Optimal regularity & Liouville property for stable solutions to semilinear elliptic equations in $\mathbb{R}^n$ with $n \geq 10$

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Abstract. Let $0 \leq f \in C^{0,1}(\mathbb{R})$. Given a domain $\Omega \subset \mathbb{R}^n$, we prove that any stable solution to the equation $-\Delta u = f(u)$ in $\Omega$ satisfies

- a BMO interior regularity when $n = 10$,
- an Morrey $M^{p_n,4+2/(p_n-2)}$ interior regularity when $n \geq 11$, where
  
  $$p_n = \frac{2(n - 2\sqrt{n-1} - 2)}{n - 2\sqrt{n-1} - 4}.$$  

This result is optimal as hinted by e.g. [3, 7, 13], and answers an open question raised by Cabré, Figalli, Ros-Oton and Serra [8]. As an application, we show a sharp Liouville property: Any stable solution $u \in C^2(\mathbb{R}^n)$ to $-\Delta u = f(u)$ in $\mathbb{R}^n$ satisfying the growth condition

$$|u(x)| = \begin{cases} o(\log|x|) & \text{as } |x| \to +\infty, \text{ when } n = 10; \\ o\left(|x|^{-\frac{n-2}{2}+\sqrt{n-1}+2}\right) & \text{as } |x| \to +\infty, \text{ when } n \geq 11 \end{cases}$$

must be a constant. This extends the well-known Liouville property for radial stable solutions obtained by Villegas [24].

1 Introduction

Let $\Omega$ be a bounded domain of $\mathbb{R}^n$ with $n \geq 2$. Given any local Lipschitz function $f: \mathbb{R} \to \mathbb{R}$ (for short $f \in C^{0,1}(\mathbb{R})$), we consider the semilinear elliptic equation

$$-\Delta u = f(u) \quad \text{in} \quad \Omega, \quad (1.1)$$

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which is the Euler-Lagrange equation for the energy functional

\[ \mathcal{E}(u) := \int_{\Omega} \left( \frac{1}{2} |Du|^2 - F(u) \right) \, dx, \tag{1.2} \]

where \( F(t) = \int_0^t f(s) \, ds \) for \( t \in \mathbb{R} \). A function \( u \in W^{1,2}(\Omega) \) is called as a weak solution to the equation (1.1) if \( f(u) \in L^1_{\text{loc}}(\Omega) \) and

\[ \int_{\Omega} Du \cdot D\xi \, dx = \int_{\Omega} f(u) \xi \, dx \quad \forall \xi \in C_0^\infty(\Omega), \]

that is, \( u \) is a critical point of the energy functional \( \mathcal{E} \). We say that a weak solution \( u \) is stable in \( \Omega \) if \( f'(u) \in L^1_{\text{loc}}(\Omega) \) and

\[ \int_{\Omega} f'(u)\xi^2 \, dx \leq \int_{\Omega} |D\xi|^2 \, dx \quad \forall \xi \in C_0^\infty(\Omega), \tag{1.3} \]

that is, the second variation of the energy functional \( \mathcal{E} \) is nonnegative. Here and below,

\[ f'(t) = \liminf_{h \to 0} \frac{f(t+h) - f(t)}{h} \quad \forall t \in \mathbb{R}, \]

and note that \( f'(t) = f'(t) \) whenever \( f \in C^1(\mathbb{R}) \).

The study of stable solutions to semilinear elliptic equations can be traced back to the seminal paper [4] by Crandall and Rabinowitz in 1975. The regularity of stable solutions provides an important way to understand the regularity of extremal solution \( u^* \) to the Gelfand-type problem

\[
\begin{cases}
- \Delta u = \lambda^* f(u) & \text{in } \Omega \\
 u > 0 & \text{in } \Omega \\
 u = 0 & \text{on } \partial \Omega
\end{cases} \tag{1.4}
\]

for some positive constant \( \lambda^* > 0 \). We refer to [2, 10, 19] and the reference therein for a comprehensive analysis of (1.4) and related topics. Note that the extremal solution \( u^* \) can be approximated by stable solutions \( \{u_{\lambda}\}_{\lambda < \lambda^*} \), see e.g. [13].

In dimension \( n \leq 9 \), Brezis [2] asked an open problem: whether the extremal solution \( u^* \) to equation (1.4) is bounded for some \( f \) and \( \Omega \). Since \( u^* \) is approximated by stable solutions \( \{u_{\lambda}\}_{\lambda < \lambda^*} \), it suffices to establish some apriori bound for stable solutions. In recent years, there were several strong efforts to study regularity for stable solutions and hence for Brezis’ open problem. In particular, a positive answer was given by Nedev [23] when \( n \leq 3 \) and by Cabrè [5] when \( n = 4 \) (see also [6] for an alternative proof).

Very recently, Cabrè, Figalli, Ros-Oton and Serra [8] provide a complete answer to Brezis’ open problem when \( f \geq 0 \) based on certain Morrey-type estimate for \( n \geq 3 \). Throughout this paper, for \( p \in [1, \infty) \) and \( \beta \in (0, n) \), we define the Morrey norm

\[ \|w\|_{M^{p,\beta}(\Omega)} := \sup_{y \in \Omega, r > 0} \left( r^{\beta-n} \int_{\Omega \cap B_r(y)} |w|^p \, dx \right)^{1/p} < \infty, \tag{1.5} \]
where $B_r(y)$ denotes the ball with center $y$ and radius $r > 0$. We simply write $B_r$ when the center of the ball is at the origin. In addition, following the convention, we denote by $C(a, b, \cdots)$ a positive constant depending only on the parameters $a, b, \cdots$.

In dimension $n \geq 10$, in particular, Cabré, Figalli, Ros-Oton and Serra [8, Theorem 1.9] established the following regularity of stable solutions to the equation (1.1).

**Theorem 1.1** ([8]). Suppose that $f \in C^{0,1}(\mathbb{R})$ is nonnegative. If $u \in C^2(B_1)$ is a stable solution to (1.1) in $B_1$, then

$$
\|u\|_{M^{p,2+\frac{4}{p-2}}(B_{1/2})} \leq C(n, p)\|u\|_{L^1(B_1)} \quad \text{for every } p < p_n.
$$

where

$$
p_n := \begin{cases}
\infty & \text{if } n = 10, \\
\frac{2(n - 2\sqrt{n-1} - 2)}{n - 2\sqrt{n-1} - 4} & \text{if } n \geq 11.
\end{cases}
$$

Moreover, suppose additionally that $f$ is nondecreasing, and $\Omega$ be a bound domain of class $C^3$. If $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ is a stable solution to (1.1) in $\Omega$ with boundary $u = 0$ on $\partial \Omega$, then

$$
\|u\|_{M^{p,2+\frac{4}{p-2}}(\Omega)} \leq C(n, p, \Omega)\|u\|_{L^1(\Omega)} \quad \text{for every } p < p_n.
$$

We remark that the exponent $n - 2\sqrt{n-1} - 4$ changes sign when $n = 10$, which was already appeared in e.g. [21].

However for the endpoint case $p = p_n$, Cabré, Figalli, Ros-Oton and Serra [8, Section 1.3] pointed out that it is an open question whether (1.6) holds.

As hinted by earlier results [7] in the radial symmetric case, when $n = 10$, instead of $L^\infty = M^{\infty,2}$ a more suitable space to consider is class of functions with bounded mean oscillations (BMO space) as remarked therein. Indeed, $u(x) = -2\log|x|$ is a stable solution to the equation (1.1) in $B_1$ with $f(u) = 2(n-2)e^u$. Obviously, $u \in \text{BMO}(B_1)$ but $u \notin L^\infty(B_1)$. Here and below, the BMO norm is defined as

$$
\|u\|_{\text{BMO}(\Omega)} := \sup_{y \in \Omega, r > 0} \inf_{c \in \mathbb{R}} \frac{1}{\Omega \cap B_r(y)} \int_{\Omega \cap B_r(y)} |u(x) - c| \, dx,
$$

where, $\frac{1}{\Omega} \int_E v \, dx$ denotes the integral average of $v$ on a measurable set $E$.

