Capacity Bounds for Broadcast Channels with Confidential Messages

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Abstract

In this paper, we study capacity bounds for discrete memoryless broadcast channels with confidential messages. Two private messages as well as a common message are transmitted; the common message is to be decoded by both receivers, while each private message is only for its intended receiver. In addition, each private message is to be kept secret from the unintended receiver where secrecy is measured by equivocation. We propose both inner and outer bounds to the rate equivocation region for broadcast channels with confidential messages. The proposed inner bound generalizes Csiszár and Körner’s rate equivocation region for broadcast channels with a single confidential message, Liu et al’s achievable rate region for broadcast channels with perfect secrecy, Marton’s and Gel’fand and Pinsker’s achievable rate region for general broadcast channels. Our proposed outer bounds, together with the inner bound, helps establish the rate equivocation region of several classes of discrete memoryless broadcast channels with confidential messages, including less noisy, deterministic, and semi-deterministic channels. Furthermore, specializing to the general broadcast channel by removing the confidentiality constraint, our proposed outer bounds reduce to new capacity outer bounds for the discrete memory broadcast channel.

I. INTRODUCTION

With the increasingly widespread wireless devices and services, the demand for reliable and secure communications is becoming more urgent due to the broadcast nature of wireless communication. Existing systems typically rely on key-based encryption schemes: the intended transceiver pair share a private key which is unknown to any unintended users. Assuming ideal transmission of encrypted messages, Shannon in his 1949 landmark paper [1] proved, using information theoretic argument, a surprising result: security is guaranteed only if the key size is at least as long as the source message. While this establishes provable security of the so-called one-time pad system, the excessive requirement on the key size essentially forebodes a negative result: any key-based encryption scheme is almost always not provably secure as the key size requirement forbids dynamic key exchange. This result motivates many secure communication scheme where provable security is sacrificed in favor of computational security;

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however, this notion of security relies on unproven intractability hypotheses. For instance, the security of RSA [2] is based on the unproven difficulty of factoring large integers.

Wyner in his seminal work in 1975 [3] demonstrated that, for noisy channels, provable secure communication (in the same sense as that of Shannon) can be achieved by exploring information theoretic limits at the physical layer. Wyner introduced the so-called wiretap channel which is in essence a degraded broadcast channel and characterized its capacity-secrecy tradeoff. It was shown that, through the use of stochastic encoding, perfect secrecy is possible in the absence of a secret key. Later, Csiszár and Körner generalized Wyner’s result [4] by considering a non-degraded discrete memoryless broadcast channel (DMBC) with a single confidential message for one of the users and a common message for both users. Following the approach of [3] and [4], information-theoretic limits of secret communications for several different wireless networks have been investigated, including multi-user systems with confidential messages [5]–[12], secret communication over fading channels [13], [14] and MIMO wiretap channels [15]–[17].

In this work, we generalize Csiszár and Körner’s model by considering discrete memoryless broadcast channels where both receivers have their own private messages as well as a common message to decode. We refer to this model as simply DMBC with two confidential messages (DMBC-2CM). The DMBC-2CM model was first studied by Liu, Maric, Spasojevic, and Yates [9], [18] where, in the absence of a common message, the authors imposed the perfect secrecy constraint and obtained inner and outer bounds for the perfect secrecy capacity region.

In this paper, we study capacity bounds to the rate equivocation region for the general DMBC-2CM. Our model generalizes that of [18] by including a common message. More importantly, we do not impose the perfect secrecy constraint and study instead the general trade-off among rates for reliable communication and secrecy for confidential messages. Study of this general model allows us to unify many existing results. Both inner and outer bounds are proposed for the general DMBC-2CM. The proposed achievable rate region generalizes Csiszár and Körner’s capacity rate region in [4] where only a single confidential message is to be communicated, Liu et al.’s achievable rate region under perfect secrecy constraint [18], and Marton and Gel’fand-Pinsker’s achievable rate region for general broadcast channels [19], [20]. The proposed outer bounds to the rate equivocation region of a DMBC-2CM also encompass existing outer bounds for various special cases of the DMBC-2CM. In particular, it reduces to Csiszár and Körner’s rate equivocation region for DMBC with only one confidential message and Liu et la’s outer bound to the capacity region with perfect secrecy. The proposed inner and outer bounds coincide with each other for the less noisy, deterministic, and semi-deterministic DMBC-2CM, which settle the rate equivocation region for these channels. Furthermore, in the absence of secrecy constraints, our proposed outer bounds specialize to new outer bounds to the capacity region of the general DMBC. Comparison with existing outer bounds in [19], [21]–[23] will be discussed.

The rest of the paper is organized as follows. In Section II, we give the channel model and review
relevant existing results. In Section III, we present an achievable rate equivocation region for our channel model and show that it coincides with various existing results under respective conditions. In section IV, we present outer bounds to the rate equivocation region of DMBC-2CM. We prove that the outer bound is tight for the less noisy, deterministic, and semi-deterministic DMBC-2CM. We also discuss the induced outer bound to the general DMBC and its subset relations with existing capacity outer bounds. Finally, we conclude in Section V.

II. PROBLEM FORMULATION AND PREVIOUS RESULTS

A. Problem Statement

A discrete memoryless broadcast channel with confidential messages $\mathcal{K}$ is a quadruple $(\mathcal{X}, p, \mathcal{Y}_1, \mathcal{Y}_2)$, where $\mathcal{X}$ is the finite input alphabet set, $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are two finite output alphabet sets, and $p$ is the channel transition probability $p(y_1, y_2|x)$. We assume that the channels are memoryless, i.e.,

$$p(y_1, y_2|x) = \prod_{i=1}^{n} p(y_{1i}, y_{2i}|x_i)$$

where,

$$x = (x_1, \cdots, x_n) \in \mathcal{X}^n,$$

$$y_1 = (y_{11}, \cdots, y_{1n}) \in \mathcal{Y}_1^n,$$

$$y_2 = (y_{21}, \cdots, y_{2n}) \in \mathcal{Y}_2^n.$$

Let $\mathcal{M}_0 = \{1, 2, \cdots, M_0\}$ be the common message set, $\mathcal{M}_1 = \{1, 2, \cdots, M_1\}$ and $\mathcal{M}_2 = \{1, 2, \cdots, M_2\}$ be user 1 and user 2’s private message sets, and $W_0, W_1, W_2$ are the respective message variables on the sets $\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_2$. We assume stochastic encoding as randomization may increase secrecy [4]. A stochastic encoder $f$ with block length $n$ for $\mathcal{K}$ is specified by $f(x|w_1, w_2, w_0)$, where $x \in \mathcal{X}^n$, $w_1 \in \mathcal{M}_1$, $w_2 \in \mathcal{M}_2$, $w_0 \in \mathcal{M}_0$ and

$$\sum_x f(x|w_1, w_2, w_0) = 1.$$  

Here $f(x|w_1, w_2, w_0)$ is the probability that the message triple $(w_1, w_2, w_0)$ is encoded as the channel input $x$. Our model involves two decoders, i.e., a pair of mappings

$$\varphi_1 : \mathcal{Y}_1^n \rightarrow \mathcal{M}_1 \times \mathcal{M}_0,$$

$$\varphi_2 : \mathcal{Y}_2^n \rightarrow \mathcal{M}_2 \times \mathcal{M}_0.$$  

The average probabilities of decoding error of this channel are defined as

$$p_{e,1}^{(n)} \triangleq \frac{1}{M_1 M_2 M_0} \sum_{w_1, w_2, w_0} P(\{\varphi_1(y_1) \neq (w_1, w_0)\}|(w_1, w_2, w_0) \text{ sent}),$$

$$p_{e,2}^{(n)} \triangleq \frac{1}{M_1 M_2 M_0} \sum_{w_1, w_2, w_0} P(\{\varphi_2(y_2) \neq (w_2, w_0)\}|(w_1, w_2, w_0) \text{ sent}).$$

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A rate quintuple \((R_1, R_2, R_0, R_e_1, R_e_2)\) is said to be achievable if there exist message sets \(\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_0\) and encoder-decoders \((f, \varphi_1, \varphi_2)\) such that \(P_{e,1}^n \to 0\) and \(P_{e,2}^n \to 0\), where for \(a = 0, 1, 2\)

\[
\lim_{n \to \infty} \frac{1}{n} \log ||\mathcal{M}_a|| = R_a
\]  

\[
\lim_{n \to \infty} \frac{1}{n} H(W_1|Y_2) \geq R_{e1}
\]

\[
\lim_{n \to \infty} \frac{1}{n} H(W_2|Y_1) \geq R_{e2}
\]

The rate equivocation region of the DMBC-2CM is the closure of union of all achievable rate quintuples \((R_0, R_1, R_2, R_{e1}, R_{e2})\). Our objective in this paper is to obtain meaningful bounds to the rate equivocation region for DMBC-2CM.

The DMBC-2CM model is illustrated in Fig. 1. We note that in the absence of \(W_2\), the model reduces to Csiszár and Körner’s model with only one confidential message [4]. On the other hand, in the absence of confidentiality constraints (i.e., \(H(W_1|Y_2)\) and \(H(W_2|Y_1)\)), our model reduces to the classical DMBC with two private messages and one common message.

Before proceeding, we introduce the following definitions. Let \(Z = (U, V_1, V_2, X, Y_1, Y_2)\) be a set of random variables such that \(X \in \mathcal{X}, Y_1 \in \mathcal{Y}_1, Y_2 \in \mathcal{Y}_2\), and the corresponding \(p(y_1, y_2|x)\) is the channel transition probability of the DMBC-2CM. Define

- \(Q_1\) to be the set of \(Z\) whose joint distribution factors as

\[
p(u, v_1, v_2, x, y_1, y_2) = p(u, v_1, v_2)p(x|u, v_1, v_2)p(y_1, y_2|x).
\]

Thus any \(Z \in Q_1\) satisfies the Markov chain condition \(UV_1V_2 \to X \to Y_1Y_2\).

- \(Q_2\) to be the set of \(Z\) whose joint distribution factors as

\[
p(u, v_1, v_2, x, y_1, y_2) = p(u)p(v_1, v_2|u)p(x|v_1, v_2)p(y_1, y_2|x);
\]

Thus any \(Z \in Q_2\) satisfies the Markov chain condition \(U \to V_1V_2 \to X \to Y_1Y_2\).

