Matter Fields and Non-Abelian Gauge Fields
Localized on Walls

Masato Arai\textsuperscript{a,b,*} Filip Blaschke\textsuperscript{b,c,**} Minoru Eto\textsuperscript{d,***} and Norisuke Sakai\textsuperscript{e,†}

\textsuperscript{a}Fukushima National College of Technology, Iwaki, Fukushima 970-8034, Japan
\textsuperscript{b}Institute of Experimental and Applied Physics, Czech Technical University in Prague, Horská 22, 128 00 Prague 2, Czech Republic
\textsuperscript{c}Institute of Physics, Silesian University in Opava, Bezručovo nám. 1150/13, 746 01 Opava, Czech Republic
\textsuperscript{d}Department of Physics, Yamagata University, Yamagata 990-8560, Japan
\textsuperscript{e}Department of Mathematics, Tokyo Woman’s Christian University, Tokyo 167-8585, Japan

Abstract

Massless matter fields and non-Abelian gauge fields are localized on domain walls in a (4+1)-dimensional $U(N)_c$ gauge theory with $SU(N)_L \times SU(N)_R \times U(1)_A$ flavor symmetry. We also introduce $SU(N)_{L+R}$ flavor gauge fields and a scalar-field-dependent gauge coupling, which provides massless non-Abelian gauge fields localized on the wall. We find a chiral Lagrangian interacting minimally with the non-Abelian gauge field together with nonlinear interactions of moduli fields as the (3+1)-dimensional effective field theory up to the second order of derivatives. Our result provides a step towards a realistic model building of brane-world scenario using topological solitons.

* E-mail: masato.arai@gmail.com
** E-mail: filip.blaschke@fpf.slu.cz
*** E-mail: meto@sci.kj.yamagata-u.ac.jp
† E-mail: norisuke.sakai@gmail.com

1 typeset using \texttt{P\!P\!T\!E\!X.cls} (Ver.0.9)
§1. Introduction

Gauge hierarchy problem is a good guiding principle to construct theories beyond the Standard Model (SM). Brane world scenario\cite{1,2,3} is one of the most attractive proposals to solve this problem, besides models with supersymmetry (SUSY)\cite{4}. In the brane world scenario, it is assumed that all fields except the graviton field are localized on (3+1)-dimensional worldvolume of a defect called 3-brane, immersed in a many-dimensional space-time called bulk. In order to realize such a scenario dynamically, we may use a topological soliton. For instance, let us consider a domain wall solution as the simplest soliton. To obtain (3+1)-dimensional worldvolume on the domain wall, we need to consider a theory in a (4+1)-dimensional space-time. Bulk fields in (4+1)-dimensions can provide massless modes localized on the domain wall, besides many massive modes in general. After integrating over massive modes, one obtains low-energy effective field theory describing the effective interactions of massless modes. Massless matter fields have been successfully localized on domain walls\cite{5}, but localization of the gauge field on domain walls in field theories has been difficult\cite{6}. It has been noted that the broken gauge symmetry in the bulk outside of the soliton inevitably makes the localized gauge field massive with the mass of the order of inverse width of the wall\cite{7,8}. To localize a massless gauge field, one needs to have the confining phase rather than the Higgs phase in the bulk outside of the soliton. Earlier attempts used a tensor multiplet in order to implement Higgs phase in the dual picture, but this approach successfully localize only $U(1)$ gauge field\cite{9}. More recently, a classical realization of the confinement\cite{10,11} through the position-dependent gauge coupling has been successfully applied to localize the non-Abelian gauge field on domain walls\cite{12}. The nontrivial profile of this position-dependent gauge coupling was naturally introduced on the domain wall background through a scalar-field-dependent gauge coupling function resulting from a cubic prepotential of supersymmetric gauge theories. The appropriate profile of the position-dependent gauge coupling was obtained from domain wall solutions using two copies of the simplest model or from a model with less fields and a particular mass assignment. However, it was still a challenge to introduce matter fields in nontrivial representations of the gauge group of the localized gauge field.

Parameters of soliton solutions are called moduli and can be promoted to fields on the world volume of the soliton. Massless fields in the low-energy effective field theory on the soliton background are generally given by these moduli fields. Moduli with non-Abelian global symmetry is often called the non-Abelian cloud, and has been explicitly realized in the case of domain walls using Higgs scalar fields with degenerate masses in $U(N)_c$ gauge theories\cite{13}. This model also has a non-Abelian global symmetry $SU(N)_L \times SU(N)_R \times U(1)_A$, \cite{2}
which is somewhat similar to the chiral symmetry of QCD. If we turn this global symmetry into a local gauge symmetry, we should be able to obtain the usual minimal gauge coupling between these moduli fields and the gauge field. Since we wish to localize the gauge field on the domain wall, it is essential to choose the global symmetry of moduli fields to be unbroken in the vacua (of both left and right bulk outside of the wall). This choice will guarantee that the bulk outside of the domain wall is not in the Higgs phase. Therefore we are led to an idea where we introduce gauge fields corresponding to a flavor symmetry group of scalar fields which will be unbroken in the vacuum. If we introduce the additional scalar-field-dependent gauge coupling function similarly to the supersymmetric model, we should be able to localize both massless matter fields and the massless gauge field at the same time on the domain wall.

The purpose of this paper is to present a (4+1)-dimensional field theory model of localized massless matter fields minimally coupled to the non-Abelian gauge field which is also localized on the domain wall with the (3+1)-dimensional world volume. We also derive the low-energy effective field theory of these localized matter and gauge fields. To introduce non-Abelian flavor symmetry (to be gauged eventually) in the domain wall sector, we replace one of the two copies of the $U(1)_c$ gauge theory with the flavor symmetry $U(1)_L \times U(1)_R$ in Ref.13, by $U(N)_c$ gauge theory with the extended flavor (global) symmetry $SU(N)_L \times SU(N)_R \times U(1)_A$. By choosing the coincident domain wall solution for this domain wall sector, we obtain the maximal unbroken non-Abelian flavor symmetry group $SU(N)_{L+R}$ which is preserved in both left and right vacua outside of the domain wall. Therefore we can introduce gauge field for the (subgroup of) the flavor $SU(N)_{L+R}$ symmetry. In order to obtain the field-dependent gauge coupling function, for the gauge field localization mechanism, we also introduce a coupling between a scalar field and gauge field strengths inspired by supersymmetric gauge theories, although we do not make the model fully supersymmetric at present. This scalar-field-dependent gauge coupling function gives appropriate profile of position-dependent gauge coupling through the background domain wall solution. With this localization mechanism for gauge field, we find massless non-Abelian gauge fields localized on the domain wall. We also obtain the low-energy effective field theory describing the massless matter fields in the non-trivial representation of non-Abelian gauge symmetry. Since our flavor symmetry resembles the chiral symmetry of QCD before introducing the gauge fields that are localized, we naturally obtain a kind of chiral Lagrangian as the effective field theory on the domain wall. We find an explicit form of full nonlinear interactions of moduli fields up to the second order of derivatives. Moreover, these moduli fields are found to interact with $SU(N)_{L+R}$ flavor gauge fields as adjoint representations. In analyzing the model, we use mostly the strong coupling limit for the domain wall sector. The strong coupling is merely to describe
our result explicitly at every stage. Even if we do not use the strong coupling, the physical
features are unchanged. It is easy to expect that (the part of) the gauge symmetry is broken
when the walls separate in each copy of the domain wall sector. Our results of the low-
energy effective field theories shows that flavor gauge symmetry $SU(N)_{L+R}$ is broken on the
non-coincident wall and the associated gauge bosons acquire masses as walls separate. This
geometrical Higgs mechanism is quite similar to D-brane systems in superstring theory. So
our domain wall system provides a genuine prototype of field theoretical D3-branes. This is
an interesting problem, which we plan to analyze more in future. We also find indications
that additional moduli will appear in the supersymmetric version of our model, which is also
an interesting future problem to study.

The organization of the paper is as follows. In section 2, we explain the localization
mechanism by taking Abelian gauge theory as an illustrative example. In section 3, we
introduce the chiral model with the non-Abelian flavor symmetry for the domain wall sector
and then also introduce gauge fields for the unbroken part of the flavor symmetry. By
introducing the scalar-field-dependent gauge coupling function, we arrive at the localized
massless gauge field interacting with the massless matter field in a nontrivial representation
of flavor gauge group. The low-energy effective field theory is also worked out. In section 4, an attempt is made to make the model supersymmetric. New additional features of
the supersymmetric models are also described. In section 5, we summarize our results and
discuss remaining issues and future directions. In Appendix A, we discuss domain wall
solution for gauged massive $\mathbb{C}P^1$ sigma model. Appendix B describes derivation of effective
Lagrangian which includes full nonlinear interactions between moduli fields. Appendix C
contains derivation of positivity condition for the potential appearing in section 4.

§2. Abelian-Higgs model of gauge field localization

2.1. The domain wall sector

Let us illustrate the localization mechanism for the gauge fields and the matter fields on
the domain walls by using a simplest model in (4+1)-dimensional spacetime: two copies
($i = 1, 2$) of $U(1)$ models, each of which has two flavors ($L, R$) of charged Higgs scalar fields
$H_i = (H_{iL}, H_{iR})$:

$$L_i = -\frac{1}{4g_i^2} (F_{MN}^i)^2 + \frac{1}{2g_i^2} (\partial_M \sigma_i)^2 + |D_M H_i|^2 - V_i,$$

$$V_i = \frac{g_i^2}{2} (|H_i|^2 - v_i^2)^2 + |\sigma_i H_i - H_i M_i|^2.$$

(2.1) 

(2.2)
We use the metric $\eta_{MN} = \text{diag}(+, -, \cdots, -)$, $M, N = 0, 1, \cdots, 4$. The Higgs field $H_i$ is charged with respect to the $U(1)_i$ gauge symmetry and the covariant derivative is given by

$$\mathcal{D}_M H_i = \partial_M H_i + i w^i_M H_i, \quad (2.3)$$

where $w^i_M$ is the $U(1)_i$ gauge field with the field strength

$$\mathcal{F}^i_{MN} = \partial_M w^i_N - \partial_N w^i_M. \quad (2.4)$$

Since we want domain walls, we will choose

$$M_i = \text{diag} (m_i, -m_i), \quad (m_i > 0), \quad (2.5)$$

resulting in the $U(1)_{iA}$ flavor symmetry. We have included the neutral scalar fields $\sigma_i$ in this Abelian-Higgs model. The gauge coupling $g_i$ appears not only in front of the kinetic terms of the gauge fields and $\sigma_i$, but also as the the quartic coupling constant of $H_i$. Both these features are motivated by the supersymmetry. Indeed, we can embed this bosonic Lagrangian into a supersymmetric model with eight supercharges by adding appropriate fermions and bosons, which will not play a role to obtain domain wall solutions. We have taken this special relation among the coupling constants only to simplify concrete computations below. One may repeat the following procedure in models with more generic coupling constants without changing essential results.

The first term of the potential is the wine-bottle type and the Higgs fields develop non-zero vacuum expectation values. There are two discrete vacua for each copy $i$

$$(H_{iL}, H_{iR}, \sigma_i) = (v_i, 0, m_i), \quad (0, v_i, -m_i). \quad (2.6)$$

Thanks to the special choice of the coupling constants in $\mathcal{L}_i$ motivated by the supersymmetry, there are Bogomol’nyi-Prasad-Sommerfield (BPS) domain wall solutions in these models. Let $y$ be the coordinate of the direction orthogonal to the domain wall and we assume all the field depend on only $y$. Then, as usual, the Hamiltonian can be written as follows

$$\mathcal{H}_i = \frac{1}{2 g_i^2} \left( \partial_y \sigma_i + g_i^2 \left( |H_i|^2 - v_i^2 \right) \right)^2 + |\mathcal{D}_y H_i + \sigma_i H_i - H_i M_i|^2$$

$$+ v_i^2 \partial_y \sigma_i - \partial_y \left( (\sigma_i H_i - H_i M_i) H_i^\dagger \right)$$

$$\geq v_i^2 \partial_y \sigma_i - \partial_y \left( (\sigma_i H_i - H_i M_i) H_i^\dagger \right). \quad (2.7)$$

---

Phantom rotation of $H_{iL}$ and $H_{iR}$ in the same direction $U(1)_i$ is gauged and the remaining global symmetry is in the opposite direction and is denoted as $U(1)_{iA}$. 

5
Thus the Hamiltonian is bounded from below. This bound is called Bogomol’nyi bound, and is saturated when the following BPS equations are satisfied

$$\partial_y \sigma_i + g_i^2 \left( |H_i|^2 - v_i^2 \right) = 0,$$

$$\mathcal{D}_y H_i + \sigma_i H_i - H_i M_i = 0. \quad (2.8)$$

In order to obtain the domain wall solution interpolating the two vacua in Eq. (2.6), we impose the boundary conditions:

$$(H_{iL}, H_{iR}, \sigma_i) = (0, v_i, -m_i), \quad y = -\infty,$$

$$(H_{iL}, H_{iR}, \sigma_i) = (v_i, 0, m_i), \quad y = \infty. \quad (2.9)$$

Tension $T_i$ of the domain wall is given by a topological charge as

$$T_i = \int_{-\infty}^{\infty} dy \left[ v_i^2 \partial_y \sigma_i - \partial_y \left( (\sigma_i H_i - H_i M_i) H_i^\dagger \right) \right]_{-\infty}^{\infty} = 2m_i v_i^2. \quad (2.10)$$

The second equation of the BPS equations (2.8) can be solved by the moduli matrix formalism$^{13,15}$ with the constant matrix (vector) $H_{i0} = (C_{iL}, C_{iR})$

$$H_i = v_i e^{-\frac{\psi_i}{2}} H_{i0} e^{M_i y}, \quad \sigma_i + i w_i = \frac{1}{2} \partial_y \psi_i. \quad (2.11)$$

For a given $H_{i0}$, the scalar function $\psi_i$ is determined by the master equation

$$\partial^2_y \psi_i = 2 g_i^2 v_i^2 \left( 1 - e^{-\psi_i} H_{i0} e^{2M_i y} H_{i0}^\dagger \right). \quad (2.12)$$

The asymptotic behavior of the field $\psi_i$ is determined by the condition that the configuration reaches the vacuum at left and right infinities:

$$\psi_i \to \log H_{i0} e^{2M_i y} H_{i0}^\dagger, \quad |y| \to \infty. \quad (2.13)$$

There exists redundancy in the decomposition in Eq. (2.11), which is called the $V$-transformation:

$$H_{i0} \to V_i H_{i0}, \quad \psi_i \to \psi_i + 2 \log V_i, \quad V_i \in \mathbb{C}^*. \quad (2.14)$$

For example, a single domain wall solution centered at $y = 0$ can be generated by a moduli matrix

$$H_{i0} = (1, 1). \quad (2.15)$$

Then the master equation is

$$\partial^2_y \psi_i = 2 g_i^2 v_i^2 \left( 1 - e^{-\psi_i} \left( e^{2m_i y} + e^{-2m_i y} \right) \right). \quad (2.16)$$
Fig. 1. The left panel shows profiles of $H_{iL}$ (solid line), $H_{iR}$ (long-dashed line), and $\sigma_i$ (dashed line) with finite gauge coupling ($g_i = 0.5$). The right panel shows a plot of $\sigma_i$: dashed curve for finite ($g_i = 0.5$) gauge coupling and solid curve for strong gauge coupling ($g_i = \infty$). The other parameters are $m_i = v_i = 1$.

No analytic solutions for the master equation have been found for finite gauge couplings $g_i$, so we must solve it numerically. The corresponding solution is shown in Fig. 1. The generic solutions of the domain wall are generated by the generic moduli matrices (after fixing the $V$-transformation)

$$H_{i0} = (C_{iL}, C_{iR}), \quad C_{iL}, C_{iR} \in \mathbb{C}^\ast. \quad (2.17)$$

The complex constants $C_{iL}, C_{iR}$ are free parameters containing the moduli parameters of the BPS solutions. The moduli parameter can be defined by

$$C_i \equiv \sqrt{\frac{C_{iR}}{C_{iL}}} = e^{i\alpha_i} e^{m_i y_i}. \quad (2.18)$$

The other degree of freedom in $C_{iL}, C_{iR}$ can be eliminated by the $V$-transformation in Eq. (2.14) and has no physical meaning. Then the master equation is found to be

$$\partial^2_y \psi_i = 2 g_i^2 v_i^2 \left(1 - e^{-\psi_i} \left( e^{2m_i(y_i - y_i)} + e^{-2m_i(y_i - y_i)} \right) \right). \quad (2.19)$$

It is obvious that the real parameter $y_i$ is the translational moduli of the domain wall. The other parameter $\alpha_i$ is an internal moduli which is the Nambu-Goldstone (NG) mode associated with the $U(1)_{iA}$ flavor symmetry spontaneously broken by the domain walls.

