Left invariant measures on locally compact fan loops.

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10 December 2018

Abstract

In this article left invariant measures and functionals on locally compact nonassociative fan loops are investigated. For this purpose necessary properties of topological fan loops, estimates and approximations of functions on them are studied. An existence of nontrivial left invariant measures on locally compact fan loops is proved. Abundant families of fan loops are provided with the help of different types of their products.

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1 Introduction.

Left invariant measures or Haar measures on locally compact groups play very important role in measure theory, harmonic analysis, representation theory, geometry, mathematical physics, etc. (see, for example, [6, 12, 18] and references therein). On the other hand, in nonassociative algebra, in noncommutative geometry, field theory, topological algebra there frequently appear binary systems which are nonassociative generalizations of groups and related with loops, quasi-groups, Moufang loops, IP-loops, etc. (see

1key words and phrases: measure; left invariant; loop; locally compact

Mathematics Subject Classification 2010: 43A05; 28C10; 20N05; 22A30
An arbitrary IP-loop $Y$ is a loop with a restriction: for each $x \in Y$ there exist elements $x_1$ and $x_2$ in $Y$ such that for each $y$ in $Y$ the identities are satisfied $x_1(xy) = y$ and $(yx)x_2 = y$, where $x_1$ and $x_2$ are also denoted by $-1x$ and $x^{-1}$ and called left and right inverses of $x$ respectively. It was investigated and proved in the 20-th century that a nontrivial geometry exists if and only if there exists a corresponding loop.

Very important role in mathematics and quantum field theory play octonions and generalized Cayley-Dickson algebras [1, 2, 3, 10]. A multiplicative law of their canonical bases is nonassociative and leads to a more general notion of a metagroup instead of a group [30]. They are used not only in algebra and geometry, but also in noncommutative analysis and PDEs, particle physics, mathematical physics (see [3, 9, 10, [13]-[17], [20]-[29] and references therein). The preposition ”meta” is used to emphasize that such an algebraic object has properties milder than a group. By their axiomatic metagroups are loops with weak relations. They were used in [30] for investigations of automorphisms and derivations of nonassociative algebras.

In this article more general binary systems such as fan loops are studied (see Definition 2.1). They also are more general than IP-loops, because in fan loops $G$ left and right inverses $^{-1}x$ and $x^{-1}$ of nonunit elements $x$ in $G$ may not exist.

This article is devoted to left invariant measures (see Definition 3.18) on locally compact fan loops. Necessary preliminary results about fan loops are given in Section 2. Specific algebraic and topological features of fan loops are studied in Lemmas 2.2-2.6, 2.12 and Propositions 2.9, 2.11. A quotient of a fan loop by its fan is investigated in Theorem 2.8. A uniform continuity of maps on topological fan loops is studied in Theorem 2.14 and Corollary 2.15.

Left invariant functionals and measures are investigated in Section 3. These properties are more complicated than for groups and IP-loops, because of the nonassociativity of fan loops and absence of left and right inverses in general. Main results are theorems 3.15, 3.16, 3.19, 3.20. For their proofs estimates of nonnegative functions with compact supports in fan loops are
investigated in Lemmas 3.2, 3.4, 3.6. Functionals on a space of nonnegative functions with compact supports in a fan loop are studied in Lemmas 3.7, 3.8, 3.10, 3.13 and Theorem 3.9. In Theorem 3.11 approximations of nonnegative functions with compact supports in the fan loop are described.

In an appendix abundant families of fan loops are provided with the help of a direct product and smashing products (see Remark 4.3 and Definition 4.5). For this purpose Theorems 4.1 and 4.4 are proved.

All main results of this paper are obtained for the first time. They can be used in harmonic analysis on nonassociative algebras and metagroups and loops, representation theory, geometry, mathematical physics, quantum field theory, particle physics, PDEs, etc.

2 Fan loops.

To avoid misunderstandings we give necessary definitions. For short it will be written fan loop instead of nonassociative fan loop.

2.1. Definition. Let $G$ be a set with a multiplication (that is a single-valued binary operation) $G^2 \ni (a,b) \mapsto ab \in G$ defined on $G$ satisfying the conditions:

(2.1.1) for each $a$ and $b$ in $G$ there is a unique $x \in G$ with $ax = b$ and

(2.1.2) a unique $y \in G$ exists satisfying $ya = b$, which are denoted by $x = a \setminus b = Div_l(a, b)$ and $y = b/a = Div_r(a, b)$ correspondingly,

(2.1.3) there exists a neutral (i.e. unit) element $e_G = e \in G$: $eg = ge = g$

for each $g \in G$.

We consider subsets in $G$:

(2.1.4) $Com(G) := \{a \in G : \forall b \in G, \ ab = ba\}$;

(2.1.5) $N_l(G) := \{a \in G : \forall b \in G, \forall c \in G, \ (ab)c = a(bc)\}$;

(2.1.6) $N_m(G) := \{a \in G : \forall b \in G, \forall c \in G, \ (ba)c = b(ac)\}$;

(2.1.7) $N_r(G) := \{a \in G : \forall b \in G, \forall c \in G, \ (bc)a = b(ca)\}$;

(2.1.8) $N(G) := N_l(G) \cap N_m(G) \cap N_r(G)$;

$Z(G) := Com(G) \cap N(G)$.

Then $N(G)$ is called a nucleus of $G$ and $Z(G)$ is called the center of $G$.
We call \( G \) a fan loop if a set \( G \) possesses a multiplication and satisfies conditions (2.1.1)-(2.1.3) and
\[
(2.1.9) \quad (ab)c = t(a, b, c)a(bc) \quad \text{and} \quad (ab)c = a(bc)p(a, b, c)
\]
for each \( a, b \) and \( c \) in \( G \), where
\[
t(a, b, c) = t_G(a, b, c) \in N(G) \quad \text{and} \quad p(a, b, c) = p_G(a, b, c) \in N(G).
\]
Then \( G \) will be called a central fan loop if in addition to (2.1.9) it satisfies the condition:
\[
(2.1.10) \quad ab = t_2(a, b)ba
\]
for each \( a \) and \( b \) in \( G \), where \( t_2(a, b) \in Z(G) \).

Let \( \tau \) be a topology on \( G \) such that the multiplication \( G \times G \ni (a, b) \mapsto ab \in G \) and the mappings \( \text{Div}_l(a, b) \) and \( \text{Div}_r(a, b) \) are jointly continuous relative to \( \tau \), then \( (G, \tau) \) will be called a topological fan loop. Henceforth it will be assumed that \( \tau \) is the \( T_1 \cap T_{3.5} \) topology, if something other will not be specified.

A minimal closed subgroup \( N_0(G) \) in the topological fan loop \( G \) containing \( t(a, b, c) \) and \( p(a, b, c) \) for each \( a, b \) and \( c \) in \( G \) will be called a fan of \( G \).

Elements of the fan loop \( G \) will be denoted by small letters, subsets of \( G \) will be denoted by capital letters. If \( A \) and \( B \) are subsets in \( G \), then \( A - B \) means the difference of them \( A - B = \{ a \in A : a \notin B \} \). Henceforward, maps and functions on fan loops are supposed to be single-valued if something other will not be specified.

2.2. Lemma. If \( G \) is a fan loop, then for each \( a, b \) and \( c \) in \( G \) the following identities are fulfilled:
\[
(2.2.1) \quad b \setminus e = t(e/b, b, b \setminus e)(e/b);
(2.2.1') \quad b \setminus e = (e/b)p(e/b, b, b \setminus e);
(2.2.2) \quad (a \setminus e)b = t(e/a, a, a \setminus e)[t(e/a, a \setminus b)]^{-1}(a \setminus b);
(2.2.2') \quad (bc) \setminus a = (c \setminus (bc)\setminus a)[p(b, c, (bc) \setminus a)]^{-1};
(2.2.2'') \quad (a \setminus b)c = (a \setminus (bc))[p(a, a \setminus b, c)]^{-1};
(2.2.2''') \quad (ab) \setminus e = (b \setminus e)(a \setminus e)[t(a, b, b \setminus e)]^{-1}t(ab, b \setminus e, a \setminus e);
(2.2.3) \quad b(e/a) = (b/a)p(b/a, a, a \setminus e)[p(e/a, a, a \setminus e)]^{-1};
(2.2.3') \quad b(a/e) = [t(b, e/a, a)]^{-1}b(e/a);
(2.2.3'') \quad b(a/e) = [t(b, e/a, a)]^{-1}b(e/a);
(2.2.3''') \quad b(a/e) = [t(b, e/a, a)]^{-1}b(e/a);
(2.2.3''') \quad b(a/e) = [t(b, e/a, a)]^{-1}b(e/a);
(2.2.3''') \quad b(a/e) = [t(b, e/a, a)]^{-1}b(e/a);

\[(2.2.3'')\ c(b/a) = t(c, b/a, a)(cb)/a;\]
\[(2.2.3'''')\ e/(ab) = [p(c/b, e/a, ab)]^{-1}p(e/a, a, b)(e/b)(e/a).\]

**Proof.** Note that \(N(G)\) is a subgroup in \(G\) due to Conditions (2.1.5)-(2.1.8) (see also [3]). Then Conditions (2.1.1)-(2.1.3) imply that
\[(2.2.4)\ b(b \setminus a) = a, \ b \setminus (ba) = a;\]
\[(2.2.5)\ (a/b)b = a, \ (ab)/b = a\]
for each \(a\) and \(b\) in any loop \(G\) (see also [3, 34]). Using Condition (2.1.9) and Identities (2.2.4) and (2.2.5) we deduce that
\[e/b = (e/b)(b(b \setminus e)) = [t(e/b, b, b \setminus e)]^{-1}(b \setminus e)\]
which leads to (2.2.1).

Let \(c = a \setminus b\), then from Identities (2.2.1) and (2.2.4) it follows that
\[(a \setminus e)b = t(e/a, a, a \setminus e)(e/a)(ac)\]
\[= t(e/a, a, a \setminus e)(t(e/a, a, a \setminus b)]^{-1}((e/a)a)(a \setminus b)\]
which taking into account (2.2.5) provides (2.2.2).

On the other hand, \(b \setminus e = ((e/b)b)(b \setminus e) = (e/b)(b(b \setminus e))p(e/b, b, b \setminus e)\)
that gives (2.2.1').

Let now \(d = b/a\), then Identities (2.2.1') and (2.2.5) imply that
\[b(e/a) = (da)(a \setminus e)[p(e/a, a, a \setminus e)]^{-1}\]
\[= (b/a)p(b/a, a, a \setminus e)[p(e/a, a, a \setminus e)]^{-1}\]
which demonstrates (2.2.3).

Next we infer from (2.1.9) and (2.2.4) that
\[b(c((bc) \setminus a)) = (bc)((bc) \setminus a)[p(b, c, (bc) \setminus a)]^{-1} = a[p(b, c, (bc) \setminus a)]^{-1},\]
\[c((bc) \setminus a) = (b \setminus a)[p(b, c, (bc) \setminus a)]^{-1}\]
that implies (2.2.2').

Symmetrically it is deduced that \((a/(bc))b)c = t(a/(bc), b, c)a\), consequently, \((a/(bc))b = t(a/(bc), b, c)(a/c)\). From the latter identity it follows (2.2.3').

Evidently, formulas
\[a((a \setminus bc) = (a(a \setminus b))c[p(a, a \setminus b, c)]^{-1} = bc[p(a, a \setminus b, c)]^{-1}\]
\[(c(b/a)a = t(c, b/a, a)cb\]
imply (2.2.3'') and (2.2.3'''') correspondingly.

From (2.1.9) we infer that
\[(ab)((b \setminus e)(a \setminus e)) = [t(ab, b \setminus e, a \setminus e)]^{-1}t(a, b, b \setminus e),\]
since by (2.2.4)
Finally applying (2.11) and (2.12) implies (2.2").

Analogously form (2.1.9) we deduce that

\[(e/b)(e/a)(ab) = [p(e/a, a, b)]^{-1}p(e/b, e/a, ab),\]

since by (2.2.5)

\[(e/b)((e/a)a)b) = e.\]

Finally applying (2.1.1) and (2.1.2) we get Identity (2.2.3").

**2.3. Lemma.** Assume that \(G\) is a fan loop. Then for every \(a, a_1, a_2, a_3\) in \(G\) and \(z_1, z_2, z_3\) in \(Z(G)\), \(b \in N(G)\):

- (2.3.1) \(t(z_1a_1, z_2a_2, z_3a_3) = t(a_1, a_2, a_3)\);
- (2.3.1') \(p(z_1a_1, z_2a_2, z_3a_3) = p(a_1, a_2, a_3)\);
- (2.3.2) \(t(a, a \setminus e, a) = ap(a, a \setminus e, a)\);
- (2.3.2') \(t(a, e/a, a) = ap(a, e/a, a)\);
- (2.3.2'') \(p(a, a \setminus e, a)t(e/a, a, a \setminus e) = e\);
- (2.3.3) \(t(a_1, a_2, a_3b) = t(a_1, a_2, a_3)\);
- (2.3.3') \(p(ba_1, a_2, a_3) = p(a_1, a_2, a_3)\);
- (2.3.4) \(t(ba_1, a_2, a_3) = bt(a_1, a_2, a_3)b^{-1}\);
- (2.3.4') \(p(a_1, a_2, a_3b) = b^{-1}p(a_1, a_2, a_3)b\).

**Proof.** Since \((a_1a_2)a_3 = t(a_1, a_2, a_3)a_1(a_2a_3)\) and \(t(a_1, a_2, a_3) \in N(G)\) for every \(a_1, a_2, a_3\) in \(G\), then

- (2.3.5) \(t(a_1, a_2, a_3) = ((a_1a_2)a_3)/(a_1(a_2a_3))\).

Therefore, for every \(a_1, a_2, a_3\) in \(G\) and \(z_1, z_2, z_3\) in \(Z(G)\) we infer that

- \(t(z_1a_1, z_2a_2, z_3a_3) = (((z_1a_1)(z_2a_2))(z_3a_3))/((z_1a_1)((z_2a_2)(z_3a_3)))\)
- \((z_1z_2z_3)((a_1a_2a_3))/((z_1z_2z_3)(a_1(a_2a_3))) = ((a_1a_2a_3)/(a_1(a_2a_3)),\) since

- (2.3.6) \(b/(qa) = q^{-1}b/a\) and \(b/q = q \setminus b = bq^{-1}\)

for each \(q \in Z(G)\), \(a\) and \(b\) in \(G\), because \(Z(G)\) is the commutative group satisfying Conditions (2.1.4) and (2.1.8). Thus \(t(z_1a_1, z_2a_2, z_3a_3) = t(a_1, a_2, a_3)\).

Symmetrically we get

- (2.3.7) \(p(a_1, a_2, a_3) = (a_1(a_2a_3)) \setminus ((a_1a_2)a_3)\) and
- \(p(z_1a_1, z_2a_2, z_3a_3) = (((z_1a_1)(z_2a_2))(z_3a_3))/(((z_1a_1)(z_2a_2))(z_3a_3)))\)
- \(((z_1z_2z_3)(a_1a_2a_3))/((z_1z_2z_3)(a_1(a_2a_3))) = (a_1(a_2a_3)) \setminus ((a_1a_2a_3))\)

that provides (2.3.1').

From Formulas (2.3.5) and (2.2.1) it follows that

\[t(a, a \setminus e, a) = ((a(a \setminus e))a)/(a((a \setminus e)a)) = a/[at(e/a, a, a \setminus e)],\] consequently,
(2.3.8) \( t(a, a \setminus e, a)at(e/a, a, a \setminus e) = a \).

Then from Formulas (2.3.7), (2.2.4) and Condition (2.1.9) we deduce that
\[
p(a, a \setminus e, a) = \langle a((a \setminus e)a) \setminus ((a(a \setminus e))a) = \{ t(a, a \setminus e, a) \} \setminus a,
\]
which implies (2.3.2). Identities (2.3.2) and (2.3.8) lead to (2.3.2). Next using (2.3.7) and (2.1.9) we deduce that
\[
p(a, e/a, a) = \langle a((e/a)a) \setminus ((a(e/a))a) = a \setminus [ t(a, e/a, a) a]
\]
that implies (2.3.2). From (2.1.9) we get that
\[
((a_1a_2)a_3) = (a_1a_2)(a_3b) = \langle t(a_1, a_2, a_3 b) a_1(a_2a_3) b,\)
from which and (2.2.5) and (2.3.5) Identity (2.3.3) follows, because \( b \in N(G) \).

Then
\[
b((a_1a_2)a_3) = ((ba_1)a_2)a_3 = b(a_1(a_2a_3)p(ba_1, a_2, a_3))
\]
and (2.2.4) and (2.3.7) imply Identity (2.3.3). Symmetrically we deduce
\[
b((a_1a_2)a_3) = \langle t(ba_1, a_2, a_3) b a_1(a_2a_3) \rangle \text{ and}
\]
\[
((a_1a_2)a_3) = (a_1(a_2a_3)b)p(a_1, a_2, a_3 b)
\]
that together with (2.3.5) and (2.3.7) imply Identities (2.3.4) and (2.3.4).

2.4. Lemma. If \((G, \tau)\) is a topological loop, then the functions \( t(a_1, a_2, a_3) \)
and \( p(a_1, a_2, a_3) \) are jointly continuous in \( a_1, a_2, a_3 \) in \( G \).

Proof. This follows immediately from Formulas (2.3.5), (2.3.7) and Definition 2.1.

2.5. Lemma. Assume that \((G, \tau)\) is a topological loop and \( U \) is an open subsets in \( G \), then for each \( b \in G \) sets \( Ub \) and \( bU \) are open in \( G \).

