Stable Solutions to the Abelian Yang–Mills–Higgs Equations on $S^2$ and $T^2$

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Abstract
We show under natural assumptions that stable solutions to the abelian Yang–Mills–Higgs equations on Hermitian line bundles over the round 2-sphere actually satisfy the vortex equations, which are a first-order reduction of the (second-order) abelian Yang–Mills–Higgs equations. We also obtain a similar result for stable solutions on a flat 2-torus. Our method of proof comes from the work of Bourguignon–Lawson (Commun Math Phys 79(2):189–230, 1981) concerning stable $SU(2)$ Yang–Mills connections on compact homogeneous 4-manifolds.

Keywords Stability · Abelian Yang–Mills–Higgs equation · Vortex equations

1 Introduction

Let $\Sigma$ be an oriented surface equipped with a Riemannian metric $g$, and suppose $L$ is a complex line bundle over $\Sigma$ equipped with a Hermitian metric $\langle \cdot, \cdot \rangle$, so that for all $x \in \Sigma$, we have

$$\langle \alpha \xi, \beta \eta \rangle = \alpha \overline{\beta} \langle \xi, \eta \rangle$$

for all $\alpha, \beta \in \mathbb{C}$ and $\xi, \eta \in L_x$.

Given $\varepsilon > 0$, we are interested in the following self-dual abelian Yang–Mills–Higgs action, which takes a section $u : \Sigma \to L$ and a metric connection $\nabla_A$ on $L$ as variables:

$$E_\varepsilon(u, \nabla_A) = \int_\Sigma \varepsilon^2 |F_A|^2 + |\nabla_A u|^2 + \frac{(1 - |u|^2)^2}{4\varepsilon^2} d\mu_g.$$
Here $F_A$ denotes the curvature of the connection $\nabla_A$, and $d\mu_g$ is the volume form on $\Sigma$. Note that since $\nabla_A$ is a $U(1)$-connection, $F_A$ is in fact a 2-form with values in $\sqrt{-1}\mathbb{R}$. Historically, functionals of this type originated with the Ginzburg–Landau theory of superconductivity, and later made their way into elementary particle physics, where $U(1)$ may be replaced by other groups depending on the situation. The reader interested in a succinct account of the physical backgrounds may consult, for example [17, Chapter 1].

A straightforward computation yields the Euler–Lagrange equations of $E_\varepsilon$:

$$\begin{align}
\nabla_A^* \nabla_A u &= \frac{1-|u|^2}{2\varepsilon^2} u, \\
\varepsilon^2 d^* F_A &= -\sqrt{-1} \Re \langle \sqrt{-1} u, \nabla_A u \rangle = \sqrt{-1} \Im \langle u, \nabla_A u \rangle,
\end{align}$$

(1.1)

where $\nabla_A^*$ is the formal adjoint of $\nabla_A$, and both sides of the second equation are 1-forms valued in $\sqrt{-1}\mathbb{R}$. Note also that the second equation is not an elliptic equation for the connection $\nabla_A$. This can be attributed to the gauge invariance of $E_\varepsilon$, where $E_\varepsilon(u, \nabla_A) = E_\varepsilon(s \cdot u, \nabla_A - ds \cdot s^{-1})$ for any $s: \Sigma \to U(1) \simeq S^1$. (1.2)

The system (1.1) and its relatives have been the subject of extensive study, and there is by now a large literature on them, which we do not attempt to survey here. The interested reader is referred to the monographs [2,25,27] and the references therein. Below, we will focus on one particular aspect of the second-order equations (1.1), namely that they admit special solutions given by first-order equations which arise from rewriting the functional $E_\varepsilon$ in a particular way. Specifically, it was observed by Bogomol’nyi [4] that, with the choice of potential $(1-|u|^2)^2$ in the definition of $E_\varepsilon$, the functional can be split into two parts whose difference, after an integration by parts, is a topological invariant, as follows:

$$\begin{align}
E_\varepsilon(u, \nabla_A) &= \int_\Sigma \frac{1}{4} |\nabla_A u - \sqrt{-1} \ast \nabla_A u|^2 + \frac{1}{2} |\varepsilon \ast \sqrt{-1} F_A - \frac{1-|u|^2}{2\varepsilon}|^2 \\
&+ \int_\Sigma \frac{1}{4} |\nabla_A u + \sqrt{-1} \ast \nabla_A u|^2 + \frac{1}{2} |\varepsilon \ast \sqrt{-1} F_A + \frac{1-|u|^2}{2\varepsilon}|^2 \\
&= \int_\Sigma \frac{1}{2} |\nabla_A u \mp \sqrt{-1} \ast \nabla_A u|^2 + |\varepsilon \ast \sqrt{-1} F_A \mp \frac{1-|u|^2}{2\varepsilon}|^2 \pm \int_\Sigma \sqrt{-1} F_A,
\end{align}$$

(1.3)

where $\ast$ denotes the Hodge star operator on $\Sigma$, and the signs are chosen so that the term $\pm \int_\Sigma \sqrt{-1} F_A$ is nonnegative. Note that if $\Sigma$ is closed, then of course

$$\frac{1}{2\pi} \int_\Sigma \sqrt{-1} F_A = \deg L.$$  (1.4)

Also, if $\Sigma = \mathbb{R}^2$, then by [17, Proposition II.3.5] and [1], under the assumption that $E_\varepsilon(u, \nabla_A)$ is finite, we still have $\int_\Sigma \sqrt{-1} F_A \in 2\pi \mathbb{Z}$, the integer being essentially the degree of $u$ at infinity.
Thus, (1.3) provides a lower bound for $E_{\epsilon}(u, \nabla A)$ in terms of a topological quantity, namely $\frac{1}{2\pi} \int_{\Sigma} \sqrt{-1} F_A$, also known as the vortex number. Configurations $(u, \nabla A)$ which attain this bound satisfy, depending on the sign of the vortex number, one of the following two first-order systems, which we will refer to collectively as the vortex equations.

\[
\begin{align*}
\nabla_A u &= \sqrt{-1} \ast \nabla_A u, \\
\ast \sqrt{-1} F_A &= \frac{1-|u|^2}{2\epsilon^2}.
\end{align*}
\]  

(1.5)

\[
\begin{align*}
\nabla_A u &= -\sqrt{-1} \ast \nabla_A u, \\
\ast \sqrt{-1} F_A &= -\frac{1-|u|^2}{2\epsilon^2}.
\end{align*}
\]  

(1.6)

Since solutions to (1.5) or (1.6) minimize $E_{\epsilon}$ among configurations with the same vortex number, they are, in particular, stable solutions to (1.1) when $\Sigma$ is closed. Here by a stable solution we mean a solution at which the second variation of $E_{\epsilon}$ is positive semi-definite. (See Sect. 3.) In view of this property of vortex solutions, it seems natural to ask whether the converse is also true; that is, given a stable solution $(u, \nabla A)$ to (1.1) on a closed $\Sigma$, must it satisfy one of (1.5) and (1.6)? Our main result gives a positive answer when $\Sigma$ is the round $S^2$ or flat $T^2$, provided $u$ is not the zero section. In the $S^2$ case, this last assumption can be dropped if $\epsilon$ is below a threshold that depends only on $\deg L$.

The reader may wonder if the vortex equations actually admit any solutions at all. Thus, we briefly digress to recall some fundamental existence and classification results for solutions to (1.5) and (1.6). When $\Sigma = \mathbb{R}^2$, these are due to Taubes [31], who showed that, up to gauge equivalence, solutions with vortex number $d$ are in one-to-one correspondence with unordered $|d|$-tuples of points on $\mathbb{R}^2$. On the other hand, if $\Sigma$ is a closed surface, a similar classification was established, using different methods, by Bradlow [6] and García-Prada [14] (see also Noguchi [24]), under the assumption that $4\pi \deg L \leq \epsilon - 2|\Sigma|$. (In fact they studied a slightly different equation, but the analysis is essentially the same.) Note that the condition

\[
4\pi \deg L \leq \epsilon - 2|\Sigma|
\]  

(1.7)

is necessary for either (1.5) or (1.6) to admit a solution, as can be seen by integrating the second lines of (1.5) or (1.6) over $\Sigma$ ([6,14]).

We now return to the relation between stable solutions to (1.1) and solutions to the vortex equations, and state our main theorem.

