Gravitational self-force corrections to tidal invariants for particles on circular orbits in a Kerr spacetime

Donato Bini and Andrea Geralico

Istituto per le Applicazioni del Calcolo “M. Picone,” CNR, I-00185 Rome, Italy

(Dated: June 25, 2018)

We generalize to the Kerr spacetime existing self-force results on tidal invariants for particles moving along circular orbits around a Schwarzschild black hole. We obtain linear-in-mass-ratio corrections to the quadratic and cubic electric-type invariants and the quadratic magnetic-type invariant in series of the rotation parameter up to the fourth order. We then construct the eigenvalues of both electric and magnetic tidal tensors and analytically compute them through high post-Newtonian orders.

I. INTRODUCTION

Consider the general relativistic description of a two-body system in the case in which one body is spinning (say, with mass \( m_2 \) and spin \( S_2 = m_2 a_2 \)) and the other is nonspinning, with mass (say \( m_1 \)) much smaller than that of the companion, i.e., \( m_1 \ll m_2 \). In this situation, the small body affects the gravitational field generated by the large body (which can be identified with a background Kerr spacetime with parameters \( m_2 \) and \( a_2 \)) by introducing a first-order perturbation proportional to the mass ratio \( q = m_1/m_2 \) of the two bodies, conveniently studied by using the Teukolsky formalism. With the aid of standard techniques one obtains the full perturbed metric, which is then suitably regularized and reconstructed along the world line of the small body and used to compute gauge-invariant quantities associated with physical observables.

Up to now it has been possible to compute in a Kerr spacetime with high accuracy (i.e., to a high post-Newtonian (PN) order) the corrections to gauge-invariant quantities which are continuous across the world line of the small body, namely its gravitational redshift in the field of the companion both in the case of circular and eccentric equatorial orbits [1–3]. Using instead the Schwarzschild spacetime as a background, a number of gauge-invariant quantities generally discontinuous across the particle’s world line has already been analytically computed, including the precession rate of a test gyroscope and the tidal curvature invariants (i.e., the tensorial contraction of the electric and magnetic parts of the Riemann tensor associated with natural observers) [4–15].

Another natural description of the two-body problem is the Hamiltonian one, where one solves the equations of motion in PN sense, order by order, starting from the flat background where the two bodies live. Their mutual interaction generates dynamical corrections to the associated gravitational potentials, equivalent to curvature effects in perturbation theory. There exists a direct correspondence between these two points of view, so that new results from black hole perturbations can be converted in the Hamiltonian formalism (e.g., improving the knowledge of the gravitational potentials).

Among the various coordinate-based Hamiltonian approaches (e.g., Arnowitt-Deser-Misner (ADM) coordinates and Harmonic coordinates), the so called “effective one-body” (EOB) model [16, 17]—in short, a properly partially PN-resummed Hamiltonian model—has proven to be very efficient (and incredibly fast in comparison with full numerical relativity simulations) in following all the dynamics of the two bodies up to their merging. More the 250 thousands of EOB-based waveform templates have been generated in the analysis of the recently discovered gravitational wave signals by the LIGO and VIRGO detectors [18–21]. So far, translating into EOB new results from Kerr perturbations has been an important contribution of GSF in the last few years, although mainly orbital effects have been taken into account. The presence of spin in fact requires some care in the modeling of the “effective” interaction and different EOB models with spin exist in the literature [22, 23].

The contribution of the present work is the analytical computation of linear-in-mass-ratio corrections to the quadratic and cubic electric-type and the quadratic magnetic-type tidal invariants for particles moving along circular equatorial orbits in a Kerr spacetime, generalizing previous results for a non-spinning black hole [10–12]. We will follow a well established procedure based on Teukolsky formalism and metric completion in a radiation gauge (see, e.g., Refs. [2–4] and references therein). Therefore, we will limit ourselves to provide the final result with a minimum of related details. This work continues some recent achievements on tidal invariants of Refs. [24, 25], where GSF corrections to them have been computed in the case of spinning bodies on circular orbits as well spinless particles on slightly eccentric orbits around a Schwarzschild black hole.

II. KERR METRIC, PERTURBATIONS AND TIDAL INVARIANTS

The Kerr metric with signature \(-2\) and parameters \( m_2 \) and \( a_2 = a \) (with \( a = a/m_2 \) dimensionless) written
in Boyer-Lindquist coordinates reads
\[
ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta
= \left(1 - \frac{2mr}{\Sigma}\right) dt^2 + \frac{4amr \sin^2 \theta}{\Sigma} dt d\phi
- \frac{\Sigma}{r} dr^2 - \Sigma d\theta^2
- \left(r^2 + a^2 + \frac{2m^2ra^2 \sin^2 \theta}{\Sigma}\right) \sin^2 \theta d\phi^2,
\]
where
\[
\Delta = r^2 + a^2 - 2m^2 r, \quad \Sigma = r^2 + a^2 \cos^2 \theta. \quad (2.2)
\]
Let the perturbation be associated with a spinless particle of mass \(m_1\) moving along a circular equatorial geodesic orbit at \(r = r_0\), with four velocity
\[
u^\alpha = \Gamma k^\alpha, \quad k = \partial_t + \zeta \partial_\phi. \quad (2.3)
\]
Here \(\zeta\) is the constant angular velocity and \(\Gamma = \nu^t\) is a normalization factor \((\nu^\alpha \nu_\alpha = 1)\), whose unperturbed values (denoted by a bar) are the following
\[
m_2 \zeta = \frac{\nu^{3/2}}{1 + a\nu^{3/2}}, \quad \Gamma = \frac{1 + a\nu^{3/2}}{\sqrt{1 - 3\nu + 2a\nu^{3/2}}}, \quad (2.4)
\]
where \(\nu = m_2/r\) is the dimensionless inverse radius of the orbit. The constant vector \(k\) generates a helical symmetry which is assumed to be a property of the (regularized) perturbed spacetime
\[
g_{\alpha\beta}^R = g_{\alpha\beta} + q h_{\alpha\beta}^R + O(q^2), \quad (2.5)
\]
namely \(k\) is supposed to be a Killing vector for the perturbed spacetime (it is also a Killing vector of the background spacetime, being a combination of \(\partial_t\), generating time translations, and \(\partial_\phi\), generating the rotational symmetry about the spin axis of the black hole).

The particle is characterized by its energy-momentum tensor
\[
T^{\alpha\beta} = \frac{m_1}{u^\nu \tau_0} \nu^\alpha \nu^\beta \delta_3, \quad (2.6)
\]
where
\[
\delta_3 = \delta(r - r_0) \delta \left(\theta - \frac{\pi}{2}\right) \delta(\phi - \zeta t), \quad (2.7)
\]
which enters the Teukolsky equation for the perturbed spin-weight \(s = 2\) Weyl scalar \(\psi_0\) as a source term. Decomposing \(\psi_0\) in spheroidal angular harmonics and using the separability property of the Teukolsky equation in the frequency domain one ends up with a single radial equation to be solved in PN sense. From \(\psi_0\) one then constructs the spin-2 Hertz potential \(\Psi\) following a procedure due to Chrzanowski-Cohen-Kegeles and, eventually (by applying a proper second-order differential operator) one obtains the perturbed metric \(h_{\alpha\beta}\) in the radiation gauge. All these steps are well established in the literature (see, e.g., Ref. [2] and references therein). Finally, upon regularization, with the reconstructed perturbed metric and the four velocity field of the particle \(u\), one can compute the perturbed Riemann tensor with associated electric and magnetic parts. The latters play the role of tidal potentials as explained below.

The geodesic condition in the perturbed spacetime implies
\[
m_2 \zeta = \frac{\nu^{3/2}}{1 + a\nu^{3/2}} \left(1 - \frac{1 + \hat{a}\nu^{3/2}}{4a^2 - m^2 [\partial_r h_{kk}]}\right), \quad (2.8)
\]
with \(h_{kk} = [\hat{h}_{\mu\nu}(x) \overline{k} x]_1\), evaluated along the world line of the perturbing body. Introducing the dimensionless frequency parameter
\[
y = (m_2 \zeta)^{2/3}, \quad (2.9)
\]
the previous equation can then be inverted to give
\[
u = \frac{y}{1 - a y^{3/2}} \left(1 + \frac{m^2 [\partial_r h_{kk}]}{6y^2 (1 - a y^{3/2})^{2/3}}\right), \quad (2.10)
\]
which is used to reexpress the radius of the orbit in terms of a gauge-invariant variable.