Proof. Take any \( c \in Ub \) and consider the equation
\[
(2.5.1) \quad xc = c.
\]
Then from (2.1.2) it follows that
\[
(2.5.2) \quad x = c/b.
\]
Thus \( x = \psi_b(c) \), where \( \psi_b(c) = c/b \) is a continuous bijective function in the variable \( c \) due to Identity (2.2.3) and Lemma 2.4. On the other hand, the right shift mapping
\[
(2.5.3) \quad R_b u := ub
\]
from \( G \) into \( G \) is continuous and bijective in \( u \) (see Definition 1). Moreover, \( \psi_b(R_bu) = u \) and \( R_b(\psi_b(c)) = c \) for each fixed \( b \in G \) and all \( u \in G \) and \( c \in G \) by Identities (2.2.5). Thus \( R_b \) and \( \psi_b \) are open mappings, consequently, \( Ub \) is open in \( G \).
Similarly for the equation
\[(2.5.4) \ by = c \text{ the unique solution is} \]
\[(2.5.5) \ y = b \setminus c \text{ by Condition (2.1.1).} \]

Therefore, \( y = \theta_b(c), \) where \( \theta_b(c) = b \setminus c \) is a continuous bijective function in \( c \) according to Lemma 2.4 and Formula (2.2.2). Next we consider the left shift mapping
\[(2.5.6) \ L_bu = bu \]
for each fixed \( b \in G \) and any \( u \in G \). This mapping \( L_b \) is continuous, since the multiplication on \( G \) is continuous. Then \( L_b(\theta_b(c)) = c \) and \( \theta_b(L_bu) = u \) for every fixed \( b \in G \) and all \( u \in G \) and \( c \in G \) by Identities (2.2.4). Therefore \( \theta_b \) and \( L_b \) are open mappings. Thus the subset \( bU \) is open in \( G \).

2.6. Lemma. Let \( (G, \tau) \) be a topological loop.

(i). Let also \( U \) and \( V \) be subsets in \( G \) such that either \( U \) or \( V \) is open, then \( UV \) is open in \( G \).

(ii). If \( A \) and \( B \) are compact subsets in \( G \), then \( AB \) is compact.

(iii). For each open neighborhood \( U \) of \( e \) in \( G \) there exists an open neighborhood \( V \) of \( e \) such that
\[(2.6.1) \ V \subseteq U, \text{ where} \]
\[(2.6.2) \  \tilde{V} = V \cup Inv_l(V) \cup Inv_r(V), \]
where \( Inv_l(a) = Div_l(a, e), \ Inv_r(a) = Div_r(a, e) \) for each \( a \in G \),
\[(2.6.3) \ DQ = \{x = ab : a \in D, b \in Q\}, \]
\[(2.6.4) \ Inv_l(D) = \{x = a \setminus e : a \in D\}, \]
\[(2.6.5) \ Inv_r(D) = \{x = e/a : a \in D\} \text{ for any subsets } D \text{ and } Q \text{ in } G. \]

Proof. (i). In view of Lemma 2.5 \( Ub \) and \( aV \) are open in \( G \) for each \( a \in U \) and \( b \in V \), consequently, \( UV = \{x : x = uv, u \in U, v \in V\} = \cup_{b \in V} Ub = \cup_{a \in U} aV \) is open in \( G \).

(ii). A subset \( AB = \{c : c = ab, a \in A, b \in B\} \) is a continuous image of a compact subset \( A \times B \) in \( G \times G \), where \( G \times G \) is supplied with the product (i.e. Tychonoff) topology, consequently, \( AB \) is a compact subset in \( G \) (see Theorem 3.1.10 and the Tychonoff Theorem 3.2.4 in [11]).

(iii). The mappings \( Inv_l \) and \( Inv_r \) are homeomorphisms of \( G \) onto itself as the topological space, since they are bijective, continuous and
\[(2.6.6) \ Inv_l(Inv_r(b)) = b \text{ and } Inv_r(Inv_l(b)) = b. \]
for each $b$ in $G$ by (2.2.4), (2.2.5). Therefore for each open neighborhood $U$ of $e$ there exists an open neighborhood of $e$ of the form

$$(2.6.7) \quad V := \hat{U}, \text{ where } \hat{U} := U \cap Inv_l(U) \cap Inv_r(U).$$

From (2.6.6) we infer that $Inv_r(Inv_l(U)) = U$ and $Inv_l(Inv_r(U)) = U$, hence $Inv_l(V) \subseteq U \cap Inv_l(U) \cap Inv_l(Inv_l(U)) \subseteq U \cap Inv_l(U)$ and $Inv_r(V) \subset U \cap Inv_r(U)$, consequently, $V \cup Inv_l(V) \cup Inv_r(V) \subseteq U$.

2.7. Definition. A subloop $H$ of a loop $G$ is called normal if it satisfies

$$(2.7.1) \quad xH = Hx \text{ and}$$

$$(2.7.2) \quad (xy)H = x(yH) \text{ and } (xH)y = x(Hy) \text{ and } H(xy) = (Hx)y$$

for each $x$ and $y$ in $G$.

A family of cosets $\{bH : b \in G\}$ will be denoted by $G/ \cdot /N_0$.

2.8. Theorem. If $G$ is a $T_1$ topological fan loop, then its fan $N_0$ is a normal subgroup and its quotient $G/ \cdot /N_0$ is a $T_1 \cap T_{3.5}$ topological group.

Proof. Let $\tau$ be a $T_1$ topology on $G$ relative to which $G$ is a topological loop. Then each point $x$ in $G$ is closed, since $G$ is the $T_1$ topological space (see Section 1.5 in [11]). From the joint continuity of the multiplication and the mappings $Div_l$ and $Div_r$ it follows that the nucleus $N = N(G)$ is closed in $G$. Therefore the subgroup $N_0$ is the closure of a subgroup $N_{0,0}(G)$ in $N$ generated by elements $t(a,b,c)$ and $p(a,b,c)$ for all $a$, $b$ and $c$ in $G$ (see Definition 2.1). According to (2.1.5)-(2.1.8) one gets that $N$ and hence $N_0$ are subgroups in $G$ satisfying Conditions (2.7.2), because $N_0 \subseteq N$ (see also [8, 34]).

Let $a$ and $b$ belong to $N$ and $x \in G$. Then $x(x \setminus (ab)) = ab$ and

$x((x \setminus a)b) = (x(x \setminus a))b = ab$, consequently,

$$(2.8.1) \quad x \setminus (ab) = (x \setminus a)b \text{ for each } a \text{ and } b \text{ in } N(G), x \in G.$$ 

Similarly it is deduced

$$(2.8.2) \quad (ab)/x = a(b/x) \text{ for each } a \text{ and } b \text{ in } N(G), x \in G.$$ 

Therefore from (2.1.9) and (2.2.4) and (2.8.1) it follows that

$$((x \setminus a)x)((x \setminus b)x) = (x \setminus a)(x((x \setminus b)x))p(x \setminus a, x, (x \setminus b)x)$$

$$= (x \setminus (ab))x[p(x, x \setminus b, x)]^{-1}p(x \setminus a, x, (x \setminus b)x),$$

since $(x \setminus a)(bx) = ((x \setminus a)b)x = (x \setminus (ab))x$. Thus

$$(2.8.3) \quad (x \setminus (ab))x = ((x \setminus a)x)((x \setminus b)x)[p(x \setminus a, x, (x \setminus b)x)]^{-1}p(x, x \setminus b, x)$$

for each $a$ and $b$ in $N(G), x \in G$.  

From Identities $(2.2.2')$ and $(2.2.2'')$ it follows that

$$(2.8.4)\ x \setminus ((u \setminus v)y) = ((ux) \setminus (vy))p(u, x, (ux) \setminus (vy))[p(u, u \setminus v, x)]^{-1}$$

for each $u, v, x$ and $y$ in $G$, since

$$x \setminus ((u \setminus v)y) = x \setminus (u \setminus (vy))[p(u, u \setminus v, y)]^{-1}.$$ 

In particular for $u = a(bc)$ and $v = (ab)c$ with any $a, b$ and $c$ in $G$ we infer using $(2.1.9)$ that $ux = (a(b(cx)))p(a, bc, x)$ and $vx = (ab)(cx)p(ab, c, x)$, hence from $(2.8.4)$ and $(2.3.7)$ it follows that

$$(2.8.5)\ x \setminus (p(a, b, c)x) = [p(b, c, x)p(a, bc, x)]^{-1}p(a, bc, x)p(u, x, (ux) \setminus (vx)),$$

since

$$x \setminus (p(a, b, c)x) = [(a(b(cx)))p(b, c, x)p(a, bc, x)] \setminus [(ab)(cx)p(ab, c, x)]$$

$$p(u, x, (ux) \setminus (vx))[p(u, u \setminus v, x)]^{-1},$$

because $u \setminus v = p(a, b, c) \in N(G)$ and $p(u, u \setminus v, x) = e$.

Notice that $(2.1.1)$, $(2.1.2)$ and $(2.1.9)$ imply $u \setminus (tu) = p$, where $t = t(a, b, c)$, $p = p(a, b, c)$, $u = a(bc)$ for any $a, b$ and $c$ in $G$. Let $z \in G$, then there exists $x \in G$ such that $z = ux$, that is $x = u \setminus z$. Therefore we deduce that

$$(2.8.6)\ z \setminus (tz) = [x \setminus (px)]p(u, u \setminus (tu), x)[p(u, u \setminus (ux) \setminus (txu))],$$

since $t \in N(G)$, $p \in N(G)$, $(u \setminus (tu))x = (u \setminus (txu))[p(u, u \setminus (tu), x)]^{-1}$ by $(2.2.2'')$: $x \setminus (u \setminus (txu)) = [(ux) \setminus (txu)]p(u, x, (ux) \setminus (txu))$ by $(2.2.2')$. Thus from Identities $(2.8.3)$, $(2.8.5)$ and $(2.8.6)$ it follows that a group $N_{0,0} = N_{0,0}(G)$ generated by $\{p(a, b, c), t(a, b, c) : a \in G, b \in G, c \in G\}$ satisfies Condition $(2.7.1)$. From the joint continuity of the multiplication and the mappings $Div_t$ and $Div_v$ it follows that the closure $N_0$ of $N_{0,0}$ also satisfies $(2.7.1)$. Thus $N_0$ is a closed normal subgroup in $G$. In view of Theorem 1.1 in Ch. IV, Section 1 in [8] a quotient loop $G/\cdot /N_0$ exists consisting of all cosets $aN_0$, where $a \in G$.

Then from Conditions $(2.1.9)$, $(2.7.1)$ and $(2.7.2)$ it follows that for each $a, b, c$ in $G$ the identities take place

$$(aN_0)(bN_0) = (ab)N_0$$

and

$$((an_0)(bn_0))(cn_0) = (an_0)((bn_0)(cn_0))$$

and $eN_0 = N_0$, since $p(a, b, c) \in N_0$ and $t(a, b, c) \in N_0$ for all $a, b$ and $c$ in $G$.

In view of Lemmas 2.2 and 2.3 $(aN_0) \setminus e = e/(aN_0)$, consequently, for each $aN_0 \in G/\cdot /N_0$ a unique inverse $(aN_0)^{-1}$ exists. Thus the quotient $G/\cdot /N_0$
of $G$ by $N_0$ is a group. Since the topology $\tau$ on $G$ is $T_1$ and $N_0$ is closed in $G$, then the quotient topology $\tau_q$ on $G/\cdot/N_0$ is also $T_1$. By virtue of Theorem 8.4 in [15] this implies that $\tau_q$ is a $T_1 \cap T_{3.5}$ topology on $G/\cdot/N_0$.

2.9. Proposition. Assume that $G$ is a $T_1$ topological fan loop and functions $t$ and $p$ on $G$ are defined by Formulas (2.1.9). Then for each compact subset $S$ in $G$ and each open neighborhood $V$ of $e$ there exists an open neighborhood $U$ of $e$ in $G$ such that

\[(2.9.1) \ t((u_1a)v_1,(u_2b)v_2,(u_3c)v_3) \in (Vt(a,b,c)) \cap (t(a,b,c)V) \text{ and} \]
\[(2.9.2) \ p((u_1a)v_1,(u_2b)v_2,(u_3c)v_3) \in (Vp(a,b,c)) \cap (p(a,b,c)V) \]

for every $a, b, c$ in $S$ and $u_j, v_j$ in $\bar{U}$ for each $j \in \{1,2,3\}$.

Proof. Take arbitrary fixed elements $f$, $g$ and $h$ in $S$. From the joint continuity of the maps $t(a,b,c)$ and $p(a,b,c)$ in the variables $a$, $b$ and $c$ in $G$ it follows that there exists an open neighborhood $U_{f,g,h}$ of $e$ in $G$ and an open neighborhood $W_{f,g,h}$ of $(f,g,h) \in S \times S \times S$ in $G \times G \times G$ such that (2.9.1) and (2.9.2) are valid for each $u_j, v_j$ in $\bar{U}_{f,g,h}$, $j \in \{1,2,3\}$, and $(a,b,c) \in W_{f,g,h}$ (see Lemmas 2.4 and 2.6). Notice that $S \times S \times S$ is compact in the Tychonoff product $G \times G \times G$ of $G$ as the topological space (see Section 2.3 and Theorem 3.2.4 in [11]). Hence an open covering $\{W_{f,g,h} : f \in S, \ g \in S, \ h \in S\}$ of $S \times S \times S$ has a finite subcovering $\{W_{f,g,h_i} : i = 1, \ldots, n\}$, where $n$ is a natural number, $n \geq 1$. That is $S \times S \times S \subseteq \bigcup_{i=1}^{n} W_{f,g,h_i}$. Then $\bigcap_{i=1}^{n} U_{f,g,h_i} =: U$ is an open neighborhood of $e$ in $G$. Therefore, Properties (2.9.1) and (2.9.2) are satisfied for every $a, b, c$ in $S$ and $u_j, v_j$ in $\bar{U}$ for each $j \in \{1,2,3\}$

We remind the following.

2.10. Definition. Let $G$ be a topological loop. For a subset $U$ in $G$ it is put:

\[(2.10.1) \ L_{U,G} := \{(x,y) \in G \times G : x \not\in U\} \text{ and} \]
\[(2.10.2) \ R_{U,G} := \{(x,y) \in G \times G : y/x \in U\}. \]

A family of all subsets $L_{U,G}$ (or $R_{U,G}$) with $U$ being an open neighborhood of $e$ will be denoted by $L_G$ (or $R_G$ correspondingly).

2.11. Proposition. Let $G$ be a $T_1$ topological locally compact fan loop. Then a family $L_G$ (or $R_G$) induces a uniform structure on $G$. A topology $\tau_1$ on $G$ provided by $L_G$ (or $R_G$ respectively) is $T_1 \cap T_{3.5}$ and equivalent to the initial topology $\tau$ on $G$. 

11
Proof. Let \((G, \tau)\) be a topological loop and let \(\mathcal{B}_e\) denote a base of its open neighborhoods at \(e\). In view of Lemma 2.5 \(\mathcal{C}_l(U) := \{xU : x \in G\}\) is an open covering of \(G\) for each \(U \in \mathcal{B}_e\). We put \(\mathcal{C}_l^0 = \{\mathcal{C}_l(U) : U \in \mathcal{B}_e\}\) and \(\mathcal{C}_l\) to be a family of all coverings for each of which there exists a refinement of the type \(\mathcal{C}_l^0\).

Below it is verified, that the family \(\mathcal{C}_l\) satisfies Conditions (UC1)-(UC4) of Section 8.1 in [11]. If \(A \in \mathcal{C}_l\), \(\mathcal{E}\) is a covering of \(G\) and \(A\) refines \(\mathcal{E}\), then there exists \(U \in \mathcal{B}_e\) such that \(\mathcal{C}_l(U)\) refines \(A\) and hence \(\mathcal{C}_l(U)\) refines \(\mathcal{E}\). Thus (UC1) is satisfied.

Let \(A_1\) and \(A_2\) belong to \(\mathcal{C}_l\). There are \(U_1\) and \(U_2\) in \(\mathcal{B}_e\) such that \(\mathcal{C}_l(U_j)\) refines \(A_j\) for each \(j \in \{1, 2\}\). We put \(U = U_1 \cap U_2\), consequently, \(U \in \mathcal{B}_e\) and hence \(\mathcal{C}_l(U)\) refines both \(\mathcal{C}_l(U_1)\) and \(\mathcal{C}_l(U_2)\). Therefore \(\mathcal{C}_l(U)\) refines \(A_1\) and \(A_2\). Thus (UC2) also is satisfied.

Condition (UC3) means that for each \(A \in \mathcal{C}_l\) there exists \(\mathcal{E} \in \mathcal{C}_l\) such that \(\mathcal{E}\) is a star refinement of \(A\). In order to prove it, it evidently is sufficient to prove that for each \(U \in \mathcal{B}_e\) there exists \(U_1 \in \mathcal{B}_e\) such that

\[
(2.11.1) \quad \text{St}(xU_1, \mathcal{C}_l(U_1)) \subseteq xU \quad \text{for each} \quad x \in G,
\]

where \(\text{St}(M, \mathcal{A})\) denotes a star of a set \(M\) with respect to \(\mathcal{A}\) (see Section 5.1 in [11]).

Note that a map \(f(x_1, x_2, x_3) = (x_1/x_2)x_3\) is the composition of jointly continuous maps \(G \times G \ni (x_1, x_2) \mapsto x_1/x_2 \in G\) and \(G \times G \ni (y, x_3) \mapsto yx_3 \in G\), hence it is jointly continuous from \(G \times G \times G\) into \(G\) and \(f(e, e, e) = e\), because \(G\) is the topological loop (see Definition 2.1). The loop \(G\) is locally compact. Notice that for each open neighborhood \(Q_1\) of \(e\) in \(G\) there exists an open neighborhood \(Q_2\) of \(e\) such that its closure \(\text{cl}_G(Q_2)\) is compact and \(\text{cl}_G(Q_2) \subseteq Q_1\) by the corresponding Theorem 3.3.2 in [11] for topological spaces. Hence for each open neighborhood \(W\) of \(e\) in \(G\) there exists an open neighborhood \(U_0\) of \(e\) in \(G\) with the compact closure \(\text{cl}_G(U_0)\) such that \(\text{cl}_G(U_0)\) is contained in \(W\) (see Lemma 2.6).

