Discretely holomorphic parafermions and integrable boundary conditions

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Abstract
In two-dimensional statistical models possessing a discretely holomorphic parafermion, we introduce a modified discrete Cauchy–Riemann equation on the boundary of the domain, and we show that the solution of this equation yields integrable boundary Boltzmann weights. This approach is applied to (i) the square-lattice $O(n)$ loop model, where the exact locations of the special and ordinary transitions are recovered, and (ii) the Fateev–Zamolodchikov $\mathbb{Z}_N$ spin model, where a new rotation-invariant, integrable boundary condition is discovered for generic $N$.

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1. Introduction

Discretely holomorphic parafermions (also called lattice parafermions) are lattice analogues for the holomorphic fields of conformal field theory (CFT). They are lattice observables which satisfy a discrete version of the Cauchy–Riemann (CR) equation, and were recently identified in various two-dimensional critical lattice models, which are either spin models with a local interaction or loop models describing self-avoiding walks (SAWs) or extended interfaces. Generally, in spin models with Kramers–Wannier duality (see [1] for a recent review), a lattice parafermion is constructed [2, 3] by taking the product of lattice spin and disorder operators, providing an exact lattice analogue for the operator product expansion of the corresponding CFT [4]. In loop models, the lattice parafermion is expressed [2, 5] in terms of a single open path attached to the boundary of the system.

Lattice parafermions have found many applications to the rigorous study of lattice models (especially the Ising and SAW models) in the scaling limit, providing mathematical proofs for the exact results obtained by the Coulomb gas approach [6, 7] thirty years ago. In particular, the lattice parafermion is a crucial ingredient in the proofs [8, 9] of convergence of Ising domain walls to the Schramm–Loewner evolution, and in the calculation of Ising correlation functions [10]. Also, for the SAW model on the honeycomb lattice, the lattice parafermion is
essential to determine rigorously the location of the critical weights, both in the bulk \[11\] and on the boundary \[12, 13\].

This paper is concerned with lattice models at finite size, and addresses the relation between lattice parafermions and quantum integrability. Previous studies \[2, 3, 5\] have shown that many (spin or loop) statistical models admit a lattice parafermion exactly at their integrable point. In other words, the Boltzmann weights \(\{W(\alpha)\}\) which satisfy the discrete CR equations on the rhombic lattice of angle \(\alpha\) are also a solution of the Yang–Baxter (YB) equations, where \(\alpha\) plays the role of the spectral parameter. In this work, it is shown that a similar relation holds between the solutions of CR equations on the boundary and of Sklyanin’s reflection equation \[14\] (or the boundary YB equation).

Two distinct models are considered in this paper. First, for the O\((n)\) loop model on the square lattice \[15\], following some ideas used in \[12\], local CR equations around boundary faces are introduced, and their solution is related to the boundary integrable weights found in \[16\]. When the rhombus angle is set to \(\alpha = \pi/3\), one recovers the result of \[12\] for the honeycomb lattice. Then, we study the Fateev–Zamolodchikov \(Z_N\) spin model \[17\]. We introduce a similar boundary CR equation, and show that its solution yields new non-trivial integrable weights, associated with free BCs. In both cases, we use the lattice parafermions constructed in \[3, 5\] which satisfy bulk CR equations.

Since any ‘Baxter lattice’ has a rhombic embedding \[18\], the YB approach is well adapted to the isoradial graphs in general (i.e. the graphs where every face is a rhombus). Similarly, most statements made on the lattice parafermions can be shown to be valid on this general class of lattices. However, for simplicity, we shall restrict here to rhombic lattices \(\Omega\) with a uniform angle, and where some of the boundary faces have a different shape from the bulk ones.

