Prime product formulas for the Riemann zeta function and related identities

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Abstract

In this article, we derive a set of prime product formulas for the Riemann zeta function $\zeta(s)$ valid for $s > 1$ and even and odd $k$th positive integer argument. We shall further give a set of generated formulas for $\zeta(k)$ up to 11th order, including Apéry’s constant, and also construct a formula for $\zeta(3/2)$. We’ll also validate these formulas numerically.

1 Main Prime Product Formula

The Euler’s prime product formula is a key connection between the Riemann zeta function and prime numbers. If $p_n$ is a sequence of $n$th prime numbers denoted such that $p_1 = 2$, $p_2 = 3$, $p_3 = 5$ and so on, then the Riemann zeta function is given as Euler prime product

$$\zeta(s) = \prod_{n=1}^{\infty} \left(1 - \frac{1}{p_n^s}\right)^{-1}$$  \hspace{1cm} (1)

which converges absolutely for $\Re(s) > 1$. Next, we shall seek to evaluate the magnitude $|\zeta(s)|$ for complex argument $s = \sigma + it$ for which $\sigma > 1$. First, by substituting complex argument

$$\zeta(\sigma + it) = \prod_{n=1}^{\infty} \left(1 - \frac{1}{p_n^{\sigma + it}}\right)^{-1}$$  \hspace{1cm} (2)

and by further algebraic simplification we arrive at

$$\zeta(\sigma + it) = \prod_{n=1}^{\infty} \frac{p_n^{\sigma} - e^{it \log p_n}}{p_n^{\sigma} + p_n^{\sigma} - 2 \cos(t \log p_n)}$$  \hspace{1cm} (3)

The magnitude can then be written as

$$|\zeta(\sigma + it)|^2 = \prod_{n=1}^{\infty} (1 + p_n^{-2\sigma})^{-1} \left(1 - \frac{2}{p_n^{\sigma} + p_n^{-\sigma}} \cos(t \log p_n)\right)^{-1}$$  \hspace{1cm} (4)

Using the identity
we obtain the main formula

$$\frac{\zeta(2\sigma)}{\zeta(\sigma)} = \prod_{n=1}^{\infty} (1 + p_n^{-\sigma})^{-1}$$  (5)

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$$|\zeta(\sigma + it)|^2 = \frac{\zeta(4\sigma)}{\zeta(2\sigma)} \prod_{n=1}^{\infty} \left(1 - \frac{2}{p_n^\sigma + p_n^{-\sigma}} \cos(t \log p_n) \right)^{-1}$$  (6)

or alternatively, the prime factors can be expressed as hyperbolic cosines, thus giving

$$|\zeta(\sigma + it)|^2 = \frac{\zeta(4\sigma)}{\zeta(2\sigma)} \prod_{n=1}^{\infty} \left(1 - \frac{\cos(t \log p_n)}{\cosh(\sigma \log p_n)} \right)^{-1}$$  (7)

This completes the derivation of the magnitude of $\zeta(\sigma + it)$. In the next section, we shall use these results and derive the integer formula for the Riemann zeta function.

## 2 Prime Product Integer Formula

Using the result of the previous section, we let $t = 0$ in Equation (6) formulation which yields

$$\zeta(\sigma)^2 = \frac{\zeta(4\sigma)}{\zeta(2\sigma)} \prod_{n=1}^{\infty} \left(1 - \frac{2}{p_n^\sigma + p_n^{-\sigma}} \right)^{-1}$$  (8)

which is valid for any $\sigma > 1$. If for a positive integer $k$ we let $\sigma = k$, and using the well-known identity for

$$\zeta(2k) = \frac{(-1)^{k+1} B_{2k} (2\pi)^{2k} }{2(2k)!}$$  (9)

where $B_k$ are Bernoulli numbers, then the zeta terms simplify as

$$\frac{\zeta(4k)}{\zeta(2k)} = \pi^{2k} \frac{(-1)^k 2^{2k} B_{4k}(2k)!}{B_{2k}(4k)!}$$  (10)

