GENERALIZED NERVES OF MONADS

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Abstract. One interpretation of the Kleisli construction (given by Miranda and related to work of Paré) is as a nerve sending a 2-monad $P$ to the Kleisli double category of $P$. In this paper we find more general nerve constructions on the 2-categories of monads, which also give fully faithful nerve 2-functors $\text{Mnd} (\mathcal{K}) \to \text{Dbl}$.

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1. Introduction

Double categories generalize categories by allowing for two different types of morphisms, called horizontal and vertical morphisms, and consist of these two types of morphism as well as squares

\[
\begin{array}{c}
X \rightarrow Y \\
\downarrow \phi \\
X' \rightarrow Y'
\end{array}
\]

satisfying some compositional axioms. One of the most the natural examples of a double category, is the Kleisli double category of a monad $P$. These double categories have been studied by Paré [6, 2], and for a given monad $P$ on a category
The double categories with horizontal morphisms those of \( \mathcal{C} \) and vertical morphisms \( X \rightsquigarrow Y \) are morphisms \( X \to PY \) in \( \mathcal{C} \) (called Kleisli arrows)\(^1\). A square \( \phi \) as on the left below

\[
\begin{array}{ccc}
X & \xrightarrow{a} & X' \\
\downarrow^{f} & \neq & \downarrow^{g} \\
Y & \xrightarrow{b} & Y'
\end{array}
\quad
\begin{array}{ccc}
X & \xrightarrow{a} & X' \\
\downarrow^{f} & \neq & \downarrow^{g} \\
PY & \xrightarrow{Pb} & PY'
\end{array}
\]

is then the condition that the right square above commutes. One view of this construction given by Miranda \(^5\) is as the nerve

\[
\begin{array}{ccc}
\text{Kl}(\mathcal{Cat}) & \xrightarrow{N} & [\Delta^{\text{op}}, \mathcal{Cat}] \\
\Downarrow^{\Delta}
\end{array}
\]

where \( \text{Kl}(\mathcal{Cat}) \) is the completion of \( \mathcal{Cat} \) under Kleisli objects as defined in \(^4\). Here the nerve factors through double categories and so may be written

\[
N: \text{Kl}(\mathcal{Cat}) \to \text{Dbl}
\]

If one switches the vertical and horizontal transformations between double functors (equivalent to considering the transpose) one recovers the embedding

\[
N^{\text{T}}: \text{Mnd}(\mathcal{Cat}) \to \text{Dbl}
\]

from the 2-category of monads \( \text{Mnd}(\mathcal{Cat}) \) \(^2\).

The goal of this paper is to find more general examples of fully faithful nerves, and thus more examples of embeddings \( \text{Mnd}(\mathcal{Cat}) \to \text{Dbl} \). To do this, we replace the Kleisli completion \( \text{Kl}(\mathcal{Cat}) \) by a 2-category \( \text{mKl}(\mathcal{Cat}) \) which is like \( \text{Kl}(\mathcal{Cat}) \) in that we have a comparison 2-functor

\[
\Phi: \text{mKl}(\mathcal{Cat}) \to \text{Kl}(\mathcal{Cat})
\]

satisfying certain axioms. This \( \Phi \) should be identity of objects and 1-cells (and is thus specified by its action on 2-cells), and should satisfy a number of axioms. Of these axioms, one is non-trivial, which is that each \( N: \text{mKl}(\mathcal{Cat}) \to \text{Dbl} \), should send a monad \( P \) to a double category containing all squares of the form

\[
\begin{array}{ccc}
X & \xrightarrow{\phi f} & PY \\
\downarrow^{f} & \neq & \downarrow^{\varepsilon_Y} \\
Y & \xrightarrow{\varepsilon_Y} & Y
\end{array}
\]

for an \( \varepsilon_Y \) in \( \text{mKl}(\mathcal{Cat}) \) where \( \Phi \varepsilon_Y \) is the identity on \( PY \) (i.e. the counit of the Kleisli adjunction). To make this clear we briefly give some examples:

\(^1\)Of course one can switch the horizontal and vertical morphisms, giving the transpose.

\(^2\)In the notation of \(^4\) this is \( \text{Mnd}(\mathcal{Cat}^{\text{op}})^{\text{op}} \).
• With \( \Phi \) the identity, and so \( mKl(\text{Cat}) = Kl(\text{Cat}) \), it is easy to check

\[
\begin{array}{c}
X \\
\downarrow f \\
PY \\
\downarrow f \\
Y \\
\end{array}
\]

with \( \varepsilon_Y = \text{id}_{P Y} \) is a square in the Kleisli double category.

• Instead of a 2-cell \( X \rightsquigarrow Y \) being a Kleisli map \( X \rightarrow PY \). Take \( mKl(\text{Cat}) \) to have 2-cells being pairs of maps \( Y \rightarrow X \rightarrow PY \) composing to the unit \( \eta_Y \). One may regard this as a Kleisli map with some extra structure. Here \( \Phi \) simply forgets the extra structure, recovering the underlying Kleisli map, and each \( \varepsilon_Y \) is the factorization of the unit as \( \text{id}_{P Y} \cdot \eta_Y : Y \rightarrow PY \rightarrow PY \). For a general 2-cell \( \tau \cdot \pi : Y \rightarrow X \rightarrow PY \), the left square below lies in \( N(P) \) since it is a square on the Kleisli arrow component \( \tau \) (as earlier) and on the extra structure \( \pi \) (by the square on the right below).

\[
\begin{array}{c}
X \\
\downarrow f \\
PY \\
\downarrow f \\
Y \\
\end{array}
\quad
\begin{array}{c}
X \\
\downarrow \tau \\
PY \\
\downarrow \pi \\
Y \\
\end{array}
\]

As this example satisfies the properties we require, it will give a transpose nerve functor

\[ N^T : \text{Mnd}(\text{Cat}) \rightarrow \text{Dbl} \]

which the reader should recognize the embedding

\[ \text{Mnd}_x(\text{Cat}) \rightarrow \text{AWFS} \rightarrow \text{Dbl} \]

given by Bourke-Garner [1] in the context of algebraic weak factorization systems (AWFS), sending a monad \( P \) to the double category of \( P \)-embeddings (also called \( P \)-split-mosons)\(^3\).

