The Frustration of being Odd: How Boundary Conditions can destroy Local Order

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A central tenant in the classification of phases is that boundary conditions cannot affect the bulk properties of a system. In this work we show striking, yet puzzling, evidence of a clear violation of this assumption. We use the prototypical example of an XYZ chain with no external field in a ring geometry with an odd number of sites and both ferromagnetic and antiferromagnetic interaction. In such setting, we can calculate directly the magnetizations that are traditionally used as order parameters to characterize the phases of the system. When the ferromagnetic interaction dominates, we recover magnetizations that in the thermodynamic limit lose any knowledge about the boundary conditions and are in complete agreement with the standard expectations. On the contrary, when the system is governed by the anti-ferromagnetic interaction, the magnetizations decay algebraically to zero with the system size and it is not staggered, despite the AFM coupling. We term this behavior ferromagnetic mesoscopic magnetization. Hence, in the antiferromagnetic regime, our results show an unexpected dependence of a local, one-point function on the boundary conditions, that is in contrast with the predictions of the general theory.

Introduction: Landau theory is one of the most impactful constructions of the last century, as it allows to distinguish different phases through different local order parameters, quantities which are finite or vanish depending on the phase of a system \cite{1, 2}. Although the new century has taught us that this classification is not complete, because certain phases of quantum matter are characterized by non-local order (for instance, topological \cite{5, 12}), Landau theory remains a cornerstone to interpret phases, directly borrowed from classical statistical mechanics.

Order parameters are supposed to capture macroscopic properties of system’s phases and thus are believed not to depend on boundary conditions. Indeed, as the boundary contributions are typically sub-extensive, they should bring a negligible effect for sufficiently large systems. Of course, depending on the system and on the type of interactions, there can be ambiguities on what “sufficiently large system” means, as sometimes boundary effects can decay just algebraically, even in phases considered gapped \cite{13, 14}. Thus, the standard prescription to characterize a phase is to take the thermodynamic limit before evaluating physical observables \cite{1, 2}.

This being said, the effects of boundary conditions have been a subject of interest in different contexts. For instance, the Kondo Effect can largely be interpreted as a boundary effect \cite{15, 16}. But additional simple examples which have received a lot of attention immediately come to mind, such as conformal field theories (CFTs) and integrable models. In the former, conformal invariance poses tight bounds on the bulk properties and it has been established that boundary condition modify the system’s equilibrium behavior \cite{17, 19}. In the latter, different boundary conditions are commonly employed to study different properties. For instance, the partition functions of 2D classical systems with domain wall boundary conditions provide the normalization of the corresponding quantum wave–functions \cite{20}. But certain boundary conditions are also known to generated the phenomenon of the “arctic curve”, which separate frozen regions (due to boundary conditions) from liquid ones \cite{21–29}.

A particularly thorny issue is represented by frustration \cite{30, 31}. This term evokes different phenomena to different ears. In fact, while it simply refers to the presence of interactions promoting incompatible orderings (hence the impossibility of simultaneously minimizing every term in the system’s Hamiltonian \cite{32, 34}), the effects of frustration are varied and complex \cite{35, 37}. Frustrated systems are a debated and very active field of research, with a rich phenomenology (different in many ways from that of non-frustrated systems) and with unique challenges \cite{39}. Nonetheless, at the heart of every frustrated system one can find one (or typically many) frustrated loops, which are the building blocks out of which the different phenomenologies arise \cite{38}. Here, we will concentrate on this simplest, and original, incarnation of geometrical frustration. This is in fact a classical concept which applies, for instance, to Ising spins coupled anti-ferromagnetically. While, locally, there is no problem in satisfying the AFM interactions, when the spins are arranged in a loop with an odd number of sites, at least one bond needs to display ferromagnetic alignment. In this case, the frustration arises because of an incompatibility between local interactions and the global structure of the system, and is due to the particular choice of boundary conditions (namely, periodic with an odd number of sites, which we term “frustrated boundary conditions”, FBC). Note that, while for loops with an even number of sites the lowest energy state is doubly degenerate (given by the two types of Neel states), with frustration the degeneracy becomes extensive, because the defect can be placed on any bond of the ring. We remark that the massive change in the degeneracy of the lowest energy states by adding a single site to the ring cannot be accounted for perturbatively and is evidence of a non–local effect.
Upon adding quantum interactions to a geometrically frustrated system, we can generally expect the degeneracy to be lifted. A perturbative approach characterizes the resulting ground state as the superposition of a delocalized excitation on top of the non frustrated ground state. This picture was recently checked in [1] for its validity beyond the perturbative regime and confirmed using the Entanglement Entropy, thus seemingly confirming the traditional interpretation on the role of boundary conditions.

In this work, we pluck a hole in this canvas by focusing on the order parameter of antiferromagnetic spin chains and by showing that FBC make it vanish. To the best of our knowledge, this is the first example of a case in which boundary conditions change a local observable in the bulk and it is in evident contrast with standard general arguments recapped above. At the moment, we do not know how to reconcile this example with the traditional paradigm, although we can speculate that, being geometrical frustration a non–local effect, some sort of topological mechanism is at play, so that we propose to represent one of the simplest, and most cited, examples of spontaneous symmetry breaking (SSB) [44]. To simplify things, let us set \( \delta = 0 \), so that eq. (1) describes the Hamiltonian in \( (H, \Pi^\alpha) \), although only two of them can be used, because one of the three parity operators can be written in terms of the other two. Moreover, since we are considering systems made by an odd number of sites \( N = 2M + 1 \), the \( \Pi^\alpha \) do not commute with one another, but rather anti–commute \( \{ \Pi^\alpha, \Pi^\beta \} = 2\delta_{\alpha\beta} \) and actually fulfill a \( SU(2) \) algebra. This structure implies that every state is exactly degenerate an even number of times, also on a finite chain. In fact, if \( |\Psi\rangle \) is an eigenstate, say, of \( \Pi_z \), then \( \Pi_z |\Psi\rangle \), that differs from \( \Pi_y |\Psi\rangle \) by a global phase factor, it is also an eigenstate of the Hamiltonian with opposite \( z \)-parity but with the same energy.