Therefore for each \(U \in \mathcal{B}_e\) there exists \(V_1 \in \mathcal{B}_e\) such that \(f(V_1, V_1, V_1) \subseteq U\) and \(\text{cl}_G(V_1)\) is compact. If for an arbitrary fixed element \(x \in G\) and some \(x_1 \in G\) the intersection \(xV_1 \cap x_1V_1 \neq \emptyset\) is non void, then there are \(h_0\) and \(h_1\) in \(V_1\) such that \(x_1 = (xh_0)/h_1\). On the other hand, \(x_1h \in x_1V_1\) for each
\[ h \in V_1 \text{ and for each } y \in x_1 V_1 \text{ there exists } h \in V_1 \text{ with } y = x_1 h, \text{ consequently,} \\
\]
\[ x_1 h = ((xh_0)/h_1)h \in ((xV_1)/V_1)V_1. \]

Using Identities (2.2.3) and (2.1.9) we get that
\[ (2.11.2) \quad x_1 h = (x(h_0(e/h_1))p(x, h_0, e/h_1) \]
\[ p(e/h_1, h_1, h_1 \setminus e)[p((xh_0)/h_1, h_1, h_1 \setminus e)^{-1}]h. \]

We choose open neighborhoods \( V \) and \( W \) of \( e \) in \( G \) such that \( V^2 \subset W \) and \( W^2 \subset V_1 \) by Lemma 2.6. In view of the inclusion (2.9.2) of Proposition 2.9 and Formula (2.11.2) there exists \( U_1 \in B_e \) such that \( U_1 \subset V \) and
\[ (2.11.3) \quad p((u_1 a)v_1, (u_2 b)v_2, (u_3 c)v_3) \in (Vp(a, b, c)) \cap (p(a, b, c)V) \]
for every \( a, b, c \) in \( cl_G(V_1) \) and \( u_j, v_j \) in \( U_1 \) for each \( j \in \{1, 2, 3\} \). This implies (2.11.1) and hence \( UC3 \), since \( p(a, b, c) = e \) if either \( a = e \) or \( b = e \) or \( c = e \).

It remains to prove that \( C_l \) also has the property \( UC4 \). That is for each \( x \neq y \) in \( G \) there exists \( \mathcal{A} \in C_l \) such that \( \{x, y\} \cap V \neq \{x, y\} \) for each \( V \in \mathcal{A} \). It is sufficient to find an open neighborhood \( U \) of \( e \) in \( G \) such that \( x/U \cap y/U = \emptyset \), because this implies \( x_0 U \cap \{x, y\} \neq \{x, y\} \) for each \( x_0 \in G \).

The loop \( G \) is \( T_1 \). By virtue of Lemmas 2.5 and 2.6 and the joint continuity of the multiplication and \( Div_r \) in \( G \) there is \( U_1 \in B_e \) such that \( y \notin (xU_1)/U_1 \), that is \( xU_1 \cap yU_1 = \emptyset \) by (2.2.5). In view of Proposition 2.9 there exists \( U \in B_e \) such that \( (e/U)p(e/U, U, U \setminus e)[p(a/U, U, U \setminus e)^{-1} \subset U_1 \) for each \( a \in \{x, y\} \), since the two-point set \( \{x, y\} \) is compact in \( G \), for each \( W \in B_e \) there exists \( W_1 \in B_e \) such that \( e/W_1 \subset W \). From (2.2.3) it follows that \( x/U \cap y/U = \emptyset \). Therefore \( \{x, y\} \cap V \neq \{x, y\} \) for each \( V \in C_l(U) \).

By virtue of Theorem 8.1.1 in [11] the uniformity \( C_l \) induces a \( T_1 \) topology \( \tau_1 \) on \( G \). Note that the family \( C_l \) consists of open coverings of \( G \) and that for each \( x \in G \) and each open neighborhood \( V \) of \( x \) in the initial topology \( \tau \) there exists \( U \in B_e \) such that \( xU \subset V \). Therefore from the latter inclusion and (2.11.1) it follows that the topology \( \tau_1 \) induced by \( C_l \) coincides with the initial topology \( \tau \) on \( G \). In view of Corollary 8.1.13 in [11] \( (G, \tau) \) is a Tychonoff space, that is \( (G, \tau) \) is a completely regular space, \( T_1 \cap T_{3.5} \). Finally note that \( C_l^0 = L_G \). Symmetrically the case \( C_l^0 = \mathcal{R}_G \) is proved.

2.12. Lemma. Suppose that \( (G, \tau) \) is a \( T_1 \) topological loop, \( S \) is a compact subset in \( G \), \( q \) is a fixed element in \( G \), \( V \) is an open neighborhood of the unit element \( e \). Then there are elements \( b_1, ..., b_m \) in \( G \) and an open
neighborhood $U$ of $e$ such that

(2.12.1) $\bar{U} \subset V$ and

(2.12.2) $\{b_1 \setminus (qU), ..., b_m \setminus (qU)\}$ is an open covering of $S$.

**Proof.** The multiplication is continuous on $G$, hence the left shift mapping $L_b(x) = bx$ is continuous on $G$ in the variable $x$. On the other hand, the mapping $Inv_l$ is continuous on $G$.

In view of (2.1.1), (2.1.2), Lemmas 2.5 and 2.6 and the compactness of $S$ for each open neighborhood $U$ of $e$ in $G$ with $\bar{U} \subset V$ there are $b_1, ..., b_m$ in $G$ such that $\{b_1 \setminus (qU), ..., b_m \setminus (qU)\}$ is an open covering of $S$.

**2.13. Corollary.** Let $G$ be a $T_1$ topological loop. Then for each open neighborhood $W$ of $e$ in $G$ there exists an open neighborhood $U$ of $e$ such that $\bar{U} \subset W$ and

(2.13.1) $(\forall x)(\forall y)((x \in G) \& (y \in G) \& (x \setminus y \in U)) \Rightarrow (y \in xW)$ and

(2.13.2) $(\forall x)(\forall y)((x \in G) \& (y \in G) \& (y/x \in U)) \Rightarrow (y \in Wx)$.

**Proof.** This follows from Lemmas 2.6 and 2.12, (2.1.1) and (2.1.2).

**2.14. Theorem.** Let $G$ and $H$ be $T_1$ topological fan loops (see Definition 2.1) and let $f : G \to H$ be a continuous map so that for each open neighborhood $V$ of a unit element $e_H$ in $H$ a compact subset $K_V$ in $G$ exists such that $f(G - K_V) \subset V$. Then $f$ is uniformly $(\mathcal{L}_G, \mathcal{L}_H)$ continuous and uniformly $(\mathcal{R}_G, \mathcal{R}_H)$ continuous (see also Definition 2.10).

**Proof.** Since the multiplication in $H$ is continuous, then for each open neighborhood $Y$ of $e_H$ there exists an open neighborhood $X$ of $e_H$ such that $X^2 \subset Y$. In view of Lemma 2.6 there exists an open neighborhood $V_1$ of $e_H$ in $H$ such that $\bar{V}_1^2 \subset V$, where $A^2 = AA$ for a subset $A$ in $H$. By the conditions of this theorem a compact subset $K_{V_1}$ in $G$ exists such that $f(G - K_{V_1}) \subset V_1$.

For a subset $A$ of the loop $G$ let

(2.14.1) $P(A) = (P_0(A) \cup \{e\})(P_0(A) \cup \{e\})$,

where $P_0(A) = A \cup Inv_l(A) \cup Inv_r(A)$,

hence $A \subset P_0(A)$ and $P_0(A) \cup \{e\} \subset P(A)$. Then $S_1 = P(K_{V_1})$ is a compact subset in $G$, since the mappings $Inv_l$ and $Inv_r$ are continuous on $G$ and the multiplication is jointly continuous on $G \times G$ (see Theorems 3.1.10, 8.3.13-8.3.15 in [Π]), hence $R_1 = P(f(S_1))$ is compact in $H$.

By virtue of Proposition 2.9 there exists an open neighborhood $V'_2$ of $e_H$
in $H$ such that

\[ (2.14.2) \quad [t_H((V_2a)V_2, (V_2b)V_2, (V_2c)V_2)] \cap [V_2t_H((V_2a)V_2, (V_2b)V_2, (V_2c)V_2)] \]
\[ \subset (V_3t_H(a, b, c)) \cap (t_H(a, b, c)V_3) \]
\[ \text{and} \]
\[ [p_H((V_2a)V_2, (V_2b)V_2, (V_2c)V_2)] \cap [V_2p_H((V_2a)V_2, (V_2b)V_2, (V_2c)V_2)] \]
\[ \subset (V_3p_H(a, b, c)) \cap (p_H(a, b, c)V_3) \]

for every $a, b, c$ in $R_1$, where $V_3^2 \subset V_1$, $V_2 = \tilde{V}_2$, $V_3$ is an open neighborhood of $e$ in $H$. For $V_2$ there exists a compact subset $K_{V_2}$ in $G$ such that $f(G - K_{V_2}) \subset V_2$ by the conditions of this theorem. If $A$ and $B$ are compact subsets in $G$, then their union $A \cup B$ is also compact. Therefore it is possible to choose $K_{V_2}$ such that $K_{V_1} \subset K_{V_2}$, since $V_2 \subset V_1$ and $(G - A) - B = G - (A \cup B) \subset G - A$. We take $S_2 = P(K_{V_2})$ by Formula (2.14.1), consequently, $S_1 \subset S_2$, since $K_{V_1} \subset K_{V_2}$.

From the continuity of the map $f$ and Lemmas 2.5, 2.6 it follows that for each $x \in G$ open neighborhoods $W_{x,l}$ and $W_{x,r}$ of $e$ in $G$ exist such that

\[ f(xW_{x,l}^2) \subset (f(x)V_2) \]
\[ \text{and} \]
\[ (2.14.3) \quad f(xW_{x,r}^2) \subset (f(x)V_2) \]

for an open neighborhood $W_x = W_{x,l} \cap W_{x,r}$ of $e$ in $G$. The compactness of $S_2$ imply that coverings \{ $xW_x : x \in S_2$ \} and \{ $W_{y,y} : y \in S_2$ \} of $S_2$ have finite subcoverings \{ $x_jW_{x_j} : x_j \in S_2, j = 1, ..., n$ \} and \{ $W_{y_i}y_i : y_i \in S_2, i = 1, ..., m$ \}. Hence

\[ (2.14.4) \quad W = \cap_{i=1}^n W_{x_j} \cap \cap_{i=1}^m W_{y_i} \]

is an open neighborhood of $e$ in $G$. Therefore according to Proposition 2.9 there exists an open neighborhood $U'$ of the unit element $e$ in $G$ such that

\[ (2.14.5) \quad [t_G((Ua)U, (Ub)U, (Uc)U)] \cap [t_G((Ua)U, (Ub)U, (Uc)U)] \]
\[ \subset [W_3t_G(a, b, c)] \cap [t_G(a, b, c)W_3] \]
\[ \text{and} \]
\[ [p_G((Ua)U, (Ub)U, (Uc)U)] \cap [p_G((Ua)U, (Ub)U, (Uc)U)] \]
\[ \subset [W_3p_G(a, b, c)] \cap [p_G(a, b, c)W_3] \]

for every $a, b, c$ in $S_2$, where $U = U'$, $W_0$ and $W_3$ are open neighborhoods of $e$ in $G$ such that $W_0^2 \subset W_0$ and $W_3^2 \subset W$.

Let now $x$ and $y$ in $G$ be such that $x \setminus y \in U$. Then Formula (2.2.4) imply that

\[ (2.14.6) \quad y \in xU. \]

There are several options. Consider at first the case $x \in K_{V_2}$. From
Formulas (2.14.4)-(2.14.6) and Corollary (2.13) it follows that there exists 
\( j \in \{1, ..., n\} \) such that \( x \in x_j W_{x_j} \) and \( y \in x_j W^2_{x_j} \). Therefore, Formulas 
(2.14.2) and (2.14.3) imply that \( f(x) \setminus f(y) \in V \).

From \( x \setminus y \in U \) and Identities (2.2.4) it follows that \( y = xu \) for a unique 
\( u \in U \). Hence 
\[
(2.14.7) \quad x = [t(y, e/u, u)]^{-1} y(e/u)
\]
according to Identities (2.2.3), (2.2.5).

If \( y \in K_{V_2} \), then similarly from Formulas (2.14.4)-(2.14.7) and Corollary 
(2.13) it follows that there exists \( k \in \{1, ..., n\} \) such that \( y \in x_k W_{x_k} \) and 
\( x \in x_k W^2_{x_k} \), since \( t(a, b, e) = t(a, e, b) = t(e, a, b) = e \) for each \( a \) and \( b \) in \( G \). Therefore, \( f(x) \setminus f(y) \in V \) by Formulas (2.14.2) and (2.14.3), since 
\( S_2 = P(K_{V_2}) \) (see Formula (2.14.1)).

It remains the case \( x \in G - K_{V_2} \) and \( y \in G - K_{V_2} \). Therefore \( f(x) \in V_2 \) 
and \( f(y) \in V_2 \). According to the choice of \( R_1 \) we have \( e_H \in R_1 \). From 
Condition (2.14.2), Identity (2.2.4) and the inclusion \( V^2_1 \subset V \), it follows that 
\( f(x) \setminus f(y) \in V \). Taking into account the inclusion \( K_{V_1} \subset K_{V_2} \) we get that \( f \) 
is uniformly \((\mathcal{L}_G, \mathcal{L}_H)\) continuous.

The uniform \((\mathcal{R}_G, \mathcal{R}_H)\) continuity is proved analogously using the finite 
subcovering \( \{W_{y_i} y_i : y_i \in S_2, i = 1, ..., m\} \) and Corollary 2.13.

2.15. Corollary. Let \( G \) be a \( T_1 \) topological locally compact fan loop 
and let \( f \in C_0(G) \) and let \( H = (\mathbb{C}, +) \) be the complex field \( \mathbb{C} \) considered as 
an additive group. Then \( f \) is uniformly \((\mathcal{L}_G, \mathcal{L}_H)\) continuous and uniformly 
\((\mathcal{R}_G, \mathcal{R}_H)\) continuous.

3 Left invariant measures.

3.1. Notation. For a completely regular topological space \( X \) by \( C_b(X) \) is 
denoted the Banach space of all continuous bounded functions \( f \) from \( X \) into 
the complex field \( \mathbb{C} \) supplied with the norm 
\[
(3.1.1) \quad \|f\|_X = \sup_{x \in X} |f(x)| < \infty.
\]
We put 
\[
(3.1.2) \quad C_0(X) := \{f \in C_b(X) : \forall \epsilon > 0, \exists S \subset X, S \text{ is compact,} 
\forall x \in X - S, |f(x)| < \epsilon \} \text{ and}
\]
Then there exists an open neighborhood of the unit element $e$.

(3.1.4) $C_{0,0}(X) = \{ f \in C_b(X) : \forall x \in X, f(x) \geq 0 \}$.

Let $G$ be a loop. For a function $f : G \to \mathbb{C}$ and an element $b \in G$ let $L_b f(x) = b f(x) = f(bx)$ and $R_b f(x) = f(bx)$ for each $x \in G$.

Consider a support $S_f := cl_G \{ x \in G : f(x) \neq 0 \}$ of $f \in C_b(G)$, where $cl_G(A)$ denotes the closure of a subset $A$ in $G$.

3.2. Lemma. Let $(G, \tau)$ be a $T_1$ topological locally compact fan loop, let also $f$ and $\phi$ belong to $C^+_0(G)$ and $\phi$ be not identically zero (see Notation 3.1). Then there exist a natural number $m > 0$, elements $b_1, ..., b_m$ in $G$ and positive constants $c_1, ..., c_m$ such that

(3.2.1) $\forall x \in G, \ f(x) \leq \sum_{j=1}^m c_j L_{b_j} \phi(x)$.

Proof. Since $f \in C^+_0(G)$, then the support $S_f$ is compact. The function $\phi$ is not null, hence there exists $q \in G$ such that $\phi(q) > 0$. From Lemma 2.5 and from the continuity of the function $\phi$ it follows that there exists an open neighborhood $qV$ of $q$ such that $\phi(x) > \phi(q)/2$ for each $x \in qV$, where $V$ is an open neighborhood of the unit element $e$. By virtue of Lemma 2.12 there exists an open neighborhood $U$ of $e$ and elements $b_1, ..., b_m$ in $G$ such that $U \subset V$ and for each $x \in S_f$ there exists $j \in \{1, ..., m\}$ such that $x \in b_j \setminus (qU)$.

Therefore,

$$f(x) \leq \|f\|_G (2/\phi(q)) \sum_{j=1}^m \phi(b_j x)$$

for each $x \in G$ according to (2.2.4), so it is sufficient to take $c_j \geq \|f\|_G (2/\phi(q))$ for each $j = 1, ..., m$.

3.3. Corollary. Let the conditions of Lemma 3.2 be satisfied and let

(3.3.1) $(f : \phi) := \inf \{ \sum_{j=1}^m c_j : \exists \{b_1, ..., b_m\} \subset G, \ \exists \{c_1, ..., c_m\} \subset (0, \infty), \ \forall x \in G, \ f(x) \leq \sum_{j=1}^m c_j L_{b_j} \phi(x) \}$.

Then $(f : \phi) \leq 2m \|f\|_G / \phi(q)$ in the notation of Lemma 3.2.

3.4. Lemma. Assume that the conditions of Lemma 3.2 are fulfilled, then for each $b \in G$

(3.4.1) $(bf : \phi) = (f : \phi^b)$.
(3.4.2) \((f : b \phi) = (f^b : \phi)\),

where \(f^b(x) = f(b \setminus x)\) for each \(x \in G\); particularly,

(3.4.1') \((\gamma f : \phi) = (f : \phi)\) and

(3.4.2') \((f : \gamma \phi) = (f : \phi)\) for each \(\gamma \in \mathcal{N}(G)\);

(3.4.3) \((\alpha f : \phi) = \alpha(f : \phi)\) for each \(\alpha \geq 0\);

(3.4.4) \(((f_1 + f_2) : \phi) \leq (f_1 : \phi) + (f_2 : \phi)\) for every \(f_1\) and \(f_2\) in \(\mathcal{C}_+^{0,0}(G)\).

(3.4.5) If \((f : \phi) \leq f_1(x)\) for each \(x \in G\), then \((f : \phi) \leq (f_1 : \phi)\).

