On strong rainbow connection number∗

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Abstract

A path in an edge-colored graph, where adjacent edges may be colored the same, is a rainbow path if no two edges of it are colored the same. For any two vertices $u$ and $v$ of $G$, a rainbow $u - v$ geodesic in $G$ is a rainbow $u - v$ path of length $d(u, v)$, where $d(u, v)$ is the distance between $u$ and $v$. The graph $G$ is strongly rainbow connected if there exists a rainbow $u - v$ geodesic for any two vertices $u$ and $v$ in $G$. The strong rainbow connection number of $G$, denoted $src(G)$, is the minimum number of colors that are needed in order to make $G$ strong rainbow connected. In this paper, we first investigate the graphs with large strong rainbow connection numbers. Chartrand et al. obtained that $G$ is a tree if and only if $src(G) = m$, we will show that $src(G) \neq m - 1$, so $G$ is not a tree if and only if $src(G) \leq m - 2$, where $m$ is the number of edge of $G$. Furthermore, we characterize the graphs $G$ with $src(G) = m - 2$. We next give a sharp upper bound for $src(G)$ according to the number of edge-disjoint triangles in graph $G$, and give a necessary and sufficient condition for the equality.

Keywords: edge-colored graph, rainbow path, rainbow geodesic, strong rainbow connection number, edge-disjoint triangle.

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1 Introduction

All graphs in this paper are finite, undirected and simple. Let $G$ be a nontrivial connected graph on which is defined a coloring $c : E(G) \to \{1, 2, \ldots, n\}$, $n \in \mathbb{N}$, of the edges of $G$, where adjacent edges may be colored the same. A path is a rainbow path if no two edges of it are colored the same. An edge-coloring graph $G$ is rainbow connected if any two vertices are connected by a rainbow path. Clearly, if a graph is rainbow connected, it must be connected. Conversely, any connected graph has a trivial edge-coloring that makes it rainbow connected; just color each edge with a distinct color. Thus, we define the rainbow connection number of a connected graph $G$, denoted $rc(G)$, as the smallest number of colors that are needed in order to make $G$ rainbow connected. Let $c$ be a rainbow coloring

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of a connected graph $G$. For any two vertices $u$ and $v$ of $G$, a rainbow $u - v$ geodesic in $G$ is a rainbow $u - v$ path of length $d(u, v)$, where $d(u, v)$ is the distance between $u$ and $v$. The graph $G$ is strongly rainbow connected if there exists a rainbow $u - v$ geodesic for any two vertices $u$ and $v$ in $G$. In this case, the coloring $c$ is called a strong rainbow coloring of $G$. Similarly, we define the strong rainbow connection number of a connected graph $G$, denoted $src(G)$, as the smallest number of colors that are needed in order to make $G$ strongly rainbow connected. A strong rainbow coloring of $G$ using $src(G)$ colors is called a minimum strong rainbow coloring of $G$. Clearly, we have $diam(G) \leq rc(G) \leq src(G) \leq m$ where $diam(G)$ denotes the diameter of $G$ and $m$ is the size of $G$. In an edge-colored graph $G$, we use $c(e)$ to denote the color of edge $e$, then for a subgraph $G_1$ of $G$, $c(G_1)$ denotes the set of colors of edges in $G_1$.

In [3], the authors investigated the graphs with small rainbow connection numbers, they determined a sufficient condition that guarantee $rc(G) = 2$.

**Theorem 1.1** ([3]) Any non-complete graph with $\delta(G) \geq n/2 + \log n$ has $rc(G) = 2$.

Let $G = G(n, p)$ denote, as usual, the random graph with $n$ vertices and edge probability $p$. For a graph property $A$ and for a function $p = p(n)$, we say that $G(n, p)$ satisfies $A$ almost surely if the probability that $G(n, p(n))$ satisfies $A$ tends to 1 as $n$ tends to infinity. We say that a function $f(n)$ is a sharp threshold function for the property $A$ if there are two positive constants $c$ and $C$ so that $G(n, cf(n))$ almost surely does not satisfy $A$ and $G(n, p)$ satisfies $A$ almost surely for all $p \geq Cf(n)$. In [3], the authors also determined the threshold function for a random graph to have $rc(G(n, p)) \leq 2$.

**Theorem 1.2** ([3]) $p = \sqrt{\log n/n}$ is a sharp threshold function for the property $rc(G(n, p)) \leq 2$.

In [2], the authors derived that the following proposition.

**Proposition 1.3** ([2]) $rc(G) = 2$ if and only if $src(G) = 2$.

That is, the problem of considering graphs with $rc(G) = 2$ is equivalent to that of considering graphs with $src(G) = 2$. So we aim to investigate the graphs with large (strong) rainbow connection numbers. In [1], we investigated the graphs with large rainbow connection numbers. In [2], Chartrand et al. obtained that $rc(G) = m$ if and only if $G$ is a tree. And in [4], we proved that $rc(G) \neq m - 1$, so $rc(G) \leq m - 2$ if and only if $G$ is not a tree. Furthermore, we characterized the graphs with $rc(G) = m - 2$. The four graph classes shown in Figure 1.1 was useful in the following result, where the paths $P_j$s may be trivial in each $G_i(1 \leq i \leq 4)$.

**Theorem 1.4** ([2]) $rc(G) \leq m - 2$ if and only if $G$ is not a tree. Furthermore, $rc(G) = m - 2$ if and only if $G$ is a 5-cycle or belongs to one of the four graph classes shown in Figure 1.1.

In this paper, we continue to investigate the graphs with large strong rainbow connection numbers with an essentially different argument. In [2], Chartrand et al. obtained that
Figure 1.1 The figure for the four graph classes.

$src(G) = m$ if and only if $G$ is a tree. Furthermore, we can show that $src(G) \neq m - 1$, so $src(G) \leq m - 2$ if and only if $G$ not is a tree. We also characterize the graphs with $src(G) = m - 2$ by showing that $src(G) = m - 2$ if and only if $G$ is a 5-cycle or belongs to one of three graph classes (Theorem 3.1).

In [2], Chartrand et al. determined the precise strong rainbow connection numbers for other special graph classes including complete graphs, wheels, complete bipartite (multipartite) graphs. However, for a general graph $G$, it is almost impossible to give the precise value for $src(G)$, so we aim to give upper bounds for it. In this paper, we will derive a sharp upper bound for $src(G)$ according to the number of edge-disjoint triangles in graph $G$, and give a necessary and sufficient condition for the equality.

