Abstract. We study the orbit types of isometries of the spherical, Euclidean and hyperbolic spaces in each finite dimension, and show that they are parameterized by a discrete algebraic invariant, the Segre symbol. In particular, we prove that the number of orbit types is finite. We study the varieties of invariant totally geodesic submanifolds of an isometry, and show that the dimensions of the connected components of these varieties determine the Segre symbol of the isometry, and thus its orbit type.

1. Introduction

Our objective is to study the isometries of the simply connected space forms. That is to say, the spherical, Euclidean and hyperbolic spaces. We will denote by $S^n$ the $n$-dimensional sphere, $E^n$ the $n$-dimensional Euclidean affine space and $H^n$ the $n$-dimensional hyperbolic space. Their isometry groups are denoted by $I(S^n)$, $I(E^n)$ and $I(H^n)$ respectively.

A very first and well known classification of isometries is based on their fixed point behavior. The translation length of an isometry $f$ is the infimum of the distances that points are moved,

$$a = \inf_{p} ||f(p) - p||.$$ 

If it is positive, then $f$ is called hyperbolic. If $f$ has a fixed point it is called elliptic. If $f$ does not fix a point but its translation length is zero, then it is called parabolic. We remark that parabolic isometries can only occur in hyperbolic spaces. All spherical and Euclidean isometries are either elliptic or hyperbolic.

A finer classification of isometries is given by their conjugacy classes. As in the case of linear endomorphisms, a key first step in studying conjugacy classes is to establish normal forms. We briefly recall the normal forms of isometries in each of the three cases.

The Lie group $I(S^n)$ is isomorphic to the group of real orthogonal matrices $O(n+1)$.

Definition 1.1. An orthogonal normal form, is a block diagonal matrix whose blocks are $\pm I_k$ or $R_\theta$, where $I_k$ denotes the $k \times k$ identity matrix and

$$R_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} ; \theta \in \mathbb{R} \setminus \{k\pi\}.$$ 

1Partially supported by the Ministerio de Ciencia e Innovación (Spain) through Project MTM2006-14575, under grant BES-2007-15178.

Keywords: Isometries, normal forms, Segre symbol, orbit types, invariant subspaces, orthogonal group, Euclidean group, Lorentz group, hyperbolic space.
The Lie group $I(\mathbb{E}^n)$ of isometries of the Euclidean affine space is isomorphic to the Euclidean group $\text{Euc}(n)$ of matrices of the form $egin{pmatrix} A & a \\ 0 & 1 \end{pmatrix}$, where $A \in O(n)$ and $a \in \mathbb{R}^n$.

**Definition 1.2.** An **Euclidean normal form** is a matrix of one of the following types:

1) $A \oplus 1$, where $A \in O(n)$ is an orthogonal normal form.
2) $A \oplus \begin{pmatrix} a \\ 1 \end{pmatrix}$, where $A \in O(n-1)$ is an orthogonal normal form and $a \in \mathbb{R}$ is a positive number (the translation length).

The direct sum of matrices denotes the block diagonal matrix whose blocks are the direct summands: $A \oplus B = \begin{pmatrix} A & B \end{pmatrix}$. Note that types 1 and 2 correspond to elliptic and hyperbolic isometries respectively.

Consider the hyperboloid model $\mathbb{H}^n$ of the hyperbolic space. The Lie group $I(\mathbb{H}^n)$ is isomorphic to the group $PO(1,n)$ of proper Lorentz matrices.

**Definition 1.3.** A **Lorentzian normal form** is a proper Lorentz matrix of one of the following types:

1) $1 \oplus A$, where $A \in O(n)$ is an orthogonal normal form.
2) $\Theta \oplus A$, where $A \in O(n-2)$ is an orthogonal normal form and
   $$\Theta = \begin{pmatrix} 3/2 & 1 & -1/2 \\ 1 & 1 & -1 \\ 1/2 & 1 & 1/2 \end{pmatrix}.$$
3) $\Omega_t \oplus A$, where $A \in O(n-1)$ is an orthogonal normal form and
   $$\Omega_t = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}, \ t \neq 0.$$

Types 1, 2 and 3 correspond to elliptic, parabolic and hyperbolic isometries respectively.

The following is an extension of the Spectral Theorem for the orthogonal group $O(n)$, to the Euclidean and Lorentz groups $\text{Euc}(n)$ and $PO(1,n)$.

**Theorem 1.4.** Every element of $O(n)$, $\text{Euc}(n)$ and $PO(1,n)$ is conjugate to an orthogonal, Euclidean and Lorentzian normal form respectively, which is unique up to block permutation.

The classification of isometries by conjugacy classes is too fine, and in all three cases there are infinitely many conjugacy classes. Moreover, classes having the same normal form structure but differing only in the sets of eigenvalues have the same geometric behavior. A more convenient classification is given by the orbit type.

Let $G$ be a Lie group acting on a manifold $M$.

**Definition 1.5.** Two points $x, y \in M$ have the same $G$-orbit type if their orbits are $G$-equivariantly isomorphic.

Thus the orbit type defines an equivalence relation in $M$. We study the orbit types of $I(\mathbb{S}^n)$, $I(\mathbb{E}^n)$ and $I(\mathbb{H}^n)$ respectively acting by conjugacy on themselves, and we show that in each case, there are
finitely many orbit types, parameterized by a discrete invariant which we describe next.

The Segre symbol of a linear endomorphism of a finite dimensional real or complex vector space is a sequence of positive integers encoding the number of distinct eigenvalues, together with the sizes of the Jordan blocks corresponding to each eigenvalue. This invariant has been studied by many authors for classification purposes. For example, in [HP], projectivities are studied by means of their Segre symbol. An application of the Segre symbol, relevant in theoretical physics, is the Petrov classification of gravitational fields, in which the Segre symbol of the curvature tensor is studied in order to stratify Einstein spaces [Pe].

Let \( \mathcal{M}(n) \) denote the set of all square matrices over \( \mathbb{C} \) and let \( GL(n) \) act on \( \mathcal{M}(n) \) by conjugation. In [Ar], Arnold suggests that the partition of \( \mathcal{M}(n) \) into Segre classes makes \( \mathcal{M}(n) \) into a stratified space. In [Gi], Gibson proves this statement, and shows that the stratification induced by the Segre symbol is a Whitney-regular stratification. Moreover in [Br] (Theorem 2.10.1) it is shown that in this case, the partition according to the Segre symbol coincides with the stratification by orbit types. Following the work of [Ar] and [Gi], in [Ci] we define the Segre symbol for an arbitrary affine endomorphism over an algebraically closed field \( k \), and we show that the stratification of the space of affine endomorphisms of an affine space according to the Segre symbol coincides with the stratification by \( GA(n) \)-orbit types, and that this stratification is Whitney regular for \( k = \mathbb{C} \). Here we extend the definition of the Segre symbol to each particular case \( I(\mathbb{S}^n) \), \( I(\mathbb{E}^n) \) and \( I(\mathbb{H}^n) \), such that the above properties will still be fulfilled. We prove the following theorem.

**Theorem 1.6.** The orbit type decomposition of \( I(\mathbb{S}^n) \), \( I(\mathbb{E}^n) \) and \( I(\mathbb{H}^n) \) respectively is parameterized by the Segre symbol. In particular the number of orbit types is finite, given in each case by

\[
\begin{align*}
\text{i)} & \quad s(n) = \sum_{j=0}^{[n+1]/2} (p(j) \left( \left\lfloor \frac{n+1}{2} \right\rfloor - j + 1 \right)), \\
\text{ii)} & \quad e(n) = \left( \sum_{j=0}^{[n/2]} p(j)(n - 2j + 1) \right) + \left( \sum_{j=0}^{[(n-1)/2]} p(j)(n - 2j) \right), \\
\text{iii)} & \quad h(n) = \left( \sum_{j=0}^{[n/2]} p(j)(n - 2j + 1) \right) + \left( \sum_{j=0}^{[(n-2)/2]} p(j)(n - 2j - 2) \right) + s(n - 2),
\end{align*}
\]

where \( p(k) \) denotes the number of partitions of \( k \) and \( \lfloor k \rfloor \) denotes the integer part of \( k \).

A future objective could be to prove that the orbit type decomposition defines a Whitney-regular stratification on each space of isometries.

The earlier classification of isometries according to their fixed point behavior is naturally extended in the Euclidean case to the study of invariant affine subspaces of \( \mathbb{E}^n \) of arbitrary dimension. A generalization of this study to isometries of the curved spaces \( \mathbb{S}^n \) and \( \mathbb{H}^n \) leads to the study of totally geodesic invariant subspaces.

We recall that a Riemannian submanifold \( N \) of a Riemannian manifold \( M \) is called **totally geodesic** if all geodesics in \( N \) are also geodesics in \( M \). For example, each closed geodesic in a Riemannian manifold defines a 1-dimensional compact totally geodesic submanifold.

Let \( M^n \) denote one of the simply connected Riemannian manifolds \( \mathbb{S}^n \), \( \mathbb{E}^n \) or \( \mathbb{H}^n \). The group of isometries \( I(M^n) \) is isomorphic to a Lie group \( G(n) \) of dimension \( n(n+1)/2 \), which given the type of
\( M^n \), is one of the following:

\[
G(n) = \begin{cases} 
O(n + 1), & \text{if } M^n = S^n. \\
\text{Euc}(n), & \text{if } M^n = E^n. \\
PO(1,n), & \text{if } M^n = H^n.
\end{cases}
\]

For all \( 0 \leq k \leq n \) define the generalized Grassmannian \( G(k, M^n) \) to be the set of closed totally geodesic submanifolds of \( M^n \) isometric to \( M^k \). It is a smooth semialgebraic variety of dimension \((k + 1)(n - k)\) and can be identified with a homogeneous space (see [Ob])

\[
G(k, M^n) = G(n)/G(k) \times O(n - k).
\]

For \( f \in I(M^n) \) we denote the set of totally geodesic \( f \)-invariant spaces \( M^k \subseteq M^n \), by \( \Gamma_f(k) \subseteq G(k, M^n) \), which is a smooth closed algebraic subvariety of \( G(k, M^n) \). The main theorem of this paper is the following.

**Main Theorem.** Let \( M^n \) be a simply connected space form of dimension \( n \), and let \( f \) be an isometry of \( M^n \). Denote by \( \Gamma_f(k) \) the subvariety of \( G(k, M^n) \) of \( f \)-invariant closed totally geodesic submanifolds of \( M^n \) isometric to \( M^k \). The dimensions of the connected components of \( \Gamma_f(k) \), for \( 0 \leq k \leq t \), determine the Segre symbol of \( f \), where

\[
t = \begin{cases} 
1, & \text{if } M^n = S^n \\
3, & \text{if } M^n = E^n \\
4, & \text{if } M^n = H^n.
\end{cases}
\]

Therefore, by the previous result, the dimensions of the connected components of the varieties of invariant subspaces in low dimensions determine the orbit type of the isometry. Furthermore, given such dimensions it is possible to recover both the Segre symbol and the varieties themselves in terms of disjoint unions of products of generalized Grassmann manifolds, as we shall see in Section 4.

We next explain the contents of the subsequent sections. In Section 2 we establish the normal forms of isometries and define the Segre symbol in each of the three cases. In particular we give a proof of Theorem 1.4. Although normal forms of isometries are known, precise statements and proofs for the Euclidean and hyperbolic cases are difficult to find in the literature. The case of Euclidean isometries can be found in [Re]. Our definition of Euclidean normal form differs from that in [Re] in that our normal forms are a natural extension of Jordan matrices. Isometries of the hyperbolic space are often studied by means of the Möbius group, by considering the upper-half plane model of the hyperbolic space, as in [Ra]. This may possibly be a cause for the lack of references concerning normal forms of \( PO(1,n) \). In [Gr] such normal forms are listed, but apparently this reference is not well enough known. For example, in [Ba] (Theorem 6.9) the list of normal forms is incomplete since parabolic isometries are missing. Also, in [GK], dynamical types of isometries of the hyperbolic space are studied, but the normal forms given in [Gr] are not used. Since we use normal forms as a key step in the study of orbit types, we present a survey of these forms in Section 2 with an elementary proof purely based on linear algebra arguments. Section 3 is concerned with orbit types. We study the isotropy subgroups of normal forms and prove Theorem 1.6. In Section 4 we study the varieties of invariant totally geodesic subspaces of an isometry. We provide a description of these varieties in terms of the Segre symbol, and show that their connected components are products of generalized Grassmannians. We then prove the main theorem of this paper.

**Acknowledgements.** I thank F. Guillén and V. Navarro for pointing me to the particular problem that gave rise to this paper, as well as for their valuable comments and suggestions.
2. Normal forms and Segre symbol

In this section we establish the normal forms of isometries of the spherical, Euclidean and hyperbolic spaces in each finite dimension, and we define the Segre symbol for each of the three cases.

2.1. Isometries of $S^n$. The standard model of the $n$-dimensional spherical space is the unit sphere $S^n$ of $\mathbb{R}^{n+1}$ defined by $S^n = \{x \in \mathbb{R}^{n+1}; |x| = 1\}$. The group $I(S^n)$ of isometries of $S^n$ is isomorphic to the real orthogonal group $O(n+1)$.

We next recall the definition of the Segre decomposition of $I(S^n)$. This decomposition is introduced in [Ar] for the space of complex square matrices. We can adapt its definition to the real orthogonal case as follows.

Throughout this paper we will denote by $\mathbb{R}^n \oplus R^n \oplus \cdots \oplus R^n$ the block diagonal matrix of size $(2n \times 2n)$ consisting of $n$ blocks equal to $R^n$, where $\theta \in \mathbb{R} \setminus \{k\pi\}$.

Let $f \in I(S^n)$ be an isometry. Then $f$ has an orthogonal normal form

$$R^\oplus_{\theta_1} \oplus \cdots \oplus R^\oplus_{\theta_s} \oplus \varepsilon_1 \mathbb{I}_{m_1} \oplus \varepsilon_2 \mathbb{I}_{m_2},$$

where we can assume that $n_1 \geq \cdots \geq n_s \geq 0$, $m_1 \geq m_2 \geq 0$, and that the eigenvalues satisfy $\theta_i \neq \theta_j$ and $\varepsilon_i \neq \varepsilon_j$, for all $i \neq j$. Since $\varepsilon_i \in \{\pm 1\}$ we have $t \leq 2$.

**Definition 2.1.** Let $f \in I(S^n)$. With the previous notation the Segre symbol of $f$ is

$$\tilde{\sigma}_f = [(n_1 m_1), \cdots, (n_s m_s), m_1, m_2].$$

For example, let $\begin{pmatrix} c & s \\ -s & c \end{pmatrix}$ be a normal form of $f$, then $\tilde{\sigma}_f = [(11), 2]$.

