Machine Learning is Abduction Inference

Marina Sapir

June 16, 2022

Abstract

Concept of Abduction with Gradated Contradictions is introduced here as a form of Peirce’s abduction inference. The general form of abduction criterion is formalized in the proposed Logic of Gradated Contradictions and Logic of Recursive Aggregation. Common steps of an abduction procedure as minimization of such a criterion are specified as well. It is demonstrated on examples of 14 popular textbook learners (from hierarchical clustering to k-NN and SVR) that each of them performs AGC. The proposed theory explains real life learners, yet it avoids any mention of statistics, so it can be considered as a logical alternative to the statistical learning theory.

Introduction

The main feature of applied Machine Learning (ML) is that it is always about learning a nondeterministic dependence from a given finite sample. Statistical learning, on another hand, is an asymptotic theory. It is based on an implicit assumption that the training set may be increased indefinitely. Essentially, applied and theoretical ML solve different problems. As a result, theoreticians cannot answer the questions practitioners ask [22].

I propose to consider applied ML as an abduction with gradated contradictions (AGC), which is a natural way to formalize Pierce’s concept of abduction inference [20].

ML is naturally associated with the concept of abduction, because we do search for the hypothesis which “explains” the observations the best. Problems of diagnostics were already considered to be an example of abduction (see [8], for example). Proposed formalization of abduction inference, AGC, adapts the concept of abduction to deal with real functions, thus expands applicability of abduction language for description of classification, regression and clustering.

AGC formalism includes concept of “badness criterion” measuring disagreement between a hypothesis and a training set, as well as certain procedures for selection the hypothesis with minimal badness. For expressing the badness criterion, logic of Gradated Contradictions (LGC) and Logic of Recursive Aggregations (LRG) are introduced here.

LGC is designed for modeling of a nondeterministic dependence. The nondeterminism here means that apparently contradictory statements \( \varphi(x) = a, \varphi(x) = b \), where \( b \neq a \)
may be both true. Here, they are not considered mutually exclusive, but “concurrent”.

I argue that, despite of this uncertainty, learning is possible in practical applications because of implicit presumptions: a hypothesis which would predict the training set better, will predict better on new data. And predictability may be evaluated by closeness of feedback on close data points. The presumption justifies using, for example, the difference $|a - b|$ to evaluate contradiction degree between the concurrent statements above.

Thus, LGC is designed to express degrees of contradictions between observations and hypothesis instead of truthfulness of the hypotheses.

A hypothesis may have multiple degrees of contradictions with observations. LRA describes the rules of combining degrees of contradictions into a single value of a badness criterion to compare hypotheses.

The definition of AGC includes definition of steps it shall perform to minimize the abduction criterion on a given class of hypotheses.

The main conjecture of this work is that each learner performs AGC. The conjecture was corroborated on diverse set of popular classification, regression, clustering learners from $k$-NN to $K$-Means.

The proposed AGC theory of ML is, to the best of my knowledge, the first theory to explain and logically justify large variety of existing learners from a single point of view. Understanding applied ML opens a new path to solving “how to” questions practitioners ask.

1 Traditional views on ML

Here I describe a traditional understanding of ML and the issues with it. I formulate the questions any theory needs to answer. Then I present the only commonly accepted theory of automatic learning (statistical learning theory) and show that it does not really answer these questions and solves a different problem.

1.1 Prediction problem

Denote $\Omega$ the set of real life objects of interest. For example, this may be patients with skin cancer, or bank clients or engine failures. There is a hidden essential quality of the objects we would like to find out (may be, a diagnosis or prognosis). Some properties (features) of the objects $\Omega$ can be always evaluated and numerically expressed. Some of them are expected to be relevant to the hidden property. Suppose, there are $n$ such features. Denote $X \in R^n$ domain of feature vectors for objects in $\Omega$. The hidden essential quality also has numerical expression from domain $Y \in R$. The value of the hidden essence in a given object is called “feedback”. We assume there is an “underlying dependence” $\varphi : X \rightarrow Y$ between feature vectors and the feedback. Yet, we can not assume that the dependence is deterministic.

This is critical to understand practical ML, and this is commonly ignored.
An applied ML scientist (practitioner) is aware that nothing is known for sure about the reality we model. For example, the features may not define completely the feedback we are trying to model, there is uncertainty in measurements, random mis-classification and so on. Objects with the same features may have different feedback, and the same object evaluated twice may have different features or even feedback.

This is not a bad luck, but a inevitability. Indeed, if there is no exact theory explaining the phenomenon we are trying to predict, we do not know what it depends on or how to measure it. If there is an established validated theory, one does not need ML. ML deals with real raw life, not an abstraction.

More over, the timing is important consideration in prediction-based decision making. The less time we spend on accumulating the data for predictions, and the less data we use for prediction, the better. So, data shortage is not a bug, it is a feature of ML.

Thus to predict future we need to model nondeterministic underlying dependence with as little data as possible.

The information about the underlying dependence $\varphi$ is given as training set: (imprecise) observations about values of feedback in certain data points $\{\varphi(x) \approx y\}$ or as set of tuples $\{(x, y)\}$.

The goal is to find a function $f : X \to Y$, which is “close enough” to the underlying dependence $\varphi$. For example, here is how the prediction problem is understood in [18]: Given a training set $S$ and data point $x$ of a new observation $(x, y)$ predict its feedback $y$.

Obviously, the Prediction problem is ill posed: knowing a finite set of observations of non-deterministic dependence $\varphi$ with non repeating data points does not imply anything about expected values of feedback $\varphi(x)$ in the same data points, let alone in other points.

Any theory of machine learning needs to answer at least two fundamental questions:

| Fundamental questions |
|-----------------------|
| 1. Q1: Why is learning from a finite training set possible? |
| 2. Q2: What shall we do with the given training set for learning the nondeterministic dependence? |

In the next subsection, I show that statistical learning theory does not ask or answer these questions, and, therefore, does not explain existence and success of applied machine learning.

1.2 Statistical Learning Theory Approach

Statistical Learning (SL) theory is the only commonly accepted theoretical approach to ML. The most popular version of this theory is also called “VC-theory”, because VC-dimension plays important role here.
There is some confusion about the term “learner” in learning theory textbooks. For example, in [24], the term is understood as a procedure for solving a problem in a finite number of steps - when actual procedures are discussed. But in chapters talking about statistical learning theory (PAC learning, VC-theory) the terms “learner” and ”algorithm” mean “functional that takes in a finite training set $S$ and outputs a function $h: X \rightarrow Y$ ” (see [1]). The authors of the same paper write: “VC-theory did not impose any requirement on the learners actually being implementable by algorithms”. For example, Empiric Risk Minimization recommended by SL, generally can not be implemented in a finite number of steps with infinite class of functions.

Thus, strictly speaking, statistical learning theory does not talk about the main subject of this work, the learners and what they are supposed to do.

This is how the proponents of the SL theory understand the problem: “Intuitively, it seems reasonable to request that a learning algorithm, when presented more and more training examples, should eventually “converge” to an optimal solution.” ([17]) The “optimal solution” here is the hypothesis having the ERM loss criterion close to minimal for the given class of functions regardless of the distribution.

The theory does not believe that

- training set is given as is, it does not change;
- we need to apply a procedure which solves the problem in a finite number of steps;
- an optimal hypothesis may not predict anything.

Thus, SL can not answer why learning from a finite training set is possible, because it does not acknowledge the finiteness of the training set. It can not answer, what shall we do with the finite training set - they only know what to do if training set goes to infinity. [27] formulated the justification for the mismatch in the most direct way

*Why do we need an asymptotic theory ⟨⋯⟩ if the goal is to construct algorithms from a limited number of observations? The answer is as follows: To construct any theory one has to use some concepts in terms of which the theory is developed ⟨⋯⟩.*

In other words, statistical learning theory assumes indefinite increase of the training set, so that the statistical approach can prove some results. Statistics has laws of large numbers, so the problem has to be about ever increasing training sets.

Perhaps, the problem should not be formulated in the probabilistic terms to begin with. The concept of probability implies existence of known, fixed sample space, (general population) or some fixed generator of samples with an established probability distribution. Most of times, neither is true. This means we can not even use the concept of probability distribution to characterize the nondeterministic dependence.
The apparatus of probability theory does not help to understand the true problem with fixed finite data and un-quantifiable uncertainty, and can not help in applied problems.

The learning theory has to make sense of the actual practice of ML. I show below that, based on popular learners, learning is possible not because of ever increasing training set, but because the underlying dependence is expected to be (and often is) “predictable” in a certain sense.

2 ML as Abduction inference

The first fundamental question (Why is learning possible?) is not trivial: as I noticed, prediction problem, as it is traditionally formulated, is incorrect, even absurd: the givens are not related with the goal.

Thus, answering the first fundamental question requires making additional assumptions. SL theory implicitly assumes that the training set can be increased indefinitely, and the goal is to approximate an optimal hypothesis in a given class of functions, regardless of a distribution or a class. Both these assumptions are false, and are unacceptable for applied learning.

2.1 Working Presumptions

From my experience as applied ML scientist ([7], [23], [21]) ML practitioners implicitly presume that whatever is necessary to predict is here:

| Working presumptions |
|-----------------------|
| 1. Permanence: The dependence does not change in time. |
| 2. Predictability: often enough, the dependence has close feedback on similar data points; |
| 3. Data sufficiency: we already have enough data to model the dependence. |

Without presuming Permanence or Data sufficiency there is no reason to start learning. Without the Predictability, we would have no way to predict the feedback in the data points of the observations, let alone in other data points. If \( \varphi(x) \approx y \) was observed, without this presumption next time the feedback in \( x \) may be arbitrary. With this presumption, we may roughly predict values of the dependence in the points close to the data points of observations.

Let us consider relationship of the Working Presumptions with the truth, knowledge, beliefs.

Working Presumptions work the same way as presumption of innocence: we act upon them as if they are true until they are proven wrong.
We do not “know”, if Working presumptions are true. We do not “believe” Working presumptions. Similarly, we do not “believe” that the decision obtained based on these presumptions is true: the decision is considered to be presumptive too.

More the over, there is no way to assign an objective truth value to the decision or the original presumptions. They may turn out to be “unacceptable” though. If and when testing of the decision on the new data turns out to be “unsatisfactory”, the decision and the Working presumptions for the given case of the prediction problem may need to be reconsidered. Since we model a nondeterministic dependence, some disagreement between predictions and observed values are inevitable. Acceptability of the presumptive decision thus is subjective, depends on the practical goals of learning.

Testing (ground truthing) the presumptive decision on the new data thus becomes a logical operation, a necessary part of the learning. Learning needs testing to succeed, because learning itself can not assign a truth value to a model of the nondeterministic dependence based on a finite training sample.

Why would one want to infer a hypothesis without knowing if it is true? I can not but quote Karl Marx here: “The philosophers have only interpreted the world in various ways; the point, however, is to change it.” Presumptive hypotheses are used not to make a statement, but to make an optimal practical decisions when there is no way to “know the truth”, which is quite common in real life.

Working presumptions allow us to turn the problem of predicting future into the problem of finding the most pragmatic explanation of the past.

### 2.2 Abduction inference and ML

Traditionally, ML is considered to be an induction inference, generalization. Induction is a method of reasoning from a part to a whole, from particulars to generals, or from the individual to the universal.

