Extensions and composites in topos quantum theory

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The extension problem asks when two bipartite states $s_{XY}$ and $s_{YZ}$ are marginals of a tripartite state $s_{XYZ}$. It is especially pertinent in quantum theory, where pairwise compatibility need not imply global compatibility. We generalise its solution for classical probability distributions to commutative monads on cartesian categories that preserve terminal objects, and then apply this to the valuation monad in topos quantum theory. In doing so, we define composite systems in topos quantum theory, analyse their states, and prove that they correspond to positive over pure tensor states.

1 Introduction

One of the salient features of quantum theory is that observables that are pairwise compatible need not be globally compatible [21], in contrast to classical probability theory. As a case in point, the Bayesian inference rule of the latter has no counterpart in the former [22, 26]: whereas a bipartite probability distribution is always described by one of its marginal distributions and a transition matrix, there exist bipartite quantum states whose correlation cannot be described by a marginal state and a completely positive trace-preserving map. This makes the extension problem pertinent: when are two bipartite states $s_{XY}$ and $s_{YZ}$ the marginals of a tripartite state $s_{XYZ}$? For probability distributions the extension problem can always be solved, thanks to the availability of conditional probability distributions [5]. For quantum states the problem is more difficult, and only partial solutions are known [29]. In this article, we analyse the solution of the extension problem for probability distributions categorically [5], before specializing to quantum theory.

We first phrase the extension problem in terms of a monad that takes the states of a system, and develop sufficient conditions for Kleisli morphisms to satisfy the extension problem (Section 2). In the classical case, the Kleisli morphisms of the distribution monad are transition matrices between probability distributions [18], and we recover the classical solution to the extension problem for probability distributions: we show that Kleisli morphisms of commutative monads on cartesian categories that preserve the terminal object satisfy the extension problem. Preservation of terminal objects is sufficient but not necessary, and in the classical case corresponds to the normalization of distributions. Commutativity of the monad corresponds to the coincidence of two maps that send local states on separate systems to a product state on the compound system. The properties of such Fubini maps are key to the extension problem.

The second half of the paper applies these general results in a quantum setting, where cartesian products become tensor products. We employ the topos approach to quantum theory [16], because that formalism by construction facilitates the passage from classical theory to quantum theory [15]. Quantum systems are represented by the Gelfand spectra of their algebras of observables, and states by valuations

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on the spectra. The relevant monad taking states is now the valuation monad \([30]\), that is still commutative and preserves terminal objects. There is no known investigation of states of composite systems in topos approaches. Our second major contribution is to define composite systems in topos quantum theory (Section \([3]\), and to analyse the correspondence between quantum states and valuations on compound systems (Section \([4]\)). We define two types of composition of systems: ‘spatial’ and ‘temporal’, both given by products of Gelfand spectra but in different toposes. There is a one-to-one correspondence between valuations on spatially composed systems and positive over pure tensor (POPT) states \([3]\). From the fact that there exist POPT states that cannot extend, we derive that the corresponding Kleisli morphisms cannot exist. We also prove that temporally composed systems lack a monogamy property, which is potentially useful to express temporally correlated states.

2 Commutative monads and extension problems

What are necessary and sufficient conditions for marginal states on multipartite systems to extend to a total state? For example, in classical probability theory: when are bipartite probability distributions \(p_{XY}\) and \(p_{XZ}\) the marginals of a tripartite probability distribution \(p_{XYZ}\)?

This version of the question is solved \([5]\): the marginals should overlap on \(X\), and the extension is then given by

\[
p_{XYZ}(x,y,z) = \frac{p_{XY}(x,y)p_{XZ}(x,z)}{p_X(x)} = p_{XY}(x,y)\frac{p_{XZ}(x,z)}{p_X(x)} = \frac{p_{XY}(x,y)p_{XZ}(x,z)}{p_X(x)},
\]

which uses conditional probabilities in the last two equations. We call the construction of extended probability distribution presented by Carlen, Lebowitz and Lieb in \((1)\) the CLL construction. The question can also be asked in other contexts. The extension problem is the case of two bipartite marginals of a tripartite quantum system, and remains open. Attempts to generalise the CLL construction to the quantum setting have failed, perhaps because of the lack of a natural notion of conditioning quantum states \([5]\).

The probabilistic case can be described by the distribution monad. This section generalises the extension problem to monads on cartesian categories, and analyses why the distribution monad allows the CLL construction.

Systems, states, and monads  Regard systems as objects \(X\) in a symmetric monoidal category \((\mathbf{C}, \otimes, I)\). Think of a monad \(T : \mathbf{C} \rightarrow \mathbf{C}\) as assigning a state space to a system, and regard states as morphisms \(I \rightarrow TX\). The composition of systems \(X\) and \(Y\) is given by product \(X \otimes Y\), so joint states are maps \(I \rightarrow T(X \otimes Y)\). Taking marginals is modelled by morphisms \(TX \leftarrow T(X \otimes Y) \rightarrow TY\). We will first restrict to cartesian categories, where canonical marginals are induced by the projections \(\pi_X : X \times Y \rightarrow X\) and \(\pi_Y : X \times Y \rightarrow Y\).

Example 1. Write \(\mathbf{Set}\) for the category of sets and functions with cartesian products. The distribution monad \(\mathcal{D} : \mathbf{Set} \rightarrow \mathbf{Set}\) is defined by (see \([18, 12]\)):

\[
\mathcal{D}(A) = \{p : A \rightarrow [0, 1] \mid \text{supp}(p) \text{ finite}, \sum_{a \in A} p(a) = 1\},
\]

\[
\mathcal{D}(f)(r_1 \delta_1 + r_2 \delta_2 + \ldots + r_n \delta_n) = r_1 \delta_{f(1)} + r_2 \delta_{f(2)} + \ldots + r_n \delta_{f(n)},
\]

for sets \(A\) and functions \(f\), where \(\delta\) denotes Kronecker’s delta function and \(\sum_{i=1}^n r_i = 1\) with \(r_i \in [0, \infty)\).
Its unit and multiplication are:

\[
\begin{align*}
\eta_A &: A \to \mathcal{D}(A) \\
\mu_A &: \mathcal{D}^2(A) \to \mathcal{D}(A) \\
\end{align*}
\]

\[
\begin{align*}
a \mapsto \delta_a \\
r_1 \delta_{p_1} + r_2 \delta_{p_2} + \ldots + r_n \delta_{p_n} \mapsto r_1 p_1 + r_2 p_2 + \ldots + r_n p_n.
\end{align*}
\]

**Definition 2.** A monad \(T\) on a monoidal category is *strong* if it has a natural transformation

\[
st_{X,Y} : X \otimes TY \to T(X \otimes Y),
\]

called its *strength*, satisfying certain conditions (see e.g. [23]). If the category is symmetric monoidal, conjugating with the swap map induces a natural transformation \(\text{cst}_{X,Y} : T(X) \otimes Y \to T(X \otimes Y)\), giving two canonical natural transformations

\[
dst_{X,Y} : TX \otimes TY \xrightarrow{\text{cst}_{X,Y}} T(X \otimes TY) \xrightarrow{T\eta_{TY}} T^2(X \otimes Y) \xrightarrow{\mu_{X,Y}} T(X \otimes Y),
\]

\[
dst'_{X,Y} : TX \otimes TY \xrightarrow{\text{cst}_{X,Y}} T(X \otimes TY) \xrightarrow{T\eta_{TY}} T^2(X \otimes Y) \xrightarrow{\mu_{X,Y}} T(X \otimes Y),
\]

called the *Fubuni maps*. The monad is *commutative* when \(\text{dst} = \text{dst}'\).

The distribution monad is commutative, with Fubuni maps given by the inclusion of product probability distributions [20]. Moreover, it satisfies \(\mathcal{D}1 \cong 1\), enabling the following key property.