### A. Tidal invariants

We briefly recall below how to define the tidal potentials for a system of \(N\) gravitationally interacting bodies through an effective action approach, as discussed in Ref. [10]. For point-mass objects with four velocities \(u^\alpha_A = dx^\alpha_A/d\tau_A\) such a description is performed in terms of the action
\[
S_0 = \frac{c^4}{16\pi G} \int \frac{d^4x}{c} \sqrt{-g} R - \sum_A \int m_A c^2 d\tau_A, \quad (2.11)
\]
where \(d\tau_A = -(u_A)_\mu dx^\mu\) is the regularized proper time along the world line \(x^\mu_A(\tau_A)\) of body \(A\). For extended objects, we have that each body feels the gravitational field of the whole system \(\sum_{A'}\)\(d\tau_A\), undergoing tidal effects which can be computed by adding other non-minimal couplings to \(S_0\), involving higher-order derivatives of the field evaluated along the world line of each body (see, e.g., Ref. [30] and references therein). The latter can be expressed in terms of the graviotelectric \((G^A_{\alpha\beta}(\tau_A) \equiv G^A_{\alpha\beta}(\tau_A))\) and gravimagnetic \((H^A_{\alpha\beta}(\tau_A) \equiv H^A_{\alpha\beta}(\tau_A))\) tidal tensors associated with the body \(A\), which are symmetric and trace-free.

The most general world line non-minimal action has then the form
\[
S_{non-min} = \sum_A S^A_{non-min}, \quad (2.12)
\]
with
\[
S^A_{non-min} = \frac{1}{4} \mu_A^{(2)} \int d\tau_A G^A_{\alpha\beta} G^A_{\alpha\beta}
+ \frac{1}{6} \sigma_A^{(2)} \int d\tau_A H^A_{\alpha\beta} H^A_{\alpha\beta} + \ldots, \quad (2.13)
\]
We will consider here only the invariants associated with the quadrupolar electric-type and magnetic-type tidal tensors $G_{\alpha\beta}, H_{\alpha\beta}$, related as follows to the spatial components of the “electric” and “magnetic” parts of the Riemann tensor

\[
G_{\alpha\beta} = -\mathcal{E}(u_A)_{\alpha\beta}, \quad H_{\alpha\beta} = 2cB(u_A)_{\alpha\beta}, \quad (2.14)
\]

where $\mathcal{E}(u_A)_{\alpha\beta}$ and $B(u_A)_{\alpha\beta}$ are defined as

\[
\mathcal{E}(u_A)_{\alpha\beta} = R_{\alpha\mu\beta\nu}u_\mu^Au_\nu^A, \quad B(u_A)_{\alpha\beta} = [R^\gamma]_{\alpha\mu\beta\nu}u_\mu^Au_\nu^A, \quad (2.15)
\]

the symbol * denoting the spacetime dual of a tensor. The associated non-minimal world line action \(^{(2.13)}\) of the body \(1\) then reads

\[
S_{\text{non-min}} = \frac{1}{4} \mu_1^{(2)} \int d\tau_1 \text{Tr} [\mathcal{E}(u_1)^2] + \frac{2}{3} \sigma_1^{(2)} \int d\tau_1 \text{Tr} [B(u_1)^2] + \ldots, \quad (2.16)
\]

where we have set $G = c = 1$. Hereafter, we will omit the body label $A = 1$ to ease notation.

In the present paper we compute first-order GSF corrections to the quadratic tidal-electric and tidal-magnetic invariants $\text{Tr} [\mathcal{E}(u)^2]$ and $\text{Tr} [B(u)^2]$ as well as to the cubic tidal-electric invariant $\text{Tr} [\mathcal{E}(u)^3]$, and evaluate the eigenvalues of the tidal tensors $\mathcal{E}(u)$ and $B(u)$. For convenience, we will work with their rescaled counterparts

\[
\mathcal{E}(u) = \Gamma^2 \mathcal{E}(k), \quad B(u) = \Gamma^2 B(k), \quad (2.17)
\]

with associated invariants

\[
J_{e^2} = m_2^4 \text{Tr} [\mathcal{E}(k)^2], \quad J_{b^2} = m_2^2 \text{Tr} [B(k)^2], \quad J_{e^3} = m_3^2 \text{Tr} [\mathcal{E}(k)^3]. \quad (2.18)
\]

### III. GSF COMPUTATION OF TIDAL INVARIANTS

The first-order self-force (ISF) accurate expansions of the electric-type and magnetic-type tidal invariants \(^{(2.13)}\) read

\[
J_{e^2} = J_{e^2}^{(0)}[1 + q \delta_{e^2}(y)] + O(q^2), \quad J_{b^2} = J_{b^2}^{(0)}[1 + q \delta_{b^2}(y)] + O(q^2), \quad J_{e^3} = J_{e^3}^{(0)}[1 + q \delta_{e^3}(y)] + O(q^2), \quad (3.1)
\]

where

\[
J_{e^2}^{(0)} = 6u_6^1 - 3u + 3u^2 - 2\tilde{a}u^{3/2}(1 + 3\tilde{a}^2u^2) + 3\tilde{a}^2u^2(1 + \tilde{a}^2u^2) + \tilde{a}^2u^3 \frac{1}{1 + \tilde{a}u^{3/2}},
\]

\[
J_{b^2}^{(0)} = 18u^7(1 - 2u + \tilde{a}^2u^2) \frac{(1 - \tilde{a}u^{1/2})^2}{(1 + \tilde{a}u^{3/2})^2},
\]

\[
J_{e^3}^{(0)} = -3u^9(1 - 3u + 2\tilde{a}u^{3/2})(2 - 3u - 2\tilde{a}u^{3/2} + 3\tilde{a}^2u^2) \frac{1 - 4\tilde{a}u^{3/2} + 3\tilde{a}^2u^2}{(1 + \tilde{a}u^{3/2})^6},
\]

\[
d\gamma_{\text{reg}} = \sum_{l=0}^{\infty} \left( \delta_{l}^0 - B(y;l) \right), \quad (3.3)
\]

where

\[
\delta_{l}^0 = \frac{1}{2} (\delta_{l}^+ + \delta_{l}^-), \quad (3.4)
\]

and the “subtraction term” is of the form

\[
B(y;l) = l(l+1)b_0(y) + b_1(y), \quad (3.5)
\]

denote the corresponding unperturbed values, with $u = y/(1 - \tilde{a}y^{3/2})^{3/3}$.

The expressions of the first order corrections $\delta_{e^2}, \delta_{b^2}$ and $\delta_{e^3}$ in terms of the components of the perturbed metric and their derivatives are listed in Appendix. Their regularized values are given by the convergent series \(^{(11)}\). We take left-right averages across the particle’s world line and only subtract the $B$-term, which is enough to have a convergent series. The completion of the metric then requires the contribution of nonradiative multipoles $l = 0, 1$, which has been recently obtained in Ref. \([34]\).

