Non-perturbative improvement of operators with Wilson fermions*

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We outline two methods of constructing improved composite operators using Wilson fermions.

1. Overview

A major source of errors in present lattice calculations is the use of a finite lattice spacing, $a$. This is particularly true for the calculations of properties of hadrons containing $c$ and $b$ quarks, which contain discretization errors proportional to $am_c$ and $am_b$, respectively. Since lattice calculations have the potential to make significant contributions to the study of D and B physics, it is important to control such errors.

One approach to solving this problem is the Symanzik improvement program. This allows one to remove discretization errors order by order in $a$. This program has been pursued by the ALPHA collaboration, who have used chiral Ward Identities to determine, non-perturbatively, the on-shell improved action, the improved vector current including $am$ terms, and the improved axial current, scalar and pseudoscalar densities in the chiral limit \cite{1,2,3}. Ward Identities for non-degenerate quarks allow the determination of some of the $O(am)$ improvement terms, at least in the quenched approximation \cite{4}. These methods do not, however, allow one to determine the improved pseudoscalar density, axial current or tensor density away from the chiral limit.

To do so we have developed two methods which remove all errors of $O(a)$, including those of $O(am)$, from the matrix elements of bilinears. These involve (1) matching correlators of bilinears to their continuum form at short Euclidean distances, and (2) matching quark and gluon correlators to their continuum form at large Euclidean momenta. Both rely on the restoration of chiral symmetry at short distances, and both work for quenched and full QCD. The first method requires only on-shell improvement, i.e. improvement of physical quantities, while the second requires improvement of off-shell correlators at an intermediate stage.\footnote{Throughout we use “improved” to denote quantities in which all errors proportional to $a$ (multiplied by any power of $\ln a$) have been removed. Errors of $O(a^2)$ remain.}

Details of the first method are given in Ref. \cite{5}. Details of the second method will be forthcoming, along with results of a numerical pilot study \cite{6}. Here we provide a sketch focusing on some important issues.

2. Improvement Program

We need first to improve the action \cite{1}. In continuum notation, the Wilson action,

$$S_W = \int_x \left[ \frac{1}{2g_0^2} \text{Tr}(F_{\mu\nu}F_{\mu\nu}) + \bar{q}(D + m_0)q \right],$$

has $O(a)$ errors due to the derivative. Improvement requires the addition of all dimension 5 operators allowed by the symmetries,

$$\mathcal{L}_{d=5} = -\frac{1}{4}c_{SW}a q_\sigma \sigma_{\mu\nu} F_{\mu\nu} q + b_2 a m \frac{\text{Tr}(F_{\mu\nu}F_{\mu\nu})}{(2g_0^2)} - b_m a m^2 \bar{q}q,$$

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with appropriately chosen, \( g_0 \)-dependent, coefficients \([2]\). Here \( m = m_0 - m_c(g_0) \) is proportional to the physical mass, with \( m_c \) the critical mass.

The role of the various terms is as follows:
1. The \( c_{SW} \) term improves dimensionless physical quantities. Its determination is clearly essential, and has been carried out in Ref. \([2]\).
2. The coefficients \( b_g \) and \( b_m \) remove \( O(am) \) terms from the relation between bare and renormalized couplings and masses, respectively. For example, \( b_g \) introduces a mass-dependence in the effective gauge coupling, \( g_{eff}^2 = g_0^2(1 - b_g am) \), in such a way that the lattice spacing remains fixed as one varies \( m \) at fixed \( g_0 \). We will need to determine \( b_g \) in our first method \([3]\).
3. The coefficients \( c'_1 \) and \( c'_2 \) are only needed for off-shell improvement. One way to see this is to note that they can be removed by a change of quark variables (and an appropriate shift in \( g_0 \)). This affects external sources, but not the spectrum. In fact, one can ignore these terms also when doing off-shell improvement, because they can be absorbed by a suitable change in the improvement coefficients for quark fields \([3]\).

The next step is to improve the operators themselves. On-shell improvement has been discussed in Ref. \([3]\); off-shell improvement requires additional terms. For example, the improved form of the bare pseudoscalar density \( P = \bar{q}\gamma_5 q \) is (for degenerate quarks)

\[
\hat{P}(x) = Z_P(g_0^2, \mu a)(1 + b_P am) \left( P(x) + a c' P \left[ \frac{\gamma_5(\gamma \bar{D} + m_0) + (-\gamma \bar{D} + m_0) \gamma_5}{2} \right] q(x) \right).
\]

In addition to the two on-shell improvement coefficients, \( Z_P \) and \( b_P \), there is new coefficient, \( c' \). This multiplies an operator which vanishes by the equations of motion, and so does not contribute to on-shell matrix elements. The pattern is the same for other bilinears—each has a single additional off-shell term \([3]\).

