ENTROPY, GEOMETRY, AND THE QUANTUM POTENTIAL

ROBERT CARROLL
UNIVERSITY OF ILLINOIS, URBANA, IL 61801

Abstract. We sketch and emphasize here the automatic emergence of a quantum potential $Q$ in e.g. classical WDW type equations upon inserting a (Bohmian) complex wave function $\psi = R \exp(iS/\hbar)$. The interpretation of $Q$ in terms of momentum fluctuations via the Fisher information and entropy ideas is discussed along with the essentially forced role of $R^2$ as a probability density. We also review the constructions of Padmanabhan connecting entropy and the Einstein equations.

Contents

1. ENTROPY 1
2. WDW 5
3. EXACT UNCERTAINTY 7
4. WDW AND THE QUANTUM POTENTIAL 10
5. FISHER INFORMATION AND ENTROPY 12
5.1. WDW AGAIN 15
6. THE ROLE OF THE QUANTUM POTENTIAL 17
References 19

1. ENTROPY

In $[58]$ one takes an entropy functional ($u^a = \bar{x}^a - x^a$ is a perturbation)

$$S = \frac{1}{8\pi} \int d^4x \sqrt{g} \left[ M^{abcd} \nabla_a u_b \nabla_c u_d + N_{ab} u^a u^b \right]$$

Extremizing with respect to $u_b$ leads to ($N_{ab} u^a u^b = N^{ab} u_a u_b$)

$$\nabla_a \left( M^{abcd} \nabla_c \right) u_d = N^{bd} u_d$$

Date: November, 2005.
email: rcarroll@math.uiuc.edu.
Note $\int d^4x \sqrt{-g} \nabla_a u_b = - \int d^4x \sqrt{-g} u_b \nabla_a f$ since via (1.2) one can write 
$\delta \sqrt{-g} = -(1/2) \sqrt{-g} g^{\mu \nu} \delta g_{\mu \nu}$ and $\nabla_a g^{\mu \nu} = 0$. Choosing $M$ and $N$ such that 
(1.2) (for all $u_d$) implies the Einstein equations entails

$$M^{abcd} = g^{ad} g^{bc} - g^{ab} g^{cd}; \quad N_{ab} = 8\pi \left( T_{ab} - \frac{1}{2} g_{ab} T \right)$$

Consequently $S$ becomes

$$S = \frac{1}{8\pi} \int d^4x \sqrt{-g} \left[ (\nabla_a u_b) (\nabla_b u^a) - (\nabla_b u^b)^2 + N_{ab} u^a u^b \right] = \frac{1}{8\pi} \int d^4x \sqrt{-g} \left[ T_r(J^2) - (Tr(J))^2 + 8\pi \left( T_{ab} - \frac{1}{2} g_{ab} T \right) u^a u^b \right]$$

where $J_a = \nabla_a u^b$. Note here

$$\int d^4x \sqrt{-g} g^{ab} g^{bc} \nabla_a u_b \nabla_c u_d = \int d^4x \sqrt{-g} (\nabla^a u_b) (\nabla^c u^d)$$

and also

$$\nabla_a \left( M^{abcd} \nabla_c \right) u_d = \nabla_a \left[ g^{ad} g^{bc} - g^{ab} g^{cd} \right] \nabla_c u_d = \nabla_a g^{ad} g^{bc} \nabla_c u_d - \nabla_a g^{ab} g^{cd} \nabla_c u_d = \nabla_a \nabla^b u^a - \nabla^b \nabla_c u^c \sim (\nabla_a \nabla^b - \nabla^b \nabla_a) u^a$$

Further (as in (1.6))

$$M^{abcd} \nabla_a u_b \nabla_c u_d = g^{ad} g^{bc} \nabla_a u_b \nabla_c u_d - g^{ab} g^{cd} \nabla_a u_b \nabla_c u_d = \nabla^d u_b \nabla^b u_d - \nabla_a u^a \nabla_c u^c$$

which confirms (1.3). We record also from (56) that

$$(\nabla_{\mu} \nabla_{\nu} - \nabla_{\nu} \nabla_{\mu}) \alpha(w) = R(\alpha, \partial_{\mu}, \partial_{\nu}, w)$$

which identifies $\nabla_{\mu} \nabla_{\nu} - \nabla_{\nu} \nabla_{\mu}$ with $R_{\mu \nu}$ and allows us to imagine (1.6) as $R_{\mu}^d u_a$ with Einstein equations

$$R_{\mu}^d u_a = N_{\mu}^b u^a \quad (= N_{\mu}^{bc} g^{ca} u_c)$$

for example, which is of course equivalent to $R_{ab} = N_{ab}$ (cf. also (59)). Note also $G_{ab} = R_{ab} - (1/2) R g_{ab}$ implies that $R_{\mu}^d = (1/2) R T_{\mu}^d$ which upon contraction gives $R = -k T$ (since $\delta_{\mu}^\mu = 4$) and hence $R_{ab} = k(T_{ab} - (1/2) T g_{ab})$.

**REMARK 1.1.** We will insert more information on entropy and fluctuations related to equation (1.1) later based on (10) [20, 37, 38, 39, 40, 51, 62, 75] (see also Section 3 on exact uncertainty).}

For completeness we sketch here a derivation of the Einstein equations from an action principle (cf. (10) [20, 39, 78]). The Einstein-Hilbert action is $A = \int_\Omega [\mathcal{L}_G + \mathcal{L}_M] d^4x$ where $\mathcal{L}_G = (1/2 \chi) \sqrt{-g} R$ (with $\chi = 8\pi$ and $4R$ is the
Ricci scalar). Following \[20\] we list a few useful facts first (generally we will write if necessary \(g_{ab}T^{ab} = T^c_a\) and \(g_{ab}T^{bc} = T^c_a\)).

1. \(\nabla_\gamma g^{\alpha\beta} = 0\) (by definitions of covariant derivative and Christoffel symbols).

2. \(\delta\sqrt{-g} = (1/2)\sqrt{-g}g^{\alpha\beta}\delta g_{\alpha\beta}\) and \((\delta g_{\alpha\beta})g^{\alpha\beta} = -(\delta g^{\alpha\beta})g_{\alpha\beta}\) (see e.g. \[78\] for the calculation).

3. For a vector field \(v^a\) one has \(\nabla_a v^a = \partial_a(\sqrt{-g}v^a)(1/\sqrt{-g})\) and \(\nabla_\beta T^\alpha_{\beta\gamma} = \partial_\beta(\sqrt{-g}T^\alpha_{\beta\gamma})(1/\sqrt{-g})+\Gamma^\alpha_{\beta\gamma}T^\tau_{\sigma\beta}(\text{from } \Gamma^\alpha_{\beta\gamma} = (1/2)(\partial_\alpha g_{\mu\nu})g^{\mu\nu}\text{ and } \delta g(\sqrt{-g}) = \Gamma^\alpha_{\beta\gamma}).\)

4. For two metrics \(g, g^*\) one shows that \(\delta \Gamma^\alpha_{\beta\gamma} = \Gamma^{*\alpha}_{\beta\gamma} - \Gamma^\alpha_{\beta\gamma}\) is a tensor.

5. \(\delta R_{\alpha\beta} = \nabla_\sigma(\delta \Gamma^\alpha_{\beta\sigma}) - \nabla_\beta(\delta \Gamma^\alpha_{\sigma\alpha})\) (see \[20\] for the calculations).