Applying an external magnetic field \( h \) along, say, the \( z \)-direction leaves only \( \Pi_z \) to commute with the Hamiltonian, thus restoring the original \( Z_2 \) symmetry the model is known for and breaking the exact degeneracy between the states \( |\Psi\rangle \). Nonetheless, up to a critical value of \( h \), it is known that the induced energy split is exponentially small in the system size \( \delta \) and thus that the degeneracy is restored in the thermodynamic limit, representing one of the simplest, and most cited, examples of spontaneous symmetry breaking (SSB) [44]. To simplify things, let us set \( \delta = 0 \), so that eq. (1) describes an anisotropic XY chain \( \Pi^\alpha \). For \( |h| < 1 \) we are in the SSB phase. This means that, although a ground state with definite \( z \)-parities necessarily has zero expectation value with respect to \( \sigma^z \) and \( \sigma^y \), in the thermodynamic limit the degeneracy allows to select a ground state which is a superposition of different \( z \)-parities and thus can develop a spontaneous magnetization in the \( x \) or \( y \) direction. In the yFM phase we expect the order parameter \( m_y \equiv \langle \sigma_y^y \rangle \) to be finite, while in the xAFM the non–vanishing order parameter should be the staggered magnetization \( m_x \equiv (-1)^z \langle \sigma_x^x \rangle \).
The ferromagnetic case: Let us now turn back to the system in eq. (1) and focus in the region $\phi \in [0, \pi/4)$. The (quasi–)long–range order represented by the order parameter can be extracted in two ways: either from the two–point function, or by selecting a suitable superposition of states at finite sizes and then following their magnetization toward the thermodynamic limit. Traditionally, the former is most suitable for analytical techniques, while the latter is easily amenable to numerical and experimental approaches.

The former takes advantage of the cluster decomposition property

$$\lim_{r \to \infty} \langle \sigma_j^\alpha \sigma_{j+r}^\alpha \rangle - \langle \sigma_j^\alpha \rangle \langle \sigma_{j+r}^\alpha \rangle = 0,$$

(2)

to extract the order parameter from the large distance behavior of the system’s two-point correlators. The XY chain is exactly mappable in a system of free fermions and thus the fundamental two–point functions can be expressed as the determinant of a Toeplitz matrix, whose asymptotic behavior can be evaluated analytically in the asymptotic limit [46]:

$$\langle \sigma_j^x \sigma_{j+r}^x \rangle \sim -\frac{(-1)^r}{4\pi \sqrt{1 - \tan^2 \phi}} \tan^r \phi + \ldots ,$$

(3)

$$\langle \sigma_j^y \sigma_{j+r}^y \rangle \sim \sqrt{1 - \tan^2 \phi} \left( 1 + \frac{1}{2\pi} \frac{\tan^{r+1} \phi}{\tan \phi} + \ldots \right),$$

(4)

$$\langle \sigma_j^z \sigma_{j+r}^z \rangle \sim -\frac{1}{2\pi} \frac{\tan^r \phi}{r^2} + \ldots .$$

(5)

From these and eq. (2) we can extract that in this phase, in the thermodynamic limit, the order parameters take value:

$$m_x = m_z = 0, \quad m_y = (1 - \tan^2 \phi)^{1/4} .$$

(6)

where $m_\alpha \equiv \langle \sigma_j^\alpha \rangle$.

However, on an odd–length chain at $h = 0$, exploiting the symmetries that we have already illustrated, we can provide a direct way to evaluate the different magnetizations even in finite systems. In fact, if $|g_x\rangle$ is one of the degenerate ground states with definite $z$–parity which can be constructed in terms of the Bogoliubov fermions [10], we can generate a ground state with definite $x$–parity ($y$–parity) as $|g_x\rangle \equiv \frac{1}{\sqrt{2}} (1 + \Pi_x) |g_x\rangle$, $|g_y\rangle \equiv \frac{1}{\sqrt{2}} (1 + \Pi_y) |g_x\rangle$. All these states have a vanishing magnetization in the orthogonal directions while along their own axes we have

$$\langle g_x | \sigma_j^x | g_x \rangle = \langle g_x | \sigma_j^y \Pi_x | g_x \rangle = \langle g_x | \Pi_x^x | g_x \rangle$$

$$\langle g_y | \sigma_j^y | g_y \rangle = \langle g_x | \sigma_j^y \Pi_y | g_x \rangle = \langle g_x | \Pi_y^y | g_y \rangle$$

(7)

where $\Pi_x^\alpha \equiv \bigotimes_{j \neq x} \sigma_j^\alpha$ with $\alpha$ running between $x$ and $y$. These states are the analytical continuation at $h = 0$ of the zero–temperature “thermal” ground state that spontaneously break the $\mathbb{Z}_2$ symmetry.

Note that in this way, we turn the calculation of a one–point function with respect to a mixed $z$–parity ground state into that of a string of an even number of operators on a definite $z$–parity state, which is a standard problem that we can evaluate. After the JWT, the RHS of eq. (7) can be written again as the determinant of a Toeplitz matrix, whose asymptotic behavior can be studied analytically, similarly to what has been done in [46]. This novel “trick” can be understood as originating from the fact that, at zero external field, the chain eq. (1) has particle/ hole dualities and that, on a chain with an odd number of sites, this relates states of different parities. More details on this direct approach on the evaluation of the different magnetizations can be found in the supplementary material. The result of such analysis reproduce eq. (6), proving the consistency of the two methods of evaluation of the order parameters.

While for $\delta = 0$ we can evaluate the one–point functions analytically, for $\delta \neq 0$ we can resort to numerics to extract the spontaneous magnetizations. The analytical results obtained for the three magnetizations can be found in the supplementary material. In Fig.1 we present some typical results for the finite size magnetizations for the XY and XYZ chain, showing a clear exponential decay to zero and a fast saturation of $m_y$ (note that each plotted magnetization $m_\alpha$ is calculated with respect to the corresponding ground state $|g_\alpha\rangle$).

The frustrated case: We now turn to the case with $\phi \in (\pi/4, \pi/2)$, where the boundary conditions induce topological frustration. The effect of frustration have been recently studied in detail in Refs. [13, 14, 48]. For $\delta = 0$, the model can be solved through the same steps used in the traditional cases and exactly mapped into a systems of free fermions. In the ferromagnetic phase, the degeneracy between the different parity states is due to the presence of a single negative energy mode (only in one of the parity sectors), whose occupation lowers the energy of those state. With frustrations, the negative energy mode moves into the other parity sector and, because of the parity selection rules, cannot be excited.
alone. Hence, the effect of frustration is that the allowed lowest energy states for each parity are not the absolute lowest energy states that could be constructed, were not for the parity requirement.