But is ML an induction? The first objection is that induction involves one hypothesis (and one or many facts), while ML generates and tests many hypotheses. In the end, the decision may be only relatively “better suited” than other hypotheses under comparison. One can not say that the selected hypothesis is “true”, while others are “false”.

The second objection is that we are dealing with a nondeterministic dependence, and knowing observations does not mean that we “know” any part of it. Rather, we observe some versions of possible feedback in some data points. The selected hypothesis is not a “whole” underlying dependence either.

Based on the Permanence presumption, time does not change the dependence. The hypothesis which satisfies the Prediction presumption better, shall not only predict better, it shall “postdict” the observations better. The “postdiction” can be understood as an explanation. Thus, in context of the Working presumptions, to find a hypothesis which will predict the best, we need to search for a hypothesis which explains observations the best.
Search for an optimal and pragmatic explanation of some “puzzling phenomena” in context of some theory of reality is a common scientific procedure, called **abduction inference**. It was studied since 1865 by C.S. Pierce (published, for example, in [20]). Pierce wrote that abduction “will include preference of any one hypothesis over others which would equally explain the facts...”

Abduction inference consists of generating hypotheses from some class \( F \), juxtaposing them against observations \( S \), evaluating their “explanation power” and some other salient properties to find the most satisfactory hypothesis \( h \in F \).

Abduction is studied in AI as an alternative approach to some specific learning problems (diagnostics, for example). There is a logical approach to formalizing the abduction [8], [15] as well as probabilistic approach [14]. Here I am interested in a logical approach only. The problem is defined in terms of predicate calculus by the triple \( \langle F, S, T \rangle \), where \( S \) are literals of manifestations, \( F \) is set of hypotheses which can be defined as conjunctions of some other literals, and the theory is implications between literals in the hypotheses and the manifestations.

In logical AI approach to abduction it may mean that one can infer the manifestation \( g \) from the hypothesis \( h \) and it is required that every manifestation in \( S \) can be derived from the hypothesis, and the hypothesis is consistent with the theory \( T \) [26].

Yet, an abduction in original definition by Pierce does not require a hypothesis to be consistent with all the observations and the theory (which he does not even mention). He wrote that a resulting hypothesis “would account for facts or some of them”. Some reasonable examples of abduction can deal with contradictions. Consider, for example, a diagnostic application which has “cancer” and “flu” as possible diagnoses, and “fever” as one of the observations. “Cancer” contradicts “fever”. But if all other symptoms of cancer are present, and no other symptom of flue is present, the patient will be diagnosed with cancer, because this hypothesis is “more consistent” with the observations than the “flu” hypothesis. The symptoms are nondeterministic manifestations of the patient’s disorder.

Let us step up from manifestations as literals to the case where the language of manifestations has one functional symbol \( \varphi \) and binary relationship = (equality). So, the manifestations have the form \( \varphi(x) = y \). Suppose, hypotheses are functions defined on the same domain as \( \varphi \). Then for every manifestation \( g \) and a hypothesis \( h \) either \( h \vdash g \) or \( h \not\vdash g \).

Something similar is happening in ML. Every observation is an atomic statement about observed value of the underlying dependence \( \varphi \). A hypothesis is usually defined on the same domain as \( \varphi \), and hypothetical instances \( H \) are all atomic statements which can be inferred from \( h \).

The main difference here is that the dependence \( \varphi \) is nondeterministic, so if it was observed that \( \varphi(x) = y \) and a hypothesis says that \( h(x) \neq y \) there is no contradiction.

This agrees with Pierce understanding of abduction. He brings up Kepler’s reasoning about a cause of planets movements as a perfect example of abduction. Pierce uses term “approximately true” characterizing Kepler’s theory which “approximately satisfies the
observations (that is, within 8'...)”.

I prefer to speak in terms of degrees of contradictions rather than degrees of truth or “approximate truth”. For example, instances \((\varphi(x) = y), \ (\varphi(x) = 1.01y)\) may be considered “less contradictory” than instances \((\varphi(x) = y), \ (\varphi(x) = 2y)\), because the first pair of statements agree with the Predictability presumption better.

I mentioned that neither Working presumptions, nor decision are considered to be true knowledge. Since the underlying dependence is nondeterministic, no observation can be considered to be true either, because it is only one possible manifestation of reality. As a matter of fact the truth values of statements in abduction are irrelevant.

As Pierce notices: “... observed facts relate exclusively to the particular circumstances that happened to exist when they were observed. They do not relate to any future occasions upon which we may be in doubt how we ought to act. They, therefore, do not, in themselves, contain any particular knowledge”.

Ability to make inferences despite apparent logical contradictions and without assigning truth values goes against traditional understanding that “The purpose of logic is to characterize the difference between valid and invalid arguments.” (Stanford Encyclopedia of Philosophy). This is because the logic, as it is understood in the Encyclopedia, is a logic of deduction. To explain ML, we need a logic of abduction.

The goal of this work is to logically formalize the concept of machine learning as abduction inference with gradated contradictions (AGC) and to show that textbook learners indeed make inference in accordance with this formalism. This will allow me to answer the second fundamental question: “What shall we do with the given training set for learning nondeterministic dependence?”

3 Existing Approaches to Logics of Uncertainty

There are plenty of well established logical approaches to reasoning under uncertainty, as well as to study of nondeterministic dependencies and modeling inconsistent data.

Modal logics are introduced to take into account some subjectivity and uncertainty. Yet, ultimately, modal logic would not tolerate the level of inconsistency and uncertainty inevitable when we model a nondeterministic dependence. When it concerns possible world semantic, for example, the worlds need to be known beforehand. In ML we can not know anything, besides a single observation in some of the data data points.

Fuzzy logic in [19] and Subjective Logic in [13] would not help either, because they assume there is an objective omnipotent observer, who can quantity degrees of certainty or belief about given statements. The systematic review [11] describes various approaches to reasoning under uncertainty as ways to quantify and exactly measure uncertainty of statements and sets of statements. Exactness about uncertainty of empirical observations appears to be a contradiction in terms. Yet, knowing a degree of truth of the given statement is used to calculate a degree of truth of the conclusion.
Since the goal of a logic, usually, to infer a valid statement, inconsistency has to be avoided. A typical approach to resolve inconsistency of knowledge is to assign some kind of “certainty” or “preference” for each formula, and then select the most preferable (“probable”, “certain”, “reliable”) subset of consistent formulas. One of the first works of this type was [10]. The main idea there is to assign reliability to each statement and remove the least reliable ones to avoid a contradiction. Here are the main issues with this approach:

1. When we are not certain about the knowledge, we can not be certain about comparative reliability of it.

2. For a nondeterministic dependence, contradictory instances create more complete picture of reality than any non-contradictory subset.

3. In case of ML, a function can not coincide with the nondeterministic dependence we are trying to predict. In this situation, excluding inconsistency will only exclude valuable information about the manifestations of this dependence.

4. The goal of ML as an abduction inference is not to eliminate inconsistency, but to find a hypothesis which agrees with the training set in the sense of Predictability presumption the best and well enough.

Nondeterministic logics in [3] and logical operations with nondeterministic tables are introduced to derive logical functions from exact but incomplete data, which is different from the situation of inconsistent observations or inconsistency between hypothesis and the training set.

There are several approaches to describe logic of learning. For example, [9] considers asymptotic learning: precise values of a function in data points are presented indefinitely, the “nature” has in mind particular function from the given class. The goal of learning is having a convergence to a stable correct rule. Unfortunately, in applied ML, all the good assumptions about this learning idea are false: the training set is finite and too small, so convergence is irrelevant; observations are known to be tentative, the underlying dependence is nondeterministic, and whatever nature has in mind, is far from the selected class of functions or, rather, is not a function of given features at all. It is interesting that the paper states explicitly all the assumptions the Statistical Learning theory implies implicitly. In addition to stating the assumptions, the paper proposes particular algorithms to achieve stated goals, but it does not say that those are the only possible algorithms.

[2] already noticed that logic where all formulas have truth values does not describe certain types of logical reasoning, particularly legal reasoning about existing norms. The logics with modalities like “it ought to be”, “you ought to do” are introduced, but to the best of my knowledge, epistemic modalities which can not be associated with truth values were not explored.
So, neither of these approaches is designed to model reasoning about nondeterministic
dependence where contradictions are inevitable and need to be measured and controlled.

4 Logic of Gradated Contradictions (LGC)

An abduction criterion has to evaluate degree to which a hypothesis corroborates the
Predictability presumption on the training set. For a hypothesis \( h \), the corroboration
will be evaluated by “degrees of contradiction” between hypothetical instances \( H \) of the
hypothesis and the observations \( S \).

The Logic of the Gradated Contradictions has to express conditions when pairs of
first order formulas have potential of being “contradictory” and evaluate a “degree of
contradiction”.

To avoid an appearance of a paradox, the instances are (i) presented in a predicate
form and (ii) no logical operations on them are defined and (iii) the truth values are not
assigned.

4.1 Language of LGC

So far, I considered an underlying dependence with a single variable. In a general case of
AGC (abduction inference with gradated contradictions), the restriction is not necessary.
For example, using two independent variables may be convenient for formalizing ranking
problem.

The first order signature has \( 6 + n \) sorts:

| Sort   | Content                      | Variables       | Constants |
|--------|------------------------------|-----------------|-----------|
| \( \mathbb{N} \) | Natural numbers            | \( i, i, k, l, m, i_1, \ldots \) | \( n \) |
| \( \mathbb{X}_1, \ldots, \mathbb{X}_n \) | Domains of the independent variables | \( x, x_1, x_1^j, \ldots \) |        |
| \( \mathbb{Y} \) | Domain of the feedback     | \( y, y_1, \ldots \) | \( \times, \times_1, \ldots \) |
| \( \mathbb{H} \) | Symbols indicating hypothetical instances | \( s, s_1, \ldots \) | \( \approx, \approx_1, \ldots \) |
| \( \mathbb{O} \) | Symbols indicating observations | \( s, s_1, \ldots \) | \( \approx, \approx_1, \ldots \) |
| \( \Psi \) | First order formulas       | \( \alpha, \beta, \alpha_1, \beta_1, \ldots \) |        |
| \( \mathbb{R} \) | Real numbers               | \( r, r_1, r_2, \ldots \) |          |

We assume the domains of the sorts \( \mathbb{X}_1, \ldots, \mathbb{X}_n \) and \( \mathbb{Y} \) are some metric spaces. For
example, the set \( \mathbb{Y} \) with two values \( \{0, 1\} \) may be considered a metric space with the norm

\[
\|y_1 - y_2\| = \begin{cases} 
0, & \text{if } y_1 = y_2 \\
1, & \text{otherwise.}
\end{cases}
\]
There is one first order predicate symbol $\psi$ of arity $n+2$ with variables from $X_1, \ldots, X_n$, $\Upsilon$, and $\mathbb{H} \cup \mathbb{O}$. The expression $\psi(x, y, s)$ with $s \in \mathbb{O}$ is interpreted as “it was observed that $\varphi(x) = y$,” where $x$ denotes the list of independent variables $x_1, \ldots, x_n$. And the same expression is interpreted as: “hypothetically, $\varphi(x) = y$,” if $s \in \mathbb{H}$. The indices of the symbols in $\mathbb{H}, \mathbb{O}$ indicate context. For example, it may be used in survival analysis, where there are at least two types of observations: censored and uncensored.

There are no logical operations on the first order formulas, they are not assigned any truth values.

