The process of student’s thinking deduction level to solve the problem of geometry

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Abstract. Each thinking level van Hiele will describe the process of thinking a people use learning the geometric concepts and types of geometric ideas thinks. This research is a qualitative descriptive research what aims to describe the thinking process of students at the level of deduction in solving geometry problems. The collection data’s method in this research is test and interview method. The test is classification of students' thinking geometry level according to van Hiele using instruments developed by Usiskin, so that the students get the level of deduction. The test of geometry to knowing the thinking process students use 1 problem from the International Mathematical Olympiad (IMO) field Geometry. Interviews were conducted to the students to confirm the student's work. The results of this study indicated that students at the level of deduction to solving geometry problems as accordance with the characteristics of van Hiele geometry thinking in its level.

1. Introduction

Geometry is a branch of mathematics which describes the forms, lines, fields, spaces, relationships between shapes, as well as the size of geometric shapes, such as length, angle, area, and volume[1–3]. Geometry is one of the topics in mathematics considered difficult by most students [4,5]. Some research results posit that students' understanding of geometry topics is still unsatisfactory. The results of the study on second-grade students of Public Junior High School 4 of Jember stated that 83.3% of students made mistakes in solving problems alternate exterior angle, 52.37% in the supplement angles, 40.5% adjacent exterior angles, 36.95% adjacent interior angles, and 33.62% inalate interior angle[6,7]. Geometry may be difficult for students to understand because there are many subjects in interconnected geometry. The difficulties in certain geometric materials will result in difficulties in other areas. Geometry is built upon abstract structures that do not directly address students' lives. This will cause learning difficulties for students. To minimize this difficulty, geometry lessons in primary and secondary education must be presented according to the level of students' understanding [8].

Research conducted to understand geometry is usually built based on van Hiele's level of geometrical understanding. This level accounts for students' understanding of geometrical material. Transitions or 'intermediate positions' from one level to another (between visual level and analysis and so on) are thought to depend on the quality of education, subject matter, and educational approaches that encourage students to perform inquiry and think critically as well as creatively. Van Hiele's theory states that students will go through every thinking level in understanding geometry, which means that the previous level is definitely passed if someone is at one particular higher level. Each level will describe the thinking process that a person employs in learning geometry. The thinking level based on van Hiele's theory includes level 0 (visualization), level 1 (analysis), level 2 (informal deduction), level 3 (deduction), and level 4 (rigor)[8–13]. There have been numerous studies on students 'thinking level using van Hiele's theory to measure students' knowledge of geometry problems. Most school
students are still at the first three levels. This is because students have not been able to link geometrical concepts in their minds [7,9,13–16].

Van Hiele geometry level thinking descriptors have been developed from level 0 (visualization) to level 4 (rigor). At level 0 (visualization), students identify and operate various figures (e.g. square, triangle) and other geometric configurations (for example lines and angles) according to the visual appearance. At level 1 (analysis), students analyse the figures based on their components and the relationships between these components, determine the properties of the figure class empirically, and use the properties to solve problems. At the level 2, informal deduction, students formulate and use definitions, provide informal arguments and formulate properties previously given in sequence, and follow and propose deductive arguments. At level (3), student’s deduction determines a system of axioms, theorems, and relationships between network theorems. At level 4, rigor, students strictly build theorems in different axiom systems and analyse or compare these systems [7,17].

The results of another study involving 458 students from 12 elementary schools in Jember City, showed that the percentage of respondents at the level of visualization, analysis, deduction and rigor were 70.09%; 28.38%; 1.75%; 0%; and 0%, respectively. In addition, 8.73% students were at the transition level and 16.16% of the students could not be clarified into any level[18]. In the same vein, another study on 576 students from 13 public junior high school in Jember showed that the percentage of respondents at the level of visualization, analysis, informal deduction, deduction, and accuracy were 44.62%; 34.55%; 6.77%; 0.17%; and 0% respectively. 14.40% of these students could not be clarified into any of these levels[6]. This means that from elementary school to junior high school level students only touch upon the level of informal deduction and only a few students are able to excel to reach the next two levels above. Owing to this issue, information is called forth to make clear how the best students (Olympiad competing students) perform their problem solving process in solving geometry problems.

Resolving a geometry problem requires particular thinking process, so students’ thinking process will be different at each level according to their learning experience in geometry. Based on the results of previous studies, most students were still at the first three levels of van Hiele theory and many students had not been able to reach the level of deduction. As such, the present study deems necessary research on the students' thinking process at the level of deduction in solving geometry problems.

