Brane gravity, higher derivative terms and non-locality

Shinji Mukohyama

Department of Physics, Harvard University
Cambridge, MA, 02138, USA

(October 30, 2018)

In brane world scenarios with a bulk scalar field between two branes it is known that 4-dimensional Einstein gravity is restored at low energies on either brane. By using a gauge-invariant gravitational and scalar perturbation formalism we extend the theory of weak gravity in the brane world scenarios to higher energies, or shorter distances. We argue that weak gravity on either brane is indistinguishable from 4-dimensional higher derivative gravity, provided that the inter-brane distance (radion) is stabilized, that the background bulk scalar field is changing near the branes and that the background bulk geometry near the branes is warped. This argument holds for a general conformal transformation to a frame in which matter on the branes is minimally coupled to the metric. In particular, Newton’s constant and the coefficients of curvature-squared terms in the 4-dimensional effective action are determined up to an ambiguity of adding a Gauss-Bonnet topological term. In other words, we provide the brane-world realization of the so called $R^2$-model without utilizing a quantum theory. We discuss the appearance of composite spin-2 and spin-0 fields in addition to the graviton on the brane and point out a possibility that the spin-0 field may play the role of an effective inflaton to drive brane-world inflation. Finally, we conjecture that the sequence of higher derivative terms is an infinite series and, thus, indicates non-locality in the brane world scenarios.

I. INTRODUCTION

In the recent development of string/M theory [1], branes have been playing many important roles. The idea that our universe is a brane in a higher dimensional spacetime has been attracting a great deal of interest [2–5]. Although the idea of the brane world had arisen at a phenomenological level already in 1983 [6], it is perhaps the discovery of the duality between M-theory and $E_8 \times E_8$ heterotic superstring theory by Horava and Witten [7] that made it more attractive. It actually gives the brane world idea a theoretical background: by compactifying six dimensions in the 11-dimensional theory, our 4-dimensional universe may be realized as a hypersurface in 5 dimensions at one of the fixed points of an $S_1/Z_2$ compactification. After compactification of 6 dimensions by a Calabi-Yau manifold, the 5-dimensional effective theory can be obtained, see e.g. [8].

Randall and Sundrum proposed two similar but distinct phenomenological brane world scenarios [4,5]. In the scenario the 5-dimensional spacetime is compactified on $S_1/Z_2$ and all matter fields are assumed to be confined on branes at fixed points of the $S_1/Z_2$ so that the bulk, or the spacetime region between two fixed points, is described by pure Einstein gravity with a negative cosmological constant. In the second scenario the fifth dimension is infinite but still the $Z_2$ symmetry is imposed. In both scenarios the existence of the branes and the bulk cosmological constant makes the bulk geometry curved, or warped. There are generalizations of their scenarios with a scalar field between two branes [9,10]. In these generalized warped brane-world scenarios, the scalar field was introduced to stabilize a modulus called the radion, which represents a separation between the two branes.

In the brane world scenarios with or without a scalar field in a warped bulk geometry, weak gravity in a static background has been extensively investigated [11–16]. It is known that weak gravity in the scenario without a bulk scalar between two branes is not Einstein but a Brans-Dicke theory at low energies. On the other hand, in scenarios with a bulk scalar field between two branes 4-dimensional Einstein gravity is restored at low energies. It is believed that the validity of Einstein gravity breaks down at a certain energy scale which can be much lower than 4-dimensional Planck energy.

Hence, it seems natural to ask 'what does gravity in brane worlds look like at high energies or at short distances?' In other words, 'how does the 4-dimensional description break down?'

