THE COHEN–MACAULAY SPACE OF TWISTED CUBICS

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Abstract. In this work, we describe the Cohen–Macaulay space $\text{CM}$ of twisted cubics parameterizing curves $C$ together with a finite map $i: C \to \mathbb{P}^3$ that is generically a closed immersion and such that $C$ has Hilbert polynomial $p(t) = 3t + 1$ with respect to $i$. We show that $\text{CM}$ is irreducible, smooth and birational to one component of the Hilbert scheme of twisted cubics.

1. Introduction

A twisted cubic is a smooth, rational curve in $\mathbb{P}^3$ of degree 3 and genus 0. It is projectively equivalent to the image of the Veronese map $\mathbb{P}^1 \to \mathbb{P}^3$ mapping a point $[u : v]$ on the line to the point in $\mathbb{P}^3$ with coordinates $[u^3 : u^2v : uv^2 : v^3]$. Being the simplest example of a space curve, these curves have been the object of interest in many problems in algebraic geometry. Here we compare two modular compactifications of the space $\mathcal{X}$ of twisted cubics.

The first, and classical, modular compactification is given by the Hilbert scheme $\text{Hilb}_{3t+1}^{3t+1} \mathbb{P}^3$ parameterizing all closed subschemes in $\mathbb{P}^3$ having Hilbert polynomial $p(t) = 3t + 1$. Piene and Schlessinger gave in [PS85] a detailed description of $\text{Hilb}_{3t+1}^{3t+1} \mathbb{P}^3$. It has two smooth irreducible components $H_0$ and $H_1$ with generic points corresponding to a twisted cubic and a smooth plane curve with an additional isolated point, respectively. The component $H_0$ actually contains all curves in $\text{Hilb}_{3t+1}^{3t+1} \mathbb{P}^3$ that do not have an embedded or isolated point, and especially all twisted cubics. Being a significantly smaller compactification of $\mathcal{X}$ than the whole Hilbert scheme $\text{Hilb}_{3t+1}^{3t+1} \mathbb{P}^3$, the component $H_0$ itself is of particular interest. Ellingsrud, Piene and Strømme described it in [EPS87] as the blow-up of the variety parameterizing nets of quadrics along a point-plane incidence relation. However, $H_0$ does not have any known modular interpretation, that is, it does not satisfy the universal property of a moduli space.

The space of Cohen–Macaulay curves that Hønsen introduced in [Høn05] gives a different modular compactification $\text{CM}$ of $\mathcal{X}$. Instead of adding degenerate schemes as in the Hilbert scheme case, one considers only curves, that is, one-dimensional schemes without embedded or isolated points. However, the curves need not be embedded into $\mathbb{P}^3$. 2010 Mathematics Subject Classification. 14H10, 14H50, 14C05.
Instead they come with a finite map to \(\mathbb{P}^3\) that is only generically a closed immersion. Explicitly, the space \(CM\) parameterizes all pairs \((C, i)\), where \(C\) is a curve and \(i: C \to \mathbb{P}^3\) is a finite map that is an isomorphism onto its image away from a finite number of closed points and such that \(C\) has Hilbert polynomial \(p(t) = 3t + 1\) with respect to \(i\). The moduli functor \(CM\) is represented by a proper algebraic space, see [Hon05] and [Hei14].

In this work, we describe the points of \(CM\). It turns out that only two cases can occur. Either the map \(i\) is a closed immersion or its scheme-theoretic image \(i(C)\) is a singular plane curve, and \(i\) induces an isomorphism away from one singular point \(p\) of \(i(C)\). Moreover, there is a bijection between the points of \(CM\) and the component \(H_0\) of the Hilbert scheme of twisted cubics such that a pair \((C, i)\) where \(i\) is not a closed immersion corresponds to the plane image \(i(C)\) augmented with an embedded point at \(p\). This bijection actually defines a birational map between the spaces. Knowing the points of \(CM\), we can moreover show that the space is smooth.

We believe that the space \(CM\) actually is isomorphic to the Hilbert scheme component, giving a modular interpretation for \(H_0\). However, this will have to be shown in future work.

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**Notation and conventions.** Throughout this paper, let \(k\) be an algebraically closed field of characteristic \(\text{char}(k) \neq 2, 3\). Unless otherwise stated, the projective space \(\mathbb{P}^3\) has coordinates \(x, y, z, w\). Moreover, we write \(k[\varepsilon] = k[t]/(t^2)\) for the Artin ring of dual numbers. All schemes considered here are locally Noetherian.

## 2. The space of Cohen–Macaulay curves

For a polynomial \(p(t) = at + b \in \mathbb{Z}[t]\), let \(CM_{\mathbb{P}^3}^{p(t)}\) be the functor \(CM_{\mathbb{P}^3}^{p(t)}: \text{(Sch}/k)^{\circ} \to \text{Sets}\) that for every \(k\)-scheme \(S\) parameterizes all equivalence classes of pairs \((C, i)\), where \(C\) is a flat scheme over \(S\), and \(i: C \to \mathbb{P}^3_S\) is a finite \(S\)-morphism such that for every \(s \in S\) we have that

(i) the fiber \(C_s\) is Cohen–Macaulay and of pure dimension 1,

(ii) the map \(i_s: C_s \to \mathbb{P}^n_{\kappa(s)}\) is an isomorphism onto its image away from finitely many closed points,

(iii) the coherent sheaf \((i_s)_*\mathcal{O}_{C_s} = (i_*\mathcal{O}_C)_s\) on \(\mathbb{P}^n_{\kappa(s)}\) has Hilbert polynomial \(p(t)\).

Two pairs \((C_1, i_1)\) and \((C_2, i_2)\) in \(CM(C)\) are equal if there exists an isomorphism \(\alpha: C_1 \to C_2\) such that \(i_2 \circ \alpha = i_1\).
Theorem 2.1 ([Høn05, Hei14]). The functor $\text{CM}^{p(t)}_{\mathbb{P}^n}$ is represented by a proper algebraic space.