Interpretations of this logic may have optional other sorts, first order functional symbols and operations, which will be specified in formalization of each problem as needed.

The next table lists the second order functional symbols.

| Symbol | Arity | Sorts | Semantic |
|--------|-------|-------|----------|
| $x$    | 2     | $\Psi \times \mathbb{N} \rightarrow X_i$ | $i$-th independent variable |
| $y$    | 1     | $\Psi \rightarrow \Upsilon$ | feedback |
| $s$    | 1     | $\Psi \rightarrow \mathbb{H} \cup \mathbb{O}$ | type symbol |
| $\rho_x$ | 3   | $\Psi \times \Psi \times \mathbb{N} \rightarrow \mathbb{R}$ | distance between $i$-th variables of two formulas |
| $\rho_y$ | 2   | $\Psi \times \Psi \rightarrow \mathbb{R}$ | distance between feedbacks of two formulas |

The second order functions are defined by the rules:

$\mathcal{R}_1 : \forall \alpha \; \psi(x(\alpha, 1), \ldots, x(\alpha, n), y, s(\alpha)) = \alpha.$

$\mathcal{R}_2 : \forall \alpha_1, \forall \alpha_2 \; \rho_x(\alpha_1, \alpha_2, i) = \|x(\alpha_1, i) - x(\alpha_2, i)\|.$

$\mathcal{R}_3 : \forall \alpha_1, \forall \alpha_2 \; \rho_y(\alpha_1, \alpha_2) = \|y(\alpha_1) - y(\alpha_2)\|.$

The second order formulas will have

- relations $\leq, <, >, \geq, =$, on real numbers,
- regular logical connectives ($\lor, \land$)
- real valued constants.
4.2 Gradated contradiction rules

The second order formulas define pairwise relationship on the first order formulas. The goal here is to define degrees of contradictions between related formulas. There may be several aspects of gradated contradictions.

1. Any second order LGC statement $\pi(\alpha, \beta)$ with two free variables $\alpha, \beta$, both over domain of the first order formulas $\Psi$ may be considered a concurrence relation; a pair of the first order formulas $\alpha, \beta$ such that

$$\pi(\alpha, \beta) \lor \pi(\beta, \alpha)$$

are said to be concurrent.

2. For a pair of first order formulas $\alpha_1, \alpha_2$ their contradiction degree $\delta(\alpha_1, \alpha_2)$ is defined as

$$\delta(\alpha_1, \alpha_2) = t(\rho_x(\alpha_1, \alpha_2), \rho_y(\alpha_1, \alpha_2)),$$

where a contradiction function $t(r_1, r_2) : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is antitone by $r_1$ and isotone by $r_2$.

The contradiction degree increases when the data points of the formulas get closer and when the distance between values of feedback increases.

Let us notice that concurrence relation does not require that one of the formulas was an observation, and another one a hypothetical instance. In some cases, it may be useful to evaluate agreement of the hypothesis itself with Predictability presumption.

5 Aggregation of contradiction degrees

Given a hypothesis $h$, training set $S$ and contradiction rule $\tau$, one can identify pairs of concurring formulas and calculate degree of contradiction for each of them.

To compare hypotheses though, we need a single number characterizing the set of contradiction degrees. The operation of mapping the set of numbers into a single number, which somehow represents the set, is usually called an aggregation.

5.1 Proper aggregation

An aggregation operation maps a multiset of real numbers into a real number.

The operation of aggregation $TOT(G)$ defined on all finite multisets of real numbers is called aggregation.

To be useful, the result of aggregation shall represent the numbers being aggregated. The aggregation may be called proper if it satisfies three axioms:
1. **Monotony**: For any two multisets $G_1, G_2$, if a mapping $q$ from $G_1$ on $G_2$ is an isomorphism then

\[
\left( \forall x \ q(x) \geq x \right) \Rightarrow TOT(G_2) \geq TOT(G_1) \quad \& \quad \left( \forall x \ q(x) > x \right) \Rightarrow TOT(G_2) > TOT(G_1)
\]

2. **Idempotence**: $TOT(G \cup \{TOT(G)\}) = TOT(G)$.

3. **Tautology**: If $G = \{x\}$ then $TOT(G) = x$.

Some natural properties of proper aggregation follow from the axioms.

**Statement 1.** Any proper aggregation $TOT(G)$ has the next properties:

1. If the multiset $G$ consists of $n$ identical elements $x$, then $TOT(G) = x$.
2. $\min(G) \leq TOT(G) \leq \max(G)$.

**Proof.**

1. Let us prove it by induction by $n = \|G\|$. It $n = 1$, it follows from the axiom Tautology. Suppose, the statement is proven for $n = k$. Then for $n = k + 1$ it follows from the axiom Idempotence.

2. Let us prove by contradiction. Suppose

\[
\exists G \forall x \ (x \in G) \Rightarrow (TOT(G) > x).
\]

Denote $n = \|G\|$. By the previous property, if the set $G_1$ consists of $n$ elements $TOT(G)$ then $TOT(G_1) = TOT(G)$. It contradicts the axiom of Monotony since every element of $G_1$ is larger than all elements of $G$. The same way we can prove that $TOT(G)$ can not be lower than all elements of $G$.

Median $\mu(G)$ of a multiset $G \subset \mathbb{R}$ shall be an option of proper aggregation. And it is.

**Statement 2.** Operation $\mu(G)$ is a proper aggregation.

**Proof.** Let us prove monotony. Denote $q$ isomorphism $G_1 \rightarrow G_2 : q(x) \geq x$ and

\[
\rho_i = \mu(G_i), \ i = 1, 2.
\]

Because the sets are isomorphic, they have the same power $\|G_1\| = \|G_2\| = n$.

For some integer $k : n = 2k$ or $n = 2k + 1$. The number of elements in $G_2$ which are larger than $\rho_2$ is the same as the number of elements larger than $\rho_1$ in $G_1$. In both cases and for both sets the number is equal $k$. 

13
Denote 
\[ G^-_i = \{ x \mid (x \in G_i) \& (x < \rho_1) \}, \text{ for } i = 1, 2. \]
\[ G^+_i = \{ x \mid (x \in G_i) \& (x > \rho_1) \}, \text{ for } i = 1, 2. \]

By definition of \( q \), for any \( x \in G^+_1 \), \( q(x) \in G^+_2 \). So, \( \|G^+_2\| \geq \|G^+_1\| \).

Suppose, \( \|G^-_2\| > \|G^+_1\| \). This means, \( \|G^-_2\| > k. \) Therefore \( \rho_2 \in G^+_2 \) and \( \rho_2 > \rho_1 \). It proves the theorem for the case \( q(x) \geq x \).

Suppose, \( \|G^-_2\| = \|G^+_1\| = k \). In this case, \( \|G^-_2\| = \|G^+_1\| = k \) and for every element \( x \in G^-_1 \) \( q(x) \in G^-_2 \).

First, suppose \( n = 2k + 1 \). Then \( \rho_1 \in G_1 \). The only element of \( G_2 \) which does not belong to \( G^-_2 \), \( G^+_2 \) is \( q(\rho_1) \). Therefore \( q(\rho_1) = \rho_2 \), and \( \rho_2 \geq \rho_1 \).

Now, suppose \( n = 2k \). In this case, for \( i = 1, 2 \)
\[ \rho_i = \mu(G_i) = \frac{\min(G^+_i) + \max(G^-_i)}{2}. \]

Since \( q(\max(G^-_1)) \geq \max(G^-_1) \) and \( q(\max(G^+_1)) \in G^+_2 \) then \( \max(G^+_2) \geq \max(G^-_1) \).

Let us notice that \( x \in G^+_1 \) if and only if \( q(x) \in G^+_2 \). For any \( x \in G^+_1 \) : \( q(x) \geq x \geq \min(G^+_1) \) therefore \( \min(G^+_2) \geq \min(G^+_1) \). It follows that \( \rho_1 \leq \rho_2 \) in this too.

This proves the monotony for the case, when \( q(x) \geq x \). The case when for every \( x : q(x) > x \) is proven similarly.

Let us prove the idempotence. Denote \( \rho = \mu(G) \), \( b < \rho < c \) are two closest elements in \( G \) to \( \rho \). Suppose, \( \|G\| = 2k \) and \( \rho = (b + c)/2 \). Then \( \mu(G \cup \{\rho\}) = \rho \). Suppose \( \|G\| = 2k + 1 \). Then \( G \cup \{\rho\} \) has has two identical elements equal \( \rho \) in the middle. And \( \mu(G \cup \{\rho\}) = \rho \).

Tautology is trivial, because median of \( \{x\} \) is \( x \).

The statement could be proven not only for median, but for any percentile. So any percentile can be used as a proper aggregation.

6 Recursive aggregation

This section is talking about the most common type of proper aggregation procedures used in ML. The advantage of these procedures over percentiles, for example, is that they require going through all the elements of the multiset \( G \) only once. This type of aggregation will be called recursive.

6.1 Logic of recursive aggregation

To describe recursive aggregation, I will use extension of the first order logic with added counting quantifiers (see, for example [16]) \( \exists^c x \), where \( x \) is a variable, and \( c \) may be a natural number or variable with values in \( \mathbb{N} \). The quantifier means: there exists exactly \( c \) of \( x \).
There are three sorts.

There is total order $\prec$ on the domain $G$. The functions in the language of aggregation are described in the next table.
6.2 Theory of recursive aggregation

The order $\prec$ on $G$ is defined as a strict total order (with axioms of irreflexivity, transitivity, anti- symmetry and total order). The relationships $\{<,>,\leq,\geq,=\}$ are defined in usual way on real numbers. The functions $\text{get}(i), \text{count}()$ are defined uniquely as $i$-th element in the order $\prec$ and the cardinality of $G$ when the domain $G$ of the sort $G$ and the order $\prec$ on it are known:

$$\forall x \forall i \ (\text{get}(i) = x) \iff (\exists y \ y < x)$$

$$\forall n \ \text{count}() = n \iff \text{count}().$$

The table shows axioms characterizing properties of other functions in the language:

| Table 5: Axioms of recursive aggregation |
|------------------------------------------|
| Axiom | Axiom | Commentary |
| $B_1$ | $\forall x \forall x_1 \ (x_1 > x) \Rightarrow (\text{scale}(x_1) \geq \text{scale}(x))$ | monotony |
| $B_2$ | $\forall x \forall y \ (\text{plus}(x,y) = \text{plus}(y,x))$ | symmetry |
| $B_3$ | $\forall x \forall y \forall x_1 \ (x_1 > x) \Rightarrow (\text{plus}(x_1,y) \geq \text{plus}(x,y))$ | monotony |
| $B_4$ | $\forall x \forall y \forall z \ (\text{plus}(x,\text{plus}(y,z)) = \text{plus}(\text{plus}(x,y),z))$ | associativity |
| $B_5$ | $\forall i \ (\text{agg}(1) = \text{scale}(\text{get}(1))$ & $(\text{agg}(i + 1) = \text{plus}(\text{agg}(i),\text{scale}(\text{get}(i + 1))))$ | recursive aggregation |
| $B_6$ | $\forall x \forall n \forall x_1 \forall n_1 \ (x_1 > x) \Rightarrow (\text{norm}(x_1,n) \geq \text{norm}(x,n)) \& (n_1 > n) \Rightarrow (\text{norm}(x,n_1) \leq \text{norm}(x,n))$ | $\text{norm}$ monotony |
| $B_7$ | $\forall i \ (\text{get}(i + 1) = \text{norm}(\text{agg}(i),i)) \Rightarrow \text{idempotence}$ |
| $B_8$ | $\forall x \ (\text{norm}(\text{scale}(x,1)) = x$ | tautology |
| $B_9$ | $\forall x_1 \forall x_2 \forall i \ (x_1 \neq x_2) \Rightarrow (\text{scale}(x_1) \neq \text{scale}(x_2) \& \text{norm}(x_1,i) \neq \text{norm}(x_2,i))$ | strict monotony |
| $B_{10}$ | $\forall x_1 \forall x_2 \forall y_1 \forall y_2 \ (x_1 < x_2) \& (y_1 < y_2) \Rightarrow \text{strict}$ |

Typical examples of the function $\text{plus}$ are

- $\text{plus}(x,y) = x + y$
- $\text{plus}(x,y) = x \cdot y$
- $\text{plus}(x,y) = \max(x,y)$.