We use $V(G), E(G)$ for the set of vertices and edges of $G$, respectively. For any subset $X$ of $V(G)$, let $G[X]$ be the subgraph induced by $X$, and $E[X]$ the edge set of $G[X]$; similarly, for any subset $E_1$ of $E(G)$, let $G[E_1]$ be the subgraph induced by $E_1$. Let $\mathcal{G}$ be a set of graphs, then $V(\mathcal{G}) = \bigcup_{G \in \mathcal{G}} V(G)$, $E(\mathcal{G}) = \bigcup_{G \in \mathcal{G}} E(G)$. A rooted tree $T(x)$ is a tree $T$ with a specified vertex $x$, called the root of $T$. Each vertex on the path $xTv$, including the vertex $v$ itself, is called an ancestor of $v$, an ancestor of a vertex is proper if it is not the vertex itself, the immediate proper ancestor of a vertex $v$ other than the root is its parent and the vertices whose parent is $v$ are its children or son. We let $P_n$ and $C_n$ be the path and cycle with $n$ vertices, respectively. $P : u_1, u_2, \ldots, u_t$ is a path, then the $u_i - u_j$ section of $P$, denoted by $u_iPu_j$, is the path: $u_i, u_{i+1}, \ldots, u_j$. Similarly, for a cycle $C : v_1, \ldots, v_t, v_1$; we define the $v_i - v_j$ section, denoted by $v_iCv_j$ of $C$, and $C$ contains two $v_i - v_j$ sections. Note the fact that if $P$ is a $u_i - u_t$ geodesic, then $u_iPu_j$ is also a $u_i - u_j$ geodesic where $1 \leq i, j \leq t$. We use $l(P)$ to denote the length of path $P$. Let $[n] = \{1, \ldots, n\}$ denote the set of the first $n$ natural numbers. For a set $S$, $|S|$ denote the cardinality of $S$. In a graph $G$ which has at least one cycle, the length of a shortest cycle is called its girth, denoted $g(G)$.

We follow the notation and terminology of [1].
2 Basic results

We first give a necessary condition for an edge-colored graph to be strong rainbow connected. If \( G \) contains at least two cut edges, then for any two cut edges \( e_1 = u_1u_2, e_1 = v_1v_2 \), there must exist some \( 1 \leq i_0, j_0 \leq 2 \), such that any \( u_{i_0} - v_{j_0} \) path must contain edge \( e_1, e_2 \). So we have:

**Observation 2.1** If \( G \) is strong rainbow connected under some edge-coloring, \( e_1 \) and \( e_2 \) are any two cut edges (if exist), then 
\[
c(e_1) \neq c(e_2).
\]

We need a lemma which will be useful in the argument of our result on graphs with large strong rainbow connection numbers.

**Lemma 2.2** \( G \) is a connected graph with at least one cycle, and \( 3 \leq g(G) \leq 5 \). Let \( C_1 \) be the smallest cycle of \( G \), and \( C_2 \) be the smallest cycle among all remaining cycles (if exist) of \( G \). If \( C_1 \) and \( C_2 \) have at least two common vertices, then we have:

1. If \( g(G) = 3 \), then \( C_1 \) and \( C_2 \) have a common edge as shown in Figure 2.1.
2. If \( g(G) = 4 \), then \( C_1 \) and \( C_2 \) have a common edge, or two common (adjacent) edges, or \( C_1 \) and \( C_2 \) are two edge-disjoint 4-cycles, as shown in Figure 2.1.
3. If \( g(G) = 5 \), then \( C_1 \) and \( C_2 \) have a common edge, or two common (adjacent) edges, as shown in Figure 2.1.

![Figure 2.1](image)

**Proof.** We only consider the case that \( g(G) = 5 \), the remaining two cases are similar. Let \( |C_i| = k_i (i = 1, 2) \), and \( V(C_1) = \{u_i : 1 \leq i \leq 5\} \), we have \( k_2 \geq k_1 = 5 \). We will consider four cases according to the value of \( |V(C_1) \cap V(C_2)| \).
Case 1. \(|V(C_1) \cap V(C_2)| = 5\), that is, \(V(C_1) \subseteq V(C_2)\). By the choice of \(C_1\) and \(C_2\), we have \(|E(C_1) \cap E(C_2)| \leq 4\). Without loss of generality, let \(u_1u_2\) doesn’t belong to \(E(C_2)\). Let \(C'_2\) be the \(u_1C_2u_2\) section of \(C_2\) which doesn’t contain \(u_3, u_4, u_5\), then \(C'_2\) and \(u_1u_2\) produce a smaller cycle than \(C_2\), a contradiction.

Case 2. \(|V(C_1) \cap V(C_2)| = 4\), say \(u_i(1 \leq i \leq 4) \in V(C_2)\). In this case, we have \(|E(C_1) \cap E(C_2)| \leq 3\).

Subcase 2.1. \(|E(C_1) \cap E(C_2)| = 3\), that is, \(u_1u_2, u_2u_3, u_3u_4 \in E(C_2)\). Let \(C'_2\) be the \(u_1C_2u_4\) section of \(C_2\) which doesn’t contain \(u_2, u_3\), then \(C'_2\) and edges \(u_1u_5, u_5u_4\) produce a smaller cycle than \(C_2\), a contradiction.

Subcase 2.2. \(|E(C_1) \cap E(C_2)| \leq 2\). Then there exists one edge, say \(u_3u_4\), which doesn’t belong to \(C_2\). Let \(C'_2\) be the \(u_3C_2u_4\) section of \(C_2\) which doesn’t contain \(u_1, u_2\), then \(C'_2\) and edge \(u_3u_4\) produce a smaller cycle than \(C_2\), a contradiction.

Case 3. \(|V(C_1) \cap V(C_2)| = 3\).

Subcase 3.1. These three vertices are consecutive, say \(u_1, u_2, u_3 \in V(C_2)\). In this case, we have \(|E(C_1) \cap E(C_2)| \leq 2\). Suppose \(|E(C_1) \cap E(C_2)| \leq 1\). Without loss of generality, we assume \(u_2u_3\) doesn’t belong to \(C_2\), and let \(C''_2\) be the \(u_2u_3\) section of \(C_2\) that doesn’t contain \(u_1\). \(C''_2\) be the \(u_1C_2u_3\) section of \(C_2\) that doesn’t contain \(u_2\). Then \(C''_2\) and edges \(u_1u_5, u_5u_4, u_4u_3\) will produce a cycle \(C''_2\) with \(k_3 = |C''_2| \geq k_2 \geq 5\), so \(l(C''_2) \geq 2\), but now \(C''_2\) and \(u_2u_3\) produce a smaller cycle than \(C_2\), a contradiction. So we have \(|E(C_1) \cap E(C_2)| = 2\).

Subcase 3.2. Two of these three vertices are not consecutive, say \(u_1, u_2, u_4\). In this case, we have \(|E(C_1) \cap E(C_2)| \leq 1\). With a similar argument to Subcase 3.1, we get a contradiction.

Case 4. \(|V(C_1) \cap V(C_2)| = 2\).

Subcase 4.1. These two vertices are adjacent in \(C_1\), say \(u_1, u_2\). If \(u_1u_2\) doesn’t belong to \(C_2\), then edge \(u_1u_2\) is a chord of \(C_2\) which divides \(C_2\) into two parts \(C'_2, C''_2\). Let \(C''_2\) \(C''_2\) be the cycle produced by edge \(u_1u_2\) and \(C'_2\) \(C'_2\). So we have \(|C''_2|, |C''_2| \geq 5, |C'_2|, |C'_2| \geq 4\), so we have \(|C''_2|, |C''_2| < |C'_2| + |C'_2| = |C_2|\) and get a contradiction, so we have \(u_1u_2 \in C_2\).