**Lemma 3.** If \(T1 \cong 1\) for a strong monad on a cartesian category, the following diagrams commute:

\[
\begin{array}{ccc}
TX & \xrightarrow{T\pi_X} & T(X \times Y) \\
\downarrow{\pi_Y} & & \downarrow{T\pi_Y} \\
TY & \xrightarrow{\text{dst}_{X,Y}} & T(X \times TY) \\
\end{array}
\]

\[
\begin{array}{ccc}
TX & \xrightarrow{T\pi_X} & T(X \times Y) \\
\downarrow{\pi_Y} & & \downarrow{T\pi_Y} \\
TY & \xrightarrow{\text{dst}'_{X,Y}} & T(X \times TY) \\
\end{array}
\]

---

**Proof.** We prove the upper left triangle, writing \(! : Y \to 1\). It follows from naturality of \(\text{dst}\) that

\[
T\pi_X \circ \text{dst}_{X,Y} = T(\pi_X \circ \text{id}_X \times !) \circ \text{dst}_{X,Y} = T\pi_X \circ \text{dst}_{X,1} \circ T\text{id}_X \times T! = T\pi_X \circ \text{dst}_{X,1} \circ \text{id}_TX \times T!,
\]

so it suffices to show \(T\pi_X \circ \text{dst}_{X,1} = \pi_{TX}\). But this follows from the proof of [19, Theorem 2.1]; notice that while that result assumes cartesian closedness, only cartesianness is sufficient for our purpose. \(\square\)

A strong monad on a cartesian *closed* category is said to be affine if \(T1 \cong 1\) [19].

**Conditioning and Kleisli morphisms** Recall that the *Kleisli category* \(\text{Kl}(T)\) of a monad \((T, \mu, \eta)\) on \(C\) has the same objects as \(C\), but morphisms \(X \to Y\) in \(\text{Kl}(T)\) are morphisms \(X \to T(Y)\) in \(C\), with composition \(g \circ_{\text{Kl}} f = (\mu \circ Tg \circ f)\) and identities \(\eta_X\). States of \(X\) now become Kleisli morphisms \(I \to X\). Similarly, joint states become Kleisli morphisms \(I \to \bigotimes_i X_i\), and marginal states are got by postcomposing with the marginal map \(\bigotimes_i X_i \to X_i\). There are many general notions of conditioning [10] [22], but the CLL construction and the Kleisli setting lead to the following definition, that will prove useful.

**Definition 4.** Let \((T, \mu, \eta)\) be a monad on a cartesian category with \(T1 \cong 1\), and let \(s_{XY}\) be a joint state on \(X \times Y\), and \(s_X\) its marginal state on \(X\). A *conditional process* for \(s_{XY}\) from \(X\) to \(Y\) is a Kleisli morphism \(f : X \to Y\) satisfying

\[
(dst_{X,Y} \circ (\eta_X, f)) \circ_{\text{Kl}} s_X = s_{XY} \quad \text{or} \quad (dst'_{X,Y} \circ (\eta_X, f)) \circ_{\text{Kl}} s_X = s_{XY}.
\]
Example 5. The Kleisli category of the distribution monad is the category of probability distributions \([25]\). A Kleisli morphism \(f: X \to Y\) is an \([X]-by-[Y]\) transition matrix with entries \(f(x)(y) \in [0,1]\) satisfying \(\sum_y f(x)(y) = 1\) for all \(x \in X\). In particular, Kleisli morphisms \(1 \to X\) are probability distributions on \(X\). Composition of transition matrices comes down to matrix multiplication. A joint probability distribution on \(X \times Y\) for a probability distribution \(p\) on \(X\) and a transition matrix \(f: X \to Y\), defined by the composition \((\text{dst}_{X,Y} \circ (f, \eta_X)) \circ K f p\), then works out to \((x,y) \mapsto p(x)f(x)(y)\). Together with \([5]\), this implies that a conditional process \(f\) for a joint probability distribution \(p_{XY}\), if it exists, must satisfy
\[
f(x)(y) = \frac{p_{XY}(x,y)}{p_X(x)} \quad \text{when } p_X(x) \neq 0.
\] (4)

Thus conditional processes for the distribution monad correspond to conditional probability matrices, uniquely specified by \((4)\).

We now extrapolate from the distribution monad.

Definition 6. Let \(T\) be a monad on a cartesian category. A state \(s_{XYZ}: 1 \to T(X \times Y \times Z)\) is an extension of \(s_{XY}: 1 \to T(X \times Y)\) and \(s_{YZ}: 1 \to T(Y \times Z)\) when \(T\pi_{X \times Y} \circ s_{XYZ} = s_{XY}\) and \(T\pi_{X \times Z} \circ s_{XYZ} = s_{XZ}\).

A necessary condition for extensions to exist is the coincidence of common marginals:

\[
\begin{array}{ccc}
1 & \xrightarrow{s_{XY}} & T(X \times Y) \\
& s_{XYZ} & \searrow T(X \times Y \times Z) \\
\swarrow T\pi_{X \times Y} & & \searrow T\pi_X \\
1 & \xrightarrow{s_{XZ}} & T(X \times Z) \\
& s_{XYZ} & \swarrow T\pi_{X \times Z} \\
\end{array}
\] (5)

For the distribution monad, this condition is also sufficient by the CLL construction \([5]\). The following theorem generalises this.

Theorem 7. Let \((T, \mu, \eta)\) be a commutative monad on a cartesian category \(C\) with \(T1 \cong 1\), and let \(s_{XY}\) and \(s_{XZ}\) be states on \(X \times Y\) and \(X \times Z\). If \(s_{XY}\) and \(s_{XZ}\) satisfy the outer square of \((5)\), and one of them has a conditional process from \(X\), then an extension exists.

Proof. Let \(f\) be a conditional process for, say, \(s_{XY}\). Consider the following diagram in \(C\):

\[
\begin{array}{ccc}
1 & \xrightarrow{s_{XZ}} & T(X \times Z) \\
& (\text{dst}_{X,Y,Z} \circ (\text{dst}_{X,Y} \circ (\eta_X, f) \times \eta_Z)) & \searrow T^2(X \times Y \times Z) \\
& T\pi_{X \times Z} & \searrow \mu_{X \times Z} \\
& T\pi_{X \times Y} & \searrow \mu_{X \times Y} \\
\end{array}
\] (6)

The lower left triangle commutes by definition of cartesian product and Lemma \([3]\). That the upper left triangle also commutes is proved in Appendix \([D]\). The right squares commute by naturality of \(\mu\). Let us denote the middle horizontal path

\[
\left(\text{dst}_{X,Y,Z} \circ (\text{dst}_{X,Y} \circ (\eta_X, f) \times \eta_Z)\right) \circ K 1 s_{XZ},
\] (7)
by $s_{XYZ}$. The top horizontal path equals $s_{XZ}$. The lower horizontal path is

$$
\mu_{X \times Y} \circ T (\text{dst}_{X,Y} \circ \langle \eta_X, f \rangle \circ \pi_X) \circ s_{XZ} = \mu_{X \times Y} \circ T (\text{dst}_{X,Y} \circ \langle \eta_X, f \rangle) \circ T (\pi_X) \circ s_{XZ}
$$

$$
= \mu_{X \times Y} \circ T (\text{dst}_{X,Y} \circ \langle \eta_X, f \rangle) \circ s_X
$$

$$
= (\text{dst}_{X,Y} \circ \langle \eta_X, f \rangle) \circ \text{Kl}_1 s_X = s_{XY}.
$$

Thus $s_{XYZ}$ is the desired extension. \hfill \square

Our generalised CLL construction is presented by (7). If applied to the distribution monad, it recovers the original CLL construction: for Kleisli morphisms $p_{XZ} : 1 \to X \times Z$ and $f : X \to Y$, direct calculation with (4) gives

$$
\mu_{X \times Y \times Z} \circ \mathcal{D} (\text{dst}_{X \times Y \times Z} \circ (\text{dst}_{X,Y} \circ \langle \eta_X, f \rangle \times \eta_Z)) \circ p_{XZ} (\ast, y, z) = p_{XZ} (x, z) \frac{p_{XY} (x, y)}{p_X (x)}.
$$

It is known that this extension for $(p_{XY}, p_{XZ})$ which is not necessarily the unique extension, has the maximum entropy over all extensions[5]. It would be interesting to see the analogue of this property in other monads, with some generalized notion of entropy. We leave this for a future work.