A Segre class of $I(S^n)$ consists of all elements in $I(S^n)$ that have a given Segre symbol. There are finitely many classes, each one being a union of $O(n+1)$-orbits that have the same discrete invariants but different eigenvalues. The following proposition is a matter of combinatorics.

**Proposition 2.2.** The number of Segre classes in $I(S^n)$ is given by

$$s(n) = \sum_{j=0}^{\left\lfloor \frac{n+1}{2} \right\rfloor} p(j) \left( \left\lfloor \frac{n+1}{2} \right\rfloor - j + 1 \right),$$

where $p(k)$ denotes the number of partitions of $k$ and $[k]$ denotes the integer part of $k$.

2.2. Isometries of $\mathbb{E}^n$. Before we provide the normal forms of Euclidean isometries let us first recall some basic definitions and facts.

Let $V = \mathbb{R}^{n+1}$ be the $n+1$-dimensional standard vector space and consider the standard affine space $\mathbb{E}^n$ as the subspace of $V$ defined by $\mathbb{E}^n = \{x_{n+1} = 1\}$. We say that the subspace of $V$ given by $V_0 = \{x_{n+1} = 0\}$ is the vector space associated to $\mathbb{E}^n$. We will assume that $V_0$ comes with the standard Euclidean metric.
An isometry \( f : \mathbb{E}^n \to \mathbb{E}^n \) is induced by a linear map \( \varphi : V \to V \) that satisfies \( \varphi(\mathbb{E}^n) \subset \mathbb{E}^n \), and for which the restriction \( \varphi|_{V_0} : V_0 \to V_0 \) is an orthogonal map with respect to the metric of \( V_0 \). The set of all isometries \( \mathbf{I}(\mathbb{E}^n) \) of \( \mathbb{E}^n \) is isomorphic to the Euclidean group, which can be described concretely as the semidirect product \( \text{Euc}(n) = O(n) \ltimes \mathbb{R}^n \).

An Euclidean reference \((\{e_i\}_{i=1}^n; p)\) of \( \mathbb{E}^n \) is an orthonormal basis \( \{e_i\}_{i=1}^n \) of \( V_0 \) and a point \( p \in \mathbb{E}^n \), and induces a basis \( \{e_i\}_{i=1}^{n+1} \) of \( V \), where \( e^{n+1} = p \).

Let \( f : \mathbb{E}^n \to \mathbb{E}^n \) be an isometry and let \((\{e_i\}_{i=1}^n; p)\) be an Euclidean reference of \( \mathbb{E}^n \). The matrix of \( f \) in this reference is the matrix of \( \varphi \) in the induced basis of \( V \). Since \( \varphi(V_0) \subset V_0 \), the last row of such a matrix will be \((0, \cdots, 0, 1)\).

We recall that an Euclidean normal form is a matrix of one of the following types:

1) \( A \oplus 1 \), where \( A \in O(n) \) is an orthogonal normal form.
2) \( A \oplus (\frac{1}{0} 1) \), where \( A \in O(n-1) \) is an orthogonal normal form and \( a \in \mathbb{R} \) is a positive number (the translation length).

The following theorem can be found in [Re]. We give a straightforward proof.

**Theorem 2.3.** Let \( f : \mathbb{E}^n \to \mathbb{E}^n \) be an isometry. There exists an Euclidean reference of \( \mathbb{E}^n \) such that the matrix of \( f \) in this reference is an Euclidean normal form.

**Proof.** Let \( x = (x_1, \cdots, x_{n+1}) \in V = \mathbb{R}^{n+1} \). Let \( w : V \to \mathbb{R} \) be the linear form defined by \( w(x) = x_{n+1} \). Hence, \( V_0 = \text{Ker}(w) \). The isometry \( f \) is determined by a linear map \( \varphi : V \to V \) such that \( w \circ \varphi = w \), and \( \varphi_0 = \varphi|_{V_0} \) is orthogonal. To prove the theorem it suffices to show that there exists a Jordan basis \( \{u_i\}_{i=1}^{n+1} \) of \( V \) with respect to \( \varphi \), satisfying \( w(u^{n+1}) = 1 \) and \( w(u^i) = 0 \) for all \( i \leq n \), and such that \( \{u^1, \cdots, u^n\} \) is a Jordan orthonormal basis of \( V_0 \) with respect to \( \varphi_0 \).

Let \( V = V_R \oplus V_1 \) be the decomposition of \( V \) into \( \varphi \)-invariant subspaces, where \( V_1 \) denotes the primary component of \( V \) corresponding to the eigenvalue 1. Since \( w \circ (\varphi - I) = 0 \), we have \( V_R \subset V_0 \).

Consequently, there is a Jordan orthonormal basis of \( V_R \) with respect to \( \varphi_0|_{V_R} \). Moreover, there is an orthogonal decomposition \( V_0 = V_R \oplus (V_1 \cap V_0) \). Therefore we can assume that \( V = V_1 \), so that \( \varphi \) is a unipotent linear map. Since \( \varphi_0 \) is orthogonal, \( \varphi_0 \) is the identity matrix. Then \( f \) is either the identity or a translation. In the first case the proof is complete. Assume that \( f \) is a translation, and let \( p = (0, \cdots, 0, 1) \). Then \( p - f(p) \neq 0 \) and we define \( u^n = (f(p) - p)/||f(p) - p|| \). We complete \( u^n \) to an orthonormal basis of \( V_0 \). Then \( \{u^1, \cdots, u^n, p\} \) is a basis of \( V \) satisfying the desired conditions. The matrix of \( f \) in the reference \((u^1, \cdots, u^n; p)\) is \( I_{n-1} \oplus (\frac{1}{0} 1) \), where \( a = ||f(p) - p|| \) is a positive real number. \( \square \)

We next define the Segre symbol of an isometry of \( \mathbb{E}^n \). The main difference with respect to the orthogonal case is that we now distinguish the elliptic from the hyperbolic case, as well as the blocks of eigenvalue 1 from the remaining blocks. This distinction will be justified in section 3.

Let \( f \in \mathbf{I}(\mathbb{E}^n) \) be an isometry induced by \( \varphi : V \to V \), and let \( V = V_R \oplus V_1 \) be the decomposition of \( V \) into \( \varphi \)-invariant subspaces, where \( V_1 \) is the primary component of eigenvalue 1, and \( V_R \) denotes
the direct sum of the remaining primary components. We denote by \( \varphi_R = \varphi|_{V_R} \) and \( \varphi_1 = \varphi|_{V_1} \) the restrictions of \( \varphi \) to \( V_R \) and \( V_1 \) respectively. Let

\[
d = \dim \ker(\varphi - I), \quad r = \dim \ker(\varphi_0 - I).
\]

By Theorem 2.3, one of the following conditions is satisfied:

1) \( d = r + 1 \), and \( f \) has a normal form \( A \oplus \mathbb{I}_{n+1} \), where \( A \) is an orthogonal normal form of \( \varphi_R \).
2) \( d = r \), and \( f \) has a normal form \( A \oplus \mathbb{I}_{r-1} \oplus \left( \begin{smallmatrix} a & 1 \\ 0 & 1 \end{smallmatrix} \right) \), where \( a > 0 \) and \( A \) is an orthogonal normal form of \( \varphi_R \).

Cases (1) and (2) correspond to elliptic and hyperbolic isometries respectively.

**Definition 2.4.** Let \( e \). For example, \( \text{Isometries of } \mathbb{H}^n \). Let \( e \).

**Proposition 2.5.** There is a finite number of Segre classes in \( I(\mathbb{E}^n) \), given by

\[
e(n) = \sum_{i=n-1}^{n} \left( \left\lfloor \frac{i}{2} \right\rfloor \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} p(j)(i - 2j + 1) \right),
\]

where \( p(k) \) is the number of partitions of \( k \), and \( \lfloor k \rfloor \) denotes the integer part of \( k \).

**Proof.** Let \( e^e(n) \), \( e^h(n) \) denote the number of Segre classes of elliptic and hyperbolic isometries of \( \mathbb{E}^n \) respectively, so that \( e(n) = e^e(n) + e^h(n) \). By the definition of Euclidean normal forms we have \( e^h(n) = e^e(n - 1) \). Therefore \( e(n) = e^e(n) + e^e(n - 1) \). A combinatoric calculation shows that

\[
e^e(n) = \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} p(j)(n - 2j + 1).
\]

For example, \( e(1) = 3, e(2) = 6, e(3) = 10 \) and \( e(4) = 16 \).

### 2.3. Isometries of \( \mathbb{H}^n \)

We next recall the normal forms of \( PO(1, n) \) obtained in [Gr], and we give an alternative proof based on elementary linear algebra arguments.

Let \( V \) be a finite dimensional vector space over \( \mathbb{R} \), and let \( Q \) be a nondegenerate symmetric bilinear form over \( V \) of signature \((1, n)\). The pair \((V, Q)\) is called a **Lorentz space**.

A linear endomorphism \( T \in \text{End}(V) \) is called an **isometry** (with respect to \( Q \)), if \( Q(Tu, Tv) = Q(u, v) \) for all \( u, v \in V \). We denote by \( O(Q) \) the isometry group of \((V, Q)\). An isometry \( T \in O(Q) \) is called **proper** if \( Q(T(v)) < 0 \) for all \( v \in V \) such that \( Q(v) < 0 \). We denote by \( PO(Q) \) the subgroup of \( O(Q) \) of proper isometries, which is a subgroup of index 2.

Let \( V = \mathbb{R}^{n+1} \) and \( Q_{1,n}(x, y) = -x_0y_0 + \sum_{i=1}^{n} x_iy_i \). Then \( Q_{1,n} \) has signature \((1, n)\) and we denote the group \( O(Q_{1,n}) \) by \( O(1, n) \). We call \((\mathbb{R}^{n+1}, Q_{1,n})\) the **standard Lorentz space**, and denote it by \( \mathbb{R}^{1,n} \). Let \( J \) denote the matrix of \( Q_{1,n} \) in the canonical basis of \( \mathbb{R}^{n+1} \). Then

\[
J = -\mathbb{I}_1 \oplus \mathbb{I}_n,
\]
and the group $O(1, n)$ is identified with the group of real square matrices $A$ of size $n + 1$ such that
\[ A^t J A = J. \]

Let $\mathbb{H}^n$ denote the connected submanifold of $\mathbb{R}^{1,n}$ defined by
\[ \mathbb{H}^n = \{ x \in \mathbb{R}^{1,n}; Q_{1,n}(x) = -1, x_0 > 0 \}. \]

With the induced metric of $\mathbb{R}^{1,n}$, $\mathbb{H}^n$ is a Riemannian manifold, and it is a model of the $n$-dimensional hyperbolic space.

The isometry group of $\mathbb{H}^n$ is isomorphic to the proper Lorentz group $PO(1, n)$, which consists of the two connected components of $O(1, n)$ leaving $\mathbb{H}^n$ invariant. There are the matrices of $O(1, n)$ having a positive entry in the upper left corner. Before providing the normal forms of $PO(1, n)$ we first establish some previous definitions and results.

A vector $v \in V$ is called space-like if $Q(v) > 0$, it is called time-like if $Q(v) < 0$, and it is called light-like or isotropic if $Q(v) = 0$.

Let $U \subset V$ be a linear subspace of $V$. Then $U$ is said to be time-like if it has a time-like vector, space-like if every nonzero vector in $U$ is space-like, and light-like otherwise.

The following is a well known result (see [Ra], p. 60).

**Lemma 2.6.** Let $(V, Q)$ be a Lorentz space and let $u, v$ be orthogonal and linearly independent vectors. If $Q(u) \leq 0$ then $Q(v) > 0$.

**Lemma 2.7.** Let $(V, Q)$ be a Lorentz space and let $T \in O(Q)$. Let $W$ be a space-like $T$-invariant subspace of $V$. Then $W^\perp$ is a $T$-invariant Lorentz space, and $V = W \oplus W^\perp$.

**Proof.** Since $W$ is space-like, $W \cap W^\perp = 0$, and $W \oplus W^\perp = V$. Since $T$ is an isometry, $W^\perp$ is $T$-invariant. Indeed, if $u \in W^\perp$ and $v \in W$, then $Q(Tu, v) = Q(u, T^{-1}v) = 0$. Therefore $Tu \in W^\perp$. Since $V$ is time-like and $W$ is space-like, $W^\perp$ must be time-like, and hence a Lorentz space. \qed

We will denote by $(-)^c$ the scalar-extension functor from $\mathbb{R}$ to $\mathbb{C}$. Then $V^c = V \otimes \mathbb{C}$, and $T^c$, $Q^c$ denote the $\mathbb{C}$-linear endomorphism of $V^c$, and the $\mathbb{C}$-bilinear form obtained by scalar extension from $T$ and $Q$ respectively. If $T$ is an isometry of $(V, Q)$ then $T^c$ is an isometry of $(V^c, Q^c)$. Moreover, conjugacy in $\mathbb{C}$ induces an $\mathbb{R}$-automorphism of $V^c$, which we denote by $v \mapsto \bar{v}$, as it is customary. The maps $\mathcal{R}, \mathcal{I}: \mathbb{C} \to \mathbb{R}$ taking the real and imaginary parts respectively, define the corresponding $\mathbb{R}$-linear maps
\[ \mathcal{R}, \mathcal{I}: V^c \to V \otimes \mathbb{R} \cong V, \]
where $\mathcal{R}(v) = \frac{1}{2}(v + \bar{v})$ and $\mathcal{I}(v) = \frac{1}{2}(v - \bar{v})$.

We next study the orthogonality of the primary components of $T^c$, for $T \in O(Q)$. The following result is possibly well known, and a proof can be found in Gongopadhyay-Kulkarni ([GK]). Here we give a more elementary proof of it.
Lemma 2.8. Let $T \in O(Q)$ and let $\lambda, \mu \in \mathbb{C}$ such that $\lambda \mu \neq 1$. The primary components $V^c_\lambda$ and $V^c_\mu$ of $V^c$ with respect to $T^c$, corresponding to the eigenvalues $\lambda$ and $\mu$ respectively, are orthogonal, i.e $Q^c(V^c_\lambda, V^c_\mu) = 0$.