2. The O\((n)\) model on the square lattice

2.1. Integrable weights and CR equation in the bulk

The O\((n)\) model on the square lattice was introduced in \[15\]. It is a loop model, defined by the local configurations shown in figure 1, and the Boltzmann weight of a loop configuration \(G\):

\[
\Pi[G] = n^{N_L(G)} u_1^{N_{u_1}(G)} \cdots w_2^{N_{w_2}(G)},
\]

where \(N_L(G)\) is the number of closed loops in \(G\), and \(N_x\) is the number of faces with configuration \(x\). The partition function is denoted by \(Z\). If the loop fugacity is set to

\[
n = -2 \cos 4\lambda,
\]
the integrable weights are given by

\[ t = \sin(3\lambda - u) \sin u + \sin 2\lambda \sin 3\lambda \]
\[ u_1 = -\sin(3\lambda - u) \sin 2\lambda \]
\[ u_2 = -\sin u \sin 2\lambda \]
\[ v = \sin(3\lambda - u) \sin u \]
\[ w_1 = \sin(3\lambda - u) \sin(2\lambda - u) \]
\[ w_2 = -\sin(\lambda - u) \sin u, \]

where \( u \) is the spectral parameter. Note that, for a given configuration of the boundary faces, both \( N_{u_1}(G) \) and \( N_{u_2}(G) \) have (possibly different) fixed parities independent of the bulk configuration, and thus the signs of weights \( u_1 \) and \( u_2 \) can be changed without affecting any of the correlation functions of the model.

We first consider a domain \( \Omega_0 \) of the rhombic lattice \( \mathcal{R}_\alpha \) (see figure 2(a)), and recall the results on the lattice parafermion for the \( O(n) \) model. For a fixed edge \( a \) on the boundary, the parafermionic observable \( F_s \) is defined on the mid-edges \( \{z\} \) of the rhombi as [5]

\[ F_s(z) := \frac{1}{Z} \sum_{G \in \Gamma(a \rightarrow z)} \Pi[G] e^{-i\theta(G)}, \]

where \( \Gamma(a \rightarrow z) \) is the set of loop configurations comprising an open path between \( a \) and \( z \), \( \theta(G) \) is the winding angle of the path included in \( G \) and \( s \) is the conformal spin. For a polygon \( P \) with vertices given by the complex numbers \((p_1, \ldots, p_m)\) ordered counter-clockwise, we define the discrete contour integral of a function \( f \) around \( P \) as

\[ \sum_P f(z)\delta z := \sum_{j=1}^{m} (p_{j+1} - p_j) f\left(\frac{p_{j+1} + p_j}{2}\right), \]

where \( p_{m+1} := p_1 \). We define the discrete CR equations as

\[ \sum_{\phi} F_{\phi}(z)\delta z = 0, \]
where the left-hand side stands for the discrete integral around any given rhombus of $\Omega_0$. Fixing the outer loop configuration, this gives the linear system

\begin{align*}
    t + \mu u_1 - \mu \tau^{-1} u_2 - v &= 0 \\
    -\tau^{-1} u_1 + nu_2 + \tau \mu v - \mu \tau^{-1} (w_1 + nw_2) &= 0 \\
    nu_1 - \tau u_2 - \mu \tau^{-2} v + \mu (nw_1 + w_2) &= 0 \\
    -\mu \tau^{-2} u_1 + \mu \tau u_2 + nv - \tau^{-2} w_1 - \tau^2 w_2 &= 0,
\end{align*}

(5)

where we have set

\begin{align*}
    \tau &:= e^{i\pi s}, \\
    \mu &:= e^{i(s+1)\alpha}.
\end{align*}

In [5] it was shown that the system (5) has a non-zero solution iff $\cos 4\pi s = \cos 12\lambda$, and one can choose the value

\begin{equation}
    s = \frac{3\lambda}{\pi} - 1.
\end{equation}

Moreover, the solution of (5) for $v \neq 0$ is given by the integrable weights (1), if one sets the spectral parameter to

\begin{equation}
    u = (s + 1)\alpha = \frac{3\lambda}{\pi} \alpha.
\end{equation}

2.2. Boundary CR equation

We now consider a domain $\Omega$ where the interior faces are rhombi with a lower angle $\alpha$, and the boundary faces are either rhombi or vertical triangles (see figure 2(b)). Moreover, we assume that the initial orientation of the open path at the boundary point $a$ is horizontal. The Boltzmann weights on a triangular face are $y, r$ (see figure 1).

