Gradient estimate of the solutions to Hessian equations with oblique boundary value

Abstract: In this paper, we study Hessian equations with the prescribed contact angle boundary value or oblique derivative boundary value and finally derive the a priori global gradient estimate for the admissible solutions.

Keywords: oblique derivative boundary value, prescribed contact angle boundary value, gradient estimate, Hessian equations

MSC code: 35J65

1 Introduction

In this paper, we consider the following Hessian equation with oblique boundary value,

\[
\begin{align*}
\sigma_k(u_0) &= f(x, u) \quad \text{in } \Omega, \\
G(x, Du) &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) with smooth boundary and \( f(x, t) \) and \( G(x, \overrightarrow{p}) \) are smooth functions defined, respectively, on \( \Omega \times \mathbb{R} \) and \( \overline{\Omega} \times \mathbb{R}^n \). We mainly study two general but important cases of \( G(x, Du) \), one is the prescribed contact angle boundary value problem and the other is the oblique derivative boundary value problem. The topic in this paper is also concentrated on the global gradient estimate, which would be one step forward to conclude the existence of the solution to problem (1).

Hessian equations including Laplace equations and Monge-Ampère equations as their special cases with various boundary values are in no doubt an interesting subject in recent years, and many topics in differential geometry, convex geometry, and optimal transport have close relations with these kind of elliptic equations. For the given boundary value, one may first be interested in the existence of the solution. In general, it is necessary to obtain the \( C^{1,a} \) estimate to conclude the existence of the solution. For instance, when the boundary value is of the Dirichlet type, one can refer to [1–3] for the existence results. For the Neumann boundary value, Trudinger [4] considered the special domain case and obtained the existence result. Also, one can solve the problem in sufficiently smooth uniformly convex domains. Recently, Ma and Qiu [22] gave a positive answer to this problem and solved the Neumann problem of \( k \)-Hessian equations in uniformly convex domains. Chen and Zhang [14] considered the Hessian quotient equation and also derived the existence results with the Neumann boundary condition.

Now, it is of natural interest to consider the existence of the solutions to Hessian equations with the other types of boundary value problems such as prescribed contact angle boundary value and oblique
derivative boundary value. It seems to be a little more complicated for these kinds of boundary values. For instance, a necessary condition for the existence of the solution to Monge-Ampère equations was exhibited in [9,10]. Till now, there are only a few progress results on this topic. In [8], the oblique derivative boundary problems for Monge-Ampère equations were considered and the existence of the solutions to two-dimensional Monge-Ampère equations was derived, and the generalized solutions for general dimension Monge-Ampère equations were also considered. In [11–13], Urbas also derived some existence results for Monge-Ampère equations with the oblique derivative boundary value. For some augmented Hessian equations with oblique boundary value, Jiang and Trudinger in [5,6] considered the existence result. Wang [7] derived the interior gradient estimate of the solutions to \(k\)-curvature equations, and Deng and Ma [25] obtained the global gradient estimate for \(k\)-curvature equations with the prescribed contact angle boundary value. It is still open for the existence of the solutions to \(k\)-curvature equations and Hessian equations with prescribed contact angle or oblique derivative boundary value. In this paper, we make an attempt for this problem and finally will derive the global gradient estimate for admissible solutions to Hessian equations with these kinds of boundary conditions, which would be considered as a little step forward to the existence of the solutions to these interesting problems.

Gradient estimate of the solutions to various partial differential equations is an important and interesting issue in the study of P.D.E. Usually, it includes interior gradient estimate and global gradient estimate, which, respectively, have close relation to Liouville type results and the existence of the solution to P.D.E. One can refer to [1,3,7,8,14–16,18,20,23–25], and the references therein for more details.

The rest of the paper is organized as follows. In Section 2, we introduce some notations and preliminaries that are necessary for the gradient estimate.

## 2 Notations and preliminaries

In this section, we list some notations and preliminaries that are necessary for the gradient estimate.

First, we denoted by \(d(x) = \text{dist}(x, \partial \Omega)\) the distance from \(x\) to \(\partial \Omega\), the boundary of a bounded smooth domain \(\Omega\). As a known fact, \(d(x)\) is also smooth near the boundary, such as on the annular domain \(\Omega_{\mu_i} = \{x \in \Omega \mid d(x) \leq \mu_i\}\), where \(\mu_i\) is a positive constant related to the domain.

Second, we give some basic properties of elementary symmetric functions, denoted by \(\sigma_k(\lambda)\) for \(\lambda \in \mathbb{R}^n\), which could be found in [1,3].

We denoted by \(\sigma_k(\lambda|i)\) the \(k\)th symmetric function with \(\lambda_i = 0\) and \(\sigma_k(\lambda|i_1, i_2)\) by the \(k\)th symmetric function with \(\lambda_i = \lambda_j = 0\). Then we have the following propositions.

**Proposition 2.1.** Assume \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}^n\), and \(k = 1, 2, \ldots, n\), then we have

\[
\sigma_k(\lambda) = \sigma_k(\lambda|i) + \lambda_i \sigma_{k-1}(\lambda|i), \quad 1 \leq i \leq n,
\]

\[
\sum_{i=1}^{n} \lambda_i \sigma_{k-1}(\lambda|i) = k \sigma_k(\lambda),
\]

\[
\sum_{i=1}^{n} \sigma_k(\lambda|i) = (n - k) \sigma_k(\lambda).
\]

Recall that the Garding’s cone is defined as follows:

\[
\Gamma_k = \{\lambda \in \mathbb{R}^n \mid \sigma_i(\lambda) > 0, \quad \forall 1 \leq i \leq k\}.
\]

**Proposition 2.2.** Assume \(k \in \{1, 2, \ldots, n\}\) and \(\lambda \in \Gamma_k\), suppose that
\[ \lambda_1 \geq \cdots \geq \lambda_k \geq \cdots \geq \lambda_n, \]

then we have

\[ \sigma_{k-l}(\lambda|n) \geq \cdots \geq \sigma_{k-l}(\lambda|k) \geq \cdots \geq \sigma_{k-l}(\lambda|1) > 0 \]

and

\[ \sigma_{k-l}(\lambda|k) \geq C(n, k) \sum_{i=1}^{n} \sigma_{k-l}(\lambda|i). \tag{3} \]

Remark that if the eigenvalues of \((u_{ij})\), denoted also by \((\lambda_1, \lambda_2, \ldots, \lambda_n)\), are located in \(\Gamma_k\), then the equation in (1) is elliptic and we will call this kind of solution as “k-admissible” solution.

