Circular symmetrization, subordination and arclength problems on convex functions

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We study the class $C(\Omega)$ of univalent analytic functions $f$ in the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$ of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ satisfying

$$1 + z \frac{f''(z)}{f'(z)} \in \Omega, \quad z \in D,$$

where $\Omega$ will be a proper subdomain of $\mathbb{C}$ which is starlike with respect to $1(\in \Omega)$. Let $\phi_\Omega$ be the unique conformal mapping of $D$ onto $\Omega$ with $\phi_\Omega(0) = 1$ and $\phi_\Omega'(0) > 0$ and $k_\Omega(z) = \int_0^1 \exp \left( \int_0^t \frac{1}{1 - \phi_\Omega(z)} - 1 \right) \frac{dt}{dt}$.

Let $L_r(f)$ denote the arclength of the image of the circle $\partial D = \{z \in \mathbb{C} : |z| = r\}$, $r \in (0, 1)$. The first result in this paper is an inequality $L_r(f) \leq L_r(k_\Omega)$ for $f \in C(\Omega)$, which solves the general extremal problem $\max_{f \in C(\Omega)} L_r(f)$, and contains many other well-known results of the previous authors as special cases. Other results of this article cover another set of related problems about integral means in the general setting of the class $C(\Omega)$.

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1 Introduction

Let $\mathbb{C}$ be the complex plane and $D(c, r) = \{z \in \mathbb{C} : |z - c| < r\}$ with $c \in \mathbb{C}$ and $r > 0$. In particular we denote the unit disk $D(0, 1)$ by $D$. Let $\mathcal{A}$ be the linear space of all analytic functions in the unit disk $D$, endowed with the topology of uniform convergence on every compact subset of $D$. Set $\mathcal{A}_0 = \{f \in \mathcal{H} : f(0) = f'(0) - 1 = 0\}$ and denote by $\mathcal{S}$ the subclass of $\mathcal{A}_0$ consisting of all univalent functions as usual. Then $\mathcal{S}$ is a compact subset of the metrizable space $\mathcal{A}$. See [5, Chap. 9] for details. For $f \in \mathcal{A}$ and $0 < r < 1$, let

$$L_r(f) = \int_{-\pi}^{\pi} r \left| f'(re^{i\theta}) \right| \, d\theta$$

denote the arclength of the image of the circle $\partial D(0, r) = \{z \in \mathbb{C} : |z| = r\}$. Many extremal problems in the class $\mathcal{S}$ have been solved by the Koebe function

$$k(z) = \frac{z}{(1 - z)^2}$$

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or by its rotation: \( k_\theta(z) = e^{-i\theta}k(e^{i\theta}z) \), where \( \theta \) is real. Note that \( k_\theta \) maps the unit disk \( \mathbb{D} \) onto the complement of a ray. In any case, since the functional \( \mathcal{A} \ni f \mapsto L_r(f) \) is continuous and the class \( \mathcal{S} \) is compact, a solution of the extremal problem

\[
\max_{f \in \mathcal{S}} L_r(f)
\]

exists and is in \( \mathcal{S} \). We remark that with a clever use of Dirichlet-finite integral and the isoperimetric inequality, Yamashita [16] obtained the upper and lower estimates for the functional (1.1)

\[
m(r) \leq L_r(k) \leq \max_{f \in \mathcal{S}} L_r(f) \leq \frac{2\pi r}{(1 - r)^2},
\]

where

\[
m(r) = \frac{2\pi r \sqrt{r^4 + 4r^2 + 1}}{(1 - r^2)^2} \geq \frac{2\pi r (1 + r) \sqrt{6}}{(1 - r^2)^2} > \frac{\pi r (1 + r)}{2(1 - r)^2}.
\]

This observation provides an improvement over the earlier result of Duren [4, Theorem 2] and [6, p. 39], and moreover,

\[
m(r) \geq \frac{\sqrt{6}}{2} \frac{\pi r}{(1 - r)^2}.
\]

The extremal problem (1.1) stimulated much research in the theory of univalent functions, and the problem of determining of the maximum value and the extremal functions in \( \mathcal{S} \) remains open. However, the extremal problem

\[
\max_{f \in \mathcal{F}} L_r(f)
\]

has been solved for a number of subclasses \( \mathcal{F} \) of \( \mathcal{S} \). In order to motivate these known results and also for our further discussion on this topic, we need to introduce some notations.