• One different example, though analogous in some ways to the previous example, is to take pairs of Kleisli maps \( e : Y \rightarrow PX \) and \( s : X \rightarrow PY \) such that

\[
\begin{array}{c}
X \\
\downarrow s \\
PY \\
\downarrow s \\
P^2X \\
\end{array}
\quad
\begin{array}{c}
P^2X \\
\downarrow P^\varepsilon \\
PX \\
\downarrow P^\varepsilon \\
P^2X \\
\end{array}
\]

which we may think of a strong version of Kleisli split-epis (hence the notation \( s \) and \( e \)) as composition by \( \mu_X : P^2X \rightarrow PX \) recovers the Kleisli identity arrow \( \eta_X : X \rightarrow PX \). Here an \( \varepsilon_Y \) is the pair \( \eta_{PY} \cdot \eta_Y : Y \rightarrow P^2Y \),

\(^3\)In order to factor through \( \text{AWFS} \) one uses monads which are suitably compatible with products.
id_{PY} : PY \to PY$, with the extra structure $e$ still giving a square since

$$PY \xrightarrow{Pe} P^2 X \quad \xleftarrow{s} \quad X \xrightarrow{\eta_P \cdot \eta_X} X$$

ensuring the required conditions also hold on the non-Kleisli components.

Lastly, we mention iterated versions of these constructions. One can iterate the Kleisli nerve to recover embeddings

$$\text{Mnd}^n (\text{Cat}) \to (n+1) \text{Fold}$$

into $(n+1)$-fold categories [5], thus sending distributive laws to triple categories and so on. We give a more general version of this, allowing for iteration of any of the examples of nerves in any order.

The original motivation for this work was to use these nerves to give simpler proofs about various results on distributive laws, however this will now be left for a future paper.

2. Background

In this section we will recall the necessary background knowledge on internal categories, double categories, 2-categories of monads, and completions under Kleisli objects.

2.1. 2-categories of monads in $\mathcal{K}$. We first recall the definition of the 2-category of monads. We warn the reader that the morphisms are defined differently to [7], with the direction of each monad morphism's 2-cell component reversed.

Remark 2.1.1. In the notation of [7] the following 2-category was denoted $\text{Mnd}(\mathcal{K}^{op})^{op}$, though we will avoid the 'op's in this paper for brevity. The choice of the direction of the 2-cell $\xi$ we are using here works most nicely with Kleisli constructions, which is unsurprising as $\xi : FP \to QF$ gives a Kleisli arrow $FP \Rightarrow F$.

Definition 2.1.2. Given a 2-category the $\mathcal{K}$, the 2-category $\text{Mnd}(\mathcal{K})$ of monads in $\mathcal{K}$ has:

- **objects**: monads which consist of data $(P : \mathcal{C} \to \mathcal{C}, \eta : 1 \Rightarrow P, \mu : P^2 \Rightarrow P)$ in $\mathcal{K}$ rendering commutative

  $$\begin{array}{ccc}
  P & \xrightarrow{\eta_P} & PP \\
  \downarrow{id} & & \downarrow{P\eta} & \downarrow{\mu} \\
  P & & P \\
  \end{array}$$

- **morphisms**: a morphism $P \to Q$ is a $\xi$ as on the left below satisfying the triangle and pentagon axioms

  $$\begin{array}{ccc}
  \mathcal{C} & \xrightarrow{FP} & \mathcal{D} \\
  \downarrow{P} & \searrow{\xi} & \downarrow{Q} \\
  \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\
  \end{array} \quad \begin{array}{ccc}
  FP \xrightarrow{\xi} QF \\
  \downarrow{F\eta_P} & \searrow{\eta_Q F} & \downarrow{\eta_Q F} \\
  FFP \xrightarrow{\xi} QFP \\
  \end{array} \quad \begin{array}{ccc}
  FFP \xrightarrow{\xi} QFP \\
  \downarrow{F\mu} & \searrow{\mu_Q F} & \downarrow{\mu_Q F} \\
  FP \xrightarrow{\xi} QP \\
  \end{array}$$
2-cells: a 2-cell \((F, \xi) \Rightarrow (F', \xi') : P \rightarrow Q\) is a map \(\alpha : F \rightarrow F'\) such that

\[
\begin{array}{ccc}
FP & \xrightarrow{\xi} & QF \\
\downarrow^{\alpha P} & & \downarrow^{Q \alpha} \\
F'P & \xrightarrow{\xi'} & QF'
\end{array}
\]

**Remark 2.1.3.** The inclusion \(\text{inc} : \text{Cat} \rightarrow \text{Mnd} (\text{Cat})\) is the middle of an adjoint triple \(\text{Kl} \dashv \text{inc} \dashv \text{und}\). Here \(\text{und}\) is the underlying functor which takes a monad \(P\) on a category \(\mathcal{C}\) to \(\mathcal{C}\), and \(\text{Kl}\) takes a monad \(P\) to the Kleisli category of \(P\) denoted \(\mathcal{C}_P\) or just \(\mathcal{C}_P\). Though trivial, we will often make use of the underlying-inclusion adjunction, which simply says that monad morphisms \((\text{id}, \mathcal{C}) \rightarrow (Q, \mathcal{D})\) out of an identity monad (which must be as below)

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{F} & \mathcal{D} \\
\downarrow^{\text{id}} & \xRightarrow{\xi = \eta_Q F} & \downarrow^{Q} \\
\mathcal{C} & \xrightarrow{F} & \mathcal{D}
\end{array}
\]

 correpond to morphisms \(F : \mathcal{C} \rightarrow \mathcal{D}\).

