ADAPTIVELY ENRICHED COARSE SPACE FOR THE DISCONTINUOUS GALERKIN MULTISCALE PROBLEMS

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Abstract. In this paper, we propose a two level overlapping additive Schwarz domain decomposition preconditioner for the symmetric interior penalty discontinuous Galerkin method for the second order elliptic boundary value problem with highly heterogeneous coefficients. An extraordinary feature of this preconditioner is that it is based on adaptively enriching the coarse space with functions created from solving generalized eigenvalue problems on a set of thin patches covering the subdomain interfaces. It is shown that the condition number of the underlined preconditioned system is independent of the contrast if adequate number of functions are used to enrich the coarse space. Numerical results are provided to confirm this claim.

Key words. Multiscale problem, Discontinuous Galerkin method, Domain decomposition preconditioner, Multiscale finite element, Generalized eigenvalue problem

AMS subject classifications. 68Q25, 68R10, 68U05

1. Introduction. We consider the symmetric interior penalty discontinuous Galerkin (SIPG) discretization of the second order elliptic boundary value problem with highly heterogeneous coefficients, representing for instance the permeability of a porous media in a reservoir simulation, and we propose a new additive Schwarz preconditioner for the iterative solution of the resulting system. It is already known in the community that standard domain decomposition methods in general have trouble dealing with the heterogeneity in a robust manner, particularly when the coefficient varies highly along subdomain boundaries. This may be explained by saying that a standard coarse space is not rich enough to capture all the worst modes in the residual, and therefore requires some form of enrichment. The main objective of this paper is to propose a new coarse space for the discontinuous Galerkin method, an adaptively enriched coarse space with functions corresponding to the bad eigenmodes of certain generalized eigenvalue problems on a set of thin patches covering the subdomain interfaces, resulting in a convergence which is independent of the variations in the coefficient.

The SIPG is a symmetric version of the discontinuous Galerkin (DG), a methodology, which in the recent years become increasingly popular in the scientific computing community. In contrast to the classical conforming and nonconforming techniques, the DG methods allow for the finite element functions to be totally discontinuous across the element boundaries, thereby allowing for additional flexibilities with regards to using irregular meshes, local mesh refinement, and different polynomial degrees for the basis functions on different elements, cf. e.g. [32, 7, 6, 1, 10]. They are also preferred when dealing with models based on the laws of conservation, because such methods are locally mass conservative, where as the classical methods only preserve a global mass balance. For an introduction to the DG methodology cf. [32], and for an overview of recent developments in the field cf. for instance [14, 7, 10]. The systems resulting from the SIPG discretization are symmetric and positive definite, and in general very large. Krylov iterations, like the conjugate gradients (CG),

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Additive Schwarz preconditioners are among the most popular preconditioners based on the domain decomposition, which are inherently parallel and are easy to implement, cf. [36, 24, 33, 31]. However, for most algorithms that exist to date, it is assumed that the coefficients are either constants or piecewise constants with respect to some partition of the domain, that is they may only have jumps across subdomain boundaries, in which case the algorithms are robust with respect to the jumps, cf. e.g. [11, 36, 24] and references therein. In case of multiscale problems, where the coefficients may vary rapidly and everywhere, in particular when the coefficients vary along subdomain boundaries, such methods fail to be robust in the general, cf. e.g. [17, 30, 28]. The preconditioner proposed in this paper is based on the abstract Schwarz framework, where nonoverlapping subdomains are used for the local subproblems, and an adaptively enriched coarse space for the coarse problem. Starting with a standard coarse space, the coarse space is enriched with functions built through solving a set of generalized eigenvalue problems on a set of thin patches each covering a subdomain interface, and including those functions that correspond to the eigenvalues that are below a given threshold.

Spectral enrichment of coarse spaces to construct robust domain decomposition methods has started to attract much interest in the recent years, focusing on the effectiveness and the optimality, cf. e.g. [15, 12, 25, 26, 21, 35, 16, 8, 9, 13] for recent applications of the approach to second order elliptic problems with heterogeneous coefficient. There also many results for FETI-DP and BDDC substructuring domain decomposition methods where this idea is used for constructing the coarse space (the primal constraints), cf. [19, 21, 22, 23, 34] in 2D and [5, 18, 20, 27, 29] in 3D.

The present work is a step in this direction, and is based on the idea of solving lower dimensional eigenvalue problems, that is 1D eigenvalue problems in the 2D and 2D eigenvalue problems in the 3D, and then appropriately extending the eigenfunctions inside in order to be included into the coarse space. This idea was first proposed for the additive Schwarz method in [16] for 2D problems, and later extended to 3D problems in [13], using continuous and piecewise linear finite elements. The present paper is an extension of the idea to the discontinuous Galerkin case, proposing a new multiscale finite element space for the DG method, inspired by the work in [17, 9, 15], and solving a set of relatively small sized eigenvalue problems on thin patches covering the subdomain interfaces in order to deal with the discontinuity of the DG functions.

The rest of the paper is organized as follows. The discrete formulation of the problem based on the SIPG is given in section 2, in section 3 we introduce the new additive Schwarz preconditioner for the discrete problem. The convergence analysis is given in section 4, while the numerical results are given in section 5.

Throughout this paper, we use the following notations: $x \lesssim y$ and $w \gtrsim z$ denote that there exist positive constants $c, C$ independent of mesh parameters and the jump of coefficients such that $x \leq cy$ and $w \geq Cz$, respectively.

2. Discrete problem. We consider the following linear variational problem: Find $u^* \in H_0^1(\Omega)$ such that

$$a(u^*, v) = f(v), \quad \forall v \in H_0^1(\Omega),$$

where $a(u, v) = \int_{\Omega} \alpha \nabla u \nabla v$ is a symmetric and positive definite bilinear form for all $u, v \in H_0^1(\Omega)$. Here $\alpha \in L^\infty(\Omega)$ is a positive function, and there exists $C_0 > 0$ such that $\alpha(x) \geq C_0$. For simplicity, since $\alpha$ can be scaled by $C_0^{-1}$, we assume that $\alpha \geq 1$,
where, however, may be highly varying.

