Second Hankel Determinant for Certain Subclass of Bi-Univalent Functions

Safa Salehian\textsuperscript{a}, Ahmad Motamednezhad\textsuperscript{b}

\textsuperscript{a}Department of Mathematics, Gorgan Branch, Islamic Azad University, Gorgan, Iran
\textsuperscript{b}Faculty of Mathematical Sciences, Shahrood University of Technology, P.O.Box 316-36155, Shahrood, Iran

Abstract. The main purpose of this paper is to obtain an upper bound for the second Hankel determinant for functions belonging to a subclass of bi-univalent functions in the open unit disk in the complex plane. Furthermore, the presented results in this work improve or generalize the recent works of other authors.

1. Introduction

Let $A$ be the class of analytic functions $f$ defined on the unit disk
\[ U = \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \} \]
of the form
\[ f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \]  

Let $S$ be the subclass of functions in $A$ that are univalent in $U$. For two functions $f$ and $g$, analytic in $U$, we say that the function $f$ is subordinate to $g$ in $U$ and we write it as $f(z) \prec g(z)$ if there exists a Schwartz function $w$, which is analytic in $U$ with $w(0) = 0$, $|w(z)| < 1$ ($z \in U$) such that
\[ f(z) = g(w(z)), \quad z \in U. \]

Indeed, it is known that $f(z) \prec g(z) \iff f(0) = g(0)$ and $f(U) \subset g(U)$.

In particular, if the function $g$ is univalent in $U$, then $f(z) \prec g(z) \iff f(0) = g(0)$ and $f(U) \subset g(U)$, [7].

The well-known Koebe One-Quarter Theorem established that the image of $U$ under every univalent function $f \in S$ contains a disk of radius $\frac{1}{4}$. Thus every univalent function $f$ has an inverse $f^{-1}$ satisfying
\[ f^{-1}(f(z)) = z \quad (z \in U) \quad \text{and} \quad f(f^{-1}(w)) = w \quad (|w| < r_0(f), \quad r_0(f) \geq \frac{1}{4}), \]

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Corresponding author: Safa Salehian

Email addresses: s.salehian84@gmail.com (Safa Salehian), a.motamedne@gmail.com (Ahmad Motamednezhad)
where
\[
g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} A_n w^n
= w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots.
\]

A function \( f \in \mathcal{A} \) is said to be bi-univalent in \( \mathbb{U} \) if both \( f \) and \( f^{-1} \) are univalent in \( \mathbb{U} \).

Let \( \Sigma \) denote the class of bi-univalent functions in \( \mathbb{U} \) given by (1). For a brief history and interesting examples of functions in the class \( \Sigma \), see [24]. In 1967, Lewin [14] introduced the concept of bi-univalent analytic functions and proved that the second coefficient satisfies \( |a_2| < 1.51 \). In the following, various subclasses of the bi-univalent functions were introduced and the first two coefficients \( |a_2|, |a_3| \) in the Taylor-Maclaurin series expansion [1, 4, 9, 16, 21–25] were estimated. But the coefficient problem for each of the following Taylor-Maclaurin coefficients

\[
|a_n| \quad (n \in \mathbb{N} - \{1, 2\}; \mathbb{N} = \{1, 2, 3, \ldots \})
\]

is still an open problem (see [15, 20]).

The \( q^{th} \) determinant for \( q \geq 1 \) and \( n \geq 0 \) is stated by Noonan and Thomas [18] as below.

\[
H_q(n) = \begin{vmatrix}
    a_0 & a_1 & \cdots & a_{n+1} \\
    a_1 & a_2 & \cdots & a_{n+2} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{n+1} & a_{n+2} & \cdots & a_{2n+2}
\end{vmatrix}
\quad (a_1 = 1).
\]

This determinant has also been considered by several authors. For example, Ehrenborg [8] studied the Hankel determinant of exponential polynomials. The Hankel transform of an integer sequence and some of its properties were discussed by Layman [13].

Note that

\[
H_2(2) = \begin{vmatrix}
    a_2 & a_3 \\
    a_3 & a_4
\end{vmatrix}
\quad \text{and} \quad H_2(1) = \begin{vmatrix}
    a_1 & a_2 \\
    a_2 & a_3
\end{vmatrix},
\]

where the Hankel determinants \( H_2(1) = a_3 - a_2^2 \) and \( H_2(2) = a_2a_4 - a_3^2 \) are well-known as Fekete-Szegö and second Hankel determinant functionals, respectively.

Recently, Çağlar et al. [2, 3], Qadeem et al. [5], Deniz et al. [6], Kanas et al. [11] and Orhan et al. [19] obtained the upper bound for the functional \( H_2(2) = a_2a_4 - a_3^2 \) for the subclasses of bi-univalent functions.

In this work, we assume that the function \( \varphi \) is an analytic function with positive real part in the unit disk \( \mathbb{U} \), satisfying \( \varphi(0) = 1, \varphi'(0) > 0 \), such that \( \varphi'(\mathbb{U}) \) is symmetric with respect to the real axis. Such a function has the power series expansion of the form

\[
\varphi(z) = 1 + B_2z + B_2z^2 + B_3z^3 + \cdots, \quad (B_1 > 0).
\]

Recently, El-Qadeem and Mamon [5] defined the subclass \( \mathcal{H}_2(\tau, \lambda, \delta; \varphi) \) of bi-univalent functions and obtained upper bound of the second Hankel determinant for functions in this class.

