OBSERVATIONS ON THE TWO DIMENSIONAL JACOBIAN CONJECTURE

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Abstract. The two dimensional Jacobian Conjecture says that a morphism $f: \mathbb{C}[x, y] \to \mathbb{C}[x, y]$ having an invertible Jacobian, is invertible. We show that a morphism $f$ having an invertible Jacobian is invertible, in each of the following two special cases: The degree of $f(x)$ is $\leq 2$; or $f$ and $D$ are fields of characteristic zero, each implies that $f$ is invertible. In each case there is no restriction on the degree of $f(y)$ nor on the parity of the degree of $f(y)$.

1 Introduction

The $n$ dimensional Jacobian Conjecture says that a morphism $f: \mathbb{C}[x_1, \ldots, x_n] \to \mathbb{C}[x_1, \ldots, x_n]$ having an invertible Jacobian, is invertible; see [13].

Wang [24, Theorem 61] showed the following: “Let $D$ be a UFD with $2 \neq 0$, and let $D[y_1, \ldots, y_n] \subseteq D[x_1, \ldots, x_n]$ be a separable ring extension (the degree of each $y_i$, considered as a polynomial in $x_1, \ldots, x_n$, is $\leq 2$, then $D[y_1, \ldots, y_n] = D[x_1, \ldots, x_n]^\ast$). If the degree of each $y_i$ is of degree $\leq 2$ and $D[y_1, \ldots, y_n] = D[x_1, \ldots, x_n]^\ast$, then $D[y_1, \ldots, y_n] = D[x_1, \ldots, x_n]$ is separable. (24, Theorem 38) shows that the converse is also true, namely, separability implies invertibility of the Jacobian). Combining the two yields Wang’s theorem [24, Theorem 62] [9, Proposition 4.3.1]: “Let $D$ be a UFD with $2 \neq 0$, and let $f: D[x_1, \ldots, x_n] \to D[x_1, \ldots, x_n]$ be a morphism that satisfies $\text{Jac}(f(x_1), \ldots, f(x_n)) \in D^\ast$. If the degree of each $f(x_i)$ is $\leq 2$, then $D[f(x_1), \ldots, f(x_n)] = D[x_1, \ldots, x_n]$ directly”. Here we will mostly deal with $n = 2$, $k$ a field of characteristic zero and $f: k[x, y] \to k[x, y]$ a morphism that satisfies $\text{Jac}(f(x), f(y)) \in k^\ast$; for convenience denote $p := f(x)$ and $q := f(y)$. In Theorem 3.3 $k = \mathbb{C}$ and in Theorem 3.4 $k = \mathbb{C}$ or $k = \mathbb{R}$.

Wang’s theorem for $n = 2$ and $D = \mathbb{C}$ says that if $p$ is of degree $\leq 2$ and $q$ is of degree $\leq 2$, then $f$ is invertible. In Theorem 3.3, we show that if $p$ is of degree $\leq 2$ (and $q$ can have any degree), then $f$ is invertible. Clearly this is a generalization of Wang’s theorem when $n = 2$ and $D = \mathbb{C}$. Notice two great things in Wang’s theorem: First, it is valid for all $n$, and second, $D$ is a UFD with $2 \neq 0$. When $n = 2$ and $D$ is a field of characteristic zero, it is not difficult to show that a morphism $f$ of Wang’s theorem is invertible, see Proposition 3.2.

Remark 1.1. There exist other nice conditions when $n = 2$ and $D$ is a field of characteristic zero, each implies that $f$ is invertible:

1. The degree of $p$ is $\leq 100$ and the degree of $q$ is $\leq 100$; Moh [16] [9, Theorem 10.2.30].
(2) The degree of \( p \) or the degree of \( q \) is a prime number; Magnus [15] [9, Corollary 10.2.25].

(3) The greatest common divisor of the degrees of \( p \) and \( q \) is \( \leq 2 \); Nakai and Baba [1].

(4) The greatest common divisor of the degrees of \( p \) and \( q \) is \( \leq 8 \) or a prime number; Appelgate, Onishi, Nagata [20] [21] [9, Theorem 10.2.26].

Given \( f : k[x_1, \ldots, x_n] \to k[x_1, \ldots, x_n] \), \( f(x_i) \) is a polynomial in \( x_1, \ldots, x_n \), so we can consider its \( x_j \)-degree, which is also called the \((0, \ldots, 1, \ldots, 0)\)-degree of \( f(x_i) \), where 1 is in the \( j \)’th place. The \((1,\ldots,1)\)-degree of \( f(x_i) \) is usually called the degree of \( f(x_i) \). In Theorem 3.4, which does not generalize a known result, but may be of some interest, we show that if the \((0,1)\)-degrees or \((1,0)\)-degrees of all monomials of \( p \) have the same parity, then \( f \) is invertible.

2 Preliminaries

Our proofs of Theorem 3.3 and Theorem 3.4 rely on results found in [18] and [19]. For the convenience of the reader, we now bring the specific results which are needed in the proofs of Theorem 3.3 and Theorem 3.4.

Let \( k \) be a field of characteristic zero. Let \( \alpha \) be the exchange involution on \( k[x, y] \), namely \( \alpha \) is the morphism of order 2 (hence invertible) defined by \( \alpha(x) = y \) and \( \alpha(y) = x \). A morphism \( g : k[x, y] \to k[x, y] \) is an \( \alpha \)-morphism if \( \alpha g = g \alpha \). If \( \delta \) is any involution on \( k[x, y] \), then a morphism \( g : k[x, y] \to k[x, y] \) is a \( \delta \)-morphism if \( g \delta = \delta g \). In those definitions there is no demand on the Jacobian of \( \delta \); however, in the results, we will always assume that a given \( \delta \)-morphism has an invertible Jacobian. We will also work with the following involutions: \( \beta(x) = x, \beta(y) = -y \), \( \gamma(x) = -x, \gamma(y) = y \), and \( \epsilon(x) = -x, \epsilon(y) = -y \). Notice that \( \alpha, \beta \), and \( \gamma \) are in the same conjugacy class: \( \alpha \beta \alpha = \gamma \), and \( g^{-1} \alpha g = \beta \) where \( g(x) = (1/2)(x + y), g(y) = y - x, g^{-1}(x) = x - (1/2)y, g^{-1}(y) = x + (1/2)y \). More generally, given two involutions on \( k[x, y] \), \( \sigma \) and \( \tau \), a morphism \( g : k[x, y] \to k[x, y] \) is a \( \sigma, \tau \)-morphism if \( g \sigma = \tau g \).

**Proposition 2.1.** There exist two conjugacy classes of involutions on \( k[x, y] \) in the group of invertible morphisms (= automorphisms) of \( k[x, y] \):

- A class which consists of all involutions having Jacobian \(-1\), denote it by \( C_{-1} \).

- A class which consists of all involutions having Jacobian \(1\), denote it by \( C_1 \).

Denote \( G_1 \) = affine automorphisms, \( G_2 \) = de Jonquieres automorphisms. Recall Jung-van der Kulk automorphism theorem which says that the group of automorphisms of \( k[x, y] \), \( k \) is any field, is the amalgamated free product of \( G_1 \) and \( G_2 \) over their intersection, see [9, Theorem 5.1.11], [12] and [14].

**Sketch of proof.** Two trivial remarks:

- Two conjugate invertible morphisms (in particular, two conjugate involutions) must have the same Jacobian. So all the members of a given conjugacy class have the same Jacobian.

- Given an involution \( \delta \), the Jacobian of \( \delta \) must equal 1 or \(-1\) (The Jacobian of \( \delta \) equals its inverse, since \( \delta \) equals its inverse).

Apriori, it may happen that there exist two or more conjugacy classes of involutions having Jacobian \(-1\), and there exist two or more conjugacy classes of involutions having Jacobian 1. However, the following arguments show that there exists only one conjugacy class of involutions having Jacobian \(-1\), denote it \( C_{-1} \), and there
exists only one conjugacy class of involutions having Jacobian 1, call it $C_1$. From direct calculations we get the following two facts:

- In $G_1$ there exist exactly two conjugacy classes of involutions, that of $\alpha$ (or $\beta$ or $\gamma$) and that of $\epsilon$.
- In $G_2$ there exist exactly two conjugacy classes of involutions, that of $\alpha$ and that of $b$, where $a(x) = -x - y^2, a(y) = y$ and $b(x) = -x - y^2, b(y) = -y$.