On the other hand, when $n \geq 11$, also hinted by the results in [7], in the range $p \leq p_n$ is the best possible to get (1.6). Besides, it was proven in [3] that the function $u(x) = |x|^\frac{n-2}{2\sqrt{n-1} - 4}$ is the extremal solution to

$$
-\Delta u = \lambda^*(1 + u)^{q_n} \quad \text{in } B_1; \quad u = 0 \quad \text{on } \partial B_1
$$

with $\lambda^* = \frac{2}{q_n}$ and $q_n := \frac{n-2\sqrt{n-1}}{n-2\sqrt{n-1} - 4}$. It is easy to see that $u \in M^{p,2+\frac{4}{p-2}}(B_{1/2})$ if and only if $p \leq p_n$. Recall that by [13, Section 3.2.2], such extremal solution can be approximated by stable solutions.
The first main purpose of this paper is to establish the following regularity at the end-point $p_n$ for stable solutions to the equation (1.1) when $n \geq 10$, and then answers the above open question by Cabré, Figalli, Ros-Oton and Serra [8].

**Theorem 1.2.** Suppose $f \in C^{0,1}(\mathbb{R})$ is nonnegative. For any stable solution $u \in C^2(B_1)$ to (1.1) in $B_1$, when $n = 10$ we have

$$
\|u\|_{\text{BMO}(B_{1/2})} \leq C(n)\|u\|_{L^1(B_1)},
$$

and when $n \geq 11$ we have

$$
\|u\|_{M^{p_n, 2+\frac{4}{p_n-2}}(B_{1/2})} \leq C(n)\|u\|_{L^1(B_1)}.
$$

Moreover, suppose additionally that $f$ is nondecreasing, and $\Omega$ is a bounded smooth convex domain. For any positive stable solution $u \in C^2(\overline{\Omega})$ to (1.1) with boundary $u = 0$ on $\partial \Omega$, when $n = 10$ we have

$$
\|u\|_{\text{BMO}(\Omega)} \leq C(n, \Omega)\|u\|_{L^1(\Omega)},
$$

and when $n \geq 11$ we have

$$
\|u\|_{M^{p_n, 2+\frac{4}{p_n-2}}(\Omega)} \leq C(n, \Omega)\|u\|_{L^1(\Omega)}.
$$

As a direct consequence of the above a priori estimates, we have the following result for stable solution in $W^{1,2}$.

**Corollary 1.3.** Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded smooth convex domain and that $f \in C^{0,1}(\mathbb{R})$ is nonnegative, nondecreasing, convex, and satisfies $f(t)/t \to +\infty$ as $t \to +\infty$. For any stable solution $u \in W^{1,2}_0(\Omega)$ to (1.1) with boundary $u = 0$ on $\partial \Omega$, we have (1.12) when $n = 10$, and (1.13) when $n \geq 11$.

**Remark 1.4.** (i) While writing this paper, via personal communication we learn that Figalli and Mayboroda have independently proved (1.10) in Theorem 1.2 with $n = 10$ via a similar argument.

(ii) In Theorem 1.2 and Corollary 1.3 we only consider bounded smooth convex domains so as to avoid technical discussions on the boundary estimate. We believe that after suitable modifications, it is possible to relax this assumption to bounded domains of $C^3$ class, as in [8].

As an application of Theorem 1.2, we prove the following Liouville property for stable solutions to the equation

$$
-\Delta u = f(u) \text{ in } \mathbb{R}^n
$$

for $f \in C^{0,1}(\mathbb{R}^n)$. 4
Theorem 1.5. Let \( n \geq 10 \) and \( 0 \leq f \in C^{0,1}_\text{loc}(\mathbb{R}) \). Suppose that \( u \in C^2(\mathbb{R}^n) \) is a nonconstant stable solution to (1.14) in \( \mathbb{R}^n \).

If \( u \) is nonconstant, then

\[
\int_{B_{4R} \setminus B_R} |u(x)| \, dx \geq \begin{cases} 
c \log R & \forall R \geq R_0, \text{ if } n = 10, 
n cR^{-\frac{n}{2} + 2 + \sqrt{n-1}} & \forall R \geq R_0, \text{ if } n \geq 11 
\end{cases}
\]  

(1.15)

for some \( R_0 \geq 2 \) and \( c > 0 \).

In particular, if \( u \) satisfies the growth condition

\[
|u(x)| = \begin{cases} 
o(\log |x|) & \text{as } |x| \to +\infty, \text{ when } n = 10; 
o(|x|^{-\frac{n}{2} + 2 + \sqrt{n-1}}) & \text{as } |x| \to +\infty, \text{ when } n \neq 10,
\end{cases}
\]  

(1.16)

then \( u \) must be a constant.

This problem has attracted a lot of attention in the literature. First of all, for radial stable solutions, Villegas [24] in 2007 obtained the following sharp Liouville property based on the monotone property by Cabré-Capella [11] (see also [24, 13]).

Theorem 1.6 ([24]). Let \( n \geq 2 \) and \( f \in C^1(\mathbb{R}) \). Suppose that \( u \in C^2(\mathbb{R}^n) \) is a radial stable solution to (1.14).

If \( u \) is not constant, then

\[
|u(x)| \geq \begin{cases} 
M \log |x| & \text{whenever } |x| \geq r_0, \text{ when } n = 10, 
M|x|^{-\frac{n}{2} + \sqrt{n-1} + 2} & \text{whenever } |x| \geq r_0, \text{ when } n \neq 10
\end{cases}
\]  

(1.17)

for some \( M > 0 \) and \( r_0 \geq 10 \).

In particular, if \( u \) satisfies the growth condition (1.16), then \( u \) must be a constant.

Note that for radial stable solutions \( u(x) \), the condition (1.15) is equivalent to (1.17). Indeed, by [24], \( u(r) = u(re) \) is always monotone, and hence

\[
\min\{|u(4r)|, |u(r)|\} \leq \int_{B_{4r} \setminus B_r} |u(x)| \, dx \leq \max\{|u(4r)|, |u(r)|\} \quad \forall r > 0,
\]

which implies the equivalence between (1.15) and (1.17).

Let \( \beta_n = -\frac{n}{2} + 2 + \sqrt{n-1} \). Then \( \beta_n < 0 \) when \( n \geq 11 \) and \( \beta_n > 0 \) when \( n \leq 9 \). The sharpness of Theorem 1.6 (and also Theorem 1.5) is demonstrated in the following sense by Villegas [24] (with a slight modification at \( n = 10 \)).

(i) When \( n \neq 10 \), the radial smooth function \((1 + |x|^2)^{\beta_n} \) is a stable solution to the equation \(-\Delta u = f_{\beta_n}(u) \) in \( \mathbb{R}^n \), where when \( n \geq 11 \),

\[
f_{\beta_n}(s) := \begin{cases} 
0, & \text{if } s \leq 0, 
\beta_n(\beta_n - 2)s^{1-\frac{4}{n}} - \beta_n(\beta_n + n - 2)s^{1-\frac{2}{n}}, & \text{if } s > 0
\end{cases}
\]
and when \( n \leq 9 \),

\[
 f_{\beta_n}(s) := \begin{cases} 
 \beta_n(\beta_n - 2)s^{1 - \frac{4}{m}} - \beta_n(\beta_n + n - 2)s^{1 - \frac{2}{m}}, & \text{if } s \geq 1, \\
 - (\beta_n - 2)(n + 2)(s - 1) - n\beta_n, & \text{if } s < 1.
\end{cases}
\]

See [24, Example 3.1] for details. Note that, when \( n \geq 11 \), by \( \beta_n < 0 \) and \( \beta_n + n - 2 > 0 \), we have \( f_{\beta_n} \geq 0 \) in \( \mathbb{R} \); while when \( n \leq 9 \), \( f_{\beta_n} \) changes sign in \( \mathbb{R} \).

