Solution of the two impurity, two channel Kondo Model

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Abstract

We solve the two-impurity two-channel Kondo model using a combination of conformal invariance and bosonisation techniques. The odd-even symmetric case is analysed in detail. The RKKY interaction turns out to be exactly marginal, resulting in a line of non-Fermi liquid fixed points. Explicit formulae are given for the critical exponents and for the finite-size spectrum, which depend continuously on a single parameter. The marginal line spans a range of values of the RKKY coupling $I$ which goes from the infinitely strong ferromagnetic point $I = -\infty$ (associated with a 4-channel spin-1 Kondo model) to a finite antiferromagnetic critical value $I_c > 0$ beyond which a Fermi liquid is recovered. We also find that, when the odd-even symmetry is broken, the marginal line is unstable for ferromagnetic $I$, while for antiferromagnetic $I$ it extends into a manifold of fixed points.

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The effect of inter-impurity interactions on quantum single-impurity models possessing a non-Fermi liquid ground-state is of crucial importance for the possible experimental realizations of such systems [1]. It is also of considerable theoretical interest for understanding non-Fermi liquid behaviour in lattice models of correlated fermions starting from a local point of view [2,3].

The two-impurity Kondo model with two channels of conduction electrons is one of the simplest model where this problem can be addressed. For a single impurity, this model is controlled by a non-trivial fixed point [4], resulting in a specific heat coefficient $C/T$ and susceptibility $\chi_{\text{imp}}$ diverging logarithmically as $T \to 0$, and a universal finite-size spectrum of excitation energies differing from the free-fermion form [5]. The corresponding two-impurity model is the simplest situation which brings in the competition between the formation of this non-trivial Kondo state and the ordering of the impurities via the RKKY interaction. It has been recently studied by numerical renormalization-group methods (NRG) [6–8].

In this letter, we present an analytic solution of the low-energy universal properties of this model using a combination of conformal field theory [5,9] and bosonisation methods [10,11]. We find that the RKKY interaction (as well as other inter-impurity couplings) is a marginal perturbation, giving rise to a continuous family of non-Fermi liquid fixed points. The finite-size spectrum and the critical properties vary continuously with the strength of the interaction. We obtain analytic formulae for this dependence. These results are in excellent agreement with recent NRG results [6–8]. They should be contrasted with the single-channel case in which, in presence of particle-hole symmetry, Kondo screening always dominate over RKKY ordering or vice-versa, resulting in two stable Fermi liquid fixed points separated by an unstable non-trivial critical point [12,9].

We formulate the model in terms of left-moving (chiral) fermions $\psi_{li\alpha}(x)$ on the full axis $-\infty < x < +\infty$. $l = 1, 2$ is an index labelling the two impurity sites, $i = 1, 2$ is a channel index and $\alpha$ a spin index. The hamiltonian is written as:

$$H = iv_F \sum_{li\alpha} \int_{-\infty}^{+\infty} dx \psi_{li\alpha}^\dagger(x) \frac{\partial}{\partial x} \psi_{li\alpha}(x)$$ (1)

$$+ J_+ (\vec{S}_1 + \vec{S}_2). (\vec{J}_1(0) + \vec{J}_2(0))$$ (2)

$$+ J_m (\vec{S}_1 - \vec{S}_2). (\vec{J}_1(0) - \vec{J}_2(0))$$ (3)

$$+ J_- (\vec{S}_1 + \vec{S}_2). (\sum_{i,\alpha,\beta} (\psi_{1i\alpha}(0) \frac{\delta_{\alpha\beta}}{2} \psi_{2i\beta}(0) + \psi_{2i\alpha}(0) \frac{\delta_{\alpha\beta}}{2} \psi_{1i\beta}(0))$$ (4)

$$+ I \vec{S}_1. \vec{S}_2$$ (5)

In these formulas, $\vec{J}_l(x) = \sum_{i,\alpha,\beta} \psi_{li\alpha}^\dagger(x) \frac{\delta_{\alpha\beta}}{2} \psi_{li\beta}(x)$ denotes the spin-current at position $x$ for species $l$. Our notations follow closely those of Ref. [9]. Alternatively, one could work in the even/odd basis $\psi_{e,o}$ with respect to the midpoint between impurities. The combinations $\psi_{1,2}$ correspond to $(\psi_e \pm \psi_o)/\sqrt{2}$, respectively, and a parity transformation amounts to exchange the indices $l = 1, 2$ for both impurity spins and conduction electrons. In order to make contact with the couplings $\Gamma_{e,o,m}$ used in Ref. [3], let us note the identifications: $J_m \propto \Gamma_m$, $J_+ \propto \Gamma_e + \Gamma_o$, $J_- \propto \Gamma_e - \Gamma_o$.