(We obtained 6 general forms of involutions of $G_2$, and then checked that some are conjugate to others).

A direct calculation shows that:

- $a$ and $\beta$ are conjugate: $\beta = h^{-1}ah$, where $h(x) = -y, h(y) = -x - (1/2)y^2$.
- $b$ and $\epsilon$ are conjugate: $\epsilon = g^{-1}bg$, where $g(x) = y, g(y) = -x - (1/2)y^2$.

We now explain why in the group of automorphisms of $k[x, y]$ (and not only in $G_1 \cup G_2$) there exist exactly two conjugacy classes of involutions: That of $\alpha$ and that of $\epsilon$.

The explanation is quite easy thanks to J. Bell [4] who told us about Serre’s theorem [23, page 6, Corollary 1]: “Every element of $G$ of finite order is conjugate to an element of one of the $G_i$”, where $G$ is the amalgamated free product of $G_1$ and $G_2$ over their intersection. Here $G$ is the group of automorphisms of $k[x, y]$ which is known to be the amalgamated free product of $G_1$ and $G_2$ over their intersection (Jung-van der Kulk). Indeed, let $\iota$ be any involution on $k[x, y]$. $\iota$ is of order 2, so from Serre’s theorem $\iota$ must be conjugate to an element, call it $e$, of $G_1$ or of $G_2$. Trivially, $e$ is also of order 2 (it is clear that two conjugate elements, each of some finite order, must have the same order), namely $e$ is an involution. $e$ is an involution of $G_1$ or of $G_2$, therefore $e$ is conjugate to $\alpha$ or $\epsilon$, so $\iota$ is conjugate to $\alpha$ or $\epsilon$. \[\square\]

**Remark 2.2.** Let $g : k[x, y] \to k[x, y]$ be a morphism such that $\text{Jac}(g(x), g(y)) \in k^*$. If $g$ is a $\sigma, \tau$-morphism, then $\sigma, \tau \in C_{-1}$ or $\sigma, \tau \in C_1$. Indeed, $g\sigma = \tau g$ combined with $\text{Jac}(g(x), g(y)) \in k^*$ implies that the Jacobian of $\sigma$ equals the Jacobian of $\tau$, so both have Jacobian $-1$ or both have Jacobian 1.

A morphism $g : k[x, y] \to k[x, y]$ is invertible if there exists a morphism $h : k[x, y] \to k[x, y]$ such that $gh = hg = 1$, where 1 is the identity morphism. Given a morphism $g : k[x, y] \to k[x, y]$ (with no restriction on its Jacobian) denote $\text{Img}(g) := k[g(x), g(y)]$. $\text{Img}(g)$ is a sub-algebra of $k[x, y]$. A morphism $g$ of $k[x, y]$ is invertible if and only if $\text{Img}(g) = k[x, y]$, see [9, Lemma 1 in the Introduction; Definition 1.1.5] and [6, page 343].

The connection between involutions and the Jacobian Conjecture begins in the following conjecture:

**Conjecture 2.3 (The $\alpha$ Jacobian Conjecture).** Let $f : k[x, y] \to k[x, y]$ be an $\alpha$-morphism that satisfies $\text{Jac}(f(x), f(y)) \in k^*$. Then $f$ is invertible.

An analogous conjecture for the first Weyl algebra, “the $\alpha$ Dixmier Conjecture”, was made before the $\alpha$ Jacobian Conjecture and was almost proved, see [17]. More specifically, we first dealt with the first Weyl algebra and found a family of $\alpha$-endomorphisms which is easily seen to be a family of $\alpha$-automorphisms, see [17, Proposition 2.8], but we were not able to show that this family includes all $\alpha$-endomorphisms. Another result about the form of an $\alpha$-endomorphism is [17, Lemma 2.6]. An almost a proof for the $\alpha$ Dixmier Conjecture is [17, Theorem 2.9]. After reading [17], C. Valqui suggested a nice proof for the $\alpha$ Jacobian Conjecture, which appears in [18, Proposition 4.1].

**Theorem 2.4 (The $\alpha$ Jacobian Conjecture is true).** Let $f : k[x, y] \to k[x, y]$ be an $\alpha$-morphism that satisfies $\text{Jac}(f(x), f(y)) \in k^*$. Then $f$ is invertible.
Proof. See [18, Proposition 4.1].

Notice that in [18, Proposition 4.1] $\text{Jac}(f(x), f(y)) = 1$; it is clear that if $\text{Jac}(f(x), f(y)) = a \in k^*$ (with $a$ not necessarily equals 1), we can consider $\tilde{f}(x) := a^{-1}f(x)$, $\tilde{f}(y) := f(y)$. Then $\text{Jac}(\tilde{f}(x), \tilde{f}(y)) = 1$, hence [18, Proposition 4.1] implies that $f$ is invertible, and then $f$ is invertible, because $k[f(x), f(y)] = k[\tilde{f}(x), \tilde{f}(y)] = k[x, y]$. □

Then it is immediate to get:

**Corollary 2.5.** Let $f : k[x, y] \to k[x, y]$ be a $\sigma, \tau$-morphism, $\sigma, \tau \in C_{-1}$, that satisfies $\text{Jac}(f(x), f(y)) \in k^*$. Then $f$ is invertible.

**Proof.** Let $\sigma, \tau \in C_{-1}$, so there exist invertible morphisms $u$ and $v$ such that $\sigma = u^{-1}\alpha u$ and $\tau = v^{-1}\alpha v$. $f$ is a $\sigma, \tau$-morphism: $f\sigma = \tau f$. So $fu^{-1}\alpha u = v^{-1}\alpha f$, then $(vfu^{-1})\alpha = \alpha(vfu^{-1})$, namely, $vfu^{-1}$ is an $\alpha$-morphism. From Theorem 2.4 [18, Proposition 4.1], we get that $vfu^{-1}$ is invertible. Indeed, Theorem 2.4 can be applied here, since from the Chain Rule, the fact that any invertible morphism has an invertible Jacobian, and the assumption that $f$ has an invertible Jacobian, we obtain: $\text{Jac}(vfu^{-1}(x), (vfu^{-1})(y)) \in k^*$. Clearly, invertibility of $vfu^{-1}$ implies invertibility of $f$. □

Similarly to the $\epsilon$ Jacobian Conjecture 2.3 we have:

**Conjecture 2.6** (The $\epsilon$ Jacobian Conjecture). Let $f : k[x, y] \to k[x, y]$ be an $\epsilon$-morphism that satisfies $\text{Jac}(f(x), f(y)) \in k^*$. Then $f$ is invertible.

We suspect that the $\epsilon$ Jacobian Conjecture is also true, though we have not yet proved it. A positive answer to the $\epsilon$ Jacobian Conjecture will immediately imply that if $f : k[x, y] \to k[x, y]$ is a $\sigma, \tau$-morphism, $\sigma, \tau \in C_1$, that satisfies $\text{Jac}(f(x), f(y)) \in k^*$, then $f$ is invertible: same proof as that of Corollary 2.5 (apply the positive answer to the $\epsilon$ Jacobian Conjecture instead of the positive answer to the Jacobian Conjecture). Whether the $\epsilon$ Jacobian Conjecture is true or not, thanks to the positive answer to the $\epsilon$ Jacobian Conjecture, Theorem 2.4 [18, Proposition 4.1], we were able to obtain additional results, some of which are brought in [19]; we now bring only those results relevant for the proofs of Theorem 3.3 and Theorem 3.4.

$k$ continues to denote a field of characteristic zero. For a morphism $g : k[x, y] \to k[x, y]$ that satisfies $\text{Jac}(g(x), g(y)) \in k^*$, $\text{Img}(g)$ is isomorphic to $k[x, y]$ (since if $\text{Jac}(g(x), g(y)) \in k^*$, then, by [22], $g(x)$ and $g(y)$ are algebraically independent over $k$).

