Expanding universe as a classical solution in the Lorentzian matrix model for nonperturbative superstring theory

Sang-Woo Kim\textsuperscript{1} \textsuperscript{*} Jun Nishimura\textsuperscript{2,3} \textsuperscript{†} and Asato Tsuchiya\textsuperscript{4}

\textsuperscript{1}Department of Physics, Osaka University, Toyonaka, Osaka 560-0043, Japan
\textsuperscript{2}KEK Theory Center, High Energy Accelerator Research Organization, Tsukuba 305-0801, Japan
\textsuperscript{3}Department of Particle and Nuclear Physics, School of High Energy Accelerator Science, Graduate University for Advanced Studies (SOKENDAI), Tsukuba 305-0801, Japan
\textsuperscript{4}Department of Physics, Shizuoka University, 836 Ohya, Suruga-ku, Shizuoka 422-8529, Japan

(Dated: October 2011; preprint: KEK-TH-1503, OU-HET-729-2011)

Recently we have shown by Monte Carlo simulation that expanding (3+1)-dimensional universe appears dynamically from a Lorentzian matrix model for type IIB superstring theory in (9+1)-dimensions. The mechanism for the spontaneous breaking of rotational symmetry relies crucially on the noncommutative nature of the space. Here we study the classical equations of motion as a complementary approach. In particular, we find a unique class of SO(3) symmetric solutions, which exhibits the time-dependence compatible with the expanding universe. The space-space noncommutativity is exactly zero, whereas the space-time noncommutativity becomes significant only towards the end of the expansion. We interpret the Monte Carlo results and the classical solution as describing the behavior of the model at earlier time and at later time, respectively.

PACS numbers: 11.25.-w; 11.25.Sq

\textit{Introduction}.— It is widely believed that the birth of our universe can be described by superstring theory, which is a natural candidate for a unified theory including quantum gravity. Indeed, a lot of insights into this issue have been obtained by string cosmology over the last decade\textsuperscript{1}. These studies are based on perturbative formulations incorporating nonperturbative effects through D-branes. An obvious drawback in such an approach, however, is that one has to choose a particular string vacuum from numerous vacua that are theoretically allowed. On the other hand, there is also a possibility that one can actually determine the true string vacuum uniquely if one uses a nonperturbative formulation.

Along this line of thought, we have studied a SO(9,1) symmetric Lorentzian matrix model, which is considered to be a nonperturbative definition of type IIB superstring theory in (9+1) dimensions\textsuperscript{3}. Surprisingly our Monte Carlo results provide clear evidence that 3 out of 9 directions start to expand at some critical time. The observed spontaneous breaking of the SO(9) rotational symmetry down to SO(3) has been understood intuitively by a mechanism, which relies crucially on the noncommutative nature of the space. While this is certainly intriguing, it also poses a crucial question whether the space-time becomes commutative at later time as we observe it now.

In this Letter we study the classical equations of motion of the model as a complementary approach. In particular, we find a unique class of SO(3) symmetric solutions, which turns out to have the time dependence compatible with the expanding universe. For this solution, the space-space noncommutativity is exactly zero, whereas the space-time noncommutativity becomes significant only towards the end of the expansion.

\textit{Lorentzian matrix model}.— The matrix model proposed as a nonperturbative formulation of type IIB superstring theory has the action $S = S_b + S_f$, where

\begin{align}
S_b &= -\frac{1}{4g^2} \text{tr} \left( [A_\mu, A_\nu] [A^\mu, A^\nu] \right), \\
S_f &= -\frac{1}{2g^2} \text{tr} \left( \Psi_\alpha (C \Gamma^\mu)_{\alpha \beta} [A_\mu, \Psi_\beta] \right),
\end{align}

with $A_\mu \ (\mu = 0, \cdots, 9)$ and $\Psi_\alpha \ (\alpha = 1, \cdots, 16)$ being $N \times N$ traceless Hermitian matrices. The Lorentz indices $\mu$ and $\nu$ are contracted using the metric $\eta = \text{diag}(-1,1, \cdots, 1)$. The $16 \times 16$ matrices $\Gamma^\mu$ are ten-dimensional gamma matrices after the Weyl projection, and the unitary matrix $C$ is the charge conjugation matrix. The action has manifest SO(9,1) symmetry, where $A_\mu$ and $\Psi_\alpha$ transform as a vector and a Majorana-Weyl spinor, respectively. The space-time is represented dynamically by the ten bosonic matrices $A_\mu$\textsuperscript{3}.