Proof. Throughout this proof we will omit the superscript $c$. Let $F^i_\lambda = \text{Ker}(T - \lambda)^i$, and $F^j_\mu = \text{Ker}(T - \mu)^j$. We want to show that $Q(u, v) = 0$ for all $u \in F^i_\lambda$ and $v \in F^j_\mu$. We will prove it by induction over $i + j$. If $i + j \leq 1$ it is trivial. The first non-trivial case is $i = j = 1$. Let $u \in F^1_\lambda$ and $v \in F^1_\mu$. Then $Tu = \lambda u$, and $Tv = \mu v$. Since $T$ is an isometry,

$$Q(u, v) = Q(Tu, Tv) = \lambda \mu Q(u, v) = 0.$$  

Assume that $i + j \geq 2$. Let $u_1 \in F^i_\lambda$ and $u_2 \in F^j_\mu$, and let $w_1 = (T - \lambda)u_1$, $w_2 = (T - \mu)u_2$. Since $w_1 \in F^{i-1}_\lambda$ and $w_2 \in F^{j-1}_\mu$, by the induction hypothesis,

$$Q(w_1, w_2) = Q(w_1, u_2) = Q(u_1, w_2) = 0.$$  

Therefore

$$Q(u_1, u_2) = Q(Tu_1, Tu_2) = Q(w_1 + \lambda u_1, w_2 + \mu u_2) = \lambda \mu Q(u_1, u_2).$$

Since $\lambda \mu \neq 1$ we have $Q(u_1, u_2) = 0$. \qed

Definition 2.9. Let $(V, Q)$ be a Lorentz space and $T \in O(Q)$. The space-time decomposition of $V$ with respect to $T$ is the decomposition $V = V_s \oplus V_t$, where $V_s$ is the direct sum of all space-like primary components of $V$ with respect to $T$ and $V_t$ is the direct sum of the remaining primary components. We will call $T_s = T|_{V_s}$ the spatial component of $T$, and $T_t = T|_{V_t}$, the temporal component of $T$.

The following properties are a consequence of Lemmas 2.7 and 2.8

Proposition 2.10. Let $T \in O(Q)$ and let $V = V_t \oplus V_s$ be the space-time decomposition of $V$ with respect to $T$.

i) $V_t$ and $V_s$ are $T$-invariant coprime orthogonal subspaces.

ii) $V_s$ is an Euclidean space, and $V_t$ is a Lorentz space.

iii) $T_s$ is an orthogonal isometry, while $T_t$ is a Lorentz isometry. The isometry $T$ is proper if and only if $T_t$ is proper.

We next study the eigenvalues of a proper Lorentz isometry. Since $T_s$ is orthogonal, we will only deal with the temporal component $T_t$.

Lemma 2.11. Let $(V, Q)$ be a Lorentz space and let $T \in PO(Q)$. Let $\lambda \in \mathbb{C}$ be an eigenvalue of $(T_t)^c$. Then $\lambda$ is real positive, and $\lambda^{-1}$ is also an eigenvalue of $(T_t)^c$.

Proof. We can assume that $V = V_t$ and $T = T_t$. Let $\lambda \in \mathbb{C}$ be an eigenvalue of $T^c$. Assume that $\lambda$ is not real, and denote by $V^c_\lambda$ the primary component of $V^c$ of eigenvalue $\lambda$. Then $\bar{V}^c_\lambda = V^c_{\bar{\lambda}}$ is the primary component corresponding to $\bar{\lambda}$. Since $\lambda^2, \bar{\lambda}^2 \neq 1$, by Lemma 2.8 we have $Q|_{V^c_\lambda} = Q|_{V^c_{\bar{\lambda}}} = 0$.

Let $V_{\lambda \bar{\lambda}} = R(V^c_{\lambda})$ denote the primary component of $V$ corresponding to $p(x) = x^2 - 2R(\lambda)x + |\lambda|^2$. If $v_1 \in V_{\lambda \bar{\lambda}}$ then there exists $w \in V^c_\lambda$ such that $v_1 = R(w)$. Let $v_2 = I(w)$. Then $v_2 \in V_{\lambda \bar{\lambda}}$, and

$$Q(v_1) = \frac{1}{4}(Q^c(w) + Q^c(\bar{w}) + 2Q^c(w, \bar{w})) = \frac{1}{2}Q(w, \bar{w})$$

$$Q(v_2) = \frac{1}{4}(Q^c(w) + Q^c(\bar{w}) - 2Q^c(w, \bar{w})) = \frac{1}{2}Q(w, \bar{w})$$
\[ Q(v_1, v_2) = Q(R(w), I(w)) = \frac{1}{4t}(Q^c(w) - Q^c(\bar{w})) = 0. \]

Since \( v_1, v_2 \) are linearly independent, by Lemma 2.6, \( Q(v_1), Q(v_2) > 0 \). Therefore \( V_{\lambda, \lambda} \) is space-like. Since \( V = V_t \) is the direct sum of all non space-like primary components, we get a contradiction. Hence \( \lambda \) must be real, and since \( T \) is a proper isometry, \( \lambda \) must be positive.

Finally, assume that \( \lambda \) is a real positive eigenvalue of \( T \), and let \( A \) be the matrix of \( T \) in an orthonormal basis. Since \( A^t J A = J \), we have \( J^{-1} A^t J = A^{-1} \). Therefore \( \lambda^{-1} \) is an eigenvalue of \( T \). \( \square \)

**Corollary 2.12.** Let \( T \in PO(1, n) \). Then one of the two following conditions is satisfied:

i) \( V_t = V_1 \).

ii) \( V_t = V_\lambda \oplus V_{\lambda^{-1}} \), where \( \lambda \neq 1 \) is real positive.

**Proof.** By Lemma 2.11 we have \( V_t \subseteq V_1 \oplus V_\lambda \oplus V_{\lambda^{-1}} \), where \( \lambda \) is a real positive eigenvalue. By Lemma 2.8 the subspaces \( V_1 \) and \( V_\lambda \oplus V_{\lambda^{-1}} \) are orthogonal. Therefore by Lemma 2.6 at least one of these subspaces must be space-like. Since \( V_t \) contains no space-like primary components, we have \( V_t = V_1 \), or \( V_t = V_\lambda \oplus V_{\lambda^{-1}} \). \( \square \)

We next study the minimal \( T \)-invariant time-like subspaces of \( V_t \). These subspaces consist of a variation of the orthogonal indecomposable subspaces of \( [GK] \). The following result corresponds to Lemma 2.1 of \( [GK] \).

**Theorem 2.13.** Let \( (V, Q) \) be a Lorentz space and let \( T \in PO(Q) \) be a proper isometry. Let \( W \subseteq V_t \) be a minimal \( T \)-invariant time-like subspace of \( V_t \). Then one of the following conditions is satisfied.

i) \( V_t = V_1 \), \( \dim W = 1 \) and \( W \) is generated by a time-like vector of eigenvalue 1.

ii) \( V_t = V_1 \), \( \dim W \geq 2 \) and \( \text{Ker}(T|_W - I) \) is generated by a light-like eigenvector.

iii) \( V_t = V_\lambda \oplus V_{\lambda^{-1}} \), where \( \lambda \neq 1 \) is real positive, \( \dim W = 2 \) and \( W \) is generated by two light-like eigenvectors corresponding to the eigenvalues \( \lambda \) and \( \lambda^{-1} \).

**Proof.** We consider the two cases given in Corollary 2.12.

Firstly we assume that \( V_t = V_1 \). Let \( U = \text{Ker}(T|_W - I) \). We first show that \( U \) has no space-like vectors. Let \( u \in U \). If \( Q(u) > 0 \) then \( \text{Sp}\{u\} \cap W \) would be a time-like \( T \)-invariant proper subspace of \( W \). Therefore it would contradict the minimality of \( W \). Consequently we have \( Q(u) \leq 0 \).

Assume there exists \( u \in U \) such that \( Q(u) < 0 \). Then \( \text{Sp}\{u\} \) is time-like and \( T \)-invariant, and by the minimality of \( W \), \( \text{Sp}\{u\} = W \). Therefore condition (i) is satisfied.

Assume that every vector in \( U = \text{Ker}(T|_W - I) \) is light-like. Then \( Q|_U = 0 \) and by Lemma 2.6 \( \dim U = 1 \). Since \( U \) is light-like, \( U \subseteq W \), and \( \dim W \geq 2 \). Therefore (ii) is satisfied.

Finally, we assume that \( V_t = V_\lambda \oplus V_{\lambda^{-1}} \), where \( \lambda \neq 1 \) is a real positive eigenvalue. Let \( U_\lambda = \text{Ker}(T - \lambda) \). Then \( \dim U_\lambda = \dim U_{\lambda^{-1}} = 1 \). Indeed, by Lemma 2.8 \( Q|_{U_\lambda} = Q|_{U_{\lambda^{-1}}} = 0 \), and by Lemma 2.6 their dimension is \( \leq 1 \). Since both \( U_\lambda \) and \( U_{\lambda^{-1}} \) are not zero, the subspace \( U_\lambda \oplus U_{\lambda^{-1}} \) has dimension 2, and
Proof. Let \( W \) be the space-time decomposition of \( \mathbb{V} \) with respect to \( T \). Since \( \mathcal{Q}(V_t, V_s) = 0 \) we can find an orthogonal basis of \( V_s \) and \( V_t \) separately. Since \( T \) is orthogonal, by the Spectral Theorem it has an orthogonal normal form. Therefore we can assume that \( V = V_t \) and \( T = T_t \). We prove the theorem by induction over \( n \). For \( n = 0 \) it is trivial. Assume that \( n > 0 \). Let \( W \) be a minimal time-like \( T \)-invariant subspace of \( V \). If \( \dim W < \dim V \) then \( W \perp \) is space-like, and the theorem is true for both \( T|W \) and \( T|_{W^\perp} \). Hence we can assume that \( W = V \). We consider the cases given in Theorem 2.13.

1) \( 1 \oplus A \), where \( A \in O(n) \) is an orthogonal normal form.

2) \( \Theta \oplus A \), where \( A \in O(n - 2) \) is an orthogonal normal form and \( \Theta = \begin{pmatrix} 3/2 & 1 & -1/2 \\ 1 & 1 & -1/2 \\ 1/2 & 1/2 & 1/2 \end{pmatrix} \).

3) \( \Omega_t \oplus A \), where \( \Omega_t = (\cosh t \sinh t, \sinh t \cosh t) \), \( t \neq 0 \). and \( A \in O(n - 1) \) is an orthogonal normal form.

**Theorem 2.14.** Every element of \( PO(1, n) \) is conjugate to a Lorentzian normal form.

Proof. Let \( V = V_t \oplus V_s \) be the space-time decomposition of \( V \) with respect to \( T \). Since \( \mathcal{Q}(V_t, V_s) = 0 \) we can find an orthogonal basis of \( V_s \) and \( V_t \) separately. Since \( T \) is orthogonal, by the Spectral Theorem it has an orthogonal normal form. Therefore we can assume that \( V = V_t \) and \( T = T_t \). We prove the theorem by induction over \( n \). For \( n = 0 \) it is trivial. Assume that \( n > 0 \). Let \( W \) be a minimal time-like \( T \)-invariant subspace of \( V \). If \( \dim W < \dim V \) then \( W \perp \) is space-like, and the theorem is true for both \( T|W \) and \( T|_{W^\perp} \). Hence we can assume that \( W = V \). We consider the cases given in Theorem 2.13.

i). Assume that \( W \) is generated by a time-like eigenvector of eigenvalue 1. Then \( T \) is the identity transformation.

ii). Assume that \( \text{Ker}(T - I) \) is generated by a light-like vector, and that \( \dim W \geq 2 \). Let \( N = T - I \). Since \( T \) is an isometry we have

\[
\mathcal{Q}(N x, N y) = -\mathcal{Q}(N x, y) - \mathcal{Q}(x, N y).
\]

Let \( F_k = \text{Ker} N^k \). Note that \( F_1 = \text{Ker}(T - I) \) has dimension 1 and it is generated by a light-like vector. Therefore the Jordan normal form of \( T \) consists of a single Jordan block, and the \( T \)-invariant subspaces are of the form \( F_k \) for some \( k \). Let us show that \( W = F_3 \).

If \( F_1 = F_2 \) then \( F_1 = W \), which is impossible, since \( \dim W \geq 2 \). Therefore \( F_1 \subsetneq F_2 \) and there exists \( v \in F_2 \setminus F_1 \). Let \( u = N(v) \). Since \( \mathcal{Q}(u) = 0 \),

\[
0 = \mathcal{Q}(u) = \mathcal{Q}(N v) = -2\mathcal{Q}(v, N v) = -2\mathcal{Q}(v, u).
\]

Consequently \( \mathcal{Q}(u, v) = 0 \) and by Lemma 2.6 \( \mathcal{Q}(v) > 0 \). We can assume that \( \mathcal{Q}(v) = 1 \). Then \( W \neq F_2 \) and \( F_2 \subsetneq F_3 \), for if \( F_2 = F_3 \) we would have \( F_2 = W \), but \( F_2 \) is not time-like. Let \( w \in F_3 \setminus F_2 \). We can assume that \( v = N(w) \). Then

\[
0 = \mathcal{Q}(u, v) = \mathcal{Q}(N^2 w, N w) = -\mathcal{Q}(w, N^2 w) - \mathcal{Q}(N w, N w) = -\mathcal{Q}(u, w) - 1.
\]

Therefore \( \mathcal{Q}(u, w) = -1 \neq 0 \). By adding to \( w \) a multiple of \( u \) we can assume that \( \mathcal{Q}(w) = 0 \). Then

\[
\mathcal{Q}(v) = \mathcal{Q}(N w) = -2\mathcal{Q}(w, N w) = -2\mathcal{Q}(w, v),
\]

and hence, \( \mathcal{Q}(v, w) = -\frac{1}{2} \). The subspace \( F_3 = Sp\{u, v, w\} \) is time-like and \( T \)-invariant. By the minimality of \( W \), \( F_3 = W \).
By construction, the matrix of $T$ in the basis $\{u, v, w\}$ is a Jordan block of size 3 and eigenvalue 1, and the matrix of $Q$ is the matrix of index $(1, 2)$ given by

$$G = \begin{pmatrix}
0 & 0 & -1 \\
0 & 1 & -\frac{1}{2} \\
-1 & \frac{1}{2} & 0
\end{pmatrix},$$

Let

$$P = \begin{pmatrix}
\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\
\frac{\sqrt{2}}{2} & 1 & -\frac{\sqrt{2}}{2} \\
-\frac{\sqrt{2}}{2} & 1 & \frac{\sqrt{2}}{2}
\end{pmatrix},$$

and let $J = -I_1 \oplus I_2$. Since $P^t GP = J$, the basis $\{e_0, e_1, e_2\}$ of $W$ defined by $(e_0 \ e_1 \ e_2) = (u \ v \ w)P$, is a Lorentz orthonormal basis of $W$. The matrix of $T$ in this basis is

$$P^{-1} \begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix} P = \begin{pmatrix}
3 & 2 & 1 \\
1 & 1 & -1 \\
-1 & 1 & 1
\end{pmatrix} = \Theta.$$

iii). Assume $W = Sp\{u, v\}$, where $u, v$ are light-like eigenvectors of eigenvalues $\lambda$ and $\lambda^{-1}$ respectively, where $\lambda \neq 1$ is real positive. By Lemma 2.6, $Q(u, v) \neq 0$, and we can choose $u, v$ such that $Q(u, v) = -1$. Then the basis of $W$ given by $\{\frac{1}{\sqrt{2}}(u+v), \frac{1}{\sqrt{2}}(u-v)\}$ is Lorentz-orthonormal and the matrix of $T$ in this basis is

$$\Omega_t = \begin{pmatrix}
\cosh t & \sinh t \\
\sinh t & \cosh t
\end{pmatrix},$$

where $t = \log \lambda$. □

Note that in $PO(1, 2)$ all parabolic isometries belong to the same conjugacy class, namely the class of the normal form $\Theta$. This class of isometries corresponds to translations $z \mapsto z + a$ in the half-plane model of the hyperbolic plane, where $a \in \mathbb{R}$. Two such translations are conjugate in $PSL_2(\mathbb{R})$ by a dilation $z \mapsto a/bz$.