**Proof.** Let \(c_1, \ldots, c_m\) in \((0, \infty)\) and \(b_1, \ldots, b_m\) in \(G\) be such that

\[
(3.4.6) \quad b f(x) \leq \sum_{j=1}^m c_j L_{b_j} \phi(x)
\]

for each \(x \in G\). From Formulas (2.2.4) and (3.4.6) by changing of a variable \(y = bx\) it follows that

\[
(3.4.7) \quad f(y) \leq \sum_{j=1}^m c_j L_{b_j} \phi(b \setminus y)
\]

for each \(y \in G\). From (3.4.7) it follows (3.4.1). Similarly from the inequality

\[
(3.4.8) \quad f(x) \leq \sum_{j=1}^m c_j L_{b_j} (L_{b} \phi(x))
\]

for each \(x \in G\) we infer that

\[
(3.4.9) \quad f(b \setminus y) \leq \sum_{j=1}^m c_j L_{b_j} \phi(y)
\]

for each \(y \in G\). Thus (3.4.9) implies Equality (3.4.2).

In particular, if \(\gamma \in \mathcal{N}(G)\), then \(b_j(\gamma \setminus y) = (b_j \gamma \gamma^{-1} y)\) and \(b_j(\gamma y) = (b_j \gamma) y\) for each \(y\) and \(b_j\) in \(G\) by Condition (2.1.8) and Formulas (2.2.2) and (2.3.1). Hence (3.4.7) transforms into to

\[
f(y) \leq \sum_{j=1}^m c_j L_{b_j} \gamma^{-1} \phi(y)
\]

and (3.4.8) into

\[
f(x) \leq \sum_{j=1}^m c_j L_{b_j} \gamma \phi(x)
\]

with \(\gamma \in \mathcal{N}(G)\) instead of \(b\). This implies Equalities (3.4.1'), (3.4.2').

Properties (3.4.3) and (3.4.4) evidently follow from Formula (3.3.1).

For proving Property (3.4.5) note that if \(f(x) \leq f_1(x)\) for each \(x \in G\), then from \(f_1(x) \leq \sum_{j=1}^m c_j L_{b_j} \phi(x)\) for each \(x \in G\) it follows that \(f(x) \leq \sum_{j=1}^m c_j L_{b_j} \phi(x)\) for each \(x \in G\). Consequently, \((f : \phi) \leq (f_1 : \phi)\).
3.5. Notation. Let $\phi$, $f_0$ and $f$ belong to $C^+_0(G)$ and $\phi$ and $f_0$ be not null, where $G$ is a $T_1$ topological locally compact fan loop. We consider a functional

\begin{equation}
J_{\phi,f_0}(f) := \frac{(f : \phi)}{(f_0 : \phi)}.
\end{equation}

Assume that

\begin{equation}
t(a,b,c) \in N_0 = N_0(G) \text{ in } N(G) \text{ such that } t(a,b,c) \in N_0 \text{ and } p(a,b,c) \in N_0 \text{ for every } a, b \text{ and } c \text{ in } G.
\end{equation}

Then we denote by $\Upsilon(G,N_0)$ a family of all non null functions $h$ in $C^+_0(G)$ such that

\begin{equation}
h(\gamma a) = h(a) \text{ for each } a \in G \text{ and } \gamma \in N_0.
\end{equation}

Evidently Condition (3.5.3) for $h \in C^+_0(G)$ is equivalent to

\begin{equation}
h(a\gamma) = h(a) \text{ for each } a \in G \text{ and } \gamma \in N_0,
\end{equation}

since $aN_0 = N_0a$ for each $a \in G$ according to Theorem 2.8.

3.6. Lemma. Let $G$ be a $T_1$ topological locally compact fan loop satisfying Condition (3.5.2), $f$ and $\phi$ be in $C^+_0(G)$ and $\omega \in \Upsilon(G,N_0)$ (see Condition (3.5.3)), $\phi$ be non null. Then

\begin{equation}
(f : \phi) \leq (f : \omega)(\omega : \phi).
\end{equation}

Proof. If $b$ is a fixed element in $G$ and there are elements $b_1, \ldots, b_m$ in $G$ and positive constants $c_1, \ldots, c_m$ such that

\begin{equation}
b \omega(x) \leq \sum_{j=1}^{m} c_j \phi(b_jx)
\end{equation}

for each $x \in G$, then

\begin{equation}
b \omega(x) \leq \sum_{j=1}^{m} c_j \phi(b_jx\gamma)
\end{equation}

for each $x \in G$ and $\gamma \in N_0$, since $N_0 \subset N(G)$ and $b \omega(x\gamma) = b \omega(x)$ for each $x \in G$ and $\gamma \in N_0$ by (3.5.4) equivalent to (3.5.3).

By the conditions of this lemma $N_0$ is a compact group. Therefore there exists a Haar measure $\lambda$ on the Borel $\sigma$-algebra $\mathcal{B}(N_0)$ of $N_0$ and with values in the unit segment $[0, 1]$ such that $\lambda(N_0) = 1$, $\lambda(sA) = \lambda(A)$ and $\lambda(As) = \lambda(A)$ for each $s \in N_0$ and $A \in \mathcal{B}(N_0)$ (see Theorems 15.5, 15.9 and 15.13 and Subsection 15.8 in [L1]). In view of this, Conditions (3.1.3) and (3.1.4) and Corollary 2.15 a function

\begin{equation}
\phi^{[\lambda]}(x) := \int_{N_0} \phi(\gamma x) \lambda(d\gamma)
\end{equation}
on $G$ is nonzero and belongs to $C_{00}^+(G)$, since $N_0S_\phi$ is a compact subset in $G$ by Lemma 2.6, where $S_\phi$ is a compact support of $\phi$. From Formula (3.6.4) it follows that

\[(3.6.5) \; \phi^{[\lambda]}(\beta x) = \phi^{[\lambda]}(x) \text{ for each } \beta \in N_0 \text{ and } x \in G,\]

since the measure $\lambda$ is left and right invariant $\lambda(\beta A) = \lambda(A) = \lambda(A\beta)$ for each $\beta \in N_0$ and each Borel subset $A$ in $N_0$. Hence $\phi^{[\lambda]} \in \Upsilon(G, N_0)$, since $S_\phi N_0$ is compact, and since Conditions (3.5.3) and (3.5.4) are equivalent, where $S_\phi$ is the support of $\phi$ (see Subsection 3.2). From (3.6.4), (3.6.5), (3.5.3), (3.5.4) and the Fubini theorem it follows that

\[(3.6.4') \; \phi^{[\lambda]}(x) = \int_{N_0} \phi(x\beta)\lambda(d\beta), \text{ since}\]

\[
\phi^{[\lambda]}(x) = \int_{N_0} \left( \int_{N_0} \phi(\gamma x\beta)\lambda(d\gamma) \right)\lambda(d\beta) \\
= \int_{N_0} \left( \int_{N_0} \phi(\gamma x\beta)\lambda(d\beta) \right)\lambda(d\gamma) = \int_{N_0} \phi(x\beta)\lambda(d\beta),
\]

because $\int_{N_0} \phi(x\gamma\beta)\lambda(d\beta) = \int_{N_0} \phi(x\beta)\lambda(d\beta)$ for each $\gamma \in N_0(G)$.

Integrating both sides of Inequality (3.6.3) and utilizing Formulas (3.6.4), (3.6.4') we infer that

\[(3.6.6) \quad \omega(x) \leq \sum_{j=1}^{m} c_j \phi^{[\lambda]}(b_j x)\]

for each $x \in G$. On the other hand,

\[
\int_{N_0} \left( \sum_{j=1}^{m} c_j b_j \phi \right)(x\gamma)\lambda(d\gamma) = \left( \sum_{j=1}^{m} c_j b_j \phi \right)^{[\lambda]}(x) = \sum_{j=1}^{m} c_j b_j \phi^{[\lambda]}(x),
\]

hence for each $x \in G$ there exists $\gamma \in N_0$ such that

\[
\left( \sum_{j=1}^{m} c_j b_j \phi \right)(x\gamma) \geq \sum_{j=1}^{m} c_j b_j \phi^{[\lambda]}(x).
\]

Thus vice versa from $\omega \in \Upsilon(G, N_0)$ and (3.6.6) it follows (3.6.3) and hence (3.6.2), consequently,

\[(3.6.7) \quad (\omega : \phi^{[\lambda]}) = (\omega : \phi).
\]

Let $a_1, ..., a_n$ in $G$ and positive constants $q_1, ..., q_n$ be such that

\[(3.6.8) \quad \omega(x) \leq \sum_{j=1}^{n} q_j \phi^{[\lambda]}(a_j x)\]

20
for each \( x \in G \) (see Lemma 3.2). From Formulas \((2.2)\), \((3.6.5)\), \((3.6.8)\) and Conditions \((3.5.2)\), \((3.5.3)\), \((3.5.4)\) we deduce that

\[
(3.6.9) \quad \omega(y) \leq \sum_{j=1}^{n} q_j \phi^{[\lambda]}((a_j(b \setminus e))y[p(a_j, b \setminus e, y)]^{-1}p(b, b \setminus e, y))
\]

\[
= \sum_{j=1}^{n} q_j \phi^{[\lambda]}(d_jy)
\]

for each \( y \in G \), where \( d_j = a_j(b \setminus e) \) for each \( j \). Therefore

\[
(b_\omega : \phi^{[\lambda]}) \leq (\omega : \phi^{[\lambda]}) \text{ for each } b \in G. \text{ Notice that}
\]

\[
L_c L_{c \setminus e} \omega(x) = \omega(x) \text{ for each } c \text{ and } x \in G \text{ by Lemmas 2.2, 2.3 and Condition (3.5.3)}. \text{ Therefore we analogously get}
\]

\[
(\omega : \phi^{[\lambda]}) \leq (\omega : \phi^{[\lambda]}) \text{ for each } c \in G. \text{ Thus}
\]

\[
(3.6.10) \quad (b_\omega : \phi^{[\lambda]}) = (\omega : \phi^{[\lambda]}) \text{ for each } b \in G.
\]

From \((3.6.7)\) and \((3.6.11)\) it follows that

\[
(3.6.12) \quad (b_\omega : \phi) = (\omega : \phi) \text{ for each } b \in G.
\]

If \( c_1, ..., c_n, h_1, ..., h_k \in (0, \infty) \) and \( a_1, ..., a_k, g_1, ..., g_n \in G \) are such that

\[
(3.6.13) \quad f(x) \leq \sum_{j=1}^{k} h_j L_{a_j} \omega(x) \text{ and}
\]

\[
(3.6.14) \quad \omega(x) \leq \sum_{i=1}^{n} c_i L_{g_i} \phi(x)
\]

for each \( x \in G \) (see Lemma 3.2). Then from \((3.5.3)\), \((3.6.7)\), \((3.6.12)-(3.6.14)\) and Lemma 2.2 we infer that

\[
(3.6.15) \quad f(x) \leq \sum_{j=1}^{k} h_j \sum_{i=1}^{n} c_i L_{g_i} L_{a_j} \phi(x) = \sum_{j=1}^{k} h_j \sum_{i=1}^{n} c_i \phi((g_i a_j)x).
\]

Apparently \((3.6.15)\) implies \((3.6.1)\).

**3.7. Lemma.** Let \( G \) be a \( T_1 \) topological locally compact fan loop, \( \phi, f_0 \) be nonzero functions belonging to \( C^+_0(G) \). Then for each functions \( f, f_1 \) in \( C^+_0(G) \) and \( \alpha \geq 0 \)

\[
(3.7.1) \quad J_{\phi, f_0}(\alpha f) = \alpha J_{\phi, f_0}(f) \text{ and}
\]

\[
(3.7.2) \quad J_{\phi, f_0}(f + f_1) \leq J_{\phi, f_0}(f) + J_{\phi, f_0}(f_1) \text{ and}
\]

\[
(3.7.3) \quad \text{if } f(x) \leq f_1(x) \text{ for each } x \in G, \text{ then } J_{\phi, f_0}(f) \leq J_{\phi, f_0}(f_1).
\]

Moreover, if \( G \) satisfies Condition \((3.5.2)\) and \( f_0 \in Y(G, N_0) \) (see Condition \((3.5.3)\)), then

\[
(3.7.4) \quad J_{\phi, f_0}(f) \leq (f : f_0).
\]
Proof. Properties (3.7.1) and (3.7.2) follow immediately from (3.4.3) and (3.4.4). Property (3.7.3) follows from Property (3.4.5).

Applying Inequality (3.6.1) and Formula (3.5.1) we infer Inequality (3.7.4), since \( J_{\phi,f_0}(f_0) = 1 \).

3.8. Lemma. Assume that \( G \) is a \( T_1 \) topological locally compact fan loop, functions \( \phi, f_0 \) and \( f \) belong to \( C^0_0(G) \) and \( \phi \) and \( f_0 \) are not null. Then mappings \( J_{\phi,f_0}(bf) \) and \( J_{\phi,f_0}(f_0) \) are continuous in the variable \( b \) in \( G \).

Proof. For each \( x, b_1 \) and \( b_2 \) in \( G \) we have \( b_1 f(x) - b_2 f(x) = f(b_1 x) - f(b_2 x) \). In view of Corollary 2.15 for each \( \epsilon > 0 \) there exists an open of the form (2.6.1) neighborhood \( U \) of \( e \) in \( G \) with a compact closure \( cl_G(U) \) for which

\[
(3.8.1) \quad |f(b_1 x) - f(b_2 x)| < \epsilon \quad \text{for each} \quad x, \ b_1 \ \text{and} \ b_2 \ \text{in} \ G \quad \text{such that} \quad (b_2 x) \setminus (b_1 x) \in U.
\]

On the other hand, a support \( S_f \) of \( f \) is compact, consequently, \( bS_f = L_b S_f \) is compact for each \( b \in G \). Let \( b_1 \) be fixed. For each \( x \in G \) with \( b_1 x \in S_f \) there exists an open of the form (2.6.1) neighborhood \( W_x \) of \( e \) in \( G \) such that \( (b_2 x) \setminus (b_1 x) \in U \) for each \( b_2 x \in (b_1 W_x) x \cap b_1 (x W_x) \) according to Lemmas 2.2, 2.4, 2.5, Proposition 2.9 and Formula (2.14.3). For an open covering \( \{(b_1 W_x) x \cap b_1 (x W_x) : b_1 x \in S_f, \ x \in G\} \) of \( S_f \) there exists a finite subcovering \( \{(b_1 W_{x_j}) x_j \cap b_1 (x_j W_{x_j}) : b_1 x_j \in S_f, \ x_j \in G, \ j = 1, ..., m\} \) (see also Lemma 2.5), since the subset \( S_f \) is compact.

We take \( W_0 = U \cap \bigcap_{j=1}^m W_{x_j} \) and choose an open of the form (2.6.1) neighborhood \( W \) of \( e \) in \( G \) with a compact closure \( cl_G(W) \) contained in \( W_0 \) (see Theorem 3.3.2 in [11] and Formula (2.14.3)), because \( G \) is locally compact.

In view of Proposition 2.9 and Lemma 2.6 there exists an open neighborhood \( V' \) of \( e \) in \( G \) with \( V = V' \) and a compact closure \( cl_G(V) \) such that

\[
(3.8.2) \quad [t((Va)V, (Vb)V, (Vc)V)V] \cup [t((Va)V, (Vb)V, (Vc)V)]
\]
\[
\subset [t(a, b, c)W_1] \cap [W_1 t(a, b, c)] \quad \text{and}
\]
\[
[p((Va)V, (Vb)V, (Vc)V)V] \cup [p((Va)V, (Vb)V, (Vc)V)]
\]
\[
\subset [p(a, b, c)W_1] \cap [W_1 p(a, b, c)]
\]

for each \( a, b \) and \( c \) in \( S \), where \( W_1^2 \subset W \), \( W_1 \) is an open neighborhood of \( e \) in \( G \),

22
\[ S = P(S_1), \quad S_1 = S_2 \cup \text{cl}_G(U), \]
\[ S_2 = \{ y \in G : \ y = (b_1 u)x, \ u \in \text{cl}_G(U), b_1 x \in S_f \} \]
(see Formula (2.14.1)), since \( S \) is compact, \( t(a, b, c) = e \) and \( p(a, b, c) = e \)
if \( e \in \{ a, b, c \} \). For \( b_1 x \notin S_f \) and \( b_2 x \notin S_f \) certainly \( f(b_1 x) - f(b_2 x) = 0 \).
So remain two cases either \( b_1 x \in S_f \) or \( b_2 x \in S_f \) which are similar to each other up to a notation. From Formulas (2.2.5) it follows that \( b_2 x \in (b_1 V)x \) is
equivalent to \( b_2 \in b_1 V \). Hence Lemma 2.2 and Inclusion (3.8.2) provide that
\((b_2 x) \setminus (b_1 x) \in U \) for each \( b_2 \in b_1 V \) and \( b_1 x \in S_f \).

Let \( w \in C^+_0(G) \) be a function such that \( w(y) = 1 \) for each \( y \in (\text{cl}_G(U)S_f)\text{cl}_G(U) \).
Then we deduce that \( |f(b_1 x) - f(b_2 x)| < \epsilon w(x) \) for each \( x, b_1 \) and \( b_2 \) in \( G \)
such that \( b_2 \in b_1 V \) and with \( b_1 x \in S_f \).

Therefore for each \( \epsilon > 0 \) there exists and open neighborhood \( V \) of \( e \) in \( G \)
such that \( |(b_1 f : \phi) - (b_2 f : \phi)| < \epsilon(w : \phi) \) for each \( b_2 \in b_1 V \),
consequently,
\[ (3.8.3) \ |J_{\phi,f_0}(b_1 f) - J_{\phi,f_0}(b_2 f)| < \epsilon J_{\phi,f_0}(w) \]
according to Formula (3.5.1), since \((f_0 : \phi) > 0 \). Thus the mapping \( J_{\phi,f_0}(b f) \)
is continuous in the parameter \( b \in G \), since \( 0 < J_{\phi,f_0}(w) < \infty \) (see Lemmas 3.2, 3.7 and Corollary 3.3).

The case \( J_{\phi,f_0}(f b) \) is proved symmetrically.

