ASYMPTOTIC COMPLETENESS FOR A SCALAR QUASILINEAR WAVE EQUATION SATISFYING THE WEAK NULL CONDITION

DONGXIAO YU

Abstract. In this paper, we prove the first asymptotic completeness result for a scalar quasilinear wave equation satisfying the weak null condition. The main tool we use in the study of this equation is the geometric reduced system introduced in [35]. Starting from a global solution \( u \) to the quasilinear wave equation, we rigorously show that well chosen asymptotic variables solve the same reduced system with small error terms. This allows us to recover the scattering data for our system, as well as to construct a matching exact solution to the reduced system.

Contents

1. Introduction 1
2. Preliminaries 14
3. Construction of the optical function 17
4. Derivatives of the optical function 40
5. The asymptotic equations and the scattering data 80
6. Gauge independence 91
7. Approximation 92
References 113

1. Introduction

This paper is devoted to the study of the long time dynamics for a scalar quasilinear wave equation in \( \mathbb{R}^{1+3}_t, x \), of the form

\[
g^{\alpha\beta}(u)\partial_\alpha \partial_\beta u = 0
\]

with small initial data

\[
(u, u_t)|_{t=0} = (\varepsilon u_0, \varepsilon u_1) \in C^\infty_c(\mathbb{R}^3), \ 0 < \varepsilon \ll 1.
\]

Here we use the Einstein summation convention, with the sum taken over \( \alpha, \beta = 0, 1, 2, 3 \) with \( \partial_0 = \partial_t, \ \partial_i = \partial_{x_i}, i = 1, 2, 3. \) We assume that \( g^{\alpha\beta}(u) \) are smooth functions of \( u \), such that \( g^{\alpha\beta} = g^{\beta\alpha} \) and \( g^{0\beta}(0) = m^{0\beta} \) where \( (m^{\alpha\beta}) = \text{diag}(-1, 1, 1, 1) \) is the Minkowski metric. Here we can assume that \( g^{00} \equiv -1. \) In fact, since we expect \( |u| \ll 1 \), we have \( g^{00}(u) < 0 \), so we can replace \( (g^{\alpha\beta}) \) with \( (g^{\alpha\beta}/(-g^{00})) \) if necessary.

2020 Mathematics Subject Classification. 35L70.

Key words and phrases. Quasilinear wave equations, the weak null condition, modified scattering theory, asymptotic completeness, a geometric reduced system.
The study of global well-posedness theory of (1.1) started with Lindblad’s paper [22]. Lindblad conjectured that the equation (1.1) has a global solution if $\varepsilon$ in (1.2) is sufficiently small. In the same paper, he proved the small data global existence for a special case

$$- \partial_t^2 u + c(u)^2 \Delta_x u = 0,$$

where $c(0) = 1$ for radially symmetric data. Later, Alinhac [1] generalized the result to general initial data for (1.3). The small data global existence result for the general case (1.1) was finally proved by Lindblad [23].

In the author’s recent paper [35], we have identified a new notion of a symptotic profile and an associated notion of scattering data for the model equation, by deriving a new reduced system. With these new notions, we have proved the existence of the modified wave operators for (1.1).

In this paper, we seek to continue the study of modified scattering by proving the asymptotic completeness for (1.1). That is, given a global solution to the Cauchy problem (1.1) and (1.2), we seek to find the corresponding asymptotic profile and scattering data associated to this global solution.

Given a global solution $u$, we start the proof with the construction of a global optical function $q = q(t, x)$. In other words, we solve the eikonal equation $g^{\alpha\beta}(u) q_\alpha q_\beta = 0$ in a spacetime region $\Omega$ contained in $\{2r \geq t \geq \exp(\delta/\varepsilon)\}$; this is where our evolution is expected to have a nonlinear behavior. Here $\delta > 0$ is a small fixed parameter. We apply the method of characteristics and then follow the idea in Christodoulou-Klainerman [5]. By viewing $(g^{\alpha\beta})$, the inverse of the coefficient matrix $(g_{\alpha\beta}(u))$, as a Lorentzian metric in $[0, \infty) \times \mathbb{R}^3$, we construct a null frame $\{e_k\}_{k=1}^4$. Then, most importantly, we define the second fundamental form $\chi_{ab}$ for $a, b = 1, 2$ which are related to the Levi-Civita connection and the null frame under the metric $(g_{\alpha\beta})$. By studying the Raychaudhuri equation and using a continuity argument, we can show that $\text{tr} \chi > 0$ everywhere. This is the key step which guarantees that the solutions to the eikonal equation are global. In addition, we can prove that $q = q(t, x)$ is smooth in some weak sense (see Section 2.4). We refer our readers to Section 3 and Section 4 for more details in the proof.

Next, following [35], we define our asymptotic variables $(\mu, U)(t, x) := (q_t - q_r, \varepsilon^{-1} ru)(t, x)$. The map

$$\Omega \to [0, \infty) \times \mathbb{R} \times S^2 : (t, x) \mapsto (\varepsilon \ln t - \delta, q(t, x), x/|x|) := (s, q, \omega)$$

is an injective smooth function with a smooth inverse, so a function $(\mu, U)(s, q, \omega)$ is obtained. It can be proved that $(\mu, U)(s, q, \omega)$ is an approximate solution to the reduced system introduced in [35], and that there is an exact solution $(\tilde{\mu}, \tilde{U})(s, q, \omega)$ to the reduced system which matches $(\mu, U)(s, q, \omega)$ as $s \to \infty$. A key step is to prove that $A(q, \omega) := -\frac{1}{2} \lim_{s \to \infty} (\mu U q)(s, q, \omega)$ is well-defined for each $(q, \omega)$. The function $A$ is called the scattering data in this paper. We also show a gauge independence result, which states that the scattering data for the solution $u$ is independent of the choice of the optical function $q$ in a suitable sense. We refer our readers to Section 5 and Section 6.

Finally, starting from the scattering data $A$, we show that we can construct an approximate solution $\tilde{u}$ to (1.1) in $\Omega$. The construction here is similar to that in Section 4 of [35]. That is, we construct a function $\tilde{q}$ by solving

$$\tilde{q}_t - \tilde{q}_r = \mu(\varepsilon \ln t - \delta, \tilde{q}(t, x), \omega)$$
by the method of characteristics, and then define
\[ \tilde{u}(t, x) := \varepsilon r^{-1} \tilde{U}(\varepsilon \ln t - \delta, \tilde{q}(t, x), \omega). \]

Then, in \( \Omega \), \( \tilde{q} \) is an approximate optical function, and \( \tilde{u} \) is an approximate solution to (1.1).

In addition, near the light cone \( t = r \), the difference \( u - \tilde{u} \), along with its derivatives, decays much faster than \( \varepsilon t^{-1+C\varepsilon} \). Since \( u \) and its derivatives is of size \( O(\varepsilon t^{-1+C\varepsilon}) \), we conclude that \( \tilde{u} \) offers a good approximation of \( u \).

1.1. Background. Let us consider a generalization of the scalar quasilinear wave equation (1.1) in \( \mathbb{R}^{1+3}_{t,x} \)
\[ \Box u = F(u, \partial u, \partial^2 u). \]

The nonlinear term is assumed to be smooth with the Taylor expansion
\[ F(u, \partial u, \partial^2 u) = \sum a_{\alpha\beta} \partial^\alpha u \partial^\beta u + O(|u|^3 + |\partial u|^3 + |\partial^2 u|^3). \]

The sum is taken over all multiindices \( \alpha, \beta \) with \( |\alpha| \leq |\beta| \leq 2 \), \( |\beta| \geq 1 \) and \( |\alpha| + |\beta| \leq 3 \). Besides, the coefficients \( a_{\alpha\beta} \)'s are all universal constants.

Since 1980’s, several results on the lifespan of the solutions to the Cauchy problem (1.4) with initial data (1.2) have been proved. For example, John [12,13] proved that (1.4) does not necessarily have a global solution; in fact, any nontrivial solution to \( \Box u = u \Delta u \) or \( \Box u = u^2 \) must blow up in finite time. In contrast, in \( \mathbb{R}^{1+d} \) with \( d \geq 4 \), Hörmander [9] proved the small data global existence for (1.4). For arbitrary nonlinearities in three space dimensions, the best result on the lifespan is the almost global existence: the solution exists for \( t \leq \exp(c/\varepsilon) \) where \( \varepsilon \ll 1 \). The almost global existence for (1.4) was proved by Lindblad [21], and we also refer to [8, 10, 14, 18] for some earlier work.

In contrast to the finite-time blowup in John’s examples, Klainerman [19] and Christodoulou [4] proved that the null condition is sufficient for small data global existence. The null condition, first introduced by Klainerman [17], states that for each \( 0 \leq m \leq n \leq 2 \) with \( m+n \leq 3 \), we have
\[ A_{mn}(\omega) := \sum_{|\alpha|=m, |\beta|=n} a_{\alpha\beta} \tilde{\omega}^\alpha \tilde{\omega}^\beta = 0, \quad \text{for all } \tilde{\omega} = (-1, \omega) \in \mathbb{R} \times \mathbb{S}^2. \]

Equivalently, we assume \( A_{mn} \equiv 0 \) on the null cone \( \{ m^\alpha \xi_\alpha \xi_\beta = 0 \} \). The null condition leads to cancellations in the nonlinear terms (1.5) so that the nonlinear effects of the equations are much weaker than the linear effects. However, note that the null condition is not necessary for small data global existence. For example, the null condition fails for (1.1) in general, but (1.4) still has small data global existence. We also refer our readers to [33] for a general introduction on the null condition.

Later, Lindblad and Rodnianski [25,26] introduced the weak null condition. To state the weak null condition, we start with the asymptotic equations first introduced by Hörmander [8]. We make the ansatz
\[ u(t, x) \approx \varepsilon r^{-1} U(s, q, \omega), \quad r = |x|, \ \omega_i = x_i/r, \ s = \varepsilon \ln(t), \ q = r - t. \]

Assuming that \( t = r \to \infty \), we substitute this ansatz into (1.4) and compare the coefficients of terms of order \( \varepsilon^2 t^{-2} \). Nonrigorously, we can obtain the following asymptotic PDE for
Here \( A_{mn} \) is defined in (1.6) and the sum is taken over \( 0 \leq m \leq n \leq 2 \) with \( m + n \leq 3 \). We say that the weak null condition is satisfied if (1.8) has a global solution for all \( s \geq 0 \) and if the solution and all its derivatives grow at most exponentially in \( s \), provided that the initial data decay sufficiently fast in \( q \). In the same papers, Lindblad and Rodnianski conjectured that the weak null condition is sufficient for small data global existence. To the best of the author’s knowledge, this conjecture remains open until today.

There are three remarks about the weak null condition and the corresponding conjecture. First, the null condition implies the weak null condition. In fact, under the null condition, (1.8) becomes \( \partial_s \partial_q U = 0 \). Secondly, though the conjecture remains open, there are many examples of (1.4) satisfying the weak null condition and admitting small data global existence at the same time. The equation (1.1) is one of several such examples: the small data global existence for (1.1) has been proved by Lindblad [23]; meanwhile, the asymptotic equation (1.8) now becomes

(1.9) \[ 2\partial_s \partial_q U = G(\omega)U \partial_q^2 U, \]

where

(1.10) \[ G(\omega) := g^{\alpha\beta} \tilde{\omega}_\alpha \tilde{\omega}_\beta, \quad g^{\alpha\beta} = \frac{d}{du} g^{\alpha\beta}(u)|_{u=0}, \quad \tilde{\omega} = (-1, \omega) \in \mathbb{R} \times S^2, \]

whose solutions exist globally in \( s \) and satisfy the decay requirements, so (1.1) satisfies the weak null condition. There are also many examples violating the weak null condition and admitting finite-time blowup at the same time. Two such examples are \( \Box u = u_2 \Delta u \) and \( \Box u = u_2^2 \): the corresponding asymptotic equations are \( (2\partial_s - U_q \partial_q) U_q = 0 \) (Burger’s equation) and \( \partial_s U_q = U_q^2 \), respectively, whose solutions are known to blow up in finite time. Thirdly, in recent years, Keir has made some further progress. In [15], he proved the small data global existence for a large class of quasilinear wave equations satisfying the weak null condition, significantly enlarging upon the class of equations for which global existence is known. His proof also applies to (1.1). In [16], he proved that if the solutions to the asymptotic system are bounded (given small initial data) and stable against rapidly decaying perturbations, then the corresponding system of nonlinear wave equations admits small data global existence.

1.2. The geometric reduced system. In [35], we have constructed a new system of asymptotic equations. Our analysis starts as in Hörmander’s derivation in [8–10], but diverges at a key point: the choice of \( q \) is different. One may contend from this work that this new system is more accurate than (1.9), in that it both describes the long time evolution and contains full information about it. In addition, if we choose the initial data appropriately, our reduced system will reduce to linear first order ODE’s on \( \mu \) and \( U_q \), so it is easier to solve it than to solve (1.9).

To derive the new equations, we still make the ansatz (1.7), but now we replace \( q = r - t \) with a solution \( q(t, x) \) to the eikonal equation related to (1.11)

(1.11) \[ g^{\alpha\beta}(u) \partial_\alpha q \partial_\beta q = 0. \]

In other words, \( q(t, x) \) is an optical function. We remark that the eikonal equations have been used in the previous works on the small data global existence for (1.11). In [11], Alinhac followed
the method used in Christodoulou and Klainerman [5], and adapted the vector fields to the characteristic surfaces, i.e. the level surfaces of solutions to the eikonal equations. In [23], Lindblad considered the radial eikonal equations when he derived the pointwise bounds of solutions to (1.1). When they derived the energy estimates, both Alinhac and Lindblad considered a weight $w(q)$ where $q$ is an approximate solution to the eikonal equation. Their works suggest that the eikonal equation plays an important role when we study the long time behavior of solutions to (1.1). Moreover, the eikonal equations have also been used in the study of the asymptotic behavior of solutions to the Einstein vacuum equations, an analogue of (1.1); we refer our readers to [5, 24]. In addition, we also refer to [30] where the eikonal equations are used to study the sharp local wellposedness for the nonlinear wave equations.

Since $u$ is unknown, it is difficult to solve (1.11) directly. Instead, we introduce a new auxiliary function $\mu = \mu(s, q, \omega)$ such that $q_t - q_r = \mu$. From (1.11), we can express $q_t + q_r$ in terms of $\mu$ and $U$, and then solve for all partial derivatives of $q$, assuming that all the angular derivatives are negligible. Then from (1.1), we can derive the following asymptotic equations for $\mu(s, q, \omega)$ and $U(s, q, \omega)$:

\[
\begin{align*}
\partial_s \mu &= \frac{1}{4} G(\omega) \mu^2 U_q, \\
\partial_s U_q &= -\frac{1}{4} G(\omega) \mu U_q^2.
\end{align*}
\]

Here $G(\omega)$ is defined by (1.10). We call this new system of asymptotic equations the geometric reduced system since it is related to the geometry of the null cone with respect to the Lorentzian metric $(g_{\alpha\beta}) = (g^{\alpha\beta}(u))^{-1}$ instead of the Minkowski metric. For a derivation of (1.12), we refer our readers to Section 3 in [35], or Chapter 2 in the author’s PhD dissertation [34]. Heuristically, one expects the solution to the quasilinear wave equation (1.1) to correspond to an approximate solution to this geometric reduced system, and to be well approximated by an exact solution to the geometric reduced system.

Note that (1.12) is a system of two ODE’s for $(\mu, U_q)$. In addition, we have $\partial_s(\mu U_q) = 0$ for each $(s, q, \omega)$. That is, if the initial data are given by

\[
(\mu, U_q)|_{s=0}(q, \omega) = (A_1, A_2)(q, \omega),
\]

then we have $\mu U_q = A_1 \cdot A_2$ at each $(s, q, \omega)$. In this paper, we define a function $A = A(q, \omega)$ for $(q, \omega) \in \mathbb{R} \times S^2$ by

\[
A(q, \omega) := -\frac{1}{2} A_1(q, \omega) \cdot A_2(q, \omega),
\]

and we call the function $A$ a scattering data associated to a solution $u$ to the quasilinear wave equation (1.1). Now (1.12) reduces to a linear system of ODE’s

\[
\begin{align*}
\partial_s \mu &= -\frac{1}{2} G(\omega) A(q, \omega) \mu, \\
\partial_s U_q &= \frac{1}{2} G(\omega) A(q, \omega) U_q,
\end{align*}
\]
whose solutions are given by

\begin{align}
\mu(s, q, \omega) &= A_1(q, \omega) \exp(-\frac{1}{2}G(\omega)A(q, \omega)s), \\
U_q(s, q, \omega) &= A_2(q, \omega) \exp(\frac{1}{2}G(\omega)A(q, \omega)s),
\end{align}

To solve for $U(s, q, \omega)$ uniquely, we assume that

$$\lim_{q \to -\infty} U(s, q, \omega) = 0 \quad \text{or} \quad \lim_{q \to \infty} U(s, q, \omega) = 0,$$

depending on which problem we are studying. For instance, in [35], to guarantee that a solution to (1.1) is zero inside a certain light cone, we assume that $\lim_{q \to -\infty} U(s, q, \omega) = 0$; in this paper, the global solution to (1.1) has localized initial data, so we assume that $\lim_{q \to \infty} U(s, q, \omega) = 0$.

We end this subsection by proposing an alternative definition of the weak null condition. In the discussion above, we define $\mu = q_t - q_r$ and derive a geometric reduced system (1.12) for $(\mu, U)(s, q, \omega)$. This method to derive a reduced system should not just work for (1.1). A derivation of the geometric reduced systems for a system of general quasilinear wave equations can be found in Chapter 2, [34]. We can make the following definition.

**Definition.** We say that a system of quasilinear wave equations satisfies the **geometric weak null condition**, if for any initial data at $s = 0$ decaying sufficiently fast in $q$, we have a global solution to the corresponding geometric reduced system for all $s \geq 0$, and if the solution and all the derivatives grow at most exponentially in $s$.

It is clear from (1.13) that (1.1) satisfies the geometric weak null condition. The author believes that it is interesting to study to what extent is the geometric weak null condition equivalent to the weak null condition, and whether this geometric weak null condition is sufficient for the global existence of general quasilinear wave equations with small and localized initial data.

1.3. **Modified scattering theory: an overview.** The objective of [34,35] and this paper is to study the long time dynamics, and more specifically, scattering theory for highly nonlinear dispersive equations. In other words, we would like to provide an accurate description of asymptotic behavior of the global solutions. For many nonlinear dispersive PDE’s, one can establish a linear scattering theory. That is, a global solution to a nonlinear PDE scatters to a solution to the corresponding linear equation as time goes to infinity. Take the cubic defocusing NLS

$$iu_t + \Delta u = u|u|^2 \quad \text{in } \mathbb{R}^{1+3}_{t,x}$$

as an example. Its corresponding linear equation is the linear Schrödinger equation (LS)

$$iw_t + \Delta w = 0 \quad \text{in } \mathbb{R}^{1+3}_{t,x}.$$ 

One can prove that for each $u_0 \in H^1$, there exists a unique $u_+ \in H^1$ such that

$$\|u(t) - w(t)\|_{H^1} \to 0 \quad \text{as } t \to \infty$$

where $u$ (or $w$) is the global solution to NLS (or LS) with data $u_0$ (or $u_+$). This result is called the **asymptotic completeness**. One can also prove that for each $u_+ \in H^1$, there exists a unique $u_0 \in H^1$ such that the same conclusion holds. This result is called the **existence**
of wave operators, where the wave operator is defined by $\Omega_+ u_+ = u_0$. We refer to Section 3.6 of [32] for this result. Some other nonlinear PDE’s have modified scattering instead of linear scattering. That is, each of their global solutions scatters to a suitable modification of a linear solution. Here the modification can be made in more than one way: we can add a phase correction term, an amplitude correction term, or a velocity correction term to the linear solution. For example, in [11], when the authors study modified scattering for the cubic 1D NLS, they make use of the following asymptotic approximation:

$$\hat{u}(t, \xi) \approx e^{-it\xi^2} W(\xi) e^{i|W(\xi)|^2 \ln t}.$$ 

That is, a phase shift term is introduced. For nonlinear wave equations, the modification often corresponds to a change of the geometry of the light cone foliation of the space-time. This point is reflected in the ansatz used in Section 1.2.

In general, the following steps are taken in order to study modified scattering. Given a nonlinear dispersive PDE, we hope to identify a good notion of asymptotic profile and an associated notion of scattering data for the model equation. This can be achieved by introducing some type of asymptotic equations. Like linear scattering, the two main problems in modified scattering theory are as follows:

1. Asymptotic completeness. Given an exact global solution to the model equation, can we find the corresponding asymptotic profile and scattering data?

2. Existence of (modified) wave operators. Given an asymptotic profile constructed for a scattering data, can we construct a unique exact global solution to the model equation which matches the asymptotic profile at infinite time?

There have been only a few previous results on the (modified) scattering for general quasi-linear wave equations and the Einstein’s equations. In [6], Dafermos, Holzegel and Rodnianski gave a scattering theory construction of nontrivial black hole solutions to the vacuum Einstein equations. That is a backward scattering problem in General Relativity. In [27], Lindblad and Schlue proved the existence of the wave operators for the semilinear models of Einstein’s equations. In [7], Deng and Pusateri used the original Hörmander’s asymptotic system (1.9) to prove a partial scattering result for (1.1). In their proof, they applied the spacetime resonance method; we refer to [28, 29] for some earlier applications of this method to the first order systems of wave equation. Recently, in [35], by using a new reduced system, the author proved the existence of the modified wave operators for (1.1).

1.4. Construction of an optical function. Let $u = u(t, x)$ be a global solution to (1.1) and (1.2) constructed in Lindblad [23]. Here we fix a constant $R > 0$ such that $\text{supp}(u_0, u_1) \subset \{|x| \leq R\}$, so we have $u \equiv 0$ for $|x| \geq t + R$ by the finite speed of propagation. Our goal in this section is to construct an optical function, i.e. a solution to the eikonal equation (1.11). Here we do not expect to solve (1.11) for all $(t, x) \in \mathbb{R}_t^{1+3}$. Instead, we solve it in a region $\Omega \subset \mathbb{R}_t^{1+3}$ which is defined by

$$\Omega := \{(t, x) : t > T_0, \ |x| > (t + T_0)/2 + 2R\}.$$ 

Here $T_0 = \exp(\delta/\varepsilon)$ and $\delta > 0$ is a fixed constant independent of $\varepsilon$. We also assign the initial data by setting $q = r - t$ on $\partial \Omega$. It is then clear that $q = r - t$ in $\Omega \cap \{r - t > R\}$, so from now on we focus on the region $\Omega \cap \{r - t < 2R\}$.

To construct an optical function, we apply the method of characteristics. In fact, the characteristics for (1.11) are the geodesics with respect to the Lorentzian metric $(g_{\alpha\beta})$ which
is the inverse of the matrix \((g^\alpha{}^\beta(u))\). Moreover, we only need to study those geodesics emanating from the cone

\[H := \partial \Omega \cap \{ t > T_0 \} = \{(t, x) : t > T_0, \ |x| = (t + T_0)/2 + 2R \}.
\]

Now we follow the idea in Christodoulou-Klainerman [3]. Fix \(T > T_0\) and suppose that the optical function exists in \(\Omega_T := \Omega \cap \{ t \leq T, r - t \leq 2R \}\). Then, every point in \(\Omega_T\) can be reached by a unique characteristic emanating from \(H\). We first define a null frame \(\{ e_k \}_{k=1}^4\) in \(\Omega_T\), such that \(e_4\) is tangent to the unique characteristic passing through that point. We then define the second fundamental form of the time slices of the null cones:

\[\chi_{ab} := \langle D_{e_a} e_4, e_b \rangle, \quad a, b \in \{1, 2\}.
\]

Here \(D\) is the Levi-Civita connection associated to the Lorentzian metric \((g_{\alpha\beta})\), and \(\langle \cdot, \cdot \rangle\) is the bilinear form associated to the metric \((g_{\alpha\beta})\). We now use a continuity argument. Suppose that in \(\Omega_T\) we have

\[\max_{a, b=1,2} |\chi_{ab} - \delta_{ab} r^{-1}| \leq A t^{-2+B\varepsilon}.
\]

The positive constants \(A\) and \(B\) are both independent of \(\varepsilon\) and \(T\). Our goal is to prove that (1.14) holds with \(A\) replaced by \(A/2\). It follows that \(\text{tr} \chi := \chi_{11} + \chi_{22}\), sometimes called the null mean curvature\(^1\) of the level sets of \(q\), is positive everywhere, and that the characteristics emanating from \(H\) will not intersect with each other. This allows us to extend the optical function to \(\Omega_{T+\varepsilon}\) for a small \(\varepsilon > 0\), such that (1.14) holds everywhere in \(\Omega_{T+\varepsilon}\). We conclude from this continuity argument that the optical function exists everywhere in \(\Omega\).

In order to prove that (1.14) holds with \(A\) replaced by \(A/2\), we make use of the Raychaudhuri equation

\[e_4(\chi_{ab}) = - \sum_{c=1,2} \chi_{ac} \chi_{cb} + \Gamma^0_{\alpha\beta} e_4^\alpha e_4^\beta \chi_{ab} + \langle R(e_4, e_a) e_4, e_b \rangle,
\]

which describes the evolution of \(\chi\) along the null geodesics foliating the light cones. In this equation, \(\Gamma^*\)’s are the Christoffel symbols, and \(\langle R(X, Y) Z, W \rangle\) is the curvature tensor, both with respect to the Lorentzian metric \((g_{\alpha\beta})\). Note that we have a decomposition

\[\langle R(e_4, e_a) e_4, e_b \rangle = e_4(f_1) + f_2
\]

where \(f_1 = O(\varepsilon t^{-2+C\varepsilon})\) and \(f_2 = O(\varepsilon t^{-3+C\varepsilon})\); see Lemma 3.11 for a more accurate statement. We also refer our readers to Corollary 5.9 in [30] for a similar decomposition of curvature tensors. Moreover, it follows from (1.11) that

\[|e_4(e_3(u)) + r^{-1} e_3(u)| \lesssim \varepsilon At^{-3+B\varepsilon}, \quad |e_4(e_3(u))| \lesssim \varepsilon t^{-2}.
\]

Combining all these estimates and the Gronwall’s inequality, we are able to prove (1.14) with \(A\) replaced by \(A/2\).

So far, we have constructed a global optical function \(q = q(t, x)\) in \(\Omega\) which is \(C^2\) by the method of characteristics. In fact, the optical function \(q = q(t, x)\) is smooth\(^2\) in \(\Omega\) in the followings sense: for each integer \(N \geq 2\), there exists \(\varepsilon_N > 0\) such that \(q\) is a \(C^N\) function in \(\Omega\) for each \(0 < \varepsilon < \varepsilon_N\). Moreover, if \(Z\) is one of the commuting vector fields: translations \(\partial_\alpha\), scaling \(t \partial_t + r \partial_r\), rotations \(x_i \partial_j - x_j \partial_i\) and Lorentz boosts \(x_i \partial_t + t \partial_i\), then in \(\Omega\) we

---

\(^1\)We will briefly explain the geometric meaning of \(\text{tr} \chi\) in Section [8]

\(^2\)See Section [24] In particular, a smooth function may not be \(C^\infty\) in this paper.
have $Z^I q = O((q) t^{C\varepsilon})$ and $Z^I \Omega_{ij} q = O(t^{C\varepsilon})$ for each multiindex $I$ and $\varepsilon \ll 1$. To prove these estimates, we introduce the commutator coefficients $\{\epsilon_{k_1 k_2}^I\}_{1 \leq k_1, k_2, l \leq 4}$ for which we have $[e_{k_1}, e_{k_2}] = \epsilon_{k_1 k_2}^I e_l$. We also introduce a weighted null frame 

$$ (V_1, V_2, V_3, V_4) := (re_1, re_2, (3R - r + t)e_3, te_4) $$

which combines the advantages of a usual null frame $\{e_k\}$ and the commuting vector fields $Z$'s. By computing $e_4(V^I \epsilon_{k_1 k_2}^I)$ for each multiindex $I$ and applying the Gronwall’s inequality, we are able to obtain several estimates for $V^I(\epsilon_{k_1 k_2}^I)$; see Proposition 4.9. These estimates for $\xi$ then imply the estimates for $q$, so we finish the proof.

We finally remark that the map

$$ \Omega \to [0, \infty) \times \mathbb{R} \times S^2: \quad (t, x) \mapsto (\varepsilon \ln t - \delta, q(t, x), x/|x|) := (s, q, \omega) $$

is an injective smooth function with a smooth inverse. This is because $q_r > 0$ everywhere in $\Omega$. Thus, a smooth function $F = F(t, x)$ induces a smooth function $F = F(s, q, \omega)$ and vice versa.

1.5. The asymptotic equations and the scattering data. For each $(t, x) \in \Omega$, we define

$$ \mu(t, x) := (q_t - q_r)(t, x), \quad U(t, x) := \varepsilon^{-1}ru(t, x). $$

We then obtain two smooth functions $\mu(s, q, \omega)$ and $U(s, q, \omega)$ as discussed at the end of the previous subsection.

To state the results in this subsection, we introduce a new notation $\mathfrak{R}_{s,p}$ for each $s, p \in \mathbb{R}$. For a function $F = F(t, x)$ defined in $\Omega \cap \{r - t < 2R\}$, we write $F = \mathfrak{R}_{s,p}$ if for each integer $N \geq 1$ and for each $\varepsilon \ll N 1$, we have

$$ \sum_{|I| \leq N} |V^I(F)| \lesssim t^{s + C\varepsilon(q)} \varepsilon^p, \quad \forall (t, x) \in \Omega \cap \{r - t < 2R\}. $$

Here recall that $\{V_s\}$ is the weighted null frame.

By the chain rule, we have

$$ \partial_s = \varepsilon^{-1}t(\partial_t - q_t q_r^{-1} \partial_r), \quad \partial_q = q_r^{-1} \partial_r, \quad \partial_{\omega_i} = r(\partial_t - q_t q_r^{-1} \partial_r). $$

Then we can express $(\partial_s, \partial_q, \partial_{\omega_i})$ in terms of the weighted null frame $\{V_s\}$. In fact, we have

$$ \partial_s = \sum_a \varepsilon^{-1} \mathfrak{R}_{-1,0} V_a + (\varepsilon^{-1} + \mathfrak{R}_{-1,0}) V_4, \quad \partial_q = \sum_k \mathfrak{R}_{0,-1} V_k, $$

$$ \partial_{\omega_i} = \sum_{k \neq 3} \mathfrak{R}_{-1,0} V_k + \sum_a e_a^i V_a = \sum_{k \neq 0} \mathfrak{R}_{0,0} V_k. $$

Meanwhile, from (2.1) and $e_4(e_3(q)) = -\Gamma_{\alpha \beta \gamma}^0 e_3^\alpha e_3^\beta e_3(q)$, we can show that

$$ e_4(e_3(u)) + r^{-1} e_3(u) = \varepsilon \mathfrak{R}_{3,0}, \quad e_4(e_3(q)) = -\frac{1}{4} e_3(u) G(\omega) e_3(q) + \varepsilon \mathfrak{R}_{2,0}. $$

Combine these estimates, and we obtain that

$$ \left\{ \begin{array}{l} 
\partial_s \mu = \frac{1}{4} G(\omega) \mu^2 U_q + \varepsilon^{-1} \mathfrak{R}_{-1,0}, \\
\partial_q U_q = -\frac{1}{4} G(\omega) \mu U_q^2 + \varepsilon^{-1} \mathfrak{R}_{-1,0}. 
\end{array} \right. $$

(1.15)
That is, \((\mu, U)(s, q, \omega)\) is an approximate solution to the geometric reduced system (1.12).

Next, we note from (1.15) that \(\partial_s (\mu U_q) = O(\epsilon^{-1} t^{-1+C\epsilon})\). By integrating the remainder term \(\epsilon^{-1} t^{-1+C\epsilon}\) (viewed as a function of \(s\)) with respect to \(s\), we can show that \(\{(\mu U_q)(s, q, \omega)\}_s\) is uniformly Cauchy for each \((q, \omega) \in \mathbb{R} \times \mathbb{S}^2\). Thus, the limit

\[
A(q, \omega) := -\frac{1}{2} \lim_{s \to \infty} (\mu U_q)(s, q, \omega)
\]

exists and the convergence is uniform in \((q, \omega)\). This function \(A\) is then the scattering data in the asymptotic completeness problem.

Similarly, we can show that for each \(m\) and \(n\), the limit

\[
A_{m,n}(q, \omega) := -\frac{1}{2} \lim_{s \to \infty} (\langle q \rangle \partial_q)^m \partial_\omega^n (\mu U_q)(s, q, \omega)
\]

exists and the convergence is uniform in \((q, \omega)\). The uniform convergences of these limits imply that the scattering data \(A\) is smooth,

\[
(\langle q \rangle \partial_q)^m \partial_\omega^n A(q, \omega) = A_{m,n}(q, \omega).
\]

Following the same method, we can define

\[
A_1(q, \omega) := \lim_{s \to \infty} \exp \left(-\frac{1}{2} G(\omega) A(q, \omega) s\right) \mu(s, q, \omega),
\]

\[
A_2(q, \omega) := \lim_{s \to \infty} \exp \left(-\frac{1}{2} G(\omega) A(q, \omega) s\right) U_q(s, q, \omega).
\]

Both of these limits exist and have derivatives of any order with respect to \(q\) and \(\omega\), as long as \(\epsilon\) is sufficiently small. It is clear that \(A_1 A_2 \equiv -2A\), so we obtain an exact solution to the geometric reduced system (1.12):

\[
\begin{cases}
\tilde{\mu}(s, q, \omega) = A_1(q, \omega) \exp \left(-\frac{1}{2} G(\omega) A(q, \omega) s\right), \\
\tilde{U}_q(s, q, \omega) = A_2(q, \omega) \exp \left(-\frac{1}{2} G(\omega) A(q, \omega) s\right),
\end{cases}
\]

(1.16)

By assuming \(\lim_{s \to \infty} \tilde{U}(s, q, \omega) = 0\), we obtain a unique function \(\tilde{U} = \tilde{U}(s, q, \omega)\). By the definition of \((A, A_1, A_2)\), we expect the \((\mu - \tilde{\mu}, U - \tilde{U})\), along with their derivatives with respect to \((q, \omega)\) of any order, decays faster than \(\mu\) and \(U\). We refer our readers to Proposition 5.1 for a complete list of estimates.

As defined, the scattering data \(A\) depends on the initialization of the optical function \(q\). In Section 6, see Proposition 6.1, we resolve this ambiguity and show a gauge independence result, which states that the scattering data is independent of the choice of \(q\) in a precise sense.

1.6. An approximation. In the previous subsection, we have discussed how to obtain an exact solution (1.16) to the geometric reduced system (1.12). Our final objective is to show that this exact solution gives a good approximation of the exact solution \(u\) to (1.1).

We first solve

\[
\tilde{q}_t - \tilde{q}_r = \tilde{\mu}(\varepsilon \ln(t) - \delta, \tilde{q}(t, x), \omega) \quad \text{in} \quad \Omega \cap \{r - t < 2R\}; \quad \tilde{q} = r - t \quad \text{when} \quad r - t \geq 2R,
\]

and set

\[
\tilde{u}(t, x) = \varepsilon^{-1} \tilde{U}(\varepsilon \ln(t) - \delta, \tilde{q}(t, x), \omega) \quad \text{in} \quad \Omega \cap \{r - t < 2R\}.
\]
Then, we can prove that \( \tilde{u} \) is an approximate solution to (1.1) in the following sense: for each integer \( N \geq 1 \) and \( \varepsilon \ll_N 1 \), we have

\[
(1.17) \quad \sum_{|I| \leq N} |Z^I (g_{\alpha \beta}(\tilde{u}) \partial_{\alpha} \partial_{\beta} \tilde{u})| \lesssim \varepsilon t^{-3 + \Gamma}, \quad \text{in } \Omega \cap \{r - t < 2R\}.
\]

Here we denote by \( Z \) any of the commuting vector fields: translations \( \partial_\alpha \), scaling \( t \partial_t + r \partial_r \), rotations \( x_i \partial_j - x_j \partial_i \) and Lorentz boosts \( x_i \partial_t + t \partial_i \). To make our proof simpler, we introduce \( A = F(q, \omega) \) such that \( F_q = -2/A_1 \). It can be shown that \( q \mapsto F(q, \omega) \) has an inverse \( q \mapsto \hat{F}(q, \omega) \) and define \( \hat{F}(q, \omega) \) by replacing \((A_1, A_2, A)\) in (1.16); see (2). Then, \( \hat{F}(t, x) := F(q(t, x), \omega) \) is a solution to

\[
\hat{q}_t - \hat{q}_r = \mu (\varepsilon \ln t - \delta, \hat{q}(t, x), \omega) \quad \text{in } \Omega \cap \{r - t < 2R\}; \quad \hat{q} = r - t \quad \text{when } r - t \geq 2R.
\]

In addition, we have

\[
\tilde{U}(\varepsilon \ln t - \delta, \tilde{q}(t, x), \omega) = \tilde{U}(\varepsilon \ln t - \delta, \hat{q}(t, x), \omega).
\]

We can now follow the proof in Section 4 of [35] to prove (1.17).

In order to estimate \( u - \tilde{u} \), we set \( p(t, x) := F(q(t, x), \omega) - \hat{q}(t, x) \) in \( \Omega \). We claim that, for each fixed \( \gamma \in (0, 1) \), an integer \( N \geq 1 \), and for each \( \varepsilon \ll_N 1 \), whenever \( (t, x) \in \Omega \) such that \( |r - t| \lesssim t^\gamma \), we have \( |Z^I p(t, x)| \lesssim t^{1 - 1 + \Gamma} \) for each \( |I| \leq N \). To show this claim, we compute \( p_t - p_r \) and apply a continuity argument. This claim then implies that, under the same assumptions on \( \gamma, N \) and \( \varepsilon \), whenever \( (t, x) \in \Omega \) such that \( |r - t| \lesssim t^\gamma \), we have \( |Z^I (u - \tilde{u})(t, x)| \lesssim \varepsilon t^{-2 + \Gamma} \) for each \( |I| \leq N \). Recall from Lindblad [23] that we only have \( Z^I u = O(\varepsilon t^{-1 + \Gamma}) \), so \( \tilde{u} \) provides a good approximation of \( u \).

1.7. The main theorem. We now state the main theorem which summarizes the outcome of the sequence of steps described in the previous subsections. In this theorem, we say that a function \( f = f(t, x) \) is smooth if for each large integer \( N \), \( f \) is \( C^N \) whenever \( \varepsilon \ll_N 1 \). See Section 2.4 for details.

**Theorem 1.** Let \( u \) be a smooth solution to the Cauchy problem (1.1) and (1.2). Fix a constant \( R > 0 \) such that \( \text{supp} (u_0, u_1) \subset \{|x| \leq R\} \), so \( u \equiv 0 \) for \( |x| > t + R \) by the finite speed of propagation. Set \( T_0 := \exp(\delta / \varepsilon) \) for a fixed constant \( \delta > 0 \). Then we have

a) There exists a smooth solution to the eikonal equation

\[
\hat{g}^{\alpha \beta}(u) \partial_{\alpha} \hat{q} \partial_{\beta} \hat{q} = 0 \quad \text{in } \Omega; \quad \hat{q} = |x| - t \quad \text{on } \partial \Omega.
\]

Here the region \( \Omega \subset \mathbb{R}^{1+3}_{t,x} \) is defined by

\[
\Omega := \{(t, x) : t > T_0, \quad |x| > (t + T_0) / 2 + 2R\}.
\]

In \( \Omega \), for each multiindex \( I \) we have

\[
|Z^I \hat{q}| \lesssim (\hat{q}) t^{\Gamma}, \quad \sum_{1 \leq i, j \leq 3} |Z^I \Omega_{ij} \hat{q}| \lesssim t^{\Gamma}.
\]

Moreover, the map

\[
\Omega \to [0, \infty) \times \mathbb{R} \times S^2 : \quad (t, x) \mapsto (\varepsilon \ln t - \delta, \hat{q}(t, x), x / |x|)
\]

is an injective smooth function with a smooth inverse. Thus, a smooth function \( F = F(t, x) \) induces a smooth function \( \tilde{F} = \tilde{F}(s, q, \omega) \) and vice versa.
b) In Ω, we set \((\mu, U)(t, x) := (q_t - q_r, \varepsilon^{-1} ru)(t, x)\) which induces a smooth function \((\mu, U)(s, q, \omega)\). Then, \((\mu, U)(s, q, \omega)\) is an approximate solution to the geometric reduced system \((1.12)\) in the sense that
\[
\begin{align*}
\partial_s \mu &= \frac{1}{4} G(\omega) \mu^2 U_q + \varepsilon^{-1} \mathfrak{R}_{1,0}, \\
\partial_s U_q &= -\frac{1}{4} G(\omega) \mu U_q^2 + \varepsilon^{-1} \mathfrak{R}_{1,0}.
\end{align*}
\]
Here the notation \(\mathfrak{R}_{\kappa, s}\) has been defined in Section 1.6. In addition, the following three limits exist for all \((q, \omega) \in \mathbb{R} \times S^2:\)
\[
\begin{align*}
A(q, \omega) &= -\frac{1}{2} \lim_{s \to \infty} (\mu U_q)(s, q, \omega), \\
A_1(q, \omega) &= \lim_{s \to \infty} \exp\left(\frac{1}{2} G(\omega) A(q, \omega) s\right) \mu(s, q, \omega), \\
A_2(q, \omega) &= \lim_{s \to \infty} \exp\left(-\frac{1}{2} G(\omega) A(q, \omega) s\right) U_q(s, q, \omega).
\end{align*}
\]
All of them are smooth functions of \((q, \omega)\) for \(\varepsilon \ll 1\), and we have \(A_1 A_2 \equiv -2A\). By setting
\[
\begin{align*}
\tilde{\mu}(s, q, \omega) &= A_1 \exp(-\frac{1}{2} GA s), \\
\tilde{U}_q(s, q, \omega) &= A_2 \exp(\frac{1}{2} GA s).
\end{align*}
\]
we obtain an exact solution to our reduced system \((1.12)\).

c) We define \(\tilde{u} = \tilde{u}(t, x)\) as in Section 1.6. Then the function \(\tilde{u} = \tilde{u}(t, x)\) is an approximate solution to \((1.1)\) in the following sense:
\[
|Z^I(g^{\alpha \beta}(\tilde{u}) \partial_\alpha \partial_\beta \tilde{u}))(t, x)| \lesssim \varepsilon t^{-3+3\varepsilon}, \quad \forall (t, x) \in \Omega, \forall I.
\]
Moreover, we fix a constant \(0 < \gamma < 1\) and a large integer \(N\). Then, for \(\varepsilon \ll \gamma, N, 1\), at each \((t, x) \in \Omega\) such that \(|r - t| \lesssim t^\gamma\), we have
\[
|Z^I(u - \tilde{u})| \lesssim \varepsilon t^{-2+3\varepsilon}(r - t), \quad \forall |I| \leq N.
\]

**Remark 1.1.** Because of the special definition of smoothness in this paper, we emphasize that our main theorem only holds in the following sense: for each large integer \(N\), there exists a sufficiently small constant \(\varepsilon_N > 0\) depending on \(N\) and the functions \(u_0, u_1 \in C_0^\infty(\mathbb{R}^3)\) given in \((1.2)\), such that the conclusions in Theorem 1 hold for all \(0 < \varepsilon < \varepsilon_N\), with all “smooth” replaced by “\(C^N\)” in the statement.

**Remark 1.2.** We expect the results above are gauge independent. That is, the scattering data \(A = A(q, \omega)\) is independent of the choice of the optical function \(q = q(t, x)\) in some suitable sense. In fact, we choose the region \(\Omega\) in a way that \(t \sim r\) in \(\Omega \cap \{r - t < 2R\}\), that \(t \geq T_0 = \exp(\delta/\varepsilon)\) in \(\overline{\Omega}\), and that \(u \equiv 0\) in \(\partial \Omega \cap \{t = T_0\}\). The proof in this paper is expected to work if we start with a different region \(\Omega\) with these three properties hold. For example, we can replace the definition of \(\Omega\) with
\[
\Omega = \Omega_{\kappa, \delta} := \{(t, x) : t > \exp(\delta/\varepsilon), |x| - \exp(\delta/\varepsilon) - 2R > \kappa(t - \exp(\delta/\varepsilon))\}
\]
for some fixed constants \(\delta > 0\) and \(0 < \kappa < 1\). For a different choice of \((\kappa, \delta)\), we do not expect to get the same scattering data. However, Proposition 6.1 states that the scattering data associated to different regions \(\Omega_{\kappa, \delta}\) are in fact related to each other in some sense.
Remark 1.3. In our construction, we fix a parameter $\delta > 0$ and solve the eikonal equation in a region contained in $\{ t > \exp(\delta / \epsilon) \}$. In fact, the proof in this paper is expected to work for each fixed $\delta > 0$. However, we do not simply set $\delta = 1$ here. Instead, we choose a sufficiently small $\delta > 0$ which depends on the pair $(u_0, u_1)$, such that the nonlinear effects of (1.1) are negligible until we reach the time $\exp(\delta / \epsilon)$. For example, we can set $\delta$ to be the small constant $c$ in the almost global existence result.

Remark 1.4. The part c) of the main theorem is an approximation result near the light cone $r = t$. Outside the light cone, we have $u \equiv 0$ whenever $r - t \geq R$ because of the finite speed of propagation. It is thus natural to ask whether we also have an approximation result in the interior region away from the light cone. For example, can we find an approximate solution $\tilde{u}$ such that $u - \tilde{u}$, along with its derivatives, decays faster than $\epsilon t^{-1+C\epsilon}$ whenever $r < t - Ct^\gamma$, where $\gamma \in (0, 1)$ is a fixed constant?

We first remark that $\tilde{u}$ constructed in Section 1.6 is not a good candidate in this case. One reason is that $\tilde{u}$ is only defined in $\Omega$. Even in the region where it is defined, it does not give a good approximation near $\partial \Omega$. Note that part c) of the main theorem implies that

$$|Z^I(u - \tilde{u})| \lesssim \epsilon t^{-2+\gamma+C\epsilon}, \quad \text{whenever} \ (t, x) \in \Omega, \ r - t = -t^\gamma/4, \ \gamma \in (0, 1).$$

If we set $\gamma = 1$ on the right hand side of this estimate, we get $\epsilon t^{-1+C\epsilon}$ which is the decay rate for the solution $u$ itself. Thus, $\tilde{u}$ does not approximate $u$ very well away from the light cone in $\Omega$. Intuitively, this is because the geometric reduced system and the Hörmander’s asymptotic PDE’s are derived under the assumption $t \approx r \to \infty$.

By the pointwise estimates for $Z^I u$ and Lemma 2.2 below, we have $|\partial^k Z^I u| \lesssim \epsilon t^{-1+C\epsilon} (r - t)^{-k}$. As a result, if $|r - t| \gtrsim t^{-\gamma}$ for some $\gamma > 0$, we have

$$|\partial^k Z^I u| \lesssim \epsilon t^{-1-k\gamma+C\epsilon}, \quad \forall (k, I).$$

So $\partial^k Z^I u$ has a decay rate better than $\epsilon t^{-1+C\epsilon}$ if $k > 0$. The case $k = 0$ seems to be more complicated, since it is unclear what would be a good approximation for $Z^I u$ in the interior region. We will not discuss this topic in this paper and we refer our readers to [4] which is a paper on the asymptotic behavior of the Maxwell-Klein-Gordon system in this direction.

Remark 1.5. We compare the results in this paper with those in Deng-Pusateri [7]. First, the approximation result (i.e. part c) in Theorem 1) is better than that in [7] (i.e. Theorem 2.3). This suggests that the geometric reduced system (1.12) gives a more accurate description of the global solutions to (1.1) than the Hörmander’s asymptotic PDE (1.9) does. Besides, the proof in this paper relies on the null geometry, and we only use estimates in physical space. In contrast, the authors in [7] made use of the spacetime resonance method which relies on estimates in frequency space.

1.8. Acknowledgement. The author would like to thank his PhD advisor, Daniel Tataru, for suggesting this problem and for many helpful discussions. The author would like to thank Sung-Jin Oh for some helpful discussions on the optical function. The author is also grateful to the anonymous reviewers for their valuable comments and suggestions on this paper.

---

We can also say that the main theorem is an asymptotic result near the null infinity. In contrast, the result in [3] is an asymptotic result near the timelike infinity. In that paper, the authors consider some limits of the form $\lim_{t \to \infty} t A_\mu(t, ty)$ for some $|y| < 1$. 

13
This research was partially supported by a James H. Simons Fellowship and by the NSF grant DMS-1800294. Most of the material here overlaps with the author’s PhD dissertation [34].

2. Preliminaries

2.1. Notations. We use $C$ to denote universal positive constants. We write $A \lesssim B$ or $A = O(B)$ if $|A| \leq CB$ for some $C > 0$. We write $A \sim B$ if $A \lesssim B$ and $B \lesssim A$. We use $C_v$ or $\lesssim_v$ if we want to emphasize that the constant depends on a parameter $v$. We make an additional convention that the constants $C$ are always independent of $\varepsilon$; that is, we would never write $C_\varepsilon$ or $\lesssim_\varepsilon$ in this paper. The values of all constants in this paper may vary from line to line.

In this paper, we always assume that $\varepsilon \ll 1$ which means $0 < \varepsilon < \varepsilon_0$ for some sufficiently small constant $\varepsilon_0 < 1$. Again, we write $\varepsilon \ll_v 1$ if we want to emphasize that $\varepsilon_0$ depends on a parameter $v$.

Unless specified otherwise, we always assume that the Latin indices $i, j, l$ take values in $\{1, 2, 3\}$ and the Greek indices $\alpha, \beta$ take values in $\{0, 1, 2, 3\}$. We also assume $a, b \in \{1, 2\}$ when we study the null frame introduced in Section 3. We use subscript to denote partial derivatives, unless specified otherwise. For example, $u_{\alpha\beta} = \partial_\alpha \partial_\beta u$, $q_r = \partial_r q = \sum_i \omega_i \partial_i q$, $A_q = \partial_q A$ and etc. For a fixed integer $k \geq 0$, we use $\partial^k$ to denote either a specific partial derivative of order $k$, or the collection of partial derivatives of order $k$.

To prevent confusion, we will only use $\partial_\omega$ to denote the angular derivatives under the coordinate $(s, q, \omega)$, and will never use it under the coordinate $(t, r, \omega)$. For a fixed integer $k \geq 0$, we will use $\partial^k_\omega$ to denote either a specific angular derivative of order $k$, or the collection of all angular derivatives of order $k$.

2.2. Commuting vector fields. We denote by $Z$ any of the following vector fields:

$\partial_\alpha$, $\alpha = 0, 1, 2, 3$; $S = t \partial t + r \partial_r$; $\Omega_{ij} = x_i \partial_j - x_j \partial_i$, $1 \leq i < j \leq 3$; $\Omega_{qi} = x_i \partial_t + t \partial_i$, $i = 1, 2, 3$.

We write these vector fields as $Z_1, Z_2, \ldots, Z_{11}$, respectively. For any multiindex $I = (i_1, \ldots, i_m)$ with length $m = |I|$ such that $1 \leq i_* \leq 11$, we set $Z^I = Z_{i_1} Z_{i_2} \cdots Z_{i_m}$. Then we have the Leibniz’s rule

$Z^I(fg) = \sum_{|J|+|K|=|I|} C^I_{JK} Z^J f Z^K g$, \hspace{1cm} \text{where } C^I_{JK} \text{ are constants.}

We have the following commutation properties.

$[S, \Box] = -2\Box$, \hspace{1cm} $[Z, \Box] = 0$ for other $Z$;

$[Z_1, Z_2] = \sum_{|I|=1} C_{Z_1, Z_2, I} Z^I$, \hspace{1cm} \text{where } C_{Z_1, Z_2, I} \text{ are constants;}

$[Z, \partial_\alpha] = \sum_\beta C_{Z, \alpha, \beta} \partial_\beta$, \hspace{1cm} \text{where } C_{Z, \alpha, \beta} \text{ are constants.}$

In Section 7, we need the following lemma related to the commuting vector fields. Here we use $f_0$ to denote an arbitrary polynomial of $\{Z^I\omega\}$. It is then clear that $Z^I f_0 = f_0$ for each $I$.  

14
Lemma 2.1. For each multiindex $I$ and each function $F$, we have
\begin{equation}
(\partial_t - \partial_r) Z^I F = Z^I (F_t - F_r) + \sum_{|J| < |I|} [f_0 Z^J (F_t - F_r) + \sum_i f_0 (\partial_t + \omega_i \partial_i) Z^J F]. 
\end{equation}

Besides, for each $1 \leq k < k' \leq 3$, we have
\begin{equation}
(\partial_t - \partial_r) Z^I \Omega_{kk'} F = Z^I \Omega_{kk'} (F_t - F_r) + \sum_{|J| < |I|} [f_0 Z^J \Omega_{kk'} (F_t - F_r) + \sum_i f_0 (\partial_t + \omega_i \partial_i) Z^J \Omega_{kk'} F].
\end{equation}

Proof. First, note that $[\partial_t - \partial_r, Z] = f_0 \cdot \partial$ and $\partial = f_0 (\partial_t - \partial_r) + \sum_i f_0 (\partial_t + \omega_i \partial_i)$. We now prove (2.6) by induction on $|I|$. If $|I| = 0$, there is nothing to prove. Now suppose we have proved (2.6) for each $|I| < n$. Now we fix a multiindex $I$ with $|I| = n > 0$. Then, by writing $Z^I = Z Z^I$, we have
\begin{align*}
(\partial_t - \partial_r) Z^I F &= [\partial_t - \partial_r, Z] Z^I F + Z ((\partial_t - \partial_r) Z^I F) \\
&= f_0 \cdot \partial Z^I F + Z (Z^I (F_t - F_r) + \sum_{|J| < n-1} [f_0 Z^J (F_t - F_r) + \sum_i f_0 (\partial_t + \omega_i \partial_i) Z^J F] \\
&= f_0 (f_0 (\partial_t - \partial_r) + \sum_j f_0 (\partial_j + \omega_j \partial_j)) Z^I F + Z^I (F_t - F_r) \\
&\quad + \sum_{|J| < n-1} Z [f_0 Z^J (F_t - F_r) + \sum_i f_0 (\partial_t + \omega_i \partial_i) Z^J F] \\
&= f_0 (\partial_t - \partial_r) Z^I F + \sum_j f_0 (\partial_j + \omega_j \partial_j) Z^I F + Z^I (F_t - F_r) \\
&\quad + \sum_{|J| < n-1} [(Z f_0) Z^J (F_t - F_r) + \sum_i (Z f_0) (\partial_t + \omega_i \partial_i) Z^J F] \\
&\quad + \sum_{|J| < n-1} [f_0 Z Z^J (F_t - F_r) + \sum_i f_0 (\partial_t + \omega_i \partial_i) Z^J F].
\end{align*}

In the second equality, we can apply (2.6) by the induction hypotheses. Moreover, we note that $[\partial_t + \omega_i \partial_i, Z] = f_0 \cdot \partial$, so
\begin{align*}
Z (\partial_t + \omega_i \partial_i) Z^J F &= (\partial_t + \omega_i \partial_i) ZZ^J F + f_0 \cdot \partial Z^J F \\
&= (\partial_t + \omega_i \partial_i) ZZ^J F + f_0 (\partial_t - \partial_r) Z^J F + \sum_j f_0 (\partial_j + \omega_j \partial_j) Z^J F.
\end{align*}

Now (2.6) follows from the induction hypotheses and the computations above.

To prove (2.7), we replace $F$ with $\Omega_{kk'}$ in (2.6) and note that
\begin{align*}
[\partial_t - \partial_r, \Omega_{kk'}] &= -\partial_r (x_k) \partial_k' + \partial_r (x_k') \partial_k + \sum_i \Omega_{kk'} (\omega_i) \partial_i \\
&= -\omega_k \partial_k' + \omega_k \partial_k + \sum_i \omega_k (\delta_{ik'} - \omega_i \omega_k') \partial_i - \sum_i \omega_{k'} (\delta_{ik} - \omega_i \omega_k) \partial_i = 0.
\end{align*}

Now, (2.7) is obvious. \qed
2.3. Several pointwise bounds. We have the pointwise estimates for partial derivatives.

Lemma 2.2. For any function \( \phi \), we have
\[
|\partial^k \phi| \leq C \langle t - r \rangle^{-k} \sum_{|I| \leq k} |Z^I \phi|, \quad \forall k \geq 1,
\]
and
\[
| (\partial_t + \partial_r) \phi | + | (\partial_t - \omega_i \partial_r) \phi | \leq C \langle t + r \rangle^{-1} \sum_{|I| = 1} |Z^I \phi |.
\]

In addition, we have the Klainerman-Sobolev inequality.

Proposition 2.3. For \( \phi \in C^\infty(\mathbb{R}^{1+3}) \) which vanishes for large \( |x| \), we have
\[
(1 + t + |x|)(1 + |t - |x||)^{1/2} |\phi(t, x)| \leq C \sum_{|I| \leq 2} \| Z^I \phi(t, \cdot) \|_{L^2(\mathbb{R}^3)}.
\]

We also state the Gronwall’s inequality.

Proposition 2.4. Suppose \( A, E, r \) are bounded functions from \([a, b]\) to \([0, \infty)\). Suppose that \( E \) is increasing. If
\[
A(t) \leq E(t) + \int_a^b r(s) A(s) \, ds, \quad \forall t \in [a, b],
\]
then
\[
A(t) \leq E(t) \exp\left( \int_a^t r(s) \, ds \right), \quad \forall t \in [a, b].
\]

The proofs of these results are standard. See, for example, [10, 23, 31] for the proofs.

2.4. A key theorem and a convention. This paper is based on the following global existence result.

Theorem 2 (Lindblad [23]). Fix a large integer \( N \gg 1 \). Then, for \( \varepsilon \ll_N 1 \), the Cauchy problem (1.1) with the initial data (1.2) has a global \( C^N \) solution \( u = u(t, x) \) for all \( t \geq 0 \). Moreover, we have pointwise decays: \( Z^I u = O_1(\varepsilon(t)^{-1} + C_\varepsilon) \) for each multiindex \( I \) such that \( |I| \leq N \). Moreover, we have \( \partial u = O(\varepsilon(t)^{-1}) \).

Most of the functions in this paper have similar properties. That is, they depend on a small parameter \( \varepsilon \), and they are \( C^N \) for any large integer \( N \) as long as \( \varepsilon \ll_N 1 \). For convenience, we make the following definition.

Definition. Fix a function \( f = f_\varepsilon(t, x) \) which depends on a small parameter \( \varepsilon \). In this paper, we say that \( f \) is smooth, if for each large integer \( N \), \( f \) is \( C^N \) whenever \( \varepsilon \ll_N 1 \).

Following the same spirits, we say that all derivatives of a function satisfy some properties, if for each large integer \( N \), all its derivatives of order \( \leq N \) exist and satisfy such properties whenever \( \varepsilon \ll_N 1 \).

We remark that under this definition, a smooth function does not need to be a \( C^\infty \) function. It will be more convenient to work with this seemingly strange definition.

Using such a convention, we can state Theorem 2 as follows: For \( \varepsilon \ll 1 \), the Cauchy problem (1.1) with the initial data (1.2) has a global smooth solution \( u = u(t, x) \) for all
$t \geq 0$. Moreover, we have pointwise decays: $Z' u = O_{I}(\varepsilon/(t)^{-1+C_{I}\varepsilon})$ for each multiindex $I$ and $\partial u = O(\varepsilon/(t)^{-1})$.

2.5. **The null condition of a matrix.** The definition and lemmas in this subsection will be used in Section 4.2. In this subsection, we assume that every matrix is in $\mathbb{R}^{4 \times 4}$ and is a symmetric constant matrix.

**Definition.** A matrix $m_{0} = (m_{0}^{\alpha\beta})_{\alpha,\beta=0,1,2,3}$ satisfies the **null condition** if

$$m_{0}^{\alpha\beta} \xi_{\alpha} \xi_{\beta} = 0,$$

whenever $\xi \in \mathbb{R}^{1+3}$ and $|\xi_{0}|^{2} + |\xi_{1}|^{2} + |\xi_{2}|^{2} + |\xi_{3}|^{2}$.

We remark that a real symmetric constant matrix $m_{0}$ satisfies the null condition if and only if $m_{0}^{\alpha\beta} \xi_{\alpha} \eta_{\beta}$ is a linear combination of $-\xi_{0} \eta_{0} + \sum_{j=1}^{3} \xi_{j} \eta_{j}$ and $\xi_{\alpha} \eta_{\beta} - \xi_{\beta} \eta_{\alpha}$.

We start with the following useful lemma.

**Lemma 2.5.** Suppose $m_{0}$ is a constant matrix satisfying the null condition. Then, for any two functions $\phi = \phi(t,x)$ and $\psi = \psi(t,x)$, we have

$$Z(m_{0}^{\alpha\beta} \phi_{\alpha} \psi_{\beta}) = m_{0}^{\alpha\beta} (\partial_{\alpha} Z \phi) \psi_{\beta} + m_{0}^{\alpha\beta} \phi_{\alpha} (\partial_{\beta} Z \psi) + m_{1}^{\alpha\beta} \phi_{\alpha} \psi_{\beta}.$$

Here $m_{1}$ is another symmetric constant matrix satisfying the null condition. Moreover, if $Z = \Omega_{ij}$ for $1 \leq i, j \leq 3$ and if $(m_{0}^{\alpha\beta}) = (m_{0}^{\alpha\beta})$ is the Minkowski metric, then $m_{1} = 0$.

We refer our readers to Lemma 6.6.5 in [10] for the proof.

In addition, we have the following pointwise estimates related to the null condition.

**Lemma 2.6.** Suppose $m_{0}$ is a matrix satisfying the null condition. Then, for any two functions $\phi = \phi(t,x)$ and $\psi = \psi(t,x)$, if $t \sim r \gg 1$, we have

$$|m_{0}^{\alpha\beta} \phi_{\alpha} \psi_{\beta}| \lesssim (t)^{-1} (|Z \phi||\partial \psi| + |Z \psi||\partial \phi|).$$

Here $|Z f| = \sum_{|I| \leq 1} |Z^{I} f|$ for a function $f = f(t,x)$.

We refer our readers to Lemma II.5.4 in [31] for the proof.

3. **Construction of the optical function.**

Let $u = u(t,x)$ be a global solution to (1.1) and (1.2) constructed in Theorem 2. If we fix a constant $R > 0$ such that supp $(u_{0},u_{1}) \subset \{|x| \leq R\}$, then $u \equiv 0$ for $|x| \geq t + R$ by the finite speed of propagation. Our goal in this section is to construct an optical function, i.e. a solution to the eikonal equation

$$g^{\alpha\beta}(u) \partial_{\alpha} q \partial_{\beta} q = 0 \text{ in } \Omega; \quad q = |x| - t \text{ on } \partial \Omega.$$

The region $\Omega \subset \mathbb{R}^{1+3}_{t,x}$ is defined by

$$\Omega := \{(t,x) : t > T_{0}, |x| > (t + T_{0})/2 + 2R\}.$$

Here $T_{0} := \exp(\delta/\varepsilon)$ for a fixed constant $\delta > 0$.

Our main result of this section is the following proposition.

**Proposition 3.1.** The eikonal equation (3.1) has a global $C^{2}$ solution in the region $\Omega$.

In Section 4 we will show that this $C^{2}$ solution is in fact smooth (in the sense defined in Section 2.4).

Here we briefly explain how the optical function is constructed. In Section 3.1 we apply the method of characteristics and solve the characteristic ODE’s. Here the characteristics
are in fact the null geodesics associated to the Lorentzian metric \((g_{\alpha\beta})\) which is the inverse of the coefficients \((g^{\alpha\beta}(u))\) in \((3.1)\). In Section \([3.2]\) assuming that the optical function \(q\) exists in some region, we prove several preliminary estimates for \(q\) by studying the characteristic ODE’s.

To finish the proof, we need to show that the characteristics, i.e. the geodesics, do not intersect with each other. This is related to the null geometry of the level sets of the optical function. In Section \([3.3]\) and \([3.4]\) we construct a null frame \(\{e_k\}_{k=1}^4\) and then define several connection coefficients under the Lorentzian metric \((g_{\alpha\beta})\). Most importantly, we define the second fundamental form

\[
\chi_{ab} := \langle D_{{}\ e_a} e_b, \rangle, \quad a, b = 1, 2.
\]

Here \(D\) is the Levi-Civita connection and \(\langle \cdot, \cdot \rangle\) is the bilinear form, both associated to \((g_{\alpha\beta})\).

One important quantity we need to estimate in our proof is the trace of the second fundamental form

\[
\text{tr} \chi > 0.
\]

We claim that it suffices to prove \(\text{tr} \chi > 0\) everywhere. In fact, for a 2-sphere \(S_{(t_0, q^0)} := \{x : q(t_0, x) = q^0\} \subset \mathbb{R}^3\), we have

\[
\frac{d}{dt_0} |S_{(t_0, q^0)}| = \int_{S_{(t_0, q^0)}} \text{tr} \chi \, dA.
\]

See, for example, Section 9.5 of \([2]\). If \(\text{tr} \chi > 0\), then it implies that the 2-sphere is expanding everywhere as the time increases. This excludes the case when two distinct characteristics intersect with each other.

We now follow the idea in Christodoulou-Klainerman \([5]\). In Section \([3.5]\) we derive an equation for \(\chi\), called the Raychaudhuri equation. In Section \([3.6]\) we use a continuity argument and the Raychaudhuri equation to prove that in the region where the optical function exists, we have

\[
\max_{a, b = 1, 2} |\chi_{ab} - \delta_{ab} r^{-1}| \lesssim t^{-2 + C \varepsilon}.
\]

We conclude that \(\text{tr} \chi > 0\) everywhere. This implies that the characteristics will not intersect with each other, so we can extend the optical function to a slightly larger region. We thus finish the proof by making using of a continuity argument.