Let us introduce the boundary CR equation

\begin{equation}
    \text{Re} \left[ \sum_{\mathcal{G}} F_{\mathcal{G}}(z) \delta z \right] = 0,
\end{equation}

where the discrete contour integral is along a triangular boundary face $T$. The configurations $G$ in (2) which contribute to (8) are those where the open path ends on an edge of the boundary face $T$. Like in the bulk, one can fix the loop configuration outside $T$ and sum over the internal configurations (see figure 3). This yields

\begin{equation}
    \text{Re} \left[ -e^{\frac{i}{2}(r+1)\alpha + \pi(1-s)} y - e^{\frac{i}{2}(-(r+1)\alpha + \pi(1-s))} r \right] = 0.
\end{equation}

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Figure 4. Left: the original square lattice $L$ for the $\mathbb{Z}_N$ spins $\sigma_j$ (full dots, full lines) and its dual lattice (empty dots, dotted lines). Right: the covering lattice $R$.

We now have to choose an appropriate parameterization for the spectral parameter. For this, we consider a particular instance of a domain $\Omega$: the infinite strip in a diagonal direction of $R_\alpha$. In [16], it is shown how to construct an integrable model on a strip with arbitrary, homogeneous spectral parameter in the bulk. If we denote $\tilde{R}$ and $K$ the solutions to the bulk and boundary YB equations, the resulting model has bulk weights given by $\tilde{R}(u)$ and boundary weights given by $K(u/2)$. This suggests that in (9), one should set

$$u = \frac{1}{2}(s + 1)\alpha$$

instead of (7). In contrast, the conformal spin belongs to the definition of the observable $F_s$, and we keep its value (6). The solution of (9) then reads

$$r(u) = -\cos \frac{1}{2}(3\lambda + 2u), \quad y(u) = \cos \frac{1}{2}(3\lambda - 2u).$$

These are exactly the boundary weights identified in [16], which satisfy the boundary YB equation

$$\tilde{R}_{12}(u - v)K_2(u)\tilde{R}_{12}(u + v)K_2(v) = K_2(v)K_2(u)\tilde{R}_{12}(u + v)\tilde{R}_{12}(u - v).$$

In [16], it is argued that (10) corresponds to the special surface transition in a certain regime of $\lambda$. Under the change $\lambda \to \lambda + \pi$, the bulk weights are invariant up to irrelevant sign factors (see the discussion below (1)), whereas the boundary weights correspond to the ordinary surface transition.

Note that, if we specialize to $u = \lambda$ ($\alpha = \pi/3$), we recover the critical bulk and boundary weights of the $O(n)$ model on the honeycomb lattice, which were found through a type of boundary CR equation in [12] for a specific domain $\Omega$.

3. The Fateev–Zamolodchikov $\mathbb{Z}_N$ model

3.1. Definitions

The $\mathbb{Z}_N$ model consists of spins $\sigma_j$ living on the sites of the square lattice $L$ (full dots in figure 4), and taking the values

$$\sigma_j \in \{1, \omega, \ldots, \omega^{N-1}\}, \quad \omega = \exp \left( \frac{2i\pi}{N} \right).$$

We shall then write $\sigma_j = \omega^{r_j}$, with $r_j \in \mathbb{Z}$. We also introduce the crossing parameter

$$\lambda := \frac{\pi}{2N}.$$

The Boltzmann weight of a spin configuration is

$$\Pi[\sigma] = \prod_{\langle ij \rangle} W(\sigma_i, \sigma_j).$$
where

\[ W(\sigma_i, \sigma_j) = W(u|r_j - r_i) \quad \text{if } (ij) \text{ is horizontal}, \]
\[ = W(\lambda - u|r_j - r_i) \quad \text{if } (ij) \text{ is vertical}, \]

and \( u \) is the spectral parameter. Moreover, we restrict ourselves to a periodic and symmetric function \( W(u|r) \):

\[ W(u|r + N) = W(u|r), \]
\[ W(u| - r) = W(u|r). \]