We assume that there exists a sequence of quasi-uniform triangulations (cf. [3, 2]) of \( \Omega \), \( \mathcal{T}_h = \mathcal{T}_h(\Omega) = \{ \tau \} \), consisting of triangles indexed by the parameter \( h = \max_{\tau \in \mathcal{T}_h} \text{diam}(\tau) \) tending to zero. Let \( \mathcal{E}_h \) denote the set of all edges of the triangles of \( \mathcal{T}_h \), which can be split into two subsets, \( \mathcal{E}_h^\partial \), the set consisting of edges that are on the boundary \( \partial \Omega \), and \( \mathcal{E}_h^0 \), the set consisting of edges that are in the interior of \( \Omega \). Let \( \mathcal{V}_h \) denote the set of all vertices of the triangles of \( \mathcal{T}_h \), and \( \mathcal{V}_h(\tau) \) the set of vertices of the triangle \( \tau \).

For each edge \( e \in \mathcal{E}_h^\partial \), the common edge between two neighboring triangles \( \tau_+ \) and \( \tau_- \), we introduce the following weights, cf. [4],

\[
\omega^e_+ = \alpha_-/(\alpha_+ + \alpha_-) \quad \text{and} \quad \omega^e_- = \alpha_+/(\alpha_+ + \alpha_-),
\]

where \( \alpha_+ \) and \( \alpha_- \) are the restrictions of \( \alpha \) to \( \tau_+ \) and \( \tau_- \), respectively. Note that

\[
\omega^e_+ + \omega^e_- = 1.
\]

Further, we introduce the following notations. For each \( e \subset \mathcal{E}_h^\partial \), define

\[
(1) \quad [u] = u_+ \cdot n_+ + u_- \cdot n_- \quad \text{and} \quad \{ u \} = \omega^e_+ \cdot u_+ + \omega^e_- \cdot u_-,
\]

where \( u_+ \) and \( u_- \) are the traces of \( u|_{\tau_+} \) and \( u|_{\tau_-} \) on \( e \), respectively, while \( n_+ \) and \( n_- \) are the unit outer normal to \( \partial \tau_+ \) and \( \partial \tau_- \), respectively. And, for each edge \( e \subset \mathcal{E}_h^\partial \), with \( e \) being an edge of some \( \tau \), we define

\[
(2) \quad [u] = u \cdot n \quad \text{and} \quad \{ u \} = u,
\]

where \( n \) is the unit outer normal to \( e \subset \partial \Omega \), and \( u \) is the trace of \( u|_\tau \) onto \( e \).

We define the \( L^2 \) inner products over the elements and the edges respectively as follows,

\[
(u, v)_{\mathcal{T}_h} = \sum_{\tau \in \mathcal{T}_h} (u, v)_{\tau} \quad \text{and} \quad (u, v)_{\mathcal{E}_h} = \sum_{e \in \mathcal{E}_h} (u, v)_e
\]

for \( u, v \in L^2(\Omega) \).

We consider a family of standard piecewise linear finite element spaces \( V^h \subset L^2(\Omega) \), built on a family of triangulations \( \{ \mathcal{T}_h \} \), given as

\[
V^h = \{ u \in L^2(\Omega) : \forall \tau \in \mathcal{T}_h \quad u|_{\tau} \in P_1(\tau) \},
\]

where \( P_1(\tau) \) is the space of linear polynomials defined over the element \( \tau \). Consequently, \( u \) is not required to be continuous across the elements. Further, without any loss of generality, we assume that \( \alpha \) is constant on each \( \tau \in \mathcal{T}_h \). Since, for a linear function, its gradient being constant over \( \tau \), we have that \( \int_{\tau} \alpha \nabla u \cdot \nabla v dx = (\nabla u \cdot \nabla v) \int_{\tau} \alpha dx \).

We can now introduce a family of discrete problems, based on the symmetric interior penalty Galerkin (SIPG) method as follows. Find \( u^*_h \in V^h \)

\[
(3) \quad a(u^*_h, v) = f(v) \quad \forall v \in V^h,
\]

where

\[
a(u, v) = (\alpha \nabla u, \nabla v)_{\mathcal{T}_h} - (\{ \alpha \nabla u \} [v])_{\mathcal{E}_h} - (\{ \alpha \nabla v \} [u])_{\mathcal{E}_h} + \gamma (S_h[u], [v])_{\mathcal{E}_h},
\]

where \( \gamma \) is a penalty parameter.
for all $u,v \in V^h$, $S_h$ is a piecewise constant function over the edges of $\mathcal{E}_h$, and
$\gamma = \text{constant} > 0$ is a penalty term, cf. [4]. $S_h$ when restricted to $e \in \mathcal{E}_h^\partial$, is defined as follows, cf. [4],
\[ S_{h|e} = h_e^{-1}(\omega^+_e \alpha_+ + \omega^-_e \alpha_-) = h_e^{-1} \frac{2}{\alpha_+ + \alpha_-} \quad \text{on} \quad \tau = \partial \tau_+ \cap \partial \tau_- , \]
with $h_e$ being the length of the edge $e \in \mathcal{E}_h$. With the harmonic average satisfying
\[ \alpha_{min} \leq \frac{2}{\alpha_+ + \alpha_-} \leq 2 \alpha_{min} \quad \text{where} \quad \alpha_{min} = \min(\alpha_+, \alpha_-) , \]
we get
\[ h_e^{-1} \alpha_{min} \ll S_{h|e} \ll h_e^{-1} \alpha_{min} . \]
For $e \in \mathcal{E}_h^\partial$ we set
\[ S_{h|e} = \alpha|_\tau . \]
If penalty parameters is large enough than the discrete problem has a unique solution, cf. Lemma 2.3 in [4].