**Definition 2.7** (The $\alpha$ restriction condition). We say that a morphism $g : k[x, y] \to k[x, y]$ satisfies the $\alpha$ restriction condition if $\alpha(g(x)) \in \text{Img}(g)$ and $\alpha(g(y)) \in \text{Img}(g)$. Equivalently, we say that $g$ satisfies the $\alpha$ restriction condition if the exchange involution $\alpha$ on $k[x, y]$ when restricted to $\text{Img}(g)$ is an involution on $\text{Img}(g)$.

We do not know if a morphism $f : k[x, y] \to k[x, y]$ that satisfies $\text{Jac}(f(x), f(y)) \in k^*$ necessarily satisfies the $\alpha$ restriction condition, but if it does, then it is invertible:

**Theorem 2.8** (The $\alpha$ restriction theorem). Let $f$ be a morphism of $k[x, y]$ that satisfies $\text{Jac}(f(x), f(y)) = 1$. $f$ satisfies the $\alpha$ restriction condition $\iff$ $f$ is invertible.

An immediate (and trivial) corollary to Theorem 2.8 is as follows: Let $f$ be a morphism of $k[x, y]$ that satisfies $\text{Jac}(f(x), f(y)) \in k^*$. $f$ satisfies the $\alpha$ restriction condition $\iff$ $f$ is invertible.
from Proposition 2.1 there exist two conjugacy classes of involutions having Jacobian 1 and involutions having Jacobian 1.

Theorem 2.12, as we have just seen). Involutions having Jacobian 1 and involutions having Jacobian 1. \( \alpha_0 \) and \( \rho_0 \) both have Jacobian 1, hence both belong to the same conjugacy class, so there exists an invertible morphism \( h_0 : \text{Im}(f) \to \text{Im}(f) \) such that \( \rho_0 = h_0^{-1} \alpha_0 h_0 \). Therefore, \( (f \alpha)(x) = f(\alpha(x)) = f(y) = \rho_0(f(x)) = (h_0^{-1} \alpha_0 h_0)(f(x)) = (h_0^{-1} \alpha_0 h_0(f))(x) \), and, \( (f \alpha)(y) = f(\alpha(y)) = f(x) = \rho_0(f(y)) = (h_0^{-1} \alpha_0 h_0)(f(y)) = (h_0^{-1} \alpha_0 h_0 f)(y) \).

Therefore, \( f \alpha = h_0^{-1} \alpha_0 h_0 f \). Then, \( h_0 f \alpha = \alpha_0 h_0 f \), so \( h_0 f \alpha = \alpha h_0 f \), namely \( h_0 f \) is an \( \alpha \)-morphism of \( k[x, y] \). Since the Jacobian of \( h_0(f(x)), h_0(f(y)) \) with respect to \( f(x), f(y) \) is a non-zero scalar \( h_0 \) is an invertible morphism of \( \text{Im}(f) \) and the Jacobian of \( f(x), f(y) \) with respect to \( x, y \) equals 1 (by assumption), we get that the Jacobian of \( (h_0 f)(x), (h_0 f)(y) \) with respect to \( x, y \) is a non-zero scalar (by the Chain Rule). By Theorem 2.4 [18, Proposition 4.1] \( h_0 f \) is an invertible morphism of \( k[x, y] \), namely, \( k[(h_0 f)(x), (h_0 f)(y)] = k[x, y] \). Then we have:

\[
x = \sum a_{ij} (h_0 f(x))^i (h_0 f(y))^j = \sum a_{ij} (h_0 f(x))^i (h_0 f(y))^j = \sum b_{ij} (h_0 f(x))^i (h_0 f(y))^j = h_0 (\sum b_{ij} (h_0 f(x))^i (h_0 f(y))^j)
\]

which satisfies the \( \alpha \)-condition, since trivially \( \alpha(f(x)) \in k[x, y] = \text{Im}(f) \) and \( \alpha(f(y)) \in k[x, y] = \text{Im}(f) \).

Theorem 2.12 and its corollary 2.13 will be applied in the proofs of Theorem 3.3 and Theorem 3.4: the proof of Theorem 2.12 relies on a theorem of Cheng-McKay-Wang [5, Theorem 1] and on the \( \alpha \)-restricion theorem 2.8 (Theorem 2.8 relies on Theorem 2.4 [18, Proposition 4.1], as we have just seen).

Before bringing Theorem 2.12, we wish to discuss Cheng-McKay-Wang’s theorem [5, Theorem 1] which says the following: “Let \( L \) be the field of complex numbers. Assume \( A, B \in L[x, y] \) satisfy \( \text{Jac}(A, B) \in L^* \). If \( R \in L[x, y] \) satisfies \( \text{Jac}(A, R) = 0 \), then \( R \in L[A] \)”. In other words, C-M-W’s theorem [5, Theorem 1] says that “the centralizer with respect to the Jacobian” of an element \( A \in C[x, y] \) which has a Jacobian mate, equals \( C[A] \). By definition, a Jacobian mate of an element \( A \in k[x, y] \) is an element \( B \in k[x, y] \) such that \( \text{Jac}(A, B) \in k^* \). Its analogous result in the first Weyl algebra over any characteristic zero field, not necessarily the field of complex numbers, can be found in [10, Theorem 2.11]; instead of the Jacobian take the commutator.

The following Lemma shows that C-M-W’s theorem over \( L = C \) implies C-M-W’s theorem over \( L = \mathbb{R} \):

**Lemma 2.9.** Assume \( A, B \in \mathbb{R}[x, y] \) satisfy \( \text{Jac}(A, B) \in \mathbb{R}^* \). If \( R \in \mathbb{R}[x, y] \) satisfies \( \text{Jac}(A, R) = 0 \), then \( R \in \mathbb{R}[A] \).

Proof. \( A, B \in \mathbb{R}[x, y] \subseteq C[x, y] \) satisfy \( \text{Jac}(A, B) \in C^* \subseteq \mathbb{R}^* \), so C-M-W’s theorem implies that \( R \in C[A] \) (of course, \( R \in \mathbb{R}[x, y] \subseteq C[x, y] \)). Hence \( R = \sum_{u=0}^t c_u A^u \) for some \( c_u \in C \). We claim that actually \( c_u \in \mathbb{R} \): On the one hand, \( R \in \mathbb{R}[x, y] \), so we can write \( R = \sum_{i=0}^k \sum_{j=0}^l r_{ij} x^i y^j \) for some \( r_{ij} \in \mathbb{R} \). On the other hand, \( R = \sum_{u=0}^t c_u A^u \) for some \( c_u \in \mathbb{R} \). Therefore \( \sum_{u=0}^t \sum_{v=0}^w a_{uv} x^v y^w \) equals \( \sum_{u=0}^t c_u A^u \). We can write \( \mathbb{R}[x, y] \ni A = \sum_{u=0}^t \sum_{v=0}^w a_{uv} x^v y^w \), where \( a_{uv} \in \mathbb{R} \). So \( \sum_{u=0}^t \sum_{v=0}^w r_{ij} x^i y^j = \sum_{u=0}^t \sum_{v=0}^w a_{uv} x^v y^w \)
∑ t u=n c u(∑ m a c w x u y w ) u. The leading coefficient on the right is c t a t m, and the leading coefficient on the left is r kl. Hence r kl = c t a t m, so c t = r kl/(a t m) ∈ ℜ, since r kl ∈ ℜ and a t m ∈ ℜ. Now consider ℜ[x, y] ∋ ∑ m r ij x i y j = c t A t = ∑ u=t-1 c u(∑ m a u x u y m ) u, and similarly obtain that c t-1 ∈ ℜ. Indeed, the leading coefficient on the right is c t-1 a t-1 m, and the leading coefficient on the left is some d ∈ ℜ. Hence c t-1 = d/(a t-1 m) ∈ ℜ. Continuing this way we obtain that c 0, c 1, ..., c t-1, c t are all ∈ ℜ. Therefore, R = ∑ u=0 t-1 c u A u ∈ ℜ[A]. □

As for the validity of C-M-W’s theorem over fields other than ℂ or ℜ, perhaps it is still valid over any field of characteristic zero, as the two following lemmas may show. However, the first lemma 2.10 does not have a complete proof, only a sketch of proof, so it may happen that the first lemma is not true.