(ii) When \( n = 10 \), the radial smooth function \(- \frac{1}{2} \log(1 + |x|^2)\) is a stable solution to the equation \(-\Delta u = f(u)\) in \( \mathbb{R}^n \), where \( f(s) = (n - 2)e^{2s} + 2e^{4s} \geq 0 \) in \( \mathbb{R} \). This is a slight modification of the [24, Example 3.1] with \( n = 10 \). See the appendix for details.

For general (nonradial) stable solutions \( u \in C^2(\mathbb{R}^n) \) to \(-\Delta u = f(u)\) in \( \mathbb{R}^n \), it is then natural to ask if certain Liouville properties similar to Theorem 1.6 hold. Namely, when \( f \) satisfies certain regularity assumption,

- if \( u \) satisfies (1.16), then is it necessary that \( u \) is a constant?
- if \( u \) is nonconstant, is it possible to give some sharp lower bound for \( |u| \) toward \( \infty \)?

Suppose that \( 0 \leq f \in C^1(\mathbb{R}) \) and \( u \in C^2(\mathbb{R}^n) \) is a stable solution to (1.14). When \( n \leq 4 \), Dupaigne-Farina [14] proved that if \( |u| \) is bounded, then \( u \) must be a constant. Recently, with the aid of Cabré et al [8], Dupaigne-Farina [15] showed that if \( n \leq 9 \) and \( u(x) \geq -C[1 + \log |x|]^{\gamma} \) for some \( \gamma \geq 1 \) and \( C > 0 \), or if \( n = 10 \) and \( u \geq -C \) for some constant \( C > 0 \), then \( u \) must be a constant. When \( n \geq 10 \), our result Theorem 1.5 finally answers the two questions above.

1.1 Ideas of the proofs

We sketch the ideas to prove Theorem 1.2 and Theorem 1.5. All of them heavily rely on the following decay estimate on the Dirichlet energy.

**Lemma 1.7.** Let \( n \geq 10 \) and \( f \in C^{0,1}(\mathbb{R}) \). For any \( y \in \mathbb{R}^n \) and \( t > 0 \), if \( u \in C^2(B_{2t}(y)) \) is a stable solution to the equation (1.1) in \( B_{2t}(y) \), one has

\[
 \left( \frac{r}{t} \right)^{-2(1 + \sqrt{n-1})} \int_{B_r(y)} |Du|^2 \, dx \leq C(n) \int_{B_t(y) \setminus B_{t/2}(y)} |Du|^2 \, dx, \quad \forall r \leq \frac{t}{2}.
\]

See Section 2 for the proof of Lemma 1.7; the key point is that we take a suitable test function in a celebrated lemma of Cabré et al [8] (see Lemma 2.1 below).

We also recall the following lemma, which was essentially established in [8], see Lemma A.2 and Proposition 2.5 with its proof therein. For the convenience of the reader, we give a sketch of the proof at the beginning of Section 3.
Lemma 1.8. Let $0 \leq f \in C^{0,1}({\mathbb{R}})$. For any stable solution $u \in C^2(B_{2^t}(y))$ to the equation (1.1) in $B_{2^t}(y)$, one has
\[
\left( \int_{B_{e/2}(y)} |Du|^2 \, dx \right)^{1/2} \leq C(n) t^{-n/2} \int_{B_{e}(y)} |Du| \, dx \tag{1.19}
\]
and
\[
\int_{B_{e/2}(y)} |Du| \, dx \leq C(n) t^{-1} \int_{B_{e}(y)} |u| \, dx. \tag{1.20}
\]

Applying Lemma 1.7, Lemma 1.8 and some known boundary estimate, we are able to prove Theorem 1.2 and Corollary 1.3. This is clarified in Section 3.

In order to prove Theorem 1.5, an auxiliary and crucial proposition is shown in Section 4, which is especially applied in the case $n = 10$.

Proposition 1.9. Let $n \geq 3$. Suppose that $u \in W^{1,1}_{\text{loc}}({\mathbb{R}}^n)$ is superharmonic, that is, $-\Delta u \geq 0$ in $\mathbb{R}^n$ in distributional sense. For any $0 < r < R < \infty$ we have
\[
\int_{B_R \setminus B_r} |Du| |x|^{-n+1} \, dx \leq C(n) \int_{B_{r/2} \setminus B_{r/4}} |u| \, dz + C(n) \int_{B_{3R} \setminus B_{2R}} |u| \, dz. \tag{1.21}
\]

The main idea of showing Proposition 1.9 goes as follows. First, it is known that
\[
Du_\delta(x) = D\Delta^{-1}[\Delta(u_\delta \eta)](x) \quad \text{for } x \in B_R \setminus B_r,
\]
where $u_\delta$ is a standard smooth mollification of $u$ and $\eta$ is a suitable cut-off function. Next, thanks to the key fact $-\Delta u_\delta \geq 0$, via some subtle kernel estimate and integration by parts, we are able to prove (1.21) for $u_\delta$, and then a standard approximation gives (1.21) as desired.

Theorem 1.5 is eventually proved in Section 5. The case $n \geq 11$ is relatively simpler. In fact, by Lemma 1.7 and Lemma 1.8, one can build up the following
\[
r^{-(1+\sqrt{n-1})} \left( \int_{B_r} |Du|^2 \, dx \right)^{1/2} \leq C(n) R^{\frac{n}{2} - 2 - \sqrt{n-1}} \int_{B_{3R} \setminus B_{3R/4}} |u| \, dx \quad \forall 0 < r << R < \infty
\]
for stable solutions, which allows us to conclude Theorem 1.5 for $n \geq 11$.

As for the case when $n = 10$, we first employ Lemma 1.7 and repeat Lemma 1.8 to get
\[
r^{-(1+\sqrt{n-1})} \left( \int_{B_r} |Du|^2 \, dx \right)^{1/2} \leq C(n) \frac{1}{\log R} \int_{B_{R^2} \setminus B_4} |Du| |x|^{-n+1} \, dx \quad \forall 0 < r << R < \infty,
\]
which, with the aid of Proposition 1.9, is then bounded from above by
\[
C(n) \frac{1}{\log R} \left( \int_{B_2 \setminus B_1} |u(z)| \, dz + \int_{B_{R^2} \setminus B_{R^2}} |u(z)| \, dz \right).
\]
From this we conclude Theorem 1.5 when $n = 10$. 

7
2 Proof of Lemma 1.7

Towards Lemma 1.7 we recall the following apriori bound by Cabré et al [8, Lemma 2.1], which is obtained by taking test function \((x \cdot Du)\eta\) in the stability condition (1.3).

**Lemma 2.1.** Let \(u \in C^2(B_1)\) be a stable solution to equation (1.1) in \(B_1\), with \(f \in C^{0,1}(\mathbb{R})\). Then for all cut-off function \(\eta \in C^{0,1}_c(B_1)\),

\[
\int_{B_1} |x \cdot Du|^2 |D\eta|^2 \; dx \\
\geq (n-2) \int_{B_1} |Du|^2 \eta^2 \; dx + 2 \int_{B_1} |Du|^2 (x \cdot D\eta) \eta \; dx - 4 \int_{B_1} (x \cdot Du)(Du \cdot D\eta) \eta \; dx. \tag{2.1}
\]

For convenience, for any \(0 < r < t < \infty\) and \(y \in \mathbb{R}^n\), write the annual \(A_{r,t}(y) := B_t(y) \setminus B_r(y)\); for simple write \(A_{r,t} = A_{r,t}(0)\).