**Theorem 1.1** Let $\Sigma$ be the round $S^2$ or a flat $T^2$ and suppose $L$ is a Hermitian line bundle over $\Sigma$ with $\deg L = d$. Let $(u, \nabla A)$ be a stable weak solution to (1.1) and assume either of the following two conditions:

(H) $u$ is not identically zero.

(H') $\Sigma$ is the round $S^2$ and $|d| \leq \epsilon^{-2}$.

Then $(u, \nabla A)$ satisfies (1.5) or (1.6).

**Remark 1.2** (1) The notion of weak solution is clarified in Sect. 3.
(2) As opposed to the case $\Sigma = \mathbb{R}^2$, where Taubes [32] showed that any finite-action solution to (1.1) in fact satisfies one of the vortex equations, Theorem 1.1 with the word “stable” removed is false in general. For instance, if $L$ is trivial and $\varepsilon$ is small enough, then the min-max construction of Pigati–Stern [26, Section 7.1] produces solutions $(u, \nabla A)$ with $E_{\varepsilon}(u, \nabla A) > 0$ and $u \not\equiv 0$. (See equation (7.1) in [26].) These cannot satisfy either of the vortex equations, for otherwise we would get the following contradiction

$$0 < E_{\varepsilon}(u, \nabla A) = \pm \int_\Sigma F_A = 0.$$ 

Here in the middle equality we use (1.3) and the right-most equality follows since here the bundle is trivial.

(3) The assumption $|d| \leq \varepsilon^{-2}$ in $(H')$ is just (1.7) since $|S^2| = 4\pi$. This threshold for $\varepsilon$ is optimal in that if $|d| > \varepsilon^{-2}$, then any solution of the form $(0, \nabla A)$ with $F_A$ harmonic is stable but does not satisfy either (1.5) or (1.6). We elaborate on this at the end of Sect. 4.

For the remainder of this introduction, we will attempt to put Theorem 1.1 into context, before briefly describing its proof. First, when $\Sigma$ is a convex domain in $\mathbb{R}^2$, in which case $L$ is necessarily trivial, Jimbo–Sternberg [19] established the constancy of stable solutions to (1.1) under a natural variational boundary condition, and for a more general class of potentials. For the Ginzburg–Landau equation $\varepsilon^2 \Delta u = (1 - |u|^2)u$ on complex-valued functions, a similar result was proved by Jimbo–Morita [18], assuming the homogeneous Neumann condition. Other related results on the classification of stable solutions to equations similar to (1.1) can be found, for instance, in [9,10,21,28].

Next, recall that several other functionals in differential geometry admit special minimizers given by first-order equations. For instance, $\pm$-holomorphic maps are homotopy minimizers for the Dirichlet energy of maps between compact Kähler manifolds, and connections with self-dual or anti-self-dual curvatures minimize the Yang–Mills functional on compact 4-manifolds. The relationship between complex subvarieties of Kähler manifolds and the area functional also falls into this framework, thanks to the Wirtinger inequality. In all these settings, “stability $\Rightarrow$ first-order reduction” results analogous Theorem 1.1 have been obtained under suitable assumptions. See for instance [8,29] (harmonic maps), [5,30] (Yang–Mills connections), [3,22] (minimal submanifolds).

Finally, we mention that the vortex equations admits various important generalizations; see for instance the surveys [7,15], and the references therein. We are particularly interested in the case of a Hermitian line bundle over a compact Kähler manifold $M$ [6,13]. Here the solutions are again stable critical points of $E_{\varepsilon}$, and are essentially in one-to-one correspondence with codimension-one complex subvarieties of $M$. In view of the recent work of Pigati–Stern [26], which revealed a close relationship between solutions to (1.1) on a Riemannian manifold and (real) codimension-two minimal submanifolds, it will be interesting to see whether a statement like Theorem 1.1 holds when $M = \mathbb{C}P^m$. The result, if true, would be an analogue of the classical theorem...
of Lawson–Simons [3], which reduces stable stationary integral currents in \( \mathbb{C} \mathbb{P}^n \) to complex subvarieties, and would serve as further evidence for the link between \( E_e \) and the volume functional in codimension two. We hope to address this question in a future work.

**Method** Here we assume \( \Sigma \) is as in the statement of Theorem 1.1. The proof of Theorem 1.1 shares a common theme with a lot of the results cited above, particularly the work of Bourguignon–Lawson [5, Section 10] on stable \( SU(2) \) Yang–Mills connections on homogeneous four-manifolds. To describe the idea in our setting, we consider the one-parameter group of diffeomorphisms generated by a vector field \( X \) on \( \Sigma \). Pulling back a solution \((u, \nabla_A)\) via these diffeomorphisms yields a one-parameter family of configurations, along which we compute the second derivative of \( E_e \). This has the same effect as computing the second variation of \( E_e \) along the path \((u + t(\nabla_A u)_X, \nabla_A + t(\i X F_A))\) (see [5, pp. 198–199]), which has to be non-negative by the stability assumption. Of course, as \( E_e \) is isometry-invariant, choosing \( X \) to be a Killing field yields no useful information since we would get zero anyway, regardless of stability. However, information can be extracted if we keep \( X \) Killing but replace \((u + t(\nabla_A u)_X, \nabla_A + t(\i X F_A))\) by \((u + t\sigma_X, \nabla_A - t\i X \varphi)\), where

\[
\sigma = \nabla_A u - \sqrt{-1} \ast \nabla_A u, \quad \varphi = \sqrt{-1} F_A - \frac{1 - |u|^2}{2\epsilon^2}.
\]

This choice is inspired by the one in [5, Section 10]. Note that, a priori, the second variation of \( E_e \) in this direction does not have to be zero, but if in addition \((u, \nabla_A)\) verifies (1.5) or (1.6), then \((\sigma_X, \i X \varphi) = (0, 0)\) or \((2(\nabla_A u)_X, 2\sqrt{-1} \i X F_A)\), and the second variation vanishes in either case as \( X \) is Killing. Thus one expects \( \sigma \) and \( \varphi \) to be helpful in detecting solutions to the vortex equations.

Computing the second variations of \( E_e \) along \((u + t\sigma_X, \nabla_A - t\sqrt{-1} \i X \varphi)\) gives rise to a quadratic form \( Q \) defined over the space \( K \) of Killing fields on \( \Sigma \), which must be positive semi-definite if \((u, \nabla_A)\) is a stable solution. As in [5], the proof then boils down to taking the trace over \( K \), and observing that when \( \Sigma \) is as in the statement of Theorem 1.1, the resulting inequalities, together with some basic estimates for solutions of (1.1), allow us to conclude the proof assuming (H). The prove the Theorem assuming \((H')\) instead, we first observe that when \(|d| = \epsilon^{-2}\), the conclusion holds when even if \( u \equiv 0 \). Then we argue that \( u \equiv 0 \) contradicts stability when \(|d| < \epsilon^{-2}\), thanks to an estimate on the lowest eigenvalue of \( d_A^* d_A \) due to Kuwabara [20].

**Notation** For the rest of the paper, \((\Sigma, g)\) will be a closed oriented surface equipped with a Riemannian metric, and \( L \) a Hermitian line bundle over \( \Sigma \). The Levi–Civita connection on \( \Sigma \) is denoted by \( \nabla \), and the volume form by \( d\mu_g \). The curvature convention we adopt is

\[
R_{X,Y,Z} = \nabla^2_{X,Y,Z} - \nabla^2_{Y,X,Z}.
\]

We use the same pointed brackets \( \langle \cdot, \cdot \rangle \) to denote any bundle metric on \( \Lambda^p T^* \Sigma \) or \( \Lambda^p T^* \Sigma \otimes L \) that is induced by \( g \) and the Hermitian metric on \( L \). As usual, we denote by \( \Omega^p(\Sigma) \) the space of \( p \)-forms on \( \Sigma \), and by \( \Omega^p(L) \) the space of sections of
\( \Lambda^p T^* \Sigma \otimes L \). Integrating the bundle metrics over \( \Sigma \) yields inner products on \( \Omega^p(\Sigma) \) and \( \Omega^p(L) \).

Fixing once and for all a smooth background metric connection \( \nabla_0 \) on \( L \), any other metric connection can be written as \( \nabla_A := \nabla_0 - \sqrt{-1} A \), where \( A \) is a real 1-form on \( \Sigma \). The curvatures of \( \nabla_A \) and \( \nabla_0 \) are then related by \( F_A = F_0 - \sqrt{-1} dA \). The exterior derivative induced by \( \nabla_A \) on \( \Omega^*(L) \) is denoted \( d_A \), and its formal adjoint \( d^*_A \).

