CONSTITUENT QUARK-BASED LINEAR σ MODEL (LσM) QUARK AND SCALAR MESONS, VECTOR MESON DOMINANCE

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Abstract

After describing the SU(2) linear sigma model (LσM), we dynamically generate it using the B.W. Lee null tadpole sum (characterizing the true vacuum) together with the dimensional regularization lemma. Next we generate the chiral-limiting (CL) nonstrange and strange constituent quark masses \( \hat{m} = 325.7 \text{ MeV}; m_s = 486 \text{ MeV} \) away from the CL. Finally, we study vector meson dominance (VMD) and the pion, kaon charge radii and the loop-order \( \rho \to \pi \gamma, \pi^0 \to \gamma \gamma \) amplitudes in the quark model. Lastly, we verify this procedure using tree-order VMD graphs.

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1 Introduction

It is well-known that hadron low-energy interactions are quite well-described by the effective chiral Lagrangians (ECL). The ECL represents the Lagrange form of an approximate chiral-symmetry realization which is also the case for quantum chromodynamics (QCD). The ECL method allows a deeper insight into the current algebra results, the low-energy theorems as well as the hypotheses of the vector meson dominance (VMD) and partial conservation of axial current (PCAC). The attractive peculiarity of the ECL formalism is the possibility to work directly with the observable (physical) particles and describe observable quantities. The study of the ECL properties is an actual problem in modern particle physics.

The derivation of ECL from ”the first principles” in QCD is a problem not solved so far. Its solution needs joint description of such effects as bosonization, spontaneous chiral symmetry breaking (SBCS) and confinement. Unfortunately at the contemporary level of development of quantum field theory, the joint description ”from first principles” of these complicated nonperturbative effects has not been achieved. That is why for the study of the ECL properties one frequently uses the effective quark models based on QCD.

The most convenient quark model, describing at the quantum field theory level such non-perturbative effects as SBCS and bosonization, is the quark-based LσM. The conversion of current quarks into constituent ones (i.e. SBCS), the elimination of quark degrees of freedom and the transition to low-lying meson states can be performed at the quantum field theory level.

This article is organized as follows. After introduction in Sec. 2 we first use four chiral couplings \( g, g', \lambda, m_q \) to describe the quark-based LσM interacting Lagrangian

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density. Then we follow B.W. Lee and characterize the true (vanishing) vacuum via the null tadpole sum of $u, d$ quark loops plus the pion loop plus the sigma meson loop. Even though these loops are all quadratically divergent, their sum must vanish independent of any renormalization scheme. We next solve this vanishing sum via dimensional analysis and then via dimensional regularization. Next in Sec. 3 we invoke a once-subtracted dispersion relation for the pion decay constant. The result is $f_\pi$ varies from $q^2 = 0$ to $m_\pi^2$ by only about 3%. Also the nonstrange constituent quark mass varies from the chiral limit (CL) to its on-shell value (to about 337.4 MeV) by 3.6%. Finally in Sec. 4 we extend the analysis to strange quarks, $SU(3)$ scalar mesons and to meson charge radii, using both the above L\(\sigma\)M and independently VMD.

2 \(SU(2)\) L\(\sigma\)M Lagrangian density

First we display the interacting part of the \(SU(2)\) L\(\sigma\)M Lagrangian density

\[
L^\text{int}_{L\sigma M} = g \bar{\psi} (\sigma + i\gamma_5 \vec{\tau} \cdot \vec{\pi}) \psi + g' \sigma (\sigma^2 + \pi^2) - \frac{\lambda}{4} (\sigma^2 + \pi^2)^2 - f_\pi g \bar{\psi} \psi
\]

for approximately $f_\pi \approx 93$ MeV. The corresponding fermion and meson Goldberger-Treiman relations (GTRs) are

\[
g = \frac{m_q}{f_\pi} \tag{2}
\]

\[
g' = \lambda f_\pi = m_\sigma^2/2 f_\pi. \tag{3}
\]

The original L\(\sigma\)M [1] involving nucleons as the fermion fields in Eq.(1) required spontaneous symmetry breaking (SSB) in order that chiral symmetry holds.

