STARLIKE FUNCTIONS ASSOCIATED WITH
A PETAL SHAPED DOMAIN

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Abstract. In this paper, we establish some radius results and inclusion relations for starlike functions associated with a petal-shaped domain.

1. Introduction

Let the open unit disk \( \{ z \in \mathbb{C} : |z| < 1 \} \) be represented by \( \mathbb{D} \) and \( \mathcal{H} \) be the class of all analytic functions in \( \mathbb{D} \). Consider \( \mathcal{A}_n \) as the class of analytic functions \( f \) in \( \mathbb{D} \) represented by

\[
f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \cdots.
\]

In particular, denote \( \mathcal{A}_1 := \mathcal{A} \) and let \( \mathcal{S} \) be the subclass of \( \mathcal{A} \) such that it involves all univalent functions \( f(z) \) in \( \mathbb{D} \). Let \( g, h \) be two analytic functions and \( \omega \) be a Schwarz function satisfying \( \omega(0) = 0 \) and \( |\omega(z)| \leq |z| \) such that \( g(z) = h(\omega(z)) \) then \( g \) is said to be subordinate to \( h \), or \( g \prec h \). If \( h \) is univalent, then \( g \prec h \) if and only if \( g(0) = h(0) \) and \( g(\mathbb{D}) \subset h(\mathbb{D}) \). Ma and Minda [11] introduced the univalent function \( \psi \) satisfying \( \operatorname{Re} \psi(\mathbb{D}) > 0 \), \( \psi(\mathbb{D}) \) starlike with respect to \( \psi(0) = 1 \) and \( \psi'(0) > 0 \) and the domain \( \psi(\mathbb{D}) \) being symmetric about the real axis. Further, they gave the definitions for the general subclasses of starlike and convex functions, respectively, as follows:

\[
\mathcal{S}^*(\psi) := \{ f \in \mathcal{S} : zf'(z)/f(z) \prec \psi(z) \}
\]

and

\[
\mathcal{K}(\psi) := \{ f \in \mathcal{S} : 1 + zf''(z)/f'(z) \prec \psi(z) \}.
\]

For different choices of \( \psi \), many subclasses of \( \mathcal{S}^* \) and \( \mathcal{K} \) can be obtained. For example, the notable classes of Janowski starlike and convex functions [8] are represented by \( \mathcal{S}^*[C, D] := \mathcal{S}^*((1 + Cz)/(1 + Dz)) \) and \( \mathcal{K}[C, D] := \mathcal{K}((1 + Cz)/(1 + Dz)) \) for \( -1 \leq D < C \leq 1 \), respectively. Further, \( \mathcal{S}^*_\alpha := \mathcal{S}^*[1 - 2\alpha, -1] \) and \( \mathcal{K}_\alpha := \mathcal{K}*[1 - 2\alpha, -1] \) represent the classes of starlike and convex functions.

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of order $\alpha \in [0, 1)$, respectively. Note that $S^* := S^*_0$ and $K := K_0$ represent the well-known classes of starlike and convex functions, respectively. We denote $SS^*(\gamma) := S^*((1+z)/(1-z)^\gamma)$ and $SK(\gamma) := K((1+z)/(1-z)^\gamma)$ representing the class of strongly starlike and strongly convex functions of order $\gamma \in (0, 1]$, respectively.

Recall that for two subfamilies $G_1$ and $G_2$ of $A$, we say that $r_0$ is the $G_1$-radius for the class $G_2$ if $r_0 (0 < r \leq r_0)$ is the greatest number which satisfies $r^{-1}g(rz) \in G_1$, where $g \in G_2$. Moreover, starlike classes $S^*(\psi)$ for different $\psi(z)$ were considered by many authors, whose works examined the geometrical properties, radius results and coefficient estimates of the functions of their respective classes. Sokół and Stankiewicz [20, 21] considered the class $S^*_2 := S^*(\sqrt{1+z})$ and Mendiratta et al. [13] worked on the class $S^*_D := S^*((\sqrt{2} - (\sqrt{2} - 1)((1 - z)/(1 + 2(\sqrt{2} - 1)z)))^{1/2})$. Sharma et al. [19] studied the class $S^*_C := S^*(1 + 4z/3 + 2z^2/3)$ while the class $S^*_G := S^*(1 + \sin z)$ was examined by Cho et al. [6]. The classes $S^*_0 := S^*(e^z)$ and $\Delta^* := S^*(z + \sqrt{1+z^2})$ were considered by Mendiratta et al. [14] and Raina et al. [15], respectively. Kargar et al. [10] introduced and studied the class $BS^*(\alpha) := S^*(1 + z/(1 - \alpha z^2))$, $\alpha \in [0, 1)$, associated with the Booth lemniscate which was also investigated by Cho et al. [4]. Some more recent work on radius problems can be found in [1,3,5,7,23].

Motivated by the classes defined in [6, 10, 13–15, 19, 21], we consider the petal shaped region $\Omega_\rho := \{w \in \mathbb{C} : |\sinh(w - 1)| < 1\}$, which is characterised functionally as $\rho(z) = 1 + \sinh^{-1}(z)$ to define our class. Clearly, $\rho(z)$ is a Ma-Minda function. See Figure 2 for its boundary curve $\gamma_0$ which is petal shaped. Note that $\sinh^{-1}(z)$ is a multivalued function and has the branch cuts along the line segments $(-i\infty, -i) \cup (i, i\infty)$, on the imaginary axis and hence it is analytic in $D$. Now we introduce a new class of starlike functions

$$S^*_\rho := \left\{ f \in A : \frac{zf'(z)}{f(z)} \prec 1 + \sinh^{-1}(z) \right\} \quad (z \in D), \tag{4}$$

which is associated with the petal-shaped domain $\rho(D)$. From the above definition, we deduce that $f \in S^*_\rho$ if and only if there exists an analytic function $q(z) \prec \rho(z)$ such that

$$f(z) = z \exp \left( \int_0^z \frac{q(t) - 1}{t} dt \right). \tag{5}$$

Table 1 presents some functions in the class $S^*_\rho$, where $q_j \prec \rho$.

Since $\rho$ is univalent in $D$, $q_j(D) \subset \rho(D)$ and $q_j(0) = \rho(0)$ $(j = 1, 2, 3)$, it follows that each $q_j \prec \rho$. Thus the functions $f_j(z)$ obtained from (5) are in the class $S^*_\rho$. In particular, if we choose

$$q(z) = 1 + \sinh^{-1}(z) = 1 + z - \frac{z^3}{6} + \frac{3z^5}{40} - \frac{5z^7}{112} + \cdots,$$
Table 1. Some functions in the class $S_\rho^*$

| j | $q_j(z)$ | $f_j(z)$ |
|---|---|---|
| 1 | $1 + z/5$ | $z \exp(z/5)$ |
| 2 | $(5 + 2z)/(5 + z)$ | $z + z^2/5$ |
| 3 | $(7 + 4z)/(7 + z)$ | $z(1 + z/7)^3$ |

then (5) gives

\[ f_0(z) = z \exp \left( \int_0^z \frac{\sinh^{-1}(t)}{t} \, dt \right) = z + z^2 + \frac{z^3}{2} + \frac{z^4}{9} - \frac{z^5}{72} - \frac{z^6}{225} + \cdots, \]

which often acts as the extremal function for the class $S_\rho^*$ yielding sharp results.