As an easy consequence of Theorem 2.14 we get the normal forms of both proper and non-proper Lorentz isometries.

**Corollary 2.15.** Every matrix in $O(1, n)$ is conjugate to a matrix $\pm A$, where $A$ is a proper Lorentzian normal form.

**Proof.** It suffices to study the case $T \notin PO(1, n)$. Then $-T \in PO(1, n)$ and we apply Theorem 2.14. □

We next define the Segre symbol of an isometry of $\mathbb{H}^n$. Let $f \in I(\mathbb{H}^n)$, and let $T_t \oplus T_s$ be the space-time decomposition of a normal form $T$ of $f$. Let $\tilde{\sigma}_s$ be the Segre symbol of $T_s$ and let $r = \dim V_t$. By Theorem 2.14, one of the following conditions is satisfied:

i) $T_t = I_r$, where $1 \leq r \leq n + 1$.

ii) $T_t = \Theta \oplus I_{r-3}$, where $3 \leq r \leq n + 1$.

iii) $T_t = \Omega_t$, and $r = 2$.

Cases i), ii) and iii) correspond to elliptic, parabolic and hyperbolic isometries respectively.

**Definition 2.16.** Let $f \in I(\mathbb{H}^n)$. With the previous notation, the **Segre symbol of $f$** is $\sigma_f = \{t; r; \tilde{\sigma}_s\}$, where $t \in \{e, p, h\}$ denotes the type of isometry (elliptic, parabolic or hyperbolic).

We have the following result.
Proposition 2.17. There is a finite number of Segre classes in $I(\mathbb{H}^n)$, given by $h(n) = h^e(n) + h^p(n) + h^h(n)$, where $h^e(n)$, $h^p(n)$ and $h^h(n)$ denote the number of Segre classes of elliptic, parabolic and hyperbolic isometries respectively, and
\[
h^e(n) = \sum_{j=0}^{[n/2]} p(j)(n - 2j + 1); \quad h^p(n) = h^e(n - 2); \quad h^h(n) = s(n - 2),
\]
where $p(k)$ denotes the number of partitions of $k$ and $s(k)$ is the number of Segre classes of $I(S^k)$.

3. Orbit types

Let $M^n$ be one of the Riemannian manifolds $S^n$, $\mathbb{E}^n$ or $\mathbb{H}^n$. In this section we study the orbit types of $I(M^n)$ acting on itself by conjugation. The main result is that the orbit types are parameterized by the Segre symbol.

Let $G$ be a Lie group acting on itself by conjugation. We recall that two elements $x, y \in G$ have the same $G$-orbit type if and only if their orbits are $G$-equivariantly isomorphic. It is easy to check that this is equivalent to the condition that $x$ and $y$ have conjugate isotropy groups. Since $G$ acts on itself by conjugation, for all $x \in G$, the isotropy group of $x$ is given by
\[
G_x = \{y \in G; xy = yx\}.
\]
Therefore when $G$ is a matrix group, the study of $G_x$ involves a problem of Frobenius: to study all elements in $G$ that commute with a given element $x \in G$ (see [Ga]).

We recall a well known result, which we will use later on. Let $k$ be an arbitrary field.

Proposition 3.1. Let $V$ be a finite dimensional vector space over $k$ and let $A \in \text{End}(V)$ be a linear map. Let $V = V_1 \oplus \cdots \oplus V_i$ be a primary decomposition of $V$ with respect to $A$. If $B \in \text{End}(V)$ is a linear map which commutes with $A$, then $V_i$ are $B$-invariant subspaces.

Proof. We denote by $\mu_i$ the minimal polynomial of $V_i$ with respect to $A$. Then $x \in V$ is such that $\mu_i(A)x = 0$ if and only if $x \in V_i$. Since $AB = BA$ we get $\mu_i(A)B = B\mu_i(A)$. Let $x \in V_i$, then $\mu_i(A)Bx = B\mu_i(A)x = 0$. Therefore $Bx \in V_i$, and $V_i$ is $B$-invariant. \qed

Let $A \in GL(n)$ be an invertible square matrix over $k$. We say that $A$ is eigenordered if $A$ can be written as the block diagonal matrix $A = A_1 \oplus \cdots \oplus A_t$, such that the minimal polynomials of $A_i$ are coprime by pairs. We call this an eigen decomposition of $A$. The following Corollary follows directly from Proposition 3.1.

Corollary 3.2. Let $A$ be an eigenordered matrix and let $A = \bigoplus_{i=1}^t A_i \in GL(n)$ be an eigendecomposition of $A$. Let $B$ be a square $(n \times n)$ matrix such that $[A, B] = 0$. Then $B = \bigoplus_{i=1}^t B_i$, and $[A_i, B_i] = 0$ for all $i = 1, \cdots , t$.

3.1. Orbit types of $I(S^n)$. In order to study the orbit types of isometries of the sphere, we first study the isotropy groups of normal forms of $O(n)$. We remark that since $O(n)$ is compact, its orbit type decomposition defines a Whitney-regular stratification [P2], and furthermore, it is the maximal stratification $O(n)$ admits, in the sense that if one defines the strata by mixing orbits from different
orbit types, the result will not be a Whitney stratification.

We denote by $U(n)$ the subgroup of $GL(n, \mathbb{C})$ of complex unitary square matrices of size $n$, which is a Lie group of dimension $n^2$, and let $j : GL(n, \mathbb{C}) \to GL(n, 2\mathbb{R})$ denote the canonical inclusion. Let $M = (m_{kl}) \in GL(n; \mathbb{C})$, where $m_{kl} = a_{kl} + ib_{kl}$. Then $j(M)$ is the block matrix $j(M) = (A_{kl})$, where $A_{kl} = \begin{pmatrix} a_{kl} - ib_{kl} \\ b_{ij} & a_{kl} \end{pmatrix}$.

**Lemma 3.3.** Let $A \in O(n)$ be a normal form with Segre symbol

$\tilde{\sigma}_A = [(n_1 \tilde{n}_1), \ldots, (n_s \tilde{n}_s), m_1, m_2]$.

Then $O(n)_A = \prod_{i=1}^s j(U(n_i)) \times \prod_{i=1}^t O(m_i)$, and

$$\dim O(n)_A = \sum_{i=1}^s (n_i)^2 + \frac{1}{2} \sum_{i=1}^t m_i(m_i - 1).$$

**Proof.** Let $A = \bigoplus_{i=1}^k A_i$ be an eigendecomposition of $A$, where $A_i \in O(k_i)$. By Corollary 3.2, the matrices commuting with $A$ are block diagonal matrices $B = \bigoplus_{i=1}^s B_i$ such that $[A_i, B_i] = 0$ for all $i$. Therefore $O(n)_A = \prod_{i=1}^k O(k_i)_A$. In particular $\dim O(n)_A = \sum_{i=1}^k \dim O(k_i)_A$. Hence it suffices to consider the case when $A$ has a single primary component. In such case, $A$ is either $\pm \mathbb{I}_n$ or $R_{\mathbb{I}}^{n/2}$.

If $A = \pm \mathbb{I}_n$ it is trivial. Assume that $A = R_{\mathbb{I}}^{n/2}$. By computing the matrices that commute with $A$ we find that

$$O(2n)_A = \left\{ B = (B_{ij}) \in O(2n); B_{ij} = \begin{pmatrix} a_{ij} & b_{ij} \\ -b_{ij} & a_{ij} \end{pmatrix} \right\}.$$ 

Therefore $O(2n)_A = j(U(n))$. In particular, $\dim O(2n)_A = \dim U(n) = n^2$. \hfill $\square$

**Lemma 3.4.** Let $A \in O(n)$ be a normal form having a single primary component, and let $B \in O(n)$ such that $O(n)_A = O(n)_B$. Then $B$ is a normal form having a single primary component, and $\tilde{\sigma}_A = \tilde{\sigma}_B$.

**Proof.** Assume that $A = \pm \mathbb{I}_n$. Then $O(n)_A = O(n)$. Since the center of $O(n)$ is $\pm \mathbb{I}_n$ it follows that $B = \pm \mathbb{I}_n$. In particular $\tilde{\sigma}_A = \tilde{\sigma}_B = [n]$.

Assume that $A = R_{\mathbb{I}}^{n/2} \in O(2n)$. By Lemma 3.3 $O(2n)_A = j(U(n))$. Since $B \in O(2n)_A$, there exists $C \in U(n)$ such that $B = j(C)$. Moreover, since $O(n)_B = j(U(n))$ we have $U(n)_C = U(n)$. Since the center of $U(n)$ are homotheties, $C = e^{i\alpha} \mathbb{I}_n$ for some $\alpha \in \mathbb{R}$. The condition $B \neq \pm \mathbb{I}_{2n}$ implies $\alpha \in \mathbb{R} \setminus \{k\pi\}$, and $B = R_{\mathbb{I}}^{n/2}$. In particular, $\tilde{\sigma}_A = \tilde{\sigma}_B = [n\tilde{n}]$. \hfill $\square$

**Theorem 3.5.** Let $f, g \in R(S^n)$. Then $\tilde{\sigma}_f = \tilde{\sigma}_g$ if and only if they have the same orbit type.

**Proof.** Assume that $\tilde{\sigma}_f = \tilde{\sigma}_g$. By Lemma 3.3 there exist normal forms $A$ and $B$ of $f$ and $g$ respectively such that $O(n)_A = O(n)_B$. Since conjugate matrices have conjugate isotropy groups it follows that $O(n)_f$ is conjugate to $O(n)_g$.

Conversely, assume that $O(n)_f$ and $O(n)_g$ are conjugate. Let $A$ be a normal form of $f$ such that $O(n)_A = O(n)_B$, where $B$ is the matrix of $g$ in some basis. Let $A = \bigoplus_{i=1}^k A_i$ be an eigendecomposition
of $A$. By Proposition 3.2 $B$ admits a block diagonal decomposition $B = \bigoplus_{i=1}^{k} B_i$, such that $O(n_i)_{A_i} = O(n_i)_{B_i}$. By Lemma 3.3 $B_i$ are normal forms, and $\tilde{\sigma}_{A_i} = \tilde{\sigma}_{B_i}$ for all $i$. Therefore if

$A = R_{\theta_1}^{n_1} \oplus \cdots \oplus R_{\theta_s}^{n_s} \oplus \varepsilon I_{m_1} \oplus -\varepsilon I_{m_2}$,

then $B$ is a normal form $R_{\theta_1}^{n_1} \oplus \cdots \oplus R_{\theta_s}^{n_s} \oplus \mu I_{m_1} \oplus \nu I_{m_2}$. In order to show that $\tilde{\sigma}_A = \tilde{\sigma}_B$ we have to see that $\alpha_i \neq \alpha_j$ for all $i \neq j$, and that $\mu \neq \nu$.

This follows directly from the description of the isotropy groups of normal forms given in Lemma 3.3 since

$$j(U(n_i)) \times j(U(n_j)) \subset j(U(n_i + n_j)), \text{ and } O(m_1) \times O(m_2) \subset O(m_1 + m_2).$$

$\square$

3.2. Orbit types of $I(\mathbb{E}^n)$. To study the orbit types of $I(\mathbb{E}^n)$ we first study the isotropy group of an Euclidean normal form.

For the rest of this section we will write Euclidean normal forms as the direct sum $f = f_o \oplus f_u$, where $f_o \in O(n_o)$ is an orthogonal normal form with no eigenvalues equal to 1, and $f_u \in \text{Euc}(n_u)$ is an unipotent Euclidean normal form. Note that $f_u$ is either $I_{n_u+1}$, or $I_{n_u-1} \oplus \left( \begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix} \right)$, for some $a > 0$.

**Lemma 3.6.** Let $f = f_o \oplus f_u \in \text{Euc}(n)$ be an Euclidean normal form. Then

$$\text{Euc}(n)_f = O(n_o)_{f_o} \times \text{Euc}(n_u)_{f_u}.$$  

Furthermore,

i) If $f_u = I_{n+1}$ then $\text{Euc}(n)_{f_u} = \text{Euc}(n)$. In particular, $\dim \text{Euc}(n)_{f_u} = n(n+1)/2$.

ii) If $f_u = I_{n-1} \oplus \left( \begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix} \right)$, where $a > 0$, then

$$\text{Euc}(n)_{f_u} = \left\{ \left( \begin{smallmatrix} B & 0 \\ 0 & 0 \end{smallmatrix} \right) : B \in O(n-1), b \in \mathbb{R}^{n-1}, c \in \mathbb{R} \right\}.$$  

In particular, $\dim \text{Euc}(n)_{f_u} = n(n-1)/2 + 1$.

**Proof.** By Corollary 3.2 since $f_o$ and $f_u$ are coprime, we have

$$\text{Euc}(n)_f = O(n_o)_{f_o} \times \text{Euc}(n_u)_{f_u}.$$  

If $f_u \in \text{Euc}(n)$ is the identity matrix, it is trivial that $\text{Euc}(n)_{f_u} = \text{Euc}(n)$. Assume that $f_u = I_{n-1} \oplus \left( \begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix} \right)$. The $(n+1)$-square matrices that commute with $f_u$ are of the form

$$A = \left( \begin{smallmatrix} B & 0 \\ 0 & 0 \end{smallmatrix} \right),$$

where $a, b \in \mathbb{R}^{n-1}$, $c, e \in \mathbb{R}$ and $B$ is a square matrix of size $n-1$. The condition $A \in \text{Euc}(n)$ implies $c = 1, B \in O(n-1)$ and $a = 0$. In particular

$$\dim \text{Euc}(n)_{f_u} = \dim O(n-1) + n = n(n-1)/2 + 1.$$  

$\square$

**Lemma 3.7.** Let $f = f_u \in \text{Euc}(n)$ be a unipotent normal form and let $g \in \text{Euc}(n)$ such that $\text{Euc}(n)_f = \text{Euc}(n)_g$. Then $g$ is unipotent and $\sigma_f = \sigma_g$. 