Subcase 4.2. These two vertices are nonadjacent in \(C_1\), say \(u_1, u_3\). Then with a similar argument to Subcase 4.1 (instead \(u_1u_2\) by \(u_1u_2, u_2u_3\)), we get a contradiction.

So by the above discussion, 3 holds.

3 Graphs with large strong rainbow connection numbers

In this section, we will give our result on graphs with large strong rainbow connection numbers. We first introduce three graph classes. Let \(C\) be the cycle of a unicyclic graph \(G\), \(V(C) = \{v_1, \ldots, v_k\}\) and \(T_G = \{T_i : 1 \leq i \leq k\}\) where \(T_i\) is the unique tree containing vertex \(v_i\) in subgraph \(G \setminus E(C)\). We say \(T_i\) and \(T_j\) are adjacent(nonadjacent) if \(v_i\) and \(v_j\) are adjacent(nonadjacent) in cycle \(C\). Then let \(G_1 = \{G : G\) is a unicyclic graph, \(k = 3\), \(T_G\) contains at most two nontrivial elements\},
\( \mathcal{G}_2 = \{ G : G \text{ is a unicyclic graph, } k = 4, \mathcal{T}_G \text{ contains two nonadjacent nontrivial elements and each nontrivial element is a path, the remaining two elements are trivial} \} \), \\
\( \mathcal{G}_3 = \{ G : G \text{ is a unicyclic graph, } k = 4, \mathcal{T}_G \text{ contains at most one nontrivial element and this nontrivial element (if exists) is a path} \} \).

The following theorem is one of our main results. During its proof, we derive that \( \text{src}(G) \neq m - 1 \).

**Theorem 3.1** \( \text{src}(G) \leq m - 2 \) if and only if \( G \) is not a tree, furthermore, \( \text{src}(G) = m - 2 \) if and only if \( G \) is a 5-cycle or belongs to one of \( \mathcal{G}_i (1 \leq i \leq 3) \).

**Proof.** In [2], the authors obtained that \( \text{src}(G) = m \) if and only if \( G \) is a tree, so \( \text{src}(G) \leq m - 1 \) if and only if \( G \) is not a tree. In order to derive our conclusion, we need the following three claims.

**Claim 1.** If \( \text{src}(G) = m - 1 \) or \( m - 2 \), then \( 3 \leq g(G) \leq 5 \).

**Proof of Claim 1.** Let \( C : v_1, \cdots, v_k, v_{k+1} = v_1 \) be a minimum cycle of \( G \) with \( k = g(G) \), and \( e_i = v_i v_{i+1} \) for each \( 1 \leq i \leq k \), we suppose that \( k \geq 6 \).

Now we give the cycle \( C \) a strong rainbow coloring the same as [2]: If \( k \) is even, let \( k = 2 \ell \) for some integer \( \ell \geq 3 \), \( c(e_i) = i \) for \( 1 \leq i \leq \ell \) and \( c(e_i) = i - \ell \) for \( \ell + 1 \leq i \leq k \); If \( k \) is odd, let \( k = 2 \ell + 1 \) for some integer \( \ell \geq 3 \), \( c(e_i) = i \) for \( 1 \leq i \leq \ell + 1 \) and \( c(e_i) = i - \ell - 1 \) for \( \ell + 2 \leq i \leq k \). We color each other edge with a fresh color. This procedure costs \( \left[ \frac{k}{2} \right] + (m - k) = m - (k - \left[ \frac{k}{2} \right]) \leq m - 3 \) colors totally.

We will show that, with the above coloring, \( G \) is strong rainbow connected, it suffices to show that there is a rainbow \( u - v \) geodesic for any two vertices \( u, v \) of \( G \). We first consider the case \( k = 2 \ell (\ell \geq 3) \). If there exists one \( u - v \) geodesic \( P \) which have at most one common edge with \( C \), then \( P \) must be a rainbow geodesic.

So we can assume that each \( u - v \) geodesic have at least two common edges with \( C \), we choose one such geodesic, say \( P : u = u_1, \cdots, v = u_t \). If there are two edges of \( P \), say \( e'_1, e'_2 \), with the same color, then they must be in \( C \), too. Without loss of generality, let \( e'_1 = v_1 v_2 \), we first consider the case that \( e'_1 = v_1 v_2 \) and \( v_1 = u_{i_1}, v_2 = u_{i_1+1} \) for some \( 1 \leq i_1 \leq t \), then we must have \( e'_2 = v_{t+1} v_{t+2} \) where \( v_{t+1} = u_j, v_{t+2} = u_{j+1} \) for some \( i_1 + 1 \leq j \leq t \) or \( v_{t+1} = u_{j_1}, v_{t+2} = u_{j_1+1} \) for some \( i_1 + 1 \leq j_1 \leq t \) (For example, see graph (\( \alpha \)) of Figure 3.1 where \( \ell = 4 \), the color of each edge is shown), then the section \( v_2 P v_{t+1} \) of \( P \) is a \( v_2 - v_{t+1} \) geodesic, so it is not longer than the section \( C' : v_2, v_3, \cdots, v_{t+1} \) of \( C \), then the length of \( v_2 P v_{t+1} \), \( l(v_2 P v_{t+1}) \), is smaller than the length of the section \( C'' : v_2, v_1, v_k, \cdots, v_{t+1} \) of \( C \). So the sections \( v_2 P v_{t+1} \) and \( C' \) will produce a smaller cycle than \( C \) (this produces a contradiction), or \( v_2 P v_{t+1} \) is the same as \( C' \) (but in this case, the section \( C''' : v_1, v_k, \cdots, v_{t+2} \) of \( C \) is shorter than \( v_1 P v_{t+2} \) which now is a \( v_1 - v_{t+2} \) geodesic, this also produces a contradiction). If \( v_{t+2} = u_{j_2}, v_{t+1} = u_{j_2+1} \) for some \( i_1 + 1 \leq j_2 \leq t \) (For example, see graph (\( \beta \)) of Figure 3.1 where \( \ell = 4 \), the color of each edge is shown), then the section \( v_1 P v_{t+2} \) of \( P \) is a \( v_1 - v_{t+2} \) geodesic, so it is not longer than the length of the section \( C'' : v_1, v_k, v_{k-1}, \cdots, v_{t+2} \) of \( C \) and its length, \( l(v_1 P v_{t+2}) \), is smaller than that of the section \( C''' : v_1, v_2, \cdots, v_{t+2} \) of \( C \). So the sections \( v_1 P v_{t+2} \) and \( C' \) will produce a smaller cycle than \( C \), this also produces a contradiction.
The remaining two subcases correspond to the case that \( v_1 = u_{i_1+1}, \) \( v_2 = u_{i_1} \) (see graphs \((\gamma)\) and \((\omega)\) in Figure 3.1 for the case of \( \ell = 4 \)), and with a similar argument, a contradiction will be produced. So \( P \) is rainbow.