The two degenerate ground states thus carry the signature of a single delocalized excitation (see the supplementary material for details) and lie at the bottom of a band of states in which this excitation moves with different momenta (with an approximate Galilean dispersion relation). Hence, another effect of frustration is to close the gap that would otherwise exists. Quite crucially, a finite transverse magnetic field splits the degeneracy between the different parity ground states with a gap that closes only polynomially in the system size (instead than exponentially in absence of frustration): thus for $h \neq 0$ the model has a unique ground state. This observation makes us question the characterization of this phase as a SSB, since a definite parity GS cannot develop a spontaneous magnetization, a point already raised in [14].

Let us then repeat the extraction of the order parameters in the xAFM phase, following the same procedure we followed for yFM. As shown in Ref. [13], the asymptotic behavior of the fundamental two point functions can be evaluated by combining the results of Ref. [10] in eq. (3) [5] with a Wiener–Hopf procedure that accounts for the presence of the extra excitation [17]. The results are:

$$
\langle \sigma^x_j \sigma^x_{j+r} \rangle \rightarrow_{r \rightarrow \infty} -(-1)^r \sqrt{1 - \cot^2 \phi} \left( 1 - \frac{2r}{N} \right) \times
$$

$$
\times \left[ 1 + \frac{\cot \phi}{2\pi} \frac{\cot \phi}{r^2} + \ldots \right], \quad (8)
$$

$$
\langle \sigma^y_j \sigma^y_{j+r} \rangle \rightarrow_{r \rightarrow \infty} - \frac{1}{2\sqrt{\pi^2(1-\cot^2 \phi)}} \left( 1 - \frac{2r}{N} \right) \frac{\cot \phi}{r^2} + \ldots, \quad (9)
$$

$$
\langle \sigma^z_j \sigma^z_{j+r} \rangle \rightarrow_{r \rightarrow \infty} - \frac{1}{2\pi} \frac{\cot \phi}{r^2} - \frac{4}{\sqrt{2\pi}} \left( -1 \right)^r \frac{\cot r^2/2}{N} \frac{1}{r}. \quad (10)
$$

While they imply quite clearly that $m_y = m_z = 0$ (in accordance to standard prediction), the extraction of $m_x$ is more subtle: using the standard prescription of taking $N \rightarrow \infty$ first, one would get $m_x = (1 - \cot^2 \phi)^{1/4}$. However, one could argue that a better procedure would be to evaluate eq. (8) at antipodal points $r \sim N/2$ in order to minimize the correlations and then take the thermodynamic limit. In this way, one would get $m_x = \frac{1}{\sqrt{N}} (1 - \cot^2 \phi)^{1/4} \rightarrow 0$.

It is thus important that we can directly access the one–point function using eq. (7). Once more, the expectation values can be casted as determinants of Toeplitz matrices, whose asymptotic behavior can be evaluated analytically using a Wiener-Hopf method (see supplementary material), yielding

$$
m_y = m_z \rightarrow_{N \rightarrow \infty} 0
$$

$$
\langle \sigma^y_j \rangle = \frac{1}{N} (1 - \cot^2 \phi)^{1/4} \rightarrow_{N \rightarrow \infty} 0,
$$

where exponentially decaying contributions have been suppressed.

Several elements are surprising in these results. The most evident one is that FBC kill the magnetization in the $x$–direction, that on an open or even-length chain would be finite, thus seemingly contradicting the independence of Landau construction from boundary conditions. Note that a finite spontaneous magnetization can be measured in any finite system, although it decreases algebraically with the system size, a phenomenon we term “mesoscopic magnetization”. Quite surprisingly, however, this finite–size spontaneous magnetization is not staggered, but rather ferromagnetic–looking (thus, we will call the AFM phase with FBC, a mesoscopic ferromagnetic phase, MFM). In hindsight, we could have expected this, since a staggered magnetization would have not been compatible with PBC with an odd number of sites (note that this is not a problem for the 2–point function, since its range naturally does not extend beyond one periodicity).

These analytical outcomes, together with the analytical behaviors for the other magnetizations reported in the supplementary material, are corroborated by numerical diagonalizations results, which can be extended to the XYZ ($\delta \neq 0$) chain. We present some of these data in Fig. 2 with behaviors clearly differing from those in Fig. 1 in the MFM phase every magnetization decays algebraically to zero. In Fig. 3 we plot the behavior of the magnetizations as a function of $\phi$ for $\delta = 0.3$ for several chain lengths $N$: while in the yFM phase there is little dependence on $N$ as the saturation values are reached quickly, in the MFM phase we observe the slow, algebraic decay toward zero of the order parameters.

It is rather surprising that a finite chain, unable to sustain AFM order, would nonetheless generate a ferromagnetic spontaneous magnetization and that in any finite system, a phase with a dominant interaction along the $x$ direction would show the weakest spontaneous magnetization in that direction, with $m_y$ being the strongest one (once more, these magnetization refers to different
states \(|g_{\alpha}\))). Finally, we remark that FBC also seem to somewhat spoil the cluster decomposition, since the non–staggered mesoscopic magnetization we find is not compatible with \([\mathbb{S}]\), although both of them vanish in the thermodynamic limit.

Conclusions: We have presented a comparative study of the ferromagnetic and AFM frustrated case for a XYZ chain, showing that, contrary to expectations, the boundary conditions are able to destroy local order. We have done so, by realizing that, with no external field, we can exploit particle/ hole duality to construct an exact ground state at finite sizes that breaks the \(\mathbb{Z}_2\) symmetry. For the XY chain we are able to express the one point function as the determinant of a Toeplitz matrix and evaluate it analytically, while for the interacting case we can numerically diagonalize the model and calculate the expectation values. We benchmarked these procedures on a ferromagnetic phase with FBC to show that they reproduce the expected results eq. \((12)\), while in a AFM phase the magnetizations, while finite in a finite chain, decay toward zero algebraically in the thermodynamic limit eq. \((12)\). Furthermore, despite a dominant AFM interaction, no magnetization shows a staggered behavior: we thus term this pseudo-phase generated by FBC a mesoscopic ferromagnetic phase (MFM). While we worked at zero field to have an exact degeneracy on any finite chain, this degeneracy is expected to be restored in the thermodynamic limit also at finite external magnetic fields (up to some threshold) and thus we expect our arguments to extend beyond \(h = 0\) (this will be subject of a future work). Thus, the existence of a MFM phase should be clearly experimentally detectable, with signatures like those in Fig. 3 easily measurable.