6. Recall also Stokes theorem \(\int_{\Omega} \nabla_\sigma v^\sigma \sqrt{-g}d^4x = \int_{\partial\Omega} \partial_\sigma (v^\sigma \sqrt{-g})d^4x = \int_{\partial\Omega} \sqrt{-g}v^\sigma d^3\Sigma_\sigma.\)

Now requiring a stationary action for arbitrary \(\delta g^{ab}\) (with certain derivatives of the \(g^{ab}\) fixed on the boundary of \(\Omega\) one obtains \((\mathcal{L}_M\text{ is the matter Lagrangian})\)

\[
\delta I = \frac{1}{2\chi} \int_{\Omega} \left( R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R \right) \sqrt{-g}\delta g^{\alpha\beta} d^4x + \frac{1}{2\chi} \int_{\Omega} g^{\alpha\beta} \sqrt{-g}\delta R_{\alpha\beta} d^4x + \int_{\Omega} \frac{\delta \mathcal{L}_M}{\delta g_{\alpha\beta}} \delta g^{\alpha\beta} d^4x = 0
\]

The second term can be written

\[
\frac{1}{2\chi} \int_{\Omega} g^{\alpha\beta} \sqrt{-g}\delta R_{\alpha\beta} d^4x = \frac{1}{2\chi} \int_{\Omega} g^{\alpha\beta} \sqrt{-g}[\nabla_\sigma(\delta \Gamma^\sigma_{\alpha\beta}) - \nabla_\beta(\delta \Gamma^\sigma_{\alpha\sigma})]d^4x = \frac{1}{2\chi} \int_{\Omega} \sqrt{-g}[\nabla_\sigma(\delta \Gamma^\sigma_{\alpha\beta}) - \nabla_\beta(\delta \Gamma^\sigma_{\alpha\sigma})]d^4x
\]

where \(\delta \Gamma^\sigma_{\alpha\beta} = (1/2)[\nabla_\sigma(\delta g_{\alpha\beta}) + \nabla_\beta(\delta g_{\alpha\gamma}) - \nabla_\gamma(\delta g_{\alpha\beta})]\). This can be transformed into an integral over the boundary \(\partial\Omega\) where it vanishes if certain derivatives of \(g_{\alpha\beta}\) are fixed on the boundary. In fact the integral over the boundary \(\partial\Omega = \sum S_i\) can be written as \(\sum \epsilon_i/2\chi \int_{S_i} \gamma_{\alpha\beta} \delta \tilde{N}^{\alpha\beta} d^3x\) where \(\epsilon_i = \hat{n}_i \cdot n_i = \pm 1\) \((n_i\text{ normal to } S_i)\) and \(\gamma_{\alpha\beta} = g_{\alpha\beta} - \epsilon_i n_\alpha \cdot n_\beta\) is the 3-metric on the hypersurface \(S_i\) (cf. \[80\]). Further

\[
\tilde{N}^{\alpha\beta} = \sqrt{|\kappa|}[K^{\alpha\beta} - K^{\alpha\beta}] = -\frac{1}{2} g^{\gamma\mu\nu} g_{\gamma\beta} \mathcal{L}_n(g^{-1}g^{\mu\nu})
\]

where \(K_{\alpha\beta} = -(1/2)\mathcal{L}_n\gamma_{\alpha\beta}\) is the extrinsic curvature of each \(S_i\) and \(\mathcal{L}_n\) is the Lie derivative. Consequently if the quantities \(\tilde{N}^{\alpha\beta}\) are fixed on the
boundary for an arbitrary $\delta g_{\alpha\beta}$ one gets from the first and last equations in (1.10) the Einstein field equations

\begin{equation}
G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}R g_{\alpha\beta} = \chi T_{\alpha\beta}; \quad T_{\alpha\beta} = -2\frac{\delta L_M}{\delta g_{ab}} + \mathcal{L}_M g_{\alpha\beta}
\end{equation}

We note here that

\begin{equation}
\delta \int L_m \sqrt{-g} d^4 x = \int \frac{\delta L_m}{\delta g_{ab}} \sqrt{-g} d^4 x + \int \mathcal{L}_m \delta (\sqrt{-g}) d^4 x = 
\end{equation}

\begin{equation}
= \int \frac{\delta L_m}{\delta g_{ab}} \sqrt{-g} d^4 x - \frac{1}{2} \int \mathcal{L}_m g_{ab} \delta g_{ab} \sqrt{-g} d^4 x
\end{equation}

A factor of 2 then arises from the $2\chi$ in (1.10).

**REMARK 1.2.** Let us rephrase some of this following [78] for clarity. Thus e.g. think of functionals $F(\psi)$ with $\psi = \psi_\lambda$ a one parameter family and set $\delta \psi = (d\psi_\lambda/d\lambda)|_{\lambda=0}$. For $F(\psi)$ one writes then $dF/d\lambda = \int \phi \delta \psi$ and sets $\phi = (\delta F/\delta \psi)|_{\psi_0}$. Then (assuming all functional derivatives are symmetric with no loss of generality) one has for $L_G = \sqrt{-g} R$ and $S_G = \int L_G d^4 x$

\begin{equation}
\frac{d\Omega_G}{d\lambda} = \sqrt{-g} (\delta R_{ab}) g^{ab} + \sqrt{-g} R_{ab} \delta g^{ab} + R \delta (\sqrt{-g})
\end{equation}

But $g^{ab} \delta R_{ab} = \nabla^a v_a$ for $v_a = \nabla^b (\delta g_{ab}) - g^{cd} \nabla_a (\delta g_{cd})$. Further $\delta \sqrt{-g} = -(1/2) \sqrt{-g} g_{ab} \delta g^{ab}$ so one has

\begin{equation}
\frac{dS_G}{d\lambda} = \int \frac{d\Omega_G}{d\lambda} d^4 x = \int \nabla^a v_a \sqrt{-g} d^4 x + \int \left( R_{ab} - \frac{1}{2} R g_{ab} \right) (\delta g^{ab}) \sqrt{-g} d^4 x
\end{equation}

Discarding the first term as a boundary integral we get the first term in (1.10).

**REMARK 1.3.** From [11 58] we see that the entropy in $S$ in (1.1) reduces to a 4-divergence when the Einstein equations are satisfied “on shell” making $S$ a surface term

\begin{equation}
S = \frac{1}{8\pi} \int_V d^4 x \sqrt{-g} \nabla_i (u^b \nabla_b u^i - u^i \nabla_b u^b) = 
\end{equation}

\begin{equation}
= \frac{1}{8\pi} \int_{\partial V} d^3 x \sqrt{h} n_i (v^b \nabla_b u^i - u^i \nabla_b u^b)
\end{equation}

Thus the entropy of a bulk region $V$ of spacetime resides in its boundary $\partial V$ when the Einstein equations are satisfied. In varying (1.1) to obtain (1.2) one keeps the surface contribution to be a constant. Thus in a semiclassical limit when the Einstein equations hold to the lowest order the entropy is contributed only by the boundary term and the system is holographic.
2. WDW

We gather now some information about the derivation of Einstein’s equations from an action principle and also discuss the Hamiltonian theory involving the Einstein-Hamilton-Jacobi (EHJ) equation and the WDW equation. One recalls from [30] that the EHJ equation

\[(2.1) \quad 3R + \frac{1}{\hbar} \left( \frac{1}{2} h_{ij} h_{k\ell} - h_{ik} h_{j\ell} \right) \left( \frac{\delta S}{\delta h_{ij}} \right) \left( \frac{\delta S}{\delta h_{k\ell}} \right) = 0 \]

\( (h_{ij} \text{ corresponds to the metric of a spatial hypersurface}) \) plus a principle of constructive interference of deBroglie waves leads to the entire set of 10 Einstein equations. The idea of Tomonaga’s multifingered time is used here (cf. also [10, 54]).

Now there are a number of derivations of the WDW equations with connections to Bohmian dynamics and the quantum potential in [10, 33, 34, 63, 64, 69, 70, 71, 72, 73, 74] and we will go directly to [33, 34] after a few comments. First let us recall the deWitt metric for which we refer to [5, 10, 22, 25, 31, 32, 27, 33, 34, 35, 43, 44, 45, 49, 63, 64, 67, 69, 70, 71, 72, 73, 74, 80] (cf. also [76, 79]). Various formulas arise for the WDW which involve a deWitt metric (or supermetric)

\[(2.2) \quad G_{abcd}^\alpha = \frac{1}{\sqrt{h}} (h_{ac} h_{bd} + h_{ad} h_{bc} - 2\alpha h_{ab} h_{cd}); \]

\[ G_{abcd}^\beta = \frac{\sqrt{h}}{2} (h_{ac} h_{bd} + h_{ad} h_{bc} - 2\beta h_{ab} h_{cd}) \]

where \( \alpha + \beta = 3\alpha\beta \). For general relativity (GR) one takes \( \beta = 1 \) and \( \alpha = 1/2 \) (see e.g. [31, 32, 43] for this form of metric). Here the WDW equation for GR is \((c = 1)\)

\[(2.3) \quad \tilde{H} \psi[h_{ab}, \phi] = [-16\pi G \hbar^2 G_{abcd} \frac{\delta^2}{\delta h_{ab} \delta h_{cd}} - \frac{\sqrt{h}}{16\pi G} (R - 2\Lambda) + \tilde{H}_m] \psi = 0 \]

where \( h_{ab} \) is a 3-metric, \( R \) the 3-D Ricci scalar, \( \Lambda \) the cosmological constant, \( G_{abcd} = G_{abcd}^{1/2} \) the deWitt metric, and \( \tilde{H}_m \) is the Hamiltonian density for non-gravitational fields. The integrated form of (2.3) is

\[(2.4) \quad \int d^3x N \tilde{H} \psi = \tilde{H}^N \psi = (\tilde{H}_G^N + \tilde{H}_m^N) \psi = 0 \]

Writing \( \psi = exp \left( i(M S_0 + S_1 + M^{-1} S_2 + \ldots) \right) \) for \( M = (32\pi G)^{-1} \) leads to a power series in \( M \) with second term

\[(2.5) \quad \tilde{H}_x = \frac{1}{2} G_{abcd} \frac{\delta S_0}{\delta h_{ab}} \frac{\delta S_0}{\delta h_{cd}} - 2\sqrt{h}(R - 2\Lambda) = 0 \]
which is the Hamilton-Jacobi (HJ) equation for the gravitational field and we refer to \[31, 32, 43\] for more details.

