A Combinatorial Cut-Based Algorithm for Solving Laplacian Linear Systems

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Abstract

Over the last two decades, a significant line of work in theoretical algorithms has been progress in solving linear systems of the form \( Lp = b \), where \( L \) is the Laplacian matrix of a weighted graph with weights \( w(i, j) > 0 \) on the edges. The solution \( p \) of the linear system can be interpreted as the potentials of an electrical flow. Kelner, Orrechia, Sidford, and Zhu [KOSZ13] give a combinatorial, near-linear time algorithm that maintains the Kirchoff Current Law, and gradually enforces the Kirchoff Potential Law. Here we consider a dual version of the algorithm that maintains the Kirchoff Potential Law, and gradually enforces the Kirchoff Current Law. We prove that this dual algorithm also runs in a near-linear number of iterations. Each iteration requires updating all potentials on one side of a fundamental cut of a spanning tree by a fixed amount. If this update step can be performed in polylogarithmic time, we can also obtain a near-linear time algorithm to solve \( Lp = b \). However, if we abstract this update step as a natural data structure problem, we show that we can use the data structure to solve a problem that has been conjectured to be difficult for dynamic algorithms, the online vector-matrix-vector problem [HKNS15]. The conjecture implies that the data structure does not have an \( O(n^{1-\epsilon}) \) time algorithm for any \( \epsilon > 0 \). Thus our dual algorithm cannot be near-linear time algorithm for solving \( Lp = b \) unless we are able to take advantage of the structure of the particular update steps that our algorithm uses.

1 Introduction

Over the last two decades, a significant line of work in theoretical algorithms has been progress in solving linear systems of the form \( Lp = b \), where \( L \) is the Laplacian matrix of a weighted graph with weights \( w(i, j) > 0 \) on the edges. Starting with the work of Spielman and Teng [ST04], researchers have devised a number of algorithms that run in near-linear time in the number of edges of the graph (corresponding to the number of non-zeros in the matrix \( L \)). The solution \( p \) of the linear system can be interpreted as the potentials of an electrical flow in which the resistance of each edge is \( r(i, j) = 1/w(i, j) \), and the current supplied to each node \( i \) is \( b(i) \). There have been many nice algorithmic ideas introduced using this interpretation of the linear system, as well as many applications of the fast algorithms for solving this system to other flow problems, such as the maximum flow problem (e.g. Christiano, Kelner, Mądry, Spielman, and Teng [CKM+11]; Mądry [Mąd16]).

Since the initial work of Spielman and Teng, a number of different near-linear time algorithms have been proposed. In this paper, we wish to focus on a particular simple, combinatorial algorithm by Kelner, Orrechia, Sidford, and Zhu [KOSZ13], hereafter known as the KOSZ algorithm; this algorithm has been the subject

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of several implementation studies [BDG16] [BDG16a] [DGM+16] [HLMW16]. The KOSZ algorithm uses the idea of an electrical flow $f$ in solving the linear system $Lp = b$ as mentioned above. An electrical flow is one that obeys the flow conservation constraints at each node $i$, saying that the net flow out of $i$ is $b(i)$ (sometimes known in this context as the Kirchoff Current Law, or KCL), and Ohm’s law, which says that there exists a potential vector $p$ such that the current flow from $i$ to $j$, $f(i, j)$, equals $(p(i) - p(j))/r(i, j)$. There exists a flow $f$ that satisfies these two properties, and the corresponding potential vector $p$ solves $Lp = b$. Ohm’s Law is known to be equivalent to the Kirchoff Potential Law (KPL), which says that the flow $f$ satisfies the property that around any directed cycle $C$, $\sum_{(i,j) \in C} r(i, j)f(i, j) = 0$. Given KPL, potentials satisfying Ohm’s law can be constructed by picking any spanning tree $T$ rooted at a vertex $r$, setting $p(r)$ to 0, and $p(k)$ to the sum of $f(i,j)r(i,j)$ on the path in $T$ from $k$ to $r$; these potentials are known as tree-induced potentials.

The KOSZ algorithm starts by picking a spanning tree $T$; for the running time of the algorithm, it is important that the tree have low stretch, whose definition is otherwise not crucial to the description of the algorithm.¹ The algorithm starts by constructing a flow $f$ that satisfies flow conservation using only the edges in $T$. For a near-linear number of iterations, the algorithm picks at random a non-tree edge $(i,j)$ and considers the fundamental cycle $C$ closed by adding $(i,j)$ to $T$; it then alters the flow $f$ along $C$ to satisfy KPL (and such that KCL continues to be satisfied). By picking $(i,j)$ with the appropriate probability, Kelner et al. show that the energy of the resulting flow decreases by a factor $1 - \frac{1}{\tau}$ in expectation, where $\tau$ is a parameter related to the stretch. The algorithm then returns the tree-induced potentials $p$ associated with $T$ and the flow $f$. Kelner et al. [KOSZ13] show that the resulting potentials are close to the potentials of the associated electrical flow for the graph. The KOSZ algorithm has the pleasing properties that it is easy to understand both the algorithm and the analysis, and it is also close in spirit to known network flow algorithms; in particular, it resembles the primal network simplex algorithm for the minimum-cost flow problem.

Deweese et al. [DGM+16] have tried implementations of the KOSZ algorithm and compared it to other Laplacian solvers. One bottleneck that they identify is the time taken to update the flow in each iteration of the graph. Potentially the number of edges in the cycle is large, but since the number of iterations is near-linear, we want an update time that is polylogarithmic at worst in order to obtain a near-linear running time. Kelner et al. give a simple data structure that performs the update in $O(\log n)$ time, but Deweese et al. identify this operation as one that makes the KOSZ algorithm uncompetitive relative to other Laplacian solvers.

One idea for making KOSZ practical would be to look for a dual version of KOSZ. Since the KOSZ algorithm maintains KCL and gradually enforces KPL, a dual version would maintain KPL/Ohm’s Law and gradually enforce KCL. While KOSZ maintains a flow $f$ and implicitly tree-induced potentials $p$, the dual KOSZ maintains potentials $p$, and then implicitly the flow $f$.² Each iteration of KOSZ picks a fundamental cycle and enforces KPL; each iteration of dual KOSZ picks a fundamental cut in the tree (the cut induced by removing an edge in the tree) and adds the same amount $\delta$ to the potentials of all vertices on one side of the cut so as to make sure that the total flow crossing the cut is what it needs to be given the supply vector $b$. An analogous situation exists in the minimum-cost flow literature in which there are both cycle-canceling algorithms (e.g. Goldberg and Tarjan [GTS9]) and cut-canceling algorithms (e.g. Ervolina and McCormick [EM93]).

In this paper, we fill in the details of this algorithm and show that it also has a near-linear number of iterations. This would imply a near-linear time algorithm if each update to the potentials could be performed in polylogarithmic time. It seems natural to abstract this update step into a data structure operation that lets us increase all the potentials on one side of a fundamental cut by a fixed amount, and query the amount of flow crossing a fundamental cut given the current potentials.

However, we show that such a data structure cannot be implemented in $O(n^{1-\epsilon})$ time for any $\epsilon > 0$ given a conjecture about the online matrix-vector multiplication problem made by Henzinger, Krinninger, Nanongkai

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¹The stretch as used in KOSZ is $\sum_{(i,j) \in T} \frac{R(P(i,j))}{r(i,j)}$, where $R(P(i,j))$ is the total resistance along the path in $T$ between $i$ and $j$.