The assumptions of Theorem[7] may be weakened. For example, $T 1 \cong 1$ can be replaced by commutativity of (2), and in fact the monoidal unit need not be a terminal object. For example, the Fock space monad $\mathcal{F}$ does not satisfy $\mathcal{F} 1 \cong 1$[4], but (2) still commutes. Similarly, even if neither of the marginal states $s_{XY}$ and $s_{XZ}$ have a conditional process, there can still be an extension, as extensibility is a property of the pair $(s_{XY}, s_{XZ})$, but having a conditional process is a property of the individual states.

3 Compound systems in topos quantum theory

The next section will apply the results from the previous one to topos quantum theory. To do so, this section first develops states of compound systems in this framework. We will work in the covariant approach[6][16][15] for the following two reasons, but see also [9][32] and references therein. First, states are represented as valuations[8], which are amenable to our monad methods[30]. Second, the relatively simple internal language lets us use any constructively valid theorems[6].

The topos framework considers the functor category $[\mathcal{C} (A), \text{Set}]$, where $\mathcal{C} (A)$ is the set of commutative unital C*-subalgebras of a unital C*-algebra $A$ modelling the observables of the system in question, ordered under inclusion. For example, $A = \mathcal{B} (H)$ could consist of all bounded operators on a Hilbert space $H$, as in the traditional formalism. As we will shortly review briefly, the system is also determined by an internal Gelfand spectrum. It is unclear how to model compound systems in this framework[33]. Simply taking $[\mathcal{C} (\mathcal{B} (H_1 \otimes H_2)), \text{Set}]$ defeats the purpose, as it needs access to the external data $H_i$. We generalise the classical solution, which takes a coproduct of the C*-algebras, or a product of their spectra, of the marginal systems, in two ways:

- the spatial composition of independent systems takes in the topos $[\mathcal{C} (A_1) \times \mathcal{C} (A_2), \text{Set}]$ the product of the images of the spectra under the injections $[\mathcal{C} (A_i), \text{Set}] \to [\mathcal{C} (A_1) \times \mathcal{C} (A_2), \text{Set}]$;

- the temporal composition of identical systems takes the product of the spectra in $[\mathcal{C} (A), \text{Set}]$. 
Bohrification and states The Bohrification of $A$ is the unital commutative C*-algebra object in $[\mathcal{C}(A), \text{Set}]$ defined by $A_1(C) = C$. By Gelfand duality, which holds constructively \[2\], $A$ is an algebra of continuous functions on its Gelfand spectrum $\Sigma_A$, which is an internal locale in the same topos. This is the object that models the system, and is accessible through the internal logic of the topos. For example, there is a bijective correspondence of quasi-states of $A$ and probability integrals on the self-adjoint part $A_{\text{sa}}$. The former are maps $\rho : A \to \mathbb{C}$ that are positive and linear on all $C \in \mathcal{C}(A)$ and satisfy $\rho(a + ib) = \rho(a) + i\rho(b)$ for all $a, b \in A_{\text{sa}}$; if $A$ is a von Neumann algebra without Type I$_2$ direct summand, as always in our finite-dimensional case, quasi-states are simply states $\rceil_{13}$. The latter are morphisms $I : A_{\text{sa}} \to \mathbb{R}$ satisfying:

- normalization: $I(1) = 1$;
- linearity: $I(\alpha a + \beta b) = \alpha I(a) + \beta I(b)$ for all $a, b \in A_{\text{sa}}$ and $\alpha, \beta \in \mathbb{R}$;
- positivity: $I(a^2) \geq 0$ for all $a \in A_{\text{sa}}$.

Riesz’s theorem guarantees any (probability) integral is represented by a measure, and holds constructively $[8, 30]$: the locale of probability integrals over $A$ is isomorphic to the locale of probability valuations over $\Sigma_A$, so that within the topos:

\[
\begin{array}{c}
\text{commutative C*-algebra} A \quad \text{Integrals over } A \\
\text{Gelfand duality} \quad \downarrow \\
\text{Gelfand spectrum } \Sigma_A \quad \text{Valuations on } \Sigma_A
\end{array}
\]

Coproducts of algebras For any small category $C$, an object $A$ in $[C, \text{Set}]$ is an internal unital commutative C*-algebra if and only if each component $A(X)$ is a unital commutative C*-algebra in Set and $A(f)$ is a unital *-homomorphism for each morphism $f : X \to Y$ $[28]$.

Theorem 8. Let $C$ be a small category and $A_1, \ldots, A_n$ be unital commutative C*-algebras in $[C, \text{Set}]$. The object $A_1 \otimes \cdots \otimes A_n \in [C, \text{Set}]$ defined by

\[
A_1 \otimes \cdots \otimes A_n(X) = A_1(X) \otimes \cdots \otimes A_n(X)
\]

\[
A_1 \otimes \cdots \otimes A_n(f) : a_1 \otimes \cdots \otimes a_n \mapsto A_1(f)(a_1) \otimes \cdots \otimes A_n(f)(a_n)
\]

is a unital commutative C*-algebra, and in fact the coproduct of $A_i$ in the category of unital commutative C*-algebras in $[C, \text{Set}]$. Here, the tensor products on the right hand side are coproducts of C*-algebras in Set; in our finite-dimensional case, they are just the algebraic tensor product.

Proof. See Appendix $A$.

Denote the internal locale of integrals over $A$ in $[C, \text{Set}]$ by $\mathcal{I}(A)$, and the internal locale of valuations on $\Sigma_A$ by $\mathcal{V}(\Sigma_A)$. Gelfand duality gives $\Sigma_{A_1 \otimes \cdots \otimes A_n} \cong \Sigma_{A_1} \times \cdots \times \Sigma_{A_n}$, where the product on the right-hand side is that of locales. Riesz’s theorem then gives $\mathcal{V}(\Sigma_{A_1 \otimes \cdots \otimes A_n}) \cong \mathcal{I}(A_1 \otimes \cdots \otimes A_n)$. It follows that integrals over a coproduct algebra correspond to valuations on the product spectrum:

\[
\mathcal{V}(\Sigma_{A_1} \times \cdots \times \Sigma_{A_n}) \cong \mathcal{I}(A_1 \otimes \cdots \otimes A_n).
\]

Thus we can analyse states on composite systems in two ways. In the rest of this section it is handier to use integrals (that more easily assign values to observables), whereas the next section uses valuations (that more easily generalise stochastic processes).
**Spatial composition** If C*-algebras $A_1, \ldots, A_n$ describe independent systems, their Bohrifications $A_i$ live in $[\mathcal{E}(A_i), \text{Set}]$. Injecting them into $[\mathcal{E}(A_1) \times \cdots \times \mathcal{E}(A_n), \text{Set}]$, we can take their coproduct there. We will show integrals then correspond to *positive over pure tensor (POPT) states* [3].

The projections $\pi_i: \mathcal{E}(A_1) \times \cdots \times \mathcal{E}(A_n) \to \mathcal{E}(A_i)$ from the product poset induce geometric morphisms $\pi_i^*: [\mathcal{E}(A_i), \text{Set}] \to [\mathcal{E}(A_1) \times \cdots \times \mathcal{E}(A_n), \text{Set}]$ by precomposition. In general geometric maps need not preserve unital commutative C*-algebras because that theory is not geometric, but in the current case the maps $\pi_i^*$ do.