Lemma 2.10. Let L be an algebraically closed field of characteristic zero. Assume A, B ∈ L[x, y] satisfy Jac(A, B) ∈ L*. If R ∈ L[x, y] satisfies Jac(A, R) = 0, then R ∈ L[A].

Sketch of proof. The proof of C-M-W’s theorem uses [25, Theorem 1] and [3, Corollary 1.5, p. 74]; except these two results, it seems that the other arguments in C-M-W’s proof are valid not only over ℂ but also over other fields (we are not sure over which fields the other arguments are valid; maybe any commutative rings). So if one wishes to generalize C-M-W’s theorem to an algebraically closed field of characteristic zero, it is enough to check that each of [25, Theorem 1] and [3, Corollary 1.5, p. 74] is valid over an algebraically closed field of characteristic zero. Indeed, in [25, Theorem 1] the base field is any ring, and in [3, Corollary 1.5, p. 74] the base field is an algebraically closed field of characteristic zero. □

If we do not have any errors in the sketch of proof of Lemma 2.10, then similarly to the proof of Lemma 2.9 (=getting the result over ℜ from the result over ℂ) we can obtain the following lemma (=getting the result over L from the result over L̄, where L is any field of characteristic zero, L is an algebraic closure of L).

Lemma 2.11. Let L be a field of characteristic zero. Assume A, B ∈ L[x, y] satisfy Jac(A, B) ∈ L*. If R ∈ L[x, y] satisfies Jac(A, R) = 0, then R ∈ L[A].

Proof. A, B ∈ L[x, y] ⊂ L̄[x, y] satisfy Jac(A, B) ∈ L̄* ⊂ L̄*, and R ∈ L[x, y] ⊂ L̄[x, y] satisfies Jac(A, R) = 0. From Lemma 2.10 we get that R ∈ L̄[A]. Exactly the same arguments as in the proof of Lemma 2.9 are valid if, instead of working with ℜ and ℂ, we work with L and L̄. □

Since in the above discussion there was no precise conclusion whether C-M-W’s theorem is valid not only over ℂ and ℜ but also over any field of characteristic zero (since Lemma 2.10 was not fully proved), we restrict the base field of Theorem 2.12 to ℂ or ℜ (because we wish to use C-M-W’s theorem in the proof of Theorem 2.12).

Theorem 2.12. Let K denote ℂ or ℜ. Let f : K[x, y] → K[x, y] be a morphism that satisfies Jac(f(x), f(y)) ∈ K*. If one of the following conditions is satisfied, then f is invertible:

1. f(x) is symmetric.
2. f(x) is skew-symmetric.
3. f(y) is symmetric.
4. f(y) is skew-symmetric.

Where by symmetric or skew-symmetric we mean symmetric or skew-symmetric with respect to α.
It is impossible to have both \( f(x) \) and \( f(y) \) symmetric with respect to \( \alpha \) or both skew-symmetric with respect to \( \alpha \), see [19, Remark 4.8]. If it happens that one of \( f(x), f(y) \) is symmetric w.r.t. \( \alpha \) and the other is skew-symmetric w.r.t. \( \alpha \), then it is quite immediate (as long as we know Theorem 2.4 [18, Proposition 4.1]) that such \( f \) is invertible, see [19, Remark 4.8].

Proof. For convenience, denote \( p := f(x), q := f(y), \) and \( T := \text{Img}(f) = K[p, q] \).

(1): Assume that \( p \) is symmetric w.r.t. \( \alpha \), namely \( \alpha(p) = p \). So \( \alpha(p) = p \in T \). Denote \( a := \text{Jac}(p, q) \in K^* \). Clearly, \( \text{Jac}(p, \alpha(q)) = \text{Jac}(\alpha(p), \alpha(q)) = -a \).

Then, \( \text{Jac}(p, q + \alpha(q)) = \text{Jac}(p, q) + \text{Jac}(p, \alpha(q)) = a - a = 0 \), so from C-M-W’s theorem (and our Lemma 2.9, in case \( K = \mathbb{R} \)) we have \( q + \alpha(q) = H(p) \) where \( H(t) \in K[t] \). So \( \alpha(q) = -q + H(p) \in T \). We have, \( \alpha(p) \in T \) and \( \alpha(q) \in T \), namely, \( f \) satisfies the \( \alpha \) restriction condition. Hence the \( \alpha \) restriction theorem 2.8 (or its immediate and trivial corollary) implies that \( f \) is invertible.

(2): Assume that \( p \) is skew-symmetric w.r.t. \( \alpha \), namely \( \alpha(p) = -p \). So \( \alpha(p) = -p \in T \). Denote \( a := \text{Jac}(p, q) \in K^* \). Clearly, \( \text{Jac}(-p, \alpha(q)) = \text{Jac}(\alpha(p), \alpha(q)) = -a \). Then, \( \text{Jac}(p, q - \alpha(q)) = \text{Jac}(p, q) + \text{Jac}(p, -\alpha(q)) = a - a = 0 \), so from C-M-W’s theorem (and our Lemma 2.9, in case \( K = \mathbb{R} \)) we have \( q - \alpha(q) = H(p) \) where \( H(t) \in K[t] \).

So \( \alpha(q) = q - H(p) \in T \). We have, \( \alpha(p) \in T \) and \( \alpha(q) \in T \), namely, \( f \) satisfies the \( \alpha \) restriction condition. Hence the \( \alpha \) restriction theorem 2.8 (or its immediate and trivial corollary) implies that \( f \) is invertible.

\( \square \)

In the proof of Theorem 2.8 we have shown that for a morphism \( f \) having an invertible Jacobian, if the restriction of \( \alpha \) to \( k[f(x), f(y)] \) is an involution on \( k[f(x), f(y)] \), then from the Chain Rule one sees that it belongs to the conjugacy class of involutions on \( k[f(x), f(y)] \); having Jacobian \(-1\). In the proof of (1) of Theorem 2.12 we have found that the restriction of \( \alpha \) to \( K[f(x), f(y)] \) is as follows: \( \alpha(f(x)) = f(x) \) and \( \alpha(f(y)) = -f(y) + H(f(x)) \), so, without using the Chain Rule, one immediately gets that the Jacobian of the restricted \( \alpha \) equals \(-1\); of course the partial derivatives are with respect to \( f(x) \) and \( f(y) \), not with respect to \( x \) and \( y \).

(In Theorem 2.8 \( k \) is any field of characteristic zero, while in Theorem 2.12 \( K \) is \( C \) or \( \mathbb{R} \).

Corollary 2.13. Let \( K \) denote \( C \) or \( \mathbb{R} \), and \( \delta \in C_{-1}. \) Let \( f : K[x, y] \to K[x, y] \) be a morphism that satisfies \( \text{Jac}(f(x), f(y)) \in K^* \). If one of the following conditions is satisfied, then \( f \) is invertible:

1. \( f(x) \) is symmetric.
2. \( f(x) \) is skew-symmetric.
3. \( f(y) \) is symmetric.
4. \( f(y) \) is skew-symmetric.

Where by symmetric or skew-symmetric we mean symmetric or skew-symmetric with respect to \( \delta \).

Proof. (1): Assume that \( f(x) \) is symmetric w.r.t. \( \delta \), namely \( \delta(f(x)) = f(x) \). Since \( \delta \in C_{-1} \), there exists an invertible morphism of \( K[x, y], g \), such that \( \delta = g^{-1} \alpha g \).

Hence \( \delta(f(x)) = f(x) \) becomes \((g^{-1} \alpha g)(f(x)) = f(x) \), so \( \alpha(g f(x)) = (g f)(x) \). It is clear that the morphism \( g f \) has an invertible Jacobian (since each of \( g \) and \( f \) has an invertible Jacobian). Therefore, we can apply Theorem 2.12 condition (1) to \( g f \), and get that \( g f \) is invertible. Then clearly \( f \) is invertible.

(2): Assume that \( f(x) \) is skew-symmetric w.r.t. \( \delta \), namely \( \delta(f(x)) = -f(x) \).

Since \( \delta \in C_{-1} \), there exists an invertible morphism of \( K[x, y], g \), such that \( \delta = g^{-1} \alpha g \).