We use the standard local nodal basis on each triangle $\tau \in \mathcal{T}_h$ to represent a function $u \in V^h$. Hence, $u \in V^h$ restricted to a triangle $\tau \in \mathcal{T}_h$ can be represented as
\[ u|_{\tau} = \sum_{x \in \mathcal{V}_h(\tau)} u(x) \phi^\tau_x \quad \text{where} \quad \mathcal{V}_h(\tau) \quad \text{is the set of vertices of} \ \tau , \ \text{and} \ \phi^\tau_x \quad \text{is a linear basis function defined over} \ \tau \quad \text{such that it takes the value one at the vertex} \ x \ \text{and zero at the other two vertices of} \ \tau . \ \phi^\tau_x \ \text{extends to the rest of the domain as zero} . \ \text{Consequently, any function} \ u \in V^h \ \text{is represented as}
\[ u = \sum_{\tau \in \mathcal{T}_h} \sum_{x \in \mathcal{V}(\tau)} u(x) \phi^\tau_x . \]
In this basis, the degrees of freedom (dofs) are the function values associated with the vertices in the set $\mathcal{F}_h := \{ \mathcal{V}_h(\tau) : \tau \in \mathcal{T}_h(\Omega) \}$. We call this set the discontinuous Galerkin or the DG vertex or node set. Vertices (or nodes) occupying the same geometric space may have different function values associated with them if the triangles they belong to are different, and therefore appear in the set $\mathcal{F}_h$ with indices of their respective triangles. So, each geometric vertex $x$ may correspond to one or several DG vertices (nodes). We use $\nu(x)$ to denote the set of all DG vertices (nodes) associated with $x$.

Below, we introduce the bilinear form $\hat{a}(\cdot, \cdot)$, and restate its equivalence to the bilinear form $a(\cdot, \cdot)$ in Lemma 1 (cf. Lemma 2.3 in [4]). For $u,v \in V^h$ let
\[ \hat{a}(u,v) = (\alpha \nabla u, \nabla v)_{\mathcal{T}_h} + (S_h[u], [v])_{\mathcal{E}_h} . \]

**Lemma 1.** The norms induced by the bilinear forms $a(\cdot, \cdot)$ and $\hat{a}(\cdot, \cdot)$ are equivalent with constants independent of the mesh parameter $h$ or the coefficient $\alpha$ if the penalty term is larger than a positive constant which is dependent only on the geometry of all triangles in the triangulation and is independent of $h$ and the contrast $\alpha$.

**Remark 2.1.** The upper bound of Lemma 1 is relatively easy to show; the difficult part is to show the lower bound, that is the bilinear form $a(\cdot, \cdot)$ is coercive in the norm induced by the bilinear form $\hat{a}(\cdot, \cdot)$. In Section 2.4 of [4]), this coercivity constant is given in a very precise way, however the formula is a bit technical, and therefore we do not present it here.
3. Additive Schwarz method. Let  be the non-overlapping decomposition of  into disjoint open substructures for each  and boundary edges .

Let  be the open interface between  and  for  and its closed interface. The geometric point or vertex where the interfaces meet are called the crosspoint and is typically denoted by . Note that  denotes the set of DG vertices or nodes associated with the crosspoint .

Now, define , the global interface, as the sum of all closed interfaces.

For each subdomain interface , we define a patch, denoted by , as the sum of all triangles having at least a vertex on . Note that  is  minus the crosspoints. For the ease of explanation, we assume that the patches are disjoint in the sense that they may share a vertex (a crosspoint) geometrically, but not a triangle; this is however not necessary in practice. Each patch  has two disjoint subpatches, for  and  with  for  and  giving . The sum of all subpatches belonging to a subdomain , called the discrete boundary layer of  and denoted by , is defined as

\[
\Omega_\delta^k = \bigcup_{\Gamma_{kl} \subset \partial\Omega_k \cap \Gamma} \Gamma_{\delta,kl}^k.
\]

**Fig. 1.** Illustrating disjoint patches (shaded regions) for both interior and boundary subdomain. Each patch covers an interface between two neighboring subdomains, extending to the boundary, as shown.

Let  be the subspace of  of functions that are zero in all  where  and let  be the subspace of  of functions that are zero on the boundary layer , that is

\[
V_{k,0} = \{ u \in V_k : u(x) = 0 \quad \forall x \in \tau, \quad \tau \subset \Omega_\delta^k \}.
\]

The bilinear form  is positive definite over both  and . We now introduce an orthogonal projection  as follows,

\[
(7) \quad a(P_k u, v) = a(u, v) \quad \forall v \in V_{k,0}.
\]

Let . Since the supports of functions in  and  are disjoint for  and functions from these spaces are zero on edges contained in , the images of  and  are orthogonal in any of the bilinear forms  and  for  and .
Further, let \( H = I - P : V^h \rightarrow V^h \) be the corresponding discrete harmonic extension operator. This operator has the minimization property stating that

\[
(8) \quad a(Hu, Hu) = \min_{u \in V^h} \left\{ a(u, u) : u(x) = H u(x) \quad \forall x \in V_h(\tau), \ \tau \subset \bigcup_k \Omega_k^\delta \right\}.
\]

### 3.1. Local spaces.

For the additive Schwarz decomposition, let \( V_k \) be the local subspace associated with the subdomain \( \Omega_k \), giving that

\[
V^h = \sum_{k=1}^{N} V_k \quad \text{with} \quad V_k \cap V_l = \{0\} \quad \text{for} \quad k \neq l.
\]

We note here that, even when two neighboring subdomains share an edge, their subspaces may not necessarily be orthogonal to each other in the inner product induced by the two bilinear forms \( a(\cdot, \cdot) \) and \( \hat{a}(\cdot, \cdot) \). This is because of the term

\[
\sum_{e \subset \Gamma_{kl}} \int_e S_h[u][v] \, ds \quad \forall u \in V_k, v \in V_l,
\]

being present in the inner product on both subspaces when \( \Gamma_{kl} \) is nonempty. In this sense, the method can be considered as an overlapping Schwarz method with the minimal overlap.

### 3.2. Coarse space.

In this section, we introduce our coarse space which consists of two components, a spectral component and a non-spectral component. The way the components are to be built play an important role in making our method robust and effective.

Let \( V^h(\Gamma_{kl}^\delta) = \{ u \in V^h : u|_\tau = 0, \ \tau \not\subset \Gamma_{kl}^\delta \} \) be the space of functions that are equal to zero on all elements which do not belong to the patch \( \Gamma_{kl}^\delta \). On \( V^h(\Gamma_{kl}^\delta) \), we introduce two symmetric bilinear forms \( a_{kl}(u, v) \) and \( b_{kl}(u, v) \) as follows:

\[
(10) \quad a_{kl}(u, v) := \sum_{\tau \subset \Gamma_{kl}^\delta} \int_\tau \alpha \nabla u \nabla v \, dx + \sum_{e \subset \Gamma_{kl}^\delta \setminus (\partial \Omega \cap \partial \Gamma_{kl}^\delta)} S_h \int_e [u][v] \, ds,
\]

\[
(11) \quad b_{kl}(u, v) := h^{-2}(u, v)_{L^2(\Gamma_{kl}^\delta)} = h^{-2} \int_{\Gamma_{kl}^\delta} \alpha u \cdot v \, dx.
\]

The second sum in the bilinear form \( a_{kl}(\cdot, \cdot) \) is taken over all edges of the triangles \( \tau \subset \Gamma_{kl}^\delta \), which are either in the interior of the patch \( \Gamma_{kl}^\delta \) or on the boundary \( \partial \Omega \cap \partial \Gamma_{kl}^\delta \). Edges that lie on the boundary of the patch and are in the interior of a subdomain at the same time, are not in this sum.