Unless otherwise stated explicitly, throughout the discussion \( \Omega \) will be a simply connected domain in \( \mathbb{C} \) with \( 1 \in \Omega \neq \mathbb{C} \) and \( \phi_\Omega \) is the unique conformal mapping of \( \mathbb{D} \) onto \( \Omega \) with \( \phi_\Omega(0) = 1 \) and \( \phi_\Omega'(0) > 0 \).

Ma and Minda [10] considered the classes \( \mathcal{S}^\alpha(\Omega) \) and \( \mathcal{C}(\Omega) \) with some mild conditions, e.g. \( \Omega \) is starlike with respect to 1 and the symmetry with respect to the real axis \( \mathbb{R} \), i.e., \( \overline{\Omega} = \Omega \):

\[
\mathcal{S}^\alpha(\Omega) = \left\{ f \in \mathcal{A}_0 : \frac{zf''(z)}{f'(z)} \in \Omega \text{ on } \mathbb{D} \right\},
\]

and

\[
\mathcal{C}(\Omega) = \left\{ f \in \mathcal{A}_0 : 1 + \frac{zf''(z)}{f'(z)} \in \Omega \text{ on } \mathbb{D} \right\}.
\]

Note that, with the special choice of \( \Omega = \mathbb{H} = \{ z \in \mathbb{C} : \text{Re } z > 0 \} \), these two classes consist of starlike and convex functions in the standard sense, and are denoted simply by \( \mathcal{S}^\alpha \) and \( \mathcal{C} \), respectively.

If \( 0 < \alpha \leq 1 \) and \( \Omega = \{ w \in \mathbb{C} : |\text{Arg } w| < 2^{-1} \pi \alpha \} \), then \( \phi_\Omega(z) = [(1 + z)/(1 - z)]^\alpha \), and hence, in this choice \( \mathcal{C}(\Omega) \) reduces to the class of strongly convex functions of order \( \alpha \). Furthermore, for \(-1/2 \leq \beta < 1 \) and \( \Omega = \{ w \in \mathbb{C} : \text{Re } w > \beta \} \) and \( \phi_\Omega(z) = (1 + (1 - 2\beta)z)/(1 - z) \), the class \( \mathcal{C}(\{ w \in \mathbb{C} : \text{Re } w > \beta \}) \) coincides with the class of convex functions of order \( \beta \). Various subclasses of \( \mathcal{C} \) can be expressed in this way. For details we refer to [10] and [17]. We notice that it may be possible that \( \mathbb{H} \subset \Omega \), and in this case we have \( \mathcal{C} \subset \mathcal{C}(\Omega) \) whenever \( 0 \leq \beta < 1 \). When \(-1/2 \leq \beta < 0 \), functions in \( \mathcal{C}(\Omega) \) are known to be convex in some direction (see [15]).

A function \( f \) in \( \mathcal{A}_0 \) is said to be close-to-convex if there exists a convex function \( g \) and a real number \( \beta \in (-\pi/2, \pi/2) \) such that

\[
e^{i\beta} \frac{f'(z)}{g'(z)} \in \mathbb{H}
\]
on $\mathbb{D}$. We denote the class of close-to-convex functions in $\mathbb{D}$ by $K$ which has been introduced by Kaplan [7]. These standard geometric classes are related by the proper inclusions

$$C \subseteq S^* \subseteq K \subseteq S.$$ 

The extremal problem (1.2) for $F = C$ has been solved by Keogh [8] who showed that

$$\max_{f \in C} L_r(f) = \frac{2\pi r}{1-r^2} = L_r(\ell_0)$$

with equality if and only if $f = \ell_0$. Here $\ell_0(z) = z/(1 - e^{i\theta}z)$, where $\theta$ is real. The extremal problem (1.2) for the cases $F = S^*$ and $F = K$ were solved by Marx [11] and Clunie and Duren [3], respectively. In both cases, the Koebe function and its rotations solve the corresponding extremal problem. That is, for $F = S^*$ and $F = K$, one has $\max_{f \in F} L_r(f) = L_r(k)$ with equality if and only if $f(z) = k_0(z)$, where $\theta$ is real. As a straightforward adaptations of the known proofs, Miller [12] extended all these three cases to the corresponding subclasses of $S$ consisting of $m$-fold convex, starlike and close-to-convex functions, respectively. Finally, by making use of the theory of symmetrization developed by Baernstein [2], Leung [9] extended the result for $F = S^*$ to the class of Bazilevič functions and the generalized functional $f^0_{-\pi}, \Phi(\log |f'(re^{i\theta})|) \, d\theta$, where $\Phi$ is a nondecreasing convex function on $\mathbb{R}$. Recently, the extremal problem (1.2) for the class of convex functions of order $-1/2$ was solved in [1].

