On a class of stable conditional measures

Eugen Mihailescu

Abstract

The dynamics of endomorphisms (smooth non-invertible maps) presents many differences from that of diffeomorphisms or that of expanding maps; most methods from those cases do not work if the map has a basic set of saddle type with self-intersections. In this paper we study the conditional measures of a certain class of equilibrium measures, corresponding to a measurable partition subordinated to local stable manifolds. We show that these conditional measures are geometric probabilities on the local stable manifolds, thus answering in particular the questions related to the stable pointwise Hausdorff and box dimensions. These stable conditional measures are shown to be absolutely continuous if and only if the respective basic set is a non-invertible repellor. We find also invariant measures of maximal stable dimension, on folded basic sets. Examples are given too, for such non-reversible systems.

MSC 2000: Primary: 37D35, 37B25, 34C45. Secondary: 37D20.

Keywords: Equilibrium measures for hyperbolic non-invertible maps, stable manifolds, conditional measures, folded repellors, pointwise dimensions of measures.

1 Background and outline of the paper.

In this paper we will study non-invertible smooth (say \( C^2 \)) maps on a Riemannian manifold \( M \), called endomorphisms, which are uniformly hyperbolic on a basic set \( \Lambda \). Here by basic set for an endomorphism \( f : M \to M \), we understand a compact topologically-transitive set \( \Lambda \), which has a neighbourhood \( U \) such that \( \Lambda = \bigcap_{n \in \mathbb{Z}} f^n(U) \).

Considering non-invertible transformations makes sense from the point of view of applications, since the evolution of a non-reversible physical system is usually given by a time-dependent differential equation \( \frac{dx(t)}{dt} = F(x(t)) \) whose solution, the flow \( (f^t)_t \), may not consist necessarily of diffeomorphisms. However if we look at the ergodic (qualitative) properties of the associated flow (equilibrium measures, Lyapunov exponents, conditional measures associated to measurable partitions), we may replace it with a discrete non-invertible dynamical system (\([5]\)). The theory of hyperbolic diffeomorphisms (Axiom A) has been studied by many authors (see for example \([4]\), \([5]\), \([6]\), \([16]\), and the references therein); also the theory of expanding maps was studied extensively (see for instance \([15]\)), and the fact that the local inverse iterates are contracting on small balls is crucial in that case.

However, the theory of smooth non-invertible maps which have saddle basic sets, is significantly different from the two above mentioned cases. Most methods of proof from diffeomorphisms or
expanding maps, do not work here due to the complicated **overlaps and foldings** that the endomorphism may have in the basic set Λ. The unstable manifolds depend in general on the **choice of a sequence** of consecutive preimages, not only on the initial point (as in the case of diffeomorphisms). So the unstable manifolds do not form a foliation, instead they may intersect each other both inside and outside Λ. Moreover the local inverse iterates do not contract necessarily on small balls, instead they will grow exponentially (at least for some time) in the stable direction. Also, an arbitrary basic set Λ is not necessarily totally invariant for f, and there do not always exist Markov partitions on Λ. We mention also that endomorphisms on Lebesgue spaces behave differently than invertible transformations even from the point of view of classifications in ergodic theory, see [13].

We will work in the sequel with a hyperbolic endomorphism f on a basic set Λ; such a set is also called a **folded basic set** (or a basic set with **self-intersections**). By n-preimage of a point x we mean a point y such that \( f^n(y) = x \). By prehistory of x we understand a sequence of consecutive preimages of x, belonging to Λ, and denoted by \( \hat{x} = (x, x_{-1}, x_{-2}, \ldots) \) where \( f(x_{-n}) = x_{-n+1}, n > 0 \), with \( x_0 = x \). And by inverse limit of \( (f, \Lambda) \) we mean the space of all such prehistories, denoted by \( \hat{\Lambda} \). For more about these aspects, see [14], [9]. By the definition of a basic set Λ, we assume that f is **topologically transitive** on Λ as an endomorphism, i.e that there exists a point in Λ whose iterates are dense in Λ.

**Hyperbolicity** is defined for endomorphisms (see [16]) similarly as for diffeomorphisms, with the crucial difference that now the unstable spaces (and thus the local unstable manifolds) depend on whole prehistories; so we have the stable tangent spaces \( E^s_x, x \in \Lambda \), the unstable tangent spaces \( E^u_{\hat{x}}, \hat{x} \in \hat{\Lambda} \), the local stable manifolds \( W^s_r(x), x \in \Lambda \) and the local unstable manifolds \( W^u_r(\hat{x}), \hat{x} \in \hat{\Lambda} \). As there may be (infinitely) many unstable manifolds going through a point, we do not have here a well defined holonomy map between stable manifolds, by contrast to the diffeomorphism case. For more details on endomorphisms, see [16], [8], [9], [11], etc.

**Definition 1.** Consider a smooth (say \( C^2 \)) non-invertible map f which is hyperbolic on the basic set Λ, such that the critical set of f does not intersect Λ. Define the **stable potential** of f as \( \Phi^s(y) := \log |Df_s(y)|, y \in \Lambda \). By stable dimension (at a point \( x \in \Lambda \)) we understand the Hausdorff dimension \( \delta^s(x) := HD(W^s_r(x) \cap \Lambda) \). We will also say that f is c-hyperbolic on Λ if f is hyperbolic on Λ, there are no critical points of f in Λ and f is conformal on the local stable manifolds.

The relations between thermodynamic formalism and the dynamics of diffeomorphisms or expanding maps form a rich field (see for instance [1], [4], [5], [7], [15], etc.) And in [10], [11], [12], we studied some aspects of the thermodynamic formalism for non-invertible smooth maps.

**Examples** of hyperbolic endomorphisms are numerous, for instance hyperbolic solenoids and horseshoes with self-intersections ([3]), polynomial maps in higher dimension hyperbolic on certain basic sets, skew products with overlaps in their fibers ([12]), hyperbolic toral endomorphisms or perturbations of these, etc.

In this non-invertible setting, a special importance is presented by **constant-to-one endomorphisms**. For such endomorphisms, we study the family of conditional measures of a certain equilib-
rium measure, family associated to a measurable partition subordinated to local stable manifolds.

If a topological condition is satisfied, namely if the number of preimages remaining in $\Lambda$ is constant along $\Lambda$, we showed in [11] the following:

**Theorem** (Independence of stable dimension). If the endomorphism $f$ is $c$-hyperbolic on the basic set $\Lambda$ (see Definition [7]) and if the number of $f$-preimages of any point from $\Lambda$, remaining in $\Lambda$ is constant and equal to $d$, then the stable dimension $\delta^s(x)$ is equal to the unique zero $t_d^s$ of the pressure function $t \to \mathcal{P}(t\Phi^s - \log d)$, for any $x \in \Lambda$. The common value of the stable dimension along $\Lambda$ will be denoted by $\delta^s$.

In fact if $f$ is open on $\Lambda$, we proved (see [11], and Proposition 1 of [10]) the following:

**Proposition** ([10], [11]). Let an endomorphism $f : M \to M$ which has a basic set $\Lambda$, disjoint from the critical set of $f$. Assume that $\Lambda$ is connected and $f|_{\Lambda} : \Lambda \to \Lambda$ is open. Then the cardinality of the set $f^{-1}(x) \cap \Lambda$ is constant, when $x$ ranges in $\Lambda$.

Examples of hyperbolic open endomorphisms on saddle sets are given in the end of the paper.

**Definition 2.** Let an endomorphism $f$ $c$-hyperbolic on the basic set $\Lambda$, such that the number of $f$-preimages of any point from $\Lambda$, remaining in $\Lambda$, is constant and equal to $d$. Then we call the equilibrium measure of $\delta^s \cdot \Phi^s$, the stable equilibrium measure of $f$ on $\Lambda$, and denote it by $\mu^s$.

