ON THE BOREL-CANTIELLI LEMMA AND ITS GENERALIZATION

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Abstract. Let \( \{A_n\}_{n=1}^{\infty} \) be a sequence of events on a probability space \((\Omega, \mathcal{F}, P)\). We show that if \( \lim_{m \to \infty} \sum_{n=1}^{m} w_n P(A_n) = \infty \) where each \( w_n \in \mathbb{R} \), then

\[
P(\limsup A_n) \geq \limsup_{n \to \infty} \frac{\left( \sum_{k=1}^{n} w_k P(A_k) \right)^2}{\sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j P(A_i \cap A_j)}.
\]

1. Introduction

Let \( \{A_n\}_{n=1}^{\infty} \) be a sequence of events on a probability space \((\Omega, \mathcal{F}, P)\). The classical Borel-Cantelli lemma states that: (a) if \( \sum_{n=1}^{\infty} P(A_n) < \infty \), then \( P(\lim sup A_n) = 0 \); (b) if \( \sum_{n=1}^{\infty} P(A_n) = \infty \) and \( \{A_n\}_{n=1}^{\infty} \) are mutually independent, then \( P(\lim sup A_n) = 1 \). Here \( \lim sup A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k \). The Borel-Cantelli lemma played an exceptionally important role in probability theory. Many investigations were devoted to the second part of the Borel-Cantelli lemma in the attempt to weaken the independence condition on \( \{A_n\}_{n=1}^{\infty} \).

Erdős and Rényi [4] proved that the mutual independence condition on \( \{A_n\}_{n=1}^{\infty} \) can be replaced by the weaker condition of pairwise independence. Indeed they [8] (see also [3, 5, 9]) proved a more general theorem: if \( \sum_{n=1}^{\infty} P(A_n) = \infty \), then

\[
P(\limsup A_n) \geq \limsup_{n \to \infty} \frac{\left( \sum_{k=1}^{n} P(A_k) \right)^2}{\sum_{i=1}^{n} \sum_{j=1}^{n} P(A_i \cap A_j)}.
\]

There are many discussions and generalizations towards the Borel-Cantelli lemma, for example see [1, 6, 7, 10]. The main purpose of this paper is to present a weighted version of the Erdős-Rényi theorem:

**Theorem 1.** Suppose \( \lim_{m \to \infty} \sum_{n=1}^{m} w_n P(A_n) = \infty \), where each \( w_n \) is a real weight (which could be negative). Then

\[
P(\limsup A_n) \geq \limsup_{n \to \infty} \frac{\left( \sum_{k=1}^{n} w_k P(A_k) \right)^2}{\sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j P(A_i \cap A_j)}.
\]
The proof of Theorem 1 is relatively easy if we further assume all terms of the weight sequence to be nonnegative. By choosing each \( w_n = 1/P(A_n) \) in Theorem 1, we obtain the following corollary:

**Corollary 2.** Suppose \( P(A_n) > 0 \) holds for all \( n \in \mathbb{N} \). Then
\[
P(\limsup_n A_n) \geq \limsup_{n \to \infty} \frac{n^2}{\sum_{i=1}^{n} \sum_{j=1}^{n} P(A_i \cap A_j) / P(A_i) P(A_j)}.
\]

2. **Proof of the main result**

For any matrix \( E = (z_{ij})_{m \times n} \), denote by \( \Gamma(E) \) the sum of all its entries, that is, \( \Gamma(E) = \sum_{i=1}^{m} \sum_{j=1}^{n} z_{ij} \).

**Lemma 3.** Given a partition of an \((m + n) \times (m + n)\) symmetric matrix \( E \):
\[
E = \begin{pmatrix}
A_{m \times m} & C_{m \times n} \\
C_{m \times n}^T & B_{n \times n}
\end{pmatrix}
\]
If \( E \) is positive semi-definite, then \( \Gamma(C)^2 \leq \Gamma(A) \Gamma(B) \).

