Harer-Zagier formulas for knot matrix models

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Abstract

Knot matrix models are defined so that the averages of characters are equal to knot polynomials. From this definition one can extract single trace averages and generation functions for them in the group rank – which generalize the celebrated Harer-Zagier formulas for Hermitian matrix model. We describe the outcome of this program for HOMFLY-PT polynomials of various knots. In particular, we claim that the Harer-Zagier formulas for torus knots factorize nicely, but this does not happen for other knots. This fact is mysteriously parallel to existence of explicit $\beta = 1$ eigenvalue model construction for torus knots only, and can be responsible for problems with construction of a similar model for other knots.

1 Introduction

Superintegrability property usually means that some complete set of averages is explicitly calculable. The famous examples of this kind begin with harmonic oscillator and the motion in Coulomb potential. Eigenvalue matrix models [1–3] seem to possess this property in the following sense [4–8]: averages of characters are explicitly known – they are again characters [9,10]. Moreover, the dependence on the size of the matrix $N$ is captured in the topological locus – the specialization of time-variables at the r.h.s. of the relation $<\text{character}> \sim \text{character}$. Therefore the $N$-dependence is polynomial in $q^N$ for $q$-deformed models and just polynomial in $N$ in the limit $q \to 1$. Laplace transform in $N$ converts this average into a rational function.

Remarkably, sometimes there is even more: not only denominator, but numerator is drastically simplified – all the roots are just plus-or-minus powers of $q$. However, this happens for single-trace averages $<P_k>$ rather than characters and one needs to derive a special formula to describe the answer. We call these expressions the Harer-Zagier formulas (HZF), because they were first discovered in [11] for the simplest Gaussian Hermitian model (see [12,13] for further developments in that case, including relation to other interesting subjects like Brezin-Hikami formulas [14], Okounkov’s exponentials [15] and $W$-representations [16]). Reversing the statement, the claim is that

$$<\text{character}> \text{ is rational for single trace averages}$$

This is a frequent property of eigenvalue matrix models and it can serve as an alternative manifestation of superintegrability phenomenon. An important question is what is the relation between the two manifestations, and, if different, which is more restrictive.

In this paper we consider the first example when the difference occurs – the hypothetical knot matrix models, where superintegrability is assumed to imply that averages of characters are the corresponding colored HOMFLY-PT polynomials. What we demonstrate is that this implies factorization of the single-trace averages in the spirit of [1] only for torus knots and only for the $Sl_N$ – which so far remain the only case, when the eigenvalue formula (the $\beta = 1$ TBEM model [17,18]) is actually known. Put differently, the statement is that the $\beta = 1$ TBEM matrix model for torus knots does possess the property [7], while [1] does not follow from assumption

$$<\text{character}> = \text{HOMFLY} - \text{PT}$$

for non-torus knots. In fact our result questions the relevance of the postulate [2] beyond the torus-knot variety, i.e. implies that the problem of knot matrix models remains open.
2 Knot matrix models

Matrix model for a knot $\mathcal{K}$ can be defined by the hypothetical/desirable superinterability property \cite{7,8,19}:

$$\langle \chi_R \rangle^\mathcal{K} := \mathcal{P}_R^\mathcal{K}(q, A = q^N)$$  \hspace{1cm} (3)

This definition is supported by existence of explicit TBEM model \cite{17,18} for torus knots, where the measure at the l.h.s. contains peculiar deformations of trigonometric Vanderomonde functions (see sec.3 below). Equally explicit formulas are not yet available even for twisted knots (see \cite{20} for explanation of related beauties and difficulties), but eq. (3) allows to bypass them. In fact, definition/ansatz (3) is very restrictive and allows for a number of non-trivial consistency checks – a possibility, which we begin to explore in the present paper.

On the other hand, ansatz (3) has some freedom. The characters $\chi_R$ can be chosen as various symmetric polynomials (Schur, Jack, Macdonald etc.) and $\mathcal{P}_R^\mathcal{K}$ can be chosen to be various knot polynomials (HOMFLY-PT, Khovanov-Rozansky, super- or hyper-polynomials etc.) in a suitably chosen framing.

All these details are crucial to the transition (3) $\rightarrow$ (1), and so current lack of explicit eigenvalue model representation for knot matrix models beyond torus knots can be, at least partly, attributed to difficulty of fixing these freedoms simultaneously in exactly the right way.