**Proof of Lemma 1.7.** It suffices to prove the following

\[
\left(\frac{r}{t}\right)^{-2(1+\sqrt{n}-1)} \int_{B_r(y)} |Du|^2 \; dx \leq C(n) \int_{A_{r,t}(y)} |Du|^2 \; dx, \quad \forall r \leq \frac{t}{2}, \tag{2.2}
\]

Indeed, applying (2.2) to \(\frac{t}{2}\) and \(t\), one has

\[
\left(\frac{1}{2}\right)^{-2(1+\sqrt{n}-1)} \int_{B_{\frac{t}{2}}(y)} |Du|^2 \; dx \leq C(n) \int_{A_{\frac{t}{2},t}(y)} |Du|^2 \; dx. \tag{2.3}
\]

If \(\frac{t}{4} \leq r < \frac{t}{2}\), by \(B_r(y) \subset B_{\frac{t}{2}}(y)\) and \(\frac{1}{4} < \frac{r}{t} \leq \frac{1}{2}\), (2.3) gives

\[
\left(\frac{r}{t}\right)^{-2(1+\sqrt{n}-1)} \int_{B_r(y)} |Du|^2 \; dx \leq C(n) \int_{A_{\frac{r}{2},t}(y)} |Du|^2 \; dx. \tag{2.4}
\]

If \(0 < r < \frac{t}{4}\), applying (2.2) to \(r\) and \(\frac{t}{2}\), and noting \(A_{r,\frac{t}{2}} \subset B_{\frac{t}{2}}\) one gets

\[
\left(\frac{r}{t^{1/2}}\right)^{-2(1+\sqrt{n}-1)} \int_{B_r(y)} |Du|^2 \; dx \leq C(n) \int_{A_{r,\frac{t}{2}}(y)} |Du|^2 \; dx \leq C(n) \int_{B_{\frac{t}{2}}(y)} |Du|^2 \; dx,
\]

which together with (2.3) yields

\[
\left(\frac{r}{t}\right)^{-2(1+\sqrt{n}-1)} \int_{B_r(y)} |Du|^2 \; dx \leq C(n) \int_{A_{\frac{r}{2},t}(y)} |Du|^2 \; dx.
\]

From this and (2.4) we conclude (1.18).
To prove (2.2), without loss of generality we may assume that \( t = 1 \) and \( y = 0 \). Indeed, if \( u(x) \) is a stable solution to \( -\Delta u = f(u) \) in \( B_2(y) \), then \( v(x) = u(tx + y) \) is the stable solution to \( -\Delta v = t^2 f(v) \) in \( B_2 \). Note that up to a change of variable \( u \) satisfies (2.2) if and only if \( v \) satisfies (2.2) with \( t = 1 \) and \( y = 0 \).

Write \( a = 2(1 + \sqrt{n - 1}) \). Let \( r \in (0, \frac{1}{2}] \) be fixed and set

\[
\eta = \begin{cases} r^{-\frac{2}{n-2}} & \text{if } 0 \leq |x| \leq r \\ |x|^{-\frac{2}{n-2}} \phi & \text{if } r < |x| \leq 1, \end{cases}
\]

where \( \phi \in C_c^\infty(B_1) \) satisfies

\[
\phi = 1 \quad \text{in } B_{3/4} \quad \text{and} \quad |D\phi| \leq 5 \chi_{B_1 \setminus B_{3/4}}.
\]

Clearly, \( \eta \in C_c^{0, 1}(B_1) \). Since \( \eta = r^{-\frac{2}{n-2}} \) in \( B_r \) and hence \( D\eta = 0 \) in \( B_r \), submitting \( \eta \) in inequality (2.1) one has

\[
\int_{A_{r,1}} |x \cdot Du|^2 |D\eta|^2 \, dx \\
\geq (n - 2)r^{-a} \int_{B_r} |Du|^2 \, dx + (n - 2) \int_{A_{r,1}} |Du|^2 \eta^2 \, dx \\
+ 2 \int_{A_{r,1}} |Du|^2 (x \cdot D\eta)\eta \, dx - 4 \int_{A_{r,1}} (x \cdot Du)(Du \cdot D\eta)\eta \, dx.
\]

Noting that

\[
D\eta = -\frac{a}{2} |x|^{-\frac{2}{n-2}} x\phi + |x|^{-\frac{2}{n-2}} D\phi \quad \text{in } A_{r,1},
\]

one has

\[
2 \int_{A_{r,1}} |Du|^2 (x \cdot D\eta)\eta \, dx - 4 \int_{A_{r,1}} (x \cdot Du)(Du \cdot D\eta)\eta \, dx \\
= -a \int_{A_{r,1}} |Du|^2 |x|^{-a} \phi^2 \, dx + 2 \int_{A_{r,1}} |Du|^2 (x \cdot D\phi)\phi |x|^{-a} \, dx \\
+ 2a \int_{A_{r,1}} (x \cdot Du)^2 |x|^{-a-2} \phi^2 \, dx - 4 \int_{A_{r,1}} (x \cdot Du)(Du \cdot D\phi)\phi |x|^{-a} \, dx.
\]

Moreover, by

\[
|D\eta|^2 = \frac{a^2}{4} |x|^{-2} \phi^2 - 2a |x|^{-2} (x \cdot D\phi)\phi + |x|^{-a} |D\phi|^2,
\]

one can write

\[
\int_{A_{r,1}} (Du \cdot x)^2 |D\eta|^2 \, dx = \frac{a^2}{4} \int_{A_{r,1}} (Du \cdot x)^2 |x|^{-2} \phi^2 \, dx + \int_{A_{r,1}} (Du \cdot x)^2 |x|^{-a} |D\phi|^2 \, dx
\]
Using (2.8) for the left hand side of (2.7) and (2.9) for the last two terms in the right hand side of (2.7), and then moving all terms including $D\phi$ to the left hand side and all other terms to the right hand side, we have

\[
\begin{align*}
&- a \int_{A_{r,1}} (Du \cdot x)^2 |x|^{-a-2} (x \cdot D\phi) \phi \, dx.
\end{align*}
\tag{2.9}
\]

Note that by $|D\phi| = 0$ in $B_{3/4}$ and $|D\phi| \leq 5$ in $B_1$ as in (2.6) and $a > 2$,

\[
\begin{align*}
0 &\geq (n-2)r^{-a} \int_{B_r} |Du|^2 \, dx + (n-2) \int_{A_{r,1}} |Du|^2 |x|^{-a} \, dx
+ a \int_{A_{r,1}} |Du|^2 |x|^{-a} \phi^2 \, dx
- \frac{a^2}{4} \int_{A_{r,1}} (Du \cdot x)^2 |x|^{-a} \phi^2 \, dx
+ \int_{A_{r,1}} \left\{ (n-2-a)|Du|^2 + \left( 2a - \frac{a^2}{4} \right) (Du \cdot x)^2 |x|^{-2} \right\} |x|^{-a} \phi^2 \, dx.
\end{align*}
\tag{2.10}
\]

Additionally, note that $n \geq 10$ implies $a = 2(1 + \sqrt{n-1}) \geq 8$, and hence

\[
2a - \frac{a^2}{4} = \frac{a}{4} (8-a) \leq 0.
\]

By $|x|^{-1} |x \cdot Du| \leq |Du|$ in $A_{r,1}$ we have

\[
(n-2-a)|Du|^2 + \left( 2a - \frac{a^2}{4} \right) (Du \cdot x)^2 |x|^{-2} \geq \left( n-2 + a - \frac{a^2}{4} \right) |Du|^2.
\]
Since
\[ n - 2 + a - \frac{a^2}{4} = -(\frac{a}{2} - [1 - \sqrt{n-1}])(\frac{a}{2} - [1 + \sqrt{n-1}]) = 0, \]
we have
\[ (n - 2 - a)|Du|^2 + \left(2a - \frac{a^2}{4}\right) (Du \cdot x)^2 |x|^{-2} \geq 0 \quad \text{in } A_{r,1}, \tag{2.12} \]
which means that the last term in the right hand side of (2.10) is nonnegative. From this, together with (2.10) and (2.11) we conclude (2.2). The proof is complete. \qed

**Remark 2.2.** Recall that in [8], Cabré et al used the test function \( \eta = |x|^{-\frac{n}{2}} \xi \) with \( \xi \in C_c^\infty(B_1) \), which was not enough to get (2.2).

### 3 Proofs of Theorem 1.2 and Corollary 1.3

In this section we prove Theorem 1.2 and Corollary 1.3. First, we show a sketch of the proof of Lemma 1.8.

