Nonlocality without inequalities has not been proved for maximally entangled states

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Two approaches to extend Hardy’s proof of nonlocality without inequalities to maximally entangled states of bipartite two-level systems are shown to fail. On one hand, it is shown that Wu and co-workers’ proof [Phys. Rev. A 53, R1927 (1996)] uses an effective state which is not maximally entangled. On the other hand, it is demonstrated that Hardy’s proof cannot be generalized by the replacement of one of the four von Neumann measurements involved in the original proof by a generalized measurement to unambiguously discriminate between non-orthogonal states. In Sec. IV a general demonstration showing that Hardy’s proof cannot be generalized in such a way will be provided. Our discussion begins in Sec. II, where Hardy’s [1] and Goldstein’s [2] versions of Hardy’s proof are reviewed. By “versions” I mean logical reasonings based on the same set of properties of certain quantum states. This distinction between versions will be useful in Sec. IV.

II. NONLOCALITY WITHOUT INEQUALITIES FOR HARDY STATES

We shall focus our attention on bipartite two-level systems initially prepared in a state of the form

$$|\psi\rangle = a |++\rangle + b (|+\rangle |\cdot\cdot\rangle + |\cdot\cdot\rangle |+\rangle),$$

(1)

where $a = \cos \theta$, and $b = \sin \theta/\sqrt{2}$, being $0 \leq \theta \leq \pi/2$. The notation $|+\rangle$ means $|+\rangle_{1} \otimes |\cdot\cdot\rangle_{2}$, being $\{|+\rangle_{j}, |\cdot\cdot\rangle_{j}\}$ an orthogonal basis for particle $j$ ($j = 1, 2$).

Now I shall explain why the study of the family of states given by Eq. (1) covers all relevant cases. For bipartite pure states, partial entropy is a good measure of entanglement [2,11] since it fulfills the following requirements [11]: to have zero value for product states, to be invariant under local unitary transformations and nonincreasing under classically coordinated local operations, and to be additive for tensor products. From the properties of partial entropy, it follows that any two pure states having the same partial entropy will give the same maximum probability for finding an event which contradicts local realism for a standard Hardy’s proof. Therefore, the conclusions reached for a state of the form (1) with partial entropy $S$, can be extended to any bipartite two-level pure state with partial entropy $S$. Partial entropy of states of the form (1) is a monotone function of the angle $\theta$, and takes the value zero, for $\theta = 0$, and the maximum allowed value, $\ln 2 \approx 0.6931$, for $\theta = \pi/2$. Therefore, states of the form (1) cover all possible values of partial entropy and thus they cover all possible cases of contradiction with local realism. Moreover, this partial entropy depends on a single parameter $\theta$: if $\theta = 0$, then $|\psi\rangle$ is a product state; if $0 < \theta < \pi/2$, then $|\psi\rangle$ is an entangled but not maximally entangled state; and if $\theta = \pi/2$, then $|\psi\rangle$ is a maximally entangled state.

Suppose $|+\rangle_{j}$ and $|\cdot\cdot\rangle_{j}$ are the eigenstates corresponding to the observable $A_{j}$, and $|\oplus\rangle_{j}$ and $|\ominus\rangle_{j}$ are the eigen-

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states corresponding to the observable \( B_j \), being

\[
|\oplus\rangle_j = N \left( a |+\rangle_j + b |-\rangle_j \right), \quad (2a)
\]

\[
|\ominus\rangle_j = N \left( b |+\rangle_j - a |-\rangle_j \right), \quad (2b)
\]

where \( j = 1, 2 \), and \( N = 1/\sqrt{1-b^2} \). Then state \( \| \) can be written in the following forms:

\[
|\psi\rangle = N \left[ (1-b^2) |\oplus\rangle + ab |\ominus\rangle + b^2 |\ominus\ominus\rangle \right], \quad (3a)
\]

\[
= N \left[ (1-b^2) |\ominus\oplus\rangle + ab |\ominus\ominus\rangle + b^2 |\ominus\ominus\ominus\rangle \right]. \quad (3b)
\]

Now, we distinguish between two versions of the proof.