- \(Q_3\) to be the set of \(Z\) whose joint distribution factors as

\[
p(u, v_1, v_2, x, y_1, y_2) = p(v_1)p(v_2)p(u|v_1, v_2)p(x|u, v_1, v_2)p(y_1, y_2|x).
\]

\(Q_3\) results in the same Markov chain as \(Q_1\) except that \(V_1\) and \(V_2\) are independent of each other. Clearly, \(Q_2 \subseteq Q_1\) and \(Q_3 \subseteq Q_1\).
B. Related Work

In the section, we review several existing results related to the present work.

Csiszár and Körner characterized the rate equivocation region \[4\] for broadcast channel with a common message for both users and a single confidential message intended for one of the two users. Without loss of generality (WLOG), we assume \(W_2\) is absent from our model. The result is summarized below.

**Proposition 1:** [4, Theorem 1] The rate equivocation region \(\mathcal{R}_{CK}\) for a DMBC with one common message for both receivers and a single confidential message intended for one of the two users is a closed convex set consisting of those triples \((R_1, R_e, R_0)\) for which there exist random variables \(U \rightarrow V \rightarrow X \rightarrow Y_1Y_2\) such that

\[
\begin{align*}
0 & \leq R_e \leq R_1 \quad (11) \\
R_e & \leq I(V;Y_1|U) - I(V;Y_2|U) \quad (12) \\
R_1 + R_0 & \leq I(V;Y_1|U) + \min[I(U;Y_1), I(U;Y_2)] \quad (13) \\
R_0 & \leq \min[I(U;Y_1), I(U;Y_2)] \quad (14)
\end{align*}
\]

We note that the Markov chain condition in Proposition 1 can be relaxed, as stated below.

**Lemma 1:** Define \(\mathcal{R}_{CK}'\) to be the convex closure of rate triples \((R_1, R_e, R_0)\) that satisfy (11)-(14) where the random variables follow the Markov chain: \(UV \rightarrow X \rightarrow Y_1Y_2\), then

\[
\mathcal{R}_{CK} = \mathcal{R}_{CK}'
\]

**Proof:** \(\mathcal{R}_{CK} \subseteq \mathcal{R}_{CK}'\) follows trivially from the fact that \(U \rightarrow V \rightarrow X \rightarrow Y_1Y_2\) implies \(UV \rightarrow X \rightarrow Y_1Y_2\). To prove \(\mathcal{R}_{CK}' \subseteq \mathcal{R}_{CK}\), assume \((R_1, R_e, R_0) \in \mathcal{R}_{CK}'\) for some \(UV \rightarrow X \rightarrow Y_1Y_2\). Define \(U' = U\) and \(V' = UV\), one can verify easily that \((R_1, R_e, R_0)\) satisfies (11)-(14) for \(U' \rightarrow V' \rightarrow X \rightarrow Y_1Y_2\), i.e., \((R_1, R_e, R_0) \in \mathcal{R}_{CK}\).

Recently, Liu et al proposed an inner bound and an outer bound to the capacity region for broadcast channels with perfect-secrecy constraint on the confidential messages [9], [18]. The model in [9], [18] is in essence a DMBC-2CM without the common message. In their model, each user has its own confidential message that is to be completely protected from the other user. The proposed achievable region and outer bound are given in Propositions 2 and 3, respectively.

**Proposition 2:** [18, Theorem 4] Let \(\mathcal{R}_{LMSY-I}\) denote the union of all \((R_1, R_2)\) satisfying

\[
\begin{align*}
0 & \leq R_1 \leq I(V_1;Y_1|U) - I(V_1;Y_2|V_2U) - I(V_1;V_2|U) \quad (16) \\
0 & \leq R_2 \leq I(V_2;Y_2|U) - I(V_2;Y_1|V_1U) - I(V_2;V_2|U)
\end{align*}
\]

over all random variables \((U, V_1, V_2, X, Y_1, Y_2) \in \mathcal{Q}_2\). Any rate pair \((R_1, R_2) \in \mathcal{R}_{LMSY-I}\) is achievable for DMBC-2CM without common message and with perfect secrecy for the confidential messages, i.e., \(R_0 = 0, R_1 = R_{e1}\), and \(R_2 = R_{e2}\).
Proposition 3: [18, Theorem 3] An outer bound to the capacity region for the DMBC-2CM with perfect secrecy constraint is the set of all \((R_1, R_2)\) satisfying

\[
0 \leq R_1 \leq \min[I(V_1; Y_1 | U) - I(V_1; Y_2 | U), I(V_1; Y_2 | V_2 U) - I(V_1; Y_2 | V_2 U)] \tag{17}
\]

\[
0 \leq R_2 \leq \min[I(V_2; Y_2 | U) - I(V_2; Y_1 | U), I(V_2; Y_2 | V_1 U) - I(V_2; Y_1 | V_1 U)]. \tag{18}
\]

for some \((U, V_1, V_2, X, Y_1, Y_2) \in Q_2\). We denote by \(R_{LMSY-O}\) this outer bound.

In the absence of secrecy constraint, the present model reduces to the DMBC first introduced by Cover [24]. The capacity region for a DMBC is only known for some special cases (see [25] and references therein). The best achievable region for general DMBC is given by Gel’fand and Pinsker in [20] which reduces to Marton’s achievable region [19, Theorem 2] for DMBC in the absence of common message. Capacity region outer bounds include Körner and Marton’s outer bound [19, Theorem 5], Liang and Kramer’s outer bound [22], [26], Nair and El Gamal’s outer bound [21], [27], and a recently proposed outer bound by Liang, Kramer and Shamai (Shitz) [23].

Marton in 1979 considered DMBC in the absence of common message and proposed the following achievable rate region [19].

Proposition 4: [19, Theorem 2] Let \(R_M\) be the union of non-negative rate pairs \((R_1, R_2)\) satisfying \(R_1, R_2 \geq 0\) and

\[
R_1 \leq I(UV_1; Y_1) \tag{19}
\]

\[
R_2 \leq I(UV_2; Y_2) \tag{20}
\]

\[
R_1 + R_2 \leq \min\{I(U; Y_1), I(U; Y_2)\} + I(V_1; Y_1 | U) + I(V_2; Y_2 | U) - I(V_1; V_2 | U) \tag{21}
\]

for some \((U, V_1, V_2, X, Y_1, Y_2) \in Q_1\). Then \(R_M\) is an achievable rate region for the DMBC without common message.

Gel’fand and Pinsker generalized Marton’s model by considering DMBC with common information. The achievable rate region they proposed [20] is summarized below.

Proposition 5: [20, Theorem 1] Let \(R_{GP}\) be the union of non-negative rate triples \((R_0, R_1, R_2)\) satisfying

\[
R_0 \leq \min[I(U; Y_1), I(U; Y_2)] \tag{22}
\]

\[
R_1 + R_0 \leq I(V_1; Y_1 | U) + \min[I(U; Y_1), I(U; Y_2)] \tag{23}
\]

\[
R_2 + R_0 \leq I(V_2; Y_2 | U) + \min[I(U; Y_1), I(U; Y_2)] \tag{24}
\]

\[
R_1 + R_2 + R_0 \leq \min[I(U; Y_1), I(U; Y_2)] + I(V_1; Y_1 | U) + I(V_2; Y_2 | U) - I(V_1; V_2 | U) \tag{25}
\]

for some \((U, V_1, V_2, X, Y_1, Y_2) \in Q_1\). Then \(R_{GP}\) is an achievable rate region for the DMBC.

We comment here that in the absence of common message, \(R_{GP}\) can be shown to be equivalent to \(R_M\) [20]. Furthermore, an equivalent definition of \(R_{GP}\) can be obtained by restricting \(Z \in Q_2\) instead of \(Q_1\), i.e.
Lemma 2: Define $\mathcal{R}^{'}_{GP}$ to be the union of non-negative rate triples $(R_0, R_1, R_2)$ satisfying (22)-(25) with $Z \in Q_2$, then

$$\mathcal{R}_{GP} = \mathcal{R}^{'}_{GP}$$

(26)

The proof is similar to that for Lemma 1 and is skipped. Similarly, $\mathcal{R}_M$ can be equivalently defined using $Z \in Q_2$.

An earlier outer bound by Körner and Marton [19, Theorem 5] for the capacity region of DMBC is subsumed by several recent outer bounds. One of the recent outer bounds was proposed by Liang and Kramer [22], [26, Theorem 6], as summarized in Proposition 6.

Proposition 6: If $(R_0, R_1, R_2)$ is achievable, then there exists $Z \in Q_1$ and

$$R_0 \leq \min[I(U;Y_1), I(U;Y_2)],$$

(27)

$$R_0 + R_1 \leq I(V_1;U;Y_1),$$

(28)

$$R_0 + R_2 \leq I(V_2;U;Y_2),$$

(29)

$$R_0 + R_1 + R_2 \leq I(X;Y_2|V_1 U) + I(V_1;Y_1|U) + \min[I(U;Y_1), I(U;Y_2)],$$

(30)

$$R_0 + R_1 + R_2 \leq I(X;Y_1|V_2 U) + I(V_2;Y_2|U) + \min[I(U;Y_1), I(U;Y_2)].$$

(31)

We denote this outer bound as $\mathcal{R}_{LK}$, i.e., $\mathcal{R}_{LK}$ is the union of non-negative rate triples $(R_0, R_1, R_2)$ satisfying (27)-(31) over $Z \in Q_1$. Furthermore, we can also restrict the Markov chain condition to be $Z \in Q_2$, i.e.,

Lemma 3: Define $\mathcal{R}^{'}_{LK}$ to be the convex closure of union of non-negative rate triples $(R_0, R_1, R_2)$ satisfying (27)-(31) with $Z \in Q_2$, then

$$\mathcal{R}_{LK} = \mathcal{R}^{'}_{LK}$$

(32)

In [21, Theorem 2.1], another outer bound to the capacity region of the general DMBC was given by Nair and El Gamal, as summarized in Proposition 7. This outer bound was shown to be strictly tighter than the Körner and Marton outer bound [19, Theorem 5].

