Irreversibility in the Halting Problem of Quantum Computer

A.E. Shalyt-Margolin, V.I. Strazhev and A.Ya. Tregubovich
National Centre of High Energy and Particle Physics,
Bogdanovich Str. 153, 220040 Minsk, Belarus,
E-mails: alexm@hep.by, a.tregub@open.by

Abstract

The Halting problem of a quantum computer is considered. It is shown that if halting of a quantum computer takes place the associated dynamics is described by an irreversible operator.

1 Introduction

In the last 10 - 15 years a quantum theory of information has developed rather intensively in various directions from mathematical aspects to different physical problems: quantum algorithms [1], quantum decoherence [2], density matrix and entropy for the entanglement states [3], measuring theory for quantum information, and a number of physical models of quantum computers based on various principles [4].

However, there are problems to be solved. Among them is the halting problem that originated in the mid eighties. This problem may be generally formulated as follows: how can a correct description of the quantum computer halting be compatible with the basic principles of a quantum theory of information [5].

This problem is studied in a number of works [6]–[9], [12], [13]. And in the present paper it is shown that halting of quantum computers is incompatible not only with unitarity but also with reversibility of the corresponding dynamics.

2 The Halting problem

In the paper [5], where the term 'halting' is firstly used, the following special qubit is chosen

\[ \hat{q} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]

to signal that the computer is halted. This means that each correctly working program sets \( \hat{q} \) to 1 when then operation is terminated, and sets \( \hat{q} \) to 0 otherwise. According to Myers [6], the program for different branches of the computing process can have different number of steps giving rise to the unitarity problem of the basic calculation operators. This problem was in principal solved in [7]. But as shown by [8], in [7] a kind of the Turing machine is used, inapplicable to realistic computers, as in this case the dynamics is unitary only for the computers having no halt. So for realistic computers the problem remains unsolved. Besides, in [8] one more problem is discovered for a quantum computer when different branches of the computation process halt at different and unknown times. And in [9] it is shown that halting of the universal quantum computer is incompatible with the unitarity constraint of quantum computations.

3 The Halting problem, unitarity and reversibility

To make the following definitions valid, we use the terminology of [9]:
1. Quantum computer is a closed quantum system controlled by the time-independent evolution operator \( U \) for each time step between the state of the input space representing some vector \( |\tau_{m}\rangle \) in a Hilbert space \( \mathcal{H} \) and the final state \( |\tau_{out}\rangle \) of the output in the same Hilbert space.

2. For halting, the dynamics is to be able to store the output that is finite in terms of qubit resources, no matter in what finite time the desirable output is computed. This reserved space, from where the output can be read out, is mathematically an invariant subspace \( V \subset \mathcal{H} \) with a component of the qubit \( \hat{q} \) equal to 1.

We intentionally weaken the requirements for the dynamics and do not consider \( U \) as obviously unitary.

Thus, any state has the form
\[
|\psi_{0}\rangle = |0_{h}\rangle \otimes |x_{0}\rangle + |1_{h}\rangle \otimes |y_{0}\rangle
\]

The information transfer matrix \( U \) is written as
\[
U = \begin{pmatrix} A & \alpha \\ 0 & B \end{pmatrix}
\]

in the basis
\[
|0_{h}\rangle \otimes |x_{0}\rangle = \begin{pmatrix} 0 \\ |x_{0}\rangle \end{pmatrix}, \quad |1_{h}\rangle \otimes |y_{0}\rangle = \begin{pmatrix} |y_{0}\rangle \\ 0 \end{pmatrix}
\]

We show that the halting conditions of a quantum computer after performance of the program with a finite number of steps
\[
\text{for } N \geq N_{0} \quad \langle 0_{h} | U^{N} | \psi_{0}\rangle = |1_{h}\rangle \otimes |y_{0}\rangle
\]

\[
\langle 0_{h} | U^{N} | \psi_{0}\rangle = B^{N} | x_{0}\rangle = 0
\]

are incompatible with the reversibility of the operator \( U \).

Actually, let \( U \) be a two-side reversible matrix and let
\[
U^{-1} = \begin{pmatrix} A_{11}^{(r)} & A_{12}^{(r)} \\ A_{21}^{(r)} & A_{22}^{(r)} \end{pmatrix}
\]

be the left-hand reciprocal for \( U \), whereas
\[
U^{-1} = \begin{pmatrix} A_{11}^{(l)} & A_{12}^{(l)} \\ A_{21}^{(l)} & A_{22}^{(l)} \end{pmatrix}
\]

be the right-hand reciprocal for \( U \). Then
\[
^{-1}UU = \begin{pmatrix} A_{11}^{(l)} & A_{12}^{(l)} \\ A_{21}^{(l)} & A_{22}^{(l)} \end{pmatrix} \begin{pmatrix} A & \alpha \\ 0 & B \end{pmatrix} = \begin{pmatrix} A_{11}^{(l)}A & A_{12}^{(l)} \alpha + A_{12}^{(l)}B \\ A_{21}^{(l)}A & A_{22}^{(l)} \alpha + A_{22}^{(l)}B \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

Thus it follows that the matrix \( A \) has the left-hand reciprocal \( A^{-1} = A_{11}^{(l)} \). Similarly, we have
\[
UU^{-1} = \begin{pmatrix} A & \alpha \\ 0 & B \end{pmatrix} \begin{pmatrix} A_{11}^{(r)} & A_{12}^{(r)} \\ A_{21}^{(r)} & A_{22}^{(r)} \end{pmatrix} = \begin{pmatrix} AA_{11}^{(r)} + \alpha A_{21}^{(r)} & AA_{12}^{(r)} + \alpha A_{22}^{(r)} \\ BA_{21}^{(r)} & BA_{22}^{(r)} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

Consequently, the matrix \( B \) is right-hand reversible. As \( B \) is right-hand reversible, \( B^{N} \) is such as well.

Using the results for examples 8 and 10 of Chapter 2 from [10], we obtain immediately that \( (B^{N})^{+} \) is also right-reversible. The condition \( (8) \) is obviously equivalent to the condition
\[
\langle x_{0} | (B^{N})^{+} = 0.
\]

Multiplying the left and right parts of the last equality on the right by \((B^{N}+)^{-1}\), we obtain for any bra - vector \( \langle x_{0} = 0 \). This is an obvious contradiction. So it follows that \( U \) is not reversible. Besides from our proof it follows that \( U \) is not even right-hand reversible. It is necessary to make two remarks:
1. The above proof is correct both for the universal quantum computer, that is the case when the Hilbert space $\hat{H}$ is infinitely dimensional, and for the realistic quantum computer when $\hat{H}$ has a finite dimension. The proof is simplified in this case due to the fact that (a) for square matrices a left-hand reciprocal is coincident with the right-hand one and (b) for the upper triangular reversible matrices the Jordan decomposition takes place [11].

$$U = \begin{pmatrix} A & \alpha \\ 0 & B \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} 1 & A^{-1}\alpha \\ 0 & 1 \end{pmatrix}$$

Then the key argument will follow directly from the condition of [5].

2. It would be natural to require that the left-hand reciprocal $-1U$ and the right-hand reciprocal $U^{-1}$ of $U$ be also elements of the dynamics of a quantum computer and should be of the upper triangular form to simplify the proof even greater.

4 Conclusion
In the work it is shown that in the general case when halting of the universal or realistic quantum computer takes place the associated dynamics is non-unitary and, what is more, irreversible.

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