World-sheet aspects of mirror symmetry

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Abstract

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1Talk given at the Oskar Klein centenary symposium 19-21 September 1994 in Stockholm, Sweden, to appear in the proceedings.
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ABSTRACT

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1. String theory, compactification and mirror symmetry

String theory is an attempt to construct a unified theory of all kinds of matter and all types of interactions, including gravity \[^1\]. It naturally incorporates many attractive ideas in particle physics, such as higher dimensions, grand unification and supersymmetry. The most remarkable accomplishment of string theory is that it seems to provide a consistent way of incorporating Einstein’s theory of gravity, i.e. general relativity, in a quantum mechanical framework. The basic idea of string theory is that the fundamental constituents of nature are one-dimensional objects, strings, rather than zero-dimensional point particles. A string sweeps out a two-dimensional world-sheet as it propagates through space-time. This is the analogue of the one-dimensional world-line of a point particle.

We can get an estimate of the typical size of a string by simple dimensional analysis: The characteristic scale of string theory, where we could expect ‘stringy’ effects to be important, is given by the Planck length \( \sqrt{\frac{\hbar G}{c}} \approx 10^{-35} \) \( \text{m} \). This is about twenty orders of magnitude shorter than the length scales relevant to ordinary particle physics, where we expect string theory to be well approximated by some low energy effective field theory.

When one examines the quantum mechanical consistency of (supersymmetric) string theory, it turns out that the dimension of space-time must be ten rather than the observed four. Inspired by the work of Kaluza and Klein \[^2\][^3\] on higher dimensional theories, we resolve this apparent paradox by postulating that the ground state of string theory takes the form

\[
10-\text{dimensional string vacuum} = M_4 \times K, \tag{1}
\]
where $M_4$ is the ordinary 4-dimensional Minkowski space and $K$ is some 6-dimensional, compact, Planck-sized space.

A closer examination reveals that $K$ must be a so called Calabi-Yau space [4]. These spaces are of great mathematical interest because of their remarkable geometric properties. Their existence was conjectured by Calabi in the 1950’s and proved by Yau in 1977 [5]. Given a Calabi-Yau space $K$, there are in general two different types of continuous deformations that preserve the Calabi-Yau conditions; we will refer to them loosely as ‘shape’ deformations and ‘size’ deformations. The number of independent such deformations depends on the particular example studied.

The topology and geometry of the Calabi-Yau space $K$ determine the properties of the observable low energy physics in Minkowski space, i.e. the unbroken gauge-group, the particle spectrum, the Yukawa couplings etcetera. It is often convenient to think of this situation as follows: The string propagating on a Calabi-Yau space defines a 2-dimensional, $N = 2$ superconformal field theory, which in its turn determines a Minkowski space low energy effective field theory.

The intermediate step involves a two-dimensional field theory defined on the world-sheet of the string. This has to be a conformal field theory for the string interpretation to be consistent. The requirement of $N = 2$ supersymmetry on the world-sheet is almost equivalent to $N = 1$ supersymmetry in Minkowski space, which is considered to be an attractive feature of the low energy theory. The picture of a string propagating on a Calabi-Yau space is intuitively appealing, although, as we will see later, one should not necessarily attach too much meaning to this space. The hard facts of this particular compactification are encoded in a more abstract way in the $N = 2$ superconformal field theory, whereas the observational consequences follow from the low energy effective field theory.

A string propagating on a Calabi-Yau space $K$ thus determines an $N = 2$ superconformal field theory (and also a low energy effective field theory), but the reverse is not true. In fact, to account for all $N = 2$ superconformal field theories we will have to generalize the concept of a Calabi-Yau compactification, as we will see later. Furthermore, we will see that different Calabi-Yau spaces (in this generalized sense) may give rise to isomorphic $N = 2$ superconformal field theories and thus equivalent low energy physics.

We will now present the canonical example of a Calabi-Yau space: A quintic hypersurface in $\mathbb{CP}^4$. We start by considering complex 5-space $\mathbb{C}^5$ with coordinates $X_1, \ldots, X_5$. To get an intuitive idea of what is going on, it might be helpful to think of an analogy with $\mathbb{C}^5$ replaced by ordinary real 3-space $\mathbb{R}^3$. Complex projective 4-space $\mathbb{CP}^4$ could be defined as the space of all complex lines through the origin in $\mathbb{C}^5$.