We notice that, since the stable foliation is Lipschitz continuous for endomorphisms (see [11]), the potential $\delta^s \cdot \Phi^s$ is Holder continuous; thus it can be shown by lifting the measure to the inverse limit $\hat{\Lambda}$, that there exists a unique equilibrium measure $\mu^s$ of $\delta^s \cdot \Phi^s$ (we can apply the results for homeomorphisms from [6] on the inverse limit $\hat{\Lambda}$, in order to get the uniqueness).

We will show in Theorem [1] that if the number of $f$-preimages in $\Lambda$ is constant, then the conditional measures of $\mu^s$ associated to a measurable partition subordinated to the local stable manifolds, are geometric probabilities of exponent $\delta^s$. This will answer then in Corollary [1] the question of the pointwise Hausdorff dimension and the pointwise box dimension of the equilibrium measure $\mu^s$ on local stable manifolds (see for instance [11] for definitions). In the constant-to-1 non-invertible case, we show in particular in Corollary [2] that these stable conditional measures are measures of maximal dimension (in the sense of [2]) on the intersections of local stable manifolds with the folded basic set $\Lambda$.

Our approach will be different both from the case of diffeomorphisms and from that of expanding maps. In Proposition [1] (which is the main ingredient for the proof of Theorem [1]), we compare the equilibrium measure on various different components of the preimage set of a small "cylinder" around an unstable manifold. We will have to carefully estimate the equilibrium measure $\mu^s$ on the different pieces of the iterates of Bowen balls, in order to get good estimates for the cylinders around local unstable manifolds, $B(W^u_r(\hat{x}), \varepsilon)$. This will be done by a process of desintegrating the measure on the various components of the preimages of borelian sets, and then by successive re-combinations. Thus we will reobtain the measure $\mu^s$ on an arbitrary open set, and then use
the essential uniqueness of the family of conditional measures of $\mu_s$; for background on conditional measures associated to measurable partitions on Lebesgue spaces, see [13].

In Corollary [3] we prove that the conditional measures of $\mu_s$ on the local stable manifolds over $\Lambda$ are absolutely continuous if and only if, the stable dimension is equal to the real dimension of the stable tangent space $\text{dim}E^s_x$; and we show that this is equivalent to $\Lambda$ being a folded repellor.

We will also give in the end Examples of hyperbolic constant-to-1 folded basic sets for which Theorem [11] and its Corollaries do apply. In particular we provide examples of folded repellors obtained for perturbation endomorphisms, which are not Anosov and for which we prove the absolute continuity of the stable conditional measures on their (non-linear) stable manifolds.

# 2 Main proofs and applications.

For our first result we assume only that $f$ is a smooth endomorphism which is hyperbolic on a basic set $\Lambda$. We will give a comparison between the values of an arbitrary equilibrium measure $\mu_\phi$ (corresponding to a Holder continuous potential $\phi$ on $\Lambda$) on the different pieces/components of the preimages of a borelian set; this will be useful when we will estimate later on, the measure $\mu_s$ on certain sets.

By a Bowen ball $B_n(x, \varepsilon)$ we understand the set $\{y \in \Lambda, d(f^i y, f^i x) < \varepsilon, i = 0, \ldots, n\}$, for $x \in \Lambda$ and $n > 1$. If $\phi$ is a continuous real function on $\Lambda$ and $m$ is a positive integer, we denote by $S_m \phi(y) := \phi(y) + \phi(f(y)) + \ldots + \phi(f^m(y))$ the consecutive sum of $\phi$ on the $n$-orbit of $y \in \Lambda$. And by $P(\phi)$ we denote the topological pressure of the potential $\phi$ with respect to the function $f|_\Lambda$.

**Proposition 1.** Let $f$ be an endomorphism, hyperbolic on a basic set $\Lambda$; consider also a Holder continuous potential $\phi$ on $\Lambda$ and $\mu_\phi$ be the unique equilibrium measure of $\phi$. Let a small $\varepsilon > 0$, two disjoint Bowen balls $B_k(y_1, \varepsilon), B_m(y_2, \varepsilon)$ and a borelian set $A \subset f^k(B_k(y_1, \varepsilon)) \cap f^m(B_m(y_2, \varepsilon))$, s.t $\mu_\phi(A) > 0$; denote by $A_1 := f^{-k}A \cap B_k(y_1, \varepsilon)$, $A_2 := f^{-m}A \cap B_m(y_2, \varepsilon)$. Then there exists a positive constant $C_\varepsilon$ independent of $k, m, y_1, y_2$ such that

$$\frac{1}{C_\varepsilon} \mu_\phi(A_2) \cdot e^{S_k \phi(y_1)} \cdot P(\phi)^{m-k} \leq \mu_\phi(A_1) \leq C_\varepsilon \mu_\phi(A_2) \cdot e^{S_k \phi(y_1)} \cdot P(\phi)^{m-k}$$

**Proof.** Let us fix a Holder potential $\phi$. We will denote the equilibrium measure $\mu_\phi$ by $\mu$ to simplify notation. We will work with $f$ restricted to $\Lambda$.

As in [6], since the borelian sets with boundaries of $\mu$-measure zero form a sufficient collection, we will assume that each of the sets $A_1, A_2$ have boundaries of $\mu$-measure zero.

From construction $f^k(A_1) = f^m(A_2)$, and assume for example that $m \geq k$. Now the equilibrium measure $\mu$ can be considered as the limit of the sequence of measures (see [6]):

$$\tilde{\mu}_n := \frac{1}{P(f, \phi, n)} \sum_{x \in \text{Fix}(f^n)} e^{S_n \phi(x)} g_x,$$

where $P(f, \phi, n) := \sum_{x \in \text{Fix}(f^n)} e^{S_n \phi(x)}, n \geq 1$. 

4
So we have

\[ \tilde{\mu}_n(A_1) = \frac{1}{P(f, \phi, n)} \sum_{x \in \text{Fix}(f^n) \cap A_1} e^{S_n \phi(x)}, \quad n \geq 1 \]  \hfill (1)

Let us consider now a periodic point \( x \in \text{Fix}(f^n) \cap A_1 \); by definition of \( A_1 \), it follows that \( f^k(x) \in A \), so there exists a point \( y \in A_2 \) such that \( f^m(y) = f^k(x) \). However the point \( y \) does not have to be periodic.

Now we will use the Specification Property \([1, 3]\) on the hyperbolic compact locally maximal set \( \Lambda \): if \( \varepsilon > 0 \) is fixed, then there exists a constant \( M_\varepsilon > 0 \) such that for all \( n >> M_\varepsilon \), there exists a \( z \in \text{Fix}(f^{n+m-k}) \) s.t \( \varepsilon \)-shadows the \((n + m - k - M_\varepsilon)\)-orbit of \( y \).

Let now \( V \) be an arbitrary neighbourhood of the set \( A_2 \) s.t \( V \subset B_m(y_2, \varepsilon) \). Consider two points \( x, \tilde{x} \in \text{Fix}(f^n) \cap A_1 \) and assume the same periodic point \( z \in V \cap \text{Fix}(f^{n+m-k}) \) corresponds to both \( x \) and \( \tilde{x} \) by the above procedure. This means that the \((n - k - M_\varepsilon)\)-orbit of \( f^m z \), \( \varepsilon \)-shadows the \((n - k - M_\varepsilon)\)-orbit of \( f^k x \) and also the \((n - k - M_\varepsilon)\)-orbit of \( f^k \tilde{x} \). Hence the \((n - M_\varepsilon - k)\)-orbit of \( f^k x \), \( 2\varepsilon \)-shadows the \((n - M_\varepsilon - k)\)-orbit of \( f^k \tilde{x} \). But recall that we chose \( x, \tilde{x} \in A_1 \subset B_k(y_1, \varepsilon) \), hence \( \tilde{x} \in B_{n-M_\varepsilon}(x, 2\varepsilon) \).