![Figure 3.1 Graphs for the example with \( \ell = 4 \).](image)

The case that \( k = 2\ell + 1 (\ell \geq 3) \) is similar. So \( G \) does not contain a cycle of length larger than 5.

Note that during the proof of Claim 1, we use the following technique: we first choose a smallest cycle \( C \) of a graph \( G \), then give it a strong rainbow coloring the same as [2], and give a fresh color to any other edge. Then for any \( u - v \) geodesic \( P \), we derive that either one section of \( P \) is the same as one section of \( C \) and then find a shorter path than the geodesic, or one section of \( P \) and one section of \( C \) produce a smaller cycle than \( C \), each of these two cases will produce a contradiction. This technique will be useful in the sequel.

Next we will show that \( G \) is a unicyclic graph under the condition that \( src(G) = m - 1 \) or \( m - 2 \).

**Claim 2.** If \( src(G) = m - 1 \) or \( m - 2 \), then \( G \) is a unicyclic graph.

**Proof of Claim 2.** Suppose \( G \) contains at least two cycles, let \( C_1 \) be the smallest cycle of \( G \) and \( C_2 \) be the smallest one among all the remaining cycles in \( G \), that is, \( C_2 \) is the smallest cycle with the exception of cycle \( C_1 \). Let \( |C_i| = k_i (i = 1, 2) \), so by the above discussion, we have \( 3 \leq k_1 \leq 5 \) and \( k_2 \geq k_1 \). We will consider two cases according to the value of \( |V(C_1) \cap V(C_2)| \).

**Case 1.** \( |V(C_1) \cap V(C_2)| \leq 1 \), that is, \( C_1 \) and \( C_2 \) have at most one common vertex. There are three subcases:
Subcase 1.1. \( k_1 = 3 \), that is, \( C_1 \) is a triangle. The following fact is trivial and will be useful:

Fact 1. For any two vertices \( u, v \) and a triangle \( T \), any \( u-v \) geodesic \( P \) contains at most one edge of \( T \).

We first give cycle \( C_2 \) a strong rainbow coloring using \( \lceil \frac{k_2}{2} \rceil \) colors the same as \([2]\); then give a fresh color to \( C_1 \), that is, edges of \( C_1 \) receive the same color; for the remaining edges, we give each of them a fresh color. With a similar procedure (technique) to that of Claim 1 and by Fact 1, we can show that the above coloring is strong rainbow, as this costs \( 1 + \lceil \frac{k_2}{2} \rceil + (m - k_2 - 3) = (m - 2) - (k_2 - \lceil \frac{k_2}{2} \rceil) \leq m - 3 \), a contradiction.

Subcase 1.2. \( k_1 = 4 \), that is, \( C_1 \) is a 4-cycle. With a similar (and a little simpler) argument to that of Claim 1, we can give the following fact,

Fact 2. For any two vertices \( u, v \), any \( u-v \) geodesic \( P \) contains at most one edge or two (adjacent) edges of \( C_1 \).

We now give a brief proof to Fact 2: Suppose it doesn’t hold, that is, there exist a geodesic \( P : a_1, \cdots, a_t \) for two vertices \( u, v \) which contains two nonadjacent edges, say \( u_1u_2, u_3u_4 \). Without loss of generality, we let \( u_1 = a_{i_1}, u_2 = a_{i_2}, u_3 = a_{i_3}, u_4 = a_{i_4} \) where \( \max\{i_1, i_2\} < \min\{i_3, i_4\} \). We only consider the case that \( i_1 < i_3 < i_4 \), the remaining three cases are similar. Then the section \( a_{i_1}Pa_{i_4} \) of \( P \) is a \( u_1 - u_4 \) geodesic whose length is at least three, but the edge \( u_1u_4 \) is a \( u_1 - u_4 \) path which is shorter than it, this produces a contradiction, so the fact holds.

We first give cycle \( C_2 \) a strong rainbow coloring using \( \lceil \frac{k_2}{2} \rceil \) colors the same as \([2]\); then give two fresh colors to \( C_1 \) in the same way; for the remaining edges, we give each of them a fresh color. With a similar procedure (technique) to that of Claim 1 and by Fact 2, we can show that the above coloring is strong rainbow, as this costs \( 2 + \lceil \frac{k_2}{2} \rceil + (m - k_2 - 4) \) colors totally, we have \( src(G) \leq 2 + \lceil \frac{k_2}{2} \rceil + (m - k_2 - 4) = (m - 2) - (k_2 - \lceil \frac{k_2}{2} \rceil) \leq m - 3 \), a contradiction.

Subcase 1.3. \( k_1 = 5 \), that is, \( C_1 \) is a 5-cycle. With a similar argument to that of Fact 2, we can give the following fact,

Fact 3. For any two vertices \( u, v \), any \( u-v \) geodesic \( P \) contains at most one edge or two (adjacent) edges of \( C_1 \).

We first give cycle \( C_2 \) a strong rainbow coloring using \( \lceil \frac{k_2}{2} \rceil \) colors the same as \([2]\); then give three fresh colors to \( C_1 \) in the same way; for the remaining edges, we give each of them a fresh color. With a similar procedure (technique) to that of Claim 1 and by Fact 3, we can show that the above coloring is strong rainbow, as this costs \( 3 + \lceil \frac{k_2}{2} \rceil + (m - k_2 - 5) \) colors totally, we have \( src(G) \leq 3 + \lceil \frac{k_2}{2} \rceil + (m - k_2 - 5) = (m - 2) - (k_2 - \lceil \frac{k_2}{2} \rceil) \leq m - 3 \), a contradiction.

Note that for each above subcase, by each corresponding fact, the cycle produced during the procedure while we use the technique of Claim 1 cannot be the cycle \( C_1 \) and must be smaller than \( C_2 \), then a contradiction will be produced.

So if \( src(G) = m - 1 \) or \( m - 2 \), Case 1 doesn’t hold, we now consider the next case:
**Case 2.** \(|V(C_1) \cap V(C_2)| \geq 2\), that is, \(C_1\) and \(C_2\) have at least two common vertices. Similarly, we need to consider three subcases:

![Figure 3.2 Graphs for Case 2 of Claim 2.](image)

**Subcase 2.1.** \(k_1 = 3\), that is, \(C_1\) is a triangle. By Lemma 2.2, \(C_1\) and \(C_2\) have a common edge as shown in Figure 2.1. Let \(V(C_2) = \{v_i : 1 \leq i \leq k_2\}\) and \(v_{k_2+1} = v_1\), where \(v_1 = u_1, v_2 = u_2\). Let \(P'\) be the subpath of \(C_2\) that doesn’t contain edge \(v_1v_2\). We now give \(G\) an edge-coloring as follows:

For the case \(l(P') = 2, 3\), we first color edges of \(C_1 \cup C_2\) as shown in Figure 3.2 (graphs \(a'\) and \(b'\)); then give each other edge of \(G\) a fresh color. This procedure costs \(m - 3\) colors totally. Then with a similar argument to **Fact 2**, we can show that any geodesic cannot contain two edges with the same color, so \(src(G) \leq m - 3\).