All these functions are used by popular learners, as I will demonstrate.

Each model $\mathcal{M}$ uniquely defines an operation

$$\text{TOT}(\mathcal{M}) = \text{norm}(\text{agg}(\text{count}), \text{count})$$
Given an interpretation of functions \textit{scale}, \textit{plus agg}, \textit{norm}, \textit{TOT}(\mathcal{M}) is defined by its finite domain \( G \) and the strict total order \( \prec \) on it.

The next theorem shows that \( \textit{TOT}(\mathcal{M}) \) does not depend on the order \( \prec \).

\textbf{Theorem 1.} Suppose \( \mathcal{M}_0, \mathcal{M}_1 \) models of recursive aggregation language are different by the orders \( \prec \) only: domains of the sort \( G \) consist of the same elements and interpretations of all the functions of the language are identical. Then \( \textit{TOT}(\mathcal{M}_0) = \textit{TOT}(\mathcal{M}_1) \).

\textit{Proof.} Suppose, the models are different by the orders on domains \( G_0, G_1 \) of sort \( G \). For the finite domain \( G \), the order \( \prec_1 \) may be considered as a permutation of order \( \prec \). Each permutation can be obtained by finite number of simple transpositions (transpositions of neighboring elements). Suppose, the order \( \prec_1 \) is obtained from order \( \prec_0 \) by \( K \) simple transpositions. Let us prove the theorem with induction by \( K \). First, suppose \( K = 1 \). Denote \( x_1, \ldots, x_i, x_{i+1}, \ldots, x_n \) elements of \( G \) ordered by \( \prec_0 \). Suppose, the order \( \prec_1 \) transposes elements \( x_i, x_{i+1} \). Denote \( \text{agg}(l) \), \( \text{agg}_1(l) \) values of the recursive aggregation function obtained on the step \( l \) with the orders \( \prec_0, \prec_1 \) respectively. Since all the elements prior to \( i \) are identical in these orders, \( \text{agg}(i - 1) = \text{agg}_1(i - 1) \). By definition

\[
\begin{align*}
\text{agg}(i) &= \text{plus}(\text{agg}(i - 1), x_i) \\
\text{agg}(i + 1) &= \text{plus}(\text{agg}(i), x_{i+1}) \\
&= \text{plus}(\text{plus}(\text{agg}(i - 1), x_i), x_{i+1}) \\
\text{agg}_1(i) &= \text{plus}(\text{agg}(i - 1), x_{i+1}) \\
\text{agg}_1(i + 1) &= \text{plus}(\text{plus}(\text{agg}(i - 1), x_{i+1}), x_i).
\end{align*}
\]

Using symmetry and associativity of the function \textit{plus} \((B_2, B_4)\) we get

\[
\begin{align*}
\text{agg}_1(i + 1) &= \text{plus}(\text{plus}(\text{agg}(i - 1), x_{i+1}), x_i) \\
&= \text{plus}(x_i, \text{plus}(\text{agg}(i - 1), x_{i+1})) \\
&= \text{plus}(\text{plus}(x_i, \text{agg}(i - 1)), x_{i+1}) \\
&= \text{plus}(\text{plus}(\text{agg}(i - 1), x_i), x_{i+1}) \\
&= \text{plus}(\text{agg}(i), x_{i+1}) \\
&= \text{agg}(i + 1).
\end{align*}
\]

All the elements in the orders \( \prec_0, \prec_1 \) after \((i+1)-\text{th}\) are identical. Therefore, \( \text{agg}_1(n) = \text{agg}(n) \). So, the constants \( \text{out}(\mathcal{M}) \) and \( \text{out}(\mathcal{M}_1) \) will be identical in this case.

Suppose, we proved the theorem for \( K = k \). Let us prove it for \( k + 1 \). Suppose, the first \( k \) simple transpositions involve elements with the indices below \( i - 1 \), and the last simple transposition involves elements \( x_i, x_{i+1} \). Then, the same considerations apply again. \( \square \)
Every interpretation of the functions \{scale, plus, agg, norm\} will have potentially infinite number of models different by the domains of the sort \(G\).

The theorem means that, given interpretation of the functions of the recursive aggregation language, the operation \(TOT(M)\) is an aggregation operation on the domain \(G\) of sort \(G: TOT(M) = TOT(G)\).

An aggregation defined by an interpretation of the recursive aggregation language may be called recursive aggregation.

The next theorem shows recursive aggregation is a proper aggregation.

**Theorem 2.** For any interpretation of the recursive aggregation language, the operation \(TOT(G)\) is a proper aggregation.

**Proof.** Let us prove tautology. If \(G = \{x\}\), then

\[
TOT(G) = \text{norm}(agg(1), 1) = \text{norm}(\text{scale}(\text{get}(1)), 1) = \text{norm}(\text{scale}(x), 1) = x,
\]

using the axiom \(B_8\).

Let us prove monotony. Suppose, two models with domains \(G_1, G_2\) of the sort \(G\) belong to the same interpretation, and \(q: G_1 \rightarrow G_2\) is isomorphism such that \(q(x) \geq x\). Suppose they are ordered in such a way that they maps \(i\)-th element of \(G_1\) into \(i\)-th element of \(G_2\).

Let us prove it by induction by \(n = \|G_1\| = \|G_2\|\). For \(n = 1\) it is true based on the axiom \(B_8\). Suppose, the statement is proven for \(n = k\). Denote \(agg_1(i), agg_2(i)\) results of aggregation on the domains \(G_1, G_2\) on the step \(i\), and denote corresponding elements of \(G_1, G_2: x_j, y_j = q(x_j), \ j = 1, \ldots, n\). By the assumption of induction, \(\text{norm}(\text{scale}(\text{agg}_1(k)), k) \geq \text{norm}(\text{scale}(\text{agg}_1(k)), k)\).

Let us prove the statement for \(n = k + 1\).

\[
\text{norm}(\text{scale}(\text{agg}_1(k + 1)), k + 1) = \text{norm}(\text{scale}(\text{plus}(agg_2(k), \text{scale}(x_{k+1}), k + 1)).
\]

\[
\text{norm}(\text{scale}(agg_2(k + 1)), k + 1) = \text{norm}(\text{scale}(\text{plus}(agg_1(k), \text{scale}(y_{k+1}), k + 1)).
\]

We need to show that

\[
\text{norm}(\text{scale}(\text{plus}(agg_2(k), \text{scale}(y_{k+1}), k + 1) \geq \text{norm}(\text{scale}(\text{plus}(agg_1(k), \text{scale}(x_{k+1}), k + 1)).
\]

The function \(\text{norm}\) is isotope by the first argument (\(B_6\)). The function \(\text{scale}\) is isotope (\(B_1\)). The function \(\text{plus}\) is isotope by the first argument (\(B_3\)) and symmetric (\(B_2\)), therefore it is isotope by both arguments. It follows that inequality holds. The strict monotony follows from monotony and the axioms \(B_9, BB_10\).

Denote \(G_0, G_1 = G_0 \cup r\). To prove idempotence, assume \(r = TOT(G_0)\). For \(G_1, get(n + 1) = r = \text{norm}(agg(n), n)\). Using axiom \(B_7\) we get

\[
TOT(G_1) = \text{norm}(agg(i + 1), i + 1) = r = TOT(G_0).
\]
It is easy to show that each of the next combinations of functions satisfies all the axioms \( B_1 - B_{10} \).

Table 6: Some interpretations of recursive aggregation

| \( \text{plus}(x, y) \) | \( \text{scale}(x) \) | \( \text{norm}(x, i) \) | \( \text{TOT}(G) = \) |
|------------------------|-----------------|----------------|----------------|
| 1 \( x + y \) | \( x \) | \( x/i \) | \( \frac{1}{n} \sum_i x_i \) |
| 2 \( x + y \) | \( x^2 \) | \( \sqrt{x/i} \) | \( \sqrt{\frac{1}{n} \sum_i x_i^2} \) |
| 3 \( \text{max}(x, y) \) | \( x \) | \( x \) | \( \text{max}(G) \) |
| 4 \( \text{min}(x, y) \) | \( x \) | \( x \) | \( \text{min}(G) \) |
| 5 \( x \cdot y \) | \( x \) | \( x^{1/i} \) | \( (\prod_i x_i)^{1/n} \) |

\( G \) is domain of the sort \( G \), \( n = \| G \| \).

**Statement 3.** Each combination of functions in the table 6 is an interpretations of the recursive aggregation language.

**Proof.** Monotonicity of all the functions is obvious. Symmetry of the function \( \text{plus} \) in all the combinations is obvious. We need to show that axioms of tautology and idempotence are true for all combinations. Let us show it for combinations from the lines (2) and (4), where it may be not obvious. If \( G = \{ x \} \), \( \text{TOT}(G) = \text{norm}(\text{agg}(1), 1) = \text{norm}(\text{scale}(x), 1) \). For the combination (2) \( \text{TOT}(G) = \sqrt{x^2/1} = x \). For the combination (4) \( \text{TOT}(G) = (\prod_i x_i)^1 = x \). Let us prove idempotence. For combinations (1) and (3) it is obvious. Let us prove it for combinations (2) and (4) again. Suppose, \( z = x_{n+1} = \text{TOT}(% \{x_1, \ldots, x_n\}) \).
For combination (2)

\[ z = TOT(\{x_1, \ldots, x_n\}) \]

\[ TOT(\{x_1, \ldots, x_n, z\}) = \sqrt{\frac{\sum_{i=1}^{n} x_i^2 + (\sum_{i=1}^{n} x_i^2)/n}{n+1}} \]

\[ = \sqrt{\frac{\sum_{i=1}^{n} x_i^2}{n}} \]

For combination (4)

\[ z = TOT(\{x_1, \ldots, x_n\}) = \left(\prod_i x_i\right)^{1/n} \]

\[ TOT(\{x_1, \ldots, x_n, z\}) = \left(\prod_i x_i \cdot z\right)^{1/n+1} \]

\[ = \left(\prod_i x_i \cdot \left(\prod_i x_i\right)^{1/n}\right)^{1/n+1} \]

\[ = \left(\prod_i x_i^{(n+1)/n}\right)^{1/n+1} \]

\[ = \prod_i x_i^{1/n} = \left(\prod_i x_i\right)^{1/n} = z. \]

\[ \square \]

7 Abduction with Gradated Contradictions

Now I am ready to formally define abduction with gradated contradiction (AGC).

