MADM For Solving Fourth-Order ODE

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Abstract. In this paper, we present the new reliable modification of Adomian decomposition method for solving fourth order ordinary deferential equations. The modification of Adomian decomposition method is an effective procedure to be applied in the singular or nonsingular problems within given the initial value problem. The results are compared with the existing exact or numerical method. Thus, the Modification of Adomian decomposition method are found to converge very quickly and more accurate compared to Adomian decomposition method. Several examples are given to show the ability and efficiency of the proposed method.

1. Introduction

Nowadays, natural phenomena that occurs in our world can be explained in mathematical modelling that handles a wide class in term of linear or nonlinear ordinary and partial differential equation. Recently, many researchers have shown an increased interest in Adomian Decomposition Method (ADM) [2][3][4][5][6] that first introduced and developed by George Adomian [1].The ADM which is can applied in ordinary differential equation (ODE) had provide several advantages which is can prove fast convergence of the solution. In this recent years, several researchers [7][8] have developed the ADM to modification to improve the accuracy and convergence which is called Modified Adomian Decomposition Method (MADM). It is the reason for this paper to present another dependable alteration of Adomian decomposition strategy for fathoming the 4th ordinary differential equation that applying to singular or non-singular problem that a new differential operator will be proposed. Likewise, the new strategy will be tried for a few illustrations and the results can be used to demonstrate the upsides of this technique.

2. MADM

Consider the underlying quality issue in the 4th-order of ODE as below

\[
\begin{align*}
y'''' + q(x)y'' + g(x,y) &= h(x) \\
y(0) &= A, \quad y'(0) = B, \quad y''(0) = C, \quad y'''(0) = D
\end{align*}
\]  

(1)
where \( g(x, y) \) is a real function, \( q(x) \) and \( h(x) \) are given functions and \( A, B, C \) and \( D \) are constants.

Here, we propose the new differential operator, as below

\[
L = e^{-\int p(x)dx} \frac{d}{dx} \left( e^{\int p(x)dx} \frac{d^3}{dx^3} \right) \tag{2}
\]

We can write problem (1) as

\[
L = h(x) - g(x, y)
\]

Therefore, \( L^{-1} \) which is the inverse operator and is there considered a 4th-fold integral operator, as below,

\[
L^{-1}(\cdot) = \int_0^x \int_0^x e^{\int p(x)dx} \int_0^x e^{-\int p(x)dx} (\cdot) dx \ dx \ dx \ dx \tag{3}
\]

By taking \( L^{-1} \) on (3), we have

\[
y(x) = \Phi(x) + L^{-1}h(x) - L^{-1}g(x, y), \tag{4}
\]

Such that

\[
L\Phi(x) = 0 \tag{5}
\]

The ADM introduce the solution \( y(x) \) and the nonlinear function \( g(x, y) \) by endless arrangement,

\[
y(x) = \sum_{n=0}^{\infty} y_n(x) \tag{6}
\]

and

\[
g(x, y) = \sum_{n=0}^{\infty} A_n \tag{7}
\]

and the parts \( y_n(x) \) of the arrangement \( y(x) \) will be resolved recurrently. Particular calculations were seen in (Wazwaz A. M. 1997) to present the Adomian Polynomials. The following algorithm
be used to construct Adomian polynomial, when \( F(u) \) is a nonlinear function. By substituting (6) and (7) into (4),

\[
\sum_{n=0}^{\infty} y_n(x) = \Phi(x) + L^{-1}h(x) + L^{-1} \sum_{n=0}^{\infty} A_n
\]  

(8)

By using ADM, the component, the components \( y_n(x) \) can be determined as

\[
\begin{align*}
&\begin{cases}
y_0 = \Phi(x) + L^{-1}h(x), \\
y_{n+1} = -L^{-1}A_n, \quad n \geq 0
\end{cases}
\end{align*}
\]

Which yields

\[
y_0 = \Phi(x) + L^{-1}h(x), \\
y_1 = -L^{-1}A_0, \\
y_2 = -L^{-1}A_1, \\
y_3 = -L^{-1}A_2, \\
\vdots
\]

From the (8) and (10), we can determine the components \( y_n(x) \), and that is the series solution of \( y(x) \) in (6) can be obtained. For the numerical needs, the \( n \)-term approximant

\[
\Psi_n = \sum_{k=0}^{n} y_k(x)
\]

(10)

can be utilized to approximate the exact solution.
3. Illustrative Example

The current part, starting with two conventional differential conditions are considered and afterward are understood by the standard and adjusted Adomian deterioration techniques.

