Nonresonant two-level transitions: Insights from quantum thermodynamics

Andreas Wacker

Mathematical Physics and NanoLund, Lund University, Box 118, 22100 Lund, Sweden

(Dated: 11 January 2022, accepted by Physical Review A)

Based on concepts from quantum thermodynamics the two-level system coupled to a single electromagnetic mode is analyzed. Focusing on the case of detuning, where the mode frequency does not match the transition frequency, effective energies are derived for the levels and the photon energy. It is shown that these should be used for energy exchange with fermionic and bosonic reservoirs in the steady state in order to achieve a thermodynamically consistent description. While recovering known features such as frequency pulling or Bloch gain, this sheds light on their thermodynamic background and allows for a coherent understanding.

I. INTRODUCTION

Two-level systems are the paradigm for the interaction of matter with light. Typically, one considers light frequencies $\omega/2\pi$, where the photon energy $\hbar \omega$ matches the energy difference $E_u - E_l$ between the upper (index $u$) and lower (index $l$) level. However, it is well known, that optical transitions are broadened due to the finite lifetimes of levels and photons, so that a certain width of frequencies can be emitted or absorbed. A straightforward question is, how energy balance is satisfied for a finite detuning $\hbar \Delta = \hbar \omega + E_l - E_u$.

Detuning is known to have a variety of practical consequences. E.g., it results in frequency pulling [4] for lasers and Bloch gain for inter-subband transitions in semiconductor heterostructures [2,3]. From a more fundamental point of view there had been discussions on the thermodynamic consistency [4] for the archetypal Scovil&Schulz-DuBois maser [5]. Detuned transitions play also an important role for certain gate operations on Qubits in quantum information [6,7].

Here, the issue of detuning is studied from a quantum-thermodynamic [8-11] perspective. Assuming local couplings of the two levels with separate reservoirs, the energy balance in the steady state allows to identify effective energies for the levels and the electromagnetic mode. These differ from the bare energies by a fraction of the total detuning $\hbar \Delta = \hbar \omega + E_l - E_u$.

It is shown that these should be used for energy exchange with fermionic and bosonic reservoirs in the steady state in order to achieve a thermodynamically consistent description. While recovering known features such as frequency pulling or Bloch gain, this sheds light on their thermodynamic background and allows for a coherent understanding.

II. METHODS

This article is organized as follows: Sec. II briefly summarizes the general concepts from quantum thermodynamics applied. The heart of the article is Sec. III where the two-level system is carefully analyzed using detailed calculations presented in appendix A and B for the classical and quantum treatment of the electromagnetic mode, respectively. Sec. IV and Sec. V consider the frequency pulling of lasers and the Bloch gain for intersubband transitions. For these examples it is shown that the effective energies introduced here provide the same features as detailed microscopic calculations performed before. Finally, Appendix C details how the effective energies can be gen-

*Andreas.Wacker@teorfys.lu.se
eralised to arbitrary systems with fermionic baths.

II. GENERAL THERMODYNAMIC POINT OF VIEW

We consider the system (e.g. the two-level system) in connection with reservoirs using the common quantum-thermodynamics treatment. Let $U_\alpha$ be the energy flow from the reservoir $\alpha$ into the system and $U_{\text{opt}}$ the energy flow from the optical field into the system. Energy conservation implies in the steady state

$$\sum_\alpha U_\alpha + U_{\text{opt}} = 0. \quad (1)$$

The entropy production in reservoir $\alpha$ reads

$$\frac{dS_\alpha}{dt} = -\frac{U_\alpha}{T_\alpha} + \dot{N}_\alpha \frac{\mu_\alpha}{T_\alpha}, \quad (2)$$

where $\dot{N}_\alpha$ is the particle transfer from the reservoir $\alpha$ into the system. Then the second law of thermodynamics in the form of positive definite entropy production in the steady state requires

$$\sum_\alpha \frac{dS_\alpha}{dt} + \frac{dS_{\text{opt}}}{dt} \geq 0. \quad (3)$$

A standard quantum kinetic treatment is provided by the time evolution of the reduced density operator $\hat{\rho}$ of the system

$$\frac{d\hat{\rho}}{dt} = \frac{1}{i\hbar} \left[ \hat{H}_S, \hat{\rho} \right] + \sum_\alpha \mathcal{L}_\alpha[\hat{\rho}], \quad (4)$$

where $\hat{H}_S$ the system Hamilton operator and $\mathcal{L}_\alpha$ describes the coupling to reservoir $\alpha$ within a Markovian treatment, see e.g. [17]. Following well established standard procedure [18-20] which is commonly used for the description of nanoscale engines, see e.g. [21], the energy and particle flows from the reservoirs into the system are

$$\dot{U}_\alpha = \text{Tr}\{\hat{H}_S \mathcal{L}_\alpha[\hat{\rho}]\} \quad \text{and} \quad \dot{N}_\alpha = \text{Tr}\{\hat{N}_S \mathcal{L}_\alpha[\hat{\rho}]\}, \quad (5)$$

respectively, where $\hat{N}_S$ is the number operator of the system. The corresponding power (work per time) transferred to the system reads

$$P_S = \text{Tr}\left\{ \hat{\rho} \frac{\partial \hat{H}_S}{\partial t} \right\} \quad (6)$$

which requires an absolute time-dependence of the Hamiltonian, as given by a classical optical field. It matches exactly the total rate of work $j : \mathcal{E}$ done by a classical electromagnetic field [22], where $j$ is the current density and $\mathcal{E}$ is the electric field.

Such thermodynamic considerations have been frequently performed for the analysis of optoelectronic systems such as light emitting diodes (LEDs), Lasers, and solar cells, see e.g., [23, 24] with a highlight on different aspects. The focus of this work is to use Eqs. (13) as necessary conditions for the choice of $\mathcal{L}_\alpha$ in the construction of master equations. As shown below, this has distinct implications on the choice of energies used in the thermal occupation functions of the reservoirs for finite detuning between the optical modes and the two-level system.

In this work, we stay within the realm of the quantum master Eq. (4) with Lindblad-type Liouvillians $\mathcal{L}_\alpha$, see Eq. (6), which is frequently applied to optical systems, see e.g. Ref. [25]. For the case of classical fields with a time-dependent Hamiltonian $H_S(t)$, thermodynamic aspects of more detailed approaches have been studied in [26, 27], where also the impact of the Rabi splitting for strong classical fields is addressed. It should be noted that the derivation of quantum master equations in the presence of time-dependent Hamiltonians $H_S(t)$ is a matter of ongoing discussion, see, e.g. [28] and references cited therein.