Each instance of AGC may be considered as an abduction learner since it takes training set and outputs an optimal hypothesis. Each abduction learner will be characterized by an abduction criterion, abduction procedure and its class of hypotheses. First, I will define components of an abduction criterion.
7.1 Badness rule

The triple of concurrence relation, contradiction function and a proper aggregation operation will be called badness rule.

Let us consider an example of a badness rule called Point-Wise rule, $T_{pw}$. It consists of

1. Concurrence relation:

$$\pi_{pw}(\alpha_1, \alpha_2) = \left( (s(\alpha_1) = \infty) \& (s(\alpha_2) = \approx) \& (x(\alpha_1) = x(\alpha_2)) \right)$$

2. Contradiction function $t(r_1, r_2) = r_2$

3. Aggregation operation: averaging.

The function $t$ is obviously isotone by the the $r_2$. The function does not depend on $r_1$, so for any $r_1, r_1^1, r_2^2 \ t(r_1, r_2) = t(r_1, r_2^2)$. Therefore, the condition of antitony by the first variable is not violated.

By definition, hypothetical instance $\alpha_1$ and an observation $\alpha_2$ are concurrent when $x(\alpha_1) = x(\alpha_2)$. For any pair of concurrent first order formulas $\alpha_1, \alpha_2$ their contradiction degree is

$$\delta(\alpha_1, \alpha_2) = t(\rho_x(\alpha_1, \alpha_2), \rho_y(\alpha_1, \alpha_2)) = \rho_y(\alpha_1, \alpha_2).$$

If there are $m$ concurrent pairs of first order formulas $\alpha_1, \alpha_2$ satisfying the condition $\pi_{pw}$ there will be $m$ contradiction degrees for a given hypothesis. Averaging is applied on all these contradiction degrees to obtain a single value, characterizing disagreement of the hypothesis with Predictability presumption on a given training set.

7.2 Regularization

An agreement between the Predictability presumption and a hypothesis $h$ may be evaluated regardless of a training set, by the scale of differences between feedback values $h(x)$ in close data points. One can see Predictability presumption applied to a hypothesis as a version of stability, which is a very important for participial applications, considering that all the data measurements are not precise.

One way to evaluate this quality of a hypothesis is by using its derivative. In ML, it is called “regularization”.

An abduction criterion may contain a formula which takes parameters of a hypothesis and outputs the value of the “regularization” component.
7.3 Abduction criterion

An abduction criterion is defined by a tuple which may contain one or more badness rules, a formula for a regularization, as well as an operator, which combines values of all the components together.

### AGC criterion scheme

Let \( h \) be a hypothesis with the set of hypothetical instances \( H \), \( S \) is a training set. Abduction criterion scheme is a tuple, consisting of:

- a sequence of badness rules \( T = \{\tau_1, \ldots, \tau_k\} \); an application of a badness rule \( \tau_i \) of \( H \cup S \) produces badness value \( B_i, i = 1 \ldots, k \);
- a formula for a regularization functional \( \Phi(h) \)
- an operator \( \Psi \) for combining set of values \( \{B_1, \ldots, B_k, \Phi(h)\} \) monotone by each element of the set.

An AGC criterion scheme defines a real-valued **abduction criterion** on the sets of instances \( H \cup S \).

7.4 Abduction learning procedure

Now it is time to define in some general terms a procedure of an abduction learner. There will be allowed only two types of procedures: “basic training” and “wrapper strategy”.

### Basic training

For an abduction learner with abduction criterion \( L(h, S) \), class of functions \( F \) and parameter \( q \) a Basic training consists of the next steps

- **Focusing** (optional): transformation \( U : S \rightarrow S_q \)
- **Fitting**: generation hypothesis \( h \in F' \subseteq F \) and evaluation of \( L(h, S_q) \)
- **Optimal selection**: output of an optimal hypothesis decision \( h_q = \arg \min_{F'} L(h, S_q) \)

Focusing may be a nonlinear transformation of the training set. Yet, typically, it is used to select observations or features or emphasize some of them with weights.

The selection of hypotheses may not go over whole class \( F \), but its finite subclass \( F' \subseteq F \).

An optional “wrapper” strategy repeats the Basic training with different parameters to come up with a single decision.
Wrapper strategy

- **Wrapper loop**: Generating parameters $q \in Q$
  - **Basic training** with parameters $q$, outputs a hypothesis $h_q$
  - **Calculating weight** $W_q$
  - **Stopping check**: evaluating conditions to exit the loop

- **Output**: $d = \Delta \left( \{h_q, W_q : q \in Q\} \right)$

Stopping check here checks a specified condition, and if it is true, the procedure exists Wrapper loop. Otherwise, the loop continues with generating new parameters.

Summarizing, we get a definition of an abduction learner

### Abduction learner

Abduction learner minimizes an abduction criterion using Basic training with or without Wrapper strategy.

The concept of an abduction learner formalizes the path of learning which can be envisioned based on the Working presumptions.

### 7.5 The main conjecture

Each popular learner (the procedures of $k$-NN, Naive Bayes, SVM, hierarchical clustering, for example) is formulated in unique terms, apparently solves its own type of problem. My conjecture is that, each popular learner for classification, regression and clustering has a criterion which is an abduction criterion, and performs an abduction learning procedure. In short,

**Main Conjecture: ML is AGC**

Every learner is an abduction learner.

The conjecture states that optimizing an agreement of a hypothesis with the Prediction presumption is the only way to learn. In other words, the Predictability presumption explains learning.

The conjecture answers the second fundamental question: **What shall we do with the given training set for learning an underlying nondeterministic dependence?** “We need to perform AGC” is the answer.
8 Popular learners support the Main Conjecture

To show that a learner is an abduction learner we need to show that it has a loss criterion (sometimes it is not obvious), the loss criterion is an abduction criterion, and its procedure may be described as Basic training with or without Wrapper strategy.

8.1 ERM-type learners

Denote $\beta_i = \psi(x_i, y_i, \approx)$ $i$-th observation in the training set $S$.

For this learner, class of functions is not specified and the procedure is not described. All we have is a loss criterion

$$L(h, S) = \frac{1}{\|S\|} \sum_i |h(x(\beta_i)) - y(\beta_i)|.$$

We need to show that the criterion is an abduction criterion.

Let us consider an abduction criterion scheme which contains only Point-wise badness rule, $T_{pw}$.

For each formula $\beta \in S$, there is exactly one concurrent hypothetical instance with the same data point $\psi(x(\beta), h(x(\beta)), \approx)$.

So, for each concurrent pair of formulas $\alpha_1, \alpha_2$ the contradiction degree is

$$\delta(\alpha_1, \alpha_2) = \rho_y(\alpha_1, \alpha_2).$$

Then, indeed,

$$L(h, S) = \frac{1}{\|S\|} \sum_{\beta \in S} \rho_y(\beta, \psi(x(\beta), h(x(\beta)), \approx)) = \frac{1}{\|S\|} \sum_{\beta \in S} \|h(x(\beta)) - y(\beta)\|.$$

Thus, the loss criterion empiric risk is a simplest abduction criterion with the scheme containing only one badness rule, $T_{pw}$.

The ERM optimization is rarely performed on the whole domain, because it checks only one aspect of agreement between a hypothesis and the Predictability presumption: closeness to observations’ feedback in the data points of observations. The criterion does not take into account closeness of the hypothesis’ values on close, but not identical data points. This shall commonly lead to overfitting on a small set of observations of the the nondeterministic dependence.

8.2 Linkage-based clustering

The learner is also popularly known as hierarchical clustering.

Clustering can be seen as modeling a nondeterministic dependence, where the cluster number is an independent variable, and data vector is a feedback.

As in classification and regression, the Working presumptions are applicable:
1. Permanence: The time is not a factor here.

2. Predictability: The dependence, mostly, has close feedback (data points) in a given cluster; and

3. Data sufficiency: We assume there is enough data to distinguish clusters.

An apparent issue with considering clustering as an abduction inference is that there is no agreed upon way to test a decision on new data. However there are at least some approach to do it, see [25].

In [24], a general concept of linkage-based clustering is introduced this way:

These algorithms proceed in a sequence of rounds. They start from trivial clustering that has each data point in a single-point cluster. Then, repeatedly, these algorithms merge “closest” clusters of the previous clustering. Input to a clustering algorithm is between-point distance, $d$. There are many ways of extending $d$ to a measure of distance between domain subsets (or clusters).

The most common ways are:

1. Single Linkage clustering, in which the between-clusters distance is defined by the minimum distance between members of the two clusters.

2. Average Linkage clustering, in which the distance between two clusters is defined to be average distance between a point in one of the clusters and a point in another.

3. Max Linkage clustering, in which the distance between two clusters is defined as maximum distance between their elements.

The last option clearly contradicts declared goal “merge ‘closest’ clusters”. But I will consider it too.

For each round, the training set is a sequence of first order formulas of LGC:

$$S = \{\psi(c_i, y_i, \approx), i = 1 : m\},$$

where $c_i \in \mathbb{N}$ is a cluster number of if $i$-th observation, and $y_i$ is the observed data point of the same observation.

Denote

$$C_i = \{y : \exists \alpha (\alpha \in S) \& (y = y(\alpha)) \& (i = x(\alpha))\},$$

the set of data points of the cluster $i$. Suppose, there are $k$ clusters.

The purpose of a round is to identify two “closest” clusters. The idea is to check the “distance” $\rho(C_i, C_j)$ for each pair of the clusters $C_i, C_j$, $j > i$ to find the two closest clusters.

The notation $h^{ij}$, $j > i$ will indicate a hypothesis that clusters $C_i, C_j$ “belong together”, are the best candidates for merging, and $\mathcal{H}_k = \{h^{ij} \mid i < j \leq k\}$ denotes the class of all the hypotheses.
All the hypothetical instances of the hypothesis \( h^{ij} \) make the set

\[
H^{ij} = \{ \psi(c_i, y, \sim) | y \in C_j \}.
\]

The loss criterion \( L(h^{ij}, S) = \rho(C_i, C_j) \) is the “distance” between the clusters \( C_i, C_j \).

To show that the learner on each round performs AGC, we need to show that, the \( L(h^{ij}, S) \) is an abduction criterion for every type of linkage clustering, and the described procedure is Basic training.

The AGC criterion scheme has a single badness rule with concurrence relation

\[
\pi(\alpha_1, \alpha_2) = (s(\alpha_1) = \sim) \& (s(\alpha_2) = \equiv) \& (x(\alpha_1) = x(\alpha_2)).
\]

The contradiction degree is defined by the rule

\[
\delta(\alpha_1, \alpha_2) = \rho_y(\alpha_1, \alpha_2).
\]

So, the concurrence relation and the contradiction degree are defined as in \( T_{p,w} \). The aggregation operation is identified by the type of clustering: minimum, average and maximum. Every aggregation operation suggested by the definition is listed in the table 6 among the most useful recursive aggregation operations. Therefore, the criterion \( L(h^{ij}, S) \) is an abduction criterion.

Now, the learning procedure can be described by the rule:

| Round of hierarchical Clustering |
|----------------------------------|
| • **Fitting**: Generation of hypotheses \( h \in H_k \) and evaluation of the loss criterion \( L(h, S) \) on each of them |
| • **Optimal selection**: Select a hypothesis \( h' \in H_k \) with the minimal value of the loss criterion \( L(h, S) \). |

This is Basic training without focusing.

This proves that each round of linkage-based clustering works as an abduction learner. The learner would agree with the main conjecture not only for the aggregation operations mentioned in the book (average, minimum, maximum), but also for any other proper aggregation operation.

