THURSTON’S FRAGMENTATION, NON-ABELIAN POINCARÉ DUALITY AND C-PRINCIPLES

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Abstract. In this paper, we generalize the original proof of Thurston of the so-called Mather-Thurston’s theorem for foliated bundles. We improve the connectivity hypothesis of his deformation lemma which also implies what is now known as the non-abelian Poincare duality theorem and its generalization using blob complexes ([MW12, Theorem 7.3.1]). We use our compactly supported c-principle theorem to prove variants of the Mather-Thurston theorem for PL homeomorphisms, contactomorphisms and leaf preserving diffeomorphisms. The first two versions answer questions posed by Gelfand and Fuchs ([GF73, Section 5]) on PL foliations and Rybicki ([Ryb10, Section 11]) on contactomorphisms. As we shall explain, the interesting point about the original Thurston’s technique compared to Segal-McDuff’s proof of the Mather-Thurston theorem is it could give a compactly supported c-principle theorem without knowing the relevant local statement.

1. Introduction

Let \( F : (\text{Mfld}^n)^{\text{op}} \to S \) be a presheaf from the category of smooth \( n \)-manifolds (possibly with nonempty boundary) with smooth embeddings as morphisms to a convenient category of spaces \( S \). For our purpose, we shall consider the category of simplicial sets or compactly generated Hausdorff spaces. Let \( F^h \) be the homotopy sheafification of \( F \) with respect to 1-good covers meaning contractible open sets whose nontrivial intersections are also contractible (see [BdBW13] for more details). One can describe the value of \( F^h(M) \) as the space of sections of the bundle \( \text{Fr}(M) \times_{\text{GL}_n(\mathbb{R})} F(\mathbb{R}^n) \to M \), where \( \text{Fr}(M) \) is the frame bundle of \( M \). We say \( F \) satisfies an h-principle if the natural map from the functor to its homotopy sheafification

\[
j : F(M) \to F^h(M),
\]

induces a weak equivalence and we say it satisfies c-principle if the above map is a homology isomorphism.

Some important examples of such presheaf in the manifold topology are the space of generalized Morse functions ([Igu84]), the space of framed functions ([Igu87]), the space of smooth functions on \( M^n \) that avoid singularities of codimension \( n + 2 \) (this is in general a c-principle theorem, see [Vas92]), the space of configuration of points with labels in a connected space ([McD75]), etc. h(c)-principle theorems (see [EM02]) come in many different forms but the general philosophy is that a space of a geometric significance \( F(M) \) which is sometimes called “holonomic solutions” is homotopy equivalent or homology isomorphic to “formal solutions”. The space of formal solutions, \( F^f(M) \), is more amenable to homotopy theory since it is often the space of sections of a fiber bundle and therefore, it is easy to check its homotopy sheaf property with respect to certain covers. Hence, from the homotopy theory
point of view, proving h-principle theorem consists of a “local statement” which is an equivalence of holonomic solutions and formal solutions on open balls and a “local to global statement” which is a homotopy sheaf property for the geometric functor of holonomic solutions.

Thurston in ([Thu74a]), however, found a remarkable compactly supported c-principle theorem without knowing the “local statement”. The main goal in this paper is to abstract his ideas to prove other compactly supported c-principle theorems. To briefly explain his compactly supported c-principle theorem, let Folc(M) := BDiffc(M) be the realization of the semisimplicial set whose k-simplices are given by the set of codimension n foliations on Δk × M that are transverse to the fibers of the projection Δk × M → Δk and the holonomies are compactly supported diffeomorphisms of the fiber M.

To describe the space of formal solutions in this case, we need to recall the notion of Haefliger classifying space which is the space of formal solutions on an open ball. Let Folc(Rn) := BFr(M) be the realization of a semisimplicial set whose k-simplices are given by the set of the germs of codimension n foliations on Δk × Rn around Δk × {0} that are transverse to the fibers of the projection Δk × Rn → Δk. After fixing a section of the space of sections of Fr(M) × Gln(R) Folc(Rn) → M, we can define the support of sections to be the set on which they take different values from the base section. Let Folc(M) be the space of compactly supported sections with respect to the fixed base section. Thurston proved that there exists a natural map Folc(M) → Folc(M) which induces a homology isomorphism.

Although Segal later proved (see [Seg78]) the local statement that BDiffc(Rn) is homology isomorphic to BFr which led to a different proof ([McD79]) of Thurston’s theorem, Thurston’s original proof of the fact that a natural map Folc(M) → Folc(M) induces a homology isomorphism did not use this local statement.

The main idea is, given a Riemannian metric on M, Thurston gives a compatible filtrations on the space of foliated M-bundles Folc(M) and the space of formal solutions Folc(M) which is a section space and compare the spectral sequences of these filtrations to prove his compactly supported c-principle theorem. These filtrations are inspired by his idea of “fragmenting” diffeomorphisms of manifolds that are isotopic to the identity. To explain his fragmentation idea, it is easier to start with the filtration that it induces on the space of sections.

1.1. Non-abelian Poincare duality and blob homology via fragmentation.
Let π : E → M be a Serre fibration over the manifold M and we suppose E is Hausdorff. Let s0 be a base section of this fiber bundle. We assume that the base section satisfies the following homotopical property: there is a fiber preserving homotopy h_t of E such that h_0 = id and h^{-1}_t(s_0(M)) is a neighborhood of s_0(M) in E and h_t(s_0(M)) = s_0(M) for all t, in other words the base section is a good base point in the space of sections. We fix a Riemannian metric on M and a positive ε so that every ball of radius ε is geodesically convex.

By the support of a section s, we mean the closure of the points on which s differs from the base section s_0. Let Sectc(π) be the space of compactly supported sections of the fiber bundle π : E → M equipped with the compact-open topology. Let Sectc(π) denote the subspace of sections s such that the support of s can be covered by k geodesically convex balls of radius 2^{-k}ε for some positive integer k.
Theorem 1.1 (Fragmentation property). If the fiber of $\pi$ is at least $(n - 1)$-connected, the inclusion

$$\text{Sect}_c(\pi) \to \text{Sect}_e(\pi),$$

is a weak homotopy equivalence.

Remark 1.2. Mather refers to the above statement as a deformation lemma in [Mat11] and he assumed that the fiber is $n$-connected but we show that $(n - 1)$-connectedness is enough.

Remark 1.3. In general, if the fiber of $\pi$ is $(n - k)$-connected, the same techniques apply to localize the support of the sections. For example, for a fixed neighborhood $U$ of the $(k-1)$-skeleton, one could show that the space $\text{Sect}_c(\pi)$ is weakly equivalent to the subspace of sections that are supported in $U$ union $s$ balls of radius $2^{-s}\epsilon$ for some non-negative integer $s$. But this is not the direction, we want to pursue in this paper.

As we shall see in Section 3.1, given a $D^k$-family of sections in $\text{Sect}_e(\pi)$, we subdivide the parameter space $D^k$ and change the family up to homotopy such on each subdivision, the new family is supported in the union of $k$ balls of radius $2^{-k}\epsilon$.

In ([MW12, Theorem 7.3.1]), Morrison and Walker found a similar statement for mapping spaces with a more relaxed notion of support but without the connectivity assumption. For a family $F : D^k \to \text{Sect}_c(\pi)$, they say $F$ is supported in $S \subset M$ if $F(p)(x)$ does not depend on $p$ for $x \notin S$. Our notion of support, however, requires $F(p)(x)$ to be equal to the value of the base section at $x$ for $x \notin S$. Without the connectivity assumption, they prove ([MW12, Lemma B.0.4]) that given a $D^k$-family of sections in $\text{Sect}_e(\pi)$, one can subdivide the parameter space $D^k$ and change the family up to homotopy such on each subdivision, the new family is supported in their more relaxed sense in the union of $k$ balls. As we shall see in Section 3.1, this deformation theorem easily follows from Thurston’s fragmentation.

Note that $\text{Sect}_e(\pi)$ which is a subspace of $\text{Sect}_c(\pi)$ has a natural filtration whose filtration quotients are similar to the filtration quotients induced by the non-abelian Poincaré duality (see [Lur16, Theorem 5.5.6.6]). We in fact show that this theorem implies the non-abelian Poincaré duality for the space of sections of $\pi : E \to M$. To recall its statement, let $U(M)$ be poset of the open subset of $M$ that are homeomorphic to a disjoint union of finitely many open disks. For an open set $U \in U(M)$, let $\text{Sect}_c(U)$ denote the subspace of sections which are compactly supported and their supports are covered by $U$.

Corollary 1.4 (Nonabelian Poincaré duality). If the fiber of the map $\pi$ is $(n - 1)$-connected, the natural map

$$\operatorname{hocolim}_{U \in U(M)} \text{Sect}_c(U) \to \text{Sect}_e(\pi),$$

is a weak homotopy equivalence.

1.2. c-principle theorems via fragmentation. Part of the method, Thurston used to prove his c-principle theorem is of course specific to foliation theory. In particular, the fact that the local statement in that case was very nontrivial and he found a way to prove the compactly supported version without the local statement are specific to foliation theory. However, we show that given the local statement (which is often the easy case unlike foliation theory), we can still apply Thurston’s method to obtain a compactly supported c-principle theorem. Then we also use
this general strategy to prove versions of Thurston’s theorem for other geometric structures that were conjectured to hold. As we shall see the local condition that is appropriate for our purpose is $F(D^n) \simeq F^I(D^n)$. Hence we define the space of formal solutions that satisfies this local condition as follows.

**Definition 1.5.** Let the space of formal solutions, $F^I(M)$, be the space of section of the fiber bundle

$$Fr(M) \times_{GL_n(R)} F(D^n) \to M.$$ 

**Remark 1.6.** In all the examples of c-principle in the introduction, the common feature is the space $F^I(D^n)$ is in fact at least $(n-1)$-connected. Therefore, the cosheaf of compactly supported sections $F^I_c(-)$ satisfies the fragmentation property and non-abelian Poincare duality.

The set-up of the c-principle theorem that we are interested in is we have a natural transformation $\iota : F(-) \to F^I(-)$ that respects the choice of base sections. Hence, for any manifold $M$, we have an induced map

$$F_c(M) \to F^I_c(M).$$

We want to find conditions under which the above map induces a homology isomorphism.

**Definition 1.7.** Let $s_0 \in F(M)$ be fixed global section. For any other element $s \in F(M)$, we define the notion of support, $\text{supp}(s)$, with respect to $s_0$ to be the closure of points in $M$ at which the stalk of $s$ and $s_0$ are different. Now similar to formal solutions, we can define the subspace of compactly supported elements, $F_c(M, s_0)$ and $\epsilon$-supported elements $F_\epsilon(M, s_0)$. We shall suppress the fixed global section $s_0$ from the notation for brevity. In the case of non-empty boundary, we assume that the supports of all elements of $F_c(M, s_0)$ and $F^I_c(M, s_0)$ are away from the boundary.

**Definition 1.8.** We say the functor $F$ satisfies the fragmentation property if the inclusion $F_c(M) \to F_c(M)$ is a weak equivalence for all $M$.

**Definition 1.9.** We say $F : (\text{Mfld}_\mathbb{R})^{op} \to S$ is good, if it satisfies

- The subspace of elements with empty support in $F(M)$ is contractible.
- For an open subset $U$ of a manifold $M$, the inclusion $F_c(U) \to F_c(M)$ is an open embedding (We will consider weaker condition in Definition 4.1).
- Let $U$ and $V$ be open disks. All embeddings $U \to V$ induces a homology isomorphism between $F_c(U)$ and $F_c(V)$.
- For each finite family of open sets $U_0, U_1, \ldots , U_k$ such that $U_1, \ldots , U_r$ are disjoint and contained in $U_0$, we have a permutation invariant map

$$\mu_{U_0, \ldots , U_k} : \prod_{i=1}^{k} F_c(U_i) \to F_c(U_0),$$

where this map satisfies the obvious associativity conditions and for $U_0 = \bigcup_{i=1}^{k} U_i$, the map $\mu_{U_0}^{U_1, \ldots , U_k}$ is a weak equivalence.
- Let $\partial_1$ be the northern-hemisphere boundary of $D^n$. Let $F(D^n, \partial_1)$ be the subspace of $F(D^n)$ that restricts to the base element in a germ of $\partial_1$ inside $D^n$. We assume $F(D^n, \partial_1)$ is acyclic.

**Theorem 1.10.** Let $F$ be a good sheaf on manifolds such that
$F^f(D^n)$ is at least $(n-1)$-connected.

$F$ has the fragmentation property.

Then for any manifold $M$, the map

$$F_c(M) \to F^f_c(M),$$

induces a homology isomorphism.

Remark 1.11. The improvement of the connectivity in Theorem 1.1, in particular, as we shall see will be useful to prove Mather-Thurston type theorems for different geometric structures. One can also use this method to give a different proof of McDuff’s theorem on the local homology of volume preserving diffeomorphisms ([McD83b, McD82]) using the methods of this paper. In that case $F(D^n)$ is at best $(n-1)$-connected (see [Hae71, Remark 2, part (a)]).

It would be interesting to see if Thurston’s method gives a different proof of Vassiliev’s c-principle theorem ([Vas92]). In the last section, we discuss how one could use Thurston’s fragmentation idea for the space of functions on $M$ avoiding singularities of codimension $\dim(M) + 2$. However, our main motivation still lies in foliation theory. People have studied fragmentation property of foliation with different transverse structures ([Tsu08, Tsu06, Ryb10]) and conjecturally expected that an analogue of Thurston’s theorem, or so called Mather-Thurston’s theory (for PL-homeomorphisms see [GF73, Section 5] and for a different version see also [Gre92], for contactomorphisms see [Ryb10]) should also hold for them. We prove in Section 5 three new variants of Mather-Thurston’s theorem for contactomorphisms, PL-homeomorphisms and foliation preserving diffeomorphisms where the first two variants were conjectured by Gelfand-Fuks and Rybicki.

Recently, there were new geometric approaches to Mather-Thurston’s theory due to Meigniez ([Mei20]) and Freedman ([Fre20]). However in this paper we follow Mather’s account ([Mat11]) of Thurston’s proof of this remarkable theorem in foliation theory. McDuff followed in [McD80, McD79] Segal’s method ([Seg78]) to find a different proof of Mather-Thurston’s theorem and she proved the same theorem for the volume preserving case ([McD82, McD83a, McD83b]). The techniques in Segal and McDuff’s approach and in particular, their group completion theorem ([MS76]) are now well-understood tools in homotopy theory. The author hopes that this paper also makes Thurston’s ideas available to a broader context.

1.3. Mather-Thurston theory for new transverse structures. We consider three different transverse structures of foliated bundles for which the fragmentation properties were known and hence conjecturally the analogues of Mather-Thurston’s theorem were posed ([Ryb10, Tsu06, Gre92]). We shall first recall these transverse structures.

Definition 1.12. Let $M$ be a smooth $n$-dimensional manifold and $\mathcal{F}$ be a codimension $q$ foliation on $M$. Let $\text{Fol}_c(M, \mathcal{F})$ be the realization of the simplicial set whose $k$-simplices are given by the set of codimension $\dim(M)$ foliations on $\Delta^k \times M$ that are transverse to the fibers of the projection $\Delta^k \times M \to \Delta^k$ and the holonomies are compactly supported diffeomorphisms of the fiber $M$ that preserve the leaves of $\mathcal{F}$.\(^1\)

\(^1\)It is important that holonomies are leaf preserving. One can also define a version that holonomies may not preserve the leaves but they preserve the foliation. As we shall explain
Let $M$ be a smooth odd dimensional manifold with a fixed contact structure $\alpha$. Let $\text{Fol}_c(M, \alpha)$ be the realization of the simplicial set whose $k$-simplices are given by the set of codimension $\dim(M)$ foliations on $\Delta^k \times M$ that are transverse to the fibers of the projection $\Delta^k \times M \to \Delta^k$ and the holonomies are compactly supported contactomorphisms of the fiber $M$.