**Proof of Lemma 1.8.** Up to considering \( v(x) = u(tx + y) \) as did in the proof of Lemma 1.4, we may assume that \( t = 1 \) and \( y = 0 \). The inequality (1.20) is given by [8, Lemma A.2]. The inequality (1.19) reads as \( \|Du\|_{L^2(B_{1/2})} \leq C(n)\|Du\|_{L^1(B_1)} \), and will follow from the proof of [8, Proposition 2.5]. Indeed, in [8, Proposition 2.5], Cabre et al proved that
\[ \|Du\|_{L^2(B_{1/2})} \leq C(n)\|u\|_{L^1(B_1)}. \tag{3.1} \]
In their proof, first they obtained a bound of \( \|Du\|_{L^2(B_{1/2})} \) via \( \|Du\|_{L^1(B_{1/2})} \) and also some other small terms. Next, they used \( \|Du\|_{L^1(B_{1/2})} \leq C(n)\|u\|_{L^1(B_1)} \). Finally, via an iteration argument, they got (3.1). If we directly apply the iteration argument without using \( \|Du\|_{L^1(B_{1/2})} \leq C(n)\|u\|_{L^1(B_1)} \), one gets \( \|Du\|_{L^2(B_{1/2})} \leq C(n)\|Du\|_{L^1(B_1)} \). \qed

Recall that \( u_E = \frac{1}{|E|} \int_E u \, dx \) denotes the integral average of \( u \) on a measurable set \( E \). The interior regularity (1.10)& (1.11) in Theorem 1.2 is a consequence of Lemma 1.7 and (1.19), together with some standard embedding argument.

**Proofs of (1.10) and (1.11) in Theorem 1.2.** Let \( u \in C^2(B_2) \) be stable solution to equation (1.1). Write \( \beta = n - 2 - 2\sqrt{n-1} \). For any \( y \in B_{1/2} \) if \( r > 1/8 \), by Lemma 1.8 we have
\[
r^{\beta - n} \int_{B_r(y) \cap B_{1/2}} |Du|^2 \, dx \leq C(n) \int_{B_{1/2}} |Du|^2 \, dx \leq C(n)\|u\|^2_{L^1(B_1)}
\]
and if \( 0 < r < 1/8 \), by Lemma 1.7 and Lemma 1.8 again we have
\[
r^{\beta - n} \int_{B_r(y) \cap B_{1/2}} |Du|^2 \, dx \leq r^\beta \int_{B_r(y)} |Du|^2 \, dx \leq C(n) \int_{B_{1/4}(y)} |Du|^2 \, dx \leq C(n)\|u\|^2_{L^1(B_1)}.
\]
This means that $Du \in M^{2,\beta}(B_{1/2})$ with $\|Du\|_{M^{2,\beta}(B_{1/2})} \leq C(n)\|u\|_{L^1(B_1)}$.

If $n = 10$, then $\beta = 2$ and $2\beta/(\beta - 2) = \infty$. Thanks to Sobolev-Poincaré inequality one can easily check that $Du \in M^{2,\beta}(B_{1/2})$ implies $u \in BMO(B_{1/2})$ with a norm bound $\|u\|_{BMO(B_{1/2})} \leq C(n)\|Du\|_{M^{2,\beta}(B_{1/2})}$. If $n \geq 11$, then $p_n = 2\beta/(\beta - 2) < \infty$ and $\beta = 4 + 2/(p_n - 2)$. By the embedding result in [1] and also [9, Section 4], $Du \in M^{2,\beta}(B_{1/2})$ implies $u \in M^{2\beta/(\beta - 2),\beta}(B_{1/2})$ with its norm bound $\|u\|_{M^{p_n,\beta}(B_{1/2})} \leq C(n)\|Du\|_{M^{2,\beta}(B_{1/2})}$. This proves (1.10) and (1.11). \hfill \square

To prove the global regularity (1.12) and (1.13) in Theorem 1.2, we need the following a priori $L^\infty$-bound in a neighborhood of $\partial \Omega$ for $C^2$ solution when $\Omega$ is a bounded smooth convex domain, see [5, Proposition 3.2] and [12, 18, 20]. For $\rho > 0$ we write

$$\Omega_\rho := \{x \in \Omega : \text{dist}(x, \partial \Omega) < \rho\}.$$

**Lemma 3.1.** Suppose that $f \in C^{0,1}([0,\infty))$, is nonnegative and $\Omega$ is a smooth convex domain in $\mathbb{R}^n$. There exist positive constants $\rho$ and $\gamma$ depending only on the domain $\Omega$ such that for any positive solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ to (1.1) one has

$$\|u\|_{L^\infty(\Omega_\rho)} \leq \frac{1}{\gamma}\|u\|_{L^1(\Omega)}.$$  \hspace{1cm} (3.2)

Note that, as $f \geq 0$, the maximal principle shows that any solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ to (1.1) with 0 boundary is always nonnegative; and the strong maximal principle further shows that $u$ is always positive in the domain $\Omega$.

**Proofs of (1.12) and (1.13) in Theorem 1.2.** Let $\beta = n - 2 - 2\sqrt{n - 1}$ and let $\rho, \gamma$ be as in Lemma 3.1. We first consider the case $n \geq 11$. For any $y \in \overline{\Omega}$ and $r > 0$, write

$$r^{\beta-n}\int_{\Omega \cap B_r(y)} |u|^{p_n} \, dx = r^{\beta-n}\int_{\Omega_{\rho} \cap B_r(y)} |u|^{p_n} \, dx + r^{\beta-n}\int_{(\Omega \setminus \Omega_{\rho}) \cap B_r(y)} |u|^{p_n} \, dx$$

$$:= \Phi_1(y, r) + \Phi_2(y, r).$$

To see (1.12), obviously, we only need to prove $\Phi_1(y, r) \leq C(n, \Omega)\|u\|_{L^1(\Omega)}^{p_n}$ and $\Phi_2(y, r) \leq C(n, \rho, \Omega)\|u\|_{L^1(\Omega)}^{p_n}$ for any $y \in \Omega$ and $r > 0$.

Note that

$$r^{\beta-n}|\Omega_{\rho} \cap B_r(y)| \leq C(n) r^{\beta-n} \leq C(n) \quad \text{when } r < 1 \text{ and } \leq |\Omega_{\rho}| \text{ when } r > 1,$$

by $2 < \beta < n$ and Lemma 3.1, we have

$$\Phi_1(y, r) \leq r^{\beta-n}|\Omega_{\rho} \cap B_r(y)|\|u\|_{L^\infty(\Omega_{\rho})}^{p_n} \leq C(n, \Omega)\|u\|_{L^1(\Omega)}^{p_n}.$$

Next, to get $\Phi_2(y, r) \leq C(n, \rho, \Omega)\|u\|_{L^1(\Omega)}^{p_n}$ for any $y \in \Omega$ and $r > 0$, we only need to consider $y \in \Omega \setminus \Omega_{\rho}$ and $0 < r < \rho/8$. Indeed, for $y \in \Omega_{\rho}$, if $r < \text{dist}(y, \Omega \setminus \Omega_{\rho})$, then
\(\Phi_2(y, r) = 0\), and if \(r \geq \Omega \setminus \Omega_\rho\), then \(\Phi_2(y, r) \leq C(n)\Phi_2(\bar{y}, 2r)\), where \(\bar{y}\) is the closest point in \(\Omega \setminus \Omega_\rho\) and \(B(y, r) \subset B(\bar{y}, 2r)\). Moreover for any \(y \in \Omega \setminus \Omega_\rho\) and \(r \geq \rho/8\),

\[
\Phi_2(y, r) \leq \rho^{-n} \int_{\Omega \setminus \Omega_\rho} |u|^{p_n} \, dx \leq \sum_{i=1}^{N} \rho^{-n} \int_{\Omega \setminus \Omega_\rho \cap B_{\rho/9}(x_i)} |u|^{p_n} \, dx = \sum_{i=1}^{N} \Phi(x_i, \rho/9),
\]

where \(\{B(x_i, \rho/9)\}_{i=1}^{N}\) is a cover of the compact set \(\Omega \setminus \Omega_\rho\), \(\{x_i\}_{i=1}^{N} \subset \Omega \setminus \Omega_\rho\) and \(N\) depending on \(\Omega\) and \(\rho\).

