Norm approximation for the Fröhlich dynamics in the mean-field regime

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Abstract

We study the time evolution of the Fröhlich Hamiltonian in a mean-field limit in which many particles weakly couple to the quantized phonon field. Assuming that the particles are initially in a Bose-Einstein condensate and that the excitations of the phonon field are initially in a coherent state we provide an effective dynamics which approximates the time evolved many-body state in norm, provided that the number of particles is large. The approximation is given by a product state which evolves according to the Landau–Pekar equations and which is corrected by a Bogoliubov dynamics. In addition, we extend the results from [27] about the approximation of the time evolved many-body state in trace-norm topology to a larger class of many-body initial states with an improved rate of convergence.

1 Introduction and main results

We are interested in the evolution of a Bose-Einstein condensate which weakly interacts with the excitations of a quantized phonon field. For this purpose we consider the Fröhlich model in the mean-field regime. It is defined on the Hilbert space

\[ H^{(N)} = \left( L^{2}(\mathbb{R}^{3}) \right)^{\otimes_{s}N} \otimes F, \]  

(1.1)

where \( \otimes_{s} \) denotes the symmetric tensor product and \( F = \bigoplus_{n=0}^{\infty} (L^{2}(\mathbb{R}^{3}))^{\otimes_{s}n} \). The state of system evolves according to the Schrödinger equation

\[ i \partial_{t} \Psi_{N,t} = H_{N,\alpha}^{F} \Psi_{N,t} \]  

(1.2)

with Fröhlich Hamiltonian

\[ H_{N,\alpha}^{F} = \sum_{j=1}^{N} \left[ -\Delta_{j} + \sqrt{\frac{\alpha}{N}} \int d k \ |k|^{-1} \left( e^{2\pi i k x_{j}} a_{k} + e^{-2\pi i k x_{j}} a_{k}^{\dagger} \right) \right] + N_{\alpha}. \]  

(1.3)

The annihilation operators \( a_{k} \) and creation operators \( a_{k}^{\dagger} \) satisfy the canonical commutation relations

\[ [a_{k}, a_{l}^{\dagger}] = \delta(k - l), \quad [a_{k}, a_{l}] = [a_{k}^{\dagger}, a_{l}^{\dagger}] = 0 \]  

(1.4)

and \( N_{\alpha} \) is the number operator defined by \( N_{\alpha} = \int d^{3}k \ a_{k}^{\dagger} a_{k} \). The coupling parameter \( \sqrt{\alpha/N} \) scales the strength of the interaction. It is chosen in a way such that all terms in the Hamiltonian are of order \( N \) if the number of phonons is of order \( N \). Let us remark that the definition (1.3) is rather formal since the form factor of the phonon field is not square integrable. Using the commutator method of Lieb
and Yamazaki [21] the Hamiltonian $H_{N,\alpha}^F$ can, however, always be defined via its associated quadratic form. More information on this account and about the domain of $H_{N,\alpha}^F$ are given in [22, 25]. We are interested in the evolution of many-body initial states of the form

$$\Psi_N \approx \psi \otimes W(\sqrt{N}\varphi)\Omega.$$  

(1.5)

Here $\psi, \varphi \in L^2(\mathbb{R}^3)$, $\Omega$ denotes the vacuum in $\mathcal{F}$ and $W(f)$ (with $f \in L^2(\mathbb{R}^3)$) is the unitary Weyl operator

$$W(f) = \exp \left( \int dk \,(f(k)a_k^* - \overline{f(k)}a_k) \right)$$  

(1.6)

satisfying

$$W^{-1}(f) = W(-f), \quad W(f)W(g) = W(g)W(f)e^{-2i\text{Im}(f,g)} = W(f + g)e^{-i\text{Im}(f,g)}.$$  

(1.7)

as well as the shift property

$$W^*(f)a_kW(f) = a_k + f(k).$$  

(1.8)

The initial datum (1.5) describes a Bose-Einstein condensate of particles with condensate wave function $\psi$ and a coherent state of phonons with mean particle number $N \|\varphi\|^2_{L^2(\mathbb{R}^3)}$. A central feature of (1.5) is that particles and phonons only have little correlations among each other. During the time evolution correlations emerge because of the interaction and the state of the system will no longer be of product type. If the number of particles $N$ is large these correlations are, however, weak enough such that the solution of the many-body Schrödinger equation (1.2) with initial datum (1.5) can be approximated in trace-norm topology (see Theorem 1.1 and [27]) by a product state

$$\psi^\otimes N \otimes W(\sqrt{N}\varphi)\Omega,$$  

(1.9)

where $(\psi_t, \varphi_t) \in L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ solves the time-dependent Landau–Pekar equations

$$\begin{aligned}
i\partial_t \psi_t(x) &= h(t)\psi_t(x), \\
i\partial_t \varphi_t(k) &= \varphi_t(k) + \sqrt{\alpha} |k|^{-1} \int dx \, e^{-2\pi ikx} \, |\psi_t(x)|^2 \
\end{aligned}$$  

(1.10)

with

$$\Phi_{\varphi}(x) = \int dk \, |k|^{-1} \left( e^{2\pi ikx}\varphi(k) + e^{-2\pi ikx}\overline{\varphi(k)} \right) \quad \text{for} \, \varphi \in L^2(\mathbb{R}^3),$$  

(1.11)

$$\mu(t) = \frac{1}{2} \sqrt{\alpha} \int dx \, \Phi_{\varphi_t}(x) |\psi(t,x)|^2$$  

(1.12)

$$h(t) = -\Delta + \sqrt{\alpha} \Phi_{\varphi_t} - \mu(t)$$  

(1.13)

and initial datum $(\psi_0, \varphi_0) = (\psi, \varphi)$. The Landau–Pekar equations model the interaction between a single quantum particle and a classical phonon field. The corresponding energy functional is given by

$$\mathcal{E}[\psi, \varphi] = \langle \psi, (-\Delta + \sqrt{\alpha} \Phi_{\varphi}) \psi \rangle_{L^2(\mathbb{R}^3)} + \|\varphi\|^2_{L^2(\mathbb{R}^3)}.$$  

(1.14)

Approximating $\Psi_{N,t}$ by (1.9) enables us to compute the many-body time evolution by a set of non-linear differential equations which involve only the condensate wave function and a classical phonon field. This reduces the complexity of the description tremendously. The validity of the approximation, however, only holds for the particle and phonon reduced density matrices of $\Psi_{N,t}$. In order to obtain an effective description which approximates $\Psi_{N,t}$ in the Hilbert space norm of $\mathcal{H}^{(N)}$ it is necessary to take the leading order of the quantum fluctuations around the Landau–Pekar equations into account. This
can be established by means of a Bogoliubov dynamics which is defined on the tensor product of two Fock spaces (see Subsection 1.2). The Bogoliubov dynamics is generated by a quadratic Hamiltonian and for this reason again much easier to analyze than the true time evolution. For a detailed discussion of this fact in the context of the Nelson model with ultraviolet cutoff we refer to [14, Chapter 3].

The goal of this work is twofold: In the first part we look at the reduced density matrices of \( \Psi_{N,t} \) and prove that their time evolutions are approximately given by the Landau–Pekar equations. The second part shows that the Bogoliubov dynamics from Subsection 1.2 takes the correlations between the particles and the phonons correctly into account, leading to a norm approximation of the time evolved many-body state.

The approximation for the reduced densities has previously been shown in [27]. The merit of Theorem 1.1 is to extend these results to a larger class of many-body initial states with an improved rate of convergence. In contrast to [27] we do not require any assumptions on the variance of the many-body energy and for this reason are able to classify the many-body initial data and to prove the results without using the Gross transform. The fact that the many-body initial data is only restricted to the form domain of the non-interacting Hamiltonian is of great importance if one would like to study the mean-field limit of the renormalized Nelson model with similar techniques and a regularization of the interaction by means of the Gross transform.\(^1\) To the best of our knowledge, Theorem 1.4 provides the first rigorous derivation of the Bogoliubov dynamics for the Fröhlich model in the mean-field regime.

In order to obtain our result we use the excitation map from [14] (a straightforward generalization of the excitation map originally introduced in [33]), the commutator method of Lieb and Yamazaki [34], an operator bound which follows from [18, Lemma 10] and energy estimates in the spirit of [32, Theorem 8] which have previously been applied in derivations of the Hartree– and Gross–Pitaevskii equations [4, 5, 6, 7, 37, 38, 39]. The norm approximation of the many-body quantum state is proven by means of the approach from [32]. The main difficulties arise from the singular interaction which requires to control the growth of the kinetic energy not only under the Bogoliubov dynamics but also under the fluctuations dynamics around the true many-body evolution. In addition, it is necessary to estimate the growth of higher moments of the number operator during the Bogoliubov time evolution. To this end we introduce a Bogoliubov dynamics which is truncated in the total number of particles. Such a truncation has previously been used in [6, 37, 38, 39, 40].

Comparison with the literature. The Fröhlich model with \( N = 1 \) was originally introduced in [19] to describe the behavior of an electron in an ionic crystal and the Landau–Pekar equations were presented in [24] as effective equations for this model in the strong coupling limit, i.e. \( H_{F,\alpha} \) with \( \alpha \gg 1 \). In recent years the rigorous derivation of the Landau–Pekar equations in the strong coupling regime was established in a series of works [16, 18, 21, 26, 31, 35]. For a comparison between the different results we refer to [26, p. 658]. In this regard let us also mention [17, 31, 15] for results on adiabatic theorems of the Landau–Pekar equations (in one and three dimensions) and on the persistence of the spectral gap during the evolution of the Landau–Pekar equations. In this work we are concerned with the Fröhlich Hamiltonian in the mean-field regime as previously considered in [27]. Here, the appearance of classical radiation rests on the fact that many weakly correlated particles in the same quantum state create radiation. This mechanism has been investigated for the Nelson model with ultraviolet cutoff [11, 12, 14, 29], the Pauli–Fierz Hamiltonian [13, 30], the renormalized Nelson model [2] and for the Nelson model with ultraviolet cutoff in a limit of many weakly interacting fermions [28]. We would like to remark that the interaction of the renormalized Nelson model is more singular than the one of the Fröhlich model and that an analysis is more complicated in this case. The results from [2] do, however, not provide explicit error estimates and we view the present article as a starting point for a derivation of the Schrödinger–Klein–Gordon equations from the renormalized Nelson model with an explicit rate of convergence. Let us also mention [8, 9, 10, 11, 20, 11] in which the classical behavior of radiation fields was shown in other scaling regimes.

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\(^1\) With this respect it is important to note that the Gross transform classifies the domain of the Fröhlich Hamiltonian while it only allows to study the form domain of the Nelson Hamiltonian [22, 23, 25].
The article is structured as follows: In the rest of this section we present our main results. Section 2 specifies the notation and provides a series of operator estimates which will be useful for the proofs. The results are proven in Section 3.

1.1 Approximation of the one-particle reduced density matrices

As in [27] we set the coupling constant $\alpha = 1$ and use the notation $H_N^F = H_{N,1}^F$. All results are, however, equally true for any $\alpha > 0$ independent of $N$. For $m \in \mathbb{N}$, let $H_m^m(\mathbb{R}^3)$ denote the Sobolev space of order $m$ and $L^2_m(\mathbb{R}^3)$ be the weighted $L^2$-space with norm $\| \varphi \|_{L^2_m(\mathbb{R}^3)} = \|(1 + |\cdot|^2)^{m/2}\varphi\|_{L^2(\mathbb{R}^3)}$. Within this work we rely on the following result about the Landau–Pekar equations which in a slightly different version was proven in [16] Lemma 2.1 and Proposition 2.2. In Appendix B it is outlined how the proof of [10] Proposition 2.2 has to be modified.

Proposition 1.1. For any $(\psi, \varphi) \in H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ there is a unique global solution $(\psi_t, \varphi_t)$ of (1.10). One has the conservation laws

$$\|\psi_t\|_{L^2(\mathbb{R}^3)} = \|\psi\|_{L^2(\mathbb{R}^3)} \quad \text{and} \quad E[\psi_t, \varphi_t] = E[\psi, \varphi] \quad \text{for all } t \in \mathbb{R}. \quad (1.15)$$

If $(\psi, \varphi) \in H^3(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ with $\|\psi\|_{L^2(\mathbb{R}^3)} = 1$ then there exists a constant $C > 0$ depending only on the initial data such that

$$\|\varphi_t\|_{L^2(\mathbb{R}^3)} \leq C \left(1 + |t|^3\right) \quad \text{and} \quad \|\psi_t\|_{H^3(\mathbb{R}^3)} \leq C \left(1 + |t|^4\right) \quad \text{for all } t \in \mathbb{R}. \quad (1.16)$$

Moreover, let

$$\gamma_{\Psi N}^{(1,0)} = \text{Tr}_{2,\ldots,N} \otimes \text{Tr}_F (|\Psi_N\rangle \langle \Psi_N|) \quad (1.17)$$

be the one-particle reduced density matrix on $L^2(\mathbb{R}^3)$ of $\Psi_N \in \mathcal{H}^{(N)}$ and $\text{Tr} |A|$ denote the trace norm of any trace class operator $A$. Our first result is the following.