### 3. Gauge-Invariant Method

This method involves only on-shell quantities, and so we do not need consider the off-shell coefficients such as \( c' \). We discuss the example of the pseudoscalar density—details for the other bilinears can be found in \([3]\).

To determine \( Z_P \) and \( b_P \), we require that the Euclidean two-point function of the improved lattice operator (evaluated using the on-shell improved action),

\[
\hat{G}_P(x) = \langle \hat{P}(x)\hat{P}(0) \rangle = Z_P^2(1 + b_P am)^2(P(x)P(0)),
\]

agrees with the continuum result up to \( O(a^2) \). The latter can be determined, at short distances, using the OPE

\[
g_P(x) = \frac{1 - 2\alpha_s \gamma_P \ln(x\mu) + \ldots}{2\pi^4 x^6} \times [1 + O(m^2 x^2) + O(m\Lambda_{QCD}^3 x^4, \Lambda_{QCD}^4 x^4)],
\]

The term on the first line is the coefficient function of the unit operator, with \( \mu \) the renormalization point and \( \gamma_P \) the one-loop anomalous dimension. The correction terms on the second line are, respectively, the perturbative contribution due to the quark mass, and the non-perturbative terms due to the operators \( \bar{q}\bar{q} \) and \( F^2 \) respectively. The crucial point is that chiral symmetry is restored at short distances (i.e. there is no \( m \) dependence if power corrections can be ignored).

Equating \( \hat{G}_P \) and \( g_P^{cont} \) in the chiral limit yields \( Z_P^2 \) in the chosen renormalization scheme. Demanding that \( \hat{G}_P \) contain no terms linear in the quark mass determines \( b_P \), i.e.

\[
(1 + b_P am)^2 = g_P(x; m = 0)/g_P(x; m),
\]

where \( g_P \) is the bare lattice two-point function. We emphasize that the determination of \( b_P \) is non-perturbative, and independent of that of \( Z_P \).

There is one subtlety in the determination of \( b_P \). The condition \([4]\) requires that the physical distance \( x \) not depend on \( m \). This in turn requires the determination of \( b_g \), as discussed above. Note that this is only an issue for full QCD—\( b_g = 0 \) in the quenched approximation. To determine \( b_g \) non-perturbatively one must hold fixed a physical quantity which itself has no dependence on \( m \). The choice we suggest is the force between a heavy \( q - \bar{q} \) pair at short distances. See Ref. \([3]\) for details.
4. Gauge Non-invariant Method

Our second idea is an extension to $O(a)$ of the non-perturbative renormalization program of Ref. [5]. We require that quark and gluon correlators agree with their renormalized continuum counterparts at large Euclidean momenta. Two major complications arise in this extension. First, since we are improving off-shell quantities, we need to include the additional improvement coefficients such as $c'_{\rho}$. Second, since we must fix the gauge, the improvement terms are no longer constrained by the gauge symmetry, but rather by the lattice BRST symmetry. This allows additional improvement terms which are gauge non-invariant (or non-covariant). The only such term which appears at $O(a)$, however, is a $\partial q$ term in the improved quark field:

$$\tilde{q} = Z_q(1 + b_q am) \left[ 1 + ac_q (\bar{D} + m_0) + ac_{NGI} \bar{q} \right] q.$$ 

Gauge invariant bilinears require non-invariant improvement terms only at $O(a^2)$.

We can determine all the on-shell and off-shell improvement coefficients by requiring that chirality violating form factors vanish at large Euclidean momenta. For example, the amputated vertex of the improved pseudoscalar, $\bar{P}$, should, for large momenta, be proportional to $\gamma_5$ with no $a(p'_1 - p'_2)\gamma_5$ term. It turns out that the off-shell terms play an essential role in this procedure: one must first determine the off-shell coefficients in order to correctly determine all the on-shell coefficients. A numerical test of this approach is underway. A similar method has been suggested in Ref. [8], although without the inclusion the gauge non-covariant $c_{NGI}$ term.

5. Conclusions

We have proposed two types of improvement condition, both non-perturbative, and sketched their use for bilinears. They provide an alternative to Ward Identities for determining improvement coefficients in the chiral limit, and they have the advantage of working also away from the chiral limit. This is particularly important for applications involving heavy quarks. The methods should be straightforward to generalize to more complicated composite operators. We are presently studying their numerical efficacy.

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