We shall start by identifying the global symmetries of the hamiltonian. For most of this paper we shall concentrate on the case $J_- = 0$, corresponding to a hamiltonian invariant under odd-even exchange ($\Gamma_e = \Gamma_o$). $H$ has a higher symmetry in that case, with independent charge and channel (or ‘flavour’) transformations allowed for $l = 1, 2$: 2
flavour' degrees of freedom. Here, we are dealing with two copies of $\Phi$, where $\Phi$ is actually an Ising model with the coset construction \( \hat{\text{A}} \), \( \hat{\text{SU}}(m) \) for the adjoint representation of the algebra. Let us first recall [13] that the spin symmetry to the diagonal $\text{SU}(2)$ is conveniently discussed by looking first at the case of two decoupled impurities (each one interacting with two conduction channels) obtained by setting $I = 0$ and $J_+ = J_m$ in addition to $J_- = 0$. In that case, $H$ has independent spin-rotation symmetry:

$$
\psi_{l\alpha} \to \sum_{\beta} V_{\alpha\beta}^{(l)} \psi_{l\beta} \, , \, S_i^\alpha \to \sum_b R^{ab}(V_{l\alpha}^{(l)}) S_i^b
$$

where $R^{ab}(V) = 1/2tr(\sigma^a V \sigma^b V^\dagger)$, \((a, b = x, y, z)\) is the adjoint representation of $V \in \text{SU}(2)$. Hence, two decoupled 2-channel Kondo models have global symmetry $(\text{SU}(2)_{\text{spin}} \otimes \text{SU}(2)_{\text{flavour}} \otimes \text{U}(1)_{\text{charge}})^2$. Coupling the two impurities \((I \neq 0, J_+ \neq J_m)\) while keeping $J_- = 0$ leaves unchanged the independent charge and flavour symmetries, but reduces the spin symmetry to the diagonal $\text{SU}(2)$ corresponding to $V^{(1)} = V^{(2)}$ in Eq. (8).

At a fixed point, these global symmetries are promoted to local conformal symmetries [13]. For decoupled impurities, the symmetry algebra consists in two copies of a product of Kac-Moody algebra for spin, channel and charge: $(\tilde{\text{SU}}_2(2)_s \otimes \tilde{\text{SU}}_2(2)_f \otimes \tilde{\text{U}}(1)_c)^2$. $(\tilde{\text{SU}}_k(2)$ stands for the level-$k$ $\text{SU}(2)$ Kac-Moody algebra, corresponding to the commutation relations of the sum of $k$ independent $\text{SU}(2)$ currents). When coupling the impurities with $J_- = 0$, the diagonal $\text{SU}(2)$ symmetry of the spin sector gives rise to a $\tilde{\text{SU}}_4(2)$ algebra. The generators of this algebra are the sum of the generators of the two $\tilde{\text{SU}}_2(2)_s$ for each impurity, that is the sum of the spin currents $\tilde{J}_1(x) + \tilde{J}_2(x)$. Hence, we must understand how the product $\tilde{\text{SU}}_2(2)_s \otimes \tilde{\text{SU}}_2(2)_s$ can be decomposed into $\tilde{\text{SU}}_4(2)_s$ plus some residual degrees of freedom. The answer is given by the so-called coset construction [14]:

$$
\tilde{\text{SU}}_2(2)_s \otimes \tilde{\text{SU}}_2(2)_s = \tilde{\text{SU}}_4(2)_s \otimes A(2, 2)
$$

The algebra $A(2, 2)$ turns out to be a $N = 1$ superconformal unitary model [13, 17] corresponding to the $m = 4$ member of the discrete series with central charge $c = \frac{3}{2}(1 - \frac{8}{m + 4})$, and thus has $c = 1$. This construction generalizes to the two-channel case the one made by Affleck and Ludwig in their solution of the one-channel two impurity problem [4]. There, the coset construction is $\tilde{\text{SU}}_1(2)_s \otimes \tilde{\text{SU}}_1(2)_s = \tilde{\text{SU}}_2(2)_s \otimes A(1, 1)$, where the algebra $A(1, 1)$ is actually an Ising model with $c = 1/2$.