Theorem 1.1. Let $(\psi, \varphi) \in H^3(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ s.t. $\|\psi\|_{L^2(\mathbb{R}^3)} = 1$, $\Psi_N \in \mathcal{D} \left(\sum_{j=1}^{N} - \Delta_j + \mathcal{N}_a\right)^{1/2} \quad \text{s.t.} \quad \|\Psi_N\|_{\mathcal{H}^{(N)}} = 1$ and define

$$a[\Psi_N, \psi] = \text{Tr} \left\{ \sqrt{1 - \Delta} (1 - |\psi\rangle \langle \psi|) \gamma_{\Psi N}^{(1,0)} (1 - |\psi\rangle \langle \psi|) \sqrt{1 - \Delta} \right\}, \quad (1.18)$$

$$b[\Psi_N, \varphi] = N^{-1} \langle W^*(\sqrt{N}\varphi) \Psi_N, \mathcal{N}_a W^*(\sqrt{N}\varphi) \Psi_N \rangle_{\mathcal{H}^{(N)}}. \quad (1.19)$$

Let $(\psi_t, \varphi_t)$ be the unique solution of (1.10) with initial datum $(\psi, \varphi)$ and $\Psi_{N,t} = e^{-it\mathcal{H}^F_N} \Psi_N$. Then there exists a constant $C > 0$ (depending only on $E[\psi, \varphi]$) and $f(t) = \int_0^t ds \left\{ \|\psi_s\|_{H^3(\mathbb{R}^3)}^2 + \|\varphi_s\|_{L^2(\mathbb{R}^3)}^2 \right\}$ such that

$$N^{-1} \langle W^*(\sqrt{N}\varphi_t) \Psi_{N,t}, \mathcal{N}_a W^*(\sqrt{N}\varphi_t) \Psi_{N,t} \rangle_{\mathcal{H}^{(N)}} \leq (a[\Psi_N, \psi] + b[\Psi_N, \varphi] + N^{-1}) C e^{Cf(t)}, \quad (1.20)$$

$$\text{Tr} \left\{ \sqrt{1 - \Delta} (\gamma_{\Psi N,t}^{(1,0)} - |\psi_t\rangle \langle \psi_t|) \sqrt{1 - \Delta} \right\} \leq \sup_{j=1,2} \left( a[\Psi_N, \psi] + b[\Psi_N, \varphi] + N^{-1}\right) \frac{1}{2} C e^{Cf(t)}. \quad (1.21)$$

Remark 1.2. Note that $\lim_{N \to \infty} (a[\Psi_N, \psi] + b[\Psi_N, \varphi]) = 0$ if initially the particles exhibit Bose-Einstein condensation, the kinetic energy of the particles outside the condensate is small compared to one and the phonons are in a coherent state. This implies that the left hand sides of (1.20) and (1.21) converge to zero as the number of particles is getting large; showing the stability of the condensate.
and the coherent structure during the time evolution. For Pekar product states of the form $\Psi_N = \psi^{\otimes N} \otimes W(\sqrt{N}\varphi)\Omega$ we get

$$N^{-1}\left\langle W^*(\sqrt{N}\varphi)\Psi_{N,t},N_0W^*(\sqrt{N}\varphi)\Psi_{N,t}\right\rangle_{\mathcal{H}^{(N)}} \leq CN^{-1}e^{CF(t)},$$

$$\text{Tr} \left|1-\Delta_0(\gamma_{\Psi_{N,t}} - |\psi_t\rangle\langle\psi_t|)\sqrt{1-\Delta}\right| \leq CN^{-1/2}e^{CF(t)}.$$  

(1.22)

Remark 1.3. A similar result has previously been proven in [27]. In contrast to [27] we do not have to ensure that the variance of $N^{-1}H_N^F$ w.r.t. $\Psi_N$ (instead only that the kinetic energy of the particles outside the condensate) is small compared to one. We can consequently consider many-body initial states in the form domain of $H_N^F$ and for this reason do not have to distinguish between initial states in the operator domain of $H_N^F$ and those in the domain of the free Hamiltonian without interaction (as it was done in [27, Theorem 2.1 and Theorem 2.2]). In comparison to [27] we obtain a better rate of convergence but a more rapid growth of the error in time. The different behavior in $t$ stems from the fact that we control the kinetic energy of the particles and the interaction by an almost conserved quantity instead of the variance of the energy per particle; which is a constant of motion due to energy conservation. Finally, let us remark that our result is slightly stronger than [27, Theorem 2.1 and Theorem 2.2] in the sense that the convergence of the one-particle reduced density matrix to the projector onto the condensate wave function holds in Sobolev and not only in trace norm. The control of the kinetic energy of the particles outside the condensate is an important ingredient in the proof of Theorem 1.4.

1.2 Excitation Fock-space and Bogoliubov Hamiltonian

Next, we provide an effective description which approximates solutions of the many-body Schrödinger equation (1.2) in the Hilbert space norm. This is obtained by modifying the Pekar product state (1.9) with the help of a Bogoliubov dynamics. The Bogoliubov dynamics describes correlations among the particles and the phonons but also between the particles and the phonons. For this reason it is convenient to describe the state of the particles on a Fock space by itself and to define the Bogoliubov dynamics on the tensor product of this space and the Fock space of the phonon field. The appearance of the Bogoliubov dynamics is motivated by means of the strategy from [32, 33]. We particularly follow the route from [14] which considers the Bogoliubov dynamics and higher order corrections of the Nelson model with ultraviolet cutoff. This means that we will factor out the condensate and the coherent state from the many-body state and study the quantum fluctuations around the mean-field dynamics. To this end we define the excitation Fock space of the particles

$$\mathcal{F}_{b,\psi_t} = \bigoplus_{k=0}^{\infty} \mathcal{F}_{b,\psi_t}^{(k)}, \quad \text{where} \quad \mathcal{F}_{b}^{(k)} = \left(L^2_{\bot \psi_t}(\mathbb{R}^3)\right)^{\otimes k}$$

(1.23)

--with $L^2_{\bot \psi_t}(\mathbb{R}^3)$ being the orthogonal complement of the one-dimensional space spanned by $\psi_t$ in $L^2(\mathbb{R}^3)$-- as well as the excitation Fock space of the phonons

$$\mathcal{F}_{a} = W^*(\sqrt{N}\varphi)\mathcal{F}.$$  

(1.24)

The truncated excitation Fock space $\mathcal{G}_{\bot \psi_t}^{\leq N}$ and the excitation Fock space $\mathcal{G}_{\psi_t}$ are then given by

$$\mathcal{G}_{\bot \psi_t}^{\leq N} = \left(\bigoplus_{k=0}^{N} \mathcal{F}_{b,\psi_t}^{(k)}\right) \otimes \mathcal{F}_{a} = \left(\bigoplus_{k=0}^{N} \mathcal{F}_{b,\psi_t}^{(k)} \otimes \mathcal{F}_{a}\right) \subset \mathcal{G}_{\psi_t} = \mathcal{F}_{b,\psi_t} \otimes \mathcal{F}_{a} \subset \mathcal{G} = \mathcal{F}_{b} \otimes \mathcal{F}_{a}$$

(1.25)

2Note that $\mathcal{F}_{b} = \mathcal{F}_{a} = \mathcal{F}$ because the Weyl operator maps $\mathcal{F}$ into itself. The notations $\mathcal{F}_{b}$ and $\mathcal{F}_{a}$ are introduced to distinguish between the Fock spaces of the particles and phonon field as well as to stress that the new vacuum $\Omega_a$ of $\mathcal{F}_{a}$ is given by $W(\sqrt{N}\varphi)\Omega$ where $\Omega$ is the vacuum of $\mathcal{F}.$
with $F_k = \bigoplus_{n=0}^\infty \{L^2(\mathbb{R}^3)\}^\otimes \cdot \cdot \cdot \cdot$. For $G_{1/\psi_t}$ and $G$ we equally use $b_x$, $b^*_x$, $N_b$ and $a_k$, $a^*_k$, $N_a$ to denote the annihilation operator, the creation operator and the number operator of the particles and the phonons respectively. For given $\psi_t$, $\varphi_t \in L^2(\mathbb{R}^3)$ such that $\|\psi_t\|_{L^2(\mathbb{R}^3)} = 1$ the unitary mapping

$$U_N(t) : \mathcal{H}^N \to G_{1/\psi_t}, \quad \Psi_N \to \left(\chi^{(k)}_{\leq N}(t)\right)_{k=0}^N$$

factors out the condensate wave function $\psi_t$ and the coherent state with mean particle number $N \|\varphi_t\|_{L^2(\mathbb{R}^2)}^2$. Here, $p_i(t) = |\psi_t\rangle \langle \psi_t|_i$ projects the coordinate of the $i$-th particle onto $\psi_t$, $q_i(t) = 1 - p_i(t)$ and the partial inner product is taken w.r.t. the coordinates $x_{k+1}, \ldots, x_N$ of the particles. The inverse of $U_N(t)$ is given by

$$\left(\chi^{(k)}_{\leq N}(t)\right)_{k=0}^N \to \Psi_N = W(\sqrt{N}\varphi_t) \sum_{k=0}^N \psi_t^{(N-k)} \otimes \chi^{(k)}_{\leq N}(t).$$

Note that the definition of $U_N(t)$ is a straightforward generalization of the unitary map originally introduced in [13]. A more detailed introduction of $U_N(t)$ and its properties is given in [14 Appendix A]. If $\Psi_{N,t}$ satisfies (1.2) and $(\psi_t, \varphi_t)$ is a solution of the Landau–Pekar equations (1.10) the time evolution of $\chi_{\leq N}(t)$ is given by the Schrödinger equation

$$i\partial_t \chi_{\leq N}(t) = H_{\leq N}(t)\chi_{\leq N}(t)$$

with Hamiltonian $H_{\leq N}(t) = U_N(t)H^F_{N,\alpha}U_N(t)^* + i\dot{U}_N(t)U_N(t)^*$. For later purposes it is convenient to define $H_{\leq N}(t) = H(t)|_{G_{1/\psi_t}^\perp}$ as a Hamiltonian $H(t)$ on $G$ whose action is restricted to elements of $G_{1/\psi_t}^\perp$.

Note that the definition of $H(t)$ is not unique since the orthogonal projection from $G$ to $G_{1/\psi_t}^\perp$ has a nontrivial kernel. We set $\alpha = 1$ and use the definition (see [14 Appendix A] for a detailed derivation in case of the Nelson model with ultraviolet cutoff)

$$H(t) = \int dx \ b^*_x h(t) b_x + N_a$$

$$+ \int dx \int dk \ K(t, k, x) \left(a^*_k + a_{-k}\right) b^*_x \left[1 - N^{-1}N_b\right]^{1/2} + \text{h.c.}$$

$$+ N^{-1/2} \int dx \ b^*_x \left(q(t) \Phi q(t) - \langle \psi_t, \Phi \psi_t \rangle_{L^2(\mathbb{R}^3)}\right) b_x,$$

where

$$\Phi(x) = \int dk \ |k|^{-1} \left(e^{2\pi i k x} a_k + e^{-2\pi i k x} a^*_k\right)$$

is interaction of the Fröhlich Hamiltonian, $q(t) = 1 - |\psi_t\rangle \langle \psi_t|$ is a projection on $L^2(\mathbb{R}^3)$ with integral kernel

$$q(t, x, y) = \delta(x - y) - \psi_t(x)\overline{\psi_t(y)}$$

and

$$K(t, k, x) = \int dy \ q(t, x, y) |k|^{-1} e^{-2\pi i k y} \psi_t(y).$$

We use $[x]_+$ to denote the positive part of $x$ and h.c. to indicate the Hermitian conjugate of the preceding term. Moreover, note that $\int dx \ b^*_x A_x = \int dx \ \int dy \ b^*_x A(x; y)b_y$ is the usual shorthand notation.
for operators on $L^2(\mathbb{R}^3)$ with integral kernel $A(x; y)$. Disregarding all terms of $H(t)$ with more than two annihilation or creation operators leads to the Bogoliubov Hamiltonian

$$H^B(t) = \int dx \, b^*_x h(t) b_x + \mathcal{N}_a + \left( \int dx \int dk \, K(t, k, x) \left( a_k^* + a_{-k} \right) b^*_x + \text{h.c.} \right).$$

The Bogoliubov equation is given by

$$i \partial_t \chi_B(t) = H^B(t) \chi_B(t) \quad \text{with} \quad \chi_B(0) \in \mathcal{G}.$$  \hfill (1.34)

Note that the Bogoliubov Hamiltonian does, in contrast to $H(t)$, not map $\{ \chi \in \mathcal{G} : 1_{\mathcal{N}_b > N} \chi = 0 \}$ into itself. For this reason it is impossible to define the Bogoliubov equation on the truncated Fock space. The well-posedness of (1.34) and the fact that $\chi \in \mathcal{G}_{\perp \psi}$ implies $\chi_B(t) \in \mathcal{G}_{\perp \psi}$, for all $t \in \mathbb{R}$ are addressed in Lemma 3.2.

In the following, we will indicate elements of $\mathcal{F}^{(k)}_{\perp \psi} \otimes \mathcal{F}_a$ and $\mathcal{F}^{(k)}_b \otimes \mathcal{F}_a$ by the superscript $(k)$. For example, we will use $\chi^{(k)}$ to denote the sector with exactly $k$ particle excitations of a state $\chi$ in $\mathcal{G}^{\perp \psi}_{\perp \psi}$, $\mathcal{G}_{\perp \psi}$, and $\mathcal{G}$.

### 1.3 Norm approximation of the many-body state

**Theorem 1.4.** Let $(\psi, \varphi) \in H^3(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ such that $\| \psi \|_{H^2(\mathbb{R}^3)} = 1$ and let $(\psi_t, \varphi_t)$ be the solution of (1.10) with initial datum $(\psi, \varphi)$. Let $T_b = -\int dx \int dy \, b^*_x A(x; y)b_y$, $\tilde{C} > 0$ and $\chi \in \mathcal{G}_{\perp \psi_0}$ satisfying $\| \chi \|_\mathcal{G} = 1$ as well as

$$\| (\mathcal{N}_a^3 + \mathcal{N}_b^3 + T_b)^{1/2} \chi \|_\mathcal{G} \leq \tilde{C}. \hfill (1.35)$$

Let $\Psi_{N,t} = W(\sqrt{N} \varphi) \sum_{k=0}^N \psi_t^\otimes (N-k) \otimes_\mathcal{F}_a \chi^{(k)}(t) \in \mathcal{H}^{(N)}$. Then, there exists a constant $C > 0$ (depending only on $\tilde{C}$ and $\mathcal{E}[\psi, \varphi]$) such that

$$\| \Psi_{N,t} - \Psi_{N,t}^B \|_{\mathcal{H}^{(N)}} \leq C e^{-f(t)} N^{-1/8}. \hfill (1.37)$$

Here, $f(t) = \int_0^t ds \left( \| \psi_s \|_{H^2(\mathbb{R}^3)} + \| \varphi_s \|_{L^2(\mathbb{R}^3)} \right)$ and

$$\Psi_{N,t}^B = W(\sqrt{N} \varphi_0) \sum_{k=0}^N \psi_t^\otimes (N-k) \otimes_\mathcal{F}_a \chi_B^{(k)}(t), \hfill (1.38)$$

where $\chi_B(t)$ is the solution of (1.34) with initial datum $\chi$.