**Hardy’s proof**

From Eqs. (3a), (4a), and (5a), respectively, it can easily be seen that any state \( |\psi\rangle \) has the following properties:

\[
P_\psi(-2 | \ominus \ominus) = 1, \quad (4a)
\]

\[
P_\psi(-1 | \ominus) = 1, \quad (4b)
\]

\[
P_\psi(-1, -2) = 0. \quad (4c)
\]

In addition, as can be easily checked,

\[
P_\psi(\ominus \ominus | \ominus) = \left( \frac{a-a^2}{1+a^2} \right)^2. \quad (4d)
\]

The proof will only run if \( a \neq 1 \) and \( a \neq 0 \), i.e., for entangled but not maximally entangled states (or Hardy states \( \| \) ). Hardy’s reasoning \( \| \) is as follows: Consider a run of the experiment for which \( B_1 \) and \( B_2 \) are measured and the results “\( \ominus \ominus \)” and “\( \ominus \ominus \)” are obtained. That this will happen sometimes follows from (4d). From the fact that we have “\( \ominus \ominus \)”, it follows from \( \| \) that if \( A_2 \) had been measured, we would have obtained the result “\( \ominus \)”. If we assume Einstein, Podolsky, and Rosen’s (EPR) condition for elements of reality \( \| \), then this prediction, with certainty and without disturbing the second particle, allows us to conclude that the second particle has an element of reality corresponding to the value “\( \ominus \)” for the observable \( A_2 \). By a similar argument, from property \( \| \), we conclude that the first particle has an element of reality corresponding to the value “\( \ominus \)” for the observable \( A_1 \). Hence, if we had measured \( A_1 \) and \( A_2 \), instead of \( B_1 \) and \( B_2 \), it follows from our assumptions that we would have obtained “\( \ominus \ominus \)” and “\( \ominus \ominus \)”. However, this contradicts \( \| \) if \( a \neq 1 \) and \( a \neq 0 \). Therefore, for a system initially prepared in a Hardy state, the assumption that local elements of reality exist is untenable.

**Goldstein’s version**

Goldstein’s version \( \| \) of Hardy’s proof is based on the same set of properties of the state \( |\psi\rangle \), arranged in a different way:

\[
P_\psi(-1, -2) = 0, \quad (5a)
\]

\[
P_\psi(\oplus | +) = 1, \quad (5b)
\]

\[
P_\psi(\ominus | +) = 1, \quad (5c)
\]

\[
P_\psi(\oplus, \ominus) = \left( \frac{a-a^2}{1+a^2} \right)^2. \quad (5d)
\]

Goldstein’s reasoning is as follows: Eq. (5a) tells us that: (G1) Either one or both of the results of measuring \( A_1 \) and \( A_2 \) must be “\( \ominus \)”. Eq. (5b) tells us that, if \( A_2 \) is “\( +\)”, then we can predict with certainty and without interacting with the other spatially separated particle, that the result “\( \ominus \ominus \)” will be found in a measurement of the observable \( B_1 \) on the first particle. Therefore, assuming EPR elements of reality, we may conclude that: (G2) If \( A_2 \) is “\( +\)”, then the first particle has an element of reality corresponding to the value “\( \ominus \ominus \)” for the observable \( B_2 \). It follows from (G1)-(G3) that: (G4) \( B_1 \) and \( B_2 \) cannot simultaneously be “\( \ominus \)”. However, (G4) contradicts the fact that state \( |\psi\rangle \) has, according to Eq. (5a), a nonvanishing probability for this to occur if \( a \neq 1 \) and \( a \neq 0 \). Therefore, for a system initially prepared in a Hardy state, the assumption that local elements of reality exist is untenable.