One of the aims of the present article is to study similar extremal problems for various subclasses $C(\Omega)$ in a unified manner. Let

$$k_\Omega(z) = \int_0^z \exp \left( \int_0^t \frac{\phi_\Omega(\xi) - 1}{\xi} \, d\xi \right) \, dt, \quad z \in \mathbb{D}. \quad (1.3)$$

Then $k_\Omega \in C(\Omega)$. When $\Omega$ is starlike with respect to 1, the extremal problem $\max_{f \in C(\Omega)} L_r(f)$ can be solved and $k_\Omega$ plays the role of the extremal function.

**Theorem 1.1** If $\Omega$ is starlike with respect to 1, then for $f \in C(\Omega)$

$$L_r(f) \leq L_r(k_\Omega) \quad (1.4)$$

with equality if and only if $f(z) = \tau k_\Omega(\varepsilon z)$ for some $\varepsilon \in \partial \mathbb{D}$.

Let $f$ and $F$ be analytic functions in $\mathbb{D}$. Then $f$ is said to be subordinate to $F$ ($f < F$, or $f(z) < F(z)$ in $\mathbb{D}$, in short) if there exists an analytic function $\omega$ in $\mathbb{D}$ with $|\omega(z)| \leq |z|$ and $f(z) = F(\omega(z))$ in $\mathbb{D}$. In particular $f(\mathbb{D}) \subset F(\mathbb{D})$ holds, if $f < F$. Notice that when $F$ is univalent in $\mathbb{D}$, $f < F$ if and only if $f(\mathbb{D}) \subset F(\mathbb{D})$ and $f(0) = F(0)$.

Furthermore, by making use of subordination and circular symmetrization, we can considerably strengthen Theorem 1.1. We note that $f \in C(\Omega)$ forces that $f'(z) \neq 0$ in $\mathbb{D}$ and the single valued branch $\log f'(z)$ with $\log f'(0) = 0$ exists on $\mathbb{D}$.

**Theorem 1.2** If $\Omega$ is starlike with respect to 1, then $\log k_\Omega'(z)$ is convex univalent in $\mathbb{D}$ and $\log f'(z) < \log k_\Omega'(z)$ holds for $f \in C(\Omega)$. Furthermore for any subharmonic function $u$ in the domain $\log k_\Omega'(\mathbb{D})$ and $r \in (0, 1)$

$$\int_{-\pi}^{\pi} u(\log f'(re^{i\theta})) \, d\theta \leq \int_{-\pi}^{\pi} u(\log k_\Omega'(re^{i\theta})) \, d\theta$$

holds with equality for some $u$ and $r \in (0, 1)$ if and only if $u$ is harmonic in $\log k_\Omega'(0, r)$ or $f(z) = \tau k_\Omega(\varepsilon z)$ for some $\varepsilon \in \partial \mathbb{D}$.

By letting $u(w)$ as particular functions we can obtain various inequalities. We shall only give typical examples. Since the functions $\log |w|, |w|^p$ with $0 < p < \infty$, $\Phi(\pm \text{Re } w)$ or $\Phi(\pm \text{Im } w)$ with a continuous convex function $\Phi$ on $\mathbb{R}$ are subharmonic functions of $w \in \mathbb{C}$, we have the following inequalities.
Corollary 1.3 If $\Omega$ is starlike with respect to 1, then for any $f \in C(\Omega)$ and $r \in (0, 1)$ the following inequalities hold:

$$\int_{-\pi}^{\pi} \log |\log f'(re^{i\theta})| \, d\theta \leq \int_{-\pi}^{\pi} \log |k_{12}'(re^{i\theta})| \, d\theta,$$

(1.5)

$$\int_{-\pi}^{\pi} |\log f'(re^{i\theta})|^p \, d\theta \leq \int_{-\pi}^{\pi} |k_{12}'(re^{i\theta})|^p \, d\theta, \quad 0 < p < \infty,$$

(1.6)

$$\int_{-\pi}^{\pi} \Phi(\pm \log |f'(re^{i\theta})|) \, d\theta \leq \int_{-\pi}^{\pi} \Phi(\pm \log |k_{12}'(re^{i\theta})|) \, d\theta,$$

