A UNITARY CUNTZ SEMIGROUP FOR C*-ALGEBRAS OF STABLE RANK ONE

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Abstract. We introduce a new invariant for C*-algebras of stable rank one that merges the Cuntz semigroup information together with the K1-group information. This semigroup, termed the Cu1-semigroup, is constructed as equivalence classes of pairs consisting of a positive element in the stabilization of the given C*-algebra together with a unitary element of the unitization of the hereditary subalgebra generated by the given positive element. We show that the Cu1-semigroup is a well-defined continuous functor from the category of C*-algebras of stable rank one to a suitable codomain category that we write Cu=K0⊙K1. Furthermore, we compute the Cu1-semigroup of some specific classes of C*-algebras. Finally, in the course of our investigation, we show that we can recover functorially Cu, K1 and Kr := K0 ⊕ K1 from Cu1.

1. Introduction

The Elliott classification program aims to find a complete invariant for nuclear separable simple C*-algebras. The original version of this invariant, written Ell(A), is based on K-theoretical information together with tracial data. As up to now, adding up decades of research, this invariant has provided satisfactory results for simple, separable, unital, nuclear, Z-stable C*-algebras satisfying the Universal Coefficient Theorem assumption (see, among many others, [15], [14], and [23]). On the other hand, the Cuntz semigroup has recently appeared to be a key tool to recover regularity properties of a (not necessarily simple) C*-algebra. As a matter of fact, it has been proved that the Cuntz semigroup of C(T) ⊗ A is naturally isomorphic to Ell(A), for any unital, simple, nuclear, finite, Z-stable C*-algebra A (see [1]).

Classification of non-simple C*-algebras has had an important resurgence in recent years. Whenever considering non-simple C*-algebras, the Cuntz semigroup, written Cu, seems to be a good candidate itself for classification. For instance, it has been shown that the Cuntz semigroup classifies any (unital) inductive limits of one-dimensional non-commutative CW complexes whose K1-group is trivial (see [20]). More concretely, the Cuntz semigroup entirely captures the complete lattice Lat(A) of ideals of any C*-algebra A, since we have a natural lattice isomorphism between Lat(A) ≃ Lat(Cu(A)), where Lat(Cu(A)) denotes the set of ideals of Cu(A). (See [3, Proposition 5.1.10].) However, a main limitation of the Cuntz semigroup lies within the fact that it fails to capture any K1 information whatsoever.

In this paper, we introduce a unitary version of the Cuntz semigroup, denoted by Cu1, for C*-algebras of stable rank one. This construction incorporates the K1 groups of the C*-algebra and its ideals to overcome this lack of information in the original construction of the Cuntz semigroup. We here establish the basic functorial properties of this construction. More concretely we show that:

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The Cu₁-semigroup is a continuous functor from the category of C*-algebras with stable rank one, that we denote C∗∗_{sr1}, to a certain subcategory of semigroups, written Cu∼, modeled after the category Cu of abstract Cuntz semigroups.

**Theorem 1.1.** The functor Cu₁ : C∗_{sr1} → Cu∼ is continuous. More precisely, given an inductive system (Ai, φij)∈I in C∗_{sr1}, then:

Cu∼ −→ lim(Cu₁(Ai), Cu₁(φij)) ≃ Cu₁(C∗_{sr1} −→ lim((Ai, φij))).

We then recover functorially the K∗-group from the Cu₁-semigroup as follows:

**Theorem 1.2.** There exists a functor H∗ : Cu∼ u −→ AbGp(S, u) ↦→ (Gr(Sc), Sc, u) α ↦→ Gr(αc)

that yields a natural isomorphism η∗ : H∗ ◦ Cu₁,u ≃ K∗.

This paper is organized as follows: In a first part, we construct our invariant, for C*-algebras of stable rank one. We show that it is an ordered monoid that satisfies the order-theoretic axioms (O1)-(O4) introduced in [11]. We also find a suitable category, called the category Cu∼, and prove that Cu₁ is a well-defined continuous functor.

Then, we give an alternative picture of our invariant, making use of the lattice of ideals of the C*-algebra, in order to compute the Cu₁-semigroup of some classes of C*-algebras, such as the simple case, AF, and some AI and AT algebras.

Finally, we explicitly define the notion of recovering an invariant from another and how one can recover classifying results. We then see that we can recover Cu, K₁ and also K. from Cu₁, to conclude that Cu₁ is a complete invariant for the class of unital AHd algebras with real rank zero.

We mention that this article is part of a twofold work. The author has been investigating further on the unitary Cuntz semigroup in [?], studying its ideal structure and exactness properties.

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2. Preliminaries

2.1. The Cuntz semigroup. We recall some definitions and properties on the Cuntz semigroup of a C*-algebra. More details can be found in [3], [5], [11], [22].

**2.1. (The Cuntz semigroup of a C*-algebra.)** Let A be a C*-algebra. We denote by A+ the set of positive elements. Let a and b be in A+. We say that a is Cuntz subequivalent to b, and we write a ≲ Cu b, if there exists a sequence (xn)∈N in A such that a = lim m→∞ xnx∗n. After antisymmetrizing this relation, we get an equivalence relation over A+, called Cuntz equivalence, denoted by ∼Cu.
Let us write $\text{Cu}(A) := (A \otimes \mathcal{K})_+. / \sim_{\text{Cu}}$, that is, the set of Cuntz equivalence classes of positive elements of $A \otimes \mathcal{K}$. Given $a \in (A \otimes \mathcal{K})_+$, we write $[a]$ for the Cuntz equivalence class of $a$. This set is equipped with an addition as follows: let $v_1$ and $v_2$ be two isometries in the multiplier algebra of $A \otimes \mathcal{K}$, such that $v_1v_1^* + v_2v_2^* = 1_{M(A \otimes \mathcal{K})}$. Consider the $^*$-isomorphism $\psi : M_2(A \otimes \mathcal{K}) \rightarrow A \otimes \mathcal{K}$ given by $\psi(0,0) = v_1av_1^* + v_2bv_2^*$, and we write $a \oplus b := [a \oplus b]$ and $[a] \leq [b]$ whenever $a \leq_{\text{Cu}} b$. In this way $\text{Cu}(A)$ is a semigroup called the Cuntz semigroup of $A$.

For any $^*$-homomorphism $\phi : A \rightarrow B$, one can define $\text{Cu}(\phi) : \text{Cu}(A) \rightarrow \text{Cu}(B)$, a semigroup map, by $[a] \mapsto [(\phi \otimes \text{id}_{\mathcal{K}})(a)]$. Hence, we get a functor from the category of $C^*$-algebras into a certain subcategory of the category PoM of positively ordered monoids, called the category Cu, that we describe next.

**Definition 2.2.** Let $(S, \leq)$ be an ordered semigroup. An auxiliary relation on $S$ is a binary relation $<$ such that:

(i) For any $a, b \in S$ such that $a < b$ we have $a \leq b$.

(ii) For any $a, b, c, d \in S$ such that $a \leq b < c \leq d$ we have $a < d$.

**2.3. (The category Cu.)** Let $(S, \leq)$ be a positively ordered semigroup and let $x, y$ in $S$. We say that $x$ is way-below $y$, and we write $x \ll y$ if, for all increasing sequences $(z_n)_{n \in \mathbb{N}}$ in $S$ that have a supremum, if $\sup z_n \geq y$, then there exists $k$ such that $z_k \geq x$. This is an auxiliary relation on $S$, called the compact-containment relation and sometimes referred to as the way-below relation. In particular $x \ll y$ implies $x \leq y$ and we say that $x$ is a compact element whenever $x \ll x$.

We say that $S$ is an abstract Cu-semigroup if it satisfies the following order-theoretic axioms:

(O1): Every increasing sequence of elements in $S$ has a supremum.

(O2): For any $x \in S$, there exists a $\ll$-increasing sequence $(x_n)_{n \in \mathbb{N}}$ in $S$ such that $\sup_{n \in \mathbb{N}} x_n = x$.

(O3): Addition and the compact containment relation are compatible.

(O4): Addition and suprema of increasing sequences are compatible.

A Cu-morphism between two Cu-semigroups is a positively ordered monoid morphism that preserves the compact containment relation and suprema of increasing sequences.

The category Cu of abstract Cuntz semigroups is the subcategory of PoM whose objects are Cu-semigroups and morphisms are Cu-morphisms.

**2.4. (Properties of the Cuntz semigroup.)** Let $S$ be a Cu-semigroup. We say that $S$ is countably-based if there exists a countable subset $B \subseteq S$ such that for any $a, a' \in S$ such that $a' \ll a$, then there exists $b \in B$ such that $a' \leq b \ll a$. An element $u \in S$ is called an order-unit of $S$ if for any $x \in S$, there exists $n \in \mathbb{N} := \mathbb{N} \cup \{\infty\}$ such that $x \leq nu$. A countably-based Cu-semigroup has a largest element or, equivalently, it is singly-generated as an ideal -for instance, by its largest element-. Let us also mention that if $A$ is a separable $C^*$-algebra, then $\text{Cu}(A)$ is countably-based. In fact, its largest element, that we write $\infty_A$, can be explicitly constructed as $\infty_A = \sup_{x \in \mathcal{S}} [x_A]$, where $x_A$ is any strictly positive element (or full positive) in $A$. A fortiori, $[x_A]$ is an order-unit of $\text{Cu}(A)$.

A notion of ideals in the category Cu has been considered in several places; we refer the reader to [3, §5.1.6] for more details. We recall that for any Cu-semigroup $S$ and any $x \in S$, the ideal generated by $x$ is
\( I_a := \{ y \in S \mid y \leq \infty x \} \). For any \( C^* \)-algebra \( A \), the assignment \( I \mapsto Cu(I) \) defines a lattice isomorphism between the lattice \( \text{Lat}(A) \) of closed two-sided ideals of \( A \) and the lattice \( \text{Lat}(Cu(A)) \) of ideals of \( Cu(A) \).

In fact, \( a \) is a full positive element in \( I \) if and only if \([a]\) is a full element in \( Cu(I) \). In this case, we have \( Cu(I_a) = I_{[a]} \).

2.2. The stable rank one context. As mentioned before, we work with \( C^* \)-algebras of stable rank one. In this context, Cuntz subequivalence of positive elements admits a nicer description easier to work with.

Let us shortly explicit this alternative picture and we refer the reader to \([15\) Proposition 4.3 - §6], \([10\) Proposition 1] and \([17\) for more details.

Let \( A \) be a \( C^* \)-algebra. We recall that an open projection is a projection \( p \in A^{**} \) such that \( p \) belongs to the strong closure of the hereditary subalgebra \( A_p := pA^{**}p \cap A \) of \( A \). These open projections are in one-to-one correspondence with the hereditary subalgebras of \( A \). For any positive element \( a \) of \( A \), we shall write \( \text{her}(a) := aAa \), the hereditary subalgebra of \( A \) generated by \( a \) and call the support projection of \( a \), the (unique) open projection \( p_a \in A^{**} \) such that \( \text{her}(a) = A_{p_a} \). We recall that \( p_a := \text{SOT} - \lim a^{1/n} \).

We also recall that two open projections \( p, q \in A^{**} \) are Peligrad-Zsidó equivalent, and we write \( p \sim_{PZ} q \) if there exists a partial isometry \( v \in A^{**} \) such that \( p = v^*v, q = vv^* \), \( vA_p \subseteq A, Aqv^* \subseteq A \). We say that \( p \preceq_{PZ} q \) if there exists an open projection \( p' \in A^{**} \) such that \( p \sim_{PZ} p' \preceq q \); see \([17\) Definition 1.1].

Suppose now that \( A \) has stable rank one. Then \( a \preceq_{Cu} b \) if and only if there exists \( x \in A \) such that \( xx^* = a \) and \( x^*x \in \text{her}(b) \). This is in turn equivalent to saying that \( p_a \preceq_{PZ} p_b \). In this case, for any partial isometry \( \alpha \in A^{**} \) that realizes the Peligrad-Zsidó equivalence between \( p_a \) and \( p_b \), we have an explicit injection as follows:

\[
\theta_{a,b;\alpha} : \text{her}(a) \hookrightarrow \text{her}(b)
\]

\[
d \mapsto a^*d
\]

The next proposition is similar to \([15\) Proposition 3.3] and \([17\) Theorem 1.4]. For the sake of completeness we will give a proof in this slightly different picture.

Proposition 2.5. Let \( A \) be a \( C^* \)-algebra. Let \( a \in A_+ \) and let \( p \in A^{**} \) be its support projection. Let \( \alpha \) be a partial isometry in \( A^{**} \) such that \( p = \alpha a^* \) and \( q = \alpha^* \alpha \) is an open projection of \( A^{**} \). Set \( x := a^{1/2} \alpha \).

Then \( p \sim_{PZ} q \) if and only if \( x \) belongs to \( A \). In this case, \( q = p_{x^*x} \).

Proof. The forward implication is coming from the definition of the Peligrad-Zsidó equivalence itself.

Conversely, let us suppose that \( x := a^{1/2} \alpha \) belongs to \( A \). Let \( d \in aAa \). Then there exists \( \delta_d \in A \) such that \( d = a\delta_d a \). Now observe that \( \alpha^*d = \alpha^* a^{1/2}d a^{1/2} \delta_d a \) belongs to \( A \). We obtain that \( \alpha^*aAa \subseteq A, \alpha^* \alpha Aa \subseteq A \), that is, \( \alpha^*A_p \subseteq A \). Now since \( p \) is a support projection and \( q = \alpha^* p a \), we deduce that \( q \) is a support projection and moreover \( \alpha^* A_p \alpha = A_q \). Finally, observe that \( \alpha A_q = \alpha A_q \alpha^* \alpha = A_q \alpha \) and that \( (\alpha^* A_p)^* = A_p \alpha, so \alpha A_q \subseteq A \). We conclude that \( p \sim_{PZ} q \) and by construction \( q = p_{x^*x} \).

Lemma 2.6. Let \( A \) be a \( C^* \)-algebra with stable rank one and let \( a \) and \( b \) be contractions in \( A_+ \) such that \( a \preceq_{Cu} b \). Let \( \alpha \) and \( \beta \) be in \( A^{**} \) such that they both realize the Peligrad-Zsidó subequivalence of \( p_a \preceq_{PZ} p_b \). For any \( u \in \mathcal{U}(\text{her}(\alpha)) \), we have:

\[
[\theta^*_{a,b;\alpha}(u)]_{\mathcal{K}(\text{her}(\beta))} = [\theta^*_{a,b;\beta}(u)]_{\mathcal{K}(\text{her}(\beta))}
\]

where \( \theta^*_{a,b;\alpha} \) (resp. \( \theta^*_{a,b;\beta} \)) is the unitized morphism of \( \theta_{a,b;\alpha} \) in Paragraph 2.2.
Proof. Since $a$ and $b$ are fixed elements, we shall write $\theta_a$ instead of $\theta_{a,a}$ (respectively $\theta_b$ for $\theta_{a,b}$). Consider the injections given by $\alpha$ and $\beta$ as in Paragraph 2.2. Define $x := a^{1/2}a$ and $y := a^{1/2}b$. We have $x, y \in A$. We first consider elements of $AA$ and the result will follow by continuity. Rewrite $\theta_a$ and $\theta_b$:

\[
\begin{align*}
\theta_a : AA & \hookrightarrow \overline{bAb} \\
\quad a & \mapsto x^{1/2}a^{1/2}x \\
\theta_b : AA & \hookrightarrow \overline{bAb} \\
\quad a & \mapsto y^{1/2}a^{1/2}y
\end{align*}
\]

Let $u$ be a unitary element of $\text{her}(a)^-$. There exists a pair $(u_0, \lambda)$ with $u_0 \in \text{her}(a)$ and $\lambda \in \mathbb{T}$ such that $u = u_0 + \lambda$.

Let $0 < \epsilon < 2$. Since $\text{her}(a) = \mathbb{K}a\mathbb{K}$, we can find $\delta \in A$ such that $\|u_0 - a\delta a\| \leq \epsilon/3$. We write $M := \|\delta\|$ and we set $\epsilon' := \epsilon/(6M)$. On the one hand, observe that $\|a^{1/2}\| \leq 1$ and hence we easily get that $\|a^{1/2}\delta a^{1/2}\| \leq M$.