The probability of obtaining an event which contradicts local realism is given in both versions by \( P_\psi(\ominus \ominus, \ominus) \). This probability has a maximum,

\[
P_{\max} (\ominus \ominus, \ominus) = \left( \frac{\sqrt{a-a^2}}{a-a^2} \right)^3, \quad (6a)
\]

\[
\approx 0.0902, \quad (6b)
\]

for \( a = \left( \frac{\sqrt{a-a^2}}{a-a^2} \right)^{3/2} \).

**III. NONLOCALITY WITHOUT INEQUALITIES IN TWO-PARTICLE INTERFEROMETRY**

In Ref. \( \| \) WXHH claim to have demonstrated a violation of local realism without using inequalities for a maximally entangled state of a bipartite two-level system. In this Section, I will show that this is not so.

WXHH’s proof uses the two-particle interferometer illustrated in Fig. 1. This arrangement is a modification of the one proposed by Horne, Shimony, and Zellinger in Ref. \( \| \). In Fig. 1 the source \( S \) emits a pair of particles into four beams \( a, b, c, \) and \( d \). Each pair is in the state

\[
|\zeta\rangle = \frac{1}{\sqrt{2}} (|ab\rangle + |cd\rangle), \quad (7)
\]

where \( |ab\rangle \) means particle 1 in beam \( a \) and particle 2 in beam \( b \), etc. Any experiment on particle 1 is assumed to be spacelike separated from any experiment on particle 2. \( M_a \) and \( M_b \) are mirrors, \( \phi_1 \) and \( \phi_2 \) are phase shifters, \( BS_1, BS_2, BS_3, \) and \( BS_4 \) are beam splitters, and \( E, \)
$F$, $G$, $H$, $K$, and $L$ are detectors whose efficiencies are assumed to be 100%.

\[ \begin{array}{c}
E & M_a & \phi_1 & a & b & \phi_2 & M_b & G \\
| & & & & & & & \xleftarrow{\downarrow}\ \\
F & BS1 & K & S & BS2 & BS3 & BS4 & L
\end{array} \]

**FIG. 1:** Two-particle interferometer considered by Wu and co-workers in Ref. [4].

On particle 1 we can perform one of two alternative experiments, $A_1$ and $B_1$. Each of them corresponds to the choice of the phase introduced by the phase shifter $\phi_1$ and the reflectance and transmittance of the beam splitter $BS1$. Similarly, on particle 2 we can perform two alternative experiments, $A_2$ and $B_2$, each of them corresponding to the choice of the phase introduced by $\phi_2$ and the reflectance and transmittance of $BS2$. WXHH choose these parameters of the experiments $A_1$, $B_1$, $A_2$, and $B_2$ such that if one selects those runs of the experiments for which particle 1 does not end in detector $K$, while in the same run particle 2 does not end in detector $L$, then, for these selected runs,

\[
\begin{align}
P(A_2 = H | B_1 = F) &= 1, \quad \text{(8a)} \\
P(A_1 = F | B_2 = H) &= 1, \quad \text{(8b)} \\
P(A_1 = F, A_2 = H) &= 0, \quad \text{(8c)} \\
P(B_1 = F, B_2 = H) &= 0, \quad \text{(8d)}
\end{align}
\]

where $P(A_2 = H | B_1 = F)$ is the probability of that particle 2 being detected in $H$ when experiment $A_2$ is performed, conditioned to the occurrence of particle 1 being detected in $F$ when experiment $B_2$ is performed. It can be immediately seen that using properties (8a)-(8d) we can develop a Hardy-like proof. However, properties (8a)-(8d) are not properties of the maximally entangled state $|\psi\rangle$ but of the state “distilled” after the selection of events stated above. In Ref. [4], it is not clear whether this selection of events takes place before or after the experiments on particles 1 and 2. In any case, it is interesting to realize that WXHH’s conclusions do not change if the selection takes place before the experiments on particles 1 and 2. In this case, the arrangement considered by WXHH is equivalent to the two-particle interferometer considered by Horne, Shimony, and Zeilinger [5] (in which beam splitters $BS3$ and $BS4$ are replaced by mirrors, and detectors $K$ and $L$ are removed), assuming that the source emits pairs in state