We will omit showing explicitly the final results for $\delta_{e^2}, \delta_{b^2}$ and $\delta_{e^3}$ after regularization, focusing only on the associated eigenvalues of the tidal tensors.
A. Eigenvalues

The eigenvalues of the tidal-electric and tidal-magnetic quadrupolar tensors \( m_2^E(u)^{\mu\nu} \) and \( m_2^B(u)^{\mu\nu} \) are such that

\[
\begin{align*}
    m_2^E(u) &= \text{diag}[(\lambda_1^{(E)}, \lambda_2^{(E)}), -(\lambda_1^{(E)} + \lambda_2^{(E)})], \\
    m_2^B(u) &= \text{diag}[(\lambda(B), -\lambda(B), 0)],
\end{align*}
\]

where we have used their traceless property, and the existence of a zero eigenvalue of \( B(u) \) \[12\]. They are related to the eigenvalues \( \sigma_a^{(E)} \) and \( \sigma(B) \) of the rescaled tidal tensors \( E(k) \) and \( B(k) \) by

\[
\begin{align*}
    \lambda_a^{(E)} &= \Gamma^2 \sigma_a^{(E)}, \\
    \lambda(B) &= \Gamma^2 \sigma(B),
\end{align*}
\]

where

\[
\Gamma = \frac{1 + \hat{a}u^{3/2}}{\sqrt{1 - 3u + 2au^{3/2}}} + q \delta U(y),
\]

with \( u = y/(1 - \hat{a}y^{3/2})^{2/3} \) and the 1SF expansion \( \delta U(y) \) has been derived in our previous work \[3\]. As usual, we will write

\[
\begin{align*}
    \lambda_a^{(E)} &= \lambda_a^{(E)0SF} + q\lambda_a^{(E)1SF}, \\
    \lambda(B) &= \lambda(B)0SF + q\lambda(B)1SF,
\end{align*}
\]

where the unperturbed (0SF) values are

\[
\begin{align*}
    \lambda_1^{(E)0SF} &= -u^3 \frac{2 - 3u - 2\hat{a}u^{3/2} + 3\hat{a}^2u^2}{1 - 3u + 2au^{3/2}}, \\
    \lambda_2^{(E)0SF} &= u^3 \frac{4\hat{a}u^{3/2} + 3\hat{a}^2u^2}{1 - 3u + 2au^{3/2}}, \\
    \lambda_0^{(E)0SF} &= 3u^{7/2}(1 - \hat{a}u^{1/2})^{\frac{\sqrt{1 - 2u + \hat{a}^2u^2}}{1 - 3u + 2au^{3/2}}}.
\end{align*}
\]

with \( u = y/(1 - \hat{a}y^{3/2})^{2/3} \). The 1SF corrections to the rescaled eigenvalues \( \sigma_a^{(E)} \) and \( \sigma(B) \) are computed following Ref. \[11\], so that one 1PN level in the analytic accuracy of \( \sigma_2^{(E)} \) is lost.

We also use the notation

\[
\begin{align*}
    \lambda_1^{(E)1SF}(y) &= \lambda_1^{(E)1SF} a^0(y) + \hat{a} \lambda_1^{(E)1SF} a^1(y) + \hat{a}^2 \lambda_1^{(E)1SF} a^2(y) + \hat{a}^3 \lambda_1^{(E)1SF} a^3(y) + \hat{a}^4 \lambda_1^{(E)1SF} a^4(y),
\end{align*}
\]

and similarly for the others.

The Schwarzschild values \( \lambda_1^{(E)1SF} a^0, \lambda_2^{(E)1SF} a^0 \) and \( \lambda_0^{(B)1SF} \) are known with high PN accuracy \[11, 13, 15\]. We recall below for completeness only the first few terms

\[
\begin{align*}
    \lambda_1^{(E)1SF} &= 2y^3 + 2y^4 - \frac{19}{4} y^5 + \left( \frac{227}{3} + \frac{593}{256} \pi^2 \right) y^6 \\
    &\quad + \left( \frac{7177}{4800} - \frac{719}{256} \pi^2 + \frac{1536}{5} \ln(2) - \frac{384}{5} \ln(y) + \frac{768}{5} \gamma \right) y^7 + O_{ln}(y^8), \\
    \lambda_2^{(E)1SF} &= -y^3 - \frac{3}{2} y^4 - \frac{23}{8} y^5 + \left( -\frac{2593}{48} + \frac{1249}{1024} \pi^2 \right) y^6 \\
    &\quad + \left( \frac{362051}{3200} - \frac{128}{5} \ln(y) + \frac{1737}{1024} \pi^2 - \frac{256}{5} \gamma - \frac{512}{5} \ln(2) \right) y^7 + O_{ln}(y^8), \\
    -\lambda_0^{(B)1SF} &= 2y^{7/2} + 3y^{9/2} + \frac{59}{4} y^{11/2} + \left( \frac{2761}{24} - \frac{41}{16} \pi^2 \right) y^{13/2} \\
    &\quad + \left( \frac{1618039}{2880} - \frac{112919}{3072} \pi^2 + \frac{1808}{15} \gamma + 240 \ln(2) + \frac{904}{15} \ln(y) \right) y^{15/2} + O_{ln}(y^{17/2}).
\end{align*}
\]

The \( O(\hat{a}) - O(\hat{a}^4) \) contributions for each eigenvalue are the main original contribution of the present work and are listed below.

Results for \( \lambda_1^{(E)1SF} \):

\[
\begin{align*}
    \lambda_1^{(E)1SF} &= 2y^3 + 2y^4 - \frac{19}{4} y^5 + \left( \frac{227}{3} + \frac{593}{256} \pi^2 \right) y^6 \\
    &\quad + \left( \frac{7177}{4800} - \frac{719}{256} \pi^2 + \frac{1536}{5} \ln(2) - \frac{384}{5} \ln(y) + \frac{768}{5} \gamma \right) y^7 + O_{ln}(y^8), \\
    \lambda_2^{(E)1SF} &= -y^3 - \frac{3}{2} y^4 - \frac{23}{8} y^5 + \left( -\frac{2593}{48} + \frac{1249}{1024} \pi^2 \right) y^6 \\
    &\quad + \left( \frac{362051}{3200} - \frac{128}{5} \ln(y) + \frac{1737}{1024} \pi^2 - \frac{256}{5} \gamma - \frac{512}{5} \ln(2) \right) y^7 + O_{ln}(y^8), \\
    -\lambda_0^{(B)1SF} &= 2y^{7/2} + 3y^{9/2} + \frac{59}{4} y^{11/2} + \left( \frac{2761}{24} - \frac{41}{16} \pi^2 \right) y^{13/2} \\
    &\quad + \left( \frac{1618039}{2880} - \frac{112919}{3072} \pi^2 + \frac{1808}{15} \gamma + 240 \ln(2) + \frac{904}{15} \ln(y) \right) y^{15/2} + O_{ln}(y^{17/2}).
\end{align*}
\]
\[
\lambda_1^{\text{EIS} a^2}(y) = -4y^{9/2} - \frac{95}{3} y^{11/2} - \frac{923}{6} y^{13/2} \\
+ \left( -\frac{44357}{72} + \frac{1165}{768} \pi^2 \right) y^{15/2} \\
+ \left( -\frac{23584579}{7200} + \frac{119515}{1536} \pi^2 - \frac{4784}{15} \gamma - \frac{2392}{15} \ln(y) - \frac{9392}{15} \ln(2) \right) y^{17/2} \\
+ \left( -\frac{22210361969}{4233600} - \frac{693016}{315} \gamma - \frac{358604}{315} \ln(y) - \frac{325576}{105} \ln(2) - \frac{20451899}{147456} \pi^2 - \frac{9234}{7} \ln(3) \right) y^{19/2} \\
- \frac{825032}{1575} \pi y^{10} \\
+ \left( -\frac{58928132968141}{228614400} + \frac{1374011}{2835} \ln(y) + \frac{3945526}{2835} \gamma - \frac{15750506}{2835} \ln(2) + \frac{21870}{7} \ln(3) \\
+ \frac{10601643869}{589824} y^2 + \frac{250702133}{524288} \pi^4 \right) y^{21/2} \\
- \frac{3849604}{525} \pi y^{11} \\
+ \left( -\frac{38054676977}{32744250} \ln(y) - \frac{18546721509346775267}{19363693600000} + \frac{1640648}{1575} \ln(y)^2 + \frac{456891023}{16372125} \gamma \\
+ \frac{5235008}{315} \gamma \ln(2) - \frac{74432}{15} \zeta(3) + \frac{1921958447}{654885} \ln(2) \\
+ \frac{224152947}{12320} \ln(3) - \frac{166015625}{9504} \ln(5) + \frac{1694532559908697}{11890851840} \pi^2 - \frac{222657808167}{33544320} \pi^4 + \frac{26137376}{1575} \ln(2)^2 \\
+ \frac{6562592}{1575} \gamma^2 + \frac{6562592}{1575} \ln(y) + \frac{2617504}{315} \ln(2) \ln(y) \right) y^{23/2} \\
+ \frac{2422993377}{6548850} \pi y^{12} + O_{\ln(y^{25/2})},
\]

