Stirling numbers with level 2 and poly-Bernoulli numbers with level 2

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Abstract
In this paper, we introduce poly-Bernoulli numbers with level 2, related to the Stirling numbers of the second kind with level 2, and study several properties of poly-Bernoulli numbers with level 2 from their expressions, relations, and congruences. Poly-Bernoulli numbers with level 2 have strong connections with poly-Cauchy numbers with level 2. In a special case, we can determine the denominators of Bernoulli numbers with level 2 by showing a von Staudt-Clausen like theorem.

Keywords: Stirling numbers, poly-Bernoulli numbers, congruences, von Staudt-Clausen theorem.

MR Subject Classifications: 11B73, 05A15, 05A19, 11A07, 11B37, 11B68, 11B75.

1 Introduction

Let \( \mathcal{S}_n \) denote the set of permutations of the set \([n] := \{1, 2, \ldots, n\} \). For \( n, k \geq 0 \), let \( \mathcal{S}_{(n,k)} \) denote the set of permutations of \( \mathcal{S}_n \) having exactly \( k \) cycles, satisfying \( \mathcal{S}_n = \bigcup_{k=0}^{n} \mathcal{S}_{(n,k)} \). Let \( s \) be a positive integer. The Stirling numbers of the first kind with level \( s \), denoted by \( \left\llbracket \begin{array}{c} n \\ k \end{array} \right\rrbracket_s \), are defined as the number of ordered \( s \)-tuples \( (\sigma_1, \sigma_2, \ldots, \sigma_s) \in \mathcal{S}_{(n,k)} \times \mathcal{S}_{(n,k)} \times \cdots \times \mathcal{S}_{(n,k)} = \mathcal{S}_s^{(n,k)} \), such that

\[
\min(\sigma_1) = \min(\sigma_2) = \cdots = \min(\sigma_s).
\]

The Stirling numbers of the first kind with higher level satisfies the recurrence relation

\[
\left\llbracket \begin{array}{c} n \\ k \end{array} \right\rrbracket_s = \left\llbracket \begin{array}{c} n-1 \\ k-1 \end{array} \right\rrbracket_s + (n-1)^s \left\llbracket \begin{array}{c} n-1 \\ k \end{array} \right\rrbracket.
\]
with the initial conditions \( {{0} \choose {0}}_s = 1 \) and \( {{n} \choose {0}}_s = {{0} \choose {n}}_s = 0 \) \((n \geq 1)\). The Stirling numbers of the first kind with higher level are yielded from the coefficients of the polynomial as
\[
 x(x + 1^s)(x + 2^s) \cdots (x + (n - 1)^s) = \sum_{k=0}^{n} \left[ {{n} \choose {k}}_s \right] x^k.
\]

When \( s = 1 \),
\[
 \left[ {{n} \choose {k}}_s \right] = \left[ {{n} \choose {k}}_1 \right] = |\mathcal{G}_{(n,k)}|
\]
are the original (unsigned) Stirling numbers of the first kind. When \( s = 2 \), the Stirling numbers of the first kind with level 2 \((8)\) are related with the central factorial numbers of the first kind \( t(n, k) \) \((11)\) as \( \left[ {{n} \choose {k}}_2 \right] = t(2n, 2k) \). Notice that the original Stirling numbers of the first kind and the Stirling numbers of the first kind with level 2 are used to express poly-Cauchy numbers \((5)\) and poly-Cauchy numbers with level 2 \((8, 10)\), respectively.

On the other hand, for \( n, k \geq 0 \), let \( \Pi_{(n,k)} \) denote the set of all partitions of \([n]\) having exactly \( k \) non-empty blocks. Given a partition \( \pi \) in \( \Pi_n \), let \( \text{min}(\pi) \) denote the set of the minimal elements in each block of \( \pi \). For a positive integer \( s \), the Stirling numbers of the second kind with level \( s \), denoted by \( \{\{ \left[ {{n} \choose {k}}_s \right] \} \} \) \((12)\), are defined as the number of ordered \( s \)-tuples \((\pi_1, \pi_2, \ldots, \pi_s) \in \Pi_{(n,k)} \times \Pi_{(n,k)} \times \cdots \times \Pi_{(n,k)} = \Pi^{s}_{(n,k)}\), such that
\[
 \text{min}(\pi_1) = \text{min}(\pi_2) = \cdots = \text{min}(\pi_s).
\]

The Stirling numbers of the second kind with higher level satisfies the recurrence relation
\[
\{\{ \left[ {{n} \choose {k}}_s \right] \} \} = \left[ {{n-1} \choose {k-1}}_s \right] + k^s \left[ {{n-1} \choose {k}}_s \right] .
\]
with the initial conditions \( \{\{ 0 \choose 0 \}_s \} = 1 \) and \( \{\{ n \choose 0 \}_s \} = \{\{ 0 \choose n \}_s \} = 0 \) \((n \geq 1)\). The Stirling numbers of the second kind with higher level are yielded from the coefficients of the polynomial as
\[
 x^n = \sum_{k=0}^{n} \left[ \{\{ \left[ {{n} \choose {k}}_s \right] \} \} x(x - 1^s)(x - 2^s) \cdots (x - (k - 1)^s)\right].
\]

When \( s = 1 \),
\[
 \left[ \{\{ \left[ {{n} \choose {k}}_s \right] \} \} \right]_1 = \left[ \{\{ \left[ {{n} \choose {k}}_s \right] \} \} \right]_1
\]
are the original Stirling numbers of the second kind. When \( s = 2 \), we have that \( \left\{ \begin{array}{l} n \\ k \end{array} \right\}_2 = T(2n, 2k) \), where \( T(n, k) \) are the central factorial numbers of the second kind (cf. [1, 2]), satisfying

\[
x^n = \sum_{k=0}^{n} T(n, k)x \left( x + \frac{k}{2} - 1 \right) \left( x + \frac{k}{2} - 2 \right) \cdots \left( x - \frac{k}{2} + 1 \right).
\]

As the original Stirling numbers of the second kind are used to express poly-Bernoulli numbers explicitly ([4]), we intend to introduce poly-Bernoulli numbers with level 2, related to the Stirling numbers of the second kind with level 2. In this paper, we study several properties of poly-Bernoulli numbers with level 2 from their expressions, relations, and congruences. Poly-Bernoulli numbers with level 2 have strong connections with poly-Cauchy numbers with level 2. In a special case, we can determine the denominators of Bernoulli numbers with level 2 by showing a von Staudt-Clausen like theorem.