Let $M$ be a PL $n$-dimensional manifold. Let $\text{Fol}_{\text{PL}}^c(M)$ be the realization of the simplicial set whose $k$-simplices are given by the set of codimension $\dim(M)$ foliations on $\Delta^k \times M$ that are transverse to the fibers of the projection $\Delta^k \times M \to \Delta^k$ and the holonomies are compactly supported PL-homeomorphisms of the fiber $M$.

Analogue of Mather-Thurston’s theorem in these cases can be summarized as follows.

**Theorem 1.13.** The functors $\text{Fol}_c(M, \mathcal{F})$, $\text{Fol}_c(M, \alpha)$ and $\text{Fol}_{\text{PL}}^c(M)$ satisfy the c-principle.\(^2\)

**Remark 1.14.** Gael Meigniez told the author that has a forthcoming paper to show that the PL case could also be obtained using his geometric proof for the smooth case ([Mei20]) and there is a work in progress to use his method in the transverse contact structure.

**Remark 1.15.** These c-principle theorems and the perfectness results in [Ryb10, Tsu08, Tsu06] can be used to improve the connectivity results of the corresponding Haefliger structures.

The paper is organized as follows: in Section 2, we use microfibration techniques to show that Thurston’s fragmentation method implies the non-abelian Poincaré duality. In Section 3, we discuss fragmentation homotopy and we improve it to prove Theorem 1.1. In Section 4, we apply Thurston’s fragmentation ideas in foliation theory in a broader context to prove Theorem 1.10. In Section 5, we prove compactly supported version of Mather-Thurston’s theorem for PL, contact and foliation preserving transverse structures. In these cases still the local statements are not known and therefore, the non-compactly supported versions are still open.

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\(^{2}\)The c-principle in the case $\text{Fol}_c(M, \mathcal{F})$ needs an explanation, see section 5 for the details.
2. Fragmentation implies non-abelian Poincaré duality

In this section, we prove Corollary 1.4 using Thurston’s fragmentation of section spaces. What makes the fragmentation property more useful in the geometric context is that it lets us deform certain spaces associated to a manifold (e.g. section spaces and spaces of foliation with certain transverse structures) to its subspace (instead of a homotopy colimit) that has natural filtrations (e.g. it deforms the section space to those sections whose supports have volume less than \(\varepsilon\)). Note that the support of sections in \(\text{Sect}_\varepsilon(\pi)\) can be covered by small balls that are not necessarily disjoint. In order to compare \(\text{Sect}_\varepsilon(\pi)\) and \(\text{hocolim}_{U \in \mathcal{O}(M)} \text{Sect}_\varepsilon(U)\), we shall define some auxiliary spaces.

**Condition 1.** Recall that for the map \(\pi : E \to M\) there exists a fiberwise homotopy \(h_t\) so that \(h^{-1}_1(s_0(M))\) is a neighborhood of \(s_0(M)\) i.e. there exists an open subset \(V \subset E\) that contains \(s_0(M)\) and \(h_1(V) = s_0(M)\).

**Definition 2.1.** We define the lax support of a section \(s \in \text{Sect}_\varepsilon(\pi)\) to be the set of points \(x\) where \(s(x)\) is not in \(V\). Let \(\widehat{\text{Sect}}_\varepsilon(\pi)\) be the subspace of \(\text{Sect}_\varepsilon(\pi)\) consisting of those sections whose lax support can be covered by \(k\) geodesically convex balls of radius \(2^{-k} \varepsilon\) for some positive integer \(k\).

**Lemma 2.2.** The inclusion \(\text{Sect}_\varepsilon(\pi) \to \widehat{\text{Sect}}_\varepsilon(\pi)\) is a weak homotopy equivalence.

**Proof.** We need to solve the following lifting problem

\[
\begin{array}{ccc}
S^k & \xrightarrow{F} & \text{Sect}_\varepsilon(\pi) \\
\downarrow & & \downarrow \\
D^{k+1} & \xrightarrow{H} & \widehat{\text{Sect}}_\varepsilon(\pi).
\end{array}
\]

But instead we change the map of pairs

\[(H, F) : (D^{k+1}, S^k) \to (\widehat{\text{Sect}}_\varepsilon(\pi), \text{Sect}_\varepsilon(\pi)),\]

up to homotopy to find the lift. For \(a \in D^{k+1}\) and \(x \in M\), we define \(H_t(a, x) \in E\) to be \(h_t(H(a, x))\). Similarly, we define \(F_t\). Note that for all \(a \in D^{k+1}\), the section \(H_1(a, -)\) in fact lies in \(\text{Sect}_\varepsilon(\pi)\) which is our desired lift.

**Definition 2.3.** Let \(\mathcal{O}_\varepsilon(M)\) be the discrete poset of open subsets of \(M\) that can be covered by a union of \(k\) geodesically convex balls of radius at most \(2^{-k} \varepsilon\) for some positive integer \(k\).

For brevity, we write \(\widehat{\text{Sect}}_\varepsilon(U)\) for \(U \in \mathcal{O}_\varepsilon(M)\) to denote those sections whose closure of their lax supports can be covered by \(U\).

**Lemma 2.4.** The natural map

\[\alpha : \text{hocolim}_{U \in \mathcal{O}_\varepsilon(M)} \widehat{\text{Sect}}_\varepsilon(U) \to \widehat{\text{Sect}}_\varepsilon(\pi),\]

is a weak homotopy equivalence.
Proof. It is enough to show that the above map is a Serre microfibration with a weakly contractible fibers (see [Wei05, Lemma 2.2]). Recall, we say the map \( \pi : T \to B \) is a Serre microfibration if for any \( k \) and any commutative diagram

\[
\begin{array}{ccc}
D^k \times \{0\} & \xrightarrow{f} & T \\
\downarrow & & \downarrow \pi \\
D^k \times [0,1] & \xrightarrow{h} & B,
\end{array}
\]

there exists an \( \epsilon > 0 \) and a microlift \( H : D^k \times [0,\epsilon) \to T \) so that \( H(x,0) = f(x) \) and \( \pi \circ H(x,t) = h(x) \). We think of \( \overline{\text{Sect}}_\epsilon(-) : \mathcal{O}_\epsilon(M) \to \text{Top} \) as a diagram of spaces. It is known (see [DI01, Appendix A]) that for the diagram of spaces, the homotopy colimit is weakly equivalent to the realization of the bar construction \( B_* (\overline{\text{Sect}}_\epsilon(-), \mathcal{O}_\epsilon(M), *) \). Note that there is a continuous injective map

\[ B_* (\overline{\text{Sect}}_\epsilon(-), \mathcal{O}_\epsilon(M), *) \to \overline{\text{Sect}}_\epsilon(\pi) \times B_*(\mathcal{O}_\epsilon(M), *), \]

where \( \alpha \) is induced by the projection to the first factor. The lax support is defined so that the subspace \( \text{Sect}_\epsilon(U) \) is open in \( \overline{\text{Sect}}_\epsilon(\pi) \) and since these spaces are Hausdorff, we could use [GRW18, Proposition 2.8] to deduce that

\[ \alpha : |B_* (\overline{\text{Sect}}_\epsilon(-), \mathcal{O}_\epsilon(M), *)| \to \overline{\text{Sect}}_\epsilon(\pi), \]

is a Serre microfibration. The fiber over a section \( s \in \overline{\text{Sect}}_\epsilon(\pi) \) can be identified with \( |B_*(\mathcal{O}_\epsilon(M)|_{\text{supp}(s)}, *) \) where \( \mathcal{O}_\epsilon(M)_{\text{supp}(s)} \) consists of those open subsets in \( \mathcal{O}_\epsilon(M) \) that contains the support of \( s \). But this sub-poset is filtered, therefore its realization is contractible. \( \square \)

Now we use a different version of the non-abelian Poincaré duality that is given by a homotopy colimit over a less rigid indexing category \( \mathbf{D}(M) \) that is defined as follows.

**Definition 2.5.** Let \( \text{Mfld}_n \) denote the topological category of smooth submanifolds of \( \mathbb{R}^\infty \) of fixed dimension \( n \) and the morphism spaces are given by the codimension zero embeddings. Let \( \text{Disk}_n \) denote the full subcategory whose objects are diffeomorphic to a finite disjoint union of Euclidean spaces of dimension \( n \). The \( \infty \)-category \( \mathbf{D}(M) \) is the over category \( \text{Disk}_n / M \) whose space of objects is given by space of embeddings \( U \to M \) as \( U \) varies in \( \text{Disk}_n \) (see [Lur16, Definition 5.2.11]). The space of morphisms \( \text{Map}_{\mathbf{D}(M)}(f,g) \) between two such embeddings \( (f : U \to M) \) and \( (g : V \to M) \) is given by the following homotopy fiber sequence

\[ \text{Map}_{\mathbf{D}(M)}(f,g) \to \text{Emb}(U,V) \to \text{Emb}(U,M), \]

where the last map is induced by precomposing with \( g \). We have the usual point-set model of \( \text{Map}_{\mathbf{D}(M)}(f,g) \) in mind which is given by the pair of embeddings \( (\iota, \iota') \) in \( \text{Emb}(U,V) \) and \( \text{Emb}(U,M) \) respectively and a specified isotopy in \( \text{Emb}(U,M) \) that maps \( \iota' \) to the image of \( \iota \).

The non-abelian Poincaré duality [Lur16, Theorem 5.5.6.6] can be reformulated as follows. Consider \( \text{Sect}_\epsilon(-) \) as a functor from \( \mathbf{D}(M) \) to \( \text{Top} \) that sends the embedding \( (U \to M) \) to \( \text{Sect}_\epsilon(U) \). Corollary 1.4 is equivalent to

\[ \text{hocolim} \overline{\text{Sect}}_\epsilon(-) \approx \overline{\text{Sect}}_\epsilon(\pi). \]
Now to show that non-abelian Poincaré duality is implied by Thurston’s deformation of the section spaces, it is enough to show

**Theorem 2.6.** The homotopy colimit $\hocolim_{\mathcal{D}(M)} \mathcal{Sect}_c(-)$ is weakly equivalent to $\hocolim_{\mathcal{O}_c(M)} \mathcal{Sect}_c(-)$.

**Proof.** To be able to compare these two homotopy colimits, we first define a functor between $\mathcal{O}_c(M)$ and $\mathcal{D}(M)$. Recall that every open set $U$ in $\mathcal{O}_c(M)$ can be covered by a union of $k$ geodesically convex balls of radius at most $2^{-k}\epsilon$ for some positive integer $k$.

**Claim 2.7.** A union of $k$ geodesically convex balls of radius at most $2^{-k}\epsilon$ can be covered by at most $k$ disjoint geodesically convex balls of radius at most $\epsilon$.

Note that if the union of $r$ geodesically convex balls of radius at most $2^{-k}\epsilon$ is connected, it can be covered by a ball of radius at most $2^{-k+r-1}\epsilon$. So we consider the connected components of the union of $k$ balls of radius at most $2^{-k}\epsilon$ and we inductively cover the connected components by bigger balls if necessary until we obtain at most $k$ disjoint balls of radius at most $\epsilon$.

Therefore, we shall define an $\infty$-functor $\gamma : N(\mathcal{O}_c(M)) \to \mathcal{D}(M)$, where $N(\mathcal{O}_c(M))$ is the nerve of the ordinary category $\mathcal{O}_c(M)$, as follows: for an the object $U$, we know each connected component of $U$ can be covered by a ball of radius at most $\epsilon$.

Let $\text{conv}(U)$ be the union of convex hulls of the connected components which is diffeomorphic to disjoint union of balls in $M$. We choose an orientation preserving parametrization of $\text{conv}(U)$ (i.e. an orientation preserving diffeomorphism from disjoint union of Euclidean spaces to $\text{conv}(U)$) to obtain an embedding $\text{conv}(U) \to M$ which is an object in $\mathcal{D}(M)$ and we denote it by $\gamma(U)$; note that for a morphism $U \to V$ in $\mathcal{O}_c(M)$, we have $\text{conv}(U) \subset \text{conv}(V)$ and the chosen parameterizations give an embedding $\gamma(U)$ to $\gamma(V)$, this embedding induces another parametrization on $U$ and since the space of orientation preserving parameterizations of $U$ is connected, we can choose an isotopy between them to obtain a morphism from $\gamma(U)$ to $\gamma(V)$ in $\mathcal{D}(M)$ (see also [Lur16, Proposition 5.5.2.13]).

To prove the theorem, we shall use Quillen’s theorem A for $\infty$-categories ([Lur09, Lemma 4.1.3.2]) to show that $\gamma$ is cofinal. Therefore, it is enough to show that the under category $\mathcal{O}_c(M)^{V/}$ has a contractible realization for every $V \in \mathcal{D}(M)$. The geometric realization of this under category is homotopy equivalent to

$$\hocolim_{U \in \mathcal{O}_c(M)} \text{Map}_{\mathcal{D}(M)}(V, \gamma(U)).$$

Recall that $\text{Map}_{\mathcal{D}(M)}(V, \gamma(U))$ can be identified with the homotopy fiber of the map

$$\text{Emb}(V, \gamma(U)) \to \text{Emb}(V, M).$$

Therefore, it is enough to show that

$$\hocolim_{U \in \mathcal{O}_c(M)} \text{Emb}(V, \gamma(U)) \to \text{Emb}(V, M),$$

is a weak equivalence. Let $\pi_0(V) = k$, then since $V$ is diffeomorphic to a disjoint union of Euclidean spaces, for any manifold $X$, the space of embeddings $\text{Emb}(V, X)$
fibers over the unordered configuration space \( \text{Conf}_k(X) \). Hence, to prove the weak equivalence \( 2.8 \), it is enough to show that

\[
\text{hocolim}_{U \in \mathcal{O}_*=\mathcal{O}_U(M)} \text{Conf}_k(\gamma(U)) \xrightarrow{\cong} \text{Conf}_k(M).
\]

Note that \( \text{Conf}_k(\gamma(U)) \) is an open subspace \( \text{Conf}_k(M) \) and as \( U \) varies in \( \mathcal{O}_*(M) \), the open subspaces \( \text{Conf}_k(\gamma(U)) \) cover \( \text{Conf}_k(M) \). Similar to the proof of Lemma 2.4, the above map is a Serre microfibration with a contractible fiber. Therefore, the above map is a weak equivalence. \( \square \)

3. Thurston’s fragmentation

In this section, we explain Thurston’s idea of fragmenting section spaces and we improve the hypothesis of the connectivity of the fiber in [Mat11, First deformation lemma] by one.

3.1. Fragmentation homotopy. Let \( \{\mu_i\}_{i=1}^{N} \) be a partition of unity with respect to an open cover of \( M \). We define a fragmentation homotopy with respect to this partition of unity. Let \( \nu_j \) be the function

\[
\nu_j(x) = \sum_{i=1}^{j} \mu_i(x).
\]

We shall write \( \Delta^q \) for the standard \( q \)-simplex parametrized by

\[
\{ t = (t_1, t_2, \ldots, t_q); 0 \leq t_1 \leq \cdots \leq t_q \leq 1 \}.
\]

We now consider the following map

\[
H_1 : M \times \Delta^q \to M \times \Delta^q,
\]

\[
H_1(x, (t_1, t_2, \ldots, t_q)) = (x, (u_1, u_2, \ldots, u_q)),
\]

\[
u_i(x, t) = \nu_{[Nt_i]}(x) + \mu_{[Nt_i]+1}(x)(Nt_i - [Nt_i]).
\]

Note that \( u_i \) only depends on \( t_i \) and \( x \). Since \( H_1(t, x) \) preserves the \( x \) coordinate, we can define a straight line homotopy \( H_t : M \times \Delta^q \to M \times \Delta^q \) from the identity to \( H_1 \). As in Figure 3.1, the map \( H_1 \) is defined so that the gray area is mapped to the bold lines in the target where the union of bold lines is a subcomplex of \( M \times \Delta^q \) of dimension \( n = \dim(M) \).