It will be important to see here how the quantum potential arises and we go to \[63, 64\] with a metric (2.6)
\[ds^2 = -(N^2 - N^i N_i)dt^2 + 2N_i dx^i dt + h_{ij} dx^i dx^j\]

(classical ADM situation - cf. \[3, 4, 49\]) and Hamiltonian (2.7)
\[H = \int d^3x (\mathcal{N} \mathcal{H} + \mathcal{N}^i \mathcal{H}_i); \quad \mathcal{H}_i = -2D_i \pi_i^j + \pi_\phi \partial_j \phi;\]

where \(\mathcal{H} = \frac{\kappa G}{2} \pi_{ij} \pi^{ij} + \frac{1}{2} h^{-1/2} \pi_\phi^2 + h^{1/2} \left[ -\kappa^{-1}(\mathcal{B} R - 2\Lambda) + \frac{1}{2} h^{ij} \partial_i \phi \partial_j \phi + U(\phi) \right]\)

where \(\kappa = 16\pi G/c^4\), \(D_k\) is the covariant derivative, and

(2.8)
\[\pi^{ij} = -h^{1/2} \left( K^{ij} - h^{ij} K \right) = G^{ijkl} [\dot{h}_{kl} - D_k N_l - D_l N_k];\]

Thus \(K_{ij}\) is the extrinsic curvature of the hypersurface and (♣) \(\pi_\phi = (h^{1/2}/N)(\dot{\phi} - N^i \partial_i \phi)\) where \(\phi\) is a matter field. The classical 4-metric above and the scalar field which are solutions of the Einstein equations can be obtained from the Hamiltonian equations of motion

(2.9)
\[\dot{h}_{ij} = \{h_{ij}, H\}; \quad \dot{\pi}^{ij} = \{\pi^{ij}, H\}; \quad \dot{\phi} = \{\phi, H\}; \quad \dot{\pi}_\phi = \{\pi_\phi, H\}\]

for some choice of \(N\) and \(N^l\), given suitable initial conditions compatible with the constraints (♣) \(\mathcal{H}_i \approx 0\) and \(\mathcal{H}_j \approx 0\) (in standard terminology). There is a standard constraint algebra involving Poisson brackets of the \(\mathcal{H}_i\) and \(\mathcal{H}_j\) (see e.g. \[63\]) and for quantization the constraints become conditions on the possible states of the quantum system yielding equations (♦) \(\hat{\mathcal{H}}_i |\psi > = 0\) and \(\hat{\mathcal{H}}_j |\psi > = 0\) leading to (♦) \(-2h_{ij} D_j [\delta \psi / \delta h_{ij}] + [\delta \psi / \delta \phi] \partial_i \phi = 0\) and the WDW equation (2.10)
\[\left\{ -\hbar^2 \left[ \kappa G^{ijkl} \frac{\delta}{\delta h_{ij}} \frac{\delta}{\delta h_{kl}} + \frac{1}{2} h^{-1/2} \delta^2 \frac{\delta}{\delta \phi^2} \right] + V \right\} \psi(h_{ij}, \phi) = 0;\]

\[V = \hbar^{1/2} \left[ -\kappa^{-1}(\mathcal{B} R - 2\Lambda) + \frac{1}{2} h^{ij} \partial_i \phi \partial_j \phi + U(\phi) \right]\]

This involves products of local operators at the same space point so regularization is indicated (we omit details).

Now for the Bohmian point of view one writes \(\psi = A exp(iS/\hbar)\) where \(A\) and \(S\) are functionals of \(h_{ij}\) and \(\phi\) leading to two equations indicating
that $A$ and $S$ are invariant under general space coordinate transformations, namely

$$-2h_{ij}D_j\frac{\delta S}{\delta h_{ij}} + \frac{\delta S}{\delta \phi} \partial_i \phi = 0; \quad -2h_{ij}D_j\frac{\delta A}{\delta h_{ij}} + \frac{\delta A}{\delta \phi} \partial_i \phi = 0$$

These could depend on factor ordering but in any event one will have e.g. the form

$$\kappa G_{ijk\ell} \frac{\delta S}{\delta h_{ij}} \frac{\delta S}{\delta h_{k\ell}} + \frac{1}{2} h^{-1/2} \left( \frac{\delta S}{\delta \phi} \right)^2 + V + Q = 0;$$

where the unregularized $Q$ above depends on the regularization and factor ordering prescribed for the WDW equation. In addition to (2.12) one has

$$\kappa G_{ijk\ell} \frac{\delta}{\delta h_{ij}} \left( A^2 \frac{\delta S}{\delta h_{k\ell}} \right) + \frac{h^{-1/2}}{2} \frac{\delta}{\delta \phi} \left( A^2 \frac{\delta S}{\delta \phi} \right) = 0$$

One can stipulate that the 3-metric of spacelike hypersurfaces, the scalar field, and their canonical momenta always exist and the metric and scalar field can be determined via guidance relations

$$\pi_{ij} = \frac{\delta S}{\delta h_{ij}}; \quad \pi_{\phi} = \frac{\delta S}{\delta \phi}$$

with $\pi_{ij}$ and $\pi_{\phi}$ given via (2.8) etc. Note that one cannot interpret (2.13) as a continuity equation for a probability density due to the hyperbolic nature of the deWitt metric. Note also that whatever may be the form of $Q$ it must be a scalar density of weight one; indeed from (2.12)

$$Q = -\frac{h^2}{A} \left( \kappa G_{ij\ell} \frac{\delta^2 A}{\delta h_{ij} \delta h_{\ell\ell}} + \frac{h^{-1/2}}{2} \frac{\delta^2}{\delta \phi^2} \right)$$

and we refer to [63] for the arguments. In addition note that $Q$ can depend only on $h_{ij}$ and $\phi$.

### 3. EXACT UNCERTAINTY

We go now to [10, 33, 34, 65, 66] and show how the WDW equation can be derived from a so called exact uncertainty principle of Hall and Reginatto. The idea here is that uncertainty can be promoted to be the fundamental element distinguishing quantum and classical mechanics. In this approach nonclassical fluctuations are added to the deterministic connection between position and momentum (via the uncertainty principle) one essentially generates the quantum potential. In [33, 34] this is applied to gravity and a WDW equation is derived and originally this approach was
used to generate the Schrödinger equation (SE) (see Remark 5.2 for additional clarifications following [65, 66]). Thus take a metric as in (2.6) and think of the metric $h_{ij}$ as being imprecise with a probability distribution $P[h_{ij}]$. Take a single field classical Hamiltonian of the form

$$H_0[h_{ij}, \pi^{ij}] = \int dx \left[ N \left( \frac{1}{2} G_{ijkl} \pi^{ij} \pi^{kl} + V(h_{ij}) \right) - 2 N_i \nabla_j \pi^{ij} \right]$$

(here $D_j \sim \nabla_j$ is the covariant derivative). As an ensemble Hamiltonian one takes now

$$\tilde{H}_c[P, S] = \int DhPH_0[h_{ij}, (\delta S/\delta h_{ij})]$$

leading to equations of motion

$$\begin{align*}
\partial_t P + \int dx \frac{\delta}{\delta h_{ij}}(P \dot{h}_{ij}) = 0; & \quad \partial_t S + H_0[h_{ij}, (\delta S/\delta h_{ij})] = 0; \\
\dot{h}_{ij} &= NG_{ijkl} \frac{\delta S}{\delta h_{kl}} - \nabla_j N^i - \nabla_i N_j
\end{align*}$$

