Totally Abelian Toeplitz operators and geometric invariants associated with their symbol curves

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Abstract: This paper mainly studies totally Abelian operators in the context of analytic Toeplitz operators on both the Hardy and Bergman space. When the symbol is a meromorphic function on $\mathbb{C}$, we establish the connection between totally Abelian property of these operators and and geometric properties of their symbol curves. It is found that winding numbers and multiplicities of self-intersection of symbol curves play an important role in this topic. Techniques of group theory, complex analysis, geometry and operator theory are intrinsic in this paper. As a byproduct, under a mild condition we provides an affirmative answer to a question raised in [2, 21], and also construct some examples to show that the answer is negative if the associated conditions are weakened.

Keywords: finite Blaschke products; local inverse; winding numbers; multiplicities of self-intersection; meromorphic functions.

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1 Introduction

In this paper, $\mathbb{D}$ denotes the unit disk in the complex plane $\mathbb{C}$, and $\mathbb{T}$ denotes the boundary of $\mathbb{D}$. Let $\mathcal{M}(\overline{\mathbb{D}})$ consist of all meromorphic functions over $\mathbb{C}$ which have no pole on the closed unit disk $\overline{\mathbb{D}}$, and $\mathcal{R}(\overline{\mathbb{D}})$ denotes the set of all rational functions without pole on $\overline{\mathbb{D}}$. It is clear that $\mathcal{M}(\overline{\mathbb{D}}) \supseteq \mathcal{R}(\overline{\mathbb{D}})$.

Let $H^\infty(\mathbb{D})$ denote the Banach algebra of all bounded holomorphic functions over $\mathbb{D}$, and $H^\infty(\overline{\mathbb{D}})$, as a subset of $H^\infty(\mathbb{D})$, consists of functions that are holomorphic on $\overline{\mathbb{D}}$. The Hardy space $H^2(\mathbb{D})$ consists of all holomorphic functions on $\mathbb{D}$ whose Taylor coefficients at 0 are square summable. The Bergman space $L^2_\alpha(\mathbb{D})$ consists of all holomorphic functions over $\mathbb{D}$ that are square integrable with respect to the normalized area measure over $\mathbb{D}$. For each function $\phi$ in $H^\infty(\mathbb{D})$, let $T_\phi$ denote the Toeplitz operator on the Hardy
Let $H$ be a Hilbert subspace. For an operator $T$ in $B(H)$, $\{T\}'$ denotes the commutant of $T$; that is,
\[ \{T\}' = \{ S \in B(H) : ST = TS \}, \]
which is a WOT-closed subalgebra of $B(H)$. The operator $T$ is called \textit{totally Abelian} if $\{T\}'$ is Abelian; equivalently, $\{T\}'$ is a maximal Abelian subalgebra of $B(H)$ [3]. Berkson and Rubel [3] completely characterized totally Abelian operators in $B(H)$ for $\dim H < \infty$: in this case they proved that $T$ is totally Abelian if and only if $T$ has a cyclic vector. In the case of $\dim H = \infty$ and $H$ being separable, they also characterized when normal operators (including unitary operators) and non-unitary isometric operators are totally Abelian. Related work on analytic Toeplitz operators on $H^2(D)$ are also initiated by Berkson and Rubel [3]. It is shown that if $\phi$ is an inner function, then $T\phi$ is totally Abelian on $H^2(D)$ if and only if there exist a unimodular constant $c$ and a point $\lambda \in \mathbb{D}$ such that $\phi(z) = c\frac{\lambda - z}{1 - \overline{\lambda}z}$ [3, Theorem 2.1].

For a function $\phi$ in $H^\infty(D)$, if there exists a point $\lambda \in \mathbb{D}$ such that the inner part of $\phi - \phi(\lambda)$ is a finite Blaschke product, then $\phi$ is called to be in Cowen-Thomson’s class, denoted by $\phi \in C\mathcal{T}(\mathbb{D})$. It is known that $C\mathcal{T}(\mathbb{D})$ contains all nonconstant functions in $H^\infty(\mathbb{D})$. Below, $\mathcal{H}$ denotes the Hardy space $H^2(\mathbb{D})$ or the Bergman space $L^2_\alpha(\mathbb{D})$. As presented below is the remarkable theorem on commutants for analytic Toeplitz operators, due to Thomson and Cowen [21, 22, 5]; also see [11, Chapter 3] for a detailed discussion and see [7, 19, 13] for related work on this line.

\textbf{Theorem 1.1.} [Cowen-Thomson] Suppose $\phi \in C\mathcal{T}(\mathbb{D})$. Then there exists a finite Blaschke product $B$ and an $H^\infty$-function $\psi$ such that $\phi = \psi(B)$ and
\{T_\phi\}' = \{T_B\}' holds on \mathcal{H}.

The identity \phi = \psi(B) in Theorem 1.1 is called a **Cowen-Thomson representation** of \phi. Note that this \(B\) is of maximal order in the following sense: if there is another finite Blaschke product \(\tilde{B}\) and a function \(\tilde{\psi}\) in \(H^\infty(\mathbb{D})\) satisfying \(\phi = \tilde{\psi}(\tilde{B})\), then

\[ \text{order } B \geq \text{order } \tilde{B}. \]

One defines a quantity \(b(\phi)\) to be the maximum of orders of \(\tilde{B}\), for which there is a function \(\tilde{\psi}\) in \(H^\infty(\mathbb{D})\) such that \(\phi = \tilde{\psi}(\tilde{B})\), and \(b(\phi)\) is called **Cowen-Thomson order** of \(\phi\). Thus for the finite Blaschke product \(B\) in Theorem 1.1 we have \(\text{order } B = b(\phi)\). Once \(\phi\) is fixed, it is not difficult to show that \(B\) is uniquely determined in the following sense. If there is another finite Blaschke product \(B_0\) satisfying one of the following:

1. order \(B_0 = b(\phi)\) and there is an \(h \in H^\infty\) such that \(\phi = h(B_0)\);
2. \(\{T_\phi\}' = \{T_{B_0}\}'\),

then there is a Moebius map \(\eta\) such that \(B_0 = \eta(B)\). This means that **Cowen-Thomson representation** of \(\phi\) is unique in the sense of modulo Moebius maps.

For convenience, we now omit the space \(\mathcal{H}\). The following is an immediate consequence of Theorem 1.1, see [2, 21].

**Corollary 1.2.** Let \(\phi\) be a nonconstant function in \(H^\infty(\overline{\mathbb{D}})\). Then there exist a finite Blaschke product \(B\) and a function \(\psi\) in \(H^\infty(\mathbb{D})\) such that \(\phi = \psi(B)\) and \(\{T_\phi\}' = \{T_B\}'\) holds. If \(\phi\) is entire, then \(\psi\) is entire and \(B(z) = z^n\) for some positive integer \(n\).

Suppose \(\phi\) belongs to Cowen-Thomson’s class \(CT(\mathbb{D})\). Then \(T_\phi\) is totally Abelian if and only if \(b(\phi) = 1\). When \(\phi\) is an entire function, expanding \(\phi\)'s Taylor series yields

\[ \phi(z) = \sum_{n=0}^{\infty} a_n z^n. \]

Set \(N = \gcd\{n : a_n \neq 0\}\), and then by Corollary 1.2 \(\{T_\phi\}' = \{T_{z^N}\}'\). In this case, \(b(\phi) = N\). Therefore, for a nonconstant entire function \(\phi\), \(T_\phi\) is totally Abelian if and only if

\[ \gcd\{n : a_n \neq 0\} = 1. \]

Therefore, for totally Abelian property of analytic Toeplitz operators \(T_\phi\) it is important to determine Cowen-Thomson order \(b(\phi)\) of \(\phi\), and it is of
interest to determine the exact form of the Blaschke product with order $b(\phi)$. As we will see, there are several ways to study $b(\phi)$. The first attack is made by Baker, Deddens and Ullman \cite{2} in the case of $\phi$ being an entire function. In what follows, for $c \notin \phi(\mathbb{T})$, let wind $(\phi, c)$ denote the winding number of the curve $\phi(z)$ ($z \in \mathbb{T}$) around the point $c$. Write $n(\phi)$ for the number

$$\min \{ \text{wind}(\phi, \phi(a)) : a \in \mathbb{D}, \phi(a) \notin \phi(\mathbb{T}) \}.$$ 

For a function $\phi \in H^\infty(\mathbb{D})$, it is obvious that $b(\phi) \leq n(\phi)$. If $b(\phi) = n(\phi)$, $\phi$ is called to satisfy Minimal Winding Number Property (MWN Property). It is shown that a nonconstant entire function $\phi$ enjoys MWN Property \cite{2}. For functions in $H^\infty(\mathbb{D})$, the problem raised in \cite{2} and \cite{21} can be reformulated as:

if $\phi$ is a nonconstant function in $H^\infty(\mathbb{D})$, then does $\phi$ have MWN Property; that is, $b(\phi) = n(\phi)$?

If the answer is yes, for a large class of analytic Toeplitz operators we can formulate their totally Abelian property in terms of winding number.

For those functions $\phi$ of MWN Property in $H^\infty(\mathbb{D})$, we can determine the exact form of $B$ appearing in Corollary \cite{1.2}. To be precise, let $a$ be a point in $\mathbb{D}$ such that $\phi - \phi(a)$ does not vanish on $\mathbb{T}$ and

$$\text{wind}(\phi, \phi(a)) = n(\phi) = b(\phi).$$

Denote the inner factor of $\phi - \phi(a)$ by $B_a$, and we will show that $B_a$ is the desired finite Blaschke product. For this, let

$$\phi = \psi(B)$$

be the Cowen-Thomson representation of $\phi$. By Corollary \cite{1.2}, $\psi$ is in $H^\infty(\mathbb{D})$. Let

$$\psi - \phi(a) = \eta F$$

be the inner-outer factorization of $\psi - \phi(a)$, where $\eta$ is inner. We see that

$$\phi - \phi(a) = (\psi - \phi(a)) \circ B = \eta \circ B F \circ B.$$

Therefore $B_a = c \eta \circ B$, where $c$ is a constant with $|c| = 1$. Since

$$\text{order } B_a = \text{wind}(\phi, \phi(a)) = b(\phi) = \text{order } B,$$

this forces that $\eta$ to be a Blaschke factor of order 1. Therefore this $B_a$ is the desired finite Blaschke products in Corollary \cite{1.2}. In this way, finding $B$
essentially reduces to finding one of these points \(a\) (in general, such points \(a\) consist of a nonempty open set). In some cases of interest this procedure is feasible (see Theorem 1.5).

MWN Property is quite restricted. We will provide some examples of functions with good smoothness on the unit circle, and they are in Cowen-Thomson’s class \(CT(\mathbb{D})\) but do not have MWN Property, see Examples 5.5 and 5.6. It is known that entire functions have MWN Property [2]. In Theorem 1.5 we extend this result to all nonconstant meromorphic functions in \(\mathcal{M}(\mathbb{D})\).

Before continuing, we introduce the finite self-intersection property (FSI property). To be precise, for a function \(\phi\) in the disk algebra \(A(\mathbb{D})\) and \(\eta \in \mathbb{T}\), let \(N(\phi - \phi(\eta), \mathbb{T})\) denote the cardinality of the set

\[
\{w \in \mathbb{T} : \phi(w) - \phi(\eta) = 0\}.
\]

called the multiplicity of self-intersection of the curve \(\phi(z) (z \in \mathbb{T})\) at the point \(\phi(\eta)\). Write

\[N(\phi) = \min \{N(\phi - \phi(\eta), \mathbb{T}) : \eta \in \mathbb{T}\},\]

called the multiplicity of self-intersection of the curve \(\phi(z) (z \in \mathbb{T})\). It is not difficult to verify that \(b(\phi) \leq N(\phi)\). A function \(\phi\) in \(A(\mathbb{D})\) is called to have FSI property if except for a finite subset of \(\mathbb{T}\) each point \(\xi \in \mathbb{T}\) satisfies \(N(\phi - \phi(\xi), \mathbb{T}) = 1\) [14].

For meromorphic functions in \(\mathcal{M}(\mathbb{D})\), we have the following result.

**Theorem 1.3.** Suppose \(\phi\) is a nonconstant function in \(\mathcal{M}(\mathbb{D})\). The following are equivalent:

1. the Toeplitz operator \(T_\phi\) is totally Abelian;
2. \(\phi\) has FSI property.
3. \(N(\phi) = 1\).

For a nonconstant function \(\phi\) in \(H^\infty(\mathbb{D})\), let \(\phi = \psi(B)\) be a Cowen-Thomson representation of \(\phi\). If \(\psi \in H^\infty(\mathbb{D})\) has FSI property, then \(\phi\) is called to have FSI-decomposable property. Quine [14] showed that each nonconstant polynomial has FSI-decomposable property. In this paper we prove that each nonconstant function in \(\mathcal{M}(\mathbb{D})\) also enjoys the same property (see Theorem 4.1).

For the characterization of geometric property of symbol curves, we introduce the semigroup \(G(\phi)\). Precisely, for each continuous function \(\phi\) on
Define $G(\phi)$ to be the set of all continuous maps $\rho$ from $T$ to $T$ satisfying $\phi(\rho) = \phi$.

For a finite Blaschke product $\phi$, $G(\phi)$ is a finite cyclic group, and furthermore $\sharp(G(\phi)) = \text{order } \phi \ [4]$. For $\phi \in H^\infty(\overline{T})$, we have the following result.

**Theorem 1.4.** Suppose $\phi$ is a nonconstant function in $H^\infty(\overline{T})$. Then $G(\phi)$ is a finite cyclic group.

Let $o(\phi)$ denote the order of $G(\phi)$; that is, $o(\phi) = \#G(\phi)$. We thus have four integer quantities for a function $\phi$: $o(\phi)$, $b(\phi)$, $n(\phi)$ and $N(\phi)$. We will prove that if $\phi$ is in $H^\infty(\overline{T})$, then $b(\phi) \leq o(\phi) \leq n(\phi)$ and $o(\phi) \leq N(\phi)$ (see Section 2).

For a finite Blaschke product $B$, $o(B) = \text{order } B = n(B) = N(B)$. More generally, we will prove that each nonconstant meromorphic function in $\mathcal{M}(\overline{T})$ enjoys this property.