Note that for \( u, v \in V^h(\Gamma_{kl}^\delta) \)

\[
ak_{kl}(u, v) = \hat{a}(u, v) - \sum_{e \subset \partial \Gamma_{kl}^\delta \setminus \partial \Omega} S_h \int_e [u][v] \, ds.
\]

In cases where \( \partial \Omega \cap \Gamma_{kl}^\delta = \emptyset \) this form is only positive semi-definite over the space \( V^h(\Gamma_{kl}^\delta) \). Therefore we introduce \( V_v(\Gamma_{kl}^\delta) \subset V^h(\Gamma_{kl}^\delta) \), the subspace of \( V^h(\Gamma_{kl}^\delta) \) of functions which are equal to zero at the nodes of \( \nu(c_\tau) \), where \( c_\tau \) are the crosspoints which are typically the endpoints of \( \Gamma_{kl} \). For all functions in this subspace we can make the following proposition
Proposition 2. The bilinear form $a_{kl}(u, v)$, where $u, v \in V_c(\Gamma_{kl}^\delta)$, is symmetric and positive definite.

Proof. Clearly the bilinear form is symmetric. To prove that it is positive definite, we only need to show that $a_{kl}(u, u) = 0$ if and only if $u = 0$. Let $u \in V_c(\Gamma_{kl}^\delta)$, and $a_{kl}(u, u)$ be equal to zero, hence $\nabla u$ is zero over each triangle $\tau \subset \Gamma_{kl}^\delta$ which means that $u$ is piecewise constant over $\Gamma_{kl}^\delta$. We also have, for each interior edge $e$, $\int_e |u|^2 \, ds = 0$. Hence, $u$ is constant over the patch $\Gamma_{kl}^\delta$, consequently, $u$ is zero at the vertices of $\nu(c_r)$ for each crosspoint $c_r$ of $\Gamma_{kl}$. By the definition of $\Gamma_{kl}$ it contains at least one crosspoint. This yields that $u = 0$ over the patch. Since $V_c(\Gamma_{kl}^\delta)$ is finite dimensional, we get that the form is positive definite over this space. \[\square\]

We first define the non-spectral component of the coarse space. It is constructed in the similar fashion as the standard multiscale finite element space is constructed. We call this component a multiscale component, and denote it by $V_{ms}$. The functions $u \in V_{ms}$ are determined by its values at the DG vertices $\nu(c_r)$ of the crosspoints $c_r$. $V_{ms} \subset V_h$ is the space of functions $u$ which satisfy the condition that

$$a_{kl}(u, v) = 0 \quad \forall v \in V_c(\Gamma_{kl}^\delta),$$

over each patch $\Gamma_{kl}^\delta$, which is guaranteed by Proposition 2, and that they are discrete harmonic in the sense that $u = \mathcal{H}u$ as defined in Section 3.

The spectral component of the coarse space is based on solving a generalized eigenvalue problem locally on each patch, which is defined as follows. On each patch $\sum_{\Gamma_{kl}}$, find all pairs $(\lambda_{j}^{kl}, \psi_j^{kl}) \in \mathbb{R}^+ \times V_c(\Gamma_{kl}^\delta)$ such that

$$a_{kl}(\psi_j^{kl}, v) = \lambda_{j}^{kl} b_{kl}(\psi_j^{kl}, v) \quad \forall v \in V_c(\Gamma_{kl}^\delta)$$

and $\|\psi_j^{kl}\|_{b_{kl}} = 1$, with $\|\cdot\|_{b_{kl}}$ as the norm induced by the bilinear form $b_{kl}(u, v)$.

We assume that the eigenvalues are indexed in the increasing order, i.e.

$$0 < \lambda_1^{kl} \leq \lambda_2^{kl} \leq \ldots \leq \lambda_{N_{kl}}^{kl}$$

where $N_{kl} = \text{dim}(V_c(\Gamma_{kl}^\delta))$. We define $\Pi_{M}^{kl} : V_c(\Gamma_{kl}^\delta) \rightarrow V_c(\Gamma_{kl}^\delta)$ as the $b_{kl}$-form orthogonal projection onto the space $\text{span}(\psi_1^{kl}, \ldots, \psi_{M}^{kl})$, as

$$\Pi_{M}^{kl} u = \sum_{j=1}^{M} b_{kl}(u, \psi_j^{kl}) \psi_j^{kl}.$$ 

with $0 \leq M \leq \text{dim}(V_c(\Gamma_{kl}^\delta))$. The integer parameter $M = M(\Gamma_{kl})$ is either preset or chosen automatically by setting a threshold for the eigenvalues. Our estimates below will depend on the choice of $M$ for the patches.

We are now ready to introduce the spectral component; it is the sum of patch subspaces $V_{kl}^{eig,M}$, $\Gamma_{kl} \subset \Gamma$, of $V_h$, defined as the following.

$$V_{kl}^{eig,M} = \text{span}(\psi_1^{kl}, \ldots, \psi_{M}^{kl}),$$

where $\Psi_j^{kl}$ is the extension of $\psi_j^{kl}$, first as zero on the triangles that are on the boundary layers minus the patch $\Gamma_{kl}^\delta$, and then as discrete harmonic further inside the subdomains in the sense as described in Section 3. The functions of this space have a support which is the union of the patch $\Gamma_{kl}^\delta$ and the interior of both $\Omega_k$ and $\Omega_l$, as shown in Figure 2.
The coarse space is defined as the sum of $V_{ms}$, the non-spectral multiscale component, and $\{V_{kl}^{\text{eig},M}\}$, the spectral component, as follows,

\begin{equation}
V_0 = V_{ms} + \sum_{\Gamma_{kl} \subset \Gamma} V_{kl}^{\text{eig},M},
\end{equation}

in other words the coarse space is a multiscale like coarse space enriched with patch spectral subspaces.