Proposition 7: If $(R_0, R_1, R_2)$ is achievable, then there exists $Z \in Q_3$ and

$$R_0 \leq \min[I(U;Y_1), I(U;Y_2)],$$

(33)

$$R_0 + R_1 \leq I(V_1;U;Y_1),$$

(34)

$$R_0 + R_2 \leq I(V_2;U;Y_2),$$

(35)

$$R_0 + R_1 + R_2 \leq I(V_2;Y_2|V_1 U) + I(V_1;Y_1|U),$$

(36)

$$R_0 + R_1 + R_2 \leq I(V_1;Y_1|V_2 U) + I(V_2;Y_2).$$

(37)

We denote by $\mathcal{R}_{NE}$ this new outer bound, i.e., $\mathcal{R}_{NE}$ is the union of non-negative rate triples $(R_0, R_1, R_2)$ satisfying (33)-(37) over $Z \in Q_3$.

The most recent outer bound to the capacity region for DMBC was proposed by Liang, Kramer, and Shamai (Shitz) [23].
Proposition 8: If \((R_0, R_1, R_2)\) is achievable, then there exist random variables \((W_0, W_1, W_2, V_1, V_2, X, Y_1, Y_2)\) whose joint distribution factors as
\[
p(w_0)p(w_1)p(w_2)p(v_1, v_2|w_0, w_1, w_2)p(x|v_1, v_2, w_0, w_1, w_2)p(y_1, y_2|x) \tag{38}
\]
such that,
\[
0 \leq R_0 \leq \min[I(W_0; Y_1|V_1), I(W_0; Y_2|V_2)] \tag{39}
\]
\[
R_1 \leq I(W_1; Y_1|V_1) \tag{40}
\]
\[
R_2 \leq I(W_2; Y_2|V_2) \tag{41}
\]
\[
R_0 + R_1 \leq \min[I(W_0W_1; Y_1|V_1), I(W_1; Y_1|W_0V_1V_2) + I(W_0V_1; Y_2|V_2)] \tag{42}
\]
\[
R_0 + R_2 \leq \min[I(W_0W_2; Y_2|V_2), I(W_2; Y_2|W_0V_1V_2) + I(W_0V_2; Y_1|V_1)] \tag{43}
\]
\[
R_0 + R_1 + R_2 \leq I(W_1; Y_1|W_0W_2V_1V_2) + I(W_0W_2V_1; Y_2|V_2) \tag{44}
\]
\[
R_0 + R_1 + R_2 \leq I(W_2; Y_2|W_0W_1V_1V_2) + I(W_0W_1V_2; Y_1|V_1) \tag{45}
\]
\[
R_0 + R_1 + R_2 \leq I(W_1; Y_1|W_0W_2V_1V_2) + I(W_2; Y_2|W_0V_1V_2) + I(W_0V_1V_2; Y_1) \tag{46}
\]
\[
R_0 + R_1 + R_2 \leq I(W_2; Y_2|W_0W_1V_1V_2) + I(W_1; Y_1|W_0V_1V_2) + I(W_0V_1V_2; Y_2), \tag{47}
\]
where \(X\) is a deterministic function of \((W_0, W_1, W_2, V_1, V_2)\), and \(W_0, W_1, W_2\) are uniformly distributed. We refer to this new outer bound as \(\mathcal{R}_{LKS}\).

III. AN ACHIEVABLE RATE EQUIVOCATION REGION

Our proposed achievable rate equivocation region for DMBC-2CM is given in Theorem 1. The coding scheme combines binning, superposition coding, and rate splitting. For the rate constraints, the binning approach in [28] is supplemented with superposition coding to accommodate the common message. An additional binning is introduced for confidentiality of private messages. We note that this double binning technique has been used by various authors for communication involving confidential messages (see, e.g., [18], [29]).

Different from that of [18], we make explicit use of rate splitting for the two private messages in order to boost the rates \(R_1\) and \(R_2\). We note that this rate splitting was implicitly used in [4] (specifically, proof of Lemma 3 in [4]). To be precise, we split the private message \(W_1 \in \{1, \ldots, 2^{nR_1}\}\) into \(W_{11} \in \{1, \ldots, 2^{nR_{11}}\}\) and \(W_{10} \in \{1, \ldots, 2^{nR_{10}}\}\), and \(W_2 \in \{1, \ldots, 2^{nR_2}\}\) into \(W_{22} \in \{1, \ldots, 2^{nR_{22}}\}\) and \(W_{20} \in \{1, \ldots, 2^{nR_{20}}\}\), respectively. \(W_{11}\) and \(W_{22}\) are only to be decoded by intended receivers while \(W_{10}\) and \(W_{20}\) are to be decoded by both receivers. Notice that this rate splitting is typically used in interference channels to achieve a larger rate region as it enables interference cancellation at the receivers. It is clear that this rate splitting is prohibited if perfect secrecy is required as in [18]. Now, we combine \((W_{10}, W_{20}, W_0)\) together into a single auxiliary variable \(U\). The messages \(W_{11}\) and \(W_{22}\) are represented by auxiliary variables \(V_1\) and \(V_2\) respectively.
The achievable rate equivocation for a DMBC-2CM is formally stated below.

**Theorem 1**: Let $\mathcal{R}_I$ be the union of all non-negative rate quintuple $(R_1, R_2, R_0, R_{e1}, R_{e2})$ satisfying

\begin{align}
R_{e1} &\leq R_1 \\
R_{e2} &\leq R_2 \\
R_0 &\leq \min[I(U;Y_1), I(U;Y_2)] \\
R_1 + R_0 &\leq I(V_1;Y_1|U) + \min[I(U;Y_1), I(U;Y_2)] \\
R_2 + R_0 &\leq I(V_2;Y_2|U) + \min[I(U;Y_1), I(U;Y_2)] \\
R_1 + R_2 + R_0 &\leq I(V_1;Y_1|U) + I(V_2;Y_2|U) - I(V_1;V_2|U) + \min[I(U;Y_1), I(U;Y_2)] \\
R_{e1} &\leq I(V_1;Y_1|U) - I(V_1;Y_1;Y_2|U) \\
R_{e2} &\leq I(V_2;Y_2|U) - I(V_2;Y_2;Y_1|U)
\end{align}

over all $(U, V_1, V_2, X, Y_1, Y_2) \in Q_2$. Then $\mathcal{R}_I$ is an achievable rate region for the DMBC-2CM.

**Proof**: See Appendix [I].

**Remark 1**: The region $\mathcal{R}_I$ remains the same if we replace $Q_2$ with $Q_1$. Formally,

**Proposition 9**: Define $\mathcal{R}'_I$ to be the union of all non-negative rate quintuple $(R_1, R_2, R_0, R_{e1}, R_{e2})$ satisfying (48)-(55) over $Z \in Q_1$, then

$$\mathcal{R}_I = \mathcal{R}'_I$$

**Proof**: The fact that $\mathcal{R}_I \subseteq \mathcal{R}'_I$ follows trivially from $Q_2 \subseteq Q_1$.

We now show $\mathcal{R}'_I \subseteq \mathcal{R}_I$. Assume $(R_1, R_2, R_0, R_{e1}, R_{e2}) \in \mathcal{R}'_I$, i.e., there exists $(U, V_1, V_2, X, Y_1, Y_2) \in Q_1$ such that $(R_1, R_2, R_0, R_{e1}, R_{e2})$ satisfies (48)-(55). The proof is completed by defining $U' = U$, $V'_1 = UV_1$, and $V'_2 = UV_2$ and observe that the same $(R_1, R_2, R_0, R_{e1}, R_{e2})$ satisfies (48)-(55) for $(U', V'_1, V'_2, X, Y_1, Y_2) \in Q_2$.

This achievable rate equivocation region unifies many existing results which we enumerate below.

**A. Csizár and Körner’s region**

In [4], Csizár and Körner characterized the rate equivocation region for broadcast channels with a single confidential message and a common message.

By setting $R_2 = 0$ and $R_{e2} = 0$ in Theorem [I] it is easy to see $\mathcal{R}_I$ reduces to Csizár and Körner’s capacity region $\mathcal{R}_{CK}$ described in Proposition [I].

**B. Liu et al’s region**

In [18], Liu et al proposed an achievable rate region for broadcast channel with confidential messages where there are two private message and no common message. In addition, the private messages are to be perfectly protected from the unintended receivers.

By setting $R_1 = R_{e1}$, $R_2 = R_{e2}$ and $R_0 = 0$ in Theorem [I] one can easily check that $\mathcal{R}_I$ reduces to Liu et al’s achievable rate region $R_{LM}$ described in Proposition [2].
C. Gel’fand and Pinsker’s region

In [20], Gel’fand and Pinsker generalized Marton’s result by proposing an achievable rate region for broadcast channels with common message. If we remove the secrecy constraints in our model by setting \( R_{e1} = 0 \) and \( R_{e2} = 0 \) in Theorem \( \text{[1]} \) we obtain an achievable rate region for the general DMBC, denoted by \( \hat{\mathcal{R}} \), with the exact expressions in (22)-(25) with \( U \rightarrow (V_1, V_2) \rightarrow X \rightarrow (Y_1, Y_2) \). From Proposition \( \text{[5]} \) and Lemma \( \text{[2]} \) \( \hat{\mathcal{R}} = \mathcal{R}_{GP} \).

Remark 2: The proofs in [19], [20] both use a corner point approach. A binning approach was used in [28] to prove a weakened version of [19, Theorem 2]. The proof introduced in the present paper, by stripping out all confidentiality constraints, provides a new way to prove the general achievable rate region of DMBC [20, Theorem 1] [19, Theorem 2] along the line of [28].