Alternatively we may think of it as $\mathbb{C}^5$ minus the origin modulo the equivalence relation $(X_1, \ldots, X_5) \sim \lambda(X_1, \ldots, X_5)$ for any non-zero complex number $\lambda$. The analogous construction for $\mathbb{R}^3$ would be real projective 2-space $\mathbb{RP}^2$, i.e. the space of lines through the origin in $\mathbb{R}^3$. Such a line could be represented by its intersection with the upper unit half sphere, so $\mathbb{RP}^2$ is homeomorphic to this half sphere with
antipodal boundary points identified. Going back to \( \mathbb{CP}^4 \), let \( W(X_1, \ldots, X_5) \) be a degree 5 polynomial, e.g.

\[
W(X_1, \ldots, X_5) = X_1^5 + \ldots + X_5^5 + \epsilon X_1 \ldots X_5,
\]

where \( \epsilon \) is a parameter. Although \( W \) does not make sense as a function on \( \mathbb{CP}^4 \), we may talk of its set of zeros. The hypersurface \( K \) in \( \mathbb{CP}^4 \) defined by \( W(X_1, \ldots, X_5) = 0 \) obviously has complex dimension 3, i.e. real dimension 6, and turns out to be a Calabi-Yau space. In our analogy, a hypersurface in \( \mathbb{RP}^2 \) is simply a curve on the upper half sphere. One should note that the only thing that really matters are the intrinsic properties of the Calabi-Yau space \( K \) and not its embedding in \( \mathbb{CP}^4 \) or the construction of this space from \( \mathbb{C}^5 \). Alternatively, we could have presented \( K \) in some other way, e.g. by an atlas and transition functions from one coordinate chart to another.

The parameter \( \epsilon \) in (2) is one of a total of 101 ‘shape’ parameters in this example. The remaining ‘shape’ parameters also amount to adding some degree 5 terms to the polynomial \( W(X_1, \ldots, X_5) \). Calabi’s conjecture (proven by Yau) amounts to the existence of a ‘Calabi-Yau’ metric with special properties on the space \( K \). This metric is unique up to an overall constant \( r \), which turns out to be the only ‘size’ parameter in this example. It should be noted that the exact form of the Calabi-Yau metric is not known.

We have already mentioned that the concept of a Calabi-Yau compactification needs to be generalized to cover all possible string compactifications. This issue is closely related to the possible values of the ‘size’ parameter \( r \), as we will now explain. The Calabi-Yau interpretation in fact works best when \( r \) is much larger than the Planck length. For an \( r \) of the order of the Planck length, we get a strongly coupled sigma model, which still makes sense, although the theory is harder to solve.

In fact, we may even continue to negative values of \( r \). Here, the geometric interpretation obviously breaks down, but interestingly enough the \( N = 2 \) superconformal field theory is still well-defined. If we continue to a negative value of \( r \) with magnitude much larger than the Planck length, we eventually reach a special point in the parameter space called the Landau-Ginzburg point. We will give an interpretation of the theory at this point shortly.

The above process has some similarity with the theory of phase transitions, and we may think of going from a ‘Calabi-Yau phase’ to a ‘Landau-Ginzburg phase’ by changing the parameter \( r \). One should note, however, that the singularity at \( r = 0 \) may be avoided by giving \( r \) an imaginary part, so there need not be any discontinuous phase transition between the two ‘phases’, much as we may go from the liquid to the steam phase of a fluid in a smooth way.

To understand the \( r \rightarrow -\infty \) limit we will first describe what might be called the \( W \) Landau-Ginzburg theory. We may think of a string propagating in complex 5-space \( \mathbb{C}^5 \) with coordinates \( X_1, \ldots, X_5 \) under the influence of a potential of the form

\[
W(X_1, \ldots, X_5) + \text{complex conjugate.}
\]
Here \( W(X_1, \ldots, X_5) \) is again a degree 5 polynomial such as (2). To complete the definition of the theory, we should also choose a kinetic energy term. Arguments based on the renormalization group show that there exists a kinetic energy term such that the resulting theory is an \( N = 2 \) superconformal field theory, but the exact form of this kinetic energy is in general not known. This is analogous to the problem with the form of the Calabi-Yau metric for a Calabi-Yau space.

