Lie algebras admitting a metacyclic Frobenius group of automorphisms

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to Victor Danilovich Mazurov on the occasion of his 70th birthday

Аннотация

Suppose that a Lie algebra $L$ admits a finite Frobenius group of automorphisms $FH$ with cyclic kernel $F$ and complement $H$ such that the characteristic of the ground field does not divide $|H|$. It is proved that if the subalgebra $C_L(F)$ of fixed points of the kernel has finite dimension $m$ and the subalgebra $C_L(H)$ of fixed points of the complement is nilpotent of class $c$, then $L$ has a nilpotent subalgebra of finite codimension bounded in terms of $m$, $c$, $|H|$, and $|F|$ whose nilpotency class is bounded in terms of only $|H|$ and $c$. Examples show that the condition of the kernel $F$ being cyclic is essential.

Key words. Frobenius groups, automorphism, Lie algebras, nilpotency class

1 Introduction

Recall that a finite Frobenius group $FH$ with kernel $F$ and complement $H$ is a semidirect product of a normal subgroup $F$ and a subgroup $H$ in which every element of $H$ acts without non-trivial fixed points on $F$, that is, $C_F(h) = 1$ for all $h \in H \setminus \{1\}$. The structure of Frobenius groups is well known. In particular, all abelian subgroup of $H$ are cyclic, and if $F$ is a cyclic group, then $H$ is also cyclic.

Mazurov’s problem 17.72 in “Kourovka Notebook” [1] gave rise to a number of recent papers, where groups $G$ are considered admitting a Frobenius group of automorphisms $FH$ with kernel $F$ and complement $H$ such that $F$ acts without fixed points, $C_G(F) = 1$. The goal of these papers [2, 3, 15, 6, 7, 8, 9, 10] are restrictions on the order, rank, nilpotent length, nilpotency class, and exponent of the group $G$ in terms of the corresponding
properties and parameters of the centralizer $C_G(H)$ and order $|H|$. For estimating the nilpotency class of the group $G$ in the case of nilpotent centralizer of the complement $C_G(H)$, Lie ring methods are used. The corresponding theorems on Lie rings and algebras $L$ with a Frobenius group of automorphisms $FH$ such that $C_L(F) = 0$ are also important in their own right.

A natural and important generalization of this situation is consideration of groups and Lie rings with a Frobenius group of automorphisms $FH$ such that its kernel $F$ has bounded cardinality or dimension of the set of fixed points. Then the goal is obtaining similar restrictions on a subgroup or a subalgebra of bounded index or codimension. In the present paper we consider the case of Lie algebras, for which strong bounds are obtained for the nilpotency class of a subalgebra of bounded codimension.

Suppose that a Lie algebra $L$ of arbitrary, not necessarily finite, dimension admits a finite Frobenius group of automorphisms $FH$ with cyclic kernel $F$ and complement $H$ such that the subalgebra $C_L(H)$ of fixed points of the complement is nilpotent of class $c$. If $C_L(F) = 0$, that is, the Frobenius kernel $F$ acts regularly (without non-trivial fixed points) on $L$, then by the Makarenko–Khukhro–Shumyatsky theorem [5, 6] the Lie algebra $L$ is nilpotent of class bounded by some function depending only on $|H|$ and $c$. In this paper we generalize the Makarenko–Khukhro–Shumyatsky theorem to the case where the Frobenius kernel $F$ acts “almost regularly” on $L$. We prove that if the dimension of $C_L(F)$ is finite and the characteristic $L$ does not divide $|H|$, then $L$ is almost nilpotent with estimates for the codimension of a nilpotent subalgebra and for its nilpotency class.

**Theorem 1.1.** Let $FH$ be a Frobenius group with cyclic kernel $F$ of order $n$ and complement $H$ of order $q$. Suppose that $FH$ acts by automorphisms on a Lie algebra $L$ of characteristic that does not divide $q$ in such a manner that the fixed point subalgebra $C_L(F)$ of the kernel has finite dimension $m$ and the fixed point subalgebra $C_L(H)$ of the complement is nilpotent of class $c$. Then $L$ has a nilpotent subalgebra of finite codimension bounded by some function depending only on $m$, $n$, $q$, and $c$, whose nilpotency class is bounded by some function depending only on $q$ and $c$.

There are examples showing that the result is not true if the kernel $F$ is not cyclic (see examples in [5]). The functions of $m$, $n$, $q$, $c$ and of $q$ and $c$ in Theorem 1.1 can be estimated from above explicitly, although we do not write out these estimates here.

The proof of Theorem 1.1 uses the method of generalized, or graded, centralizers, which was originally created in [11] for almost regular automorphisms of prime order, see also [12, 13, 14, 15] and Ch. 4 in [16]. This method consists in the following. In the proof of Theorem 1.1 we can assume that the ground field contains a primitive $n$th root of unity $\omega$. Let $F = \langle \varphi \rangle$. Then $L$ decomposes into the direct sum of eigenspaces $L_j = \{a \in L \mid a^\varphi = \omega^j a\}$, which are also components of a $(\mathbb{Z}/n\mathbb{Z})$-grading: $[L_s, L_t] \subseteq L_{s+t}$, where $s + t$ is calculate modulo $n$. In each of the $L_i$, $i \neq 0$, certain subspaces $L_i(k)$ of bounded codimension — “graded centralizers” — of increasing levels $k$ are successively constructed, and simultaneously certain elements (representatives) $x_i(k)$ are fixed, all this up to a certain $(c, q)$-bounded level. Elements of $L_j(k)$ have a centralizer property with respect to the fixed elements of lower levels: if a commutator (of bounded weight) that involves exactly one element $y_j(k) \in L_j(k)$ of level $k$ and some fixed elements $x_i(s) \in L_i(s)$ of lower levels $s < k$ belongs to $L_0$, then this commutator is equal to 0. The sought-for subalgebra is
the subalgebra $Z$ generated by all the $L_i(T)$, $i \neq 0$, of highest level $T$. The proof of the fact that the subalgebra $Z$ is nilpotent of bounded class is based on Proposition 3.3 which is a combinatorial consequence of the Makarenko–Khukhro–Shumyatsky theorem \[5, 6\] and reduces the question of nilpotency to consideration of commutators of a special form. Various collecting processes applied here and other arguments are based precisely on the aforementioned centralizer property.

Results on Lie algebras (rings) with Frobenius groups of automorphisms are applicable to various classes of groups. In particular, it follows from the Makarenko–Khukhro–Shumyatsky theorem \[5\] that if a finite group (or a locally nilpotent group, or a Lie group) $G$ admits a Frobenius group of automorphisms $FH$ with cyclic kernel $F$ of order $n$ and complement $H$ of order $q$ such that $C_G(F) = 1$ and $C_G(H)$ is nilpotent of class $c$, then $G$ is nilpotent of $(c, q)$-bounded class. Theorem 1.1 is also applicable to locally nilpotent torsion-free groups with a metacyclic Frobenius group of automorphisms (Theorem 7.2).

We briefly describe the plan of the paper. After recalling definitions and introducing notation in § 2 we firstly prove in § 3 combinatorial consequences of the Makarenko–Khukhro–Shumyatsky theorem (Theorem 3.2 and Proposition 3.3), which are key in the proof of the theorem. Then in § 4 and § 5 generalized centralizers and fixed elements are constructed and their basic properties are proved. This is based on the original construction in \[11\], which, however, had to be considerably modified in accordance with the hypotheses of the problem. In § 6 the sought-for subalgebra is constructed and the nilpotency of this subalgebra is proved. In § 7 Theorem 7.2 on locally nilpotent torsion-free groups is proved.

## 2 Preliminaries

We recall some definitions and notions. For brevity we say that a certain quantity is $(c, q)$-bounded (or, say, $(m, n, q, c)$-bounded) if it is bounded above by some function depending only $c$ and $q$ (respectively, only on $m$, $n$, $q$, and $c$).

Products in a Lie algebra are called “commutators”. We denote by $\langle S \rangle$ the Lie subalgebra generated by a subset $S$.

Terms of the lower central series of a Lie algebra $L$ are defined by induction: $\gamma_1(L) = L$; $\gamma_{i+1}(L) = [\gamma_i(L), L]$. By definition a Lie algebra $L$ is nilpotent of class $h$ if $\gamma_{h+1}(L) = 0$.

