Reciprocal ontological models show indeterminism of the order of quantum theory

Somshubro Bandyopadhyay, Manik Banik, Some Sankar Bhattacharya, Sibasish Ghosh, Guruprasad Kar, Amit Mukherjee, and Arup Roy

1Department of Physics and Center for Astroparticle Physics and Space Science, Bose Institute, Block EN, Sector V, Bidhan Nagar, Kolkata 700091, India
2Physics and Applied Mathematics Unit, Indian Statistical Institute, 203 B. T. Road, Kolkata 700108, India
3Optics & Quantum Information Group, The Institute of Mathematical Sciences, C.I.T Campus, Tharamani, Chennai 600 113, India

The question whether indeterminism in quantum measurement outcomes is fundamental or is there a possibility of constructing a finer theory underlying quantum mechanics that allows no such indeterminism, has been debated for a long time. We show that within the class of ontological models due to Harrigan and Spekkens, those satisfying preparation-measurement reciprocity must allow indeterminism of the order of quantum theory. Our result implies that one can design quantum random number generator, for which it is impossible, even in principle, to construct a reciprocal deterministic model.

I. INTRODUCTION

Quantum mechanics is believed to be fundamentally random. According to quantum theory, measurement outcomes on a quantum system prepared in a known state cannot be definitively predicted as long as the state is not an eigenstate of the observable being measured [1, 2]. This is said to be quantum indeterminism, which gives rise to quantum randomness [3–5]. The significance of quantum randomness lies in the fact that true randomness is hard to characterize mathematically [6, 7], and also cannot be obtained from classical physics [8]. Thus quantum randomness, regarded as the only form of true randomness in nature, becomes crucial as a resource for applications ranging from cryptography to numerical simulation of physical and biological systems.

On the other hand, we know that classical statistical physics also allows indeterminism, and therefore, randomness of some form. This randomness, however, cannot be considered genuine because the underlying theory, i.e., Newtonian physics is deterministic. In fact, the source of randomness in a purely classical theory can be attributed to our lack of knowledge. This observation alone makes room for a similar argument in the case of quantum theory. That is, if there is a finer deterministic theory that underlies quantum mechanics, then quantum randomness ceases to be fundamental. Indeed, for two level quantum systems such theories do exist, e.g., Bell-Mermin model [9, 10], and Kochen-Specker model [11]. Therefore, the question of existence of a finer deterministic theory, which nonetheless should reproduce quantum statistics of measurement outcomes, becomes important.

The above question can be addressed within the framework of ontological models due to Harrigan and Spekkens [12]. It is well known that one classification of ontological models arise from the probable interpretations of quantum state $|\psi\rangle$; i.e., whether $|\psi\rangle$ represents the physical reality or merely the observer’s knowledge about the quantum system. In fact, the question of interpretation of quantum state has been strongly debated since the inception of quantum theory [9, 13–18].

An ontological model is said to be $\psi$-epistemic if it considers $|\psi\rangle$ to represent observer’s knowledge about the system, and $\psi$-ontic if it considers $|\psi\rangle$ to represent reality of the system, e.g., [19–22]. Thus, in a $\psi$-epistemic model we can find distinct quantum states with overlapping probability distributions in the space of ontic states, whereas in a $\psi$-ontic model distinct quantum states correspond to probability distributions that do not overlap. Moreover, we say that a model is maximally $\psi$-epistemic if the quantum overlap between two state vectors can be completely accounted for by the overlap of their probability distributions in the ontic space.

The above ontological models can be characterized by the so called “degree of epistemicity” $0 \leq \Omega(\psi, \phi) \leq 1$ defined for a pair of quantum states $(|\psi\rangle, |\phi\rangle)$ [23, 24]. In particular, we have $\Omega(\psi, \phi) = 0$ for a $\psi$-ontic model, $\Omega(\psi, \phi) = 1$, for a maximally $\psi$-epistemic model and $0 < \Omega(\psi, \phi) < 1$ for models that are neither ontic nor maximally $\psi$-epistemic, also called non-maximally $\psi$-epistemic [25, 26].