In this paper we investigate weak gravity in brane world scenarios with a bulk scalar field between two branes at higher energies. For this purpose we use the gauge-invariant perturbation formalism developed in ref. [16]. In this formalism all quantities and equations are Fourier transformed with respect to the 4-dimensional coordinates and classified into scalar, vector and tensor perturbations so that the problem is reduced to a set of purely 1-dimensional problems. We also adopt expansion in a parameter $\mu \equiv l^2 \eta^{\mu\nu} k_\mu k_\nu$, where $l$ is a characteristic length scale of the model and $k_\mu$ is the 4-dimensional momentum (or the Fourier parameter). In the lowest order in $\mu$, 4-dimensional Einstein
gravity is restored on either brane \[16\]. In the next order it is shown that gravity on either brane is indistinguishable from a higher derivative gravity whose action includes the Einstein term and curvature-squared terms. Equipped with the result for this order, we conjecture that in the order \(\mu^N\), gravity on either brane is indistinguishable from a higher derivative gravity whose action includes terms of up to the \((N + 1)\)-th power of curvature tensors. Noting that the expansion in \(\mu\) is in principle an infinite series, this conjecture indicates that gravity on either brane is non-local at high energies even at the linearized level. This explains how the 4-dimensional description breaks down at high energies. Physically, the non-locality is due to gravitational and scalar waves in the bulk.

This paper is organized as follows. In section II we summarize the basic equations by reviewing the formulation given in ref. \[16\]. In section III we perform the low energy expansion to investigate the system. In section IV we review linear perturbations in 4-dimensional higher derivative gravity so as to compare it with gravity in the brane world. In section V we discuss some physical implications. Finally, section VI is devoted to a summary of results.

II. BASIC EQUATIONS

The model and basic equations we shall investigate in this paper are exactly the same as those in ref. \[16\]. We now summarize them briefly.

A. Model description and background

We consider a 5-dimensional spacetime \(\mathcal{M}\) of the topology \(\mathcal{M}_4 \otimes S^1/\mathbb{Z}_2\), where \(\mathcal{M}_4\) represents 4-dimensional spacetime. We denote two timelike hypersurfaces corresponding to fixed points of the \(S^1/\mathbb{Z}_2\) compactification by \(\Sigma_\pm\).

Each hypersurface can be considered as the world volume of a 3-brane. In order to describe \(\Sigma_\pm\) we use the parametric equations

\[\Sigma_\pm : x^M = Z^M_\pm (y^\mu_\pm),\]

where \(x^M (M = 0, \ldots, 4)\) are 5-dimensional coordinates in \(\mathcal{M}\) and each \(y^\mu_\pm\) denotes four parameters \(\{y^\mu_\pm\}\) \((\mu = 0, \ldots, 3)\). The four parameters play the role of 4-dimensional coordinates on each hypersurface. It is notable that the 4-dimensional coordinates \(y^\mu_\pm\) are not necessarily a part of the 5-dimensional coordinates. Actually, in the following we shall consider a 4-dimensional gauge transformation (a 4-gauge transformation) on each brane and a 5-dimensional gauge transformation (a 5-gauge transformation) in the bulk independently, where we call the 5-dimensional region between \(\Sigma_\pm\) the bulk (and we shall denote it by \(\mathcal{M}_b\)). In particular, a quantity invariant under the latter (a 5-gauge invariant variable) is not necessarily invariant under the former (4-gauge invariant).

We consider a theory described by the action

\[I_{\text{tot}} = I_{\text{EH}} + I_\Psi + I_{\text{matter}},\]

where \(I_{\text{EH}}\) is the 5-dimensional Einstein-Hilbert action

\[I_{\text{EH}} = \frac{1}{2\kappa_5^2} \int_\mathcal{M} d^5x \sqrt{-g} R,\]

\(I_\Psi\) is the action of a scalar field \(\Psi\)

\[I_\Psi = -\int_{\mathcal{M}_b} d^5x \sqrt{-g} \left[\frac{1}{2} g^{MN} \partial_M \Psi \partial_N \Psi + U(\Psi)\right] - \sum_{\sigma = \pm} \int_{\Sigma_\sigma} d^4y_\sigma \sqrt{-q_\sigma} V_\sigma(\Psi_\sigma),\]

and \(I_{\text{matter}}\) is the action of matter fields confined on the branes

\[I_{\text{matter}} = \sum_{\sigma = \pm} \int_{\Sigma_\sigma} d^4y_\sigma \mathcal{L}_{\pm}[\bar{q}_{\pm\mu\nu}, \text{matter}],\]