On the other hand, for any \(y \in \Omega \setminus \Omega_\rho\) and \(0 < r < \rho/8\), since \(u\) is a stale solution in \(B_{\rho}(y) \subset \Omega\), by (1.11) with a scaling argument we have \(u \in M^{p_n, \beta}(B_{\rho/8}(y))\) with \(\|u\|_{M^{p_n, \beta}(B_{\rho/8}(y))} \leq C(n, \rho)\|u\|_{L^1(B_{\rho/2}(y))}\), in particular

\[
\Phi_2(y, r) \leq r^{\beta} \int_{B_r(y)} |u|^{p_n} \, dx \leq C(n, \rho)\|u\|_{L^1(\Omega)}^{p_n}
\]

as desired. This proves (1.13).

In the case \(n = 10\), for any \(y \in \Omega\), if \(r > \frac{1}{9} \rho\), we have

\[
r^{-n} \int_{\Omega \cap B_r(y)} |u| \, dx \leq C(n, \rho)\|u\|_{L^1(\Omega)}.
\]

Below we assume that \(0 < r < \frac{1}{9} \rho\). If \(y \in \Omega \setminus \Omega_{8\rho/9}\), we have \(\rho < \frac{9}{8}\) \(\text{dist} (y, \partial \Omega)\). Since \(0 < r < \frac{1}{9} \text{dist} (y, \partial \Omega)\) and \(u\) is a stale solution in \(B_{\text{dist} (y, \partial \Omega)}(y) \subset \Omega\), by (1.10) with a scaling we have

\[
\int_{B_r(y)} |u - u_{B_r(y)}| \, dx \leq C(n, \rho)\|u\|_{L^1(B_{\text{dist} (y, \partial \Omega)}(y))} \leq C(n, \rho)\|u\|_{L^1(\Omega)}.
\]

For \(y \in \Omega_{8\rho/9}\), noting \(0 < r < \frac{1}{9} \rho \leq \text{dist} (y, \partial \Omega)\), one has \(\Omega \cap B_r(y) \subset \Omega \setminus \Omega_\rho\). Thus

\[
r^{-n} \int_{\Omega \cap B_r(y)} |u| \, dx = r^{-n} \int_{\Omega \cup B_r(y)} |u| \, dx \leq C(n, \rho)\|u\|_{L^1(\Omega)}.
\]

Combining these estimates, we obtain (1.12).

We finally prove Corollary 1.3.

**Proof of Corollary 1.3.** Let \(u \in W^{1,2}_0(\Omega)\) be a stable solution to (1.1) with 0 boundary. By [13, Corollary 3.2.1](see also the proof in [8, Theorem 4.1] and [14, Theorem 5]), there is a nonnegative, nondecreasing sequence \((f_k)\) of convex functions in \(C^1(\mathbb{R})\) such that \(f_k \to f\) pointwise in \([0, \infty)\) and a nondecreasing sequence \((u_k)\) in \(C^2(\overline{\Omega}) \cap W^{1,2}_0(\Omega)\) such that \(u_k\) is a weak stable solution to

\[
-\Delta u_k = f_k(u_k) \quad \text{in} \; \Omega; \; u_k = 0 \quad \text{on} \; \partial \Omega.
\]  

(3.3)
and 

\[ u_k \to u \text{ in } W^{1,2}(\Omega) \text{ as } k \to +\infty. \]

If \( n = 10 \), applying (1.12) to \( u_k \) one has

\[ \int_{\Omega \cap B_r(y)} \left| u_k(x) - \int_{\Omega \cap B_r(y)} u_k \, dx \right| \, dx \leq \| u_k \|_{BMO(\Omega)} \leq C(n, \Omega) \int_{\Omega} |u_k| \, dx, \quad \forall r > 0, \forall y \in \overline{\Omega}. \]

By \( u_k \to u \) in \( W^{1,2}(\Omega) \) as \( k \to +\infty \) we conclude that \( \| u \|_{BMO(\Omega)} \leq C(n)\| u \|_{L^1(\Omega)} \) as desired.

If \( n \geq 11 \), applying (1.13) to \( u_k \) we have

\[ r^{\beta-n} \int_{\Omega \cap B_r(y)} |u_k|^{pn} \, dx \leq C(n, \Omega, \rho)(\| u_k \|_{L^1(\Omega)})^{pn}, \quad \forall y \in \overline{\Omega}, \quad \forall r > 0, \quad (3.4) \]

where \( \beta = \frac{2pn}{p_n-2} \in (0, n) \). By \( u_k \to u \) in \( W^{1,2}(\Omega) \) as \( k \to +\infty \) we deduce that \( u_k \in L^{p_n}(\Omega) \) uniformly in \( k \geq 0 \), and hence, \( u_k \to u \) weakly in \( L^{p_n}(\Omega) \). Thus, letting \( k \to +\infty \) in (3.4) we conclude \( \| u \|_{M^{p_n, \beta}(\Omega)} \leq C(n)\| u \|_{L^1(\Omega)} \) as desired. \( \square \)

### 4 Proof of Proposition 1.9

Let \( 0 < r < R < \infty \). Let \( \eta \in C_c^\infty(A_{7/4R}) \) satisfy that

\[ 0 \leq \eta \leq 1 \text{ in } A_{7/4R} \quad \text{and} \quad \eta = 1 \text{ in } A_{7/2R}, \quad (4.1) \]

\[ |D\eta|^2 + |D^2\eta| \leq \frac{C}{r^2} \text{ in } A_{7/2} \quad \text{and} \quad |D\eta|^2 + |D^2\eta| \leq \frac{C}{R^2} \text{ in } A_{2R,4R}, \quad (4.2) \]

where \( C > 0 \) is a universal constant.

Let \( u_\delta = u + \phi_\delta \) for \( \delta > 0 \), where \( \phi_\delta \) is the standard smooth mollifier and supported in \( B(0, \delta) \). Recall that \( u \in W^{1,1}_{\text{loc}}(\mathbb{R}^n) \) and \( u_\delta \to u \) in \( W^{1,1}_{\text{loc}}(\mathbb{R}^n) \). Since \( -\Delta u \geq 0 \) is a locally finite measure, we have \( -\Delta u_\delta = (-\Delta u) \ast \phi_\delta \geq 0 \) everywhere. By \( u_\delta \eta \in C_c^\infty(\mathbb{R}^n) \), one has

\[ u_\delta \eta(x) = \Delta^{-1}[\Delta(u_\delta \eta)](x) = c(n) \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-2}} \Delta(u_\delta \eta)(y) \, dy \quad \forall x \in \mathbb{R}^n \]

and hence

\[ D(u_\delta \eta)(x) = D\Delta^{-1}[\Delta(u_\delta \eta)](x) = c(n)(2-n) \int_{\mathbb{R}^n} \frac{x-y}{|x-y|^{n-1}} \Delta(u_\delta \eta)(y) \, dy \quad \forall x \in \mathbb{R}^n. \]

Noting

\[ \Delta(u_\delta \eta)(y) = \Delta u_\delta(y) \eta(y) + u_\delta(y) \eta(y) + Du_\delta(y) \cdot D\eta(y), \]

for \( 0 < \delta << r/8 \) we write

\[ \int_{A_{r,R}} |Du_\delta||x|^{-n+1} \, dx = \int_{A(r,R)} |D(u_\delta \eta)||x|^{-n+1} \, dx \]
Employing the triangle inequality, for $y \in \mathbb{R}^n$, we have

$$
\int_{A_{r,R}} \left| \int_{\mathbb{R}^n} \frac{x-y}{|x-y|^n} \Delta(u_\delta \eta)(y) \, dy \right| |x|^{-n+1} \, dx 
\leq C(n) \int_{A_{r,R}} \left| \int_{\mathbb{R}^n} \frac{x-y}{|x-y|^n} \Delta u_\delta(y) \eta(y) \, dy \right| |x|^{-n+1} \, dx 
+ C(n) \int_{A_{r,R}} \left| \int_{\mathbb{R}^n} \frac{x-y}{|x-y|^n} u_\delta(y) \Delta \eta(y) \, dy \right| |x|^{-n+1} \, dx 
+ C(n) \int_{A_{r,R}} \left| \int_{\mathbb{R}^n} \frac{x-y}{|x-y|^n} Du_\delta(y) \cdot D\eta(y) \, dy \right| |x|^{-n+1} \, dx 
= I_1 + I_2 + I_3.
$$