### 3.3. Preconditioned system

We define the coarse and the local projection like operators $\{T_k\}_{k=0}^N$ as $T_k : V^h \to V_k$, $k = 0, 1, \ldots, N$, satisfying

\begin{equation}
a(T_k u, v) = a(u, v) \quad \forall v \in V_k,
\end{equation}

and the corresponding additive Schwarz operator $T$ as

\begin{equation}
T = T_0 + \sum_{k=1}^N T_k.
\end{equation}

Now, following the Schwarz framework, cf. e.g. [36], the discrete formulation (3) can be written equivalently as

\begin{equation}
T u_h^* = g,
\end{equation}

which is a preconditioned version of the original system, where $g = \sum_{k=0}^N g_k$ and $g_k = T_k u_h^*$.

**Remark 3.1.** If we replace the exact bilinear form $a(u, v)$ by the inexact bilinear form $\hat{a}(u, v)$, on the left of the equality in the definition of $T_k$, cf. (18), we will get a second variant of the preconditioner with inexact solvers for the sub problems, but a similar convergence estimate as the exact version.

### 3.4. Condition number

We get the following condition number bound for the preconditioned system.

**Theorem 3.** For any $u \in V^h$ it holds for the additive Schwarz operator $T$ that

\begin{equation}
\left(1 + \max_{\Gamma_{kl} \subset \Gamma} \frac{1}{\lambda_{M+1}^M}\right)^{-1} a(u, u) \lesssim a(Tu, u) \lesssim a(u, u)
\end{equation}
where $M = M(\Gamma_{kl})$ is the number of enrichment used on each edge $\Gamma_{kl}$.

The proof of this theorem is given in the next section.

4. Proof of Theorem 3. The proof is based on the abstract Schwarz framework, see [33, 36, 24] for more details. Accordingly, there are three key assumptions that need to be verified, these are the assumptions on the stability of the decomposition, the strengthened Cauchy-Schwarz inequality between the local subspaces, and the local stability of the inexact bilinear forms if any. The second assumption is verified using a simple coloring argument. The third assumption needs to be verified if $\tilde{a}(\cdot, \cdot)$ is used instead of the exact bilinear form $a(\cdot, \cdot)$, cf. Remark 3.1, in which case it is a simple consequence of the equivalence between the two as given in Lemma 1. We are then left with the assumption on the stability of the decomposition, which needs to be verified, and which is shown in Lemma 10 below.

We need a few technical tools. The first one is the following set of local inverse inequalities, cf. Lemma 4.

**Lemma 4.** Let $u$ be a function such that $u \in V^h$. Then for any $\tau \in T_h$ or $e \in \mathcal{E}_h$ we have

$$
\int_{\tau} \alpha |\nabla u|^2 \, dx \lesssim h^{-2} \int_{\tau} |u|^2 \, dx,
$$

$$
\int_e S_h |u|^2 \, ds \lesssim h^{-2} \int_{\tau_+} \alpha_+ |u|^2 \, dx + h^{-2} \int_{\tau_-} \alpha_- |u|^2 \, dx, \quad e \in \mathcal{E}_h^0,
$$

$$
\int_e S_h |u|^2 \, ds \lesssim h^{-2} \int_{\tau} |u|^2 \, dx, \quad e \in \mathcal{E}_h^\partial.
$$

where $e \in \mathcal{E}_h^0$ in the second inequality, is the edge shared by the elements $\tau_+$ and $\tau_- \in T_h$.

**Proof.** The first inequality is the classical local inverse inequality, cf. e.g. [3]. The last two inequalities follow from the trace theorem over $e$, a scaling argument, and the fact that $h_e S_h \leq 2 \min\{\alpha_+, \alpha_-\} h$ for $e \in \mathcal{E}_h^0$, cf. (5).

The next technical tool is a corollary of Lemma 4.

**Corollary 5.** Let $u = \mathcal{H}u \in V^h$ be a discrete harmonic function as defined in Section 3. Then

$$
a(u, u) \lesssim \sum_{k=1}^N h^{-2} \|\alpha^{1/2} u\|^2_{L^2(\Omega_k^h)}.
$$

**Proof.** Let $\hat{u} \in V^h$ be equal to $u$ on all the boundary layers, that is the vertices of $\Omega_k^h$ for $k = 1, \ldots, N$, and be extended by zero further inside the subdomains, out of the boundary layers.

By the minimizing property of the discrete harmonic extension, cf. (8), and Lemma 1, we get

$$
|u|^2 \lesssim |\hat{u}|^2 \lesssim |\check{u}|^2 = \sum_{k=1}^N \left( \sum_{\tau \subset \Omega_k^h} \|\alpha^{1/2} \nabla \check{u}\|^2_{L^2(\tau)} + \sum_{e \subset \mathcal{E}_k^\partial} \int_e S_h |\check{u}|^2 \, ds \right).
$$

By the first inequality of Lemma 4, the first sum above can be estimated by the square of $L^2$ norm of $\hat{u}$ scaled by $h^{-2}$ over all boundary layers. Thus, since $\hat{u} = u$ on the
Further, adding (24) to (23) and summing over the subdomains we get
\[ \sum_{\tau \subset \Omega_k} \| \alpha^{1/2} \nabla \hat{u} \|_{L^2(\tau)}^2 \lesssim \sum_{\tau \subset \Omega_k} h^{-2} \| \alpha^{1/2} \hat{u} \|_{L^2(\tau)}^2 = h^{-2} \| \alpha^{1/2} u \|_{L^2(\Omega_k^e)}^2. \]

For the edge term, \( \sum_{e \subset \Omega_k} \int_{e} S_h[\hat{u}]^2 \, ds \), we estimate it by estimating its edge integral term separately for each case of \( e \) in the sum.