For the remaining case, that is, \(l(P') \geq 4\) and \(k_2 \geq 5\). We first give cycle \(C_1\) a color, say \(a\), that is, three edges of \(C_1\) receive the same color. Then in \(C_2\), if \(k_2 = 2\ell\) for some \(\ell \geq 2\), then let \(c(v_2v_3) = c(v_{\ell+2}v_{\ell+3})\) be a new color, say \(b\); if \(k_2 = 2\ell + 1\) for some \(\ell \geq 2\), then let \(c(v_2v_3) = c(v_{\ell+3}v_{\ell+4})\) be a new color, say \(b\). For the remaining edges, we give each of them a fresh color. This procedure costs \(m - 3\) colors totally. For any two vertices \(u, v\), \(P\) is a \(u-v\) geodesic, by **Fact 1**, \(P\) cannot contain two edges with color \(a\); for the two edges with color \(b\), with a similar argument to that of **Claim 1** (Note that now, by **Fact 1**, the cycle produced during the procedure cannot be \(C_1\) and must be shorter than \(C_2\), then a contradiction will be produced), we can show \(P\) contains at most one of them. So \(P\) is rainbow and \(src(G) \leq m - 3\).

**Subcase 2.2.** \(k_1 = 4\), that is, \(C_1\) is a 4-cycle. By Lemma 2.2, \(C_1\) and \(C_2\) have a common edge, or two common (adjacent) edges, or \(C_1\) and \(C_2\) are two edge-disjoint 4-cycles, as shown in Figure 2.1

**Subsubcase 2.2.1.** If \(C_1\) and \(C_2\) have a common edge, say \(u_1u_2\) (see the left one of the three graphs with \(g(G) = 4\) in Figure 2.1). We let \(V(C_2) = \{v_i : 1 \leq i \leq k_2\}\), where \(v_1 = u_1, v_2 = u_2\). We let \(c(v_2v_3) = c(u_1v_1) = a, c(v_2u_3) = c(v_1v_2) = b, c(u_1v_2) = c(u_3u_4) = c\). For the remaining edges, we give each of them a fresh color. This procedure costs \(m - 3\) colors totally. For any two vertices \(u, v\), \(P\) is a \(u-v\) geodesic, then by **Fact 2**, \(P\) contains at most one of the two edges with color \(c\); for the two edges with color \(a(b)\), with a similar argument to that of **Fact 2**, we can show that there exists one \(u-v\) geodesic which contains
at most one of them. So we have \( src(G) \leq m - 3 \).

**Subsubcase 2.2.2.** If \( C_1 \) and \( C_2 \) have two (adjacent) common edges, say \( u_1u_2, u_2u_3 \) (see the middle one of the three graphs with \( g(G) = 4 \) in Figure 2.1). We let \( V(C_2) = \{ v_i : 1 \leq i \leq k_2 \} \), where \( v_1 = u_1, v_2 = u_2, v_3 = u_3 \). Let \( P' \) be the subpath of \( C_2 \) which doesn’t contain edges \( u_1u_2, u_2u_3 \).

For the case \( l(P') = 2, 3 \), we first color edges of \( C_1 \cup C_2 \) as shown in Figure 3.2 (graphs \( c' \) and \( d' \)); then give each other edge of \( G \) a fresh color. This procedure costs \( m - 3 \) colors totally. Then with a similar argument to **Fact 2**, we can show that any geodesic contains at most one edge with the same color, so we have \( src(G) \leq m - 3 \).

For the case \( l(P') \geq 4 \), that is \( k_2 \geq 6 \). We let \( c(u_4v_1) = c(v_3v_4) = a, c(v_1v_2) = c(v_3u_4) = b \); for edge \( v_2v_3 \), we give a similar treatment to that of **Subcase 2.1** and let \( c(v_2v_3) = c \); we then give each other edge of \( G \) a fresh color. This procedure costs \( m - 3 \) colors totally. For any two vertices \( u, v \), \( P \) is a \( u - v \) geodesic, then by **Fact 2**, \( P \) contains at most one of the two edges with color \( b \); for the two edges with color \( a \), with a similar argument to that of **Fact 2**, we can show that there exists one \( u - v \) geodesic which contains at most one of them. With a similar argument to that of **Claim 1** (Note that now, by **Fact 2**, the cycle produced during the procedure cannot be \( C_1 \) and must be shorter than \( C_2 \), then a contradiction will be produced), we can show any geodesic contains at most one edge with color \( c \). So we have \( src(G) \leq m - 3 \).

**Subsubcase 2.2.3.** The remaining case, the right graph of the three graphs with \( g(G) = 4 \) in Figure 2.1. We let \( c(u_1u_2) = c(u_3u_4) = a, c(u_2u_3) = c(u_1u_4) = b, c(u_1v_2) = c(v_3v_4) = c \); \( c(v_2u_3) = c(u_1v_4) = d \), where \( a, b, c, d \) are four distinct colors; for the remaining edges, we give each of them a fresh color. This procedure costs \( m - 4 \) colors totally. As now both \( C_1 \) and \( C_2 \) are the smallest cycle of \( G \), by **Fact 2**, any geodesic contains at most one of the two edges with the same color, so \( src(G) \leq m - 4 \).

**Subcase 2.3.** \( k_1 = 5 \), that is, \( C_1 \) is a 5-cycle. By Lemma 2.2, \( C_1 \) and \( C_2 \) have a common edge, or two common (adjacent) edges, as shown in Figure 2.1. The following discussion will use **Fact 3**.

**Subsubcase 2.3.1.** If \( C_1 \) and \( C_2 \) have a common edge, say \( u_1u_2 \) (see the left one of the two graphs with \( g(G) = 5 \) in Figure 2.1). We let \( V(C_2) = \{ v_i : 1 \leq i \leq k_2 \} \), where \( v_1 = u_1, v_2 = u_2 \). We let \( c(u_4u_5) = c(v_2v_3) = a, c(v_1u_5) = c(v_2u_3) = b, \) and \( c(v_1v_2) = c(u_3u_4) = c \); for the remaining edges, we give each of them a fresh color. This procedure costs \( m - 3 \) colors totally. With a similar argument to above, we can show that \( src(G) \leq m - 3 \).

**Subsubcase 2.3.2.** If \( C_1 \) and \( C_2 \) have two common (adjacent) edges, say \( u_1u_2, u_2u_3 \) (see the right one of the two graphs with \( g(G) = 5 \) in Figure 2.1). We let \( c(v_1u_5) = c(v_3v_4) = a, c(v_1v_2) = c(v_3u_4) = b, \) and \( c(v_2v_3) = c(u_4u_5) = c \); for the remaining edges, we give each of them a fresh color. This procedure costs \( m - 3 \) colors totally. With a similar argument to above, we can show that \( src(G) \leq m - 3 \).

With the above discussion, **Claim 2** holds. To complete our proof, we still need the following claim.