III. TWO-LEVEL SYSTEM WITH ELECTRONIC RESERVOIRS

We consider an electronic two-level system with energies $E_u > E_l$ as given by the bare Hamiltonian

$$\hat{H}_0 = E_u \hat{c}_u\hat{c}_u + E_l \hat{c}_l\hat{c}_l \quad (7)$$

which is coupled to electron reservoirs $\alpha \in \{u, l\}$ with temperatures $T_\alpha$ and electrochemical potential $\mu_\alpha$ via Lindblad operators [30, 31]

$$\mathcal{L}_\alpha[\hat{\rho}] = \gamma_\alpha f_\alpha D_{\hat{c}_\alpha} + \gamma_\alpha (1 - f_\alpha) D_{\hat{c}_\alpha}^\dagger \hat{c}_\alpha \hat{\rho} + \hat{\rho} D_{\hat{c}_\alpha}^\dagger \gamma_\alpha \frac{1}{2} \left( \sigma^\dagger \sigma + \sigma \sigma^\dagger \right) \quad (8)$$

where $\gamma_\alpha$ denotes the coupling strength between the level $\alpha$ and its connected reservoir. This is known as the local approach [12, 13]. Commonly, one assumes that only the energy $E_\alpha$ of the isolated level is relevant for the transition to bath $\alpha$, which results in the occupation functions $f_\alpha = f_\alpha^\text{common}$ with the Fermi function

$$f_\alpha^\text{common} = \frac{1}{\exp[(E_\alpha - \mu_\alpha)/k_B T_\alpha] + 1}. \quad (9)$$

However, it is known that such an approach can lead to violations of the thermodynamic rules [11, 12, 32], which is also the case for the system studied here (see below). The key point of this work is to trace such violations to the use of Eq. (9). Therefore, we keep the occupations $f_\alpha$ undefined until we can identify effective energies [14, 21] which are suggested to replace the localized level energies $E_\alpha$ in Eq. (9). These effective energies reflect the coupling of the local levels to the other levels in the system (here by the light field), which is disregarded in the common use of the local approach.

Transitions between the states $u$ and $l$ are possible by coupling to an optical field resulting in a net rate $R$ for
transitions $u \rightarrow l$. Below, we provide detailed calculations of $R$ in the steady state (superscript$^{ss}$) both for classical and quantum fields in Sects. III A and III B respectively. In particular, we investigate the sign of $R^{ss}$ obtained by quantum kinetics and compare it with the sign determined by thermodynamic considerations as outlined in Sec. III.

Typically, such systems are treated under resonance ($\Delta = 0$) in the literature (see, e.g., [33, 34]). Here the focus is on detuning with a finite value of $\Delta$.

### A. Classical field

Within the common rotating wave approximation (RWA) we set $H_S = H_0 + \hat{V}_c(t)$ with

$$\hat{V}_c(t) = \hbar \omega \hat{a}^\dagger \hat{a} e^{-i\omega t} + \hbar \omega^* \hat{a}^\dagger \hat{a} e^{i\omega t}$$

(10)

with the coupling strength $\omega$. A standard density matrix calculation, as detailed in Sec. A, provides the following results in the steady state, where $R^{ss}$ is the transition rate between the upper and lower level, as given by Eq. (A12): The net power absorbed from the electromagnetic field $\hat{P}^{ss}_{u}$ is given by

$$P^{ss}_{u} = -\hbar \omega R^{ss}$$

(11)

and the particle and energy flows from the contacts $\hat{N}^{ss}_{u}$ and $\hat{N}^{ss}_{l}$ in the steady state read

$$\hat{N}^{ss}_{u} = R^{ss} \quad \text{and} \quad \hat{N}^{ss}_{l} = -R^{ss}$$

(12)

$$\hat{U}^{ss}_{u} = \hat{E}_{u} R^{ss} \quad \text{and} \quad \hat{U}^{ss}_{l} = -\hat{E}_{l} R^{ss}$$

(13)

with the effective energies

$$\hat{E}_{u} = E_{u} + \frac{\gamma_{u}\hbar \Delta}{\gamma_{u} + \gamma_{l}} \quad \text{and} \quad \hat{E}_{l} = E_{l} - \frac{\gamma_{l}\hbar \Delta}{\gamma_{u} + \gamma_{l}}.$$ 

(14)

Note that this is a rigorous result for the system considered based on the general thermodynamic framework of section [11] within the Markovian approximation and the local coupling to the reservoir [8]. The effective energies can be interpreted as a distribution of the total detuning $\hbar \Delta$ to the bare levels according to their relative coupling strengths $\gamma_{\alpha}$, so that they satisfy $\hat{E}_{u} - \hat{E}_{l} = \hbar \omega$. This reflects the fact that the sum of energy flows, $\hat{U}^{ss}_{u} + \hat{U}^{ss}_{l} + \hat{P}^{ss}_{S} = 0$ conserves energy [11].

Based on these results the entropy production [2] provides

$$\hat{S}^{ss}_{u} + \hat{S}^{ss}_{l} = R^{ss} \left( \frac{\hat{E}_{l} - \mu_{l}}{T_{l}} - \frac{\hat{E}_{u} - \mu_{u}}{T_{u}} \right).$$

(15)

Up to now, we performed all calculations without specifying the values of $f_{\alpha}$, which need to reflect the reservoir properties. Now we use the criterion of positive definite entropy production [3] to identify these. According to Eq. (A12), $R^{ss}$ has the same sign as $(f_{u} - f_{l})$. Thus, positivity of entropy production [3] implies that $f_{u} - f_{l}$ needs to have the same sign as $\left( \frac{E_{u} - \mu_{u}}{T_{u}} - \frac{E_{l} - \mu_{l}}{T_{l}} \right)$. This can be guaranteed if we choose

$$f_{\alpha} = \frac{1}{e(\mu_{\alpha} - \mu_{\alpha})/k_{B} T_{\alpha} + 1}$$

(16)

as $(e^{1/\gamma_{\alpha} - 1} - e^{1/\gamma_{\alpha} + 1})$ has always the same sign as $(X_{l} - X_{u})$ due to the strong monotonic decrease of $e^{1/\gamma_{\alpha} + 1}$. However, violations of Eq. (3) are possible if we choose the bare energy levels $E_{\alpha}$ in the bath Fermi functions following the tempting guess [3]. We conclude, that the effective energies [14] should be used for the occupation functions in the fermionic reservoirs in full analogy to the bosonic case treated in [16].

For equal temperatures, $T_{u} = T_{l} = T$, Eq. (15) shows that $R^{ss}$ has the same sign as $\mu_{u} - \mu_{l} - \hbar \omega$. This implies operation as an LED if the bias $\mu_{u} - \mu_{l}$ surpasses the photon energy $\hbar \omega$ and operation as a solar cell in the opposite case. This is a consequence of the fact that no entropy is produced in the light field, which is described by a classical field in Eq. (10). Thus, the only source of entropy production is the generation of heat from the excess energy $\mu_{u} - \mu_{l} > \hbar \omega$ for emission or $\hbar \omega > \mu_{u} - \mu_{l}$ for absorption, which is transferred to the reservoirs.