Hence \( \delta(f(x)) = -f(x) \) becomes \((g^{-1} \alpha g)(f(x)) = -f(x) \), so \( \alpha(g f(x)) = -(g f)(x) \). It is clear that the morphism \( g f \) has an invertible Jacobian (since each
of $g$ and $f$ has an invertible Jacobian). Therefore, we can apply Theorem 2.12 condition (2) to $gf$, and get that $gf$ is invertible. Then clearly $f$ is invertible. □

3 Our two observations

In theorem 3.3 we take $\mathbb{C}$ as a base field, while in Theorem 3.4 we take $\mathbb{C}$ or $\mathbb{R}$ as a base field; both proofs rely on Corollary 2.13 which is over $\mathbb{C}$ or $\mathbb{R}$, but in the proof of Theorem 3.3 we also wish that square roots of elements of the base field belong to the base field, so we dismiss of $\mathbb{R}$.

If it will turn out that Lemma 2.11 (which relies on Lemma 2.10) is true, then Theorem 2.12 and its corollary 2.13 are valid over any field of characteristic zero, not only over $\mathbb{C}$ or $\mathbb{R}$, and then Theorem 3.4 is valid over any field of characteristic zero and Theorem 3.3 is valid over any field of characteristic zero which has the property that square roots of elements of it belong to it (for example, an algebraically closed field of characteristic zero).

$k$ continues to denote a field of characteristic zero.

Lemma 3.1 (Degree 1 case). Let $f : k[x, y] \to k[x, y]$ be a morphism that satisfies $\text{Jac}(f(x), f(y)) \in k^*$. If one of $f(x), f(y)$ is of degree 1, then $f$ is invertible.

Proof. Write $f(x) = ax + by + e, f(y) = \sum_{i=0}^{n} \sum_{j=0}^{m} c_{ij} x^i y^j$, where $a, b, c, c_{ij} \in k, c_{nm} \neq 0$. If both $a$ and $b$ are non-zero, then we can define $g(x) := (1/a)(x - y), g(y) := (1/b)y$, and get $\text{Jac}(x - y, Q) \cap (1, 1) - \text{degree of } f = \text{Jac}(x + e, Q_0 + Q_1 + \ldots + Q_r) = \text{Jac}(x + e, Q_1) + \ldots + \text{Jac}(x, Q_r)$. From considerations of degree (the $(1, 1)$-degree of $\text{Jac}(x + e, Q_j)$ equals $j - 1, 1 \leq j \leq r$) we have: $\text{Jac}(x + e, Q_1) \in k^*$, $\text{Jac}(x + e, Q_j) = 0$ for $2 \leq j \leq r$. Apply C-M-W’s theorem to $\text{Jac}(x + e, Q_j) = 0$ for $2 \leq j \leq r$, and get that $Q_j \in k[x + e] = k[x] + e$ for $2 \leq j \leq r$. So, $Q_j = H_j(x), H_j(t) \in k[t]$ for $2 \leq j \leq r$. If $\text{Jac}(x + e, Q_1) = a \in k^*$, then $Q_1 = ay + H_1(x)$, where $H_1(t) \in k[t]$. Concluding that $(gf)(y) = Q_0 + Q_1 + \ldots + Q_r = Q_0 + ay + H_1(x) + H_2(x) + \ldots + H_r(x)$. So, $(gf)(y) = ay + H(x)$. $(H(x) = Q_0 + H_1(x) + H_2(x) + \ldots + H_r(x)$ is a polynomial in $x$ over $k$). Clearly such $gf$ is invertible.

Third proof: This proof is valid over any field of characteristic zero, since we use [9, Lemma 10.2.4 (i)] which is valid over any field of characteristic zero. Somewhat similar to the second proof, use [9, Lemma 10.2.4 (i)] (see also [11, Proposition 2.1 (b)]) instead of C-M-W’s theorem. □

3.1 First observation

First we wish to prove the following proposition, which was mentioned in the introduction:

Proposition 3.2 (A special case of Wang’s theorem). Let $f : k[x, y] \to k[x, y]$ be a morphism that satisfies $\text{Jac}(f(x), f(y)) \in k^*$. If the degree of $f(x)$ is $\leq 2$ and the degree of $f(y)$ is $\leq 2$, then $f$ is invertible.
Proof. Write: \( f(x) = p_0 + p_1 + p_2, \) \( f(y) = q_0 + q_1 + q_2, \) where \( p_i, q_i \) is the \((1,1)\)-homogeneous component of \( f(x), f(y) \) respectively, having \((1,1)\)-degree \(i\). \( k^* \ni \text{Jac}(f(x), f(y)) = \text{Jac}(p_0 + p_1 + p_2, q_0 + q_1 + q_2). \) From considerations of \((1,1)\)-degrees, \( \text{Jac}(p_2, q_2) = 0 \), hence from [9, Lemma 10.2.4 (i)] or [11, Proposition 2.11 (b)] it follows that there exists a \((1,1)\)-homogeneous element \( R \in k[x, y] \), such that:

1. If \( R \) is of degree 1, then \( p_2 = \lambda R^2 \) and \( q_2 = \mu R^2 \) for some \( \lambda, \mu \in k^* \).
2. If \( R \) is of degree 2, then \( p_2 = \lambda R \) and \( q_2 = \mu R \) for some \( \lambda, \mu \in k^* \).

In each case it is not difficult to obtain that \( f \) is invertible; just take \((fg)(x) = f(g(x)) = f(x - (\lambda/\mu)y) = f(x) - (\lambda/\mu)f(y) = p_0 + p_1 + p_2 - (\lambda/\mu)(q_0 + q_1 + q_2) = p_0 + p_1 + \lambda R - (\lambda/\mu)(q_0 + q_1 + \mu R) = p_0 + p_1 - (\lambda/\mu)(q_0 + q_1) \), where \( R = R^2 \) for (1), \( R = R \) for (2), and \( g \) is the invertible morphism defined by: \( g(x) := x - (\lambda/\mu)y \) and \( g(y) := y \). (Notice that if \( \mu = 0 \), then \( f(y) \) is of degree 1, and then Lemma 3.1 immediately shows that \( f \) is invertible). We can apply Lemma 3.1 to \( fg \) \((fg)(x)\) is of degree 1), get that \( fg \) is invertible, and then \( f \) is invertible.

Now we are finally in the position to bring:

**Theorem 3.3** (Our first observation). Let \( f : \mathbb{C}[x, y] \to \mathbb{C}[x, y] \) be a morphism that satisfies \( \text{Jac}(f(x), f(y)) \in \mathbb{C}^* \). If the degree of \( f(x) \) or the degree of \( f(y) \) is \( \leq 2 \), then \( f \) is invertible.

The proof uses Theorem 2.12 and its corollary 2.13 (Theorem 2.12 is based on the a restriction theorem 2.8, which is based on Theorem 2.4 [18, Proposition 4.1]). There is really nothing more than that, only some quite easy calculations.

**Proof.** For convenience denote: \( p := f(x) \) and \( q := f(y) \).

**First option:** The degree of \( p \) is 1, so Lemma 3.1 shows that \( f \) is invertible.

Another argument is as follows; we bring it since it is in the spirit of the arguments which we will immediately bring in the second option: Write \( p = ax + by + c \), for some \( a, b, c \in \mathbb{C} \) with at least one of \( a, b \) non-zero.

**I** If \( b = 0 \), then \( f(x) = p = ax + c \) is symmetric with respect to \( \beta \). By Corollary 2.13, \( f \) is invertible.

**II** If \( a = 0 \), then \( f(x) = p = by + c \) is symmetric with respect to \( \gamma \). By Corollary 2.13, \( f \) is invertible.

**III** If both \( a \) and \( b \) are non-zero, then we can define the invertible morphism \( g(x) = bx, g(y) = ay \). The morphism \( gf \) has an invertible Jacobian (since the Jacobian of each of \( f \) and \( g \) is invertible, and by the Chain Rule) and \( (gf)(x) = g(f(x)) = g(p) = g(ax + by + c) = ag(x) + bg(y) + c = a(bx) + b(ay) + c = ab(x + y) + c \) is symmetric with respect to the exchange involution \( \alpha \). By Theorem 2.12, \( gf \) is invertible. Then clearly \( f \) is invertible \((f = g^{-1}(gf))\) is a composition of two invertible morphisms: \( g^{-1} \) and \( gf \).

