Riemann-Hilbert method and soliton solutions in the system of two-component Hirota equations

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1. Introduction

Soliton theory is a crucial research content in nonlinear science. Due to the significant application of the soliton theory in mathematics and physics, its research has received universal attention by physicists and mathematicians, such as, soliton theory provides a series of methods for solving integrable linear and nonlinear evolution partial differential equations (PDEs in brief) in mathematics and the solitons are often used to describe solitary waves with elastic scattering properties in physics. In many disciplines, there are problems related to soliton theory, which makes it paramount to establish soliton equation model and construct its analytical solution, especially soliton solution. With the development of the soliton theory, increasing methods for solving soliton equations have emerged, such as, the inverse scattering transform (IST in brief) [1–4], the Hirota method [5–9], the Bäcklund transformation method [10–12], the Darboux transformation (DT in brief) method [13–15] and other methods [16–20]. Recently, a new powerful method, the Riemann-Hilbert (RH in brief) method has been developed to examine the N-soliton solutions [21–23]. Through this method, the N-soliton solution for an ocean of integrable systems are obtained [24–26]. In particular, the RH method is an effective way to working the initial-boundary value problem of the integrable nonlinear evolution PDEs [24, 25, 26].

It is well known that the two-component Hirota (TH in brief) equations can be effect explains pulse propagation in single mode fibers which reads [27]

\[
\begin{align*}
q_{1,t} &+ 2A_1q_{1,x} + 4k_1^2A_1(q_1^2 + |q_2|^2)q_1 + i\epsilon|q_1q_{1,x} + 3ik_1^2(|q_1|^2 + |q_2|^2)q_1 + 3ik_1^2q_1(q_1^2q_{1,x} + q_2^2q_{2,x}) = 0, \\
q_{2,t} &+ 2A_1q_{2,x} + 4k_1^2A_1(|q_1|^2 + |q_2|^2)q_2 + i\epsilon|q_1q_{2,x} + 3ik_1^2(|q_1|^2 + |q_2|^2)q_2 + 3ik_1^2q_1(q_1^2q_{1,x} + q_2^2q_{2,x}) = 0,
\end{align*}
\]

\(1.1\)

where \(q(x,t)\) is the complex smooth envelops, and \(\epsilon\) represent the strength of high-order effects. Indeed, when \(k_1 = 1, A_1 = -\frac{1}{2}\), the above system (1.1) is the bright soliton version of the TH equations, which the Lax pair and the IST method were reported in [28, 29], and N-soliton solutions has been discussed via Hirota bilinear form [29], and rogue wave solutions were obtained by using of DT [30], and the bright soliton solitons are discussed by RH formulation in [31].

On the other hand, when \(k_1 = i, A_1 = \frac{1}{2}\), that the above system (1.1) is the dark soliton version of the TH equations, which the Painlevé analysis, the dark soliton solutions and the Lax pair for the N-coupled Hirota equations have been studied [27]. However, to the best of the author’s knowledge, the soliton solutions of the dark soliton version of the TH equations via the RH method have never been investigated by any authors.

The letter is organized as follows. In section 2, we establish a specific RH problem based on the inverse scattering transformation. In section 3, we compute N-soliton solutions of the TH equations from a specific RH problem, which possesses the identity jump matrix on the real axis. Finally, quiet a few discussions and conclusions are given in section 4.
2. The Riemann-Hilbert problem

In what follows, we choose \( k_1 = i, A_1 = \frac{i}{2} \) for the convenient of the analysis. Then, system (1.1) possess the following Lax pair (2.7)

\[
\psi_x = U \psi = \left( \frac{i}{2} \sigma_3 + iQ \right) \psi, \quad \psi_t = V \psi = [-\frac{i}{2} (\epsilon \lambda^2 + \epsilon^2 \lambda^3) \sigma_3 + G] \psi.
\]

(2.1)

where \( G = -i \epsilon \lambda^2 Q + \lambda (i \epsilon Q^2 \sigma_3 - \epsilon \sigma_3 Q_4 - i Q) - \sigma_3 Q_4 + i \epsilon Q_{31} + i \epsilon Q_3 \sigma_3 + 2i \epsilon Q^4 + \epsilon (Q_4 Q - Q Q_4) \), and

\[
\sigma_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & -q_1 & -q_2 \\ q_1^* & 0 & 0 \\ q_2^* & 0 & 0 \end{pmatrix}.
\]

(2.2)

Direct computations display that the zero-curvature equation \( U_x - V_t + [U, V] = 0 \) exactly gives system (1.1).