and hence, the integer formula is obtained

$$\zeta(k) = \pi^k \sqrt{\frac{(-1)^k 2^{2k} B_{4k}(2k)!}{B_{2k}(4k)!}} \prod_{n=1}^{\infty} \left(1 - \frac{2}{p_n^k + p_n^{-k}} \right)^{-1/2}$$  (11)

or alternatively,

$$\zeta(k) = \pi^k \sqrt{\frac{(-1)^k 2^{2k} B_{4k}(2k)!}{B_{2k}(4k)!}} \prod_{n=1}^{\infty} \sqrt{\frac{p_n^{2k} + 1}{p_n^k - 1}}$$  (12)

which can be used to evaluate $\zeta(k)$ for any positive integer greater than unity as will be presented in the next section.
3 Evaluation of $\zeta(2)$

By letting $k = 2$ into the above formula yields

$$\zeta(2) = \frac{\pi^2}{\sqrt{105}} \prod_{n=1}^{\infty} \left(1 - \frac{2}{p_n^2 + p_n^{2\_2}}\right)^{-1/2} \tag{13}$$

The numerical computation of this formula clearly converges to the correct result. We also note that prime product term converges to a constant factor

$$\prod_{n=1}^{\infty} \left(1 - \frac{2}{p_n^2 + p_n^{2\_3}}\right)^{-1/2} \to \frac{\sqrt{105}}{6} \tag{14}$$

where we obtain the familiar result

$$\zeta(2) = \frac{\pi^2}{6} \tag{15}$$

In the next example, we can evaluate the magnitude of $\zeta(2+i)$ using Equation (7) as

$$|\zeta(2 + i)| = \frac{\pi^2}{\sqrt{105}} \prod_{n=1}^{\infty} \left(1 - \frac{\cos(p_n\log p_n)}{\cosh(2\log p_n)}\right)^{-1/2} \to 1.23075241321861 \tag{16}$$

which also converges to a correct result.

4 Evaluation of $\zeta(3)$

Perhaps of greatest interest is the formula for $\zeta(3)$, or Apery’s constant, for which there is no known formula such as for an even order case $\zeta(2k)$, without some other factor. To carry out this analysis, we let $k = 3$ into the formula which directly results in

$$\zeta(3) = \pi^3 \sqrt{\frac{691}{675675}} \prod_{n=1}^{\infty} \left(1 - \frac{2}{p_n^3 + p_n^{3\_3}}\right)^{-1/2} \tag{17}$$

Numerical computation also validate its convergence. The prime product term converges to a constant factor,

$$\prod_{n=1}^{\infty} \left(1 - \frac{2}{p_n^3 + p_n^{3\_3}}\right)^{-1/2} \to 1.21228661439701 \tag{18}$$

which is approximately equal to $\zeta(3)$ to within 1 decimal place, but it is not known if it can be expressed in closed-form. As another example, we use Equation (7) to compute the magnitude of $\zeta(3 + i)$ as
\[ |\zeta(3 + i)| = \pi^3 \sqrt{\frac{691}{675675}} \prod_{n=1}^{\infty} \left(1 - \frac{\cos(\log p_n)}{\cosh(3\log p_n)}\right)^{-1/2} \rightarrow 1.11710067922572 \] (19)

One can carry out these computations for any \( k \). In Appendix A, we summarized these formulas up to 11th order. And in Appendix B, we summarized the numerical validation of these formulas in Table 1 up to 15 decimal places.