These results are surprising, because they seemingly contradict the expectation that boundary conditions cannot influence the bulk behavior of a system and therefore certainly not destroy local order. We do not know at the moment how to reconcile this apparent paradox and we invite the community to help us in looking for a general explanation. For the moment we can contribute with a couple of observations. The first is that FBC provide a non–local contribution to the system, since frustration arises from an incompatibility between local and global order. Thus, it is possible that the problem we consider can have a topological origin that defies Landau paradigm. Another, somewhat more technical angle, is that in our class of models, the one point function is dual to a non–local correlator, see eq. \((12)\). From this point of view, it is not surprising that a non–local function is sensitive to the boundary conditions. Nonetheless, we have to admit that it seems to us rather paradoxical to consider a one–point function non–local.

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Supplementary Material

The Hamiltonian of the topologically frustrated XY chain \((\delta = 0)\) in a zero magnetic field can be written as

\[ H = \sin \phi \sum_{j=1}^{N} \sigma_j^x \sigma_{j+1}^x - \cos \phi \sum_{j=1}^{N} \sigma_j^y \sigma_{j+1}^y, \quad (12) \]

where \(\phi \in [0, \pi/2]\) is the parameter that allows us to change the relative weight of the ferromagnetic and antiferromagnetic Hamiltonian terms, \(\sigma_j^x\) are the Pauli operators, \(N = 2M + 1\) is the odd number of lattice sites in the system and we assume periodic boundary conditions, i.e. \(\sigma_{N+j}^x = \sigma_j^x\).

Due to the absence of any external field, the Hamiltonian in eq. \((12)\) commutes with all the three parity op-
operators $\Pi_\alpha = \bigotimes_{j=1}^N \sigma_j^\alpha$. Because we are considering only systems made by an odd number of spins $N$, such parity operators do not commute with each other. Indeed we have

$$[\Pi_\alpha, \Pi_\beta] = i \varepsilon_{\alpha,\beta,\gamma} 2(-1)^{\frac{N+1}{2}} \Pi_\gamma,$$

(13)

where $\varepsilon_{\alpha,\beta,\gamma}$ is the Levi-Civita symbol. The existence of several operators that commute with the Hamiltonian but do not commute with each other induces a degeneracy in all the eigenstates of the Hamiltonian. We have that to any eigenvalue is associated a $2d$-times degenerate manifold ($d$ positive integer). For any of the parity operators $\Pi_\alpha$, each eigenstate manifold will contain $d$ eigenstates belonging to the even sector and $d$ to the odd one. This is also valid for the ground state manifold which always has a minimum size equal to two. As we shall see, this property plays a fundamental role in the analytical evaluation of magnetizations in the different spin directions.

Solution of the topologically frustrated XY model It is possible to diagonalize analytically the spin model in eq. [12], employing the well-known techniques based on a Jordan–Wigner transformation that maps spins into spinless fermions. Once we have obtained a spinless fermionic model, a Fourier transformation followed by a Bogoliubov rotation allows us to separate it in $N$ non-interacting fermionic problems that can be analytically treated [40]. At the end of this process, the Hamiltonian in eq. [12] can be written as

$$H = \frac{1 + \Pi_\sigma}{2} + \frac{1 + \Pi_\pi}{2} + \frac{1 - \Pi_\pi}{2} H - \frac{1 - \Pi_\sigma}{2},$$

(14)

where

$$H^\pm = \sum_{q \in \Gamma^\pm} \epsilon(q) \left( a_q^\dagger a_{-q} - \frac{1}{2} \right),$$

(15)

and

$$\epsilon(q) = |\sin \phi| e^{\pm q r} - \cos \phi, \quad q \neq 0, \pi,$$

$$\epsilon(0) = -\epsilon(\pi) = \sin \phi - \cos \phi,$$

(16)

with the two sets of momenta given, respectively, by $\Gamma^- = \{2 \pi k/N \}$ and $\Gamma^+ = \{2 \pi (k + \frac{1}{2})/N \}$ with $k$ running on all integers between 0 and $N - 1$. It is worth to note that the momenta $0 \in \Gamma^-$ and $\pi \in \Gamma^+$ (if $N$ is even $\pi \in \Gamma^-$), are different from the others because a) they do not have a corresponding opposite momentum; b) their energies can be negative.

From eqs. [14][16] it is easy to determine the ground states of the system starting from the vacuum of Bogoliubov fermions in the two sectors ($|0^\pm\rangle$) and taking into account the negative energy modes. When $0 < \phi < \pi/4$ the 0–mode has negative energy while the $\pi$–mode a positive one. Therefore, the state with the minimum energy in the odd sector ($a_0^\dagger |0^-\rangle$) has an odd number of fermions while the one in the even sector is characterized by an even number of fermions ($|0^+\rangle$). Having, both states the right parity and the same energy they represent a basis for the two-fold degenerate ground states manifold of the Hamiltonian, that is separated, from the rest of the eigenstates, by a finite energy gap that does not close as $N$ increases.

On the contrary, when $\pi/4 < \phi < \pi/2$ the energy of the 0–mode becomes negative while the one of the $\pi$–mode becomes positive. As a consequence, the state with the minimum energy in the even sector ($a_0^\dagger |0^+\rangle$) has now an odd number of fermions while the one in the odd sector ($|0^-\rangle$) has an even one. Therefore they cannot represent physical states of our system, as they violate the parity constraint of their relative sectors. On the contrary, the ground states can be recovered from such states with the minimum energy by adding the lightest possible excitation. Because we are considering $\phi \in [0, \pi/2]$, the smallest excitations are associated with the $\pi$ and the 0–mode respectively for the even and the odd sector. Therefore in the region $\phi \in [0, \pi/2]$ the two ground states of the Hamiltonian that are also eigenstates of $\Pi_\sigma$ are

$$|g^+\rangle = |0^+\rangle \quad \text{even sector}$$

$$|g^-\rangle = a_0^\dagger |0^-\rangle \quad \text{odd sector}$$

(17)

Note that, although the expressions in eq. [17] describe the ground states in both the xFM and xAFM phases, they in fact characterize quite different structures. For instance, in the frustrated case, since the GS is obtained as the lightest excitation on top of the lowest possible energy state (as just explained), adding different excitations provides states with an almost continuum of energy, which become a dense, gapless band in the thermodynamic limit [13][48].