\[
\lambda_2^{\text{EIS} a^2}(y) = 5y^{5} + \frac{53}{2} y^{6} + \frac{9539}{72} y^{7} + \left( \frac{131101}{144} - \frac{3343}{512} \pi^2 \right) y^{8} \\
+ \left( \frac{13469583}{3200} - \frac{292797}{8192} \pi^2 + \frac{4112}{5} \ln(2) + \frac{1032}{5} \ln(y) + \frac{2064}{5} \gamma \right) y^{9} \\
+ \left( \frac{151504}{315} \ln(y) + \frac{9439449583}{1209600} + \frac{30308}{315} \gamma + \frac{161312}{315} \ln(2) + \frac{9477}{7} \ln(3) + \frac{1053261163}{1179648} \pi^2 \right) y^{10} \\
+ \frac{219992}{525} \pi y^{21/2} \\
+ \left( \frac{3779243}{2835} \ln(y) + \frac{29633535218351}{101606400} + \frac{529502}{567} \gamma + \frac{4608}{5} \zeta(3) + \frac{36439126}{2835} \ln(2) - \frac{79623}{14} \ln(3) \\
- \frac{21842364000733}{1101004800} \pi^2 - \frac{7984116587}{67108864} \pi^4 \right) y^{11} \\
+ \frac{15603907}{11025} \pi y^{23/2} \\
+ \left( \frac{24488109508}{606375} \gamma - \frac{280768}{21} \gamma \ln(2) + \frac{1323559510617011}{6442509440} \pi^4 + \frac{1986757084}{72765} \ln(2) + \frac{289495377}{24640} \ln(3) \\
- \frac{585504}{175} \gamma^2 - \frac{2383592}{175} \ln(2)^2 + \frac{283203125}{19008} \ln(5) + \frac{55872}{5} \zeta(3) + \frac{25199471184842137}{79272345600} \pi^2 \\
+ \frac{14829961154}{606375} \ln(y) - \frac{585504}{175} \gamma \ln(y) - \frac{140384}{21} \ln(2) \ln(y) - \frac{146376}{175} \ln(y)^2 \\
- \frac{16911655702097901527}{3520661760000} \right) y^{12} + O_{\ln(y^{25/2})},
\]
\[ \lambda_1^{(E)} \text{SF } a^3(y) = \left( \frac{5y^{13/2} - 55y^{15/2}}{648} - \frac{324305}{y^{17/2}} \right) + \left( \frac{-10722523}{4050} \right) + \left( \frac{-34019}{1536} \right) - \frac{768}{5} \right) - \left( \frac{5}{5} \right) \left( \ln(y) - \frac{576}{5} \ln(2) \right) y^{19/2} \]
\[ + \left( \frac{-854990743}{28800} \right) + \left( \frac{60387577}{49152} \right) - \frac{608 \ln(y)}{5} - \frac{704}{5} \left( \frac{-304}{5} \zeta(3) - \frac{3904}{5} \ln(2) \right) y^{21/2} \]
\[ + \left( \frac{-2142752}{315} \right) \ln(y) - \left( \frac{-2235738608977}{7620480} \right) - \frac{2255776}{315} \left( \frac{7}{5} \zeta(3) - \frac{2043824}{135} \ln(2) \right) \]
\[ - \left( \frac{15552}{7} \right) \ln(3) + \left( \frac{3535975103}{165888} \right) \pi^2 \left( 1 - 203368 \right) \pi y^{12} + O_\ln(y^{25/2}) \]
\[
\lambda_2^{(E)}^{1\text{SF}} a^2(y) = -5y^5 - 21y^6 - \frac{3647}{24}y^7 + \left( -\frac{34247}{36} + \frac{12427}{2048}\pi^2 \right)y^8 \\
+ \left( -\frac{40143929}{9600} + \frac{582859}{65536}\pi^2 - \frac{772}{5}\ln(y) - \frac{1544}{5}\gamma - 616\ln(2) \right)y^9 \\
+ \left( -\frac{68134}{315}\ln(y) + \frac{719136473}{8064} - \frac{136268}{315}\gamma - \frac{25804}{315}\ln(2) - 729\ln(3) - \frac{863548149}{786432}\pi^2 \right)y^{10} \\
- \frac{4708}{15}\pi y^{21/2} \\
+ \left( -\frac{1144565}{567}\gamma - \frac{42961473011}{536870912}\pi^4 - \frac{24381769}{2835}\ln(2) + \frac{17820}{7}\ln(3) - \frac{1536}{5}\zeta(3) \\
- \frac{1286325724676179}{39636172800}\pi^2 - \frac{1531637}{1134}\ln(y) + \frac{63382286411599}{20321280} \right)y^{11} + O_\ln(y^{23/2}),
\]

\[
\lambda_2^{(E)}^{1\text{SF}} a^3(y) = -\frac{5}{3}y^{13/2} + 62y^{15/2} + \frac{100469}{216}y^{17/2} \\
+ \left( \frac{22598287}{8100} + \frac{192}{5}\ln(y) + \frac{256}{5}\zeta(3) + \frac{192}{5}\ln(2) + \frac{283177}{12288}\pi^2 \right)y^{19/2} \\
+ \left( \frac{278665423}{5760} + \frac{1536}{5}\ln(y) + \frac{1568}{5}\zeta(3) + 632\ln(2) - \frac{49870393}{16384}\pi^2 \right)y^{21/2} \\
+ O_\ln(y^{23/2}),
\]

\[
\lambda_2^{(E)}^{1\text{SF}} a^4(y) = -\frac{263}{18}y^3 - \frac{625}{8}y^9 + \left( -\frac{683999}{648} - \frac{135495}{65536}\pi^2 \right)y^{10} \\
+ \left( -\frac{24615820919}{2721600} - \frac{264}{5}\gamma + \frac{54784}{23625}\pi^4 - \frac{2216}{5}\ln(2) - \frac{31232}{15}\zeta(3) + 2048\zeta(5) + \frac{2919530863}{58982400}\pi^2 \\
- \frac{276}{5}\ln(y) \right)y^{11} + O_\ln(y^{23/2}).
\]

Results for \(\lambda^{(B)}^{1\text{SF}}\): 

\[ (3.14) \]
\[-\lambda^{(B)1SF a^7}(y) = -4y^4 - \frac{28}{3} y^5 - \frac{121}{2} y^6 + \left( -384 + \frac{41}{8} \pi^2 \right) y^7 \\
+ \left( -\frac{1690523}{1440} + \frac{497}{192} \pi^2 - \frac{516}{5} \ln(y) - \frac{1032}{5} \gamma - \frac{2056}{5} \ln(2) \right) y^8 \\
+ \left( \frac{498497509}{50400} - \frac{763429}{512} \pi^2 + \frac{9956}{21} \ln(2) + \frac{318}{35} \ln(y) + \frac{636}{35} \gamma - \frac{2916}{7} \ln(3) \right) y^9 \\
- \frac{109996}{525} \pi y^{19/2} \\
+ \left( \frac{223267}{2835} \gamma + \frac{21059401}{262144} \pi^4 - \frac{217951}{81} \ln(2) + \frac{13365}{7} \ln(3) - \frac{3314946773}{442368} \pi^2 \\
+ \frac{1067166582679}{25401600} \ln(y) \right) y^{10} \\
- \frac{971542}{11025} \pi y^{21/2} \\
+ \left( \frac{586496084291}{32744250} \gamma + \frac{140384}{21} \ln(y) - \frac{603286332419}{201326592} \pi^4 - \frac{6239060383}{261954} \ln(2) \\
- \frac{24619869}{6160} \ln(3) + \frac{292752}{175} \gamma^2 + \frac{1169296}{175} \ln(2)^2 - \frac{9765625}{4752} \ln(5) - \frac{16416}{5} \zeta(3) \\
+ \frac{58939924322641}{928972800} \pi^2 + \frac{73188}{175} \ln(y)^2 + \frac{292752}{175} \gamma \ln(y) + \frac{70192}{21} \ln(2) \ln(y) \\
- \frac{88892510118308059}{220041360000} \ln(y) \right) y^{11} \\
+ \frac{7620479}{66825} \pi y^{23/2} + O_{ln}(y^{12}), \]