Proof. Assume that \( f = I_{n+1} \). Since the center of \( \text{Euc}(n) \) is the identity it follows that \( g = I_{n+1} = f \).

Assume that \( f = I_{n-1} \oplus (\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}) \). If \( n = 1 \) then \( g = (\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}) \), for some \( b \in \mathbb{R} \). Since \( g \) can not be the identity, \( b \neq 0 \) and \( \sigma_f = \sigma_g \).

Assume that \( n > 1 \), and let \( h = h_1 \oplus (\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}) \in \text{Euc}(n) \), where \( h_1 \in O(n-1) \) and \( b \in \mathbb{R} \) are arbitrary. By Lemma 3.6, \( h \in \text{Euc}(n) \), and by hypothesis \( h \in \text{Euc}(n) \). In particular, \([h, g] = 0\). Therefore \( g \) is a block diagonal matrix \( g = g_1 \oplus g_2 \), where \( g_1 \in O(n-1) \) and \( g_2 \in \text{Euc}(1) \) are such that \( O(n)g_1 = O(n) \) and \([g_2, (\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix})] = 0\). Therefore \( g_1 = I_{n-1} \) and by the case \( n = 1 \), \( g_2 = (\begin{smallmatrix} 1 & c \\ 0 & 1 \end{smallmatrix}) \), for some \( c \in \mathbb{R} \). Since \( g \) must not be the identity we have \( c \neq 0 \), hence \( \sigma_f = \sigma_g \).

**Theorem 3.8.** Let \( f, g \in I(\mathbb{E}^n) \). Then \( \sigma_f = \sigma_g \) if and only if they have the same orbit type.

Proof. Assume that \( \sigma_f = \sigma_g \). By Lemma 3.6, there exist normal forms \( A \) and \( B \) of \( f \) and \( g \) respectively, such that \( \text{Euc}(n)_A = \text{Euc}(n)_B \). Therefore since conjugate matrices have conjugate isotropy groups, \( \text{Euc}(n)_f \) is conjugate to \( \text{Euc}(n)_g \).

Conversely, assume that \( \text{Euc}(n)_f \) is conjugate to \( \text{Euc}(n)_g \). There exists a normal form \( A \in \text{Euc}(n) \) of \( f \) and a matrix \( B \) of \( g \) such that \( \text{Euc}(n)_A = \text{Euc}(n)_B \). Let \( A = A_o \oplus A_u \) be the decomposition of \( A \) into its orthogonal and unipotent components. Then \( B \) is a block diagonal matrix \( B = B_1 \oplus B_2 \), where \( B_1 \in \text{O}(n_o) \) and \( B_2 \in \text{Euc}(n_u) \) are such that \( \text{O}(n_o)B_1 = \text{O}(n_o)A_o \) and \( \text{Euc}(n_u)B_2 = \text{Euc}(n_u)f_u \). By Theorem 3.5, \( \tilde{\sigma}_{A_o} = \tilde{\sigma}_{B_1} \), and by Lemma 3.7, \( \sigma_{A_o} = \sigma_{B_1} \). In particular, \( B_2 \) is unipotent. To prove that \( \sigma_A = \sigma_B \) it remains to show that \( 1 \) is not an eigenvalue of \( B_1 \).

Let \( A_o = R \oplus -I_m \), where \( R \in \text{O}(n_o - m) \) is a rotation matrix of the form \( R = R_{\theta_1}^{m_1} \oplus \cdots \oplus R_{\theta_s}^{m_s} \). Since \( \tilde{\sigma}_{A_o} = \tilde{\sigma}_{B_1} \), \( B_1 \) is of the form \( R' \oplus \varepsilon I_m \), where \( \tilde{\sigma}_R = \tilde{\sigma}_{R'} \). Assume that \( \varepsilon = 1 \). Then

\[
\begin{align*}
d_B &= \dim \text{Euc}(n)_B = \dim \text{O}(n_o - m)_R + \dim \text{Euc}(n_u + m)_{I_m \oplus A_u}, \\
d_A &= \dim \text{Euc}(n)_A = \dim \text{O}(n_o - m)_R + \dim \text{O}(m) + \dim \text{Euc}(n_u)_{A_u},
\end{align*}
\]

and by Lemma 3.6, we would have \( d_B - d_A > 0 \), which is impossible. Therefore \( \varepsilon = -1 \) and \( \sigma_B = \sigma_A \). Since \( A \) and \( B \) are matrices of \( f \) and \( g \) respectively, we have \( \sigma_f = \sigma_g \). \( \square \)

### 3.3. Orbit types of \( I(\mathbb{H}^n) \).

Let us study the isotropy group of a Lorentzian normal form.

**Lemma 3.9.** Let \( T \in \text{PO}(1, n) \) be a Lorentzian normal form. Then

\[
\text{PO}(1, n)_T = \text{PO}(1, n_t)_T \times O(n_s)_T.
\]

Furthermore,

- \( i \) If \( T_t = \mathbb{I}_{n+1} \) then \( \text{PO}(1, n)_{T_t} = \text{PO}(1, n) \). In particular, \( \dim \text{PO}(1, n)_{T_t} = n(n + 1)/2 \).
- \( ii \) If \( T_t = \Theta \oplus \mathbb{I}_{n-2} \), then

\[
\text{PO}(1, n)_T = \left\{ \begin{pmatrix} 1+d & e \\ e & 1-d \end{pmatrix} : D \in O(n-2); a \in \mathbb{R}^{n-2}; c \in \mathbb{R}, \right. \\
\left. \begin{array}{c}
\frac{1}{2}(c^2 + ||b||^2); a = bD.
\end{array} \right.
\]

In particular, \( \dim \text{PO}(1, n)_{T_t} = 1 + (n - 2)(n - 1)/2 \).

- \( iii \) If \( T_t = \Omega \) then \( \text{PO}(1, 1)_{T_t} \) is the connected component of the identity in \( \text{PO}(1, 1) \). In particular, \( \dim \text{PO}(1, 1)_{T_t} = 1 \).
Proof. Let \(T = T_t \oplus T_s\) be the space-time decomposition of \(T\). By Corollary \([3.2]\)
\[
\text{PO}(1, n)_T = \{ A \oplus B; A \in \text{PO}(1, n_t)_{T_t}, B \in O(n_s)_{T_s} \}.
\]
Therefore \(\text{PO}(1, n)_T = \text{PO}(1, n_t)_{T_t} \times O(n_s)_{T_s}\).

Assume that \(T = T_t\). The only non trivial case is when \(T = \Theta \oplus \mathbb{I}_{n-2}\). Let \(M \in \text{PO}(1, n)_T\). We write \(M\) as a block matrix \(M = \begin{pmatrix} C & A \\ B & D \end{pmatrix}\). The condition \([M, T] = 0\) implies
\[
[\Theta, C] = 0, (\Theta - I_3)A = 0, B(\Theta - I_3) = 0.
\]
An easy computation shows that \(A\) and \(B\) must be of the form
\[
A = \begin{pmatrix} a_1 & \cdots & a_{n-2} \\ 0 & \cdots & 0 \\ a_1 & \cdots & a_{n-2} \end{pmatrix}, \quad B = \begin{pmatrix} b_1 & 0 & -b_1 \\ \vdots & \vdots & \vdots \\ b_{n-2} & 0 & -b_{n-2} \end{pmatrix}.
\]
On the other hand, recall from Section \([2.3]\) that \(\Theta\) is conjugate to a unipotent Jordan block in \(GL(3)\).
Since the matrices commuting with a Jordan block are regular upper triangular matrices (see \([Ga]\)),
we find that \(C\) is a matrix of the form
\[
C = \begin{pmatrix} e+d & c & -d \\ c & e & -c \\ d & c & e-d \end{pmatrix}.
\]
The condition \(M^tQM = Q\) implies:
1) \(C^tQC + B^tB = Q\)
2) \(A^tQA + D^tD = I_{n-2}\)
3) \(C^tQA + B^tD = 0\)
From the first condition we deduce that \(e = \pm 1\), and that \(d = \frac{1}{2n}(c^2 + \sum_{i=1}^{n-2} b_i^2)\). Furthermore, for \(M\) to be proper, the upper left entry of \(C\) must be positive. Hence \(e = 1\). From the second condition, since \(A^tQA = 0\) we obtain \(D^tD = I_{n-2}\), therefore \(D \in O(n-2)\). The third condition implies that
\[
a_j = \sum_{i=1}^{n-2} b_i d_{ij}, \text{ where } d_{ij} \text{ are the coefficients of the matrix } D.
\]
Therefore the dimension of \(\text{PO}(1, n)_T\) is
\[
\dim \text{PO}(1, n)_T = \dim O(n-2) + n - 1 = 1 + \frac{1}{2}(n-1)(n-2).
\]
\[\square\]

Lemma 3.10. Let \(T \in \text{PO}(1, n)\) be a normal form with no spatial component, and let \(A \in \text{PO}(1, n)\)
such that \(\text{PO}(1, n)_T = \text{PO}(1, n)_A\). Then \(\sigma_T = \sigma_A\).

Proof. The elliptic and hyperbolic cases are trivial. Assume that \(T\) is parabolic. Then \(T = \Theta \oplus \mathbb{I}_{n-2}\).
Let \(B = \mathbb{I}_3 \oplus C\), where \(C \in O(n-2)\). Then \(B \in \text{PO}(1, n)_T\) and by hypothesis, \([B, A] = 0\). Since \(C\) is arbitrary we have, by Corollary \([3.2]\) that \(A\) is a block diagonal matrix \(A = A_1 \oplus A_2\), where \([A_1, \Theta] = 0\) and \([A_2, C] = 0\) for all \(C \in O(n-2)\). Therefore \(A_2 = \varepsilon I_{n-2}\), and
\[
A_1 = \begin{pmatrix} a & d & b \\ b & a & -b \\ d & b & a-d \end{pmatrix}.
\]
The condition \(A \in \text{PO}(1, n)\) implies \(a = 1\) and \(d = b^2/2\). Therefore
\[
A_1 = \begin{pmatrix} 1+b^2/2 & b & -b^2/2 \\ b & 1 & -b \\ b^2/2 & 1 & b^2/2 \end{pmatrix}.
\]
If \(b = 0\) then \(A_1\) is the identity matrix and by Lemma \([3.9]\) \(\text{PO}(1, 2)_A \neq \text{PO}(1, n)_T\). Therefore \(b \neq 0\). In such case, \(A_1\) is conjugate to \(\Theta_1\), and \(A\) has a normal form \(\Theta \oplus \varepsilon I_{n-2}\). To see that \(\sigma_A = \sigma_T\) it remains to show that for all \(n > 2\) we have \(\varepsilon = 1\).
Assume that $n > 2$ and $\varepsilon = -1$. Then by Lemma 3.9, $PO(1, n) \cong PO(1, 2) \times O(n - 2)$. In particular we have
\[
\begin{align*}
d_A &= \dim PO(1, n) = 1 + \frac{1}{2}(n - 2)(n - 3), \\
d_T &= \dim PO(1, n) = 1 + \frac{1}{2}(n - 1)(n - 2).
\end{align*}
\]
Therefore $d_T - d_A = n - 2 > 0$, which is impossible, since $PO(1, n) = PO(1, n)$. Hence $\varepsilon = 1$ and $\sigma_A = \sigma_T$. Since $T$ and $A$ are matrices of $f$ and $g$ respectively, we have $\sigma_f = \sigma_g$.

**Theorem 3.11.** Let $f, g \in \mathcal{I}(\mathbb{H}^n)$. Then $\sigma_f = \sigma_g$ if and only if they have the same orbit type.

**Proof.** Assume that $\sigma_f = \sigma_g$. By Lemma 3.9, there exist normal forms $A$ and $B$ of $f$ and $g$ respectively such that $PO(1, n)_A = PO(1, n)_B$. Therefore the isotropy groups of $f$ and $g$ are conjugate to each other.

Conversely, assume that $PO(1, n)_f$ is conjugate to $PO(1, n)_g$. Then there exists a normal form $T$ of $f$ such that $PO(1, n)_T = PO(1, n)_B$, where $B$ is the matrix of $g$ in some Lorentz basis. Let $T = T_1 \oplus T_s$ be the space-time decomposition of $T$. Then $B$ is a block diagonal matrix $B = B_1 \oplus B_s$, where $PO(1, n_1)T_1 = PO(1, n_1)B_1$ and $O(n_s)T_s = O(n_s)B_2$. By Theorem 3.5, $\tilde{\sigma}_{T_1} = \tilde{\sigma}_{B_1}$. To prove that $\sigma_B = \sigma_T$ it remains to show that if $T$ is elliptic or parabolic, then 1 is not an eigenvalue of $B_2$.

Let $T_s = R \oplus -\mathbb{I}_k$, where $R$ is a composition of rotations. Since $\tilde{\sigma}_{T_s} = \tilde{\sigma}_{B_2}$, $B_2$ has a normal form $R' \oplus \varepsilon \mathbb{I}_k$, where $\varepsilon = 1$. Then
\[
\begin{align*}
d_B &= \dim PO(1, n)_B = \dim PO(1, n_t + k)T_1 \oplus \mathbb{I}_k + \dim O(n_s - k)_R, \\
d_T &= \dim PO(1, n)_T = \dim PO(1, n_t)T_1 + \dim O(n_s - k)_R + \dim O(k).
\end{align*}
\]
By Lemma 3.9, we find that in both elliptic and parabolic cases, $d_B - d_T > 0$, which is impossible. Therefore $\varepsilon = -1$, and 1 is not an eigenvalue of $B_2$. Hence $\sigma_T = \sigma_B$ and consequently $\sigma_f = \sigma_g$. 

4. Invariant subspaces

Let $M^n$ denote one of the Riemannian manifolds $\mathbb{S}^n$, $\mathbb{E}^n$ or $\mathbb{H}^n$. Recall that the **generalized Grassmannian** $\mathcal{G}(k, M^n)$ is the set of closed totally geodesic submanifolds of $M^n$ isometric to $M^k$, which is an homogeneous space of dimension $(k + 1)(n - k)$. (see [OH]).