3.1. Example 1

Let us take the linear non-singular initial value problem in 4th-order ODE:

\[
\begin{align*}
  y^{(4)} + y''' &= 2(x+1) \\
  y(0) &= 0, \quad y'(0) = 0, \quad y''(0) = 0, \quad y'''(0) = 0 \\
\end{align*}
\]  \hspace{1cm} (11)

Standard ADM: we put

\[ L(.) = \frac{d^4}{dx^4}(.), \]  \hspace{1cm} (12)

so,

\[ L^{-1}(.) = \int \int \int \int (. \bigga dx \bigga dx \bigga dx \bigga dx. \]  \hspace{1cm} (13)

Then, Eq. (11) yields

\[ Ly = -y''' + 2x + 2 \]  \hspace{1cm} (14)

Applied \( L^{-1} \) to both sides of (14) we have

\[ y(0) + y'(0)x + y''(0)x^2 + y'''(0)x^3 - L^{-1}(y''') + L^{-1}(2x + 2) \]  \hspace{1cm} (15)

Continuing as like before we acquired the successive recursive relationship

\[
\begin{align*}
  y_0 &= y(0) + y'(0)x + y''(0)x^2 + y'''(0)x^3 + L^{-1}(2x + 2) \\
  y_{n+1} &= -L^{-1}y'''_n, \quad n \geq 0.
\end{align*}
\]  \hspace{1cm} (16)

Thus, we have

\[
\begin{align*}
  y_0 &= \frac{1}{60}x^5 + \frac{1}{12}x^4 \\
  y_1 &= -\frac{1}{360}x^6 + \frac{1}{60}x^5 \\
  y_2 &= -\frac{1}{2520}x^7 + \frac{1}{360}x^6 \\
  y_3 &= -\frac{1}{20160}x^8 - \frac{1}{2520}x^7 \\
  y_4 &= \frac{1}{181440}x^9 + \frac{1}{20160}x^8 \\
  y_5 &= -\frac{1}{1814400}x^{10} + \frac{1}{181440}x^9
\end{align*}
\]  \hspace{1cm} (16)
Cancelling these noise term gives the exact solution

\[ y(x) = \frac{x^4}{12} \]  \hspace{1cm} (17)

MADM: Based to (2), we construct

\[ L = e^x \frac{d}{dx}(e^{-x}) \frac{d^3}{dx^3}, \]  \hspace{1cm} (18)

so

\[ L^{-1}(.) = \int_0^x \int_0^x \int_0^x e^x \int_0^x e^{-x} d\cdot dx \cdot dx, \]  \hspace{1cm} (19)

Again with the same method, Eq (11) takes the form

\[ Ly = 2x + 2, \]  \hspace{1cm} (20)

Applied \( L^{-1} \) to both sides of (20), we have

\[ L^{-1}Ly = \int_0^x \int_0^x \int_0^x \int_0^x e^x \int_0^x e^{-x}(2x + 2) d\cdot dx \cdot dx, \]  \hspace{1cm} (21)

And it implies that

\[ y(x) = y(0) + y'(0)x + \frac{x^2}{2} y''(0) + \frac{x^3}{6} y'''(0) + \frac{x^4}{12} \Rightarrow y(x) = \frac{x^4}{12} \]  \hspace{1cm} (22)

So, here we can prove that by using MADM the exact solution can be obtained easily and less computational work between ADM.

### 3.2. Example 2

Let us take the following linear non-singular with initial value problem:

\[
\begin{align*}
\{ y^{iv} + y''' &= x \\
y(0) = 1, y'(0) = 0, y''(0) = 0, y'''(0) = 0
\end{align*}
\]  \hspace{1cm} (23)

Standard ADM: we put

\[ L(\cdot) = \frac{d^4}{dx^4}(\cdot), \]  \hspace{1cm} (24)

so,
\[ L(.) = \int_0^x \int_0^x \int_0^x \int_0^x (.) \, dx \, dx \, dx \, dx. \]  
(25)

Then, Eq. (23) becomes
\[ Ly = -y'' + x^3 \]  
(26)