**Definition 9.** The *spatial composition* of $A_i \in [\mathcal{E}(A_i), \text{Set}]$ for $i = 1, \ldots, n$ is the coproduct $A_1 \otimes \cdots \otimes A_n$ of $\pi_i^*A_i$ in $[\mathcal{E}(A_1) \times \cdots \times \mathcal{E}(A_n), \text{Set}]$ according to Theorem 8:

$$\pi_1^*A_1 \otimes \cdots \otimes \pi_n^*A_n(C_1, \ldots, C_n) = \pi_1^*A_1(C_1) \otimes \cdots \otimes \pi_n^*A_n(C_n) = C_1 \otimes \cdots \otimes C_n.$$  \hspace{1cm} (8)

Properties of binary spatial compositions are analysed in [33]. That definition allows nontrivial overlap $A_1 \wedge A_2$, and replaces $\mathcal{E}(A_1) \times \mathcal{E}(A_2)$ with the pullback $\mathcal{E}(A_1) \times_{\mathcal{E}(A_1 \wedge A_2)} \mathcal{E}(A_2)$, and $C_1 \otimes C_2$ with $C_1 \otimes_{C_1 \wedge C_2} C_2$. Here we focus on the case $A_1 \wedge A_2 \equiv C$ and leave the rest to future work.

Next we analyse states of spatially compound systems. Recall that for any poset $P$, and any unital commutative C*-algebra $A$ in the topos $[P, \text{Set}]$ with real number object $\mathbb{R} : P \to \text{Set}$, an integral over $A$ is a natural transformation whose components $I_C : A_{sa}(C) \to \mathbb{R}(C)$ at $C \in P$ are integrals.

**Definition 10.** An *integral* over $A_1 \otimes \cdots \otimes A_n$ is a family $\{I_{(C_1, \ldots, C_n)} : (C_1 \otimes \cdots \otimes C_n)_{sa} \to \mathbb{R}(C)\}$ indexed by $(C_1, \ldots, C_n) \in \mathcal{E}(A_1) \times \cdots \times \mathcal{E}(A_n)$ satisfying:

- each $I_{(C_1, \ldots, C_n)} : (C_1 \otimes \cdots \otimes C_n)_{sa} \to \mathbb{R}(C_1, \ldots, C_n)$ is an integral in $\text{Set}$;
- if $(C_1, \ldots, C_n) \leq (D_1, \ldots, D_n)$, then $I_{(D_1, \ldots, D_n)}(X) = I_{(C_1, \ldots, C_n)}(X)$ for all $X \in (C_1 \otimes \cdots \otimes C_n)_{sa}$.

Thus an integral on $A_1 \otimes \cdots \otimes A_n$ is a family of integrals in $\text{Set}$ mapping self-adjoint operators to reals.

**Definition 11.** Let $H_1, \ldots, H_n$ be Hilbert spaces. A bounded linear operator $\omega$ on $H_1 \otimes \cdots \otimes H_n$ is a *positive over pure tensor (POPT) state* if $\text{Tr}(\omega) = 1$ and for any positive operators $e_i$ on $H_i$:

$$\text{Tr}(e_1 \otimes \cdots \otimes e_n \omega) \geq 0.$$  

Thus a POPT state is either a quantum state or a trace-one *entanglement witness*. Write $\mathcal{W}(H_1 \otimes \cdots \otimes H_n)$ for the set of all POPT states.

**Theorem 12.** Let $H_1, \ldots, H_n$ be Hilbert spaces of dimension at least three, and $A_i = \mathcal{B}(H_i)$. POPT states $\omega$ on $H_1 \otimes \cdots \otimes H_n$ are in bijective correspondence with integrals over $A_1 \otimes \cdots \otimes A_n$ via

$$I_{(C_1, \ldots, C_n)}^\omega(a) = \text{Tr}(\omega a).$$

**Proof.** See Appendix [3].

Observe that the previous theorem also gives a correspondence between $\mathcal{W}(H_1 \otimes \cdots \otimes H_m)$ and integrals over $A_1 \otimes \cdots \otimes A_m$ in $[\mathcal{E}(A_1) \times \cdots \times \mathcal{E}(A_n), \text{Set}]$ for any $m \leq n$.

**Temporal composition** In classical mechanics a state evolution in a single system can be represented by a joint state over time. There the underlying composite system for the joint state is the product of copied systems. Here we consider the analogous type of composition to the topos approach.
Definition 13. We call the n-time coproduct of identical Bohrification algebra $A$ in $[\mathcal{C}(A), \text{Set}]$ the n-time temporal composition of $A$, and denote it by $A^1 \otimes \cdots \otimes A^n$ with the superscripts to distinguish the components. According to Theorem 8, $A^1 \otimes \cdots \otimes A^n(C) = C \otimes \cdots \otimes C$. (9)

We also denote the Gelfand spectrum of $A^1 \otimes \cdots \otimes A^n$ by $\Sigma^1_A \times \cdots \times \Sigma^n_A$, where $\Sigma^n_A$ represents the spectra of $A_i$ for $i = 1, \ldots, n$.

Note the difference between temporal and spatial compositions. Each component of the temporally composed algebra (9) is the $n$th tensor product $C \otimes \cdots \otimes C$ of the defining context $C$, while that of spatially composed algebra (8) is the tensor product $C_1 \otimes \cdots \otimes C_n$ of the defining contexts $C_1, \ldots, C_n$.

As a consequence states on temporally composed systems are incomparable to those on spatially composed ones. We will discuss in Section 4 that the temporally composed system can express certain states over time, and this is why we call (9) “temporal” composition.

4 The valuation monad

This section applies the extension results of Section 2 to the topos quantum systems of Section 3, by considering the valuation monad on the category of locales in the topos. This category is cartesian, and the valuation monad is commutative and satisfies $\forall 1 \cong 1$ [30], making this application possible. The valuation monad constructively generalises the Radon monad, that takes the space of Radon measures of a given compact Hausdorff space [11]; constructively, topological spaces are replaced by locales, and measures are replaced by valuations.

We will first analyse spatial composition. If $\Sigma^1_A \times \cdots \times \Sigma^n_A$ describes a spatially composed system, then a valuation, or rather a morphism $1 \to V(\Sigma^1_A \times \cdots \times \Sigma^n_A)$, corresponds to a POPT state. If there is a Kleisli morphism $\Sigma^1_A \to \Sigma^2_A$, then we can construct extensions for the pair of bipartite POPT states, one of which has this Kleisli morphisms as a conditional process. On the other hand, a convexity argument about POPT states reveals that there are bipartite POPT states that can only be extended with product states, such as entangled pure states. The resulting contradiction with Theorem 7 proves that nonproduct states do not have conditional processes.

Next we will analyse temporal composition. In that case there are Kleisli morphisms, such as the unit of the valuation monad. We will construct extensions by combining these morphisms, and discuss how this property describes correlations of temporally ordered quantum systems.

Conditional processes for POPT states  Consider the Kleisli category of valuation monad in a tripartite topos quantum system $[\mathcal{C}(A_1) \times \mathcal{C}(A_2) \times \mathcal{C}(A_3), \text{Set}]$. As in Section 3, joint valuations on spatially composed systems correspond to POPT states. Thus a conditional process for a bipartite joint valuation, if it exists, is a conditional process for the corresponding bipartite POPT state. We start outlining the possibilities by exhibiting a pair of bipartite POPT states with overlapping marginal but no extension.

The set of bipartite POPT states is convex, the extremal points of which can be characterized [24]. Examples of extremal bipartite POPT states are pure quantum states, and partial transpositions of those.