One can take, if one wishes, the strong gauge coupling limit of the Lagrangian $\mathcal{L}_i$. As is well-known, the $U(1)$ gauge theory with two flavors of Higgs scalars in the strong gauge coupling limit becomes a non-linear sigma model whose target space is $\mathbb{C}P^1$:

$$|H_i|^2 = |H_{iL}|^2 + |H_{iR}|^2 = v_i^2. \quad (2.20)$$

The gauge fields and the neutral scalar field become infinitely massive and lose their kinetic terms. They are mere Lagrange multipliers in the limit, and are solved as

$$w^i_M = -\frac{i}{2v_i^2} \left( H_i \partial_M H_i^\dagger - \partial_M H_i H_i^\dagger \right), \quad \sigma_i = \frac{1}{v_i^2} H_i M_i H_i^\dagger. \quad (2.21)$$
Plugging these into $\mathcal{L}_i$, we get
\[
\mathcal{L}_i^\infty = \partial_M H_i P_i \partial^M H_i^\dagger - H_i M_i P_i M_i H_i^\dagger, \tag{2.22}
\]
with a projection operator
\[
P_i \equiv 1 - \frac{1}{v_i} H_i^\dagger H_i. \tag{2.23}
\]
Let us introduce an inhomogeneous coordinate $\phi_i$ of $\mathbb{C}P^1$ by
\[
H_{iL} = \frac{v_i}{\sqrt{1 + |\phi_i|^2}}, \quad H_{iR} = \frac{v_i \phi_i}{\sqrt{1 + |\phi_i|^2}}. \tag{2.24}
\]
Then the Lagrangian of the $\mathbb{C}P^1$ model in terms of $\phi_i$ is
\[
\mathcal{L}_i^\infty = v_i^2 |\partial_M \phi_i|^2 - 4m_i^2 |\phi_i|^2 \left(1 + |\phi_i|^2\right)^2. \tag{2.25}
\]
Let us reconsider the domain wall solutions in this limit. The Hamiltonian can be written as
\[
\mathcal{H}_i^\infty = \frac{v_i^2}{(1 + |\phi_i|^2)^2} \left[\partial_y \phi + 2m_i \phi_i\right]^2 + 2m_i v_i^2 \frac{d}{dy} \frac{1}{1 + |\phi_i|^2} \geq 2m_i v_i^2 \frac{d}{dy} \frac{1}{1 + |\phi_i|^2}. \tag{2.26}
\]
The BPS equation and the boundary conditions are given by
\[
\partial_y \phi_i + 2m_i \phi_i = 0, \quad \phi(y = -\infty) = \infty, \quad \phi(y = \infty) = 0, \tag{2.27}
\]
corresponding to the boundary conditions in Eq. (2.9). The BPS equation can be easily solved by
\[
\phi_i = C^{-1}_{iL} C_{iR} e^{-2m_i y} = C_i^2 e^{-2m_i y}. \tag{2.28}
\]
The tension of the domain wall is
\[
T_i = \int_{-\infty}^{\infty} dy \mathcal{H}_i^\infty = 2m_i v_i^2. \tag{2.29}
\]
This is the same as the one in the finite gauge coupling model.

In this way, the strong gauge coupling limit has a great advantage compared to the finite gauge coupling case. One can exactly solve the BPS equation and see the moduli parameter in the analytic solutions. Furthermore, there is no important differences between domain wall
solutions in the finite coupling (Abelian-Higgs model) and the strong coupling (non-linear sigma model). Both solutions have the same tension of domain wall and the same number of the moduli parameters. To see the difference explicitly, let us compare the configuration of the neutral scalar field $\sigma_i$. In the strong gauge coupling limit, it can be written as

$$\sigma_i = m_i \frac{1 - |\phi_i|^2}{1 + |\phi_i|^2} = m_i \tanh 2m_i(y - y_i),$$

where we have used

$$C_i = e^{i\alpha} e^{m_i y_i}. \tag{2.31}$$

In Fig. 1 we show the configurations of $\sigma_i$ in two cases, the one in the small finite gauge coupling and the one in the strong gauge coupling limit. As can be seen from the figure, there are no significant differences.

Let us next derive the low energy effective theory on the domain wall. We integrate all the massive modes while keeping the massless modes. We use the so-called moduli approximation where the dependence on (3+1)-dimensional spacetime coordinates comes into the effective Lagrangian only through the moduli fields:

$$C_i \rightarrow C_i(x^\mu), \quad \phi_i(y) \rightarrow \phi_i(y, C_i(x^\mu)) = C_i(x^\mu) e^{-2m_i y}. \tag{2.32}$$

The effective Lagrangian for the moduli field $C_i(x^\mu)$ can be obtained by plugging this into the Lagrangian $L_i$ and integrate it over $y$. This can be done explicitly as follows.

$$L_{i, \text{eff}} = \int_{-\infty}^{\infty} dy \frac{v_i^2}{(|C_i|^{-2} e^{2m_i y} + |C_i|^2 e^{-2m_i y})^2} \frac{[\partial_\mu C_i]^2}{|C_i|^2} = \frac{v_i^2}{4m_i} \frac{[\partial_\mu C_i]^2}{|C_i|^2}. \tag{2.33}$$

With Eq. (2.31), the effective Lagrangian is given by

$$L_{i, \text{eff}} = \frac{2m_i v_i^2}{2} (\partial_\mu y_i)^2 + \frac{v_i^2}{m_i} (\partial_\mu \alpha_i)^2, \tag{2.34}$$

where energy of soliton solution is neglected since it does not contribute to dynamics of moduli. Note that $2m_i v_i^2$ is precisely the domain wall tension. This is the free field Lagrangian.

Although we have derived this effective Lagrangian in the strong gauge coupling limit, we can obtain the same Lagrangian in the finite gauge coupling constant. In other words, the effective Lagrangian cannot distinguish the infinite versus finite coupling cases at least in the quadratic order of the derivative expansion.
2.2. Localization of the Abelian gauge fields

In the previous subsection, we have seen that the NG modes of the translation and $U(1)$
global symmetry are the only massless modes in the Abelian-Higgs model. They are localized
on the domain wall. There are no massless gauge field on the domain wall and all the modes
contained in the gauge field are massive. The mass of the lightest mode of the gauge field
is of the order of the inverse of the width of the domain wall, since the bulk outside of the
domain wall is in the Higgs phase. The low energy effective Lagrangian for the massless
fields is obtained after integrating out the massive modes including gauge fields.

In order to obtain the massless gauge field to be localized on the domain wall, we need
a new gauge symmetry which is unbroken in the bulk. Recently, a new mechanism was
proposed to localize gauge fields on domain walls.\textsuperscript{12}

A key ingredient is the so-called dielectric coupling constant for the new gauge
symmetry. To illustrate the new localization mechanism, let us introduce a new $U(1)$ gauge
field $a_M$ which we wish to localize on the domain wall. Since this gauge symmetry should
be unbroken in the bulk, we consider the case where all the Higgs fields are neutral under
this newly introduced $U(1)$ gauge symmetry. The gauge field $a_M$ is assumed to couple to
the neutral scalar fields $\sigma_i$ only in the following particular combination

$$L = L_1 + L_2 - \frac{\lambda}{2} \left( \frac{\sigma_1}{m_1} - \frac{\sigma_2}{m_2} \right) (G_{MN})^2,$$

(2.35)

where a real constant $\lambda$ with the unit mass dimension, in accordance with the (4+1)-
dimensional spacetime and the field strength is defined by

$$G_{MN} = \partial_M a_N - \partial_N a_M.$$

(2.36)

The field-dependent gauge coupling function is given by

$$\frac{1}{4e^2(\sigma)} = \frac{\lambda}{2} \left( \frac{\sigma_1}{m_1} - \frac{\sigma_2}{m_2} \right),$$

(2.37)

which depends on the position $y$ through fields $\sigma_i$. Thus the field-dependent gauge coupling
function $e(\sigma)$ plays the role of the dielectric coupling constant. Furthermore, the special
choice in Eq. (2.37) is chosen for the gauge interaction to become strongly coupled in the
bulk ($\sigma_i \rightarrow \pm m_i$ as $y \rightarrow \pm \infty$).

Let us again consider a double copy of domain walls as a background configuration in the
Abelian-Higgs model in Eq. (2.35). Since Lagrangian has no term linear in $a_M$, the equations
of motion for $a_M$ is trivially solved by $a_M = 0$, and the rest of the equations of motion are
explicitly the same as those in the previous subsection. Therefore the domain wall solution
in the previous subsection together with $a_M = 0$ is still a solution of the equations of motion. Clearly, the low energy effective Lagrangian on the domain wall is also unchanged

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{1,\text{eff}} + \mathcal{L}_{2,\text{eff}} - \frac{1}{4e_4^2} (G_{\mu\nu})^2, \quad (2.38)$$

except for the additional kinetic term (the last term) of the (3+1)-dimensional gauge field $w_{\mu}$, which is the zero mode ($y$-independent mode) of the (4+1)-dimensional field $w_{\mu}$. The (3+1)-dimensional gauge coupling constant is given by

$$\frac{1}{4e_4^2} = \frac{\lambda}{2} \int_{-\infty}^{\infty} dy \left( \frac{\sigma_1}{m_1} - \frac{\sigma_2}{m_2} \right) = \frac{\lambda}{4} \left[ \frac{\psi_1}{m_1} - \frac{\psi_2}{m_2} \right]_{-\infty}^{\infty} = \lambda(y_2 - y_1), \quad (2.39)$$

where we have used the asymptotic behavior $\psi_i \to \log 2 \cosh 2m_i(y - y_i)$ as $|y| \to \infty$. Note that this result is again independent of the gauge couplings $g_i$ in the domain wall sector. In summary, the low energy effective Lagrangian is

$$\mathcal{L}_{\text{eff}} = \sum_{i=1,2} \left[ \frac{2m_iv_i^2}{2} (\partial_\mu y_i)^2 + \frac{v_i^2}{m_i} (\partial_\mu \alpha_i)^2 \right] - \lambda(y_2 - y_1)(G_{\mu\nu})^2. \quad (2.40)$$

Now we separate the quantum fields (fluctuations) from the classical background moduli parameters by

$$y_i(x^\mu) = y_i^0 + \delta y_i, \quad \alpha_i(x^\mu) = \alpha_i^0 + \delta \alpha_i. \quad (2.41)$$

Then the effective Lagrangian up to the second order of the small quantum fluctuations is given by

$$\mathcal{L}_{\text{eff}}(y_i^0, \alpha_i^0) = \sum_{i=1,2} \left[ \frac{2m_iv_i^2}{2} (\partial_\mu \delta y_i)^2 + \frac{v_i^2}{m_i} (\partial_\mu \delta \alpha_i)^2 \right] - \lambda(y_2^0 - y_1^0)(G_{\mu\nu})^2. \quad (2.42)$$

We note that the massless gauge field $a_\mu$ has a positive finite gauge coupling squared\(^\text{*}) $1/(4\lambda(y_2^0 - y_1^0))$ provided $y_2^0 - y_1^0 > 0$.

Although we succeeded in localizing the massless $U(1)$ gauge field $a_\mu$ on the domain walls, the Lagrangian Eq. (2.42) has no charged matter fields minimally coupled with the localized gauge field $a_\mu$. To obtain matter fields interacting with the localized gauge field, one may be tempted to identify the Higgs fields $H_i^0 = (H_{iL}^0, H_{iR}^0)$ as matter fields\(^\text{**}) with

\(^*\) Here we are content with the fact that the positivity of the gauge kinetic term is assured at least in finite region of moduli space, instead of just at a point. However, it is possible to make a more economical model where one has less moduli, and the positivity of the gauge kinetic term is assured\(^\text{12})

\(^{**}\) We consider the diagonal subgroup $U(1)_A$ of $U(1)_{1,A}$ and $U(1)_{2,A}$. Actually the $U(1)_{1,A}$ global symmetries are broken by the domain wall solution, we consider this gauging to leading order of gauge coupling only to illustrate the Higgs mechanism for the broken symmetry.

11
charges (1, −1). The minimal gauge interaction of Higgs fields with the $a_M$ is introduced through the modified covariant derivatives as

$$\tilde{D}_M H_{iL} = \partial_M H_{iL} + i w_M^i H_{iL} + i a_M H_{iL}, \quad (2.43)$$

$$\tilde{D}_M H_{iR} = \partial_M H_{iR} + i w_M^i H_{iR} - i a_M H_{iR}. \quad (2.44)$$

Since the moduli field $C_i$ is charged, the derivatives in the low energy effective theory Eq. (2.33) should be replaced by the covariant derivative

$$\partial_\mu C_i \rightarrow D_\mu C_i = \partial_\mu C_i + i a_\mu C_i. \quad (2.45)$$

It is straightforward task to derive the effective Lagrangian with the covariant derivative above along the same line of reasoning for the previous case

$$L_{\text{eff}}(y_0^i, \alpha_0^i) = \sum_{i=1,2} \left[ \frac{2m_i v_i^2}{2} (\partial_\mu \delta y_i)^2 + \frac{v_i^2}{m_i} (\partial_\mu \delta \alpha_i + q_i a_\mu)^2 \right] - \lambda (y_2^0 - y_1^0)(G_{\mu\nu})^2. \quad (2.46)$$

This clearly shows that the new gauge field $a_\mu$ is not massless due to the Higgs mechanism, and should be integrated out together with the other massive fields. Namely the low energy effective Lagrangian does not include the massless gauge fields, since the $U(1)$ symmetry which we gauged is broken by the domain wall. A more explicit example at the strong gauge coupling limit is described in Appendix A. Thus the Abelian-Higgs model in this section gives an important lesson that we should not gauge a symmetry which is broken by the domain wall solution, since the corresponding gauge fields may be localized on the domain walls but they become massive and should be integrated out from the low energy effective theory. In the next section, we will give a model with a non-Abelian global symmetry whose unbroken subgroup can be gauged to yield massless localized gauge fields on the domain wall.

### §3. The chiral model

In this section we study domain walls in the chiral model which is a natural extension of the Abelian-Higgs model in the previous section. This chiral model leads to two important consequences 1) massless non-Abelian gauge fields are localized on the domain wall and moreover 2) the scalar fields which are non-trivially interacting are also localized on the domain walls.
3.1. The domain walls in the chiral model

As a natural extension of the domain wall sector in the previous section, we consider the Yang-Mills-Higgs model with $SU(N_c) \times U(1)$ gauge symmetry with $S[U(N)_L \times U(N)_R] = SU(N)_L \times SU(N)_R \times U(1)_A$ flavor symmetry. To localize the gauge field in a simple manner, we again introduce two sectors $\mathcal{L}_1$ and $\mathcal{L}_2$, but only the former is extended to Yang-Mills-Higgs system and the latter is the same form as (2.16). The second sector couples to the first sector through the coupling as described in (2.35) after gauging the flavor symmetry it plays a role as localization of gauge fields, combined with the first sector. The matter contents are summarized in Table I. Since the presence of two factors of $SU(N)$ global symmetry resembles the chiral symmetry of QCD, we call this Yang-Mills-Higgs system as the chiral model.

The Lagrangian is then given by

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2,$$

$$\mathcal{L}_1 = \text{Tr} \left[ -\frac{1}{2g_1^2}(F_{1MN})^2 + \frac{1}{g_1^2}(D_M \Sigma_1)^2 + |D_M H_1|^2 \right] - V_1,$$

$$V_1 = \text{Tr} \left[ \frac{g_1^2}{4} \left( H_1 H_1^\dagger - v_1^2 1_N \right)^2 + |\Sigma_1 H_1 - H_1 M_1|^2 \right],$$

with $H_1 = (H_{1L}, H_{2L})$. $\mathcal{L}_2$ is the same form as (2.16) with $i = 2$. Gauge fields of $U(N)_c = (SU(N)_c \times U(1)_1)/Z_N$ are denoted as $W_{1M}$, and adjoint scalar as $\Sigma_1$. The covariant derivative and the field strength are denoted as $D_M \Sigma_1 = \partial_M \Sigma_1 + i[W_{1M}, \Sigma_1]$, $D_M H_1 = \partial_M H_1 + iW_{1M} H_1$, and $F_{1MN} = \partial_M W_{1N} - \partial_N W_{1M} + i[W_{1M}, W_{1N}]$. The mass matrix is given by $M_1 = \text{diag}(m_1 1_N, -m_1 1_N)$. Let us note that the chiral model reduces to the Abelian-Higgs model in the limit of $N \to 1$, by deleting all the $SU(N)$ groups.

The second sector is just necessary to realize the field-dependent gauge coupling function similar to (2.35) as we will discuss in the subsequent subsection. In the rest of this subsection,
we focus only on the first sector \((i = 1)\) and suppress the index \(i = 1\). The symmetry transformations act on the fields as

\[
H = (H_L, H_R) \rightarrow U_c(H_L, H_R)\begin{pmatrix} U_L e^{i\alpha} \\ U_R e^{-i\alpha} \end{pmatrix},
\]

\[
\Sigma \rightarrow U_c\Sigma U_c^\dagger,
\]

with \(U_c \in U(N)_c\), \(U_L \in SU(N)_L\), \(U(N)_R \in SU(N)_R\) and \(e^{i\alpha} \in U(1)_A\).