Similarly, \( \nabla_A^* \) denotes the adjoint of \( \nabla_A \). For instance, for \( \sigma \in \Omega^1(L) \), we have

\[
\nabla_A^* \sigma = - (\nabla_A \sigma)_{e_i, e_i},
\]

where the right-hand side is summed over an orthonormal basis \( e_1, e_2 \) of \( T_x \Sigma \) at each \( x \in \Sigma \). (Below, unless otherwise stated, repeated indices are always summed.) Also, for a section \( u \) of \( L \), we will use \( dAu \) and \( \nabla Au \) interchangeably.

Next, by \( \Delta_1 \) we will always mean the Hodge Laplacian \( d^* d + d d^* \), even when it is acting on \( \Omega^0(\Sigma) \). Thus, for example, in this notation a real function \( f \) is sub-harmonic if \( \Delta f \leq 0 \). Similarly, using \( dA \) and \( d^*_A \), the Hodge Laplacian acting on \( \Omega^*(L) \) is given by

\[
\Delta_A = dA d^*_A + d^* A dA.
\]

When there is no danger of confusion, we will sometimes drop the subscripts in \( \nabla_A \), \( \Delta_A \), \( d_A \), \( d^*_A \), etc. and simply write them as \( \nabla \), \( \Delta \), \( d \), \( d^* \), etc.

Finally, by a configuration we mean a pair \((u, \nabla_A)\) where \( u \) is a section of \( L \) and \( \nabla_A \) is a metric connection on \( L \), with regularity to be specified depending on the context. Given \( \varepsilon > 0 \) and a configuration \((u, \nabla_A)\), we define

\[
\begin{align*}
    h(u, \nabla_A) &= \frac{1 - |u|^2}{2\varepsilon^2} (|u| \text{ is computed using the bundle metric on } L), \\
    f(u, \nabla_A) &= \ast \sqrt{-1} F_A, \\
    \sigma(u, \nabla_A) &= \nabla_A u - \sqrt{-1} \ast \nabla_A u, \\
    \varphi(u, \nabla_A) &= \sqrt{-1} F_A \ast h = \ast (f - h).
\end{align*}
\]

The subscripts \((u, \nabla_A)\) will be dropped when it’s clear from the context which configuration we mean. Also, note that \( h, f, \varphi \) stay unchanged when we switch from \((u, \nabla_A)\) to a gauge equivalent configuration \((e^{\sqrt{-1} \theta} u, \nabla_A - \sqrt{-1} d\theta)\), whereas \( \sigma \) transforms by

\[
\sigma(e^{\sqrt{-1} \theta} u, \nabla_A - \sqrt{-1} d\theta) = e^{\sqrt{-1} \theta} \sigma(u, \nabla_A).
\]

Nonetheless, \( |u|, |\sigma| \) and \( \langle u, \sigma \rangle \) are still gauge invariant. Other notation and terminology will be introduced when needed.

**Organization** In Sect. 2 we review a couple of Weitzenböck-type formulas and note some consequences which are important for the computations to follow. Section 3 collects a number of basic facts about \( E_\varepsilon \) and (1.1), including the first and second
variation formula, regularity of weak solutions up to change of gauge, and some basic pointwise estimates which help us distinguish vortices from other solutions of (1.1). At the end we also recall how to derive (1.3). In Sect. 4, we prove Theorem 1.1 and elaborate on Remark 1.2(2).

Remark Concerning the discussion on p. 16 below about solutions of the form $(0, \nabla A)$ to (1.1) on $S^2$, I was informed after completing this work that their instability (stability, resp.) when $|d| < \varepsilon^{-2}$ ($|d| \geq \varepsilon^{-2}$, resp.) also follows from the work of Nagy in [23].

2 Review of Weitzenböck Formulas and Some Consequences

Both of the Weitzenböck formulas recalled below are standard and the proofs can be found essentially in [5, Section 3]. Note that because $U(1)$ is abelian, the formulas simplify somewhat in our case.

**Proposition 2.1** Let $\sigma \in \Omega^1(L)$ and $\varphi \in \Omega^2(\Sigma)$. Then the following hold.

(a) (See also [5, Theorem 3.2])

$$(\Delta_A \sigma)_X = (\nabla^*_A \nabla_A \sigma)_X + (F_A)_{e_i, X} \varphi_{e_i} + \sigma_{\text{Ric}(X), e_i},$$

where the second term on the right-hand side is summed over an orthonormal basis $\{e_i\}$ of $T_x \Sigma$.

(b) ([5, Theorem 3.10])

$$(\Delta \varphi)_{e_i, X} = (\nabla^* \nabla \varphi)_{X, e_j}. \text{ Note that the curvature terms all cancel since } \varphi \text{ is a top-dimensional form.}$$

Besides the Weitzenböck formulas, we also need to know how differential operators like $d$, $d^*$ and $d_A$, $d^*_A$ interact with the operation of contracting with Killing vector fields on $\Sigma$. We don’t think these formulas are new, but we still include their proofs for the reader’s convenience.

**Lemma 2.2** Let $\sigma$ and $\varphi$ be as in Proposition 2.1 and suppose $X$ is a Killing vector field on $\Sigma$. Then the following hold.

(a) $d^* d_A (\sigma_X) = (\Delta_A \sigma)_X - (F_A)_{e_i, X} \varphi_{e_i} - \langle dX^b, d_A \sigma \rangle$, where $X^b$ is the 1-form dual to $X$, and $\langle dX^b, d_A \sigma \rangle \in \Omega^0(L)$ is given by

$$\langle dX^b, d_A \sigma \rangle = 2 \sum_{i<j} \langle \nabla_{e_i} X, e_j \rangle \langle d_A \sigma \rangle_{e_i, e_j}.$$

(b) $d^* (\iota_X \varphi) = - (d^* \varphi)_X + \langle dX^b, \varphi \rangle$.

(c) $(\Delta \iota_X \varphi)_{e_j} = (\Delta \varphi)_{X, e_j} - \varphi_{e_i, R_{X, e_j, e_i}} - 2 \langle \nabla \varphi \rangle_{e_i, e_j, X, e_j}$.

(d) $(d^* d \iota_X \varphi)_{e_j} = (\Delta \varphi)_{X, e_j} + (dX d^* \varphi)_{e_j}$.

**Proof** Fix a point $p \in \Sigma$ and let $\{e_i\}$ be a local orthonormal frame near $p$ with $\nabla e_i = 0$ at $p$ for all $i$. To see (a), we start by computing (below we drop the subscripts in $\nabla_A$, $d_A$, etc.)

$$d^* d (\sigma_X) = - \nabla_{e_i} \nabla_{e_i} (\sigma_X) = - \nabla_{e_i} \left( (\nabla \sigma)_{e_i, X} + \sigma_{\nabla_{e_i} X} \right)$$

$$= - (\nabla^2 \sigma)_{e_i, e_i, X} - 2 \langle \nabla \sigma \rangle_{e_i, e_i, X} - \sigma_{\nabla^2_{e_i} e_i, X}$$

$$= (\nabla^* \nabla \sigma)_X - 2 \langle \nabla_{e_i} X, e_j \rangle \langle \nabla \sigma \rangle_{e_i, e_j} - \sigma_{\text{Ric}_{e_i, X} e_i}.$$
In getting the last line we wrote $\nabla_{e_i} X = \langle \nabla_{e_i} X, e_j \rangle e_j$ and also used the fact that, when $X$ is a Killing vector field, we have

$$\nabla^2_{V,W} X = RV_X W.$$

(The identity (2.1) will be frequently used in what follows, sometimes without further comment.) To continue, we use Proposition 2.1(a) to replace $\nabla^* \nabla \sigma$ and also note that

$$\sigma Rei_{X e_i} = -\sigma Ric(X).$$

Then we obtain

$$d^* d(\sigma X) = (\Delta \sigma)_X - Fei_{X} \sigma_{ei} - 2\langle \nabla_{e_i} X, e_j \rangle (\nabla \sigma)_{ei,e_j}.$$

We get (a) upon noting that since $X$ is a Killing vector field, $\langle \nabla V X, W \rangle$ is skew-symmetric in $V, W$, and hence we have

$$2 \sum_{i,j} \langle \nabla_{ei} X, e_j \rangle (\nabla \sigma)_{ei,e_j} = 2 \sum_{i<j} \langle \nabla_{ei} X, e_j \rangle (\nabla \sigma)_{ei,e_j} = (dX^b, d\sigma).$$