**Proof:** This lemma follows from the following inequality: \( \forall x, y \in \mathbb{R}, \)
\[
(x, \ldots, x, y \ldots, y)^T E(x, \ldots, x, y \ldots, y)^T = \Gamma(A)x^2 + 2\Gamma(C)xy + \Gamma(B)y^2 \geq 0.
\]
\[ \square \]

**Lemma 4.** Let \( \{A_i\}_{i=1}^{n} \) be finitely many events on a probability space \((\Omega, \mathcal{F}, P)\). Then the matrix \((P(A_i \cap A_j))_{n \times n}\) is positive semi-definite.

**Proof:** Let \( \mathbb{E}(\cdot) \) be the expectation function and \( \chi_{A_i} \) be the indicator function of the event \( A_i \). Then \( P(A_i) = \mathbb{E}(\chi_{A_i}) \) and \( P(A_i \cap A_j) = \mathbb{E}(\chi_{A_i} \chi_{A_j}) \). For each \( (s_1, s_2, \ldots, s_n) \in \mathbb{R}^n \),
\[
(s_1, s_2, \ldots, s_n)^T (P(A_i \cap A_j))(s_1, s_2, \ldots, s_n)^T = \sum_{i=1}^{n} \sum_{j=1}^{n} s_i s_j P(A_i \cap A_j) = \mathbb{E}(\sum_{i=1}^{n} \sum_{j=1}^{n} s_i s_j \chi_{A_i} \chi_{A_j})
\]
\[ = \mathbb{E}(\sum_{i=1}^{n} \sum_{j=1}^{n} s_i s_j \chi_{A_i} \chi_{A_j}) = \mathbb{E}(\sum_{i=1}^{n} s_i \chi_{A_i})^2 \geq 0.
\]
This proves the lemma. \[ \square \]

**Lemma 5.** Suppose \( \lim_{m \to \infty} \sum_{n=1}^{m} w_n P(A_n) = \infty \), where each \( w_n \in \mathbb{R} \). Then
\[
\lim_{n \to \infty} \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j P(A_i \cap A_j)}{\sum_{i=2}^{n} \sum_{j=2}^{n} w_i w_j P(A_i \cap A_j)} = 1.
\]
Proof: By Lemma 4, \( E_n = \left( w_i w_j P(A_i \cap A_j) \right)_{n \times n} \) is positive semi-definite. Define 
\[ A = \left( w_1 w_1 P(A_1 \cap A_1) \right), \quad B_n = \left( w_i w_j P(A_i \cap A_j) \right)_{2 \leq i, j \leq n}, \quad C_n = \left( w_i w_j P(A_i \cap A_j) \right)_{2 \leq j \leq n} . \]

By Lemma 3, \( \Gamma(C_n)^2 \leq \Gamma(A) \Gamma(B_n) \) \((\forall n \geq 2)\). By the Cauchy-Schwarz inequality, 
\[ \left( \sum_{i=2}^{n} w_i P(A_i) \right)^2 = \left( E \left( \sum_{i=2}^{n} w_i \chi_{A_i} \right) \right)^2 \leq P \left( \bigcup_{i=2}^{n} A_i \right) \cdot E \left( \left( \sum_{i=2}^{n} w_i \chi_{A_i} \right)^2 \right) \]
\[ = P \left( \bigcup_{i=2}^{n} A_i \right) \cdot \left( \sum_{i=2}^{n} \sum_{j=2}^{n} w_i w_j P(A_i \cap A_j) \right) \leq \Gamma(B_n) . \]

Since \( \lim_{n \to \infty} \sum_{i=2}^{n} w_i P(A_i) = \infty \), we have \( \Gamma(B_n) \to \infty \) as \( n \) approaches to infinity. Hence 
\[ \lim_{n \to \infty} \frac{\Gamma(A) + \Gamma(B_n) + 2\Gamma(C_n)}{\Gamma(B_n)} = 1 + \lim_{n \to \infty} \frac{2\Gamma(C_n)}{\Gamma(B_n)} = 1 . \]

This proves the lemma. \( \square \)

Remark 1. We obtained the following by-product from the proof of the above lemma:

(1) \[ \left( \sum_{i=1}^{n} w_i P(A_i) \right)^2 \leq P \left( \bigcup_{i=1}^{n} A_i \right) \cdot \left( \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j P(A_i \cap A_j) \right) . \]

This formula can be viewed as a weighted version of the Chung-Erdős inequality \((\cite{2})\).