IV. OUTER BOUNDS

Define \( \mathcal{R}_{O1} \) to be the union, over all \( Z \in Q_1 \), of non-negative rate quintuple \((R_0, R_1, R_2, R_{e1}, R_{e2})\) satisfying

\[
\begin{align*}
R_{e1} &\leq R_1 \quad (57) \\
R_{e2} &\leq R_2 \\
R_0 &\leq \min[I(U; Y_1), I(U; Y_2)] \quad (58) \\
R_0 + R_1 &\leq I(V_1; Y_1 | U) + \min[I(U; Y_1), I(U; Y_2)] \quad (59) \\
R_0 + R_2 &\leq I(V_2; Y_2 | U) + \min[I(U; Y_1), I(U; Y_2)] \quad (60) \\
R_0 + R_1 + R_2 &\leq I(V_2; Y_2 | V_1 U) + I(V_1; Y_1 | U) + \min[I(U; Y_1), I(U; Y_2)] \quad (61) \\
R_0 + R_1 + R_2 &\leq I(V_1; Y_1 | V_2 U) + I(V_2; Y_2 | U) + \min[I(U; Y_1), I(U; Y_2)] \quad (62) \\
R_{e1} &\leq \min[I(V_1; Y_1 | U) - I(V_1; Y_2 | U), I(V_1; Y_1 | V_2 U) - I(V_1; Y_2 | V_2 U)] \quad (63) \\
R_{e2} &\leq \min[I(V_2; Y_2 | U) - I(V_2; Y_1 | U), I(V_2; Y_2 | V_1 U) - I(V_2; Y_1 | V_1 U)]. \quad (64)
\end{align*}
\]

Similarly, define \( \mathcal{R}_{O2} \) and \( \mathcal{R}_{O3} \) in exactly the same fashion except with \( Q_1 \) replaced by \( Q_2 \) and \( Q_3 \), respectively. We have

Theorem 2: \( \mathcal{R}_{O1}, \mathcal{R}_{O2}, \) and \( \mathcal{R}_{O3} \) are all outer bounds to the rate equivocation region of the DMBC-2CM.

Proof: The proof that \( \mathcal{R}_{O2} \) and \( \mathcal{R}_{O3} \) are outer bounds is given in Appendix \( \text{[1]} \) That \( \mathcal{R}_{O1} \) is an outer bound follows directly from Proposition \( \text{[10]} \).

Proposition 10:

\[
\mathcal{R}_{O3} \subseteq \mathcal{R}_{O1} = \mathcal{R}_{O2}. \quad (66)
\]

Proposition \( \text{[10]} \) can be established by simple algebra whose proof is skipped. While \( \mathcal{R}_{O3} \) subsumes both \( \mathcal{R}_{O1} \) and \( \mathcal{R}_{O2} \), the latter expressions are often easier to use in establishing capacity results or comparing
with existing bounds. For example, it is straightforward to show that \(R_{O2}\) is tight for Csiszár and Körner’s model [4], i.e., DMBC with only one confidential message.

Below, we discuss various implications of Theorem 2

A. The rate equivocation region of less noisy DMBC-2CM

For the DMBC defined in Section II-A channel 1 is said to be less noisy than channel 2 [30] if for every \(V \to X \to Y_1Y_2\),

\[
I(V;Y_1) \geq I(V;Y_2).
\]

(67)

Furthermore, for every \(U \to V \to X \to Y_1Y_2\), the above less noisy condition also implies

\[
I(V;Y_1|U) \geq I(V;Y_2|U).
\]

(68)

Using Theorems 1 and 2 we can establish the rate equivocation region for less noisy DMBC-2CM as in Theorem 3.

**Theorem 3:** If channel 1 is less noisy than channel 2, then the rate equivocation region for this less noisy DMBC-2CM is the set of all non-negative \((R_0, R_1, R_2, R_{e1}, R_{e2})\) satisfying

\[
\begin{align*}
R_{e1} &\leq R_1 & (69) \\
R_0 + R_2 &\leq I(U;Y_2) & (70) \\
R_0 + R_1 + R_2 &\leq I(V;Y_1|U) + I(U;Y_2) & (71) \\
R_{e1} &\leq I(V;Y_1|U) - I(V;Y_2|U) & (72) \\
R_{e2} &\leq 0. & (73)
\end{align*}
\]

for some \((U, V, X, Y_1, Y_2)\) such that \(U \to V \to X \to Y_1Y_2\).

**Proof:** The achievability is established by setting \(V_2 = \text{const}\) in Theorem 1 and using Eqs. (67) and (68). To prove the converse, we need to show that for any rate quintuple satisfying Eqs. (67)-(65) in Theorem 2, we can find \((U', V', X, Y_1, Y_2)\) such that \(U' \to V' \to X \to Y_1Y_2\) and (69)-(73) are satisfied. This can be accomplished using simple algebra and by defining \(U' = U V_2\) and \(V' = V_1\) where \((U, V_1, V_2, X, Y_1, Y_2) \in Q_2\) are the variables used in Theorem 2.

**Remark 3:** The fact that \(R_{e2} = 0\) is a direct consequence of the less noisy assumption: receiver 1 can always decode anything that receiver 2 can decode.

B. The rate equivocation region of semi-deterministic DMBC-2CM

Theorem 2 also allows us to establish the rate equivocation region of the semi-deterministic DMBC-2CM. WLOG, let channel 1 be deterministic.
Theorem 4: If \( p(y_1|x) \) is a \((0,1)\) matrix, then the rate equivocation region for this DMBC-2CM, denoted by \( \mathcal{R}_{sd} \), is the set of all non-negative \((R_0, R_1, R_2, R_{e1}, R_{e2})\) satisfying

\[
R_{e1} \leq R_1 \tag{74}
\]

\[
R_{e2} \leq R_2 \tag{75}
\]

\[
R_0 \leq \min[I(U;Y_1), I(U;Y_2)] \tag{76}
\]

\[
R_0 + R_1 \leq H(Y_1|U) + \min[I(U;Y_1), I(U;Y_2)] \tag{77}
\]

\[
R_0 + R_2 \leq I(V_2;Y_2|U) + \min[I(U;Y_1), I(U;Y_2)] \tag{78}
\]

\[
R_0 + R_1 + R_2 \leq H(Y_1|V_2U) + I(V_2;Y_2|U) + \min[I(U;Y_1), I(U;Y_2)] \tag{79}
\]

\[
R_{e1} \leq H(Y_1|Y_2V_2U) \tag{80}
\]

\[
R_{e2} \leq I(V_2;Y_2|U) - I(V_2;Y_1|U), \tag{81}
\]

for some \((U, Y_1, V_2, X, Y_1, Y_2) \in \mathcal{Q}_2\).

Proof: The direct part of this theorem follows trivially from Theorem \[\square\] by setting \( V_1 = Y_1 \).

The proof is therefore complete by showing \( \mathcal{R}_{SD-O2} \subseteq \mathcal{R}_{sd} \), where \( \mathcal{R}_{SD-O2} \) is the outer bound \( \mathcal{R}_{O2} \) specializing to the semi-deterministic DMBC-2CM. That is, for any \( Z \in \mathcal{Q}_2 \) and \((R_0, R_1, R_2, R_{e1}, R_{e2})\) satisfying \((77)-(81)\), we need to show that \((R_0, R_1, R_2, R_{e1}, R_{e2})\) also satisfies \((74)-(81)\). We note that Eqs. \((74)-(76), (78),\) and \((81)\) can be trivially established. That the sum-rate bound Eq. \((77)\) is satisfied follows easily from the fact

\[
H(Y_1|U) \geq I(V_1;Y_1|U). \tag{82}
\]

The sum-rate bound for \( R_0 + R_1 + R_2 \) in Eq. \((62)\) and \((63)\) can be re-written as

\[
R_0 + R_1 + R_2 \leq \min[I(V_2;Y_2|V_1U) + I(V_1;Y_1|U), I(V_1;Y_1V_2U) + I(V_2;Y_2|U)] \tag{83}
\]

\[
+ \min[I(U;Y_1), I(U;Y_2)]. \tag{84}
\]

Thus \((79)\) is satisfied since

\[
H(Y_1|V_2U) + I(V_2;Y_2|U) \geq I(V_1;Y_1V_2U) + I(V_2;Y_2|U). \tag{85}
\]

For Eq. \((80)\), we only need to show (cf. \((64)\))

\[
H(Y_1|Y_2V_2U) \geq I(V_1;Y_1|Y_2V_2U) - I(V_1;Y_2|V_2U). \tag{86}
\]

We have

\[
H(Y_1|Y_2V_2U) \geq I(V_1;Y_1|Y_2V_2U) \geq I(V_1;Y_1Y_2|V_2U) - I(V_1;Y_2|V_2U) \tag{87}
\]

\[
= I(V_1;Y_1Y_2|V_2U) - I(V_1;Y_2|V_2U) \tag{88}
\]

\[
\geq I(V_1;Y_1|V_2U) - I(V_1;Y_2|V_2U). \tag{89}
\]
The proof of Theorem 4 is therefore complete.

Similarly, the rate equivocation region of deterministic DMBC-2CM can be established as follows.

**Proposition 11:** If \( p(y_1|x) \) and \( p(y_2|x) \) are both \((0,1)\) matrices, then the rate equivocation region for this deterministic DMBC-2CM is the set of all \((R_0, R_1, R_2, R_{e1}, R_{e2})\) satisfying

\[
0 \leq R_{e1} \leq R_1 \quad (90)
\]
\[
0 \leq R_{e2} \leq R_2 \quad (91)
\]
\[
0 \leq R_0 \leq \min[I(U;Y_1), I(U;Y_2)] \quad (92)
\]
\[
R_0 + R_1 \leq H(Y_1|U) + \min[I(U;Y_1), I(U;Y_2)] \quad (93)
\]
\[
R_0 + R_2 \leq I(Y_2|U) + \min[I(U;Y_1), I(U;Y_2)] \quad (94)
\]
\[
R_0 + R_1 + R_2 \leq H(Y_1Y_2|U) + \min[I(U;Y_1), I(U;Y_2)] \quad (95)
\]
\[
R_{e1} \leq H(Y_1|Y_2U) \quad (96)
\]
\[
R_{e2} \leq H(Y_2|Y_1U) \quad (97)
\]

for some \((U, Y_1, Y_2, X, Y_1, Y_2) \in Q_2\).

**C. Outer bound for DMBC-2CM with perfect secrecy**

By setting \( R_0 = 0 \), \( R_{e1} = R_1 \) and \( R_{e2} = R_2 \) in Theorem 2 we obtain outer bounds for DMBC-2CM with perfect secrecy, denoted respectively by \( \mathcal{R}_{PS−O1} \), \( \mathcal{R}_{PS−O2} \), and \( \mathcal{R}_{PS−O3} \) for \( Z \in Q_1 \), \( Z \in Q_2 \), and \( Z \in Q_3 \). Clearly,

\[
\mathcal{R}_{PS−O1} = \mathcal{R}_{PS−O2} \supseteq \mathcal{R}_{PS−O3} \quad (98)
\]

In addition, from Proposition 3 we have

\[
\mathcal{R}_{PS−O2} = \mathcal{R}_{LMSY−O}. \quad (99)
\]

i.e., \( \mathcal{R}_{PS−O2} \) coincides with Liu et al’s outer bound in Proposition 3. Finally, all these outer bounds are tight for the semi-deterministic DMBC-2CM with perfect secrecy.