### 2.2. The Kleisli-object completion of a 2-category \(\mathcal{K}\).

Since \(\text{Kl}\) is a left adjoint, its existence may be defined by a representability condition. That is to give for each monad \(P\) a representing object \(\text{Kl}(P)\) for the copresheaf \(\mathcal{K} \rightarrow \text{Cat} : \mathcal{A} \mapsto \text{Mnd}(\mathcal{K}) (P, \text{id}_\mathcal{A})\)

The category \(\text{Mnd} (\mathcal{K}) (P, \text{id}_\mathcal{A})\) is simply the category of \(P\)-opalgebras on \(\mathcal{A}\), and so \(\text{Kl}(P)\) is the universal \(P\)-opalgebra. The Kleisli-object completion \(\text{Kl}(\mathcal{K})\) of a 2-category \(\mathcal{K}\) freely completes a 2-category \(\mathcal{K}\) under these Kleisli objects (universal opalgebras) [4].

**Definition 2.2.1.** [4] Given a 2-category the \(\mathcal{K}\), the 2-category \(\text{Kl}(\mathcal{K})\) called the Kleisli-object completion of \(\mathcal{K}\) has the same objects and 1-cells of \(\text{Mnd}(\mathcal{K})\), but a 2-cell \((F, \xi) \Rightarrow (F', \xi') : P \rightarrow Q\) is now a map \(\alpha : F \rightarrow QF'\) such that

\[
\begin{array}{ccc}
FP & \xrightarrow{\xi} & QF \\
\downarrow^{\alpha P} & & \downarrow^{Q \alpha} \\
QF'P & \xrightarrow{\xi'} & QF'
\end{array}
\]

The horizontal composite of a pair of 2-cells \(\alpha : (F, \xi) \Rightarrow (F', \xi') : P \rightarrow Q\) and \(\beta : (G, \phi) \Rightarrow (G', \phi') : Q \rightarrow R\) is the map

\[
\begin{array}{ccc}
GF & \xrightarrow{G \alpha F} & QF' \\
\downarrow^{R \phi F'} & & \downarrow^{R \beta F'} \\
RGF & \xrightarrow{R \phi' F'} & RGF'
\end{array}
\]

**Example 2.2.2.** Disregarding that \(\text{Spn}(\text{Set})\) is only a bicategory, the 2-category \(\text{Cat}\) is equivalent to \(\text{Kl}, (\text{Spn}(\text{Set}))\) where * denotes that we only take monad morphisms whose underlying 1-cell is a left adjoint span.

---

4Note in [7] the direction of 2-cells is chosen differently, thus giving a slightly different triple involving the Eilenberg-Moore construction.
2.3. **Kleisli double categories.** We now recall the basic definitions of internal categories and double categories, so that we may recall the notion of Kleisli double categories and how to see them as a nerve.

**Definition 2.3.1.** A category internal to a category $\mathcal{E}$ with pullbacks consists of objects $C_0$ and $C_1$, source and target maps $s, t: C_1 \rightarrow C_0$ a unit map $i: C_0 \rightarrow C_1$ and a composition map $\circ: C_1 \times_{C_0} C_1 \rightarrow C_1$ satisfying

\[
\begin{array}{ccc}
C_0 & \xrightarrow{i} & C_1 \\
\downarrow{\pi_2} & & \downarrow{\pi_1} \\
C_0 \times_{C_0} C_1 & \cong & C_1 \times_{C_0} C_1 \\
\downarrow{\circ} & & \downarrow{\circ} \\
C_1 & \xrightarrow{i} & C_1 \\
\end{array}
\]

and

\[
\begin{array}{ccc}
C_0 \times_{C_0} C_1 & \xrightarrow{1 \times 1} & C_0 \times_{C_0} C_1 \\
\downarrow{\pi_2} & & \downarrow{\pi_1} \\
C_1 & \xrightarrow{1 \times 1} & C_1 \\
\end{array}
\]

We omit the definitions of the relevant pullbacks.

**Remark 2.3.2.** Categories internal to $\mathcal{E}$ embed into the category of simplicial objects of $\mathcal{E}$ denoted $[\Delta^{op}, \mathcal{E}]$. In the case $\mathcal{E} = \text{Set}$ this is the usual nerve functor

\[
\begin{array}{ccc}
\text{Cat} & \xrightarrow{} & [\Delta^{op}, \text{Set}] \\
\downarrow{\Delta} & & \\
\end{array}
\]

**Definition 2.3.3.** A double category is a category internal to $\text{Cat}$. In more detail, this consists of a set of objects, a category of horizontal morphisms between these objects, a category of vertical morphisms between these objects, and squares of horizontal and vertical morphisms

\[
\begin{array}{ccc}
X & \xrightarrow{a} & X' \\
\downarrow{f} & & \downarrow{g} \\
Y & \xrightarrow{b} & Y' \\
\end{array}
\]

which may be composed vertically and horizontally (satisfying compositional axioms). In terms of the above definition, $C_0$ is the category of objects and horizontal morphisms, and $C_1$ is the category of vertical morphisms and squares.