Now we can split the set \( B_{n-M_\varepsilon}(x, 2\varepsilon) \) in at most \( N_\varepsilon \) smaller Bowen ball of type \( B_n(\zeta, 2\varepsilon) \). In each of these \((n, 2\varepsilon)\)-Bowen balls \( B_n(\zeta, 2\varepsilon) \) we may have at most one fixed point for \( f^n \). This holds since fixed points for \( f^n \) are solutions to the equation \( f^n \xi = \xi \), and on tangent spaces we have that \( Df^n - Id \) is a linear map without eigenvalues of absolute value 1. Thus if \( d(f^i \xi, f^i \zeta) < 2\varepsilon, i = 0, \ldots, n \) and if \( \varepsilon \) is small enough, we can apply the Inverse Function Theorem at each step. Therefore there exists only one fixed point for \( f^n \) in each Bowen ball \( B_n(\zeta, 2\varepsilon) \). Hence there exist at most \( N_\varepsilon \) periodic points from \( \text{Fix}(f^n) \cap \Lambda \) having the same periodic point \( z \in V \) attached to them by the above procedure.

Let us notice also that, if \( x, \tilde{x} \) have the same point \( z \in V \cap \text{Fix}(f^{n+m-k}) \) attached to them, then as before, \( \tilde{x} \in B_{n-M_\varepsilon}(x, 2\varepsilon) \). So the distances between iterates are growing exponentially in the unstable direction, and decrease exponentially in the stable direction. Thus we can use the Holder continuity of \( \phi \) and a Bounded Distortion Lemma to prove that:

\[ |S_n \phi(x) - S_n \phi(\tilde{x})| \leq \tilde{C}_\varepsilon, \]

for some positive constant \( \tilde{C}_\varepsilon \) depending on \( \phi \) (but independent of \( n, x \)). This can be used then in the estimate for \( \tilde{\mu}_n(A_1) \), according to \([11]\). We use the fact that if \( z \in B_{n+m-k-M_\varepsilon}(y, \varepsilon) \), then \( f^m(z) \in B_{n-M_\varepsilon-k}(f^m y, \varepsilon) \); also recall that \( f^k x = f^m y \), so \( f^m z \in B_{n-M_\varepsilon-k}(f^k x, \varepsilon) \). Then from the Holder continuity of \( \phi \) and the fact that \( x \in A_1 \subset B_m(y_1, \varepsilon) \), it follows again by a Bounded Distortion Lemma that there exists a constant \( \tilde{C}_\varepsilon \) (denoted as before without loss of generality) satisfying:

\[ |S_{n+m-k} \phi(z) - S_n \phi(x)| \leq |S_k \phi(y_1) - S_m \phi(y_2)| + \tilde{C}_\varepsilon, \]  \hfill (2)

for \( n > n(\varepsilon, m) \).

But from Proposition 20.3.3 of \([9]\) (which extends immediately to endomorphisms), we have
that there exists a positive constant \(c_\varepsilon\) such that for sufficiently large \(n\):

\[
\frac{1}{c_\varepsilon} e^{nP(\phi)} \leq P(f, \phi, n) \leq c_\varepsilon e^{nP(\phi)},
\]

where the expression \(P(f, \phi, n)\) was defined immediately before (1). Hence in our case, if \(n > n(\varepsilon, m)\) we obtain:

\[
\frac{1}{c_\varepsilon} e^{(n+m-k)P(\phi)} \leq P(f, \phi, n + m - k) \leq c_\varepsilon e^{(n+m-k)P(\phi)}, \quad \text{and} \quad \frac{1}{c_\varepsilon} e^{nP(\phi)} \leq P(f, \phi, n) \leq c_\varepsilon e^{nP(\phi)} \tag{3}
\]

Recall also that there are at most \(N_\varepsilon\) points \(x \in \text{Fix}(f^n)\) which have the same attached \(z \in V \cap \text{Fix}(f^n)\). Therefore, by using (1), (2) and (3) we can infer that there exists a constant \(C_\varepsilon > 0\) such that for \(n\) large enough \((n > n(\varepsilon, m))\),

\[
\bar{\mu}_n(A_1) \leq C_\varepsilon \bar{\mu}_{n+m-k}(V) \cdot \frac{e^{S_k\phi(y_1)}}{e^{S_m\phi(y_2)}} \cdot P(\phi)^{m-k}, \tag{4}
\]

where we recall that \(A_1 \subset B_m(y_1, \varepsilon), A_2 \subset B_m(y_2, \varepsilon)\). But since \(\partial A_1, \partial A_2\) have \(\mu\)-measure zero, we obtain:

\[
\mu(A_1) \leq C_\varepsilon \mu(V) \cdot \frac{e^{S_k\phi(y_1)}}{e^{S_m\phi(y_2)}} \cdot P(\phi)^{m-k}
\]

But \(V\) has been chosen arbitrarily as a neighbourhood of \(A_2\), hence

\[
\mu(A_1) \leq C_\varepsilon \mu(A_2) \frac{e^{S_k\phi(y_1)}}{e^{S_m\phi(y_2)}} P(\phi)^{m-k}
\]

Similarly we prove also the other inequality, hence we are done.

Let us recall a few notions about measurable partitions (see [14]). Let \(\zeta\) be a partition of a Lebesgue space \((X, \mathcal{B}, \mu)\) with \(\mathcal{B}\)-measurable sets. Subsets of \(X\) that are unions of elements of \(\zeta\) are called \(\zeta\)-sets. For an arbitrary point \(x \in X\) (modulo \(\mu\)), we denote the unique set which contains \(x\), by \(\zeta(x)\). By basis for \(\zeta\) we understand a countable collection \(\{B_\alpha, \alpha \in A\}\) of measurable \(\zeta\)-sets so that for any two elements \(C, C' \in \zeta\), there exists some \(\alpha \in A\) with \(C \subset B_\alpha, C' \cap B_\alpha = \emptyset\) or viceversa, i.e \(C \cap B_\alpha = \emptyset, C' \subset B_\alpha\). A partition \(\zeta\) is called measurable if it has a basis as above.

Now we remind briefly the notion of family of conditional measures associated to a measurable partition \(\zeta\). Assume we have an endomorphism \(f\) on a compact set \(\Lambda\), and let a probability borelian measure \(\mu\) on \(\Lambda\) which is \(f\)-invariant. If \(\zeta\) is a measurable partition of \((\Lambda, \mathcal{B}, \mu)\) denote by \((\Lambda/\zeta, \mu_\zeta)\) the factor space of \(\Lambda\) relative to \(\zeta\). Then we can attach an essentially unique collection of conditional measures \(\{\mu_C\}_{C \in \zeta}\) satisfying two conditions (see [14]):

i) \((C, \mu_C)\) is a Lebesgue space

ii) for any measurable set \(B \subset \Lambda\), the set \(B \cap C\) is measurable in \(C\) for \(\mu_\zeta\)-almost all points \(C \in \Lambda/\zeta\), the function \(C \to \mu_C(B \cap C)\) is measurable on \(\Lambda/\zeta\) and \(\mu(B) = \int_{\Lambda/\zeta} \mu_C(B \cap C) d\mu_\zeta(C)\).