In cases when characters $\chi_R$ are Schur functions (i.e. for $\beta = 1$ eigenvalue models), the single-trace operators (times) are expressed entirely through single-hook characters:

$$P_k = \sum_{i=0}^{k-1} (-)^i \text{Schur}_{[k-i,1]^i}(P)$$ \hspace{1cm} (4)

while the simplest possibility for knot polynomials \cite{21-28} are HOMFLY-PT polynomials. This implies that

$$\langle P_k \rangle^\mathcal{K}_N = \sum_{i=0}^{k-1} (-)^i \mathcal{H}_R^\mathcal{K}_{[k-i,1]}(q, q^N)$$ \hspace{1cm} (5)

One can further perform a Laplace transform in $N$ to get a (1-point) Harer-Zagier function

$$Z_k^\mathcal{K} := \sum_{N=0}^{\infty} \lambda_N^N \cdot \langle P_k \rangle^\mathcal{K}_N$$ \hspace{1cm} (6)

Also various generation functions w.r.t. the $k$-variables can be introduced, but while they lead to simplification at $q = 1$, they seem to blur matters in the $q$-deformed case \cite{29}.

3 Torus knots matrix model

TBEM model \cite{17,18} provides explicit measure in (3) for torus knot $\mathcal{K} = \text{Torus}_{m,n}$ and for the gauge group $SU(N)$ (HOMFLY-PT invariants \cite{21,28}):

$$\langle F \rangle^\text{Torus}_{m,n} \sim \oint_{\text{unit circle}} F(e^{x_i}) \prod_{i<j}^N \sinh \frac{x_i - x_j}{m} \sinh \frac{x_i - x_j}{n} \prod_{i=1}^N e^{-x_i^2/(2g)} \, dx_i$$ \hspace{1cm} (7)

where $q = e^{\frac{2\pi i}{mn}}$, then (3) becomes a non-trivial theorem (proved in above mentioned papers). Generalization of this explicit formula for other knots is still an open problem (see \cite{20} for discussion). For our purposes in this paper we can use just (3), even if the measure is unknown. However, for torus knots this consideration is better grounded since (7) provides an explicit realization of the average $\langle \ldots \rangle$.

In the torus case we can define HOMFLY-PT polynomials as functions of arbitrary time variables \cite{30}:

$$H^\text{Torus}_{m,n}(p) \sim q^{2n \sum_{(i,j) \in Q} (j-i)} \cdot \text{Schur}_R\{p_{mk}\} = \sum_{Q \in m|R} q^{2n \sum_{(i,j) \in Q} (j-i)} \cdot C^Q_R \cdot \text{Schur}_Q\{p_k\}$$ \hspace{1cm} (8)

where $\gamma_Q := \sum_{(i,j) \in Q} (j-i)$ and Adams coefficients $C^Q_R$ describe expansion of Schur functions

$$\text{Schur}_R\{p_{mk}\} = \sum_{Q \in m|R} C^Q_R \cdot \text{Schur}_Q\{p_k\}$$ \hspace{1cm} (9)
These polynomials become topological invariants, in particular, acquire \( m - n \) symmetry, when time variables are restricted to topological locus:

\[
p_k = p_k^* := \frac{A^k - A^{-k}}{q^k - q^{-k}}
\]  

(10)

For the particular gauge group \( SL(N) \) the variables \( A = q^N \), so that \( p_k^* = \frac{[kN]}{[k]} \) becomes a ratio of \( q \)-numbers \([x] := \frac{q^x - q^{-x}}{q - q^{-1}}\), and we get

\[
(S\text{chur}_R)^{\text{Torus}_{m,n}}_{SL(N)} = A^{n|R} \sum_{q \in \mathbb{Z}[R]} q^{2nq_\text{SL}} \cdot C_R^Q \cdot \text{Schur}_Q \left\{ \frac{[N]}{[k]} \right\}
\]  

(11)

where proportionality factor \( A^{n|R} \) ensures agreement with (11). We can finally convert them into Harer-Zagier functions

\[
Z_k^{\text{Torus}_{m,n}} := \sum_{N=0}^{\infty} \lambda^N \cdot (P_k^*)^{\text{Torus}_{m,n}}_{SL(N)}
\]  

(12)

At this point it is important to comment on the framing of HOMFLY polynomials here. It is often convenient to deal with knot polynomials which are reduced and are in topological framing – then they satisfy simple differential-expansion identities like \( H_{[k]}(q, A) - 1 \sim \{Aq^{\ast}\}\{A/q\} \) in symmetric representations, where overline denotes reduced knot polynomial \( H_R = D_R \cdot H_R \) and \( D_R \) is quantum dimension of representation \( R \). However the framing of HOMFLY polynomials obtained here from TBEM model is generally not topological: instead it is so-called vertical or spectral framing.