\[
|\eta\rangle = P(|ab\rangle + Q|cd\rangle), \quad \text{with} \quad |Q| < 1, \quad \text{(9)}
\]

instead of in state $|\psi\rangle$. However, state $|\psi\rangle$ is not a maximally entangled state but a Hardy state. Therefore, I conclude that while WXHH’s proof of nonlocality is correct, it is not a proof for a maximally entangled state but for a Hardy state distilled from a maximally entangled state. Such a distillation is always possible by selecting a subset of events, since the degree of entanglement of the maximally entangled state $|\psi\rangle$ is higher than the degree of entanglement of the Hardy state $|\eta\rangle$.

**IV. NONLOCALITY WITHOUT INEQUALITIES USING UNAMBIGUOUS DISCRIMINATION BETWEEN NON-ORTHOGONAL STATES**

Any attempt to extend Hardy’s proof to cover maximally entangled states requires finding a subset of events, all of them referring to a maximally entangled state, so that the correlations exhibited by such subset cannot be reproduced by any local realistic theory. As becomes clear after our analysis of WXHH’s proof, this subset cannot be selected before the local measurements involved in the proof. Therefore, it is interesting to investigate what would happen in a set-up in which the selection of events necessarily occurs after the local measurements. A possible scenario which fulfills this requisite is the one in which one of the four von Neumann local measurements involved in the original proof is replaced by a generalized measurement which unambiguously discriminate between non-orthogonal states $|\phi\rangle$. This scenario was considered by Cheffes and Barnett for a different purpose: restoring local realism in the Goldstein’s version of Hardy’s proof [10]. In the following, I will demonstrate that it is impossible to develop a proof of nonlocality in this scenario, except in the particular case considered by Hardy, in which the generalized measurement discriminates between orthogonal states.

Consider the state $|\psi\rangle$ defined in Eq. (1), and the following change of basis for the states of the first particle:

\[
|\tilde{+}\rangle_1 = \cos \alpha |+\rangle_1 + \sin \alpha |-\rangle_1, \quad \text{(10a)}
\]

\[
|\tilde{-}\rangle_1 = -\sin \beta |+\rangle_1 + \cos \beta |-\rangle_1, \quad \text{(10b)}
\]

being $\alpha - \beta \neq \left(\frac{1}{2} + n\right)\pi$, with $n$ integer. Note that $\{ |\tilde{+}\rangle_1, |\tilde{-}\rangle_1 \}$ is a non-orthogonal basis since

\[
\langle \tilde{-} | \tilde{+} \rangle_1 = \sin (\alpha - \beta). \quad \text{(11)}
\]

Consider the following change of basis for the second particle:

\[
|\tilde{+}\rangle_2 = \cos \gamma |+\rangle_2 + \sin \gamma |-\rangle_2, \quad \text{(12a)}
\]

\[
|\tilde{-}\rangle_2 = -\sin \gamma |+\rangle_2 + \cos \gamma |-\rangle_2. \quad \text{(12b)}
\]

These changes of basis are illustrated in Fig. 2. The inverse transformations are:
\(|\rangle_1 = M \left( \cos \beta |\downarrow\rangle_1 - \sin \alpha |\uparrow\rangle \right),
\]
\(|\rangle_2 = M \left( \sin \beta |\downarrow\rangle_1 + \cos \alpha |\uparrow\rangle \right),
\]
where
\[M = \sec (\alpha - \beta),
\]
and
\[|\rangle_2 = \cos \gamma |\downarrow\rangle_2 - \sin \gamma |\uparrow\rangle_2,
\]
\[|\rangle_2 = \sin \gamma |\downarrow\rangle_2 + \cos \gamma |\uparrow\rangle_2.
\]
\{ |\downarrow\rangle_2, |\uparrow\rangle_2 \} is an orthonormal basis for the second particle. With the changes of basis given by Eqs. (13a), (15b), (15c), (15b), and choosing \(\gamma\) such that
\[\cot \gamma = \sqrt{2} \cot \theta - \cot \alpha,
\]
the state given by Eq. (4) can be rewritten as
\[|\psi\rangle = M \left( \left( q |\downarrow\rangle + r |\uparrow\rangle \right) + s |\uparrow\rangle \right),
\]
where
\[q = a \cos \beta \cos \gamma + b \sin (\beta - \gamma),
\]
\[r = -a \cos \beta \sin \gamma + b \cos (\beta - \gamma),
\]
\[s = -a \sin \alpha \cos \gamma + b \cos (\alpha + \gamma).
\]