The partition function is then defined as

\[ Z = \sum_{\{\sigma\}} \Pi[\sigma], \]

and a typical correlation function of \( n \) spin operators is of the form

\[ \langle \sigma_{j_1}^{k_1} \sigma_{j_2}^{k_2} \cdots \sigma_{j_k}^{k_k} \rangle := \frac{1}{Z} \sum_{\{\sigma\}} \Pi[\sigma] \sigma_{j_1}^{m_1} \sigma_{j_2}^{m_2} \cdots \sigma_{j_k}^{m_k}, \]

where \( \{j_1, \ldots, j_k\} \) are \( n \) sites of the lattice \( \mathcal{L} \) and \( \{m_1, \ldots, m_k\} \) are \( n \) arbitrary integers. As a consequence of the symmetries \((13)\) of the Boltzmann weights, the model enjoys Kramers–Wannier duality, and in particular one may define dual spin operators, also called disorder operators \( \mu^{(m)} \). These live on the dual lattice (empty dots in figure 4), and are defined through their correlation functions. For the purposes of this study, it is sufficient to define correlation functions involving two \( \mu \) operators. Let \( k \) and \( \ell \) be sites of the dual lattice \( \mathcal{L}' \), and \( \gamma_{k\ell} \) an oriented path from \( k \) to \( \ell \) on \( \mathcal{L}' \). One then defines

\[ \langle \mu_k^{(-m)} \mu_\ell^{(m)} \cdots \rangle := \frac{1}{Z} \sum_{\{\sigma\}} \prod_{(i,j) \in \gamma_{k\ell}^+} W(\sigma_i, \sigma_j) \prod_{(i,j) \in \gamma_{k\ell}^-} W(\sigma_i, \sigma_j, \omega^m \sigma_j) \cdots, \]

where the dots denote any function of the spins, \( \gamma_{k\ell}^+ \) is the set of edges of \( \mathcal{L} \) which cross \( \gamma_{k\ell} \), and we have taken the convention that \( \sigma_i (\sigma_j) \) is on the left (right) of \( \gamma_{k\ell} \). The symmetries \((13)\) imply that \( \langle \mu_k^{(-m)} \mu_\ell^{(m)} \rangle \) is independent of the choice of the path \( \gamma_{k\ell} \).

In the following, we shall consider a deformation of the square lattice \( \mathcal{L} \) into a rectangular lattice of aspect ratio \( \tan(\alpha/2) \). The union of \( \mathcal{L} \) and its dual is then a rhombic lattice of angle \( \alpha \), called the covering lattice and denoted \( R_u \).

Later, in this section, we will use the discrete Fourier transform

\[ \hat{f}(k) := \sum_{r=0}^{N-1} \omega^{kr} f(r), \quad f(r) := \frac{1}{N} \sum_{r=0}^{N-1} \omega^{-kr} \hat{f}(k). \]

The second condition in \((13)\) is equivalent to saying that \( \hat{W}(u|k) \) is real for all values of \( u \) and \( k \).

### 3.2. Integrable weights and CR equation in the bulk

The integrability condition for a periodic system (i.e. the commutation of transfer matrices on the cylinder) has the form of a star–triangle equation

\[ \sum_{r'=0}^{N-1} W(\lambda - u|r_1 - r') W(u + v|r_2 - r') W(\lambda - v|r_3 - r') = C(u, v) W(u|r_2 - r_1) W(\lambda - u - v|r_1 - r_3) W(v|r_1 - r_2), \]

where

\[ W(\sigma_i, \sigma_j) = W(u|r_j - r_i) \quad \text{if } (ij) \text{ is horizontal}, \]
\[ = W(\lambda - u|r_j - r_i) \quad \text{if } (ij) \text{ is vertical}, \]

and \( u \) is the spectral parameter. Moreover, we restrict ourselves to a periodic and symmetric function \( W(u|r) \):

\[ W(u|r + N) = W(u|r), \]
\[ W(u| - r) = W(u|r). \]
The observable $\mathcal{F}_s$ adjacent sites. Consider a point $C$ where

$$
\sum_{s'} r_1
$$

Figure 5. The star–triangle relation as a special case of the Yang–Baxter (YB) equation.