### 8.3 \( k \)-NN

This classification method is intended for the observations with binary feedback in \( Y = \{0, 1\} \). Given a new data point \( x_0 \in \chi \), the goal is to predict \( \varphi(x_0) \).

The procedure selects between two hypotheses \( f(x) = 0 \) and \( f(x) = 1 \) defined on the neighborhood of \( O(x_0, k) \), which includes \( k \) closest neighbors, using subset \( S(x_0, k) \) of the
training set. Each hypothesis is evaluated by the frequency of its errors in the neighborhood \( L(f, S(x_0, k)) \).

This loss criterion \( L(f, S(x_0, k)) \) has the only badness rule, point-wise \( T_{pw} \).

The procedure of the learner can be described in these steps.

| k-NN |
|---|
| **Focusing:** selecting focus training set \( S(x_0, k) \) of \( k \) observations with data points closest to \( x \). |
| **Fitting:** Generating hypotheses \( f(x) = 0, f(x) = 1 \) and evaluating their error rate \( L(f, S(x_0, k)) \) |
| **Optimal selection:** Selection of the hypothesis with minimal error rate. |

The procedure is the Basic training.

Thus, \( k \)-NN is an abduction learner.

The learner uses the same criterion as general ERM method. However, ERM takes into account only a narrow concept of Predictability presumption: close of feedback in the same data point. Because of added focusing, this learner takes into account frequency of of mismatch in feedback in close data points.

The procedure may be considered as a smoothing procedure: it assigns the feedback closest to the average in the neighborhood. On another hand, point-wise \( k \)-point smoothing may be considered as an abduction learner with neighborhood-type focusing, where we generate only one hypothesis for the given data point: average feedback on the neighborhood.

### 8.4 Two \( k \)-NN learners with adaptive choice of \( k \)

The parameter \( k \) defines the size of the focus training sample. For binary feedback. Predictability presumption means that small area around a given data point, including the data point itself, is presumed to have the same feedback “often enough”. Optimally, for most of data points \( \xi \in \chi \), \( k \) shall be small enough to have majority of the points \( O(\xi, k) \) of the same class, and large enough that random outliers in the finite sample \( S(\xi, k) \) do not play much of a role.

Here I discuss two approaches to select \( k \) optimally for every new data point. The first is described in [4], the second is my new algorithm. Both learners find prevalent class \( y \) in the focus sample, calculate its frequency \( p_k(y) \) and the error rate \( r_k(y) = 1 - p_k(y) \) the same as \( k \)-NN.

[4] propose, given a data point \( x \), start with a small \( k \) and gradually increase it while calculating bias \( t_k(y) = p_k(y) - 0.5 \) of the prevalent class with every \( k \). The procedure stops when the bias reaches certain threshold. If the threshold was not ever reached, they don’t output any answer.

27
The threshold they propose to use is:

$$\Delta(n, k, \delta, c_1) = c_1 \sqrt{\frac{\log(n) + \log(\frac{1}{\delta})}{k}},$$

where $n$ is size of the training sample, $\delta$ and $c_1$ are some user-selected parameters. The learner uses the same criterion as $k$-NN.

The procedure can be described like this:

| Ada $k$-NN |
|-------------|
| **Wrapper Loop:** Generating parameter $k := k + 1$  |
| - **Basic training** with parameter $k$  |
| * **Focusing:** Transformation $S \rightarrow S(x_0, k)$  |
| * **Fitting:** Generation of two constant hypotheses and evaluation of their loss $L(f, S(x_0, k))$  |
| * **Optimal selection:** Outputs the constant hypothesis $f'$ with minimal loss.  |
| - **Stopping check:** $(L(h', S(x_0, k)) > \Delta(S, k, \delta, c_1))$ or $(k = n)$  |
| **Output:** If $k < n$, output $f'$ as decision. Otherwise, refuse to output a decision.  |

Thus, the learner performs the basic training of the original $k$-NN with wrapper for parameter selection, corroborating the main conjecture.

This learner is developed within the statistical learning paradigm, where the training set is expected to be arbitrary large. As $n$ increases, so does the threshold $\Delta(n, k, \delta, c_1)$. Therefore, the selected value $k$, the size of the focus training set, will go to infinity with $n$. And thus, by the law of large numbers, the solution will converge asymptotically to the expectation of the class in the given neighborhood. At the same time, the ratio of $k$ to $n$ is expected to decrease, thus the size the $k$-neighborhood will tend to 0. If the distribution is continuous in $x$, then the learner will likely find the solution as $n$ tends to infinity.

The issue here is that $n$ is not going to infinity or anywhere. For a fixed $n$, the learner favors smaller $k$, where the evaluation of prevalent class is subject to random fluctuations caused by small sample.

To alleviate this issue, I propose an alternative approach which uses Hoeffding inequality (see, for example, [24]) to select $k$.

The Hoeffding inequality can be written as

$$P\left[ |p - E| > t \right] \leq 2 \exp(-2kt^2),$$

(2)
where \( p \) is observed frequency of an event, \( E \) is the expected frequency (probability) of the same event, and \( t \) is an arbitrary threshold, and \( k \) is the sample size.

Suppose, \( p \) evaluates observed frequency of class 1 (rate of the class 1 among the neighbors), \( E \) is the probability of the class 1 in the neighborhood of a given point. If \( p \) is above 0.5, then observations of the class 1 prevail, and we pick hypothesis 1 out of two. Otherwise, we pick hypothesis 0.

Let \( t = |0.5 - p| \). If \( |p - E| > t \) the expected prevalent class is different from the observed prevalent class. If it is the case, we selected the wrong hypothesis. In this case, the right side of the inequality gives us an upper limit of probability that we picked the prevalent class wrong.

For selection of \( k \) we use the weight, calculated as the right part of (2):

\[
W(y, S, k) = 2 \cdot \exp(-2k |p - 0.5|^2).
\]

Obviously, the larger is \( k \), and the further is the frequency \( p \) from 0.5, the lower is the weight. The weight will serve well for the selection of the parameters \( k \), because we need to find the neighborhood where \( p \) is far from uncertainty, 0.5, yet, the size of the neighborhood is not too small.

Here is the description of the learner’s procedure for the given data point \( x \).

### Hoeffding k-NN

- **Generation of parameter** \( k := k + 1 \)
  - **Basic training:**
    * **Focusing:** Select focus training set \( Q_k(x) \) of \( k \) observations with data points closest to \( x \).
    * **Fitting:** Evaluate error rate \( r_k(c) \) of hypotheses \( c \in \{0, 1\} \) in \( Q_k(x) \)
    * **Optimal selection:** Select the hypothesis \( c'(k) \) with minimal error rate \( r_k(c'(k)) \).
  - **Calculating weight** \( W(x, S, k) \).
  - **Stopping check:** \( k = n - 1 \)
- **Output:** \( k' = \arg \min W(x, S, k); \) output \( c'(k') \).

The proper learning procedure in both \( k \)-NN wrappers minimizes empirical risk, the same as original \( k \)-NN. Thus, this learner is an an abduction learner as well.

### 8.5 Decision trees

For this learner, the features are expected to be “ordinal”: every feature has finite number of ordered values; there are no operations on feature values. The feedback of observations
is binary.

The learner starts with whole domain, split it in two subdomains by a value of some feature. Then, the procedure is repeated for every of the subdomains until a subdomain called "leaf" is reached. The decision is selected for this subdomain. The navigation over the tree of subdomains continues until some stopping criterion is reached. The algorithm has a precise rule for generating the parameters of the next subdomain based on the previous trajectory and the obtained results.

There are two criteria of a leaf:

1. Number of observations in the subdomain is below a threshold $N$.
2. Percentage of observations of the prevalent class in the subdomain is above the threshold $q$.

The procedure may be described as a Basic training with Wrapper strategy:

**Decision Tree**

- **Generating parameters** $g$ of the next subdomain
  - **Basic training:**
    - **Focusing**: select subdomain $G(g)$ and subset of the training set $S(g)$ with parameters $g$
    - **Fitting**: evaluate error rate of constant hypotheses $\{0, 1\}$ in $S(g)$
    - **Optimal selection**: select the hypothesis $d(g)$ with minimal error rate.
  - **Calculating weight**: If $G(g)$ is a leaf, $W(g) = 1$, otherwise $W(g) = 0$.
  - **Stopping check**: End of tree
- **Output**: For each $g : W(g) = 1$ output $d(g)$ as decision on $G(g)$.

In this case, we do not need to analyze whole tree before we create a wrapper decision: the decision is prevalent value on each leaf. For the points outside of any leaf the decision is not defined.

The error of a constant function in a subdomain $Q$ is defined as empiric risk. And we demonstrated that empiric risk is an overall badness criterion.

Therefore, the Decision tree is an AGC inference as well.

### 8.6 Naive Bayes

The underlying dependence has $n$ independent variables and binary feedback. The learner works as if it deals with nominal data: the only relationship between data points is equivalence.
The procedure defines decision function on one data point at the time. For a given data point \( z = (z_1, \ldots, z_n) \) the procedure selects \( n \) subsets of the training set. Subset \( S_j \) includes all the observations with \( j \)-th variable equal \( z_j \). For each subset \( S_j \), the learner evaluates error rate \( e(c, S_j) \) of each hypothesis \( c \in \{0, 1\} \). Then for each hypothesis it calculates criterion
\[
\Delta(c, S) = \prod_j (1 - e(c, S_j)).
\]

The learner selects a hypothesis with the maximal value of the criterion.

The abduction criterion scheme for this learner has \( n \) badness rules as well as a functional \( \psi \) to aggregate values of all \( n \) badness criteria.

The \( i \) badness rule for a hypothesis \( c \) is
\[
\pi_i(\alpha, \beta, c) = (s(\alpha) = \approx) \& (s(\beta) = \approx) \& (x_i(\alpha) = x_i(\beta)) \& (y(\alpha) = c)
\]
\[
\delta(\alpha, \beta, c) = \rho_y(\alpha, \beta) = |c - y(\beta)|.
\]
\[
e_i(c) = R_i(\{\delta(\alpha, \beta) | \pi_i(\alpha, \beta, c)\}) = mean_{\pi_i(\alpha, \beta, c)}(\delta(\alpha, \beta, c)).
\]

The aggregating functional is
\[
\psi(\{b_1, \ldots, b_n\}) = 1 - \prod_i (1 - b_i)
\]
It is isotone by each component.

Substituting \( e_i(c) \) for variable \( b_i \), we get
\[
\psi(\{e_1(c), \ldots, e_n(c)\}) = \Delta(c, S).
\]
Thus, the loss criterion of Naive Bayes is an abduction criterion.

Now the procedure of the learner with given data point \( z = (z_1, \ldots, z_n) \) may be described by the rule

| Naive Bayes |
|----------------|
| **Fitting**: generating hypotheses \( c \in \{0, 1\} \) and calculating their loss criterion \( \Delta(c, S) \) |
| **Optimal selection** Select a hypothesis with minimal \( \Delta(c, S) \). |

On the Fitting step, the procedure calculates \( n \) badness values (corresponding \( n \) badness rules) for each hypotheses, then aggregates these values into the abduction criterion for a given hypothesis. Interesting that calculation of each individual badness on this way requires focusing. However, as always, a process involved in calculation of a criterion is not reflected in the scheme of the learner.
This proves that Naive Bayes selects the hypothesis with the lowest abduction criterion, performing the Basic training. This means, Naive Bayes is an abduction learner.

