Three-body problem in 3D space: ground state, (quasi)-exact-solvability

Alexander V Turbiner\textsuperscript{1,2}, Willard Miller Jr\textsuperscript{3} and Adrian M Escobar-Ruiz\textsuperscript{1,3}

\textsuperscript{1} Instituto de Ciencias Nucleares, UNAM, México DF 04510, Mexico
\textsuperscript{2} IHES, Bures-sur-Yvette, France
\textsuperscript{3} School of Mathematics, University of Minnesota, Minneapolis, MN, United States of America

E-mail: turbiner@nucleares.unam.mx, miller@ima.umn.edu and mauricio.escobar@nucleares.unam.mx

Received 27 November 2016, revised 14 March 2017
Accepted for publication 12 April 2017
Published 2 May 2017

Abstract
We study aspects of the quantum and classical dynamics of a 3-body system in 3D space with interaction depending only on mutual distances. The study is restricted to solutions in the space of relative motion which are functions of mutual distances only. It is shown that the ground state (and some other states) in the quantum case and the planar trajectories in the classical case are of this type. The quantum (and classical) system for which these states are eigenstates is found and its Hamiltonian is constructed. It corresponds to a three-dimensional quantum particle moving in a curved space with special metric. The kinetic energy of the system has a hidden $sl(4,\mathbb{R})$ Lie (Poisson) algebra structure, alternatively, the hidden algebra $h^{(3)}$ typical for the $H_3$ Calogero model. We find an exactly solvable three-body generalized harmonic oscillator-type potential as well as a quasi-exactly-solvable three-body sextic polynomial type potential; both models have an extra integral.

Keywords: three-body problem, (quasi)-exact-solvability, hidden algebra

1. Introduction

The Hamiltonian for 3-body quantum system of 3-dimensional particles with translation-invariant potential, which depends on relative distances between particles only, is of the form,

$$\mathcal{H} = -\sum_{i=1}^{3} \frac{1}{2m_i} \Delta_i^{(3)} + V(r_{12}, r_{13}, r_{23}).$$

(1)
with coordinate vector of $i^{th}$ particle $r_i \equiv r_i^{(3)} = (x_{i,1}, x_{i,2}, x_{i,3})$, where
\[ r_{ij} = |r_i - r_j|, \tag{2} \]
is the (relative) distance between particles $i$ and $j$. We consider the case when all masses are assumed to be equal: $m_i = m = 1$. In this case the kinetic energy operator is $-\Delta^{(9)}$, where $\Delta^{(9)}$ is nine-dimensional Laplacian. The number of relative distances is equal to the number of edges of the triangle formed by taking the body positions as vertices. We call this triangle the triangle of interaction. Here, $\Delta^{(3)}$ is the 3-dimensional Laplacian,
\[ \Delta^{(d)}_i = \frac{\partial^2}{\partial r_i \partial r_i}, \]
associated with the $i^{th}$ body. The configuration space for $\mathcal{H}$ is $\mathbb{R}^9$. The center-of-mass motion described by vectorial coordinate
\[ R_0 = \frac{1}{\sqrt{3}} \sum_{k=1}^{3} r_k, \]
can be separated out; this motion is described by a 3-dimensional plane wave.

The spectral problem is formulated in the space of relative motion $R_r \equiv \mathbb{R}^6$; it is of the form,
\[ \mathcal{H}_r \Psi(x) \equiv \left(-\frac{1}{2} \Delta^{(6)}_r + V(r_{12}, r_{13}, r_{23}) \right) \Psi(x) = E \Psi(x), \quad \Psi \in L^2(R_r), \tag{3} \]
where $\Delta^{(6)}_r$ is the flat-space Laplacian in the space of relative motion. If the space of relative motion $R_r$ is parameterized by two, 3-dimensional vectorial Jacobi coordinates
\[ r_j^{(F)} = \frac{1}{\sqrt{j(j+1)}} \sum_{k=1}^{j} k (r_{k+1} - r_k), \quad j = 1, 2, \]
the flat-space 6-dimensional Laplacian in the space of relative motion becomes diagonal
\[ \Delta^{(6)}_r = \sum_{j=1}^{2} \frac{\partial^2}{\partial r_j^{(F)} \partial r_j^{(F)}}. \tag{4} \]

1.1. Observation [1]

There exists a family of the eigenstates of the Hamiltonian (1), including the ground state, which depends on three relative distances $\{r_{ij}\}$ only. The same is correct for $n$ body problem.

Our primary goal is to find the differential operator in the space of relative distances $\{r_{ij}\}$ for which these states are eigenstates. In other words, to find a differential equation depending only on $\{r_{ij}\}$ for which these states are solutions. This implies a study of the evolution of the triangle of interaction.
2. Generalities

As a first step let us change variables in the space of relative motion $R_\mathbf{r} : (r_i^{(F)}) \leftrightarrow (r_j, \Omega)$, where the number of (independent) relative distances $r_j$ is equal to 3 and $\Omega$ is a collection of three angular variables. Thus, we split $R_\mathbf{r}$ into a sum of the space of relative distances $\tilde{R}$ and a space parameterized by angular variables, essentially those on the sphere $S^3$. There are known several ways to introduce variables in $R_\mathbf{r}$: the perimetric coordinates by Hylleraas [2], the scalar products of vectorial Jacobi coordinates $r_i^{(F)}$ [3] and the relative (mutual) distances $r_{ij}$ (see e.g. [4]). We follow the last one. In turn, the angular variables are introduced as the two Euler angles on the $S^2$ sphere defining the normal to the interaction plane (triangle) and the azimuthal angle of rotation of the interaction triangle around its barycenter, see e.g. [3].

A key observation is that in new coordinates $(r_{ij}, \Omega)$ the flat-space Laplace operator (the kinetic energy operator) in the space of relative motion $R_\mathbf{r}$ takes the form of the sum of two second-order differential operators

$$\frac{1}{2}\Delta^{(6)}_r = \Delta_r(r_j) + \tilde{\Delta}(r_{ij}, \Omega, \partial_{\Omega}),$$

(5)

where the first operator depends on relative distances only, while the second operator depends on angular derivatives in such a way that it annihilates any angle-independent function,

$$\tilde{\Delta}(r_{ij}, \Omega, \partial_{\Omega}) \Psi(r_j) = 0.$$  

In general, the commutator

$$[\Delta_r(r_j), \tilde{\Delta}(r_{ij}, \Omega, \partial_{\Omega})] \neq 0.$$  

If we look for angle-independent solutions of (3), the decomposition (5) reduces the general spectral problem (3) to a particular spectral problem

$$\tilde{H}_R \Psi(r_{ij}) \equiv \left(-\Delta_r(r_j) + V(r_{12}, r_{13}, r_{23})\right) \Psi(r_j) = E \Psi(r_j), \quad \Psi \in L^2(\tilde{R}),$$

(6)

where $\tilde{R}$ is the space of relative distances. Surprisingly, one can find the gauge factor $\Gamma(r_j)$ such that the operator $\Delta_r(r_j)$ takes the form of the Schrödinger operator,

$$\Gamma^{-1} \Delta_r(r_j) \Gamma = \Delta_{LB}(r_j) - \tilde{V}(r_j) \equiv -\tilde{H}_R,$$

(7)

where $\Delta_{LB}$ is the Laplace–Beltrami operator with contravariant metric $g^{\mu\nu}(r)$, in general, on some non-flat, (non-constant curvature) manifold. It makes sense of the kinetic energy. Here $\tilde{V}(r_{ij})$ is the effective potential. The potential $\tilde{V}$ becomes singular at the boundary of the configuration space, where the determinant $D = \det g^{\mu\nu}(r)$ vanishes. The operator $\tilde{H}_R$ is Hermitian with measure $D^{-\frac{1}{2}}$. Eventually, we arrive at the spectral problem for the Hamiltonian

$$H_R = -\Delta_{LB}(r_j) + V(r_j) + \tilde{V}(r_j).$$

(8)

Following the de-quantization procedure of replacement of the quantum momentum (derivative) by the classical momentum

$$-i \partial \rightarrow p,$$

one can get a classical analogue of (8),

$$H_R^{(c)} = g^{\mu\nu}(r) p_\mu p_\nu + V(r_j) + \tilde{V}(r_j).$$

(9)
It describes the motion of a 3-dimensional body with tensor of inertia \((g^{\mu\nu})^{-1}\).