where $C(u, v)$ is an arbitrary function. The star–triangle equation can be formulated as the YB equation on the covering lattice (see figure 5), and a solution to this equation was given in [17]:

$$
W(u(0)) = 1
$$

$$
W(u(r)) = \prod_{p=0}^{r-1} \frac{\sin((2p+1)\lambda - u)}{\sin((2p+1)\lambda + u)} \quad \text{for } 0 < r < N.
$$

(18)

This solution is self-dual, in the sense that

$$
\tilde{W}(u(k)) = W(\lambda - u(k)),
$$

(19)

The construction [3] of the discrete parafermion in the $\mathbb{Z}_N$ model is the lattice version of the parafermionic current in the $\mathbb{Z}_N$ CFT [4]. In the lattice model, as well as in the CFT, the parafermion $\psi$ is obtained as the product of a spin operator $\sigma$ and a disorder operator $\mu$ on adjacent sites. Consider a point $z$ on the middle of an edge of $\mathcal{R}_a$, and denote $j(z), k(z)$ the sites of $\mathcal{L}, \mathcal{L}'$ adjacent to $z$. We shall use the short-hand notations

$$
\sigma(z) := \sigma_{j(z)}, \quad \mu(z) := \mu_{k(z)}, \quad \mu^*(z) := \mu_{k(z)}^{-1}.
$$

The lattice parafermion $\psi(z)$ is defined as [3]

$$
\psi(z) := \sigma(z) \mu(z).
$$

(20)

The observable $\mathcal{F}_s(z)$ is then defined by fixing a point $a$ on the boundary and setting

$$
\mathcal{F}_s(z) := \langle e^{-i\theta(a,z)} \psi^*(a) \psi(z) \rangle = \langle e^{-i\theta(a,z)} \sigma^*(a) \sigma(z) \mu^*(a) \mu(z) \rangle,
$$

(21)

where $\theta(a, z)$ is the angle between the vectors $k(a) \overrightarrow{j(a)}$ and $k(z) \overrightarrow{j(z)}$, and $s$ is the conformal spin. As in the $O(n)$ model, one defines the discrete CR equation on $\mathcal{R}_a$ as

$$
\sum_\phi \mathcal{F}_s(z) \delta \phi = 0,
$$

(22)

which can be written (see figure 6) as

$$
e^{-i\theta(a)} \{ \sigma^*(a) \sigma_1 \mu^*(a) \mu_1 [ (e^{-i\pi s'/2} - e^{i\pi s'/2}) \sigma_1^* \sigma_2 ] - (e^{-i\pi s'/2} - e^{i\pi s'/2}) \sigma_1^* \sigma_2 \} \mu^{-1} \mu_2 \} = 0,
$$

where $s' = 1 - s$ and $\theta(a)$ is the angle between $k(a) \overrightarrow{j(a)}$ and the horizontal. From the definition of $\mu$, the above equation reads

$$
\frac{1}{Z} \sum_{\{\sigma\}} \sigma^*(a) \sigma_1 \prod_{(j,j') \in \{1,2\}} W(\sigma_i, \sigma_j) \prod_{(j,j') \in \{1,2\}} W(\sigma_i, \omega \sigma_j) [(e^{-i\pi s'/2} - e^{i\pi s'/2}) \sigma_1^* \sigma_2 ] W(\sigma_1, \sigma_2)
$$

$$
- (e^{-i\pi s'/2} - e^{i\pi s'/2}) \sigma_1^* \sigma_2 \} W(\sigma_1, \omega \sigma_2) = 0,
$$