A simple commutator $[a_1, a_2, \ldots, a_s]$ of weight (length) $s$ is by definition the commutator $\ldots[[a_1, a_2], a_3], \ldots, a_s]$. By the Jacobi identity $[a, [b, c]] = [a, b, c] - [a, c, b]$ any (complex, repeated) commutator in some elements in any Lie algebra can be expressed as a linear combination of simple commutators of the same weight in the same elements. Using also the anticommutativity $[a, b] = -[b, a]$, one can make sure that in this linear combination all simple commutators begin with some pre-assigned element occurring in the original commutator. In particular, if $L = \langle S \rangle$, then the space $L$ is generated by simple commutators in elements of $S$.

Let $A$ be an additively written abelian group. A Lie algebra $L$ is $A$-graded if

$$L = \bigoplus_{a \in A} L_a \quad \text{and} \quad [L_a, L_b] \subseteq L_{a+b}, \quad a, b \in A,$$
where \( L_a \) are subspaces of \( L \). Elements of the subspaces \( L_a \) are called \textit{homogeneous}, and commutators in homogeneous elements \textit{homogeneous commutators}. A subspace \( H \) of the space \( L \) is said to be \textit{homogeneous} if \( H = \bigoplus_a (H \cap L_a) \); then we set \( H_a = H \cap L_a \). Obviously, any subalgebra or an ideal generated by homogeneous subspaces is homogeneous. A homogeneous subalgebra and the quotient algebra by a homogeneous ideal can be regarded as \( A \)-graded algebras with induced grading.

Suppose that a Frobenius group \( FH \) with cyclic kernel \( F = \langle \varphi \rangle \) of order \( n \) and complement \( H \) of order \( q \) acts on a Lie algebra \( L \) in such a way that the subalgebra of fixed points \( C_L(F) \) has finite dimension \( \dim C_L(F) = m \), and the subalgebra of fixed points \( C_L(H) \) is nilpotent of class \( c \).

Let \( \omega \) be a primitive \( n \)th root of unity. We extend the ground field by \( \omega \) and denote by \( \tilde{L} \) the algebra over the extended field. The group \( FH \) naturally acts on \( \tilde{L} \), and the subalgebra of fixed points \( C_{\tilde{L}}(H) \) is nilpotent of class \( c \), while the subalgebra of fixed points \( C_{\tilde{L}}(F) \) has dimension \( m \).

**Definition.** We define \textit{\( \varphi \)-homogeneous components} \( L_k \) for \( k = 0, 1, \ldots, n-1 \) as the eigensubspaces

\[ L_k = \{ a \in L \mid a^{\varphi} = \omega^k a \}. \]

It is known that if the characteristic of the field does not divide \( n \), then

\[ L = L_0 \oplus L_1 \oplus \cdots \oplus L_{n-1} \]

(see, for example, Ch. 10 in the book [17]). This decomposition is a \((\mathbb{Z}/n\mathbb{Z})\)-grading due to the obvious inclusions

\[ [L_s, L_t] \subseteq L_{s+t \mod n}, \]

where \( s + t \) is calculated modulo \( n \).

**Index Convention.** Henceforth a small letter with index \( i \) will denote an element of the \( \varphi \)-homogeneous component \( L_i \), here the index will only indicate the \( \varphi \)-homogeneous component to which this element belongs: \( x_i \in L_i \). To lighten the notation we will not use numbering indices for elements in \( L_i \), so that different elements can be denoted by the same symbol when it only matters to which \( \varphi \)-homogeneous component these elements belong. For example, \( x_1 \) and \( x_1 \) can be different elements of \( L_1 \), so that \( [x_1, x_1] \) can be a nonzero element of \( L_2 \). These indices will be usually considered modulo \( n \); for example, \( a_{-i} \in L_{-i} = L_{n-i} \).

Note that in the framework of the Index Convention a \( \varphi \)-homogeneous commutator belongs to the \( \varphi \)-homogeneous component \( L_s \), where \( s \) is the sum modulo \( n \) of the indices of all the elements occurring in this commutator.

### 3 Combinatorial theorem

In this section we prove a certain combinatorial fact that follows from the following Makarenko–Khukhro–Shumyatsky theorem [5].

**Theorem 3.1** (Makarenko–Khukhro–Shumyatsky [5]). Let \( FH \) be a Frobenius group with cyclic kernel \( F \) of order \( n \) and complement \( H \) of order \( q \). Suppose that \( FH \) acts
by automorphisms on a Lie algebra $L$ in such a way that $C_L(F) = 0$ and the subalgebra of fixed points $C_L(H)$ is nilpotent of class $c$. Then for some $(q,c)$-bounded number $f = f(q,c)$ the algebra $L$ is nilpotent of class at most $f$.

We consider a Frobenius group $FH$ with cyclic kernel $F = \langle \varphi \rangle$ of order $n$ and complement $H$ of order $q$ that acts on a Lie algebra $L$ in such a way that the subalgebra of fixed points $C_L(F)$ has finite dimension $m$ and the subalgebra of fixed points $C_L(H)$ is nilpotent of class $c$. Since the kernel $F$ of the Frobenius group $FH$ is a cyclic subgroup, the subgroup $H$ is also cyclic. Let $H = \langle h \rangle$ and $\varphi^{h^{-1}} = \varphi^r$ for some $1 \leq r \leq n - 1$. Then $r$ is a primitive $q$th root of unity in the ring $\mathbb{Z}/n\mathbb{Z}$ and, moreover, the image of the element $r$ in $\mathbb{Z}/d\mathbb{Z}$ is a primitive $q$th root of unity for every divisor $d$ of the number $n$, since $h$ acts without non-trivial fixed points on every subgroup of the group $F$.

The group $H$ permutes the homogeneous components $L_i$ as follows: $L_i^h = L_{ri}$ for all $i \in \mathbb{Z}/n\mathbb{Z}$. Indeed, if $x_i \in L_i$, then $(x_i^h)^r = x_i^{h\varphi^{-1}^h} = (x_i^{\varphi^r})^h = \omega^r x_i^h$.

In what follows, for a given $u_k \in L_k$ we denote the element $u_k^h$ by $u_{rk}$ in the framework the Index Convention, since $L_k^{h^s} = L_{rk^s}$. Since the sum over any $H$-orbit belongs to the centralizer $C_L(H)$, we have $u_k + u_{rk} + \cdots + u_{rn-1k} \in C_L(H)$.

**Theorem 3.2.** Let $FH$ be a Frobenius group with cyclic kernel $F = \langle \varphi \rangle$ of order $n$ and complement $H = \langle h \rangle$ of order $q$ and let $\varphi^{h^{-1}} = \varphi^r$ for some positive integer $1 \leq r \leq n - 1$. Let $f(q,c)$ be the function in Theorem 3.1, let $F$ be a field containing a primitive $n$th root of unity the characteristic of which does not divide $q$ and $n$, and let $L$ be a Lie algebra over $F$. Suppose that $FH$ acts by automorphisms on $L$ in such a way that the subalgebra of fixed points $C_L(H)$ is nilpotent of class $c$ and $L = \bigoplus_{i=0}^{n-1} L_i$, where $L_i = \{ x \in L \mid x^\varphi = \omega^ix \}$ are $\varphi$-homogeneous components (eigensubspaces for eigenvalues $\omega^i$). Then any $\varphi$-homogeneous commutator $[x_{i_1}, x_{i_2}, \ldots, x_{i_T}]$ with non-zero indices of weight $T = f(q,c) + 1$ can be represented as a linear combination of $\varphi$-homogeneous commutators of the same weight $T$ each of which, for every $s = 1,\ldots,T$, includes exactly the same number of elements of the orbit

$$O(x_{i_s}) = \{x_{i_s}, x_{i_s}^h = x_{ri_s}, \ldots, x_{i_s}^{h^{s-1}} = x_{ri_s-1}\}$$

as the original commutator, and contains a subcommutator with zero sum of indices modulo $n$.