\[-\lambda^{(B)1SF a^7}(y) = \frac{10}{3} y^{11/2} + \frac{604}{9} y^{13/2} + \frac{1457}{4} y^{15/2} + \left( \frac{64885}{36} + \frac{9631}{1024} \pi^2 \right) y^{17/2} \\
+ \left( \frac{281701261}{14400} - \frac{12600593}{12288} \pi^4 + \frac{976}{5} \ln(y) + \frac{1952}{5} \gamma + \frac{2272}{3} \ln(2) \right) y^{19/2} \\
+ \left( \frac{147786}{35} \gamma + \frac{716126}{105} \ln(2) + \frac{44469}{28} \ln(3) - \frac{9838313339}{196008} \pi^2 + \frac{226607}{105} \ln(y) + \frac{23409038221}{44100} \right) y^{21/2} \\
+ \frac{207652}{315} \pi y^{11} \\
+ \left( \frac{36534419}{8505} \gamma + \frac{199119651947}{50331648} \pi^4 + \frac{124957859}{8505} \ln(2) - \frac{99387}{280} \ln(3) + \frac{3072}{5} \zeta(3) \\
- \frac{197722622556217}{928972800} \pi^2 + \frac{48690899}{17010} \ln(y) + \frac{860650678007317}{457228800} \right) y^{23/2} + O_{ln}(y^{12}), \]

\[-\lambda^{(B)1SF a^3}(y) = -3y^6 - \frac{463}{18} y^7 - \frac{96337}{648} y^8 + \left( -\frac{20683}{16} + \frac{429}{128} \pi^2 \right) y^9 \\
+ \left( -\frac{2035430657}{259200} + \frac{1101311}{18432} \pi^2 - \frac{1828}{5} \gamma - \frac{3812}{5} \ln(2) - \frac{512}{5} \zeta(3) - \frac{1106}{5} \ln(y) \right) y^{10} \\
+ \left( -\frac{25730}{21} \gamma - \frac{49566}{315} \ln(2) + \frac{8019}{7} \ln(3) - \frac{3776}{5} \zeta(3) - \frac{25089030215}{663552} \pi^2 \\
- \frac{100949}{105} \ln(y) + \frac{729564502223}{2177280} \right) y^{11} \\
- \frac{38734}{105} \pi y^{23/2} + O_{ln}(y^{12}), \]
\[-\lambda^{(B)\text{1SF}} a^4 (y) = \frac{15}{4} y^{15/2} + \frac{10627}{648} y^{17/2} + \frac{2983073}{7776} y^{19/2} + \left(\frac{12732217}{4800} + \frac{384}{5} \ln(2) + \frac{2048}{15} \zeta(3) + \frac{32339}{2048} \pi^2 + \frac{384}{5} \ln(y)\right) y^{21/2} + \left(\frac{-406}{3} - \frac{89024}{23625} \pi^4 + \frac{298}{3} \ln(2) - \frac{11840}{3} \zeta(3) - 3328 \zeta(5) - \frac{4998352104167}{928972800} \pi^2 + \frac{5033}{15} \ln(y) + \frac{60509611189}{806400}\right) y^{23/2} + O_m(y^{12}). \] (3.15)

In order to associate a theoretical error to our analytical expressions we follow the discussion of Ref. [11] (see, e.g., Eq. (4.18) there). The 1SF corrections to the eigenvalues are given as series expansion with respect to the black hole rotation parameter \( \hat{a} \), so that we expect that each term generally diverges at the Schwarzschild light-ring \( y = 1/3 \). Therefore, we use the following estimate of the theoretical error \[ \sigma_{\text{th}}^N (y) \simeq C_N + \frac{1}{2} (3y)^{N+\frac{1}{2}} (1-3y)^{\alpha_N}, \] (3.16)

where \( N \) is the maximum power appearing in the PN expansion of each term \( \lambda^{(\text{1SF})}_N a^n \), and \( \alpha_N \) is an adjustable parameter which can be suitably chosen in order to improve the agreement with numerical data, if available. In this case no numerical data exist and hence we assume it to be zero. Furthermore, the (positive) coefficients \( C_N + \frac{1}{2} \) are roughly of the order of unity, typically between 1 and 10. For example, the estimated error on \( \lambda^{(E)\text{1SF}} a^1 \) at \( y = 0.1 \) turns out to be \( \sigma_{12}^\text{th} (0.1) \approx 10^{-7} \), so that \( \lambda^{(E)\text{1SF}} a^1 (0.1) \approx -0.0003077 \). This estimate can be also checked by considering the various successive PN-approximants and identifying the digits which stabilize as the PN order increases, as summarized in Table 1. Similar considerations hold for the other coefficients \( \lambda^{(\text{1SF})}_N a^n \).

IV. CONCLUDING REMARKS

We have computed the first-order GSF corrections to both the electric-type and magnetic-type tidal eigenvalues for particles moving along circular orbits in a Kerr spacetime, generalizing previous results valid for the Schwarzschild case. The computation is performed as a power series of the black hole rotation parameter \( \hat{a} \) (up to \( O(\hat{a}^4) \) included) and through a high-PN order in terms of the gauge-invariant frequency variable \( y \).

These results are ready to be converted into other formalisms, like the EOB model, entering the \( S_1 \) corrections to the tidal part of the Hamiltonian, eventually Padé-resummed. However, this would require a specific treatment which goes beyond the scopes of the present paper. We will leave it for future works.

Appendix A: Tidal invariants in a perturbed Kerr metric

We list below the expressions of the rescaled tidal invariants in terms of the components of the perturbed metric and their derivatives. All quantities are evaluated at \( u = y/(1 - \hat{a}y^{3/2})^{2/3} \).

1. Quadratic electric-type invariant \( \text{Tr} [\mathcal{E}(k)]^2 \)

The \( O(q) \) perturbation to \( m_4^2 \text{Tr} [\mathcal{E}(k)]^2 \) is given by
TABLE I: A list of numerical values of $\lambda_{1}^{(E) 1SF} a^n$ for $y = 0.1$.

| PN | $\lambda_{1}^{(E) 1SF} a^0 (0.1)$ | $\lambda_{1}^{(E) 1SF} a^2 (0.1)$ | $\lambda_{1}^{(E) 1SF} a^3 (0.1)$ | $\lambda_{1}^{(E) 1SF} a^4 (0.1)$ |
|----|---------------------------------|---------------------------------|---------------------------------|---------------------------------|
| 7  | -0.0002942847                  | 0.0000897486                    | -0.0000012122                   | -                                |
| 8  | -0.0003030081                  | 0.0000982084                    | -0.0000027948                   | 0.000001461                     |
| 9  | -0.0003058106                  | 0.0001023980                    | -0.0000037009                   | 0.000002371                     |
| 10 | -0.0003070667                  | 0.0001042302                    | -0.0000042609                   | 0.000003382                     |
| 11 | -0.0003077993                  | 0.0001050993                    | -0.0000045455                   | 0.000004350                     |
| 12 | -0.0003076977                  | 0.0001054684                    | -0.0000045492                   | 0.000004958                     |