The Hamiltonians (8) and (9) are the main objects of study of this paper.

3. Three-body case: concrete results

After straightforward calculations the operator \(\Delta_R(r_{ij})\) in decomposition (5) is found to be

\[
2 \Delta_R(r_{ij}) = \left[ 2 \left( \partial_{r_{12}}^2 + \partial_{r_{13}}^2 + \partial_{r_{23}}^2 \right) + \frac{4}{r_{12}} \partial_{r_{12}} + \frac{4}{r_{23}} \partial_{r_{23}} + \frac{4}{r_{13}} \partial_{r_{13}} + \frac{r_{12}^2 - r_{13}^2 + r_{23}^2}{r_{12} r_{23}} \partial_{r_{13}} \partial_{r_{23}} + \frac{r_{12}^2 + r_{13}^2 - r_{23}^2}{r_{12} r_{13}} \partial_{r_{12}} \partial_{r_{13}} + \frac{r_{13}^2 + r_{23}^2 - r_{12}^2}{r_{13} r_{23}} \partial_{r_{23}} \partial_{r_{13}} \right],
\]

(10)

see e.g. [4]. It does not depend on the choice of the angular variables \(\Omega\). Its configuration space is

\[0 < r_{12}, r_{13}, r_{23} < \infty, \quad r_{23} < r_{12} + r_{13}, \quad r_{13} < r_{12} + r_{23}, \quad r_{12} < r_{13} + r_{23}.\]

(11)

In the space with Cartesian coordinates \((x, y, z) \equiv (r_{12}, r_{13}, r_{23})\) the configuration space lies in the first octant and is the interior of the inverted tetrahedral-shaped object with base at infinity, vertex at the origin and edges \((t, t, 2t), (t, 2t, t)\) and \((2t, t, t), 0 \leq t < \infty\).

Formally, the operator (10) is invariant under reflections \(Z_2 \oplus Z_2 \oplus Z_2\),

\[
r_{12} \rightarrow -r_{12}, \quad r_{13} \leftrightarrow -r_{13}, \quad r_{23} \leftrightarrow -r_{23},
\]

and w.r.t. \(S_3\)-group action. If we introduce new variables,

\[r_{12}^2 = \rho_{12}, \quad r_{13}^2 = \rho_{13}, \quad r_{23}^2 = \rho_{23},\]

(12)

the operator (10) becomes algebraic,

\[
\Delta_R(\rho) = 4(\rho_{12} \partial_{\rho_{12}}^2 + \rho_{13} \partial_{\rho_{13}}^2 + \rho_{23} \partial_{\rho_{23}}^2) + 6(\partial_{\rho_{12}} + \partial_{\rho_{13}} + \partial_{\rho_{23}}) + 2 \left( (\rho_{12} + \rho_{13} - \rho_{23}) \partial_{\rho_{12}} \partial_{\rho_{13}} + (\rho_{12} + \rho_{23} - \rho_{13}) \partial_{\rho_{12}} \partial_{\rho_{23}} + (\rho_{13} + \rho_{23} - \rho_{12}) \partial_{\rho_{13}} \partial_{\rho_{23}} \right).
\]

(13)

From (11) and (12) it follows that the corresponding configuration space in \(\rho\) variables is given by the conditions

\[0 < \rho_{12}, \rho_{13}, \rho_{23} < \infty, \quad \rho_{23} < (\sqrt{\rho_{12}} + \sqrt{\rho_{13}})^2, \quad \rho_{13} < (\sqrt{\rho_{12}} + \sqrt{\rho_{23}})^2, \quad \rho_{12} < (\sqrt{\rho_{13}} + \sqrt{\rho_{23}})^2.\]

We remark that

\[
\rho_{12}^2 + \rho_{13}^2 + \rho_{23}^2 - 2\rho_{12}\rho_{13} - 2\rho_{12}\rho_{23} - 2\rho_{13}\rho_{23} < 0,
\]

(14)

because the left-hand side (lhs) is equal to

\[-(r_{12} + r_{13} - r_{23})(r_{12} + r_{23} - r_{13})(r_{13} + r_{23} - r_{12})(r_{12} + r_{13} + r_{23})\]

and conditions (11) should hold. Therefore, following the Heron formula, lhs is proportional to the square of the area of the triangle of interaction \(S_{\Delta}^2\).

The associated contravariant metric for the operator \(\Delta_R(\rho)\) defined by coefficients in front of second derivatives is remarkably simple
\[ g^{\mu\nu}(\rho) = \begin{vmatrix} 4\rho_{12} & \rho_{12} + \rho_{13} - \rho_{23} & \rho_{12} + \rho_{23} - \rho_{13} \\ \rho_{12} + \rho_{13} - \rho_{23} & 4\rho_{13} & \rho_{13} + \rho_{23} - \rho_{12} \\ \rho_{12} + \rho_{23} - \rho_{13} & \rho_{13} + \rho_{23} - \rho_{12} & 4\rho_{23} \end{vmatrix}, \quad (15) \]

it is linear in \( \rho \)-coordinates(!) with factorized determinant

\[
\det g^{\mu\nu}(\rho) = -6(\rho_{12} + \rho_{13} + \rho_{23})(\rho_{12}^2 + \rho_{13}^2 + \rho_{23}^2 - 2\rho_{12}\rho_{13} - 2\rho_{12}\rho_{23} - 2\rho_{13}\rho_{23}) \equiv D(\rho) > 0, \quad (16)
\]

and is positive definite. It is worth noting a remarkable factorization property of the determinant

\[
D(\rho) = 6 \tau_1(4\tau_2 - \tau_1^2), \quad (17)
\]

where

\[
P = \text{Tr} g^{\mu\nu} = r_{12}^2 + r_{13}^2 + r_{23}^2,
\]

— the sum of squares of the sides of the interaction triangle.

The determinant can be rewritten in terms of elementary symmetric polynomials \( \sigma_{1,2} \),

\[
\tau_1 = \sigma_1(\rho_{12}, \rho_{13}, \rho_{23}) = \rho_{12} + \rho_{13} + \rho_{23},
\]

\[
\tau_2 = \sigma_2(\rho_{12}, \rho_{13}, \rho_{23}) = \rho_{12}\rho_{13} + \rho_{12}\rho_{23} + \rho_{13}\rho_{23},
\]

\[
\tau_3 = \sigma_3(\rho_{12}, \rho_{13}, \rho_{23}) = \rho_{12}\rho_{13}\rho_{23},
\]

which are invariant w.r.t. \( S_3 \)-group action, as follows,

\[
D(\rho) = 6 \tau_1(4\tau_2 - \tau_1^2), \quad (18)
\]

where

\[
16\tau_1^2 = (4\tau_2 - \tau_1^2),
\]

in terms of the elementary symmetric polynomials \( \tau_{1,2} \). When \( \det g^{\mu\nu}(\rho) = 0 \), hence, either \( \tau_1 = 0 \), or \( \tau_1^2 = 4\tau_2 \)—it defines the boundary of the configuration space, see (14).

3.1. Integral

It can be shown that there exists the 1st order symmetry operator

\[
L_1 = (\rho_{13} - \rho_{23})\partial_{\rho_{12}} + (\rho_{23} - \rho_{12})\partial_{\rho_{13}} + (\rho_{12} - \rho_{13})\partial_{\rho_{23}}, \quad (19)
\]

for the operator (13),

\[
[\Delta R(\rho), L_1] = 0.
\]

Here, \( L_1 \) is an algebraic operator, which is anti-invariant under the \( S_3 \)-group action. The existence of the symmetry operator \( L_1 \) implies that in the space of relative distances one variable can be separated out in (13).