### 2 Some expressions

In [12], the Stirling numbers of the second kind with higher level are expressed explicitly as

\[
\left\{ \begin{array}{l} n \\ k \end{array} \right\}_s = \sum_{j=1}^{k} \frac{j^{ns}}{\prod_{i=0, i \neq j}^{k} (j^{s} - i^{s})} \quad (1 \leq k \leq n).
\]

When \( s = 1 \), by

\[
(k - j)!j! = (-1)^{k-j} \prod_{i=0}^{k} (j - i),
\]

this is reduced to a famous expression of the original Stirling numbers of the second kind:

\[
\left\{ \begin{array}{l} n \\ k \end{array} \right\} = \frac{1}{k!} \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} j^n \quad (1 \leq k \leq n).
\]

When \( s = 2 \), ([3]) is reduced to an expression

\[
\left\{ \begin{array}{l} n \\ k \end{array} \right\}_2 = \frac{2}{(2k)!} \sum_{j=1}^{k} (-1)^{k-j} \binom{2k}{k-j} j^{2n} \quad (1 \leq k \leq n)
\]
However, no explicit expression for $s \geq 3$ has not been found yet. It implies that unfortunately,

$$\binom{n}{k} \equiv_{3} \neq \frac{3}{(3k)!} \sum_{j=1}^{k} (-1)^{k-j} \left(\frac{3k}{k-j}\right)^{3n}$$
or something like this.

There exists a different explicit expression from $[4]$.

**Theorem 1.** For integers $n$ and $k$ with $2 \leq k \leq n$, 

$$\binom{n}{k} \equiv_{s} = \sum_{j=1}^{k-1} j^{k-1-s} (j(n-k+1)s - k(n-k+1)s) \prod_{i=1, i \neq j}^{k} (j^s - i^s)$$

with $\binom{n}{1} \equiv_{s} = 1$ ($n \geq 1$).

**Remark.** When $s = 1$ in Theorem 1 by (5) and

$$\sum_{j=1}^{k} (-1)^{j} \binom{k}{j} j^{k-1} = 0,$$

we have (4) again.

In [12], the ordinary generating function of the Stirling numbers of the second kind with higher level is given by

$$\sum_{n=k}^{\infty} \binom{n}{k} s x^n = \frac{x^k}{(1-x)(1-2^sx)\cdots(1-k^sx)}.$$

The exponential generating function of the Stirling numbers of the second kind with higher level is given as follows.

**Theorem 2.** For $k \geq 1$,

$$\sum_{n=k}^{\infty} \binom{n}{k} \frac{x^n}{n!} = \sum_{j=1}^{k} \frac{e^{j^s x}}{\prod_{i=1, i \neq j}^{k} (j^s - i^s)} + \frac{(-1)^k}{(k!)^s}.$$

**Remark.** 1) When $s = 1$ in Theorem 2 we have the exponential generating function of the original Stirling numbers of the second kind:

$$\sum_{n=k}^{\infty} \binom{n}{k} \frac{x^n}{n!} = \frac{1}{k!} \left( \sum_{j=1}^{k} \frac{k!}{(k-j)!j!} (e^x)^j (-1)^{k-j} + (-1)^k \right)$$
\[ (e^x - 1)^k \quad k! \]

2) Another variation is similarly shown as follows.

\[
\sum_{n=k}^{\infty} \left\{ \binom{n}{k} \right\}_s \frac{x^{ns}}{(ns)!} = \frac{1}{s} \sum_{\ell=0}^{s-1} \sum_{j=1}^{k} \frac{\zeta^{jx}}{\prod_{i=1, i \neq j}^{k} (j^s - i^s)} + \frac{(-1)^k}{(k!)^s}.
\]

where \( \zeta := e^{2\pi i/s} \) is the \( s \)-th root of unity.

**Proof of Theorem 2**

1) By (4), we have

\[
\sum_{n=k}^{\infty} \left\{ \binom{n}{k} \right\}_s \frac{x^{ns}}{n!} = \sum_{n=k}^{\infty} \frac{k}{\prod_{i=0, i \neq j}^{k} (j^s - i^s)} \sum_{n=0}^{\infty} j^{ns} x^n \frac{1}{n!}
\]

\[
- \sum_{n=k}^{\infty} j^{ns} x^n \frac{1}{n!} - \sum_{n=0}^{\infty} j^{ns} x^n \frac{1}{n!}
\]

\[
= \sum_{n=0}^{\infty} j^{ns} x^n \frac{1}{n!} + \frac{(-1)^k}{(k!)^s}.
\]

Here,

\[
\sum_{j=1}^{k} \frac{1}{\prod_{i=0, i \neq j}^{k} (j^s - i^s)} = \frac{(-1)^{k-1}}{(k!)^s}
\]

and

\[
\sum_{j=1}^{k} j^{ns} \frac{1}{\prod_{i=0, i \neq j}^{k} (j^s - i^s)} = 0 \quad (1 \leq n \leq k - 1).
\]

2) By Theorem 1 we have

\[
\sum_{n=k}^{\infty} \left\{ \binom{n}{k} \right\}_s \frac{x^{ns}}{n!} = \sum_{n=k}^{\infty} \frac{k}{\prod_{i=0, i \neq j}^{k} (j^s - i^s)} \sum_{n=0}^{\infty} j^{ns} x^n \frac{1}{n!}
\]

\[
= \sum_{j=1}^{k} \frac{1}{\prod_{i=0, i \neq j}^{k} (j^s - i^s)} \sum_{n=0}^{\infty} j^{ns} x^n \frac{1}{n!}
\]

\[
= \sum_{j=1}^{k} \frac{e^{jx}}{\prod_{i=0, i \neq j}^{k} (j^s - i^s)} + \frac{(-1)^k}{(k!)^s}.
\]

\[
= \frac{\zeta^{jx}}{\prod_{i=0, i \neq j}^{k} (j^s - i^s)} + \frac{(-1)^k}{(k!)^s}.
\]
3 Poly-Bernoulli numbers with level 2

Poly-Bernoulli numbers are defined by the generating function
\[
\frac{\text{Li}_k(1 - e^{-x})}{1 - e^{-x}} = \sum_{n=0}^{\infty} \frac{B_n^{(k)} x^n}{n!}
\]
\[\text{(8)}\]
where
\[
\text{Li}_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k}
\]
is the polylogarithm function.