5 Evaluation of \( \zeta(3/2) \)

We can use the same approach to obtain a formula for \( \zeta(3/2) \), by substituting \( \sigma = 3/2 \) into Equation (11) which yields

\[ \zeta(3/2)^2 = \frac{\zeta(6)}{\zeta(3)} \prod_{n=1}^{\infty} \left(1 - \frac{2}{p_n^{3/2} + p_n^{-3/2}}\right)^{-1} \] (20)

Using \( \zeta(3) \) formula obtained earlier, and carrying out the calculation further yields

\[ \zeta(3/2) = \pi^{3/2} \sqrt{\frac{675675}{617080275}} \prod_{n=1}^{\infty} \left(1 - \frac{2}{p_n^{3/2} + p_n^{-3/2}}\right)^{1/2} \left(1 - \frac{2}{p_n^3 + p_n^{-3}}\right)^{1/4} \] (21)

which retains the leading \( \pi^n \) factor and other terms. Numerical validation is also summarized in Appendix B.

6 Other Prime Product Formulas

Another set of similar prime product formulas can be obtained as such. Using the identity in Equation (5) and multiplying both sides by \( \zeta(s) \) Euler prime product Equation (1) results in

\[ \zeta(s)^2 = \zeta(2s) \prod_{n=1}^{\infty} (1 + p_n^{-s})(1 - p_n^{-s})^{-1} \] (22)

and so

\[ \zeta(s) = \sqrt{\zeta(2s)} \prod_{n=1}^{\infty} \left(\frac{p_n^s + 1}{p_n^s - 1}\right)^{1/2} \] (23)

and if \( s \) is a positive integer \( k \), then

\[ \zeta(k) = \pi^k \sqrt{\frac{(-1)^{k+1} B_{2k} 2^{2k-1}}{(2k)!}} \prod_{n=1}^{\infty} \left(\frac{p_n^k + 1}{p_n^k - 1}\right)^{1/2} \] (24)
This results in

$$\zeta(2) = \frac{\pi^2}{\sqrt{90}} \prod_{n=1}^{\infty} \left(\frac{p_n^2 + 1}{p_n^2 - 1}\right)^{1/2}$$

and similarly, for Apéry’s constant

$$\zeta(3) = \frac{\pi^3}{\sqrt{945}} \prod_{n=1}^{\infty} \left(\frac{p_n^3 + 1}{p_n^3 - 1}\right)^{1/2}$$

And just as before,

$$\zeta(3/2) = \frac{\pi^{3/2}}{\sqrt{945}} \prod_{n=1}^{\infty} \left(\frac{p_n^{3/2} + 1}{p_n^{3/2} - 1}\right)^{1/2} \left(\frac{p_n^3 + 1}{p_n^3 - 1}\right)^{1/4}$$

Higher order formulas can be generated in a similar fashion.

7 Conclusion

A simple formula for the magnitude of the Riemann zeta function was derived based on Euler prime products, which naturally yield similar formulas, such as for positive integer argument $k > 1$, and that allowed to generate a formula for Apéry’s constant and higher order $\zeta(k)$. We also notice that for an even order case, the prime product term

$$\prod_{n=1}^{\infty} \left(1 - \frac{2}{p_n^2 + p_n^2}\right)^{-1/2} \rightarrow \frac{\sqrt{105}}{6}$$

converges to a constant such that when multiplied by the constants in Equation (11) simplifies to the standard $\zeta(2k)$ result given by Equation (9), such as $\pi^2/6$ for $k = 2$. For an odd order case however, the prime product term converges to a value that is not known to be expressed in closed-form, such as in the $\zeta(2k)$ case. The usefulness of these formulas is that the magnitude of $\zeta(s)$ for complex argument with $\Re(s) > 1$ can be evaluated by using the $\cos(t \log p_n)$ term in Equations (6)(7) and (11). We also derived a similar set of formulas from Equation (5) identity, which itself is derived by multiplying the Euler prime product by $\prod_{n=1}^{\infty}(1 + p_n)^{-1}$. The essence of these formulas is that $\pi^k$ term is extracted from the Euler prime product formula and a close-form representation of $\zeta(2k)$, and by combining multiple formulas many different varieties can be obtained.
8 Appendix A