The lack of conjugate momenta for the lapse and shift components $N$ and $N_i$ places constraints on the classical equations of motion which in the ensemble formalism take the form

$$\begin{align*}
\frac{\delta P}{\delta N} = \frac{\delta P}{\delta N_i} = \frac{\partial P}{\partial t} = 0; & \quad \nabla_j \left( \frac{\delta P}{\delta h_{ij}} \right) = 0; \\
\frac{\delta S}{\delta N} = \frac{\delta S}{\delta N_i} = \frac{\partial S}{\partial t} = 0; & \quad \nabla_j \left( \frac{\delta S}{\delta h_{ij}} \right) = 0
\end{align*}$$

This corresponds to invariance of the dynamics with respect to $N$, $N_i$, and the initial time; also to invariance of $P$ and $S$ under arbitrary spatial coordinate transformations. Applying these constraints to the above classical equations for the “Gaussian” choice $N = 1$ and $N_i = 0$ yields

$$\frac{\delta}{\delta h_{ij}} \left( PG_{ijkl} \frac{\delta S}{\delta h_{kl}} \right) = 0; \quad \frac{1}{2} G_{ijkl} \frac{\delta S}{\delta h_{ij}} \frac{\delta S}{\delta h_{kl}} + V = 0; \quad V \sim c\sqrt{h}(2\Lambda - 3R)$$

Now the exact uncertainty approach involves writing (★) $\pi^{ij} = (\delta S/\delta h_{ij}) + f^{ij}$ where $f^{ij}$ vanishes on average for all configurations. This adds a kinetic term to the average ensemble energy leading to

$$\tilde{H}_q = \langle E \rangle = \tilde{H}_c + \frac{1}{2} \int DhP \int dx NG_{ijkl} f^{ij} f^{kl}$$

Note here that the term in (3.1) which is linear in the derivative of $\pi^{ij}$ can be integrated by parts giving a term directly proportional to $\pi^{ij}$ which remains unchanged when the fluctuations are added and averaged. Now
using some general properties of causality, independence, invariance, and exact uncertainty (cf. [10, 33, 34]) one arrives at

\[ \tilde{H}_q[P, S] = \tilde{H}_c[P, S] + \frac{c}{2} \int \mathcal{D}h \int dx \, N G_{ijkl} \frac{\delta P}{\delta h_{ij}} \frac{\delta P}{\delta h_{kl}} \]

where \( C \) is a positive universal constant. Now if one defines \( \hbar = 2\sqrt{c} \) and \( \psi[h_{ij}] = \sqrt{P} \exp(iS/\hbar) \) then, calculating as above, we obtain a WDW equation for quantum geometry in the form

\[ \left[ -\frac{\hbar^2}{2} \frac{\partial^2}{\partial h_{ij}} G_{ijkl} \frac{\partial}{\partial h_{kl}} + V \right] \psi = 0 \]

with a Q term \( -(\hbar^2/2P)G_{ijkl}(\delta^2 P/\delta h_{ij}\delta h_{kl}) \) added in the Hamiltonian equation (cf. (2.12)). Note further

\[ \frac{\delta \psi}{\delta N} = \frac{\delta \psi}{\delta N_i} = \frac{\partial \psi}{\partial t} = 0; \quad \nabla_j \left( \frac{\delta \psi}{\delta h_{ij}} \right) = 0 \]

An important feature of this WDW equation is that it is obtained with a particular operator ordering. Indeed \( G_{ijkl} \) is sandwiched between the two functional derivatives and thus ambiguity is removed in this respect. One recalls that the same thing happens with the SE which is derived in the form (\( \bullet\bullet \) \( \hbar \partial_t \psi = -(\hbar^2/2)(\nabla \cdot (1/m)\nabla \psi + V\psi) \).  

Now in the theory of Schrödinger equations there is a strong connection between terms of the form

\[ I = \frac{1}{2} g^{ik} \int \frac{1}{P} \frac{\partial P}{\partial y^i} \frac{\partial P}{\partial y^k} d^n y \]

and concepts of Fisher information, entropy, and quantum potential (see [10] for an extensive development). Classical Fisher information is known to be connected to various forms of entropy via formulas like (cf. [10, 28])

\[ \frac{\partial S}{\partial t} = \frac{h}{2m} \frac{\delta}{\delta \bar{\psi}} = \frac{h}{2m} \int (\nabla \rho)^2 \rho = \frac{4}{h} \int \rho Q; \quad Q = -\frac{h^2}{2m} \frac{\delta \sqrt{\rho}}{\sqrt{\rho}} \]

Here \( \mathcal{S} \sim -\int \rho \log(\rho) \) is a so-called differential entropy and \( \rho \) here corresponds to \( P \) or \( A \) in the notation of this paper (note \( P \) and \( A \) refer to 3-space quantities); \( \mathcal{S} \) is a Fisher information measure. There are relations between differential entropy and Shannon-Boltzmann entropy for example and we refer to [10, 11, 12, 30] for details. We remark also that Olavo in [57] derives Schrödinger equations using entropy ideas where the entropy in [57] is of Shannon-Boltzmann type \( \mathcal{S} = k_B \log(W) = -k_B \log(P) \) where \( P = 1/W \) is the probability of a microstate occurrence. One deals with momentum fluctuations \( (\delta p)^2 \) and assumes \( (\delta p)^2(\delta x)^2 = h^2/4 \). There results \( (\delta p)^2 \sim -(h^2/4)\partial^2 \log(\rho) \) where \( \rho \) is a probability density. Then
$S_{\text{equilib}} \sim k_B \log(\rho)$ implies $(\delta p)^2 \sim -(h^2/4k_B)\partial^2 S_{\text{equilib}}$. Note also here that (calculating in 1-D for convenience) $\partial^2 \log(\rho) = (\rho''/\rho) - (\rho'/\rho)^2$ and (3.12)

$$Q = -\frac{h^2}{2m} \frac{\partial^2 \sqrt{\rho}}{\sqrt{\rho}} = \frac{h^2}{8m} \left[ 2\frac{\rho''}{\rho} - \left( \frac{\rho'}{\rho} \right)^2 \right] = \frac{h^2}{8m} \left[ 2\partial^2 S_{\text{equilib}} + (\partial S_{\text{equilib}})^2 \right]$$

The theme here is to relate entropy, the quantum potential, and geometry in the relativistic context. One can think of entropy or of quantum fluctuations as generating quantum behavior (often via a quantum potential) and we want to connect these matters to the Einstein equations in a Bohmian spirit. Most of this is already done and sketched in [10] for example and we want to make it more explicit here (see Section 5.1 for some clarifications of the exact uncertainty method and WDW).

**REMARK 3.1.** We call attention here to [17, 18] (cf. also [47, 48, 82]) where Fisher information is related to uncertainty relations and a differential Shannon entropy is introduced.

### 4. WDW AND THE QUANTUM POTENTIAL

We have seen already how the quantum potential arises in the WDW equation in Section 2 following [63, 64]. Let us now approach this from another point of view following [72, 73, 74] (see also [10]). One takes $16\pi G = 1$ here for convenience and uses the standard ADM decomposition. Assuming first that there is no matter field the Lagrangian density of GR is

$$\mathcal{L} = \sqrt{-g}R = \sqrt{h}N[\mathcal{L}]$$

($K_{ij}$ is the extrinsic curvature). The canonical momentum of the 3-metric is

$$p^{ij} = \frac{\partial L}{\partial \dot{h}^{ij}} = \sqrt{h}[K^{ij} - h^{ij}Tr(K)]$$

The classical Hamiltonian is $H = \int d^3x \mathcal{H}$ where $\mathcal{H} = \sqrt{h}(NC + N^iC_i)$ and the constraints are

$$C = -3R + \frac{1}{h} \left[ Tr(p^2) - \frac{1}{2}(Tr(p))^2 \right] = -2G_{\mu\nu}n^\mu n^\nu;$$

$$C_i = -2\nabla^j \left( \frac{p_{ij}}{\sqrt{h}} \right) = -2G_{\mu i}n^\mu$$

where $n^\mu$ is the normal to the spatial hypersurfaces $n^\mu = (1/N, -(1/N)N)$. As usual in a Bohmian theory one adds a quantum potential to the Hamiltonian in order to obtain the correct equations of motion so $H \rightarrow H + Q$
or \( \mathcal{H} \rightarrow \mathcal{H} + Q \) where \( Q = \int d^3x \Omega \) and the quantum potential is

\[
(4.4) \quad \Omega = \hbar^2 NG_{ijkl} \frac{1}{|\psi|} \frac{\delta^2 |\psi|}{\delta h_{ij} \delta h_{k\ell}}
\]