We also list the generalized Newton-MacLaurin inequality in the following, which includes the Newton inequality and the MacLaurin inequality as the special cases.

**Proposition 2.3.** Assume \(\lambda \in \Gamma_k\), and \(k, l, r, s \in \{0, 1, 2, \ldots, n\}\) with \(k > l \geq 0\), \(r > s \geq 0\), \(k \geq r\), \(l \geq s\), we have

\[
\begin{bmatrix}
\frac{\partial^{(k)}}{\partial x^k} & \frac{\partial^{(l)}}{\partial x^l} \\
\frac{\partial^{(k)}}{\partial x^k} & \frac{\partial^{(l)}}{\partial x^l}
\end{bmatrix} 
\leq
\begin{bmatrix}
\frac{\partial^{(r)}}{\partial x^r} & \frac{\partial^{(s)}}{\partial x^s} \\
\frac{\partial^{(r)}}{\partial x^r} & \frac{\partial^{(s)}}{\partial x^s}
\end{bmatrix}^{rs},
\tag{4}
\]

and the equality holds if and only if \(\lambda_1 = \lambda_2 = \cdots = \lambda_n > 0\).

As the last point of this section, we also state that the universal constant \(C\) during the whole paper may change from line to line.

### 3 Prescribed contact angle boundary data

In this section, we set out to obtain the gradient estimate of the admissible solution to Hessian equations with the prescribed contact angle boundary value. In a word, we will prove the following theorem.

**Theorem 3.1.** Let \(\Omega\) be a smooth bounded domain in \(\mathbb{R}^n(n \geq 2)\) and \(u\) be the admissible solution to the following Hessian equations with the prescribed contact angle boundary value,

\[
\begin{aligned}
\sigma_{t}(u_{ij}) &= f(x, u) \quad \text{in } \Omega, \\
\frac{\partial u}{\partial n} &= -\cos \theta \sqrt{1 + |Du|^2} \quad \text{on } \partial \Omega.
\end{aligned}
\tag{5}
\]

Assume that \(f(x, t)\) is a positive smooth function defined on \(\Omega \times \mathbb{R}\) with \(f_i \geq 0\) and \(\theta(x)\) is a smooth function defined on \(\bar{\Omega}\) with \(|\cos \theta| \leq 1 - b < 1\) for some positive constant \(b\). \(v\) is denoted to be the inward unit normal along \(\partial \Omega\). Also we assume that we have already obtained the \(C^0\) estimate as \(|u| \leq M\). Then, there exists a positive constant \(C = C(M, n, \Omega, b, |\theta|_{C^{1}([\Omega])}, |f|_{C^{1}([\Omega \times [-M, M])}})\) such that

\[
|Du| \leq C. \tag{6}
\]

**Proof.** Due to [1], we have already known the interior gradient estimate, so we only need to obtain the gradient estimate near boundary, denoted by \(\Omega_{\mu}\), where \(\mu \leq \mu_1\) is a positive constant to be determined later.

Let \(v = \sqrt{1 + |Du|^2}, \ w = v + \sum_{i=1}^{n} u_i d_i \cos \theta\) and let \(h(t)\), and \(\tau\) be a smooth function and a positive constant, respectively, to be determined later. We choose the auxiliary function

\[ \Phi = \log w + h(t) + \tau d.\]
Assume $\Phi$ achieves its maximum on the domain $\Omega_{\mu}$ at the point $x_0$, according to the interior gradient estimate, we can only consider the following two cases.

**Case I:** $x_0 \in \partial \Omega$.

For convenience, we choose a coordinate around $x_0$ such that $v = \frac{\partial}{\partial x_0}$, assume $\frac{\partial}{\partial x_i} (i = 1, 2, \ldots, n - 1)$ are tangent to $\partial \Omega$. Under this coordinate, we have

$$\frac{\partial d}{\partial x_i} = 0, \quad \frac{\partial d}{\partial x_a} = 1, \quad \frac{\partial^2 d}{\partial x_0 \partial x_a} = 0, \quad \frac{\partial^2 d}{\partial x_0 \partial x_j} = -\kappa_0 \delta_{ij},$$

where $1 \leq i, j < n - 1, 1 \leq a \leq n$ and $\kappa_i (i = 1, 2, \ldots, n - 1)$ are the principal curvatures of $\partial \Omega$ at $x_0$.

By the fact that $x_0$ is the maximum point on the boundary, we have

$$0 = \Phi_i = \frac{w_i}{w} + h' u_i + \tau d_i = \frac{w_i}{w} + h' u_i, \quad i = 1, 2, \ldots, n - 1, \quad (7)$$

and

$$0 \geq \Phi_n = \frac{w_n}{w} + h' u_n + \tau d_n = \frac{w_n}{w} + h' u_n + \alpha. \quad (8)$$

By a direct computation, we have

$$w_n = v_n + u_n \cos \theta + u_n (\cos \theta) n$$

$$= \sum_{i=1}^{n-1} u_i u_n v + u_n \cos \theta + u_n (\cos \theta) n$$

$$= \sum_{i=1}^{n-1} u_i u_n \cos \theta + u_n \cos \theta + u_n (\cos \theta) n$$

$$= \sum_{i=1}^{n-1} u_i u_n \cos \theta + \sum_{i,j=1}^{n-1} u_i k_{ij} u_j + u_n (\cos \theta) n, \quad (9)$$

where we denote by $k_{ij}$ the Weingarten matrix of the boundary with respect to $v$.