Alternatively, if the fermion fields in Eq.(1) correspond to constituent $u$ and $d$ quarks, SSB is replaced by dynamical symmetry breaking (DSB) [2], characterized by the true vacuum determined by the B.W. Lee null tadpole sum [3] of Fig.1:

\[
\begin{aligned}
&\text{u,d} \quad \text{+} \quad \pi \quad \text{+} \quad \sigma \\
&\text{l}\sigma \quad \text{l}\sigma \quad \text{l}\sigma \quad = \quad 0
\end{aligned}
\]

Fig.1.

Here the average quark mass $\hat{m} = \frac{1}{2}(m_u + m_d)$ is generated by the GTR $\hat{m} = f_\pi g$. The dimensionless meson-quark coupling $g$ and quartic meson coupling $\lambda$ obey the GTRs of Eqs.(2,3). The massive cubic meson coupling $g'$ is $m_\sigma^2/2 f_\pi$ in Eq.(3).

In order to "solve" the null tadpole sum condition of Fig.1. [2] we characterize these quadratic divergent tadpole integrals using dimensional analyses:

\[
\int \frac{d^4 p}{p^2 - m^2} \propto m^2, \quad \int \frac{d^4 p}{p^2} = 0, \quad \int \frac{d^4 p}{p^2 - m_\sigma^2} \propto m_\sigma^2. \tag{4}
\]
Then we invoke the GTRs Eqs.(2,3) and also the \( \sigma - \sigma - \sigma \) combinatoric factor of 3 to express Fig.1. in the chiral limit (CL) as [2]

\[
N_c(2m_q)^4 = 3m_\sigma^4,
\]

(5)

where \( N_c \) is the color number of quarks (presumably \( N_c = 3 \)).

Next we invoke the dimensional regularization lemma [2]

\[
\lim_{l \to 2} \int d^2l p \left[ \frac{m^2}{(p^2 - m^2)^2} - \frac{1}{p^2 - m^2} \right] = \lim_{l \to 2} \frac{im^{2l-2}}{(4\pi)^l} \left[ \Gamma(2 - l) + \Gamma(1 - l) \right] = -\frac{im^2}{(4\pi)^2}.
\]

(6)

This dimensional regularization condition follows because of the gamma function identity

\[
\Gamma(2 - l) + \Gamma(1 - l) = \Gamma(3 - l) \quad \text{as} \quad l \to 2.
\]

(7)

In fact, this dimensional regularization lemma Eq.(6) also is valid by involving a partial-fraction identity combined with the massless quadratic tadpole \( \int d^4pp^{-2} = 0 \) [4].

Finally one expresses the squared \( \sigma \) mass via the bubble plus tadpole graphs of Fig.2:

\[
m_\sigma^2 = -\sigma + \sigma.
\]

Fig.2.

Since the bubble graph is logarithmically divergent and the tadpole graph of Fig.2 is quadratic divergent, invoking the dimensional regularization lemma of Eq.(6), Fig.2 implies

\[
m_\sigma^2 = N_c g^2 \frac{m_q^2}{\pi^2}.
\]

(8)

Simultaneously solving Eqs.(5)(6)(8), one is led to in the CL

\[
N_c = 3, \quad m_\sigma = 2m_q \text{ (NJL)},
\]

(9)

\[
g = \frac{2\pi}{\sqrt{3}} \approx 3.6276 \quad (Z = 0 \text{ cc}).
\]

(10)

Eq.(9) is the NJL condition [5] and Eq.(10) is the \( Z = 0 \) compositeness condition [6], the latter also following from the infrared limit of QCD [7]. These conditions Eqs.(9)(10) hold for the quark-level SU(2) LσM [2] without first referring to the nonlinear NJL model. Once the above B.W. Lee null tadpole condition is satisfied (leading to Eqs.(9)(10)), we suggest no further renormalization is required.
3 Dispersion relation for the pion decay constant