The evaluation of Equation (11) in Mathematica up to the 11th order

\[ \zeta(2) = \frac{\pi^2}{\sqrt{105}} \prod_{n=1}^{\infty} \left( 1 - \frac{2}{p_n^2 + p_n^{-2}} \right)^{-1/2} \]  
(29)

\[ \zeta(3) = \pi^3 \sqrt{\frac{691}{675675}} \prod_{n=1}^{\infty} \left( 1 - \frac{2}{p_n^3 + p_n^{-3}} \right)^{-1/2} \]  
(30)

\[ \zeta(4) = \pi^4 \sqrt{\frac{3617}{34459425}} \prod_{n=1}^{\infty} \left( 1 - \frac{2}{p_n^4 + p_n^{-4}} \right)^{-1/2} \]  
(31)

\[ \zeta(5) = \pi^5 \sqrt{\frac{174611}{16368226875}} \prod_{n=1}^{\infty} \left( 1 - \frac{2}{p_n^5 + p_n^{-5}} \right)^{-1/2} \]  
(32)

\[ \zeta(6) = \pi^6 \sqrt{\frac{236364091}{218517792968475}} \prod_{n=1}^{\infty} \left( 1 - \frac{2}{p_n^6 + p_n^{-6}} \right)^{-1/2} \]  
(33)

\[ \zeta(7) = \pi^7 \sqrt{\frac{3392780147}{30951416768146875}} \prod_{n=1}^{\infty} \left( 1 - \frac{2}{p_n^7 + p_n^{-7}} \right)^{-1/2} \]  
(34)

\[ \zeta(8) = \pi^8 \sqrt{\frac{7709321041217}{69409790159240930625}} \prod_{n=1}^{\infty} \left( 1 - \frac{2}{p_n^8 + p_n^{-8}} \right)^{-1/2} \]  
(35)

\[ \zeta(9) = \pi^9 \sqrt{\frac{263152715530535773}{233833764946960915287281703125}} \prod_{n=1}^{\infty} \left( 1 - \frac{2}{p_n^9 + p_n^{-9}} \right)^{-1/2} \]  
(36)

\[ \zeta(10) = \pi^{10} \sqrt{\frac{261082718496449122051}{2289686345687357378035370971875}} \prod_{n=1}^{\infty} \left( 1 - \frac{2}{p_n^{10} + p_n^{-10}} \right)^{-1/2} \]  
(37)

\[ \zeta(11) = \pi^{11} \sqrt{\frac{2530297234481911294093}{21901247025838384016431785453125}} \prod_{n=1}^{\infty} \left( 1 - \frac{2}{p_n^{11} + p_n^{-11}} \right)^{-1/2} \]  
(38)
9 Appendix B

In the following table, numerical validation of Equation (11) in Mathematica to 15 decimal places for the first 1000 prime products is summarized. We note that convergence is slower for smaller arguments, such as $k = 1.5$ or $k = 2$, and essentially all the higher order converged faster.

| n  | $\zeta(k)$  | $\zeta(k)$ Equation 11 |
|----|--------------|-------------------------|
| 1.5| 2.61237534868549 | 2.60691093229650 |
| 2  | 1.64493406684823  | 1.64491317470628 |
| 3  | 1.20205690315959  | 1.20205690215259 |
| 4  | 1.08232323371114  | 1.08232323371106 |
| 5  | 1.03692775514337  | 1.03692775514337 |
| 6  | 1.01734306198445  | 1.01734306198445 |
| 7  | 1.00834927738192  | 1.00834927738192 |
| 8  | 1.00407735619794  | 1.00407735619794 |
| 9  | 1.00200839282608  | 1.00200839282608 |
| 10 | 1.0009457512782   | 1.0009457512782 |
| 11 | 1.00049418860412  | 1.00049418860412 |

References

[1] H.M. Edwards. *Riemann's Zeta Function*. Dover Publication, Mineola, New York 1974.

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