When \( k = 1 \), it is reduced to the generating function of the original Bernoulli numbers:
\[
\frac{x}{1 - e^{-x}} = \sum_{n=0}^{\infty} \frac{B_n^{(1)} x^n}{n!}
\]
with \( B_1^{(1)} = 1/2 \). Another definition is given by
\[
\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n x^n}{n!}
\]
\[\text{(9)}\]
with $B_1 = -1/2$.

Then, poly-Bernoulli numbers can be expressed explicitly in terms of the Stirling numbers of the second kind (4, Theorem 1):

$$B_n^{(k)} = \sum_{m=0}^{n} \left\{ \begin{array}{c} n \\ m \end{array} \right\} \frac{(-1)^{n-m}m!}{(m+1)^k}.$$  \hspace{1cm} (10)

There have been many generalizations of Bernoulli or poly-Bernoulli numbers. In this paper, we introduced poly-Bernoulli numbers with level 2 by using higher-level Stirling numbers (11, 12).

In 10, poly-Cauchy numbers $c_n^{(k)}$ with level 2 are defined by

$$\text{Lif}_{2,k}(\text{arcsinh}x) = \sum_{n=0}^{\infty} c_n^{(k)} \frac{x^n}{n!},$$  \hspace{1cm} (11)

where arcsinh$x$ is the inverse hyperbolic sine function and

$$\text{Lif}_{2,k}(z) = \sum_{m=0}^{\infty} \frac{z^{2m}}{(2m)!(2m+1)^k}.$$  \hspace{1cm} (12)

The function $\text{Lif}_{2,k}(z)$ is an analogue of polylogarithm factorial or polyfactorial function $\text{Lif}_k(z)$ [5, 6], defined by

$$\text{Lif}_k(z) = \sum_{m=0}^{\infty} \frac{z^m}{m!(m+1)^k}.$$  \hspace{1cm} (13)

By using the polyfactorial function, poly-Cauchy numbers (of the first kind) $c_n^{(k)}$ are defined as

$$\text{Lif}_k(\log(1 + x)) = \sum_{n=0}^{\infty} c_n^{(k)} \frac{x^n}{n!}.$$  \hspace{1cm} (14)

When $k = 1$, by $\text{Lif}_1(z) = (e^z - 1)/z$, $c_n = c_n^{(1)}$ are the original Cauchy numbers defined by

$$\frac{x}{\log(1 + x)} = \sum_{n=0}^{\infty} c_n \frac{x^n}{n!}.$$  \hspace{1cm} (15)

Define the polylogarithm function $\text{Li}_{2,k}(z)$ with level 2 by

$$\text{Li}_{k,2}(z) = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)^k}.$$  \hspace{1cm} (16)
Then, poly-Bernoulli numbers $\mathfrak{B}^{(k)}_n$ with level 2 are defined by

$$\frac{\text{Li}_{2,k}(2\sin(x/2))}{2\sin(x/2)} = \sum_{n=0}^{\infty} \frac{\mathfrak{B}^{(k)}_n x^n}{n!}. \quad (14)$$

Note that $\mathfrak{B}^{(k)}_n = 0$ for odd $n$.

The generating function of the poly-Cauchy numbers with level 2 can be written in the form of iterated integrals ([10, Theorem 2.1]):

$$\text{arcsinh} \frac{x}{2} \int_0^x \text{arcsinh} \sqrt{1 + x^2} \cdot \cdots \cdot \int_0^x \text{arcsinh} \sqrt{1 + x^2} k^{-1} x \, dx \cdots \, dx = \sum_{n=0}^\infty \frac{C^{(k)}_n x^n}{n!} \quad (k \geq 1).$$

We can also write the generating function of the poly-Bernoulli numbers with level 2 in (14) in the form of iterated integrals.

**Theorem 3.** For $k \geq 1$, we have

$$\frac{1}{2 \sin \frac{x}{2}} \int_0^x \frac{1}{2 \tan \frac{x}{2}} \cdots \int_0^x \frac{1}{2 \tan \frac{x}{2}} k^{-1} \frac{1}{2 \log \frac{1 + 2 \sin \frac{x}{2}}{1 - 2 \sin \frac{x}{2}}} \, dx \cdots \, dx = \sum_{n=0}^\infty \frac{\mathfrak{B}^{(k)}_n x^n}{n!}.$$

**Proof.** Since

$$\frac{d}{dz} \text{Li}_{2,k}(z) = \frac{1}{z} \text{Li}_{2,k-1}(z),$$

we have

$$\text{Li}_{2,k}(z) = \int_0^z \frac{\text{Li}_{2,k-1}(z_1)}{z_1} \, dz_1$$

$$= \int_0^z \frac{dz_1}{z_1} \int_0^{z_1} \frac{\text{Li}_{2,k-2}(z_2)}{z_2} \, dz_2$$

$$= \int_0^z \frac{dz_1}{z_1} \int_0^{z_2} \frac{dz_2}{z_2} \cdots \int_0^{z_{k-1}} \frac{\text{Li}_{2,1}(z_{k-1})}{z_{k-1}} \, dz_{k-1}$$

$$= \int_0^z \frac{dz_1}{z_1} \int_0^{z_2} \frac{dz_2}{z_2} \cdots \int_0^{z_{k-2}} \frac{1}{z_{k-1}} \frac{1}{2 \log \frac{1 + z_{k-1}}{1 - z_{k-1}}} \, dz_{k-1}.$$
Putting \( z = z_1 = \cdots = z_{k-1} = 2 \sin(x/2) \), we get
\[
\frac{\text{Li}_{2,k}(2 \sin(x/2))}{2 \sin(x/2)} = \frac{1}{2 \sin(x/2)} \int_0^x \frac{\cos(x/2)}{2 \sin(x/2)} dx \int_0^x \frac{\cos(x/2)}{2 \sin(x/2)} dx \\
\cdots \int_0^x \frac{\cos(x/2)}{2 \sin(x/2)} \frac{1}{2} \log \frac{1 + 2 \sin(x/2)}{1 - 2 \sin(x/2)} dx.
\]

