Note on a polytropic $\beta$–model to fit the X-ray surface brightness of clusters of galaxies

Stefano Ettori
Institute of Astronomy, Madingley Road, Cambridge CB3 0HA
settori@ast.cam.ac.uk

MNRAS, 311, 313 (2000)

ABSTRACT
In this note, I suggest that the $\beta$–model used to fit the X-ray surface brightness profiles of extended sources, like groups and clusters of galaxies, has to be corrected when the counts are collected in a wide energy band comparable to the mean temperature of the source and a significant gradient in the gas temperature is observed. I present a revised version of the $\beta$–model for the X-ray brightness that applies to an intracluster gas with temperature and density related by a polytropic equation and extends the standard version that is strictly valid for an isothermal gas. Given a temperature gradient observed through an energy window of 1–10 keV typical for the new generation of X-ray observatories, the $\beta$ parameter can change systematically up to 20 per cent from the value obtained under isothermal assumption, i.e. by an amount larger that any statistical uncertainty obtained from the present data. Within the virial regions of typical clusters of galaxies, these systematic corrections affect the total gravitating mass estimate by 5–10 per cent, when compared to the measurements obtained under the isothermal assumption.

Key words: galaxies: clustering – X-ray: galaxies.

1 INTRODUCTION
The $\beta$–model is widely used in the X-ray astronomy to parametrise the gas density profile in groups and clusters of galaxies fitting their surface brightness profile. Cavaliere & Fusco-Femiano (1976, 1978) note that “... since both gas and galaxies distributions conforms to the same gravitational potential, the former can be directly related to the latter; the galaxies may be considered as tracers of the total potential well ...”:

$$\frac{1}{\rho_{\text{gal}}} \frac{dP_{\text{gal}}}{dr} = \frac{1}{\rho_{\text{gas}}} \frac{dP_{\text{gas}}}{dr}$$

Using the King approximation (1962) to the inner portions of an isothermal sphere (Lane-Emden equation in Binney & Tremaine 1987, note that the King approximation is proportional to $r^{-3}$ at the outer radii, whereas the isothermal sphere is proportional to $r^{-2}$; cf. Figure 1);

$$\rho_{\text{gal}} = \rho_{0,\text{gal}}(1 + x^2)^{-3/2}, \quad x = r/r_c$$

and the perfect gas law, one obtains the formula:

$$\rho_{\text{gas}} = \rho_{0,\text{gas}}(1 + x^2)^{-3/2},$$

where the different energy distribution of the gas and galaxies is parameterized by using the parameter $\beta \sim (\sigma_{\text{gal}}^2/T_{\text{gas}})$, where $\sigma_{\text{gal}}$ is the galaxies velocity dispersion and $T_{\text{gas}}$ is the temperature of the gas.

The surface brightness profile observed at the projected radius $b$, $S(b)$, is the projection on the sky of the plasma emissivity, $\epsilon(r)$:

$$S(b) \equiv S_b = \int_b^\infty \frac{\epsilon \, dr^2}{\sqrt{r^2 - b^2}}.$$  (4)

(Hereafter I adopt $r$ as symbol for the projected radius $b$).

The emissivity is equal to

$$\epsilon(r) = \Lambda(T_{\text{gas}}) \, n_p^2 \, \text{erg} \, s^{-1} \, \text{cm}^{-3},$$

where $n_p = \rho_{\text{gas}}/(2.21 \, \mu \text{m} \, \text{p})$ is the proton density and the cooling function, $\Lambda(T_{\text{gas}})$, depends upon the mechanism of the emission and can be represented as

$$\Lambda(T_{\text{gas}}) = \frac{\overline{\epsilon}}{\alpha} T_{\text{gas}}^{\beta - 1},$$

where $\overline{\epsilon}$ is the velocity averaged Gaunt factor that is equal to about 1.2 within an accuracy of 20% for a bolometric emissivity (for example, at $T_{\text{gas}} > 2.5$ keV, the emission is mainly due to bremsstrahlung and the cooling function can be written with $\lambda \sim 10^{-23}$ and $\alpha = 0.5$; see, e.g., Sarazin 1988).