(1.7)

$$\int_{-\pi}^{\pi} \Phi(\pm \arg f'(re^{i\theta})) \, d\theta \leq \int_{-\pi}^{\pi} \Phi(\pm \arg k_{12}'(re^{i\theta})) \, d\theta.$$  

(1.8)

Equality holds in (1.5) or (1.6) if and only if $f(z) = e Q_{12}(\varepsilon z)$ for some $\varepsilon \in \partial \mathbb{D}$. Furthermore, when $\Phi(\pm r)$ is not linear in the interval

$$\left( \min_{-\pi \leq \theta \leq \pi} \log |k_{12}'(re^{i\theta})|, \ max_{-\pi \leq \theta \leq \pi} \log |k_{12}'(re^{i\theta})| \right)$$  

(1.9)

or

$$\left( \min_{-\pi \leq \theta \leq \pi} \arg k_{12}'(re^{i\theta}), \ max_{-\pi \leq \theta \leq \pi} \arg k_{12}'(re^{i\theta}) \right),$$  

(1.10)

equality holds respectively in (1.7) or (1.8) if and only if $f(z) = e k_{12}(\varepsilon z)$ for some $\varepsilon \in \partial \mathbb{D}$.

In contrast to the above corollary we need to assume that $\Phi$ is nondecreasing in the following theorem.

Theorem 1.4 If $\Omega$ is starlike with respect to 1 and symmetric with respect to $\mathbb{R}$, then $\log |k_{12}'(re^{i\theta})|$ is a symmetric function of $\theta$ and nonincreasing on $[0, \pi]$, and for any convex and nondecreasing function $\Phi$ in $\mathbb{R}$ and any Lebesgue measurable set $E \subset [-\pi, \pi]$ of Lebesgue measure $\theta$, we have

$$\int_{E} \Phi(\log |f'(re^{i\theta})|) \, ds \leq \int_{0}^{\theta} \Phi(\log |k_{12}'(re^{i\theta})|) \, ds.$$  

In particular

$$\int_{E} r |f'(re^{i\theta})| \, d\theta \leq \int_{0}^{\theta} r |k_{12}'(re^{i\theta})| \, ds,$$

i.e., the length of $\{f(re^{i\theta}) : s \in E\}$ does not exceed that of $\{k_{12}(re^{i\theta}) : |s| \leq \theta\}$.

2 Subordination

First we state a variant of the Littlewood subordination theorem (see [5, Theorem 1.7]) and give a proof for completeness.

Lemma 2.1 Let $f, F \in A$ with $f \prec F$. Then for any subharmonic function $u$ in $F(\mathbb{D})$ and $r \in (0, 1)$

$$\int_{-\pi}^{\pi} u(\log f'(re^{i\theta})) \, d\theta \leq \int_{-\pi}^{\pi} u(F(re^{i\theta})) \, d\theta$$  

(2.1)

with equality if and only if $f(z) = F(\varepsilon z)$ for some $\varepsilon \in \partial \mathbb{D}$ or $u$ is harmonic in $F(\mathbb{D}(0, r))$.

Proof. Let $\omega \in A$ with $|\omega(z)| \leq |z|$ and $f(z) = F(\omega(z))$ in $\mathbb{D}$. Let $U$ be the continuous function on $\overline{\mathbb{D}}(0, r)$ such that $U = u \circ F$ on $\partial \mathbb{D}(0, r)$ and harmonic in $\mathbb{D}(0, r)$. Since $u \circ F$ is subharmonic in $\mathbb{D}$, it follows from the maximum principle that $(u \circ F)(z) \leq U(z)$ on $\overline{\mathbb{D}}(0, r)$. Thus

$$(u \circ f)(z) = (u \circ F \circ \omega)(z) \leq U \circ \omega(z).$$
and
\[
\int_{-\pi}^{\pi} u \left( f \left( r e^{i\theta} \right) \right) d\theta = \int_{-\pi}^{\pi} \left( u \circ F \circ \omega \right) \left( r e^{i\theta} \right) d\theta \\
\leq \int_{-\pi}^{\pi} \left( u \circ F \right) \left( r e^{i\theta} \right) d\theta \\
= 2\pi \left( U \circ \omega \right)(0) = 2\pi U(0) = \int_{-\pi}^{\pi} U \left( r e^{i\theta} \right) d\theta = \int_{-\pi}^{\pi} u \left( F \left( r e^{i\theta} \right) \right) d\theta.
\]

If \( f(z) = F(\varepsilon z) \) for some \( \varepsilon \in \partial \mathbb{D} \), then equality trivially holds in (2.1). Also if \( u \) is harmonic in \( F(\mathbb{D}(0, r)) \), then \( u \circ F \) and \( u \circ f \) are harmonic in \( \mathbb{D}(0, r) \), and hence it follows from \( f(0) = F(0) = 0 \) and the mean value property of harmonic functions that both hand sides of (2.1) reduces to \( 2\pi u(0) \).

