Estimate the parameters of Weibull distribution by using nonlinear membership function by Gaussian function

Suhaila N. Abdullah¹                      Iden Hasan Huessian²

1. ALKARKH University of Science, College of Energy & Environmental Science, Baghdad, Iraq Email: Sss333iraq@yahoo.com
2. University of Baghdad, College of Science for Women, Baghdad, Iraq Email: Iden_Alkani@yahoo.com

Abstract. The main aim of the presented study is estimating the parameters of Weibull distribution by utilizing simulation to generated the samples size when $n=10, 50, 100$. Considering in the current study the parameters estimator of Weibull membership function, then using the nonlinear membership function for Gaussian function to find the fuzzy number for these parameters estimator. After that utilizing the ranking function to transform the fuzzy number to crisp number.

Keywords— Weibull distribution, simulation technique, nonlinear membership function, the ranking function.

1. Introduction

In 1951, Waloddi Weibull (Swedish mathematician) was the first to describe Weibull distribution. With regard to the probability theory and statistics, Weibull distribution is specified as the continuous probability distribution. Weibull distribution has been a useful density function in reliability and survival analysis, which is a famous distribution in medical and industry.

In the case when certitudes happen, individuals will usually look back to the acquired data and attempt to evaluate future events. Usually, a major methodology; that is commonly applied as the probability theory has fulfilled such necessities to handle im-precision and un-certainty. Yet, with regard to the full-uncertainty conditions, the probability theory could not be adequate, also there must be integration between fuzzy logic and probability theory for enhancing the robustness. Full-uncertainty could be defined as nobody has the data on the incidence of likely conditions furthermore, in certain conditions cases no one knows anything regarding such becoming true possible events.

Zadeh in 1968 was first structure probability measures in fuzzy sets. Kwakernaak defined fuzzy random variables (FRVs), also the authors indicated many concepts related to the independent fuzzy variables for the first time in his paper [3]. After one year, he added algorithms related to the fuzzy random variable, also the author provided examples related to discrete case [4].

Pak et al. in (2013) [5] developed inferential procedures according to the fuzzy environment with Weibull distribution.

Shafiq and Viertl in (2014) [9] calculated characterizing function related to fuzzy parameter estimate for Weibull parameters based on censored fuzzy data.
Pak et al. in (2016) [7] applied the algorithm of Newton-Raphson for determining the maximum probability estimate of shape parameter of lognormal distribution when the observations are fuzzy.

Al-Sultany in (2016)[10]; evaluated the performance of moment maximum likelihood and Bayes estimation method for estimating the unknown parameters and reliability function of inverse Weibull distribution according to the fuzzy data.

In this paper, generating the data of Weibull distributions by utilizing the Monte carol technique through using simulation method, with different sample size when \( n=10, 50, 100 \) respectively. After that finding the unknown parameters of Weibull distribution by using the nonlinear membership function for Gaussian function to get the lower and the upper fuzzy parameters number for inconsistent sample size utilizer, this method enables the construction of normal fuzzy number which can be adapted to have Gaussian shape.

Finally, using the ranking function to transform the fuzzy parameters number to crisp for various sample size.

I. Fuzzy set

Fuzzy sets have been a main field of focus in many ways starting from its initiation in the year 1965. Some applications related to Fuzzy sets can be seen in robotics, operation research, logic, decision theory, medicine, AI, computer science, control engineering, expert systems management science, and pattern recognition.

The development in mathematics increased largely in the past as well as huge advances recently. In the presented study, an elementary mathematical system related to the Fuzzy set is going to be discussed, in addition to the major applications of Fuzzy sets to the other methods and concepts. Soft computing or computational intelligence was the name used to describe Fuzzy sets, neural nets theory, and evolutionary programming since the year 1992.

The associated between such extents has become mainly close. In the presented study, the Fuzzy sets will be the main focus. The applications related to the fuzzy sets to real problems abound. Certain references are going to be provided. To describe even a part of them will definitely exceed the extent of the presented study; John Wiley & Sons [2].