Now consider an additional change of basis for the first particle:
\[|\hat{\phi}\rangle_1 = M \left[ \cos (\delta - \beta) |\downarrow\rangle + \sin (\delta - \alpha) |\uparrow\rangle \right],
\]
\[|\hat{\phi}\rangle_1 = M \left[ -\sin (\delta - \beta) |\downarrow\rangle + \cos (\delta - \alpha) |\uparrow\rangle \right].
\]
\{ |\hat{\phi}\rangle_1, |\hat{\phi}\rangle_1 \} is an orthonormal basis. The relation between this basis and the previous one is shown in Fig. 2. The inverse transformations are:
\[|\hat{+}\rangle_1 = \cos (\delta - \alpha) |\hat{\phi}\rangle_1 - \sin (\delta - \alpha) |\hat{-}\rangle_1,
\]
\[|\hat{-}\rangle_1 = \sin (\delta - \beta) |\hat{\phi}\rangle_1 + \cos (\delta - \beta) |\hat{+}\rangle_1.
\]
In this new basis, and choosing \(\delta\) such that,
\[\tan \delta = \frac{\sin \alpha + \frac{\pi}{2} \cos \beta}{\cos \alpha - \frac{\pi}{2} \sin \beta},
\]
the state \(|\psi\rangle\) has the form
\[|\psi\rangle = M \left( \left( q \cos (\delta - \alpha) + s \sin (\delta - \beta) \right) |\hat{+}\rangle \right.
\[+ r \cos (\delta - \alpha) |\hat{-}\rangle \right) - r \sin (\delta - \alpha) |\hat{-}\rangle \}.
\]

Now consider an additional change of basis for the second particle:
\[|\hat{\phi}\rangle_2 = \cos \epsilon |\hat{+}\rangle_2 + \sin \epsilon |\hat{-}\rangle_2,
\]
\[|\hat{\phi}\rangle_2 = -\sin \epsilon |\hat{+}\rangle_2 + \cos \epsilon |\hat{-}\rangle_2.
\]
\{ |\hat{\phi}\rangle_2, |\hat{\phi}\rangle_2 \} is an orthonormal basis, as illustrated in Fig. 2. The inverse transformations are:
\[|\hat{+}\rangle_2 = \cos \epsilon |\hat{\phi}\rangle_2 - \sin \epsilon |\hat{\phi}\rangle_2,
\]
\[|\hat{-}\rangle_2 = \sin \epsilon |\hat{\phi}\rangle_2 + \cos \epsilon |\hat{\phi}\rangle_2.
\]
In this new basis, and choosing \(\epsilon\) such that
\[\tan \epsilon = \frac{\sin \alpha}{\cos \alpha} = \frac{r}{q},
\]
the state \(|\psi\rangle\) has the form
\[|\psi\rangle = M \left[ \left( q \cos \epsilon + r \sin \epsilon \right) |\hat{+}\rangle + s \cos \epsilon |\hat{-}\rangle \right.
\[+ s \sin \epsilon |\hat{-}\rangle \}.
\]

In addition, as can be easily checked,
\[P_{\psi} (\hat{\phi}_1, \hat{\phi}_2) = \left| M s \sin \epsilon \cos (\delta - \beta) \right|^2 \]
\[= \left| M r \cos \epsilon \cos (\delta - \alpha) \right|^2.
\]

\(P_{\psi} (\hat{\phi}_1, \hat{\phi}_2)\) is only a function of \(\theta\) (the angle that characterizes the degree of entanglement of the state we are considering), and \(\alpha\) and \(\beta\) (the angles that characterize the type of basis —orthogonal or not— we are using to describe the state of the first particle). The angles \(\gamma\), \(\delta\), and \(\epsilon\) are fixed by, respectively, Eqs. (16), (25), and (27).