The motivation of the present work stems from a recent result by Maroney [23]. He proved a powerful theorem, which states that for quantum systems of dimension greater than two it is impossible to construct a maximally $\psi$-epistemic ontological model. Maroney’s result stands out because it was proved without auxiliary assumptions unlike results [27, 28] claiming $|\psi\rangle$ must be ontic. For example, the authors of [28] have shown that a quantum system’s wave function is in one-to-one correspondence with its elements of reality, which implies that quantum indeterminism is irreducible; however, this was derived under a strong freedom of choice assumption whose characterization is unrealistic and unsatisfactory [29, 30]. Also note that [27] requires additional property, namely, preparation-independence, and without it...
explicit counter-examples show that epistemic models underlying quantum theory can be formulated [25, 26].

To understand the implication of Maroney’s theorem, we first need to briefly describe the structural features associated with ontological models. These features were explicitly discussed by Ballentine [30]. He introduced the property preparation-measurement reciprocity, which is satisfied by quantum theory but not necessarily holds in an ontological model. It’s given by two conditions: The first one, termed as “Quantum certainty” states that a quantum state |ψ⟩ will always pass the measurement filter |ψ⟩⟨ψ| (here we note that any ontological model that aims to reproduce quantum statistics must satisfy quantum certainty). The second condition is simply the converse, i.e., |ψ⟩ is the only state that passes the filter |ψ⟩⟨ψ| with probability one. An ontological model that satisfies these two conditions is said to be reciprocal, and non-reciprocal if the converse does not hold (this is because we always require that any ontological model should reproduce quantum statistics). Furthermore, an ontological model is outcome-deterministic if the measurement outcomes, at the ontological level, can be predicted with certainty, else the model is said to be outcome-indeterministic.

Ballentine [30] showed that Maroney’s theorem rules out ontological models that are both reciprocal and outcome-deterministic. In other words, a reciprocal ontological model must be indeterministic. Naturally, the question is, To what extent such a model is indeterministic? In this paper, we focus on this question.

We prove that indeterminism in a reciprocal ontological model is of the order of quantum theory. This is shown in two steps. For a quantum state |ψ⟩ and a projective observable |φ⟩⟨φ|, let $I_{\text{ont}}(\psi, \phi)$ and $I_{\text{Q}}(\psi, \phi)$ (precisely defined later) quantify indeterminism associated with outcome φ in a reciprocal ontological model and quantum theory respectively. We first show that the following relation holds:

\[ I_{\text{ont}}(\psi, \phi) = [1 - \Omega(\psi, \phi)] I_{\text{Q}}(\psi, \phi), \tag{1} \]

where $\Omega(\psi, \phi)$ is the degree of epistemicity of the pair of quantum states (|ψ⟩, |φ⟩). Note that, although Eq. (1) connects $I_{\text{ont}}$ and $I_{\text{Q}}$ via the degree of epistemicity of the states, it does not tell us how $I_{\text{ont}}$ compares with $I_{\text{Q}}$. To compare, we therefore need to know $\Omega(\psi, \phi)$ for a given pair of states (|ψ⟩, |φ⟩).

In the next step, we make use of a basis independent upper bound on the degree of epistemicity [23] to show that for all pairs of non-orthogonal quantum states in Hilbert spaces of dimension $d \geq 3$,

\[ I_{\text{ont}} \geq \frac{(d - 2)^2}{d^2 + (d - 2)^2} I_{\text{Q}}. \tag{2} \]

For example, in three dimension, $I_{\text{ont}} \geq \frac{1}{10} I_{\text{Q}}$, and in the limit of large dimension of the Hilbert space, $I_{\text{ont}}(\psi, \phi) \geq \frac{1}{10} I_{\text{Q}}(\psi, \phi)$. We therefore conclude that indeterminism in a reciprocal ontological model is of the order of quantum theory. Note that our result implies that it is possible to design quantum random number generator for which it is not possible, even in principle, to construct a deterministic ontological model.