Suppose that equality holds in (2.1). Then for almost every \( \theta \), \( u \circ F \left( \omega \left( r e^{i\theta} \right) \right) = U \left( \omega \left( r e^{i\theta} \right) \right) \) holds. Without loss of generality we may assume that \( F \) is not constant. By the classical Schwarz lemma it suffices to show that \( u \) is harmonic in \( F(\mathbb{D}(0, r)) \) when \( |\omega(z)| < |z| \) for all \( z \in \mathbb{D} \), since otherwise \( \omega(z) = \varepsilon z \) in \( \mathbb{D} \) for some \( \varepsilon \in \partial \mathbb{D} \).

Therefore for any fixed real \( \theta \), \( \omega(\varepsilon r e^{i\theta}) \) is an interior point of \( \mathbb{D}(0, r) \). It follows from the maximum principle for subharmonic functions that \( u \circ F = U \) in \( \mathbb{D}(0, r) \).

Now we show that \( u \) is harmonic in \( F(\mathbb{D}(0, r)) \). Let \( w_0 \in F(\mathbb{D}(0, r)) \) and choose \( z_0 \in \mathbb{D}(0, r) \) with \( F(z_0) = w_0 \). If \( F'(z_0) \neq 0 \), then \( F \) maps a neighborhood \( V_{z_0} \) of \( z_0 \) conformally onto a neighborhood \( F(V_{z_0}) \) of \( w_0 \), and hence \( u = U \circ \left( F \left| V_{z_0} \right. \right)^{-1} \) is harmonic in \( F(V_{z_0}) \). Even if \( F'(z_0) = 0 \), \( u \) is at least continuous at \( z_0 \). In fact, for each \( \eta > 0 \) there exists \( \delta > 0 \) such that \( |U(z) - U(z_0)| < \delta \) holds for \( z \in \mathbb{D}(z_0, \delta) \).

Since \( F \) is nonconstant, \( F(\mathbb{D}(z_0, \delta)) \) is an open neighborhood of \( w_0 \) and \( |u(w) - u(w_0)| < \eta \) holds for \( w \in F(\mathbb{D}(z_0, \delta)) \). Thus \( u \) is continuous at \( w_0 \).

We have shown that \( u \) is continuous in \( F(\mathbb{D}(0, r)) \) and harmonic in \( F(\mathbb{D}(0, r)) \) except at each point in the set of critical values
\[
B = \{ w_0 = F(z_0) : z_0 \in \mathbb{D}(0, r) \} \text{ with } F(z_0) = 0.
\]

Since \( B \) is finite, each point in \( B \) is isolated and hence is a removable singularity of \( u \). Thus \( u \) is harmonic in \( F(V_{z_0}) \).

**Proof of Theorem 1.2** Let \( f \in \mathcal{C}(\Omega) \) and \( h(z) = 1 + zf''(z)/f'(z) \). Then \( h \) satisfies \( h(0) = 1, h < \phi_{z_0} \). By the starlikeness of \( \Omega \) with respect to 1 it follows from the Suffridge lemma (see [14]) that
\[
\log f'(z) = \int_{0}^{\pi} \frac{h(\zeta) - 1}{\zeta} d\zeta < \int_{0}^{\pi} \frac{\phi_{z_0}(\zeta) - 1}{\zeta} d\zeta = \log k'_{z_0}(z),
\]
where \( k'_{z_0} \) is defined by (1.3). Also it follows from the starlikeness of \( \Omega \) with respect to 1 that \( \log k'_{z_0} \) is convex univalent in \( \mathbb{D} \). Now the latter half of the statement is a direct consequence of Lemma 2.1.