Note, that this procedure also applies to tunneling problems, which correspond to $\omega = 0$. Thereby it resolves the violation of the second law for the local approach in Ref. [12].

### B. Quantized field

Within the RWA we set in the spirit of the Jaynes-Cummings model [35]

$$\hat{V}_{JC} = \hbar g \hat{a}^\dagger \hat{c} \hat{a} + \hbar g^* \hat{a} \hat{c}^\dagger \hat{a}^\dagger + \hbar \omega \hat{a}^\dagger \hat{a}^\dagger$$

with the bosonic annihilation operator $a$ for the photon mode. Now the detuning is given by $\hbar \Delta_{av} = \hbar \omega_{av} + E_{l} - E_{u}$. In order to allow for a steady state we add the interaction of the photon mode with a thermal bosonic reservoir with average occupation $n_{b}$ and transition rate $\gamma_{b}$ via the Lindblad operator

$$\mathcal{L}_{b}[\hat{a}] = \gamma_{b} (n_{b} + 1) \mathcal{D}_{b}[\hat{a}] + \gamma_{b} n_{b} \mathcal{D}_{b}[\hat{a}^\dagger][\hat{a}].$$

(17)

As for the fermionic occupation factors $f_{\alpha}$, the value of $n_{b}$ is specified at a later stage. This provides the equation of motion for the density operator

$$\frac{d\hat{\rho}}{dt} = \frac{1}{i\hbar} [\hat{H}_{0} + \hat{V}_{JC}, \hat{\rho}] + \mathcal{L}_{b}[\hat{\rho}] + \mathcal{L}_{l}[\hat{\rho}] + \mathcal{L}_{u}[\hat{\rho}].$$

(18)

As described in Appendix B this provides the following results in the steady state with the net transition rate $R^{ss}$, for which we do not have a closed expression:

$$\hat{N}^{ss}_{u} = R^{ss} \quad \text{and} \quad \hat{N}^{ss}_{l} = -R^{ss},$$

(19)
\[
\dot{U}^{ss} = E_u R^{ss}, \quad \dot{U}_l^{ss} = -E_l R^{ss},
\]
and
\[
\dot{U}_b^{ss} = -E_{ph} R^{ss}
\]
with the effective energies
\[
E_u = E_u + \frac{\gamma_u \hbar \Delta_{cav}}{\gamma_u + \gamma_l + \gamma_b},
\]
\[
E_l = E_l - \frac{\gamma_l \hbar \Delta_{cav}}{\gamma_u + \gamma_l + \gamma_b},
\]
and
\[
E_{ph} = \hbar \omega_{cav} - \frac{\gamma_b \hbar \Delta_{cav}}{\gamma_u + \gamma_l + \gamma_b},
\]
where \(E_u\) and \(E_{ph}\) satisfy Eq. (20), so that the fluxes fulfill energy conservation [1]. This is analogous to Eq. (14).

In Appendix B it is shown within a Hartree-Fock like approximation, that

\[
\text{The sign of } R^{ss} \text{ equals the sign of } \frac{f_u}{1 - f_u} - \frac{f_l}{1 - f_l} = \frac{n_b}{1 + n_b},
\]

Applying the Fermi and Bose distributions
\[
f_u = e^{(E_u - \mu_u)/\kappa_B T_u} + 1, \quad f_l = e^{(E_l - \mu_l)/\kappa_B T_l} + 1, \quad \text{and } n_{ph} = e^{E_{ph}/\kappa_B T_u} - 1,
\]
Eq. (23) becomes
\[
e^{-(E_u - \mu_u)/\kappa_B T_u} - e^{-E_{ph}/\kappa_B T_u} \times e^{-(E_l - \mu_l)/\kappa_B T_l}
\]
determines the sign of \(R^{ss}\)
which guarantees the positivity in Eq. (22) due to the monotonic increase of the exponential function. Here it is crucial, that the effective energies enter the respective occupation functions. Otherwise violations of Eq. (22) can be easily constructed for particular values of temperatures and electrochemical potentials.

For equal temperatures, \(T_u = T_l = T\), Eq. (22) shows that \(R^{ss}\) has the same sign as \(\mu_u - \mu_l - E_{ph}(1 - T/T_u)\). Thus, light emission (\(R^{ss} > 0\)) is even possible for \(\mu_u - \mu_l < E_{ph}\) provided that \(T_u\) is not too large. In this case the LED emits more light than it consumes electrical energy, which has been experimentally verified [36]. The excess energy is taken from the reservoirs which provides cooling [37]. For operation as a solar cell (\(R^{ss} < 0\)) this shows that the extracted electrical power \(P_{el} = -R^{ss}(\mu_u - \mu_l)\) is limited by
\[
P_{el} \leq U_b^{ss} \frac{T_b - T}{T_b}
\]
which is just Carnot’s law for the incoming heat \(U_b^{ss}\) from the warm reservoir with \(T_b > T\).

**IV. RELATION TO FREQUENCY PULLING IN A LASER**

For laser operation, the coherent states are a better approximation for the optical field than number states [39]. Technically, this can be done by the semi-classical approximation [39], where \(\langle \hat{a} \rangle\) is treated as a classical field. Therefore, in Eqs. (B1) [B2], the approximation
\[
Y = g^* \text{Tr}\{\hat{c}^\dagger \hat{c} \hat{a}^\dagger \hat{\rho}\} \approx g^* a^* \sigma_{ul}
\]
is used with
\[
\sigma_{ul} = e^{i \omega t} \text{Tr}\{\hat{c}^\dagger \hat{c} \hat{a} \hat{\rho}\} \quad \text{and} \quad a = e^{i \omega t} \text{Tr}\{\hat{\rho}\}
\]
where a natural oscillation with frequency \(\omega\) is assumed. Without detuning, \(\omega = \omega_{cav} = (E_u - E_l)/\hbar\) is the natural choice, which corresponds to the interaction picture. For the case of detuning, \(\omega\) is determined below. In Eq. (18) provides
\[
\frac{d}{dt} \sigma_{ul} = i \left( \frac{\omega - E_u - E_l}{\hbar} \right) \sigma_{ul} + i g^* a^* (\sigma_{uu} - \sigma_{ul})
\]
\[
- \frac{\gamma_u + \gamma_l}{2} \sigma_{ul} + \gamma_{ul} a^* \sigma_{ul}
\]
(25)
\[
\frac{d}{dt} a = i \left( \omega - \omega_{cav} \right) a - i g^* \sigma_{ul} - \frac{\gamma_b}{2} a
\]
(26)
which, together with [B1] [B2], give a closed set of equations. In the steady state Eq. (26) provides
\[
\sigma_{ul}^{ss} = \frac{\omega - \omega_{cav} + i \gamma_b/2}{g^2} g a^{ss}
\]
(27)
With the semiclassical approximation \(24\), the set of equations for the quantum case \(21, 22, 25\) equal the classical treatment \(A1, A2, A3\) with \(\epsilon = ga\). Thus we obtain from Eqs. \(A8, A11\)

\[
\sigma_{ul}^{ss} = -\frac{H((ga^{ss})^2)(f_u - f_l)}{\Delta + i(\gamma_u + \gamma_l)/2}ga^{ss}
\]

