Regularization of identity based solution in string field theory

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Abstract: We demonstrate that an Erler-Schnabl type solution in cubic string field theory can be naturally interpreted as a gauge invariant regularization of an identity based solution. We consider a solution which interpolates between an identity based solution and ordinary Erler-Schnabl one. Two gauge invariant quantities, the classical action and the closed string tadpole, are evaluated for finite value of the gauge parameter. It is explicitly checked that both of them are independent of the gauge parameter.

Keywords: String Field Theory, Tachyon Condensation
1. Introduction

Identity based solutions, which are constructed upon the identity string field, have been mysterious objects in string field theory. Typically a solution takes form of
\[ \Psi = CI, \]
where \( C \) is certain linear combination of ghost number one operators and \( I \) is the identity string field, a surface state which represents an open string world sheet of vanishing width. Such solutions have been considered since early days of string field theory [1–3]. More elaborated versions have been investigated [4–15] to describe tachyon condensation and marginal deformation. Even after Schnabl’s discovery of the analytic solution [16], it had been recognized that an identity based string field is useful to construct regular solutions [17–20]. However, even though much efforts have been done in past, identity based solutions have not yet been widely accepted as a regular solution. A major problem is indefiniteness of physical quantities which originates from the inner product between identity based string fields. Naive evaluation of the classical action in terms of CFT method tends to be indefinite since it corresponds to a correlator on vanishing strip. Various attempts of regularization in terms of a strip of infinitesimal width still seem to fail to give definite value of the classical action [5, 21, 22].

In this paper, we consider one-parameter family of classical solutions in open string field theory given by
\[ \Psi_\lambda = U_\lambda Q_B U_\lambda^{-1} + U_\lambda \Psi_I U_\lambda^{-1}, \]
where \( \Psi_I \) is an identity based solution and \( U_\lambda = 1 + \lambda cBK \) is an element of gauge group [21]. Since \( U_\lambda \) approaches to identity as \( \lambda \) goes to zero, it is clear that the above string field naturally defines gauge invariant regularization which interpolates between an identity based solution and non identity one. We will see that suitable choice of \( \Psi_I \) yields an Erler-Schnabl type solution.

Rest of the paper is devoted to evaluation of two gauge invariant quantities, the classical action and the closed string tadpole, following to a method used in [22,24]. It will be checked that both of them are independent of the gauge parameter \( \lambda \).
2. Regularization of an identity based solution

To begin with, let us briefly review the notation used in [17, 18, 23, 24] which will be extensively used in this paper. The building block of our solution is elements of ‘$KBc$ subalgebra’ which is defined by

\[ \{B, c\} = 1, \quad [B, K] = 0, \quad \{B, B\} = 0, \quad \{c, c\} = 0. \]  

(2.1)

The action of the BRST charge on these elements is

\[ Q_Bc = cKc, \quad Q_BB = K, \quad Q_BK = 0. \]  

(2.2)

In this notation, the star multiplication between elements is understood. The BPZ inner product of elements is denoted as ‘trace’. Next, let us regularize a simple identity based solution appeared in [21, 24, 25]. A clue to regularization is Arroyo’s observation made in [21], where it was shown that the identity based solution

\[ \Psi_A = c(1 - K) \]  

(2.3)

is gauge equivalent to the Erler-Schnabl solution.

\[ \Psi_{ES} = c(1 - K)Bc \frac{1}{1 + K} = Uc(1 - K)U^{-1} + UQ_BU^{-1}, \]  

(2.4)

where $U$ is a gauge transformation given by

\[ U = 1 + cBK, \quad U^{-1} = 1 - cBK \frac{1}{1 + K}. \]  

(2.5)

Here we consider a slight modification of (2.5) in which a real parameter is inserted in front of the $cBK$ piece in the gauge transformation.

\[ U_{\lambda} = 1 + \lambda cBK, \quad U_{\lambda}^{-1} = 1 - \lambda cBK \frac{1}{1 + \lambda K}. \]  

(2.6)

Then, one-parameter family of solutions is obtained by performing the above gauge transformation on the identity bases solution (2.3) \(^1\).

\[ \Psi_{\lambda} = U_{\lambda}Q_BU_{\lambda}^{-1} + U_{\lambda}c(1 - K)U_{\lambda}^{-1} = c(1 + \lambda K)Bc \frac{1 + (\lambda - 1)K}{1 + \lambda K}. \]  