![Figure 1. Fragmentation map for \( N = 3 \) and \( q = 1 \). The thick lines are the images of \( M \times \{0\}, M \times \{1/3\}, M \times \{2/3\} \) and \( M \times \{1\} \) under the map \( H_1 \).](attachment:image.png)
It is easy to check that $H_t$ is compatible with the face maps $d_i : \Delta^{q-1} \to \Delta^q$. Therefore, for any simplicial complex $K$, we still can define the homotopy $H_t : M \times K \to M \times K$.

**Definition 3.1.** To define the analogue of bold lines in Figure 3.1 for the simplicial complex $K$, let $V(\Delta^q)$ be the set $t \in \Delta^q$ such that $Nt$ is a vector with integer coordinates. Let $V(K)$ be the union of $V(\Delta^q)$, where the union is taken over simplices of $K$. The analogue of thick lines is $L(K) = H_t(M \times V(K)) \subset M \times K$.

Note that the topological dimension of the subcomplex $L$ is $n$. But if we choose a small open ball $B_r(K)$ of radius say $2^{-q-1} \epsilon$ where $q = \text{dim}(K)$, then the homotopical dimension of $L_r(K) = H_t((M \setminus B_r(K)) \times V(K))$ is $n-1$ because the $n$-dimensional manifold $M \setminus B_r$ has homotopical dimension $n-1$ by which we mean it has the homotopy type of the CW complex of dimension $n-1$. The fragmentation map $H_1$ has the following useful property.

**Lemma 3.2.** Let $H_1 : M \times \Delta^q \to M \times \Delta^q$ be the fragmentation map. For each $t \in \Delta^q$, the space $(M \times t) \setminus H_1^{-1}(L_r(\Delta^q))$ can be covered by the support of at most $q$ functions among the partition of unity functions and the ball $B_r(\Delta^q)$.

**Proof.** This is straightforward from the definitions. As in Figure 3.1, the complement of the gray area in each slice $M \times t$ can be covered by the support of a partition of unity function. In general the complement of $H_1^{-1}(L(\Delta^q))$ in the slice $M \times t$ can be covered by the support of at most $q$ functions (one for each coordinate of $\Delta^q$) among the partition of unity functions. Given that $H_1$ preserves the $M$ factor, to cover the complement of $H_1^{-1}(L_r(\Delta^q))$ in the slice $M \times t$, we only need to add $B_r(\Delta^q)$.

Now we want to use this lemma to prove Theorem 1.1. To deform a family of sections of $\pi : E \to M$, parametrized by a map $g : K \to \text{Sect}_c(\pi)$, we consider its adjoint as a map $G : M \times K \to E$. We also define the support of $g$ over $K$ with respect to the base section $s_0$ as follows.

**Definition 3.3.** Let $\text{supp}(g)|_K$ consist of those points $x \in M$ for which there exists at least one $t \in K$ such that $G(x,t) = s_0(x)$.

We shall need the following lemma that uses the fiber of the map $\pi : E \to M$ is $(n-1)$-connected to prove Theorem 1.1.

**Lemma 3.4.** Given a family $g : D^q \to \text{Sect}_c(\pi)$, there exists a homotopy $g_s : D^q \to \text{Sect}_c(\pi)$ so that for all $t \in D^q$ and $s \in [0,1]$, we have $\text{supp}(g_s(t)) \subset \text{supp}(g(t))$ and at time 1, the adjoint $G_1$ of $g_1$ satisfies $G_1(L_r(D^q)) = s_0(M)$.

**Proof.** We think of the desired homotopy $G_t : M \times D^q \to E$ as a section of the pullback of $\pi : E \to M$ over $M \times D^q \times [0,1]$. The map $G_0$ is the adjoint of $g$. Let $N \subset M \times D^q$ be the subcomplex consisting of points $(x,t)$ so that $G_0(x,t) = s_0(x)$. By the homotopy extension property, we will obtain the desired homotopy $G_t$, if we show that $G_0$ can be extended to a section $\tilde{G}$ over

$$M \times D^q \times \{0\} \cup N \times [0,1] \cup L_r(D^q) \times [0,1],$$

so that $\tilde{G}$ on $M \times D^q \times \{0\}$ is the same as $G_0$, on $N \times [0,1]$ is given by $\tilde{G}(x,t,s) = s_0(x)$ and on $L_r(D^q) \times \{1\}$ is also given by $\tilde{G}(x,t,1) = s_0(x)$. So far, we know how to define $\tilde{G}$ on $M \times D^q \times \{0\} \cup N \times [0,1]$. We extend it over $L_r(D^q) \times [0,1]$ with a prescribed...
value on \( L_\epsilon(D^q) \times \{ 1 \} \), by obstruction theory. Note that the homotopical dimension of \( L_\epsilon(D^q) \times [0,1] \) is \( n \) and the fiber of the pullback of \( \pi \) over \( M \times D^q \times [0,1] \) is \((n-1)\)-connected. Hence all obstruction classes that live in \( H^*(L_\epsilon(D^q) \times [0,1]; \pi_{*1} \) (fiber)) vanish and we obtained the desired extension \( \tilde{G} \).

3.2. Proof of Theorem 1.1. The idea is roughly as follows. To deform a family \( g : D^q \to \text{Sect}_e(\pi) \) to a family of sections in \( \text{Sect}_e(\pi) \), we use Lemma 3.4 to assume that for the family \( g \), we have \( G(L_\epsilon(D^q)) = s_0(M) \). We then use the fragmentation homotopy to deform this family so that for each \( s \in D^q \), the section \( g(s) \) sends the most part of \( M \) to \( G(L_\epsilon(D^q)) \). For example in Figure 3.1, for each \( s \in D^1 \), the support of the section \( g(s) \) lies inside the support of one of the partition unity functions which can be chosen to be very small.

More precisely, we shall prove that homotopy groups of the pair \( (\text{Sect}_e(\pi), \text{Sect}_e(\pi)) \) are trivial. To do so, we show that for any commutative diagram

\[
\begin{array}{c}
S^{q-1} \xrightarrow{f} \text{Sect}_e(\pi) \\
\downarrow \\
D^q \xrightarrow{g} \text{Sect}_e(\pi).
\end{array}
\]

there exists a homotopy of pairs \( (g_1, f_1) : (D^q, S^{q-1}) \to (\text{Sect}_e(\pi), \text{Sect}_e(\pi)) \) so that \( f_0 = f \), \( g_0 = g \) and \( g_1 : D^q \to \text{Sect}_e(\pi) \) factors through \( \text{Sect}_e(\pi) \). We first use Condition 1 to satisfy the following.

Claim 3.5. For sufficiently fine triangulation of \( S^{q-1} \), we can assume that for every simplex \( \sigma \subset S^{q-1} \), we can cover \( \text{supp}(f|_\sigma) \) by at most \( k \) balls of radius \( 2^{-k} \epsilon \) for some \( k \).

This is because there exists a fiberwise homotopy \( h_t : E \to E \) that is the identity on \( s_0(M) \) and maps a neighborhood of \( s_0(M) \) onto \( s_0(M) \). So we can define a homotopy \( F_t(x, s) = h_t(F(x, s)) \), \( G_t(x, s) = h_t(G(x, s)) \) where \( F \) and \( G \) are adjoints of \( f \) and \( g \) respectively. Now it is easy to see for every \( s \in S^{q-1} \), there exists a neighborhood \( \sigma \) of \( s \) so that \( \text{supp}(F_t|_{\sigma}) \subset \text{supp}(f)(s) \). So from now on we assume that \( f \) satisfies the claim.

To deform the family \( g : D^q \to \text{Sect}_e(\pi) \) to a family in \( \text{Sect}_e(\pi) \), we choose a partition of unity \( \{ \mu_i \} \) for a neighborhood of \( \text{supp}(g|_{D^q}) \) so that each \( \text{supp}(\mu_i) \) can be covered by a ball of radius \( 2^{-q-1} \epsilon \). Let \( H_t : M \times D^q \to M \times D^q \) be the fragmentation homotopy associated to this partition of unity.

By Lemma 3.4, there exists a homotopy \( G' : M \times D^q \times [0,1/2] \to E \) so that \( G_0' \) is the adjoint of \( g \), for all \( s \in D^q \) and \( t \in [0,1/2] \), we have \( \text{supp}(G'_t(s)) \subset \text{supp}(g(s)) \) and at time \( 1/2 \), we have \( G_{1/2}(L_\epsilon(D^q)) = s_0(M) \). Note that if \( s \in S^{q-1} \), then \( G'_t(s) \) lies in \( \text{Sect}(\pi) \). Therefore, \( G'_t \) gives a homotopy of the pairs \( (D^q, S^{q-1}) \to (\text{Sect}_e(\pi), \text{Sect}_e(\pi)) \).

Now we use the fragmentation homotopy to define \( G_t : M \times D^q \to E \)

\[
G_t = \begin{cases} 
G'_t & 0 \leq t \leq 1/2, \\
G'_1 \circ H_{2t-1} & 1/2 \leq t \leq 1.
\end{cases}
\]

To show that \( G_t \) is the desired homotopy, we first need to show that \( G_t(\cdot, S^{q-1}) \) is also in \( \text{Sect}_e(\pi) \) for \( 1/2 \leq t \leq 1 \). Recall that by the claim, for every \( x \in S^{q-1} \),
there exists a simplex $\sigma$ containing $x$ so that $\text{supp}(f|_{\sigma})$ is contained in at most $k$ balls of radius $2^{-k}\epsilon$ for some $k$. We showed that $\text{supp}(G'_t|_{\sigma})$ also has the same property. Since the fragmentation homotopy preserves the $M$ factor, $\text{supp}(G'_t \circ H_{2i-1}|_{\sigma})$ also has the same property. Hence, $G_t(\cdot, S^{q-1})$ lies in $\text{Sect}_c(\pi)$. So $G_t$ induces a homotopy of the pair of maps $(g, f)$.

Now it is left to show that $G_1(\cdot, s)$ lies in $\text{Sect}_c(\pi)$ for all $s \in D^q$. Note that the section $G_1(\cdot, s)$ is the same as the base section on $H_{1-1}(L_{\cdot}(D^n)) \cap M \times \{s\}$. Hence, by Lemma 3.2 the support of $G_1(\cdot, s)$ can be covered by $q+1$ balls of radius $2^{-q-1}\epsilon$. Therefore, $G_1(\cdot, s)$ is $\text{Sect}_c(\pi)$ for all $s \in D^q$.

**Remark 3.6.** As we mentioned in the introduction, Morrison and Walker in their blob homology paper ([MW12]) dropped the connectivity assumption but relaxed the notion of support to prove a key deformation lemma ([MW12, Lemma B.0.4]). Note that when we drop the connectivity hypothesis, we no longer have Lemma 3.4. However, for each $t \in \Delta^q$, by Lemma 3.2, we know that $(M \times t) \setminus H_{1-1}(L_{\cdot}(\Delta^q))$ is covered by at most $q$ open sets. Therefore, the same deformation $G_t$ as above, deforms a $\Delta^q$-family of sections to sections whose supports, in the sense of ([MW12, Lemma B.0.4]), can be covered by $q$ open balls.

## 4. On h-principle theorems whose formal sections have highly connected fibers

Let us recall the set-up from the introduction. Let $F : (\text{Mfld}_n^{\partial})^{\text{op}} \to S$ be a topologically invariant sheaf in the sense of [Kup19, Section 2] from the category of smooth $n$-manifolds (possibly with nonempty boundary) with smooth embeddings as morphisms to a convenient category of spaces $S$ (see [Kup19, Appendix A]). For our purpose, it is enough to consider the category of simplicial sets or compactly generated Hausdorff spaces. For brevity, when we refer to a simplicial set as a space, we mean the geometric realization of it. Recall that we defined the space of formal solutions $F^f(M)$ to be the space of sections of the bundle $\text{Fr}(M) \times_{\text{GL}_n(\mathbb{R})} F(D^n) \to M$, where $\text{Fr}(M)$ is the frame bundle of $M$. We say $F$ satisfies an h-principle if the natural map from the functor to its homotopy sheafification (see [BdBW13, Proposition 7.6])

$$j : F(M) \to F^f(M),$$

induces a weak equivalence and we say $F$ satisfies c-principle if the above map induces a homology isomorphism.

Often in proving $h(c)$-principle theorems, proving the local statement $F(D^n) \xrightarrow{\sim} F^f(D^n)$ which is statement for 0-handles is the easy step. The hard step often is to inductively deduce the statement for higher handles relative to their attaching maps. Then one could prove the statement for compact manifold using handle decompositions. Thurston, however, proved a c-principle theorem in foliation theory (see [Mat11] and [Ser79]) using his fragmentation idea without using the corresponding local statement. Proving the local statement in this c-principle theorem, is surprising very subtle and it was later proved by Segal ([Seg78]) for smooth foliations and McDuff ([McD81]) for foliations with transverse volume form when the codimension is larger than 2!

Let us first recall Thurston’s theorem in this language. Let $F : (\text{Mfld}_n^{\partial})^{\text{op}} \to \text{sSet}$ be the functor from manifolds with possibly non-empty boundary to simplicial sets so that the $q$-simplices $F_q(M)$ is the set of codimension $n$ foliations on $M \times \Delta^q$. Then one could prove the statement for compact manifold using handle decompositions.
that are transverse to the fibers of $M \times \Delta^q \to \Delta^q$. Let $F_c : \text{Mfld}_n^0 \to \text{sSet}$ be the compactly supported version of $F$ meaning that we impose the condition that the foliations on $M \times \Delta^q$ are horizontal near the boundary $\partial M \times \Delta^q$. Since the homotopy sheafification $F^I(M)$ is given by the section space of a bundle over $M$, one could make sense of the compactly supported version by choosing a base section. In fact, there is a canonical choice of the base section so that we could define a map

$$j : F_c(M) \to F^I_c(M).$$

Thurston uses his fragmentation technique on the closed disk $D^n$ to show directly (instead of induction on handles and inductively deloop) that

$$|F_{c,\bullet}(D^n)| \to |F^I_{c,\bullet}(D^n)| \simeq \Omega^n|F_{\bullet}(D^n)|,$$

is a homology isomorphism. Then he showed that this statement and the fragmentation on $M$ implies that $|F_{\bullet}(M)| \to |F^I_{\bullet}(M)|$ is a homology isomorphism for all compact manifolds $M$. If $M$ has a boundary, there is a version relative to the boundary. His fragmentation technique avoids the usual delooping steps in other approaches to go inductively from the statement for a handle of index $i$ to that of a handle of index $i + 1$ and also avoids the step for 0-handles.