Moreover, the correlations in the different phases are different. In fact, knowing the two ground states that are simultaneously eigenstates of $\Pi_\sigma$, it is possible to calculate, on them, the spin correlation functions. To this goal it is useful to introduce the following (quasi-)Majorana fermion operators [40]

$$A_j = \bigotimes_{l=1}^{j-1} \sigma_i^l, \quad B_j = \bigotimes_{l=1}^{j-1} \sigma_i^l.$$ (18)

Exploiting Wick’s theorem, any non-vanishing spin correlation functions of an eigenstate of $\Pi_\sigma$ can be written in terms of the two-body Majorana correlation functions:

$$\langle g^+ | A_j A_j | g^+ \rangle = \delta_{00}$$

(19)

$$\langle g^+ | A_j B_j | g^+ \rangle = \frac{i}{N} \sum_{q \in \Gamma^\pm} e^{i2q} e^{-iqr} + \frac{2i}{N} \Gamma^\pm(r)$$

In eqs. [19] $\theta_q$ stands for the Bogoliubov angle satisfying

$$e^{i2q} = e^{i\theta_q} \sin \phi - \cos \phi e^{-i2q}$$

(20)
while the function \( f^\pm (r) \) is zero for \( 0 < \phi < \pi/4 \), while for \( \pi/4 < \phi < \pi/2 \) we have \( f^+(r) = (-1)^r \) and \( f^-(r) = -1 \).

**Magnetizations along \( x \) and \( y \) directions** In this section, we show how it is possible to exploit the particular symmetries of the model in eq. (12), to evaluate, for any odd \( N \), the magnetization along the \( x \) and the \( y \) directions. For sake of simplicity, we limit ourselves to illustrate the method for the magnetization along the \( x \) direction and we report the results for both at the end.

As we have seen, in the region that we are analyzing, the ground state manifold has always dimension equal to two. Therefore the set made by \( |g^+\rangle \) and \( |g^-\rangle \) represents a good basis for the ground state manifold and, hence, all its elements can be written as a linear combination of \( |g^+\rangle \) and \( |g^-\rangle \). But in our case, we can say more. As we have already shown, the Hamiltonian in eq. (12) commutes not only with \( \Pi_x \) but also with \( \Pi_x \) and \( \Pi_y \). This fact implies that the state \( \Pi_x |g^+\rangle \) is also a ground state of the system. On the other hand, taking into account the anticommutation rules of the spin operators on the same site and the fact that we are considering a system with odd \( N \), it is easy to see that, while \( |g^+\rangle \) is in the even sector of \( \Pi_x \), \( \Pi_x |g^+\rangle \) lives in the odd one. As a consequence we have that \( |g^-\rangle = \Pi_x |g^+\rangle \), up to a global multiplicative phase factor, and hence the generic ground state can be written as

\[
|g\rangle = (\cos(\theta) + \sin(\theta)e^{i\varphi} \Pi_x) |g^+\rangle ,
\]

or equivalently using \( \Pi_y \), since \( |g^-\rangle \) and \( \Pi_y |g^+\rangle \) only differ by a global phase factor.

Let us now choose a generic site \( j \) of the system. For the generic ground state in eq. (21) the magnetization along \( x \) on the \( j \)-th spin is

\[
m_x(j) = \langle g | \sigma^x | g \rangle = \cos^2(\theta) \langle g^+ | \sigma^x | g^+ \rangle + \sin^2(\theta) \langle g^+ | \Pi_x \sigma^x \Pi_x | g^+ \rangle + \frac{1}{2} \sin(2\theta) \left[ e^{i\varphi} \langle g^+ | \sigma^x \Pi_x | g^+ \rangle + e^{-i\varphi} \langle g^+ | \Pi_x \sigma^x | g^+ \rangle \right] .
\]

Being both \( |g^+\rangle \) and \( \Pi_x |g^+\rangle \) eigenstates of \( \Pi_x \), the two expectation values \( \langle g^+ | \sigma^x | g^+ \rangle \) and \( \langle g^+ | \Pi_x \sigma^x \Pi_x | g^+ \rangle \) vanish. On the contrary, because the number of spins in the system is odd, the operator \( \Pi_x \sigma^x \Pi_x = \sigma^x \Pi_x \), that is equal to \( \Pi_x \Pi_x \Pi_x = \bigotimes_{j \neq j} \sigma^x \), is an operator that commutes with \( \Pi_x \) and hence can have a non–vanishing expectation value on \( |g^+\rangle \). Therefore we have

\[
m_x(j) = \cos(\varphi) \sin(2\theta) \langle g^+ | \Pi_x | g^+ \rangle .
\]

which reaches the maximum for \( \psi = 0 \) and \( \theta = \frac{\pi}{4} \), that is, the state on which we focus in the letter.

Hence, to evaluate the magnetization, we only need to determine the expectation value \( \langle g^+ | \Pi_x | g^+ \rangle \). Since \( [\Pi_x, \Pi_y] = 0 \), the magnetization can be easily evaluated exploiting the representation of \( \Pi_x \) in terms of the Majorana operators in eq. (18) and Wick’s theorem. Without loss of generality, let us set \( j = 1 \). From the definition of the Majorana operators in eq. (18), the operator \( \Pi_x \) can be written as

\[
\Pi_x = i \sum_{l=1}^{N-1} A_{2l+1} B_{2l} ,
\]

and, exploiting Wick’s theorem, we obtain that the expectation value \( \langle g^+ | \Pi_x | g^+ \rangle \) is

\[
\langle g^+ | \Pi_x | g^+ \rangle = (-1)^{N-1} \Delta(\rho_x) ,
\]

where \( \Delta(\rho_x) \) is the determinant of the \( \frac{N-1}{2} \times \frac{N-1}{2} \) Toeplitz matrix \( \rho_x \) that reads

\[
\rho_x = \begin{pmatrix}
G(1) & G(-1) & G(-3) & \cdots & G(4-N) \\
G(3) & G(1) & G(-1) & \cdots & G(6-N) \\
G(5) & G(3) & G(1) & \cdots & G(8-N) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
G(N-2) & G(N-4) & G(N-6) & \cdots & G(1)
\end{pmatrix}
\]

with \( G(r) = -i \langle g^+ | A_{1+r} B_r | g^+ \rangle \).