There exist \(N + 1\) vacua in which the fields develop the following VEV

\[
H = (H_L, H_R) = v\begin{pmatrix} 1_{N-r} \\ 0_r \end{pmatrix} 0_{N-r} 1_r,
\]

\[
\Sigma = m\begin{pmatrix} 1_{N-r} \\ -1_r \end{pmatrix},
\]

with \(r = 0, 1, 2, \cdots, N\). We refer these vacua with the label \(r\). In the \(r\)-th vacuum, both the local gauge symmetry \(U(N)_c\) and the global symmetry are broken, but a diagonal global symmetries are unbroken (color-flavor-locking)

\[
U(N)_c \times SU(N)_L \times SU(N)_R \times U(1)_A \rightarrow 
SU(N - r)_{L+c} \times SU(r)_L \times SU(r)_{R+c} \times SU(N - r)_R \times U(1)_{A+c}.
\]

As in the Abelian-Higgs model, the BPS equations for the domain walls can be obtained through the Bogomol’nyi completion of the energy density with the assumption that all the fields depend on only the fifth coordinate \(y\) and \(W_\mu = 0\):

\[
\mathcal{H} = \text{Tr} \left[ \frac{1}{g^2} \left( D_y \Sigma - \frac{g^2}{2} (v^2 1_N - HH^\dagger) \right)^2 + |D_y H + \Sigma H - HM|^2 \right] + \partial_y \left\{ \text{Tr} \left[ v^2 \Sigma - (\Sigma H - HM) H^\dagger \right] \right\} \\
\geq \partial_y \left\{ \text{Tr} \left[ v^2 \Sigma - (\Sigma H - HM) H^\dagger \right] \right\}.
\]

This bound is saturated when the following BPS equations are satisfied

\[
D_y \Sigma - \frac{g^2}{2} (v^2 1_N - HH^\dagger) = 0,
\]

\[
D_y H + \Sigma H - HM = 0.
\]

The tension of the domain wall is given by

\[
T = \int_{-\infty}^{\infty} dy \partial_y \left\{ \text{Tr} \left[ v^2 \Sigma - (\Sigma H - HM) H^\dagger \right] \right\} \\
= v^2 \text{Tr} [\Sigma(\infty) - \Sigma(-\infty)].
\]
Let us concentrate on the domain wall which connects the 0-th vacuum at \( y \to \infty \) and the \( N \)-th vacuum at \( y \to -\infty \). Its tension can be read as

\[
T = 2Nv^2m,
\]

from Eq. (3.12). Since there are \( N + 1 \) possible vacua, the maximal number of walls is \( N \) at various positions. The simplest domain wall solution corresponding to the coincident walls is given by making an ansatz that \( H_L, H_R, \Sigma \) and \( W_y \) are all proportional to the unit matrix. Then the BPS equations (3.10) and (3.11) can be identified with the BPS equations in Eq. (2.8) in the Abelian-Higgs model. Thus the domain wall solution can be solved as

\[
H_L = ve^{-\frac{y}{2}e^{my}} 1_N, \quad H_R = ve^{-\frac{y}{2}e^{-my}} 1_N, \quad \Sigma + iW_y = \frac{1}{2}\partial_y \psi 1_N,
\]

where \( \psi \) is the solution of the master equation (2.12) in the Abelian-Higgs model. Eq. (3.8) shows that the unbroken global symmetry for \( N \)-th vacuum \((H_L = 0, H_R = v1_N \) and \( \Sigma = -m1_N \)) at the left infinity \( y \to -\infty \) is \( SU(N)_L \times SU(N)_{R+c} \times U(1)_{A+c} \), whereas that for the 0-th vacuum \((H_L = v1_N, H_R = 0 \) and \( \Sigma = m1_N \)) at the right infinity \( y \to \infty \) is \( SU(N)_{L+c} \times SU(N)_R \times U(1)_{A+c} \).

The domain wall solution further breaks these unbroken symmetries because it interpolates the two vacua. The breaking pattern by the domain wall is

\[
U(N)_c \times SU(N)_L \times SU(N)_R \times U(1)_A \to SU(N)_{L+R+c}.
\]

This spontaneous breaking of the global symmetry gives NG modes on the domain wall as massless degrees of freedom valued on the coset similarly to the chiral symmetry breaking in QCD:

\[
\frac{SU(N)_L \times SU(N)_R}{SU(N)_{L+R+c}} \times U(1)_A.
\]

Since our model can be embedded into a supersymmetric field theory, these NG modes (\( U(N) \) chiral fields) appear as complex scalar fields accompanied with additional \( N^2 \) pseudo-NG modes.\(^*\)

\(^*\) The unbroken generators of \( U(1)_{A+c} \) for \( r \)-th vacuum contains different combination of \( U(N)_c \) generators depending on \( r \). Therefore the right and left vacua preserve actually different \( U(1)_{A+c} \), and the wall solution does not preserve any of these \( U(1)_{A+c} \).

\(^*\) One of them is actually a genuine NG mode corresponding to the broken translation.
3.2. Localization of the matter fields

In the remainder of this subsection, we will give the low-energy effective Lagrangian on the domain walls where the massless moduli fields (the matter fields) are localized. The best way to parametrize these massless moduli fields is to use the moduli matrix formalism\cite{13, 14, 15}

\[
H_L = ve^{my}S^{-1}, \quad (3.19)
\]
\[
H_R = ve^{-my}S^{-1}e^\phi, \quad (3.20)
\]
\[
\Sigma + iW_y = S^{-1}\partial_y S, \quad (3.21)
\]

where \( S \in GL(N, \mathbb{C}) \) and \( \Omega = SS^\dagger \) is the solution of the following master equation

\[
\partial_y (\Omega^{-1}\partial_y \Omega) = g^2v^2(1_N - \Omega^{-1}\Omega_0), \quad (3.22)
\]

where

\[
\Omega_0 = e^{2my}1_N + e^{-2my}e^\phi e^{\phi^\dagger}. \quad (3.23)
\]

We have used the V-transformation to identify the moduli \( e^\phi \), which is a complex \( N \times N \) matrix. It can be parametrized by an \( N \times N \) hermitian matrix \( \hat{x} \) and a unitary matrix \( U \) as\cite{13}

\[
e^\phi = e^{\hat{x}}U^\dagger, \quad (3.23)
\]

where \( U \) is nothing but the \( U(N) \) chiral fields associated with the spontaneous symmetry breaking Eq. (3.18) and \( \hat{x} \) is the pseudo-NG modes whose existence we promised above.

In the strong gauge coupling limit \( g \to \infty \), solution of master equation is simply \( \Omega = \Omega_0 \). After fixing the \( U(N)_c \) gauge, we obtain

\[
S = e^{\hat{y}/2}\sqrt{2\cosh(2my - \hat{x})}. \quad (3.24)
\]

Let us denote, for brevity

\[
\hat{y} = 2my - \hat{x}, \quad (3.25)
\]

the Higgs fields are then given as

\[
H_L = v\frac{e^{\hat{y}/2}}{\sqrt{2\cosh \hat{y}}}, \quad (3.26)
\]
\[
H_R = v\frac{e^{-\hat{y}/2}U^\dagger}{\sqrt{2\cosh \hat{y}}}. \quad (3.27)
\]

From this solution, one can easily recognize that eigenvalues of \( \hat{x} \) correspond to the positions of the \( N \) domain walls in the \( y \) direction. Now we promote moduli parameters \( \hat{x} \) and \( U \) to
fields on the domain wall world volume, namely functions of world volume coordinates $x^\mu$.

We plug the domain wall solutions $H_{L,R}(y; \hat{x}(x^\mu), U(x^\mu))$ into the original Lagrangian $L$ in Eq. (3.2) at $g \to \infty$ and pick up the terms quadratic in the derivatives. Thus the low energy effective Lagrangian is given by

$$L_{\text{eff}} = \int_{-\infty}^{\infty} dy \, \text{Tr} \left[ \partial_\mu H_L^\dagger \partial_\mu H_L^\dagger + \partial_\mu H_R^\dagger \partial_\mu H_R^\dagger - v^2 W_\mu W_\mu \right], \quad (3.28)$$

where

$$W_\mu = \frac{i}{2v^2} \left[ \partial_\mu H_L^\dagger H_L^\dagger - H_L \partial_\mu H_L^\dagger + (L \leftrightarrow R) \right]. \quad (3.29)$$

Here we have eliminated the massive gauge field $W_\mu$ by using the equation of motion. Using the solutions for $H_L$ and $H_R$ we have found a closed formula for the effective Lagrangian up to the second order of derivatives but with full nonlinear interactions involving moduli fields $\hat{x}$ and $U$. Detailed derivation is given in Appendix B.

Here we exhibit the result only in the leading orders of $U^{-1}$ and $\hat{x}$:

$$L_{\text{eff}} = \frac{v^2}{2m} \text{Tr} \left( \partial_\mu U^\dagger \partial_\mu U + \partial_\mu \hat{x} \partial_\mu \hat{x} \right) + \ldots \quad (3.30)$$

When $N = 1$ and with the redefinitions $U = e^{2i\alpha_1}$, and $\hat{x} = 2my_1$, this coincides with the effective Lagrangian $L_{i=1,\text{eff}}$ in Eq. (2.34) of the Abelian-Higgs model, which we obtained in the previous section.

### 3.3. Localization of the gauge fields

Let us next introduce the gauge fields which are to be massless and localized on the domain walls. As we learned in section 2, the associated gauge symmetry should not be broken by the domain walls. Therefore, the symmetry which we can gauge is the unbroken symmetry $SU(N)_{L+R+c}$ itself or its subgroup.

Let us gauge $SU(N)_{L+R} \equiv SU(N)_V$ and let $A_\mu^a$ be the $SU(N)_{L+R}$ gauge field. The Higgs fields are in the bi-fundamental representation of $U(N)_c$ and $SU(N)_{L+R}$. The covariant derivatives of the Higgs fields are modified by

$$\hat{D}_M H_{1L} = \partial_M H_{1L} + iW_{1M} H_{1L} - iH_{1L} A_M, \quad (3.31)$$

$$\hat{D}_M H_{1R} = \partial_M H_{1R} + iW_{1M} H_{1R} - iH_{1R} A_M. \quad (3.32)$$

The quantum numbers are summarized in Table III.

We now introduce a field-dependent gauge coupling function $g^2(\Sigma)$ for $A_M$, which is inspired by the supersymmetric model in Ref.[12].

$$\frac{1}{2e^2(\Sigma)} = \frac{\lambda}{2} \left( \frac{\text{Tr} \Sigma_1}{Nm_1} - \frac{\Sigma_2}{m_2} \right). \quad (3.33)$$
The Lagrangian is given by
\[ \mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 - \frac{1}{2e^2(\Sigma)} \text{Tr} [G_{MN}G^{MN}] . \] (3.34)

The equations of motion for the gauge field and the moduli are derived and applied to get the final result. The spectrum of massless NG modes is unchanged by switching on the gauge fields, and the gauged chiral model admits the same domain wall solutions as those in the ungauged chiral model.

Table II. Quantum numbers of the domain wall sectors in gauged chiral model

|                  | $SU(N)_c$ | $U(1)_1$ | $U(1)_2$ | $SU(N)_V$ | $U(1)_{1A}$ | $U(1)_{2A}$ | mass       |
|------------------|-----------|-----------|-----------|-----------|-------------|-------------|------------|
| $H_{1L}$         | □         | 1         | 0         | □         | 1           | 0           | $m_11_N$   |
| $H_{1R}$         | □         | 1         | 0         | □         | −1          | 0           | $−m_11_N$  |
| $\Sigma_1$      | adj ⊕ 1   | 0         | 0         | 1         | 0           | 0           | 0          |
| $H_{2L}$         | 1         | 0         | 1         | 1         | 0           | 1           | $m_2$      |
| $H_{2R}$         | 1         | 0         | 1         | 1         | 0           | −1          | $−m_2$     |
| $\Sigma_2$      | 1         | 0         | 0         | 1         | 0           | 0           | 0          |
We just repeat the similar computation to those in section 3.2. Again we shall focus on the first sector \( L_1 \) and suppress the index \( i = 1 \) of fields. Since color gauge fields \( W_\mu \) becomes auxiliary fields and eliminated through their equations of motion, it is convenient to define the covariant derivative only for the flavor \((SU(N)_{L+R})\) gauge interactions as

\[
\hat{D}_\mu H = \partial_\mu H - iH A_\mu.
\]

Then we obtain the effective Lagrangian of the first sector as

\[
L_{1,\text{eff}} = \int_{-\infty}^{\infty} dy \text{Tr} \left[ \hat{D}_\mu H_L (\hat{D}^\mu H_L)^\dagger + \hat{D}_\mu H_R (\hat{D}^\mu H_R)^\dagger - v^2 W_\mu W^\mu \right. \\
- \frac{1}{2\varepsilon^2(\Sigma)} G_{MN} G^{MN} \left. \right], \quad (3.37)
\]

with

\[
W_\mu = \frac{i}{2v^2} \left[ \hat{D}_\mu H_L H_L^\dagger - H_L (\hat{D}_\mu H_L)^\dagger + (L \leftrightarrow R) \right]. \quad (3.38)
\]

Eliminating \( W_\mu \), we obtain the following expression for the integrand of the effective Lagrangian after some simplification

\[
L_{\text{eff}} = \frac{1}{2v^2} \int_{-\infty}^{\infty} dy \text{Tr} \left[ D_\mu H_{ab} D^\mu H_{ba} \right], \quad (3.39)
\]

where we defined fields \( H_{ab} \) with the label \( ab \) of adjoint representation of the flavor gauge group \( SU(N)_{L+R+c} \) and the covariant derivative as

\[
D_\mu H_{ab} = \partial_\mu H_{ab} + i[A_\mu, H_{ab}], \quad H_{ab} \equiv H^\dagger_a H_b, \quad a, b = L, R. \quad (3.40)
\]

In Appendix B we will describe fully the procedure to derive the effective Lagrangian by substituting (3.26) and (3.27) and rewriting in terms of moduli fields \( \hat{x} \) and \( U \). Here we merely state the result:

\[
L_{1,\text{eff}} = \frac{v^2}{2m} \text{Tr} \left[ D_\mu \hat{x} \frac{\cosh(L_{\hat{x}}) - 1}{L_{\hat{x}} \sinh(L_{\hat{x}})} \ln \left( \frac{1 + \tanh(L_{\hat{x}})}{1 - \tanh(L_{\hat{x}})} \right) (D^\mu \hat{x}) \right. \\
+ U^\dagger D_\mu U \frac{\cosh(L_{\hat{x}}) - 1}{L_{\hat{x}} \sinh(L_{\hat{x}})} \ln \left( \frac{1 + \tanh(L_{\hat{x}})}{1 - \tanh(L_{\hat{x}})} \right) (D^\mu \hat{x}) \\
+ \left. \frac{1}{2} D_\mu U^\dagger U \frac{1}{\tanh(L_{\hat{x}})} \ln \left( \frac{1 + \tanh(L_{\hat{x}})}{1 - \tanh(L_{\hat{x}})} \right) (U^\dagger D^\mu U) \right], \quad (3.41)
\]

\(^{(*)}\) Tree level mass spectra are unchanged even though the chiral symmetry \( SU(N)_L \times SU(N)_R \) is broken by the \( SU(N)_{L+R} \) gauge interactions.
where
\[ \mathcal{L}_A(B) = [A, B] \] (3.42)
is a Lie derivative with respect to \( A \). The covariant derivative \( \mathcal{D}_\mu \) is defined by
\[ \mathcal{D}_\mu U = \partial_\mu U + i [A_\mu, U]. \] (3.43)

The above result suggests that the chiral fields \( U(x^\mu) \) and hermitian fields \( \hat{x}(x^\mu) \) are in the adjoint representation of \( SU(N)_{L+R} \). Let us now examine the transformation property of \( U \) and \( \hat{x} \) under the \( SU(N)_{L+R} \) flavor gauge transformation on the domain wall background in order to demonstrate that they are in the adjoint representation. The domain wall solution only preserves the diagonal subgroup \( SU(N)_{L+R+c} \). Eqs. (3.4) and (3.5) shows the fields transform under the \( SU(N)_{L+R+c} \) transformations \( U \) as
\[ H'_L = U H_L U^\dagger, \quad H'_R = U H_R U^\dagger, \quad \Sigma' = U \Sigma U^\dagger. \] (3.44)

Eqs. (3.19) and (3.20) show that
\[ S' = U S U^\dagger, \quad e^{\phi'} = U e^{\phi} U^\dagger, \quad \Omega' = U \Omega U^\dagger. \] (3.45)
The complex moduli \( e^{\phi} \) is decomposed into hermitian part \( e^{\hat{x}} \) and unitary part \( U \) in Eq. (3.23). Since we can express \( e^{2\hat{x}} = e^{\phi} e^{\hat{x}} \), and \( U = e^{-\phi} e^{\hat{x}} \), we find that they transform as adjoint representations
\[ e^{2\hat{x}'} = U e^{2\hat{x}} U^\dagger, \quad U' = U U U^\dagger. \] (3.46)

By expanding (3.41), we here illustrate nonlinear interactions of \( \hat{x} \) up to fourth orders in the fluctuations \( \hat{x} \) and \( U - 1 \)
\[ \mathcal{L}_{1,\text{eff}} = \frac{v^2}{2m} \text{Tr} \left( \mathcal{D}_\mu U^\dagger \mathcal{D}^\mu U + \mathcal{D}_\mu \hat{x} \mathcal{D}^\mu \hat{x} + U^\dagger \mathcal{D}_\mu U \left[ \hat{x}, \mathcal{D}^\mu \hat{x} \right] - \frac{1}{12} \left[ \mathcal{D}_\mu \hat{x}, \hat{x} \right] \left[ \hat{x}, \mathcal{D}^\mu \hat{x} \right] + \frac{1}{3} \left[ \mathcal{D}_\mu U^\dagger U, \hat{x} \right] \left[ \hat{x}, U^\dagger \mathcal{D}^\mu U \right] + \cdots \right). \] (3.47)

Similarly to Eq. (2.39), we can define the (3+1)-dimensional non-Abelian gauge coupling \( e_4 \) by integrating (3.33) and find
\[ \frac{1}{2e_4^2} = \int dy \frac{1}{2e_4^2 (\Sigma)} = \lambda (y_2 - y_1), \] (3.48)
where \( y_i \) is the wall position for the \( i \)-th domain wall sector. Summarizing, we obtain the following effective Lagrangian
\[ \mathcal{L}_{\text{eff}} = \mathcal{L}_{1,\text{eff}} + \mathcal{L}_{2,\text{eff}} - \frac{1}{2e_4^2} \text{Tr} \left[ G_{\mu\nu} G^{\mu\nu} \right]. \] (3.49)
where $\mathcal{L}_{2,\text{eff}}$ is given in (2.34). This is the main result of this paper. We have succeeded in constructing the low energy effective theory in which the matter fields (the chiral fields) and the non-Abelian gauge fields are localized with the non-trivial interaction. We show the profile of "wave functions" of localized massless gauge field and massless matter fields as functions of the coordinate $y$ of the extra dimension in Fig. 2.