**Remark 1.1.** Note that $\sinh^{-1}(z) = \ln(z + \sqrt{1 + z^2})$. Let $w = zf'(z)/f(z)$, then the class $S_\rho^*$ can be alternatively represented by $\exp(w - 1) \preceq z + \sqrt{1 + z^2}$, where $z + \sqrt{1 + z^2}$ represents the Crescent shaped domain [15]. Thus, there exists an exponential relation among the functions in the classes $S_\rho^*$ and $\Delta^\psi$.

In the present investigation, the geometrical properties of the function $1 + \sinh^{-1}(z)$ are studied and certain inclusion properties as well as radius problems are established for the class $S_\rho^*$.

**2. Properties of the function $1 + \sinh^{-1}(z)$**

The current section deals with the study of some geometric properties of the function $1 + \sinh^{-1}(z)$.

**Theorem 2.1.** The function $\rho(z) = 1 + \sinh^{-1}(z)$ is a convex univalent function.

**Proof.** Let $h(z) = \sinh^{-1}(z)$. Clearly, $h(0) = 0$. Since $h'(z) = 1/\sqrt{1 + z^2}$ and $\sqrt{1 + z^2} \preceq \sqrt{1 + z} \in \mathcal{P}$, where $\mathcal{P}$ is the Carathéodory class. Therefore, $1/\sqrt{1 + z^2} \in \mathcal{P}$ which implies that $\Re h'(z) > 0$. Hence $\rho$ is univalent. Now a calculation yields

\[ 1 + z h''(z) \frac{1}{h'(z)} = \frac{1}{1 + z^2}. \]

Since

\[ \frac{1}{1 + z^2} \preceq \frac{1}{1 + z} \in \mathcal{P}, \]

Therefore, $\Re(1 + z h''(z)/h'(z)) > 0$ which implies that $h$ (and thus $\rho$) is a convex univalent function. \hfill \Box

**Remark 2.2.** Note that $\rho'(0) > 0$ and the function $\varphi(z) = z + \sqrt{1 + z^2}$ satisfies $\varphi(z) = \overline{\varphi(z)}$. Therefore, $\rho(z) = \overline{\rho(z)}$ and hence, the domain $\Omega_\rho = \rho(\mathbb{D})$ is symmetric about the real axis.
Theorem 2.3. The domain $\Omega_\rho$ is symmetric about the line $\text{Re}(w) = 1$.

Proof. Since $\Omega_\rho$ is symmetric about the real axis, the condition $0 \leq \theta \leq \pi/2$ is sufficient to prove our result. As we know that symmetry along imaginary axis for $f \in A$ holds if $\text{Re}(f(\theta)) = -\text{Re}(f(\pi - \theta))$ and $\text{Im}(f(\theta)) = \text{Im}(f(\pi - \theta))$. Now let $h(z) = \sinh^{-1}(z) = \ln(z + \sqrt{1 + z^2})$. Then $\text{Im}(h(z)) = \arg(z + \sqrt{1 + z^2})$. For $z = re^{it}$, $t \in [0, \pi]$ and fixed $r \in (0, 1)$, we have the following expressions for $t \to \theta$

$$I_1 = \arg \left( r(\cos \theta + i \sin \theta) + \sqrt{1 + r^2(\cos(2\theta) + i \sin(2\theta))} \right)$$
$$= \arg \left( z + \sqrt{1 + z^2} \right),$$

and for $t \to \pi - \theta$

$$I_2 = \arg \left( r(\cos(\pi - \theta) + i \sin(\pi - \theta)) \right.$$
$$+ \sqrt{1 + r^2(\cos(2(\pi - \theta)) + i \sin(2(\pi - \theta)))} \bigg)$$
$$= \arg \left( r(-\cos \theta + i \sin \theta) + \sqrt{1 + r^2(\cos(2\theta) - i \sin(2\theta))} \right)$$
$$= \arg \left( -z + \sqrt{1 + z^2} \right).$$

Now let us consider $(z + \sqrt{1 + z^2})/(-\overline{z} + \sqrt{1 + \overline{z}^2})$. On rationalising the denominator, we get

$$\frac{z + \sqrt{1 + z^2}}{-\overline{z} + \sqrt{1 + \overline{z}^2}} = \frac{(z + \sqrt{1 + z^2})(-z + \sqrt{1 + z^2})}{(-\overline{z} + \sqrt{1 + \overline{z}^2})(-z + \sqrt{1 + z^2})} = \frac{1}{|z + \sqrt{1 + z^2}|^2} = k,$$

where $k > 0$ is some real positive constant. Thus,

$$\arg \left( \frac{z + \sqrt{1 + z^2}}{-\overline{z} + \sqrt{1 + \overline{z}^2}} \right) = \arg(k) = 0$$
$$\Rightarrow \arg \left( z + \sqrt{1 + z^2} \right) = \arg \left( -\overline{z} + \sqrt{1 + \overline{z}^2} \right)$$
$$\Rightarrow I_1 = I_2.$$

Similarly, $\text{Re}(h(\theta)) = -\text{Re}(h(\pi - \theta))$ for $0 \leq \theta \leq \pi/2$. Hence, $h(z)$ is symmetric about the imaginary axis and thus, by translation property, $\rho(z)$ is symmetric about the line $\text{Re}(w) = 1$.\hfill \Box

Now using Theorem 2.3, we obtain the next result:

Corollary 2.4. The disk $\{w : |w - 1| \leq \sinh^{-1}(r)\}$ is contained in $\rho(|z| \leq r)$ and is maximal.

Proof. Since $\min_{|z|=r} |\sinh^{-1}(z)| = |\sinh^{-1}(-r)| = \sinh^{-1}(r)$ and hence the conclusion can be drawn at once.\hfill \Box

Theorem 2.5. We find that the following properties hold for $\rho(z) = 1 + \sinh^{-1}(z)$:
Figure 1. \( \rho(D) \) lies in the annular region bounded between the circles \( C_1 \) and \( C_2 \).

(i) \( \rho(-r) \leq \text{Re} \rho(z) \leq \rho(r) \) (\( |z| \leq r < 1 \));

(ii) \( |\text{Im} \rho(z)| \leq \pi/2 \) (\( |z| \leq 1 \));

(iii) \( \rho(-r) \leq |\rho(z)| \leq \rho(r) \) (\( |z| \leq r < 1 \));

(iv) \( |\arg \rho(z)| \leq \tan^{-1}(1/t) \), where \( t = \frac{4}{\pi} \sqrt{\sinh^{-1}(1)(1 - \sinh^{-1}(1))} \).