**Theorem 1.5.** Suppose $\phi$ is a nonconstant function in $\mathcal{M}(\overline{T})$. Then $n(\phi) = b(\phi) = o(\phi) = N(\phi)$.

In particular, for $\phi \in \mathcal{M}(\overline{T})$ the Toeplitz operator $T_\phi$ is totally Abelian if and only if $o(\phi) = 1$; equivalently, the identity map is the only continuous map $\rho : T \to T$ satisfying $\phi(\rho) = \phi$.

This paper is arranged as follows. Section 2 first provides some basic properties of the group $G(\phi)$ for $\phi$ in $H^\infty(\overline{T})$ and gives the proof of Theorem 1.4. Section 3 focuses on Toeplitz operators with meromorphic symbols, discusses the MWN property, $o(\phi)$, $N(\phi)$ and Cowen-Thomson order $b(\phi)$ of $\phi$ for $\phi \in \mathcal{M}(\overline{T})$, and gives the proof of Theorem 1.5. Section 4 first presents the proof of Theorem 1.3 and then give further results on FSI and FSI-decomposable properties. Section 5 constructs some examples. On one hand, we give some totally Abelian Toeplitz operators defined by symbols in $\mathcal{M}(\overline{T})$. On the other hand, some examples show that conclusion of Theorem 1.3 can fail even if the associated functions have good smoothness on $T$.

2 The group $G(\phi)$

This section provides some basic properties of $G(\phi)$.

For a function $\phi$ holomorphic on the closure of a domain $\Omega$, $N(\phi, \Omega)$ or $N(\phi, \overline{T})$ denotes the number of zeros of $\phi$ on $\Omega$ or $\overline{T}$ respectively, counting multiplicity. The winding number of $\phi$ is defined to be $\text{wind } (\phi, 0)$. 

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The following shows that each member in $G(\phi)$ has very strong restriction.

**Lemma 2.1.** Suppose $\phi$ is a nonconstant function in $H^\infty(\bar{D})$. Then every $\rho \in G(\phi)$ is an automorphism of $T$ with winding number $1$.

**Proof.** For each $\rho \in G(\phi)$, define
\[
\Lambda = \{ t \in [0, 2\pi) : \phi'(e^{it})\phi'(\rho(e^{it})) = 0 \}.
\]
We will show that $\Lambda$ is a finite set. In fact, let $Z'$ denote the zero of $\phi'$ on $D$ and put
\[
F = \phi^{-1}(\phi(Z')) \cap T.
\]
Since $\phi \in H^\infty(D)$, $F$ is a finite set. If $\phi'(\rho(e^{it})) = 0$, then $\rho(e^{it}) \in F$.

Therefore,
\[
e^{it} \in \bigcup_{\zeta \in F} \{ z \in T : \phi(\rho(z)) - \phi(\zeta) = 0 \} = \bigcup_{\zeta \in F} \{ z \in T : \phi(z) - \phi(\zeta) = 0 \}.
\]
Since $\phi$ is holomorphic on $D$ and nonconstant, the right hand side is a union of finitely many finite sets. Hence $\{ t \in [0, 2\pi) : \phi'(e^{it}) = 0 \}$ is a finite set, and so is $\Lambda$.

Write $(0, 2\pi) \setminus \Lambda = \bigcup_{k=0}^{n-1} (t_k, t_{k+1})$, where $0 = t_0 < t_1 < \cdots < t_n = 2\pi$.

Since $\rho$ is continuous, there exists a real continuous function $\theta$ on $[0, 2\pi]$ such that $\rho(e^{it}) = e^{i\theta(t)}$. Then
\[
\phi(e^{it}) = \phi(e^{i\theta(t)}),
\]
the Inverse Function Theorem implies that $\theta$ is differentiable on $(0, 2\pi) \setminus \Lambda$.

Taking derivatives of $t$ yields that
\[
\theta'(t) = \frac{e^{it}\phi'(e^{it})}{e^{i\theta(t)} \phi'(e^{i\theta(t)})} \neq 0, \ t \in (0, 2\pi) \setminus \Lambda. \quad (2.1)
\]
Hence for $0 \leq k \leq n - 1$, $\theta$ is strictly monotonic on each interval $(t_k, t_{k+1})$.

Since $\phi(T)$ is of zero area measure and $\phi(D)$ is open, one can pick $\lambda \in D$ such that $\phi(\lambda) \notin \phi(T)$. By Argument Principle, we have
\[
N(\phi - \phi(\lambda), D) = \frac{1}{2\pi i} \int_T \frac{\phi'(\xi)}{\phi(\xi) - \phi(\lambda)} \, d\xi
\]
\[
= \frac{1}{2\pi} \int_{0}^{2\pi} \frac{\phi'(e^{it})}{\phi(e^{it}) - \phi(\lambda)} e^{it} \, dt.
\]
\[ \frac{1}{2\pi} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \frac{\phi'(e^{it})}{\phi(e^{it}) - \phi(\lambda)} e^{it} dt \]

\[ \frac{1}{2\pi} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \frac{\phi'(e^{i\theta(t)})}{\phi(e^{i\theta(t)}) - \phi(\lambda)} e^{i\theta(t)} \theta'(t) dt \]

\[ \frac{1}{2\pi} \sum_{k=0}^{n-1} \int_{\theta(t_k)}^{\theta(t_{k+1})} \frac{\phi'(e^{i\theta})}{\phi(e^{i\theta}) - \phi(\lambda)} e^{i\theta} d\theta \]

\[ \frac{1}{2\pi} \int_{\theta(0)}^{\theta(2\pi)} \frac{\phi'(e^{i\theta})}{\phi(e^{i\theta}) - \phi(\lambda)} e^{i\theta} d\theta \]

\[ \frac{\phi'(\xi)}{\phi(\xi) - \phi(\lambda)} d\xi \]

where \( \sharp \phi = \text{wind}(\rho, 0) \). Since \( N(\phi - \phi(\lambda), \mathbb{D}) \) is a positive integer, we have \( \sharp \phi = 1 \). This implies that \( \rho \) is surjective.

It remains to show that \( \rho \) is injective. Otherwise, there exist two points \( \xi_1 \) and \( \xi_2 \) in \( \mathbb{T} \) such that \( \rho(\xi_1) = \rho(\xi_2) = \eta \). Let \( A \) be the set of zeros of \( \phi - \phi(\eta) \) in \( \mathbb{T} \), and \( \rho|_A : A \to A \) is surjective as \( \rho \) is surjective. Since \( A \) is a finite set, \( \rho|_A \) is actually a bijection, which is a contradiction to \( \rho(\xi_1) = \rho(\xi_2) \). The proof is complete. \( \square \)

**Corollary 2.2.** Suppose that \( \phi \) is a nonconstant function in \( H^\infty(\mathbb{D}) \) and both \( \rho_1 \) and \( \rho_2 \) belong to \( G(\phi) \). If \( \rho_1(\xi_0) = \rho_2(\xi_0) \) for some point \( \xi_0 \in \mathbb{T} \), then \( \rho_1 = \rho_2 \).

**Proof.** Let \( \lambda \) be an arbitrary point in \( \mathbb{T} \setminus \{\xi_0\} \). Let \( A \) be the zero set of \( \phi - \phi(\lambda) \) in \( \mathbb{T} \). Arrange the points of \( \{\xi_0\} \cup A \) in the anti-clockwise direction:

\[ \xi_0, \xi_1, \ldots, \xi_{n_0} \quad (n_0 \geq 1) \]

It is clear that \( \rho_1(\{\xi_0\} \cup A) = \rho_2(\{\xi_0\} \cup A) \). By Lemma 2.1, both \( \rho_1 \) and \( \rho_2 \) are automorphisms of \( \mathbb{T} \) with winding number \( \#\rho_1 = \#\rho_2 = 1 \). Thus, when \( \xi \) moves along \( \mathbb{T} \) in the positive direction, the images \( \rho_1(\xi) \) and \( \rho_2(\xi) \) run in the same direction. As \( \xi \) goes from \( \xi_0 \) to \( \xi_1 \), \( \rho_1 \) and \( \rho_2 \) must coincide at the point \( \xi_1 \). By induction, we have \( \rho_1(\xi_k) = \rho_2(\xi_k), 1 \leq k \leq n_0 \). In particular, \( \rho_1(\lambda) = \rho_2(\lambda) \). The proof is finished. \( \square \)

In the case of finite Blaschke products, \( G(\phi) \) is a finite cyclic group. In what follows we will prove that this result also is true for functions in \( H^\infty(\mathbb{D}) \). Now we come to the proof of Theorem 1.4 (=Theorem 2.3), which is represented as below.
Theorem 2.3. Suppose $\phi$ is a nonconstant function in $H^\infty(D)$. Then $G(\phi)$ is a finite cyclic group.

Proof. By Lemma 2.1, $G(\phi)$ is a group. We will show that $G(\phi)$ is a finite cyclic group. Note that for a fixed point $\zeta \in \mathbb{T}$, \{\(z \in \mathbb{T} : \phi(z) = \phi(\zeta)\}\} is a finite set, and by Corollary 2.2 $G(\phi)$ is a finite group.

Let $\xi_0$ be a point on $\mathbb{T}$, and let $\xi_0, \cdots, \xi_{n_0-1}$ be all zeros of $\phi - \phi(\xi_0)$ on $\mathbb{T}$ in the anti-clockwise direction. Then define $\{\xi_j\}_{j=0}^\infty$ to be the infinite sequence $

\xi_0, \cdots, \xi_{n_0-1}; \xi_0, \cdots, \xi_{n_0-1}; \cdots.

That is, for all $j$

$$\xi_j = \xi_{[j]}, \text{ where } j \equiv [j] \pmod{n_0},$$

where $0 \leq [j] \leq n_0 - 1$ Let $d$ be the minimal positive integer $l$ for which there is a member $\rho_0$ in $G(\phi)$ satisfying $\rho_0(\xi_0) = \xi_l$. By Lemma 2.1 $\rho_0$ maps each circular arc $\xi_i \xi_{i+1}$ to $\xi_j \xi_{j+1}$ for some $j$, preserving the orientation. By continuity, if $\rho_0(\xi_0) = \xi_l$, then one has

$$\rho_0(\xi_i) = \xi_{i+l}, \quad 0 \leq i \leq n_0 - 1. \quad (2.2)$$

By definition of $d$, there is a member $\tau$ in $G(\phi)$ satisfying $\tau(\xi_0) = \xi_d$. To finish the proof of Theorem 2.3 it suffices to show that for each member $\rho$ in $G(\phi)$, there is an integer $m$ such that $\rho = \tau^m$(in the sense of composition). Write $\rho(\xi_0) = \xi_l$, and there are two integers $k \geq 0$ and $l_0$ such that $0 \leq l_0 < d$ and

$$l = kd + l_0.$$

Letting $\sigma = \tau^{-k} \rho$, we have $\sigma \in G(\phi)$, and by (2.2) $\sigma(\xi_0) = \xi_{l_0}$. By definition of $d$ we have $l_0 = 0$. By Corollary 2.2 $\sigma = id$, forcing $\rho = \tau^k$ to complete the proof.

For two positive integers $m$ and $n$, write $m|n$ to denote that $m$ divides $n$. By Theorem 2.3, if $\phi$ is a nonconstant function in $H^\infty(D)$, $G(\phi)$ is a finite cyclic group. If $\phi = \psi(B)$ is the Cowen-Thomson representation of $\phi$, then $G(B)$ is a subgroup of $G(\phi)$, and hence $o(B)|o(\phi)$. Since $o(B) = b(\phi)$, we have

$$b(\phi) | o(\phi).$$

The following gives some properties of the order $o(\phi)$ of $G(\phi)$. 

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Corollary 2.4. Suppose \( \phi \) is a nonconstant function in \( H^\infty(\mathbb{T}) \). Then for each \( \xi \in \mathbb{T} \), \( o(\phi) | N(\phi - \phi(\xi), \mathbb{T}) \). Besides, for each \( a \in \mathbb{D} \), if \( \phi(a) \notin \phi(\mathbb{T}) \), then
\[ o(\phi) | \text{wind}(\phi, \phi(a)) . \]
In particular,
\[ o(\phi) | N(\phi) \quad \text{and} \quad o(\phi) | n(\phi) , \]
where \( N(\phi) = \min \{ N(\phi - \phi(\xi), \mathbb{T}) : \xi \in \mathbb{T} \} \) and
\[ n(\phi) = \min \{ \text{wind}(\phi, \phi(a)) : a \in \mathbb{D}, \phi(a) \notin \phi(\mathbb{T}) \} . \]

Proof. Let us have a close look at the proof of Theorem 2.3. Fix \( \xi_0 \in \mathbb{T} \), and let \( \xi_0, \cdots, \xi_{n_0-1} \) be all zeros of \( \phi - \phi(\xi_0) \) on \( \mathbb{T} \) in anti-clockwise direction. Let \( d \) be the minimal positive integer \( l \) so that there is a member \( \rho \) in \( G(\phi) \) satisfying \( \rho(\xi_0) = \xi_1 \), and \( \tau \) denotes the generator in \( G(\phi) \) satisfying \( \tau(\xi_0) = \xi_d \). Then by Lemma 2.1 we have
\[ \tau(\xi_i) = \xi_{i+d}, \quad 0 \leq i \leq n_0 - 1 , \]
and \( d | n_0 \). Write
\[ n_0 = jd . \]
For \( k > 0 \), \( \tau^k(\xi_0) = \xi_{kd} \). By Corollary 2.2 we have that \( \tau^k \) is the identity map if and only if \( \xi_{kd} = \xi_0 \). Therefore \( j \) is the minimal positive number \( k \) such that \( \tau^k \) is the identity map, and then
\[ j = o(\phi) . \]
Since \( n_0 = jd, \ j | n_0 \); that is, \( o(\phi) | N(\phi - \phi(\xi_0), \mathbb{T}) \). The first statement is proved.