**Remark 2.3.4.** One may iterate this process, taking a triple category to be a category internal to $\text{Dbl}$ (the category of double categories) and so on. By iterating this process $n$ times one recovers the notion of $n$-fold categories.

One class of examples of double categories, is that for a monad $P$ on a category $\mathcal{C}$ we may form the Kleisli double category of $P$, consisting of the two types of maps: arrows of $\mathcal{C}$ and $P$-Kleisli arrows of $\mathcal{C}$. 
Definition 2.3.5. The Kleisli double category of a monad $P$ on a category $C$ consists of the objects of $C$, has horizontal arrows those of $C$ and vertical morphisms $X \rightsquigarrow Y$ are morphisms $X \to P Y$ in $C$ (called Kleisli arrows). A square $\phi$ as on the left below

\[
\begin{array}{ccc}
X & \xrightarrow{\alpha} & X' \\
\Downarrow f & \Downarrow \phi & \Downarrow g \\
Y & \xrightarrow{b} & Y'
\end{array}
\quad
\begin{array}{ccc}
X & \xrightarrow{\alpha} & X' \\
\Downarrow f & \Downarrow g & \Downarrow P b \\
PY & \xrightarrow{P b} & PY'
\end{array}
\]

is the condition that the square on the right commutes.

Remark 2.3.6. One may switch the horizontal and vertical morphisms of a double category, giving the transpose of a double category.

Remark 2.3.7. As noted by Miranda [5], the (transpose) Kleisli double category of a monad $P$ can be seen as the nerve 2-functor

\[
\text{Kl}(\text{Cat}) \xrightarrow{N} [\Delta^{op}, \text{Cat}]
\]

which is 2-fully faithful. The nerve here factors through double categories and sends a monad $P$ to the double category with object of objects $C_{P,\text{kl}}$ and object of morphisms $[2, C]_{[2,P],\text{kl}}$ where 'kl' denotes the Kleisli category of a given monad $P$. Taking the transpose of the nerve (equivalent to switching vertical and horizontal transformations between double functors) we recover the embedding $N^T : \text{Mnd}(\text{Cat}) \to \text{Dbl}$.

Example 2.3.8. As the nerve functor $N : \text{Kl}(\text{Cat}) \to \text{Dbl}$ is fully faithful, and we have a Kleisli adjunction for a monad $P$ (now between the double category of squares and Kleisli double category), we may consider the corresponding adjunction in the preimage. This is given by the adjunction of monad morphisms

\[
\begin{array}{ccc}
\Delta & \xrightarrow{\text{Kl}} & [\Delta^{op}, \text{Cat}] \\
\Delta & \xrightarrow{\text{Kl}} & [\Delta^{op}, \text{Cat}]
\end{array}
\]

This pair of monad morphisms allows one to go back and forth between identity monads, and will be useful later on.\footnote{In the more general situation later on involving modified Kleisli completions, we still have a $\varepsilon : P \to 1$, but no $\eta : 1 \to P$ and thus no adjunction of monad morphisms.}

3. General Nerves

In this section we consider more general examples of fully faithful nerves of monads, aside from Kleisli double categories. The simple idea is that instead of the 2-category $\text{Kl}(\text{Cat})$, we use a 2-category $\text{mKl}(\text{Cat})$ which is similar to $\text{Kl}(\text{Cat})$ in
that one has a comparison 2-functor $\Phi: \text{mKl}(\mathbf{Cat}) \to \text{Kl}(\mathbf{Cat})$ satisfying certain axioms. The notation 'mkl' is to denote a modified version of the Kleisli completion. Moreover, to be more general and also allow for iteration we will replace $\mathbf{Cat}$ by a more general 2-category $K$ satisfying some properties.

The following describes a set of axioms which will suffice to construct such a fully faithful nerve functor on 2-categories of monads. Whilst it may seem a lot of axioms, the majority of the conditions are trivial. The central axiom is the existence of squares

$$
\begin{array}{ccc}
X & \xrightarrow{\phi f} & PY \\
\downarrow{f} & & \downarrow{\varepsilon Y} \\
Y & \xrightarrow{} & Y
\end{array}
$$

in the below, and this is what one checks when looking for examples. It would not be surprising if some of the other axioms turn out to be redundant.

**Theorem 3.0.1.** Suppose we are given a 2-category $K$, a 2-category $\text{mKl}(K)$, a 2-functor $\Delta \to K$ and an identity on objects, identity on 1-cells 2-functor $\Phi: \text{mKl}(K) \to \text{Kl}(K)$ with nerve functor denoted $N: \text{mKl}(K) \to [\Delta^{\text{op}}, \mathbf{Cat}]$. Suppose that:

1. the 2-functor $\Delta \to K$ is nice in that:
   a. the nerve $n: K \to [\Delta^{\text{op}}, \mathbf{Cat}]$ factors through $\text{Dbl}$ and is fully faithful;
   b. there exists double functors natural in $C$ of the form\(^6\)
      $\text{Sq} K(1, C) \to nC \to \text{Sq} (\text{ob} K(1, C) : \text{ob} K(2, C))$
2. the 2-functor $\Phi: \text{mKl}(K) \to \text{Kl}(K)$ is nice in that:
   a. for all monads $P$ there exists a 2-cell $\varepsilon: P \Rightarrow \text{id}_C$ in $\text{mKl}(K)$ such that $\Phi\varepsilon: P \Rightarrow \text{id}_C$ in $\text{Kl}(K)$ is the identity on $P$;
   b. for all monads $P$ on $C$ and $f: X \Rightarrow Y: 1 \to (P, C)$, there exists squares

$$
\begin{array}{ccc}
X & \xrightarrow{\phi f} & PY \\
\downarrow{f} & & \downarrow{\varepsilon Y} \\
Y & \xrightarrow{} & Y
\end{array}
$$