Доказательство. The idea of the proof consists in application of Theorem 3.1 to a free Lie algebra with operators $FH$. Let $F$ be a field containing a primitive $n$th root of unity the characteristic of which does not divide $q$ and $n$, and let $n, q, r, T$ be the numbers in the hypothesis of Theorem 3.2. In the ring $\mathbb{Z}/n\mathbb{Z}$ we choose arbitrary non-zero (not necessarily distinct) elements $i_1, i_2, \ldots, i_T \in \mathbb{Z}/n\mathbb{Z}$. We consider a free Lie algebra $K$ over the field $F$ with $qT$ free generators in the set

$$Y = \{y_{i_1}, y_{ri_1}, \ldots, y_{ri_1-1}, y_{i_2}, y_{ri_2}, \ldots, y_{ri_2-1}, \ldots, y_{i_T}, y_{ri_T}, \ldots, y_{ri_T-1}\}$$

where the subsets $O(y_{i_s}) = \{y_{i_s}, y_{ri_s}, \ldots, y_{ri_s-1}\}$ are called the $r$-orbits of the elements $y_{i_1}, y_{i_2}, \ldots, y_{i_T}$. Here, as in the Index Convention, we do not use numbering indices, that
is, all elements $y_{ri}$ are by definition different free generators, even if indices coincide. (The Index Convention will come into force in a moment.) For every $i = 0, 1, \ldots, n - 1$ we define the subspace $K_i$ of the algebra $K$ generated by all commutators in the generators $y_{ri}$ in which the sum of indices of the elements occurring in them is equal to $i$ modulo $n$. Then $K = K_0 \oplus K_1 \oplus \cdots \oplus K_{n-1}$. It is also obvious that $[K_i, K_j] \subseteq K_{i+j \pmod n}$; therefore this is a $(\mathbb{Z}/n\mathbb{Z})$-grading. The Lie algebra $K$ also has the natural $\mathbb{N}$-grading with respect to the generating set $Y$:

$$K = \bigoplus_i G_i(Y),$$

where $G_i(Y)$ is the subspace generated by all commutators of weight $i$ in elements of the generating set $Y$.

We define an action of the Frobenius group $FH$ on $K$. We set $k_i^\varphi = \omega^i k_i$ for $k_i \in K_i$ and extend this action to $K$ by linearity. Since $K$ is the direct sum of homogeneous $\varphi$-components and the characteristic of the ground field does not divide $n$, we have

$$K_i = \{ k \in K \mid k^\varphi = \omega^i k \},$$

that is, $K_i$ is the eigensubspace for the eigenvalue $\omega^i$. An action of the subgroup $H$ is defined on the generating set $Y$ as follows: $H$ cyclically permutes the elements of the $r$-orbits $O(y_{ri})$, $s = 1, \ldots, T$:

$$(y_{r+ki})^h = y_{r+k+1i}, \quad k = 0, \ldots, q - 2; \quad (y_{r+s-1i})^h = y_{ri}.$$ 

Thus, the $r$-orbit of an element $y_{ri}$ is also the $H$-orbit of this element. Clearly, $H$ permutes the components $K_i$ according to the following rule: $K_i^h = K_{ri}$ for all $i \in \mathbb{Z}/n\mathbb{Z}$.

Let $J = \text{id}(K_0)$ be the ideal generated by the $\varphi$-homogeneous component $K_0$. By definition the ideal $J$ consists of all linear combinations of commutators in elements of $Y$ each of which contains a subcommutator with zero sum of indices modulo $n$. Clearly, the ideal $J$ is generated by homogeneous elements with respect to the gradings $K = \bigoplus G_i(Y)$ and $K = \bigoplus_{i=0}^{n-1} K_i$ and, consequently, is homogeneous with respect to both gradings, that is,

$$J = \bigoplus_i J \cap G_i(Y) = \bigoplus_{i=0}^{n-1} J \cap K_i.$$

Note also that the ideal $J$ is obviously $FH$-invariant.

Let $I = \text{id}(\gamma_{c+1}(C_K(H)))^F$ be the smallest $F$-invariant ideal containing the subalgebra $\gamma_{c+1}(C_K(H))$ (this ideal can be called the $F$-closure of the ideal generated by this subalgebra). We claim that the ideal $I$ is homogeneous with respect to the grading $K = \bigoplus_{i=0}^{n-1} K_i$. Since $q$ is not divisible by the characteristic of the ground field $F$, we have the equality $C_K(H) = \{ a + a^h + \cdots + a^{h^{q-1}} \mid a \in K \}$. It is easy to see that the ideal $I$ consists of linear combinations of all possible elements of the form

$$\left[ (u_a + u_a^h + \cdots + u_a^{h^{q-1}}), (v_b + \cdots + v_b^{h^{q-1}}), \ldots, (w_d + \cdots + w_d^{h^{q-1}}), y_{j_1}, y_{j_2}, \ldots \right]^\varphi,$$

where $u_a, v_b, \ldots, w_d$ are $\varphi$-homogeneous commutators (possibly, of different weights) in elements of $Y$ and $y_{j_1}, y_{j_2}, \ldots \in Y$. 

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We continue using the fact that $H$ permutes the components $K_i$ by the rule $K_i^h = K_{ri}$ for all $i \in \mathbb{Z}/n\mathbb{Z}$, and denote $a_{ik}^h$ by $a_{ik}$ (under the Index Convention). It is important that then the image of a commutator in elements of the generating set $Y$ under the action of the automorphism $h$ is again a commutator in elements of $Y$. Rewriting (1) in the new notation we obtain that the ideal $I$ consists of linear combinations of all possible elements of the form

$$\left[(u_a + \cdots + u_{n-1}), (v_b + \cdots + v_{n-1}), \ldots, (w_d + \cdots + w_{n-1}), y_{j_1}, y_{j_2}, \ldots\right]^\varphi,$$

where $u_a, v_b, \ldots, w_d$ are homogeneous commutators (possibly, of different weights) in elements of the set $Y$ and $y_{j_1}, y_{j_2}, \ldots \in Y$.

We denote the element (2) by $z$ and represent it as a sum of $\varphi$-homogeneous elements of the form $k_0 + k_1 + \cdots + k_{n-1}$, where $k_i \in K_i$. For every $i = 0, \ldots, n-1$ we set $z_i = \sum_{s=0}^{n-1} \omega^{-is}z^s$. It is easy to verify that $z_i$ belongs to the eigensubspace for the eigenvalue $\omega^i$, that is, in $K_i$. Furthermore, $nz = \sum_{s=0}^{n-1} z_i$. Since the characteristic of the field does not divide $n$, the element $n$ is invertible in the field $\mathbb{F}$, that is, $z = 1/n \sum_{s=0}^{n-1} z_i$. By comparing the two representations of $z$ we obtain that $k_i = (1/n)z_i = (1/n)\sum_{s=0}^{n-1} \omega^{-is}z^s$. But the element $(1/n)\sum_{s=0}^{n-1} \omega^{-is}z^s$, being a linear combination of the elements $z, z^2, \ldots, z^{n-1}$ in $I$, also belongs to $I$. Consequently, $k_i \in I$, that is, the ideal $I$ is homogeneous with respect to the grading $K = \bigoplus_{i=0}^{n-1} K_i$.

Note that $I$ is also homogeneous with respect to the grading $K = \bigoplus_{i=0}^{n} G_i(Y)$ and is $FH$-invariant.