![Wave functions of the zero modes](image)

Fig. 2. The wave functions of the zero modes. DW1 and DW2 stand for the wave functions of the massless matter fields of the $i = 1$ domain wall and $i = 2$ domain wall, respectively for strong gauge coupling limit $g_i = \infty$ and $m_i = 1$. The gauge fields are localized between the domain walls.

As is seen from Eq.(3.47), the flavor gauge symmetry $SU(N)_{L+R+c}$ is further (partly) broken and the corresponding gauge field $A_\mu$ becomes massive, when the fluctuation $\phi = \hat{e}^x U$ develops non-zero vacuum expectation values. Especially, $\hat{x}$ is interesting because its non-vanishing (diagonal) values of the fluctuation has the physical meaning as the separation between walls away from the coincident case. For instance, if all the walls are separated, $SU(N)_{L+R+c}$ is spontaneously broken to the maximal $U(1)$ subgroup $U(1)^{N-1}$. However, if $r$ walls are still coincident and all other walls are separated, we have an unbroken gauge symmetry $SU(r) \times U(1)^{N-r+1}$. Then, a part of the pseudo-NG modes $\hat{x}$ turn to NG modes associated with the further symmetry breaking $SU(N)_{L+R+c} \to SU(r) \times U(1)^{N-r+1}$, so that the total number of zero modes is preserved. These new NG modes, called the non-Abelian cloud, spread between the separated domain walls. The flavor gauge fields eat the non-Abelian cloud and get masses which are proportional to the separation of the domain walls. This is the Higgs mechanism in our model. This geometrical understanding of the Higgs mechanism is quite similar to D-brane systems in superstring theory. So our domain wall system provides a genuine prototype of field theoretical D3-branes.

---

*In Ref[13], the authors argued that the non-Abelian clouds spreading between walls become massive contrary to the results of Ref[13].*
4. Embedding into supersymmetric theory

A crucial point to localize gauge field around domain wall is the coupling between scalar and gauge kinetic term. Such a coupling is naturally realized in (4+1)-dimensional supersymmetric gauge theory. This theory generally consists of hypermultiplet part and vector multiplet part. The latter is specified by the so-called prepotential. In (4 + 1)-dimensional theory the prepotential generally allows up to cubic terms in vector multiplets, which serves interactions among vector multiplets such as $\Sigma^3$.

4.1. Supersymmetric model

In embedding the model into supersymmetric gauge theories in (4+1) dimensions, we will give non-Abelian global flavor symmetry $SU(N_i)_V$ for each copy ($i = 1, 2$) of the domain wall sector, instead of only one copy as in $(3.34)$ of the previous section. This contains the model $(3.34)$ as a limiting case of $N_2 \rightarrow 1$, and may offer more general situation phenomenologically.

To formulate supersymmetric gauge theories, we need to introduce $Y_i$ as auxiliary fields of $U(N_i)_c$ vector multiplet, and $\Phi_i$ and $\Sigma_i$ as adjoint scalar fields and auxiliary fields of $SU(N_i)_V$ vector multiplet. As bosonic fields of theories with eight supercharges, we also need to double the scalar fields $H_i$, by introducing another set $\tilde{H}_i = (\tilde{H}_i^L, \tilde{H}_i^R)$ with masses $(m_i 1_{N_i}, -m_i 1_{N_i})$. They are in the same representations as $H_i$ under $U(N_i)_c$ and $U(1)_{iA}$.

Explicit charge assignments for hypermultiplets matter fields and adjoint scalar fields are summarized in Table III. The resultant supersymmetric Lagrangian is written as

$$\mathcal{L} = a_{\alpha\beta}(\Sigma) \left( -\frac{1}{4} F^{\alpha}_{MN} F_{\beta MN} + \frac{1}{2} \mathcal{D}_M \Sigma^\alpha \mathcal{D}^M \Sigma^\beta + \frac{1}{2} Y^\alpha Y^\beta \right) - c_\alpha Y^\alpha$$

$$+ \sum_{i=1}^2 \text{Tr} \left\{ \left( \mathcal{D}_M H_{iL} \mathcal{D}^M H_{iL} + \mathcal{D}_M \tilde{H}_{iL} \mathcal{D}^M \tilde{H}_{iL} + (L \leftrightarrow R) \right) - V_{iF} + \mathcal{L}_{iY} + \mathcal{L}_{iCS} + \mathcal{L}_{i\text{fermion}} \right\}.$$  

(4.1)
where

\[
\mathcal{L}_{iY} = \text{Tr} \left[ H_{iL}^1 Y_i H_{iL} - H_{iL}^1 H_{iL} Y_i + H_{iL}^1 Y_i H_{iL} + Y_i H_{iL} H_{iL}^1 + (L \leftrightarrow R) \right],
\]

\[
V_{iF} = \text{Tr} \left[ \Sigma_i H_{iL} - H_{iL} (\Phi_i + m_i 1_{N_i})^2 + |H_{iL} \Sigma_i - (\Phi_i + m_i 1_{N_i}) H_{iL}|^2 
+ |\Sigma_i H_{iR} - H_{iR} (\Phi_i - m_i 1_{N_i})|^2 + |H_{iR} \Sigma_i - (\Phi_i - m_i 1_{N_i}) H_{iR}|^2 \right],
\]

where \(\alpha, \beta \cdots\) denote all gauge groups and their generators collectively. We label them with the ordering

\[
\alpha, \beta = 0_1, I_1, A_1; 0_2, I_2, A_2,
\]

where \(0_i\) denotes \(U(1)_i\) parts of \(U(N_i)\) gauge group, while \(I_i = 1, \cdots, N_i^2 - 1\) are color indices of \(SU(N_i)\) and \(A_i = 1, \cdots, N_i^2 - 1\) denotes flavor indices of \(SU(N_i)\) gauge group.

The scalar fields \(\Sigma^\alpha\) and auxiliary fields \(Y^\alpha\) are explicitly written by

\[
\Sigma^\alpha = (\Sigma^{0_1}, \Sigma^{I_1}, \Phi^{A_1}, \Sigma^{0_2}, \Sigma^{I_2}, \Phi^{A_2}),
\]

\[
Y^\alpha = (Y^{0_1}, Y^{I_1}, Y^{A_1}, Y^{0_2}, Y^{I_2}, Y^{A_2}),
\]

and similarly the field strength \(F_{MN}^\alpha\) and gauge field \(W_M^\alpha\) are written by

\[
F_{MN}^\alpha = (F_{MN}^{0_1}, F_{MN}^{I_1}, F_{MN}^{A_1}, F_{MN}^{0_2}, F_{MN}^{I_2}, F_{MN}^{A_2}),
\]

\[
W_M^\alpha = (W_M^{0_1}, W_M^{I_1}, W_M^{A_1}, W_M^{0_2}, W_M^{I_2}, W_M^{A_2}).
\]

We adopt the convention of \(U(N_i)\) and \(SU(N_i)\) matrices such as

\[
\Sigma_i = \Sigma_i^{0_1} 1_{N_i} + \Sigma_i^{I_1} T_i^{I_1}, \quad \text{Tr}(T_i^{I_1} T_j^{I_1}) = \frac{1}{2} \delta^{I_1 I_1},
\]

\[
\Phi_i = \Phi_i^{A_1} T_i^{A_1}, \quad \text{Tr}(T_i^{A_1} T_j^{A_1}) = \frac{1}{2} \delta^{A_1 A_1}, \quad (\text{no sum for } i).
\]

Covariant derivatives for \(H_{iL}\) and \(H_{iR}\) are given as \((3.31)\) and \((3.32)\) with identical definition for \(\tilde{H}_{iL}^1 \tilde{H}_{iR}^1\). Covariant derivatives of \(\Sigma^I_i, \Phi_i^A\) are defined as the adjoint representation. We will not display the Chern-Simons term \(\mathcal{L}_{iCS}\) and the fermionic term \(\mathcal{L}_{\text{fermion}}\), since we do not need them for our analysis.

Functions \(a_{\alpha\beta}(\Sigma)\) are gauge coupling functions, which are given as second derivative of the prepotential

\[
a(\Sigma) = \sum_{\alpha=1}^{2} \left[ \frac{1}{2g_1^2} (\Sigma^{0_1})^2 + \frac{1}{2g_2^2} (\Sigma^{I_1})^2 + \frac{\lambda_1}{2} \left( \frac{\Sigma^{0_1}}{m_1} - \frac{\Sigma^{0_2}}{m_2} \right) (\Phi^{A_1})^2 \right],
\]

\[
a_{\alpha\beta}(\Sigma) = \frac{\partial^2 a(\Sigma)}{\partial \Sigma^\alpha \partial \Sigma^\beta}.
\]
From the above prepotential, we see the coupling constants of $U(1)_i$ and $SU(N_i)_c$ are given by $\hat{g}_i$ and $g_i$, respectively. We denote the coupling function of $SU(N_i)_V$ corresponding to $\Sigma^\alpha = \Phi^A_i$ and $\Sigma^\beta = \Phi^B_i$ as $e_i(\Sigma)$,

$$\frac{1}{e_i^2(\Sigma)} = \lambda_i \left( \frac{\Sigma^{0_1}}{m_1} - \frac{\Sigma^{0_2}}{m_2} \right),$$

(4.13)

but will suppress the argument $\Sigma$ to write $e_i$ in the following.

The constants $c_{\alpha}$ are coefficients of the Fayet-Iliopoulos (FI) terms, allowed to be non-zero only for the $U(1)$ part of the gauge groups.\(^{(*)}\)

$$c_{\alpha} Y^\alpha = c_{0_1} Y^{0_1} + c_{0_2} Y^{0_2}. \quad (4.14)$$

We have assumed both the FI parameters $c_{0_1}$ and $c_{0_2}$ to be positive in the same direction in $SU(2)_R$, which is chosen to be along the third component. In this setup, the $\tilde{H}$ fields will vanish in the classical solution. Moreover, they do not contribute to the desired order of effective Lagrangian. Similarly we have neglected the auxiliary fields $Y$ other than the third component in $SU(2)_R$, and we have denoted as $Y^\alpha$. Hence we can call the potential after eliminating the auxiliary fields $Y$'s to be D-term potential.

The F-term potential $V_{iF}$ can be worked out from the following superpotential

$$W_i = \text{Tr} \left[ \{ \Sigma_i H_{iL} - H_{iL}(\Phi + m_i) \} \tilde{H}_{iL} + \{ \Sigma_i H_{iR} - H_{iR}(\Phi - m_i) \} \tilde{H}_{iR} \right], \quad (4.15)$$

where we restored the tilde fields $\tilde{H}$'s to facilitate writing the superpotential. After eliminating the auxiliary fields $F$'s, and with the use of

$$V_{iF} = -\mathcal{L}_{iF} = |F_{iL}|^2 + |F_{iR}|^2 + |\tilde{F}_{iL}|^2 + |\tilde{F}_{iR}|^2, \quad (4.16)$$

we have (4.13).

Finally, let us work out explicit forms of the D-term potential $V_D$. Collecting terms containing the auxiliary fields $Y$'s, we obtain

$$-V_D = \sum_{i=1}^{2} \left[ \frac{1}{2\hat{g}_i^2}(Y^{0_1})^2 + \frac{1}{2g_i^2}(Y^L)^2 + r_i^0 Y^L_i + (r_i^0 - c_{0_1}) Y^{0_1}_i + \frac{1}{2e_i^2}(Y^A_i)^2 + s_A^i Y^A_i \right. \nonumber$$

$$+ \left. \lambda_i \left( \frac{Y^{0_1}_i}{m_1} - \frac{Y^{0_2}_i}{m_2} \right) \Phi^A_i Y^A_i \right], \quad (4.17)$$

\(^{*})\) The $U(1)_i$ coupling is in principle unrelated to the $SU(N_i)_c$ coupling. In section 3 we made a simplifying assumption $\hat{g}_i = g_i/\sqrt{2N_i}$, which allows simple solutions.

\(^{**})\) The FI parameters $c_{0_1}$ are related to the parameters $v_i^2$ in Eqs. (3.1)-(3.3) in section 3 as $c_{0_1} = N_i v_i^2$. 

24
where

\[
\begin{align*}
    r_i &= H_i H_i^\dagger - \tilde{H}_i H_i + H_i R H_i^\dagger - \tilde{H}_i R \tilde{H}_i, \\
    s_i &= -H_i H_i^\dagger + \tilde{H}_i H_i - H_i R \tilde{H}_i^\dagger - \tilde{H}_i R H_i^\dagger,
\end{align*}
\]

are Hermitian matrices, with the decomposition

\[
    r_i = \frac{1}{N_i} r_0^i 1_{N_i} + 2 r_i T^i, \quad s_i = \frac{1}{N_i} s_0^i 1_{N_i} + 2 s^i T^A_i.
\]

We observe, that in the potential \((4.17), Y^i\), do not couple to the rest of auxiliary fields and can be easily eliminated. Having this done, we collect the \(U(1)_i\) and \(SU(N_i)_\nu\) terms into a matrix form labeled by \(\alpha, \beta = 0_1, 0_2, A_1, A_2\)

\[
    -V_D = -\frac{1}{2} \sum_{i=1}^2 g_i^2 (r_i)^2 + \frac{1}{2} G_{\alpha \beta} Y^\alpha Y^\beta + (r - c)_\alpha Y^\alpha,
\]

\[
    (r - c)_\alpha \equiv r_0^\alpha - c_\alpha, \quad (r - c)_A \equiv s^A_i.
\]

Eliminating remaining auxiliary fields we obtain:

\[
    V_D = \frac{1}{2} \sum_{i=1}^2 g_i^2 (r_i)^2 + \frac{1}{2} (G^{-1})^{\alpha \beta} (r - c)_\alpha (r - c)_\beta.
\]