This means that one must modify the classical constraints via (★★) \( C \rightarrow C + (Q/\sqrt{\hbar}N) \) and \( C_i \rightarrow C_i \). For the constraint algebra one considers

\[
(4.5) \quad C(N) = \int d^3x \sqrt{\hbar}NC; \; \tilde{C}(N) = \int d^3x \sqrt{\hbar}N^iC_i
\]

and there results (cf. [10, 72, 73])

\[
(4.6) \quad \{\tilde{C}(N), \tilde{C}(N')\} = \tilde{C}(N \cdot \nabla N' - N' \cdot \nabla N);
\]

\[
\{\tilde{C}(N), C(N)\} = C(N \cdot \nabla N); \; \{C(N), C(N')\} \sim 0
\]

The first 3-diffeomorphism subalgebra has no change relative to the classical equation and the second, representing the fact that the Hamiltonian constraint is a scalar under 3-diffeomorphisms, is also the same as the classical situation. In the third case the quantum potential changes the Hamiltonian constraint algebra dramatically (making it weakly equal to zero). The details are written out in [72, 73] using the Bohm-Hamilton-Jacobi equation

\[
(4.7) \quad G_{ijkl}p^i p^k + \sqrt{\hbar}(\bar{\alpha} R - 2\Lambda) + \Omega = 0
\]

which is differentiated to obtain

\[
(4.8) \quad \frac{1}{N} \frac{\delta}{\delta h_{ij}} \frac{\Omega}{\sqrt{\hbar}} = \frac{3}{4\sqrt{\hbar}} h_{k\ell}p^i p^k \delta(x - z) - \frac{\sqrt{\hbar}}{2} h_{ij}(\bar{\alpha} R - 2\Lambda) \delta(x - z) - \sqrt{\hbar} \frac{\delta^3 R}{\delta h_{ij}}
\]

Putting this information in the Poisson brackets one obtains the last relation in (4.6). The existence of the quantum potential shows that the quantum algebra is a 3-diffeomorphism algebra times an Abelian subalgebra and the only difference with [50] for example is that this algebra is weakly closed (this will eventually mean closed on the Bohmian trajectories). Thus the algebra (4.6) is a projection of general coordinate transformations to the spatial and temporal diffeomorphisms and the equations of motion are invariant under such transformations. The important point here is that, although the form of the quantum potential depends on the regularization and ordering, nevertheless in the quantum constraints algebra the form of the quantum potential is not important; the equations are correct independently of the definition of quantum potential. Note that for \( 16\pi G = 1 \) and \( c = 1 \) (so \( \kappa = 1 \)) we can relate (4.7) and (2.12) provided there is no matter field, \( N = 1 \), and we assume a different signature in the metric.

Now one can derive the quantum corrections to the Einstein equations.
For the dynamical part consider
\[(4.9) \dot{h}_{ij} = \{H, h_{ij}\}; \quad \dot{p}_{ij} = \{H, p_{ij}\}\]
and some calculation which we omit leads to \((\star \star)\) \(G_{ij} = -(1/N)(\delta Q/\delta h_{ij})\) which means that the quantum force modifies the dynamical parts of the Einstein equations. For the nondynamical parts one uses the constraint equations \((4.3)\) to obtain \(G_{00} = \Omega/2N^3\sqrt{\hbar}\) and \(G_{0i} = -\Omega N_i/2N^3\sqrt{\hbar}\), which can be written in the form \((\bullet \bullet)\) \(G_{0\mu} = (\Omega/2\sqrt{\hbar})g^{0\mu}\). The equations obtained via the Hamiltonian will also agree with those given by the phase of the wave function and the guidance formula. Indeed from the Bohmian HJ equation \((4.7)\) one has
\[(4.10) G_{ijk\ell} \frac{\delta S}{\delta h_{ij}} \frac{\delta S}{\delta h_{k\ell}} - \sqrt{\hbar}(3R - \Omega) = 0\]
To get the equation of motion one must differentiate the HJ equation with respect to \(h_{ab}\) and use the guidance formula \(p_{k\ell} = \sqrt{\hbar}(K_{k\ell} - h_{k\ell}K) = \delta S/\delta h_{k\ell}\) and doing this leads again to \((\star \star)\) so the evolution generated by the Hamiltonian is compatible with the guidance formula.

Inclusion of matter fields is straightforward; one simply adds the matter quantum potential and writes
\[(4.11) G_{ij} = -\kappa \Sigma_{ij} - \frac{1}{N} \delta(Q_G + Q_m); \quad G_{0\mu} = -\kappa \Sigma_{0\mu} + \frac{\Omega G + \Omega_m}{2\sqrt{-g}}g^{0\mu}\]
Here \(\phi\) is the matter field
\[(4.12) \Omega_m = \hbar^2 N\sqrt{\hbar} \frac{1}{2} \frac{\delta^2}{\delta \phi^2} \psi^2; \quad \Omega_G = \hbar^2 N\hbar G_{ijk\ell} \frac{1}{\psi^2} \frac{\delta^2}{\delta h_{ij}\delta h_{k\ell}} \psi\]
and \(Q_G = \int d^3x Q_G\) with \(Q_m = \int d^3x Q_m\). The equations \((\bullet \bullet \bullet)\) are the Bohm-Einstein equations and are the quantum version of the Einstein equations. Since regularization only affects the quantum potential the quantum Einstein equations are the same for any regularization. They are invariant under temporal \(\otimes\) spatial diffeomorphisms and can be written also in the form
\[(4.13) G^{\mu\nu} = -\kappa \Sigma^{\mu\nu} + \mathcal{G}^{\mu\nu}; \quad \mathcal{G}^{\mu\nu} = \frac{\Omega}{2\sqrt{-g}}g^{0\mu}; \quad \mathcal{G}^{ij} = -\frac{1}{N} \frac{\delta Q}{\delta g_{ij}}\]
We refer to \([10, 72, 73, 74]\) for further discussion.

5. FISHER INFORMATION AND ENTROPY

We will connect up here various ideas of entropy and Fisher information (following \([10, 23]\)). First recall \(N_{ab}\) in Section 1 corresponds to \(T_{ab} - (1/2)g_{ab}T\) and one can imagine this arising from a matter Lagrangian \(L_m\) as in (1.10)-(1.13) where \(G_{ab} = R_{ab} - (1/2)Rg_{ab} = \chi T_{ab} \sim \chi(-2(\delta \mathcal{L}/\delta g^{ab}) + \chi^2 (\delta^2 \mathcal{L}/\delta g_{ab}^2))\).
We recall that $R = -\chi T$ so $R_{ab} = \chi(T_{ab} - (1/2)g_{ab}T)$ and note that

$$T = g^{ab}\mathcal{L}_m g_{ab} - 2g^{ab}\frac{\delta \mathcal{L}_m}{\delta g^{ab}} = \mathcal{L}_m - 2g^{ab}\frac{\delta \mathcal{L}_m}{\delta g^{ab}}$$

(5.1)

Hence

$$R_{ab} = \chi \left( \mathcal{L}_m g_{ab} - \frac{2}{\chi} \frac{\delta \mathcal{L}_m}{\delta g^{ab}} - \frac{1}{2} \left[ \mathcal{L}_m - 2g^{ab}\frac{\delta \mathcal{L}_m}{\delta g^{ab}} \right] \right) =$$

$$= \chi \left( \frac{1}{2} g_{ab}\mathcal{L}_m - \frac{\delta \mathcal{L}_m}{\delta g^{ab}} \right)$$

(5.2)

and in the situation of Section 1 we have $R_{ab}u^b \sim N_{ab}u^b$. It is clear however from Sections 2-4 that one does not need a matter potential in order to discuss the quantum potential in general spaces.