### 3.1. The method of characteristics

Now we use the method of characteristics to solve \((3.1)\). We have the characteristic ODE’s

\[
\begin{aligned}
\dot{x}^\alpha(s) &= 2g^{\alpha\beta}(x(s))p_\beta(s), \\
\dot{z}(s) &= 2g^{\alpha\beta}(x(s))p_\beta(s)p_\alpha(s) = 0, \\
\dot{p}_\alpha(s) &= -(\partial_\alpha g^{\mu\nu})(x(s))p_\mu(s)p_\nu(s).
\end{aligned}
\]

Here we write \(g^{\alpha\beta}(t, x) = g^{\alpha\beta}(u(t, x))\) with an abuse of notation. We expect that \(z(s) = q(x(s))\) and \(p(s) = (\partial q)(x(s))\) for some optical function \(q(t, x)\). By differentiating the first equation, we obtain the geodesic equation

\[
\ddot{x}^\alpha(s) + \Gamma^\alpha_{\mu\nu} \dot{x}^\mu(s) \dot{x}^\nu(s) = 0.
\]

Here \(\Gamma\) is the Christoffel symbol of the Levi-Civita connection \(D\) of the Lorentzian metric \((g_{\alpha\beta})\). Thus, in this paper, the curve \(x(s)\) is either called a characteristic curve, or a geodesic.

To solve the eikonal equation \((3.1)\), we only need to consider the geodesics emanating from the surface

\[
H := \{(t, x) : t \geq T_0, \ r = (t + T_0)/2 + 2R\} \subset \partial \Omega.
\]

\[18\]
From these geodesics, later we will construct a solution \(q(t, x)\) in the region \(\Omega \cap \{r - t < 2R\}\) such that \(q = r - t\) in \(\Omega \cap \{R < r - t < 2R\}\). Since \(u \equiv 0\) in the region \(r - t > R\), we can then extend our solution to the whole region \(\Omega\) by defining \(q = r - t\) when \(r > t + R\).

To solve the characteristic ODE's (3.3) and the geodesic equation (3.4), we need to first determine \((\partial q)|_H\). Fix \((t, x) \in H\) and recall that \(q = r - t\) on \(H\). Since \(X_i := \partial_t + 2\omega_i\partial_t\) is tangent to \(H\), we have \(X_iq = X_i(r - t) = -\omega_i\) on \(H\). Thus, for \((t, x) \in H\) we have 

\[
q_i = X_iq - 2\omega_iq_t = -\omega_i - 2\omega_iq_t \quad \text{and} \quad 0 = -q_i^2 + 2g^{0i}q_t(-\omega_i - 2\omega_iq_t) + g^{ij}(-\omega_i - 2\omega_iq_t)(-\omega_j - 2\omega_jq_t)
\]

\[
= (-1 - 4g^{0i}\omega_i + 4g^{ij}\omega_i\omega_j)q_t^2 + (4g^{ij}\omega_i\omega_j - 2g^{0i}\omega_i)q_t + g^{ij}\omega_i\omega_j.
\]

Since \(g^{\alpha\beta}(u) = m^{\alpha\beta} + O(|u|)\), we have

\[
0 = (-1 + 4m^{ij}\omega_i\omega_j + O(|u|))q_t^2 + (4m^{ij}\omega_i\omega_j + O(|u|))q_t + (m^{ij}\omega_i\omega_j + O(|u|))
\]

\[
= (3 + O(|u|))q_t^2 + (4 + O(|u|))q_t + (1 + O(|u|)).
\]

Since \(|u| \ll 1\), by the root formula we can uniquely determine \(q_t = -1 + O(|u|)\) at \((t, x)\) (the other root \(q_t = -1/3 + O(|u|)\) is discarded since we expect \(q\) to behave like \(r - t\)). We also have \(q_t = -\omega_i\) and \(q_r = \omega_i\) for \((t, x) \in H\). Now fix \(x(0) \in H\). We set

\[
z(0) = r(x(0)) - x^0(0), \quad p_\alpha(0) = (\partial_\alpha q)(x(0))
\]

where we set

\[
r(V) := \left(\sum_{i=1}^{3} (V^i)^2\right)^{1/2}, \quad \text{for a vector } V = (V^\alpha)_{\alpha=0}^3.
\]

We have the following lemma.

**Lemma 3.2.** Fix \(x(0) \in H\) and construct \(z(0), p(0)\) as above. Then the system (3.3) along with the initial data \((x(0), z(0), p(0))\) has a unique solution \((x(s), z(s), p(s))\) on \([0, \infty)\). In addition, we have \(z^3(s) > 0\) for all \(s \geq 0\), and \(x^0(s) \to \infty\) as \(s \to \infty\).

If moreover we have \(x(0) \in H \cap \{t < T_0 + 2R\}\), then \(x(s) = (2s, 2s\omega + x(0))\). In other words, the geodesics emanating from \(H \cap \{t < T_0 + 2R\}\) are straight lines. Thus \(q = r - t\) whenever \(r > t + R\).

**Proof.** We apply the Picard existence and uniqueness theorem, e.g. Theorem 1.17 in [32], to (3.3). From the theorem, we obtain a unique solution \((x(s), z(s), p(s))\) for all \(0 \leq s < s_{\text{max}}\).

By the blowup criterion in the theorem, either we have \(s_{\text{max}} < \infty\) and \(|x(s)| + |z(s)| + |p(s)| \to \infty\) as \(s \to s_{\text{max}}\) or we have \(s_{\text{max}} = \infty\). Here \(|x(s)| + |z(s)| + |p(s)| \to \infty\) is equivalent to \(|x(s)| + |\dot{x}(s)| \to \infty\) due to \(z(s) = z(0)\) and the first equation in (3.3).

We claim that, along each geodesic, for all \(s \geq 0\) we have

\[
4g^{\alpha\beta}(x(s))p_\alpha(s)p_\beta(s) = 2\dot{x}^\alpha(s)p_\alpha(s) = g_{\alpha\beta}(x(s))\dot{x}^\alpha(s)\dot{x}^\beta(s) = 0.
\]

In other words, the geodesics \(x(s)\) are null curves. The first two equations follow from the first equation in (3.3), so here we only prove the last one. Note that the equality holds for
s = 0 by the construction of \((\partial q)|_H\). In addition,
\[
\frac{d}{ds}(g^{\alpha\beta}(x(s))p_\alpha(s)p_\beta(s)) = 2g^{\alpha\beta}(x(s))\dot{p}_\alpha(s)p_\beta(s) + (\partial_\mu g^{\alpha\beta})(x(s))\dot{x}^\mu(s)p_\alpha(s)p_\beta(s)
= \dot{x}^\alpha(s)\dot{p}_\alpha(s) - \dot{p}_\mu(s)\dot{x}^\mu(s) = 0.
\]
In the last line we use the third equation in (3.3). This ends the proof of (3.6).

Next we claim that \(\dot{x}^0(s) > 0\) for all \(s\). Since \(g^{\alpha\beta}(u) = m^{\alpha\beta} + O(|u|)\) for \(|u| \ll 1\), its inverse \((g_{\alpha\beta}(u))\) is also a small perturbation of the Minkowski metric, i.e. \(g_{\alpha\beta} = m_{\alpha\beta} + O(|u|)\). Thus, (3.6) implies
\[
0 = g_{00}(\dot{x}^0)^2 + 2g_{0i}\dot{x}^i\dot{x}^i + g_{ij}\dot{x}^i\dot{x}^j = -\dot{x}^0(s)^2 + \sum_i (\dot{x}^i(s))^2 + O(|u(x(s))||\dot{x}|^2).
\]
We first show that \(\dot{x}^0(s) \neq 0\) for all \(s\). If \(\dot{x}^0(s_0) = 0\) for some \(s_0 > 0\), then we have \(g_{ij}\dot{x}^i\dot{x}^j = 0\) at \(s = s_0\). Since \(g_{ij} = \delta_{ij} + O(|u|)\), the symmetric matrix \((g_{ij})\) is positive definite. Then \(\dot{x}(s_0) = 0\). However, recall that \(x(s)\) is a geodesic, and the only geodesic passing through \(x(s_0)\) with \(\dot{x}(s_0) = 0\) is the constant curve \(x(s) = x(s_0)\). This leads to a contradiction. In addition, since \(g_0 = -1 + O(|u|)\) on \(H\) and \(\dot{x}^0(0) = 2g^{0\beta}\dot{p}_\beta(0)\), we have \(\dot{x}^0(0) = 2 + O(|u|)\). Thus \(\dot{x}^0(s) > 0\) for all \(s\).

Moreover, since \(u = O(\varepsilon(t)^{-1+C\varepsilon})\), we have
\[
| - (\dot{x}^0(s))^2 + \sum_i (\dot{x}^i(s))^2 | \leq C\varepsilon(\dot{x}^0(s))^{-1+C\varepsilon}(|\dot{x}^0(s)|^2 + \sum_i (\dot{x}^i(s))^2).
\]
By choosing \(\varepsilon \ll 1\), we can make \(C\varepsilon \leq 1/2\). Thus, for \(\varepsilon \ll 1\), we have
\[
\sum_i (\dot{x}^i(s))^2 \leq (\dot{x}^0(s))^2 + \frac{1}{2}(\dot{x}^0(s))^2 + \sum_i (\dot{x}^i(s))^2 \implies \sum_i (\dot{x}^i(s))^2 \lesssim (\dot{x}^0(s))^2.
\]
Thus, for each \(i\) we have
\[
|x^i(s)| = |x^i(0) + \int_0^s \dot{x}^i(\tau) \, d\tau| \leq |x^i(0)| + C\int_0^s \dot{x}^0(\tau) \, d\tau = |x^i(0)| + Cx^0(s).
\]
In conclusion, if \(|x(s)| + |\dot{x}(s)| \to \infty\), then we must have \(x^0(s) + \dot{x}^0(s) \to \infty\).

If we differentiate the first equation in (3.3) and use the third one, we obtain
\[
|\dot{x}^0(s)| \leq |2g^{0\beta}\dot{p}_\beta| + |2(\partial_\mu g^{0\beta})\dot{x}^\mu p_\beta| \lesssim |\partial u(x(s))||\dot{x}(s)|^2 \lesssim \varepsilon(\dot{x}^0(s))^{-1}(\dot{x}^0(s))^2.
\]
The last inequality follows since \(|\dot{x}^i(s)| \lesssim \dot{x}^0(s)\) and since \(\partial u = O(\varepsilon(t)^{-1})\). Since \(\dot{x}^0 > 0\), we then have
\[
\frac{d}{ds} \ln \dot{x}^0 = \frac{\dot{x}^0}{\dot{x}^0} \leq C\varepsilon \frac{\dot{x}^0}{\dot{x}^0} = C\varepsilon \frac{d}{ds} \ln x^0,
\]
which implies that
\[
|\ln \dot{x}^0(s) - \ln \dot{x}^0(0)| \lesssim \varepsilon(\ln x^0(s) - \ln x^0(0)).
\]
The last inequality is equivalent to
\[
x^0(0)(\frac{x^0(s)}{x^0(0)})^{-C\varepsilon} \leq x^0(0)(\frac{x^0(s)}{x^0(0)})^{C\varepsilon}.
\]
It follows that
\[
\frac{d}{ds}((x^0(s))^{1-C\varepsilon}) = (1 - C\varepsilon)(x^0(s))^{-C\varepsilon} \frac{\partial}{\partial s}x^0(s) \leq \dot{x}^0(0)(x^0(0))^{-C\varepsilon},
\]
and thus
\[
\frac{d}{ds}((x^0(s))^{1+C\varepsilon}) = (1 + C\varepsilon)(x^0(s))^{C\varepsilon} \frac{\partial}{\partial s}x^0(s) \geq x^0(0)(x^0(0))^{C\varepsilon} > 0,
\]
and thus
\[
(x^0(s))^{1-C\varepsilon} \leq (x^0(0))^{1-C\varepsilon} + \dot{x}^0(0)s(x^0(0))^{-C\varepsilon},
\]
\[
(x^0(s))^{1+C\varepsilon} \geq (x^0(0))^{1+C\varepsilon} + \dot{x}^0(0)s(x^0(0))^{C\varepsilon}.
\]
If \(s_{\text{max}} < \infty\), then \(x^0(s) \to \infty\) as \(s \to s_{\text{max}}\) fails because of (3.7). On the other hand, if \(s_{\text{max}} < \infty\), then \(x^0(s) + \dot{x}^0(s) \to \infty\) as discussed above. But since \(\dot{x}^0(s) \leq \dot{x}^0(0)(x^0(s)/x^0(0))^{C\varepsilon}\), we must have \(x^0(s) \to \infty\) as \(s \to s_{\text{max}}\), which is a contradiction. Thus, \(s_{\text{max}} = \infty\). We thus conclude \(x^0(s) \to \infty\) as \(s \to \infty\) by (3.8).

The proof of the second half of this lemma is easy. We simply use the fact that \(g^{\alpha\beta}(u) = m^{\alpha\beta}\) when \(r \geq t + R\).  

\[\square\]

Remark 3.2.1. We let \(A\) denote the set of all the geodesics constructed in this lemma.

3.2. Estimates for the optical function. Fix a time \(T > T_0 = \exp(\delta/\varepsilon)\) and we set \(\Omega_T = \Omega \cap \{t \leq T, \ r - t \leq 2R\}\). Note that \(r \sim t\) in \(\Omega_T\). From now on, we assume that the optical function \(q = q(t,x)\) exists in \(\Omega_T\), that \(q\) is \(C^2\) and that \(q_t < 0\) everywhere. We remark that the assumptions are true for \(T = T_0 + 2R\) since \(g^{\alpha\beta} = m^{\alpha\beta}\) in \(\Omega_{T_0+2R}\). Our goal is to derive some estimates which allow us to extend the optical function to \(\Omega_{T+\varepsilon}\) for some \(\varepsilon > 0\).

First of all, we claim that each point in \(\Omega_T\) lies on exactly one geodesic in \(A\) (which is defined in Remark 3.2.1). A direct corollary is that to define a function \(F(t,x)\) in \(\Omega_T\), we can define \(F(x(s))\) along each geodesic in \(A\). To prove this claim, we define a vector field \(L = L^\alpha \partial_\alpha\) by \(L^\alpha := 2g^{\alpha\beta}q_\beta\). Note that \(L^0 > 0\) everywhere. In fact, we have
\[
g_{ij}L^iL^j = 4g_{ij}g^{\alpha\beta}g^{\gamma\delta}q_\alpha q_\beta q_\gamma q_\delta = 4g^{\alpha\beta}q_\alpha q_\beta = 0.
\]
If \(L^0 = 0\), then \(g_{ij}L^iL^j = 0\). But \(g_{ij} = \delta_{ij} + O(|u|)\), so \((g_{ij})\) is positive definite for \(\varepsilon \ll 1\). Thus, \(L^0 = 0\) and \(q_t = 1/2g_{ij}L^iL^j = 0\). This contradicts with the assumption that \(q_t < 0\). And since \(L^0 = -2q_t + O(|u\partial q|) = 2 + O(|u|) > 0\) on \(\partial\Omega\), we have \(L^0 > 0\) in \(\Omega_T\). Moreover, because of the characteristic ODE’s (3.3), a curve in \(\Omega_T\) is a geodesic in \(A\) if and only if it is an integral curve of \(L\) emanating from \(H\). By the existence and uniqueness of integral curves, we finish the proof of the claim.

We also claim that each geodesic emanating from \(H \cap \partial\Omega_T\) must stay in \(\Omega_T\) until it intersects with \(\{t = T\}\). This claim simply follows from the fact that the optical function remains constant along each geodesic and that the optical function is injective when restricted to \((\partial\Omega_T) \setminus \{t = T\}\).

Here a useful lemma which follows directly from the chain rule and the pointwise estimates in Theorem 2 (also see Proposition 6.1 in Lindblad [23]).

Lemma 3.3. For each \(k \geq 0\) and \(\varepsilon \ll k\), we have
\[
\sum_{|t| \leq k} (|Z^I(g^{\alpha\beta} - m^{\alpha\beta})| + |Z^I(g_{ij} - m_{ij})|) \lesssim_k \sum_{|t| \leq k} |Z^I u| \lesssim_k \varepsilon^{-1+C_k\varepsilon}.
\]
Moreover, 
\[ |\partial g^{\alpha\beta}| + |\partial g_{\alpha\beta}| + |\Gamma^\alpha_{\mu\nu}| \lesssim |\partial u| \lesssim \varepsilon (t)^{-1}. \]

Now we can prove several useful estimates for \( q \) in \( \Omega_T \).

**Lemma 3.4.** In \( \Omega_T \), we have \( |Sq| + \sum_i |\Omega_{ai}| \lesssim |q| + t^{C\varepsilon} \), \( |\partial q| + \sum_{i,j} |\Omega_{ij}q| \lesssim t^{C\varepsilon} \) and \( \sum_i |q_i - \omega_i q_r| \lesssim t^{-1+C\varepsilon} \).

**Proof.** If we apply a vector field \( Z \) defined by (2.1) to the eikonal equation, we obtain 
\[ 0 = (Z g^{\alpha\beta})q_{\alpha}q_{\beta} + 2g^{\alpha\beta}q_{\alpha}Zq_{\beta} = (Z g^{\alpha\beta})q_{\alpha}q_{\beta} + 2g^{\alpha\beta}q_{\alpha}\partial_{\beta}Zq + 2g^{\alpha\beta}q_{\alpha}[Z, \partial_{\beta}]q. \]

It is easy to check that \( 2m^{\alpha\beta}q_{\alpha}[Z, \partial_{\beta}]q = 0 \) if \( Z \neq S \) and \([S, \partial_{\beta}] = -\partial_{\beta} \). Thus, for some geodesic \( x(s) \), we have 
\[ |d/ds (Zq(x(s)))| \lesssim (|Z g^{\alpha\beta}| + |g^{\alpha\beta} - m^{\alpha\beta}|) |p(s)|^2 \lesssim \varepsilon (x^0(s))^{-1+C\varepsilon} |\dot{x}(s)|^2 \lesssim \varepsilon (x^0(s))^{-1+C\varepsilon} \dot{x}^0(s). \]

Recall that \( p(s) = (\partial q)(x(s)) \) and that we have \( |\dot{x}^i(s)| \lesssim \dot{x}^0(s) \lesssim (x^0(s))^{C\varepsilon} \) from the proof of Lemma 3.2. Since \( \partial q = (-1, \omega) + O(|u|) \) on \( H \), we have \( |Sq| + |\Omega_{0j}q| = O(|q| + \varepsilon t^{C\varepsilon}) \) and \( |\Omega_{ij}q| = O(\varepsilon t^{C\varepsilon}) \) on \( H \). By integrating the inequality, we have 
\[ |Zq(x(s)) - Zq(x(0))| \lesssim \int_0^s \varepsilon (x^0(\tau))^{-1+C\varepsilon} x^0(\tau) \, d\tau \lesssim (x^0(s))^{C\varepsilon}, \]
so we have 
\[ |Zq(x(s))| \lesssim |Zq(x(0))| + (x^0(s))^{C\varepsilon} \lesssim 1 + |q(x(0))| + (x^0(s))^{C\varepsilon} = 1 + |q(x(s))| + (x^0(s))^{C\varepsilon}. \]

In conclusion, we have \( |Zq| = O(|q| + t^{C\varepsilon}) \) in \( \Omega_T \). For \( Z = \partial_\alpha \) or \( \Omega_{ij} \) we have better bounds \( |\Omega_{ij}q| + |\partial q| = O(t^{C\varepsilon}) \), since the estimates for \( \partial q|_H \) and \( \Omega_{ij}q|_H \) are better. In addition, we have \( |q_i - \omega_i q_r| = r^{-1} |\sum_j \omega_j \Omega_{ij}q| \lesssim t^{-1+C\varepsilon}. \)

**Lemma 3.5.** For each \((t, x) \in \Omega_T\), we have \( q_r \geq C^{-1} t^{-C\varepsilon} \), \( -q_t \geq C^{-1} t^{-C\varepsilon} \) and \( |q_t + q_r| \lesssim \varepsilon t^{-1+C\varepsilon} \).

**Proof.** Recall that from the proof of Lemma 3.2, we have \( |\dot{x}^i(s)| \lesssim \dot{x}^0(s) \) and
\[ (x^0(s))^{-C\varepsilon} \leq \dot{x}^0(0) \left( \frac{x^0(s)}{x^0(0)} \right)^{-C\varepsilon} \leq \dot{x}^0(s) \leq \dot{x}^0(0) \left( \frac{x^0(s)}{x^0(0)} \right)^{C\varepsilon} \leq (x^0(s))^{C\varepsilon} \]
along each geodesic \( x(s) \) in \( \mathcal{A} \). At \( (t_0, x_0) = x(s_0) \) for some geodesic \( x(s) \) in \( \mathcal{A} \), we have (3.9)
\[ q_t = \frac{1}{2} g_{0\alpha} \dot{x}^\alpha(s_0) = -\frac{1}{2} x^0(s_0) + O(|u(x(s_0))||\dot{x}(s_0)||) \leq -\frac{1}{2} t_0^{-C\varepsilon} + C \varepsilon t_0^{1+C\varepsilon} \leq -\frac{1}{4} t_0^{-C\varepsilon}. \]

Here we take \( \varepsilon \ll 1 \) as usual.

To prove the estimate for \( q_r \), we first prove that \( q_r > 0 \) in \( \Omega_T \). Assume \( q_r = 0 \) at some \((t_0, x_0) \in \Omega_T \). By the eikonal equation (3.11) and the previous lemma, at \((t_0, x_0) \) we have
\[ 0 = g^{00} q_t^2 + 2 g^{0i} q_t (q_i - q_i \omega_i) + g^{ij} (q_i - \omega_i q_r) (q_j - \omega_j q_r) \]

\[ = -q_t^2 + O(|u||q_i| \sum_i |q_i - q_r \omega_i|) + O(\sum_i |q_i - \omega_i q_r|^2) \]

\[ = -q_t^2 + O(t_0^{-2+C\varepsilon}). \]

22
Plug (3.9) into (3.10), and we conclude that \( t_0^{2C\varepsilon} \lesssim q_t^2 \lesssim t_0^{2+2C\varepsilon} \) and \( t_0^{2-3C\varepsilon} \lesssim 1 \). This is impossible, since \( t_0^{2-3C\varepsilon} \geq t_0 \geq T_0 = \exp(\delta/\varepsilon) \gg 1 \) for \( \varepsilon \ll 1 \). So we have \( q_r \neq 0 \) everywhere in \( \Omega_T \). Since \( q_r = 1 + O(|u|) > 0 \) on \( H \), we have \( q_r > 0 \) everywhere in \( \Omega_T \). By (3.9), we have \(-q_t + q_r \geq -q_t \geq \frac{1}{4} t^{-C\varepsilon} \). Then since
\[
0 = -q_t^2 + \sum_i q_i^2 + O(|u||\partial q|^2) = (q_t + q_r)(-q_t + q_r) + \sum_i (q_i - q_r w_i)^2 + O(\varepsilon t^{-1+C\varepsilon}|\partial q|^2)
\]
and since \( t^{-1} \leq T_0^{-1} \ll \varepsilon \), we have
\[
|q_t + q_r| = (-q_t + q_r)^{-1}O(\varepsilon t^{-1+C\varepsilon}) \lesssim \varepsilon t^{-1+C\varepsilon} \lesssim \varepsilon t^{-1+C\varepsilon}.
\]
Then we have \( q_r = -q_t + (q_t + q_r) \geq C^{-1} t^{-C\varepsilon} - C\varepsilon t^{-1+C\varepsilon} \geq C^{-1} t^{-C\varepsilon} \). \( \square \)

3.3. A null frame. We construct a null frame \( \{e_1, e_2, e_3, e_4\} \) in \( \Omega_T \) as follows. Define two vector fields \( e_3, e_4 \) by
\[
e_4 := (L^0)^{-1} L, \quad e_3 := e_4 + 2 g^{0a} \partial_a.
\]
Since \( g^{00} \equiv -1 \), we have \( e_3^0 \equiv 1 \) and \( e_3^0 \equiv -1 \). Moreover, we have
\[
\langle e_4, e_4 \rangle = (L^0)^{-2} \langle L, L \rangle = (L^0)^{-2} g_{\alpha\beta} L^\alpha L^\beta = 0,
\]
\[
\langle e_4, e_3 \rangle = \langle e_3, e_4 \rangle = \langle 2 g^{00} \partial_0, e_4 \rangle = 2 g_{\alpha\beta} g^{00} e_4^\beta = 2 e_4^0 = 2,
\]
\[
\langle e_3, e_3 \rangle = \langle e_4, e_3 \rangle + \langle 2 g^{00} \partial_0, e_3 \rangle = 2 + 2 g_{\alpha\beta} g^{00} e_3^\beta = 2 + 2 e_3^0 = 0.
\]
Here \( \langle \cdot, \cdot \rangle \) is the bilinear form defined by the Lorentzian metric \( (g_{\alpha\beta}) = (g^{\alpha\beta})^{-1} \).

Next we define \( \{e_a\}_{a=1,2} \). When restricted to the 2-sphere \( H \cap \{t = t'\} \) for some \( t' \geq T_0 \), the metric \( (g_{\alpha\beta}) \) is positive definite. Thus, we can choose a smooth orthonormal basis \( \{E_a\}_{a=1,2} \) locally on this 2-sphere. Here we make our choice such that \( E_0|_H \) depends only on \( \omega \) and not on \( t \). Note that \( E_a \) is tangent to \( H \cap \{t = t'\} \), that \( E_0 = 0 \) and that \( \langle E_a, E_b \rangle = \delta_{ab} \). Then we take the parallel transport of \( E_a \) along the geodesics. That is, we consider the equations \( D_4 E_a = 0 \) for \( a = 1,2 \). Here \( D \) is the Levi-Civita connection of the Lorentzian metric, and \( D_4 := D_{e_4} \). Since \( e_4 \) is tangent to the geodesic, equivalently we need to solve the ODE's
\[
\frac{d}{ds} E^a_\alpha (x(s)) + \dot{x}^\mu(s) E^\nu_\alpha (x(s)) \Gamma^\alpha_{\mu\nu} (x(s)) = 0.
\]
By the existence and uniqueness for linear ODE's (e.g. Theorem 4.12 in [20]), these ODE's admit a unique solution for all \( 0 \leq s \leq s_0 \). Finally, we define
\[
e_a := E_a - E^0_a e_4, \quad a = 1,2.
\]
Thus \( e_0^0 = 0 \). Unlike \( e_3, e_4 \), the vector fields \( e_1, e_2 \) cannot be defined globally in \( \Omega_T \). This is because there is no global orthonormal basis on a 2-sphere. In the rest of this paper, when we state a property of \( e_a \) on \( \Omega_T \), we mean that any locally defined \( e_a \) satisfies this property.

We conclude that \( \{e_k\}_{k=1,2,3,4} \) is a null frame by (3.11) and the following lemma.

Lemma 3.6. In \( \Omega_T \) we have \( \langle e_a, e_b \rangle = \delta_{ab} \) and \( \langle e_4, e_a \rangle = \langle e_3, e_a \rangle = 0 \) for each \( a, b = 1,2 \).
Proof. We first prove that \( \langle E_a, E_b \rangle = \delta_{ab} \) and \( \langle e_4, E_a \rangle = 0 \) on \( H \). The first equality follows directly from the construction of \( \{E_a\} \). To prove the second one, we recall that \( q_t = q_s \omega_t \) on \( H \); see the computations right above Lemma 3.2. Moreover, note that \( \sum_i x^i(0)E_a^i = 0 \) since \( E_a \) is tangent to the sphere on \( H \). Thus, on \( H \), we have
\[
\langle L, E_a \rangle = g_{\alpha\beta}L^\alpha E_a^\beta = 2g_{\beta}E_a^\beta = 2q_tE_a^i = 2q_t\omega_t E_a^i = 0.
\]
And since \( e_4 = (L^0)^{-1}L \), we have \( \langle e_4, E_a \rangle = 0 \) at \( x(0) \).

Along each geodesic \( x(s) \) in \( A \), we have
\[
e_4\langle E_a, E_b \rangle = \langle D_4E_a, E_b \rangle + \langle E_a, D_4E_b \rangle = 0,
\]
\[
e_4\langle E_a, E_a \rangle = \langle D_4L, E_a \rangle + \langle L, D_4E_a \rangle = 0.
\]
Because of the equalities at \( s = 0 \), we conclude that \( \langle E_a, E_b \rangle = \delta_{ab} \) and \( \langle L, E_a \rangle = 0 \) (and thus \( \langle e_4, E_a \rangle = 0 \)) along each geodesic.

Finally, note that
\[
\langle e_a, e_b \rangle = \langle E_a, E_b \rangle - E_0^0\langle e_4, E_b \rangle - E_0^0\langle E_a, e_4 \rangle + E_0^0E_0^0\langle e_4, e_4 \rangle = \delta_{ab},
\]
\[
\langle e_4, E_a \rangle = \langle e_4, E_a \rangle - E_0^0\langle e_4, e_4 \rangle = 0,
\]
\[
\langle e_4, e_a \rangle = \langle e_4, E_a \rangle - E_0^0\langle e_4, e_4 \rangle = 0,
\]
\[
\langle e_3, e_a \rangle = 2g^{\alpha\beta}\partial_\alpha \epsilon_a + \langle e_4, e_a \rangle = 2g_{\alpha\beta}g^{\alpha\beta}e_\alpha = 2e_a^0 = 0.
\]
This finishes the proof.

Before we move on to the next lemma, we summarize some important properties of a null frame. First, any vector field \( X \) can be uniquely expressed as a linear combination of the null frame:
\[
X = \sum_{a=1,2} \langle X, e_a \rangle e_a + \frac{1}{2} \langle X, e_4 \rangle e_3 + \frac{1}{2} \langle X, e_3 \rangle e_4.
\]

In addition, for each \( k = 1, 2, 3, 4 \) we have
\[
\langle g^{\alpha\beta} \partial_\beta, e_k \rangle = g^{\alpha^\beta}g_{\beta\mu}e_k^\mu = e_k^\alpha,
\]
so we obtain
\[
g^{\alpha\beta} \partial_\beta = \sum_{a=1,2} e_a^\alpha e_a + \frac{1}{2} e_3^\alpha e_3 + \frac{1}{2} e_4^\alpha e_4 \implies g^{\alpha\beta} = \sum_{a=1,2} e_a^\alpha e_a + \frac{1}{2} e_4^\alpha e_3 + \frac{1}{2} e_3^\alpha e_4.
\]

Finally, we have \( e_1(q) = e_2(q) = e_4(q) = 0 \) and \( e_3(q) = L^0 \) in \( \Omega_T \). In fact, since \( q_\alpha = \frac{1}{2}g_{\alpha\beta}L^\beta \), we have \( Xq = \frac{1}{2}\langle X, L \rangle = \frac{1}{2}L^0\langle e_4, X \rangle \) for each vector field \( X \). Then we use the properties of a null frame. The equality \( e_1(q) = e_2(q) = e_4(q) = 0 \) implies that \( e_1, e_2, e_4 \) are tangent to the level set of \( q \), so \( e_1, e_2, e_4 \) are sometimes called the tangential derivatives.

The next lemma shows several better estimates for the tangential derivatives.

**Lemma 3.7.** In \( \Omega_T \), we have \( e_4 = \partial_t + \partial_r + O(t^{-1+C_\epsilon})\partial_r \), \( e_3 = e_4 + 2g^{\alpha\beta}\partial_\alpha = -\partial_t + \partial_r + O(t^{-1+C_\epsilon})\partial_r \) and \( e_a = O(1)\partial_t \). Then, for all \( I, s, l \), we have
\[
\sum_{k=1,2,4} (|e_k(\partial^s Z^I u)| + |e_k(\partial^s Z^I g^{\alpha\beta})| + |e_k(\partial^s Z^I g_{\alpha\beta})|) \lesssim \varepsilon t^{-2+C_\epsilon}(r-t)^{-s}.
\]
Here we use the convention given in Section 2.4. Moreover, we have
\[
|e_1(\partial_\alpha g_{\mu\nu})e_1^\alpha| + |e_2(\partial_\alpha g_{\mu\nu})e_1^\alpha| + |e_1(\partial_\alpha g_{\mu\nu})e_3^\alpha - e_2(\partial_\alpha g_{\mu\nu})e_3^\alpha| \lesssim \varepsilon t^{-3+C_\epsilon}.
\]
Proof. By the lemmas in Section 3.2 we have
\[ e^i - \omega_i = \frac{L^i - L^0 \omega_i}{L^0} = \frac{2q_i + 2q_i \omega_i + O(|u||\partial q|)}{-2q_i + O(|u||\partial q|)} = \frac{2(q_i - q_r \omega_i) + 2(q_r + q_i)\omega_i + O(|u||\partial q|)}{-2q_i + O(|u||\partial q|)}. \]

By Lemma 3.3 and Lemma 3.5 the denominator has a lower bound \( C^{-1}t^{-C\varepsilon} - C\varepsilon t^{-1+C\varepsilon} \geq (2C)^{-1}t^{-1+C\varepsilon} \) and the numerator is \( O(t^{-1+C\varepsilon}) \). In conclusion, \( e^i = \partial_t + \partial_r + O(t^{-1+C\varepsilon})\partial_r \). It follows that for each \( I \),
\[ |e_4(\partial^s Z^I u)| \lesssim |(\partial_t + \partial_r) \partial^s Z^I u| + t^{-1+C\varepsilon} |\partial^s Z^I u| \]
\[ \lesssim \langle t + r \rangle^{-1} \sum_{|J|=1} |Z^I \partial^s Z^I u| + t^{-1+C\varepsilon} \langle r - t \rangle^{-s-1} \sum_{|J| \leq s+1} |Z^I \partial^s Z^I u| \]
\[ \lesssim \langle t + r \rangle^{-1} \sum_{|J| \leq s+1} |\partial^s Z^I u| + t^{-1+C\varepsilon} \langle r - t \rangle^{-s} \epsilon \langle r - t \rangle^{1+C\varepsilon} \]
\[ \lesssim \langle t + r \rangle^{-1} \cdot \epsilon \langle r - t \rangle^{-s} + \epsilon \langle t - 2+C\varepsilon \rangle \langle r - t \rangle^{-s-1} \]
\[ \lesssim \epsilon \langle t^{-2+C\varepsilon} \rangle \langle r - t \rangle^{-s}. \]

Here we apply Lemma 2.2, the pointwise decays in Theorem 2 and (2.3). By the chain rule and Leibniz’s rule, we can express \( e_4(\partial^s Z^I (g^{a\beta}, g_{a\beta})) \) as a linear combination of terms of the form
\[ \frac{d^m}{d^m u^m} (g^{a\beta}, g_{a\beta})(u) \cdot (\partial^s Z^I u) \cdot (\partial^s Z^I,w) \cdot e_4(\partial^s Z^I u) \]
where \( \sum s_\ast = s, \sum |I_\ast| = |I| \) and \( m > 0 \). These terms have an upper bound
\[ \epsilon \langle t^{-1+C\varepsilon} \rangle \langle r - t \rangle^{-s} \ldots \epsilon \langle t^{-2+C\varepsilon} \rangle \langle r - t \rangle^{-s_m-1} \cdot \epsilon \langle t^{-2+C\varepsilon} \rangle \langle r - t \rangle^{-s_m} \lesssim \epsilon \langle t^{-2+C\varepsilon} \rangle \langle r - t \rangle^{-s}. \]

We thus have \( e_4(\partial^s Z^I (g^{a\beta}, g_{a\beta})) = O(\epsilon \langle t^{-2+C\varepsilon} \rangle \langle r - t \rangle^{-s}) \).

Next we fix \((t_0, x_0) \in \Omega_T\). Without loss of generality, we assume \(|q_3| = \max\{|q_j| : j = 1, 2, 3\} \) at \((t_0, x_0)\). For \( i = 1, 2 \), we define
\[ Y_i := q_i \partial_t - q_3 \partial_r = r^{-1} q_r \Omega_3 + (q_i - \omega_i q_r) \partial_3 - (q_3 - \omega_3 q_r) \partial_i = r^{-1} q_r \Omega_3 + O(t^{-1+C\varepsilon}) \partial_r. \]
Here \( \{Y_1, Y_2\} \) is a basis of the tangent space of the 2-sphere \( \Sigma_{(t_0, x_0)} = \{ t = t_0, q = q(t_0, x_0) \} \) at \((t_0, x_0)\). Since \( e_a \) lies in the tangent space (as \( e_a^0 = 0 \) and \( e_a(q) = 0 \)), we can write \( e_a = \sum_{i=1,2} c_{ai} Y_i \) in a unique way. Since
\[ \langle Y_i, Y_j \rangle = q_i q_j q_3 + q^2_i q_3 q_j - q_i q_3 q_j - q_j q_3 q_i = q_i q_j + q^2_i \delta_{ij} + O(|q^3_j|), \quad i, j = 1, 2, \]
we have
\[ 1 = \langle e_a, e_a \rangle = \sum_{i,j} c_{ai} c_{aj} \langle Y_i, Y_j \rangle = (\sum_i c_{ai} q_i)^2 + (1 + O(|u|)) q^2_3 \sum_i c^2_{ai}. \]

Then, for \( \varepsilon \ll 1 \) we have
\[ 1 \geq 0 + (1 + O(\varepsilon t^{-1+C\varepsilon})) q^2_3 \sum_i c^2_{ai} \geq \frac{1}{2} q^2_3 \sum_i c^2_{ai}. \]
Thus, we have \[|q_3 c_{ai}| \lesssim 1 \] for each \( a, i \) and thus \[ e_a = \sum_i c_{ai} Y_i^\alpha = O(|c_{ai}q_3|) = O(1). \] And since \( C^{-1} t^{-C \varepsilon} \leq |q_r| = |\sum_i \omega_i q_i| \leq \sum_i |q_i| \leq 3|q_3| \), for each multiindex \( I \), we have

\[
|e_a(\partial^s Z^I u)| \leq \sum_i |c_{ai} Y_i(\partial^s Z^I u)| \lesssim \sum_i |c_{ai}| (r^{-1}|q_r|\Omega \partial^s Z^I u + t^{-1+C \varepsilon}\partial \partial^s Z^I u) \\
\lesssim \varepsilon t^{-2+C \varepsilon}(r - t)^{-s}.
\]

By the chain rule and Leibniz's rule, we finish the proof of the first estimate.

In addition,

\[
0 = \langle e_1, e_1 \rangle - \langle e_2, e_2 \rangle = \left( \sum_i c_{1i}q_i \right)^2 - \left( \sum_i c_{2i}q_i \right)^2 + q_3^2 \sum_i (c_{1i}^2 - c_{2i}^2) + O(|u|q_3^2 \sum_{a,i} c_{ai}^2) \\
(3.15)
= \sum_{i,j} (c_{1i}c_{1j} - c_{2i}c_{2j}) q_i q_j - \sum_i c_{2i}q_i^2 + q_3^2 \sum_i (c_{1i}^2 - c_{2i}^2) + O(|u|),
\]

\[
0 = \langle e_1, e_2 \rangle = \sum_{i,j} c_{1i}c_{2j} \langle Y_i, Y_j \rangle = \sum_{i,j} c_{1i}c_{2j} q_i q_j + \sum_i c_{1i}c_{2i}q_3^2 + O(|u|q_3^2 \sum_{i,j} |c_{1i}c_{2j}|) \\
(3.16)
= \sum_{i,j} c_{1i}c_{2j} q_i q_j + \sum_i c_{1i}c_{2i}q_3^2 + O(|u|).
\]

Then, we have

\[
Y_i(\Omega g) = r^{-1} q_r \Omega \alpha_3 g + O(t^{-1+C \varepsilon}|\partial g|) = O(\varepsilon t^{-2+C \varepsilon}),
\]

\[
Y_i(\partial_\alpha g) Y^\alpha_j = (r^{-1} q_r \Omega \alpha_3 (\partial_\alpha g) + (q_i - \omega_i q_r) \partial_3 \partial_\alpha g - (q_3 - \omega_3 q_r) \partial_i \partial_\alpha g) Y^\alpha_j \\
= r^{-1} q_r \langle Y^\alpha_j \Omega \alpha_3, \partial_\alpha g \rangle + (q_i - \omega_i q_r) \langle Y^\alpha_j, (\partial_3 \partial_\alpha g) - (q_3 - \omega_3 q_r) Y^\alpha_j \partial_i \partial_\alpha g \rangle \\
= r^{-1} q_r (\delta_{ij} q_3 \partial_3 g + q_j \partial_3 g) + r^{-1} q_r Y^\alpha_j \Omega \alpha_3 g + O(t^{-1+C \varepsilon}|Y^\alpha_j \partial g|) \\
= r^{-1} q_r (\delta_{ij} q_3 \partial_3 g + q_j \partial_3 g) + O(\varepsilon t^{-3+C \varepsilon}),
\]

\[
e_a(\partial_\alpha g)e^\alpha_b = \sum_{i,j} c_{ai} Y_i(\partial_\alpha g) c_{bj} Y^\alpha_j = \sum_{i,j} c_{ai}c_{bj} (r^{-1} q_r (\delta_{ij} q_3 \partial_3 g + q_j \partial_3 g) + O(\varepsilon t^{-3+C \varepsilon})) \\
= \sum_i r^{-1} c_{ai} c_{3b} q_r q_3 \partial_3 g + \sum_{i,j} r^{-1} c_{ai} c_{bj} q_r q_j \partial_3 g + O(\sum_{i,j} |c_{ai}c_{bj}| |q_3| \varepsilon t^{-3+C \varepsilon}) \\
= \sum_i r^{-1} c_{ai} c_{3b} q_r q_3 \partial_3 g + \sum_{i,j} r^{-1} c_{ai} c_{bj} q_r q_j \partial_3 g + O(\varepsilon t^{-3+C \varepsilon}).
\]
When $a \neq b$, by (3.16) we have
\[
e_a(\partial_\alpha g)e_\beta^a = r^{-1}q_iq_3^{-1}(-\sum_{i,j} c_{ai}c_{bj}q_ig_j + O(|u|)\partial_3 g + \sum_{i,j} r^{-1}c_{ai}c_{bj}q_iq_j\partial_1 g + O(\varepsilon t^{-3+C\varepsilon})
\]
\[
= r^{-1}q_iq_3^{-1}\sum_{i,j} c_{ai}c_{bj}q_j(-q_i\partial_3 g + q_3\partial_i g) + O(r^{-1}q_iq_3^{-1}|u|\partial g)| + O(\varepsilon t^{-3+C\varepsilon})
\]
\[
= r^{-1}q_3^{-1}\sum_{i,j} c_{ai}c_{bj}q_j(-Y_i g) + O(\varepsilon t^{-3+C\varepsilon}) = O(\varepsilon t^{-3+C\varepsilon}).
\]

By (3.15) we have
\[
e_1(\partial_\alpha g)e_1^a - e_2(\partial_\alpha g)e_2^a
\]
\[
= \sum_i r^{-1}(c_{1i}^2 - c_{2i}^2)q_iq_3\partial_3 g + \sum_i r^{-1}(c_{1i}c_{1j} - c_{2i}c_{2j})q_iq_j\partial_1 g + O(\varepsilon t^{-3+C\varepsilon})
\]
\[
= r^{-1}q_iq_3^{-1}(-\sum_{i,j} (c_{1i}c_{1j} - c_{2i}c_{2j})q_iq_j)\partial_3 g + \sum_i r^{-1}(c_{1i}c_{1j} - c_{2i}c_{2j})q_iq_j\partial_1 g + O(\varepsilon t^{-3+C\varepsilon})
\]
\[
= \sum_i r^{-1}q_iq_3^{-1}q_j(c_{1i}c_{1j} - c_{2i}c_{2j})(-Y_i g) + O(\varepsilon t^{-3+C\varepsilon}) = O(\varepsilon t^{-3+C\varepsilon}).
\]

It is clear that our proof would still work if we assume $|q_1| = \max\{|q_j| : j = 1, 2, 3\}$ or $|q_2| = \max\{|q_j| : j = 1, 2, 3\}$. This ends the proof.

\[\square\]

**Lemma 3.8.** In $\Omega_T$, we have $|q - (r - t)| \lesssim t^{C\varepsilon}$.

**Proof.** By the previous lemma and Lemma 3.5 we have
\[
e_4^i = \omega_i = \frac{2(q_i - q_\tau\omega_i) + 2(q_r + q_\tau)\omega_i + O(|u|\partial q)|}{L^0} = 2(L^0)^{-1}(q_i - q_\tau\omega_i) + O(\varepsilon t^{-1+C\varepsilon}).
\]

Thus,
\[
e_4(q - r + t) = (\partial_t + \partial_\tau)(-r + t) - 2(L^0)^{-1}\sum_i(q_i - q_\tau\omega_i)\omega_i + O(\varepsilon t^{-1+C\varepsilon}) = O(\varepsilon t^{-1+C\varepsilon}).
\]

Suppose $(t, x) \in \Omega_T$ lies on a geodesic $x(s)$ in $\Omega_T$. Since $q - r + t = 0$ on $H$, by integrating $e_4(q - r + t)$ along this geodesic, we have
\[
|q - r + t| \lesssim \int_{x^0(0)}^{x^0(t)} \varepsilon t^{-1+C\varepsilon} \, d\tau \lesssim t^{C\varepsilon}.
\]

\[\square\]

3.4. **The connection coefficients.** From now on, we write $D_k = D_{e_k}$ for $k = 1, 2, 3, 4$ for simplicity.

**Lemma 3.9.** In $\Omega_T$, we have
\[
D_4 e_k = (\Gamma_{\alpha\beta}^0 e_\alpha^\alpha e_\beta^\beta) e_4, \quad k = 1, 2, 4.
\]

As a result, we have $e_4(e_k^\alpha) = O(\varepsilon t^{-2+C\varepsilon})$ for each $k = 1, 2, 3, 4$.  

Proof. Since a geodesic in $A$ is an integral curve of $L$, we have $L^\alpha = \dot{x}^\alpha(s)$ at $x(s)$. Then, the geodesic equation \[ L(L^0) = \dot{x}^\alpha(s)(\partial_\alpha L^0) = \frac{d}{ds}L^0(x(s)) = \ddot{x}^0(s) = -\Gamma^0_{\mu\nu}L^\mu L^\nu, \] at $x(s)$.

Divide both sides by $L^0$, and we conclude $e_4(L^0) = -\Gamma^0_{\mu\nu}e^\mu_4 L^\nu$ in $\Omega_T$ and thus $e_4(\ln L^0) = -\Gamma^0_{\mu\nu}e^\mu_4 e^\nu_4$. Similarly, from (3.12) we obtain $e_4(E^0_a) = -\Gamma^0_{\mu\nu}e^\mu_4 E^\nu_a$. Thus, we have

$$D_4e_4 = D_4((L^0)^{-1}L) = -(L^0)^{-2}e_4(L^0)L + (L^0)^{-1}D_4L = -(L^0)^{-1}e_4(L^0)e_4 = (\Gamma^0_{\mu\nu}e^\mu_4 e^\nu_4)e_4.$$

For $a = 1, 2$, since $D_4E_a = 0$, we have

$$D_4e_a = D_4(E_a - E^0_a e_4) = -D_4(E^0_a e_4) = -e_4(E^0_a) - e_4(D_4 e_4) = (\Gamma^0_{\mu\nu}e^\mu_4 E^\nu_a)e_4 - (E^0_a\Gamma^0_{\mu\nu}e^\mu_4 e^\nu_4)e_4 = \Gamma^0_{\mu\nu}e^\mu_4 (E^\nu_a - E^0_a e^\nu_4)e_4 = (\Gamma^0_{\mu\nu}e^\mu_4 e^\nu_4)e_4.$$

In addition, $D_4e_k = e_4(e^0_4)\partial_{\alpha} + \Gamma^0_{\mu\nu}e^\mu_4 e^\nu_4 \partial_{\alpha}$. If we consider the coefficients of $\partial_{\alpha}$ in $D_4e_k$ for $k = 1, 2, 4$, we have $e_4(e^0_k) = \Gamma^0_{\mu\nu}e^\mu_4 e^\nu_4 - \Gamma^0_{\mu\nu}e^\nu_4 e^\mu_4$. By Lemma 3.4, we have

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}(\partial_\mu g_{\nu\beta} + \partial_\nu g_{\mu\beta} - \partial_\beta g_{\mu\nu}),$$

(3.17)

$$= \frac{1}{2}g^{\alpha\beta}(\partial_\mu g_{\nu\beta} + \partial_\nu g_{\mu\beta}) - \frac{1}{2}(\sum_{\alpha} e^\alpha_4 e_a(g_{\mu\nu}) + \frac{1}{2}(e^\alpha_4 e_4(g_{\mu\nu}) + e^\alpha_4 e_3(g_{\mu\nu})))$$

$$= \frac{1}{2}g^{\alpha\beta}(\partial_\mu g_{\nu\beta} + \partial_{\nu} g_{\mu\beta}) - \frac{1}{4}e^\alpha_4 e_3(g_{\mu\nu}) + O(\varepsilon t^{-2+C\varepsilon}).$$

Then, since $e^0_4 = 1$, for $k = 1, 2, 4$ we have

$$e_4(e^0_k) = (\frac{1}{2}g^{03}(\partial_\mu g_{\nu\beta} + \partial_\nu g_{\mu\beta}) - \frac{1}{4}e^0_4 e_3(g_{\mu\nu}) + O(\varepsilon t^{-2+C\varepsilon}))e^\nu_4 e^\alpha_4$$

$$= (\frac{1}{2}g^{03}(\partial_\mu g_{\nu\beta} + \partial_\nu g_{\mu\beta}) - \frac{1}{4}e^0_4 e_3(g_{\mu\nu}) + O(\varepsilon t^{-2+C\varepsilon}))e^\nu_4 e^\alpha_4$$

$$= \frac{1}{2}g^{03}(e_4(g_{\mu\nu})e^\nu_4 e^\alpha_4 + e_k(g_{\mu\beta})e^\mu_4 e^\alpha_4) + \frac{1}{2}g^{03}(e_4(g_{\mu\nu})e^\nu_4 e^\alpha_4 + e_k(g_{\mu\beta})e^\mu_4 e^\alpha_4)$$

$$= \frac{1}{4}e_3(g_{\mu\nu})(e^\mu_4 e^\nu_4 e^\alpha_4 e^0_4 - e^\nu_4 e^\alpha_4 e^\mu_4) + O(\varepsilon t^{-2+C\varepsilon})$$

$$= O(\varepsilon t^{-2+C\varepsilon}).$$

It follows that $e_4(e^0_3) = e_4(e^0_4) + e_4(2g^{03}) = O(\varepsilon t^{-2+C\varepsilon})$. This finishes the proof. \[ \square \]

Remark 3.9.1. Since $e_3(q) = L^0$, we have

$$e_4(e_3(q)) = e_4(L^0) = -\Gamma^0_{\alpha\beta}e^\alpha_4 L^\beta = -\Gamma^0_{\alpha\beta}e^\alpha_4 e^\beta_3(q).$$

This equality is useful in the rest of this paper.

Next, we set $\chi_{ab} := \langle D_a e_4, e_b \rangle$ for $a, b = 1, 2$.

Lemma 3.10. In $\Omega_T$, we have

(a) $\chi_{12} = \chi_{21}$.

(b) $\text{tr} \chi := \chi_{11} + \chi_{22}$ is independent of the choice of $e_1$ and $e_2$. 

28
(c) \[
[e_4, e_a] = -\sum_b \chi_{ab} e_b, \quad D_a e_4 = \sum_b \chi_{ab} e_b + (e_4^\mu e^\nu \Gamma^0_{\mu\nu}) e_4, \quad e_a (e_4^a) = \sum_b \chi_{ab} e_b^a + O(\varepsilon t^{-2 + C\varepsilon}).
\]

Proof. (a) Since \( e_a(q) = 0 \), we have
\[
\langle e_4, [e_1, e_2] \rangle = (L^0)^{-1} \langle L, [e_1, e_2] \rangle = 2(L^0)^{-1} [e_1, e_2] q = 2(L^0)^{-1} (e_1 (e_2(q)) - e_2 (e_1(q))) = 0.
\]
And since
\[
\langle D_k e_l, e_m \rangle = e_k (\langle e_l, e_m \rangle) - \langle e_l, D_k e_m \rangle = -\langle e_l, D_k e_m \rangle, \quad k, l, m = 1, 2, 3, 4,
\]
we have
\[
\chi_{12} - \chi_{21} = \langle D_1 e_4, e_2 \rangle - \langle D_2 e_4, e_1 \rangle = \langle e_4, -D_1 e_2 + D_2 e_1 \rangle = -\langle e_4, [e_1, e_2] \rangle = 0.
\]
(b) Suppose that \( \{e'_k\} \) is another null frame with \( e_3 = e'_3 \) and \( e_4 = e'_4 \). Then we have \( e'_a = \sum_b \langle e'_a, e_b\rangle e_b \) and thus
\[
e_a = \sum_b \langle e_a, e'_b \rangle e'_b = \sum_{b,c} \langle e_a, e'_b \rangle \langle e'_b, e'_c \rangle e_c \implies \sum_{b,c} \langle e_a, e'_b \rangle \langle e'_b, e'_c \rangle = \delta_{ac}.
\]
Then,
\[
\chi'_{11} + \chi'_{22} = \sum_a \langle D'_a e_4, e'_a \rangle = \sum_a \sum_{b,c} \langle e'_a, e'_b \rangle \langle e'_b, e'_c \rangle \langle D_b e_4, e_c \rangle
\]
\[
= \sum_{b,c} \sum_a \langle e'_a, e'_b \rangle \langle e'_b, e'_c \rangle \chi_{bc} = \sum_{b,c} \delta_{bc} \chi_{bc} = \chi_{11} + \chi_{22}.
\]
(c) Since \( D_4 e_k = (\Gamma^0_{a\beta} e_4^a e_k^\beta) e_4 \) for \( k = 1, 2, 4 \), we have \( \langle D_4 e_k, e_a \rangle = 0 \) for \( k = 1, 2, 4 \) and thus
\[
\langle e_4, [e_4, e_a] \rangle = \langle e_4, D_4 e_a - D_a e_4 \rangle = -\langle D_4 e_4, e_a \rangle - \frac{1}{2} e_a (e_4, e_4) = 0,
\]
\[
\langle e_0, [e_4, e_a] \rangle = \langle e_0, D_4 e_a - D_a e_4 \rangle = \langle e_0, D_4 e_4 \rangle - \chi_{ab} = -\chi_{ab}.
\]
Since \( e_4^0 = 1 \) and \( e_a^0 = 0 \), we have \( [e_4, e_a]^0 = 0 \) (where \( [e_4, e_a] = [e_4, e_a]^\alpha \partial_\alpha \)) and thus
\[
\langle e_3, [e_4, e_a] \rangle = \langle e_4, [e_4, e_a] \rangle + 2g^{0\alpha} g_{a\beta} [e_4, e_a]^\beta = 0 + 2[e_4, e_a]^0 = 0.
\]
By (3.13) we conclude that \( [e_4, e_a] = -\sum_{b=1,2} \chi_{ab} e_b \). The second equality follows from \( D_a e_4 = [e_a, e_4] + D_4 e_a \). The third one follows from \( e_a (e_4^a) - e_4 (e_a^a) = [e_a, e_4]^a \) and the previous lemma. \( \square \)

3.5. The Raychaudhuri equation. It turns out that the estimates for \( \chi_{ab} \) are crucial in the proof of the global existence of the optical function. To obtain such estimates, we need the Raychaudhuri equation
\[
e_4 (\chi_{ab}) = -\sum_c \chi_{ac} \chi_{cb} + \Gamma^0_{a\beta} e_4^a e_4^\beta \chi_{ab} + \langle R(e_4, e_a) e_4, e_b \rangle.
\]
(3.18)
Here $\langle R(X, Y)Z, W \rangle := \langle D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z, W \rangle$ is the curvature tensor. In fact, since $2\langle D_a e_4, e_4 \rangle = e_4 \langle e_4, e_4 \rangle = 0$, we have

$$e_4(\chi_{ab}) = e_4 \langle D_a e_4, e_b \rangle = \langle D_a D_4 e_4, e_b \rangle + \langle D_4 e_4, D_a e_4 \rangle = \langle D_a (\Gamma^0_{\alpha\beta} e_4^\alpha e_4^\beta), e_b \rangle - \sum_c \chi_{ac} \langle D_c e_4, e_b \rangle + \langle R(e_4, e_a) e_4, e_b \rangle = e_a (\Gamma^0_{\alpha\beta} e_4^\alpha e_4^\beta) (e_4, e_b) + \Gamma^0_{\alpha\beta} e_4^\alpha e_4^\beta \chi_{ab} - \sum_c \chi_{ac} \chi_{cb} + \langle R(e_4, e_a) e_4, e_b \rangle = \Gamma^0_{\alpha\beta} e_4^\alpha e_4^\beta \chi_{ab} - \sum_c \chi_{ac} \chi_{cb} + \langle R(e_4, e_a) e_4, e_b \rangle.$$

From (3.18), we can compute $e_4(\chi_{11} - \chi_{22})$, $e_4(\chi_{12})$ and $e_4(\text{tr} \chi)$. Note that

$$\sum_c \chi_{1c} \chi_{c1} - \sum_c \chi_{2c} \chi_{c2} = \chi_{11}^2 - \chi_{22}^2 = \text{tr} \chi (\chi_{11} - \chi_{22}),$$

$$\sum_c \chi_{1c} \chi_{c2} = \sum_c \chi_{2c} \chi_{c1} = \chi_{11} \chi_{12} + \chi_{12} \chi_{22} = \chi_{12} \text{tr} \chi,$$

$$\sum_c \chi_{1c} \chi_{c1} + \sum_c \chi_{2c} \chi_{c2} = \chi_{11}^2 + \chi_{22}^2 + 2 \chi_{12}^2 = \frac{1}{2} (\text{tr} \chi)^2 + \frac{1}{2} (\chi_{11} - \chi_{22})^2 + 2 \chi_{12}^2.$$  

As for the curvature tensor, we have the following lemma.

**Lemma 3.11.** In $\Omega_T$, we have

$$\langle R(e_4, e_a) e_4, e_b \rangle = e_4 (f_{ab}) + \frac{1}{2} e_4^\alpha e^\beta e_4^\mu e_4^\nu \partial_\beta \partial_\nu g_{\alpha\mu} + O(\varepsilon^2 t^{-3+C\varepsilon})$$

where

$$f_{ab} := \frac{1}{2} (e_4^\beta e_4^\nu e_4 (g_{\beta\nu}) - e_4^\beta e_4^\nu e_4 (g_{\beta\mu})) - \frac{1}{2} e_4^\alpha e_a (g_{\alpha\nu}) e_b^\nu = O(\varepsilon t^{-2+C\varepsilon}).$$

Moreover,

$$\langle R(e_4, e_1) e_4, e_1 \rangle - \langle R(e_4, e_2) e_4, e_2 \rangle = e_4 (f_{11} - f_{22}) + O(\varepsilon t^{-3+C\varepsilon}),$$

$$\langle R(e_4, e_1) e_4, e_2 \rangle = e_4 (f_{12}) + O(\varepsilon t^{-3+C\varepsilon}),$$

$$\langle R(e_4, e_1) e_4, e_1 \rangle + \langle R(e_4, e_2) e_4, e_2 \rangle = e_4 (\text{tr} f - \frac{1}{2} e_4^\alpha e_4^\mu e_4 g_{\alpha\mu} + O(\varepsilon^2 t^{-3+C\varepsilon}).$$

**Proof.** We have $\langle R(e_4, e_a) e_4, e_b \rangle = e_4^\alpha e^\beta e_4^\mu e_4^\nu R_{\alpha\beta\mu\nu}$ where $R_{\alpha\beta\mu\nu}$ is given by

$$R_{\alpha\beta\mu\nu} := \langle R(\partial_\alpha, \partial_\beta) e_4, \partial_\mu e_4, \partial_\nu e_4 \rangle = g_{\sigma\nu} (\partial_\alpha \Gamma^\sigma_{\beta\mu} - \partial_\beta \Gamma^\sigma_{\alpha\mu} + \Gamma^\delta_{\beta\mu} \Gamma^\sigma_{\alpha\delta} - \Gamma^\delta_{\alpha\mu} \Gamma^\sigma_{\beta\delta}) = \partial_\alpha \Gamma^\nu_{\beta\mu} - \partial_\beta \Gamma^\nu_{\alpha\mu} - \Gamma^\sigma_{\beta\mu} \partial_\alpha g_{\sigma\nu} + \Gamma^\sigma_{\alpha\mu} \partial_\beta g_{\sigma\nu} + \Gamma^\delta_{\beta\mu} \Gamma^\nu_{\alpha\delta} - \Gamma^\delta_{\alpha\mu} \Gamma^\nu_{\beta\delta} = \partial_\alpha \Gamma^\nu_{\beta\mu} - \partial_\beta \Gamma^\nu_{\alpha\mu} - \Gamma^\delta_{\beta\mu} \Gamma^\nu_{\alpha\delta} + \Gamma^\delta_{\alpha\mu} \Gamma^\nu_{\beta\delta} = \frac{1}{2} (\partial_\alpha \partial_\mu g_{\beta\nu} - \partial_\alpha \partial_\nu g_{\beta\mu} - \partial_\beta \partial_\mu g_{\alpha\nu} + \partial_\beta \partial_\nu g_{\alpha\mu}) - \Gamma^\delta_{\beta\mu} \Gamma^\nu_{\alpha\delta} + \Gamma^\delta_{\alpha\mu} \Gamma^\nu_{\beta\delta}. $$
Here for simplicity we set $\Gamma^{\alpha\mu\nu} := g_{\alpha\beta} \Gamma^{\beta}_{\mu\nu} = \frac{1}{2}(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu})$. Then
\[
\frac{1}{2} e_4^\alpha e_4^\mu e_4^\nu e_b^\gamma (\partial_\alpha \partial_\mu g_{\beta\nu} - \partial_\alpha \partial_\nu g_{\beta\mu} - \partial_\beta \partial_\mu g_{\alpha\nu} + \partial_\beta \partial_\nu g_{\alpha\mu}) \\
= \frac{1}{2} e_4(\partial_\mu g_{\beta\nu} - \partial_\nu g_{\beta\mu}) e_a^\beta e_4^\mu e_b^\nu - \frac{1}{2} e_4^\alpha e_4 e_4(\partial_\beta g_{\alpha\nu} + \partial_\beta g_{\alpha\mu}) = e_4(\partial_\mu g_{\beta\nu} - \partial_\nu g_{\beta\mu}) e_a^\beta e_4^\mu e_b^\nu + \frac{1}{8} e_4^\alpha e_4^\beta e_4^\mu e_4^\nu \partial_\beta \partial_\nu g_{\alpha\mu} + O(|\partial g|) = O(1) + O(t^{-1} \varepsilon) + O(\varepsilon^2 t^{-3+C \varepsilon}).
\]

To finish the proof of the first part, we note that
\[
\Gamma^{\sigma}_{\mu\beta} \Gamma_{\alpha\nu} = g^{\sigma\delta} \Gamma^{\delta}_{\sigma\mu} \Gamma_{\nu\alpha} = \frac{1}{4} g^{\sigma\delta} (\partial_\beta g_{\sigma\mu} + \partial_\mu g_{\beta\sigma} - \partial_\sigma g_{\beta\mu}) (\partial_\alpha g_{\delta\nu} + \partial_\nu g_{\alpha\delta} - \partial_\delta g_{\alpha\nu}).
\]

By (3.14), we have
\[
e_4^\alpha e_4^\beta e_4^\mu e_b^\gamma \Gamma_{\mu\beta}^{\delta \alpha \nu} = \frac{1}{8} e_4^\alpha e_4^\beta e_4 e_b^\gamma g^{\sigma\delta} \partial_\sigma g_{\beta\gamma} + \sum_{k=1,2,4} O(1) e_k(g) \partial g
\]
\[
= \frac{1}{4} \sum_{c=1,2} e_c(g) e_c(g) + \frac{1}{8} e_3(g) e_4(g) + \frac{1}{8} e_4(g) e_3(g) + O(\sum_{k=1,2,4} |e_k(g)||\partial g|)
\]
\[
= O(\varepsilon t^{-2+C \varepsilon} \cdot t^{-1} + C \varepsilon) = O(\varepsilon^2 t^{-3+C \varepsilon}).
\]

Similarly, we have $e_4^\alpha e_4^\beta e_4^\mu e_b^\gamma \Gamma_{\nu \beta}^{\delta \alpha \mu} = O(\varepsilon^2 t^{-3+C \varepsilon})$.