In the special case $n = 3$ and $p(t) = 3t + 1$, we write $\text{CM}$ instead of $\text{CM}^{3t+1}_{\mathbb{P}^3}$.

3. The points of $\text{CM}$

In this section, we classify the points $(C, i)$ in $\text{CM}((\text{Spec}(k)))$ according to the scheme-theoretic image $i(C)$. Moreover, we present in Subsection 3.5 some specialization relations between them.

3.1. The scheme-theoretic image. We start by giving a description of the curves in $\mathbb{P}^3_k$ that can occur as the scheme-theoretic image of a point $(C, i) \in \text{CM}((\text{Spec}(k)))$.

Proposition 3.1. Let $(C, i)$ be a $k$-rational point of $\text{CM}$. Then one of the following two cases occurs.

(i) The morphism $i$ is a closed immersion, and the embedded curve corresponds to a point on the Hilbert scheme $\text{Hilb}^{3t+1}_{\mathbb{P}^3}$ of twisted cubics.

(ii) The scheme-theoretic image $i(C)$ is a plane curve of degree 3, and $i$ induces an isomorphism onto the image away from one closed point in $i(C)$.

Proof. The finite morphism $i$ factors through the scheme-theoretic image $i(C) \subset \mathbb{P}^3_k$, and we have an induced short exact sequence

$$0 \rightarrow \mathcal{O}_{i(C)} \rightarrow i_* \mathcal{O}_C \rightarrow \mathcal{K} \rightarrow 0$$

of coherent $\mathcal{O}_{\mathbb{P}^3_k}$-modules, where the cokernel $\mathcal{K}$ is supported on the finitely many closed points where $i(C)$ is not isomorphic to $C$. The Hilbert polynomial $p_{\mathcal{K}}(t)$ of $\mathcal{K}$ is constant, equal to a nonnegative integer $l$, and we have

$$p_{i(C)}(t) = p_{i_* \mathcal{O}_C}(t) - p_{\mathcal{K}}(t) = 3t + 1 - l.$$

In particular, we see that $i(C) \subset \mathbb{P}^3_k$ is a curve of degree $d = 3$. Hence, by [Har94, Theorem 3.1], its arithmetic genus $g_{i(C)}$ is bounded from above by $g_{i(C)} \leq \frac{1}{2}(d - 1)(d - 2) = 1$. As also $g_{i(C)} = l \geq 0$, it follows that there are only two possibilities, namely $l = 0$ and $l = 1$.

Suppose first that $l = 0$. Then $\mathcal{K} = 0$ and $i$ induces an isomorphism between $C$ and $i(C)$, that is, the map $i$ is a closed immersion.

If $l = 1$, then the scheme-theoretic image $i(C)$ is a curve of degree $d = 3$ and genus $g_{i(C)} = 1 = \frac{1}{2}(d - 1)(d - 2)$. Again by [Har94, Theorem 3.1], it follows that the curve $i(C)$ lies in a plane and does not have any embedded or isolated points. Moreover, $p_{\mathcal{K}}(t) = 1$ implies that the non-isomorphism locus consists of a single point in $i(C)$. □
Furthermore, we can show that the non-isomorphism locus is contained in the singular locus of the scheme-theoretic image $i(C)$.

**Lemma 3.2.** Let $(C, i) \in CM_{\mathbb{P}^n}(\text{Spec}(k))$ with scheme-theoretic image $i(C)$, and let $U \subseteq i(C)$ be an reduced open subscheme. Then the normalization $\nu: \tilde{U} \to U$ factors through the restriction $i_U: i^{-1}(U) \to U$.

**Proof.** Observe that the morphism $i_U$ is integral and birational. Then the statement is a special case of [Aut, Tag 035Q]. □

**Proposition 3.3.** Let $(C, i) \in CM_{\mathbb{P}^n}(\text{Spec}(k))$. Then the zero-dimensional locus $Y \subset i(C)$ where $C$ and $i(C)$ are not isomorphic is contained in the singular locus of $i(C)$. In particular, if the scheme-theoretic image $i(C)$ is smooth, then $i$ is a closed immersion.

**Proof.** Let $U = \text{Spec}(A) \subset i(C)$ be an open affine subscheme contained in the regular locus of $i(C)$, and let $i^{-1}(U) = \text{Spec}(B)$. As $\tilde{U} = U$, the factorization of Lemma 3.2 induces a sequence of injective maps $A \hookrightarrow B \hookrightarrow A$. It follows that $i$ induces an isomorphism between $U$ and $i^{-1}(U)$, and the non-isomorphism locus $Y$ is contained in the singular locus of $i(C)$.

In particular, the locus $Y$ is empty if $i(C)$ is smooth, that is, $i$ is a closed immersion. □

This allows us to give a complete list of the possibilities for the points of the Cohen–Macaulay space of twisted cubics $CM$.

**Proposition 3.4.** Let $(C, i) \in CM(\text{Spec}(k))$ be such that the map $i$ is not a closed immersion. Then the scheme-theoretic image $i(C)$ is a plane curve of degree 3 and $i$ induces an isomorphism between $C$ and $i(C)$ away from one singular point $p \in i(C)$. Moreover, $i(C)$ and $p$ have to be as in one of the following cases:

- (I) a plane nodal curve, and $p$ is the singular point,
- (II) a plane cuspidal curve, and $p$ is the singular point,
- (III) a plane conic intersecting a line twice, and $p$ is one of the intersection points,
- (IV) a plane conic with a tangent line through $p$ that lies in its plane,
- (V) three coplanar lines with three different points of pairwise intersection, and $p$ is one of these intersection points,
- (VI) three coplanar lines with one common point of intersection $p$,
- (VII) a plane double line meeting a line in its plane, and $p$ is a point on the double line other than the intersection point,
- (VIII) a double line meeting a line as in (VII) and $p$ is the point of intersection,
- (IX) a planar triple line, and $p$ is any point on it.