Of course in getting the last equality we used the fact that, at $p$,

$$(dX^b)_{ei,e_j} = \nabla_{ei} (X^b(e_j)) - \nabla_{ej} (X^b(e_i)) = 2\langle \nabla_{ei} X, e_j \rangle.$$

To see (b), we compute

$$d^*(\iota_X \varphi) = -\nabla_{ei} (\varphi X, e_i) = -\langle \nabla \varphi \rangle_{ei, X, ei} - \varphi \nabla_{ei} X, e_i$$

$$= \langle \nabla \varphi \rangle_{ei, ei} + \sum_{i,j} \langle \nabla_{ei} X, e_j \rangle \varphi_{ei,e_j}$$

$$= -(d^* \varphi)_X + 2 \sum_{i<j} \langle \nabla_{ei} X, e_j \rangle \varphi_{ei,e_j}$$

$$= -(d^* \varphi)_X + \langle dX^b, \varphi \rangle.$$

To see (c), we first note by part (b) that

$$dd^*(\iota_X \varphi) = -d(\iota X d^* \varphi + d \langle dX^b, \varphi \rangle)$$

$$= -L_X d^* \varphi + \iota_X \Delta \varphi + d \langle dX^b, \varphi \rangle,$$

where $L_X$ denotes the Lie derivative, and we’ve used the identity $L_X = d\iota_X + \iota_X d$ and the fact that $d\varphi = 0$ to get, respectively, the second and third lines. Now because $X$ is a Killing field, the operator $L_X$ commutes with $*$ and hence with $d^*$. Consequently,

$$L_X d^* \varphi = d^* L_X \varphi = d^*(d\iota_X + \iota_X d) \varphi = d^* \iota_X \varphi,$$

$$\square$$
Putting this back into (2.2) and rearranging yield
\[
\Delta (\iota_X \varphi) = \iota_X \Delta \varphi + d \langle d X^b, \varphi \rangle. \tag{2.4}
\]

For the last term on the right-hand side, we recall its definition and compute
\[
(d \langle d X^b, \varphi \rangle)_{e_j} = \nabla_{e_j} (\varphi_{e_i}, \nabla_{e_i} X)
= (\nabla \varphi)_{e_j, e_i, \nabla_{e_i} X, \varphi_{e_i}, \nabla_{e_j} X}
= -(\nabla \varphi)_{e_i, \nabla_{e_i} X, e_j} - (\nabla \varphi)_{e_j x, e_j, e_i} + \varphi_{e_i, R_{e_j, e_i}}.
\]

where the last line follows from (2.1) and the fact that \(d \varphi = 0\). The anti-symmetry of (\(\nabla X, \cdot\)) then implies that
\[
(\nabla \varphi)_{e_i, \nabla_{e_i} X, e_j} + (\nabla \varphi)_{e_j, \nabla_{e_j} X, e_i} = 2 (\nabla \varphi)_{e_i, \nabla_{e_i} X, e_j} = 2 (\nabla \varphi)_{e_i, \nabla_{e_j} X, e_j}.
\]

Therefore \((d \langle d X^b, \varphi \rangle)_{e_j} = -2 (\nabla \varphi)_{e_i, \nabla_{e_i} X, e_j} - \varphi_{e_i, R_{e_j, e_i}}\). Combining this with (2.4) finishes the proof of (c).

Finally, to see (d), we simply use (2.3) to write
\[
d^* d \iota_X \varphi = \mathcal{L}_X d^* \varphi = \iota_X dd^* \varphi + d \iota_X d^* \varphi,
\]
and note that \(dd^* \varphi = \Delta \varphi\) since \(\varphi\) is closed. \(\square\)

3 Review of Some Basic Facts About the Abelian Yang–Mills–Higgs and Vortex Equations

We begin by reviewing the first and second variation formulas of \(E_\varepsilon\). Let \(C\) to be the set of configurations \((u, \nabla_A)\) where \(u\) is a section of \(L\) of class \(L^\infty \cap W^{1,2}\) and \(\nabla_A = \nabla_0 - \sqrt{-1}A\) is a metric connection of class \(W^{1,2}\). (Recall that \(\nabla_0\) is our fixed reference connection on \(L\).) The latter means that the real 1-form \(A\) lies in \(W^{1,2}\). Note that we then have \(F_A = F_0 - \sqrt{-1}dA\).

Given \((u, \nabla_A) \in C\), and a pair \((v, a) \in \Omega^0(L) \times \Omega^1(S)\), recall that the first variation of \(E_\varepsilon\) is given by
\[
\delta E_\varepsilon (u, \nabla_A) (v, a) := \frac{d}{dt} \bigg|_{t=0} E_\varepsilon (u + tv, \nabla_A - t \sqrt{-1} a)
= \int_\Sigma 2 \varepsilon^2 \langle \sqrt{-1} F_A, da \rangle + 2 \text{Re} \langle \nabla_A u, \nabla_A v - \sqrt{-1} au \rangle
+ \frac{|u|^2 - 1}{\varepsilon^2} \text{Re} \langle u, v \rangle d \mu_g \tag{3.1}
\]
A configuration \((u, \nabla A) \in C\) is said to be a weak solution to \((1.1)\) if \(\delta E_\epsilon(u, \nabla A)(v, a) = 0\) for all \((v, a) \in \Omega^0(L) \times \Omega^1(\Sigma)\). Moreover, any weak solution to \((1.1)\) are locally gauge equivalent to a smooth solution. (See Proposition 3.3 below for a more precise statement.) Next we recall the second variation formula for \(E_\epsilon\), which for instance may be found in [16]. Assuming that \((u, \nabla A) \in C\) is a weak solution to \((1.1)\), we have:

\[
\delta^2 E_\epsilon(u, \nabla A)(v, a) := \frac{d^2}{dt^2} \bigg|_{t=0} E_\epsilon(u + tv, \nabla A - t \sqrt{1} a) \\
= \int_\Sigma 2\epsilon^2 |da|^2 + 2|d_A v|^2 - 4(a, u \times d_A v + v \times d_A u) + 2|u|^2 |a|^2 \\
+ \frac{|u|^2 - 1}{\epsilon^2} |v|^2 + \frac{2(\text{Re} \langle u, v \rangle)^2}{\epsilon^2} \mu_g. 
\tag{3.2}
\]

Here the cross product “\(\times\)” has the following meaning

\[
\xi \times \eta = \text{Re}(\sqrt{-1} \xi, \eta), \text{ for } \xi, \eta \in L_x \text{ and for all } x \in \Sigma. 
\]

The following fact justifies the term “cross product”:

\[
\text{Re}(\sqrt{-1} \xi, \eta) = - \text{Re}(\xi, \sqrt{-1} \eta) = - \text{Re}(\sqrt{-1} \eta, \xi). 
\tag{3.3}
\]

Polarizing \((3.2)\) and then formally integrating by parts, we get the following Jacobi operators \(J^1_{(u, \nabla A)}, J^2_{(u, \nabla A)}\), which already appeared for example in [16, Section 3]:

\[
J^1_{(u, \nabla A)}(v, a) := d^*_A d_A v + 2(a, \sqrt{-1} d_A u) - (d^*_a) \sqrt{-1} u + \frac{|u|^2 - 1}{2\epsilon^2} v + \frac{\text{Re} \langle u, v \rangle}{\epsilon^2} u. 
\tag{3.4}
\]

\[
J^2_{(u, \nabla A)}(v, a) := \epsilon^2 d^* da - u \times d_A v - v \times d_A u + |u|^2 a. 
\tag{3.5}
\]

These have the property that when \(u, \nabla, v, a\) are sufficiently regular, we have

\[
\delta^2 E_\epsilon(u, \nabla A)(v, a) = 2 \int_\Sigma \text{Re} \langle J^1_{(u, \nabla A)}(v, a), v \rangle + \langle J^2_{(u, \nabla A)}(v, a), a \rangle \mu_g. 
\]

**Definition 3.1** A solution \((u, \nabla A) \in C\) to \((1.1)\) is said to be stable if \(\delta^2 E_\epsilon(u, \nabla A)(v, a)\) as defined in \((3.2)\) is non-negative for any \((v, a) \in \Omega^0(L) \times \Omega^1(\Sigma)\).