If \(\alpha - \beta = n \pi\), with \(n\) integer, the scalar product in Eq. (11) vanishes, and then we recover a standard Hardy’s
proof using orthogonal basis for each particle. In particular, if \( \alpha = \beta = 0 \), then \( P_\psi (\hat{\psi}_1, \hat{\psi}_2) \) gives the probability of obtaining an event which contradicts local realism given by Eqs. (3d) and (4d).

Now let me introduce some notations: Let \( \hat{A}_2 \) be the von Neumann measurement to discriminate between the orthogonal states of the second particle \( |+\rangle_2 \) and \( |-\rangle_2 \). The only possible results of measuring \( \hat{A}_2 \) are “+” and “−”. Analogously, let \( \hat{B}_1 (\hat{B}_2) \) be the von Neumann measurement which discriminates between the orthogonal states of particle 1 (2) \( |\hat{\psi}_1\rangle_1 (|\hat{\psi}_2\rangle_2) \) and \( |\hat{\phi}_1\rangle_1 (|\hat{\phi}_2\rangle_2) \). On the other hand, the states \( |+\rangle_1 \) and \( |-\rangle_1 \) are not orthogonal. To unambiguously discriminate between them, we define a positive operator valued measure \( [6–9] \).

Then, the possible results of measuring \( \hat{A}_1 \) are “+1”, “−1”, or an inconclusive result “⊕”. Hardy’s proof is based on four incompatible experiments. As seen in Sec. II, three of them are used to make predictions with certainty, to define, via EPR’s condition, certain elements of reality that cannot be reconciled with some results of the fourth experiment. Then the proof only applies to some runs of the fourth experiment. In the following, I will refer to those events as “events for which local realism leads to a contradiction”. On the other hand, the presence of a generalized measurement introduces a new element in our analysis. In particular, the possibility of an inconclusive result implies that Hardy’s (or Goldstein’s) reasoning cannot be applied to a certain subset of events. I will refer to those events as “events for which the proof cannot be applied to”. In fact, these subsets of events are different in Hardy’s and Goldstein’s versions of the proof.

Hardy-like reasoning

If one selects all runs of the experiment except those in which the result of measuring \( \hat{A}_1 \) is inconclusive and the result of measuring \( \hat{B}_2 \) is “−2”, then, for these selected runs,

\[
P (\hat{\psi}_2 | \hat{\psi}_1) = 1, \quad P (\hat{\psi}_1 | \hat{\psi}_2) = 1, \quad P (\hat{\psi}_1, \hat{\psi}_2) > 0. \tag{30d}
\]

Property (30d) only occurs for certain combinations of \( \theta, \alpha, \) and \( \beta \). Therefore, for these selected runs a Hardy-like reasoning like the one in Sec. II can be applied. Hardy’s reasoning cannot be applied to those events in which the result of measuring \( \hat{A}_1 \) is inconclusive and the result of measuring \( \hat{B}_2 \) is “−2”. Note, however, that Hardy’s reasoning still applies if the result of measuring \( \hat{A}_1 \) is inconclusive and the result of measuring \( \hat{B}_2 \) is “−2”.

Goldstein-like reasoning

If one selects all runs of the experiment except those in which the result of measuring \( \hat{A}_1 \) is inconclusive and the result of measuring \( \hat{A}_2 \) is “−2”, then, for these selected runs,

\[
P (\hat{\psi}_1, \hat{\psi}_2) > P_\psi \left( \hat{\psi}_1, \hat{\psi}_2 \right). \tag{31d}
\]

Property (31d) only occurs for certain combinations of \( \theta, \alpha, \) and \( \beta \). Therefore, for these selected runs a Goldstein-like reasoning like the one in Sec. II can be applied. Goldstein’s reasoning cannot be applied to those events in which the result of measuring \( \hat{A}_1 \) is inconclusive and the result of measuring \( \hat{A}_2 \) is “−2”. Note that Goldstein’s reasoning still goes through if the result of \( \hat{A}_1 \) is inconclusive and the result of \( \hat{A}_2 \) is “−2”.