Let \( e \subset \Omega_k \) be an edge of some \( \tau \subset \Omega_k^e \), and be lying on \( \partial \Omega_k \cap \partial \Omega \), then by the third inequality of Lemma 4, we estimate the edge integral as
\[ \int_{e} S_h[\hat{u}]^2 \, ds \lesssim h^{-2} \int_{\tau} \alpha \hat{u}^2 \, dx. \]

Let \( e \subset \Omega_k^e \) be the edge common to two triangles \( \tau_+ \) and \( \tau_- \), then by the second inequality of Lemma 4, we get
\[ \int_{e} S_h[\hat{u}]^2 \, ds \lesssim h^{-2} \int_{\tau_+ \cup \tau_-} \alpha \hat{u}^2 \, dx. \]

There are three cases to consider in this case. In the first, both \( \tau_+ \subset \Omega_k \) and \( \tau_- \subset \Omega_k^e \). The edge integral is then bounded by the sum of the squares of the \( L^2 \) norm scaled by \( h^{-2} \) over the two triangles. In the second, \( e \subset \Gamma_{kl} \), in which case one of the two triangles, \( \tau_+ \) and \( \tau_- \), is in the boundary layer \( \Omega_k^e \) and the other one in the boundary layer \( \Omega_k^e \). The edge integral is then bounded by the sum of the squares of the \( L^2 \) norm scaled by \( h^{-2} \) over the two triangles. Finally, the case when \( e \subset \partial \Omega_k \setminus \partial \Omega_k \), in which case we have \( \tau_+ \subset \Omega_k^e \) and \( \tau_- \subset \Omega_k \setminus \Omega_k^e \) with \( \hat{u} = 0 \) on \( \tau_- \). We estimate the edge integral as
\[ \int_{e} S_h[\hat{u}]^2 \, ds \lesssim h^{-2} \int_{\tau_+ \cup \tau_-} \alpha \hat{u}^2 \, dx = h^{-2} \int_{\tau_+} \alpha \hat{u}^2 \, dx, \]
which is again the square of the \( L^2 \) norm scaled by \( h^{-2} \) over the triangle in the boundary layer \( \Omega_k^e \). Now adding all contributions from the edge integrals, and noting that \( \hat{u} = 0 \) on the boundary layers, we can estimate the edge term as follows.
\[ \sum_{e \subset \Omega_k^e} \int_{e} S_h[\hat{u}]^2 \, ds \lesssim h^{-2} \| \alpha^{1/2} u \|_{L^2(\Omega_k^e)}^2 + \sum_{\Gamma_{kl} \subset \Gamma \cap \partial \Omega_k} h^{-2} \| \alpha^{1/2} u \|_{L^2(\Omega_k^e)}^2. \]

Further, adding (24) to (23) and summing over the subdomains we get the right hand side of (22), and the proof then follows.

The following result can be obtained through a standard algebraic reasoning, see for instance [16, 35] for similar use.

**Lemma 6.** Let \( \Pi_M^{kl} \) be as defined in (15), then it is both the \( b_{kl} \)- and \( a_{kl} \)-orthogonal projection onto some subspace, and
\[ \| u - \Pi_M^{kl} u \|_{a_{kl}} \leq \| u \|_{a_{kl}}, \quad \| \Pi_M^{kl} u \|_{a_{kl}} \leq \| u \|_{a_{kl}}, \]
and
\[ \| u - \Pi_M^{kl} u \|_{b_{kl}}^2 \leq \frac{1}{\lambda_{M+1}} \| u - \Pi_M^{kl} u \|_{a_{kl}}^2. \]
The next lemma states certain properties of the patch bilinear form \( a_{kl}(u,v) \).
We have used the fact that \( P_u \) is a multiscale function interpolating \((28)\) following bound for the coarse interpolant.

\[ \||\Sigma|u\|_0^2 \leq \sum_{\tau \subset \Omega_h \cup \Omega_l} \int_{\tau} |\nabla u|^2 \, dx + \sum_{e \subset \Omega_h \cup \Omega_l} \int_e S_h[u]^2 \, ds. \]

**Proof.** We first note that, in the bilinear form \( a_{kl}(u, u) \), the edge integrals are taken over the edges of \( \Gamma_{kl}^3 \), that are either in the interior of the patch \( \Gamma_{kl}^3 \) or on the outer boundary \( \partial \Omega \). Thus the proof follows directly from the definition of the bilinear forms as all triangle terms and edge terms of the form \( a_{kl} \) are contained in the right hand side of \((25)\).

For \( u \in V^h \), we introduce the following interpolation operators, the multiscale interpolant and the coarse interpolant, respectively, as

\[ I_{ms} : V^h \rightarrow V_{ms} \quad \text{and} \quad I_0 : V^h \rightarrow V_0. \]

For \( u \in V^h \), let \( I_{ms} u \in V_{ms} \) be the function such that

\[ I_{ms} u(x) = u(x) \quad \forall x \in V(c_r) \quad \forall c_r \in \Gamma, \]

that is a multiscale function interpolating \( u \) at the DG vertices \( \nu(c_r) \) of all cross-points \( c_r \) in \( \Gamma \). This yields that \( u - I_{ms} u \) is zero at the DG vertices of crosspoints and thus \( u - I_{ms} u \) restricted to a patch \( \Gamma_{kl}^3 \) is in the \( V_v(\Gamma_{kl}^3) \).

Next we define \( I_0 u \in V_0 \) on each patch \( \Gamma_{kl}^h \) as

\[ I_0 u = I_{ms} u + \Pi_{ms}^{kl}(u - I_{ms} u) \quad \text{on} \quad \Gamma_{kl}^3, \]

with \( \Pi_{ms}^{kl} \) as defined in \((15)\). \( I_0 u \) is extended inside as discrete harmonic. We have the following bound for the coarse interpolant.

**Lemma 8.** For the coarse interpolant \( I_0 \) it holds that

\[ a(I_0 u, I_0 u) \lesssim \left( 1 + \max_{\Gamma_{ki} \subset \Gamma} \frac{1}{\lambda_{M+1}} \right) a(u, u) \quad \forall u \in V^h. \]

**Proof.** By a triangle inequality we get immediately that

\[ \| I_0 u \|_a \leq \| I_0 u - u \|_a + \| u \|_a. \]

It suffices therefore to prove the bound for the first term on the right hand side of this inequality. Define \( w = u - I_0 u \) for \( u \in V^h \). Note that \( P w = P u \) and \( H w = H u - H I_0 u = H u - I_0 u \) as \( I_0 u \) is discrete harmonic in the way described in Section 3. Hence

\[ \| I_0 u - u \|_a = \| H w \|_a + \| P u \|_a \leq \| H u - I_0 u \|_a + \| P u \|_a \leq \| H u - I_0 u \|_a + \| u \|_a. \]

We have used the fact that \( P \) is the orthogonal projection with respect to \( a(\cdot, \cdot) \).