For \( 0 \leq i \leq j - 1 \) let \( \gamma_i \) denote the positively oriented circular arc \( \xi_i d \xi_{i+1} d, \ (\xi_{jd} = \xi_0) \). Then \( \tau^i(\gamma_0) = \gamma_i \), and \( \phi(\tau^i) = \phi \). Also noting that \( \phi(\gamma_i) \) are closed curves, we have
\[ \text{wind}(\phi(\gamma_i), \lambda) = \text{wind}(\phi(\gamma_0), \lambda), \lambda \in \mathbb{C} \setminus \phi(\mathbb{T}), 0 \leq i \leq j - 1 . \]
Since
\[ \mathbb{T} = \bigcup_{i=0}^{j-1} \gamma_i \quad (\text{as curves}) , \]
\[ \text{wind}(\phi(\mathbb{T}), \lambda) = j \cdot \text{wind}(\phi(\gamma_0), \lambda), \lambda \in \mathbb{C} \setminus \phi(\mathbb{T}) . \]
Thus \( o(\phi) | \text{wind}(\phi, \lambda) \). In particular, we have
\[ o(\phi) | \text{wind}(\phi, \phi(a)) \]
for each \( a \in \mathbb{D} \) such that \( \phi(a) \notin \phi(\mathbb{T}) \). The proof is complete. \( \square \)
Recall that a Jordan curve in \( C \) is the image of a continuous injective map from the unit circle \( T \) into \( C \). For \( \phi \in H^\infty(\Omega) \), in the case of \( \phi(T) \) being a Jordan curve, we have the following.

**Proposition 2.5.** Suppose \( \phi \in H^\infty(\Omega) \) and its image on \( \Omega \) is a Jordan curve. Then there is a univalent function \( h \) on \( \Omega \) and a finite Blaschke product \( B \) satisfying \( \phi = h(B) \). In this case, we have

\[
\begin{align*}
n(\phi) &= b(\phi) = o(\phi) = N(\phi) = \text{order } B. \\
\end{align*}
\]

**Proof.** Write \( \Gamma = \phi(T) \), the image of \( T \) under \( \phi \). Then \( \Gamma \) is a Jordan curve. We will prove that \( \Gamma = \partial \phi(\Omega) \). For this, note that \( \partial \phi(\Omega) \subset \Gamma \). Assume conversely that \( \partial \phi(\Omega) \neq \Gamma \). Since \( \Gamma \) is a Jordan curve, \( \mathbb{C} \setminus \partial \phi(\Omega) \) is connected. A fact from topology states that a domain \( \Omega \) in \( \mathbb{C} \) is a component of \( \mathbb{C} \setminus \partial \Omega \).

Letting \( \Omega = \phi(\Omega) \), we have \( \Omega = \mathbb{C} \setminus \partial \phi(\Omega) \). However, this can not happen since \( \mathbb{C} \setminus \partial \phi(\Omega) \) is not bounded. Therefore, \( \Gamma = \partial \phi(\Omega) \).

The Jordan curve \( \Gamma \) divides the complex plane \( \mathbb{C} \) to an interior region and an exterior region. By \( \Gamma = \partial \phi(\Omega) \), we know that \( \phi \) is a proper map and \( \phi(\Omega) \) is the interior region of \( \Gamma \), a simply connected domain. Let \( h \) be a conformal map from \( \Omega \) onto \( \phi(\Omega) \), and write \( \psi = h^{-1}(\phi) \). Then \( \psi \) is a holomorphic proper map from \( \Omega \) to \( \Omega \). Hence \( \psi \) is a finite Blaschke product [17, Theorem 7.3.3]. Also, we have \( \phi = h(\psi) \) as desired.

By Carathéodory’s Theorem, a conformal map from \( \Omega \) onto a Jordan domain \( \Omega \) extends to a continuous bijection from \( \overline{\Omega} \) onto \( \overline{\Omega} \). Thus \( h \) is bijective on \( \Gamma \). Rewrite \( \psi = B \). By \( \phi = h(B) \), we get \( o(\phi) = o(B) = \text{order } B \), \( n(\phi) = n(B) \), and \( N(\phi) = N(B) \). Since \( B : T \to T \) is a covering map,

\[
\begin{align*}
o(B) &= n(B) = N(B) = \text{order } B,
\end{align*}
\]

forcing \( o(\phi) = n(\phi) = N(\phi) = \text{order } B \). Since \( \text{order } B \leq b(\phi) \leq o(\phi) \), we have \( o(\phi) = n(\phi) = b(\phi) = N(\phi) = \text{order } B \). The proof is finished.

**Remark 2.6.** The last theorem in [6] says that if \( \phi : \Omega \to \phi(\Omega) \) is an \( n \)-to-1 analytic map, then there is a finite Blaschke product \( B \) and a univalent function \( h \) so that \( \phi = h(B) \). Using this one can prove the former part of Proposition 2.5.

### 3 Toeplitz operators with meromorphic symbols

This section focuses on the class \( M(\Omega) \), consisting of all meromorphic functions on \( \mathbb{C} \) whose poles are outside \( \Omega \). In this interesting case, we give the proof of Theorem 1.5.
3.1 Some preparations

Before going on, let us introduce some notions and lemmas.

We begin with the notion of analytic continuation [18, Chapter 16]. A function element is an ordered pair \((f, D)\), where \(D\) is an open disk and \(f\) is a holomorphic function on \(D\). Two function elements \((f_0, D_0)\) and \((f_1, D_1)\) are called direct continuations if \(D_0 \cap D_1\) is not empty and \(f_0 = f_1\) holds on \(D_0 \cap D_1\). By a curve, we mean a continuous map from \([0, 1]\) into \(\mathbb{C}\).

Given a function element \((f_0, D_0)\) and a curve \(\gamma\) with \(\gamma(0) \in D_0\), if there is a partition of \([0, 1]\):

\[0 = s_0 < s_1 < \cdots < s_n = 1\]

and function elements \((f_j, D_j)(0 \leq j \leq n)\) such that

1. \((f_j, D_j)\) and \((f_{j+1}, D_{j+1})\) are direct continuations for all \(j\) with \(0 \leq j \leq n - 1\);
2. \(\gamma([s_j, s_{j+1}] \subseteq D_j(0 \leq j \leq n - 1)\) and \(\gamma(1) \in D_n\),

then \((f_n, D_n)\) is called an analytic continuation of \((f_0, D_0)\) along \(\gamma\).

Suppose \(\Omega\) is a domain satisfying \(D_0 \cap \Omega \neq \emptyset\). A function element \((f_0, D_0)\) is called to admit unrestricted continuation in \(\Omega\) if for any curve \(\gamma\) in \(\Omega\) such that \(\gamma(0) \in D_0\), \((f_0, D_0)\) admits an analytic continuation along \(\gamma\). Furthermore, analytic continuation along a curve is essentially unique; that is, if \((g, U)\) is another analytic continuation of \((f_0, D_0)\) along \(\gamma\), then on \(U \cap D_n\) we have \(f_n = g\). We denote by \(f_0(\gamma, s)\) the value of analytic continuation of \(f_0\) along \(\gamma\) at the endpoint \(\gamma(s)\) of \(\gamma_s : t \mapsto \gamma(st), 0 \leq t \leq 1\). In particular, \(f_0(\gamma, 1) = f_n(\gamma(1))\).

For example, let \(D = \{z \in \mathbb{C} : |z - 1| < 1\}\) and define

\[f(z) = \ln z, z \in D\]

with \(\ln 1 = 0\). Let \(\gamma(t) = \exp(2t\pi i)\). Then \((f, D)\) admits analytic continuation along \(\gamma\). We have \(f(\gamma, 0) = f(1) = 0\), and in general

\[f(\gamma, t) = 2t\pi i, 0 \leq t \leq 1.\]

Note that \(f(\gamma, 1) = 2\pi i \neq f(\gamma, 0)\), but \(\gamma(1) = \gamma(0)\).

For a holomorphic function \(f\) on a domain \(V\), if there is a subdomain \(V\) of \(U\) and a holomorphic function \(\rho : V \to U\) such that

\[f(z) = f(\rho(z)), z \in V,\]

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then $\rho$ is called a local inverse of $f$ on $V$. The analytic continuation of a local inverse of $f$ is also a local inverse.

Some notations will be frequently used. For each $z \in \mathbb{C} \setminus \{0\}$, define

$$z^* = 1/z.$$  

Let $A$ be a subset of the complex plane, and define $A^* = \{z^* : z \in A \setminus \{0\}\}$. For a meromorphic function $f$ on domain $\Omega$, define $f^*$ by

$$f^*(z) = (f(z^*))^*, \ z \in \Omega^*.$$  

Note that $f^*$ is a meromorphic function if $f \not\equiv 0$.

In the sequel, we need the following lemma.

**Lemma 3.1.** Suppose that $f$ is a holomorphic function on a convex domain $\Omega$ and $x_0 \in \Omega \cap \mathbb{R}$. If there exists a sequence $\{x_k\}$ in $\mathbb{R} \setminus \{x_0\}$ such that $x_k \to x_0(k \to \infty)$, and $f(x_k) \in \mathbb{R}$, then $f(\Omega \cap \mathbb{R}) \subseteq \mathbb{R}$.

**Proof.** Write $U = \Omega \cap \{z : z \in \Omega\}$, which is itself a domain. Then define $g(z) = f(z) - f(\overline{z})$ on $U$. For each $k$, we have

$$g(x_k) = f(x_k) - f(\overline{x_k}) = f(x_k) - (f(x_k)) = 0.$$  

Since $x_0 \in U$ and $x_0$ is the accumulation point of $\{x_k\}$, $g \equiv 0$. In particular, for each $x$ in $\Omega \cap \mathbb{R}$, $f(x) = \overline{f(\overline{x})} = f(x)$. That is, $f(x) \in \mathbb{R}$ to finish the proof. \qed

Since there is a Moebius map mapping the real line to the unit circle, we get a translation of the Lemma 3.1.

**Corollary 3.2.** Assume $f$ is holomorphic in $O(\zeta_0, \delta)$ where $\zeta_0 \in \mathbb{T}$ and $\delta > 0$. Suppose that there exists a sequence $\{\zeta_k\}$ in $\mathbb{T} \setminus \{\zeta_0\}$ such that $\zeta_k \to \zeta_0(k \to \infty)$ and $f(\zeta_k) \in \mathbb{T}$. Then $f(O(\zeta_0, \delta) \cap \mathbb{T}) \subseteq \mathbb{T}$.

An observation is in order. Let $f$ be a nonconstant function holomorphic at $a$. By complex analysis, there is a neighborhood $W$ and a holomorphic function $\psi$ on $W$ such that $f(z) - f(a) = (z-a)^n\psi(z)$, $z \in W$ and $\psi(a) \neq 0$. For enough small $W$, $g(z) = (z-a)\sqrt[n]{\psi(z)}$ is univalent on $W$, and we have

$$f(z) - f(a) = g(z)^n.$$  

Furthermore, we can require $W$ to be a Jordan domain such that $g(W)$ is a disk centered at $0$. Therefore, we immediately get the following.
Lemma 3.3. Suppose \( f \) is a nonconstant holomorphic function over a domain containing both \( a \) and \( b \), and \( f(a) = f(b) \). Then for each enough small number \( \varepsilon > 0 \), there are two Jordan neighborhoods \( W_1 \) and \( W_2 \) of \( a \) and \( b \), such that both \( f|_{W_1} \) and \( f|_{W_2} \) are proper maps onto \( f(a) + \varepsilon \mathbb{D} \).

Furthermore, in this case for each pair \( (z, w) \) satisfying \( f(z) = f(w) \), \( z \in W_1 \setminus \{a\} \), and \( w \in W_2 \setminus \{b\} \), there is a local inverse \( \rho \) of \( f \) such that \( \rho(z) = w \) and \( \rho \) admits analytic continuation along any curve in \( W_1 \setminus \{a\} \), with values in \( W_2 \setminus \{b\} \).

3.2 Proof of Theorem 1.5

A problem raised in [2] and [21] asks whether each nonconstant function in \( H^\infty(\mathbb{D}) \) has MWN Property. Under a mild condition this is answered by the following result (=Theorem 1.5).

Theorem 3.4. Suppose \( \phi \) is a nonconstant function in \( \mathfrak{M}(\mathbb{D}) \). Then
\[
n(\phi) = b(\phi) = o(\phi) = N(\phi).
\]

The proof of Theorem 3.4 is long and thus it is divided into several parts. In what follows we will establish some lemmas and corollaries and then prove Theorem 3.4 at the end of this subsection.

In this section, let \( \phi \) be a nonconstant function in \( \mathfrak{M}(\mathbb{D}) \). We write \( P \) for the set of poles of \( \phi \) in \( \mathbb{C} \), and \( Z' \) for the set of zeros of \( \phi' \). Let \( X = P \cup \phi^{-1}(\phi(0)) \cup \phi^{-1}(\phi(Z')) \), \( Y = X \cup X^* \), and write
\[
\tilde{Y} = \phi^{-1}(\phi(Y)).
\]
Note that \( \tilde{Y} \) is a countable set containing \( Y \) and \( \phi^{-1}(\phi(\tilde{Y})) = \tilde{Y} \). Recall that a planar domain minus a countable set is path-connected.

Lemma 3.5. Suppose that there is a point \( \xi \) on \( T \setminus \tilde{Y} \) and a local inverse \( \rho \) of \( \phi \) at \( \xi \) such that \( \rho = \rho^* \) on some neighborhood of \( \xi \). If \( \gamma \) is a curve in \( \mathbb{C} \setminus \tilde{Y} \) such that \( \gamma(0) = \xi \), then \( \rho \) admits analytic continuation along \( \gamma \).

Proof. To reach a contradiction, assume that \( \rho \) admits no analytic continuation along \( \gamma \). Write \( \gamma_s(t) = \gamma(st) \), \( t \in [0, 1] \) and put
\[
s_0 = \sup \{ s \in [0, 1] : \rho \text{ admits an analytic continuation along } \gamma_s \}.
\]
Then it is clear that \( \rho \) admits no analytic continuation along \( \gamma_{s_0} \); otherwise there is some \( s_1 > s_0 \) such that \( \rho \) admits an analytic continuation along
γ_{s_1} to derive a contradiction. Recall that ρ(γ, s) denotes the value of the analytic continuation of ρ at the endpoint γ(s) of γs, where 0 ≤ s < s_0.