The curves listed above are displayed in Figure 1.
Proof. We showed in Proposition 3.1 that $i(C)$ is a plane curve of degree 3 and that the non-isomorphism locus is one closed point $p$ in $i(C)$. Moreover, it follows from Proposition 3.3 that $p$ is a singular point.

The list consists of all types, up to projective equivalence, of singular plane curves of degree 3 and the possibilities of choosing a singular point on it. □

Figure 1. The possible scheme-theoretic images and the non-isomorphism point.

3.2. Existence. All the curves listed above actually occur as scheme-theoretic images, that is, for every choice of plane curve $D$ and singular point $p$ as in Proposition 3.4, there exists at least one point $(C, i)$ in $CM(Spec(k))$ such that $i(C) = D$ and $p$ is the non-isomorphism locus.

Theorem 3.5. For every plane cubic $D \subseteq \mathbb{P}^3_k$ of degree 3 with singular point $p \in D$, there exists $(C, i) \in CM(Spec(k))$ with the following properties:

(i) The scheme-theoretic image of $C$ in $\mathbb{P}^3_k$ is $D$, and the induced map $C \to D$ is an isomorphism away from $p$.

(ii) The curve $C$ is the flat degeneration of a twisted cubic, and it has an embedding $h:C \subseteq \mathbb{P}^3_k$ such that $i^*\mathcal{O}_{\mathbb{P}^3_k}(1) = h^*\mathcal{O}_{\mathbb{P}^3_k}(1)$.

Proof. Without loss of generality, we can assume that the curve $D$ is contained in the plane $z = 0$ and that it is given by a cubic form $q(x, y, w)$ with a singularity at the point $p = [0 : 0 : 0 : 1]$. For every type of curve as in Proposition 3.4 it suffices to consider one particular example for $q(x, y, w)$ as they all are projectively equivalent.

In all cases, the curve is given as $C = \text{Proj}(k[x, y, w, u]/I)$ for some ideal $I$, and the morphism $i:C \to \mathbb{P}^3_k$ is induced by the homomorphism of graded rings $\varphi:k[x, y, z, w] \to k[x, y, w, u]/I$ with $\varphi(x) = x$, $\varphi(y) = y$, $\varphi(z) = 0$ and $\varphi(w) = w$. 
(I) With \( I = (xu - y, yu - x(x + w), u^2 - w(x + w)) \), the curve \( C \) is a twisted cubic, and the scheme-theoretic image \( i(C) \) is the plane nodal curve defined by the ideal \( \ker(\varphi) = (z, x^3 + x^2w - y^2w) \). Note moreover that \( i \) is an isomorphism onto the image away from the node \( p = [0 : 0 : 0 : 1] \).

(II) Let \( I = (xu - yw, yu - x^2, u^2 - xw) \). Then \( C \) is a twisted cubic, and the scheme-theoretic image is the plane cuspidal curve defined by the ideal \( \ker(\varphi) = (z, x^3 - y^2w) \).

(III) The conic intersecting a line not in its plane having the ideal \( I = (xu, yu - (x^2 + yw), u^2 - uw) \), has scheme-theoretic image given by \( \ker(\varphi) = (z, x^3 + xyw) \), that is, a conic meeting a line in two points, one of them being the non-isomorphism point \( p = [0 : 0 : 0 : 1] \).

(IV) With \( I = (xu - (x^2 + yw), yu, u^2 - (x^2 + yw)) \), the curve \( C \) is a conic intersecting a line that does not lie in its plane, and the image is the conic with tangent line, given by the ideal \( \ker(\varphi) = (z, x^2y + y^2w) \).

(V) For \( I = (xu, yu - yw, u^2 - uw) \), the curve \( C \) consists of three noncoplanar lines with two intersection points. The scheme-theoretic image, given by the ideal \( \ker(\varphi) = (z, xyw) \), is three coplanar lines such that two of them intersect in the non-isomorphism locus \( p = [0 : 0 : 0 : 1] \).

(VI) With \( I = (xu - xy, yu - xy, u^2 - yu) \), we have that \( C \) consists of three concurrent but not coplanar lines. The scheme-theoretic image is three concurrent and coplanar lines, and it is given by the ideal \( \ker(\varphi) = (z, x^2y - xy^2) \).

(VII) For \( I = (xu, yu - x^2, u^2) \), the curve \( C \) is a double line of genus \(-1\) meeting a line. The image is a planar double line and a line in its plane, given by the ideal \( \ker(\varphi) = (z, x^2w) \). The curves \( C \) and \( i(C) \) are isomorphic away from the point \( p = [0 : 0 : 0 : 1] \) that lies on the double line but is not the intersection point.

(VIII) Similarly, the ideal \( I = (xu - x^2, yu, u^2 - xu) \) describing a planar double line and a line not in its plane gives as image the planar double line and the line in its plane defined by the ideal \( \ker(\varphi) = (z, x^2y) \). In this case the non-isomorphism point is the intersection point \( p = [0 : 0 : 0 : 1] \).

(IX) Finally, if \( C \) is the nonplanar triple line defined by the ideal \( I = (xu, yu - x^2, u^2) \), the image is the planar triple line given by the ideal \( (z, x^3) \).

In all cases the curve \( C \) is given with an embedding into the projective space \( \mathbb{P}^2_k = \text{Proj}(k[x, y, w, u]) \) so that \( i^*\mathcal{O}_{\mathbb{P}^2_k}(1) = \mathcal{O}_C(1) \). Moreover, every curve is the specialization of a twisted cubic. \( \square \)
Remark 3.6. Let for example $D$ be the plane curve given by the ideal $(z, x^3 + x^2w - y^2w)$ as in case (I). Then we consider the flat one-parameter family $Z \subset \mathbb{P}^3_{k[t]}$ generated by the homogeneous polynomials $f_1 = xz - tyw$, $f_2 = yz - tx(x + w)$, $f_3 = z^2 - t^2w(x + w)$ and $q = x^3 + x^2w - y^2w$ in $k[t][x, y, z, w]$. Note that $yf_1 - xf_2 = tq$. For $t \neq 0$ the fiber $Z_t$ is a twisted cubic, whereas $Z_0$ is the plane nodal curve $D$ with an embedded point at the singularity given by the ideal $(xz, yz, z^2, q)$. Then the ideal $I$ of the curve $C$ is generated by the polynomials $g_1, g_2, g_3 \in k[x, y, w, u]$ that are obtained by dividing $f_1$ and $f_2$ by $t$ and $f_3$ by $t^2$ and setting $u = t^{-1}z$.