To prove the second half, we only need to consider the term $\frac{1}{2} e_4^\alpha e_4^\beta e_4 e_b^\gamma \partial_\beta \partial_\nu g_{\alpha\mu}$. By Lemma 3.7 we have
\[
\frac{1}{2} e_4^\alpha e_4^\beta e_4 e_b^\gamma \partial_\beta \partial_\nu g_{\alpha\mu} = \frac{1}{2} e_4^\alpha e_4 e_b^\gamma \partial_\beta \partial_\nu g_{\alpha\mu} = O(\varepsilon t^{-3+C \varepsilon}).
\]

Finally, note that
\[
\sum_{a} e_4^\alpha e_4^\beta e_4^\mu e_a^\nu \partial_\nu \partial_\beta g_{\alpha\mu} = \frac{1}{2} e_4^\alpha e_4 e_b^\gamma \partial_\beta \partial_\nu g_{\alpha\mu} - \frac{1}{2} e_4^\alpha e_4 e_b^\gamma \partial_\beta \partial_\nu g_{\alpha\mu} = 0 + F''(u)(\sum_{c} e_c(u) e_c(u) + e_3(u) e_4(u)) = O(\varepsilon t^{-3+C \varepsilon}).
\]

We briefly explain how we obtain the third estimate here. If $F = F(u)$ is a function of $u$ which is a solution to (1.11), then by (3.14)
\[
g^{\beta\nu} \partial_\beta \partial_\nu (F(u)) = F'(u) g^{\beta\nu} u_{\beta\nu} + F''(u) g^{\beta\nu} u_{\beta\nu} u_{\beta\nu} = 0 + F''(u) (\sum_{c} e_c(u) e_c(u) + e_3(u) e_4(u)) = O(\varepsilon t^{-3+C \varepsilon}).
\]
We thus have $e_1^a e_4^\mu g^{\beta\nu} \partial_\beta \partial_\nu g_{\alpha\mu} = O(\varepsilon t^{-3+C\varepsilon})$. To handle the other term, we note that

$$e_4(\frac{1}{2} e_1^a e_4^\mu e_2^\beta \partial_\beta g_{\alpha\mu}) - \frac{1}{2} e_1^a e_4^\mu e_3^\beta e_4(\partial_\beta g_{\alpha\mu}) = \frac{1}{2} e_4(e_1^a e_4^\mu e_3^\beta) \partial_\beta g_{\alpha\mu} = O(\varepsilon^2 t^{-3+C\varepsilon}).$$

Thus, it follows from (3.18) that

(3.19)$$e_4(\chi_{11} - \chi_{22}) = -\text{tr} \chi (\chi_{11} - \chi_{22}) + \Gamma^0_{\alpha\beta} e_4^\alpha e_4^\beta (\chi_{11} - \chi_{22}) + e_4(f_{11} - f_{22}) + O(\varepsilon t^{-3+C\varepsilon}),$$

$$e_4(\chi_{12}) = -\chi_{12} \text{tr} \chi + \Gamma^0_{\alpha\beta} e_4^\alpha e_4^\beta \chi_{12} + e_4(f_{12}) + O(\varepsilon t^{-3+C\varepsilon}),$$

$$e_4(\text{tr} \chi) = \frac{1}{2} (\text{tr} \chi)^2 - \frac{1}{2} (\chi_{11} - \chi_{22})^2 - 2\chi_{12}^2 + \Gamma^0_{\alpha\beta} e_4^\alpha e_4^\beta \text{tr} \chi + e_4(\text{tr} f - \frac{1}{2} e_4^a e_4^\rho e_3(g_{\alpha\mu})) + O(\varepsilon^2 t^{-3+C\varepsilon}).$$

It turns out to be more convenient to work with (3.19) instead of (3.18).

3.6. **Continuity argument.** Fix a geodesic $x(s)$ in $\mathcal{A}$ with $x^0(0) \in H \cap \{ t < T \}$. Since $\dot{x}^0(s) > 0$ for all $s \geq 0$ and $\lim_{s \to \infty} x^0(s) = \infty$, there exists a unique $0 < s_0 < \infty$ such that $x^0(s_0) = T$. Also fix some $s_1 \in [0, s_0]$. Our assumption is that for all $s \in [0, s_1]$, at $(t, x) = x(s) \in \Omega_T$ we have

(3.20)$$\max_{a, b = 1, 2} | \chi_{ab} - \delta_{ab} r^{-1} | \leq A t^{-2+B \varepsilon}.$$

Here $A$ and $B$ are large constants which are independent of $T, \varepsilon, s_1, s_0$ and the geodesic $x(s)$. In the derivation below, we always assume that the constants $C$ in the inequalities are given before we choose $A, B$, and that the constants $C$ are also independent of $T, \varepsilon, s_1, s_0$ and $x(s)$. Note that for $A, B \gg 1$, we have (3.20) for $s_1 = 0$ by the next lemma.

**Lemma 3.12.** On $H$, we have $| \partial^2 q | \lesssim t^{-1}$ and $\max_{a, b = 1, 2} | \chi_{ab} - \delta_{ab} r^{-1} | \lesssim t^{-2+C \varepsilon}$.

**Proof.** Recall from Section 3.1 that on $H$ we have

$$(-1 - 4g^{0i} \omega_i + 4g^{ij} \omega_i \omega_j)q_i^2 + (4g^{ij} \omega_i \omega_j - 2g^{0i} \omega_i)q_i + g^{ij} \omega_i \omega_j = 0.$$

To compute $X_i q_t$ where $X_i = \partial_i + 2\omega_i \partial_t$, we apply $X_i$ to the equation and then solve for $X_i q_t$. Then,

$$X_i q_t = \frac{-q_i^2 X_i(-1 - 4g^{0i} \omega_i + 4g^{ij} \omega_i \omega_j) + q_t X_i(4g^{ij} \omega_i \omega_j - 2g^{0i} \omega_i) + X_i(g^{ij} \omega_i \omega_j)}{2q_t(-1 - 4g^{0i} \omega_i + 4g^{ij} \omega_i \omega_j) + 4g^{ij} \omega_i \omega_j - 2g^{0i} \omega_i}.$$

Note that every term on the right hand side is known. The denominator is equal to $-2 + O(|u|)$ on $H$, so it is nonzero for $\varepsilon \ll 1$. In addition, we have $X_i \omega_j = O(r^{-1}) = O(t^{-1})$ and $X_i u = O(|\partial u|) = O(\varepsilon t^{-1})$, so $X_i q_t = O(t^{-1})$. Next, we have

$$X_i q_j = X_i(-\omega_j - 2\omega_j q_t) = -(\partial_i \omega_j)(1 + 2q_t) - \omega_j X_i q_t = O(t^{-1}).$$

By applying $\partial_i$ to the eikonal equation, we have

$$0 = 2g^{\alpha\beta} q_\beta q_\alpha + (\partial_i g^{\alpha\beta}) q_\alpha q_\beta = 2g^{\alpha\beta} q_\beta q_t + 2g^{\alpha\beta} (X_i q_t - 2\omega_i q_t) + (\partial_i g^{\alpha\beta}) q_\alpha q_\beta.$$
And since \((q_t, q_i) = (-1, \omega) + O(|u|)\) on \(H\), we have
\[
q_{tt} = -\frac{2g^{ij}q_{3i}X_iq_t + (\partial_t g^{ij})q_{o}q_{j}}{2g^{ij}q_{3j} - 4g^{ij}\omega_i q_{j}} = -\frac{O(|\partial q| t^{-1} + \varepsilon t^{-1} |\partial q|^2)}{-2q_t - 4q_r + O(|u| |\partial q|)} = O(t^{-1}).
\]
Finally we note that \(q_{it} = X_iq_t - 2\omega_i q_{tt} = O(t^{-1})\) and \(q_{ij} = X_iq_j - 2\omega_i q_{jt} = O(t^{-1})\).

We move on to the estimates for \(\chi\). By definition, we have
\[
\chi_{ab} = \langle Da e_4, e_b \rangle = (e_a(e_4^a) + e_a^\mu e_4^{\Gamma_a}_{\mu b})e_b g_{\alpha\beta}.
\]
As computed in Lemma [3.7], we have
\[
e_a^\mu e_4^{\Gamma_a}_{\mu b} e_b g_{\alpha\beta} = \left(\frac{1}{2} g^{\alpha\gamma}(\partial_\mu g_{\nu\gamma} + \partial_\nu g_{\mu\gamma}) - \frac{1}{4} e_4^a e_3(g_{\mu\nu}) + O(\varepsilon t^{-2+C\varepsilon})\right)e_a^\mu e_4^{\Gamma_a}_{\mu b} e_b g_{\alpha\beta}
\]
\[
= \frac{1}{2} (e_a(g_{\nu\beta}) e_4^{\nu b} e_b g_{\alpha\beta} + e_4(g_{\mu\beta}) e_a^\mu e_b g_{\alpha\beta}) - \frac{1}{4} e_3(g_{\mu\nu}) e_a^\mu e_4^{\nu b} e_b g_{\alpha\beta} + O(\varepsilon t^{-2+C\varepsilon})
\]
\[
= O(\varepsilon t^{-2+C\varepsilon}).
\]
In addition, recall from Section [3.1] that \(q_i = \omega_i q_r\) on \(H\). Since \(e_a\) is tangent to \(H\), on \(H\) we have
\[
e_a(q_i) = e_a(\omega_i q_r) = e_a^i r^{-1}(\delta_{ij} - \omega_i \omega_j) q_r + \omega_i e_a(q_r) = e_a^i r^{-1} - \omega_i q_r r^{-1} e_a(r) + \omega_i e_a(q_r).
\]
Since \(e_a\) is tangent to the 2-sphere \(\{t = t_0, q = q(t_0, x_0)\} = \{t = t_0, |x| = |x_0|\}\) at \((t_0, x_0) \in H\), we have \(e_a(r) = e_a^i \omega_i = 0\) on \(H\). Thus, on \(H\) we have
\[
e_b^a e_a(q_r) = e_b^i e_a(q_i) = \sum_i\delta_{ij} e_b^i (e_a^i r^{-1} - 0 + \omega_i e_a(q_r))
\]
\[
= r^{-1} g_{ij} e_b^i e_a(r) - r^{-1} (g_{ij} - \delta_{ij}) e_b^i e_a(r) + 0 = r^{-1} \delta_{ab} + O(\varepsilon t^{-2+C\varepsilon}).
\]
It follows that
\[
e_a(e_4^a) = e_a(L^\alpha \frac{L^a}{L^0}) = \frac{L^0 e_a(2g^{\alpha\gamma} q_{\gamma}) - L^a e_a(2g^{\alpha\gamma} q_{\gamma})}{(L^0)^2} = \frac{2(g^{\alpha\gamma} e_a^\gamma e_4^{\alpha\gamma}) e_a(q_{\gamma})}{L^0} = O(\varepsilon t^{-2+C\varepsilon})
\]
\[
e_a(e_4^a) e_b g_{\alpha\beta} = \frac{2(e_b^\gamma - \langle e_4, e_b \rangle g^{\beta\gamma}) e_a(q_{\gamma})}{-2q_t + O(|u| |\partial q|)} + O(\varepsilon t^{-2+C\varepsilon}) = \frac{2e_b^\gamma e_a(q_{\gamma})}{2 + O(|u|)} + O(\varepsilon t^{-2+C\varepsilon})
\]
\[
= r^{-1} \delta_{ab} + O(\varepsilon t^{-2+C\varepsilon}).
\]
This finishes the proof. \(\square\)

To complete the continuity argument, we need to prove (3.20) with \(A\) replaced by \(A/2\). We start with \(\chi_{12}\) and \(\chi_{11} - \chi_{22}\). By (3.12), we have
\[
e_4(r^2(\chi_{12} - f_{12})) = 2r e_4(r)(\chi_{12} - f_{12}) + r^2 e_4(\chi_{12} - f_{12})
\]
\[
= 2r e_4(r)(\chi_{12} - f_{12}) + r^2 ((- \text{tr}\chi + \Gamma^0_{\alpha\beta} e_4^\beta e_4^\alpha) \chi_{12} + O(\varepsilon t^{-3+C\varepsilon}))
\]
\[
= r(2e_4(r) - r \text{tr}\chi + r \Gamma^0_{\alpha\beta} e_4^\beta e_4^\alpha) \chi_{12} - 2r e_4(r) f_{12} + O(\varepsilon t^{-1+C\varepsilon}).
\]
Recall that $e_4(r) = 1 + O(t^{-1+C\varepsilon})$, $f_{12} = O(\varepsilon t^{-2+C\varepsilon})$ and $r\Gamma^0_{\alpha\beta}e_4^\alpha e_4^\beta = O(r|\partial g|) = O(\varepsilon)$. By (3.20), we have $|2 - r \text{ tr } \chi| \leq 2Art^{-2+B\varepsilon}$. In conclusion,
\[
|e_4(r^2(\chi_{12} - f_{12}))| \leq 2(2Art^{-2+B\varepsilon} + C\varepsilon + Ct^{-1+C\varepsilon}) \cdot At^{-2+B\varepsilon} + C\varepsilon t^{-1+C\varepsilon}
\]
\[
\leq CA^2t^{-2+2B\varepsilon} + CA\varepsilon t^{-1+B\varepsilon} + CAT^{-2+(B+C)\varepsilon} + C\varepsilon t^{-1+C\varepsilon}
\]
\[
\leq CA^2t^{-2+2B\varepsilon} + CA\varepsilon t^{-1+B\varepsilon}.
\]

By choosing $A, B \gg C$, we obtain the last inequality. On $H$, we have $|r^2(\chi_{12} - f_{12})| \leq Ct^{C\varepsilon}$ by the previous lemma. Thus, by integrating $e_4(r^2(\chi_{12} - f_{12}))$ along the geodesic, we have
\[
|e_4(r^2(\chi_{12} - f_{12}))| \leq C(x^0(0))^{C\varepsilon} + CA^2(x^0(0))^{-1+2B\varepsilon} + CAB^{-1}t^{B\varepsilon}
\]
\[
\leq Ct^{C\varepsilon} + CA^2T_0^{-1+2B\varepsilon} + CAB^{-1}t^{B\varepsilon}.
\]

Since $T_0 \gg \varepsilon^{-1}$, we have $A^2T_0^{-1+2B\varepsilon} \leq 1$ for $\varepsilon \ll 1$. In addition, by choosing $B \geq A$, we have
\[
|\chi_{12}| \leq r^{-2}(|f_{12}| + Ct^{C\varepsilon} + C + Ct^{B\varepsilon}) \leq Ct^{-2+B\varepsilon}.
\]

Here $C$ is independent of $A$ and $B$, so if we choose $A \geq 4C$, we obtain with $|\chi_{12}| \leq \frac{1}{4}At^{-2+B\varepsilon}$. The proof for $|\chi_{11} - \chi_{22}| \leq \frac{1}{4}At^{-2+B\varepsilon}$ is essentially the same.

To finish the continuity argument, we need to prove that $|\text{ tr } \chi - 2r^{-1}| \leq \frac{1}{4}At^{-2+B\varepsilon}$. For $h = \text{ tr } \chi - \text{ tr } f = \text{ tr } \chi + O(\varepsilon t^{-2+C\varepsilon})$, by (3.20) we have $h = 2r^{-1} + O(At^{-2+B\varepsilon}) \sim 2r^{-1}$. Then, for $\varepsilon \ll 1$, by the last equation in (3.19) we have
\[
e_4(h^{-1}) = -h^{-2}e_4(h)
\]
\[
= -h^{-2}(-\frac{1}{2}(\text{ tr } \chi)^2 + \Gamma^0_{\alpha\beta}e_4^\alpha e_4^\beta \text{ tr } \chi - \frac{1}{2}e_4(e_4^\alpha e_4^\beta e_3(g_{\alpha\beta})) + O(\varepsilon^2 t^{-3+C\varepsilon} + (\chi_{11} - \chi_{22})^2 + \chi_{12}^2))
\]
\[
= -h^{-2}(-\frac{1}{2}h^2 + \Gamma^0_{\alpha\beta}e_4^\alpha e_4^\beta h - \frac{1}{2}e_4(e_4^\alpha e_4^\beta e_3(g_{\alpha\beta})) + O(\varepsilon t^{-3+C\varepsilon} + \varepsilon^2 t^{-3+C\varepsilon} + A^2t^{-4+2B\varepsilon}))
\]
\[
= \frac{1}{2} - \Gamma^0_{\alpha\beta}e_4^\alpha e_4^\beta h^{-1} + \frac{1}{2}h^{-2}e_4^\alpha e_4^\beta e_3(g_{\alpha\beta}) + O(\varepsilon t^{-1+C\varepsilon}).
\]

In the last line we use the product rule and the estimate $e_4(e_4^\alpha) = O(\varepsilon t^{-2+C\varepsilon})$. In addition, we have
\[
|h^{-1} - r/2| = \frac{|2 - r(\text{ tr } \chi - \text{ tr } f)|}{2h} \lesssim r(|2 - r \text{ tr } \chi| + |r \text{ tr } f|) \lesssim At^{B\varepsilon};
\]
by (3.17), we have
\[
\Gamma^0_{\alpha\beta}e_4^\alpha e_4^\beta = \frac{1}{2}g^{0\gamma}(e_4^\alpha e_4^\beta e_4(g_{\beta\gamma}) + e_4^\alpha e_4(g_{\alpha\gamma})) - \frac{1}{4}e_4^\alpha e_3(g_{\alpha\beta})e_4^\alpha e_4^\beta + O(\varepsilon t^{-2+C\varepsilon})
\]
\[
= -\frac{1}{4}e_3(g_{\alpha\beta})e_4^\alpha e_4^\beta + O(\varepsilon t^{-2+C\varepsilon}).
\]

Thus, we have
\[
e_4(h^{-1}) = \frac{1}{2} + \frac{1}{4}e_4^\alpha e_4^\beta e_3(g_{\alpha\beta})h^{-1} + \frac{1}{4}r h^{-1}e_4^\alpha e_4^\beta e_4(e_3(g_{\alpha\beta}))
\]
\[
+ O(\varepsilon t^{-1+C\varepsilon} + h^{-1} \varepsilon^{-2+2C\varepsilon} + At^{B\varepsilon}h^{-1}|e_4(e_3(g)|)
\]
\[
= \frac{1}{2} + \frac{1}{4}h^{-1}e_4^\alpha e_4^\beta (e_3(g_{\alpha\beta}) + e_4(e_3(g_{\alpha\beta}))) + O(At^{1+B\varepsilon}|e_4(e_3(g_{\alpha\beta}))| + \varepsilon t^{-1+C\varepsilon}).
\]
The next three lemmas are necessary for us to control $e_3(g_{\alpha\beta}) + re_4(e_3(g_{\alpha\beta}))$ and $e_4(e_3(g_{\alpha\beta}))$.

**Lemma 3.13.** Under the assumption (3.20), in $\Omega_T$ we have $|e_a(e_3(q))| + |e_a(\partial q)| \lesssim t^{-1+C_\varepsilon}$, $|e_a(\Omega_{ij}q)| \lesssim At^{-1+B_\varepsilon}|e_3(q)| + t^{-1+C_\varepsilon}$ and $|\partial^2 q| \lesssim t^{C_\varepsilon}$.

Proof. We have (assuming $\{a, a'\} = \{1, 2\}$)

$$e_4(e_a(e_3(q))) = [e_4, e_a]e_3(q) + e_a(e_4(e_3(q))) = -\sum_b \chi_{a b e_b}(e_3(q)) - e_a(\Gamma^0_{\mu\nu} e^\mu_4 e^\nu_4 e_3(q))$$

$$= -\sum_b \chi_{a b e_b}(e_3(q)) - 2\Gamma^0_{\mu\nu} \sum_b \chi_{a b e_b} + O(\varepsilon t^{-2+C_\varepsilon}) e^\nu_4 e_3(q)$$

$$\leq |(1 + O(t^{-1+C_\varepsilon})) e_a(e_3(q)) - r(\chi_{a a} + \Gamma^0_{\mu\nu} e^\mu_4 e^\nu_4) e_a(e_3(q)) - r\chi_{12} e_{a'}(e_3(q))| + C\varepsilon t^{-1+C_\varepsilon}$$

$$\leq |r^{-1} - \chi_{a a} + O(t^{-1+C_\varepsilon})| |\chi_{a a} + \Gamma^0_{\mu\nu} e^\mu_4 e^\nu_4| + O(t^{-2+C_\varepsilon})| e_a(e_3(q))| + |r\chi_{12} e_{a'}(e_3(q))| + C\varepsilon t^{-1+C_\varepsilon}$$

$$\leq (At^{-2+B_\varepsilon} + C\varepsilon t^{-1} + Ct^{-2+C_\varepsilon})|e_a(e_3(q))| + CAt^{-2+B_\varepsilon}|e_{a'}(e_3(q))| + C\varepsilon t^{-1+C_\varepsilon}$$

$$\leq C\varepsilon t^{-1} \sum_b |e_b(e_3(q))| + C\varepsilon t^{-1+C_\varepsilon}.$$

In the last line, we choose $\varepsilon \ll 1$ so that $C\varepsilon t^{-1} \geq At^{-2+B_\varepsilon} + t^{-2+C_\varepsilon}$ for $t \geq T_0 = \exp(\delta/\varepsilon)$. Since $e_a$ is tangent to $H$, on $H$ we have $e_a(e_3(q)) = e_a(2g^{0\alpha} g_{\alpha a}) = O(|\partial^2 q| + |e_a(q)\partial q|) = O(t^{-1})$ by Lemma 3.12. In conclusion, if $(t, x) \in \Omega_T$ lies on a geodesic $x(s)$ in $\mathcal{A}$, at $(t, x)$ we have

$$\sum_a |e_a(e_3(q))| \leq C + \int_{x(0)}^t C\varepsilon t^{-1} \sum_a |e_a(e_3(q))| |(\tau, \bar{\tau}(\tau))| d\tau + Ct^{C_\varepsilon}$$

Here $(\tau, \bar{\tau}(\tau))$ is a reparametrization of the geodesic $x(s)$. We conclude that $\sum_a |e_a(e_3(q))| \lesssim Ct^{C_\varepsilon}$ by the Gronwall’s inequality. In addition, in $\Omega_T$ we have

$$e_a(g_{\alpha}) = e_a(\frac{1}{2}(\partial_{\alpha}, e_4) e_3(q)) = e_a(\frac{1}{2} e_4^\beta g_{\alpha\beta} e_3(q))$$

$$= \frac{1}{2} e_a(e_4^\beta) g_{\alpha\beta} e_3(q) + \frac{1}{2} e_4^\beta e_a(g_{\alpha\beta}) e_3(q) + \frac{1}{2} e_4^\beta g_{\alpha\beta} e_a(e_3(q)) = O(t^{-1+C_\varepsilon}).$$

Next we compute $e_a(\Omega_{ij}q)$. Note that

$$\Omega_{ij}q = \frac{1}{2}(\Omega_{ij}, e_4) e_3(q) = \frac{1}{2}(x_i g_{j\beta} - x_j g_{i\beta}) e_4^\beta e_3(q) = \frac{1}{2} r(\omega_i g_{j\beta} e^\beta_4 - \omega_j g_{i\beta} e^\beta_4) e_3(q).$$
We have
\[ \omega_i g_{j\beta} e^\beta_4 - \omega_j g_{i\beta} e^\beta_4 = \omega_i e^i_4 - \omega_j e^j_4 + O(|u|) = O(\sum_j |e^j_4 - \omega_j|) + O(|u|) = O(t^{-1+C\varepsilon}), \]
so
\[ r(\omega_i g_{j\beta} e^\beta_4 - \omega_j g_{i\beta} e^\beta_4)e_a(e_3(q)) = O(t^{-1+C\varepsilon}). \]
In addition,
\[
e_a((x_i g_{j\beta} - x_j g_{i\beta}) e^\beta_4)
= (e^i_4 g_{j\beta} - e^j_4 g_{i\beta}) e^\beta_4 + (x_i g_{j\beta} - x_j g_{i\beta}) e_a(e^\beta_4) + O(|e_a(g)|)
= e^i_a e^j_4 - e^j_a e^i_4 + (x_i g_{j\beta} - x_j g_{i\beta}) \sum_b (\chi_{ab} e^\beta_b + O(\varepsilon t^{-2+C\varepsilon})) + O(|e_a(g)| + |u|)
= e^i_a e^j_4 - e^j_a e^i_4 + \sum_b \chi_{ab} (x_i e^j_b - x_j e^i_b) + O(r|u|) + O(\varepsilon t^{-1+C\varepsilon})
= e^i_a e^j_4 - e^j_a e^i_4 + r^{-1} (x_i e^j_a - x_j e^i_a) + O(r(|\chi_{aa} - r^{-1}| + |\chi_{12}|)) + O(\varepsilon t^{-1+C\varepsilon})
= e^i_a (e^j_4 - \omega_j) - e^j_a (e^i_4 - \omega_i) + O(A t^{-1+2\varepsilon}) + O(\varepsilon t^{-1+C\varepsilon}) = O(A t^{-1+2\varepsilon}).
\]
By the product rule we obtain the second estimate.

Finally, we consider \( \partial^2 q \). Recall that \( e^a_4 = L^a / L^0 \) and that \( |\partial q| \sim |q_r| \sim |q_t| \sim e_3(q) \). By the characteristic ODE’s, we have
\[
e_4(q_a) = -(\partial_a g^{\mu\nu}) q_\mu q_\nu / e_3(q) = O(\varepsilon t^{-1}) e_3(q)
\]
and thus
\[
\partial_a(e_4(q_\beta)) = -(\partial_a((\partial_\beta g^{\mu\nu}) q_\mu q_\nu) e_3(q) + (\partial_\beta g^{\mu\nu}) q_\mu q_\nu \cdot 2 \partial_\alpha(g^\beta_\gamma q_\gamma) / (e_3(q))^2
= -2(\partial_\beta g^{\mu\nu}) q_\mu q_\nu e_3(q) + (\partial_\beta g^{\mu\nu}) q_\mu q_\nu \cdot 2 g^{\alpha\gamma} q_\gamma / (e_3(q))^2 + O(\varepsilon t^{-1+C\varepsilon})
= O(|\partial q| |\partial^2 q|) + O(\varepsilon t^{-1+C\varepsilon}) = O(\varepsilon t^{-1} |\partial^2 q|) + O(\varepsilon t^{-1+C\varepsilon}).
\]
In the second line, we take out those terms without \( \partial^2 q \) and control them using the estimates for \( g \) and \( \partial q \). In the last line, we use the estimate \( |\partial q| \sim e_3(q) \). Besides, we have
\[
\partial_a e^\beta_4 = \partial_a(L^\beta) L^0 / L^\beta \partial_a(L^0) = 2 \partial_a(g^{\beta\nu} q_\nu) - 2 e^\beta_4 \partial_a(g^{\beta\nu} q_\nu) / e_3(q)
= 2(g^{\beta\nu} - e^\beta_4 g^{\beta\nu}) q_\nu / e_3(q) + O(|\partial q| |\partial q| (e_3(q))^{-1})
= (\sum \partial_a e^\beta_4 q_\nu + e^\beta_4 e^\nu_4 + e^\beta_4 e^4_4 q_\nu) / e_3(q) + O(\varepsilon t^{-1})
= 2 \sum \partial_a e^\beta_4 q_\nu + (e^\beta_3 + e^\beta_4) e_4(q_a) / e_3(q) + O(\varepsilon t^{-1}) = 2 \sum \partial_a e^\beta_4 q_\nu / e_3(q) + O(\varepsilon t^{-1}).
\]
Thus, we have
\[ e_4(q_{a\beta}) = [e_4, \partial_a]q_\beta + \partial_a(e_4(q_\beta)) = -\partial_a(e_4^\mu)\partial_\mu(q_\beta) + \partial_a(e_4(q_\beta)) \]
\[ = O((e_3(q))^{-1}\sum_a |e_3(q_\beta)e_a(q_\beta)|) + O(\varepsilon t^{-1}|\partial^2q|) + O(\varepsilon t^{-1+C\varepsilon}) \]
\[ = O(\varepsilon t^{-1}|\partial^2q|) + O(\varepsilon t^{-1+C\varepsilon} + t^{-2+C\varepsilon}). \]
In the last line we use the estimate \( e_3(q) \geq C^{-1}t^{-C\varepsilon} \). Since \( \partial^2q = O(t^{-1}) \) on \( H \), we conclude \( \partial^2q = O(t^{C\varepsilon}) \) by the Gronwall’s inequality.

**Lemma 3.14.** Set \( h_i := r(\partial_i(\nu u) - q_iq_r^{-1}\partial_r(\nu u)) \). Under the assumption (3.20), in \( \Omega_T \) we have \( |h_i| \lesssim \varepsilon t^{C\varepsilon} \), \( |e_a(h_i)| \lesssim A\varepsilon t^{-1+B\varepsilon} \) and \( e_a(\nu u) = \sum_i e_a(\omega_i)h_i \).

**Proof.** We have
\[ h_i = r(\omega_i u + \nu u_i - q_iq_r^{-1}u_i - q_iq_r^{-1}r u_r) = ruq_i^{-1}(q_i \omega_i - q_i) + r^2(u_i - q_iq_r^{-1}u_r) \]
\[ = (ru + r^2u_r)q_i^{-1}(q_i \omega_i - q_i) + r^2(u_i - \omega_i u_r) = (u + ru_r)q_i^{-1}\sum_j \omega_j \Omega_{ij} q + \sum_j x_j \Omega_{ij} u. \]
Since \( |u| + |u_r| \lesssim \varepsilon t^{-1+C\varepsilon}, |q_i - \omega_i \nu r| \lesssim t^{-1+C\varepsilon} \) and \( |u_i - \omega_i u_r| \lesssim \varepsilon t^{-2+C\varepsilon} \), we obtain \( |h_i| \lesssim \varepsilon t^{C\varepsilon} \). Moreover,
\[ e_a(x_j \Omega_{ij} u) = e_a^i \Omega_{ij} u + x_j e_a(\Omega_{ij} u) = O(\varepsilon t^{-1+C\varepsilon}), \]
\[ e_a((u + ru_r)q_i^{-1}\omega_j \Omega_{ij} q) = e_a(u + ru_r)q_i^{-1}\omega_j \Omega_{ij} q = (u + ru_r)q_i^{-2}e_a(\nu)\omega_j \Omega_{ij} q \]
\[ + (u + ru_r)q_i^{-1}e_a(\omega_j)\Omega_{ij} q + (u + ru_r)q_i^{-1}\omega_j e_a(\Omega_{ij} q) \]
\[ = O(\varepsilon t^{-1+C\varepsilon}) + O(\varepsilon |q_r|^{-1}|e_a(\nu)|) \]
\[ = O(\varepsilon t^{-1+C\varepsilon}) + O(A\varepsilon t^{-1+B\varepsilon}e_3(q)/q_r) = O(A\varepsilon t^{-1+B\varepsilon}). \]
Here we apply many estimates such as \( e_3(r) = O(1), e_a(\omega_i) = O(r^{-1}), \Omega_q = O(t^{C\varepsilon}), q_r \geq C^{-1}t^{-C\varepsilon} \) and etc. In particular, we apply \( e_a(\Omega_q) = O(A\varepsilon t^{-1+B\varepsilon}e_3(q) + t^{-1+C\varepsilon}) \) from the previous lemma. Thus, we have \( e_a(h_i) = O(A\varepsilon t^{-1+B\varepsilon}). \)

Finally, we have
\[ \sum_i e_a(\omega_i)h_i = \sum_{i,j} e_a^i r^{-1}(\delta_{ij} - \omega_i \omega_j)h_i \]
\[ = \sum_i e_a^i (\partial_i(\nu u) - q_iq_r^{-1}\partial_r(\nu u)) - \sum_{i,j} e_a^i \omega_i \omega_j (\partial_i(\nu u) - q_iq_r^{-1}\partial_r(\nu u)) \]
\[ = e_a(\nu u) - \sum_{i,j} e_a^i \omega_j \sum_1 (\omega_i \partial_i(\nu u) - \omega_i q_iq_r^{-1}\partial_i(\nu u)) \]
\[ = e_a(\nu u). \]

**Lemma 3.15.** Under the assumption (3.20), in \( \Omega_T \) we have \( |r^{-1}e_3(u) + e_4(e_3(u))| \lesssim \varepsilon At^{-3+B\varepsilon} \) and \( |e_4(e_3(u))| \lesssim \varepsilon t^{-2} \).
Proof. The second inequality follows directly from the first one. To prove the first one, we note that for each function $F = F(t, x)$, we have

$$g^{\alpha\beta}\partial_\alpha\partial_\beta F = \left(\sum_a e^\alpha_a e^\beta_a + \frac{1}{2} e^\alpha_4 e_3 + \frac{1}{2} e^\beta_3 e^\alpha_4\right)\partial_\alpha\partial_\beta F$$

$$= \sum_a \left( e_a(e_a(F)) - e_a(e^\alpha_a F_\alpha) + e_4(e_3(F)) - e_4(e^\alpha_3 F_\alpha) \right)$$

$$= \sum_a \left( e_a(e_a(F)) - (D_a e_a) F + e^\mu_a e^\nu_a \Gamma^\alpha_{\mu\nu} F_\alpha \right) + e_4(e_3(F)) - (D_4 e_3) F + e^\mu_4 e^\nu_3 \Gamma^\alpha_{\mu\nu} F_\alpha.$$  

By (3.17), we have

$$e^\mu_a e^\nu_\alpha \Gamma^\alpha_{\mu\nu} F_\alpha = \frac{1}{2} g^{\alpha\beta} F_\alpha (e^\nu_\alpha e_a(g_\nu\beta) + e^\mu_a e_a(g_\mu\beta)) - \frac{1}{4} e_3(g_\mu\nu) e^\mu_a e^\nu_a e_4(F) + O(\varepsilon t^{-2+C\varepsilon} |\partial F|)$$

$$= O(\varepsilon t^{-2+C\varepsilon} |\partial F| + \varepsilon t^{-1} |e_4(F)|),$$

$$e^\mu_4 e^\nu_3 \Gamma^\alpha_{\mu\nu} F_\alpha = \frac{1}{2} g^{\alpha\beta} F_\alpha (e^\nu_\alpha e_3(g_\nu\beta) + e^\mu_4 e_3(g_\mu\beta)) - \frac{1}{4} e_4 e^\mu_3 e_3(g_\mu\nu) e_4(F) + O(\varepsilon t^{-2+C\varepsilon} |\partial F|)$$

$$= \frac{1}{2} \sum_a \left( e^\beta_a e_3(F) + \frac{1}{2} e^\beta_3 e_3(F) + \frac{1}{2} e^\beta_4 e_3(F) \right) e^\mu_a e_3(g_\mu\beta) + O(\varepsilon t^{-2+C\varepsilon} |\partial F| + \varepsilon t^{-1} |e_4(F)|)$$

$$= \frac{1}{4} e_3(F) e^\beta_4 e^\mu_3 e_3(g_\mu\beta) + O(\varepsilon t^{-2+C\varepsilon} |\partial F| + \varepsilon t^{-1} \sum_{k=1,2,4} |e_k(F)|).$$

Moreover, since

$$D_a e_a = \langle D_a e_a, e_{a'} \rangle e_{a'} + \frac{1}{2} \langle D_a e_a, e_4 \rangle e_3 + \frac{1}{2} \langle D_a e_a, e_3 \rangle e_3$$

$$= \langle D_a e_a, e_{a'} \rangle e_{a'} + \frac{1}{2} (-\chi_{aa}) e_3 - \frac{1}{2} \chi_{aa} + \frac{1}{4} e^\mu_a e^\nu_a \Gamma^\alpha_{\mu\nu} e_4, \quad a \neq a'$$

$$D_4 e_3 = \sum_b \langle D_4 e_3, e_b \rangle e_b + \frac{1}{2} \langle D_4 e_3, e_4 \rangle e_3 + \frac{1}{2} \langle D_4 e_3, e_3 \rangle e_4$$

$$= -2 \sum_b \Gamma^0_{\mu\nu} e^\mu_4 e^\nu_b e_b - \Gamma^0_{\mu\nu} e^\mu_4 e^\nu_4 e_3,$$

we have

$$\sum_a (D_a e_a) F = \langle D_1 e_1, e_2 \rangle e_2(F) + \langle D_2 e_2, e_1 \rangle e_1(F) - \frac{1}{2} (\text{tr} \chi)(e_3(F) + e_4(F)) + \sum_a e^\mu_a e^\nu_a \Gamma^\alpha_{\mu\nu} e_4(F)$$

$$= \langle D_1 e_1, e_2 \rangle e_2(F) + \langle D_2 e_2, e_1 \rangle e_1(F) - \frac{1}{2} (\text{tr} \chi)(e_3(F) + O(\varepsilon t^{-1} |e_4(F)|))$$

$$= \langle D_1 e_1, e_2 \rangle e_2(F) + \langle D_2 e_2, e_1 \rangle e_1(F) - r^{-1} e_3(F) + O(\varepsilon t^{-1} |e_4(F)| + At^{-2+B\varepsilon} |e_3(F)|),$$

$$\langle D_4 e_3 \rangle F = -2 \sum_b \Gamma^0_{\mu\nu} e^\mu_4 e^\nu_b e_b(F) - \Gamma^0_{\mu\nu} e^\mu_4 e^\nu_4 e_3(F)$$

$$= \frac{1}{4} e_3(g_{a\beta}) e^\alpha_4 e^\beta_4 e_3(F) + O(\varepsilon t^{-1} \sum_b |e_b(F)| + \varepsilon t^{-2+C\varepsilon} |e_3(F)|).$$
Here we use the assumption (3.20) and $|e_3(u)| \lesssim |\partial u| \lesssim \varepsilon t^{-1}$. In conclusion, we have
\[
g^{\alpha \beta} \partial_\alpha \partial_\beta F = \sum_a e_a(e_a(F)) - \langle D_1 e_1, e_2 \rangle e_2(F) - \langle D_2 e_2, e_1 \rangle e_1(F) + e_4(e_3(F)) + r^{-1} e_3(F) + O(t^{-1}|e_4(F)|) + At^{-2+B\varepsilon}|e_3(F)|) + O(\varepsilon t^{-2+C\varepsilon}|\partial F|) + \varepsilon t^{-1} \sum_{k=1,2,4} |e_k(F)|.
\]
By taking $F = u$, we obtain
\[
0 = g^{\alpha \beta} \partial_\alpha \partial_\beta u = \sum_a e_a(e_a(u)) - \langle D_1 e_1, e_2 \rangle e_2(u) - \langle D_2 e_2, e_1 \rangle e_1(u) + r^{-1} e_3(u) + e_4(e_3(u)) + O(A\varepsilon t^{-3+B\varepsilon}).
\tag{3.22}
\]
In addition, note that
\[
e_4(e_3(F)) + r^{-1} e_3(F) = e_4(2g^{0\alpha} + e_\alpha^0)F_\alpha + (2g^{0\alpha} + e_\alpha^0)e_4(F_\alpha) + r^{-1} e_3(F) = O((|e_4(g^{0\alpha})| + |e_4(e_\alpha^0)|)|\partial F| + |e_4(F_\alpha)| + r^{-1}|e_3(F)|)
= O(\varepsilon t^{-2+C\varepsilon}|\partial F| + |e_4(\partial F)| + r^{-1}|e_3(F)|).
\]
Thus, we have
\[
|\sum_a e_a(e_a(F)) - \langle D_1 e_1, e_2 \rangle e_2(F) - \langle D_2 e_2, e_1 \rangle e_1(F)|
\lesssim |\partial^2 F| + \varepsilon t^{-2+C\varepsilon}|\partial F| + r^{-1}|e_3(F)| + t^{-1}|e_4(F)| + At^{-2+B\varepsilon}|e_3(F)|) + \varepsilon t^{-1} \sum_{k=1,2,4} |e_k(F)|.
\]
When $F = r^{-1}$, the right hand side has an upper bound $Ct^{-3+C\varepsilon}$. When $F = \omega_i$, the right hand side has an upper bound $Ct^{-2+C\varepsilon}$. Here we choose $\varepsilon \ll_{A,B}$ so that $At^{-2+B\varepsilon}|e_3(r^{-1})| \lesssim At^{-4+B\varepsilon} \lesssim t^{-3}$ and $At^{-2+B\varepsilon}|e_3(\omega_i)| \lesssim At^{-3+B\varepsilon} \lesssim t^{-2}$.

We set $U(t, x) = ru(t, x)$. Then, by the previous lemma,
\[
e_a(u) = e_a(r^{-1}U) = e_a(r^{-1})U + r^{-1} e_a(U) = e_a(r^{-1})U + r^{-1} \sum_i e_a(\omega_i)h_i,
\]
\[
e_a(e_a(u)) = e_a(e_a(r^{-1}))U + 2e_a(r^{-1}) \sum_i e_a(\omega_i)h_i + r^{-1} \sum_i e_a(e_a(\omega_i))h_i + r^{-1} \sum_i e_a(\omega_i)e_a(h_i)
= e_a(e_a(r^{-1}))U + r^{-1} \sum_i e_a(e_a(\omega_i))h_i + O(A\varepsilon t^{-3+B\varepsilon} + \varepsilon t^{-3+C\varepsilon}).
\]
Thus, we have
\[
\sum_a e_a(e_a(u)) - \langle D_1 e_1, e_2 \rangle e_2(u) - \langle D_2 e_2, e_1 \rangle e_1(u)
= (\sum_a e_a(e_a(r^{-1})) - \langle D_1 e_1, e_2 \rangle e_2(r^{-1}) - \langle D_2 e_2, e_1 \rangle e_1(r^{-1}))U + r^{-1} \sum_i (\sum_a e_a(e_a(\omega_i)) - \langle D_1 e_1, e_2 \rangle e_2(\omega_i) - \langle D_2 e_2, e_1 \rangle e_1(\omega_i))h_i + O(A\varepsilon t^{-3+B\varepsilon} + \varepsilon t^{-3+C\varepsilon})
= O(t^{-3+C\varepsilon}|ru| + t^{-2+C\varepsilon}r^{-1}|h_i| + At^{-3+B\varepsilon} + \varepsilon t^{-3+C\varepsilon}) = O(A\varepsilon t^{-3+B\varepsilon}).
\]
We finish the proof by this estimate and (3.22).
We now finish the continuity argument. By writing $g'_{\alpha\beta} := \frac{d}{dt}\big|_{u=0} g^{\alpha\beta}(u)$, we have
\[
e_3(g_{\alpha\beta}) = g'_{\alpha\beta}(u) e_3(u),
\]
\[
e_4(e_3(g_{\alpha\beta})) = g'_{\alpha\beta}(u) e_4(e_3(u)) + g''_{\alpha\beta}(u) e_4(u) e_3(u)
\]
and thus
\[
e_3(g_{\alpha\beta}) + re_4(e_3(g_{\alpha\beta})) = g'_{\alpha\beta}(u) (e_3(u) + re_4(e_3(u))) + g''_{\alpha\beta}(u) e_4(u) e_3(u)
\]
\[= O\left(\varepsilon t^{-2} + \varepsilon t^{-2+C\varepsilon} \cdot \varepsilon t^{-1}\right) = O\left(\varepsilon t^{-2}\right),
\]
Thus, by (3.21),
\[
|e_4(h^{-1}) - 1/2| \lesssim t \cdot A\varepsilon t^{-2+B\varepsilon} + At^{1+B\varepsilon} \cdot \varepsilon t^{-2} + \varepsilon t^{-1+C\varepsilon} \lesssim A\varepsilon t^{-1+B\varepsilon}.
\]
By the initial condition, on $H$ we have
\[
|h^{-1} - r/2| = \frac{|2 - r(\text{tr } \chi - \text{tr } f)|}{2h} \lesssim r(|2 - r \text{ tr } \chi| + |\text{r tr } f|) \lesssim t^{C\varepsilon}
\]
where the constants are known before we choose $A, B$. Now, suppose that $(t, x) \in \Omega_T$ lies on a geodesic $x(s)$ in $A$. At $x(0)$, we have $h^{-1}|x(0)| = r(x(0))/2 + O((x^0(0))^{C\varepsilon})$. Thus,
\[
|h^{-1}|_{(t, x)} - \frac{1}{2}r(x(0)) - \frac{1}{2}(t - x^0(0))| \leq |h^{-1}|_{(t, x)} - h^{-1}|_{x(0)} - \frac{1}{2}(t - x^0(0))| + Ct^{C\varepsilon}
\]
\[\lesssim \int_{x^0(0)}^t A\varepsilon^{t^{-1+B\varepsilon}} d\tau + t^{C\varepsilon} \lesssim B^{-1} At^{B\varepsilon} + t^{C\varepsilon}.
\]
Also note that $r(x(0)) - x^0(0) + t = q(t, x) + t = r + O(t^{C\varepsilon})$ by Lemma 3.8. In conclusion, $|h^{-1} - r/2| \lesssim t^{C\varepsilon} + B^{-1} At^{B\varepsilon}$ at $(t, x)$. This implies that $h^{-1} \sim r$ and
\[
|\text{tr } \chi - \frac{2}{r}| \leq |h - \frac{2}{r}| + C\varepsilon t^{-2+C\varepsilon} \leq \left|\frac{r - 2h^{-1}}{rh^{-1}}\right| + C\varepsilon t^{-2+C\varepsilon}
\]
\[\leq Ct^{-2}(Ct^{C\varepsilon} + CB^{-1} At^{B\varepsilon}) + C\varepsilon t^{-2+C\varepsilon} \leq Ct^{-2+C\varepsilon} + CB^{-1} At^{-2+B\varepsilon}.
\]
By choosing $B \geq A \gg C \ 1$, we conclude that $|\text{tr } \chi - 2/r| \leq \frac{1}{4} At^{-2+B\varepsilon}$. This finishes the continuity argument as we have proved that (3.20) holds with $A$ replaced by $A/4$.

4. Derivatives of the optical function

In this section, we aim to prove that $q$ is smooth in $\Omega$, where smoothness is defined in Section 2.4. Our main result is the following proposition.

Proposition 4.1. The optical function $q = q(t, x)$ constructed in Proposition 3.4 is a smooth function in $\Omega$. Moreover, in $\Omega$, we have $Z^I q = O((q)t^{C\varepsilon})$ and $Z^I \Omega_{ij} q = O(t^{C\varepsilon})$ for each multiindex $I$ and $1 \leq i < j \leq 3$.

In Section 4.1 we define the commutator coefficients $\xi_{ij}$ with respect to the null frame $\{e_k\}$, and derive several differential equations for $\xi$ and their derivatives. Note that the estimates for these $\xi$ would imply the estimates for $q$ in Proposition 4.1. We also define a weighted null frame $\{V_k\}$ which will be used in the rest of this paper. In Section 4.2, we focus on the estimates for $q$ on the surface $H$ where the initial data of $q$ are assigned. In Section 4.3, we prove Proposition 4.9 which gives several important estimates for $\xi$. Here
we make use of the differential equations and the estimates on $H$ proved in the first two subsections. Finally, in Section 4.4 we conclude the proof of Proposition 4.1 by applying Proposition 4.9.

To end this section, in Section 4.5 we derive two equations (4.31) and (4.32) for $e_3(u)$ and $e_3(q)$, respectively. In these two equations, we have estimates for all derivatives of the remainder terms. While they are not related to the proof of Proposition 4.1, they will be very useful in the next section.

4.1. Setup. As a convention, we use $k, l$ to denote a number in $\{1, 2, 3, 4\}$, and we use $a, b, c$ to denote a number in $\{1, 2\}$. For a finite sequence of indices $K = (k_1, \ldots, k_m)$, we set $|K| = m$, $n_{K,k} = \{j : k_j = k\}$ and $e_K = e_{k_1}e_{k_2} \cdots e_{k_m}$.

4.1.1. Commutator coefficients. We define

$$\xi^a_{kl} = \langle [e_k, e_l], e_a \rangle,$$ $$\xi^a_{kkl} = \frac{1}{2} \langle [e_k, e_l], e_a \rangle,$$ $$\xi^4_{kl} = \frac{1}{2} \langle [e_k, e_l], e_3 \rangle.$$ 

By (3.13) we have $[e_k, e_{k_2}] = \xi^l_{k_1 k_2} e_l$. Thus these $\xi^4_{k_k}$ are also called commutator coefficients in this paper.

We now derive several equations for $\xi$. Note that $\xi^l_{k_1 k_2} = -\xi^l_{k_2 k_1}$ (so $\xi^l_{k_k} = 0$) and that $\xi^3_{kl} = \xi^4_{kl}$ since $[e_k, e_l]$ never contains $\partial_t$. Thus, we only need to study those $\xi^l_{k_1 k_2}$'s with $k_1 < k_2$ and $l \leq 3$.

We start with $[e_3, e_4]$. By Lemma 3.9 we have

$$\langle [e_3, e_4], e_4 \rangle = \langle D_3 e_4 - D_4 e_3, e_4 \rangle = -\langle D_4 e_3, e_4 \rangle = \langle e_3, D_4 e_4 \rangle = 2\Gamma^0_{\alpha\beta} e_4^a e_4^\beta,$$

so $\epsilon^3_{34} = \Gamma^0_{\alpha\beta} e_4^a e_4^\beta$. For $\xi^a_{34}$, we have the following equation

$$e_4(\xi^a_{34}) = e_4(\langle D_3 e_4 - D_4 e_3, e_a \rangle) = e_4(\langle D_3 e_4, e_a \rangle) + e_4(\langle e_3, D_4 e_4 \rangle)$$

$$= \langle D_3 D_3 e_4, e_a \rangle + \langle D_3 e_4, D_4 e_a \rangle + 2e_4(\Gamma^0_{\alpha\beta} e_4^a e_4^\beta)$$

$$= \langle D_3 D_4 e_4, e_a \rangle + \langle D_4 e_4, e_4 \rangle + \langle R(e_4, e_4) e_4, e_a \rangle + \langle D_3 e_4, \ldots e_4 \rangle + 2e_4(\Gamma^0_{\alpha\beta} e_4^a e_4^\beta)$$

$$= \langle D_3(\langle \Gamma^0_{\alpha\beta} e_4^a e_4^\beta \rangle e_4), e_a \rangle - \xi^l_{34} \langle D_4 e_4, e_a \rangle + \langle R(e_4, e_4) e_4, e_a \rangle + 2e_4(\Gamma^0_{\alpha\beta} e_4^a e_4^\beta)$$

$$= -\chi_{ac} \xi^a_{34} + \langle R(e_4, e_4) e_4, e_a \rangle + 2e_4(\Gamma^0_{\alpha\beta} e_4^a e_4^\beta).$$

Next we consider $[e_a, e_4]$. From Lemma 3.10, we have $\xi^b_{a4} = \chi_{ab}$ and $\xi^c_{a4} = 0$. Thus we have the Raychaudhuri equation

$$e_4(\chi_{ab}) = \Gamma^0_{\alpha\beta} e_4^a e_4^\beta \chi_{ab} - \sum_c \chi_{ac} \chi_{cb} + \langle R(e_4, e_a) e_4, e_b \rangle.$$

Next we consider $[e_1, e_2]$. Note that $\xi^c_{12} = 0$ as $\langle [e_1, e_2], e_4 \rangle = 0$. For $\xi^a_{12} = \langle D_1 e_2 - D_2 e_1, e_1 \rangle = \langle D_1 e_2, e_1 \rangle$ and $\xi^b_{12} = \langle D_1 e_2 - D_2 e_1, e_2 \rangle = -\langle D_2 e_1, e_2 \rangle = \langle D_2 e_1, e_2 \rangle$. So, $\xi^a_{12} = \langle D_a e_2, e_1 \rangle$ and

$$e_4(\xi^a_{12}) = e_4(\langle D_a e_2, e_1 \rangle) = \langle D_4 D_a e_2, e_1 \rangle + \langle D_a e_2, D_4 e_1 \rangle$$

$$= \langle D_a D_4 e_2, e_1 \rangle + \langle D_4 e_2, e_1 \rangle + \langle R(e_4, e_2) e_2, e_1 \rangle + \Gamma^0_{\alpha\beta} e_4^a e_4^\beta \langle D_a e_2, e_4 \rangle$$

$$= \Gamma^0_{\alpha\beta} e_4^a e_4^\beta \chi_{12} - \Gamma^0_{\alpha\beta} e_4^a e_1^\beta \chi_{a2} - \chi_{ac} \xi_{12}^c + \langle R(e_4, e_2) e_2, e_1 \rangle.$$
We end with \([e_a, e_3]\). Note that
\[
\xi^3_{a3} = \frac{1}{2} \langle D_a e_3 - D_3 e_a, e_4 \rangle = -\frac{1}{2} \langle e_3, D_a e_4 \rangle + \frac{1}{2} \langle e_a, D_3 e_4 \rangle \\
= -\frac{1}{2} \xi^4_{a3} - \frac{1}{2} \langle e_3, D_4 e_a \rangle + \frac{1}{2} \xi^a_{34} + \frac{1}{2} \langle e_a, D_4 e_3 \rangle = -\langle e_3, D_4 e_a \rangle + \frac{1}{2} \xi^a_{34} \\
= -2\Gamma^0_{\alpha\beta} e_4^0 e_a^\beta + \frac{1}{2} \xi^a_{34},
\]
\[
\xi^a_{33} = \langle D_a e_3 - D_3 e_a, e_a \rangle = \langle D_a e_3, e_a \rangle = \chi_{aa} + \langle D_a (2g^{0\alpha}\partial_\alpha), e_a \rangle \\
= \chi_{aa} + 2e_a (g^{0\alpha} g_{\alpha\beta} e_a^\beta + 2g^{0\alpha} e_a^\gamma \Gamma^\mu_\beta_\gamma g_{\mu\nu} e_\nu).\]

For \(\xi^b_{a3}\) where \(a \neq b\), we have
\[
e_4(\xi^b_{a3}) = e_4(\langle D_a e_3 - D_3 e_a, e_b \rangle) = e_4(\chi_{ab} + \langle D_a (2g^{0\alpha}\partial_\alpha), e_b \rangle - \langle D_3 e_a, e_b \rangle) \\
= e_4(\chi_{ab} + 2e_a (g^{0\alpha} g_{\alpha\beta} e_b^\beta + 2g^{0\alpha} e_a^\gamma \Gamma^\mu_\beta_\gamma g_{\mu\nu} e_\nu)) - \langle D_4 D_3 e_a, e_b \rangle - \langle D_3 D_4 e_a, e_b \rangle - \langle D_{[e_4, e_3]} e_a, e_b \rangle \\
= \langle R(e_4, e_3) e_a, e_b \rangle - \Gamma^0_{\alpha\beta} e_4^0 e_3^\beta \langle D_3 e_a, e_4 \rangle \\
= \langle e_4 + \Gamma^0_{\mu\nu} e_4^\mu e_4^\nu \rangle \chi_{ab} + 2e_a (g^{0\alpha} g_{\alpha\beta} e_b^\beta + 2g^{0\alpha} e_a^\gamma \Gamma^\mu_\beta_\gamma g_{\mu\nu} e_\nu) - \Gamma^0_{\mu\nu} e_4^\mu e_4^\nu e_3^b - \sum_c \xi^c_{34} e^c_e a^d.
\]

Given \(\xi\), we can express \(e_k (e^a_k)\) in terms of \(e^*_a\) and \(\xi^*_a\). In fact, the formulas for \(e_4 (e^a_k)\) follow from Lemma 3.3. Besides,
\[
e_k (e^4_a) = [e_k, e^4_a] + e_4 (e^a_k) = \xi_{k4} e^a_i + e_4 (e^a_k), \\
e_k (e^3_a) = [e_k, e^3_a] + 2e_k (g^{0\alpha}), \\
e_3 (e^a_k) = [e_3, e^a_k] + e_3 (e^a_k) = \xi_{3k} e^a_i + e_3 (e^a_k), \\
e_a (e^b_k) = (D_a e_b) e^a_k = e_a e^b_k \Gamma^\mu_\gamma g_{\mu\nu} \\
= \sum_e \langle D_a e_b, e_e \rangle e^a_c + \frac{1}{2} \langle D_a e_b, e_3 \rangle e^a_4 + \frac{1}{2} \langle D_a e_b, e_4 \rangle e^a_3 - e^a_e e^b_k \Gamma^\mu_\gamma.
\]
\[
= \sum_c \xi^a_e e^a_c - \frac{1}{2} \chi_{ab} (e^a_4 + e^a_3) - \langle e_b, D_a (g^{0\beta} \partial_\beta) \rangle e^a_4 - e^a_e e^b_k \Gamma^\mu_\gamma \\
= -\sum_c \xi^a_e e^a_c - \frac{1}{2} \chi_{ab} (e^a_4 + e^a_3) - (e^a_\mu g_{\mu\beta} e_a (g^{0\beta}) + e^a_\mu g_{\mu\nu} g^{0\beta} e^a_\gamma \Gamma^\nu_\beta) e^a_4 - e^a_e e^b_k \Gamma^\mu_\gamma.
\]

4.1.2. A weighted null frame. A new frame \(\{V_k\}\) defined below turns out to be very useful in this section.

**Definition.** We define a new frame \(\{V_k\}_{k=1}^4\) by \(V_a = re_a\) for \(a = 1, 2\) and \(V_3 = (3R-r+t)e_3\) and \(V_4 = te_4\). We call \(\{V_k\}_{k=1}^4\) a **weighted null frame**, since \(V_k\) is a multiple of \(e_k\) for each \(k\).

As usual, for each multiindex \(K = (k_1, \ldots, k_m)\) with \(k_\ast \in \{1, 2, 3, 4\}\), we define \(V^I = V_{k_1} \cdots V_{k_m}\) as the product of \(|I|\) vector fields.
It is easy to see that

\[
\begin{align*}
V_4 &= (t+r)^{-1}S + (t+r)^{-1}t\omega_j\Omega_{0j} + t(e_4^j - \omega^i)\partial_i, \\
V_3 &= (3R-r+t)\partial^{-1}V_4 + 2g^{0\alpha}(3R-r+t)\partial_\alpha, \\
V_a &= V_a(r)\omega_i\partial_i + e^i_\alpha\omega_j\Omega_{ji};
\end{align*}
\]

(4.1)

\[
Z = r^{-1}\sum_a (Z,e_a)V_a + \frac{1}{2}t^{-1}(Z,e_3)V_4 + \frac{1}{2}(3R-r+t)^{-1}(Z,e_4)V_3.
\]

(4.2)

These formulas illustrate the connection between the weighted null frame and the commuting vector fields.

Here we briefly explain why we work with \(\{V_k\}\). First, we note that

\[
Z \approx \sum_{k \neq 3} O(t)e_k + O((r-t))e_3 \approx \sum_k O(1)V_k.
\]

If we work with a usual null frame, then in order to prove \(Z^Iq = O((q)t^{C_\varepsilon})\), we might need to prove

\[
|e_I(q)| \lesssim (r-t)^{1-n_{I,*}^1 - n_{I,1} - n_{I,2} - n_{I,4} + C_\varepsilon}
\]

(4.3)

where \(e_I\) and \(n_{I,*}\) are defined at the beginning of Section 4.1. In contrast, if we work with a weighted null frame, then we can prove

\[
|V^Iq| \lesssim (r-t)t^{C_\varepsilon}.
\]

(4.4)

Since (4.3) is much more complicated than (4.4), we expect the proof to be much simpler if we choose to work with the new weighted null frame.

Next, to prove an estimate for \(V^Iq\), we need to compute

\[
e_4(V^Iq) = t^{-1}\sum_{I=(j,j',j'')} V^J[V_4,V_j]V^{j'}q.
\]

Since \(V_k\) is a multiple of \(e_k\) for each \(k\), we expect \([V_4,V_k]\) to be relatively simple. If we choose to work with the commuting vector fields defined in (2.1), then we need to compute either \([e_4,Z]\) or \([V_4,Z]\). Neither of these two terms has a simple form.

4.2. Estimates on \(H\). We start with the estimates on the surface \(H\). Recall that the vector fields \(X_i = \partial_i + 2\omega_i\partial_t\) are tangent to \(H\) for \(i = 1,2,3\). For a multiindex \(I = (i_1,\ldots,i_m)\) where \(i_j \in \{1,2,3\}\), we write \(X^I = X_{i_1}\cdots X_{i_m}\) and \(|I| = m\).

In this subsection, we keep using the convention stated in Section 2.4.

We have the following pointwise estimate. We ask our readers to compare this lemma with Lemma 2.2.

**Lemma 4.2.** Suppose that \(F = F(t,x)\) is a smooth function whose domain is contained in \(\{(t,x) \in \mathbb{R}^{1+3} : r \sim t \gtrsim 1\}\). Then, for nonnegative integers \(m,n\), we have

\[
\sum_{|I|=m, |J|=n} |Z^IX^JF| \lesssim (r-t)^{-n} \sum_{|I| \leq m+n} |Z^IF|.
\]
Proof. We induct first on $m + n$ and then on $n$. There is nothing to prove when $n = 0$. If $m = 0$ and $n = 1$, we simply apply Lemma 2.2. In general, we fix multiindices $I, J$ such that $|I| = m$ and $|J| = n$, such that $m + n > 1$ and $n > 0$. We can write $X^I = X^J X^J$. Then, by our induction hypotheses, we have

$$|Z^I X^J F| \leq |Z^I X^J \partial_I F| + |Z^I X^J (\omega_j \partial_I F)|$$

$$\lesssim \langle r-t \rangle^{1-n} \sum_{|K| \leq n+m-1} (|Z^K \partial F| + |Z^K(\omega_j \partial_I F)|).$$

Since $Z^K \omega = O(1)$ for each $|K| \geq 0$, by the Leibniz’s rule we have

$$|Z^I X^J F| \lesssim \langle r-t \rangle^{-n} \sum_{|K| \leq n+m-1} |Z^K F|$$

In the second inequality here we use the commutation property $[Z, \partial] = C \partial$.

The next lemma is a variant of Lemma 2.5 with $Z$ replaced by $X$. Note that we do not need to assume that $(m_0^{\alpha \beta})$ satisfies the null condition defined in Section 2.

**Lemma 4.3.** Fix two functions $\phi(t, x)$ and $\psi(t, x)$. Let $(m_0^{\alpha \beta})$ be a constant matrix. Then,

$$X_i (m_0^{\alpha \beta} \phi \psi) = m_0^{\alpha \beta}(\partial_\alpha X_i \phi) \psi + m_0^{\alpha \beta} \phi (\partial_\beta X_i \psi) + r^{-1} \sum_{\alpha, \beta} f_0 \phi \psi.$$

Here $f_0$ denotes a polynomial of $\omega$; we allow $f_0$ to vary from line to line.

**Proof.** We have $[X_i, \partial_\alpha] = -2(\partial_\alpha \omega_i) \partial_i$. By the Leibniz’s rule, we have

$$X_i (m_0^{\alpha \beta} \phi \psi) = m_0^{\alpha \beta}(\partial_\alpha X_i \phi) \psi + m_0^{\alpha \beta} \phi (\partial_\beta X_i \psi) - 2m_0^{\alpha \beta}(\partial_\alpha \omega_i) \phi \psi + 2m_0^{\alpha \beta}(\partial_\beta \omega_i) \phi \psi$$

$$= m_0^{\alpha \beta}(\partial_\alpha X_i \phi) \psi + m_0^{\alpha \beta} \phi (\partial_\beta X_i \psi) - 2r^{-1}[m_0^{\alpha \beta}(\delta_{ji} - \omega_j \omega_i) \phi \psi + m_0^{\alpha \beta}(\delta_{ji} - \omega_j \omega_i) \psi \phi]$$

$$= m_0^{\alpha \beta}(\partial_\alpha X_i \phi) \psi + m_0^{\alpha \beta} \phi (\partial_\beta X_i \psi) + r^{-1} \sum_{\alpha, \beta} f_0 \phi \psi.$$

Using the previous two lemmas, we can now prove the estimates for $Z^I q$ on $H$. In the next two lemmas, $\Omega^I$ denotes the product of $|I|$ vector fields in $\{\Omega_{12}, \Omega_{23}, \Omega_{13}\}$. In the rest of Section 4.2 we would use $\Omega$ to denote any vector field in $\{\Omega_{12}, \Omega_{23}, \Omega_{13}\}$ instead of the region. There should be no confusion as we focus on estimates on $H$.

**Lemma 4.4.** On $H$, for all multiindices $I$, we have $Z^I q = O(\langle q \rangle t^{C \epsilon})$ and $Z^I \Omega q = O(t^{C \epsilon})$.

**Proof.** For convenience, we set

$$O_{m,n,p} = O_{m,n,p}(t, x) := \sum_{|I|=m, |J|=n, |K|=p} |Z^I X^J \Omega^K q|.$$

On $H$, we claim that

$$O_{m,n,0} \lesssim \langle q \rangle^{1-n} t^{C \epsilon}, \forall m, n \geq 0; \quad O_{m,n,p} \lesssim \langle q \rangle^{-n} t^{C \epsilon}, \forall m, n \geq 0, p > 0.$$
We first assume $m = 0$. Since $\Omega$ and $X$ are tangent to $H$ and since $q|_H = r - t$, we have $X^J\Omega^K q = X^J\Omega^K (r - t)$ for all multiindices $J, K$. If $|K| > 0$, we have $X^J\Omega^K (r - t) = 0$; if $|J| > 0$, we have $X^J (r - t) = O(r^{-1-|J|}) = O(\langle q \rangle^{-1-|J|})$. Then, on $H$ we have $O_{0,0,0} = |q|$, $O_{0,n,p} = 0$ for $p > 0$, and $O_{0,n,0} = O(\langle q \rangle^{-n})$ for $n > 0$. So the claim is true for $m = 0$.

In general, we fix $(m, n, p)$ with $m > 0$. Suppose we have proved

\begin{equation}
O_{m', n', 0} \lesssim \langle q \rangle^{1-n'} t^{C\varepsilon}, \quad \forall m', n' \geq 0 \text{ such that } m' + n' < m + n + p
\end{equation}

or $m' + n' = m + n + p$, $m' < m$;

\begin{equation}
O_{m', n', p} \lesssim \langle q \rangle^{-n'} t^{C\varepsilon}, \quad \forall m', n' \geq 0, p' > 0 \text{ such that } m' + n' + p' < m + n + p
\end{equation}

or $m' + n' + p' = m + n + p$, $m' < m$.

From now on, we fix three multiindices $I, J, K$ such that $|I| = m$, $|J| = n$, and $|K| = p$.