On the other hand, since $a = xx^* = yy^*$, by [10, Lemma 2.4] we know there exists a unitary element $u_e$ of $\text{her}(b)^-$ such that $\|y - xu_e\| \leq \epsilon'$ (equivalently $\|u_e^*x^* - y^*\| \leq \epsilon'$). Now, we compute:

\[
\begin{align*}
\|u_e^*\theta_a(a\delta a + \lambda)u_e - \theta_b^*(a\delta a + \lambda)\| &= \|u_e^*x^{1/2}\delta a^{1/2}xu_e - y^{1/2}\delta a^{1/2}y\| \\
&\leq \|u_e^*x^{1/2}\delta a^{1/2}xu_e - y^{1/2}\delta a^{1/2}y\| \\
&\quad + \|y^*a^{1/2}\delta a^{1/2}y - y^{1/2}\delta a^{1/2}y\| \\
&\leq \|u_e^*x^* - y^*\| \|a^{1/2}\delta a^{1/2}\| \|xu_e\| + \|y - xu_e\| \|a^{1/2}\delta a^{1/2}\| \|y^*\| \\
&\leq \epsilon'M + \epsilon'M \\
&\leq \epsilon/3.
\end{align*}
\]

Combining the fact that $u$ and $a\delta a + \lambda$ are close up to $\epsilon/3$ with the fact that $\theta_a$ and $\theta_b^*$ are contractive maps, we conclude that $\|u_e^*\theta_a(u)u_e - \theta_b^*(u)\| \leq \epsilon < 2$. On the other hand, it is well-known that unitary elements that are close enough (i.e. $\|u - v\| < 2$) are homotopic. We conclude that $u_e^*\theta_a(u)u_e \sim_h \theta_b^*(u)$ and the result follows.

\[\square\]

3. The $\text{Cu}_1$ semigroup

In this section we define the invariant and establish its first properties. The unitary Cuntz semigroup consists of classes of pairs of element $(a, u)$, where $a$ is a positive element of $A \otimes \mathcal{K}$ and $u$ is a unitary of $\text{her}(a)^-$, under a suitable equivalence relation, written $\sim$, that is built using the Cuntz subequivalence to compare positive elements and using Lemma 2.6 to compare unitary elements. Our main result here focuses on the continuity of this invariant.

3.1. The $\lesssim_1$ binary relation. Let $A$ be a $C^*$-algebra of stable rank one. Let $a, b \in A_+$ and let $u, v$ be unitary elements of $\text{her}(a)^-$ and $\text{her}(b)^-$ respectively.

We say that $(a, u)$ is unitarily Cuntz subequivalent to $(b, v)$, and we write $(a, u) \lesssim_1 (b, v)$ if,

\[
\begin{align*}
\begin{cases}
\quad a \lesssim_{\text{Cu}_0} b \\
\quad \theta_{ab_a}(u) = [v] \text{ in } K_1(\text{her}(b)^-)
\end{cases}
\end{align*}
\]

where $\theta_{ab_a}$ is the injection given by a partial isometry $a$ as constructed in Paragraph 2.2.

Lemma 3.1. The relation $\lesssim_1$ is reflexive and transitive.
Proof. Reflexivity of $\leq_1$ follows from the fact that $\leq_{Cu}$ is reflexive and that $id_{\text{her}(a)} = \theta_{u, pu}$.

Now let $a$, $b$, and $c$ be in $A_+$ and let $u_a$, $u_b$, and $u_c$ be unitary elements of $\text{her}(a)^*$, $\text{her}(b)^*$, and $\text{her}(c)^*$ respectively. Assume that $(a, u_a) \leq (b, u_b)$ and $(b, u_b) \leq (c, u_c)$. By hypothesis, we know that $a \leq_{Cu} b$ and $b \leq_{Cu} c$. Since $A$ has stable rank one, there exist $x, y \in A$ such that $a = xx^*$, $b = yy^*$, $x^*x \in \text{her}(b)$ and $y^*y \in \text{her}(c)$. Let us consider the polar decompositions of $x$ and $y$. That is, $x = a^{1/2}a = b^{1/2}b$, for some partial isometries $\alpha, \beta$ of $A^{**}$. Using Paragraph 2.2 we get $p_a = a\alpha^* \sim p_z \; \alpha^* \alpha \leq p_b$ and also $p_b \sim p_z \; \beta^* \beta \leq p_c$. We set $q_a := \alpha^* \alpha$, $q_b := \beta^* \beta$. One can check that $\gamma := \alpha \beta$ is a partial isometry of $A^{**}$ and that $p_a = \gamma \gamma^*$.

Let us write $z := a^{1/2} \gamma$. Observe that $zz^* = a$ and also $z = x \beta$. We hence compute that $z^*z = \beta^* x^* x \beta \in \text{her}(c)$. We deduce that $zz^* = a$ and $z^*z \in \text{her}(c)$. By [5, Proposition 2.12] we may write $x := u'(x^* x)^{1/3}$ for some element $u$ of $A$. Since $(x^* x) \in A_{p_b}$ and $\beta^* A_{p_b} \subset A$, we deduce that $\beta^* x^*$ is in $A$, and hence $z \in A$.

Using Proposition 2.3 we obtain that $q_c := \gamma' \gamma$ is the support projection of $z^* z$ and is Peligrad-Zsidő equivalent to $p_{u'}$. Finally, Lemma 2.6 tells us that $\theta_{u'(x^* x)^{1/3}} = \theta_{bc} \circ \theta_{ab, \alpha}$ is one of the morphisms described in Paragraph 2.2, from which the transitivity of $\leq_1$ follows. \hfill $\Box$

3.2. Standard maps. We have seen that for any unitary element $u$ of $\text{her}(a)^*$, and any partial isometry $a \in A^+$ such that $p_u = a^\alpha a^*$ and $a^\alpha \leq p_u$, the $K_1$-class of $\theta_{ab, \alpha}(u)$ does not depend on the $a$ chosen. In the sequel, whenever $a \leq_{Cu} b$, we will refer to the maps $\theta_{ab, \alpha}$ as standard maps and will rewrite them as $\theta_{ab}$. In particular, whenever $a \leq b$ observe that the canonical inclusion map is a standard map. Also, notice that every standard morphism between $a$ and $b$ gives rise to the same group morphism at the $K_1$-level, that we will denote by $\chi_{ab}$. That is, $\chi_{ab} := K_1(\theta_{ab}) : K_1(\text{her}(a)) \rightarrow K_1(\text{her}(b))$.

3.3. The $\text{Cu}_1$-semigroup. We construct the unitary Cuntz semigroup of a stable rank one $C^*$-algebra in a similar fashion to the original Cuntz semigroup, using the $\leq_1$ relation and the standard maps.

Let $A$ be a $C^*$-algebra of stable rank one. By antisymmetrizing the $\leq_1$ relation, we define an equivalence relation $\sim$ on the set of pairs $(a, u)$ where $a \in (A \otimes \mathcal{K})_+$ and $u \in \mathcal{U}(\text{her}(a)^*)$. The equivalence relation $\sim$ is called the unitary Cuntz equivalence and we denote by $[(a, u)]$ the equivalence class of $(a, u)$. We construct the unitary Cuntz semigroup of $A$ as follows:

$$\text{Cu}_1(A) := \{(a, u) \in (A \otimes \mathcal{K})_+ : u \in \mathcal{U}(\text{her}(a)^*)]/\sim_1\}.$$ 

The set $\text{Cu}_1(A)$ naturally inherits a partial order induced by the relation $\leq_{1}$. More concretely, for any two $[(a, u)], [(b, v)] \in \text{Cu}_1(A)$, we say that $[(a, u)] \leq [(b, v)]$ if and only if $(a, u) \leq_1 (b, v)$.

The addition on $\text{Cu}_1(A)$ is defined component-wise and mimics the construction of the addition in $\text{Cu}(A)$: given any two elements $a, b \in (A \otimes \mathcal{K})_+$, we know that $a \oplus b := \psi(a \otimes 0)$ is a positive element of $A \otimes \mathcal{K}$, where $\psi : M_2(A \otimes \mathcal{K}) \rightarrow A \otimes \mathcal{K}$ (see Paragraph 2.1). Given unitary elements $u \in \mathcal{U}(\text{her}(a)^*)$ and $v \in \mathcal{U}(\text{her}(b)^*)$, we first observe that their respective scalar part can be assumed to be equal without loss of generality, since the equivalence relation $\sim_1$ identifies unitary elements up to homotopy equivalence. Besides, $\psi(\theta_{0, 0}(u \oplus v)) \in \text{her}(a \oplus b)$ and hence $u \oplus v := \psi(\theta_{0, 0}(u \oplus v))$ is a unitary element of $\text{her}(a \oplus b)^*$. We conclude that $[(a, u)] + [(b, v)] := [(a \oplus b, u \oplus v)]$ is a well-defined element in $\text{Cu}_1(A)$.

Therefore, we obtain a partially ordered monoid $(\text{Cu}_1(A), +, \leq)$ whose neutral element is $[(0_A, 1)]$ and the proof is left to the reader. By positive elements, we mean elements that are greater or equal to...
the neutral element. It is easy to see that the positive elements of \( Cu_1(A) \) are those of the form \((a, 1)\) for some \( a \in (A \otimes K)_+ \), and thus \( Cu_1(A) \) is in general not positively ordered. We now show that \((Cu_1(A), \leq)\) satisfies the order-theoretic axioms (O1)-(O4) mentioned in Paragraph 2.3.

**Proposition 3.2.** Let \( A \) be a \( C^* \)-algebra of stable rank one. Let \((a_n)\) be a sequence in \( A_+ \) such that \( a_n \leq Cu_m a_m, \) for any \( n \leq m. \) Let \( a \in A_+ \) be any representative of sup\([a_n]\) \( \in Cu(A) \) obtained from axiom (O1) and the stable rank one hypothesis. Then, for any unitary element \( u \in her(a)^- \), there exists a unitary element \( u_n \in her(a_n)^- \) for some \( n \in \mathbb{N} \) such that \([a_n, u_n] \leq [a, u]\) in \( Cu_1(A) \).

**Proof.** For any \( n \in \mathbb{N} \), consider \( b_n := (a - 1/n)_+ \). It is well-known that \((b_n)_n\) is a \( \ll \)-increasing sequence in \( Cu(A) \) whose supremum is \([a]\); see e.g. [22] Proposition 2.61. Also, it is not hard to check that \( AbGp - \lim K_1(her(b_n)), \chi_{b_nb_m}) = (K_1(her(a)), \chi_{b_nb_m}). \) Since we are in the category AbGp, for any \([a] \in K_1(her(a))\), we can find \( n \) and \([u_n] \in K_1(her(b_m))\) such that \( \chi_{b_nb_m}([u_n]) = [u]. \) Since \( A \) has stable rank one, then so does \( her(b_m)^- \). Hence using \( K_1\)-surjectivity (see [19] Theorem 2.10), we can find a unitary element \( u_n \) of \( her(b_m)^- \) whose \( K_1\)-class is \([u_n]\). On the other hand, \([a] \in K_1(her(a))\) is an increasing sequence in \( Cu(A) \) whose supremum is \([a]\) and hence there exists \( m \in \mathbb{N} \) such that \([b_n] \leq [a_m] \) in \( Cu(A) \). So we can consider the unitary element \( \theta_{a,a_m}(u_n) \) in \( her(a_m)^- \). By transitivity of \( \leq_1\), we obtain that \( \chi_{a,a_m}(\theta_{a,a_m}(u_n)) = \chi_{a,a_m}([u_n]) = \chi_{b,b_m}(1) = [u] \) and the result follows. \( \square \)

**Lemma 3.3.** Let \( A \) be a \( C^* \)-algebra of stable rank one. Then any increasing sequence \([([a_n, u_n])_{n \in \mathbb{N}}\) in \( Cu_1(A) \) has a supremum \([a, u]\) in \( Cu_1(A) \). In particular, \([a] = \sup[a_n]\) in \( Cu_1(A) \) and there exists \( n \in \mathbb{N} \) large enough such that \([u] = \chi_{a,a_m}([u_n]) \in K_1(her(a)).\)

**Proof.** Let \( ([a_n, u_n])_{n \in \mathbb{N}} \) be an increasing sequence in \( Cu_1(A) \). Then \( ([a_n])_{n \in \mathbb{N}} \) is an increasing sequence in \( Cu(A) \). By (O1) in \( Cu(A) \), the sequence \( ([a_n])_{n \in \mathbb{N}} \) has a supremum \([a]\) in \( Cu(A) \). Now, let \( n \leq m. \) Since \( ([a_n, u_n]) \leq ([a_m, u_m]) \), we get that \( \chi_{a,a_m}([u_n]) = [u_n]. \) By transitivity of \( \leq_1\), we obtain that \( \chi_{a,a_m}([u_m]) = \chi_{a,a_m}([u_n]) \in K_1(her(a)). \) Write \([u] := \chi_{a,a_m}([u_n])\). We deduce that \([a, u]\) \( \geq ([a_n, u_n]) \) in \( Cu_1(A) \) for any \( n \in \mathbb{N}. \)

Let us check that \([a, u]\) is in fact the supremum of the sequence \(([a_n, u_n])_{n \in \mathbb{N}}\). Let \([b, v] \in Cu_1(A) \) such that \([b, v] \geq ([a_n, u_n]) \) for every \( n \in \mathbb{N}. \) Since \([a] = \sup[a_n]\), we have \([b] \geq [a]\) in \( Cu(A) \). Using transitivity of \( \leq_1\), the following diagram is commutative:

```
\[
\begin{array}{ccc}
K_1(her(a)^-) & \xrightarrow{\chi_{a,a}} & K_1(her(b)^-) \\
\xrightarrow{\chi_{a,a_n}} & \ & \xrightarrow{\chi_{a,a_m}} \\
K_1(her(a_n)^-) & \xrightarrow{\chi_{a,a_n}} & K_1(her(b_m)^-) \\
\end{array}
\]
```

Hence for every \( n \) and \( m \) in \( \mathbb{N} \), we have \( \chi_{a,a_n}([u_n]) = \chi_{a,a_m}([u_n]) = \chi_{ab}([u]) \) in \( K_1(her(b)). \) We deduce that \( \chi_{ab}([u]) = [v] \) in \( K_1(her(b)) \) and hence \([a, u]\) \( \leq ([b, v]) \).

**Proposition 3.4.** Let \( A \) be a \( C^* \)-algebra of stable rank one and let \([a, u], [b, v] \in Cu_1(A) \). Then \([a, u] \ll [b, v]\) if and only if \([a] \ll [b]\) in \( Cu(A) \) and \( \chi_{ab}([u]) = [v] \) in \( K_1(her(b)). \)
Proof. Suppose that $[(a, u)] \ll [(b, v)]$. A fortiori $[(a, u)] \leq [(b, v)]$, so $\chi_{ab}[u] = [v]$. Now let $([c_n])_n$ be an increasing sequence in $\text{Cu}(A)$ whose supremum $[c]$ satisfies $[c] \geq [b]$. Write $w := \theta_{b,c}(v)$ and consider $s := [(c, w)] \in \text{Cu}_1(A)$. By [Proposition 3.2] we know that there exists $n \in \mathbb{N}$ and a unitary element $w_n$ of $\text{her}(c_n)^-$ such that $\chi_{c,e}([w_n]) = [w]$. Now define $s_k := [c_{n+k}, \theta_{b,c_k}(w_n)]$. Then $(s_k)_k$ is an increasing sequence in $\text{Cu}_1(A)$. By the description of supreme obtained in [Lemma 3.3] we know that $(s_k)_k$ admits $s$ as a supremum. Further, $s \geq [(b, v)]$ and since $[(a, u)] \ll [(b, v)]$, we deduce that there exists $k \in \mathbb{N}$ such that $[(a, u)] \leq s_k$ and hence that $[a] \leq [c_{a+k}]$. We conclude that $[a] \ll [b]$ in $\text{Cu}(A)$.

Conversely, let $[(a, u)], [(b, v)] \in \text{Cu}_1(A)$ such that $[a] \ll [b]$ in $\text{Cu}(A)$ and $\chi_{ab}[u] = [v]$ in $\text{K}_1(\text{her}(b))$. Let $((c_n, w_n))_n$ be an increasing sequence in $\text{Cu}_1(A)$ that has a supremum in $\text{Cu}_1(A)$, say $[(c, w)]$. Also suppose that $[(b, v)] \leq [(c, w)]$. First, by transitivity of $\leq_1$, observe that $\chi_{aw}([u]) = \chi_{bc} \circ \chi_{ab}([u]) = [w]$ in $\text{K}_1(\text{her}(c))$.