In fact, (2.1) is equivalent to

\[
\Phi_x = i \lambda \sigma_3 \Phi = i Q \Phi, \quad \Phi_t = i (\lambda^2 + \epsilon \lambda^3) \sigma_3 \Phi = G \Phi.
\]

(2.3)

It is easy to see that \( \tilde{A}(x, t, \lambda) = e^{\frac{i}{2} (\lambda - \epsilon \lambda^2 + \epsilon^2 \lambda^3) i t} \sigma_3 \) is a solution of the (2.3). Introducing a new function \( \Psi(x, t, \lambda) = J(x, t, \lambda) \tilde{A}(x, t, \lambda) \), and by simple calculation, we know that the spectral problems about \( J(x, t, \lambda) \) satisfies

\[
J_x - i \lambda \sigma_3, J = i Q J, \quad J_t + i (\lambda^2 + \epsilon \lambda^3) \sigma_3, J = G J.
\]

(2.4)

Now, we are construct two Jost solutions \( J_\pm = J_\pm(x, \lambda) \) of the first formula in (2.4) for \( \lambda \in \mathbb{R} \)

\[
J_+ = ([J_+], [J_+], [J_+]), \quad J_- = ([J_-], [J_-], [J_-]),
\]

(2.5)

with the boundary conditions

\[
J_\pm \rightarrow 1, \quad x \rightarrow \mp \infty,
\]

(2.6)

where \( [[J_\pm]]_{a_1} \) represent the \( n \)-th column vector of \( J_\pm \), \( 1 = \operatorname{diag}(1, 1, 1) \) is a \( 3 \times 3 \) unit matrix, and the subscripts of \( J(x, \lambda) \) indicate which end of the \( x \)-axis the boundary conditions are set. In fact, one can using the following Volterra integral equations to define these two Jost solutions \( J_\pm = J_\pm(x, \lambda) \) of the first formula in (2.4) for \( \lambda \in \mathbb{R} \)

\[
J_\pm(x, \lambda) = 1 + \int_{\infty}^{x} e^{-i \epsilon \lambda (x-\xi)} i Q J_\pm(x, \xi) d\xi,
\]

(2.7)

where \( \hat{\sigma}_3 \) is a matrix operator which acting on \( 3 \times 3 \) matrix \( X \) as \( \hat{\sigma}_3 X = [\sigma_3, X] \) and \( e^{ib \sigma_3} X = e^{ib} X e^{-ib \sigma_3} \).

Moreover, after simple analysis, we find that \( [J_+], [J_-], [J_-] \) and \( [J_-] \) admits analytic extensions to \( C_- \). Similarly, \( [J_+], [J_-] \) and \( [J_-] \) admits analytic extensions to \( C_+ \), here \( C_- \) and \( C_+ \) represents the upper half \( \lambda \)-plane and the lower half \( \lambda \)-plane, respectively.