Next we dynamically generate the SU(2) and SU(3) constituent quark masses by first invoking a once-subtracted dispersion relation for the pion decay constant [8][9]:

\[
\frac{f_\pi}{f_\pi^{CL}} - 1 = \frac{m_\pi^2}{8\pi^2 f_\pi^2} \left( 1 + \frac{m_\pi^2}{10\hat{m}^2} \right) \approx 2.946 \%. \tag{11}
\]

Note that the order \( m_\pi^2 \) term 2.88% [8] increases to 2.95% including \( m_\pi^4 \) corrections [9]. Now invoking the presumably observed pion decay constant [10] using Eq.(11):

\[
f_\pi \approx 92.42 \text{ MeV} & \quad f_\pi^{CL} = \frac{f_\pi}{1.02946} \approx 89.775 \text{ MeV}. \tag{12}
\]

Then the quark-level GTR in the CL is (via Eq.(10))

\[
\hat{m}^{CL} = f_\pi^{CL} g = 325.7 \text{ MeV}, \tag{13}
\]

near \( m_N/3 \approx 313 \text{ MeV} \) as expected. Away from the CL the nonstrange constituent quark mass can be estimated via the proton magnetic dipole moment as [11][12]

\[
\hat{m} \approx \frac{m_N}{\mu_p} \approx \frac{938.9}{2.7928} \text{ MeV} \approx 336.2 \text{ MeV}, \tag{14}
\]

just 3.2% greater than the CL quark mass in Eq.(13). Note that Eq.(14) is near the GTR quark mass away from the CL:

\[
m_q = f_\pi g \approx 93 \text{ MeV} \times \frac{2\pi}{\sqrt{3}} \approx 337.4 \text{ MeV}, \tag{15}
\]

where we again have invoked the L\( \sigma \)M [2] or the \( Z = 0 \) cc \( \implies g = 2\pi/\sqrt{3} \) [6].

4 The extension of the analysis to strange quarks

Because of the continued consistency with these SU(2) relations, we next extend this theory to SU(3). To introduce the strange constituent quark, we extend the nonstrange GTR \( \hat{m} = f_\pi g = (1/2)(m_u + m_d) \) to strange quarks obeying \( f_K g = (1/2)(m_s + \hat{m}) \) with ratio (independent of \( g \)):

\[
\frac{m_s}{\hat{m}} = \frac{2f_K}{f_\pi} - 1 \approx 1.44 \tag{16}
\]

because [10] \( f_K/f_\pi \approx 1.22 \). Then invoking \( \hat{m} \approx 337.4 \text{ MeV} \) from Eq.(15), the strange quark mass in Eq.(16) is

\[
m_s \approx 1.44 \hat{m} \approx 485.9 \text{ MeV}, \tag{17}
\]

only 5% less than the original estimate [11, 12] \( m_s \approx 510 \text{ MeV} \). Another test of this \( m_s \) mass scale is via the kaon GTR \( m_s = 2f_K g - \hat{m} = (823.2 - 337.4) \text{ MeV} = 485.8 \text{ MeV} \), in excellent agreement with Eq.(17) - but now depending on the \( g = 2\pi/\sqrt{3} \) scale.
Note that the analogue NJL (or LσM) condition in the CL implies

\[ m_{f_0}(980) = 2m_s \approx 972 \text{ MeV}, \quad (18) \]

then compatible because data says [10] \( m_{f_0}(980) = 980 \pm 10 \) MeV, suggesting that the \( f_0(980) \) is scalar mainly \( \bar{ss} \). This is consistent with the nonstrange \( a_1(1260) \) (having \( a_1 \to \sigma\pi \) seen, but \( a_1 \to f_0(980)\pi \) not seen [10]).