In order to control $I_1$ from above, first by $-\Delta u_\delta \geq 0$ and (4.1) one has

$$
I_1 \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |x-y|^{-n+1} |x|^{-n+1} \, dx(-\Delta u_\delta)(y) \eta(y) \, dy.
$$

Employing the triangle inequality, for $y \in \mathbb{R}^n$, we further get

$$
\int_{\mathbb{R}^n} |x-y|^{-n+1} |x|^{-n+1} \, dx \leq 2^{n-1} \int_{\{ |x| > 2|y| \}} |x|^{-2n+2} \, dx + 2^{n-1} \int_{\{ |x| \leq \frac{1}{2} |y| \}} |x|^{-n+1} |y|^{-n+1} \, dx 
+ \int_{\{ \frac{1}{2} |y| \leq |x| \leq 2 |y| \}} |x-y|^{-n+1} |y|^{-n+1} \, dx 
\leq C(n) |y|^{-n+2} + C(n) |y|^{-n+2} + \int_{\{ |y-x| \leq 3 |y| \}} |x-y|^{-n+1} |y|^{-n+1} \, dx 
\leq C(n) |y|^{-n+2}.
$$

(4.3)

This together with $-\Delta u_\delta \geq 0$ again gives

$$
I_1 \leq C(n) \int_{\mathbb{R}^n} (-\Delta u_\delta)|y|^{-n+2} \eta(y) \, dy.
$$

Via integration by parts and using $\eta \in C^\infty_c(A_{\frac{3}{4}A_4}^R)$, we have

$$
\int_{\mathbb{R}^n} (-\Delta u_\delta)|y|^{-n+2} \eta(y) \, dy = \int_{A_{\frac{3}{4}A_4}^R} u_\delta(-\Delta |y|^{-n+2} \eta(y) + D|y|^{-n+2} \cdot D\eta(y) - |y|^{-n+2} \Delta \eta(y)) \, dy.
$$

Observing that $\Delta |y|^{-n+2} = 0$ in $A_{\frac{3}{4}A_4}^R$, and using (4.1)&(4.2) we arrive at

$$
I_1 \leq C(n) \int_{A_{\frac{3}{4}A_4}^R} u_\delta(y)((2 - n)|y|^{-n} \cdot D\eta(y) - |y|^{-n+2} \Delta \eta(y)) \, dy 
\leq C(n) \int_{A_{\frac{3}{4}A_4}^R} |u_\delta(y)||r^{-n}\chi_{A_{\frac{3}{4}A_4}^R} + R^{-n}\chi_{A_{2R,4R}^R}| \, dy
$$

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As a consequence, since

For any we obtain

\[ \int_{\mathbb{R}^n} |x-y|^{-n} |x-y|^{-n+1} dx \leq C(n) \int_{\mathbb{R}^n} |x-y|^{-2n+1} dx \leq C(n) r^{-n+1}; \]

For \( I_2 \), by (4.3) and (4.1)

\[ I_2 \leq \int_{\mathbb{R}^n} \int_{A(r, R)} |x-y|^{-n-1} |x-y|^{n+1} dx u_\delta(y) |\Delta \eta(y)| dy \]

\[ \leq C(n) \int_{\mathbb{R}^n} \int_{A(r, R)} |x-y|^{-n+2} |u_\delta(y)\Delta \eta(y)| dy \]

\[ \leq C(n) \int_{\mathbb{R}^n} \int_{A(r, R)} |u_\delta(y)\chi_{A_{\frac{R}{2}}} + R^{-n}\chi_{A_{2R,A_R}}| dy \]

\[ \leq C(n) \int_{A_{\frac{R}{2}}} |u_\delta| dz + C(n) \int_{A_{2R,A_R}} |u_\delta| dz. \]

Now let us estimate \( I_3 \). First via integration by parts one gets

\[ \int_{\mathbb{R}^n} |x-y|^{-n} (x-y) Du_\delta(y) \cdot D\eta(y) dy \]

\[ = \int_{\mathbb{R}^n} |x-y|^{-n} (x-y) u_\delta(y) \Delta \eta(y) dy + \int_{\mathbb{R}^n} u_\delta(y) D(|x-y|^{-n} (x-y)) D\eta(y) dy. \]

Since \( |D(|x-y|^{-n} (x-y))| \leq C(n) |x-y|^{-n} \), we obtain

\[ \left| \int_{\mathbb{R}^n} |x-y|^{-n} (x-y) Du_\delta(y) \cdot D\eta(y) dy \right| \]

\[ \leq C(n) \int_{\mathbb{R}^n} |x-y|^{-n} u_\delta(y) \Delta \eta(y) dy + C(n) \int_{\mathbb{R}^n} |x-y|^{-n} u_\delta(y) D\eta(y) dy. \]

As a consequence,

\[ I_3 \leq C(n) I_2 + C(n) \int_{\mathbb{R}^n} \int_{A_{r,R}} |x-y|^{-n} |x-y|^{-n+1} dx |u_\delta(y)||D\eta(y)| dy =: C(n) I_2 + C(n) \tilde{I}_3. \]

In order to estimate \( \tilde{I}_3 \), first we note that (4.1) gives

\[ \tilde{I}_3 \leq C(n) \int_{\mathbb{R}^n} \int_{A_{r,R}} |x-y|^{-n} |x-y|^{-n+1} dx |u_\delta(y)||r^{-n}\chi_{A_{\frac{R}{2}}} + R^{-n}\chi_{A_{2R,A_R}}| dy \]

For any \( x \in A_{r,R} \), if \( y \in A_{\frac{R}{2}} \) we have \( |x-y| \geq |x|/2 \) and hence

\[ \int_{A_{r,R}} |x-y|^{-n} |x-y|^{-2n+1} dx \leq C(n) \int_{A_{r,R}} |x|^{-2n+1} dx \leq C(n) r^{-n+1}; \]
if \( y \in A_{2R,4R} \), then \( |x - y| \geq R \) and hence

\[
\int_{A_{r,R}} |x - y|^{-n} |x|^{-n+1} dx \leq C(n)R^{-n} \int_{A_{r,R}} |x|^{-n+1} dx \leq C(n)R^{-n+1}.
\]

Thus it follows that

\[
\tilde{I}_3 \leq C(n) \int_{\mathbb{R}^n} |u_\delta(y)||r^{-n}\chi_{A_{r,R}} + R^{-n}\chi_{A_{2R,4R}}| dy \leq C(n) \int_{A_{r,R}} |u_\delta| dz + C(n) \int_{A_{2R,4R}} |u_\delta| dz.
\]

To conclude,

\[
\int_{A_{r,R}} |Du_\delta||x|^{-n+1} dx \leq C(n) \int_{A_{r,R}} |u_\delta| dz + C(n) \int_{A_{2R,4R}} |u_\delta| dz.
\]

By letting \( \delta \to 0 \) and noting \( u_\delta \to u \) in \( W^{1,1}_\text{loc} \), we conclude (1.21).