²To make matters somewhat confusing, the Laplacian solver literature would regard such an algorithm as a primal algorithm for working directly in the space of potentials rather than in the space of flows, which it regards as the dual space.
and Saramurak [HKNS15]. They have conjectured that this problem does not have any algorithm that can carry out an online sequence of \( n \) Boolean matrix-vector multiplications in time \( O(n^{3 - \epsilon}) \), and show that if the conjecture is false, then various long-standing combinatorial problems will have faster algorithms. We show that a single Boolean matrix-vector multiply can be carried out as a sequence of \( O(n) \) operations of our desired data structure. Given the conjecture, then, we cannot implement the data structure operations in \( O(n^{1 - \epsilon}) \) time. Thus a near-linear time version of the dual KOSZ algorithm appears unobtainable unless the algorithm’s calls to the data structure have a special structure that can be exploited.

Our paper is structured as follows. In Section 2 we introduce notation and concepts in electrical flow that we will use. In Section 3 we introduce the algorithm. In Section 4 we perform the analysis of the algorithm, and in Section 5 we show that a near-linear number of iterations will find nearly optimal potentials. In Section 6 we discuss the complexity of each iteration, and show the reduction to the online vector-matrix-vector problem that implies that we cannot perform each iterate in sublinear time. We conclude in Section 7.

2 Notation and Problem Statement

We are given an undirected graph \( G = (V, E) \), with positive resistances \( r \in \mathbb{R}^E \). Although our graph is undirected, it will be helpful for us notationally to direct the edges. To that end, fix an arbitrary orientation of the edges, and denote the set of directed edges by \( \vec{E} \).

In addition to the graph \( G \) and the resistances \( r \), we are given a supply vector \( b \in \mathbb{R}^V \).

The Laplacian matrix of \( G \) with respect to the resistances \( r \) is the matrix \( L \in \mathbb{R}^{V \times V} \) defined by

\[
L = \sum_{(i, j) \in E} \frac{1}{r(i, j)}(e_i - e_j)(e_i - e_j)^\top,
\]

where \( e_i \) is the \( i \)th unit basis vector. We note then that \( x^\top L x = \sum_{(i, j) \in E} \frac{1}{r(i, j)}(x(i) - x(j))^2 \) for any vector \( x \).

Our goal is to solve the system of linear equations \( Lp = b \) for \( p \). However, we will not be able to solve \( Lp = b \) exactly, so we will solve it approximately instead. It is usual to measure the quality of a solution \( p \) in terms of the matrix norm induced by \( L \). In other words, if \( p \) is the vector of potentials returned by our algorithm and \( p^* \) is an actual solution to \( Lp^* = b \), then the error of our solution \( p \) is

\[
||p^* - p||_L^2 := (p^* - p)^\top L (p^* - p).
\]

Hence, our objective is to find \( p \in \mathbb{R}^V \) that minimizes \( ||p^* - p||_L^2 \). A precise statement of this is given below.

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**Goal:** Given \( \epsilon > 0 \), find potentials \( p \in \mathbb{R}^V \) that satisfy \( ||p^* - p||_L^2 \leq \epsilon ||p^*||_L^2 \).

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Of course, the algorithm does not know the actual solution \( p^* \). The place where \( p^* \) appears is in the analysis.

Equations of the form \( Lp = b \), where \( L \) is the Laplacian matrix of a graph, are called Laplacian systems, and are found in a wide variety of applications in computer science and other fields. They also have a nice physical interpretation, in terms of electrical flows.

**Definition 1 (Electrical Flows).** Let \( G = (V, E) \) be an undirected graph, let \( r \in \mathbb{R}^E \) be positive resistances on the edges of \( G \), and let \( b \in \mathbb{R}^V \) be a supply vector with \( \sum_{i \in V} b(i) = 0 \). A \( b \)-flow is one such that Kirchhoff’s Current Law (KCL) is obeyed; that is, the net flow leaving each node is equal to its supply: \( \sum_{j: (i,j) \in E} f(i,j) - \sum_{j: (j,i) \in E} f(j,i) = b(i) \). The electrical \( b \)-flow in \( G \) defined by the resistances \( r \) is the (unique) \( b \)-flow that minimizes the quantity \( \sum_{e \in E} r(e) f(e)^2 \) over all \( b \)-flows \( f \).

For a \( b \)-flow \( f \), the quantity \( \sum_{e \in E} r(e) f(e)^2 \) is called the energy of \( f \), and is denoted \( E(f) \).

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\(^3\)In a personal communication to the third author, Sherman [She17] said he also had worked out a dual version of the KOSZ algorithm, but was unable to solve the data structure problem for the updates to potentials. Our paper explains why this might be difficult to do.
The following fact relates the solution $p$ of $Lp = b$ to the electrical $b$-flow in $G$.

**Fact 1 (Electrical Flow Duality).** Let $G = (V, E)$ be a graph with positive resistances $r \in \mathbb{R}^E$. Let $b \in \mathbb{R}^V$ be a supply vector with $\sum_{i \in V} b(i) = 0$, and let $f$ be a $b$-flow in $G$. The following are equivalent:

1. $f$ is the electrical $b$-flow in $G$;

2. (Ohm’s Law) There exist potentials $p \in \mathbb{R}^V$ such that $f(i, j) = \frac{p(i) - p(j)}{r(i, j)}$ for all $(i, j) \in E$. Moreover, $p$ is the solution to $Lp = b$, up to adding a constant to every component of $p$.\footnote{Note that the all-ones vector is in the null space of $L$.}

Thus solving $Lp = b$ and finding the electrical $b$-flow in $G$ are equivalent: Given a solution $p$ to $Lp = b$, we can calculate the electrical flow $f$ using $f(i, j) = \frac{p(i) - p(j)}{r(i, j)}$. On the other hand, given the electrical flow $f$, we can recover corresponding potentials $p$ by setting $p(v) = 0$ for some arbitrary vertex $v$, and using the equation $f(i, j) = \frac{p(i) - p(j)}{r(i, j)}$ to solve for the potentials on every other vertex.

A $b$-flow satisfies Ohm’s Law if and only if it satisfies the Kirchoff Potential Law (KPL): KPL states that for every directed cycle $C$, $\sum_{(i, j) \in C} f(i, j)r(i, j) = 0$.

Both the Kelner et al. algorithm and our algorithm use a low-stretch spanning tree $T$. Given resistances $r$, the stretch of a tree is defined as

$$st_T(G) = \sum_{(i, j) \in E} st_T(i, j) = \sum_{(i, j) \in E} \frac{1}{r(i, j)} \sum_{(k, \ell) \in P(i, j)} r(k, \ell),$$

where $P(i, j)$ is the unique path from $i$ to $j$ in $T$. We can find a spanning tree $T$ with total stretch $st_T(G) = O(m \log n \log \log n)$ in $O(m \log n \log \log n)$ time \footnote{AN12}.

We use the notation $\mathbb{1}$ to stand for the vector of all 1s, and $\mathbb{1}_X$ to be the characteristic vector of a set $X$ that has 1s in the entries corresponding to the elements of $X$ and 0s elsewhere.