**Second option:** The degree of \( p \) is 2, so \( p = ax^2 + bxy + cy^2 + dx + cy + r \), for some \( a, b, c, d, e, r \in \mathbb{C} \) with at least one of \( a, b, c \) non-zero.

We consider three main cases: \( a = c = 0 \); one of \( a, c \) is zero and the other is non-zero; both \( a \) and \( c \) are non-zero. Each of the three main cases is further divided into sub-cases.

In each sub-case we get that \((g_2 g_1 f)(x)\) or \((g_1 f)(x)\) or \( f(x) \) is symmetric with respect to some involution \( \in C_{-1} \), where \( g_i \) are some invertible morphisms. Then by Corollary 2.13, \( \prod g_i f \) is invertible, hence \( f \) is invertible.

**Case 1** \( a = c = 0 \) (hence \( b \neq 0 \)).

**I(1)** If \( d = e = 0 \), then \( p = bxy + r \) is already symmetric with respect to \( \alpha \).

**I(2)** If \( e = 0, d \neq 0 \), then \( p = bxy + dx + r \). Since \( b \neq 0 \), we can define the following morphism which is obviously invertible: \( g_1(x) = x, g_1(y) = y/b - d/b \).
Then \((g_1 f)(x) = bx/y - d/b\) + \(dx + r = xy - dx + dx + r = xy + r\) is symmetric w.r.t. \(\alpha\).

I(3) If \(d = 0, e \neq 0\), then \(p = bxy + ey + r\). Define the following invertible morphism: \(g_1(x) = x/b - e/b, g_1(y) = y\). Then \((g_1 f)(x) = b(x/b - e/b) + ey + r = xy - ey + ey + r = xy + r\) is symmetric w.r.t. \(\alpha\).

I(4) If \(d \neq 0\) and \(e \neq 0\), then define the following invertible morphism: \(g_1(x) = ex, g_1(y) = dy\). Then \((g_1 f)(x) = b(ey)(dy) + de(x) + e(dy) + r = bdexy + de(x + y) + r\) is symmetric w.r.t. \(\alpha\).

Case II One of \(a, c\) is zero and the other is non-zero, without loss of generality assume \(c = 0\) and \(a \neq 0\).

II(1) \(b = 0\): So \(p = ax^2 + dx + ey + r\).
- If \(d = e = 0\), then \(p = ax^2 + r\) is symmetric w.r.t. \(\beta\) (or \(\gamma\)).
- If \(e = 0\), then \(p = ax^2 + r\) is symmetric w.r.t. \(\beta\).
- If \(d = 0, e \neq 0\), then \(p = ax^2 + ey + r\) is symmetric w.r.t. \(\gamma\).
- If \(d \neq 0\) and \(e \neq 0\), then define the following invertible morphism: \(g_1(x) = ex, g_1(y) = dy\). Then \((g_1 f)(x) = ae^2x^2 + de(x + y) + r\). Take \(g_2(x) = x, g_2(y) = -x + y\), and get: \((g_2 g_1 f)(x) = ae^2x^2 + dey + r\) which is symmetric w.r.t. \(\gamma\).

II(2) \(b \neq 0\): So \(p = ax^2 + bxy + dx + ey + r\). Define the following invertible morphism: \(g_1(x) = x/\sqrt{a} - iy/\sqrt{a}, g_1(y) = 2\sqrt{a}y/b\). Then \((g_1 f)(x) = x^2 + y^2 + Dx + Ey + r\), where \(D = d/\sqrt{a}, E = -di/\sqrt{a} + 2ey/\sqrt{a}/b\). Take \(g_2(x) = x - D/2, g_2(y) = y - E/2\). Then \((g_2 g_1 f)(x) = x^2 + y^2 - D^2/4 - E^2/4 + r\) is symmetric w.r.t. \(\alpha\).

Case III Both \(a\) and \(c\) are non-zero.

III(1) \(b = 0\): So \(p = ax^2 + cy^2 + dx + ey + r\). Since \(a \neq 0\) and \(c \neq 0\), we can define: \(g_1(x) = x/\sqrt{a}, g_1(y) = y/\sqrt{c}\). Then \((g_1 f)(x) = x^2 + y^2 + dx/\sqrt{a} + ey/\sqrt{c} + r\).

Remark: \(a, c \in \mathbb{C}\), so there exist \(\sqrt{a}, \sqrt{c}\). This shows that the same proof will not necessarily work over \(\mathbb{R}\); we shall have to at least add the condition that both coefficients \(a\) and \(c\) are positive or both are negative; if both are negative, we will have to consider \(-p\) instead of \(p\), show that \(-f\) is invertible, hence \(f\) is invertible.

Now define: \(g_2(x) = x - d/2\sqrt{a}, g_2(y) = y - e/2\sqrt{c}\). Then \((g_2 g_1 f)(x) = x^2 + y^2 - d^2/4a - e^2/4c + r\) is symmetric w.r.t. \(\alpha\).

III(2) \(b \neq 0\): Again we can define: \(g_1(x) = x/\sqrt{a}, g_1(y) = y/\sqrt{c}\). Then \((g_1 f)(x) = x^2 + Bxy + y^2 + Dx + Ey + r\), where \(B = b/\sqrt{ac}, D = d/\sqrt{a}, E = e/\sqrt{c}\). Now, \(x^2 + Bxy + y^2 + Dx + Ey + r = (x + By/2)^2 + ty^2 + Dx + Ey + r\), where \(t = 1 - B^2/4\).

- If \(t \neq 0\), then we can define \(g_2(x) = x - By/2\sqrt{t}, g_2(y) = y/\sqrt{t}\). So \((g_2 g_1 f)(x) = x^2 + y^2 + Dx + Ey + r\), where \(D = D\) and \(E = -DB/2\sqrt{t} + E/\sqrt{t}\), which is of the form of III(1), so \(g_2 g_1 f\) is invertible.
- If \(t = 0\), then \(B = 2\) or \(B = -2\): If \(B = 2\), then \((g_1 f)(x) = (x + y)^2 + Dx + Ey + r\). Take \(g_2(x) = x + y, g_2(y) = -y\). Then \((g_2 g_1 f)(x) = x^2 + Dx + (D - E)y + r\), which is of the form of III(1), so \(g_2 g_1 f\) is invertible.
- If \(B = -2\), then \((g_1 f)(x) = (x - y)^2 + Dx + Ey + r\). Take \(g_2(x) = x + y, g_2(y) = y\). Then \((g_2 g_1 f)(x) = x^2 + Dx + (D + E)y + r\), which is of the form of III(1), so \(g_2 g_1 f\) is invertible.

3.2 Second observation

Theorem 3.4 (Our second observation). Let \(K\) denote \(\mathbb{C}\) or \(\mathbb{R}\). Let \(f : K[x, y] \to K[x, y]\) be a morphism that satisfies \(\text{Jac}(f(x), f(y)) \in K^*\). If one of the following two conditions is satisfied, then \(f\) is invertible:
(1) The \((0,1)\)-degrees of all monomials in \(f(x)\) are of the same parity.
(2) The \((1,0)\)-degrees of all monomials in \(f(x)\) are of the same parity.

Proof. If all monomials in \(f(x)\) are of odd \((0,1)\)-degrees then \(f(x)\) is skew-symmetric w.r.t. \(\beta\). If all monomials in \(f(x)\) are of even \((0,1)\)-degrees then \(f(x)\) is symmetric w.r.t. \(\beta\). If all monomials in \(f(x)\) are of odd \((1,0)\)-degrees then \(f(x)\) is skew-symmetric w.r.t. \(\gamma\). If all monomials in \(f(x)\) are of even \((1,0)\)-degrees then \(f(x)\) is symmetric w.r.t. \(\gamma\). In each case, Corollary 2.13 shows that \(f\) is invertible \(\Box\)

4 Generalizations

We suggest the following ideas how to generalize Theorems 3.3 and 3.4. We have not yet seriously checked each idea, so maybe some of them will not work; we hope that it will turn out that some of them will work.