Next, we estimate \( \| H w \|_a = \| H w - I_0 u \|_a. \) By Lemma 1 and Corollary 5 we have

\[ \| H u - I_0 u \|_a \lesssim \| H u - I_0 u \|_a \lesssim \sum_{k=1}^N h^{-2} \| \alpha^{1/2}(H u - I_0 u) \|_{L^2(\Omega^k_0)}^2 \]

\[ \lesssim \sum_{\Gamma_{ki} \subset \Gamma} h^{-2} \| \alpha^{1/2}(u - I_0 u) \|_{L^2(\Omega^k_0)}^2. \]
Note that \( u - I_0 u = (I - \Pi_{M}^{kl})(u - I_{ms} u) \in V_e(\Gamma_{kl}^3) \). Hence by Lemma 6 we get
\[
\| \alpha^{1/2} (u - I_0 u) \|^2_{L^2(\Gamma_{kl}^3)} = \| (I - \Pi_{M}^{kl})(u - I_{ms} u) \|^2_{bkl} \\
\leq \frac{1}{\lambda_{M+1}^{kl}} \| (I - \Pi_{M}^{kl})(u - I_{ms} u) \|^2_{akl} \\
\leq \frac{1}{\lambda_{M+1}^{kl}} \| (u - I_{ms} u) \|^2_{akl} \\
\leq \frac{1}{\lambda_{M+1}^{kl}} \| u \|^2_{akl}.
\]

The last inequality follows from the fact that \( u - I_{ms} u \) restricted to the patch \( \Gamma_{kl}^3 \) is \( a_{kl} \)-orthogonal to the space \( V_e(\Gamma_{kl}^3) \), what follows from the definitions of \( V_{ms} \) and \( I_{ms} \), cf. (12).

Now, utilizing Lemma 7 for each patch, summing over the interfaces, and finally using Lemma 1, the proof then follows.

**Lemma 9.** Let \( w_k \) be the restriction of \( u - I_0 u \) to \( \Omega_k \), and extended by zero to the other subdomains, then
\[
\sum_{k=1}^{N} a(w_k, w_k) \lesssim \left( 1 + \max_{k=1}^{N} \frac{\lambda_{M+1}^{kl}}{\lambda_{M+1}^{kl}} \right) a(u, u)
\]

**Proof.** Using Lemma 1 we get
\[
a(w_k, w_k) \lesssim \tilde{a}(w_k, w_k) \\
= \sum_{\tau \subset \Omega_k} \alpha^{1/2} \| \nabla w_k \|^2_{L^2(\tau)} + \sum_{\epsilon \subset \Omega_k} \| S_{h}^{1/2} [w_k] \|^2_{L^2(\epsilon)} \\
+ \sum_{\epsilon \subset \partial \Omega \cap \partial \Omega_k} \| S_{h}^{1/2} [w_k] \|^2_{L^2(\epsilon)} + \sum_{\epsilon \subset \Omega_{kl} \cap \partial \Omega_k} \| S_{h}^{1/2} [w_k] \|^2_{L^2(\epsilon)},
\]

where the second sum in the right hand side is over all edges of elements \( \tau \subset \Omega_k \) that are not on the boundary of \( \Omega_k \).

The first three terms, when summed over \( k = 1, \ldots, N \), yield
\[
\sum_{k=1}^{N} \left( \sum_{\tau \subset \Omega_k} \alpha^{1/2} \| \nabla w_k \|^2_{L^2(\tau)} + \sum_{\epsilon \subset \Omega_k} \| S_{h}^{1/2} [w_k] \|^2_{L^2(\epsilon)} + \sum_{\epsilon \subset \partial \Omega \cap \partial \Omega_k} \| S_{h}^{1/2} [w_k] \|^2_{L^2(\epsilon)} \right) \\
= \sum_{k=1}^{N} \left( \sum_{\tau \subset \Omega_k} \| \alpha^{1/2} \nabla (u - I_0 u) \|^2_{L^2(\tau)} + \sum_{\epsilon \subset \Omega_k \cup \partial \Omega \cap \partial \Omega_k} \| S_{h}^{1/2} [u - I_0 u] \|^2_{L^2(\epsilon)} \right) \\
\leq \tilde{a}(u - I_0 u, u - I_0 u),
\]

since all terms in the left of inequality are also in the right. We can bound this term in the same way as in the proof of Lemma 8.

It remains to bound the fourth term, that is the sum of integrals over the edges that are on \( \Gamma \). Let \( \epsilon \subset \Gamma \cap \partial \Omega_k \) be an edge on the interface \( \Gamma_{kl}^3 \), so it is the common edge of two triangles \( \tau_+ \subset \Omega_k^3 \) and \( \tau_- \subset \Omega_l^3 \). Note that on \( \tau_- \) the function \( w_k \) is zero,
hence, by Lemma 4, we get
\[
\sum_{e \subset \Gamma \cap \partial \Omega_k} \| S^{1/2}_h [w_k] \|_{L^2(e)}^2 \lesssim h^{-2} \| \alpha^{1/2} (u - I_0 u) \|_{L^2(\Omega^k)}^2 \lesssim \sum_{e \subset \Gamma \cap \partial \Omega_k} h^{-2} \| \alpha^{1/2} (u - I_0 u) \|_{L^2(e)}^2.
\]
This term was already bounded in the proof of Lemma 8. Following the lines of proof there, we end the proof of the lemma here.

**Lemma 10.** For any \( u \in V^h \) there is a stable decomposition, that is, there exists a coarse function \( u_0 \in V_0 \) and local functions \( u_k \in V_k \) for \( k = 1, \ldots, N \), such that
\[
u = u_0 + \sum_{k=1}^N u_k, \quad (30) \quad a(u_0, u_0) + \sum_{k=1}^N a(u_k, u_k) \leq \left( 1 + \max_{\Gamma_{kl} \subset \Gamma \cap \partial \Omega_k} \frac{1}{\lambda_{M+1}} \right) \lesssim a(u, u).
\]

**Proof.** For \( u \in V^h \), define its coarse component as the coarse interpolant of \( u \), \( u_0 = I_0 u \in V_0 \). The local decomposition \( u - u_0 \) as given as \( u_k = (u - u_0)|_{\Omega_k} \in V_k \), \( k = 1, \ldots, N \), is defined uniquely. Accordingly, \( u_k \) equals to \( u - u_0 \) on all triangles of \( \Omega_k \) and to zero on all triangles of the remaining subdomains. Clearly,
\[
u_0 + \sum_{k=1}^N u_k = u_0 + \sum_{k=1}^N (u - u_0)|_{\Omega_k} = u_0 + u - u_0 = u.
\]

The stability estimate (30) follows from the lemmas 8-9.