Matrix \(G = (G_{\alpha \beta})\) is explicitly given by

\[
    G = \begin{pmatrix}
        \frac{1}{g_1^2} & 0 & \frac{\lambda_1}{m_1} \phi^{A_1} & \frac{\lambda_2}{m_1} \phi^{A_2} \\
        0 & \frac{1}{g_2^2} & -\frac{\lambda_1}{m_2} \phi^{A_1} & -\frac{\lambda_2}{m_2} \phi^{A_2} \\
        \frac{\lambda_1}{m_1} \phi^{B_1} & \frac{\lambda_1}{m_2} \phi^{B_1} & \frac{1}{e_1} \delta^{A_1 B_1} & 0 \\
        \frac{\lambda_2}{m_1} \phi^{B_2} & -\frac{\lambda_2}{m_2} \phi^{B_2} & 0 & \frac{1}{e_2} \delta^{A_2 B_2}
    \end{pmatrix},
\]

with the inverse

\[
    G^{-1} = \frac{1}{1 - \tilde{g}^2 \phi^2} \times
\]

\[
    \begin{pmatrix}
        \tilde{g}_1^2 - \tilde{g}_1 \tilde{g}_2 m_1^2 \tilde{\phi}^2 & -\tilde{g}_2 \tilde{g}_2 m_1 m_2 \tilde{\phi}^2 & -\tilde{g}_2 e_1 m_2 \tilde{\phi}^{A_1} & -\tilde{g}_2 e_2 m_2 \tilde{\phi}^{A_2} \\
        -\tilde{g}_1 \tilde{g}_2 m_1 m_2 \tilde{\phi}^2 & \tilde{g}_2^2 - \tilde{g}_1 \tilde{g}_2 m_2^2 \tilde{\phi}^2 & \tilde{g}_2 e_1 m_1 \tilde{\phi}^{A_1} & \tilde{g}_2 e_2 m_1 \tilde{\phi}^{A_2} \\
        -\tilde{g}_1 e_1 m_2 \tilde{\phi}^{B_1} & \tilde{g}_2 e_1 m_1 \tilde{\phi}^{B_1} & e_1^2 \delta^{A_1 B_1} - e_1 \tilde{g}_2 \tilde{\phi}^{A_1 B_1} & \tilde{g}_2 e_2 \tilde{\phi}^{A_2 B_1} \\
        -\tilde{g}_1 e_2 m_2 \tilde{\phi}^{B_2} & \tilde{g}_2 e_2 m_1 \tilde{\phi}^{B_2} & \tilde{g}_2 e_1 \tilde{\phi}^{A_1 B_2} & e_2^2 \delta^{A_2 B_2} - e_2 \tilde{g}_2 \tilde{\phi}^{A_2 B_2}
    \end{pmatrix},
\]

\[25\]
where we abbreviated:

\[ \tilde{g}^2 = \tilde{g}_1^2m_2^2 + \tilde{g}_2^2m_1^2, \quad (4.26) \]

\[ \tilde{\Phi}^{A_i} = \frac{\lambda_i e_i}{m_1m_2}, \quad (4.27) \]

\[ \tilde{g}^2 = \tilde{\Phi}^{A_1}\tilde{\Phi}^{A_1} + \tilde{\Phi}^{A_2}\tilde{\Phi}^{A_2}, \quad (4.28) \]

\[ \tilde{\Phi}^{A_iB_i} = \tilde{\Phi}^{A_1}\delta^{A_2B_i} - \tilde{\Phi}^{A_i}\tilde{\Phi}^{B_i}, \quad (4.29) \]

4.2. Positivity of Potential

The F-term potential (4.3) is manifestly positive. The D-term potential (4.21) is positive definite under certain conditions. To find the condition we shall decompose (4.21) to:

\[ V_D = V_{1D} + V_{2D}, \quad (4.30) \]

\[ V_{1D} = \frac{1}{2}\tilde{g}_1^2(r^1)^2 + \frac{1}{2}\tilde{g}_2^2(r^2)^2, \quad (4.31) \]

\[ V_{2D} = \frac{1}{2}(G^{-1})^{\alpha\beta}(r - c)_\alpha(r - c)_\beta. \quad (4.32) \]

It is clear that the \( V_{1D} \) is positive definite by itself. Therefore we can only focus on \( V_{2D} \), which is positive if and only if \( G \) is positive definite.

It is easy to recognize that positivity of \( G \) is manifest once the adjoint scalars vanish \( \Phi_i = 0 \). Nevertheless, it is instructive and assuring if we consider the potential as well as the BPS equations keeping the adjoint scalars \( \Phi_i \) nonzero.

To ascertain positivity of \( G \) we need to compute its eigenvalues. This is most easily done by looking at its determinant (We leave the derivation of this result to the Appendix C):

\[ \det G = \left[ \frac{1}{\tilde{g}_1\tilde{g}_2} - \left( \frac{m_1^2}{\tilde{g}_1^2} + \frac{m_2^2}{\tilde{g}_2^2} \right) \tilde{g}^2 \right] \left( \frac{1}{e_1^2} \right)^{N_1} \left( \frac{1}{e_2^2} \right)^{N_2}. \quad (4.33) \]

Requiring \( \det G > 0 \), we have

\[ \frac{1}{\tilde{g}_1\tilde{g}_2} - \left( \frac{m_1^2}{\tilde{g}_1^2} + \frac{m_2^2}{\tilde{g}_2^2} \right) \tilde{g}^2 > 0. \quad (4.34) \]

In Appendix C we show that this condition is both necessary and sufficient to ensure positivity of matrix \( G \) in Eq.(4.24).

4.3. BPS equations

Let us denote the codimension of the domain wall as \( y \). Since we assume Lorentz invariance for other dimensions, we obtain gauge field to vanish for component other than \( y \).
The energy density $\mathcal{H}$ for domain walls is given by

$$
\mathcal{H} = \frac{1}{2} G_{\alpha\beta} D_y \Sigma^\alpha D_y \Sigma^\beta + \frac{1}{2} (G^{-1})^{\alpha\beta} (r - c)_\alpha (r - c)_\beta
$$

(4.35)

$$
+ \sum_{i=1}^2 \text{Tr} \left\{ \left( \bar{D}_y H_{iL} \bar{D}_y H_{iL}^\dagger + \bar{D}_y H_{iR} \bar{D}_y H_{iR}^\dagger + \bar{D}_y \tilde{H}_{iL} \bar{D}_y \tilde{H}_{iL} + \bar{D}_y \tilde{H}_{iR} \bar{D}_y \tilde{H}_{iR} \right) + V_I \right\},
$$

where color-flavor indices $\alpha, \beta$ span all values as in Eq. (4.4) and we have incorporated color sector $\alpha = I_1, I_2$ into the definition of matrix $G$ for brevity. Accordingly, we have incorporated the definition, $(r - c)_L = r^I$. Since there is no mixing of color sector with the rest, the inverse is calculated trivially and non-color part remains the same as in (4.1).

Now we observe that the mixing due to the cubic prepotential occurs only in the kinetic term and potential of the vector multiplets. Moreover, they appear as $G$ and $G^{-1}$ respectively. Therefore the cross term coming out of the Bogomol’nyi completion has no dependence on the metric $G$. This fact implies that the cancellation of cross terms to give topological charge goes through unaffected by the mixing of the vector multiplets.

More explicitly, we obtain the Bogomol’nyi completion as

$$
\mathcal{H} = \frac{1}{2} \left( G_{\alpha\gamma} D_y \Sigma^\gamma + (r - c)_\alpha \right) (G^{-1})^{\alpha\beta} \left( G_{\beta\delta} D_y \Sigma^\delta + (r - c)_\beta \right)
$$

$$
+ \sum_{i=1}^2 \text{Tr} \left[ \left| \bar{D}_y H_{iL} + \Sigma_i H_{iL} - H_{iL} (\Phi_i + m_i 1_{N_i}) \right|^2 
\right.
$$

$$
\left. + \left| \bar{D}_y \tilde{H}_{iL} - \Sigma_i \tilde{H}_{iL} + \tilde{H}_{iL} (\Phi_i + m_i 1_{N_i}) \right|^2 
\right.
$$

$$
\left. + \left| \bar{D}_y H_{iR} + \Sigma_i H_{iR} - H_{iR} (\Phi_i - m_i 1_{N_i}) \right|^2 
\right.
$$

$$
\left. + \left| \bar{D}_y \tilde{H}_{iR} - \Sigma_i \tilde{H}_{iR} + \tilde{H}_{iR} (\Phi_i - m_i 1_{N_i}) \right|^2 
\right]
$$

$$
- \sum_{i=1}^2 \partial_y \text{Tr} \left[ \Sigma_i r_i + \Phi_i s_i - m_i (H_{iL}^\dagger H_{iL} - H_{iR}^\dagger H_{iR} - \tilde{H}_{iL} \tilde{H}_{iL}^\dagger + \tilde{H}_{iR} \tilde{H}_{iR}^\dagger) \right]
$$

$$
+ c_i \partial_y \text{Tr} \Sigma_i.
$$

(4.36)

The last term gives the usual Bogomol’nyi bound and becomes the topological charge. The line before that is the total derivative which give vanishing contribution for an infinite line $-\infty < y < \infty$.

BPS equations for $H$’s and $\tilde{H}$’s of hypermultiplets are

$$
\bar{D}_y H_{iL} + \Sigma_i H_{iL} - H_{iL} (\Phi_i + m_i 1_{N_i}) = 0,
$$

(4.37)

$$
\bar{D}_y \tilde{H}_{iL} - \Sigma_i \tilde{H}_{iL} + \tilde{H}_{iL} (\Phi_i + m_i 1_{N_i}) = 0,
$$

(4.38)

$$
\bar{D}_y H_{iR} + \Sigma_i H_{iR} - H_{iR} (\Phi_i - m_i 1_{N_i}) = 0,
$$

(4.39)

$$
\bar{D}_y \tilde{H}_{iR} - \Sigma_i \tilde{H}_{iR} + \tilde{H}_{iR} (\Phi_i - m_i 1_{N_i}) = 0.
$$

(4.40)
BPS equations for vector multiplets are

\[ G_{\alpha\beta} D_y \Sigma^\beta + (r - c)_\alpha = 0 . \]  

(4.41)

More explicitly,

\[ \frac{1}{g_i^2} \partial_y \Sigma^0_i + \sum_{j,k=1}^2 \frac{\lambda_i \varepsilon_{ijk} m_k}{m_1 m_2} \Phi_{A_j} D_y \Phi_{A_j} + r^0_i - c_0_i = 0 , \]  

(4.42)

\[ \frac{1}{g_i^2} D_y \Sigma^I_i + r^I_i = 0 , \]  

(4.43)

\[ \frac{1}{c_i^2} D_y \Phi_{A_i} + \sum_{j,k=1}^2 \frac{\lambda_i \varepsilon_{jkm} m_k}{m_1 m_2} \Phi_{A_j} \partial_y \Sigma^{0_j} + s^{A_i} = 0 . \]  

(4.44)

We can easily solve the BPS equation for hypermultiplets, by using the moduli matrix approach. We define \( S_{ic}, S_{iF} \) and \( \psi_i \) as

\[ \Sigma_i(y) + i W_{iy}(y) = S_{ic}^{-1}(y) \partial_y S_{ic}(y) + \frac{1}{2} \partial_y \psi_i(y) , \]  

(4.45)

\[ \Phi_i(y) + i A_{iy}(y) = S_{iF}^{-1}(y) \partial_y S_{iF}^{-1}(y) . \]  

(4.46)

Then the hypermultiplets BPS equations \((4.37)-(4.40)\) are solved by the constant moduli matrices \( H_{iL}^0 \) and \( H_{iR}^0 \)

\[ H_{iL} = e^{-\psi_i/2} S_{ic}^{-1} H_{iL}^0 S_{ic}^{-1} e^{m_i y} , \]  

(4.47)

\[ H_{iR} = e^{-\psi_i/2} S_{ic}^{-1} H_{iR}^0 S_{ic}^{-1} e^{-m_i y} , \]  

(4.48)

where \( S_{ic}, S_{iF} \in SL(N_i, \mathbb{C}) \). The hypermultiplet fields \( \hat{H}_{iL} \) and \( \hat{H}_{iR} \) do not contribute to domain wall solution and they are therefore vanishing. We write down \((4.42)-(4.44)\) in terms of the gauge invariant fields

\[ \Omega_{ic} = S_{ic} S_{ic}^\dagger , \quad \Omega_{iF} = S_{iF}^\dagger S_{iF} , \quad \eta_i = \frac{1}{2} (\psi_i + \psi_i^*) . \]  

(4.49)

The adjoint scalar fields of the vector multiplets are given by

\[ \Sigma_i = \frac{1}{2} S_{ic}^{-1} (\partial_y \Omega_{ic} \Omega_{ic}^{-1}) S_{ic} + \frac{1}{2} \partial_y \eta_i , \]  

(4.50)

\[ \Phi_i = -\frac{1}{2} S_{iF}^{-1} (\partial_y \Omega_{iF} \Omega_{iF}^{-1}) S_{iF}^\dagger . \]  

(4.51)

Also, we have

\[ D_y \Sigma_i = \partial_y \Sigma_i + i [W_{iy}, \Sigma_i] = \frac{1}{2} S_{ic}^{-1} \partial_y (\partial_y \Omega_{ic} \Omega_{ic}^{-1}) S_{ic} + \frac{1}{2} \partial_y^2 \eta_i , \]  

(4.52)

\[ D_y \Phi_i = \partial_y \Phi_i + i [A_{iy}, \Phi_i] = -\frac{1}{2} S_{iF}^{-1} \partial_y (\partial_y \Omega_{iF} \Omega_{iF}^{-1}) S_{iF}^\dagger . \]  

(4.53)
BPS equations for vector multiplets (4.42)-(4.44) can be now rewritten as the following master equations:

\[
\frac{1}{2g_i^2} \partial_y^2 \eta_i + \frac{\varepsilon_{ikm_k}}{2m_1m_2} \text{Tr} \left[ \lambda_j (\partial_y \Omega_j) \Omega_j^{-1} \right]^2 = c_0 - e^{-\eta_i} \text{Tr} \left[ (H_{iL}^0 \Omega_{iF}^{-1} H_{iL}^0 e^{2m_i y} + H_{iR}^0 \Omega_{iF}^{-1} H_{iR}^0 e^{-2m_i y}) \Omega_{iF}^{-1} \right],
\]

\[
\frac{1}{g_i^2} \partial_y \partial_y (\eta_i \Omega_{iF}) = -e^{-\eta_i} \left\langle \left( H_{iL}^0 \Omega_{iF}^{-1} H_{iL}^0 e^{2m_i y} + H_{iR}^0 \Omega_{iF}^{-1} H_{iR}^0 e^{-2m_i y} \right) \Omega_{iF}^{-1} \right\rangle.
\]

Here we have used a notation

\[
\langle X \rangle \equiv X - \frac{\text{Tr}[X]}{N_1} 1_N.
\]

We make a comment about a possibility of additional moduli. At present we cannot say definitely if there are additional moduli other than the moduli matrices $H_0$'s, since we cannot solve these master equations. We have several clues at hand. The BPS equations for domain walls and other solitons in gauge theories with scalar fields in the fundamental representations are in the Higgs phase where all the gauge symmetries are broken in the vacuum. In that situation, we learned that all the moduli are contained in the moduli matrix. On the other hand, instantons are solitons in the pure Yang-Mills theory without scalar fields, where gauge symmetry is unbroken in the vacuum. In this case, moduli reside in the BPS equation for gauge fields. In our present case, unbroken gauge symmetry $SU(N_i c + V)$ remains. This feature is indicative of additional moduli coming from the vector multiplet.

Irrespective of the possible additional moduli, we can demonstrate that the BPS equations admit the coincident wall solution. Since the hypermultiplet parts are already solved as in (4.47)-(4.48), our main task is to solve the master equations (4.54)-(4.56) associated to the vector multiplet. In order to solve them explicitly, we take strong gauge coupling limit $g_i, g_i \to \infty$, where the master equations give just the algebraic constraints for $\Omega_{ic}, \Omega_{iF}$ and $\eta_i$. In principle, they can be solved algebraically. Furthermore, Eq. (4.34) with the limit $g_i \to \infty$ tells us that positivity is maintained only if $\Phi_i$ vanishes. In the following we will, therefore, consider a special point in the solution space where

\[
\Phi_i = 0, \quad i = 1, 2
\]
which implies from Eq. (4.51) that $\Omega_iF$ are constant matrices. Then the differential equations (4.55) - (4.56) reduce to the set of algebraic equations:

$$
H_{iL}^0 \Omega_i^{-1} H_{iL}^{0\dagger} e^{2m_iy} + H_{iR}^0 \Omega_i^{-1} H_{iR}^{0\dagger} e^{-2m_iy} = \frac{c_i}{N_i} e^{\eta} \Omega_i c , \quad (4.59)
$$

$$
H_{iL}^{0\dagger} \Omega_i^{-1} H_{iL}^0 e^{2m_iy} + H_{iR}^{0\dagger} \Omega_i^{-1} H_{iR}^0 e^{-2m_iy} = \frac{c_i}{N_i} e^{\eta} \Omega_i F . \quad (4.60)
$$

Notice, that for both sectors $i = 1, 2$ these equations are the same and do not couple to each other. We can, therefore, focus our discussion only on one sector, since all results are equivalent in both of them. So in the remaining discussion we will drop the index $i$ from all fields.

Now we consider moduli matrix for the coincident walls corresponding to the most symmetric point of the moduli space.

$$
(H_{L}^0, H_{R}^0) = (1_N, 1_N) . \quad (4.61)
$$

Eqs. (4.59) and (4.60) show that these two constant matrices commute and only the product $\Omega_c \Omega_F = \Omega_F \Omega_c$ can be determined.

$$
e^{\eta} \Omega_c \Omega_F = \frac{N}{c} (e^{2my} + e^{-2my}) 1_N . \quad (4.62)
$$

Since we have chosen the matrices $S_c, S_F$ in $SL(N, \mathbb{C})$, we find that $\det(\Omega_c \Omega_F) = 1$ and we can separate the $U(1)$ part.

$$
e^{\eta} = \frac{N}{c} (e^{2my} + e^{-2my}) , \quad \Omega_c \Omega_F = 1_N . \quad (4.63)
$$

The $U(1)$ part gives the usual domain wall solution. Without affecting the physical quantities, we can choose $\Omega_c = 1, S_c = 1$, and finally we obtain the coincident wall solution for (4.61) with (4.58).

$$
\Phi = \langle \Sigma \rangle = 0 , \quad (4.64)
$$

$$
\eta = \log \frac{N}{c} (e^{2m(y-y_0)} + e^{-2m(y-y_0)}) , \quad (4.65)
$$

$$
\Sigma^0 = \frac{1}{2} \partial_y \eta = m \tanh(2m(y - y_0)) , \quad (4.66)
$$

$$
H_L = \sqrt{\frac{c}{N}} \frac{e^{m(y-y_0)} 1_N}{\left( e^{2m(y-y_0)} + e^{-2m(y-y_0)} \right)^{1/2}} , \quad (4.67)
$$

$$
H_R = \sqrt{\frac{c}{N}} \frac{e^{m(y-y_0)} 1_N}{\left( e^{2m(y-y_0)} + e^{-2m(y-y_0)} \right)^{1/2}} . \quad (4.68)
$$

*) This is due to the special choice of the moduli matrix in Eq. (4.61), since Eqs. (4.47) and (4.48) imply that only the product $S_{iF}S_{ic}$ can enter into the physical fields such as $H_{iL}, H_{iR}$. 