An important feature of the Lorentzian model is that the bosonic part of the action is proportional to

\[ \text{tr} (F_{\mu \nu} F^{\mu \nu}) = -2 \text{tr} (F_{00})^2 + \text{tr} (F_{ij})^2, \]

where $F_{\mu \nu} = -i [A_\mu, A_\nu]$ are Hermitian matrices, and hence the two terms in (2) have opposite signs. A common approach to study the nonperturbative dynamics of this model was to make the Wick rotation $A_0 = iA_0$ and to study the SO(10) symmetric Euclidean model, which is proved to have finite partition function \textsuperscript{4, 5}. See Ref.\textsuperscript{8} and references therein for studies of the spontaneous symmetry breaking (SSB) of the SO(10) in the Euclidean model\textsuperscript{9}. On the other hand, it is suggested that the Lorentzian signature of the metric plays an important role in the dynamics of quantum gravity\textsuperscript{12, 13}.

In Ref.\textsuperscript{3} we studied, for the first time, the nonperturbative dynamics of the Lorentzian model defined by

\[ Z = \int dA d\Psi e^{iS} = \int dA e^{iS_b + \text{PfM}(A)}, \]
where the Pfaffian PfM(A) appears from integrating out the fermionic matrices Ψµ. We made the partition function \( Z = \text{Pf} M(A) \) finite by introducing infrared cutoffs in both the spatial and temporal directions instead of making the Wick rotation. It was shown by Monte Carlo simulation that one can remove these cutoffs in the large-N limit, and that the theory thus obtained has no parameters other than one scale parameter.

The classical solutions.— Taking account of the infrared cutoffs introduced in the Lorentzian model, we search for stationary points of the bosonic action \( S_0 \) for fixed \( \frac{1}{N} \text{tr} (A_0)^2 \) and \( \frac{1}{N} \text{tr} (A_i)^2 \). Then the problem reduces to solving the classical equations of motion

\[
\begin{align*}
[ A_0, [ A_0, A_i ]] + [ A_j, [ A_j, A_i ]] - \lambda A_i &= 0, \\
[ A_j, [ A_j, A_0 ]] - \tilde{\lambda} A_0 &= 0, 
\end{align*}
\]

where \( \lambda \) and \( \tilde{\lambda} \) represent the Lagrange multipliers corresponding to the constraints. We look for solutions, which are given by a unitary representation of a Lie algebra \( [ A_n, A_\nu ] = if_{\mu\nu\lambda}A_\lambda \), which guarantees automatically that the Jacobi identity is satisfied. (See Ref. \[14\] for an analogous study in the Euclidean model.) Motivated by the Monte Carlo results mentioned above, we restrict ourselves to solutions with \( A_1 = 0 \) (4 \( \leq I \leq 9 \)) and with SO(3) symmetry corresponding to rotations in the \( i = 1, 2, 3 \) directions. From the complete list of real Lie algebras with four generators \( e_\mu \) (0 \( \leq \mu \leq 3 \)), the one with SO(3) symmetry is given uniquely by \( [ e_0, e_i ] = -ie_i \) for \( i = 1, 2, 3 \), whereas all the other commutators vanish. (This corresponds to the algebra \( A_1^{ab} \) in Table I of Ref. \[15\] for \( a = b = 1 \).

The unitary irreducible representations of the above algebra are classified into two categories. One consists of the trivial one-dimensional representations given by \( e_0 = a \) and \( e_i = 0 \), where \( a \) is a real parameter. The other consists of the infinite-dimensional representations given by the operators \( e_0 = -\frac{i}{2} \frac{d}{dx} \) and \( e_i = a_i \exp(x) \) on the space of functions of \( x \) with \( L^2 \) integrability, where the three real parameters \( a_i \) specify a representation. As the basis of the functional space, we use the eigenfunctions of the Hamiltonian of a one-dimensional harmonic oscillator, which are given as

\[
f_n(x) = c_n H_n(x) e^{-\frac{1}{2}x^2}, \quad c_n = (\pi^{1/4} \sqrt{n!} 2^{n/2})^{-1}.
\]

The representation matrices of \( e_0 \) and \( e_i/a_i \), which we denote as \( \tilde{P} \) and \( \tilde{K} \), respectively, have the following elements.

\[
P_{nm} = \int dx f_n(x)^* (-i) \frac{d}{dx} f_m(x) = -\frac{1}{\sqrt{2}}(\sqrt{m} \delta_{n,m-1} - \sqrt{m+1} \delta_{n,m+1}),
\]

\[
K_{nm} = \int dx f_n(x)^* e^{x} f_m(x) = c_n c_m e^{1/4} \left[ 1 - x^2 \right] H_m(x) + \frac{1}{2} H_n(x) + \frac{1}{2} H_m(x) = e^{1/4} 2^{-|n-m|/2} \sqrt{n!m!} \sum_{l=0}^{M} \left( 2^l \right) (M-l)! (|n-m|+l)!^{-1},
\]

where \( M = \min(n, m) \). In the last equality, we have used the property \( \frac{d}{dx} H_n(x) = 2n H_{n-1}(x) \) of the Hermite polynomials.