**Claim 3.** \( src(G) \neq m - 1 \). Furthermore, \( src(G) = m - 2 \) if and only if \( G \) is a 5-cycle or belongs to one of \( G_5s(1 \leq i \leq 3) \).
Proof of Claim 3. Let $G$ be a unicyclic graph and $C$ be its cycle, $|C| = k$. We consider three cases according to the length $k$ of cycle $C$.

Case 1. $k = 3$.

Subcase 1.1. All $T_i$s are nontrivial. We first give each edge of $G \setminus E(C)$ a fresh color, then let $c(v_1v_2) \in c(T_3)$, $c(v_2v_3) \in c(T_1)$, $c(v_1v_3) \in c(T_2)$, it is easy to show, with this coloring, $G$ is strong rainbow connected, so $\text{src}(G) \leq m - 3$ in this case.

Subcase 1.2. At most two $T_i$s are nontrivial, that is, $G \in G_1$. We first consider the case that there are exactly two $T_i$s which are nontrivial, say $T_1$ and $T_2$. We first give each edge of $G \setminus E(C)$ a fresh color, then let $c(v_1v_2) = c(v_2v_3) = c(v_1v_3)$, it is easy to show, with this coloring, $G$ is strong rainbow connected, so now $\text{src}(G) \leq m - 2$. On the other hand, by Observation 1.3 and the definition of rainbow geodesic, we know that $c(T_1) \cap c(T_2) = \emptyset$ and $c(v_1v_2)$ doesn’t belong to $c(T_1) \cup c(T_2)$. So we have $\text{src}(G) = m - 2$ in this case. With a similar argument, we can derive $\text{src}(G) = m - 2$ for the case that at most one $T_i$ is nontrivial. So $\text{src}(G) = m - 2$ if $G \in G_1$.

Case 2. $k = 4$.

Subcase 2.1. There are at least three nontrivial $T_i$s, say $T_1, T_3, T_4$. We first give each edge of $G \setminus E(C)$ a fresh color, then let $c(v_1v_2) \in c(T_3)$, $c(v_3v_4) \in c(T_1)$, $c(v_1v_4) \in c(T_2)$ and we give edge $v_2v_3$ a fresh color. It is easy to show, with this coloring, $G$ is strong rainbow connected, so $\text{src}(G) \leq m - 3$ in this case.

Subcase 2.2. There are exactly two nontrivial $T_i$s, say $T_{i_1}$ and $T_{i_2}$.

Subsubcase 2.2.1. $T_{i_1}$ and $T_{i_2}$ are adjacent, say $T_1$ and $T_2$. We first give each edge of $G \setminus E(C)$ a fresh color, then let $c(v_2v_3) \in c(T_1)$, $c(v_1v_4) \in c(T_2)$ and we color edges $v_1v_2$ and $v_3v_4$ with the same new color. It is easy to show, with this coloring, $G$ is strong rainbow connected, so $\text{src}(G) \leq m - 3$ in this case.

Subsubcase 2.2.2. $T_{i_1}$ and $T_{i_2}$ are nonadjacent, say $T_1$ and $T_3$. We can consider $T_i$ as rooted tree with root $v_i$ ($i = 1, 3$). If there exists some $T_i$, say $T_1$, that contains a vertex, say $u_1$, with at least two sons, say $u_1', u_1''$ (see Figure 3.3). We first color each edge of $\bigcup_{i=1,3} T_i \cup \{v_1v_2\}$ with a distinct color, this costs $m - 3$ colors, then we let $c(v_1v_4) = c(v_1v_2)$, $c(v_2v_3) = c(u_1u_1')$, $c(v_3v_4) = c(u_1u_1'')$. It is easy to show that this coloring is strong rainbow and we have $\text{src}(G) \leq m - 3$ in this case. If $G \in G_2$, we first give each edge of $G \setminus E(C)$ a fresh color, then let $c(v_1v_2) = c(v_3v_4) = a$ and $c(v_2v_3) = c(v_1v_4) = b$ where $a$ and $b$ are two new colors. It is easy to show, with this coloring, $G$ is strong rainbow connected,
so \( \text{src}(G) \leq m - 2 \) in this case. On the other hand, \( \text{src}(G) \geq m - 2 = \text{diam}(G) \). So \( \text{src}(G) = m - 2 \) if \( G \in \mathcal{G}_2 \).

**Subcase 2.3.** There are at most one nontrivial \( T_i \). Then with a similar argument to **Subsubcase 2.2.2**, we can derive that \( \text{src}(G) = m - 2 \) if \( G \in \mathcal{G}_3 \).

**Case 3.** \( k = 5 \).

If there are at least one nontrivial \( T_i \), say \( T_1 \), then we give each edge of \( G \setminus E(C) \) a fresh color, let \( v_3v_4 \in c(T_1) \), \( c(v_1v_2) = c(v_4v_5) = a \) and \( c(v_2v_3) = c(v_1v_5) = b \), where \( a \) and \( b \) are two new colors. It is easy to show, with this coloring, \( G \) is strong rainbow connected, so now we have \( \text{src}(G) \leq m - 3 \). On the other hand, we know \( \text{src}(G) = m - 2 = 3 \) if \( G \cong C_5 \) from [2].

By Claim 1 and Claim 2, we have that if \( \text{src}(G) = m - 1 \) or \( m - 2 \), then \( G \) is a unicyclic graph with the cycle of length at most 5. By the discussion from **Case 1** to **Case 3** of Claim 3, we know that if \( G \) is a unicyclic graph with the cycle of length at most 5, then \( \text{src}(G) \neq m - 1 \). So \( \text{src}(G) \neq m - 1 \) for any graph \( G \) and Claim 3 holds. By our three claims, our theorem holds.

**4 Upper bound for \( \text{src}(G) \) according to edge-disjoint triangles**

In this section, we give an upper bound for \( \text{src}(G) \) according to their edge-disjoint triangles in graph \( G \).

Recall that a *block* of a connected graph \( G \) is a maximal connected subgraph without a cut vertex. Thus, every block of graph \( G \) is either a maximal 2-connected subgraph or a bridge (cut edge). We now introduce a new graph class. For a connected graph \( G \), we say \( G \in \mathcal{G}_t \), if it satisfies the following conditions:

**C_1.** Each block of \( G \) is a bridge or a triangle;

**C_2.** \( G \) contains exactly \( t \) triangles;

**C_3.** Each triangle contains at least one vertex of degree two in \( G \).

By the definition, each graph \( G \in \mathcal{G}_t \) is formed by (edge-disjoint) triangles and paths (may be trivial), these triangles and paths fit together in a treelike structure, and \( G \) contains no cycles but the \( t \) (edge-disjoint) triangles. For example, see Figure 4.1 here \( t = 2 \), \( u_1, u_2, u_6 \) are vertices of degree 2 in \( G \). If a tree is obtained from a graph \( G \in \mathcal{G}_t \) by deleting one vertex of degree 2 for each triangle, then we call this tree is a \( D_2 \)-tree of \( G \), denoted \( T_G \). For example, in Figure 4.1, \( T_G \) is a \( D_2 \)-tree of \( G \). Clearly, the \( D_2 \)-tree is not unique, since in this example, we can obtain another \( D_2 \)-tree by deleting vertex \( u_1 \) instead of \( u_2 \). On the other hand, we can say any element of \( \mathcal{G}_t \) can be obtained from a tree by adding \( t \) new vertices of degree 2. It is easy to show that number of edges of \( T_G \) is \( m - 2t \) where \( m \) is the number of edges of \( G \).
Figure 4.1 An example of $G \in \overrightarrow{G}_t$ with $t = 2$.