Set
\[ w_1 = \rho_{12} + \rho_{13} + \rho_{23}, \quad w_2 = 2 \sqrt{\rho_{12}^2 + \rho_{13}^2 + \rho_{23}^2 - \rho_{12}\rho_{13} - \rho_{12}\rho_{23} - \rho_{13}\rho_{23}}. \]  

(20)

where \( w_2 = 2 \sqrt{(\rho_1^2 - 3\rho_2)} \) as well, which are invariant under the action of \( L_1 \), and

\[ w_3 = \frac{\sqrt{3}}{9} \left( \text{sgn} \left( \rho_{23} - \rho_{13} \right) \arcsin \left( \frac{2\rho_{12} - \rho_{23} - \rho_{13}}{w_2} \right) + \text{sgn} \left( \rho_{13} - \rho_{12} \right) \arcsin \left( \frac{2\rho_{23} - \rho_{13} - \rho_{12}}{w_2} \right) 
+ \text{sgn} \left( \rho_{12} - \rho_{23} \right) \arcsin \left( \frac{2\rho_{13} - \rho_{23} - \rho_{12}}{w_2} \right) - \frac{3\pi}{4} \right), \]  

(21)

with \( \text{sgn}(x) = \frac{x}{|x|} \) for nonzero \( x \). These coordinates are invariant under a cyclic permutation of the indices on the \( \rho_{ab} \): \( 1 \to 2 \to 3 \to 1 \). Under a transposition of exactly two indices, see e.g. (12) and (3) we see that \( w_1, w_2 \) remain invariant, and \( w_3 \to -w_3 - \sqrt{\frac{\pi}{3}} \). (For the method used to compute \( w_3 \) see [5].) Expressions for \( w_3 \) vary, depending on which of the 6 non-overlapping regions of \( (\rho_{12}, \rho_{13}, \rho_{23}) \) space we choose to evaluate them:

1. 
   \( (a): \rho_{23} > \rho_{13} > \rho_{12}, \quad (b): \rho_{13} > \rho_{12} > \rho_{23}, \quad (c): \rho_{12} > \rho_{23} > \rho_{13}, \) 

2. 
   \( (d): \rho_{13} > \rho_{23} > \rho_{12}, \quad (e): \rho_{12} > \rho_{13} > \rho_{23}, \quad (f): \rho_{23} > \rho_{12} > \rho_{13}, \) 

The regions in class 1 are related by cyclic permutations, as are the regions in class 2. We map between regions by a transposition. Thus it is enough to evaluate \( w_3 \) in the region \( (a): \rho_{23} > \rho_{13} > \rho_{12} \). The other 5 expressions will then follow from the permutation symmetries. In this case we have

\[ (a): \quad w_3 = -\frac{\sqrt{3}}{9} \arcsin \left[ \frac{2\sqrt{2}}{w_2^2} \left( (2 - \sqrt{3})\rho_{13} - \rho_{23} + (\sqrt{3} - 1)\rho_{12} \right) \times (2\rho_{23} - (1 + \sqrt{3})\rho_{13} + (\sqrt{3} - 1)\rho_{12})((2 + \sqrt{3})\rho_{12} - (1 + \sqrt{3})\rho_{13} - \rho_{23}) \right]. \]

(The special cases where exactly two of the \( \rho_{ab} \) are equal can be obtained from these results by continuity. Here, \( w_3 \) is a single-valued differentiable function of \( \rho_{12}, \rho_{13}, \rho_{23} \) everywhere in the physical domain (configuration space), except for the points \( \rho_{12} = \rho_{13} = \rho_{23} \) where it is undefined.)

In these coordinates, the operators (19) and (13) take the form

\[ L_1(w) = \partial_{w_1}, \]

\[ \frac{1}{6} \Delta w = w_1 \partial_{w_1}^2 + w_2 \partial_{w_2}^2 + \frac{w_1}{3w_2^2} \partial_{w_3}^2 + 2w_2 \partial_{w_1w_2}^2 + 3 \partial_{w_1} + \frac{w_1}{w_2} \partial_{w_2}, \]

respectively. It is evident that for the \( w_3 \)-independent potential

\[ V(w_1, w_2, w_3) = 6g(w_1, w_2), \]
the operator \( L_1 \) is still an integral for an arbitrary function \( g \): 
\[
[-\Delta_R(w) + 6g(w_1, w_2)]\Psi = E\Psi,
\]
where \( \Psi = \psi(w_1, w_2) \xi(w_3) \) is defined by the differential equations,
\[
\partial_{w_3}\xi = im\xi, \quad \xi = e^{im\theta},
\]
\[
6 \left( w_1\partial^2_{w_1} + w_1\partial^2_{w_2} + 2w_2\partial^2_{w_1w_2} + 3\partial_{w_1} + \frac{w_1}{w_2}\partial_{w_2} - \frac{m^2}{3}\frac{w_1}{w_2^3} - g(w_1, w_2) \right) \psi
\]
\[
= -E\psi.
\]

Note that the integral \( L_1 \) is the integral for the three-dimensional quantum problem (8). As for the original 3-body problem (3) this integral is a particular integral: it commutes with the Hamiltonian (3) over the space of relative distances \( \tilde{R} \) only. As for operators \( H_r \) and \( L_1 \) they do not commute.

3.2. The Representations of \( \mathfrak{sl}(4, \mathbb{R}) \)

Both operators (13) and (19) are \( \mathfrak{sl}(4, \mathbb{R}) \)-Lie algebraic—they can be rewritten in terms of the generators of the maximal affine subalgebra \( b_4 \) of the algebra \( \mathfrak{sl}(4, \mathbb{R}) \), see e.g. [6, 7]

\[
\begin{align*}
\mathcal{J}^-_i &= \frac{\partial}{\partial u_i}, & i &= 1, 2, 3, \\
\mathcal{J}^0_j &= u_j \frac{\partial}{\partial u_j}, & i, j &= 1, 2, 3, \\
\mathcal{J}^0(N) &= \sum_{i=1}^3 u_i \frac{\partial}{\partial u_i} - N, \\
\mathcal{J}^+(N) &= u_i \mathcal{J}^0(N) = u_i \left( \sum_{j=1}^3 u_j \frac{\partial}{\partial u_j} - N \right), & i &= 1, 2, 3,
\end{align*}
\]

where \( N \) is a parameter and
\[
u_1 \equiv \rho_{12}, \quad \nu_2 \equiv \rho_{13}, \quad \nu_3 \equiv \rho_{23}.
\]

If \( N \) is a non-negative integer, a finite-dimensional representation space occurs,
\[
\mathcal{P}_N = \langle u_1^{p_1} u_2^{p_2} u_3^{p_3} | 0 \leq p_1 + p_2 + p_3 \leq N \rangle.
\]

Explicitly, these operators look as
\[
\Delta_R(\mathcal{J}) = 4(\mathcal{J}^0_1 \mathcal{J}^-_1 + \mathcal{J}^0_2 \mathcal{J}^-_2 + \mathcal{J}^0_3 \mathcal{J}^-_3) + 6(\mathcal{J}^-_1 + \mathcal{J}^-_2 + \mathcal{J}^-_3)
\]
\[
+ 2 \left( \mathcal{J}^0_1 (\mathcal{J}^-_2 + \mathcal{J}^-_3) + \mathcal{J}^0_2 (\mathcal{J}^-_1 + \mathcal{J}^-_3) + \mathcal{J}^0_3 (\mathcal{J}^-_1 + \mathcal{J}^-_2) - \mathcal{J}^0_1 \mathcal{J}^-_2 - \mathcal{J}^0_2 \mathcal{J}^-_1 - \mathcal{J}^0_3 \mathcal{J}^-_3 \right).
\]
\[ L_1 = \mathcal{J}^0_{21} - \mathcal{J}^0_{11} + \mathcal{J}^0_{12} - \mathcal{J}^0_{13} + \mathcal{J}^0_{23}. \]  

(27)

### 3.3. The Laplace Beltrami operator, underlying geometry

The remarkable property of the algebraic operator \( \Delta_R(\rho) \) (13) is its gauge-equivalence to the Schrödinger operator. Making the gauge transformation with determinant (16) and (18) as the factor,

\[ \Gamma = D^{-1/4} \sim \frac{1}{\tau_1^{1/4} (4\tau_2 - \tau_1^{1/2})}, \]

see also (17), we find that

\[ \Gamma^{-1} \Delta_R(\rho) \Gamma = \Delta_{LB}(\rho) - \tilde{V}, \]

(28)

where the effective potential

\[ \tilde{V}(\rho) = \frac{9}{8 (\rho_{12} + \rho_{13} + \rho_{23})} + \frac{(\rho_{12} + \rho_{13} + \rho_{23})}{2 (\rho_{12}^2 + \rho_{13}^2 + \rho_{23}^2 - 2\rho_{12}\rho_{13} - 2\rho_{12}\rho_{23} - 2\rho_{13}\rho_{23})} \]

\[ = \frac{9}{8P} - \frac{P}{32 S_\Delta}. \]