Proof. (i) Since \( \rho(z) \) is convex and typically real, the value of \( \text{Re} \rho(z) \) falls between \( \lim_{\theta \to 0} \rho(re^{i\theta}) \) and \( \lim_{\theta \to \pi} \rho(re^{i\theta}) \), thus the result follows.

(ii) Using Theorem 2.3, it suffices to take \( \theta \in [0, \pi/2] \). Then the inequality follows by letting \( r \) tending to \( 1^- \) and observing that the function

\[
\text{Im} \rho(z) = \text{arg} \left( r \cos(\theta) + \sqrt{1 + r^2(\cos(2\theta) + i \sin(2\theta))} + ir \sin(\theta) \right)
\]

is strictly increasing in the interval \( [0, \pi/2] \) and hence the result follows at once.

(iii) The radially farthest and nearest points in \( \rho(D) \) from origin are respectively \( B \) and \( A \) (see Figure 1) and therefore the result obviously holds. Moreover we observe that these points \( A \) and \( B \) lie on the real line and hence the bounds of \( |\rho(z)| \) and \( \text{Re} \rho(z) \) coincide.

The proof of (iv) is evident from Theorem 3.1(iii) so skipped here. \( \square \)

Next we have the following important result:

Lemma 2.6. For \( 1 - \sinh^{-1}(1) < a < 1 + \sinh^{-1}(1) \), let \( r_a \) be given by

\[
r_a = \begin{cases} 
  a - (1 - \sinh^{-1}(1)), & 1 - \sinh^{-1}(1) < a \leq 1; \\
  1 + \sinh^{-1}(1) - a, & 1 \leq a < 1 + \sinh^{-1}(1).
\end{cases}
\]

Then \( \{w : |w - a| < r_a\} \subset \Omega_\rho \).

We omit the proof of Lemma 2.6 as it directly follows from Theorem 2.3 and Corollary 2.4.
Remark 2.7. Evidently the domain $\Omega_\rho$ is contained inside the disk $\{w : |w - 1| < \pi/2\}$.

3. Inclusion relations

This section establishes some inclusion results involving the class $S^*_\rho$ with some well-known classes.

We consider the class $M(\beta)$, first studied by Uralegaddi et al. [22], given by

$$M(\beta) := \left\{ f \in A : \text{Re} \left( \frac{zf'(z)}{f(z)} \right) < \beta, \quad z \in \mathbb{D}, \quad \beta > 1 \right\},$$

and another interesting class introduced by Kanas and Wiśniowska [9] of $k$-starlike functions, denoted by $k^{-ST}$ and defined by

$$k^{-ST} := \left\{ f \in A : \text{Re} \left( \frac{zf'(z)}{f(z)} \right) > k \left| \frac{zf'(z)}{f(z)} - 1 \right|, \quad z \in \mathbb{D}, \quad k \geq 0 \right\}.$$

Note that $S^* = 0^{-ST}$ and $S^*_p = 1^{-ST}$, where $S^*_p$ is the class of parabolic starlike functions [17].

We establish the following inclusion relations for the class $S^*_\rho$.

**Theorem 3.1.** The class $S^*_\rho$ satisfies the following relationships:

(i) $S^*_\rho \subset S^*_0 \subset S^*_{\alpha}$ for $0 \leq \alpha \leq 1 - \sinh^{-1}(1)$;

(ii) $S^*_\rho \subset M(\beta)$ for $\beta \geq 1 + \sinh^{-1}(1)$;

(iii) $S^*_\rho \subset \Sigma S^*(\gamma)$ for $(2/\pi) \tan^{-1}(1/t) \leq \gamma \leq 1$,

where $t = \frac{4}{\pi \sqrt{\sinh^{-1}(1)(1 - \sinh^{-1}(1))}}$;

(iv) $k^{-ST} \subset S^*_\rho$ for $k \geq 1 + 1/\sinh^{-1}(1)$.

**Proof.** Consider $f \in S^*_\rho$ which implies $zf'(z)/f(z) \prec 1 + \sinh^{-1}(z)$. By Theorem 2.5, it is evident that for $z \in \mathbb{D},$

$$1 - \sinh^{-1}(1) = \min_{|z|=1} \text{Re}(1 + \sinh^{-1}(z)) \leq \text{Re} \left( \frac{zf'(z)}{f(z)} \right)$$

and

$$\text{Re} \left( \frac{zf'(z)}{f(z)} \right) \leq \max_{|z|=1} \text{Re}(1 + \sinh^{-1}(z)) = 1 + \sinh^{-1}(1).$$

This proves (i) and (ii).

For (iii), let $w \in \mathbb{C}$, $X = \text{Re}(w)$, $Y = \text{Im}(w)$, and $b = 1 - \sinh^{-1}(1)$. Now consider the parabolic domain $\Gamma_p$ with the boundary curve $\partial \Gamma_p = \gamma_p : Y^2 = 4a(X - b)$. Then the focus $a$ of the smallest parabola $\gamma_p$ which contains $\Omega_\rho$ will touch the peak points $1 \pm i\pi/2$ of $S^*_\rho$ is $\pi^2/(16 \sinh^{-1}(1))$. Let $P$ be any point on the parabola $\gamma_p$ with parametric coordinates $(b + at^2, 2at)$ such that the tangent $OE$ at $P$ passes through origin for some parameter $t$. Let the equation
of the tangent OE be $y = mx$, where $m = dy/dx = (dy/dt)/(dx/dt) = 1/t$. Therefore at $P$, we have

$$m = \frac{y}{x} \Rightarrow \frac{1}{t} = \frac{2at}{b + at^2},$$

which yields

$$t = \sqrt{\frac{b}{a}} = \frac{4}{\pi} \sqrt{\sin^{-1}(1)(1 - \sin^{-1}(1))}$$

(7)

and the argument of the tangent at $P$ of $\gamma_p$ is $\tan^{-1}(1/t)$. Since $\Omega_p \subset \Gamma_p$, it gives

$$\left| \arg \frac{zf'(z)}{f(z)} \right| \leq \max_{|z|=1} \arg(\rho(z)) = \max_{|z|=1} \arg(\gamma_p) = \tan^{-1}(1/t),$$

which demonstrates $f \in SS^* ((2/\pi) \tan^{-1}(1/t))$, where $t$ is given by (7).