4 Explicit formulae and recurrence relations

From the definition in (14), we see that
\[
\mathcal{B}_0^{(k)} = 1,
\]
\[
\mathcal{B}_2^{(k)} = \frac{2}{3^k},
\]
\[
\mathcal{B}_4^{(k)} = -\frac{2}{3^k} + \frac{24}{5^k},
\]
\[
\mathcal{B}_6^{(k)} = \frac{2}{3^k} - \frac{120}{5^k} + \frac{720}{7^k},
\]
\[
\mathcal{B}_8^{(k)} = -\frac{2}{3^k} + \frac{504}{5^k} - \frac{10080}{7^k} + \frac{40320}{9^k},
\]
\[
\mathcal{B}_{10}^{(k)} = \frac{2}{3^k} - \frac{2040}{5^k} + \frac{105840}{7^k} - \frac{32659200}{9^k} + \frac{3628800}{11^k}.
\]

In this section, we shall show some explicit formulae and some recurrence relations.

Poly-Cauchy numbers with level 2 can be expressed explicitly in terms of the Stirling numbers of the second kind with level 2 ([8, Theorem 1]):
\[
\mathcal{C}_{2n}^{(k)} = \sum_{m=0}^{n} \left[ \begin{array}{c} n \\ m \end{array} \right]_2 \frac{(-4)^{n-m}}{(2m+1)^k}.
\]

Poly-Bernoulli numbers with level 2 can be expressed explicitly in terms of the Stirling numbers of the second kind with level 2. It is a natural extension of the expression in [10].

**Theorem 4.** For \( n \geq 0 \),
\[
\mathcal{B}_{2n}^{(k)} = \sum_{m=0}^{n} \left\{ \begin{array}{c} n \\ m \end{array} \right\}_2 \frac{(-1)^{n-m}(2m)!}{(2m+1)^k}.
\]
Proof. We use the power series of powers of trigonometric functions

\[
(2 \sin \frac{x}{2})^{2m} = \sum_{n=m}^{\infty} (-1)^{n-m}(2m)! \binom{n}{m} x^{2n}
\]

(see [1, Theorem 4.1.1 (4.1.1)]). Then, by [13] and [14], we have

\[
\sum_{n=0}^{\infty} \mathfrak{B}_n^{(k)} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \mathfrak{B}_{2n}^{(k)} \frac{x^{2n}}{(2n)!}
= \sum_{m=0}^{\infty} \frac{(2 \sin(x/2))^{2m}}{(2m + 1)^k}
= \sum_{m=0}^{\infty} \frac{1}{(2m + 1)^k} \sum_{n=m}^{\infty} (-1)^{n-m}(2m)! \binom{n}{m} \binom{2n}{2m} x^{2n}
= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \binom{n}{m}^2 (-1)^{n-m}(2m)! x^{2n}
\]

(2m + 1)^k

Comparing the coefficients on both sides, we get the desired result.

Next, we shall show an explicit formula without Stirling numbers.

Theorem 5. For integers \( n \) and \( k \) with \( n \geq 0 \),

\[
\mathfrak{B}_{2n}^{(k)} = \frac{n}{(2m + 1)^k} \sum_{i_1 + \cdots + i_{2m} = n-m}^{2n} \left( \frac{1}{4} \right)^{n-m} \binom{2n}{2i_1 + 1, \cdots, 2i_{2m} + 1}
\]

where

\[
\binom{2n}{2i_1 + 1, \cdots, 2i_{2m} + 1} = \frac{(2i_1 + 1) + \cdots + (2i_{2m} + 1)!}{(2i_1 + 1)! \cdots (2i_{2m} + 1)!}
\]

is the multinomial coefficient.

Proof. Since

\[
2 \sin \frac{x}{2} = \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}}{(2\ell + 1)!} x^{2\ell+1}
\]

we have

\[
\sum_{n=0}^{\infty} \mathfrak{B}_{2n}^{(k)} \frac{x^{2n}}{(2n)!}
\]
Comparing the coefficients on both sides, we get the desired result. \(\square\)

There exists a recurrence formula for \(B_n^{(k)}\) in terms of \(B_n^{(k-1)}\) and the original Bernoulli numbers \(B_n\) in (9). In fact, \(B_n = B_n^{(1)}\) for even \(n\).

**Theorem 6.** For integers \(n\) and \(k\) with \(n \geq 0\) and \(k \geq 1\),

\[ B_n^{(k-1)} = B_n^{(k)} + (2n)! \sum_{m=0}^{n-1} \frac{4((-1)^{n-m} - (-4)^{n-m}) B_{2n-2m} B_{2m+2}}{(2n-2m)!(2m+1)!}. \]

**Proof.** From the definition in (14), we see

\[ \text{Li}_{2,k}(2 \sin(x/2)) = 2 \sin(x/2) \sum_{n=0}^{\infty} B_n^{(k)} \frac{x^{2n}}{(2n)!}. \]

Differentiating both sides by \(x\), we have

\[ \frac{\cos(x/2)}{2 \sin(x/2)} \text{Li}_{2,k-1}(2 \sin(x/2)) = \cos \frac{x}{2} \sum_{n=0}^{\infty} \frac{B_n^{(k)} x^{2n}}{(2n)!} + 2 \sin \frac{x}{2} \sum_{n=1}^{\infty} \frac{B_n^{(k)} x^{2n-1}}{(2n-1)!}. \]

Hence,

\[ \sum_{n=0}^{\infty} B_n^{(k-1)} \frac{x^{2n}}{(2n)!} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{x^{2n}}{(2n)!} + 2 \tan \frac{x}{2} \sum_{n=0}^{\infty} B_n^{(k)} \frac{x^{2n+1}}{(2n+1)!}. \quad (15) \]
Since
\[
2 \tan \sum_{n=0}^{\infty} \frac{B^{(k)}_{2n+2} x^{2n+1}}{(2n+1)!} = \left( \sum_{n=0}^{\infty} \frac{4((-1)^n - (-4)^n) B_{2n}}{(2n)!} \right) \left( \sum_{n=0}^{\infty} \frac{B^{(k)}_{2n+2} x^{2n+1}}{(2n+1)!} \right)
\]
\[
= \sum_{n=1}^{\infty} \sum_{m=0}^{n-1} \frac{4((-1)^{n-m} - (-4)^{n-m}) B_{2n-2m} B^{(k)}_{2m+2} x^{2n}}{(2n-2m)!(2m+1)!},
\]
comparing the coefficients of both sides of (15), we get the desired result. \(\square\)

5 Relations with poly-Cauchy numbers with level 2

There exist some strong reasons why we define poly-Bernoulli numbers with level 2 as in (14). Poly-Cauchy numbers with level 2 can be expressed in terms of poly-Bernoulli numbers with level 2.