Eqs. \(27, 28\) are only consistent, if

\[
(\omega - \omega_{cav} + i\gamma_b/2)[\Delta + i(\gamma_u + \gamma_l)/2] = -|g|^2H((|a^{ss}|^2)(f_u - f_l)
\]

While, the real part of the equation provides the field strength \(a^{ss}\) of laser light in the cavity due to gain saturation, its imaginary part part determines \(\omega\). As the right-hand side is purely real, the imaginary part needs to be zero on the left-hand side. Using \(\Delta = \omega - (E_u - E_l)/\hbar\), one obtains

\[
\hbar\omega = (\gamma_u + \gamma_l)\omega_{cav} + \gamma_b(E_u - E_l)
\]

Straightforward algebra shows, that \(\hbar\omega = \hat{E}_{ph}\) from Eq. \(21\). Thus the effective photon energy obtained in Sec. \(III\ B\) matches the actual oscillation frequency in the laser. Furthermore, the effective energies \(\hat{E}_u\) and \(\hat{E}_l\) from the classical field \(13\) and the quantum treatment \(21\) agree with each other for this choice of \(\hbar\omega\).

\[
R_{DM}(k_0) \propto \frac{\gamma_u + \gamma_l}{\Delta^2 + (\gamma_u + \gamma_l)^2/4} \left\{ \frac{\gamma_l [f_u(E_{k_0}) - f_l(E_{k_0} - h\Delta)]}{\gamma_u + \gamma_l} + \frac{\gamma_u [f_u(E_{k_0} + h\Delta) - f_l(E_{k_0})]}{\gamma_u + \gamma_l} \right\}
\]

for states with a particular value of \(k = k_0\) (which is essentially conserved in the optical transition), see Eq. \(20\) or Ref. \(2\). The first factor provides the common Lorentzian broadening of the line and the second factor shows, that the transitions are driven by differences between the occupations of the subbands. This factor is not just \(f_u(E_{k_0}) - f_l(E_{k_0})\) as frequently assumed, but has different energy arguments. This leads to a particular gain spectrum for the case of equal occupation of both subbands \(f_u(E) = f_l(E)\), called dispersive gain or Bloch gain due to its relation to Bloch oscillations in superlattices \(11, 42\). This type of gain could be observed in Quantum Cascade Lasers \(3\) and has been recently suggested to be relevant for the generation of frequency combs \(43\).

Analysing the occupation factors of Eq. \(31\) reveals that the occupations of the upper level are taken at an average energy \(\hat{E}_{k_0}^u = E_{k_0} + h\Delta\gamma_u/(\gamma_u + \gamma_l)\). Correspondingly, the occupations of the lower subband are taken at an average energy \(\hat{E}_{k_0}^l = E_{k_0} - h\Delta\gamma_l/(\gamma_u + \gamma_l)\), so that \(E_u + \hat{E}_{k_0}^u - (E_l + \hat{E}_{k_0}^l) = \hbar\omega\). This fully corresponds to

Eq. \(30\) shows that for detuning between the cavity frequency and the optical transition frequency, the laser operates at a frequency in between. This is known as frequency pulling \(11\). Actually, the expression \(12, 23\) of \(11\) is directly obtained from Eq. \(30\) by introducing the Q-factors \(Q_c = \omega/\gamma_b\) and \(Q_a = \omega/(\gamma_u + \gamma_l)\) for the cavity and the atomic transition, respectively. Thus, the treatment by heat flows suggested here, provides the same result as a detailed optoelectronic study.

V. RELATION TO BLOCH GAIN

For layered semiconductor structures, optical transitions occur between subbands with a well-defined transition energy \(E_u - E_l\) due the quantized energies in growth direction \(30\). In addition, the lateral free-particle motion (with energy \(E_k\) described by wave vector \(k\)) provides a continuous degree of freedom with occupation function \(f_u(E_k)\) and \(f_l(E_k)\) in each of the subbands. Due to intra-subband scattering, the levels are broadened and thus light emission/absorption is possible for detuning, with a finite value of \(\Delta = \omega - (E_u - E_l)/\hbar\). A microscopic density-matrix approach for the scattering \(2\) provided the steady state net transition rate between the upper and the lower subband.

VI. CONCLUSION

Applying a quantum thermodynamic approach, effective energies were identified to describe optical transition under detuning, as given in Eqs. \(14, 21, 24\) for the classical and quantum treatment, respectively. These effective energies satisfy energy conservation by distributing the detuning to the bare energies according to their respective contribution to the broadening of the transition. Applying these effective energies for transitions with thermal reservoirs provides thermodynamic consistency within the generally accepted heat and work definitions \(53\), where both the first and second law are satisfied for steady state operation.
The effective energy of the photon could be attributed to the pulled frequency in a laser cavity. Similarly, the effective level energies agree with the average energies associated with inter-subband transitions. Thus, the approach used here provides a new comprehensive view of features obtained from different sophisticated microscopic treatments.

All results obtained are rigorous within the Markovian Lindblad master equation used. The only exception is the fulfillment of the second law for the quantum treatment of the cavity mode, where a Hartree-Fock like approximation was required. It would be interesting, how far the results can be generalized. Another open issue is whether these effective energies can also be applied for noise calculations, where different types of averages apply.

It is interesting to note, that the effective energies appear independently whether the coupling to the electromagnetic field is treated classically by a time-dependent Hamiltonian and quantum-mechanically within a time-independent Hamiltonian. Thus, these findings appear not to be affected by the question, in how far the quantum master equation Eq. (1) is applicable for time-dependent Hamiltonians.

For the classical case, the results can be generalized to arbitrary fermionic systems with a time-periodic Hamiltonian, see Appendix C. However, the identification of the effective energies relies then on a self-consistent solution of the system dynamics with the occupation functions, which may limit practical applications. (The recently developed thermodynamically consistent local approach [53] is an interesting alternative at the cost of a limited resolution for heat.) Albeit the approach is thermodynamic consistent for arbitrary system parameters, one has to remember that the local approach [5] restricts to a single transition energy, which does not allow the detailed description of photon-assisted reservoir transitions or Coulomb blockade phenomena.

VII. ACKNOWLEDGMENT

The author thanks Alex Kalaee, Patrick Potts, and Peter Samuelsson for helpful discussions and collaboration on related issues. Financial support from the Knut and Alice Wallenberg Foundation (project 2016.0089), the Swedish Research Council (project 2017-04287), and NanoLund is gratefully acknowledged.