To show (iv), consider $f \in k - ST$ along with the conic domain $\Gamma_k = \{ w \in \mathbb{C} : \Re w > k|w - 1| \}$). For $k > 1$, let $\partial \Gamma_k$ represent the horizontal ellipse

$$\gamma_k : x^2 = k^2(x - 1)^2 + k^2y^2$$

which may be rewritten as

$$\frac{(x - x_0)^2}{a^2} + \frac{(y - y_0)^2}{b^2} = 1,$$

where $x_0 = k^2/(k^2 - 1)$, $y_0 = 0$, $a = k/(k^2 - 1)$ and $b = 1/\sqrt{k^2 - 1}$. For $\gamma_k \subset \Omega_p$, the condition $x_0 + a \leq 1 + \sinh^{-1}(1)$ must hold, or equivalently $k \geq 1 + 1/\sinh^{-1}(1)$. Since $\Gamma_{k_1} \subset \Gamma_{k_2}$ for $k_1 \geq k_2$, it follows that for $k \geq 1 + 1/\sinh^{-1}(1)$, $k - ST \subset S_p$. Figure 2 clearly depicts these relations. □

For our next result, we consider $\mathcal{P}_n[C, D]$, the class of functions $p(z)$ of the form $1 + \sum_{k=1}^{\infty} c_k z^k$, satisfying $p(z) < (1 + Cz)/(1 + Dz)$, where $-1 \leq D < C \leq 1$. Denote by $\mathcal{P}_n(\alpha) := \mathcal{P}_n[1 - 2\alpha, -1]$ and $\mathcal{P}_n := \mathcal{P}_n(0)$. For $n = 1$, $\mathcal{P} = \mathcal{P}_1$ is the Carathéodory class. We need the following lemmas:

**Lemma 3.2** ([18]). For $p \in \mathcal{P}_n(\alpha)$, we have

$$\left| \frac{zp'(z)}{p(z)} \right| \leq \frac{2(1 - \alpha)n\rho^n}{(1 - r^n)(1 + (1 - 2\alpha)n\rho^n)}, \quad (|z| = r).$$

**Lemma 3.3** ([16]). For $p \in \mathcal{P}_n[C, D]$, we have

$$\left| p(z) - \frac{1 - CD\rho^{2n}}{1 - D^2 \rho^{2n}} \right| \leq \frac{(C - D)\rho^n}{1 - D^2 \rho^{2n}}, \quad (|z| = r).$$

Especially, for $p \in \mathcal{P}_n(\alpha)$, we have

$$\left| p(z) - \frac{1 + (1 - 2\alpha)\rho^{2n}}{1 - \rho^{2n}} \right| \leq \frac{2(1 - \alpha)\rho^n}{1 - \rho^{2n}}, \quad (|z| = r).$$

**Theorem 3.4.** Let $-1 < D < C \leq 1$. If either of the following two conditions holds:

(i) $(1 - \sinh^{-1}(1))(1 - D^2) < 1 - CD \leq 1 - D^2$ and $C - D \leq (1 - D) \sinh^{-1}(1)$;

(ii) $1 - D^2 \leq 1 - CD < (1 + \sinh^{-1}(1))(1 - D^2)$ and $C - D \leq (1 + D) \sinh^{-1}(1)$. 

Let $a + D$.

In this section, radius results for various subclasses of $A$ are established. We begin by determining sharp $S^*$ radii ($k \geq 0$) for the class $S^*_p$. Using Theorem 3.1, we can establish that $R_{S^*_p} = R_M(\beta)(S^*_p) = 1$ for $0 \leq \alpha < 1 - \sinh^{-1}(1)$ and $\beta > 1 + \sinh^{-1}(1)$.

**Theorem 4.1.** If $f \in S^*_p$, then the following results hold:

Then $S^*[C, D] \subset S^*_p$.

**Proof.** Let $f \in S^*[C, D]$ which implies $zf'(z)/f(z) \in P[C, D]$. Using Lemma 3.3 we have

$$
\frac{|zf'(z)|}{f(z)} = \frac{1 - CD}{1 - D^2} \leq \frac{(C - D)}{(1 - D^2)}.
$$

Let $a = (1 - CD)/(1 - D^2)$ and assume that (i) holds. Now multiplying $1 + D$ and dividing by $(1 - D^2)$ on either side of the inequality $(C - D) \leq (1 - D)\sinh^{-1}(1)$ gives $(C - D)/(1 - D^2) \leq a - (1 - \sinh^{-1}(1))$ on simplification. Also, the inequality $(1 - \sinh^{-1}(1))/(1 - D^2) < 1 - CD \leq 1 - D^2$ is equivalent to $1 - \sinh^{-1}(1) < (1 - CD)/(1 - D^2) \leq 1$. Therefore, from (8) we find $w = zf'(z)/f(z)$ is contained inside the disk $|w - a| < r_a$, where $r_a = a - (1 - \sinh^{-1}(1))$ and $1 - \sinh^{-1}(1) < a \leq 1$. Hence $f \in S^*_p$ by Lemma 2.6. A similar proof can be shown when (ii) holds.

4. Radius problems

In this section, radius results for various subclasses of $A$ are established. We begin by determining sharp $S^*_p$ ($0 \leq \alpha < 1$), $M(\beta)$ ($\beta > 1$) and $k - ST$-radii ($k \geq 0$) for the class $S^*_p$. Using Theorem 3.1, we can establish that $R_{S^*_p} = R_M(\beta)(S^*_p) = 1$ for $0 \leq \alpha < 1 - \sinh^{-1}(1)$ and $\beta > 1 + \sinh^{-1}(1)$.

**Theorem 4.1.** If $f \in S^*_p$, then the following results hold:

Figure 2. Boundary curves, depicting some inclusion relations for $w = 1 + \sinh^{-1}(z)$.
(i) For $1 - \sinh^{-1}(1) \leq \alpha < 1$, we have $f \in S^*_\alpha$ in $|z| \leq \sinh(1 - \alpha)$.
(ii) For $1 < \beta \leq 1 + \sinh^{-1}(1)$, we have $f \in M(\beta)$ in $|z| \leq \sinh(\beta - 1)$.
(iii) For $k > 0$, we have $f \in k - ST$ in $|z| \leq \sinh(1/(k + 1))$.

The results are sharp.

Proof. Since $f \in S^*_\rho$, $zf'(z)/f(z) < 1 + \sinh^{-1}(z)$ and hence for $|z| = r < 1$ Theorem 2.5 gives

$$1 - \sinh^{-1}(r) \leq \Re \frac{zf'(z)}{f(z)} \leq 1 + \sinh^{-1}(r),$$

thereby validating the first two parts. Also, the constants $\sinh(1 - \alpha)$ and $\sinh(\beta - 1)$ are optimal for the function $f_0$ given by (6). Now to prove (iii), note that $f \in k - ST$ in $|z| < r$, if

$$\Re(1 + \sinh^{-1}(w(z))) \geq k|1 + \sinh^{-1}(w(z)) - 1| = k|\sinh^{-1}(w(z))|.$$ 

Here $w$ denotes the Schwarz function. Since $\Re(1 + \sinh^{-1}(w(z))) \geq 1 - \sinh^{-1}(r)$ and $|\sinh^{-1}(w(z))| \leq \sinh^{-1}(r)$, the inequality $\Re(1 + \sinh^{-1}(w(z))) \geq k|\sinh^{-1}(w(z))|$ holds whenever $1 - \sinh^{-1}(r) \geq k\sinh^{-1}(r)$, which implies $r \leq \sinh(1/(1 + k))$. For the function $f_0$ given by (6) and for $z_0 = -\sinh(1/(1 + k))$, we have

$$\Re \frac{z_0 f_0(z_0)}{f_0(z_0)} = \Re(1 + \sinh^{-1}(z_0)) = \frac{k}{k + 1} = k|\sinh^{-1}(z_0)| = k\left|z_0 f_0(z_0) - 1\right|. $$

This concludes the proof. \qed

Corollary 4.2. Substituting $k = 1$ in part (iii) above, we find that $f \in S^*_\rho$ is parabolic starlike [17] in $|z| \leq \sinh(1/2)$.

