THE DESCENT STATISTIC OVER 123-AVOIDING PERMUTATIONS

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Abstract We exploit Krattenthaler’s bijection between 123-avoiding permutations and Dyck paths to determine the Eulerian distribution over the set $S_n(123)$ of 123-avoiding permutations in $S_n$. In particular, we show that the descents of a permutation correspond to valleys and triple falls of the associated Dyck path. We get the Eulerian numbers of $S_n(123)$ by studying the joint distribution of these two statistics on Dyck paths.

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1. Introduction

A permutation $\sigma \in S_n$ avoids a pattern $\tau \in S_k$ if $\sigma$ does not contain a subsequence that is order-isomorphic to $\tau$. The subset of $S_n$ of all permutations avoiding a pattern $\tau$ is denoted by $S_n(\tau)$. Pattern avoiding permutations have been intensively studied in recent years from many points of view (see e.g. [1], [4] and references therein).

In the case $\tau \in S_3$, it has been shown that the cardinality of $S_n(\tau)$ equals the $n$-th Catalan number, for every pattern $\tau$, and hence the set $S_n(\tau)$ is in bijection with the set of Dyck paths of semilength $n$. Indeed, the six patterns in $S_3$ are related as follows:

- $321 = 123^{rev}$,
- $231 = 132^{rev}$,
- $213 = 132^c$,
- $312 = (132^c)^{rev}$,

where $rev$ and $c$ denote the usual reverse and complement operations. Hence, in order to determine the distribution of the descent statistic over $S_n(\tau)$, for every $\tau \in S_3$, it is sufficient to examine the distribution of descents over two sets $S_n(132)$ and $S_n(123)$.

In both cases, the two bijections due to Krattenthaler [1] allow to translate the descent statistic into some appropriate statistics on Dyck paths.
In the case $\tau = 132$, the descents of a permutation are in one-to-one correspondence with the valleys of the associated Dyck path (see [5]).

In this paper we investigate the case $\tau = 123$. In particular, we exploit a variation of Krattenthaler’s map to translate the descents of a permutation $\sigma \in S_n(123)$ into peculiar subconfigurations of the associated Dyck path, namely, valleys and triple falls.

For that reason, we study the joint distribution of valleys and triple falls over the set $\mathcal{P}_n$ of Dyck paths of semilength $n$, and we give an explicit expression for its trivariate generating function

$$A(x, y, z) = \sum_{n \geq 0} \sum_{D \in \mathcal{P}_n} x^n y^v(D) z^{tf(D)} = \sum_{n,p,q \geq 0} a_{n,p,q} x^n y^p z^q,$$

where $v(D)$ denotes the number of valleys in $D$ and $tf(D)$ denotes the number of triple falls in $D$. This series specializes into some well known generating functions, such as the generating function of Catalan numbers, Motzkin numbers, Narayana numbers, and seq. A092107 in [5] (see also [2]).

2. DYCK PATHS

A Dyck path is a lattice path in the integer lattice $\mathbb{N} \times \mathbb{N}$ starting from the origin, consisting of up-steps $U = (1, 1)$ and down steps $D = (1, -1)$, never passing below the x-axis, and ending at ground level.

We recall that a return of a Dyck path is a down step ending on the x-axis. An irreducible Dyck path is a Dyck path with exactly one return.

We observe that a Dyck path $\mathcal{D}$ can be decomposed according to its last return (last return decomposition) into the juxtaposition of a (possibly empty) Dyck path $\mathcal{D}'$ of shorter length and an irreducible Dyck path $\mathcal{D}''$.

For example, the Dyck path $\mathcal{D} = U^5 D^2 U D U D U^3 D U D^3$ decomposes into $\mathcal{D}' \oplus \mathcal{D}''$, where $\mathcal{D}' = U^5 D^2 U D U D$ and $\mathcal{D}'' = U^3 D U D^3$, as shown in Figure 1.

3. KRATTENTHALER’S BIJECTION

In [1], Krattenthaler describes a bijection between the set $S_n(123)$ and the set $\mathcal{P}_n$ of Dyck paths of semilength $n$. We present a slightly modified version of this bijection.
Let $\sigma = \sigma(1) \ldots \sigma(n)$ be a 123-avoiding permutation. Recall that a left-to-right minimum of $\sigma$ is an element $\sigma(i)$ which is smaller than $\sigma(j)$, with $j < i$ (note that the first entry $\sigma(1)$ is a left-to-right minimum). Let $x_1, \ldots, x_s$ be the left-to-right minima in $\sigma$. Then, we can write

\[
\sigma = x_1 w_1 \ldots x_s w_s,
\]

where $w_i$ are (possibly empty) words. Moreover, since $\sigma$ avoids 123, the word $w_1 w_2 \ldots w_s$ must be decreasing.