In this paper, we improve the estimates of second Hankel determinant which obtained by El-Qadeem and Mamon [5], Çağlar et al. [2], Murugusundaramoorthy and Vijaya [17] and Khani et al. [12].

1.1. Preliminaries

Lemma 1.1. [11] If \( w(z) = \sum_{n=1}^{\infty} w_n z^n, z \in \mathbb{U} \), is a Schwarz function, then

\[
w_2 = h(1 - w_1^2)
\]

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and
\[ w_3 = (1 - w_1^2)(1 - |h|^2)s - w_1(1 - w_1^2)h^2, \]
for some \( h, s \), with \(|h| \leq 1 \) and \(|s| \leq 1 \).

**Lemma 1.2.** [7] Let \( v \) be analytic function in the unit disk \( U \), with \( v(0) = 0 \), and \(|v(z)| < 1 \) for all \( z \in U \), with the power series expansion
\[ v(z) = \sum_{n=1}^{\infty} c_n z^n. \]
Then, \(|c_n| \leq 1\) for all \( n \in \mathbb{N} \). Furthermore, \(|c_n| = 1\) for some \( n \in \mathbb{N} \) if and only if \( v(z) = e^{\theta z^n}, \theta \in \mathbb{R} \).

2. Coefficient bounds

**Definition 2.1.** [5] A function \( f \in \Sigma \) given by (1) is said to be in the class \( \mathcal{H}_\Sigma(\tau, \lambda, \delta; \varphi) \) \((\lambda \geq 1, \delta \geq 0, \tau \in \mathbb{C} \setminus \{0\})\), if the following conditions are satisfied:
\[ 1 + \frac{1}{\tau} \left( 1 - \lambda \frac{f(z)}{z} + \lambda f'(z) + \delta z f''(z) - 1 \right) < \varphi(z), \quad (z \in U) \]
and
\[ 1 + \frac{1}{\tau} \left( 1 - \lambda \frac{g(w)}{w} + \lambda g'(w) + \delta wg''(w) - 1 \right) < \varphi(w), \quad (w \in U), \]
where the function \( g \) is the inverse of \( f \) given by (2).

**Remark 2.2.** For special choices of the parameters \( \lambda, \tau, \delta \) and the function \( \varphi \), we can obtain the following classes as below.

(I) By putting \( \tau = 1, \delta = 0 \) and \( \varphi(z) = \frac{1+(1-2\beta)z}{1-z} \) \((0 \leq \beta < 1)\), we have
\[ \mathcal{H}_\Sigma(1, \lambda, 0; \frac{1+(1-2\beta)z}{1-z}) = \mathcal{F}_\Sigma(\beta, \lambda), \]
where the bi-univalent function class consists of functions \( f \) satisfying the following conditions:
\[ \Re((1 - \lambda)\frac{f(z)}{z} + \lambda f'(z)) > \beta \quad \text{and} \quad \Re((1 - \lambda)\frac{g(w)}{w} + \lambda g'(w)) > \beta. \]
The bi-univalent function class \( \mathcal{F}_\Sigma(\beta, \lambda) \) was studied by Frasin and Aouf [10].

(II) By putting \( \tau = \lambda = 1 \) and \( \varphi(z) = \frac{1+(1-2\beta)z}{1-z} \) \((0 \leq \beta < 1)\), we have
\[ \mathcal{H}_\Sigma(1, 1, \delta; \frac{1+(1-2\beta)z}{1-z}) = \mathcal{H}_\Sigma(\delta, \beta), \]
where the bi-univalent function class consists of functions \( f \) satisfying the following conditions:
\[ \Re(f'(z) + \delta z f''(z)) > \beta \quad \text{and} \quad \Re(g'(w) + \delta wg''(w)) > \beta. \]
The bi-univalent function class \( \mathcal{H}_\Sigma(\delta, \beta) \) was studied by Frasin [9].
(III) By putting $\tau = \lambda = 1, \delta = 0$ and $\varphi(z) = \frac{1+2\beta}{1-z}$ ($0 \leq \beta < 1$), we have

$$\mathcal{H}_c(1, 1, 0; \frac{1+(1-2\beta)z}{1-z}) = \mathcal{N}_c(\beta),$$

where the bi-univalent function class consists of functions $f$ satisfying the following conditions:

$$\Re(f'(z)) > \beta \quad \text{and} \quad \Re(f'(\omega)) > \beta.$$

The bi-univalent function class $\mathcal{N}_c(\beta)$ was studied by Srivastava et al. [24].

(IV) By putting $\tau = \lambda = 1, \delta = 0$ and $\varphi(z) = \left(\frac{1+z}{1-z}\right)^{\alpha}$ ($0 < \alpha \leq 1$), we have

$$\mathcal{H}_c(1, 1, 0; \left(\frac{1+z}{1-z}\right)^{\alpha}) = \mathcal{N}_c^\alpha,$$

where the bi-univalent function class consists of functions $f$ satisfying the following conditions:

$$\left|\arg(f'(z))\right| < \frac{\alpha\pi}{2} \quad \text{and} \quad \left|\arg(f'(\omega))\right| < \frac{\alpha\pi}{2}.$$

The bi-univalent function class $\mathcal{N}_c^\alpha$ was studied by Srivastava et al. [24].