Lemma 14. Let $H_1, H_2, H_3$ be Hilbert spaces, and $\omega \in \mathcal{W}(H_1 \otimes H_2 \otimes H_3)$ be a POPT state. If $\text{Tr}_3(\omega)$ is an extremal bipartite POPT state on $H_1 \otimes H_2$, then $\omega = \text{Tr}_3(\omega) \otimes \text{Tr}_{12}(\omega)$. 
Proof. If \( R \) is an orthogonal projection on \( H_3 \), then

\[
\omega|_R = \frac{\text{Tr}_3(\omega R)}{\text{Tr}(\omega R)} \quad \text{and} \quad \omega|_{R^\perp} = \frac{\text{Tr}_3(\omega R^\perp)}{\text{Tr}(\omega R^\perp)}
\]

are POPT states on \( H_1 \otimes H_2 \). The marginal is then represented by \( \text{Tr}_3(\omega) = \text{Tr}(\omega R) \omega|_R + \text{Tr}(\omega R^\perp) \omega|_{R^\perp} \). Since \( \text{Tr}(\omega R) \) and \( \text{Tr}(\omega R^\perp) \) are nonnegative, and \( \text{Tr}(\omega R) + \text{Tr}(\omega R^\perp) = \text{Tr}(\omega) = 1 \), it follows that the marginal equals \( \omega|_R = \omega|_{R^\perp} \). As this holds for any projection \( R \) on \( H_3 \), the claim follows. \( \square \)

Lemma 15. If either of a pair of nonproduct POPT states \( \omega_{12} \in \mathcal{W}(H_1 \otimes H_2) \) and \( \omega_{13} \in \mathcal{W}(H_1 \otimes H_3) \) is extremal, the pair \( (\omega_{12}, \omega_{13}) \) have no extension.

Proof. Without loss of generality we assume that \( \omega_{12} \) is a extremal nonproduct POPT state. If there is an extension \( \omega \) of \( (\omega_{12}, \omega_{13}) \), it follows from the previous lemma that \( \omega \) is expressed as \( \omega = \omega_{12} \otimes \text{Tr}_{12}(\omega) \). Then \( \text{Tr}_2(\omega) \) is a product state, contradicting the assumption that \( \omega_{13} \) is a nonproduct state. \( \square \)

Now consider the valuation monad \( \mathcal{V} \) on the category of locales in \([\mathcal{C}(A_1) \times \mathcal{C}(A_2) \times \mathcal{C}(A_3), \text{Set}]\) for \( A_i = \mathcal{B}(H_i) \). If Hilbert spaces \( H_1, H_2, H_3 \) have dimension at least 3, then Theorem 12 and Riesz’s theorem imply that a Kleisli morphism \( 1 \rightarrow \prod_{i \in I} \Sigma_{A_i} \) represents a POPT state on \( \bigotimes_{i \in I} H_i \) for any \( I \subseteq \{1, 2, 3\} \). Note that taking marginals of a valuation on a spatially composed system is equivalent to taking partial traces of the corresponding POPT state (for proof see Appendix C).

Theorem 16. Let \( H_1, H_2, H_3 \) be Hilbert spaces with \( \dim(H_1) = \dim(H_2) = \dim(H_3) \geq 3 \), and set \( A_i = \mathcal{B}(H_i) \). A bipartite valuation \( v: 1 \rightarrow \mathcal{V}(\Sigma_{A_1} \times \Sigma_{A_2}) \) in \([\mathcal{C}(A_1) \times \mathcal{C}(A_2) \times \mathcal{C}(A_3), \text{Set}]\) has a conditional process if and only if it corresponds to a product POPT state.

Proof. Assume \( v \) corresponds to a nonproduct state and there is a conditional process \( f \) for \( v \) from \( \Sigma_{A_1} \) to \( \Sigma_{A_2} \). By purifying we can always construct a pure bipartite state \( \omega \) on \( H_1 \otimes H_3 \) with the same marginal state on \( H_1 \) as \( v \). Then \( \omega \) is not a product state as \( v \) does not corresponds to a product state. Let \( v': 1 \rightarrow \mathcal{V}(\Sigma_{A_1} \times \Sigma_{A_3}) \) be a valuation corresponding to \( \omega \). Theorem 7 now constructs an extension of \( (v, v') \), contradicting Lemma 15. So if \( v \) has a conditional process, \( v \) must correspond to a product state.

Conversely, if \( v \) corresponds to a product state, it is a product valuation \( 1 \rightarrow \mathcal{V}(\Sigma_{A_1} \times \Sigma_{A_3}) \) of valuations \( v_1: 1 \rightarrow \mathcal{V}(\Sigma_{A_1}) \) and \( v_2: 1 \rightarrow \mathcal{V}(\Sigma_{A_3}) \). Precomposing with the unique morphism to the terminal object gives a conditional process \( \Sigma_{A_1} \rightarrow 1 \Rightarrow \mathcal{V}(\Sigma_{A_3}) \) for \( v \) from \( \Sigma_{A_1} \) to \( \Sigma_{A_2} \). \( \square \)

Thus nonextensibility of POPT states implies triviality of Kleisli morphisms for the valuation monad.

Valuations on temporally composed system Since the states on temporally composed systems do not correspond to POPT states, there may still be nontrivial Kleisli morphisms. We now investigate valuations on such systems, and discuss a possibility to use the temporal composition for describing “temporal correlation.”

Let us first consider the 2-time temporally composed system \( \Sigma_1^A \times \Sigma_2^A \) in \([\mathcal{C}(A), \text{Set}]\). The unit morphisms are denoted by \( \eta_{i \rightarrow i}: \Sigma_i^A \rightarrow \mathcal{V}(\Sigma_i^A) \) for \( i = 1, 2 \). There is a trivial *-isomorphism \( \alpha: \Sigma_1^A \rightarrow \Sigma_2^A \) and we denote the composition \( \eta_{2 \rightarrow 2} \circ \alpha: \Sigma_1^A \rightarrow \mathcal{V}(\Sigma_2^A) \) by \( \eta_{1 \rightarrow 2} \). Thus we now have at least one Kleisli morphism \( \eta_{1 \rightarrow 2} \) other than those of the form \( \Sigma_i^A \rightarrow 1 \rightarrow \mathcal{V}(\Sigma_j^A) \). The joint state on \( \Sigma_1^A \times \Sigma_2^A \) having \( \eta_{1 \rightarrow 2} \) as the conditional process and the marginal state \( v_1: 1 \rightarrow \mathcal{V}(\Sigma_1^A) \) is \((\text{dist}_{1,2} \circ (\eta_{1 \rightarrow 1}, \eta_{1 \rightarrow 2})) \circ \text{KL} v_1 \). By the definition of \( \eta_{1 \rightarrow 2} \), the marginals of this state on \( \Sigma_1^A \) and \( \Sigma_2^A \) both represents the same quantum state \( v_1 \).

Next we analyse extensions constructed through Theorem 7 by using \( \eta_{1 \rightarrow 2} \) as the conditional process. Rather than the 3-time temporal composition, let us consider the spatial composition \( (\Sigma_1^A \times \Sigma_2^A) \times \Sigma_A \in \).
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$[\mathcal{C}(A) \times \mathcal{C}(A_3), \text{Set}]$ of $\Sigma_1 \times \Sigma_3 \in [\mathcal{C}(A), \text{Set}]$ and $\Sigma_A \in [\mathcal{C}(A_3), \text{Set}]$. Valuations on $\Sigma_1 \times \Sigma_3$ and $\Sigma_3 \times \Sigma_A$ correspond to POPT states on the respective systems.

**Theorem 17.** Let $v_{13} : 1 \to \mathcal{V}(\Sigma_1 A \times \Sigma_A)$ be a valuation corresponding to a POPT state $\omega$. Let $v_{123} : 1 \to \mathcal{V}(\Sigma_1 A \times \Sigma_2 A \times \Sigma_A)$ be a valuation defined by

$$v_{123} = (\text{dst}_{1\times 2, 3} \circ (\text{dst}_{1, 2} \circ (\eta_{1\to 1} \times \eta_{1\to 2} \times \eta_{2\to 3}))) \circ \text{KI} v_{13},$$

namely, $v_{123}$ is an extension of $v_{13}$ constructed by a conditional process $\eta_{1\to 2}$ through (7). Then the marginal valuation of $v_{123}$ on $\Sigma_2 A \times \Sigma_A$ corresponds to $\omega$ (see Figure 1 for a schematic presentation of this Theorem).