**Remark 1.5.** If we choose $\chi$ to be the vacuum of $\mathcal{G}$ we obtain a many-body initial state $\Psi_N = \psi^\otimes_\mathcal{F}_a \otimes W(\sqrt{N} \varphi) \Omega$ of Pekar product type.

**Remark 1.6.** Since $\chi = (\chi^{(k)})_{k \geq 0}$ is normalized to one, $\Psi_N$ is not necessarily normalized. Assumption (1.35) and $\| \psi \|_{L^2(\mathbb{R}^3)} = 1$, however, imply that

$$\| \Psi_N \|_{\mathcal{H}^{(N)}}^2 = \| (\chi^{(k)})_{k \geq 0} \|^2_{\mathcal{G}^{\otimes N}} \leq \| \chi \|_{\mathcal{G}}^2 - \| 1_{\mathcal{N}_b > N} \chi \|_{\mathcal{G}}^2 \rightarrow 1 \quad \text{as} \quad N \rightarrow \infty \hfill (1.39)$$

because $\| 1_{\mathcal{N}_b > N} \chi \|_{\mathcal{G}} \leq N^{-3/2} \left\| \mathcal{N}_b^{3/2} \chi \right\|_{\mathcal{G}}$.  

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2 Preliminaries

2.1 Notation

We introduce the usual bosonic creation and annihilation operators \( f \in L^2(\mathbb{R}^3) \)

\[
\begin{align*}
    a(f) &= \int d^3 k \, f(k) a_k, \\
    a^*(f) &= \int d^3 k \, f(k) a^*_k, \\
    b(f) &= \int d^3 x \, f(x) b_x, \\
    b^*(f) &= \int d^3 x \, f(x) b^*_x.
\end{align*}
\] (2.1)

They satisfy the well-known inequalities

\[
\begin{align*}
    &\|a(f)\chi\|_G \leq \|f\|_{L^2(\mathbb{R}^3)} \left\| N_a^{1/2} \chi \right\|_G, \\
    &\|a^*(f)\chi\|_G \leq \|f\|_{L^2(\mathbb{R}^3)} \left\| (N_a + 1)^{1/2} \chi \right\|_G, \\
    &\|b(f)\chi\|_G \leq \|f\|_{L^2(\mathbb{R}^3)} \left\| N_b^{1/2} \chi \right\|_G, \\
    &\|b^*(f)\chi\|_G \leq \|f\|_{L^2(\mathbb{R}^3)} \left\| (N_b + 1)^{1/2} \chi \right\|_G,
\end{align*}
\] (2.2)

for any \( \chi \in G \). We, moreover, define the total number of excitations operator

\[
N = N_a + N_b
\] (2.3)

and recall the second quantization of the particle’s kinetic energy

\[
T_b = -\int dx \, b^*_x \Delta b_x.
\] (2.4)

In addition, it is convenient to introduce

\[
G_x(k) = e^{-2i\pi k x} |k|^{-1},
\] (2.5)

allowing to write the interaction of the Fröhlich Hamiltonian as \( \hat{\Phi}(x) = a(G_x) + a^*(G_x) \). It holds that

\[
\sup_{x \in \mathbb{R}^3} \left\| \frac{1}{|x|} \hat{\chi} \hat{G}_x \right\|_{L^2(\mathbb{R}^3)} = \left\| \frac{1}{|x|} \hat{\chi} \hat{\Lambda} \right\|_{L^2(\mathbb{R}^3)} = \sqrt{4\pi \Lambda}
\] (2.6)

and

\[
\left\| \langle \psi, G_x(k) \hat{\psi} \rangle_{L^2(\mathbb{R}^3)} \right\| \leq \frac{1 + |k|}{|k| (1 + k^2)} \left( \| \psi \|_{L^2(\mathbb{R}^3)} \| \hat{\psi} \|_{L^2(\mathbb{R}^3)} + \| \psi \|_{L^2(\mathbb{R}^3)} \| \hat{\psi} \|_{H^1(\mathbb{R}^3)} \right),
\] (2.7)

where the latter is obtained by means of (2.1) and integration by parts.

The norm and scalar product of \( G \) will be denoted by \( \| \cdot \| \) and \( \langle \cdot, \cdot \rangle \). The symbol \( \text{Tr} \) is used to denote the trace over \( L^2(\mathbb{R}^3) \) and the trace norm of a trace class operator \( \omega : L^2(\mathbb{R}^3) \rightarrow L^2(\mathbb{R}^3) \) is written as \( \text{Tr} |\omega| \). We use the notation \( \hat{\omega} \) to denote the derivative of a function \( \omega \) with respect to time. Moreover, recall that \( H^m(\mathbb{R}^3) \) with \( m \in \mathbb{N} \) denotes the Sobolev space of order \( m \) and \( L^2_m(\mathbb{R}^3) \) is a weighted \( L^2 \)-space with norm \( \| \varphi \|_{L^2_m(\mathbb{R}^3)} = \|(1 + | \cdot |^2)^{m/2} \varphi \|_{L^2(\mathbb{R}^3)} \).

2.2 Interaction and Hamiltonian estimates

In this section we provide preliminary estimates which are needed to prove the main results. We start with the interaction terms of the Hamiltonian \( H(t) \). The part with two annihilation and creation operators can be controlled by the following bounds.
Lemma 2.1. Let $\Lambda \geq 1$, $\Psi, \chi \in \mathcal{G}$, $\psi_t \in H^1(\mathbb{R}^3)$ such that $\|\psi_t\|_{L^2(\mathbb{R}^3)} = 1$. Then,

$$
\left| \left\langle \chi, \int dx \int_{|k| \geq \Lambda} dk \psi_t(x)G_x(k) (a^*_k + a_{-k}) b^*_x \Psi \right\rangle \right| \leq \frac{C}{\sqrt{\Lambda}} \left\| \psi_t \right\|_{H^1(\mathbb{R}^3)} \left\| (\mathcal{N}_b + T_b)^{1/2} \chi \right\| (\mathcal{N}_a + 1)^{1/2} \Psi, \right.
$$

(2.8)

$$
\left| \left\langle \chi, \int dx \int_{|k| \leq \Lambda} dk \psi_t(x)G_x(k) (a^*_k + a_{-k}) b^*_x \Psi \right\rangle \right| \leq C \left\| \mathcal{N}_a^{1/2} \chi \right\| \left[ \sqrt{\Lambda} \| \Psi \| + \| \psi_t \|_{H^1} \right] \left\| \mathcal{N}_a^{1/2} \Psi \right\|, \right.
$$

(2.9)

$$
\left| \left\langle \chi, \int dx \int dk K(t, k, x) (a^*_k + a_{-k}) b^*_x \Psi \right\rangle \right| \leq C \left\| \psi_t \right\|_{H^1(\mathbb{R}^3)} \left\| (\mathcal{N}_b + T_b)^{1/2} \chi \right\| (\mathcal{N}_a + 1)^{1/2} \Psi, \right.
$$

(2.10)

The cubic interaction term of $H(t)$ is treated by means of

Lemma 2.2. Let $\Lambda \geq 1$, $\Psi, \chi \in \mathcal{G}$, $\psi_t \in H^1(\mathbb{R}^3)$ such that $\|\psi_t\|_{L^2(\mathbb{R}^3)} = 1$. Then,

$$
\left| \left\langle \chi, \int dx b^*_x \left( q(t) \hat{\Phi} q(t) - \left\langle \psi_t, \hat{\Phi} \psi_t \right\rangle_{L^2(\mathbb{R}^3)} \right) b_x \Psi \right\rangle \right| \leq C \left\| \psi_t \right\|_{H^1(\mathbb{R}^3)} \left( \Lambda^{1/2} \left\| (\mathcal{N}_b + T_b)^{1/2} \chi \right\| (\mathcal{N}_a + 1)^{1/2} \mathcal{N}_0^{1/2} \Psi \right) + \Lambda^{-1/2} \left\| (\mathcal{N}_a + 1)^{1/2} \mathcal{N}_a^{1/2} \chi \right\| (\mathcal{N}_b + T_b)^{1/2} \Psi, \right.
$$

(2.11)

For $\varepsilon > 0$ this implies

$$
\pm i \int dx b^*_x \left( q(t) \hat{\Phi} q(t) - \left\langle \psi_t, \hat{\Phi} \psi_t \right\rangle_{L^2(\mathbb{R}^3)} \right) b_x \leq \varepsilon \left( \mathcal{N}_b + T_b \right) + C \varepsilon^{-1} \| \psi_t \|^2_{H^1(\mathbb{R}^3)} \left( \mathcal{N}_a + 1 \right) \mathcal{N}_b. \right.
$$

(2.12)

The proofs of Lemma 2.1 and Lemma 2.2 are given in Appendix A. Inequalities (2.8) and (2.11) are obtained by means of the commutator method of Lieb and Yamazaki [34]. Estimate (2.9) is derived with the help of [38, Lemma 10]. The advantage in comparison to estimates by means of the commutator method of Lieb and Yamazaki is that the kinetic energy of the particles, $T_b$, does not appear in (2.7). The two remaining inequalities follow almost immediately from the previous estimates.

By means of Lemma 2.1 we obtain the following estimates for $H^B(t)$ which will later be used to prove the well-posedness of the Bogoliubov equation.

Lemma 2.3. Let $\varepsilon > 0$ be arbitrary. Then, there exists a constant $C_{\varepsilon} > 0$, depending on $\varepsilon$, such that

$$
\pm i \left[ \mathcal{N}, H^B(t) \right] \leq \varepsilon T_b + C_{\varepsilon} \left\| \psi_t \right\|^2_{H^1(\mathbb{R}^3)} (\mathcal{N} + 1), \right.
$$

(2.13)

$$
\left( H^B(t) - T_b \right) \leq \varepsilon T_b + C_{\varepsilon} \left( \left\| \psi_t \right\|^2_{H^1(\mathbb{R}^3)} + \left\| \phi \right\|^2_{L^2(\mathbb{R}^3)} \right) (\mathcal{N} + 1), \right.
$$

(2.14)

$$
\pm \frac{d}{dt} H^B(t) \leq \varepsilon T_b + C_{\varepsilon} \left( \left\| \psi_t \right\|^2_{H^1(\mathbb{R}^3)} + \left\| \phi \right\|^2_{L^2(\mathbb{R}^3)} \right) (\mathcal{N} + 1). \right.
$$

(2.15)

Proof. By the shifting property of the annihilation and creation operators we have

$$
i \left[ \mathcal{N}, H^B(t) \right] = 2i \int dx \intdk K(t, k, x) a^*_k b^*_x + \text{h.c.}. \right.
$$

(2.16)

By the same estimates as in the proof of Lemma 2.1 we get (2.13). Note that

$$
\left| \mu(t) \right| = \frac{1}{2} \left\| \psi_t, \Phi \phi, \psi_t \right\|_{L^2(\mathbb{R}^3)} \leq C \left( \left\| \psi_t \right\|^2_{H^1(\mathbb{R}^3)} + \left\| \phi \right\|^2_{L^2(\mathbb{R}^3)} \right), \right.
$$

(2.17)

because of (2.7). Using (2.1) we get

$$
\int dx b^*_x \Phi(x) \left( \mathcal{N} + 1 \right) \leq \int dx \intdk G_x(k) \phi_t(k) \left\{ \int \frac{1}{1 + k^2} G_x(k) \phi_t(k) \right\} b_x \right.
$$

(2.18)
By the Cauchy–Schwarz inequality and Young’s inequality for products we obtain
\[
\pm \left( \int dx b_x^* h(t) b_x - T_b \right) \leq \varepsilon T_b + C \varepsilon \left( \|\psi_t\|^2_{H^1(\mathbb{R}^3)} + \|\varphi_t\|^2_{L^2(\mathbb{R}^3)} \right) N_b. \tag{2.19}
\]

Inequality (2.14) then follows from Lemma 2.1. Note that \( \frac{d}{dt} \Phi(t) \leq C \|\varphi_t\|_{L^2(\mathbb{R}^3)} \) because \( \frac{d}{dt} \Phi = 2\text{Im} \langle G_x \varphi_t \rangle_{L^2(\mathbb{R}^3)} \). Together with \( \|h(t)\psi_t\|_{L^2(\mathbb{R}^3)} \leq \|\psi_t\|_{H^2(\mathbb{R}^3)} + C \|\varphi_t\|_{L^2(\mathbb{R}^3)} \) we get
\[
\frac{d}{dt} \|h(t)\|_{L^2(\mathbb{R}^3)} \leq C \left( \|\psi_t\|^2_{H^2(\mathbb{R}^3)} + \|\varphi_t\|^2_{L^2(\mathbb{R}^3)} \right) \tag{2.20}
\]

Using
\[
\frac{d}{dt} K(t, k, x) = \hat{\psi}_t(x) \left( G_x(k) - \langle \psi_t, G_x(k) \psi_t \rangle_{L^2(\mathbb{R}^3)} \right)
+ \psi_t \left( \langle \hat{\psi}_t, G(k) \psi_t \rangle_{L^2(\mathbb{R}^3)} + \langle \hat{\psi}_t, G(k) \psi_t \rangle_{L^2(\mathbb{R}^3)} \right) \tag{2.21}
\]
and similar estimates as in the proof of Lemma 2.1 we obtain
\[
\pm \frac{d}{dt} \left( \int dx \int dk K(t, k, x) (a_k^* + a_{-k}) b_x^* + \text{h.c.} \right) \leq \varepsilon T_b + C \varepsilon \left( \|\hat{\psi}_t\|^2_{H^1(\mathbb{R}^3)} + \|\hat{\psi}_t\|_{L^2(\mathbb{R}^3)} \right) (N + 1). \tag{2.22}
\]

Since
\[
\|\hat{\psi}_t\|_{L^2(\mathbb{R}^3)} \leq \|\psi_t\|_{H^2(\mathbb{R}^3)} + C \|\varphi_t\|_{L^2(\mathbb{R}^3)} \quad \text{and} \quad \|\hat{\psi}_t\|_{H^1(\mathbb{R}^3)} \leq \|\psi_t\|_{H^3(\mathbb{R}^3)} + C \|\psi_t\|_{H^1(\mathbb{R}^3)} \|\varphi_t\|_{L^2(\mathbb{R}^3)} \tag{2.23}
\]
this proves (2.16). □

If we, in addition, use Lemma 2.2 we obtain similar inequalities for the Hamiltonian \( H(t) \).