The potential (3) with \( W \) a degree 5 polynomial such as (2) is invariant under \( \mathbb{Z}_5 \) acting as

\[(X_1, \ldots, X_5) \rightarrow (\alpha X_1, \ldots, \alpha X_5)\]  \hspace{1cm} (4)

for \( \alpha \) a fifth root of unity, \( i.e. \alpha^5 = 1 \). Given a quantum field theory invariant under some group of symmetries, we may construct a new theory by taking the orbifold of the original theory with respect to the symmetry group \( \mathbb{Z}_5 \). In a Lagrangian framework, where a theory is given by a path integral over some fields and an action functional of the fields, this amounts extending the domain of integration in the path integral by allowing the fields to be well-defined only modulo the action of the symmetry group. In our example, we may construct the Landau-Ginzburg orbifold \( W/\mathbb{Z}_5 \) by allowing the fields \( X_1, \ldots, X_5 \) to transform by an element of \( \mathbb{Z}_5 \) as we traverse a non-contractible, closed curve on the world-sheet.

We may now state the Landau-Ginzburg Calabi-Yau correspondence \[6\] for our example:

*The \( W/\mathbb{Z}_5 \) Landau-Ginzburg orbifold theory is equivalent to the \( r \rightarrow -\infty \) limit of the Calabi-Yau theory given (for \( r > 0 \)) by the hypersurface \( W = 0 \) in \( \mathbb{CP}^4 \).*

We have already mentioned the fact that two different Calabi-Yau models (or Landau-Ginzburg generalizations thereof as discussed above) may give rise to isomorphic \( N = 2 \) superconformal field theories and thus equivalent low energy physics. Mirror symmetry is the best known example of this phenomenon \[8\]. It can be made plausible by the following argument \[3\] \[10\]: Consider again the ‘shape’ and ‘size’ deformations of a Calabi-Yau space. At the level of \( N = 2 \) superconformal field theory, these deformations correspond to a perturbation of the theory by some operator.

The operators in the theory are characterized by a quantum number that we may call ‘charge’. It turns out that a ‘shape’ or a ‘size’ deformation corresponds to a perturbing operator of charge \(+1\) or \(-1\) respectively. (At the level of the low energy effective field theory, these deformations of course correspond to a continuous change of some coupling constants.) It now seems unnatural that the profound difference between a ‘shape’ and a ‘size’ deformation of a Calabi-Yau space should correspond to the purely conventional difference between operators of charge \(+1\) or \(-1\). This observation naturally leads us to postulate the mirror hypothesis:
To every Calabi-Yau space $K$ corresponds a mirror space $\tilde{K}$ such that the corresponding $N = 2$ superconformal field theories are isomorphic (with the isomorphism amounting to a change of sign of the charge quantum number).

Next, we note that a ‘shape’ deformation of $K$ is equivalent to a charge $-1$ perturbation of the corresponding $N = 2$ superconformal field theory and thus, by the isomorphism, a charge $-1$ perturbation of the mirror $N = 2$ superconformal field theory, which corresponds to a ‘size’ deformation of $\tilde{K}$. The deformed spaces are still mirror partners, and we may thus conclude that it suffices to establish mirror symmetry between two particular Calabi-Yau spaces $K$ and $\tilde{K}$ to establish it in the whole parameter space. It would therefore be very interesting to find an example where mirror symmetry may be rigorously proven.

Before giving such an example, we must briefly discuss the so called $A_k$ minimal models, which in a sense are the simplest examples of $N = 2$ superconformal field theories [11]. The $A_k$ minimal model has a Lagrangian representation as a Landau-Ginzburg theory with a single field $X$ and the potential $X_5^{k+2} [12]$, but it may also be defined in an abstract algebraic way in terms of its Hilbert space and algebra of observables. It is essentially exactly solvable by the representation theory of the $N = 2$ superconformal algebra. The following facts are of particular importance to us [13][14]:

The $A_k$ minimal model is invariant under a symmetry group isomorphic to $\mathbb{Z}_{k+2}$. Furthermore, the orbifolds of the model with respect to $\mathbb{Z}_m \text{ and } \mathbb{Z}_{\tilde{m}}$ subgroups of this group are isomorphic if $m\tilde{m} = k + 2$.