This coset construction can be understood more explicitly when dealing with spin currents (i.e for the adjoint representation of the algebra). Let us first recall [13] that the $\tilde{\text{SU}}_2(2)$ spin current $\tilde{J}_l^a(x)$ \((a = x, y, z)\) for a given \(l = 1, 2\) can be represented in terms of three Majorana (i.e real) fermions $\chi_l^{x,y,z}$ as follows:

$$
\tilde{J}_l^a(x) = i\epsilon_{abc} \chi_l^b \chi_l^c
$$

This is particularly transparent when using the Emery-Kivelson bosonisation approach to the two-channel Kondo model [14, 11], in which case $\chi^x = \sin \Phi_s, \chi^y = \cos \Phi_s, \chi^z = \cos \Phi_{sf}$ where $\Phi_s, \Phi_{sf}$ are the boson field introduced in ref. [11] corresponding to spin and ‘spin-flavour’ degrees of freedom. Here, we are dealing with two copies of $\tilde{\text{SU}}_2(2)$ and hence with
six Majorana fermions. We combine them into three Dirac fermions and bosonize these new degrees of freedom as:

$$\chi_1^a(x) + i\chi_2^a(x) = e^{i\Phi_a(x)}, \ a = x, y, z$$

(11)

Our notations are such that the free boson correlator reads: $$\langle \Phi(r)\Phi(0) >= -ln(r)$$, so that $$e^{ik\Phi}$$ has dimension $$k^2/2$$. In terms of these fields, the total spin current corresponding to the diagonal $$\hat{S}U_4(2)$$ algebra reads:

$$J^x = J^x_1 + J^x_2 = \cos(\Phi_y - \Phi_z)$$ (and cyclic permutations).

It is convenient to introduce three linear combinations of boson fields as follows:

$$\Phi = \frac{1}{\sqrt{3}}(\Phi_x + \Phi_y + \Phi_z), \ \mu = \frac{1}{\sqrt{2}}(\Phi_x - \Phi_y), \ \nu = \frac{1}{\sqrt{6}}(\Phi_x + \Phi_y - 2\Phi_z)$$

(12)

In term of these combinations, the components of the total spin current read:

$$J^x = \cos(\frac{\mu - \sqrt{3}\nu}{\sqrt{2}}), \ J^y = \cos(\frac{\mu + \sqrt{3}\nu}{\sqrt{2}}), \ J^z = \cos(\sqrt{2}\mu)$$

(13)

Note that $$\Phi$$ does not enter these expressions. Hence the two bosons $$\mu, \nu$$ are sufficient to describe the $$\hat{S}U_4(2)$$ algebra (as expected from its central charge $$c = 2$$) and $$\Phi$$ corresponds to the residual $$A(2, 2)$$ degree of freedom ($$c = 1$$). (The central charge $$c = 3/2 + 3/2$$ has thus been distributed as $$c = 2 + 1$$ in the coset construction Eq.(9).