**Lemma 2.4.** Let \( \varepsilon > 0 \) be arbitrary. Then, there exists a constant \( C \varepsilon > 1 \), depending on \( \varepsilon \), such that
\[
\pm i[N, H(t)] \leq \varepsilon T_b + C \varepsilon \|\psi_t\|^2_{H^1(\mathbb{R}^3)} (N + 1 + N^{-1} (N_a + 1) N_b) \tag{2.24}
\]
\[
\pm (H(t) - T_b) \leq \varepsilon T_b + C \varepsilon \left( \|\psi_t\|^2_{H^1(\mathbb{R}^3)} + \|\varphi_t\|^2_{L^2(\mathbb{R}^3)} \right) (N + 1 + N^{-1} (N_a + 1) N_b) \tag{2.25}
\]
\[
\pm \frac{d}{dt} H(t) \leq \varepsilon T_b + C \varepsilon \left( \|\psi_t\|^2_{H^3(\mathbb{R}^3)} + \|\psi_t\|^2_{H^1(\mathbb{R}^3)} \|\varphi_t\|^2_{L^2(\mathbb{R}^3)} \right) (N + 1 + N^{-1} (N_a + 1) N_b). \tag{2.26}
\]

**Proof.** Using \( \left[1 - N^{-1} N_a^{1/2} \right]^2 \leq 1 \) the first three terms of \( H(t) \) on the right hand side of (1.29) can be estimated in exactly the same way as in Lemma 2.3. We consequently only have to consider the term with three annihilation and creation operators. Since
\[
\left[ N, \int dx b_x^* \left( q(t) \hat{\Phi} q(t) - \langle \psi_t, \hat{\Phi} \psi_t \rangle_{L^2(\mathbb{R}^3)} \right) b_x \right]
= \int dx b_x^* \left( a^*(G.) - a(G.) \right) q(t) - \langle \psi_t, (a^*(G.) - a(G.)) \psi_t \rangle \right) b_x \tag{2.27}
\]
we obtain (2.24) by similar estimates as in the proof of Lemma 2.2 and (2.13). Inequality (2.14) in combination with Lemma 2.2 leads to (2.25). Using \( \dot{\psi}(t; x; y) = -\dot{\psi}(x)\overline{\psi}(y) - \dot{\psi}(x)\psi(y) \) we compute

\[
\frac{d}{dt} \int dx \, b_x^* \left( q(t) \hat{\Phi} q(t) - \left< \psi_t, \hat{\Phi} \psi_t \right>_{L^2(\mathbb{R}^3)} \right) b_x
\]

\[
= - \left< b^* (\psi_t) \int dy \, \overline{\psi}(y) (G_y) \psi_t, b^* (\psi_t) \int dy \, \overline{\psi}(y) \hat{\Phi} (G_y) b_y + \text{h.c.} \right>
\]

\[
+ b^* (\psi_t) \sum_{\vec{x} \neq \{0, \cdots, n \}} a^2 \left( \left< \psi_t, \hat{G} \psi_t \right>_{L^2(\mathbb{R}^3)} \right) b(\psi_t) + \text{h.c.}
\]

\[
+ \sum_{\vec{x} \neq \{0, \cdots, n \}} \left( b^* (\psi_t) \left( a^2 \left( \left< \psi_t, \hat{G} \psi_t \right>_{L^2(\mathbb{R}^3)} \right) + a^2 \left( \left< \psi_t, \hat{G} \psi_t \right>_{L^2(\mathbb{R}^3)} \right) \right) b(\psi_t)
\]

\[
- \frac{d}{dt} a^2 \left( \left< \psi_t, \hat{G} \psi_t \right>_{L^2(\mathbb{R}^3)} \right) \left( N_\psi \right).
\]

By means of (A.1) and integration by parts, i.e. the commutator method of Lieb and Yamazaki [34], we obtain

\[
\pm (2.28) \leq \varepsilon (N_\psi + T_b) + C \varepsilon^{-1} \left( \| \psi_t \|^2_{H^1(\mathbb{R}^3)} + \| \dot{\psi}_t \|^2_{L^2(\mathbb{R}^3)} \| \dot{\psi}_t \|^2_{L^1(\mathbb{R}^3)} \right) (N_\psi + 1) N_\psi.
\]

Using (2.7) we get

\[
\pm (2.29) \leq C \| \psi_t \|^2_{H^1(\mathbb{R}^3)} \| \dot{\psi}_t \|^2_{L^2(\mathbb{R}^3)} (N_\psi + 1)^{1/2} N_\psi
\]

and

\[
\pm (2.30) \leq C \left( \| \psi_t \|^2_{H^1(\mathbb{R}^3)} \| \dot{\psi}_t \|^2_{L^2(\mathbb{R}^3)} \| \dot{\psi}_t \|^2_{L^1(\mathbb{R}^3)} \right) (N_\psi + 1)^{1/2} N_\psi.
\]

Together with (2.20) this leads to

\[
\pm N^{-1/2} \frac{d}{dt} \int dx \, b_x^* \left( q(t) \hat{\Phi} q(t) - \left< \psi_t, \hat{\Phi} \psi_t \right>_{L^2(\mathbb{R}^3)} \right) b_x
\]

\[
\leq \varepsilon T_b + C \varepsilon \left( \| \psi_t \|^2_{H^1(\mathbb{R}^3)} \| \dot{\psi}_t \|^2_{L^2(\mathbb{R}^3)} \| \dot{\psi}_t \|^2_{L^1(\mathbb{R}^3)} \right) (N_\psi + N^{-1} (N_\psi + 1) N_\psi).
\]

Thus if we combine this estimate with (2.16) we obtain (2.26).

3

3 Proofs

In this section we prove the main results of the article. We start with Theorem [1.1]. Afterwards, we discuss the well-posedness of the Bogoliubov dynamics, introduce for technical reasons a Bogoliubov evolution which is truncated in the total number of excitations and finally derive Theorem [1.3]. It is convenient to consider solutions \( \chi_{\leq N}(t) \) of the Schrödinger equation (1.28) rather on \( \mathcal{G} \) than on the time dependent truncated excitation space \( \mathcal{G}_{\leq N} \). We therefore define \( \chi(t) \in \mathcal{G} \) by \( \chi \)

\[
\chi^{(k)}(t) = \begin{cases} 
\chi^{(k)}_{\leq N}(t) & \text{if } k \in \{1, 2, \ldots, N\}, \\
0 & \text{else}
\end{cases}
\]

which satisfies the Schrödinger equation

\[
i \partial_t \chi(t) = H(t) \chi(t) \quad \text{with} \quad \chi^{(k)}(0) = \begin{cases} 
(U_N(0) \Psi_N,0)^{(k)} & \text{if } k \in \{1, 2, \ldots, N\}, \\
0 & \text{else}
\end{cases}
\]

\[
\text{Note that we refrain from indicating the dependence of } \chi(t) \text{ on } N \text{ to simplify the notation.}
\]
3.1 Convergence of reduced density matrices

Proof of Theorem (1.7) Note that
\[ \|\psi_t\|_{L^2(\mathbb{R}^3)} + \|\varphi_t\|_{L^2(\mathbb{R}^3)} \leq 2 (\mathcal{E}[\psi_t, \varphi_t] + C) = 2 (\mathcal{E}[\psi, \varphi] + C) \leq C (\|\psi\|_{H^1(\mathbb{R}^3)}^2 + \|\varphi\|_{L^2(\mathbb{R}^3)}^2) \] (3.3)
holds because of (2.7) and the conservation of energy, see Proposition (1.1). According to Lemma 2.4 there exists \( \tilde{C} > 0 \) which only depends on \( \mathcal{E}[\psi, \varphi] \) such that the operator
\[ A(t) = H(t) + \tilde{C} (N + 1) \] (3.4)
satisfies
\[ \mathbb{1}_{N_b \leq N} (N + T_b + 1) \mathbb{1}_{N_b \leq N} A(t) \mathbb{1}_{N_b \leq N} \leq 2 \mathbb{1}_{N_b \leq N} A(t) \mathbb{1}_{N_b \leq N}, \] (3.5)
\[ \mathbb{1}_{N_b \leq N} A(t) \mathbb{1}_{N_b \leq N} \leq 3\tilde{C} \mathbb{1}_{N_b \leq N} (N + T_b + 1) \mathbb{1}_{N_b \leq N}, \] (3.6)
\[ \pm \mathbb{1}_{N_b \leq N} i [H(t), A(t)] \mathbb{1}_{N_b \leq N} \leq \tilde{C} \mathbb{1}_{N_b \leq N} A(t) \mathbb{1}_{N_b \leq N}, \] (3.7)
\[ \pm \mathbb{1}_{N_b \leq N} \frac{d}{dt} A(t) \mathbb{1}_{N_b \leq N} \leq \left( \|\psi_t\|_{H^1(\mathbb{R}^3)} + \|\varphi_t\|_{L^2(\mathbb{R}^3)}^2 \right) \mathbb{1}_{N_b \leq N} A(t) \mathbb{1}_{N_b \leq N}. \] (3.8)

Let \( \Psi_{N,t} \) be the solution of the Schrödinger equation of Theorem 1.1 and \( \chi(t) \) be defined as in (3.1). By means of (3.2), (3.7), (3.8) and \( \chi(t) = \mathbb{1}_{N_b \leq N} \chi(t) \) we estimate
\[ \left| \frac{d}{dt} \langle \chi(t), A(t) \chi(t) \rangle \right| \leq \left| \langle \chi(t), A(t) \chi(t) \rangle \right| + \left| \langle \chi(t), [H(t), A(t)] \chi(t) \rangle \right| \] (3.9)
\[ \leq \tilde{C} \left( \|\psi_t\|_{H^1(\mathbb{R}^3)} + \|\varphi_t\|_{L^2(\mathbb{R}^3)}^2 \right) \langle \chi(t), A(t) \chi(t) \rangle. \] (3.10)

Using Gronwall’s lemma we get
\[ \langle \chi(t), A(t) \chi(t) \rangle \leq e^{\tilde{C} \int_0^t ds \left( \|\psi_s\|_{H^1(\mathbb{R}^3)} + \|\varphi_s\|_{L^2(\mathbb{R}^3)}^2 \right)} \langle \chi(0), A(0) \chi(0) \rangle. \] (3.11)

Inequalities (3.9) and (3.10) then lead to
\[ \langle \chi(t), (N + T_b + 1) \chi(t) \rangle \leq 6\tilde{C} e^{\tilde{C} \int_0^t ds \left( \|\psi_s\|_{H^1(\mathbb{R}^3)} + \|\varphi_s\|_{L^2(\mathbb{R}^3)}^2 \right)} \langle \chi(0), (N + T_b + 1) \chi(0) \rangle. \] (3.12)

Note that
\[ \langle \chi(t), (N + T_b + 1) \chi(t) \rangle = \langle \chi_{\leq N}(t), (N_a + N_b + T_b + 1) \chi_{\leq N}(t) \rangle_{L^2_{\mathbb{R}^3} \otimes L^2_{\mathbb{R}^3}} \] (3.13)
and that the unitary mapping (1.20) can be written as \( U_N(t) = \tilde{U}_N(t) \otimes W^*(\sqrt{N} \varphi_t) \) where \( \tilde{U}_N(t) \) is the excitation map from Chapter 2.5. Since \( [\tilde{U}_N(t) \otimes \mathbb{1}_{L^2(\mathbb{R}^3)} \otimes_{L^2(\mathbb{R}^3)} \otimes W^*(\sqrt{N} \varphi_t)] = 0 \) we have
\[ \langle \chi_{\leq N}(t), N_a \chi_{\leq N}(t) \rangle_{L^2_{\mathbb{R}^3} \otimes L^2_{\mathbb{R}^3}} = \langle \Psi_{N,t}, W(\sqrt{N} \varphi_t)N_a W^*(\sqrt{N} \varphi_t) \Psi_{N,t} \rangle_{L^2_{\mathbb{R}^3}} \] (3.14)
and
\[ U_N(t) \psi(f) g(t) U_N(t)^* = \psi(f) g(t) \text{ for all } f, g \in L^2_{\mathbb{R}^3}(\mathbb{R}^3) \] (3.15)
by means of (3.3) Proposition 4.2. For the reduced density \( \gamma_{\leq N}(t) \) with integral kernel
\[ \gamma_{\leq N}(t)(x, y) = N^{-1} \langle \chi_{\leq N}(t), b_x^* b_y \chi_{\leq N}(t) \rangle \] (3.16)
\footnote{We refer to [13 Lemma A.1]] for a thorough introduction to \( U_N(t) \) and its properties.}
Likewise one can replace 

and 

proceeds in analogy to the proof of [32, Theorem 8] and considers a regularized version of

Using

and 

like to remark that it is rather formal because the second term on the right hand side of

Bogoliubov equation

Moreover, there exists a constant

Lemma 3.2.

introduce a truncated Bogoliubov dynamics which will be used in the proof of Theorem 1.4.