30
Note that in this solution we restore a moduli parameter \( y_0 \) corresponding to the position of the coincident wall. A similar construction of domain wall solution works for the second sector \((i = 2)\), besides the first sector \((i = 1)\) given above.

Let us note that the field-dependent gauge coupling function similarly to (3.33) is automatically obtained as a bosonic part of the Lagrangian specified by the cubic prepotential in Eq. (4.11). Restoring the index \(i = 1, 2\) for both of the domain wall sectors, and by using (4.65) with (4.50), we finally conclude that the appropriate profile of the field-dependent gauge coupling function \(\Sigma_0^i/m_i - \Sigma_0^2/m_2\), similarly to (3.33) is achieved. When we make (a part of) the global flavor symmetry as a local gauge symmetry, we can have several options. Since the first flavor group \(SU(N_1)\) is generally different from the second flavor group \(SU(N_2)\), we can naturally introduce two different gauge fields for the \(i = 1\) and 2. This option leads to two decoupled sectors in the low-energy effective Lagrangian, which can only be coupled by higher derivative terms induced by massive modes. Another interesting option is to introduce a gauge field only for the diagonal subgroup of isomorphic subgroups of two different flavor groups, such as \(SU(\tilde{N}) \in SU(N_1), SU(\tilde{N}) \in SU(N_2)\) with \(\tilde{N} \leq N_1, N_2\). This option is interesting in the sense that the massless gauge field exchange will communicate between two domain wall sectors. We hope to come back to these issues in near future.

Let us make a few comments. First we have shown that the chiral model analyzed in section 3 can be extended to a supersymmetric gauge theory with eight supercharges and that the field-dependent gauge coupling function which is a clue for localization is naturally explained by taking the cubic prepotential. Second, there may be more moduli not contained in \((H^0_L, H^0_R)\), which require further studies. Third, here we have presented a solution at a special point \(\Phi = 0\). It would be interesting to consider the case or \(\Phi \neq 0\), but in this case, we need to take a finite gauge coupling limit, on which we will investigate in future work.

\section*{§5. Conclusions and discussion}

In this paper we have successfully localized both massless non-Abelian gauge fields and massless matter fields in non-trivial representation of the gauge group. We first considered a (4+1)-dimensional \(U(N)\) gauge theory with additional \(SU(N)_L \times SU(N)_R \times U(1)_A\) flavor symmetry. We introduced the flavor gauge field for the diagonal flavor group \(SU(N)_{L+R}\), which is unbroken in the coincident wall background. The flavor gauge fields are localized on the wall by introducing the scalar-field-dependent gauge coupling function. Then we studied the low-energy effective Lagrangian and showed that massless localized matter fields interact minimally with localized \(SU(N)_{L+R}\) gauge field as adjoint representations. Moreover, full nonlinear interaction between the moduli containing up to the second derivatives, was worked
Main result of this paper is the effective Lagrangian (3.49). The moduli field \( U \) appearing in the effective theory, is a chiral \( N \times N \) matrix field like a pion, since it is a NG boson of spontaneously broken chiral symmetry. Other moduli in (3.49), denoted by \( N \times N \) Hermitian matrix \( \hat{x} \), has the physical meaning of positions of \( N \) domain walls as its diagonal elements. We argued that the fluctuations of moduli field \( \hat{x} \), can develop VEV corresponding to splitting of walls, and the Higgs mechanism will occur as a result. Namely, the flavor gauge fields get masses by eating the non-Abelian cloud. Therefore, in this model, Higgs mechanism has a geometrical origin like low energy effective theories on D-branes in superstring theory.

Amongst the possible future investigations, we would like to study non-coincident solution to further clarify this geometrical Higgs mechanism.

We have noticed that our effective moduli fields resemble the pion in QCD. Similar attempts have been quite successful using D-branes. We believe that our methods can provide more insight in various aspects of low-energy hadron physics. We plan to explore this direction more fully in subsequent studies.

In the discussion of supersymmetric extension of our model in section 4, we employed a general setup where both sectors possessed their own domain wall solution, preserving the same half of the supercharges. But another alternative approach is also possible. We can consider a model, where different halves of supercharges are preserved at each sector (BPS and anti-BPS walls), and the SUSY is completely broken in the system as a whole. It has been proposed that the coexistence of BPS and anti-BPS walls gives the supersymmetry breaking in a controlled manner. In our present case, BPS and anti-BPS sectors interact only weakly. If we choose flavor gauge field for each sector separately, we have only higher derivative interactions induced by massive modes. If we choose the diagonal subgroup of (subgroups of) each sector as flavor gauge group, we have a more interesting possibility of the massless gauge field as a messenger between two sectors. We plan to address this issue elsewhere.

In order to construct a realistic brane-world scenario with the SM fields on the domain wall, we need the localization of fields in the fundamental representation of the gauge group. This is still an open problem and one of the priorities of our future investigations. In particular, the SM contains chiral fermions. Localization of chiral fermions is a particularly challenging problem. Anomaly associated with the chiral fermion is also an interesting issue to be addressed. We would also like to clarify these problems in subsequent studies.

Two more issues remain to be addressed. First is the question of sign of gauge kinetic
In our present model, the positivity of the gauge coupling function is assured only when positions of walls are properly ordered (see Eq. (3.48)), namely only in a region of the moduli space. More economical models such as given in Ref. 12) may not have such moduli and, therefore, the effective gauge coupling may be always positive. And lastly, as discussed in section 4, we have not succeeded in exhausting all moduli in the supersymmetric extension of our model. We would also like to investigate these aspects in the future.

Acknowledgements

This work is supported in part by Japan Society for the Promotion of Science (JSPS) and Academy of Sciences of the Czech Republic (ASCR) under the Japan - Czech Republic Research Cooperative Program, and by Grant-in-Aid for Scientific Research from the Ministry of Education, Culture, Sports, Science and Technology, Japan No.21540279 (N.S.), No.21244036 (N.S.), and No.23740226 (M.E.). The work of M.A. and F.B. is supported in part by the Research Program MSM6840770029 and by the project of International Cooperation ATLAS-CERN of the Ministry of Education, Youth and Sports of the Czech Republic.

Appendix A

Domain walls in the gauged massive $\mathbb{C}P^1$ sigma model

Here we consider the domain wall solutions in the gauged massive $\mathbb{C}P^1$ sigma model. The model is obtained as the strong gauge coupling limit of a model similar to that we have studied in section 2.2. Namely, we start with the Lagrangian which has $U(1) \times U(1)$ gauge symmetry with two flavors

$$\mathcal{L} = -\frac{1}{4g^2}(F_{MN})^2 + \frac{1}{2g^2}((\partial_M \sigma)^2 - \frac{1}{4e^2}(G_{MN})^2 + |\mathcal{D}_M H|^2 - V, $$

$$V = \frac{g^2}{8}(|H|^2 - v^2)^2 + |\sigma H - HM|^2, $$

where $H = (H_L, H_R)$. The covariant derivative is given by

$$\mathcal{D}_M H = \partial_M H + iw_M H + ia_M Hq, \quad q = \text{diag}(q_L, q_R).$$

The mass matrix is chosen $M = \text{diag}(m, -m)$ as before.

We next take the strong gauge coupling limit $g \to \infty$ of only one of the gauge coupling which results in the non-linear sigma model coupled to the other gauge field with the finite gauge coupling $e$. In the limit the gauge field $w_M$ and the neutral scalar field $\sigma$ become Lagrange multipliers. After solving their equations of motion, we have

$$w_M = \frac{i}{v^2} \mathcal{D}_M H H^\dagger, \quad \sigma = \frac{1}{v^2} H M H^\dagger, $$

33
where we have introduced the covariant derivative
\[ \hat{D}_M H = \partial_M H + i a_M H q. \] (A-5)

Plugging these into the original Lagrangian at \( g \to \infty \), we get the gauged massive \( \mathbb{C}P^1 \) sigma model
\[ L_{g\to\infty} = -\frac{1}{4e^2} (G_{MN})^2 + \hat{D}_M H P \hat{D}^M H^\dagger - H M P M H^\dagger, \] (A-6)
with the projection operator
\[ P = 1 - \frac{1}{v^2} H^\dagger H. \] (A-7)

As before, let us rewrite this Lagrangian with respect to the inhomogeneous coordinate
\[ H = \frac{v}{\sqrt{1 + |\phi|^2}} (1, \phi), \quad \phi \in \mathbb{C}. \] (A-8)

Then the charge matrix should be chosen as
\[ q = \text{diag}(0, 1), \] (A-9)
which leads to a natural expression that the complex scalar field \( \phi \) has the \( U(1) \) charge 1 for the gauge field \( a_M \):
\[ \hat{D}_M H = -\frac{v}{2(1 + |\phi|^2)^{3/2}} \left( \partial_M |\phi|^2, \phi \partial_M |\phi|^2 - 2(1 + |\phi|^2) \hat{D}_M \phi \right), \] (A-10)
\[ \hat{D}_M \phi = (\partial_M + i a_M) \phi. \] (A-11)

Plugging these into Eq.(A-6), we finally get the Lagrangian
\[ L_{g\to\infty} = -\frac{1}{4e^2} (G_{MN})^2 + v^2 \frac{|\hat{D}_M \phi|^2}{(1 + |\phi|^2)^2} - v^2 \frac{4m^2 |\phi|^2}{(1 + |\phi|^2)^2}. \] (A-12)

Let us next consider a domain wall solution in this model. We assume all the fields depend on only the extra-dimensional coordinate \( y \). Then the four dimensional components of the Maxwell equation
\[ \partial_N G^{NM} = i e^2 v^2 \frac{\hat{D}^M \phi \phi^* - \phi \hat{D}^M \phi^*}{(1 + |\phi|^2)^2}, \] (A-13)
can be immediately solved by
\[ a_\mu = 0, \quad \mu = 0, 1, 2, 3. \] (A-14)
The fifth component is

\[ 0 = i e^2 v^2 \frac{\hat{D}^y \phi \phi^* - \phi \hat{D}^y \phi^*}{(1 + |\phi|^2)^2}. \]  

(A.15)

Now the Hamiltonian reduces to the following form

\[ H = \frac{\nu^2}{(1 + |\phi|^2)^2} \left( |\hat{D}_y \phi|^2 + 4 m^2 |\phi|^2 \right) \]

\[ = \frac{\nu^2}{(1 + |\phi|^2)^2} \left( |\hat{D}_a \phi + 2 m \phi|^2 - 2 m \partial_y |\phi|^2 \right) \]

\[ \geq 2 m v^2 \frac{d}{dy} \frac{1}{1 + |\phi|^2}. \]  

(A.16)

Thus the reduced Hamiltonian is minimized when the following first order equation is satisfied

\[ \hat{D}_y \phi = -2 m \phi. \]  

(A.17)

Since the mass parameter \( m \) is real, Eq. (A.15) is also satisfied. Let us take the gauge where

\[ a_y = 0. \]  

(A.18)

Then we have the explicit domain wall solution

\[ \phi = C^2 e^{-2 m y}, \quad C^2 = e^{2i \alpha + 2 m y_0}. \]  

(A.19)

This is completely the same as the domain wall solution given in Eq. (2.28) in the ungauged massive \( \mathbb{C}P^1 \) sigma model.

The final step is to obtain a low energy effective theory on the domain wall. The effective Lagrangian is given by

\[ \mathcal{L}_{\nu \rightarrow \infty}^{\text{eff}} = \int dy \left[ -\frac{1}{4 e^2} (G_{\mu \nu})^2 - \frac{1}{2 e^2} (G_{\mu y})^2 + v^2 \frac{|\hat{D}_\mu \phi|^2}{(1 + |\phi|^2)^2} \right] \]

\[ = \int dy \left[ -\frac{1}{4 e^2} (G_{\mu \nu})^2 - \frac{1}{2 e^2} (G_{\mu y})^2 + v^2 \frac{m^2 (\partial_\mu y_0)^2 + (\hat{D}_\mu \alpha)^2}{\cosh^2 2 m (y - y_0)} \right], \]  

(A.20)

where we have promoted the moduli parameter \( y_0, \alpha \) to the fields \( y_0(x^\mu), \alpha(x^\mu) \) on the wall, and we have introduced the covariant derivative

\[ \hat{D}_\mu \alpha = \partial_\mu \alpha + \frac{a_\mu}{2}, \]  

(A.21)

where \( \alpha \) is the function of the (3+1)-dimensional coordinate \( x^\mu \). Assuming \( a_\mu \) to be \( y \)-independent (zero mode), we finally obtain

\[ \mathcal{L}_{\nu \rightarrow \infty}^{\text{eff}} = -\frac{1}{4 e^2} (G_{\mu \nu})^2 + \frac{v^2}{m} \left( m^2 (\partial_\mu y_0)^2 + (\hat{D}_\mu \alpha)^2 \right). \]  

(A.22)
Thus we find that the gauge field \( a_\mu(x) \) absorbs the scalar field \( \alpha(x) \) to become massive via the Higgs mechanism. Since that the \( U(1) \) gauge field \( a_\mu \) is massive in the effective Lagrangian, we have to integrate it out according to the spirit of the low energy effective theory.

**Appendix B**

--- **Effective Lagrangian on the domain wall** ---

In this appendix we derive our main result (3.41) of the effective Lagrangian for the gauged Chiral model introduced in §3.

**B.1. Compact form of gauged nonlinear model**

Starting from the Lagrangian using the Einstein summation convention for \( a = \{L, R\} \)

\[
\mathcal{L}_{\text{eff}} = \int_{-\infty}^{\infty} dy \, \text{Tr} \left[ \hat{D}_\mu H_a \hat{D}^\mu H_a^\dagger - v^2 W_\mu W^\mu \right],
\]

with the constraint

\[
H_a H_a^\dagger = v^2 1_N,
\]

we first eliminate the gauge fields \( W_\mu \) to obtain a simple expression for gauged nonlinear sigma model. Gauge fields \( W_\mu \) are given by equations of motion as

\[
W_\mu = \frac{i}{2v^2} \left[ \hat{D}_\mu H_a H_a^\dagger - H_a \hat{D}_\mu H_a^\dagger \right],
\]

and

\[
\hat{D}_\mu H = \partial_\mu H - i H A_\mu.
\]

The effective Lagrangian \( \text{[B-1]} \) should also contain kinetic term for gauge field \( A_\mu \), but we will not explicitly write it here, for brevity. Eq. \( \text{[B-1]} \) can be further simplified by using the following identities

\[
H_a \hat{D}_\mu H_b^\dagger = \partial_\mu (H_a H_b^\dagger) - \hat{D}_\mu H_a H_b^\dagger,
\]

\[
H_a^\dagger \hat{D}_\mu H_b = -\hat{D}_\mu H_a H_b^\dagger + D_\mu H_{ab},
\]

where

\[
D_\mu H_{ab} = \partial_\mu H_{ab} + i [A_\mu, H_{ab}] , \quad H_{ab} \equiv H_a^\dagger H_b.
\]

After some algebra we find:

\[
W_\mu W^\mu = \frac{1}{v^2} \left( \hat{D}_\mu H_a \hat{D}^\mu H_a^\dagger \right) - \frac{1}{2v^2} (D_\mu H_{ab} D^\mu H_{ba}) .
\]
Plugging above expression back into the (B.1) we arrive at:

\[ \mathcal{L}_{\text{eff}} = \frac{1}{2v^2} \int_{-\infty}^{\infty} dy \text{Tr} \left[ \mathcal{D}_\mu H_{ab} \mathcal{D}^\mu H_{ba} \right]. \]  

(B-8)

### B.2. Effective Lagrangian

Now we are ready to compute effective Lagrangian. Using a solution (with \( \hat{y} = my \mathbf{1}_N - \hat{x} \)):

\[ H = (H_L, H_R) = \left( \frac{v}{\sqrt{2}} \frac{e^{\hat{y}/2}}{\sqrt{\cosh(\hat{y})}}, \frac{v}{\sqrt{2}} \frac{e^{-\hat{y}/2} U^\dagger}{\sqrt{\cosh(\hat{y})}} \right). \]  

(B-9)

Our new fields \( H_{ab} \) are given as:

\[ H_{LL} = \frac{v^2}{2} \frac{e^{\hat{y}}}{\cosh(\hat{y})}, \]  