Differentiating $u_n$ along $\partial \Omega$, we obtain for $i = 1, 2, \ldots, n - 1$ that

$$u_{ni} = (-v \cos \theta)_i = -v \cos \theta - v (\cos \theta)_i$$

$$= -\left( w_i - u_n \cos \theta - \sum_{i=1}^{n-1} u_i d_i \cos \theta - u_n (\cos \theta)_i \right) \cos \theta - v (\cos \theta)_i$$

$$= -w_i \cos \theta + u_n \cos^2 \theta + \sum_{i=1}^{n-1} u_i d_i \cos^2 \theta + u_n \cos \theta (\cos \theta)_i - v (\cos \theta)_i,$$

furthermore, using (7), we can obtain

$$u_{ni} = \frac{h' w_i \cos \theta + \sum_{i=1}^{n-1} u_i d_i \cos^2 \theta - v (1 + \cos^2 \theta)(\cos \theta)_i}{\sin^2 \theta}.$$  \hspace{1cm} (10)

Substituting (10) into (9), we then have

$$w_n = \frac{\sum_{i=1}^{n-1} u_i u_n}{v} + \frac{\sum_{i,j=1}^{n-1} u_i k_{ij} u_j}{v} + u_n (\cos \theta) n$$

$$= \frac{\sum_{i=1}^{n-1} \left[ h' w_i \cos \theta + \sum_{i=1}^{n-1} u_i d_i \cos^2 \theta - v (1 + \cos^2 \theta)(\cos \theta)_i \right]}{v \sin^2 \theta} + \frac{\sum_{i,j=1}^{n-1} u_i k_{ij} u_j}{v} + u_n (\cos \theta) n.$$

Then
Without the loss of generality, we may assume that $v$ is large such that if $\tau$ is chosen large enough determined by $\theta$ and the geometry of $\partial \Omega$, the right hand of the above inequality will be positive, which shows that this case will not occur at all.

**Case II:** $x_0 \in \Omega$. At this point, we can assume that $|Du|$ is large enough such that $|Dw|, w$ and $v$ are equivalent with each other. Remark that the Einstein summation convention will be adopted during all the calculations if no otherwise specified.

Since $x_0$ is the maximum point, we then have

$$0 = \Phi_i = \frac{W_i}{w} + h'u_i + \tau d_i,$$

and it follows that

$$w_i = -w(h'u_i + \tau d_i). \tag{11}$$

By the definition of $w$, we have

$$w = \frac{u_0u_i}{v} + u_0d_i \cos \theta + u_0d_i \cos \theta + u_0d_i \cos \theta = \left( \frac{u_0}{v} + d_0 \frac{cos \theta}{v} \right) u_i + u_0d_i \cos \theta + u_0d_i \cos \theta.$$  

Therefore,

$$-w(h'u_i + \tau d_i) = \left( \frac{u_0}{v} + \cos \theta d_i \right) u_i + u_0d_i \cos \theta + u_0d_i \cos \theta. \tag{12}$$

We now come to deal with $\Phi_{ij}$. By (11), we derive that

$$\Phi_{ij} = \frac{W_{ij}}{w} - \frac{WW_{ij}}{w^2} + h'u_{ij} + h'u_{ij} + \tau d_{ij}$$

$$= \frac{W_{ij}}{w} - \left( h'u_{ij} + \tau d_{ij}(h'u_{ij} + \tau d_{ij}) + h'u_{ij} + h'u_{ij} + \tau d_{ij} \right)$$

$$= \frac{W_{ij}}{w} - \tau h'u_{ij} - \tau h'u_{ij} - \tau^2 d_{ij} + h'u_{ij} + [h'' - (h')^2] u_{ij} + \tau d_{ij}.$$

Following [25], we take the coordinate around $x_0$ such that $(u_{ij})$ is diagonal at this point, and all the following calculation will be done at this point. Denoted by $F^i$ the derivative $\frac{\partial (u_{ij})}{\partial u_{ij}}$ and $F$ the sum $\sum_{i=1}^{n} F^i$.

We then have

$$0 \geq F^i \Phi_{ij} = \frac{F^i W_{ij}}{w} + [h'' - (h')^2] F^i u_{ij} + h' F^i u_{ij} + \tau F^i d_{ij} - 2 h' F^i u_{ij} d_{ij} - \tau^2 F^i d_{ij} d_{ij} = I + II + III, \tag{13}$$

where
\[ I = F^{ij} w_{ij}, \]
\[ II = [h' - (h')^2] F^{ij} u_{ij}, \]
\[ III = h' F^{ij} u_{ij} + \tau F^{ij} d_{ij} - 2r h' F^{ij} u_{ij} d_{ij} - \tau^2 F^{ij} d_{ij}. \]

For the last term, we can easily have
\[ III = h' F^{ij} u_{ij} + \tau F^{ij} d_{ij} - 2r h' F^{ij} u_{ij} d_{ij} - \tau^2 F^{ij} d_{ij} \geq -C|\partial u F|. \quad (14) \]

In the following, we come to deal with the first term \( I \). The key point is to calculate \( F^{ij} w_{ij} \). By a direct calculation, we can deduce that
\[
F^{ij} w_{ij} = \left( \frac{u_i}{v} + d_i \cos \theta \right) u_{ij} + \left( \frac{u_j}{v} + d_j \cos \theta \right) u_{ij} + u_{ij} (d_i \cos \theta) + u_{ij} (d_j \cos \theta) \]
\[ = \left( \frac{u_i}{v} + d_i \cos \theta \right) u_{ij} + \left( \frac{u_j}{v} - \frac{u_i u_j}{v^3} \right) u_{ij} + (d_i \cos \theta) u_{ij} + (d_j \cos \theta) u_{ij} + u_{ij} (d_i \cos \theta) + u_{ij} (d_j \cos \theta). \]