Next, one can discuss the properties of \( J_\pm \). It follows from the Abel’s identity and \( \det(Q) = 0 \) that the determinants of \( J_\pm \) are constants for all \( x \), then from boundary conditions (2.8) yields

\[
\det J_\pm = 1, \quad \lambda \in \mathbb{R}.
\]

(2.8)

In addition, we introducing another new function \( A(x, \lambda) = e^{i \epsilon \lambda x} \), we find that spectral problem of the first formula in (2.4) exists two fundamental matrix solutions \( J_\pm A \) and \( J_\pm A \), which are not independent of each other but rather to enjoy a linearly correlation by a \( 3 \times 3 \) scattering matrix \( S(\lambda) \), that is

\[
J_\pm A = J_\pm A \cdot S(\lambda), \quad \lambda \in \mathbb{R}.
\]

(2.9)

It follows from Eq. (2.8) and (2.9) that

\[
\det S(\lambda) = 1.
\]

(2.10)

Moreover, let \( x \) go to \( +\infty \), the \( 3 \times 3 \) scattering matrix \( S(\lambda) \) is given as

\[
S(\lambda) = (s_{ij})_{3 \times 3} = \lim_{x \rightarrow +\infty} A^{-1} \quad J_\pm A = 1 + \int_{-\infty}^{+\infty} e^{-i \epsilon \lambda x} i Q J_\pm d\xi, \quad \lambda \in \mathbb{R}.
\]

(2.11)

Indeed, it follows from analytic property of \( J_\pm \) that the scattering data \( s_{22}, s_{23}, s_{32}, s_{33} \) allow analytic extensions to \( C_+ \), and \( s_{11} \) admits analytic extensions to \( C_- \). Generally speaking, the other scattering data \( s_{12}, s_{13}, s_{21} \) and \( s_{31} \) cannot be extended off the real \( x \)-axis.
So as to discuss behavior of Jost solutions for very large \( \lambda \), we suppose

\[
J = J_0 + \frac{J_1}{\lambda} + \frac{J_2}{\lambda^2} + \frac{J_3}{\lambda^3} + \cdots \quad \lambda \to \infty,
\]

and substituting the above expansion into the first formula of (2.4) and comparing the coefficients of the same power of \( \lambda \) yields

\[
\begin{align*}
O(\lambda^1) : & \frac{1}{2}(\sigma_3, J_0) = 0, \\
O(\lambda^0) : & J_{0,1} - \frac{1}{2}(\sigma_3, J_1) - iQJ_0 = 0, \\
O(\lambda^{-1}) : & J_{1,1} - \frac{1}{2}(\sigma_3, J_2) - iQJ_1 = 0,
\end{align*}
\]

From \( O(\lambda^1) \) and \( O(\lambda^0) \) we have

\[
-\frac{i}{2}(\sigma_3, J_1) = iQJ_0, \quad J_{0,1} = 0.
\]

In order to construct the RH problem of the TH equations, we must define another new Jost solution for the first formula of (2.4) by

\[
P_+ = ([J_{-1}], [J_{+1}], [J_{+3}]) = J_0AS_A^{-1} = J_0A \begin{pmatrix} s_{11} & 0 & 0 \\ s_{21} & 1 & 0 \\ s_{31} & 0 & 1 \end{pmatrix} A^{-1},
\]

which is analytic for \( \lambda \in C_+ \) and admits asymptotic behavior for very large \( \lambda \) as

\[
P_+ \to 1, \quad \lambda \to +\infty, \quad \lambda \in C_+.
\]

Furthermore, we also need consider the adjoint scattering equation of the first formula (2.4), that is

\[
\Phi_+ - \frac{i}{2} \lambda [\sigma_3, \Phi] = -iQ\Phi.
\]

for the convenient of the analysis, we denote the analytic counterpart of \( P_+ \) in \( C_- \) by \( P_- \). Obviously, the inverse matrices \( J_{-1} \) defined as

\[
J_{-1} = ([J_{-1}]^1, [J_{-1}]^2, [J_{-1}]^3)^T, \quad [J_{-1}]^n = ([J_{-1}]^1, [J_{-1}]^2, [J_{-1}]^3)^T,
\]

satisfy this adjoint equation (2.17), here \( [J_{-1}]^n \) denote the \( n \)-th row vector of \( J_{-1}^{-1} \). Then we can see that \( [J_{+1}]^1, [J_{+1}]^2 \) and \( [J_{+1}]^3 \) admits analytic extensions to \( C_+ \). On the other hand, \( [J_{-1}]^1, [J_{-1}]^2 \) and \( [J_{-1}]^3 \) admits analytic extensions to the \( C_+ \).