## 3 The Algorithm

The algorithm of Kelner et al. \cite{KOSZ13} works by maintaining a flow that always satisfies Kirchhoff’s Current Law, and iteratively updates the flow along cycles to satisfy Kirchhoff’s Potential Law on the cycle. It starts by choosing a spanning tree $T$ that has low stretch, and computes a $b$-flow $f^0$ using only edges in the tree $T$. Then for a number of iterations $K$ that depends on the stretch of the tree, it chooses a non-tree edge $(i, j) \in E - T$ according to a probability distribution, and for the fundamental cycle closed by adding edge $(i, j)$ to $T$, it modifies the flow $f$ so that Kirchoff’s Potential Law is satisfied on the cycle. The probability $P_{ij}$ that edge $(i, j)$ gets chosen is proportional to the total resistance around the cycle closed by $(i, j)$ divided by $r(i, j)$. Given the tree $T$ with root $r$ and the current flow $f^t$ in iteration $t$, there is a standard way to define a set of potentials $p^t$ (called the tree-induced or tree-defined potentials): set $p(r)$ to 0, and $p(k)$ to the sum of $f(i, j)r(i, j)$ on the path in $T$ from $k$ to $r$; these potentials are known as tree-induced potentials. We summarize the algorithm in Algorithm 1.

Our algorithm, which we will call Dual KOSZ, is dual to the KOSZ algorithm. It works by maintaining a set of potentials that always satisfies Kirchhoff’s Potential Law (via Ohm’s Law), and iteratively samples cuts in the graph, updating potentials on one side of the cut to satisfy flow conservation across that cut. Following KOSZ, we choose a spanning tree $T$ of low stretch. Then for a number of iterations $K$ that depends on the stretch of tree $T$, we repeatedly sample a fundamental cut from the spanning tree (i.e. a cut induced by removing one of the tree edges). We update all of the potentials on one side of the cut by an amount $\Delta$ so that the amount of flow crossing the cut via Ohm’s Law is what is required by the supply vector.

Next we give the algorithm in somewhat more detail. We set up a probability distribution $P_{ij}$ on edges $(i, j)$ in the spanning tree $T$; we will show how to pick this probability distribution in Section \footnote{13}. We initialize...
Compute a tree $T$ with low stretch with respect to resistances $r$
Find flow $f^0$ in $T$ satisfying supplies $b$
Let $p^0$ be tree-defined potentials for $f^0$ with respect to tree $T$
for $t \leftarrow 1$ to $K$ do
  Pick an $(i, j) \in E - T$ with probability $P_{ij}$; Update $f^{t-1}$ to satisfy KCL on the fundamental cycle corresponding to $(i, j)$
  Let $f^t$ be resulting flow
  Let $p^t$ be tree-defined potentials for $f^t$
return $f^K, p^K$

Algorithm 1: The KOSZ algorithm for solving $Lp = b$.

potentials $p^0(i)$ to 0 for all nodes $i \in V$. Then for $K$ iterations (where $K$ is specified later), we sample edge $(i, j) \in T$ according to the probabilities $P_{ij}$. Let $C = C(i, j)$ be the set of vertices on one side of the fundamental cut defined by $(i, j)$, such that $i \in C(i, j)$ and $j \notin C(i, j)$. Let $S(C) = b^\top 1_C$ be the total supply of the nodes in $C$. Note that $S(C)$ is also the amount of flow that should be flowing out of $C$ in any feasible $b$-flow. Let
\[
R(C) = \left( \sum_{(k, \ell) \in \delta(C)} \frac{1}{r(k, \ell)} \right)^{-1}.
\]

Note that

- this sum is over all (undirected) edges crossing the cut $C$;
- $R(C)^{-1}$ is the amount by which the flow out of $C$ increases, if we increase the potential of every node in $C$ by 1;
- $R(C)$ has units of resistance.

Let $f^t(C)$ be the total amount of flow going out of $C$ in the flow induced by $p^t$. That is,
\[
f^t(C) = \sum_{i \in C, j \notin C} \frac{p^t(i) - p^t(j)}{r(i, j)}
\]

Note that $f^t(C)$ can be positive or negative. In any feasible $b$-flow, the amount of flow leaving $C$ should be equal to $b^\top 1_C = S(C)$. Hence, we define
\[
\Delta^t = (S(C) - f^t(C)) \cdot R(C).
\]

Observe that $\Delta^t$ is precisely the quantity that we need to increase the potentials of every node in $C$ by, so that flow conservation is satisfied on $\delta(C)$. We then update the potentials, so that
\[
p^{t+1}(v) = \begin{cases} p^t(v) + \Delta^t, & \text{if } v \in C, \\ p^t(v), & \text{if } v \notin C. \end{cases}
\]

Once we have completed $K$ iterations, we return the final potentials $p^K$. We summarize our algorithm in Algorithm 2.

Our main theorem is the following. We define
\[
\tau := \sum_{(i,j) \in T} \frac{r(i,j)}{R(C(i,j))}.
\]

Theorem 1. After $K = \tau \ln(\frac{1}{\epsilon})$ iterations, the algorithm returns $p \in \mathbb{R}^V$ such that $\mathbb{E} \|p^* - p\|_L^2 \leq \epsilon \|p^*\|_L^2$. 

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Compute a tree $T$ with low stretch with respect to resistances $r_e$
Set $p^0(i) = 0$ for all $i \in V$
\begin{algorithmic}
  \For {$t \leftarrow 1$ to $K$}
    \State Pick an edge $(i, j) \in T$ with probability $P_{ij}$
    \State $\Delta^t \leftarrow (S(C) - f^t(C)) \cdot R(C)$
    \State $p^{t+1}(v) \leftarrow \begin{cases} 
      p^t(v) + \Delta^t, & \text{if } v \in C, \\
      p^t(v), & \text{if } v \notin C.
    \end{cases}$
  \EndFor
  \State return $p^K$
\end{algorithmic}

Algorithm 2: Algorithm Dual KOSZ for solving $Lp = b$.

4 Key Lemmas of the Analysis

The analysis is based on a potential energy argument. For a vector of potentials $p \in \mathbb{R}^V$, we define a potential bound on $p$ to be the quantity $B(p)$, given by

$$B(p) := 2b^\top p - p^\top Lp.$$ 

The potential bound is a lower bound on the energy, because of the following fact.

Fact 2 ([Wil19, Lemma 8.9]). For a vector $p \in \mathbb{R}^V$, $p$ satisfies $Lp = b$ if and only if $p$ maximizes $B(p)$ over all $p \in \mathbb{R}^V$, and at its maximum $B(p) = E(f)$ for the electrical $b$-flow $f$.

Thus solving the Laplacian system $Lp = b$ is equivalent to finding a vector of potentials that maximizes the potential bound. In the next few subsections, we prove the lemmas that form the bulk of the analysis. The lemmas and their proofs are similar to their counterparts in [KOSZ13], because everything that appears here is dual to what appears there.

4.1 Potential Bound Always Increases

In this subsection, we show that each iteration of the algorithm increases the potential bound of our vector of potentials $p$. The lemma below quantifies this increase.

Lemma 1. Let $p \in \mathbb{R}^V$ be a vector of potentials and let $C \subset V$. Let $p'$ be the potentials obtained from $p$ as in the algorithm (that is, by adding $\Delta$ to the potential of every vertex in $C$ so that flow conservation is satisfied across $\delta(C)$). Then

$$B(p') - B(p) = \Delta^2 / R(C).$$

Proof. The way we update $p$ is by adding a constant $\Delta$ to the potentials of every vertex in $C$, where

$$\Delta = (S(C) - f(C)) \cdot R(C)$$

Recall that $f(C)$ is the net amount of flow going out of $C$ in the flow induced by $p$. That is,

$$f(C) = \sum_{i,j \in E, i \in C, j \notin C} \frac{p(i) - p(j)}{r(i,j)}$$

Note that the new potentials $p'$ can be expressed as $p' = p + \Delta \mathbb{1}_C$. 