We further observe that Eq. (1) establishes the following correspondence:

\[ I_{\text{ont}}(\psi, \phi) = I_{\text{Q}}(\psi, \phi) \iff \text{ont-istic}; \]
\[ I_{\text{ont}}(\psi, \phi) = 0 \iff \text{maximally \text{epistemic};} \]
\[ I_{\text{ont}}(\psi, \phi) < I_{\text{Q}}(\psi, \phi) \iff \text{nonmaximally \text{epistemic.}} \]

The paper is arranged in the following way. Sections II and III contain the necessary background material. In Sec. II we first define the notion of indeterminism in an operational theory. We then describe, in some detail, quantum indeterminism and the condition of preparation-measurement reciprocity. In Sec. III we first briefly review the general theory of ontological models, and the ontological models of quantum theory following Ballentine [30]. We also discuss Maroney’s no-go theorem [23], and its implication as pointed out by Ballentine [30]. In Sec. IV we define the notion of indeterminism in an ontological model of operational quantum theory, and prove our results given by Eqs. (1) and (2). Sec. V concludes with a discussion on related aspects.

## II. INDETERMINISM IN OPERATIONAL QUANTUM THEORY

The primitive elements of an operational theory are preparation procedures $P \in \mathcal{P}$, transformations $T \in \mathcal{T}$, and measurement procedures $M \in \mathcal{M}$, where $\mathcal{P}, \mathcal{T}$ and $\mathcal{M}$ denote collection of all permissible preparations, transformations and measurements respectively. An operational theory specifies the probabilities of different outcomes of a measurement performed on a system prepared according to some definite procedure. Let $p(k|P, M) \in [0, 1]$ denote the probability of outcome $k$ when a measurement $M \in \mathcal{M}$ is performed on a system prepared according to some procedure $P \in \mathcal{P}$ [31]. We therefore have,

\[ \sum_{k \in K_M} p(k|P, M) = 1 \quad \forall P, M, \tag{3} \]

where $K_M$ denotes the set of all measurement outcomes corresponding to the measurement $M$.

**Definition.** An operational theory is deterministic if and only if $p(k|P, M)$ takes values either 0 or 1 for every $k, P, M$. Otherwise the theory is indeterministic.

In operational quantum theory, a preparation corresponds to the state of a quantum system described by a vector $|\psi\rangle \in \mathcal{H}$, where $\mathcal{H}$ is the Hilbert space associated
with the system; a measurement corresponds to an hermitian operator – an observable \( \mathcal{O} \) – acting on \( \mathcal{H} \), whose eigenvalues represent the measurement outcomes. Suppose \( \{ \phi_k \} \) is the set of eigenvalues of \( \mathcal{O} \) with the corresponding set of eigenvectors \( \{ |\phi_k\rangle \} \). When observable \( \mathcal{O} \) is measured on the system prepared in state \( |\psi\rangle \), the probability of obtaining the outcome \( \phi_k \) is given by the Born rule:

\[
p(\phi_k|\psi,\mathcal{O}) = |\langle \phi_k|\psi \rangle|^2,
\]

where \( p(\phi_k|\psi,\mathcal{O}) \in [0,1] \). For a specific outcome, the probability is one if and only if \( |\psi\rangle \) is the corresponding eigenstate and is zero if and only if \( |\psi\rangle \) is orthogonal to the corresponding eigenstate. It therefore follows that quantum theory in general is indeterministic.

To define indeterminism in quantum theory we proceed as follows. We first observe that a measurement (observable \( \mathcal{O} \)) is, in fact, a collection of projective measurements (observables) \( \{ \phi_k \} \). Therefore, without loss of generality, we can represent a measurement by a single projective observable: \( M_\phi = |\phi\rangle \langle \phi| \). Stated in this language, the indeterminism in quantum mechanics for a preparation-measurement pair \( (|\psi\rangle,M_\phi) \) is defined as,

\[
\mathcal{I}_Q(\psi,\phi) := |\langle \psi|\phi \rangle|^2,
\]

where \( \mathcal{I}_Q \in [0,1] \), and the outcome is indeterministic if and only if \( \mathcal{I}_Q \neq 0,1 \).

It is now easy to see that when measurement of an observable \( \mathcal{O} \) on a system in state \( |\psi\rangle \) results in indeterministic outcomes, the preparation-measurement pair \( (\psi,\mathcal{O}) \) can be used to design quantum random number generator, where randomness can be quantified by the guessing probability \( G := \max_{\phi_k} p(\phi_k|\psi,\mathcal{O}) \) [32]. Note that the amount of randomness can also be quantified in terms of bit, i.e., \( H_{\mathcal{O}} = -\log_2 G \), which is the expression for min entropy [33].