**Theorem 2.3.** If $f \in \Sigma$ of the form (1) belongs to the class $\mathcal{H}_c(\tau, \lambda, \delta; \varphi)$, then

$$|a_2a_4 - a_3^2| \leq B_1|\tau|^2 \left\{ \begin{array}{ll}
\frac{B_1}{(1+\tau^2+6\delta)}; & T \leq 0, S \leq -T \\
\left|\frac{B_3}{(1+\lambda+2\delta)(1+3\lambda+12\delta)} - \frac{\tau^2B_1^2}{(1+\lambda+2\delta)^2}\right|; & (T \geq 0, S \geq -\frac{T}{2}) \text{ or } (T \leq 0, S \geq -T) \\
\frac{4S(U-T^2)}{4S}; & T > 0, S \leq -\frac{T}{2}
\end{array} \right.$$ 

where

$$S = \left|\frac{B_3}{(1+\lambda+2\delta)(1+3\lambda+12\delta)} - \frac{\tau^2B_1^2}{(1+\lambda+2\delta)^2}\right| - \frac{|\tau|B_1^2}{2(1+\lambda+2\delta)^2(1+2\lambda+6\delta)} - \frac{2|B_2|+B_1}{(1+\lambda+2\delta)(1+3\lambda+12\delta)} + \frac{B_1}{(1+2\lambda+6\delta)^2},$$

$$T = \frac{|\tau|B_1^2}{2(1+\lambda+2\delta)^2(1+2\lambda+6\delta)} + \frac{2|B_2|+B_1}{(1+\lambda+2\delta)(1+3\lambda+12\delta)} - \frac{2B_1}{(1+2\lambda+6\delta)^2}$$

and

$$U = \frac{B_1}{(1+2\lambda+6\delta)^2}.$$

**Proof.** Since $f \in \mathcal{H}_c(\tau, \lambda, \delta; \varphi)$, there exist two Schwartz functions $u, v$ in $\mathcal{U}$, of the form $u(z) = \sum_{n=1}^{\infty} c_n z^n$ and $v(z) = \sum_{n=1}^{\infty} d_n w^n$, with $u(0) = 0, v(0) = 0$ and $|u(z)| < 1, |v(w)| < 1$ such that

$$1 + \frac{1}{\tau} \left(1 - \lambda \right) \frac{f(z)}{z} + \lambda f'(z) + \delta f''(z) - 1 = \varphi(u(z))$$

(3)
and

\[ 1 + \frac{1}{\tau} \left( (1 - \lambda) \frac{g(w)}{w} + \lambda g'(w) + \delta wg''(w) - 1 \right) = \varphi(v(w)), \quad (4) \]

where

\[ \varphi(u(z)) = 1 + B_1 c_1 z + (B_1 c_2 + B_2 c_1^2)z^2 + (B_1 c_3 + 2B_2 c_1 c_2 + B_3 c_1^3)z^3 + \cdots \quad (5) \]

and

\[ \varphi(v(w)) = 1 + B_1 d_1 w + (B_1 d_2 + B_2 d_1^2)w^2 + (B_1 d_3 + 2B_2 d_1 d_2 + B_3 d_1^3)w^3 + \cdots. \quad (6) \]

Since \( f \in \Sigma \) has the Taylor series expansion (1) and \( g = f^{-1} \) the series (2), we have

\[ 1 + \frac{1}{\tau} \left( (1 - \lambda) \frac{f(z)}{z} + \lambda f'(z) + \delta zf''(z) - 1 \right) = 1 + \sum_{n=2}^{\infty} \left( \frac{1 + (n - 1)(\lambda + n\delta)}{\tau} \right) a_n z^n \quad (7) \]

and

\[ 1 + \frac{1}{\tau} \left( (1 - \lambda) \frac{g(w)}{w} + \lambda g'(w) + \delta wg''(w) - 1 \right) = 1 + \sum_{n=2}^{\infty} \left( \frac{1 + (n - 1)(\lambda + n\delta)}{\tau} \right) A_n w^n. \quad (8) \]

Now, from (3), (5) and (7), we get

\[ \frac{(1 + \lambda + 2\delta)}{\tau} d_2 = B_1 c_1, \quad (9) \]

\[ \frac{(1 + 2\lambda + 6\delta)}{\tau} d_3 = B_1 c_2 + B_2 c_1^2, \quad (10) \]

\[ \frac{(1 + 3\lambda + 12\delta)}{\tau} d_4 = B_1 c_3 + 2B_2 c_1 c_2 + B_3 c_1^3. \quad (11) \]

Similarly, from (4), (6) and (8), we have

\[ -\frac{(1 + \lambda + 2\delta)}{\tau} a_2 = B_1 d_1, \quad (12) \]

\[ \frac{(1 + 2\lambda + 6\delta)}{\tau} (2a_2^2 - a_3) = B_1 d_2 + B_2 d_1^2, \quad (13) \]

\[ -\frac{(1 + 3\lambda + 12\delta)}{\tau} (5a_2^3 - 5a_3 + a_4) = B_1 d_3 + 2B_2 d_1 d_2 + B_3 d_1^3. \quad (14) \]