**D. New outer bounds for the general DMBC**

Specializing Theorem 2 to the general DMBC, i.e, setting \( R_{e1} = R_{e2} = 0 \), we obtain the following outer bounds for the general DMBC.
Theorem 5: For any $Z \in Q_1$, let $S_{BC}(Z)$ be the set of all $(R_0, R_1, R_2)$ of non-negative numbers satisfying

\begin{align*}
R_0 & \leq \min[I(U; Y_1), I(U; Y_2)] \quad (100) \\
R_0 + R_1 & \leq I(V_1; Y_1 | U) + \min[I(U; Y_1), I(U; Y_2)] \quad (101) \\
R_0 + R_2 & \leq I(V_2; Y_2 | U) + \min[I(U; Y_1), I(U; Y_2)] \quad (102) \\
R_0 + R_1 + R_2 & \leq I(V_2; Y_1 | V_1 U) + I(V_1; Y_1 | U) + \min[I(U; Y_1), I(U; Y_2)] \quad (103) \\
R_0 + R_1 + R_2 & \leq I(V_1; Y_1 | V_2 U) + I(V_2; Y_2 | U) + \min[I(U; Y_1), I(U; Y_2)]. \quad (104)
\end{align*}

Then

$$R_{BC-O1} = \bigcup_{Z \in Q_1} S_{BC}(Z) \quad (105)$$

constitutes an outer bound to the capacity region for the DMBC.

One can establish in a similar fashion two other outer bounds for the general DMBC, denoted by $R_{BC-O2}$ and $R_{BC-O3}$, by replacing $Q_1$ in Theorem 5 with $Q_2$ and $Q_3$, respectively. Similar to Proposition 10, we have

$$R_{BC-O3} \subseteq R_{BC-O1} = R_{BC-O2}. \quad (106)$$

Remark 4: It is interesting to observe that the inequalities of our outer bound $R_{BC}$ are all identical to those of the existing inner bound [20], described in Proposition 5 except for the bound on $R_0 + R_1 + R_2$, for which there is a gap of

$$\gamma = \min[I(V_1; V_2 Y_1 U), I(V_1; V_2 Y_2 U)]. \quad (107)$$

Remark 5: It is easy to show that $R_{BC-O2}$ subsumes the outer bound in [22, Theorem 6] since

\begin{align*}
I(V_1; Y_1 | V_2 U) & \leq I(X; Y_1 | V_2 U), \quad (108) \\
I(V_2; Y_2 | V_1 U) & \leq I(X; Y_2 | V_1 U). \quad (109)
\end{align*}

Remark 6: The new outer bound $R_{BC-O3}$ is also a subset of the outer bound proposed in [21, Theorem 2.1], as described in Proposition 7. More precisely, we have

Proposition 12: $R_{BC-O3} \subseteq R_{NE}$, where the equality holds when 1) $R_0 = 0$; or 2) $R_1 = 0$; or 3) $R_2 = 0$.

Proof: See Appendix III.

Remark 7: Note that the conditions in Proposition 12 are only sufficient conditions, i.e., there may be other instances when the two bounds are equivalent. It is also possible that $R_{BC-O3} = R_{NE}$ though we have not been successful in proving (or disapproving) it.
Remark 8: One can easily verify that the outer bound proposed in [23], $\mathcal{R}_{LKS}$ in Proposition 8, subsumes all the above outer bounds. To summarize, we have

$$\mathcal{R}_{LKS} \subseteq \mathcal{R}_{BC-OS} \subseteq \left\{ \mathcal{R}_{LK}, \mathcal{R}_{NE} \right\} \quad (110)$$

It remains unknown if any of the above subset relations can be strict or not.

The fact that $\mathcal{R}_{LKS}$ subsumes existing outer bounds can be attributed to the way auxiliary random variables are defined in [23]. By further splitting auxiliary random variables and isolating those corresponding to the message variables, one can keep the terms in the rate upper bounds which are otherwise dropped if only three auxiliary variables are used as in Theorem 2 or [21]. Finally, we remark that the approach in [23] can be adopted to the problem involving secrecy constraint in a straightforward manner to obtain a new outer bound to the rate equivocation region for DMBC-2CM.

V. CONCLUSION

We proposed inner and outer bounds for the rate equivocation region of discrete memoryless broadcast channels with two confidential messages (DMBC-2CM). The proposed inner bound combines superposition, rate splitting, and double binning and unifies existing known results for broadcast channels with or without confidential messages. These include Csiszár and Körner’s capacity rate region for broadcast channel with single private message [4], Liu et al’s rate region for broadcast channel with perfect secrecy [18], Marton and Gel’fand-Pinsker’s achievable rate region for general broadcast channels [19], [20]. The proposed outer bounds also generalize several existing results. In addition, the proposed inner and outer bounds settle the rate equivocation region of less noisy, deterministic, and semi-deterministic DMBC-2CM. In the absence of the equivocation constraints, the proposed outer bounds reduce to outer bounds for the general broadcast channel. General subset relations with other known outer bounds were established.

VI. ACKNOWLEDGMENT

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APPENDIX I

PROOF FOR THEOREM 1

We prove that if $(R_0, R_1, R_2, R_{e1}, R_{e2})$ is achievable, then it must satisfy Eqs. (48)-(55) in Theorem 1 for some $(U, V_1, V_2, X, Y_1, Y_2) \in Q_2$. We first prove the case when

$$R_1 \geq R_{e1} = I(V_1; Y_1 | U) - I(V_1; Y_2 V_2 | U) \geq 0, \quad (111)$$

$$R_2 \geq R_{e2} = I(V_2; Y_2 | U) - I(V_2; Y_1 V_1 | U) \geq 0. \quad (112)$$
Rate splitting, as described in Section [III] gives rise to the following five message variables:

\[
W_0 \in \{1, 2, \ldots, 2^{nR_0}\} \\
W_{10} \in \{1, 2, \ldots, 2^{nR_{10}}\} \\
W_{11} \in \{1, 2, \ldots, 2^{nR_{11}}\} \\
W_{20} \in \{1, 2, \ldots, 2^{nR_{20}}\} \\
W_{22} \in \{1, 2, \ldots, 2^{nR_{22}}\}
\]

where \(R_{10} + R_{11} = R_1\) and \(R_{20} + R_{22} = R_2\). We remark here that (111) and (112) combined with the rate splitting and the fact that \(W_{10}\) and \(W_{20}\) are decoded by both receivers ensures that,

\[
R_{11} \geq R_{\epsilon 1} = I(V_1; Y_1|U) - I(V_1; Y_2V_2|U) \geq 0, \quad (113)
\]

\[
R_{22} \geq R_{\epsilon 2} = I(V_2; Y_2|U) - I(V_2; Y_1V_1|U) \geq 0. \quad (114)
\]

**Auxiliary Codebook Generation:** Fix \(p(u), p(v_1|u), p(v_2|u)\) and \(p(x|v_1, v_2)\). For arbitrary \(\epsilon_1 > 0\), Define

\[
L_{11} = I(V_1; Y_1|U) - I(V_1; Y_2V_2|U), \quad (115)
\]

\[
L_{12} = I(V_1; Y_2|V_2U), \quad (116)
\]

\[
L_{21} = I(V_2; Y_2|U) - I(V_2; Y_1V_1|U), \quad (117)
\]

\[
L_{22} = I(V_2; Y_1|V_1U), \quad (118)
\]

\[
L_3 = I(V_1; Y_2|U) - \epsilon_1. \quad (119)
\]

Note that

\[
L_{11} + L_{12} + L_3 = I(V_1; Y_1|U) - \epsilon_1, \quad (120)
\]

\[
L_{21} + L_{22} + L_3 = I(V_2; Y_2|U) - \epsilon_1. \quad (121)
\]

- Generate \(2^{n(R_{10}+R_{20}+R_0)}\) independent and identically distributed (i.i.d.) codewords \(u(k)\), with \(k \in \{1, \ldots, 2^{n(R_{10}+R_{20}+R_0)}\}\), according to \(\prod_{t=1}^{n} p(u_t)\).
- For each codeword \(u(k)\), generate \(2^{n(L_{11}+L_{12}+L_3)}\) i.i.d. codewords \(v_1(i, i', i'')\), with \(i \in \{1, \ldots, 2^{nL_{11}}\}\), \(i' \in \{1, \ldots, 2^{nL_{12}}\}\) and \(i'' \in \{1, \ldots, 2^{nL_3}\}\), according to \(\prod_{t=1}^{n} p(v_{1t}|u_t)\). The indexing allows an alternative interpretation using binning. We randomly place the generated \(v_1\) vectors into \(2^{nL_{11}}\) bins indexed by \(i\); for the codewords in each bin, randomly place them into \(2^{nL_{12}}\) sub-bins indexed by \(i'\); thus \(i''\) is the index for the codeword in each sub-bin.
- Similarly, for each codeword \(u\), generate \(2^{n(L_{21}+L_{22}+L_3)}\) i.i.d. codewords \(v_2(j, j', j'')\) according to \(\prod_{t=1}^{n} p(v_{2t}|u_t)\), where \(j \in \{1, \ldots, 2^{nL_{21}}\}\), \(j' \in \{1, \ldots, 2^{nL_{22}}\}\) and \(j'' \in \{1, \ldots, 2^{nL_3}\}\).

**Encoding:** Encoding involves the mapping of message indices to channel input, which is facilitated by the auxiliary codewords generated above.
To send message \((w_{10}, w_{20}, w_0)\), we first calculate the corresponding message index \(k\) and choose the corresponding codeword \(u(k)\). Given this \(u(k)\), we have \(2^{n(L_1 + L_2 + L_3)}\) codewords of \(\nu_1(i, i', i'')\) to choose from for message \(w_{11}\). Evenly map \(2^{n_{R_1}}\) messages \(w_{11}\) to \(2^{n_{L_1}}\) bins, then, given (113), each bin corresponds to at least one message \(w_{11}\). Thus, given \(w_{11}\), the bin index \(i\) can be decided.