**Remark 3.2** Note that if \((u, \nabla A)\) is stable, then in fact \(\delta^2 E_\epsilon(u, \nabla A)(v, a) \geq 0\) even if \(v\) is merely a section of \(L\) of class \(W^{1,2} \cap L^\infty\), and \(a\) is a \(1\)-form of class \(W^{1,2}\). This follows by inspecting the integrands in \((3.2)\) and noting that \(v\) and \(a\) can be smoothly approximated, respectively, in the \(W^{1,2} \cap L^p\) and \(W^{1,2}\) topology. (Here \(p < \infty\) is arbitrary.)

Next we give a more precise statement of the regularity of weak solutions mentioned before. The result is due to Taubes [32]. Let \(U \subset \Sigma\) be an open set, and assume that \(L\)
has a local, non-vanishing section over $U$, then we have a unitary trivialization of $L|_U$, under which sections are identified with complex-valued functions, and each metric connection can be written as $\nabla = d - \sqrt{-1}B$ for some real-valued 1-form $B$ on $U$.

**Proposition 3.3** [32, Proposition 4.1] Let $(u, \nabla) \in C$ be a weak solution to (1.1) on $\Sigma$ and let $U \subset \Sigma$ be a connected open set over which $L$ can be trivialized as above. Write $\nabla = d - \sqrt{-1}B$, and let $\theta \in W^{2,2}(\Sigma; \mathbb{R})$ be the unique solution to

\[
\begin{cases}
-d^*d\theta = d^*B \text{ in } U, \\
\theta = 0 \text{ on } \partial U.
\end{cases}
\]

Then $(e^{\sqrt{-1}\theta}u, \nabla - \sqrt{-1}d\theta)$ is a smooth solution to (1.1) on $U$.

Based on Proposition 3.3, one can in fact show that a weak solution $(u, \nabla) \in C$ is gauge-equivalent to a smooth solution over all of $\Sigma$. This was pointed out to the author by the reviewer of [11]. Specifically, we have

**Proposition 3.4** Let $(u, \nabla)$ be as in Proposition 3.3. There exists $\varphi \in W^{2,2}(\Sigma; \mathbb{R})$ such that $(e^{\sqrt{-1}\varphi}u, \nabla - \sqrt{-1}d\varphi)$ is a smooth solution to (1.1) on $\Sigma$.

**Proof** Take any $p \in \Sigma$ and concentric geodesic balls $B_r(p) \subset B_{4r}(p)$. Define $U_1 = B_{4r}(p)$ and $U_2 = \Sigma \setminus B_r(p)$. Then the line bundle $L$ is trivializable over both $U_1$ and $U_2$. ($U_1$ is contractible. Also, since we are on a compact surface, $U_2$ deformation retracts onto a wedge sum of finitely many circles, over which any complex line bundle is trivializable.) Hence we may apply Proposition 3.3 to get $\theta_i \in W^{2,2}(U_i; \mathbb{R})$ such that $(u_i, \nabla_i) := (e^{\sqrt{-1}\theta_i}u, \nabla - \sqrt{-1}d\theta_i)$ is smooth on $U_i$ for $i = 1, 2$. Now note that on $U_1 \cap U_2$, the 1-form

\[
d(\theta_1 - \theta_2) = \sqrt{-1}[(\nabla - \sqrt{-1}d\theta_1) - (\nabla - \sqrt{-1}d\theta_2)]
\]

is smooth, and hence the $W^{2,2}$-function $\theta_1 - \theta_2$, having distributional gradient that is smooth, is itself smooth on $U_1 \cap U_2$. To finish, let $\xi \in C_c^\infty(B_{3r}(p))$ be a cut-off function which is identically 1 on $B_{2r}(p)$, and define

\[
\varphi = \xi\theta_1 + (1 - \xi)\theta_2,
\]

Then we check that $(\tilde{u}, \tilde{\nabla}) := (e^{\sqrt{-1}\varphi}u, \nabla - \sqrt{-1}d\varphi)$ is equal to $(u_1, \nabla_1)$ on $B_{2r}(p)$, and to $(u_2, \nabla_2)$ on $\Sigma \setminus B_{3r}(p)$. It remains to check that $(\tilde{u}, \tilde{\nabla})$ is smooth on $U_1 \cap U_2$. To see that, note that on $U_1 \cap U_2 = B_{4r}(p) \setminus \overline{B_r(p)}$, we have

\[
(\tilde{u}, \tilde{\nabla}) = (e^{\sqrt{-1}\xi(\theta_1 - \theta_2)}u_2, \nabla_2 - \sqrt{-1}d(\xi(\theta_1 - \theta_2))),
\]

which is smooth on $U_1 \cap U_2$ since $(u_2, \nabla_2)$, $\xi$ and $\theta_1 - \theta_2$ all are. $\square$

Thanks to Proposition 3.4, from now on we can just work with smooth solutions $(u, \nabla_A)$ instead of weak solutions. Below, we recall some pointwise identities which are valid for solutions of (1.1). The proofs in the case $\Sigma = \mathbb{R}^2$ and $\varepsilon = 1$ are contained
in [17, Chapter III.6], and the general case requires no essential modification. For the reader’s convenience we sketch the argument for some of the parts.

**Proposition 3.5** (see also [17], Chapter III.6) Let \((u, \nabla A)\) be a smooth solution to (1.1) on \(\Sigma\). Then, with \(h, f, \sigma\) and \(\varphi\) defined as in the introduction, the following hold.

(a) \(\Delta h = \frac{1}{\varepsilon^2} |d_Au|^2 - \frac{1}{\varepsilon^2} |\nabla A|^2 \).

(b) \(\Delta f = \frac{1}{\varepsilon^2} * (d_Au \times d_Au) - \frac{1}{\varepsilon^2} |u|^2 f\).

(c) \(\Delta_A d_Au = -\frac{1}{\varepsilon^2} (d_Au, u) + \sqrt{-1} f (\ast d_Au) + h d_Au\).

(d) \(d^* \varphi = \frac{1}{\varepsilon^2} \text{Re} \langle \sqrt{-1} u, \sigma \rangle\).

(e) \(d_A \sigma = -\varphi \sqrt{-1} u\).

In (b), the 2-form \(d_Au \times d_Au\) is defined by

\[
(d_Au \times d_Au)_{V, W} = 2 (d_Au)_V \times (d_Au)_W = 2 \text{Re} \langle \sqrt{-1} (d_Au)_V, (d_Au)_W \rangle.
\]

**Proof** For the proofs of parts (a), (b), (c) we refer the reader to [17, Proposition III.6.1]. The remaining parts are immediate consequences of (1.1), but we include the proofs for completeness. To get (d), we recall that, since \(\Sigma\) is a surface, we have

\[d^* = - \ast d \ast\] for forms of any degree. \hspace{1cm} (3.7)

Now we have

\[
d^* \varphi = d^* \sqrt{-1} F_A - d^* \ast h
= \frac{1}{\varepsilon^2} \text{Re} \langle \sqrt{-1} u, d_Au \rangle + *d \ast h
= \frac{1}{\varepsilon^2} \text{Re} \langle \sqrt{-1} u, d_Au \rangle + \ast d h
= \frac{1}{\varepsilon^2} \text{Re} \langle \sqrt{-1} u, d_Au \rangle - \frac{1}{2\varepsilon^2} (\langle u, d_Au \rangle + \langle d_Au, u \rangle).
\]

In getting the second line we used (1.1) and (3.7). To continue, we have

\[
\frac{1}{\varepsilon^2} \text{Re} \langle \sqrt{-1} u, d_Au \rangle - \frac{1}{2\varepsilon^2} (\langle u, d_Au \rangle + \langle d_Au, u \rangle)
= \frac{1}{2\varepsilon^2} (\langle \sqrt{-1} u, d_Au \rangle + \langle d_Au, \sqrt{-1} u \rangle) - \frac{1}{2\varepsilon^2} (\langle u, \ast d_Au \rangle + \langle \ast d_Au, u \rangle)
= \frac{1}{2\varepsilon^2} (\langle \sqrt{-1} u, d_Au \rangle + \langle d_Au, \sqrt{-1} u \rangle)
- \frac{1}{2\varepsilon^2} (\langle \sqrt{-1} u, \sqrt{-1} \ast d_Au \rangle + \langle \sqrt{-1} \ast d_Au, \sqrt{-1} u \rangle).
\]

The last line follows because the metric on \(L\) is Hermitian. Combining the two strings of computations above, simplifying, and recalling the definition of \(\sigma\), we get

\[
d^* \varphi = \frac{1}{2\varepsilon^2} (\langle \sqrt{-1} u, \sigma \rangle + \langle \sigma, \sqrt{-1} u \rangle),
\]
which is exactly what we want to prove.