From (9) and (12), we have

\[ c_1 = -d_1 \quad (15) \]

and

\[ a_2 = \frac{B_1 c_1 \tau}{1 + \lambda + 2\delta}. \quad (16) \]

Now from (10) and (13), we get that

\[ d_3 = \frac{B_1^2 c_1^2 \tau^2}{(1 + \lambda + 2\delta)^2} + \frac{B_1 \tau (c_2 - d_2)}{2(1 + 2\lambda + 6\delta)} \quad (17) \]

Also from (11) and (14), we get that

\[ a_4 = \frac{5B_1^2 c_1^2 \tau^2 (c_2 - d_2)}{4(1 + \lambda + 2\delta)(1 + 2\lambda + 6\delta)} + \frac{B_1 \tau (c_3 - d_3)}{2(1 + 3\lambda + 12\delta)} + \frac{B_3 c_1^2 \tau}{1 + 3\lambda + 12\delta} + \frac{B_2 c_1 \tau (c_2 + d_2)}{1 + 3\lambda + 12\delta}. \quad (18) \]
Thus we can easily obtain that

$$a_2 a_4 - a_3^2 = \left( \frac{B_1 B_3 \tau^2}{(1 + \lambda + 2 \delta)(1 + 3 \lambda + 12 \delta)} - \frac{B_1^4 \tau^4}{(1 + \lambda + 2 \delta)^4} \right) c_1^4$$

$$+ \frac{B_1^2 c_1^2 \tau^2(c_2 - d_2)}{4(1 + \lambda + 2 \delta)(1 + 2 \lambda + 6 \delta)} + \frac{B_1 B_2 c_1^2 \tau^2(1 - c_1^2)}{(1 + \lambda + 2 \delta)(1 + 3 \lambda + 12 \delta)}$$

$$+ \frac{B_2^2 c_1 \tau^2 c_3 - d_3}{2(1 + \lambda + 2 \delta)(1 + 3 \lambda + 12 \delta)} - \frac{B_2^2 \tau^2(c_2 - d_2)^2}{4(1 + 2 \lambda + 6 \delta)^2}. \quad (19)$$

According to Lemma 1.1, we have

$$c_2 = h(1 - c_1^2), \quad d_2 = j(1 - d_1^2),$$

$$c_3 = (1 - c_1^2)(1 - |h|^2)s - c_1(1 - c_1^2)h^2,$$

$$d_3 = (1 - d_1^2)(1 - |j|^2)w - d_1(1 - d_1^2)j^2, \quad (20)$$

$$\text{for some } h, j, s, w \text{ with } |h| \leq 1, |j| \leq 1, |s| \leq 1 \text{ and } |w| \leq 1.$$ 

Hence by (20), (21) and (22), we have

$$c_2 + d_2 = (1 - c_1^2)(h + j), \quad c_2 - d_2 = (1 - c_1^2)(h - j),$$

$$c_3 - d_3 = (1 - c_1^2) \left( (1 - |h|^2)s - (1 - |j|^2)w - c_1(h^2 + j^2) \right). \quad (23)$$

By substituting the relations (23), (24) in (19), we obtain

$$a_2 a_4 - a_3^2 = \left( \frac{B_1 B_3 \tau^2}{(1 + \lambda + 2 \delta)(1 + 3 \lambda + 12 \delta)} - \frac{B_1^4 \tau^4}{(1 + \lambda + 2 \delta)^4} \right) c_1^4$$

$$+ \frac{B_1^2 c_1^2 \tau^2(c_2 - d_2)}{4(1 + \lambda + 2 \delta)(1 + 2 \lambda + 6 \delta)} + \frac{B_1 B_2 c_1^2 \tau^2(1 - c_1^2)}{(1 + \lambda + 2 \delta)(1 + 3 \lambda + 12 \delta)}$$

$$+ \frac{B_2^2 c_1 \tau^2(c_2 - d_2)(1 - c_1^2)^2}{2(1 + \lambda + 2 \delta)(1 + 3 \lambda + 12 \delta)} - \frac{B_2^2 \tau^2(c_2 - d_2)(1 - c_1^2)^2}{4(1 + 2 \lambda + 6 \delta)^2}. \quad (25)$$

It follows that

$$|a_2 a_4 - a_3^2| = \left| \left( \frac{B_1 B_3 \tau^2}{(1 + \lambda + 2 \delta)(1 + 3 \lambda + 12 \delta)} - \frac{B_1^4 \tau^4}{(1 + \lambda + 2 \delta)^4} \right) c_1^4$$

$$+ \frac{B_1^2 c_1^2 \tau^2(c_2 - d_2)(1 - c_1^2)}{4(1 + \lambda + 2 \delta)(1 + 2 \lambda + 6 \delta)} + \frac{B_1 B_2 c_1^2 \tau^2(1 - c_1^2)}{(1 + \lambda + 2 \delta)(1 + 3 \lambda + 12 \delta)}$$

$$+ \frac{B_2^2 c_1 \tau^2(c_2 - d_2)(1 - c_1^2)(1 - |h|^2)s - (1 - |j|^2)w - c_1(h^2 + j^2)}{2(1 + \lambda + 2 \delta)(1 + 3 \lambda + 12 \delta)}$$