(2.7)

A check of equation of motion for the above solution is straightforward. It should be noticed that the solution resembles the non-real form of Erler-Schnabl’s solution of [24]. Then we are interested in how general such solution is. Fortunately, a wider class of

\(^1\)A real solution can be obtained by putting $\sqrt{(1 + \lambda K)/(1 + (\lambda - 1)K)}$ on both sides of the solution. The author thank H. Isono for discussion.
solutions is already considered in [24] and further explored in [25]. Here we focus on a class of solutions of the form

$$\Psi = cF Bc G,$$  \hfill (2.8)

where $F$ and $G$ are functions of $K$. Equation of motion for this field becomes

$$c \{(FG + K)cF - Fc(FG + K)\} BcG = 0.$$  \hfill (2.9)

There are two nontrivial solutions of (2.9),

$$FG + K = F, \quad FG + K = 0,$$  \hfill (2.10)

and corresponding string fields

$$\Psi_1 = cF Bc \left(1 - \frac{K}{F}\right), \quad \Psi_2 = -cF Bc K \frac{F}{F}.$$  \hfill (2.11)

It can be easily seen that the solution (2.7) corresponds to $\Psi_1$ with $F = 1 + \lambda K$.

Next, let us evaluate classical action in CFT method. As for a classical solution of the equation of motion, we only need to evaluate

$$E = \frac{1}{6} \text{Tr}[\Psi_\lambda Q_B \Psi_\lambda].$$  \hfill (2.12)

We evaluate this quantity following to [22, 24]. As similar to the case of [24], the second term of (2.7) which begins from $c\lambda K$ does not contribute to $E$. Introduction of Schwinger parameters leads

$$\text{Tr}[\Psi_\lambda Q_B \Psi_\lambda] = \int_0^\infty \int_0^\infty dt_1 dt_2 e^{-t_1-t_2} \times \text{Tr}[c\{1 + (\lambda - 1)K\} \Omega^{\lambda t_1} c^K \{1 + (\lambda - 1)K\} \Omega^{\lambda t_2}].$$  \hfill (2.13)

All terms in (2.13) can be obtained by differentiating the basic trace $h(\lambda t_1, \lambda t_2)$ with respect to $t_1$ or $t_2$. Here, the basic trace $h(s, t) = \text{Tr}[c \Omega^s cK \Omega^t]$ is already known to be

$$h(s, t) = -\left(\frac{s + t}{\pi}\right)^2 \sin^2 \left(\frac{\pi s}{s + t}\right).$$  \hfill (2.14)

The trace in the right hand side of (2.13) is can be written as

$$(\text{trace}) = \left(1 + \frac{1 - \lambda}{\lambda} \partial_{t_1}\right) \left(1 + \frac{1 - \lambda}{\lambda} \partial_{t_2}\right) h(\lambda t_1, \lambda t_2).$$  \hfill (2.15)

Remaining process is completely parallel to that of [24]. We change variables as $t_1 \to uv, \quad t_2 \to u(1 - v)$ and perform $v$ integral. This gives

$$\text{Tr}[\Psi_\lambda Q_B \Psi_\lambda] = -\frac{1}{2\pi^2} \int_0^\infty du e^{-u} \left\{\lambda^2 u^3 + 6l(1 - l)u^2 + 6(\lambda - 1)^2 u\right\}.$$  \hfill (2.16)
At this stage, the integrand still depends on $\lambda$. However, it turns out that the $\lambda$ dependence disappears once we perform $u$ integration. More precisely, with the help of the formula
\[
\int_0^\infty du e^{-u} u^n = \Gamma(n+1) = n!,
\]
(2.17)
(2.16) is evaluated as
\[
\text{Tr}[\Psi_\lambda Q_B \Psi_\lambda] = -\frac{1}{2\pi^2} \left(6\lambda^2 + 12\lambda(1-\lambda) + 6(\lambda-1)^2\right)
= -\frac{3}{\pi^2},
\]
(2.18)
which corresponds to the D-brane tension as
\[
E = \frac{1}{6} \text{Tr}[\Psi_\lambda Q_B \Psi_\lambda] = -\frac{1}{2\pi^2}.
\]
(2.19)
Our major concern is the $\lambda \to 0$ limit in which the solution approaches to the identity based configuration. Let us see whether singularity occurs in each step of our calculation. First, we find negative powers of $\lambda$ in (2.15) which diverges in vanishing $\lambda$ limit. Therefore $\lambda$ should be kept finite at this stage. Once the trace is evaluated explicitly, inverse of $\lambda$ disappears so we can take the limit. In fact, setting $\lambda$ to zero in (2.16) gives correct answer. This nicely explains why taking $\lambda \to 0$ limit before evaluation of the trace yields singular answer.