We consider the quotient Lie algebra $M = K/(J + I)$. Since the ideals $J$ and $I$ are homogeneous with respect to the gradings $K = \bigoplus_{i=0}^{n} G_i(Y)$ and $K = \bigoplus_{i=0}^{n-1} K_i$, the quotient algebra $M$ has the corresponding induced gradings. The group $FH$ acts on $M$ in such a way that $C_M(F) = 0$ and $\gamma_{c+1}(C_M(H)) = 0$. By Theorem 3.1 the quotient algebra $K/(J + I)$ is nilpotent of $(q, c)$-bounded class $f = f(q, c)$. Consequently,

$$[y_{i_1}, y_{i_2}, \ldots, y_{i_T}] \in J + I = \text{id}(K_0) + \text{id}(\gamma_{c+1}(C_K(H)))^{F}.$$

This means that the commutator $[y_{i_1}, y_{i_2}, \ldots, y_{i_T}]$ can be represented modulo the ideal $I$ as a linear combination of commutators of weight $T$ in elements of $Y$ belonging to the $\varphi$-homogeneous component $K_{i_1+i_2+\cdots+i_T}$ that contain a subcommutator with zero sum of indices modulo $n$. It is claimed that for every $s = 1, \ldots, T$ any such commutator includes exactly one element of the orbit

$$O(y_{i_s}) = \{y_{i_s}, y_{i_s}^h, \ldots, y_{i_s}^{h^{s-1}}\}.$$

For every $s = 1, \ldots, T$ we consider the homomorphism $\theta_s$ extending the mapping

$$O(y_{i_s}) \to 0; \quad y_{i_k} \to y_{i_k} \quad \text{if} \quad k \neq s.$$

Clearly, the kernel $\text{Ker} \theta_s$ is equal to the ideal generated by the orbit $O(y_{i_s})$. Furthermore, clearly, the ideal $I$ is invariant under $\theta_s$ (as any homogeneous ideal). We apply the homomorphism $\theta_s$ to the commutator $[y_{i_1}, y_{i_2}, \ldots, y_{i_T}]$ and its representation modulo
As a linear combination of commutators in elements of $O$ of weight $T$ that contain a subcommutator with zero sum of indices modulo $n$. We obtain that the image of $\theta_j([y_{i_1}, y_{i_2}, \ldots, y_{i_T}])$ is equal to 0, as well as the image of any commutator containing elements of the orbit $O(y_{i_s})$. Hence the sum of all those commutators in the representation of the element $[y_{i_1}, y_{i_2}, \ldots, y_{i_T}]$ that do not contain elements of the orbit $O(y_{i_s})$ is equal to zero, and we can exclude all these commutators from our consideration. By applying consecutively $\theta_s$, $s = 1, \ldots, T$, we have obtained modulo $I$ a linear combination of commutators each of which contains at least one element from every orbit $O(y_{i_s})$, $s = 1, \ldots, T$. Since under these transformations the weight of commutators remains the same and is equal to $T$, no other elements can appear, and every commutator will contain exactly one element in every orbit $O(y_{i_s})$, $s = 1, \ldots, T$.

Thus, we have proved that in the free Lie algebra $K$ generated by elements of the set $Y$ the commutator $[y_{i_1}, y_{i_2}, \ldots, y_{i_T}]$ can be represented modulo the ideal $I$ as a linear combination of commutators of weight $T$ in elements of $O$, and for every $s = 1, \ldots, T$ any of commutators of this linear combination contains exactly one element in the orbit $O(y_{i_s})$ and has a subcommutator with zero sum of indices modulo $n$.

Now suppose that $L$ is an arbitrary Lie algebra satisfying the hypothesis of Theorem 3.2. Let $x_{i_1}, x_{i_2}, \ldots, x_{i_T}$ be arbitrary $\varphi$-homogeneous elements with non-zero indices in the subspaces $L_{i_1}, L_{i_2}, \ldots, L_{i_T}$, respectively. We define the following homomorphism $\delta$ from the free Lie algebra $K$ into $L$:

$$\delta(y_{i_s}) = x_{i_s}, \ \delta(y_{i_s}^{x_{i_s}}) = x_{i_s}^{h_k} \text{ for } s = 1, \ldots, T; \ k = 1, \ldots, q - 1.$$

Then

$$\delta[y_{i_1}, y_{i_2}, \ldots, y_{i_T}] = [x_{i_1}, x_{i_2}, \ldots, x_{i_T}]; \ \delta(I) = 0; \ \delta(J) = \text{id}(L_0); \ \delta(O(y_{i_s})) = O(x_{i_s}).$$

By applying $\delta$ to the representation of the commutator $[y_{i_1}, y_{i_2}, \ldots, y_{i_T}]$ constructed above, as the image we obtain a representation of the commutator $[x_{i_1}, x_{i_2}, \ldots, x_{i_T}]$ as a linear combination of commutators in elements of the set $X = O(x_{i_1}) \cup O(x_{i_2}) \cup \cdots \cup O(x_{i_T})$. Since $\delta(I) = 0$, every commutator in this linear combination has weight $T$, contains exactly the same number of elements from every orbit $O(x_{i_s})$, $s = 1, \ldots, T$, as the original commutator, and has a subcommutator with zero sum of indices modulo $n$. The theorem is proved.

We define a MKhSh-transformation of a commutator $[x_{i_1}, x_{i_2}, \ldots, x_{i_T}]$ its representation according to Theorem 3.2 as linear combination of simple commutators in elements of $X = O(x_{i_1}) \cup O(x_{i_2}) \cup \cdots \cup O(x_{i_T})$ that contain exactly the same number of elements from every orbit $O(x_{i_s})$, $s = 1, \ldots, T$, as the original commutator and have initial segment from $L_0$ of weight $\leq T = f(q, c) + 1$, that is, commutators of the form

$$[c_0, y_{j_{w+1}}, \ldots, y_{j_w}],$$

where

$$c_0 = [y_{j_1}, \ldots, y_{j_w}] \in L_0, \ w \leq T, \ j_1 + j_2 + \cdots + j_w = 0 \ (\text{mod } n), \ y_{j_k} \in X.$$
with subsequent re-denoting
\[ z_{i_1} = -[c_0, y_{j_{w+1}}], \quad z_{i_s} = y_{j_{w+s}} \text{ for } s > 1. \]

The following assertion is obtained by repeated application of the MKhSh-transformation.

**Proposition 3.3.** Let \( FH \) be a Frobenius group with cyclic kernel \( F = \langle \varphi \rangle \) of order \( n \) and complement \( H = \langle h \rangle \) of order \( q \), and let \( \varphi^{h^{-1}} = \varphi^r \) for some \( 1 \leq r \leq n-1 \). Let \( \mathbb{F} \) be a field containing a primitive \( n \)-th root of unity the characteristic of which does not divide \( q \) and \( n \), and let \( L \) be a Lie algebra over \( \mathbb{F} \). Suppose that \( FH \) acts by automorphisms on \( L \) in such a way that the subalgebra of fixed points \( C_L(H) \) is nilpotent of class \( c \) and \( L = \bigoplus_{i=0}^{n-1} L_i \), where \( L_i = \{ x \in L \mid x^\varphi = \omega^i x \} \) are eigensubspace for eigenvalues \( \omega^i \) of the automorphism \( \varphi \). Then for any positive integers \( t_1 \) and \( t_2 \) there exists a \((t_1, t_2, q, c)\)-bounded positive integer \( V = V(t_1, t_2, q, c) \) such that any commutator in \( \varphi \)-homogeneous elements \([x_{i_1}, x_{i_2}, \ldots, x_{i_V}]\) with non-zero indices of weight \( V \) can be represented as a linear combination of \( \varphi \)-homogeneous commutators in elements of the set \( X = \bigcup_{s=1}^{V} O(x_{i_s}) \), where

\[ O(x_{i_s}) = \{ x_{i_s}, \ x_{i_s}^h = x_{r_{i_s}}, \ldots, x_{i_s}^{h^{s-1}} = x_{r_{i_s}^{-1}} \}, \]

and every such commutator either has a subcommutator of the form

\[ [u_{k_1}, \ldots, u_{k_s}], \tag{3} \]

where there are \( t_1 \) different initial segments with zero sum of indices modulo \( n \), that is,

\[ k_1 + k_2 + \cdots + k_{r_i} \equiv 0 \pmod{n}, \quad i = 1, 2, \ldots, t_1, \]

\[ 1 < r_1 < r_2 < \cdots < r_{t_1} = s, \]

or has a subcommutator of the form

\[ [u_{k_0}, c_{i_1}, \ldots, c_{i_2}], \tag{4} \]

where \( u_{k_0} \in X \), every \( c_i \) belongs to \( L_0 \), \( i = 1, \ldots, t_2 \), and has the form

\[ [x_{k_1}, \ldots, x_{k_j}], \quad x_{k_j} \in X \]

with zero sum of indices modulo \( n \)

\[ k_1 + \cdots + k_i \equiv 0 \pmod{n}. \]

Here we can set \( V(t_1, t_2, c, q) = \sum_{i=1}^{t_1} ((f(q, c) + 1)^2 t_2)^i + 1. \)

Доказательство. The proof practically word-for-word repeats the proof of a proposition in [11] (see also Proposition 4.4.2 in [16]), but instead of the HKK-transformation one should repeatedly apply the MKhSh-transformation. In contrast to the HKK-transformation, which always produces commutator in the original elements \( x_{i_1}, x_{i_2}, \ldots, x_{i_V} \), in our case after the MKhSh-transformation in commutators there may appear some images of the elements \( x_{i_1}, x_{i_2}, \ldots, x_{i_V} \) under the action of the automorphism \( h \). This detail does not affect the course of the proof, but it precisely why the conclusion of Proposition 3.3 involves commutators in elements of the \( h \)-orbits of the original elements. 

\[ \square \]
4 Representatives and generalized centralizers

Let $FH$ be a Frobenius group with kernel $F = \langle \varphi \rangle$ of order $n$ and complement $H = \langle h \rangle$ of order $q$, and let $\varphi^{h^{-1}} = \varphi^r$ for some $1 \leq r \leq n - 1$. Suppose that the group $FH$ acts by automorphisms on a Lie algebra $L$, and the subalgebra $C_L(H)$ of fixed points of the complement is nilpotent of class $c$, while the subalgebra $L_0 = C_L(\varphi)$ of fixed points of the kernel has finite dimension $m$. First suppose that $L = \bigoplus_{i=0}^{q-1} L_i$, where $L_i = \{ x \in L \mid x^r = \omega^i x \}$ are eigensubspace for eigenvalues $\omega^i$ of the automorphism $\varphi$ (which is actually the main case).