II. Gaussian function

The membership definition for a Gaussian function \( G: \rightarrow [0, 1] \) is given by two parameters as:
\[
G(\alpha, k) = e^{-k(\alpha-x)^2}
\]

Where \( \alpha \) is the midpoint and \( K \) reflects the slop value. Note that \( K \) must be positive and that the function never reaches zero. The Gaussian function can also be extended to have different left and right slops. We then have three parameters in
\[
G(x; \alpha, k_1, k_2) = \begin{cases} 
  e^{-k_1(x-\alpha)^2} & x \leq \alpha \\
  e^{-k_2(x-\alpha)^2} & x > \alpha 
\end{cases}
\]

Where \( k_1 \) and \( k_2 \) are, respectively, left and right slopes.

1- G-membership function
Theorem 1:
Let \( \mu(x) \) by fuzzy number with Gaussian membership function as
\[
\mu(x) = e^{-k(x-m)^2}
\]
Let \( \mu(x) = x \) and \( x = u(x) \) then
\[
\frac{\ln x}{k} = \left[ u(x) - m \right]^2
\]
By taken root
\[
\sqrt{-\ln x} = \pm \left[ u(x) - m \right]
\]
If \( \tilde{u}(x) - m > 0 \) then \( +[\tilde{u}(x) - m] > 0 \)
\[
\frac{1}{\sqrt{k}} \sqrt{-\ln x} = \tilde{u}(x) - m
\]
If \( \tilde{u}(x) - m < 0 \) then \( -[\tilde{u}(x) - m] < 0 \)
\[
\frac{1}{\sqrt{k}} \sqrt{-\ln x} = -[u(x) - m]
\]
These are a fuzzy number in a parametric form where \( 0 \leq x \leq 1 \).
If we inspection that the membership function \( \mu(x) = e^{-k(x-m)^2} \) is Gaussian function substituting \( k = 1, m = 1 \) then the fuzzy number of Gaussian membership function becomes as follow:
\[
\mu(x) = e^{-(x-1)^2}
\]
Let \( \mu(x) = x \) and \( x = u(x) \) then
\[
\frac{\ln x}{-1} = \left[ u(x) - 1 \right]^2\]
By taken root
\[
\sqrt{-\ln x} = \pm \left[ u(x) - 1 \right]
\]
If \( \tilde{u}(x) - 1 > 0 \) then \( +[\tilde{u}(x) - 1] > 0 \)
\[
\sqrt{-\ln x} = \tilde{u}(x) - 1
\]
If \( \tilde{u}(x) - 1 < 0 \) then \( -[\tilde{u}(x) - 1] < 0 \)
\[
\sqrt{-\ln x} = -[u(x) - 1]
\]
These are a fuzzy number in a parametric form where \( 0 \leq x \leq 1 \).
IV. Ranking function

Ranking fuzzy numbers are of high importance in data analysis, they are applied in forecasting in addition to the decision-making optimization. The approach of ranking has been initially suggested via Jain (1976). Yager (1981) suggested 4 indices which could be used to order the fuzzy quantities in [0,1]. Ghen and Ghen (2007) suggested an approach for ranking the generalized fuzzy trapezoidal fuzzy number generalized fuzzy numbers. Compos and Gonzales (1989) suggested a subjective method for ranking fuzzy numbers. Ramil and Mohamad suggested a complete survey related to various approaches for ranking of fuzzy numbers.

Ranking of fuzzy is of high importance in forecasting, making optimizations, and risk analysis decision. Fuzzy sets have been developed via Zadeh, that is a tool of high power to handle real-life cases. From the time when Yager suggested centroid system in the ranging method.

The advantage of Ranking function which applies in operation research, statistics and mathematics is to transform the fuzzy number to crisp number.