3. the trivial axioms that:
   a. for all squares

$$
\begin{array}{ccc}
X & \xrightarrow{} & X \\
\downarrow{f} & & \downarrow{g} \\
Y & \xrightarrow{} & Y
\end{array}
$$

\(^6\)In $\mathbf{Cat}$ 2-cells between 1-cells $1 \to C$ are the same as arrows (that is maps $2 \to C$). However, for a general $K$ this may not be true, hence the need for this axiom. The notation $(\text{ob} K(1, C) : \text{ob} K(2, C))$ refers to the category with objects $\text{ob} K(1, C)$ and morphisms $\text{ob} K(2, C)$. 

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**Diagram:**

- $\xrightarrow{\phi f}$
- $\xrightarrow{\varepsilon Y}$
we have $f = g$;
(b) every $\varepsilon_X$ comprises a square

\[
\begin{array}{ccc}
P X & \xrightarrow{\eta_X} & P X \\
\varepsilon_X & \downarrow & P \varepsilon_X \\
X & \xrightarrow{\eta_X} & P X
\end{array}
\]

(c) the 2-functor

\[m\text{Kl}(K)[2, -] : m\text{Kl}(K) \to \text{Cat}\]

is locally fully faithful;\(^7\)

(d) the nerve 2-functor $N : m\text{Kl}(K) \to \text{Dbl}$ is fully faithful on underlying 1-cells.\(^8\)

Then the nerve $N : m\text{Kl}(K) \to \text{Dbl}$ is $\varepsilon$-fully faithful. That is, full on double functors $F : \mathcal{C}_{P, \text{mkl}} \to \mathcal{D}_{Q, \text{mkl}}$ which are determined by their action on the class of morphisms $\{PX \xrightarrow{\varepsilon_X} X \xrightarrow{f} Y\} \cup \{PX \xrightarrow{P \varepsilon_X} PY\}.\(^9\)

Proof. We first consider the nerve $N : m\text{Kl}(\text{Cat}) \to \text{Dbl}$ on 1-cells. This is defined by the assignation

\[
P \xrightarrow{(F, \xi)} Q \quad \mapsto \quad F_{\text{mkl}} \quad X \xrightleftharpoons{f} Y
\]

that is whiskering with $(F, \xi)$, and this has the candidate inverse assignation

\[
\begin{array}{ccc}
\mathcal{C}_{P, \text{stkl}} & \xrightarrow{\Phi_{F, \text{mkl}}} & \mathcal{C} \\
F_{\text{mkl}} & \quad & F^\ast
\end{array}
\]

\[FX \xrightleftharpoons{(F, \xi) f} FY
\]

\[\begin{array}{ccc}
\mathcal{D}_{Q, \text{stkl}} & \xrightarrow{\Phi_{F, \text{mkl}}} & \mathcal{D} \\
F_{\text{mkl}} & \quad & F^\ast
\end{array}
\]

\[QFX
\]

\(^7\)This axiom will hold in most examples where the 2-cells of $m\text{Kl}(K)$ have a natural definition.

\(^8\)These are the monad morphisms out of identity monads, which appear in the underlying-inclusion adjunction. This is easy to check as these are all trivial monad morphisms.

\(^9\)The class of morphisms including those of the form $PX \xrightarrow{\varepsilon_X} X \xrightarrow{f} Y$ and $PX \xrightarrow{P \varepsilon_X} PY$ is quite large, and so it is expected that any reasonable double functor will be determined by its action on this class. It is likely that all such double functors are determined on this class, in which case $\varepsilon$-fully faithfulness is the same as fully faithfulness (though we have no proof of this). Any counterexample would need to involve a very unusual double functor.
We must check that $\xi := \Phi F_{dbl}\varepsilon$ is a well defined 2-cell $FP \Rightarrow QF$ in $\mathcal{K}$. To do this, note we have the double natural transformation

$$
\begin{array}{c}
\mathcal{C}_{1-mkl} \xrightarrow{C} \mathcal{C}_{P-mkl} \\
\varepsilon \downarrow \downarrow \\
\mathcal{C}_{P-mkl} \xrightarrow{F} \mathcal{D}_{Q-mkl}
\end{array}
$$

which consists of components

$$mKl(2, m) \xrightarrow{\Phi F\varepsilon} mKl(2, Q) \quad \text{in Cat}$$

which corresponds to (using fullness on underlying monad morphisms and local fully faithfulness at 2)

$$
\begin{array}{c}
1\mathcal{C} \xrightarrow{FP} \mathcal{Q} \\
\varepsilon \downarrow \downarrow \\
\mathcal{Q} \xrightarrow{\Phi \varepsilon} \mathcal{Q}
\end{array}
$$

in $mKl(\mathcal{K})$

giving by $\Phi$ the Kleisli map $FP \Rightarrow F$ i.e. 2-cell $FP \Rightarrow QF$. The monad morphism axioms follow from applying $F$ and $\Phi$ to

$$
\begin{array}{c}
X \xrightarrow{\eta X} PX \\
\varepsilon_X \downarrow \downarrow \\
X \xrightarrow{\varepsilon X} X
\end{array} \quad \quad \quad
\begin{array}{c}
P^2X \xrightarrow{\mu X} PX \\
\varepsilon_{PX} \downarrow \downarrow \\
PX \xrightarrow{\varepsilon X} X
\end{array}
$$

and using that $n$ is fully faithful. For faithfulness, we must show $(F, \xi) = (F, \Phi F_{dbl}\varepsilon)$ for any $F_{dbl}$ which is equal to whiskering by some $(F, \xi)$. Clearly the underlying $F$ is sent to $F$, moreover for all $X \in \mathcal{C}$

$$\Phi F_{dbl}\varepsilon_X = \Phi [(F, \xi) \varepsilon_X] = \Phi (F, \xi) \Phi \varepsilon_X = (F, \xi) \text{id}_X = \xi_X$$