Further note that the kappa scalar obeys the NJL condition

\[ m_\kappa = 2[m_\sigma\hat{m}]^{1/2} \approx 809 \text{ MeV} \quad (19) \]

for \( \hat{m} \approx 337 \) MeV, \( m_\sigma \approx 486 \) MeV, is then also compatible with the observed E 791 data [13] \( m_\kappa \approx 797 \pm 19 \) MeV.

Lastly we estimate the nonstrange NJL - LσM scalar mass in the CL as

\[ m_\sigma^{CL} = 2\hat{m}_CL \approx 651.4 \text{ MeV}, \quad (20) \]

invoking the nonstrange CL quark mass 325.7 MeV as obtained in Eq.(13) above. In fact Eq.(20) implies the LσM mass (squared) away from the CL:

\[ m_\sigma^2 - m_\pi^2 = (m_\sigma^{CL})^2 \quad \text{or} \quad m_\sigma \approx 666 \text{ MeV} \quad (21) \]

very near the model independent [14] \( m_\sigma \approx 665 \) MeV based on a coupled-channel \( \pi\pi \to \pi\pi, K\bar{K} \) dispersion analysis. An analogous coupled-channel analysis was earlier used to estimate a kappa scalar mass in the 730 - 800 MeV region [15].

In fact a still earlier infinite momentum frame [IMF] approach [16] estimates the quadratic meson mass SU(3) relations

\[ m_K^2 - m_\pi^2 = m_{K^*}^2 - m_\rho^2 = m_\Phi^2 - m_{K^*}^2 \approx 0.22 \text{ GeV}^2. \quad (22) \]

These \( \bar{q}q \) meson \( \Delta S = 1 \) mass splittings are about 1/2 the \( qqq \) baryon mass splittings

\[ \Delta m_{qqq}^2(\text{octet}) \approx m_{\Sigma^+}^2 - m_N^2 \approx m_{\Xi^-}^2 - m_{\Sigma^0}^2 \approx 0.43 \text{ GeV}^2, \quad (23) \]

\[ \Delta m_{qqq}^2(\text{decuplet}) \approx m_{\Sigma^0}^2 - m_\Delta^2 \approx m_{\Xi^+}^2 - m_{\Sigma^-}^2 \approx m_{\Omega}^2 - m_{\Xi^-}^2 \approx 0.43 \text{ GeV}^2 \quad (24) \]

because there are two \( \Delta S = 1 \) transitions for \( qqq \) baryons as opposed to \( \bar{q}q \) mesons [16].

Finally we study VMD. Pions and kaons are tightly bound, so the CL \( \bar{q}q \) pion charge radius is

\[ r_{CL}^{\pi} = \frac{\hbar c}{m_{CL}} = \frac{197.3 \text{ MeV}}{325.7 \text{ MeV}} \approx 0.606 \text{ fm}, \quad (25) \]

\[ r_{VMD}^{\pi} = \sqrt{6}\frac{\hbar c}{m_\rho} \approx 0.623 \text{ fm}, \quad (26) \]

\[ r_{\pi}^{\exp} = 0.672 \pm 0.008 \text{ fm}. \quad (27) \]

The above pattern also holds for kaons:

\[ r_{CL}^{K} = \frac{2\hbar c}{m_s + \hat{m}|_{CL}} \approx 0.497 \text{ fm}, \quad (28) \]
\[ r_{K}^{VMD} = \frac{\sqrt{6} \hbar c}{m_{K^*}} \approx 0.541 \text{ fm}, \]  
\[ r_{K}^{\text{exp}} = 0.560 \pm 0.031 \text{ fm}. \]  

Such tight \( \bar{q}q \) binding for pions and kaons should be extended to loosely bound \( qqq \) baryons such as protons \([17, 18]\):

\[ R_{p} \approx [1 + \sin 30^0] r_{\pi} \approx 0.9 \text{ fm}, \]  

near data \([10]\)

\[ R_{p} = 0.870 \pm 0.008 \text{ fm}. \]  