We start with commenting on the well-posedness of the Bogoliubov dynamics. Afterwards, we will

Next, we are going to show that (1.38) approximates the time evolved many-body state in norm.

3.2 Well-posedness of the Bogoliubov dynamics

Combining (3.12) with (3.13), (3.14), (3.18) and (3.20) proves (1.20) and (1.21).

Remark 3.1. The derivation of (3.12) from above is in our opinion the most insightful but we would like to remark that it is rather formal because the second term on the right hand side of (3.9) is not well defined for all \( \chi(t) \in \mathcal{D}((\sum_{j=1}^{N} - \Delta_j + N_\gamma)^{1/2}) \). A rigorous derivation is obtained if one proceeds in analogy to the proof of [32, Theorem 8] and considers a regularized version of (3.9). Likewise one can replace \( H(t) \) in (3.2) by \( 1_{N_\gamma < N} H(t) 1_{N_\gamma < N} \) and directly apply [32, Theorem 8] with

\( A = 1_{N_\gamma < N} (N + T_b) 1_{N_\gamma < N} + 1 \) and \( B = 1_{N_\gamma < N} N 1_{N_\gamma < N} + 1 \).

3.2 Well-posedness of the Bogoliubov dynamics

Next, we are going to show that (1.38) approximates the time evolved many-body state in norm. We start with commenting on the well-posedness of the Bogoliubov dynamics. Afterwards, we will introduce a truncated Bogoliubov dynamics which will be used in the proof of Theorem 1.4.

Lemma 3.2. Let \((\psi, \varphi) \in H^3(\mathbb{R}^3) \times L^2(\mathbb{R}^3)\) such that \( \|\psi\|_{L^2(\mathbb{R}^3)} = 1 \) and \((\psi_t, \varphi_t)\) be the unique solution of (1.10) with initial datum \((\psi, \varphi)\). For every \( \chi \in \mathcal{G} \cap Q(N + T_b) \) there exists a unique solution to the Bogoliubov equation (1.34) with \( \chi_B(0) = \chi \) such that \( \chi_B \in C^0([0, \infty) \cap \mathcal{G}) \cap L^\infty_{loc}([0, \infty) \cap Q(N + T_b)). \) Moreover, there exists a constant \( C > 0 \) depending only on \( E[\psi, \varphi] \) such that

\[
\langle \chi_B(t), (N + T_b + 1) \chi_B(t) \rangle \leq C e^{\int_0^t ds \left( \|\psi_s\|_{H^3(\mathbb{R}^3)}^2 + \|\varphi_s\|_{L^2(\mathbb{R}^3)}^2 \right)} \langle \chi, (N + T_b + 1) \chi \rangle
\]

and the condition \( \chi \in \mathcal{G} \) implies \( \chi_B(t) \in \mathcal{G} \), for all \( t \in \mathbb{R} \).

Proof. Let \( A = N + T_b + 1 \) and \( B = N + 1 \). By Lemma 2.3 and (3.3) there exists a constant \( C > 0 \) depending only on \( E[\psi, \varphi] \) such that

\[
C^{-1} A - C B \leq H(t) (N + T_b + 1) \chi_B(t) \leq C A, \\
\pm i[H(t), B] \leq CA, \\
\frac{d}{dt}|H(t) \leq C \left( \|\psi_t\|_{H^3(\mathbb{R}^3)}^2 + \|\varphi_t\|_{L^2(\mathbb{R}^3)}^2 \right) A.
\]
The statement of Lemma 3.2 until \[3.21\] then follows from \([32, \text{Theorem } 8]\). Note that the time dependence of \([3.22]\) must be tracked in the proof of \([32, \text{Theorem } 8]\) to obtain the explicit form of the exponent in \([3.21]\). Let us define \(\Gamma_t : \mathcal{G} \to \mathcal{G}_L \psi_t\) by \(\Gamma_t|_{\mathcal{F}_B(t) \otimes \mathcal{F}_A} = q(t)^{\otimes_7} \otimes \mathbb{1}_{\mathcal{F}_A}\) and compute
\[
\frac{d}{dt} \left| \langle \chi_B(t), i \left[H^B(t), \Gamma_t \right] \chi_B(t) \rangle \right| + \left| \langle \chi_B(t), \dot{\Gamma}_t H^B(t) \rangle \right| = 0. \tag{3.23}
\]
Here, we have used that the relations
\[
\dot{\Gamma}_t = -b^*(\psi_t)b(q(t)\dot{\psi}_t)\Gamma_t - \Gamma_t b^*(q(t)\dot{\psi}_t)b(\psi_t) \tag{3.24}
\]
and
\[
i \left[H^B(t), \Gamma_t \right] = b^*(\psi_t)b(q(t)\dot{\psi}_t)\Gamma_t + \Gamma_t b(q(t)\dot{\psi}_t)b(\psi_t) \tag{3.25}
\]
can obtained (in analogy to \([7, \text{p. 1588}]\)) by a direct calculation on the Fock space sector with \(k\) particles. This leads to \(\| (1 - \Gamma_t) \chi_B(t) \| = \| (1 - \Gamma_0) \chi_B(0) \|^2\) and shows that \(\chi \in \mathcal{G}_L \psi_t\) implies \(\chi_B(t) \in \mathcal{G}_L \psi_t\) for all \(t \in \mathbb{R}\).

As a technical tool we introduce (for \(M \in \mathbb{N}\) arbitrary but fixed) the truncated Bogoliubov dynamics
\[
i \tilde{\xi}_t \chi_{B,M}(t) = 1_{N \leq M} H^B(t) 1_{N \leq M} \chi_{B,M}(t), \quad \chi_{B,M}(0) = 1_{N \leq M} \chi_B(0). \tag{3.26}
\]
In the following, we use the shorthad notations \(1_{M} = 1_{N \leq M}\) and \(1_{M} = 1_{N > M}\). The truncated dynamics satisfies the same existence result as the original Bogoliubov equation.

**Lemma 3.3.** Let \((\psi, \varphi) \in H^3(\mathbb{R}^3) \times L^2(\mathbb{R}^3)\) such that \(\| \psi \|_{L^2(\mathbb{R}^3)} = 1\) and \((\psi_t, \varphi_t)\) be the unique solution of \([1.10]\) with initial datum \((\psi, \varphi)\). For every \(\chi \in \mathcal{G} \cap Q(\mathcal{N} + T_b)\) there exists a unique solution to the truncated Bogoliubov equation \([3.26]\) with \(\chi_B(0) = \chi\) such that \(\chi_{B,M} \in C^0([0, \infty) \cap \mathcal{G}) \cap L^\infty_{loc}([0, \infty); Q(\mathcal{N} + T_b))\). Furthermore, \(1_{N > M} \chi_{B,M}(t) = 0\) holds for all \(t \in \mathbb{R}\) and \(\chi \in \mathcal{G}_L \psi_t\) implies \(\chi_{B,M}(t) \in \mathcal{G}_L \psi_t\) for all \(t \in \mathbb{R}\).

In addition, assume the existence of a constant \(\tilde{C} > 0\) such that \(\| (\mathcal{N}^{3/2} + T_b^{1/2} + 1) \chi \| \leq \tilde{C}\). Then, there exists a constant \(C > 0\) (depending only on \(\tilde{C}\) and \(\mathcal{E}[\psi, \varphi]\)) such that
\[
\sup \left\{ \left\| (\mathcal{N} + T_b + 1)^{1/2} \chi_{B,M}(t) \right\|, M^{-1/4} \left\| (\mathcal{N} + 1) \chi_{B,M}(t) \right\|, M^{-5/8} \left\| (\mathcal{N} + 1)^{3/2} \chi_{B,M}(t) \right\| \right\} \leq Ce^{Cf(t)} \tag{3.27}
\]
with \(f(t) = \int_0^t ds \left( \| \psi_t \|_{H^3(\mathbb{R}^3)} + \| \varphi_s \|_{L^2(\mathbb{R}^3)} \right)\).

**Proof.** Note that \(1_{N \leq M} \chi_B(t) \) satisfies the same estimates \([3.22]\) as \(H^B(t)\) if one replaces \(A\) and \(B\) by \(1_{N \leq M}(\mathcal{N} + T_b)1_{N \leq M} + 1\) and \(1_{N \leq M}N_{1 \leq M} + 1\). By \([32, \text{Theorem } 8]\) it follows that for every \(\chi \in \mathcal{G} \cap Q(\mathcal{N} + T_b)\) there exists a unique solution to the truncated Bogoliubov equation \([3.26]\) with \(\chi_B(0) = \chi\) such that \(\chi_{B,M} \in C^0([0, \infty) \cap \mathcal{G}) \cap L^\infty_{loc}([0, \infty); Q(1_{N \leq M}(\mathcal{N} + T_b))1_{N \leq M})\) and \(\| (\mathcal{N} + T_b)^{1/2} 1_{N \leq M} \chi_B(t) \| \leq Ce^{Cf(t)}\). Since \(\frac{d}{dt} \| 1_{N > M} \chi_{B,M}(t) \|^2 = 0\) we get \(1_{N > M} \chi_{B,M}(t) = 0\) for all \(t \in \mathbb{R}\). Together with the previous inequality this implies \(\| (\mathcal{N} + T_b + 1)^{1/2} \chi_{B,M}(t) \| \leq Ce^{Cf(t)}\). Using \([1_{N \leq M}, \Gamma_t]\) we conclude by similar means as in the proof of Lemma \([3.2]\) that \(\chi_{B,M}(t) \in \mathcal{G}_L \psi_t\), if \(\chi \in \mathcal{G}_L \psi_t\). In total, this proves the first part of the lemma. Below we will prove
\[
\left\| (\mathcal{N} + 1)^{3/2} \chi_{B,M}(t) \right\|^2 \leq C \int_0^t ds \left[ \| \psi_t \|^2 \| \mathcal{N} + 1 \chi_{B,M}(s) \|^2 \right] + \int_0^t ds \left( \Lambda \| (\mathcal{N} + 1)^{1/2} \chi_{B,M}(s) \|^2 + \Lambda^{-1} M_{k-1} \| (\mathcal{N} + T_b + 1)^{3/2} \chi_{B,M}(s) \|^2 \right) \tag{3.28}
\]

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for \( k \in \mathbb{N} \) satisfying \( k \geq 2 \) and \( \Lambda \geq 1 \). Choosing \( \Lambda = M^{\frac{1}{2}} \) for \( k = 2 \) and \( \Lambda = M^{\frac{1}{2}} \) for \( k = 3 \) proves (3.27). It remains to show (3.25). With this regard note that \( \{N, 1_{N \leq M}\} = 0 \) and that \( \mathcal{N} \) is a bounded operator on the subspace \( \{\chi \in \mathcal{F}; 1_{N > M}\chi = 0\} \). Using the shifting properties of \( \mathcal{N} = \mathcal{N}_a + \mathcal{N}_b \) we calculate

\[
\frac{d}{dt} \left( (N+1)^{\frac{3}{2}} \chi_{B,M}(t) \right)^2
= i \left\langle \chi_{B,M}(t), H^B(t), (N+1)^k \chi_{B,M}(t) \right\rangle
= 2\text{Im} \left\langle \chi_{B,M}(t), (N+3)^k - (N+1)^k \int dx \int dk \psi_t(x) \overline{\langle \psi_t, G(k) \psi_t \rangle_{L^2(\mathbb{R}^3)}} a_k b_x \chi_{B,M}(t) \right\rangle
- 2\text{Im} \left\langle \chi_{B,M}(t), (N+3)^k - (N+1)^k \int dx \int dk \psi_t(x) G_x(k) a_k b_x \chi_{B,M}(t) \right\rangle.
\]

(3.29)

Using again the shifting property of the number operator, (2.7) and the Cauchy–Schwarz inequality we bound the first term by

\[
2 \left\| (N+3)^k - (N+1)^k \chi_{B,M}(t) \right\| \leq C \left\| \psi_t \right\|_{H^1(\mathbb{R}^3)} \left\| (N+1)^{\frac{3}{2}} \chi_{B,M}(t) \right\|^2.
\]

(3.30)

To obtain the ultimate inequality we have, in addition, used that

\[
\left\| (N+3)^k - (N+1)^k \chi_{B,M}(t) \right\| \leq 2k \left\| (N+3)^{k-1} \chi_{B,M}(t) \right\|
\]

(3.32)

holds by the spectral theorem because \( |y^k - x^k| \leq ky^{k-1}(y-x) \) for all \( y \geq x \geq 0 \). Next, we write (3.30) as

\[
-2\text{Im} \left\langle \chi_{B,M}(t), (N+3)^k - (N+1)^k \int dx \int_{|k| \leq \Lambda} dk \psi_t(x) G_x(k) a_k b_x (N+1)^{\frac{k-1}{2}} \chi_{B,M}(t) \right\rangle
- 2\text{Im} \left\langle \chi_{B,M}(t), (N+3)^k - (N+1)^k \int dx \int_{|k| \geq \Lambda} dk \psi_t(x) G_x(k) a_k b_x \chi_{B,M}(t) \right\rangle.
\]

(3.34)