In addition, it is not difficult to find that the inverse matrices \( J_{-1} \) and \( J_{-1}^{-1} \) satisfy the following boundary conditions

\[
J_{-1}^{-1} \to 1, \quad x \to +\infty.
\]

Therefore, one can define a matrix function \( P_- \) is expressed as follows:

\[
P_- = ([J_{-1}]^1, [J_{-1}]^2, [J_{-1}]^3)^T.
\]

Through an analysis similar to the above, one can manifest that the \( P_- \) analytic in \( C_- \) and

\[
P_- \to 1, \quad \lambda \to -\infty, \quad \lambda \in C_-.
\]

Assume that \( R(k) = S^{-1}(k) \), we have

\[
J_{-1}^{-1} = AR(\lambda)A^{-1}J_{+1}^{-1},
\]

and

\[
P_- = \begin{pmatrix} [J_{-1}]^1 \\ [J_{-1}]^2 \\ [J_{-1}]^3 \end{pmatrix} = AR(\lambda)A^{-1}J_{+1}^{-1} = A \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A^{-1}J_{+1}^{-1}.
\]

So far, we have established two matrix-value functions \( P_+(x, \lambda) \) and \( P_-(x, \lambda) \) which are analytic for \( \lambda \) in \( C_\pm \), respectively. In fact, these two matrix-value functions \( P_+(x, k) \) can be construct a RH problem:

\[
P_-(x, \lambda)P_+(x, \lambda) = T(x, \lambda), \quad \lambda \in C_-
\]

where

\[
T(x, \lambda) = ARsA^{-1} = \begin{pmatrix} 1 & r_{12}e^{-ikx} & r_{13}e^{-ikx} \\ s_{21}e^{ikx} & 1 & 0 \\ s_{31}e^{ikx} & 0 & 1 \end{pmatrix}, \quad \lambda \in C_-
\]
Here we have adopted the identity $r_{11}s_{11} + r_{12}s_{21} + r_{13}s_{31} = 1$, and the jump contour is real $x$-axis.

Furthermore, since $J_-$ satisfies the temporal part of spectral equation

$$J_- + \frac{i}{2}(\lambda^2 + 4\lambda^3)[\sigma_3, J_-] = GJ_-,$$

we have

$$(\tilde{\Lambda}^{-1}J_-\tilde{\Lambda}) = \tilde{\Lambda}^{-1}GJ_-\tilde{\Lambda} = e^{i\kappa x - \beta \kappa \lambda r_1 t}$$

suppose $u$ and $v$ sufficient smoothness and decay as $x \to \infty$, we have $Q_0 \to 0$ as $x \to \pm\infty$. Then taking the limit $x \to \pm\infty$ of Eq. (2.22) yields

$$S_{j} = -\frac{i}{2}(\lambda^2 + 4\lambda^3)[\sigma_3, S_j].$$

This above equation imply that the scattering data $s_{11}, s_{22}, s_{33}, s_{32}$ are time independent, and the other scattering data satisfies

$$s_{1j}(t, \lambda) = s_{1j}(0, \lambda)e^{i(\xi_2 + \xi_3\lambda)t}, \quad s_{j1}(t, \lambda) = s_{j1}(0, \lambda)e^{-i(\xi_2 + \xi_3\lambda)t}, \quad j = 2, 3.$$