Arguing as in the proof of [8, Lemma 4.3], since $A$ has stable rank one, we can find a strictly decreasing sequence $(e_n)_n$ in $\mathbb{R}_+$ and unitary elements $(u_n)_n$ in $(A \otimes \mathcal{K})^-$ such that 

$$\text{her}(c_1 - e_1) \subseteq u_1(\text{her}(c_2 - e_2))u_1^* \subseteq \ldots \subseteq u_{n-1}(\text{her}(c_n - e_n))u_{n-1}^* \subseteq \ldots$$

and such that $\sup_n |(c_n - e_n)_n| = [c]$ in $\text{Cu}(A)$. Hence, by [Proposition 3.2] we can find $k \in \mathbb{N}$ and a unitary element $w_k^\prime$ of $\text{her}(c_k - e_k)^-$ such that $\chi_{(c_k - e_k)}(w_k^\prime) = [w_k]$ in $\text{K}_1(\text{her}(c_k))$. Now, using the same argument as in the proof of [Proposition 3.2] we observe that

$$\text{AbGp} - \lim(K_1(\text{her}(c_n - e_n)_n, \chi_{(c_n - e_n)_n})) = (K_1(\text{her}(c)), \chi_{(c_n - e_n)_n}) \in \text{K}_1(\text{her}(c))$$

On the other hand, since $[a] \ll [b] \leq \sup_n |(c_n - e_n)_n|$, there exists $l \in \mathbb{N}$ such that $[a] \leq [(c_l - e_l)_n]$ in $\text{Cu}(A)$. Without loss of generality, $l \geq k$. Using transitivity of $\leq_1$ again, we have that $\chi_{(c_l - e_l)_n}(w_l^\prime) = \chi_{(c_l - e_l)_n}(w_l) = [w] = \chi_{aw}([u]) = \chi_{bc} \circ \chi_{ab}([u])$ in $\text{K}_1(\text{her}(c))$. Since we are in the category $\text{AbGp}$, there exists some $l \geq l$ such that $\chi_{(c_l - e_l)_n}(w_l^\prime) = \chi_{aw}([u]) = \chi_{bc} \circ \chi_{ab}([u])$. Composing with $\chi_{(c_l - e_l)_n}$ on both sides, we finally obtain that $[w_l] = \chi_{aw}([u])$ and hence $[(a, u)] \leq [(c_l, w_l)]$, which completes the proof.

Corollary 3.5. Let $A$ be a $C^*$-algebra of stable rank one and let $[(a, u)] \in \text{Cu}_1(A)$. Then $[(a, u)]$ is compact if and only if $[a]$ is compact in $\text{Cu}(A)$.

Theorem 3.6. Let $A$ be a $C^*$-algebra of stable rank one. Then $(\text{Cu}_1(A), \leq)$ satisfies axioms (O1), (O2), (O3), and (O4).

Proof. (O1) follows from [Lemma 3.3].

(O2): Let $s := [(a, u)] \in \text{Cu}_1(A)$. We want to write $s$ as the supremum of a $\ll$-increasing sequence in $\text{Cu}_1(A)$. By (O2), we can find a $\ll$-increasing sequence $([a_n])_n$ in $\text{Cu}_1(A)$ such that $\sup_n [a_n] = [a]$. Let $a_0$ be any representative of $[a_0]$ in $(A \otimes K)_+$. Using [Proposition 3.2] we know that we can find a unitary element $u_n$ of $\text{her}(a_0)^-$ for some $n \in \mathbb{N}$ such that $[(a_n, u_n)] \leq [(a, u)]$. Now we consider $s_k := [(a_{n+k}, \theta_{a_n, a_k}(u_n))]$, for any $k \in \mathbb{N}$. Then, by [Proposition 3.4] we deduce that $(s_k)_k$ is a $\ll$-increasing sequence in $\text{Cu}_1(A)$. By the description of supreme obtained in [Lemma 3.3] $\sup_k s_k = s$. 

\[ \square \]
(O3): Let \([(a_1, u_1)] \ll [(b_1, v_1)] \text{ and } [(a_2, u_2)] \ll [(b_2, v_2)]\). We already know that \([(a_1, u_1)] + [(a_2, u_2)] \leq [(b_1, v_1)] + [(b_2, v_2)]\) and that \([a_1] + [a_2] \ll [b_1] + [b_2]\) in \(Cu(A)\). The conclusion follows from Proposition 3.4.

(O4): Let \([(a_n, u_n))]_{n \in \mathbb{N}}\) and \([(b_n, v_n))]_{n \in \mathbb{N}}\) be two increasing sequences in \(Cu_1(A)\). Let \([(a, u)] := sup [(a_n, u_n)] \text{ and } [(b, v)] := sup [(b_n, v_n)]\). Now we define \([(c_n, w_n)] := [(a_n, u_n)] + [(b_n, v_n)]\) for any \(n \in \mathbb{N}\). Since \([c_n] = [a_n] + [b_n]\) in \(Cu(A)\) and \(Cu(A)\) satisfies (O4), we have \(sup [c_n] = [a \oplus b]\). Also, we know that \(\chi_{a_n, b_n}([u_n]) = [u]\) and \(\chi_{b_n, b_n}([v_n]) = [v]\), and hence we obtain \(\chi_{c_n, c_n}([u_n] \oplus [v_n]) = [u \oplus v]\). We conclude that \(sup\) and addition are compatible in \(Cu_1(A)\), using Lemma 3.3.

3.4. The \(Cu_1\)-semigroup as a functor. From now on, let us write \(C_{sr1}^\sim\) to denote the category of \(C^*\)-algebras of stable rank one. Also, we denote by Mon\(_\leq\) the category of ordered monoids, in contrast to the category of positively ordered monoids, that we write PoM. Finally, the category of monoids is denoted by Mon. We have just proved that \(Cu_1(A)\) is a semigroup satisfying the axioms (O1)-(O4). The aim is to define a functor \(Cu_1\) from the category \(C_{sr1}^\sim\) to a suitable category of semigroups as was done for the \(Cu\)-semigroup; see [3, Chapter 3], [11]. Since \(Cu_1(A)\) is usually not positively ordered, we need to adjust the definition of the codomain category. In the sequel, we show that \(Cu_1: C_{sr1}^\sim \longrightarrow Cu^\sim\) is a well-defined functor that is continuous.

Definition 3.7. The unitary Cuntz category, written \(Cu^\sim\) is the subcategory of Mon\(_\leq\) whose objects are ordered monoids satisfying the axioms (O1)-(O4) and such that \(0 \ll 0\). Morphisms in \(Cu^\sim\) are Mon\(_\leq\)-morphisms that respect suprema of increasing sequences and the compact-containment relation.

Definition 3.8. Let \(M \in\) Mon\(_\leq\) and let \(S \in Cu^\sim\). We define their positive cones, that we write \(M_+\) and \(S_+\) respectively, as the subset of positive elements. Observe that \(M_+ \in\) PoM and \(S_+ \in Cu\).

Lemma 3.9. The category \(Cu\) (respectively PoM) is a coreflective subcategory of \(Cu^\sim\) (respectively Mon\(_\leq\)). More precisely, the assignment \(S \longrightarrow S_+\) defines a coreflector \(\nu_+ : Cu^\sim \longrightarrow Cu\).

Proof. Since \(Cu^\sim\)-morphisms respect \(\leq\), we deduce that \(\nu_+\) is a well-defined functor. Moreover, one can check that \(\text{Hom}_{Cu^\sim}((S, T), T) = \text{Hom}_{Cu}(S, \nu_+(T))\) for any \(S \in Cu\) and \(T \in Cu^\sim\). We get that the inclusion functor \(i : Cu \hookrightarrow Cu^\sim\) is left adjoint to \(\nu_+\), which implies that \(Cu\) is a full (obviously faithful) coreflective subcategory of \(Cu^\sim\).

Proposition 3.10. Let \(\varphi : A \longrightarrow B\) a \(C^*\)-homomorphism between \(C^*\)-algebras \(A\), \(B\) of stable rank one. We denote by \(\varphi^\sim\) the unitized morphism between \((A \otimes K)^\sim \longrightarrow (B \otimes K)^\sim\). Then:

\[Cu_1(\varphi) : Cu_1(A) \longrightarrow Cu_1(B)\]

\[[(a, u)] \longmapsto [(\varphi(a), \varphi^\sim(u))]\]

is a \(Cu^\sim\)-morphism.

Proof. Let \(a \in A \otimes K\). The restriction \(\varphi|_{\text{her}(a)} : \text{her}(a) \longrightarrow \text{her}(\varphi(a))\) of \(\varphi\) gives us the following commutative square:

\[\begin{array}{ccc}
\text{her}(a) & \xrightarrow{\varphi} & \text{her}(\varphi(a)) \\
\downarrow & & \downarrow \\
\text{her}(a)^\sim & \xrightarrow{\varphi^\sim} & \text{her}(\varphi(a))^\sim
\end{array}\]
Hence, \( \varphi^-(u) \) is a unitary element of \((\text{her}(\varphi(a))^\sim, \varphi^-(u)) \) and we deduce that \( [(\varphi(a), \varphi^-(u))] \in \mathcal{C}_1(B) \). Let us check it does not depend on the representative \((a, u)\) chosen. Let \([[(a, u)], [[b, v]] \in \mathcal{C}_1(A)\) such that \( [(a, u)] \leq [[b, v]] \). Then we get \( a \leq b \) in \( A \otimes K \). Since \( \varphi \) is a \(*\)-homomorphism, we deduce that \( \varphi(a) \leq \varphi(b) \) in \( B \otimes K \). Further, if \( \alpha \) is a partial isometry of \((A \otimes K)^\ast\) that realizes one of our standard morphisms \( \theta_{\alpha, b} \) (see \textit{Paragraph 3.2}) between \text{her}(a) and \text{her}(b), then \( \varphi^*(\alpha) \) is a partial isometry of \((B \otimes K)^\ast\) that realizes \( \theta_{\varphi^*(\alpha), b} \) between \text{her}(\varphi(a)) and \text{her}(\varphi(b)), since \( \varphi^* \) is a \(*\)-homomorphism. We get that the following diagram is commutative:

\[
\begin{array}{ccc}
\text{her}(a) & \xrightarrow{\theta_{a,b}} & \text{her}(b) \\
\varphi \downarrow & & \varphi' \downarrow \\
(\text{her}(\varphi(a))^\sim) & \xrightarrow{\theta_{\varphi(a), b}} & (\text{her}(\varphi(b))^\sim)
\end{array}
\]

from which we deduce that \( \theta_{\varphi(a), b}(\varphi^- (u)) \sim \varphi^- (v) \) and thus \( [(\varphi(a), \varphi^- (u))] \leq [(\varphi(b), \varphi^- (v))] \). So \( \mathcal{C}_1(\varphi) \) is indeed well-defined, respects \( \leq \) and it is easy to check that \( \mathcal{C}_1(\varphi) \) also respects addition. We conclude that \( \mathcal{C}_1(\varphi) \) is a Mon\(_{\mathcal{L}}\)-morphism. By \textit{Proposition 3.4}, \( \mathcal{C}_1(\varphi) \) preserves the compact containment relation. Finally, we leave to the reader to check that \( \mathcal{C}_1(\varphi) \) preserves suprema of increasing sequences. \( \square \)

**Corollary 3.11.** The assignment \( A \mapsto \mathcal{C}_1(A) \) from \( C^*_\text{tr} \) to \( \mathcal{C}_1^{\ast} \) is a functor.

It has been shown that the functor \( \mathcal{C}_1 \) from the category of \( C^\ast \)-algebras to \( \mathcal{C}_1 \) is continuous (\textit{Corollary 3.2.9}), generalizing the result of \textit{Theorem 2} that established sequential continuity. We shall expect a similar result for the functor \( \mathcal{C}_1 \). In the sequel, we shall prove that \( \mathcal{C}_1 : C^*_\text{tr} \rightarrow \mathcal{C}_1^{\ast} \) is a continuous functor, using a process analogous to that in \textit{Chapter 2 and 3} and \textit{Section 2.2} for the Cuntz semigroup.

To do so, we are going to consider a pre-completed version of \( \mathcal{C}_1 \), that we will denote by \( \mathcal{W}_1 \), to then extend the result to \( \mathcal{C}_1 \) using Category Theory techniques. We first introduce an analogous category to \( \mathcal{W} \) defined in \textit{Definition 2.5} that we shall call \( \mathcal{W}^\ast \). The main difference of our context lies in the fact that binary/auxiliary relations considered need not be positive, and similarly, the underlying ordered monoids involved are not necessarily positively ordered. Still, most of the proofs from \textit{3} and \textit{4} remain valid. (We give additional details when needed.)

### 3.5. The category \( \mathcal{W}^\ast \)

Let \( S \in \text{Mon} \) and consider a transitive binary relation \( \prec \) on \( S \). (Again, we do not require \( \prec \) to be positive, in the sense that there may exist \( s \in S \) such that \( 0 \not\prec s \).) For any \( s \in S \) we denote \( s_\prec := \{ s' \in S \mid s' \prec s \} \). Let us recall the \( \mathcal{W} \)-axioms from \textit{Definition 2.2}:

- \((W1)\): For any \( s \in S \), there exists a \( \prec \)-increasing sequence \( (s_k)_k \) in \( s_\prec \) such that for any \( s' \in s_\prec \), there exists some \( k \) such that \( s' \prec s_k \).
- \((W3)\): Addition and \( \prec \) are compatible.
- \((W4)\): For any \( s, t, x \in S \) such that \( x < s + t \), we can find \( s', t' \in S \) such that \( s' < s, t' < t \) and \( x < s' + t' \).

A \( \mathcal{W}^\ast \)-\textit{semigroup} is a pair \((S, \prec)\), where \( S \in \text{Mon} \) and \( \prec \) is a transitive binary relation (not necessarily positive) on \( S \) such that \((S, \prec)\) satisfies axioms \((W1)-(W3)-(W4)\) and such that \( 0 < 0 \).

A \( \mathcal{W}^\ast \)-\textit{morphism} between any two \( S, T \in \mathcal{W} \) is a Mon-morphism \( g : S \rightarrow T \) that respects the transitive binary relation and satisfies the following \( \mathcal{W}^\ast \)-\textit{continuity axiom}:

- \((M)\): For any \( s \in S \) and \( t \in T \) such that \( t < g(s) \), there exists \( s' \in s_\prec \) such that \( t < g(s') \).
The category $W^-$ has inductive limits. More precisely, let $(S_i, \varphi_{ij})_{i,j}$ be an inductive system in $W^-$ and let $S := \text{Mon} - \text{lim}(S_i, \varphi_{ij})$. Then $(S, \prec) \approx W^- - \text{lim}(S_i, \varphi_{ij})$, where $\prec$ is the following transitive binary relation on $S$: $s \prec t$ in $S$ if $\varphi_k(s_i) \prec \varphi_k(t_j)$, where $s_i \in S_i$, $t_j \in S_j$ are representatives of $s, t$ respectively and $k \geq i, j$.

Now that we have a well-defined categorical setup, we define a pre-completed version of $\text{Cu}_1$ and show that it is continuous. More precisely, we build a functor from the category $C^*_\text{loc}$ of local $C^*$-algebras to the category $W^-$, termed $W_1$. See [3] §2.2] for more details.

3.6. Local $C^*$-algebras. A local $C^*$-algebra $A$ is an upward-directed union of $C^*$-algebras. That is, $A = \bigcup A_i$, where $\{A_i\}_i$ is a family of complete $^*$-invariant subalgebras such that for any $i, j$, there exists $k \geq i, j$ such that $A_i \subseteq A_j \subseteq A_k$.