More generally, all curves and maps in the proof of Theorem 3.5 were constructed in a similar way: We consider a flat one-parameter family $Z \subset \mathbb{P}^3_{k[t]}$ such that the fiber $Z_t$ is a Cohen–Macaulay curve with Hilbert polynomial $p(n) = 3n + 1$ for $t \neq 0$, and $Z_0$ is the plane curve $D$ with an embedded point supported at $p$. Suitable generators $f_1, f_2, f_3, q \in k[t][x, y, z, w]$ of the ideal defining $Z$ give then rise to the generators $g_1, g_2, g_3 \in k[x, y, w, u]$ of $I$.

3.3. Uniqueness. In the next step, we show that the curves constructed in the proof of Theorem 3.5 are the unique solutions, see Theorem 3.8.

Lemma 3.7. Let $(C, i) \in CM(\text{Spec}(k))$ be such that the map $i$ is not a closed immersion. Assume that the scheme-theoretic image $i(C)$ is contained in the plane $z = 0$ and that $i$ induces an isomorphism between the curve $C$ and the image away from the singular point $[0 : 0 : 0 : 1]$ on $i(C)$. Let further $A$ be the $k$-algebra such that $i(C) \cap D_+(w) = \text{Spec}(A)$, and let $i^{-1}(\text{Spec}(A)) = \text{Spec}(B)$. Then the map $i$ corresponds to an inclusion $A \subset B$ of rings such that

(i) $\dim_k(B/A) = 1$,

(ii) $xB \subseteq A$ and $yB \subseteq A$, and

(iii) if $a \in A$ is not a zero divisor in $A$, then $a$ is not a zero divisor in $B$.

Proof. The first property (i) follows directly since the Hilbert polynomials of $C$ and $i(C)$ differ by $1$ and the non-isomorphism locus is contained in $\text{Spec}(A)$.

Note that the quotient $B/A$ is only supported at the maximal ideal $m = (x, y)$ of $A$. By property (i), it follows that $\text{Ann}_A(B/A) = m$ and property (ii) holds.

For property (iii), assume that $a$ is a zero divisor of $B$, that is, that $a$ is contained in an associated prime ideal $p$ of $B$. As the curve $C$ is Cohen–Macaulay without isolated points, it follows that $p$ is a minimal prime ideal that is not maximal. The restriction $p \cap A$ is then a minimal prime ideal in $A$ that contains $a$. This implies that $a$ is a zero divisor. □
Theorem 3.8. For every plane cubic curve $D \subset \mathbb{P}^3_k$ of degree 3 with singular point $p \in D$, there exists at most one $k$-rational point $(C, i)$ on CM such that $i(C) = D$ and the induced map $C \to D$ is an isomorphism away from $p$.

Proof. We prove the statement individually for the different possibilities of $D$ as listed in Proposition 3.3. In the cases [I] of a nodal curve, [II] of a cuspidal curve, [III] of a conic and a line intersecting twice and [IV] of three coplanar lines, the point $p$ is an isolated singular point. Lemma 3.2 and comparison of the Hilbert polynomials imply that locally around $p$ the map $C \to D$ has to be the normalization, and hence it is unique.

In the remaining cases, we can without loss of generality assume that $D$ is contained in the plane $z = 0$ and that $p = [0 : 0 : 0 : 1]$, that is, $D$ is given by an ideal $I = (z, q(x, y, w))$, where $q(x, y, w)$ is a cubic form with singularity at $p$. Then we show that for $D \cap D_+(w) = \text{Spec}(A)$, there exists, up to $A$-algebra isomorphism, only one $k$-algebra extension $A \subset B$ satisfying the properties of Lemma 3.7.

In case [IV], the curve $D$ consists of a conic and a tangent line, say $q(x, y, w) = x^2y + y^2w$. Note that all such curves are projectively equivalent, and hence it suffices to show the claim for one particular choice of cubic form $q(x, y, w)$. In the ring $A = k[x, y]/(x^2y + y^2)$ we have that $y^n = y(-x^2)^n$ for every $n \in \mathbb{N}$. In particular, every element $a \in A$ can be written uniquely as $a = f(x) + yg(x)$ with $f(x), g(x) \in k[x]$. Now let $A \subset B$ be as above, and let $b \in B \setminus A$. As $xb, yb \in A$ and $y(xb) = x(yb)$, one can show that there are polynomials $g_1(x), g_2(x) \in k[x]$ such that

$$\begin{align*}
&\begin{cases}
  xb = xg_2(x) + (x^2 + y)g_1(x) \\
  yb = yg_2(x).
\end{cases}
\end{align*}$$

We can write $g_1(x) = c + xu(x)$ for $c \in k$ and $u(x) \in k[x]$. Replacing $b$ by $b - g_2(x) - (x^2 + y)u(x)$, we get that

$$\begin{align*}
&\begin{cases}
  xb = c(x^2 + y) \\
  yb = 0.
\end{cases}
\end{align*}$$

Moreover, it follows that $xb^2 = xc^2(x^2 + y)$. As $x$ is not a zero divisor in $B$ by property [iii] in Lemma 3.7, we can conclude that $b^2 = c^2(x^2 + y)$ and $c \neq 0$. After replacing $b$ by $c^{-1}b$, we can consider the $A$-algebra $B' := A[b]/(xb - (x^2 + y), yb, b^2 - (x^2 + y))$ that lies between $A$ and $B$. As $\dim_k(B'/A) = 1 = \dim_k(B/A)$, it follows that $B \cong B'$.

