New, simple models of “black hole interiors”, namely spherically symmetric solutions of the Einstein field equations in matter matching the Schwarzschild vacuum at spacelike hypersurfaces “$R < 2M$” are constructed. The models satisfy the weak energy condition and their matter content is specified by an equation of state of the elastic type.
Spherically symmetric, static models of stars in General Relativity are composed by a regular solution of the Einstein field equations in matter matching smoothly the Schwarzschild vacuum at a surface which lies outside the event horizon, while the Schwarzschild black hole is usually regarded per se as a singular solution of the Einstein field equations; its physical acceptability from the astrophysical point of view relies in the fact that the singularity is covered by the horizon. However, of course, the problem of singularities is far from being solved, and it is generally believed that quantum effects should restrict the curvature to finite values. As a consequence, in recent years the possibility of an alternative picture in which also black holes have non–singular interiors has been actively investigated [1-6]. For “interior structure” of a (non–rotating) black hole one means a solution of the Einstein field equations in matter matching smoothly the Schwarzschild solution in a region which lies inside the Schwarzschild radius. Of course, speculations of this kind are a priori academic because no one can communicate to an external observer the discovery of a “closed world” inside a black hole. We consider, however, interesting to check whether General Relativity allows the existence of this kind of objects at a purely classical level.

Investigating the possible “interior structure” of black holes, one is immediately faced with rigorous theorems [7-9] which impose severe restrictions: no solutions satisfying the strong or the dominant energy condition may exist. Therefore, the physical acceptability of such models must be restricted to the weak energy condition only. Models of regular black holes interiors satisfying this condition have been found by Mars, Martín-Prats and Senovilla [6]. Recently, Borde [10] remarked that a model of this kind is already known from unpublished work by J. Bardeen (see also [11]). Borde has shown that the mechanism responsible for avoidance of singularities in all regular black holes is topology change.

Taken together, the above mentioned results give us a quite complete understanding of “regular black hole physics”. It is, however, worth mentioning that, if we really want to check if nature allows the existence of such objects, we should find models satisfying an equation of state, so that a clear physical interpretation of the matter content of their source can be given. As far as the present author is aware, no examples of this kind are known, and our aim here is to fill this gap. Our starting point is that strongly collapsed, relativistic bodies such as neutron stars typically exhibit elastic–solid properties due to a process of crystallization of matter at very high densities (see e.g. [12]). Therefore, there is a possible interpretation of isolated, anisotropic sources of strong gravitational fields as
highly collapsed, elastic solid objects. In the present paper, we propose a very simple class of “elastic–solid” black hole interiors.

Following Ref.[6], we consider the spherically symmetric line element in Eddington–Finkelstein coordinates:

$$ds^2 = -e^{4\beta} \chi du^2 + 2e^{2\beta} dudr + r^2(d\theta^2 + \sin^2(\theta)d\phi^2) ,$$  

where $\beta$ and $\chi$ are functions of $r$ only. It is useful to represent $\chi$ as

$$\chi(r) = 1 - \frac{2\mu(r)}{r} .$$

The form (1) of the line element is very convenient in dealing with the black hole interior problem. In fact, observe that one can always use the transformation between the null coordinate $u$ and the Schwarzschild “time” $t$ given by

$$dt(u, r) = du - \frac{e^{-2\beta(r)}}{\chi(r)} dr$$

which brings the line element in its Schwarzschild form, but this transformation is allowed only separately within regions in which the function $\chi$ has a constant sign. It is worthwhile to note that, rewriting (1) as

$$ds^2 = -e^{4\beta} \chi \left( du - \frac{e^{-2\beta}}{\chi} dr \right)^2 + \frac{1}{\chi} dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\phi^2) ,$$  

the line element in Schwarzschild form is static if $\chi$ is positive, while it is non–static if $\chi$ is negative. We shall refer towards to such regions as to $R$– and $T$– regions respectively [13]. The line element (1) can be used to describe spacetimes which contain both $R$– and $T$– regions.