3.9. Theorem. Assume that \( G \) is a \( T_1 \) topological locally compact fan
loop satisfying Condition (3.5.2), \( \phi, f \) and \( f_1 \) are nonzero functions belonging
to \( C^+_0(G) \) and \( f_0 \in \Upsilon(G, N_0) \) (see (3.5.3)), then
\[ (3.9.1) \quad (f_0 : f)^{-1} \leq J_{\phi,f_0}(f) \leq (f : f_0) \quad \text{and} \]
\[ (3.9.2) \quad (f_1 : f_0)^{-1}(f_0 : f)^{-1} \leq J_{\phi,f_1}(f) \leq (f : f_0)(f_0 : f_1). \]

Proof. The right inequality in (3.9.1) follows from the inequality (3.7.4).
Formulas (3.6.7) and (3.6.12) imply that
\[ (3.9.3) \quad (b f_0 : f) = (f_0 : f[\lambda]) \quad \text{and} \quad (b f[\lambda] : \phi) = (f[\lambda] : \phi[\lambda]) \]
for each \( b \in G \).
Let \( c_1, \ldots, c_k, h_1, \ldots, h_n \) in \( (0, \infty) \) and \( a_1, \ldots, a_k, g_1, \ldots, g_n \) in \( G \) be such that
\[ (3.9.4) \quad f_0(x) \leq \sum_{i=1}^{k} c_j f^{[\lambda]}(a_j x) \quad \text{and} \]
\[ (3.9.5) \quad f^{[\lambda]}(x) \leq \sum_{i=1}^{n} h_i \phi^{[\lambda]}(g_i x) \]
for each \( x \in G \) (see Lemma 3.2). Then from Identity (2.1.9), Inequalities (3.9.4), (3.9.5) and Conditions (3.5.3), (3.5.4) we deduce that

\[
(3.9.6) \quad f_0(x) \leq \sum_{j=1}^{k} c_j \sum_{i=1}^{n} h_i \phi_i^{|x|}((g_i a_j)x[p(g_i, a_j, x)]^{-1}) = \sum_{j=1}^{k} c_j \sum_{i=1}^{n} h_i \phi_i^{|x|}((g_i a_j)x).
\]

Suppose that there are \( y_1, \ldots, y_k \in G \) and \( q_1, \ldots, q_k \in (0, \infty) \) such that

\[
(3.9.7) \quad f(x) \leq \sum_{i=1}^{k} q_i \phi(y_i x)
\]

for each \( x \in G \). Taking the integral \( \int_{N_0} f(x \gamma) \lambda(d\gamma) \) and similarly for the right side (see Formulas (3.6.4) and (3.6.4')) we get from Inequality (3.9.7) that

\[
f(\lambda)|x| \leq \sum_{i=1}^{k} q_i \phi(\lambda)(y_i x)
\]

for each \( x \in G \) (see Lemma 3.2). Hence

\[
(3.9.8) \quad (f : \phi[\lambda]) \leq (f : \phi).
\]

Utilizing Formulas (3.6.1), (3.9.3) and (3.9.8) we infer that

\[
(3.9.9) \quad (f_0 : \phi) \leq (f_0 : f)(f : \phi[\lambda]) \leq (f_0 : f)(f : \phi) \quad \text{for each } f_0 \in \Upsilon(G, N_0) \text{ and nonzero functions } f \text{ and } \phi \text{ in } C_0(G).
\]

Using (3.5.1) and (3.9.9) we infer that

\[
(f_0 : f)J_{\phi, f_0}(f) = \frac{\langle f_0 : f \rangle \langle f : \phi \rangle}{\langle f_0 : \phi \rangle} \geq \frac{\langle f_0 : f \rangle}{\langle f_0 : \phi \rangle} = 1,
\]

consequently, \( J_{\phi, f_0}(f) \geq (f_0 : f)^{-1} \). Thus the left inequality in (3.9.1) also is proved.

From Inequalities (3.9.1) for \( J_{\phi, f_0}(f) \) and \( J_{\phi, f_0}(f_1) \) and Formula (3.5.1) it follows (3.9.2).

**3.10. Lemma.** Let \( G \) be a \( T_1 \) topological locally compact fan loop satisfying Condition (3.5.2), let \( f_0 \in \Upsilon(G, N_0) \) (see Condition (3.5.3)) and let \( f_1, \ldots, f_m \) be nonzero functions belonging to \( C_0(G) \), let also \( 0 < \delta < \infty \), \( 0 < \delta_1 < \infty \). Then there exists an open neighborhood \( V \) of \( e \) in \( G \) such that for each nonzero function \( \phi \) in \( C_0^+(G) \) with a support \( S_\phi \) contained in \( V \) and \( 0 \leq q_j \leq \delta_1 \) for each \( j = 1, \ldots, m \) the following inequality is satisfied:

\[
(3.10.1) \quad \sum_{j=1}^{m} q_j J_{\phi, f_0}(f_j) \leq J_{\phi, f_0}(\sum_{j=1}^{m} q_j f_j) + \delta.
\]

24
Proof. The loop $G$ is locally compact. Let $S_{f_0,...,f_m} = \bigcup_{j=0}^{m} S_{f_j}$ be a common compact support of these functions, where $S_{f_j}$ denotes a closed support of $f_j$ (see also Subsection 3.1). We choose any function $g_1 \in C_{0,0}^{+}(G)$ such that $g_1 : G \to [0,1]$ and $g_1(S_{f_0,...,f_m}cl_G(Y_1)) = \{1\}$, where $Y'_1$ is an open neighborhood of $e$ in $G$ with $Y_1 = Y'_1$ and a compact closure $cl_G(Y_1)$ (see Lemma 2.6). Consider arbitrary fixed positive numbers $0 < \delta < \infty$, $0 < \delta_1 < \infty$ and $0 < \epsilon < M$ such that

$$\epsilon \delta_1 \sum_{j=1}^{m}(f_j : f_0) + \epsilon(1+\epsilon)(g_1 : f_0) \leq \delta, \text{ where } M = \delta_1 m \max_{j=1,...,m} \|f_j\|_G.$$ 

By virtue of Corollary 2.15 the functions $f_0, ..., f_m$ are uniformly $(\mathcal{L}_G, \mathcal{L}_H)$ continuous, where $H = (C, +)$. Therefore there exists an open neighborhood $W'$ of $e$ with $W = \tilde{W}'$ and a compact closure $cl_G(W)$ in $G$ and $W \subset Y_1$, since $G$ is locally compact, such that

$$(3.10.2) \ |f_j(s) - f_j(x)| < \epsilon^3 [4M m \delta_1]^{-1}$$

for each $s \setminus x \in W$. Next we take a function $g \in C_{0,0}^{+}(G)$ such that $g : G \to [0,1]$ and $g(S_{f_0,...,f_m}cl_G(W)) = \{1\}$ and $g(x) \leq g_1(x)$ for each $x \in G$, because $W \subset Y_1$. Hence $(g : f_0) \leq (g_1 : f_0)$ by Inequality (3.4.5).

Let $S = P((S_{f_0,...,f_m} \cup S_g)cl_G(W))$ (see Formula (2.14.1)). Since $cl_G(V)$, $S_{f_0,...,f_m}$ and $S_g$ are compact, then $S$ is a compact subset in $G$. For each open neighborhood $Y$ of $e$ in $G$ there exists an open neighborhood $X$ of $e$ in $G$ such that $X^2 \subset Y$, since the multiplication in $G$ is continuous. In view of Proposition 2.9 and Corollary 2.15 there exist open neighborhoods $U'_k$ of $e$ in $G$ such that $U_k = \tilde{U}'_k$ and

$$(3.10.3) \ |t(U_k)U_k, (U_kb)U_k, (U_kc)U_k)U_k]\cup[U_k t((U_ka)U_k, (U_kb)U_k, (U_kc)U_k)]$$

$$\subset \{t(a, b, c)W_{k-1} \} \cap [W_{k-1} t(a, b, c)]$$

$$[p((U_ka)U_k, (U_kb)U_k, (U_kc)U_k)U_k] \cup [U_k p((U_ka)U_k, (U_kb)U_k, (U_kc)U_k)]$$

$$\subset [p(a, b, c)W_{k-1}] \cap [W_{k-1} p(a, b, c)]$$

for every $a, b, c$ in $S$ and $k \in \{1, 2\}$ with $U_0 = W$ and an open of the form (2.6.1) neighborhood $W_{k-1}$ of $e$ in $G$ such that $W_{k-1}^2 \subset U_{k-1}$ and

$$(3.10.4) \ |g(s) - g(x)| < \epsilon^2 [4M]^{-1}$$

for each $s$ and $x$ in $G$ such that $s \setminus x \in U_1$, where $t = t_G$.

Take any $0 \leq q_j \leq \delta_1$ for each $j = 1, ..., m$ and put

$$(3.10.5) \ \Psi = \epsilon g + \sum_{j=1}^{m} q_j f_j$$

and

$$(3.10.6) \ h_j(x) = q_j f_j(x) [\Psi(x)]^{-1}$$

for each $x \in S_{f_1,...,f_m}$ and
where $S_{f_1,\ldots,f_m} = \bigcup_{j=1}^m S_{f_j}$. Therefore the function $\Psi$ belongs to $C^+_{0,0}(G)$ and $\sum_{j=1}^m h_j(x) \leq 1$ for each $x \in G$.

From Inequalities (3.10.2) and (3.10.4) it follows that

\[(3.10.7) \quad |\Psi(s) - \Psi(x)| \leq \epsilon^3 [2M]^{-1}\]

for each $s$ and $x$ in $G$ such that $s \setminus x \in U_1$. Moreover, $\|\Psi\|_G \leq M + \epsilon < 2M$.

Let $s$ and $x$ belong to $S_{f_1,\ldots,f_m} c_G(W)$ and $s \setminus x \in U_1$. The latter inclusion is equivalent to $x \in sU_1$ and also to $s \in x/U_1$. Then from (3.10.2) and (3.10.7) we deduce that

\[(3.10.8) \quad |h_j(s) - h_j(x)| \leq \epsilon/m.\]

Next we consider the following case: $s \setminus x \in U_1$ and $x \notin S_{f_1,\ldots,f_m} c_G(W)$. Suppose that $s \in S_{f_1,\ldots,f_m}$, then Condition (3.10.3), Lemmas 2.2, 2.3 imply that $x \in S_{f_1,\ldots,f_m} c_G(W)$ contradicting an assumption $x \notin S_{f_1,\ldots,f_m} c_G(W)$. Hence $s \notin S_{f_1,\ldots,f_m}$ and consequently, $h_j(s) = 0$ and $h_j(x) = 0$. Thus Inequality (3.10.8) takes place in this case as well.

In the case $s \setminus x \in U_1$ and $s \notin S_{f_1,\ldots,f_m} c_G(W)$ Condition (3.10.3), Lemmas 2.2, 2.3 imply that $x \notin S_{f_1,\ldots,f_m}$. Therefore the inequality (3.10.8) is fulfilled in this case also. Thus the estimate (3.10.8) is satisfied for each $s$ and $x$ in $G$ such that $s \setminus x \in U_1$.

Next we choose any fixed function $\phi \in C^+_{0,0}(G)$ such that $\phi$ is not identically zero on $G$ and $\phi(y) = 0$ for each $y \in G - U'$. By virtue of Lemma 3.2 there are $m \in \mathbb{N}$, $c_j > 0$ and $b_j \in G$ for each $j \in \{1, \ldots, m\}$ such that

\[(3.10.9) \quad \Psi(x) \leq \sum_{j=1}^m c_j \phi(b_j x)\]

for each $x \in G$ and

\[(3.10.10) \quad -\epsilon + \sum_{j=1}^m c_j \leq (\Psi : \phi) \leq \sum_{j=1}^m c_j.\]

Then Formulas (3.10.3), (3.10.8), (3.10.9) and Lemma 2.2 imply that for each $x \in G$

\[\Psi(x)h_l(x) \leq \sum_{j=1}^m c_j \phi(b_j x) [h_l(b_j \setminus \epsilon) + \epsilon/m]\]

for each $l$. Hence for each $x \in G$ we get

\[q_l f_i(x) = \Psi(x)h_l(x) \leq \sum_{j=1}^m c_j [h_l(b_j \setminus \epsilon) + \epsilon/m] \phi(b_j x)\]
and consequently, \((q_1 f_1 : \phi) \leq \sum_{j=1}^{m} c_j [h_1(b_j \setminus e) + \epsilon/m]\). From \(\sum_{i=1}^{m} h_i \leq 1\) we deduce that \(\sum_{i=1}^{m} (q_i f_i : \phi) \leq (1 + \epsilon) \sum_{j=1}^{m} c_j\). Together with Inequalities (3.10.10) this leads to the following estimate:
\[
\sum_{j=1}^{m} (q_j f_j : \phi) \leq (1 + \epsilon)(\Psi : \phi).
\]

Dividing both sides of it on \((f_0 : \phi)\) we get the inequality
\[
(3.10.11) \quad \sum_{j=1}^{m} q_j J_{\phi,f_0}(f_j) \leq (1 + \epsilon)J_{\phi,f_0}(\Psi).
\]

Then from (3.7.1), (3.7.2), (3.10.5) and (3.10.11) we infer that
\[
(3.10.12) \quad \sum_{j=1}^{m} q_j J_{\phi,f_0}(f_j) \leq J_{\phi,f_0}(\sum_{j=1}^{m} q_j f_j) + \epsilon \sum_{j=1}^{m} q_j J_{\phi,f_0}(f_j) + (1 + \epsilon)J_{\phi,f_0}(g).
\]

Therefore from Inequalities (3.9.1), (3.10.12), (3.4.5) and for \(\epsilon\) as above it follows that
\[
\sum_{j=1}^{m} q_j J_{\phi,f_0}(f_j) \leq J_{\phi,f_0}(\sum_{j=1}^{m} q_j f_j) + \epsilon \delta_1 \sum_{j=1}^{m} (f_j : f_0)
\]
\[
+ \epsilon (1 + \epsilon)(g : f_0) \leq J_{\phi,f_0}(\sum_{j=1}^{m} q_j f_j) + \delta.
\]

This implies the estimate (3.10.1) with \(V = U’_2\).

### 3.11. Theorem

Let \(G\) be a \(T_1\) topological locally compact fan loop, let \(0 < \epsilon \) and \(f\) in \(C_{0,0}^+(G)\) be a nonzero function, \(S_f = \text{cl}_G\{x \in G : f(x) \neq 0\}\). Let also \(V\) be an open neighborhood of \(e\) in \(G\) such that \(V = V’\) and
\[
(3.11.1) \quad |f(x) - f(y)| < \epsilon \text{ for each } x \text{ and } y \text{ in } G \text{ with } x \setminus y \in V. \quad \text{Let } g \in C_{0,0}^+(G) \text{ be a nonzero function such that } g(x) = 0 \text{ for each } x \in G - V’.
\]

Then for each \(\delta > \epsilon\) and each open neighborhood \(W_e\) of \(e\) in \(G\) with \(W_e = \bar{W}_e\) and a compact closure \(\text{cl}_G(W_e)\) contained in \(V\) there is an open neighborhood \(U’\) of \(e\) in \(G\) such that \(U = \bar{U}\) and for each nonzero function \(\phi\) in \(C_{0,0}^+(G)\) with a support \(S_\phi\) contained in \(U’\) there are positive constants \(c_1, \ldots, c_n\) and elements \(b_1, \ldots, b_n\) in \(S_f \text{cl}_G(W_e)\) such that for each \(x \in G\) and \(\gamma \in N(G)\):
\[
(3.11.2) \quad |f(\gamma x) - \sum_{j=1}^{n} c_j J_{\phi,f_0}^v(g(v \setminus x)) g(b_j \setminus \gamma x)| \leq \delta,
\]
where an expression \(J_{\phi,f_0}^v(g(v \setminus x))\) means that a functional \(J_{\phi,f_0}\) is taken in the \(v\) variable.
Proof. The continuous functions $f$ and $g$ are with compact supports, hence they are uniformly $(\mathcal{L}_G, \mathcal{L}_H)$ continuous and uniformly $(\mathcal{R}_G, \mathcal{R}_H)$ continuous on $G$ by Corollary 2.15, where $H = (\mathbb{C}, +)$. For each $y \in G$ the right translation operator $R_y$ is the homeomorphism of $G$ as the topological space onto itself (see also Subsection 2.5). Therefore a function $\nu(y) := (f(x) : g(x \setminus y))$ is continuous on the loop $G$ and consequently, uniformly continuous on the compact subset $S_f$, hence $\sup_{y \in S_f} \nu(y) < \infty$, where $(f(x) : g(x \setminus y)) = (f : z)$ is calculated in the $x$ variable with $z(x) = g(x \setminus y)$ for a fixed parameter $y$. We take any fixed $\delta$ such that $\epsilon < \delta < \infty$. Evidently there exists $0 < \eta$ such that

$$(3.11.3) \quad \eta \sup_{y \in S_f} \nu(y) < \delta - \epsilon.$$ 

Therefore take any fixed open neighborhood $W'_e$ of $e$ in $G$ such that $W'_e = \hat{W}'_e$ and $cl_G(W'_e)$ is compact and $cl_G(W'_e) \subset V$ (see Lemma 2.6). By virtue of Corollary 2.15 the functions $g$ and $h$ are uniformly $(\mathcal{L}_G, \mathcal{L}_H)$ continuous and uniformly $(\mathcal{R}_G, \mathcal{R}_H)$ continuous. Hence there exists an open neighborhood $W'_1$ of $e$ in $G$ such that $W_1 = \hat{W}'_1$ and $cl_G(W_1)$ is compact and $cl_G(W_1) \subset W'_e$ and for each $x$ and $y$ in $G$ with $x \setminus y \in W_1$:

$$(3.11.4) \quad |g(x) - g(y)| < \eta.$$ 

Therefore, a subset $S_f cl_G(W_1)$ is compact in $G$ (see Theorems 3.1.10, 8.3.13-8.3.15 in [11], Lemma 2.6). Then we take compact subsets $S_1 = S_f cl_G(W_1)$ and $S = P(S_f cl_G(W_1))$ in $G$ (see Formula (2.14.1)). In view of Lemma 2.6 they contain open subsets $S_f W_1$ and $P(S_f W_1)$ respectively, since $W_1$ is open in $G$. Mention that the topological spaces $S_1$ and $S$ are normal, since they are compact and $T_1 \cap T_{3.5}$ (see Theorem 3.1.9 in [11]). Using Proposition 2.9 we take an open neighborhood $W'_2$ of $e$ in $G$ with $W_2 = \hat{W}'_2$ such that

$$(3.11.5) \quad [t((W_2 a) W_2, (W_2 b) W_2, (W_2 c) W_2) W_2] \cup \{W_2 t((W_2 a) W_2, (W_2 b) W_2, (W_2 c) W_2)\} \subset \{t(a, b, c) W_3\} \cap \{W_3 t(a, b, c)\}$$

and

$$[p((W_2 a) W_2, (W_2 b) W_2, (W_2 c) W_2) W_2] \cup \{W_2 p((W_2 a) W_2, (W_2 b) W_2, (W_2 c) W_2)\} \subset \{p(a, b, c) W_3\} \cap \{W_3 p(a, b, c)\}$$

for every $a, b, c$ in $S$, where $W_3$ is an open neighborhood of $e$ in $G$ such that $\hat{W}_3^2 \subset W_1$.