Hence,
\[
F^{ij} w_{ij} = \left( \frac{u_i}{v} + d_i \cos \theta \right) D_i f + \left( \frac{1}{v} - \frac{u_i^2}{v^3} \right) F^{ii} u_{ii}^2 + (d_i \cos \theta) F^{ii} u_{ii} \]
\[ + F^{ij} d_{ij} (d_i \cos \theta) + F^{ij} d_{ij} (d_j \cos \theta) + F^{ij} d_{ij} (d_i \cos \theta) + F^{ij} d_{ij} (d_j \cos \theta). \quad (15) \]

For the choice of the coordinate and (12), we have at \( x_0 \) that
\[
-w(h' u_i + \tau d_i) = \left( \frac{u_i}{v} + d_i \cos \theta \right) u_{ii} + u_{ij} (d_i \cos \theta), \quad i = 1, 2, \ldots, n. \quad (16) \]

Setting
\[
K = \left\{ i \in I \mid |d_i \cos \theta| + \frac{b}{8n} \leq \left| \frac{u_i}{v} \right| \right\}, \]
where \( I = \{1, 2, \ldots, n\} \). It is obvious that the index set \( K \) is not empty and if we further assume that \( v \) is large enough, we can assume that
\[
|\tau d_i| \leq \frac{1}{2} h' |u_i|, |u_i (d_i \cos \theta)| \leq \frac{1}{4} |h' w u| \quad \text{for} \ i \in K. \]

Note that we here need \( h' \) have a positive bound, which will be satisfied later. Under these assumptions, we have
\[
-C h' |w u| \leq u_i \leq 0 \quad \text{for} \ i \in K. \quad (17) \]

Then for \( i \in K \), we have by (3) that
\[
F^{ii} \geq F^{kk} \geq C F. \]

Hence,
\[
F^{ii} w_{ii} \geq \sum_{i=1}^{n} \left( \frac{1}{v} - \frac{u_i^2}{v^3} \right) F^{ii} u_{ii}^2 - 2 (d_i \cos \theta) F^{ii} u_{ii} \]
\[ - C|\partial u F| - C|\partial u| \]
\[ = \sum_{i \in K} \left( \frac{1}{v} - \frac{u_i^2}{v^3} \right) F^{ii} u_{ii}^2 - 2 (d_i \cos \theta) F^{ii} u_{ii} + \sum_{i \in K} \left( \frac{1}{v} - \frac{u_i^2}{v^3} \right) F^{ii} u_{ii}^2 - 2 (d_i \cos \theta) F^{ii} u_{ii} \]
\[ - (C|\partial u F| + C|\partial u|) \]
\[ = T_1 + T_2 + T_3. \quad (18) \]
For the term $T_1$, according to (17), we have
\[
T_1 = \sum_{i \in K} \left( \frac{1}{v} - \frac{u_i^2}{v^3} \right) F^i u_i^2 - 2(d_i \cos \theta_i) F^i u_i \geq \sum_{i \in K} (-2(d_i \cos \theta_i) F^i u_i) \geq -Cv^2F
\] (19)
and for the term $T_2$, because of the definition of $K$ and the fact $ax^2 + bx \geq -\frac{b^2}{4a}$ for $a > 0$, we have
\[
T_2 \geq \sum_{i \in K} \left( \frac{1}{v} - \frac{u_i^2}{v^3} \right) F^i u_i^2 - 2(d_i \cos \theta_i) F^i u_i \geq \sum_{i \in K} \left( \frac{C}{vF^i} (F^i u_i)^2 - 2(d_i \cos \theta_i) F^i u_i \right) \geq -CvF.
\] (20)

It follows that
\[
I = \frac{F^i u_i}{w} \geq -CvF - CF - C.
\] (21)

For the term $II$,
\[
II = [H' - (H')^2] F^i u_i u_j = [H' - (H')^2] \sum_{i=1}^n F^i u_i^2 \geq [H' - (H')^2] \sum_{i \in K} F^i u_i^2 \geq C[H' - (H')^2]v^2F.
\] (22)

By the Newton-MacLaurin inequality stated in Proposition 2.3, we have
\[
F \geq C > 0,
\] (23)
and therefore,
\[
0 \geq \frac{F^i u_i}{F} = I + II + III \geq C[H' - (H')^2]v^2 - CV - C - \frac{C}{F} \geq C[H' - (H')^2]v^2 - CV - C.
\] (24)

If we take $h(t) = \frac{1}{2} \ln \frac{1}{(M-t)^{\delta}}$, then $h' - (h')^2 = (h')^2$ and $h(t)$ satisfies all the assumptions we have set in advance. Thus, we bound the gradient at this point such that $v \leq C$, then we derive the gradient estimate near the boundary by a standard discussion. Thus, we complete the proof of Theorem 3.1.

\[\square\]

## 4 Oblique derivative boundary value

In this section, we will obtain the a priori gradient estimate of the solution to Hessian equations with the oblique derivative boundary value. Specifically, we will show the following result.