(B-10)

\[ H_{LR} = \frac{v^2}{2} \frac{1}{\cosh(\hat{y})} U^\dagger = H_{RL}^\dagger, \]  

(B-11)

\[ H_{RR} = \frac{v^2}{2} U \frac{e^{-\hat{y}}}{\cosh(\hat{y})} U^\dagger. \]  

(B-12)

It can be checked, that (B.8) is given as:

\[ \mathcal{L}_{\text{eff}} = \frac{v^2}{4} \int_{-\infty}^{\infty} dy \text{Tr} \left\{ \mathcal{D}_\mu \frac{e^{\hat{y}}}{\cosh(\hat{y})} \mathcal{D}^\mu \frac{e^{\hat{y}}}{\cosh(\hat{y})} + \mathcal{D}_\mu \frac{1}{\cosh(\hat{y})} \mathcal{D}^\mu \frac{1}{\cosh(\hat{y})} \right. \]  

\[ + U^\dagger \mathcal{D}_\mu U \left( \frac{e^{-\hat{y}}}{\cosh(\hat{y})}, U^\dagger \mathcal{D}_\mu \left( \frac{e^{-\hat{y}}}{\cosh(\hat{y})} \right) \right) + U^\dagger \mathcal{D}_\mu U \left[ \frac{1}{\cosh(\hat{y})} \mathcal{D}^\mu \frac{1}{\cosh(\hat{y})} \right] \]  

\[ + \mathcal{D}_\mu U^\dagger \mathcal{D}^\mu U \frac{1}{\cosh^2(\hat{y})} \right\}. \]  

(B-13)

In the following we would like to carry out the integration over the extra-dimensional coordinate \( y \). This can be done in two steps. First, we must factorize all quantities depending on \( y \) (or on \( \hat{y} \)) to one term inside the trace, effectively reducing our problem to fit the following form:

\[ \int_{-\infty}^{\infty} dy \text{Tr} \left[ f(my \mathbf{1}_N - \hat{x}) M \right], \]  

(B-14)

where \( M \) is some matrix, independent of \( y \) and \( f \) some function. In the second step we diagonalize \( \hat{x} \):

\[ \hat{x} = P^{-1} \text{diag}(\lambda_1, \ldots, \lambda_N) P, \]  

and use the fact that \( f(P^{-1} \hat{y} P) = P^{-1} f(\hat{y}) P \). This transformation leads to

\[ \int_{-\infty}^{\infty} dy \text{Tr} \left[ f(my \mathbf{1}_N - \text{diag}(\lambda_i)) PMP^{-1} \right] = \int_{-\infty}^{\infty} dy \sum_{i=1}^{\lambda} f(my - \lambda_i)(PMP^{-1})_{ii}. \]
For every term in the sum we can perform substitution \( \tilde{y} = my - \lambda_i \). The key observation is that in each term the integration will be the same and independent on a particular value of \( \lambda_i \). Thus we arrive at an identity

\[
\int_{-\infty}^{\infty} dy \, \text{Tr} \left[ f(\tilde{y}) M \right] = \frac{1}{m} \text{Tr}(M) \int_{-\infty}^{\infty} d\tilde{y} \, f(\tilde{y}).
\]  

(B-15)

It appears as if we just made a substitution \( \hat{y} = \tilde{y} \frac{1}{N} \). This is possible, of course, only thanks to the diagonalization trick and properties of the trace. In the subsequent subsections, however, we will refer to this procedure as if it is just a ‘substitution’, for brevity. Let us decompose the effective Lagrangian (B.13) into three pieces

\[
\mathcal{L}_{\text{eff}} = \mathcal{T}_z + \mathcal{T}_U + \mathcal{T}_{\text{mixed}}
\]

(B-16)

and see the outlined procedure for each term.

B.2.1. Kinetic term for \( U \)

First, let us concentrate only on terms containing double derivatives of \( U \), which we denote \( \mathcal{T}_U \):

\[
\mathcal{T}_U = \frac{\nu^2}{4} \int_{-\infty}^{\infty} dy \, \text{Tr} \left\{ \mathcal{D}_\mu U^\dagger \mathcal{D}_\mu U - \frac{1}{\cosh^2(y)} + \mathcal{D}_\mu U^\dagger \left[ \frac{e^{\hat{y}}}{\cosh(\hat{y})}, U^\dagger \mathcal{D}_\mu U \right] \frac{e^{-\hat{y}}}{\cosh(\hat{y})} \right\},
\]

where we have used the fact that inside commutator it is possible to freely interchange

\[
\frac{e^{-\hat{y}}}{\cosh(\hat{y})} \rightarrow -\frac{e^{\hat{y}}}{\cosh(\hat{y})},
\]

since the difference is just a constant matrix. In this way we made \( \mathcal{T}_U \) manifestly invariant under exchange \( \hat{y} \rightarrow -\hat{y} \).

Since in the first factor of \( \mathcal{T}_U \) all \( \hat{y} \)-dependent quantities are on the right side, we can, according to our previous discussion, make use of the identity (B.15) and carry out the integration:

\[
\frac{\nu^2}{4} \int_{-\infty}^{\infty} dy \, \text{Tr} \left[ \mathcal{D}_\mu U^\dagger \mathcal{D}_\mu U - \frac{1}{\cosh^2(y)} \right] = \frac{\nu^2}{2m} \text{Tr} \left[ \mathcal{D}_\mu U^\dagger \mathcal{D}_\mu U \right].
\]

For the second term, however, we first use the identity:

\[
[f(A), B] = \sum_{k=1}^{\infty} \frac{1}{k!} \mathcal{L}_A^k(B) f^{(k)}(A),
\]

(B-17)

where \( \mathcal{L}_A(B) = [A, B] \) is a Lie derivative with respect to \( A \). Thus

\[
\left[ \frac{e^{\hat{y}}}{\cosh(\hat{y})}, U^\dagger \mathcal{D}_\mu U \right] = \sum_{k=1}^{\infty} \frac{(-1)^k k!}{k!} \mathcal{L}_z^k(U^\dagger \mathcal{D}_\mu U) \left( \frac{e^{\hat{y}}}{\cosh(\hat{y})} \right)^{(k)}.
\]

38
Now all \( \hat{y} \)-dependent factors are standing on the right and we can formally exchange \( \hat{y} \to \tilde{y} \).

The summation can be carried out to get:

\[
\sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \mathcal{L}_x^k (U^\dagger D^\mu U) \left( \frac{e^{\tilde{y}}}{\cosh(\tilde{y})} \right)^{(k)} = \frac{e^{\tilde{y} - \mathcal{L}_x}}{\cosh(\tilde{y} - \mathcal{L}_x)} (U^\dagger D^\mu U) - \frac{e^{\tilde{y}}}{\cosh(\tilde{y})} U^\dagger D^\mu U. \tag{B.18}
\]

The formula for \( \mathcal{T}_U \) now reads:

\[
\mathcal{T}_U = \frac{c}{4m} \int_{-\infty}^{\infty} d\tilde{y} \text{Tr} \left[ \frac{e^{-\mathcal{L}_x}}{\cosh(\tilde{y} - \mathcal{L}_x) \cosh(\tilde{y})} (U^\dagger D^\mu U) D_\mu U^\dagger U \right]. \tag{B.19}
\]

Since we started with \( \mathcal{T}_U \) invariant under the transformation \( \hat{y} \to -\hat{y} \), we should take only even part of the above formula (under exchange \( L_x \to -L_x \)) as the final result:

\[
\mathcal{T}_U = \frac{c}{4m} \int_{-\infty}^{\infty} d\tilde{y} \text{Tr} \left[ \frac{\cosh(L_x)}{\cosh(\tilde{y} - L_x) \cosh(\tilde{y})} (U^\dagger D^\mu U) D_\mu U^\dagger U \right]. \tag{B.20}
\]

Now we can carry out the integration using primitive function

\[
\int \frac{dy}{\cosh(y - \alpha) \cosh(y)} = \frac{1}{\sinh(\alpha)} \ln \frac{1}{1 - \tanh(\alpha) \tanh(y)}.
\]

Therefore we obtain the result to all orders in \( \hat{x} \) as:

\[
\mathcal{T}_U = \frac{v^2}{4m} \text{Tr} \left[ D_\mu U^\dagger U \frac{1}{\tanh(L_x)} \ln \left( \frac{1 + \tanh(L_x)}{1 - \tanh(L_x)} \right) (U^\dagger D^\mu U) \right]. \tag{B.21}
\]

Performing the Taylor-expansion of the function

\[
\frac{1}{\tanh(x)} \ln \left( \frac{1 + \tanh(x)}{1 - \tanh(x)} \right) = 2 + \frac{2x^2}{3} - \frac{2x^4}{45} + \frac{4x^6}{945} - \frac{2x^8}{4725} + \frac{4x^{10}}{93555} + O(x^{12}), \tag{B.22}
\]

we can easily read off coefficients of terms beyond the leading one. For example, the first three terms reads:

\[
\mathcal{T}_U = \frac{v^2}{2m} \text{Tr} \left( D_\mu U^\dagger D^\mu U \right) - \frac{v^2}{6m} \text{Tr} \left( [\hat{x}, U^\dagger D_\mu U] [\hat{x}, D^\mu U^\dagger U] \right)
\]
\[
- \frac{v^2}{90m} \text{Tr} \left( [\hat{x}, [\hat{x}, U^\dagger D_\mu U]] [\hat{x}, [\hat{x}, D^\mu U^\dagger U]] \right) + \ldots \tag{B.23}
\]

B.2.2. Mixed term

Mixed term between \( \hat{x} \) and \( U \) is given by

\[
\mathcal{T}_{mixed} = \frac{v^2}{4} \int_{-\infty}^{\infty} dy \text{Tr} \left\{ U^\dagger D_\mu U \left( \frac{1}{\cosh(\tilde{y})} D^\mu \frac{1}{\cosh(\tilde{y})} \right) - \left[ \frac{e^{\tilde{y}}}{\cosh(\tilde{y})} D^\mu \frac{e^{-\tilde{y}}}{\cosh(\tilde{y})} \right] \right\}. \]
With use of the identity (B.17) and
\[
\mathcal{D}_\mu f(\hat{x}) = \sum_{k=0}^{\infty} \mathcal{L}_x^k(\mathcal{D}_\mu \hat{x}) \frac{f^{(k+1)}(\hat{x})}{(k+1)!}, \tag{B.24}
\]
one can prove the following:
\[
[f(\hat{x}), \mathcal{D}_\mu g(\hat{x})] = \sum_{n=2}^{\infty} \frac{1}{n!} \mathcal{L}_x^{n-1}(\mathcal{D}_\mu \hat{x}) \left[ \left( f(\hat{x})g(\hat{x}) \right)^{(n)} - f^{(n)}(\hat{x})g(\hat{x}) - f(\hat{x})g^{(n)}(\hat{x}) \right].
\]
We can use this result to factorize all \( \hat{y} \)-dependent quantities to the right and make the substitution \( \hat{y} = \hat{y}_1 \):\[
T_{\text{mixed}} = \frac{v^2}{4m} \int_{-\infty}^{\infty} dy \sum_{n=2}^{\infty} \frac{(-1)^n}{n!} \text{Tr} \left[ U^\dagger \mathcal{D}_\mu U \mathcal{L}_x^{n-1}(\mathcal{D}_\mu \hat{x}) \right] \times \left[ \left( \frac{e^{\hat{y}}}{\cosh(\hat{y})} \right)^{(n)} \frac{e^{-\hat{y}}}{\cosh(\hat{y})} + \left( \frac{e^{\hat{y}}}{\cosh(\hat{y})} \right)^{(n)} \frac{e^{-\hat{y}}}{\cosh(\hat{y})} - 2 \left( \frac{1}{\cosh(\hat{y})} \right)^{(n)} \frac{1}{\cosh(\hat{y})} \right].
\]
Now we are free to perform summation and integration to obtain:
\[
T_{\text{mixed}} = \frac{v^2}{2m} \text{Tr} \left[ U^\dagger \mathcal{D}_\mu U \cosh(\mathcal{L}_x) - 1 \cosh(\mathcal{L}_x) \ln \left( \frac{1 + \tanh(\mathcal{L}_x)}{1 - \tanh(\mathcal{L}_x)} \right) \mathcal{D}_\mu \hat{x} \right]. \tag{B.25}
\]
Performing the Taylor-expansion of the function
\[
\frac{\cosh(x) - 1}{x \sinh(x)} \ln \left( \frac{1 + \tanh(x)}{1 - \tanh(x)} \right) = x - \frac{x^3}{12} + \frac{x^5}{120} - \frac{17x^7}{20160} + \frac{31x^9}{362880} + O(x^{11}), \tag{B.26}
\]
we can easily read off coefficients of terms beyond the leading order in the series expansion:
\[
T_{\text{mixed}} = \frac{v^2}{2m} \text{Tr} \left[ U^\dagger \mathcal{D}_\mu U [\hat{x}, \mathcal{D}_\mu \hat{x}] \right] - \frac{v^2}{24m} \text{Tr} \left[ U^\dagger \mathcal{D}_\mu U [\hat{x}, [\hat{x}, [\hat{x}, \mathcal{D}_\mu \hat{x}]]] \right] + \ldots \tag{B.27}
\]
B.2.3. Kinetic term for \( \hat{x} \)

Kinetic term for \( \hat{x} \) is given by
\[
\mathcal{T}_x = \frac{\phi^2}{4} \int_{-\infty}^{\infty} dy \text{Tr} \left\{ \mathcal{D}_\mu \frac{1}{\cosh(\hat{y})} \mathcal{D}_\mu \frac{1}{\cosh(\hat{y})} - \mathcal{D}_\mu \frac{e^{\hat{y}}}{\cosh(\hat{y})} \mathcal{D}_\mu \frac{e^{-\hat{y}}}{\cosh(\hat{y})} \right\}. \tag{B.28}
\]
We are going to need the identity
\[
\text{Tr} \left[ \mathcal{D}_\mu f(\hat{x}) \mathcal{D}_\mu g(\hat{x}) \right] = \sum_{n=2}^{\infty} \frac{(-1)^n}{n!} \text{Tr} \left\{ \mathcal{L}_x^{n-2}(\mathcal{D}_\mu \hat{x}) \mathcal{D}_\mu \hat{x} \times \left[ \left( f(\hat{x})g(\hat{x}) \right)^{(n)} - f^{(n)}(\hat{x})g(\hat{x}) - f(\hat{x})g^{(n)}(\hat{x}) \right] \right\}. \tag{B.29}
\]
With the aid of this we arrive at

\[ T_{st} = \frac{v^2}{4m} \int_{-\infty}^{\infty} dy \sum_{n=2}^{\infty} \frac{1}{n!} \text{Tr} \left[ L_{\hat{x}}^{n-2}(D_{\mu} \hat{x})D_{\mu} \hat{x} \right] \times \left[ \left( \frac{e^{\tilde{y}}}{\cosh(\tilde{y})} \right)^{(n)} - \left( \frac{e^{-\tilde{y}}}{\cosh(\tilde{y})} \right)^{(n)} - 2 \left( \frac{1}{\cosh(\tilde{y})} \right)^{(n)} \right], \]

where we again employed diagonalization trick and identity (B.15). Let us carry out the summation and the integration to obtain:

\[ T_{st} = \frac{v^2}{2m} \text{Tr} \left[ D_{\mu} \hat{x} \frac{\cosh(L_{\hat{x}}) - 1}{L_{\hat{x}} \sinh(L_{\hat{x}})} \ln \left( \frac{1 + \tanh(L_{\hat{x}})}{1 - \tanh(L_{\hat{x}})} \right) (D_{\mu} \hat{x}) \right], \quad (B.30) \]

leading to the power series:

\[ T_{st} = \frac{v^2}{2m} \text{Tr} \left[ D_{\mu} \hat{x} D_{\mu} \hat{x} \right] + \frac{v^2}{24m} \text{Tr} \left[ [\hat{x}, D_{\mu} \hat{x}] [\hat{x}, D_{\mu} \hat{x}] \right] + \ldots \quad (B.31) \]

Putting all pieces together as \( L_{\text{eff}} = T_{st} + T_U + T_{\text{mixed}} \), we obtain our final result (3.41).