To proceed further, we assume as a source an isotropic elastic medium. The energy–momentum tensor of an elastic–solid sphere has been thoroughly discussed in Ref.[14]. In particular, it was shown that a spherically symmetric, relativistic body can be characterized by three quantities $\epsilon, p, \Omega$. In terms of such quantities the eigenvalues of the energy–momentum tensor may be written as

$$\lambda_0 = -\epsilon ,$$  

$$\lambda_1 = p + 2\Omega ,$$  

$$\lambda_2 = \lambda_3 = p - \Omega .$$
In $R$–regions, $\lambda_0$ corresponds to the time–like eigenvector and therefore $\epsilon$ is the energy density, while $\lambda_1$ is the radial stress, which the sum of two terms: the isotropic part $p$ and the quantity $\Omega$ which is a measure of the response of the body to strains that change its shape without changing its volume: if $\Omega$ vanishes, the elastic medium is simply a perfect fluid. The eigenvalue $\lambda_2$ always corresponds to a space–like eigenvector, and therefore is the tangential stress, while, in $T$–regions, the role of $\lambda_0$ and $\lambda_1$ is reversed.

The Einstein field equations may be written as follows:

$$
\mu' = 4\pi r^2 \epsilon ,
$$

$$
\chi \beta' = 2\pi r (\epsilon + p + 2\Omega) ,
$$

$$
2\chi \left( p' + 2\Omega' + \frac{6\Omega}{r} \right) = -(\epsilon + p + 2\Omega)(4\chi \beta' + \chi') ,
$$

where a dash denotes the derivative with respect to $r$. This system is composed by three equations for the five unknowns ($\chi, \beta, \epsilon, p, \Omega$) and therefore it is not closed until we supplement it with an equation of state. Generally speaking, to specify the equation of state for an isotropic elastic body one gives the energy density as a function of the three invariants of the strain tensor. The stresses may then be calculated as derivatives of the energy with respect to such invariants (in the case of a perfect fluid, this obviously amounts to say that the equation of state relates energy and specific volume $v$, and that the pressure may be obtained as the derivative of the energy with respect to $v$). In the particular case of spherical symmetry, it has been shown [14] that assigning the state equation of the body consists in giving the energy density as a function of only two invariants of the strain tensor, one of them being the specific volume (so that the derivative of $\epsilon$ with respect to it governs the isotropic response $p$) and the other one being the quadratic invariant of the deformation, the “response” associated to this invariant being $\Omega$.

The weak energy condition requires

$$
T_{\mu\nu} v^\mu v^\nu \geq 0 ,
$$

for any timelike vector $v^\mu$. Taking into account (3), the above inequality leads to

$$
\epsilon \geq 0 , \ \epsilon + p + 2\Omega \geq 0 , \ \epsilon + p - \Omega \geq 0 ,
$$

in $R$–regions, and to

$$
p + 2\Omega \leq 0 , \ \epsilon + p + 2\Omega \leq 0 , \ \Omega \leq 0 ,
$$
A very simple way to simplify the above conditions is to consider that particular class of elastic materials satisfying
\[ \varepsilon + p + 2\Omega = 0. \tag{6} \]
In fact, the weak energy condition for such materials reduces to
\[ \varepsilon \geq 0, \quad \Omega \leq 0, \tag{7} \]
in any region. We stress that the above choice, besides being very convenient from the mathematical point of view, has a clear physical meaning. It is, in fact, equivalent to a “constitutive” partial differential equation for \( \varepsilon \). The solution of such equation identifies a well-defined class of elastic materials which depends on the choice of an arbitrary function of one variable only (we refer the reader to Ref. [14] for details). This arbitrariness may be used to assign the “on shell” value \( \varepsilon = \varepsilon(r) \). The system (4) then becomes closed and reduces to
\[ \mu' = 4\pi r^2 \varepsilon, \]
\[ \beta' = 0, \]
\[ \epsilon' = 6 \Omega \frac{r}{r}. \tag{8} \]
Therefore, the function \( \chi \) is given by
\[ \chi(r) := 1 - \frac{8\pi}{r} \int_0^r s^2 \varepsilon(s) ds, \]
the function \( \beta \) is a constant which may be set equal to zero without loss of generality, and the line element assumes the simple form
\[ ds^2 = -\chi(r) du^2 + 2 du dr + r^2 (d\theta^2 + \sin^2(\theta) d\phi^2). \tag{9} \]
This solution may be seen as a “variation of the mass” of the Schwarzschild solution and has many interesting features. For example, it is a Kerr–Schild geometry and it arises naturally as the spherically symmetric limit of a recently found class of sources of the Kerr metric [15]. It arises also in completely different contexts, for example in general-relativistic electron models (see e.g. [16] and references therein) or in vacuum polarization [5].