In the next result, we find the $K_\alpha$-radius for the class $S^*_\rho$.

Theorem 4.3. Let $f \in S^*_\rho$. Then $f \in K_\alpha$ in $|z| < r_\alpha$, where $r_\alpha$ is the least positive root of

$$9(1-r^2)\sqrt{1+r^2} \left(1 - \sinh^{-1}(r)\right) \left(1 - \alpha - \sinh^{-1}(r)\right) - r = 0 \quad (0 \leq \alpha < 1).$$

Proof. Let $f \in S^*_\rho$ and $w$ be a Schwarz function. Then $zf'(z)/f(z) = 1 + \sinh^{-1}(w(z))$ such that

$$1 + \frac{zf''(z)}{f'(z)} = 1 + \sinh^{-1}(w(z)) + \frac{zw'(z)}{(1 + \sinh^{-1}(w(z)))\sqrt{1 + w^2(z)}}$$

which yields

$$\Re \left(1 + \frac{zf''(z)}{f'(z)}\right) \geq \Re \left(1 + \sinh^{-1}(w(z))\right) - \left|\frac{zw'(z)}{(1 + \sinh^{-1}(w(z)))\sqrt{1 + w^2(z)}}\right|. $$
We know for the Schwarz function $w$, the inequality $|w'(z)| \leq (1 - |w(z)|^2)/(1 - |z|^2)$ holds. Thus we observe that

$$\Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) \geq 1 - \sinh^{-1}(|z|) - \frac{|z|(1 - |w(z)|^2)}{(1 - \sinh^{-1}(|z|))(1 - |z|^2)\sqrt{1 + |z|^2}}$$

$$\geq 1 - \sinh^{-1}(|z|) - \frac{|z|}{(1 - \sinh^{-1}(|z|))(1 - |z|^2)\sqrt{1 + |z|^2}}.$$  

Let $q(r) := 1 - \sinh^{-1}(r) - r/(1 - \sinh^{-1}(r))(1 - r^2)\sqrt{1 + r^2}$. We find $q(r)$ is a decreasing function in $[0, 1]$ with $q(0) = 1$. Therefore $\Re(1 + zf''(z)/f'(z)) > \alpha$ in $|z| < r_\alpha < 1$, where $r_\alpha$ is given as the least positive root of the equation $q(r) = \alpha$, which is same as (0) and hence the result.

**Remark 4.4.** Note for $\alpha = 0$, $r_0 \approx 0.37198$ which is not sharp, so the result can be further improved. The sharp $K_\alpha$-radius for the class $S^*_\alpha$ is $r_0 \approx 0.400435$, which we can guess graphically but a mathematical proof is yet to derive.

For our next Theorems 4.5–4.8, the following subclasses are required:

Let $S^*_\alpha[C, D] := \{f \in A : zf'(z)/f(z) \in P_{\alpha}(C, D)\}$. Also, let $S^*_\alpha[1 - 2\alpha, -1] = A_{\alpha} \cap S^*_\alpha$ and $S^*_\alpha := A_{\alpha} \cap S^*_\alpha$. Further, Ali et al. [2] studied the three classes $S_n := \{f \in A_{\alpha} : f(z)/z \in P_{\alpha}\}$, $S^*_\alpha[C, D]$ and $CS_n(\alpha) := \left\{ f \in A_{\alpha} : \frac{f(z)}{g(z)} \in P_{\alpha}, g \in S^*_\alpha(\alpha) \right\}$.

Now we obtain the $S^*_\rho,n$-radii for the classes defined above.

**Theorem 4.5.** For the class $S_n$, the sharp $S^*_\rho,n$-radius is given by:

$$R_{S^*_\rho,n}(S_n) = \left( \frac{\sinh^{-1}(1)}{n + \sqrt{n^2 + (\sinh^{-1}(1))^2}} \right)^{1/n}.$$  

**Proof.** Let $f \in S_n$. Define $s : D \to \mathbb{C}$ by $s(z) = f(z)/z$. Then $s \in P_{\alpha}$ and we can obtain $zf'(z)/f(z) - 1 = zs'(z)/s(z)$ from the above definition of $s$. Using Lemma 2.6 and Lemma 3.2, the following holds

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| = \left| \frac{zs'(z)}{s(z)} \right| \leq \frac{2nr^n}{1 - r^{2n}} \leq \sinh^{-1}(1),$$

or equivalently $(\sinh^{-1}(1))r^{2n} + 2nr^n - \sinh^{-1}(1) \leq 0$. Therefore, the $S^*_\rho,n$-radius of $S_n$ is the least positive root of $(\sinh^{-1}(1))r^{2n} + 2nr^n - \sinh^{-1}(1) = 0$ for $r \in (0, 1)$. We can verify $\Re(f_0(z)/z) > 0$ holds in $D$, where $f_0(z) = z(1 + z^n)/(1 - z^n)$. Thus $f_0 \in S_n$ and $zf_0'(z)/f_0(z) = 1 + 2nz^n/(1 - z^{2n})$. Moreover, the result is sharp since at $z = R_{S^*_\rho,n}(S_n)$, we obtain

$$\frac{zf_0'(z)}{f_0(z)} - 1 = \frac{2nz^n}{1 - z^{2n}} = \sinh^{-1}(1).$$

The proof is complete.  

□
Let $\mathcal{F}$ define the class of functions $f \in \mathcal{A}$ satisfying $f(z)/z \in \mathcal{P}$. The radius of univalence and starlikeness of the class $\mathcal{F}$ is $\sqrt{2} - 1$, as shown in [12].