Appendix A: Calculations for the classical case

The quantum master equation (4) with \( \hat{H}_S = \hat{H}_D + \hat{V}_C(t) \) based on the operators (7,10) can be mapped to a time-independent problem in the rotating frame where \( \hat{A}^R = \hat{U}(t) \hat{A} \hat{U}^\dagger(t) \) with \( \hat{U}(t) = e^{i[(E_0 + \hbar \omega)t - E_0 \hat{t}_C]t/\hbar} \) and provides the equation of motion for the density operator

\[
\frac{d\hat{\rho}^R}{dt} = -i[\Delta \hat{c}_u \hat{c}_u + \epsilon \hat{c}_l \hat{c}_l + \epsilon^* \hat{c}_l \hat{c}_l, \hat{\rho}^R] + \mathcal{L}_u[\hat{\rho}^R] + \mathcal{L}_l[\hat{\rho}^R]
\]

with \( \Delta = \omega - (E_u - E_l)/\hbar \). This gives the equations of motion for the reduced density matrix \( \sigma_{ij} = \text{Tr}\{\hat{c}^\dagger_i \hat{c}_j \hat{\rho}^R\} \)

\[
\frac{d}{dt}\sigma_{uu} = \gamma_u(f_u - \sigma_{uu}) + i(\epsilon^* \sigma_{ul} - \epsilon \sigma_{lu}) \quad (A1)
\]

\[
\frac{d}{dt}\sigma_{ul} = \gamma_l(f_l - \sigma_{ul}) + i(\epsilon \sigma_{lu} - \epsilon^* \sigma_{ul}) \quad (A2)
\]

\[
\frac{d}{dt}\sigma_{ul} = i\Delta \sigma_{ul} + i\epsilon(\sigma_{uu} - \sigma_{ll}) - \frac{\gamma_u + \gamma_l}{2} \sigma_{ul} \quad (A3)
\]

From the changes of the populations, we identify the net transitions rate \( u \rightarrow l \)

\[
R = -i(\epsilon^* \sigma_{ul} - \epsilon \sigma_{ul}) = 23\{\epsilon^* \sigma_{ul}\} . \quad (A4)
\]

Furthermore the work flow [3] becomes

\[
P_S = \text{Tr}\{i\hbar \omega \left(e^{+ \hat{c}_l \hat{c}_l e^{i\omega t} - e^{+ \hat{c}_l \hat{c}_l e^{-i\omega t}}} \right) \hat{\rho}\} = \text{Tr}\left\{\left(e^{+ \hat{c}_l \hat{c}_l - e^{+ \hat{c}_l \hat{c}_l}} \right) \hat{\rho}^R \right\} = -\hbar \omega R . \quad (A5)
\]

This shows that each transition from the upper to the lower level is associated with the energy portion \( \hbar \omega \) removed from the system even if \( E_u - E_l \neq \hbar \omega \). This agrees with the conception that the transition is associated with the creation of one photon with angular frequency \( \omega \) at the optical field in treated as a classical variable in Eq. (10). In the steady state, Eq. (A5) directly provides Eq. (11).

The energy and particle flows from the leads [5] are

\[
\hat{U}_\alpha = \text{Tr}\{\hat{H}\mathcal{L}_\alpha[\hat{\rho}]\} = \text{Tr}\{\hat{H}^R\mathcal{L}_\alpha[\hat{\rho}^R]\} = E_\alpha \gamma_\alpha (f_\alpha - \sigma_{\alpha\alpha}) - \frac{\hbar \gamma_\alpha}{2} (\epsilon \sigma_{lu} + \epsilon^* \sigma_{ul}) \quad (A6)
\]

and

\[
\hat{N}_\alpha = \text{Tr}\{(\hat{c}^\dagger \hat{c}_u + \hat{c}_l \hat{c}_l) \mathcal{L}_\alpha[\hat{\rho}]\} = \gamma_\alpha (f_\alpha - \sigma_{\alpha\alpha}) . \quad (A7)
\]

In the steady state, Eq. (A3) provides

\[
\sigma_{ss}^{ul} = -\frac{\epsilon(\sigma_{uu} - \sigma_{ll})}{\Delta + i(\gamma_u + \gamma_l)/2} \quad (A8)
\]

Thus, we can relate the transition rate (A4) to the real part of \( \epsilon^* \sigma_{ss}^{ul} \) by

\[
\mathbb{R}\{\epsilon^* \sigma_{ss}^{ul}\} = -\frac{2\Delta}{\gamma_u + \gamma_l} \mathbb{R}\{\epsilon^* \sigma_{ss}^{ul}\} = -\frac{\Delta}{\gamma_u + \gamma_l} R_{ss} \quad (A9)
\]

In the steady state, Eqs. (A1,A2) become

\[
\gamma_u(f_u - \sigma_{uu}) = R_{ss} \quad \text{and} \quad \gamma_l(f_l - \sigma_{ll}) = -R_{ss} \quad (A10)
\]
and thus Eq. (A7) provides Eq. (12) of the main text. Inserting Eqs. (A9, A10) into the energy flows (A6) from the leads provides Eqs. (13, 14), which are the key result. Eqs. (A4, A8) provide the steady state transition rate

\[ R^{ss} = \alpha (\sigma_{uu} - \sigma_{ll}) \quad \text{with} \quad \alpha = \frac{|\epsilon|^2(\gamma_u + \gamma_l)}{(\gamma_u + \gamma_l)^2/4 + \Delta^2} \]

so that

\[ \sigma_{uu}^{ss} - \sigma_{ll}^{ss} = H(|\epsilon|^2)(f_u - f_l) \quad \text{with} \quad H(|\epsilon|^2) = \frac{\gamma_u \gamma_l}{\gamma_u \gamma_l + \alpha(\gamma_l - \gamma_l)} \quad \text{and} \quad H(|\epsilon|^2) = \frac{\gamma_u \gamma_l}{\gamma_u \gamma_l + \alpha(\gamma_u - \gamma_l)} \quad \text{(A11)} \]

and we find

\[ R^{ss} = \alpha H(f_u - f_l) \quad \text{(A12)} \]

with positive \( \alpha \) and \( H \).