1) If \(R_{11} \leq L_{11} + L_{12}\), each bin corresponds to \(2^{R_{11} - L_{11}}\) messages \(w_{11}\). Evenly place the \(2^{n_{L_1}}\) sub-bins into \(2^{R_{11} - L_{11}}\) cells. Given \(w_{11}\), we can find the corresponding cell, and randomly choose a sub-bin from that cell, thus the sub-bin index \(i'\) can be decided. The codeword \(\nu_1(i, i', i'')\) will be chosen from that sub-bin.

2) If \(L_{11} + L_{12} < R_{11} \leq L_{11} + L_{12} + L_3\), then each sub-bin is mapped to at least one message \(w_{11}\), and \(i'\) is decided given \(w_{11}\). In each sub-bin, there are \(2^{R_{11} - L_{11} - L_{12}}\) messages. Evenly place those \(2^{n_{L_3}}\) codewords \(\nu_1\) into \(2^{R_{11} - L_{11} - L_{12}}\) cells. Given \(w_{11}\), we can find the corresponding cell. The codeword \(\nu_1(i, i', i'')\) will be chosen from that cell.

Given \(w_{22}\), the selection of \(\nu_{j, j', j''}\) is carried in exactly the same manner. From the given sub-bins or cells, the encoder chooses the codeword pair \((\nu_1(i, i', i''), \nu_2(j, j', j''))\) that satisfies

\[
(\nu_1(i, i', i''), \nu_2(j, j', j''), u(k)) \in A_e^{(n)}(V_1, V_2, U),
\]

where \(A_e^{(n)}(\cdot)\) denotes the jointly typical set. If there are more than one such pair, randomly choose one; if there is no such pair, an error is declared.

Given \(\nu_1\) and \(\nu_2\), we generate the channel input \(x\) according to i.i.d. \(p(x|v_1, v_2)\), i.e., \(x \sim \prod_{i=1}^n p(x_i|v_{1i}, v_{2i})\) where \(v_{1i}\) and \(v_{2i}\) are respectively the \(i\)th element of the vectors \(v_1\) and \(v_2\).

Decoding: Receiver \(Y_1\) looks for \(u(k)\) such that

\[
(u(k), y_1) \in A_e^{(n)}(U, Y_1).
\]

(123)

If such \(u(k)\) exists and is unique, set \(\hat{k} = k\); otherwise, declare an error. Upon decoding \(k\), receiver \(Y_1\) looks for sequences \(\nu_1(i, i', i'')\) such that

\[
(\nu_1(i, i', i''), u(k), y_1) \in A_e^{(n)}(V_1, U, Y_1).
\]

(124)

If such \(\nu_1(i, i', i'')\) exists and is unique, set \(i = i, \hat{i}' = i'\) and \(\hat{i}'' = i''\); otherwise, declare an error. From the values of \(\hat{k}, \hat{i}, \hat{i}'\) and \(\hat{i}''\), the decoder can calculate the message index \(\hat{w}_0, \hat{w}_{10}\) and \(\hat{w}_{11}\). The decoding for receiver \(Y_2\) is symmetric.

Analysis of Error Probability: We only consider \(P_{e,1}^{(n)}\) since \(P_{e,2}^{(n)}\) can be analyzed symmetrically. WLOG, we assume the transmitted codeword indices are \(k = i = i' = i'' = 1\). If an error is declared, one or more of the following events occur.

- \(A_1\): There is no pair \((\nu_1, \nu_2)\) such that (122) holds.
- \(A_2\): \(u(1, 1)\) does not satisfy (123).
- \(A_3\): \(u(k, k')\) satisfies (123), where \((k, k') \neq (1, 1)\).
- \(A_4\): \(\nu_1(1, 1, 1)\) does not satisfy (124).
- \(A_5\): \(\nu_1(i, i', i'')\) satisfies (124), where \((i, i', i'') \neq (1, 1, 1)\).
The fact that \( Pr\{A_2\} \leq \epsilon \) and \( Pr\{A_4\} \leq \epsilon \) for sufficiently large \( n \) follows directly from the asymptotic equipartition property. We now examine error events \( A_1, A_3, A_5 \).

Let \( E(v_1, v_2, u) \) denote the event \([122]\). Then

\[
Pr\{E(v_1, v_2, u)\} = \sum_{(u, v_1, v_2) \in A^n} p(u)p(v_1|u)p(v_2|u) \tag{126}
\]

\[
\geq |A^n_\epsilon| 2^{-n(H(U)+\epsilon)} 2^{-n(H(V_1|U)+\epsilon)} 2^{-n(H(V_2|U)+\epsilon)} \tag{127}
\]

\[
\geq 2^{-n(H(U)+H(V_1|U)+H(V_2|U)-H(UV_1V_2)+4\epsilon)} \tag{128}
\]

\[
\geq 2^{-n(I(V_1;V_2|U)+4\epsilon)} \tag{129}
\]

So,

\[
Pr\{A_1\} \leq \prod_{(v_1, v_2|k)} (1 - Pr\{E(v_1, v_2, u)\}) \tag{130}
\]

\[
\leq \prod_{(v_1, v_2|k)} (1 - 2^{-n(I(V_1;V_2|U)+4\epsilon)}) \tag{131}
\]

From [28], [31], it is clear that if

\[
I(V_1; Y_1|U) - \epsilon_1 - R_{11} + I(V_2; Y_2|U) - \epsilon_2 - R_{22} \geq I(V_1; V_2|U) \tag{132}
\]

\( Pr\{A_1\} \leq \epsilon \).

For \( A_3 \), we have, from the decoding rule, \( Pr\{A_3\} \leq \epsilon \) if

\[
R_0 + R_{10} + R_{20} \leq I(U; Y_1). \tag{133}
\]

For \( A_5 \), we first note that for \((i, i', i'') \neq (1, 1, 1),

\[
P\{v_i(i, i', i''), u(k), y_1\} \in A_i^{(n)}(V_1, U, Y_1) \leq 2^{-n(I(V_i; Y_i|U)-4\epsilon)} \tag{134}
\]

Given that the total number of codewords for \( v_1 \) is \( L_{11} + L_{12} + L_3 = I(V_1; Y_1|U) - \epsilon_1 \), it is easy to show that if

\[
R_{11} \leq I(V_1; Y_1|U) - \epsilon_1 \tag{135}
\]

then \( P\{A_5\} < \epsilon \) for \( n \) sufficiently large.

Since

\[
P_{e1}^{(n)} \leq Pr\left\{ \bigcup_{i=1}^{5} A_i \right\} \leq \sum_{i=1}^{5} Pr\{A_i\}, \tag{136}
\]

\( P_{e1}^{(n)} \leq 5\epsilon \) when \([53], \tag{133} \) and \([135] \) hold.

Symmetrically, for \( P_{e2}^{(n)} \leq 5\epsilon \) as \( n \) is sufficiently large, we need \([132], \tag{133} \) and

\[
R_0 + R_{10} + R_{20} \leq I(U; Y_2) \tag{137}
\]

\[
R_{22} \leq I(V_2; Y_2|U) - \epsilon_1 \tag{138}
\]
Apply Fourier-Motzkin elimination on (132), (133), (135), (137) and (138) with the definition \( R_1 = R_{11} + R_{10} \) and \( R_2 = R_{22} + R_{20} \), we get (50)-(53).

**Equivocation:** Now, we prove the bound on equivocation rate (54). Eq. (55) follows by symmetry.

\[
H(W_1|Y_2) \geq H(W_1|Y_2, V_2, U) \geq H(W_{11}, W_{10}|Y_2, V_2, U) \geq H(W_{11}|Y_2, V_2, U)
\]  
(139)

\[
= H(W_{11}, V_1, Y_2|V_2, U) - H(Y_2|V_2, U)
\]  
(140)

\[
\overset{(a)}{=} H(W_{11}, Y_2|V_2, U) - H(Y_2|V_2, U)
\]  
(141)

\[
= H(W_{11}, V_1, Y_2|V_2, U) - H(Y_2|V_2, U) - H(V_1|Y_2, V_2, U, W_{11})
\]  
(142)

\[
= H(W_{11}, V_1, V_2, U) - H(V_1|Y_2, V_2, U, W_{11})
\]  
(143)

\[
\overset{(b)}{=} H(V_1|V_2, U) - I(V_1; Y_2|V_2, U) - H(V_1|Y_2, V_2, U, W_{11})
\]  
(144)

where (a) follows from the fact that given \( U, W_{10} \) is uniquely determined, and (b) follows from the fact that given \( V_1, W_{11} \) is uniquely determined.

Consider the first term in (144), the codeword generation ensures that

\[
H(V_1|U) = \log 2^{n(L_{11}+L_{12}+L_3)} = nI(V_1; Y_1|U) - n\epsilon_1.
\]  
(145)

For the second and third terms in (144), using the same approach as that in [18, Lemma 3], we obtain

\[
I(V_1; V_2|U) \leq nI(V_1; V_2|U) + n\epsilon_2
\]  
(146)

\[
I(V_1; Y_2|V_2, U) \leq nI(V_1; Y_2|V_2 U) + n\epsilon_3
\]  
(147)

Now, we consider the last term of (144). We first prove that, given \( V_2, U \) and \( W_{11} \), the probability of error for \( Y_2 \) to decode \( V_1 \) satisfies \( P_e \leq \epsilon \) for \( n \) sufficiently large. \( Y_2 \) looks for \( v_1 \) such that

\[
(v_1, v_2, u, y_2) \in A_k^{(n)}(V_1, V_2, U, Y_2).
\]  
(148)

Since \( R_{11} \geq L_{11} \), and given the knowledge of \( W_{11} \), the total number of possible codewords of \( v_1 \) is

\[
N_1 \leq 2^{n(L_{12}+L_3)} = 2^{n(I(V_1; V_2 Y_2|U) - \epsilon_1)}
\]  
(149)

Now define \( E(v_1, v_2, u, y_2) \) the event in (148). We have

\[
Pr\{E(v_1, v_2, u, y_2)\} = \sum_{(u, v_1, v_2, y_2) \in A_k^{(n)}} p(u)p(v_1|u)p(v_2, y_2|u) \leq |A_k^{(n)}| 2^{-n(H(U) - \epsilon)} 2^{-n(H(V_1|U) - \epsilon)} 2^{-n(H(V_2 Y_2|U) - \epsilon)} \leq 2^{-n(H(U) + H(V_1|U) + H(V_2 Y_2|U) - H(UV_1 V_2 Y_2) - 4\epsilon)} \leq 2^{-n(I(V_1; V_2 Y_2|U) - 4\epsilon)}
\]  
(150-153)
Now, the probability of error for $Y_2$ to decode $V_1$ is
\[ P_e \leq \epsilon + N_1 \cdot 2^{-n(I(V_1;V_2|U) - 4\epsilon)} \leq \epsilon + 2^{-n(\epsilon_1 - 4\epsilon)} \leq 2\epsilon \]
(154)

where the first $\epsilon$ accounts for the error that the true $V_1$ is not jointly typical with $V_2, U, Y_2$ while the second term accounts for the error when a different $V_1$ is jointly typical with $V_2, U, Y_2$. By Fano’s inequality [32], we get
\[ H(V_1|Y_2, V_2, U, W_{11}) \leq n\epsilon_1. \]
(157)

Combine (145), (146), (147) and (157), we have the bound (54).