If $A$ is a local $C^*$-algebra, then so is $M_k(A)$ for any $k \in \mathbb{N}$. In fact, $M_k(A)$ sits as upper-left corner inside $M_k'(A)$ for any $k' \geq k$ and we can picture any $M_k(A)$ as a corner of $M_\infty(A) := \bigcup_k M_k(A)$, which is again a local $C^*$-algebra. Observe that the completion of a local $C^*$-algebra $A$, that we write $\overline{A}$, is a $C^*$-algebra. In particular, we have $\overline{M_k(A)} \cong M_k(\overline{A})$ for any $k \in \mathbb{N}$ and $\overline{M_\infty(A)} \cong \overline{A} \otimes \mathcal{K}$. Further $A$ is closed under functional calculus. Moreover, for any local $C^*$-algebra $A := \bigcup A_i$, if each $A_i$ has stable rank one, then by [18] Theorem 5.1, we get that $\overline{A}$ has stable rank one. We may abuse the language and say that $A$ has stable rank one.

We now consider $C^*_\text{loc}$, the category whose objects are local $C^*$-algebras and morphisms are $^*$-homomorphisms. Obviously, $C^*$ is a full subcategory of $C^*_\text{loc}$. In fact, $C^*$ is a reflective subcategory of $C^*_\text{loc}$ and the assignment $A \mapsto \overline{A}$ defines a reflector from $C^*_\text{loc}$ to $C^*$ that we denote by $\gamma$. As for $C^*$-algebras, we denote $C^*_\text{loc,sr1}$ the full subcategory of $C^*_\text{loc}$ consisting of local $C^*$-algebras whose completion have stable rank one.

Finally, let $(A_i, \varphi_{ij})_{i,j}$ be an inductive system in $C^*_\text{loc}$. As in [3] §2.2.8, we consider the algebraic inductive limit $A_{\text{alg}} := \bigcup_i A_i/\sim$ with the pre-norm: $\|x\| := \inf \{\|\varphi_j(x)\| \}$, for $x \in A_i$ and we define:

$$C^*_\text{loc} - \text{lim}(A_i, \varphi_{ij}) := (A_{\text{alg}}/N, \|\|)$$

where $N := \{a \in A_{\text{alg}} \mid \|a\| = 0\}$. Besides, $\varphi_{ij}$ induces a $^*$-homomorphism that we also write $\varphi_{ij} : M_\infty(A_i) \longrightarrow M_\infty(A_j)$ and we have $C^*_\text{loc} - \text{lim}(M_\infty(A_i), \varphi_{ij}) \approx M_\infty(C^*_\text{loc} - \text{lim}(A_i, \varphi_{ij}))$. See [3] §2.2.8.

3.7. The precompleted unitary Cuntz semigroup. We briefly recall the definition of the precompleted Cuntz semigroup $W(A)$ of a $C^*$-algebra $A$ and we refer the reader to [3] §2.2 for details. In fact, we give an equivalent definition that can be found in [3] Remark 3.2.4; see also [3] Lemma 3.2.7.

Let $A \in C^*_\text{loc}$. We define $W(A) := \{[a] \in \text{Cu}(\overline{A}) \mid a \in M_\infty(A)\}$. Obviously, $(W(A), +) \in \text{Mon}$ as a submonoid of $\text{Cu}(\overline{A})$. Given $[a], [b] \in W(A)$, we write $[a] < [b]$ if $a \preceq (b - \epsilon)_+$ in $M_\infty(A)$ for some $\epsilon > 0$. This defines a (positive) transitive binary relation on $W(A)$, hence we have that $(W(A), <) \in W$.

(See [3] Proposition 2.2.5) and [4] Section 2.2.)

**Lemma 3.12.** Let $A \in C^*_\text{loc}$ and let $B := \overline{A}$ be its completion in $C^*$. Then, for any $a \in A$, we have $aAa = aBa$. 


\textbf{Proof.} The inclusion $\subseteq$ is trivial. Now let $x \in a\overline{B}a$. Then there exists a sequence $(b_k)_k$ in $B$ such that $x = \lim a b_k a$. Furthermore, for any $k \in \mathbb{N}$, there exists a sequence $(a_{k,i})_i$ in $A$ such that $b_k = \lim_i a_{k,i}$. We deduce that $x = \lim_k (\lim_i a_{k,i}) a = \lim_i (a a b_k a a)$. Thus $x \in a\overline{A}a$. \hfill $\square$

**Definition 3.13.** Let $A \in C^*_{loc}$ and let $B := \overline{A}$ be its completion as a $C^*$-algebra. For $a \in A_+$, we define the hereditary subalgebra generated by $a$ as her$(a) := a\overline{B}a$.

We have now all the tools to define a precompleted version of $Cu_1$ that we will denote by $W_1(A)$, as a submonoid of $Cu_1(\overline{A})$.

**Definition 3.14.** Let $A \in C^*_{loc, str}$. We define $W_1(A) := \{(a, u) \in Cu_1(\overline{A}) \mid a \in M_{\infty}(A_+)\}$. Obviously, $(W_1(A), +) \in \text{Mon}$ as a submonoid of $Cu_1(\overline{A})$. We now equip $W_1(A)$ with the following binary relation: for any two $[(a, u)], [(b, v)]$ in $W_1(A)$, we say $[(a, u)] < [(b, v)]$ if:

\[
\begin{align*}
& a \leq_{Cu} (b - \epsilon)_+ \text{ in } M_{sr}(A), \text{ for some } \epsilon > 0, \\
& [\theta_{\text{loc}}(u)] = [v] \text{ in } K_1(\text{her}(b)^\sim).
\end{align*}
\]

**Proposition 3.15.** Let $A \in C^*_{loc, str}$. Let $a \in A_+$ and let $(a_n)_n$ be a sequence in $A_+$ such that $[(a_n)]_n$ is a $\prec$-increasing cofinal sequence in $[a]_+$ (obtained from $(W_1)$ applied to $[a]$.)

For any unitary element $u \in \text{her}(a)^\sim$, there exists $n \in \mathbb{N}$ and a unitary element $u_n \in \text{her}(a_n)^\sim$ such that 

\[ [(a_n, u_n)] < [(a, u)] \text{ in } W_1(A). \]

**Proof.** Combine the fact that $\overline{A}$ has stable rank one, with Definition 3.13 and the result follows from Proposition 3.2. \hfill $\square$

**Proposition 3.16.** (cf \cite[Proposition 2.2.5]{3}). Let $A \in C^*_{loc, str}$. The relation defined in Definition 3.14 is a transitive binary relation and $(W_1(A), \prec)$ satisfies axioms (W1), (W3) and (W4). That is, $(W_1(A), \prec) \in W^-$. We may omit the reference to $<$ and simply write $W_1(A) \in W^-$. \hfill $\square$

**Proof.** Let us check that $\prec$ is transitive. If $[(a, u)] < [(b, v)] < [(c, w)]$, then we have $\chi_{\infty}([u]) = [w]$ and we also know that $a \leq_{Cu} (b - \epsilon)_+$, $b \leq_{Cu} (c - \epsilon')_+$, for some $\epsilon > 0$. We conclude that $[(a, u)] < [(c, w)]$.

If $[(b, v)] < [(a, u)]$, then, by Proposition 3.3 we have $[(b, v)] \ll [(a, u)]$ in $Cu_1(A)$ and thus $[(b, v)] < [(a - 1/n)_+, u_n]$ for some $n \in \mathbb{N}$. Hence (W1) holds. To check (W3) and (W4) is routine. \hfill $\square$

**Proposition 3.17.** Let $\varphi : A \rightarrow B$ be a $\ast$-homomorphism between $A, B \in C^*_{loc, str}$, and denote by $\varphi$ its extension to $M_{\infty}(A)$. We write $\varphi := \gamma(\varphi)$ and $\overline{\varphi} : M_{\infty}(A) \rightarrow M_{\infty}(B)$ its unitization. Then the map:

\[ W_1(\varphi) : W_1(A) \rightarrow W_1(B), \]

\[ [(a, u)] \mapsto [(\varphi(a), \overline{\varphi}(u))] \]

is a $W^-$-morphism.

**Proof.** Using the same argument as in Proposition 3.10 we easily deduce that $W_1(\varphi)$ is a Mon-morphism that respects $\ll$. Further, we have to check that $W_1(\varphi)$ satisfies the $W^-$-continuity axiom (see Paragraph 3.5).

Let us write $f := W_1(\varphi)$. Let $x := [(a, u)] \in W_1(A)$ and $y := [(b, v)] \in W_1(B)$ such that $y < f(x)$. We have to find $x' \in W_1(A)$ such that $x' < x$ and $y < f(x')$. 

Observe that \(\{(a - 1/n)_n\}_n\) is one of the \(\prec\)-increasing sequences obtained from axiom (W1) applied to \([a]\) in \(W(A)\). Thus, by Proposition 3.15, we can find some \(n \in \mathbb{N}\) and a unitary element \(u_n \in \text{her}(a - 1/n)_n^{-}\) such that \([((a - 1/n)_n, u_n)] < [(a, u)]\) in \(W_1(A)\). Similarly, \([((\varphi(a) - 1/k)_k)_k\) is one of the \(\prec\)-increasing sequences obtained from (W1) applied to \([\varphi(a)]\) in \(W(B)\). Therefore, there exists \(k \in \mathbb{N}\) such that \([b] < [((\varphi(a) - 1/k)_k)_k]\) in \(W(B)\). We deduce that there exists \(m \in \mathbb{N}\) large enough \((m \geq k, n)\) such that:

\[
[b] < [((\varphi(a) - 1/m)_m)_m] \text{ in } W(B).
\]

By transitivity of \(\leq_1\), we obtain:

\[
[\theta_{\varphi(a)}(\varphi(a)) \circ \theta_{\varphi(a)}^{-1/m}_m(v)] = [\theta_{\varphi(a)}^{-1/m}_m, \varphi(a) \circ \theta_{\varphi(a)}(\varphi(a))^{-1/m}_m, (\varphi(u))_m] \text{ in } K_1(\text{her}(\varphi(a))).
\]

Finally, since \(\text{AbGp}_j - \text{lim}_i (K_1(\text{her}(\varphi(a) - 1/m)_m))_m, \chi_{\varphi(a) - 1/m}_m, \chi_{\varphi(a) - 1/m}_m \times (K_1(\text{her}(a)), \chi_{\varphi(a) - 1/m}, \varphi(a)), \text{ we conclude that there exists } l \geq m \text{ such that:}
\]

\[
[\theta_{\varphi(a) - 1/l}_l(v)] = [\theta_{\varphi(a) - 1/l}_l, \varphi(a) \circ \theta_{\varphi(a) - 1/l}_l, (\varphi(u))_l] \text{ in } K_1(\text{her}(\varphi(a) - 1/l)_l).
\]

Write \(x' := [(a - 1/l)_l, \theta_{a - 1/n}_n, (a - 1/l)_l, (u_n)]\). Then we already know that \(x' < x\) in \(W_1(A)\) and the above exactly states that \(y < f(x')\) in \(W_1(B)\).

\[\square\]

**Corollary 3.18.** The assignment \(A \mapsto W_1(A)\) from \(C^*_{\text{loc}, r_1}\) to \(W^-\) is a functor.

**Theorem 3.19.** The functor \(W_1 : C^*_{\text{loc}, r_1} \rightarrow W^-\) is continuous.

**Proof.** This proof is an adapted version of [3] Theorem 2.2.9 and [4] Theorem 2.9. Let \((A_i, \varphi_{ij})_{ij}\) be an inductive system in \(C^*_{\text{loc}, r_1}\) and let \((A_{alg}/N, \varphi_{i alg}/N)\) be its inductive limit. Without loss of generality, we can suppose that each \(A_i \cong M_{\infty}(A_i)\); see Paragraph 3.6. Thus, we may suppose that each element of \(W(A_i)\) is realized by a positive element of \(A_i\).

Let \(\sigma_{ij} := W_1(\varphi_{ij})\) and consider the inductive system \((W_1(A_i), \psi_{ij})_{ij}\) in \(W^-\). We denote by \((S, \sigma_{i alg})\) its inductive limit in \(W^-\). Observe that \((W_1(A_{alg}/N), W_1(\varphi_{i alg}/N))\) is a co-cone for the inductive system. Hence from universal properties, we deduce that there exists a unique \(W^-\)-morphism \(w_1 : S \rightarrow W_1(A_{alg}/N)\) such that for all \(i, j \in I\) with \(i \leq j\), the following diagram commutes:

To complete the proof, let us show that \(w_1\) is a \(W^-\)-isomorphism. First, we start to show that \(w_1\) is surjective. Let \([\varphi(a)] \in W_1(A_{alg}/N)\). Since \(a \in A_{alg}/N\), we know that there exists \(a_k \in (A_{alg})_+\) such that \(\varphi_{i alg}(a_k) = a\). Also, \(u\) is a unitary element of \(\text{her}(a)^{-} = \varphi_{i alg}(A_{alg}/N)\varphi_{k alg}(a_k)\). Now, observe that
$C^* - \lim \phi_k_j(a_k) \overset{\cong}{\to} (\text{her}(a), \text{her}(b))$. Hence for any $\epsilon > 0$, there exists $j \geq k$ and a unitary element $u_j$ of her $\varphi_k_j(a_k)$ such that $[u - \varphi_k_j(u)] < \epsilon$. In particular, for $\epsilon < 2$, we obtain a unitary element $u_j$ of her $\varphi_k_j(a_k)$ such that $[u] = [\varphi_k_j(u)]$ in $K_1(\text{her}(a))$. We compute that $W_1(\varphi_j)([(\varphi_k_j(a_k), u_j)]) = [(\varphi_k_j(a_k), \varphi_j(u))].$

Thus, by the commutativity of the diagram above we obtain

$$w_1 \circ \sigma_j(\varphi_k_j(a_k), u_j) = W_1(\varphi_j)([(\varphi_k_j(a_k), u_j)]) = [(a, u)]$$

as desired. We conclude that $w_1$ is surjective.

Finally, let us show that $w_1$ is injective. Let $s, t \in S$ such that $w_1(s) = w_1(t)$. Since the inductive limit $S$ is algebraic, there exists some $k \in \mathbb{N}$ and $s_k, t_k$ in $W_1(A_k)$ such that $\sigma_{k_0}(s_k) = s$ and $\sigma_{k_0}(t_k) = t$.

Now choose $a, b \in \{A_k\}_\omega$ and unitary elements $a, v$ in the respective hereditary subalgebras such that $s_k = [(a, u)]$ and $t_k = [(b, v)]$. We know that $w_1(s) = w_1(t)$ and using the commutativity of the above diagram, we deduce that

$$\left\{ \begin{array}{l}
[\varphi_{k_0}(a)] = [\varphi_{k_0}(b)] \in W(A_{alg}/N), \\
[\theta_{k_0}(\varphi_{k_0}(a), v)] = [\varphi_{k_0}(\varphi_{k_0}(u))] \in K_1(\text{her}(\varphi_{k_0}(b))).
\end{array} \right.$$  

Again, since the inductive limits are algebraic, we conclude that there exists $i \geq k$ such that:

$$\left\{ \begin{array}{l}
[\varphi_{i}(a)] = [\varphi_{i}(b)] \in W(A_i), \\
[\theta_{i}(\varphi_{i}(a), v)] = [\varphi_{i}(\varphi_{i}(u))] \in K_1(\text{her}(\varphi_{i}(b))).
\end{array} \right.$$  

We conclude that $\sigma_{k}(s_k) = \sigma_{k}(t_k)$ for some $i \geq k$. Thus $s = t$, which ends the proof.  

3.8. **Continuity of the functor $C^*_1$.** We now have all the tools to conclude that $C^*_1 : C^\sim \longrightarrow W^\sim$ is a continuous functor, using the same techniques as in [3] Chapter 3. First of all, using a similar argument as in [3] Proposition 3.1.6], we easily deduce the following:

Let $(S, \prec)$ be a $W^\sim$-semigroup. Then there exists a $C^\sim$-semigroup $\gamma^\sim(S)$ together with a $W^\sim$-morphism $\alpha_S : S \longrightarrow \gamma^\sim(S)$ satisfying the following conditions:

(i) The morphism $\alpha_S$ is an ‘$\prec$-embedding’ in the sense that $s' \prec s$ whenever $\alpha(s') \ll \alpha(s)$.

(ii) The morphism $\alpha_S$ has a ‘dense image’ in the sense that for any two $t', t \in \gamma^\sim(S)$ such that $t' \ll t$ there exists $s \in S$ such that $t' \leq \alpha(s) \leq t$.