$$- \frac{B_2^2 \tau^2(c_2 - d_2)(1 - c_1^2)(h^2 + j^2)}{4(1 + 2 \lambda + 6 \delta)^2}. \quad (26)$$
Thus, we have

\[
|a_2a_4 - a_3^2| \leq B_1|\tau|^2 \left\{ \frac{B_3}{(1 + \lambda + 2\delta)(1 + 3\lambda + 12\delta)} - \frac{B_1^3\tau^2}{(1 + \lambda + 2\delta)^4} \right\} \left(1 + \lambda + 2\delta\right)^2(1 + 2\lambda + 6\delta) + \frac{|B_3|\tau^2}{(1 + \lambda + 2\delta)(1 + 3\lambda + 12\delta)} \left(1 - \lambda^2\right)(|\lambda| + |\mu_i|) + \frac{B_1c(1 - \lambda^2)(1 - |\lambda|^2)(|\lambda| + |\lambda|^2)}{2(1 + \lambda + 2\delta)(1 + 3\lambda + 12\delta)} + \frac{B_1c(1 - \lambda^2)^2(|\lambda| + |\lambda|^2)}{4(1 + 2\lambda + 6\delta)^2} \right). \tag{27}
\]

Since \(|s| \leq 1\) and \(|w| \leq 1\), we get

\[
|a_2a_4 - a_3^2| \leq B_1|\tau|^2 \left\{ \frac{B_3}{(1 + \lambda + 2\delta)(1 + 3\lambda + 12\delta)} - \frac{B_1^3\tau^2}{(1 + \lambda + 2\delta)^4} \right\} \left(1 + \lambda + 2\delta\right)^2(1 + 2\lambda + 6\delta) + \frac{|B_3|\tau^2}{(1 + \lambda + 2\delta)(1 + 3\lambda + 12\delta)} \left(1 - \lambda^2\right)(|\lambda| + |\mu_i|) + \frac{B_1c(1 - \lambda^2)(1 - |\lambda|^2)(|\lambda| + |\lambda|^2)}{2(1 + \lambda + 2\delta)(1 + 3\lambda + 12\delta)} + \frac{B_1c(1 - \lambda^2)^2(|\lambda| + |\lambda|^2)}{4(1 + 2\lambda + 6\delta)^2} \right) \tag{28}
\]

Now, for \(\mu = |\lambda| \leq 1\) and \(\gamma = |\mu_i| \leq 1\), we get

\[
|a_2a_4 - a_3^2| \leq B_1|\tau|^2 \left[ T_1 + (\mu + \gamma)T_2 + (\mu^2 + \gamma^2)T_3 + (\mu + \gamma)^2T_4 \right] = B_1|\tau|^2F(\mu, \gamma), \tag{29}
\]

where

\[
T_1 = T_1(c) = \frac{B_3}{(1 + \lambda + 2\delta)(1 + 3\lambda + 12\delta)} - \frac{B_1^3\tau^2}{(1 + \lambda + 2\delta)^4} \left(1 + \lambda + 2\delta\right)^2(1 + 2\lambda + 6\delta) + \frac{B_1c(1 - \lambda^2)}{(1 + \lambda + 2\delta)(1 + 3\lambda + 12\delta)} \geq 0
\]

\[
T_2 = T_2(c) = c^2 \left(1 - \lambda^2\right) \left(\frac{|\tau|B_3^2}{(1 + \lambda + 2\delta)^2(1 + 2\lambda + 6\delta)} + \frac{|B_2|}{(1 + 3\lambda + 12\delta)} \right) \geq 0
\]

\[
T_3 = T_3(c) = \frac{B_1c(1 - \lambda^2)}{2(1 + \lambda + 2\delta)(1 + 3\lambda + 12\delta)} \leq 0
\]

\[
T_4 = T_4(c) = \frac{B_1(1 - \lambda^2)^2}{4(1 + 2\lambda + 6\delta)^2} \leq 0
\]

Now, we need to maximize \(F(\mu, \gamma)\) in the closed square \(S = [0, 1] \times [0, 1]\) for \(c \in [0, 1]\). We must investigate the maximum of \(F(\mu, \gamma)\) according to \(c = 0, c = 1, \) and \(c \in (0, 1)\) taking into account the sign of \(F_{\mu\mu}F_{\gamma\gamma} - (F_{\mu\gamma})^2\).

First, for \(c = 0\), we have

\[
F(\mu, \gamma) = \frac{B_1(\mu + \gamma)^2}{4(1 + 2\lambda + 6\delta)^2}. \tag{30}
\]

Thus, we have

\[
\max \{F(\mu, \gamma) : (\mu, \gamma) \in [0, 1] \times [0, 1]\} = F(1, 1) = \frac{B_1}{(1 + 2\lambda + 6\delta)^2}. \tag{31}
\]
Second, for $c = 1$, we have
\[ F(\mu, \gamma) = \left| \frac{B_3}{(1 + \lambda + 2\delta)(1 + 3\lambda + 12\delta)} - \frac{B_1^2\gamma^2}{(1 + \lambda + 2\delta)^2} \right|. \] (32)

Thus, we get
\[ \max \{ F(\mu, \gamma) : (\mu, \gamma) \in [0, 1] \times [0, 1] \} = \left| \frac{B_3}{(1 + \lambda + 2\delta)(1 + 3\lambda + 12\delta)} - \frac{B_1^2\gamma^2}{(1 + \lambda + 2\delta)^2} \right|. \] (33)

At the end, let $c \in (0, 1)$. Since $T_3 < 0$ and $T_3 + 2T_4 > 0$, we conclude that
\[ F_{\mu\mu}F_{\gamma\gamma} - (F_{\gamma\gamma})^2 < 0. \]

Thus, the function $F(\mu, \gamma)$ can’t have a local maximum in the interior of the square $S$. So, we investigate the maximum of $F(\mu, \gamma)$ on the boundary of the square $S$.