Thus, a very useful explicit realization of the algebra $$A(2, 2)$$ as a free field theory of a single compact chiral boson $$\Phi$$ has been found. Since $$\Phi_{x,y,z}$$ have periodicity $$2\pi$$, the radius of $$\Phi$$ is found from eq.(12) to be $$R = \sqrt{3}$$, which means that $$\Phi$$ and $$\Phi + 2\pi R$$ are identified. However, this construction based on currents in the adjoint representation does not reveal the full structure of the $$\hat{S}U_2(2)$$ and $$A(2, 2)$$ algebra. Indeed, the three Majorana fermions involved in the currents of $$\hat{S}U_2(2)$$ are not fully independent. For example, if we change their boundary conditions from periodic to antiperiodic, all three must be changed simultaneously so that the boundary condition on the current remain periodic. The boundary conditions on the two sets of Majorana fermions $$\chi^a_l$$, $$l = 1, 2$$ are independent however, which means that the fermions constructed in Eq.(11) are not strictly speaking Dirac fermions but rather that $$\Phi$$ and $$-\Phi$$ must be identified. $$A(2, 2)$$ is thus really a (chiral) ‘orbifold’ theory. This is of crucial importance in determining the operator content of this algebra, and thus the finite size spectrum of the present model. It implies that, in addition to the operators $$e^{\pm i(\nu\sqrt{3}+m/2\sqrt{3})\Phi}$$ and $$\partial^n\Phi$$ (with $$n, m$$ integers), $$A(2, 2)$$ contains two operators of dimension $$1/16$$ which do not have a simple boson representation. These operators are analogous to the (twist) operator associated with the order parameter for the case of the Ising model, which change the boundary condition of the Majorana fermion from periodic to antiperiodic [16]. For convenience, the full set of primary operators of the $$A(2, 2)$$ algebra and their boson representation (when it exists) is given in Table 1.

We now consider the effect of turning on the ‘RKKY’ coupling $$I$$, and/or setting $$J_+ \neq J_m$$, starting from decoupled impurities (but keeping $$J_- = 0$$, i.e $$\Gamma_e = \Gamma_o$$). An order by order perturbative calculation in these couplings (e.g of the free energy) involves correlation functions of the operator $$\vec{S}_1.\vec{S}_2$$ which is a product of spin correlators at the decoupled impurities fixed point. As will be detailed in a longer publication [17], it can be shown that, for the purpose of calculating these correlations, the impurity spin $$\vec{S}_1$$ (resp. $$\vec{S}_2$$)
can be replaced by $a_1 \chi_1$ (resp. $a_2 \chi_2$), where $a_{1,2}$ are local real fermions needed to ensure proper commutations. Hence, the perturbing term of lower dimension associated with the RKKY interaction reads $\int dt a_1 a_2 \chi_1 \cdot \chi_2$. In the bosonic language above, this translates into an induced boundary term in the $A(2,2)$ sector of the hamiltonian:

$$H_{A(2,2)} = \frac{v_F}{4} \int dx (\Pi(x) - \frac{\partial \Phi}{\partial x})^2 + \bar{I}(d^+ d - \frac{1}{2}) \frac{\partial \Phi}{\partial x}(0)$$

where we have set $d^+ \equiv (a_1 + i a_2)/\sqrt{2}$, $\Pi$ is the field conjugate to $\Phi$ and $\bar{I}$ is some (non-universal) function of $I$ and $J_+ - J_m$. Hence the RKKY coupling is associated with a dimension 1 operator and is an exactly marginal perturbation.

The hamiltonian $[4]$ is very similar to the X-ray edge hamiltonian (in presence of the charged core) in the bosonised form $[8]$. In the X-ray edge problem, the boundary term gives a phase-shift, which changes the dimensions of the operators $e^{\pm ik \Phi}$ from $\frac{k^2}{2}$ to $\frac{1}{2}(k \pm \frac{\pi}{2})^2$. This can be obtained by formally fusing $e^{\pm ik \Phi}$ with $U_\delta \equiv \exp(i \frac{\pi}{2}(d^+ d - 1/2)\Phi(0))$. This is the analogue of the "fusion hypothesis" introduced by Affleck and Ludwig for the Kondo problem $[4]$. This fusion rule can be generalised to the present problem (with $\delta$ some non-universal function of $I$ and $J_+ - J_m$) provided we know how to generalise it for the operators of dimension $\frac{1}{16}$. It turns out $[7]$ that the fusion gives back the $\frac{1}{16}$ operators plus operators of dimension $\frac{1}{16} + \frac{1}{2}$ and conformal descendants of these two operators.