Using the second inequality of Lemma 2.1 (note that the two summands on the left hand side of (2.9) are estimated separately) let us bound the first summand by

\[
\left\| (N+3)^k - (N+1)^k \chi_{B,M}(t) \right\| \leq C \left\| (N+1)^{\frac{3}{2}} \chi_{B,M}(t) \right\|^2 + \left\| \psi_t \right\|_{H^1(\mathbb{R}^3)} \left\| (N+1)^{\frac{k-1}{2}} \chi_{B,M}(t) \right\| + \left\| \psi_t \right\|_{H^1(\mathbb{R}^3)} \left\| (N+1)^{\frac{k}{2}} \chi_{B,M}(t) \right\|
\]

(3.35)
By the first inequality of Lemma 2.1 and (3.32) we obtain
\begin{align*}
\| \chi_B(t) - \chi_B(t) \| & \leq C \| \psi_0 \|_{H^1(\mathbb{R}^3)} \Lambda^{-\frac{1}{2}} \bigg( (N + T_b) \frac{1}{2} \chi_B(t) \bigg) \bigg( (N_a + 1)^{\frac{1}{2}} \left( (N + 3)^k - (N + 1)^k \right) \chi_B(t) \bigg) \\
& \leq C \| \psi_0 \|_{H^1(\mathbb{R}^3)} \Lambda^{-\frac{1}{2}} \bigg( (N + T_b) \frac{1}{2} \chi_B(t) \bigg) \bigg( (N + 1)^{\frac{k-1}{2}} \chi_B(t) \bigg) \\
& \leq C \| \psi_0 \|_{H^1(\mathbb{R}^3)} \Lambda^{-\frac{1}{2}} \bigg( (N + T_b) \frac{1}{2} \chi_B(t) \bigg) \bigg( (N + 1)^{\frac{k-1}{2}} \chi_B(t) \bigg) \\
& \leq C \| \psi_0 \|_{H^1(\mathbb{R}^3)} \Lambda^{-\frac{1}{2}} \bigg( (N + T_b) \frac{1}{2} \chi_B(t) \bigg) \bigg( (N + 1)^{\frac{k-1}{2}} \chi_B(t) \bigg). 
\end{align*}
In total, we get
\begin{align}
\frac{d}{dt} \left( (N + 1)^{\frac{k}{2}} \chi_B(t) \right) & \leq C \| \psi_0 \|_{H^1(\mathbb{R}^3)} \left( (N + 1)^{\frac{k}{2}} \chi_B(t) \right) + \Lambda^{1} \left( (N + 1)^{\frac{k-1}{2}} \chi_B(t) \right) \\
& \quad + \Lambda^{-1} (N + 1)^{\frac{k-1}{2}} \chi_B(t). 
\end{align}
Inequality (3.28) then follows by Gronwall’s lemma.

The following Lemma compares the Bogoliubov dynamics to the one with cutoff in the total number of particles.

**Lemma 3.4.** Let $M \in \mathbb{N}$ such that $M \geq 3$, $\tilde{C} > 0$ and $\chi \in \mathcal{G}$ such that $\| (\Lambda^{3/2} + T_b^{1/2} + 1) \chi \| \leq \tilde{C}$. Let $\chi_B(t)$ and $\chi_{B,M}(t)$ be the unique solutions of (1.31) and (3.29) with $\chi_B(0) = \chi$. Then, there exists a constant $C > 0$ (depending only on $\tilde{C}$ and $\mathcal{E}[\psi, \varphi]$) such that
\begin{equation}
\| \chi_B(t) - \chi_{B,M}(t) \| \leq C e^{C f(t)} M^{-3/8}. 
\end{equation}
**Proof.** We have
\begin{align}
\frac{d}{dt} \| \chi_B(t) - \chi_{B,M}(t) \|^2 & = 2 \text{Im} \left( \langle \chi_B(t), (B_B(t) - 1_{\leq M} B_B(t) 1_{\leq M}) \chi_{B,M}(t) \rangle \right) \\
& = 2 \text{Im} \left( \langle \chi_B(t), 1_{\geq M} \int dx \int dk K(t, k, x) a^*_k b_k \|_{\leq M} \chi_{B,M}(t) \rangle \right) \\
& = 2 \text{Im} \left( \langle \chi_B(t), 1_{\geq M} \int dx \int dk K(t, k, x) a^*_k b_k \|_{M-2 \leq N \leq M} \chi_{B,M}(t) \rangle \right) 
\end{align}
because $\chi_{B,M}(t) = 1_{\leq M} \chi_{B,M}(t)$ and other contributions of $B_B(t)$ map $\{ \chi \in \mathcal{G} : 1_{N>\geq M} \chi = 0 \}$ into itself. Note that
\begin{equation}
\left| \langle \chi, \int dx \int dk K(t, k, x) a^*_k b_k \varphi \rangle \right| \leq C \| \psi \|_{H^1} \left( (T_b + N_b)^{1/2} \chi \right) \left( (N_a + 1)^{1/2} \Psi \right) \end{equation}
can be shown in complete analogy to (2.10). Together with $\| (T_b + N_b) 1_{N>M} \|$, Lemma 3.2 and Lemma 3.3 we get
\begin{align}
\frac{d}{dt} \| \chi_B(t) - \chi_{B,M}(t) \|^2 & \leq C \| \psi \|_{H^1} \left( (T_b + N_b)^{1/2} 1_{N>M} \chi_B(t) \right) \left( (N_a + 1)^{1/2} 1_{M-2 \leq N \leq M} \chi_B(t) \right) \\
& \leq C \| \psi \|_{H^1} \left( (T_b + N_b)^{1/2} \chi_B(t) \right) \left( (N_a + 1)^{1/2} 1_{M-2 \leq N \leq M} \chi_B(t) \right) \\
& \leq C (M-2)^{-1} \| \psi \|_{H^1} \left( (T_b + N_b)^{1/2} \chi_B(t) \right) \left( (N_a + 1)^{3/2} \chi_B(t) \right) \\
& \leq C e^{C f(t)} M^{-3/8}. 
\end{align}
Here, $f(t) = \int_0^t ds \left( \| \psi_s \|_{H^3(\mathbb{R}^3)}^2 + \| \varphi_s \|_{L^2(\mathbb{R}^3)}^2 \right)$ and $C$ depends only on $\tilde{C}$ and $\mathcal{E}[\psi, \varphi]$. Using
\begin{equation}
\| \chi_B(0) - \chi_{B,M}(0) \|^2 \leq M^{-3} \| 1_{N>M} \chi \|^2 \leq \tilde{C} M^{-3}
\end{equation}
and Duhamel’s formula shows the claim.
3.3 Norm approximation

Proof of Theorem 3.2. Since $\chi \in G_{\perp \psi_0}$ we have $\chi_{\leq N}(t), (\chi_B^{(k)}(t))_{k=0}^N \in G_{\perp \psi_i}$ for all $t \in \mathbb{R}$. Using (1.27) and (1.38) we estimate

$$
\|\Psi_{N,t} - \Psi_{N,t}^B\|_{2^N(N)} = \left\| \left( \chi_{\leq N}(t) - \chi_B^{(k)}(t) \right)_{k=0}^N \right\|_{G_{\leq N}} \\
\leq \|\chi(t) - \chi_B(t)\|_G \\
\leq \|\chi(t) - \chi_{B,M}(t)\| + \|\chi_B(t) - \chi_{B,M}(t)\| \tag{3.43}
$$

with $\chi(t)$ and $\chi_{B,M}(t)$ being defined as in (3.1) and (3.20). Because of (3.2) and $\chi_{B,M}(t) = \mathbb{1}_{\leq M} \chi_{B,M}(t)$ we get

$$
\frac{d}{dt} \|\chi(t) - \chi_{B,M}(t)\|^2 = 2\text{Im} \left\langle \chi(t), (H(t) - \mathbb{1}_{\leq M} H_B(t) \mathbb{1}_{\leq M}) \chi_{B,M}(t) \right\rangle \\
= 2\text{Im} \left\langle \chi(t), (H(t) - \mathbb{1}_{\leq M} H_B(t)) \mathbb{1}_{\leq M} \chi_{B,M}(t) \right\rangle. \tag{3.44}
$$

Note that

$$
H(t) - \mathbb{1}_{\leq M} H_B(t) \\
= \mathbb{1}_{> M} \left( \int dx b_x^* b(t)b_x + \mathcal{N}_b \right) \tag{3.45a} \\
+ \mathbb{1}_{> M} \left( \int dx \int dk K(t, k, x)(a_k^* + a_{-k}) b_x^* \left[ 1 - N^{-1} \mathcal{N}_b \right]_{1^+}^{1/2} + \text{h.c.} \right) \tag{3.45b} \\
+ \mathbb{1}_{\leq M} \left( \int dx \int dk K(t, k, x)(a_k^* + a_{-k}) b_x^* \left[ 1 - N^{-1} \mathcal{N}_b \right]_{1^+}^{1/2} - 1 \right) + \text{h.c.} \tag{3.45c} \\
+ N^{-1/2} \int dx b_x^* \left( q(t) \hat{\Phi} q(t) - \left\langle \psi_t, \hat{\Psi} \psi_t \right\rangle_{L^2(\mathbb{R}^3)} \right) b_x. \tag{3.45d}
$$

The contribution from (3.45a) vanishes because the operators in the brackets leave the total number of excitations invariant. We, moreover, have

$$
(3.45b) \mathbb{1}_{\leq M} = \mathbb{1}_{> M} \int dx \int dk K(t, k, x)(a_k^* + a_{-k}) b_x^* \left[ 1 - N^{-1} \mathcal{N}_b \right]_{1^+}^{1/2} \mathbb{1}_{M-1 \leq N \leq M}. \tag{3.46}
$$

This leads to

$$
\frac{1}{2} \frac{d}{dt} \|\chi(t) - \chi_{B,M}(t)\|^2 = \text{Im} \left\langle \chi(t), \mathbb{1}_{> M} \int dx \int dk K(t, k, x)(a_k^* + a_{-k}) b_x^* \left[ 1 - N^{-1} \mathcal{N}_b \right]_{1^+}^{1/2} \mathbb{1}_{M-1 \leq N \leq M} \chi_{B,M}(t) \right\rangle \tag{3.47a} \\
+ \text{Im} \left\langle \chi(t), \mathbb{1}_{\leq M} \left( \int dx \int dk K(t, k, x)(a_k^* + a_{-k}) b_x^* \right. \right. \tag{3.47b} \right. \\
\times \left. \left. \left[ 1 - N^{-1} \mathcal{N}_b \right]_{1^+}^{1/2} - 1 \right) + \text{h.c.} \right\rangle \chi_{B,M}(t) \right\rangle \tag{3.47c} \\
+ N^{-1/2} \text{Im} \left\langle \chi(t), \int dx b_x^* \left( q(t) \hat{\Phi} q(t) - \left\langle \psi_t, \hat{\Psi} \psi_t \right\rangle_{L^2(\mathbb{R}^3)} \right) b_x \chi_{B,M}(t) \right\rangle. \tag{3.47c}
$$

In the following we estimate each term separately.
The term \((3.47a)\) Using Lemma 2.1 we bound the first term by
\[
|3.47a| \leq C \|\psi_t\|_{H^1} \left\| (\mathcal{N}_b + T_b)^{1/2} \mathbb{1}_{M>\lambda} \chi(t) \right\| \left( |\mathcal{N}_a + 1\rangle + \frac{1}{2} \mathbb{1}_{M<\lambda} \chi(t) \right) \left( |(\mathcal{N}_a + 1)^{1/2} (1 - N^{-1} \mathcal{N}_b)^{1/2} \mathbb{1}_{M<\lambda} \chi(t) \right) \\
\leq C \|\psi_t\|_{H^1} \left\| (\mathcal{N}_b + T_b)^{1/2} \chi(t) \right\| \left( |\mathcal{N}_a + 1\rangle + \frac{1}{2} \mathbb{1}_{M<\lambda} \chi(t) \right) \left( |(\mathcal{N}_a + 1)^{1/2} \mathbb{1}_{M<\lambda} \chi(t) \right) \\
\leq CM^{-1} \|\psi_t\|_{H^1} \left\| (\mathcal{N}_b + T_b)^{1/2} \chi(t) \right\| \left( |\mathcal{N} + 1\rangle \right)^{3/2} \chi_B(t) \\
(3.48)
\]
By means of \((1.35), (3.12)\) and Lemma 3.3 we get
\[
|3.47a| \leq C e^{Cf(t)} \|\psi_t\|_{H^1(\mathbb{R}^3)} M^{-3/8}.
(3.49)
\]