Corollary 4.6. For the class $\mathcal{F}$, the $S^*_p$-radius is stated as

$$R_{S^*_p}(\mathcal{F}) = -e + \sqrt{1 + e^2} \approx 0.178105.$$  

Theorem 4.7. For the class $\mathcal{CS}_n(\alpha)$, the sharp $S^*_p,n$-radius is given by

$$R_{S^*_p,n}(\mathcal{CS}_n(\alpha)) = \left(\frac{\sinh^{-1}(1)}{n - \alpha + 1 + \sqrt{(n - \alpha + 1)^2 + (\sinh^{-1}(1) + 2(1 - \alpha))\sinh^{-1}(1)}}\right)^{1/n}.$$

Proof. Let $f \in \mathcal{CS}_n(\alpha)$ and $g \in S^*_p(\alpha)$. Considering $s(z) = f(z)/g(z)$, clearly indicates $s \in \mathcal{P}_n$. Also, it gives

$$\frac{zf'(z)}{f(z)} = \frac{zs'(z)}{s(z)} + \frac{zg'(z)}{g(z)}.$$  

The use of Lemmas 3.2–3.3 gives us

$$|zf'(z)/f(z) - 1 + (1 - 2\alpha)r^{2n}|/1 - r^{2n} | \leq \frac{2(n - \alpha + 1)r^n}{1 - r^{2n}}.$$  

(10)

Considering $(1 + (1 - 2\alpha)r^{2n})/(1 - r^{2n}) \geq 1$, the relation $f \in S^*_p,n$ follows from (10) and Lemma 2.6 if the subsequent inequality is true:

$$\frac{1 + 2(n - \alpha + 1)r^n + (1 - 2\alpha)r^{2n}}{1 - r^{2n}} \leq 1 + \sinh^{-1}(1)$$

or equivalently, $(2 - 2\alpha + \sinh^{-1}(1))r^{2n} + 2(n - \alpha + 1)r^n - \sinh^{-1}(1) = 0$ holds. Thus, the least positive root of

$$(2 - 2\alpha + \sinh^{-1}(1))r^{2n} + 2(n - \alpha + 1)r^n - \sinh^{-1}(1) = 0$$

gives the $S^*_p,n$-radius for the class $\mathcal{CS}_n(\alpha)$. Next examine the following functions

$$f_0(z) = \frac{z(1 + z^n)}{(1 - z^n)(n + 2\alpha)/n} \text{ and } g_0(z) = \frac{z}{(1 - z^n)^{2(1 - \alpha)/n}},$$

which implies $f_0(z)/g_0(z) = (1 + z^n)/(1 - z^n)$ and $zg_0'(z)/g_0(z) = (1 + (1 - 2\alpha)z^n)/(1 - z^n)$. Moreover, $\text{Re}(f_0(z)/g_0(z)) > 0$ and $\text{Re}(zg_0'(z)/g_0(z)) > \alpha$ in the unit disk $D$ is obvious. Hence $f_0 \in \mathcal{CS}_n(\alpha)$. At $z = R_{S^*_p,n}(\mathcal{CS}_n(\alpha))$, the function $f_0$ defined in (11) satisfies

$$\frac{zf_0'(z)}{f_0(z)} = \frac{1 + 2(n - \alpha + 1)z^n + (1 - 2\alpha)z^{2n}}{1 - z^{2n}} = 1 + \sinh^{-1}(1),$$

which accomplish sharpness of the result. □

Theorem 4.8. For the class $S^*_n[C, D]$, the $S^*_p,n$-radius is given by

$$R_{S^*_p,n}(S^*_n[C, D]) = \begin{cases} \min\{1; R_1\}, & -1 \leq D < 0 < C \leq 1; \\ \min\{1; R_2\}, & 0 < D < C \leq 1, \end{cases}$$
where

\[ R_1 := \left( \frac{2 \sinh^{-1}(1)}{C - D + \sqrt{(C - D)^2 + 4(D^2(1 + \sinh^{-1}(1)) - CD)\sinh^{-1}(1)}} \right)^{\frac{1}{n}} \]

and

\[ R_2 := \left( \frac{2 \sinh^{-1}(1)}{C - D + \sqrt{(C - D)^2 + 4(D^2(\sinh^{-1}(1) - 1) + CD)\sinh^{-1}(1)}} \right)^{\frac{1}{n}}. \]

Proof. Let \( f \in S_\ast_n[C, D] \). From Lemma 3.3, we have

\[ \left| \frac{zf'(z)}{f(z)} - b \right| \leq \frac{(C - D)r^n}{1 - D^2r^{2n}}, \tag{12} \]

where \( b = \frac{(1 - CDr^{2n})}{(1 - D^2r^{2n})} \), \( |z| = r \), represents the center of the disk. We infer \( b \geq 1 \) for \(-1 \leq D < 0 < C \leq 1\). From Lemma 2.6, \( f \in S_\ast \rho,n \) depends on whether the following condition is true:

\[ \frac{1 + (C - D)r^n - CDr^{2n}}{1 - D^2r^{2n}} \leq 1 + \sinh^{-1}(1), \]

which reduces to

\[ r \leq \left( \frac{2 \sinh^{-1}(1)}{C - D + \sqrt{(C - D)^2 + 4(D^2(1 + \sinh^{-1}(1)) - CD)\sinh^{-1}(1)}} \right)^{\frac{1}{n}} = R_1. \]

Further, taking \( D = 0 \), we get \( b = 1 \). Then (12) yields

\[ \left| \frac{zf'(z)}{f(z)} - 1 \right| \leq Cr^n, \ (0 < C \leq 1). \]

Now applying Lemma 2.6 with \( a = 1 \) gives \( f \in S_\ast \rho,n \) if \( r \leq ((\sinh^{-1}(1))/C)^{1/n} \).

For \( 0 < D < C \leq 1 \), we have \( b < 1 \). Thus, using Lemma 2.6 and (12), we have \( f \in S_\ast \rho,n \) if the following holds:

\[ \frac{CDr^{2n} + (C - D)r^n - 1}{1 - D^2r^{2n}} \leq \sinh^{-1}(1) - 1, \]

or equivalently, if

\[ r \leq \left( \frac{2 \sinh^{-1}(1)}{C - D + \sqrt{(C - D)^2 + 4(D^2(\sinh^{-1}(1) - 1) + CD)\sinh^{-1}(1)}} \right)^{\frac{1}{n}} = R_2. \]

This concludes the proof. \( \square \)

The next theorem establishes radius results for some well-known classes mentioned earlier.
Theorem 4.9. The sharp $S^*_p$-radii for the classes $S^*_L$, $S^*_RL$, $S^*_C$, $S^*_C$, $\Delta^*$ and $\mathcal{B}S^*(\alpha)$ are:

(i) $R_{S^*_p}(S^*_L) = \sinh^{-1}(1)(2 - \sinh^{-1}(1)) \approx 0.985928$.

(ii) $R_{S^*_p}(S^*_RL) = \frac{(2 + (1 + \sqrt{2}) \sinh^{-1}(1)) \sinh^{-1}(1)}{5 - 3\sqrt{2} + 4(\sqrt{2} - 1) + 2 \sinh^{-1}(1)} \sinh^{-1}(1) \approx 0.964694$.

(iii) $R_{S^*_p}(S^*_C) = \frac{1}{2} \left( \sqrt{2 (2 + 3 \sinh^{-1}(1))} - 2 \right) \approx 0.523831$.

(iv) $R_{S^*_p}(S^*_C) = \ln(1 + \sinh^{-1}(1)) \approx 0.632002$.