To prove (e), note that

\[
\begin{align*}
  d_A \sigma &= d_A (d_A u - \sqrt{-1} \ast d_A u) \\
  &= FAu - \sqrt{-1} d_A \ast d_A u \\
  &= FAu + \sqrt{-1} \ast d_A^* d_A u,
\end{align*}
\]

where, as in the proof of (d), in the last line we used the fact that $d_A^* = - \ast d_A^*$. Using (1.1), we may continue the computation

\[
FAu + \sqrt{-1} \ast d_A^* d_A u = FAu + \sqrt{-1}(\ast h)u = -\sqrt{-1}(\sqrt{-1}FA - \ast h)u,
\]

and (e) is proved.

From the identities above we deduce the following properties for solutions to (1.1) which help us detect when they are solutions to the vortex equations.

**Proposition 3.6** Let $(u, \nabla A)$ be a smooth solution to (1.1) on $\Sigma$ with $u \not\equiv 0$. Then

(a) (See also [17, Lemma III.8.4]) We have $\pm f \leq h$. Moreover, if equality is achieved at some point on $\Sigma$ then the two sides are identically equal on $\Sigma$.

(b) (See also [17, p. 97]) $f = h$ ($f = -h$, resp.) if and only if $d_A u = \sqrt{-1} \ast d_A u$ ($d_A u = -\sqrt{-1} \ast d_A u$, resp.).

**Proof** As in the proof of [17, Lemma III.8.4], from Proposition 3.5(a), (b) we deduce that

\[
\Delta(\pm f - h) + \frac{|u|^2}{\varepsilon^2}(\pm f - h) \leq 0.
\]

The first conclusion of part (a) now follows immediately from multiplying both sides with $(\pm f - h)_+$, integrating by parts, and recalling that $u \not\equiv 0$. The second conclusion then follows from the strong maximum principle.

For part (b), we recall the following two additional identities.

\[
|d_A u|^2 = \left| \frac{d_A u - \sqrt{-1} \ast d_A u}{2} \right|^2 + \left| \frac{d_A u + \sqrt{-1} \ast d_A u}{2} \right|^2.
\]

(3.8)

\[
\ast(d_A u \times d_A u) = \left| \frac{d_A u + \sqrt{-1} \ast d_A u}{2} \right|^2 - \left| \frac{d_A u - \sqrt{-1} \ast d_A u}{2} \right|^2.
\]

(3.9)

Both can be easily verified by direct computation. For instance, to get (3.9), we expand the right-hand side to get

\[
\left| \frac{d_A u + \sqrt{-1} \ast d_A u}{2} \right|^2 - \left| \frac{d_A u - \sqrt{-1} \ast d_A u}{2} \right|^2 = \text{Re}(d_A u, \sqrt{-1} \ast d_A u)
\]
Fixing \( p \in \Sigma \) and letting \( e_1, e_2 \) be an orthonormal basis for \( T_p \Sigma \), we have
\[
(*dAu)_{e_1} = -(dAu)_{e_2}, \quad (*dAu)_{e_2} = (dAu)_{e_1}.
\]
Hence we find that
\[
\text{Re} \langle dAu, \sqrt{-1} * dAu \rangle = - \text{Re} \langle (dAu)_{e_1}, \sqrt{-1}(dAu)_{e_2} \rangle + \text{Re} \langle (dAu)_{e_2}, \sqrt{-1}(dAu)_{e_1} \rangle
\]
\[
= 2 \text{Re} \langle \sqrt{-1}(dAu)_{e_1}, (dAu)_{e_2} \rangle
\]
\[
= (dAu \times dAu)_{e_1,e_2} = *(dAu \times dAu),
\]
where in getting the second line we used (3.3). This proves (3.9).

Continuing with part (b), that \( f = h \) implies \( dAu = \sqrt{-1} * dAu \) now follows, as in p.97 of [17], from Proposition 3.5(a), (b) along with (3.8) and (3.9). For the converse, assume \( dAu = \sqrt{-1} * dAu \). Then by (3.8) and (3.9) we have \( |dAu|^2 = *(dAu \times dAu) \), which by Proposition 3.5 implies that
\[
\left( \Delta + \frac{|u|^2}{\varepsilon^2} \right) (f - h) = 0.
\]
Recall by part (a) that \( f - h \leq 0 \), and hence we get \( \Delta (f - h) \geq 0 \). Since \( \Sigma \) is closed, this means \( f - h \) is constant, and hence
\[
|u|^2 (f - h) = 0,
\]
which forces \( f - h \) to vanish identically since, by assumption, \( |u| \) is not identically zero.

For the reader’s convenience, we close this section by briefly recalling how to derive (1.3) for closed \( \Sigma \). For simplicity we assume \( \varepsilon = 1 \), as the computation is the same for other cases. The first equality of (1.3) is straightforward. As for the second equality, it suffices to show that
\[
\int_{\Sigma} \frac{1}{4} |dAu + \sqrt{-1} * dAu|^2 + \frac{1}{2} |f + h|^2 d\mu_g
\]
\[
- \int_{\Sigma} \frac{1}{4} |dAu - \sqrt{-1} * dAu|^2 + \frac{1}{2} |f - h|^2 d\mu_g = \int_{\Sigma} \sqrt{-1} F_A.
\]
To that end, note that
\[
\frac{1}{2} \int_{\Sigma} |f + h|^2 d\mu_g - \frac{1}{2} \int_{\Sigma} |f - h|^2 d\mu_g = \int_{\Sigma} (1 - |u|^2) \sqrt{-1} F_A \tag{3.10}
\]
Combining this with (3.9) gives

\[
\int_{\Sigma} \frac{1}{4} |d_A u + \sqrt{-1} \ast d_A u|^2 + \frac{1}{2} |f + h|^2 d\mu_g \\
- \int_{\Sigma} \frac{1}{4} |d_A u - \sqrt{-1} \ast d_A u|^2 + \frac{1}{2} |f - h|^2 d\mu_g \\
= \int_{\Sigma} d_A u \times d_A u - \sqrt{-1} |u|^2 F_A + \int_{\Sigma} \sqrt{-1} F_A \\
= \int_{\Sigma} d(u \times d_A u) + \int_{\Sigma} \sqrt{-1} F_A = \int_{\Sigma} \sqrt{-1} F_A.
\]

In the second-to-last equality we used the identity

\[
d(u \times d_A u) = d_A u \times d_A u - \sqrt{-1} |u|^2 F_A,
\]

and the last equality follows because \( \Sigma \) is closed by assumption.

### 4 Smooth Stable Solutions of the Abelian Yang–Mills–Higgs Equations

Throughout this section we assume that \( \Sigma \) is either the round \( S^2 \) or a flat \( T^2 \), and that \((u, \nabla_A)\) is a smooth solution to (1.1) on all of \( \Sigma \). Moreover, we write \( h, f, \sigma \) and \( \varphi \), respectively, for \( h(u, \nabla_A), f(u, \nabla_A), \sigma(u, \nabla_A) \) and \( \varphi(u, \nabla) \), whose definitions we recall below.

\[
h(u, \nabla_A) = \frac{1 - |u|^2}{2\varepsilon^2},
\]

\[
f(u, \nabla_A) = *\sqrt{-1} F_A,
\]

\[
\sigma(u, \nabla_A) = d_A u - \sqrt{-1} \ast d_A u,
\]

\[
\varphi(u, \nabla_A) = \sqrt{-1} F_A - *h = *(f - h).
\]

Next, we define the following real quadratic form defined over the space of smooth vector fields \( X \) on \( \Sigma \).

\[
Q(X) := \delta^2 E_\varepsilon(u, \nabla_A)(\sigma_X, \iota_X \varphi).
\]

As in [5, Section 10], the key step to proving Theorem 1.1 consists in computing the trace of \( Q \) restricted to the space \( K \) of Killing vector fields. We begin with the following lemma.

**Lemma 4.1** Let \( X \in K \). Then the following hold.

(a) \( J^1_{(u, \nabla_A)}(\sigma_X, \iota_X \varphi) = -2 f (d_A u)_X + 2 h \sqrt{-1} (\ast d_A u)_X \).