For $\mu = 0$ and $0 \leq \gamma \leq 1$ (similarly $\gamma = 0$ and $0 \leq \mu \leq 1$), we obtain
\[ F(0, \gamma) = H(\gamma) = T_1 + \gamma T_2 + \gamma^2(T_3 + T_4). \] (34)

In order to find the maximum of $H(\gamma)$, we consider the situation of the function $H(\gamma)$ as increasing or decreasing as follows:
\[ H'(\gamma) = T_2 + 2\gamma(T_3 + T_4). \]

(i) Suppose that $T_3 + T_4 \geq 0$. In this case $H'(\gamma) > 0$; that is, $H(\gamma)$ is an increasing function. Hence the maximum of $H(\gamma)$ occurs at $\gamma = 1$ and
\[ \max \{ H(\gamma) : \gamma \in [0, 1] \} = H(1) = T_1 + T_2 + T_3 + T_4. \]

(ii) Suppose that $T_3 + T_4 < 0$. Then we consider for critical point $\gamma = \frac{T_2}{2(T_3 + T_4)}$ where $\theta = -(T_3 + T_4) > 0$, the following two cases:

**Case 2.4.** Let $\gamma = \frac{T_3 + T_4}{2\theta} > 1$. Then $\theta < \frac{T_2}{2} \leq T_2$, and $T_2 + T_3 + T_4 \geq 0$. Therefore
\[ H(0) = T_1 \leq T_1 + T_2 + T_3 + T_4 = H(1). \]

**Case 2.5.** Let $\gamma = \frac{2T_3}{4\theta} \leq 1$. Since $T_2 / 2 \geq 0$, we get $\frac{T_3}{2\theta} \leq \frac{T_2}{2} \leq T_2$. Also, we have $H(1) = T_1 + T_2 + T_3 + T_4 \leq T_1 + T_2$. Therefore,
\[ H(0) = T_1 \leq T_1 + \frac{T_3^2}{2\theta} = H(\frac{T_3}{2\theta}) \leq T_1 + T_2. \]

Thus, we observe that the maximum of $H(\gamma)$ occurs when $T_3 + T_4 \geq 0$, it means
\[ \max \{ H(\gamma) : \gamma \in [0, 1] \} = H(1) = T_1 + T_2 + (T_3 + T_4) \] (35)

for any fixed $c \in (0, 1)$.

For $\mu = 1$ and $0 \leq \gamma \leq 1$ (similarly $\gamma = 1$ and $0 \leq \mu \leq 1$), we obtain
\[ F(1, \gamma) = G(\gamma) = T_1 + T_2 + T_3 + T_4 + \gamma(T_2 + 2T_4) + \gamma^2(T_3 + T_4). \] (36)
In order to obtain the maximum of $G(\gamma)$, we consider the situation of the function $G(\gamma)$ as increasing or decreasing as follows:

$$G'(\gamma) = T_2 + 2T_4 + 2(T_3 + T_4)\gamma.\ 
(iii)$$

Suppose that $T_3 + T_4 \geq 0$. In this case $G'(\gamma) > 0$; that is, $G(\gamma)$ is an increasing function. Hence the maximum of $G(\gamma)$ occurs at $\gamma = 1$ and

$$\max \{G(\gamma) : \gamma \in [0,1]\} = G(1) = T_1 + 2T_2 + 2T_3 + 4T_4.$$

$$(iv)$$

Suppose that $T_3 + T_4 < 0$. Then we consider for critical point $\gamma = \frac{T_3 + T_4}{-2(T_3 + T_4)} = \frac{T_2 + 2T_4}{-4\theta}$ where $\theta = -(T_3 + T_4) > 0$, the following two cases:

**Case 2.6.** Let $\gamma = \frac{T_2 + 2T_4}{2\theta} > 1$. Then $\theta < \frac{T_2 + 2T_4}{2} \leq T_2 + 2T_4$, and $T_2 + T_3 + 3T_4 \geq 0$. Therefore

$$G(0) = T_1 + T_2 + T_3 + T_4 \leq T_1 + T_2 + T_3 + T_4 + (T_2 + T_3 + 3T_4) = G(1) = T_1 + 2T_2 + 2T_3 + 4T_4.$$

**Case 2.7.** Let $\gamma = \frac{T_2 + 2T_4}{2\theta} \leq 1$. Since $\frac{T_2 + 2T_4}{2} \geq 0$, we get that

$$\frac{(T_2 + 2T_4)^2}{4\theta} \leq \frac{T_2 + 2T_4}{2} \leq T_2 + 2T_4.$$

Therefore,

$$G(0) = T_1 + T_2 + T_3 + T_4 \leq T_1 + T_2 + T_3 + T_4 + \frac{(T_2 + 2T_4)^2}{4\theta} = G\left(\frac{T_2 + 2T_4}{2\theta}\right) \leq T_1 + T_2 + T_3 + T_4 + T_2 + 2T_4 = T_1 + 2T_2 + T_3 + 3T_4 + 2T_4.$$

Thus, the function $G(\gamma)$ gets its maximum when $T_3 + T_4 \geq 0$, it means

$$\max \{G(\gamma) : \gamma \in [0,1]\} = G(1) = T_1 + 2T_2 + 2(\overbrace{T_3 + T_4}^{\geq 0} + 2T_4)$$

for any fixed $c \in (0, 1)$.