**Definition 3.** If \(f\) is a hyperbolic map on a basic set \(\Lambda\) and if \(\mu\) is an \(f\)-invariant borelian measure on \(\Lambda\), then a measurable partition \(\zeta\) of \((\Lambda, B(\Lambda), \mu)\) is said to be **subordinated to the local stable manifolds** if for \(\mu\)-a.e \(x \in \Lambda\), we have \(\zeta(x) \subset W^s_{\text{loc}}(x)\), and \(\zeta(x)\) contains an open neighbourhood of \(x\) in \(W^s_{\text{loc}}(x)\) (with respect to the topology induced on the local stable manifold).
Let us fix an \( f \)-invariant borelian measure \( \mu \) on \( \Lambda \). Since we work with a uniformly hyperbolic endomorphism, we can **construct a measurable partition** \( \xi \) (w. r. t \( \mu \)) subordinated to the local stable manifolds, in the following way: first we know that there is a small \( r_0 > 0 \). t for each \( x \in \Lambda \) there exists a local stable manifold \( W^s_{r_0}(x) \). Then it is possible to take a countable partition \( \mathcal{P} \) of \( \Lambda \) (modulo \( \mu \)) with open sets, each having diameter less than \( r_0 \) and such that the boundary of each set from \( \mathcal{P} \) has \( \mu \)-measure zero (see for example [9]). Now for every open set \( U \in \mathcal{P} \), and \( x \in U \subset \Lambda \), we consider the intersection between \( U \) and the unique local stable manifold going through \( x \); denote this intersection by \( \xi(x) \). It is clear that \( \xi(x) = \xi(y) \) if and only if both \( x, y \) are in the same set \( U \in \mathcal{P} \) and they are on the same local stable manifold \( W^s_{r_0}(z) \) for some \( z \in \Lambda \). Now take the collection \( \xi \) of all the borelian sets \( \xi(x), x \in U, U \in \mathcal{P} \). We see easily that \( \xi \) is a partition of \( \Lambda \) (modulo sets of \( \mu \)-measure zero) and that \( \xi \) is measurable, since \( \mathcal{P} \) was assumed countable and, inside each member \( U \in \mathcal{P} \), we can separate any two local stable manifolds with the help of a countable collection of \( \xi \)-sets (which are neighbourhoods of local stable manifolds). Therefore we have concluded the construction of the measurable partition \( \xi \) which is subordinated to the local stable manifolds. Modulo a set of \( \mu \)-measure zero we have thus a partition with pieces of local stable manifolds, \( \xi(x) \subset W^s_{r(y(x))}(y(x)), x \in \Lambda \). In fact without loss of generality, we may assume that for each member \( A \in \xi \), there exists some \( x(A) \in \Lambda \) and \( r(A) \in (0, r_0) \) so that

\[
W^s_{r(A)/2}(x(A)) \cap \Lambda \subset A \subset W^s_{r(A)}(x(A)) \cap \Lambda.
\]

**Remark 1:** From the construction above it follows that, outside a set of \( \mu \)-measure zero, the radius \( r(A) \) can be taken to vary continuously, i.e there exists a constant \( \chi > 0 \) s. t for each \( x \) in a set of full \( \mu \)-measure in \( \Lambda \), there exists a neighbourhood \( U(x) \) of \( x \) with \( r(\xi(z)) \leq \chi, z, z' \in U(x) \).

\[\square\]

**Notation:** In our uniformly hyperbolic setting, with the partition \( \xi \) constructed above, we denote the conditional measure \( \mu_A \) by \( \mu^s_A \), for \( W^s_{r(A)/2}(x(A)) \cap \Lambda \subset A \subset W^s_{r(A)}(x(A)) \cap \Lambda, A \in \xi \). We will also denote the set of centers \( \{x(A), A \in \xi\} \) by \( S \). In particular, if \( \mu = \mu_s \), we denote the conditional measures by \( \mu^s_A \) for \( A \in \xi \), or by \( \mu^s_{x,A} \) when \( \xi(x) = A \) for \( \mu_s \)-a.e \( x \in \Lambda \).

Now, if \( f \) is a d-to-1 c-hyperbolic endomorphism on the basic set \( \Lambda \), we showed in [11] that the stable dimension \( \delta^s(x) \) at any point \( x \in \Lambda \) is independent of \( x \), and is equal to the unique zero of the pressure function \( t \to P(t \Phi^s - \log d) \). Thus we can talk in this case about the **stable dimension** of \( \Lambda \) and will denote it by \( \delta^s \).

**Theorem 1.** Let \( f \) be a smooth endomorphism on a Riemannian manifold \( M \), and assume that \( f \) is c-hyperbolic on a basic set of saddle type \( \Lambda \). Let us assume moreover that \( f \) is d-to-1 on \( \Lambda \). Assume that \( \Phi^s(y) := \log |Df_s(y)|, y \in \Lambda \), that \( \delta^s \) is the stable dimension of \( \Lambda \), and that \( \mu_s \) is the equilibrium measure of the potential \( \delta^s \Phi^s \) on \( \Lambda \). Then the conditional measures of \( \mu_s \) associated to the partition \( \xi \), namely \( \mu^s_{x,A} \), are geometric probabilities, i.e for every set \( A \in \xi \) there exists a positive constant \( C_\Lambda \) such that

\[
C_\Lambda^{-1} \rho^s \leq \mu^s_{x,A}(B(y, \rho)) \leq C_\Lambda \rho^\delta^s, y \in A \cap \Lambda, 0 < \rho < \frac{r(A)}{2}
\]

**Proof.** By using the partition \( \xi \) subordinated to local stable manifolds from above, we can associate
conditional measures of \( \mu_s \), denoted by \( \mu_{s,A}^\ast, A \in \xi \). We want to estimate the measure \( \mu_{s,A}^\ast \) of a small arbitrary ball \( B(y, \rho) \) centered at some \( y \in A \), where \( W^s_{r(A)}/2)(x) \cap \Lambda \subset A \subset W^s_{r(A)}(x) \cap \Lambda, x = x(A) \).

Let us first consider an arbitrary set \( f^n(B_n(z, \varepsilon)) \), where we remind that \( B_n(z, \varepsilon) \) denotes a Bowen ball, and where \( \varepsilon > 0 \) is arbitrary but small. This set is actually a neighbourhood of the unstable manifold \( W^s(f^n z) \) corresponding to a prehistory \( (f^n z, f^{n-1} z, \ldots, z, \ldots) \). We will estimate the \( \mu_s \)-measure of a cross section of such a set \( f^n(B_n(z, \varepsilon)) \), i.e an intersection of type

\[
B(n, z; k, x; \varepsilon) := f^n(B_n(z, \varepsilon)) \cap B_k(x, \varepsilon),
\]

for arbitrary \( z, x \in \Lambda \) and positive integers \( n, k \). We see that if we vary \( z, x, k, n, x \), we can write any open set in \( \Lambda \) as a union of mutually disjoint sets of type \( B(n, z; k, x; \varepsilon) \).

So let us estimate the \( \mu_s \)-measure of \( B(n, z; k, x; \varepsilon) \). Notice that \( B(n, z; k, x; \varepsilon) \) is contained in \( f^n(B_{n+k}(z, \varepsilon)) \). Without loss of generality we can assume that \( z = x_{-n} \), i.e that \( z \) itself is the unique \( n \)-preimage of \( x \) inside \( B_n(z, \varepsilon) \); if not, then we can replace \( z \) by a point \( x_{-n} \) which is \( \varepsilon \)-shadowed by \( z \) up to order \( n + k \), and thus the dynamical behaviour of \( z \) up to order \( n + k \) will be the same as that of \( x_{-n} \).

Let us denote the positive quantity \(|Df^n_s(z)| \cdot \varepsilon \) by \( \rho \). Since the endomorphism \( f \) is conformal on local stable manifolds, the diameter of the intersection \( f^n(B_n(z, \varepsilon)) \cap W^s(f^n z) \) is equal to \( 2\rho \).

Now recall that we assumed without loss of generality that \( f^n z = x \), and consider all the finite prehistories of the point \( x \), in \( \Lambda \). We will call then \( \rho \)-maximal prehistory of \( x \) any finite prehistory \( (x, x_{-1}, \ldots, x_{-p}) \) so that \( |Df^p(x_{-p+1})| \cdot \varepsilon \geq \rho \) but \( |Df^p(x_{-p})| \cdot \varepsilon < \rho \). Clearly, given any prehistory \( \hat{x} = (x, x_{-1}, \ldots) \) of \( x \), there exists some positive integer \( n(\hat{x}, \rho) \) such that \( (x, x_{-1}, \ldots, x_{-n(\hat{x}, \rho)}) \) is a \( \rho \)-maximal prehistory. Let us denote by

\[
N(x, \rho) := \{n(\hat{x}, \rho), \hat{x} \text{ prehistory of } x \text{ from } \Lambda\}
\]