### Preparation-measurement reciprocity

A consequence of (5) is an interesting reciprocal relationship of state preparation and measurement as observed by Ballentine [30]:

- **Quantum Certainty**: A system that is prepared in the state \( |\psi\rangle \) will always pass the test of measuring the projector \( |\psi\rangle \langle \psi| \).
- **Converse**: The state \( |\psi\rangle \) is the only state that will pass the projective measurement filter \( |\psi\rangle \langle \psi| \) with certainty.

Thus quantum mechanics admits preparation-measurement reciprocity, or reciprocity for short. It is important to recognize that a finer theory, which may or may not be deterministic, must satisfy “Quantum Certainty” otherwise it will fail to reproduce quantum statistics of measurement outcomes. However, it is possible that in such a theory the converse may not hold.

### III. ONTOLOGICAL MODELS AND DEGREE OF EPISTEMICITY

In an ontological model of an operational theory, the primitives of description are the properties of microscopic systems [12], where a preparation procedure prepares a system with definite properties and a measurement procedure tells us something about those properties. A complete specification of the properties of a system is referred to as the ontic state – the state of reality of that system and is denoted by \( \lambda \in \Lambda \), where \( \Lambda \) denotes the space of ontic states. A particular preparation \( P \) actually yields an ontic state \( \lambda \), and in general the same preparation, when repeated, produces a different ontic state. We therefore have a probability distribution \( 0 \leq \mu(\lambda|P) \leq 1 \) over the ontic states \( \lambda \in \Lambda \), said to be the epistemic state – the state of knowledge, which satisfies:

\[
\int_{\Lambda} \mu(\lambda|P) = 1 \quad \forall P.
\]

For a measurement \( M \), the probability of obtaining an outcome \( k \) is determined by a response (or indicator) function \( 0 \leq \xi(k|\lambda, M) \leq 1 \) satisfying

\[
\sum_k \xi(k|\lambda, M) = 1 \quad \forall k, \lambda, M.
\]

**Definition.** If the response function always takes values either 0 or 1 then the ontological model is outcome-deterministic. Otherwise it is outcome-indeterministic.

The ontological model, however, is required to reproduce the predictions of the operational theory for every preparation-measurement pair \( (P, M) \). We therefore must have,

\[
p(k|P, M) = \int_{\Lambda} \xi(k|\lambda, M) \mu(\lambda|P) d\lambda
\]

**Remark.** If the ontological model is outcome-deterministic then the operational theory may or may not be deterministic. However, if the operational theory is deterministic, then it necessarily implies that at the ontological level, the theory is outcome-deterministic.

### Ontological models of operational quantum theory

In an ontological model of operational quantum theory we have, the epistemic state \( 0 \leq \mu(\lambda|\psi) \leq 1 \quad \forall \lambda \), satisfying

\[
\int_{\Lambda} \mu(\lambda|\psi) d\lambda = 1 \quad \forall |\psi\rangle,
\]
and the response function $0 \leq \xi (\phi | \lambda, \mathcal{O}) \leq 1$, satisfying
\[
\sum_{\phi} \xi (\phi | \lambda, \mathcal{O}) = 1 \forall \lambda, \mathcal{O} \tag{10}
\]

We require that the ontological model must reproduce the predictions of operational quantum theory. Therefore, for any projective observable $M_\phi = |\phi\rangle \langle \phi|$, the ontological model must satisfy
\[
\int_\Lambda \xi (\phi | \lambda) \mu (\lambda | \psi) d\lambda = |\langle \psi | \phi \rangle|^2, \tag{11}
\]
where for simplicity the following notation: $\xi (\phi | \lambda, M_\phi) = \xi (\phi | \lambda)$ is adopted and will hold for the rest of the paper. In the ontic state space $\Lambda$ we can identify the following subsets $\{ \}$:
\[
\Lambda_\phi := \{ \lambda \in \Lambda | \mu (\lambda | \psi) > 0 \}, \tag{12}
\]
\[
\text{Supp} [\xi (\psi | \lambda)] := \{ \lambda \in \Lambda | \xi (\psi | \lambda) > 0 \}, \tag{13}
\]
\[
\text{Core} [\xi (\psi | \lambda)] := \{ \lambda \in \Lambda | \xi (\psi | \lambda) = 1 \}. \tag{14}
\]