Also note the PVV quark triangle amplitude magnitude for \( \rho \to \pi \gamma \) decay (weighted by a Levi-Civita factor) for color number \( N_{c} = 3 \)

\[ |F_{\rho \pi \gamma}| = \frac{e g_{\rho}}{8\pi^{2} f_{\pi}} \approx 0.206 \text{ GeV}^{-1}, \]  
as obtained via the quark triangle graph for \( g_{\rho} \approx 4.97 \) as found below. This amplitude magnitude is near data \([10]\)

\[ |F_{\rho \pi \gamma}|^{\text{exp}} = 0.222 \pm 0.012 \text{ GeV}^{-1}. \]  

Similarly the \( \pi^{0} \gamma \gamma \) quark triangle magnitude is

\[ |F_{\pi^{0} 2\gamma}| = \frac{e^2}{4\pi^{2} f_{\pi}} \approx 0.0251 \text{ GeV}^{-1}, \]  

also near data \([10]\)

\[ |F_{\pi^{0} 2\gamma}|^{\text{exp}} = 0.0252 \pm 0.0009 \text{ GeV}^{-1}. \]  

Note that VMD tree graphs require

\[ F_{\rho \pi \gamma} \frac{e}{g_{\rho}} = \frac{F_{\pi^{0} 2\gamma}}{2} \approx 0.01256 \text{ GeV}^{-1} \]  
as first noted by ref.\([19]\), also near \( F_{\rho \pi \gamma}e/g_{\rho} \) data at 0.0140 \( \pm 0.0007 \) GeV\(^{-1}\).

To point out the close link between VMD and the quark level \( \sigma \)M \([2, 18]\), we first note that the \( \sigma \)M requires \( g_{\rho \pi \pi} = \sqrt{3}g = 2\pi \approx 6.28 \) as obtained by many authors \([2, 20]\).

The \( \rho \pi \pi \) decay rate is for \( q = 364 \text{ MeV} \)

\[ \Gamma_{\rho \pi \pi} = \frac{g_{\rho \pi \pi}^{2}}{6\pi m_{\rho}^{2}} g^{2} \approx 150.3 \text{ MeV} \implies g_{\rho \pi \pi} \approx 5.95, \]  

while the smaller \( \rho e^{+}e^{-} \) rate is via VMD \([10]\)

\[ \Gamma_{\rho e^{+}e^{-}} = \frac{e^{4} m_{\rho}}{12\pi g_{\rho}^{2}} \approx 7.02 \text{ keV} \implies g_{\rho} \approx 4.96. \]  

VMD universality suggests \([21]\) \( g_{\rho \pi \pi} \approx g_{\rho} \), which is extended in the quark-level \( \sigma \)M via the \( \pi - \sigma - \pi \) meson loop \([22]\):

\[ g_{\rho \pi \pi} = g_{\rho} + \frac{1}{6} g_{\rho \pi \pi} \implies \frac{g_{\rho \pi \pi}}{g_{\rho}} = \frac{6}{5}, \]  

\[ 6 \]
very close to the data ratio in Eqs.(38)(39). The 1/6 factor in Eq.(40) corresponds to
\(\lambda/(16\pi^2)\) with \(\lambda = 2g^2\) and \(g = 2\pi/\sqrt{3}\) as obtained in the L\(\sigma\)M relation Eq.(10) above.

In this paper we have first reviewed the quark-level SU(2) L\(\sigma\)M and then dynamically generated it. Next we obtained the CL pion decay constant and found the nonstrange and strange constituent CL quark masses. Then we found the pion and kaon charge radii and the quark loop values for the \(\rho \to \pi \gamma\) and \(\pi^0 \to \gamma \gamma\) amplitudes - all fitting data without introducing arbitrary parameters.

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3Appendices *Nucl. Phys.* A 724 (2003) 391 summarize the quark-level LσM.