**Theorem 7.** For integers \(n\) and \(k\) with \(n \geq 1\),
\[
\mathcal{C}^{(k)}_{2n} = \sum_{m=1}^{n} \sum_{l=1}^{m} \frac{(-4)^{n-m}}{(2m)!} \left[ \begin{array}{c} n \\ m \end{array} \right] \left[ \begin{array}{c} m \\ l \end{array} \right] B^{(k)}_{m-2l}.
\]

**Remark.** Poly-Cauchy numbers can be expressed in terms of poly-Bernoulli numbers ([9, Theorem 2.2]):
\[
c^{(k)}_{n} = \sum_{m=1}^{n} \sum_{l=1}^{m} \frac{(-1)^{n-m}}{m!} \left[ \begin{array}{c} n \\ m \end{array} \right] \left[ \begin{array}{c} m \\ l \end{array} \right] B^{(k)}_{l}.
\]

**Proof of Theorem 7** By Theorem 4 and the orthogonal relation
\[
\sum_{l=j}^{m} (-1)^{l-j} \left[ \begin{array}{c} m \\ l \end{array} \right] \left\{ \begin{array}{c} l \\ j \end{array} \right\}_{s} = \begin{cases} 1 & m = j; \\ 0 & m \neq j \end{cases} \quad (s \geq 1)
\]
([12, Theorem 5.1]) together with \(\left[ \begin{array}{c} n \\ 0 \end{array} \right]_{2} = 0 \quad (n \geq 1)\), we have
\[
\text{RHS} = \sum_{m=1}^{n} \sum_{l=1}^{m} \frac{(-4)^{n-m}}{(2m)!} \left[ \begin{array}{c} n \\ m \end{array} \right] \left[ \begin{array}{c} m \\ l \end{array} \right] \sum_{j=0}^{l} \left\{ \begin{array}{c} l \\ j \end{array} \right\}_{2} (-1)^{l-j} \frac{(2j)!}{(2j+1)^{k}}
\]

12
\[
\sum_{m=1}^{n} \frac{(-4)^{n-m}}{(2m)!} \left[ \begin{array}{c} n \\ m \end{array} \right] \sum_{j=0}^{m} \frac{(2j)!}{(2j+1)^k} \sum_{l=j}^{m} (-1)^{l-j} \left[ \begin{array}{c} m \\ l \end{array} \right] \left[ \begin{array}{c} l \\ j \end{array} \right]_2 \\
= \sum_{m=1}^{n} \frac{(-4)^{n-m}}{(2m)!} \left[ \begin{array}{c} n \\ m \end{array} \right] \frac{(2m)!}{(2m+1)^k} \\
= \sum_{m=1}^{n} \frac{(-4)^{n-m}}{(2m+1)^k} \left[ \begin{array}{c} n \\ m \end{array} \right]_2 \\
= \mathfrak{C}^{(k)}_{2n} \quad (\text{Theorem 1}).
\]

On the contrary, poly-Bernoulli numbers can be expressed in terms of poly-Cauchy numbers ([5, 9]):
\[
B_n^{(k)} = \sum_{m=1}^{n} \sum_{l=1}^{m} (-1)^{n-m} m! \left\{ \begin{array}{c} n \\ m \end{array} \right\} \left\{ \begin{array}{c} m \\ l \end{array} \right\} \mathfrak{c}^{(k)}_l. 
\]
Similarly, poly-Bernoulli numbers with level 2 can be expressed in terms of poly-Cauchy numbers with level 2.

**Theorem 8.** For integers \(n\) and \(k\) with \(n \geq 1\),
\[
\mathfrak{B}^{(k)}_{2n} = \sum_{m=1}^{n} \sum_{l=1}^{m} (-1)^{n-m} 4^{m-l} (2m)! \left\{ \begin{array}{c} n \\ m \end{array} \right\} \left\{ \begin{array}{c} m \\ l \end{array} \right\} \mathfrak{C}^{(k)}_l. 
\]

**Proof.** Using another orthogonal relation
\[
\sum_{l=j}^{m} (-1)^{l-j} \left\{ \begin{array}{c} m \\ l \end{array} \right\} \left\{ \begin{array}{c} l \\ j \end{array} \right\}_s = \begin{cases} 1 & m = j; \\ 0 & m \neq j \end{cases} \quad (s \geq 1) 
\]
(12 Theorem 5.1)), we have
\[
\text{RHS} = \sum_{m=1}^{n} (-1)^{n-m} (2m)! \left\{ \begin{array}{c} n \\ m \end{array} \right\} \sum_{l=0}^{m} 4^{m-l} \left\{ \begin{array}{c} m \\ l \end{array} \right\} \sum_{j=1}^{l} \frac{(-4)^{l-j}}{(2j+1)^k} \left\{ \begin{array}{c} l \\ j \end{array} \right\}_2 \\
= \sum_{m=1}^{n} (-1)^{n-m} (2m)! \left\{ \begin{array}{c} n \\ m \end{array} \right\} \sum_{j=0}^{m} \frac{4^{m-j}}{(2j+1)^k} \sum_{l=j}^{m} (-1)^{l-j} \left\{ \begin{array}{c} m \\ l \end{array} \right\} \left\{ \begin{array}{c} l \\ j \end{array} \right\}_2 \\
= \sum_{m=1}^{n} \frac{(-1)^{n-m} (2m)!}{(2m+1)^k} \left\{ \begin{array}{c} n \\ m \end{array} \right\}_2
\]
\[ B_{2n}^{(k)} = \sum_{m=0}^{n} \left\{ \begin{array}{c} n \\ m \end{array} \right\}_2 2^m \sum_{l=0}^{m} \left( \begin{array}{c} m \\ l \end{array} \right) \frac{(-1)^{m-l}(2l)!}{(2l+1)^k} \]

Other relations with Stirling numbers with level 2 are given as follows.