Eq. (11) implies that an ontological model underlying quantum theory must satisfy Quantum Certainty, i.e., for the ontic states $\lambda$’s that appear with a non-zero probability in the preparation of the state $|\psi\rangle$, the response function $\xi (\psi | \lambda) = 1 \forall \lambda \in \Lambda_\phi$. Consequently,
\[
\int_{\Lambda_\phi} \xi (\psi | \lambda) \mu (\lambda | \psi) d\lambda = 1. \tag{15}
\]

Therefore, among the subsets defined in Eqs. (12-14) the following set inclusion relation holds,
\[
\Lambda_\phi \subseteq \text{Core} [\xi (\psi | \lambda)] \subseteq \text{Supp} [\xi (\psi | \lambda)]. \tag{16}
\]

The above inclusion relations immediately lead to the following classification.

**Definition.** An ontological model for quantum theory that satisfies preparation-measurement reciprocity (Quantum Certainty and its Converse) is said to be reciprocal. If the Converse does not hold the model is said to be non-reciprocal.

**Remark.** For a reciprocal ontological model the relation
\[
\Lambda_\phi = \text{Core} [\xi (\psi | \lambda)] \forall |\psi\rangle \tag{17}
\]
holds. Note that quantum theory also satisfies the preparation-measurement reciprocity relation.

**Definition.** An ontological model for quantum theory is said to be outcome-deterministic if and only if $\text{Core} [\xi (\psi | \lambda)] = \text{Supp} [\xi (\psi | \lambda)] \forall |\psi\rangle$, otherwise it is said to be outcome-indeterministic.

**Remark.** If an ontological model assigns indeterministic response function to some outcomes, then at least for those outcomes quantum randomness exists at ontological level.

Thus essentially we have four kinds of ontological models. An ontological model is either (a) reciprocal and deterministic/indeterministic or (b) non-reciprocal and deterministic/indeterministic. We, however, require all models to satisfy the condition of "Quantum Certainty".

**Maroney’s no go theorem and its implication**

Recently Maroney introduced the concept of degree of epistemicity [23, 24] for ontological models of quantum theory. He showed that in any ontological model the following relation holds:
\[
\int_{\Lambda_\phi} \mu (\lambda | \psi) d\lambda = \Omega (\psi, \phi) |\langle \psi | \phi \rangle|^2, \tag{18}
\]
where $0 \leq \Omega (\psi, \phi) \leq 1$ is defined as the “degree of epistemicity” associated with the states $|\psi\rangle$ and $|\phi\rangle$. As noted earlier, the ontological models can be characterized by the degree of epistemicity.

Without making auxiliary assumptions, Maroney proved that for Hilbert spaces of dimension greater than two it is not possible to construct a maximally $\psi$-epistemic ontological model $\{ \Omega (\psi, \phi) = 1 \}$ for quantum theory [23]. In a very recent work [30] Ballentine showed that if the ontological model is maximally $\psi$-epistemic, then for any quantum state $|\psi\rangle$ the following relation must hold:
\[
\Lambda_\phi = \text{Core} [\xi (\psi | \lambda)] = \text{Supp} [\xi (\psi | \lambda)]. \tag{19}
\]

The above equation tells us that a maximally $\psi$-epistemic model must be reciprocal and outcome-deterministic. He further showed that the converse also holds; therefore, for all $|\psi\rangle$

Maximally $\psi$-epistemic $\iff$ Reciprocal AND Outcome-deterministic. \tag{20}

It follows that any ontological model can fail to be maximally $\psi$-epistemic in only one of the three possible ways; it can be outcome-indeterministic or non-reciprocal or both.

Maroney’s result implies that a reciprocal ontological model must necessarily be indeterministic. In the following section we show that the indeterminism is of the order of quantum theory.