We now check fullness, and suppose we are given a double functor $F_{dbl} : \mathcal{C}_{P-mkl} \rightarrow \mathcal{D}_{Q-mkl}$ which is completely determined by its action on the class $\left\{ PX \xrightarrow{\varepsilon_X} X \xrightarrow{f} Y \right\} \cup \left\{ PX \xrightarrow{f} PY \right\}$. To show this double functor lies in the image of the nerve, we must check that $F_{dbl}$ is equal to the operation of whiskering by $(F, \xi)$ (with $\xi := \Phi F_{dbl}\varepsilon_X$). We need only show this for compisites $PX \xrightarrow{\varepsilon_X} X \xrightarrow{f} Y$. Firstly, we see
that \( F_{dbl} \varepsilon_X = (F, \xi) \varepsilon_X \) from the diagram

\[
\begin{array}{cccc}
FPX & FP^2X & FPX \\
\downarrow & \downarrow & \downarrow \\
FX & FPX & (F, \xi) \\
\downarrow & \downarrow & \downarrow \\
FX & FX & FX \\
\end{array}
\]

and the fact that \( FP \) lies in the image, because it factors through an underlying as

\[
(3.2) \quad \mathcal{C}_{P-mkl} \hookrightarrow \mathcal{C}_{1-mkl} \hookrightarrow \mathcal{C}_{P-mkl} \rightarrow \mathcal{D}_{Q-mkl}
\]

Note also that by middle four interchange we have

\[
\begin{array}{ccc}
PX & PX \\
\downarrow Pf & \downarrow \varepsilon \\
PY & X \\
\downarrow \varepsilon Y & \downarrow f \\
Y & Y \\
\end{array}
\]

Now as \( F_{dbl} \) is determined on \( \varepsilon \) (by the above argument) as well as \( Pf \) (by \( 3.2 \)), it is thus determined on composites as on the right above. Hence any double functor which is determined by its action on the class \( \{ PX \xrightarrow{\varepsilon} X \xrightarrow{f} Y \} \) lies in the image.

We now consider the 2-cell aspect of

\[
N : \text{mKl}(\mathcal{K}) \to \text{Dbl}
\]

and show that it is bijective on 2-cells (locally fully faithful). A 2-cell \( \alpha \) in \( \text{mKl}(\mathcal{K}) \) is assigned as below

\[
\begin{array}{c}
P \\
\downarrow (F, \xi) \xrightarrow{\underline{\alpha}_{dbl}} (F', \xi') \\
Q \\
\end{array}
\]

\[
\begin{array}{c}
\mathcal{C}_{P-stkl} \\
\downarrow \alpha_{dbl} \\
\mathcal{D}_{Q-stkl} \\
\end{array}
\]

where \( \alpha_{dbl} \) is comprised of (recall \( \mathcal{C}_{P-stkl} \) has object of objects \( \text{mKl}(\mathcal{K})[1, \mathcal{C}] \) and object of morphisms \( \text{mKl}(\mathcal{K})[2, \mathcal{C}] \)) the natural transformations

\[
\begin{array}{cccc}
\text{mKl}(\mathcal{K})[1, \mathcal{C}] & \text{mKl}(\mathcal{K})[2, \mathcal{C}] \\
\circ (F, \xi) \xrightarrow{\circ \alpha} \circ (F', \xi') \\
\text{mKl}(\mathcal{K})[1, \mathcal{D}] & \text{mKl}(\mathcal{K})[2, \mathcal{D}] \\
\circ (F, \xi) \xrightarrow{\circ \alpha} \circ (F', \xi') \\
\end{array}
\]

and note the first is determined by the second (as we can restrict along \( 1 \to 2 \)). The nerve being assumed locally fully faithful at \( 2 \) then gives the result. \( \square \)
Remark 3.0.2. It is useful to first consider the restriction to the identity monads
\[ m(K) \xrightarrow{\Phi} K \]
\[ m\text{Kl}(K) \xrightarrow{\phi} \text{Kl}(K) \]
in checking the axioms and looking for examples. In particular, one first checks horizontal composition of 2-cells in the simpler 2-category \( m(K) \).

Remark 3.0.3. An additional functoriality axiom one might assume is
\[
\begin{array}{c}
X \xrightarrow{\Phi f} PY \xrightarrow{P \Phi f = \Phi P g} P^2 Z \xrightarrow{\mu_Z} PZ \\
\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\
Y \xrightarrow{\Phi g} PZ \\
\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\
Z \xrightarrow{\Phi \rho} \mu_Z Z \\
\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\
\end{array}
\]
\[
\begin{array}{c}
X \xrightarrow{\Phi f} PZ \\
\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\
X \xrightarrow{\Phi g} PZ \\
\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\
X \xrightarrow{\Phi \rho} \mu_Z Z \\
\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\
\end{array}
\]
though this does not appear to be a necessary condition.

Finally, we mention the transpose of the nerve which arises from switching the horizontal and vertical transformations between double functors. Provided that this functor \( N^T: \text{Mnd}(K) \to \text{Dbl} \) is well defined, it will be fully faithful.