Another gauge invariant quantity, the closed string tadpole, can be evaluated in a similar way as in [24]. Again, the second term in (2.7) does not contribute to the tadpole due to the BRST invariance of the tadpole. Then we would like to evaluate
\[
\text{Tr}[V \Psi_\lambda] = \text{Tr} \left[ V \frac{1+\lambda K}{1+\lambda K} \right],
\]
(2.20)
where $V = \bar{c}\bar{c} V_{\text{matter}}$ is a closed string vertex operator insertion at open string midpoint. The first term of (2.20) is evaluated as
\[
\text{Tr} \left[ V \frac{1}{1+\lambda K} \right] = \text{Tr}[Vc\Omega] \times \int_0^\infty dt e^{-t}(t\lambda)
\]
(2.21)
While the second term is given by
\[
\text{Tr} \left[ V \frac{(\lambda-1)K}{1+\lambda K} \right] = \frac{1-\lambda}{\lambda} \int_0^\infty dt e^{-t}\partial_t \text{Tr}[Vc\Omega^t]
= \frac{1-\lambda}{\lambda} \int_0^\infty dt e^{-t}\partial_t \text{Tr}[(t\lambda)Vc\Omega]
= \text{Tr}[Vc\Omega] \times \int_0^\infty dt e^{-t}(1-\lambda),
\]
(2.22)
where we perform scale transformation in the second line of (2.22). Then sum of above two terms gives
\[
\text{Tr}[V \Psi_\lambda] = \text{Tr}[Vc\Omega] \times \int_0^\infty dt e^{-t}(1-\lambda + t\lambda).
\]
(2.23)
As similar to the case of the classical action, the last integral in right hand side of (2.23) does not depends on $\lambda$. Furthermore, it coincides with an expected answer of closed string tadpole on the disk [26].

$$\text{Tr}[V\Psi_\lambda] = \text{Tr}[Ve\Omega] \times 1 = (\mathcal{V}(i\infty)c(0))_{C_1}. \quad (2.24)$$

3. Simpler solution

It is known that there is more simpler identity based solution [22, 24, 25].

$$\Psi_S = -cK$$
$$= \Psi_A - c. \quad (3.1)$$

Let us examine the gauge transformation $U_\lambda$ for this solution. Applying it to (3.1) we have

$$\Psi'_\lambda = \Psi_\lambda - U_\lambda c U^{-1}_\lambda$$
$$= -c(1 + \lambda K)Bc\frac{K}{1 + \lambda K} \quad (3.2)$$

It can be seen that the above string field belongs to the second class of solutions, i.e., $\Psi_2$ in (2.11).

Evaluation of gauge invariant quantities is almost similar to that of Sec. 2, so we only quote results here. First, the kinetic term of SFT action is evaluated as

$$\text{Tr}[\Psi'_\lambda Q_B \Psi'_\lambda] = -\frac{3}{\pi^2} \int_0^\infty du u e^{-u}$$
$$= \frac{3}{\pi^2}. \quad (3.3)$$

Surprisingly, the integrand does not depend on $\lambda$ even before preforming $u$ integration! Similar phenomena also occurs for the closed string tadpole.

$$\text{Tr}[Ve\frac{-K}{1 + \lambda K}] = \frac{1}{\lambda} \int_0^\infty dte^{-t} \partial_t \text{Tr}[Ve^{\lambda t}]$$
$$= \frac{1}{\lambda} \int_0^\infty dte^{-t} \partial_t \text{Tr}[\lambda Ve\Omega]$$
$$= \int_0^\infty dte^{-t} \partial_t \text{Tr}[Ve\Omega]$$
$$= \text{Tr}[Ve\Omega]. \quad (3.4)$$