**Theorem 9.** For \( n \geq 1 \),

\[
\frac{1}{(2n)!} \sum_{m=0}^{n} \left\{ \begin{array}{c} n \\ m \end{array} \right\}_2 B_{2m}^{(k)} = \frac{1}{(2n+1)^k},
\]

(16)

\[
\sum_{m=0}^{n} \left\{ \begin{array}{c} n \\ m \end{array} \right\}_2 4^{n-m} c_{2m}^{(k)} = \frac{1}{(2n+1)^k}.
\]

(17)

**Remark.** For poly-Bernoulli and poly-Cauchy numbers ([5, Theorem 3]), we have

\[
\frac{1}{n!} \sum_{m=0}^{n} \left\{ \begin{array}{c} n \\ m \end{array} \right\} B_{m}^{(k)} = \frac{1}{(n+1)^k},
\]

\[
\sum_{m=0}^{n} \left\{ \begin{array}{c} n \\ m \end{array} \right\}_c c_{m}^{(k)} = \frac{1}{(n+1)^k}.
\]

**Proof of Theorem 9** By the orthogonal relations, we have

\[
\frac{1}{(2n)!} \sum_{m=0}^{n} \left\{ \begin{array}{c} n \\ m \end{array} \right\}_2 B_{2m}^{(k)} = \frac{1}{(2n)!} \sum_{m=0}^{n} \left\{ \begin{array}{c} n \\ m \end{array} \right\}_2 \sum_{l=0}^{m} \left\{ \begin{array}{c} m \\ l \end{array} \right\}_2 \frac{(-1)^{m-l}(2l)!}{(2l+1)^k}
\]

\[
= \frac{1}{(2n)!} \sum_{l=0}^{n} \frac{(2l)!}{(2l+1)^k} \sum_{m=l}^{n} (-1)^{m-l} \left\{ \begin{array}{c} n \\ m \end{array} \right\}_2 \left\{ \begin{array}{c} m \\ l \end{array} \right\}_2
\]

\[
= \frac{1}{(2n)!} \sum_{l=0}^{n} \frac{(2l)!}{(2n+1)^k} = \frac{1}{(2n+1)^k}
\]

and

\[
\sum_{m=0}^{n} \left\{ \begin{array}{c} n \\ m \end{array} \right\}_2 4^{n-m} c_{2m}^{(k)} = \sum_{m=0}^{n} \left\{ \begin{array}{c} n \\ m \end{array} \right\}_2 4^{n-m} \sum_{l=0}^{m} \left\{ \begin{array}{c} m \\ l \end{array} \right\}_2 \frac{(-4)^{m-l}}{(2l+1)^k}
\]

\[
= \sum_{l=0}^{n} \frac{4^n}{(2l+1)^k} \left( \frac{-1}{4} \right)^l \sum_{m=l}^{n} (-1)^{m} \left\{ \begin{array}{c} n \\ m \end{array} \right\}_2 \left\{ \begin{array}{c} m \\ l \end{array} \right\}_2
\]

\[
= \frac{4^n}{(2n+1)^k} \left( \frac{-1}{4} \right)^n (-1)^n = \frac{1}{(2n+1)^k}.
\]
6 Double summation formula

The poly-Bernoulli numbers satisfy the duality formula $B_n^{(-k)} = B_k^{(-n)}$ for $n, k \geq 1$, because of the symmetric formula

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} B_n^{(-k)} \frac{x^n y^k}{n! k!} = \frac{e^{x+y}}{e^x + e^y - e^{x+y}}.$$

Though the corresponding duality formula does not always hold for other cases, we still have the double summation formula for poly-Bernoulli numbers with level 2.

**Theorem 10.**

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \mathfrak{B}_{2n}^{(-2k)} \frac{x^{2n} y^{2k}}{(2n)! (2k)!} = \frac{\cos x \cosh y}{2(1 - \cos x)(1 - \cosh 2y) + \cos^2 x}.$$

**Proof.** We have

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \mathfrak{B}_{2n}^{(-2k)} \frac{x^{2n} y^{2k}}{(2n)! (2k)!}$$

$$= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \left(2 \sin \frac{x}{2}\right)^{2m} \frac{2^{2k} (2m+1) y^{2k}}{(2k)!}$$

$$= \sum_{m=0}^{\infty} \left(2 \sin \frac{x}{2}\right)^{2m} \cosh((2m+1)y)$$

$$= \frac{e^y}{2} \sum_{m=0}^{\infty} \left(2 \sin \frac{x}{2}\right)^{2m} e^{2my} + \frac{e^{-y}}{2} \sum_{m=0}^{\infty} \left(2 \sin \frac{x}{2}\right)^{2m} e^{-2my}$$

$$= \frac{e^y}{2} \frac{1}{1 - (2e^y \sin(x/2))^2} + \frac{e^{-y}}{2} \frac{1}{1 - (2e^{-y} \sin(x/2))^2}$$

$$= \frac{1}{2} \left(\frac{e^y}{1 - e^{2y}(1 - \cos x)} + \frac{e^{-y}}{1 - e^{-2y}(1 - \cos x)}\right)$$

$$= \frac{1}{2} \left(\frac{e^y \cos x + e^y \cos x}{2(1 - \cos x)}\right)$$

$$= \frac{2}{2} \frac{(1 - \cos x) - (e^{2y} + e^{-2y})(1 - \cos x) + \cos^2 x}{\cos x \cosh y}$$

$$= \frac{2(1 - \cos x)(1 - \cosh 2y) + \cos^2 x}{2(1 - \cos x)(1 - \cosh 2y) + \cos^2 x}.$$
7 Congruences

In this section, we shall show some congruent relations for \( B^{(k)}_n \) for negative \( k \).