2. Methods
This research was descriptive in nature with qualitative approach, projected to describe students’ thinking processes at the level of deduction in solving geometry problems. Subjects in this study were 2 students, selected from two classes, the first and second grade of senior high school. The students met the criteria corresponding to deduction level, based on van Hiele's theory. The first subject was encoded as S1, who was a student of the second grade, and the other one was coded as a S2 who was a student of the first grade.

Data collection involved test and interview. The test was divided into two parts, with the first test aimed to determine van Hiele level [11] and the second test projected to explore the thinking processes of S1 and S2. One problem from the International Mathematical Olympiad (IMO) in the field of geometry was operative in both tests. Interviews were used to explore the information on students’ thinking processes or to reveal whatever students had done when solving problems given. The results of student answers and interview results were analyzed based on van Hiele's descriptors at the level of deduction. Subsequently, the researchers drew conclusion based on the analysis results.

3. Result and Discussion
S1 and S2 were at the same thinking level, deduction level. Both were in different classes (S1 was a second-grade student and S2 was first-grade student). As both of them were at the same level, namely deduction, the class difference had no effect on the study. Increasing students' thinking level depends more on their experience and understanding in learning geometry. This is in line with the characteristics of van Hiele's theory[10,19,20]. Thinking level in van Hiele’s theory does not depend on age, but geometric experience, which is the most determinant factor. In other words, regardless of
how old one is, when he lacks the learning experience in geometry, it is very likely that he belongs to lower level.

The geometry problem given to the students was related to the proof of tangent circles, that was "Suppose R and S are two different points in circle O, so RS is not diameter. Suppose the line ℓ is adjacent with O in R. Given point T so S is the midpoint of the RT segment. Point J is selected in RS arc which is shorter at O, so the outer circle Q of the triangle JST intersects ℓ at two different points. Suppose A is intersection of Q and ℓ is closer to R. The AJ line intersects O again at K. Prove that KT line is adjacent with Q".

Based on the results of data analysis, it was clear that when solving geometry problems S1 and S2 deployed a series of thinking processes in accordance with the level deduction descriptors. The analysis also demonstrated that the thinking process of S1 and S2 was similar. Hereunder are the results of proof-checking done by S1 and S2.

![Diagram](attachment:geometry_problem.png)

**Figure 1.** The work of subject one

The excerpt of interview with S1 is presented in the Table 1.

| P08 | Coba jelaskan penyelesaian Anda |
|-----|---------------------------------|
| S108 | Penyelesaiannya itu yang pertama ∠ORA inikan 90° karena yang tertera pada soal bahwa diluar garis O akan ada suatu garis ℓ yang menyinggung O. Seperti yang kita tahu bahwa ketika ada suatu garis menyinggung lingkaran maka akan tegak lurus dengan jari-jarinya oleh karena itu karena RA adalah suatu segmen dari garis ℓ maka ∠ORA pada penjelasan ini besarnya 90°. Begitu juga misalnya ada AS besarnya 90° mengingat S juga salah satu bagian dari lingkaran. Dari step-step tersebut itu akan membentuk beberapa segitiga dari kedua lingkaran dan dapat diketahui juga bahwa ternyata titik J ataupun titik S merupakan titik potong kedua lingkaran jadi sebenarnya masalah ini disesuaikan dengan hubungan antar sudut-sudut yang diawali dengan permisalan salah satu sudut. Saya disini memisalkan ∠ORS sebagai sudut x. Dari hubungan tersebut sebenarnya kalau kita ingin mengkaji lebih lanjut dapat diketahui besar setiap sudut seperti besar sudut yang ini adalah 90° + x. Derajat dan sebagainya hingga diketahui pada akhirnya nanti pada ∠QTK akan membentuk 90°.
Sebenarnya cara yang saya gunakan ini belum selesai, apabila dilanjutkan kemungkinan terbesar dari analisis yang saya lakukan adalah memang akan terjadi suatu titik singgung di T dari garis TK yang membuktikan bahwa garis tersebut akan menyinggung lingkaran. Seperti yang saya katakan apabila dikaji lebih lanjut mungkin akan terjadi pembuktian yang valid. Tapi menurut saya punya saya ini kurang valid karena memang...
What follows is the result of S1’s work regarding the employment of basic elements, postulate, definition, and theorem.