We will consider now the various components of the \( p \)-preimages of \( B(n, z; k, x; \varepsilon) \), when \( p \) ranges in \( N(x, \rho) \). We extended the stable diameter of \( B(n, z; k, x; \varepsilon) \) in backward time until we reach a diameter of at most \( \varepsilon \). As the maximum expansion in backward time is realized on the stable manifolds (local inverse iterates contract all the unstable directions), it follows that for any prehistory \( \hat{x} \) of \( x \), there exists a component of \( f^{-n(\hat{x}, \rho)}(B(n, z; k, x; \varepsilon)) \) inside the Bowen ball \( B_n(\hat{x}, \rho)(x_{-n(\hat{x}, \rho)}; \varepsilon) \); denote this component by \( A(\hat{x}, \rho) \). We see that all these components \( A(\hat{x}, \rho) \) are mutually disjoint if \( \varepsilon << \varepsilon_0 \), where \( \varepsilon_0 \) is the local injectivity constant of \( f \) on \( \Lambda \) (recall that there are no critical points in \( \Lambda \)). Indeed if the sets \( A(\hat{x}, \rho) \) and \( A(\hat{x}', \rho) \) would intersect for some prehistories \( \hat{x} = (x, x_{-1}, \ldots), \hat{x}' = (x, x'_{-1}, \ldots) \) of \( x \) then, since they are contained in Bowen balls, their forward iterates would be \( 2\varepsilon \)-close. But then we get a contradiction since the prehistories \( \hat{x}, \hat{x}' \) must contain different preimages \( x_p, x_{-p'} \) at some level \( p \), and these different preimages must be at a distance of at least \( \varepsilon_0 \) from each other. Hence either \( A(\hat{x}, \rho) = A(\hat{x}', \rho) \), or \( A(\hat{x}, \rho) \cap A(\hat{x}', \rho) = \emptyset \).

Now we will use the \( f \)-invariance of the equilibrium measure \( \mu_s \) in order to estimate the \( \mu_s \)-measure of the set \( B(n, z; k, x; \varepsilon) \). Recall that \( f^n z = x \), and \( \varepsilon |Df^n_s(z)| =: \rho \). Then we have

\[
\mu_s(B(n, z; k, x; \varepsilon)) = \sum_{\hat{x} \text{ prehistory of } x} \mu_s(A(\hat{x}, \rho)),
\]
since we showed above that the sets $A(\hat{x}, \rho)$ either coincide or are disjoint.

Now let us take two sets $A(\hat{x}, \rho), A(\hat{x}', \rho)$, one of them with $n(\hat{x}, \rho) = p$ and the other with $n(\hat{x}', \rho) = p'$. We proved in [1] that for a $d$-to-1 c-hyperbolic endomorphism $f$ on the basic set $\Lambda$, we have $\delta^s = t_d^s$, where $t_d^s$ is the unique zero of the pressure function $t \rightarrow P(t\Phi^s - \log d)$. Therefore we can use that

$$P(\delta^s \Phi^s) = \log d \quad (5)$$

Then from the definition of the sets $A(\hat{x}, \varepsilon)$ and by using Proposition [1] we can compare the measure $\mu_s$ on two sets $A(\hat{x}, \rho), A(\hat{x}', \rho)$ as follows:

$$\frac{1}{C_{\varepsilon}} \mu_s(A(\hat{x}', \rho)) \frac{|Df^p_s(x_{-p})|^{\delta_s}}{|Df^p_s(x_{-p}')}^{\delta_s} \leq \mu_s(A(\hat{x}, \rho)) \leq C_{\varepsilon} \mu_s(A(\hat{x}', \rho)) \frac{|Df^p_s(x_{-p})|^{\delta_s}}{|Df^p_s(x_{-p}')|^{\delta_s}} \cdot d^{p'-p} \quad (6)$$

In general, if for two variable quantities $Q_1, Q_2$, there exists a positive universal constant $c$ such that $\frac{1}{2}Q_2 \leq Q_1 \leq cQ_2$, we say that $Q_1, Q_2$ are comparable, and will denote this by $Q_1 \approx Q_2$; the constant $c$ is called the comparability constant.

But from the definition of $n(\hat{x}, \rho)$ above (as being the length of the $\rho$-maximal prehistory along $\hat{x}$), and since $n(\hat{x}, \rho) = p, n(\hat{x}', \rho) = p'$ we obtain:

$$\frac{1}{C} |Df^p_s(x_{-p}')| \leq |Df^p_s(x_{-p})| \leq C |Df^p_s(x_{-p}')|$$

Therefore, from relation (6) we obtain

$$\frac{1}{C_{\varepsilon}} \mu_s(A(\hat{x}', \rho)) d^{p'-p} \leq \mu_s(A(\hat{x}, \rho)) \leq C_{\varepsilon} \mu_s(A(\hat{x}', \rho)) d^{p'-p}, \quad (7)$$

where we used the same constant $C_{\varepsilon}$ as in (5), without loss of generality. Hence the proof will now be reduced to a combinatorial argument about the different pieces/components, of the preimages of various orders of $B(n, z; k, x; \varepsilon)$.

However we assumed that every point from $\Lambda$ has exactly $d$ $f$-preimages inside $\Lambda$. We use (7) in order to compare the $\mu_s$-measures of the different pieces $A(\hat{x}, \rho)$, which will then be added successively. Recall that one of these components $A(\hat{x}, \rho)$ is precisely $B_{n+k}(z, \varepsilon)$. The comparisons will always be made with respect to this component $B_{n+k}(z, \varepsilon)$. Let us order the integers from $N(x, \rho)$ as:

$$n_1 > n_2 > \ldots > n_T$$

We shall add first the measures $\mu_s(A(\hat{x}, \rho))$ over all the sets corresponding to $\hat{x}$ with $n(\hat{x}, \rho) = n_1$, then over those prehistories with $n(\hat{x}, \rho) = n_2$, etc. And will use that any point from $\Lambda$ has exactly $d^n m$-preimages belonging to $\Lambda$ for any $m \geq 1$. Therefore by such successive additions and by using (7) we obtain:

$$\mu_s(B_{n+k}(z, \varepsilon)) \cdot d^n \leq \mu_s(B(n, z; k, x; \varepsilon)) = \sum_{\text{prehistory of } x} \mu_s(A(\hat{x}, \rho)) \leq \mu_s(B_{n+k}(z, \varepsilon)) \cdot d^n,$$

with the positive constant $C_{\varepsilon}$ independent of $n, k, z, x$.  

9
We use now Theorem 1 of [9] which gave estimates for equilibrium measures on Bowen balls, similar to those from the case of diffeomorphisms (see [9] for example); this was done by lifting to an equilibrium measure on \( \hat{\Lambda} \). Hence from the last displayed formula and (5), we obtain:

\[
\frac{1}{C_{\varepsilon}} d^n \cdot \frac{|Df_{s+k}^n(z)|^{\delta^s}}{d^{v+k}} \leq \mu_{s}(B(n, z; k, x; \varepsilon)) \leq C_{\varepsilon} d^n \cdot \frac{|Df_{s+k}^n(z)|^{\delta^s}}{d^{v+k}} \tag{8}
\]

Let us now study in more detail the conditions from the definition of conditional measures. From the construction of the measurable partition \( \xi \) we have that \( W_{r(A)/2}^s(x) \cap \Lambda \subset A \subset W_{r(A)}^s(x) \cap \Lambda, x = x(A) \in S \) and the radii \( r(A) \) vary continuously with \( A \). So from Remark 1 we can split an arbitrary set \( U \in \mathcal{P} \), modulo \( \mu_s \), into a disjoint union of open sets \( V \), each being a \( \xi \)-set, so there exists \( r = r(V) > 0 \) s.t for all \( A \in \xi \) intersecting \( V \), we have \( W_{r/2}^s(x(A)) \cap \Lambda \subset A \subset W_r^s(x(A)) \cap \Lambda \). Hence locally, on a subset \( V \subset U \in \mathcal{P} \), we can consider that \( \xi \) is, modulo a set of \( \mu_s \)-measure zero, a foliation with local stable manifolds \( W_{r}^s(x) \) of the same size \( r = r(V) \). The intersections of these local stable manifolds with \( \Lambda \) are then identified with points in the factor space \( \Lambda / \xi \). We will work for the rest of the proof on an open set \( V \) as above, i.e.~where the sets \( A \in \xi \) can be assumed to be of type \( W_r^s(x) \), of the same size \( r = r(V) \). Take also \( \varepsilon = r \).