(23)
Figure 6. A face of the rhombic covering lattice $R_\alpha$ in the $\mathbb{Z}_N$ model. The dotted line represents an arbitrary path $\gamma$ on $L'$ from $\mu(a)$ to $\mu_1$. 

where $\gamma$ is an arbitrary path on $L'$ going from $\mu(a)$ to $\mu_1$ (see figure 6). Denoting $\sigma_1^* \sigma_2 = \omega^r$, the bracket on the left-hand side of (23) may be written as

$$I(r) := (e^{-\frac{i\pi}{2}+iu} - e^{i\pi/2-iu} \omega^r)W(u|r) - (e^{-\frac{i\pi}{2}-iu} - e^{i\pi/2+iu} \omega^r)W(u|r + 1),$$

where we have set

$$u = \frac{1}{2}(1 - s)\alpha. \quad (24)$$

Therefore, a sufficient condition for (23) to hold is that $I(r)$ vanishes for all $r$, which yields the recursion relation

$$W(u|r+1) = W(u|r) \times \frac{\sin\left[\frac{\pi r}{N} + \frac{\pi(1-s)}{2} - u\right]}{\sin\left[\frac{\pi r}{N} + \frac{\pi(1-s)}{2} + u\right]}.$$

(25)

Compatibility of (25) with the symmetries (13) fixes the value of the spin

$$s = 1 - \frac{1}{N}, \quad (26)$$

and for this value, the solution of (25) is given by the integrable weights (18).

As explained in [3], each of the $(N-1)$ discrete parafermions $\psi^{(m)}(z) = \sigma^{m}(z) \mu^{(m)}(z)$ with charge $m \in \{1, \ldots, N-1\}$ satisfies the CR equations, but for a different function $W^{(m)}$, and a conformal spin $s_m = m(N-m)/N$. In fact, $W^{(m)}$ is simply related to $W$ (18) by the global transformation $\sigma_j \rightarrow \sigma_j^m$. Hence, one can restrict him/herself to the case $m = 1$ exposed above, without loss of generality.

3.3. Boundary YB equation

Integrable boundary conditions (BCs) are generally introduced to construct a solvable model on a strip of finite width. Hence, in this paragraph, we shall consider this particular geometry, but the boundary YB equation is a local relation, which may be used on a general domain of the lattice.

Since it is less standard than for vertex or loop models, let us recall the conditions for the commutation of two-row transfer matrices (see figure 7), as they are given in [19] for instance. We denote $Y_{R,L}(\sigma_i, \sigma_j)$ the Boltzmann weight on the right and left boundaries, and the partition function on the strip reads

$$Z_{\text{strip}} = \sum_{\{\sigma\}} \prod_{\langle ij \rangle \in \text{bulk}} W(\sigma_i, \sigma_j) \prod_{\langle ij \rangle \in \text{right bound}} Y_R(\sigma_i, \sigma_j) \prod_{\langle ij \rangle \in \text{left bound}} Y_L(\sigma_i, \sigma_j).$$
In [19], fixed-spin integrable BCs (i.e. \( Y(\sigma_i, \sigma_j) = \delta_{r_i,0} \delta_{r_j,0} \) with \( \alpha, \beta \) fixed) were considered, and the associated surface free energies were obtained. In this work, we address the boundary YB equation for rotation-invariant BCs, i.e. an interaction of the form

\[
Y(\sigma_i, \sigma_j) = Y(u, \xi | r_i - r_j),
\]

where \( \xi \) is a complex parameter associated with each boundary (boundary field) and \( Y \) satisfies the symmetries (13). The boundary YB equation then reads (see figure 8)

\[
W(u - v | r_1 - r_2) \sum_{r' = 0}^{N-1} Y(u, \xi | r_1 - r') W(u + v | r_2 - r') Y(v, \xi | r_3 - r') = W(u - v | r_2 - r_3) \sum_{r' = 0}^{N-1} Y(v, \xi | r_2 - r') W(u + v | r_3 - r') Y(u, \xi | r_3 - r').
\]