In view of the Dieudonné theorem 3.1 in [18] there exists a partition of
unity on $S_1$. Together with Theorem 3.3.2 in [11] and Lemma 2.5 this implies that there are functions $q_1, ..., q_n$ in $C^+_{0,0}(G)$ and elements $w_1, ..., w_n$ in $S_1$ such that $S_1 \subset \bigcup_{j=1}^{n} w_jW_2$ and
\begin{equation}
(3.11.6) \sum_{j=1}^{n} q_j(x) = 1 \text{ for each } x \in S_1 \text{ and }
(3.11.7) q_j(y) = 0 \text{ for each } y \in G - (w_jW_2).
\end{equation}

The conditions of this theorem imply that for each $x$ and $y$ in $G$ with $y \setminus x \in V$ the following inequalities are satisfied:
\begin{equation}
(3.11.8) [f(x) - \varepsilon]g(y \setminus x) \leq f(y)g(y \setminus x) \leq [f(x) + \varepsilon]g(y \setminus x),
\end{equation}
since for $y \setminus x \in V$ Inequality (3.11.1) is fulfilled; for $u = y \setminus x \notin V$ the function $g$ is nil, $g(u) = 0$.

Certainly $y \in w_jW_2$ if and only if there exists $b \in W_2$ such that $y = w_jb$. Then $(y \setminus x) \setminus (w_j \setminus x) \in W_1$ if and only if there exists $c \in W_1$ such that $w_j \setminus x = ((w_jb) \setminus x)c$. For $w_j \setminus x = v \in V$ this gives $c = ((w_jb) \setminus (w_jv)) \setminus v$.

In view of (2.2.2'), (2.2.4), (2.1.8) and (2.1.9)
\begin{equation*}
((w_jb) \setminus (w_jv)) \setminus v = p(w_j, b, (w_jb) \setminus (w_jv))((b \setminus v) \setminus v).
\end{equation*}

Therefore, from Conditions (3.11.5)-(3.11.7) it follows that for each $x$ and $y$ in $G$ and $j = 1, ..., n$:
\begin{equation}
(3.11.9) q_j(y)f(y)[g(y \setminus x) - \eta] \leq q_j(y)f(y)g(w_j \setminus x)
\leq q_j(y)f(y)[g(y \setminus x) + \eta].
\end{equation}

Summing by $j$ in (3.11.9), using (3.11.8) we infer that for each $x$ and $y$ in $G$:
\begin{equation}
(3.11.10) [f(x) - \varepsilon]g(y \setminus x) - \eta f(y)
\leq \sum_{j=1}^{n} q_j(y)f(y)g(w_j \setminus x) \leq [f(x) + \varepsilon]g(y \setminus x) + \eta f(y).
\end{equation}

Next we take any $\phi$ and $f_0$ in $C^+_{0,0}(G)$ such that $\phi$ and $f_0$ are not identically zero. From Inequalities (3.11.10) after dividing on $J_{\phi,f_0}(y \setminus x)$ and Lemma 3.7 it follows that for each $x$ in $G$:
\begin{equation}
(3.11.11) [f(x) - \varepsilon] - \eta \frac{J_{\phi,f_0}(f)}{J_{\phi,f_0}(y \setminus x)} \leq J^y_{\phi,f_0}(\sum_{j=1}^{n} g(w_j \setminus x)q_j(y)f(y))
\leq [f(x) + \varepsilon] + \eta \frac{J_{\phi,f_0}(f)}{J_{\phi,f_0}(y \setminus x)},
\end{equation}
where $J^y_{\phi,f_0}(y \setminus u) = J_{\phi,f_0}(z)$ means that the functional $J^y_{\phi,f_0}$ is taken in the $y$ variable in $G$, where $z(y) = g(y \setminus u)$ for each $y \in G$ and a fixed parameter $u$ in $G$. 

29
Notice that the function \( g(y \setminus x) \) is jointly continuous in \((x, y) \in G \times G\).

On the other hand, in view of Lemmas 2.2, 2.4, 2.6

\[ \{u = y \setminus x : x \in S_f, u \in S_g\} \text{ is a compact subset in } G, \]

since \( \text{Inv}_1(S_f), S_g, S_fS_g \) and \( t(S_f, \text{Inv}_1(S_f), S_fS_g) \) are compact subsets in \( G\).

By virtue of Lemma 3.8 a mapping \( \psi \) is continuous in the variable \( x \in S_f \), \( \psi : S_f \to (0, \infty) \). Hence

\[
(3.11.12) 0 < K_0 = \inf_{x \in S_f} \psi(x) \leq \sup_{x \in S_f} \psi(x) = K_1 < \infty.
\]

Apparently in Formula (3.11.3) the parameter \( \eta > 0 \) can be taken sufficiently small, because Inequalities (3.11.3) and (3.11.12) are independent.

Then from (3.11.11) and (3.11.12) we deduce that for each \( \beta > \epsilon \) there exist \( q_j \) and \( w_j \) (see above) such that

\[
\eta J_{\phi,f_0}(f) < (\beta - \epsilon) \min(1, K_0), \quad \text{consequently,}
\]

\[
(3.11.13) f(x) - \beta \leq J_{\phi,f_0}^g\left(\frac{\sum_{j=1}^n g(w_j \setminus x)q_j(y)f(y)}{J_{\phi,f_0}^g(g(v \setminus x))}\right) \leq f(x) + \beta
\]

for each \( x \in G \).

In view of Lemmas 3.7 and 3.10 for each \( \delta > \delta_1 > \beta > \epsilon \) there exists an open of the form (2.6.1) neighborhood \( U \) of \( e \) in \( G \) such that \( U \subset W_2 \) and

\[
(3.11.14) \left|J_{\phi,f_0}^g\left(\frac{\sum_{j=1}^n g(w_j \setminus x)q_j(y)f(y)}{J_{\phi,f_0}^g(g(v \setminus x))}\right) - \sum_{j=1}^n J_{\phi,f_0}(q_jf)g(w_j \setminus x)\right| < \delta_1 - \beta
\]

for each \( x \in S_f \). We put \( c_j = J_{\phi,f_0}(q_jf) \) and \( b_j = w_j \) for each \( j = 1, ..., n \). Thus the estimates (3.11.13) and (3.11.14) and Formula (3.4.1)' imply the assertion of this theorem.

**3.12. Definition.** Let \( W \) be an open neighborhood of \( e \) in a locally compact loop \( G \) and a nonzero function \( \phi_W \in C_{0,0}^+(G) \) be such that \( \phi_W(x) = 0 \) for each \( x \in G - W \). A family \( \{\phi_W\} \) of these functions will be directed by:

\[
(3.12.1) \phi_{W_1} \preceq \phi_{W_2} \text{ if and only if } W_2 \subseteq W_1 \text{ and } \phi_{W_2}(x) = 0 \text{ implies } \phi_{W_1}(x) = 0.
\]

If \( \phi_{W_1} \preceq \phi_{W_2} \) and \( \phi_{W_1} \) and \( \phi_{W_2} \) are different functions, then it will be written \( \phi_{W_1} < \phi_{W_2} \).

**3.13. Lemma.** Let \( G \) be a \( T_1 \) topological locally compact fan loop satisfying Condition (3.5.2) and let a family of nonzero functions \( \{\phi_U\} \in C_{0,0}^+(G) \) be directed by Condition (3.12.1). Let also \( f_0 \in \mathcal{Y}(G, N_0) \) (see (3.5.3)) and \( f \in C_{0,0}^+(G) \). Then the limit exists:

\[
(3.13.1) \lim_{\phi \to \phi_U} J_{\phi, f_0}(f) =: J_{f_0}(f).
\]
Proof. It is sufficient to prove that a net \( \{J_{\phi_U, f_0}(f) : \phi_U\} \) is fundamental (i.e. Cauchy) in \( \mathbb{R} \), where a net \( \{\phi_U\} \) is directed by Condition (3.12.1).

We take any fixed open neighborhood \( U' \) of \( e \) in \( G \) with \( U_0 = U' \) and a compact closure \( cl_G(U_0) \). Let \( A = S_{f+fo}cl_G(U_0) \), where \( S_{f+fo} = cl_G\{x \in G : f(x) + f_0(x) \neq 0\} \). Therefore, a subset \( S = P(A) \) is compact (see Formula (2.14.1) and Lemma 2.6), since \( S_{f+fo} \) is compact.

We choose any function \( z \in C^+_0(G) \) such that \( z|_A = 1 \). Let \( 0 < \epsilon < 1 \) and \( \xi_1 = \epsilon(16[1 + (z : f_0)][1 + (f : f_0)])^{-1} \). From Corollary (2.15) it follows that there exists an open neighborhood \( W' \) of \( e \) in \( G \) with \( W = W' \) such that

\[
(3.13.2) \quad |f(x) - f(y)| < \xi_1/2 \quad \text{and} \quad |f_0(x) - f_0(y)| < \xi_1/2
\]

for each \( x \) and \( y \) in \( G \) with \( x \setminus y \in W \).

In view of Proposition 2.9 there exists an open neighborhood \( U'_2 \) of \( e \) in \( G \) with \( U_2 = U'_2 \) such that

\[
(3.13.4) \quad [t((U_2a)U_2, (U_2b)U_2, (U_2c)U_2)U_2] \cup [U_2t((U_2a)U_2, (U_2b)U_2, (U_2c)U_2)] \subset [t(a, b, c)B_1] \cap [B_1t(a, b, c)] \quad \text{and} \quad [p((U_2a)U_2, (U_2b)U_2, (U_2c)U_2)U_2] \cup [U_2p((U_2a)U_2, (U_2b)U_2, (U_2c)U_2)] \subset [p(a, b, c)B_1] \cap [B_1p(a, b, c)]
\]

for every \( a, b, c \) in \( S \), where \( B_1 \) is an open neighborhood of \( e \) in \( G \) such that \( \hat{B}_1 \cup U_1, U_1 = U'_0 \cap W' \) (see Lemma 2.6). Next we take a nonzero function \( g \in C^+_0(G) \) such that \( g(x) = 0 \) for each \( x \in G \setminus U'_2 \).

By virtue of Theorem 3.11 for any fixed \( 0 < \delta < \xi_1 \) and each open neighborhood \( W'_e \) of \( e \) in \( G \) with \( W_e = W'_e \) and a compact closure \( cl_G(W_e) \) contained in \( U'_2 \) there is an open neighborhood \( U'_{3,f} \) of \( e \) in \( G \) with \( U_{3,f} = U'_{3,f} \) such that for each nonzero function \( \phi \) in \( C^+_0(G) \) with a support \( S_\phi \) contained in \( U'_{3,f} \) there are positive constants \( c_1, ..., c_n \) and elements \( b_1, ..., b_n \) in \( S_f cl_G(W_e) \) such that for each \( x \in G \) and \( \gamma \in N(G) \):

\[
(3.13.5) \quad |f(\gamma x) - \sum_{j=1}^n \frac{c_j}{J_{\phi, f_0}(g(v \setminus x))}g(b_j \setminus \gamma x)| \leq \delta.
\]

Taking \( U_{3,f} \subset U'_2 \) we get \( f(x) = 0 \) and \( g(b_j \setminus x) = 0 \) for each \( x \in G - A \) according to the choice of \( b_j \) in the proof of Theorem 3.11, consequently,

\[
(3.13.6) \quad |f(\gamma x) - \sum_{j=1}^n \frac{c_j}{J_{\phi, f_0}(g(v \setminus x))}g(b_j \setminus \gamma x)| \leq \delta z(\gamma x)
\]

31
for each $x \in G$ and $\gamma \in N(G)$. From the latter estimate and Lemma 3.7 we infer that

\[(3.13.7) \quad |J_{\phi,f_0}(f) - K_{\phi,f_0}(f; g)| \leq \delta J_{\phi,f_0}(z) \leq \delta(z : f_0),\]

where

\[K_{\phi,f_0}(f; g) = J_{\phi,f_0}^v \left( \sum_{j=1}^n \frac{c_j}{J_{\phi,f_0}(g(v \setminus x))} g(b_j \setminus x) \right).\]

From Estimate (3.13.7) and the right Inequality (3.9.1) it follows that

\[(3.13.7') \quad \sup_{\{\phi v\}} K_{\phi v,f_0}(f; g) \leq (1 + \delta)(f : f_0) + \delta(z : f_0) < \infty.\]

Applying the proof above to $f_0$ instead of $f$ we get and open neighborhood $U'_{3,f_0}$ of $e$ with $U_{3,f_0} = U'_{3,f_0}$ and $U_{3,f_0} \subset U_2'$ such that for each nonzero function $\phi$ in $C_{0,0}^+(G)$ with a support $S_\phi$ contained in $U'_{3,f_0}$ there are positive constants $d_1, \ldots, d_m$ and elements $v_1, \ldots, v_m$ in $S_{f_0}c_{LG}(W_e)$ such that

\[(3.13.8) \quad |f_0(\gamma x) - \sum_{j=1}^m J_{\phi,f_0}(g(v \setminus x)) d_j g(v_j \setminus \gamma x)| \leq \delta(\gamma x)\]

for each $x \in G$ and $\gamma \in N(G)$, consequently,

\[(3.13.9) \quad |1 - K_{\phi,f_0}(f_0; g)| \leq \delta(z : f_0),\]

where

\[K_{\phi,f_0}(f_0; g) = J_{\phi,f_0}^v \left( \sum_{j=1}^m \frac{d_j}{J_{\phi,f_0}(g(v \setminus x))} g(v_j \setminus \gamma x) \right),\]

since $J_{\phi,f_0}(f_0) = 1$. Moreover,

\[(3.13.10) \quad \sup_{\{\phi v\}} K_{\phi v,f_0}(f_0; g) \leq (1 + \delta) + \delta(z : f_0) < \infty.\]

Then $U'_3 = U'_{3,f} \cap U'_{3,f_0}$ is an open neighborhood of $e$ in $G$. From (3.13.7), (3.13.9) and (3.13.10) we deduce that

\[(3.13.11) \quad |J_{\phi,f_0}(f) - K_{\phi,f_0}(f; g)| \leq \delta_2 + [1 + \delta + \delta_2] \delta_2 (1 - \delta_2)^{-1},\]

where $\delta_2 = \delta(z : f_0) < \xi_1(z : f_0) < 1/16$. In view of Lemmas 3.7 and 3.10, Formulas (3.13.5) and (3.13.6) there exists an open neighborhood $U'_4$ of $e$ with $U_4 = U'_4$ and $U_4$ contained in $U'_3$ such that for each nonzero $\phi$ in $C_{0,0}^+(G)$ with $S_\phi \subset U'_4$ there are inequalities:

\[(3.13.12) \quad |K_{\phi,f_0}(f; g) - \sum_{j=1}^n c_j J_{\phi,f_0}(g(b_j \setminus x))| \leq \delta \quad \text{and}\]

32
(3.13.13) \[ |K_{\phi,f_0}(f_0; g) - \sum_{j=1}^{m} d_j J_{\phi,f_0}^x(\frac{g(v_j \setminus x)}{J_{\phi,f_0}(g(v \setminus \gamma x))})| \leq \delta \]

for each \( \gamma \in N(G) \). On the other hand, Formulas (3.5.1), (3.6.4), (3.6.4'), (3.6.12) and (3.4.3) imply that

(3.13.14) \[ J_{\phi,f_0}(\frac{g(b_j \setminus x)}{J_{\phi,f_0}(g(v \setminus \gamma x))}) = \int_{N_0} \left( \frac{g(b_j \setminus x)}{g(v \setminus \gamma x)} : \phi(x) \right) \lambda(d\gamma) \]

\[ = \left( \frac{g(b_j \setminus x) : \phi(x)}{g(b_j \setminus x) : \phi(x)} \right) \]

Then from Proposition (2.9.1) and Formulas (3.4.2), (3.4.2') it follows that for each \( b \in G \) and each \( 0 < \delta_3 \leq \delta \) there exists an open neighborhood \( U'_{5,b} \) of \( e \) in \( G \) with \( U_{5,b} = \bar{U}'_{5,b} \) such that for each nonzero \( \phi_U \in C_{0,0}^+(G) \) with \( S_{\phi_U} \subset U \subset U'_{5,b} \)

(3.13.15) \[ |\left( \frac{g(b \setminus x)}{g(b \setminus x)} : \phi_U(x) \right) - \left( \frac{g(x)}{g(x)} : \phi_U(x) \right)| < \delta_3, \]

since \( S_{\phi_U} \subset U \) and \( t(a,b,e) = t(a,e,b) = t(e,a,b) = e \) and \( p(a,b,e) = p(a,e,b) = p(e,a,b) = e \) for each \( a \) and \( b \) in \( G \). Therefore we take \( U'_5 = \cap_{j=1}^{n} U'_{5,b_j} \cap \cap_{k=1}^{m} U'_{5,v_k} \cap U'_{4} \) and \( \phi = \phi_Y \) with \( Y = U'_5 \). We put \( c = \sum_{j=1}^{n} c_j \) and \( d = \sum_{k=1}^{m} d_k \). From (3.13.11) - (3.13.15) and (3.9.1) it follows that

\[ \frac{c}{d} < K_1, \] where \( K_1 = 3[1 + (f : f_0)](1 + \delta)(1 - \delta)^{-1} < 4[1 + (f : f_0)]. \]

Then we deduce from Formulas (3.13.11)-(3.13.15) for each \( \phi_U \) with an open neighborhood \( U \) of \( e \) in \( G \) such that \( U \subset U'_5 \):

\[ |J_{\phi_U,f_0}(f) - \frac{c}{d}| < \delta(1 - \delta)^{-1}[1 + (f : f_0)] + \delta_2 + [1 + \delta + \delta_2]\delta_2(1 - \delta_2)^{-1}, \]

consequently,

(3.13.16) \[ |J_{\phi_{V_1},f_0}(f) - J_{\phi_{V_2},f_0}(f)| < 8\delta(1 - \delta)^{-1}[1 + (f : f_0)] \]

\[ + 2\delta_2 + 2[1 + \delta + \delta_2]\delta_2(1 - \delta_2)^{-1} < \epsilon \]

for each open neighborhoods \( V_1 \) and \( V_2 \) of \( e \) in \( G \) such that \( V_1 \subset U'_5 \) and \( V_2 \subset U'_5 \). Thus the net \( \{J_{\phi_U,f_0}(f) : \phi_U\} \) is fundamental, where the net \( \{\phi_U\} \) is directed by Condition (3.12.1).