We begin construction of generalized centralizers by induction on the level — a parameter taking integer values from 0 $T$, where the number $T = T(q, c) = f(q, c) + 1$ is defined in Theorem 3.2. A generalized centralizer $L_j(s)$ of level $s$ is a certain subspace of the $\varphi$-homogeneous component $L_j$. Simultaneously with construction of generalized centralizers we fix certain elements of them — representatives of various levels, — the total number of which is $(m, n, q, c)$-bounded.

**Definition.** The pattern of a commutator in $\varphi$-homogeneous elements (in $L_i$) is defined as its bracket structure together with the arrangement of indices under the Index Convention. The weight of a pattern is the weight of the commutator. The commutator itself is called the value of its pattern on given elements.

**Definition.** Let $\vec{x} = (x_{i_1}, \ldots, x_{i_k})$ be some ordered tuple of elements $x_{i_s} \in L_{i_s}$, $i_s = 1, \ldots, n - 1$, such that $i_1 + \cdots + i_k \equiv 0 \pmod{n}$. We set $j = -i_1 - \cdots - i_k \pmod{n}$ and define the mappings

$$\vartheta_{\vec{x}} : y_j \rightarrow [y_j, x_{i_1}, \ldots, x_{i_k}].$$

(5)

By linearity they all are homomorphisms of the subspace $L_j$ into $L_0$. Since $\dim L_0 = m$, we have $\dim(L_j/\ker\vartheta_{\vec{x}}) \leq m$.

**Notation.** Let $U = U(q, c)$ denote the number $V(T, T - 1, q, c)$, where $V$ is the function in the conclusion of Proposition 3.3.

**Definition of level 0.** At level 0 we only fix representatives of level 0. First, for every pattern $P$ of a simple commutator of weight $\leq U$ with indices $i \neq 0$ and zero sum of indices, among all values of this pattern $P$ on $\varphi$-homogeneous elements in $L_i$, $i \neq 0$ we choose commutators $c$ that form a basis of the subspace spanned by all values of this pattern on $\varphi$-homogeneous elements in $L_i$, $i \neq 0$. The elements of $L_j$, $j \neq 0$, occurring in these fixed representations of the commutators $c$ are called representatives of level 0. Representatives of level 0 are denoted by $x_j(0)$ under the Index Convention (Recall that the same symbol can denote different elements). Furthermore, together with every representative $x_j(0) \in L_j$, $j \neq 0$, we also fix all elements of the orbit $O(x_j(0))$ of this element under the action of the automorphism $h$

$$O(x_j(0)) = \{ x_j(0), x_j(0)^h, \ldots, x_j(0)^{h^{r-1}} \},$$

and also call them representatives of level 0. Elements of these orbits are denoted by $x_{rj}(0) := x_j(0)^{h^r}$ under the Index Convention (since $L_i^h \leq 0_{ri}$).
Since the total number of pattern $\mathbf{P}$ under consideration is $(n,q,c)$-bounded, the dimension of $L_0$ is at most $m$, and the number of elements in every $h$-orbit is equal to $q$, it follows that the number of representatives of level 0 is $(m,n,q,c)$-bounded.

**Definition of level 1.** We define the *generalized centralizers* $L_j(1)$ of level 1 by setting, for every $j \neq 0$,

$$L_j(1) = \bigcap_{\vec{x}} \ker \vartheta_{\vec{x}},$$

where $\vec{x} = (x_{i_1}(0), \ldots, x_{i_k}(0))$ runs over all possible ordered tuples of length $k$ for all $k \leq U$ consisting of representatives of level 0 such that $j + i_1 + \cdots + i_k \equiv 0 \pmod n$. Since the number of representatives of level 0 is $(m,n,q,c)$-bounded, the intersection here is taken over a $(m,n,q,c)$-bounded number of subspaces of codimension $\leq m$ in $L_j$. Hence $L_j(1)$ is a subspace of $(m,n,q,c)$-bounded codimension in $L_j$. For brevity we also call elements of $L_j(1)$ *centralizers of level 1* and fix for then the notation $y_j(1)$ (under the Index Convention).

By construction every element $y_j(1) \in L_j(1)$ has the centralizer property with respect to representatives of level 0:

$$[y_j(1), x_{i_1}(0), \ldots, x_{i_k}(0)] = 0,$$

as soon as $k \leq U$ and $j + i_1 + \cdots + i_k \equiv 0 \pmod n$.

We now fix representatives of level 1. For every pattern $\mathbf{P}$ of a simple commutator of weight $\leq U$ with nonzero indices and zero sum of indices modulo $n$, among all values of the pattern $\mathbf{P}$ on homogeneous elements in $L_i(1)$, $i \neq 0$, we choose commutators that form a basis of the subspace spanned by all values of this pattern on homogeneous elements in $L_i(1)$, $i \neq 0$. The elements occurring in these commutators are called *representatives of level 1* and are denoted by $x_j(1)$ (under the Index Convention). Furthermore, for every (already fixed) representative $x_j(1)$ of level 1 we fix all elements of the $h$-orbit

$$O(x_j(1)) = \{x_j(1), x_j(1)^h, \ldots, x_j(1)^{h^{r-1}}\},$$

and also call them *representatives of level 1*. These elements are denoted by $x_{r \kappa j}(1) := x_j(1)^{h^r}$ under the Index Convention (since $L_i^r \leq L_i$).

Since the number of pattern under consideration is $(n,q,c)$-bounded, and the dimension of the subspace $L_0$ is equal to $m$, the total number of representatives of level 1 is $(m,n,q,c)$-bounded.

**Definition of level $t > 1$.** Suppose that we have already fixed a $(m,n,q,c)$-bounded number of representatives of levels $< t$. We define *generalized centralizers* of level $t$ by setting, for every $j \neq 0$,

$$L_j(t) = \bigcap_{\vec{x}} \ker \vartheta_{\vec{x}},$$

where $\vec{x} = (x_{i_1}(\varepsilon_1), \ldots, x_{i_k}(\varepsilon_k))$ runs over all possible ordered tuples of all lengths $k \leq U$ consisting of representatives of (possibly different) levels $< t$ such that

$$j + i_1 + \cdots + i_k \equiv 0 \pmod n.$$ 

For brevity we also call elements of $L_j(t)$ *centralizers of level $t$* and fix for them the notation $y_j(t)$ (under the Index Convention).
The number of representatives of all levels \(< t\) is \((m,n,q,c)\)-bounded and \(\dim L_j/\ker \vartheta_x \leq m\) for all \(x\). Hence the intersection here is taken over a \((m,n,q,c)\)-bounded number of subspaces of codimension \(\leq m\) in \(L_j\), and therefore \(L_j(t)\) also has \((m,n,q,c)\)-bounded codimension in the subspace \(L_j\).

By definition a centralizer \(y_j(t)\) of level \(t\) has the following centralizer property with respect to representatives of lower levels:

\[
[y_j(t), x_{i_1}(\varepsilon_1), \ldots, x_{i_k}(\varepsilon_k)] = 0,
\]

as soon as \(j + i_1 + \cdots + i_k \equiv 0 \pmod{n}\), \(k \leq U\), and the elements \(x_{i_s}(\varepsilon_s)\) are representatives of any (possibly different) levels \(\varepsilon_s < t\).