We write $Z^I = ZZ^I$ and apply $Z^I X^J \Omega^K$ to the eikonal equation. We have

$$0 = 2g^{\alpha \beta} q_{\beta} (\partial_\alpha Z^I X^J \Omega^K q) + R_1 + R_2 + R_3$$

where the remainders are given by

\begin{align*}
R_1 &= Z^I X^J \Omega^K (m^{\alpha \beta} q_{\alpha} q_{\beta}) - 2m^{\alpha \beta} (\partial_\alpha Z^I X^J \Omega^K q) q_{\beta}, \\
R_2 &= Z^I X^J \Omega^K ((g^{\alpha \beta} - m^{\alpha \beta}) q_{\alpha} q_{\beta}) - 2(g^{\alpha \beta} - m^{\alpha \beta}) q_{\beta} (Z^I X^J \Omega^K q), \\
R_3 &= 2(g^{\alpha \beta} - m^{\alpha \beta}) q_{\beta} (Z^I X^J \Omega^K q - \partial_\alpha Z^I X^J \Omega^K q)
\end{align*}

We start with $R_3$. Recall that $g - m = O(\varepsilon t^{-1+C\varepsilon})$ and $q_{\beta} = O(1)$ on $H$. Besides, $Z^I X^J \Omega^K q_{\alpha} - \partial_\alpha Z^I X^J \Omega^K q$ is a linear combination of terms of the following forms

\begin{align*}
Z^I [Z, \partial_\alpha] Z^J X^K q &= CZ^I \partial Z^J X^K q, \\
Z^I X^{J_1} [X, \partial_\alpha] X^{J_2} \Omega^K q &= CZ^I X^{J_1} ((\partial_\alpha \omega) \partial_\alpha X^{J_2} \Omega^K q), \\
Z^I X^J \Omega^K [\Omega, \partial_\alpha] \Omega^{K_2} q &= CZ^I X^J \Omega^K_1 \partial \Omega^{K_2} q, \\
\Omega^{K_1} \Omega^{K_2} &= \Omega^K.
\end{align*}

The first row has an upper bound

$$\sum_{|K'| \leq |I_1| + |J_2|} |\partial Z^{K'} X^{J'} \Omega^{K'} q| \lesssim \langle r - t \rangle^{-1} \sum_{|K'| \leq m - 1} |Z^{K'} X^{J} \Omega^K q| = \langle q \rangle^{-1} \sum_{m' \leq m - 1} O_{m', n, p} \lesssim \langle q \rangle^{-n} t^{C\varepsilon} \lesssim \langle q \rangle^{-n} t^{C\varepsilon}.$$

We can use the induction hypotheses (4.5) to control the sum $\sum_{m' \leq m - 1} O_{m', n, p}$, since $m' + n + p \leq m - 1 + n + p < m + n + p$. The second row has an upper bound

$$\sum_{|I_1| + |J_2| = m - 1} |Z^I X^{J_1} \partial \omega| \cdot |Z^{J_2} X^{J_2} \partial X^{J_2} \Omega^K q| \lesssim \langle q \rangle^{-1-n} \sum_{m' \leq m - 1 + n} O_{m', 0, p} \lesssim \langle q \rangle^{-n} t^{C\varepsilon}.$$
In the first inequality we apply Lemma 2.2 and Lemma 4.2. In the second line, we apply (4.5). The third row has an upper bound
\[
(r-t)^{-n} \sum_{|K'| \leq m-1+n} |Z^{K'} \Omega^{K_1} \partial \Omega^{K_2} q| \lesssim (r-t)^{-1-n} \sum_{|K'| \leq m-1+n+|K_1|+1} |Z^{K'} \Omega^{K_2} q|
\lesssim \langle q \rangle^{-1-n} \sum_{m' \leq m-1+n+p} O_{m',0,0} \lesssim \langle q \rangle^{-n} t^{C_\varepsilon}.
\]

In conclusion, \( \mathcal{R}_3 = O(\varepsilon t^{-1+C_\varepsilon} \langle q \rangle^{-n}) \).

We move on to \( \mathcal{R}_2 \). By the Leibniz’s rule, we can express \( \mathcal{R}_2 \) as a linear combination of terms of the form
\[
Z^{I_1} X^{J_1} \Omega^{K_1} (g^{\alpha \beta} - m^{\alpha \beta}) \cdot Z^{I_2} X^{J_2} \Omega^{K_2} q_{\alpha} \cdot Z^{I_3} X^{J_3} \Omega^{K_3} q_{\beta},
\]
where \( \sum |I_i| = m - 1, \sum |J_i| = n, \sum |K_i| = p, \max_{i=2,3} \{|I_i| + |J_i| + |K_i|\} < m + n + p - 1 \).

On \( H \), by Lemma 4.2 and (4.5) we have
\[
|Z^{I_2} X^{J_2} \Omega^{K_2} q_{\alpha}| \lesssim \langle q \rangle^{-|J_2|} \sum_{|K'| \leq |I_2|+|J_2|+|K_2|} |Z^{K'} q_{\alpha}| \lesssim \langle q \rangle^{-|J_2|-1} \sum_{|K'| < m+n+p} |Z^{K'} q_{\alpha}| \lesssim \langle q \rangle^{-|J_2|} t^{C_\varepsilon}.
\]
We can estimate \( Z^{I_3} X^{J_3} \Omega^{K_3} q_{\beta} \) in the same way. And since \( Z^{I_1} X^{J_1} \Omega^{K_1} (g^{\alpha \beta} - m^{\alpha \beta}) = O(\varepsilon \langle q \rangle^{-|J_1|} t^{-1+C_\varepsilon}) \) by Lemma 4.2 we conclude that \( \mathcal{R}_2 = O(\varepsilon \langle q \rangle^{-n} t^{-1+C_\varepsilon}) \) on \( H \).

We move on to \( \mathcal{R}_1 \). By Lemma 2.5, we can write \( \Omega^K (m^{\alpha \beta} q_{\alpha} q_{\beta}) \) as a linear combination (with real constant coefficients) of terms of the form
\[
(4.6) \quad m^{\alpha \beta} (\partial_\alpha \Omega^{K_1} q) (\partial_\beta \Omega^{K_2} q), \quad \min\{1,p\} \leq |K_1| + |K_2| \leq p.
\]
Here \( m^{\alpha \beta} \) is the usual Minkowski metric. In fact, if \( p = 0 \), then (4.6) is \( m^{\alpha \beta} q_{\alpha} q_{\beta} \) so there is nothing to prove; if \( p > 0 \), then we guarantee that \( |K_1| + |K_2| > 0 \) in (4.6) since
\[
\Omega^K (m^{\alpha \beta} q_{\alpha} q_{\beta}) = \Omega^{K'} (m^{\alpha \beta} (\partial_\alpha \Omega) q_{\beta} + m^{\alpha \beta} q_{\alpha} (\partial_\beta \Omega)), \quad \Omega^K = \Omega^{K'} \Omega.
\]
Next we consider \( X^J \Omega^K (m^{\alpha \beta} q_{\alpha} q_{\beta}) \), so we apply \( X^J \) to (4.6). By Lemma 4.3 we can write \( X^J \Omega^K (m^{\alpha \beta} q_{\alpha} q_{\beta}) \) as a linear combination (with real constant coefficients) of terms of the form
\[
\left\{ \begin{array}{ll}
m^{\alpha \beta} (\partial_\alpha X^{J_1} \Omega^{K_1} q) (\partial_\beta X^{J_2} \Omega^{K_2} q), & |J_1| + |J_2| = n, \\
& \min\{1,p\} \leq |K_1| + |K_2| \leq p;
\end{array} \right.
\]
\[
X^{J_1} (r^{-1} f_0) \cdot (X^{J_2} \partial X^{J_2} \Omega^{K_1} q) (X^{J_1} \partial X^{J_2} \Omega^{K_2} q), \quad \sum |J_4| + |J_5| = n - 1, \\
& \min\{1,p\} \leq |K_1| + |K_2| \leq p.
\]
Again \( m^{\alpha \beta} \) is the Minkowski metric. We finally apply \( Z^\nu \) to each of these terms. By Lemma 2.5 and the Leibniz’s rule, we can write \( \mathcal{R}_1 \) as a linear combination (with real constant coefficients) of terms of the form
\[
(4.7) \quad \left\{ \begin{array}{ll}
m^{\alpha \beta} (\partial_\alpha Z^{I_1} X^{J_1} \Omega^{K_1} q) (\partial_\beta Z^{I_2} X^{J_2} \Omega^{K_2} q), & |I_1| + |I_2| \leq m - 1, |J_1| + |J_2| = n, \\
& \min\{1,p\} \leq |K_1| + |K_2| \leq p
\end{array} \right.
\]
\[
Z^{I_3} X^{J_3} (r^{-1} f_0) \cdot (Z^{I_1} X^{J_1} \partial X^{J_1} \Omega^{K_1} q) (Z^{I_2} X^{J_2} \partial X^{J_2} \Omega^{K_2} q), \quad \sum |I_4| = m - 1, \quad \sum |J_4| + |J_5| = n - 1, \\
& \min\{1,p\} \leq |K_1| + |K_2| \leq p.
\]
Here \((m_0^{\alpha\beta})\) is some constant matrix satisfying the null condition defined in Section 2. It follows from Lemma 2.6 that on \(H\) the terms of the first type in (4.7) has an upper bound
\[
\langle t \rangle^{-1} \sum_{|L| = 1} (|ZZ^I X^I X^J \Omega^K q| + |\partial Z^I X^I X^J \Omega^K q|)|Z^{L'} X^J \Omega^K q|)
\]
\[
\lesssim t^{-1} \langle q \rangle^{-1} \sum_{|L_1| = |L_2| = 1} |Z^{L_1} Z^I X^J \Omega^K q||Z^{L_2} Z^I X^J \Omega^K q| \lesssim t^{-1} \langle q \rangle^{-1} \mathcal{O}_{1 + |I_1|, |J_1|, |K_1|} \mathcal{O}_{1 + |I_2|, |J_2|, |K_2|}.
\]
Since \(\min_{t=1,2} \{|I| + |J| + |K| + 1\} < m + n + p\) and since \(|J_1| + |J_2| = n\), we can apply (4.5) to conclude that on \(H\)
\[
|m_0^{\alpha\beta} (\partial_\alpha Z^I X^I \Omega^K q)(\partial_\beta Z^I X^J \Omega^K q)| \lesssim t^{-1} C_\varepsilon \langle q \rangle^{1-n}, \quad \text{if } p = 0;
\]
\[
|m_0^{\alpha\beta} (\partial_\alpha Z^I X^I \Omega^K q)(\partial_\beta Z^I X^J \Omega^K q)| \lesssim t^{-1} C_\varepsilon \langle q \rangle^{1-n}, \quad \text{if } p > 0.
\]
Meanwhile, by Lemma 4.2 and (4.3), on \(H\) we have
\[
|Z^I X^J (r^{-1} f_0)| \lesssim t^{-1} C_\varepsilon \langle q \rangle^{1-n} |J_3|,
\]
\[
|Z^I X^J \partial X^J \Omega^K q| \lesssim \langle q \rangle^{-1-|I| - |J|} \sum_{m'} \mathcal{O}_{m', 0, |K|},
\]
\[
Z^I X^J \partial X^J \Omega^K q \lesssim \langle q \rangle^{-1-|J| - |J|} \sum_{m'} \mathcal{O}_{m', 0, |K|}.
\]
Here we can apply (4.6) as \(\max_{t=1,2} \{|I| + |J| + |J'| + |K| + 1\} < m + n + p\). Thus, the product of these terms is \(O(t^{-1} C_\varepsilon \langle q \rangle^{1-n})\) if \(p = 0\), or \(O(t^{-1} C_\varepsilon \langle q \rangle^{-n})\) if \(p > 0\). Thus, on \(H\) we have \(R_1 = O(t^{-1} C_\varepsilon \langle q \rangle^{1-n})\) if \(p = 0\), and \(R_1 = O(t^{-1} C_\varepsilon \langle q \rangle^{-n})\) if \(p > 0\). In conclusion, we have
\[
2g^{\alpha\beta} q_\beta (\partial_\alpha Z^I X^J \Omega^K q) = O(t^{-1} C_\varepsilon \langle q \rangle^{1-n}), \quad \text{if } p = 0;
\]
\[
2g^{\alpha\beta} q_\beta (\partial_\alpha Z^I X^J \Omega^K q) = O(t^{-1} C_\varepsilon \langle q \rangle^{-n}), \quad \text{if } p > 0.
\]
Next, we note that
\[
X^I Z^J X^I \Omega^K q = Z^I X^J X^I \Omega^K q + \sum_{I' = (I_1, i, I_2)} Z^I X^J X^I \Omega^K q + \sum_{I' = (I_1, i, I_2)} Z^I X^J X^I \Omega^K q,
\]
\[
\Omega_{kk'} Z^I X^J \Omega^K q = Z^I X^J \Omega_{kk'} \Omega^K q + \sum_{I' = (I_1, i, I_2)} Z^I X^J \Omega_{kk'} \Omega^K q + \sum_{I' = (I_1, i, I_2)} Z^I X^J \Omega_{kk'} \Omega^K q
\]
\[
+ \sum_{J = (J_1, J_2)} Z^I X^I \Omega_{kk'} X^J X^J \Omega^K q.
\]
Recall that \([\Omega, Z] = \sum f_0 Z\) and \([X, Z] = \sum f_0 \partial\) where \(f_0\) denotes any function such that \(Z^{K'} f_0 = O(1)\) for all \(K'\). By Lemma 2.2 we have
\[
|X^I Z^J X^I \Omega^K q| \lesssim \mathcal{O}_{m-1, n+1, p} + \sum_{I' = (I_1, i, I_2)} |Z^I_1 (f_0 \partial Z^I_2 X^I \Omega^K q)|
\]
\[
\lesssim \mathcal{O}_{m-1, n+1, p} + \langle q \rangle^{-1} \sum_{m' \leq m-1} \mathcal{O}_{m', n, p},
\]
47
\[
|\Omega_{kk'}Z'X^J\Omega^Kq| 
\lesssim \mathcal{O}_{m-1,n,p+1} + \sum_{l'=(I_1,i_1),l_2} \left|Z^{I_1}(f_0ZZ^JX^J\Omega^Kq)\right| + \sum_{J=(J_1,J_2)} \left|Z^{I_1}(f_0\partial X^J_2\Omega^Kq)\right|
\]
\[
\lesssim \mathcal{O}_{m-1,n,p+1} + \sum_{m'\leq m-1} \mathcal{O}_{m',n,p} + \sum_{|J_1|+|J_2|=n-1} \langle q \rangle^{-|J_1|} ||Z^{I_1}Z^{I_2}(f_0\partial X^J_2\Omega^Kq)||
\]
\[
\lesssim \mathcal{O}_{m-1,n,p+1} + \sum_{m'\leq m-1} \mathcal{O}_{m',n,p} + \langle q \rangle^{-n} \sum_{m'\leq m+n-1} \mathcal{O}_{m',0,p}.
\]

In conclusion, on \( H \) we have
\[
|XZ^J\Omega^Kq| \lesssim \langle q \rangle^{-ntC\varepsilon}, \quad \text{if } p = 0;
\]
\[
|XZ^J\Omega^Kq| \lesssim \langle q \rangle^{-1-n}tC\varepsilon, \quad \text{if } p > 0;
\]
\[
|\Omega Z^J\Omega^Kq| \lesssim \langle q \rangle^{1-n}tC\varepsilon, \quad \text{if } p = 0;
\]
\[
|\Omega Z^J\Omega^Kq| \lesssim \langle q \rangle^{-ntC\varepsilon}, \quad \text{if } p > 0.
\]

We now end the proof. By setting \( L^n = 2g^{n3}q_3 \) and \( L = L^n \partial_n \), we have
\[
\partial_t = \frac{L - L^nX^n}{L^n - 2\omega^nL^n} = \frac{1}{2} L + \sum_i \omega_i X_i + O(|u|)L + \sum_i O(|u|)X_i,
\]
\[
\partial_J = X_J - 2\omega_J \partial_t = \omega_J L + X_J - 2\omega_J \sum_i \omega_i X_i + O(|u|)L + \sum_i O(|u|)X_i.
\]

Note that \( L^n = 2 + O(|u|) \) and \( L^i = 2\omega_i + O(|u|) \) on \( H \). Then, we have
\[
S = (-\frac{1}{2} t + r)L + (t - r) \sum_i \omega_i X_i + O((r + t)|u|)L + \sum_i O((r + t)|u|)X_i
\]
\[
= O(t + \varepsilon tC\varepsilon)L + \sum_i O(\langle q \rangle + \varepsilon tC\varepsilon)X_i.
\]

And since \( \Omega_{kk'} = x_k x_{k'} - x_k x_{k'} \), we have \( \sum_k r^{-1} \omega_k \Omega_{k,k'} = X_{k'} - \sum_k \omega_{k'} \omega_k X_k \). Thus,
\[
\Omega_{ij} = (-\frac{1}{2} x_j + t\omega_j)L + tX_j + (x_j - 2t\omega_j) \sum_i \omega_i X_i + O((r + t)|u|)L + \sum_i O((r + t)|u|)X_i
\]
\[
= t(X_j - \omega_J \omega_i X_i) + O(t + \varepsilon tC\varepsilon)L + \sum_i O(\langle q \rangle + \varepsilon tC\varepsilon)X_i
\]
\[
= tr^{-1} \sum_i \omega_i \Omega_{ij} + O(t + \varepsilon tC\varepsilon)L + \sum_i O(\langle q \rangle + \varepsilon tC\varepsilon)X_i.
\]

In conclusion, for each \( Z \in \{\partial_n, S, \Omega_{ij}\} \), we have
\[
|ZZ^J\Omega^Kq| \lesssim \sum_{1 \leq i < j \leq 3} |\Omega_{ij}ZZ^J\Omega^Kq| + t|LZ^J\Omega^Kq| + \langle q \rangle + tC\varepsilon \sum_i |X_iZ^J\Omega^Kq|.
\]

If \( p = 0 \), the right hand side has an upper bound \( \langle q \rangle^{1-n}tC\varepsilon \); if \( p > 0 \), the right hand side has an upper bound \( \langle q \rangle^{-ntC\varepsilon} \). We finish the proof by induction. \( \square \)

**Lemma 4.5.** On \( H \), we have \( Z^I(q_i - \omega_i q_r) = O(t^{-1+C\varepsilon}) \) and \( Z^I(q_i + q_r) = O(\varepsilon t^{-1+C\varepsilon}) \) for each \( I \). As a result, \( Z^I(q_i + \omega_i q_r) = O(t^{-1+C\varepsilon}) \).
Proof. Recall that $q_t - \omega_i q_r = \sum_j r^{-1} \omega_j \Omega_j q$. By Lemma 4.4 and the Leibniz’s rule, for each $I$ we have

$$|Z^I(r^{-1} \omega_j \Omega_j q)| \lesssim \sum_{|I_1| = |I_2| = |I|} |Z^{I_1}(r^{-1} \omega_j)| \cdot |Z^{I_2} \Omega q| \lesssim t^{-1+C\varepsilon}.$$ 

So $Z^I(q_t - \omega_i q_r) = O(t^{-1+C\varepsilon})$. Moreover, by the eikonal equation we have

$$-(q_t + q_r)(q_t - q_r) + \sum_i (q_t - \omega_i q_r)^2 + (g^\alpha(u) - m^\alpha \beta)q_\alpha q_\beta = 0,$$

so

$$q_t + q_r = \frac{\sum_i (q_t - \omega_i q_r)^2 + (g^\alpha(u) - m^\alpha \beta)q_\alpha q_\beta}{q_t - q_r}.$$ 

Thus, $Z^I(q_t + q_r)$ is a linear combination of terms of the form

$$(q_t - q_r)^{-1-s} \cdot Z^{I_1}(q_t - q_r) \cdots Z^{I_s}(q_t - q_r) \cdot Z^{I_0}(\sum_i (q_t - \omega_i q_r)^2 + (g^\alpha(u) - m^\alpha \beta)q_\alpha q_\beta)$$

where $\sum |I_i| = |I|$. It is clear that $Z^{I_1}(q_t - q_r) = O(t^{C\varepsilon})$ and that $q_t - q_r = -2 + O(\varepsilon t^{-1+C\varepsilon}) \leq -1$ on $H$. Moreover, since $Z^I(r^{-1} \Omega q) = O(t^{-1+C\varepsilon})$ for each $I$, we have $Z^{I_0}((q_t - \omega_i q_r)^2) = O(t^{-2+C\varepsilon})$. Finally, for each $I$ we have

$$|Z^I((g^\alpha - m^\alpha \beta)q_\alpha q_\beta)| \lesssim \sum_{|I_1| = |I_2| = |I|} |Z^{I_1}(g - m)||Z^{I_2} \partial q||Z^{I_3} \partial q| \lesssim \varepsilon t^{-1+C\varepsilon}.$$ 

In conclusion, $Z^I(q_t + q_r) = O(t^{-2+C\varepsilon} + \varepsilon t^{-1+C\varepsilon}) = O(\varepsilon t^{-1+C\varepsilon})$, as $t \geq T_0 = \exp(\delta/\varepsilon)$. Since $q_i + \omega_i q_t = q_t - \omega_i q_r + \omega_i(q_t + q_r)$, we can easily show $Z^I(q_t + \omega_i q_t) = O(t^{-1+C\varepsilon})$ by the Leibniz’s rule.

We move on to estimates for $e^*_k$ and $\xi^*_a$ on $H$.

**Lemma 4.6.** On $H$, we have $Z^I e^\alpha_k = O(t^{C\varepsilon})$ and $Z^I (e^\alpha_a - \omega_i, e^i_a - \omega_i) = O(t^{-1+C\varepsilon})$ for each $I$.

**Proof.** Since $e^0_a = 1$, $e^0_a = -1$ and $e^0_0 = 0$, we can ignore the case $\alpha = 0$. We write

$$e^\alpha_a - \omega_i = (g^\mu_\alpha q_\mu)^{-1}(g^\beta_\alpha q_\beta - \omega_i g^\beta_\alpha q_\beta)$$

$$= (g^\mu_\alpha)^{-1}(q_\alpha + \omega_i q_\alpha + (g^\beta_\alpha - m^\beta)q_\beta - \omega_i (g^\beta_\alpha - m^\beta)q_\beta)$$

$$=: (g^\mu_\alpha)^{-1} Q.$$ 

By Lemma 4.4 and the Leibniz’s rule, we have

$$Z^I Q = O(t^{-1+C\varepsilon}), \quad Z^I (g^\mu_\alpha q_\mu) = O(t^{C\varepsilon}), \quad g^\mu_\alpha q_\mu = 1 + O(\varepsilon t^{-1+C\varepsilon}) \geq 1/2.$$ 

Besides, $Z^I(e^\alpha_a - \omega_i)$ is a linear combination of terms of the form

$$(g^\mu_\alpha q_\mu)^{-1-s} Z^{I_1}(g^\mu_\alpha q_\mu) \cdots Z^{I_s}(g^\mu_\alpha q_\mu) Z^{I_0} Q,$$

where $\sum |I_i| = |I|, |I_j| > 0$ for $j \neq 0$.

We conclude that $Z^I(e^\alpha_a - \omega_i) = O(t^{-1+C\varepsilon})$. Since $Z^I \omega = O(1)$ on $H$, we conclude that $Z^I e^\alpha_a = O(t^{C\varepsilon})$. And since $Z^I (e^\alpha_a - e^\alpha_a) = 2 Z^I g^\mu_\alpha = O(\varepsilon t^{-1+C\varepsilon})$, we conclude that $Z^I(e^\alpha_a - \omega_i) = O(t^{-1+C\varepsilon})$ and $Z^I e^\alpha_a = O(t^{C\varepsilon})$ on each $H$. The proofs of these estimates do not rely on the estimates for $Z^I e^*_a$, so we can use them freely in the following proof.

Next, we claim that $Z^I X^I \Omega^K e^i_a = O(|I|^4 |t^{C\varepsilon})$ on $H$ for all $I, J, K$ and $a = 1, 2$. Recall that $\Omega^K$ is the product of $|K|$ vector fields in $\{\Omega_2, \Omega_{23}, \Omega_{13}\}$. We induct first on $|I| + |J| + |K|$, and then on $|I|$. When $|I| + |J| + |K| = 0$, there is nothing to prove. When $|I| = 0$ and
$|J| + |K| > 0$, we have $X^J \Omega^K e^i_a = O(r^{-|K|})$ on $H$, since $e^i_a|_H$ is a locally defined function of $\omega$ and it is independent of $t$.

In general, we fix $I, J, K$ such that $|I| > 0$. Suppose we have proved the claim for all $(I', J', K')$ such that $|I'| + |J'| + |K'| < |I| + |J| + |K|$, or $|I'| + |J'| + |K'| = |I| + |J| + |K|$, and $|I'| < |I|$. We write $Z' = ZZ'$. For $a = 1, 2$ we have

$$Z' X^J \Omega^K e^i_a = Z' X^J \Omega^K (e^i_4 \Gamma_0 e^i_4 - e^i_4 e^i_4).$$

Since we can write $\Gamma = g \cdot \partial g$, for each $K'$, we have $Z^K' \Gamma = O(\varepsilon^{-1 + C\varepsilon}(q)^{-l})$ on $H$. By induction hypotheses, Lemma 4.2 and the Leibniz’s rule, we conclude that

$$Z' X^J \Omega^K e^i_a = O(\varepsilon^{-1 + C\varepsilon}q^{-|I|}).$$

Moreover, $Z' X^J \Omega^K e^i_a$ is equal to the sum of $e_4(Z' X^J \Omega^K e^i_a)$ and a linear combination of terms of the form

$$Z^I[e_4, Z^I] Z^J X^J \Omega^K e^i_a, \quad (I_1, I_2, I_3) = I', \quad |I_2| = 1;$$

$$Z^J X^J [e_4, X^J] X^J \Omega^K e^i_a, \quad (J_1, J_2, J_3) = J, \quad |J_2| = 1;$$

$$Z^I X^I \Omega^K [e_4, \Omega^K] \Omega^K e^i_a, \quad (K_1, K_2, K_3) = K, \quad |K_2| = 1.$$

Note that

$$[e_4, Z] = e_4(\nu^r) \partial_r - Z(e^r_4) \partial_r = e_4(\nu^r) \partial_r - Z(e^r_4 - \omega_j) \partial_r,$$

$$[e_4, X_l] = e_4(2\omega_l) \partial_l - X_l(e^r_4) \partial_l = 2r^{-1}(e^r_4 - \omega_l - (\omega_l - e^r_4)\omega_l) \partial_l - (\partial_l \omega_l) \partial_l - X_l(e^r_4 - \omega_l) \partial_l$$

where we write $Z = z'(t, x) \partial_r$. We have

$$e_4(\nu^r) \partial_r = Z(\omega_j) \partial_j = \begin{cases} -\partial(\omega_j) \partial_j, & Z = \partial; \\ (r + t)^{-1} S + (r + t)^{-1} \omega_l \Omega_{0l} + (\omega^r_4 - \omega_j) \partial_j, & Z = S; \\ r^{-1} \omega_j + (e^r_4 - \omega_l) \partial_l - (e^r_4 - \omega_j) \partial_l - r^{-1} \Omega_{ij}, & Z = \Omega_{ij}; \\ r^{-1} \omega_l + r^{-1}(t - r) \partial_l + (e^r_4 - \omega_l) \partial_l - t r^{-2} \omega_l \Omega_{li}, & Z = \Omega_{0l}. \end{cases}$$

In conclusion,

$$[e_4, Z] = f_1 \cdot Z, \quad [e_4, X] = f_1 \cdot \partial$$

where $f_1$ denotes any function satisfying $Z' f_1 = O(t^{-1 + C\varepsilon})$ for each $J'$ on $H$. Thus, the first row in (4.8) has an upper bound

$$|Z^I_1(f_1 Z Z^I_3 X^J \Omega^K e^i_a)| \lesssim \sum_{|J'| \leq |I|} t^{-1 + C\varepsilon} |Z' Z Z^I_3 X^J \Omega^K e^i_a| \lesssim t^{-1 + C\varepsilon} (q)^{-|I|}.$$

We can use the induction hypotheses here as

$$|J'| + 1 + |I_3| + |J| + |K| \leq |I_1| + 1 + |I_2| + |J| + |K| = |I'| + |J| + |K| < |I| + |J| + |K|.$$
We can use the induction hypotheses here as

$$|J'| + |J_3| + |K| \leq |I'| + |J| + |K| < |I| + |J| + |K|.$$  

The third row in (4.8) has an upper bound

$$|Z^'X^J\Omega^K_i(f_1Z\Omega^K_i e^i_a)| \lesssim \sum_{|J'| \leq |I'| + |J|} \langle q \rangle^{-|J|} |Z^'\Omega^K_i(f_1Z\Omega^K_i e^i_a)|$$

$$\lesssim \sum_{|J'| \leq |I'| + |J| + |K_1|} \langle q \rangle^{-|J|} t^{-1+C_\varepsilon} |Z^'Z\Omega^K_i e^i_a| \lesssim \langle q \rangle^{-|J|} t^{-1+C_\varepsilon}.$$

We can use the induction hypotheses here as

$$|J'| + |K_3| + 1 \leq |I'| + |J| + |K_1| + 1 + |K_3| = |I'| + |J| + |K| < |I| + |J| + |K|.$$  

In conclusion, on $H$ we have

$$e_4(Z^'X^J\Omega^K_i e^i_a) = Z^'X^J\Omega^K_i e_4(e^i_a) + O(t^{-1+C_\varepsilon}\langle q \rangle^{-|J|}) = O(t^{-1+C_\varepsilon}\langle q \rangle^{-|J|}).$$

We recall from the proof of Lemma 4.2 that $[Z,\Omega] = C \cdot Z$ and $[Z,X] = f_0 \cdot \partial$ where $f_0$ denotes any function such that $Z^K_i f_0 = O(t^{C_\varepsilon})$ on $H$ for each $K$. If we keep commuting $\Omega$ with each vector field in $Z^'X^J$ and applying the Leibniz's rule, we get $\Omega Z^'X^J\Omega^K_i e^i_a = O(t^{C_\varepsilon}\langle q \rangle^{-|J|})$. If we keep commuting $X_I$ with each vector field in $Z^I$ and applying the Leibniz's rule, we get $X_I Z^I X^J\Omega^K_i e^i_a = O(t^{C_\varepsilon}\langle q \rangle^{1-|J|})$. Finally, we recall from the proof of Lemma 4.2 that we can write

$$(\partial, S, \Omega_{0j}) = O(t)L + O(1) \cdot \Omega + O(\langle q \rangle + \varepsilon t^{C_\varepsilon}) \cdot X$$

where $L = 2g^{\alpha\beta}q_{\beta}\partial_{\alpha} = O(1)e_4$ on $H$. In conclusion, when $Z = \partial, S, \Omega_{0j}$, we have

$$|Z Z^'X^J\Omega^K_i e^i_a| \lesssim |t| e_4(Z^'X^J\Omega^K_i e^i_a) + |\Omega Z^'X^J\Omega^K_i e^i_a| + \langle q \rangle t^{C_\varepsilon} |X Z^'X^J\Omega^K_i e^i_a| \lesssim t^{C_\varepsilon}\langle q \rangle^{-|J|}.$$

We finish the proof by induction. \( \square \)

We now prove the following lemma which illustrates the connection between the weighted null frame and the commuting vector fields.

**Lemma 4.7.** Let $F = F(t,x)$ be a smooth function defined near $H$. Then, on $H$ we have

$$|V^J F| \lesssim \sum_{|I| \leq |J|} t^{C_\varepsilon} |Z^I F|.$$  

**Proof.** We induct on $|I|$. When $|I| = 0$, there is nothing to prove. Suppose we have proved the estimate for each function $F$ and for each multiindex $I'$ such that $|I'| < |I|$. Then, by writing $V^I = V^I V_k$ and applying the induction hypotheses, we have

$$|V^J F| \lesssim \sum_{|I| \leq |I| - 1} t^{C_\varepsilon} |Z^I (V_k F)|.$$

We then apply 4.11. When $k = 4$, we have $V_4 F = f_0 \cdot Z F$. Here $f_0$ denotes any function such that $Z^J f_0 = O(t^{C_\varepsilon})$ on $H$ for each $J$. In particular, since $Z^J(e^i_a - \omega_i) = O(t^{-1+C_\varepsilon})$
for each $J'$ by Lemma 4.6 we have $Z'^{(t)(e'_{i} - \omega_{i}))} = O(t^{-1+C\varepsilon})$ and thus $t(e'_{i} - \omega_{i}) = f_{0}$. By the Leibniz’s rule, we have

$$|V^{I}F| \lesssim \sum_{|J| \leq |I| - 1} t^{C\varepsilon} |Z^{J}(f_{0} \cdot ZF)| \lesssim \sum_{|J| \leq |I| - 1} t^{C\varepsilon} |Z^{J}ZF| \lesssim \sum_{|J| \leq |I|} t^{C\varepsilon} |Z^{J}F|.$$  

The proof for $k = 3$ follows from the case $k = 4$ and the estimate $Z'^{(t)(r - t)} = O((r - t))$ for all $J'$. Finally, when $k = a \in \{1, 2\}$, we note that

$$V_{a}(r) = re^{j}a_{j} = re_{a}^{j}(-g^{\alpha\beta} + m_{\alpha\beta})e^{j}_{\beta} + re_{a}^{j}m^{j}_{\beta}(-e^{i}_{\beta} + \omega_{i}).$$

By Lemma 4.6 we have $Z'^{(\omega_{a}, e_{a}^{*})} = O(t^{C\varepsilon})$ and thus $Z'^{(V_{a}(r))} = O(t^{C\varepsilon})$ on $H$ for each $|J'|$. Thus, for all $|J| \leq |I| - 1$, we have

$$|Z^{J}(V_{a}F)| \lesssim |Z^{J}(V_{a}(r)\omega_{i}\partial F)| + |Z^{J}(e_{a}^{j}\omega_{j}\Omega_{ji}F)| \lesssim t^{C\varepsilon} \sum_{|K| \leq |J|} |Z^{K}\partial F| + t^{C\varepsilon} \sum_{|K| \leq |J|} |Z^{K}F| \lesssim t^{C\varepsilon} \sum_{|K| \leq |J|} |Z^{K}F|.$$  

This finishes the proof. □

**Remark 4.7.1.** With the help of this lemma, we conclude immediately that

$V^{I}(g - m) = O(\varepsilon^{1+C\varepsilon})$, $V^{I}((3R - r + t)^{-1}) = O((q)^{-1}t^{C\varepsilon})$, $V^{I}(r^{-1}, t^{-1}) = O(t^{-1+C\varepsilon})$,

$V^{I}(q) = \langle q \rangle t^{C\varepsilon}$, $V^{I}(e_{k}^{\alpha}) = O(t^{C\varepsilon})$, $V^{I}(e_{i}^{0} - \omega_{i}, e_{i}^{1} - \omega_{i}) = O(t^{-1+C\varepsilon})$

on $H$ for each $I$.

**Lemma 4.8.** For each $I$, on $H$ we have $V^{I}(\xi_{13}^{2}, \xi_{23}^{2}) = O((q)^{-1}t^{C\varepsilon})$, $V^{I}(\xi_{31}^{2}) = O(t^{-1+C\varepsilon}(q)^{-1})$ and $V^{I}(\xi_{k_{1},k_{2}})$ for all other $k_{1} < k_{2}$ and $a \in \{1, 2\}$; $V^{I}(\xi_{k_{1},k_{2}}^{3}) = O(t^{-1+C\varepsilon}(q)^{-1})$ for all $k_{1} < k_{2}$; $V^{I}((X_{ab} - r^{-1}\delta_{ab}) = O(t^{-2+C\varepsilon})$.

**Proof.** First, for any function $F = F(t, x)$ and for each $1 \leq k \leq 4$, on $H$ we have

$|V^{I}(e_{k}(F))| \lesssim \langle q \rangle^{-1}t^{C\varepsilon} \sum_{|J| \leq |I| + 1} |V^{J}(F)|$.

This inequality easily follows from the Leibniz’s rule, Remark 4.7.1 and the estimate $\langle r - t \rangle \lesssim t$ on $H$.

Since $e_{l}(e_{k_{1}, e_{k_{2}}}) = 0$ for each $k_{1}, k_{2}, l$, we have

$$2e_{k_{1},k_{2}}^{3} = \langle [e_{k_{1}, e_{k_{2}}}], e_{4} \rangle = e_{k_{1}}(e_{k_{2}}^{\alpha}g_{\alpha\beta}e_{4}^{\beta} - e_{k_{2}}(e_{k_{2}}^{\alpha}g_{\alpha\beta}e_{4}^{\beta} = -e_{k_{2}}^{\alpha}e_{k_{1}}(g_{\alpha\beta}e_{4}^{\beta} - e_{k_{2}}^{\alpha}g_{\alpha\beta}e_{k_{1}}(e_{4}^{\beta}) + e_{k_{1}}^{\alpha}e_{k_{2}}^{\beta}(g_{\alpha\beta}e_{4}^{\beta} + e_{k_{1}}^{\alpha}g_{\alpha\beta}e_{k_{2}}^{\beta}).$$

We assume $k_{1} \neq k_{2}$ as $\xi_{k_{1}, k_{2}}^{3} \equiv 0$. By (4.9) and the Leibniz’s rule, on $H$ for each $I$ we have

$|V^{I}(-e_{k_{2}}^{\alpha}e_{k_{1}}(g_{\alpha\beta}e_{4}^{\beta} + e_{k_{1}}^{\alpha}e_{k_{2}}^{\beta}(g_{\alpha\beta}e_{4}^{\beta})| \lesssim \varepsilon t^{-1+C\varepsilon}(q)^{-1}.$

Moreover, since $e_{l}^{3} \equiv 1$, we have

$$e_{k_{2}}^{\alpha}g_{\alpha\beta}e_{k_{1}}(e_{4}^{\beta}) = e_{k_{2}}^{\alpha}g_{\alpha\beta}e_{k_{1}}(e_{4}^{\beta} - \omega_{j}) + e_{k_{2}}^{\alpha}g_{\alpha\beta}e_{k_{1}}(e_{4}^{\beta} - e_{k_{1}}^{l}\omega_{j}).$$

Again, by (4.9) and the Leibniz’s rule, on $H$ for each $I$ we have

$|V^{I}(e_{k_{2}}^{\alpha}g_{\alpha\beta}e_{k_{1}}(e_{4}^{\beta} - \omega_{j}))| \lesssim t^{-1+C\varepsilon}(q)^{-1}.$
If $k_1 = 3$ or $4$, then

$$e^i_{k_1} - e^i_{k_2} = e^i_{k_1} - \omega_j + (1 - e^i_{k_1}) \omega_j = e^i_{k_1} - \omega_j + \sum_l (\omega_l - e^i_{k_1}) \omega_l \omega_j,$$

by the Leibniz’s rule and the estimate $V^I(e^i_{3} - \omega_i, e^i_{4} - \omega_i) = O(t^{-1+c \varepsilon})$ for each $I$, we conclude that

$$|V^I(r^{-1} e^\alpha_{k_2} g_{\alpha j} (e^i_{k_1} - \omega_j + (1 - e^i_{k_1}) \omega_j))| \lesssim t^{-2+c \varepsilon}, \quad k_1 \geq 3.$$ 

If $k_1 = 1$ or $2$, then $e^0_{k_1} = 0$.

$$r^{-1} e^\alpha_{k_2} g_{\alpha j} (e^i_{k_1} - e^i_{k_2} \omega_j) = r^{-1} e^i_{k_2} e^i_{k_1} - r^{-1} e^\alpha_{k_2} g_{\alpha j} e^i_{k_1} \omega_j = -r^{-1} e^\alpha_{k_2} g_{\alpha j} e^i_{k_1} \omega_j.$$

Note that

$$e^l_{k_1} \omega_j = e^l_{k_1} \delta_{I} e^\nu e^\mu e^\nu - e^l_{k_1} (g_{\mu \nu} - m_{\mu \nu}) e^\nu + e^l_{k_1} \delta_{I} (\omega_\nu - e^\nu).$$

Thus, by the Leibniz’s rule, we have $V^I(e^l_{k_1} \omega_j) = O(t^{-1+c \varepsilon})$ and thus

$$|V^I(r^{-1} e^\alpha_{k_2} g_{\alpha j} (e^i_{k_1} - e^i_{k_2} \omega_j))| \lesssim t^{-2+c \varepsilon}, \quad k_1 \leq 2.$$ 

In conclusion, for each $I$, on $H$ we have

$$|V^I(\xi^B_{k_2 k_1})| \lesssim t^{-1+c \varepsilon} (q)^{-1} + t^{-2+c \varepsilon} \lesssim t^{-1+c \varepsilon} (q)^{-1}.$$

Next, we have

$$\xi^B_{k_2 k_1} = \{e^i_{k_1} e^j_{k_2}, e^c, c\} = e^i_{k_1} (e^\alpha_{k_2}) g_{\alpha j} e^c_{\beta} - e^j_{k_2} (e^\alpha_{k_1}) g_{\alpha j} e^\beta_c.$$

We first prove some estimates for $e^a_{k_1} e^\alpha_{k_2} g_{\alpha j} e^\beta_c$ with $k_1 \neq k_2$. If $k_1 = a \in \{1, 2\}$ and $k_2 = b \in \{1, 2\}$, we have $e^a_{k_1} = r^{-1} V^a$ and thus $V^I(e^a_{k_2} e^\alpha_{k_1} e^\beta_c) = O(t^{-1+c \varepsilon})$ on $H$. If $k_2 = 3$ and $k_1 = a \in \{1, 2\}$, then

$$e^a_{3} (e^\alpha_{a}) g_{\alpha j} e^\beta_c = e^a_{3} (e^\alpha_{a}) g_{\alpha j} e^\beta_c = e^a_{3} - (e^i_{a} - \omega_i) g_{\alpha j} e^\beta_c - e^a_{3} (e^i_{a} - \omega_i) g_{\alpha j} e^\beta_c = e^a_{3} - (e^i_{a} - \omega_i) g_{\alpha j} e^\beta_c = e^a_{3} (e^\alpha_{a}) g_{\alpha j} e^\beta_c.$$

Recall that $V^I(e^a_{k_2} \omega_j) = O(t^{-1+c \varepsilon})$ on $H$. By Remark 1.7.1, we have $V^I(e^a_{k_2} e^\alpha_{k_1} e^\beta_c) = O(t^{-2+c \varepsilon})$ on $H$. Following the same proof, we can show that $V^I(e^a_{k_2} e^\alpha_{k_1} e^\beta_c) = O(t^{-2+c \varepsilon})$ on $H$. Next, for $k_1 \neq 3$ we have

$$e^A_{4} e^\alpha_{k_2} g_{\alpha j} e^\beta_c = e^A_{4} e^\alpha_{k_2} (\Gamma^0_{\mu \nu} e^0_4 - \Gamma^a_{\mu \nu}) g_{\alpha j} e^\beta_c = -e^A_{4} e^\alpha_{k_2} (\Gamma^0_{\mu \nu} g_{\alpha j} e^\beta_c.$$

Then, on $H$ we have $V^I(e^a_{k_2} e^\alpha_{k_1} e^\beta_c) = O(t^{-2+c \varepsilon})$. Next, we have

$$e^a_{3} (e^\alpha_{a}) g_{\alpha j} e^\beta_c = e^a_{3} (e^\alpha_{a}) g_{\alpha j} e^\beta_c + (3 R - r + t)^{-1} V^a (e^i_{a} - \omega_j) g_{\alpha j} e^\beta_c = e^a_{3} (e^\alpha_{a}) g_{\alpha j} e^\beta_c + (3 R - r + t)^{-1} V^a (e^i_{a} - \omega_j) g_{\alpha j} e^\beta_c.$$

Then, on $H$ we have $V^I(e^a_{k_2} e^\alpha_{k_1} e^\beta_c) = O(t^{-2+c \varepsilon})$. Next, we have

$$e^a_{3} (e^\alpha_{a}) g_{\alpha j} e^\beta_c = e^a_{3} (e^\alpha_{a}) g_{\alpha j} e^\beta_c + (3 R - r + t)^{-1} V^a (e^i_{a} - \omega_j) g_{\alpha j} e^\beta_c = e^a_{3} (e^\alpha_{a}) g_{\alpha j} e^\beta_c + (3 R - r + t)^{-1} V^a (e^i_{a} - \omega_j) g_{\alpha j} e^\beta_c.$$

53
Then, on $H$ we have $V^I(e_3^c e_c^β e_3 e_β^c) = O(t^{-1+4ε}(q)^{-1})$. Besides, we have

$$e_3^c e_β^c e_β e_c^c = - e_β^c e_3 e_3^c e_c^β e_β^c e_β e_c^c = -rac{1}{2}(3R - r + t)^{-1} e_c^β V_3(g_β) e_β^c,$$

so we have $V^I(e_3^c e_β^c e_β e_c^c) = O(t^{-1+Cε}(q)^{-1})$ on $H$. If $c' \neq c$, then

$$e_3^c e_β^c e_β e_c^c = (3R - r + t)^{-1} V_3(g_β) e_β^c,$$

so we have $V^I(e_3^c e_β^c e_β e_c^c) = O((q)^{-1} t^{Cε})$ on $H$ if $c \neq c'$. All these estimates imply that on $H$, we have

$$V^I(ξ^c_{ab}, ξ^a_{ab}; ξ^c_{3c}) = O(t^{-1+Cε}); \quad V^I(ξ^c_{a3}) = O((q)^{-1} t^{Cε}), c \neq c'; \quad V^I(ξ^c_{34}) = O(t^{-1+Cε}(q)^{-1}).$$

Moreover,

$$|V^I(χ_{ab} - r^{-1} δ_{ab})| ≤ |V^I(e_a(e_α^0 g^α β e_β^c - r^{-1} δ_{ab})| + |V^I(e_a(e_α^0 g^α β e_β^c)| ≤ t^{-2+Cε}.$$

4.3. Estimates in $Ω$. Recall that we defined a weighted null frame $\{V_k\}^4_{k=1}$ in Section 4.1. Our goal in this section is to prove the following proposition. Note that the estimates here are the same as those in Lemma 4.8.

**Proposition 4.9.** In $Ω ∩ \{r - t < 2R\}$, for each $I$ we have the following estimates:

$$|V^I(ξ^2_{13})| + |V^I(ξ^1_{23})| ≤ (q)^{-1} t^{Cε},$$

and for all other $(k_1, k_2, a)$ such that $k_1 < k_2$ and $a = 1, 2$, we have

$$|V^I(ξ^a_{k_1 k_2})| ≤ (q)^{-1} (q)^{-1} t^{1+Cε},$$

for all $k_1 < k_2$, we have

$$|V^I(ξ^b_{k_1 k_2})| ≤ t^{-1+Cε}(q)^{-1},$$

for $ξ^b_{34} = χ_{ab}$, we have

$$|V^I(χ_{ab} - r^{-1} δ_{ab})| ≤ t^{-2+Cε}.$$

In this proposition we use the convention given in Section 2.4. That is, for each fixed integer $N > 0$, we can choose $ε ≪ N$ 1, such that the estimates in this proposition hold for all multiindices $I$ with $|I| ≤ N$.

Since it is known that $q = r - t$ for $r - t > R$, we only care about the region where $r - t < 2R$ in this subsection. Recall that every point in $Ω ∩ \{r - t < 2R\}$ lies on exactly one geodesic in $A$ emanating from $H$. The following lemma would be the key lemma in the proof of Proposition 4.9.

**Lemma 4.10.** Fix $0 < ε ≪ 1$. Let $Q_1, \ldots, Q_m$ be $m$ functions defined in $Ω ∩ \{r - t < 2R\}$. For each $i = 1, \ldots, m$, suppose in $Ω ∩ \{r - t < 2R\}$ we have

$$e_4(Q_i) = (-n_0 r^{-1} + n_1 e_4(\ln(3R - r + t)))Q_i + O(ε t^{-1} \sum_j |Q_j|) + O(f(t)).$$

54
Here \( n_0, n_1 \geq 0 \) are two fixed real numbers which do not depend on \( i \). Moreover, for some fixed \( s \geq 1 \), we suppose that \( Q_i|_H = O(h(t)) \) for each \( i \). Then, in \( \Omega \cap \{ r - t < 2R \} \) we have

\[
\sum_i |Q_i| \lesssim \epsilon^{-n_0 + C\epsilon} ((x^0(0))^{n_0} h(x^0(0)) + \int_{x^0(0)}^t \tau^{n_0 + C\epsilon} f(\tau) \, d\tau).
\]

Here we suppose that \( (t, x) \) lies on the geodesic \( x(s) \) in \( A \) and that the integral is taken along the geodesic \( x(s) \).

**Proof.** Recall that \( e_4(r) = 1 + O(t^{-1+C\epsilon}). \) If we define \( Q'_i = (3R - r + t)^{-n_1} r^{n_0} Q_i \), then by (4.16), we have

\[
e_4(Q'_i) = -n_1 (3R - r + t)^{-n_1} e_4(3R - r + t) r^{n_0} Q_i + n_0 (3R - r + t)^{-n_1} r^{n_0} e_4(r) Q_i
\]

\[
\quad \quad \quad \quad + (3R - r + t)^{-n_1} r^{n_0} e_4(Q_i)
\]

\[
= n_0 r^{-1} (e_4(r) - 1) Q'_i + O(\epsilon t^{-1} \sum_j |Q'_j| + (3R - r + t)^{-n_1} r^{n_0} f(t))
\]

\[
= O(\epsilon t^{-1} \sum_j |Q'_j| + (3R - r + t)^{-n_1} r^{n_0} f(t)).
\]

To get the last equality, we note that \( r^{-1} (e_4(r) - 1) = O(t^{-2+C\epsilon}) = O(\epsilon t^{-1}) \) as \( t \geq \exp(\delta/\epsilon) \).

In addition, we have \( \langle q \rangle / \langle r - t \rangle = t^{O(\epsilon)} \). In fact, by Lemma 3.8, we have \( |q - (r - t)| \lesssim t^{C\epsilon} \) and thus

\[
1 + |q| \lesssim 1 + |r - t| + t^{C\epsilon} \lesssim t^{C\epsilon} |r - t| \implies \langle r - t \rangle^{-1} \lesssim \langle q \rangle^{-1} t^{C\epsilon}
\]

\[
1 + |r - t| \lesssim 1 + |q| + t^{C\epsilon} \lesssim t^{C\epsilon} \langle q \rangle \implies \langle q \rangle^{-1} \lesssim \langle r - t \rangle^{-1} t^{C\epsilon}.
\]

Thus, in \( \Omega \cap \{ r - t < 2R \} \) we have

\[
(3R - r + t)^{-n_1} r^{n_0} f(t) \lesssim \langle q \rangle^{-n_1} t^{n_0 + C\epsilon} f(t).
\]

Fix a point \((t_0, x_0)\) in \( \Omega \cap \{ r - t < 2R \} \), and let \( x(s) \) be the unique geodesic in \( A \) passing through \((t_0, x_0)\). Note that \( t_0 \geq x^0(0) \geq T_0 \) and that \( q \) remains constant along each geodesic in \( A \). Then by integrating \( e_4(Q'_i) \), we have

\[
\sum_i |Q'_i(t_0, x_0)| \lesssim \sum_i |Q'_i(x(0))| + \int_{x^0(0)}^{t_0} \epsilon \tau^{-1} \sum_j |Q'_j(\tau, y(\tau))| + \langle q \rangle^{-n_1} \tau^{n_0 + C\epsilon} f(\tau) \, d\tau
\]

\[
\lesssim \langle q \rangle^{-n_1} (x^0(0))^{n_0} h(x^0(0)) + \int_{x^0(0)}^{t_0} \epsilon \tau^{-1} \sum_j |Q'_j(\tau, y(\tau))| + \langle q \rangle^{-n_1} \tau^{n_0 + C\epsilon} f(\tau) \, d\tau.
\]

Here \((\tau, y(\tau))\) is a reparameterization of \( x(s) \) such that \( y(t_0) = x_0 \). By the Gronwall’s inequality, we conclude that

\[
\sum_i |Q'_i(t_0, x_0)| \lesssim t_0^{C\epsilon} \langle q \rangle^{-n_1} ((x^0(0))^{n_0} h(x^0(0)) + \int_{x^0(0)}^{t_0} \tau^{n_0 + C\epsilon} f(\tau) \, d\tau).
\]

To end the proof, we multiply both sides by \( r^{-n_0} (3R - r + t)^{n_1} \), and recall that \( t \sim r \) in \( \Omega \cap \{ r - t < 2R \} \).

\[\Box\]

To prove Proposition 4.9, we induct on \(|I|\).
4.3.1. The base case $I = 0$. From Section 4.1 in $\Omega \cap \{r - t < 2R\}$ we already have the following estimates: $\xi_{34}^3 = O(||\Gamma||) = O(\min\{t^{-1}, e^{t+1+C\epsilon}\})$, $\xi_{a4}^6 = \lambda_{ab} = \delta_{ab}p^{-1} + O(t^{-2+C\epsilon}) = O(t^{-1})$, $\xi_{a3}^3 = \lambda_{aa} + O(\epsilon t^{-1}) = O(t^{-1})$, $\xi_{a3}^3 = \xi_{12}^3 = 0$. To control the rest $\xi$, we recall that

$$\langle R(\epsilon_t, e_t) e_r, e_s \rangle = e_k^a e_l^b e_s^\nu R_{\alpha\beta\mu\nu}$$

(17)

$$e_k^a e_l^b e_s^\nu \left( \partial_{\alpha} \partial_{\mu} g_{\beta\nu} - \partial_{\mu} \partial_{\beta} g_{\alpha\nu} + \partial_{\beta} \partial_{\nu} g_{\alpha\mu} - \Gamma_{\beta\mu}^\gamma \partial_{\alpha\nu} + \Gamma_{\alpha\mu}^\gamma \partial_{\beta\nu} \right) - \Gamma_{\beta\mu}^\gamma \partial_{\alpha\nu} + \Gamma_{\alpha\mu}^\gamma \partial_{\beta\nu}$$

If at most one of $k, l, r, s$ is equal to 3, then we have $\langle R(\epsilon_t, e_t) e_r, e_s \rangle = O(\epsilon t^{-2+C\epsilon}(t - r)^{-1})$ by Lemma 3.7. From the equations in Section 4.1 we have

$$|e_4(\xi_{34}^a) + r^{-1}\xi_{34}^a \lesssim t^{-2+C\epsilon} \sum_b |\xi_{34}^b| + \epsilon t^{-2+C\epsilon}(q)^{-1},$$

$$|e_4(\xi_{12}^a) + r^{-1}\xi_{12}^a \lesssim t^{-2+C\epsilon} \sum_b |\xi_{12}^b| + \epsilon t^{-2+C\epsilon}(q)^{-1}.$$}

By Lemma 4.10 with $n_0 = 1, n_1 = 0$ and $f(t) = \epsilon t^{-2+C\epsilon}(q)^{-1}$, we have

$$|\xi_{34}^a \lesssim t^{-1+C\epsilon}(q)^{-1} + t^{-1+C\epsilon}(q)^{-1} \lesssim t^{-1+C\epsilon}(q)^{-1},$$

$$|\xi_{12}^a \lesssim t^{-1+C\epsilon}(q)^{-1} + t^{-1+C\epsilon}(q)^{-1} \lesssim t^{-1+C\epsilon}.$$}

Here we get different estimates for $\xi_{34}^a$ and $\xi_{12}^a$ because their estimates on $H$ are different; see Lemma 4.8.

It follows from Section 4.1 that $\xi_{34}^a = \frac{1}{2} e_4^a + O(\epsilon t^{-1}) = O(t^{-1+C\epsilon})$. It remains to estimate $\xi_{a3}^a$ where $a \neq a'$. Note that

$$e_4(\xi_{a3}^a) = \langle R(e_4, e_3) e_a, e_a' \rangle = \Gamma_{\alpha\beta}^\gamma e_4^\alpha e_3^\beta e_a^\gamma + 2 \Gamma_{\alpha\beta}^\gamma e_4^\alpha e_3^\beta e_a^\gamma \Gamma_{\gamma\alpha}^\mu g_{\beta\mu} e_{a'}^\nu - \Gamma_{\alpha\beta}^\gamma e_3^\alpha e_4^\beta e_a^\gamma - \sum_c \xi_{34}^c \xi_{aa'}^c \rangle$$

By Lemma 4.10 with $n_0 = n_1 = 0$ and $f(t) = t^{-2+C\epsilon}$ we have $|\xi_{a3}^a \lesssim (x^0(0))^{-1} + t^{-2+C\epsilon}(q)^{-1}$. Here note that if $(t, x)$ lies on a geodesic $x(s)$ in $A$, then

$$q(t, x) = q(x(0)) = r(x(0)) = t - x^0(0) = \frac{T_0 - x^0(0)}{2} + 2R \implies x^0(0) = T_0 - 2(q - 2R).$$

And since we only care about the region where $q < 2R$, we have $t \geq x^0(0) \sim (T_0 + \langle q \rangle) \geq \langle q \rangle$. In conclusion, we prove Proposition 4.9 in the case $I = 0$.

4.3.2. The general case. Fix $m > 0$. Suppose we have proved Proposition 4.9 for all $|I| < m$. Our goal is to prove Proposition 4.9 for $|I| = m$.

Under the induction hypotheses, we can prove a key lemma which is Lemma 4.11 below. For convenience, we introduce the following notation.

**Definition.** Let $F = F(t, x)$ be a function with domain $\Omega \cap \{r - t < 2R\}$. For any integer $m \geq 0$ and any real numbers $s, p$, we write $F = R_{s, p}^m$ if for $\epsilon \ll s, p, m$ 1 we have

$$\sum_{|I| \leq m} |V^I(\epsilon)| \lesssim t^{s+C\epsilon}(q)^p \quad \text{in} \quad \Omega \cap \{r - t < 2R\}.$$
By the Leibniz’s rule, we can easily prove that $\Re^{m_{1,p_1}}_{s_1,p_1} \cdot \Re^{m_{2,p_2}}_{s_2,p_2} = \Re^{\min\{m_1,m_2\}}_{s_1+s_2,p_1+p_2}$. In addition, under the induction hypotheses, we have

$$
(4.20) \quad \Re^{m_{1-1}}_{\alpha,0,1}; \quad \Re^{m_{2-1}}_{\alpha,0,1}; \quad \Re^{m_{3-1}}_{\alpha,0,1} \quad \text{for all other } k_1 < k_2 \text{ and } a = 1, 2;
$$

$$
(4.21) \quad \Re^{m_{2-1}}_{\alpha,0,1} \quad \text{for all } k_1 < k_2; \quad \chi_{ab} - r^{-1}\delta_{ab} = \Re^{m_{-2,0}}_{-2,0}.
$$

**Lemma 4.11.** For $\varepsilon \ll m 1$, we have

$$
(4.19) \quad e_k^\alpha = \Re^{m}_{0,0};
$$

$$
(4.20) \quad (e_i^4 - \omega_i, e_3^j - \omega_i) = \Re^{m}_{-1,0};
$$

$$
(4.21) \quad (g^{\alpha\beta} - m^{\alpha\beta}, g_{\alpha\beta} - m_{\alpha\beta}) = \varepsilon \Re^{m+1}_{-1,0}, \quad \Gamma^{\alpha}_{\mu\nu} = \varepsilon \Re^{m+1}_{-1,-1},
$$

for each fixed $s \in \mathbb{R}$, we have

$$
(4.22) \quad \omega_i = \Re^{m+1}_{0,0}, \quad (t^s, r^s) = \Re^{m+1}_{s,0}, \quad (3R - r^s t)^s = \Re^{m+1}_{0,0}.
$$

**Proof.** We prove by induction. First, since $e^*_s = O(1)$, we have $e^*_k = \Re^{m}_{0,0};$ by Lemma 3.7, we have $(e^*_k - \omega_i, e^*_3 - \omega_i) = \Re^{0}_{-1,0}$. Besides, $(g_{**} - m_{**}, g_{**} - m_{**}) = O(\varepsilon t^{-1} + C\varepsilon)$ and

$$
|\Gamma| \lesssim |g^2| |\partial g| \lesssim \varepsilon t^{-1+C\varepsilon} (r-t)^{-1} \lesssim \varepsilon t^{-1+C\varepsilon} (q^{-1}).
$$

Here we use the estimate $\langle r-t \rangle / \langle q \rangle = t^{O(\varepsilon)}$. Besides,

$$
\sum_k |V_k(g)| \lesssim \sum_{k \neq 3} (t+r)|e_k(g)| + \langle r-t \rangle |\partial g| \lesssim \varepsilon t^{-1+C\varepsilon}.
$$

Since $\Gamma$ is a linear combination of terms of the form $g \cdot \partial g$ with constant real coefficients, by Lemma 3.7, we have

$$
\sum_k |V_k(\Gamma)| \lesssim \varepsilon t^{-1+C\varepsilon} \cdot \varepsilon r-t-1 t^{-1+C\varepsilon} + \sum_{k \neq 3} (t+r)|e_k(g)| + \langle r-t \rangle |\partial^2 g|
$$

$$
\lesssim \varepsilon (q^{-1}) t^{-1+C\varepsilon}.
$$

We thus obtain (4.21) with $m = 0$. Since $3R - r + t \sim (r-t)$ in $\Omega \cap \{r-t < 2R\}$, (4.22) with $m + 1$ replaced by 0 is obvious. In addition, by writing $Vf := (V_1f, V_2f, V_3f, V_4f)$, we have

$$
(4.23) \quad \begin{cases}
V(t) = (0, 0, -(3R - r + t), t); \\
V(r) = (re_1(r), re_2(r), (3R - r + t)(e_3^j \omega_i), te_3^j \omega_i); \\
V(\omega_i) = (e_i^1 - \omega_i e_1(r), e_i^2 - \omega_i e_2(r), r^{-1}(3R - r + t)(e_i^3 - \omega_i e_3^j \omega_j), r^{-1}t(\omega_i e_4^j \omega_j)); \\
V(3R - r + t) = (-re_1(r), -re_2(r), (3R - r + t)(-1 - e_3^j \omega_i), t(1 - e_4^j \omega_i))
\end{cases}
$$
Since $e_3, e_4 = \pm \partial_t + \partial_r + O(t^{-1+C\varepsilon})\partial_r$, we have
\begin{align}
e_a(r) &= e_a^i \omega_i = \sum_i e_a^i e_4^i + \sum_i \omega_i (e_i^r - e_i^4) \\
      &= (e_a, e_4) - (g^{a\beta} - m^{a\beta})e_4^\alpha e_a^\beta + \sum_i e_i^r (\omega_i - e_i^4) = O(t^{-1+C\varepsilon}), \\
1 - e_i^4 \omega_i &= - \sum_i (e_i^4 - \omega_i) \omega_i = O(t^{-1+C\varepsilon}).
\end{align}

(4.24)

Also note that for each fixed $s \in \mathbb{R}$ and for each function $\phi(t, x)$, $V(\phi^s) = s \phi^{s-1} V(\phi)$. Then, we have $V(\omega) = O(t^{C\varepsilon}), V(t^s, r^s) = O(t^{s+C\varepsilon}), V((3R - r + t)^s) = O((r - t)^{s+C\varepsilon})$. We thus obtain (4.22) with $m = 0$. This finishes the proof in the base case.

In general, we assume that we have proved (4.19)-(4.22) with $m$ replaced by $n$ where $0 \leq n < m$. We first prove (4.19) with $m$ replaced by $n + 1$. Fix a multiindex $I$ such that $|I| = n + 1$. If $I = (I', 4)$, note that $t e_4(e_k^\alpha)$ is a linear combination (with constant real coefficients) of terms of the form $t I_{**}(e_4^\alpha)(e_4^\beta)(e_4^\gamma)$ and $V_4(g^0\alpha)$. By the induction hypothesis, we notice that
\begin{align}
t I_{**}(e_4^\alpha)(e_4^\beta)(e_4^\gamma) &= \mathcal{R}_1^{n+1} \cdot \mathcal{R}_{-1,-1}^{n+1} \cdot \mathcal{R}_0^{n} \cdot \mathcal{R}_{0,0}^{n} \cdot \mathcal{R}_{0,0}^{n} = \varepsilon \mathcal{R}_0^{n-1}
\end{align}
and similarly
\begin{align}
t I_{**}(e_4^\alpha)(e_4^\beta) &= \varepsilon \mathcal{R}_0^{n-1}.
\end{align}

Besides,
\begin{align}
g^0\alpha - m^0\alpha &= \varepsilon \mathcal{R}_0^{n+1} \implies V_k(g^0\alpha) = \varepsilon \mathcal{R}_0^{n}.
\end{align}

So in conclusion,
\begin{align}
V_4(e_k^\alpha) &= \varepsilon \mathcal{R}_0^{n-1} \implies V_l(e_k^\alpha) = O(\varepsilon (q)^{-1} t^{C\varepsilon}).
\end{align}

If $I = (I', k')$ where $k' \neq 4$, then by the formulas at the end of Section 4.1, we have
\begin{align}
V_{k'}(e_a^\alpha) &= r \xi_{a4} e_a^\alpha + rt^{-1} V_4 (e_a^\alpha) \\
&= \mathcal{R}_1^{n+1} \cdot \mathcal{R}_{-1,0}^{n+1} \cdot \mathcal{R}_0^{n} \cdot \mathcal{R}_{0,0}^{n} = \mathcal{R}_0^{n}, \quad k' = a = 1, 2;
\end{align}
\begin{align}
V_3(e_a^4) &= (3R - r + t)^2 \xi_{a4} e_a^4 + t^{-1} (3R - r + t) V_4 (e_a^4) \\
&= \mathcal{R}_0^{n+1} \cdot \mathcal{R}_{-1,0}^{n+1} \cdot \mathcal{R}_0^{n} \cdot \mathcal{R}_{0,0}^{n} = \mathcal{R}_0^{n}.
\end{align}

In addition, note that $e_3^\alpha = e_4^\alpha + 2g^0\alpha$, so
\begin{align}
V_{k'}(e_a^\alpha, e_3^\alpha) &= \mathcal{R}_0^{n} \implies V_l(e_a^\alpha, e_3^\alpha) = O(t^{C\varepsilon}).
\end{align}

If $I = (I', 3)$, we have
\begin{align}
V_3(e_a^\alpha) &= (3R - r + t)^2 \xi_{a4} e_a^\alpha + r^{-1} (3R - r + t) V_a (e_a^\alpha) \\
&= \mathcal{R}_0^{n+1} \cdot \mathcal{R}_{-1,0}^{n+1} \cdot \mathcal{R}_0^{n} \cdot \mathcal{R}_{0,0}^{n} = \mathcal{R}_0^{n}.
\end{align}
Here we recall that $t \geq x^0(0) \sim (q) + T_0$, so $\mathcal{R}_{a,s} = \mathcal{R}_0^{n}$ for each $s > 0$. Thus,
\begin{align}
V_l(e_a^\alpha) = O(t^{C\varepsilon}).
\end{align}

If $I = (I', a)$, then
\begin{align}
V_a(e_b^\alpha) &= - \sum_c r \xi_{bc} e_c^\alpha - \frac{1}{2} r \chi_{ab} (e_4^\alpha + e_3^\alpha) - (e_b^\mu g_{\mu\beta} V_a (g^0\beta) + r e_b^\mu g_{\mu\nu} g^0\beta e_a^\gamma \Gamma_{\gamma\nu}) e_4^\alpha - re_b^\alpha e_6^\gamma \Gamma_{\gamma\nu}.
\end{align}
Again, by our induction hypotheses, we conclude that
\[ V_a(e_\beta^a) = \mathcal{R}_{0,0}^n \implies V^I(e_\beta^a) = O(t^{C_\epsilon}). \]

Summarize all the results above and we conclude that \( e_*^i = \mathcal{R}_{0,0}^{n+1} \). Note that the computations above work as long as \( n \leq m - 1 \).