A product in the aggregation of the badness values is chosen in Naive Bayes because it is sensitive to the low frequencies of a class: if some value $1 - e(c, S_j)$ is close to 0, the product will be affected much more than the sum of the frequencies, for example. If some feature value almost never happens in a given class $c$, the hypothesis $c$ will have no chance of being selected, regardless of other feature values of $z$. It justifies choice of product for aggregation.

The loss criterion of Naive Bayes is traditionally interpreted as evaluation of posterior probabilities with “naive” assumption that the features are independent. There are several issues with this narrative.

First, it works for only one learner: if NB learner is based on naive idea about Bayes rule, other learners would need different foundations.

Another issue is that it creates an impression that the learner needs an improvement, is not sophisticated enough.

I hope, I demonstrated that interpretation of the learner as “naive” and “Bayesian” misses the point. The procedure is driven by its specific data type, and it is explained as performing AGC inference, the same as majority of other learners.

### 8.7 Logistic Regression

This learner assumes the features are continuous, the feedback of the observations is binary, but the feedback of the decision is continuous: so, it is required some rounding up in each data point. The decision is defined on the domain $\chi$. The procedure of generating the hypotheses is not specified.

The class of functions associated with logistic regression is

$$F = \left\{ \frac{1}{1 + exp(-\langle w, x \rangle)} \right\}.$$

The functions have values in the interval $(0, 1)$.

The learner minimizes criterion

$$\Delta(f, S) = \frac{1}{m} \sum_{s \in S} \log \left( |y(s) - f(x(s))| \right).$$

Let us define the abduction criterion scheme for this learner. The scheme contains only one badness rule, $T_{br}$. The concurrence condition is the same as $\pi_{pw}$ of $T_{pw}$.

$$\pi(\alpha_1, \alpha_2) = (s(\alpha_1) = \approx) \& (s(\alpha_1) = \approx) \& (x(\alpha_1) = x(\alpha_2)).$$

The contradiction degree is

$$\delta(\alpha_1, \alpha_2) = \log(\rho_y(\alpha_1, \alpha_2)).$$
The function is isotone by \( \rho_y(\alpha_1, \alpha_2) \) and does not depend on \( \rho_x(\alpha_1, \alpha_2) \).

The proper aggregation operation is averaging. It is obvious that this badness rules defines the criterion \( \Delta(f, S) \).

So, the logistic regression supports the main conjecture as well.

The issue with this learner is that, the same as ERM, it does not take into account the similarity of feedback in close (not identical) data points, as presumed in the Predictability presumption, and, therefore, has a tendency of overfitting.

### 8.8 Linear SVM for classification

All the previous learners belong to machine learning “folklore”. Their authors are not known, or, at least, not famous.

SVM is one of the first learners associated with a known author: it is invented by V. Vapnik. His earliest English publications on this subject appeared in early nineties [5], [6].

Let us start with linear SVM for binary classification. The observations

\[
S = \{\beta_i, i = 1 : m\}
\]

have two class labels: \( \{-1, 1\} \) with data points \( x \in \mathbb{R}^n \).

The class of hypotheses \( F \) consists of linear functions \( f(x) \) with \( n \) variables. For a \( f \in F \), \( f(x) = x^T \beta + \beta_0 \). denote \( w(f) = \beta, b(f) = \beta_0 \).

The problem is formulated as minimization of the criterion

| Linear SVM |
|---|
| \[ L(f, S, \xi) = \alpha \|w(f)\|^2 + \frac{1}{m} \sum_{\beta \in S} \xi(\beta) \] (3) |
| s.t. for all \( \beta \in S \), \( y(\beta) \cdot f(x(\beta)) \geq 1 - \xi(\beta) \) and \( \xi(\beta) \geq 0 \). (4) |

The criterion looks intimidating, but it may be simplified though. For this, we want to switch to narrower class of functions, which shall contain all the same decisions.

The observations \( \beta \in S \) satisfying condition \( y(\beta) \cdot f(x(\beta)) > 0 \). are considered correctly classified by the function \( f \). Denote \( S^\oplus(f) \) all correctly classified observations by the function \( f \), and \( S^\ominus(f) = S \setminus S^\oplus(f) \) the rest of the observations.

Let us consider all the functions \( f \in F \) such that \( S^\oplus(f) \neq \emptyset \) and

\[
\min_{S^\oplus(f)} |f(x(\beta))| = 1.
\]

Denote this class of function \( F'(S) \). The class \( F'(S) \) is not empty. Indeed, if for some \( f, f \neq 0 \), \( S^\oplus(f) = \emptyset \), then, \( S^\oplus(-f) = S \). If

\[
q = \min_{S^\oplus(f)} |f(x(\beta))| \neq 1,
\]

33
then the function $f' = \frac{1}{q} f$ satisfies the condition
\[
\min_{S^\oplus(f)} |f'(x(\beta))| = 1.
\]

The last consideration implies that if $f$ is the decision of the problem, then the problem has a decision $f'$ in the class $F'(S)$ with the same set of correctly recognized observations $S^\oplus(f') = S^\oplus(f)$.

Therefore, we can restrict the search for a decision in the class $F'(S)$ only.

**Theorem 3.** The linear SVM classification problem minimizes the loss criterion
\[
L_{svm}(f, S) = \alpha \|w(f)\|^2 + \frac{1}{m} \sum_{\beta \in S^\oplus(f)} |y(\beta) - f(x(\beta))|,
\]
for $f \in F'(S)$.

**Proof.** The conditions (4) can be rewritten as $\forall \beta, \beta \in S :$

\[
\begin{cases}
\xi(\beta) \geq 1 - y(\beta) \cdot f(x(\beta)) \\
\xi(\beta) \geq 0.
\end{cases}
\]

or
\[
\xi(\beta) \geq \max \{1 - y(\beta) \cdot f(x(\beta)), 0\}.
\]

The values $\xi(\beta), \beta \in S$ do not depend on each other, so the minimum of their sum is achieved when every variable $\xi(\beta)$ equals its lowest possible value. Let us find these lowest values for $\xi(\beta)$ depending on if $\beta \in S^\oplus(f)$ or $\beta \in S^\ominus(f)$.

If $\beta \in S^\oplus(f)$,
\[
y(\beta) \cdot f(x(\beta)) = |f(x(s))|.
\]

By definition of $F'(S)$, $|f(x(s))| \geq 1$. Then
\[
\xi(\beta) \geq \max \{1 - y(\beta) \cdot f(x(\beta)), 0\} = 0.
\]

In this case, the lowest possible value for $\xi(\beta)$ is 0.

If $\beta \in S^\ominus(f)$,
\[
y(\beta) \cdot f(x(\beta)) = -|f(x(\beta))|.
\]

Then
\[
\xi(\beta) \geq \max \{1 - y(\beta) \cdot f(x(\beta)), 0\} = 1 + |f(x(\beta))|.
\]

In this case, the lowest possible value for $\xi(\beta)$ is $1 + |f(x(\beta))|$. So,
\[
\min_{\xi} \frac{1}{m} \sum_{S} \xi(\beta) = \sum_{\beta \in S^\oplus(f)} (1 + |f(x(\beta))|).
\]
We still need to prove that for $\beta \in S \ominus (f)$

$$1 + |f(x(\beta))| = |y(\beta) - f(x(\beta))|.$$ 

Let us take $\beta \in S \ominus (f)$. If $y(\beta) = 1$, then $f(x(\beta)) < 0$ and $|f(x(\beta))| = -f(x(\beta))$. So,

$$(1 + |f(x(\beta))|) = 1 - f(x(\beta)) = |y(\beta) - f(x(\beta))|.$$ 

If $y(\beta) = -1$, then $f(x(\beta)) > 0$ and $|f(x(\beta))| = f(x(\beta))$. So,

$$1 + |f(x(\beta))| = 1 + f(x(\beta)) = -y(\beta) + f(x(\beta)) = |y(\beta) - f(x(\beta))|.$$ 

Now to prove that $L_{\text{svm}}(f, S)$ is an abduction criterion, I need to define corresponding AGC criterion scheme.

The distance between the feedback of observations and the function value is defined here by the rule

$$\rho_y(\alpha_1, \alpha_2) = \begin{cases} 
0, & \text{if } y_1 \cdot y_2 \geq 0 \\
|y_2 - y_1|, & \text{otherwise.}
\end{cases}$$ 

The only badness rule coincides with point-wise rule $T_{pw}$.

The formula $Z(f) = \|w(f)\|^2$ of the criterion is a regularization component: $w(f)$ is the gradient of the hypothesis $f$, and $\|w(f)\|^2$ is the square of its norm. The functional $\Psi$ which combines values of these two criteria is $\Psi(x_1, x_2) = \alpha x_1 + x_2$.

This confirms that linear SVM for classification minimizes an abduction criterion.

### 8.9 Linear Support vector regression

The learner ([12]) minimizes criterion

$$L_{\text{svr}}(f, S) = \sum_{i=1}^{m} V_{\epsilon}(y(\beta_i) - f(x(\beta_i))) + \lambda \|w(f)\|^2,$$

where

$$V_{\epsilon}(r) = \begin{cases} 
0, & \text{if } |r| < \epsilon \\
|r| - \epsilon, & \text{otherwise}
\end{cases}$$

and $S = \{\beta_1, \ldots, \beta_m\}$.

The class of hypothesis is the class of all linear functions.

The second component of the loss criterion is regularization, the same as in the SVM. The distance between feedback of an observation and the value of a hypothesis is defined through the function $V$ : for $\alpha_1 = (\preceq (\varphi(x_1) = y_1))$, $\alpha_2 = (\approx (\varphi(x_2) = y_2))$

$$\rho_y(\alpha_1, \alpha_2) = V(y_1 - y_2).$$
Then the AGC criterion scheme here coincides with the scheme for linear SVM for classification.

So, the linear support vector regression supports the main conjecture as well.

8.10 Support Vector Regression with Kernels

The learner is defined as in \( \text{(12)} \). Suppose we have a set of basis functions \( H = \{h_i(x), i = 1, \ldots, k\} \). We are looking for a hypotheses

\[
f(x) = \sum_{i=1}^{k} w_i h_i(x) + b.
\]

The loss criterion used here is

\[
L(f, S) = \sum_{i=1}^{m} V(y(\beta_i) - f(x(\beta_i)) + \lambda\|w(f)\|^2,
\]

where

\[
V(r) = \begin{cases} 0, & \text{if } |r| < \epsilon \\ |r| - \epsilon, & \text{otherwise}. \end{cases}
\]

Here the transformation \( x \to \langle h_1(x), \ldots, h_k(x) \rangle \) from a \( n \)-dimensional space \( R^n \) into \( k \)-dimensional space \( H(x) \) may be called focusing. Then the problem is reduced to solving a linear SVM regression in the transformed space. Thus, SVR with kernel supports the main conjecture as well.

8.11 Ridge Regression

The learner finds the solution in the class of all linear hyperplanes \( F = \{f : f = wx + b\} \), and it has the criterion

\[
L_{rr}(f, S) = \alpha\|w(f)\|^2 + \frac{1}{m} \sum_{\beta \in S} (f(x(\beta)) - y(\beta))^2.
\]

The first component of the loss criterion is regularization component, the same as in SVM, SVR.

Unlike SVR, in this case the distance on \( Y \) is \( \|y_1 - y_2\| = (y_1 - y_2)^2 \). While SVR does not count small errors, Ridge regression counts all errors, but small errors have low influence because of the square in the norm.