Assuming isothermality and a $\beta$-model for the gas density (eq. b), the surface brightness profile has an analytic solution:

$$S_b = \sqrt{\pi n_0^2 r_c} \Lambda(T_{\text{gas}}) \frac{\Gamma(3\beta - 0.5)}{\Gamma(3\beta)} (1 + x^2)^{0.5 - 3\beta}$$

$$= S_0(1 + x^2)^{0.5 - 3\beta},$$  (7)

* Throughout this note, a Hubble constant of $50 \, h_{50} \, \text{km} \, \text{s}^{-1} \, \text{Mpc}^{-1}$ is considered.
that is strictly valid under the condition that \(3\beta > 0.5\) and the cooling function \(\Lambda(T_{\text{gas}})\) does not change radially.

In this note I will focus on energies typical for clusters of galaxies, considering the fact that there is now evidence for a decrease in the gas temperature in the outer parts of clusters (Markevitch et al. 1998). Even if these results conflict with studies from other groups which indicate that clusters are generally isothermal (e.g. Irwin et al. 1999, Kikuchi et al. 1999, White 1999), I highlight how a temperature gradient can affect the estimate of the \(\beta\) parameter.

In the following discussion, I will assume that the cluster gas density, \(n_{\text{gas}}\), is well described by a \(\beta\) - model and the gas is in the polytropic state, so that

\[
\frac{T_{\text{gas}}}{T_0} = \left( \frac{n_{\text{gas}}}{n_0} \right)^{\gamma - 1},
\]

where the polytropic index \(\gamma\) ranges between 1 and 5/3, the limits corresponding to the gas being isothermal and adiabatic, respectively.

\[\alpha = \log(\epsilon/\epsilon^{\max}) / \log(T_{\text{gas}}/T_{\text{gas}}^{\max}).\]

\[\Delta E (\text{keV}) \Delta T_{\text{gas}} (\text{keV}) \quad \alpha (Z = 0.3Z_\odot) \quad \alpha (Z = 1Z_\odot)\]

\begin{tabular}{ccc}
\hline
bol & 5-10 & 0.45 & 0.39 \\
5-7 & 0.44 & 0.37 \\
3-6 & 0.43 & 0.36 \\
3-5 & 0.42 & 0.34 \\
0.5-2 & 5-10 & -0.13 & -0.20 \\
5-7 & -0.10 & -0.20 \\
3-6 & -0.08 & -0.21 \\
3-5 & -0.06 & -0.22 \\
1-10 & 5-10 & 0.25 & 0.16 \\
5-7 & 0.37 & 0.27 \\
3-6 & 0.47 & 0.36 \\
3-5 & 0.54 & 0.41 \\
\hline
\end{tabular}

2 A POLYTROPIC \(\beta\) - MODEL

When the cluster temperature is above the energy range of the detector with a narrow bandpass (e.g. ROSAT), the emission measure will have a negligible dependence on the temperature (Figure 2).

However, this dependence becomes significant when the energy band is wide and its mean energy range is comparable to the
mean temperature of the cluster. For example, the emissivity due to free-free radiation for a plasma temperature of 10 keV is 50 per cent larger than one at 4 keV in the energy range [1-10] keV, whereas it changes by 1 per cent in the ROSAT bandpass [0.5-2] keV. In Figure 2, I show how the total cluster emissivity convolved with a given energy bandpass of a X-ray telescope depends upon the plasma temperature.

A proper deprojection analysis (cf. White et al. 1997, Ettori & Fabian 1999) will be necessary to evaluate the emissivity in differential volume shells and recover both the gas density and temperature profile. On the other hand, this technique is computationally very expensive and a simple fitting procedure can be preferred in most cases where the gas and total mass distributions are investigated.