(v) $R_{S^*_p}(\Delta^*) = \frac{\sinh^{-1}(1)(2 + \sinh^{-1}(1))}{2(1 + \sinh^{-1}(1))} \approx 0.674924$.

(vi) $R_{S^*_p}(\mathcal{B}S^*(\alpha)) = \frac{-1 + \sqrt{1 + \alpha(2 \sinh^{-1}(1))^2}}{2\alpha \sinh^{-1}(1)}$, $\alpha \in [0, 1]$.

Proof. (i) Suppose $f \in S^*_L$. We have $z f'(z)/f(z) \prec \sqrt{1 + z}$. When $|z| = r$, we obtain

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1 - \sqrt{1 - r} \leq \sinh^{-1}(1),$$

such that $r \leq (2 - \sinh^{-1}(1)) \sinh^{-1}(1) = R_{S^*_p}(S^*_L)$ holds. Next examine the function

$$f_0(z) = \frac{4z}{(1 + \sqrt{1 + z})^2} e^{2(\sqrt{1 + z} - 1)}.$$

Since $zf_0'(z)/f_0(z) = \sqrt{1 + z}$, it follows that $f_0 \in S^*_L$. As $zf_0'(z)/f_0(z) - 1 = -\sinh^{-1}(1)$ is obtained at $z = -R_{S^*_p}(S^*_L)$, the result is sharp.

(ii) Suppose $f \in S^*_RL$, we obtain

$$\frac{zf'(z)}{f(z)} < \sqrt{2} - (\sqrt{2} - 1) \sqrt{1 - \frac{1 - z}{(1 + 2(\sqrt{2} - 1))z}}.$$

For $|z| = r$, the subsequent inequality holds

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1 - \sqrt{2} + (\sqrt{2} - 1) \sqrt{\frac{1 + r}{(1 - 2(\sqrt{2} - 1)r)}},$$

provided

$$r \leq \frac{(2 + (1 + \sqrt{2}) \sinh^{-1}(1)) \sinh^{-1}(1)}{5 - 3\sqrt{2} + 4(\sqrt{2} - 1) + 2 \sinh^{-1}(1)} = R_{S^*_p}(S^*_RL).$$

Next observe the following function defined as

$$f_0(z) = z \exp \left( \int_0^z \frac{g_0(t) - 1}{t} dt \right),$$

where

$$g_0(t) = \sqrt{2} - (\sqrt{2} - 1) \sqrt{\frac{1 - t}{(1 + 2(\sqrt{2} - 1)t)}}.$$
From the definition of $f_0$, at $z = -R_{S^*_C}(S^*_R L)$, we have
\[
\frac{zf_0'(z)}{f_0(z)} = \sqrt{2} - (\sqrt{2} - 1)\sqrt{1 - z \over 1 + 2(\sqrt{2} - 1)z} = 1 - \sinh^{-1}(1),
\]
which confirms the sharpness.

(iii) Suppose $f \in S^*_C$. So $zf'(z)/f(z) < 1 + 4z/3 + 2z^2/3$. This gives
\[
\left| z \frac{f'(z)}{f(z)} - 1 \right| \leq \frac{4r}{3} + 2r^2 / 3 \leq \sinh^{-1}(1), \ |z| = r,
\]
for $r \leq \frac{1}{2} \left( \sqrt{2(2 + 3 \sinh^{-1}(1))} - 2 \right) = R_{S^*_C}(S^*_C)$. The sharpness of the result is established using the subsequent function
\[
f_0(z) = z \exp \left( \frac{4z + z^2}{3} \right),
\]
where $zf_0'(z)/f_0(z) = 1 + (4z + 2z^2)/3$ yields $f_0 \in S^*_C$, and substituting $z = R_{S^*_C}(S^*_C)$ gives $zf_0'(z)/f_0(z) = 1 + \sinh^{-1}(1)$, thereby proving the sharpness.

(iv) Suppose $f \in S^*_C$, we have $zf'(z)/f(z) < e^z$, which yields
\[
\left| z \frac{f'(z)}{f(z)} - 1 \right| \leq e^z - 1 \leq \sinh^{-1}(1) \text{ holds in } |z| = r,
\]
provided $r \leq \ln(1 + \sinh^{-1}(1)) = R_{S^*_C}(S^*_C)$. Now consider
\[
f_0(z) = z \exp \left( \int_0^z e^t - 1 \ dt \right).
\]
Since $zf_0'(z)/f_0(z) = e^z$, where $f_0 \in S^*_C$, so at $z = R_{S^*_C}(S^*_C)$, we have
\[
zf_0'(z)/f_0(z) = 1 + \sinh^{-1}(1),
\]
which shows the sharpness of the result.

(v) Suppose $f \in \Delta^*$ which gives $zf'(z)/f(z) < z + \sqrt{1 + z^2}$. Then,
\[
\left| z \frac{f'(z)}{f(z)} - 1 \right| \leq r + \sqrt{1 + r^2} - 1 \leq \sinh^{-1}(1), \ |z| = r,
\]
for $r \leq \sinh^{-1}(1)/2(1 + \sinh^{-1}(1)) = R_{S^*_C}(\Delta^*)$. For sharpness, define $f_0$ as
\[
f_0(z) = z \exp \left( \int_0^z t \left( t + \sqrt{1 + t^2} - 1 \right) dt \right).
\]
Since $zf_0'(z)/f_0(z) = z + \sqrt{1 + z^2}$, $f_0 \in \Delta^*$, so at $z = R_{S^*_C}(\Delta^*)$, we have
\[
zf_0'(z)/f_0(z) = 1 + \sinh^{-1}(1) \text{ which shows the sharpness of the result}.
\]

(vi) Suppose $f \in B^{S^*_C}(\alpha)$, $\alpha \in [0, 1]$, which gives $zf'(z)/f(z) < 1 + z/(1 - \alpha z^2)$. Then,
\[
\left| z \frac{f'(z)}{f(z)} - 1 \right| \leq \frac{r}{1 - \alpha r^2} \leq \sinh^{-1}(1), \ |z| = r,
\]
for \( r \leq \frac{-1+\sqrt{1+\alpha(2\sinh^{-1}(1))^2}}{2\alpha\sinh^{-1}(1)} = R_{S_p^*}^*(\alpha_1), \alpha_1 \in (0,1] \). For \( \alpha = 0, r \leq \sinh^{-1}(1) \). Next examine the function \( f_0 \) defined as

\[
f_0(z) = z \left( \frac{1 + \sqrt{\alpha z}}{1 - \sqrt{\alpha z}} \right)^{1/(2\sqrt{\alpha})}.
\]

Since \( zf_0'(z)/f_0(z) = 1 + z/(1 - \alpha z^2) \), where \( f_0 \in (BS^*(\alpha_1)) \), so at \( z = -R_{S_p^*}^*(BS^*(\alpha_1)) \), we have \( zf_0'(z)/f_0(z) = 1 - \sinh^{-1}(1) \), which ensures sharpness of the result.