To recall the main theorem, let $F$ be a topologically invariant sheaf enriched over $\mathbb{S}$. Suppose that there is a canonical base element in $F(N)$ for each manifold $N$ so that for a manifold with boundary $M$, we can define the relative version $F(M, \partial)$ to be the subspace of those elements in $F(M)$ that restrict to the base element in the germ of the boundary. We can also define the compactly supported version $F_c(M)$ to be the subspace of $F(M)$ consisting of those elements that restrict to the base element outside of a compact subset of $M$. Similarly, we can define the relative and compactly supported versions for $F^I$ so that we have a map $F_c(M) \to F^I_c(M)$. Similar to the previous section, we can define $\epsilon$-supported versions $F_{c,\epsilon}(M)$ and $F_{\epsilon}^I(M)$. We need to impose a homotopy theory condition on $F$ similar to Lemma 2.2. It is easy to see that this condition is satisfied for all geometric examples in the introduction and it will be necessary to find a simplicial resolution for $F$ in Lemma 4.15 and as we shall explain this is also a technical oversight in Mather’s note.

**Condition 2.** Suppose that $(F(M), d)$ is a metric space for each manifold $M$. For each $\delta > 0$, we can define the notion of lax compactly supported elements $F_c(M, \delta)$ in $F(M)$ to be those elements $s$ such that outside of a compact set $K$ in $M$, we have $d(s|_K, s_0|_K) < \delta$. We assume for small enough $\delta$, the inclusion $F_c(M) \to F_c(M, \delta)$ is a weak equivalence.

**Definition 4.1.** We say $F$ is good, if it satisfies

1. The subspace of elements with empty support in $F(M)$ is contractible.
2. For each open subset $U$ of a manifold $M$ and small enough $\delta$ the inclusion $F_c(U, \delta) \to F_c(M, \delta)$ be an open embedding.
3. For each finite family of open sets $U_0, U_1, \ldots, U_k$ such that $U_1, \ldots, U_r$ are disjoint and contained in $U_0$, we have a permutation invariant map

$$\mu^{U_0, \ldots, U_k}_{U_1, \ldots, U_k} : \prod_{i=1}^{k} F_c(U_i) \to F_c(U_0),$$

where this map satisfies the obvious associativity conditions and for $U_0 = \bigcup_{i=1}^{k} U_i$, the map $\mu^{U_0}_{U_1, \ldots, U_k}$ is a weak equivalence.
(4) Let $U$ and $V$ be open disks. All embeddings $U \to V$ induces a homology isomorphism between $F_c(U)$ and $F_c(V)$.

(5) Let $\partial_1$ be the northern-hemisphere boundary of $D^n$. Let $F(D^n, \partial_1)$ be the subspace of $F(D^n)$ that restricts to the base element in a germ of $\partial_1$ inside $D^n$. We assume $F(D^n, \partial_1)$ is contractible.

**Theorem 4.2.** Let $F$ be a good functor such that $F(D^n)$ is at least $(n-1)$-connected and $F$ has the fragmentation property meaning that

$$F_c(M) \to F_c(M),$$

is a weak homotopy equivalence, then for any compact manifold $M$, the map

$$F_c(M) \to F_c^f(M),$$

is a homology isomorphism.

**Example 4.3.** Let $\Gamma_{vol}^n$ denote the topological Haefliger groupoid whose objects are $\mathbb{R}^n$ with the usual topology and the space of morphisms are local symplectomorphisms of $\mathbb{R}^n$ with respect to the standard symplectic form (see [Hae71] for more details on how this groupoid is topologized). We shall write $BG_{vol}^n$ to denote its classifying space. There is a map

$$\theta : BG_{vol}^n \to BSL_n(\mathbb{R}),$$

which is induced by the functor $\Gamma_{vol}^n \to SL_n(\mathbb{R})$ that sends a local diffeomorphism to its derivative at its source. We denote the homotopy fiber of $\theta$ by $BG_{vol}^n$. Let $M$ be an $n$-dimensional manifold with possibly non-empty boundary with a fixed volume form $\omega$. Let $\tau^*(\theta)$ be the bundle over $M$ given by the pullback of $\theta$ via the map $\tau$

$$\begin{array}{ccc}
BG_{vol}^n & \xrightarrow{\theta} & BSL_n(\mathbb{R}) \\
M & \xrightarrow{\tau} & BSL_n(\mathbb{R}).
\end{array}$$

which is the classifying map for the tangent bundle. The space of sections of $\tau^*(\theta)$ has a natural base point $s_0$. Let $\text{Sect}(\tau^*(\theta), \partial)$ to be those sections that are equal to $s_0$ in the germ of the boundary (see [Nar17, Section 5.1] for more details). It was proved by Haefliger that the fiber of $\tau^*(\theta)$ is $(n-1)$-connected. Note that $\text{Sect}(\tau^*(\theta), \partial)$ is not connected.

Now let $BDiff_\omega(M, \partial)$ denote the homotopy fiber of the map

$$BDiff_\omega^\delta(M, \partial) \to BDiff_\omega(M, \partial),$$

induced by the identity homomorphism. This space can be thought of the space of foliated $M$ bundles with a transverse volume form. It is easy to check the conditions in Definition 4.1 except the second condition which is proved by McDuff in [McD83a]. McDuff ([McD81]) showed that $BDiff_\omega(\mathbb{R}^n) \to BG_{vol}^n$ is a homology isomorphism for $n > 2$ and it still not known for $n = 2$. She used this fact to show that when $\dim(M) > 2$

$$BDiff_\omega(M, \partial) \to \text{Sect}(\tau^*(\theta), \partial),$$
induces a homology isomorphism into the connected component that it hits. She also found a different proof for \( \dim(M) = 2 \) in [McD82]. But with this method, one could give a uniform proof for the compactly supported version without using her local statement in dimension 3 and higher.

**Example 4.5.** Consider \( F(M) \) to be the labeled configuration space ([B87], [Seg73]) for which proving the fragmentation property is easy. To recall the definition from [B87], let \( X \) be a fixed connected CW complex with a base point \( x_0 \). Let \( C(M; X) \) to be configuration space of finite number of points with labels in \( X \) and the topology such that points can vanish if their label is \( x_0 \) (For a precise definition of the topology see [B87], [Seg73]). We shall write a point \( \xi \in C(M; X) \) as a formal sum \( \sum x_i m_i \) where \( m_i \in M \) are distinct points and \( x_i \in X \) satisfying the relation \( \sum x_i m_i \sim \sum x_i m_i + x_0 m \). For a subspace \( N \subset M \), we let \( C(M, N; X) \) to be the quotient of \( C(M; X) \) by the relation \( \sum x_i m_i \sim \sum x_i m_i + xn \) where \( n \in N \). We define the support of \( \sum x_i m_i \) be the set of the points \( m_i \) whose label \( x_i \) is not the base point \( x_0 \). Note that, similar to section spaces, we can define the subspace \( C_c(M; X) \) to be those labeled configuration of points whose support can be covered by \( k \) balls of radius \( 2^{-k} \) for some \( k \). But obviously we have \( C_c(M; X) = C(M; X) \).

It is easy to show that \( C(D^n, \partial D^n; X) \) is homotopy equivalent to the reduced suspension \( \Sigma^n X \) which is at least \( n \)-connected. Fragmentation method implies that the natural scanning map

\[
C(D^n; X) \to \Omega^n C(D^n, \partial D^n; X),
\]

is a homology isomorphism (it is in fact a weak homotopy equivalence by [Seg73]). Using fragmentation again for \( C(M; X) \), we could obtain the homological version of McDuff’s theorem ([McD75]) that for any closed manifold \( M \), the natural map

\[
C(M; X) \to \text{Sect}_c(\text{Fr}(M) \times_{\text{GL}_n(\mathbb{R})} \Sigma^n X) \to M,
\]

induces a homology isomorphism.

4.1. \( n \)-fold delooping via fragmentation. The key step in proving Theorem 4.2 is to show that if \( F \) has a fragmentation property then the map

\[
F(D^n, \partial) \to F^f(D^n, \partial) \simeq \Omega^n F(D^n),
\]

is a homology isomorphism. To do so, we filter \( F(D^n) \) and \( F^f(D^n) \). Since \( D^n \) is compact, the fragmentation property for \( F \) and \( F^f \) implies that

\[
F_i(D^n) \xrightarrow{\sim} F(D^n),
\]

\[
F^f_i(D^n) \xrightarrow{\sim} F^f(D^n).
\]

The spaces \( F_i(D^n) \) and \( F^f_i(D^n) \) are naturally filtered by the number of balls that cover the supports. We shall denote these filtrations and the corresponding maps between them by

\[
\begin{array}{ccccccc}
F_1(D^n) & \to & F_2(D^n) & \to & \cdots & \to & F_i(D^n) & \to & F(D^n) \\
\downarrow j_1 & & \downarrow j_2 & & & & \downarrow j & & \downarrow \\
F^f_1(D^n) & \to & F^f_2(D^n) & \to & \cdots & \to & F^f_i(D^n) & \to & F^f(D^n).
\end{array}
\]

(4.6)

Note that the last vertical map is a weak equivalence because \( F^f(D^n) \) is a section space of a bundle over contractible space \( D^n \) with the fiber \( F(D^n) \). Therefore, the map \( j \) in the diagram 4.6 also is a weak homotopy equivalence.
Remark 4.7. We dropped $\epsilon$ from our notations for filtrations $F_k(-)$ and $F^f_k(-)$ but if we want to emphasize our choice of $\epsilon$, we shall instead use $F_k(-, \epsilon)$ and $F^f_k(-, \epsilon)$.

Proposition 4.8. If $j_1$ induces a homology isomorphism, so does the map
\[ F(D^n, \partial) \rightarrow F^f(D^n, \partial) \cong \Omega^nF(D^n). \]

We first explain the strategy to prove that $j_1$ is a homology isomorphism before we embark on proving Proposition 4.8. We have the following general lemma about filtered spaces ([Mat11, Lemma 2, Section 27]):

Lemma 4.9. Consider the commutative diagram of spaces

\[
\begin{array}{ccccccccc}
X_1 & \longrightarrow & X_2 & \longrightarrow & \cdots & \longrightarrow & X_{\infty} & \longrightarrow & X \\
\downarrow f_1 & & \downarrow f_2 & & \cdots & & \downarrow f_\infty & & \downarrow f \\
Y_1 & \longrightarrow & Y_2 & \longrightarrow & \cdots & \longrightarrow & Y_{\infty} & \longrightarrow & Y.
\end{array}
\]

Suppose:
- $f$, $\iota$ and $\iota'$ are weak homotopy equivalences.
- The filtration is so that if $f_1$ is $k$-acyclic (we say $f : A \rightarrow B$ is $k$-acyclic if it induces a homology isomorphism for homological degrees less than $k$ and surjection on degree $k$) for some $k$ then the induced map $f_N : X_N/X_{N-1} \rightarrow Y_N/Y_{N-1}$, is $(2N+k-2)$-acyclic.
then $f_1$ induces a homology isomorphism.

Proof. We can assume that the maps $f_i$ are inclusions by replacing them with the mapping cylinder of $f_i$. Therefore, the filtration $(Y_p, X_p)$ of $(Y_{\infty}, X_{\infty})$ gives rise to a spectral sequence whose first page is
\[ E^1_{p,q} = H_p(Y_p, Y_{p-1} \cup X_p). \]
It converges to the homology of the pair $(Y_{\infty}, X_{\infty})$ but this pair is weakly homotopy equivalent to the pair $(Y, X)$. Since by the first condition $f$ is a weak homotopy equivalence, the spectral sequence converges to zero. Now we suppose the contrary

\[ \begin{array}{cccccc}
q & & & & & k \\
& & & & & k-1 \\
& & & & & k-2 \\
& & & & & \vdots \\
0 & & & & & 0
\end{array} \]

Figure 2. The first page of the homology spectral sequence
that \( f_1 \) is not a homology isomorphism and we choose the smallest \( k \) so that \( E_{1,k}^1 = H_{k+1}(Y_1, X_1) \neq 0 \). Therefore, \( f_1 \) is \( k \)-acyclic and by the second condition \( \overline{f_p} \) is \((2p+k-2)\)-acyclic which implies that \( E_{p,q}^1 = H_{p+q}(Y_p, Y_{p-1} \cup X_p) = 0 \) for \( q \leq p + k - 2 \).

Hence, as is indicated in the Figure 2, no nontrivial differentials can possibly hit \( E_{1,k}^1 \) which contradicts the fact that the spectral sequence converges to zero in all degrees. \( \square \)

In order to apply Lemma 4.9 to the diagram 4.6, we need to establish the second condition of Lemma 4.9 for the diagram 4.6. The subtlety here is in the filtrations \( F_k(-) \) and \( F^l_k(-) \) where we know that the support is covered by \( k \) small balls but the data of these balls are not given. We shall define certain auxiliary spaces by adding the data of covering balls.

4.1.1. Semisimplicial resolutions. To study the filtration quotients in the diagram 4.6, we shall define auxiliary semisimplicial spaces.

For the definition of semisimplicial spaces and the relevant techniques, we follow [ERW17]. We fix a Riemannian metric and a small positive \( \epsilon \) so that all balls of radius \( \epsilon \) is geodesically convex. We say that a subset \( U \) of \( M \) is \( \epsilon \)-admissible if it is open, geodesically convex and it can be covered by an open ball of radius \( \epsilon \).

**Definition 4.10.** Let \( CF_k(M) \) be the subspace of \( F(M)^k \) consisting of \( k \)-tuples so that each one has a support in a ball of radius \( 2^{-k}\epsilon \). We define the subspace \( DF_k(M) \) of \( CF_k(M) \) to be **degenerate** \( k \)-tuples that is the union of their supports can be covered by \( k_0 \) balls of radius \( 2^{-k_0}\epsilon \) for some \( k_0 < k \). We denote the quotient \( CF_k(M)/DF_k(M) \) by \( NF_k(M) \). Similarly we can define \( CF^l_k(M) \), \( DF^l_k(M) \) and \( NF^l_k(M) \).

The natural maps \( NF_k(M) \to F_k(M)/F_{k-1}(M) \) and \( NF^l_k(M) \to F^l_k(M)/F^l_{k-1}(M) \) are \((k!)\)-sheeted covers. Therefore, the spectral sequence for covering spaces implies the following.

**Lemma 4.11.** If the induced map \( NF_k(M) \to NF^l_k(M) \) is \( j \)-acyclic so is the map between the filtration quotients

\[
F_k(M)/F_{k-1}(M) \to F^l_k(M)/F^l_{k-1}(M).
\]

Hence, to establish the second condition of Lemma 4.9 for the diagram 4.6, it is enough to study the acyclicity of the map \( NF_k(M) \to NF^l_k(M) \). To do so, we shall use the following semisimplicial spaces.

**Definition 4.12.** \( CF_k(M) \) be a semisimplicial space whose space of \( q \)-simplices is given by \( (\sigma, (B_{ij})) \) where \( \sigma = (\sigma_1, \ldots, \sigma_k) \in CF_k(M) \) and \( (B_{ij}) \) is a \( k \times (q+1) \) matrix of \((2^k\epsilon)\)-admissible sets such that \( B_{ij} \) contains the support of \( \sigma_i \) for all \( j \) (if the support of \( \sigma_i \) is empty \( B_{ij} \)’s are just \((2^k\epsilon)\)-admissible sets). We topologize the \( q \)-simplices as a subspace of \( F_k(M) \times \mathcal{O}(M)^{k(q+k)} \).

**Definition 4.13.** We define subsemisimplicial space \( DF_k(M) \) so that its \( q \)-simplices are given by pairs \( (\sigma, (B_{ij})) \) so that for each \( 0 \leq j \leq q \), the closure of \( U_i B_{ij} \) is covered by \( k_0 \) balls of radius \( 2^{-k_0}\epsilon \) for some \( k_0 < k \).

We define similarly \( CF^l_k(M) \) and \( DF^l_k(M) \).