One will show that \{ρ(γ, s) : s ∈ [0, s_0]\} is bounded. Note that ρ(γ, s) is continuous in s. If \{ρ(γ, s) : s ∈ [0, s_0]\} is not bounded, then there exists a sequence \{s_n\} ⊆ [0, s_0) such that \{s_n\} tends to s_0, and

$$\lim_{n→∞} ρ(γ, s_n) = \infty. \quad (3.1)$$

Since γ \cap \widetilde{Y} = \emptyset, γ has no intersection with φ^{-1}(φ(0)), and then the local inverse \rho^* of φ admits an analytic continuation along γ^*_s, where

$$γ^*_s(t) = (γ_s(t))^*, \quad t ∈ [0, 1].$$

Then by (3.1)

$$\lim_{n→∞} ρ^*(γ^*_s, s_n) = 0,$$

forcing

$$\lim_{n→∞} φ(γ^*_s(s_n)) = φ(0).$$

That is, φ(γ(s_0)^*) = φ(0), and hence γ(s_0)^* ∈ φ^{-1}(φ(0)). But γ has no intersection with the set φ^{-1}(φ(0))^*, which is a contradiction. Therefore \{ρ(γ, s) : s ∈ [0, s_0]\} is bounded by a positive number C.

Let \{z_i\}_{i=1}^m be all the zeros of \phi - φ(γ(s_0)) in \mathbb{C}D. Since γ has no intersection with φ^{-1}(φ(Z^*)), we have that \phi'(γ(s_0)) \neq 0 and

$$φ'(z_i) \neq 0, \quad i = 1, \ldots, m.$$ Then one can find a connected neighborhood U of γ(s_0) and disjoint connected neighborhoods U_i(i = 1, \ldots, m) of z_i such that φ|_U and φ|_{U_i} are univalent. Since φ(z_i) = φ(γ(s_0)) for 1 ≤ i ≤ m, using Lemma 3.3 and contracting U and U_i we have that

$$φ(U_i) = φ(U) = O(φ(γ(s_0)), ε), \quad 1 ≤ i ≤ m,$$

for some ε > 0, and that

$$φ^{-1}(O(φ(γ(s_0)), ε)) \cap \mathbb{C}D ⊆ \bigcup_{i=1}^m U_i.$$

(3.2)

By continuity of φ, there exists a positive number δ < s_0 such that

$$φ(γ[s_0 - δ, s_0]) \subseteq O(φ(γ(s_0)), ε).$$
By \[\text{(3.2)}\] \(\{\rho(\gamma, s) : s \in (s_0 - \delta, s_0)\}\) is a connected set in \(\bigcup_{i=1}^{m} U_i\), and thus it is contained in a single \(U_j\) for some \(1 \leq j \leq m\). Letting
\[
\tau = (\phi|_{U_j})^{-1} \circ (\phi|_U),
\]
we have that \(\tau\) is a local inverse of \(\phi\) such that \(\tau(\gamma(s_0)) = z_j\) and \(\tau(U) = U_j\).

For each \(s \in (s_0 - \delta, s_0)\), let \(\rho_s\) be the analytic continuation for \(\rho\) along \(\gamma_s\), and then \(\rho_s\) is a direct continuation of \(\tau\). Then by combining \(\rho_s\) with \(\tau\), we have that \(\rho\) admits analytic continuation along \(\gamma_{s_0}\) to derive a contradiction.

The proof is complete.

**Lemma 3.6.** Suppose \(\phi \in \mathcal{M}(\overline{D})\) is not a rational function. Then for each positive number \(C\), there exist two points \(a\) and \(a'\) in \(\mathbb{C}\) such that \(|\phi(a)| > C\), \(|\phi(a')| < \frac{1}{C}\), and \(\min\{|a|, |a'|\} > C\).

**Proof.** Since \(\phi \in \mathcal{M}(\overline{D})\), \(\phi\) is a meromorphic function over \(\mathbb{C}\), and then the infinity \(\infty\) is either an isolated singularity or the limit of poles. If \(\infty\) is an isolated singularity, \(\infty\) is a removable singularity, a pole or an essential singularity. If \(\infty\) were either a removable singularity or a pole, then \(\phi\) would have finitely many singularities (poles), and by complex analysis \(\phi\) is a rational function. This is a contradiction to our assumption. Therefore, \(\infty\) is an essential singularity of \(\phi\). By Weierstrass’ theorem in complex analysis, for each point \(w \in \mathbb{C} \cup \{\infty\}\) there is a sequence \(\{z_n\}\) tending to \(\infty\) such that \(\{\phi(z_n)\}\) tends to \(w\). Hence the conclusion of Lemma 3.6 follows.

If \(\infty\) is the limit of poles of \(\phi\), then for a fixed number \(C > 0\), one can find a point \(a\) satisfying \(|a| > C\) and \(|\phi(a)| > C\). To complete the proof, we will show that there exists a point \(a'\) such that \(|a'| > C\) and \(|\phi(a')| < \frac{1}{C}\). If this were not true, then we would have
\[
\frac{1}{|\phi(z)|} \leq C, |z| > C,
\]
where \(\frac{1}{\phi(z)}\) equals zero if \(z\) is one pole of \(\phi\). Since \(\frac{1}{\phi}\) is bounded at a neighborhood of \(\infty\), \(\infty\) is a removable singularity of \(\frac{1}{\phi}\). Then \(\frac{1}{\phi}\) has only finitely many poles in \(\mathbb{C} \cup \{\infty\}\). Then by complex analysis \(\frac{1}{\phi}\) is a rational function, and so is \(\phi\). This derives a contradiction to finish the proof.

For a nonconstant function \(\phi\) in \(\mathcal{M}(\overline{D})\) and for a local inverse \(\rho\) of \(\phi\), let \(\rho^-\) be the inverse of \(\rho\). We will use Lemmas 3.5 and 3.6 to prove the following.
Lemma 3.7. Suppose \( \phi \in \mathcal{M}(\overline{\mathbb{D}}) \) is not a rational function. Then there exists a bounded domain \( \Omega \supseteq \overline{\mathbb{D}} \) having the following property: if \( \rho \) is a local inverse of \( \phi \) at a point \( \xi \in T \setminus \tilde{Y} \) such that \( \rho = \rho^* \) on some neighborhood of \( \xi \), then for each curve \( \gamma \) in \( \Omega \setminus \tilde{Y} \) with \( \gamma(0) = \xi \), we have \( \rho(\gamma, 1) \in \overline{\Omega} \), i.e. the value of the analytic continuation \( \tilde{\rho} \) of \( \rho \) along \( \gamma \) at endpoint \( \gamma(1) \) lies in \( \overline{\Omega} \).

Proof. Suppose \( \phi \in \mathcal{M}(\overline{\mathbb{D}}) \) is not a rational function. First we give the construction of \( \Omega \). By comments above Lemma 3.3 there exists a small neighborhood \( V \) of 0 biholomorphic function \( g : V \to r \mathbb{D}(r > 0) \) such that \( \phi(z) - \phi(0) = g(z)^k \) on \( V \) for some positive integer \( k \). One can require that \( V \) is a Jordan domain and \( \partial V \) is contained in \( \overline{D} \). Put

\[
\Gamma = (\partial V)^* = \{ \frac{1}{z} : z \in \partial V \},
\]

which is a closed Jordan curve outside \( \overline{\mathbb{D}} \). Let \( \Omega \) be the interior of \( \Gamma \), and then

\[
V^* = \mathbb{C} \setminus \overline{\Omega}.
\]

Let \( \xi \in T \setminus \tilde{Y} \), and let \( \gamma \) be a curve in \( \Omega \setminus \tilde{Y} \) with \( \gamma(0) = \xi \). Suppose that \( \rho \) is a local inverse of \( \phi \) at \( \xi \) such that \( \rho = \rho^* \) on some neighborhood of \( \xi \). To reach a contradiction, we assume \( \tilde{\rho}(\gamma(1)) = \rho(\gamma, 1) \in \mathbb{C} \setminus \overline{\Omega} = V^* \). Let

\[
\gamma_s(t) = \gamma(st), \ t \in [0, 1]
\]

and by Lemma 3.5 \( \rho \) admits an analytic continuation \( \rho_s \) along \( \gamma_s \). Recall that \( \rho(\gamma, s) \) is the value of \( \rho_s \) at the endpoint \( \gamma_s(1) = \gamma(s) \), and let

\[
\sigma(s) = \rho(\gamma, s), \ s \in [0, 1]. \tag{3.3}
\]

Then \( \sigma \) is a curve in \( \mathbb{C} \setminus \tilde{Y} \). Since \( \gamma \) has no intersection with \( \phi^{-1}(\phi(Z')) \), \( \rho^- \) admits an analytic continuation \( \tilde{\rho}^- \) along \( \sigma \), and by (3.3) we have

\[
\rho^-(\sigma, t) = \gamma(t), \ t \in [0, 1].
\]

In particular, we get

\[
\tilde{\rho}^-(\sigma(1)) = \rho^-(\sigma, 1) = \gamma(1). \tag{3.4}
\]

Let \( \{p_i\}_{i=1}^m \) be all the poles of \( \phi \) on \( \overline{\Omega} \). One can construct disjoint connected neighborhoods \( U_i \) \((i = 1, \ldots, m)\) of \( p_i \) such that

1. \( \phi \) has no zeros in \( \overline{U_i} \) for \( 1 \leq i \leq m \);
(2) \( \overline{U_i} \cap \phi^{-1}(f(0))^* \subseteq \{p_i\} \) for \( 1 \leq i \leq m \);

(3) For such \( i \) that \( \phi(p_i^*) = \phi(0) \), there exists an enough small connected neighborhood \( V_i \subseteq V \) of \( 0 \), such that \( \phi|_{U_i^*}, \phi|_{V_i} \) are proper maps satisfying \( \phi(U_i^*) = \phi(V_i) \); for other \( i \), let \( V_i = V \).

In fact, Condition (1) is easy to fulfill. Since \( \phi^{-1}(\phi(0)) \) is discrete and \( \phi^{-1}(\phi(0))^* \) has at most one accumulation point \( 0 \), Condition (2) is fulfilled if we let \( U_i \) be enough small. By Lemma 3.3 we can choose \( U_i \) and \( V_i \) to satisfy (3) and be as small as possible thus to meet (1) and (2). Therefore, one has (1)-(3) as desired.

Let

\[ M = \max_{z \in \overline{\mathbb{R}} \setminus \bigcup_{i=1}^{m} U_i} |\phi(z)|, \]

and define

\[ \varepsilon_i = \text{dist}(\phi(0), \phi(U_i^*)), \quad i = 1, \ldots, m. \]

If each \( \varepsilon_i \) equals zero, set \( \varepsilon = +\infty \); otherwise, write

\[ \varepsilon = \min \{ \varepsilon_i : \varepsilon_i > 0, \ 1 \leq i \leq m \}. \quad (3.5) \]

Then there exists a number \( \delta > 0 \) such that

\[ \phi(\delta \mathbb{D}) \subseteq O(\phi(0), \varepsilon) \quad \text{and} \quad \delta \mathbb{D} \subseteq \bigcap_{i=1}^{m} V_i. \quad (3.6) \]

By Lemma 3.6 we get a point \( a \notin \tilde{Y} \) satisfying

\[ |a| > \frac{1}{\delta} \quad \text{and} \quad |\phi(a)| > M. \]

Since \( V^* \setminus \tilde{Y} \) is path-connected, we can choose a curve \( \zeta \) in \( V^* \setminus \tilde{Y} \) connecting \( \widetilde{\rho}(\gamma(1)) = \sigma(1) \) with \( a \). By Lemma 3.5, \( \rho^{-1} \) admits an analytic continuation \( \tau \) along \( \sigma \zeta \), where \( \sigma \zeta \) is defined by

\[ \sigma \zeta(t) = \begin{cases} 
\sigma(2t) & 0 \leq t \leq \frac{1}{2} \\
\zeta(2t - 1) & \frac{1}{2} < t \leq 1.
\end{cases} \]

Then \( \rho^{-1} \) admits an analytic continuation \( \tau^* \) along \( \sigma^* \zeta^* \). Note that both \( \tau \) and \( \tau^* \) are local inverses of \( f \). Since

\[ |\phi(\tau(a))| = |\phi(a)| > M, \]

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by the definition of $M$ we get either $\tau(a) \in V^*$ or $\tau(a) \in U_i$ for some $i$. As follows, we will distinguish two cases to derive contradictions.

**Case I.** $\tau(a) \in V^*$. For $w, z \in V$, let $\phi(w) = \phi(z)$. Recall that on $V$ we have

$$\phi(z) - \phi(0) = g(z)^k,$$

and then $g(w)^k = g(z)^k$. Since $g|_V$ is biholomorphic, we get

$$w = g^{-1} \circ (\lambda g(z)), \ z \in V,$$

where $\lambda = \exp(\frac{2\pi i j}{k})$ for some integer $j$ in $\{1, \ldots, k\}$. Rewriting $\rho_j$ for the map $g^{-1} \circ (\lambda g(z))$, we have $\rho_j(V) = V$ and $\rho_j(0) = 0$.

Note that $\tau^*(a^*) = (\tau(a))^* \in (V^*)^* = V$.

Since $\phi(\tau^*(a^*)) = \phi(a^*)$ and $a^* \in V$, there exists a $j_0$ such that

$$\rho_{j_0}(a^*) = \tau^*(a^*).$$

Therefore $\tau = (\tau^*)^*$ extends analytically to

$$\rho_{j_0}^*: V^* \to V^*.$$  

Recall that $\tau$ and $\tilde{\rho}^-$ are analytic continuations of $\rho^-$ along $\sigma_\zeta$ and $\sigma$, respectively. Thus $\tilde{\rho}^-$ also extends analytically to $\rho_{j_0}^*$. Then by (3.4)

$$\gamma(1) = \tilde{\rho}^-(\sigma(1)) = \rho_{j_0}^*(\sigma(1)) \in V^*.$$

This contradicts with the fact that $\gamma \subseteq \Omega$.