**Theorem 4.1.** Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^n (n \geq 2)$ and $u$ be the admissible solution to the following Hessian equations with the oblique derivative boundary value,
\[
\begin{aligned}
\sigma_i(u) &= f(x, u) \quad \text{in } \Omega, \\
\frac{\partial u}{\partial \beta} &= \varphi(x, u) \quad \text{on } \partial \Omega,
\end{aligned}
\] (25)
where $f(x, t)$ is a positive smooth function defined on $\Omega \times \mathbb{R}$ with $f_t \geq 0$, $\varphi(x, t)$ is a smooth function defined on $\bar{\Omega} \times \mathbb{R}$ and $\beta$ is a smooth unit vector field along $\partial \Omega$ with $\langle \beta, v \rangle \geq c_0 > 0$ for some positive constant $c_0$, and $v$ is denoted to be the inward unit normal along $\partial \Omega$. Also we assume that we have already obtained the $C^0$ estimate as $|u| \leq M$. Then, there exists a positive constant $C = C(M, n, \Omega, c_0, |\beta|_{C^0(\Omega)}, |f|_{C^0(\Omega \times [-M, M])}, |\varphi|_{C^0(\Omega \times [-M, M])})$ such that
\[
| Du | \leq C.
\] (26)

**Proof.** First, we say some words about the boundary value.
Taking a unit normal moving frame along $\partial \Omega$, denoted by $\{e_1, e_2, \ldots, e_{n-1}, \nu\}$, then $\beta$ can be represented as
\begin{equation}
\beta = \beta_n \nu + \sum_{l=1}^{n-1} \beta_l e_l,
\end{equation}
where $\beta_n = \langle \beta, \nu \rangle = \cos \theta$, which is bounded from below by the positive constant $c_0$ according to the conditions of Theorem 4.1.

By the boundary data, we have
\begin{equation}
\phi(x, u) = \frac{\partial u}{\partial \nu} = \langle Du, \beta \rangle = \frac{\partial u}{\partial \nu} \beta_n + \sum_{l=1}^{n-1} \beta_l u_l.
\end{equation}

Setting $w = u - \frac{qd}{\cos \theta}$, we then have
\begin{equation}
\phi(x, u) = \frac{\partial (w + \frac{qd}{\cos \theta})}{\partial \nu} \cos \theta + \sum_{l=1}^{n-1} \beta_l \left( w + \frac{qd}{\cos \theta} \right)_l,
\end{equation}
which indicates that
\begin{equation}
0 = \frac{\partial w}{\partial \nu} \beta_n + \sum_{l=1}^{n-1} \beta_l w_l.
\end{equation}

Therefore, we have
\begin{equation}
\frac{\partial w}{\partial \nu} = -\sum_{l=1}^{n-1} \frac{\beta_l}{\beta_n} w_l,
\end{equation}
and it follows by Cauchy inequality and the fact $\sum_{l=1}^{n-1} \beta_l^2 = 1$ that
\begin{equation}
\left( \frac{\partial w}{\partial \nu} \right)^2 \leq |Dw|^2 \cdot \sin^2 \theta.
\end{equation}

As before, we only need to obtain the gradient estimate near boundary, denoted by $\Omega_\mu$, where $\mu \leq \mu_i$ is a positive constant to be determined later. We extend $\beta$ smoothly to $\Omega_\mu$, also denoted by $\beta$, such that $\langle \beta, Dd \rangle = \cos \theta \geq c_0$ is also assumed to be still valid. Denote by
\begin{equation}
\phi = |Dw|^2 - \left( \sum_{a=1}^{n} w_a d_a \right)^2 = \sum_{a, \delta=1}^{n} (\delta_{a\delta} - d_a d_\delta) w_a w_\delta = \sum_{a, \delta=1}^{n} C_{a\delta} w_a w_\delta
\end{equation}
and take the auxiliary function
\begin{equation}
\Phi = \log \phi + h(u) + \tau d,
\end{equation}
where $h(t)$ is a smooth function and $\tau$ is a positive constant. Both of them will be determined later.

Assume the maximum of $\Phi$ on $\Omega_\mu$ is achieved at $x_0$. Also by the interior gradient estimate, which has been derived in [1], we only need to consider the two following cases.

**Case I**: $x_0 \in \partial \Omega$.

As in Section 3, we choose a coordinate around $x_0$ such that $\nu = \frac{\partial}{\partial x_n}$ and $\frac{\partial}{\partial x_i} (i = 1, 2, \ldots, n-1)$ are tangent to $\partial \Omega$. We also have that
\begin{equation}
\frac{\partial d}{\partial x_i} = 0, \quad \frac{\partial d}{\partial x_n} = 1, \quad \frac{\partial^2 d}{\partial x_i \partial x_n} = 0, \quad \frac{\partial^2 d}{\partial x_i \partial x_j} = -\kappa_i \delta_{ij},
\end{equation}
where $1 \leq i, j < n-1, 1 \leq \alpha \leq n$ and $\kappa_i (i = 1, 2, \ldots, n-1)$ are the principal curvatures of $\partial \Omega$ at $x_0 \in \partial \Omega$.

By the fact that $x_0$ is the maximum point of $\Phi$ on the boundary, it follows that
0 = \Phi_i = \frac{\phi_i}{\phi} + h'u_i, \quad i = 1, 2, \ldots, n - 1 \tag{33}

and

0 \geq \Phi_n = \frac{\phi_n}{\phi} + h'u_n + \alpha d_n = \frac{w_n}{w} + h'u_n + \tau. \tag{34}

From (33), we obtain

\begin{align*}
-\phi'hu_i &= (|D\phi|^2)_{i} - \left[\sum_{a=1}^{n} w_a d_a\right]_{i} = 2 \sum_{j=1}^{n-1} w_j w_j - 2 w_n \sum_{j=1}^{n-1} d_j w_j, \quad i = 1, 2, \ldots, n - 1. \tag{35}
\end{align*}

We then deal with the term $\phi_n$ as follows:

\begin{align*}
\phi_n &= 2 \sum_{a=1}^{n} w_a w_n - 2 w_n w_n = 2 \sum_{i=1}^{n-1} w_i w_n = 2 \sum_{i=1}^{n-1} w_i w_n + 2 \sum_{i,j=1}^{n-1} \kappa_{ij} w_i w_j \\
&= -2 \sum_{i=1}^{n-1} w_i \left(\frac{\beta_i}{\beta_n}\right) + 2 \sum_{i,j=1}^{n-1} \kappa_{ij} w_i w_j \\
&= -2 \sum_{i=1}^{n-1} w_i \left(\frac{\beta_i}{\beta_n}\right) + 2 \sum_{i,j=1}^{n-1} \kappa_{ij} w_i w_j \\
&= \frac{\phi'hu_n}{\beta_n} \sum_{i=1}^{n} u_i \beta_i - 2w_n \sum_{i,j=1}^{n-1} d_j w_j \beta_i + 2 \sum_{i,j=1}^{n-1} \kappa_{ij} w_i w_j.
\end{align*} \tag{36}

Note that the last equality comes from (35), and we denote by $\kappa_{ij}$ the Weingarten matrix of the boundary with respect to $\nu$.