**Remark 4.14.** If we keep track of the the choice of \( \epsilon \) in our notations we have the useful identifications \( CF_k(M, \epsilon) = CF_k(M, 2^{-k}\epsilon) \) and the same for \( F^l \).
Lemma 4.15. The natural maps
\[ ||CF_k(M)|| \to CF_k(M), ||DF_k(M)|| \to DF_k(M), \]
are all weak homotopy equivalencies. Similarly the corresponding statement holds for \( F^i \).

Proof. This is [Mat11, Lemma in section 20] for the functor defined by Thurston. But there is an oversight in that proof that Mather assumes that the augmentation map from the realization of semisimplicial sets to \( F_k(M) \) is a fibration and says that it is enough to show that their fibers are contractible. To fix this oversight, we need Condition 2. The idea is to show that the augmentation maps are microfibrations with contractible fibers.

For \( \delta > 0 \), let \( CF_k(\delta) \) be the subspace of \( F(M)^k \) consisting of \( k \)-tuples such that each one has a lax support in ball of radius \( 2^{-k}\epsilon \). Similarly, we define \( CF_k(\delta) \). Suppose that \( \delta \) satisfies Condition 2. Hence, it is enough to show that
\[ \alpha:||CF_k(M,\delta)|| \to CF_k(M,\delta). \]

Let \( S_q \) be the simplicial set whose \( q \)-simplices are given by \( q + 1 \) ordered \((2^{-k}\epsilon)\)-admissible sets. By the second condition of goodness of the functor (see Definition 4.1), it is clear that \( CF_k(M,\delta) \subset CF_k(\delta) \times (S_q)^k \) is open. Similar to the proof of Lemma 2.4, this inclusion satisfies the conditions of [GRW18, Proposition 2.8]. Therefore, the map \( \alpha \) induced by the projection to the first factor is a microfibration. To identify the fiber over \( \sigma = (\sigma_1, \ldots , \sigma_k) \) let \( S_i \) be the set of \((2^{-k}\epsilon)\)-admissible sets containing the support of \( \sigma_i \). Let \( S_{\bullet} \) be the simplicial set whose \( q \)-simplices are given by mappings \( [q] \to S_i \). Therefore, the realization of this simplicial set is contractible. The fiber over \( \sigma \) can be identified with the fat realization of \( S_1 \times \cdots \times S_k \). Since the fat realization and the realization for the simplicial sets are weakly equivalent and the realization commutes with products ([Mil57]), we deduce that the fiber over \( \sigma \) is contractible. The proof for the other augmentation map is similar. \( \square \)

Now the strategy to check the second condition of Lemma 4.9 for the diagram 4.6 is as follows. We define a functor \( \nu_N \) on spaces so that when we apply it to a \( k \)-acyclic map \( f: X \to Y \), we obtain a \((2N + k - 2)\)-acyclic map \( \nu_N(f): \nu_N(X) \to \nu_N(Y) \). And then we construct a homotopy commutative diagram
\[ \begin{array}{ccc}
||CF_k(D^n)||/||DF_k(D^n)|| & \longrightarrow & \nu_k(F(D^n, \partial)) \\
||CF_k(D^n)||/||DF_k(D^n)|| & \longrightarrow & \nu_k(F^i(D^n, \partial)),
\end{array} \]
where the horizontal maps induce homology isomorphisms. We shall define a suitable functor \( \nu_k \) satisfying the desired properties in the next section.

4.1.2. A thick model of the suspension of a based space. To define the functor \( \nu_k \) that receives a map from above semisimplicial resolutions, we need to modify the definition of the suspension of a space. First we define auxiliary simplicial sets associated to the manifold \( M \) with the fixed choice of \( \epsilon \).
**Definition 4.18.** Let $S(k)$ be the set of $(2^{-k} \epsilon)$-admissible sets in $M$. Let $\Delta_\bullet(M, k)$ denote the simplicial set whose $q$-simplices are given by mappings $[q]$ into $S(k)$ i.e. $(q + 1)$-tuple of elements in $S(k)$. Let $M_\bullet(k)$ be the subsimplicial set of $\Delta_\bullet(M, k)$ whose $q$-simplices consist of those admissible sets that have nontrivial intersections. And let $\partial M_\bullet(k)$ be the subsimplicial set of $\Delta_\bullet(M, k)$ whose $q$-simplices consist of those admissible sets that have nontrivial intersections with the boundary $\partial M$.

**Remark 4.17.** The geometric realizations of $\Delta_\bullet(M, k)$, $M_\bullet(k)$ and $\partial M_\bullet(k)$ have the homotopy type of point, $M$ and $\partial M$ respectively.

Our modification of the suspension of a space $X$ is

**Definition 4.18.** Let $\tilde{\Sigma}^n X$ be the realization of the following semisimplicial space

$$\tilde{\Sigma}^n X := \frac{(\Delta_\bullet(D^n) \times \{\ast\}) \cup (D^n \times X)}{(t, x) \sim (t, x') \text{ if } t \in \partial D^n}.$$  

It is easy to see that $\tilde{\Sigma}^n X$ has the same homotopy type of the suspension $\Sigma^n X$. We have a natural projection $\pi : \tilde{\Sigma}^n X \to \|\Delta_\bullet(D^n)\|$.

**Definition 4.19.** Let $T_k, \bullet(M)$ be the subsimplicial set of $(\Delta_\bullet(M))^k$ whose $q$-simplices are given by matrices $(B_{ij})$, $i = 0, 1, \ldots, q$, $j = 1, \ldots, k$ of admissible sets so that for each $i$, the union $\cup_j B_{ij}$ can be covered by $k_0$ open balls of radius $2^{-k_0} \epsilon$ for some $k_0 < k$. For $k = 1$, we define $T_1, \bullet = \ast$.

**Definition 4.20.** We define $\theta_k(X)$ to be the pair

$$((\tilde{\Sigma}^n X)^k, (\pi^k)^{-1}([T_k, \bullet(D^n)])),$$

where $\pi^k : (\tilde{\Sigma}^n X)^k \to \|\Delta_\bullet(D^n)\|^k$ is the natural projection. Let $\nu_k(X)$ denote the quotient

$$\tilde{\Sigma}^n X^k / (\pi^k)^{-1}([T_k, \bullet(D^n)]).$$

**Remark 4.21.** Note that for $k = 1$, the space $\nu_1(X)$ has the homotopy type of $\Sigma^n X$.

We suppress $n$ which is the dimension from the notations $\theta_k(X)$ and $\nu_k(X)$ as it is fixed throughout. The following technical lemma is the main property of the functor $\nu_k$.

**Lemma 4.22.** If $f : X \to Y$ is $j$-acyclic, the induced map of pairs $\nu_k(f) : \nu_k(X) \to \nu_k(Y)$ is $(j + n + 2k - 2)$-acyclic.

**Proof.** Recall that the reduced suspension of $X$ for a based space $(X, \ast)$, is the smash product $S^n \wedge X$ and we represent points in this smash product by a pair $(s, x)$ where $s \in S^n$ and $x \in X$. First it is not hard to see ([Mat11, Section 24]) that the space $\nu_k(X)$ is homotopy equivalent to

$$(S^n \wedge X)^k / \Delta_{\text{fat}, k}(S^n, X),$$

where $\Delta_{\text{fat}, k}(S^n, X)$ consists of tuples $((s_1, x_1), (s_2, x_2), \ldots, (s_k, x_k))$ such that $s_i = s_j$ for some $i \neq j$. We can further simplify the homotopy type of $\nu_k(X)$ by separating $S^n$ and $X$ in the above quotient to obtain

$$\nu_k(X) \simeq (S^n / \Delta_{\text{fat}, k}(S^n)) \wedge X^{\wedge k}.$$  

Note that if $f : X \to Y$ is $j$-acyclic, the long exact sequence for homology of a pair implies that the induced map $f^{\wedge k} : X \wedge X \to Y \wedge Y$ is $(j + 1)$-acyclic. Hence, one can inductively show that the induced map $f^{\wedge k} : X^{\wedge k} \to Y^{\wedge k}$ is $(j + k - 1)$-acyclic.
Thus, it is enough to show that \( (S_n^k/\Delta_{\text{fat},k}(S^n)) \) is \((n+k-2)\)-acyclic. Using again the long exact sequence for homology of a pair, we need to show that \( \Delta_{\text{fat},k}(S^n) \) is \((n+k-3)\)-acyclic.

For \( i \neq j \), let \( \Delta_{(i,j)}(S^n,k) \subset (S^n)^{\text{fat}} \) be the subspace given by tuples \((s_1,s_2,\ldots,s_k)\) where \( s_i = s_j \). Note that \( \Delta_{(i,j)}(S^n,k) \cong (S^n)^{\text{fat}k-1} \). The fat diagonal \( \Delta_{\text{fat},k}(S^n) \) is the union of \( \Delta_{(i,j)}(S^n,k) \subset (S^n)^{\text{fat}k} \) for all pairs \((i,j)\) where \( i \neq j \). These are not open subsets but they are sub-CW complexes, so we still can apply Mayer-Vietoris spectral sequence for this cover to compute the homology of \( \Delta_{\text{fat},k}(S^n) \). Let \( \Delta_{(i_1,j_1),\ldots,(i_r,j_r)}(S^n,k) \) denote the intersection \( \Delta_{(i_1,j_1)}(S^n,k) \cap \cdots \cap \Delta_{(i_r,j_r)}(S^n,k) \). Hence, we have

\[
E^1_{p,q} = \bigoplus_{(i_m,j_m)} H_q(\Delta_{(i_0,j_0),\ldots,(i_r,j_r)}(S^n,k)) \Rightarrow H_{p+q}(\Delta_{\text{fat},k}(S^n)),
\]

where the sum is over different tuples of pairs \((i_m,j_m)\). Since the intersection \( \Delta_{(i_0,j_0),\ldots,(i_r,j_r)}(S^n,k) \) is a \( n(k-p-1) \)-connective space, \( E^1_{p,q} = 0 \) for \( q < n(k-p-1) \). Note that \( p \) is at most \( k-2 \) so we have \( n + k - 3 < n(k-p-1) + p \). On the other hand, if \( p + q < n(k-1) - pn + p \), we have \( E^1_{p,q} = 0 \) which implies that \( \Delta_{\text{fat},k}(S^n) \) is \((n+k-3)\)-acyclic.

Now we are ready to prove the second condition of Lemma 4.9 for the diagram 4.6.

4.1.3. Proof of Theorem 4.2 for \( M = D^n \). So we want to prove that the natural map

\[
F(D^n,\partial) \to F^f(D^n,\partial) \cong \Omega^n F(D^n),
\]

induces a homology isomorphism. To do this, we show that

**Lemma 4.23.** There exists a commutative diagram of pairs

\[
\begin{array}{ccc}
\|CF_k(D^n)\|_\bullet & |DF_k(D^n)\|_\bullet \\
\downarrow & \downarrow \\
\|CF_k'(D^n)\|_\bullet & |DF_k'(D^n)\|_\bullet
\end{array}
\]

so that the horizontal maps are homology isomorphisms (by which we mean homology isomorphism on each member of the pair).

Before, we prove this lemma, let us explain how the above lemma finishes the proof of Theorem 4.2 for \( M = D^n \). By Lemma 4.15, Lemma 4.22 and Lemma 4.23, the second condition of Lemma 4.9 for the diagram 4.6 holds. Hence, \( j_1 \) in the diagram 4.6 is a homology isomorphism. Recall that for \( k = 1 \), the pair \( \tilde{\theta}_1(X) \) has the homotopy type of \((\Sigma^n X, \ast)\). Therefore, Proposition 4.8 follows from Lemma 4.23 for \( k = 1 \).

**Construction 4.25.** To define the horizontal map in the diagram 4.24, we first define a semi-simplicial map

\[
f_* : CF_1(D^n) \to \Sigma^n F(D^n,\partial) = \left( \Delta_\bullet(D^n) \times \{\ast\} \right) \cup \left( D^n_\bullet \times F(D^n,\partial) \right) / \left( t, x \right) \sim (t, x') \text{ if } t \in \partial D^n_\bullet.
\]

For a \( q \)-simplex \((\sigma,B_0,\ldots,B_q)\) on the left hand side, we know that \( c_i B_i \) contains \( \text{supp}(\sigma) \).
• If \((B_0, \ldots, B_q)\) is a \(q\)-simplex in \(\partial D^n_\bullet\), then we send \((\sigma, B_0, \ldots, B_q)\) to the base point on the right hand side.

• If \((B_0, \ldots, B_q)\) is a \(q\)-simplex in \(D^n_\bullet\) but not in \(\partial D^n_\bullet\), then the support of \(\sigma\) lies inside \(D^n\). Therefore \(\sigma \in F(D^n, \partial)\), so we send \((\sigma, B_0, \ldots, B_q)\) to the corresponding element in \(F(D^n, \partial) \times D^n_\bullet\).

• And if \((B_0, \ldots, B_q)\) is in \(\Delta_\bullet(D^n)\) but not in \(D^n_\bullet\), we send \((\sigma, B_0, \ldots, B_q)\), to \((B_0, \ldots, B_q)\) in \(\Delta_q(D^n) \times \{\ast\}\).

Since the above map is semisimplicial map, we could take the realization to obtain

\[
f : \|CF_1(D^n)\| \to \tilde{\Sigma}^n F(D^n, \partial).
\]

Recall from Remark 4.14, that \(CF_k(M, \epsilon)_\ast = CF_1(M, 2^{-k}\epsilon)^k\). Therefore, the above construction gives rise to maps

\[
\|CF_k(D^n)\| \to (\tilde{\Sigma}^n F(D^n, \partial))^k.
\]

Definition 4.19 and Definition 4.13 are so that the above map induces a map

\[
\|DF_k^f(D^n)\| \to (\pi^k)^{-1}(\|T_k(\bullet(D^n))\|).
\]

The bottom horizontal map in diagram 4.24 is similarly defined.

**Proof of Lemma 4.25.** From the naturality of the construction, we obtain the commutative diagram 4.24. We show that the top horizontal map is a homology isomorphism. The proof for the bottom horizontal map is similar. We first show that the map \(f\) in 4.26 induces a homology isomorphism. Recall that for all semisimplicial spaces \(X_\bullet\), there is a spectral sequence \(E_{p,q}(X_\bullet) = H_q(X_p)\) that converges to \(H_{p+q}(\|X_\bullet\|)\). The map \(f\) induces a comparison map between spectral sequence

\[
\begin{array}{cccc}
H_q(CF_1(D^n)_p) & (f_p)_\ast & H_q(\tilde{\Sigma}^n_p F(D^n, \partial)) \\
\downarrow & & \downarrow \\
H_{p+q}(\|CF_1(D^n)\|) & f_\ast & H_{p+q}(\|\tilde{\Sigma}^n_p F(D^n, \partial)\|).
\end{array}
\]

So to prove \(f_\ast\) is an isomorphism, we need to show that \(f_p\) induces a homology isomorphism. Note that we have the following commutative diagram

\[
\begin{array}{cccc}
CF_1(D^n)_p & f_p & \tilde{\Sigma}^n_p F(D^n, \partial) \\
\uparrow & & \uparrow \\
\Delta_\bullet(D^n) & \equiv & \Delta_p(D^n).
\end{array}
\]

where \(\pi\) and \(\tau\) are natural projection to the simplicial set \(\Delta_\bullet(D^n)\). Hence to show that \(f_p\) induces a homology isomorphism, it is enough to prove that that \(f_p\) induces a homology isomorphism on the fibers of \(\tau\) and \(\pi\).