Proposition 6. Suppose \( \lim_{n \to \infty} \sum_{n=1}^{m} w_n P(A_n) = \infty \), where each \( w_n \in \mathbb{R} \). Then for all \( s \in \mathbb{N} \),
\[ \limsup_{n \to \infty} \frac{\left( \sum_{k=1}^{n} w_k P(A_k) \right)^2}{\sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j P(A_i \cap A_j)} = \limsup_{n \to \infty} \frac{\left( \sum_{k=s}^{n} w_k P(A_k) \right)^2}{\sum_{i=s}^{n} \sum_{j=s}^{n} w_i w_j P(A_i \cap A_j)} . \]

Proposition 6 is an immediate corollary of Lemma 5. With all the above preparation in hand, we are ready to prove Theorem 1.

Proof of Theorem 1: By (1) and Proposition 6,
\[ P \left( \limsup A_n \right) = P \left( \bigcap_{s=1}^{\infty} \bigcup_{k=s}^{\infty} A_k \right) = \lim_{s \to \infty} P \left( \bigcup_{k=s}^{\infty} A_k \right) = \lim_{s \to \infty} \left( \lim_{n \to \infty} P \left( \bigcup_{k=s}^{n} A_k \right) \right) \]
\[ \geq \lim_{s \to \infty} \left( \limsup_{n \to \infty} \frac{\left( \sum_{k=s}^{n} w_k P(A_k) \right)^2}{\sum_{i=s}^{n} \sum_{j=s}^{n} w_i w_j P(A_i \cap A_j)} \right) = \limsup_{n \to \infty} \frac{\left( \sum_{k=1}^{n} w_k P(A_k) \right)^2}{\sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j P(A_i \cap A_j)} . \]
This completes the proof of Theorem 1.

**Corollary 7.** Let \{w_n \geq 0\}_{n=1}^{\infty} be a bounded sequence with \(\sum_{n=1}^{\infty} w_n P(A_n) = \infty\). Then

\[
P(\limsup A_n) \geq \limsup_{n \to \infty} \frac{\sum_{1 \leq i < j \leq n} w_i w_j P(A_i \cap A_j)}{\sum_{1 \leq i < j \leq n} w_i w_j P(A_i \cap A_j)}.
\]

**Proof.** By the weighted version of the Chung-Erdős inequality (1),

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j P(A_i \cap A_j) \geq \left(\sum_{i=1}^{n} w_i P(A_i)\right)^2 \quad (\forall n \in \mathbb{N}),
\]

which yields

\[
\lim_{n \to \infty} \frac{\sum_{i=1}^{n} w_i P(A_i \cap A_i)}{\sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j P(A_i \cap A_j)} = 0
\]

by considering \(\{w_n \geq 0\}_{n=1}^{\infty}\) is a bounded sequence with \(\sum_{n=1}^{\infty} w_n P(A_n) = \infty\). Note also

\[
\left(\sum_{k=1}^{n} w_k P(A_k)\right)^2 \geq 2 \cdot \sum_{1 \leq i < j \leq n} w_i w_j P(A_i \cap A_j)
\]

Combining (2), (3) and Theorem 1 yields the desired result. \(\Box\)

**Example 1.** Let \((\Omega, \mathcal{F}, P)\) be a probability space, \(A, B \in \mathcal{F}\), \(P(A \cap B) > 0\). For all \(n \in \mathbb{N}\), let

\[A_{3n-2} = A, A_{3n-1} = B, A_{3n} = A \cap B.\]

By the Erdős-Rényi theorem,

\[
P(\limsup A_n) \geq \frac{(P(A) + P(B) + P(A \cap B))^2}{P(A) + P(B) + 7P(A \cap B)}.
\]

Applying Theorem 1 with the weight sequence \(1, 1, -1, 1, 1, -1, 1, 1, -1, \ldots\), we obtain

\[
P(\limsup A_n) \geq P(A) + P(B) - P(A \cap B) = P(A \cup B).
\]

In fact \(P(\limsup A_n) = P(A \cup B)\). So Theorem 1 is best possible for this example.

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