Here, \(\Psi_\pm\) and \(q_{\pm\mu\nu}\) represent the pullback of \(\Psi\) and the induced metric on \(\Sigma_\pm\), and \(\bar{q}_{\pm\mu\nu}\) is the physical metric on \(\Sigma_\pm\), which is not necessarily equivalent to \(q_{\pm\mu\nu}\). We assume that the physical metric is related to the induced metric by a conformal transformation depending on \(\Psi_\pm\):

\[\bar{q}_{\pm\mu\nu} = \exp[-\alpha_\pm(\Psi_\pm)] q_{\pm\mu\nu},\]
where \( \alpha \pm \) is a function of \( \Psi \pm \), respectively. As shown in ref. \cite{17} the variational principle based on the action (2) gives the correct set of equations of motion, including variations of \( Z_\pm^M \), \( g_{MN} \) and \( \Psi \). It is essential that the region of integration in the Einstein action (4) is not \( M_b \) but \( M \): the integration across \( \Sigma \) gives the so-called Gibbons-Hawking term correctly.

We consider general perturbations around a background with 4-dimensional Poincaré symmetry:

\[
\begin{align*}
  g_{MN} &= g_{MN}^{(0)} + \delta g_{MN}, \\
  \Psi &= \Psi^{(0)} + \delta \Psi, \\
  Z_\pm^M &= Z_\pm^{(0)M} + \delta Z_\pm^M, \\
  \bar{S}_{\pm \mu \nu} &= \bar{S}_{\pm \mu \nu}^{(0)} + \delta \bar{S}_{\pm \mu \nu},
\end{align*}
\]

where the background is given by

\[
\begin{align*}
  g_{MN}^{(0)} dx^M dx^N &= e^{-2A(w)}(\eta_{\mu \nu} dx^\mu dx^\nu + dw^2), \\
  \Psi^{(0)}(w), \\
  Z_\pm^{(0)\mu} = y_\pm^\mu, \\
  Z_\pm^{(0)w} = w_\pm, \\
  \bar{S}_{\pm \mu \nu}^{(0)} &= 0,
\end{align*}
\]

\{\( x^\mu \) (\( \mu = 0, \cdots, 3 \)) represent first four of 5-dimensional coordinates \{\( x^M \) (\( M = 0, \cdots, 4 \)) in \( M \), \( w \) represents the fifth coordinate \( x^4 \), and \( w_\pm (w_- < w_+) \) are constants. Here, \( \bar{S}_{\pm \mu \nu} \) is the physical surface energy momentum tensor defined by

\[
\bar{S}_{\pm \mu \nu} \equiv \frac{2}{\sqrt{-q_{\pm \mu \nu}}} \frac{\delta}{\delta q_{\pm \mu \nu}} \int_{\Sigma_\pm} d^4 y_\pm L_\pm[q_{\pm \mu \nu}, \text{matter}],
\]

and we have redefined \( V_\pm \) and \( L_\pm \) so that \( \bar{S}_{\pm \mu \nu}^{(0)} \) vanishes. Hereafter, we assume that the brane at \( w = w_+ \) is our brane and that there is no excitation of matter on the other brane (\( \bar{S}_{\pm \mu \nu} = 0 \)).

The equations of motion for the background are as follows.

\[
\begin{align*}
  3 \ddot{A} + 3 \dot{A}^2 &= \kappa_5^2 \bar{\Psi}^{(0)2}, \\
  3 \ddot{A} - 9 \dot{A}^2 &= 2\kappa_5^2 e^{-2A} U(\Psi^{(0)}),
\end{align*}
\]

B. Gauge-invariant variables

Let us now construct gauge-invariant variables from the metric perturbation \( \delta g_{MN} \), the scalar field perturbation \( \delta \Psi \), the brane fluctuation \( \delta Z_\pm^M \) and the matter perturbation \( \delta \bar{S}_{\pm \mu \nu} \). There are actually two types of gauge-invariant variables as there are two types of gauge-transformations: the 5-dimensional gauge transformation in the bulk (5-gauge transformation)
\[ x^M \rightarrow x^M + \xi^M(x), \]  
and the 4-dimensional gauge transformation on each brane \( \Sigma_\pm \) (4-gauge transformation)

\[ y^\mu_\pm \rightarrow y^\mu_\pm + \tilde{\gamma}^\mu_\pm(y_\pm). \]  

As pointed out in ref. [37], these two kinds of gauge-transformation are independent.