The cases [VI] to [IX] are shown in the same way. For [VI], the plane curve $D$ consists of three concurrent lines, and we can assume that $q(x, y, w) = x^2y - xy^2$ and get $B \cong A[b]/(xb - xy, yb - xy, b^2 - xy)$. If the scheme-theoretic image is given by $q(x, y, w) = x^2w$, as in the situation of [VII], the extension is $B \cong A[b]/(xb, yb - x, b^2)$. If $p$ is
the intersection point of a double line and a line as in [VIII] we can assume that $q(x, y, w) = x^2y$ and get $B \cong A[b]/(xb - x^2, yb, b^2 - x^2)$. In the last case [IX] the curve $D$ is the triple line given by $q(x, y, w) = x^3$. We show then that $B \cong A[b]/(xb, yb - x^2, b^2)$.

Remark 3.9. Note that the extensions $B$ constructed in the proof are affine charts of the curves $C$ listed in the proof of Theorem 3.5.

3.4. Classification of the points of $CM$. We summarize the results of the previous subsections as follows.

**Theorem 3.10.** There is a one-to-one correspondence between the $k$-rational points of $CM$ and the union of the set of equidimensional Cohen–Macaulay curves in $\mathbb{P}_{k}^3$ with Hilbert polynomial $3t + 1$ and the set of singular plane curves in $\mathbb{P}_{k}^3$ together with a singular point $p$ on it.

**Proof.** We have seen in Proposition 3.1 that for every pair $(C, i)$ in $CM(\text{Spec}(k))$, the map $i$ is either a closed immersion or an isomorphism onto a plane curve away from one point $p$ that has to be singular by Proposition 3.4.

Conversely, every embedding of an equidimensional Cohen–Macaulay curve with Hilbert polynomial $3t + 1$ gives a point on $CM$. Moreover, we have seen in Theorem 3.5 and Theorem 3.8 that for every plane curve $D$ with singular point $p$ the exists a unique point $(C, i)$ on $CM$ such that $i$ induces an isomorphism between $C$ and $D$ away from $p$. □

3.5. Specializations in $CM$. Comparing the ideals in the proof of Theorem 3.5 we can see that all points of $CM$ specialize to a point corresponding to a pair $(C, i)$ where the scheme-theoretic image is a triple line.

**Example 3.11.** Let $(C, i) \in CM(\text{Spec}(k[t]))$ be a family of Cohen–Macaulay curves where $C \subset \mathbb{P}_{k[t]}^3 = \text{Proj}(k[t][x, y, w, u])$ is given by the ideal $I = (xu, yu - x(x + t y), u^2)$, and the map $i$ corresponds to the homomorphism of graded rings

$$\varphi: k[t][x, y, z, w] \to k[t][x, y, w, u]/I$$

given by $\varphi(x) = x$, $\varphi(y) = y$, $\varphi(z) = 0$ and $\varphi(w) = w$.

For $t \neq 0$, the scheme-theoretic image $i_t(C_t)$ consists of the double line intersecting a line $(z, x^3 + tx^2y)$, and $i_t$ induces an isomorphism away from the intersection point.

The scheme-theoretic image $i_0(C_0)$, on the contrary, is the plane triple line $(z, x^3)$.

In a similar way, we can show that all types [I] to [VIII] specialize to the case of a triple line [IX]. Specifically, we have the chart of specializations as shown in Figure 2.

A similar diagram of specializations for the component $H_0$ of the Hilbert schemes of twisted cubics can be found in [Har82, p. 40].
Figure 2. Specializations between points of $CM$ with scheme-theoretic image and non-isomorphism locus of types [I] to [IX] as in Proposition 3.4.

4. THE HILBERT SCHEME OF TWISTED CUBICS

Knowing the points of $CM$, we can now establish a bijection with the points of one component of the Hilbert scheme of twisted cubics.

**Theorem 4.1 (PSS85)**. The Hilbert scheme $\text{Hilb}^{3t+1}_{\mathbb{P}^3}$ consists of two components $H_0$ and $H_1$. The points of $H_0$ are the degenerations of a twisted cubic, namely all equidimensional Cohen–Macaulay curves in $\mathbb{P}^3$ with Hilbert polynomial $3t + 1$ and all singular, plane curves with an embedded point that is supported at a singularity and emerges from the plane. The component $H_0$ is smooth and has dimension 12.
Proposition 4.2. There is a bijection between the set of \( k \)-rational points of the space of Cohen–Macaulay curves \( CM \) and the set of \( k \)-rational points of the component \( H_0 \) of the Hilbert scheme \( \text{Hilb}^3_{\mathbb{P}^3} \). Moreover, the open subfunctor \( U \) of \( CM \) corresponding to closed immersions is isomorphic to the open subscheme of \( H_0 \) corresponding to Cohen–Macaulay curves.

Proof. The locus \( U \) in \( CM \) coincides with the Cohen–Macaulay locus in \( \text{Hilb}^3_{\mathbb{P}^3} \). Moreover, every equidimensional Cohen–Macaulay curve in \( \mathbb{P}^3 \) corresponds to a point of \( H_0 \), see the proof of [PS85, Lemma 1]. The remaining points in both \( CM(\text{Spec}(k)) \) and \( H_0(\text{Spec}(k)) \) are in bijection with the set of pairs consisting of a plane curve of degree 3 in \( \mathbb{P}^3 \) and a singular point on it, see Theorem 3.10 and Theorem 4.1. \( \square \)

Corollary 4.3. The space of Cohen–Macaulay curves \( CM \) has an irreducible open dense subscheme that is smooth of dimension 12. In particular, \( CM \) is irreducible and has dimension 12.