**Case II.** There is some $i$ such that $\tau(a) \in U_i$. First we show $\phi(p_i^*) = \phi(0)$. In fact, since $a^* \in \delta \mathbb{D}$, by (3.6) we have

$$|\phi(0) - \phi(\tau^*(a^*))| = |\phi(0) - \phi(a^*)| < \varepsilon.$$

Since $\tau^*(a^*) = (\tau(a))^* \in U_i^*$,

$$\varepsilon_i = \text{dist}(\phi(0), \phi(U_i^*)) < \varepsilon,$$

which along with (3.5) gives $\varepsilon_i = 0$. This shows that $U_i^* \cap \phi^{-1}(\phi(0))$ is not empty. By condition (2) we immediately get $U_i^* \cap \phi^{-1}(\phi(0)) \subseteq \{p_i^*\}$, and thus

$$\phi(p_i^*) = \phi(0).$$

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Since Condition (1) shows that
\[ \min\{|\phi(z)| : z \in \overline{U}_i, 1 \leq i \leq m\} > 0, \]
bymLemma 3.6there is a point \( a' \notin \tilde{\mathcal{Y}} \) satisfying
\[ |\phi(a')| < \min\{|\phi(z)| : z \in \overline{U}_i, 1 \leq i \leq m\} \] (3.7)
and \(|a'| > \frac{1}{\delta}\). By 3.6 \( a' \in V_i^* \). Let \( \zeta \) be a curve in \( V_i^* \setminus \tilde{\mathcal{Y}} \) joining \( a \) with \( a' \), and let \( \tilde{\tau} \) be the analytic continuation of \( \tau \) along \( \zeta \). Since both \( \tau \) and \( \tau^* \) are local inverses of \( \phi \), so are \( \tilde{\tau} \) and \( \tilde{\tau}^* \), and \( \tilde{\tau}^* \) is the analytic continuation of \( \tau^* \) along \( \zeta^* \). By Condition (3) and Lemma 3.3 along any curve in \( V_i \setminus \{0\} \), \( \tau^* \) admits analytic continuation with values in \( U_i^* \setminus \{p_i^*\} \). Thus we have
\[ \tilde{\tau}^*(a^*) \in U_i^*. \]
Since \((\tilde{\tau}(a'))^* = \tilde{\tau}^*(a^*)\), \( \tilde{\tau}(a') \) lies in \( U_i \), and hence \( \phi(a') = \phi(\tilde{\tau}(a')) \in \phi(U_i) \). This is a contradiction to (3.7). In either case, we conclude a contradiction thus to finish the proof of Lemma 3.7.

Suppose \( \phi \) is a function in \( H^\infty(\overline{\mathbb{D}}) \). For \( \xi \in \mathbb{T} \), define
\[ m(\xi) = \lim_{\delta \to 0^+} \min_{\eta \in O(\xi, \delta) \cap \mathbb{T}} N(\phi - \phi(\eta), \mathbb{T}). \]
Clearly, \( m(\xi) \leq N(\phi - \phi(\xi), \mathbb{T}) \). Write
\[ S = \{ \xi \in \mathbb{T} : m(\xi) < N(\phi - \phi(\xi), \mathbb{T}) \}. \]
We need the following lemma.

**Lemma 3.8.** Let \( \phi \) be a nonconstant function in \( H^\infty(\overline{\mathbb{D}}) \). Then \( S \) is countable.

**Proof.** To reach a contradiction, assume \( S \) is uncountable. Let \( Z' \) denote the zero set of \( \phi' \) on \( \overline{\mathbb{D}} \) and
\[ F = \phi^{-1}(\phi(Z')). \]
For each positive integer \( j \), put
\[ S_j = \{ \xi \in S : N(\phi - \phi(\xi), \mathbb{T}) = j \}. \]
Then there exists at least a positive integer \( l \) such that \( S_l \) is uncountable. Recall that an uncountable set in \( \mathbb{C} \) has infinitely many accumulation points. One can pick an accumulation point \( \xi_0 \) of \( S_l \) such that \( \xi_0 \notin F \).
For $r \in (0,1)$, let $A_r$ denote the annulus
\[
\{ z \in \mathbb{C} : r < |z| < \frac{1}{r} \}.
\]

Since $\phi - \phi(\xi_0)$ has finitely many zeros on $\mathbb{T}$ and $\phi - \phi(\xi_0)$ is holomorphic on $\mathbb{T}$, one can pick an $r(0 < r < 1)$ close to 1 such that all zeros of $\phi - \phi(\xi_0)$ in $A_r$ lie on $\mathbb{T}$. By Rouché’s Theorem, there exists a positive number $\delta$ such that for each $z$ in $O(\xi_0, \delta)$
\[
N(\phi - \phi(z), A_r) = N(\phi - \phi(\xi_0), A_r) = N(\phi - \phi(\xi_0), \mathbb{T}) = l.
\]

On the other hand, there is a sequence $\{\xi_k\}$ in $S_\delta \cap [O(\xi_0, \delta) \setminus \{\xi_0\}]$, such that $\xi_k \to \xi_0 (k \to \infty)$. Thus,
\[
l = N(\phi - \phi(\xi_k), A_r) \geq N(\phi - \phi(\xi_k), \mathbb{T}) = l.
\]

This means that each zero of $\phi - \phi(\xi_k)$ in $A_r$ lies on $\mathbb{T}$. Since $\xi_0 \notin F$, there exist $l$ local inverses $\rho_0, \ldots, \rho_{l-1}$ of $\phi$ defined on $O(\xi_0, \delta)$; that is,
\[
\phi(\rho_i) = \phi, \; i = 0, \ldots, l - 1.
\]

Note that $\rho_0(\xi_0), \ldots, \rho_{l-1}(\xi_0)$ are exactly $l$ zeros of $\phi - \phi(\xi_0)$ on $\mathbb{T}$, and for all $k$ we have
\[
\rho_i(\xi_k) \in \mathbb{T}, \; i = 0, \ldots, l - 1.
\]

By Corollary 3.2, $\rho_i(O(\xi_0, \delta) \cap \mathbb{T}) \subseteq \mathbb{T}, \; i = 0, \ldots, l - 1$. We can require $\delta$ to be enough small such that $\rho_i(O(\xi_0, \delta) \cap \mathbb{T})$ are pairwise disjoint. Hence for each $\xi \in O(\xi_0, \delta) \cap \mathbb{T}$, $\phi - \phi(\xi)$ has $l$ distinct zeros on $\mathbb{T}$, $\rho_0(\xi), \ldots, \rho_{l-1}(\xi)$.

Therefore
\[
m(\xi_0) \geq l = N(\phi - \phi(\xi_0), \mathbb{T}),
\]

which derives a contradiction to $\xi_0 \in S$, finishing the proof.

By using Lemmas 3.7 and 3.8 one can prove the following.

**Proposition 3.9.** Suppose $\phi$ is a nonconstant function in $\mathfrak{M}(\mathbb{D})$. Then
\[
o(\phi) = b(\phi).
\]

**Proof.** We will first construct some local inverses of $\phi$ that maps some arc of $\mathbb{T}$ into $\mathbb{T}$. For this, by Lemma 3.8 $\phi(S)$ is countable as well as $S$, and then $\partial \phi(\mathbb{D}) \setminus \phi(S)$ is uncountable. Then there is a point $\xi_0$ in $\mathbb{T} \setminus (\bar{Y} \cup S)$ satisfying
\[
\phi(\xi_0) \in \partial \phi(\mathbb{D}).
\]
Rewrite \( n_0 = N(\phi - \phi(\xi_0), T) = m(\xi_0) \). Then one can find an \( r \in (0, 1) \) close to 1 such that
\[
n_0 = N(\phi - \phi(\xi_0), T) = N(\phi - \phi(\xi_0), A_r).
\]

By application of Rouché’s theorem, there exists a positive number \( \delta > 0 \) satisfying
\[
N(\phi - \phi(\xi), T) \leq N(\phi - \phi(\xi), A_r) = N(\phi - \phi(\xi_0), A_r) = n_0, \; \xi \in O(\xi_0, \delta) \cap T.
\]

By definition of \( m(\xi_0) \), \( N(\phi - \phi(\xi), T) \geq n_0 \), forcing
\[
N(\phi - \phi(\xi), T) = m(\xi_0) = n_0, \; \xi \in O(\xi_0, \delta) \cap T. \tag{3.8}
\]

As done in Lemma 3.3 one can find \( n_0 \) holomorphic functions \( \rho_0, \ldots, \rho_{n_0-1} \) on \( O(\xi_0, \delta) \) (\( \delta \) can be decreased if necessary) such that

1. for \( z \in O(\xi_0, \delta) \),
   \[
   N(\phi - \phi(z), A_r) = N(\phi - \phi(\xi_0), A_r) = N(\phi - \phi(\xi_0), T) = n_0;
   \]
2. \( \phi(\rho_i) = \phi, \; 0 \leq i \leq n_0 - 1; \)
3. \( \rho_i(O(\xi_0, \delta)) \subseteq A_r, \; 0 \leq i \leq n_0 - 1. \)

In particular, \( \rho_0(\xi_0), \ldots, \rho_{n_0-1}(\xi_0) \) are exactly those \( n_0 \) zeros of \( \phi - \phi(\xi_0) \) on \( T \). Then by (1)
\[
n_0 = N(\phi - \phi(\xi), A_r) \geq N(\phi - \phi(\xi), T) = n_0, \; \xi \in O(\xi_0, \delta) \cap T,
\]
forcing all zeros of \( \phi - \phi(\xi) \) in \( A_r \) to fall onto \( T \). Hence by Conditions (2) and (3) we get
\[
\rho_i(\xi) \in T, \; i = 0, \ldots, n_0 - 1.
\]

Hence there exists a neighborhood of \( \xi_0 \) where we have \( \rho_i = \rho_i^* \) for \( i = 0, \ldots, n_0 - 1 \), as they are equal on some arc of \( T \).

By Lemma 3.3 for each curve \( \varphi \) in \( \mathbb{C} \setminus \bar{Y} \) such that \( \varphi(0) = \xi_0 \), each member in \( \{\rho_i : i = 0, \ldots, n_0 - 1\} \) admits analytic continuation along \( \varphi \). We will see that the family \( \{\rho_i : i = 0, \ldots, n_0 - 1\} \) is closed under analytic continuation. For this, assume that \( \gamma \) is a loop in \( \mathbb{C} \setminus \bar{Y} \) with \( \gamma(0) = \gamma(1) = \xi_0 \). Let \( \tilde{\rho}_i \) (\( 0 \leq i \leq n_0 - 1 \)) be the analytic continuation of \( \rho_i \) along \( \gamma \). Clearly, all these \( \tilde{\rho}_i \) are local inverses of \( \phi \), i.e. \( \phi(\tilde{\rho}_i) = \phi \). Since \( \phi(\xi_0) \in \partial \phi(\mathbb{D}) \),
\[
\tilde{\rho}_i(\xi_0) \notin \mathbb{D}.
\]
Besides, we have \( \rho_i = \rho_i^* \) on some neighborhood of \( \xi_0 \), and then

\[
\phi(\rho_i) = \phi(\rho_i^*) = \phi.
\]

Write \( \gamma^*(t) = (\gamma(t))^* \) \( (t \in [0,1]) \) and define \( \tilde{\rho}_i^* \) along \( \gamma^* \). Hence

\[
\phi(\tilde{\rho}_i^*) = \phi, \quad 0 \leq i \leq n_0 - 1.
\]

By similar reasoning as above, \( \tilde{\rho}_i^*(\xi_0) \notin \mathbb{D} \). Also noting \( \tilde{\rho}_i(\xi_0) \notin \mathbb{D} \) gives \( \tilde{\rho}_i(\xi_0) \in \mathbb{T} \). Then it follows that \( \{\tilde{\rho}_i(\xi_0) : i = 0, \ldots, n_0 - 1\} \) is a permutation of \( \{\rho_i(\xi_0) : i = 0, \ldots, n_0 - 1\} \). If two local inverses are equal at one point \( \xi_0 \notin \phi^{-1}(\phi(Z')) \), by the Implicit Function Theorem they are equal on a neighborhood of this point. Thus we have

\[
\{\tilde{\rho}_i : i = 0, \ldots, n_0 - 1\} = \{\rho_i : i = 0, \ldots, n_0 - 1\}.
\]

Give two curves \( \gamma_1 \) and \( \gamma_2 \) with \( \gamma_1(0) = \gamma_2(0) = \xi_0 \) and \( \gamma_1(1) = \gamma_2(1), \gamma_1 \gamma_2 \) is a loop with endpoints \( \xi_0 \). Therefore, we have that analytic continuations of the family \( \{\rho_i : i = 0, \ldots, n_0 - 1\} \) along \( \gamma_1 \) are the same as those along \( \gamma_2 \). Thus analytic continuations of the family \( \{\rho_i : i = 0, \ldots, n_0 - 1\} \) does not depend on the choice of the curve. Define

\[
B(z) = \prod_{i=0}^{n_0-1} \tilde{\rho}_i(z), \quad z \in \mathbb{C} \setminus \tilde{Y}, \quad (3.9)
\]

where we use analytic continuations. In what follows, we will show that \( B \) extends analytically to a finite Blaschke product and there are two cases to distinguish:

\( \phi \in \mathfrak{R}(\mathbb{D}) \) \text{ or } \( \phi \in \mathfrak{M}(\mathbb{D}) \setminus \mathfrak{R}(\mathbb{D}) \).

**Case I.** \( \phi \in \mathfrak{R}(\mathbb{D}) \). Thus \( \phi \) is a rational function, and then \( \tilde{Y} \) is a finite set. Assume that the infinity \( \infty \) is a pole of \( \phi \), without loss of generality. Otherwise, one can compose \( \phi \) with some \( \eta \in \text{Aut}(\mathbb{D}) \) defined by

\[
\eta(z) = \frac{\alpha - z}{1 - \alpha z},
\]

mapping \( \infty \) to a pole \( 1/\alpha \) of \( \phi \). Replacing \( \phi \) with \( \phi(\eta) \) reduces to the desired case. Since \( \phi \in \mathfrak{R}(\mathbb{D}) \), there is a constant \( C_1 > 1 \) such that \( \phi \) is holomorphic on some neighborhood of \( C_1 \mathbb{D} \). Let

\[
M = \max\{|\phi(z)| : |z| \leq C_1\} < +\infty.
\]
Since \( \phi(\infty) = \infty \), there exists a constant \( C_2 > 0 \) satisfying
\[
|\phi(z)| > M, |z| > C_2.
\]
For \( z \in C_1 \mathbb{D} \setminus \tilde{Y} \), we have
\[
|\phi(\tilde{\rho}_i(z))| = |\phi(z)| \leq M, 0 \leq i \leq n_0 - 1,
\]
and then each \( \tilde{\rho}_i(z) \) is bounded by \( C_2 \). Hence \( B \) is an analytic function bounded by \( C_2^{n_0} \). Therefore, \( B \) extends analytically to \( C_1 \mathbb{D} \) since \( \tilde{Y} \) is a finite set. Besides, all \( \tilde{\rho}_i(z) \) are unimodular on the circular arc \( O(\xi_0, \delta) \cap \mathbb{T} \), and so is \( B \). By Corollary \( 3.2 \), \( B \) is unimodular on \( \mathbb{T} \), and hence \( B \) is a finite Blaschke product \( \{17\} \).