(29)

Note that in \( r \)-coordinates

\[ 4 \frac{(r_{12} + r_{13} - r_{23})(r_{12} + r_{23} - r_{13})(r_{13} + r_{23} - r_{12})}{(r_{23} + r_{13} - r_{12})(r_{23} + r_{12} - r_{13})(r_{13} + r_{12} - r_{23})(r_{12} + r_{13} + r_{23})} \]

\[ = \frac{1}{r_{12}r_{13}r_{23}} \left( \frac{r_{23}}{r_{12} + r_{13} - r_{23}} + \frac{r_{13}}{r_{12} + r_{23} - r_{13}} + \frac{r_{12}}{r_{13} + r_{23} - r_{12}} + 1 \right), \]

and

\[ \frac{1}{r_{23} + r_{13} - r_{12}} \left( \frac{1}{r_{23} + r_{13} - r_{12}} + \frac{1}{r_{23} + r_{12} + r_{13}} \right) \]

\[ + \frac{1}{8r_{23}r_{13}r_{12}} \left( \frac{1}{r_{12} + r_{13} - r_{23}} + \frac{1}{r_{12} + r_{13} - r_{23}} \right), \]

thus, the effective potential can be written differently,

\[ \tilde{V}(r_i) = \frac{9}{8 (r_{12}^2 + r_{13}^2 + r_{23}^2)} \]

\[ + \frac{r_{12}^2 + r_{13}^2 + r_{23}^2}{16} \left( \frac{1}{r_{12}r_{13}r_{23}} \left( \frac{1}{r_{13} + r_{23} - r_{12}} + \frac{1}{r_{12} + r_{13} + r_{23}} \right) \right) \]

\[ + \frac{1}{r_{12}r_{13}r_{23}} \left( \frac{1}{r_{12} + r_{23} - r_{13}} + \frac{1}{r_{12} + r_{13} - r_{23}} \right) . \]
In turn,
\[ \Delta_{LB}(\rho) = 4(\rho_{12} \partial_{\rho_{12}}^2 + \rho_{13} \partial_{\rho_{13}}^2 + \rho_{23} \partial_{\rho_{23}}^2) \]
\[ + 2 \left( (\rho_{12} + \rho_{13} - \rho_{23}) \partial_{\rho_{12}} \partial_{\rho_{13}} + (\rho_{12} + \rho_{23} - \rho_{13}) \partial_{\rho_{12}} \partial_{\rho_{23}} + (\rho_{13} + \rho_{23} - \rho_{12}) \partial_{\rho_{13}} \partial_{\rho_{23}} \right) \]
\[ -3 \left( \frac{\rho_{12} \partial_{\rho_{12}} + \rho_{13} \partial_{\rho_{13}} + \rho_{23} \partial_{\rho_{23}}}{\rho_{12} + \rho_{13} + \rho_{23}} \right) + 4 \left( \partial_{\rho_{12}} + \partial_{\rho_{13}} + \partial_{\rho_{23}} \right) \]
is the Laplace–Beltrami operator,
\[ \Delta_{LB}(\rho) = \frac{1}{\sqrt{D(\rho)}} \partial_{\mu} g^{\mu\nu} \partial_{\nu}, \quad \partial_{\nu} \equiv \frac{\partial}{\partial \rho_{\nu}}, \]
see (15) and (16). Eventually, taking into account (28) we arrive at the Hamiltonian
\[ H_{cl}(r) = -\Delta_{LB}(r) + \tilde{V}(r_{12}, r_{13}, r_{23}), \]
in the space of relative distances \( \{ r \} \) with the Laplace–Beltrami operator \( \Delta_{LB}(r) \), see (1), or
\[ H_{cl}(\rho) = -\Delta_{LB}(\rho) + \tilde{V}(\rho) + V(\rho), \]
in \( \rho \)-space, see (12). The Hamiltonian (31), or (32) describes the three-dimensional quantum particle of mass \( M = 1/2 \) moving in the curved space with metric \( g^{\mu\nu} \). The Ricci scalar, see e.g. [8], for this space is equal to
\[ R_{\text{S}} = -\frac{41 (\rho_{12} + \rho_{13} + \rho_{23})^2 - 84 (\rho_{12} \rho_{13} + \rho_{12} \rho_{23} + \rho_{23} \rho_{13})}{12 \left( \rho_{12} + \rho_{13} + \rho_{23} \right) \left( (\rho_{12} + \rho_{13} + \rho_{23})^2 - 4 (\rho_{12} \rho_{13} + \rho_{12} \rho_{23} + \rho_{23} \rho_{13}) \right)}, \]
\[ = \frac{-84 \tau_2 + 41 \tau_1^2}{12 \tau_1 (4 \tau_2 - \tau_1^2)} = \frac{-84 S_\Delta^2 + 5 P^2}{48 S_\Delta^2} = -\frac{7}{4P} + \frac{5P}{48 S_\Delta^2} , \]

it has a structure similar to the effective potential (28). It is singular at the boundary of the configuration space. The Cotton tensor, see e.g. [8], for this metric is nonzero, so the space is not conformally flat.

3.4. Classical mechanics

Making the de-quantization of (32) we arrive at a three-dimensional classical system which is characterized by the Hamiltonian,
\[ H_{cl}(\rho) = g^{\mu\nu}(\rho) P_{\mu} P_{\nu} + \tilde{V}(\rho) + V(\rho), \]
where \( P_{\mu}, \mu = 1, 2, 3 \) is classical momenta in \( \rho \)-space and \( g^{\mu\nu}(\rho) \) is given by (15). Here the underlying manifold (zero-potential case) admits an \( so(3) \) algebra of constants of the motion linear in the momenta, i.e. Killing vectors. Thus, the free Hamilton–Jacobi equation is integrable. However, it admits no separable coordinate system.

3.5. (Quasi)-exact-solvability

Let us take the function
\[ \Psi_0(\rho_{12}, \rho_{13}, \rho_{23}) = \tau_1^{1/4} (4 \tau_2 - \tau_1^2)^{7/2} e^{-\omega \tau_1 - \frac{5}{4} \tau_1^2}, \]
where \(\gamma, \omega > 0\) and \(A \geq 0\) are constants and \(\tau^{s}\) are given by (17), and seek the potential for which this (34) is the ground state function for the Hamiltonian \(H_{r}(\rho)\), see (32). This potential can be found immediately by calculating the ratio

\[
\frac{\Delta_{LB}(\rho)}{\Psi_{0}} = V_{0} - E_{0}.
\]

The result is

\[
V_{0}(\tau_{1}, \tau_{2}) = \frac{9}{8\tau_{1}} + \gamma(\gamma - 1) \left( \frac{2\tau_{1}}{4\tau_{2} - \tau_{1}} \right)
+ 6\omega^{2}\tau_{1} + 6A\tau_{1}(2\omega\tau_{1} - 3\gamma - 2\gamma - 3) + 6A^{2}\tau_{1}^{3},
\]

(35)

with the energy of the ground state

\[
E_{0} = 12\omega(1 + \gamma).
\]

(36)

Now, let us take the Hamiltonian \(H_{rd0} \equiv -\Delta_{LB} + V_{0}\), see (32), with potential (35), subtract \(E_{0}\) (36) and make the gauge rotation with \(\Psi_{0}\) (34). As the result we obtain the \(sl(4, R)\)-Lie-algebraic operator with additional potential \(\Delta V_{N}\) [6, 7] (and references therein)

\[
\Psi_{0}^{-1}(-\Delta_{LB} + V_{0} - E_{0})\Psi_{0} = -\Delta R(J) + 2(1 - 2\gamma)(J_{1}^{-} + J_{2}^{-} + J_{3}^{-})
+ 12\omega(J_{11}^{0} + J_{22}^{0} + J_{33}^{0}) + 12A(J_{1}^{+}(N) + J_{2}^{+}(N) + J_{3}^{+}(N)) + \Delta V_{N}
\]

\[
\equiv h^{(qes)}(J) + \Delta V_{N},
\]

(37)

see (26), where

\[
\Delta V_{N} = 12A\tau_{1}N.
\]