The AGC criterion scheme for Ridge Regression is not any different from the scheme for SVR. The only difference between the learners is in the distance on \( Y \).

Thus, Ridge regression corroborates the main conjecture too.
8.12 Neural Network (NN)

Let us consider single hidden layer NN for two class classification as it is described in [12].

First, the learner transforms $n$-dimensional metric space of inputs $\mathbb{R}$ into $k$-dimensional space $Z$ using non-linear transformation:

$$Z_i(x) = \delta(g_i(x)), i = 1, \ldots, k,$$

where $\delta(r)$ is delta function and $g_i$ are linear functions. Denote $z(x)$ the vector with coordinates $\langle Z_1(x), \ldots, Z_k(x) \rangle$.

Then, for each class $c \in \{0, 1\}$, the learner builds linear voting function $f_c(z(x))$. Denote $G = \{g_1, \ldots, g_k\}$ and $F = \{f_0, f_1\}$.

For each $x \in \mathbb{R}^n$ the class is selected as $C(x, G, F) = \arg\max_c f_c(z(x))$.

The learner uses the loss criterion

$$L_{nn}(G, F, S) = \sum_{\beta \in S} (y(\beta) - C(x(\beta), G, F)).$$

It is obvious that the loss criterion is a point-wise badness criterion $T_{pw}$.

The learner optimizes simultaneously parameters of the functions $G$ and $F$. For selection of parameters of these functions the learner uses gradient descent, which is called “back propagation” in this case. The learner uses some additional stopping criterion.

So, the procedure does not have a focusing stage. If calculates loss for a given set of parameters, evaluates gradients by each parameter, and then updates parameters based on the gradients. After the stopping criterion is achieved, the algorithm outputs the decision with the lowest loss criterion.

The procedure has only two types of steps:

1. fitting, which includes
   - generation of the $C(x, G, F)$ hypothesis based on previous value of loss criterion and gradients
   - evaluation of loss criterion $L_{nn}(G, F, S)$. of the current hypothesis

2. optimal selection: selection of the hypothesis with the lowest loss criterion.

Thus, NN is also an adaptive learner.

8.13 $K$ Means Clustering

The learner is different from hierarchical clustering in that it does not combine clusters, rather, for each observation, it chooses the proper cluster. It is assumed that the distance on the domain of data points is Euclidean.

Here is the description of the learner from [12].
1. Given the current set of means of clusters $M = \{m_1, \ldots, m_K\}$, each observation is assigned to the cluster with the closest mean.

2. The rounds of assignment of all observations are repeated until clusters do not change.

The learning happens when we search for the cluster for the given observed data point. Denote $C(x)$ the assignment of a cluster to a data point $x$. Given the set of observed data points $S_x = \{x_1, \ldots, x_m\}$, $K$ clusters with cluster centers $M$ of the sizes $\{l_1, \ldots, l_K\}$ the procedure assigns a new class to an observed data point to minimize sum of all pairwise distances within each cluster

$$W(C, S) = \frac{1}{2} \sum_{k=1}^{K} \sum_{C(\xi)=k} \sum_{C(\zeta)=k} ||\xi - \zeta||^2$$

(7)

$$= \sum_{k=1}^{K} l_k \sum_{C(\xi)=k} ||\xi - \overline{x}_k||^2,$$

(8)

where $\xi, \zeta \in S$, and $\overline{x}_k$ is mean of the $k$-th cluster. I use the form (7) to prove that the learner agrees with the main conjecture.

Denote $x_0$ a data point, $x_0 \in S_x$, which we need to assign a cluster on this step.

As in the case of hierarchical clustering, we consider underlying dependence $\varphi$ as a function from cluster index $k$ to the observed data point $x$. There are $K$ hypotheses $H(x_0) = \{h_1, \ldots, h_K\}$. Each hypothesis $h_i$ has a single hypothetical case $\psi(x_0, i, \infty)$.

We assume, before current run of the learner, the clusters are already assigned to each observed data point besides $x_0$. So, the run starts with the training set having observations

$$S(x_0) = \{\beta \in S : x(\beta) \neq x_0\}.$$

Let us define an AGC criterion scheme for the 7. There is only one badness rule with the concurrence condition

$$\pi(\alpha_1, \alpha_2) = (x(\alpha_1) = x(\alpha_2)),$$

which says that we evaluate contradiction degree for each pair of formulas with the same cluster, regardless if it is an observation or a hypothetical instance. The concurrence condition is symmetrical, therefore for each pair of formulas $\alpha_1, \alpha_2$ which satisfies the condition, the pair $\alpha_2, \alpha_1$ satisfies the condition as well. In effect, every pair is counted twice.

The contradiction function is

$$\delta(\alpha_1, \alpha_2) = \rho_y(\alpha_1, \alpha_2)^2$$

For the proper aggregation we use the averaging.

The learner generates all hypotheses $H(x_0)$, evaluates the loss criterion for each of them and selects the hypothesis with the lowest loss criterion. Thus this learner corroborates the main conjecture as well.
9 Conclusions

I introduced the concept of Abduction with Gradated Contradictions (AGC) and suggested that machine learning process may be explained as AGC inference.

Each instance of AGC inference is defined by its criterion and by its procedure. To formalize the general concept of AGC criterion, I introduced Logic of Observations and Hypotheses, LOH, and the Logic of Recursive Aggregation (LRA).

LOH defines the concept of the gradated contradictions between a hypothesis and a training set. LRA provides language to describe aggregation of the contradiction degrees for the AGC criterion.

I defined certain steps which every instance of AGC inference shall perform. To corroborate my conjecture about ML as AGC inference I demonstrated that 14 popular learners, including learners for classification, regression and clustering, indeed minimize some version of AGC criterion and implement the specified steps, thus performing AGC inference.

The concept of AGC thus generalizes the big part of applied ML as we know it.

To justify AGC approach to learning, I explicitly state informal Working Presumptions, every practitioner has to make implicitly. They explain why learning shall work as an abduction inference.

Here are some advantages of the proposed ML paradigm over traditional statistical one.

• The framework provides unified logical justification and explanation for large variety of real life learners used by practitioners. To the best of my knowledge this is the first theory of ML which can do it.

• As far as I know there was no definition of ML which would include classification, regression and clustering. I demonstrated inner similarity of these types of problems.

• The proposed approach allows to understand regularization component of loss criteria as an aspect of “predictability”.

• The proposed formalism of AGC allows to describe wider variety of learners than are commonly used. It opens new ways to design new learners, optimize existing ones, and facilitate automatic learning.

• As an example of such learner customization, I proposed a version of adaptive $k$-NN learner based on Hoeffding inequality. The learner shall have advantages over ADA $k$-NN for small data.

References

[1] Sushant Agarwal, Nivasini Ananthakrishnan, Shai Ben-David, and Tosca Lechner. On learnability with computable learners. Proceedings of Machine Learning Research, 117:1 – 13, 2020.
[2] Carlos E. Alchorrón and Antonio A. Martino. Logic without truth. *Ratio juris*, 3:46–67, 1990.

[3] Arnon Avron and Anna Zamansky. Non-deterministic multi-valued logics - a tutorial. In *40th IEEE International Symposium on Multiple-Valued Logic, ISMVL*, 2010.

[4] A. Balsubramani, S. Dasgupta, and Y. Freund. An adaptive nearest neighbor rule for classification. In *33rd Conference on Neural Information Processing Systems (NeurIPS 2019), Vancouver, Canada.*, 2019.

[5] B.E. Boser, I. M. Guyon, and V.N. Vapnik. A training algorithm for optimal margin classifiers. In *COLT ’92: Proceedings of the fifth annual workshop on Computational learning theory*, 1992.

[6] C. Cortes and V. Vapnik. Support vector networks. *Machine Learning*, 20:273 – 297, 1995.

[7] M. J. Donovan, F. Khan, G.R. Fernandez, R. .and Mesa-Tejada, M. Sapir, et al. Personalized prediction of tumor response and cancer progression on prostate needle biopsy., 2009.

[8] Thomas Eiter and Georg Gottlob. The complexity of logic-based abduction. In *Tenth Symposium on Theoretical Aspects of Computing (STACS), LNCS 665*, pages 70–79. Springer, 1993.

[9] Nina Gierasimczuk. Bridging learning theory and dynamic epistemic logic. *Synthese*, 169:371–384, 2009.

[10] John Grant. Measuring inconsistency in generalized propositional logic. *Logica Universalis*, 14:331 – 356, 2020.

[11] J. Halpern. *Reasoning about uncertainty*. MIT Press, 2005.

[12] T. Hastie, R. Tibshirani, and J. Friedman. *Elements of statistical learning*. Springer, 2009.

[13] A. Jinsang. *Subjective Logic: A formalism for reasoning under uncertainty*. Springer-Verlag, 2016.

[14] Rohit J. Kate and Raymond J. Mooney. Probabilistic abduction using markov logic networks. In *IJCAI-09 Workshop on Plan, Activity, and Intent Recognition (PAIR-09)*, pages 1–6. Springer, 2009.

[15] Paolo Liberatore and Marco Schaerf. On the complexity of second-best abductive explanations. *International Journal of Approximate Reasoning*, Volume 63:22–31, 2015.
[16] Leonid Libkin. *Elements of Finite Model Theory*. Springer, 2012.

[17] Ulrike von Luxburg and Bernhard Schölkopf. Statistical learning theory: models, concepts and results. In Dov M. Gabbay, Stephan Hartmann, and John Woods, editors, *Handbook of the History of Logic. Volume 10: Inductive Logic*, pages 651–706. Elsevier BV, 2009.

[18] Sayan Mukherjee, Partha Niyogi, Tomaso Poggio, and Ryan Rifkin. Statistical learning: stability is sufficient for generalization and necessary and sufficient for consistency. *Advances in Computational Mathematics*, 25:161–193, 2006.

[19] V. Novák, I. Perfilieva, and J. Močkoř. *Mathematical principles of fuzzy logic*. Dordrecht: Kluwer Academic, 1999.

[20] C.S. Pierce. Abduction and induction. In *Philosophical writings of Pierce*, pages 150–156. Routledge and Kegan Paul Ltd, 1955.

[21] M. Sapir. Optimal choice : New machine learning problem and its solution. *International Journal of Computational Science and Information Technology*, 5:1 – 9, 2017.

[22] Marina Sapir. *Papaya Orchard: Comedy in one act*, 2018. https://www.academia.edu/35254962/Papaya_Orchard_Comedy_in_one_act.

[23] Marina Sapir, Faisal M Khan, Yuliya Vengrenyuk, et al. Improved automated localization and quantification of protein multiplexes via multispectral fluorescence imaging in heterogenous biopsy samples. In 2010 IEEE International Symposium on Biomedical Imaging, pages 157–160. 2010.

[24] Shai Shalev-Shwartz and Shai Ben-David. *Understanding Machine Learning*. Cambridge University Press, NY, 2014.

[25] Robert Tibshirani and Guenther Walter. Cluster validation by prediction strength. *Journal of Computational and Graphical Statistics*, 14(3):511–528, 2005.

[26] Pietro Torasso, Luca Console, Luigi Portinale, and Theseider Dupre. On the role of abduction. *ACM Computing Surveys*, 27:353 –355, 1995.

[27] V. N. Vapnik. *The nature of statistical learning theory*. Springer - Verlag, 1995.