The term \((3.47b)\) Note that
\[
|3.47b| \leq \left\langle \chi(t), \mathbb{1}_{M} \left( \int dk \langle \psi_t, G(k) \psi_t \rangle (a_k^+ + a_{-k}) b^*_k (\psi_t) \left( |1 - N^{-1} \mathcal{N}_b\rangle + \mathbb{1}_{M} \chi_B(t) \right) \right) \chi_B(t) \right\rangle \\
(3.50a)
\]
\[
+ \left\langle \chi(t), \mathbb{1}_{M} \int dx \int dk \psi_t(x) G_x(k) (a_k^+ + a_{-k}) b^*_k \left( |1 - N^{-1} \mathcal{N}_b\rangle + \mathbb{1}_{M} \chi_B(t) \right) \right\rangle \\
(3.50b)
\]
\[
+ \left\langle \chi(t), \mathbb{1}_{M} \int dx \int dk \psi_t(x) G_x(k) (a_k^+ + a_{-k}) b^*_k \left( |1 - N^{-1} \mathcal{N}_b\rangle + \mathbb{1}_{M} \chi_B(t) \right) \right\rangle \\
(3.50c)
\]
follows directly from the definition of \(K\). Due to \((2.2)\) and the shifting property of the number operator we have
\[
|3.50a| \leq C \|\psi_t\|_{H^1(\mathbb{R}^3)} \left\| (\mathcal{N} + 1)^{1/2} \chi(t) \right\| \left( |\mathcal{N} + 1\rangle \right)^{1/2} \left| \mathbb{1}_{M} \chi_B(t) \right| \\
+ C \|\psi_t\|_{H^1(\mathbb{R}^3)} \left\| (\mathcal{N} + 1)^{1/2} \chi(t) \right\| \left( |\mathcal{N} + 1\rangle \right)^{1/2} \left| \mathbb{1}_{M} \chi_B(t) \right|.
(3.51)
\]
Using \((1 - N^{-1} \mathcal{N}_b)\rangle + \mathbb{1}_{M} \chi_B(t) \leq C N^{-1} \mathcal{N}_b\rangle + \mathbb{1}_{M} \chi_B(t) \chi_B(t) + \mathbb{1}_{M} \chi_B(t) \chi_B(t) \leq CN^{-1} \mathcal{N}_b\rangle + \mathbb{1}_{M} \chi_B(t) \chi_B(t)\), which is a consequence of \([1 - x]_+^2 - 1 \leq x\) for all \(x \geq 0\) and the spectral calculus, we obtain
\[
|3.50a| \leq C N^{-1} \|\psi_t\|_{H^1(\mathbb{R}^3)} \left\| (\mathcal{N} + 1)^{1/2} \chi(t) \right\| \left( |\mathcal{N} + 1\rangle \right)^{1/2} \chi_B(t).
(3.52)
\]
By means of Lemma 2.1 we estimate
\[
|3.50b| \leq C \|\psi_t\|_{H^1(\mathbb{R}^3)} \left\| (\mathcal{N}_b + T_b)^{1/2} \chi(t) \right\| \left( |\mathcal{N}_a + 1\rangle \right)^{1/2} \left( |1 - N^{-1} \mathcal{N}_b\rangle + \mathbb{1}_{M} \chi_B(t) \right) \\
\leq C N^{-1} \|\psi_t\|_{H^1(\mathbb{R}^3)} \left\| (\mathcal{N}_b + T_b)^{1/2} \chi(t) \right\| \left( |\mathcal{N} + 1\rangle \right)^{1/2} \chi_B(t).
(3.53)
\]
Before estimating \((3.50c)\) we shift the term involving the number operator to the right hand side of the scalar product and split the integral in \(k\) by means of a cutoff parameter \(\Lambda_1 \geq 1\). Applying \((2.8)\) and \((2.9)\) with \(\Lambda = \Lambda_1\) then leads to
\[
|3.50c| \leq C \|\psi_t\|_{H^1(\mathbb{R}^3)} \left\| (\mathcal{N}_a + 1)^{1/2} \chi(t) \right\| \left( |\Lambda_1\rangle \right)^{1/2} \left( |1 - N^{-1} \mathcal{N}_b\rangle + \mathbb{1}_{M} \chi_B(t) \right) \\
\leq C \|\psi_t\|_{H^1(\mathbb{R}^3)} \left\| (\mathcal{N} + 1)^{1/2} \chi(t) \right\| \left( |\mathcal{N}_a + 1\rangle \right)^{1/2} \chi_B(t) \\
\leq C \|\psi_t\|_{H^1(\mathbb{R}^3)} \left\| (\mathcal{N} + 1)^{1/2} \chi(t) \right\| \left( |\mathcal{N}_a + 1\rangle \right)^{1/2} \chi_B(t) \\
\leq C \|\psi_t\|_{H^1(\mathbb{R}^3)} \left\| (\mathcal{N} + 1)^{1/2} \chi(t) \right\| \left( |\mathcal{N}_a + 1\rangle \right)^{1/2} \chi_B(t) \\
\leq C \|\psi_t\|_{H^1(\mathbb{R}^3)} \left\| (\mathcal{N} + 1)^{1/2} \chi(t) \right\| \left( |\mathcal{N}_a + 1\rangle \right)^{1/2} \chi_B(t).
(3.54)
Summing up, we get
\begin{equation}
\|3.47b\| \leq C \|\psi_t\|_{H^1(\mathbb{R}^3)} N^{-1/2} \Lambda_1^{1/2} \left\| (\mathcal{N} + T_b + 1)^{1/2} \chi(t) \right\| \left\| (\mathcal{N} + 1)^{3/2} \chi_{B,M}(t) \right\| \\
+ C \|\psi_t\|_{H^1(\mathbb{R}^3)} \Lambda_1^{1/2} \left\| (\mathcal{N} + 1)^{1/2} \chi(t) \right\| \left\| (\mathcal{N}_0 + T_b)^{1/2} \chi_{B,M}(t) \right\|. \tag{3.55}
\end{equation}
Using (1.35), (3.12), as well as Lemma 3.3 and setting \( \Lambda_1 = N M^{-5/8} \) leads to
\begin{equation}
\|3.47b\| \leq C e^{Cf(t)} \|\psi_t\|_{H^1(\mathbb{R}^3)} \left( N^{-1} M^{5/8} \Lambda_1^{1/2} + \Lambda_1^{-1/2} \right) = C e^{Cf(t)} \|\psi_t\|_{H^1(\mathbb{R}^3)} N^{-1/2} M^{5/16}. \tag{3.56}
\end{equation}

The term (3.47c) By means of \( \chi_{B,M}(t) = 1_{\leq M} \chi_{B,M}(t) \), (2.11) with \( \Lambda = \Lambda_2 \geq 1 \), (1.35), (3.12) and Lemma 3.3 we get
\begin{equation}
\|3.47c\| \leq N^{-1/2} \left\| 1_{\leq M+1} \chi(t) \right\| \int dx b_x^2 \left\langle q(t) \hat{\Phi} q(t) - \langle \psi_t, \hat{\Phi} \psi_t \rangle \right\rangle b_x 1_{\leq M} \chi_{B,M}(t) \right\|
\leq C \|\psi_t\|_{H^1(\mathbb{R}^3)} N^{-1/2} \left[ \Lambda_2^{1/2} \left\| (\mathcal{N}_0 + T_b)^{1/2} \chi(t) \right\| \left\| (\mathcal{N}_0 + T_b)^{1/2} \chi_{B,M}(t) \right\| \\
+ \Lambda_2^{-1/2} \left\| (\mathcal{N}_0 + T_b)^{1/2} \chi(t) \right\| \left\| (\mathcal{N}_0 + T_b)^{1/2} \chi_{B,M}(t) \right\| \right]
\leq C \|\psi_t\|_{H^1(\mathbb{R}^3)} N^{-1/2} \left[ \Lambda_2^{1/2} \left\| (\mathcal{N}_0 + T_b)^{1/2} \chi(t) \right\| \left\| (\mathcal{N}_0 + T_b)^{1/2} \chi_{B,M}(t) \right\| \\
+ \Lambda_2^{-1/2} \left\| (\mathcal{N}_0 + T_b)^{1/2} \chi_{B,M}(t) \right\| \right]
\leq C e^{Cf(t)} \|\psi_t\|_{H^1(\mathbb{R}^3)} N^{-1/2} \left( M^{1/4} \Lambda_2^{1/2} + M^{1/2} \Lambda_2^{-1/2} \right). \tag{3.57}
\end{equation}
Choosing \( \Lambda_2 = M^{1/4} \) leads to
\begin{equation}
\|3.47c\| \leq C e^{Cf(t)} \|\psi_t\|_{H^1(\mathbb{R}^3)} N^{-1/2} M^{3/8}. \tag{3.58}
\end{equation}
If we collect the estimates and use (3.3) to bound the \( H^1 \)-norm of the condensate wave function we obtain
\begin{equation}
\frac{d}{dt} \|\chi(t) - \chi_{B,M}(t)\|^2 \leq C e^{Cf(t)} \left( M^{-3/8} + N^{-1/2} M^{3/8} \right). \tag{3.59}
\end{equation}
Since
\begin{equation}
\|\chi(0) - \chi_{B,M}(0)\| \leq \|\chi(0) - \chi\| + \|\chi - \chi_{B,M}(0)\| \leq \|1_{\mathcal{N} > N} \chi\| + \|1_{\mathcal{N} > M} \chi\| \leq C \left( N^{-3/2} + M^{-3/2} \right) \tag{3.60}
\end{equation}
we get
\begin{equation}
\|\chi(t) - \chi_{B,M}(t)\|^2 \leq C e^{Cf(t)} \left( M^{-3/8} + N^{-1/2} M^{3/8} + N^{-3} \right) \tag{3.61}
\end{equation}
by Duhamel’s formula. Plugging this estimate and (3.38) into (3.43) and choosing \( M = N^{2/3} \) concludes the proof.

A Proof of Lemma 2.1 and Lemma 2.2

Proof of Lemma 2.1 Using the commutator method of Lieb and Yamazaki [31], i.e.
\begin{equation}
(1 + |k|^2) G_x(k) = G_x(k) + (2\pi)^{-1} [k \cdot \nabla_x, G_x(k)] \tag{A.1}
\end{equation}
and integration by parts, we write the left hand side of the first inequality as

\[
\chi \langle dx \int_{|k| \geq \Lambda} dk \psi_t(x) G_x(k) (a_k^* + a_{-k}) b_x^* \Psi \rangle
= \int dx \int_{|k| \geq \Lambda} dk \frac{1}{1 + k^2} \langle b_x \chi, G_x(k) (a_k^* + a_{-k}) \psi_t(x) \Psi \rangle
+ \frac{1}{2\pi} \int dx \int_{|k| \geq \Lambda} dk \frac{k}{1 + k^2} \langle i \nabla_x b_x \chi, G_x(k) (a_k^* + a_{-k}) \psi_t(x) \Psi \rangle
- \frac{1}{2\pi} \int dx \int_{|k| \geq \Lambda} dk \frac{k}{1 + k^2} \langle b_x \chi, G_x(k) (a_k^* + a_{-k}) i \nabla_x \psi_t(x) \Psi \rangle.
\]

(A.2)

By means of the Cauchy–Schwarz inequality we obtain

\[
\left| \chi \langle dx \int_{|k| \geq \Lambda} dk \psi_t(x) G_x(k) (a_k^* + a_{-k}) b_x^* \Psi \rangle \right| 
\leq C \| \psi_t \|_{H^1(\mathbb{R}^3)} \left[ \| 1 - |.|^2 \|_{L^2(\mathbb{R}^3)} \right] \left[ (N_a + 1)^{1/2} \| \psi_t \|_{L^2(\mathbb{R}^3)} \right].
\]

(A.3)

Together with \( \| 1 - |.|^2 \|_{L^2(\mathbb{R}^3)} \leq \sqrt{4\pi/\Lambda} \) this shows (2.8). Using

\[
\left\| \int_{|k| \leq \Lambda} dk G_x(k) a_k^* \psi_t(x) \Psi \right\|^2 = \left\| \int_{|k| \leq \Lambda} dk G_x(k) a_{-k} \psi_t(x) \Psi \right\|^2 + 4\pi \Lambda \| \psi_t \|^2 \| \Psi \|^2
\]

and the Cauchy–Schwarz inequality we estimate

\[
\left| \chi \langle dx \int_{|k| \leq \Lambda} dk \psi_t(x) G_x(k) (a_k^* + a_{-k}) b_x^* \Psi \rangle \right| 
\leq \int dx \| b_x \chi \| \left[ \left\| \int_{|k| \leq \Lambda} dk G_x(k) a_{-k} \psi_t(x) \Psi \right\| + \left\| \int_{|k| \leq \Lambda} dk G_x(k) a_k^* \psi_t(x) \Psi \right\| \right]
\leq C \| N_a^{1/2} \chi \| \left[ \left( \int dx \left\| \int_{|k| \leq \Lambda} dk G_x(k) a_{-k} \psi_t(x) \Psi \right\|^2 \right]^{1/2} + \Lambda^{1/2} \| \psi_t \|_{L^2(\mathbb{R}^3)} \| \Psi \| \right].
\]

(A.5)

Using that [18, Lemma 10] implies \( a^* \left( \mathbb{1}_{|.| \leq \Lambda} G_x \right) a \left( \mathbb{1}_{|.| \leq \Lambda} G_x \right) \leq C (1 - \Delta_x) N_a \) (see [27, (6.8)]) we estimate

\[
\left\| \int_{|k| \leq \Lambda} dk G_x(k) a_{-k} \psi_t(x) \Psi \right\| \leq C \| (1 - \Delta_x)^{1/2} \psi_t(x) N_a^{1/2} \|.
\]

(A.6)

Altogether this shows (2.9). Using \( K(t, k, x) = \psi_t(x) \left( G_x(k) - \langle \psi_t, G_x(k) \psi_t \rangle_{L^2(\mathbb{R}^3)} \right) \) we bound the left hand side of (2.10) by

\[
\left| \chi \langle dx \int dk K(t, k, x) (a_k^* + a_{-k}) b_x^* \Psi \rangle \right| 
\leq \left| \chi \langle dx \int dk \psi_t(x) \langle \psi_t, G_x(k) \psi_t \rangle_{L^2(\mathbb{R}^3)} (a_k^* + a_{-k}) b_x^* \Psi \rangle \right|
+ \left| \chi \langle dx \int dk \psi_t(x) G_x(k) (a_k^* + a_{-k}) b_x^* \Psi \rangle \right|.
\]

(A.7)
Due to (2.7) we have that
\[ \left\langle \chi, \int dx \int dk \psi(x) \langle \psi_k, G(k) \psi \rangle_{L^2(\mathbb{R}^3)} (a^*_k + a_k) b^*_x \Psi \right\rangle \lesssim C \| \psi \|_{H^1} \| N_b^{1/2} \chi \| (N_\alpha + 1)^{1/2} \Psi. \]  
(A.8)

Together with the previous estimates this shows (A.10).