Proof. The subspace \( U \) in Proposition 4.2 has the required properties. This implies that \( CM \) itself is irreducible and has dimension 12. \( \square \)

5. Deformations

The goal of this section is to show that \( CM \) is smooth. In particular we compute the dimension of the tangent space of \( CM \) at one certain point.

A first-order deformation of a point \((C, i) \in CM^{at+b}_{\mathbb{P}^n}(\text{Spec}(k))\) is an element \((\tilde{C}, \tilde{i}) \in CM^{at+b}_{\mathbb{P}^n}(\text{Spec}(k[\varepsilon]))\) such that the diagram

\[
\begin{array}{ccc}
C & \xrightarrow{i} & \mathbb{P}^n_k \\
\downarrow & & \downarrow \\
\tilde{C} & \xrightarrow{\tilde{i}} & \mathbb{P}^n_{k[\varepsilon]}
\end{array}
\]

is Cartesian. The space of these first-order deformations is isomorphic to the tangent space of \( CM^{at+b}_{\mathbb{P}^n} \) at the point \((C, i)\).

We show first that the curve \( C \) can be embedded in a projective space \( \mathbb{P}^N_k \) in such a way that the scheme \( \tilde{C} \) is given as deformation of \( C \) in \( \mathbb{P}^N_k \).

Proposition 5.1 ([Hei14, Proposition 4.14]). There exist \( m, N \in \mathbb{N} \) such that for every field \( k \) and every \((C, i) \in CM^{at+b}_{\mathbb{P}^n}(\text{Spec}(k))\) there exists a closed immersion \( j: C \hookrightarrow \mathbb{P}^N_k \) such that \( j^*\mathcal{O}_{\mathbb{P}^n_k}(1) = i^*\mathcal{O}_{\mathbb{P}^n_k}(m) \) and \( j^*x_0, \ldots, j^*x_N \) form a basis of \( H^0(C, i^*\mathcal{O}_{\mathbb{P}^n_k}(m)) \).

Proposition 5.2. Let \((\tilde{C}', \tilde{i}') \in CM^{at+b}_{\mathbb{P}^n}(\text{Spec}(k[\varepsilon]))\) be a first-order deformation of the point \((C, i) \in CM^{at+b}_{\mathbb{P}^n}(\text{Spec}(k))\). Suppose that the curve \( C \) is given as a closed subscheme \( j: C \subset \mathbb{P}^N_k \) as in Proposition 5.1.
Then \((\tilde{C}', \tilde{i}') = (\tilde{C}, \tilde{i})\) in \(CM_{\mathbb{P}^3}^{at+b}(\text{Spec}(k[\varepsilon]))\) for a first-order deformation \(\tilde{C}\) of the closed subscheme \(C\) of \(\mathbb{P}^3_k\).

**Proof.** Let \(m, N \in \mathbb{N}\) be as in Proposition 5.1. By [Hei14, Proposition 4.16], the \(k[\varepsilon]\)-module \(H^0(\tilde{C}', (\tilde{i}')^*O_{\mathbb{P}^3_k}(m))\) isfree of rank \(N + 1\) and

\[
H^0(C, i^*O_{\mathbb{P}^3_k}(m)) = H^0(\tilde{C}', (\tilde{i}')^*O_{\mathbb{P}^3_k}(m)) \otimes_{k[\varepsilon]} k[\varepsilon]/(\varepsilon).
\]

Therefore we can choose a basis \(\tilde{s}_0, \ldots, \tilde{s}_N\) of \(H^0(\tilde{C}', (\tilde{i}')^*O_{\mathbb{P}^3_k}(m))\) that lifts the basis \(j^*x_0, \ldots, j^*x_N\) of \(H^0(C, i^*O_{\mathbb{P}^3_k}(m))\). Then, by [Hei14, Proposition 4.19], the choice of global sections induces a closed immersion \(\tilde{j}: \tilde{C}' \hookrightarrow \mathbb{P}^3_{\mathbb{P}^3_k}\). Note that the commutative diagram

\[
\begin{array}{ccc}
\mathbb{P}^3_k & \rightarrow & C' \\
\downarrow & & \downarrow \\
\mathbb{P}^3_{\mathbb{P}^3_k} & \rightarrow & \tilde{C}'
\end{array}
\]

is Cartesian. Now let \(\tilde{C}\) be the scheme-theoretic image of the closed immersion \(\tilde{j}\) and \(\tilde{\alpha}: \tilde{C} \twoheadrightarrow \tilde{C}'\) the induced isomorphism. Then \(\tilde{C} \subset \mathbb{P}^3_{\mathbb{P}^3_k}\) is flat over \(\text{Spec}(k[\varepsilon])\), and its restriction modulo \(\varepsilon\) is \(C\). Hence \(\tilde{C}\) is a first-order deformation of \(C \subset \mathbb{P}^3_k\). The restriction \(\alpha\) of \(\tilde{\alpha}\) modulo \(\varepsilon\) is an automorphism of \(C\) such that \(i \circ \alpha = i\). Then, by [Hei14, Theorem 2.19], the map \(\alpha\) is the identity. With \(\tilde{i} := \tilde{j} \circ \tilde{\alpha}\), it follows that the restriction of \(\tilde{i}\) modulo \(\varepsilon\) is \(i\) and that \((\tilde{C}', \tilde{i}') = (\tilde{C}, \tilde{i})\) in \(CM_{\mathbb{P}^3}^{at+b}(\text{Spec}(k[\varepsilon]))\). \(\Box\)

From now on we treat the special case \(n = 3\) and \(p(t) = 3t + 1\). We show that \(CM\) is smooth by proving that the tangent space at every point has dimension 12. In Section 3.5, we have seen that all maps \(i: C \rightarrow \mathbb{P}^3_k\) specialize to a map such that the scheme-theoretic image is a plane triple line. Hence it suffices to study the \(k[\varepsilon]\)-deformations at such a point of \(CM\).