We have
\[
E(p') - B(p) = 2b^\top p' - (p')^\top Lp' - (2b^\top p - p^\top Lp) \\
= 2b^\top (p + \Delta \cdot 1_C) - 2b^\top p - (p')^\top Lp' + p^\top Lp \\
= 2\Delta \cdot b^\top 1_C - \sum_{(i,j) \in E} \frac{1}{r(i,j)} [(p'(i) - p'(j))^2 - (p(i) - p(j))^2] \\
= 2\Delta \cdot b^\top 1_C - \sum_{i \in C, j \notin C \cup \delta(C)} \frac{1}{r(i,j)} [(p(i) + \Delta - p(j))^2 - (p(i) - p(j))^2] \\
= 2\Delta \cdot b^\top 1_C - \sum_{i \in C, j \notin C \cup \delta(C)} \frac{1}{r(i,j)} [2\Delta \cdot (p(i) - p(j)) + \Delta^2] \\
= 2\Delta \cdot b^\top 1_C - 2\Delta \cdot f(C) - \Delta^2 \sum_{(i,j) \in \delta(C)} \frac{1}{r(i,j)} \\
= 2\Delta \cdot b^\top 1_C - 2\Delta \cdot f(C) - \Delta^2 \cdot R(C)^{-1} \\
= 2\Delta \cdot S(C) - 2\Delta \cdot f(C) - \Delta^2 \cdot R(C)^{-1} \\
= 2\Delta^2 R(C)^{-1} - \Delta^2 \cdot R(C)^{-1} \\
= \Delta^2 / R(C).
\]

\[\square\]

### 4.2 Measuring Progress in terms of the Duality Gap

In the previous subsection, we showed that each iteration of the algorithm increases the potential bound \(B(p)\). The second ingredient in the analysis is to introduce an upper bound on how large this potential bound can become. This will allow us to bound the number of iterations the algorithm takes.

**Definition 2 (Gap).** Let \(f\) be a feasible \(b\)-flow and let \(p\) be any vertex potentials. Define
\[
gap(f, p) := E(f) - B(p) = \sum_{e \in E} r(e) f(e)^2 - (2b^\top p - p^\top Lp).
\]

This same notion of a gap was introduced in the analysis of the Kelner et al. algorithm, and was also used to bound the number of iterations of the algorithm.

The electrical flow \(f^*\) minimizes \(E(f)\) over all \(b\)-flows \(f\), and the corresponding vertex potentials \(p^*\) maximize \(B(p^*)\) over all vertex potentials \(p\). Moreover, \(E(f^*) = B(p^*)\). Therefore, for any feasible flow \(f\), \(\gap(f, p)\) is an upper bound on optimality:
\[
\gap(f, p) \geq B(p^*) - B(p).
\]

The lemma below gives us another way to write \(\gap(f, p)\), and will be useful to us later. This relation is shown in Kelner et al. [KOSZ13, Lemma 4.4], but we reprove it here for completeness.

**Lemma 2.** Another way to write \(\gap(f, p)\) is
\[
\gap(f, p) = \sum_{(i,j) \in \bar{E}} r(i,j) \left( f(i,j) - \frac{p(i) - p(j)}{r(i,j)} \right)^2.
\]
Proof. By definition, we have

$$\text{gap}(f, p) = \sum_{e \in E} r(e) f(e)^2 - (2b^\top p - p^\top Lp)$$

Note that

$$b^\top p = \sum_{i \in V} b(i)p(i) = \sum_{i \in V} p(i) \left( \sum_{j: (i, j) \in E} f(i, j) - \sum_{j: (j, i) \in E} f(j, i) \right) = \sum_{(i, j) \in E} f(i, j)(p(i) - p(j))$$

and

$$p^\top Lp = \sum_{(i, j) \in E} \frac{(p(i) - p(j))^2}{r(i, j)}$$

Plugging these into our expression for $\text{gap}(f, p)$, we obtain

$$\text{gap}(f, p) = \sum_{(i, j) \in E} \left[ r(i, j)f(i, j)^2 - 2f(i, j)(p(i) - p(j)) + \frac{(p(i) - p(j))^2}{r(i, j)} \right]$$

$$= \sum_{(i, j) \in E} r(i, j) \left( f(i, j) - \frac{p(i) - p(j)}{r(i, j)} \right)^2$$

which is what we wanted to show. \hfill \Box

The analysis of Kelner et al. \cite{KOSZ13} relies on measuring progress in terms of the above-defined duality gap between primal flow energy and dual potential bound. The high-level idea of the analysis is that one can show that the duality gap decreases by a constant factor each iteration, which implies a linear convergence rate. In the analysis of their algorithm, they maintain a feasible $b$-flow $f$ at each iteration, and measure $\text{gap}(f, p)$ against corresponding tree-defined potentials $p$.

One difference between their algorithm and ours is that we do not maintain a feasible $b$-flow at each iteration. However, for $\text{gap}(f, p)$ to be a valid bound on distance to optimality, we need $f$ to be a feasible $b$-flow. To this end, we introduce the definition of “tree-defined flows” below. This is only used in the analysis; the algorithm does not need to compute these flows.

**Definition 3** (Tree-defined flow). Let $T$ be a spanning tree, $p \in \mathbb{R}^V$ vertex potentials, and $b \in \mathbb{R}^V$ satisfying $1^\top b = 0$ be a supply vector. The tree-defined flow with respect to $T$, $p$ and $b$ is the flow $f_T$ defined by

$$f_T(i, j) = \frac{p(i) - p(j)}{r(i, j)} \quad \text{if} \ (i, j) \not\in T,$$

and for $(i, j) \in T$, $f_T(i, j)$ is the unique value such that the resulting $f_T$ is a feasible $b$-flow. That is, for $(i, j) \in T$, if $C = C(i, j)$ is the fundamental cut defined by $(i, j)$ and $S(C) = b^\top 1_C$ is the amount of flow that should be flowing out of $C$ in a feasible $b$-flow, then

$$f_T(i, j) = S(C) - \sum_{k \in C, \ell \notin C \atop k \ell \in E-i-j} f_T(k, \ell)$$

$$= S(C) - \sum_{k \in C, \ell \notin C \atop k \ell \in E-i-j} \frac{p(k) - p(\ell)}{r(k, \ell)}$$

In other words, $f_T$ is a potential-defined flow outside of the tree $T$, and routes the unique flow on $T$ to make it a feasible $b$-flow.
The below lemma expresses \( \text{gap}(f_T, p) \) in a nice way.