Since $H(1) \leq G(1)$ for $c \in [0, 1]$, then

$$\max \{F(\mu, \gamma) : (\mu, \gamma) \in \{0, 1\} \times [0, 1]\} = F(1, 1) = T_1 + 2T_2 + 2T_3 + 4T_4.$$

Let $K : [0, 1] \to \mathbb{R}$,

$$K(c) = B_1|\tau|^2 \max \{F(\mu, \gamma) : (\mu, \gamma) \in \{0, 1\} \times [0, 1]\} = B_1|\tau|^2 F(1, 1)$$

$$= B_1|\tau|^2(T_1 + 2T_2 + 2T_3 + 4T_4). \quad \quad \text{(37)}$$

Now putting $T_1, T_2, T_3$ and $T_4$ in the function $K$, we have

$$K(c) = B_1|\tau|^2 \left\{\frac{B_3}{(1 + \lambda + 2\delta)(1 + 3\lambda + 12\delta)} - \frac{r^2B_1^3}{(1 + \lambda + 2\delta)^2} - \frac{|\tau|B_1^2}{2(1 + \lambda + 2\delta)^2(1 + 2\lambda + 6\delta)} - \frac{2|B_2| + B_1}{(1 + \lambda + 2\delta)(1 + 3\lambda + 12\delta)} + \frac{B_1}{(1 + 2\lambda + 6\delta)^2}\right\}^4$$

$$+ \left\{\frac{|\tau|B_1^2}{2(1 + \lambda + 2\delta)^2(1 + 2\lambda + 6\delta)} + \frac{2|B_2| + B_1}{(1 + \lambda + 2\delta)(1 + 3\lambda + 12\delta)} + \frac{2B_1}{(1 + 2\lambda + 6\delta)^2}\right\}^2$$

$$+ \frac{B_1}{(1 + 2\lambda + 6\delta)^2}. \quad \quad \text{(38)}$$
By putting $c^2 = t$ in equation (38), it may have the form

$$K(t) = B_1|\tau|^2(S^2 + Tt + U), \quad t \in [0, 1]$$

where

$$S = \left| \frac{B_3}{(1 + \lambda + 2\delta)(1 + 3\lambda + 12\delta)} - \frac{\tau^2 B_3}{(1 + \lambda + 2\delta)^2} - \frac{|\tau|B_1^2}{2(1 + \lambda + 2\delta)(1 + 2\lambda + 6\delta)} \right|$$

$$- \left( \frac{2|B_2| + B_1}{(1 + \lambda + 2\delta)(1 + 3\lambda + 12\delta)} \right) + \frac{B_1}{(1 + 2\lambda + 6\delta)^2}$$

$$T = \frac{|\tau|B_1^2}{2(1 + \lambda + 2\delta)^2(1 + 2\lambda + 6\delta)} + \frac{2|B_2| + B_1}{(1 + 2\lambda + 6\delta)^2} - \frac{2B_1}{(1 + 2\lambda + 6\delta)^2}$$

$$U = \frac{B_1}{(1 + 2\lambda + 6\delta)^2}.$$

Since

$$\max_{0 \leq t \leq 1} (S^2 + Tt + U) = \begin{cases} U; & T \leq 0, S \leq -T \\ S + T + U; & (T \geq 0, S \geq -\frac{T}{2}) \text{ or } (T \leq 0, S \geq -T) \\ \frac{4SU - T^2}{4S}; & T > 0, S \leq -\frac{T}{2}, \end{cases}$$

it gives,

$$|a_2a_4 - a_3^2| \leq B_1|\tau|^2 \begin{cases} U; & T \leq 0, S \leq -T \\ S + T + U; & (T \geq 0, S \geq -\frac{T}{2}) \text{ or } (T \leq 0, S \geq -T) \\ \frac{4SU - T^2}{4S}; & T > 0, S \leq -\frac{T}{2}. \end{cases}$$

This completes the proof.

3. Corollaries and Consequences

By taking

$$\tau = 1, \delta = 0 \text{ and } \varphi(z) = \frac{1 + (1 - 2\beta)z}{1 - z} \quad (0 \leq \beta < 1)$$

in Theorem 2.3, we conclude the following corollary.