Using a direct sum of the non-trivial representations, we find a set of SO(3) symmetric solutions to \[4\], which is given by

\[
A_0 = \sqrt{\tilde{P} \otimes \mathbf{1}_k}, \quad A_i = \tilde{K} \otimes \text{diag}(x_11, \cdots, x_{k1}).
\]

The parameters \( x_{ai} \equiv (x_a)_i \) should be chosen such that the points \( x_a = (a = 1, \cdots, k) \) have spherically symmetric distribution in the 3-dimensional space. One of the Lagrange multiplier is fixed as \( \lambda = 0 \).

In the following analysis, the \( k \times k \) matrices that appear in \[5\] and \[6\] are omitted since they only give an irrelevant constant factor. Also we consider only one spatial direction \( i = 1 \) for simplicity since it turns out that the number of spatial directions does not play any role.

The space-time structure.— In order to extract the space-time structure from the solution, we first need to diagonalize \( A_0 \). We do this numerically by truncating the functional space to the \( N \)-dimensional space spanned by \( f_n(x) \) with \( |n| \leq N-1 \). Let us define the eigenvector \( |t_a \rangle \) corresponding to the eigenvalues \( t_a \) of \( A_0 = (a = 1, \cdots, N) \) with the specific order \( t_1 < \cdots < t_N \). The spatial matrix \( (t_1 |A_i| t_1) \) in that basis is not diagonal. However, it turns out that the off-diagonal elements decay exponentially in the direction orthogonal to the diagonal line.

To see it explicitly, let us consider the \( N \times N \) matrix \( Q_{ij} = (t_i |A_j| t_j) \). Plotting \( \sqrt{Q_{ij}/Q_{N/2,N/2}} \) against \( I-J \) for a fixed value of \( (I+J)/2 \), we find that it decreases exponentially with \( |I-J| \). The half width is largest for \( (I+J)/2 = N/2 \), and we denote it as \( n \) for later convenience. For \( N = 16, 32, 64, 128 \), we obtain \( n = 11, 15, 23, 33 \).

The above observation motivates us to define \( n \times n \) matrices \( A^{(ab)}_i(t) \equiv (t_{\nu+a}|A_i|t_{\nu+b}) \) with \( 1 \leq a, b \leq n \) and \( t = t \sum_{a=1}^{n} t_{\nu+a} \) for \( \nu = 0, \cdots, (N-n) \). These matrices represent the space structure at fixed time \( t \). Let us define the extent of space at the time \( t \) as \( R(t)^2 \equiv \frac{1}{n} \text{tr} A_i(t)^2 \). In Fig. \[7\] we plot \( R(t)/R(0) \) for \( N = 16, 32, 64, 128 \). It is symmetric under the time reflection \( t \to -t \) as one can prove analytically even at finite \( N \). For each \( N \), we have chosen the Lagrange multiplier \( \lambda \), which determines the scale of \( t \), so that \( R(t) \) scales around \( t = 0 \). We have fixed
\[ \lambda = 1 \text{ for } N = 16 \text{ without loss of generality. Then we obtain } \lambda = 0.92, 0.72, 0.59 \text{ for } N = 32, 64, 128, \text{ respectively.} \]

As we increase \( N \), the scaling region extends to larger \(|t|\). The solid line is a fit to the Gaussian function. Thus we find that the time-evolution of the space is compatible with the expanding behavior observed in the Monte Carlo simulation [3].

![Figure 1](image1.png)

**FIG. 1:** The extent of space \( R(t)/R(0) \) is plotted as a function of \( t \) for four values of \( N \). The block size \( n \) is determined from the the decay rate of the off-diagonal elements of \( A_1 \) in the basis which diagonalizes \( A_0 \). The value of \( \lambda \) is chosen for each \( N \) in such a way that the results scale in \( N \). The solid line represents \( y = \exp(-0.034 t^2) \), which is obtained by fitting the \( N = 128 \) data to the Gaussian function.

Let us next turn our attention to the space-time noncommutativity. We define the dimensionless parameter

\[ \chi(t) = -\frac{1}{n} \text{tr} \left[ A_0(t), A_1(t) \right] \]

\[ \frac{1}{n} \text{tr} A_0(t)^2 \cdot \frac{1}{n} \text{tr} A_1(t)^2 \]

(7)

as an estimate on the space-time noncommutativity [16]. In Fig. 2 we plot \( \chi(t) \) for \( N = 16, 32, 64, 128 \). We find that it is of \( O(1) \) at \( t = 0 \) and decreases as \( \sim |t|^{-p} \) at large \(|t|\), where we obtain \( p = 1.708(3) \) by fitting the \( N = 128 \) data within \(-6 < t < -2\). Therefore, the space-time noncommutativity is significant only around \( t \sim 0 \), and it becomes smaller as we go back in time.