5. **Numerical Results.** We consider our model problem to be defined on the unit square with zero Dirichlet boundary condition and a constant force function, and solve it as proposed, using the discontinuous Galerkin discretization (SIPG [32]), and the conjugate gradients iteration with the additive Schwarz preconditioner of this paper. For the Schwarz preconditioner, we decompose the domain into \( 8 \times 8 = 64 \) non-overlapping square sub-domains, each with a regular triangulation of \( 2 \times 16 \times 16 \) triangles resulting in a total of 32768 triangles. There are a total of 112 patches, each corresponding to the interior edges, and 49 cross points, each being connected to the four different patches meeting at the point. In all our experiments, the penalty parameter \( \eta \) was chosen to be equal to four, and the iterations stopped as soon as the relative residual norm became less than \( 10^{-6} \). In order to test our method on multiscale problems, we have chosen high contrast media which are represented by the values and jumps of the coefficient \( \alpha \), as shown in Fig. 3 and Fig. 4 where \( \alpha \) equals one on the background and a much higher value \( \alpha_0 \) on inclusions and channels.

As described in the paper, our coarse space has one spectral component whose basis functions are related to the first few eigenfunctions of the generalized eigenvalue problems on the patches, and one non-spectral component whose basis functions are associated with the degrees of freedom associated with the crosspoints. For the spectral component, a threshold is set for the eigenvalues; eigenfunctions corresponding to eigenvalues below the threshold are then chosen to construct the basis functions for the spectral component. For the non-spectral component, in order to keep the
number of its basis functions to a minimal, we include exactly one basis function per crosspoint per patch, as a consequence, two basis functions of non-spectral type per patch. It should be noted here that this rearrangement of the algorithm does not change the outcome of our theory. We have observed in this case a slight decrease in the condition number as opposed to including multiple basis functions per crosspoint per patch.

In our first experiment we consider a somewhat structured but complex distribution of the coefficient \( \alpha \), which includes both channels and inclusions (smaller channels) across subdomains, cf. Fig. 3. The results are shown in Table 1, reporting for each test case a condition number estimate and the corresponding number of iterations required to converge. The rows in the table correspond to different jumps in the coefficient, and the columns correspond to different numbers of enrichment of the spectral component of the coarse space, either applying a fixed number of enrichment on every patch, starting with no enrichment to a maximum number of enrichment (the first three columns in the table), or applying fully adaptive enrichment where the threshold of 0.18 has been used (the fourth column in the table). This choice of threshold has resulted in a total of 140 eigenfunctions for the jump \( \alpha = 1 \), and 212 eigenfunctions for the other jumps.

In our next experiment, we consider a problem with rather complex distribution of the coefficient with channels crossing each other as well as subdomains in a random fashion, as shown in Figure 4. The problem has been tested under the same experimental setup as the first experiment, that is with the same threshold and jumps in \( \alpha \), and the results are presented in Table 2. The fully adaptive choice of the coarse space has required a total of 140 eigenfunctions for \( \alpha = 1 \), 227 eigenfunctions for \( \alpha = 10^2 \), and 234 eigenfunctions for \( \alpha \in \{10^4, 10^6\} \).

Our experiments have shown that the algorithm can handle problems with high contrast and complex distribution (network of channels and inclusions) with a modest number of spectral enrichment, demonstrating the power of our method.
Adaptively enriched coarse space for the DG multiscale

Additive Schwarz with enriched coarse space for the problem of Fig. 3

| Jump in $\alpha$ | Fixed enrichment | Adaptive enrich. |
|------------------|------------------|------------------|
| $10^0$           | $5.73 \times 10^4$ (53) | 2.87 $\times 10^4$ (401) |
| $10^2$           | $3.34 \times 10^2$ (141) | 1.42 $\times 10^2$ (83) |
| $10^4$           | $2.86 \times 10^6$ (832) | 1.43 $\times 10^6$ (79) |
| $10^6$           | $2.86 \times 10^6$ (832) | 1.43 $\times 10^6$ (79) |

Table 1

Numerical results showing condition number estimates and iteration counts (in parentheses), corresponding to problem of Fig. 3. As we continue to enrich the spectral component of the coarse space with additional functions, the condition number improves, and becomes independent of the variations in $\alpha$ when all eigenvalues below the threshold has been used.

Fig. 4. The domain with $8 \times 8$ square subdomains and a complex distribution of $\alpha$ with channels crossing each other, and stretching across subdomains. The coefficient $\alpha$ equals one on the background and $\alpha_0$ on channels.

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Additive Schwarz with enriched coarse space for the problem of Fig. 4

| Jump in | Fixed enrichment | Adaptive enrich. |
|---------|-------------------|-----------------|
| \(\alpha\) | none | 2 | 4 | 4 |
| \(10^0\) | \(5.73 \times 10^4\) (53) | \(1.57 \times 10^4\) (31) | 9.64 (24) | 15.65 (31) |
| \(10^2\) | \(3.18 \times 10^2\) (124) | \(1.87 \times 10^2\) (80) | 19.17 (37) | 27.90 (44) |
| \(10^4\) | \(2.98 \times 10^4\) (328) | \(1.01 \times 10^4\) (122) | 22.04 (39) | 28.57 (47) |
| \(10^6\) | \(2.98 \times 10^6\) (508) | \(2.96 \times 10^4\) (131) | 22.07 (42) | 28.56 (50) |

Table 2
Numerical results showing condition number estimates and iteration counts (in parentheses), corresponding to Fig. 4. As we continue to enrich the spectral component of the coarse space with additional functions, the condition number improves, and becomes independent of the variations in \(\alpha\) when all eigenvalues below the threshold has been used.
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