As discussed above, Cavaliere & Fusco-Femiano have introduced the $\beta$-model for an isothermal gas distribution. But if, as usually done, the gas density distribution is obtained from the parameters of the best fit of eqn. 9 for the surface brightness profile, any temperature gradient present in the plasma will not be taken properly into account. To consider this correction, I assume that the gas density profile is well described by a $\beta$-model (eqn. 3) and use the temperature dependence of the emissivity (given in eqn. 4 through the parameter $\alpha$) and a polytropic relation between gas density and temperature (represented by the index $\gamma$ in eqn. 5) in the definition of the surface brightness (eqn. 6). Then, eqn. 6 can be re-written as:

$$S_b = \int_{r_0}^{\infty} \rho \, d\rho,$$

where $\beta' = \beta + \frac{1}{2}$. The uncorrected measured value.

In Table 1, I calculate the $\alpha$ values for a set of interesting cases.

It is worth noting that, even if the observed gas temperature, $T_{\text{gas}}$, is the projection on the sky of the real temperature, $T_{\text{gas}}^{\text{real}}$, weighted by the cluster emission, $T_{\text{gas}}$ does not depend upon the parameter $\alpha$ (see also Markevitch et al. 1999):

$$T_{\text{gas}}^{\text{proj}}(b) \equiv T_{\text{gas}} = \int_{r_0}^{\infty} \rho \, d\rho,$$

$$\propto \frac{(1 + x^2)^{0.5 - 1.5\beta(2 + \gamma - 1)(\alpha + 1)}}{(1 + x^2)^{0.5 - 1.5\beta(2 + \alpha)(\gamma - 1)}} \propto (1 + x^2)^{-1.5\beta(\gamma - 1)}.$$

I can then estimate the corrections that a polytropic temperature profile produces on the uncorrected values $(\beta', \gamma')$. To do this, I solve the system of equations given by the surface brightness and temperature profiles modeled with a $\beta$-model:

$$\begin{align*}
S_0(1 + x^2)^{0.5 - 3\beta'} &= S_0(1 + x^2)^{0.5 - 3\beta(1 + \alpha^{2\beta - 1})} \\
T_0(1 + x^2)^{-1.5\beta'(\gamma' - 1)} &= T_0(1 + x^2)^{-1.5\beta(\gamma - 1)}
\end{align*}$$

that can be simplified to:

$$\begin{align*}
\beta' &= \beta \left(1 + \alpha^{2\beta - 1}\right) \\
\gamma' &= \frac{\gamma + \alpha^{2\beta - 1}}{\gamma - 1}
\end{align*}$$

This system has the following solution:

$$\begin{align*}
\beta &= \beta' \left(1 + \alpha/2 - \alpha\gamma'/2\right) \\
\gamma &= \frac{\gamma' + \alpha^{2\beta - 1}}{\gamma - 1}
\end{align*}$$

Figure 3 shows the relative systematic corrections that affect the values of $(\beta, \gamma)$ for a given set of $(\alpha, \gamma')$.

In practice, once a temperature profile is measured in a known energy bandpass, the $\alpha$ parameter can be defined. The conversion from the count rate to the flux can be done assuming the central, highest temperature (i.e. with the largest emissivity, cf. Table 1). The functional forms, $T_0 = S_0(1 + x^2)^b$ and $T_{\text{gas}} = T_0(1 + x^2)$, can be then fitted to the surface brightness and temperature profiles, respectively. If they still represent a good model of the data, the correct $(\beta, \gamma)$ values can be estimated:

$$\begin{align*}
\beta &= \frac{\beta'}{2 - \gamma'} \frac{2\beta}{2 - 2\beta - \gamma'} \\
\gamma &= \frac{1 + \alpha^{2\beta - 1}}{2 - \gamma'}
\end{align*}$$

The consequences of these corrections are discussed below.