In order to construct the Dyck path $\kappa(\sigma)$ corresponding to $\sigma$, read the decomposition (1) from left to right. Any left-to-right minimum $x_i$ is translated into $x_{i-1} - x_i$ up steps (with the convention $x_0 = n + 1$) and any subword $w_i$ is translated into $l_i + 1$ down steps, where $l_i$ denotes the number of elements in $w_i$.

For example, the permutation $\sigma = 5 7 2 6 4 3 1$ in $S_7(123)$ corresponds to the path in Figure 2.
4. The descent statistic

We say that a permutation $\sigma$ has a descent at position $i$ if $\sigma(i) > \sigma(i + 1)$. We denote by $\text{des}(\sigma)$ the number of descents of the permutation $\sigma$.

In this section we determine the generating function

$$E(x, y) = \sum_{n \geq 0} \sum_{\sigma \in S_n(123)} x^n y^{\text{des}(\sigma)} = \sum_{n \geq 0} \sum_{k \geq 0} e_{n,k} x^n y^k,$$

where $e_{n,k}$ denotes the number of permutations in $S_n(123)$ with $k$ descents.

**Proposition 1.** Let $\sigma$ be a permutation in $S_n(123)$, and $D = \kappa(\sigma)$. The number of descents of $\sigma$ is

$$\text{des}(\sigma) = v(D) + tf(D),$$

where $v(D)$ is the number of valleys in $D$ and $tf(D)$ is the number of triple falls in $D$, namely, the number of occurrences of $DDD$ in $D$.

**Proof** Let $\sigma = x_1 w_1 \ldots x_s w_s$ be a 123-avoiding permutation. The descents of $\sigma$ occur precisely in the following positions:

1. between two consecutive symbols in the same word $w_i$ (we have $l_i - 1$ of such descents),
2. before every left-to-right minimum $x_i$, except for the first one.

The proof is completed as soon as we remark that:

1. every word $w_i$ is mapped into a descending run of $\kappa(\sigma)$ of length $l_i + 1$. Such descending run contains $l_i - 1$ triple falls, that are therefore in bijection with the descents contained in $w_i$,
2. every left-to-right minimum $x_i$ with $i \geq 2$ corresponds to a valley in $\kappa(\sigma)$.

⋄

The preceding result implies that we can switch our attention from permutations in $S_n(123)$ with $k$ descents to Dyck paths of semilength $n$ with $k$ among valleys and triple falls. Hence, we study the joint distribution of valleys and triple falls over $P_n$, namely, we analyze the generating function

$$A(x, y, z) = \sum_{n \geq 0} \sum_{\varnothing \in P_n} x^n y^{v(\varnothing)} z^{tf(\varnothing)} = \sum_{n,p,q \geq 0} a_{n,p,q} x^n y^p z^q.$$
We determine the relation between the function $A(x, y, z)$ and the generating function

$$B(x, y, z) = \sum_{n \geq 0} \sum_{\mathcal{D} \in \mathcal{IP}_n} x^n y^{v(\mathcal{D})} z^{f(\mathcal{D})} = \sum_{n, p, q \geq 0} b_{n,p,q} x^n y^p z^q$$

of the same joint distribution over the set $\mathcal{IP}_n$ of irreducible Dyck paths in $\mathcal{P}_n$.

**Proposition 2.** For every $n > 2$, we have:

\begin{equation}
    b_{n,p,q} = a_{n-1,p,q-1} - a_{n-2,p-1,q-1} + a_{n-2,p-1,q}.
\end{equation}

**Proof** An irreducible Dyck path of semilength $n$ with $p$ valleys and $q$ triple falls can be obtained by prepending $U$ and appending $D$ to a Dyck path of semilength $n-1$ of one of the two following types:

1. a Dyck path with $p$ valleys and $q$ triple falls, ending with $UD$,
2. a Dyck path with $p$ valleys and $q-1$ triple falls, not ending with $UD$.