From the definition of the factor space \( \Lambda / \xi \), the \( (\mu_s)_{\xi} \)-measure induced on the quotient space \( \Lambda / \xi \) is given by \( (\mu_s)_{\xi}(E) = \mu_s(\pi_{\xi}^{-1}(E)) \), where \( \pi_{\xi} : \Lambda \rightarrow \Lambda / \xi \) is the canonical projection which collapses a set from \( \xi \) to a point. Hence from (5) with \( \xi \approx \pi_{\xi}^{-1}(\pi_{\xi}(B(n, z; k, x; r))) \) (9) and by using again the estimates of equilibrium states on Bowen balls, we obtain as in (8) that \( \mu_s(B_k(x, r)) \) is comparable to \( \frac{|Df_k^s(x)|^{\delta^s}}{d^n} \) (with a comparability constant \( c = c(V) \)).

But, from the definition of conditional measures we have

\[
\mu_s(B(n, z; k, x; r)) = \int_{B_k(x, r)/\xi} \mu_s^A(A \cap B(n, z; k, x; r)) d(\mu_s)_{\xi}(\pi_{\xi}(A)) \tag{9}
\]

Now by (8) and recalling that \( f^n z = x \), we infer that \( \mu_s(B(n, z; k, x; r)) \) is comparable to \( \frac{|Df_k^s(x)|^{\delta^s}}{d^n} \cdot r^{\delta^s} \), i.e. to \( \mu_s(B_k(x, r)) \cdot r^{\delta^s} \), where in our case \( r := |Df_k^s(x)| r \) (the comparability constant being denoted by \( C_{V} \)). And we showed above that \( (\mu_s)_{\xi}(B_k(x, r)/\xi) = \mu_s(B_k(x, r)) \approx \frac{|Df_k^s(x)|^{\delta^s}}{d^n} \) (with the comparability constant \( C_{V} \)).

In addition, we notice that the sets of type \( B(n, z; k, x; r) \) where \( n, z, k, x \) vary, form a basis for the open sets in \( V \); also, if we vary \( n \), the radius \( \rho := |Df_k^s(z)| r \) can be made arbitrarily small. Therefore from the essential uniqueness of the system of conditional measures associated to \( (\mu_s, \xi) \) and since any borelian set in \( V \) can be written modulo \( \mu_s \) as a union of disjoint sets of type \( B(n, z; k, x; r) \), we conclude that the conditional measure \( \mu_{s,A}^s \) is a geometric probability of exponent \( \delta^s \). Hence for all \( \rho, 0 < \rho < r/2 \), we have

\[
\frac{1}{C_{V}} \rho^{\delta^s} \leq \mu_{s,A}^s(B(y, \rho)) \leq C_{V} \rho^{\delta^s}, y \in A,
\]

for \( A \subset V, A \in \xi \). The comparability factor \( C_{V} \) is constant on \( V \); in general it can be taken locally constant on the complement in \( \Lambda \) of a set of \( \mu_s \)-measure zero. The proof is thus finished. \( \square \)
**Definition 4.** Let $f$ be a hyperbolic endomorphism on the folded basic set $\Lambda$, $\mu$ a borelian probability measure on $\Lambda$ and $\xi$ a measurable partition subordinated to local stable manifolds. Then the conditional measure $\mu^s_A$ corresponding to $A \in \xi$ will be called the stable conditional measure of $\mu$ on $A$. When $\mu = \mu_s$ we denote this stable conditional measure by $\mu^s_{s,A}$.

**Remark 2.** We notice from the proof of Theorem 1 that, in fact, the stable conditional measures of $\mu_s$ do not depend on the measurable partition $\xi$ constructed above, subordinated to local stable manifolds. Therefore there exists a set $\Lambda(\mu_s)$ of full $\mu_s$-measure inside $\Lambda$, such that for every $x \in \Lambda(\mu_s)$ there exists some small $r(x) > 0$ so that $W^s_r(x)$ is contained in a set $A$ from a measurable partition of type $\xi$ (subordinated to local stable manifolds); then one can construct the stable conditional measure $\mu^s_{s,A}$. We denote this conditional measure also by $\mu^s_{s,x}, x \in \Lambda(\mu_s)$.

We recall now the notions of lower, respectively upper pointwise dimension of a finite borelian measure $\mu$ on a compact space $\Lambda$ (see for example [1]). For $x \in \Lambda$, they are defined by

$$d_\mu(x) := \liminf_{\rho \to 0} \frac{\log \mu(B(x, \rho))}{\log \rho}, \quad \text{and} \quad \bar{d}_\mu(x) := \limsup_{\rho \to 0} \frac{\log \mu(B(x, \rho))}{\log \rho}$$

If the lower pointwise dimension at $x$ coincides with the upper pointwise dimension at $x$, we denote the common value by $d_\mu(x)$ and call it simply the pointwise dimension at $x$.

One can also define the Hausdorff dimension, lower box dimension and upper box dimension of $\mu$ respectively by:

$$HD(\mu) := \inf \{ HD(Z), \mu(\Lambda \setminus Z) = 0 \}$$

$$\dim_B(\mu) := \liminf_{\delta \to 0} \{ \dim_B(Z), \mu(\Lambda \setminus Z) \leq \delta \}$$

$$\overline{\dim}_B(\mu) := \liminf_{\delta \to 0} \{ \overline{\dim}_B(Z), \mu(\Lambda \setminus Z) \leq \delta \}$$

Assume now in general that $f$ is a hyperbolic endomorphism on $\Lambda$ and $\mu$ a probability measure on $\Lambda$, and let $\xi$ be a measurable partition subordinated to local stable manifolds of $f$ on $\Lambda$. We define then the lower/upper stable pointwise dimension of $\mu$ at $y$, for $\mu$-a.e $y \in \Lambda$, as the lower/upper pointwise dimension of the stable conditional measure $\mu^s_A$ at $y$, for $y \in A$, namely:

$$d^s_\mu(y) := \liminf_{\rho \to 0} \frac{\log \mu^s_A(B(y, \rho))}{\log \rho} \quad \text{and} \quad \bar{d}^s_\mu(y) := \limsup_{\rho \to 0} \frac{\log \mu^s_A(B(y, \rho))}{\log \rho}$$

Similarly we define the stable Hausdorff dimension of $\mu$ on $A \in \xi$, and the stable lower/upper box dimension of $\mu$ on $A$, respectively, as the quantities:

$$HD^s(\mu, A) := HD(\mu^s_A), \quad \dim_B^s(\mu, A) := \dim_B(\mu^s_A), \quad \overline{\dim}_B^s(\mu, A) := \overline{\dim}_B(\mu^s_A), A \in \xi$$

When $\mu = \mu_s$ we denote $HD^s(\mu_s, x) := HD(\mu^s_{s,x}), \dim_B^s(\mu_s, x) := \dim_B(\mu^s_{s,x})$, and $\overline{\dim}_B^s(\mu_s, x) := \overline{\dim}_B(\mu^s_{s,x})$, for $x \in \Lambda(\mu_s)$.

Recall now the stable dimension $\delta^s$ from Definition 4 and the Theorem of Independence of the Stable Dimension given afterwards.
Corollary 1. Let $f$ be a c-hyperbolic, d-to-1 endomorphism on a basic set $\Lambda$, and $\mu_s$ be the equilibrium measure of the potential $\delta^s \Phi^s$. Then the stable pointwise dimension of $\mu_s$ exists $\mu_s$-almost everywhere on $\Lambda$ and is equal to the stable dimension $\delta^s$.

Also the stable Hausdorff dimension of $\mu_s$, stable lower box dimension of $\mu_s$ and stable upper box dimension of $\mu_s$ are all equal to $\delta^s$.

Proof. The proof follows from Theorem 1 since we proved that the stable conditional measures of the equilibrium measure $\mu_s$ are geometric probabilities.

