin and \( b : [0, 1] \times \mathbb{R} \to \mathbb{R} \) a compactly supported smooth function such that \( \| \hat{b}(t, z) \| \leq 1 \) with \( \hat{b}(t, z) := \frac{b(t, z)}{1 + |z|} \), \( (t, z) \in [0, 1] \times \mathbb{R} \). Set \( B_\sigma(t) = \sum_{i=1}^d \sigma_i B_i(t) \simeq N(0, t \| \sigma \|^2) \) for \( \sigma \in \mathbb{R}^d \). Then there exists a constant \( C \) depending on \( \sigma \) such that for any even positive number \( n \), we have

\[
E \left[ \int_{t_0}^t b'(t, x + B_\sigma(t))dt \right]^n \leq C_n^2 (1 + |x|^n)(\frac{n}{2})! (t - t_0)^{n/2}.
\]  

(B.1)

**Proof.** As in [7], the proof is split into several parts.

Let \( P_\sigma(t, z) = (2\pi t \| \sigma \|^2)^{-1/2}e^{-|z|^2/2t \| \sigma \|^2} \) be the Gaussian kernel, then using the joint distribution of \( B(t_1), \ldots, B(t_n) \), the left hand side of (B.1) can be written as

\[
n! \int_{t_0 < t_1 < \cdots < t_n < t} \prod_{i=1}^n b'(t_i, x + z_i) P_\sigma(t - t_{i-1}, z_i - z_{i-1})dz_1 \ldots dz_n dt_1 \ldots dt_n.
\]

Define

\[
J_n(t_0, t, x, z_0) := \int_{t_0 < t_1 < \cdots < t_n < t} \prod_{i=1}^n b'(t_i, x + z_i) P_\sigma(t - t_{i-1}, z_i - z_{i-1})dz_1 \ldots dz_n dt_1 \ldots dt_n,
\]

The proposition will be proved if we show that

\[
|J_n(t_0, t, 0)| \leq C_n^2 (t - t_0)^{n/2} (1 + |x|^n)/\Gamma(n/2 + 1).
\]

Note that the above comes from Proposition 4.10. in Menoukeu-Pamen and Mohammed [28]. The result then follows. \( \square \)

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