3.14. Remark. Suppose that \( G \) is a \( T_1 \) topological locally compact fan loop and Condition (3.5.2) is fulfilled and \( f_0 \in \mathcal{T}(G, N_0) \) (see (3.5.3)), functions \( f \) and \( g \) belong to \( C_{0,0}^+(G) \) and \( g \) is nonzero. Then in view of Lemma 3.13 a functional exists

33
(3.14.1) \( J_g(f) = J_{f_0}(f)/J_{f_0}(g) \).

As a consequence of Lemma 3.13 and Formulas (3.5.1) and (3.14.1) we get that

(3.14.2) the functional \( J_g(f) \) is independent of \( f_0 \).

Then Formula (3.9.2) and Lemma 3.13 imply that

(3.14.3) \((g : f_0)^{-1}(f_0 : f)^{-1} \leq J_g(f) \leq (f : f_0)(f_0 : g)\)

for each \( f_0 \in \Upsilon(G, N_0) \) and a nonzero function \( f \in C^+_{0,0}(G) \).

3.15. **Theorem.** Let \( G \) be a \( T_1 \) topological locally compact fan loop fulfilling Condition (3.5.2) and a functional \( J = J_g \) be defined by Formula (3.14.1). Then \( J \) possesses the following properties:

(3.15.1) \( J(f) \geq 0 \) for each \( f \in C^+_{0,0}(G) \); and if a function \( f \) is nonzero, then \( J(f) > 0 \);

(3.15.2) \( J(\alpha_1 f_1 + \ldots + \alpha_n f_n) = \alpha_1 J(f_1) + \ldots + \alpha_n J(f_n) \) for each \( f_1, \ldots, f_n \) in \( C^+_{0,0}(G) \) and \( \alpha_1 \geq 0, \ldots, \alpha_n \geq 0 \);

(3.15.3) \( J(b f) = J(f) \) for each \( b \in G \) and \( f \in C^+_{0,0}(G) \).

**Proof.** Property (3.15.1) follows from Formula (3.14.3). On the other hand, Lemmas 3.7, 3.10, 3.13 imply Equality (3.15.2).

Then Formulas (3.6.4), (3.6.4'), (3.6.12), (3.14.1) and Lemma 3.13 imply

(3.15.4) \( J(b f^{[\lambda]}) = J(f^{[\lambda]}) \) for each \( b \in G \) and \( f \) in \( C^+_{0,0}(G) \).

As a topological space \( G \) is locally compact. According to the measure theory on locally compact spaces (see Chapter 3, Section 11 in [18]) a functional \( J \) on \( C^+_{0,0}(G) \) satisfying Conditions (3.15.1) and (3.15.2) induces

(3.15.5) a regular \( \sigma \)-additive measure \( \mu \) on a Borel \( \sigma \)-algebra \( B(G) \) of \( G \) such that \( \mu(U) = \sup \{ \mu(X) : X \text{ is compact, } X \subset U \} \)

for each open subset \( U \) in \( G \) and

(3.15.6) \( \mu(A) = \inf \{ \mu(V) : V \text{ is open, } A \subset V \subset G \} \)

for each \( A \in B(G) \) and

(3.15.7) \( J(f) = \int_G f(x) \mu(dx) \) for each \( f \in C^+_{0,0}(G) \) and

(3.15.8) the functional \( J \) has an extension \( \tilde{J} \) such that \( \tilde{J}(f) = \int_G f(x) \mu(dx) \)

for each nonnegative \( \mu \)-measurable function \( f \) on \( G \), where

\( \tilde{J}(f) = \inf \{ \tilde{J}(h) : h \geq f, \text{ } h \text{ is lower semicontinuous} \} \),

\( \tilde{J}(h) = \sup \{ J(p) : p \in C^+_{0,0}(G), p \leq h \} \)

(see Theorems 11.22, 11.23, 11.36 and Corollary 11.37 in [18]).
On the other hand, for each \( \gamma \in N(G) \) Formulas (3.4.1’) and (3.4.2’) give
\[
(3.15.9) \quad (\gamma f : \phi_U) = (f : \gamma \phi_U) = (f : \phi_U).
\]
From Lemma 3.13, Formulas (3.14.1) and (3.15.9) we deduce that
\[
(3.15.10) \quad J(\gamma f) = J(f) \quad \text{for each} \quad \gamma \in N_0(G).
\]
By virtue of the Fubini theorem 13.8 in [18], (3.15.10) gives
\[
\int_G J(\gamma f) \lambda(d\gamma) = \int_G J(f) \lambda(d\gamma) = J(f),
\]
since \( \lambda(N_0) = 1 \) and \( N_0 \subset N(G) \). Thus the last assertion of this theorem also is proved.

3.16. Theorem. If \( G \) is a \( T_1 \) topological locally compact fan loop fulfilling Condition (3.5.2), then there exists
\begin{enumerate}
  \item[(3.16.1)] a regular \( \sigma \)-additive measure \( \mu \) on a Borel \( \sigma \)-algebra \( \mathcal{B}(G) \) of \( G \),
  \item[(3.16.2)] \( \mu(U) > 0 \) for each open subset in \( G \);
  \item[(3.16.3)] \( \mu(A) < \infty \) for each compact subset \( A \) in \( G \);
  \item[(3.16.4)] \( \mu(bB) = \mu(B) \) for each \( B \in \mathcal{B}(G) \) and \( b \in G \).
\end{enumerate}

Such \( \mu \) can be chosen corresponding to a functional \( J \) satisfying Conditions (3.15.1)-(3.15.3).

Proof. This is an immediate consequence of (3.15.1)-(3.15.3), (3.15.5)-(3.15.8). In particular \( \mu(A) = J(\chi_A) \) for the characteristic function \( \chi_A \) of a Borel subset \( A \) in \( G \), where \( \chi_A(x) = 1 \) for each \( x \in A \), \( \chi_A(y) = 0 \) for each \( y \in G - A \).

3.17. Remark. Each function \( f \) in \( C_{0,0}(G) \) can be represented as
\[
f = f^+ - f^-,
\]
where \( f^+(x) = \max(0, f(x)) \), \( f^+ \) and \( f^- \) belong to \( C_{0,0}^+(G) \). Therefore, a functional \( J \) satisfying Conditions (15.1) and (15.2) can be extended to a linear functional on \( C_{0,0}(G) \) such that \( J(f) = J(f^+) - J(f^-) \). Hence Property (3.15.3) extends onto \( C_{0,0}(G) \).

3.18. Definition. A linear functional \( J \) on \( C_{0,0}(G) \) satisfying Property (3.15.3) is called left invariant.

A measure \( \mu \) on the Borel \( \sigma \)-algebra \( \mathcal{B}(G) \) of a topological fan loop \( G \) such that \( \mu \) satisfies Condition (3.16.4) is called left invariant.
3.19. Theorem. Let $G$ be a $T_1$ topological locally compact fan loop fulfilling Condition (3.5.2) and let $\mu$ be a measure possessing Properties (3.16.1)-(3.16.4). Then $\mu(G) < \infty$ if and only if $G$ is compact.

Proof. If $G$ is compact, then by (3.16.3) $\mu(G) < \infty$.

Vice versa suppose that $\mu(G) < \infty$ and consider the variant that $G$ is not compact and take an open neighborhood $U'$ of $e$ in $G$ with $U = U'$ such that $U = N_0U$ and its closure $cl_G(U)$ is compact, hence $0 < \mu(U) < \infty$ (see also Condition (3.5.2)). By virtue of Theorem 2.8 there exists an open neighborhood $V'$ of $e$ in $G$ with $V = V'$ such that $V = N_0V$ and $[cl_G(V)]^2 \subset U'$. In view of Lemma 2.5 a subset $xU$ is open in $G$ for each $x \in G$.

At first we take some fixed $x_1 \in G$. Then we construct a sequence $\{x_j : j \in \mathbb{N}\}$ by induction. Let $x_1, ..., x_n$ be constructed such that if $n \geq 2$, then $x_jV \cap x_kV = \emptyset$ for each $1 \leq j < k \leq n$. There exists

$$y \in G - \bigcup_{j=1}^n U_j,$$

where $U_j := x_jU p(x_jU, V, V)p(V, V, V)[p(x_jU, V, V)]^{-1}$, since $G$ is not compact and $U_j$ is open by Lemma 2.6 and $cl_G(U_j)$ is compact by Theorem 3.1.10 in [11] and Lemmas 2.4, 2.6. Put $x_{n+1} = y$ with this $y$.

Suppose that there is $z \in x_jV \cap x_{n+1}V$ for some $1 \leq j \leq n$. Therefore there would be $v$ and $u$ in $V$ for which $z = x_jv = x_{n+1}u$, consequently,

$$(x_jv)/u = (x_{n+1}u)/u = x_{n+1}$$

by Condition (2.1.2) and Formula (2.2.5). Therefore by Formulas (2.2.3), (2.3.3) and Condition (2.1.9)

$$x_{n+1} = x_j(v/e/u)p(x_jv, v, e/u)p(e/u, u, u \setminus e)p(x_jv(e/u), u, u \setminus e)]^{-1}$$

contradicting the choice of $x_{n+1}$, since $[cl_G(V)]^2 \subset U'$. Thus $x_jV \cap x_kV = \emptyset$ for each $1 \leq j < k \leq n + 1$. This would mean by (3.16.4) that $\mu(G) \geq \sum_{j=1}^n \mu(x_jV) = n\mu(V)$ for each $n$, contradicting $0 < \mu(G) < \infty$.

3.20. Theorem. Assume that $G$ is a $T_1$ topological locally compact fan loop satisfying Condition (3.5.2) and functionals $J$ and $H$ on $C^+_0(G)$ satisfy Conditions (3.15.1)-(3.15.3).

Then a positive constant $\kappa$ exists such that

$$(3.20.1) \quad H(f) = \kappa J(f) \text{ for each } f \in C^+_0(G).$$

Proof. By virtue of Theorem 3.16 there exist two measures $\mu_1$ and $\mu_2$ corresponding to $J$ and $H$. We consider a subalgebra $C(G) := \theta^{-1}(B(G/\cdot/N_0))$ in $B(G)$, where $\theta : G \to G/\cdot/N_0$ is the quotient homomorphism, $B(G)$ denotes the Borel $\sigma$-algebra on $G$. Put $\nu_j(A) = \mu_j(\theta^{-1}(A))$ for each $j$ and
A ∈ B(G/ · /N₀).

From Theorems 2.8 and 3.16 it follows that the measure νⱼ on the group G/ · /N₀ is such that νⱼ(V) > 0 for each open subset V in G/ · /N₀, νⱼ(A) < ∞ for each compact subset A in G/ · /N₀, νⱼ(cB) = νⱼ(B) for each c ∈ G/ · /N₀ and B ∈ B(G/ · /N₀), j ∈ {1, 2}. By virtue of Theorem 15.6 in [IS] there are positive constants pⱼ such that νⱼ = pⱼη, where η is a left invariant Haar measure on G/ · /N₀. Thus J(f[λ]) = p₁H(f[λ])/p₂ for each f ∈ C⁺₀,₀(G).

We consider η₁(b, f) = Jₜ(bf)/J(f[λ]) and η₂(b, f) = Hₜ(bf)/H(f[λ]) for each b ∈ G and a nonzero function f in C⁺₀,₀(G). According to Property (3.15.3) we get the identities ηⱼ(b, f) = ηⱼ(e, f[λ]) = 1 for each j ∈ {1, 2}. This implies that for each nonzero function f ∈ C⁺₀,₀(G) and b ∈ G:

\[(3.20.2) \ Jₜ(bf)/Hₜ(bf) = p₁/p₂.\]

The measures µ₁ and µ₂ possess Properties (3.16.1)-(3.16.4). In view of the Lebesgue-Radon-Nikodym theorem (see Theorem (12.17) in [IS] or see [5]) there exists a µ₁ measurable nonnegative function h(x) such that ∫ₖ g(x)µ₂(dx) = ∫ₖ g(x)h(x)µ₁(dx) for each g ∈ C⁺₀,₀(G). Therefore from Formulas (3.15.8) and (3.20.2) it follows that h(x) is a positive constant. Thus (3.20.1) is proved.

4 Appendix. Products of fan loops.

The main subject of this paper are measures on fan loops. Nevertheless, in this section it is shortly demonstrated that there are abundant families of fan loops besides those which appear in areas described in the introduction.

4.1. Theorem. Let (Gⱼ, τⱼ) be a family of topological T₁ fan loops (see Definition 2.1), where j ∈ J, J is a set. Then their direct product G = \( \prod_{j \in J} G_j \) relative to the Tychonoff product topology τ is a topological T₁ fan loop and

\[(4.1.1) \ Z(G) = \prod_{j \in J} Z(G_j) \text{ and } N(G) = \prod_{j \in J} N(G_j).\]

Proof. The direct product of topological loops is a topological loop (see [8, 11, 19]). Thus conditions (2.1.1)-(2.1.3) are satisfied.

Each element a ∈ G is written as a = \( \{a_j : \forall j ∈ J, a_j ∈ G_j\} \). From (2.1.4)-(2.1.7) we infer that
(4.1.2) $\text{Com}(G) := \{a \in G : \forall b \in G, \ ab = ba\} =$

\[
\{a \in G : \ a = \{a_j : \forall j \in J, a_j \in G_j\}; \forall b \in G, \ b = \{b_j : \forall j \in J, b_j \in G_j\}; \forall j \in J, a_j b_j = b_j a_j = \prod_{j \in J} \text{Com}(G_j),
\]

(4.1.3) $N_l(G) := \{a \in G : \forall b \in G, \ \forall c \in G, \ (ab)c = a(bc)\} = \{a \in G : a = \{a_j : \forall j \in J, a_j \in G_j\}; \forall b \in G, \ b = \{b_j : \forall j \in J, b_j \in G_j\}; \forall c \in G, \ c = \{c_j : \forall j \in J, c_j \in G_j\}; \forall j \in J, \ (a_j b_j)c_j = a_j(b_j c_j) = \prod_{j \in J} N_l(G_j)$

and similarly

(4.1.4) $N_m(G) = \prod_{j \in J} N_m(G_j)$ and

(4.1.5) $N_r(G) = \prod_{j \in J} N_r(G_j)$.

Therefore (4.1.3)-(4.1.5) and (2.1.8) imply that

(4.1.6) $N(G) = \prod_{j \in J} N(G_j)$. Thus

(4.1.7) $Z(G) := \text{Com}(G) \cap N(G) = \prod_{j \in J} Z(G_j)$.

Let $a$, $b$ and $c$ be in $G$, then

\[
(ab)c = \{(a_j b_j)c_j : \forall j \in J, \ a_j \in G_j, b_j \in G_j, c_j \in G_j\} = \{t_{G_j}(a_j, b_j, c_j) : \forall j \in J, \ a_j \in G_j, b_j \in G_j, c_j \in G_j\} = t_G(a, b, c) a(bc)
\]

and analogously $(ab)c = a(bc)p_G(a, b, c)$, where

(4.1.8) $t_G(a, b, c) = \{t_{G_j}(a_j, b_j, c_j) : \forall j \in J, \ a_j \in G_j, b_j \in G_j, c_j \in G_j\}$

and

(4.1.9) $p_G(a, b, c) = \{p_{G_j}(a_j, b_j, c_j) : \forall j \in J, \ a_j \in G_j, b_j \in G_j, c_j \in G_j\}$.

Therefore, Formulas (4.1.7)-(4.1.9) imply that Conditions (2.1.9) also are satisfied. Thus $G$ is a topological fan loop. By virtue of Theorem 2.3.11 in [11] a product of $T_1$ spaces is a $T_1$ space, hence $G$ is the $T_1$ topological fan loop.

4.2. Corollary. (1). Let conditions of Theorem 4.1 be satisfied and for each $j \in J$ a fan loop $G_j$ satisfies Condition (3.5.2). Then the product fan loop $G$ satisfies Condition (3.5.2).

(2). Moreover, if $G_j$ is compact for all $j \in J_0$ and locally compact for each $j \in J \setminus J_0$, where $J_0 \subset J$ and $J \setminus J_0$ is a finite set, then $G$ is locally compact.

Proof. Using Formulas (4.1.8) and (4.1.9) it is sufficient to take $N_0(G) = \prod_{j \in J} N_0(G_j)$, since the direct product of compact groups $N_0(G_j)$ is a compact group $N_0(G)$ (see the Tychonoff theorem 3.2.4 in [11] or [18]). The last
assertion (2) follows from the known fact that $G$ as a topological space is locally compact under the imposed above conditions (see Theorem 3.3.13 in [11]).

4.3. Remark.

(4.3.1) Let $A$ and $B$ be two fan loops and let $N$ be a group such that $N_0(A) \hookrightarrow N$, $N_0(B) \hookrightarrow N$, $N \hookrightarrow \mathcal{N}(A)$ and $N \hookrightarrow \mathcal{N}(B)$ and let $N$ be normal in $A$ and in $B$ (see also Sections 2.1, 2.7 and 3.5).

Using direct products it is always possible to extend either $A$ or $B$ to get such a case. In particular, either $A$ or $B$ may be a group. On $A \times B$ an equivalence relation $\Xi$ is considered such that

\[(4.3.2) \ (v\gamma, b) \Xi (v, \gamma b) \]

for every $v$ in $A$, $b$ in $B$ and $\gamma$ in $N$.