Let $f \in \mathcal{I}(M^n)$. We denote by $\Gamma_f$ the set of $f$-invariant closed totally geodesic submanifolds of $M^n$ isometric to $M^k$ for some $k \leq n$, and by $\Gamma_f(k)$, the subset of $\Gamma_f$ consisting of those submanifolds of dimension $k$. Then $\Gamma_f(k)$ is a closed subset of $\mathcal{G}(k, M^n)$, and $\Gamma_f \subseteq \bigsqcup_{k=0}^n \mathcal{G}(k, M^n)$.

In this section, we describe the sets $\Gamma_f(k)$ and relate them to the Segre symbol of $f$. We show that $\Gamma_f(k)$ are closed submanifolds of $\mathcal{G}(k, M^n)$ and that the dimensions of their connected components determine the Segre symbol, and thus the orbit type, of the isometry.

We begin by giving some basic properties of invariant subspaces for a linear endomorphism of a finite dimensional vector space, which will be needed in the later parts of the section. These have been studied by Shayman in [Sh].
Let $V$ be an $n$-dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$. For all $k \leq n$ we denote by $G(k, V)$ the Grassmann manifold of all $k$-dimensional linear subspaces of $V$.

Let $A \in \text{End}(V)$. A subspace $U \subseteq V$ is $A$-invariant if $A(U) \subseteq U$. We denote by $S_A$ the set of all $A$-invariant subspaces of $V$, and by $S_A(k)$ the subset of $S_A$ consisting of those invariant subspaces of dimension $k$, which is a closed algebraic subvariety of $G(k, V)$.

Lemma 4.1. If $A, B \in \text{End}(V)$ are conjugate, there is an isomorphism of algebraic varieties $S_A(k) \cong S_B(k)$, for all $k$.

Proof. Let $B = \alpha A \alpha^{-1}$ for some $\alpha \in GL(n)$, and let $U \in S_A(k)$. Then $\alpha U \in S_B(k)$. The map $U \mapsto \alpha U$ defines an isomorphism $S_A(k) \to S_B(k)$. \hfill $\square$

Lemma 4.2. Let $A \in \text{End}(V)$ and let $V = V_1 \oplus \cdots \oplus V_s$ be a primary decomposition of $V$ with respect to $A$. Then every $A$-invariant subspace $U \subset V$ admits a primary decomposition $U = U_1 \oplus \cdots \oplus U_s$ with respect to $A|_U$, such that $U_i = U \cap V_i$.

Proof. Let $V_i = \text{Ker}(A - \varepsilon_i I)^{n_i}$. Then $U_i = \text{Ker}(A|_U - \varepsilon_i I)^{m_i}$, where $m_i \leq n_i$, so $U_i \subset V_i \cap U$. Conversely, $V_i \cap U = \text{Ker}(A|_U - \varepsilon_i I)^{n_i} = \text{Ker}(A|_U - \varepsilon_i I)^{m_i} = U_i$. \hfill $\square$

The following proposition is a consequence of the previous lemma (see [Sh], Sec.3, Teo.2).

Proposition 4.3. [Sh] Let $A \in \text{End}(V)$ and $V = V_1 \oplus \cdots \oplus V_s$ be a primary decomposition of $V$ with respect to $A$. For all $i = 1, \ldots, s$ we denote by $\pi_i : V \to V_i$ the natural projection, and by $A_i$ the restriction of $A$ to $V_i$. The map

$$S_A \to S_{A_1} \times \cdots \times S_{A_s}; U \mapsto (\pi_1(U), \ldots, \pi_s(U))$$

is an isomorphism of algebraic varieties. In particular,

$$S_A(k) \cong \bigsqcup_{k_1 + \cdots + k_s = k} S_{A_1}(k_1) \times \cdots \times S_{A_s}(k_s).$$

Therefore in order to study the varieties of invariant subspaces of a linear endomorphism, it suffices to do it in the case when the endomorphism has a single primary component.

We remark that the varieties $S_A(k)$ are, in general singular varieties (see [Sh]). In the particular case of orthogonal maps the situation is much more simple, since the normal forms of such transformations are diagonal over $\mathbb{C}$. In particular we will see that the varieties $S_A$ are smooth.

4.1. Invariant totally geodesic subspheres of $S^n$. The spherical Grassmannian $G(k, S^n)$ is the set of $k$-dimensional totally geodesic subspaces of $S^n$. Such subspaces are precisely the intersections of $S^n$ with the linear subspaces of $\mathbb{R}^{n+1}$. This follows directly from the description of the geodesics in $S^n$. There is an identification

$$G(k, S^n) = G(k + 1, \mathbb{R}^{n+1}).$$

Therefore $G(k, S^n)$ is a smooth projective variety of dimension $(k + 1)(n - k)$.

Let $f \in I(S^n)$. By the above identification, $S^k \in \Gamma_f(k)$ if and only if there exists $U \in S_f(k)$ such that $S^k = S^n \cap U$. We get the following proposition.
Proposition 4.4. Let \( f \in \mathcal{I}(S^n) \). Then \( \Gamma_f(k) \cong S_f(k + 1) \).

Therefore the study of \( \Gamma_f(k) \) reduces to the study of the varieties of invariant linear subspaces of a real orthogonal linear map.

Proposition 4.5. Let \( V \) be an \( n \)-dimensional Euclidean vector space and let \( A \in \text{End}(V) \) be an orthogonal map having a single primary component.

1. If \( A = \pm I_n \) then \( S_A(k) \cong G(k, \mathbb{R}^n) \).
2. If \( A = R_{\theta}^{\otimes p} \in O(n) \), with \( n = 2p \) then \( S_A(k) = \emptyset \) if \( k \) is odd and \( S_A(k) \cong G(k, \mathbb{C}^p) \) if \( k \) is even.

Proof. The first case is trivial. Assume that \( A = R_{\theta}^{\otimes n} \). Let \( V = \mathbb{R}^2n \), and denote by \( V^c \) the complexification of \( V \). Let \( A^c : V^c \to V^c \) denote the complexification of \( A \). Then \( A^c \) has a pair of conjugate eigenvalues \( \lambda \) and \( \overline{\lambda} \). There is a primary decomposition \( V^c = V^c_\lambda \oplus V^c_{\overline{\lambda}} \), where \( V^c_\lambda \) is the eigenspace of eigenvalue \( \lambda \) and \( V^c_{\overline{\lambda}} \) is the eigenspace of eigenvalue \( \overline{\lambda} \). Let \( A_\lambda = A^c|_{V^c_\lambda} \) denote the restriction of \( A^c \) to \( V^c_\lambda \), and let \( \pi_\lambda : V^c \to V^c_\lambda \) be the projection of \( V^c \) along \( V^c_{\overline{\lambda}} \). By Lemma 4.2, there is an isomorphism \( S_A \to S_{A_\lambda} \) of real algebraic varieties given by \( W \mapsto \pi_\lambda(W^c) \), where \( W^c \) denotes the complexification of the subspace \( W \). Note that the normal form of \( A_\lambda \) is \( \lambda I_{2n} \), where \( \lambda \in \mathbb{C} \). Therefore \( S_{A_\lambda}(k) \cong G(k, \mathbb{C}^n) \). By the above isomorphism we get \( S_A(k) \cong S_{A_\lambda}(k) \cong G(k, \mathbb{C}^n) \).

Theorem 4.6. Let \( A \in \text{End}(V) \) be an orthogonal map.

1. If \( \tilde{\sigma}_A = [(n_1, \overline{m}_1), \cdots, (n_s, \overline{m}_s), m_1, m_i] \) is the Segre symbol of \( A \) then
   \[
   S_A(k) \cong \bigcup_{2 \sum_{i=1}^{s} k_i + \sum_{i=1}^{s} r_i = k} G(k_1, \mathbb{C}^{n_1}) \times \cdots \times G(k_s, \mathbb{C}^{n_s}) \times G(r_1, \mathbb{R}^{m_1}) \times G(r_s, \mathbb{R}^{m_i}).
   \]
2. The dimensions of the connected components of \( S_A(1) \) and \( S_A(2) \) determine the Segre symbol of \( A \).

Proof. The first statement is a consequence of Propositions 4.3 and 4.5. Let us prove the second. Let \( d_i = (d_1^i, \cdots, d_s^i) \) denote the vector formed by the dimensions of the connected components of \( S_A(i) \), such that \( d_1^i \geq d_2^i \geq \cdots \geq d_s^i \). Let
   \[
   \tilde{\sigma}_A = [(m_1, \overline{m}_1), \cdots, (n_s, \overline{m}_s), m_1, m_i]
   \]
be the Segre symbol of \( A \). We need to show that \( d_i = (d_1^i, \cdots, d_s^i) \), for \( i = 1, 2 \), determine \( n_1, \cdots, n_s \) and \( m_1, m_i \).

We first study the variety of \( A \)-invariant lines. By the first statement of the theorem we have \( S_A(1) \cong \bigsqcup_{k=1}^n \mathbb{P}^{m_i-1}_\mathbb{R} \). We consider three cases.

1) If \( c_1 = 0 \) then \( t = 0 \).
2) If \( c_1 = 1 \) then \( t = 1 \) and \( m_1 = d_1^1 + 1 \).
3) If \( c_2 = 2 \) then \( t = 2 \), and \( m_i = d_1^2 + 1 \).

Therefore \( d_1 \) determines \( t \) and \( m_i \), for \( i \leq t \).

Assume that \( t \) and \( m_i \) are known. Let us study the variety of \( A \)-invariant planes. Again, there are three cases to consider.
1) If \( t = 0 \), \( S_A(2) \cong \bigcup_{i=1}^{c_2} \mathbb{P}^{m_i-1} \). Then \( s = c_2 \).

2) If \( t = 1 \), \( S_A(2) \cong \left( \bigcup_{i=1}^{c_2} \mathbb{P}^{m_i-1} \right) \sqcup \mathbb{G}(2, \mathbb{R}^{m_1}) \). Since \( \mathbb{G}(2, \mathbb{R}^m) = \emptyset \), for \( m < 2 \), we have

\[
\begin{cases}
  c_2 - 1, & \text{if } m_1 \geq 2, \\
  c_2, & \text{if } m_1 < 2.
\end{cases}
\]

3) If \( t = 2 \), \( S_A(2) \cong \left( \bigcup_{i=1}^{c_2} \mathbb{P}^{m_i-1} \right) \sqcup \left( \mathbb{P}^{m_1-1} \times \mathbb{P}^{m_2-1} \right) \sqcup \mathbb{G}(2, \mathbb{R}^{m_1}) \sqcup \mathbb{G}(2, \mathbb{R}^{m_2}) \). Therefore in this case we have,

\[
\begin{cases}
  c_2 - 3, & \text{if } m_1, m_2 \geq 2, \\
  c_2 - 2, & \text{if } m_1 \geq 2 > m_2, \\
  c_2 - 1, & \text{if } m_1, m_2 < 2.
\end{cases}
\]

In all three cases \( s \) is determined. Note as well that \( \dim \mathbb{P}^k = 2k \). Therefore since \( m_i \) are known, \( n_1, \ldots, n_s \) are determined by \( d_2 \).

**Theorem 4.7.** Let \( f \in I(S^n) \). The dimensions of the connected components of \( \Gamma_f(0) \) and \( \Gamma_f(1) \) determine the Segre symbol of \( f \).

**Proof.** It follows from Proposition 4.3 and Theorem 4.6.

Note that the elements of \( \mathbb{G}(0, S^n) \) are 0-dimensional subspheres of \( S^n \), that is, pairs of antipodal points of \( S^n \) which are invariant for \( f \), but each point of the pair need not be fixed. One could also consider a finer classification taking into account the sets of fixed points. Then every Segre class of isometries having a real eigenvalue \( \varepsilon = \pm 1 \) would split into subclasses depending on the value of \( \varepsilon \). These subclasses are precisely the connected components of the strata defined by the orbit type decomposition.

Tables 1, 2 and 3 show a normal form representative of each Segre class, for isometries of \( S^1 \), \( S^2 \) and \( S^3 \), together with the varieties of invariant subspheres of each dimension.

Let \( f \in I(S^n) \) and \( c_k \) be the number of connected components of \( \Gamma_f(k) \). Denote by \( d_k = (d_{k,1}, \ldots, d_{k,c_k}) \) the vector formed by the dimensions of the connected components of \( \Gamma_f(k) \), in decreasing order. Then \( d = [d_0; \ldots; d_{n-1}] \). The last column of each table shows the vector \( d \) formed by the dimensions of the connected components of \( \Gamma_f(k) \). In all tables a collection of points is denoted by \( \{\ast \cdot \cdot \cdot \ast\} \).

We remark that \( S^1 \) is not a space of constant curvature, since it has no curvature defined. However, we include the table of \( I(S^1) \) for completeness.

**Table 1.** Isometries of \( S^1 \)

| \( \tilde{\sigma}_f \) | \( f \) | \( \Gamma_f(0) \) | \( d \) |
|----------------------|--------|----------------|-----|
| [2] \( (\ast, \ast) \) | \( \mathbb{P}^1 \) | [1] |
| [1, 1] \( (\ast, \ast) \) | \( \{\ast \} \) | [0, 0] |
| [(11)] \( (\ast, \ast) \) | \( \emptyset \) | [-1] |
4.2. Invariant affine subspaces of $\mathbb{R}^n$. Let $V = \mathbb{R}^{n+1}$. Recall that the standard Euclidean affine space is given by $\mathbb{E}^n = \{x_{n+1} = 1\} \subset V$, and that $V_0 = \{x_{n+1} = 0\} \subset V$ is its associated vector space. The canonical inclusion $V_0 \to V$ induces an inclusion on the respective Grassmannian varieties $G(k, V_0) \to G(k, V)$. The bijection between $k$-dimensional affine subspaces of $\mathbb{E}^n$ and $k+1$-dimensional linear subspaces of $V$ not contained in $V_0$ induces an isomorphism of algebraic varieties

$$G(k, \mathbb{E}^n) \cong G(k+1, V) \setminus G(k+1, V_0).$$

Since it is an open Zariski connected subset of $G(k+1, V)$, $G(k, \mathbb{E}^n)$ is a quasiprojective algebraic variety of dimension $(k+1)(n-k)$.

Let $f : \mathbb{E}^n \to \mathbb{E}^n$ be an isometry, and let $\varphi_0 : V_0 \to V_0$ be its associated linear map. Let $\pi : G(k, \mathbb{E}^n) \to G(k, V_0)$ denote the natural projection sending each affine subspace to its associated vector space. Recall that an affine subspace $p + V \subset \mathbb{E}^n$ is $f$-invariant if and only if $\varphi_0(V) \subset V$ and $f(p) - p \in V$. Therefore we have $\pi(\Gamma_f(k)) \subset S_{\varphi_0}(k)$.