**Lemma 3.** Let \( T \) be a spanning tree, \( p \) vertex potentials, and \( b \) a supply vector. Let \( f_T \) be the associated tree-defined flow. Then

\[
\text{gap}(f_T, p) = \sum_{(i,j) \in E} r(i,j) \cdot \frac{\Delta(C(i,j))^2}{R(C(i,j))^2}
\]

Recall that \( C(i,j), \Delta(C(i,j)) \) and \( R(C(i,j)) \) were defined as follows:

- \( C(i,j) \) is the set of vertices on the side of the fundamental cut of \( T \) determined by \( (i,j) \) containing \( i \). In other words, \( C(i,j) \) consists of the vertices in the component of \( T - ij \) with \( i \in C(i,j) \) and \( j \notin C(i,j) \).
- \( R(C(i,j)) = \left( \sum_{ij \in E} \frac{1}{R(i,j)} \right)^{-1} \).
- \( \Delta(C(i,j)) = (S(C(i,j)) - f(C(i,j)))R(C(i,j)) \), where
  
  \[
  - S(C(i,j)) = b^T \mathbb{1}_C, \text{ and } \quad - f(C(i,j)) = \sum_{k \in C(i,j), \ell \notin C(i,j)} \frac{p(k) - p(\ell)}{r(k, \ell)}
  \]

**Proof.** We have

\[
\text{gap}(f_T, p) = \sum_{(i,j) \in E} r(i,j) \left( f_T(i,j) - \frac{p(i) - p(j)}{r(i,j)} \right)^2
\]

\[
= \sum_{(i,j) \in T} r(i,j) \left( f_T(i,j) - \frac{p(i) - p(j)}{r(i,j)} \right)^2
\]

\[
= \sum_{(i,j) \in T} r(i,j) \left[ \left( S(C(i,j)) - \sum_{k \in C(i,j), \ell \notin C(i,j)} \frac{p(k) - p(\ell)}{r(k, \ell)} \right) - \frac{p(i) - p(j)}{r(i,j)} \right]^2
\]

\[
= \sum_{(i,j) \in T} r(i,j) \left[ S(C(i,j)) - \sum_{k \in C(i,j), \ell \notin C(i,j)} \frac{p(k) - p(\ell)}{r(k, \ell)} \right]^2
\]

\[
= \sum_{(i,j) \in T} r(i,j) |S(C(i,j)) - f(C(i,j))|^2
\]

\[
= \sum_{(i,j) \in T} r(i,j) \cdot \frac{\Delta(C(i,j))^2}{R(C(i,j))^2}
\]

\[
\square
\]

### 4.3 Sampling Edges from a Probability Distribution

Suppose we have a probability distribution \((P_{ij} : (i,j) \in T)\) on the edges in \( T \). If the algorithm samples an edge \((i,j) \in T\) from this distribution, then the expected increase in potential bound is

\[
\mathbb{E}[\mathcal{B}(p')] - \mathcal{B}(p) = \sum_{(i,j) \in T} P_{ij} \cdot \frac{\Delta(C(i,j))^2}{R(C(i,j))^2}
\]

by Lemma 1.

\[9\]
Hence, if $K$ is desired. In other words, in expectation, each iteration decreases the distance to the optimal potential bound by a multiplicative factor of $(1 - \frac{1}{\tau})$.

**Proof.** We know from the discussion above that

$$E[B(p')] - B(p) = \frac{1}{\tau} \sum_{(i,j) \in T} r(i,j) \frac{\Delta(C(i,j))^2}{R(C(i,j))^2} = \frac{1}{\tau} \text{gap}(f_T, p),$$

where $f_T$ is the tree-defined flow associated with potentials $p$. As a consequence of this, we have the following.

**Lemma 4.** If each iteration of the algorithm samples an edge $(i, j) \in T$ according to the probabilities $P_{ij} = \frac{1}{\tau} \frac{r(i,j)}{R(C(i,j))}$, then we have

$$E[B(p^*)] - E[B(p^{t+1})] \leq \left(1 - \frac{1}{\tau}\right) (B(p^*) - B(p^t)).$$

In other words, in expectation, each iteration decreases the distance to the optimal potential bound by a multiplicative factor of $(1 - \frac{1}{\tau})$.

**Proof.** We know from the discussion above that

$$E[B(p^{t+1})] - B(p^t) = \frac{1}{\tau} \text{gap}(f_T, p^t),$$

where $f_T$ is the tree-defined flow associated with potentials $p^t$. Since $\text{gap}(f_T, p^t) \geq B(p^*) - B(p^t)$, we get

$$E[B(p^{t+1})] - B(p^t) \geq \frac{1}{\tau} (B(p^*) - B(p^t)).$$

Rearranging gives

$$E[B(p^*)] - E[B(p^{t+1})] \leq \left(1 - \frac{1}{\tau}\right) (B(p^*) - B(p^t)), $$

as desired. 

**Corollary 1.** After $K = \tau \ln(\frac{1}{\epsilon})$ iterations, we have $B(p^*) - E[B(p^T)] \leq \epsilon \cdot B(p^*)$.

**Proof.** Define the random variable $D_t := B(p^*) - B(p^t)$. By Lemma 4, we know that

$$E[D_{t+1} | p^t] \leq \left(1 - \frac{1}{\tau}\right) E[D_t | p^t]$$

for all possible vectors of potentials $p^t$. This implies that $E[D_{t+1}] \leq (1 - \frac{1}{\tau}) E[D_t]$ unconditionally. By induction on $t$, it then follows that

$$E[D^K] \leq \left(1 - \frac{1}{\tau}\right)^K E[D^0] = \left(1 - \frac{1}{\tau}\right)^K (B(p^*) - B(p^0)) = \left(1 - \frac{1}{\tau}\right)^K B(p^*).$$

Thus,

$$B(p^*) - E[B(p^K)] \leq \left(1 - \frac{1}{\tau}\right)^K B(p^*).$$

Using the inequality $1 - x \leq e^{-x}$, we obtain

$$B(p^*) - E[B(p^K)] \leq e^{-K/\tau} B(p^*).$$

Hence, if $K \geq \tau \ln(\frac{1}{\epsilon})$, then we will have $B(p^*) - E[B(p^K)] \leq \epsilon \cdot B(p^*)$, as desired. 

\[\square\]
5 Convergence Rate

In this section we use the lemmas proved in the previous section to bound the number of iterations Dual KOSZ takes. The analysis of the convergence rate depends on two things.

1. What is the value of $\tau$? We want $\tau$ to be as small as possible, because the smaller $\tau$ is, the faster the potential energies $B(p_t)$ converge to the optimal energy $B(p^*)$. Note that

$$\tau = \sum_{(i,j) \in T} \frac{r(i,j)}{R(C(i,j))}$$

is solely a function of the spanning tree $T$ (and the graph and its resistances). So the question is, how do we pick a spanning tree $T$ to make $\tau$ as small as possible?

2. Lemma 4 shows that the quantity $B(p^*) - B(p_t)$ decreases multiplicatively by $(1 - \frac{1}{\tau})$ each iteration. Initially, this quantity is $B(p^*) - B(p_0)$ = $B(p^*)$. We want to relate this decrease in energy to $\|p^* - p_t\|^2_L$, because our goal is to find potentials $p$ such that $\|p^* - p\|^2_L \leq \epsilon$.

5.1 Low-Stretch Spanning Trees Still Work

Low-stretch spanning trees were the right spanning trees to consider in the cycle-based Laplacian solver of Kelner et al. It turns out that they are exactly what we need, too.

Lemma 5. We have $\tau = st_T(G)$.

Proof. We write out the definitions of $\tau$ and $st_T(G)$:

$$\tau = \sum_{(i,j) \in T} \frac{r(i,j)}{R(C(i,j))} = \sum_{(i,j) \in T} r(i,j) \sum_{(k,\ell) \in \delta(C(i,j))} \frac{1}{r(k,\ell)}$$

and

$$st_T(G) = \sum_{(i,j) \in E} st_T(i,j) = \sum_{(i,j) \in E} \frac{1}{r(i,j)} \sum_{(k,\ell) \in P(i,j)} r(k,\ell),$$

where $P(i,j)$ is the unique path from $i$ to $j$ in $T$.