It is evident that for integer \(N\) the operator \(h(J)\) has a finite-dimensional invariant subspace \(P_{N}\), (25), with \(\dim P_{N} \sim N^{3}\) at large \(N\). Finally, we arrive at the quasi-exactly-solvable Hamiltonian in the space of relative distances:

\[
H_{rd}^{(qes)}(\rho) = -\Delta_{LB}(\rho) + V_{N}^{(qes)}(\rho),
\]

(38)

cf.(8), where

\[
V_{N}^{(qes)}(\tau_{1}, \tau_{2}) = \frac{9}{8\tau_{1}} + \gamma(\gamma - 1) \left( \frac{2\tau_{1}}{4\tau_{2} - \tau_{1}} \right)
+ 6\omega^{2}\tau_{1} + 6A\tau_{1}(2\omega\tau_{1} - 3\gamma - 2\gamma - 2N - 3) + 6A^{2}\tau_{1}^{3}
\]

(39)

or, in geometrical terms,

\[
= \frac{9}{8P} + \gamma(\gamma - 1) \left( \frac{P}{8S} \right) + 6\omega^{2}P + 6A\omega P(2\omega - 2\gamma - 2N - 3) + 6A^{2}P^{3}.
\]

For this potential \(\sim N^{3}\) eigenstates can be found by algebraic means. They have the factorized form of the polynomial multiplied by \(\Psi_{0}\) (34),
\[ \text{Pol}_N(\rho_{12}, \rho_{13}, \rho_{23}) \ \Psi_0(\tau_1, \tau_2). \]

(Note that for given \( N \) we can always choose appropriate values of \( \gamma \) such that the boundary terms vanish for polynomials in the invariant subspace and the Hamiltonian (38) acts as a self-adjoint operator.) These polynomials are the eigenfunctions of the quasi-exactly-solvable algebraic operator

\[ h^{(\text{qes})}(\rho; A) \]

\[ = -4(\rho_{12} \partial_{\rho_{12}}^2 + \rho_{13} \partial_{\rho_{13}}^2 + \rho_{23} \partial_{\rho_{23}}^2) \]

\[ -2 \left( (\rho_{12} + \rho_{13} - \rho_{23}) \partial_{\rho_{12}} \partial_{\rho_{13}} + (\rho_{12} + \rho_{23} - \rho_{13}) \partial_{\rho_{12}} \partial_{\rho_{23}} + (\rho_{13} + \rho_{23} - \rho_{12}) \partial_{\rho_{13}} \partial_{\rho_{23}} \right) \]

\[ -4(1 + \gamma)(\partial_{\rho_{12}} + \partial_{\rho_{13}} + \partial_{\rho_{23}}) + 12 \omega(\rho_{12} \partial_{\rho_{12}} + \rho_{13} \partial_{\rho_{13}} + \rho_{23} \partial_{\rho_{23}}) \]

\[ -12A(\rho_{12} + \rho_{13} + \rho_{23})(\rho_{12} \partial_{\rho_{12}} + \rho_{13} \partial_{\rho_{13}} + \rho_{23} \partial_{\rho_{23}} - N), \]  
(40)

which is the quasi-exactly-solvable \textit{sl}(4, \mathbb{R})-Lie-algebraic operator

\[ h^{(\text{qes})}(\mathbf{J}; A) \]

\[ = -4(\mathbf{J}_1^0 \mathbf{J}_1^- + \mathbf{J}_2^0 \mathbf{J}_2^- + \mathbf{J}_3^0 \mathbf{J}_3^-) \]

\[ -2 \left( \mathbf{J}_1^0 (\mathbf{J}_2^- + \mathbf{J}_3^-) + \mathbf{J}_2^0 (\mathbf{J}_1^- + \mathbf{J}_3^-) + \mathbf{J}_3^0 (\mathbf{J}_1^- + \mathbf{J}_2^-) - \mathbf{J}_1^0 \mathbf{J}_2^- - \mathbf{J}_2^0 \mathbf{J}_1^- - \mathbf{J}_3^0 \mathbf{J}_2^- \right) \]

\[ -4(1 + \gamma)(\mathbf{J}_1^- + \mathbf{J}_2^- + \mathbf{J}_3^-) + 12 \omega(\mathbf{J}_1^0 + \mathbf{J}_2^0 + \mathbf{J}_3^0) \]

\[ + 12A(\mathbf{J}_1^+(N) + \mathbf{J}_2^+(N) + \mathbf{J}_3^+(N)), \]
(41)

see (37).

As for the original problem (6) in the space of relative motion

\[ \tilde{H}_R \Psi(r_0) \equiv \left( -\Delta_R(r) + V(r_0) \right) \Psi(r_0) = E \Psi(r_0), \ \Psi \in L^2(\tilde{\mathbf{R}}), \]

the potential for which quasi-exactly-solvable and polynomial solutions occur of the form

\[ \text{Pol}_N(\rho_{12}, \rho_{13}, \rho_{23}) \ \Gamma \ \Psi_0(\tau_1, \tau_2), \]

where \( \Gamma \sim D^{-1/4} \), see (18), is given by

\[ V^{(\text{qes})}_N(\tau) = \left( \gamma - \frac{1}{2} \right)^2 \left( \frac{2 \tau_1}{4\tau_2 - \tau_1^2} \right) \]

\[ + 6 \omega^2 \tau_1 + 6A \tau_1 (2 \omega \tau_1 - 2 \gamma - 2N - 3) + 6A^2 \tau_1^3, \]  
(42)

see (39); unlike (39) this potential does not depend on \( \gamma_3 \), it does not contain the singular, a type of centrifugal term \(-1/\tau_1\) and at \( \gamma = 1/2 \) becomes polynomial.

If the parameter \( A \) vanishes in (34), (39) and (37), (41) we will arrive at the exactly-solvable problem, where \( \Psi_0 \) (34) at \( A = 0 \), plays the role of the ground state function.
\[
\Psi_0(\rho_{12}, \rho_{13}, \rho_{23}) = \tau_1^{\frac{1}{14}}(4\tau_2 - \tau_1^2)^2 e^{-\omega \tau_1},
\]
which does not depend on \(\tau_3\). In this case the \(\mathfrak{sl}(4, \mathbb{R})\)-Lie-algebraic operator (41) contains no raising generators \(\{\mathcal{J}^+ (N)\}\) and becomes
\[
h^{(\text{exact})} = -\Delta_R(\mathcal{J}) - 4(1 + \gamma)(\mathcal{J}_1^- + \mathcal{J}_2^- + \mathcal{J}_3^-) + 12 \omega (\mathcal{J}_{11}^0 + \mathcal{J}_{22}^0 + \mathcal{J}_{33}^0),
\]
see (26), and, hence, preserves the infinite flag of finite-dimensional invariant subspaces \(\mathcal{P}_N\) (25) at \(N = 0, 1, 2 \ldots\). The potential (39) becomes
\[
V^{(\text{es})}(\tau_1, \tau_2) = \frac{9}{8\tau_1} + \gamma(\gamma - 1) \left( \frac{2\tau_1}{4\tau_2 - \tau_1^2} \right) + 6 \omega^2 \tau_1
\]
\[
= \frac{9}{8(\rho_{12} + \rho_{13} + \rho_{23})} + 6 \omega^2 (\rho_{12} + \rho_{13} + \rho_{23})
- \gamma(\gamma - 1) \left( \frac{2(\rho_{12} + \rho_{13} + \rho_{23})}{\rho_{12}^2 + \rho_{13}^2 + \rho_{23}^2 - 2\rho_{12}\rho_{13} - 2\rho_{12}\rho_{23} - 2\rho_{13}\rho_{23}} \right)
= \frac{9}{8 (r_{12}^2 + r_{13}^2 + r_{23}^2)} + 6 \omega^2 (r_{12}^2 + r_{13}^2 + r_{23}^2)
+ \gamma(\gamma - 1) \left( \frac{r_{12}^2 + r_{13}^2 + r_{23}^2}{16} \right)
+ \frac{1}{r_{12}r_{13}r_{23}} \left( \frac{1}{r_{12} + r_{23} - r_{13}} + \frac{1}{r_{12} + r_{13} - r_{23}} \right).
\]
Eventually, we arrive at the exactly-solvable Hamiltonian in the space of relative distances
\[
\mathcal{H}^{(\text{es})}_N(\rho) = -\Delta_{\text{LB}}(\rho) + V^{(\text{es})}(\rho),
\]
where the spectra of energies
\[
E_n = 12 \omega(n + \gamma + 1), \quad n = 0, 1, 2 \ldots
\]
is equidistant. All eigenfunctions have the factorized form of a polynomial multiplied by \(\Psi_0\) (43),
\[
\text{Pol}_N(\rho_{12}, \rho_{13}, \rho_{23}) \Psi_0(\tau_1, \tau_2), \quad N = 0, 1, \ldots
\]
These polynomials are eigenfunctions of the exactly-solvable algebraic operator
\[
h^{(\text{exact})}(\rho) = h^{(\text{qes})}(\rho; A = 0),
\]
see (40), or, equivalently, of the exactly-solvable \(\mathfrak{sl}(4, \mathbb{R})\)-Lie-algebraic operator
\[
h^{(\text{exact})}(J) = h^{(\text{qes})}(J; A = 0),
\]
see (41). Those polynomials are orthogonal w.r.t. \(\Psi_0^2\), (34) at \(A = 0\), their domain is given by (14). Being written in variables \(w_{1,2,3}\), see above, they are factorizable, \(F(w_1, w_2) f(w_3)\). To the best of our knowledge these orthogonal polynomials have not been studied in literature.
The Hamiltonian with potential (44) can be considered as a three-dimensional generalization of the 3-body Calogero model [9], see also [10, 11], with loss of the property of pairwise interaction. Now the potential of interaction contains two- and three-body interaction terms. If \( \gamma = 0, 1 \) in (44), or equivalently, \( \gamma = 1/2 \) in (42) at \( A = 0 \), we arrive at the celebrated harmonic oscillator potential (with singular term) in the space of relative distances. In turn, in the space of relative motion this potential contains no singular terms and becomes,