Therefore, it follows that

\begin{align*}
0 \geq \Phi_n &= \frac{\phi'hu_n}{\beta_n} \sum_{i=1}^{n} u_i \beta_i - 2w_n \sum_{i,j=1}^{n-1} d_j w_j \beta_i + 2 \sum_{i,j=1}^{n-1} \kappa_{ij} w_i w_j + h'u_n + \tau \\
&= \frac{\phi'hu_n}{\beta_n} + h'u_n + \tau.
\end{align*} \tag{37}

We may assume in advance that

\begin{align*}
0 < h'(t) < 1, \quad \forall t \in [-M, M]. \tag{38}
\end{align*}

Thus, if we set $\tau$ large enough, depending on $C_0, |\beta|_{C^0(\partial \Omega)}, n$ and the geometry of $\partial \Omega$, we can conclude that this case does not occur at all.

**Case II:** $x_0 \in \Omega_\rho$.

All the calculations will proceed at this point, and the Einstein summation convention will be adopted during all the calculations if no otherwise specified. Also, we denoted by $F^i$ the derivative $\frac{\partial g(u_n)}{\partial u_i}$ and $F$ the sum $\sum_{i=1}^{n} F^i$.

According to [1], we know that

\begin{align*}
\sup_{\Omega} |Du| &\leq C_4 \left(1 + \sup_{\partial \Omega} |Du| \right), \tag{39}
\end{align*}

where $C_4$ is a positive constant depending only on $\Omega, n, k, |D_x f|_{C^0(\Omega \times [-M, M])}$. One can verify this point by setting a auxiliary function $\chi = \log |Du|^2 + d|x|^2$ and checking that $F^\chi_{x_0} \geq 0$ once we set $\alpha$ to be small and $|Du|$ to be large enough. Remark that we have supposed with out loss of generality that the point $0$ is located out of $\Omega$. 

Now we assume that the maximum value of $|Du|$ on $\partial \Omega$ is achieved at the point $x_1$, without loss of generality, we can suppose that

$$|Du|^2(x_1) \geq 4 \sup_{\partial \Omega} \left( \left| \frac{\phi}{\cos \theta} \right| \right)^2,$$

otherwise we have finish the estimate of the gradient of the solutions.

By the fact that $\Phi(x_0) \geq \Phi(x_1)$, it follows that

$$\phi(x_0) \geq C(\tau, \mu) e^{-2Mh} \phi(x_1) = C(\tau, \mu) e^{-2Mh} \left( |Dw|^2 - \left( \frac{\partial w}{\partial \nu} \right)^2 \right)(x_1)$$

$$\geq C(\tau, \mu) e^{-2Mh} |Dw|^2 \cos^2 \theta (x_1)$$

$$\geq c_0^2 C(\tau, \mu) e^{-2Mh} |Dw|^2 (x_1)$$

$$= c_0^2 C(\tau, \mu) e^{-2Mh} |Du - \frac{\phi}{\cos \theta} y|^2 (x_1)$$

$$\geq \frac{c_0^2 C(\tau, \mu) e^{-2Mh}}{4} |Du|^2 (x_1),$$

remark that the last inequality above comes from (40) and the fact that $(x - y)^2 \geq \frac{x^2}{2} - y^2$.

Joining with (39) and assuming once again that

$$0 < h'(t) < \frac{1}{2M}, \quad \forall t \in [-M, M],$$

we then derive

$$\phi(x_0) \geq \frac{c_0^2 C(\tau, \mu) e^{-2Mh}}{4 C_1} \left( \sup_{\Omega} |Du|^2 - C_1 \right)$$

$$\geq \frac{c_0^2 C(\tau, \mu) e^{-2Mh}}{8 C_1} \sup_{\Omega} |Du|^2$$

$$\geq \frac{c_0^2 C(\tau, \mu) e^{-2Mh}}{8 C_1} |Du|^2 (x_0)$$

$$\geq \frac{c_0^2 C(\tau, \mu) |Dw|^2 (x_0)}{9 C_1} \equiv C_0 |Dw|^2 (x_0).$$

Without the loss of generality, we can assume that $C_0 \in (0, 1)$.

At $x_0$, we also follow [25] to choose the coordinate such that $(u_0)$ is diagonal.

For $k = 1, 2, \ldots, n$, denote by $T_k = \sum_{i=1}^n C_i w_i$ and $\overrightarrow{T} = (T_1, T_2, \ldots, T_n)$, it is obvious to observe that $|\overrightarrow{T}| \leq |Dw|$ and

$$\phi = \sum_{i,j=1}^n C_i w_i w_j = \sum_{j=1}^n T_j w_j = \langle \overrightarrow{T}, Dw \rangle.$$  

(44)

Considering the lower bound we just derived in (43), we obtain

$$C_0 |Dw| \leq |T| \leq |Dw|.$$  

(45)

Without the loss of generality, we further assume by the Pigeon-Hole Principle that

$$T_j w_j \geq \frac{C_0}{n} |Dw|^2,$$  

(46)

and therefore,
\[ \frac{w_i}{T_i} \geq \frac{C_0}{n}, \quad (47) \]

and we can set \( \mu \) is small such that
\[ \frac{u_i}{T_i} \geq \frac{C_0}{3n}. \quad (48) \]