For the purpose of construction of gauge-invariant variables we expand all perturbations by harmonics in 4-dimensional Minkowski spacetime. This strategy is convenient since the background has 4-dimensional Poincare symmetry and the induced (and physical) metric on each brane is 4-dimensional Minkowski metric. In appendix A we define scalar harmonics \( Y \), vector harmonics \( V_{(T,L)\mu} \) and tensor harmonics \( T_{(T,LT,LL,Y)\mu\nu} \). By using those harmonics we can expand all perturbations as

\[
\delta g_{MN} dx^M dx^N = (h(T) T_{\mu\nu} + h(LT) T_{(LT)\mu\nu} + h(LL) T_{(LL)\mu\nu} + h(Y) T_{(Y)\mu\nu}) dx^\mu dx^\nu + 2(h(T) w V_{(T)\mu} + h(L) w V_{(L)\mu}) dx^\mu dw + h_{ww} Y dw^2,
\]

\[ \delta \Psi = \psi Y, \]

\[ \delta Z_{\pm M} dx^M = (z_{\pm(T)} V_{(T)\mu} + z_{\pm(L)} V_{(L)\mu}) dx^\mu + z_{\pm w} Y dw, \]  

and

\[ \delta \bar{S}_{\pm \mu \nu} = \bar{\tau}_{\pm(T)} T_{(T)\mu\nu} + \bar{\tau}_{\pm(LT)} T_{(LT)\mu\nu} + \bar{\tau}_{\pm(LL)} T_{(LL)\mu\nu} + \bar{\tau}_{\pm(Y)} T_{(Y)\mu\nu}, \]

where we omitted dependence of harmonics and the corresponding coefficients on the 4-dimensional momentum \( k_\mu \) and the integration with respect to \( k_\mu \). The \( k \)-dependent Fourier coefficients \( h_{(T,LT,LL,Y)} \), \( h_{(L)w} \), \( h_{ww} \), and \( \psi \) are functions of the fifth coordinate \( w \) only, and the other coefficients \( z_{\pm(T,L)} \), \( z_{\pm w} \), and \( \bar{\tau}_{\pm(T,LT,LL,Y)} \) are constants.

Now we can analyze 5-gauge transformation of the coefficients of the harmonic expansion and construct 5-gauge-invariant variables, or those linear combinations of perturbations that are invariant under the 5-gauge transformation. The result is

\[
F_{(T)} = h(T),
\]

\[
F_w = h(T) w - e^{-2A} (e^{2A} h(LT)) ,
\]

\[
F = h(Y) + 2A X_w + \frac{1}{2} \eta^\mu_\nu k_\mu k_\nu h(LL),
\]

\[
F_{ww} = h_{ww} - 2e^{-A} (e^{2A} X_w) ,
\]

\[
\varphi = \psi - e^{2A} \psi(0) X_w ,
\]

\[
\phi_{\pm(T)} = z_{\pm(T)} + h(LT) |_{w = w_\pm} ,
\]

\[
\phi_{\pm(L)} = z_{\pm(L)} + h(LL) |_{w = w_\pm} ,
\]

\[
\phi_{\pm w} = z_{\pm w} + X_w |_{w = w_\pm} ,
\]  

(17)

where \( X_w = h(L) w - e^{-2A} (e^{2A} h(LL)) \). They form a maximal set of independent 5-gauge invariant variables constructed from the metric perturbation \( \delta g_{MN} \), the scalar field perturbation \( \delta \Psi \) and the brane fluctuation \( \delta Z^M_{\pm \mu \nu} \). (The matter perturbation \( \delta \bar{S}_{\pm \mu \nu} \) is not included here since it is a 4-dimensional object.)