We now fix representatives of level \(t\). For every pattern \(P\) of a simple commutator of weight \(\leq U\) with nonzero indices and zero sum of indices, among all values of the pattern \(P\) on \(\varphi\)-homogeneous elements in \(L_i(t), i \neq 0\), we choose commutators that form a basis of the subspace spanned by all the values of the pattern \(P\) on \(\varphi\)-homogeneous elements in \(L_i(t), i \neq 0\). The homogeneous elements occurring in these commutators are called representatives of level \(t\) and are denoted by \(x_j(t)\) (under the Index Convention). Next, for every (already fixed) representative \(x_j(t)\) of level \(t\), we fix the elements of the \(h\)-orbit \(O(x_j(t)) = \{x_j(t), x_j(t)^h, \ldots, x_j(t)^{h^{q-1}}\}\), and call them also representatives of level \(t\). These elements are denoted by \(x_{r,j}(t) := x_j(t)^{h^r}\) under the Index Convention (since \(L_j^{h^r} \leq L_{r,j}\)). Since the number of patterns under consideration is \((n,q,c)\)-bounded and the dimension of the subspace \(L_0\) is equal to \(m\), the total number of representatives of level \(t\) is \((m,n,q,c)\)-bounded. The construction of centralizers and representatives of levels \(\leq T\) is complete.

5 Properties of centralizer and representatives

Recall that we fixed the notation \(T = T(q,c) = f(q,c) + 1\) (for the maximal level) and \(U = V(T, T-1, q, c)\), where \(f, V\) are functions in the conclusion of Theorem 3.1 and Proposition 3.3, respectively.

It is clear from the construction of generalized centralizers that

\[
L_j(k + 1) \leq L_j(k)
\]

for all \(j \neq 0\) and all \(k = 1, \ldots, T\).

The following lemma follows immediately from the definitions of level 0 and levels \(t > 0\) and from the inclusions (7); we shall usually refer to this lemma as the “freezing” procedure.

**Lemma 5.1** (freezing procedure). Every simple commutator

\[
[y_{j_1}(k_1), y_{j_2}(k_2), \ldots, y_{j_w}(k_w)]
\]

of weight \(w \leq U\) in centralizers of levels \(k_1, k_2, \ldots, k_w\) with zero modulo \(n\) sum of indices

\(j_1 + \cdots + j_w \equiv 0 \pmod{n}\)
can be represented (frozen) as a linear combination of commutators $[x_{j_1}(s), x_{j_2}(s), \ldots, x_{j_w}(s)]$ of the same pattern in representatives of any level $s$ satisfying $0 \leq s \leq \min\{k_1, k_2, \ldots, k_w\}$.

**Definition.** We define a quasirepresentative of weight $w$ and level $k$ to be any commutator of weight $w \geq 1$ which involves exactly one representative $x_i(k)$ of level $k$ and $w - 1$ representatives $x_s(\varepsilon_s)$ of any lower levels $\varepsilon_s < k$. Quasirepresentatives of level $k$ (and only they) are denoted by $\hat{x}_j(k) \in L_j$ under the Index Convention; here, obviously, the index $j$ is equal modulo $n$ sum of indices of all the elements occurring in the quasirepresentative. Quasirepresentatives of weight 1 are precisely representatives.

**Lemma 5.2.** If $y_j(t) \in L_j(t)$ is a centralizer of level $t$, then $(y_j(t))^h$ is a centralizer of level $t$. If $\hat{x}_j(t)$ is a quasirepresentative of level $t$, then $(\hat{x}_j(t))^h$ is a quasirepresentative of level $t$.

**Доказательство.** Since $(y_j(t))^h \in L_{rj}$, we can denote $(y_j(t))^h$ by $y_{rj}$. Let $x_{t_1}(\varepsilon_1), \ldots, x_{t_k}(\varepsilon_k), k \leq U$, be arbitrarily chosen representatives of any (possibly different) levels $\varepsilon_s < t$ such that $rj + t_1 + t_2 + \cdots + t_k \equiv 0 \pmod n$. By construction the elements $(x_{t_i}(\varepsilon_s))^{h^{-1}} = x_{t_i-1}(\varepsilon_s), s = 1, \ldots, k$, are also representatives of the corresponding levels $\varepsilon_s$. By hypothesis the element $y_j(t) = (y_{rj})^{h^{-1}}$ is a centralizer of level $t$; therefore it has the centralizer property (6) with respect to representatives of lower levels:

$$[y_j(t), x_{r^{-1}i_1}(\varepsilon_1), \ldots, x_{r^{-1}i_k}(\varepsilon_k)] = 0,$$

since $j + r^{-1}t_1 + \cdots + r^{-1}t_k \equiv 0 \pmod n$ and $k \leq U$. By applying the automorphism $h$ to the last equation we obtain that

$$[y_{rj}, x_{t_1}(\varepsilon_1), \ldots, x_{t_k}(\varepsilon_k)] = 0,$$

that is, the element $(y_j(t))^h = y_{rj}$ is a centralizer of level $t$.

We now consider a quasirepresentative $\hat{x}_j(t)$ of weight $k$ of level $t$. By definition this element has the form

$$\hat{x}_j(t) = [x_{t_1}(t), x_{t_2}(\varepsilon_2), \ldots, x_{t_k}(\varepsilon_k)],$$

where $x_{t_1}$ is a representative of level $t$ and $x_{t_2}(\varepsilon_2), \ldots, x_{t_k}(\varepsilon_k)$ are representatives of any (possibly different) levels $\varepsilon_s < t$ such that $t_1 + t_2 + \cdots + t_k \equiv j \pmod n$. By construction the elements $(x_{t_1}(\varepsilon_s))^{h} = x_{r^{-1}t_1}(\varepsilon_s), s = 1, \ldots, k$, are also representatives of the same levels $\varepsilon_s$. Therefore,

$$(\hat{x}_j(t))^h = [(x_{t_1}(t))^h, (x_{t_2}(\varepsilon_2))^h, \ldots, (x_{t_k}(\varepsilon_k))^h] = [x_{r^{-1}t_1}(t), x_{r^{-1}t_2}(\varepsilon_2), \ldots, x_{r^{-1}t_k}(\varepsilon_k)],$$

whence $(\hat{x}_j(t))^h$ is also a quasirepresentative of level $t$. \qed

In what follows, when using Lemma 5.2 we shall by default denote the elements $y_j(t)^{h^s}$ by $y_{r_j}(t)$, and the elements $\hat{x}_j(t)^{h^s}$ by $\hat{x}_{r_j}(t)$.

Lemma 5.2 also implies that representatives of level $t$, elements $x_j(t), x_j(t)^h, \ldots, x_j(t)^{h^{s-1}}$, are centralizers of level $t$.  

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Lemma 5.3. Any commutator involving exactly one centralizer $y_i(t)$ of level $t$ and quasirepresentatives of level $< t$ is equal to 0 if the sum of indices of the elements occurring in it is equal to 0 and the sum of weight of all these elements is $t$ most $U + 1$.

Доказательство. Based on the definitions, by the Jacobi and anticommutativity identities we can represent this commutator as a linear combination of simple commutators of weight $\leq U + 1$ beginning with the centralizer of level $t$ and involving in addition only some representatives of levels $< t$. Since the sum of indices of all these elements is also equal to 0, all these commutators are equal to 0 by (6).

6 Main theorem

In the proof of Theorem 1.1 the main case is when $L$ is a $\varphi$-homogeneous $\mathbb{Z}/n\mathbb{Z}$-graded Lie algebra, that is, $L = L_0 \oplus L_1 \oplus \cdots \oplus L_{n-1}$.

Proposition 6.1. Theorem 1.1 holds for $\varphi$-homogeneous $\mathbb{Z}/n\mathbb{Z}$-graded Lie algebras $L = L_0 \oplus L_1 \oplus \cdots \oplus L_{n-1}$.

Доказательство. Recall that $T$ is the fixed notation for the highest level, which is a $(q, c)$-bounded number.