We may now go back to the problem of finding an example of a mirror pair of string compactifications. Consider the Landau-Ginzburg model with potential

$$W(X_1, \ldots, X_5) = X_5^5 + \cdots + X_5^5. \quad (5)$$

We will refer to this particular model as the ‘Fermat point’. It is invariant under $\mathbb{Z}_5$ acting as

$$(X_1, \ldots, X_5) \rightarrow (\alpha_1 X_1, \ldots, \alpha_5 X_5), \quad (6)$$

where the $\alpha_i$ are possibly different fifth roots of unity, i.e. $\alpha_1^5 = \cdots = \alpha_5^5 = 1$. We may thus construct Landau-Ginzburg orbifolds $W/S$ for any subgroup $S$ of $\mathbb{Z}_5^5$. As before, the orbifold $W/\mathbb{Z}_5$ for the ‘diagonal’ embedding of $\mathbb{Z}_5$ in $\mathbb{Z}_5^5$ is the $r \rightarrow -\infty$ limit of the corresponding Calabi-Yau theory. We see that the Landau-Ginzburg model with the potential (5) is ‘separable’ into five $A_3$ minimal models, each described by a single field $X$ and the potential $X^5$. Using this relationship and the previous result about the orbifolds of the minimal models, it is possible to prove that

$$W/\mathbb{Z}_5 \simeq (W/\mathbb{Z}_5^5)/\mathbb{Z}_5^3 = W/\mathbb{Z}_5^4 \quad (7)$$

6
for a certain embedding of $\mathbb{Z}_5^3$ and $\mathbb{Z}_5^4$ in $\mathbb{Z}_5^5$ [13]. This isomorphism is an example of mirror symmetry. From our previous arguments, mirror symmetry now follows for other values of the ‘shape’ and ‘size’ parameters in the whole parameter space of quintic Calabi-Yau and Landau-Ginzburg models.

2. Mirror symmetry for the Kazama-Suzuki models

We have seen that a crucial ingredient in the proof of mirror symmetry for the quintic Calabi-Yau model and its Landau-Ginzburg relatives is the pairwise equivalence of the orbifolds of the $A_k$ minimal models. As already mentioned, this equivalence may be rigorously established by algebraic means using the abstract definition of these models. However, it would be interesting to understand the mirror symmetry property of the minimal models in a Lagrangian framework, where the models are defined by an action functional of some fields and a path-integral over these fields. We have already given an example of such a Lagrangian realization of the $A_k$ minimal model, namely a Landau-Ginzburg theory with a single field $X$ and the potential $X^{k+2}$.

In this particular case, there is also a well supported conjecture for the correct kinetic energy term [16]. However, the Landau-Ginzburg theory is a strongly interacting quantum field theory and quite difficult to work with. To find another Lagrangian representation, more suited to our needs, we will first generalize the minimal models to the so called Kazama-Suzuki models [17]. Such a model is specified by a Lie group $G$ with a subgroup $H$ of the form $H \cong U(1) \times H'$ and a positive integer $k$. The corresponding $N=2$ superconformal field theory is referred to as the $G/H$ Kazama-Suzuki model at level $k$.

It should be noted that the construction only works for certain groups $G$ and $H$, though. The $A_k$ minimal model is equivalent to the $SU(2)/U(1)$ Kazama-Suzuki model at level $k$. A general Kazama-Suzuki model has a Lagrangian representation which could be thought of as describing a string propagating on the group manifold of $G$ such that the theory has a gauge symmetry with the gauge group isomorphic to $H$ [18] [19].

We will now sketch how mirror symmetry for the Kazama-Suzuki models may be understood using this Lagrangian representation. The detailed calculation is presented in [20]. The formulation of mirror symmetry for the $A_k$ minimal model involves a symmetry group isomorphic to $\mathbb{Z}_{k+2}$. Our first task is therefore to understand how a discrete symmetry group arises in the Lagrangian formulation of the $G/H$ Kazama-Suzuki model at level $k$. Locally in field space, we may describe the model in terms of a scalar field $\phi$ which is periodic with period $2\pi$, a gauge field $A$ for the $U(1)$-part of the gauge group $H \cong U(1) \times H'$, and some other fields which need not concern us here. We now consider transformations of the form

$$\phi \rightarrow \phi + \gamma,$$  

(8)
where the parameter $\gamma$ is a constant. The effective action $S$ of the theory is not invariant under this transformation, but transforms as

$$
S \rightarrow S + \gamma(k + Q) \frac{1}{2\pi} \int dA.
$$

(9)