We can also analyze 4-gauge transformation and construct 4-gauge-invariant variables, or those linear combinations of perturbations that are invariant under the 4-gauge transformation. For this purpose we first need to obtain expressions of various 4-dimensional quantities in terms of the 5-dimensional quantities \( g_{MN} \), \( \Psi \) and \( Z^M_{\pm \mu \nu} \). What we need are the physical metric \( \bar{q}_{\pm \mu \nu} \) defined by (13), the extrinsic curvature \( K_{\pm \mu \nu} \), the pull back \( \Psi_{\pm} \) of the scalar field \( \Psi \), and the normal derivative \( \partial_1 \Psi_{\pm} \) of \( \Psi \) on \( \Sigma_{\pm} \):

\[
\bar{q}_{\pm \mu \nu}(y_\pm) = \exp \left[ -\alpha_{\pm}(\Psi_{\pm}) \right] e^M_{\pm \mu} e^N_{\pm \nu} g_{MN} \bigg|_{x = Z_{\pm}(y_\pm)} ,
\]

\[
K_{\pm \mu \nu}(y_\pm) = \frac{1}{2} e^M_{\pm \mu} e^N_{\pm \nu} \mathcal{L}_{\pm \mu \nu} g_{MN} \bigg|_{x = Z_{\pm}(y_\pm)} ,
\]

\[
\Psi_{\pm}(y_\pm) = \Psi \bigg|_{x = Z_{\pm}(y_\pm)} ,
\]

\[
\partial_1 \Psi_{\pm} = n^M_{\pm} \partial_M \Psi \bigg|_{x = Z_{\pm}(y_\pm)} ,
\]  

(18)

where
\[ e_{\pm \mu}^M = \frac{\partial Z_{\pm}^M}{\partial y_{\pm}^\mu}. \] (19)

and \( \mathcal{L} \) represents a 5-dimensional Lie derivative. Using the harmonic expansions (13), we can obtain the corresponding harmonic expansion of \( \delta q_{\pm \mu \nu}, \delta K_{\pm \mu \nu}, \delta \Psi_{\pm} \) and \( \delta \partial_{\perp} \Psi_{\pm} \). The result is

\[
\begin{align*}
\delta q_{\pm \mu \nu} &= \bar{\sigma}_{\pm} (T) T_{(T)\mu\nu} + \bar{\sigma}_{\pm} (LT) T_{(LT)\mu\nu} + \bar{\sigma}_{\pm} (LL) T_{(LL)\mu\nu} + \bar{\sigma}_{\pm} (Y) T_{(Y)\mu\nu}, \\
\delta K_{\pm \mu \nu} &= k_{\pm} (T) T_{(T)\mu\nu} + k_{\pm} (LT) T_{(LT)\mu\nu} + k_{\pm} (LL) T_{(LL)\mu\nu} + k_{\pm} (Y) T_{(Y)\mu\nu}, \\
\delta \Psi_{\pm} &= \psi_{\perp} Y, \\
\delta \partial_{\perp} \Psi_{\pm} &= \psi_{\perp} Y,
\end{align*}
\]

where \( \delta \tilde{K}_{\pm \mu \nu} \) is defined by

\[ \delta \tilde{K}_{\pm \mu \nu} \equiv \delta K_{\pm \mu \nu} - \frac{1}{2} (K_{\pm \mu \nu} (0) \delta q_{\pm \rho \nu} + K_{\pm \rho \nu} (0) \delta q_{\pm \mu \rho}). \] (21)