\[
V^{(c)} = 6\omega^2 \tau_1 = 6\omega^2 (\rho_{12} + \rho_{13} + \rho_{23}) = 6\omega^2 (r_{12}^2 + r_{13}^2 + r_{23}^2),
\]

see e.g. [12].

The quasi-exactly-solvable \( sl(4, \mathbb{R}) \)-Lie-algebraic operator \( h^{(\text{qes})}(J; A) \), (41) as well as the exactly-solvable operator as a degeneration at \( A = 0 \), written originally in \( \rho \) variables (40), can be rewritten in \( \tau \) variables (17). Surprisingly, this operator is algebraic (!) as well

\[
h^{(\text{qes})}(\tau; A) = -6 \tau_1 \partial_1 - 2\tau_1(7\tau_2 - \tau_1^2)\partial_2^2 - 2\tau_2(6\tau_2 - \tau_1^2)\partial_1^2 - 24 \tau_2\partial_{1,2}^2 - 36\tau_3\partial_{1,3}^2
\]

\[
- 2(4\tau_2^2 + 9\tau_1\tau_3 - \tau_1\tau_2^2)\partial_{2,3}^2 - 18\partial_1 - 14\tau_1\partial_2 - 2(7\tau_2 - \tau_1^2)\partial_3
\]

\[
- 4(1 + \gamma)(3\partial_1 + 2\tau_1\partial_2 + 7\tau_2\partial_3) + 12\omega (\tau_1\partial_1 + 2\tau_2\partial_2 + 3\tau_3\partial_3)
\]

\[
+ 12A\tau_1(\tau_1\partial_1 + 2\tau_2\partial_2 + 3\tau_3\partial_3 - N).
\]

and evidently, it remains algebraic at \( A = 0 \),

\[
h^{(\text{exact})}(\tau) = h^{(\text{qes})}(\tau; A = 0),
\]

becoming the exactly-solvable one. Note that the (quasi)-exactly-solvable operator (48) (and (49)) admits the integral

\[
-L_1^2 = (27\tau_2^2 + 4\tau_3\tau_1^2 - 18\tau_3\tau_2 + 4\tau_3^2 - \tau_2^2\tau_1^2)\partial_{\tau_1}^2 + (27\tau_3 + 2\tau_3^2 - 9\tau_1\tau_2)\partial_{\tau_2},
\]

see (19), \([h^{(\text{qes})}(\tau), L_1^2] = 0 \). It involves derivatives w.r.t. \( \tau_3 \) only.

It can be immediately checked that the quasi-exactly-solvable operator (48) has the finite-dimensional invariant subspace in polynomials,

\[
P^{(1,2,3)}_N = \langle \tau_1^{p_1}\tau_2^{p_2}\tau_3^{p_3} | 0 \leq p_1 + 2p_2 + 3p_3 \leq N \rangle,
\]

see (25). This finite-dimensional space appears as a finite-dimensional representation space of the algebra of differential operators \( h^{(3)} \) which was discovered in the relation with \( H_1 \) (non-crystallographic) rational Calogero model [13] as its hidden algebra.

The algebra \( h^{(3)} \) is infinite-dimensional but finitely-generated, for discussion see [13]. Their generating elements can be split into two classes. The first class of generators (lowering and Cartan operators) act in \( P^{(1,2,3)}_N \) for any \( N \) and therefore they preserve the flag \( P^{(1,2,3)}_N \). The second class operators (raising operators) act on the space \( P^{(1,2,3)}_N \) only.

Let us introduce the following notation for the derivatives:

\[
\partial_i \equiv \frac{\partial}{\partial \tau_i}, \quad \partial_{ij} \equiv \frac{\partial^2}{\partial \tau_i \partial \tau_j}, \quad \partial_{ijk} \equiv \frac{\partial^3}{\partial \tau_i \partial \tau_j \partial \tau_k}.
\]

The first class of generating elements consist of the 22 generators where 13 of them are the first order operators.
\[ T_0^{(1)} = \partial_1, \quad T_0^{(2)} = \partial_2, \quad T_0^{(3)} = \partial_3, \]
\[ T_1^{(1)} = \tau_1 \partial_1, \quad T_2^{(2)} = \tau_2 \partial_2, \quad T_3^{(3)} = \tau_3 \partial_3, \]
\[ T_1^{(3)} = \tau_1 \partial_3, \quad T_1^{(1)} = \tau_1 \partial_1, \quad T_1^{(3)} = \tau_1 \partial_3, \]
\[ T_1^{(2)} = \tau_1 \partial_2, \quad T_2^{(1)} = \tau_2 \partial_1, \quad T_2^{(3)} = \tau_2 \partial_3, \]
\[ T_{12}^{(3)} = \tau_1 \partial_2, \]
\[ \text{the 6 are of the second order} \]
\[ T_2^{(11)} = \tau_2 \partial_1, \quad T_2^{(13)} = \tau_2^2 \partial_3, \quad T_2^{(33)} = \tau_2^3 \partial_3, \]
\[ T_3^{(12)} = \tau_3 \partial_2, \quad T_3^{(22)} = \tau_3 \partial_2, \quad T_3^{(32)} = \tau_2 \partial_3, \]
\[ \text{and 2 are of the third order} \]
\[ T_3^{(11)} = \tau_3 \partial_1, \quad T_3^{(22)} = \tau_3^2 \partial_2. \]

The generators of the second class consist of 8 operators, where one of them is of the first order
\[ T_1^+ = \tau_1 T_0. \]
\[ \text{four are of the second order} \]
\[ T_{2-1}^+ = \tau_2 \partial_1 T_0, T_{2-2}^+ = \tau_3 \partial_2 T_0, \]
\[ T_{22-3}^+ = \tau_2 \partial_3 T_0, \quad T_2^+ = \tau_2 T_0(T_0 + 1), \]
\[ \text{and 3 are of the third order} \]
\[ T_{3-11}^+ = \tau_3 \partial_1 T_0, \quad T_{3-2}^+ = \tau_3 \partial_2 T_0(T_0 + 1), \quad T_3^+ = \tau_3 T_0(T_0 + 1)(T_0 + 2), \]
where we have introduced the diagonal operator (the Euler-Cartan generator)
\[ T_0 = \tau_1 \partial_1 + 2 \tau_2 \partial_2 + 3 \tau_3 \partial_3 - N, \]
for a convenience. This operator is of the first order, hence, it belongs to the first class.