**Theorem 4.1** $G$ is a graph with $m$ edges and $t$ edge-disjoint triangles, then

$$\text{src}(G) \leq m - 2t,$$

the equality holds if and only if $G \in \overrightarrow{G}_t$.

**Proof.** Let $\mathcal{T} = \{T_i : 1 \leq i \leq t\}$ be a set of $t$ edge-disjoint triangles in $G$.

We color each triangle with a fresh color (this means that three edges of each triangle receive the same color), then we give each other edge a fresh color. For any two vertices $u, v$ of $G$, let $P$ be any $u - v$ geodesic, then $P$ contains at most one edge from each triangle by Fact 1, so $P$ is rainbow under the above coloring. As this procedure costs $m - 2t$ colors totally, we have $\text{src}(G) \leq m - 2t$.

**Claim 1.** If the equality holds, then for any set $\mathcal{T}$ of edge-disjoint triangles of $G$, we have $|\mathcal{T}| \leq t$.

**Proof of Claim 1.** We suppose there is a set $\mathcal{T}'$ of $t'$ edge-disjoint triangles in $G$ with $t' > t$, then with a similar procedure, we have $\text{src}(G) \leq m - 2t' < m - 2t$, a contradiction.

**Claim 2.** If the equality holds, then $G$ contains no cycle but the above $t$ (edge-disjoint) triangles.

**Proof of Claim 2.** We suppose that there are at least one cycles distinct with the above $t$ triangles. Let $\mathcal{C}$ be the set of these cycles and $C_1$ be the smallest element of $\mathcal{C}$ with $|C_1| = k$. We will consider two cases:

**Case 1.** $E(C_1) \cap E(\mathcal{T}) = \emptyset$, that is, $C_1$ is edge-disjoint with each of the above $t$ triangles. With a similar argument to Lemma 2.2, we know $C_1$ has at most one common vertex with each of the above $t$ triangles. In this case $k \geq 4$ by Claim 1. We give $G$ an edge coloring as follows: we first color edges of cycle $C_1$ the same as [2] (this is shown in the proof of Claim 1 of Theorem 3.1); then we color each triangle with a fresh color; for the remaining edges, we give each one a fresh color. Recall the fact that any geodesic contains at most one edge from each triangle and with a similar procedure to the proof of Claim 1 of Theorem 3.1, we know the above coloring is strong rainbow, as this procedure costs $\lceil \frac{k}{2} \rceil + t + (m - k - 3t) = (m - 2t) + (\lceil \frac{k}{2} \rceil - k) < m - 2t$, we have $\text{src}(G) < m - 2t$, this produces a contradiction.
Case 2. \( E(C_1) \cap E(T) \neq \emptyset \), that is, \( C_1 \) have common edges with the above \( t \) triangles, in this case \( k \geq 3 \). By the choice of \( C_1 \), we know \( |E(C_1) \cap E(T_i)| \leq 1 \) for each \( 1 \leq i \leq t \). We will consider two subcases according to the parity of \( k \).

Subcase 2.1. \( k = 2\ell \) for some \( \ell \geq 2 \). For example, see graph (\( \alpha \)) of Figure 4.2, here \( T = \{ T_1, T_2, T_3 \} \), \( V(C_1) = \{ u_i : 1 \leq i \leq 6 \} \), \( E(C_1) \cap E(T_1) = \{ u_1 u_2 \} \), \( E(C_1) \cap E(T_2) = \{ u_4 u_5 \} \). Without loss of generality, we assume that there exists a triangle, say \( T_1 \), which contains edge \( u_1 u_2 \) and let \( V(T_1) = \{ u_1, u_2, w_1 \} \), \( G' = G \setminus E(T_1) \). If there exists some triangle, say \( T_2 \), which contains edge \( u_{\ell+1} u_{\ell+2} \), we let \( V(T_2) = \{ u_{\ell+1}, u_{\ell+2}, w_2 \} \).

![Figure 4.2 Graphs of two examples in Theorem 4.1.](image)

We first consider the case that \( \ell = 2 \), see Figure 4.3, we first give each triangle of \( G' \) a fresh color; for the remaining edges of \( G' \), we give each of them a fresh color; for edges of \( T_1 \), let \( c(u_1 w_1) = c(u_2 u_3), c(u_2 w_1) = c(u_1 u_4), c(u_1 w_2) = c(u_3 u_4) \). Then with a similar argument to that of Fact 2, we can show that there is a \( u-v \) geodesic which contains at most one edge from any two edges with the same color for \( u, v \in G \), so the above coloring is strong rainbow. As this procedure costs \( m - 2t - 1 < m - 2t \) colors totally, we have \( src(G) < m - 2t \), a contradiction.

We next consider the case that \( \ell \geq 3 \). Let \( G'' = G \setminus (E(T_1) \cup E(T_2)) \). We give \( G \) an edge-coloring as follows: We first give each triangle of \( G'' \) a fresh color; then give a fresh color to each of the remaining edges of \( G'' \); for the edges of \( T_1 \) and \( T_2 \), let \( c(u_1 w_1) = c(u_2 u_3) = a, c(u_2 w_1) = c(u_1 u_5) = b, c(u_1 w_2) = c(u_{\ell+1} u_{\ell+2}) = c, c(u_2 u_{\ell+1}) = c(u_{\ell+2} u_{\ell+3}) = d, c(u_2 u_{\ell+2}) = c(u_1 u_{\ell+1}) = e \) where \( a, b, c, d, e \) are five new colors. Then with a similar argument to that of Fact 2, we can show that there is a \( u-v \) geodesic which contains at most one edge from any two edges with the same color for \( u, v \in G \), so the above coloring is strong rainbow. As this procedure costs \( m - 2t - 1 < m - 2t \) colors totally, we have \( src(G) < m - 2t \), a contradiction.

Subcase 2.2. \( k = 2\ell + 1 \) for some \( \ell \geq 1 \).

We first consider the case that \( \ell \geq 2 \). For example, see graph (\( \beta \)) of Figure 4.2, here \( T = \{ T_1, T_2 \} \), \( V(C_1) = \{ u_i : 1 \leq i \leq 5 \} \), \( E(C_1) \cap E(T_1) = \{ u_1 u_2 \} \), \( E(C_1) \cap E(T_2) = \{ u_3 u_4 \} \). Without loss of generality, we assume that there exists a triangle, say \( T_1 \), which contains edge \( u_1 u_2 \) and let \( V(T_1) = \{ u_1, u_2, w_1 \} \). If there exists some triangle, say \( T_2 \), which contains edge \( u_{\ell+1} u_{\ell+2} \), we let \( V(T_2) = \{ u_{\ell+1}, u_{\ell+2}, w_2 \} \) and \( G' = G \setminus (E(T_1) \cup E(T_2)) \).