4 Fundamental representation \( R = [1] \)

The ansatz (4), (5) becomes very simple for the first single-trace average \( \langle p_1 \rangle \). Namely, only the fundamental character \( \chi_{\square} \) does contribute. This provides a quick test for various options to insert in ansatz (4), as the Harer-Zagier factorization (11), if present, should occur already at this level. In this section we evaluate different possibilities using this test.

Let us begin with the simplest case non-normalized HOMFLY-PT, where the answer for \( \langle p_1 \rangle \) is given by a single fundamental HOMFLY-PT

\[
Z_1^{\text{Torus}_{m,n}} := \sum_{N=0}^{\infty} \lambda^N \cdot H_{[1]}^{\text{Torus}_{m,n}}
\]  

(13)

For example, in the case of the trefoil \((m, n) = (2, 3)\) the fundamental HOMFLY-PT is

\[
H_{[1]}^{\text{Torus}_{2,3}} = A^6 \frac{\{A\}}{\{q\}} (1 - A^{-2} \{Aq\} \{A/q\}) = \frac{(q^2 + q^{-2}) q^5 - (q^2 + 1 + q^{-2}) q^3 + q^N}{q - q^{-1}}
\]  

(14)

and we get a Harer-Zagier formula

\[
Z_1^{\text{Torus}_{2,3}} = \frac{1}{q - q^{-1}} \left( \frac{q^2 + q^{-2}}{1 - q^3 \lambda} - \frac{q^2 + 1 + q^{-2}}{1 - q^3 \lambda} + \frac{1 - q \lambda}{1 - q \lambda}(1 - q \lambda)(1 - q^3 \lambda)(1 - q^5 \lambda) \right).
\]  

(15)

which is a nicely factorized expression. It begins from \( \lambda \), because the contribution of \( N = 0 \) (Alexander polynomial) is nullified by the factor \( D_R \).

This answer can be easily generalized to other torus knots and other single-trace averages, for instance, for 2-strand torus knots one has

\[
Z_1^{\text{Torus}_{2,n}} = \frac{\lambda q^n (q^n - \lambda)}{(1 - q^{n+2} \lambda)(1 - q^{n+2} \lambda)(1 - q^{n+2} \lambda)} = \lambda q^{2n} \cdot \frac{(q^{-n} \lambda; q^2)_{\infty}}{(q^{-n} \lambda; q^2)_{\infty}} \cdot \frac{(q^{n+4} \lambda; q^2)_{\infty}}{(q^{n+2} \lambda; q^2)_{\infty}}
\]  

(16)

with the standard notation for the \( q \)-Pochhammer symbol

\[
(a; q)_{\infty} = \prod_{k=0}^{\infty} (1 - q^k a)
\]  

(17)
5 Failures beyond torus fundamental HOMFLY

In this section we list the cases, when factorization does not occur already in the simplest HZF – for the knot polynomial in the fundamental representation. What unifies these case – for torus and non-torus knots – is the lack of explicit eigenvalue matrix model, which converts characters into knot polynomials.

5.1 Reduced polynomials and Alexander polynomials

If we define averages with the help of reduced polynomials, things would be different. In particular at \( N = 0 \) we get at the r.h.s. the Alexander polynomials, which depend on representation only through the size of the single-hook diagram \[31\]

\[
\langle P_0 \rangle_{N=0}^K = \sum_{i=0}^k (-i)^i A_{K_{k-i,1}}^i(q) = A_{[1]}^i(q^k) \cdot \left( \sum_{i=0}^k (-i)^i \right) = A_{[1]}^i(q^k) \cdot \delta_{k, odd} \quad (18)
\]