**Proof.** See Appendix D. \qed

Thus $v_{123}$ is a symmetric extension of $\omega$. It is impossible to express $v_{123}$ as a tripartite POPT state if $\omega$ is an extremal bipartite POPT state (Lemma 15).

For example, consider three spacetime regions as in Figure 2. The quantum states on pairs of space-like separated systems $(A_1, A_3)$ and $(A_2, A_3)$ are individually well-defined, but we usually do not consider the composition of these three systems and states. If the bipartite state on $(A_1, A_3)$ is $\omega$ and the map from $A_1$ to $A_2$ is the identity, it is natural to consider a tripartite ‘state’ on $(A_1, A_2, A_3)$ with both marginals $(A_1, A_3)$ and $(A_2, A_3)$ given by $\omega$. Temporal composition can express this kind of state.

However, we have not found any nontrivial Kleisli morphisms $f : \Sigma A_1 \to \Sigma A_2$, but if there is one, we can replace $\eta_{1\to 2}$ in (10) with $f$ to model a nontrivial process from $A_1$ to $A_2$ in Figure 2.

![Figure 1: Schematic presentation of Theorem 17](image1.png)

![Figure 2: Spacetime configuration: $A_1$ and $A_3$, and $A_2$ and $A_3$ are spatially separated, $A_2$ is after $A_1$.](image2.png)

5 Concluding remarks

We generalised extension problems to monads on cartesian categories, and investigated when CLL solution is available. To do so, we defined conditional processes as Kleisli morphisms generalizing conditional probability matrices. For commutative monads with $T 1 \cong 1$, if either of the bipartite joint states $s_{XY}$ and $s_{XZ}$ with overlapping marginal $s_X$ has a conditional processes from $X$, we have constructed an extension.

It is often said that cartesian produces are not suitable to compose quantum systems, for example because of the no-cloning theorem [7]. Nevertheless, the Kleisli category of a monad on a cartesian category may have a quantum-like structure. One example is the category of $\text{Rel}$ of sets and relations, which is
the Kleisli category of the power set monad on Set \([7]\). Another example is the Fock space monad on the category of Banach spaces (for bosons) or finite dimensional vector spaces (for fermions) \([4]\); both underlying categories have finite products (direct sums). The exponential law \(\mathcal{F}(H_1 \oplus H_2) \cong \mathcal{F}(H_1) \otimes \mathcal{F}(H_2)\) gives composition \([1]\), but the ensuing extension problem is somewhat trivial, as the marginal maps \(\mathcal{F}\pi\) are not partial traces. Another example still is the valuation monad in topos quantum theory.

Section \([5]\) investigated composite topos quantum systems independently of the extension problem. Both spatial and temporal composition comes down to taking coproducts of marginal algebras, or, equivalently, taking products of their spectra.

We proved that valuations on a spatially composed system correspond bijectively with POPT states. This presents an attack on the open problem of when POPT states restrict to quantum states, as follows. It is known that \(\mathcal{C}(A)\) is generally not enough to reconstruct a C*-algebra \(A\) \([15, 14]\). The existence of POPT states which are not quantum states would imply that \(\mathcal{C}(A_1) \times \mathcal{C}(A_2)\) does not suffice to define states for \(A_1 \otimes A_2\), whereas \(\mathcal{C}(A)\) does for \(A\). Adding active lattice structure \([17]\) to \(\mathcal{C}(A)\) does suffice to reconstruct \(A\); translating this condition to POPT states is therefore a good candidate for restricting to quantum states.

Finally, we applied the extension results to compound topos quantum systems. We showed that spatially composed systems need not extend even if they coincide on the overlap, and thus that conditional processes cannot exist for nonproduct joint valuations. For temporally composed systems, we exhibited an extension induced by a non-trivial Kleisli morphism, and we have discussed the use of extended states to describe temporal correlations. A quantitative analysis on the temporal correlations will require an explicit description of the lattice structure of the composite Gelfand spectra, and we leave it to future works.

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A Coproducts of internal unital commutative C*-algebras

This appendix proves Theorem 8 by generalizing [33].

The object \( A_1 \otimes \cdots \otimes A_n \) is an internal unital commutative C*-algebra, since each component \( A_1 \otimes \cdots \otimes A_n(X) \) is a unital commutative C*-algebra in \( \text{Set} \), and a tensor product of unital *-homomorphisms is again a unital *-homomorphism. Interpretation of theories for unital *-homomorphisms by the Kripke-Joyal semantics on \([\mathcal{C}, \text{Set}]\) reveals that a natural transformation \( \alpha : A \to B \) between internal unital commutative C*-algebra objects \( A, B \) in \([\mathcal{C}, \text{Set}]\) is an internal *-homomorphism if and only if all the components are *-homomorphisms of unital commutative C*-algebras in \( \text{Set} \). For example, linearity

\[
\forall a, b \in A, \alpha(a) + \alpha(b) = \alpha(a + b)
\]

holds

iff \( \forall X \in \mathcal{C}, X \models \forall a, b \in A : \alpha(a) + \alpha(b) = \alpha(a + b) \)

iff \( \forall X \in \mathcal{C}, \forall f : X \to Y, \forall a, b \in A(Y), Y \models \alpha(a) + \alpha(b) = \alpha(a + b) \)

iff \( \forall X \in \mathcal{C}, \forall f : X \to Y, \forall a, b \in A(Y), \forall g : Y \to Z, \alpha_Z(A(g)(a)) + \alpha_Z(A(g)(b)) = \alpha_Z(A(g)(a + b)) \).

Since \( A(g) \) is a *-homomorphism, the last line holds if and only if

\[
\forall X \in \mathcal{C}, \forall f : X \to Y, \forall a, b \in A(Y), \forall g : Y \to Z, \alpha_Z(A(g)(a)) + \alpha_Z(A(g)(b)) = \alpha_Z(A(g)(a) + A(g)(b)),
\]

which is equivalent to the simpler statement

\[
\forall X \in \mathcal{C}, \forall a, b \in A(X), \alpha_X(a) + \alpha_X(b) = \alpha_X(a + b). \tag{11}
\]

In other words, linearity of a *-homomorphism just means linearity of all its components. The other axioms for *-homomorphisms are interpreted in the same manner.

Define natural transformations \( \alpha^i : A_i \to A_1 \otimes \cdots \otimes A_n \) as the candidate coproduct injections by

\[
\alpha^i_X(a_i) = \mathbb{I}_1 \otimes \cdots \otimes \mathbb{I}_{i-1} \otimes a_i \otimes \mathbb{I}_{i+1} \otimes \cdots \otimes \mathbb{I}_n. \tag{12}
\]

These natural transformations are internal *-homomorphisms because each component is.

Now let \( A \) be any internal unital commutative C*-algebra, with internal *-homomorphisms \( \beta^i : A_i \to A \). Consider morphisms \( \gamma_X : A_1(X) \otimes \cdots \otimes A_n(X) \to A(X) \) defined for \( X \in \mathcal{C} \) by

\[
\gamma_X(a_1 \otimes \cdots \otimes a_n) = \beta^1_X(a_1)\beta^2_X(a_2)\cdots \beta^n_X(a_n).
\]
This is a natural transformation since for \( f : X \rightarrow Y \) we have
\[
\gamma_f(A_1 \otimes \cdots \otimes A_n(f)(a_1 \otimes \cdots \otimes a_n)) = \gamma_f(A_1(f)(a_1) \otimes \cdots \otimes A_n(f)(a_n))
\]
\[
= \beta_1(1(A_1(f)(a_1)) \beta_2(A_2(f)(a_2)) \cdots \beta_n(A_n(f)(a_n))
\]
\[
= A(f)\beta_1(A_1(f)(a_1))A(f)\beta_2(A_2(f)(a_2)) \cdots A(f)\beta_n(A_n(f)(a_n))
\]
\[
= A(f)\beta_1(A_1(f)(a_1))\beta_2(A_2(f)(a_2)) \cdots \beta_n(A_n(f)(a_n))
\]
\[
= A(f)(\gamma_f(a_1 \otimes \cdots \otimes a_n)).
\]
Clearly \( \gamma \circ \beta = \alpha \), and since each component is unique, \( \gamma \) is the unique mediating map satisfying this.