### Appendix C

**Determinant of** \( G \)**

In order to calculate determinant of matrix \( G \) (4.24), we will use the following recurrence formula, which relates determinant of a symmetric matrix \( M \) of rank \( N + 1 \) to determinant of its \( N \times N \) submatrix \( M \):

\[ M = \begin{pmatrix} \alpha & u^T \\ u & M \end{pmatrix}, \quad \det M = (\alpha - u^T M^{-1} u) \det M. \quad (C.1) \]

After double application of formula (C.2), we get

\[
\det G = \left[ \frac{1}{g_1^2} - \left( \frac{\lambda_1}{m_1} \phi_{B_1}, \frac{\lambda_2}{m_1} \phi_{B_2} \right) \left( \begin{array}{c} G_1 \\ \frac{\lambda_1}{m_1} \phi_{A_1} \\ \frac{\lambda_2}{m_1} \phi_{A_2} \end{array} \right) \right]^{-1} \left[ \begin{array}{c} 0 \\ \frac{\lambda_1}{m_1} \phi_{A_1} \\ \frac{\lambda_2}{m_1} \phi_{A_2} \end{array} \right] \left[ \frac{1}{g_2^2} - \left( \frac{\lambda_1}{m_2} \phi_{B_1}, \frac{\lambda_2}{m_2} \phi_{B_2} \right) \left( \begin{array}{c} G_2 \\ \frac{\lambda_1}{m_2} \phi_{A_1} \\ \frac{\lambda_2}{m_2} \phi_{A_2} \end{array} \right) \right]^{-1} \left( \begin{array}{c} \frac{\lambda_1}{m_2} \phi_{A_1} \\ \frac{\lambda_1}{m_2} \phi_{A_1} \end{array} \right) \times \left( \frac{1}{\epsilon_1^2} \right)^{N_1} \times \left( \frac{1}{\epsilon_2^2} \right)^{N_2}, \quad (C.3) \]
where

\[
G_1 = \begin{pmatrix}
-\frac{1}{g_2^2} g B_1 & -\frac{\lambda_1}{m_2} \phi A_i & -\frac{\lambda_2}{m_2} \phi A_2 \\
\frac{\lambda_1}{m_2} \phi B_1 & \frac{1}{c_1} \delta A_1 B_1 & 0 \\
\frac{\lambda_2}{m_2} \phi B_2 & 0 & \frac{1}{c_2} \delta A_2 B_2
\end{pmatrix},
\]

(C.4)

\[
G_2 = \begin{pmatrix}
\frac{1}{c_1^2} \delta A_1 B_1 & 0 \\
0 & \frac{1}{c_2^2} \delta A_2 B_2
\end{pmatrix}.
\]

(C.5)

Inverse of \(G_1\) is given as

\[
G_1^{-1} = \frac{1}{1 - \frac{g_2^2}{g_2^2} \phi^2} \times
\begin{pmatrix}
\frac{g_2^2}{g_2^2} g B_1 & e_1 m_1 g_2^2 \phi A_i & e_2 m_1 g_2^2 \phi A_2 \\
e_1 m_1 g_2^2 \phi B_1 & e_1^2 \delta A_1 B_1 & m_1^2 e_1 e_2 g_2^2 \phi B_2 \phi A_2 \\
e_2 m_1 g_2^2 \phi B_2 & m_1^2 e_1 e_2 g_2^2 \phi B_2 \phi A_1 & e_2^2 \delta A_2 B_2 - m_1^2 e_1^2 g_2^2 \phi A_2 B_2 \end{pmatrix},
\]

(C.6)

where we have used (C.26)-(C.29).

Straightforward calculation leads us to

\[
det G = \left[ \frac{1}{g_1^2} - \frac{m_2^2 \phi^2}{1 - \frac{g_2^2}{g_2^2} \phi^2} \right] \left[ \frac{1}{g_2^2} - m_2^2 \phi^2 \right] \left( \frac{1}{c_1^2} \right)^{N_1} \left( \frac{1}{c_2^2} \right)^{N_2}.
\]

(C.7)

After multiplying both brackets we obtain the result (4.33):

\[
det G = \left[ \frac{1}{g_1^2 g_2^2} - \left( \frac{m_1^2}{g_1^2} + \frac{m_2^2}{g_2^2} \right) \phi^2 \right] \left( \frac{1}{c_1^2} \right)^{N_1} \left( \frac{1}{c_2^2} \right)^{N_2}.
\]

(C.8)

Next we would like to find condition, which ensures positive definiteness of \(G\). In other words, we require that all eigenvalues of \(G\) are non-negative. We can easily turn (4.33) into characteristic equation by replacing \(\tilde{g}_1^{-2}, \tilde{g}_2^{-2}, e_1^{-2}, e_2^{-2}\) with \(\tilde{g}_1^{-2} - \lambda, \tilde{g}_2^{-2} - \lambda, e_1^{-2} - \lambda, e_2^{-2} - \lambda\). However, since \(\tilde{\phi}^2\) consists of terms proportional to either \(e_1^2\) or \(e_2^2\) instead of \(e_1^{-2}\) or \(e_2^{-2}\), we should first multiply the term in the square bracket by a factor \(e_1^{-2} e_2^{-2}\). Then, after the replacement and denoting \(\tilde{\phi}_i^2 = \phi A_i^2, i = 1, 2\), we obtain a characteristic equation of the forth order:

\[
\left[ \left( \frac{1}{g_1^2} - \lambda \right) \left( \frac{1}{g_2^2} - \lambda \right) \left( \frac{1}{c_1^2} - \lambda \right) \left( \frac{1}{c_2^2} - \lambda \right) \right. \\
\left. - \left( \frac{m_1^2}{g_1^2} + \frac{m_2^2}{g_2^2} - \lambda (m_1^2 + m_2^2) \right) \frac{\tilde{\phi}^2}{c_1^2 c_2^2} - \lambda \frac{\lambda_1^2 \phi_1^2 + \lambda_2^2 \phi_2^2}{m_1^2 m_2^2} \right] \left( \frac{1}{c_1^2} \right)^{N_1} \left( \frac{1}{c_2^2} \right)^{N_2}.
\]

(C.9)

times the factor

\[
\left( \frac{1}{c_1^2} - \lambda \right)^{N_1-1} \left( \frac{1}{c_2^2} - \lambda \right)^{N_2-1},
\]

(C.10)
which clearly leads to positive eigenvalues. Expanding the brackets we obtain explicit coefficients of the characteristic polynomial:

\[ \lambda^4 - \left( \frac{1}{g_1^2} + \frac{1}{g_2^2} + \frac{1}{e_1^2} + \frac{1}{e_2^2} \right) \lambda^3 \]

\[ + \left( \frac{1}{g_1^2 g_2^2} + \frac{1}{g_1^2 e_1^2} + \frac{1}{g_2^2 e_2^2} + \frac{1}{g_1^2 e_2^2} + \frac{1}{g_2^2 e_1^2} + \frac{1}{e_1^2 e_2^2} - \frac{m_1^2 + m_2^2}{m_1^2 m_2^2} (\lambda^2 \phi_1^2 + \lambda^2 \phi_2^2) \right) \lambda^2 \]

\[ - \left( \frac{\hat{g}_1^2 + \hat{g}_2^2 + e_1^2 + e_2^2}{g_1^2 g_2^2 e_1^2 e_2^2} - \hat{g}_1^2 \phi_1^2 \hat{g}_2^2 \phi_2^2 \right) \sum \lambda^N \left( \frac{m_1^2 + m_2^2}{m_1^2 m_2^2} - \frac{m_1^2 + m_2^2}{e_1^2 e_2^2} \right) \lambda + \frac{1 - \hat{g}^2 \phi^2}{g_1^2 g_2^2 e_1^2 e_2^2}. \] (C.11)

In order to see the non-negativeness of eigenvalues it is not necessary to solve the characteristic equation. Generally speaking, characteristic equation of a real symmetric matrix can be always put into the form

\[ (\lambda - \lambda_1)(\lambda - \lambda_2) \ldots (\lambda - \lambda_N) = 0, \] (C.12)

where all roots \( \lambda_1, \ldots, \lambda_N \) are real numbers. Multiplying all parentheses we see that the coefficients \( c_k \) of characteristic polynomial are given by the sum of all possible \( k \)-tuples of \( \lambda \)'s with alternating sign:

\[ (\lambda - \lambda_1)(\lambda - \lambda_2) \ldots (\lambda - \lambda_N) = \lambda^N - \left( \sum_i \lambda_i \right) \lambda^{N-1} + \left( \sum_{i>j} \lambda_i \lambda_j \right) \lambda^{N-2} - \ldots - (-1)^N \lambda_1 \ldots \lambda_N \]

\[ = \sum_{i=0}^{N} (-1)^i c_i \lambda^{N-i}. \] (C.13)

The positivity of all coefficients \( c_k \) turns out to be equivalent to the positivity of all eigenvalues \( \lambda_k \). To ensure positivity of the eigenvalues, we now demand that all terms in (C.11) inside brackets are positive. This gives us three conditions:

\[ \hat{g}_1^2 \hat{g}_2^2 + \hat{g}_1^2 e_1^2 + \hat{g}_1^2 e_2^2 + \hat{g}_2^2 e_1^2 + \hat{g}_2^2 e_2^2 + e_1^2 e_2^2 \]

\[ - \hat{g}_1^2 \hat{g}_2^2 (m_1^2 + m_2^2) (e_1^2 \phi_1^2 + e_2^2 \phi_2^2) \geq 0, \] (C.14)

\[ \hat{g}_1^2 + \hat{g}_2^2 + e_1^2 + e_2^2 - \hat{g}^2 (e_1^2 \phi_1^2 + e_2^2 \phi_2^2) - \hat{g}_1^2 \hat{g}_2^2 (m_1^2 + m_2^2) \phi^2 \geq 0, \]

\[ 1 - \hat{g}^2 \phi^2 \geq 0. \] (C.15)

\[ 1 - \hat{g}^2 \phi_1^2 + \hat{g}_1^2 \phi_2^2 \geq 0, \] (C.16)

These can be put into the convenient form:

\[ 1 - \hat{g}_1^2 \phi_1^2 - \hat{g}_2^2 \phi_2^2 \geq 0, \] (C.17)

\[ 1 - \hat{g}_1^2 \phi_1^2 - \hat{g}_2^2 \phi_2^2 \geq 0, \] (C.18)

\[ 1 - \hat{g}^2 \phi_1^2 - \hat{g}^2 \phi_2^2 \geq 0, \] (C.19)
where
\[\hat{g}_{21}^2 = \frac{g_1^2 g_2^2 (m_2^2 + m_2^2) e_2^2}{g_1^2 g_2^2 + g_1^2 e_1^2 + g_1^2 e_2^2 + g_2^2 e_1^2 + g_2^2 e_2^2 + e_1^2 e_2^2},\]
\[\hat{g}_{22}^2 = \frac{g_1^2 g_2^2 (m_2^2 + m_2^2) e_2^2}{g_1^2 g_2^2 + g_1^2 e_1^2 + g_1^2 e_2^2 + g_2^2 e_1^2 + g_2^2 e_2^2 + e_1^2 e_2^2},\]
\[\hat{g}_{11}^2 = \frac{g_2^2 e_2^2 + g_2^2 g_1^2 (m_1^2 + m_2^2)}{g_1^2 + g_2^2 + e_1^2 + e_2^2},\]
\[\hat{g}_{12}^2 = \frac{g_2^2 e_2^2 + g_2^2 g_1^2 (m_1^2 + m_2^2)}{g_1^2 + g_2^2 + e_1^2 + e_2^2}.
\]

We are going to argue, that the last condition \(1 - \hat{g}^2 \hat{g}^2 \geq 0\) is the strongest one and, therefore, only one important. This can be true if and only if parameter \(\hat{g}^2\) is always greater then \(\hat{g}_{11}^2,\) \(\hat{g}_{12}^2,\) \(\hat{g}_{21}^2\) and \(\hat{g}_{22}^2\) for all possible values of involved parameters. This is indeed so. Let us demonstrate this fact by showing, for example
\[\hat{g}^2 \geq \hat{g}_{11}^2.\] (C.24)

Multiplying both sides by \(\hat{g}_1^2 + \hat{g}_2^2 + e_1^2 + e_2^2\) and expanding our notation, we get
\[m_2^2 \hat{g}_2^2 (\hat{g}_2^2 + e_2^2) - e_1^2 + e_2^2) \geq e_2^2 (m_2^2 \hat{g}_2^2 + m_2^2 \hat{g}_1^2) + \hat{g}_1^2 \hat{g}_2^2 (m_1^2 + m_2^2),\] (C.25)
leading to
\[m_1^2 \hat{g}_1^2 (\hat{g}_2^2 + e_1^2) + m_2^2 \hat{g}_1^2 (\hat{g}_1^2 + e_1^2) \geq 0.\] (C.26)

Last line is obviously always true. In the same way, one can show that \(\hat{g}^2 \geq \hat{g}_{12}^2,\) \(\hat{g}^2 \geq \hat{g}_{21}^2\) and \(\hat{g}^2 \geq \hat{g}_{22}^2.\) This proves our claim, that condition (4.34) is both necessary and sufficient to ensure positivity of the matrix \(G.\)

References

1) P. Horava and E. Witten, Nucl. Phys. B 460 (1996) 506 [arXiv:hep-th/9510209].
2) N. Arkani-Hamed, S. Dimopoulos and G. R. Dvali, Phys. Lett. B 429 (1998) 263 [hep-ph/9803315]; I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. R. Dvali, Phys. Lett. B 436 (1998) 257 [hep-ph/9804398].
3) L. Randall and R. Sundrum, Phys. Rev. Lett. 83 (1999) 3370 [hep-ph/9905221]; Phys. Rev. Lett. 83 (1999) 4690 [hep-th/9906064].
4) S. Dimopoulos and H. Georgi, Nucl. Phys. B 193 (1981) 150.
N. Sakai, Z. Phys. C 11 (1981) 153.
E. Witten, Nucl. Phys. B 188 (1981) 513.
5) V. A. Rubakov and M. E. Shaposhnikov, Phys. Lett. B 125 (1983) 136.
6) S. L. Dubovsky and V. A. Rubakov, Int. J. Mod. Phys. A 16 (2001) 4331 [hep-th/0105243].
7) G. R. Dvali and M. A. Shifman, Phys. Lett. B 396 (1997) 64 [Erratum-ibid. B 407 (1997) 452] [hep-th/9612128].
8) N. Maru and N. Sakai, Prog. Theor. Phys. 111 (2004) 907 [arXiv:hep-th/0305222].
9) Y. Isozumi, K. Ohashi and N. Sakai, JHEP 0311 (2003) 061 [arXiv:hep-th/0310130].
10) J. B. Kogut and L. Susskind, Phys. Rev. D 9 (1974) 3501.
11) R. Fukuda, Phys. Lett. B 73 (1978) 305 [Erratum-ibid. B 74 (1978) 433]; Mod. Phys. Lett. A 24 (2009) 251 [arXiv:0805.3864 [hep-th]].
12) K. Ohta and N. Sakai, Prog. Theor. Phys. 124 (2010) 71 [arXiv:1004.4078 [hep-th]].
13) M. Eto, T. Fujimori, M. Nitta, K. Ohashi and N. Sakai, Phys. Rev. D 77 (2008) 125008 [arXiv:0802.3135 [hep-th]].
14) Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, “Construction of non-Abelian walls and their complete moduli space,” Phys. Rev. Lett. 93 (2004) 161601 [arXiv:hep-th/0404198]; Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, “Non-Abelian walls in supersymmetric gauge theories,” Phys. Rev. D 70 (2004) 125014 [arXiv:hep-th/0405194].
15) M. Eto, Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, J. Phys. A A 39 (2006) R315 [hep-th/0602170].
16) N. S. Manton, Phys. Lett. B 110 (1982) 54; N. S. Manton and P. Sutcliffe, Topological Solitons (Cambridge University Press, Cambridge, England, 2004).
17) N. Seiberg, Phys. Lett. B 388 (1996) 753 [hep-th/9608111].
18) M. Shifman and A. Yung, Phys. Rev. D 70 (2004) 025013 [hep-th/0312257].
19) M. Bando, T. Kuramoto, T. Maskawa and S. Uehara, Phys. Lett. B 138, 94 (1984); Prog. Theor. Phys. 72, 313 (1984); Prog. Theor. Phys. 72, 1207 (1984); K. Higashijima, M. Nitta, K. Ohta and N. Ohta, Prog. Theor. Phys. 98, 1165 (1997) [arXiv:hep-th/9706219]; K. Higashijima and M. Nitta, Prog. Theor. Phys. 103, 635 (2000) [arXiv:hep-th/9911139]; Prog. Theor. Phys. 103, 833 (2000) [arXiv:hep-th/9911225]; M. Nitta, Nucl. Phys. B 711, 133 (2005) [arXiv:hep-th/0312025].
20) Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, Phys. Rev. Lett. 93 (2004) 161601 [arXiv:hep-th/0404198]; Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, Phys. Rev. D 70 (2004) 125014 [arXiv:hep-th/0405194]; M. Eto, Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, Phys. Rev. Lett. 96 (2006) 161601 [arXiv:hep-th/0511088].
21) M. Eto, Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, Phys. Rev. D 73 (2006)
22) M. Eto, M. Nitta, K. Ohashi and D. Tong, Phys. Rev. Lett. 95, 252003 (2005) [hep-th/0508130].

23) T. H. R. Skyrme, Proc. Roy. Soc. Lond. A 260, 127 (1961).

24) G. S. Adkins, C. R. Nappi and E. Witten, Nucl. Phys. B 228, 552 (1983).

25) T. Sakai and S. Sugimoto, Prog. Theor. Phys. 113 (2005) 843 [hep-th/0412141].

26) N. Maru, N. Sakai, Y. Sakamura and R. Sugisaka, Phys. Lett. B 496 (2000) 98 [hep-th/0009023]; N. Maru, N. Sakai, Y. Sakamura and R. Sugisaka, Nucl. Phys. B 616 (2001) 47 [hep-th/0107204].