In § 4 we constructed the generalized centralizer $L_j(T)$. We set

$$Z = \langle L_1(T), L_2(T), \ldots, L_{n-1}(T) \rangle.$$ 

For every $k = 0, 1, \ldots, n - 1$ we denote the subspace $Z \cap L_k$ by $Z_k$. Clearly,

$$Z = \bigoplus_{k=0}^{n-1} Z_k,$$

and, in particular, $Z$ is generated by the subspaces $Z_k$. Furthermore, the subalgebra $Z$ is $H$-invariant by Lemma 5.2 and $(Z_k)^h = Z_{rh}$, since $(L_i)^h = L_{ri}$, $i \neq 0$.

Every subspace $L_j(T)$ has $(m, n, q, c)$-bounded codimension in $L_j$, while the dimension of $L_0$ is equal to $m$ by hypothesis. Since $L = \bigoplus_{i=0}^{n-1} L_i$ and the subalgebra $Z$ is generated by the subspaces $L_j(T)$, $j \neq 0$, it follows that $Z$ has $(m, n, q, c)$-bounded codimension in $L$. We claim that the subalgebra $Z$ is in addition nilpotent of $(c, q)$-bounded class and therefore is a required one.

Let $U = V(T, T - 1, q, c)$, where $V$ is the function in the conclusion of Proposition 3.3. It is sufficient to prove that every simple commutator of weight $U$ of the form

$$[y_{i_1}(T), \ldots, y_{i_U}(T)],$$

where $y_{i_j}(T) \in L_{i_j}(T)$, is equal to zero. Let $X$ be the union of the $h$-orbits of the elements $y_{i_1}(T), \ldots, y_{i_U}(T)$, that is,

$$X = \bigcup_{j=1}^{U} O(y_{i_j}(T)),$$
where, recall,
\[
O(y_j(T)) = \{y_j(T), \quad y_j(T)^h = y_{rj}(T), \quad \ldots \quad y_j(T)^{h^{q-1}} = y_{r^{q-1}j}(T) \}.
\]

By Proposition 3.3 the commutator (8) can be represented as a linear combination of \(\varphi\)-homogeneous commutators in elements belonging to the set \(X\) each of which either has a subcommutator of the form (3) in which there are \(T\) distinct initial segments in \(L_0\), or has a subcommutator of the form (4) in which there are \(T - 1\) occurrences of elements from \(L_0\). It is sufficient to prove that the commutators (3) and (4) are equal to zero.

We firstly consider the commutator
\[
[u_{k_0}, c_1, \ldots, c_{T-1}],
\]
where \(u_{k_0} \in X\), every \(c_i \in L_0\) with numbering indices \(i = 1, \ldots, T - 1\) has the form
\[
[x_{k_1}, \ldots, x_{k_i}],
\]
where \(x_{k_i} \in X\) and \(k_1 + \cdots + k_i \equiv 0 \pmod{n}\).

Using Lemma 5.1 we “freeze” every element \(c_k\), where \(k = 1, \ldots, T - 1\), as a linear combination of commutators of the same pattern of weight \(< U\) in representatives of level \(k\). Expanding then the inner brackets by the Jacobi identity \([a, [b, c]] = [a, b, c] - [a, c, b]\) we represent the commutator (9) as a linear combination of commutators of the form
\[
[u_{k_0}(T), \quad x_{j_1}(1), \ldots, x_{j_k}(1), \quad x_{j_{k+1}}(2), \ldots, x_{j_u}(2), \ldots, \quad x_{j_{v+1}}(T-1), \ldots, x_{j_v}(T-1)].
\]

We subject the commutator (10) to a certain collecting process. Our aim is a representation of the commutator as a linear combination of commutators with initial segments consisting of representatives of different levels \(1, 2, \ldots, T - 1\) and the element \(u_{k_0}(T)\). For that, by the formula \([a, b, c] = [a, c, b] + [a, [b, c]]\), in the commutator (10) we begin moving the element \(x_{j_{k+1}}(2)\) (the first from the left element of level 2) to the left, aiming at placing it right after the element \(x_{j_1}(1)\). In the course of these transformations there will appear additional summands of special form. At first step, say, we obtain a sum
\[
[u_{k_0}(T), \ldots, x_{j_{k+1}}(2), x_{j_k}(1), \ldots, ] + [u_{k_0}(T), \ldots, [x_{j_k}(1), x_{j_{k+1}}(2)], \ldots].
\]

In the first summand we continue transferring the element \(x_{j_{k+1}}(2)\) to the left, over all representatives of level 1. In the second summand we replace the subcommutator \([x_{j_k}(1), x_{j_{k+1}}(2)]\) by the quasirepresentative \(\hat{x}(2) = \hat{x}_{j_k+j_{k+1}}(2)\) and move already this quasirepresentative to the left over all representatives of level 1. Since we transfer a quasirepresentative of level 2 over representatives of level 1, in additional summands every time there appear subcommutators that are quasirepresentatives of level 2, which assume the role of the element that is being transferred. As a result we obtain a linear combination of commutators of the form
\[
[[u_{k_0}(T), x(1), \hat{x}(2)], x(1), \ldots, x(1), \quad x(2), \ldots, x(2), \ldots, x(T-1), \ldots, x(T-1)]
\]
with collected initial segment \([u_{k_0}(T), x(1), \hat{x}(2)].\) (For simplicity we omitted indices in the formula.) Next we begin moving to the beginning the first from the left representative
of level 3 aiming at placing it in the fourth place. It is important that this element is also transferred only over representatives of lower level, and subcommutators in additional summands are quasirepresentatives of level 3. Replacing these subcommutators by quasirepresentatives of level 3, we keep moving them to the left and so on. At the end of this process we obtain a linear combination of commutator with initial segment of the form
\[ [y_{k_0}(T), \hat{x}_{k_1}(1), \hat{x}_{k_2}(2), \ldots, \hat{x}_{k_{T-1}}(T-1)]. \] (11)
By Theorem 3.2 the commutator (11) of weight \( T \) is equal to a linear combination of \( \varphi \)-homogeneous commutators of the same weight \( T \) in elements of the \( h \)-orbits of the elements \( y_{k_0}(T), \hat{x}_{k_1}(1), \hat{x}_{k_2}(2), \ldots, \hat{x}_{k_{T-1}}(T-1) \) that have subcommutators with zero sum of indices modulo \( n \). By Lemma 5.2 every element \((\hat{x}_{k_i}(i))^{h_i} \) is a quasirepresentative of the form \( \hat{x}_{r_i\cdot k_i}(i) \) of level \( i \) and any \((y_{k_0}(T))^{h_i} \) is a centralizer of the form \( y_{r_i\cdot k_0}(T) \) of level \( T \). Since every level appears only once in (11) and there is an initial segment with zero sum of indices, every commutator of this linear combination is equal to 0 by Lemma 5.3.

We now show that a commutator of the form
\[ [y_{k_1}(T), \ldots, y_{k_s}(T)], \] (12)
where \( y_{k_j} \in X \) and there are \( T \) distinct initial segments with zero sum of indices modulo \( n \):
\[ k_1 + k_2 + \cdots + k_i \equiv 0 \pmod{n}, \quad i = 1, 2, \ldots, T, \]
\[ 1 < r_1 < r_2 < \cdots < r_T = s, \]
is equal to zero. The commutator (12) belongs to \( L_0 \) and is a commutator in elements of centralizers \( L_i(T) \) of level \( T \); therefore by Lemma 5.1 it can be “frozen” in level \( T \), that is, represented as a linear combination of commutators of the same pattern of weight \( \leq U \) in representatives of level \( T \):
\[ [x_{k_1}(T), \ldots, x_{k_s}(T)]. \] (13)
Next, the initial segment of the commutator (13) of length \( r_{T-1} \) also belongs to \( L_0 \) and is a commutator in elements of the centralizers \( L_i(T-1) \) of level \( T-1 \), since \( L_i(T-1) \leq L_i(T) \); therefore by Lemma 5.1 it can be “frozen” in level \( T-1 \), that is, represented in the form of a linear combination of commutators of the pattern of weight \( \leq U \) in representatives of level \( T-1 \), and so on. As a result we obtain a linear combination of commutator of the form
\[ [x(1), \ldots, x(1), x(2), \ldots, x(2), \ldots, \ldots, x(T), \ldots, x(T)]. \] (14)
(We omitted here indices for simplicity.) We subject the commutator (14) to exactly the same transformations as the commutator (10). First we transfer the left-most element of level 2 to the left to the second place, then the left-most element of level 3 to the third place, and so on. In additional summands the emerging quasirepresentatives \( \hat{x}(i) \) assume the role of the element being transferred and are also transferred to the left to the \( i \)th place. In the end we obtain a linear combination of commutators of the form
\[ [\hat{x}_{k_1}(1), \hat{x}_{k_2}(2), \ldots, \hat{x}_{k_T}(T)]. \] (15)
By Theorem 3.2 the commutator (15) of weight \( T \) is equal to a linear combination of \( \varphi \)-homogeneous commutators of the same weight \( T \) in elements of the \( h \)-orbits of the elements
\( \hat{x}_k(1), \hat{x}_k(2), \ldots, \hat{x}_k(T) \) that have subcommutators with zero sum of indices modulo \( n \).
By Lemma 5.2 every element \((\hat{x}_k(i))^H\) is a quasirepresentative of the form \( \hat{x}_{r_k}(i) \) of level \( i \). Let \( \hat{x}_k(s) \) be a quasirepresentative of maximal level \( s \) occurring in the initial segment with zero sum of indices. By representing the commutator as a linear combination of simple commutators beginning with \( \hat{x}_k(s) \) and with the same set of elements occurring in it, we obtain 0 by Lemma 5.3.