## 5 Proof of Theorem 1.5

Since \( u \) satisfies (1.16), then \( u \) does not satisfy (1.15). We only need to show that if \( u \) is nonconstant, then (1.15) holds. Equivalently, it suffices to show that if \( u \) does not satisfy (1.15), then \( u \) is a constant. Namely, there exists a sequence \( \{R_j\}_{j \in \mathbb{N}} \) towards \( \infty \) so that

\[
\frac{1}{\log R_j} \int_{A_{R_j,4R_j}} |u(z)| dz \to 0 \quad \text{as} \quad j \to \infty \quad \text{when} \quad n = 10 \quad (5.1)
\]

and

\[
R_j^{\frac{n}{2} - 2 - \sqrt{n-1}} \int_{A_{R_j,4R_j}} |u(x)| dx \to 0 \quad \text{as} \quad j \to \infty \quad \text{when} \quad n \geq 11. \quad (5.2)
\]

On the other hand, given any \( 0 < r < \infty \), applying (1.18) for any \( R > 4r \) we have

\[
r^{-\left(1+\sqrt{n-1}\right)} \left( \int_{B_r} |Du|^2 \right)^{1/2} \leq C(n)R^{-\left(1+\sqrt{n-1}\right)} \left( \int_{A_{R,2R}} |Du|^2 \right)^{1/2}.
\]

Observe that the annual \( A_{1,2} \) can be covered by \( \{B_{\frac{1}{2}}(y_i)\}_{i=1}^N \) with \( y_1, \cdots, y_N \in A_{1,2} \) and \( N \leq C(n) \).

\[
\chi_{A_{1,2}} \leq \sum_{i=1}^N \chi_{B_{\frac{1}{2}}(y_i)} \leq \sum_{i=1}^N \chi_{B_{\frac{1}{2}}(y_i)} \leq C(n)\chi_{A_{\frac{3}{4},3}}.
\]

Below we consider the case \( n \geq 11 \) and the case \( n = 10 \) separately.
Case $n \geq 11$. For each $i$, applying (1.19) and (1.20), one attains

$$
\left( \int_{B_{\frac{R}{4}}(Ry_i)} |Du|^2 \, dx \right)^{1/2} \leq C(n)R^{-\frac{n+3}{2}} \int_{B_{\frac{R}{4}}(Ry_i)} |u| \, dx \leq C(n)R^{-\frac{n+2}{2}} \int_{A_{\frac{2R}{3}}} |u| \, dx.
$$

Thus by summing over all these balls,

$$
\int_{A_{R,2R}} |Du|^2 \, dx \leq C(n)R^{n-2} \left( \int_{A_{\frac{3R}{4}}} |u| \, dx \right)^2,
$$

and we eventually obtain

$$
r^{-(1+\sqrt{n-1})} \left( \int_{B_r} |Du|^2 \, dx \right)^{1/2} \leq C(n)R^{\frac{n-2}{2}-\sqrt{n-1}} \int_{A_{\frac{3R}{4}}} |u| \, dx.
$$

Taking $R = \frac{4R_j}{3}$, applying (5.2) and letting $j \to \infty$, one concludes

$$
\int_{B_r} |Du|^2 \, dx = 0.
$$

By the arbitrariness of $r > 0$, we obtain $\|Du\|_{L^2(\mathbb{R}^n)} = 0$, which implies that $u$ is a constant.

Case $n = 10$. For each $i$, applying (1.19), one attains

$$
\left( \int_{B_{\frac{R}{4}}(Ry_i)} |Du|^2 \, dx \right)^{1/2} \leq C(n)R^{-\frac{n}{2}} \int_{B_{\frac{R}{4}}(Ry_i)} |Du| \, dx \leq C(n)R^{\frac{n-2}{2}} \int_{A_{\frac{3R}{4}}} |Du| |x|^{-n+1} \, dx.
$$

Thus

$$
\int_{A_{R,2R}} |Du|^2 \, dx \leq C(n)R^{n-2} \left( \int_{A_{\frac{3R}{4}}} |Du| |x|^{-n+1} \, dx \right)^2.
$$

We therefore obtain

$$
r^{-(1+\sqrt{n-1})} \left( \int_{B_r} |Du|^2 \, dx \right)^{1/2} \leq C(n)R^{\frac{n}{2}-2-\sqrt{n-1}} \int_{A_{\frac{3R}{4}}} |Du| |x|^{-n+1} \, dx
$$

$$
= C(n) \int_{A_{\frac{2R}{3}}} |Du| |x|^{-n+1} \, dx,
$$

where in the last identity we use $\frac{n}{2} - 2 - \sqrt{n-1} = 5 - 2 - 3 = 0$. 

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For $R > 2^5 + r > 4$, let $m$ be the largest integer so that $m \leq \log_2 R - 3$. Applying the (5.3) to $2^j R$ with $j = 1, \ldots, m$, one has

$$\left(\int_{B_r} |Du|^2 \, dx\right)^{1/2} \leq C(n) \frac{1}{m} \sum_{j=1}^{m} \int_{A_{2^j R, A(2^j R)}} |Du||x|^{-n+1} \, dx$$

$$\leq C(n) \frac{1}{m} \int_{A_{R,2^m+2 R}} |Du||x|^{-n+1} \, dx$$

$$\leq C(n) \frac{1}{\log R} \int_{A_{R,2^m+2 R}} |Du||x|^{-n+1} \, dx.$$  

By (1.21), one has

$$\left(\int_{B_r} |Du|^2 \, dx\right)^{1/2} \leq C(n) \frac{1}{\log R} \int_{A_{1,2}} |u(z)| \, dz + C(n) \frac{1}{\log R^2} \int_{A_{R,2^m+2 R}} |u(z)| \, dz.$$  

Taking $R = \sqrt{R_j}$ and letting $j \to \infty$, by (5.1) one concludes

$$\int_{B_r} |Du|^2 \, dx = 0.$$  

Then the arbitrariness of $r > 0$ implies $\|Du\|_{L^2(\mathbb{R}^n)} = 0$, which further implies that $u$ is a constant.

**Appendix A radial stable solution when $n = 10$**

Suppose $n = 10$ in this appendix. Villegas [24] proved that $\frac{1}{2} \log(1 + |x|^2)$ is a stable solution to equation $-\Delta u = -(n - 2)e^{-2u} - 2e^{-4u}$ in $\mathbb{R}^n$. Note that $-(n - 2)e^{-2s} - 2e^{-4s} \leq 0$ in $\mathbb{R}$.

Below, we show that $u = -\frac{1}{2} \log(1 + |x|^2)$ is a stable solution to the equation

$$-\Delta u = f(u) \quad \text{in} \quad \mathbb{R}^n,$$

where $f(s) = (n - 2)e^{2s} + 2e^{4s} \geq 0$ in $\mathbb{R}$.

First we show that $u$ is a solution. Indeed, for any $x \in \mathbb{R}^n$ a direct calculation gives

$$-\Delta u(x) = \left((1 + |x|^2)^{-1}x_i\right)_{x_i} = \frac{n}{1 + |x|^2} + 2\frac{|x|^2}{(1 + |x|^2)^2} = (n - 2)\frac{1}{1 + |x|^2} + 2\frac{1}{(1 + |x|^2)^2}.$$  

By $e^{2u(x)} = (1 + |x|^2)^{-1}$, we have

$$-\Delta u(x) = (n - 2)e^{2u(x)} + 2e^{4u(x)} = f(u(x)).$$
Next, we show that $u$ is stable. Note that $f'(s) = 2(n-2)e^{2s} + 8e^{4s}$ for $s \in \mathbb{R}$. Given any $x \neq 0$, writing $r = |x|$ and noting $e^{2u(x)} = (1 + |x|^2)^{-1}$, we have

$$f'(u(x)) = 2(n-2)e^{2u(x)} + 8e^{4u(x)} = \frac{2(n-2)}{1 + r^2} + \frac{8}{(1 + r^2)^2}$$

By $n = 10$ we have

$$f'(u(x)) = \frac{16r^2(1 + r^2) + 8r^2}{r^2(1 + r^2)^2} = \frac{16r^4 + 24r^2}{r^2(1 + r^2)^2} < \frac{16(1 + r^2)^2}{r^2(1 + r^2)^2} = \frac{(n-2)^2}{4|x|^2}.$$

By this, and the Hardy inequality, we have

$$\int_{\mathbb{R}^n} f'(u(x)) \xi^2 \, dx \leq \frac{(n-2)^2}{4} \int_{\mathbb{R}^n} \frac{\xi^2}{|x|^2} \, dx \leq \int_{\mathbb{R}^n} |D\xi|^2 \, dx \quad \forall \xi \in C_0^\infty(\mathbb{R}^n).$$

Thus $u$ is a stable solution to $-\Delta u = f(u)$ in $\mathbb{R}^n$.

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