On the other hand, the magnetization along \( y \) on the spin 1 becomes

\[
m_y(1) = \cos(\varphi) \sin(2\theta) \langle g^+ | \Pi_y | g^+ \rangle .
\]

where \( \Pi_y = \bigotimes_{l=2}^{N} \sigma^y \). Also in this case the maximum of the magnetization is equal to \( \langle g^+ | \Pi_y | g^+ \rangle \), which in turn can be written as

\[
\langle g^+ | \Pi_y | g^+ \rangle = (-1)^{N-1} \Delta(\rho_y) ,
\]

where \( \Delta(\rho_y) \) is the determinant of the \( \frac{N-1}{2} \times \frac{N-1}{2} \) Toeplitz matrix \( \rho_y \)

\[
\rho_y = \begin{pmatrix}
G(-1) & G(-3) & G(-5) & \cdots & G(2-N) \\
G(1) & G(-1) & G(-3) & \cdots & G(4-N) \\
G(3) & G(1) & G(-1) & \cdots & G(6-N) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
G(N-4) & G(N-6) & G(N-8) & \cdots & G(1)
\end{pmatrix}
\]

**Magnetizations in the ferromagnetic phase** If we are in the yFM phase, we have that in eq. (19) the function \( f^+(r) = 0 \) and hence \( G(r) \) becomes

\[
G(r) = \frac{1}{N} \sum_{q \in \Gamma^+} \sin \phi - \cos \phi e^{-iq} e^{-iq(r-1)} ,
\]

and for large \( N \) we can approximate the sum with an integral, hence obtaining

\[
G(r) = \frac{1}{2\pi} \int_{0}^{2\pi} \sin \phi - \cos \phi e^{-iq} e^{-iq(r-1)} dq .
\]

To evaluate the determinants \( \Delta(\rho_{x,y}) \) of the Toeplitz matrices in eqs. (26,29) we introduce

\[
D_n = G(2n-1) = \frac{1}{2\pi} \int_{0}^{2\pi} \frac{1 - \tan \phi e^{iq}}{1 - \tan \phi e^{iq}} e^{-iq(r-1)} dq ,
\]
and rewrite eqs. (26,29) as

$$\Delta_r(x) = \begin{vmatrix} D_1 & D_0 & \ldots & D_{2-r} \\ D_2 & D_1 & \ldots & D_{3-r} \\ \vdots & \vdots & \ddots & \vdots \\ D_r & D_{r-1} & \ldots & D_1 \end{vmatrix}, \quad r = N - \frac{1}{2},$$

and

$$\Delta_r(y) = \begin{vmatrix} D_0 & D_{-1} & \ldots & D_{1-r} \\ D_1 & D_0 & \ldots & D_{2-r} \\ \vdots & \vdots & \ddots & \vdots \\ D_{r-1} & D_{r-2} & \ldots & D_0 \end{vmatrix}, \quad r = N - \frac{1}{2} .$$

The latter can be evaluated straightforwardly for large $N$ using Szeg"o theorem [19], yielding, to leading order,

$$\langle g^+ | \Pi_N^1 | g^+ \rangle = (1 - \tan^2 \phi)^{\frac{1}{2}} .$$  \hspace{1cm} (35)

The magnetization in the x direction, instead, is more complicated, because the generating function of the corresponding Toeplitz matrix has a non–zero winding number. To overcome this problem, we proceed as in Ref. [50] and notice that the determinant in eq. (33) can be seen as the minor of $\Delta_{r+1}(\rho_y)$ in eq. (34) obtained removing the first row and the last column. To calculate this minor, we use Cramer’s rule and consider the following linear problem:

$$\sum_{m=0}^{r} D_{n-m} x_m = \delta_{n,0}, \quad n = 0, \ldots, r .$$  \hspace{1cm} (36)

Then,

$$\Delta_r(x) = (-1)^r x_r \Delta_{r+1}(\rho_y) ,$$

where $\Delta_{r+1}(\rho_y)$ is a Toeplitz determinant satisfying the conditions for Szegö theorem. For large $r$, $x_r$ can be evaluated following the standard Wiener–Hopf procedure as in Ref. [50]. The result is

$$x_r \sim -\frac{1}{2\pi i} \int \frac{\xi^{r-1} d\xi}{\sqrt{(1 - \tan \phi \xi)(1 - \tan \phi \xi^{-1})}} = -\frac{1}{\pi} \int_0^{\tan \phi} \frac{x^{r-1/2} dx}{\sqrt{(1 - \tan \phi x)(\tan \phi - x)}} ,$$  \hspace{1cm} (38)

where we deformed the contour of integration around the branch cut. Up to now, everything has been similar to the standard calculations usually performed in the XY model, but now we have to proceed anew because, unlike the generating functions for the two–point correlators which have two pairs of poles and zero when extended to the complex plane, the generating function (symbol) in eq. (29) we have for the magnetization only has one movable pole/zero.

Fortunately, the integral in eq. (38) can be expressed in terms of hypergeometric functions:

$$x_r \sim -\frac{1}{\sqrt{\pi r}} \frac{\tan^r \phi}{\sqrt{1 - \tan^2 \phi}} ,$$  \hspace{1cm} (39)

whose asymptotic behavior gives to leading order

$$x_r \sim -\frac{1}{\sqrt{\pi r}} \frac{\tan^r \phi}{\sqrt{1 - \tan^2 \phi}} ,$$  \hspace{1cm} (40)

since the $_2F_1$ tends to 1 for large $r$. Combining (40) with (37) and (35) we arrive at

$$\Delta_r(\rho_x) = \frac{\tan^r \phi}{(1 - \tan^2 \phi)^{\frac{1}{2}} \sqrt{\pi r}} ,$$  \hspace{1cm} (41)

which means that the magnetization in the x direction decays exponentially with the system size:

$$\langle g^+ | \Pi_N^1 | g^+ \rangle \sim \frac{[-\tan \phi]^{\frac{N}{2}}}{(1 - \tan^2 \phi)^{\frac{1}{2}} \sqrt{\pi(N - 1)/2}} 0 .$$  \hspace{1cm} (42)

Note that, despite the $x$ interaction being AFM, the corresponding magnetization is not staggered.

Finally, the magnetization in the z–direction is just equal to the Majorana two point function in eq. (19) and thus its exponential decay to zero arises as to the difference between the finite sum in eq. (19) and vanishing of the corresponding integral in the $N \to \infty$ limit.