Next we prove (4.20) with \( m \) replaced by \( n + 1 \). It suffices to consider \( e_i^4 - \omega_i \) as \( e_i^3 - e_i^4 = 2g^{0i} = \varepsilon \mathcal{R}_{-1,0}^{n+1} \). Fix a multiindex \( I \) with \( |I| = n + 1 \). Note that

\[
V_a(e_i^4 - \omega_i) = re_a(e_i^4 - \omega_i) = r((\xi_{a4}^i e_i^4 + e_4(a)) - r^{-1}(e_a^i - \omega_i e_a(r)))
= r(\chi_{ab} - \delta_{ab}r^{-1})e_i^a + re_4(e_i^4) + r^{-1}\omega_i V_a(r)
= \mathcal{R}_{1,0}^{n+1} \cdot \mathcal{R}_{-2,0}^{m-1} \cdot \mathcal{R}_{0,0}^n + re_4(e_i^4) + \mathcal{R}_{-1,0}^{n+1} = re_4(e_i^4) + \mathcal{R}_{-1,0}^{n+1},
\]
\[
V(e_i^4 - \omega_i) = te_4(e_i^4 - \omega_i) = t(e_i^4 - \omega_i) - (e_i^4 - \omega_j)\partial_j \omega_i
= te_4(e_i^4) - tr^{-1}(e_i^4 - \omega_i - \omega_j e_i^4 - \omega_j)
= te_4(e_i^4) + \mathcal{R}_{0,0}^{n+1} \cdot \mathcal{R}_{-1,0}^{n+1} \cdot \mathcal{R}_{-1,0}^n = te_4(e_i^4) + \mathcal{R}_{-1,0}^n,
\]
\[
V_3(e_i^4 - \omega_i) = (3R - r + t)e_4(e_i^4 - \omega_i) = (3R - r + t)(\xi_{i4}^k e_k^i + e_4(e_i^4) - (e_i^3 - \omega_j)\partial_j \omega_i)
= (3R - r + t)(\xi_{i4}^k e_k^i + e_4(e_i^4) + 2t^{-1}V_4(g^{0i}) - r^{-1}(e_i^3 - \omega_i - (e_i^3 - \omega_j)\omega_j))
= (3R - r + t)e_4(e_i^4) + \mathcal{R}_{0,0}^{n+1} \cdot \mathcal{R}_{-1,0}^{n+1} \cdot \mathcal{R}_{-1,0}^n + \varepsilon \mathcal{R}_{-2,0}^{n+1} + \mathcal{R}_{-1,0}^{n+1} \cdot \mathcal{R}_{-1,0}^n
= (3R - r + t)e_4(e_i^4) + \mathcal{R}_{-1,0}^n.
\]

Here we use (4.18). To finish the proof, we note that for \( k \neq 3 \),
\[
2e_4(e_i^4) = 2e_4 \varepsilon e_k^\beta(\Gamma_\alpha^0 e_4^i - \Gamma_{\alpha\beta}^i) = e_4 \varepsilon e_k^\beta(\partial_\alpha g_{\delta\beta} + \partial_\beta g_{\delta\alpha} - \partial_\delta g_{\alpha\beta})
= (g^{0i} e_4^i - g^{i4})(e_4(g_{\beta\delta}) e_4^\beta + e_4(g_{\delta\beta}) e_4^\alpha + e_4 e_4^\alpha(-\frac{1}{2} e_4(g_{\alpha\beta}) e_4^i + e_3^i) - \sum_b e_b^i e_b(g_{\alpha\beta}))
= \mathcal{R}_{0,0}^{n+1} t^{-1}V_4(g) + \mathcal{R}_{0,0}^{n+1} r^{-1}V_a(g) = \varepsilon \mathcal{R}_{-2,0}^{n+1}.
\]

Also note that \( e_4(g) = t^{-1}V_4(g) = \varepsilon \mathcal{R}_{-2,0}^{n+1} \) and that \( e_0^i \) is a constant, so we have \( e_4(e_0^k) = \varepsilon \mathcal{R}_{-2,0}^{n+1} \) for each \( k, \alpha \). Thus,
\[
V(e_i^4 - \omega_i) = \mathcal{R}_{-1,0}^{n+1} \implies e_i^4 - \omega_i = \mathcal{R}_{-1,0}^{n+1}.
\]

Finally, we prove (4.21) and (4.22) with \( m + 1 \) replaced by \( n + 2 \). Fix a multiindex \( I \) such that \( |I| = n + 2 \). Note that
\[
(3R + t - r)\partial_t = 3R\partial_t + \frac{tS - x_i \Omega_{0i}}{r + t} = \mathcal{R}_{0,0}^{n+1} \cdot Z,
(3R + t - r)\partial_r = 3R\partial_r + \frac{t\omega_i \Omega_{0i} - rS}{r + t} = \mathcal{R}_{0,0}^{n+1} \cdot Z,
(3R + t - r)\partial_i = 3R\partial_i + (t - r)\omega_i \partial_r + (t - r)r^{-1}\omega_j \Omega_{ji} = \mathcal{R}_{0,0}^{n+1} \cdot Z.
\]
Thus, \( \partial = (3R + t - r)^{-1} \mathfrak{R}_{0,0}^{n+1} \cdot Z = \mathfrak{R}_{0,0}^{n+1} \cdot Z \). Since we have just proved \( e^*_i = \mathfrak{R}_{0,0}^{n+1} \) and \( e^i - \omega_i = \mathfrak{R}_{-1,0}^{n+1} \), by (4.24) we have \( e_a(r) = \mathfrak{R}_{-1,0}^{n+1} \). In conclusion, by (4.1) we have

\[
V_4 = t(t + r)^{-1}S + (t + r)^{-1}t_0j_0 + t(e^*_i - \omega_i)\partial_i = \mathfrak{R}_{0,0}^{n+1} \cdot Z,
\]
\[
V_5 = (3R - r + t)^{-1}V_4 + 2g^{0a}(3R - r + t)\partial_a = \mathfrak{R}_{0,0}^{n+1} \cdot Z,
\]
\[
V_a = r e_a(r)\omega_i\partial_i + e^*_i o_j \partial_j = \mathfrak{R}_{-1,0}^{n+1} \cdot \mathfrak{R}_{0,0}^{n+1} \cdot Z + \mathfrak{R}_{0,0}^{n+1} \cdot Z = \mathfrak{R}_{0,0}^{n+1} \cdot Z.
\]

Now, given a function \( F = F(t, x) \), if \( |I| = n + 2 \), we can write \( V^I F \) as a linear combination of terms of the form

(4.25) \[
V^{I^i}(\mathfrak{R}_{0,0}^{n+1}) \cdots V^{I^s}(\mathfrak{R}_{0,0}^{n+1}) Z^s F, \quad \sum |I_s| + s = n + 2, \ s > 0.
\]

Since \( |I_j| < n + 2 \) for each \( j \), we have \( V^{I_j}(\mathfrak{R}_{0,0}^{n+1}) = O(t^C) \). Note that for each \( J \) with \( |J| > 0 \), we have \( Z^j g = O(\varepsilon t^{-(1-C)}), \ Z^j \omega = O(1), \ Z^j (t^s) = O(t^s), \ Z^j ((3R - r + t)^s) = O((r - t)^s) \) and \( Z^j (\Gamma) = O(\varepsilon t^{-(1-C)}(g)^{-1}) \). The last one is true because \( Z^j \Gamma \) is a linear combination (with constant real coefficients) of terms of the form \( (Z^j g) \cdot (Z^j \partial g) = O(\varepsilon t^{-(1-C)}(r - t)^{-1}) \). By plugging these estimates into (4.25), we conclude (4.21) and (4.22) with \( m + 1 \) replaced by \( n + 2 \).

**Remark 4.11.1.** We have \( Z^j \partial^k g = \varepsilon \mathfrak{R}_{-1,0-k}^{m+1} \) for each \( I \) and \( k \), as long as \( \varepsilon \ll_{I, k} 1 \). This follows directly from (4.25), Lemma 2.2 and \( [Z, \partial] = C \cdot \partial \).

From the proof, we note that \( e_4(e^a_k) = \varepsilon \mathfrak{R}_{-2,0}^{m} \) and \( e_a(r) = \mathfrak{R}_{-1,0}^{m} \). These estimates are better than what we can get from (4.19) and (4.22).

By Lemma 4.11 we have \( e^*_i \omega_i - 1 = (e^*_i - \omega_i) \omega_i = \mathfrak{R}_{-1,0}^{m} \). This result can be improved as shown in the next lemma.

**Lemma 4.12.** For \( \varepsilon \ll_{m} 1 \), we have \( e^*_i \omega_i - 1 = \varepsilon \mathfrak{R}_{-1,0}^{m} \).

**Proof.** By Lemma 4.11 we have

\[
e^a_j \omega_j = -(g^{a\beta} - m^{a\beta}) e^a_e e^e_j + \sum_i e^i_j (\omega_i - e^i_j) = \mathfrak{R}_{-1,0}^{m}.
\]

Recall that

\[
g^{a\beta} = \sum_a e^a e^a + \frac{1}{2}(e^0 e^0 + e^3 e^3).
\]

Then,

\[
g^{a\beta} (\partial_\alpha (r - t))(\partial_\beta (r - t)) = \sum_a (e^0 \omega_i)(e^0 \omega_j) + (e^3 \omega_i - 1)(e^3 \omega_j + 1)
\]
\[
= \mathfrak{R}_{-2,0}^{m} + (e^0 \omega_i - 1)(2 + (e^3 - \omega_j) \omega_j).
\]

Meanwhile, we have

\[
g^{a\beta} (\partial_\alpha (r - t))(\partial_\beta (r - t)) = g^{00} - 2g^{0i} \omega_i + g^{ij} \omega_j \omega_j
\]
\[
= -2g^{0i} \omega_i + (g^{ij} - m^{ij}) \omega_j \omega_j = \varepsilon \mathfrak{R}_{-1,0}^{m+1}.
\]

Thus,

\[
e^*_i \omega_i - 1 = (2 + (e^0_i - \omega_j) \omega_j)^{-1}(\varepsilon \mathfrak{R}_{-1,0}^{m} + \mathfrak{R}_{-2,0}^{m}) = (2 + (e^3 - \omega_j) \omega_j)^{-1} \cdot \varepsilon \mathfrak{R}_{-1,0}^{m}.
\]

Here we note that \( \mathfrak{R}_{-2,0}^{m} = \varepsilon \mathfrak{R}_{-1,0}^{m} \) as \( t \geq \exp(\delta/\varepsilon) \).
Fix a multiindex $I$ with $|I| \leq m$. Then, $V_I(e^i_4\omega_i - 1)$ is a linear combination of terms of the form
\[
(2 + (e^i_3 - \omega_j)\omega_j)^{-s-1}V^{I_0}(\varepsilon\mathbb{R}^m_{-1,0})V^{I_2}(2 + (e^i_3 - \omega_j)\omega_j) \cdots V^{I_s}(2 + (e^i_3 - \omega_j)\omega_j)
\]
where $\sum |I_s| = |I| \leq m$ such that $|I_k| > 0$ for each $k > 0$. Thus, we can replace $V^{I_s}(2 + (e^i_3 - \omega_j)\omega_j)$ with $V^{I_s}((e^i_3 - \omega_j)\omega_j)$ in the product. By Lemma 4.11 we have $(e^i_3 - \omega_j)\omega_j = \mathbb{R}^m_{-1,0}$. Since $e^i_3 - \omega_j = O(t^{-1+C\varepsilon})$, we have $2 + (e^i_3 - \omega_j)\omega_j \geq 1$ for $\varepsilon \ll 1$. In conclusion, we have
\[
|V_I(e^i_4\omega_i - 1)| \lesssim \varepsilon t^{-1+C\varepsilon} \max_{0 \leq s \leq m} \{(t^{-1+C\varepsilon})^s\} \lesssim \varepsilon t^{-1+C\varepsilon}.
\]
Thus, $e^i_4\omega_i - 1 = \varepsilon\mathbb{R}^m_{-1,0}$.

We can now control the curvature tensor terms.

Lemma 4.13. We have $\langle R(e_4, e_k)e_l, e_p \rangle = \varepsilon\mathbb{R}^m_{-2,-1}$ if $l, p \neq 3$.

Proof. By (4.117), we can express $e^\alpha_4 e^\beta_4 e^\mu_4 e^\nu_4 R_{\alpha\beta\mu\nu}$ as a linear combination of terms of the form
\[
e_4(\partial_g g_{\beta\nu} - \partial_\nu g_{\beta\mu})e^\mu_4 e^\nu_4, \quad e_4(\partial_\beta g_{\alpha\nu})e^\mu_4 e^\nu_4, \quad e_4(\partial_\beta g_{\alpha\mu})e^\alpha_4 e^\mu_4, \quad e^\alpha_4 e^\beta_4 e^\mu_4 e^\nu_4 \cdot \Gamma \cdot (g \cdot \Gamma).
\]
By Lemma 4.11 and Remark 4.11.1, we have
\[
e_4(\partial_g g_{\beta\nu} - \partial_\nu g_{\beta\mu})e^\mu_4 e^\nu_4 = t^{-1}V_I(\partial g) \cdot \mathbb{R}^m_{-1,0} = \mathbb{R}^m_{-1,0} \cdot Z(\partial g) = \varepsilon\mathbb{R}^m_{-2,-1}.
\]
Since $l \neq 3$, either have $e_l = t^{-1}V_I$ or $e_l = r^{-1}V_I$. In both cases, we can follow the same proof as above to conclude that
\[
e_l(\partial_\beta g_{\alpha\nu})e^\alpha_4 e^\mu_4 = \varepsilon\mathbb{R}^m_{-2,-1}.
\]
Similarly, we also have
\[
e_p(\partial_\beta g_{\alpha\mu})e^\alpha_4 e^\mu_4 = \varepsilon\mathbb{R}^m_{-2,-1}.
\]
Finally, note that
\[
e^\alpha_4 e^\beta_4 e^\mu_4 e^\nu_4 \cdot \Gamma \cdot (g \cdot \Gamma) = (\varepsilon\mathbb{R}^m_{-1,-1})^2 \cdot \mathbb{R}^m_{0,0} = \varepsilon^2 \mathbb{R}^m_{-2,-2}.
\]
Thus, $\langle R(e_4, e_k)e_l, e_p \rangle = \varepsilon\mathbb{R}^m_{-2,-1}$.

Lemma 4.13 can be improved in a special case.

Lemma 4.14. (a) We have
\[
\langle R(e_4, e_a)e_4, e_b \rangle = e_4(f_{ab}) + \frac{1}{4} e^\alpha_4 e^\mu_4 r^{-1}\delta_{ab} e_3(g_{\alpha\mu}) + \varepsilon\mathbb{R}^m_{-3,0}.
\]
Here we set
\[
f_{ab} = \frac{1}{2}(e^\beta_4 e^\nu_4 e_4(g_{\beta\nu}) - e^\beta_4 e^\mu_4 e_4(g_{\beta\mu})) - \frac{1}{2} e^\alpha_4 e_4(g_{\alpha\nu}) e^\nu_4 = \varepsilon\mathbb{R}^m_{-2,0}.
\]
(b) Assume that $\chi_{ab} = \mathbb{R}^m_{-1,0}$. Then we have
\[
\Gamma^0_{\alpha\beta} e^\alpha_4 e^\beta_4 \chi_{ab} + \frac{1}{4} e^\alpha_4 e^\beta_4 e_3(g_{\alpha\beta}) \chi_{ab} = \varepsilon\mathbb{R}^m_{-3,0}.
\]
Proof. (a) Recall that \( (R(e_4, e_α)e_4, e_β) = e_α^α e_β^β e_4^μ e_4^ν R_{αβμν} \) where \( R_{αβμν} \) is given by

\[
R_{αβμν} = \frac{1}{2}(\partial_α \partial_μ g_{βν} - \partial_α \partial_ν g_{βμ} - \partial_β \partial_μ g_{αν} + \partial_β \partial_ν g_{αμ}) - \Gamma^α_{βμ} \Gamma_δ^ν + \Gamma^α_{μν} \Gamma_δ^β.
\]

Note that (for simplicity we take the sum over all the indices without writing the summation)

\[
\frac{1}{2} e_α^α e_β^β e_4^μ e_4^ν \partial_α \partial_μ g_{αν} = \frac{1}{2} e_α^α e_β^β e_4^μ e_4^ν (\partial_α \partial_ν g_{αμ} + \partial_ν \partial_μ g_{αμ})
\]

\[
= \frac{1}{2} e_α^α e_β^β e_4^μ e_4^ν (\partial_α \partial_ν g_{αμ}) + \frac{1}{2} e_α^α e_β^β e_4^ν (\partial_α \partial_μ g_{αμ})
\]

\[
= \frac{1}{2} e_α^α e_β^β e_4^ν (\partial_α \partial_μ g_{αμ}) + \frac{1}{2} e_α^α e_β^β e_4^μ (\partial_α \partial_ν g_{αμ})
\]

\[
= \frac{1}{2} e_α^α e_β^β e_4^μ (\partial_α \partial_ν g_{αμ}) + \frac{1}{2} e_α^α e_β^β e_4^ν (\partial_α \partial_μ g_{αμ})
\]

Recall that in Lemma 4.11 we have proved that \( e_α(r) = \mathfrak{R}^m_{α1,0} \). Thus, we have

\[
\frac{1}{2} e_α^α e_β^β e_4^μ e_4^ν \partial_α \partial_μ g_{αν} = \frac{1}{2} e_α^α e_β^β e_4^ν (\partial_α \partial_ν g_{αμ}) + \mathfrak{R}^m_{α3,0}
\]

\[
= \frac{1}{2} e_α^α e_β^β e_4^ν (\partial_α \partial_ν g_{αμ}) + \frac{1}{2} e_α^α e_β^β e_4^μ (\partial_α \partial_μ g_{αμ})
\]

Next, we note that

\[
\frac{1}{2} e_α^α e_β^β e_4^μ e_4^ν (\partial_α \partial_μ g_{βν} - \partial_α \partial_ν g_{βμ} - \partial_β \partial_μ g_{αν})
\]

\[
= \frac{1}{2} e_α^α e_β^β e_4^ν (\partial_α \partial_μ g_{βν} - \partial_α \partial_ν g_{βμ} - \partial_β \partial_μ g_{αν})
\]

\[
e_4(\mathfrak{f}_{αβ}) - \frac{1}{2} e_4(\mathfrak{f}_{α}^α e_4^λ (\partial_α \partial_λ g_{νβ} - \partial_λ \partial_ν g_{βμ} - \partial_β \partial_μ g_{αν})
\]

In Lemma 4.11 we have proved that \( e_4(\mathfrak{f}_k) = \mathfrak{R}^m_{k2,0} \). By Lemma 4.11 we can easily prove that \( \mathfrak{f}_{αβ} = \mathfrak{R}^m_{αβ3,0} \). This implies that

\[
\frac{1}{2} e_α^α e_β^β e_4^ν (\partial_α \partial_μ g_{βν} - \partial_α \partial_ν g_{βμ} - \partial_β \partial_μ g_{αν}) = e_4(\mathfrak{f}_{αβ}) + \mathfrak{R}^m_{αβ3,0}.
\]
Finally, we note that
\[
e^{\alpha}_{4}e^{\beta}_{4}e^{\mu}_{4}e^{\nu}_{b}(-\Gamma^{\delta}_{\beta\mu}\Gamma^{\gamma}_{\delta\nu}+\Gamma^{\delta}_{\alpha\mu}\Gamma^{\gamma}_{\delta\nu})
\]
\[
= -\frac{1}{2}e^{\alpha}_{4}e^{\beta}_{4}e^{\mu}_{4}e^{\nu}_{b}\Gamma^{\delta}_{\beta\mu}\Gamma^{\gamma}_{\delta\nu} + \frac{1}{2}e^{\alpha}_{4}e^{\beta}_{4}e^{\mu}_{4}e^{\nu}_{b}\Gamma^{\delta}_{\alpha\mu}\Gamma^{\gamma}_{\delta\nu}
\]
\[
= -\frac{1}{2}e^{\alpha}_{4}e^{\beta}_{4}e^{\mu}_{4}e^{\nu}_{b}\delta^{\gamma}(\partial_{\gamma}g_{\mu\nu}+\partial_{\mu}g_{\beta\nu} - \partial_{\nu}g_{\beta\mu})(\partial_{\alpha}g_{\nu\delta} + \partial_{\nu}g_{\alpha\delta} - \partial_{\delta}g_{\alpha\nu})
\]
\[
+ \frac{1}{2}e^{\alpha}_{4}e^{\beta}_{4}e^{\mu}_{4}e^{\nu}_{b}\delta^{\gamma}(\partial_{\alpha}g_{\mu\nu}+\partial_{\mu}g_{\alpha\sigma} - \partial_{\sigma}g_{\alpha\mu})(\partial_{\beta}g_{\nu\delta} + \partial_{\beta}g_{\delta\beta} - \partial_{\delta}g_{\beta\nu}).
\]
Note that in the expansion of the right hand side, each term contains a product $e_k(g)\cdot e_l(g)$ where $l \neq 3$, except
\[
I := -\frac{1}{2}e^{\alpha}_{4}e^{\beta}_{4}e^{\mu}_{4}e^{\nu}_{b}\delta^{\gamma}\partial_{\gamma}g_{\mu\nu}\partial_{\delta}g_{\alpha\nu} + \frac{1}{2}e^{\alpha}_{4}e^{\beta}_{4}e^{\mu}_{4}e^{\nu}_{b}\delta^{\gamma}\partial_{\gamma}g_{\mu\nu}\partial_{\delta}g_{\beta\nu}.
\]
Now we apply $g^{\beta\nu} = \sum_{a} e^{\beta}_{a}e^{a}_{\beta} + \frac{1}{2}(e^{3\beta}_{3} + e^{4\beta}_{4})$. Then, we can also write $I$ as a sum of several terms containing $e_k(g)\cdot e_l(g)$ where $l \neq 3$. Since $e_l(g) = V_l(g) \cdot \mathcal{R}_{m+1}$, the whole sum is $\varepsilon \mathcal{R}_{m-3,0}$. Combine all the discussion above and we finish the proof.

(b) We have
\[
\Gamma^{\gamma}_{\alpha\beta}e^{\alpha}_{4}e^{\beta}_{4}\chi_{ab} = \frac{1}{2}g^{\gamma\mu}(e^{\beta}_{4}e_{4}(g_{\beta\mu}) + e^{\alpha}_{4}e_{4}(g_{\alpha\mu}) - e^{\alpha}_{4}e_{4}\partial_{\mu}g_{\alpha\beta})\chi_{ab}
\]
\[
= -\frac{1}{2}g^{\gamma\mu}e^{\alpha}_{4}e_{4}\partial_{\mu}g_{\alpha\beta}\chi_{ab} + \mathcal{R} = -\frac{1}{4}e^{\alpha}_{4}e_{4}(e_{3}(g_{\alpha\beta}) - e_{4}(g_{\alpha\beta}))\chi_{ab} + \mathcal{R}
\]
\[
= -\frac{1}{4}e^{\alpha}_{4}e_{4}(e_{3}(g_{\alpha\beta})\chi_{ab} + \mathcal{R}.
\]
Here the remainder $\mathcal{R}$ is a linear combination of $g \cdot (e^{\alpha}_{\ast}) \cdot e_{4}(g) \cdot \chi$ or $(e^{\alpha}_{\ast}) \cdot (e^{\beta}_{\ast}) \cdot e_{4}(g) \cdot \chi$. Since $e_{4}(g) = t^{-1}V_{4}(g) = \varepsilon \mathcal{R}_{m-2,0}$ and $(g, e^{\ast}) = \mathcal{R}_{m}$, under our assumption on $\chi$, it follows from the Leibniz’s rule that $\mathcal{R} = \varepsilon \mathcal{R}_{m-3,0}$.

Remark 4.14.1. Note we only have $\chi = \mathcal{R}^{m-1}$ from our induction hypotheses, so we cannot apply (b) directly assuming (4.18) only.

We now prove Proposition 4.13 for $|I| = m$. Fix a multiindex $I$ such that $|I| = m$. We have
\[
[V_{4}, V_{4}] = 0,
\]
\[
[V_{4}, V_{6}] = t(e^{i}_{4} - \omega_{i})\omega_{i}e_{a} - t(r\chi_{ab} - \delta_{ab})e_{b},
\]
\[
[V_{4}, V_{3}] = -t(e^{i}_{4} - \omega_{i})\omega_{i}e_{3} + (3R - r + t)e_{4} - t(3R - r + t)\xi^{i}_{34}e_{i}.
\]
We write $[V_4, V_k] := \eta_k V_i$. Then by Lemma 4.11, Lemma 4.12 and the induction hypotheses (4.18), we have

\[
\begin{align*}
\eta_a &= (e_i - \omega_i)\omega_i t r^{-1} - t(\chi_{aa} - r^{-1}) = \Omega^{m-1}_{-1,0}; \\
\eta_a' &= -t\chi_{12} = \Omega^{m-1}_{-1,0}; \quad a \neq a' \\
\eta_3^3 &= -t(e_i - \omega_i)(3R - r + t)^{-1} - t\xi_{34}^3 = \varepsilon\Omega^m_{0,-1}; \\
\eta_3^4 &= (3R - r + t)^{-1} - (3R - r + t)\xi_{34}^4 = \Omega^m_{-1,1}; \\
\eta_3^a &= -(3R - r + t)\xi_a^3 r t^{-1} = \Omega^{m-1}_{-1,0}; \\
\eta^*_a &= 0 \text{ in all other cases.}
\end{align*}
\]

(4.26)

In summary we have $\eta^*_a = \Omega^{m-1}_{-1,1}$. Here we briefly explain why $\eta^*_3 = \varepsilon\Omega^m_{0,-1}$, since all other estimates are clear. Note that $(e_i - \omega_i)\omega_i = \varepsilon\Omega^m_{-1,0}$ by Lemma 4.12. Also note that $\xi_{34}^4 = \xi_{34}^3 = e_i^4 e_i^3 \Gamma^0_{\alpha\beta} = \varepsilon\Omega^m_{0,-1}$. Thus,

\[
\eta_3^3 = -(3R - r + t)^{-1} t e_i (3R - r + t) - t\xi_{34}^3 = V_4(\ln(3R - r + t)) + O(\varepsilon).
\]

Next, we note that

\[
\begin{align*}
V_4(V^I(\xi_{k1k2})) &= \sum_{(j,k,k')} V^J[V_4, V_k] V^J(\xi_{k1k2}) + V^I(V_4(\xi_{k1k2})) \\
&= \sum_{(j,k,k')} V^J(\eta_k^I V_i(V^J(\xi_{k1k2}))) + V^I(V_4(\xi_{k1k2})) \\
&= \sum_{(j,k,k')} \eta^I_k V^J(\xi_{k1k2}) + \sum_{|J_1| + |J_2| = m, 0 < |J_1| < m} C_{J_1, J_2} V^{J_1}(\eta_k^I) V^{J_2}(\xi_{k1k2}) + V^I(V_4(\xi_{k1k2})) \\
&= Q_1 + Q_2 + Q_3.
\end{align*}
\]

In $Q_1$, we note that if $\eta^I_k \neq 0$, then we must have $n_{(j,l,l'),3} \leq n_{(j,k,k'),3}$. Recall that $n_{I,3}$ denotes the number of $V_3$ in the product $V^J$. This is because $\eta^*_3 \equiv 0$ for $k \neq 3$. In addition, we note that $n_{(j,l,l'),3} < n_{(j,k,k'),3}$ if $k = 3$ and $l \neq 3$. Then,

\[
\begin{align*}
Q_1 &= (n_{I,3}\eta^*_3 - \sum_{a} n_{I,a} n_{a}^0) V^I(\xi_{k1k2}) + O((|\eta_1^2| + |\eta_2^1|) \sum_{|J| = m} |V^J(\xi_{k1k2})|) \\
&\quad + O(\sum_{I \neq 3} |\eta^*_3| \sum_{(j,3,k2)} |V^J(J_1,l_2)(\xi_{k1k2})|) \\
&\quad + n_{I,3} V_4(\ln(3R - r + t)) V^I(\xi_{k1k2}) + O((- + t^{-1} + \varepsilon) \sum_{|J| = m, n_{J,3} = n_{I,3}} |V^J(\xi_{k1k2})|) \\
&\quad + O((q)t^{-1} + \varepsilon) \sum_{|J| = m, n_{J,3} < n_{I,3}} |V^J(\xi_{k1k2})|).
\end{align*}
\]

(4.28)
In $Q_2$, we have $|J_1|, |J_2| < m$. Since $\eta_s^* = \mathcal{R}_{-1,1}^{m-1}$, we have

\begin{equation}
|Q_2| \lesssim \sum_{|J_1|+|J_2|=m \atop 0<|J|<m} |V^{J_1}(\mathcal{R}_{-1,1}^{m-1})V^{J_2}(\mathcal{R}_{-1,1}^{m})| \lesssim t^{-1+C\varepsilon(q)} \sum_{0<|J|<m} |V^{J}(\mathcal{R}_{-1,1}^{m})|.
\end{equation}

Now we combine (4.27) with Section 4.1. First, note that $\xi_{04}^3 = \Gamma_{0\alpha\beta}^0 e_4^\alpha e_4^\beta = \varepsilon \mathcal{R}_{-1,1}^{m}$ by Lemma 4.11 so $|V^{J}(\xi_{04}^3)| \lesssim t^{-1+C\varepsilon(q)}$ whenever $|I| \leq m$. There is no need to apply (4.27).

Next, we consider $\chi_{ab} = \xi_{ab}^{04}$.

**Proposition 4.15.** Under our induction hypotheses (4.18), for $|I| = m$ we have

$$|V^{J}(\chi_{ab})| \lesssim t^{-1+C\varepsilon}, \quad |V^{J}(\chi_{ab} - r^{-1}\delta_{ab})| \lesssim t^{-2+C\varepsilon}.$$ 

So $\chi_{ab} = \mathcal{R}_{-1,0}^{m}$ and $\chi_{ab} - r^{-1}\delta_{ab} = \mathcal{R}_{-2,0}^{m}$.

**Proof.** We first prove that $V^{J}(\chi_{ab}) = O(t^{-1+C\varepsilon})$ whenever $|I| = m$. Fix $I$ such that $|I| = m$ and $n_{1,3} = n \leq m$. Recall from (4.18) that $\chi_{ab} = \mathcal{R}_{-1,0}^{m-1}$ and $\chi_{ab} - r^{-1}\delta_{ab} = \mathcal{R}_{-2,0}^{m-1}$. Suppose that we have proved $V^{J}(\chi_{ab}) = O(t^{-1+C\varepsilon})$ for all $J$ such that $|J| = m$ and $n_{1,3} < n$. Note that

$$\chi_{ac}\chi_{cb} = \delta_{ab}r^{-2} + 2(\chi_{ab} - \delta_{ab}r^{-1})r^{-1} + (\chi_{ac} - \delta_{ac}r^{-1})(\chi_{cb} - \delta_{cb}r^{-1}).$$

By Lemma 4.11 we have $r^{-1} = \mathcal{R}_{-1,0}^{m+1}$ and $t = \mathcal{R}_{1,0}^{m+1}$. Also note that $V(tr^{-1}) = V((t-r)r^{-1}) = \mathcal{R}_{m+1,1}^{m}$. Thus, the Raychaudhuri equation, we have

$$V^{J}(V^{J}_{4}(\chi_{ab})) = V^{J}(t\Gamma_{0\alpha\beta}^0 e_4^\alpha e_4^\beta \chi_{ab}) - \sum_c \chi_{ac}\chi_{cb} + V^{J}(t R(e_4, e_4)e_4) + O(t^{-1+C\varepsilon})$$

$$= t\Gamma_{0\alpha\beta}^0 e_4^\alpha e_4^\beta V^{J}(\chi_{ab}) + O(\sum_{|J_1|+|J_2|=m \atop |J|<m} |V^{J_1}(\mathcal{R}_{0,1}^{m})V^{J_2}(\chi_{ab})|)$$

$$- 2t r^{-1}V^{J}(\chi_{ab} - \delta_{ab}r^{-1})V^{J}(\chi_{ab} - \delta_{ab}r^{-2})t + O((\varepsilon t^{-3+C\varepsilon} + t^{-1+C\varepsilon}) |V^{J}(\chi_{ab} - \delta_{ab}r^{-1})|)$$

$$+ V^{J}(\varepsilon \mathcal{R}_{-2,1}^{m})$$

$$= -2t r^{-1}V^{J}(\chi_{ab}) + O((\varepsilon + t^{-1+C\varepsilon}) |V^{J}(\chi_{ab})|) + O(t^{-1+C\varepsilon}).$$
Besides, by (4.28) and our induction hypotheses, we have

\[
|Q_1 - nV_4(\ln(3R - r + t))V^I(\chi_{ab})| \lesssim \varepsilon \sum_{|J|=m, n_{I,J} = n} |V^J(\chi_{ab})| + \langle q \rangle t^{-1+C\varepsilon} \sum_{|J|=m, n_{I,J} < n} |V^J(\chi_{ab})| 
\]

\[
\lesssim \varepsilon \sum_{|J|=m, n_{I,J} = n} |V^J(\chi_{ab})| + \langle q \rangle t^{-2+C\varepsilon}.
\]

By (4.29) and our induction hypotheses, we have

\[
|Q_2| \lesssim t^{-1+C\varepsilon} \langle q \rangle \sum_{|J|<m} |V^J(\chi_{ab})| \lesssim t^{-2+C\varepsilon} \langle q \rangle.
\]

In conclusion, by (4.27) we have

\[
|e_4(V^I(\chi_{ab}) + (-ne_4(\ln(3R - r + t)) + 2r^{-1})V^I(\chi_{ab})| 
\]

\[
\lesssim t^{-1}(|Q_1 - nV_4(\ln(3R - r + t))V^I(\chi_{ab})| + |Q_2| + |V^I(V_4(\chi_{ab})| + 2t^{-1}V^I(\chi_{ab}))| 
\]

\[
\lesssim \varepsilon^{-1} \sum_{c,c'} \sum_{n_{I,J} = n} |V^J(\chi_{cc'})| + t^{-2+C\varepsilon} + \langle q \rangle t^{-3+C\varepsilon} \leq \varepsilon^{-1} \sum_{c,c'} \sum_{n_{I,J} = n} |V^J(\chi_{cc'})| + t^{-2+C\varepsilon}.
\]

The last inequality holds as \( \langle q \rangle \lesssim t \). By Lemma 4.10 with \( n_0 = 2 \), \( n_1 = n \) and Lemma 4.8 we conclude that

\[
\sum_{a,b} \sum_{|I|=m, n_{I,J} = n} |V^I(\chi_{ab})| \lesssim t^{-2+C\varepsilon}(x^0(0)^2 \cdot x^0(0)^{-1+C\varepsilon} + \int_{x^0(0)}^{t} t^{2+C\varepsilon} \cdot t^{-1-C\varepsilon} d\tau)
\]

\[
\lesssim t^{-2+C\varepsilon} \cdot t^{1+C\varepsilon} \lesssim t^{-1+C\varepsilon}.
\]

By induction we obtain \( \chi_{ab} = \Phi_{-1,0}^m \).

Next we prove \( V^I(\chi_{ab} - r^{-1} \delta_{ab}) = O(t^{-2+C\varepsilon}) \) whenever \( |I| = m \). Again fix \( I \) such that \( |I| = m \) and \( n_{I,J} = n \leq m \). Suppose we have proved that \( V^J(\chi_{ab} - r^{-1} \delta_{ab}) = O(t^{-2+C\varepsilon}) \) for \( |J| = m \) and \( n_{I,J} < n \). Now we can apply Lemma 4.14. We have

\[
V^I(V_4(\chi_{ab})) = V^I(t \Gamma_{\alpha\beta}^a e_4^\alpha e_4^\beta t \chi_{ab} + t \Phi_{-3,0}^m ) + V^I(V_4(f_{ab})) + \frac{1}{4} e_4^\alpha e_4^\beta t^{-1} \delta_{ab} e_3(g_{ab}) + t \Phi_{-3,0}^m 
\]

\[
- 2t^{-1} V^I(\chi_{ab} - \delta_{ab}r^{-1}) - V^I(\delta_{ab}r^{-2}) + O(t^{-3+C\varepsilon} \langle q \rangle + t^{-1+C\varepsilon} |V^I(\chi_{**} - r^{-1} \delta_{**})|)
\]

\[
= V^I(-\frac{1}{4} e_4^\alpha e_4^\beta e_3(g_{ab}) t \chi_{ab} + t \Phi_{-3,0}^m ) + V^I(V_4(f_{ab})) + O(t^{-2+C\varepsilon})
\]

\[
- 2t^{-1} V^I(\chi_{ab} - \delta_{ab}r^{-1}) - V^I(\delta_{ab}r^{-2}) + O(t^{-3+C\varepsilon} \langle q \rangle + t^{-1+C\varepsilon} |V^I(\chi_{**} - r^{-1} \delta_{**})|).
\]

Also note that

\[
V^I(V_4(r^{-1})) = V^I(te_4(r^{-1})) = V^I(-tr^{-2}e_4(r))
\]
and that $e_4(r) - 1 = \varepsilon \mathfrak{R}^{m}_{1,0}$ by Lemma 4.12. In conclusion,

$$V^I(V_4(\chi_{ab} - r^{-1}\delta_{ab} - f_{ab}))$$

$$= V^I(-\frac{1}{4}e_4^\alpha e_4^\beta e_3(\alpha\beta)t(\chi_{ab} - r^{-1}\delta_{ab})) - 2tr^{-1}V^I(\chi_{ab} - \delta_{ab}r^{-1}) + V^I(\delta_{ab}r^{-2}t(e_4(r) - 1))$$

$$+ O(t^{-3+\varepsilon}\langle q \rangle + \varepsilon t^{-2+C\varepsilon} + t^{-1+C\varepsilon}|V^I(\chi_{**} - r^{-1}\delta_{**})|)$$

$$= V^I(-\frac{1}{4}e_4^\alpha e_4^\beta e_3(\alpha\beta)t(\chi_{ab} - r^{-1}\delta_{ab})) - 2tr^{-1}V^I(\chi_{ab} - \delta_{ab}r^{-1})$$

$$+ O(t^{-3+\varepsilon}\langle q \rangle + \varepsilon t^{-2+C\varepsilon} + t^{-1+C\varepsilon}|V^I(\chi_{**} - r^{-1}\delta_{**})|).$$

Besides, we note that

$$V^I(-\frac{1}{4}e_4^\alpha e_4^\beta e_3(\alpha\beta)t(\chi_{ab} - r^{-1}\delta_{ab})) + \frac{1}{4}e_4^\alpha e_4^\beta e_3(\alpha\beta)tV^I(\chi_{ab} - r^{-1}\delta_{ab})$$

is a linear combination of terms of the form

$$V^{I_1}(e_4^\alpha e_4^\beta t(3R - r + t)^{-1}V_3(\alpha\beta))V^{I_2}(\chi_{ab} - r^{-1}\delta_{ab})$$

where $|I_1| + |I_2| = |I| = m$ and $|I_2| < m$. By the induction hypotheses and since

$$e_4^\alpha e_4^\beta t(3R - r + t)^{-1}V_3(\alpha\beta) = \mathfrak{R}^{m}_{1,1} \cdot \varepsilon \mathfrak{R}^{m}_{1,0} = \varepsilon \mathfrak{R}^{m}_{0,1}$$

by Lemma 4.11, we conclude that

$$V^I(-\frac{1}{4}e_4^\alpha e_4^\beta e_3(\alpha\beta)t(\chi_{ab} - r^{-1}\delta_{ab})) + \frac{1}{4}e_4^\alpha e_4^\beta e_3(\alpha\beta)tV^I(\chi_{ab} - r^{-1}\delta_{ab}) = O(\varepsilon t^{-2+C\varepsilon}\langle q \rangle^{-1}).$$

Thus, by setting $F_{ab} = \chi_{ab} - r^{-1}\delta_{ab} - f_{ab} = \mathfrak{R}^{m-1}_{-2,0}$ and noting that $f_{ab} = \varepsilon \mathfrak{R}^{m-1}_{-2,0}$, we have

$$V^I(V_4(F_{ab})) = -2tr^{-1}V^I(F_{ab} + f_{ab}) + O(\varepsilon |V^I(F_{ab} + f_{ab})|)$$

$$+ O(\varepsilon t^{-2+C\varepsilon} + t^{-3+C\varepsilon}\langle q \rangle + t^{-1+C\varepsilon}|V^I(F_{**} + f_{**})|)$$

$$= -2tr^{-1}V^I(F_{ab}) + O(\varepsilon |V^I(F_{ab})|) + \varepsilon t^{-2+C\varepsilon} + t^{-3+C\varepsilon}\langle q \rangle + t^{-1+C\varepsilon}|V^I(F_{**})|).$$

In (4.27), (4.28) and (4.29), we can replace $\xi_{k_1k_2}$ with $F_{ab}$. Thus, we have $V_4(V^I(F_{ab})) = Q_1 + Q_2 + V^I(V_4(F_{ab}))$, where by the induction hypotheses we have

$$Q_1 = nV_4(\ln(3R - r + t))V^I(F_{ab}) + O(\varepsilon \sum_{|J|=m \atop n,j_3=n} |V^J(F_{ab})|) + O(\langle q \rangle t^{-1+C\varepsilon} \sum_{|J|=m \atop n,j_3<n} |V^J(F_{ab})|)$$

$$= nV_4(\ln(3R - r + t))V^I(F_{ab}) + O(\varepsilon \sum_{|J|=m \atop n,j_3=n} |V^J(F_{ab})|) + O(\langle q \rangle t^{-3+C\varepsilon}),$$

$$|Q_2| \lesssim \langle q \rangle t^{-1+C\varepsilon} \sum_{0<|J|<m} |V^J(F_{ab})| \lesssim \langle q \rangle t^{-3+C\varepsilon}.$$
By Lemma \[4.10\] with \( n_0 = 2, n_1 = n \) and Lemma \[4.8\] we have
\[
\sum_{a,b} \sum_{n_{I,3}=n}^{[I]=m} |V^I(F_{ab})| \lesssim t^{-2+C\varepsilon}(x^0(0)^{C\varepsilon} + \int_{x^0(0)}^t \langle q \rangle \tau^{-2+C\varepsilon} + \varepsilon \tau^{-1+C\varepsilon} \, d\tau)
\]
\[
\lesssim t^{-2+C\varepsilon}(x^0(0)^{C\varepsilon} + \langle q \rangle (x^0(0))^{-1+C\varepsilon} + t^{C\varepsilon}) \lesssim t^{-2+C\varepsilon}.
\]
Here we recall that \( t \geq x^0(0) \sim T_0 + \langle q \rangle \). We then finish the proof by induction. \( \square \)

Next, we consider \( \xi^a_{12} \).

**Proposition 4.16.** Under our induction hypotheses \( 4.18 \), for \( |I| = m \), we have
\[
|V^I(\xi^a_{12})| \lesssim t^{-1+C\varepsilon}.
\]
So \( \xi^a_{12} = \mathcal{R}^{-m}_{-1,0} \).

**Proof.** Fix \( I \) such that \( |I| = m \) and \( n_{I,3} = n \leq m \). Recall from \( 4.18 \) that \( \xi^a_{12} = \mathcal{R}^{-m+1}_{-1,0} \).

Suppose that \( V^I(\xi^a_{12}) = O(t^{-1+C\varepsilon}) \) for \( |J| = m \) and \( n_{J,3} < n \). By the equation in Section \( 4.1 \) we have
\[
(4.30) \quad V^I(V_4(\xi^a_{12})) = V^I(t\Gamma^0_{\alpha\beta}e^a_2\chi_1 - t\Gamma^0_{\alpha\beta}e^a_1\chi_2) - V^I(t\chi_{ac}\xi^c_{12}) + V^I(tR(e_4, e_a)e_2, e_1)).
\]
By Lemma \( 4.13 \) the last term is \( O(\varepsilon \langle q \rangle^{-1}t^{-1+C\varepsilon}) \). By Lemma \( 4.11 \) and Proposition \( 4.15 \) we note that
\[
t\Gamma^0_{\alpha\beta}e^a_2\chi_1 - t\Gamma^0_{\alpha\beta}e^a_1\chi_2 = \mathcal{R}^{-m+1}_{1,0} \cdot \varepsilon \mathcal{R}^{-m+1}_{-1,-1} \cdot \mathcal{R}^m_{0,0} \cdot \mathcal{R}^m_{-1,0} = \varepsilon \mathcal{R}^{-m+1}_{-1,-1}.
\]

Thus, the first term in \( 4.30 \) is also \( O(\varepsilon \langle q \rangle^{-1}t^{-1+C\varepsilon}) \). Next, by the Leibniz’s rule we have
\[
|V^I(t\chi_{ac}\xi^c_{12}) - t\chi_{ac}V^I(\xi^c_{12})| \lesssim \sum_{|J_1|+|J_2|=m \atop |J_1|>0} |V^{J_1}(t\chi_{ac})V^{J_2}(\xi^c_{12})|
\]
\[
\lesssim \sum_{|J_1|+|J_2|=m \atop |J_1|>0} (|V^{J_1}(t\chi_{ac} - \delta_{ac}r^{-1})|V^{J_2}(\xi^c_{12})) + |V^{J_1}(t\chi_{ac} - \delta_{ac}r^{-1})V^{J_2}(\xi^c_{12})|).
\]
By Proposition \( 4.15 \) we have \( t(\chi_{ac} - \delta_{ac}r^{-1}) = \mathcal{R}^{-m}_{m-1,1} \). Also recall that \( V(tr^{-1}) = V((t - r)r^{-1}) = \mathcal{R}^{-m}_{m-1,1} \). Thus,
\[
|V^I(t\chi_{ac}\xi^c_{12}) - tr^{-1}V^I(\xi^c_{12})| \lesssim |V^I(t\chi_{ac}\xi^c_{12}) - t\chi_{ac}V^I(\xi^c_{12})| + |t\chi_{ac} - r^{-1}\delta_{ac}V^I(\xi^c_{12})|
\]
\[
\lesssim t^{-2+C\varepsilon} \langle q \rangle + t^{-1+C\varepsilon|V^I(\xi^c_{12})|}.
\]
In conclusion, we have
\[
V^I(V_4(\xi^a_{12})) = -tr^{-1}V^I(\xi^a_{12}) + O(t^{-1+C\varepsilon}|V^I(\xi^c_{12})| + t^{-2+C\varepsilon}\langle q \rangle + \varepsilon \langle q \rangle^{-1}t^{-1+C\varepsilon}).
\]
Moreover, by \( 4.28 \), we have
\[
|Q_1 - nV_4(\ln(3R - r + t))V^I(\xi^c_{12})| \lesssim \varepsilon \sum_{|J|=m \atop n_{J,3}<n} |V^{J}(\xi^c_{12})| + \langle q \rangle t^{-1+C\varepsilon} \sum_{|J|=m \atop n_{J,3}<n} |V^{J}(\xi^c_{12})|
\]
\[
\lesssim \varepsilon \sum_{|J|=m \atop n_{J,3}<n} |V^{J}(\xi^c_{12})| + \langle q \rangle t^{-2+C\varepsilon}.
\]
By \( (4.29) \), we have
\[
|Q_2| \lesssim t^{-1+C\varepsilon} \langle q \rangle \sum_{0<|J|<m} |V^J(\xi_{12}^a)| \lesssim t^{-2+C\varepsilon} \langle q \rangle.
\]
Thus,
\[
|e_4(V^I(\xi_{12}^a)) + (-n e_4(\ln(3R - r + t)) + r^{-1})V^I(\xi_{12}^a)| \\
\lesssim \varepsilon t^{-1} \sum_{|J|=m \atop n_{i,j}=n} |V^J(\xi_{12}^a)| + t^{-2+C\varepsilon} |V^I(\xi_{12}^a)| + t^{-3+C\varepsilon} \langle q \rangle + \varepsilon \langle q \rangle^{-1} t^{-2+C\varepsilon}.
\]
We now apply Lemma 4.10 with \( n_0 = 1, n_1 = n \) and Lemma 4.8. Then,
\[
\sum_a \sum_{|I|=m \atop n_{i,j}=n} |V^I(\xi_{12}^a)| \lesssim t^{-1+C\varepsilon} (x^0(0)C\varepsilon + \int_{x^0(0)}^t \tau^{-2+C\varepsilon} \langle q \rangle + \varepsilon \langle q \rangle^{-1} \tau^{-1+C\varepsilon} \, d\tau) \\
\lesssim t^{-1+C\varepsilon} (x^0(0)C\varepsilon + x^0(0)^{-1+C\varepsilon} \langle q \rangle + \varepsilon \langle q \rangle^{-1} t^{C\varepsilon}) \lesssim t^{-1+C\varepsilon}.
\]
Again recall that \( t \geq x^0(0) \sim \langle q \rangle + T_0 \). We finish the proof by induction. \( \square \)

Next we study \( \xi_{34}^a \). The proof of the following proposition is very similar to that of the previous one.

**Proposition 4.17.** Under our induction hypotheses \( (4.18) \), for \( |I| = m \), we have
\[
|V^I(\xi_{34}^a)| \lesssim t^{-1+C\varepsilon} \langle q \rangle^{-1}.
\]
So \( \xi_{34}^a = \mathfrak{R}_{m-1,-1} \).

**Proof.** Fix \( I \) such that \( |I| = m \) and \( n_{i,j} = n \leq m \). Recall from \( (4.18) \) that \( \xi_{34}^a = \mathfrak{R}_{m-1,-1} \).
Suppose that \( V^J(\xi_{34}^a) = O(t^{-1+C\varepsilon} \langle q \rangle^{-1}) \) for \( |J| = m \) and \( n_{i,j} < n \). By the equation in Section 4.1 we have
\[
V^I(V_4(\xi_{34}^a)) = -V^I(t \chi_{ba} \xi_{34}^b) + V^I(t R(e_4, e_4, e_a)) + 2V^I(V_4(\Gamma_{a}^{0} e_{4}^{a} e_{4}^{a})).
\]
By Lemma 4.13 the second term is \( O(\varepsilon t^{-1+C\varepsilon} \langle q \rangle^{-1}) \). In the third term, we note that
\[
V_4(\Gamma_{a}^{0} e_{4}^{a} e_{4}^{a}) = V_4(\Gamma_{a}^{0} e_{4}^{a} e_{a}^{a} + \Gamma_{a}^{0} V_4 e_{4}^{a} e_{a}^{a} + \Gamma_{a}^{0} e_{4}^{a} V_4 e_{a}^{a}) \\
= \varepsilon \mathfrak{R}_{m-1,-1} \cdot \varepsilon \mathfrak{R}_{-1,0} \cdot \varepsilon \mathfrak{R}_{m-1,-1} \cdot \varepsilon \mathfrak{R}_{-1,0} = \varepsilon \mathfrak{R}_{m-1,-1}.
\]
We recall from Remark 4.11.1 that \( e_4(e_4^*) = \varepsilon \mathfrak{R}_{m-2,0} \). Thus, \( V^I(V_4(\Gamma_{a}^{0})) = O(\varepsilon \langle q \rangle^{-1} t^{-1+C\varepsilon}) \).
Following the computation in Proposition 4.10, we can prove that
\[
|V^I(t \chi_{ba} \xi_{34}^b) - tr^{-1}V^I(\xi_{34}^a)| \lesssim |V^I(t \chi_{ab} \xi_{34}^b) - t \chi_{ab} V^I(\xi_{34}^b)| + |t(\chi_{ab} - r^{-1}\delta_{ab})V^I(\xi_{34}^b)| \\
\lesssim \sum_{|J|+|J|+m \atop |J| > 0} |(|V^J(t(\chi_{ab} - \delta_{ab}r^{-1}))V^J(\xi_{34}^b)| + |V^J(t(r^{-1})V^J(\xi_{34}^b)) + t^{-1+C\varepsilon}|V^I(\xi_{34}^b)| \\
\lesssim t^{-2+C\varepsilon} + t^{-1+C\varepsilon}|V^I(\xi_{34}^a)|.
\]
Moreover, by (4.28) we have
\[ |Q_1 - nV_1(\ln(3R - r + t))V^I(\xi_{a3}^a)| \lesssim \varepsilon \sum_{|J| = m} \sum_{n, J, m = n} |V^J(\xi_{a3}^a)| + |q|t^{-1+C\varepsilon} \sum_{|J| = m} \sum_{n, J, m = n} |V^J(\xi_{a3}^a)| \]
\[ \lesssim \varepsilon \sum_{|J| = m} \sum_{n, J, m = n} |V^J(\xi_{a3}^a)| + t^{-2+C\varepsilon}. \]
By (4.29), we have
\[ |Q_2| \lesssim t^{-1+C\varepsilon} |q| \sum_{0 < |J| < m} |V^J(\xi_{a3}^a)| \lesssim t^{-2+C\varepsilon}. \]
Thus,
\[ |e_4(V^I(\xi_{a3}^a)) + (-ne_4(\ln(3R - r + t)) + r^{-1})V^I(\xi_{a3}^a)| \]
\[ \lesssim \varepsilon t^{-1} \sum_{|J| = m} |V^J(\xi_{a3}^a)| + t^{-2+C\varepsilon} |V^I(\xi_{a3}^a)| + t^{-3+C\varepsilon} + \varepsilon |q|^{-1}t^{-2+C\varepsilon}. \]
We now apply Lemma 4.10 with \( n_0 = 1, n_1 = n \) and Lemma 4.8. Then,
\[ \sum_a \sum_{|J| = m} \sum_{n, J, m = n} |V^I(\xi_{a12}^a)| \lesssim t^{-1+C\varepsilon} (x^0(0)C\varepsilon |q|^{-1} + \int t \tau^{-2+C\varepsilon} + \varepsilon |q|^{-1}\tau^{-1+C\varepsilon} d\tau) \]
\[ \lesssim t^{-1+C\varepsilon} (x^0(0)C\varepsilon |q|^{-1} + x^0(0)^{-1+C\varepsilon} + |q|^{-1}t^{-2+C\varepsilon}) \lesssim t^{-1+C\varepsilon} |q|^{-1}. \]
Again recall that \( t \geq x^0(0) \sim |q| + T_0. \) We finish the proof by induction. \( \square \)

Finally, we consider \( \xi_{a3}^l. \) The case when \( l \in \{a, 3\} \) is easy.

**Proposition 4.18.** Under our induction hypotheses (4.18), for \( |I| = m, \) we have
\[ |q|V^I(\xi_{a3}^3) + |q|V^I(\xi_{a3}^a) \lesssim t^{-1+C\varepsilon}. \]
So \( \xi_{a3}^3 = \mathcal{R}_{-1,-1} \) and \( \xi_{a3}^a = \mathcal{R}_{-1,0}. \)

**Proof.** Recall from Section 4.11 that
\[ \xi_{a3}^3 = -2\Gamma^a_{\alpha\beta} \varepsilon_4 e_4 e_\beta + \frac{1}{2} \xi_{a3}^a, \quad \xi_{a3}^a = \chi_{aa} + 2e_a(g^{0\alpha})g_{\alpha\beta}e_\beta + 2g^{0\alpha}e_\beta \Gamma^\mu_{\beta\alpha}g_{\mu\nu}e_\nu. \]

Now we apply Lemma 4.11. Since \( \Gamma = \varepsilon \mathcal{R}_{-1,-1}^{n+1} \) and \( (g, e^*_a) = \mathcal{R}_{-1,0} \), we have \( \Gamma^0_{\alpha\beta} \varepsilon_4 e_4 e_\beta = \varepsilon \mathcal{R}_{-1,-1} \) and \( g^{0\alpha}e_\beta \Gamma^\mu_{\beta\alpha}g_{\mu\nu}e_\nu = \varepsilon \mathcal{R}_{-1,-1}. \) Since \( e_4(g^{0\alpha}) = t^{-1}V_4(g) = \varepsilon \mathcal{R}_{-2,0} \) and \( e_a(g^{0\alpha}) = r^{-1}V_a(g) = \varepsilon \mathcal{R}_{-2,0} \), we have \( e_a(g^{0\alpha})g_{\alpha\beta}e_\beta = \varepsilon \mathcal{R}_{-2,0}. \) We thus conclude that
\[ (\xi_{a3}^3, \xi_{a3}^a) = \left( \frac{1}{2} \xi_{a3}^a, \chi_{aa} \right) + \varepsilon \mathcal{R}_{-1,-1}. \]
We finally apply Proposition 4.15, Proposition 4.16 and Proposition 4.17 to conclude that \( \xi_{a3}^3 = \mathcal{R}_{-1,-1} \) and \( \xi_{a3}^a = \mathcal{R}_{-1,0}. \)

The case \( l = a' \) where \( \{a, a'\} = \{1, 2\} \) is harder.
Proposition 4.19. Under our induction hypotheses (4.18), for \(|I| = m\), we have

\[ |V^I(\xi'_{a_3})| \lesssim \langle q \rangle^{-1} t^{C\varepsilon}. \]

So \( \xi'_{a_3} = \mathcal{N}_{0,-1}^m \).

Proof. Fix \( I \) such that \(|I| = m\) and \( n_{I,3} = n \leq m \). Recall from (4.18) that \( \xi'_{a_3} = \mathcal{N}_{0,-1}^m \).

Suppose that \( V^J(q_{a_3}) = O(\langle q \rangle^{-1} t^{C\varepsilon}) \) for \(|J| = m\) and \( n_{J,3} < n \). By the equation in Section 4.1 we have

\[
V^I(V_4(\xi'_{a_3})) = V^I((V_4 + t\Gamma_{\mu\nu}^0 e_4^\mu e_4^\nu)(\chi_{a_0} + 2e_a (g^{0\alpha}) g_{\alpha\beta} e_{a'}^\beta + 2g^{0\alpha} e_a^\beta \Gamma_{\beta\mu}^\alpha g_{\mu\nu} e_{a'}^\nu)) - V^I(t\Gamma_{\mu\nu}^0 e_4^\mu e_4^\nu(\mathcal{N}_{0,-1}^m)) \\
- \sum_c V^I(t\xi_{a_{34}}(\mathcal{N}_{0,-1}^m)) - V^I(t(R(e_4, e_3) e_a, e_{a'})) - V^I(t\Gamma_{\alpha\beta}^0 e_a^\alpha e_a^\beta + t\Gamma_{\alpha\beta}^0 e_a^\alpha e_a^\beta(\mathcal{N}_{-1,-1}^m)).
\]

By the Leibniz's rule and all the previous results, we conclude that the second line has an upper bound

\[ t^{-1+C\varepsilon} \langle q \rangle^{-1} + \varepsilon \langle q \rangle^{-1} t^{-1+C\varepsilon} \lesssim t^{-1+C\varepsilon} \langle q \rangle^{-1}. \]

In the first line, we note that

\[
t\Gamma_{\mu\nu}^0 e_4^\mu e_4^\nu(2e_a (g^{0\alpha}) g_{\alpha\beta} e_{a'}^\beta + 2g^{0\alpha} e_a^\beta \Gamma_{\beta\mu}^\alpha g_{\mu\nu} e_{a'}^\nu) = \varepsilon \mathcal{N}_{0,-1}^m \cdot (\varepsilon \mathcal{N}_{-2,0}^m + \varepsilon \mathcal{N}_{-1,1}^m) = \varepsilon^2 \mathcal{N}_{-1,-2}^m.
\]

Besides, since \( \chi_{a_{00}} = \mathcal{N}_{-2,0}^m \) and since \( \sum_c \chi_{a_{00} a_{00}} = \chi_{12} \text{ tr } \chi \), we have

\[
|V^I(V_4(\chi_{a_0} + t\Gamma_{\mu\nu}^0 e_4^\mu e_4^\nu(\chi_{a_0}))| \\
\lesssim |V^I(2t\Gamma_{\alpha\beta}^0 e_4^\alpha e_4^\beta(\chi_{a_0}))| + |V^I(t(R(e_4, e_3) e_a, e_{a'}))| \\
\lesssim |V^I(\varepsilon \mathcal{N}_{-2,-1}^m)| + |V^I(t\mathcal{N}_{1,0}^m \cdot \mathcal{N}_{-2,0}^m \cdot \mathcal{N}_{-1,0}^m)| + |V^I(\varepsilon \mathcal{N}_{-1,-1}^m)| \lesssim t^{-2+C\varepsilon} + \varepsilon t^{-1+C\varepsilon} \langle q \rangle^{-1} \lesssim t^{-1+C\varepsilon} \langle q \rangle^{-1}.
\]

Moreover, recall that \( V_4(\xi_{a_3}) = \varepsilon \mathcal{N}_{-1,0}^m \). We also have \( \partial g = \varepsilon \mathcal{N}_{-1,-1}^m \) by Remark 4.11. Thus, we have

\[
V_4(2e_a (g^{0\alpha}) g_{\alpha\beta} e_{a'}^\beta + 2g^{0\alpha} e_a^\beta \Gamma_{\beta\mu}^\alpha g_{\mu\nu} e_{a'}^\nu) = 2V_4(e_a (g^{0\alpha}) g_{\alpha\beta} e_{a'}^\beta + \varepsilon \mathcal{N}_{-1,-1}^m \\
= 2e_a V_4(\partial g^{0\alpha}) g_{\alpha\beta} e_{a'}^\beta + 2V_4(e_a^\sigma \partial g^{0\alpha}) g_{\alpha\beta} e_{a'}^\beta + \varepsilon \mathcal{N}_{-1,-1}^m = \varepsilon \mathcal{N}_{-1,-1}^m.
\]

In conclusion,

\[
|V^I(V_4(\xi'_{a_3})| \lesssim |V^I(t\Gamma_{\mu\nu}^0 e_4^\mu e_4^\nu(\mathcal{N}_{0,-1}^m))| + t^{-1+C\varepsilon} \langle q \rangle^{-1} \\
\lesssim |t\Gamma_{\mu\nu}^0 e_4^\mu e_4^\nu V^I(V_4(\xi_{a_3}'))| + \sum_{|J_1| + |J_2| = m, |J_2| < m} |V^J(\varepsilon \mathcal{N}_{0,-1}^m) V^J_{J_2}(\xi_{a_3}'))| + t^{-1+C\varepsilon} \langle q \rangle^{-1} \\
\lesssim \varepsilon |V^I(\xi_{a_3}')| + \varepsilon \langle q \rangle^{-2+C\varepsilon} + t^{-1+C\varepsilon} \langle q \rangle^{-1}.
\]

Next, by (4.28), we have

\[
|Q_1 - nV_4(\ln(3R - r + t)) V^I(\xi_{a_3}')| \lesssim \varepsilon \sum_{|J| = m, n_{J,3} = n} |V^J(\xi_{a_3}')}| + \langle q \rangle^{-1+C\varepsilon} \sum_{|J| = m, n_{J,3} < n} |V^J(\xi_{a_3}')| \\
\lesssim \varepsilon \sum_{|J| = m, n_{J,3} = n} |V^J(\xi_{a_3}')}| + t^{-1+C\varepsilon}.
\]
By (4.29), we have

\[ |Q_2| \lesssim t^{-1+C_\varepsilon(q)} \sum_{0<|J|<m} |V^J(\xi_{\alpha_3}')| \lesssim t^{-1+C_\varepsilon}. \]

Thus,

\[ \left| e_4(V^I(\xi_{\alpha_3}')) - ne_4(\ln(3R - r + t))V^I(\xi_{\alpha_3}') \right| \lesssim \varepsilon t^{-1} \sum_{|J|=m \atop n,J,3=n} |V^J(\xi_{\alpha_3}')| + \varepsilon \langle q \rangle^{-2}t^{1+C_\varepsilon} + t^{-2+C_\varepsilon}. \]

By Lemma 4.10 with \( n_0 = 0, n_1 = n \) and Lemma 4.8 we have

\[ \sum_{a,a'} \sum_{|I|=m \atop n,I,3=n} |V^I(\xi_{a,a}')| \lesssim t^{C_\varepsilon} \langle q \rangle^{-1} x^0(0)^{C_\varepsilon} + \int_0^t \varepsilon \langle q \rangle^{-2} \tau^{-1+C_\varepsilon} + \tau^{-2+C_\varepsilon} d\tau \]

\[ \lesssim t^{C_\varepsilon} \langle q \rangle^{-1} t^{C_\varepsilon} + \langle q \rangle^{-2} t^{C_\varepsilon} + (x^0(0))^{-1+C_\varepsilon} \lesssim \langle q \rangle^{-1} t^{C_\varepsilon}. \]

We finish the proof by induction. \( \square \)

Combining Proposition 4.13 and 4.19 we finish the proof of Proposition 4.9 by induction.

4.4. Estimates for higher derivatives of \( q \). Now we can prove the estimates for higher derivatives of \( q \). We first note that (4.26) holds for each \( m \geq 1 \), as long as \( \varepsilon \ll_m 1 \). This is because (4.26) is a result of (4.18) which then results from Proposition 4.9.