**Case II.** \( \phi \in \mathfrak{M}(\mathbb{D}) \setminus \mathfrak{R}(\mathbb{D}) \), then \( \phi \) is not a rational function. By Lemma \( 3.7 \) there exists a bounded domain \( \Omega \supseteq \overline{\mathbb{D}} \) such that for \( z \in \Omega \setminus \tilde{Y} \), we have
\[
\tilde{\rho}_i(z) \in \overline{\Omega}, 0 \leq i \leq n_0 - 1.
\]
For these \( \rho_i \), each analytic continuation along a curve in \( \Omega \setminus \tilde{Y} \) is defined by a chain of disks, and hence by \( 3.9 \), \( B \) extends naturally to an open set \( V(V \subseteq \Omega) \) containing \( \Omega \setminus \tilde{Y} \). Since \( \tilde{Y} \) is countable, there is a relatively closed countable set \( Y_0 \) such that
\[
V = \Omega \setminus Y_0.
\]
Since \( \Omega \setminus \tilde{Y} \) is dense in \( \Omega \setminus Y_0 \), for \( z \in \Omega \setminus Y_0 \) we have
\[
\tilde{\rho}_i(z) \in \overline{\Omega}, 0 \leq i \leq n_0 - 1.
\]
Therefore, \( B \) is a well-defined bounded analytic function on \( \Omega \setminus Y_0 \). Since \( Y_0 \) is a countable relatively closed set in \( \Omega \), \( Y_0 \) is \( H^\infty \)-removable, and thus \( B \) extends analytically on \( \Omega \). In particular, \( B \) is analytic on a neighborhood of \( \overline{\mathbb{D}} \). Since each \( \tilde{\rho}_i(z) \) is unimodular on the circular arc \( O(\xi_0, \delta) \cap \mathbb{T} \), so is \( B \). By Corollary \( 3.2 \), \( B \) is unimodular on \( \mathbb{T} \), forcing \( B \) to be a finite Blaschke product.

In both cases we have shown that \( B \) extends analytically to a finite Blaschke product. All local inverses of \( B \) are exactly \( \{\tilde{\rho}_i : i = 0, \ldots, n_0 - 1\} \), and clearly, order \( B = n_0 \). By Corollary \( 2.2 \), each member \( \rho \) in \( G(\phi) \) is uniquely determined by the value \( \rho(\xi_0) \). Thus
\[
o(\phi) \leq N(\phi - \phi(\xi_0), \mathbb{T}).
\]
Note that \( \rho_0(\xi_0), \ldots, \rho_{n_0-1}(\xi_0) \) are all zeros of \( \phi - \phi(\xi_0) \) on \( \mathbb{T} \), and thus
\[
o(\phi) \leq n_0 = \text{order } B = o(B). \tag{3.10}
\]
On the other hand, \( \{ \tilde{\rho}_i : i = 0, \ldots, n_0 - 1 \} \) are local inverses of \( \phi \), and then \( B(z) \mapsto \phi(z) \) is a well-defined analytic function, denoted by \( h \). Since \( h \) is bounded on \( C_1 \mathbb{D} \) minus a finite set where \( C_1 > 1 \), \( h \) extends to a function in \( H^\infty(\mathbb{D}) \), and

\[
\phi = h(B).
\]

This gives \( G(\phi) \supseteq G(B) \). Noting (3.10), we have \( o(\phi) = o(B) \). By Section 2,

\[
o(\phi) \geq b(\phi) \geq \text{order } B = o(B) = o(\phi).
\]

forcing \( b(\phi) = o(\phi) = n_0 \).

To establish Theorem 3.4, we also need the following.

**Corollary 3.10.** For a nonconstant function \( \phi \in \mathfrak{M}(\overline{\mathbb{D}}) \), except for a countable set each point \( \xi \) in \( \mathbb{T} \) satisfies \( N(\phi - \phi(\xi), \mathbb{T}) = b(\phi) \). Furthermore, \( N(\phi) = b(\phi) \).

**Proof.** By Proposition 3.9, we write

\[
n_0 = b(\phi) = o(\phi).
\]

By Corollary 2.4 we have

\[
N(\phi - \phi(\xi), \mathbb{T}) \geq n_0, \xi \in \mathbb{T}.
\]

Write

\[
A = \{ \xi \in \mathbb{T} : N(\phi - \phi(\xi), \mathbb{T}) > n_0 \},
\]

and it suffices to show that \( A \) is countable. Assume conversely that \( A \) is uncountable. Since \( A \) contains uncountable accumulation points in itself, one can pick an accumulation point \( \eta_0 \) in \( \mathbb{T} \setminus (\tilde{Y} \cup S) \). Write

\[
l = N(\phi - \phi(\eta_0), \mathbb{T}) > n_0.
\]

In the first paragraph of the proof of Proposition 3.9 by replacing \( \xi_0 \) with \( \eta_0 \) we get \( l \) local inverses on some neighborhood of \( \eta_0 \), which maps an arc in \( \mathbb{T} \) into \( \mathbb{T} \). Let \( \gamma \) be a curve in \( \mathbb{C} \setminus (\tilde{Y} \cup S) \) connecting \( \eta_0 \) and \( \xi_0 \), and these \( l \) local inverses admit analytic continuations along \( \gamma \), denoted by \( \tilde{\tau}_0, \ldots, \tilde{\tau}_{l-1} \). Also \( \tilde{\tau}_0^*, \ldots, \tilde{\tau}_{l-1}^* \) are exactly analytic continuations along \( \gamma^* \) of the local inverses \( \tau_0^*, \ldots, \tau_{l-1}^* \) at \( \eta_0 \). For \( 0 \leq i \leq l - 1 \), neither \( \tilde{\tau}_i(\xi_0) \) nor \( \tilde{\tau}_i^*(\xi_0) \) belongs to \( \mathbb{D} \) as \( \phi(\xi_0) \in \partial\phi(\mathbb{D}) \). Therefore \( \{ \tilde{\tau}_i(\xi_0) : 0 \leq i \leq l - 1 \} \) are \( l \) distinct zeros of \( \phi - \phi(\xi_0) \) on \( \mathbb{T} \). But by (3.8),

\[
N(\phi - \phi(\xi_0), \mathbb{T}) = n_0 < l,
\]

which derives a contradiction. Hence \( A \) is countable, as desired. \( \Box \)
Now we proceed to present the proof of Theorem 3.4.

**Proof of Theorem 3.4.** Suppose that $\phi$ is a nonconstant meromorphic function in $\mathbb{C}$ without pole on $\overline{D}$. By Proposition 3.9 and Corollary 3.10 we have

$$N(\phi) = b(\phi) = o(\phi).$$

It remains to show that

$$n(\phi) = b(\phi).$$

Recall that

$$n(\phi) = \min_{z \in D, \phi(z) \notin \phi(T)} \text{wind} (\phi(T), \phi(z)).$$

By Corollary 2.4 $o(\phi) \leq n(\phi)$. Since $b(\phi) | o(\phi)$, $b(\phi) \leq n(\phi)$. It remains to prove that

$$n(\phi) \leq b(\phi).$$

Recall that $Z'$ is the zero of $\phi$, and let $F = \phi^{-1}(\phi(Z' \cap \overline{D}))$. By Corollary 3.10 there is a point $w_0 \in \mathbb{T} \setminus F$ such that $\phi(w_0) \in \partial \phi(\mathbb{D})$ and

$$N(\phi - \phi(w_0), \mathbb{T}) = b(\phi).$$

Since the zeros of $\phi$ are isolated in $\mathbb{C}$, there exists a positive constant $t > 1$ satisfying

$$N(\phi - \phi(w_0), t\mathbb{D}) = N(\phi - \phi(w_0), \overline{\mathbb{D}}) = b(\phi).$$

By Rouché’s Theorem, there is a positive number $\delta$ such that

$$N(\phi - \phi(z), t\mathbb{D}) = N(\phi - \phi(w_0), t\mathbb{D}) = b(\phi), \quad z \in O(w_0, \delta).$$

Let $z_0$ be a point in $O(w_0, \delta) \cap \mathbb{D}$ such that $\phi(z_0) \notin \phi(\mathbb{T})$, and by Argument Principle we get

$$\text{wind} (\phi(T), \phi(z_0)) = N(\phi - \phi(z_0), \mathbb{D}) \leq N(\phi - \phi(z_0), t\mathbb{D}) = b(\phi).$$

Thus $n(\phi) \leq b(\phi)$, forcing $n(\phi) = b(\phi)$. This finishes the proof of Theorem 3.4. \qed

4 FSI and FSI-decomposable properties

In this section it is shown that each nonconstant function in $\mathfrak{M}(\mathbb{D})$ has FSI-decomposable property. Based on this, the proof of Theorem 1.3 is furnished.

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4.1 Proof of Theorem 1.3

Recall that $\mathcal{M}(\overline{D})$ denotes the class of all meromorphic functions in $\mathbb{C}$ without pole on $\overline{D}$. One main aim of this section is to prove the following.

**Theorem 4.1.** For a nonconstant function $\phi \in \mathcal{M}(\overline{D})$, suppose $\phi = \psi(B)$ is the Cowen-Thomson representation of $\phi$. Then $\psi$ has FSI property.

Later, by using Theorem 4.1 one will get Theorem 1.3 restated as follows.

**Theorem 4.2.** Suppose $\phi$ is a nonconstant function in $\mathcal{M}(\overline{D})$. The following are equivalent:

1. the Toeplitz operator $T_\phi$ is totally Abelian;
2. $\phi$ has FSI property.
3. $N(\phi) = 1$.

Recall that a point $\lambda$ in $\mathbb{C}$ is called a point of self-intersection of the curve $\phi(z)(z \in T)$ [14] if there exist two distinct points $w_1$ and $w_2$ on $T$ such that

$$\phi(w_1) = \phi(w_2) = \lambda;$$
equivalently, $N(\phi - \lambda, T) > 1$. To prove Theorem 4.1 we need the following.

**Lemma 4.3.** Suppose $\phi \in H^\infty(\overline{D})$. Then the cardinality of points of self-intersections of the curve $\phi(z)(z \in T)$ is either finite or $\aleph$, the continuum.

**Proof.** Suppose $\phi \in H^\infty(\overline{D})$. Denote the set of all points of self-intersection of $\phi(T)$ by $A$. If $A$ is finite, the proof is finished.

Assume $A$ is an infinite set. Then $\phi^{-1}(A) \cap T$ must have an accumulation point $\xi_0$ on $T$. By the definition of points of self-intersection, there is a sequence $\{\xi_k\}$ in $T \setminus \{\xi_0\}$ and a sequence $\{\eta_k\}$ in $T$ such that $\xi_k \to \xi_0 (k \to \infty)$, and

$$\phi(\xi_k) = \phi(\eta_k), \xi_k \neq \eta_k, \forall k.$$Without loss of generality, one assumes that $\{\eta_k\}$ itself converges to a point $\eta_0$ on $T$. Thus we have

$$\phi(\xi_0) = \phi(\eta_0) \equiv \lambda_0.$$Note that $\xi_0$ may be equal to $\eta_0$.

Since $\phi$ is not constant, by Lemma 3.3 there are two simply-connected neighborhoods $U$ of $\xi_0$, $V$ of $\eta_0$ and a positive number $\varepsilon$ such that

$$\phi(U) = \phi(V) = \varepsilon \mathbb{D} + \lambda_0,$$
and $\phi|_U, \phi|_V$ are holomorphic proper maps, whose multiplicities are equal to the multiplicities of zero of $\phi - \lambda_0$ at $\xi_0$ and $\eta_0$ respectively. Write
\[ \hat{U} = U \setminus \{\xi_0\} \quad \text{and} \quad \hat{V} = V \setminus \{\eta_0\}. \]

Let $N$ be the multiplicity of the zero of $\phi - \lambda_0$ at the point $\eta_0$. By Lemma 3.3, for each $z \in \hat{U}$ we have the following:

1. there exist exactly $N$ distinct local inverses of $\phi$ on a connected neighborhood $U_z$ of $z$ with values in $\hat{V}$ and $U_z \subseteq \hat{U}$;

2. each local inverse in (1) admits analytic continuation along any curve in $\hat{U}$ starting from the point $z$.

Note that analytic continuation of a local inverse of $\phi$ in (1) is also a local inverse, with values in $\hat{V}$.