**Magnetizations in the frustrated phase** If we are in the xAFM phase we have that in eq. (19) the function $f^+(r) \neq 0$ and hence the generating function $G(r)$ becomes

$$G(r) = \frac{2}{N} (-1)^r + \frac{1}{N} \sum_{q \in \Gamma^*} \frac{\sin \phi - \cos \phi e^{-i2q}}{\sin \phi - \cos \phi e^{-i2q}} e^{-iq(r-1)} ,$$  \hspace{1cm} (43)

in which, for large $N$, we can approximate the sum with an integral, hence obtaining

$$G(r) = \frac{2}{N} (-1)^r + \frac{1}{2\pi} \int_0^{2\pi} \frac{\sin \phi - \cos \phi e^{-i2q}}{\sin \phi - \cos \phi e^{-i2q}} e^{-iq(r-1)} dq .$$  \hspace{1cm} (44)

This generating function reflects the fact that, effectively, the ground states of the frustrated case have a single, delocalized excitation. The determinant of Toeplitz matrices of this type can be evaluated using the extension of Szegö theorem considered in Refs. [13, 48, 51], which uses a Wiener–Hopf approach to account for the extra excitation.
Thus, in this phase, we write the Toeplitz determinants [26 29] as
\[
\Delta_r(\rho_x) = \begin{vmatrix} \tilde{D}_0 & \tilde{D}_1 & \ldots & \tilde{D}_{1-r} \\ \tilde{D}_1 & \tilde{D}_0 & \ldots & \tilde{D}_{2-r} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{D}_{1-r} & \tilde{D}_{2-r} & \ldots & \tilde{D}_0 \end{vmatrix}, \quad r = \frac{N - 1}{2}, \quad (45)
\]
and
\[
\Delta_r(\rho_y) = \begin{vmatrix} \tilde{D}_0 & \tilde{D}_1 & \ldots & \tilde{D}_{1-r} \\ \tilde{D}_1 & \tilde{D}_0 & \ldots & \tilde{D}_{1-r} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{D}_{1-r} & \tilde{D}_{1-r} & \ldots & \tilde{D}_0 \end{vmatrix}, \quad r = \frac{N - 1}{2}, \quad (46)
\]
where
\[
\tilde{D}_n \equiv G(2n + 1) = -\frac{2}{N}(-1)^n + \frac{1}{2\pi} \int_0^{2\pi} D(e^{i\theta}) e^{-i\theta n} d\theta,
\]
with
\[
D(e^{i\theta}) \equiv \frac{1 - \cot \phi e^{-i\theta}}{1 - \cot \phi e^{-i\theta}}.
\]

Note that, compared to the definitions employed for the yFM phase, we changed the definition of the generating function by shifting its Fourier series, so that [48] has zero winding number.

In this way, the magnetization in the x direction can be calculated directly borrowing the results in (51), because both the generating function \(D(e^{i\theta})\) and \(\ln D(e^{i\theta})\) are continuous on the unit circle \(|e^{i\theta}| = 1\). We write (47) as
\[
\tilde{D}_n = D_n + \frac{x}{N} e^{ikn},
\]
with
\[
D_n \equiv \frac{1}{2\pi} \int_0^{2\pi} D(e^{i\theta}) e^{-i\theta n} d\theta, \quad (50)
\]
\[
k = \pi, \quad x = -2.
\]

Then, the large r behavior of the determinant in (45) is given by
\[
\Delta_r(\rho_x) \overset{r \gg 1}{=} \tilde{\Delta}_r(\rho_x) \left(1 + \frac{x r}{N D(e^{-ik})}\right), \quad (52)
\]
where \(\tilde{\Delta}_r(\rho_x)\) is the determinant [45] with \(D_n\) instead of \(\tilde{D}_n\), which can be directly evaluated using Szegö theorem. We thus have
\[
\langle g^+ \mid \Pi_x^+ \mid g^+ \rangle \overset{N \gg 1}{=} (1 - \cot^2 \phi)^{\frac{1}{2}} + \frac{x}{N} (1 - \cot^2 \phi)^{\frac{1}{2}} \quad (53)
\]

The factor \((-1)^{\frac{N-1}{2}}\) is irrelevant and can be removed, for instance, by taking \(\theta = (-1)^{\frac{N-1}{2}} \pi/4\) in eq. [23].

The evaluation of the asymptotic behavior of (46) is more complicated, because the shift in the matrix entries gives a non-zero winding number to the generating function. Thus, we resorted to numerics to check that it decays algebraically as the magnetization in the x-direction.

Finally, the magnetization in the z is simply equal to \(\pm \frac{2}{N}\).

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1. L.D. Landau, E.M. Lifshitz, & L.P. Pitaevskij, Statistical Physics, Pergamon Press, Oxford (1978).
2. P.W. Anderson, Basic Notions Of Condensed Matter Physics, Addison-Wesley (1997).
3. D. Chandler, Introduction to Modern Statistical Mechanics, Oxford University Press; 1 edition (1987).
4. P. Coleman, Introduction to Many-Body Physics, Cambridge University Press (2016).
5. M. Stone (Ed.), Quantum Hall Effect, World Scientific (1992).
6. X.-G. Wen, Quantum Field Theory of Many-body Systems: From the Origin of Sound to an Origin of Light and Electronics, Oxford University Press (2004).
7. E. Fradkin, Field theories of condensed matter physics, Cambridge University Press (2013).
8. B. A. Bernevig & T.L. Hughes, Topological Insulators And Topological Superconductors, Princeton University Press (2013).
9. B. Zeng, X. Chen, D.-L. Zhou, & X.-G. Wen, Quantum Information Meets Quantum Matter: From Quantum Entanglement to Topological Phases of Many-Body Systems, Springer (2019).
10. C. Nayak, S.H. Simon, A. Stern, M. Freedman, & S. Das Sarma, Rev. Mod. Phys. 80, 1083 (2008).
11. M.Z. Hasan & C.L. Kane, Rev. Mod. Phys. 82, 3045 (2010).
12. M. Franz, in: Generalized quantum entanglement, ed. by V. Vedral, R.1741 (2010).
13. E. Witten, Rev. Mod. Phys. 88, 35001 (2016).
14. J.-J. Dong, P. Li, & Q.-H. Chen, J. Stat. Mech. P113102 (2016).
15. J.-J. Dong, P. Li, & Q.-H. Chen, J. Stat. Mech. P113102 (2016).
16. The A-Cycle Problem for Transverse Ising Ring.
17. S. M. Giampaolo, F. B. Ramos, & F. Franchini, J. Phys. Commun. 3, 081001 (2019).
18. The A-Cycle Problem for Transverse Ising Ring.
19. J. Cardy, Acta Phys. Polon B 26, 1860 (1995).
20. P. D. Fradkin, Phys. Rev. B 53, R8322(R) (1996).
21. Kondo effect in a Luttinger liquid: A boundary-conformal-field-theory approach.
22. J. Cardy, Nucl. Phys. B 240, 4 (1984).
23. Conformal invariance and surface critical behavior.
24. J. Cardy, arXiv:hep-th/0411120 (2004).
25. Boundary Conformal Field Theory.
26. P. Di Francesco, P. Mathieu, & D. Senechal, Conformal Field Theory, Springer (1999).
27. V.E. Korepin, N.M. Bogoliubov, & A.G. Izergin, Quan-
tum Inverse Scattering Method and Correlation Functions, Cambridge University Press (1997).