We have three cases:

• If \(\beta = (B_0, \ldots, B_p)\) be in \(\Delta_p(D^n)\) but not in \(D^n_p\), then the fiber of \(\tau\) consists of those elements in \(F(D^n)\) that have empty support which is a contractible space by Definition 4.1. The fiber of \(\pi\) over \(\beta\) is a point.
• If $\beta = (B_0, \ldots, B_p)$ be in $D^n_p$ but not in $\partial D^n_p$, then the fiber of $\tau$ over $\beta$ is the subspace $F_\epsilon(c, B_j, \beta)$. The fiber of $\pi$ over $\beta$ is $F(D^n, \partial)$. Note that by the second condition in Definition 4.1, the inclusion $F_\epsilon(c, B_j, \beta) \to F(D^n, \partial)$ is a homology isomorphism.

• If $\beta = (B_0, \ldots, B_p)$ be in $\partial D^n_p$, then the fiber of $\tau$ over $\beta$ is acyclic by the third condition of Definition 4.1 and the fiber of $\pi$ over $\beta$ is a point. Therefore, $f_p$ induces a homology isomorphism which in turn implies that $\tilde{f}$ induces a homology isomorphism.

Since $CF_k(M, \epsilon_\bullet) = CF_1(M, 2^{-k}\epsilon)^k$. Therefore, the fact that $f$ induces a homology isomorphism implies that the map

$$||CF_\epsilon(D^n)\| \to (\Sigma^n F(D^n, \partial))^k.$$  

is also a homology isomorphism. Similar to the diagram 4.28, one can fiber the map

$$DF_\epsilon^f(D^n) \to (\pi^k)^{-1}(T_k, \bullet(D^n)),$$

over $T_k, \bullet(D^n)$, to prove that

$$||DF_\epsilon^f(D^n)\| \to (\pi^k)^{-1}(||T_k, \bullet(D^n)||).$$

is also a homology isomorphism. \(\square\)

Remark 4.29. Let $U$ be an open subset of $M$ that is homeomorphic to disjoint union of Euclidean spaces of dimension $n$. The same proof as the case of $D^n$, implies that

$$F_\epsilon(U) \to F^f_\epsilon(U) \simeq \prod_{\pi_0(U)} \Omega^n F(D^n, \partial),$$

is a homology isomorphism.

4.1.4. Proof of Theorem 4.2. Since both $F$ and $F^f$ satisfy fragmentation property, the spaces $F_\epsilon(M) \simeq F_\epsilon(M)$ and $F^f_\epsilon(M) \simeq F^f_\epsilon(M)$ can be filtered and the natural map $F_\epsilon(M) \to F^f_\epsilon(M)$ respects the filtration. Hence, it is enough to show that the induced map between filtration quotients induces a homology isomorphism. Using Lemma 4.11, it is enough to prove that the induced map between pairs

$$(F_k(M), DF_k(M)) \to (F^f_k(M), DF^f_k(M)),$$

induces a homology isomorphism. Let us first show that $F_k(M) \to F^f_k(M)$ induces a homology isomorphism using the same idea as in the proof of Lemma 4.23. We use Lemma 4.15 to resolve $F_k(M)$ and $F^f_k(M)$ by $F_k(M)_\bullet$ and $F^f_k(M)_\bullet$. Recall that $F_k(M, \epsilon) = (F_1(M, 2^{-k}\epsilon))^k$ and $F^f_k(M, \epsilon) = (F^f_1(M, 2^{-k}\epsilon))^k$. Therefore, it is enough to show that

$$F_1(M)_p \to F^f_1(M)_p,$$

induces a homology isomorphism for each simplicial degree $p$. To do so, we consider the commutative diagram

$$\begin{align*}
F_1(M)_p \xrightarrow{f_p} F^f_1(M)_p \\
\downarrow \tau & \downarrow \pi \\
\Delta_p(M) & \xrightarrow{=} \Delta_p(M),
\end{align*}$$

(4.30)
where $\pi$ and $\tau$ are natural projection to the simplicial set $\Delta_\bullet(M)$. Hence to show that $f_p$ induces a homology isomorphism, it is enough to prove that $f_p$ induces a homology isomorphism on the fibers of $\tau$ and $\pi$. Let $\sigma_p = (B_0, B_1, \ldots, B_p)$ be a $p$-simplex in $\Delta_p(M)$. There are two cases for the fibers of $\tau$ and $\pi$ over $\sigma_p$.

- The intersection of $B_i$’s is empty. Therefore, the preimages of $\sigma_p$ under $\tau$ and $\pi$ are contractible by the first condition in Definition 4.1.
- The intersection of $B_i$’s is not empty. Given that the disks $B_i$’s are geodesically convex, their intersection is homeomorphic to a disk. Hence, the induced map on fibers over $\sigma_p$ is

$$F_c(\cap_i B_i) \to F^\ell_c(\cap_i B_i),$$

which is a homology isomorphism as we proved the main theorem for the disks relative to their boundaries in section 4.1.3.

Similarly, we could see that $DF_k(M) \to DF^\ell_k(M)$ induces a homology isomorphism by fibering the semisimplicial resolutions $DF_\bullet(M)$ and $DF^\ell_\bullet(M)$ over the semisimplicial set $T_\bullet(M)$. Hence, we conclude that the map between pairs

$$(F_k(M), DF_k(M)) \to (F^\ell_k(M), DF^\ell_k(M)),$$

induces a homology isomorphism. \qed

**Remark 4.31.** One could give a different proof of Theorem 4.2 using Condition 2 and goodness of $F$ directly. Recall in Section 2, we used the notion of lax support to show that

$$\hocolim_{U \in D(M)} F^\ell_c(U) \xrightarrow{\simeq} F^\ell_c(M).$$

Since $F$ satisfies Condition 2, the same proof implies that

$$\hocolim_{U \in D(M)} F_c(U) \xrightarrow{\simeq} F_c(M).$$

Using Remark 4.29 for $U \in D(M)$, we know that $F_c(U) \to F^\ell_c(U)$ is a homology isomorphism. Using the spectral sequence to compute the homology of the homotopy colimits and the bar construction model for the homotopy colimits, it is enough to prove that that natural maps

$$B_\bullet(F_c(-), D(M), *) \to B_\bullet(F^\ell_c(-), D(M), *),$$

is a homology isomorphism which easily follows from $F_c(U) \to F^\ell_c(U)$ being homology isomorphism for all $U \in D(M)$.

5. **Mather-Thurston’s Theory for New Transverse Structures**

In this section, we use Theorem 1.10 to prove Mather-Thurston’s type Theorem 1.13 for foliated bundles with new transverse structures. We shall first explain in more details, what it means for the functors $\text{Fol}_c(M, F)$, $\text{Fol}_c(M, \alpha)$ and $\text{Fol}^{\text{PL}}_c(M)$ to satisfy the c-principle (see Definition 1.12 to recall definitions of foliated bundle with different transverse structures).

- **$\text{Fol}_c(M, F)$**: Let $F$ be a codimension $q$ foliation on $n$-dimensional manifold $M$. Let $\text{Diff}_c(M, F)$ be the subgroup of $\text{Diff}_c(M)$ consisting of compactly supported diffeomorphisms that preserve the leaves of $F$. Let $\text{Diff}^\delta(M, F)$
be the same group with the discrete topology. Let $\text{BDiff}_c(M, F)$ be the homotopy fiber of the natural map between classifying spaces

$$\text{BDiff}_c^\delta(M, F) \to \text{BDiff}_c(M, F).$$

The space $\text{BDiff}_c(M, F)$ is homotopy equivalent to $\text{Fol}_c(M, F)$ as is defined in Definition 1.12. To define the space of formal sections in this case, note that the foliation $F$ on $M$ gives a lifting of the classifying map for the tangent bundle of $M$ to $B\Gamma_q \times B\text{GL}_{n-q}(\mathbb{R})$ where $\Gamma_q$ is the Haefliger groupoid of germs of diffeomorphisms $\mathbb{R}^q$. Now consider the diagram

$$\begin{array}{ccc}
M & \xrightarrow{\tau_F} & B\Gamma_q \times B\text{GL}_{n-q}(\mathbb{R}) \\
\downarrow & & \downarrow \\
\text{BDiff}_c(M, F) \times M & \xrightarrow{\theta} & B\Gamma_{n,q}
\end{array}$$

(5.1)

where $\Gamma_{n,q}$ is the subgroupoid $\Gamma_n$ given by germs of diffeomorphisms of $\mathbb{R}^n$ that preserve the standard codimension $q$ foliation on $\mathbb{R}^n$ (see [LM16, Section 1.1] for more details). Let $B\Gamma_{n,q}$ denote the homotopy fiber of $\theta$. Let $\text{Fol}_c^f(M, F)$ be the space of lifts of $\tau_F$ to $B\Gamma_{n,q}$ up to homotopy.

Since the trivial $M$-bundle $\text{BDiff}_c(M, F) \times M$ is the universal trivial foliated $M$-bundle whose holonomy preserves the leaves of $F$, we have a homotopy commutative diagram

$$\begin{array}{ccc}
\text{BDiff}_c(M, F) \times M & \xrightarrow{\theta} & B\Gamma_{n,q} \\
\downarrow & & \downarrow \\
M & \xrightarrow{\tau_F} & B\Gamma_q \times B\text{GL}_{n-q}(\mathbb{R})
\end{array}$$

The adjoint of the top horizontal map induces the map $F_c(M, F) \to F_c^f(M, F)$.

- **Fol$_c(M, \alpha)$**: Let $(M, \alpha)$ be a contact manifold where $M$ is a manifold of dimension $2n + 1$ and $\alpha$ is a smooth 1-form such that $\alpha \wedge (d\alpha)^n$ is a volume form. The group of $C^\infty$-contactomorphisms consists of $C^\infty$-diffeomorphisms such that $f^*\alpha = \lambda_f \alpha$ where $\lambda_f$ is a non-vanishing smooth function on $M$ depending on $f$. Since we are working with orientation preserving automorphisms, we assume that $\lambda_f$ is a positive function. Let $\text{Cont}_c(M, \alpha)$ denote the group of compactly supported contactomorphisms with induced topology from $C^\infty$-diffeomorphisms. It is known that this group is also locally contractible ([Tsu08]). Let $\text{Cont}_c^\delta(M, \alpha)$ denote the same group with the discrete topology.

The functor $\text{Fol}_c(M, \alpha)$ is homotopy equivalent to $B\text{Cont}_c(M, \alpha)$ which is the homotopy fiber of the natural map

$$B\text{Cont}_c^\delta(M, \alpha) \to B\text{Cont}_c(M, \alpha).$$

The space of formal sections in this case is easier to describe.

---

3The classifying spaces in the diagram 5.1 are defined up to homotopy but if we fix models for them so that $\theta$ is a Serre fibration, $\text{Fol}_c^f(M, F)$ is homotopy equivalent to the space of lifts of $\tau_F$ along $\theta$ in that model.
Let $\Gamma_{2n+1,ct}$ be the etale groupoid whose space of objects is $\mathbb{R}^{2n+1}$ and the space of morphisms is given by the germ of contactomorphisms of $(\mathbb{R}^{2n+1}, \alpha_{st})$ where $\alpha_{st}$ is the standard contact form $dx_0 + \sum_{i=1}^{n} x_i dx_i$.

Note that the subgroup of $GL_{2n+1}(\mathbb{R})$, formed by orientation preserving linear transformations that preserve $\alpha_{st}$ has $U_n$ as a deformation retract.

Hence, the derivative of morphisms in $\Gamma_{2n+1,ct}$ at their sources induces the map

$$\nu: B\Gamma_{2n+1,ct} \to BU_n.$$ 

Let $\tau_M$ be the map $M \to BU_n$ that classifies the tangent for the contact manifold $(M, \alpha)$. The space of formal sections, $\text{Fol}^f_c(M, \alpha)$, is the space of lifts of the map $\tau_M$ to $B\Gamma_{2n+1,ct}$

$$M \xrightarrow{\tau_M} BU_n.$$ 

The universal foliated $M$-bundle $B\text{Cont}_c(M, \alpha) \times M \to B\text{Cont}_c(M, \alpha)$ with the transverse contact structure (i.e. the holonomy maps respect the contact structure of the fibers) induces a classifying map $B\text{Cont}_c(M, \alpha) \times M \to B\Gamma_{2n+1,ct}$. The adjoint of this classifying map induces a map $\text{Fol}^f_c(M, \alpha) \to \text{Fol}^f(M, \alpha)$. Rybicki [Ryb10, Section 11] mentioned that an analogue of Mather-Thurston theorem is not known for smooth contactomorphisms and he continues saying that “it seems likely that such a version could be established, but a possible proof seems to be hard”. We show that the above adjoint map induces a homology isomorphisms in the compactly supported case. The non-compactly supported version remains open. In particular, it is unknown whether the map $B\text{Cont}(\mathbb{R}^{2n+1}, \alpha_{st}) \to B\Gamma_{2n+1,ct}$ induces a homology isomorphism where $B\Gamma_{2n+1,ct}$ is the homotopy fiber of the map $\nu$. The original Thurston’s technique is useful to avoid such subtle local statement to get the compactly supported version.

- **Fol^PL_c(M)**: Let $M$ be an $n$-dimensional manifold that has a PL structure. Let $\text{PL}_c(M)$ be the simplicial group of PL homeomorphisms of $M$. The set of $k$-simplices, $\text{PL}_k(M)$, is the group of PL homeomorphisms $\Delta^k \times M$ that commute with projection to $\Delta^k$. The topological group, $\text{PL}(M)$, of PL homeomorphisms of $M$ is the geometric realization of $\text{PL}_c(M)$. Hence, the 0-simplices of $\text{PL}_c(M)$ is $\text{PL}(M)^0$ which is the group of PL homeomorphisms of $M$ as a discrete group. Therefore, we have a map

$$B\text{PL}(M)^0 \to B\text{PL}(M),$$

whose homotopy fiber is denoted by $B\text{PL}(M)$. This space is homotopy equivalent to $\text{Fol}^PL(M)$. The space of formal sections is defined similar to the contact case. Let $\Gamma^PL_n$ denote the etale groupoid whose space of objects is $\mathbb{R}^n$ and whose space morphisms is given by germs of PL homeomorphisms of $\mathbb{R}^n$. Note that a germ of PL homeomorphism at its sources in $\mathbb{R}^n$ uniquely extends to a PL homeomorphism of $\mathbb{R}^n$. Hence, we obtain a map

$$B\Gamma^PL_n \to B\text{PL}(\mathbb{R}^n).$$
Let $\tau_M : M \to \text{BPL}(\mathbb{R}^n)$ be a map that classifies the tangent microbundle of $M$. The space $\text{Fol}^\text{c,PL}(M)$ is the space of lifts of $\tau_M$ in the diagram

\[
\begin{array}{ccc}
M & \xrightarrow{\nu} & \text{BPL}(\mathbb{R}^n) \\
\downarrow^{\tau_M} & & \\
\text{B}\Gamma_n^{\text{PL}} & \xleftarrow{\text{Fol}^\text{c,PL}(M)}
\end{array}
\]

Similar to the previous cases, the universal foliated $M$ bundle with transverse PL structure induces a map $\text{BPL}(M) \times M \to \text{B}\Gamma_n^{\text{PL}}$ whose adjoint gives the map $\text{Fol}^\text{c,PL}(M) \to \text{Fol}^\text{c,PL}(M)$. We show that this map induces a homology isomorphism which answers a question of Gelfand and Fuks in [GF73, Section 5]. However, the non-compactly supported version even for $M = \mathbb{R}^n$ is not known.

In each case, to show that the corresponding $F$ satisfies c-principle, by Theorem 1.10, we need to show that the fragmentation properties and the goodness conditions (Definition 4.1) for $F$ are satisfied.