\[
J_{v}^{(y)} \delta_{v}(y) = u^{3}(1 - 2u + \hat{a}^{2}u^{2})(2 - 3u - 2\hat{a}u^{3/2} + 3\hat{a}^{2}u^{2}) \frac{m_{2}^{2}\partial_{rr}h_{kk}}{(1 + \hat{a}u^{3/2})^{2}}
\]
\[
- u^{5} \frac{1 - 4\hat{a}u^{3/2} + 3\hat{a}^{2}u^{2}}{(1 + \hat{a}u^{3/2})^{2}} \partial_{\theta\theta}h_{kk}
\]
\[
- u^{5} \frac{(1 - 3u + 2\hat{a}u^{3/2})^{2}}{(1 + \hat{a}u^{3/2})^{4}(1 - 2u + \hat{a}^{2}u^{2})^{4}} \partial_{\phi\phi}h_{kk}
\]
\[
+ u^{5} \frac{2}{(1 + \hat{a}u^{3/2})^{3}}(1 - 3u)(2 - 3u - 3\hat{a}u^{3/2}(1 - 3u^{2}) + 3\hat{a}^{2}u^{2}(3 - 3u - 2u^{2}) - 2\hat{a}^{3}u^{7/2}(6 - 5u)
\]
\[
+ 12\hat{a}^{4}u^{4}(1 - 2u) + 12\hat{a}^{7}u^{11/2})m_{2}\partial_{r}h_{kk}
\]
\[
- 2 u^{11/2} \frac{1}{(1 + \hat{a}u^{3/2})^{4}}[1 - 3u - 2\hat{a}u^{3/2}(1 - 6u) + \hat{a}^{2}u^{2}(3 - 17u) + 6\hat{a}^{3}u^{7/2}]\partial_{r}h_{t\phi}
\]
\[
- 2u^{7} \frac{(1 - 3u + 2\hat{a}u^{3/2})(1 - 4\hat{a}u^{3/2} + 3\hat{a}^{2}u^{2})}{(1 + \hat{a}u^{3/2})^{5}} \frac{1}{m_{2}}(\partial_{r}h_{\phi\phi} - \partial_{\phi}h_{\theta\phi})
\]
\[
+ 2u^{11/2} \frac{(1 - 3u + 2\hat{a}u^{3/2})(1 - 4\hat{a}u^{3/2} + 3\hat{a}^{2}u^{2})}{(1 + \hat{a}u^{3/2})^{4}} \partial_{r}h_{t\tau}
\]
\[
+ 2u^{7} \frac{(1 - 2\hat{a}u^{3/2} + \hat{a}^{2}u^{2})(1 - 4\hat{a}u^{3/2} + 3\hat{a}^{2}u^{2})}{(1 - 2u + \hat{a}^{2}u^{2})(1 + \hat{a}u^{3/2})^{4}} h_{kk}
\]
\[
+ 4u^{15/2}(1 - 3u + 2\hat{a}u^{3/2})(1 - 2\hat{a}u^{3/2} + \hat{a}^{2}u^{2})(1 - 4\hat{a}u^{3/2} + 3\hat{a}^{2}u^{2}) \frac{1}{m_{2}} h_{t\phi}
\]
\[
- 2u^{6} \frac{1 - 2u + \hat{a}^{2}u^{2}}{(1 + \hat{a}u^{3/2})^{4}} [5 - 18u + 18u^{2} - 4\hat{a}u^{3/2} + 2\hat{a}^{2}u^{2}(6 - 5u) - 12\hat{a}^{3}u^{7/2} + 9\hat{a}^{4}u^{4}] h_{t\phi}
\]
\[
- 2u^{8} \frac{(1 - 4\hat{a}u^{3/2} + 3\hat{a}^{2}u^{2})}{(1 + \hat{a}u^{3/2})^{4}} \frac{1}{m_{2}} h_{\theta\theta}
\]
\[
+ 2u^{8} \frac{(1 - 3u + 2\hat{a}u^{3/2})(1 - u + 2\hat{a}u^{3/2})(1 - 2u) + 2\hat{a}^{2}u^{3})(1 - 4\hat{a}u^{3/2} + 3\hat{a}^{2}u^{2})}{(1 - 2u + \hat{a}^{2}u^{2})(1 + \hat{a}u^{3/2})^{6}} \frac{1}{m_{2}} h_{\phi\phi}.
\]
2. Quadratic magnetic-type invariant $\text{Tr} [B(k)]^2$

The $O(q)$ perturbation to $m^2 \text{Tr} [B(k)]^2$ is given by

\[
\mathcal{J}^{(0)}_{\alpha \nu} \delta_{\nu \nu}(y) = 3u^4 \frac{1 - \hat{a} u^{1/2}}{(1 + \hat{a} u^{3/2})^3} (1 - 2 \hat{a} u^{3/2} + \hat{a}^2 u^2)(1 - 2u + \hat{a}^2 u^2)m^2 \partial_{\tau r} h_{kk}
\]

\[
-3u^6 \frac{1 - \hat{a} u^{1/2}}{(1 + \hat{a} u^{3/2})^3} (1 - 2 \hat{a} u^{3/2} + \hat{a}^2 u^2) \partial_{\theta \phi} h_{kk}
\]

\[
+3u^9/2 \frac{1 - \hat{a} u^{1/2}}{(1 + \hat{a} u^{3/2})^4} (1 - 3u + 2 \hat{a} u^{3/2})(1 - 2u + \hat{a}^2 u^2) \left( m_2 \partial_{\tau r} h_{1 \phi} + \frac{u^{3/2}}{1 + \hat{a} u^{3/2}} \partial_{\phi \theta} h_{\phi \phi} \right)
\]

\[
-3u^{13/2} \frac{1 - \hat{a} u^{1/2}}{(1 + \hat{a} u^{3/2})^4} (1 - 3u + 2 \hat{a} u^{3/2})(1 - 2u + \hat{a}^2 u^2) \left( m_2 \partial_{\tau \phi} h_{i \tau} + \frac{u^{3/2}}{1 + \hat{a} u^{3/2}} \partial_{\phi \phi} h_{\tau \phi} \right)
\]

\[
+3u^{13/2} \frac{1 - \hat{a} u^{1/2}}{(1 + \hat{a} u^{3/2})^4} (1 - 3u + 2 \hat{a} u^{3/2}) \left( m_2 \partial_{\theta \phi} h_{i \theta} + \frac{u^{3/2}}{1 + \hat{a} u^{3/2}} \partial_{\phi \phi} h_{\theta \phi} \right)
\]

\[
-3u^5 \frac{1 - \hat{a} u^{1/2}}{(1 + \hat{a} u^{3/2})^4} (1 - 3u + 2 \hat{a} u^{3/2})(3 - 5u + 4 \hat{a} u^{3/2})(1 - 2u) + 2 \hat{a}^2 u^2(1 + u) + 2 \hat{a}^3 u^{7/2} \left( m_2 \partial_{\theta \phi} h_{i \theta} + \frac{u^{3/2}}{1 + \hat{a} u^{3/2}} \partial_{\phi \phi} h_{\theta \phi} \right)
\]

\[
-3u^5 \frac{1 - \hat{a} u^{1/2}}{(1 + \hat{a} u^{3/2})^4} (1 - 3u + 2 \hat{a} u^{3/2})(1 - 2u + \hat{a}^2 u^2) \left( m_2 \partial_{\phi \phi} h_{r r} + \frac{u^{3/2}}{1 + \hat{a} u^{3/2}} \partial_{\theta \phi} h_{\phi r} \right)
\]

\[
-3u^5 \frac{1 - \hat{a} u^{1/2}}{(1 + \hat{a} u^{3/2})^4} (1 - 3u + 2 \hat{a} u^{3/2})(1 - 2u + \hat{a}^2 u^2) \left( m_2 \partial_{\phi \phi} h_{r r} + \frac{u^{3/2}}{1 + \hat{a} u^{3/2}} \partial_{\theta \phi} h_{\phi r} \right)
\]

\[
+6u^{11/2} \frac{1 - \hat{a} u^{1/2}}{(1 + \hat{a} u^{3/2})^4} (1 - 3u + 2 \hat{a} u^{3/2})(1 - 2u + \hat{a}^2 u^2) \left( m_2 \partial_{\phi \phi} h_{r r} + \frac{u^{3/2}}{1 + \hat{a} u^{3/2}} \partial_{\theta \phi} h_{\phi r} \right)
\]

\[
+6u^6 \frac{1 - \hat{a} u^{1/2}}{(1 + \hat{a} u^{3/2})^4} (1 - 3u + 2 \hat{a} u^{3/2})(1 - 2u + \hat{a}^2 u^2) \left( m_2 \partial_{\phi \phi} h_{r r} + \frac{u^{3/2}}{1 + \hat{a} u^{3/2}} \partial_{\theta \phi} h_{\phi r} \right)
\]