and \( k \)-dependent Fourier coefficients are

\[
\begin{align*}
\bar{\sigma}_{\pm} (T) &= e^{-\alpha_{\pm}^{(0)}} F_{(T)}, \\
\bar{\sigma}_{\pm} (LT) &= e^{-\alpha_{\pm}^{(0)}} \phi_{\pm} (T), \\
\bar{\sigma}_{\pm} (LL) &= e^{-\alpha_{\pm}^{(0)}} \phi_{\pm} (L), \\
\bar{\sigma}_{\pm} (Y) &= e^{-\alpha_{\pm}^{(0)}} \left[ -2 A \dot{\phi}_{\pm w} - \frac{1}{2} \eta_{\mu \nu} k_{\mu} k_{\nu} \phi_{\pm} (L) - \epsilon_{\pm} (0) (e^{-2 A} \varphi + \phi_{\pm w} \dot{\Psi}) \right], \\
k_{\pm} (T) &= \frac{1}{2} e^{-A} \left( e^{2 A} F_{(T)} \right), \\
k_{\pm} (LT) &= \frac{1}{2} e^{A} F_{w}, \\
k_{\pm} (LL) &= \frac{1}{2} e^{A} \phi_{\pm w}, \\
k_{\pm} (Y) &= \frac{1}{2} \left\{ e^{-A} \left( e^{2 A} F + (e^{A})' \right) F_{ww} + \left[ \frac{1}{2} e^{A} \eta_{\mu \nu} k_{\mu} k_{\nu} - 2 (e^{A})' \right] \phi_{\pm} \right\}, \\
\dot{\psi}_{\pm} &= \varphi + e^{2 A} \dot{\Psi} (0) \phi_{\pm w}, \\
\psi_{\perp \perp} &= \frac{1}{2} \left[ -e^{3 A} \dot{\Psi} (0) F_{ww} + 2 e A \dot{\phi} + 2 e^{2 A} \left( e^{A} \dot{\Psi} (0) \right) \phi_{\pm w} \right].
\end{align*}
\] (22)

Here, the right hand sides of (22) are evaluated at \( w = w_{\pm} \), respectively, and have been written in terms of 5-gauge invariant variables \( \bar{F}, \phi, F_{ww}, F_{wT}, F_{TT} \). The matter perturbation \( \delta \bar{S}_{\pm \mu \nu} \) on each brane can also be expanded by harmonics as

\[ \delta \bar{S}_{\pm \mu \nu} = \bar{\tau}_{\pm (T)} T_{(T)\mu\nu} + \bar{\tau}_{\pm (LT)} T_{(LT)\mu\nu} + \bar{\tau}_{\pm (LL)} T_{(LL)\mu\nu} + \bar{\tau}_{\pm (Y)} T_{(T)\mu\nu}. \] (23)

We can now analyze 4-gauge transformation of coefficients the harmonic expansion and construct the following 4-gauge invariant variables from the physical metric perturbation \( \delta q_{\pm \mu \nu} \).

\[
\begin{align*}
\tilde{f}_{\pm (T)} &= \bar{\sigma}_{\pm} (T) = e^{-\alpha_{\pm}^{(0)}} F_{(T)}, \\
\tilde{f}_{\pm} &= \bar{\sigma}_{\pm} (Y) + \frac{1}{2} \eta_{\mu \nu} k_{\mu} k_{\nu} \bar{\sigma}_{\pm} (LL) = e^{-\alpha_{\pm}^{(0)}} \left[ -2 A \dot{\phi}_{\pm w} - \alpha_{\pm}^{(0)} (e^{-2 A} \varphi + \phi_{\pm w} \dot{\Psi}) \right].
\end{align*}
\] (24)

It is easily shown that \( k_{\pm (T,LT,LL,Y)}, \psi_{\perp \perp}, \psi_{\perp \perp} \) and \( \bar{\tau}_{(T,LT,LL,Y)} \) are invariant under the 4-gauge transformation. Moreover, they are at the same time 5-gauge invariant since all except for \( \bar{\tau}_{(T,LT,LL,Y)} \) are written in terms of 5-gauge invariant variables. This fact illustrates that 4-gauge transformation is not a part of 5-gauge transformation and that these two kinds of gauge transformations are independent.