It is easy to see that these functors satisfy the first and the third conditions in Definition 4.1. Since the subspace of foliations with empty support is a point, hence contractible. And the third condition is obvious. The second condition is also satisfied for these spaces of foliations because there exists a metric on the space of foliations that makes them complete metric spaces (see [Hir73, Section 2]). Hence, it is easy to see that the base point in these spaces which is the horizontal foliation is a good base point, in particular it is a strong neighborhood deformation retract. Therefore, similar to Lemma 2.2, all these functors satisfy Condition 2 meaning that enlarging the subspace of compactly supported foliations to lax compactly supported (which is an open subspace) does not change the homotopy type. Hence, to prove the goodness for these functors we need to check the last two conditions in Definition 4.1.

The case of the contactomorphisms and PL homeomorphisms are similar and given what we already know about the connectivity of the corresponding Haefliger spaces, as we shall see, we have all the ingredients to check the above conditions. Hence, we prove the c-principle for $\text{Fol}^\text{c}(M, \alpha)$ and $\text{Fol}^\text{c,PL}(M)$ first.

### 5.1. The case of the contactomorphisms and PL homeomorphisms.

Haefliger’s argument in [Hae71, Section 6] implies that $\text{B}\Gamma_n^{\text{PL}}$ is $(n - 1)$-connected. Haefliger showed ([Hae70, Theorem 3]) that Phillips’ submersion theorem in the smooth category implies that $\text{B}\Gamma_n$ is $n$-connected. Given that Phillips’ submersion theorem also holds in the PL category ([HP64]), one could argue exactly similar to the smooth case that $\text{B}\Gamma_n^{\text{PL}}$ is in fact $n$-connected. On the other hand, McDuff in [McD87, Proposition 7.4] also proved that $\text{B}2n+1,ct$ is $(2n + 1)$-connected which is even one degree higher than what we need. Hence, $\text{Fol}^f,\text{PL}(D^n)$ is $n$-connected and $\text{Fol}^f(D^{2n+1}, \alpha)$ is $(2n + 1)$-connected. Therefore, the space of formal sections satisfy the fragmentation property. To prove the fragmentation property for $\text{Fol}^f,\text{PL}(D^n)$ and $\text{Fol}(D^{2n+1}, \alpha)$, we shall use the following lemma.

---

4Epstein ([Eps77, Section 6] showed that in the case of smooth foliations the topology induced by such metric is the same as the subspace topology of space of plane fields.
Lemma 5.2 (McDuff). Let $G(M)$ be a topological group of compactly supported automorphisms of $M$ with a transverse geometric structure (e.g. PL homeomorphisms, contactomorphisms, volume preserving diffeomorphisms and foliation preserving diffeomorphisms). We assume that

- $G(M)$ with its given topology is locally contractible.
- For every isotopy $h_t$ as a path in $G(M)$ and every open cover $\{B_i\}_{i=1}^k$ of $M$, we can write $h_t = h_{t,1} \circ \cdots \circ h_{t,k}$ where each $h_{t,i}$ is an isotopy supported in $B_i$.

Let $F_c(M)$ be the realization of the semisimplicial set whose $p$ simplices are the set of foliations on $\Delta^p \times M$ transverse to fibers of the projection $\Delta^p \times M \to \Delta^p$ and whose holonomies lie in $G(M)$. Then the functor $F_c(M)$ which is homotopy equivalent to $BG(M)$ has the fragmentation property in the sense of Definition 1.8.

Proof. See section 4 of [McD83a] and the discussion in the subsection 4.15. □

The PL homeomorphism groups are known to be locally contractible ([Gau76]) and as is proved by Hudson ([Hud69, Theorem 6.2]) they also satisfy an appropriate isotopy extension theorem which gives the second condition in Lemma 5.2. Therefore, $\text{Fol}_c^{\text{PL}}(M)$ satisfies the fragmentation property. On the other hand, the group of contactomorphisms is also locally contractible ([Tsu08, Section 3]) and it satisfies the second condition ([Ryb10, Lemma 5.2]). Hence, $\text{Fol}_c(M, \alpha)$ also satisfies the fragmentation property in the sense of Definition 1.8.

Now we are left to show that $\text{Fol}_c(\cdot, \alpha)$ and $\text{Fol}_c^{\text{PL}}(\cdot)$ are good functors in the sense of Definition 4.1.

Lemma 5.3. $\text{Fol}_c(\cdot, \alpha)$ and $\text{Fol}_c^{\text{PL}}(\cdot)$ are good functors.

Proof. Recall that we need to check the last two conditions in Definition 4.1. We focus on $\text{Fol}_c(\cdot, \alpha)$ and we mention where $\text{Fol}_c^{\text{PL}}(\cdot)$ is different.

We may assume that $U$ and $V$ are balls $B(r)$ and $B(R)$ of radii $r < R$ in $\mathbb{R}^{2n+1}$. And we want to show that the induced map

$$\nu : \text{BCont}_c(B(r), \alpha_{st}) \to \text{BCont}_c(B(R), \alpha_{st}),$$

is a homology isomorphism.

Note that for any topological group $G$, the homology of $BG$ can be computed by subchain complex $S_*(BG)$ of singular chains $\text{Sing}_*(G)$ of the group $G$ given by smooth chains $\Delta^* \to G$ that send the first vertex to the identity (see section 1.4 of [Hal98] for more detail). Given a chain $c$ in $S_*(\text{BCont}_c(B(R), \alpha_{st}))$, to find a chain homotopy to a chain in $\text{BCont}_c(B(r), \alpha_{st})$, we need an easy lemma ([Hal98, Lemma 1.4.8]) which says that for every contactomorphism $h \in \text{Cont}_c,0(B(R), \alpha_{st})$ that is isotopic to the identity, the conjugation by $h$ induces a self-map of $\text{BCont}_c(B(R), \alpha_{st})$ which is the identity on homology. Hence, it is enough to show that there exists $h$ a contactomorphism, isotopic to the identity, that shrinks the support of the given chain to lie inside $B(r)$.

To find such compactly supported contraction, consider the following family of contactomorphisms $\rho_t : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$

$$\rho(x_0, x_1, \cdots, x_{2n+1}) = (t^2 x_0, t x_1, \cdots, t x_{2n+1}).$$

For $t < 1$, it is a contracting contactomorphism but it is not compactly supported. To cut it off, we use the fact that the family $\rho_t$ is generated by a vector field $\rho_1$. 

Let $\lambda$ be a bump function that is positive on the support of the chain $c$ and zero near the boundary of $B(R)$. One wants to consider the flow of the vector field $\dot{\rho}_{\lambda,t}$ to cut off $\rho_1$ but $\dot{\rho}_{\lambda,t}$ may not be a contact vector field. However, there is a retraction $\pi$ from the Lie algebra of smooth vector fields to contact vector fields on every contact manifold ([Ban97, Section 1.4]). Briefly, the reason that this retraction exists is that there is an isomorphism (see [Ban97, Proposition 1.3.11]) between contact vector fields on a contact manifold $(M, \alpha)$ and smooth functions by sending a contact vector field $\xi$ to $\iota_\xi(\alpha)$, the contraction of $\alpha$ by $\xi$. The retraction $\pi$ is defined by sending a smooth vector field to the contact vector field associated with the function $\iota_\xi(\alpha)$. Therefore, the flow of $\pi(\dot{\rho}_{\lambda,t})$ gives a family of compactly supported contactomorphisms of $B(R)$ that shrinks the support of the chain $c$.

Hence, by conjugating by such contactomorphisms that are isotopic to the identity, we conclude that $\iota$ induces a homology isomorphism. The case of the PL foliations is much easier, because the existence of such contracting PL homeomorphisms that are isotopic to the identity is obvious.

To check the last condition, we want to show that any chain $c$ in $S_\bullet(B\text{Cont}(D^{2n+1}, \partial_1))$ is chain homotopic to the identity. Recall that the chain complex $S_\bullet(B\text{Cont}(D^{2n+1}, \partial_1))$ is generated by the set of smooth maps from $\Delta^\bullet \times [0,1]$ to $\text{Cont}(D^{2n+1}, \partial_1)$ that send the first vertex to the identity contactomorphism. Note that this set is in bijection with the set of foliations on the total space of the projection $\Delta^\bullet \times M \to \Delta^\bullet$ that are transverse to the fibers $M$ and whose holonomies lie in $\text{Cont}(D^{2n+1}, \partial_1)$. Thus, it is enough to show that each of these generators are chain homotopically trivial. Geometrically, this means that for each such foliation $c$ on $\Delta^\bullet \times M$, there is a foliation on $\Delta^\bullet \times [0,1] \times M$ transverse to the projection to the first two factors (i.e. it is a concordance) such that on $\Delta^\bullet \times \{0\} \times M$, it is given by $c$ and on $\Delta^\bullet \times \{1\} \times M$, it is the horizontal foliation. The idea is to “push” the support of the foliation $c$ towards the free boundary until the foliation becomes completely horizontal.

To do so, consider a small neighborhood $U$ of $\partial_1$ in $D^{2n+1}$ that is in the complement of the support of the foliation $c$. Note that as in the previous case, there is a contact contraction that maps $D^{2n+1}$ to $U$ and is isotopic to the identity. Let us denote this contact isotopy by $h_t$ such that $h_0 = \text{id}$. And let $F: \Delta^\bullet \times [0,1] \times D^{2n+1} \to \Delta^\bullet \times D^{2n+1}$ be the map that sends $(s, t, x)$ to $(s, h_t(x))$ and $F_i$ be the map $F$ at time $t$. Since $h_t$ is a contact isotopy, for each $t$, the pullback foliation $F^*\{c\}$ on $\Delta^\bullet \times D^{2n+1}$ also gives an element in $S_\bullet(B\text{Cont}(D^{2n+1}, \partial_1))$. Therefore, the pullback foliation $F^*\{c\}$ on $\Delta^\bullet \times [0,1] \times D^{2n+1}$ is a concordance from $c$ to the horizontal foliation which means that $c$ is chain homotopic to the identity in the chain complex $S_\bullet(B\text{Cont}(D^{2n+1}, \partial_1))$.

Note that we only used that for each foliation $c$ on $\Delta^\bullet \times D^{2n+1}$, there exists a neighborhood $U$ away from the support of $c$ and there is a contact embedding $h$ that maps $D^{2n+1}$ into $U$ which is also isotopic to the identity. Such embeddings isotopic to the identity also exist in the PL case. Therefore, $\text{Fol}_c(-, \alpha)$ and $\text{Fol}^{PL}_c(-)$ also satisfies the 5th condition.

As an application of this theorem, we could improve the connectivity of $B\text{Cont}_{2n+1, ct}$.

**Corollary 5.4.** The classifying space $B\text{Cont}_{2n+1, ct}$ is at least $(2n + 2)$-connected.

**Proof.** We already know by McDuff’s theorem ([McD87, Proposition 7.4]) that $B\text{Cont}_{2n+1, ct}$ is $(2n+1)$-connected. To improve the connectivity by one, note that Rybicki proved ([Ryb10]) that $B\text{Cont}_c(\mathbb{R}^{2n+1})$ has a perfect fundamental group. Therefore, its first
homology vanishes. On the other hand, by Theorem 1.13, the space \( B\text{Cont}_c(\mathbb{R}^{2n+1}) \) is homology isomorphic to \( \Omega^{2n+1}_c B\text{GL}_{2n+1,ct} \). Hence, we have

\[
0 = H_1(\Omega^{2n+1}_c B\text{GL}_{2n+1,ct}; \mathbb{Z}) = \pi_1(\Omega^{2n+1}_c B\text{GL}_{2n+1,ct}) = \pi_{2n+2}(B\text{GL}_{2n+1,ct}),
\]

which shows that \( B\text{GL}_{2n+1,ct} \) is \((2n+2)\)-connected. \( \square \)

5.2. **The case of foliation preserving diffeomorphisms.** First, let see why \( \text{Fol}_c(M, \mathcal{F}) \) and \( \text{Fol}_c^q(M, \mathcal{F}) \) satisfy the fragmentation property in the sense of Definition 1.8.

Recall \( \text{Fol}_c(M, \mathcal{F}) \) is homotopy equivalent to \( B\text{Diff}_c(M, \mathcal{F}) \). Thus, to show that \( B\text{Diff}_c(M, \mathcal{F}) \) has fragmentation property, it is enough to check if \( \text{Diff}_{c,0}(M, \mathcal{F}) \) satisfies the two conditions in Lemma 5.2. By [Tsu06, Section 2], every element of the group \( \text{Diff}_{c,0}(M, \mathcal{F}) \) can be fragmented. Hence, this group satisfies the second condition in Lemma 5.2. On the other hand, it is a consequence of a theorem of Omori ([MNR18, Proposition 2.7]) that \( \text{Diff}_{c,0}(M, \mathcal{F}) \) is a Fréchet Lie group. Hence, it is locally contractible. It is worth noting, that if we consider a larger group of foliation preserving diffeomorphisms that do not necessarily preserve each leaf, then we might have a group that is not locally contractible (see [MNR18, Theorem 2.9]). Therefore, \( \text{Fol}_c(M, \mathcal{F}) \) has the fragmentation property.

Since \( \text{Fol}_c^q(M, \mathcal{F}) \) is homotopy equivalent to the space of sections of a fiber bundle over \( M \) whose fiber is \( B\Gamma_{n,q} \), it satisfies fragmentation property by Theorem 1.1, if the fiber \( B\Gamma_{n,q} \) is at least \((n-1)\)-connected. Sithanantham mentioned in [S’84, Remark at page 493] that he proved in his thesis that \( B\Gamma_{n,q} \) is \((n-1)\)-connected. Here, we give a proof that we learned from Gael Meigniez using Thurston’s jiggling and civilization techniques ([Thu74b, section 5 and 6]).

**Theorem 5.5.** \( B\Gamma_{n,q} \) is \( n \)-connected.

**Proof.** Equivalently, we show that the map \( \theta : B\Gamma_{n,q} \to B\Gamma_q \times B\text{GL}_{n-q}(\mathbb{R}) \) is \((n+1)\)-connected by solving the following lifting problem

\[
\begin{array}{ccc}
P & \xrightarrow{\theta} & B\Gamma_{n,q} \\
\downarrow & & \downarrow \\
Q & \xrightarrow{\theta} & B\Gamma_q \times B\text{GL}_{n-q}(\mathbb{R}),
\end{array}
\]

where \( Q \) is a manifold of dimension \( m \) where \( m \leq n+1 \) and \( P \) is its submanifold. Similar to [Thu74b, Section 1], we interpret the data of a map to the classifying space of a Haefliger groupoid as a foliated microbundle. So the above lifting problem can be geometrically set up as follows.