3. The soliton solutions

From the $P_+$ and $P_-$ defined in section 2 as well as the $J_+$ and $J_-$ satisfies the scattering relationship (2.9), it is easy to find that

$$\det P_+(x, \lambda) = s_{11}(\lambda), \quad \det P_-(x, \lambda) = r_{11}(\lambda),$$

where $r_{11} = s_{22}s_{33} - s_{32}s_{32}$. Indeed, owing to the $s_{11}$ and $r_{11}$ are time independent, then the zeros of $s_{11} = 0$ and $r_{11} = 0$ are also time independent. Moreover, due to $\sigma_3Q\sigma_3 = -Q$ and $Q^\dagger = -Q$, it is not difficult to find that

$$J'(x, t, \lambda') = \sigma_3J^{-1}(x, t, \lambda)\sigma_3, \quad S'(t) = \sigma_3S^{-1}(\lambda)\sigma_3,$$

then

$$P_+(\lambda)\psi_+ = \sigma_3P_-(\lambda).$$

Suppose that $s_{11}$ possess $N \geq 0$ possible zeros in $C_+$ denoted by $\{\lambda_j, 1 \leq j \leq N\}$, and $r_{11}$ possess $N \geq 0$ possible zeros in $C_-$ denoted by $\{\lambda_j, 1 \leq j \leq N\}$. For the sake of simplicity, one can suppose that all zeros $(\lambda_j, \lambda_j), j = 1, 2, ..., N$ of $s_{11}$ and $r_{11}$ are simple zeros. In this case, each of kernel $P_+(\lambda)$ and kernel $P_-(\lambda)$ include only a single column vector $\psi_+$ and row vector $\psi_+$, respectively, such that

$$P_+(\lambda)\psi_+ = \sigma_3P_-(\lambda)\psi_+ = 0.$$

Owing to $P_+(\lambda)$ is the solution of the first formula of (2.4), we assume that the asymptotic expansion of $P_+(\lambda)$ at large $\lambda$ as

$$P_+ = 1 + \frac{P_+^{(1)}}{\lambda} + O(\lambda^{-2}), \quad \lambda \to \infty,$$

substitute the above expansion into (2.4) and compare $O(1)$ terms obtain the potential functions $q_1$ and $q_2$ can be reconstructed by

$$q_1 = -(P_+^{(1)})_{12}, \quad q_2 = -(P_+^{(1)})_{13},$$

where $P_+^{(1)} = (P_+^{(1)})_{33}$ and $(P_+^{(1)})_{ij}$ is the $(i; j)$-entry of $P_+^{(1)}$, $i, j = 1, 2, 3$.

In order to obtain the spatial evolutions for vectors $\psi_+(x, t)$, on the one hand, we taking the $x$-derivative to equation $P_+\psi_+ = 0$ and using the first formula of (2.4) obtain

$$v_{kk} = \frac{i}{2}\lambda_k\sigma_3v_k,$$

on the other hand, we also taking the $t$-derivative to equation $P_+\psi_+ = 0$ and using the second formula of (2.4) obtain

$$v_{kl} = -\frac{i}{2}(\lambda_k^\dagger + 4\lambda_k^\dagger\sigma_3v_k,$$

By solving (3.7) and (3.8) explicitly, we get

$$v_k(x, t) = e^{\frac{i}{2}(\lambda_k^\dagger x - \xi_2 t + \xi_3\lambda_k t)}\psi_{k0}, \quad \psi_k(x, t) = \frac{1}{\lambda_k^\dagger}e^{\frac{i}{2}(\lambda_k^\dagger x - \xi_2 t + \xi_3\lambda_k t)}\sigma_3.$$
where \( v_{00} \) and \( \tilde{v}_{00} \) are constant vectors.

In order to obtain multi-soliton solutions for the TH equations (1.1), one can choose the jump matrix \( T = I \) is a \( 3 \times 3 \) unit matrix in (2.23). That is to say, the discrete scattering data \( r_{12} = r_{13} = s_{21} = s_{31} = 0 \), consequently, the unique solution to this special RH problem have been solved in [16], and the result is

\[
P_{+}(k) = I - \sum_{j=1}^{N} \sum_{k=1}^{N} \frac{v_{j}(M^{-1})_{jk}\tilde{v}_{k}}{\lambda - \lambda_{j}}.
\]  

(3.10)

where \( M = (M_{jk})_{N \times N} \) is a matrix whose entries are

\[
M_{jk} = \frac{v_{j}(M^{-1})_{jk}}{\lambda_{j} - \lambda_{k}}, \quad 1 \leq j, k \leq N.
\]  