It is not surprising that the algebraic operator \( h^{(\text{qes})}(T) \) (48) can be rewritten in terms of generators of the \( h^{(3)} \)-algebra,
\[
 h^{(\text{qes})}(T; A) = -\left[ 6 T_1^{(1)} T_0^{(1)} + 2 (7 T_2^{(2)} - T_1^{(2)}) T_1^{(2)} + T_0^{(3)} (6 T_2^{(3)} - T_1^{(3)}) + T_0^{(1)} (24 T_2^{(2)} + 36 T_3^{(3)}) + 2 (4 T_2^{(3)} T_2^{(2)} + 9 T_3^{(2)} T_3^{(3)} - T_1^{(3)})^{(2)} + 2 (9 T_0^{(1)} + 7 T_1^{(2)}) + 2 (7 T_2^{(3)} - T_1^{(3)}) \right] + 4(1 + \gamma) (T_2^{(3)} + 2 T_1^{(2)} + 3 T_0^{(1)}) + 12 \omega (T_0 + N) + 12 A T_1^+,
\]
as well as the algebraic operator \( h^{(\text{exact})}(T) \) (49), which occurs at \( A = 0 \), can be rewritten in terms of generators of the \( h^{(3)} \)-algebra,
\[
 h^{(\text{es})}(T) = h^{(\text{qes})}(T; A = 0),
\]
where without a loss of generality we put \( N = 0 \).
It can be immediately verified that with respect to the action of the operator (48) the finite-dimensional invariant subspace (50) is reducible: it preserves
\[ \mathcal{P}_N^{(1,2)} \equiv \langle \tau_1^{p_1} \tau_2^{p_2} | 0 \leq p_1 + 2p_2 \leq N \rangle \subset \mathcal{P}_N^{(1,2,3)} \].
(60)
The operator which acts on \( \mathcal{P}_N^{(1,2)} \) has the form,
\[ h^{\text{qes}}(\tau_1, \tau_2) = -6 \tau_1 \partial_1^2 - 2\tau_1 (7\tau_2 - \tau_1) \partial_2^2 - 24 \tau_2 \partial_{1,2}^2 - 6 (5 + 2\gamma) \partial_1 - 2 (11 + 4\gamma) \tau_1 \partial_2 
+ 12 \omega (\tau_1 \partial_1 + 2\tau_2 \partial_2) + 12 A \tau_1 (\tau_1 \partial_1 + 2\tau_2 \partial_2 - N), \]
(61)
see (23). It has \( \sim N^2 \) polynomial eigenfunctions which depend on two variables \( \tau_{1,2} \) only. Note that the space \( \mathcal{P}_N^{(1,2)} \) is a finite-dimensional representation space of the non-semi-simple Lie algebra \( gl(2, \mathbb{R}) \oplus \mathbb{R}^3 \) realized by the first order differential operators, [14] (see also [15–17]),
\[ t_1 = \partial_{\tau_1}, \]
\[ t_2(N) = \tau_1 \partial_{\tau_1} - \frac{N}{3}, \quad t_3(N) = 2\tau_2 \partial_{\tau_2} - \frac{N}{3}, \]
\[ t_4(N) = \tau_1^2 \partial_{\tau_1} + 2\tau_1 \tau_2 \partial_{\tau_2} - N \tau_1, \]
\[ r_i = \tau_i \partial_{\tau_i}, \quad i = 0, 1, 2. \]
(62)
The operator (61) can be rewritten in terms of \( gl(2, \mathbb{R}) \oplus \mathbb{R}^3 \) operators
\[ h^{\text{qes}}(t, r) = -6 r_1 t_1 - 14(t_3 + \frac{N}{3}) r_1 + 2r_2 t_1 - 24 t_4(t_3 + \frac{N}{3}) \]
\[ - 6 (5 + 2\gamma) t_1 - 2 (11 + 4\gamma) r_1 + 12 \omega(t_2 + t_3 + N) + 12 A t_4. \]

The space (50) is reducible further: the operator (48) (and also the operator (61)) preserves the space
\[ \mathcal{P}_N^{(1)} \equiv \langle \tau_1^{p_1} | 0 \leq p_1 \leq N \rangle \subset \mathcal{P}_N^{(1,2)} \subset \mathcal{P}_N^{(1,2,3)}, \]
(63)
as well. The operator, which acts on \( \mathcal{P}_N^{(1)} \), has the form,
\[ h^{\text{qes}}(\tau_1) = -6 \tau_1 \partial_1^2 - 6 (5 + 2\gamma) \partial_1 + 12 \omega \tau_1 \partial_1 + 12 A \tau_1 (\tau_1 \partial_1 - N). \]
(64)
It can be rewritten in terms of \( sl(2, \mathbb{R}) \) algebra generators,
\[ J^+(N) = \tau_1^2 \partial_{\tau_1} - N \tau_1, \quad J^0(N) = 2\tau_1 \partial_{\tau_1} - N, \quad J^-(N) = \partial_{\tau_1}. \]
(65)
It can be immediately recognized that the spectra of polynomial eigenfunctions of (64) corresponds to the spectra of QES sextic polynomial potential with singular term \( \sim 1/\tau_1 \), see [7]. Case VII.

Eventually, it can be stated that among \( \sim N^3 \) polynomial eigenfunctions in \( \tau \) variables of the quasi-exactly-solvable operator (48) there are \( \sim N^2 \) polynomial eigenfunctions of the quasi-exactly-solvable operator (61) in variables \( \tau_{1,2} \) only and \( \sim N \) polynomial eigenfunctions of the quasi-exactly-solvable operator (64) in single variable \( \tau_1 \). A similar situation occurs for the exactly-solvable operator (49), see (48) at \( A = 0 \), for which there exist infinitely-many polynomial eigenfunctions in \( \tau \)'s variables. Among these eigenfunctions there exists the infinite family of the polynomial eigenfunctions in \( \tau_{1,2} \) variables, which are eigensolutions of the operator.
(66)
\[ h^{(es)}(\tau_1, \tau_2) = -6 \tau_1 \partial_1^2 - 2 \tau_1 (7 \tau_2 - \tau_1^2) \partial_2^2 - 24 \tau_2 \partial_{1,2}^2 - 30 \partial_1 - 22 \tau_1 \partial_2 \]

Besides that there exists the infinite family of the polynomial eigenfunctions in \( \tau_1 \) variable, which are eigenfunctions of the operator
\[ h^{(es)}(\tau_1) = -6 \tau_1 \partial_1^2 + 6(2\omega \tau_1 - 2\gamma - 5) \partial_1, \] (67)

they are nothing but the Laguerre polynomials. The spectra of polynomial eigenfunctions is equidistant,
\[ E_N = 12\omega N, \]

and it corresponds to the spectra of harmonic oscillator (with a singular term \( \sim 1/\tau_1 \) in the potential).

4. Conclusions

In this paper we found the Schrödinger type equation in the space \( \tilde{\mathbf{R}} \) of relative distances \( \{r_{ij}\} \),
\[ \mathcal{H}_{rd} \Psi(r_{12}, r_{13}, r_{23}) = E \Psi(r_{12}, r_{13}, r_{23}), \quad \mathcal{H}_{rd} = -\Delta_{LB}(r) + V(r_{12}, r_{13}, r_{23}), \] (68)

where the Laplace–Beltrami operator \( \Delta_{LB} \), see e.g. (30), makes sense as the kinetic energy of a three-dimensional particle in curved space with metric (15). This equation describes angle-independent solutions of the original 3-body problem in three-dimensional space (1), including the ground state. Hence, finding the ground state involves the solution of the differential equation in three variables, contrary to the original six-dimensional Schrödinger equation of the relative motion. Since the Hamiltonian \( \mathcal{H}_{rd} \) is Hermitian, the variational method can be employed with only three-dimensional integrals involved.

The above formalism admits a natural generalization to the case of arbitrary \( d \)-dimensional three body problem. The Laplace–Beltrami operator remains unchanged, the effective potential (28) is changed but not dramatically; (quasi)-exactly-solvable integrable models continue to exist. It will be presented in our forthcoming paper [19].