This equation is invariant under a global rotation of spins \( r_1, r_2, r_3 \), and thus, one may set \( r_2 = 0 \) and apply the Fourier transform (16)

\[
\sum_{r_1, r_3 = 0}^{N-1} \omega^{(r_1 + kr) (\cdot)}
\]

on both sides of the equation. One obtains

\[
\tilde{Y}(v, \xi | k) \sum_{m=0}^{N-1} \tilde{W}(u - v | m) \tilde{Y}(u, \xi | \ell - m) \tilde{W}(u + v | k + \ell - m) = \tilde{Y}(v, \xi | \ell) \sum_{m=0}^{N-1} \tilde{W}(u - v | m) \tilde{W}(u + v | k + \ell - m) \tilde{Y}(u, \xi | k - m).
\]
These functional equations for $Y$ are difficult to solve for general $N$, but we shall now show that a non-trivial solution can be found by the use of the boundary CR equation.

3.4. Boundary CR equation

We now come back to a general lattice domain $\Omega$ as shown in figure 2(b) and consider boundary faces in the shape of a pentagon, determined by the angles $\alpha$ and $\beta$ (see figure 9). The contour integral of $F_s$ around such a face $P$ reads

$$\sum_P F_s(z) \delta z = e^{-i\theta(a)} \langle \psi^* (a) \sigma_1 \mu_1 \{ (e^{-2iu} - e^{2iu} \sigma_1^* \sigma_2 ) - (e^{2i\xi - i\pi'} - e^{-2i\xi + i\pi'} \sigma_1^* \sigma_2 ) \mu_1^* \mu_2 \} \rangle,$$

(30)

where, again, $s' := 1 - s$, and we have set

$$u = \frac{1}{4} (1 - s) (\pi - \alpha), \quad \text{and} \quad \xi = \frac{1}{4} (1 - s) (\pi - \beta).$$

(31)

Like for the bulk CR equation, but changing the notation to $\sigma_1^* \sigma_2 := \omega^{-1}$, (30) is rewritten in the form

$$\sum_P F_s(z) \delta z = \frac{e^{-i\theta(a)}}{Z} \sum_{\sigma} \sigma^* (a) \sigma_1 \prod_{i,j \in \psi^+} W(\sigma_i, \sigma_j) \prod_{i,j \in \psi^-} W(\sigma_i, \omega \sigma_j) \times J(r),$$

where $J(r)$ is defined as

$$J(r) := (e^{-2iu} - e^{2iu} \omega r^{-1}) Y(u, \xi | r - 1) - (e^{2i\xi - i\pi'} - e^{-2i\xi + i\pi'} \omega r^{-1}) Y(u, \xi | r).$$

To define the boundary CR equation for the $\mathbb{Z}_N$ model, we proceed by analogy with the $O(n)$ model. In the latter, this equation can be viewed as a relation between the discrete integral $\sum_{\sigma} F_s(z) \delta z$ and its counterpart after a reflection about a horizontal axis. Similarly, we define the ‘reflected quantity’ $J^*(r)$, and introduce the boundary CR equation

$$e^{i\phi} J(r) + e^{-i\phi} J^*(r) = 0,$$

(32)

where $\phi$ is a constant angle, to be determined.