(3.11)

Therefore, from (3.10), we obtain

\[
P_{+}^{(1)} = \sum_{j=1}^{N} \sum_{k=1}^{N} v_{j}(M^{-1})_{jk}\tilde{v}_{k}.
\]  

(3.12)

It follows from (3.12) that the general N-soliton solution for the TH equations (1.1) reads

\[
q_{1} = -\sum_{j=1}^{N} \sum_{k=1}^{N} m_{j}^{*}e^{\theta_{j} + \lambda_{k}}(M^{-1})_{jk}, \quad q_{2} = -\sum_{j=1}^{N} \sum_{k=1}^{N} n_{j}^{*}e^{\theta_{j} + \lambda_{k}}(M^{-1})_{jk}.
\]  

(3.13)

and \( M = (M_{jk})_{N \times N} \) is given by

\[
M_{jk} = \frac{-e^{-\theta_{j} + \lambda_{k}} + (m_{j}m_{k} + n_{j}n_{k})e^{\theta_{j} + \lambda_{k}}}{\lambda_{j} - \lambda_{k}}, \quad 1 \leq j, k \leq N.
\]  

(3.14)

with \( \theta_{k} = \frac{1}{2}[\lambda_{k}x - (\lambda_{k}^{2} + \epsilon \lambda_{k}^{1})t] \), we have chosen \( v_{00} = [1, m_{k}, n_{k}]^{T} \).

Then, on the one hand, as a special example, one can choose \( N = 1 \) in formula (3.13) and with (3.11), we obtain the one-soliton solution as follows:

\[
q_{1}(x,t) = \frac{-m_{1}^{*}e^{\theta_{1} - \lambda_{1}}(\lambda_{1}^{2} - \lambda_{1})}{-e^{-\theta_{1} + \lambda_{1}} + (|m_{1}|^{2} + |n_{1}|^{2})e^{\theta_{1} - \lambda_{1}}}, \quad q_{2}(x,t) = \frac{-n_{1}^{*}e^{\theta_{1} - \lambda_{1}}(\lambda_{1}^{2} - \lambda_{1})}{-e^{-\theta_{1} + \lambda_{1}} + (|m_{1}|^{2} + |n_{1}|^{2})e^{\theta_{1} - \lambda_{1}}}.
\]  

(3.15)

Letting \( \lambda_{1} = \lambda_{11} + i\lambda_{12} \), then the one-soliton solution (3.15) can be written as

\[
q_{1}(x,t) = -i\lambda_{12}m_{1}^{*}e^{\theta_{1} - \lambda_{1}} \csc \theta_{1} - \theta_{1} + \xi_{1}), \quad q_{2}(x,t) = -i\lambda_{12}n_{1}^{*}e^{\theta_{1} - \lambda_{1}} \csc \theta_{1} - \theta_{1} + \xi_{1}.
\]  

(3.16)

where \( \theta_{1} = -i(\lambda_{11} - \lambda_{12}t - e(\lambda_{11}^{2} - 3\lambda_{12}^{2}t), \quad \theta_{1} + \xi_{1} = 2\lambda_{11}12t + e(3\lambda_{12}^{2}12t - \lambda_{11}t + \lambda_{12}t) \) and \( \xi_{1} \) satisfy \( e^{\xi_{1}} = |m_{1}|^{2} + |n_{1}|^{2} \).

On the other hand, as another special example, one can choose \( N = 2 \) in formula (3.13) and with (3.11), we arrive at the two-soliton solution as follows:

\[
q_{1}(x,t) = -[m_{1}^{*}e^{\theta_{1} - \lambda_{1}}(M^{-1})_{11} + m_{2}^{*}e^{\theta_{2} - \lambda_{2}}(M^{-1})_{21} + m_{3}^{*}e^{\theta_{3} - \lambda_{3}}(M^{-1})_{12} + m_{4}^{*}e^{\theta_{4} - \lambda_{4}}(M^{-1})_{22}], \quad q_{2}(x,t) = -[n_{1}^{*}e^{\theta_{1} - \lambda_{1}}(M^{-1})_{11} + n_{2}^{*}e^{\theta_{2} - \lambda_{2}}(M^{-1})_{21} + n_{3}^{*}e^{\theta_{3} - \lambda_{3}}(M^{-1})_{12} + n_{4}^{*}e^{\theta_{4} - \lambda_{4}}(M^{-1})_{22}].
\]  