Applied \( L^{-1} \) to (26) we obtain
\[ y(0) + y'(0)x + y''(0)x^2 + y'''(0)x^3 - L^{-1}(y'''(0)) + L^{-1}(x) \]
(27)

Thus, we have
\[
\begin{align*}
\{ y_0 &= y(0) + y'(0)x + y''(0)x^3 + L^{-1}(x) \\
y_{n+1} &= -L^{-1}y'''_{n}, & n \geq 0
\end{align*}
\]

By cancelling the noise term, we have the exact solution
\[ y(x) = 1 + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \frac{x^8}{8!} y(x) = e^x \]  
(28)

MADM: Based to (2), we construct
\[ L = e^x \frac{d}{dx} (e^x) \frac{d^3}{dx^3} \]  
(29)

so
\[ L^{-1}(.) = \int_0^x \int_0^x \int_0^x e^x \int_0^x e^{-x(.)} \, dx \, dx \]  
(30)

Again with the same method, Eq (23) becomes
\[ Ly = x \]  
(31)

Applied \( L^{-1} \) to both sides of (31), we have
\[
L^{-1}Ly = \int_0^x \int_0^x \int_0^x e^x \int_0^x e^{-x}(x)dx \, dx \, dx
\]  
(32)

And it implies that

\[
y(x) = y(0) + y'(0)x + y''(0)x^2 + y'''(0)x^3 + e^x \Rightarrow y(x) = e^x.
\]  
(33)

Hence, we have the exact solution easily through the proposed ADM with less computational work.

3.3. Example 3

Let us take the following linear non-singular with initial value problem:

\[
\begin{align*}
    y^{iv} + y''' &= \cos x \\
    y(0) &= 1, y'(0) = 0, y''(0) = 0, y'''(0) = 0
\end{align*}
\]  
(34)

exact solution given by:

\[
h(x) = 1 - \frac{1}{2}x^2 + \frac{1}{2}x^3 + \frac{1}{48}x^4
\]

Standard ADM: we put

\[
L(.) = \frac{d^4}{dx^4}(.),
\]  
(35)

so,

\[
L^{-1}(.) = \int_0^x \int_0^x \int_0^x \int_0^x . \, dx \, dx \, dx \, dx.
\]  
(36)

Then, Eq (34) becomes

\[
Ly = -\cos x - y'''
\]  
(37)

Applied \(L^{-1}\) to both sides of (37) we obtain

\[
y(0) + y'(0)x + y''(0)x^2 + y'''(0)x^3 - L^{-1}(y''') + L^{-1}(\cos x)
\]  
(38)

Continuing as before we acquired the recursive relationship.
\[
\begin{cases}
y_0 = 1 + L^{-1}(\cos x) \\
y_{n+1} = -L^{-1}y_n'', \quad n \geq 0
\end{cases}
\]

Thus, we have
\[
\begin{align*}
y_0 &= \frac{1}{2}x^2 + \cos x \\
y_1 &= x - \sin x \\
y_2 &= 1 - \frac{1}{2}x^2 - \cos x \\
y_3 &= -x + \sin x \\
&\vdots
\end{align*}
\]

(39)

By cancelling the noise term we have the exact solution

\[
y(x) = \frac{1}{2}x^4 + \cos x
\]

(40)

MADM: Based on (2), we construct

\[
L = e^x \frac{d}{dx} \left( e^x \right) \frac{d^3}{dx^3}
\]

(41)

so

\[
L^{-1}(.) = \int_0^x \int_0^x \int_0^x e^{-x} \int_0^x e^{x}(.) \, dx \, dx \, dx
\]

(42)

Then, Eq (34) becomes

\[
Ly = \cos x.
\]

(43)

Applied \(L^{-1}\) to both sides of (43) and it becomes

\[
L^{-1}Ly = \int_0^x \int_0^x \int_0^x e^{-x} \int_0^x e^{x}(\cos x) \, dx \, dx \, dx
\]

(44)

and

\[
y(x) = y(0) + y'(0)x + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + x^2 + \frac{1}{3}x^3 - \frac{1}{24}x^4
\]

(45)

\[
\Rightarrow y(x) = 1 + x^2 + \frac{1}{3}x^3 - \frac{1}{24}x^4 - \frac{1}{120}x^5
\]

(46)
Figure 1. Graph for example 3 between exact solution, ADM and MADM.