Lemma 4.8. If $f, g \in I(\mathbb{E}^n)$ are conjugate, there is an isomorphism of algebraic varieties $\Gamma_f \cong \Gamma_g$.

Proof. Let $\alpha \in \text{Euc}(n)$ such that $g = \alpha f \alpha^{-1}$. Then $p + V \mapsto \alpha(p + V)$ is an isomorphism. \qed
Let \( f : \mathbb{E}^n \to \mathbb{E}^n \) be an isometry induced by \( \varphi : V \to V \), and let \( \varphi_0 : V_0 \to V_0 \) be its associated linear map. Let \( V = V_R \oplus V_I \) be the decomposition of \( V \) into \( \varphi \)-invariant subspaces, where \( V_I \) denotes the primary component of eigenvalue 1 and \( V_R \) is the direct sum of the remaining primary components. Denote by \( \varphi_R = \varphi|_{V_R} \) and \( \varphi_I = \varphi|_{V_I} \) the restrictions of \( \varphi \) to \( V_R \) and \( V_I \) respectively.

Let \( \mathbb{B} = \mathbb{E}^n/V_R \) be the quotient affine space with associated vector space

\[
V_{01} := V_0/V_R \cong V_0 \cap V_I.
\]

Note that \( \mathbb{B} \) is the affine space defined by \( \mathbb{B} = \{ x \in V_1 ; x_{n+1} = 1 \} \), and that the restriction of \( \varphi_0 \) to \( V_{01} \) is the identity transformation. Since the map \( \varphi_1 : V_1 \to V_1 \) is unipotent, it induces an isometry \( f_1 : \mathbb{B} \to \mathbb{B} \), which is either the identity or a translation.

**Proposition 4.9.** Let \( f \in I(\mathbb{E}^n) \). With the previous notation, \( \Gamma_f \cong S_{\varphi_R} \times f_{1} \). In particular,

\[
\Gamma_f(k) \cong \bigcup_{k_1+k_2=k} S_{\varphi_R}(k_1) \times S_{f_1}(k_2).
\]

**Proof.** Note that there is an isomorphism \( \Gamma_f \cong S_{\varphi} \setminus S_{\varphi_0} \). By Proposition 4.3, we have an isomorphism \( S_{\varphi} \cong S_{\varphi_R} \times S_{\varphi_I} \). Furthermore, since \( V_R \subset V_0 \) we have \( S_{\varphi_0} \cong S_{\varphi_R} \times S_{\varphi_0} \). Therefore

\[
\Gamma_f \cong (S_{\varphi_R} \times S_{\varphi_1}) \setminus (S_{\varphi_R} \times S_{\varphi_0}) \cong S_{\varphi_R} \times (S_{\varphi_1} \setminus S_{\varphi_0}) \cong S_{\varphi_R} \times f_{1}.
\]

\( \square \)

**Proposition 4.10.** Let \( f : \mathbb{E}^n \to \mathbb{E}^n \) be an isometry, and let \( f_1 : \mathbb{E}^r \to \mathbb{E}^r \) denote its unipotent part.

i) If \( f \) is elliptic then \( \Gamma_{f_1}(k) \cong \mathbb{G}(k, \mathbb{E}^r) \) for all \( k \geq 0 \).

ii) If \( f \) is hyperbolic then \( \Gamma_{f_1}(0) = \emptyset \), and \( \Gamma_{f_1}(k) \cong \mathbb{G}(k-1, \mathbb{E}^{r-1}) \), for all \( k \geq 1 \).

**Proof.** If \( f \) is elliptic then \( f_1 \) is the identity transformation and every subspace is \( f_1 \)-invariant.

Assume that \( f \) is hyperbolic. Let \( \{ e^i \}_{i=1}^r \); \( p \) be an Euclidean reference of \( \mathbb{E}^r \) such that the matrix of \( f_1 \) in this reference is a normal form \( \mathbb{I}_{r-1} \oplus \left( \begin{smallmatrix} 1 & q \\ 0 & 1 \end{smallmatrix} \right) \). For all \( p \in \mathbb{E}^r \) we have \( f(p) - p = ae^r \). Let \( p + V \) be an \( f \)-invariant subspace of dimension \( k \) and define \( L := Sp\{ e^r \} \). Then \( L \subset V \), since the invariance of \( p + V \) implies \( f(p) - p = ae^r \subset V \).

Let \( L^\perp \) denote the orthogonal complement of \( L \) in \( V_0 \cap V_I \). Then \( \mathbb{B} = p + L^\perp \) is an Euclidean affine space of dimension \( r - 1 \). We denote by \( \pi : \mathbb{E}^r \to \mathbb{B} \) the orthogonal projection of \( \mathbb{E}^r \) along \( L \), and \( i : \mathbb{B} \to \mathbb{E}^r \) denotes the canonical inclusion. Let \( g = \pi \circ f \circ i : \mathbb{B} \to \mathbb{B} \). Then \( g \) is the identity transformation. There is a commutative diagram

\[
\begin{array}{ccc}
\mathbb{E}^n & \xrightarrow{f} & \mathbb{E}^n \\
\uparrow{i} & & \downarrow{\pi} \\
\mathbb{B} & \xrightarrow{g} & \mathbb{B}
\end{array}
\]

Since \( L \subset V \), \( \pi(p + V) \) is a \( g \)-invariant subspace of dimension \( k - 1 \). We get an isomorphism \( \Gamma_f(k) \to \Gamma_g(k-1) \) given by \( (p + V) \to \pi(p + V) \). Therefore, if \( k \geq 1 \),

\[
\Gamma_{f_1}(k) \cong \Gamma_{g}(k-1) = \mathbb{G}(k-1, \mathbb{E}^{r-1}).
\]

\( \square \)
Theorem 4.11. Let \( f \in I(\mathbb{E}^n) \).

i) Let \( \sigma_f = [t; r; \tilde{\sigma}] \) be the Segre symbol of \( f \), and let \( \varphi_R \in O(n - r) \) be an orthogonal map with Segre symbol \( \tilde{\sigma} \). Let \( d = 0 \) if \( t = e \), and \( d = 1 \) if \( t = h \). Then

\[
\Gamma_f(k) \cong \bigcup_{k_1 + k_2 = k} S_{\varphi_R}(k_1) \times \mathbb{G}(k_2 - d, \mathbb{E}^{r-d}).
\]

ii) The dimensions of the connected components of \( \Gamma_f(k) \), for \( k \leq 2 + d \) determine the Segre symbol of \( f \), where \( d = 0 \) if \( f \) is elliptic and \( d = 1 \) if \( f \) is hyperbolic.

Proof. i). We can assume that \( f \) is a normal form. Let \( f = \varphi_R \oplus f_1 \) be the decomposition of \( f \) into its orthogonal and unipotent components. By Proposition 4.9, \( \Gamma_f \cong S_{\varphi_R} \times \Gamma_{f_1} \). Consider two cases:

1) If \( t = e \) then \( f_1 = \mathbb{I}_{r+1} \), and hence \( \Gamma_{f_1}(k) \cong \mathbb{G}(k, \mathbb{E}^r) \).
2) If \( t = h \) then \( f_1 = \mathbb{I}_{r-1} \oplus \left( \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right) \), and by Proposition 4.10, \( \Gamma_{f_1}(k) \cong \mathbb{G}(k - 1, \mathbb{E}^{r-1}) \).

ii). Let \( \sigma_f = [t; r; \tilde{\sigma}] \). If \( \Gamma_f(0) = \emptyset \) then \( t = h \), while if \( \Gamma_f(0) \neq \emptyset \) we have \( t = e \). Therefore we can study each case separately.

Assume that \( t = e \). Then \( \Gamma_f(0) \cong \mathbb{E}^r \), hence the dimension of the variety of fixed points determines \( r \). Since \( r \) is known, the dimensions of the connected components of \( \Gamma_f(k) \), for \( k = 1, 2 \), determine the dimensions of \( S_{\varphi_R}(k) \), for \( k = 1, 2 \), and by Theorem 4.6, these determine \( \tilde{\sigma} \).

Assume that \( t = h \). Then for \( k \geq 1 \), \( \Gamma_f(k) \cong \Gamma_g(k - 1) \), where \( \sigma_g = [e; r; \tilde{\sigma}] \). By the previous case, \( \sigma_g \) is determined by the dimensions of \( \Gamma_g(k) \), for \( k \leq 2 \). Therefore \( \sigma_f \) is determined by the dimensions of \( \Gamma_f(k) \), for \( k \leq 3 \).

\( \square \)

Corollary 4.12. Let \( f, g \in I(\mathbb{E}^n) \). Then \( f \) and \( g \) have the same Segre symbol if and only if the dimensions of the connected components of the varieties of their fixed points, invariant lines and invariant planes coincide.

Tables 4 to 6 show a normal form representative of each Segre class, for isometries of \( \mathbb{E}^1 \), \( \mathbb{E}^2 \) and \( \mathbb{E}^3 \), together with the varieties of invariant affine subspaces of each dimension.

Table 4. Isometries of \( \mathbb{E}^1 \)

| \( \sigma_f \) | \( f \) | \( \Gamma_f(0) \) | \( d \) |
|-----------------|-------|-----------------|---|
| [e; 1; 0]       | \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) | \( \mathbb{E}^1 \) | [1] |
| [e; 0; 1]       | \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) | \{\} | [0] |
| [h; 1; 0]       | \( \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} \) | \( \emptyset \) | [-1] |
Table 5. Isometries of $E^2$

| $\sigma_f$ | $f$ | $\Gamma_f(0)$ | $\Gamma_f(1)$ | $d$ |
|------------|-----|----------------|----------------|-----|
| $[e; 2; 0]$ | $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ | $E^2$ | $G(1, E^2)$ | $[2, 2]$ |
| $[e; 1; 1]$ | $\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$ | $E^1$ | $E^1 \sqcup \{\ast\}$ | $[1, (1, 0)]$ |
| $[e; 0; 2]$ | $\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$ | $\{\ast\}$ | $\mathbb{P}^1$ | $[0, 1]$ |
| $[e; 0; (11)]$ | $\begin{pmatrix} c & s \\ -s & c \end{pmatrix}$ | $\{\ast\}$ | $\emptyset$ | $[0, -1]$ |
| $[h; 2; 0]$ | $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ | $\emptyset$ | $E^1$ | $[-1, 1]$ |
| $[h; 1; 1]$ | $\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$ | $\emptyset$ | $\{\ast\}$ | $[-1, 0]$ |

Table 6. Isometries of $E^3$

| $\sigma_f$ | $f$ | $\Gamma_f(0)$ | $\Gamma_f(1)$ | $\Gamma_f(2)$ | $d$ |
|------------|-----|----------------|----------------|----------------|-----|
| $[e; 3; 0]$ | $I_4$ | $E^3$ | $G(1, E^3)$ | $G(2, E^3)$ | $[3, 4, 3]$ |
| $[e; 2; 1]$ | $-I_1 \oplus I_3$ | $E^2$ | $G(1, E^2) \sqcup E^2$ | $G(1, E^2) \sqcup \{\ast\}$ | $[2, (2, 2), (2, 0)]$ |
| $[e; 1; 2]$ | $-I_2 \oplus I_2$ | $E^1$ | $(\mathbb{P}^1 \times E^1) \sqcup \{\ast\}$ | $\mathbb{P}^1 \sqcup E^1$ | $[1, (2, 0), (1, 1)]$ |
| $[e; 1; (11)]$ | $R_\theta \oplus I_2$ | $E^1$ | $\{\ast\}$ | $E^1$ | $[1, 0, 1]$ |
| $[e; 0; 3]$ | $-I_3 \oplus I_1$ | $\{\ast\}$ | $\mathbb{P}^2$ | $\mathbb{P}^2$ | $[0, 2, 2]$ |
| $[e; 0; (11), 1]$ | $R_\theta \oplus -I_1 \oplus I_1$ | $\{\ast\}$ | $\{\ast\}$ | $\{\ast\}$ | $[0, 0, 0]$ |
| $[h; 3; 0]$ | $I_2 \oplus \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ | $\emptyset$ | $E^2$ | $G(1, E^2)$ | $[-1, 2, 2]$ |
| $[h; 2; 1]$ | $-I_1 \oplus I_1 \oplus \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ | $\emptyset$ | $E^1$ | $E^1 \sqcup \{\ast\}$ | $[-1, 1, (1, 0)]$ |
| $[h; 1; 2]$ | $-I_2 \oplus \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ | $\emptyset$ | $\{\ast\}$ | $\mathbb{P}^1$ | $[-1, 0, 1]$ |
| $[h; 1; (11)]$ | $R_\theta \oplus \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ | $\emptyset$ | $\{\ast\}$ | $\emptyset$ | $[-1, 0, -1]$ |

4.3. Invariant totally geodesic hyperbolic subspaces of $H^n$. The hyperbolic Grassmannian $G(k, H^n)$ is the set of all $k$-dimensional totally geodesic hyperbolic subspaces $H^k$ of $H^n$. These are precisely the intersections of $H^n$ with $k + 1$-dimensional time-like subspaces of $\mathbb{R}^{n+1}$ (see [Ra], p. 70). Therefore $G(k, H^n)$ is an open connected subset of $G(k + 1, \mathbb{R}^{n+1})$, but not a Zariski open set. Hence it is a semialgebraic manifold of dimension $(k + 1)(n - k)$. 
Note that if \( f \in I(\mathbb{H}^n) \), the elements of \( \Gamma_f(k) \) are in bijection with the \( k + 1 \)-dimensional \( f \)-invariant time-like subspaces of \( \mathbb{R}^{n+1} \).

**Lemma 4.13.** If \( f, g \in I(\mathbb{H}^n) \) are conjugate there is an isomorphism of semialgebraic varieties \( \Gamma_f(k) \cong \Gamma_g(k) \).

**Proof.** It follows from the fact that Lorentz isometries preserve time-like subspaces. \( \square \)

**Proposition 4.14.** Let \( T \in PO(1,n) \). Then

\[
\Gamma_T(k) \cong \bigcup_{k_1+k_2=k} \Gamma_T_1(k_1) \times S_{T_2}(k_2).
\]

**Proof.** Let \( U \subset \mathbb{R}^{n+1} \) be a time-like \( T \)-invariant subspace of dimension \( k > 0 \). Then by Lemma 4.12 \( U = U_t \oplus U_s \), where \( U_t \subset V_t \) is \( T_t \)-invariant and \( U_s \subset V_s \) is \( T_s \)-invariant. Moreover \( U_s \) is space-like, so for \( U \) to be time-like, \( U_t \) must be time-like. Therefore \( U_t \in \Gamma_{T_t}(k_1) \), for some \( k_1 \geq 0 \), and \( U_s \in S_{T_s}(k_2) \), such that \( k_1 + k_2 = k \).