It turns out that the expressions for $\tau$ and $st_T(G)$ are summing exactly the same terms, just in different ways. Indeed, we have

$$\tau = \sum_{(i,j) \in T} \sum_{(k,\ell) \in \delta(C(i,j))} \frac{r(i,j)}{r(k,\ell)} = \sum_{(k,\ell) \in E} \sum_{(i,j) \in P(k,\ell)} \frac{r(i,j)}{r(k,\ell)} = st_T(G).$$

To switch the order of summation from the first line to the second line, we used the fact that for an edge $(k,\ell) \in E$, we have $(k,\ell) \in \delta(C(i,j))$ if and only if $(i,j) \in P(k,\ell)$. This is because $T$ is a spanning tree.
5.2 Convergence of Energy Implies Convergence of Potentials

By Corollary 1, we know that the potentials \( p^t \) found by the algorithm satisfy the property that \( B(p^t) \) converges to \( B(p^*) \) at a linear rate, in expectation. The following lemma shows that if \( p \) is a set of potentials such that \( B(p) \) is close to \( B(p^*) \), then \( p \) is close to \( p^* \) as a vector (measured in the matrix norm defined by the Laplacian \( L \)).

**Lemma 6.** Let \( p \) be any vector of potentials. Then
\[
\|p^* - p\|_L^2 = B(p^*) - B(p).
\]
In particular, if \( B(p^*) - B(p) \leq \epsilon \cdot B(p^*) \), then
\[
\|p^* - p\|_L^2 \leq \epsilon \cdot B(p^*) = \epsilon \|p^*\|_L^2.
\]
This is because \( B(p^*) = 2b^\top p^* - (p^*)^\top Lp^* = (p^*)^\top Lp^* = \|p^*\|_L^2 \).

**Proof.** We have
\[
\|p^* - p\|_L^2 = (p^* - p)^\top L(p^* - p) = (p^*)^\top Lp^* - 2p^\top Lp^* + p^\top Lp = B(p^*) - 2p^\top b + p^\top Lp = B(p^*) - B(p).
\]

5.3 Overall Running Time

By Corollary 1 and Lemma 6 it suffices to run the algorithm for \( K = \tau \ln(\frac{1}{\epsilon}) \) iterations, where \( \tau = \text{st}_T(G) \) is the total stretch of the spanning tree \( T \) used by the algorithm. After this many iterations, the algorithm returns potentials \( p^K \) that satisfy \( \mathbb{E} \|p^* - p^K\|_L^2 \leq \epsilon \|p^*\|_L^2 \).

We can find a spanning tree \( T \) with total stretch \( \tau = O(m \log n \log \log n) \) in \( O(m \log n \log \log n) \) time [AN12]. For concreteness, fix an arbitrary vertex to be the root of \( T \), and direct all edges in \( T \) towards the root. The set of fundamental cuts we consider will be the vertex sets of subtrees of \( T \).

To compute \( S(C) \) for these \( n - 1 \) fundamental cuts \( C \), we can work our way from the leaves up to the root. If \( C = \{v\} \) is a leaf of \( T \), then \( S(C) = b(v) \). Otherwise, \( C \) is a subtree rooted at \( v \), and \( S(C) = b(v) + \sum_{C'} S(C') \), where the sum is over the subtrees of \( C \). Hence we can compute \( S(C) \) for all fundamental cuts \( C \) in \( O(n) \) time.

To compute \( R(C) \) for the fundamental cuts \( C \), we can maintain \( n - 1 \) variables, one for each fundamental cut. The variable corresponding to cut \( C \) will represent \( R(C)^{-1} = \frac{1}{\tau_C} \sum_{e \in \delta(C)} \frac{1}{\tau(e)} \), and the variables are initialized to 0. We then iterate through all the edges in the graph, and for each such edge \( e \), add \( \frac{1}{\tau(e)} \) to the value of each variable that represents a cut \( C \) such that \( e \in \delta(C) \). Although this naive implementation takes \( O(mn) \) time in the worst-case, it is possible to improve this running time to \( O(m \log n) \) using link-cut trees [STS3]. One can also achieve this running time using the same data structure as the one used in [KOSZ13].

The last part of the running time is the time it takes to run a single iteration of the algorithm. In each iteration of the algorithm, we need to compute \( \Delta = (S(C) - f(C)) \cdot R(C) \), where \( C \) is the fundamental cut selected at that iteration. In the above two paragraphs, we described how to precompute the values of \( S(C) \) and \( R(C) \) for every fundamental cut \( C \); note that these values are fixed at the beginning and do not change during the course of the algorithm. One way to compute \( f(C) \) is to simply iterate over all the edges in \( \delta(C) \), summing each edge’s contribution to \( f(C) \). This takes time proportional to \(|\delta(C)|\), which could be \( O(m) \) in the worst case. We can get this down to \( O(n) \) per iteration by maintaining the values of \( f(C) \) for every
fundamental cut $C$, and updating these values each time the algorithm updates potentials. Since there are $n - 1$ cuts, to do this in $O(n)$ time requires us to be able to update $f(C)$ for a single cut in $O(1)$ time. To do this, we can precompute an $(n - 1) \times (n - 1)$ table with a row/column for each fundamental cut, where the $(C_1, C_2)$ entry is the amount by which the flow out of $C_2$ increases if we add 1 to the potentials of every node in $C_1$. With this table, updating the value of $f(C)$ after a potential update step essentially reduces to a single table lookup, which takes $O(1)$ time. Finally, note that one can construct the table in $O(mn)$ time. This is because each row of the table can be filled in $O(m)$ time, by starting from the leaves of the tree and working up to the root (in the same way we computed $S(C)$ for all the cuts $C$). In more detail, suppose we are computing the row corresponding to some cut $C$. That is, we want to compute how much the flow out of every cut $C'$ changes, if we increase the potentials of every node in $C$ by 1. Let $H(C, C')$ denote this value. Then we can compute $H(C, C')$ recursively as follows:

$$H(C, C') = \begin{cases} \sum_{w:v,w \in E} \frac{1_{w \in C} - 1_{w \in C'}}{r(v, w)}, & \text{if } C' = \{v\} \text{ is a leaf} \\
\sum_{w:v,w \in E} \frac{1_{w \in C} - 1_{w \in C'}}{r(v, w)} + \sum_{C} H(C, \bar{C}), & \text{if } C' \text{ is subtree rooted at } v
\end{cases}$$

In the second part of the above expression, the sum is over all subtrees $\bar{C}$ of $C'$. The above recursive formula allows us to compute $H(C, C')$ for all cuts $C'$ in $O(m)$ time (for a fixed $C$). Hence computing $H(C, C')$ for all pairs $(C, C')$ (which corresponds to filling out the table), takes $O(mn)$ preprocessing time.

To summarize, we can run each iteration of the algorithm in $O(m)$ time, which can be reduced to $O(n)$ time with an additional overhead of $O(n^2)$ storage and $O(mn)$ preprocessing time.

Suppose each iteration of the algorithm takes $O(\Delta)$ time, and the algorithm uses $O(L)$ preprocessing time (not including the time needed to compute the low-stretch spanning tree). Then the total running time of the algorithm is $O(L + \Delta T + m \log n \log \log n) = O(L + m\Delta \log n \log \log n \log \log \frac{1}{\epsilon})$.

Hence, if we use the version which uses $O(mn)$ preprocessing time and $O(n)$ time per iteration, the running time of Dual KOSZ is $O(mn \log n \log \log n \log \log \frac{1}{\epsilon})$. In the next section, we argue that given a natural abstraction of the data structure problem we use in computing $f(C)$ and updating potentials, it appears unlikely that we can implement each iteration in $o(n^{1-\epsilon})$ time, unless we can take advantage of the particular sequence of updates that we perform.