Note that the construction of such a completion is similar in every way except that we do not impose the transitive binary relation on $S$ to be positive. This implies that the ordered monoid obtained respects the axioms (O1)-(O4) but need not be positively ordered, whence $\gamma^\sim(S)$ belongs to $C^\sim$ instead of $C^\sim$. Again, arguing as in [3] Theorem 3.1.8], we deduce that $C^\sim$ is a (full) reflective subcategory of $W^\sim$ with reflector $\gamma^\sim$. In particular, $C^\sim$ has inductive limits. Finally, observe that for any $A \in C^\sim_{\text{st}}$, the compact-containment relation on $Cu_1(A)$ and the $\prec$ relation on $W_1(A \otimes K)$ agree; see [3] Remark 3.2.4]. Thus, we have that $Cu_1(A) = W_1(A \otimes K)$ as $C^\sim$-semigroups.

**Theorem 3.20.** There exists a natural isomorphism $\gamma^\sim \circ W_1 \simeq Cu_1 \circ \gamma$, where $\gamma$ is the reflector from $C^\sim_{\text{loc.st}}$ to $C^\sim_{\text{tr}}$ defined in [Paragraph 3.6]. In particular, for any $C^\sim$-algebra $A$ of stable rank one, there is a (natural) $C^\sim$-isomorphism between $Cu_1(A) \simeq \gamma^\sim(W_1(A))$. 
Proof. The aim of the proof is to show that $$(\text{Cu}_i(y(A)), W_i(i))$$ is a $\text{Cu}^-$-completion of $W_i(A)$ for any $A \in C_{\text{loc}, r}^*$, where $W_i(i)$ is built as follows:

Let $A \in C_{\text{loc}, r}^*$, write $B := M_\infty(A) \subset C_{\text{loc}, r}^*$. Consider the canonical inclusion $i : B \hookrightarrow \overline{B} \cong \overline{A} \otimes \mathcal{K}$. Then $i$ induces a $W^*$-morphism $W_i(i) : W_i(B) \longrightarrow W_i(\overline{B})$. On the other hand, we know that $W_i(\overline{B}) = W_i(A)$ and that $W_i(\overline{B}) \cong \text{Cu}_i(\overline{A})$. Thus, we obtain a $W^*$-morphism $W(i) : W_i(A) \longrightarrow \text{Cu}_i(\overline{A})$ (we use the same notation). By the argument in [3, Theorem 3.1.8], we only have to check that $W_i(i)$ is an $\prec$-embedding and that it has a dense image.

Let $s, s' \in W_i(A)$ such that $W_i(i)(s') \ll W_i(i)(s')$. We deduce that $W_i(i)(s') \prec W_i(i)(s')$. Also, observe that $W_i(i)$ is in fact an order embedding (even more, it is the canonical injection). Thus, we conclude that $s \prec s'$ and hence $W_i(i)$ is an ‘$\prec$-embedding’.

Let $t, t' \in \text{Cu}_i(y(A))$ such that $t' \ll t$. Now pick $a, a' \in (\gamma(A) \otimes \mathcal{K})_+$ and unitary elements $u, u'$ in the respective hereditary subalgebras of $a, a'$, such that $t := [(a, u)]$ and $t' := [(a', u')]$. Then, we know that $[a'] \ll [a]$ in $\text{Cu}(\overline{A})$ and that $\gamma_{a,a'}([u')] = [u]$. Using the argument in [3, Lemma 3.2.7], there exists $b \in M_\infty(A)$, such that $[a'] \leq [b] \leq [a]$ in $\text{Cu}(\overline{A})$. Now consider $s := [(b, \delta_{\text{ub}}(u))] \in W_i(A)$ and we get that $t' \leq W_i(i)(s) \leq t$ in $\text{Cu}(\overline{A})$. It follows that $W_i(i)$ has a ‘dense image’ and hence that $(W_i(i), \text{Cu}_i(y(A)))$ is a $\text{Cu}^-$-completion of $W_i(A)$. □

Corollary 3.21. The functor $\text{Cu}_i : C_{\text{loc}}^* \longrightarrow \text{Cu}^-$ is continuous. More precisely, given an inductive system $(A_i, \phi_{ij})_{i,j} \in C_{\text{loc}}^*$, then:

$$
\text{Cu}^- \lim (\text{Cu}_i(A_i), \text{Cu}_i(\phi_{ij})) \cong \text{Cu}_i(\text{Cu}^- \lim (A_i, \phi_{ij})) \cong \gamma^-(\text{Cu}^- \lim (W_i(A), W_i(\phi_{ij}))).
$$

3.9. Algebraic $\text{Cu}^-$-semigroups and $\text{Mon}_{\infty}$-completion. In this last subsection, we will briefly introduce algebraic $\text{Cu}^-$-semigroups in order to link the notion of real rank zero for a $C^*$-algebra $A$ of stable rank one, that ensures an abundance of projections, with the notion of ‘density’ of compact elements in $\text{Cu}_i(A)$. In fact, as compact elements of $\text{Cu}_i(A)$ are entirely determined by the ones of its positive cone $\text{Cu}_i(A)$ (see Corollary 3.5), all results from $\text{Cu}_i(A)$ will apply here. These can be found in [3, §5.5].

Let $S \in \text{Cu}^-$. We denote by $S_c := \{s \in S : s \ll s\}$. It is easily shown that $S_c \subset \text{Mon}_{\infty}$ and that for any $\text{Cu}^-$-morphism $f : S \longrightarrow T$ between $S, T \in \text{Cu}^-$, we have $f(S_c) \subset T_c$. Thus, $f$ induces a $\text{Mon}_{\infty}$-morphism $f_c := f|_{S_c} : S_c \longrightarrow T_c$. Hence, unlike $\nu_+$ that recovers the positive cone of a $\text{Cu}^-$-semigroup (see Lemma 3.9), we obtain a functor $\nu_c$ that recovers the compact elements of a $\text{Cu}^-$-semigroup:

$$
\nu_c : \text{Cu}^- \longrightarrow \text{Mon}_{\infty}
$$

$S \mapsto S_c$

$\nu_c : \text{Cu}^- \longrightarrow \text{Mon}_{\infty}$

$\nu_c : \text{Cu}^- \longrightarrow \text{Mon}_{\infty}$

$\nu_c : \text{Cu}^- \longrightarrow \text{Mon}_{\infty}$

Conversely, let $M \in \text{Mon}_{\infty}$. Then, $\ll$ is a natural transitive binary relation on $M$ such that $(M, \ll) \in \text{W}^-$. We denote $\text{Cu}^-(M) := \gamma^-(M, \ll)$ the $\text{Cu}^-$-completion of $(M, \ll)$. Any $\text{Mon}_{\infty}$-morphism $f : M \longrightarrow N$ between $M, N \in \text{Mon}_{\infty}$ induces a $\text{Cu}^-$-morphism $\gamma^-(f) : \gamma^-(M) \longrightarrow \gamma^-(N)$. Thus we obtain a functor:

$$
\text{Cu}^- : \text{Mon}_{\infty} \longrightarrow \text{Cu}^-
$$

$M \mapsto \text{Cu}^-(M)$

$f \mapsto \gamma^-(f)$
Definition 3.22. Let $S \in \mathbb{C}^\kappa$. We say that $S$ is an algebraic $\mathbb{C}^\kappa$-semigroup if every element in $S$ is the supremum of an increasing sequence of compact elements, that is, an increasing sequence in $S_c$. We denote by $\mathbb{C}^\kappa_{\text{alg}}$ the full subcategory of $\mathbb{C}^\kappa$ consisting of algebraic $\mathbb{C}^\kappa$-semigroups (see [3] §5.5).

Proposition 3.23. (cf [3] Proposition 5.5.4)
(i) Let $M \in \text{Mon}_\kappa$. Then $\mathbb{C}^\kappa(M)$ is an algebraic $\mathbb{C}^\kappa$-semigroup and, moreover, there is a natural identification between $M$ and the ordered monoid of compact elements of $\mathbb{C}^\kappa(M)$.
(ii) For any algebraic $\mathbb{C}^\kappa$-semigroup $S$, we have $\mathbb{C}^\kappa(S_c) \cong S$ as $\mathbb{C}^\kappa$-semigroups.

Proposition 3.24. (cf [11] Corollary 5, [3] Remark 5.5.2). Whenever $A$ has real rank zero, $\mathbb{C}u(A)$ is an algebraic $\mathbb{C}^\kappa$-semigroup. If moreover $A$ has stable rank one, then the converse is true.

Corollary 3.25. Let $A$ be a $C^\ast$-algebra of stable rank one. Then $A$ has real rank zero if and only if $\mathbb{C}u_1(A) \in \mathbb{C}^\kappa_{\text{alg}}$ if and only if $\mathbb{C}u(A) \in \mathbb{C}^\kappa_{\text{alg}}$.

Proof. Using the characterization of compact elements of $\mathbb{C}u_1(A)$ by compact elements of $\mathbb{C}u(A)$ as in Corollary 3.5 we get that $\mathbb{C}u(A)$ is algebraic if and only if $\mathbb{C}u_1(A)$ is algebraic.

We end this section by observing that $\nu_c$ and $\nu_s$ satisfy the following: $\nu_s \circ \nu_c = \nu_c \circ \nu_s$. Hence, we sometimes consider $\nu_{s,c} : \mathbb{C}^\kappa \to \text{PoM}$ as the composition of $\nu_s$ and $\nu_c$. Naturally, for any $S \in \mathbb{C}^\kappa$, we denote by $S_{s,c} := \nu_{s,c}(S)$ the positively ordered monoid of positive compact elements of $S$.

4. SOME COMPUTATIONS $\mathbb{C}u_1$-SEMIGROUPS

This section is aiming to compute the unitary Cuntz semigroup of certain $C^\ast$-algebras, such as simple $C^\ast$-algebras of stable rank one, AF algebras, and some AT, AI algebras. We first give another picture of the $\mathbb{C}u_1$-semigroup and its morphisms using the lattice of ideals of the $C^\ast$-algebra that makes these computations easier.

4.1. ALTERNATIVE PICTURE OF THE INVARIANT. We start by recalling some well-known facts about (closed two-sided) ideals of a $C^\ast$-algebra. Let $A$ be a $C^\ast$-algebra, the set of closed two-sided ideals, that we write $\text{Lat}(A)$, has a complete lattice structure given by $I \cap J = I \cap J$ and $I \vee J = I + J$. Furthermore, it has been pointed out [3] Section 5.1] that the set of closed two-sided ideals that contain a full, positive element, that we write $\text{Lat}_f(A)$, is also of interest since it is not only a sublattice of $\text{Lat}(A)$ but also a $C$-semigroup. In fact, there exists a complete lattice isomorphism between $\text{Lat}(A)$ and $\text{Lat}(\mathbb{C}u(A))$ that maps $\text{Lat}_f(A)$ onto $\text{Lat}_f(\mathbb{C}u(A))$, where $\text{Lat}_f(\mathbb{C}u(A))$ denotes the sublattice of singly-generated $\mathbb{C}$-ideals in $\mathbb{C}u(A)$. (See [3] Section 5.1 and Paragraph 2.4 for more details.)

It is not hard to see that any $\sigma$-unital ideal belongs to $\text{Lat}_f(A)$, and the converse is not true in general. However, in order to construct the alternative picture of the unitary Cuntz semigroup, we will need the extra-hypothesis that $\text{Lat}_f(A) = \{\sigma$-unital ideals of $A\}$. Observe that if $A$ is a separable $C^\ast$-algebra, then $A$ satisfies this extra-hypothesis.

Let $A$ be a $C^\ast$-algebra of stable rank one such that $\text{Lat}_f(A) = \{\sigma$-unital ideals of $A\}$ and let $a \in (A\oplus K)_+$. Recall that for any $a \in A_+$, we write $I_a := \overline{AaaA}$ the ideal generated by $a$ and $\text{her}(a) := \overline{aAa}$ the hereditary subalgebra generated by $a$. Then $a$ is obviously a full positive element in $I_a$. By the hypothesis can find
a strictly positive element of $I_a$, that we write as $s_a$. Since $a \in \text{her}(s_a)$, we know that $a \leq_{Cu} s_a$. Observe that the canonical inclusion $i : \text{her}(a) \hookrightarrow \text{her}(s_a)$ is $I_a$ is one of our standard morphisms (see Paragraph 3.2).
That is, in the notation of Paragraph 3.2, $\delta_a : K_1(I_a) \rightarrow K_1(I_a)$ is in fact an abelian group isomorphism and $\delta_a(\{a\}) = \{a\}$ for any unitary element $a \in \text{her}(a)$. 

**Proposition 4.1.** Let $A$ be a $C^*$-algebra of stable rank one such that $\text{Lat}(A) = \{\sigma\text{-unital ideals of } A\}$. Let $a, b \in (A \otimes K_i)_+$, be such that $a \leq_{Cu} b$. Let $s_a, s_b$ be strictly positive elements of the ideals $I_a, I_b$ respectively. Then the following diagram is commutative:

$$
\begin{array}{ccc}
\mathcal{U}(\text{her}(a)) & \xrightarrow{\theta_{ab}} & K_1(\text{her}(a)) \\
\downarrow{\delta_a} & & \downarrow{\chi_a} \\
\mathcal{U}(\text{her}(b)) & \xrightarrow{\theta_{ab}} & K_1(\text{her}(b))
\end{array}
$$

In particular, for any other strictly positive element $s_a$ of $I_a$, we have $\text{her}(s_a) = \text{her}(s_a)$ and hence $\chi_{s_a s_b} = \delta_{s_a s_b} = \chi_{s_a s_b}$.

**Proof.** By definition, $\chi_{ab} := K_1(\theta_{ab})$ and hence the left-square is commutative. Furthermore, by transitivity of $\leq$ (see Paragraph 3.1), we know that $\chi_{s_a s_b} \circ \chi_{s_a s_b} = \chi_{s_a s_b} = \chi_{s_b s_b} \circ \chi_{ab}$. That is, the right square is commutative, which ends the proof. 

**Notation 4.2.** Let $A$ be a $C^*$-algebra of stable rank one such that $\text{Lat}(A) = \{\sigma\text{-unital ideals of } A\}$. Let $a \in (A \otimes K_i)_+$ and let $s_a$ be any strictly positive element of $I_a$. By Proposition 4.1, $\delta_a : K_1(\text{her}(a)) \approx K_1(I_a)$ is a well-defined group isomorphism that does not depend on the strictly positive element $s_a$ chosen. We write $\delta_a := \delta_a$.

Let $I, J \in \text{Lat}(A)$ and let $s_I, s_J$ be any strictly positive elements of $I, J$ respectively. Suppose that $I \subseteq J$ or, equivalently $[s_I] \leq [s_J]$ in $\text{Cu}(A)$. By Proposition 4.1, $\chi_{s_I s_J} : K_1(I) \rightarrow K_1(J)$ is a well-defined group morphism that does not depend on the strictly positive elements chosen. We write $\delta_{IJ} := \chi_{s_I s_J}$.

Observe that $\delta_{IJ} = K_1(i)$, where $i : I \hookrightarrow J$ is the canonical inclusion. In particular, $\delta_{I} := \delta_{I} := id_{K_1(I)}$.

**Proposition 4.3.** Let $A$ be a $C^*$-algebra of stable rank one such that $\text{Lat}(A) = \{\sigma\text{-unital ideals of } A\}$. Let $a, b \in (A \otimes K_i)_+$, such that $[a] \leq [b]$ in $\text{Cu}(A)$. Let $u, v$ be unitary elements of $\text{her}(a)^\vee, \text{her}(b)^\vee$ respectively. We write $[a] := [u]_{K_1(\text{her}(a))}$ and $[v] := [v]_{K_1(\text{her}(b))}$. Then the following are equivalent:

(i) $\theta_{ab}(u) \sim_h v$ in $\text{her}(b)^\vee$.
(ii) $\chi_{ab}([u]) = [v]$ in $K_1(\text{her}(b))$.
(iii) $\delta_{IJ}([u]) = \delta_{IJ}([v])$ in $K_1(I_J)$, that is, $\delta_{IJ}([u])_{K_1(I_J)} = [v]_{K_1(I_J)}$.