We now complete the proof of Theorem 1.1. We shall need the following lemma.

**Lemma 6.2.** Let \( p \) be a prime number and let \( \psi \) be a linear transformation of finite order \( p^k \) of a vector space \( V \) over a field of characteristic \( p \) the space of fixed points of which has finite dimension \( m \). Then the dimension \( \dim V \) is finite and does not exceed \( mp^k \).

**Доказательство.** This is a well-known fact, the proof of which is based on considering the Jordan form of the transformation \( \psi \); see, for example, \([16, 1.7.4]\). \( \Box \)

First suppose that the characteristic of the field \( \mathbb{F} \) is equal to a prime divisor \( p \) of the number \( n \). Let \( \langle \psi \rangle \) be the Sylow \( p \)-subgroup of the group \( \langle \varphi \rangle \), and let \( \langle \varphi \rangle = \langle \psi \rangle \times \langle \chi \rangle \), where the order of \( \chi \) is not divisible by \( p \). Consider the subalgebra of fixed points \( A = C_L(\chi) \). It is \( \psi \)-invariant and \( \dim C_A(\psi) \subseteq C_L(\varphi) \). Therefore, \( \dim C_A(\psi) \leq m \), and by Lemma 6.2 the dimension \( \dim A = \dim C_L(\chi) \) is bounded by some \((m, n)\)-bounded number \( u(m, n) \). Furthermore, \( \chi \) is a semisimple automorphism of the Lie algebra \( L \) of order \( \leq n \). Thus, \( L \) admits the Frobenius group of automorphisms \( \langle \chi \rangle H \) and \( \dim C_L(\chi) \leq u(m, n) \). Replacing \( F \) by \( \langle \chi \rangle \) we can assume that \( p \) does not divide \( n \).

Let \( \omega \) be a primitive \( n \)th root of unity. We extend the ground field by \( \omega \) and denote by \( \bar{L} \) the algebra over the extended field. The group \( FH \) naturally acts on \( \bar{L} \), and the subalgebra of fixed points \( C_{\bar{L}}(H) \) is nilpotent of the same class \( c \), while the subalgebra of fixed points \( C_{\bar{L}}(F) \) has the same dimension \( m \). Since the characteristic of the field does not divide \( n \), we have

\[
\bar{L} = L_0 \oplus L_1 \oplus \cdots \oplus L_{n-1},
\]
where

\[
L_k = \left\{ a \in \bar{L} \mid a^s = \omega^k a \right\},
\]
and this decomposition is a \((\mathbb{Z}/n\mathbb{Z})\)-grading, since

\[
[L_s, L_t] \subseteq L_{s+t \text{ (mod } n)},
\]
where \( s + t \) is calculated modulo \( n \).

By Proposition 6.1 the algebra \( \bar{L} \) has a nilpotent subalgebra \( Z \) of \((m, n, q, c)\)-bounded codimension and of \((q, c)\)-bounded nilpotency class. Obviously, the subalgebra \( L \cap Z \) is the sought-for subalgebra of \((m, n, q, c)\)-bounded codimension and of \((q, c)\)-bounded nilpotency class in \( L \). The theorem is proved.

### 7 Locally nilpotent torsion-free groups

Every locally nilpotent torsion-free group \( G \) can be embedded into a divisible group \( \sqrt{G} \) consisting of all roots of non-trivial element of \( G \), the so-called Mal’cev completion.
of $G$ (see, for example, [18, Ch. 10]). Every automorphism of the group $G$ can be uniquely extended to an automorphism of the group $\sqrt{G}$. Divisible torsion-free groups can be regarded as $\mathbb{Q}$-groups with additional operations of extracting rational roots. The Mal’cev correspondence given by the Baker–Hausdorff formula and its inversions establishes an equivalence of the category of locally nilpotent $\mathbb{Q}$-groups and the category of locally nilpotent Lie $\mathbb{Q}$-algebras (see, for example, [18, Ch. 10]). We can assume that the corresponding objects in these two categories have the same underlying set. Let $G$ and $L$ be category equivalent a $\mathbb{Q}$-group and a Lie $\mathbb{Q}$-algebra, respectively, with the same underlying set. Then $\mathbb{Q}$-subgroups of $G$ (that is, divisible subgroups) are (as subsets) $\mathbb{Q}$-subalgebras of the algebra $L$ and vice versa; normal $\mathbb{Q}$-subgroups of $G$ are precisely ideals of $L$, and so on. The nilpotency class of a subgroup of $G$ coincides with its nilpotency class as a Lie subalgebra of $L$.

Recall that a group has finite rank $r$ if every finitely generated subgroup of it is generated by $r$ elements (and $r$ the smallest number with this property). By Mal’cev’s theorem [19, Theorem 5] a locally nilpotent torsion-free group $G$ has finite rank if and only if it is nilpotent and has finite sectional rank. We shall need the following version of Mal’cev’s theorem proved in [14].

**Lemma 7.1** ([14], Lemma 9). If a locally nilpotent torsion-free group $C$ has finite rank $r$, then the Lie $\mathbb{Q}$-algebra $U$ that is equivalent to $\sqrt{C}$ under the Mal’cev category correspondence has finite $r$-bounded dimension.

**Theorem 7.2.** Let $FH$ be a Frobenius group with cyclic kernel $F$ of order $n$ and with complement $H$ of order $q$. If $FH$ acts by automorphisms on a locally nilpotent torsion-free group $G$ in such a way that the subgroup of fixed points $C_G(F)$ has finite rank $r$ and the subgroup of fixed points $C_G(H)$ is nilpotent of class $c$, then $G$ has a nilpotent subgroup $T$ of nilpotency class bounded by some function depending only on $q$ and $c$ such that $T$ has finite “corank” $t = t(r,n,q,c)$ in $G$ bounded above in terms of $r$, $n$, $q$, $c$ in the sense that there are $t$ element $g_1, \ldots, g_t$ such that every element of $G$ is a root of an element of the subgroup $\langle g_1, \ldots, g_t, T \rangle$.

**Доказательство.** We denote by the same letters $F$ and $H$ the extensions of the groups of automorphisms to the group $\sqrt{G}$. Let $L$ be the Lie $\mathbb{Q}$-algebra with the same underlying set $\sqrt{G}$ constructed by the Mal’cev correspondence. The automorphisms of the Lie algebra $L$ are the automorphisms of the group $\sqrt{G}$ acting on the same set in exactly the same way. Since

$$\sqrt{C_G(H)} = C_{\sqrt{G}}(H) = C_L(H), \quad \sqrt{C_G(F)} = C_{\sqrt{G}}(F) = C_L(F),$$

the subalgebra $C_L(H)$ is nilpotent of class $c$ and the subalgebra $C_L(F)$ has $r$-bounded dimension by Lemma 7.1. By Theorem [14] the algebra $L$ has a nilpotent subalgebra $Z$ of $(c,q)$-bounded nilpotency class and of $(r,n,q,c)$-bounded codimension. The intersection $Z \cap G$ is a sought-for subgroup of $G$.

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