(3.17)

where \( \theta_{k} = \frac{1}{2}[\lambda_{k}x - (\lambda_{k}^{2} + \epsilon \lambda_{k}^{1})t] \), \( M = (M_{jk})_{2 \times 2} \) with

\[
M_{11} = \frac{2\lambda_{12}}{\lambda_{12}^{2} - \lambda_{12}^{2}} \sinh(\theta_{1} + \theta_{1} + \xi_{1}), \quad M_{12} = \frac{2\lambda_{12}}{\lambda_{12}^{2} - \lambda_{12}^{2}} \sinh(\theta_{1} + \theta_{2} + \xi_{2}),
\]  

\[
M_{21} = \frac{2\lambda_{12}}{\lambda_{12}^{2} - \lambda_{12}^{2}} \sinh(\theta_{1} + \theta_{2}^{2} + \xi_{3}), \quad M_{22} = \frac{2\lambda_{12}}{\lambda_{12}^{2} - \lambda_{12}^{2}} \sinh(\theta_{2} + \theta_{2} + \xi_{5}),
\]

and \( e^{\xi_{1}} = m_{1}^{*}m_{2} + n_{1}^{*}n_{2} \) and \( e^{\xi_{2}} = |m_{1}|^{2} + |n_{2}|^{2} \), respectively.

4. Discussions and conclusions

In fact, as a promotion, the integrable two-component Hirota equations can be extended to the integrable generalized multi-component Hirota equations:

\[
k_{1}q_{l} + 2A_{k}k_{1}q_{lx} + 4k_{1}A_{l} \sum_{j=1}^{N} q_{j}q_{l} - i e[-k_{1}q_{lxx} - 3i k_{1}] \sum_{j=1}^{N} q_{j}^{2}q_{lx} - 3i k_{1}^{3} q_{l} \sum_{j=1}^{N} q_{j}^{3}q_{lx} = 0, \quad l = 1, 2, \ldots, N
\]  

(4.1)
which possess the following Lax pair

$$\psi_x = U \psi = \left( i \sigma_3 + iQ \right) \psi, \quad \psi_t = V \psi = \left( \frac{i}{2}( \epsilon \sigma_3 + A^2 ) \sigma_3 + G \right) \psi,$$

(4.2)

where $G = -ie\lambda Q + \lambda(ieQ^2 \sigma_3 - \epsilon \sigma_3 \sigma_3 - iQ) - \sigma_3 Q_x + ieQ_{xx} + 2ieQ^3 + \epsilon( Q_x Q - Q \sigma_3 )$, and

$$\sigma_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & -q^T \\ q & 0 \end{pmatrix},$$

(4.3)

with $q = (q_1, q_2, \ldots, q_N)^T$. When $k_1 = 1, A_1 = -\frac{i}{2}$, which means to the bright soliton version of the multi-component Hirota equations, when $k_1 = i, A_1 = -\frac{i}{2}$, which means to the dark soliton version of the multi-component Hirota equations. Accordingly, one can also examine the N-soliton solutions to the integrable generalized multi-component Hirota equations by the same way in above two Section. However, we don’t examine them here since the procedure is mechanical.

Moreover, based on the $3 \times 3$ matrix RH problem of the TH equations discussed by authors in [25], one can examine the long-time asymptotic behavior for the solutions of the TH equations via the nonlinear steepest descent method introduce by Deift and Zhou [32].

Acknowledgements

The work was supported by the NSF of China under Grant Nos.11601055, 11805114, NSF of Anhui Province under Grant No.1408085QA06.

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