From \[30\] for torus knots

\[
A_{\text{Torus}_{m,n}}^i = \frac{(q - q^{-1})(q^{mn} - q^{-mn})}{(q^m - q^{-m})(q^n - q^{-n})} \quad (19)
\]

so \( \langle P_0 \rangle_{N=0}^K \) does have the structure \( \langle 1 \rangle \), but this does not generalize to \( N \neq 0 \). Namely, the needed average would be

\[
(\text{Schur}^1_{\text{Toruss},n})_{SL(N)} = \sum_{Q \in \text{m}[R]} q^{2mn} \cdot C_{R}^{Q} \cdot \text{Schur}^Q \left\{ \frac{[N]}{[K]} \right\} \quad (20)
\]

and the corresponding Harer-Zagier formula does not factorize already for trefoil

\[
Z^1_{\text{Torus}_{2,3}} = \frac{q^2 + q^{-2}}{1 - q^4 \lambda} - \frac{1}{1 - q^2 \lambda} = \frac{q^2 - 1 + q^{-2} - \lambda q^2}{(1 - q^4 \lambda)(1 - q^2 \lambda)} \quad (21)
\]

Factorization, however, occurs for the \( q \)-derivative

\[
Z^1_{\text{Torus}_{2,3}} = \frac{Z^1_{\text{Torus}_{2,3}}(\lambda q) - Z^1_{\text{Torus}_{2,3}}(\lambda/q)}{q - q^{-1}} \quad (22)
\]

but there is no clear reason for it, besides pure technical on one hand and explicit existence of the TBEM eigenvalue model on the other hand.

5.2 First non-torus knot: figure 8

It is easy to check that at least some other knots in the fundamental representation do not possess such HZ factorization property. For example, for the figure-eight knot \( 4_1 \) the Harer-Zagier functions (for both normalized and non-normalized HOMFLY-PT) do not factorize:

\[
H^4_{[1]} = \frac{A^2 + \frac{1}{A^2} - q^2 - \frac{1}{q^2} + 1}{1 - t - t^{-1}} \quad (23)
\]

\[
Z^4_{[1]} = \frac{A - A^{-1}}{t - t^{-1}} \quad (24)
\]

5.3 Superpolynomials

Likewise, the torus superpolynomials are also not suitable:

\[
P^{(2,3)}_{[1]} = A^2 (A^2 t^2 + A^2 q^{-2} - 1) \frac{A - A^{-1}}{t - t^{-1}} \quad (25)
\]

\[
Z^{(2,3)}_{[1]} = \frac{1}{t - t^{-1}} \left( \frac{t^2 + q^{-2}}{1 - \lambda t^2} - \frac{t^2 + 1 + q^{-2}}{1 - \lambda t^3} + \frac{1}{1 - \lambda t} \right) = \frac{\lambda q^3(q^3 - \lambda)}{(1 - q^2)(1 - q^3)(1 - q^4)}
\]

Perhaps, for things to work, one needs to modify \( t \)-deform the prescription \[12\] on how to do the Laplace transform. However, at this point this is pure speculation, and this line of development will be pursued elsewhere.
5.4 Kauffman polynomials

Kauffman polynomials are the analogues of HOMFLY-PT for the \( SO(N) \) gauge groups. They are also described by an analog of Rosso-Jones formula

\[
K_{R}^{\text{Torus}_{n,m}}(q) = q^{mn(N-1)} x_{R}^{nm|\lambda|} \sum_{|\nu| \leq n|R|} b_{R,n}^{\beta} q^{-\frac{a}{2}(N-1) x_{\nu}} q^{-\frac{a}{2} |\nu|} d_{\nu} \tag{26}
\]

where \( d_{\nu} \) is the quantum dimension of \( SO(N) \) representation, corresponding to diagram \( \nu \) and coefficients \( b_{R,n}^{\beta} \) are the direct analog of Adams coefficients. The details of their calculation can be found in [32].

In topological framing unreduced fundamental Kauffman satisfies

\[
K_{R}^{\text{Torus}_{n,m}}(A-q) (A-q) (A+1) (A-1) \tag{27}
\]

From (26) one can deduce Kauffman polynomial for trefoil in fundamental representation

\[
K_{R}^{\text{Torus}_{2,3}} = -A^{2} \cdot A^{3}(q^{3} - q) + A^{2}(q^{4} - q^{2} + 1) - A(q^{3} - q) - (q^{4} + 1) \tag{28}
\]

but its Laplace transform does not factorize. Moreover, this time it is not cured by multiplication with any reasonable functions of \( A \), even different from dimension. One can wonder what this means from the point of view of our matrix model – factorization hypothesis. The answer is that the \( SO(N) \) models are associated with \( \beta = 2 \) rather than \( \beta = 1 \) (roughly speaking, \( 2 \beta \) is the power of Vandermonde-like factor in the measure). TBEM formula with \( \beta = 1 \) is easily generalized to simply-laced groups, which includes \( D_{i} \), but not \( B_{i} \) – while (28) for Kauffman invariant unifies even and odd \( N \) through \( A = q^{N-1} \) and the unifying matrix model should give \( \beta = 2 \). This puts Kauffmann example into intermediate position, and once again calls for the proper understanding in group theoretic terms of the Laplace transform, and its proper generalization.