**Proof of Theorem 1.1** Let \( u(w) = re^{Re w} \) and \( L \) be the Laplace operator. Since \( L(re^{Re w}) = re^{Re w} \neq 0 \), \( u \) is subharmonic in \( \mathbb{C} \) and (1.4) easily follows from Theorem 1.2. Furthermore the subharmonic function \( re^{Re w} \) is not harmonic in any domain in \( \mathbb{C} \). Thus if equality holds in (1.4), then \( \log f'(z) = \log k'_{z_0}(\varepsilon z) \) for some \( \varepsilon \in \partial \mathbb{D} \).

**Proof of Corollary 1.3** Inequalities (1.5), (1.6), (1.7) and (1.8) are consequences of Theorem 1.2 and the subharmonicity of the functions \( \log |w|, |w|^p, \phi(\pm Re w) \) and \( \phi(\pm Im w) \), respectively.

Notice \( \log |w| \) is not harmonic in \( \log k'_{z_0}(\mathbb{D}(0, r)) \) for any \( r \in (0, 1) \) because \( \log k'_{z_0}(0) = 0 \). Also \( |w|^p \) is not harmonic in any domain in \( \mathbb{C} \). Furthermore, \( \phi(\pm Re w) \) and \( \phi(\pm Im w) \) are not harmonic in \( \log k'_{z_0}(\mathbb{D}) \), since \( \phi(\pm t) \) is not linear in the interval given by (1.9) or by (1.10). Thus if equality holds in (1.5), (1.6), (1.7) or (1.8), then by Lemma 2.1 we have \( \log f'(z) = \log k'_{z_0}(\varepsilon z) \) for some \( \varepsilon \in \partial \mathbb{D} \) and hence \( f(z) = \pi k'_{z_0}(\varepsilon z) \).

**3 Circular symmetrization**

We summarize without proofs some of the standard facts on the theory of \( \ast \)-functions developed by Baernstein [2]. For more on \( \ast \)-functions we refer to Duren [5].
Let \(|E|\) denote the Lebesgue measure of a Lebesgue measurable set \(E \subset \mathbb{R}\). Let \(h : [-\pi, \pi] \to \mathbb{R} \cup \{\pm \infty\}\) be a Lebesgue measurable function which is finite-valued almost everywhere. Then the distribution function \(\lambda_h\) defined by

\[
\lambda_h(t) = |\{\theta \in [-\pi, \pi] : h(\theta) > t\}|
\]

is nonincreasing and right continuous on \(\mathbb{R}\), and satisfies \(\lim_{t \to -\infty} \lambda_h(t) = 2\pi\) and \(\lim_{t \to \infty} \lambda_h(t) = 0\). Let \(\hat{h}(\theta) = \inf\{t \in \mathbb{R} : \lambda_h(t) \leq 2|\theta|\}\) for \(|\theta| < \pi\),

\[
\text{ess inf} \ h \quad \text{for} \quad |\theta| = \pi.
\]

It is easy to see that \(\hat{h}\) is symmetric, i.e., \(\hat{h}(-\theta) = \hat{h}(\theta)\), and satisfies the following conditions:

(i) \(\text{ess inf} \ h \leq \hat{h}(\theta) \leq \text{ess sup} \ h\),
(ii) \(\hat{h}\) is right continuous and nonincreasing on \([0, \pi]\),
(iii) \(\lim_{\theta \to 0} \hat{h}(\theta) = \hat{h}(0) = \text{ess sup} \ h \) and \(\lim_{\theta \to \pi} \hat{h}(\theta) = \hat{h}(\pi) = \text{ess inf} \ h\),
(iv) \(\hat{h}\) is equimeasurable with \(h\), i.e., \(\lambda_{\hat{h}}(t) = \lambda_h(t)\) for all \(t \in \mathbb{R}\).

The function \(\hat{h}\) is called the symmetric nonincreasing rearrangement of \(h\). Notice that \(\hat{h}\) is unique in the sense that if \(\tilde{h}\) is also a symmetric function on \([-\pi, \pi]\), nonincreasing and right continuous on \([0, \pi]\) with \(\hat{h}(\pi) = \text{ess inf} \ h \) and equimeasurable with \(h\), then \(\tilde{h} = \hat{h}\).