Discussion

In contrast to WXHH’s set up, in the scenario examined in this Section, the selection of events can only take place after the local experiments on particles 1 and 2. This raises the new problem of whether this postselection is legitimate in a proof of nonlocality. The only way to develop such proof, without making any additional assumptions, would be to show that, considering all runs of the experiment, the probability of obtaining an event for which local realism leads to a contradiction using a Hardy-like (or a Goldstein-like) reasoning is greater than the probability of obtaining an event for which the reasoning cannot be applied. In both versions of the proof, the probability of obtaining an event for which local realism leads to a contradiction is \( P_\psi (\hat{\psi}_1, \hat{\psi}_2) \). However, the probability of finding an event which the proof cannot be applied to is different for each version. Hardy’s reasoning cannot be applied to those events in which the result of measuring \( \hat{A}_1 \) is inconclusive and the result of measuring \( \hat{B}_2 \) is “−2”. Therefore, we can prove the impossibility of local realism using Hardy’s reasoning if

\[
P_\psi (\hat{\psi}_1, \hat{\psi}_2) > P_\psi (\hat{\psi}_1, \hat{\psi}_2). \tag{32}
\]

On the other hand, Goldstein’s reasoning cannot be applied to those events in which the result of measuring \( \hat{A}_1 \) is inconclusive and the result of measuring \( \hat{A}_2 \) is “−2”. Therefore, we can prove the impossibility of local realism using Goldstein’s reasoning if

\[
P_\psi (\hat{\psi}_1, \hat{\psi}_2) > P_\psi (\hat{\psi}_1, \hat{\psi}_2). \tag{33}
\]

Therefore, to elucidate whether a proof of nonlocality is possible, we have to find out whether Eqs. (32) and
are satisfied. For this purpose we shall use the result obtained for $P_\psi (\hat{\gamma}_1, \hat{\gamma}_2)$ in Eq. (29a) or Eq. (29b). On the other hand, $P_\psi (\hat{\gamma}_1, \hat{\gamma}_2)$ can be calculated as

$$P_\psi (\hat{\gamma}_1, \hat{\gamma}_2) = \text{Tr} \left[ (\hat{O}_1 \otimes |\hat{\gamma}_2\rangle \langle \hat{\gamma}_2|) |\psi\rangle \langle \psi| \right],$$

(34)

where

$$\hat{O}_1 = \mathbb{I} - 2\mathbb{I} - |\hat{\gamma}_1\rangle \langle \hat{\gamma}_1| + |\hat{\gamma}_1\rangle \langle \hat{\gamma}_1|,$$

(35)

is a positive operator associated with the inconclusive answer which belongs to a positive operator valued measure $\mathfrak{P}$. An alternative way to calculate $P_\psi (\hat{\gamma}_1, \hat{\gamma}_2)$ is:

$$P_\psi (\hat{\gamma}_1, \hat{\gamma}_2) = P_\psi (\hat{\gamma}_1, \hat{\gamma}_2) - P_\psi (\hat{\gamma}_1, \hat{\gamma}_2) - P_\psi (\hat{\gamma}_1, \hat{\gamma}_2),$$

(36)

where $P_\psi (\hat{\gamma}_1, \hat{\gamma}_2)$ is zero according to Eq. (17), and

$$P_\psi (\hat{\gamma}_1, \hat{\gamma}_2) = P_\psi (\hat{\gamma}_1, \hat{\gamma}_2) + P_\psi (\hat{\gamma}_1, \hat{\gamma}_2),$$

(37a)

$$= (-a \sin \beta + b \cos \beta)^2 + (b \sin \beta)^2.$$

(37b)