Let us now consider the conditions which have to be imposed on the free function \( \varepsilon \) in order to obtain a regular black hole model.
First of all, the weak energy condition (7) requires $\epsilon$ to be a non-negative function, while $\Omega = r\epsilon'/6$ must be negative, so that $\epsilon$ must be a decreasing function of $r$. The regularity of the spacetime at $r = 0$ requires $\mu \sim r^3$ as $r$ tends to zero, and therefore $\epsilon$ must be finite there. Finally, in order to obtain a black–hole model, we want to match smoothly the interior solution (9) with the Schwarzschild solution at a space–like hypersurface $r = R$. The matching between two metrics is smooth (no surface distributions of matter–energy arise) if the first and the second fundamental form are continuous at the matching surface. Continuity of the metric requires the mass of the vacuum solution to be given by

$$M := \mu(R).$$

To obtain a black–hole model, we require $R$ to belong to a $T$–region:

$$1 - \frac{2M}{R} < 0.$$  \hfill (10)

Continuity of the second fundamental form requires vanishing of the energy density since $r = R$ is a space–like hypersurface. The energy density is $-(p + 2\Omega)$ in a $T$–region, but finally in our model this equals $\epsilon$, so that we require $\epsilon(R) = 0$.

Regarding the structure of the solutions, it is easy to check that the function $\chi$ must vanish at at least one value $r_o$ of $r$, because $\chi(r)$ is continuous and satisfies $\chi(0) = 1$, $\chi(R) < 0$. This is a characteristic shared by all regular black holes interiors satisfying the weak energy condition [6]. A nice feature of our solutions is that there exist always only one value $r_o$, as a graphic comparison between $r$ and the monotone increasing function $2\mu(r)$ immediately shows. A simple calculation also shows that the Riemann invariants are finite, and therefore the solution is non–singular and $r_o$ is a Cauchy horizon. At the horizon, the Plebański type of the energy–momentum tensor degenerates from $[2S_1 - S_2 - T]$ to $[2S - 2N]$ (the timelike and the radial spacelike eigenvector degenerate into a double null eigenvector). It may, however, be easily shown that the weak energy condition remains satisfied there.

The maximal analytic extension of our solutions is similar to that already well known for others “black hole interiors” (see e.g. [5-6]). It closely resembles that of the Reissner–Nordström spacetime with the key difference that the singularity at $r = 0$ is replaced by the matter–filled region.

It may be instructive to discuss some explicit examples. Considering a power-law distribution of density

$$\epsilon = \alpha \left[ 1 - \left( \frac{r}{R} \right)^n \right],$$  \hfill (11)
(where $\alpha$ and $n$ are positive quantities), the Schwarzschild mass is
\[
M = \frac{4\pi}{3(n+3)} n^3 R^3 \alpha,
\]
and the solution is a black hole interior provided that
\[
R > R_{\text{min}} := \left[\frac{3(n+3)}{8\pi n \alpha}\right]^{1/2},
\]
therefore, at fixed $n$, the minimal required value of $r$ at the matching hypersurface decreases as $\alpha$ increases. The solution is given by (9) with
\[
\chi(r) = 1 - \frac{2M}{nR} \left(\frac{r}{R}\right)^2 \left[ n + 3 - 3 \left(\frac{r}{R}\right)^n \right].
\]

Since the central value $\alpha = \epsilon(0)$ is really the density at the centre because $r = 0$ always belongs to a $R$–region, it seems reasonable to assume it around the typical values for neutron stars, namely of order $10^{15}$ g/cm$^3$. Considering, for instance, the case of a linear function $\epsilon$ ($n = 1$ in (11)) we can calculate $R_{\text{min}}$. Since we are using relativistic units, we can express it in Kilometers (it is about 25 Km), although it would be more correct to express it in time–units (about $8.2 \times 10^{-5}$ seconds) because it is in fact an instant of time, belonging to a $T$–region. In any case, of course, the unique quantity which is really measurable by an external observer is the mass of the Schwarzschild black–hole. Assuming $R$ to be 26 Km one obtains this mass to be $M \approx 8.5M_\odot$, with a Schwarzschild radius $2M \approx 28$ Km.

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