**Theorem 11.** For \( n, k \geq 1 \), \( B^{(-k)}_{2n} \equiv 0 \) (mod 6).

**Proof.** For \( n \geq 1 \), from Theorem 4 together with \( \left\{ \binom{n}{m} \right\}_2 = 0 \) (\( n \geq 1 \)).

\[
B^{(-k)}_{2n} = \sum_{m=1}^{n} \left\{ \binom{n}{m} \right\}_2 (-1)^{n-m} (2m)! (2m + 1)^k.
\]

Since \( 2 | (2m)! \) and \( 3 | (2m + 1)^k \) for \( m = 1 \), and \( 6 | (2m)! \) for \( m \geq 2 \), together with the fact that the Stirling numbers of the second kind with level 2 are integers, we get the desired result. \( \square \)

**Theorem 12.** For \( n, k \geq 1 \), the values of \( B^{(-k)}_{2n} \) (mod 5) are given in the following table.

| \( n \) | \( k \) | 0 | 1 | 2 | 3 | (mod 4) |
|------|------|---|---|---|---|--------|
| 0    | 0    | 3 | 4 | 2 | 1 |        |
| 1    | (mod 2) | 2 | 1 | 3 | 4 |        |

**Proof.** In the terms of the summation expression of \( B^{(-k)}_{2n} \), \( 5 | (2m + 1)^k \) for \( m = 2 \), and \( 5 | (2m)! \) for \( m \geq 3 \). Hence,

\[
B^{(-k)}_{2n} \equiv (-1)^{n-1} 2 \cdot 3^k \pmod{5}.
\]

Since \( 3^4 \equiv 1 \pmod{5} \) by Fermat’s little theorem, it is sufficient to check the cases for \( k \equiv 0, 1, 2, 3 \pmod{4} \). When \( k \equiv 0 \pmod{4} \),

\[
B^{(-k)}_{2n} \equiv (-1)^{n-1} 2 \equiv \begin{cases} 3 & (n \equiv 0 \pmod{2}) \\ 2 & (n \equiv 1 \pmod{2}) \end{cases} \pmod{5}.
\]

When \( k \equiv 1 \pmod{4} \),

\[
B^{(-k)}_{2n} \equiv (-1)^{n-1} 6 \equiv \begin{cases} 4 & (n \equiv 0 \pmod{2}) \\ 1 & (n \equiv 1 \pmod{2}) \end{cases} \pmod{5}.
\]
When $k \equiv 2 \pmod{4}$,

$$\mathfrak{B}_{2n}^{(-k)} \equiv (-1)^{n-1} \cdot 18 \equiv \begin{cases} 2 & (n \equiv 0 \pmod{2}) \\ 3 & (n \equiv 1 \pmod{2}) \end{cases} \pmod{5}.$$

When $k \equiv 3 \pmod{4}$,

$$\mathfrak{B}_{2n}^{(-k)} \equiv (-1)^{n-1} \cdot 154 \equiv \begin{cases} 1 & (n \equiv 0 \pmod{2}) \\ 4 & (n \equiv 1 \pmod{2}) \end{cases} \pmod{5}.$$

\hfill \Box

**Theorem 13.** For $n, k \geq 1$, the values of $\mathfrak{B}_{2n}^{(-k)} \pmod{7}$ are given in the following table.

| $n \pmod{6}$ | 0 | 1 | 2 | 3 | 4 | 5 |
|----------|---|---|---|---|---|---|
| 0        | 6 | 6 | 0 | 1 | 1 | 0 |
| 1        | 2 | 6 | 4 | 5 | 1 | 3 |
| 2        | 1 | 2 | 1 | 6 | 5 | 6 |
| 3        | 1 | 1 | 0 | 6 | 6 | 0 |
| 4        | 5 | 1 | 3 | 2 | 6 | 4 |
| 5        | 6 | 5 | 6 | 1 | 2 | 1 |

Proof. In the terms of the summation expression of $\mathfrak{B}_{2n}^{(-k)}$, $7|(2m+1)^k$ for $m = 3$, and $7|(2m)!$ for $m \geq 4$. Since

$$\left\lfloor \frac{n}{2} \right\rfloor_2 = \frac{4^{n-1} - 1}{3},$$

$$\mathfrak{B}_{2n}^{(-k)} \equiv (-1)^{n-1} \cdot 2^k + (-1)^n \cdot 4^{n-1} \cdot \frac{1}{3} \cdot 5^k$$

$$= (-1)^{n-1} \cdot 2^k + (-1)^n (4^{n-1} - 1) \cdot 5^k \pmod{7}.$$

Since $a^6 \equiv 1 \pmod{7}$ ($a = 3, 4, 5$) by Fermat’s little theorem, it is sufficient to check the cases for $k \equiv 0, 1, 2, 3, 4, 5 \pmod{6}$ and $n \equiv 0, 1, 2, 3, 4, 5 \pmod{6}$. When $n \equiv 2 \pmod{6}$,

$$\mathfrak{B}_{2n}^{(-k)} \equiv -2 \cdot 3^k + 3 \cdot 5^k$$
\[
\begin{align*}
-2 + 3 &= 1 \quad (k \equiv 0 \pmod{6}) \\
-2 \cdot 3 + 3 \cdot 5 &= -6 + 1 \equiv 2 \quad (k \equiv 1 \pmod{6}) \\
-6 \cdot 3 + 1 \cdot 5 &= -4 + 5 = 1 \quad (k \equiv 2 \pmod{6}) \\
-4 \cdot 3 + 5 \cdot 5 &= -5 + 4 \equiv 6 \quad (k \equiv 3 \pmod{6}) \\
-5 \cdot 3 + 4 \cdot 5 &= -1 + 6 = 5 \quad (k \equiv 4 \pmod{6}) \\
-1 \cdot 3 + 6 \cdot 5 &= -3 + 2 \equiv 6 \quad (k \equiv 5 \pmod{6}) 
\end{align*}
\] (mod 7).