$P_\psi (\hat{\gamma}_1, \hat{\gamma}_2)$ is the probability to unambiguously discriminate between the states $|\hat{\gamma}_1\rangle$ and $|\hat{\gamma}_1\rangle$ (given by $1 - |\langle \hat{\gamma}_1 | \hat{\gamma}_1\rangle|$), times the probability to obtain “$\hat{\gamma}_1$, $\hat{\gamma}_2$” when the discrimination succeeds,

$$P_\psi (\hat{\gamma}_1, \hat{\gamma}_2) = [1 - |\sin (\alpha - \beta)|] (Mr)^2.$$  

(38)

Analogously, $P_\psi (\hat{\gamma}_1, \hat{\gamma}_2)$ can be calculated as

$$P_\psi (\hat{\gamma}_1, \hat{\gamma}_2) = \text{Tr} \left[ (\hat{O}_2 \otimes |\hat{\gamma}_2\rangle \langle \hat{\gamma}_2|) |\psi\rangle \langle \psi| \right],$$

(39)

where $\hat{O}_2$ is the positive operator defined in Eq. (39). As before, an alternative way to calculate $P_\psi (\hat{\gamma}_1, \hat{\gamma}_2)$ would be as follows:

$$P_\psi (\hat{\gamma}_1, \hat{\gamma}_2) = P_\psi (\hat{\gamma}_2) + P_\psi (\hat{\gamma}_2) + P_\psi (\hat{\gamma}_2),$$

(40)

where $P_\psi (\hat{\gamma}_1, \hat{\gamma}_2)$ is zero according to Eq. (28), and

$$P_\psi (\hat{\gamma}_2) = P_\psi (\hat{\gamma}_2) + P_\psi (\hat{\gamma}_2),$$

(41a)

$$= [-a \sin (\gamma + \epsilon) + b \cos (\gamma + \epsilon)]^2 + [b \sin (\gamma + \epsilon)]^2.$$  

(41b)

$P_\psi (\hat{\gamma}_1, \hat{\gamma}_2)$ is the probability to unambiguously discriminate between the states $|\hat{\gamma}_1\rangle$ and $|\hat{\gamma}_1\rangle$, times the probability to obtain “$\hat{\gamma}_1$, $\hat{\gamma}_2$” when the discrimination succeeds,

$$P_\psi (\hat{\gamma}_1, \hat{\gamma}_2) = [1 - |\sin (\alpha - \beta)|] (Ms \sin \epsilon)^2.$$  

(42)

As can be checked, in the limit in which we recover Hardy’s proof (i.e., if $\alpha - \beta = n\pi$, with $n$ integer) both $P_\psi (\hat{\gamma}_1, \hat{\gamma}_2)$ and $P_\psi (\hat{\gamma}_1, \hat{\gamma}_2)$ are zero. However, a detailed numerical examination reveals that for every $\theta, \alpha$ or $\beta$, Eqs. (32) and (33) are never satisfied. Therefore, assuming that the argument developed in this section is the most comprehensive based on the idea of replacing a von Neumann measurement with a measurement which discriminates between non-orthogonal states, I conclude that no proof of Bell’s theorem without inequalities based on such idea can work, except in the particular case considered by Hardy, in which the generalized measurement discriminates between orthogonal states. In particular, no proof of nonlocality without inequalities for maximally entangled states of bipartite two-level systems can be developed in this scenario.

V. CONCLUSIONS

There is a proof of nonlocality without inequalities for bipartite three-level maximally entangled states [17]. However, so far, no attempt to extend Hardy’s proof to bipartite two-level maximally entangled states works. In the proof by Wu and co-workers [4], the source emits a maximally entangled state. However, the state after the selection is, as in Hardy’s proof, entangled but nonmaximally entangled. On the other hand, it has been proved that it is impossible to generalize Hardy’s proof by replacing one of the four von Neumann measurements with a measurement to unambiguously discriminate between non-orthogonal states. Therefore, neither this scenario can be used to extend the proof to maximally entangled states.

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