The bottom map gives a vector bundle \( V \to Q \) of dimension \( n-q \) and a codimension \( q \) Haefliger structure \( \xi \). The Haefliger structure \( \xi \) is a germ of a foliation \( \mathcal{F}_\xi \) in a neighborhood of the zero section of a \( q \)-dimensional vector bundle \( \nu_\xi \to Q \), called the normal bundle of the Haefliger structure, that is transverse to the \( \mathbb{R}^q \) fibers. Now let \( \mathcal{F} \) be the pullback of \( \mathcal{F}_\xi \) via the projection \( V \oplus \nu_\xi \to \nu_\xi \). This gives a codimension \( q \) foliation on \( V \oplus \nu_\xi \) that is transverse to the fibers and contains the \( \mathbb{R}^{n-q} \) factor of the fibers. This factor induces a \((n-q)\)-dimensional vector bundle on each leaf of \( \mathcal{F} \). To find a lift in the above diagram amounts to finding a flag of foliations
are satisfied. We need to modify the last two conditions such that every simplex of $s(M)$ intersects $H$ in terms of cubes. To do so, let $\sigma$ be a simplex of $s(M)$ that is $C^0$-close to the zero section of the bundle $H$. Hence, if $m \leq n + 1$, the intersection of $H$ with $\sigma \cap \mathcal{L}$ is either 0 or 1 dimensional which is integrable. Therefore, after a $C^0$-small perturbation of $H$, for each leaf $\mathcal{L}$ of $\mathcal{F}$, the plane field $H$ is integrable in a neighborhood of $\sigma \cap \mathcal{L}$ in $\mathcal{L}$ which is what we wanted. \hfill \Box

Now that we know $\text{Fol}_c(M, \mathcal{F})$ and $\text{Fol}_{c}^l(M, \mathcal{F})$ satisfy the fragmentation property, to prove that $\text{Fol}_c(M, \mathcal{F})$ satisfies a $c$-principle, we are left to check if it is a good functor. Note similar to the PL case and the contact case the first three conditions in Definition 4.1 are satisfied. We need to modify the last two conditions of goodness in this case. Let $I^n_r$ denote an $n$-dimensional cube in $\mathbb{R}^n$ whose edges have length $r$. It is easy to see that in the proof of Theorem 1.10, in the case of $D^n$ relative to its boundary (see Section 4.1), we can replace balls of radius $\epsilon$ with cubes $I^n_\epsilon$. Therefore, it is enough to reformulate the last two conditions of Definition 4.1 in terms of cubes. To do so, let $\mathcal{H}_q$ be the foliation of $\mathbb{R}^n$ by horizontal $\mathbb{R}^{n-2} \times \{\ast\}$ planes. We shall denote the induced foliation on any cube $I^n_\epsilon$ by the same notation.

Theorem 5.6. $\text{Fol}_c(M, \mathcal{F})$ satisfies the following reformulation of the last two conditions of goodness:

- For every embeddings of cubes $I^n_\alpha \hookrightarrow I^n_\epsilon$ a cube into the interior of a larger cube, the induced map $\text{BDiff}_c(I^n_\alpha, \mathcal{H}_q) \to \text{BDiff}_c(I^n_\epsilon, \mathcal{H}_q)$ induces a homology isomorphism.
- Suppose $I^n_\alpha$ intersects the boundary of $I^n_\epsilon$ nontrivially. Let $\partial_1$ denote the part of the boundary of $I^n_\alpha$ that lies in $I^n_\epsilon$. The space $\text{BDiff}(I^n_\alpha, \mathcal{H}_q, \text{rel } \partial_1)$

\(^5\)Thurston gives the argument for one plane field but exactly the same argument works for a pair of plane fields.
is acyclic where $\text{Diff}(I^n_\partial, \mathcal{H}_q, \text{rel } \partial_1)$ is the group of leaf preserving diffeomorphisms of $I^n_\partial$ that are the identity near the part $\partial_1$.

**Proof.** Let us first prove that $\text{BDiff}(I^n_\partial, \mathcal{H}_q, \text{rel } \partial_1)$ is acyclic. Recall from the proof of Lemma 5.3, every generator of the abelian group $S_\bullet(\text{BDiff}(I^n_\partial, \mathcal{H}_q, \text{rel } \partial_1))$ can be represented by a foliation on the total space of the projection $\Delta^\bullet \times I^n_\partial \to \Delta^\bullet$ that are transverse to fibers and whose holonomies lie in $\text{Diff}(I^n_\partial, \mathcal{H}_q, \text{rel } \partial_1)$. So it is enough to prove that each generator $c$ is chain homotopically trivial which geometrically means that the foliation associated to $c$ is concordant to the horizontal foliation on $\Delta^\bullet \times I^n_\partial$. The idea is similar to Lemma 5.3. Let $U$ be a neighborhood of $\partial_1$ that is in the complement of the support of the foliation $c$. There exists an isotopy $h_t$ of $I^n_\partial$ of leaf preserving diffeomorphisms of the foliation $\mathcal{H}_q$ such that $h_1(I^n_\partial) \subset U$.

Note that here it is important to work with cubes, since if we consider round disks in $\mathbb{R}^n$ with horizontal foliation $\mathcal{H}_q$, such leaf preserving diffeomorphisms may not exist. Now consider the map

$$H : \Delta^\bullet \times [0, 1] \times I^n_\partial \to \Delta^\bullet \times I^n_\partial,$$

where it sends $(s, t, x)$ to $(s, h_t(x))$. Since the image of $h_1$ is away from the support of the foliation $c$, the pullback foliation $h_t^\ast(c)$ is the horizontal foliation. Therefore, the foliation $H^\ast(c)$ on $\Delta^\bullet \times [0, 1] \times I^n_\partial$ gives a concordance from $c$ to the horizontal foliation.

To prove the first property, we need to do several steps. For simplicity, assume $I^n_\partial = [-1, 1]^n$ and $I^n_\partial = [-2, 2]^n$. First, let us observe that the map

$$\text{BDiff}_c((-1, 1)^{n-q} \times (-2, 2)^q, \mathcal{H}_q) \to \text{BDiff}_c((-2, 2)^n, \mathcal{H}_q),$$

induces a homology isomorphism.

Recall from the proof of Lemma 5.3, conjugation by a diffeomorphism in $\text{Diff}_c((-2, 2)^n, \mathcal{H}_q)$ that is isotopic to the identity, induces the identity on homology of $\text{BDiff}_c((-2, 2)^n, \mathcal{H}_q)$. Hence, it is enough to show that for any chain $c$ in $S_\bullet(\text{BDiff}_c((-2, 2)^n, \mathcal{H}_q))$, we can find such a diffeomorphism that shrinks the support of $c$ to lie inside $(-1, 1)^{n-q} \times (-2, 2)^q$.

To find such a leaf preserving diffeomorphism, note that for a given chain $c$, we can find $\epsilon > 0$ such that the support of $c$ lies in the cube $(-2 + \epsilon, 2 - \epsilon)^n$. Note that there exists a diffeomorphism of $(-2, 2)$ that is isotopic to the identity and maps $(-2 + \epsilon, 2 - \epsilon)$ to $(-1, 1)$ and is the identity in the complement $(-2 + \epsilon/2, 2 - \epsilon/2)$. Hence, using the same diffeomorphism on the first $(n-q)$ coordinates gives a leaf preserving diffeomorphism of $(-2, 2)^n$ that sends the support of $c$ to $(-1, 1)^{n-q} \times (-2, 2)^q$. Therefore, the map (5.7) induces a homology isomorphism. Since the map from $\text{BDiff}_c((-1, 1)^n, \mathcal{H}_q)$ to $\text{BDiff}_c((-2, 2)^n, \mathcal{H}_q)$ factors through the space $\text{BDiff}_c((-1, 1)^{n-q} \times (-2, 2)^q, \mathcal{H}_q)$, it is enough to show that the map

$$\text{BDiff}_c((-1, 1)^n, \mathcal{H}_q) \to \text{BDiff}_c((-1, 1)^{n-q} \times (-2, 2)^q, \mathcal{H}_q),$$

induces a homology isomorphism.

Note that the group $\text{Diff}_c((-1, 1)^n, \mathcal{H}_q)$ can be identified with the space of compactly supported smooth maps $\text{Map}_c((-1, 1)^q, \text{Diff}_c((-1, 1)^{n-q}))$ and similarly $\text{Diff}_c((-1, 1)^{n-q} \times (-2, 2)^q, \mathcal{H}_q)$ is homeomorphic to the space compactly supported smooth maps $\text{Map}_c((-2, 2)^q, \text{Diff}_c((-1, 1)^{n-q}))$. The group structure is induced by the pointwise multiplication on the target $\text{Diff}_c((-1, 1)^{n-q})$. Hence, it is enough to prove the following claim.
Claim 5.9. Let $G$ be a topological group and $U \to V$ be an embedding of two open disks in $\mathbb{R}^2$. Then the map

$$\nu:\operatorname{BMap}_c(U, G) \to \operatorname{BMap}_c(V, G),$$

induces a homology isomorphism.

To prove the claim, we shall define a map

$$\eta:\operatorname{BMap}_c(V, G) \to \operatorname{BMap}_c(U, G)$$

such that both $\eta \circ \iota$ and $\iota \circ \eta$ induce homology isomorphisms. Let $e$ be a self-embedding of $V$ whose image is $U$ and is isotopic to the identity. Precomposing with $e$ gives a map $\operatorname{BMap}_c(V, G) \to \operatorname{BMap}_c(U, G)$ and since the group structure on these mapping spaces are given by pointwise multiplication, this map is also a group homomorphism. Therefore, precomposing with $e$ induces the map $\eta$. We prove that the composition

$$\operatorname{BMap}_c(U, G) \xrightarrow{\iota} \operatorname{BMap}_c(V, G) \xrightarrow{\eta} \operatorname{BMap}_c(U, G),$$

induces a homology isomorphism. The same argument works for the map $\iota \circ \eta$. To prove surjectivity and injectivity on homology, it is enough to show that any chain $c$ in $S_\ast\left(\operatorname{BMap}_c(U, G)\right)$ is homotopic to a chain in the image of $\eta \circ \iota$.

It turns out to be easier to work with cubical complexes to represent chains in $S_\ast\left(\operatorname{BMap}_c(U, G)\right)$ (see [McD83a, Section 2] and [Ban97, Remark 2.1.14] for details on cubical cycles in $BG$). Let $K$ be a cubical complex with ordered vertices. For any topological group $G$, a compatible map $f : K \to BG$ is a set of maps $f_\kappa : \kappa \to G$ where $\kappa \subset K$ is a cube and $\nu_\kappa$ is the first vertex of $\kappa$ such that $f_\kappa(\nu_\kappa) = \text{id}$ and for nested cubes $\kappa \subset \theta$ we have the compatibility condition

$$f_\kappa(x) = f_\theta(x) \cdot f_\theta^{-1}(\nu_\theta),$$

for every $x \in \kappa$. Every chain in $S_\ast\left(BG\right)$ can be represented by such cubical chains.

Note that for every cubical chain $c$ given by a map $g : I^\ast \to \operatorname{Map}_c(U, G)$, we can define $\operatorname{supp}(g) \subset U$ to be the set

$$\{x \in U | g(s)(x) \neq \text{id} \text{ for some } s \in I^\ast\}.$$

Hence, it is enough to show that every $g : I^\ast \to \operatorname{Map}_c(U, G)$ is chain homotopic to a cubical chain $g'$ whose support lies in $e(U)$. Recall that $e$ was the self embedding of $V$ that sends $V$ to $U$ and is isotopic to the identity. Now let $W$ be an open disk containing the support of $g$. There is an isotopy $f_t \in \operatorname{Diff}_c(U)$ such that $f_0 = \text{id}$ and $f_1(e(U)) = W$. Let $f_t(g) : I^\ast \to \operatorname{Map}_c(U, G)$ be the cubical chain given by $g(s)(f_t(x))$ where $s \in I^\ast$ and $x \in U$. The isotopy $f_t$ is chosen so that the support of $f_1(g)$ lies in $e(U)$. Now we associate to every cubical chain $g$ the cubical chain $H(g) : I^\ast \times I \to \operatorname{Map}_c(U, G)$ where $H(s, t) = f_t(g(s))$. It is easy to see that $\partial H(g) + H(\partial g) = g - f_1(g)$. Therefore, the chain $g$ is homotopic to a chain whose support lies in $e(U)$. Hence, this claim finishes the proof of the second part of theorem.

Similar to Corollary 5.4, the perfectness result of Tsuboi ([Tsu06]) improves the connectivity range.

Corollary 5.10. $\mathbb{B}T_{n,q}$ is $(n + 1)$-connected.
6. Further Discussion

It would be interesting to see if the space of smooth functions on $M$ not having certain singularities satisfies the fragmentation property. In particular, it would give a different proof of Vassiliev’s c-principle theorem ([Vas92, Section 3]) using the fragmentation method. Let $S$ be a closed semi-algebraic subset of the jet space $J^n(R^n;R)$ of codimension $n+2$ which is invariant under the lift of Diff$(R^n)$ to the jet space. We denote the space of functions over $M$ avoiding the singularity set $S$ by $F(M,S)$. It is easy to check that $F$ is a good functor satisfying the conditions 4.1. To prove that $F$ satisfies c-principle, we need to check whether the functors $F$ and $F^f$ satisfy the fragmentation property. It seems plausible to the author, as we shall explain, that using appropriate transversality argument for stratified manifolds ought to prove fragmentation property for $F(M)$ but since we still do not know if fragmenting the space of functions $F(M)$ is independently interesting, we do not pursue it further in this paper.

Recall, to check that the space of formal sections $F^f$ has fragmentation property, we have to show that $F(R^n,S)$ is at least $(n-1)$-connected. But it is easy to see that $F(R^n,S)$ is homotopy equivalent $J^n(R^n,\mathbb{R})\backslash S$ (see [Kup19, Lemma 5.13]) and this space by Thom’s jet transversality is at least even $n$-connected. Therefore, the space of formal sections $F^f$ has the fragmentation property.

It is still not clear to the author how to check whether $F$ has the fragmentation property but here is an idea inspired by the fragmentation property for foliations. We want to solve the following lifting problem up to homotopy.

\[
\begin{array}{ccc}
\mathcal{P} & \longrightarrow & F_\epsilon(M,S) \\
\downarrow & & \downarrow \\
\mathcal{Q} & \longrightarrow & F(M,S),
\end{array}
\]

where $\mathcal{Q}$ is a simplicial complex and $\mathcal{P}$ is a subcomplex. Let $\sigma \in \mathcal{Q}$ be a simplex. We can think of the restriction of $g$ to each $\sigma$ by adjointness as a map $g: M \times \sigma \rightarrow \mathbb{R}$. In Section 3.1, we defined the fragmentation homotopy $H_1: M \times \sigma \rightarrow M \times \sigma$ after fixing a partition of unity $\{\mu_i\}_i \subseteq \mathbb{R}$. We have a flexibility to choose this partition of unity. Note that for each point $t \in \sigma$, the space $H_1(M \times \{t\})$ is diffeomorphic to $M$ (see Section 3.1). So the restriction of the map $g$ to this space gives a smooth function on $M$. By jet transversality, we can choose the triangulation of $\mathcal{Q}$ fine enough so that each simplex $\sigma$ and each point $t \in \sigma$, the restriction of $g$ to $H_1(M \times \{t\})$ avoids the singularity type $S$.

Let $f_0$ be a function in $F(M,S)$ that we fix as a base section to define the support of other functions with respect to $f_0$. Similar to the proof of Theorem 1.1, consider the subcomplex $L(\sigma)$ which is an $n$-dimensional subcomplex of $M \times \sigma$ which is the union of finitely many manifolds $L_i$ that are canonically diffeomorphic to $M$. In fact $L(\sigma)$ is a union of the graphs of finitely many functions $M \rightarrow \sigma$ inside $M \times \sigma$. It is easy to choose the partition of unity so that $L(\sigma)$ is a stratified manifold. The goal is to find a homotopy of family of functions in $F(M,S)$ denoted by $g_i: M \times \sigma \rightarrow \mathbb{R}$ so that $g_0 = g$, and $g_i$ restricted to $H_1(M \times \{t\})$ for each $t \in \sigma$ is in $F(M,S)$, and most importantly the restriction of $g_i$ to each $L_i$ is given by the
base function $f_0$. If we can find such homotopy then the rest of the proof, is similar to proving fragmentation property for space of sections in Theorem 1.1.

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