The following discussions are based on the upper half plane $\mathbb{H}$ rather than on the unit disk, and this will be more convenient. Let $\varphi$ be a Moebius transformation mapping $\mathbb{D}$ onto $\mathbb{H}$, its pole being distinct from $\xi_0$ and $\eta_0$. Rewrite
\[ x_k = \varphi(\xi_k) \quad \text{and} \quad y_k = \varphi(\eta_k), \quad k \geq 0. \]

Here by no means we indicate that $x_k$ and $y_k$ are the real or imaginary part of some complex number. Let $\delta$ be a positive number such that $O(x_0, \delta) \subseteq \varphi(U)$, and we define four simply connected domains:
\[ D_0 = \{z \in O(x_0, \delta) : \text{Re}(z-x_0) > 0\}, \quad D_1 = \{z \in O(x_0, \delta) : \text{Im}(z-x_0) > 0\}; \]
\[ D_2 = \{z \in O(x_0, \delta) : \text{Re}(z-x_0) < 0\}, \quad D_3 = \{z \in O(x_0, \delta) : \text{Im}(z-x_0) < 0\}. \]

Note that $D_1, D_2$ and $D_3$ can be obtained by a rotation of $D_1$. Since $\varphi^{-1}(D_i)$ is simply connected for $i = 0, 1, 2, 3$, by (1) and (2) we get $N$ local inverses of $\phi$ with values in $\hat{V}$; and by the Monodromy Theorem these local inverse are all analytic on $\varphi^{-1}(D_i)$ for fixed $i$. Let
\[ \tilde{\phi} = \phi \circ \varphi^{-1}, \]
and we obtain $N$ local inverses of $\tilde{\phi}$, which are analytic on each domain $D_i$ for $i = 0, 1, 2, 3$. With no loss of generality, assume there are infinitely many points of $\{x_k\}$ lying in $D_0$. Then there exists at least one local inverse $\sigma_0$ of $\tilde{\phi}$ defined on $D_0$ so that $\sigma_0$ maps $x_k$ to $y_k$, for infinitely many $k$. Define
\[ D_{4j+i} = D_i, \quad 0 \leq i \leq 3, \quad j \in \mathbb{Z}_+. \]
Take analytic continuations $\left(\sigma_i, D_i\right) (i = 1, \cdots, 4N)$ of $\left(\sigma_0, D_0\right)$ along the chain $\{D_0, D_1, \ldots, D_{4N}\}$. Note that

$$D_0 = D_4 = D_8 = \cdots,$$

and there are only finitely many distinct local inverses of $\tilde{\phi}$ on $D_0$. There must be a minimal positive integer $n_0 \leq N$ satisfying $\sigma_{4n_0} = \sigma_0$. As follows, we will use function elements $\left(\sigma_i, D_i\right) (i = 0, \ldots, 4n_0 - 1)$ to construct a holomorphic function on a disk $D$. Precisely, write $D = O(0, \sqrt{\delta})$, and for $z \in D \setminus \{0\}$ define

$$\omega(z) = \begin{cases} 
\sigma_0(z^{n_0} + x_0), & 0 \leq \arg z < \frac{\pi}{2n_0}; \\
\sigma_1(z^{n_0} + x_0), & \frac{\pi}{2n_0} \leq \arg z < \frac{\pi}{n_0}; \\
\vdots \\
\sigma_{4n_0-1}(z^{n_0} + x_0), & \frac{\pi}{2n_0}(4n_0 - 1) \leq \arg z < 2\pi.
\end{cases}$$

Then $\omega$ is well-defined and holomorphic in $D \setminus \{0\}$. Observe that as $z$ tends to $x_0$ in $D_i (i = 0, \ldots, 4n_0 - 1)$, each $\sigma_i(z)$ tends to $y_0$. Therefore $\omega$ is bounded near 0, and hence 0 is a removable singularity of $\omega$. By setting $\omega(0) = y_0$ we get a holomorphic function $\omega$ on $D$.

Since $\omega|_{D \cap \mathbb{R}^+}(x) = \sigma_0(x^{n_0} + x_0)$, and $\sigma_0(x_k) = y_k$ holds for infinitely many $k$, we have $\omega\left(\sqrt{x_k - x_0}\right) = y_k \in \mathbb{R}$ as $x_k > x_0$. By Lemma 3.1

$$\omega(D \cap \mathbb{R}) \subseteq \mathbb{R},$$

forcing $\sigma_0(D_0 \cap \mathbb{R}) \subseteq \mathbb{R}$. Letting

$$\gamma = \varphi^{-1}(D_0 \cap \mathbb{R}) \subseteq \mathbb{T},$$

and

$$\tilde{\sigma}_0(w) = \varphi^{-1} \circ \sigma_0 \circ \varphi(w), w \in \varphi^{-1}(D_0) \subseteq U,$$

we have $\tilde{\sigma}_0(\gamma) \subseteq \mathbb{T}$. Clearly $\sigma_0$ is not the identity map, and neither is $\tilde{\sigma}_0$. Let

$$W = \{z \in \varphi^{-1}(D_0) \cap \mathbb{T} : \tilde{\sigma}_0(z) = z\},$$

and $W$ is at most countable. Since the cardinality of $\phi(\gamma)$ is $\mathbb{N}$, so is $\phi(\gamma \setminus W)$, finishing the proof of Lemma 4.3.

Now we are ready to give the proof of Theorem 4.1.
Proof of Theorem 4.1. Suppose that \( \phi \) is a nonconstant function in \( M(D) \), and \( \phi = \psi(B) \) is the Cowen-Thomson representation. Corollary 1.2 says that \( \psi \) is in \( H^\infty(D) \). By comments below Theorem 1.1, \( B \) is of maximal order and thus \( \psi \) can not be written as a function of a finite Blaschke product of order larger than 1. Again by Corollary 1.2 we have \( b(\psi) = 1 \). Corollary 3.10 implies that 

\[
\{ w \in \mathbb{T} : N(\psi - \psi(w), \mathbb{T}) > 1 \}
\]

is countable, as well as \( \{ \psi(w) \in \mathbb{T} : N(\psi - \psi(w), \mathbb{T}) > 1 \} \). But by Lemma 4.3 the cardinality of self-intersections of \( \psi(z)(z \in \mathbb{T}) \) is a natural number. Thus \( \psi \) has FSI property as desired. \( \square \)

We are ready to give the proof of Theorem 4.2 (=Theorem 1.3).

Proof of Theorem 4.2. Note (2) \( \Rightarrow \) (3) is trivial. To show (3) \( \Rightarrow \) (1), assume \( N(\phi) = 1 \). By Corollary 3.10 we have \( b(\phi) = N(\phi) = 1 \). Then Theorem 1.1 gives that \( T_\phi \) is totally Abelian.

For (1) \( \Rightarrow \) (2), let \( \phi = \psi(B) \) be a Cowen-Thomson representation. Then by Theorem 4.1 \( \psi \) has FSI property. Since \( T_\phi \) is totally Abelian,

\[
\{ T_z \}' = \{ T_\phi \}' \supseteq \{ T_B \}' \supseteq \{ T_z \}'.
\]

Then \( \{ T_B \}' = \{ T_z \}' \), forcing order \( B = 1 \). Since \( \phi = \psi(B) \), \( \phi \) has FSI property as desired. The proof of Theorem 4.2 is complete. \( \square \)

It is straightforward to get equivalent formulations for (1)-(3) in Theorem 4.2: (4) there is a point \( \xi \in \mathbb{T} \) satisfying \( N(\phi - \phi(\xi), \mathbb{T}) = 1 \); and (5) except for a countable or finite set every point \( \xi \in \mathbb{T} \) satisfies \( N(\phi - \phi(\xi), \mathbb{T}) = 1 \).

Theorem 4.1 shows that each function \( \phi \) in \( M(D) \) has FSI-decomposable property; that is, for the Cowen-Thomson representation \( \phi = \psi(B) \), \( \psi \) has FSI-property. In fact, we will see that \( \psi \) has quite special form (see Lemma 4.4 and Theorem 4.6).

Recall that \( \Re(D) \) consists of all rational functions which have no pole on \( \overline{D} \). If \( P \) and \( Q \) are two co-prime polynomials, order \( \frac{P}{Q} \) is defined to be \( \max \{ \deg P, \deg Q \} \). The following is of independent interest.

Lemma 4.4. If \( f \) is in \( \Re(D) \) and there is a function \( h \) on \( D \) such that

\[
f = h(B),
\]

where \( B \) is a finite Blaschke product, then \( h \) is \( \Re(D) \). In this case, we have order \( f = \text{order } h \times \text{order } B \).
Proof. Suppose \( f \) is a function in \( \mathfrak{M}(\overline{D}) \) and \( h \) is a function on \( D \) satisfying

\[
f = h(B),
\]
where \( B \) is a finite Blaschke product. Let \( n = \text{order } B \), and denote \( n \) local inverses of \( B \) by \( \rho_0, \cdots, \rho_{n-1} \). Let \( Z' \) denote the zero set of \( B' \) in \( \mathbb{C} \), and by Bochner’s Theorem \[23\] \( Z' \) is a finite subset of \( D \). Write

\[
\mathcal{E} = B^{-1}(B(Z')).
\]

It is known that all local inverses admit unrestricted continuation in \( \overline{D} \setminus \mathcal{E} \). For each \( j (0 \leq j \leq n - 1) \), define \( \rho_j^*(z) = (\rho_j(z^*))^* \), which admits unrestricted continuation on \( \mathbb{C} \setminus \mathcal{D} \) minus a finite set. Recall that the derivative \( B' \) of \( B \) does not vanish on \( T \), \( \rho_j \) is analytic on \( T \) and \( \rho_j^* = \rho_j \) on \( T \). Thus each \( \rho_j \) admits unrestricted continuation on \( \mathbb{C} \) minus a finite set, say \( F_1 \).

For each \( z \in D \), by \( h(B(z)) = f(z) \) we get

\[
f(z) = f(\rho_j(z)), \quad 0 \leq j \leq n - 1.
\]

By analytic continuation, the above also holds for all \( z \in \mathbb{C} \setminus F_1 \). Let \( P \) denote the poles of \( f \), a finite set in \( \mathbb{C} \). Then by \([4.1]\) \( B(z) \mapsto f(z) \) defines a holomorphic function \( h \) on \( \mathbb{C} \setminus (B(F_1) \cup P) \). Thus on the complex plane minus discrete points, we have

\[
f = h(B).
\]

If \( f \) is a rational function, then its only possible isolated singularities (including \( \infty \)) are poles. By \([4.2]\), \( h \) has at most finitely many singularities including \( \infty \), which are either removable singularity or poles. Hence \( h \) is a rational function. Since \( f \) is holomorphic on \( D \), by \([4.2]\) \( h \) is bounded on a neighborhood of \( \overline{D} \) with finitely many singularities possible. Thus \( h \) extends analytically on \( \overline{D} \), forcing \( h \in \mathfrak{M}(\overline{D}) \).

Suppose \( f \) is a rational function. Noting that \( f \) can be written as the quotient of two co-prime polynomials, by computations we have that \( f \) is a covering map on \( \mathbb{C} \setminus (f^{-1}(f(\infty)) \cup \mathcal{E}) \), and the multiplicity is exactly order \( f \). Since both \( h \) and \( B \) are rational functions, they can be regarded as covering maps on \( \mathbb{C} \) minus some finite set. This leads to the conclusion that

\[
\text{order } f = \text{order } h \times \text{order } B,
\]

to complete the proof.

By Theorem \([4.1]\) and Lemma \([4.4]\) we get the following.
Corollary 4.5. Suppose $R$ is a rational function in $\mathcal{R}(\mathbb{D})$ with prime order. Then either $R$ is a composition of a Moebius transformation and a finite Blaschke product, or $R$ has FSI property. In the later case, $T_R$ is totally Abelian.

Below we come to functions in $\mathcal{M}(\mathbb{D}) \setminus \mathcal{R}(\mathbb{D})$. The main theorem in [2] says that each entire function $\phi$ has the Cowen-Thomson representation $\phi(z) = \psi(z^n)$ for some entire function $\psi$ and some integer $n$. The following theorem generalizes the theorem in [2] to functions in $\mathcal{M}(\mathbb{D}) \setminus \mathcal{R}(\mathbb{D})$, and is of independent interest.

Theorem 4.6. Suppose that $f \in \mathcal{M}(\mathbb{D})$ is not a rational function. Then there is a positive integer $n$ and a function $h$ in $\mathcal{M}(\mathbb{D})$ such that $f(z) = h(z^n)$ and $\{T_f\}' = \{T_{z^n}\}'$. Furthermore, $n = o(f) = b(f) = n(f) = N(f)$.

Proof. To prove Theorem 4.6, we begin with an observation from complex analysis. By Lemma 3.3 and comments above it, for a function $\varphi$ holomorphic on a neighborhood of $\lambda$, let $k = \text{order} (\varphi, \lambda)$, the multiplicity of the zero of $\varphi - \varphi(\lambda)$ at $\lambda$. Then there is a Jordan neighborhood $W$ of $\lambda$ such that $\varphi|W$ is a $k$-to-1 proper map onto a neighborhood of $\varphi(\lambda)$. This is right even if $\lambda = \infty$ or $\lambda$ is a pole of $\varphi$ (for $\lambda = \infty$, $\text{order} (\varphi, \infty) = \text{order} (\varphi(1/z), 0)$).

Based on this, we will show that if $B_0$ is a finite Blaschke product of order $k$, and order $(B_0, \infty)$ equals $k$, then $B_0$ is a function of $z^k$. In fact, either $B_0(\infty) = \infty$ or $|B_0(\infty)| > 1$. If $|B_0(\infty)| > 1$, by letting $\psi(z) = \frac{1}{B_0(\infty)} - \frac{z}{1 - z/B_0(\infty)}$ we have $\psi \circ B_0(\infty) = \infty$. Then we can assume $B_0(\infty) = \infty$. Note that order $(1/B_0(1/z), 0) = \text{order} (B_0, \infty) = k$. Write

$$B_0(z) = c \prod_{j=1}^{k} \frac{\alpha_j - z}{1 - \overline{\alpha_j}z},$$

where $|c| = 1$ and $\alpha_j \in \mathbb{D}$ for all $j$, and then

$$1/B_0(1/z) = \tau \prod_{j=1}^{k} \frac{\overline{\alpha_j} - z}{1 - \alpha_j z}.$$
Since order \((1/B_0(1/z), 0) = k, \alpha_j = 0\) for all \(j\). Then \(B_0\) is a function of \(z^k\). This fact will be used later.

Suppose that \(f \in \mathfrak{M}(\mathbb{D})\) is not a rational function. By Theorem 1.1 there is a function \(h \in H^\infty(\mathbb{D})\) and a finite Blaschke product \(B\) such that
\[
f(z) = h(B(z)), z \in \mathbb{D},
\]
and \(\{T_f\}' = \{T_B\}'\). Without loss of generality, assume order \(B = n \geq 2\).

In the proof of Lemma 4.4, we have shown that there is a finite set \(F_1\) such that each local inverse \(\rho_j\) of \(B\) admits unrestricted continuation on \(\mathbb{C} \setminus F_1\).