Here $Q$ is an integer called the dual Coxeter number of the group $G$. For the case of $G \simeq SU(2)$, which is relevant for the minimal models, we have $Q = 2$. The integral in (9) is over the world-sheet. Naively, the integrand is a total derivative, so since the world-sheet has no boundary one would expect the integral to vanish by Stokes’ theorem. However, the gauge field $A$ need not be globally well-defined over the world-sheet, but may be given by different one-forms over different patches related by gauge transformations on the overlaps. The integrand is in fact a representative of the first Chern class of the line bundle on which $A$ is a connection, and from the theory of characteristic classes it follows that the integral, including the prefactor $(2\pi)^{-1}$, may take any integer value.

Let us now take the parameter $\gamma$ to be a multiple of $2\pi(k + Q)$. From the above discussion, it follows that the effective action $S$ then changes by a multiple of $2\pi$. In a path integral, the action only appears as $\exp iS$, and this quantity is thus invariant under the transformation. Since $\gamma = 2\pi$ acts trivially (because of the $2\pi$ periodicity of $\phi$), we have identified a discrete symmetry group isomorphic to $\mathbb{Z}_{k+Q}$. For the case of the $A_k$ minimal models, this is the $\mathbb{Z}_{k+2}$ symmetry that we discussed in the previous section.

Let us now consider a $\mathbb{Z}_m$ orbifold of the $G/H$ Kazama-Suzuki model at level $k$. According to our general discussion of orbifolds, we may construct the $\mathbb{Z}_m$ orbifold by changing the periodicity of $\phi$ from $2\pi$ to $2\pi/m$. The theory is given by an action functional of the form

$$
S = \frac{k + Q}{2\pi} \int d^2 z \left( \partial_z \phi \partial_{\bar{z}} \phi + B_z \partial_{\bar{z}} \phi + B_{\bar{z}} \partial_z \phi \right) + C,
$$

(10)

where $z$ and $\bar{z}$ are local coordinates on the world-sheet and $B_z$, $B_{\bar{z}}$ and $C$ are some expressions independent of $\phi$. The important point about this action is that the field $\phi$ only appears as $\partial_z \phi$ and $\partial_{\bar{z}} \phi$. We may therefore write an equivalent first order action by introducing a vector field $V$ and replacing $\partial_z \phi$ and $\partial_{\bar{z}} \phi$ by $V_z$ and $V_{\bar{z}}$ respectively. The relationship between $V$ and $\phi$ may be enforced by introducing a Lagrange multiplier $\tilde{\phi}$, which we take to be periodic with period $2\pi/\tilde{m}$. Here $m$ and $\tilde{m}$ are related through $m\tilde{m} = k + Q$. For $\mathbb{Z}_m$ to be a subgroup of $\mathbb{Z}_{k+Q}$, $m$ must divide $k + Q$ so that $\tilde{m}$ is an integer. The first order action is

$$
S_1 = \frac{k + Q}{2\pi} \int d^2 z \left( V_z V_{\bar{z}} + B_z V_{\bar{z}} + B_{\bar{z}} V_z \right) + C + \frac{k + Q}{2\pi} \int d^2 z \tilde{\phi} \left( \partial_z V_{\bar{z}} - \partial_{\bar{z}} V_z \right).
$$

(11)

By performing the path integral over $\tilde{\phi}$ in this action we retrieve the original action (10). Indeed, $\tilde{\phi}$ acts as a Lagrange multiplier for the constraint $dV = 0$, which we may solve locally, by the Poincaré lemma, as $V = d\phi$. Furthermore, since $\tilde{\phi}$ is $2\pi/\tilde{m}$
periodic, it may have non-trivial winding around non-contractible closed curves on the world-sheet. The path integral over $\phi$ contains a sum over the winding number around each such curve, and this sum constrains the holonomy of $V$ around the curve, i.e. $\oint V$, to be a multiple of $2\pi/m$ \cite{21}. This amounts to the field $\phi$ being $2\pi/m$ periodic, so the action (11) indeed describes the $\mathbb{Z}_m$ orbifold.