Other cases are similarly proved and omitted. \[\square\]

8 Bernoulli numbers with level 2

When \( k = 1 \), Bernoulli numbers \( B_n = B_n^{(1)} \) with level 2 are given by the generating function

\[
\frac{1}{4 \sin(x/2)} \log \frac{1 + 2 \sin(x/2)}{1 - 2 \sin(x/2)} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}.
\] (18)

First several values of Bernoulli numbers with level 2 are

\[
\{B_{2n}\}_{0 \leq n \leq 10} = \frac{1}{3}, \frac{2}{15}, \frac{62}{21}, \frac{1670}{15}, \frac{47102}{33}, \frac{6936718}{1365}, \frac{92167388522}{2}, \frac{9208191626}{3}, \frac{7637588708954836042}{255}, \frac{58943788779804242}{399}, \frac{150996747969694}{33}, \frac{29167388522}{165}.
\]

Table 1: Fractional parts of Bernoulli numbers with level 2

| \( n \) | 0 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 |
|-----|---|---|---|---|---|----|----|----|----|----|----|
| \( B_n \pmod{1} \) | 0 | 1 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 |

Note that the sequence of the denominators is the same as those of cosecant numbers

\[-2(2^{2n-1} - 1)B_{2n}\]

(15).

The von Staudt-Clausen theorem \([3, 17]\) states that for every \( n > 0 \),

\[
B_{2n} + \sum_{(p-1)|2n} \frac{1}{p}.
\]

18
is an integer. The sum extends over all primes \( p \) for which \( p - 1 \) divides \( 2n \). In [7][13][16], von Staudt-Clausen’s type formulas for poly-Euler numbers, Euler numbers of the second kind and poly-Bernoulli numbers have been shown.

We can also determine the denominators of Bernoulli numbers with level 2 completely.

**Theorem 14.** For every \( n > 0 \),

\[
\mathfrak{B}_{2n} + \sum_{(p-1)2n} \frac{(-1)^{n-\frac{p-1}{2}}}{p}
\]

is an integer. The sum extends over all odd primes \( p \) for which \( p - 1 \) divides \( 2n \).

**Examples.** For \( n = 8 \) and \( n = 9 \),

\[
(\mathfrak{B}_{16} \mod 1) - \frac{1}{3} + \frac{1}{5} + \frac{1}{17} = 0,
\]

\[
(\mathfrak{B}_{18} \mod 1) + \frac{1}{3} + \frac{1}{7} + \frac{1}{19} = 1.
\]

**Proof of Theorem 14.** From Theorem 4 and (7), notice that

\[
\mathfrak{B}_{2n}^{(1)} = \sum_{m=0}^{n} \left\{ \binom{n}{m} \right\} \frac{(-1)^{n-m}(2m)!}{2m+1}
\]

\[
= \sum_{m=0}^{n} \frac{2(-1)^{n}}{2m+1} \sum_{j=1}^{m} (-1)^{j} \left( \binom{2m}{m-j} \right) j^{2m}.
\]

For \( n \geq 1 \), we see \( m \geq 1 \) in the above summations.

**Case 1.** When \( 2m+1 \) is composite, as \( m \geq 4 \), \((2m+1)(2m)!\). Since \( \left\{ \binom{n}{m} \right\} \) is an integer, every such a term of

\[
\left\{ \binom{n}{m} \right\} \frac{(-1)^{n-m}(2m)!}{2m+1}
\]

is an integer.

**Case 2.** When \( 2m+1 = p \) is prime, \( p \geq 3 \) and \( m = (p-1)/2 \) is an integer. Now, consider the summation

\[
\sum_{j=1}^{(p-1)/2} (-1)^{j} \left( \binom{p-1}{(p-1)/2-j} \right) j^{2m}.
\]
Case 2.1. If $\frac{p-1}{2} | n$, then by Fermat’s little theorem,

$$j^{2n} \equiv 1 \pmod{p} \quad (j = 1, 2, \ldots, p - 1).$$

Hence,

$$\sum_{j=1}^{(p-1)/2} (-1)^j \binom{p-1}{(p-1)/2 - j} j^{2n} \equiv \sum_{j=1}^{(p-1)/2} (-1)^j \binom{p-1}{(p-1)/2 - j} \pmod{p}$$

$$= - \binom{p-2}{(p-1)/2}.$$

The central binomial coefficient modulo prime yields

$$\binom{p-2}{k} \equiv \frac{(-2)(-3)\cdots(-k-1)}{1 \cdot 2 \cdots k} = (-1)^k (k + 1)$$

$$= (-1)^{\frac{p-1}{2}} \frac{p+1}{2} \pmod{p} \quad \left(k = \frac{p-1}{2}\right).$$

Thus, when $p \equiv 1 \pmod{4}$, since

$$\binom{p-2}{(p-1)/2} \equiv \frac{p+1}{2} \pmod{p},$$

$$2(-1)^n \sum_{j=1}^{m} (-1)^j \binom{2m}{m-j} j^{2n} = - \frac{2(-1)^n p+1}{p}$$

$$\equiv -\frac{(-1)^n}{p} \pmod{1}.$$

When $p \equiv 3 \pmod{4}$, since

$$\binom{p-2}{(p-1)/2} \equiv \frac{p-1}{2} \pmod{p},$$

$$2(-1)^n \sum_{j=1}^{m} (-1)^j \binom{2m}{m-j} j^{2n} \equiv - \frac{2(-1)^n p-1}{p}$$

$$\equiv -\frac{(-1)^{n-1}}{p} \pmod{1}.$$
Case 2.2. If \( \frac{p-1}{2} \nmid n \), then by Fermat’s little theorem,
\[
j^{2n} \equiv j^{2n-\nu(p-1)} \pmod{p} \quad (j = 1, 2, \ldots, p-1)
\]
for \( 0 < 2n-\nu(p-1) < p-1 \) with \( \nu = \lfloor (2n)/(p-1) \rfloor \). Notice that \( 2n-\nu(p-1) \) is even. Hence,
\[
\sum_{j=1}^{(p-1)/2} (-1)^j \left( \frac{p-1}{(p-1)/2 - j} \right) j^{2n} \equiv \sum_{j=1}^{(p-1)/2} (-1)^j \left( \frac{p-1}{(p-1)/2 - j} \right) j^{2n-\nu(p-1)} \equiv 0 \pmod{p}.
\]

9 Open problems

In [11, 12], Stirling numbers of both kinds with higher level are discussed. One may wonder if poly-Bernoulli numbers with level 3 or higher can be introduced by using the Stirling numbers with level 3 or higher. However, the situation becomes very complicated for the case with level 3 or higher.

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