**Proof.** Since $K_1(\theta_{ab}) = \chi_{ab}$, we trivially obtain that (i) is equivalent to (ii). Furthermore, by the right-square of the commutative diagram in Proposition 4.1, we know that $\delta_{IJ} \circ \delta_{IJ}([u]) = \delta_{IJ} \circ \chi_{ab}([u])$. And since $\delta_{IJ}$ is an isomorphism, we obtain that (ii) is equivalent to (iii).
Corollary 4.4. Let $A$ be a $C^*$-algebra of stable rank one and let $[(a, u)], [(b, v)] \in Cu_1(A)$. Then $[(a, u)] \preceq [(b, v)]$ in $Cu_1(A)$ if and only if
\[
\begin{aligned}
[a] &\preceq [b] \text{ in } Cu(A) \\
\delta_{tL_b}([u])_{K_1(I_b)} &\preceq [v]_{K_1(I_b)} \text{ in } K_1(I_b)
\end{aligned}
\]

where $\delta_{tL_b}$ is as in [Proposition 4.1].

We will now use all the above to get a new picture of the $Cu_1$-semigroup and its elements.

Definition 4.5. Let $A$ be a $C^*$-algebra of stable rank one. Let $I \in \text{Lat}_I(A)$ be an ideal of $A$ that contains a full positive element. We recall that $Cu(I)$ is a singly-generated ideal of $Cu(A)$. We also recall that for $x \in Cu(A)$, we write $I_x := \{ y \in Cu(A) \mid y \leq x \}$ the ideal of $Cu(A)$ generated by $x$.

Define $Cu_f(I) := \{ [a] \in Cu(A) \mid I_x = I \}$. Equivalently, $Cu_f(I) := \{ x \in Cu(A) \mid I_x = Cu(I) \}$. In other words, $Cu_f(I)$ consists of the elements of $Cu(A)$ that are full in $Cu(I)$.

One could define $Cu_f(I)$ for any ideal $I \in \text{Lat}(A)$. However, it is easily seen that $Cu_f(I) \neq \emptyset$ if and only if $I \in \text{Lat}_I(A)$. We also mention that whenever $A$ is separable, we have that $\text{Lat}_I(A) = \text{Lat}(A)$.

For notational purposes, we will indistinguishably use $I_a$ or $I[a]$, referring to one or the other; see [Paragraph 2.4] For instance, we might consider objects such as $\delta_{tL_b}$ or $K_1(I)$, where $x, y \in Cu(A)$, when we really mean $\delta_{tL_b}$ or $K_1(I)$, where $a, b \in (A \otimes K)_+$ are representatives of $x, y$ respectively.

Definition 4.6. Let $A$ be a $C^*$-algebra of stable rank one such that $\text{Lat}_I(A) = \{\sigma\text{-unital ideals of } A\}$. Let us consider
\[
S := \bigsqcup_{I \in \text{Lat}_I(A)} Cu_f(I) \times K_1(I).
\]

We equip $S$ with addition and order as follows: For any $(x, k) \in Cu_f(I_x) \times K_1(I)$ and $(y, l) \in Cu_f(I_y) \times K_1(I_l)$, then
\[
\begin{aligned}
(x, k) \leq (y, l) \text{ if: } x \leq y \text{ and } \delta_{tL_x}(k) = l. \\
(x, k) + (y, l) = (x + y, \delta_{tL_x}(k) + \delta_{tL_y}(l)).
\end{aligned}
\]

Lemma 4.7. Let $S$ be a $Cu^-$-semigroup and let $T$ be a $Mon_2$. Let $f : S \rightarrow T$ be a $Mon_2$-isomorphism. Then, $T$ is a $Cu^-$-semigroup and $f$ is a $Cu^-$-isomorphism. A fortiori, $S \cong T$ as $Cu^-$-semigroups.

Proof. We recall that suprema and the compact-containment relation are entirely determined by the order-structure. Thus, existence of suprema and axioms (O1)-(O4) in $T$ are directly obtained from the surjective order-embedding $f$. More concretely, for any increasing sequence $(t_k)_k$ in $T$, we have that $\sup_k t_k = f(\sup s_k)$ where $s_k$ is the (unique) element in $S$ such that $f(s_k) = t_k$. It is now routine to check that $T$ is a $Cu^-$-semigroup and that $f$ is a $Cu^-$-isomorphism. \qed

Theorem 4.8. Let $A$ be a $C^*$-algebra of stable rank one such that $\text{Lat}_I(A) = \{\sigma\text{-unital ideals of } A\}$. Let $(S, +, \leq)$ be the object defined in [Definition 4.2]. Then $(S, +, \leq)$ is a $Cu^-$-semigroup and the following map is a $Cu^-$-isomorphism:
\[
\xi : Cu_1(A) \rightarrow S \\
[a, u] \mapsto ([a], \delta_{\sigma}[u])
\]

where $[a] := [a]_{Cu(A)}$ and $[u] := [u]_{K_1(\text{her}(a))}$.
Whenever convenient, and many times in the sequel, we will describe elements of $Cu(A)$ as a pair $(x, \alpha)$ where $x \in Cu(A)$ and $\alpha \in K_1(A)$. Let $I \in Lat_A$ and let $\phi: A \rightarrow B$ be a $\ast$-homomorphism. Write $J := B\phi(I)B$ the smallest ideal of $B$ containing $\phi(I)$. Also write $\alpha := Cu(\phi)$, $\alpha_0 := Cu(\phi)$ and $\alpha_I := K_1(\phi_I)$, where $\phi_I : I \rightarrow J$.

(i) For any $x \in Cu_I(J)$, we have $\alpha_0(x) \in Cu_I(J)$. That is, $I_{\alpha_0(x)} = Cu(I)$ is the smallest ideal of $Cu(B)$ containing $\alpha_0(Cu(I))$ and $Cu(I) \subseteq Cu_I(J)$.

(ii) For any $(x, k)$ with $x \in Cu_I(J)$ and $k \in K_1(I)$, we have $\alpha(x^{\ast^{-1}}(x, k)) = (\alpha_0(x), \alpha_I(k))$, where $\xi_A, \xi_B$ are the $Cu^\ast$-isomorphism as in [Theorem 4.3.8] for $A, B$ respectively.

Proof. By [Notation 4.2 and Definition 4.5] the map $Cu_I(A) \rightarrow \bigcup_{I \in Lat_A} Cu_I(J) \times K_1(I)$ is well-defined. Further, by construction, addition and order are well-defined in $S$. Now let $a \in (A \otimes K)_+$. Since $A$ has stable rank one, then so has $\text{her}(a)$. Hence, by $K_1$-surjectivity, we know that any element of $K_1(\text{her}(a))$ lifts to a unitary in $\text{her}(a)^\ast$ and that any two of those lifts are homotopic. Also $\delta_\alpha$ is an isomorphism and obviously any two representatives of $x \in (A \otimes K)_+$ are Cuntz equivalent. Thus for any $(x, k) \in Cu(A) \times K_1(I_2)$, there exist $a \in (A \otimes K)_+$ and $u \in \mathcal{U}(\text{her}(a)^\ast)$ such that $[a] = x$ and $\delta_\alpha(u) = k$. Moreover, for any other lift $(a', u')$, we have $[(a', u')] = [(a, u)]$. So we conclude that $\xi$ is a set bijection.

Now, using [Proposition 4.3 and Corollary 4.4] we know that $[(a, u)] \subseteq [(b, v)]$ if and only if $\xi([(a, u)]) \leq \xi([(b, v)])$. Moreover, using [Proposition 4.1] we have $\xi([(a, u)] + [(b, v)]) = \xi([(a, u)]) + \xi([(b, v)])$. In the end, we obtain that $\xi$ is a $\text{Mon}_2$-isomorphism. We finally conclude that $S$ is a $Cu^\ast$-semigroup and that $\xi$ is a $Cu^\ast$-isomorphism using [Lemma 4.7].}$
We now compute the $\Cu_1$-semigroup in some specific settings. In the process, we will remind the reader about lower semicontinuous functions which play a key role in the computation of $\Cu$-semigroups of certain $C^*$-algebras.

### 4.11. (Lower semicontinuous functions.)

Let $X$ be a topological space and $S$ be a $\Cu$-semigroup. Let $f : X \to S$ be a map. We say that $f$ is lower semicontinuous if for any $s \in S$, the set $\{t \in X \mid s \preceq f(t)\}$ is open in $X$. We write $\Lsc(X, S)$ for the set of lower-semicontinuous functions from $X$ to $S$.

Also, we recall that if $A$ is a separable $C^*$-algebra of stable rank one such that $K_1(I) = 0$ for every ideal of $A$ and $X$ is a locally compact Hausdorff space that is second countable and of covering dimension at most one, then $\Cu(C_0(X) \otimes A) \cong \Lsc(X, \Cu(A))$; see [2, Theorem 3.4].

Finally, $V \mapsto I_{1V}$ defines a one-to-one correspondence between the open subsets of $X$, that we write $O(X)$, and the ideals of $\Lsc(X, \overline{\mathbb{N}})$. Note that for any $f \in \Lsc(X, \overline{\mathbb{N}})$, $I_f := I_{\supp(f)}$, where $\supp(f) := \{x \in X \mid f(x) \neq 0\}$ is an open set of $X$.

### 4.2. The simple case.

Let $A$ be a simple $\sigma$-unital $C^*$-algebra of stable rank one. Then $\Cu_1(A)$ can be described in terms of $\Cu(A)$ and $K_1(A)$ as follows:

\[
\Cu_1(A) \xrightarrow{\phi} (\Cu(A) \setminus \{0\}) \times K_1(A) \sqcup \{0\}
\]

\[
(x, k) \mapsto \begin{cases} 0 & \text{if } x = 0 \\ (x, k) & \text{otherwise} \end{cases}
\]

**Proof.** Since $A$ is simple, we know that $\Lat(A) = \{0, A\}$. Therefore, in the description of the $\Cu_1$-semigroup of [Notation 4.10], we have $\Cu_f(\{0\}) = \{0\}$ and $\Cu_f(A) = \Cu(A) \setminus \{0\}$. The result follows. □

### 4.3. The case of no $K_1$-obstructions.

**Definition 4.12.** We say that a $C^*$-algebra $A$ has no $K_1$-obstructions, if $A$ has stable rank one and $K_1(I)$ is trivial for any $I \in \Lat(A)$.

**Proposition 4.13.** Let $A$ be a $C^*$-algebra with no $K_1$-obstructions such that $\Lat_f(A) = \{\sigma$-unital ideals of $A\}$. Then $\Cu_1(A) \cong \Cu(A)$. In particular, for any separable AF algebra $A$, $\Cu_1(A) \cong \Cu(A)$.

**Proof.** By assumption, we know that $K_1(I)$ is trivial for any $I \in \Lat(A)$. Therefore, using again the description of the $\Cu_1$-semigroup of [Theorem 4.8] (see [Notation 4.10]), we have $\Cu_1(A) \cong \Cu(A) \times \{0\}$. The result follows. □

### 4.4. AI and AT algebras: The case of $C([0,1])$ and $C(\mathbb{T})$.

Here we compute the $\Cu_1$-semigroup of the interval algebra and the circle algebra. Using the continuity of $\Cu_1$, we also give an explicit computation of the $\Cu_1$-semigroup of AI-algebras (respectively AT-algebras), constructed as the tensor product of the interval algebra (respectively the circle algebra) with any UHF algebra of infinite type.

**Notation 4.14.** Let $X$ be the interval or the circle and let $f \in \Lsc(X, \overline{\mathbb{N}})$. The open set $V_f := \supp(f)$ of $X$ can be (uniquely) decomposed into a countable disjoint union $\bigcup_{k \geq 1} V_{f_k}$ of open arcs of $X$. In other words, $V_f = \bigcup_{k \geq 1} V_{f_k}$ for some $n_f \in \mathbb{N}$, where $\{V_{f_k}\}_{k \geq 1}$ are pairwise disjoint open arcs of $X$. For the specific case of the interval, we also define $m_f := n_f - (1_{V_f}(0) + 1_{V_f}(1))$. That is, $m_f$ is the number of open intervals of the decomposition $\{V_{f_k}\}_{k \geq 1}$ of $V_f$ that are strictly contained in $]0, 1[$. (Therefore $m_f$ also belongs to $\mathbb{N}$.)
The $C([0,1])$ case.

**Lemma 4.15.** Let $I \in \text{Lat}(C([0,1]))$ and let $f_i := 1_{V_i}$ be the indicator map on the unique open set $V_i$ of $[0,1]$ corresponding to $I$. We have

\[
\begin{align*}
\text{Cu}(I) & \cong \text{Lsc}(V_i, \overline{\mathbb{N}}).
\text{K}_1(I) & \cong \oplus \mathbb{Z}.
\end{align*}
\]

**Proof.** We know that $\text{Cu}(I) = I_{f_i} \cong \text{Lsc}(V_i, \overline{\mathbb{N}})$ and we obtain that $\text{Cu}(I) \cong \text{Lsc}(V_i, \overline{\mathbb{N}})$. Then, we observe that open arcs of $[0,1]$ are of the following the form:

$$[a,b] \quad [0,1] \quad [a,1] \quad [0,a] \quad \emptyset$$

and the $K_1$ groups of continuous maps over these open arcs are, respectively:

$$\mathbb{Z} \quad [0] \quad [0] \quad [0] \quad [0]$$

Furthermore, for any two disjoint open arcs $V,W$ of the open interval $]0,1[$, the canonical inclusion $i : I_{V_i} \subseteq I_{V_{i+1}}$ induces an injection $K_1(i) : \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$. Now, let us decompose $V = \bigcup_{i=1}^{n_I} V_i$ as in

**Notation 4.14** Equivalently, we have that $f_i = \sum_{k=1}^{n_I} 1_{V_i}$. Using all the above, we compute that $K_1(I) \cong \lim_{n \to \infty} (\sum_{k=1}^{n_I} 1_{V_i}) \subseteq I_{\text{Lsc}(V_i)}$ is the canonical inclusion. □

**Theorem 4.16.** Let $W_0 := ]0,1[$ and $W_1 := ]0,1[:$

(i) $\text{Cu}(C([0,1])) = \bigcup_{V \in \partial([0,1])} \text{Lsc}(V, \overline{\mathbb{N}}) \times (\oplus \mathbb{Z})$

$\cong \text{Cu}(C([0,1])) \cup (\bigcup_{i=0,1} \text{Lsc}(W, \overline{\mathbb{N}}) \times \{0\}) \cup \text{Lsc}([0,1], \overline{\mathbb{N}}) \times \{0\}.$

(ii) $\text{Cu}(C([0,1])) \cong ([1\{0,1\}] \times \mathbb{Z}) \times \{0\} \cong \mathbb{N}.$

**Proof.** (i) Combine [Theorem 4.8] with [Lemma 4.15] and [Paragraph 4.11]

(ii) From [Corollary 3.5] we know that $(x,k) \in \text{Cu}(C([0,1]))$ is a compact element if and only if it is compact in $\text{Lsc}([0,1], \overline{\mathbb{N}})$, if and only if $x$ is constant on $[0,1]$ and $x \ll \infty$. □

The $C(\mathbb{T})$ case.

**Lemma 4.17.** Let $I \in \text{Lat}(C(\mathbb{T}))$ and let $f_i := 1_{V_i}$ be the indicator map on the unique open set $V_i$ of $\mathbb{T}$ corresponding to $I$. We have

\[
\begin{align*}
\text{Cu}(I) & \cong \text{Lsc}(V_i, \overline{\mathbb{N}}).
\text{K}_1(I) & \cong \oplus \mathbb{Z}.
\end{align*}
\]

**Proof.** We know that $\text{Cu}(I) = I_{f_i} \cong \text{Lsc}(V_i, \overline{\mathbb{N}})$ and we obtain that $\text{Cu}(I) \cong \text{Lsc}(V_i, \overline{\mathbb{N}})$. Then, we observe that open arcs of $\mathbb{T}$ are of the following the form:

$$[a,b] \quad \mathbb{T} \quad \emptyset$$

and the $K_1$ groups of continuous maps over these open arcs are, respectively:

$$\mathbb{Z} \quad \mathbb{Z} \quad \{0\}$$
Furthermore, for any two disjoint open arcs $V, W$ of $T$, the canonical inclusion $i : I_{V} \subseteq I_{V+1}$ induces a injection $K_{i} : \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$. Now, let us decompose $V_{i} = \bigoplus_{k=1}^{n_{i}} V_{i}$ as in [Notation 4.13]. Equivalently, we have that $f_{i} = \sum_{k=1}^{n_{i}} 1_{V_{i}}$. Using all the above, we get $K_{i}(I) = \lim_{n \to \infty} K_{i}(I_{\sum_{k=1}^{n_{i}} 1_{V_{i}}}) \approx \oplus \mathbb{Z}$, where $l_{n} : \bigoplus_{k=1}^{n_{i}} \subseteq I_{\sum_{k=1}^{n_{i}} 1_{V_{i}}}$ is the canonical inclusion.