**Proof of Lemma 2.4.** By means of
\[ \int dx b^*_x \langle \psi_t, \hat{\Phi} \psi \rangle_{L^2(\mathbb{R}^3)} b_x = N_b \left( a \left( \langle \psi_t, G \psi \rangle_{L^2(\mathbb{R}^3)} \right) + a^* \left( \langle \psi_t, G \psi \rangle_{L^2(\mathbb{R}^3)} \right) \right) \]  
(A.9)

and (2.7) we get
\[ \left\langle \chi, \int dx b^*_x q(t) \Phi(t) b_x \Psi \right\rangle \lesssim C \| \psi \|_{H^1(\mathbb{R}^3)} \| N_b^{1/2} \chi \| (N_\alpha + 1)^{1/2} N_b^{1/2} \Psi. \]  
(A.10)

Moreover, note that
\[ \left\langle \chi, \int dx b^*_x q(t) \Phi(t) b_x \Psi \right\rangle \leq \sum_{\# \in \{1, \ldots, \# \}} \left\langle \chi, \int dx b^*_x q(t) a^* \left( \mathbb{1}_{| \cdot | \leq \Lambda} G \right) q(t) b_x \Psi \right\rangle 
+ \sum_{\# \in \{1, \ldots, \# \}} \left\langle \chi, \int dx b^*_x q(t) a^* \left( \mathbb{1}_{| \cdot | > \Lambda} G \right) q(t) b_x \Psi \right\rangle. \]  
(A.11)

Using
\[ \int dx b^*_x q(t) a^* \left( \mathbb{1}_{| \cdot | \leq \Lambda} G \right) q(t) b_x \]
\[ = b^* (\psi_t) a^* \left( \mathbb{1}_{| \cdot | \leq \Lambda} \langle \psi_t, G \psi \rangle_{L^2(\mathbb{R}^3)} \right) b (\psi_t) + \int dx b^*_x a^* \left( \mathbb{1}_{| \cdot | \leq \Lambda} G_x \right) b_x 
- b^* (\psi_t) \int dx \overline{\psi_t(x)} a^* \left( \mathbb{1}_{| \cdot | \leq \Lambda} G_x \right) b_x - \int dx b^*_x a^* \left( \mathbb{1}_{| \cdot | \leq \Lambda} G_x \right) \psi_t(x) b (\psi_t), \]
as well as (2.7) and (2.8) we obtain
\[ \left\langle \chi, \int dx b^*_x q(t) a^* \left( \mathbb{1}_{| \cdot | \leq \Lambda} G \right) q(t) b_x \Psi \right\rangle \leq C \sup_{x \in \mathbb{R}^3} \left\| \mathbb{1}_{| \cdot | \leq \Lambda} G_x \right\|_{L^2(\mathbb{R}^3)} \| N_b^{1/2} \chi \| (N_\alpha + 1)^{1/2} N_b^{1/2} \Psi \leq C \Lambda^{1/2} \| N_b^{1/2} \chi \| (N_\alpha + 1)^{1/2} N_b^{1/2} \Psi. \]  
(A.14)

Similarly,
\[ \left\langle \chi, \int dx b^*_x q(t) a^* \left( \mathbb{1}_{| \cdot | > \Lambda} G \right) q(t) b_x \Psi \right\rangle 
\leq \left\langle b(\psi_t) \chi, a^* \left( \mathbb{1}_{| \cdot | > \Lambda} \langle \psi_t, G \psi \rangle_{L^2(\mathbb{R}^3)} \right) b (\psi_t) \Psi \right\rangle + \left\| b_x \chi, a^* \left( \mathbb{1}_{| \cdot | > \Lambda} G_x \right) b_x \Psi \right\| 
+ \left\langle dx \psi_t(x) b (\psi_t) \chi, a^* \left( \mathbb{1}_{| \cdot | > \Lambda} G_x \right) b_x \Psi \right\rangle + \left\| dx \psi_t(x) b (\psi_t) \chi, a^* \left( \mathbb{1}_{| \cdot | > \Lambda} G_x \right) \psi_t(x) b (\psi_t) \Psi \right\|. \]  
(A.15)

By means of (A.11) and integration by parts we get
\[ \left\langle \chi, \int dx b^*_x q(t) a^* \left( \mathbb{1}_{| \cdot | > \Lambda} G \right) q(t) b_x \Psi \right\rangle \leq C \| \psi \|_{H^1(\mathbb{R}^3)} A^{-1/2} \left( (N_\alpha + T_b)^{1/2} \chi \| (N_\alpha + 1)^{1/2} N_b^{1/2} \Psi \right) 
+ \left( (N_\alpha + 1)^{1/2} N_b^{1/2} \chi \| (N_\alpha + T_b)^{1/2} \Psi \right) \right) \leq C \| \psi \|_{H^1(\mathbb{R}^3)} A^{-1/2} \left( (N_\alpha + T_b)^{1/2} \chi \| (N_\alpha + 1)^{1/2} N_b^{1/2} \Psi \right) + \left( (N_\alpha + 1)^{1/2} N_b^{1/2} \chi \| (N_\alpha + T_b)^{1/2} \Psi \right). \]  
(A.16)
In total, this shows (2.11). Setting \( \Lambda = 1 \) and \( \chi = \Psi \) in (2.11) and applying Young’s inequality for products leads to (2.12).

\[\Box\]

### B Proof of Proposition 1.1

Proposition 1.1 is, except of bound for \( \|\psi_t\|_{H^1(\mathbb{R}^3)} \), a direct consequence [16, Lemma 2.1 and Proposition C.2]. In order to obtain the missing estimate we modify the proof of [16, Proposition 2.2]. There, the \( H^4 \)-norm of \( \psi_t \) was estimated by means a functional which is better controllable during the time evolution than \( \|\psi_t\|_{H^4(\mathbb{R}^3)} \). We will rely on the following results.

**Proposition B.1** (Part of Proposition C.2 in [16]). If \( \alpha = 1 \), \((\psi, \varphi) \in H^2(\mathbb{R}^3) \times L^1_t(\mathbb{R}^3)\), then \((\psi_t, \varphi_t) \in H^2(\mathbb{R}^3) \times L^1_t(\mathbb{R}^3)\) for all \( t \in \mathbb{R} \) and there exists a constant \( C > 0 \) depending only on the initial data such that

\[
\|\psi_t\|_{H^2(\mathbb{R}^3)} \leq C (1 + |t|) \quad \text{and} \quad \|\varphi_t\|_{L^2_t(\mathbb{R}^3)} \leq C (1 + |t|). \tag{B.1}
\]

If, in addition, \( \varphi \in L^2_0(\mathbb{R}^3) \) then \( \varphi_t \in L^2_0(\mathbb{R}^3) \) for all \( t \in \mathbb{R} \) and there exists a constant \( C > 0 \) depending only on the initial data such that

\[
\|\varphi_t\|_{L^2_0(\mathbb{R}^3)} \leq C \left( 1 + |t|^3 \right). \tag{B.2}
\]

**Lemma B.1.** There exists a constant \( C > 0 \) such that

\[
\|\partial_\beta \Phi_{\varphi_t} \|_{L^\infty(\mathbb{R}^3)} \leq C \|\varphi\|_{L^2_t(\mathbb{R}^3)}, \quad \|\partial_\beta \Phi_{\varphi_t} \|_{L^\infty(\mathbb{R}^3)} \leq C \|\varphi_t\|_{L^2_{|\beta|+1}(\mathbb{R}^3)} \tag{B.3}
\]

for all \( \beta \in \mathbb{N}_0^3 \) and such that

\[
1 \leq -\Delta + \Phi_{\varphi_t} + C (E[\psi, \varphi] + C). \tag{B.4}
\]

**Proof of Lemma [B.1]**. If we insert \((1 + |t|^2)^\frac{1+|\beta|}{2}\) and it’s inverse to the right hand side of \( \partial_\beta \Phi_{\varphi_t}(x) = 2\text{Re} \left< |t|^{-1} \partial_\beta \varphi e^{-2\pi i x \cdot \varphi}, \varphi_t \right>_{L^2(\mathbb{R}^3)} \) and apply the Cauchy–Schwarz inequality we obtain the first estimate.

The second inequality is derived by similar means because \( \Phi_{\varphi_t} = 2\text{Im} \left< |t|^{-1} e^{-2\pi i x \cdot \varphi_t}, \varphi_t \right>_{L^2(\mathbb{R}^3)} \). By (2.4) and the Cauchy–Schwarz inequality we get

\[
\left| \left< \xi, \Phi_{\varphi_t} \xi \right>_{L^2(\mathbb{R}^3)} \right| \leq C \|\xi\|_{H^1(\mathbb{R}^3)} \|\xi\|_{L^2(\mathbb{R}^3)} \|\varphi_t\|_{L^2(\mathbb{R}^3)} \leq \frac{1}{2} \|\xi\|_{H^1(\mathbb{R}^3)}^2 + C \|\varphi_t\|_{L^2(\mathbb{R}^3)}^2 \|\xi\|_{L^2(\mathbb{R}^3)}^2 \tag{B.5}
\]

for \( \xi \in H^1(\mathbb{R}^3) \). Together with (3.3) this leads to (B.4).

\[\Box\]

**Bound for \( \|\psi_t\|_{H^3(\mathbb{R}^3)} \)**. The local well-posedness of solution in \( H^3(\mathbb{R}^3) \times L^2_0(\mathbb{R}^3) \) can be shown by a standard fixed-point argument. In order to derive a bound on the \( H^3 \)-norm of \( \psi_t \) we define the functional

\[
E^{(3)}[\psi_t, \varphi_t] = \left\| (-\Delta + \Phi_{\varphi_t} + M)^{3/2} \psi_t \right\|_{L^2(\mathbb{R}^3)}, \tag{B.6}
\]

where \( M \geq 1 \) is a constant depending only on \( E[\psi, \varphi] \) such that \( 1 \leq -\Delta + \Phi_{\varphi_t} + M \). Note that the existence of \( M \) is guaranteed by (B.4). The functional satisfies the inequalities

\[
\left| E^{(3)}[\psi_t, \varphi_t] - \left\| (-\Delta + M)^{3/2} \psi \right\|_{L^2(\mathbb{R}^3)} \right| \leq \frac{1}{2} E^{(3)}[\psi_t, \varphi_t] + CM^2 \left( 1 + |t|^6 \right), \tag{B.7}
\]

\[
\sqrt{E^{(3)}[\psi_t, \varphi_t]} \leq \sqrt{E^{(3)}[\psi, \varphi]} + CM \left( 1 + |t|^4 \right) \tag{B.8}
\]
with $C > 0$ depending only on the initial data. Combining the estimates let us obtain
\[\|\psi_t\|_{H^3(\mathbb{R}^3)} \leq \|(-\Delta + M)^{3/2} \psi_t\|_{L^2(\mathbb{R}^3)} \leq C \left(\|(-\Delta + M)^{3/2} \psi\|_{L^2(\mathbb{R}^3)} + M \left(1 + |t|^4\right)\right). \tag{B.9}\]

This proves the second inequality in (1.16). It remains to prove the inequalities from above. Using
\[-\Phi_{\psi_t} (-\Delta + \Phi_{\psi_t} + M) \Phi_{\psi_t} + (-\Delta + \Phi_{\psi_t} + M)^2 \Phi_{\psi_t} + \Phi_{\psi_t} (-\Delta + \Phi_{\psi_t} + M)^2 \Phi_{\psi_t} \leq 0 \tag{B.10}\]

and the Cauchy–Schwarz inequality let us estimate
\[\left| \mathcal{E}^{(3)}[\psi_t, \varphi_t] - \|(-\Delta + M)^{3/2} \psi_t\|_{L^2(\mathbb{R}^3)}^2 \right| \leq 2 \sqrt{\mathcal{E}^{(3)}[\psi_t, \varphi_t]} \left\|(-\Delta + \Phi_{\psi_t} + M)^{1/2} \Phi_{\psi_t} \psi_t\right\|_{L^2(\mathbb{R}^3)}^2 + \left\|(-\Delta + \Phi_{\psi_t} + M)^{1/2} \Phi_{\psi_t}\right\|_{L^2(\mathbb{R}^3)}^2 \tag{B.11} \]

Inequality (B.7) then follows from Proposition [B.1] and Lemma [B.1]. Next, we estimate
\[
\frac{d}{dt} \mathcal{E}^{(3)}[\psi_t, \varphi_t] = 2 \text{Re} \left\langle \psi_t, \Phi_{\psi_t} (-\Delta + \Phi_{\psi_t} + M)^2 \psi_t \right\rangle_{L^2(\mathbb{R}^3)} + \left\langle \psi_t, (-\Delta + \Phi_{\psi_t} + M) \Phi_{\psi_t} (-\Delta + \Phi_{\psi_t} + M) \psi_t \right\rangle_{L^2(\mathbb{R}^3)} \leq \sqrt{\mathcal{E}^{(3)}[\psi_t, \varphi_t]} \left\|(-\Delta + \Phi_{\psi_t} + M)^{1/2} \Phi_{\psi_t} \psi_t\right\|_{L^2(\mathbb{R}^3)} + \left\|\Phi_{\psi_t} (-\Delta + \Phi_{\psi_t} + M) \psi_t\right\|_{L^2(\mathbb{R}^3)} \leq \sqrt{\mathcal{E}^{(3)}[\psi_t, \varphi_t]} CM \left(\|\nabla \Phi_{\psi_t}\|_{L^\infty(\mathbb{R}^3)} + \|\Phi_{\psi_t}\|_{L^\infty(\mathbb{R}^3)} \left(\|\psi_t\|_{H^2(\mathbb{R}^3)} + \|\Phi_{\psi_t}\|_{L^\infty(\mathbb{R}^3)}\right)\right), \tag{B.12}\]

By means of Proposition [B.1] and Lemma [B.1] we get
\[\frac{d}{dt} \mathcal{E}^{(3)}[\psi_t, \varphi_t] \leq \sqrt{\mathcal{E}^{(3)}[\psi_t, \varphi_t]} CM \left(1 + |t|^3\right), \tag{B.13}\]

which implies (B.8). \hfill \square

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