## B Integrals and POPT states

This appendix proves Theorem \([12]\) We first review unentangled frame functions \([31][27]\), and a generalisation of Gleason’s theorem called the unentangled Gleason’s theorem, which the proof uses.

**Definition 18.** Let \( H_1, \ldots, H_n \) be Hilbert spaces, and Prod\( (H_1, \ldots, H_n) \) be the set of all product unit vectors on \( H_1 \otimes \cdots \otimes H_n \). An unentangled frame function is a function \( f : \text{Prod}(H_1, \ldots, H_n) \rightarrow [0, \infty) \) such that \( \sum f(\xi_j) = w \in [0, \infty) \) whenever \( \{ \xi_j \} \) is an orthonormal basis of \( H_1 \otimes \cdots \otimes H_n \) with each \( \xi_j \in \text{Prod}(H_1, \ldots, H_n) \). We call \( w \) the weight of \( f \).

Write \( \text{UFF}^1(H_1, \ldots, H_n) \) for the set of unit-weight unentangled frame functions. For composite systems, we have the following theorem \([31]\).

**Theorem 19.** Let \( H_1, \ldots, H_n \) be finite-dimensional Hilbert spaces each of dimension at least 3, and let \( f : \text{Prod}(H_1, \ldots, H_n) \rightarrow [0, \infty) \) be an unentangled frame function. There exists a self-adjoint operator \( \omega_f \in \mathcal{B}(H_1 \otimes \cdots \otimes H_n) \) such that whenever \( v_1 \otimes \cdots \otimes v_n \in \text{Prod}(H_1, \ldots, H_n) \), and \( p_i \) is the projection of \( H_i \) onto the one-dimensional subspace generated by \( v_i \), then:

\[
f(v_1 \otimes \cdots \otimes v_n) = \text{Tr}(p_1 \otimes \cdots \otimes p_n \omega_f).
\]  
(13)

In fact, the operator \( \omega_f \) is unique to \( f \) \([27]\). This implies a bijective correspondence between POPT states and unit-weight unentangled frame functions, that under the translation of Theorem \([12]\) becomes an injective map from \( \text{UFF}^1(H_1, \ldots, H_n) \) to integrals defined by

\[
f \mapsto \begin{cases} \{ I_{(C_1, \ldots, C_n)}^f : (C_1 \otimes \cdots \otimes C_n)_{\text{sa}} \rightarrow \mathbb{R} \}_{(C_1, \ldots, C_n)_{\text{sa}} \in \mathcal{C}(A_1) \times \cdots \times \mathcal{C}(A_n)} & \text{if } p_i \in C_i \text{ for all } i, \\
I_{(C_1, \ldots, C_n)}(p_1 \otimes \cdots \otimes p_n) = f(v_1 \otimes \cdots \otimes v_n) & \text{(14)}
\end{cases}
\]

where \( A_i = \mathcal{B}(H_i) \). We show that the map from integrals to \( \text{UFF}^1(H_1, \ldots, H_n) \) presented by

\[
\begin{align*}
\{ I_{(C_1, \ldots, C_n)} : (C_1 \otimes \cdots \otimes C_n)_{\text{sa}} \rightarrow \mathbb{R} \}_{(C_1, \ldots, C_n)_{\text{sa}} \in \mathcal{C}(A_1) \times \cdots \times \mathcal{C}(A_n)} & \mapsto f^I, \\
f^I(v_1 \otimes \cdots \otimes v_n) & = I_{(C_1, \ldots, C_n)}(p_1 \otimes \cdots \otimes p_n) & \text{(15)}
\end{align*}
\]

is well-defined and injective. First observe that for any integral \( \{ I_{(C_1, \ldots, C_n)} \}_{(C_1, \ldots, C_n)_{\text{sa}} \in \mathcal{C}(A_1) \times \cdots \times \mathcal{C}(A_n)} \), and for all product projectors \( p_1 \otimes \cdots \otimes p_n \in (C_1 \otimes \cdots \otimes C_n)_{\text{sa}} \cap (C_1' \otimes \cdots \otimes C_n')_{\text{sa}} \)

\[
I_{(C_1, \ldots, C_n)}(p_1 \otimes \cdots \otimes p_n) = I_{(C_1 \cup C_1' \cap C_2, \ldots, C_n)_{\text{sa}}}(p_1 \otimes \cdots \otimes p_n) = I_{(C_1, \ldots, C_n)}(p_1 \otimes \cdots \otimes p_n),
\]

since \( p_1 \otimes \cdots \otimes p_n \in (C_1 \otimes \cdots \otimes C_n)_{\text{sa}} \cap (C_1' \otimes \cdots \otimes C_n')_{\text{sa}} \) implies \( p_1 \otimes \cdots \otimes p_n \in (C_1 \cap C_1' \otimes \cdots \otimes C_n \cap C_n')_{\text{sa}} \). Thus the value of integration does not depend on the context, and the map \([15]\) is well-defined. Furthermore, the map is injective, since the context-wise linearity of integrals shows that an integral is uniquely determined by its value on product projectors. Finally, the map \([15]\) is the inverse of the map \([14]\). This completes the proof of Theorem \([12]\).
C Partial trace and projections

This appendix proves that the projection \( \mathcal{Y}(\pi_1): \mathcal{Y}(\Sigma_{A_1} \times \Sigma_{A_2}) \to \Sigma_{A_1} \) represents the partial trace on POPT states, if \( \Sigma_{A_1} \times \Sigma_{A_2} \) represents a spatial composition. We will extend taking integrals and valuations on commutative \( C^* \)-algebras to contravariant functors \( \mathcal{F}: \mathbf{cCstar} \to \mathbf{Loc} \) and \( \mathcal{Y} \circ \Sigma: \mathbf{cCstar} \to \mathbf{Loc} \) to the category of locales. This will let us formulate Riesz’s theorem as an isomorphism between these functors [3].

Definition 20. The functor \( \mathcal{F}: \mathbf{cCstar} \to \mathbf{Loc} \) acts a *-homomorphism \( f: A_1 \to A_2 \) as

\[
\mathcal{F}(f)(a_1) = f(a_1),
\]

where \( I \in \mathcal{F}(A_2) \) and \( a_1 \in A_1 \). The functor \( \mathcal{Y} \circ \Sigma: \mathbf{cCstar} \to \mathbf{Loc} \) acts on a morphism \( f: A_1 \to A_2 \) as

\[
(\mathcal{Y} \circ \Sigma f)(\mu)(r < a_1 < s) = \mu((\Sigma f)^*(r < a_1 < s)) = \mu(r < f(a_1) < s),
\]

where \( \mu \in (\mathcal{Y} \circ \Sigma)(A_2) \), and “\( r < a < s \)” (for \( a \in A_1 \) and \( r, s \in \mathbb{Q} \) represents an open of the spectrum \( \Sigma A_1 \) as defined in [3] (the same open is denoted “\( a \in (r, s) \)” in [2]).

Lemma 21. The locale isomorphism \( \mathcal{F}(A) \cong (\mathcal{Y} \circ \Sigma)(A) \) extends to a natural isomorphism \( \mathcal{F} \cong \mathcal{Y} \circ \Sigma \).