By a direct calculation, we have
\[ w_i = u_i \left( 1 - \frac{\varphi_i d}{\cos \theta} \right) + \varphi_i \left( \frac{d}{\cos \theta} \right) + \varphi_i \left( \frac{d}{\cos \theta} \right)_i; \]
\[ w_{ij} = u_{ij} \left( 1 - \frac{\varphi_i d}{\cos \theta} \right) - \frac{\varphi_{ij} d}{\cos \theta} u_{ij} - \varphi_{ij} \left( \frac{d}{\cos \theta} \right) - \varphi_{ij} \left( \frac{d}{\cos \theta} \right)_j + \varphi_{ij} \left( \frac{d}{\cos \theta} \right)_i \]
\[ + \frac{d}{\cos \theta} \varphi_{ij} + \varphi_{ij} \left( \frac{d}{\cos \theta} \right) + \left( \frac{d}{\cos \theta} \right) \varphi_j + \left( \frac{d}{\cos \theta} \right) \varphi_j + \left( \frac{d}{\cos \theta} \right) \varphi. \quad (49) \]

By the assumption that \( x_0 \) is the maximum point, we then have \( \Phi_i = 0 \) for \( i = 1, 2, \ldots, n \), and it follows that
\[ \frac{\phi_i}{\phi} + h'u_i + \tau d_i = 0, \quad (50) \]

especially for \( i = 1, \)
\[ \sum_{i=1}^{n} T_i w_i = -\frac{\phi}{2} (h'u_i + \tau d_i) - \sum_{k,l=1}^{n} C_{kl} w_{kl}, \quad (51) \]
then by (45)–(48), we have
\[ u_{ij} \left( 1 - \frac{\varphi_i d}{\cos \theta} \right) \leq -\frac{u_i}{2T_i} h' \phi + C |D w|^2 + C |D w|. \quad (52) \]

If we assume that \( h' \) has a positive lower bound and \( |D w| \) is large enough, and \( \mu \) is small enough, then we can obtain
\[ u_{i1} < 0, \quad (53) \]
and thus,
\[ F^{11} \geq F^{kk} \geq C(n, k) F. \quad (54) \]

Now, it is turn for us to deal with the second order derivatives of \( \Phi \). With the help of the first-order condition (50), it follows that
\[ \Phi_{ij} \left( \frac{\sum_{k,l=1}^{n} C_{kl} w_{kl}}{\phi} \right) - \left( h'u_i + \tau d_i \right) (h'u_j + \tau d_j) + h'u_{ij} + h''u_{ij} + \tau d_{ij} \]
\[ = \left( \frac{\sum_{k,l=1}^{n} C_{kl} w_{kl}}{\phi} \right) - h'u_i d_j - h'u_j d_i - \tau d_i d_j + h'u_{ij} + [h'' - (h')^2] u_{ij} + \tau d_{ij}. \quad (55) \]

Hence, we have at \( x_0 \) that
\[ 0 \geq F^i_\Phi \eta = \frac{\sum_{k,l=1}^n C^{kl} W_k W_l}{\phi} - 2h' \sum_{i,j=1}^n F^i_{uj} d_j - \tau^2 \sum_{i,j=1}^n F^i_{dijd_j} + h' k f + [h'' - (h')^2] \sum_{i,j=1}^n F^i_{uijd_j} + \tau \sum_{i,j=1}^n F^i_{dijd_j} \]
\[ \geq \sum_{i,j=1}^n \frac{F^i_{(C^{kl} W_k W_l)}\psi}{\phi} - (\tau^2 + 1) \sum_{i,j=1}^n F^i_{dijd_j} + h' k f + [h'' - 2(h')^2] \sum_{i,j=1}^n F^i_{uijd_j} + \tau \sum_{i,j=1}^n F^i_{dijd_j} \]
\[ = I + II + III + IV + V. \]

It is a simple and direct calculation to deal with the last four terms. According to (45)–(48) and (54), we have

\[ II = -(\tau^2 + 1) F^i_{dijd_j} \geq - (\tau^2 + 1) F, \]

\[ III = h' k f \geq 0, \]

\[ IV = [h'' - 2(h')^2] \sum_{i,j=1}^n F^i_{uijd_j} \geq [h'' - 2(h')^2] F^i_{uijd_j} \geq C [h'' - 2(h')^2] ||Dw||^2 F, \]

\[ V = \tau F^i_{dijd_j} \geq -k_0 \tau \sum_{i=1}^n F^i = -k_0 F, \]

where \( k_0 \) is a positive constant related to the geometry of \( \partial \Omega \).

To deal with the term \( I \), we have

\[ I = \sum_{i,j,k,l=1}^n F^i_{C^{kl} W_k W_l} \phi + 2 \sum_{i,j,k,l=1}^n F^i_{C^{kl} W_k W_l} \phi + 4 \sum_{i,j,k,l=1}^n F^i_{C^{kl} W_k W_l} \phi + 2 \sum_{i,j,k,l=1}^n F^i_{C^{kl} W_k W_l} \phi \]
\[ = I_1 + I_2 + I_3 + I_4. \]

We consider these four terms one by one in the following text.