One goes to the deWitt 6-dimensional “superspace” with metric $G_{ijkl}$ (cf. [22]) (here $G_{ijkl} = (1/2\sqrt{h})(h_{ik}h_{jk} + h_{ik}h_{jk} - h_{ij}h_{kl})$ following [22] - cf. also (2.2) which differs by a factor of 2). The Fisher information will have a general form

$$I = 4 \int dx \int D\psi \sum_{ijkl} G_{ijkl} \frac{\partial \psi^*}{\partial h_{ij}} \frac{\partial \psi}{\partial h_{kl}}$$

(5.3)

where $dg \sim \prod dg_{ij}$ (cf. [31 [10 [23 [24 [60]). To motivate and clarify this one thinks of a probability density function $f(y|\theta)$ used in estimating a parameter $\theta$ based on imperfect observations $y = \theta + x$ ($x \sim$ noise). Assume unbiased estimates, namely (♦ ♦) $< \hat{\theta}(y) - \theta >= 0 = \int dy (\hat{\theta} - \theta)p(y|\theta)$ where $p(y|\theta)$ is the probability for $y$ in the presence of one parameter value $\theta$. Differentiate (♦ ♦) to get $\int dy (\hat{\theta} - \theta)\partial_{\theta}p - \int dyp = 0$ and via $\int p = 1$ and $\partial_{\theta} = p\partial_{\theta}log(p)$ one arrives at

$$\int dy (\hat{\theta} - \theta)\partial_{\theta}log(p) = 1 = \int dy[\sqrt{p}\partial_{\theta}log(p)]((\hat{\theta} - \theta)\sqrt{p})$$

(5.4)

The Schwartz inequality gives then

$$\int dy (\partial_{\theta}log(p))^2p \int dy (\hat{\theta} - \theta)^2p \geq 1$$

(5.5)

(Cramer-Rao inequality) which links the mean square estimate $e^2$ (second factor) to the Fisher information $I$ (first factor). In [24] one writes $p = q^2$ so (♦ ♦) $I \sim 4 \int dx(q)^2$ and various quadratic Lagrangians in physics are considered, e.g. (1) $(1/2)m(q)^2 - V$, (2) $-\nabla\psi \cdot \nabla\psi^* + \cdots$, (3) $-(h^2/2m)\nabla\psi \cdot \nabla\psi^* + \cdots$, (4) $\sum g_{mn}(q)\partial_r q_m \partial_r q_n$, etc. A principle of extreme physical information (EPI) is enunciated (in a game theoretic context) and, setting e.g. $x_1 = ix$, $x_2 = iy$, $x_3 = iz$, and $x_4 = ct$ with $(x_1, x_2, x_3) \sim r$, one posits modes $q_n = q_n(r, t)$ with $\psi_n = q_{2n-1} + iq_{2n}$ ($n = 1, \cdots, N/2$). Then
take \( \sum_{n}^{N/2} \psi_n^* \psi_n = \sum q_n^2 = p(r, t) \) with \( I \sim 4 \sum_{n}^{N/2} \int d\tau \nabla q_n \cdot \nabla q_n \) (cf. \( \clubsuit \)). Then physical content is introduced via Fourier transform momentum-energy variables \((ir, ct) \leftrightarrow [(i\mu/\hbar), (E/ct)]\) with \( \psi_n \leftrightarrow \phi_n \) so that \((\nabla \psi_n, \partial \psi_n) \leftrightarrow [(-i\mu \phi_n/\hbar), (iE \phi_n/\hbar)]\) (only \( E = mc^2 \) will be assumed physically below). \( I \) is regarded as information obtained by an observer and this is to be balanced by the physical payoff \( J \) by a "demon" expressed in physical terms. The net information change \( \Delta I = I - J \) should be zero (as in zero sum game) and EPI specifies that \( I = J \) which means here

\[
\sum_{n}^{N/2} \phi_n^* \phi_n = P(\mu, E)
\]

\[I = 4c \sum_{n}^{N/2} \int \int d\tau dt \left[ -(\nabla \psi_n)^* \cdot \nabla \psi_n + \frac{1}{c^2} \left( \frac{\partial \psi_n}{\partial t} \right)^* \left( \frac{\partial \psi_n}{\partial t} \right) \right] = I = J = \frac{4c}{\hbar^2} \int \int d\mu dE P(\mu, E)(-\mu^2 + (E^2/c^2)) = \frac{4c}{\hbar^2} \left< -\mu^2 + \frac{E^2}{c^2} \right>
\]

Some argument then gives \(-\mu^2 + (E^2/c^2) = m^2c^2\) and minimizing \( \Delta I \) (\( \Delta I = 0 \)) leads to the Klein-Gordon equation. This approach seems a little silly but it is also cute; it does in any case sort of motivate the use of (5.3) as a Fisher information.

**REMARK 5.1.** The entropy in Section 1 is of course contrived via perturbations in displacement and their derivatives (elastic deformation) and is not designed for quantization (cf. however [57]). We note also the apparent denial of an entropy functional for gravity without sources in [19]. However in [19] an entropy is introduced via a fluid stress energy tensor. The theme of [58] does not seem to be threatened; the Einstein equations arise as a consistency condition indicating that spacetime structure (as defined by the Einstein equations!) is robust under fluctuations. The quantum potential in Section 2 \((Q = Q_G + Q_M)\) arises via the Hamiltonian context when one looks at a complex wave function \( \psi = Aexp(iS/\hbar) \) with \( A \) and \( S \) functionals of \( h_{ij} \) and a matter field. The interesting fact here is that \( Q_G \) automatically arises once a complex (quantum) solution is sought (cf. (2.12)). The same feature arises in Section 4 where the introduction of a Bohmian context corresponds to the entrance of quantum theory and the quantum potential automatically appears. No matter potential is needed here and thus it seems that space time automatically contains a quantum aspect which emerges when one looks at a Hamiltonian formulation with a complex wave function (implicitly introducing a probability). The exact uncertainty approach of Section 3 introduces perturbations or fluctuations in momentum based on fluctuations in \( h_{ij} \) and exhibits the associated quantum potential. The perturbations here are quite general in an explicit way and essentially generate the amplitude of the wave function. The form
(3.7) in terms of Fisher information automatically gives the fluctuations an entropic character (cf. [57] where one derives the SE on entropy ideas and see also Section 5.1 below); we will expand on this in Section 6.

5.1. WDW AGAIN. We will rephrase some of this now following [65, 66] which clarifies the exact uncertainty treatment of Sections 3-4. We remark first that there seem to be strong relations between the exact uncertainty method of deriving the SE and a technique of Olavo via entropy methods (cf. [11, 13, 57]). We sketch first the exact uncertainty method for the SE following [65] (cf. also [10]). For an ensemble of classical nonrelativistic particles of mass $m$ moving in a potential $V$ one has an ensemble Hamiltonian

$$\tilde{H}_c[P, S] = \int dx \, P \left( \frac{\nabla S}{2m} + V \right)$$

The ensuing equations of motion ($\dagger$) $\partial_t P = (\delta \tilde{H}_c/\delta S)$ and $\partial_t S = -(\delta \tilde{H}_c/\delta P)$ take the form

$$\partial_t P + \nabla \cdot \left( P \frac{\nabla S}{m} \right) = 0; \quad \partial_t S + \frac{|\nabla S|^2}{2m} + V = 0$$

Given stochastic perturbations ($\ast$) $p = \nabla S + f$ with $\overline{f} = 0$ and $\overline{\nabla S}$ with ($\dagger\dagger$) $< E > = \int dx \, P(\frac{|\nabla S + f|^2}{2m} + V) = \tilde{H}_c + \int dx \, P(\frac{\overline{f}^2}{2m}$.

one asks for conditions on $f$ leading to quantum equations of motion and this is described in [10] for example via four principles including exact uncertainty. The quantum ensemble Hamiltonian is then ($\ast\ast$) $\tilde{H}_q = \tilde{H}_c + c \int dx (1/P)(|\nabla P|^2/2m)$ where the last term is a form of Fisher information.