We give \( G \) an edge-coloring as follows: We first give each triangle of \( G' \) a fresh color; then give a fresh color to each of the remaining edges of \( G' \); for the edges of \( T_1 \) and \( T_2 \), let \( c(u_1 w_1) = c(u_2 u_3), c(u_2 w_1) = c(u_1 u_5), c(u_{\ell+1} w_1) = c(u_{\ell+2} u_{\ell+3}) \) and let \( c(u_1 w_2) = \)
c(u_{t+1}u_{t+2}) = c(w_2u_{t+2}) be a fresh color. With a similar procedure to the proof of Fact 1 and Claim 1 of Theorem 3.1, we can show that $G$ is strong rainbow connected, and so $src(G) \leq (t-1) + (m-3t) = (m-2t) - 1 < m - 2t$, this produces a contradiction.

For the case that $\ell = 1$, that is, $C_1$ is a triangle. See Figure 4.3, we color the three edges (if exist) with color 1, these edges are shown in the figure; the remaining edges of these three triangles (if exist) all receive color 2; each of other triangles receive a fresh color; for the remaining edges, we give each one a fresh color. It is easy to show that the above coloring is strong rainbow, so we have $src(G) < m - 2t$ in this case, a contradiction. So the claim holds.

Figure 4.3 Edge coloring for the case that $C_1$ is a triangle and the case that $C_1$ a 4-cycle in Theorem 4.1

Claim 3. If the equality holds, then $G \in \overline{G}_t$.

Proof of Claim 3. If the equality holds, to prove that $G \in \overline{G}_t$, it suffices to show that each triangle contains at most one vertex of degree 2 in $G$. Suppose it doesn’t holds, without loss of generality, let $T_1$ be the triangle with $deg_G(v_i) \geq 3$, where $V(T_1) = \{v_i : 1 \leq i \leq 3 \}$. By Claim 2, it is easy to show that $E(T_1)$ is an edge-cut of $G$, let $H_i$ be the subgraph of $G \setminus E(T_1)$ containing vertex $v_i$ (1 $\leq i \leq 3$), by the assumption of $T_1$, we know each $H_i$ is nontrivial. We now give $G$ an edge-coloring: for the $t-1$ (edge-disjoint) triangles of $G \setminus E(T_1)$, we give each of them a fresh color; for the remaining edges of $G \setminus E(T_1)$ (by Claim 2, each of them must be a cut edge), we give each of them a fresh color; for the edges of $E(T_1)$, let $c(v_1v_3) \in c(H_2)$, $c(v_1v_2) \in c(H_3)$, $c(v_2v_3) \in c(H_1)$. It is easy to show, with the above coloring, $G$ is strong rainbow connected, and we have $src(G) < m - 2t$, a contradiction, so the claim holds.

Claim 4. If $G \in \overline{G}_t$, then the equality holds.

Proof of Claim 4. Let $T_G$ be a $D_2$-tree of $G$, the result clearly holds for the case $|E(T_G)| = 1$. So now we assume $|E(T_G)| \geq 2$. We will show, for any strong rainbow coloring of $G$, $c(e_1) \neq c(e_2)$ where $e_1,e_2 \in T_G$, that is, each edge of $T_G$ receive a distinct color, so edges of $T_G$ cost $m - 2t$ colors totally, recall that $|E(T_G)| = m - 2t$, then $src(G) \geq m - 2t$, by the above claim, Claim 4 holds.

For any two edges, say $e_1,e_2$, of $T_G$, let $e_1 = u_1u_2$, $e_2 = v_1v_2$. Without loss of generality, we assume $d_{T_G}(u_1,v_2) = \max\{d_{T_G}(u_i,v_j) : 1 \leq i, j \leq 2 \}$ where $d_{T_G}(u,v)$ denote the distance between $u$ and $v$ in $T_G$. As $T_G$ is a tree, the (unique) $u_1 - v_2$ geodesic, say $P$, in $T_G$ must contains edges $e_1,e_2$. Moreover, it is easy to show $P$ is also an unique $u_1 - v_2$ geodesic in $G$, so $c(e_1) \neq c(e_2)$ under any strong rainbow coloring.
By Claim 3 and Claim 4, the equality holds if and only if \( G \in \overline{G}_t \). Then our result holds.

Next we give an application to Theorem 4.1, we consider the strong rainbow connection numbers of line graphs of connected cubic graphs. Recall that the line graph of a graph \( G \) is the graph \( L(G) \) whose vertex set \( V(L(G)) = E(G) \) and two vertices \( e_1, e_2 \) of \( L(G) \) are adjacent if and only if they are adjacent in \( G \). The star, denoted \( S(v) \), at a vertex \( v \) of graph \( G \), is the set of all edges incident to \( v \). Let \( \langle S(v) \rangle \) be the subgraph of \( L(G) \) induced by \( S(v) \), clearly, it is a clique of \( L(G) \). A clique decomposition of \( G \) is a collection \( \mathcal{C} \) of cliques such that each edge of \( G \) occurs in exactly one clique in \( \mathcal{C} \). An inner vertex of a graph is a vertex with degree at least two. For a graph \( G \), we use \( V_2 \) to denote the set of all inner vertices of \( G \). Let \( \mathcal{C}_0 = \{ \langle S(v) \rangle : v \in V(G) \} \), \( \mathcal{C} = \{ \langle S(v) \rangle : v \in V_2 \} \). It is easy to show that \( \mathcal{C}_0 \) is a clique decomposition of \( L(G) \) and each vertex of the line graph belongs to at most two elements of \( \mathcal{C}_0 \). We know that each element \( \langle S(v) \rangle \) of \( \mathcal{C}_0 \setminus \mathcal{C} \), a single vertex of \( L(G) \), is contained in the clique induced by \( u \) that is adjacent to \( v \) in \( G \). So \( \mathcal{C} \) is a clique decomposition of \( L(G) \).

**Corollary 4.2** Let \( L(G) \) be the line graph of a connected cubic graph with \( n \) vertices, then \( src(L(G)) \leq n \).

**Proof.** Since \( G \) is a connected cubic graph, each vertex is an inner vertex and the clique \( \langle S(v) \rangle \) in \( L(G) \) corresponding to each vertex \( v \) is a triangle. We know that \( \mathcal{C} = \{ \langle S(v) \rangle : v \in V_2 \} = \{ \langle S(v) \rangle : v \in V \} \) is a clique decomposition of \( L(G) \). Let \( \mathcal{T} = \mathcal{C} \). Then \( \mathcal{T} \) is a set of \( n \) edge-disjoint triangles that cover all edges of \( L(G) \). As there are \( 3n \) edges in \( L(G) \), by Theorem 4.1 we have \( src(L(G)) \leq 3n - 2n = n \).

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