(b) \( J^2_{(u, \nabla_A)}(\sigma_X, \iota_X \varphi) = -|d_A u|^2 \iota_X (\ast 1) - 2 \text{Re}(\ast d_A u)_X, d_A u). \)
Proof We start with (a). First, by Proposition 3.5(c) and the fact that $\sqrt{-1} \ast$ commutes with $\Delta_1$, we have
\[
\Delta_1 \sigma = -\frac{1}{\varepsilon^2} \langle \sigma, u \rangle u + h \sigma + f \sqrt{-1} \ast \sigma,
\]
and consequently by Lemma 2.2(a) we see that
\[
\Delta_1 (\sigma X) = -\frac{1}{\varepsilon^2} \langle \sigma X, u \rangle u + h \sigma X + 2 f (\sqrt{-1} \ast \sigma)_X - \langle d X^b, d A \sigma \rangle,
\]
where in getting the second line we used the fact that
\[
f \sqrt{-1} (\ast \sigma) = -(F_A)_{e_1, e_2} (\ast \sigma) = -(F_A)_{e_i} \sigma_{e_i}.
\]
Adding $\frac{|u|^2 - 1}{2 \varepsilon^2} \sigma_X + \frac{\Re \langle u, \sigma_X \rangle}{\varepsilon^2} u$ to both sides of (4.1) gives
\[
\Delta_1 (\sigma X) - h \sigma_X + \frac{\Re \langle u, \sigma_X \rangle}{\varepsilon^2} u
\]
\[
= -\frac{1}{\varepsilon^2} \langle \sigma X, u \rangle u + \frac{1}{2 \varepsilon^2} (\langle u, \sigma_X \rangle - \langle \sigma_X, u \rangle)u + 2 f (\sqrt{-1} \ast \sigma)_X - \langle d X^b, d A \sigma \rangle
\]
\[
= \frac{1}{2 \varepsilon^2} (\langle u, \sigma_X \rangle - \langle \sigma_X, u \rangle)u + 2 f (\sqrt{-1} \ast \sigma)_X - \langle d X^b, d A \sigma \rangle.
\]
To see what $J^1_{(u, \nabla_A)} (\sigma X, i_X \varphi)$ actually is, we still need to compute $2 \langle i_X \varphi, \sqrt{-1} d_A u \rangle - (d^* i_X \varphi) \sqrt{-1} u$. To that end, we recall Proposition 3.5(d), which together with Lemma 2.2(b) gives
\[
d^* \varphi = \frac{1}{\varepsilon^2} \Re \langle \sqrt{-1} u, \sigma \rangle = \frac{\sqrt{-1}}{2 \varepsilon^2} \left( \langle u, \sigma \rangle - \langle \sigma, u \rangle \right).
\]
\[
\implies - (d^* i_X \varphi) \sqrt{-1} u = \frac{1}{2 \varepsilon^2} (\langle \sigma X, u \rangle - \langle u, \sigma X \rangle)u - \langle d X^b, \varphi \rangle \sqrt{-1} u.
\]
Putting everything together, we arrive at
\[
J^1_{(u, \nabla_A)} (\sigma X, i_X \varphi) = 2 f (\sqrt{-1} \ast \sigma)_X - \langle d X^b, d A \sigma + \varphi \sqrt{-1} u \rangle + 2 \langle i_X \varphi, \sqrt{-1} d_A u \rangle.
\]
To finish the proof of (a), we note that the second term on the right-hand side vanishes because of Proposition 3.5(e). Furthermore, picking an orthonormal frame $\{e_i\}$ on $\Sigma$, we compute
\[
\langle i_X \varphi, \sqrt{-1} d_A u \rangle = \varphi_{X, e_i} \sqrt{-1} (d_A u)_{e_i} = (f - h) (\ast 1)_{X, e_i} \sqrt{-1} (d_A u)_{e_i}
\]
\[
= - \sqrt{-1} (f - h) (\ast d_A u)_X.
\]
Substituting this back into (4.4), recalling the definition of $\sigma$ and observing a cancellation, we obtain

$$J^1_{(u, \nabla_A)}(\sigma_X, \iota_X \varphi) = 2\sqrt{-1} h (\ast d_A u)_X - 2 f (d_A u)_X,$$

as asserted.

To prove (b), we first note by Proposition 3.5(a), (b) we have

$$\Delta \varphi = \ast \Delta (f - h) = \frac{1}{\varepsilon^2} d_A u \times d_A u - \frac{|d_A u|^2}{\varepsilon^2} (\ast 1) - \frac{|u|^2}{\varepsilon^2} \varphi.$$

Thus by Lemma 2.2(d), we have

$$\varepsilon^2 (d^* d \iota_X \varphi)_{e_j} + |u|^2 (\iota_X \varphi)_{e_j} = (d_A u \times d_A u)_{X, e_j} - |d_A u|^2 (\ast 1)_{X, e_j} + \varepsilon^2 (d \iota_X d^* \varphi)_{e_j}. \quad (4.5)$$

By Proposition 3.5(d) and the Leibniz rule, we compute

$$\varepsilon^2 (d \iota_X d^* \varphi)_{e_j} = \text{Re} (\sqrt{-1} (d_A u)_{e_j}, \sigma_X) + \text{Re} (\sqrt{-1} u, (d_A (\sigma_X))_{e_j}).$$

Plugging this back into (4.5) and combining $\text{Re} (\sqrt{-1} (d_A u)_{e_j}, \sigma_X)$ with $(d_A u \times d_A u)_{X, e_j}$, we get

$$\varepsilon^2 (d^* d \iota_X \varphi)_{e_j} + |u|^2 (\iota_X \varphi)_{e_j} = \text{Re} (\sqrt{-1} (d_A u + \sqrt{-1} * d_A u)_{X, (d_A u)_{e_j}} - |d_A u|^2 (\ast 1)_{X, e_j}$$

$$+ \text{Re} (\sqrt{-1} u, (d_A (\sigma_X))_{e_j}).$$

Next, noting that

$$(u \times d_A (\sigma_X) + \sigma_X \times d_A u)_{e_j} = \text{Re} (\sqrt{-1} u, (d_A (\sigma_X))_{e_j}) + \text{Re} (\sqrt{-1} \sigma_X, (d_A u)_{e_j}),$$

and recalling (3.5), we arrive at the formula asserted in (b). Namely,

$$\left( J^2_{(u, \nabla_A)}(\sigma_X, \iota_X \varphi) \right)_{e_j} = -2 \text{Re} ((\ast d_A u)_{X, (d_A u)_{e_j}} - |d_A u|^2 (\ast 1)_{X, e_j}.$$

The proof of Lemma 4.1 is now complete. \qed

Now we’d like to use Lemma 4.1 to compute the trace over $\mathcal{K}$ of the quadratic form $Q$ defined at the beginning of the section, and hence we need to fix an appropriate inner product on $\mathcal{K}$. To that end we use the fact that if $\Sigma = S^2$ or $T^2$, an inner product on $\mathcal{K}$ can be chosen so that at each $x \in \Sigma$ there exists an orthonormal basis $X_1, \ldots, X_q$ of $\mathcal{K}$ such that $X_1(x), X_2(x)$ form an orthonormal basis for $T_x \Sigma$, and $X_1(x) = 0$ for
\[ (V, W)_{\mathcal{K}} := \langle V(x), W(x) \rangle \text{ for } V, W \in \mathcal{K}, \]

the choice of \( x \in T^2 \) being irrelevant because \( V, W \) are parallel. On the other hand, if \( \Sigma = S^2 \), then \( \mathcal{K} \) is isomorphic to \( \mathfrak{so}(3) \) and hence gets an induced inner product from the pairing \( \langle A, B \rangle = \frac{1}{2} \text{tr}(AB^T) \) on the latter. To see that this latter choice of inner product has the property described above, it suffices, by homogeneity, to find the desired \( X_1, X_2, X_3 \) at the point \((0,0,1)\). Recalling that the isomorphism between \( \mathfrak{so}(3) \) and \( \mathcal{K} \) associates to a given \( A \in \mathfrak{so}(3) \) the restriction to \( S^2 \) of the vector field \( x \mapsto Ax \), it’s not hard to see that we can let \( X_1, X_2 \) and \( X_3 \) be, respectively, the Killing fields corresponding to
\[
\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},
\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}
\text{ and }
\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

**Proposition 4.2** Let \( X_1, \ldots, X_q \) be any orthonormal basis for \( \mathcal{K} \). Then we have
\[
\sum_{i=1}^{q} \left[ \text{Re} \langle J_{u,v}^1(\sigma X_i, \iota X_i \phi), \sigma X_i \rangle + \langle J_{u,v}^2(\sigma X_i, \iota X_i \phi), \iota X_i \phi \rangle \right] = -(f + h)|\sigma|^2,
\]

(4.6)
as functions on \( \Sigma \).