4. SFT around the solution

In this section, we give some remarks about the string field theory around the solutions discussed in this paper. The new kinetic operator $Q'_B$ for the background field $\Psi_\lambda$,

$$Q'_B \Psi = Q_B \Psi + \Psi_\lambda \Psi + \Psi \Psi_\lambda, \quad (4.1)$$
characterize the spectrum in this background. First, the homotopy operator for (2.7) is given by
\[ A_\lambda = \frac{B}{1 + \lambda K}. \] (4.2)

It can be easily checked that it satisfies
\[ Q'_B(A_\lambda) = Q_BA_\lambda + \Psi_\lambda A_\lambda + A_\lambda \Psi_\lambda = 1. \] (4.3)

In particular, the homotopy operator for the identity based solution is simply given by
\[ A_{\lambda=0} = B. \] (4.4)

This gives very simple prescription of gauge fixing. The existence of homotopy operator tells us that
\[ Q'_B\Psi = 0 \rightarrow \Psi = Q'_B(B\Psi). \] (4.5)

This means that \( \Psi \) is always exact if \( B\Psi \neq 0 \). On the other hand, when \( B\Psi = 0 \) holds, (4.3) implies
\[ Q'_B\Psi = 0 \rightarrow \Psi = 0. \] (4.6)

Therefore, \( B\Psi = 0 \) completely fixes gauge so as to \( \Psi \) being zero.

The action of the kinetic operator \( Q'_B \) on \( KBc \) subalgebra is also interesting. It is given by
\[ Q'_Bc = 0, \] (4.7)
\[ Q'_BB = 1, \] (4.8)
\[ Q'_BK = [c, K](1 - K). \] (4.9)

Transformation of \( c \) implies that it loses the role of infinitesimal vector of conformal transformation. \( Q'_Bc = 0 \), which already appeared as the cohomology operator, implies that it also does not give a generator of conformal transformation \( K \). At first look, these two transformation seems to imply \( Q'_B \sim c \), which is a reminiscent of the vacuum string field theory. However, things are not so simple since the third equation is nontrivial with respect to \( K \).

Algebra for the simplest identity based solution \( \Psi = -cK \) is more interesting. It is given by
\[ Q'_Bc = 0, \] (4.10)
\[ Q'_BB = 0, \] (4.11)
\[ Q'_BK = [K, c]K. \] (4.12)

\( Q'_B \) vanishes for \( B \) and \( c \) both.
5. Discussion

In this paper, we investigate an one-parameter family of classical solutions in SFT which interpolates between identity based solution and the Erler-Schnabl solution. Classical action does not depend on the gauge parameter and also gives correct value of D-brane tension. The closed string tadpole is also confirmed to give expected answer. Since the solution can be made arbitrary close to the identity based one, it can be regarded as a consistent regularization of an identity based solution. To our knowledge, this is the first example of regularization which correctly reproduce the D-brane tension.

It is very valuable to discuss why our regularization works well and earlier attempts fail. Clearly, a key feature of our regularization is gauge invariance. In gauge theory, gauge invariant regularization plays crucial role in evaluation of physical quantities. Since our regularization is realized by a gauge transformation, the equation of motion is ensured and the value of classical action is kept unchanged as long as the gauge transformation is regular. One the contrary, earlier attempts to attach world sheets with small width to a solution violate gauge invariance thus yield indefinite result.

It is also interesting to notice that the final form of the gauge invariant quantity is given by

$$\int_0^\infty du e^{-u} f(\lambda, u),$$

where $f(\lambda, u)$ is a polynomial such that the $\lambda$ dependence disappears after $u$ integration. It should be stressed that the $u$ integration runs thorough all width even in $\lambda \to 0$ limit. In other words, a gauge invariant contraction between identity based string fields can be calculated as an correlator on world sheet with non zero width! This fact is trivial from the point of view of gauge invariance, but also tells us that a reason why a naive regularization of attaching infinitesimal piece of world sheet has been failed in past. It is also expected that other gauge invariant quantity, which is not yet known, takes form of (5.1). Therefore it will be interesting to investigate possible form of $f(\lambda, u)$ to classify gauge invariant observable.

Existence of consistent regularization of an identity based solution will play important role in feature developments in string field theory. In particular, it will be very useful to explore the physics around close string vacuum since the description of the theory becomes much simpler. It will also be useful to find other solutions which are not gauge equivalent to the Erler-Schnabl solution.

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