To summarize, we illustrated the distinguished role of HOMFLY-PT polynomials in torus family from the point of view of factorizability of HZF: any deviation seems to violate it. Instead, as we claim in the next section, for torus HOMLY-PT factorization is true for all single-trace HZF \( \langle p_{k} \rangle \), not only for \( \langle p_{1} \rangle \).

6 Other single-trace correlators

For higher representations of \( SU(N) \) the issue of framing becomes important, because we get a linear combination of different representation in \( R^{\otimes m} \), which transform differently under a change of framing. Factorization takes place in the so-called spectral (or vertical) framing, which was used by Rosso and Jones in their original formula for colored HOMFLY [33] and is exactly the framing, assumed in the Tierz-Brini-Eynard-Marino matrix model [17][18], and in which the Ooguri-Vafa partition function is nicely expressed in terms of the free-fermion formalism [34][35].

Namely, the Rosso-Jones formula in this framing reads

\[
H_{R}^{\text{Torus}_{n,m}}(A,q) = A^{n|R|} \sum_{Q \subseteq m|R|} q^{2 \lambda_{Q}} C_{R}^{Q} \cdot \text{Schur}_{Q} \left\{ p_{k} = \frac{A^{k} - A^{-k}}{q^{k} - q^{-k}} \right\} \tag{29}
\]

where \( C_{R}^{Q} \) are, again, Adams coefficients from [9]. Note that vertical framing explicitly breaks the \( n \leftrightarrow m \) symmetry.

With this convention, the Laplace transform of single-trace correlator becomes nicely factorized:

\[
Z_{R}^{\text{Torus}_{n,m}} = \lambda q^{2nm} \prod_{i=0}^{d_{m}-2} \left( 1 - q^{d(n-m)+2i+2} \lambda \right) \prod_{i=0}^{d_{m}} \left( 1 - q^{d(n-m)+2i} \right) = \lambda q^{2nm} \cdot \left( \lambda q^{d(n-m)+2} ; q^{2} \right)_{\infty} \left( \lambda q^{d(n+m)+2} ; q^{2} \right)_{\infty} \tag{29a}
\]

where we once again use the \( q \)-Pochhammer symbols [17].

This factorization formula for the Laplace transform in \( N \) of the peculiar combination of torus HOMFLY-PT polynomials is the main result of this paper.
The proof of this identity is calculational and combinatorial, very similar in spirit to that of [29]. The main point is seen already at $q = 1$: according to [31], in this case we deal just with the dimension of representation $\text{Schur}_R \{N\}$, whose Laplace transform is factorizable only for single-hook $R$, while factorization is lost beyond one-hook (and thus for multi-trace averages), e.g. $\sum_N \lambda^N \cdot \text{Schur}_{[3,3]} \{N\} = \frac{\lambda^2 (\lambda^2 + 3 \lambda + 1)}{(\lambda - 1)^3}$. Moreover, the single-hook factorization is preserved by the $q$-deformed Adams rule [30] and the action of $\bar{W}$ operator [16], which is responsible for the factor $q^{-2n \varepsilon_{\bar{Q}}}$ in [29]. This is technically straightforward:

$$Z_d \big|_{\text{Torus}_{n,m}} = \sum_{N=0}^{\infty} \lambda^N \sum_{i=0}^{d-1} (-1)^i \frac{q^{N \delta_{0,m}}}{{\mathcal{H}}_{d-i,1]} \big|_\text{Torus}_{n,m} (q, q^N) \big)$$

$$= \sum_{N=0}^{\infty} \lambda^N \sum_{Q \cap dm} q^{2 \varepsilon_{\bar{Q}} \varepsilon_{[d-m,1]}^T} \sum_{i=0}^{d-1} (-1)^i \left. C_{[d-i,1]}^Q \right|_\text{Schur}_Q \left( p_k = \frac{q^{Nk} - q^{-Nk}}{q^{k} - q^{-k}} \right)$$