**Proof** Fix an arbitrary \( x \in \Sigma \) and let \( X_1, \cdots X_q \) be an orthonormal basis for \( \mathcal{K} \) such that \( e_1 := X_1(x), e_2 := X_2(x) \) is an orthonormal basis for \( T_x \Sigma \), and \( X_i(x) = 0 \) for \( i > 2 \). Observe that, to prove the Proposition, it suffices to verify (4.6) for this particular choice of the \( X_i \)'s. This is because the left-hand side of (4.6) as a function on \( \Sigma \) is invariant when we change to another orthonormal basis for \( \mathcal{K} \), since \( J_{u,v}^1 \) and \( J_{u,v}^2 \) are linear operators.

Now, by our choice of the \( X_i \)'s, and the fact that they are Killing fields, we can apply Lemma 4.1 to see that the left-hand side of (4.6) evaluated at \( x \) is equal to
\[
2h \text{Re} \langle \sqrt{-1} \ast d_A u, \sigma \rangle - 2f \text{Re} \langle d_A u, \sigma \rangle - (f - h)(\ast 1)_{e_i,e_j} \left[ |d_A u|^2 (\ast 1)_{e_i,e_j} \right] + 2 \text{Re} \langle (\ast d_A u)_{e_i}, (d_A u)_{e_j} \rangle
\]
\[
= \left[ 2h \text{Re} \langle \sqrt{-1} \ast d_A u, \sigma \rangle - 2f \text{Re} \langle d_A u, \sigma \rangle \right] .
\]

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where in getting the last line we made a cancellation with the help of the following identities:

\[
2 \sum_{i,j=1}^{2} (*1)_{e_i,e_j} \text{Re}\langle(*d_A u)_{e_i}, (d_A u)_{e_j}\rangle = -|d_A u|^2;
\]

\[
\sum_{i,j=1}^{2} (*1)_{e_i,e_j} (*1)_{e_i,e_j} = 2.
\]

The first identity can be checked by recalling that, from the definition of the Hodge star operator, we have

\[
(*d_A u)_{e_1} = -(d_A u)_{e_2},
\]

\[
(*d_A u)_{e_2} = (d_A u)_{e_1}.
\]

Recalling the definition of \(\sigma\) and expanding the inner products, we get

\[
2h \text{Re}\langle\sqrt{-1} * d_A u, \sigma\rangle - 2f \text{Re}\langle d_A u, \sigma\rangle
\]

\[
= (f + h)(2 \text{Re}\langle\sqrt{-1} * d_A u, d_A u\rangle - 2|d_A u|^2) = -(f + h)|\sigma|^2,
\]

as asserted. \(\square\)

**Corollary 4.3** We have

\[
\text{tr}_{\mathcal{K}} Q = -\int_{\Sigma} (f + h)|\sigma|^2 d\mu_g.
\]

**Proof** By definition of \(Q\), we have

\[
\text{tr}_{\mathcal{K}} Q = \int_{\Sigma} \sum_{i=1}^{q} \left[ \text{Re}\langle J_{(u,\nabla_A)}^{1}(\sigma X_i, iX_i\varphi), \sigma X_i \rangle + \langle J_{(u,\nabla_A)}^{2}(\sigma X_i, iX_i\varphi), iX_i\varphi \rangle \right] d\mu_g,
\]

where \(X_1, \ldots, X_q\) is an orthonormal basis for \(\mathcal{K}\). The result now follows from Proposition 4.2. \(\square\)

**Proof of Theorem 1.1** We first deal with the case where (H) is assumed. By Proposition 3.4 we may assume that \((u, \nabla_A)\) is smooth. Since \((u, \nabla_A)\) is stable by assumption, from Corollary 4.3 we have

\[
\int_{\Sigma} (f + h)|\sigma|^2 d\mu_g = -\text{tr}_{\mathcal{K}} Q \leq 0.
\]

By Proposition 3.6(a), we have that \(f + h \geq 0\), so the above inequality implies that

\[
(f + h)|\sigma|^2 = 0 \text{ everywhere on } \Sigma.
\]

Now recall, again by Proposition 3.6(a), that \(f + h\) is either identically zero or everywhere positive. In the former case, we are done by Proposition 3.6(b). In the latter case, we get \(\sigma \equiv 0\) and again we are done by Proposition 3.6(b).
To prove Theorem 1.1 assuming (H’) instead, we need only consider the case \( u \equiv 0 \), since otherwise we reduce to the previous case. Note that since \( u \equiv 0 \), we have from (1.1) that

\[
d^* F_A = 0.
\]

Consequently \( \sqrt{-1} F_A \) is a real harmonic 2-form on \( \Sigma \) and hence a constant multiple of \( d \mu_g \). Recalling (1.4), we see that we must have

\[
* \sqrt{-1} F_A = \frac{d}{2}.
\]

(4.7)

In particular, in the case \( |d| = \varepsilon^{-2} \), since \( u \equiv 0 \), we have

\[
* \sqrt{-1} F_A = \pm \frac{1}{2 \varepsilon^2} = \pm \frac{1 - |u|^2}{2 \varepsilon^2}.
\]

Hence \((u, \nabla A)\) solves (1.5) or (1.6) by Proposition 3.6(b).

On the other hand, if \( |d| < \varepsilon^{-2} \), then \( u \equiv 0 \) contradicts stability and thus cannot occur. Indeed, observe that in this case the second variation formula (3.2) gives

\[
\delta^2 E_\varepsilon(0, \nabla A)(v, 0) = \int_{S^2} 2|d_A v|^2 - \frac{|v|^2}{\varepsilon^2}.
\]

(4.8)

Now because \( \Sigma = S^2 \) and because the connection satisfies (4.7), we can invoke [20, Theorem 5.1] to see that

\[
\inf \left\{ \int_{S^2} |d_A v|^2 d \mu_g \mid v \in \Omega^0(L) \mid \int_{S^2} |v|^2 d \mu_g = 1 \right\} = \frac{|d|}{2}.
\]

(4.9)

Indeed, (4.7) implies that the connection \( A \) coincides with the connection defined at the top of [20, p. 9] with \( m = -d \). Hence [20, Theorem 5.1] tells us that the eigenvalues of \( d_A^* d_A \) acting on sections of \( L \) are given by

\[
\left( \frac{|d|}{2} + k \right) \left( \frac{|d|}{2} + k + 1 \right) - \frac{d^2}{4} = \frac{|d|}{2} (2k + 1) + k(k + 1), k = 0, 1, 2, \ldots
\]

In particular, the lowest eigenvalue is \( \frac{|d|}{2} \), giving us (4.9), which together with (4.8) and the assumption \( |d| < \varepsilon^{-2} \) shows that \((0, \nabla_A)\) is unstable, a contradiction. \( \square \)

**Remark 4.4** Here we clarify Remark 1.2(2). Note that if (4.7) holds for \( \nabla A \), then \((0, \nabla_A)\) solves (1.1). Moreover, since \( \Sigma = S^2 \), we get (4.9) thanks to [20]. By (3.2), this implies that for all \((v, a) \in \Omega^0(L) \times \Omega^1(S^2)\), we have
\[
\delta^2 E_\varepsilon(0, \nabla_A)(v, a) = \int_{S^2} 2\epsilon^2 |da|^2 + 2|d_A v|^2 - \frac{|v|^2}{\epsilon^2} \geq \int_{S^2} 2|d_A v|^2 - \frac{|v|^2}{\epsilon^2} \\
\geq \left(|d| - \frac{1}{\epsilon^2}\right) \int_{S^2} |v|^2 \geq 0,
\]

provided \(|d| \geq \varepsilon^{-2}\). In other words, \((0, \nabla_A)\) is stable when \(|d| \geq \varepsilon^{-2}\). However, for \((0, \nabla_A)\) to be a solution to (1.5) or (1.6), we must have \(|d| = \varepsilon^{-2}\). Thus, there exist stable solutions to (1.1) on \(S^2\) which do not satisfy either of the vortex equations when \(|d| > \varepsilon^{-2}\), because we can certainly find a connection on \(L\) whose curvature satisfies (4.7), for example by looking at the Hodge decomposition of \(F_0\), the curvature of the background connection \(\nabla_0\). In fact, since \(\Sigma = S^2\), on each Hermitian line bundle there is a unique such connection up to gauge transformations [12, Proposition 2.2.6].

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