\[
+6u^8 \frac{1 - \hat{a} u^{1/2}}{(1 + \hat{a} u^{3/2})^4} (1 - 3u + 2 \hat{a} u^{3/2})(1 - 2u + \hat{a}^2 u^2) \left( m_2 \partial_{\phi \phi} h_{r r} + \frac{u^{3/2}}{1 + \hat{a} u^{3/2}} \partial_{\theta \phi} h_{\phi r} \right)
\]

\[
+ \hat{a}^3 u^{7/2} (3 - 4u) + 6 \hat{a}^4 u^5 \frac{1}{m_2} h_{\phi \phi}
\]

\[
-18u^9 \frac{1 - \hat{a} u^{1/2}}{(1 + \hat{a} u^{3/2})^4} (1 - 2u + \hat{a}^2 u^2) \frac{1}{m_2} h_{\theta \theta}
\]

\[
+12u^{15/2} \frac{1 - \hat{a} u^{1/2}}{(1 + \hat{a} u^{3/2})^5} (1 - 2u + \hat{a}^2 u^2) \left( (1 - 3u)(2 - 3u) + 6 \hat{a} u^{5/2}(2 - 3u) + \hat{a}^2 u^2(5 - 30u + 33u^2) \right)
\]

\[
+8 \hat{a}^3 u^{7/2} (1 - u) + \hat{a}^4 u^4 (3 - 13u) + 6 \hat{a}^5 u^{11/2} \frac{1}{m_2} h_{i \phi}
\]

\[
+6u^{15/2} \frac{1 - \hat{a} u^{1/2}}{(1 + \hat{a} u^{3/2})^4} (1 - 2u + \hat{a}^2 u^2) \left( h_{\phi \phi} + \hat{a}(3 - 18u + 20u^2) + \hat{a}^3 u^{3/2}(3 - 3u - 4u^2) \right)
\]

\[
+2 \hat{a}^3 u^2 (3 - 14u + 12u^2) + \hat{a}^4 u^{7/2}(12 - 11u) + \hat{a}^5 u^4 (3 - 10u) + 6 \hat{a}^6 u^{11/2} h_{kk}
\]
The $O(q)$ perturbation to $m_{2}\text{Tr}[E(k)]^3$ is given by

\[ J_{c}(y) = -\frac{3}{2} \frac{u^6}{(1 + \hat{a}u^{3/2})^4} (1 - 2u + \hat{a}u^2)(1 - 2u + 2\hat{a}u^{3/2} + 3\hat{a}^2u^2)^2 m_2^2 \partial_{rr} h_{kk} \]

\[ -\frac{3}{2} \frac{u^8}{(1 + \hat{a}u^{3/2})^4} (1 - 4\hat{a}u^{3/2} + 3\hat{a}^2u^2) \partial_{\theta\phi} h_{kk} \]

\[ -\frac{3}{2} \frac{u^8}{(1 + \hat{a}u^{3/2})^6} (1 - 3u + 2\hat{a}u^{3/2}) \partial_{\phi\phi} h_{kk} \]

\[ -3 \frac{u^7}{(1 + \hat{a}u^{3/2})^6} (1 - 3u + 2\hat{a}u^{3/2} + 3\hat{a}u^{3/2}(1 - u)(5 - 3u) + 18\hat{a}^2u^2(1 - u + u^2) \]

\[ -2\hat{a}^3u^{7/2}(15 - u) + 3\hat{a}^4u^3(7 - 9u) + 12\hat{a}^5u^{11/2}]m_2 \partial_{\theta} h_{kk} \]

\[ +3 \frac{u^{17/2}}{(1 + \hat{a}u^{3/2})^6} (1 - 3u + 2\hat{a}u^{3/2})[5 - 18u + 18u^2 - 4\hat{a}u^{3/2} + 2\hat{a}^2u^2(6 - 5u) \]

\[ -12\hat{a}^3u^{7/2} + 9\hat{a}^4u^4 \left[ \frac{1}{m_2^2} \left( m_2 - 3\hat{a}u^{3/2} + \hat{a}^2u^2 \right) - \frac{lu^4}{(1 + \hat{a}u^{3/2})^6} \left[ 5 - 18u + 18u^2 - 4\hat{a}u^{3/2} + 2\hat{a}^2u^2(6 - 5u) - 12\hat{a}^3u^{7/2} + 9\hat{a}^4u^4 \right] \right] \]

\[ -3 \frac{u^{10}}{(1 + \hat{a}u^{3/2})^6} \left[ 1 - 3u + 2\hat{a}u^{3/2} \left( 1 - u + 2\hat{a}u^{3/2}(1 - 2u) + 2\hat{a}^2u^2 \right) \right][5 - 18u + 18u^2 - 4\hat{a}u^{3/2} + 2\hat{a}^2u^2(6 - 5u) \]

\[ +2\hat{a}^3u^{2}(6 - 5u) - 12\hat{a}^3u^{7/2} + 9\hat{a}^4u^4 \frac{1}{m_2^2} h_{\phi\phi} \]

\[ -6 \frac{u^{21/2}}{(1 + \hat{a}u^{3/2})^7} \left[ 1 - 3u + 2\hat{a}u^{3/2} \left( 1 - 2\hat{a}u^{3/2} + \hat{a}^2u^2 \right) \right][5 - 18u + 18u^2 - 4\hat{a}u^{3/2} + 2\hat{a}^2u^2(6 - 5u) \]

\[ -12\hat{a}^3u^{7/2} + 9\hat{a}^4u^4 \frac{1}{m_2^2} h_{\phi\phi} \]

\[ -3 \frac{u^{11}}{(1 + \hat{a}u^{3/2})^6} (1 - 4\hat{a}u^{3/2} + 3\hat{a}^2u^2)^3 \frac{1}{m_2^2} \partial_{\phi\theta} h_{kk} \]

\[ +3 \frac{u^9}{(1 + \hat{a}u^{3/2})^6} (1 - 4\hat{a}u^{3/2} + 3\hat{a}^2u^2)[1 - 2u + \hat{a}u^2]^2(15 - 23u)] \]

\[ -6\hat{a}^3u^{7/2} + 9\hat{a}^4u^4 \partial_{rr} h_{kk} \].

(A3)

3. Cubic electric-type invariant \( \text{Tr}[E(k)]^3 \)

Acknowledgments

The authors thank T. Damour for useful discussions.

D.B. thanks the Naples Section of the Italian Istituto Nazionale di Fisica Nucleare (INFN) and the International Center for Relativistic Astrophysics Network (ICRANet) for partial support.

[1] A. G. Shah, talk delivered at the 14th Marcel Grossmann Meeting, “Sapienza” University of Rome, Rome (IT).
[2] A. G. Shah, J. L. Friedman and T. S. Keidel, “EMRI corrections to the angular velocity and redshift factor of a mass in circular orbit about a Kerr black hole,” Phys. Rev. D 86, 084059 (2012) doi:10.1103/PhysRevD.86.084059 [arXiv:1207.5595] [gr-qc].
[3] D. Bini, T. Damour and A. Geralico, “Spin-dependent two-body interactions from gravitational self-force computations,” Phys. Rev. D 92, no. 12, 124058 (2015) Erratum: [Phys. Rev. D 93, no. 10, 109902 (2016)] doi:10.1103/PhysRevD.93.109902, 10.1103/PhysRevD.92.124058 [arXiv:1510.06230] [gr-qc].
[4] C. Kavanagh, A. C. Ottewill and B. Wardell, “Analytical high-order post-Newtonian expansions for spinning extreme mass ratio binaries,” Phys. Rev. D 93, no. 12, 124038 (2016) doi:10.1103/PhysRevD.93.124038
[5] D. Bini, T. Damour and A. Geralico, “High post-Newtonian order gravitational self-force analytical results for eccentric equatorial orbits around a Kerr black hole,” Phys. Rev. D 93, no. 12, 124058 (2016) doi:10.1103/PhysRevD.93.124058 [arXiv:1602.08282 [gr-qc]].