[21] V.E. Korepin & P. Zinn-Justin, J. Phys. A 33, 7053 (2000)
Thermodynamic limit of the six-vertex model with domain wall boundary conditions.

[22] P. Zinn-Justin, (2002), arXiv:cond-mat/0205192 (2002). The influence of boundary conditions in the six-vertex model.

[23] F. Colomo & A. G. Pronko, J. Stat. Phys. 138, 662 (2010)
The arctic curve of the domain-wall six-vertex model.

[24] P. Bleher & K. Liechty, Random Matrices and the Six-Vertex Model, CRM monographs series, vol. 32, American Mathematical Society, Providence (2013).

[25] F. Colomo & A. Sportiello, J. Stat. Phys. 164, 1488 (2016)
Arctic curves of the six-vertex model on generic domains: the Tangent Method.

[26] N. Allegra, J. Dubail, J.-M. Stéphan, & J. Viti, J. Stat. Mech. 2016, 053108 (2016).
Inhomogeneous field theory inside the arctic circle.

[27] N. Reshetikhin & A. Sridhar, Commun. Math. Phys. 356, 535 (2017)
Integrability of limit shapes of the six-vertex model.

[28] P. Di Francesco & E. Guitter, J. Phys. A: Math. Theor. 51, 355201 (2018)
Arctic curves for paths with arbitrary starting points: a tangent method approach.

[29] F. Colomo, A.G. Pronko, & A. Sportiello, J. Stat. Phys. 174, 1 (2018)
Arctic Curve of the Free-Fermion Six-Vertex Model in an L-Shaped Domain.

[30] G. Toulouse, Commun. Phys. 2, 115 (1977)
Theory of the frustration effect in spin glasses: I.

[31] J. Vannimenus & G. Toulouse, J. Phys. C 10, L537 (1977)
Theory of the frustration effect. II. Ising spins on a square lattice.

[32] M.M. Wolf, F. Verstraete & J.I. Cirac, Int. Journal of Quantum Information 1, 465 (2003)
Entanglement and Frustration in Ordered Systems.

[33] S. M. Giampaolo, G. Gualdi, A. Monras, & F. Illuminati, Phys. Rev. Lett. 107, 260602 (2011)
Characterizing and quantifying frustration in quantum many-body systems.

[34] U. Marzolino, S. M. Giampaolo, & F. Illuminati, Phys. Rev. A 88, 020301(R) (2013)
Frustration, entanglement, and correlations in quantum many body systems.

[35] J. F. Sadoc & R. Mosseri, Geometrical frustration. Cambridge University Press (2007).

[36] C. Lacroix, P. Mendels, & F. Mila (eds), Introduction to Frustrated Magnetism: Materials, Experiments, Theory. Springer Series in Solid-State Sciences, Vol. 164 (2011).

[37] H. T. Diep, Frustrated Spin Systems, World Scientific (2013).

[38] G.H. Wannier, Phys. Rev. 79, 357 (1950)
Antiferromagnetism. The Triangular Ising Net.

[39] E. Ercolessi, S. Evangelisti, F., & F. Ravanini, Phys. Rev. B 88, 104418 (2013)
Modular invariance in the gapped XYZ spin-1/2 chain.

[40] F. Franchini, An introduction to integrable techniques for one-dimensional quantum systems, Lecture Notes in Physics 940, Springer (2017).

[41] V. Marić, S. M. Giampaolo, D. Kuć, & F. Franchini, In preparation
The Frustration in being Odd: Exact finite size degeneracies.

[42] V. Marić, S. M. Giampaolo, D. Kuć, & F. Franchini, In preparation
The Frustration in being Odd: Incommensurate Site-Dependent Spontaneous Magnetization.

[43] B. Damski & M. M. Rams, J. Phys. A 47, 025303 (2014)
Exact results for fidelity susceptibility of the quantum Ising model: The interplay between parity, system size, and magnetic field.

[44] S. Sachdev, Quantum Phase Transitions, Cambridge University Press (2011).

[45] E. Lieb, T. Schultz, & D. Mattis, Ann. of Phys. 16, 407-466 (1961)
Two Soluble Models of an Antiferromagnetic Chain.

[46] E. Barouch & B.M. McCoy, Phys. Rev. A 3, 786 (1971)
Statistical Mechanics of the XY Model. II. Spin-Correlation Functions.

[47] The 2-point function in the y-direction is an educated empirical fit.

[48] J.-J. Dong & P. Li, Mod. Phys. Lett. B 31, 1750061 (2017)
Rigorous proof for the non-local correlation functions in the antiferromagnetic seamed transverse Ising ring.

[49] I.I. Hirschman, Jr., Amer. J. Math. 88, 577 (1966).
The Strong Szegő Limit Theorem for Toeplitz Determinants.

[50] T.T. Wu, Phys. Rev. 149, 380 (1966).
Theory of Toeplitz Determinants and the Spin Correlations of the Two-Dimensional Ising Model. I.

[51] J.-J. Dong, Z.-Y. Zhen, P. & Li, Phys. Rev. E 97, 012133 (2018) Rigorous proof for the non-local correlation functions in the antiferromagnetic seamed transverse Ising ring.
The A-Cycle Problem in XY model with Ring Frustration.