For \(h \in L^1[-\pi, \pi]\), the \(*\)-function of \(h\) is the function defined by

\[
h^*(\theta) = \sup_{|E| = 2\theta} \int_E h(s) \, ds, \quad 0 \leq \theta \leq \pi,
\]

where supremum is taken over all Lebesgue measurable subsets of \([-\pi, \pi]\) with \(|E| = 2\theta\). Then it is known that

\[
h^*(\theta) = \int_{-\theta}^{\theta} \hat{h}(s) \, ds, \quad 0 \leq \theta \leq \pi.
\]

**Lemma 3.1** ([2, p. 150]) For \(h, H \in L^1[-\pi, \pi]\), the following statements are equivalent:

(a) For every convex nondecreasing function \(\Phi\) on \(\mathbb{R}\)

\[
\int_{-\pi}^{\pi} \Phi(h(\theta)) \, d\theta \leq \int_{-\pi}^{\pi} \Phi(H(\theta)) \, d\theta.
\]

(b) For every \(t \in \mathbb{R}\)

\[
\int_{-\pi}^{\pi} (h(\theta) - t)^+ \, d\theta \leq \int_{-\pi}^{\pi} (H(\theta) - t)^+ \, d\theta.
\]

(c) \(h^*(\theta) \leq H^*(\theta)\) for \(0 \leq \theta \leq \pi\),

where \((h(\theta) - t)^+ = \max\{h(\theta) - t, 0\}\).

Let \(v\) be a subharmonic function in the unit disk \(\mathbb{D}\). Then for each fixed \(r \in (0, 1)\), \(v(re^{i\theta})\) is an integrable function of \(\theta \in [-\pi, \pi]\). Let \(\hat{v}(re^{i\theta})\) and \(v^*(re^{i\theta})\) be the symmetrically nonincreasing rearrangement and the \(*\)-function of the function \([-\pi, \pi] \ni \theta \mapsto v(re^{i\theta})\), respectively. The function \(\hat{v}\) is called the circular symmetrization of \(v\).

We now can conclude an inequality concerning \(*\)-functions from Lemma 2.1. The following lemma is not new and it is an equivalent variant of Lemma 2 in Leung [9].

**Lemma 3.2** Let \(f, F \in \mathcal{A}\) with \(f \ll F\). Then for any subharmonic function \(u\) in \(F(\mathbb{D})\) and \(r \in (0, 1)\)

\[
(u \circ f)^*(re^{i\theta}) \leq (u \circ F)^*(re^{i\theta}), \quad 0 \leq \theta \leq \pi.
\]
Proof. Since \((u(w) - t)^+\) is also subharmonic in \(F(\mathbb{D})\) for any \(t \in \mathbb{R}\), we have by Lemma 2.1 that
\[
\int_{\pi} (u \circ f (re^{i\theta}) - t)^+ d\theta \leq \int_{\pi} (u \circ F (re^{i\theta}) - t)^+ d\theta.
\]
Therefore it follows from Lemma 3.1 that \((u \circ f)^+ (re^{i\theta}) \leq (u \circ F)^+ (re^{i\theta})\) for \(0 \leq \theta \leq \pi\).

Proof of Theorem 1.4 Let \(\Phi\) be a nondecreasing convex function in \(\mathbb{R}\). Then \(\Phi(\text{Re} \, w)\) is a subharmonic function of \(w \in \mathbb{C}\). By Lemma 3.2 we have
\[
(\Phi(\log |f'|)^+ (re^{i\theta}) \leq (\Phi(\log |k_{\Omega}'|)^+ (re^{i\theta}))
\]
for all \(r \in [0, 1) \) and \(\theta \in [0, \pi]\). Now we temporarily suppose that for fixed \(r \in (0, 1)\), \(\log |k_{\Omega}'(re^{i\theta})|\) is a symmetric function of \(\theta\) and nonincreasing on \([0, \pi]\). Then so is \(\Phi(\log |k_{\Omega}'(re^{i\theta})|), \) since \(\Phi\) is nondecreasing. Thus the symmetrically nonincreasing rearrangement of \(\Phi(\log |k_{\Omega}'|)\) coincides with itself, i.e., \((\Phi(\log |k_{\Omega}'|))(re^{i\theta}) = \Phi(\log |k_{\Omega}'(re^{i\theta})|)\). Therefore we have by (3.1) and (3.2) that for any Lebesgue measurable set \(E \subset [-\pi, \pi]\) with \(|E| = 2\theta\)
\[
\int_{E} \Phi(\log |f'(re^{i\theta})|) \, ds \leq \Phi(\log |f'|)^+ (re^{i\theta})
\]
\[
\leq \Phi(\log |k_{\Omega}'|)^+ (re^{i\theta})
\]
\[
= \int_{-\theta}^{\theta} \Phi(\log |k_{\Omega}'|)(re^{i\theta}) \, ds = \int_{-\theta}^{\theta} \Phi(\log |k_{\Omega}'(re^{i\theta})|) \, ds.
\]