**IV. INDETERMINISM IN RECIPROCAL ONTOLOGICAL MODEL OF QUANTUM THEORY**

We begin by defining the notion of indeterminism in an ontological model of quantum theory. To denote the subsets Core and Support associated with a
for a preparation-measurement pair comes. Noting that the contribution to the integral state be outcome-indeterministic; i.e., for every quantum which simply captures the fact that an ontological model tion) we know that a reciprocal ontological model must reproduced below ,

for quantum systems in dimensions greater than two.

for a preparation $\psi$ and a measurement $M_\psi = \langle \psi | \phi \rangle$, define the region $\Lambda_{\psi \cap C_\phi}$, indeterminism in an ontological model is therefore defined as

$$I_{\text{ont}}(\psi, \phi) := \int_{\lambda \in \Lambda_{\psi \cap C_\phi}} \xi(\phi | \lambda)\mu(\lambda | \psi)d\lambda.$$  \hspace{1cm} (21)

Observe that if the model is outcome-deterministic, then $S_\phi = C_\phi$, and therefore, $I_{\text{ont}} = 0$, i.e., at the ontological level indeterminism is absent. It is also important to recognize that while Eq. (5) quantifies indeterminism at an operational level, Eq. (21) does so at the ontological level.

We now consider only reciprocal ontological models for quantum systems in dimensions greater than two. From Maroney’s theorem (discussed in the previous section) we know that a reciprocal ontological model must be outcome-indeterministic; i.e., for every quantum state $|\eta\rangle$ there exists a non-trivial region $S_\phi \cap C_\eta \neq \emptyset$ in the ontic state space, where $0 < \xi(\eta | \lambda) < 1$. Thus for a preparation-measurement pair $(\psi, M_\psi)$, we have $\Lambda_{\psi} := \Lambda_\psi \cap (S_\phi \cap C_\psi) \neq \emptyset$. This means, whenever the system is prepared in state $|\psi\rangle$, outcome $\phi$ is indeterministic at the ontological level only for those $\lambda \in \Lambda_{\psi}$. Since every $\lambda \in \Lambda_{\psi}$ gives indeterministic response for the outcome $\phi$, $I_{\text{ont}}(\psi, \phi) := \int_{\lambda \in \Lambda_{\psi}} \xi(\phi | \lambda)\mu(\lambda | \psi)d\lambda > 0$. \hspace{1cm} (22)

We now prove Eq. (1). We begin by recalling Eq. (11) reproduced below,

$$\int_{\Lambda} \mu(\lambda | \psi)\xi(\phi | \lambda)d\lambda = |\langle \psi | \phi \rangle|^2,$$ \hspace{1cm} (23)

which simply captures the fact that an ontological model must reproduce quantum statistics of measurement outcomes. Noting that the contribution to the integral comes only from the region $\Lambda_\phi \cap S_\phi$ (as it vanishes elsewhere), we can write Eq. (23) as

$$\int_{\Lambda_\phi \cap S_\phi} \mu(\lambda | \psi)\xi(\phi | \lambda)d\lambda = |\langle \psi | \phi \rangle|^2.$$ \hspace{1cm} (24)

We can now break up the region of the above integration in the following way (see Fig.1),

$$\int_{\Lambda_\phi \cap C_\phi} \mu(\lambda | \psi)\xi(\phi | \lambda)d\lambda + \int_{\Lambda_\phi} \mu(\lambda | \psi)\xi(\phi | \lambda)d\lambda = |\langle \psi | \phi \rangle|^2.$$ \hspace{1cm} (25)

Using the fact that $\xi(\phi | \lambda) = 1 \forall \lambda \in C_\phi$, we have

$$\int_{\Lambda_\phi \cap C_\phi} \mu(\lambda | \psi)d\lambda + \int_{\Lambda_\phi} \mu(\lambda | \psi)\xi(\phi | \lambda)d\lambda = |\langle \psi | \phi \rangle|^2.$$ \hspace{1cm} (26)

Because the concerned ontological model is reciprocal, i.e., $C_\phi = \Lambda_\phi$,

$$\int_{\Lambda_\phi \cap C_\phi} \mu(\lambda | \psi)d\lambda + \int_{\Lambda_\phi} \mu(\lambda | \psi)\xi(\phi | \lambda)d\lambda = |\langle \psi | \phi \rangle|^2.$$ \hspace{1cm} (27)