**Appendix B: Calculations the quantum case**

Defining the average occupation of the photon mode \( n_{ph} = \text{Tr}\{a^\dagger a\} \) and \( Y = g^* \text{Tr}\{c^\dagger \hat{c}_u a^\dagger \hat{\rho}\} \), Eq. (18) provides the equations of motion

\[
\frac{d}{dt} \sigma_{uu} = \gamma_u (f_u - \sigma_{uu}) + i(Y - Y^*) \quad \text{(B1)}
\]
\[
\frac{d}{dt} \sigma_{ll} = \gamma_l (f_l - \sigma_{ll}) - i(Y - Y^*) \quad \text{(B2)}
\]
\[
\frac{d}{dt} n_{ph} = \gamma_b (n_b - n_{ph}) - i(Y - Y^*) \quad \text{(B3)}
\]
\[
\frac{d}{dt} Y = i\Delta_{cav} Y + i|g|^2 F - \frac{\gamma_u + \gamma_l + \gamma_b}{2} Y \quad \text{(B4)}
\]

with \( F = \text{Tr}\{c^\dagger \hat{c}_u (1 - c^\dagger \hat{c}_l) \hat{\rho}\} \) and \( Y^{ss} = -\frac{|g|^2 F}{\Delta_{cav} + i\Delta_{cav} + \gamma_u + \gamma_l + \gamma_b} \), which implies

\[
\Re\{Y^{ss}\} = -\frac{2\Delta_{cav} \Im\{Y^{ss}\}}{\gamma_u + \gamma_l + \gamma_b} = -\frac{\Delta_{cav} R^{ss}}{\gamma_u + \gamma_l + \gamma_b} \quad \text{(B10)}
\]

Furthermore, Eqs. (B1, B2, B3) provide

\[
\gamma_u (f_u - \sigma_{uu}) = R^{ss}, \quad \gamma_l (f_l - \sigma_{ll}) = -R^{ss}, \quad \text{and} \quad \gamma_b (n_b - n_{ph}) = -R^{ss} \quad \text{(B11)}
\]

Inserting into Eq. (A7) (which is the same here), we obtain Eq. (19). Inserting the relations (B10, B11) into the energy flows (B7, B8), we find in the steady state Eqs. (20, 21).

From Eqs. (B6, B9) we get \( R^{ss} = \alpha Q F \) with

\[
\alpha Q = \frac{|g|^2(\gamma_u + \gamma_l + \gamma_b)}{\Delta_{cav}^2 + (\gamma_u + \gamma_l + \gamma_b)/4} > 0
\]
so that the sign of $R^{ss}$ equals the sign of $F$. Using a Hartree-Fock like approximations, Eq. (B5) provides

$$F \approx \sigma_{uu}(1 - \sigma_{ll}) + (\sigma_{uu} - \sigma_{ll})n_{ph}. \quad (B12)$$

Then straightforward algebra shows

$$R^{ss} = \alpha Q(1 - \sigma_{ll})(1 - \sigma_{uu})(1 + n_{ph}) \left( \frac{\sigma_{uu}}{1 - \sigma_{uu}} - \frac{\sigma_{ll}}{1 - \sigma_{ll}} \times \frac{n_{ph}}{1 + n_{ph}} \right) \quad (B13)$$

where we assume $f_u, f_l, \sigma_{uu}, \sigma_{ll} < 1$ as appropriate for fermionic states, which are not entirely occupied due to their contribution in a dynamical process. Similarly, Eqs. (B11) can be rewritten as

$$R^{ss} = \gamma_u(f_u - \sigma_{uu}) = \gamma_u(1 - \sigma_{uu})(1 - f_u) \left( \frac{f_u}{1 - f_u} - \frac{\sigma_{uu}}{1 - \sigma_{uu}} \right) \quad (B14)$$

$$R^{ss} = \gamma_l(\sigma_{ll} - f_l) = \gamma_l(1 - \sigma_{ll})(1 - f_l) \left( \frac{\sigma_{ll}}{1 - \sigma_{ll}} - \frac{f_l}{1 - f_l} \right) \quad (B15)$$

$$R^{ss} = \gamma_b(n_{ph} - n_b) = \gamma_b(1 + n_{ph})(1 + n_b) \left( \frac{n_{ph}}{1 + n_{ph}} - \frac{n_b}{1 + n_b} \right) \quad (B16)$$

Applying Eqs. (B14 B15 B16) and subsequently Eq. (B13) provides

$$R^{ss} = 0 \Rightarrow \frac{\sigma_{uu}}{1 - \sigma_{uu}} = \frac{f_u}{1 - f_u}, \quad \frac{\sigma_{ll}}{1 - \sigma_{ll}} = \frac{f_l}{1 - f_l} \quad \text{and} \quad \frac{n_{ph}}{1 + n_{ph}} = \frac{n_b}{1 + n_b} \Rightarrow \frac{f_u}{1 - f_u} = \frac{f_l}{1 - f_l} \times \frac{n_b}{1 + n_b}$$

$$R^{ss} > 0 \Rightarrow \frac{\sigma_{uu}}{1 - \sigma_{uu}} < \frac{f_u}{1 - f_u}, \quad \frac{\sigma_{ll}}{1 - \sigma_{ll}} > \frac{f_l}{1 - f_l} \quad \text{and} \quad \frac{n_{ph}}{1 + n_{ph}} > \frac{n_b}{1 + n_b} \Rightarrow \frac{f_u}{1 - f_u} > \frac{f_l}{1 - f_l} \times \frac{n_b}{1 + n_b}$$

$$R^{ss} < 0 \Rightarrow \frac{\sigma_{uu}}{1 - \sigma_{uu}} > \frac{f_u}{1 - f_u}, \quad \frac{\sigma_{ll}}{1 - \sigma_{ll}} < \frac{f_l}{1 - f_l} \quad \text{and} \quad \frac{n_{ph}}{1 + n_{ph}} < \frac{n_b}{1 + n_b} \Rightarrow \frac{f_u}{1 - f_u} < \frac{f_l}{1 - f_l} \times \frac{n_b}{1 + n_b}$$

As the right hand-side needs to satisfy one of the three relations, we find equivalence for all relations. This provides the condition (23).

---

**Appendix C: Generalization for arbitrary systems with classical fields**

The approach for the 2-level system used in the main article can be generalized to arbitrary fermionic systems with a time-periodic Hamiltonian $H_S(t) = \hat{H}_S(t + \tau)$ as characteristic for a classical optical field. This includes an arbitrary number of reservoirs, which are coupled to the system in the local form (8), where each reservoir $\alpha$ provides transitions to a unique level $\alpha$ of the system. Here, we assume that the system eventually reaches a steady state $\dot{\rho}^{ss}(t)$ which is periodic with period $\tau$ following the external driving in $\hat{H}_S(t)$ after initial conditions died out due the dissipative terms in the quantum evolution. We define

$$\dot{U}^{\alpha}_{av} = \left\langle \text{Tr}\{\hat{H}_S(t)\mathcal{L}_\alpha[\hat{\rho}^{ss}(t)]\} \right\rangle \quad (C1)$$

$$\dot{N}^{\alpha}_{av} = \left\langle \text{Tr}\{\hat{\alpha}_\alpha^\dagger \hat{\alpha}_\alpha \mathcal{L}_\alpha[\hat{\rho}^{ss}(t)]\} \right\rangle \quad (C2)$$

where the averaging

$$\langle f(t) \rangle = \frac{1}{\tau} \int_{t}^{t+\tau} f(t')dt'$$

is taken over the common period of $\dot{\rho}^{ss}(t)$ and $\hat{H}_S(t)$. Thus $\dot{U}^{\alpha}_{av}$ and $\dot{N}^{\alpha}_{av}$ do not depend on time. This allows for the definition of effective energies

$$\tilde{E}_\alpha = \frac{\dot{U}^{\alpha}_{av}}{N^{\alpha}_{av}} \quad (C3)$$

which are the average energies taken from the reservoir $\alpha$ per particle transferred into the system. It appears natural to apply these energies $\tilde{E}_\alpha$ for the energy dependence (10) of the occupation function $f_\alpha$ (and possibly also $\gamma_\alpha$, if the wide band limit is not applicable). This requires a self-consistent solution for the steady state of the dynamical equation (4) with its input parameters.