### 6 Lower Bound on the Per-Iteration Complexity of the Algorithm

In this section we give evidence against the possibility of implementing the algorithm described in Section 3 in near-linear time. Concretely, we give a specification of a natural data structure, which we call the TreeFlow data structure, that would allow us to implement the algorithm. The TreeFlow data structure will have two operations, and the algorithm will make one call to each operation in each of its iterations. Then, we prove that assuming the online matrix-vector multiplication (OMv) conjecture, it is impossible to implement the TreeFlow data structure such that each operation of the data structure takes $O(n^{1-\epsilon})$ time.

To simplify the reduction we reduce from a closely related problem called the Online Vector-Matrix-Vector Multiplication Problem (OvMv) \cite{HKNS15}. It was shown in \cite{HKNS15} that the OMv problem and the OvMv problem have the same asymptotic time complexity. Thus, we state our presentation below in terms of OvMv instead of OMv.

**Definition 4** (Online Vector-Matrix-Vector Multiplication Problem). We are given a positive integer $n$, and a Boolean $n \times n$ matrix $M$. At each time step $t = 1, \ldots, n$, we are shown a pair of Boolean vectors $(u_t, v_t)$, each of length $n$. Our task is to output $u_t^T M v_t$ using Boolean matrix-vector operations. Specifically, “addition” is replaced by the OR operation, so that $0+0=0$, and $0+1=1+0=1+1=1$. Hence, $u_t^T M v_t$ is always either 0 or 1.

The OM conjecture implies that no algorithm for this problem can do substantially better than naively multiplying $u_t^T M v_t$ at time step $t$. 

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Lemma 7 ([HKNS15]). Let \( \epsilon > 0 \) be any constant. Assuming the OMv conjecture, there is no algorithm for the online vector-matrix-vector multiplication problem that uses preprocessing time \( O(n^{3-\epsilon}) \) and takes total time \( O(n^{3-\epsilon}) \) with error probability at most \( 1/3 \) in the word-RAM model with \( O(\log n) \) bit words.

We will reduce the online vector-matrix-vector multiplication problem to the \TreeFlow data structure such that computing \( u_i^T M v_i \) requires 2 operations in the \TreeFlow data structure. The reduction will work as follows.

1. We define a data structure, such that any implementation of the data structure where each operation takes \( O(\Delta) \) time gives an implementation of Dual KOSZ which takes \( O(\Delta) \) time per iteration.
2. Assuming the OMv conjecture, we prove that no implementation of the data structure exists where each operation takes \( O(n^{1-\epsilon}) \) time.

6.1 The \TreeFlow Data Structure

The \TreeFlow data structure is given as input (1) an undirected graph \( G = (V, E) \), (2) a spanning tree \( T \) of \( G \) that is rooted at an arbitrary but fixed vertex, (3) a value \( r(u, v) \) for each edge \( (u, v) \in E \) (representing the resistance of \( (u, v) \)), and (4) a value \( b(v) \) for each vertex \( v \in V \) (representing the supply at \( v \)). The quantities \( r(u, v) \) and \( b(v) \) are given at the beginning and will remain unchanged throughout the operations. Furthermore, each vertex \( v \) has a non-negative value, denoted \( \text{value}(v) \), which is initially 0 and which is modified by one of the operations. One should think of \( \text{value}(v) \) as the potential of \( v \).

For any subset \( C \) of vertices we define the flow out of \( C \) to be the quantity

\[
 f(C) := \sum_{(u, v) \in E, u \in C, v \notin C} (\text{value}(u) - \text{value}(v)) / r(u, v).
\]

Furthermore, let \( S(C) := \sum_{v \in C} b(v) \).

The \TreeFlow data structure supports the following operations.

- \text{addvalue} (vertex \( v \), real \( x \)): Add \( x \) to the value of every vertex in the subtree of \( T \) rooted at \( v \).
- \text{findflow} (vertex \( v \)): Return \( S(C) - f(C) \), where \( C \) is the set of vertices in the subtree of \( T \) rooted at \( v \).

The \TreeFlow data structure is a natural abstraction of the operations we require to implement the Dual KOSZ algorithm: The \text{addvalue} operation allows us to update the potentials on a fundamental cut, and \text{findflow} computes \( S(C) - f(C) \), thereby allowing us to compute \( \Delta \) at each iteration. Note that if all \( b(v) \)-values are zero, the \TreeFlow data structure simply returns \( -f(C) \), which gives it its name.

6.2 The Reduction

In this section we will show that exact or approximate \TreeFlow data structure takes near-linear time per operation assuming the OMv conjecture. In the \( \alpha \)-approximate \TreeFlow data structure, the \text{addvalue} operations are executed as described above and each \text{findflow}(v) operation returns a value that is within a multiplicative factor \( \alpha \geq 1 \) of the correct answer, i.e., a value that is at between \( (S(C) - f(C))/\alpha \) and \( (S(C) - f(C)) \cdot \alpha \). Note that in the following lemma, \( \alpha \) does not need to a be a constant and can depend on \( n \).

Lemma 8. Let \( \epsilon > 0 \) be any constant and let \( \alpha \geq 1 \) be any value. Assuming the OMv conjecture, no implementation of the \( \alpha \)-approximate \TreeFlow data structure exists that uses preprocessing time \( O(n^{3-\epsilon}) \) and where the two operations \text{addvalue} and \text{findflow} both take \( O(n^{1-\epsilon}) \) time, such that over a polynomial number of operations the error probability is at most \( 1/3 \) in the word-RAM model with \( O(\log n) \) bit words.
Proof. Given an $n \times n$ Boolean matrix $M$, we create the following TreeFlow data structure. The graph contains $2n + 1$ nodes, namely a special node $x$, one node $c_j$ for each column $j$ with $1 \leq j \leq n$ and one node $d_i$ for each row $i$ with $1 \leq i \leq n$. There is an edge $(d_i, c_j)$ if entry $M_{ij} = 1$. Additionally, every node $c_j$ and every node $d_i$ has an edge to $x$. These edges are added to guarantee that the graph is connected. We set $r(c, d) = 1$ for every edge $(c, d)$ and denote this graph by $G$. Let $T$ be the spanning tree of $G$ that is rooted at $x$ and consists of all the edges incident to $x$. Note that the subtree of $T$ root at any node $y \neq x$ consists of a single node $y$.

Now consider the sequence of $n$ vector pairs $(u, v)$ of the OvMv problem. Let $(u, v)$ be any such pair. We show below how to compute $u^\top Mv$ with $O(n)$ operations in the TreeFlow data structure. Thus the sequence of $n$ vector pairs leads to $O(n^2)$ operations. It then follows from the OMv conjecture and Lemma 7 that this sequence of $O(n^2)$ operations in the TreeFlow data structure cannot take time $O(n^{3-\epsilon})$, i.e., that it is not possible that the complexity of both the addvalue operation and the findflow operation are $O(n^{3-\epsilon})$.

It remains to show how to compute $u^\top Mv$ with $O(n)$ operations in the TreeFlow data structure. Initially the value $\text{value}(v)$ of all nodes $v$ is 0. Let $K$ be a large enough constant that we will specify later.

First, increase the value of $x$ to $K$ by calling addvalue$(x, K)$. When given $(u, v)$ we increase the value of each row node $d_i$ with $u_i = 0$ by $K$ by calling addvalue$(d_i, K)$. Then, we perform the following two operations for each column node $c_j$ with $v_j = 1$: addvalue$(c_j, K)$ and findflow$(c_j)$. Afterwards we decrease the value again for all nodes with value $K$, so that every node has value 0 again. (Alternatively, we could also increase the value of every node to $K$, in which case we never execute an addvalue operation with negative second parameter.)