**Corollary 3.1.** Let $f$ given by (1) be in the class $\mathcal{F}_\Sigma(\beta, \lambda)$. Then

$$|a_2a_4 - a_3^2| \leq \frac{(1 - \beta)^2}{(1 + 2\lambda)^2} \left( 4 - \frac{1}{(1 + 3\lambda)} \frac{\xi}{\zeta} \right)$$

where

$$\xi = \left[ (1 + 2\lambda)(1 + 3\lambda)(1 - \beta) + (1 + \lambda)(1 + 4\lambda + 6\lambda^2) \right]^2,$$

$$\zeta = \left( 1 + 2\lambda \right)^2 \left[ (1 + \lambda)^3 - 4(1 + 3\lambda)(1 - \beta)^2 \right] + (1 + \lambda)^2 \left[ (1 + 2\lambda)(1 + 3\lambda) \right. \beta$$

$$\left. - (9\lambda^3 + 23\lambda^2 + 15\lambda + 3) \right].$$
Remark 3.2. The bound on $|a_2a_4 - a_3^2|$ given in Corollary 3.1 is better than that given in [17, Concluding Remarks]. Because
\[
\frac{(1 - \beta)^2}{(1 + 2\lambda)^2} \left( 4 - \frac{1}{1 + 3\lambda} \right) \xi \leq \frac{9(1 + \lambda)^2(1 - \beta)^2}{2(1 + 3\lambda)(1 + \lambda)^2 - 2(1 - \beta)^2(1 + 3\lambda)},
\]
\[
(1 - \beta)^2 \geq \frac{9(1 + 2\lambda)^2(1 + \lambda)^2}{16(1 + 3\lambda)^2} \left( \frac{2}{\delta + [1 + 5\lambda + 8\lambda^2 + 3\lambda^3]} \right),
\]
where
\[
\delta = (1 + 3\lambda)(1 + \lambda)^2(1 + 5\lambda + 9\lambda^2 + 5\lambda^3).
\]

By taking
\[
\tau = \lambda = 1 \text{ and } \phi(z) = \frac{1 + (1 - 2\beta)z}{1 - z} \quad (0 \leq \beta < 1)
\]
in Theorem 2.3, we conclude the following corollary.

Corollary 3.3. Let $f$ given by (1) be in the class $H_2(\delta, \beta)$. Then
\[
|a_2a_4 - a_3^2| \leq \frac{(1 - \beta)^2}{(1 + 2\delta)^2} \left( 4 - \frac{1}{1 + 3\delta} \right) \rho \nu
\]
where
\[
\rho = \left[ 6(1 + 2\delta)(1 + 3\delta)(1 - \beta) + (1 + \delta)(11 + 44\delta + 60\delta^2) \right]^2,
\]
\[
\nu = 9(1 + 2\delta)^2 \left[ (1 + \delta)^3 - 2(1 + 3\delta)(1 - \beta)^2 \right] + (1 + \delta)^2 \left[ 6(1 + 2\delta)(1 + 3\delta) \beta \right.
\]
\[
- (25 + 125\delta + 196\delta^2 + 84\delta^3) \right].
\]

By taking $\delta = 0$ in Corollary 3.3, we conclude the following corollary.

Corollary 3.4. Let $f$ given by (1) be in the class $N_2(\delta)$. Then
\[
|a_2a_4 - a_3^2| \leq (1 - \beta)^2 \left( 4 - \frac{1}{36\lambda^2 \left[ \frac{1}{2} - (1 - \beta)^2 \right] + \frac{2\beta}{1 - \beta} \right) \right)
\]
Remark 3.5. The bound on $|a_2a_4 - a_3^2|$ given in Corollary 3.4 is better than that given in [2, Theorem 1]. Because
\[
(1 - \beta)^2 \left( 4 - \frac{1}{36\lambda^2 \left[ \frac{1}{2} - (1 - \beta)^2 \right] + \frac{2\beta}{1 - \beta} \right) \right) \leq \frac{(1 - \beta)^2}{2} \left( 2(1 - \beta)^2 + 1 \right), \quad \beta \leq 1 - \sqrt[4]{\frac{240}{83}}.
\]

By taking
\[
\tau = \lambda = 1, \quad \delta = 0 \text{ and } \phi(z) = \left( \frac{1 + z}{1 - z} \right)^{\alpha} \quad (0 < \alpha \leq 1)
\]
in Theorem 2.3, we conclude the following corollary.

Corollary 3.6. Let $f$ given by (1) be in the class $N_2^\alpha$. Then
\[
|a_2a_4 - a_3^2| \leq \begin{cases} \frac{4\alpha^2}{\alpha^2}, & 0 < \alpha \leq \frac{7}{24} \\ \alpha^2 \left( \frac{2}{\alpha} - \frac{(i\alpha - 2)^2}{2\left[ \frac{1}{2} - \frac{2\alpha}{1 - \alpha} \right]} \right), & \frac{7}{24} < \alpha \leq 1. \end{cases}
\]
Remark 3.7. The bound on $|a_2a_4 - a_3^2|$ given in Corollary 3.6 is better than that given in [2, Theorem 2] and [12, Corollary 2.6]. Because
\[
\alpha^2 \left(\frac{4}{9} - \frac{\left(\frac{7}{3}a - \frac{7}{36}\right)^2}{2\left[\frac{1}{12} - \frac{7}{36} - \frac{5}{9}a - \frac{5}{36}\right]}\right) \leq \frac{\alpha^2(8a^2 + 1)}{6}; \quad \alpha \geq \sqrt[4]{406} \frac{32}{3^2}.
\]

Remark 3.8. The bound on $|a_2a_4 - a_3^2|$ given in Theorem 2.3 is better than that given in [5, Theorem 2.1].

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