The above proof is only for the case when (111) and (112) are satisfied. By using the same convexity argument as in Lemma 5 and Lemma 6 in [4], we can easily show that the region (48)-(55) is also achievable. This completes the proof for Theorem 1.

**APPENDIX II**

**PROOF OF THE OUTER BOUNDS IN THEOREM 2**

We only prove $\mathcal{R}_{O2}$ and $\mathcal{R}_{O3}$ are outer bounds in this section. The proof of Theorem 2 is complete by the fact that $\mathcal{R}_{O1} = \mathcal{R}_{O2}$ (cf. Proposition 10).

We first define the following notations/quantities. All vectors involved are assumed to be length $n$.

\[ X^i \triangleq (X_1, \cdots, X_i); \]
\[ \tilde{X}^i \triangleq (X_i, \cdots, X_n); \]
\[ \Sigma_1 = \sum_{i=1}^{n} I(\tilde{Y}_2^{i-1}; Y_1^i|Y_1^{i-1}W_0); \]
\[ \Sigma_1^* = \sum_{i=1}^{n} I(Y_1^{i-1}; Y_2^i|\tilde{Y}_2^{i+1}W_0); \]
(158)
(159)
(160)
(161)

and $(\Sigma_2, \Sigma_2^*), (\Sigma_3, \Sigma_3^*), (\Sigma_4, \Sigma_4^*)$ are analogously defined by replacing $W_0$ with $W_0W_1, W_0W_2$ and $W_0W_1W_2$ in Eqs. (160) and (161), respectively. In exactly the same fashion as in [4, Lemma 7], one can establish, for $a = 1, 2, 3, 4$,
\[ \Sigma_a = \Sigma_a^*. \]
(162)

We begin by Fano’s Lemma,
\[ H(W_0, W_1|Y_1^n) \leq n\epsilon_n, \]
\[ H(W_0, W_2|Y_2^n) \leq n\epsilon_n. \]
where \( \epsilon_n \to 0 \) as \( n \to \infty \). Eqs. (57) and (58) follow trivially from
\[
0 \leq H(W_1|Y_2^n) \leq H(W_1), \tag{163}
\]
\[
0 \leq H(W_2|Y_1^n) \leq H(W_2). \tag{164}
\]

Next we check bound for \( R_0 \).
\[
nR_0 = H(W_0) = I(W_0; Y_1^n) + H(W_0|Y_1^n) \leq \sum_{i=1}^n I(W_0; Y_1^i) + n\epsilon_n \tag{165}
\]
\[
= \sum_{i=1}^n (I(W_0 Y_1^{i-1}; Y_1^i) - I(Y_1^{i-1}; Y_1^i)) + n\epsilon_n \tag{166}
\]
\[
\leq \sum_{i=1}^n (I(W_0 Y_1^{i-1} Y_2^{i+1}; Y_1^i) - I(Y_1^{i-1} W_1^n; Y_1^i Y_2^{i+1} W_0)) + n\epsilon_n \tag{167}
\]
\[
= \sum_{i=1}^n I(W_0 Y_1^{i-1} Y_2^{i+1}; Y_1^i) - \Sigma_1 + n\epsilon_n \tag{168}
\]
\[
\leq \sum_{i=1}^n I(W_0 Y_1^{i-1} Y_2^{i+1}; Y_1^i) + n\epsilon_n \tag{169}
\]

Similarly,
\[
nR_0 \leq \sum_{i=1}^n I(W_0 Y_1^{i-1} Y_2^{i+1}; Y_2^i) - \Sigma_1 + n\epsilon_n \tag{171}
\]
\[
\leq \sum_{i=1}^n I(W_0 Y_1^{i-1} Y_2^{i+1}; Y_2^i) + n\epsilon_n \tag{172}
\]

Therefore
\[
nR_0 \leq \min \left[ \sum_{i=1}^n I(W_0 Y_1^{i-1} Y_2^{i+1}; Y_1^i), \sum_{i=1}^n I(W_0 Y_1^{i-1} Y_2^{i+1}; Y_2^i) \right] + n\epsilon_n. \tag{173}
\]

Consider the sum rate bound for \( R_0 + R_1 \).
\[
n(R_0 + R_1) = H(W_0, W_1) = H(W_0) + H(W_1|W_0) \tag{174}
\]
\[
= H(W_0) + I(W_1; Y_1^n|W_0) + H(W_1 Y_1^n W_0) \tag{175}
\]
\[
\leq H(W_0) + I(W_1; Y_1^n|W_0) + n\epsilon_n \tag{176}
\]
where

\[
I(W_1; Y_1^n | W_0) = \sum_{i=1}^{n} I(W_1; Y_1 | Y_1^{i-1} W_0) \tag{177}
\]

\[
= \sum_{i=1}^{n} (I(W_1; \tilde{Y}_2^{i+1}; Y_1 | Y_1^{i-1} W_0) - I(\tilde{Y}_2^{i+1}; Y_1 | Y_1^{i-1} W_0 W_1)) \tag{178}
\]

\[
= \sum_{i=1}^{n} (I(W_1; Y_1 | Y_1^{i-1} \tilde{Y}_2^{i+1} W_0) + I(\tilde{Y}_2^{i+1}; Y_1 | Y_1^{i-1} W_0) - I(\tilde{Y}_2^{i+1}; Y_1 | Y_1^{i-1} W_0 W_1)) \tag{179}
\]

\[
= \sum_{i=1}^{n} I(W_1; Y_1 | Y_1^{i-1} \tilde{Y}_2^{i+1} W_0) + \Sigma_1 - \Sigma_2. \tag{180}
\]

Combine (168), (176), and (181), we have

\[
n(R_0 + R_1) \leq \sum_{i=1}^{n} I(W_0 Y_1^{i-1} \tilde{Y}_2^{i+1}; Y_1) + \sum_{i=1}^{n} I(W_1; Y_1 | Y_1^{i-1} \tilde{Y}_2^{i+1} W_0) - \Sigma_2 + 2n\epsilon_n. \tag{182}
\]

On the other hand, combining (171), (176), (181), and (162) yields

\[
n(R_0 + R_1) \leq \sum_{i=1}^{n} I(W_0 Y_1^{i-1} \tilde{Y}_2^{i+1}; Y_2) + \sum_{i=1}^{n} I(W_1; Y_1 | Y_1^{i-1} \tilde{Y}_2^{i+1} W_0) - \Sigma_2 + 2n\epsilon_n. \tag{183}
\]

Thus,

\[
n(R_0 + R_1) \leq \min \left[ \sum_{i=1}^{n} I(W_0 Y_1^{i-1} \tilde{Y}_2^{i+1}; Y_1), \sum_{i=1}^{n} I(W_0 Y_1^{i-1} \tilde{Y}_2^{i+1}; Y_2) \right]
\]

\[
+ \sum_{i=1}^{n} I(W_1; Y_1 | Y_1^{i-1} \tilde{Y}_2^{i+1} W_0) - \Sigma_2 + 2n\epsilon_n \tag{184}
\]

\[
\leq \min \left[ \sum_{i=1}^{n} I(W_0 Y_1^{i-1} \tilde{Y}_2^{i+1}; Y_1), \sum_{i=1}^{n} I(W_0 Y_1^{i-1} \tilde{Y}_2^{i+1}; Y_2) \right]
\]

\[
+ \sum_{i=1}^{n} I(W_1; Y_1 | Y_1^{i-1} \tilde{Y}_2^{i+1} W_0) + 2n\epsilon_n \tag{185}
\]

In an analogous fashion, we can get

\[
n(R_0 + R_2) \leq \min \left[ \sum_{i=1}^{n} I(W_0 Y_1^{i-1} \tilde{Y}_2^{i+1}; Y_1), \sum_{i=1}^{n} I(W_0 Y_1^{i-1} \tilde{Y}_2^{i+1}; Y_2) \right]
\]

\[
+ \sum_{i=1}^{n} I(W_1; Y_2 | Y_1^{i-1} \tilde{Y}_2^{i+1} W_0) - \Sigma_3 + 2n\epsilon_n \tag{186}
\]

\[
\leq \min \left[ \sum_{i=1}^{n} I(W_0 Y_1^{i-1} \tilde{Y}_2^{i+1}; Y_1), \sum_{i=1}^{n} I(W_0 Y_1^{i-1} \tilde{Y}_2^{i+1}; Y_2) \right]
\]

\[
+ \sum_{i=1}^{n} I(W_1; Y_2 | Y_1^{i-1} \tilde{Y}_2^{i+1} W_0) + 2n\epsilon_n \tag{187}
\]
Consider the sum rate bound for $R_0 + R_1 + R_2$.

$$n(R_0 + R_1 + R_2) = H(W_0, W_1) + H(W_2|W_1W_0)$$

$$= H(W_0, W_1) + I(W_2; Y^n_2 | W_1, W_0) + H(W_2|Y^n_2 W_0 W_1)$$

$$\leq H(W_0, W_1) + I(W_2; Y^n_2 | W_1, W_0) + n\epsilon_n,$$

(188) (189) (190)

$$n(R_0 + R_1 + R_2) = H(W_0, W_2) + H(W_1|W_2W_0)$$

$$= H(W_0, W_2) + I(W_1; Y^n_1| W_2, W_0) + H(W_1|Y^n_1 W_0 W_2)$$

$$\leq H(W_0, W_2) + I(W_1; Y^n_1| W_2, W_0) + n\epsilon_n.$$  

(191) (192) (193)