Note that for \( \alpha = 0, R_{S_p^*}^*(BS^*(0)) = \sinh^{-1}(1) \approx 0.881374 \) and for \( \alpha = 1, R_{S_p^*}^*(BS^*(1)) = \left(-1 + \sqrt{1 + (2\sinh^{-1}(1))^2}\right)/(2\sinh^{-1}(1)) \approx 0.58241. \)

Next we present some radius problems for certain classes of functions expressed as ratio of functions:

\[
F_1 := \left\{ f \in A_n : \text{Re} \left( \frac{f(z)}{g(z)} \right) > 0 \text{ and } \text{Re} \left( \frac{g(z)}{z} \right) > 0, \, g \in A_n \right\},
\]

\[
F_2 := \left\{ f \in A_n : \text{Re} \left( \frac{f(z)}{g(z)} \right) > 0 \text{ and } \text{Re} \left( \frac{g(z)}{z} \right) > 1/2, \, g \in A_n \right\},
\]

and

\[
F_3 := \left\{ f \in A_n : \left| \frac{f(z)}{g(z)} - 1 \right| < 1 \text{ and } \text{Re} \left( \frac{g(z)}{z} \right) > 0, \, g \in A_n \right\}.
\]

**Theorem 4.10.** For functions in the classes \( F_1, F_2 \) and \( F_3 \), the sharp \( S_{p,n}^* \)-radius, respectively, are:

(i) \( R_{S_{p,n}^*}(F_1) = \left( \frac{4n^2 + (\sinh^{-1}(1))^2 - 2n}{\sinh^{-1}(1)} \right)^{1/n} \).

(ii) \( R_{S_{p,n}^*}(F_2) = \left( \frac{4n^2 + 4\sinh^{-1}(1) (n + \sinh^{-1}(1)) - 2n}{2(n + \sinh^{-1}(1))} \right)^{1/n} \).

(iii) \( R_{S_{p,n}^*}(F_3) = R_{S_{p,n}^*}(F_2) \).

**Proof.** (i) Let \( f \in F_1 \) and consider the functions \( s,d : \mathbb{D} \to C \), where \( s(z) = f(z)/g(z) \) and \( d(z) = g(z)/z \). Clearly, \( s,d \in P_n \). As \( f(z) = zd(z)s(z) \), applying Lemma 3.2 here gives

\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \frac{4nr^n}{1 - r^{2n}} \leq \sinh^{-1}(1)
\]

such that

\[
r \leq \left( \frac{4n^2 + (\sinh^{-1}(1))^2 - 2n}{\sinh^{-1}(1)} \right)^{1/n} = R_{S_{p,n}^*}(F_1)
\]
The sharpness can be verified as follows. Examine the functions

\[ f_0(z) = z \left( \frac{1 + z^n}{1 - z^n} \right)^2 \quad \text{and} \quad g_0(z) = z \left( \frac{1 + z^n}{1 - z^n} \right). \]

Evidently, \( \Re(f_0(z)/g_0(z)) > 0 \) and \( \Re(g_0(z)/z) > 0 \), which implies \( f_0 \in F_1 \).

Further calculation yields at \( z = R_{S_{p,n}}(F_1)e^{\pi i/n} \)

\[ \frac{zf'(z)}{f(z)} = 1 + \frac{4nz^n}{1 - z^{2n}} = 1 - \sinh^{-1}(1), \]

which validates the result is sharp.

(ii) Let \( f \in F_2 \) and consider the functions \( s, d : \mathbb{D} \to \mathbb{C} \), where \( s(z) = f(z)/g(z) \) and \( d(z) = g(z)/z \). Clearly, \( s \in P_n(1/2) \) and \( d \in P_n \). As \( f(z) = zd(z)s(z) \), applying Lemma 3.2 here gives

\[ \frac{|zf'(z)|}{f(z)} - 1 \leq \frac{2nr^n}{1 - r^{2n}} + \frac{nr^n}{1 - r^n} = \frac{3nr^n + nr^{2n}}{1 - r^{2n}} \leq \sinh^{-1}(1), \]

whenever

\[ r \leq \left( \frac{\sqrt{9n^2 + 4\sinh^{-1}(1)(n + \sinh^{-1}(1)) - 3n}}{2(n + \sinh^{-1}(1))} \right)^{1/n} = R_{S_{p,n}}(F_2). \]

Therefore, \( f \in S_{p,n} \) holds for \( r \leq R_{S_{p,n}}(F_2) \). Next see that \( \Re(g_0(z)/z) > 1/2 \)
while \( \Re(f_0(z)/g_0(z)) > 0 \) for the functions

\[ f_0(z) = z(1 + z^n)/(1 - z^n)^2 \quad \text{and} \quad g_0(z) = z/(1 - z^n). \]

Therefore \( f_0 \in F_2 \) which verifies the sharpness for \( z = R_{S_{p,n}}(F_2) \) such that

\[ \frac{zf'(z)}{f(z)} - 1 = \frac{3nz^n + nz^{2n}}{1 - z^{2n}} = \sinh^{-1}(1). \]

(iii) Let \( f \in F_3 \) and consider the functions \( s, d : \mathbb{D} \to \mathbb{C} \), where \( s(z) = g(z)/f(z) \) and \( d(z) = g(z)/z \). Then \( d \in P_n \). We can verify that \( \frac{|1/s(z) - 1|}{1} < 1 \) holds whenever \( \Re(s(z)) > 1/2 \) and therefore \( s \in P_n(1/2) \). As \( f(z) = zd(z)/s(z) \), on applying Lemma 3.2, we obtain

\[ \frac{|zf'(z)|}{f(z)} - 1 \leq \frac{3nr^n + nr^{2n}}{1 - r^{2n}} \leq \sinh^{-1}(1). \]

The rest of the proof is omitted as it is analogous to proof of Theorem 4.10(ii).

The sharpness can be verified as follows. Examine the functions

\[ f_0(z) = z(1 + z^n)/(1 - z^n) \quad \text{and} \quad g_0(z) = z/(1 - z^n). \]

Using above definitions of \( f_0 \) and \( g_0 \), we see that

\[ \Re \left( \frac{g_0(z)}{f_0(z)} \right) = \Re \left( \frac{1}{1 + z^n} \right) > 1/2 \quad \text{and} \quad \Re \left( \frac{g_0(z)}{z} \right) = \Re \left( \frac{1 + z^n}{1 - z^n} \right) > 0, \]
and therefore, \( f_0 \in F_3 \). Now at \( z = R_{S, n} (F_3) e^{i \pi/n} \), we obtain
\[
\frac{zf_0'(z)}{f_0(z)} - 1 = \frac{3nz^n - n_z^{2n}}{1 - z^{2n}} = -\sinh^{-1}(1),
\]
which serves as validation for the sharp result.

This concludes the proof. \( \square \)

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