Figure 2. The Use of Basic elements, Postulate, Definition, and Theorems.

Table 2. Interview Excerpt of S2 Concerning The Use of Basic elements, Postulate, Definition, and Theorem

| P14  | Dalam belajar geometri, apakah Anda pernah membaca dan mempelajari definisi-definisi formal, postulat, teorema yang ada di geometri? |
| S114 | Pernah. Menurut saya kalau misal seperti itu pasti perlu mempelajari karena tanpa ada definisi, teorema atau postulat yang bersangkutan dengan geometri tidak mungkin kita membuktikan sesuatu, karena pada dasarnya itu kan dibuat untuk memberi pengetahuan atau petunjuk untuk mengerjakan suatu soal. Jadi pada awalnya harus tahu hubungan antara ini dan ini dan sebagainya. |

Hereunder is the work of S2 in generating evidence for problem given.

Figure 3. The Result of S2’s Work

Interview excerpt related to the result of S2’s work is presented in the Table 3.

Table 3. Interview Excerpt of S2

| P07  | Dari gambar yang sudah Anda buat, penyelesaian seperti apa yang Anda pikirkan? |
| S207 | Disini saya menuju ke gambar layang-layang. Disini Q ditaris garis ke S, karena Q=Q=Q=QB=r misalkan jari-jari yang besar itu R besar jadi disini R disini R (menunjuk gambar) Terus karena TS sama dengan jari-jari yang besar. |
| P08  | Anda menyimpulkan bahwa itu bangun layang-layang dari mana? |
| S208 | Dari segitiga TSB sama TRB itu saya anggap sama atau sebangun, jadi ini 2x ini R ini 2R karena sini 2R sini 2R sini x, terus dari sudut juga. Kalau misalkan BSD ini segaris maka dia layang-layang tapi itu belum dibuktikan tapi saya anggap kalau dia segaris. |

Interview excerpt concerned with the congruence of triangle and the use of postulate on tangent is presented in the Table 4.
Based on the results of the students’ works, actually they solve the problem by employing the basic elements, postulates, definitions, and theorems involved in geometry. Although explicitly they do not understand the actual geometry of the base, postulates, definitions and theorems, it is implicitly evident that they understand its meaning. What the students perform are in fact in line with the characteristics of deduction level. S1 and S2 recognize the need for undefined terms, postulates, and definitions in solving geometry problems. Both apply base elements (points and lines), postulates (tangents of circles), definitions (circles, triangles, and kites) in solving problems. They are able to recognize the characteristics of formal definitions and equivalence definitions. This finding corroborates the results of research conducted by [8], which posits that students at deduction level are able to understand the equivalent form of a given definition. Moreover, they are able to use and choose the apt theorem. They are also able to prove the relationship between theorems and related statements, test changes in the initial definition or postulate in logical sequences, and bring forward formal deductive arguments. Both are able to garner evidence by employing direct deductive arguments. This is also in line with the results of research conducted by [10], which puts forward that students’ at deduction level are even able to compile evidence, rather than merely accepting and anticipating evidence [21].

One of the characteristics of van geometry thinking level is the sequential level [22]. In learning geometry, one has to go through particular learning sequences. To reach a higher level, students have to proceed through all previous levels. This implies that students at deduction level are also included in the previous level (informal visualization, analysis and deduction). Both of the research subjects proceed through the visualization level, where students are familiar with geometric structures based on visual characteristics. This is indicated by the fact that both garnered evidence based on the sketches they have made. Both of them demonstrate the ability at analysis level, that is students already know the properties of geometrical figures by analysing the figure properties. This is also indicated by the fact that both take geometrical properties into consideration upon problem solving. They analyse the properties of circles, triangles, and kites. At the informal deduction level, they can relate the properties of a figure and propose their deductive arguments. That is evident inasmuch as they are able to connect the properties of circles and triangles in solving problems. In addition, they are able to collect evidence by using direct deductive arguments.

4. Conclusion
Based on the results of the analysis and discussion, it can be concluded that in solving geometry problems, students acknowledge the importance of basic elements (undefined terms), postulates, and definitions in solving problems, recognizing the characteristics of formal definitions (for example, necessary and sufficient conditions) and equivalence definition, proving the relationship between theorems and related statements, testing changes in the initial definition or postulate in logical sequences, and proposing formal deductive arguments in solving problems.
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