Let \(P\) denote the poles of \(f\). By \(B(\rho_j) = B\) on \(\mathbb{C} \setminus F_1\) and \(f(z) = h(B(z))\), we can define a holomorphic function
\[
h(z) : B(z) \mapsto f(z)
\]
on \(\mathbb{C} \setminus (B(F_1) \cup P)\). So \(h\) has only isolated singularities. Letting
\[
F_0 = F_1 \cup B^{-1}(P),
\]
we have
\[
f(z) = h(B(z)), z \in \mathbb{C} \setminus F_0
\]
(4.3)

If order \((B, \infty) = n\), then by the second paragraph \(B\) is a function of \(z^n\), and hence
\[
\{T_f\}' = \{T_B\}' = \{T_{z^n}\}'.
\]
By similar reasoning as (4.3), there is a function \(\tilde{h}\) such that \(f(z) = \tilde{h}(z^n)\) holds on \(\mathbb{C}\) minus a discrete set. In this case, it is straightforward to show \(\tilde{h}\) is in \(\mathfrak{M}(\mathbb{D})\). By Theorem 3.4, we have \(n = o(f) = b(f) = n(f) = N(f)\) to complete the proof.

As follows, we assume that order \((B, \infty) < n\) to reach a contradiction. Since order \((B, \infty) < n\) and \(B\) is an \(n\)-to-1 map, we have a point \(a \in \mathbb{C}\), two neighborhoods \(N_1\) of \(a\) and \(N_2\) of \(\infty\) such that \(B(a) = B(\infty), B|_{N_1}\) and \(B|_{N_2}\) are proper maps, and their images are equal. There are two cases to distinguish: either \(P\) is a finite set or \(P\) is an infinite set.

**Case I.** \(P\) is a finite set. Then \(F_0\) is a finite set. By (4.3), \(f\) has similar behaviors at \(a\) and at \(\infty\). Since \(f\) is a meromorphic function and not a rational function, \(\infty\) is an essential singularity of \(f\), and so is \(a\). But this is a contradiction to the fact that \(f\) has no isolated singularities other than poles in \(\mathbb{C}\).

**Case II.** \(P\) is an infinite set. Let \(\infty\) be the limit of all poles \(\{w_k\}\) of \(f\). Note that \(F_0\) contains only finitely many accumulation points, that is, poles.
of $B$. There is an integer $k_0$ such that $w_k \in \mathcal{N}_2$ for $n \geq k_0$. Then there is a sequence $\{w'_k : k \geq k_0\}$ in $\mathcal{N}_1$ such that

$$B(w'_k) = B(w_k) \quad \text{and} \quad w'_k \to a.$$ 

Since the only accumulation points of $F_0$ are poles of $B$, these points $w'_k$ are isolated singularities of $f$. Noting (1.3), we have that $w'_k$ are poles of $f$ because $w_k$ are poles of $f$. But $\{w'_k\}$ tends to the finite point $a$, and thus $a$ is not an isolated singularity of $f$. This is a contradiction to $f \in \mathfrak{M}(\mathbb{D})$.

Therefore, in both cases we derive a contradiction to finish the proof. □

Some comments are in order. Quine [14] showed that each polynomial has FSI-decomposible property. Precisely, he proved that a nonconstant polynomial can always be written as $p(z^m)(m \geq 1)$ where $p$ is a polynomial of FSI property. For decomposition of rational functions, we call the reader’s attention to [15, 16].

The following result gives some equivalent conditions for an entire-symbol Toeplitz operator to be totally Abelian, and it follows from Theorems 4.1 and 4.2. The reader can consult related work in [20].

**Proposition 4.7.** Suppose $\phi(z) = \sum_{k=0}^{\infty} c_k z^k$ is a nonconstant entire function. Then the following are equivalent:

1. $T_\phi$ is totally Abelian;
2. $n(\phi) = \min \{\text{wind}(\phi, \phi(a)) : a \in \mathbb{D}, \phi(a) \notin f(\mathbb{T})\} = 1$.
3. there is a point $w$ on $\mathbb{T}$ such that $\phi(w)$ is not a point of self-intersection;
4. $\phi$ has only finitely many points of self-intersection on $\mathbb{T}$;
5. there is a point $w$ in $\mathbb{D}$ such that $\phi - \phi(w)$ has exactly one zero in $\mathbb{D}$, counting multiplicity.
6. there is a point $\lambda \in \mathbb{C}$ such that $\phi - \lambda$ has exactly one zero in $\mathbb{D}$, counting multiplicity;
7. $\gcd\{c_k : c_k \neq 0\} = 1$.  

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5 Some examples

This section provides some examples. Some of them are examples of totally Abelian Toeplitz operators, and others will show that the MWN Property is quite restricted even for functions of good “smooth” property on $\mathbb{T}$.

We begin with a rational function in $\mathcal{R}(\mathbb{D})$.

**Example 5.1.** Let $Q$ be a polynomial without zero on $\mathbb{D}$ and of prime degree $q$. Let $P$ be a nonconstant polynomial satisfying $\deg P < q$.

Suppose that $P$ has at least one zero in $\mathbb{D}$ and let $R = \frac{P}{Q}$. We will show that $T_R$ is totally Abelian. For this, assume $R$ has $k$ zeros in $\mathbb{D}$, counting multiplicity. We have

$$1 \leq k \leq \deg P < q. \quad (5.1)$$

If $T_R$ were not totally Abelian, then by Corollary 4.5 there would be a finite Blaschke product $B$ and a Moebius map $\tilde{R}$ such that

$$R = \tilde{R} \circ B,$$

and order $B=q$. But by $R = \tilde{R} \circ B$, we have $k \geq \text{order } B = q$, which is a contradiction to (5.1). Therefore $T_R$ is totally Abelian.

The following two examples arise from the Riemann-zeta function and the Gamma function. It is shown that under a translation or a dilation of the variable, the corresponding Toeplitz operators are totally Abelian.

**Example 5.2.** The Riemann-zeta function $\zeta(z)$ is defined as the analytic continuation of the following:

$$z \mapsto \sum_{n=1}^{\infty} \frac{1}{n^z}, \text{Re}z > 1.$$

It is a meromorphic function in $\mathbb{C}$ and the only pole is $z = 1$. Write $f(z) = \zeta(\frac{1}{z})$, and then $f(z) \in \mathcal{M}(\mathbb{D})$. We claim that $T_f$ is totally Abelian.

For this, by Theorem 4.6 it suffices to show that there is no meromorphic function $g$ on $\mathbb{C}$ such that $f(z) = g(z^k)$ for some integer $k \geq 2$. Otherwise, taking $\omega \neq 1$ and $\omega^k = 1$, we have $f(\omega z) = f(z)$. This gives $\zeta(\frac{1}{\omega z}) = \zeta(\frac{1}{z})$, and thus

$$\zeta(\omega z) = \zeta(z).$$

Then $\zeta$ has at least two poles 1 and $\omega$. This is a contradiction. Therefore, $T_f$ is totally Abelian.
Example 5.3. The Gamma function $\Gamma(z)$ is a meromorphic function with only poles at non-positive integers

$$0, -1, -2, \cdots .$$

Let $f(z) = \Gamma(z + 2)$ and $f(z) \in \mathfrak{M}(\mathbb{D})$. Then $T_f$ is totally Abelian.

Otherwise, by Theorem 4.6, there is a function $g \in \mathfrak{M}(\mathbb{D})$ such that $f(z) = g(z^k)$ for some integer $k \geq 2$. Let $\omega \neq 1$ and $\omega^k = 1$, and we have $f(z) = f(\omega z)$; that is,

$$\Gamma(z + 2) = \Gamma(\omega z + 2).$$

The poles of $\Gamma(z + 2)$

$$-2, -3, \cdots$$

must be the poles of $\Gamma(\omega z + 2)$

$$-2\omega, -3\omega, \cdots.$$

This is impossible. Hence $T_f$ is totally Abelian.

Before continuing, recall that a Jordan domain is the interior of a Jordan curve. We need Caratheodory’s theorem, which can be found in a standard textbook of complex analysis, see [1] for example.

Lemma 5.4. [Caratheodory’s theorem] Suppose that $\Omega$ is a Jordan domain. Then the inverse Riemann mapping function $f$ from $\mathbb{D}$ onto $\Omega$ extends to a 1-to-1 continuous function $F$ from $\overline{\mathbb{D}}$ onto $\overline{\Omega}$. Furthermore, the function $F$ maps $\mathbb{T}$ 1-to-1 onto $\partial \Omega$.

In what follows, we provide some examples to show that in general a function $f$ in the disk algebra $A(\mathbb{D})$ may not satisfy MWN Property, even if $f$ has good smoothness on $\mathbb{T}$.

Example 5.5. First, we present an easy example of a function in $A(\mathbb{D})$ with good smoothness on $\mathbb{T}$, but not satisfying MWN Property. Put

$$\Omega_0 = \{z : 0 < |z| < 1, 0 < \arg z < \pi\},$$

and write $g(z) = z^8, z \in \Omega_0$. Let $\phi_0$ be a conformal map from the unit disk $\mathbb{D}$ onto $\Omega_0$. Precisely, write $u(z) = \sqrt{\frac{z+1}{z-1}}$ with $\sqrt{1} = 1$ and

$$\phi_0(z) = \frac{1 - 2u(z)}{1 + 2u(z)}, z \in \mathbb{D}.$$
and put \( \phi_1 = g \circ \phi_0 \). Note that \( \phi_1 - \phi_1(0) \) has finitely many zeros in \( \mathbb{D} \) and is away from zero on \( \mathbb{T} \). One can then show that the inner part of \( \phi_1 - \phi_1(0) \) is a finite Blaschke product, and hence \( \phi_1 \in \mathcal{CT}(\mathbb{D}) \).

For smoothness of \( \phi_1 \), by Lemma 5.4 we have that \( \phi_1 \in A(\mathbb{D}) \). In addition, by using Schwarz Reflection Principle we see that except for at most three points on \( \mathbb{T} \), \( \phi_1 \) extends analytically across \( \mathbb{T} \).

However, \( \phi_1 \) does not satisfy MWN Property. In fact, for each point \( a \in \mathbb{D} \), \( \phi_1 - \phi_1(a) \) has at least 3 zeros in \( \mathbb{D} \). Thus, \( n(\phi_1) \geq 3 \).

On the other hand, since \( N(\phi_1, \mathbb{T}) = 1 \), \( \phi_1 \) can not be written as a function of a finite Blaschke product of order larger than 1. Then by Theorem 1.7 \( \{T_{\phi_1}\}' = \{T_z\}' \). That is, \( b(\phi_1) = 1 \). But \( n(\phi_1) > 1 \), forcing \( n(\phi_1) \neq b(\phi_1) \).

Inspired by this example, put \( \Omega_1 = \{z \in \mathbb{C} : |z| < 1, |z - 1| < 1\} \), and let \( h : \mathbb{D} \rightarrow \Omega \) be a conformal map. Note that by Caretheodory’s theorem, \( h \) extends continuously to a bijective map from \( \overline{\mathbb{D}} \) onto \( \overline{\Omega} \). Furthermore, noting that \( \Omega_1 \) has two cusp points, one can show that except for two cusp points, \( h \) can be analytically extended across \( \mathbb{T} \), as well as \( h^9 \). Also, \( \{T_{h^9}\}' = \{T_z\}' \).

The next example shows that Theorems 1.3 and 1.5 are restricted.

Example 5.6. By Schwarz-Christoffel formula, one can construct a conformal map \( f \) form the upper plane onto the rectangle \( \Omega \) with vertices \( \{-K, K, K + iK', -K + iK'\} \), where \( K, K' > 0 \). Precisely, it is defined by

\[
f(z) = C \int_0^z \frac{1}{\sqrt{(\lambda^2 - 1)(\lambda^2 - t^2)}} d\lambda, \quad z \in \Omega,
\]

where \( \sqrt{1} = 1 \), \( C > 0 \) and \( t \) is a parameter in \( (0, 1) \) [9, Section 2.5]. We can specialize \( K' = 2k\pi \) for some integer \( k \geq 100 \).

Define \( h(z) = \exp(z - K/2), \quad z \in \mathbb{C} \) and let \( g(z) \) be a conformal map from the unit disk onto the upper plane. Write

\[
\phi(z) = h \circ f \circ g(z), \quad z \in \mathbb{D}.
\]

It is not difficult to see that

\[
n(\phi) \geq k,
\]
and for each $\xi \in \mathbb{T}$, $N(\phi - \phi(\xi), \mathbb{T}) \geq 2$. Moreover, we have $N(\phi) = 2$.

Next we show that $o(\phi) = 1$. For this, note that $f \circ g$ maps the unit disk $\mathbb{D}$ conformally onto the rectangle $\Omega$, and $f \circ g$ extends to a continuous bijection from $\overline{\mathbb{D}}$ onto $\overline{\Omega}$, and $f \circ g(\mathbb{T}) = \partial \Omega$. Thus by definition of $o(\phi)$, the assertion $o(\phi) = 1$ is equivalent to that the only continuous map $\rho : \partial \Omega \to \partial \Omega$ satisfying $h(\rho) = h$ is the identity map. For this, let

$$h(\rho(z)) = h(z), z \in \partial \Omega.$$ 

Then for each $z$ in $\partial \Omega$, $\rho(z) = z + 2k(z)\pi i$ for some integer $k(z)$. But $\rho$ is continuous, forcing $k(z)$ to be a constant integer $k$. Hence

$$\rho(z) = h(z) + 2k\pi i, z \in \partial \Omega.$$ 

Since $\rho(\partial \Omega) \subseteq \partial \Omega$, we have $k = 0$ and $\rho$ is the identity map, forcing $o(\phi) = 1$.

Since $b(\phi) | o(\phi)$, $b(\phi) = 1$. By Theorem 1.1, $T_\phi$ is totally Abelian. But for this function $\phi$ we have

$$b(\phi) = o(\phi) < N(\phi) < n(\phi).$$

We conclude this section by showing that the function $\phi$ defined in Example 6.6 has good smoothness property on $\mathbb{T}$. Rewrite $h(z) = v(z)2^n$ where $v(z) = \exp\left[\frac{1}{2K}(z - \frac{K}{2})\right]$. Note that $v \circ f \circ g$ defines a conformal map from $\mathbb{D}$ onto the domain

$$\{z \in \mathbb{C} : \exp\left(-\frac{K}{2k}\right) < |z| < 1, \arg z \in (0, \pi)\},$$

whose boundary contains only four “cusp points”. By Lemma 5.4 we have $v \circ f \circ g \in A(\mathbb{D})$, and by Schwarz Reflection Principle $v \circ f \circ g$ extends analytically across $\mathbb{T}$ except for these cusp points. The same is true for $\phi$.

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