For the second part of the Corollary, we use Theorem 2.1.6 of [1]. Indeed since the stable conditional measures of $\mu_s$ are geometric probabilities of exponent $\delta^s$, we conclude that the stable Hausdorff, lower/upper dimensions coincide, and are all equal to the stable dimension $\delta^s$. \square

Definition 5. We will say that a measure $\mu$ on $\Lambda$ has maximal stable dimension on $A \in \xi, A \subset W^s_{r(x)}(x)$ if:

$$HD^s(\mu, A) = \sup\{HD^s(\nu, A), \nu \text{ is an } f|_\Lambda \text{ invariant probability measure on } \Lambda\}$$

This definition is similar to that of measure of maximal dimension; see [1], [2] where measures of maximal dimension on hyperbolic sets of surface diffeomorphisms were studied. Our setting/methods for the maximal stable dimension in the non-invertible case, are however different.

Now, since the stable Hausdorff dimension of any $f$-invariant probability measure $\nu$ on $\Lambda$ is bounded above by $\delta^s := HD(W^s_r(x) \cap \Lambda)$, we see from Corollary 1 that:

Corollary 2. In the setting of Theorem 1 it follows that the stable equilibrium measure $\mu_s$ of $f$, is of maximal stable dimension on $W^s_{r(x)}(x) \cap \Lambda$ among all $f$-invariant probability measures on $\Lambda$, for $\mu_s$-a.e $x \in \Lambda$. And $\mu_s$ maximizes in a Variational Principle for stable dimension on $\Lambda$, i.e:

$$\delta^s = HD^s(\mu_s, x) = \sup\{HD^s(\nu, x), \nu \text{ is an } f|_\Lambda \text{ invariant probability measure on } \Lambda\}, \mu_s - a.e x$$

We say now that the basic set $\Lambda$ is a repellor (or folded repellor) if there exists a neighbourhood $U$ of $\Lambda$ such that $\bar{U} \subset f(U)$. And that $\Lambda$ is a local repellor if there are local stable manifolds of $f$ contained inside $\Lambda$ (see [10] for more on these notions in the case of endomorphisms).

Corollary 3. Let an open c-hyperbolic endomorphism $f$ on a connected basic set $\Lambda$. Then we have that the stable conditional measures $\mu_{s,x}^s$ of $\mu_s$, are absolutely continuous with respect to the induced Lebesgue measures on $W^s_{r(x)}(x), x \in \Lambda(\mu_s)$, if and only if $\Lambda$ is a non-invertible repellor.

Proof. If $f$ is open on a connected $\Lambda$ we saw in Section 1 that $f$ is constant-to-1 on $\Lambda$.

The first part of the proof follows from Theorem 1 and from Theorem 1 of [10]. Indeed in [10] we showed that in the above setting, if none of the stable manifolds centered at $x$ is contained in $\Lambda$, then $\delta^s$ is strictly less than the real dimension $d_s$ of the manifold $W^s_{r(x)}(x)$ (the result in [10], given for the case when $d_s$ is 2, can be generalized easily to other dimensions as long as the condition of conformality on stable manifolds is satisfied). Thus in order to have absolute continuity of the stable conditional measures we must have some local stable manifolds contained in $\Lambda$, equivalent
to $\Lambda$ being a local repellor (in the terminology of [10]). But we proved in Proposition 1 of [10] that when $f|\Lambda : \Lambda \to \Lambda$ is open, then $\Lambda$ is a local repellor if and only if $\Lambda$ is a repellor.

The converse is clearly since, if $\Lambda$ is a repellor, then the local stable manifolds are contained inside $\Lambda$, and thus the stable dimension $\delta^s$ is equal to the dimension $d_s$ of the manifold $W^s_r(x)$. Hence from Theorem 4 it follows that the stable conditional measures of $\mu_s$ are geometric of exponent $d_s$; thus they are absolutely continuous with respect to the respective induced Lebesgue measures. □

Let us give in the end some examples of $c$-hyperbolic endomorphisms which are constant-to-1 on basic sets, for which we will apply Theorem 1 and its Corollaries.

**Example 1.** The first and simplest example is that of a product

$$f(z, w) = (f_1(z), f_2(w)), (z, w) \in \mathbb{C}^2$$

where $f_1$ has a fixed attracting point $p$ and $f_2$ is expanding on a compact invariant set $J$. Then the basic set that we consider is $\Lambda := \{p\} \times J$. For instance take $f(z, w) = (z^2 + c, w^2), c \neq 0, |c|$ small, on the basic set $\Lambda = \{p_c\} \times S^1$, where $p_c$ denotes the unique fixed attracting point of $z \to z^2 + c$. The stable dimension here is equal to zero and the intersections of type $W^s_r(x) \cap \Lambda$ are singletons.

**Example 2.** We can take a hyperbolic toral endomorphism $f_A$ on $\mathbb{T}^2$, where $A$ is an integer-valued matrix with one eigenvalue of absolute values strictly less than 1, and another eigenvalue of absolute value strictly larger than 1. In this case we can take $\Lambda = \mathbb{T}^2$, and we have the stable dimension equal to 1. We see that $f_A$ is $|\det(A)|$-to-1 on $\mathbb{T}^2$.

We may take also $f_{A,\varepsilon}$ a perturbation of $f_A$ on $\mathbb{T}^2$. Then again $f_{A,\varepsilon}$ is $|\det(A)|$-to-1 on $\mathbb{T}^2$, and $c$-hyperbolic on $\mathbb{T}^2$. The stable dimension is equal to 1, but the stable potential $\Phi^s$ is not necessarily constant now. From Corollary 3 we see that the stable conditional measures of the equilibrium measure $\mu_s$ are absolutely continuous.

**Example 3.** We construct now examples of folded repellors which are not necessarily Anosov endomorphisms.

We remark first that if $\Lambda$ is a repellor for an endomorphism $f$, with neighbourhood $U$ so that $\bar{U} \subset f(U)$, then $f^{-1}(\Lambda) \cap U = \Lambda$. Therefore if $\Lambda$ is in addition connected, it follows easily that $f$ is constant-to-1 on $\Lambda$. Let us show now that constant-to-1 repellors are stable under perturbations.

**Proposition 2.** Let $\Lambda$ be a connected repellor for an endomorphism $f$ so that $f$ is hyperbolic on $\Lambda$, and let a perturbation $f_\varepsilon$ which is $C^1$-close to $f$. Then $f_\varepsilon$ has a connected repellor $\Lambda_\varepsilon$ close to $\Lambda$, and such that $f_\varepsilon$ is hyperbolic on $\Lambda_\varepsilon$. Moreover for any $x \in \Lambda_\varepsilon$, the number of $f_\varepsilon$-preimages of $x$ belonging to $\Lambda_\varepsilon$, is the same as the number of $f$-preimages in $\Lambda$ of a point from $\Lambda$.

**Proof.** Since $\Lambda$ has a neighbourhood $U$ so that $\bar{U} \subset f(U)$, it follows that for $f_\varepsilon$ close enough to $f$, we will obtain $\bar{U} \subset f_\varepsilon(U)$. If $f_\varepsilon$ is $C^1$-close to $f$, then we can take the set $\Lambda_\varepsilon := \bigcap_{n \in \mathbb{Z}} f_\varepsilon^n(U)$, and it is quite well-known that $f_\varepsilon$ is hyperbolic on $\Lambda_\varepsilon$ (for example [16], etc.)