**Lemma 5.3.** Let \((C, i) \in CM(\text{Spec}(k))\) be a point of \(CM\). Then the following holds.

(i) The coherent sheaf \(i_*O_C\) is 1-regular.

(ii) \(h^0(C, i^*O_{\mathbb{P}^3_k}(1)) = 4\).

(iii) The global sections of \(i^*O_{\mathbb{P}^3_k}(1)\) separate points and tangent vectors.

(iv) Every choice of basis of \(H^0(C, i^*O_{\mathbb{P}^3_k}(1))\) gives a closed immersion \(j: C \hookrightarrow \mathbb{P}^3_k\).

**Proof.** We have seen in Theorem 3.5 and 3.8 that the curve \(C\) is a Cohen–Macaulay specialization of a twisted cubic and that it has an embedding \(h: C \hookrightarrow \mathbb{P}^3_k\) such that \(i^*O_{\mathbb{P}^3_k}(1) = h^*O_{\mathbb{P}^3_k}(1)\). Let \(\mathcal{I}\) be the
sheaf of ideals describing $C$ as a subscheme in $\mathbb{P}^3_k$. Then by [Ell75, Exemple 1] there exists a short exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3_k}(-3)^{\oplus 2} \longrightarrow \mathcal{O}_{\mathbb{P}^3_k}(-2)^{\oplus 3} \longrightarrow I \longrightarrow 0.$$  

From the corresponding long exact sequence in cohomology we conclude that $H^r(I(d)) = 0$ for all $d$ and $r = 1$ and $r \geq 4$. Moreover, we get that $H^0(I(d)) = 0$ for $d < 2$, $H^2(I(d)) = 0$ for $d \geq 0$ and $H^3(I(d)) = 0$ for $d \geq -1$. In particular, it follows that $I$ is 2-regular. Applying these results on the cohomology of $I$ to the short exact sequence

$$0 \longrightarrow I \longrightarrow \mathcal{O}_{\mathbb{P}^3_k} \longrightarrow h^*\mathcal{O}_C \longrightarrow 0,$$

we conclude that $h^0(\mathbb{P}^3_k, (h_*\mathcal{O}_C)(1)) = h^0(\mathbb{P}^3_k, \mathcal{O}_{\mathbb{P}^3_k}(1)) = 4$ and that $h_*\mathcal{O}_C$ is 1-regular. Finally, due to the projection formula and the fact that the maps $i$ and $h$ are finite, we have that

$$H^r(\mathbb{P}^3_k, (i_*\mathcal{O}_C)(d)) = H^r(C, i^*\mathcal{O}_{\mathbb{P}^3_k}(d)) = H^r(\mathbb{P}^3_k, (h_*\mathcal{O}_C)(d))$$

for all $d$ and $r \geq 0$, and we have shown properties (i) and (ii).

By [Har77, Proposition II.7.3], the global sections of the invertible sheaf $i^*\mathcal{O}_{\mathbb{P}^3_k}(1) = h^*\mathcal{O}_{\mathbb{P}^3_k}(1)$ separate points and tangent vectors, hence we have property (iii). In particular, every basis of global sections of $i^*\mathcal{O}_{\mathbb{P}^3_k}(1)$ separates points and tangent vectors and induces a closed immersion $j: C \hookrightarrow \mathbb{P}^3_k$, again by [Har77, Proposition II.7.3]. This shows property (iv) and concludes the proof.

In terms of the notation of Proposition 5.1, the lemma says that we have that $m = 1$ and $N = 3$ in the twisted cubic case.

**Proposition 5.4.** Let $(C, i)$ be a point in $CM(\text{Spec}(k))$ such that the scheme-theoretic image is a plane triple line. Then the tangent space at this point has dimension 12.

**Proof.** Without loss of generality, we can assume that the curve $C$ is given by the ideal $I = (xu, yu - x^2, w^2)$ in $\mathbb{P}^3_k = \text{Proj}(k[x, y, w, u])$ and that $i$ corresponds to the the homomorphism of graded rings

$$\varphi: k[x, y, z, w] \rightarrow k[x, y, w, u]/I$$

with $\varphi(x) = x$, $\varphi(y) = y$, $\varphi(z) = 0$ and $\varphi(w) = w$.

As in Proposition 5.2, we study first the deformations of $C$ as a subscheme of the projective space $\mathbb{P}^3_k$. These deformations are in one-to-one correspondence with the elements of $H^0(C, \mathcal{N}_{C/\mathbb{P}^3_k})$, see for example [Har10, Theorem 2.4], and we can compute them from the exact sequence

$$0 \longrightarrow \mathcal{N}_{C/\mathbb{P}^3_k} \longrightarrow \mathcal{O}_C(2)^{\oplus 3} \longrightarrow \mathcal{O}_C(3)^{\oplus 2}$$
induced by a resolution of the ideal \( I \). It follows that the space of deformations has dimension 12, and for every \( \mathbf{a} = (a_1, \ldots, a_{12}) \in k^{12} \) we get a deformation \( \tilde{C}_\mathbf{a} \subset \mathbb{P}_k^3 \) defined by the ideal \( \tilde{I}_\mathbf{a} \) generated by the polynomials

\[
p_{1,\mathbf{a}}(x, y, w, u) = xu + \varepsilon(a_1 x^2 + a_2 xy + a_3 xw + a_4 y^2 + a_5 yw + a_6 u),
\]

\[
p_{2,\mathbf{a}}(x, y, w, u) = yu - x^2 + \varepsilon(a_7 x^2 + a_8 xy + a_9 xw + a_{10} y^2 + a_{11} yw + a_{12} u),
\]

\[
p_{3,\mathbf{a}}(x, y, w, u) = u^2 + \varepsilon((a_2 + a_{10}) x^2 + a_4 xy + a_5 xw + (a_3 + a_{11}) yw).
\]