For the term \( I_1 \), it is easy to deduce that

\[ I_1 = \sum_{i,j,k,l=1}^n F^i_{C^{kl} W_k W_l} \phi \geq -CF. \]

For the term \( I_2 \), we need a subtle operation as follows:

\[ \phi I_2 = 2 \sum_{i,j,k,l=1}^n F^i_{C^{kl} W_k W_l} \]
\[ = 2 \sum_{i,j,k,l=1}^n F^i_{T_{ijl}} \left[ u - \frac{\varphi(x, u)}{\cos \theta} \right]_{ijl} \]
\[ = 2 \sum_{i,j,k,l=1}^n T_{ijl} F^i_{ijl} + F^i_{ijl} \left[ \varphi(x, u) \frac{d}{\cos \theta} \right]_{ijl} \]
\[ = 2 \sum_{l=1}^n T_l \left[ F^i_{ijl} + \sum_{i,j=1}^n F^i_{ijl} \left[ \varphi(x, u) \frac{d}{\cos \theta} \right]_{ijl} \right]. \]

To proceed, we should compute \( \left( \varphi(x, u) \frac{d}{\cos \theta} \right)_{ijl} \). By a direct calculation,

\[ \left( \varphi(x, u) \frac{d}{\cos \theta} \right)_{ijl} = (\varphi)_{ijl} \left( \frac{d}{\cos \theta} \right)_{ijl} + (\varphi)_{ijl} \left( \frac{d}{\cos \theta} \right)_{ijl} + (\varphi)_{ijl} \left( \frac{d}{\cos \theta} \right)_{ijl} + (\varphi)_{ijl} \left( \frac{d}{\cos \theta} \right)_{ijl}, \]

where
\((\varphi)_i = \varphi_i + \varphi_{,i} u_i,\)
\((\varphi)_{ij} = (\varphi)_{i,j} + \varphi_{,i} u_j + \varphi_{,j} u_i + \varphi_{,ij} u_i u_j, \)
\((\varphi)_{iji} = (\varphi)_{ij,i} + \varphi_{,i} u_j + \varphi_{,j} u_i + \varphi_{,ij} u_{i,j} + \varphi_{,iji} u_i u_j u_i u_j.\) (62)

Note that
\[
\sum_{i,j=1}^n F_{ij} u_{ij} = k f, \quad \sum_{i,j=1}^n F_{ij} u_{ij} = D_i f, \quad \sum_{j=1}^n F_{ij} u_{ij} = F_{ii} u_{ii} \quad \text{(fixed } i), \quad 0 < \sum_{j=1}^n F_{ij} u_{ij} \leq |Du|^2 F,
\]
and therefore, we have
\[
\phi I_2 \geq - C d |Dw|^4 F - C |Dw|^3 F - C d |Dw|^2 \sum_{i=1}^n |F_{ii} u_{ii}| - C |Dw| \sum_{i=1}^n |F_{ii} u_{ii}| - C |Dw|^2.
\] (63)

Almost the same procedure, we can settle the remained two terms.
\[
\phi I_3 = 4 \sum_{i,j,p,l=1}^n F_{ij} C_{ij} w_{ij} w_{ij} \geq 2 \sum_{i,j=1}^n w_{ij} F_{ii} u_{ij} - C |Dw|^2 F \geq - C |Dw|^3 F - C |Dw| \sum_{i=1}^n |F_{ii} u_{ii}|, (64)
\]
and
\[
\phi I_4 = 2 \sum_{i,j,p,l=1}^n F_{ij} C_{ij} w_{ij} w_{ij} \geq 2 \sum_{i,j=1}^n F_{ii} C_{ii} u_{ii}^2 - C d |Dw|^4 F - C |Dw|^3 F - C |Dw|^2.
\] (65)

Taking into account (59), (63), (64), and (65), we can obtain
\[
\phi I \geq 2 \sum_{i=1}^n F_{ii} C_{ii} u_{ii}^2 - C d |Dw|^2 \sum_{i=1}^n |F_{ii} u_{ii}| - C |Dw| \sum_{i=1}^n |F_{ii} u_{ii}| - C d |Dw|^4 F - C |Dw|^3 F - C |Dw|^2.
\] (66)

Denoting by
\[
H = 2 \sum_{i=1}^n F_{ii} C_{ii} u_{ii}^2 - C d |Dw|^2 \sum_{i=1}^n |F_{ii} u_{ii}| - C |Dw| \sum_{i=1}^n |F_{ii} u_{ii}|,
\] (67)
and we will bound \(H\) from below in the following.

Let \(C_{\text{bic}}\) be the smallest of \(\{C_{ii}\}_{i=1}^n\), without the loss of generality, we can assume \(i_0 = 1\). Then, we have \(C_{ii} \geq \frac{1}{2}\) for any \(i \geq 2\), otherwise it follows that \(\sum_{i=1}^n C_{ii} < 1 + (n - 2) = n - 1\), which contradicts with \(\sum_{i=1}^n C_{ii} = n - 1\). Then by the equation, we can obtain
\[
F_{ii} u_{ii} = k f - \sum_{a=2}^n F_{ai} u_{ai},
\] (68)
and therefore, we have by the simple fact \(ax^2 + bx \geq - \frac{b^2}{4a}\) if \(a > 0\) that
\[
H \geq \sum_{i=2}^n F_{ii} u_{ii}^2 - C (d |Dw|^2 + |Dw|) \sum_{i=2}^n |F_{ii} u_{ii}| - k f \geq - C \sum_{i=2}^n F_{ii} (d |Dw|^2 + |Dw|^2) - C.
\] (69)

Plugging this into (66) and joining with (43), we can derive
\[
I \geq - C d |Dw|^2 F - C |Dw| F.
\] (70)

Therefore, combining (57) and (70), we can obtain
\[
0 \geq \frac{F_{ii} \Phi_{ij}}{F} \geq C d h'^2 - 2 (h'^2)|Dw|^2 - C d |Dw|^2 - C |Dw|,
\] (71)
where we use once again the fact \(F \geq C > 0\).
Now, we set
\[ h(t) = \frac{1}{4} \ln \frac{1}{(3M - t)} , \]  
and it satisfies all the assumptions we have made in advance. Let \( \mu \) be small enough so that \( C R \leq C R(h)^2 \), we then obtain
\[ C R \left( \frac{1}{16M} \right)^2 |Dw|^2 - C R |Dw| \leq 0 , \]  
and this will lead to the universal bound of \( |Dw| \) at \( x_0 \) and we then obtain the global gradient estimate of \( u \) on \( \Omega \) by a standard discussion, and this finishes the whole proof of Theorem 4.1. \( \square \)

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