Now observe that, $\int_{\Lambda_\phi \cap C_\phi} \mu(\lambda | \psi)d\lambda = \int_{\Lambda_\phi} \mu(\lambda | \psi)d\lambda$, using this equality and Maroney’s relation given by Eq. (18), Eq. (27) can be expressed as,

$$\int_{\Lambda_\phi} \mu(\lambda | \psi)d\lambda + \int_{\Lambda_\phi} \mu(\lambda | \psi)\xi(\phi | \lambda)d\lambda = |\langle \psi | \phi \rangle|^2.$$ \hspace{1cm} (28)

Noting that the left hand side is simply the definition of $I_{\text{ont}}(\psi, \phi)$ given by Eq. (22) and $|\langle \psi | \phi \rangle|^2 = I_Q(\psi, \phi)$, we finally arrive at our desired result,

$$I_{\text{ont}}(\psi, \phi) = |1 - \Omega(\psi, \phi)| I_Q(\psi, \phi).$$ \hspace{1cm} (29)

The above equation is quite satisfying as it connects the notion of indeterminism in quantum theory and reciprocal ontological model through the degree of epistemicity. In order to compare $I_{\text{ont}}(\psi, \phi)$ with $I_Q(\psi, \phi)$ for a given pair of states $(|\psi\rangle, |\phi\rangle)$, the knowledge of $\Omega(\psi, \phi)$ is required. We now obtain inequality (2), which allows us to compare $I_{\text{ont}}$ with $I_Q$.

In [23], Maroney noted that a basis independent measure of degree of epistemicity allows one to set $\Omega(\psi, \phi) = \Omega(d)$, where $\Omega(d)$ is a constant for all pairs of non-orthogonal quantum states in dimensions $d \geq 3$, and satisfies,

$$\Omega(d) \leq \frac{d^2}{2d^2 - 4d + 4}.$$ \hspace{1cm} (30)

Substituting the above in Eq. (29) leads us to

$$I_{\text{ont}} \geq \frac{(d - 2)^2}{d^2 + (d - 2)^2} I_Q$$ \hspace{1cm} (31)

for all pairs of non-orthogonal states $(|\psi\rangle, |\phi\rangle)$ in dimensions $d \geq 3$. For instance, for quantum systems.
of dimension three, we have $I_{\text{ont}} \geq \frac{1}{3} I_Q$, and in the limit of large dimension $I_{\text{ont}} \geq \frac{1}{4} I_Q$. It may be noted that if a basis independent measure of $\Omega(\psi, \phi)$ is not assumed, then one can find pairs of states $\{\psi, \phi\}$ giving tighter upper bound on $\Omega(\psi, \phi)$, and consequently a lower bound on $I_{\text{ont}}(\psi, \phi)$, which is closer to $I_Q(\psi, \phi)$. Thus we have shown that indeterminism in a reciprocal ontological model must be of the order of quantum theory.

V. CONCLUSIONS

Quantum theory is known to be fundamentally random. This randomness is due to the indeterminism associated with measurement outcomes when an observable is being measured on a quantum system prepared in some known state. Quantum randomness is significant as it is believed to represent true randomness, and more so because true randomness does not exist in classical physics. However, if it is possible to construct a deterministic theory that at an operational level reproduces quantum predictions then quantum indeterminism, and hence quantum randomness, is not fundamental any more. Thus the question of existence of such a theory is of considerable importance.

The ontological models developed by Harrigan and Spekkens [12] provide the framework to address the above question in a meaningful way. Our work was motivated by a recent no-go result by Maroney [23], and its subsequent analysis by Ballentine [30]. Maroney proved, without additional assumptions, that it’s impossible to construct a maximally $\psi$-epistemic theory in dimensions greater than two; i.e., there cannot be an ontological model, where the quantum overlap between two state vectors can be completely accounted for by the overlap of their respective probability distributions in the space of ontic states.

Ballentine showed that Maroney’s result rules out ontological models that are both reciprocal and deterministic. In other words, a reciprocal ontological model is necessarily indeterministic. In this paper, we proved that indeterminism in a reciprocal ontological model must be of the order of quantum theory. Therefore, if we want to hold on to objective reality then we should adopt an interpretation of wave function close to $\psi$-ontic.

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* som@jcbose.ac.in
† manik1ljju@gmail.com
‡ somesankar@gmail.com
§ sibasis@imsc.res.in
¶ gkar@isical.res.in
** amitisiphys@gmail.com
†† arup145.roy@gmail.com

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