Proof. We check the commutativity of the following diagram

\[
\begin{array}{ccc}
A_1 & \xrightarrow{\mathcal{F}(f)} & (\mathcal{Y} \circ \Sigma)(A_1) \\
\mathcal{F}(I) & \xrightarrow{\cong} & (\mathcal{Y} \circ \Sigma)(A_2) \\
\mathcal{F}(I) & \xrightarrow{\cong} & (\mathcal{Y} \circ \Sigma)(A_2) \\
\end{array}
\]

The isomorphism \( (\mathcal{Y} \circ \Sigma)(A_2) \xrightarrow{\cong} \mathcal{F}(A_2) \) sends a valuation \( \mu \) to an integral \( I_\mu \in \mathcal{F}(A_2) \) defined by

\[
I_\mu(a_2) = \left( \sup_{(s_i)} \sum s_i \mu(s_i < a_2 < s_{i+1}), \inf_{(s_i)} \sum s_{i+1}(1 - \mu(a_2 < s_i) - \mu(s_{i+1} < a_2)) \right),
\]

which is further transformed to an integral \( \mathcal{F}(I_\mu) \in \mathcal{F} A_1 \) explicitly given by

\[
\mathcal{F}(I_\mu)(a_1) = \left( \sup_{(s_i)} \sum s_i \mu(s_i < f(a_1) < s_{i+1}), \inf_{(s_i)} \sum s_{i+1}(1 - \mu(f(a_1) < s_i) - \mu(s_{i+1} < f(a_1))) \right).
\]

On the other hand, \( (\mathcal{Y} \circ \Sigma)(f) \) sends \( \mu \) to \( (\mathcal{Y} \circ \Sigma)(f)(\mu) \in (\mathcal{Y} \circ \Sigma)(A_1) \) by [16], which is further transformed by the isomorphism \( (\mathcal{Y} \circ \Sigma)(A_1) \xrightarrow{\cong} \mathcal{F}(A_1) \) to an integral \( I_{(\mathcal{Y} \circ \Sigma)(f)(\mu)} \) given explicitly by

\[
I_{(\mathcal{Y} \circ \Sigma)(f)(\mu)}(a_1) = \left( \sup_{(s_i)} \sum s_i (\mathcal{Y} \circ \Sigma)(f)(\mu)(s_i < a_1 < s_{i+1}), \inf_{(s_i)} \sum s_{i+1}(1 - (\mathcal{Y} \circ \Sigma)(f)(\mu)(a_1 < s_i) - (\mathcal{Y} \circ \Sigma)(f)(\mu)(s_{i+1} < a_1)) \right)
\]

\[
= \left( \sup_{(s_i)} \sum s_i \mu(s_i < f(a_1) < s_{i+1}), \inf_{(s_i)} \sum s_{i+1}(1 - \mu(f(a_1) < s_i) - \mu(s_{i+1} < f(a_1))) \right).
\]

Thus \( \mathcal{F}(f)(I_\mu) \) and \( I_{(\mathcal{Y} \circ \Sigma)(f)(\mu)} \) are the same integral over \( A_1 \), and [17] commutes. \( \square \)
Gelfand duality and Lemma 2 now imply that the projection \( \gamma \pi_1 : \gamma(\Sigma_1 \times \Sigma_2) \to \Sigma_1 \) is equivalent to \( \mathcal{S}(\alpha^1) : \mathcal{S}(A_1 \otimes A_2) \to \mathcal{S}A_1 \), for the injection \( \alpha^1 : A_1 \to A_1 \otimes A_2 \) of the coproduct. Injections for a coproduct of C*-algebras internal to the toposes in question are context-wise injections by (12).

By the definition of the coproduct injection, an integral \( I \) over \( A_1 \otimes A_2 \) and its marginal \( \mathcal{S}(\alpha^1(I)) \) over \( A_1 \) give the same values to observables in \( A_1 \). From the bijection between integrals and POPT states, \( \mathcal{S}(\alpha^1) : \mathcal{S}(A_1 \otimes A_2) \to \mathcal{S}A_1 \) amounts to taking a partial trace on the corresponding POPT states. Since \( \gamma'(\pi_1) : \gamma(\Sigma_1 \times \Sigma_2) \to \Sigma_1 \), is equivalent to \( \mathcal{S}(\alpha^1) : \mathcal{S}(A_1 \otimes A_2) \to \mathcal{S}A_1 \), the projection \( \gamma'(\pi_1) \) is also equivalent to the partial trace.

D Proof of Theorem 17 and the upper left triangle of (6)

This appendix proves that the diagram

\[
\begin{array}{ccc}
W \times Z & \xrightarrow{\text{dst}_{X,Y} \circ (f_{W \to X}, f_{W \to Y}) \times \eta_Z} & T(X \times Y) \\
& \downarrow T \pi_X \times \text{tid}_Z & \downarrow T \pi_X \times Z \\
T X \times T Z & \xrightarrow{\text{dst}_{X,Z}} & T(X \times Z)
\end{array}
\]

(18)

commutes for any monad \( (T, \mu, \eta) \) on a cartesian category \( C \) such that \( T1 \cong 1 \), any objects \( X, Y \) and \( Z \), and any morphisms \( f_{W \to X} : W \to TX \) and \( f_{W \to Y} : X \to TY \). If we substitute \( X \) for \( W \), \( \eta_X \) for \( f_{X \to X} \), the outer triangle of \( (18) \) is the upper left triangle of (6). If we substitute \( \Sigma^1 \) for \( W \) and \( Y, \Sigma^2 \) for \( X, \Sigma^3 \) for \( Z, \eta_1 \to 2 \) for \( f_{W \to X}, \eta_1 \to 1 \) for \( f_{W \to Y} \) and \( \eta_3 \to 3 \) for \( \eta_Z \), the outer triangle of \( (18) \) states

\[
\gamma'((\pi_2 \circ \eta_3) \circ ((\pi_1 \circ (\eta_1 \to 1, \eta_1 \to 2) \times (\eta_3 \to 3))) = \text{dst}_{X,Y} \circ (\eta_1 \to 2 \times (\eta_3 \to 3) = \eta_2 \times (\eta_3 \to 3) \circ (\alpha \times \text{id}_3),}
\]

from which Theorem 17 follows. (This reasoning uses the fact that \( \text{dst}_{W,V} \circ (\eta_W \times \eta_V) = \eta_{W \times V} \) since \( \eta \) is a monoidal natural transformation if \( T \) is commutative.)

The left triangle decomposes into two triangles

\[
\begin{array}{ccc}
W & \xrightarrow{\text{dst}_{X,Y} \circ (f_{W \to X}, f_{W \to Y})} & T(X \times Y) \\
& \downarrow T \pi_X & \downarrow \eta_Z \\
T X & \xrightarrow{T \pi_X} & T Z
\end{array}
\]

The right triangle clearly commutes, and commutativity of the left one follows from Lemma 3.

To prove that the square of \( (18) \) commutes, denote the domain of a projections on its right shoulder, e.g. \( \pi^X_{X \times Y} : X \times Y \to X \). By definition of the product \( X \times Z \), the projection \( \pi^X_{X \times Z} \) is the unique morphism making the following diagram commute:

\[
\begin{array}{ccc}
\pi^X_{X \times Y} & \xrightarrow{\pi^X_{X \times Z} \circ \pi^X_{X \times Z}} & X \times Y \times Z \\
\pi^X_{X \times Z} & \xrightarrow{\pi^X_{X \times Z} \circ \pi^X_{X \times Z}} & X \times Z \\
\pi^X_{X \times Z} & \xrightarrow{\pi^X_{X \times Z} \circ \pi^X_{X \times Z}} & Z
\end{array}
\]

On the other hand, \( \pi^X_{X \times Y} \times \text{id}_Z \) is by definition \( (\pi^X_{X \times Y} \circ \pi^X_{X \times Z} \circ \text{id}_Z \circ \pi^X_{X \times Z}) = (\pi^X_{X \times Y} \times \pi^X_{X \times Y} \times \text{id}_Z) \). Thus \( \pi^X_{X \times Y} \times \text{id}_Z \) equals \( \pi^X_{X \times Y} \times \text{id}_Z \). The square in \( (18) \) commutes, because if we replace the right-most projection \( T \pi^X_{X \times Z} \) by \( T(\pi^X_{X \times Y} \times \text{id}_Z) \), the square represents naturality of the Fubini map dst. This completes the proof.