[6] S. R. Dolan, N. Warburton, A. I. Harte, A. L. Tiec, B. Wardell and L. Barack, “Gravitational self-torque and spin precession in compact binaries,” Phys. Rev. D 89, 064011 (2014) [arXiv:1312.0775 [gr-qc]].

[7] D. Bini and T. Damour, “Two-body gravitational spin-orbit interaction at linear order in the mass ratio,” Phys. Rev. D 90, no. 2, 024039 (2014) doi:10.1103/PhysRevD.90.024039 [arXiv:1404.2747 [gr-qc]].

[8] C. Kavanagh, D. Bini, T. Damour, S. Hopper, A. C. Ottewill and B. Wardell, “Spin-orbit precession along eccentric orbits for extreme mass ratio black hole binaries and its effective-one-body transcription,” Phys. Rev. D 96, no. 6, 064012 (2017) doi:10.1103/PhysRevD.96.064012 [arXiv:1706.00459 [gr-qc]].

[9] D. Bini, T. Damour and A. Geralico, “Spin-orbit precession along eccentric orbits: improving the knowledge of self-force corrections and of their effective-one-body counterparts,” Phys. Rev. D 97, 104046 (2018) doi:10.1103/PhysRevD.97.104046 [arXiv:1801.03704 [gr-qc]].

[10] D. Bini, T. Damour and G. Faye, “Effective action approach to higher-order relativistic tidal interactions in binary systems and their effective one body description,” Phys. Rev. D 85, 124034 (2012) doi:10.1103/PhysRevD.85.124034 [arXiv:1202.3565 [gr-qc]].

[11] D. Bini and T. Damour, “Gravitational self-force corrections to two-body tidal interactions and the effective one-body formalism,” Phys. Rev. D 90, no. 12, 124037 (2014) doi:10.1103/PhysRevD.90.124037 [arXiv:1409.6933 [gr-qc]].

[12] S. R. Dolan, P. Nolan, A. C. Ottewill, N. Warburton and B. Wardell, “Tidal invariants for compact binaries on quasicircular orbits,” Phys. Rev. D 91, no. 2, 023009 (2015) doi:10.1103/PhysRevD.91.023009 [arXiv:1406.4890 [gr-qc]].

[13] C. Kavanagh, A. C. Ottewill and B. Wardell, “Analytical high-order post-Newtonian expansions for extreme mass ratio binaries,” Phys. Rev. D 92, no. 8, 084025 (2015) doi:10.1103/PhysRevD.92.084025 [arXiv:1503.02334 [gr-qc]].

[14] A. G. Shah and A. Pound, “Linear-in-mass-ratio contribution to spin precession and tidal invariants in Schwarzschild spacetime at very high post-Newtonian order,” Phys. Rev. D 91, no. 12, 124022 (2015) doi:10.1103/PhysRevD.91.124022 [arXiv:1503.02414 [gr-qc]].

[15] P. Nolan, C. Kavanagh, S. R. Dolan, A. C. Ottewill, N. Warburton and B. Wardell, “Octupolar invariants for compact binaries on quasicircular orbits,” Phys. Rev. D 92, no. 12, 123008 (2015) doi:10.1103/PhysRevD.92.123008 [arXiv:1505.04447 [gr-qc]].

[16] A. Buonanno and T. Damour, “Effective one-body approach to general relativistic two-body dynamics,” Phys. Rev. D 59, 084006 (1999) doi:10.1103/PhysRevD.59.084006 [gr-qc/9811091].

[17] A. Buonanno and T. Damour, “Transition from inspiral to plunge in binary black hole coalescences,” Phys. Rev. D 62, 064015 (2000) doi:10.1103/PhysRevD.62.064015 [gr-qc/0001013].

[18] B. P. Abbott et al. [LIGO Scientific and Virgo Collaborations], “Observation of Gravitational Waves from a Binary Black Hole Merger,” Phys. Rev. Lett. 116, no. 6, 061102 (2016) doi:10.1103/PhysRevLett.116.061102 [arXiv:1602.03837 [gr-qc]].

[19] B. P. Abbott et al. [LIGO Scientific and Virgo Collaborations], “GW151226: Observation of Gravitational Waves from a 22-Solar-Mass Binary Black Hole Coalescence,” Phys. Rev. Lett. 116, no. 24, 241103 (2016) doi:10.1103/PhysRevLett.116.241103 [arXiv:1606.04855 [gr-qc]].

[20] B. P. Abbott et al. [LIGO Scientific and Virgo Collaborations], “GW170814: A Three-Detector Observation of Gravitational Waves from a Binary Black Hole Coalescence,” Phys. Rev. Lett. 119, no. 14, 141101 (2017) doi:10.1103/PhysRevLett.119.141101 [arXiv:1709.09660 [gr-qc]].

[21] B. P. Abbott et al. [LIGO Scientific and Virgo Collaborations], “GW170817: Observation of Gravitational Waves from a Binary Neutron Star Insipr,” Phys. Rev. Lett. 119, no. 16, 161101 (2017) doi:10.1103/PhysRevLett.119.161101 [arXiv:1710.05852 [gr-qc]].

[22] T. Damour and A. Nagar, “New effective-one-body description of coalescing nonprecessing spinning black-hole binaries,” Phys. Rev. D 90, no. 4, 044018 (2014) doi:10.1103/PhysRevD.90.044018 [arXiv:1406.6913 [gr-qc]].

[23] A. Boh et al., “Improved effective-one-body model of spinning, nonprecessing binary black holes for the era of gravitational-wave astrophysics with advanced detectors,” Phys. Rev. D 95, no. 4, 044028 (2017) doi:10.1103/PhysRevD.95.044028 [arXiv:1611.03703 [gr-qc]].

[24] D. Bini and A. Geralico, “Gravitational self-force corrections to tidal invariants for spinning particles on circular orbits in a Schwarzschild spacetime,” arXiv:1806.03496 [gr-qc].

[25] D. Bini and A. Geralico, “Gravitational self-force corrections to tidal invariants for particles on eccentric orbits in a Schwarzschild spacetime,” arXiv:1806.06635 [gr-qc].

[26] T. Damour, Gravitational Radiation And The Motion Of Compact Bodies, in Gravitational Radiation, edited by N. Deruelle and T. Piran (North-Holland, Amsterdam, 1983), pp. 59-144.

[27] Xiao-He Zhang, Multipole expansions of the general-relativistic gravitational field of the external universe, Phys. Rev. D 34, 991 (1986).

[28] T. Damour, M. Soffel and C.-m. Xu, General relativistic celestial mechanics. 1. Method and definition of reference systems, Phys. Rev. D 43, 3272 (1991).

[29] T. Damour, M. Soffel and C.-m. Xu, General relativistic celestial mechanics. 2. Translational equations of motion, Phys. Rev. D 45, 1017 (1992).

[30] M. Levi and J. Steinhoff, “Spinning gravitating objects in the effective field theory in the post-Newtonian scheme,” JHEP 1509, 219 (2015) doi:10.1007/JHEP09(2015)219 [arXiv:1501.04956 [gr-qc]].

[31] T. Damour, M. Soffel and C.-m. Xu, General relativistic
celestial mechanics. 3. Rotational equations of motion, Phys. Rev. D 47, 3124 (1993).

[32] T. Damour, M. Soffel and C.-m. Xu, General relativistic celestial mechanics. 4: Theory of satellite motion, Phys. Rev. D 49, 618 (1994).

[33] A. Heffernan, A. Ottewill and B. Wardell, “High-order expansions of the Detweiler-Whiting singular field in Schwarzschild spacetime,” Phys. Rev. D 86, 104023 (2012) doi:10.1103/PhysRevD.86.104023

[arXiv:1204.0794 [gr-qc]].

[34] A. Pound and M. van de Meent, “Quasi-invariants from radiation gauge self-force calculations,” in preparation; see also D. Bini, T. Damour, A. Geralico, C. Kavanagh, M. van de Meent, “Gravitational self-force corrections to gyroscope precession along circular orbits in the Kerr spacetime,” in preparation.