Acknowledgments

AVT is thankful to the University of Minnesota, USA for kind hospitality extended to him where this work was initiated and IHES, France where it was completed. He is deeply grateful to IE Dzyaloshinsky (Irvine), T Damour, M Kontsevich (Bures-sur-Yvette) and VlG Tyuterev.
Consider the general case of the particles located at points \( \mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3 \) of masses \( m_1, m_2, m_3 \), respectively. Then the operator (13) becomes (in terms of the relative coordinates \( \rho_{ij} = \mathbf{r}_i - \mathbf{r}_j \)):

\[
\Delta R'(\rho_{ij}) = \frac{2}{\mu_{ij}} \rho_{i3} \frac{\partial^2}{\partial \rho_{i1}^2} + \frac{2}{\mu_{23}} \rho_{23} \frac{\partial^2}{\partial \rho_{i2}^2} + \frac{2}{\mu_{12}} \rho_{12} \frac{\partial^2}{\partial \rho_{i1}^2} + \frac{2}{m_3} \left( \rho_{13} + \rho_{12} - \rho_{23} \right) \frac{\partial}{\partial \rho_{i1} \rho_{i2}} + \frac{2}{m_2} \left( \rho_{23} + \rho_{12} - \rho_{13} \right) \frac{\partial}{\partial \rho_{i2} \rho_{i3}} + \frac{3}{\mu_{13}} \frac{\partial}{\partial \rho_{i1}} + \frac{3}{\mu_{23}} \frac{\partial}{\partial \rho_{i2}} + \frac{3}{\mu_{12}} \frac{\partial}{\partial \rho_{i3}},
\]

(A.1)

where

\[
\frac{1}{\mu_{ij}} = \frac{m_i + m_j}{m_i m_j},
\]

is the reduced mass for particles \( i \) and \( j \); it is in agreement with (13) for \( m_1 = m_2 = m_3 = 1 \) or, equivalently, \( \mu_{ij} = 1/2 \). This operator has the same algebraic structure as \( \Delta R(\rho) \) but lives on a different manifold in general. It can be rewritten in terms of the generators of the maximal affine subalgebra \( b_4 \) of the algebra \( sl(4, \mathbb{R}) \), see (24), see (26). The determinant of the contravariant metric tensor is

\[
D_m = \det g^{\mu
u} = 2 \frac{m_1 + m_2 + m_3}{m_1 m_2 m_3} \times (m_1 m_2 \rho_{12} + m_1 m_3 \rho_{13} + m_2 m_3 \rho_{23}) \times (2 \rho_{13} \rho_{13} + 2 \rho_{12} \rho_{23} + 2 \rho_{13} \rho_{23} - \rho_{12} - \rho_{13} - \rho_{23}),
\]

(A.2)

and is positive definite. It is worth noting a remarkable factorization property

\[
D_m = 2 \frac{m_1 + m_2 + m_3}{m_1^2 m_2^2 m_3^2} (m_1 m_2 r_{12}^2 + m_1 m_3 r_{13}^2 + m_2 m_3 r_{23}^2) \times (r_{12} + r_{13} - r_{23})(r_{12} + r_{23} - r_{13})(r_{13} + r_{23} - r_{12})(r_{12} + r_{13} + r_{23}) = 32 \frac{m_1 + m_2 + m_3}{m_1^2 m_2^2 m_3^2} P_m S_\Delta^2,
\]

where \( P_m = m_1 m_2 r_{12}^2 + m_1 m_3 r_{13}^2 + m_2 m_3 r_{23}^2 \)—the weighted sum of squared of sides of the interaction triangle and \( S_\Delta \) is their area. Hence, \( D_m \) remains still proportional to \( S_\Delta^2 \), see (18).

Making the gauge transformation of (A.1) with determinant (A.2) as the factor,
we find that
\[
D_m^{-1/4} \Delta_{\rho}^{\prime} (\rho) D_m^{-1/4} = \Delta_{LB}^{\prime}(\rho) - \tilde{V}_m, \tag{A.3}
\]
is the Laplace–Beltrami operator with the effective potential
\[
\tilde{V}_m = \frac{3(m_1 + m_2 + m_3)}{8 \left( m_1 m_2 r_{12}^2 + m_1 m_3 r_{13}^2 + m_2 m_3 r_{23}^2 \right)} + \frac{(m_1 m_2 r_{12}^2 + m_1 m_3 r_{13}^2 + m_2 m_3 r_{23}^2)}{2 m_1 m_2 m_3 \left( \rho_{12}^2 + \rho_{13}^2 + \rho_{23}^2 - 2 \rho_{12} \rho_{13} - 2 \rho_{12} \rho_{23} - 2 \rho_{13} \rho_{23} \right)},
\]
or, in geometric terms,
\[
\tilde{V}_m = \frac{3(m_1 + m_2 + m_3)}{8P_m} + \frac{(d - 2)(d - 4)}{2m_1 m_2 m_3 \Delta^2} \frac{P_m}{S^2},
\]
see (29). The Laplace–Beltrami operator plays a role of the kinetic energy of a three-dimensional quantum particle moving in curved space. It seems evident the existence of (quasi)-exactly-solvable problems with such a kinetic energy, see e.g. [18] as for the example of an exactly-solvable problem.

References

[1] Ter-Martirosyan K A 1972 Lectures on quantum field theory (Moscow: ITEP)
[2] Hylleraas E A 1929 Neue berechnung der energie des Heliums im grundzustande, sowie des tiefsten terms von Ortho-Helium Z. Phys. 54 347–66
[3] Gu X-Y, Duan B, Ma Z-Q 2002 Quantum three-body system in D dimensions J. Math. Phys. 43 2895–906
[4] Loos P-F, Bloomfield N J and Gill P M W 2015 Communication: three-electron coalescence points in two and three dimensions J. Chem. Phys. 143 181101
[5] Ince E L 1956 Ordinary Differential Equations (New York: Dover) pp 45–7
[6] Turbiner A V 1988 Quasi-exactly-solvable problems and the sl(2,R) algebra Commun. Math. Phys. 118 467–74
[7] Turbiner A V 2016 One-dimensional quasi-exactly-solvable Schrödinger equations Phys. Rep. 642 1–71
[8] Eisenhart L 1964 Riemannian Geometry (Princeton, NJ: Princeton University Press) (2nd printing)
[9] Calogero F 1969 Solution of a three-body problem in one dimension J. Math. Phys. 10 2191–6
Calogero F 1971 Solution of the one-dimensional N-body problem with quadratic and/or inversely quadratic pair potentials J. Math. Phys. 12 419–36
[10] Rühl W and Turbiner A V 1995 Exact solvability of the Calogero and Sutherland models Mod. Phys. Lett. A10 2213–22
[11] Sokolov V V and Turbiner A V 2015 Quasi-exact-solvability of the $A_2/G_2$ Elliptic model: algebraic forms, sl(3)/$g^{(2)}$ hidden algebra, polynomial eigenfunctions J. Phys. A48 155201
[12] Green H S 1964 Structure and energy levels of light nuclei Nucl. Phys. 54 505
[13] Garcia M A G and Turbiner A V 2010 The quantum $H_3$ integrable system Int. J. Mod. Phys. A 25 5567–94
[14] González-Lópe A, Kamran N and Olver P J 1991 Quasi-exactly-solvable Lie algebras of the first order differential operators in two complex variables J. Phys. A: Math. Gen. 24 3995–4008
González-Lópe A, Kamran N and Olver P J 1992 Lie algebras of differential operators in two complex variables Am. J. Math. 114 1163–85
[15] Turbiner A V 1994 Lie-algebras and linear operators with invariant subspaces Lie Algebras, Cohomologies and New Findings in Quantum Mechanics (AMS Contemporary Mathematics series vol 160) ed N Kamran and P Olver (Providence, RI: American Mathematical Society) pp 263–310
[16] González-Lópe A, Hurtubise J, Kamran N and Olver P J 1993 Quantification de la cohomologie des algèbres de Lie de champs de vecteurs et fibrés en droites sur des surfaces complexes compactes C. R. Acad. Sci., Paris I 316 1307–12
[17] Turbiner A 1998 Hidden algebra of three-body integrable systems Mod. Phys. Lett. A 13 1473–83
[18] Crandall R, Bettega R, Whitnell R 1985 A class of exactly-solvable three body problems J. Chem. Phys. 83 698–702
[19] Turbiner A V, Miller W Jr, Escobar-Ruiz M A Three-body problem in d-dimensional space: ground state, (quasi)-exact-solvability II (in preparation)