3 DISCUSSION AND CONCLUSIONS

I suggest that the $\beta$-model used to fit the X-ray surface brightness profiles of clusters of galaxies has to be corrected when the data from the next generation of X-ray observatories will be available. In fact, to include any temperature gradient that will affect the cluster emissivity as observed in a large energy window, we have to extend the use of the $\beta$-model to the polytropic case. The new form of this polytropic $\beta$-model is given in eqn. 11.
The spatially resolved surface brightness profiles obtained with the ROSAT observatory allow estimates of $\beta$ with an accuracy (1σ) of about 2 per cent (e.g., Mohr et al. 1999, Neumann & Arnaud 1999). However, the limited energy band pass of ROSAT does not arise any problem on the application of the $\beta$-model when a temperature gradient is observed in the intracluster medium. This is also true for any surface brightness profile that is obtained collecting photons within an instrumental band pass at energies lower than the mean plasma temperature. In particular, considering that the effective area of the present X-ray detectors is larger at $\sim$1–2 keV, an energy window around these values can be a good choice for a temperature-independent emissivity for hot clusters of galaxies. But an energy range around few keV is still problematic for cool clusters and groups of galaxies that present temperature gradients.

The new generation of X-ray observatories, e.g., Chandra and XMM will operate in a wider energy band (e.g. [1-10] keV) than ROSAT and will provide a more accurate estimates of the parameters of the $\beta$-model. Therefore, as shown above, the presence of a plasma temperature gradient will affect the use of the $\beta$-model with a systematic uncertainty comparable or larger than any statistical error. In particular, the estimate of (i) the total gravitating mass, $M_{\text{tot}} \propto \beta \gamma (1 + x^2)^{-1.5\beta/(\beta - 1)}$ (cf. eqn. [A1]), (ii) the gas mass, $M_{\text{gas}} \propto \int x^3 (1 + x^2)^{-1.5\beta} x^2 dx$, and (iii) the consequent gas fraction, $f_{\text{gas}}$, can be quite significantly affected. For example, given a cluster with a typical core radius, $r_c$, of 0.3 Mpc and a radial decrease of the plasma temperature from 6 to 3 keV with uncorrected parameters $\beta = 2/3$ and $\gamma = 1.20$ (e.g. Markevitch et al. 1999), corrections of +8, -4, +12 per cent on $M_{\text{gas}}, M_{\text{tot}}, f_{\text{gas}}$, respectively, will be necessary at $r = 1$ Mpc. At the more physically meaningful radius $r_{200}$, where the mean cluster density is $\Delta = 200$ times the critical value (cf. eqn. [2] in Appendix), the corrections are +11, -5, +17 per cent, respectively. These corrections increase considerably up to +56, -10, +74 per cent at $r = 2$ Mpc for $M_{\text{gas}}, M_{\text{tot}}, f_{\text{gas}}$, respectively, when $\alpha = 0.5, \beta = 2/3, \gamma = 5/3$ (cf. Table 2).

TABLE 2. Changes of the derived quantities after that the corrections of the best-fit parameters are considered: unc(%) = \( 100 \times (Q_{\text{corr}} - Q_{\text{unc-corr}}) / Q_{\text{unc-corr}} \).

| $r$ (Mpc) | $\Delta M_{\text{gas}}$ | $\Delta M_{\text{tot}}$ | $\Delta f_{\text{gas}}$ |
|-----------|-------------------------|-------------------------|-------------------------|
| 1         | +8%                     | -4%                     | +12%                    |
| 2         | +13%                    | -4%                     | +18%                    |
| $r_{200}$ | +11%                    | -5%                     | +17%                    |
| $\alpha = 0.47, \beta = 2/3, \gamma = 1.20$ |
| 1         | +31%                    | -10%                    | +46%                    |
| 2         | +56%                    | -10%                    | +74%                    |
| $r_{200}$ | +34%                    | -10%                    | +48%                    |

Acknowledgements

I acknowledge the support of the Royal Society. Andy Fabian and David White are thanked for an useful reading of the manuscript.