We remark that:

1. the paths of the first kind are in bijection with Dyck paths of semilength $n-2$ with $p-1$ valleys and $q$ triple falls, enumerated by $a_{n-2,p-1,q}$.

2. in order to enumerate the paths of the second kind we have to subtract from the integer $a_{n-1,p,q-1}$ the number of Dyck paths of semilength $n-1$ with $p$ valleys and $q-1$ triple falls, ending with $UD$. Dyck paths of this kind are in bijection with Dyck paths of semilength $n-2$ with $p-1$ valleys and $q-1$ triple falls, enumerated by $a_{n-2,p-1,q-1}$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{dyck_path}
\caption{The Dyck path $U^5D^3U^2D^3UD^2$ with 2 valleys and 2 triple falls is obtained by appending $UD$ to the path $U^4D^3U^2D^3$ with 1 valley and 2 triple falls, and then elevating.}
\end{figure}
Figure 4. The Dyck path $U^5D^3U^2D^2UD^3$ with 2 valleys and 2 triple falls is obtained by elevating the path $U^4D^3U^2D^2UD^2$ with 2 valleys and 1 triple fall.

Proposition 3. For every $n > 0$, we have:

\[ a_{n,p,q} = b_{n,p,q} + \sum_{i=1}^{n-1} \sum_{j,s>0} b_{i,j,s} a_{n-i,p-j-1,q-s}. \]

Proof Let $\mathcal{D}$ be a Dyck path of semilength $n$ and consider its last return decomposition $\mathcal{D} = \mathcal{D}' \oplus \mathcal{D}''$. If $\mathcal{D}'$ is empty, then $\mathcal{D}$ is irreducible. Otherwise:

- $v(\mathcal{D}) = v(\mathcal{D}') + v(\mathcal{D}'') + 1$,
- $tf(\mathcal{D}) = tf(\mathcal{D}') + tf(\mathcal{D}'')$.

Identities (2) and (3) yield the following relations between the two generating functions $A(x, y, z)$ and $B(x, y, z)$:

Proposition 4. We have:

\[ B(x, y, z) = (A(x, y, z) - 1)(xz + x^2y - x^2yz) + 1 + x + x^2 - x^2z, \]
\[ A(x, y, z) = B(x, y, z) + y(B(x, y, z) - 1)(A(x, y, z) - 1) \]

Proof Observe that recurrence (2) holds for $n > 2$. This fact gives rise to the correction terms of degree less than 3 in Formula (4).

Combining Formulæ (4) and (5) we obtain the following:

Theorem 5. We have:

\[ A(x, y, z) = \frac{1}{2xy(xyz - z - xy)} \left(-1 + xy + 2x^2y - 2x^2y^2 + xz - 2xyz - 2x^2yz + 2x^2y^2z + \sqrt{1 - 2xy - 4x^2y + x^2y^2 - 2xz + 2x^2yz + x^2z^2}\right) \]
This last result allows us to determine the generating function $E(x, y)$ of the Eulerian distribution over $S_n(123)$. In fact, previous arguments show that

$$E(x, y) = A(x, y, y)$$

and hence:

**Theorem 6.** We have:

$$E(x, y) = \frac{-1 + 2xy + 2x^2y - 2xy^2 - 4x^2y^2 + 2x^2y^3 + \sqrt{1 - 4xy - 4x^2y + 4x^2y^3}}{2xy^2(xy - 1 - x)}.$$

The first values of the sequence $e_{n,d}$ are shown in the following table:

| n/d | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|-----|---|---|---|---|---|---|---|
| 0   | 1 |
| 1   | 1 |
| 2   | 1 | 1 |
| 3   | 0 | 4 | 1 |
| 4   | 0 | 2 | 11 | 1 |
| 5   | 0 | 0 | 15 | 26 | 1 |
| 6   | 0 | 0 | 5  | 69 | 57 | 1 |
| 7   | 0 | 0 | 0  | 56 | 252| 120| 1 |

Needless to say, the series $A(x, y, z)$ specializes into some well known generating functions. In particular, $A(x, 1, 1)$ is the generating function of Catalan numbers, $A(x, 1, 0)$ the generating function of Motzkin numbers, $yA(x, y, 1)$ the generating function of Narayana numbers, and $A(x, 1, z)$ the generating function of seq. A092107 in [5].

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