Lemma 4.20. In \( \Omega \cap \{ r - t < 2R \} \), we have \( V^I q = O(\langle q \rangle t^{C_\varepsilon}) \) for each multiindex \( I \).

Proof. We induct on \(|I|\). If \(|I| = 0\), there is nothing to prove. If \(|I| = 1\), the estimates are clear since \( V_1(q) = V_2(q) = V_4(q) = 0 \) and \( V_3(q) = O((3R - r + t)\partial q) = O(\langle q \rangle t^{C_\varepsilon}) \).

In general, we fix an integer \( m \geq 1 \). By choosing \( \varepsilon \ll_m 1 \), we can assume that Proposition 4.9 holds for all \(|I| \leq m \). Suppose we have proved the estimates for \(|I| < m \), so \( q = R_{0,1}^{m-1}. \)

Fix a multiindex \( I \) such that \(|I| = m \). If \( n_{I,4} > 0 \), we can write \( I = (J',4,J) \). Here we can assume \(|J| > 0\) since otherwise we have \( V^I(q) = V_{J'}(V_4(q)) = 0 \). By (4.26), we have

\[ V^I(q) = V^{J'}(V_4(V^J(q))) = \sum_{J=(J_1,k,J_2)} V^{(J',J_1)}[V_4,V_k]V^J(q) \]

\[ = \sum_{J=(J_1,k,J_2)} V^{(J',J_1)}(\eta_k^{J_1} V^{(l,J_2)}(q)) = \sum_{J=(J_1,k,J_2)} V^{(J',J_1)}(R_{0,1}^{m-1}, R_{0,1}^{m-1}(1+|J_2|)) \]

\[ = \sum_{J=(J_1,k,J_2)} V^{(J',J_1)}(R_{0,1}^{m-1}(1+|J_2|)) = O(\langle q \rangle^{2} t^{1+C_\varepsilon}) = O(\langle q \rangle t^{C_\varepsilon}). \]

Here we note that \(|J_2| + 1 = |J| - |J_1| = m - 1 - |J'| - |J'_1|\), so we are able to apply the definition of \( R_{***} \) here.

Next suppose \( n_{I,3} < m \) and \( n_{I,4} = 0 \). Thus we can write \( I = (J',a,J) \) where \( n_{I,3} = |J| \). Here we can assume \(|J| > 0\) since \( V_a(q) = 0 \). Then

\[ V^I(q) = V^{J'}V_a(V^J(q)) = \sum_{J=(J_1,3,J_2)} V^{(J',J_1)}[V_a,V_3]V^J(q). \]
Note that
\[ [V_a, V_3]F = V_a((3R - r + t) e_3(F)) - V_3(r e_a(F)) \]
\[ = V_a(3R - r + t) e_3(F) - V_3(r) e_a(F) + (3R - r + t) r [e_a, e_3](F) \]
\[ = -(3R - r + t)^{-1} V_a(r) V_3(F) - r^{-1} V_3(r) V_a(F) \]
\[ + (3R - r + t) \xi_{a3} V_b(F) + r \xi_{a3} V_3(F) + (3R - r + t) r t^{-1} \xi_{a3} V_4(F). \]

By Lemma 4.11 and Remark 4.11.1 we have \( V_a(r) = R_{01}^m, V_3(r) = (3R - r + t) e_3^i \omega_i = R_{01}^m. \)

By Proposition 4.9 we have
\[ [V_a, V_3] = \sum_{k=1}^4 R_{001}^m, V_k = R_{001}^m, V. \]

Thus,
\[ V^I(q) = \sum_{J=(J_1, J_2)} V^{(J_1, J_2)}(R_{001}^m, V(V^{J_2}(q))) \]
\[ = \sum_{J=(J_1, J_2)} V^{(J_1, J_2)}(R_{001}^m, R_{01}^{m-1}(1+|J_2|)) = O(t^C\varepsilon(q)). \]

Again, we have \( m - 1 = 1 + |J_2| + |J_1| + |J|. \)

Finally, suppose \( n_{I, 3} = |I|. \) We have
\[ V_4(V^I(q)) = \sum_{I=(I_3, J_2)} V^{J_1}[V_4, V_3][V^{J_2}(q)] = \sum_{I=(I_3, J_2)} V^{J_1}(\eta_3^I V^{(I, J_2)}(q)). \]

By the Leibniz’s rule, we can express \( V^{J_1}(\eta_3^I V^{(I, J_2)}(q)) \) as a linear combination of terms of the form \( V^{K_1}(\eta_3^I) V^{K_2}(q) \), where \( |K_1| + |K_2| = m, K_2 \) contains \( I \), and \( (K_1, K_2) \) is an rearrangement of \( (J_1, I, J_2). \) Now recall from (4.26) that \( \eta_3^I = R_{-11}^{m-1} + \varepsilon R_{00}^m. \) Since \( V^I(q) = O((q)t^C\varepsilon) \) for \( |J| = m \) and \( n_{I, 3} < |J| \), we have
\[ V^{J_1}(\eta_3^I V^{(I, J_2)}(q)) \]
\[ = \eta_3^I V^I(q) + O(\sum_{|K_1| + |K_2| = m, 0 < |K_1| < m} |V^{K_1}(\eta_3^I) V^{K_2}(q)|) \]
\[ + O(\sum_{K_1, K_2} |V^{K_1}(\eta_3^I) V^{K_2}(q)|) \]
\[ = (t e_4(\ln(3R - r + t)) + O(\varepsilon)) V^I(q) + O(\sum_{0 < |K_1| < m} |V^{K_1}(\varepsilon R_{00}^m + R_{-11}^{m-1})| \cdot t^C\varepsilon(q)) \]
\[ + O(\sum_{|K_1| < m} |V^{K_1}(\varepsilon R_{00}^m + R_{-11}^{m-1})| \cdot (q)t^C\varepsilon) \]
\[ = t e_4(\ln(3R - r + t)) V^I(q) + O(\varepsilon |V^I(q)|) + O(\varepsilon t^C\varepsilon + t^{-1+C\varepsilon}(q)^2). \]

Thus,
\[ |e_4(V^I(q)) - m e_4(\ln(3R - r + t)) V^I(q)| \lesssim t^{-1} |V^I(q)| + \varepsilon t^{-1+C\varepsilon} + t^{-2+C\varepsilon}(q)^2. \]
Recall from Remark 4.7.1 that \( V^I(q) = O(t^{C \varepsilon}(q)) \) on \( H \). Then, by Lemma 4.10 with \( n_0 = 0 \) and \( n_1 = |I| \), we have

\[
|V^I(q)| \lesssim t^{C \varepsilon}(\langle q \rangle x^0(0) t^{C \varepsilon} + \int_{x^0(0)}^t \varepsilon \tau^{-1+C \varepsilon} + \tau^{-2+C \varepsilon} (q)^2 \, d\tau) \\
\lesssim t^{C \varepsilon}(\langle q \rangle t^{C \varepsilon} + t^{C \varepsilon} + (x^0(0))^{-1+C \varepsilon} (q)^2) \lesssim \langle q \rangle t^{C \varepsilon}.
\]

\[\square\]

We have the following important corollary.

**Corollary 4.21.** The function \( q(t, x) \) is a smooth function (in the sense defined in Section 2.4) in \( \Omega \). Moreover, we have \( Z^I q = O(\langle q \rangle t^{C \varepsilon}) \) and \( Z^I \Omega_{ij} q = O(t^{C \varepsilon}) \) for each multiindex \( I \) and \( 1 \leq i < j \leq 3 \).

**Proof.** Fix an integer \( m > 1 \). We seek to prove that for \( \varepsilon \ll m \), \( q \) is a \( C^m \) function and \( Z^I q = O(\langle q \rangle t^{C \varepsilon}) \) for \( |I| \leq m \). By writing \( Z = z^\nu(t, x) \partial_\nu \), we have

\[
\langle Z, e_\alpha \rangle = r^{-1} z^\alpha e_\alpha^\beta g_{\alpha \beta} = \mathcal{A}_{0,0}^m, \\
\langle t^{-1} Z, e_3 \rangle = t^{-1} z^\alpha e_3^\beta g_{\alpha \beta} = \mathcal{A}_{0,0}^m.
\]

Moreover,

\[
\langle Z, e_4 \rangle = z^\alpha e_4^\beta g_{\alpha \beta} = z^\alpha e_4^\beta (g_{\alpha \beta} - m_{\alpha \beta}) + z^\alpha e_4^\beta m_{\alpha \beta} \\
= \varepsilon \mathcal{A}_{0,0}^m - z^0 + z^i (e_i^\beta - \omega_i) + z^i \omega_i = \mathcal{A}_{0,1}^m + Z(r - t).
\]

We can easily check that \( Z(r - t) = \mathcal{A}_{0,1}^m \), so \( (3R - r + t)^{-1} \langle Z, e_4 \rangle = \mathcal{A}_{0,0}^m \). Then, by (4.2), \( Z = \mathcal{A}_{0,0}^m \cdot V \), so \( Z^I q \) is a linear combination of terms of the form

\[
Z^{I_1} (\mathcal{A}_{0,0}^m) \cdots Z^{I_s} (\mathcal{A}_{0,0}^m) V^s(q), \\
\sum |I_s| + s = |I|, \ s > 0.
\]

Each of such terms is \( O(t^{C \varepsilon}(q)) \) if \( |I| \leq m \), so we have \( Z^I q = O(t^{C \varepsilon}(q)) \) for \( |I| \leq m \).

Moreover, for each \( m > 1 \), as long as \( \varepsilon \ll m \), we have \( q = \mathcal{A}_{0,1}^{m+1} \) by Lemma 4.20. Then we have

\[
\Omega_{ij} q = \frac{1}{2} \langle \Omega_{ij}, e_3 \rangle e_3(q) = \frac{1}{2} (x_i g_{j \beta} - x_j g_{i \beta}) e_3^\beta e_3(q) \\
= \frac{1}{2} (x_i m_{jk} - x_j m_{ik}) \omega_k e_3(q) + \frac{1}{2} (x_i (g_{jk} - m_{jk}) - x_j (g_{ik} - m_{ik})) \omega_k e_3(q) \\
+ \frac{1}{2} (x_i g_{jk} - x_j g_{ik}) (e_4^k - \omega_k) e_3(q) \\
= 0 + \varepsilon \mathcal{A}_{0,0}^m + \mathcal{A}_{0,0}^m = \mathcal{A}_{0,0}^m.
\]

Again, for each multiindex \( I \) with \( |I| \leq m \), we can write \( Z^I \Omega_{ij} q \) as a linear combination of terms of the form

\[
Z^{I_1} (\mathcal{A}_{0,0}^m) \cdots Z^{I_s} (\mathcal{A}_{0,0}^m) V^s \Omega_{ij}(q), \\
\sum |I_s| + s = m, \ s > 0.
\]

Each of such terms is \( O(t^{C \varepsilon}) \), so we have \( Z^I \Omega_{ij} q = O(t^{C \varepsilon}) \) for \( |I| \leq m \). 

\[\square\]
4.5. More estimates. We end this section with some estimates derived from our original wave equation (1.1). We first introduce a new definition.

**Definition.** Let $F = F(t, x)$ be a function with domain $\Omega \cap \{r - t < 2R\}$. For any integer $m \geq 0$ and any real numbers $s, p$, we have defined $F = \mathcal{R}_s^m$ in Section 4.3 prior to Lemma 4.11. We now define $F = \mathcal{R}_s^m$, if $F = \mathcal{R}_s^m$ for each $m \geq 0$.

Again, by the Leibniz’s rule, we have $V^I(\mathcal{R}_s^p) = \mathcal{R}_s^p$ and $\mathcal{R}_s^1 \cdot \mathcal{R}_s^2 = \mathcal{R}_s^{1+2}$. In addition, by Proposition 4.9 we have

\[
(q^2, \xi^1) = \mathcal{R}_s^{0, -1}; \quad (\xi^2, \xi^3) = \mathcal{R}_s^{0, 0}; \quad (\xi^k, \xi^{k'}) = \mathcal{R}_s^{0, 0} \quad \text{for all} \quad k < k' \quad \text{and} \quad a = 1, 2;
\]

\[
\xi^3 = \mathcal{R}_s^{0, -1} \quad \text{for all} \quad k_1 < k_2;
\]

\[
\chi_{ab} - \delta_{ab} = \mathcal{R}_s^{0, 0}.
\]

There are many other estimates in Section 4.3 involving $\mathcal{R}_s^m$. They would still hold if all the superscripts are removed, because they all rely on Proposition 4.9. For example, by Lemma 4.11 we have

\[
e_* = \mathcal{R}_s^{0, 0}, \quad (e_* - \omega, e_* - \omega) = \mathcal{R}_s^{0, 0}; \quad \partial^* Z^{(g - m)} = \varepsilon \mathcal{R}_s^{0, s}; \quad \Gamma_* = \varepsilon \mathcal{R}_s^{0, 0};
\]

\[
\omega = \mathcal{R}_s^{0, 0}, \quad (t^*, r^*) = \mathcal{R}_s^{0, 0}, \quad (3R - r + t)^s = \mathcal{R}_s^{0, s}.
\]

We remark that this definition follows the spirit of the convention in Section 2.4. In the definition of $\mathcal{R}_s^m$, we require some estimates to hold for all $\varepsilon \ll s, p, m$. The dependence on $m$ here should be emphasized.

Our goal in this subsection is to prove that

\[
e_4(e_3(u)) + r^{-1}e_3(u) = \varepsilon \mathcal{R}_s^{0, 0}; \quad e_4(e_3(u)) = \varepsilon \mathcal{R}_s^{0, 0};
\]

\[
e_4(e_3(q)) = -\frac{1}{4} e_3(u) G(\omega) e_3(q) + \varepsilon \mathcal{R}_s^{0, 0}.
\]

We start our proof with the following lemma.

**Lemma 4.22.** We have the following estimates.

(a) $q_s = \mathcal{R}_s^{0, 0}$, $q^{-1}_s = \mathcal{R}_s^{0, 0}$; $e_k(q) = \mathcal{R}_s^{0, 0}$, $e_k(q^{-1}) = \mathcal{R}_s^{0, 0}$ for $k \neq 3$.

(b) $q_t + \omega_t q_t = \mathcal{R}_s^{0, 0}$, $u_t + \omega_t u_t = \mathcal{R}_s^{0, 0}$.

(c) $e_k(q_t + \omega_t q_t) = \mathcal{R}_s^{0, 0}$, $e_k(u_t + \omega_t u_t) = \mathcal{R}_s^{0, 0}$, for $k \neq 3$.

(d) In (b) and (c) we can replace $q_t + \omega_t q_t$ with $q_t + q_r$ or $q_t - \omega_t q_r$, and replace $u_t + \omega_t u_t$ with $u_t + u_r$ or $u_t - \omega_t u_r$. The results are the same.

**Proof.** (a) By Lemma 4.20 we have $V_3(q) = \mathcal{R}_s^{0, 0}$ and $e_3(q) = \mathcal{R}_s^{0, 0}$. Then,

\[
q_s = \frac{1}{2} g_{\alpha \beta} e_{\alpha} e_{\beta} = \mathcal{R}_s^{0, 0}; \quad \mathcal{R}_s^{0, 0}; \quad \mathcal{R}_s^{0, 0}; \quad \mathcal{R}_s^{0, 0}.
\]

Since $\omega = \mathcal{R}_s^{0, 0}$, we have $q_r = \mathcal{R}_s^{0, 0}$. Since $q_r \geq C^{-1} t^{-C_\varepsilon}$ and since $V^I(q^{-1})$ is a linear combination of terms of the form

\[
q_r^{-s-1} V^I(q_r) \cdots V^I(q_r), \quad \text{where} \quad \sum |I_j| = |I|, \quad |I_j| > 0,
\]

we conclude that $V^I(q^{-1}) = O(t^{C_\varepsilon})$ for each $I$ and thus $q^{-1}_s = \mathcal{R}_s^{0, 0}$. Besides, we have

\[
e_k(e_3(q)) = [e_k, e_3] q = \xi_k e_3(q), \quad k = 1, 2, 3, 4;
\]

\[
2\omega_t g_{i \beta} e_{i \beta} = (e_3 + e_4, e_4) = (2\omega_t - e_4 - e_3) g_{i \beta} e_{i \beta} = 2 + \mathcal{R}_s^{0, 0}.
\]
Thus, for $k \neq 3$,
\[
e_k(q_r) = e_k\left(\frac{1}{2}\omega_i g_i \beta e_4^4 e_3(q)\right) = e_k\left(\frac{1}{2}\omega_i g_i \beta e_4^4 e_3(q) + \frac{1}{2} \omega_i g_i \beta e_4^4 e_k(e_3(q))\right)
\]
\[
= e_k\left(\frac{1}{2} + \Re_{-1,0}\right) e_3(q) + \left(\frac{1}{2} + \Re_{-1,0}\right) e_k^3 e_3(q)
\]
\[
= \Re_{-1,0} \cdot V_k(\Re_{-1,0}) \cdot \Re_{0,0} + \Re_{-1,-1} = \Re_{-1,-1}.
\]

Now if we expand $V'(e_k(q_{r-1}))$, each term is still of the form $\Re_{s,0}$ with $s > 0$ and $V'(e_k(q_r))$ replaced by $V'(e_k(q_r))$. We thus conclude that $e_k(q_{r-1}) = \Re_{-1,-1}$ for $k \neq 3$.

(b) We have
\[
q_i + \omega_i q_t = \frac{1}{2} (g_i + \omega_i g_0) e_4^4 e_3(q)
\]
and
\[
u_i + \omega_i u_t = \frac{1}{2} (g_i + \omega_i g_0) e_4^4 e_3(u) + \frac{1}{2} (g_i + \omega_i g_0) e_4^4 e_4(u) + \sum_a (g_i + \omega_i g_0) e_4^4 e_a(u)
\]
\[
= \frac{1}{2} (g_i + \omega_i g_0) e_4^4 (3R - r + t)^{-1} V_3(u) + \varepsilon \Re_{-2,0}.
\]
Here we have
\[
(g_i + \omega_i g_0) e_4^4 = e_4^4 - \omega_i + ((g_i - m_i) + \omega_i (g_0 - m_0)) e_4^4 = \Re_{-1,0}.
\]
We thus conclude that $q_i + \omega_i q_t = \Re_{-1,0}$ and $u_i + \omega_i u_t = \varepsilon \Re_{-2,0}$.

(c) Recall that $e_a(r) = \Re_{-1,0}$, $e_4(\omega_i) = r^{-1}(e_4^4 - \omega_i + (1 - e_4^4 \omega_i) \omega_i) = \Re_{-2,0}$ and $e_4(e_a^4) = \varepsilon \Re_{-2,0}$ by Lemma 4.11 and Lemma 4.12. Besides, note that
\[
e_a(\omega_i) = r^{-1}(e_a^4 - e_a(\omega_i) \omega_i) = r^{-1} e_a^4 + \Re_{-2,0},
\]
\[
e_4(\omega_i) = (e_4^4 - \omega_j) \partial_j \omega_i = r^{-1}(e_4^4 - \omega_i - (e_4^4 - \omega_j) \omega_j \omega_i) = \Re_{-2,0}.
\]
Thus we have
\[
e_a((g_i + \omega_i g_0) e_4^4) = e_a(g_i + \omega_i g_0) e_4^4 + (g_i + \omega_i g_0) e_a(e_4^4)
\]
\[
= (e_a(g_i) + \omega_i e_a(g_0)) e_4^4 + (g_i + \omega_i g_0)(\xi_a^4 e_4^4 + e_4(e_a^4))
\]
\[
= (\varepsilon \Re_{-2,0} + (r^{-1} e_a^4 + \Re_{-2,0}) g_0) e_4^4 + (g_i + \omega_i g_0)(\xi_a^4 e_b^4 + \varepsilon \Re_{-2,0})
\]
\[
= r^{-1} e_a^4 g_0 \beta e_4^4 + r^{-1}(g_i + \omega_i g_0) e_a^4 + (g_i + \omega_i g_0)(\xi_a^4 e_b^4 + \varepsilon \Re_{-2,0})
\]
\[
= r^{-1}(-e_a^4 + e_a^4 (g_i + \omega_i g_0) e_4^4 + \xi_a^4 e_a^4 + (g_i + \omega_i g_0) (\xi_a^4 e_a^4)) + \Re_{-2,0} = \Re_{-2,0},
\]
and
\[
e_4((g_i + \omega_i g_0) e_4^4) = e_4(g_i + \omega_i g_0) e_4^4 + (g_i + \omega_i g_0) e_4(e_4^4)
\]
\[
= (e_4(g_i) + \omega_i e_4(g_0) + e_4(\omega_i) g_0) e_4^4 + \varepsilon \Re_{-2,0}
\]
\[
= \Re_{-2,0} + \varepsilon \Re_{-2,0} = \Re_{-2,0}.
\]
Since $(g_i + \omega_i g_0) e_4^4 = \Re_{-1,0}$ and $e_k(e_3(q)) = \xi_k e_3(q) = \Re_{-1,-1}$, we conclude from the Leibniz’s rule that for $k \neq 3$,
\[
e_k(q_i + \omega_i q_t) = \frac{1}{2} e_k((g_i + \omega_i g_0) e_4^4 e_3(q) + \frac{1}{2}(g_i + \omega_i g_0) e_4^4 e_k(e_3(q))
\]
\[
= \Re_{-2,0} \cdot \Re_{0,0} + \Re_{-1,0} \cdot \Re_{-1,-1} = \Re_{-2,0}.
\]
Besides,
\[ u_i + \omega_i u_t = r^{-1} \sum_j \omega_j \Omega_{ij} u + r^{-1} \omega_i S u + r^{-1} \omega_i (t + r)^{-1} (t S u - \sum_j x_j \Omega_{0j} u) = \mathcal{R}_{-1,0} \cdot Zu. \]

Note that \( Zu = \varepsilon \mathcal{R}_{-1,0} \) and \( e_k = \mathcal{R}_{-1,0} \cdot V \) for \( k \neq 3 \). We conclude that
\[
e_k(u_i + \omega_i u_t) = e_k(\mathcal{R}_{-1,0}) \cdot Zu + \mathcal{R}_{-1,0} \cdot e_k(Zu)
= \mathcal{R}_{-1,0} \cdot V_k(\mathcal{R}_{-1,0}) \cdot \varepsilon \mathcal{R}_{-1,0} + \mathcal{R}_{-1,0} \cdot \mathcal{R}_{-1,0} \cdot V_k(\varepsilon \mathcal{R}_{-1,0}) = \varepsilon \mathcal{R}_{-3,0}.
\]

(d) This part follows directly from
\[
\partial_t + \partial_r = \sum \omega_i (\partial_i + \omega_i \partial_t), \quad \partial_t - \omega_i \partial_r = \partial_t + \omega_i \partial_t - \sum \omega_j \omega_j (\partial_j + \omega_j \partial_t).
\]

\[ \square \]

**Proposition 4.23.** We have \( e_4(e_3(u)) + r^{-1} e_3(u) = \varepsilon \mathcal{R}_{-3,0} \) and \( e_4(e_3(ru)) = \varepsilon \mathcal{R}_{-2,0} \).

**Proof.** Note that
\[
g^{\alpha \beta}(u) \partial_{\alpha} \partial_{\beta} \mathcal{R}_{-3,0} = \sum_a e_a^\alpha e_a^\beta \partial_{\alpha} \partial_{\beta} u + \frac{1}{2} e_a^\alpha e_a^\beta \partial_{\alpha} \partial_{\beta} u + \frac{1}{2} e_a^\alpha e_a^\beta \partial_{\alpha} \partial_{\beta} u
= \sum_a \left( e_a(e_a(u)) - e_a^\alpha e_a^\beta \partial_{\alpha} u + e_4(e_3(u)) - e_4(e_3^\alpha) \partial_{\alpha} u. \right)
\]

Here we have
\[
e_a(e_a^\alpha) \partial_{\alpha} u
= -\xi_{a\beta} e_a^\alpha e_a^\beta(u) - \frac{1}{2} \chi_{\alpha \beta} (e_3(u) + e_4(u)) - (e_a \cdot e_a (g^{\alpha \beta}) \partial_{\beta} + g^{\alpha \beta} e_a^\alpha e_\beta e_{\mu} e_{\nu} e_{a e_{\mu} e_{\nu} e_{a \mu} e_{\nu} u_{\alpha}}
= -\xi_{a\beta} e_a^\alpha e_a^\beta(u) - \frac{1}{2} \chi_{\alpha \beta} (e_3(u) + e_4(u)) - (e_a g_{\alpha \beta} e_a (g^{\alpha \beta}) + e_a g_{\mu \nu} g^{\alpha \beta} e_a e_{\mu \nu} e_{\alpha \beta} e_4(u) - e_a e_\alpha e_{\mu \nu} u_{\alpha}
= -\frac{1}{2} \chi_{\alpha \beta} e_3(u) - e_a e_\alpha e_{\mu \nu} u_{\alpha} + \varepsilon \mathcal{R}_{-3,0}
\]
and
\[
e_4(e_3^\alpha) \partial_{\alpha} u = \varepsilon \mathcal{R}_{-2,0} \cdot \varepsilon \mathcal{R}_{-1,1} = \varepsilon^2 \mathcal{R}_{-3,1}.
\]

In addition, for \( k, l \neq 3 \), we have
\[
e_k^\mu e_l^\nu \Gamma_{\mu \nu} e_{a \alpha} = \frac{1}{2} g^{\alpha \beta} (\partial_{\mu} g_{\nu \beta} + \partial_{\nu} g_{\mu \alpha} - \partial_{\beta} g_{\mu \nu}) e_k^\mu e_l^\nu u_{\alpha}
= \frac{1}{2} g^{\alpha \beta} e_k(g_{\mu \beta} e_l^\mu u_{\alpha}) + \frac{1}{2} g^{\alpha \beta} e_l(g_{\mu \alpha}) e_k^\mu u_{\alpha} - \frac{1}{2} g^{\alpha \beta} \partial_{\beta} g_{\mu \nu} e_k^\mu e_l^\nu u_{\alpha}
= \varepsilon^2 \mathcal{R}_{-3,1} - \frac{1}{2} \sum c e_c(g_{\mu \nu}) e_c(u) e_k^\mu e_l^\nu - \frac{1}{4} e_3(g_{\mu \nu}) e_4(u) e_k^\mu e_l^\nu - \frac{1}{4} e_4(g_{\mu \nu}) e_3(u) e_k^\mu e_l^\nu
= \varepsilon^2 \mathcal{R}_{-3,1}.
\]
Since $\chi_{ab} - \delta_{ab} r^{-1} = \mathcal{R}_{-2,0}$ and $e_3(u) = (3R - r + t)^{-1}V_3(u) = \varepsilon \mathcal{R}_{-1,-1}$, their product is $\varepsilon \mathcal{R}_{-3,-1}$. Thus we have

$$0 = \sum_a e_a(e_a(u)) + e_4(e_3(u)) + \frac{1}{2}\text{tr}\,\chi e_3(u) + \varepsilon \mathcal{R}_{-3,0}$$

$$= \sum_a e_a(e_a(u)) + e_4(e_3(u)) + r^{-1}e_3(u) + \varepsilon \mathcal{R}_{-3,0}.$$ Next, as in Lemma 3.14 we set

$$h_i := r(\partial_i(\rho u) - q_i q^{-1}_r \partial_r(\rho u)) = -r(u + ru_r) q^{-1}_r(q_i - \omega_i q_r) + r^2(u_i - \omega_i u_r).$$

Recall from Lemma 3.14 that

$$e_a(\rho u) = \sum_i e_a(\omega_i) h_i.$$ We claim that $h_i = \varepsilon \mathcal{R}_{0,0}$ and $e_a(h_i) = \varepsilon \mathcal{R}_{-1,0}$. In fact, note that $u + ru_r = \varepsilon \mathcal{R}_{-1,0} + \mathcal{R}_{1,0} \cdot \varepsilon \mathcal{R}_{-1,-1} = \varepsilon \mathcal{R}_{0,-1}$. We also recall that $e_a(r) = \mathcal{R}_{-1,0}$, so $e_a(r^{-1}) = -r^{-2}e_a(r) = \mathcal{R}_{-3,0}$. Thus by Lemma 4.22 we have $h_i = \varepsilon \mathcal{R}_{0,0}$ and $e_a(h_i) = \varepsilon \mathcal{R}_{-1,0}$. We thus have

$$e_4(e_3(u)) = e_4(\rho e_3(u)) + e_4(e_3(r)u) = e_4(e_3(u)) + e_4(r)e_3(u) + e_3(r)e_4(u) + e_4(e_3(r)u)$$

$$= -e_3(u) + e_4(r)e_3(u) + e_4(e_3\omega_i)u + \varepsilon r \mathcal{R}_{-3,0} + r \mathcal{R}_{-2,0}$$

$$= (e_4(r) - 1)e_3(u) + t^{-1}V_4(1 + (e_3\omega_i) u) + \varepsilon \mathcal{R}_{-2,0}$$

$$= \mathcal{R}_{-1,0} \cdot \varepsilon \mathcal{R}_{-1,-1} + \mathcal{R}_{-1,0} \cdot V_4(\mathcal{R}_{-1,0}) \cdot \varepsilon \mathcal{R}_{-1,0} + \varepsilon \mathcal{R}_{-2,0} = \varepsilon \mathcal{R}_{-2,0}. \quad \square$$

Next we prove an estimate for $e_3(q)$. We start with the following lemma.

**Lemma 4.24.** Fix a function $f \in C^\infty(\mathbb{R})$. Then, for $\varepsilon \ll 1$, $f(u) - f(0) - f'(0)u = \varepsilon^2 \mathcal{R}_{-2,0}$ where $u$ is a solution to (1.1).
Proof. For ε ≪ 1, we have \( f(u) - f(0) - f'(0)u = O(|u|^2) = O(\varepsilon^2 t^{-2+C\varepsilon}) \). Now, for each \( I \) with \( |I| > 0 \), we can write \( V^I(f(u)) - f'(u)(V^I u) \) as a linear combination of terms of the form
\[
f^{(s)}(u)V^{I_1}u \cdots V^{I_s}u, \quad \sum |I_s| = |I|, \ s \geq 2, \ |I_s| > 0.
\]
Since \( u = \varepsilon \Re_{-1,0} \), we can prove that each of these terms are \( O((\varepsilon t^{-1+C\varepsilon})^s) = O(\varepsilon^2 t^{-2+C\varepsilon}) \). Finally, note that \( f'(u)V^I u - f'(0)V^I u = O(|u| \cdot |V^I u|) = O(\varepsilon^2 t^{-1+C\varepsilon}) \). This finishes the proof.

Our main result is as follows.

**Proposition 4.25.** In \( \Omega \cap \{ r - t < 2R \} \), we have
\[
e_4(e_3(q)) = -\frac{1}{4}e_3(u)G(\omega)e_3(q) + \varepsilon \Re_{-2,0}.
\]

**Proof.** We recall that
\[
e_4(e_3(q)) = -\Gamma^0_{\alpha\beta}e_4^\alpha e_4^\beta e_3(q) = -\frac{1}{2}g^{0\nu}(e_4^\beta e_4(g_{\nu\beta}) + e_4^\alpha e_4(g_{\alpha\nu}))e_3(q) + \frac{1}{2}g^{0\nu}\partial_\nu g_{\alpha\beta}e_4^\alpha e_4^\beta e_3(q).
\]
Here \( e_3(q) = (3R - r + t)^{-1}V_3(q) = \Re_{0,0} \) and \( e_4(g) = t^{-1}V_4(g) = \varepsilon \Re_{-2,0} \). Thus,
\[
e_4(e_3(q)) = \frac{1}{2}g^{0\nu}\partial_\nu g_{\alpha\beta}e_4^\alpha e_4^\beta e_3(q) + \varepsilon \Re_{-2,0} = \frac{1}{4}(e_3 - e_4)(g_{\alpha\beta})e_4^\alpha e_4^\beta e_3(q) + \varepsilon \Re_{-2,0}
\]
\[
= \frac{1}{4}e_3(g_{\alpha\beta})e_4^\alpha e_4^\beta e_3(q) + \varepsilon \Re_{-2,0}.
\]
Recall that the coefficients \( (g^{\alpha\beta}(v)) \) in (1.1) are known smooth functions, and that for all \( |v| \ll 1 \) the matrix \( (g^{\alpha\beta}(v)) \) has a smooth inverse \( (g_{\alpha\beta}(v)) \). We differentiate \( g^{\alpha\sigma}(v)g_{\sigma\beta}(v) = \delta_{\alpha\beta} \) with respect to \( v \) and then set \( v = 0 \). Thus,
\[
\frac{d}{dv}g^{\alpha\sigma}|_{v=0} \cdot m_{\sigma\beta} + m^{\alpha\sigma} \cdot \frac{d}{dv}g_{\sigma\beta}|_{v=0} = 0.
\]
By setting \( g_{\alpha\beta}^0 = \frac{d}{dv}g_{\alpha\beta}|_{v=0} \) and \( g_{\alpha\beta}^0 = \frac{d}{dv}g_{\alpha\beta}|_{v=0} \), we conclude that
\[
g_{\alpha\beta}^0 = -m_{\alpha\alpha}m_{\beta\beta}g_{\alpha\beta}^0.
\]
Here we do not take sum over \( \alpha, \beta \). Thus we have
\[
g_{\alpha\beta}^0 e_4^\alpha e_4^\beta = -g_{00}^0 e_4^0 e_4^0 + 2g_0^0 e_4^j e_4^0 - g_0^j e_4^i e_4^j
\]
\[
= -G(\omega) + 2g_0^j (e_4^i - \omega_i) - g_0^ij e_4^i (e_4^j - \omega_j) - g_0^ij (e_4^i - \omega_i) \omega_j = -G(\omega) + \Re_{-1,0}.
\]
By the previous lemma we have
\[
e_4(e_3(q)) = \frac{1}{4}e_3(g_{\alpha\beta}^0 u)e_4^\alpha e_4^\beta e_3(q) + \varepsilon \Re_{-2,0} = -\frac{1}{4}e_3(u)G(\omega)e_3(q) + \varepsilon \Re_{-2,0}.
\]
5. The asymptotic equations and the scattering data

In Section 3, we have constructed a global optical function \( q(t, x) \) in \( \Omega \) such that \(-q_t, q_r \geq C^{-1}t^{-C_\varepsilon} > 0\). By setting

\[
\Omega' := \{(s, q, \omega) : s > 0, q > (\exp(\delta/\varepsilon) - \exp((s + \delta)/\varepsilon))/2 + 2R, \omega \in S^2\},
\]

we have an invertible map from \( \Omega \) to \( \Omega' \), defined by

\[
\Phi(t, r, \omega) = (s, q, \omega) := (\varepsilon \ln(t) - \delta, q(t, r\omega), \omega).
\]

In fact, we have \( t = \exp((s + \delta)/\varepsilon) \) and the map \( r \mapsto q(t, r\omega) \) is strictly increasing for each fixed \((t, \omega)\). Thus, \( \Phi \) is injective. Since \( q = r - t \) when \( r \geq t + 2R \), we have \( \lim_{r \to \infty} q(t, r\omega) = \infty \). Thus, \( \Phi \) is surjective. This gives us a new coordinate system \((s, q, \omega)\) on \( \Omega \).

In addition, \( \Phi \) is smooth since \( q \) is a smooth function. Its inverse \( \Phi^{-1} \) is also smooth, since we have \( q_r > 0 \). So, any smooth function \( F(t, x) \) induces a smooth function \( F \circ \Phi^{-1} \). With an abuse of notation, we still write \( F \circ \Phi^{-1}(s, q, \omega) \) as \( F(s, q, \omega) \).

We define

\[
(\mu, U)(t, x) = (q_t - q_r, \varepsilon^{-1}ru)(t, x), \quad (t, x) \in \Omega.
\]

Since \( q \) and \( u \) are both smooth, \( \mu(t, x) \) and \( U(t, x) \) are smooth. As discussed above, we also obtain two smooth functions \( \mu(s, q, \omega) \) and \( U(s, q, \omega) \) in \( \Omega' \). Our goal in this section is to derive a system of asymptotic equations for \((\mu, U)\) in the coordinate set \((s, q, \omega)\). Our main result is the following proposition.

**Proposition 5.1.** Let \((\mu, U)(s, q, \omega)\) be defined as above. Then, by writing \( t = \exp(\varepsilon^{-1}(s+\delta)) \) we have

\[
\begin{align*}
\partial_s \mu &= \frac{1}{4} G(\omega)\mu^2 U_q + \varepsilon^{-1} R_{-1,0}, \\
\partial_s U_q &= -\frac{1}{4} G(\omega)\mu U_q^2 + \varepsilon^{-1} R_{-1,0}.
\end{align*}
\]

In addition, the following three limits exist for all \((q, \omega) \in \mathbb{R} \times S^2\):

\[
\begin{align*}
A(q, \omega) &:= -\frac{1}{2} \lim_{s \to \infty} (\mu U_q)(s, q, \omega), \\
A_1(q, \omega) &:= \lim_{s \to \infty} \exp\left(\frac{1}{2} G(\omega) A(q, \omega) s\right) \mu(s, q, \omega), \\
A_2(q, \omega) &:= \lim_{s \to \infty} \exp\left(-\frac{1}{2} G(\omega) A(q, \omega) s\right) U_q(s, q, \omega).
\end{align*}
\]

All of them are smooth functions of \((q, \omega)\) for \( \varepsilon \ll 1 \). By setting

\[
\begin{align*}
\tilde{\mu}(s, q, \omega) &:= A_1 \exp\left(-\frac{1}{2} G As\right), \\
\tilde{U}_q(s, q, \omega) &:= A_2 \exp\left(\frac{1}{2} G As\right).
\end{align*}
\]

we obtain an exact solution to our reduced system

\[
\begin{align*}
\tilde{\mu}_s &= \frac{1}{4} G(\omega)\tilde{\mu}^2 \tilde{U}_q, \\
\tilde{U}_{sq} &= -\frac{1}{4} G(\omega)\tilde{\mu} \tilde{U}_q^2.
\end{align*}
\]
which satisfies the following estimates:

\[
\begin{align*}
(q)_{\partial_q}^m \partial^p_{\omega} (\mu U + 2A) &= O(t^{-1+C\varepsilon}), \\
(q)_{\partial_q}^m \partial^p_{\omega} (\exp(\frac{1}{2} G A s) \mu - A_1) &= O(t^{-1+C\varepsilon}), \\
(q)_{\partial_q}^m \partial^p_{\omega} (\exp(-\frac{1}{2} G A s) U - A_2) &= O(t^{-1+C\varepsilon}), \\
\partial^p_{\omega} ((q)_{\partial_q}^m \partial^p_{\omega} (\mu - \tilde{U} - U)) &= O(\varepsilon^{-p} t^{-1+C\varepsilon}).
\end{align*}
\]

Remark 5.1.1. Here \( A \) is called the scattering data.

After some preliminary computations in the new coordinate set \((s, q, \omega)\) in Section 5.1, we derive the asymptotic equations for \( \mu \) and \( U \) in Section 5.2 and Section 5.3, respectively. Next, in Section 5.4, we make use of the asymptotic equations to construct our scattering data. The main propositions in this subsection are Proposition 5.5 and Proposition 5.7.

Finally, in Section 5.5, we define an exact solution \((\tilde{\mu}, \tilde{U})(s, q, \omega)\) to our reduced system and we show that it provides a good approximation of \((\mu, U)(s, q, \omega)\).

5.1. Derivatives under the new coordinate. For convenience, from now on we make the following convention. For a function \( F = F(s, q, \omega) \) where \( \omega \in S^2 \), we extend it to all \( \omega \neq 0 \) by setting \( F(s, q, \lambda \omega) = F(s, q, \omega) \) for each \( \lambda > 0 \). Under such a setting, it is easy to compute the angular derivatives of \( F \) since we can now define \( \partial_{\omega_i} \). To avoid ambiguity, we will only use \( \partial_{\omega_i} \) in the coordinate \((s, q, \omega)\) and will never use it in the coordinate \((t, r, \omega)\).

First we explain how to compute the derivatives of \( U \) in \((s, q, \omega)\). Note by the chain rule, for any function \( F = F(s, q, \omega) = F(t, r, \omega) \) we have

\[
\begin{align*}
F_t &= \varepsilon t^{-1} F_s + q F_q \\
F_r &= q F_q
\end{align*}
\]

In addition, by the homogeneity, we have \( F(s, q, \omega) = F(s, q, \lambda \omega) \) and \( \partial_{\omega_i} F(s, q, \omega) = \lambda \partial_{\omega_i} F(s, q, \lambda \omega) \) for each \( \lambda > 0 \). At \((t, x)\), we set \( \lambda = |x| \) which gives

\[
F_i = q_i F_q + r^{-1} F_{\omega_i} \Rightarrow F_{\omega_i} = r(F_i - q_i r^{-1} F_r).
\]

Now we can explain the meaning of the function \( h_i \) defined in Lemma 3.14: it is the derivative of \( ru \) with respect to \( \omega_i \) under the coordinate \((s, q, \omega)\).

To simplify our future computations, we note that \( \partial_q, \partial_s \) and \( \partial_{\omega_i} \) commute with each other. In fact,

\[
[\partial_q, \partial_{\omega_i}] = [q^{-1} \partial_r, r \partial_t - r q q^{-1} \partial_r] = q^{-1} \partial_t - q^{-1} \partial_r(q q^{-1} \partial_r) - r \partial_t(q q^{-1} \omega)_j \partial_j + q q^{-1} \partial_r(q q^{-1}) \partial_r,
\]

\[
[\partial_s, \partial_q] = [\varepsilon^{-1} t \partial_t - \varepsilon^{-1} t q q^{-1} \partial_r, q^{-1} \partial_r] = \varepsilon^{-1} t \partial_t(q q^{-1}) \partial_r - \varepsilon^{-1} t q q^{-1} \partial_r(q q^{-1}) \partial_r + \varepsilon^{-1} t q q^{-1} \partial_r(q q^{-1}) \partial_r = 0,
\]

\[
[\partial_s, \partial_{\omega_i}] = [\varepsilon^{-1} t \partial_t - \varepsilon^{-1} t q q^{-1} \partial_r, q^{-1} \partial_r] = \varepsilon^{-1} t \partial_t(q q^{-1}) \partial_r - \varepsilon^{-1} t q q^{-1} \partial_r(q q^{-1}) \partial_r + \varepsilon^{-1} t q q^{-1} \partial_r(q q^{-1}) \partial_r = 0.
\]

\(81\)
\[ [\partial_\alpha, \partial_\omega] = [\varepsilon^{-1} t \partial_t - \varepsilon^{-1} t q r_1^{-1} \partial_r, r \partial_t - r q r_1^{-1} \partial_r] \]
\[ = -\varepsilon^{-1} t r \partial_t (q_r r_1^{-1}) \partial_r - \varepsilon^{-1} t q r_1^{-1} (\partial_t - \partial_r (r q r_1^{-1}) \partial_r) \]
\[ + \varepsilon^{-1} t r \partial_t (q_r r_1^{-1} \omega_3) \partial_j - \varepsilon^{-1} t r q r_1^{-1} \partial_r (q_t r_1^{-1}) \partial_r \]
\[ = -\varepsilon^{-1} t r q_t q_r^{-1} \partial_t - \varepsilon^{-1} t q r_1^{-2} \partial_t + \varepsilon^{-1} t q r_1^{-2} \partial_t (q_t \partial_r) \]
\[ + \varepsilon^{-1} t r q_t q_r^{-2} \partial_t (q_t \omega_t \partial_r) + \varepsilon^{-1} t q r_1^{-1} (\partial_t - \omega_t \partial_r) \]
\[ = \varepsilon^{-1} t q_t q_r^{-2} \partial_t - \varepsilon^{-1} t q r_1^{-2} (q_t - \omega_t q_r) \partial_r - \varepsilon^{-1} t q r_1^{-1} \omega_t \partial_r = 0. \]

Moreover, we can express \((\partial_\alpha, \partial_\beta, \partial_\omega)\) in terms of the weighted null frame \(\{V_k\}\).

**Lemma 5.2.** We have
\[
\partial_\alpha = \sum_a \varepsilon^{-1} \mathcal{R}_{-1,0} V_a + (\varepsilon^{-1} + \mathcal{R}_{-1,0}) V_4, \\
\partial_\omega = \sum_a \mathcal{R}_{-1,0} V_a + \sum_{k \neq 3} e^a_\alpha V_a = \sum_{k \neq 3} \mathcal{R}_{0,0} V_k, \\
\partial_\beta = \sum_k \mathcal{R}_{0,1} V_k. 
\]

**Proof.** We can express \(\partial_\alpha, \partial_\omega\) in terms of the null frame:
\[
\partial_\alpha = \varepsilon^{-1} t (g_{0\beta} e_\alpha^\beta e_a + \frac{1}{2} g_{0\beta} e_3 e_3 + \frac{1}{2} g_{0\beta} e_3 e_3) - \varepsilon^{-1} t q r_1^{-1} (\omega_t g_{1\beta} e_\alpha^\beta e_a + \frac{1}{2} \omega_t g_{1\beta} e_4 e_3 + \frac{1}{2} \omega_t g_{1\beta} e_3 e_4) \\
= \varepsilon^{-1} t ((g_{0\beta} - q_t q_r^{-1} \omega_t g_{1\beta}) e_\alpha^\beta e_a + \frac{1}{2} (g_{0\beta} - q_t q_r^{-1} \omega_t g_{1\beta}) e_3 e_4), \\
\partial_\omega = r (g_{i\beta} e_\alpha^\beta e_a + \frac{1}{2} g_{i\beta} e_3 e_3 + \frac{1}{2} g_{i\beta} e_3 e_4) - r q_t q_r^{-1} (\omega_t g_{j\beta} e_\alpha^\beta e_a + \frac{1}{2} \omega_t g_{j\beta} e_3 e_3 + \frac{1}{2} \omega_t g_{j\beta} e_3 e_4) \\
= r ((g_{i\beta} - q_t q_r^{-1} \omega_t g_{j\beta}) e_\alpha^\beta e_a + \frac{1}{2} (g_{i\beta} - q_t q_r^{-1} \omega_t g_{j\beta}) e_3 e_4). 
\]

We note that there is no term with \(e_3\) in \(\partial_\alpha\) and \(\partial_\omega\), since
\[
(g_{0\beta} - q_t q_r^{-1} \omega_t g_{1\beta}) e_4 = q_r^{-1} (g_{0\beta} - q_t q_r^{-1} \omega_t g_{1\beta}) e_4 = \frac{1}{2} q_r^{-1} e_3 (q_t (\omega_t g_{i\beta} e_4 g_{0\beta} e_3 - g_{0\beta} e_4 \omega_t g_{i\beta} e_4) = 0, \\
(g_{i\beta} - q_t q_r^{-1} \omega_t g_{j\beta}) e_4 = q_r^{-1} (g_{i\beta} - q_t q_r^{-1} \omega_t g_{j\beta}) e_4 = \frac{1}{2} q_r^{-1} e_3 (q_t (\omega_t g_{j\beta} e_4 g_{i\beta} e_3 - g_{i\beta} e_4 \omega_t g_{j\beta} e_4) = 0. 
\]

In these computations we use the equality \(q_\alpha = \frac{1}{2} g_{\alpha\beta} e_4^\beta e_3 (q)\). In addition, we have
\[
\varepsilon^{-1} t (g_{0\beta} - q_t q_r^{-1} \omega_t g_{1\beta}) e_a^\beta = \varepsilon^{-1} t ((g_{0\beta} - q_t q_r^{-1} \omega_t (g_{ij} - m_{ij})) e_a^\beta - \varepsilon^{-1} t q r_1^{-1} e_a (r) \\
= \mathcal{R}_{0,0} + \varepsilon^{-1} \mathcal{R}_{0,0} = \varepsilon^{-1} \mathcal{R}_{0,0}, \\
r (g_{i\beta} - q_t q_r^{-1} \omega_t g_{j\beta}) e_a^\beta = r ((g_{i\beta} - m_{ij}) - q_t q_r^{-1} \omega_t (g_{ij} - m_{ij})) e_a^\beta + r (e_a^\beta - q_t q_r^{-1} e_a (r)) \\
= \mathcal{R}_{0,0} + r e_a^\beta. 
\]
Besides, since \( e_i^3 \omega_i = 2g^0_i \omega_i + e_i^1 \omega_i = 1 + \varepsilon R_{-1,0} \), we have

\[
\varepsilon^{-1}t(g_{03} - q_t q_r^{-1} \omega_i g_{ij}) e_3^3 = \varepsilon^{-1}t((g_{03} - m_{03}) - q_t q_r^{-1} \omega_i (g_{ij} - m_{ij})) e_3^3 + \varepsilon^{-1}t(1 - q_t q_r^{-1} e_i^3 \omega_i) \\
= R_{0,0} + \varepsilon^{-1}t q_r^{-1}(2q_r - (q_t + q_r) - q_t (e_i^3 \omega_i - 1)) = R_{0,0} + 2\varepsilon^{-1}t,
\]

\[
r(g_{ij} - q_t q_r^{-1} \omega_j g_{ij}) e_3^3 = r((g_{ij} - m_{ij}) - q_t q_r^{-1} \omega_j (g_{ij} - m_{ij})) e_3^3 + r(e_i^3 - q_t q_r^{-1} \omega_j e_3^3) \\
= \varepsilon R_{0,0} + r q_r^{-1}((e_i^3 - \omega_i) q_r - (\delta_i - \omega_i q_r) - q_i (e_i^3 \omega_j - 1)) = R_{0,0}.
\]

Thus,

\[
\partial_s = \sum_a \varepsilon^{-1} R_{0,0} e_a + (\varepsilon^{-1}t + R_{0,0}) e_4 = \sum_a \varepsilon^{-1} R_{-1,0} V_a + (\varepsilon^{-1} + R_{-1,0}) V_4,
\]

\[
\partial_{\omega_i} = \sum_{k \neq 3} R_{0,0} e_k + \sum_a r e_a e_a = \sum_{k \neq 3} R_{-1,0} V_k + \sum_a e_a V_a = \sum_{k \neq 3} R_{0,0} V_k.
\]

It is also clear that

\[
\partial_q = \sum_k R_{0,0} e_k = \sum_k R_{0,-1} V_k.
\]

We end this subsection with the following estimates for \( U \).

**Lemma 5.3.** We have

\[
(U, U_q, U_s, U_{\omega_i}) = (R_{0,0}, \varepsilon^{-1} R_{0,0}, \varepsilon^{-1} R_{0,0}, 0).
\]

In conclusion, we have \( \mu U_q = R_{0,-1} \).

**Proof.** We have

\[
U = \varepsilon^{-1} ru,
\]

\[
U_q = q_r^{-1} \partial_r (\varepsilon^{-1} ru) = \varepsilon^{-1} q_r^{-1} (u + ru_r),
\]

\[
U_s = \varepsilon^{-2} tr(u_t + u_r - q_t q_r^{-1} (q_t + q_r) u_t) - \varepsilon^{-2} t q_r q_r^{-1} u,
\]

\[
U_{\omega_i} = -\varepsilon^{-1} r (q_i - \omega_i q_r) q_r^{-1} (u + ru_r) + \varepsilon^{-1} r^2 (u_i - \omega_i u_r).
\]

It follows directly from Lemma 4.11, Lemma 4.22 and the proof of Proposition 4.23 that \( (U, U_q, U_s, U_{\omega_i}) = (R_{0,0}, \varepsilon^{-1} R_{0,0}, R_{0,0}, 0) \). Finally, since \( \mu = R_{0,0} \), we have \( \mu U_q = R_{0,-1} \). \( \square \)

**Remark 5.3.1.** Note that we have \( U_q = R_{0,-1} \) which is stronger than \( (U, \varepsilon U_q, U_{\omega_i}) = R_{0,0} \). This is because we gain an additional factor \( (r-t)^{-1} \) when we estimate \( u_r \) and its derivatives compared to \( u \). We refer our readers to Lemma 2.2.

We also remark that it is important for us to obtain \( U_q = R_{0,-1} \) instead of \( R_{0,0} \) here, because \( U_q = R_{0,-1} \) is necessary for the scattering data to be defined later.
5.2. The asymptotic equation for \( \mu \). We start with several estimates for \( \mu = q_t - q_r \). By Proposition 4.25, we have

\[
e_4(e_3(q)) = -\frac{1}{4} e_3(u)G(\omega)e_3(q) + \varepsilon \mathfrak{R}_{-2,0}
\]

Moreover,

\[
e_3(q) = -\mu + \mathfrak{R}_{-1,0} \cdot \partial q = -\mu + \mathfrak{R}_{-1,0}.
\]

Since \( e_i^3 - \omega_i = \mathfrak{R}_{-1,0} \), we have

\[
e_3(q) = -\mu + \mathfrak{R}_{-1,0} \cdot \partial q = -\mu + \mathfrak{R}_{-1,0}.
\]

To get the last equality, we use the following estimates: \( e_4(e_3(q)) = \xi^3_{12} e_3(q) = \varepsilon \mathfrak{R}_{-1,-1} \), and

\[
q_t + q_r = \frac{1}{2} (\varepsilon g_{ij} + \omega_i g_{i3}) e_3(q) = \frac{1}{2} \left( -1 + e_i^3 \omega_i \right) e_3(q) + (g_{ij} - m_{ij}) \cdot \mathfrak{R}_{0,0} = \varepsilon \mathfrak{R}_{-1,0}.
\]

Besides, by the chain rule, we have

\[
e_3(U) = e_3(q) U_q - \varepsilon t^{-1} U_s + \sum_i e_3(\omega_i) U_{\omega_i} = -\mu U_q + \mathfrak{R}_{-1,0}.
\]

Here we apply Lemma 5.3 and we note that \( e_3(\omega_i) = (e_i^3 - \omega_i)^{-1} (\delta_{ij} - \omega_i \omega_j) = \mathfrak{R}_{-2,0} \). Thus, we have

\[
e_4(-\mu) + \varepsilon \mathfrak{R}_{-2,0} = -\frac{\varepsilon}{4r} G(\omega)(-\mu U_q + \mathfrak{R}_{1,-1})(-\mu + \mathfrak{R}_{-1,0}) + \varepsilon \mathfrak{R}_{-2,0}
\]

Then,

\[
e_4(\mu) = \frac{\varepsilon}{4r} G(\omega) \mu^2 U_q + \varepsilon \mathfrak{R}_{-2,0}.
\]

By Lemma 5.2, we have

\[
\mu_s = \varepsilon^{-1} t e_4(\mu) + \sum_{k \neq 3} \varepsilon^{-1} \mathfrak{R}_{-1,0} V_k(\mu) = \varepsilon^{-1} t \left( \frac{\varepsilon}{4r} G(\omega) \mu^2 U_q + \varepsilon \mathfrak{R}_{-2,0} \right) + \sum_{k \neq 3} \varepsilon^{-1} \mathfrak{R}_{-1,0} V_k(\mathfrak{R}_{0,0})
\]

\[
= \frac{t}{4r} G(\omega) \mu^2 U_q + \varepsilon^{-1} \mathfrak{R}_{-1,0} = \frac{1}{4} G(\omega) \mu^2 U_q + \frac{\varepsilon (t - r)}{4r} G(\omega) \mu^2 U_q + \varepsilon^{-1} \mathfrak{R}_{-1,0}
\]

\[
= \frac{1}{4} G(\omega) \mu^2 U_q + \varepsilon \mathfrak{R}_{-1,1} \cdot \mathfrak{R}_{0,0} \cdot \mathfrak{R}_{0,-1} + \varepsilon^{-1} \mathfrak{R}_{-1,0} = \frac{1}{4} G(\omega) \mu^2 U_q + \varepsilon^{-1} \mathfrak{R}_{-1,0}.
\]
We thus obtain the first asymptotic equation

\begin{equation}
\mu_s = \frac{1}{4}G(\omega)\mu^2 U_q + \varepsilon^{-1}R_{-1,0}.
\end{equation}

5.3. The asymptotic equation for $U$. By Proposition 4.23, we have

\[ e_4(e_3(U)) = \varepsilon^{-1}e_4(ru) = R_{-2,0}. \]

Meanwhile, by Lemma 5.3 we have

\[ e_4(e_3(U)) = e_4(e_3(q)U_q + \varepsilon t^{-1}U_s + e_3(\omega_i)U_{\omega_i}) \]

\[ = -e_4(\mu U_q) + e_4((e_i^3 - \omega_i)q U_q + \varepsilon t^{-1}U_s + (e_i^3 - \omega_i)q U_{\omega_i}) \]

\[ = -e_4(\mu U_q) + R_{-1,0} \cdot V_4(R_{-1,-1} + \varepsilon t^{-1} \cdot \varepsilon^{-1}R_{0,0} + R_{-1,0} \cdot r^{-1} \cdot R_{0,0}) \]

\[ = -e_4(\mu U_q) + R_{-2,0}. \]

Thus, $e_4(\mu U_q) = R_{-2,0}$.

Now, we compute $\partial_s(\mu U_q)$. By Lemma 5.2 we have

\[ \partial_s(\mu U_q) = \sum_a \varepsilon^{-1}R_{-1,0}V_a(\mu U_q) + (\varepsilon^{-1} + R_{-1,0})V_4(\mu U_q) \]

\[ = \sum_a \varepsilon^{-1}R_{-1,0}V_a(R_{0,-1}) + (\varepsilon^{-1} + R_{-1,0})R_{-1,0} = \varepsilon^{-1}R_{-1,0}. \]

Thus, we have

\[ \mu U_s = \partial_s(\mu U_q) - \mu_s U_q = \varepsilon^{-1}R_{-1,0} - (\frac{1}{4}G(\omega)\mu^2 U_q + \varepsilon^{-1}R_{-1,-1} + R_{-1,0})U_q \]

\[ = -\frac{1}{4}G(\omega)\mu^2 U_q + \varepsilon^{-1}R_{-1,0}. \]

Since $|\mu| > C^{-1}t^{-C}$, we have $\mu^{-1} = R_{0,0}$. Thus we obtain the second asymptotic equation

\begin{equation}
U_s = -\frac{1}{4}G(\omega)\mu U_q^2 + \varepsilon^{-1}R_{-1,0}.
\end{equation}

In summary, by (5.2) and (5.3), we have proved the following proposition.

**Proposition 5.4.** We have

\begin{equation}
\begin{cases}
\partial_s \mu = \frac{1}{4}G(\omega)\mu^2 U_q + \varepsilon^{-1}R_{-1,0}, \\
\partial_s U_q = -\frac{1}{4}G(\omega)\mu U_q^2 + \varepsilon^{-1}R_{-1,0}.
\end{cases}
\end{equation}

In other words, $(\mu, U_q)(s, q, \omega)$ is an approximate solution to the reduced system of ODE’s

\begin{equation}
\begin{cases}
\partial_s \mu = \frac{1}{4}G(\omega)\mu^2 U_q, \\
\partial_s U_q = -\frac{1}{4}G(\omega)\mu U_q^2.
\end{cases}
\end{equation}

We remark that this proposition verifies the nonrigorous derivation in Section 3 of the author’s previous paper [35].
5.4. **The scattering data.** From the previous subsections, we have proved that \((\mu, U_q)(s, q, \omega)\) is an approximate solution to the reduced system \((5.5)\). In this subsection, we seek to construct an exact solution \((\bar{\mu}, \bar{U}_q)\) to \((5.5)\) which is a good approximation of \((\mu, U_q)\).

We start with the following key proposition. In this proposition, we define the scattering data \(A = A(q, \omega)\) for each \((q, \omega) \in \mathbb{R} \times S^2\) and we show that it is a smooth function (in the sense defined in Section 2.4).

**Proposition 5.5.** In \(\Omega'\), we have

\[
(q\partial_q)^m \partial^n_s (\mu U_q) = O(q^{-1}\varepsilon^C), \quad \partial^n_s (\langle q \partial_q \rangle^m \partial^n_\omega (\mu U_q)) = O(\varepsilon^p t^{-1+C\varepsilon}), \quad p \geq 1.
\]

Moreover, for each \(m, n\), the limit

\[
A_{m,n}(q, \omega) := -\frac{1}{2} \lim_{s \to \infty} (q \partial_q)^m \partial^n_s (\mu U_q) (s, q, \omega)
\]

exists for all \((q, \omega) \in \mathbb{R} \times S^2\), and the convergence is uniform in \((q, \omega)\). So \(A(q, \omega) := A_{0,0}(q, \omega)\) is a smooth function of \((q, \omega)\) in \(\mathbb{R} \times S^2\) such that \((q \partial_q)^m \partial^n_s A = A_{m,n}\). We call this function \(A\) the scattering data. It is clear that \(A \equiv 0\) for \(q > R\).

Finally, we have

\[
(q \partial_q)^m \partial^n_\omega (\mu U_q + 2A) = O(t^{-1+C\varepsilon}), \quad (q \partial_q)^m \partial^n_\omega A = O(q^{-1+\varepsilon}).
\]

**Proof.** First we note that in the region \(r - t > R\), we have \(q = r - t\) and \(u = 0\). In this case, every estimate in the statement of this proposition is equal to 0, so there is nothing to prove. Thus, we can assume that \(q < 2R\) and \(r - t < 2R\) in the rest of this proof.

We need to derive an estimate for \(\partial_s \partial^n_q \partial^n_\omega (\mu U_q)\). Here we apply Lemma 5.2. Recall that \(\mu U_q = \mathcal{R}_{0,-1}\) and \(V_4(\mu U_q) = \mathcal{R}_{-1,0}\). By the Leibniz’s rule, we have

\[
(q \partial_q)^m \partial^n_\omega (\mu U_q) = \left( \sum_k \mathcal{R}_{0,0} V_k \right)^{m+n}(\mathcal{R}_{0,-1}) = O((q)^{-1+\varepsilon} t^{C\varepsilon}) = O((q)^{-1+C\varepsilon}).
\]

In addition, for \(p \geq 1\) we have

\[
\partial^n_s (\langle q \partial_q \rangle^m \partial^n_\omega (\mu U_q)) = \partial^n_s \left( \langle q \partial_q \rangle^m \partial^n_\omega \partial_s (\mu U_q) \right)
= \partial^n_s \left( \langle q \partial_q \rangle^m \partial^n_\omega \left( \sum_{k \neq 3} \varepsilon^{-1} \mathcal{R}_{-1,0} \cdot V_k (\mu U_q) + \varepsilon^{-1} V_4 (\mu U_q) \right) \right)
= \partial^n_s \left( \langle q \partial_q \rangle^m \partial^n_\omega \left( \sum_{k \neq 3} \varepsilon^{-1} \mathcal{R}_{-1,0} \cdot \mathcal{R}_{0,-1} + \varepsilon^{-1} \mathcal{R}_{-1,0} \right) \right)
= \varepsilon^{-p} \left( \sum_k \mathcal{R}_{0,0} V_k \right)^{p+m+n-1}(\varepsilon^{-1} \mathcal{R}_{-1,0}) = O(\varepsilon^{-p} t^{-1+C\varepsilon}).
\]

In both these estimates, we view \(t\) as a function of \(s\).

For fixed \(q < 2R\) and \(\omega \in S^2\), by the definition of \(\Omega'\), we have \((s, q, \omega) \in \Omega'\) if and only if \(s > 0\) and

\[
\exp((s + \delta)/\varepsilon) > \exp(\delta/\varepsilon) - 2q + 4R.
\]

We can write this condition as \(s > s_{q,\delta,\varepsilon}\) where \(s_{q,\delta,\varepsilon} \geq 0\) is a constant depending on its subscripts, such that \((s_{q,\delta,\varepsilon}, q, \omega) \in \partial \Omega'\) corresponds with a point on \(H\). Thus, for each fixed
(q, ω) and \( s_2 > s_1 \geq s_{q,δ,ε} = \exp(δ/ε) - 2q + 4R \), by (5.6) with \( p = 1 \), we have
\[
|\langle (q)\partial_q \rangle^m \partial^p_ω (μU_q)(s_2, q, ω) - (\langle q \rangle \partial_q )^m \partial^p_ω (μU_q)(s_1, q, ω)|
\leq \int_{s_1}^{s_2} \varepsilon^{-1} \exp((-1 + Cε)ε^{-1}(s + δ)) \, ds \lesssim \exp((-1 + Cε)ε^{-1}(s + δ)).
\]
In conclusion, \( \{(\langle q \rangle \partial_q )^m \partial^p_ω (μU_q)(s, q, ω)\}_{s \geq s_{q,δ,ε}} \) is uniformly Cauchy for each \( (q, ω) \). Thus, the limit
\[
A_{m,n}(q, ω) := \frac{1}{2} \lim_{s \to \infty} (\langle q \rangle \partial_q )^m \partial^p_ω (μU_q)(s, q, ω)
\]
exists, and the convergence is uniform in \( (q, ω) \). Besides, for each \( s \geq s_{q,δ,ε} \), we have
\[
|\langle q \rangle \partial_q )^m \partial^p_ω (μU_q)(s, q, ω) + 2A_{m,n}| \lesssim t^{-1+Cε} = \exp((-1 + Cε)ε^{-1}(s + δ)).
\]
By evaluating (5.8) at \( (s,q,ω) \), we have
\[
|A_{m,n}(q, ω)| \lesssim |\langle (q)\partial_q \rangle^m \partial^p_ω (μU_q) + 2A_{m,n}| + |\langle (q)\partial_q \rangle^m \partial^p_ω (μU_q)|
\leq (\exp(δ/ε) - 2q + 4R)^{-1+Cε} + \langle q \rangle^{-1}(\exp(δ/ε) - 2q + 4R)^{Cε} \lesssim \langle q \rangle^{-1+Cε}.
\]
In the last inequality, we note that \( (a + b)Cε \leq 2Cε \max\{a, b\}Cε \leq 2(aCε + bCε) \) for each pairs \( a, b \geq 0 \). Since the convergence is uniform in \( (q, ω) \), if we define \( A := A_{0,0} \), then we have
\[
\langle (q)\partial_q \rangle^m \partial^p_ω A = A_{m,n} = O((\langle q \rangle)^{-1+Cε}).
\]
Note that each function of \( (s,q,ω) \) can be viewed as a function of \( (t, x) \). We then have the following lemma.