Note that $u^\top Mv = 1$ iff there exists an edge between a column node $c_j$ with $v_j = 1$ (i.e. $\text{value}(c_j) = K$) and a row node $d_i$ with $u_i = 1$ (i.e. $\text{value}(d_i) = 0$).

We now show that $u^\top Mv = 1$ iff $f(c_j) > 0$ for some column node $c_j$ with $v_j = 1$. (a) Assume first that $u^\top Mv = 1$ and let $c^*$ denote a node $c_j$ and $d^*$ denote a node $d_i$ such that $v_j = 1$, $u_i = 1$ and $M_{ij} = 1$. We will show that $f(c^*) > 0$. Recall that the subtree of $c^*$ consists only of $c^*$. The edge $(c^*, d^*)$ leaves the subtree of $c^*$, contributing a positive amount to $f(c^*)$ because $\text{value}(c^*) = K$ and $\text{value}(d^*) = 0$. All other edges leaving the subtree of $c^*$ contribute a non-negative amount to $f(c^*)$, since $\text{value}(c^*) = K$ and $\text{value}(d_k)$ for other $k \neq i$ is either $K$ or 0. Thus $f(c^*) > 0$. (b) Assume next that $u^\top Mv = 0$. In this case every node $c_j$ with $u_j = 1$ has edges to nodes $d_i$ with $v_i = 0$ (and value $K$). As before the subtree of every node $c_j$ only consists of $c_j$ and, thus, all edges leaving the subtree of $c_j$ contribute 0 to the flow out of the subtree. Thus, for every node $c_j$ with $u_j = 1$ we have $f(c_j) = 0$.

To summarize we have shown above that $u^\top Mv = 1$ iff $f(c_j) > 0$ for some column node $c_j$ with $\text{value}(c_j) = K$. We will now show how to use the results of the findflow queries returned by an $\alpha$-approximate TreeFlow data structure to determine if $f(c_j)$ is positive or zero.

Here is where we will choose the value of $K$. The idea is to make $K$ large enough so that if $f(c_j) > 0$, then $f(c_j)$ is very large. The idea is that this will allow us to distinguish between $f(c_j) = 0$ versus $f(c_j) > 0$, even if we only have access to an $\alpha$-approximation of $S(c_j) - f(c_j) = b(c_j) - f(c_j)$.

It will suffice to choose $K$ large enough so that if $f(c_j) > 0$, then $f(c_j) > \max\{b(c_j), b(c_j)(1 - \alpha^2)\}$ (As $(1 - \alpha^2) < 0$, the second term makes sense if $b(c_j) < 0$.) The value of $K$ depends on $\alpha$, the supplies $b$, and the resistances $r$. For instance, it suffices to choose $K > \|r\|_{\infty} \|b\|_{\infty} \alpha^2$. For this choice of $K$, we have that if $f(c_j) > 0$ then (since it must have an edge to some $d_i$ with $\text{value}(d_i) = 0$),

$$f(c_j) \geq \frac{\text{value}(c_j) - \text{value}(d_i)}{r(c_j, d_i)} = \frac{K - 0}{r(c_j, d_i)} > |b(c_j)| \alpha^2 > \max\{b(c_j), b(c_j)(1 - \alpha^2)\}.$$ 

Having chosen $K$ this way, we have the following:

- If $b(c_j) \geq 0$, then $b(c_j) - f(c_j)$ is non-negative if $f(c_j) = 0$, and negative otherwise (because $f(c_j) > b(c_j)$ when $f(c_j) > 0$). Any $\alpha$-approximation of $b(c_j) - f(c_j)$ allows us to correctly deduce the sign of $b(c_j) - f(c_j)$, hence also whether $b(c_j) - f(c_j) \geq 0$ or whether $b(c_j) - f(c_j) < 0$. From this we can deduce whether $f(c_j) = 0$ or $f(c_j) > 0$. 

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Suppose \( b(c_j) < 0 \). If \( f(c_j) = 0 \), the approximate data structure returns an answer in the interval \([b(c_j) \cdot \alpha, \frac{b(c_j)}{\alpha}]\). If \( f(c_j) > 0 \), it returns an answer in the interval \([(b(c_j) - f(c_j)) \cdot \alpha, \frac{b(c_j) - f(c_j)}{\alpha}]\). Note that the left endpoint of the first interval is to the right of the right endpoint of the second interval as \( f(c_j) > b(c_j) \left(1 - \alpha^2\right)\) implies that

\[
\implies b(c_j) \cdot \alpha > \frac{b(c_j) - f(c_j)}{\alpha}.
\]

Since the two intervals for \( f(c_j) = 0 \) and \( f(c_j) > 0 \) do not overlap, we can correctly distinguish the two cases using the approximate data structure.

To summarize, each \texttt{findflow} query on \( c_j \) allows us to determine if \( f(c_j) > 0 \) or \( f(c_j) = 0 \). If the flow is positive for some \( c_j \), then the answer is \( u^\top Mv = 1 \), otherwise it is 0. Note that it requires \( O(n) \) operations in the \textit{TreeFlow} data structure to determine one \( u^\top Mv \) value, which completes the proof.

\[ \square \]

**Remark** We believe this reduction to OMv provides strong evidence against the possibility of implementing the algorithm in nearly-linear time. Note also that

1. The proof can be adapted so that the values stored at the nodes are only increased.

2. The proof can be modified so that after each \texttt{addvalue} operation a \texttt{findflow} operation is executed (whose answer might be ignored), leading to an alternating sequence of \texttt{addvalue} and \texttt{findflow} operations. This is interesting as Dual KOSZ also alternates between \texttt{addvalue} and \texttt{findflow} operations.

3. The hardness of approximation of the difference \( S(C) - f(C) \) is interesting, because it turns out that even an approximation of this quantity is sufficient to obtain an algorithm for \( L_p = \mathbf{b} \). (Albeit with a convergence rate that deteriorates with the approximation factor.)

However, this is not an impossibility proof for the following reasons:

1. It is possible that there is an implementation of the algorithm which does not use the data structure in Section 6.1 (or a similar data structure for which the reduction to OMv applies).

2. Our proof crucially required to choose a particularly “worst-case” spanning tree \( T \) (which has, however, very small stretch), namely a star. As the choice of the tree \( T \) is determined by the algorithm it might choose a different spanning tree to which the lower bound on the running time of the \textit{TreeFlow} data structure cannot be applied.

### 7 Conclusion

We propose a cut-based combinatorial algorithm to solve Laplacian systems approximately. This algorithm is dual to the cycle-based algorithm by Kelner et al. \[\text{KOSZ13}\]. We show that our algorithm converges in a near-linear number of iterations.

To achieve a near-linear running time, we would further need each iteration to run in polylogarithmic time. We give evidence against this, by presenting a reduction from the OMv conjecture. This is in contrast to the algorithm in \[\text{KOSZ13}\], which uses a data structure such that each iteration of the algorithm runs in \( O(\log n) \) time. In order to obtain a better running time, one would need to show it is possible take advantage of the particular structure of updates in the algorithm to implement the data structure. Note that our reduction crucially needs that a very specific spanning tree (albeit with very small stretch) is chosen. Is it possible for the algorithm to choose a different small-stretch spanning tree that is amendable to a polylogarithmic time implementation?
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