Every deformation of \((C, i)\) is then given by a map \( \tilde{\iota}_{\mathbf{a}, \mathbf{b}} \) associated to

\[
\tilde{\varphi}_{\mathbf{a}, \mathbf{b}}: k[\varepsilon][x, y, z, w] \to k[\varepsilon][x, y, w, u]/\tilde{I}_\mathbf{a}
\]

defined by

\[
\tilde{\varphi}_{\mathbf{a}, \mathbf{b}}(x) = x + \varepsilon(b_1 x + b_2 y + b_3 w + b_4 u)
\]

\[
\tilde{\varphi}_{\mathbf{a}, \mathbf{b}}'(y) = y + \varepsilon(b_5 x + b_6 y + b_7 w + b_8 u)
\]

\[
\tilde{\varphi}_{\mathbf{a}, \mathbf{b}}(z) = 0 + \varepsilon(b_9 x + b_{10} y + b_{11} w + b_{12} u)
\]

\[
\tilde{\varphi}_{\mathbf{a}, \mathbf{b}}(w) = w + \varepsilon(b_{13} x + b_{14} y + b_{15} w + b_{16} u)
\]

for \( \mathbf{b} = (b_1, \ldots, b_{16}) \in k^{16} \). Thus the pairs \((\tilde{C}_\mathbf{a}, \tilde{\iota}_{\mathbf{a}, \mathbf{b}})\) give all deformations, and the dimension of the tangent space at the point \((C, i)\) is at most 12 + 16 = 28.

Recall that in \( CM \) we only consider isomorphism classes of pairs. In particular, we have \((\tilde{C}_\mathbf{a}, \tilde{\iota}_{\mathbf{a}, \mathbf{b}}) = (\tilde{C}_\mathbf{a}', \tilde{\iota}_{\mathbf{a}', \mathbf{b}'})\) in \( CM(Spec(k[\varepsilon])) \) for \( \mathbf{a}, \mathbf{a}' \in k^{12} \) and \( \mathbf{b}, \mathbf{b}' \in k^{16} \) if and only if there exists an isomorphism \( \tilde{\alpha}: \tilde{C}_\mathbf{a} \cong \tilde{C}_\mathbf{a}' \) such that \( \tilde{\iota}_{\mathbf{a}, \mathbf{b}} = \tilde{\iota}_{\mathbf{a}', \mathbf{b}'} \circ \tilde{\alpha} \). As the restriction \( \alpha \) of \( \tilde{\alpha} \) to \( k = k[\varepsilon]/(\varepsilon) \) is an automorphism of \( C \) such that \( i = i \circ \alpha \), it follows from [Hei14, Theorem 2.19] that \( \alpha \) is the identity morphism.

We consider the particular case that \( \tilde{\alpha} \) is induced by a homomorphism of graded rings \( \tilde{\sigma}_s: k[\varepsilon][x, y, w, u] \to k[\varepsilon][x, y, w, u] \) with

\[
\tilde{\sigma}_s(x) = x + \varepsilon(s_1 x + s_2 y + s_3 w + s_4 u)
\]

\[
\tilde{\sigma}_s(y) = y + \varepsilon(s_5 x + s_6 y + s_7 w + s_8 u)
\]

\[
\tilde{\sigma}_s(w) = w + \varepsilon(s_9 x + s_{10} y + s_{11} w + s_{12} u)
\]

\[
\tilde{\sigma}_s(u) = u + \varepsilon(s_{13} x + s_{14} y + s_{15} w + s_{16} u).
\]

for \( s = (s_1, \ldots, s_{16}) \in k^{16} \). Then one can compute that \( \tilde{\sigma}_s(C_\mathbf{a}) = C_{\mathbf{a}'} \) and \( \tilde{\sigma}_s \circ \tilde{\varphi}_{\mathbf{a}, \mathbf{b}} = \tilde{\varphi}_{\mathbf{a}', \mathbf{b}'} \) for some \( s = (s_1, \ldots, s_{16}) \in k^{16} \) if and only if the
following 12 conditions hold:

\begin{align*}
    a_2 - a_{10} &= a'_2 - a'_{10}, \\
    a_3 - a_{11} &= a'_3 - a'_{11}, \\
    a_4 &= a'_4, \\
    a_5 &= a'_5, \\
    b_2 + \frac{1}{3}(a_8 - a_1) &= b'_2 + \frac{1}{3}(a'_8 - a'_1), \\
    b_3 + \frac{1}{2}a_9 &= b'_3 + \frac{1}{2}a'_9, \\
    b_4 - a_6 &= b'_4 - a'_6, \\
    b_7 - a_{12} &= b'_7 - a'_7, \\
    b_9 &= b'_9, \\
    b_{10} &= b'_{10}, \\
    b_{11} &= b'_{11}, \\
    b_{12} &= b'_{12}.
\end{align*}

So the equivalence class of the element \((\tilde{C}_a, \tilde{i}_{a,b})\) in \(CM(Spec(k[\varepsilon]))\) depends only on the values of \(a_2 - a_{10}, a_3 - a_{11}, a_4, a_5, b_2 + \frac{1}{3}(a_8 - a_1), b_3 + \frac{1}{2}a_9, b_4 - a_6, b_7 - a_{12}, b_9, b_{10}, b_{11}, b_{12}\).

It follows that the dimension of the space of first-order deformations of the point \((C, i)\) is at most 12. Since, by Corollary 4.3, the space \(CM\) is of dimension 12, this concludes the proof.

**Theorem 5.5.** The Cohen–Macaulay space \(CM\) of twisted cubics is irreducible, smooth and it has dimension 12.

**Proof.** We only have to show that \(CM\) is smooth, that is, the tangent space has dimension 12 at every point. We have seen in Corollary 4.3, that the open subscheme \(U\) of \(CM\), consisting of all points \((C, i)\) where \(i\) is a closed immersion, is smooth. Hence it remains to consider the most specialized ones among the remaining points, namely those having a triple line as the scheme-theoretic image. This case was treated in Proposition 5.4. \(\square\)

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