**Lemma 5.6.** By viewing each function of \( (s,q,ω) \) as a function of \( (t, x) \in Ω \cap \{r - t < 2R\} \), we have \( A = A_{0,0} = μU_q + 2A = A_{0,0} \) and \( \exp(±\frac{1}{2}G(ω)As) - 1 = A_{0,0} \).

**Proof.** Note that \( V^I A \) is a linear combination of terms of the form
\[
\partial^m_ω \partial^p_ω A \cdot V^{I_1}q \cdots V^{I_m}q \cdot V^{J_1}ω \cdots V^{J_n}ω, \quad \sum |I_s| + \sum |J_s| = |I|.
\]
Each of these terms is \( O((\langle q \rangle)^{-1-m+Cε} \cdot (q)^m \partial^p_ω) = O((\langle q \rangle)^{-1+Cε}) \), so \( A = A_{0,0} \). The proof of \( \partial^m_ω A = A_{0,0} \) is essentially the same.

Moreover, \( V^I(μU_q + 2A) \) is a linear combination of terms of the form
\[
\partial^m_ω (μU_q + A) \cdot V^{I_1}q \cdots V^{I_m}q \cdot V^{J_1}ω \cdots V^{J_n}ω, \quad \sum |I_s| + \sum |J_s| = |I|;
\]
\[
\partial^m_ω \partial^p_ω (μU_q) \cdot V^{K_1}s \cdots V^{K_p}s \cdot V^{I_1}q \cdots V^{I_m}q \cdot V^{J_1}ω \cdots V^{J_n}ω, \quad \sum |I_s| + \sum |J_s| + \sum |K_s| = |I|, \quad p > 0.
\]
By applying (5.8) to the first row and (5.6) to the second row, we conclude that \( V^I(μU_q + 2A) = O(t^{-1+Cε}) \) and thus \( μU_q + 2A = A_{0,0} \).

Finally, by the chain rule, for each \( |I| > 0 \) we can write \( V^I(\exp(±\frac{1}{2}G(ω)As) - 1) \) as a linear combination of terms of the form
\[
\exp(±\frac{1}{2}G(ω)As) \cdot V^{I_1}(±\frac{1}{2}G(ω)As) \cdots V^{I_m}(±\frac{1}{2}G(ω)As), \quad \sum |I_s| = |I|, \quad |I_s| > 0.
\]
The first term in this product is \( O(t^{C\varepsilon}) \), and each of the rest terms are \( O(V^I \cdot (\Re_{0,-1})) = O(\langle q \rangle^{-1+C\varepsilon}) \), so we conclude that \( V^I (\exp(\pm \frac{1}{2} G(\omega) A_s) - 1) = O(\langle q \rangle^{-1+C\varepsilon}) \) for \( |I| > 0 \). When \( |I| = 0 \), since \(|e^\theta - 1| \lesssim |\rho| e^{\rho |t|} \), we have

\[
| \exp(\frac{1}{2} G(\omega) A_s) - 1 | \lesssim \langle q \rangle^{-1+C\varepsilon} s \exp(C \langle q \rangle^{-1+C\varepsilon} s) \lesssim \langle q \rangle^{-1+C\varepsilon}.
\]

Here we note that \( s = \varepsilon \ln(t) - \delta = O(t^{C\varepsilon}) \). In conclusion, \( \exp(\pm \frac{1}{2} G A_s) - 1 = \Re_{0,-1} \). \( \square \)

By \( 5.4 \) and Lemma \( 5.6 \), we have

\[
\begin{cases}
\partial_s \mu = -\frac{1}{2} G(\omega) A(q, \omega) \mu + \varepsilon^{-1} \Re_{-1,0}, \\
\partial_s U_q = \frac{1}{2} G(\omega) A(q, \omega) U_q + \varepsilon^{-1} \Re_{-1,0}.
\end{cases}
\]

With the remainder terms omitted, we obtain two linear ODE’s for \( \mu \) and \( U_q \). They motivate us to define

\[
\begin{align*}
\bar{V}_1 &:= \exp(\frac{1}{2} G(\omega) A(q, \omega) s) \mu, \\
\bar{V}_2 &:= \exp(-\frac{1}{2} G(\omega) A(q, \omega) s) U_q.
\end{align*}
\]

Now we can prove the following proposition.

**Proposition 5.7.** We have

\[
\langle q \rangle \partial_q^m \partial^p_\omega \bar{V}_1 = O(t^{C\varepsilon}), \quad \partial^p_\omega \langle q \rangle \partial_q^m \partial^p_\omega \bar{V}_1 = O(\varepsilon^{-p} t^{-1+C\varepsilon}), \quad p \geq 1;
\]

\[
\langle q \rangle \partial_q^m \partial^p_\omega \bar{V}_2 = O(\langle q \rangle^{-1+C\varepsilon}), \quad \partial^p_\omega \langle q \rangle \partial_q^m \partial^p_\omega \bar{V}_2 = O(\varepsilon^{-p} t^{-1+C\varepsilon}), \quad p \geq 1.
\]

Moreover, for each \( m, n \), the limit

\[
A_{j,m,n}(q, \omega) := \lim_{s \to \infty} \bar{V}_j(s, q, \omega), \quad j = 1, 2
\]

exists for all \( (q, \omega) \in \mathbb{R} \times \mathbb{S}^2 \), and the convergence is uniform in \( (q, \omega) \). So, for \( j = 1, 2 \), \( A_j := A_{j,0,0} \) is smooth functions of \( (q, \omega) \) in \( \mathbb{R} \times \mathbb{S}^2 \) such that \( \langle q \rangle \partial_q^m \partial^p_\omega A_j = A_{j,m,n} \). It is clear that \( A_1 \equiv -2 \) and \( A_2 \equiv 0 \) for \( q > R \). Besides, we have \( A_1 A_2 = -2A \) everywhere.

Finally, we have

\[
\langle q \rangle \partial_q^m \partial^p_\omega (\bar{V}_1 - A_1) = O(t^{-1+C\varepsilon}), \quad \langle q \rangle \partial_q^m \partial^p_\omega A_1 = O(\langle q \rangle^{C\varepsilon}),
\]

\[
\langle q \rangle \partial_q^m \partial^p_\omega (\bar{V}_2 - A_2) = O(t^{-1+C\varepsilon}), \quad \langle q \rangle \partial_q^m \partial^p_\omega A_2 = O(\langle q \rangle^{-1+C\varepsilon}).
\]

**Proof.** By \( 5.1 \) and since \( t/r = 1 + \Re_{-1,1} \), we have

\[
V_4(\mu) = \frac{\varepsilon t}{4 \tau} G(\omega) \mu^2 U_q + \varepsilon \Re_{-1,0} = \frac{\varepsilon}{4} G(\omega) \mu^2 U_q + \varepsilon \Re_{-1,0}.
\]

Moreover, by viewing \( (s, q, \omega) \) as functions of \( (t, x) \), we have

\[
e_4(G(\omega) A(q, \omega) s) = \varepsilon G(\omega) A t^{-1} + e_4(\omega_j) \partial_{\omega_j} (GA) s = \varepsilon G(\omega) A t^{-1} + \Re_{-2,-1}.
\]
Here we note that $\partial_{\omega_i}(GA) = \mathcal{R}_{0,-1}$ by Lemma [5.6] and $e_4(\omega_i) = (e^j_4 - \omega_j)\partial_j\omega_i = \mathcal{R}_{-2,0}$. Then, by Lemma [5.6] we have $V_1 = \mathcal{R}_{0,0} \cdot \mathcal{R}_{0,0} = \mathcal{R}_{0,0}$ and

$$V_4(\tilde{V}_1) = \frac{1}{2} V_4(GA_4) \tilde{V}_1 + \exp(2GA_4) V_4(\mu) = \frac{1}{4} (2\varepsilon GA + \varepsilon G \mu U_q + \mathcal{R}_{-1,-1}) \tilde{V}_1 + \varepsilon \mathcal{R}_{1,0} \cdot \exp(\frac{1}{2} GA)$$

$$= \frac{1}{4} (\varepsilon \mathcal{R}_{-1,0} \cdot \mathcal{R}_{1,-1}) \cdot \mathcal{R}_{0,0} + \varepsilon \mathcal{R}_{1,0} \cdot \mathcal{R}_{0,0} = \varepsilon \mathcal{R}_{-1,0} + \mathcal{R}_{-1,1} = \mathcal{R}_{-1,0}.$$ 

Next, we have $\tilde{V}_1 \tilde{V}_2 = \mu U_q$ and $\mu U_q = \mathcal{R}_{0,-1}$, $V_4(\mu U_q) = \mathcal{R}_{-1,0}$ from Proposition [5.3] Since $\mu = q_t - q_r \leq -2C^1 t^{-C^2}$ and $\exp(\frac{1}{2} GA) \geq \exp(-C) = \exp(C) t^{-C}$, we have $|\tilde{V}_1| = -\tilde{V}_1 \geq C^1 t^{-C^2}$. We can express $V(\tilde{V}_2) = V(\mu U_q)/\tilde{V}_1)$ as a linear combination of terms of the form

$$\tilde{V}_1 \cdot V^{I_1}(\tilde{V}_1) \cdots V^{I_m}(\tilde{V}_1) \cdot V^{I_0}(\mu U_q), \quad \sum |I_s| = |I|.$$ 

It is easy to conclude that $\tilde{V}_2 = \mathcal{R}_{0,-1}$ and $V_4(\tilde{V}_2) = \mathcal{R}_{-1,0}$.

Now we can follow the proof in Proposition [5.5] to prove every estimate involving $A_2$ in the statement. As for $A_1$, we note that

$$(\langle q \rangle \partial_q)^m \partial_{\omega}^n (\tilde{V}_1) = (\sum_k \mathcal{R}_{0,0} V_k)^{m+n} (\mathcal{R}_{0,0}) = O(t^{C^2}).$$

In addition, for $p \geq 1$ we have

$$\partial^p_s(\langle q \rangle \partial_q)^m \partial_{\omega}^n (\tilde{V}_1) = \partial^p_s(\langle q \rangle \partial_q)^m \partial_{\omega}^n \partial_s(\tilde{V}_1)$$

$$= \partial^p_s(\langle q \rangle \partial_q)^m \partial_{\omega}^n (\sum_{k \neq 3} \varepsilon^{-1} \mathcal{R}_{-1,0} \cdot V_k(\tilde{V}_1) + \varepsilon^{-1} V_4(\tilde{V}_1))$$

$$= \partial^p_s(\langle q \rangle \partial_q)^m \partial_{\omega}^n (\sum a \cdot \mathcal{R}_{-1,0} \cdot \mathcal{R}_{0,0} + \varepsilon^{-1} \mathcal{R}_{-1,0})$$

$$= \varepsilon^{-p} (\sum \mathcal{R}_{0,0} V_k)^{p+m+n-1} (\varepsilon^{-1} \mathcal{R}_{-1,0}) = O(t^{-p t + 1+C^2}).$$

It is then clear that the estimates for $\tilde{V}_1 - A_1$ are the same as those for $\mu U_q + 2A$. Finally, at $(s,q,\omega) = (s,q,\delta \varepsilon, q, \omega)$ we have

$$||\langle q \rangle \partial_q \partial_{\omega} \partial_\omega A_1(q,\omega)|| \lesssim (\langle q \rangle \partial_q \partial_{\omega} \partial_\omega (\tilde{V}_1 - A_1) (s,q,\omega) + ||\langle q \rangle \partial_q \partial_{\omega} (\tilde{V}_1) || (s,q,\omega))$$

$$\lesssim (\exp(\delta/\varepsilon) - 2q + 4R)^{-1+C^2} + \exp(\delta/\varepsilon) - 2q + 4R)^{C^2} \lesssim (\langle q \rangle)^{C^2}.$$ 

In the last inequality, we note that $(a+b)^{C^2} \leq 2^{C^2} \max \{a,b\}^{C^2} \leq 2(a^{C^2} + b^{C^2})$ for each pairs $a,b \geq 0$. \hfill $\Box$

Remark 5.7.1. Following the proof of Lemma [5.6] we can show that $(A_1, \partial_{\omega} A_1) = \mathcal{R}_{0,0}$, $\tilde{V}_1 - A_1 = \mathcal{R}_{-1,0}$, $(A_2, \partial_{\omega} A_2) = \mathcal{R}_{0,-1}$ and $\tilde{V}_2 - A_2 = \mathcal{R}_{-1,0}$.

Moreover, we note that $A_1 \approx -2$ in the following sense.

Lemma 5.8. Fix $0 < \kappa < 1$. For $\varepsilon \ll 1$ and for all $(q,\omega) \in \mathbb{R} \times \mathbb{S}^2$, we have $|A_1(q,\omega) + 2| \leq \kappa (q)^{-1+C^2}$. The constant in the power may depend on $\kappa$. As a result, we have $A_1(q,\omega) < -1 < 0$.
Proof. Since \( A_1 \equiv -2 \) for \( q > R \), we can assume \( q < 2R \) in the proof. Recall from the proof of Proposition 5.7 that

\[
e_4(\tilde{V}_1) = \varepsilon \mathbb{R}_{-2,0} + \mathbb{R}_{-2,-1} = O(\varepsilon t^{-2+C\varepsilon} + t^{-2+C\varepsilon} \langle q \rangle^{-1}).
\]

Next we consider \( \tilde{V}_1|_H \). On \( H \) we have \( \mu = -2 + O(\lvert u \rvert) = -2 + O(\varepsilon t^{-1+C\varepsilon}) \). As computed in Lemma 5.6 on \( H \) we have

\[
\lvert (\exp(\frac{1}{2} G As) - 1) \mu \rvert \lesssim \langle q \rangle^{-1+C\varepsilon} s \exp(C \langle q \rangle^{-1+C\varepsilon}) \cdot (2 + O(\varepsilon t^{-1+C\varepsilon}))
\approx \langle q \rangle^{-1+C\varepsilon} s \exp(C \langle q \rangle^{-1+C\varepsilon}).
\]

Thus, \( \tilde{V}_1|_H = -2 + O(\varepsilon t^{-1+C\varepsilon} + \langle q \rangle^{-1+C\varepsilon} \exp(C \langle q \rangle^{-1+C\varepsilon})) \).

We integrate \( e_4(\tilde{V}_1) \) along the geodesic in \( \mathcal{A} \) passing through \( (t, x) \in \Omega \cap \{ r - t < 2R \} \). Then,

\[
|\tilde{V}_1(t, x) + 2| \lesssim \varepsilon (x^0(0))^{-1+C\varepsilon} + \langle q \rangle^{-1+C\varepsilon} (\varepsilon \ln x^0(0) - \delta) \exp(C \langle q \rangle^{-1+C\varepsilon} \varepsilon \ln x^0(0) - \delta))
\approx \varepsilon (x^0(0))^{-1+C\varepsilon} + \langle q \rangle^{-1+C\varepsilon} (\varepsilon \ln x^0(0) - \delta) \exp(C \langle q \rangle^{-1+C\varepsilon} \varepsilon \ln x^0(0) - \delta))
\approx (\varepsilon + \langle q \rangle^{-1})(x^0(0))^{-1+C\varepsilon}.
\]

If \( \varepsilon \ln x^0(0) - \delta \leq c \) for some small constant \( c > 0 \), we have

\[
|\tilde{V}_1(t, x) + 2| \leq C \varepsilon (q)^{-1+C\varepsilon} + C \langle q \rangle^{-1+C\varepsilon} \exp(C \langle q \rangle^{-1+C\varepsilon}) + C(\varepsilon + \langle q \rangle^{-1})(\langle q \rangle + \exp(1/\varepsilon))^{-1+C\varepsilon}
\leq C \varepsilon (q)^{-1+C\varepsilon} + C \langle q \rangle^{-1+C\varepsilon}.
\]

By choosing \( c, \varepsilon \ll \kappa \), we can make \( Cc + C\varepsilon < \kappa \). Thus, \( |\tilde{V}_1(t, x) + 2| \leq \kappa (q)^{-1+C\varepsilon} \). If \( \varepsilon \ln(x^0(0)) - \delta \geq c \), we have \( x^0(0) \geq \exp((c + \delta)/\varepsilon) \) and thus \( g = (\exp(\delta/\varepsilon) - x^0(0))/2 + 2R < -C^{-1} \exp((c + \delta)/\varepsilon) \) for \( \varepsilon \ll 1 \). Then we have \( \langle q \rangle \geq C^{-C\varepsilon} \exp(C'(c + \delta)) \) and thus

\[
|\tilde{V}_1(t, x) + 2| \lesssim (\varepsilon + \langle q \rangle^{-1})(x^0(0))^{-1+C\varepsilon} + \langle q \rangle^{-1+C\varepsilon} (x^0(0))^{C\varepsilon}
\approx (\varepsilon + \langle q \rangle^{-1})(\exp(\delta/\varepsilon) + \langle q \rangle)^{C\varepsilon} + \langle q \rangle^{-1+C\varepsilon} (\exp(\delta/\varepsilon) + \langle q \rangle)^{C\varepsilon}
\approx (\varepsilon + \exp(C\varepsilon) \exp(-C'c)).
\]

The second last inequality holds since \( a^{C\varepsilon} + b^{C\varepsilon} \leq (2 \max\{a, b\})^{C\varepsilon} \leq 2^{C\varepsilon} (a^{C\varepsilon} + b^{C\varepsilon}) \) for \( a, b > 0 \). By choosing \( C' \gg \kappa \) and \( \varepsilon \ll \kappa \), again we have \( |\tilde{V}_1(t, x) + 2| \leq \kappa (q)^{-1+C\varepsilon} \).

We finish the proof by sending \( s \to \infty \). \( \square \)

5.5. An exact solution to the reduced system. For each \( (s, q, \omega) \in \mathbb{R} \times \mathbb{R} \times \mathbb{S}^2 \), we define

\[
\mu(s, q, \omega) = A_1(q, \omega) \exp(-\frac{1}{2} G(\omega)A(q, \omega)s),
\]

\[
\bar{U}_q(s, q, \omega) = A_2(q, \omega) \exp(-\frac{1}{2} G(\omega)A(q, \omega)s).
\]

Since \( \bar{\mu} \bar{U}_q = A_1 A_2 = -2A \), it is easy to show that \( (\bar{\mu}, \bar{U}_q) \) is indeed a solution to the reduced system (5.5). To solve for \( \bar{U} \) uniquely, we assume that \( \lim_{q \to \infty} \bar{U}(s, q, \omega) = 0 \) (since
\[ \lim_{q \to \infty} U(s, q, \omega) = 0. \] This also implies that \( \tilde{U} \equiv 0 \) for \( q \geq 2R \). At \((s, q, \omega) \in \Omega' \cap \{ q < 2R \}\) we have

\[ \tilde{\mu} = \mathfrak{R}_{0,0} \cdot (1 + \mathfrak{R}_{0,-1}) = \mathfrak{R}_{0,0}, \quad \tilde{U}_q = \mathfrak{R}_{0,-1}(1 + \mathfrak{R}_{0,0}) = \mathfrak{R}_{0,-1}, \]

\[ \tilde{\mu} - \mu = \exp(-\frac{1}{2}G(\omega)A(q, \omega)s)(A_1 - \tilde{V}_1) = \mathfrak{R}_{-1,0}, \]

\[ \tilde{U}_q - U_q = \exp(-\frac{1}{2}G(\omega)A(q, \omega)s)(A_2 - \tilde{V}_2) = \mathfrak{R}_{-1,0}. \]

Thus, for each \( p, m, n \), we have

\[ \partial_s^p (\langle q \rangle \partial_q)^m \partial_\omega^n \tilde{\mu} = \varepsilon^{-p} \left( \sum_k \mathfrak{R}_{0,0} V_k \right)^{p+m+n} \mathfrak{R}_{0,0} = O(\varepsilon^{-p} C\varepsilon), \]

\[ \partial_s^p (\langle q \rangle \partial_q)^m \partial_\omega^n \tilde{U}_q = \varepsilon^{-p} \left( \sum_k \mathfrak{R}_{0,0} V_k \right)^{p+m+n} \mathfrak{R}_{0,-1} = O(\varepsilon^{-p} \langle q \rangle^{-1} C\varepsilon), \]

\[ \partial_s^p (\langle q \rangle \partial_q)^m \partial_\omega^n (\tilde{\mu} - \mu, \tilde{U}_q - U_q) = \varepsilon^{-p} \left( \sum_k \mathfrak{R}_{0,0} V_k \right)^{p+m+n} \mathfrak{R}_{-1,0} = O(\varepsilon^{-p} t^{-1} C\varepsilon). \]

Moreover, since \( U = \varepsilon^{-1} ru = \mathfrak{R}_{0,0} \), we can also show that \( \partial_s^p (\langle q \rangle \partial_q)^m \partial_\omega^n U = O(\varepsilon^{-p} t C\varepsilon) \). Now, by integrating \( \partial_s^p \partial_\omega^n (\tilde{U}_q - U_q) \) with respect to \( q \), we have

\[ \partial_s^p \partial_\omega^n (\tilde{U} - U) = O(\varepsilon^{-p} \langle q \rangle t^{-1} C\varepsilon), \quad \partial_s^p \partial_\omega^n \tilde{U} = O(\varepsilon^{-p} \langle q \rangle t^{-1} C\varepsilon + \varepsilon^{-p} t C\varepsilon) = O(\varepsilon^{-p} t C\varepsilon). \]

Here we note that \( \langle q \rangle \lesssim t \) in \( \Omega' \cap \{ q < 2R \} \). The estimates \((5.11)\) and \((5.12)\) will be used in Section 7.

### 6. Gauge independence

At the beginning of Section 3, we define a region \( \Omega \) by \((3.2)\) and then construct an optical function in \( \Omega \). If we replace \((3.2)\) with

\[ \Omega_{\kappa, \delta} := \{(t, x) : t > \exp(\delta\varepsilon), |x| - \exp(\delta\varepsilon) - 2R > \kappa(t - \exp(\delta\varepsilon))\} \]

for some fixed constants \( \delta > 0 \) and \( 0 < \kappa < 1 \), we are still able to construct an optical function in \( \Omega_{\kappa, \delta} \) by following the proofs in Section 3 and Section 4. We are also able to construct a scattering data by following the proofs in Section 5. We do not expect that the scattering data to be independent of \((\kappa, \delta)\), but we have the next proposition.

**Proposition 6.1.** Suppose \( q(t, x) \) and \( \tilde{q}(t, x) \) are two solutions to the same eikonal equation

\[ g^{a\beta}(u) q^a q^\beta = 0 \]

in different regions \( \Omega_{\kappa, \delta} \) and \( \Omega'_{\kappa, \delta} \), respectively, as constructed in Section 3 and Section 4. Let \( A(q, \omega) \) and \( \tilde{A}(\tilde{q}, \omega) \) be the corresponding scattering data constructed in Section 5 and Section 4. Under the change of coordinates \((s, q, \omega) = (\varepsilon \ln(t) - \delta, q(t, x), \omega)\), we can view \( \tilde{q}(t, x) \) as a function of \((s, q, \omega) \) in \( \Omega_{\kappa, \delta} \cap \Omega_{\kappa' \delta} \). Then, the limit \( \tilde{q}_\infty(q, \omega) \) exists, and we have

\[ A(q, \omega) = \tilde{A}(\tilde{q}_\infty(q, \omega), \omega). \]
Proof. We first recall several notations and estimates in Section 3. For example, we have $\mu = q_t - q_r = O(t^{2\varepsilon})$, $\nu = q_t + q_r = O(t^{-1+2\varepsilon})$, and we have similar definitions and estimates for $\bar{\mu}$ and $\bar{\nu}$. By viewing $\bar{q}(t,x)$ as a function of $(s,q,\omega) = (\varepsilon \ln(t) - \delta, q(t,x), \omega)$, we have

$$\partial_s \bar{q} = \varepsilon^{-1} t (\bar{q}_t - q_r \bar{q}_r^{-1}) = t \varepsilon^{-1} \bar{q}_r (\bar{v} q_r^{-1} - \nu q_r^{-1}).$$

By the eikonal equation, we have

$$0 = -(q_r - q_r)(q_r + q_r) + O(t^{-2+2\varepsilon}) + (g^{\alpha\beta}(u) - m^{\alpha\beta})q_{\alpha}q_{\beta} = -\nu \mu + \frac{1}{4} U G(\omega) \mu^2 + O(t^{-2+2\varepsilon}).$$

Since $\mu \leq -C^{-1} t^{-2\varepsilon}$, we have

$$\nu = \frac{1}{4} U G(\omega) \mu + O(t^{-2+2\varepsilon})$$

and thus

$$\frac{\nu}{q_r} = \frac{1}{4} U G(\omega) \frac{\mu}{q_r} + O(t^{-2+2\varepsilon}) = \frac{1}{4} U G(\omega) (\frac{\nu}{q_r} - 2) + O(t^{-2+2\varepsilon}) = -\frac{1}{2} U G(\omega) + O(t^{-2+2\varepsilon}).$$

We conclude that

$$\partial_s \bar{q} = t \varepsilon^{-1} \bar{q}_r^{-1} (\frac{1}{2} U G(\omega) + O(t^{-2+2\varepsilon}) - (\frac{1}{2} U G(\omega) + O(t^{-2+2\varepsilon})))$$

$$= O(\varepsilon^{-1} t^{-1+C\varepsilon}) = O(\varepsilon^{-1} \exp((-\varepsilon^{-1} + C)(s + \delta))).$$

As computed in Section 5.4, we can show that $\bar{q}_\infty(q,\omega) := \lim_{s \to \infty} \bar{q}(s,q,\omega)$ exists for all $(q,\omega)$. Moreover, we can show that

$$|\bar{q}(s,q,\omega) - \bar{q}_\infty(q,\omega)| \leq t^{-1+C\varepsilon}.$$
The initial data is given by (system of ODE’s 5.5, Proposition 5.7 and Lemma 5.8, we have |Z(t)µ| ≲ t−3+Cε, \( \forall t,x \in \Omega, \forall I \).

Moreover, if we fix a constant 0 < γ < 1 and a large integer N, then for \( \varepsilon \ll \gamma, N \), at each \( (t,x) \in \Omega \) such that \( |r-t| < t^\gamma \), we have \( |Z(t)\tilde{u}| \leq \gamma t^{-2+C_\varepsilon} (r-t) \) for each \( |I| \leq N \).

The estimates for \( u - \tilde{u} \) in this proposition is better than the estimates for \( u \) itself.

After making several definitions in Section 7.1, we introduce a simplification in Section 7.2. Instead of \( (\tilde{u}, \tilde{U}) \), the simplification in Section 7.2 allows us work with \( (\hat{\mu}, \hat{U}) \) which is an exact solution to the reduced system \( (5.10) \) with initial data \( (-2, \hat{A}) \). We thus get a new function \( \hat{q} \) which is a solution to \( \hat{q}_t - \hat{q}_r = \hat{\mu} \). In Section 7.3, we follow Section 4 of [35] to prove several estimates for \( \hat{q} \) and \( \hat{U} \). The most important result here is Proposition 7.1 which states that \( \tilde{u} = \hat{u} \) is indeed an approximate solution to \( (1.1) \). In Section 7.4, we show that \( \hat{q} \) approximates the optical function \( q \) in a certain sense. Finally, in Section 7.5, we make use of the estimates in Section 7.4 to prove Proposition 7.1.

7.1. Definitions. We first define a function \( \tilde{q}(t,x) \) in \( \Omega \) by solving the following equation
\[
\tilde{q}_t - \tilde{q}_r = \tilde{\mu}(\varepsilon \ln(t) - \delta, \tilde{q}(t,x), \omega) \quad \text{in} \quad \Omega \cap \{ r-t < 2R \}; \quad \tilde{q} = r-t \quad \text{when} \quad r-t \geq 2R.
\]

Recall that \( \hat{\mu} \) is defined by
\[
\tilde{\mu}(s,q,\omega) := A_1(q,\omega) \exp(-\frac{1}{2}G(\omega)A(q,\omega)s), \quad \forall(s,q,\omega) \in \mathbb{R} \times \mathbb{R} \times S^2.
\]

In this section, when we write \( q \), we usually mean a variable instead of the optical function \( q(t,x) \).

As in [35], we can use the method of characteristics to solve \( (7.1) \). We fix \( (t,x) \in \Omega \cap \{ r-t < 2R \} \) and set \( z(\tau) := \tilde{q}(\tau, r+t-\tau, \omega) \). Then, the function \( z(\tau) \) is a solution to the autonomous system of ODE’s
\[
\dot{z}(\tau) = \tilde{\mu}(\varepsilon s(\tau) - \delta, z(\tau), \omega), \quad \dot{s}(\tau) = \varepsilon \tau^{-1}.
\]

The initial data is given by \( (z, s)((r+t)/2-R) = (2R, \varepsilon \ln((r+t)/2-R) - \delta) \). By Proposition 5.2, Proposition 5.7 and Lemma 5.8, we have \( |A_1 + 2| = O(q^{-1+C_\varepsilon}) \), \( (A_2, A)(q,\omega) = O(q^{-1+C_\varepsilon}) \) and \( A_1 < -1 \) for all \( (q,\omega) \). Thus,
\[
0 \geq \mu(\varepsilon s(\tau) - \delta, z(\tau), \omega) = A_1(z(\tau), \omega) \exp(-\frac{1}{2}G(\omega)A(z(\tau), \omega)(\varepsilon s(\tau) - \delta))
\]
\[
\geq -C_\tau^{C_\varepsilon} |z(\tau) - 1 + C_\varepsilon| \geq -C_\tau^{C_\varepsilon}.
\]

Then, \( -C_\tau^{C_\varepsilon} \leq \dot{z}(\tau) \leq 0 \), so \( |z(\tau)| \) cannot blow up in finite time. By the Picard’s theorem, the system of ODE’s above has a solution for all \( (r+t)/2-R \leq \tau < \frac{1}{\varepsilon}([2(r+t)-4R - \exp(\delta/\varepsilon))] \). The upper bound here guarantees that \( (\tau, r+t-\tau, \omega) \in \Omega \). Thus, \( (7.1) \) has a solution \( \tilde{q}(t,x) \) in \( \Omega \).

Next, we define \( \tilde{U}(s,q,\omega) \) by
\[
\tilde{U}(s,q,\omega) = -\int_q^\infty A_2(p,\omega) \exp\left(-\frac{1}{2}G(\omega)A(p,\omega)s\right) dp.
\]

(7.2)
Note that $A_2(q, \omega) = 0$ whenever $q > R$, so when $q < R$, we can replace $\infty$ with $R$ in (7.2). In $\Omega$ we set

$$\tilde{u}(t, x) = \varepsilon r^{-1} \tilde{U}(\varepsilon \ln(t) - \delta, \tilde{q}(t, x), \omega).$$

We seek to prove that $\tilde{u}(t, x)$ provides a good approximation of $u(t, x)$.

### 7.2. Simplification

We aim to introduce some simplification in this subsection. Define a new function $F(q, \omega)$ on $\mathbb{R} \times S^2$ by

$$F(q, \omega) := 2R - \int_{2R}^q \frac{2}{A_1(p, \omega)} dp.$$ 

Then, we have

a) $F$ is defined everywhere, and $2(q - R) \leq F(q, \omega) \leq 2(q + R)/3$ for all $q < 2R$. This is because $A_1 \in [-3, -1]$ by Lemma [5.8].

b) $F$ is a smooth function of $(q, \omega)$, in the sense that for each large integer $N$ and $\varepsilon \ll N$, $F$ is in $C^N$. This is because $A_1 \in [-3, -1]$ and by Proposition [5.7].

c) $F(q, \omega) = q$ for $q > R$, and $(F(q, \omega)) \sim (q)$. This is because $A_1 \equiv -2$ for $q > R$.

d) For each fixed $\omega$, the map $q \mapsto F(q, \omega)$ has an inverse denoted by $\tilde{F}(q, \omega)$ which is also smooth (in the same sense as in a) above) in $\mathbb{R} \times S^2$. This is because $F_q = -2/A_1 \in [2/3, 2]$.

e) $\partial_q \omega A_1 = O((q)^{1-a+C\varepsilon})$. Recall that $A_1 < -1$ and $\partial_q \omega A_1 = O((q)^{-a+C\varepsilon})$. If $a = 0$, then $|\partial_q \omega A_1| \lesssim \int_{[q, 2R]} (p)^{C\varepsilon} dp \lesssim (q)^{1+C\varepsilon}$. If $a \geq 1$, then $|\partial_q \omega A_1| \lesssim \int_{[q, 2R]} (2/A_1) \lesssim (q)^{1-a+C\varepsilon}$.

For each $(s, q, \omega)$, we set

$$\hat{A}(q, \omega) := A(\hat{F}(q, \omega), \omega)$$

and

$$\begin{cases} 
\hat{\mu}(s, q, \omega) := -2 \exp(-\frac{1}{2}G(\omega)\hat{A}(q, \omega)s), \\
\hat{U}(s, q, \omega) := -\int_s^\infty \hat{A}(p, \omega) \exp(-\frac{1}{2}G(\omega)\hat{A}(p, \omega)s) dp.
\end{cases}$$

(7.3)

It is clear that $(\hat{\mu}, \hat{U})$ is a solution to the reduced system (5.5).

For each $(t, x) \in \Omega$, we set

$$\hat{q}(t, x) := F(\tilde{q}(t, x), \omega), \quad \hat{u}(t, x) := \varepsilon r^{-1} \hat{U}(\varepsilon \ln t - \delta, \tilde{q}(t, x), \omega).$$

We then have the next key lemma.

**Lemma 7.2.** In $\Omega$, we have

$$\hat{q}_t - \hat{q}_r = \hat{\mu}(\varepsilon \ln t - \delta, \hat{q}(t, x), \omega)$$

and $\hat{q} = r - t$ whenever $r - t > R$. Moreover, we have $\hat{u}(t, x) = \tilde{u}(t, x)$ everywhere.

**Proof.** At $(t, x) \in \Omega$, we first have

$$\tilde{q}(t, x) = \tilde{F}(F(\tilde{q}(t, x), \omega), \omega) = \tilde{F}(\hat{q}(t, x), \omega).$$
Thus,
\[ \dot{q}_t - \dot{q}_r = (\partial_t - \partial_r) F(\tilde{q}(t,x), \omega) = F_q(\tilde{q}(t,x), \omega) \cdot \tilde{\mu}(\varepsilon \ln t - \delta, \tilde{q}(t,x), \omega) \]
\[ = (-2/A_1 \cdot A_1 \exp(-1/2 GA_s)) (\varepsilon \ln t - \delta, \tilde{q}(t,x), \omega) \]
\[ = -2 \exp(-1/2 G(\omega) A(\tilde{q}(t,x), \omega)(\varepsilon \ln t - \delta)) \]
\[ = -2 \exp(-1/2 G(\omega) A(\hat{F}(\tilde{q}(t,x), \omega), \omega)(\varepsilon \ln t - \delta)) \]
\[ = -2 \exp(-1/2 G(\omega) \hat{A}(\tilde{q}(t,x), \omega)(\varepsilon \ln t - \delta)) = \hat{\mu}(\varepsilon \ln t - \delta, \tilde{q}(t,x), \omega). \]

Since \( F(q, \omega) = q \) for all \( q > R \), we have \( \dot{q}(t,x) = \tilde{q}(t,x) = r - t \) whenever \( r - t > R \).

Moreover, if \( \rho = \hat{F}(p, \omega) \), then we have \( p = F(\rho, \omega) \) and thus
\[ A(\rho, \omega) = A(\hat{F}(p, \omega), \omega) = \hat{A}(p, \omega). \]

Then by the change of variables \( (\rho = \hat{F}(p, \omega) \) and thus \( p = F(\rho, \omega) \)), we have
\[ \dot{U}(s, \tilde{q}, \omega) = - \int_{\tilde{q}}^{\infty} \hat{A}(p, \omega) \exp(1/2 G(\omega) \hat{A}(p, \omega)s) \, dp \]
\[ = - \int_{\tilde{q}}^{\infty} A(\rho, \omega) \exp(1/2 G(\omega) A(\rho, \omega)s) F_\rho(p, \omega) \, dp \]
\[ = - \int_{\tilde{q}}^{\infty} A_2(\rho, \omega) \exp(1/2 G(\omega) A(\rho, \omega)s) \, dp = \tilde{U}(s, \tilde{q}, \omega). \]

Here we note that \( AF_q = -2A_1 = A_2 \). That is, for each \( (s, q, \omega) \) (not viewed as functions of \( (t, x) \)),
\[ (7.4) \quad \dot{U}(s, q, \omega) = \tilde{U}(s, \hat{F}(q, \omega), \omega). \]

We thus have \( \tilde{u}(t, x) = \hat{u}(t, x). \)

Because of Lemma 7.2, we can work with \( (\hat{u}, \hat{q}) \) instead of \( (\tilde{u}, \tilde{q}) \).

We end this subsection with several useful estimates for \( (\hat{A}, \hat{\mu}, \hat{U}) \).

**Proposition 7.3.** For each \( (q, \omega) \), we have
\[ ((\langle q \rangle \partial_q)^a \partial_\omega \hat{F}(q, \omega) = O((\langle q \rangle)^{1+C(\varepsilon)}), \quad (\langle q \rangle \partial_q)^a \partial_\omega \hat{A}(q, \omega) = O((\langle q \rangle)^{-1+C(\varepsilon)}). \]

Besides, for each \( (s, q, \omega) \in \Omega' \cap \{ q < 2R \} \), we have
\[ \partial_s^a (\langle q \rangle \partial_q)^a \partial_\omega \hat{U} = O(\varepsilon^{-b} t^{C(\varepsilon)}), \quad \partial_s^b (\langle q \rangle \partial_q)^a \partial_\omega \hat{\mu} = O(t^{C(\varepsilon)}); \]
\[ \hat{\mu} = O(t^{C(\varepsilon)}), \quad \partial_s^b (\langle q \rangle \partial_q)^a \partial_\omega \hat{\mu} = O(\langle q \rangle^{-1+C(\varepsilon)} t^{C(\varepsilon)} |\hat{\mu}|), \quad a + b + |c| > 0. \]

**Proof.** First, it is clear that \( \langle \hat{F}(q, \omega) \rangle \sim \langle q \rangle \) and that \( \hat{F}_q(q, \omega) = 1/(\hat{F}(q, \omega), \omega) = -A_1(\hat{F}(q, \omega), \omega)/2 \sim (\langle q \rangle)^{C(\varepsilon)}. \) In general we induct on \( m + |n| \). By differentiating \( q = F(\hat{F}(q, \omega), \omega) \) for \( (a, c) \notin \{(0, 0), (1, 0)\} \), we have
\[ 0 = F_q(\hat{F}(q, \omega), \omega) \cdot \partial_q^a \partial_\omega \hat{F}(q, \omega) + \sum \frac{\partial^n}{\partial_q \partial_\omega^n} \hat{F}(q, \omega) \cdot \prod_{j=1}^{m} (\partial_q^a \partial_\omega^b \hat{F}(q, \omega)). \]

95
Here the sum on the right hand side is taken over all \((m, c', a_s, c_a)\) such that \(\sum a_j = a, c' + \sum c_j = c, a_j + |c_j| < a + |c|\). We can now apply the induction hypotheses to conclude that
\[
0 = F_q(\hat{F}(q, \omega), \omega) \cdot \partial_q^a \partial_\omega^c \hat{F}(q, \omega) + \sum O((\langle \hat{F}(q, \omega) \rangle)^{1-m+C_\varepsilon} \cdot \langle \hat{q} \rangle^{m-\sum a_j+C_\varepsilon})
\]
\[
= F_q(\hat{F}(q, \omega), \omega) \cdot \partial_q^a \partial_\omega^c \hat{F}(q, \omega) + O(\langle q \rangle^{1-a+C_\varepsilon}).
\]

And since \(F_q \sim 1\), we conclude that \(\partial_q^a \partial_\omega^c \hat{F}(q, \omega) = O(\langle q \rangle^{1-a+C_\varepsilon})\).

Next, recall that
\[
\hat{A}(q, \omega) = A(\hat{F}(q, \omega), \omega), \quad \hat{U}(s, q, \omega) = \tilde{U}(s, \hat{F}(q, \omega), \omega).
\]
Then, \(\partial_q^a \partial_q q \partial_\omega^c \hat{U}(s, q, \omega)\) is a linear combination of terms of the form
\[
\partial_q^a \partial_\omega^c \hat{U}(s, q, \omega) = \prod_{j=1}^m \partial_q q_j \partial_\omega^c \hat{F}(q, \omega), \quad \sum a_j = a, c' + \sum c_j = c.
\]
By (5.11) and (5.12), we conclude that each of these terms are controlled by
\[
\varepsilon^{-b} \langle \hat{F}(q, \omega) \rangle^{-m_1 t C_\varepsilon} \cdot \langle q \rangle^{m-\sum a_j+C_\varepsilon} \lesssim \varepsilon^{-b} \langle q \rangle^{-a t C_\varepsilon}.
\]
Thus, \(\partial_q^a ((q) \partial_q)^a \partial_\omega \hat{U}(s, q, \omega) = O(\varepsilon^{-b} t C_\varepsilon)\).

Finally, by (7.3), we can write \(\partial_q^a \partial_q q \partial_\omega^c \hat{U}(s, q, \omega)\) as a linear combination of terms of the form
\[
\partial_q^a \partial_\omega^c \hat{A}(q, \omega) \cdot \exp(\frac{1}{2}G \hat{A}s) \prod_{j=1}^m \partial_q q_j \partial_\omega^c \hat{F}(q, \omega)
\]
where \(a' + \sum a_j = a, \sum b_j = b, c' + \sum c_j = c\). Each of these terms are controlled by
\[
\langle q \rangle^{-a'+c+C_\varepsilon} \cdot t C_\varepsilon \cdot \langle q \rangle^{-\sum a_j t C_\varepsilon} \lesssim \langle q \rangle^{-a t C_\varepsilon}.
\]
In conclusion, we have \(\partial_q^a ((q) \partial_q)^a \partial_\omega \hat{U}(s, q, \omega) = O(t C_\varepsilon)\). Here we do not have the factor \(\varepsilon^{-b}\) which is better. Moreover, we have \(\hat{\mu} = O(t C_\varepsilon)\) and
\[
(\hat{\mu}, (q) \hat{\mu}_q, \hat{\mu}_x) = -\frac{1}{2} (G A, (q) G A q_s, \partial_\omega (G A) s) \hat{\mu}.
\]

Following the same proof, we can show that \(\partial_q^a ((q) \partial_q)^a \partial_\omega \hat{\mu}(s, q, \omega) = O((\langle q \rangle)^{-1+C_\varepsilon} t C_\varepsilon |\hat{\mu}|)\) if \(a + b + |c| > 0\).

7.3. Estimates for \(\hat{q}\) and \(\hat{U}\). We now follow Section 4 in [35] to prove several useful estimates. In this subsection, all functions of \((s, q, \omega) \in [0, \infty) \times \mathbb{R} \times S^2\) are viewed as functions of \((t, x) \in \Omega\) by setting \((s, q, \omega) = (\varepsilon \ln t - \delta, \hat{q}(t, x, \omega)\). This setting is different from that in the previous sections of this paper, where we take \(q = q(t, x)\).

**Lemma 7.4.** In \(\Omega \cap \{r - t < 2R\}\), we have \(\langle \hat{q} \rangle / \langle r - t \rangle = t^{O(\varepsilon)}\) and \(\hat{q}(t, x) - r + t = O(\min\{\varepsilon^{-1}, \langle \hat{q} \rangle\}t C_\varepsilon)\).
Proof. Fix \((t, x) \in \Omega \cap \{r - t < 2R\}\). Then, we have

\[
|\dot{q}(t, x) - 2R| = \int_{(r+t)/2-R}^{t} (-\dot{\mu}(\varepsilon \ln \tau - \delta, \dot{q}(\tau, r + t - \tau, \omega), \omega)) \, d\tau
\]

\[
\lesssim \int_{(r+t)/2-R}^{t} \exp(C\langle \dot{q} \rangle^{-1+C\varepsilon} s)(\tau, r + t - \tau, \omega) \, d\tau
\]

\[
\lesssim \frac{(r - t)}{2 + R} t^{C\varepsilon} \lesssim \langle r - t \rangle t^{C\varepsilon};
\]

\[
|\dot{q}(t, x) - 2R| = \int_{(r+t)/2-R}^{t} (-\dot{\mu}(\varepsilon \ln \tau - \delta, \dot{q}(\tau, r + t - \tau, \omega), \omega)) \, d\tau
\]

\[
\geq \int_{(r+t)/2-R}^{t} \exp(-C\langle \dot{q} \rangle^{-1+C\varepsilon} s)(\tau, r + t - \tau, \omega) \, d\tau
\]

\[
\geq \frac{(r - t)}{2 + R} t^{-C\varepsilon} \gtrsim \langle r - t \rangle t^{-C\varepsilon}.
\]

Thus, we have \(t^{-C\varepsilon} \langle \dot{q} \rangle \lesssim \langle r - t \rangle \lesssim t^{C\varepsilon} \langle \dot{q} \rangle\). It follows that

\[
|\dot{q}(t, x) - (r - t)| \leq |\dot{q} - 2R| + |r - t - 2R| \lesssim t^{C\varepsilon} \langle \dot{q} \rangle + \langle r - t \rangle \lesssim \langle \dot{q} \rangle t^{C\varepsilon}.
\]

To improve the estimate above, we note that

\[
\dot{q}(t, x) = 2R + \int_{(r+t)/2-R}^{t} \dot{\mu}(\varepsilon \ln \tau - \delta, \dot{q}(\tau, r + t - \tau, \omega), \omega) \, d\tau
\]

\[
= r - t + \int_{(r+t)/2-R}^{t} (\dot{\mu}(\varepsilon \ln \tau - \delta, \dot{q}(\tau, r + t - \tau, \omega), \omega) + 2) \, d\tau.
\]

For each \((s, q, \omega) \in [0, \infty) \times \mathbb{R} \times S^2\), by Proposition 5.6 and Lemma 5.8 we have

\[
|\dot{\mu}(s, q, \omega) + 2| \lesssim |1 - \exp(-\frac{1}{2} GAs)| \lesssim \langle q \rangle^{-1+C\varepsilon} |s| \exp(C\langle q \rangle^{-1+C\varepsilon} s).
\]

By setting \((s, q, \omega) = (\varepsilon \ln \tau - \delta, \dot{q}(\tau, r + t - \tau, \omega), \omega)\), we have

\[
|\dot{\mu} + 2| \lesssim \langle r + t - 2\tau \rangle^{-1+C\varepsilon} \tau^{C\varepsilon} \lesssim (3R - r - t + 2\tau)^{-1+C\varepsilon} \tau^{C\varepsilon}
\]

and then

\[
|\dot{q} - r + t| \lesssim t^{C\varepsilon} \int_{(r+t)/2-R}^{t} (3R - r - t + 2\tau)^{-1+C\varepsilon} \, d\tau \lesssim \varepsilon^{-1} t^{C\varepsilon} (3R - r + t)^{C\varepsilon}.
\]

And since \(0 \leq 3R - r + t \lesssim 1 + t \lesssim t\), we have \(|\dot{q} - r + t| \lesssim \varepsilon^{-1} t^{C\varepsilon}\). \qed

Lemma 7.5. In \(\Omega\) we have

\[
\dot{\nu} := \dot{q} + \dot{r} = O(t^{-1+C\varepsilon}), \quad \dot{\lambda}_i := \dot{q}_i - \omega_i \dot{q}_r = O((1 + \ln \langle r - t \rangle) t^{-1+C\varepsilon}).
\]

It follows that \(\dot{q}_r = (\dot{\nu} - \dot{\mu})/2 > C^{-1} t^{-C\varepsilon}\) and \(\dot{q}_t = (\dot{\nu} + \dot{\mu})/2 < -C^{-1} t^{-C\varepsilon}\). Thus, for each fixed \((t, \omega)\) the function \(r \mapsto \dot{q}(t, r, \omega)\) is continuous and strictly increasing.

Proof. There is nothing to prove when \(r - t > R\). Fix \((t, x) \in \Omega \cap \{r - t < 2R\}\). Then,

\[
(\partial_t - \partial_r) \dot{\nu} = (\partial_t + \partial_r) \dot{\mu} = \mu_q \dot{\nu} + \varepsilon t^{-1} \dot{\mu}_s = \mu_q \dot{\nu} - \frac{\varepsilon}{2t} G(\omega) A(\dot{q}, \omega) \dot{\mu}
\]

\[
= -\frac{1}{2} G \dot{A}_q s \dot{\mu} - \frac{\varepsilon}{2t} G \dot{A} \dot{\mu}.
\]
By setting $z(\tau) := \hat{q}(\tau, r + t - \tau, \omega)$, we have $\dot{z} = \hat{\mu} < 0$ and thus

$$
\int_{(r+t)/2-R}^t |G\hat{A}_q \hat{\mu}|(\tau, r + t - \tau, \omega) \, d\tau \lesssim \int_{(r+t)/2-R}^t (\varepsilon \ln \tau + 1) (\hat{q})^{-2+C\varepsilon} (-\hat{\mu}) \, d\tau
$$

$$
\lesssim (\varepsilon \ln t + 1) \int_{(r+t)/2-R}^t (z)^{-2+C\varepsilon} (-\dot{z}) \, d\tau \lesssim \varepsilon \ln t + 1,
$$

$$
\int_{(r+t)/2-R}^t |\varepsilon \tau^{-1} G\hat{A}_Q \hat{\mu}|(\tau, r + t - \tau, \omega) \, d\tau \lesssim \varepsilon ((r + t)/2 - R)^{-1} \int_{(r+t)/2-R}^t (\hat{q})^{-1+C\varepsilon} (-\hat{\mu}) \, d\tau
$$

$$
\lesssim \varepsilon^{-1} \int_{(r+t)/2-R}^t (z)^{-1+C\varepsilon} (-\dot{z}) \, d\tau \lesssim t^{-1} (\hat{q})^{-C\varepsilon} \lesssim t^{-1+C\varepsilon}.
$$

Here we note that $\langle \hat{q} \rangle \lesssim \langle r - t \rangle t^{C\varepsilon} \lesssim t^{1+C\varepsilon}$. Since $\dot{\nu} = 0$ at $\tau = (r + t)/2 - R$, by the Gronwall’s inequality we conclude that $\dot{\nu} = O(t^{-1+C\varepsilon})$.

Next, we have

$$
(\partial_t - \partial_\tau) \hat{\lambda}_i = (\partial_t - \omega_\tau \partial_\tau) \hat{\mu} + r - 1 \hat{\lambda}_i = (\hat{\mu}_q + r - 1) \hat{\lambda}_i + \sum_l (\partial_\omega_q \hat{\lambda}_l (\hat{\omega}_q + r - 1) \hat{\lambda}_i)
$$

$$
= (\hat{\mu}_q + r - 1) \hat{\lambda}_i + \frac{1}{2} \sum_l (\partial_\omega_q (G\hat{A})) (\varepsilon \ln t - \delta) \hat{\mu_r}^{-1} (\delta_q - \omega_q i)
$$

$$
= (\hat{\mu}_q + r - 1) \hat{\lambda}_i + O(\langle \hat{q} \rangle^{-1+C\varepsilon} t^{-1+C\varepsilon} |\hat{\mu}|).
$$

We have proved that $\int_{(r+t)/2-R}^t |\mu_q| \, d\tau \lesssim \varepsilon \ln t + 1$. Integrate along the characteristic $(\tau, r + t - \tau, \omega)$ and we have

$$
\int_{(r+t)/2-R}^t (r + t - \tau)^{-1} \, d\tau = \ln \frac{(r + t)/2 + R}{r} = O(1),
$$

$$
\int_{(r+t)/2-R}^t \langle \hat{q} \rangle^{-1+C\varepsilon} (-\mu) \tau^{-1+C\varepsilon} \, d\tau \lesssim \int_{(r+t)/2-R}^t \langle \hat{q} \rangle^{-1} (-\hat{\mu}) \tau^{-1+C\varepsilon} \, d\tau
$$

$$
\lesssim t^{-1+C\varepsilon} \int_{(r+t)/2-R}^t (z)^{-1} (-\dot{z}) \, d\tau
$$

$$
\lesssim (1 + \ln \langle \hat{q} \rangle) t^{-1+C\varepsilon} \lesssim (1 + \ln(r - t)) t^{-1+C\varepsilon}.
$$

Here note that $\langle \hat{q} \rangle \lesssim t^{1+C\varepsilon}$ and $\ln \langle \hat{q} \rangle \lesssim \ln(r - t) + C\varepsilon \ln t$ in $\Omega \cap \{r - t < 2R\}$. Since $\hat{\lambda}_i = 0$ at $\tau = (r + t)/2 - R$, by Gronwall’s inequality we conclude that $\hat{\lambda}_i = O((1 + \ln(r - t)) t^{-1+C\varepsilon})$. □

**Lemma 7.6.** In $\Omega$, we have

$$
\dot{\nu} = \frac{\varepsilon G(\omega)}{4t} \hat{\mu} \hat{U} + O(\varepsilon t^{-2+2C\varepsilon} \langle r - t \rangle),
$$

$$
\hat{\nu}_q = \frac{\varepsilon G(\omega)}{4t} (\hat{\mu} \hat{U}_q + \hat{\mu}_q \hat{U}) + O(\varepsilon (1 + \ln(r - t)) t^{-2+2C\varepsilon}).
$$
Proof. We have

\[
(\partial_t - \partial_r)(\hat{\nu} - \frac{\varepsilon G(\omega)}{4t} \hat{\mu} U) = \hat{\mu}_q \hat{\nu} - \frac{\varepsilon}{2t} G \hat{A} \hat{\mu} + \frac{\varepsilon G \hat{\mu} \hat{U}}{4t^2} - \frac{\varepsilon G}{4t} (\hat{\mu}_q \hat{U} + \hat{\mu} \hat{U}_q) \hat{\mu} - \frac{\varepsilon G}{4t} (\hat{\mu}_s \hat{U} + \hat{\mu} \hat{U}_s) \varepsilon t^{-1}
\]

\[
= \hat{\mu}_q \hat{\nu} - \frac{\varepsilon}{2t} G \hat{A} \hat{\mu} + \frac{\varepsilon G \hat{\mu} \hat{U}}{4t^2} - \frac{\varepsilon G}{4t} (\hat{\mu}_q \hat{U} - 2 \hat{A} \hat{\mu}) - \frac{\varepsilon G}{4t} (-\frac{1}{2} G \hat{A} \hat{\mu} \hat{U} + \hat{\mu} \hat{U}_s) \varepsilon t^{-1}
\]

\[
= \hat{\mu}_q \hat{\nu} - \frac{\varepsilon G}{4t} \hat{\mu} \hat{U} + \frac{\varepsilon G \hat{\mu} \hat{U}}{4t^2} - \frac{\varepsilon^2 G}{4t^2} (-\frac{1}{2} G \hat{A} \hat{U} + \hat{U}_s) \mu.
\]

Since \( \hat{U} = O(t^{C\varepsilon}) \) and \( \hat{U}_s = O(\varepsilon^{-1} t^{C\varepsilon}) \) by Proposition 7.3, we have

\[
|\frac{\varepsilon G \hat{\mu} \hat{U}}{4t^2} - \frac{\varepsilon^2 G}{4t^2} (-\frac{1}{2} G \hat{A} \hat{U} + \hat{U}_s) \mu| \lesssim \varepsilon t^{-2+C\varepsilon}.
\]

Besides, we have

\[
\int_t^{(r+t)/2-R} \varepsilon t^{-2+C\varepsilon} \lesssim ((r+t)/2 - R)^{-2+C\varepsilon} \cdot \varepsilon ((t-r)/2 - R) \lesssim \varepsilon t^{-2+C\varepsilon} (r-t).
\]

And since \( \hat{\nu} - \frac{\varepsilon G}{4t} \hat{\mu} \hat{U} = 0 \) at \( \tau = (r+t)/2 - R \), by Gronwall’s inequality we conclude that

\[
\hat{\nu} - \frac{\varepsilon G}{4t} \hat{\mu} \hat{U} = O(\varepsilon t^{-2+C\varepsilon} (r-t)).
\]

Next, we have

\[
(\partial_t - \partial_r)(\hat{\nu} - \frac{\varepsilon G(\omega)}{4t} \hat{\mu} U) = \partial_r (\hat{\nu} - \frac{\varepsilon G(\omega)}{4t} \hat{\mu} U) - \frac{\varepsilon G}{4t} \hat{\mu} U + \frac{\varepsilon G \hat{\mu} \hat{U}}{4t^2} - \frac{\varepsilon^2 G}{4t^2} (-\frac{1}{2} G \hat{A} \hat{U} + \hat{U}_s) \mu
\]

\[
= \hat{\mu}_q \partial_r (\hat{\nu} - \frac{\varepsilon G}{4t} \hat{\mu} U) + \frac{\varepsilon G \hat{\mu} \hat{U}}{4t^2} - \frac{\varepsilon^2 G}{4t^2} (-\frac{1}{2} G \hat{A} \hat{U} + \hat{U}_s) \mu
\]

\[
= \hat{\mu}_q \partial_r (\hat{\nu} - \frac{\varepsilon G}{4t} \hat{\mu} U) + \frac{\varepsilon G \hat{\mu} \hat{U}}{4t^2} - \frac{\varepsilon^2 G}{4t^2} (-\frac{1}{2} G \hat{A} \hat{U} + \hat{U}_s) \mu - \frac{\varepsilon G}{4t} \hat{\mu} \hat{U} + \frac{\varepsilon G \hat{\mu} \hat{U}}{4t^2} - \frac{\varepsilon^2 G}{4t^2} (-\frac{1}{2} G \hat{A} \hat{U} + \hat{U}_s) \mu.
\]

By Proposition 7.3, we have

\[
|\hat{\mu}_q \partial_r (\hat{\nu} - \frac{\varepsilon G}{4t} \hat{\mu} U)| \lesssim |\hat{\mu}_q \partial_r (G \hat{A} \hat{\mu} s \hat{\mu})| \cdot \varepsilon t^{-2+C\varepsilon} (r-t) \lesssim \varepsilon t^{-2+C\varepsilon} (\hat{q})^{-2+C\varepsilon},
\]

\[
|\partial_r (\hat{\mu} U)| \lesssim |\hat{\mu} U| + |2 \hat{A}| \lesssim t^{C\varepsilon} (\hat{q})^{-2+C\varepsilon} + (\hat{q})^{-1+C\varepsilon} \lesssim t^{C\varepsilon} (\hat{q})^{-1+C\varepsilon},
\]

\[
|(-\frac{1}{2} G \hat{A} \hat{U} + \hat{U}_s) \mu| \lesssim (|\hat{q}|^{-1+C\varepsilon} t^{C\varepsilon} + \varepsilon^{-1} t^{C\varepsilon}) \cdot \varepsilon^{-2+C\varepsilon} t^{C\varepsilon} \lesssim \varepsilon^{-1} (\hat{q})^{-2+C\varepsilon} t^{C\varepsilon},
\]

\[
|(-\frac{1}{2} G \partial_q (\hat{A} \hat{U}) + \hat{U}_s) \mu| \lesssim |(-\frac{1}{2} G \partial_q (\hat{A} \hat{U}) + \frac{1}{2} G \hat{A} \hat{U} \hat{q}) \mu| \lesssim |(-\frac{1}{2} G \hat{A} \hat{U} \hat{q}) \mu| \lesssim (\hat{q})^{-2+C\varepsilon} t^{C\varepsilon}.
\]
In conclusion,
\[
\left( \partial_t - \partial_r \right) \partial_r (\hat{v} - \frac{\varepsilon G(\omega)}{4t} \hat{\mu} \hat{U}) = \hat{\mu}_q \partial_r (\hat{v} - \frac{\varepsilon G}{4t} \hat{\mu} \hat{U}) + O(|\hat{q}_r| \varepsilon (\hat{q})^{-1+\gamma} t^{-2+\gamma} \\
= \mu_q \partial_r (\hat{v} - \frac{\varepsilon G}{4t} \hat{\mu} \hat{U}) + O((-\hat{\mu}) \varepsilon (\hat{q})^{-1+\gamma} t^{-2+\gamma} + |\hat{v}| \varepsilon (\hat{q})^{-1+\gamma} t^{-2+\gamma})
\]
\[
= \mu_q \partial_r (\hat{v} - \frac{\varepsilon G}{4t} \hat{\mu} \hat{U}) + O((-\hat{\mu}) \varepsilon (\hat{q})^{-1+\gamma} t^{-2+\gamma} + \varepsilon (\hat{q})^{-1+\gamma} t^{-3+\gamma}).
\]
Take integral of the remainder terms along a characteristic \((\tau, r + t - \tau, \omega)\) for \((r + t)/2 - R \leq \tau \leq t\). We have
\[
\int_{(r+t)/2-R}^{t} \tau^{-2+C_\varepsilon} \varepsilon(z)^{-1+C_\varepsilon} (-\hat{z}) + \varepsilon \tau^{-3+C_\varepsilon} d\tau \lesssim \varepsilon (1 + \ln(r-t)) t^{-2+C_\varepsilon}.
\]
The proof of this estimate can be found in the proof of Lemma 7.5. Since \(\hat{v} - \frac{\varepsilon G(\omega)}{4t} \hat{\mu} \hat{U} = 0\) whenever \(r - t > R\), we have \(\partial_r (\hat{v} - \frac{\varepsilon G(\omega)}{4t} \hat{\mu} \hat{U}) = 0\) at \(\tau = (r+t)/2 - R\). By Gronwall’s inequality, we conclude that \(\partial_r (\hat{v} - \frac{\varepsilon G(\omega)}{4t} \hat{\mu} \hat{U}) = O(\varepsilon (1 + \ln(r-t)) t^{-2+C_\varepsilon})\). To end the proof, we recall that \(\partial_r = \hat{q}_r \partial_q\) where \(\hat{q}_r > \zeta^{-1} t^{-\gamma - 2\varepsilon}\) in \(\Omega \cap \{r - t < 2R\}\).}

Before we state the next lemma, we introduce the following notation.

**Definition.** Fix \(s, p \in \mathbb{R}\). We say that a function \(F = F(t, x)\) with domain \(\Omega \cap \{r - t < 2R\}\) belongs to \(S^{s, p}\), if for \(\varepsilon \ll s, p\) 1, we have \(Z^I F = O_I (t^{s+C_\varepsilon (r-t)^p})\) for all multiindices \(I\) in \(\Omega \cap \{r - t < 2R\}\).

It follows directly that \(S^{s, p} + S^{s', p'} \in S^{\max(s, s'), \max(p, p')}\), that \(S^{s, p} \cdot S^{s', p'} \in S^{s+s', p+p'}\), and that \(Z^I S^{s, p} \in S^{s, p}\).

Following the proof of Corollary 4.21 we can show that \(\mathcal{R}_{s, p} \in S^{s, p}\). Here we prefer this new notation \(S^{s, p}\) since it does not rely on the optical function \(q(t, x)\) and the corresponding null frames.

**Lemma 7.7.** We have \(\hat{q} \in S^{0,1}\). We also have \(\Omega_{k_k''} \hat{q} \in S^{0, \gamma}\) for each \(1 \leq k < k' \leq 3\) and \(0 < \gamma < 1\). In other words, in \(\Omega \cap \{r - t < 2R\}\), for each \(I\) we have
\[
(7.5) \quad |Z^I \hat{q}| \lesssim_t (r - t) t^{C_\varepsilon},
\]
\[
(7.6) \quad |Z^I \Omega_{k_k''} \hat{q}| \lesssim_t t^{C_\varepsilon (r - t)^\gamma}.
\]
As a result, we have \(\partial_q^m \partial_w \hat{A} \in S^{0, -1 - m}\), \(\hat{\mu} \in S^{0, 0}\), \(\partial^p_q \partial^m_w \hat{\mu} \in S^{0, -1 - m}\) for \(m + n + p > 0\), \(\partial_q^p \partial_w \hat{U} \in \varepsilon^{-p} S^{0, 0}\) and \(\partial^p_q \partial^m_w \partial^p w \hat{U} \in S^{0, -1 - m}\). All functions here are of \((s, q, w) = (\varepsilon \ln t - \delta, \hat{q} (t, x), \omega)\).