1 Like for the $O(n)$ model, for a given angle $\alpha$, the spectral parameter for boundary weights is half the bulk value (24). Moreover, on a triangular face, the angle opposite to the diagonal $\sigma_1 \sigma_2$ is $\pi - \alpha$ rather than $\alpha$ in the bulk.
Equation (32) is easier to deal with after Fourier transform, where, using the value $s = 1 - 1/N$, it becomes
\[
\sin[(2k + 1)\lambda - u - \xi] \sin[(2k + 1)\lambda + \varphi - u + \xi] \hat{Y}(u, \xi | k) - \sin[(2k - 1)\lambda + u + \xi] \times \sin[(2k - 1)\lambda + \varphi + u - \xi] \hat{Y}(u, \xi | k + 1) = 0.
\] (33)
This is a simple recursion relation on $k$, and one finds that, for $\hat{Y}$ to satisfy (13) (i.e. $Y$ is periodic in $r$ and takes real values), the angle in (32) must be $\varphi = 2\lambda$. The solution of (32) then reads
\[
\hat{Y}(u, \xi | k) = W(u + \xi | k)W(u - \xi | k),
\] (34)
where $W$ was defined in (18). A remarkable fact is that the weights (34) are also a solution of the boundary YB equation (29). The solutions $W^{(m)}, Y^{(m)}$ associated with the other parafermions $\psi^{(m)}$ also satisfy the boundary YB equation, and they are related to (18)–(34) by a simple transformation $\sigma_j \rightarrow \sigma_j^m$.

3.5. Physical interpretation of the integrable BC

To indentify the BC corresponding to (34) in the scaling limit, we consider the diagonal-to-diagonal transfer matrix, which is constructed from the two-row transfer matrix by the procedure described in [16]. If the two-row transfer matrix with horizontal spectral parameter $u$, vertical spectral parameters $(v_1, \ldots, v_N)$ and boundary parameters $\xi_{LR}$ is denoted
\[
t(u, \xi_L, \xi_R | v_1, \ldots, v_N),
\]
then the diagonal-to-diagonal transfer matrix is given by
\[
t_d(u, \xi_L, \xi_R) := t \left( \frac{u}{2}, \xi_L, \xi_R | \frac{u}{2}, -\frac{u}{2}, \ldots, -\frac{u}{2} \right).
\]
In this setting, the bulk weights are given by $W(u | \cdot)$ for horizontal edges and $W(\lambda - u | \cdot)$ for vertical edges, whereas the left and right boundary weights read
\[
Y(\lambda - u/2, \xi_L | r)W(\lambda - u | r) \quad \text{and} \quad Y(u/2, \xi_R | r).
\]

For the value $\xi_L = \xi_R = u/2$, the latter coincide\(^2\), and both are proportional to $W(\lambda - u | r)$. Hence, for this choice of boundary parameters $\xi_L, \xi_R$, the boundary weights are equal to the bulk weights in the vertical direction, and thus (34) corresponds to free BCs for the spins $\{\sigma_j\}$. Although these BC are already known to be critical in the scaling limit, we have shown here that they satisfy both the boundary YB and CR equation, as part of a family of BC labelled by $\xi$.

4. Conclusions

In the two examples that we have studied, we have shown that the integrable boundary weights can be obtained by imposing a simple linear condition—which we have called the boundary Cauchy–Riemann (CR) equation—on the discrete contour integral
\[
\sum_P F_P(z) dz,
\]
where $P$ stands for a modified boundary face. In the case of the $O(n)$ model, we have recovered the integrable boundary weights which were obtained [16] through the mapping to the 19-vertex model, whereas for the $\mathbb{Z}_N$ model, we have found new integrable boundary weights. However, unlike the bulk CR equations, where the conformal spin $s$ is generally obtained

\(^2\) For this value, the bulk CR equation automatically ensures the boundary CR equation, since $\alpha = \beta$ in figure 9.
by simple consistency conditions, it is not clear yet how to extract conformal data from the boundary CR equations.

Our approach is quite general and is likely to extend to any lattice model where a solution of the bulk CR equation has been identified. Also, an interesting continuation of this work would be to study the boundary CR equations in loop models with 'non-trivial' integrable boundary conditions, i.e. where loops touching the boundary get a different weight [20].

A physical interpretation of boundary CR equations such as (8) and (32) is still lacking, and would be very useful to generalize it systematically to other models.

Finally, an important possible application of our results would be to set up proofs of convergence to variants of SLE as in [8, 9], in the presence of various integrable boundary conditions.

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