Where

\[ \mu(x) = e^{-(x-1)^2} \]

Then the Ranking function becomes as follow:

\[ \alpha \int_0^1 R^{-1}(x) \, dx + (1 - \alpha) \int_0^1 L^{-1}(x) \, dx \]

Where \( L^{-1}(x) \) is left shape function of \( \bar{u}(x) \)

\( R^{-1}(x) \) is left shape function of \( u(x) \)

Then

\[ L^{-1}(x) = \bar{u}(x) \quad \text{and} \quad R^{-1}(x) = u(x) \]

\( \alpha \) is weight function then \( \alpha \in [0,1] \) and GMF equal to Gaussian membership function.
Theorem: if $\tilde{\mu}$ is a GMF in a parametric form then

$$Lw^\alpha(\tilde{\mu}) = m + \frac{\sqrt{n}(2\alpha-1)}{2\sqrt{k}}$$

Proof: the parametric form in Gaussian membership function is as follow:

$$\tilde{\mu} = \left[ \mu(x), \tilde{\mu}(x) \right] = \left[ m - \frac{1}{\sqrt{k}} \sqrt{-lnx}, m + \frac{1}{\sqrt{k}} \sqrt{-lnx} \right]$$

$$Lw^\alpha(\tilde{\mu}) = a \int_0^1 R^{-1}(x) \, dx + (1-a) \int_0^1 L^{-1}(x) \, dx$$

$$Lw^\alpha(\tilde{\mu}) = a \int_0^1 \left( m + \frac{1}{\sqrt{k}} \sqrt{-lnx} \right) \, dx + \int_0^1 \left( m - \frac{1}{\sqrt{k}} \sqrt{-lnx} \right) \, dx$$

$$Lw^\alpha(\tilde{\mu}) = a \int_0^1 \left( \frac{1}{\sqrt{k}} \sqrt{-lnx} \right) \, dx + \int_0^1 \left( \frac{1}{\sqrt{k}} \sqrt{-lnx} \right) \, dx$$

$$a \int_0^1 \frac{1}{\sqrt{k}} \sqrt{-lnx} \, dx + \int_0^1 \frac{1}{\sqrt{k}} \sqrt{-lnx} \, dx + \int_0^1 \left( \frac{1}{\sqrt{k}} \sqrt{-lnx} \right) \, dx$$

$$a \int_0^1 \frac{1}{\sqrt{k}} \sqrt{-lnx} \, dx + \int_0^1 \frac{1}{\sqrt{k}} \sqrt{-lnx} \, dx + \int_0^1 \left( \frac{1}{\sqrt{k}} \sqrt{-lnx} \right) \, dx$$

$$Lw^\alpha(\tilde{\mu}) = m + \frac{2\alpha}{\sqrt{k}} \int_0^1 \left( \sqrt{-lnx} \right) \, dx$$

$$Lw^\alpha(\tilde{\mu}) = m + \frac{2\alpha}{\sqrt{k}} \int_0^1 \left( \sqrt{-lnx} \right) \, dx$$

Recall that

$$\int_0^1 (-lnx)^{\frac{1}{2}} \, dx = \int_0^1 y \, dy = 2y dy = d(x) = -2xy dy$$

$$\int_0^1 (-lnx)^{\frac{1}{2}} \, dx = \int_0^1 (y^2)(-2xy) \, dy = \int_0^1 2y^2 \, dy = \int_0^1 2e^{-y^2} \, dy$$

$$\int_0^1 2y^2 e^{-y^2} \, dy$$

$$u = y^2 \quad , \quad \int du = \int y^2 \, dy$$

$$\int udv = uv - \int v \, du$$
The method of Maximum Likelihood Estimation as this technique gives a simpler Estimation as compared to the Method of Moments and the Local frequency ratio method of estimation. Now we are estimating the parameter of the Weibull distribution from which the sample comes. Let \(X_1, X_2, \ldots, X_n\) be a random sample of \(n\) observations from the Weibull population with pdf

\[
f(x; \beta_1, \beta_2) = \frac{\beta_1}{\beta_2} x^{\beta_1 - 1} e^{-\left(\frac{x}{\beta_2}\right)^\beta_2}
\]