**Proof.** We prove (7.5) by induction on \(|I|\). The case \(|I| = 0\) has been proved in Lemma 7.4. In general, suppose (7.5) holds for all \(|I| \leq k\), and fix a multiindex \(I\) with \(|I| = k + 1\). By the chain rule and Leibniz’s rule, we express \(Z^I \hat{\mu}\) as a linear combination of terms of the form
\[
(7.7) \quad (\partial^p_q \partial^m_w \hat{\mu}) \cdot Z^I \hat{q} \cdot Z^{J_1} (\varepsilon \ln t - \delta) \cdots Z^{J_n} (\varepsilon \ln t - \delta) \cdot \prod_l Z^{K_{l,\varepsilon}} \omega_l \cdots Z^{K_{l,\varepsilon}} \omega_l
\]

where \( a + b + |c| > 0, |I_\ast|, |J_\ast|, |K_{\ast, \ast}| \) are nonzero, and the sum of all these multiindices is \( k + 1 \). The only term with some \(|I_\ast| > k\) is \( \tilde{\mu}_q Z^I \tilde{q} \). All the other terms have an upper bound

\[
\langle \tilde{q} \rangle^{-1-a+C\varepsilon t C} |\tilde{\mu}| \cdot ((r-t) t C)^a \cdot \varepsilon^b \cdot 1 \lesssim \langle \tilde{q} \rangle^{-1} t C \langle \tilde{\mu} \rangle.
\]

Here we apply Proposition 7.3 and the induction hypotheses to control \( Z^I \tilde{q} \). In summary, we have \( Z^I \tilde{\mu} = \tilde{\mu}_q Z^I \tilde{q} + O(\langle \tilde{q} \rangle^{-1+C\varepsilon t C} |\tilde{\mu}|) \). Following the same proof, we also have

\[
\sum_{0 < |I| \leq k} |Z^I \tilde{\mu}| = O(\langle \tilde{q} \rangle^{-1+C\varepsilon t C} |\tilde{\mu}|).
\]

In addition, by the induction hypotheses and Lemma 2.2 we have

\[
\sum_{|J| < |I|} |(\partial_i + \omega_i \partial_t) Z^J \tilde{q}| \lesssim \sum_{|J| < k+1} (1 + t + r)^{-1} |ZZ^J \tilde{q}|
\]

\[
\lesssim (1 + t + r)^{-1} \sum_{|J| = k+1} |Z^J \tilde{q}| + t^{-1+C\varepsilon} \langle r - t \rangle.
\]

In summary, by (2.6) in Lemma 2.1 we have

\[
|\langle \partial_i - \partial_r \rangle Z^I \tilde{q}| \lesssim |\tilde{\mu}_q Z^I \tilde{q}| + (1 + t + r)^{-1} \sum_{|J| = k+1} |Z^J \tilde{q}| + t^{C\varepsilon}(-\tilde{\mu}) + t^{-1+C\varepsilon}(r - t).
\]

Here we note that

\[
\sum_{|J| \leq k} |Z^J \tilde{\mu}| \lesssim |\tilde{\mu}| + \langle \tilde{q} \rangle^{-1+C\varepsilon t C} |\tilde{\mu}| \lesssim t^{C\varepsilon}(-\tilde{\mu}).
\]

Now, we fix \((t, x) \in \Omega \cap \{r - t < 2R\}\), integrate \((\partial_i - \partial_r) Z^I \tilde{q}\) along the characteristic \((\tau, \rho + \tau - \tau, \omega)\) for \((\rho + \tau) / 2 - R \leq \tau \leq t\), and sum over all \(|I| = k + 1\). We then have

\[
\sum_{|I| = k+1} |Z^I \tilde{q}(t, x) - Z^I \tilde{q}|_{\tau = (r+t)/2 - R} \lesssim \int_{(r+t)/2 - R}^{t} \left( |\tilde{\mu}_q| + (1 + t + r)^{-1} \sum_{|I| = k+1} |Z^I \tilde{q}| + t^{C\varepsilon}(-\tilde{\mu}) + t^{C\varepsilon} \right) \, d\tau.
\]

Moreover, we have \( \tilde{q} = r - t \) for \( r - t > R \) and \( \tilde{q} = 2R \) at \( \tau = (r+t)/2 - R \), so

\[
|Z^I \tilde{q}|_{\tau = (r+t)/2 - R} = |Z^I (r-t)|_{\tau = (r+t)/2 - R} \lesssim t^{C\varepsilon}.
\]

By Gronwall’s inequality, we conclude that \( \sum_{|I| = k+1} |Z^I \tilde{q}(t, x)| \lesssim \langle r - t \rangle t^{C\varepsilon} \).

Fix \( \gamma > 0 \). Now we prove (7.6) by induction on \(|I|\). First, in Lemma 7.5 we have proved \( \hat{\lambda}_i = O((1 + \ln(r-t)) t^{-1+C\varepsilon}) = O_\gamma(t^{-1+C\varepsilon}) \) for \( r - t > t^{-1+C\varepsilon} \). So we have \( \Omega_{kk'} \tilde{q} = x_k \lambda_{k'} - x_{k'} \lambda_k = O((r - t)^{C\varepsilon} t^{-1+C\varepsilon}) \), so the case \(|I| = 0\) is proved. In general, we fix \( I \) with \(|I| > 0\). As computed above, we have

\[
Z^I \Omega_{kk'} \tilde{\mu} = \tilde{\mu}_q Z^I \Omega_{kk'} \tilde{q} + O(\langle \tilde{q} \rangle^{-1+C\varepsilon t C} |\tilde{\mu}|), \quad \sum_{|J| \leq |I|} |Z^J \tilde{\mu}| = O(\langle \tilde{q} \rangle^{-1+C\varepsilon t C} |\tilde{\mu}|);
\]
In addition, recall that $|\partial_t + \omega_q \partial_t| Z^J \Omega_{kk'} \hat{q} | \lesssim (1 + t + r)^{-1} \sum_{|J| \leq |I|} |Z^J \Omega_{kk'} \hat{q}|

\lesssim (1 + t + r)^{-1} \sum_{|J| = |I|} |Z^J \Omega_{kk'} \hat{q}| + t^{-1+C \varepsilon} (r - t)^\gamma.

Thus, by (2.1), we have

$$
|\partial_t - \partial_t| Z^J \Omega_{kk'} \hat{q} | \lesssim |\mu_q Z^J \Omega_{kk'} \hat{q}| + (1 + t + r)^{-1} \sum_{|J| = |I|} |Z^J \Omega_{kk'} \hat{q}|

+ \langle \hat{q} \rangle^{-1+C \varepsilon} t^{C \varepsilon} (-\hat{\mu}) + t^{-1+C \varepsilon} (r - t)^\gamma.
$$

Fix $(t, x) \in \Omega \cap \{ r - t < 2R \}$ and take integrals along a geodesic $(\tau, r + t - \tau, \omega)$. We note that

$$
\int_{(r+t)/2 - R}^{t} \langle \hat{q}(\tau) \rangle^{-1+C \varepsilon} \tau^{C \varepsilon} (-\hat{\mu}(\tau)) + \tau^{-1+C \varepsilon} \langle r + t - 2\tau \rangle^\gamma d\tau

\lesssim t^{C \varepsilon} \int_{(r+t)/2 - R}^{t} \langle z(\tau) \rangle^{-1} \langle \hat{z}(\tau) \rangle d\tau + t^{-1+C \varepsilon} \langle r - t \rangle^{1+\gamma}

\lesssim (1 + \langle r - t \rangle) t^{C \varepsilon} + t^{C \varepsilon} \langle r - t \rangle^\gamma \lesssim t^{C \varepsilon} \langle r - t \rangle^\gamma.
$$

In addition, recall that $Z^J q |_{\tau=(r+t)/2-2R} = O(t^{C \varepsilon})$. We finish the proof by applying Gronwall.

Finally, if $Q = Q(s, q, \omega)$ is a given function of $(s, q, \omega)$ and if we take $(s, q, \omega) = (\varepsilon \ln t - \delta, \hat{q}(t, x), 4\omega)$, then $Z^J Q$ is a linear combination of terms of the form (7.7) with $\hat{\mu}$ replaced by $\hat{Q}$. Thus,

$$
|Z^J Q| \lesssim \sum_{a+b+c \leq |I|} \varepsilon^b (r - t)^{a+C \varepsilon} |\hat{\partial}^a \hat{\partial}^b \hat{\partial}^c Q|.
$$

We combine this inequality with Proposition 7.8. As a result, we have $\partial^m \partial^n \hat{A} \in S^{0,-1-m}, \hat{\mu} \in S^{0,0}, \partial^m \partial^n \partial^m \hat{\mu} \in S^{0,-1-m}$ for $m + n + p > 0$, $\partial^p \partial^q F \hat{\mu} \in \varepsilon^{-p} S^{0,0}$ and $\partial^n \partial^m \partial^m \hat{U} q \in S^{0,-1-m}$. □

**Lemma 7.8.** Fix $\gamma \in (0, 1)$. We have $\hat{\nu} \in \varepsilon S^{1,-1}, \hat{\nu} q \in \varepsilon S^{-1,-1}, \hat{\lambda} \in S^{1,-1}$ and

$$
\hat{\nu} - \frac{\varepsilon}{1+t} G(\omega) \mu \hat{U} \in \varepsilon S^{-2,1}, \quad \hat{\nu} q - \frac{\varepsilon}{1+t} G(\omega) (\mu \hat{U} - 2\hat{A}) \in \varepsilon S^{-2,0}.
$$

All functions here are of $(s, q, \omega) = (\varepsilon \ln t - \delta, \hat{q}(t, x), \omega)$.

**Proof.** First, we have

$$
\hat{\lambda} = \sum_{j} r^{-1} \omega_j \Omega_{ji} \hat{q} \in S^{1,-1} \cdot S^{0,\gamma} \subset S^{1,-1,\gamma}.
$$

Next, we set $Q := \hat{\nu} - \varepsilon G(\omega) \mu \hat{U} / (4t)$. We have proved $Q = O(\varepsilon t^{-2+C \varepsilon} (r - t))$ in Lemma 7.6. In general, we fix $I$ with $|I| > 0$ and suppose $Z^J Q = O(\varepsilon t^{-2+C \varepsilon} (r - t))$ whenever $|J| < |I|$. As computed in Lemma 7.6, we have

$$
Q_t - Q_r = \mu q Q + \frac{\varepsilon G \hat{U}}{4t^2} - \frac{\varepsilon^2 G}{4t^2} \left( - \frac{1}{2} \hat{A} \hat{U} + \hat{u} q \right) \mu = \mu q Q + \varepsilon S^{-2,0}.
$$
By (2.6) in Lemma 2.1, we have
\[ |(\partial_t - \partial_r)Z^I Q| \lesssim |Z^I (\hat{\mu}_q Q + \varepsilon S^{-2,0})| + \sum_{|J|<|I|} [|Z^J (\hat{\mu}_q Q + \varepsilon S^{-2,0})| + (1 + t + r)^{-1}|ZZ^I Q|] \]
\[ \lesssim |\hat{\mu}_q Z^I Q| + (1 + t + r)^{-1} \sum_{|J|=|I|} |Z^J Q| + \sum_{|K_1|+|K_2|\leq |I| \atop |K_2|<|I|} (|Z^{K_1} \hat{\mu}_q| + t^{-1})|Z^{K_2} Q| + \varepsilon t^{-2+C\varepsilon} \]
\[ \lesssim |\hat{\mu}_q Z^I Q| + (1 + t + r)^{-1} \sum_{|J|=|I|} |Z^J Q| + \varepsilon t^{-2+C\varepsilon} (r - t)^{-1} + \varepsilon t^{-2+C\varepsilon}. \]

The last estimate follows from \( \hat{\mu}_q \in S^{0,-2} \) and the induction hypotheses. Since \( Q \equiv 0 \) near \( \tau = (r + t)/2 - R \), and since
\[ \int_{(r+t)/2-R}^t \varepsilon_t^{-2+C\varepsilon} \, d\tau \lesssim \varepsilon t^{-2+C\varepsilon} (r - t), \]
we conclude by Gronwall that \( Z^I Q = O(\varepsilon t^{-2+C\varepsilon} (r - t)) \). So \( Q \in \varepsilon S^{-2,1} \).

Since \( \hat{\mu}, \hat{U} \in S^{0,0} \) and since \( \langle r - t \rangle \lesssim t \in \Omega \cap \{ r - t < 2R \} \), we have \( \hat{v} = Q + \varepsilon G(\omega) \hat{\mu} \hat{U} / (4t) \in \varepsilon S^{-2,1} + \varepsilon S^{-1,0} \subset \varepsilon S^{-1,0} \). Moreover, for each \( I \) we have
\[ |Z^I Q_q| \lesssim |Z^I (\hat{q}_r^{-1} \omega \cdot \partial Q)| \lesssim \sum_{|J|\leq |I|} t^{C\varepsilon} |Z^J \partial Q| \]
\[ \lesssim \sum_{|J|\leq |I|} t^{C\varepsilon} |\partial Z^J Q| \lesssim \langle r - t \rangle^{-1} t^{C\varepsilon} \sum_{|J|\leq |I|+1} |Z^I Q| \lesssim \varepsilon t^{-2+C\varepsilon}. \]

Here we use the estimate \( \hat{q}_r^{-1} \in S^{0,0} \) which follows from \( \hat{q}_r \in S^{0,0} \) and \( \hat{q}_r > C^{-1} t^{-C\varepsilon} \). Thus,
\[ Q_q = \hat{v}_q - \frac{\varepsilon}{4t} G(\omega)(\hat{\mu}_q \hat{U} - 2\hat{A}) \in \varepsilon S^{-2,0}. \]

Since \( \hat{\mu}_q \hat{U} \in S^{0,-2} \) and \( \hat{A} \in S^{0,-1} \), we conclude that \( \hat{v}_q \in \varepsilon S^{-1,-1} + \varepsilon S^{-2,0} = \varepsilon S^{-1,-1} \).

Now we prove that \( \hat{q} \) is an approximate optical function.

**Proposition 7.9.** We have
\[ g^{\alpha\beta}(\hat{u}) \hat{q}_\alpha \hat{q}_\beta \in S^{-2,1}. \]

**Proof.** Fix \( \gamma \in (0,1/2) \) and suppose we have obtained \( \hat{\lambda}_i \in S^{-1,\gamma} \) from the pervious lemma. We note that \( \hat{q}_t = \frac{1}{2}(\hat{\mu} + \hat{v}) \in S^{0,0} \) and \( \hat{q}_t = \frac{1}{2}(-\hat{\mu} + \hat{v}) \omega_i + \hat{\lambda}_i \in S^{0,0} \). Thus,
\[ g^{\alpha\beta}_0 \hat{q}_\alpha \hat{q}_\beta = \frac{1}{4} \left( \frac{1}{2}(\hat{\mu} + \hat{v})^2 + \frac{1}{2} g^{0i}(\hat{\mu} + \hat{v})(\hat{\mu} + \hat{v}) \omega_i + 2\hat{\lambda}_i \right) \]
\[ + \frac{1}{4} g^{ij}((\hat{\mu} + \hat{v}) \omega_i + 2\hat{\lambda}_i)((\hat{\mu} + \hat{v}) \omega_j + 2\hat{\lambda}_j) \]
\[ = \frac{1}{4} G(\omega) \hat{\mu}^2 + \frac{1}{2} g^{00} \mu \hat{v} + \frac{1}{4} g^{0i} (2\mu \hat{\lambda}_i + \hat{v}^2 \omega_i + 2\hat{v} \lambda_i) \]
\[ + \frac{1}{4} g^{ij}(\hat{\mu}(2\hat{v} \omega_j \omega_i + 2\hat{\lambda}_j \omega_i + 2\hat{\lambda}_i \omega_j) + (\hat{v} \omega_i + 2\hat{\lambda}_i)(\hat{v} \omega_j + 2\hat{\lambda}_j)). \]
Since $\hat{\nu} \in \varepsilon S^{-1,0}$ and $\hat{\lambda}_i \in S^{-1,\gamma}$, we have $\hat{\nu}^2, \hat{\nu} \hat{\lambda}_i, \hat{\lambda}_i \hat{\lambda}_j \in S^{-2,2\gamma}$ and thus
\[
g^{\alpha\beta}_0 \hat{q}_\alpha \hat{q}_\beta = \frac{1}{4} G(\omega) \hat{\mu}^2 + \frac{1}{2} (g_0^{00} - g_0^{ij} \omega_i \omega_j) \hat{\mu} \hat{\nu} + g_0^{0i} \hat{\mu} \hat{\lambda}_i - \frac{1}{2} g_0^{ij} \hat{\mu} (\hat{\lambda}_j \omega_i + \hat{\lambda}_i \omega_j) \mod S^{-2,2\gamma}
\]
\[
= \frac{1}{4} G(\omega) \hat{\mu}^2 \mod S^{-1,\gamma}.
\]
If we replace $(g_0^{\alpha\beta})$ with $(m^{\alpha\beta})$ in the computations, we have
\[
-\hat{q}_i^2 + \sum_j \hat{q}_j^2 = -\hat{\mu} \hat{\nu} - \frac{1}{2} m^{ij} \hat{\mu} (\hat{\lambda}_j \omega_i + \hat{\lambda}_i \omega_j) \mod S^{-2,2\gamma} = -\hat{\mu} \hat{\nu} \mod S^{-2,2\gamma}.
\]
Here we note that $m^{ij} \hat{\lambda}_j \omega_i = m^{ij} \hat{\lambda}_i \omega_j = \sum_j \omega_j (\hat{q}_j - \omega_j \hat{q}_r) = 0$.

Moreover, note that $\hat{u} = \varepsilon r^{-1} \hat{U} \in \varepsilon S^{-1,0}$. Following the proof of Lemma 1.24 with $V$ replaced by $Z$, we can prove that $f(\hat{u}) = f(0) - f'(0) \hat{u} \in \varepsilon^2 S^{-2,0}$ for each smooth function $f$. Thus,
\[
g^{\alpha\beta}(\hat{u}) \hat{q}_\alpha \hat{q}_\beta = -\hat{q}_i^2 + \sum_j \hat{q}_j^2 + g_0^{\alpha\beta} \hat{u} \hat{q}_\alpha \hat{q}_\beta + (g^{\alpha\beta}(\hat{u}) - g_0^{\alpha\beta} \hat{u} - m^{\alpha\beta}) \hat{q}_\alpha \hat{q}_\beta
\]
\[
= -\hat{\mu} (\hat{\nu} - \frac{\varepsilon}{4r} G(\omega) \hat{\mu} \hat{U}) \mod S^{-2,2\gamma}
\]
\[
= -\hat{\mu} \hat{\nu} - \frac{\varepsilon}{4t} G(\omega) \hat{\mu} \hat{U} + \frac{\varepsilon(t-r)}{4rt} G(\omega) \hat{\mu}^2 \hat{U} \mod S^{-2,2\gamma}
\]
\[
= \varepsilon S^{-2,1} \mod S^{-2,2\gamma}.
\]
Since $\gamma \in (0,1/2)$, we have $\varepsilon S^{-2,1} \subset S^{-2,1}$ and $S^{-2,2\gamma} \subset S^{-2,1}$.

In order to prove that $\hat{u}$ is an approximate solution to (1.1), we need the following lemma.

**Lemma 7.10.** For each $\gamma \in (0,1/2)$, we have
\[
g^{\alpha\beta}(\hat{u}) \partial_\alpha \partial_\beta \hat{q} = -r^{-1} \hat{\mu} + \frac{\varepsilon}{2t} G \hat{A} \hat{\mu} \mod S^{-2,\gamma}.
\]

**Proof.** Fix $\gamma \in (0,1/2)$ and suppose we have obtained $\hat{\lambda}_i \in S^{-1,\gamma}$. First we note that
\[
\varepsilon t^{-1} \hat{\nu}_s = \hat{\nu}_t - \hat{\nu}_q \hat{q}_t = \hat{\nu}_t + \hat{\nu}_r - \hat{\nu}_q,
\]
\[
\sum_j (\partial_i \omega_j) \hat{\nu}_s = \hat{\nu}_i - \hat{\nu}_q \hat{q}_i = \hat{\nu}_i - \omega_i \hat{\nu}_r - \hat{\lambda}_i \hat{\nu}_r.
\]
Note that
\[
\partial_t + \partial_r = \frac{\sum_j \omega_j \Omega_{0j} + S}{r + t}, \quad \partial_t - \omega_i \partial_r = r^{-1} \sum_j \omega_j \Omega_{ji},
\]
and that $\hat{\nu} \in \varepsilon S^{-1,0}$. Thus, we conclude that $\hat{\nu}_t + \hat{\nu}_r, \hat{\nu}_i - \omega_i \hat{\nu}_r, \in \varepsilon S^{-2,0}$. Besides, we have $\hat{\nu}_q \in \varepsilon^2 S^{-2,-1}$ and $\hat{\lambda}_i \hat{\nu}_q \in \varepsilon S^{-2,-1+\gamma}$. We conclude that $\varepsilon t^{-1} \hat{\nu}_s, \sum_j (\partial_i \omega_j) \hat{\nu}_s \in \varepsilon S^{-2,0}$.

Now, we have
\[
\hat{q}_{st} = \partial_t \left( \frac{1}{2} (\hat{\mu} + \hat{\nu}) \right) = \frac{1}{2} ((\hat{\mu}_q + \hat{\nu}_q) \cdot \frac{1}{2} (\hat{\mu} + \hat{\nu}) + \varepsilon t^{-1} \hat{\mu}_s + \varepsilon t^{-1} \hat{\nu}_s)
\]
\[
= \frac{1}{4} \hat{\mu}_q \hat{\mu} + \frac{1}{4} \hat{\nu}_q \hat{\mu} + \frac{\varepsilon}{2t} \hat{\mu}_s \mod \varepsilon S^{-2,0} = \frac{1}{4} \hat{\mu}_q \hat{\mu} \mod \varepsilon S^{-1,-1},
\]
\[
\begin{align*}
\hat{q}_{ti} &= \partial_t (\frac{1}{2}(\hat{\mu} + \hat{\nu})) = \frac{1}{2}((\hat{\mu}_q + \hat{\nu}_q) \cdot (\frac{1}{2}(\hat{\nu} - \hat{\mu})\omega_i + \hat{\lambda}_i) + \sum_j (\partial_i \omega_j) \hat{\mu}_j + \sum_j (\partial_j \omega_j) \hat{\nu}_j) \\
&= -\frac{1}{4} \hat{\mu}_q \omega_i \mod S^{-1,-1},
\end{align*}
\]
\[
\begin{align*}
\hat{q}_{ij} &= \partial_t (\frac{1}{2}(\hat{\nu} - \hat{\mu})\omega_j + \hat{\lambda}_j) \\
&= \frac{1}{2} (\hat{\nu}_q - \hat{\mu}_q) (\frac{1}{2}(\hat{\nu} - \hat{\mu})\omega_i + \hat{\lambda}_i) \omega_j + \frac{1}{2} \sum_k (\hat{\nu}_k - \hat{\mu}_k)(\partial_i \omega_k) \omega_j + \frac{1}{2}(\hat{\nu} - \hat{\mu})\partial_i \omega_j + \partial_i \hat{\lambda}_j \\
&= \frac{1}{4} (\hat{\mu}_q - \hat{\nu}_q \hat{\nu} - \hat{\nu}_q \hat{\mu}) \omega_i \omega_j - \frac{1}{2} \sum_k \hat{\nu}_k (\partial_i \omega_k) \omega_j - \frac{1}{2} \sum_k \hat{\mu}_k (\partial_i \omega_k) \omega_j - \frac{1}{2} \hat{\mu} \partial_i \omega_j + \partial_i \hat{\lambda}_j \mod \epsilon S^{-2,0} \\
&= \frac{1}{4} \hat{\mu}_q \omega_i \omega_j \mod S^{-1.0}.
\end{align*}
\]

In the last estimate, we note that \(\partial_i \hat{\lambda}_j \in S^{-1,0}\) since for each \(I,\)
\[
\begin{align*}
|Z^I \partial_i \hat{\lambda}_j| &\lesssim \sum_{|J| \leq |I|} |\partial Z^J \hat{\lambda}_j| \lesssim (r-t)^{-1} \sum_{|J| \leq |I|+1} |Z^J \hat{\lambda}_j| \\
&\lesssim (r-t)^{-1} \cdot t^{-1+C\epsilon} (r-t)^\gamma \lesssim t^{-1+C\epsilon} (r-t)^{1-\gamma}.
\end{align*}
\]

Thus, we have \(\partial^2 \hat{q} \in S^{0,-2} + S^{-1,-1} = S^{0,-2}\) and
\[
g_{0}^{\alpha \beta} q_{\alpha \beta} = \frac{1}{4} G(\omega) \hat{\mu}_q \hat{\mu} \mod S^{-1.0}.
\]

In addition,
\[
\begin{align*}
\square \hat{q} &= -\frac{1}{4} \hat{\mu}_q \hat{\mu} + \frac{1}{4} \hat{\mu}_q \hat{\nu} + \frac{1}{4} \hat{\nu}_q \hat{\mu} + \frac{\epsilon}{2t} \hat{\mu}_s + \frac{1}{4} (\hat{\mu}_q - \hat{\nu}_q \hat{\nu} - \hat{\nu}_q \hat{\mu}) - r^{-1} \hat{\mu} + \sum_i \partial_i \hat{\lambda}_i \mod \epsilon S^{-2.0} \\
&= -\frac{1}{2} \hat{\nu}_q \hat{\mu} + \frac{\epsilon}{2t} \hat{\mu}_s - r^{-1} \hat{\mu} + \sum_i \partial_i \hat{\lambda}_i \mod \epsilon S^{-2.0}.
\end{align*}
\]

Since \(\sum_i \omega_i \hat{\lambda}_i = 0,\) we have \(0 = \partial_t (\sum_i \omega_i \hat{\lambda}_i) = \sum_i \omega_i \partial_t \hat{\lambda}_i.\) And since \(\hat{\lambda}_i \in S^{-1,\gamma},\) we have
\[
\sum_i \partial_t \hat{\lambda}_i = \sum_i (\partial_i - \omega_i \partial_r) \hat{\lambda}_i = \sum_{i,j} r^{-1} \omega_i \Omega_{ij} \hat{\lambda}_i \in S^{-2,\gamma}
\]

Finally, we have
\[
g^{\alpha \beta} (\hat{u}) \partial_{\alpha} \partial_{\beta} \hat{q} = \square \hat{q} + g_{0}^{\alpha \beta} \hat{u} \partial_{\alpha} \partial_{\beta} \hat{q} + (g^{\alpha \beta} (\hat{u}) - g_{0}^{\alpha \beta} \hat{u} - m^{\alpha \beta}) \partial_{\alpha} \partial_{\beta} \hat{q}
\]
\[
= -\frac{1}{2} \hat{\mu}_q \hat{\nu} + \frac{\epsilon}{2t} \hat{\mu}_s - r^{-1} \hat{\mu} + \frac{\epsilon}{4r} G(\omega) \hat{\mu}_q \hat{U} \mod S^{-2,\gamma} \\
= -\frac{1}{2} \hat{\mu} \cdot \frac{\epsilon}{4t} G(\hat{\mu}_q \hat{U}) - \frac{1}{2} \hat{\mu} \cdot \frac{\epsilon}{4t} G(\hat{\mu}_q \hat{U} - 2\hat{A}) + \frac{\epsilon}{4t} G \hat{A} \hat{\mu} - r^{-1} \hat{\mu} \\
+ \frac{\epsilon}{4t} G \hat{\mu}_q \hat{U} + \frac{\epsilon(t-r)}{4tr} G \hat{\mu}_q \hat{U} \mod S^{-2,\gamma} \\
= -r^{-1} \hat{\mu} + \frac{\epsilon}{2t} G \hat{A} \hat{\mu} \mod S^{-2,\gamma}.
\]
Now we claim that \( \hat{u} = \varepsilon r^{-1} \hat{U}(\varepsilon \ln t - \delta, \hat{q}(t, x), \omega) \) is an approximate solution to (1.1).

**Proposition 7.11.** We have

\[
g^{\alpha \beta}(\hat{u}) \partial_\alpha \partial_\beta \hat{u} \in \varepsilon S^{-3,0}.
\]

**Proof.** We have

\[
\hat{u}_t = \varepsilon r^{-1}(\varepsilon t^{-1} \hat{U}_s + \hat{q}_t \hat{U}_q), \quad \hat{u}_i = -\varepsilon r^{-2} \omega_i \hat{U} + \varepsilon r^{-1}(\hat{U}_q \hat{q}_i + \sum_k \hat{U}_{\omega_k} \omega_i \omega_k).
\]

By Lemma 7.7 we have \( \partial^k \omega \hat{U} \in \varepsilon^{-b} S^{0,0} \). Thus we have

\[
\hat{u}_{tt} = \varepsilon r^{-1}(-\varepsilon t^{-2} \hat{U}_s + \varepsilon^2 t^{-2} \hat{U}_{ss} + 2\varepsilon t^{-1} \hat{q}_t \hat{U}_{sq} + \hat{q}_{tt} \hat{U}_q + \hat{q}_t^2 \hat{U}_{qq})
\]

\[
= \varepsilon r^{-1}(2\varepsilon t^{-1} \hat{q}_t \hat{U}_{sq} + \hat{q}_{tt} \hat{U}_q + \hat{q}_t^2 \hat{U}_{qq}) \mod \varepsilon S^{-3,0}
\]

\[
= \varepsilon r^{-1}(\hat{q}_{tt} \hat{U}_q + \hat{q}_t^2 \hat{U}_{qq}) \mod \varepsilon S^{-2,1},
\]

\[
\hat{u}_{ti} = -\varepsilon r^{-2} \omega_i (\varepsilon t^{-1} \hat{U}_s + \hat{q}_t \hat{U}_q)
\]

\[
+ \varepsilon r^{-1}(\varepsilon t^{-1} \hat{U}_{sq} \hat{q}_i + \varepsilon t^{-1} \sum_k \hat{U}_{\omega_k} \omega_i \omega_k + \hat{q}_{ti} \hat{U}_q + \hat{q}_t \hat{U}_{qq} \hat{q}_i + \hat{q}_t \sum_k \hat{U}_{\omega_k} \omega_i \omega_k)
\]

\[
= \varepsilon r^{-1}(\hat{q}_{ti} \hat{U}_q + \hat{q}_t \hat{U}_{qq} \hat{q}_i) \mod \varepsilon S^{-2,1},
\]

\[
\hat{u}_{ij} = -\varepsilon \partial_i(r^{-2} \omega_j) \hat{U} - \varepsilon r^{-2} \omega_j (\hat{U}_q \hat{q}_i + \sum_k \hat{U}_{\omega_k} \omega_i \omega_k) - \varepsilon r^{-2} \omega_i (\hat{U}_q \hat{q}_j + \sum_k \hat{U}_{\omega_k} \omega_j \omega_k)
\]

\[
+ \varepsilon r^{-1}[\hat{U}_{qq} \hat{q}_i \hat{q}_j + \sum_k \hat{U}_{\omega_k} (\partial_j \omega_k) \hat{q}_i + \hat{U}_q \hat{q}_{ij}]
\]

\[
+ \sum_k (\hat{U}_{\omega_k} \hat{q}_i \partial_j \omega_k + \hat{U}_{\omega_k} \omega_i \partial_j \omega_k) + \sum_{k,k'} \hat{U}_{\omega_k \omega_{k'}} (\partial_i \omega_k)(\partial_j \omega_{k'})
\]

\[
= -\varepsilon r^{-2} \omega_j \hat{U}_q \hat{q}_i - \varepsilon r^{-2} \omega_i \hat{U}_q \hat{q}_j
\]

\[
+ \varepsilon r^{-1}[\hat{U}_{qq} \hat{q}_i \hat{q}_j + \sum_k \hat{U}_{\omega_k} ((\partial_i \omega_k) \hat{q}_j + (\partial_j \omega_k) \hat{q}_i) + \hat{U}_q \hat{q}_{ij}] \mod \varepsilon S^{-3,0}
\]

\[
= \varepsilon r^{-1}(\hat{U}_{qq} \hat{q}_i \hat{q}_j + \hat{U}_q \hat{q}_{ij}) \mod \varepsilon S^{-2,1}.
\]
Since \( g^{\alpha \beta}(\hat{u}) - m^{\alpha \beta} = g_0^{\alpha \beta} \hat{u} \mod \varepsilon^2 S^{-2,0} \in \varepsilon S^{-1,0} \), we have

\[
\begin{align*}
g^{\alpha \beta}(\hat{u}) \partial_\alpha \partial_\beta \hat{u} &= \Box \hat{u} + (g^{\alpha \beta}(\hat{u}) - m^{\alpha \beta}) \partial_\alpha \partial_\beta \hat{u} \\
&= -\varepsilon r^{-1} (2 \varepsilon t^{-1} \hat{q}_t \hat{U}_{sq} + \hat{q}_t \hat{U}_q + \hat{q}_t^2 \hat{U}_{qq}) - 2 \varepsilon r^{-2} \hat{U}_q \hat{q}_r \\
&+ \varepsilon r^{-1} \sum_i [\hat{U}_{qq} \hat{q}_i^2 + \sum_k 2 \hat{U}_{q\omega_k} (\partial_i \omega_k) \hat{q}_i + \hat{q}_i \hat{q}_{ii}] \\
&+ (g^{\alpha \beta}(\hat{u}) - m^{\alpha \beta}) \cdot \varepsilon r^{-1} (\hat{q}_\alpha \hat{q}_\beta \hat{U}_q + \hat{q}_\alpha \hat{q}_\beta \hat{U}_{qq}) \mod \varepsilon S^{-3,0} \\
&= -\varepsilon^2 (tr)^{-1} \hat{q}_t G \hat{A} \hat{U}_q - 2 \varepsilon r^{-2} \hat{U}_q \hat{q}_r + \varepsilon r^{-1} \sum_i \sum_k 2 \hat{U}_{q\omega_k} (\partial_i \omega_k)(\hat{\lambda}_i + \omega_i \hat{q}_r) \\
&+ \varepsilon r^{-1} (g^{\alpha \beta}(\hat{u}) \hat{q}_\alpha \hat{q}_\beta \hat{U}_q + g^{\alpha \beta}(\hat{u}) \hat{q}_\alpha \hat{q}_\beta \hat{U}_{qq}) \mod \varepsilon S^{-3,0} \\
&= -\varepsilon^2 (rt)^{-1} \hat{q}_t G \hat{A} \hat{U}_q - 2 \varepsilon r^{-2} \hat{U}_q \hat{q}_r - \varepsilon r^{-2} \hat{\mu} \hat{U}_q + \varepsilon^2 (2tr)^{-1} G \hat{A} \hat{\mu} \hat{U}_q \mod \varepsilon S^{-3,0} \\
&= -\frac{1}{2} \varepsilon^2 r^{-2} \hat{\nu} G \hat{A} \hat{U}_q - \varepsilon r^{-2} \hat{\nu} \hat{U}_q \mod \varepsilon S^{-3,0} \in \varepsilon S^{-3,0}.
\end{align*}
\]

In the third equality, we note that

\[
\varepsilon r^{-1} [g^{\alpha \beta}(\hat{u}) \hat{q}_\alpha \hat{q}_\beta + r^{-1} \hat{\mu} - \frac{\varepsilon}{2t} G \hat{A} \hat{\mu}] \hat{U}_q \in \varepsilon S^{-1,0} \cdot S^{-2,\gamma} \cdot S^{0,-1} \subset \varepsilon S^{-3,0},
\]

\[
\varepsilon r^{-1} g^{\alpha \beta}(\hat{u}) \hat{q}_\alpha \hat{q}_\beta \hat{U}_{qq} \in \varepsilon S^{-1,0} \cdot S^{-2,1} \cdot S^{0,-2} \subset \varepsilon S^{-3,0}
\]

and that

\[
\varepsilon r^{-1} \sum_i \sum_k 2 \hat{U}_{q\omega_k} (\partial_i \omega_k)(\hat{\lambda}_i + \omega_i \hat{q}_r) = \varepsilon r^{-1} \sum_i \sum_k 2 \hat{U}_{q\omega_k} (\partial_i \omega_k) \hat{\lambda}_i + \varepsilon r^{-1} \sum_k 2 \hat{U}_{q\omega_k} (\partial_r \omega_k) \hat{q}_r
\]

\[
\in \varepsilon S^{-1,0} \cdot S^{0,-1} \cdot S^{-1,0} + 0 \subset \varepsilon S^{-3,0}.
\]

\[
\Box
\]

7.4. Approximation of the optical function. We set \( p(t, x) := F(q(t, x), \omega) - \hat{q}(t, x) \) in \( \Omega \), where \( q(t, x) \) is the optical function constructed in Section \( \text{Section 3} \).

**Proposition 7.12.** Fix a constant \( \gamma \in (0, 1) \). Then, for \( \varepsilon \ll \gamma 1 \), at each \( (t, x) \in \Omega \) such that \( |r - t| \lesssim t^\gamma \), we have \( |p(t, x)| \lesssim t^{-1+\gamma} (r - t) \).

**Proof.** It is clear that \( p \equiv 0 \) in the region \( \{ r - t > R \} \). In \( \Omega \cap \{ r - t < 2R \} \), by setting \( s = \varepsilon \ln t - \delta \) we have

\[
p_t - p_r = F_q \mu(s, q(t, x), \omega) - \hat{\mu}(s, \hat{q}(t, x), \omega)
\]

\[
= [F_q \mu(s, q(t, x), \omega) - \hat{\mu}(s, F(q(t, x), \omega), \omega)] + [\hat{\mu}(s, F(q(t, x), \omega), \omega) - \hat{\mu}(s, \hat{q}(t, x), \omega)]
\]

\[
=: \mathcal{R}_1 + \mathcal{R}_2.
\]
Since $\hat{A}(F(q, \omega), \omega) = A(q, \omega)$, we have

$$
\mathcal{R}_1 = -\frac{2}{A_1(q(t, x), \omega)} \tilde{V}_1(s, q(t, x), \omega) \exp(-\frac{1}{2} G(\omega) A(q(t, x), \omega) s) \\
+ 2 \exp(-\frac{1}{2} G(\omega) \hat{A}(F(q(t, x), \omega), \omega) s) \\
= (-\frac{2}{A_1(q(t, x), \omega)} \tilde{V}_1(s, q(t, x), \omega) + 2) \exp(-\frac{1}{2} G(\omega) A(q(t, x), \omega) s) \\
= -\frac{2}{A_1(q(t, x), \omega)} (\tilde{V}_1(s, q(t, x), \omega) - A_1(q(t, x), \omega)) \exp(-\frac{1}{2} G(\omega) A(q(t, x), \omega) s).
$$

(7.9)

By Proposition 5.7, we have

$$
|\mathcal{R}_1| \lesssim |\tilde{V}_1(s, q(t, x), \omega) - A_1(q(t, x), \omega)| \exp(C(q)_{-1+C} s) \lesssim t^{-1+C}. 
$$

Moreover,

$$
|\mathcal{R}_2| = \left| \int_{\tilde{q}}^{F(q, \omega)} \tilde{\mu}_p(s, \rho, \omega) \, dp \right| \lesssim \left| \int_{\tilde{q}}^{F(q, \omega)} \langle p \rangle^{-2+C} s |\tilde{\mu}(s, \rho, \omega)| \, dp \right| \\
\lesssim (\varepsilon \ln t - \delta) |p| \cdot \max_{\kappa \in [0, 1]} [\langle \tilde{q} + \kappa p \rangle^{-2+C} \exp(-\frac{1}{2} G(\omega) \hat{A}(\tilde{q} + \kappa p, \omega) s)].
$$

We now use a continuity argument to end the proof. Fix $(t, x) \in \Omega \cap \{r-t < 2R, |r-t| \lesssim t^\gamma\}$. Suppose that for some $t_0 \in [(r+t)/2 - R, t)$, we have

$$
|p(\tau, r-t-\tau, \omega)| \leq \frac{\delta}{10 \varepsilon \ln \tau}, \quad \forall \tau \in [(r+t)/2 - R, t_0].
$$

(7.10)

Note that (7.10) holds for $t_0 = (r+t)/2 - R$, since $p((r+t)/2 - R, (r+t)/2 + R, \omega) = 0$. At $(\tau, r-t-\tau, \omega)$ for $(r+t)/2 - R \leq \tau \leq t_0$ and for each $\kappa \in [0, 1]$, we have

$$
\langle \tilde{q} + \kappa p \rangle \sim 1 + |\tilde{q} + \kappa p| \geq 1 + |\tilde{q}| - |\kappa p| \geq 1 + |\tilde{q}| - \frac{1}{10} \gtrsim \langle \tilde{q} \rangle.
$$

In the second last inequality we note that $\tau \gtrsim \exp(\delta/\varepsilon)$, so $\varepsilon \ln \tau > \delta$ and thus $|p| \leq 1/10$. Moreover,

$$
\exp(-\frac{1}{2} G(\omega)(\hat{A}(\tilde{q} + \kappa p, \omega) - \hat{A}(\tilde{q}, \omega)) s) \lesssim \exp(C\kappa |p| s) \lesssim \exp(\delta/10) \lesssim 1.
$$

In conclusion, at $(\tau, r-t-\tau, \omega)$ for $(r+t)/2 - R \leq \tau \leq t_0$, we have

$$
|\mathcal{R}_2| \lesssim (\varepsilon \ln \tau - \delta) [M(p)(\tilde{q})^{-2+C} \exp(-\frac{1}{2} G(\omega) \hat{A}(\tilde{q}, \omega) s)](\tau, r-t-\tau, \omega) \\
\lesssim (\varepsilon \ln \tau - \delta) [M(p)(\tilde{q})^{-2+C} (\tilde{\mu})](\tau, r-t-\tau, \omega).
$$

If we fix any $t_1 \in [(r+t)/2 - R, t_0]$, then

$$
\int_{(r+t)/2 - R}^{t_1} (\varepsilon \ln \tau - \delta) (\tilde{q})^{-2+C} (\tilde{\mu})(\tau, r-t-\tau, \omega) \, d\tau \lesssim \varepsilon \ln t_1 \int_{(r+t)/2 - R}^{t_1} (\tilde{z})^{-2+C} (\tilde{\mu}) \, d\tau \\
\lesssim \varepsilon \ln t_1
$$
and
\[
\int_{(r+t)/2-R}^{t_1} |\mathcal{R}_1|((\tau, r + t - \tau, \omega)) \, d\tau \lesssim \int_{(r+t)/2-R}^{t_1} \tau^{-1+\xi} \, d\tau \\
\lesssim ((r + t)/2 - R)^{-1+\xi}(t_1 - (r + t)/2 + R) \\
\lesssim t_1^{-1+\xi}(r - t).
\]

Here we recall that \([(r + t)/2 - R] \sim t \sim t_1\). And since \(p = 0\) at \(\tau = (r + t)/2 - R\), by applying the Gronwall’s inequality to \(p_t - p_r = \mathcal{R}_1 + \mathcal{R}_2\), we conclude that
\[
|p(t_1, r + t - t_1, \omega)| \lesssim t_1^{-1+\xi}(r - t) \cdot \exp(\xi \ln(Ct_1)) \lesssim t_1^{-1+\xi}(r - t),
\]
\forall t_1 \in [(r + t)/2 - R, t_0].

For \(\epsilon \ll \gamma 1\) (where \(\epsilon\) does not depend on \((t, x)\)) and \(t_1 \in [(r + t)/2 - R, t_0]\), we have \(|r - t| \lesssim t^\gamma \sim t_1^\gamma\) and thus
\[
t_1^{-1+\xi}(r - t) \lesssim t_1^{-1+\gamma + \xi} \leq t_1^{(\gamma - 1)/2} \leq \delta/(20 \epsilon \ln t_1).
\]

And since \(\tau \mapsto \epsilon(\ln \tau)p(\tau, r + t - \tau, \omega)\) is a continuous function, (7.10) holds with \(t_0\) replaced by some \(t_0' > t_0\). By the continuity argument we conclude that \(|p(t, x)| \lesssim t^{-1+\xi}(r - t)\). The constants here do not depend on \((t, x)\).

Next we consider \(Z^I p\). We need the following lemma.

**Lemma 7.13.** Let \(\mathcal{R}_1\) and \(\mathcal{R}_2\) be defined as in (7.8). Then, we have \(\mathcal{R}_1 \in S^{-1,0}\) and for \(|I| > 0\) we have
\[
|Z^I \mathcal{R}_2| \lesssim (r - t)^{-2t^\xi} \sum_{|J| < |I|} |Z^J p| + |\hat{\mu} Z^I p|.
\]

**Proof.** By (7.9), Remark 5.7.1 and Lemma 5.6 and since \(A_1 < -1\) everywhere, we have \(\mathcal{R}_1 = \mathcal{R}_{0,0} \cdot \mathcal{R}_{-1,0} \cdot \mathcal{R}_{0,0} = \mathcal{R}_{-1,0} \in S^{-1,0}\).

To estimate \(\mathcal{R}_2\), we fix an arbitrary multiindex \(I\) with \(|I| > 0\). By the chain rule and Leibniz’s rule, we can express \(Z^I \hat{\mu}(s, F(q(t, x), \omega), \omega) - Z^I \hat{\mu}(s, \hat{q}(t, x), \omega)\) as a linear combination of terms of the form
\[
[(\partial^a_q \partial^b_{\omega} \partial^c_s \hat{\mu})(s, F(q, \omega), \omega) \cdot \prod_{i=1}^a Z^I_i (F(q, \omega)) - (\partial^a_q \partial^b_{\omega} \partial^c_s \hat{\mu})(s, \hat{q}, \omega) \cdot \prod_{i=1}^a Z^I_i \hat{q}]
\]
\[
\cdot \prod_{j=1}^b Z^J_j (\epsilon \ln t - \delta) \cdot \prod_{l=1}^c Z^{K_l} \omega_l \cdot \cdots \cdot Z^{K_{l_1}} \omega_l
\]

where \(|I_*|, |J_*|, |K_{s,*}| \) are nonzero, and the sum of all these multiindices is \(|I|\). The only term with \(|I_j| = |I|\) for some \(j\) is \(\hat{\mu} Z^I p\), so from now on we assume \(|I_j| < |I|\) for each \(j\) in (7.12).

Here the second row in (7.12) is \(O(\epsilon^9)\). The first row is equal to the sum of
\[
[(\partial^a_q \partial^b_{\omega} \partial^c_s \hat{\mu})(s, F(q, \omega), \omega) - (\partial^a_q \partial^b_{\omega} \partial^c_s \hat{\mu})(s, \hat{q}, \omega) \cdot \prod_{i=1}^a Z^I_i (F(q, \omega))
\]
\[
\cdot \prod_{j=1}^b Z^J_j (\epsilon \ln t - \delta) \cdot \prod_{l=1}^c Z^{K_l} \omega_l \cdot \cdots \cdot Z^{K_{l_1}} \omega_l
\]

\[
= O(\epsilon^9).
\]
and for each \( j = 1, 2, \ldots, a \)

\[
(7.14) \quad (\partial_s^a \partial_q^b \partial_s^c \mu)(s, q, \omega) \cdot \prod_{i=1}^{j-1} Z^i(F(q, \omega)) \cdot Z^j p \cdot \prod_{i=j+1}^a Z^i \hat{q}.
\]

Since \(|I| > 0\), we must have \( a > 0 \) if \((7.14)\) does appear.

To control \((7.13)\) and \((7.14)\), we first recall from Lemma 7.7 and Proposition 7.3 that

\[
Z^I(\hat{q}(t, x), F(q(t, x), \omega)) = O((r - t)t^{C \varepsilon});
\]

\[
(\partial_s^a \partial_q^b \partial_s^c \mu)(s, q, \omega) = O((q)^{-a+1+C \varepsilon}) = O((r - t)^{-a-1+C \varepsilon}), \quad \text{when } a + b + |c| > 0.
\]

It follows immediately that \((7.14)\) is \( O(\sum_{|J| < |I|} t^{C \varepsilon} (r - t)^{-2} |Z^J p|) \). In addition, we have \( \langle F(q, \omega) \rangle / (r - t) \sim \langle q \rangle / (r - t) = t^{O(\varepsilon)} \) and \( \langle \hat{q} \rangle / (r - t) = t^{O(\varepsilon)} \). Thus, for each \( \tau \in [0, 1] \),

\[
(7.15) \quad \langle \tau \hat{q} + (1 - \tau) F(q, \omega) \rangle \sim \tau \langle q \rangle + (1 - \tau) \langle F(q, \omega) \rangle \gtrsim \langle r - t \rangle t^{-C \varepsilon}.
\]

Then, we have

\[
| (\partial_s^a \partial_q^b \partial_s^c \mu)(s, F(q, \omega), \omega) - (\partial_s^a \partial_q^b \partial_s^c \mu)(s, q, \omega) | = \left| \int_q^{F(q, \omega)} (\partial_s^a \partial_q^b \partial_s^c \mu)(s, p, \omega) \, dp \right|
\]

\[
\lesssim \left| \int_q^{F(q, \omega)} (p)^{-2-a+C \varepsilon} \exp(Cs) \, dp \right| \lesssim |p(t, x)| t^{C \varepsilon} (r - t)^{-a-2}.
\]

Thus, \((7.13)\) is \( O(|p| t^{C \varepsilon} (r - t)^{-2}) \).

In conclusion, for \(|I| > 0\) we have

\[
|Z^I R_2| \lesssim \langle r - t \rangle^{-2+C \varepsilon} \sum_{|J| < |I|} |Z^J p| + |\hat{\mu} Z^I p|.
\]

\[\square\]

**Proposition 7.14.** Fix a constant \( \gamma \in (0, 1/2) \) and a large integer \( N \). Then, for \( \varepsilon \ll \gamma, N, 1 \), at each \((t, x)\) \( \in \Omega \) such that \(|r - t| \lesssim t^{\gamma} \), we have \(|Z^I p(t, x)| \lesssim \gamma^{-1+C \varepsilon} (r - t) \) for each \(|I| \leq N\).

**Proof.** We prove by induction on \(|I|\). The case \(|I| = 0\) has been proved in Proposition 7.12.

Fix a multiindex \( I \) with \(|I| > 0\), and suppose that we have proved the proposition for all \(|J| < |I|\). By Lemma 7.11, we have

\[
(\partial_t - \partial_r) Z^I p = Z^I (p_t - p_r) + \sum_{|J| < |I|} f_0 Z^J (p_t - p_r) + \sum_i f_0 (\partial_i + \omega_i \partial_r) Z^J p.
\]

By Lemma 7.13 and our induction hypotheses, in \( \Omega \cap \{ r - t < 2R, \ |r - t| \lesssim t^{\gamma} \} \) we have

\[
| (\partial_t - \partial_r) Z^I p | \lesssim |Z^I (R_1 + R_2)| + \sum_{|J| < |I|} |Z^J (R_1 + R_2)| + t^{-1} |ZZ^J p|
\]

\[
\lesssim t^{-1+C \varepsilon} + \langle r - t \rangle^{-2} t^{C \varepsilon} \sum_{|J| < |I|} |Z^J p| + |\hat{\mu} Z^I p| + \sum_{|J| < |I|} t^{-1} |Z^J p|
\]

\[
\lesssim t^{-1+C \varepsilon} + \langle r - t \rangle^{-2} t^{-1+C \varepsilon} (r - t) + |\hat{\mu} Z^I p| + \sum_{|J| = |I|} t^{-1} |Z^J p| + t^{-2+C \varepsilon} (r - t)
\]

\[
\lesssim t^{-1+C \varepsilon} + |\hat{\mu} Z^I p| + \sum_{|J| = |I|} t^{-1} |Z^J p|.
\]
The integral of $|\mu_q|$ and $t^{-1}$ along a characteristic $(\tau, r + t - \tau, \omega)$, $\tau \in [(r + t)/2 - R, t]$, is $O(\varepsilon \ln t + 1)$. Moreover,

$$\int_{(r + t)/2 - R}^{t} \tau^{-1 + C\varepsilon} d\tau \lesssim ((r + t)/2 - R)^{-1 + C\varepsilon}((t - r)/2 + R) \lesssim t^{-1 + C\varepsilon}(r - t).$$

Since $Z^I p \equiv 0$ in the region $\Omega \cap \{r - t > R\}$, by Gronwall’s inequality we conclude that $|Z^I p| \lesssim t^{-1 + C\varepsilon}(r - t)$. \hfill \Box

### 7.5. Approximation of the solution to $(1.1)$. We can now discuss the difference $u - \hat{u}$ where $u$ is a solution to $(1.1)$ and $\hat{u}$ is defined in Section 7.2. Again, we fix a point in region $\Omega \cap \{|r - t| \lesssim \tau\}$ for some $0 < \gamma < 1$. Note that

$$u - \hat{u} = \varepsilon r^{-1} U(s, q(t, x), \omega) - \varepsilon r^{-1} \hat{U}(s, \hat{q}(t, x), \omega)$$

$$= \varepsilon r^{-1} U(s, q(t, x), \omega) - \varepsilon r^{-1} \hat{U}(s, F(q(t, x), \omega), \omega)$$

$$+ \varepsilon r^{-1} \hat{U}(s, F(q(t, x), \omega), \omega) - \varepsilon r^{-1} \hat{U}(s, \hat{q}(t, x), \omega)$$

$$=: R_3 + R_4.$$

Now we estimate $R_3$ and $R_4$ separately.

#### Lemma 7.15. Fix a constant $0 < \gamma < 1$ and a large integer $N$. Then, for $\varepsilon \ll \gamma, N$ 1, at each $(t, x) \in \Omega$ such that $|r - t| \lesssim \tau$, we have $|Z^I R_3| \lesssim \varepsilon t^{-2 + C\varepsilon}(r - t)$ for each $|I| \leq N$.

**Proof.** As computed in Lemma 7.2, by change of variables we can prove that

$$\hat{U}(s, F(q(t, x), \omega), \omega) = \tilde{U}(s, q(t, x), \omega).$$

Thus,

$$R_3 = \varepsilon r^{-1} (U(s, q(t, x), \omega) - \tilde{U}(s, q(t, x), \omega)).$$

By (5.12), we have $|U - \tilde{U}| \lesssim \langle q \rangle t^{-1 + C\varepsilon}$ at $(s, q, \omega) = (\varepsilon t - \delta, q(t, x), \omega)$, so

$$|R_3| \lesssim \varepsilon t^{-2 + C\varepsilon}(q) \lesssim \varepsilon t^{-2 + C\varepsilon}(r - t).$$

Next, we fix a multiindex $I$ with $|I| > 0$. Then, $Z^I R_3$ can be expressed as a linear combination of terms of the form

$$(7.16) \quad Z^I (\varepsilon r^{-1}) \cdot (\partial_s^a \partial_q^b \partial_\omega^c (U - \tilde{U}))(s, q, \omega) \cdot \prod_{i=1}^{a} Z^I_i q \cdot \prod_{i=1}^{b} Z^J_i s \cdot \prod_{i=1}^{c} Z^K_i \omega.$$

The sum of all the $|I|, |I_s|, |J_s|, |K_s|$ is $|I|$. If $a \geq 1$, by (5.11), we have

$$|\partial_s^a \partial_q^b \partial_\omega (U - \tilde{U})| \lesssim \varepsilon^{-b} \langle q \rangle^{-1-a} t^{-1+C\varepsilon}.$$

Thus, the terms (7.16) with $a > 0$ have an upper bound

$$\varepsilon t^{-1} \cdot \varepsilon^{-b} \langle q \rangle^{-1-a} t^{-1+C\varepsilon} \cdot \langle q \rangle^{C\varepsilon} \cdot \varepsilon^b \lesssim \varepsilon \langle q \rangle t^{-2+C\varepsilon} \lesssim \varepsilon \langle r - t \rangle^{-2+C\varepsilon}.$$

Moreover, by (5.12), we have

$$|\partial_s^a \partial_q^b \partial_\omega (U - \tilde{U})| \lesssim \varepsilon^{-b} \langle q \rangle t^{-1+C\varepsilon}.$$

Thus, the terms (7.16) with $a = 0$ have an upper bound

$$\varepsilon t^{-1} \cdot \varepsilon^{-b} \langle q \rangle t^{-1+C\varepsilon} \cdot \varepsilon^b \lesssim \varepsilon \langle q \rangle t^{-2+C\varepsilon} \lesssim \varepsilon \langle r - t \rangle^{-2+C\varepsilon}.$$

In conclusion, $|Z^I R_3| \lesssim \varepsilon t^{-2+C\varepsilon}(r - t)$ for $|I| > 0$. \hfill \Box
Lemma 7.16. Fix a constant $0 < \gamma < 1$ and a large integer $N$. Then, for $\varepsilon \ll_{\gamma,N} 1$, at each $(t, x) \in \Omega$ such that $|r - t| \lesssim t^\gamma$, we have $|Z^t R_4| \lesssim \varepsilon t^{-2+\varepsilon} (r-t)$ for each $|I| \leq N$.

Proof. First we consider the case $|I| = 0$. We have

$$|R_4| \lesssim \varepsilon t^{-1} |\tilde U(s, F(q, t, x), \omega) - \varepsilon t^{-1} \tilde U(s, \hat q(t, x), \omega)|$$

$$\lesssim \varepsilon t^{-1} \int_\hat q^F(q, \omega) |\tilde U_\rho(s, \rho, \omega)| \, d\rho \lesssim \varepsilon t^{-1} \int_\hat q^F(q, \omega) \left( |\partial \rho A_2| + |A_2| |\partial \rho A| \right) t^{-\varepsilon} \, d\rho$$

$$\lesssim \varepsilon (r-t)^{-2} t^{-1+\varepsilon} |p(t, x)| \lesssim \varepsilon t^{-2+\varepsilon} (r-t)^{-1}.$$ 

In the last inequality we apply Proposition 7.12. In the last inequality we apply Proposition 7.12.

In general, fix a multiindex $I$ with $|I| > 0$. Then, we can express $Z^t R_4$ as a linear combination of terms of the form

$$[\partial^p \partial^q \partial^r \hat U](s, F(q, \omega), \rho) \cdot \prod_{i=1}^a Z_i^I(F(q, \omega)) - (\partial^p \partial^q \partial^r \hat U)(s, \hat q, \omega) \cdot \prod_{i=1}^a Z_i^I\hat q]$$

$$\cdot Z^I(\varepsilon r^{-1}) \cdot \prod_{j=1}^b Z^J_j(\varepsilon \ln t - \delta) \cdot \prod_{l=1}^c Z^{K_l}\omega$$

where the sum of all these multiindices is $|I|$. The estimates for such terms are similar to those for 7.12. The second row is $O(\varepsilon^{b+1} t^{-1+\varepsilon})$ while the first row is equal to the sum of

$$[\partial^p \partial^q \partial^r \hat U](s, F(q, \omega), \rho) \cdot \prod_{i=1}^a Z^I_i(F(q, \omega))$$

and for each $j = 1, 2, \ldots, a$

$$\prod_{i=1}^{j-1} Z^I_i(F(q, \omega)) \cdot Z^I_j p \cdot \prod_{i=j+1}^a Z^I_i\hat q.$$ 

Since $|I| > 0$, we must have $a + b + |c| > 0$ if (7.19) appears.

Note that

$$Z^I_i(\hat q, F(q, \omega)) = O((r-t) t^{\varepsilon}), \quad Z^I_j p = O(t^{-1+\varepsilon} t^{\varepsilon})$$

$$\left(\partial^p \partial^q \partial^r \hat U\right)(s, \hat q, \omega) = O(\varepsilon^{-b}(r-t)^{1-a+\varepsilon} t^{\varepsilon}), \quad \left(\partial^p \partial^q \partial^r \hat U\right)(s, \hat q, \omega) = O(\varepsilon^{-b}(r-t)^{1-a+\varepsilon} t^{\varepsilon})$$

when $a + b + |c| > 0$.

So (7.19) has an upper bound

$$\varepsilon^{-b}(r-t)^{1-a+\varepsilon} t^{\varepsilon} \cdot (r-t)^{a-1} \cdot t^{-1+\varepsilon} (r-t) \lesssim \varepsilon t^{-1+\varepsilon} (r-t).$$

Besides, by applying Proposition 7.13 and 7.15, we have

$$\left| (\partial^p \partial^q \partial^r \hat U)(s, F(q, \omega), \rho) - (\partial^p \partial^q \partial^r \hat U)(s, \hat q, \omega) \right| \lesssim \int_{\hat q}^F(q, \omega) \left| \partial^p \partial^q \partial^r \hat U \right|(s, \rho, \omega) \, d\rho$$

$$\lesssim \int_{\hat q}^F(q, \omega) (\rho)^{-a-1+\varepsilon} t^{\varepsilon} \, d\rho \lesssim |p(t, x)| \cdot (r-t)^{-a-1+\varepsilon} t^{\varepsilon} \lesssim t^{-1+\varepsilon} \cdot (r-t)^{-a}.$$ 

In conclusion, (7.18) has an upper bound

$$t^{-1+\varepsilon} (r-t)^{-a} \cdot (r-t)^{a} \lesssim t^{-1+\varepsilon}.$$
Combine all the estimates above and we conclude that $|Z^I R_4| \lesssim \epsilon t^{-2+\gamma} (r - t)$. □

We thus conclude the following approximation result.

**Proposition 7.17.** Fix a constant $0 < \gamma < 1$ and a large integer $N$. Then, for $\epsilon \ll \gamma, N$, at each $(t, x) \in \Omega$ such that $|r - t| \lesssim t^\gamma \langle r - t \rangle$, we have $|Z^I (u - \tilde{u})| \lesssim \gamma \epsilon t^{-2+\gamma} (r - t)$ for each $|I| \leq N$.

**References**

[1] Serge Alinhac. An example of blowup at infinity for a quasilinear wave equation. *Astérisque*, 283:1–91, 2003.
[2] Serge Alinhac. *Geometric analysis of hyperbolic differential equations: an introduction*, volume 374. Cambridge University Press, 2010.
[3] Timothy Candy, Christopher Kauffman, and Hans Lindblad. Asymptotic behavior of the Maxwell-Klein-Gordon system. *Comm. Math. Phys.*, 367(2):683–716, 2019.
[4] Demetrios Christodoulou. Global solutions of nonlinear hyperbolic equations for small initial data. *Communications on Pure and Applied Mathematics*, 39(2):267–282, 1986.
[5] Demetrios Christodoulou and Sergiu Klainerman. *The global nonlinear stability of the Minkowski space*, volume 41 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1993.
[6] Mihalis Dafermos, Gustav Holzegel, and Igor Rodnianski. A scattering theory construction of dynamical vacuum black holes. *to appear in J. Diff. Geom.*, 2013.
[7] Yu Deng and Fabio Pusateri. On the global behavior of weak null quasilinear wave equations. *Communications on Pure and Applied Mathematics*, 73(5):1035–1099, 2020.
[8] Lars Hörmander. On the fully nonlinear Cauchy problem with small data. II. In *Microlocal analysis and nonlinear waves (Minneapolis, MN, 1988–1989)*, volume 30 of *IMA Vol. Math. Appl.*, pages 51–81. Springer, New York, 1991.
[9] Lars Hörmander. Lectures on nonlinear hyperbolic differential equations, volume 26. Springer Science & Business Media, 1997.
[10] Mihaela Ifrim and Daniel Tataru. Global bounds for the cubic non linear Schrödinger equation (NLS) in one space dimension. *Nonlinearity*, 28(8):2661, 2015.
[11] Fritz John. Blow-up for quasi-linear wave equations in three space dimensions. *Communications on Pure and Applied Mathematics*, 34(1):29–51, 1981.
[12] Fritz John. Blow-up of radial solutions of $u_{tt} = c^2(u_t) \Delta u$ in three space dimensions. *Matemática Aplicada e Computacional*, 4(1):3–18, 1985.
[13] Fritz John and Sergiu Klainerman. Almost global existence to nonlinear wave equations in three space dimensions. *Communications on Pure and Applied Mathematics*, 37(4):443–455, 1984.
[14] Joseph Keir. The weak null condition and global existence using the $p$-weighted energy method. *arXiv preprint arXiv:1806.01649*, 2018.
[15] Joseph Keir. Global existence for systems of nonlinear wave equations with bounded, stable asymptotic systems. *arXiv preprint arXiv:1906.01649*, 2019.
[16] Sergiu Klainerman. Long time behaviour of solutions to nonlinear wave equations. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Warsaw, 1983)*, pages 1209–1215. PWN, Warsaw, 1984.
[17] Sergiu Klainerman. Uniform decay estimates and the Lorentz invariance of the classical wave equation. *Communications on Pure and Applied Mathematics*, 38(3):321–332, 1985.
[18] Sergiu Klainerman. The null condition and global existence to nonlinear wave equations. In *Nonlinear systems of partial differential equations in applied mathematics, Part 1 (Santa Fe, N.M., 1984)*, volume 23 of *Lectures in Appl. Math.*, pages 293–326. Amer. Math. Soc., Providence, RI, 1986.
[19] John M Lee. *Riemannian manifolds: an introduction to curvature*, volume 176. Springer Science & Business Media, 2006.
[21] Hans Lindblad. On the lifespan of solutions of nonlinear wave equations with small initial data. *Communications on Pure and Applied Mathematics*, 43(4):445–472, 1990.

[22] Hans Lindblad. Global solutions of nonlinear wave equations. *Communications on Pure and Applied Mathematics*, 45(9):1063–1096, 1992.

[23] Hans Lindblad. Global solutions of quasilinear wave equations. *American Journal of Mathematics*, 130(1):115–157, 2008.

[24] Hans Lindblad. On the asymptotic behavior of solutions to the Einstein vacuum equations in wave coordinates. *Comm. Math. Phys.*, 353(1):135–184, 2017.

[25] Hans Lindblad and Igor Rodnianski. The weak null condition for Einstein’s equations. *Comptes Rendus - Mathématique*, 336(11):901–906, 2003.

[26] Hans Lindblad and Igor Rodnianski. Global existence for the Einstein vacuum equations in wave coordinates. *Communications in Mathematical Physics*, 256(1):43–110, 2005.

[27] Hans Lindblad and Volker Schlue. Scattering from infinity for semilinear wave equations satisfying the null condition or the weak null condition. *arXiv preprint arXiv:1711.00822*, 2017.

[28] Fabio Pusateri. Space-time resonances and the null condition for wave equations. *Boll. Unione Mat. Ital. (9)*, 6(3):513–529, 2013.

[29] Fabio Pusateri and Jalal Shatah. Space-time resonances and the null condition for first-order systems of wave equations. *Communications on Pure and Applied Mathematics*, 66(10):1495–1540, 2013.

[30] Hart F. Smith and Daniel Tataru. Sharp local well-posedness results for the nonlinear wave equation. *Ann. of Math. (2)*, 162(1):291–366, 2005.

[31] Christopher Donald Sogge. *Lectures on non-linear wave equations*, volume 2. International Press Boston, MA, 1995.

[32] Terence Tao. *Nonlinear dispersive equations: local and global analysis*. Number 106. American Mathematical Soc., 2006.

[33] Daniel Tataru. Nonlinear wave equations. In *Proceedings of the International Congress of Mathematicians, Vol. III (Beijing, 2002)*, pages 209–220. Higher Ed. Press, Beijing, 2002.

[34] Dongxiao Yu. Modified scattering for a scalar quasilinear wave equation satisfying the weak null condition. 2021. Dissertation (Ph.D.)–University of California, Berkeley.

[35] Dongxiao Yu. Modified wave operators for a scalar quasilinear wave equation satisfying the weak null condition. *Communications in Mathematical Physics*, 382(3):1961–2013, 2021.

Department of Mathematics, University of California at Berkeley

Email address: yudx@math.berkeley.edu