Theorem 4.18. We have the following:

(i) \[ \text{Cu}_{1}(C(T)) \approx \bigsqcup_{\nu \in \mathbb{Z}} \text{Lsc}(V, \mathbb{N}) \times (\oplus \mathbb{Z}) \approx \text{Cu}_{1}(C([0, 1])) \cup \text{Lsc}(T, \mathbb{N}) \times \mathbb{Z}. \]

(ii) \[ \text{Cu}_{1}(C(T)) \approx (\{n1_{T}\}_{n \in \mathbb{Z}}) \times \mathbb{Z} \approx \mathbb{N} \times \mathbb{Z}. \]

Proof. (i) Combine Theorem 4.8 with Lemma 4.15 and Paragraph 4.11.

(ii) From Corollary 3.5, we know that $(x, k) \in \text{Cu}_{1}(C(T))$ is a compact element if and only if $x$ is compact in $\text{Lsc}(T, \mathbb{N})$, if and only if $x$ is constant on $T$ and $x \ll \infty$.

Now that we have computed the $\text{Cu}_{1}$-semigroup of the interval algebra and the circle algebra, we are able to obtain the $\text{Cu}_{1}$-semigroup of any AI and AT algebra, using Corollary 3.2. Actually, we will next compute a concrete example of an AT algebra that is constructed as $C(T) \otimes \text{UHF}$. Let $q$ be a supernatural number and consider $M_{q}$ the UHF algebra associated to $q$. Consider any sequence of prime numbers $(q_{n})$, such that $q = \prod_{n} q_{n}$. Write $(A_{n}, \phi_{nm})$ the inductive system associated to $(q_{n})$. Now consider the following AT algebra: $A := \lim_{n} C([0, 1]) \otimes A_{n}, id \otimes \phi_{nm}$. In fact, $A \approx C(T) \otimes M_{q}$.

Theorem 4.19. Let $M_{q}$ be a UHF algebra. Then:

\[ \text{Cu}_{1}(C(T) \otimes M_{q}) \approx \bigsqcup_{\nu \in \mathbb{Z} \times \mathbb{Z}} \text{Lsc}(V, (\text{Cu}(M_{q}) \setminus \{0\}) \times (\oplus \mathbb{Z}) \approx \text{Cu}_{1}(C([0, 1])) \cup \text{Lsc}(T, \mathbb{N}) \times \mathbb{Z}. \]

In particular, for any UHF algebra of infinite type $M_{p}$, we get:

\[ \text{Cu}_{1}(C(T) \otimes M_{q}) \approx \bigsqcup_{\nu \in \mathbb{Z} \times \mathbb{Z}} \text{Lsc}(V, (\mathbb{N}^{ \frac{1}{p}}) \setminus \{0\}) \times (\oplus \mathbb{Z} \setminus \{1\}). \]

Proof. Since UHF algebras are simple, we know that all ideals of $C(T) \otimes M_{q}$ are of the form $C_{0}(U) \otimes M_{q}$ for some $U \in \mathcal{O}(T)$. Hence, using the Künneth formula (see [6] Theorem 23.1.3), we obtain that $K_{1}(C_{0}(U) \otimes M_{q}) \approx (\oplus \mathbb{Z}) \otimes K_{0}(M_{q}) \approx \oplus K_{0}(M_{q})$. On the other hand, by [2] Theorem 3.4, we compute that $\text{Cu}(C_{0}(U) \otimes M_{q}) \approx \text{Lsc}(U, \text{Cu}(M_{q}))$. The result follows from Theorem 4.8.

5. Relation of $\text{Cu}_{1}$ with existing $K$-theoretical invariants

The aim of this section is to recover existing invariants functorially. We have already seen that the positive cone of $\text{Cu}_{1}(A)$ is isomorphic to $\text{Cu}(A)$. Our first step is to capture the $K_{1}$ group information. To that end, we define a well-behaved set of maximal elements of a $C^{*}$-semigroup $S$, written $S_{\text{max}}$, and we prove that $\text{Cu}_{1}(A)_{\text{max}}$ is isomorphic to $K_{1}(A)$. Subsequently, we recover functorially $\text{Cu}$, $K_{1}$ and finally the $K_{2}$ group. As before, we shall assume that $A$ is a $C^{*}$-algebra that has stable rank one and denote the category of such $C^{*}$-algebras by $C_{sr}^{*}$. 
5.1. An abelian group of maximal elements: $v_{\text{max}}$.

**Definition 5.1.** Let $S$ be a Cu$^*$-semigroup. We say that $S$ is positively directed if, for any $x \in S$, there exists $p_x \in S$ such that $x + p_x \geq 0$.

**Lemma 5.2.** Let $A$ be a $C^*$-algebra of stable rank one. Then $\text{Cu}_1(A)$ is positively directed.

**Proof.** Using the picture of Theorem 4.8 (see Notation 4.10), consider $(x, k) \in \text{Cu}_1(A)$, where $x \in \text{Cu}(A)$ and $k \in K_1(I_c)$. Observing that $(x, k) + (x, -k) = (2x, 0) \geq 0$, we deduce that $\text{Cu}_1(A)$ is positively directed. □

**Definition 5.3.** Let $S$ be a Cu$^*$-semigroup. We define $S_{\text{max}} := \{ x \in S \mid \text{if } y \geq x, \text{ then } y = x \}$.

**Proposition 5.4.** Let $S$ be a positively directed Cu$^*$-semigroup. Then $S_{\text{max}}$ is either empty or an absorbing abelian group in $S$ whose neutral element $e_{\text{max}}$ is positive.

**Proof.** The empty case is trivial. Let us suppose that $S_{\text{max}}$ is not empty. By assumption, for any $x \in S$, there exists at least one element $p_x \in S$, such that $x + p_x \geq 0$. We first show that $S_{\text{max}}$ is closed under addition.

Let $y, z$ be elements in $S_{\text{max}}$ and let $x \in S$ be such that $x \geq y + z$. We first have $x + p_x \geq y + z + p_x \geq y$ and $x + p_y \geq z + y + p_y \geq z$, which gives us the following equalities: $x + p_x = y + z + p_x = y$ and $x + p_y = z + y + p_y = z$. Obviously $x \leq x + p_x + z = x + p_x + x + p_y = y + z$ and since $x \geq y + z$, we have $x = y + z$ which tells us that $S_{\text{max}}$ is closed under addition.

Now, let us show that $S_{\text{max}}$ has neutral element. We first prove that for any $z \in S_{\text{max}}$ and any $p_z \in S$ such that $z + p_z \geq 0$, the element $z + p_z$ is a positive element of $S_{\text{max}}$ that does not depend on $z$ nor $p_z$.

Let $z$ and $p_z$ be such elements and let $x \in S$ be such that $x \geq z + p_z$. We know that for any $y \in S_{\text{max}}$, $y + z + p_z = y$. In particular, $2z + p_z = z$. Also, $x + z \geq 2z + p_z = z$. Hence $x + z = z$. Finally compute that $x \leq x + z + p_z = z + p_z$. Therefore $x = z + p_z$, that is, $z + p_z \in S_{\text{max}}$. Further, for any $y, z$ elements of $S_{\text{max}}$, we have $y + p_z + z + p_z \geq z + p_z, y + p_z$, which by what we have just proved gives us $y + p_z = y + p_z + z + p_z = z + p_z$. Hence, the positive element $e_{\text{max}} := y + p_z$ belongs to $S_{\text{max}}$ and does not depend on $y$ and $p_z$. Now let $z \in S_{\text{max}}$. Since $e_{\text{max}} \geq 0$, we obtain $z + e_{\text{max}} \geq z$ and we get that $z + e_{\text{max}} = z$ for any $z \in S_{\text{max}}$. Thus, $e_{\text{max}}$ is the neutral element for $(S_{\text{max}}, +)$ and the unique positive element of $S_{\text{max}}$.

In other words, $S_{\text{max}}$ is an abelian monoid with neutral element its unique positive element $e_{\text{max}}$.

Then on, let us prove that any element has an additive inverse. We already know that $z + (2p_z + z) = e_{\text{max}}$ for any $z \in S_{\text{max}}$. Let us show that $2p_z + z$ belongs to $S_{\text{max}}$, for any $z \in S_{\text{max}}$ and any $p_z \in S$ such that $z + p_z \geq 0$. Let $x \geq 2p_z + z$. Then $x + z \geq e_{\text{max}}$, hence $x + z = e_{\text{max}}$. On the other hand, $x \leq x + z + p_z = e_{\text{max}} + p_z = 2p_z + z$. Therefore $2p_z + z$ belongs to $S_{\text{max}}$ and is the (unique) inverse of $z$, which ends the proof that $S_{\text{max}}$ is an abelian group.

Lastly, let us show the absorption property. Let $x \in S$ and let $p \in S_{\text{max}}$, we know there exists $y \in S$ such that $x + y \geq 0$. Hence $x + y + p \geq p$. Let $z \in S$ be such that $z \geq x + p$. We have $z + y \geq x + y + p = p$ and hence $z + y = p$. Now since $x + y \geq 0$, we have $z \geq x + p = x + z + y \geq z$ which gives us $z = x + p$, that is, $x + p \in S_{\text{max}}$ for any $x \in S$ and $p \in S_{\text{max}}$. □
We note that a positively directed $C^+$-semigroup $S$ might not have maximal elements. However if it does, then $S$ a unique positive maximal element which is the neutral element for $S_{\text{max}}$. Also, whenever $S$ is simple or countably-based, the existence of such a maximal positive element is ensured and $S_{\text{max}}$ is not empty.

As a result, whenever $A$ is either a simple or separable $C^*$-algebra of stable rank one, then $Cu_1(A)_{\text{max}}$ is an abelian group whose neutral element is $e_{S_{\text{max}}} := (\infty_{Cu(A)}, 0_{K_1(A)})$. In fact, we will see that under such hypothesis, we have $Cu_1(A)_{\text{max}} \cong K_1(A)$.

**Proposition 5.5.** Let $\alpha : S \longrightarrow T$ be a $C^+$-morphism between positively directed $C^+$-semigroups $S, T$ that have maximal elements. Then $\alpha_{\text{max}} := \alpha|_{S_{\text{max}}} + e_{T_{\text{max}}}$ is a AbGp-morphism from $S_{\text{max}}$ to $T_{\text{max}}$.

**Proof.** Let us first show that $\alpha_{\text{max}}$ is a group morphism. For any $s \in S_{\text{max}}$, we know that $(\alpha(s) + e_{T_{\text{max}}}) \in T_{\text{max}}$. Now, since $\alpha$ is a $C^+$-morphism, we have $\alpha_{\text{max}}(s_1) + \alpha_{\text{max}}(s_2) = \alpha(s_1) + \alpha(s_2) + 2e_{T_{\text{max}}} = \alpha(s_1 + s_2) + e_{T_{\text{max}}} = \alpha_{\text{max}}(s_1 + s_2)$, for any $s_1, s_2$ elements of $S_{\text{max}}$. \[\Box\]

As with $\nu_+$ and $\nu_e$, we define a functor $\nu_{\text{max}}$ that recovers the maximal elements of a positively directed $C^+$-semigroup as follows:

$$\nu_{\text{max}} : C^+ \longrightarrow \text{AbGp}$$

$$S \longmapsto S_{\text{max}}$$

$$\alpha \longmapsto \alpha_{\text{max}}$$

It is left to the reader to prove that $\nu_{\text{max}}$ is a well-defined functor. Also, we specify that to be thoroughly defined as a functor, $\nu_{\text{max}}$ should have as domain the full subcategory of positively directed $C^+$-semigroups that have maximal elements, that we also denote $C^+$. Observe that $Cu_1(C^+_{\text{st},1,\sigma})$ belongs to the latter full subcategory, where $C^+_{\text{st},1,\sigma}$ is the full subcategory of separable $C^*$-algebras of stable rank one.

5.2. **Link with $Cu_1$ and $K_1$.** Recall that for a positively directed $C^+$-semigroup $S$ that has maximal elements, we have $S_+ \in Cu$ and that $S_{\text{max}} \in \text{AbGp}$; see [Proposition 5.4]. In fact, both categories $Cu$ and $\text{AbGp}$ can be seen as subcategories of $C^+$-by defining an order as the equality for the case of groups. Therefore, in what follows, we consider $\nu_+$ and $\nu_{\text{max}}$ as functors with codomain $C^+$.

**Definition 5.6.** Let $S$ be a positively directed $C^+$-semigroup that has maximal elements. Let us define two $C^+$-morphisms that link $S$ to $S_+$ on the one hand, and to $S_{\text{max}}$ on the other hand, as follows:

$$i : S_+ \longhookrightarrow S$$

$$s \longmapsto s$$

$$j : S \twoheadrightarrow S_{\text{max}}$$

$$s \longmapsto s + e_{S_{\text{max}}}$$

In the next theorem, we use the picture of the $Cu_1$-semigroup obtained from [Theorem 4.8] see [Notation 4.10].

**Theorem 5.7.** Let $A$ be either a separable or a simple $\sigma$-unital $C^*$-algebra of stable rank one. We have the following natural isomorphisms in $Cu$ and $\text{AbGp}$ respectively:

$$Cu_1(A)_+ \cong Cu(A)$$

$$(x, 0) \longmapsto x$$

$$Cu_1(A)_{\text{max}} \cong K_1(A)$$

$$(\infty_A, k) \longmapsto k$$

In fact, we have the following natural isomorphisms: $\nu_+ \circ Cu_1 \cong Cu$ and $\nu_{\text{max}} \circ Cu_1 \cong K_1$.  

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Proof. Let us prove the theorem for $A$ separable and the simple case is proven similarly. We know that any positive element of $\Cu_1(A)$ is of the form $(x, 0)$ for some $x \in \Cu(A)$ and that $\infty_A := \sup_{n \in \mathbb{N}} [s_n A]$ is the largest element of $\Cu(A)$, where $s_n A$ is a strictly positive element of $A$. We also know that any maximal element of $\Cu_1(A)$ is of the form $(\infty_A, k)$ for some $k \in K_1(A)$. Hence we easily get the two canonical isomorphisms of the statement. Now let $\phi : A \to B$ be a $^*$-homomorphism, let $(x, 0) \in \Cu_1(A)_+$ and let $(\infty_A, k) \in \Cu_1(A)_{\text{max}}$. We have that $\Cu_1(\phi)_+ (x, 0) = (\Cu(\phi)(x), 0)$ and that

$$\Cu_1(\phi)_{\text{max}}(\infty_A, k) = (\Cu(\phi)(\infty_A), \Cu_1(\phi)_A(k)) + (\infty_B, 0)$$

$$= (\infty_B, \delta_{\infty_B} B \circ \Cu_1(\phi)_A(k))$$

$$= (\infty_B, K_1(\phi)(k)).$$

This exactly gives us that

$$\begin{array}{ccc}
\Cu_1(A)_+ & \xrightarrow{\sim} & \Cu_1(A) \\
\Cu_1(B)_+ & \xrightarrow{\sim} & \Cu_1(B) \\
\Cu_1(\phi)_+ & \xrightarrow{\sim} & \Cu_1(\phi)
\end{array}
$$

are commutative squares.

\[ \square \]

5.3. Recovering an invariant. We will now define the categorical notion of ‘recovering’ a functor. This allows us to check whether information and classification results of an invariant can be recovered from another one. To that end, we introduce the notion of weakly-complete invariant: an isomorphism at the level of the codomain category implies an isomorphism at the level of $C^*$-algebras without knowing whether it actually corresponds to a lift.

**Definition 5.8.** Let $C, D$ be arbitrary categories and let $I : C^* \to C$ and $J : C^* \to D$ be (covariant) functors. Let $H : D \to C$ be a functor such that there exists a natural isomorphism $\eta : H \circ J \cong I$. Then we say we can recover $I$ from $J$ through $H$.

**Theorem 5.9.** Let $C, D$ be arbitrary categories and let $I : C^* \to C$ and $J : C^* \to D$ be (covariant) functors. Suppose that there exists a functor $H : D \to C$ such that we recover $I$ from $J$ through $H$.