**Proposition 4.15.** Let \( T \in PO(1,n) \) and let \( r = \dim V_t \).

i) If \( T \) is elliptic then \( \Gamma_{T_t}(k) \cong \mathbb{G}(k, \mathbb{H}^{r-1}) \) for all \( 0 \leq k \leq r - 1 \).

ii) If \( T \) is parabolic then \( \Gamma_{T_t}(k) = \emptyset \) for \( k = 0, 1 \) and \( \Gamma_{T_t}(k) \cong \mathbb{G}(k-2, \mathbb{R}^{r-3}) \) for \( k \geq 2 \).

iii) If \( T \) is hyperbolic then \( \Gamma_{T_t}(0) = \emptyset \) and \( \Gamma_{T_t}(1) = \{V_t\} \).

**Proof.** i). If \( T \) is elliptic then \( T_t \) is the identity transformation. Therefore every subspace is \( T_t \)-invariant and the result follows.

ii). Assume that \( T \) is parabolic, and let \( \{e_i\}_{i=0}^{r-1} \) be a Lorentz basis of \( V_t \) such that the matrix of \( T_t \) is \( \Theta \oplus I_{r-3} \). To compute \( \Gamma_{T_t}(k) \) we study the \( T_t \)-invariant time-like linear subspaces of \( V_t \). We first show that if \( U \subset V_t \) is a time-like \( T_t \)-invariant linear subspace of \( V_t \) then \( Sp\{e_1, e_0 + e_2\} \subset U \).

Let \( \{v_i\}_{i=0}^{r-1} \) be the basis of \( V_t \) defined by \( v_2 = \frac{1}{2}(e_0 + e_2) \) and \( v_i = e_i \) for all \( i \neq 2 \). Then

\[
T_t v_0 = v_0 + v_1 + v_2, \\
T_t v_1 = v_1 + 2v_2, \\
T_t v_i = v_i, \forall i > 1.
\]

Let \( U \subset V_t \) be a time-like \( T_t \)-invariant linear subspace of \( V_t \), and let \( u = \sum_{i=0}^{r-1} a_i v_i \notin U \) be a time-like vector of \( U \). Then

\[
(T-I)u = a_0 v_1 + (a_0 + 2a_1)v_2, \\
(T-I)^2u = 2a_0 v_2.
\]

Since \( Q(u) < 0 \), we have \( a_0 \neq 0 \), and since \( (T-I)^2u \in U \) it follows that \( v_2 \in U \). Therefore since \( (T-I)u \in U \) we have \( v_1 \in U \). Hence \( Sp\{v_1, v_2\} \subset U \), as claimed.

Note that since \( Sp\{v_1, v_2\} \) is not time-like, every time-like \( T_t \)-invariant subspace of \( V_t \) has dimension of at least 3. Hence \( \Gamma_{T_t}(k) = \emptyset \), for \( k \leq 1 \).

For \( k \geq 2 \), the set of \( k \)-dimensional subspaces \( U \subset V_t \) such that \( Sp\{v_1, v_2\} \subset U \) is isomorphic to \( \mathbb{G}(k-2, V_t/Sp\{v_1, v_2\}) \cong \mathbb{G}(k-2, \mathbb{R}^{r-2}) \). Furthermore, every such subspace is \( T_t \)-invariant. We next
show that the set of those $T_i$-invariant subspaces that contain $Sp\{v_1, v_2\}$ and are not time-like, is isomorphic to $\mathbb{G}(k - 2, \mathbb{R}^{r - 3})$.

Let $V_0 = \{x_0 = 0\} \cap V_i \subset V_i$ and let $U \subset V_i$ such that $Sp\{v_1, v_2\} \subset U$. Then $Q|_U \geq 0$ if and only if $U \subset V_0$. Indeed, assume that $Q|_U \geq 0$ and let $u = \sum_{i=0}^{r} a_i v_i \in U$. Then $Q(u) = -a_0(a_0 + 2a_2) + \sum_{i=2}^{r} a_i^2$. Therefore if $a_0 \neq 0$, and since $v_2 \in U$, we can choose $a_2$ such that $Q(u) < 0$, which is a contradiction. Hence $a_0 = 0$ and $U \subset V_0$. Conversely since $V_0$ is space-like, if $U \subset V_0$ then $Q|_U \geq 0$.

Therefore the set of $T_i$-invariant $k$-dimensional time-like subspaces of $V_i$ is isomorphic to $\mathbb{G}(k - 2, \mathbb{R}^{r - 2}) \setminus \mathbb{G}(k - 2, \mathbb{R}^{r - 3})$, and we have

$$\Gamma_T(k) \cong \mathbb{G}(k - 1, \mathbb{R}^{r - 2}) \setminus \mathbb{G}(k - 1, \mathbb{R}^{r - 3}) \cong \mathbb{G}(k - 2, \mathbb{R}^{r - 3}).$$

iii). If $T$ is hyperbolic then $r = 2$ and $T_i$ has a normal form $\Omega = (c d \, d c)$, where $c, d \in \mathbb{R}$ are such that $c^2 - d^2 = 1, d \neq 0$. It follows from an easy computation that the only $T_i$-invariant proper subspaces of $V_i$ are light-like lines. Therefore $\Gamma_T(0) = \emptyset$ and $\Gamma_T(1) = \{V_i\}$.  

\begin{proof}
The first statement follows from Propositions 4.14 and 4.15. Let us prove the second.

Let $c_i$ denote the number of connected components of $\Gamma_T(i)$. By the above proposition applied to the case $k = 0$ we know that if $c_0 > 0$ then $T$ is elliptic, if $c_0 = 0$ and $c_1 > 0$ then $T$ is hyperbolic, and if $c_0 = c_1 = 0$ then $T$ is parabolic. Therefore we can study each type of isometry separately. Let $r = \dim V_i$.

i) If $T$ is elliptic then

$$\Gamma_T(k) \cong \bigcup_{k_1 + k_2 = k} \mathbb{G}(k_1, \mathbb{H}^{r - 1}) \times S_{T_1}(k_2).$$

If $T$ is parabolic then $\Gamma_T(k) = \emptyset$ for $k = 0, 1$ and

$$\Gamma_T(k) \cong \bigcup_{k_1 + k_2 = k - 2} \mathbb{G}(k_1, \mathbb{H}^{r - 3}) \times S_{T_1}(k_2).$$

If $T$ is hyperbolic then $\Gamma_T(0) = \emptyset$ and $\Gamma_T(k) \cong S_{T_1}(k - 1)$ for $k \geq 1$.

ii) The dimensions of the connected components of $\Gamma_T(k)$, for $k \leq 2 + d$ determine the Segre symbol of $T$, where $d = 0, 1$ or 2 if $T$ is elliptic, hyperbolic or parabolic respectively.

\end{proof}
iii) If $T$ is parabolic then $\Gamma_T(2) \cong \mathbb{H}^{r-2}$, so its dimension determines $r$. Moreover,
\[\Gamma_T(3) \cong \mathbb{G}(1, \mathbb{H}^{r-2}) \sqcup (\mathbb{H}^{r-2} \times S_{T_s}(1)),\]
so the dimensions of $\Gamma_T(3)$ determine the dimensions of $S_{T_s}(1)$. Finally,
\[\Gamma_T(4) \cong \mathbb{G}(2, \mathbb{H}^{r-2}) \sqcup (\mathbb{G}(1, \mathbb{H}^{r-2}) \times S_{T_s}(1)) \sqcup (\mathbb{H}^{r-2} \times S_{T_s}(2)),\]
so the dimensions of $\Gamma_T(4)$ determine the ones of $S_{T_s}(2)$. Again by Theorem 4.6 we get the result.

\[\square\]

Tables 7 to 9 show a normal form representative of each Segre class, for isometries of $\mathbb{H}^1$, $\mathbb{H}^2$ and $\mathbb{H}^3$, together with the varieties of invariant hyperbolic subspaces of each dimension.

**Table 7. Isometries of $\mathbb{H}^1$**

| $\sigma_f$ | $f$ | $\Gamma_f(0)$ | $d$ |
|------------|-----|---------------|-----|
| [e; 2; 0]  | $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ | $\mathbb{H}^1$ | [1] |
| [e; 1; 1]  | $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ | $\ast$ | [0] |
| [h; 2; 0]  | $\begin{pmatrix} 1 & c \\ d & c \end{pmatrix}$ | $\emptyset$ | [-1] |

**Table 8. Isometries of $\mathbb{H}^2$**

| $\sigma_f$ | $f$ | $\Gamma_f(0)$ | $\Gamma_f(1)$ | $d$ |
|------------|-----|---------------|--------------|-----|
| [e; 3; 0]  | $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ | $\mathbb{H}^2$ | $\mathbb{G}(1, \mathbb{H}^2)$ | [2; 2] |
| [e; 2; 1]  | $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ | $\mathbb{H}^1$ | $\mathbb{H}^1 \sqcup \{\ast\}$ | [1; (1, 0)] |
| [e; 1; 2]  | $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ | $\ast$ | $\mathbb{P}^1$ | [0; 1] |
| [e; 1; (1\overline{1})] | $\begin{pmatrix} 1 & a & b \\ -a & 1 & -b \\ -a & -b & 1 \end{pmatrix}$ | $\ast$ | $\emptyset$ | [0; -1] |
| [h; 2; 1]  | $\begin{pmatrix} 1 & c \\ d & c \end{pmatrix}$ | $\ast$ | $\emptyset$ | [-1; 0] |
| [p; 3; 0]  | $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ | $\emptyset$ | $\emptyset$ | [-1; -1] |
Table 9. Isometries of $\mathbb{H}^3$

| $\sigma_f$ | $f$ | $\Gamma_f(0)$ | $\Gamma_f(1)$ | $\Gamma_f(2)$ | $d$ |
|------------|-----|---------------|---------------|---------------|-----|
| $[e; 4; 0]$ | $I_4$ | $\mathbb{H}^3$ | $G(1, \mathbb{H}^3)$ | $G(2, \mathbb{H}^3)$ | $[3; 4; 3]$ |
| $[e; 3; 1]$ | $I_3 \oplus -I_1$ | $\mathbb{H}^2$ | $G(1, \mathbb{H}^2) \sqcup \mathbb{H}^2$ | $G(1, \mathbb{H}^2) \sqcup \{\ast\}$ | $[2; (2, 2); (2, 0)]$ |
| $[e; 2; 2]$ | $I_2 \oplus -I_2$ | $\mathbb{H}^1$ | $(\mathbb{P}^1 \times \mathbb{H}^1) \sqcup \{\ast\}$ | $\mathbb{H}^1 \sqcup \mathbb{P}^1$ | $[1; (2, 0); (1, 1)]$ |
| $[e; 2; (11)]$ | $I_2 \oplus R_\theta$ | $\mathbb{H}^1$ | $\{\ast\}$ | $\mathbb{H}^1$ | $[1; 0; 1]$ |
| $[e; 1; 3]$ | $I_1 \oplus -I_3$ | $\{\ast\}$ | $\mathbb{P}^2$ | $\mathbb{P}^2$ | $[0; 2; 2]$ |
| $[h; 2; (11), 1]$ | $I_1 \oplus R_\theta \oplus -I_1$ | $\{\ast\}$ | $\{\ast\}$ | $\{\ast\}$ | $[0; 0; 0]$ |
| $[h; 2; 2]$ | $\Omega_t \oplus eI_2$ | $\emptyset$ | $\{\ast\}$ | $\mathbb{P}^1$ | $[-1, 0, 1]$ |
| $[h; 2; 1, 1]$ | $\Omega_t \oplus eI_1 \oplus -eI_1$ | $\emptyset$ | $\{\ast\}$ | $\{\ast \ast\}$ | $[-1, 0, (0, 0)]$ |
| $[p; 4; 0]$ | $\Theta \oplus I_1$ | $\emptyset$ | $\emptyset$ | $\mathbb{E}^1$ | $[-1, -1, 1]$ |
| $[p; 3; 1]$ | $\Theta \oplus -I_1$ | $\emptyset$ | $\emptyset$ | $\{\ast\}$ | $[-1, -1, 0]$ |

References

[Ar] Arnold, V.I. On matrices depending on parameters. Russian Math. Surveys. 26 29, (1971).
[Ba] Baker, A. Matrix groups, an introduction to Lie groups. Springer. (2002).
[Br] Broer, A. Lectures on decomposition classes. Representation Theories and Algebraic Geometry, Proceedings of the NATO Advanced Study Institute. Kluwer Academic, Vol.514, p.39-83 (1998).
[Ci] Cirici, J. Orbit types of affine endomorphisms. In preparation.
[Co] Coxeter, H.S.M. Introduction to geometry. John Wiley and Sons. (1989).
[Ga] Gantmacher, F.R. The theory of matrices. Chelsea Publishing Company. Vol I. (1977).
[Gi] Gibson, C.G. Regularity of the Segre stratification. Math. Proc. Camb. Phil. Soc. 80 (1976).
[GK] Gonopadhyay, K ; Kulkarni, R. Dynamical types of isometries of the hyperbolic space. Preprint arXiv: [math/0511144v4]. (2008).
[Gr] Greenberg, L Discrete subgroups of the Lorentz group. Math. Scand. (1962).
[HP] Hodge, W.V.D. ; Pedoe, D. Methods of algebraic geometry I. Cambridge Mathematical Library. (1994).
[Ob] Obata, M. The Gauss map of immersions of Riemannian manifolds in spaces of constant curvature. Journal of Differential Geometry . (1968).
[Pe] Petrov, A.Z. Einstein spaces. Pergamon. (1969).
[Pf] Pfaff, M.J. Analytic and geometric study of Stratified Spaces. Springer-Verlag. (2001).
[Ra] Ratcliffe, J.G. Foundations of hyperbolic manifolds Springer-Verlag (1994).
[Re] Reventós, A. Afirmats, moviments i quàdriques U.A.B. Servei de Publicacions, Bellaterra. (2008)
[Sh] Shayman, M.A. On the variety of invariant subspaces of a finite-dimensional linear operator. Transactions of the American Mathematical Society. 274 (1982).

Departament d’Àlgebra i Geometria, Universitat de Barcelona, Gran Via 585, 08007 Barcelona (Spain)

E-mail address: jcirici@ub.edu