4.1 First idea

The following theorem is true, as a special case of theorems (1), (2), (4) of Remark 1.1. However, we wish to prove it in a way similar to the way we proved Theorem 3.3, namely, using Corollary 2.13.

Theorem 4.1. Let \(f : \mathbb{C}[x,y] \rightarrow \mathbb{C}[x,y]\) be a morphism that satisfies \(\text{Jac}(f(x),f(y)) \in \mathbb{C}^*\). If the degree of \(f(x)\) or the degree of \(f(y)\) is \(\leq 3\), then \(f\) is invertible.

Sketch of proof. If the degree of \(f(x)\) is \(\leq 2\), then \(f\) is invertible by Theorem 3.3. If the degree of \(f(x)\) is \(3\), then \(f(x)\) is of the following form: \(f(x) = ax^3 + bx^2y + cxy^2 + dy^3 + Ex^2 + Fxy + Gy^2 + Dx + Ey + R\), where \(a, b, c, d, A, B, C, D, E, R \in \mathbb{C}\) with at least one of \(a, b, c, d\) non-zero. We guess it is possible to find invertible morphisms \(g_1, \ldots, g_l\), such that \((\prod g_if)(x)\) is symmetric or skew-symmetric w.r.t. some involution \(\in C_{-1}\). If so, then by Corollary 2.13, \(\prod g_if\) is invertible, hence so is \(f\).

Actually, what is important in the proof of Theorem 3.3 and in the sketch of proof of Theorem 4.1 is not the degree of \(f(x)\), but the existence of invertible morphisms \(g_1, \ldots, g_l\) such that \((\prod g_if)(x)\) becomes symmetric or skew-symmetric w.r.t. some involution \(\in C_{-1}\) (When the degree of \(f(x)\) is \(\leq 2\), it is not difficult to find such \(g_i\)’s, as we have seen in the proof of Theorem 3.3. When the degree of \(f(x)\) is \(3\), the situation becomes more complicated, and we have not yet found such \(g_i\)’s, only in some special cases, one of them is brought in Example ??). Therefore we make the following conjecture:

Conjecture 4.2. Let \(K\) denote \(\mathbb{C}\) or \(\mathbb{R}\). Assume \(A \in K[x,y]\) has a Jacobian mate. Then there exist invertible morphisms \(g_1, \ldots, g_l\) such that \(\prod g_iA\) is symmetric or skew-symmetric w.r.t. some involution on \(K[x,y]\) that has Jacobian \(-1\).

Example 4.3. Let \(A = x + y^3\). Clearly, \(A\) is skew-symmetric w.r.t. \(\epsilon\), but the Jacobian of \(\epsilon\) equals \(1\), so \(\epsilon\) is not relevant for Conjecture 4.2. However, we can take \(g_1(x) := x - y^3, g_1(y) := y\), and get \(g_1A = g_1(x + y^3) = g_1(x) + g_1(y^3) = x - y^3 + y^3 = x\), which is symmetric w.r.t. \(\beta\) (and is skew-symmetric w.r.t. \(\gamma\)).

We suspect that Conjecture 4.2 is true, or at least the following weaker version of it is true, in which we allow the involution to have either Jacobian \(-1\) or \(1\):

Conjecture 4.4. Let \(K\) denote \(\mathbb{C}\) or \(\mathbb{R}\). Assume \(A \in K[x,y]\) has a Jacobian mate. Then there exist invertible morphisms \(g_1, \ldots, g_l\) such that \(\prod g_iA\) is symmetric or skew-symmetric w.r.t. some involution on \(K[x,y]\).
Of course, what is nice in Conjecture 4.2 is that a positive answer to it implies a positive answer to the two dimensional Jacobian Conjecture:

**Theorem 4.5.** If Conjecture 4.2 is true, then the two dimensional Jacobian Conjecture (over \( \mathbb{C} \) or \( \mathbb{R} \)) is true.

**Proof.** Let \( K \) denote \( \mathbb{C} \) or \( \mathbb{R} \). Let \( f : K[x, y] \to K[x, y] \) be a morphism that satisfies \( \text{Jac}(f(x), f(y)) \in K^* \). If \( f(x) \) has a Jacobian mate, so Conjecture 4.2 implies that there exist invertible morphisms \( g_1, \ldots, g_t \) such that \( \prod g_i f(x) \) is symmetric or skew-symmetric w.r.t. some involution on \( K[x, y] \) that has Jacobian \(-1\). Apply Corollary 2.13 to \( \prod g_i f \) and obtain that \( \prod g_i f \) is invertible. Then \( f \) is invertible. \( \square \)

If Conjecture 4.2 is not true, but its weak version Conjecture 4.4 is true, we do not have an analogous result to Theorem 4.5 yet (hopefully, we will have an analogous result to Theorem 4.5); this is because we have not yet proved the \( \epsilon \) Jacobian Conjecture 2.6.

Indeed, if the \( \epsilon \) Jacobian Conjecture is true, then it is not difficult to obtain analogous results to the \( \alpha \) restriction theorem 2.8, Theorem 2.12 and its corollary 2.13, with \( \alpha \) replaced by \( \epsilon \) and \( \delta \) of Corollary 2.13, which is any involution that is conjugate to \( \alpha \), replaced by any involution that is conjugate to \( \epsilon \).

### 4.2 Second idea

If the following “proof” has no errors, then the following theorem is a generalization of Wang’s theorem [24, Theorem 62] (when \( D = \mathbb{C} \) in Wang’s theorem).

**Theorem 4.6.** Let \( f : \mathbb{C}[x_1, \ldots, x_n] \to \mathbb{C}[x_1, \ldots, x_n] \) be a morphism having an invertible Jacobian \((n \geq 2)\). If there exists \( 1 \leq i_0 \leq n \) such that the degree of \( f(x_{i_0}) \) is \( \leq 2 \), then \( f \) is invertible.

**Proof.** First idea of a proof: \( f(x_{i_0}) \) is of degree \( \leq 2 \), then \( f(x_{i_0}) = ax_1^2 + bx_1 x_2 + cx_2^2 + ux_1 + vx_2 + w \), where \( a, b, c \in \mathbb{C}, u, v, w \in \mathbb{C}[x_3, \ldots, x_n] \). So now we wish to work with \( R[x_1, x_2] \), where \( R := \mathbb{C}[x_3, \ldots, x_n] \). Notice that taking \( \sqrt{a}, \sqrt{c} \) is still possible, since here also \( a, c \in \mathbb{C} \).

Caution: Working over \( R = \mathbb{C}[x_3, \ldots, x_n] \) instead of working over a field of characteristic zero may cause problems:

- The positive answer to the \( \alpha \) Jacobian Conjecture, Theorem 2.4 [18, Proposition 4.1], is over a field of characteristic zero, but maybe it is still valid over \( R \).
- C-M-W’s theorem is originally over \( \mathbb{C} \); we have seen that probably a more general field than \( \mathbb{C} \) can be taken (Lemmas 2.9, 2.10 and 2.11). We have not yet checked if C-M-W’s theorem is still valid over an integral domain (of characteristic zero). Maybe a way to overcome this is to consider the field of fractions of \( R \), denote it \( Q(R) \), apply C-M-W’s theorem over \( Q(R) \) and then consider \( R \subset Q(R) \) (similarly to what we have done in Lemma 2.9).

However, even if those problems can be solved, considering \( R[x_1, x_2] \) may not help, since the Jacobian of \( f(x_{i_0}) \) and \( f(x_j) \), \( j \neq i_0 \), with respect to \( x_1, x_2 \), does not necessarily belong to \( R^* = \mathbb{C}^* \).

**Second idea of a proof:** To prove that the \( \alpha \) Jacobian Conjecture has a positive answer, and to prove the \( \alpha \) restriction theorem, both for \( k[x_1, \ldots, x_n], n > 2 \). Then to check how to generalize Theorem 2.12 and its corollary 2.13 to \( n > 2 \); we will have to find the correct generalization of C-M-W’s theorem (Theorem 2.12 relies on C-M-W’s theorem). Perhaps we will have to demand that for \( n - 1 \) indices
the associate \( n - 1 \) \( f(x_j) \)'s are all symmetric w.r.t. some involution having Jacobian \(-1\), in case demanding for only one index will not be enough.