(i) If $I$ is a complete invariant for a class $C_1^*$ of $C^*$-algebras, then $J$ is a weakly-complete invariant for $C_1^*$.

(ii) If $I$ classifies homomorphisms from a class $C_1^*$ of $C^*$-algebra to another class $C_2^*$ of $C^*$-algebra, then $J$ weakly classifies homomorphisms from $C_1^*$ to $C_2^*$.

If moreover $H$ is faithful, then $J$ is a complete invariant for $C_1^*$ and $J$ classifies homomorphisms from $C_1^*$ to $C_2^*$. In this case, we say that we can fully recover $I$ from $J$ through $H$.

**Proof.** Let $I, J$ and $H$ be functors as in the theorem.

(i) Suppose that $I$ is a complete invariant for $C_1^*$. Take any two $C^*$-algebras $A, B \in C_1^*$. If there exists an isomorphism $\alpha : J(A) \cong J(B)$, by functoriality, we get an isomorphism $H(\alpha) : H \circ J(A) \cong H \circ J(B)$. Using the natural isomorphism $H \circ J \cong I$, we know that $H(\alpha)$ gives us an isomorphism $\beta : I(A) \cong I(B)$.
By hypothesis, we can lift $\beta$ to an isomorphism in the category $C^*$. That is, there exists a $^*$-isomorphism $\phi : A \cong B$ such that $I(\phi) = \beta$. We have just shown that $J$ is weakly-complete for $C^*_I$.

Suppose now that $H$ is faithful. Then the natural isomorphism exactly gives us that $H \circ J(\phi) = H(\alpha)$. Now since $H$ is faithful, we conclude that $J(\phi) = \alpha$. That is, $J$ is a complete invariant for $C^*_I$.

(ii) Suppose that $I$ classifies homomorphisms from $A$ to $B$. Let $\alpha : J(A) \rightarrow J(B)$ be any morphism in $\mathcal{D}$. If $\phi, \psi : A \rightarrow B$ are $^*$-homomorphisms such that $J(\phi) = J(\psi) = \alpha$, then composing with $H$, we get $H \circ J(\phi) = H \circ J(\psi) = H(\alpha)$. Thus, $I(\phi) = I(\psi)$, which gives us, by hypothesis, that $\phi \sim_{\text{new}} \psi$. Hence $J$ weakly classifies homomorphisms from $A$ to $B$.

Finally if $H$ is faithful, then for any $\alpha : J(A) = J(B)$, using again the natural isomorphism $H \circ J = I$, we obtain: For any lift $\phi : A \rightarrow B$ of $\beta : I(A) \rightarrow I(B)$, where $\beta$ is the morphism obtained from $H(\alpha)$ as in the proof of (i) above, we have $H \circ J(\phi) = H(\alpha)$. Since $H$ is faithful, we get that $\alpha = J(\phi)$, from which we deduce that $J$ classifies homomorphisms from $A$ to $B$. $\square$

We illustrate all the above with the following results:

**Proposition 5.10.** By [Theorem 5.7] we can recover $Cu$ and $K_A$ from $Cu_1$ through $\nu_+$ and $\nu_{\text{max}}$ respectively. As to be expected, neither $\nu_+$ nor $\nu_{\text{max}}$ are faithful functors.

**Proof.** Use the natural isomorphisms of [Theorem 5.7] $\square$

**Corollary 5.11.** Let $\phi, \psi : A \rightarrow B$ be two $^*$-homomorphism between separable $C^*$-algebras of stable rank one. If $Cu_1(\phi) = Cu_1(\psi)$ then $Cu(\phi) = Cu(\psi)$ and $K_A(\phi) = K_A(\psi)$.

5.4. **Recovering the $K_*$ invariant.** We now study a concrete use of [Theorem 5.9] to recover existing classifying functors from $Cu_1$, and in the process, recall some classification results that have been obtained in the past. Here, we give some insight on $K_* := K_0 \oplus K_1$. Although notations might slightly differ, all of this can be found in [12] and [13].

An approximately homogeneous dimensional algebra, written $AH_\mathbb{D}$ algebra, is an inductive limit of finite direct sums of the form $M_{n_i}(I_{q_i})$ and $M_n(C(X))$, where $I_{q_i} := \{f \in M_{q_i}(C([0, 1])) \mid f(0), f(1) \in \mathbb{C}_{q_i}\}$ is the Elliott-Thomsen dimension-drop interval algebra and $X$ is one of the following finite connected CW complexes: $[\ast, \mathbb{T}, [0, 1]$. Observe that we have the following inclusions: $AF \subseteq AI$, $AT \subseteq AH_\mathbb{D} \subseteq 1$-NCCW, where 1-NCCW are the inductive limits of 1-dimensional non-commutative CW complexes (abbreviated one dimensional NCCW complexes).

The category of ordered groups with order-unit, written $\text{AbG}_\mathbb{r}$, is the category whose objects are ordered groups with order-unit and morphisms are ordered group morphisms that preserve the order-unit.

**Definition 5.12.** (cf [13] Definition 1.2.1) Let $A$ be a (unital) $C^*$-algebra. We define $K_*(A) := K_0(A) \oplus K_1(A)$. We also define $K_*(A)_+ := \{(p|_{K_0(A)}, \left[v|_{K_1(A)}\right]) \subseteq K_0(A) \oplus K_1(A), \text{where } p \text{ is a projection in } A \otimes \mathcal{K}\text{ and } v \text{ is a unitary in the corner } p(A \otimes \mathcal{K})p\text{.}$ Notice that we look at the $K_1$ class of $v$ in $A$, that is, $[v + (1 - p)]_{K_1(A)}$. Finally, we define $1_{K_*(A)} := \{1, 0\}_{K_0(A)}$.

**Proposition 5.13.** (cf [13] §1.2.2) Let $A$ and $B$ be unital $C^*$-algebras of stable rank one. Then $(K_*(A), K_*(A)_+)$ is an ordered group and $1_{K_*(A)} \in K_*(A)_+$ is an order-unit of $K_*(A)$. 

Thus, \((K_\star(A), K_\star(A)_+, \iota_{K_\star(A)}) \in \text{AbGp}_u\). Moreover, any \(^*\)-homomorphism \(\phi : A \to B\) induces an ordered group morphism \(K_0(\phi) \otimes K_1(\phi) : K_\star(A) \to K_\star(B)\) that preserves the order-unit. Thus, we obtain a covariant functor \(K_\star : \text{AH}_{d,1} \to \text{AbGp}_u\) where \(\text{AH}_{d,1}\) is the category of unital \(\text{AH}_d\) algebras.

We do not give a proof of the above, but we remind the reader that whenever a \(C^*\)-algebra \(A\) has stable rank one (which is the case of any \(\text{AH}_d\) algebra), then the monoid \(V(A)\) has cancellation and hence \(K_0(A)_+\) can be identified with \(V(A)\) and thus \((K_0(A), V(A))\) is an ordered group.

We also recall that in the stable rank one case, the Murray-von Neumann equivalence and the Cuntz equivalence agree on the projections of \(A \otimes K\) and that \(V(A) \simeq (\text{Cu}(A))_+\). That is, any compact element of \(\text{Cu}(A)\) is the class of some projection of \(A \otimes K\).

We now recall two notable classification results by means of \(K_\star\) that catch our interest:

**Theorem 5.14.** ([13] Corollary 4.9], [12] Theorem 7.3 - Theorem 7.4)

(i) The functor \(K_\star\) is a complete invariant for (unital) \(\text{AH}_d\) algebras of real rank zero.

(ii) Let \(A, B\) be (unital) \(\text{AH}\) algebras of real rank zero and let \(\alpha : K_\star(A) \to K_\star(B)\) be a scaled ordered group morphism. Then there exists a unique \(^*\)-homomorphism (up to approximate unitary equivalence) \(\phi : A \to B\) such that \(K_\star(\phi) = \alpha\).

The aim now is to recover \(K_\star\) from \(\text{Cu}_1\) and thus show that \(\text{Cu}_1\) contains more information than \(K_\star\). For that purpose, we first define the category of \(\text{Cu}^+\)-semigroups with order-unit, that we denote by \(\text{Cu}^+_u\). Further, we create a functor \(H_\star : \text{Cu}^+_u \to \text{AbGp}_u\) such that \(H_\star \circ \text{Cu}_1 = K_\star\) as functors. Moreover, restricting to an adequate subcategory of \(\text{Cu}^+_u\), we will see that \(H_\star\) is faithful.

**Definition 5.15.** Let \(S\) be a \(\text{Cu}^+\)-semigroup. We say that \(S\) has weak cancellation if \(x + z \ll y + z\) implies \(x \leq y\) for \(x, y, z \in S\). We say that \(S\) has cancellation of compact elements if \(x + z \leq y + z\) implies \(x \leq y\) for any \(x, y \in S\) and \(z \in S_c\).

The following property is proved using the same argument as in [20, Proposition 2.1.3].

**Proposition 5.16.** Let \(A\) be a \(C^*\)-algebra of stable rank one. Then \(\text{Cu}_1(A)\) has weak cancellation and a fortiori \(\text{Cu}_1(A)\) has cancellation of compact elements.

**Definition 5.17.** Let \(S\) be a positively directed \(\text{Cu}^+\)-semigroup. Suppose that \(S\) has cancellation of compact elements. Also suppose that \(S_+\) admits a compact order-unit.

We say that \((S, u)\) is a \(\text{Cu}^+\)-semigroup with compact order-unit. Now, a \(\text{Cu}^+\)-morphism between two \(\text{Cu}^+\)-semigroups with compact order-unit \((S, u), (T, v)\) is a \(\text{Cu}^+\)-morphism \(\alpha : S \to T\) such that \(\alpha(u) \leq v\).

We define the category of \(\text{Cu}^+\)-semigroups with compact order-unit, denoted \(\text{Cu}^+_u\), as the category whose objects are \(\text{Cu}^+\)-semigroups with order-unit and morphisms are \(\text{Cu}^+\)-morphisms that preserve the order-unit.

**Lemma 5.18.** The assignment \(\text{Cu}_{1,u} : C^*_{sr1,1} \to \text{Cu}^+_u\)

\[
A \mapsto (\text{Cu}_1(A), ([1_A], 0))
\]

\[
\phi \mapsto \text{Cu}_1(\phi)
\]

from the category of unital \(C^*\)-algebras of stable rank one, denoted by \(C^*_{sr1,1}\), to the category \(\text{Cu}^+_u\) is a covariant functor.
Proof. We know that $Cu_1(A)_c$ has cancellation of compact elements. Further, we know that $(\{1_A\},0)$ is a compact order-unit of $Cu_1(A)_c$, so it easily follows that $Cu_1(A) \in Cu_1^-$. Finally, it is trivial to see that $Cu_1(\phi)(\{1_A\}) \leq [1_B]$, which ends the proof. □

Lemma 5.19. The assignment $H_* : Cu^-_u \rightarrow AbGp_u$
$(S,u) \mapsto (\text{Gr}(S,c), S,c,u)$
$\alpha \mapsto \text{Gr}(\alpha_c)$
from the category $Cu^-_u$ to the category $AbGp_u$ is a covariant functor.

Moreover, if we restrict the domain of $H_*$ to the category of algebraic $Cu^-_u$-semigroups with compact order-unit, denoted by $Cu^-_{ulg}$, then $H_*$ becomes a faithful functor.

Proof. Let $(S,u) \in Cu^-_u$. By [Proposition 5.16] we know that $S,c$ is a monoid with cancellation and hence, using the Grothendieck construction, one can check that $(\text{Gr}(S,c), S,c,u)$ is an ordered group with order-unit. Now let $\alpha : S \rightarrow T$ be a $Cu^-_u$-morphism between two $Cu^-_u$-semigroups with order-unit $(S,u),(T,v)$. By functoriality of $\text{Gr}(\cdot,c)$, it follows that $\text{Gr}(\alpha_c) : \text{Gr}(S,c) \rightarrow \text{Gr}(T,c)$ is a group morphism such that $\text{Gr}(\alpha_c)(S,c) \subseteq T,c$. Finally, using that $\alpha(u) \leq \alpha(v)$, we obtain $\text{Gr}(\alpha_c)(u) \leq v$. We conclude that $H_*$ is a well-defined functor.

Now, we have to show that if we restrict the domain of $H_*$ to $Cu^-_{ulg}$, then $H_*$ becomes faithful. Let $\alpha, \beta : (S,u) \rightarrow (T,v)$ be two scaled $Cu^-_u$-morphisms between $(S,u),(T,v) \in Cu^-_{ulg}$ such that $H_*(\alpha) = H_*(\beta)$. In particular, $\alpha_c = \beta_c$, and since we are in the category of algebraic $Cu^-_u$-semigroups, any element is the supremum of an increasing sequence of compact elements. Thus any morphism is entirely determined by its restriction to compact elements. One can conclude that $\alpha = \beta$ and the proof is complete. □

Theorem 5.20. The functor $H_* : Cu^-_u \rightarrow AbGp_u$ yields a natural isomorphism $\eta_* : H_* \circ Cu_1u \cong K_*$.

Proof. First we prove that $K_*(A)_c \cong Cu_1(A)_c$ as monoids and the result will follow from the Grothendieck construction.

We know that $Cu_1(A)_c$ is a monoid. Now consider $[(a,u)] \in Cu_1(A)_c$. By [Corollary 3.3] we know that $[a]$ is a compact element of $Cu(A)$. Besides, since $A$ has stable rank one, we know that we can find a projection $p \in A \otimes K$ such that $[p] = [a]$ in $Cu(A)$. So without loss of generality, we now describe compact elements of $Cu_1(A)$ as classes $[(p,u)]$ where $p$ is projection in $A \otimes K$ and $u$ is a unitary element in $\text{her}(p)$.

On the other hand, by [Theorem 5.7] we have $Cu_1(A)_\text{max} \cong K_1(A)$, where the AbGP-isomorphism is given by $[(s_{A\otimes K},u)] \mapsto [u]$, where $s_{A\otimes K}$ is any strictly positive element of $A \otimes K$. Combined with [Definition 5.6] we get a monoid morphism $j : Cu_1(A) \rightarrow K_1(A)$. Now set:

$$\alpha : Cu_1(A)_c \rightarrow K_1(A)_c,$$
$$[(p,u)] \mapsto (j\{p\}, j\{[p,u]\}).$$

It is routine to check that $\alpha$ is monoid morphism. Further, observe that $j\{p,u\} = \delta_{i_\text{max}}([u])$ for any $[(p,u)] \in Cu_1(A)_c$, where $\delta_{i_\text{max}} : K_1(\text{her}(p)) \rightarrow K_1(A)$ (see [Notation 4.2]). Thus, $j\{p,u\} = [u + (1 - p)]_{K_1(A)}$. Now, since $A$ has stable rank one, Murray-von Neumann equivalence and Cuntz equivalence
agree on projections. It is now clear that $\alpha$ is an isomorphism and hence $\text{Cu}_1(A) \cong K_*(A)_+$ as monoids. From the Grothendieck construction, one can check that $(K_*(A), K_*(A)_+)$ is an ordered group. Finally, it is routine to check that $[[1_A, 1_A]]$ is a compact order-unit for $\text{Cu}_1(A)$ (a fortiori, an order-unit for $(\text{Gr} (\text{Cu}_1(A)_+), \text{Cu}_1(A)_+)$) and that $\alpha([[1_A, 1_A]]) = 1_{K_*(A)}$.

We conclude that for any $A \in C^*_\text{sr,II}$, there exists a natural ordered group isomorphism $\eta_{u,A} : H_+ \circ \text{Cu}_{1,u}(A) \cong (K_*(A), K_*(A)_+, 1_{K_*(A)})$ that preserves the order-unit and hence there exists a natural isomorphism $\eta_A : H_+ \circ \text{Cu}_{1,u} \cong K_*$. 

**Corollary 5.21.** By restricting to the category $\text{Cu}_{1,u}^\text{alg}$, we can fully recover $K_*$ from $\text{Cu}_{1,u}$ through $H_+$. A fortiori, we have:

(i) $\text{Cu}_{1,u}$ is a complete invariant for unital $\text{AH}_2$ algebras of real rank zero.

(ii) $\text{Cu}_{1,u}$ classifies homomorphisms of unital AT algebras with real rank zero.

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