In both Figs. 1 and 2 we consider it very important to study the larger \(|t|\) region by increasing \( N \). In fact some preliminary results suggest certain deviation from the Gaussian behavior of \( R(t) \) and the power-law behavior of \( \chi(t) \). We hope to report on it in future publications.

**Summary and discussions.**— We have studied the classical equations of motion in the Lorentzian matrix model for type IIB superstring theory. Restricting ourselves to the class of solutions that are written in terms of Lie algebras with four generators, we find that the solution with \( \text{SO}(3) \) symmetry is essentially unique. The space-time structure extracted from the solution exhibits the time dependence, which is compatible with the behavior observed in our previous Monte Carlo results. The space-time noncommutativity becomes significant only towards the end of the expansion, whereas the space-space noncommutativity is zero. These results suggest the appearance of an expanding (3+1)-dimensional (almost commutative) space-time from the Lorentzian matrix model.

We speculate that the noncommutativity of space which plays a crucial role in making three directions expand at earlier time, somehow disappears at some point due to some dynamical reason. For instance, let us use the model obtained after integrating out the scale factor [2]. In that model we have a constraint that requires the quantity [2] to vanish. If the expansion with large space-space noncommutativity in the early time continues for too long a period, the second term of (2) will be too large to satisfy the constraint [2] = 0. Such an effect may lead to an end of the noncommutative expansion. One might speculate that this corresponds to the end of “inflation”.

Our classical solution is symmetric under time reflection, and the size of the space becomes maximum at \( t = 0 \), after which it has a contracting behavior. At \( t = 0 \), the dimensionless space-time noncommutativity becomes maximum, too, and it is of the order one. Hence the physics there will be quite exotic. This may be taken as a prediction on the fate of our universe from the Lorentzian matrix model given that our classical solution is valid around \( t = 0 \).

Obviously one can generalize our solution to \( \text{SO}(d) \) symmetric ones with \( 1 \leq d \leq 9 \). The time evolution of the size of the space and that of the space-time non-
emphasize that the nonzero dimensions even without the Myers-like term. Here we crucial for realizing a nontrivial structure in the extra extended directions and the extra dimensions. This is is that there exists noncommutativity between the four extended directions and the extra dimensions. This is crucial for realizing a nontrivial structure in the extra dimensions even without the Myers-like term. Here we emphasize that the nonzero $\lambda$, which is introduced in our work, is crucial for the expanding behavior. Let us recall that $\lambda$ is the Lagrange multiplier corresponding to the covariant derivative on a curved space. It would be interesting to clarify the relationship to our work.

Ref. [18] reports on interesting solutions to the classical equations of motions [19] with $\lambda = \lambda = 0$. They represent a flat Minkowski space with extra dimensions described by fuzzy spheres. An interesting feature of these solutions is that there exists noncommutativity between the four extended directions and the extra dimensions. This is crucial for realizing a nontrivial structure in the extra dimensions even without the Myers-like term. Here we emphasize that the nonzero $\lambda$, which is introduced in our work, is crucial for the expanding behavior. Let us recall that $\lambda$ is the Lagrange multiplier corresponding to the covariant derivative on a curved space. It would be interesting to clarify the relationship to our work.

Ref. [19] the Matrix theory [20] has been applied to cosmology. A classical solution with three expanding (commutative) directions and six oscillating (noncommutative) directions was discussed. (The number of expanding directions does not have to be three.) In order to have such a solution, the authors introduced a SO(9) symmetric tachyonic mass term, which was interpreted as the cosmological term. The relationship to our solution is not clear, though, since the time is treated in a different way. The idea to use the matrices to avoid the big-bang singularity is also pursued in Refs. [22].

It is tempting to imagine that the rapid growth of $R(t)$ observed in the present solution has something to do with the accelerating expansion confirmed by recent cosmological observations. The power-law expansion at earlier time may be understood by considering the quantum corrections around the classical solution. It is also expected that the gauge interactions and the matter content in the (3+1)-dimensional space are determined by the structure in the extra dimensions $[23][24]$ analogously to the case of intersecting D-brane models. We hope the present model provides a new perspective on particle physics beyond the standard model as well as on cosmological models for inflation, modified gravity etc..

Acknowledgments.— We thank H. Aoki, S. Iso, H. Kawai, Y. Kitazawa, Y. Sekino and H. Steinacker for discussions. S.-W.K. is supported by Grant-in-Aid for Scientific Research from the Ministry of Education, Culture, Sports, Science and Technology in Japan (No. 20105002). J.N. and A.T. is supported in part by Grant-in-Aid for Scientific Research (No. 19340066, 19540294, 20540286 and 23244057) from JSPS.