V. Parameter Estimation by MLE

Let

\[
g_1(\beta_1) = \frac{n}{\beta_1} - n\ln \beta_2 + \sum_{i=1}^{n} \ln x_i - \sum_{i=1}^{n} \left(\frac{x_i}{\beta_2}\right)^{\beta_1}
\]

\[
g_2(\beta_2) = -n\frac{\beta_1}{\beta_2} + \sum_{i=1}^{n} \frac{x_i}{\beta_2^{\beta_1 + 1}}
\]

where

\[
J^{-1} = \begin{bmatrix}
\frac{\partial g_1(\beta_1)}{\partial \beta_1} & \frac{\partial g_1(\beta_1)}{\partial \beta_2} \\
\frac{\partial g_2(\beta_2)}{\partial \beta_1} & \frac{\partial g_2(\beta_2)}{\partial \beta_2}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\beta_{1,k+1} \\
\beta_{2,k+1}
\end{bmatrix}
= J^{-1} \begin{bmatrix}
g_1(\beta_{1,k}) \\
g_2(\beta_{2,k})
\end{bmatrix}
\]
I. Practical side:

In this section, we will create a simulated environment to find the fuzzy parameters of Weibull distribution as described in the following steps.

Based on $F(x) = 1 - \exp\left(-\left(\frac{x}{\beta_2}\right)^{\beta_1}\right)$ we can write the Matlab program where $F(x) = u \to \text{uniform distribuion}(0,1)$

then

$u = 1 - \exp\left(-\left(\frac{x}{\beta_2}\right)^{\beta_1}\right)$

$-\exp\left(-\left(\frac{x}{\beta_2}\right)^{\beta_1}\right) = 1 - u$

$\ln \exp\left(-\left(\frac{x}{\beta_2}\right)^{\beta_1}\right) = -\ln(1 - u)$

$x = \beta_2[-\ln(1 - u)]^{1/\beta_1}$. And the Initial value $\beta_1=0.9$, $\beta_2=0.1$ we got,

$n=10$ then $\bar{\beta_1} = 0.2174$, $\bar{\beta_2} = 32.6844$

$n=50$ then $\bar{\beta_1} = 0.33665$, $\bar{\beta_2} = 0.0331$

$n=100$ then $\bar{\beta_1} = 1.9541$, $\bar{\beta_2} = 0.1422$

2- to calculate the lower and upper fuzzy parameters using the eq(1),eq(2) Where $n=10$, then $0.001 = 0.2174 - \frac{1}{\sqrt{k}} (\sqrt{-0.001})$

$0.999 = 0.2174 - \frac{1}{\sqrt{k}} (\sqrt{-0.999})$

Then $K=34.602$, in the same way

Where $n=50 \to K=0.149$, and Where $n=100 \to K=0.427$

|   | $\beta_1$ | $\beta_2$ | $\beta_3$ | $\beta_4$ | $\beta_5$ | $\beta_6$ |
|---|---|---|---|---|---|---|
| $n=10$ | $u_1 | u_2 | u_3 | u_4 | u_5 | u_6 |
| 0.1 | -0.04 | 0.48 | -0.45 | 70.82 | -0.59 | 7.28 | 0.015 | 47.3 | -0.37 | 4.28 | -0.025 | 0.31 |
| 0.2 | 0.001 | 0.43 | 0.81 | 64.56 | 0.08 | 6.65 | 0.017 | 39.54 | 0.014 | 3.89 | 0.003 | 0.28 |
| 0.3 | 0.04 | 0.4 | 5.13 | 60.24 | 0.53 | 6.21 | 0.019 | 34.03 | 0.28 | 3.63 | 0.02 | 9.26 |
| 0.4 | 0.05 | 0.38 | 8.63 | 56.47 | 0.89 | 5.85 | 0.022 | 29.85 | 0.49 | 3.42 | 0.04 | 0.25 |
| 0.5 | 0.1 | 0.56 | 11.77 | 55.8 | 1.21 | 5.53 | 0.023 | 25.95 | 0.68 | 3.23 | 0.051 | 0.22 |
| 0.6 | 0.12 | 0.54 | 14.73 | 58.63 | 1.52 | 5.52 | 0.025 | 22.28 | 0.86 | 3.05 | 0.06 | 0.21 |
| 0.7 | 0.14 | 0.32 | 17.77 | 47.66 | 1.82 | 4.91 | 0.026 | 18.6 | 1.04 | 2.87 | 0.08 | 0.20 |
| 0.8 | 0.142 | 0.29 | 20.82 | 44.55 | 2.14 | 4.59 | 0.027 | 14.74 | 1.23 | 2.68 | 0.09 | 0.19 |
| 0.9 | 0.15 | 0.28 | 22.95 | 42.41 | 2.36 | 4.37 | 0.028 | 12.09 | 1.36 | 2.55 | 0.1 | 0.18 |
To determine the Ranking value by using the eq(3)
Where \( n=10, 50, 100, \) and let \( k=1 \) and \( \alpha \geq 0.5 \) we get:

Table 2: Ranking value for parameters

| \( \beta_1 \) | \( \beta_2 \) | \( \beta_1 \) | \( \beta_2 \) | \( \beta_1 \) | \( \beta_2 \) |
|---|---|---|---|---|---|
| 0.5 | 0.217 | 32.684 | 3.367 | 0.0331 | 1.954 | 0.1422 |
| 0.6 | 0.394 | 32.86 | 3.544 | 0.21 | 2.131 | 0.16 |
| 0.7 | 0.571 | 33.04 | 3.721 | 0.387 | 2.308 | 0.496 |
| 0.8 | 0.748 | 433.26 | 3.898 | 0.564 | 2.485 | 0.673 |
| 0.9 | 0.925 | 33.39 | 4.075 | 0.741 | 2.662 | 0.85 |

Conclusion:
We propose a nonlinear membership function for Gaussian function in statistic for first time in Iraq. In table (1) the lower fuzzy parameters number are increasing and the upper fuzzy parameters number are decreasing. In table (2) the ranking function of parameters \( \beta_1, \beta_2 \) number are increasing we take \( \alpha \geq 0.5 \) because the value of \( \alpha \) greater than and equal to (0.5) it is normal concave.

References
1- Chan S.J and Chan S.M.; (2007);"Fuzzy Risk analysis based on the ranking of generalized Trapezoidal fuzzy numbers;Applied intelligence;vol.1;pp1-11.
2- John Wiley & Sons, (2010)," Fuzzy set theory";vol 2, issue 3;pp 317-332.
3- Kwakernaak, H., (1978)," Fuzzy random variables-I. Definitions and theorems." Inf, Sci. 15, 1–29.
4- Kwakernaak, H., (1979)," Fuzzy random variables-II. Algorithms and examples for the discrete case", Inf. Sci. 17, 253–278.
5- Pak A., Parham G.A and Saraj M., (2013),"Inference for the weibull distribution based on fuzzy data" Revista colombiana deEstadistica, VOL 36, NO.2,PP339-358.
6- Pak A., Parham G.A and Saraj M., (2014),"Inference on the competing Risk Reliability problem for exponential distribution based on fuzzy data" IEEE Transactions on Reliability, VOL.63, NO.1, PP 2-12.
7- Pak A., Parham G.A and Saraj M., (2016),"Inference for the shape parameter of lognormal distribution in presence of fuzzy data", Pakistan Journal of statistic and operation Research, VOL.12, NO.1, PP 89-99.
8- Ramli N. and Mohamad D.; (2009);"A Comparative analysis of centroid method in ranking fuzzy numbers "; European Journal of scientific research;vol.28;pp 492-501.
9- Shafiq M. and Viertl R., (2014),"Maximum likelihood Estimation for weibull distribution in case of censored fuzzy lifetime datd", Vienna university of Ftechnology, pp1-17.
10- Al-Sultany  S.A-K., (2016),"Estimate the parameters and Reliability function of Inverse weibull distribution based on linear membership, Ph.D. Thesis of submitted to college of science at the University of Mustansiyah.