(4.3.3) Let $\phi : A \rightarrow \mathcal{A}(B)$ be a single-valued mapping, where $\mathcal{A}(B)$ denotes a family of all bijective surjective single-valued mappings of $B$ onto $B$ subjected to the conditions given below. If $a \in A$ and $b \in B$, then it will be written shortly $b^a$ instead of $\phi(a)b$, where $\phi(a) : B \rightarrow B$. Let also

\[\eta_\phi : A \times A \times B \rightarrow N, \ \kappa_\phi : A \times B \times B \rightarrow N\]

and $\xi_\phi : ((A \times B)/\Xi) \times ((A \times B)/\Xi) \rightarrow N$ be single-valued mappings written shortly as $\eta$, $\kappa$ and $\xi$ correspondingly such that

\[(4.3.4) \ (b^a)^u = b^{u \eta(v, u, b)}, \ \gamma^u = \gamma, \ b^\gamma = b;\]

\[(4.3.5) \ \eta(v, u, (\gamma_1 b)\gamma_2) = \eta(v, u, b);\]

if $\gamma \in \{v, u, b\}$ then $\eta(v, u, b) = e$;

\[(4.3.6) \ (cb)^u = c^{u \kappa}(u, c, b);\]

\[(4.3.7) \ \kappa(u, (\gamma_1 c)\gamma_2, (\gamma_3 b)\gamma_4) = \kappa(u, c, b)\]

and if $\gamma \in \{u, c, b\}$ then $\kappa(u, c, b) = e$;

\[(4.3.8) \ \xi(((\gamma u)\gamma_1, (\gamma_2 c)\gamma_3), ((\gamma_4 v)\gamma_5, (\gamma_6 b)\gamma_7)) = \xi((u, c), (v, b))\]

\[\xi((e, c), (v, b)) = e \text{ and } \xi((u, c), (e, e)) = e\]

for every $u$ and $v$ in $A$, $b$, $c$ in $B$, $\gamma$, $\gamma_1$, ..., $\gamma_7$ in $N$, where $e$ denotes the neutral element in $N$ and in $A$ and $B$.

We put

\[(4.3.9) \ (a_1, b_1)(a_2, b_2) = (a_1a_2, b_1b_2^\eta, \xi((a_1, b_1), (a_2, b_2)))\]

for each $a_1$, $a_2$ in $A$, $b_1$ and $b_2$ in $B$.
The Cartesian product $A \times B$ supplied with such a binary operation (4.3.9) will be denoted by $A \rtimes^\phi_{\eta,\kappa,\xi} B$.

**4.4. Theorem.** Let the conditions of Remark 4.3 be fulfilled. Then the Cartesian product $A \times B$ supplied with a binary operation (4.3.9) is a fan loop.

**Proof.** From the conditions of Remark 4.3 it follows that the binary operation (4.3.9) is single-valued. The group $N$ is normal in the loops $A$ and $B$ by Conditions (4.3.1). Hence for each $a \in A$ and $\beta \in N$ there exists $(a\beta)/a \in N$ and $a \setminus (\beta a) \in N$, since $aN = Na$ for each $a \in A$. Similarly it is for $B$. Thus there are single-valued mappings

$$r_{A,a}(\beta) = (a\beta)/a, \quad r_{B,b}(\beta) = b \setminus (\beta b),$$

where $r_{A,a} : N \to N$, $r_{B,b} : N \to N$, $r_{A,a} : N \to N$, $r_{B,b} : N \to N$ for each $a \in A$ and $b \in B$. Evidently

$$r_{A,a}(\hat{r}_{A,a}(\beta)) = \beta$$

and $\hat{r}_{A,a}(r_{A,a}(\beta)) = \beta$ for each $a \in A$ and $\beta \in N$, and similarly for $B$.

Let $I_1 = ((a_1, b_1)(a_2, b_2))(a_3, b_3)$ and $I_2 = (a_1, b_1)((a_2, b_2)(a_3, b_3))$, where $a_1$, $a_2$, $a_3$ belong to $A$, $b_1$, $b_2$, $b_3$ belong to $B$. Then we infer that

$$I_1 = ((a_1 a_2 a_3, b_1 b_2^{a_1})\xi((a_1, b_1), (a_2, b_2))b_3^{a_1 a_2}\xi((a_1 a_2, b_1 b_2^{a_1}), (a_3, b_3)))$$

and

$$I_2 = (a_1(a_2 a_3), b_1(b_2^{a_1} b_3^{a_2}))$$

with

$$\beta = \eta(a_1, a_2, b_3)\kappa(a_1, b_1, b_3^{a_2}) [\xi((a_2, b_2), (a_3, b_3))]^{a_1}\xi((a_1, b_1), (a_2 a_3, b_2 b_3^{a_2})).$$

Hence

$$I_1 = (a, ba)$$

and

$$I_2 = (a, b\beta),$$

where $a = a_1(a_2 a_3)$ and $b = b_1(b_2^{a_1} b_3^{a_2})$.

$$\alpha = \hat{r}_{B,b}(p_A(a_1, a_2, a_3))$$

$$p_B(b_1, b_2^{a_1}, b_3^{a_2}) \hat{r}_{B,b}(\xi((a_1, b_1), (a_2, b_2))) \xi((a_1 a_2, b_1 b_2^{a_1}), (a_3, b_3))).$$

Therefore

$$(4.4.1) \quad I_1 = I_2 p \quad \text{with } p = p_A \rtimes^\phi_{\eta,\kappa,\xi} B((a_1, b_1), (a_2, b_2), (a_3, b_3))$$

and

$$(4.4.2) \quad p = \beta^{-1} \alpha \quad \text{and } t = r_{A,a}(r_{B,b}(p)).$$

Apparently $t_A \rtimes^\phi_{\eta,\kappa,\xi} B((a_1, b_1), (a_2, b_2), (a_3, b_3)) \in N$ and $p_A \rtimes^\phi_{\eta,\kappa,\xi} B((a_1, b_1), (a_2, b_2), (a_3, b_3)) \in N$ for each $a_j \in A$, $b_j \in B$, $j \in \{1, 2, 3\}$, since $a$ and $\beta$ belong to the group $N$.

If $\gamma \in N$ and either $(\gamma, e)$ or $(e, \gamma)$ belongs to $\{(a_1, b_1), (a_2, b_2), (a_3, b_3)\}$,
then from the conditions of Section 4.3 and Formulas (4.4.1) and (4.4.2) it follows that
\[ p_A \otimes^{\phi,\eta,\kappa,\xi}_B ((a_1, b_1), (a_2, b_2), (a_3, b_3)) = e \quad \text{and} \]
\[ t_A \otimes^{\phi,\eta,\kappa,\xi}_B ((a_1, b_1), (a_2, b_2), (a_3, b_3)) = e, \]
consequently, \((N, e) \cup (e, N) \subset N(A \otimes^{\phi,\eta,\kappa,\xi} B)\).

Apparently (2.1.3) follows from (4.3.8) and (4.3.9).

Next we consider the following equation
\[ (4.4.3) \quad (a_1, b_1)(a, b) = (e, e), \text{ where } a \in A, b \in B. \]

From (2.1.2) for fan loops \(A\) and \(B\), (4.3.8) and (4.3.9) we deduce that
\[ (4.4.4) \quad a_1 = e/a, \]
consequently, \(b_1 b^{e/a} \xi ((e/a, b_1), (a, b)) = e\) and hence
\[ (4.4.5) \quad b_1 = e/[b^{e/a} \xi ((e/a, b^{e/a}), (a, b))]. \]
Thus \(a_1 \in A\) and \(b_1 \in B\) given by (4.4.4) and (4.4.5) provide a unique solution of (4.4.3).

Similarly from the following equation
\[ (4.4.6) \quad (a, b)(a_2, b_2) = (e, e), \text{ where } a \in A, b \in B \]
we infer that
\[ (4.4.7) \quad a_2 = a \setminus e, \]
consequently, \(bb_2^e \xi ((a, b), (a \setminus e, b_2)) = e\) and hence
\[ b_2^e = b \setminus [\xi ((a, b), (a \setminus e, b_2))]^{-1} \]
by Conditions (2.1.1), (2.1.2) and (4.3.3) for fan loops \(A\) and \(B\). On the other hand, \((b_2^e)^{e/a} = b_2 \eta (e/a, a, b_2)\), consequently, by Lemmas 2.2, 2.3 and the conditions of Section 4.3
\[ (4.4.8) \quad b_2 = (b \setminus [\xi ((a, b), (a \setminus e, b \setminus e)](e/a))]^{-1}(e/a)/\eta (e/a, a, (b \setminus e)^{e/a}). \]
Thus Formulas (4.4.7) and (4.4.8) provide a unique solution of (4.4.6).

Next we put \((a_1, b_1) = (e, e)/(a, b)\) and \((a_2, b_2) = (a, b) \setminus (e, e)\) and
\[ (4.4.9) \quad (a, b) \setminus (e, d) = ((a, b) \setminus (e, e))(c, d)p((a, b), (a, b) \setminus (e, e), (c, d)); \]
\[ (4.4.10) \quad (c, d)/(a, b) = [t((c, d), (e, e)/(a, b), (a, b))]^{-1}(c, d)((e, e)/(a, b)) \]
and \(e_G = (e, e)\), where \(G = A \otimes^{\phi,\eta,\kappa,\xi} B\). Therefore Properties (2.1.1)-(2.1.3) and (2.1.9) are fulfilled for \(A \otimes^{\phi,\eta,\kappa,\xi} B\).

4.5. Definition. The fan loop \(A \otimes^{\phi,\eta,\kappa,\xi} B\) provided by Theorem 4.4 we call a smashed product of fan loops \(A\) and \(B\) with smashing factors \(\phi, \eta, \kappa\) and \(\xi\).

4.6. Corollary. Suppose that the conditions of Remark 4.3 are fulfilled
and $A$ and $B$ are topological $T_1$ fan loops and smashing factors $\phi$, $\eta$, $\kappa$, $\xi$ are jointly continuous by their variables. Suppose also that $A \otimes^{\phi, \eta, \kappa, \xi} B$ is supplied with a topology induced from the Tychonoff product topology on $A \times B$. Then $A \otimes^{\phi, \eta, \kappa, \xi} B$ is a topological $T_1$ fan loop.

4.7. Corollary. If the conditions of Corollary 4.6 are satisfied and loops $A$ and $B$ are locally compact, then $A \otimes^{\phi, \eta, \kappa, \xi} B$ is locally compact. Moreover, if $A$ and $B$ satisfy Condition (3.5.2) and ranges of $\eta$, $\kappa$, $\xi$ are contained in $N_0(A)N_0(B)$, then $A \otimes^{\phi, \eta, \kappa, \xi} B$ satisfies Condition (3.5.2).

Proof. Corollaries 4.6 and 4.7 follow immediately from Theorems 2.3.11, 3.2.4, 3.3.13 in [11] and Theorem 4.4.

4.8. Remark. From Theorems 4.1, 4.4 and Corollaries 4.2, 4.6, 4.7 it follows that taking nontrivial $\phi$, $\eta$, $\kappa$ and $\xi$ and starting even from groups with nontrivial $N(G_j)$ or $N(A)$ and $G_j/\cdot /N(G_j)$ or $A/\cdot /N(A)$ it is possible to construct new fan loops with nontrivial $N_0(G)$ and ranges $t_G(G, G, G)$ and $p_G(G, G, G)$ of $t_G$ and $p_G$ may be infinite and nondiscrete. With suitable smashing factors $\phi$, $\eta$, $\kappa$ and $\xi$ and with nontrivial fan loops or groups $A$ and $B$ it is easy to get examples of fan loops in which $e/a \neq a \setminus e$ for an infinite family of elements $a$ in $A \otimes^{\phi, \eta, \kappa, \xi} B$.

4.9. Conclusion. The results of this article can be used for further studies of measures on homogeneous spaces and noncommutative manifolds related with loops. Besides applications of left invariant measures on loops outlined in the introduction it is interesting to mention possible applications in mathematical coding theory and its technical applications [4, 31, 35], because frequently codes are based on topological-algebraic binary systems and measures. Another very important applications are in representation theory of loops and harmonic analysis on loops, mathematical physics, quantum field theory, quantum gravity, gauge theory, etc.

References

[1] H. Albuquerque, S. Majid. "Quasialgebra structure of the octonions". J. of Algebra 220: 1 (1999), 188-224.
[2] D. Allcock. "Reflection groups and octave hyperbolic plane". J. of Algebra 213: 2 (1998), 467-498.

[3] J.C. Baez. "The octonions". Bull. Amer. Mathem. Soc. 39: 2 (2002), 145-205.

[4] R.E. Blahut. "Algebraic codes for data transmission" (Cambridge: Cambridge Univ. Press, 2003).

[5] V.I. Bogachev. "Measure theory". V. 1, 2 (Berlin: Springer-Verlag, 2007).

[6] N.N. Bogolubov, A.A. Logunov, A.I. Oksak, I.T. Todorov. "General principles of quantum field theory" (Moscow: Nauka, 1987).

[7] N. Bourbaki. "Algebra" (Berlin: Springer, 1989).

[8] R.H. Bruck. "A survey of binary systems" (Berlin: Springer-Verlag, 1971).

[9] O.A. Castro-Alvaredo, B. Doyon, D. Fioravanti. "Conical twist fields and null polygonal Wilson loops". Nuclear Physics B931 (2018), 146-178.

[10] L.E. Dickson. "The collected mathematical papers". Volumes 1-5 (New York: Chelsea Publishing Co., 1975).

[11] R. Engelking. "General topology". 2-nd ed., Sigma Series in Pure Mathematics, V. 6 (Berlin: Heldermann Verlag, 1989).

[12] J.M.G. Fell, R.S. Doran. "Representations of *-algebras, locally compact groups, and Banach *-algebraic bundles". V. 1 and V. 2 (Boston: Acad. Press, 1988).

[13] E. Frenod, S. V. Ludkowski. "Integral operator approach over octonions to solution of nonlinear PDE". Far East J. of Mathem. Sci. (FJMS). 103: 5 (2018), 831-876; DOI: 10.17654/MS103050831.
[14] J.E. Gilbert, M.A.M. Murray. "Clifford algebras and Dirac operators in harmonic analysis". Cambr. studies in advanced Mathem. 26 (Cambridge: Cambr. Univ. Press, 1991).

[15] P.R. Girard. "Quaternions, Clifford algebras and relativistic Physics" (Basel: Birkhäuser, 2007).

[16] K. Gürlebeck, W. Sprössig. "Quaternionic and Clifford calculus for physicists and engineers" (Chichester: John Wiley and Sons, 1997).

[17] F. Gürsey, C.-H. Tze. "On the role of division, Jordan and related algebras in particle physics" (Singapore: World Scientific Publ. Co., 1996).

[18] E. Hewitt, K.A. Ross. "Abstract harmonic analysis" (Berlin: Springer-Verlag, 1979).

[19] V. Kakkar. "Boolean loops with compact left inner mapping groups are profinite." Topology and Its Appl. 244 (2018), 51-54.

[20] I.L. Kantor, A.S. Solodovnikov. "Hypercomplex numbers" (Berlin: Springer-Verlag, 1989).

[21] H. Kiechle. "Theory of K-loops", (Berlin: Springer-Verlag, 2002).

[22] S.V. Ludkowski. "Decompositions of PDE over Cayley-Dickson algebras". Rendic. dell'Istit. di Matem. dell'Univer. di Trieste. Nuova Ser. 46 (2014), 1-23.

[23] S.V. Ludkowski. "Integration of vector Sobolev type PDE over octonions". Complex Variab. and Elliptic Equat. 61: 7 (2016), 1014-1035.

[24] S.V. Ludkowski. "Manifolds over Cayley-Dickson algebras and their immersions". Rendic. dell'Istit. di Matem. dell'Univer. di Trieste. Nuova Ser. 45 (2013); 11-22.

[25] S.V. Ludkovsky. "Normal families of functions and groups of pseudo-conformal diffeomorphisms of quaternion and octonion variables". J. Mathem. Sci., N.Y. (Springer) 150: 4 (2008), 2224-2287.
[26] S.V. Ludkovsky. "Functions of several Cayley-Dickson variables and manifolds over them", J. Mathem. Sci.; N.Y. (Springer) **141**: 3 (2007), 1299-1330.

[27] S.V. Ludkovsky, W. Sprössig. "Ordered representations of normal and super-differential operators in quaternion and octonion Hilbert spaces". Adv. Appl. Clifford Alg. **20**: 2 (2010), 321-342.

[28] S.V. Ludkovsky, W. Sprössig. "Spectral theory of super-differential operators of quaternion and octonion variables". Adv. Appl. Clifford Alg. **21**: 1 (2011), 165-191.

[29] S.V. Ludkovsky. "Integration of vector hydrodynamical partial differential equations over octonions". Complex Variab. and Elliptic Equat. **58**: 5 (2013); 579-609.

[30] S.V. Ludkowski. "Automorphisms and derivations of nonassociative $C^*$ algebras". Linear and Multilinear Algebra, (2018), 1-8, DOI: 10.1080/03081087.2018.1460794.

[31] A.B. Petrov, S.V. Bagrov, A.I. Sycheva. "Approaches to sustainable operation of complex information systems for government and corporate purpose". Russian Technological J. **2**: 4 (2015), 175-183.

[32] G. Pickert. "Projektive Ebenen" (Berlin: Springer-Verlag, 1955).

[33] Yu.P. Razmyslov. "Identities of algebras and their representations". Series "Modern Algebra". **14** (Moscow: Nauka, 1989).

[34] J.D.H. Smith. "An introduction to quasigroups and their representations" (Boca Raton: Chapman and Hall/CRC, Taylor and Francis Group, 2007).

[35] K.P. Shum, X. Ren, Y. Wang. "Semigroups on semilattice and the constructions of generalized cryptogroups". Southeast Asian Bull. of Mathem. **38** (2014), 719-730.

[36] P. Vojtěchovský. "Bol loops and Bruch loops of order $pq$ up to isotopism". Finite Fields and Their Appl. **52** (2018), 1-9.