Since the first order action (11) is at most bilinear in the vector field $V$ and the coefficient before the bilinear term is independent of the other fields, the path integral over $V$ may be performed exactly. The result is a new action $S_{\text{dual}}$. A detailed computation yields that $S_{\text{dual}}$ equals the original action $S$ in (10) with the field $\phi$ replaced by $\tilde{\phi}$. Since the latter field is $2\pi/\tilde{m}$ periodic, this describes the $\mathbb{Z}_{\tilde{m}}$ orbifold. We have thus shown that the $\mathbb{Z}_m$ and the $\mathbb{Z}_{\tilde{m}}$ orbifolds of the $G/H$ Kazama-Suzuki model at level $k$ are equivalent if $m\tilde{m} = k + Q$, where $Q$ is the dual Coxeter number of $G$.

In the above discussion we have omitted an important point: There exist two in general inequivalent versions of the $G/H$ Kazama-Suzuki model at level $k$. They are usually referred to as the vectorially and the axially gauged model. The mirror transformation described above not only takes a $\mathbb{Z}_m$ orbifold into a $\mathbb{Z}_{\tilde{m}}$ orbifold, but also interchanges the vector model and the axial model. However, for the case of $G/H \simeq SU(2)/U(1)$ the two models are related by a change of variables under which the measure in the path-integral is invariant, so they both describe the same $\mathcal{N} = 2$ superconformal field theory, i.e. the $A_k$ minimal model. We have thus proven the mirror symmetry property of the $A_k$ minimal models.

3. Conjugate Landau-Ginzburg models

The Fermat point Landau-Ginzburg model with potential

$$W(X_1, \ldots, X_5) = X_1^5 + \ldots + X_5^5 \quad (12)$$

is conjugate to itself in the sense that the orbifolds $W/S$ and $W/\tilde{S}$ are equivalent when $S$ and $\tilde{S}$ are ‘dual’ subgroups of the symmetry group $\mathbb{Z}_5$. We have already discussed a particular example of such equivalences when we constructed the mirror partner of (the $r \to -\infty$ limit of) the quintic Calabi-Yau model.

It is now natural to consider more general Landau-Ginzburg models described by a set of fields $X_1, \ldots, X_n$ and a (quasi)homogeneous potential $W(X_1, \ldots, X_n)$. This means that if we assign a degree to each of the fields $X_1, \ldots, X_n$ then all the terms in the potential $W(X_1, \ldots, X_n)$ should be of the same degree. We then look for pairs of potentials $W(X_1, \ldots, X_n)$ and $\tilde{W}(\tilde{X}_1, \ldots, \tilde{X}_n)$ with isomorphic symmetry groups such that the orbifolds $W/S$ and $\tilde{W}/\tilde{S}$ are equivalent when $S$ and $\tilde{S}$ are ‘dual’ subgroups of the symmetry group.

To implement this program, we need a way to compare two Landau-Ginzburg orbifolds and determine if they might be equivalent. For example, one might try to compare the Hilbert spaces of the corresponding $\mathcal{N} = 2$ superconformal field
theories. Each state in the Hilbert space is characterized by its ‘energy’ and some other quantum numbers. If the two theories have identical spectra of all quantum numbers, it would be a strong indication that they are in fact equivalent.

Unfortunately, it is in general not possible to calculate the spectrum of a Landau-Ginzburg theory. One difficulty is that the action of the theory is not known, since the exact form of the kinetic energy is in general unknown. Furthermore, the Landau-Ginzburg theories are strongly interacting quantum field theories, and therefore difficult to analyze. We must therefore content ourselves with less complete information than the complete spectrum. A tractable calculation is to restrict our attention to the spectrum of zero-energy states and count bosonic and fermionic states with opposite sign. This is usually referred to as calculating the ‘Poincaré polynomial’ \cite{22} or the ‘elliptic genus’ \cite{12,23} of the theory.

The reason that these quantities are effectively calculable may be found by considering the supersymmetry algebra \cite{24}. Schematically, this algebra can be thought of as being generated by a bosonic energy generator $H$ and a fermionic supersymmetry generator $Q$. The generators $H$ and $Q$ commute, and the square of $Q$ equals $H$. Since $H$ may be written as the square of a self-adjoint operator, we may conclude that all energy eigenvalues are non-negative. Suppose now that we have a bosonic state $|\text{bos}\rangle$ of strictly positive energy, \textit{i.e.} $H|\text{bos}\rangle = h|\text{bos}\rangle$ for some $h > 0$. We may then define a fermionic state $|\text{ferm}\rangle$ by acting on $|\text{bos}\rangle$ with the supersymmetry generator, \textit{i.e.} $|\text{ferm}\rangle = Q|\text{bos}\rangle$.