In the remaining part of this section, it is shown, that this procedure is thermodynamic consistent. The energy is conserved and the entropy production is semi-positive for the steady state averaged over a period. This follows concepts used in [60] for periodically driven systems in a related context.

The general expressions (56) provide

$$\frac{d}{dt} \text{Tr}\{\hat{H}_S(t)\dot{\rho}(t)\} = \sum_{\alpha} \dot{U}^{\alpha}_{av} + P_S.$$ 

For the steady state the average change $\langle \frac{d}{dt} \text{Tr}\{\hat{H}_S(t)\dot{\rho}^{ss}(t)\} \rangle$ vanishes due to the integration over one period and we directly get $\sum_{\alpha} \dot{U}^{\alpha}_{av} + P_S^{av} = 0$ showing that the average external power and energy...
Using Eq. (C2) we find upon averaging in the steady state we get its temporal change in heat transfer $\dot{Q}$, entropy of the system and the reservoirs (changing by up to zero. This is just energy conservation for the currents from the reservoirs going into the system add.

The total entropy $S_{\text{tot}}$ is given by the von Neumann entropy of the system and the reservoirs (changing by $\dot{Q}$.

$$\frac{dS_{\text{tot}}}{dt} = -k_B \frac{d}{dt} \text{Tr}\{\hat{\rho}(t) \ln \hat{\rho}(t)\} - \sum_{\alpha} \dot{U}_{\alpha} - \mu_{\alpha} \dot{N}_{\alpha}$$

Upon averaging in the steady state we get

$$\dot{S}_{\text{av}} = \left\langle \frac{dS_{\text{tot}}}{dt} \right\rangle_{\text{av}} = -\sum_{\alpha} \dot{E}_{\alpha} - \mu_{\alpha} \dot{N}_{\alpha}$$

Using Eq. (C2) we find

$$\dot{S}_{\text{av}} = k_B \left\langle \sum_{\alpha} \text{Tr}\{\mathcal{L}_{\alpha} [\hat{\rho}^\alpha(t)] \ln \hat{\rho}^\alpha(t)\} \right\rangle$$

with the locally thermal operator $\hat{\rho}^\mu = \frac{1}{Z^\mu} \prod_{\alpha} \text{exp} \left( -\frac{\mathcal{E}_{\alpha} - \mu_{\alpha} \hat{a}_{\alpha}^\dagger \hat{a}_{\alpha}}{k_B T_{\alpha}} \right)$ where the number $Z^\mu$ renormalises the trace to unity (and drops out as $\text{Tr}\{\mathcal{L}_{\alpha} [\hat{\rho}]\} = 0$). As Eq. (4) gives

$$\frac{d}{dt} \text{Tr}\{\hat{\rho}(t) \ln \hat{\rho}(t)\} = \sum_{\alpha} \text{Tr}\{\mathcal{L}_{\alpha} [\hat{\rho}(t)] \ln \hat{\rho}(t)\}$$

we find for steady-state averaging

$$0 = k_B \left\langle \sum_{\alpha} \text{Tr}\{\mathcal{L}_{\alpha} [\hat{\rho}^\alpha(t)] \ln \hat{\rho}^\alpha(t)\} \right\rangle$$

which we can subtract from Eq. (C4) resulting in

$$\dot{S}_{\text{av}} = k_B \left\langle \sum_{\alpha} \text{Tr}\{\mathcal{L}_{\alpha} [\hat{\rho}^\alpha(t)] (\ln \hat{\rho}^\mu - \ln \hat{\rho}^\alpha(t))\} \right\rangle$$

As $\hat{\rho}^\mu$ satisfies $\mathcal{L}_{\alpha} [\hat{\rho}^\mu] = 0$ if the occupations (16) are applied, we find $S_{\text{av}} \geq 0$ by Spohn’s inequality [19, 47, 48]. Thus, entropy production is positive semi-definite for the average steady state.

References:

[1] A. E. Siegman, *Lasers* (University Science Books, Mill Valley, 1986).
[2] H. Willenberg, G. H. Döhler, and J. Faist, Intersubband gain in a Bloch oscillator and quantum cascade laser, Phys. Rev. B **67**, 085315 (2003).
[3] R. Terazzi, T. Gresch, M. Giovannini, N. Hoyler, N. Sekine, and J. Faist, Bloch gain in quantum cascade lasers, Nat. Phys. **3**, 329 (2007).
[4] E. Boukobza and D. J. Tannor, Three-level systems as amplifiers and attenuators: A thermodynamic analysis, Phys. Rev. Lett. **98**, 240601 (2007).
[5] H. E. D. Scovil and E. O. Schulz-DuBois, Three-level masers as heat engines, Phys. Rev. Lett. **2**, 262 (1959).
[6] I. Roos and K. Mölmer, Quantum computing with an inhomogeneously broadened ensemble of ions: Suppression of errors from detuning variations by specially adapted pulses and coherent population trapping, Physical Review A **69**, 022321 (2004).
[7] M. Saffman, T. G. Walker, and K. Mölmer, Quantum information with Rydberg atoms, Rev. Mod. Phys. **82**, 2313 (2010).
[8] F. Binder, L. A. Correa, C. Gogolin, J. Anders, and G. Adesso, eds., *Thermodynamics in the Quantum Regime: Fundamental Aspects and New Directions* (Springer International Publishing, Cham, 2018).
[9] A. Streitsova, G. Adesso, and M. B. Plenio, Colloquium: Quantum coherence as a resource, Rev. Mod. Phys. **89**, 041003 (2017).
[10] G. Benenti, G. Casati, K. Saito, and R. S. Whitney, Fundamental aspects of steady-state conversion of heat to work at the nanoscale, Phys. Rep. **694**, 1 (2017).
[11] S. Deffner and S. Campbell, *Quantum Thermodynamics*, 2053-2571 (Morgan and Claypool, 2019).
[12] A. Levy and R. Kosloff, The local approach to quantum transport may violate the second law of thermodynamics, Europhys. Lett. **107**, 20004 (2014).
[13] P. P. Hofer, M. Perarnau-Llobet, L. D. M. Miranda, G. Haack, R. Silva, J. B. Brask, and N. Brunner, Markovian master equations for quantum thermal machines: local versus global approach, New Journal of Physics **19**, 123037 (2017).
[14] J. O. González, L. A. Correa, G. Nocerino, J. P. Palao, D. Alonso, and G. Adesso, Testing the validity of the ‘local’ and ‘global’ GKLS master equations on an exactly solvable model, Open Systems & Information Dynamics **24**, 1740010 (2017).
[15] W. Khan, P. P. Potts, S. Lehmann, C. Thelander, K. A. Dick, P. Samuelsson, and V. F. Maisi, Efficient and continuous microwave photoconversion in hybrid cavity-semiconductor nanowire double quantum dot diodes, Nat. Commun. **12**, 5130 (2021).
[16] A. A. S. Kalaee and A. Wacker, Positivity of entropy production for the three-level maser, Phys. Rev. A **103**, 012202 (2021).
[17] R. Alicki and R. Kosloff, Introduction to quantum thermodynamics: History and prospects, in *Thermodynamics in the Quantum Regime: Fundamental Aspects and New Directions* edited by F. Binder, L. A. Correa, C. Gogolin, J. Anders, and G. Adesso (Springer International Publishing, Cham, 2018) pp. 1–33.
[18] W. Pusz and S. L. Woronowicz, Passive states and KMS states for general quantum systems, Comm. Math. Phys. **82**, 329 (2007).
10