Using this method, we have derived the finite-size spectrum of the model for $I \neq 0$, $J_+ \neq J_m$ (but $J_- = 0$) as a function of the single parameter $\delta$, starting from the (known $[4]$) finite-size spectrum of two decoupled two channel Kondo models. This is displayed in Table 2 (where the parameter $x \equiv 2\sqrt{3}_x$ has been used). Note that the ground-state is the triplet of lowest energy for $x > 0$ (ferromagnetic coupling), and the singlet of lowest energy for $x < 0$ (antiferromagnetic coupling). Accordingly, the normalised excitation energy of a given state, $\frac{L \Delta E}{\pi v_F}$ (with $L$ the radial length of the bulk system and $v_F$ the Fermi velocity), is obtained from the total dimension $\Delta_{tot}$ given in Table 2 by the formula:

$$\frac{L \Delta E}{\pi v_F} = \Delta_{tot} - \frac{1}{3} - \frac{(1 - x)^2}{24}, \quad (\text{for } x < 0)$$

$$\frac{L \Delta E}{\pi v_F} = \Delta_{tot} - \frac{3}{8}(1 + \frac{x}{3})^2, \quad (\text{for } x > 0)$$

These formulae are in excellent agreement with recent numerical renormalization group results of K.Ingersent and B.Jones obtained by the numerical renormalization group method $[5]$. They correspond to a one-parameter family of (non-Fermi liquid) fixed points (when symmetry between odd and even channels is preserved) which can be explored by varying either $I$ or $J_+ - J_m$. One of the endpoints of this line is the strong ferromagnetic fixed point of Ref. $[5]$, obtained for $I \rightarrow -\infty$ (corresponding to $x = 1$). At this point, the impurity spins bind into a triplet state, and the spectrum of the 4-channel spin-1 Kondo problem is found, as conjectured in Ref. $[8]$. The $A(2,2)$ operator associated with each state at this fixed-point is also displayed in Table 2. The marginal line includes the point corresponding to two decoupled Kondo problems ($x = 0$) and extends over to a region of antiferromagnetic coupling. As discussed below, we expect it to end at $x = \sqrt{6} - 3 \simeq -0.55$ on this side.

The above fusion principle can also be applied to find the dimensional change of any operator $\mathcal{O}$ for a non-zero $x$ ($\mathcal{O}$ transforms as $U_\delta \mathcal{O} U_\delta^+$), and hence the operator content at
a given fixed point along the line. Spin correlations can be obtained from the identification $S_1^a \pm S_2^a \propto de^{\pm i\Phi^a} + h.c$, which leads to a singular behaviour of the uniform susceptibility $\chi_{imp} \propto T^{-\theta(x)}$ on the ferromagnetic side $x > 0$ with a continuously varying exponent $\theta(x) = x(2 - x)/3$. Similarly, the staggered susceptibility behaves as $\chi_{st} \simeq T^{-\theta(-x)}$ on the antiferromagnetic side $x < 0$. $\chi_{st}$ (resp. $\chi_{imp}$) is finite for $x > 0$ (resp. $x < 0$). The low-temperature behaviour of the specific heat is governed by the leading irrelevant perturbations compatible with all symmetries of the model. At the decoupled impurities point, one has two such operators, of dimension $3/2$ which read $(\vec{S}_1 \pm \vec{S}_2).((\vec{J}_\infty \pm \vec{J}_c)$. In the bosonised form, they involve $de^{-i\sqrt{3}\Phi} + h.c$ and $d^+e^{-i\Phi}/\sqrt{3} \sum_a \xi_a J_a + h.c$, where $\xi_a$ is the adjoint operator of the $\tilde{SU}_4(2)$ algebra. For a non-zero $x$, they are changed into $de^{-i(\sqrt{3}+x)/\sqrt{3}\Phi} + h.c$ and $d^+e^{-i\Phi(1-x)}/\sqrt{3} \sum_a \xi_a J_a + h.c$, respectively. Hence the leading contribution to $C/T$ is due to the latter for $x > 0$, and to the former for $x < 0$. This leads to $C/T \simeq T^{-\theta(x)}$ and to a universal ($x$-dependent) Wilson ratio for $x > 0$, while a different behaviour $C/T \simeq T^{-\alpha(x)}$ with $\alpha(x) = -x(6 + x)/3$ is found for $x < 0$. Note that this yields $\alpha = 0$ at $x = x_c = \sqrt{6} - 3$, corresponding to the first operator above becoming marginal. We expect that, for $x < x_c$, the system flows to the ‘strong-antiferromagnetic’ (Fermi liquid) fixed point where the two impurities bind into a singlet state, as found in [6].