From the properties of the supersymmetry algebra, it easily follows that $|\text{ferm}\rangle$ has the same energy eigenvalue as $|\text{bos}\rangle$, \textit{i.e.} $H|\text{ferm}\rangle = h|\text{ferm}\rangle$. States of non-zero energy thus appear in Bose-Fermi pairs with the same values of all other quantum numbers which commute with $Q$. The argument fails for zero-energy states, because $Q$ annihilates such a state rather than creating a new state of opposite statistics. We may therefore have an unequal number of bosonic and fermionic zero-energy states for a given set of values of the remaining quantum numbers. If we now deform the theory smoothly, the energy eigenvalues will change continuously.

It might then happen a Bose-Fermi pair of non-zero energy descends down to zero energy. However, this does not change the number of bosonic zero-energy states minus the number of fermionic zero-energy states for given values of the other quantum numbers. This number may therefore be calculated by deforming the model to some well understood theory, typically by turning of all interactions so that we get a free field theory. We may then search for possible mirror pairs by comparing the zero-energy spectra of two theories in this way. Obviously, this rather crude method is not sensitive to continuous deformations such as the ‘shape’ and ‘size’ deformations.

We will now briefly present the main results of this investigation. For a more detailed review see \cite{25} or the original paper \cite{26}. Our main result is that we found two classes of possible pairs of conjugate Landau-Ginzburg potentials. (These models were first proposed in \cite{27}.) The first class of examples relates the potentials

$$W(X_1, \ldots, X_n) = X_1^{\alpha_1} + X_1 X_2^{\alpha_2} + \ldots + X_{n-1} X_n^{\alpha_n}$$
\[ \hat{W}(\tilde{X}_1, \ldots, \tilde{X}_n) = \tilde{X}_1^{\alpha_1} + \tilde{X}_1 \tilde{X}_2^{\alpha_2} + \ldots + \tilde{X}_{n-1} \tilde{X}_n^{\alpha_n}. \]  \hspace{1cm} (13)

Both of these potentials are invariant under a symmetry group isomorphic to \( \mathbb{Z}_D \) for \( D = \alpha_1 \ldots \alpha_n \). A comparison of the spectra of zero-energy states shows that suitable deformations of the Landau-Ginzburg orbifolds \( W/\mathbb{Z}_m \) and \( \hat{W}/\mathbb{Z}_{\tilde{m}} \) might be each others mirror partners when \( m \tilde{m} = D \). We note that the case \( n = 1 \) gives the \( A_k \) minimal models, for which mirror symmetry has been rigorously established by algebraic methods and by the method described in the previous section.

Our second class of examples relates the potentials
\[
\begin{align*}
W(X_1, \ldots, X_n) &= X_n X_1^{\alpha_1} + X_1 X_2^{\alpha_2} + \ldots + X_{n-1} X_n^{\alpha_n} \\
\hat{W}({\tilde{X}}_1, \ldots, {\tilde{X}}_n) &= {\tilde{X}}_1 {\tilde{X}}_2^{\alpha_2} + {\tilde{X}}_1 X_2^{\alpha_1} + \ldots + {\tilde{X}}_{n-1} X_n^{\alpha_n}. \\
\end{align*}
\]  \hspace{1cm} (14)

The symmetry groups are isomorphic to \( \mathbb{Z}_D \) for \( D = \alpha_1 \ldots \alpha_n + (-1)^{n-1} \), and the zero-energy spectra indicate a possible mirror symmetry between deformations of the Landau-Ginzburg orbifolds \( W/\mathbb{Z}_m \) and \( \hat{W}/\mathbb{Z}_{\tilde{m}} \) when \( m \tilde{m} = D \). Again, the \( n = 1 \) case gives the \( A_k \) minimal models.

Finally, one may take any combination of these examples. The total symmetry group is then the product of the symmetry groups of the simple theories, and the zero-energy spectra of the Landau-Ginzburg orbifolds with respect to ‘dual’ subgroups of the total symmetry group are consistent with mirror symmetry. We have in fact already given an example of this procedure when we regarded the quintic Landau-Ginzburg theory at the Fermat point as a product of five \( A_3 \) minimal models.

This work is supported by DOE grant DE-FG02-92ER40704.

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