[19] H. Spohn and J. L. Lebowitz, Irreversible thermodynamics for quantum systems weakly coupled to thermal reservoirs, in *Advances in Chemical Physics* Vol. 38, edited by S. A. Rice (John Wiley & Sons, Ltd, 1978) Chap. 2, pp. 109–142.

[20] R. Alicki, The quantum open system as a model of the heat engine, *J. Phys. A* **12**, L103 (1979).

[21] R. Kosloff and Y. Rezek, The quantum harmonic otto cycle, *Entropy* **19**, 10.3390/e1904136 (2017).

[22] J. D. Jackson, *Classical Electrodynamics*, 3rd ed. (John Wiley & Sons, New York, 1998).

[23] K. E. Dorfman, K. R. Chapin, C. H. R. Ooi, A. A. Svidzinsky, and M. O. Scully, Quantum thermodynamics of photo and solar cells, *AIP Conference Proceedings* **1411**, 256 (2011).

[24] R. Alicki, D. Gelbwaser-Klimovsky, and A. Jenkins, A thermodynamic cycle for the solar cell, *Ann. Phys.* **378**, 71 (2017).

[25] J. Restrepo, C. Ciuti, and I. Favero, Single-polariton optomechanics, *Phys. Rev. Lett.* **112**, 013601 (2014).

[26] E. Geva and R. Kosloff, The quantum heat engine and heat pump: An irreversible thermodynamic analysis of the three-level amplifier, *The Journal of Chemical Physics* **104**, 7681 (1996).

[27] K. Szczygieliski, D. Gelbwaser-Klimovsky, and R. Alicki, Markovian master equation and thermodynamics of a two-level system in a strong laser field, *Phys. Rev. E* **87**, 012120 (2013).

[28] C. Elouard, D. Herrera-Martí, M. Esposito, and A. Auffèves, Thermodynamics of optical Bloch equations, *New Journal of Physics* **22**, 103039 (2020).

[29] R. Hotz and G. Schaller, Coarse-graining master equation for periodically driven systems, *Phys. Rev. A* **104**, 052219 (2021).

[30] G. Lindblad, On the generators of quantum dynamical semigroups, *Comm. Math. Phys.* **48**, 119 (1976).

[31] H.-P. Breuer and F. Petruccione, *Open Quantum Systems* (Oxford University Press, Oxford, 2006).

[32] J. T. Stockburger and T. Motz, Thermodynamic deficiencies of some simple Lindblad operators, *Fortschritte der Physik* **65**, 1600067 (2017).

[33] C. Bergenfeldt, P. Samuelsson, B. Sothmann, C. Flindt, and M. Büttiker, Hybrid microwave-cavity heat engine, *Phys. Rev. Lett.* **112**, 076803 (2014).

[34] W. Niedenzu, M. Huber, and E. Boukobza, Concepts of work in autonomous quantum heat engines, *Quantum* **3**, 195 (2019).

[35] E. T. Jaynes and F. W. Cummings, Comparison of quantum and semiclassical radiation theories with application to the beam maser, *Proc. IEEE* **51**, 89 (1963).

[36] P. Santhanam, D. J. Gray, and R. J. Ram, Thermoelectrically pumped light-emitting diodes operating above unity efficiency, *Phys. Rev. Lett.* **108**, 097403 (2012).

[37] T. Sadi, I. Radevici, and J. Oksanen, Thermophotonic cooling with light-emitting diodes, *Nat. Photonics* **14**, 205 (2020).

[38] L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics* (Cambridge University Press, 1995).

[39] J. H. Shirley, Validity of the semiclassical approximation in maser theory, *Phys. Rev.* **181**, 600 (1969).

[40] M. Helm, The basic physics of intersubband transitions, in *Intersubband Transitions in Quantum wells* Semiconductor and Semimetals, Vol. 62, edited by H. Liu and F. Capasso (Elsevier, 1999) pp. 1 – 99.

[41] S. A. Ktitorov, G. S. Simin, and V. Y. Sindalovskii, Bragg reflections and the high-frequency conductivity of an electronic solid-state plasma, *Sov. Phys.–Sol. State* **13**, 1872 (1972), [Fizika Tverdogo Tela **13**, 2230 (1971)].

[42] A. Wacker, Semiconductor superlattices: a model system for nonlinear transport, *Phys. Rep.* **357**, 1 (2002).

[43] N. Opačak, S. D. Cin, J. Hillbrand, and B. Schwarz, Frequency comb generation by bloch gain induced giant kerr nonlinearity, *Phys. Rev. Lett.* **127**, 093902 (2021).

[44] A. Wacker, Lasers: Coexistence of gain and absorption, *Nat. Phys.* **3**, 298 (2007).

[45] P. P. Potts, A. A. S. Kalaee, and A. Wacker, A thermodynamically consistent markovian master equation beyond the secular approximation, *New Journal of Physics* **23**, 123013 (2021).

[46] R. Kosloff, Quantum thermodynamics: A dynamical viewpoint, *Entropy* **15**, 2100 (2013).

[47] H. Spohn, Entropy production for quantum dynamical semigroups, *Journal of Mathematical Physics* **19**, 1227 (1978).

[48] S.-W. Li, Production rate of the system-bath mutual information, *Phys. Rev. E* **96**, 012139 (2017).