It remains to show that the function \(\theta \mapsto \log |k_{\Omega}'(re^{i\theta})|\) is symmetric and strictly decreasing on \([0, \pi]\). Since \(\Omega\) is symmetric with respect to \(\text{Re}\), we have \(\frac{d}{d\theta} \log |k_{\Omega}'(z)| = \frac{d}{d\theta} \log |k_{\Omega}'(\overline{z})|\). This implies \(\log |k_{\Omega}'(re^{i\theta})| = \log |k_{\Omega}'(re^{-i\theta})|\). Furthermore from \(\phi_{\Omega}(0) > 0\) it follows that \(\phi_{\Omega}\) maps the upper half disk \(\mathbb{D} \cap \{z \in \mathbb{C} : \text{Im} \, z > 0\}\) conformally onto \(\Omega \cap \{w \in \mathbb{C} : \text{Im} \, w > 0\}\). Thus for \(\theta \in (0, \pi)\)
\[
\frac{d}{d\theta} \left(\frac{\log |k_{\Omega}'(re^{i\theta})|}{\log |k_{\Omega}'(re^{i\theta})|}\right) = \frac{\log |k_{\Omega}'(re^{i\theta})|}{\text{Re} \, \log |k_{\Omega}'(re^{i\theta})|}
\]
\[
= \text{Re} \left\{ \frac{d}{d\theta} \log |k_{\Omega}'(re^{i\theta})| \right\}
\]
\[
= \text{Re} \left\{ i \phi_{\Omega}'(re^{i\theta}) - 1 \right\} = -\text{Im} \, \phi_{\Omega}'(re^{i\theta}) < 0.
\]

References

[1] Y. Abu Muhanna, L. Li, and S. Ponnusamy, Extremal problems on the class of convex functions of order \(-1/2\), Arch. Math. 103(6), 461–471 (2014).
[2] A. Baernstein, Integral means, univalent functions and circular symmetrization, Acta Math. 133, 139–169 (1974).
[3] J. Clunie and P. L. Duren, Addendum: An arclength problem for close-to-convex functions, J. London Math. Soc. 41, 181–182 (1966).
[4] P. L. Duren, An arclength problem for close-to-convex function, J. London Math. Soc. 39, 757–761 (1964).
[5] P. L. Duren, Theory of \(H^p\) Spaces, Pure and Applied Mathematics Vol. 38 (Academic Press, New York, London, 1970).
[6] P. L. Duren, Univalent functions, Grundlehren der mathematischen Wissenschaften Band 259 (Springer-Verlag, New York, Berlin, Heidelberg, Tokyo, 1983).
[7] W. Kaplan, Close-to-convex schlicht functions, Michigan Math. J. 1, 169–185 (1952).
[8] F. R. Keogh, Some inequalities for convex and starshaped domains, J. London Math. Soc. 29, 121–123 (1954).
[9] J. Y. Leung, Integral means of the derivatives of some univalent functions, Bull. London Math. Soc. 11(3), 289–294 (1979).
[10] W. Ma and D. Minda, A unified treatment of some special classes of univalent functions, in Proc. Inter. Conf. on Complex Anal. of the Nankai Inst. of Math. (1992), 157–169.
[11] A. Marx, Untersuchungen über schlichte Abbildungen, Math. Ann. \textbf{107}, 40–67 (1932).
[12] S. S. Miller, An arclength problem for $m$-fold symmetric univalent functions, Kodai Math. J. Sem. Rep. \textbf{24}, 196–202 (1972).
[13] Ch. Pommerenke, Univalent Functions (Vandenhoek and Ruprecht, Göttingen, 1975).
[14] T. J. Suffridge, Some remarks on convex maps of the unit disk, Duke Math. J. \textbf{37}, 775–777 (1970).
[15] T. Umezawa, Analytic functions convex in one direction, J. Math. Soc. Japan \textbf{4}, 194–202 (1952).
[16] S. Yamashita, Area and length maxima for univalent functions, Bull. Austral. Math. Soc. \textbf{41}, 435–439 (1990).
[17] H. Yanagihara, Variability regions for families of convex functions, Comput. Methods Funct. Theory \textbf{10}, 291–302 (2010).