Only after having all these generalizations, we can see if it is possible to prove the current theorem; perhaps we will have to demand in the current theorem that for \( n - 1 \) indices \( \subset \{ 1, \ldots, n \} \) the associate \( n - 1 \) \( f(x_j) \)'s are all of degree \( \leq 2 \).

(Remark: If in the generalized Theorem 2.12, demanding for only one index will be enough, namely demanding that there exists \( 1 \leq i_0 \leq n \) such that \( f(x_{i_0}) \) is symmetric w.r.t. some involution having Jacobian \(-1\), then we may also have a similar conjecture to Conjecture 4.2).

\[ \square \]

### 4.3 Third idea

We hope that it is possible to generalize our results here and show that cubic homogeneous polynomial maps (also called Yagzhev maps) are invertible, see Bass, Connell and Wright [2], Yagzhev [26], [9, Theorem 6.3.1]. It may be less difficult to show that cubic linear polynomial maps (also called Druzkowski maps) are invertible, see [7], [9, Theorem 6.3.2].

**Proposition 4.7.** When \( n = 2 \), Druzkowski maps are invertible.

**Proof.** Let \( d \) be a Druzkowski map. It is easy to find invertible morphisms such that composing them an \( d(x) \) yields a morphism which takes \( x \) to \( \lambda x + \mu y^3 \). We have seen in Example 4.3 that such a morphism is invertible. \[ \square \]

Proposition 4.7 is not enough, since we must prove that for all \( n \geq 2 \), Druzkowski maps are invertible, not just for \( n = 2 \). However, we strongly suspect that it is possible to show, with the help of the results brought in this paper, that cubic linear polynomial maps (Druzkowski maps) are invertible:

If the generalizations we suggested in the “proof” of Theorem 4.6 (second idea) are indeed possible, namely, if it is possible to get a generalized version (\( n > 2 \)) of the positive answer to the \( \alpha \) Jacobian Conjecture, the \( \alpha \) restriction theorem, Theorem 2.12 and its corollary 2.13, then we hope that it is not difficult to show that Druzkowski maps are invertible; this depends on what exactly a generalization of Corollary 2.13 will be:

- “A good generalization”: Invertibility of \( f \) can be obtained by demanding that there exists \( 1 \leq i_0 \leq n \) such that \( f(x_{i_0}) \) is symmetric w.r.t. some involution having Jacobian \(-1\). Denote a given Druzkowski map by \( d \). Then it seems possible, in a similar way to what we have seen in the proof of Proposition 4.7, to find invertible morphisms \( g_1, \ldots, g_l \) such that \( (\prod g_i d)(x_1) \) is symmetric w.r.t. some involution having Jacobian \(-1\). (Of course, we can take any \( (\prod g_i d)(x_1) \), not necessarily \( (\prod g_i d)(x_1) \), since all \( d(x_i) \) are of the same form).

- “Not as good as the above generalization”: Invertibility of \( f \) can be obtained by demanding that there exist \( n - 1 \) indices \( \subset \{ 1, \ldots, n \} \) such that the associate \( n - 1 \) \( f(x_j) \)'s are all symmetric w.r.t. some involution having Jacobian \(-1\). Now it seems less easy (but hopefully still possible) to find invertible morphisms \( g_1, \ldots, g_l \) such that for \( n - 1 \) indices \( j \), the \( n - 1 \) \( (\prod g_i d)(x_j) \)'s are symmetric w.r.t. some involution having Jacobian \(-1\).

### 4.4 Fourth idea:

We suggest to generalize Theorem 3.4 to all \( n > 2 \). We wonder if one can find new proofs for known results (for example [8]) based on Corollary 2.13.
4.5 Fifth idea:

In the introduction we mentioned two great things in Wang’s theorem. The first thing seems more important than the second, at least from the point of view of trying to solve the Jacobian Conjecture, since it reminds us of the above mentioned Theorems of Bass, Connell and Wright [2], Yagzhev [26] and Druzkowski [7]. We wonder if one can adjust the proofs of those theorems to show that it suffices to prove the Jacobian Conjecture for all $n \geq 2$ and all quadratic polynomial maps (in order to do so, maybe one should consider a new notion of “stable equivalence” which involves degree 2 instead of degree 3, see [9, pages 119-124]).

References

[1] K. Baba, Y. Nakai, A generalization of Magnus’ theorem, Osaka J. Math 14 (1977), 403-409.
[2] H. Bass, E. Connell and D. Wright, The Jacobian Conjecture: reduction of degree and formal expansion of the inverse, Bull. Amer. Math. Soc. (New Series) 7 (1982), 287-330.
[3] H. Bass, Differential structure of etale extensions of polynomial algebras, Commutative Algebra: Proceedings of a Microprogram (June 15–July 2, 1987, MSRI, Berkeley, CA), Math. Sci. Res. Inst. Publ., 15, Springer-Verlag, New York, (1989), 69-108.
[4] J. Bell, a private note sent by email, 8 Apr 2014.
[5] C. C.-A. Cheng, J. H. Mckay and S. S.-S. Wang, Younger mates and the Jacobian conjecture, Proc. Amer. Math. Soc. 123 (1995), no. 10, 2939-2947.
[6] P. M. Cohn, Free rings and their relations, London Mathematical Society Monograph No. 19, Second edition, 1985.
[7] L. M. Druzkowski, An effective approach to Keller’s Jacobian Conjecture, Math. Ann., 264 (1983), 303-313.
[8] A. van den Essen, E.-M.G.M. Hubbers, A new class of invertible polynomial maps, J. Algebra 187 (1997), 214-226.
[9] A. van den Essen, Polynomial automorphisms and the Jacobian conjecture, Progress in Mathematics 190, Birkhäuser Verlag, Basel, 2000.
[10] J. A. Guccione, J. J. Guccione and C. Valqui, On the centralizers in the Weyl algebra, Proc. Amer. Math. Soc. 140 (2012), no. 4, 1233-1241.
[11] J.A. Guccione, J.J. Guccione, and C. Valqui, On the shape of possible counterexamples to the Jacobian Conjecture, arXiv:1401.1794 [math.AC], 8 Jan 2014
[12] H.W.E. Jung, Uber Ganze birationale Transformationen der Ebene, J. Reine Angew. Math., 184 (1942), 161-174.
[13] O. H. Keller, Ganze Cremona-transformationen, Monatsh. Math. Phys. 47 (1939), 299-306.
[14] W. van der Kulk, On polynomial rings in two variables, Nieuw Archief voor Wiskunde, 3 (1953), no. 1, 33-41.
[15] A. Magnus, On polynomial solutions of a differential equation, Math. Scand. 3 (1955), 255-260.
[16] T. Moh, On the global Jacobian Conjecture and the configuration of roots, J. Reine Angew. Math. 340 (1983), 140-212.
[17] V. Moskowicz, The starred Dixmier’s conjecture, arXiv:1310.7562v4 [math.RA] 18 Feb 2014.
[18] V. Moskowicz, C. Valqui, The starred Dixmier conjecture for $A_1$, Comm. Algebra 43 (2015), no. 8, 3073-3082.
[19] V. Moskowicz, Involutions and the Jacobian Conjecture, arXiv:1410.7705v1 [Math.RA] 28 Oct 2014.
[20] M. Nagata, Two dimensional Jacobian Conjecture, Proceedings of the third KIT Mathematics Workshop (M.H. Kim and K.H. Ko, eds.), Korean Institute of Technology, 1988, 77-89.
[21] M. Nagata, Some remarks on the two-dimensional Jacobian Conjecture, Chin. J. Math. 17 (1989), 1-7.
[22] L. H. Rowen, Graduate algebra: Commutative view, Graduate Studies in Mathematics, volume 73, Amer. Math. Soc., 2006.
[23] J.P. Serre, Trees, Springer Monographs in Mathematics, 1980.
[24] S. S.-S. Wang, A Jacobian criterion for separability, J. Algebra 65 (1980), 453-494.
[25] S. S.-S. Wang, Extension of derivations, J. of Algebra 69 (1981), 240-246.
[26] A. V. Yagzhev, On Keller’s problem, Siberian Math. J. 21(1980), 747-754.