We know that there exists a conjugating homeomorphism $H : \hat{\Lambda} \to \hat{\Lambda}_\varepsilon$ which commutes with $\hat{f}$ and $\hat{f}_\varepsilon$. The natural extension $\hat{\Lambda}$ is connected iff $\Lambda$ is connected. Hence $\hat{\Lambda}_\varepsilon$ is connected and so $\Lambda_\varepsilon$ is also connected. Moreover since $\bar{U} \subset f_\varepsilon(U)$, we obtain that $\Lambda_\varepsilon$ is a connected repellor for $f_\varepsilon$. 13
Now assume that \( x \in \Lambda \) has \( d \) \( f \)-preimages in \( \Lambda \). Then if \( C_f \cap \Lambda = \emptyset \) and if \( f_\varepsilon \) is \( C^1 \)-close enough to \( f \), it follows that the local inverse branches of \( f_\varepsilon \) are close to the local inverse branches of \( f \) near \( \Lambda \). Therefore any point \( y \in \Lambda_\varepsilon \) has exactly \( d \) \( f_\varepsilon \)-preimages in \( U \), denoted by \( y_1, \ldots, y_d \). Any of these \( f_\varepsilon \)-preimages from \( U \) has also an \( f_\varepsilon \)-preimage in \( U \) since \( \bar{U} \subset f_\varepsilon(U) \), etc. Thus \( y_i \in \Lambda_\varepsilon = \bigcap_{n \in \mathbb{Z}} f_\varepsilon^n(U), i = 1, \ldots, d \); hence any point \( y \in \Lambda_\varepsilon \) has exactly \( d \) \( f_\varepsilon \)-preimages belonging to the repellor \( \Lambda_\varepsilon \).

Let us now take the hyperbolic toral endomorphism \( f_A \) from Example 2, and the product \( f(z, w) = (z^k, f_A(w)), (z, w) \in \mathbb{P}^1 \mathbb{C} \times \mathbb{T}^2 \), for some fixed \( k \geq 2 \). And consider a \( C^1 \)-perturbation \( f_\varepsilon \) of \( f \) on \( \mathbb{P}^1 \mathbb{C} \times \mathbb{T}^2 \). Since \( f \) is \( c \)-hyperbolic on its connected repellor \( \Lambda := S^1 \times \mathbb{T}^2 \), it follows from Proposition 2 that the perturbation \( f_\varepsilon \) also has a connected folded repellor \( \Lambda_\varepsilon \), on which it is \( c \)-hyperbolic. Also it follows from above that \( f_\varepsilon \) is constant-to-1 on \( \Lambda_\varepsilon \), namely it is \( (k + |\det(A)|) \)-to-1. The stable dimension \( \delta^s(f_\varepsilon) \) of \( f_\varepsilon \) on \( \Lambda_\varepsilon \) is equal to 1 in this case. We can form the stable potential of \( f_\varepsilon \), namely \( \Phi^s(f_\varepsilon)(z, w) := \log |D(f_\varepsilon)_w|(z, w), (z, w) \in \Lambda_\varepsilon \), and the equilibrium measure \( \mu_s(f_\varepsilon) \) of \( \delta^s(f_\varepsilon) \cdot \Phi^s(f_\varepsilon) \), like in Theorem 1. Since the basic set \( \Lambda_\varepsilon \) is a repellor, we obtain from Corollary 3 that the stable conditional measures of \( \mu_s(f_\varepsilon) \) are absolutely continuous on the local stable manifolds of \( f_\varepsilon \) (which in general are non-linear submanifolds).

One actual example can be constructed by the above procedure, if we consider first the linear toral endomorphism \( f_A(w) = (3w_1 + 2w_2, 2w_1 + 2w_2), w = (w_1, w_2) \in \mathbb{R}^2/\mathbb{Z}^2 \). The associated matrix \( A \) has one eigenvalue of absolute value less than 1 and the other eigenvalue larger than 1, hence \( f_A \) is hyperbolic on \( \mathbb{T}^2 \). And as above we can take the product \( f(z, w) = (z^k, f_A(w)) \) for some \( k \geq 2 \). Then we consider the perturbation endomorphism:

\[
f_\varepsilon(z, w) := (z^k, 3w_1 + 2w_2 + \varepsilon \sin(2\pi(w_1 + 5w_2)), 2w_1 + 2w_2 + \varepsilon \cos(2\pi w_2) + \varepsilon \sin^2(\pi(w_1 - 2w_2))),
\]

defined for \( z \in \mathbb{P}^1 \mathbb{C}, w \in \mathbb{T}^2 \). We see that \( f_\varepsilon \) is well defined as an endomorphism on \( \mathbb{P}^1 \mathbb{C} \times \mathbb{T}^2 \) and that it has a repellor \( \Lambda_\varepsilon \) close to \( S^1 \times \mathbb{T}^2 \), given by Proposition 2 namely there exists a neighbourhood \( U \) of \( S^1 \times \mathbb{T}^2 \) so that

\[
\Lambda_\varepsilon = \bigcap_{n \in \mathbb{Z}} f_\varepsilon^n(U)
\]

Then \( f_\varepsilon \) is \( c \)-hyperbolic on \( \Lambda_\varepsilon \) (see Definition 1) and it is \( (k + 2) \)-to-1 on \( \Lambda_\varepsilon \). The stable potential \( \Phi^s(f_\varepsilon) \) is not necessarily constant in this case. We obtain as before that the stable conditional measures of \( \mu_s(f_\varepsilon) \) are absolutely continuous, and that the stable pointwise dimension of \( \mu_s(f_\varepsilon) \) is essentially equal to 1, on \( \mu_s \)-a.a local stable manifolds over \( \Lambda_\varepsilon \).

Acknowledgements: Partial support for this work was provided by PN II Project ID-1191.

References

[1] L. Barreira, Dimension and recurrence in hyperbolic dynamics, Birkhauser, Basel-Boston-Berlin, 2008.

[2] L. Barreira and C. Wolf, Measures of maximal dimension for hyperbolic diffeomorphisms, Commun. Math. Phys. 239, 2003, 93-113.
[3] H. G. Bothe, Shift spaces and attractors in noninvertible horseshoes, Fundamenta Math., 152 (1997), no. 3, 267-289.

[4] R. Bowen, Equilibrium states and the ergodic theory of Anosov diffeomorphisms, Lecture Notes in Mathematics, 470, Springer 1973.

[5] J. P. Eckmann and D. Ruelle, Ergodic theory of chaos, Rev. Modern Physics, 57, no. 3, 1985, 617-656.

[6] A. Katok and B. Hasselblatt, Introduction to the Modern Theory of Dynamical Systems, Cambridge Univ. Press, London-New York, 1995.

[7] F. Ledrappier, L.S Young, The metric entropy of diffeomorphisms I. Characterization of measures satisfying Pesin’s entropy formula, Ann. of Math. (2) 122 (1985), 509–539.

[8] P. D Liu, Invariant measures satisfying an equality relating entropy, folding entropy and negative Lyapunov exponents, Commun. Math. Physics, vol. 284, no. 2, 2008, 391-406.

[9] E. Mihailescu, Unstable manifolds and Holder structures associated to noninvertible maps, Discrete and Cont. Dynam. Syst., 14, no.3, 2006, 419-446.

[10] E. Mihailescu, Metric properties of some fractal sets and applications of inverse pressure, to appear in Math. Proc. Cambridge, 2010, available at the journal website and at www.imar.ro/~mihailescu

[11] E. Mihailescu and M. Urbanski, Inverse pressure estimates and the independence of stable dimension for non-invertible maps, Canadian J. Math., 60, no. 3, 2008, 658-684.

[12] E. Mihailescu and M. Urbanski, Transversal families for hyperbolic skew products, Discrete and Cont. Dynam. Syst. 21, 3, 2008, 907-928.

[13] W. Parry and P. Walters, Endomorphisms of a Lebesgue space, Bull. AMS, 78, 1972, 272-276.

[14] V. A. Rokhlin, Lectures on the theory of entropy of transformations with invariant measures, Russ. Math. Surveys, 22, 1967, 1-54.

[15] D. Ruelle, The thermodynamical formalism for expanding maps, Commun. Math. Physics 125, 1989, 239-262.

[16] D. Ruelle, Elements of differentiable dynamics and bifurcation theory, Academic Press, New York, 1989.

**Email:** Eugen.Mihailescu@imar.ro

Institute of Mathematics of the Romanian Academy, P. O. Box 1-764, RO 014700, Bucharest, Romania.

**Webpage:** www.imar.ro/~mihailes