REFERENCES

Arnaud K.A., 1996, "Astronomical Data Analysis Software and Systems V", eds. Jacoby G. and Barnes J., ASP Conf. Series vol. 101, 17
Binney J., Tremaine S., 1987, *Galactic Dynamics*, Princeton University Press
Cavaliere A., Fusco-Femiano R., 1976, A&A, 49, 137
Cavaliere A., Fusco-Femiano R., 1978, A&A, 70, 677
Cowie L.L., Henriksen M.J., Mushotzky R.F., 1987, ApJ, 317, 593
Ettori S., Fabian A.C., 1999, MNRAS, 305, 834
Henriksen M.J., Mushotzky R.F., 1986, ApJ, 302, 287
Hughes J.P., Yamashita K., Okumura Y., Tsunemi H., Matsuoka M., 1988, ApJ, 327, 615
Kaastra J.S., 1992, *An X-Ray Spectral Code for Optically Thin Plasmas* (Internal SRON-Leiden Report, updated version 2.0)
Kikuchi K., Furusho T., Ezawa H., Yamasaki N.Y., Ohashi T., Fukazawa Y., Ikebe Y., 1999, PASJ, 51, 301
King, I. R., 1962, AJ, 67, 471
Irwin J.A., Bregman J.N. & Evrard A.E., 1999, ApJ, 519, 518
Liedahl D.A., Osterheld A.L., Goldstein W.H., 1995, ApJ, 438, L115
Markevitch M., Forman W.R., Sarazin C.L., Vikhlinin A., 1998, ApJ, 503, 77
Markevitch M., Vikhlinin A., Forman W.R., Sarazin C.L., 1999, ApJ, submitted [astro-ph/9904383]
Mohr J.J., Mathiesen B., Evrard A.E., 1999, ApJ, 517, 627
Navarro J.F., Frenk C.S. & White S.D.M., 1995, MNRAS, 275, 720
Neumann D.M., Arnaud M., 1999, A&A, 348, 711
Sarazin C.L., 1986, *X-ray emission from clusters of galaxies*, Cambridge University Press
White D.A., 1999, MNRAS, in press [astro-ph/9909467]

APPENDIX A: OTHER ANALYTIC FORMULAE

I write here the analytic expressions for the derived quantities like the total gravitating mass, $M_{\text{tot}}$, and the radius at which the mean density within a cluster at redshift $z$ is $\Delta$ times the background, $r_\Delta$, when the gas density is assumed well described by a $\beta$-model and presents a polytropic ($1 \leq \gamma \leq 5/3$) dependence upon the gas temperature:

$$M_{\text{tot}}(r) = \frac{-T(r) r}{G \mu m_p} \left( \frac{\partial \ln \mu}{\partial \ln r} + \frac{\partial \ln T}{\partial \ln r} \right)$$

$$= 3 \beta \gamma T_0 r_c \frac{x^3}{G \mu m_p (1 + x^2)^{3/2}}$$

$$= 1.060 \times 10^{14} \mu^{-3/2} \beta \gamma T_0 r_c \frac{x^3}{(1 + x^2)^{3/2}} \ M_\odot$$

([Henriksen & Mushotzky 1986; see also Cowie, Henriksen & Mushotzky 1987, Hughes et al. 1988] for a critical discussion on the presence of an artificial cutoff in the parameter space, i.e. $2B < 3$ or $\gamma < 1 + (1/3\beta)$, to avoid the related virial density falling to zero (or negative values) at small radii) and, given that $M_{\text{tot}}(r_\Delta) = \frac{4}{3} \pi \rho_0 (1 + z)^3 r_\Delta^4 \Delta$.

$$r_\Delta = \sqrt[3]{\frac{3 \beta \gamma T_0}{G \mu m_p (4/3) \pi \rho_0 (1 + z)^3 r_c^3 \Delta}} - 1$$

$$= \sqrt[3]{\frac{229.5 \beta \gamma T_0}{\mu (1 + z)^3 r_c^3 \Delta}} - 1,$$

where the exponent $B = 1.5\beta(\gamma - 1) + 1$, $T_0$ is the central temperature in keV, $r_c$ the core radius in $h_{50}^{-1}$ Mpc, $\mu$ is the mean molecular
weight in a.m.u. and the numerical values include the gravitational constant $G$, the mass of the proton $m_p$, and all the unit conversions.