For more general situations following [66] one looks at the Fisher information matrix

$$I_{kl} = \int P(x') \left( \frac{\partial \log(P(x'))}{\partial x^k} \frac{\partial \log(P(x'))}{\partial x^l} \right) d\mu(x')$$

based on $y^i = \theta^i + x^i$ etc. Using a standard 3-D metric $g_{ij}$ one obtains ($\ast\ast$) $\partial_t P + \sum g^{ik} \partial_i (P \partial_k - k S) = 0$ from a variational principle via ($\dagger\dagger\dagger$) $\Phi_A = \int P(\partial_t S + (1/2) \sum g^{ik} \partial_i S \partial_k S) d^3x dt$ (varying $S$). This leads trivially to the classical HJ equation for a free particle upon variation in $P$ so variation in $S$ and $P$ leads to equations of motion for an ensemble of particles. Now one can define the information in $P$ using the Fisher matrix via

$$\Phi_B = \sum g^{ik} \int \frac{1}{P} \partial_i P \partial_k P d^3x dt = \sum g^{ik} I_{ik}$$
Set then \( \Phi = \Phi_A + \Phi_B \) and for variations of \( S \) and \( P \) vanishing at the boundary one obtains

\[
(5.11) \quad \partial_t P + \sum g^{ik} \partial_i (P \partial_k S) = 0; \quad \partial_t S + \sum (1/2) g^{ik} \partial_i S \partial_k S - \lambda \sum g^{ik} \left( \frac{2}{P} \partial_i \partial_k P - \frac{1}{P^2} \partial_i P \partial_k P \right) = 0
\]

This is equivalent to the free particle SE (\( \bullet \bullet \bullet \)) \( i \hbar \partial_t \psi = (\hbar^2 / 2m) \sum g^{ik} \partial_i \partial_k \psi \) provided \( \lambda = \hbar^2 / 8 \) and \( \psi = P^{1/2} \exp(iS / \hbar) \). The connection to the quantum potential comes here through (cf. (3.11))

\[
(5.12) \quad \int P Q d^3 x d t = -\frac{\hbar^2}{8} \sum g^{ik} \int P \left( \frac{2}{P} \partial_i \partial_k P - \frac{1}{P^2} \partial_i P \partial_k P \right) d^3 x dt = \frac{\hbar^2}{8} \sum g^{ik} \int \frac{1}{P} \partial_i P \partial_k P d^3 x dt \sim \frac{\hbar^2}{8} \Phi_B
\]

which involves dropping a boundary term (note e.g. \( \int_\Omega \Delta P dV = \int_{\partial \Omega} \nabla P \cdot \mathbf{n} d \Sigma \)); this could become important in considerations of entropy and holography (cf. Remark 1.3).

Now going to [65] one considers

\[
(5.13) \quad H = \frac{1}{2} G_{ijkl} \frac{\delta k S}{\delta h_{ij}} \frac{\delta S}{\delta h_{kl}} - \sqrt{h} R = 0; \quad H_i = -2D_j \left( \delta_{ik} \frac{\delta S}{\delta h_{kj}} \right) = 0
\]

where \( D_j \) is the covariant derivative and (\( \bullet \bullet \bullet \bullet \)) \( G_{ijkl} = (1/\sqrt{h})(h_{ik} h_{j\ell} + h_{il} h_{jk} - h_{ij} h_{k\ell}) \) with \( G = 1/16 \pi \) (this differs by a factor of 2 from a previous \( G_{ijkl} \)). As a consequence of the constraint \( H = 0, \) \( \Phi_B \), and \( H_i = 0, S \) must satisfy various constraints (including \( \partial_t S = 0 \) - cf. [7]) and this is all subsumed in the invariance of the HJ functional \( S \) under spatial coordinate transformations. Hence one can keep the Hamiltonian constraint, ignore the momentum constraints, and require that \( S \) be invariant under the gauge group of spatial coordinate transformations. Now to define ensembles for gravitational fields one needs a measure \( D h \) and a probability functional \( P[h_{ij}] \) and this is discussed in some detail in [65] following [36, 51] (we omit details here). One is led to an ensemble Hamiltonian and derived equations

\[
(5.14) \quad \tilde{H}_c = \int d^3 x \int D h P H; \quad \partial_t P = \frac{\Delta \tilde{H}_c}{\Delta S}, \quad \partial_t S = -\frac{\Delta \tilde{H}_c}{\Delta P}
\]

With \( \partial_t S = \partial_t P = 0 \) the equations take the form

\[
(5.15) \quad H = 0; \quad \int d^3 x \frac{\delta}{\delta h_{ij}} \left( P G_{ijkl} \frac{\delta S}{\delta h_{kl}} \right) = 0
\]
The latter equation corresponds to a continuity equation and in this spirit some argument shows that it implies the standard rate equation

\[ \partial_t h_{ij} = N G_{ij k\ell} \frac{\delta S}{\delta h_{k\ell}} + D_i N_j + D_j N_i \]

(as follows from the ADM formalism with N the lapse function and N\(_j\) the shift vector). Now writing \( \pi^{k\ell} = (\delta S/\delta h_{k\ell}) + f^{k\ell} \) with \( f^{k\ell} = 0 \) one obtains an ensemble Hamiltonian (3.6) and an equation (3.7) as before. Again putting \( h = 2\sqrt{c} \) and \( \psi[h_{k\ell}] = \sqrt{P} \exp(iS/h) \) leads to the WDW equation (cf. (3.8))

\[ \left[ -\frac{\hbar^2}{2} \frac{\partial}{\partial h_{ij}} G_{ij k\ell} \frac{\delta}{\delta h_{k\ell}} - \sqrt{h} R \right] \psi = 0 \]

The procedures here suggest also replacing (5.16) by

\[ \partial_t h_{ij} = N G_{ij k\ell} \left( \frac{\delta S}{\delta h_{k\ell}} + f^{k\ell} \right) + D_i N_j + D_j N_i \]

Since the field momenta are subject to fluctuations so must be the extrinsic curvature \( K_{ij} = (1/2)G_{ij k\ell} (\delta S/\delta h_{k\ell}) \) yielding then

\[ K_{ij} = \frac{1}{2} G_{ij k\ell} \left( \frac{\delta S}{\delta h_{k\ell}} + f^{k\ell} \right) \]

6. THE ROLE OF THE QUANTUM POTENTIAL

We will sketch here an approach to quantum gravity based on the quantum potential. Generally one could start with WDW (which implies the Einstein equations with a few assumptions (cf. [30]). The introduction of a complex wave function \( A \exp(iS/\hbar) \) automatically leads to a quantum potential \( \mathcal{Q} \) and can be thought of as introducing a statistical element into the picture via the amplitude A which should create a probability density via \( A^2 = P \). A Hamiltonian term arises then via \( Q = \int \mathcal{Q} P \) which is proportional to a Fisher information and this specifies an ensemble of metric coefficients \( h_{ij} \) with probabilities \( P[h_{ij}] \). This does not involve a matter Lagrangian but arises gratuitously from the metric term \( P \) since \( \mathcal{Q} \) can be written entirely in terms of \( P \) and the \( h_{ij} \). Thus the introduction of a complex wave function into a classical problem is enough in itself to generate a quantum theory via information (or equivalently entropy) ideas.

The probability \( P \) can be thought of in various ways and the quantum potential term derived from various points of view. Thus in particular one can imagine momentum perturbations \( \pi^{ij} = (\delta S/\delta h_{ij}) + f^{ij} \) as in (\( \star \)) of Section 3 which via exact uncertainty requires the \( f^{ij} \) to be provided as in
(3.7) producing a term

\begin{equation}
\Phi_B \sim \frac{\hbar^2}{8} \int D\hbar \int dx NG_{ij\ell} \frac{\delta P}{\delta h_{ij}} \frac{\delta P}{\delta h_{\ell\ell}}
\end{equation}

We recall that the exact uncertainty theory for the SE involves a relation \((\textbf{EU})\) \(\delta X \Delta P_{nc} = \hbar/2\) where \(P_{nc}\) with \(< P_{nc} >= 0\) is a nonclassical component of momentum (cf. \[10, 33, 34, 65\]). This is very similar in spirit to a formula of the form \((\textbf{O})\) \((\delta p)^2(\delta x)^2 = \hbar^2/4\) used by Olavo in deriving the SE from entropy considerations (cf. \[13, 57\]). The approach of Olavo is especially interesting since it builds entropy explicitly into the theory and this could be identified as its information content.

In connection with the role of a complex wave function we recall that a complex velocity has been emphasized by Castro, Mahecha, Nottale, and the author (cf. \[10, 14, 15, 16, 55\]) and in a Weyl geometry with Weyl field \(\phi_\mu \sim A_\mu = -\partial_\mu \log(P)\) (where \(P \sim \rho\) is a density) a complex velocity \(p_\mu + i\lambda A_\mu\) leads to

\begin{equation}
|p_\mu + i\sqrt{\lambda} A_\mu|^2 = p_\mu^2 + \lambda A_\mu^2 \sim g^{\mu\nu} \left( \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu} + \frac{\lambda}{P^2} \frac{\partial P}{\partial x^\mu} \frac{\partial P}{\partial x^\nu} \right)
\end{equation}

which generates again a Fisher matrix (cf. also \[51\]).