Corollary 3.0.4. Suppose further that for all \( \alpha: (F, \xi) \to (G, \xi'): \mathcal{C} \to \mathcal{D} \) (a monad 2-cell), and \( \rho: X \Rightarrow Y: 1 \to \mathcal{C} \) there exists a family of squares
\[
\begin{array}{c}
FX \xrightarrow{\alpha_X} GX \\
\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\
(F, \xi)_\rho \xrightarrow{\alpha_\rho} (G, \xi')_\rho \\
\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\
FY \xrightarrow{\alpha_Y} GY \\
\end{array}
\]
which compose vertically and horizontally, then the transpose \( N^T: \text{Mnd}(K) \to \text{Dbl} \) is also \( \varepsilon \)-fully faithful.\(^{10}\)

4. Examples

We now give some examples of various nerves on 2-categories of monads. This means we must give examples of 2-categories \( \text{mKl}(K) \) satisfying these given axioms. We will take \( K = \text{Cat} \) here for simplicity. Of course the simplest example is to take \( \text{mKl}(\text{Cat}) \) to be \( \text{Kl}(\text{Cat}) \), in which case we have the fully faithful nerve \( N: \text{Kl}(\text{Cat}) \to \text{Dbl} \) and its transpose \( N^T: \text{Mnd}(\text{Cat}) \to \text{Dbl} \) sending a monad \( P \) to the Kleisli double category of \( P \).

Recall that \( \text{mKl}(\text{Cat}) \) has the same objects and 1-cells as \( \text{Kl}(\text{Cat}) \), thus we need only specify the 2-cells of \( \text{mKl}(\text{Cat}) \). Typically in defining this 2-category, horizontal composition of 2-cells is the most difficult calculation to verify, though it

\[^{10}\text{If } K \text{ is not assumed to be } \text{Cat}, \text{ it can be useful to upgrade } \text{Dbl} \text{ to } \text{Tpl} \text{ as the nerve will forget some structure.}\]
is true in each of our examples satisfying axiom (3.1). It is left as an open question if condition (3.1) yields closure of the 2-cells under horizontal composition in general.

4.1. Double categories of $P$-embeddings. Let us now consider the construction that turns up in the setting of AWFS [1]. Here we take a 2-cell of $\text{mKl}(\text{Cat})$ of the form $f: X \rightarrow Y$: $(T, A) \rightarrow (P, C)$ to be a pair of maps $\pi: Y \rightarrow X$ and $\tau: X \rightarrow PY$ composing to the unit $\eta_Y$ (such pairs factoring a unit map are also called $P$-split monos). The Kleisli arrow component $\tau$ must satisfy condition (2.1).

Remark 4.1.1. Another presentation of this data (which appears in [8]) is as a pair of maps $L: Y \rightarrow X$ and $\text{res}_L: PX \rightarrow PY$ where $\text{res}_L \cdot PL = \text{id}$ and $\text{res}_L$ is a $P$-homomorphism. This corresponds the choice of the free algebra presentation or Kleisli arrow presentation of Kleisli categories. It is clear from this presentation we have closure under vertical composition of 2-cells.

We leave it to the reader to check that the 2-cells are also closed under horizontal composition and whiskering by monad morphisms. The map $\Phi: \text{mKl}(\text{Cat}) \rightarrow \text{Kl}(\text{Cat})$ simply forgets the extra structure sending $(\pi, \tau)$ to $\tau$, and each $\varepsilon_Y$ is the factorization of the unit as $\text{id}_{PY} \cdot \eta_Y: Y \rightarrow PY \rightarrow PY$.

A general square in the double category $N(P)$ denoted as on the left below, is such that both squares on the right below commute

\[
\begin{array}{ccc}
X & \xrightarrow{f} & X' \\
\downarrow^{(\pi, \tau)} & & \downarrow^{(\pi', \tau')} \\
Y & \xrightarrow{g} & Y'
\end{array}
\quad
\begin{array}{ccc}
X & \xrightarrow{f} & X' \\
\downarrow^{\tau} & & \downarrow^{\tau'} \\
PY & \xrightarrow{\text{res}_L} & PY'
\end{array}
\quad
\begin{array}{ccc}
X & \xrightarrow{f} & X' \\
\downarrow^{\pi} & & \downarrow^{\pi'} \\
Y & \xrightarrow{\text{res}_L} & Y'
\end{array}
\]

and clearly we have a square of the form (3.1) for any $(\pi, \tau)$. The other axioms are all trivial conditions with any reasonable example, and so we omit their verification. It follows that we have a fully faithful nerve and nerve-transpose $N: \text{mKl}(\text{Cat}) \rightarrow \text{Dbl}$ and $N^T: \text{Mnd}(\text{Cat}) \rightarrow \text{Dbl}$.

Remark 4.1.2. One possible generalization is to take a 2-cell the be the data $\pi_i: Y \rightarrow X$ for $1 \leq i \leq n$ and $\tau: X \rightarrow PY$ where every $\tau \cdot \pi_i$ is equal to the unit $\eta_Y$. We mention this generalization only to point out there are infinitely many examples of fully faithful nerves of monads.