The Comparison of the results between ADM, MADM and the exact solution are depicted in figure 1. As shown in the graph, MADM is very close and converges to exact solutions between ADM. It shows that, the propose method of ADM is more accurate than standard ADM.

3.4. Example 4

Let us take the 4th-order linear singular with initial value problem:

\[
\begin{cases}
y^{iv} + 2y''' = 4x - 8 \\
y(0) = 0, y'(0) = 1, y''(0) = 0, y'''(0) = 1
\end{cases}
\]  

(47)

exact solution given by :

\[h(x) = x - \frac{1}{4}x^2 + \frac{5}{12}x^3 + \frac{1}{12}x^4\]

Standard ADM: we put

\[L(.) = \frac{d^4}{dx^4}(.),\]

(48)

so,

\[L^{-1}(.) = \int_0^x \int_0^x \int_0^x .
\]

dx dx dx,

(49)
Then, eq (47) becomes

\[ Ly = (4x - 8) + 2y''' \]  

(50)

Applied \( L^{-1} \) to both sides of (50) we obtain

\[ y(0) + y'(0)x + \frac{x^2}{2!} y''(0) + \frac{x^3}{3!} y'''(0) - L^{-1}(2y''') + L^{-1}(4x - 8) \]  

(51)

Continuing as like before we acquired the successive recursive relationship

\[
\begin{aligned}
y_0 &= x + \frac{x^3}{3!} + L^{-1}(4x - 8) \\
y_{n+1} &= -L^{-1}2y_n''' , n \geq 0
\end{aligned}
\]  

(52)

Thus, we have:

\[
\begin{aligned}
y_0 &= x + \frac{x^3}{3!} - \frac{1}{3} x^4 + \frac{1}{30} x^5 \\
y_1 &= \frac{1}{12} x^4 - \frac{x^5}{15} + \frac{x^6}{90} \\
y_2 &= \frac{1}{30} x^5 - \frac{2}{45} x^6 + \frac{1}{315} x^7 \\
y_3 &= \frac{1}{90} x^6 - \frac{4}{315} x^7 + \frac{1}{1260} x^8 \\
&\vdots
\end{aligned}
\]

(53)

By cancelling the noise term we have the exact solution

\[ y(x) = x + \frac{1}{6} x^3 - \frac{1}{4} x^4 - \frac{1}{15} x^5 - \frac{1}{45} x^6 \]  

(54)

MADM: Based to (2), we construct

\[ L = e^x \frac{d}{dx} \left( e^x \right)^{d^3/dx^3} \]  

(55)

so,

\[ L^{-1}(.) = \int_0^x \int_0^x \int_0^x e^{2x} \int_0^x e^{-2x}(.) \ dx \ dx \ dx \]

Then, Eq (47) yields
\[ Ly = 4x - 8 \] (56)

Applied \( L^{-1} \) to both sides of (56), we have

\[ L^{-1}Ly = \int_0^x \int_0^x \int_0^x \int_0^x e^{2x} e^{-2x} (4x - 8) \, dx \, dx \, dx \, dx \] (57)

And it implies that

\[ y(x) = y(0) + y'(0)x + \frac{x^2}{2!} y''(0) + \frac{x^3}{3!} y'''(0) - \frac{x^4}{12} - \frac{x^3}{8} + \frac{3x^2}{8} \] (58)

\[ \Rightarrow y(x) = x + \frac{x^3}{6} - \frac{x^4}{12} - \frac{x^3}{8} + \frac{3x^2}{8} \] (59)

In Figure 2, we compare the solution obtained by ADM, MADM and exact solution by using efficient numerical solution in the form of infinite series that is obtained iteratively and usually converges to the exact solution. As shown, MADM is more accurate and fast converges toward the exact solution between ADM.

**Figure 2.** Graph for example 4 between Exact solution, ADM and MADM

**Conclusion**

MADM has been applied to solve many differential equations that require less calculation in solving than the standard Adomian. Here we used this proposed method to solved particular Fourth-Order differential equations. We usually derive a very good approximation to the solutions, and sometimes the
exact solution can be found. In above examples, it demonstrated solving this method initial value problem of order four that can show this proposed method has the ability to solve both linear and nonlinear ordinary differential equation.

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