Following similar procedure as in (178)-(181), we can obtain

$$I(W_2; Y^n_2 | W_1, W_0) = \sum_{i=1}^{n} I(W_2; Y_{2i}|Y_{1i}^{i-1}Y_{2i+1}^{i+1}W_0 W_1) + \Sigma_2^n - \Sigma_4^n.$$  

(194)

$$I(W_1; Y^n_1| W_2, W_0) = \sum_{i=1}^{n} I(W_1; Y_{1i}|Y_{1i}^{i-1}Y_{2i+1}^{i+1}W_0 W_2) + \Sigma_3 - \Sigma_4,$$  

(195)

Combine (184), (190), (194), and (162), we get

$$n(R_0 + R_1 + R_2) \leq \min \left[ \sum_{i=1}^{n} I(W_0 Y_{1i}^{i-1}Y_{2i+1}^{i+1}; Y_{1i}), \sum_{i=1}^{n} I(W_0 Y_{1i}^{i-1}Y_{2i}^{i+1}; Y_{2i}) \right]$$

$$+ \sum_{i=1}^{n} I(W_1; Y_{1i}|Y_{1i}^{i-1}Y_{2i+1}^{i+1}W_0) + \sum_{i=1}^{n} I(W_2; Y_{2i}|Y_{1i}^{i-1}Y_{2i+1}^{i+1}W_0 W_1) + 3n\epsilon_n.$$  

(196)

Alternatively, combining (186), (193), (198), and (162) yields

$$n(R_0 + R_1 + R_2) \leq \min \left[ \sum_{i=1}^{n} I(W_0 Y_{1i}^{i-1}Y_{2i+1}^{i+1}; Y_{1i}), \sum_{i=1}^{n} I(W_0 Y_{1i}^{i-1}Y_{2i}^{i+1}; Y_{2i}) \right]$$

$$+ \sum_{i=1}^{n} I(W_2; Y_{2i}|Y_{1i}^{i-1}Y_{2i+1}^{i+1}W_0) + \sum_{i=1}^{n} I(W_1; Y_{2i}|Y_{1i}^{i-1}Y_{2i+1}^{i+1}W_0 W_2) + 3n\epsilon_n.$$  

(197)

We now consider the equivocation rate bound.

$$R_{e1} \leq H(W_1|Y^n_2)$$

$$= H(W_1|Y^n_2 W_0) + I(W_1; W_0|Y^n_2)$$

$$\leq H(W_1|W_0) - I(W_1; Y^n_2|W_0) + H(W_0|Y^n_2)$$

$$= I(W_1; Y^n_1|W_0) - I(W_1; Y^n_2|W_0) + H(W_1|Y^n_1 W_0) + H(W_0|Y^n_2)$$

$$\leq I(W_1; Y^n_1|W_0) - I(W_1; Y^n_2|W_0) + 2n\epsilon_n,$$  

(198) (199) (200) (201) (202)

$$R_{e1} \leq H(W_1|Y^n_2)$$

$$= H(W_1|Y^n_2 W_0 W_2) + I(W_1; W_0 W_2|Y^n_2)$$

$$\leq H(W_1|W_0 W_2) - I(W_1; Y^n_2|W_0 W_2) + H(W_0 W_2|Y^n_2)$$

(203) (204) (205)
Of the terms involved in (202) and (207), only determined. Similar to (178)-(181), we can get

\[ I(W_1; Y^n_1 | W_0 W_2) = \sum_{i=1}^{n} I(W_1; Y_{2i}|Y_{1i}^{-1}Y_{i+1}^{j+1} W_0) + \Sigma_1^* - \Sigma_2^*, \]  

(208)

\[ I(W_1; Y^n_2 | W_0 W_2) = \sum_{i=1}^{n} I(W_1; Y_{2i}|Y_{1i}^{-1}Y_{i+1}^{j+1} W_0 W_2) + \Sigma_3^* - \Sigma_4^*. \]  

(209)

Therefore we get

\[ R_{e1} \leq \sum_{i=1}^{n} I(W_1; Y_{1i}|Y_{1i}^{-1}Y_{i+1}^{j+1} W_0) - \sum_{i=1}^{n} I(W_1; Y_{2i}|Y_{1i}^{-1}Y_{i+1}^{j+1} W_0) + 2n\epsilon_n, \]  

(210)

\[ R_{e1} \leq \sum_{i=1}^{n} I(W_1; Y_{1i}|Y_{1i}^{-1}Y_{i+1}^{j+1} W_0 W_2) - \sum_{i=1}^{n} I(W_1; Y_{2i}|Y_{1i}^{-1}Y_{i+1}^{j+1} W_0 W_2) + 2n\epsilon_n. \]  

(211)

Bounds on \( R_{e2} \) are analogously obtained:

\[ R_{e2} \leq \sum_{i=1}^{n} I(W_2; Y_{2i}|Y_{i}^{j-1}Y_{i+1}^{j} W_0) - \sum_{i=1}^{n} I(W_2; Y_{1i}|Y_{i}^{-1}Y_{i+1}^{j+1} W_0) + 2n\epsilon_n, \]  

(212)

\[ R_{e2} \leq \sum_{i=1}^{n} I(W_2; Y_{2i}|Y_{1i}^{-1}Y_{i+1}^{j+1} W_0 W_1) - \sum_{i=1}^{n} I(W_2; Y_{1i}|Y_{1i}^{-1}Y_{i+1}^{j+1} W_0 W_1) + 2n\epsilon_n. \]  

(213)

Let us introduce a random variable \( J \), independent of \( W_0 W_1 W_2 X^n Y^n_1 Y^n_2 \), uniformly distributed over \( \{1, \cdots, n\} \). Set

\[ U \triangleq W_0 Y_{1}^{j-1}Y_{j+1}^{j}, \quad V_1 \triangleq W_1 U, \quad V_2 \triangleq W_2 U, \]

\[ X \triangleq X J, \quad Y_1 \triangleq Y_{1} J, \quad Y_2 \triangleq Y_{2} J. \]

Substituting these definitions into Eqs. (173), (185), (187), (196), (197), and (210)-(213), we obtain, through standard information theoretic argument, the desired bounds as in Eqs. (57)-(65). The memoryless property of the channel guarantees \( U \rightarrow V_1 V_2 \rightarrow X \rightarrow Y_1 Y_2 \). This completes the proof.

To prove \( R_{O3} \) is also an outer bound, we follow exactly the same procedure except that auxiliary random variables are defined differently. Specifically,

\[ U \triangleq W_0 Y_{1}^{j-1}Y_{j+1}^{j}, \quad V_1 \triangleq W_1, \quad V_2 \triangleq W_2. \]

APPENDIX III

PROOF OF PROPOSITION 12

By simple algebra, one can show \( R_{BC-O3} \subseteq R_{NE} \). The fact that \( R_{BC-O3} = R_{NE} \) when \( R_0 = 0 \) can also be verified by direct substitution.
We now prove the equivalence under $R_2 = 0$, and the case for $R_1 = 0$ can be established by index swapping. With $R_2 = 0$, Eqs. (100)-(104) of $\mathcal{R}_{BC-O3}$ can be easily shown to be equivalent to

\[
R_0 \leq \min[I(U;Y_1), I(U;Y_2)], \quad (214)
\]
\[
R_0 + R_1 \leq I(V_1;Y_1|U) + \min[I(U;Y_1), I(U;Y_2)], \quad (215)
\]

We note this is precisely the capacity region for DMBC with degraded message set [4, Corollary 5].

With $R_2 = 0$, $\mathcal{R}_{NE}$ in Proposition 7 reduces to

\[
R_0 \leq \min[I(U;Y_1), I(U;Y_2)], \quad (216)
\]
\[
R_0 + R_1 \leq I(V_1 U; Y_1), \quad (217)
\]
\[
R_0 + R_1 \leq I(V_1 Y_1 | V_2 U) + I(U V_2; Y_2). \quad (218)
\]

Apparently $\mathcal{R}_{BC-O3} \subseteq \mathcal{R}_{NE}$, and it remains to check $\mathcal{R}_{NE} \subseteq \mathcal{R}_{BC-O3}$. Assume $(R_0, R_1) \in \mathcal{R}_{NE}$ and $(U, V_1, V_2, X, Y_1, Y_2) \in Q_3$ are the variables such that Eqs. (216)-(218) are satisfied. Consider three cases for analysis.

1) $I(U;Y_1) \leq I(U;Y_2)$. The proof of $(R_0, R_1) \in \mathcal{R}_{BC-O3}$ is trivial.

2) $I(U;Y_1) \geq I(U;Y_2)$ and $I(V_2, U; Y_1) \geq I(V_2, U; Y_2)$.

Define $V'_1 = V_1, U' = UV_2$. From (216),

\[
R_0 \leq \min[I(U;Y_1), I(U;Y_2)] \quad (219)
\]
\[
\leq \min[I(U V_2; Y_1), I(U V_2; Y_2)] \quad (220)
\]
\[
= \min[I(U';Y_1), I(U';Y_2)] \quad (221)
\]

From (218),

\[
R_0 + R_1 \leq I(V_1 Y_1 | UV_2) + I(UV_2; Y_2) \quad (222)
\]
\[
= I(V'_1 Y_1 | U') + I(U'; Y_2) \quad (223)
\]

Thus $(R_0, R_2)$ also satisfies (214) and (215) for $U'V'_1 \rightarrow X \rightarrow Y_1 Y_2$.

3) $I(U;Y_1) \geq I(U;Y_2)$ and $I(V_2, U; Y_1) \leq I(V_2, U; Y_2)$.

For this case, we can always find a function $g(\cdot)$ such that

\[
I(U g(V_2); Y_1) = I(U g(V_2); Y_2). \quad (224)
\]

Define $V'_1 = V_1, U' = U g(V_2)$ and we can verify that $(R_0, R_1)$ satisfies (214) and (215) for $U'V'_1 \rightarrow X \rightarrow Y_1 Y_2$.

The above argument completes the proof of Proposition 12.
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