4.2. Double categories of (strong) Kleisli split-epis. In looking for examples of suitable 2-categories $\text{mKl}(\text{Cat})$, one thing to consider is that 2-cells should be closed under horizontal composition. One example of such 2-cells are split epimorphisms, and this motivates the following. Take a 2-cell of $\text{mKl}(\text{Cat})$ of the form $f: X \rightarrow Y$: $(T, A) \rightarrow (P, C)$ to be pairs of maps $s: Y \rightarrow PX$ and $e: X \rightarrow PY$ rendering commutative

\[
\begin{array}{ccc}
Y & \xrightarrow{s} & PX \\
& \underset{\eta_Y}{\searrow} & \text{Pscr}_{PY} \\
& & \text{Pscr}_{PY}
\end{array}
\]

Note that such arrows are split epimorphisms in the sense of Kleisli composition, seen by composing with the multiplication $\mu_Y: P^2Y \rightarrow PY$. A general square in
the double category \( N(P) \) denoted as on the left below, is data such that both squares on the right below commute

\[
\begin{array}{c}
X \xrightarrow{f} X' \\
\downarrow \pi \quad \downarrow \pi' \\
Y \xrightarrow{g} Y'
\end{array}
\quad:

\begin{array}{c}
P X \xrightarrow{Pf} PX' \\
\downarrow \pi \quad \downarrow \pi' \\
P Y \xrightarrow{Pg} P Y'
\end{array}
\quad:

\begin{array}{c}
X \xrightarrow{e} X' \\
\downarrow s \quad \downarrow s' \\
Y \xrightarrow{g} Y'
\end{array}
\quad:

\begin{array}{c}
P X \xrightarrow{e} PX' \\
\downarrow s \quad \downarrow s' \\
P Y \xrightarrow{g} P Y'
\end{array}
\]

It is an easy exercise to check this notion of 2-cell is closed under vertical composition. A more interesting exercise is to check that this definition of 2-cell is closed under horizontal composition and whiskering by monad morphisms, a fact which is much less obvious (though simple and tedious to check). Again this yields fully faithful nerve functors \( N: mKl(Cat) \to Dbl \) and \( N^T: Mnd(Cat) \to Dbl \).

5. Iteration

Given a pair of examples of fully faithful nerve functors \( N^T_1: Mnd(Cat) \to Dbl \) and \( N^T_2: Mnd(Cat) \to Dbl \), we may iterate to construct a fully faithful functor from \( Mnd(Mnd(Cat)) \to Tpl \) sending distributive laws to triple categories, and more generally functors \( Mnd^n(Cat) \to (n+1)\text{fold} \) sending \( n \)-ary distributive laws to \( (n+1)\text{fold} \) categories. This works by first noting that when \( K = Dbl \), a nerve of the form \( Mnd(Dbl) \to Dbl \) factors through \( Tpl \), as both the object of objects and object of morphisms of this double category, are themselves double categories. Thus we have the composite

\[
\begin{array}{c}
Mnd(Mnd(Cat)) \xrightarrow{Mnd(N^T_1)} Mnd(Dbl) \xrightarrow{N^T_2} Tpl
\end{array}
\]

Iterating this process yields embeddings \( Mnd^n(Cat) \to (n+1)\text{fold} \), and one can mix any family of examples of nerves of monads in any order.

Example 5.0.1 (The triple category of embeddings of a distributive law). Given two copies of the nerve functor sending a monad \( P \) to the double category of \( P \)-embeddings \( \mathcal{E}_{P-emb} \), we have an embedding of distributive laws into triple categories. We now work out what this triple category is.

Suppose we are given monads \( T \) and \( P \), and a distributive law \( \lambda: TP \to PT \). We have since \( Mnd(Cat) \to Dbl \) is fully faithful, a double category \( \mathcal{E}_{P-emb} \) and a double monad \( \tilde{T} \) on it. Given the 1-category \( \mathcal{E}^1_{P-emb} \) (using “1” to refer to the object of morphisms and “0” the object of objects), we get from the monad \( \tilde{T} \) a double category \( \mathcal{E}_{\tilde{T}-emb} \). This gives a diagram

\[
\begin{array}{c}
\mathcal{E}^1_{\tilde{T}-emb} \leftrightarrow \mathcal{E}^1_{P-emb} \\
\downarrow \downarrow \\
\mathcal{E}^0_{\tilde{T}-emb} \leftrightarrow \mathcal{E}^0_{P-emb}
\end{array}
\]
as a double-category internal to \textbf{Cat}, i.e. a triple category. This may also be written

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\mathcal{P}_{\text{emb}}} & \mathcal{C} \\
\mathcal{C}_{T_{1}\text{-emb}} & \xrightarrow{\mathcal{C}_{T_{\text{emb}}}} & \mathcal{C}_{P_{\text{emb}}} \\
\end{array}
\]

To make this clear, we write down what general objects in these categories are (using “res” notation for \(P\)-embeddings and “unit-factorization” notation for \(T\)-embeddings to easily distinguish them)

\[
(L, \text{res}_L) \to (K, \text{res}_K) \to T_1(L, \text{res}_L) \\
L \to Y \to TX
\]

Thus a distributive law \(\lambda: TP \to PT\) allows us to assemble together the double category of \(P\)-embeddings and double category of \(T\)-embeddings into a single triple category.

6. Future work

The original motivation of this work was to construct various examples of embeddings out of 2-categories of monads \(N: \text{Mnd} (\text{Cat}) \to \text{DbI}\) to prove results about distributive laws. As such an embedding is fully faithful, a distributive law may be identified with a double monad on a double category \(N(P)\). This approach is most useful in reducing the coherence axioms for pseudodistributive laws involving KZ pseudomonads \([3]\) (using the “\(P\)-embedding” nerve which shows up in AWFS \([1]\)). However, this would require a generalization of these nerves to the setting of pseudomonads.

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\(^{11}\)In the talk \([6]\) a related construction with distributive laws was mentioned, with two Kleisli double categories being combined with a distributive law. The iterated version on the simpler Kleisli case was also seen by Miranda \([5]\).