Finally, we report on some results on the effect of a non-zero coupling $J_-$ (even-odd asymmetry). It can be shown that it gives rise to a relevant perturbation on the ferromagnetic side ($x > 0$), thus destabilizing the marginal line in favour of a spin-1, 2-channel Fermi liquid fixed point, as found in [8]. For antiferromagnetic RKKY interactions ($x < 0$), the leading operators generated are marginal, and the line extends into a surface of non-Fermi liquid fixed points with continuously varying properties, in agreement with recent NRG findings [7]. These will be analysed in detail in a forthcoming work [17], together with the effect of particle-hole symmetry breaking.

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Table Captions

Table 1
Primary operators of the $A(2,2)$ algebra. $(\Delta)$ labels an operator of dimension $\Delta$. The second line displays the bosonic representation of the operator (when it exists). $NS$ and $R$ stand for the Neveu-Schwarz and Ramond sectors of the algebra.

Table 2
Finite-size spectrum of low-lying states. $j$ is the total spin quantum number, $j_1$ (resp. $j_2$) is the $SU(2)_{flavour}$ quantum number for $l = 1$ (resp. $l = 2$), and $Q_1$ (resp. $Q_2$) is the charge. The sixth column displays the $A(2,2)$ operator associated with each eigenstate at the decoupled impurities fixed point ($I = 0$, i.e $x = 0$), whereas the eighth column displays the corresponding operator at the strong ferromagnetic fixed point ($I = -\infty$ i.e $x = 1$). The degeneracy of each state is displayed in the last column, while $\Delta_{tot}$ is the total conformal dimension at arbitrary $x$ entering the formula for the excitation energy given in the text (eq.(16).
\[
e^{\pm i \frac{\Phi}{\sqrt{3}}} \quad e^{\pm i \frac{2\Phi}{\sqrt{3}}} \quad e^{\pm i \frac{\sqrt{3}\Phi}{2}} \quad \partial \Phi
\]

Table 1

| j | j_1 | Q_1 | Q_2 | A(2, 2) decoupled | \(\Delta_{\text{tot}}\) | A(2, 2) strongferro. | Deg. |
|---|---|---|---|-----------------|----------------|---------------------|-----|
| 0 | 0 | 0 | 0 | 0 | (\frac{3}{8}) | \(\frac{3}{8}(1 + \frac{x}{3})^2\) | (\frac{1}{6} + \frac{1}{2}) | 1 |
| 1 | 0 | 0 | 0 | 0 | (\frac{1}{24}) | \(\frac{1}{3} + \frac{(1-x)^2}{24}\) | (0) | 3 |
| 0 | 0 | 0 | 0 | 0 | (\frac{1}{16}) | \frac{1}{2} | (\frac{1}{16}) | 8 |
| 0 | 0 | 0 | 0 | 0 | (\frac{1}{16}) | \frac{1}{2} | (\frac{1}{16}) | 8 |
| 0 | 0 | 0 | 0 | 0 | (\frac{5}{8} + \frac{x^2}{24}) | (\frac{1}{24}) | 8 |
| 1 | 0 | 0 | 0 | 0 | (\frac{5}{8} + \frac{x^2}{24}) | (\frac{1}{24}) | 8 |
| 0 | 0 | 0 | 0 | 0 | (\frac{5}{8} + \frac{x^2}{24}) | (\frac{1}{24}) | 8 |
| 0 | 0 | 0 | 0 | 0 | (\frac{5}{8} + \frac{x^2}{24}) | (\frac{1}{24}) | 8 |
| 1 | 0 | 0 | 0 | 0 | (\frac{5}{8} + \frac{x^2}{24}) | (\frac{1}{24}) | 8 |
| 0 | 0 | 0 | 0 | 0 | (\frac{5}{8} + \frac{x^2}{24}) | (\frac{1}{24}) | 8 |

Table 2