The quantum potential also arises as a stress tensor in a quantum fluid (cf. \[10, 21, 77\]) and in a diffusion context following \[10, 53, 52\] (cf. \[10\] for a survey of the quantum potential). There is also a quantum potential connected with a quantum matter field \(\phi\) as in Sections 2 and 4. In such cases the nature of A as a probability or density is less clear since it depends on \(\phi\) and the \(h_{ij}\). One should be able to form a Fisher information based on perturbations of both terms. We see that the quantum WDW equation is formed via a Fisher information type term in the Hamiltonian and in view of the strong connection between entropy and Fisher information the idea of having an entropy functional to extremize as in Section 1 seems eminently reasonable.
References

[1] R. Adler, M. Bazin, and M. Schiffer, Introduction to general relativity, McGraw-Hill, 1965
[2] A. Anderson and J. York, gr-qc 9807041
[3] R. Arnowitt, S. Deser, and C. Misner, gr-qc 0405109
[4] J. Baez and J. Muninain, Gauge fields, knots, and gravity, World Scientific, 1994
[5] A. Barvinsky and C. Kiefer, gr-qc 9711037
[6] G. Beretta, quant-ph 0112046
[7] P. Bergmann, Phys. Rev., 144 (1966), 1078-1080
[8] J. Calmet and X. Calmet, math-ph 0403043
[9] X. Calmet and J. Calmet, cond-mat 0410552
[10] R. Carroll, Fluctuations, information, gravity, and the quantum potential, Lecture notes, 2005
[11] R. Carroll, Gravity, entropy, and the quantum potential, Lecture notes, 2005, 65 pages
[12] R. Carroll, Bohmian gravity, Lecture notes, 2005, 38 pages
[13] R. Carroll, Quantum theory, deformation, and integrability, North-Holland, 2000
[14] R. Carroll, Found. Phys., 35 (2005), 131-154
[15] C. Castro, Found. Phys., 22 (1992), 569-615; Found. Phys. Lett., 4 (1991), 81
[16] C. Castro and J. Mahecha, Physica D, to appear
[17] I. Chakrabarty, quant-ph 0511169
[18] C. Chakrabarti and I. Chakrabary, quant-ph 0511171
[19] Y. Choquet-Bruhat and J. York, gr-qc 0511032
[20] J. Halliwell, Phys. Rev. D, 38 (1988), 2468-2481
[21] M. Hall, K. Kumar, and M. Reginatto, quant-ph 0103041; Jour. Phys. A, 36 (2003), 9779-9794 (hep-th 0206235 and 0307259)
[22] B. deWitt, Phys. Rev., 160 (1967), 1113-1148
[23] B Frieden, Physics from Fisher information, Cambridge Univ. Press, 1999
[24] B. Frieden and B. Soffer, Phys. Rev. E, 52 (1995), 2274-2286
[25] J. Friedman and A. Higuchi, Phys. Rev. D, 41 (1990), 2479-2486
[26] V. Frolov, D. Fursaev, and A. Zelnikov, hep-th 9607104
[27] R. Garattini, gr-qc 9508060 and 9604004
[28] P. Garbaczewski, cond-mat 0211362, 0301044, and 0504115; quant-ph 0408192, 0504098, and 0509215
[29] A. Gentle, N. George, A. Kheyfets, and W. Miller, gr-qc 0302044 and 0302051
[30] U. Gerlach, Phys. Rev., 117 (1960), 1929-1941
[31] D. Giulini, gr-qc 9311017
[32] D. Giulini and C. Kiefer, gr-qc 9409004 and 9505040
[33] M. Hall, K. Kumar, and M. Reginatto, quant-ph 0103041 Jour. Phys. A, 36 (2003), 9779-9794 (hep-th 0206235 and 0307259)
[34] M. Hall, gr-qc 0408008
[35] J. Halliwell, Phys. Rev. D, 38 (1988), 2468-2481
[36] H. Hamber and R. Williams, Phys. Rev. D, 59 (1999), 064014
[37] B. Hu and A. Roura, gr-qc 0402029 and 0508010
[38] B. Hu and N. Phillips, gr-qc 0004006
[39] B. Hu and E. Verdaguer, gr-qc 0307032
[40] B. Hu, gr-qc 9902064 and 0204069
[41] B. Hu, A. Roura, S. Sinha, and E. Verdaguer, gr-qc 0304057
[42] A. Kheyfets and W. Miller, gr-qc 9412037, 9406031, and 0006001
[43] C. Kiefer, Quantum gravity, Oxford Univ. Press, 2004
[44] C. Kiefer, gr-qc 9312015
[45] C. Kiefer and T. Lück, gr-qc 0505158
[46] D. Kribs and F. Markopoulou, gr-qc 0510052
[47] S. Luo, Jour. Phys. A, 35 (2002), 5181-5187
[48] V. Majernik and L. Richterek, Euro. Jour. Phys., 18 (1997), 79-89
[49] C. Misner, K. Thorne, and J. Wheeler, Gravitation, Freeman, 1973
[50] F. Markopoulou, gr-qc 9610138
[51] E. Mottola, Jour. Math. Phys., 36 (1995), 2470-2511
[52] M. Nagasawa, Schrödinger equations and diffusion theory, Birkhäuser, 1993; Stochastic processes in quantum physics, Birkhäuser, 2000
[53] E. Nelson, Quantum fluctuations, Princeton Univ. Press, 1985
[54] H. Nikolić, hep-th 0501046
[55] L. Nottale, Fractal spacetime and microphysics: A theory of scale relativity, World Scientific, 1993
[56] H. Ohanian and R. Ruffini, Gravitation and spacetime, Norton, 1994
[57] L. Olavo, Physica A, 262 (1999), 197-214 and 271 (1999), 260-302; Phys. Rev. E, 64 (2001), 036125
[58] T. Padmanabhan, gr-qc 0408051
[59] T. Padmanabhan, hep-th 0205078 gr-qc 0204199 0412068, and 0510015; Class. and Quant. Gravity, 21 (2004), 4485-4494
[60] F. Pennini and A. Plastino, cond-mat 0405033
[61] H. Pfeiffer and J. York, gr-qc 0207095
[62] W. Pietsch, cond-mat 0509212
[63] N. Pinto-Neto, gr-qc 0410117
[64] N. Pinto-Neto and E. Santini, gr-qc 0009080 and 0302112; General Relativ. Gravitation, 34 (2002), 505; Phys. Lett. A, 315 (2003), 36; Phys. Rev. D, 59 (1999), 123517 (gr-qc 9811067)
[65] M. Reginatto, gr-qc 0501030
[66] M. Reginatto, cond-mat 9910039 quant-ph 9909065 Phys. Rev. A, 58 (1998), 1775-1778
[67] C. Rovelli, Quantum gravity, Cambridge Univ. Press, 2004
[68] E. Santamato, Phys. Rev. D, 29 (1984), 216-222 and 32 (1985), 2615-2621; Jour. Math. Phys., 25 (1984), 2477-2480
[69] A. and F. Shojai and N. Dadhich, gr-qc 0504137
[70] F. Shojai, Phys. Rev. D, 60 (1999), 124001
[71] F. and A. Shojai, gr-qc 0105102
[72] A. and F. Shojai, gr-qc 0306100 and 0311076
[73] F. and A. Shojai, gr-qc 0109052
[74] A. and F. Shojai, gr-qc 0409053 Class. Quant. Grav., 21 (2004), 1-9
[75] R. Sorkin and D. Sudarsky, gr-qc 9902051
[76] N. Straumann, astro-ph 0006423
[77] T. Takabayashi, Prog. Theor. Phys., 11 (1954), 341
[78] R. Wald, General relativity, Univ. Chicago Press, 1984
[79] S. Weinberg, Gravitation and cosmology, Wiley,1972
[80] J.A. Wheeler, Battelle Rencontres, Benjamin, 1967, pp. 242-307
[81] J. Wheeler, Phys. Rev. D, 41 (1990), 431
[82] T. Yamano cond-mat 0009078 and 0010074
[83] J. York, Phys. Rev. Lett., 28 (1972), 1082-1085; Jour. Math. Phys., 14 (1973), 456-464
[84] J. York, gr-qc 0405005
[85] J. York, gr-qc 9307022
[86] J. York, Found. Phys., 16 (1986), 249-257