$$= \sum_{N=0}^{\infty} \lambda^N \sum_{L=0}^{dm-1} (-1)^L \frac{q^{2 \varepsilon_{\bar{Q}} \varepsilon_{[d-m,1]}^T} \sum_{N=0}^{\infty} \lambda^N \sum_{s=0}^{L-1} \left( q^{N-L+s} - q^{-N-L-s} \right) \prod_{L=0}^{dm-L-1} (q^{dm-L+s} - q^{-dm-L+s}) \prod_{L=0}^{L-1} (q^{L-s} - q^{-L+s})}$$

$$= \sum_{L=0}^{dm-1} (-1)^L \frac{q^{2 \varepsilon_{\bar{Q}} \varepsilon_{[d-m,1]}^T} \sum_{L=0}^{dm-L-1} \frac{q^{dm-L+s} - q^{-dm-L+s}}{q^{L-s} - q^{-L+s}} \prod_{L=0}^{dm-L-1} (q^{dm-L+s} - q^{-dm-L+s}) \prod_{L=0}^{L-1} (q^{L-s} - q^{-L+s})} \sum_{s=0}^{L-1} \frac{(q^s - q^{-s})}{(1 - q^{d(n-m)+2s \lambda})}$$

$$= \lambda q^{2 \varepsilon_{\bar{Q}} \varepsilon_{[d-m,1]}^T} \sum_{s=0}^{L-1} \frac{(q^s - q^{-s})}{(1 - q^{d(n-m)+2s \lambda})}$$

The main point here is the double application of the projection rule [1] to the definition [9] of Adams coefficients:

$$\sum_R (-)^i \cdot \delta_{R,[d-i,1]} \left( \sum_{Q \cap dm} C_{[d-i,1]}^Q \cdot \text{Schur}_R \{p_k\} \right) \Rightarrow \text{Schur}_R \{p_{km}\}$$

$$\sum_Q \left( \sum_i (-)^i C_{[d-i,1]}^Q \right) \text{Schur}_Q \{p_k\} = \sum_i (-)^i \text{Schur}_{[d-i,1]} \{p_{km}\} \Rightarrow \sum_Q \left( \sum_i (-)^i C_{[d-i,1]}^Q \right) \text{Schur}_{[d-m-L,1]} \{p_k\}$$

$$\sum_{i=0}^{d-1} (-)^i C_{[d-i,1]}^Q = \sum_{L=0}^{dm-1} \delta_{q,[d-m-L,1]}$$

(30)

Despite apparent simplicity of this calculation, a conceptual proof is highly desirable, applicable to the whole variety of $\beta = 1$ matrix models.

7 Conclusion

In this paper we proposed to view the factorization of the Laplace transform of single-trace average as an alternative manifestation of superintegrability.

Remarkably, this factorization turns out to be present for torus knots’ HOMFLY-PT polynomials, where the eigenvalue model (and free-fermion representation) is explicitly known, but the very naive attempts to observe similar factorization for slight deformations of this setting: to superpolynomials, to other knots and even to Kauffman polynomials – all fail.
This seems to suggest, that the problem of finding proper matrix models for families of knot polynomials is more tricky and rigid than was first thought. One of the ways around this situation would be to relax the prescription

\[
\langle \text{character} \rangle = \text{knot polynomial}
\]  

(31)
in some yet unknown way. As was recently demonstrated\cite{36} similar broadening of the point of view can be very fruitful in discovering new character expansion formulas.

Another possibility would be to understand the group-theoretic meaning of the Laplace transform, and to adjust it, accordingly. At the moment this part of the Harer-Zagier construction seems completely \textit{ad hoc}.

There is a number of possible directions/questions to pursue, which we would like to point out

- What is the formula for the double-trace correlators? Is it, in some sense, similar to the one for \(q\)-deformed Hermitian Gaussian matrix model?
- HOMFLY-PT polynomials for torus knots have a well-known generalization from \(S^3\) to Seifert spaces. The knot matrix model is known for this case. Does the Harer-Zagier factorization persist as well?
- Superpolynomials, which we considered in this paper, are not Khovanov-Rozansky polynomials. Instead, they coincide for large enough \(N\) (which is knot-dependent). Therefore, the Laplace transformed sums for superpolynomials and actual KR polynomials differ by some polynomial in \(\lambda\). Can it be, that this polynomial transforms non-factorizable Harer-Zagier function into factorizable?
- Last but not least, there is a question about the relation of superintegrability (in the form of Harer-Zagier factorization) to other well-known, and undergoing rapid development, knot-theoretical structures: the knots-quivers correspondence\cite{37,38} and theory of \(q\)-Virasoro localization\cite{39,40}.

We hope to address some, or all of these questions in future.

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