\footnote{The reason why they can be expressed in terms of 5-gauge-invariant variables only is that they by themselves are 5-gauge-invariant \( \bar{F}, \phi, F_{ww}, F_{wT}, F_{TT} \). This fact illustrates that 4-gauge transformation is not a part of 5-gauge transformation and that these two kinds of gauge transformations are independent.}
invariant variables and $\delta S_{\pm, \mu \nu}$ is a 4-dimensional object. Hence, we have the set $(\hat{f}_{\pm}(T), \tilde{f}_{\pm}, k_{\pm(T,LT,LL,Y)}, \psi_{\pm}, \psi_{\pm, \perp}, \tau_{(T,LT,LL,Y)})$ of doubly-gauge invariant variables.

From the point of view of observers on each brane, all observable quantities must be doubly-gauge invariant. However, all doubly-gauge invariant variables are not necessarily observable quantities. They can observe physical metric perturbation $(\hat{f}_{\pm}(T), \tilde{f}_{\pm})$ and matter perturbation $\tilde{\tau}_{(T,LT,LL,Y)}$ only. The remaining doubly-gauge invariant variables $(k_{\pm(T,LT,LL,Y)}, \psi_{\pm}, \psi_{\pm, \perp})$ shall be used to write down junction conditions of 5-dimensional quantities in a doubly-gauge invariant way.

Our remaining task in this section is to give 5-gauge invariant equations in the bulk and doubly-gauge invariant junction conditions on each brane. Our final aim in this paper is to seek doubly-gauge invariant equations governing the physical metric perturbations and matter perturbations on our brane and to compare the resulting equations with the corresponding equations in 4-dimensional higher derivative gravity.

Since there are many coefficients in the above harmonic expansions, let us divide these into three classes. The first class is the scalar perturbations and consists of coefficients of $Y, V_{(L)\mu}, T_{(LL)\mu\nu}$ and $T_{(Y)\mu\nu}$. The second is the vector perturbations and consists of coefficients of $V_{(T)\mu}$ and $T_{(LT)\mu\nu}$. The last is the tensor perturbations and consists of coefficients of $T_{(T)\mu\nu}$. Perturbations in different classes are decoupled from each other at the linearized level. Hence, in the following we analyze perturbations in each class separately. Decomposition into scalar, vector and tensor modes is commonly used in cosmology. However, usually in cosmology we use scalar, vector and tensor representations of the isometry group related to the symmetry of 3-dimensional space. Meanwhile, here we will use scalar, vector and tensor representations of the isometry group of 4-dimensional space-time.

In ref. [16] it was shown that vector type perturbations vanish unless matter fields on the hidden brane are excited. In this paper we assume that there is no matter excitation on the hidden brane and, thus, we shall consider scalar and tensor type perturbations only.

### C. Scalar perturbations

For scalar perturbations, we have three 5-gauge invariant variables from metric perturbation and scalar field perturbation in the bulk: $F$, $F_{ww}$ and $\varphi$. The first two are from metric perturbation and the last one is from scalar field perturbation. The Einstein equation leads to two relations among them

$$ F_{ww} = -2F, $$

$$ \varphi = -\frac{3e^{2A}}{2\kappa^2} \dot{F}, $$

and a wave equation

$$ \ddot{A} + \ddot{A}^2 \left[ \frac{\dot{A}_0^2 e^{3A}}{A + \dot{A}} \right] - \eta^{\mu\nu} k_{\mu} k_{\nu} \left( \frac{F}{A e^A} \right) = 0, $$

where a dot denotes derivative with respect to $w$ and $k_{\mu}$ is the 4-dimensional momentum in the coordinate $y^{0\mu}$. Throughout this paper we consider modes with $k_{\mu} \neq 0$ for scalar perturbations since a scalar mode with $k_{\mu} = 0$ preserves 4-dimensional Poincare symmetry and, thus, represents just a change of the background within the ansatz $[9]$. Of course we shall consider modes with $\eta^{\mu\nu} k_{\mu} k_{\nu} = 0$ as long as $k_{\mu} \neq 0$.

The boundary condition at $w = w_{\pm}$, respectively, is given by the junction condition for the scalar field as

$$ \mp \left[ -e^{3A} \dot{\phi}_{w} + 2e^A \phi_{w} + 2e^{2A} \left( e^A \dot{\psi}_{w}(0) \right) \phi_{w} \right] = V^{(0)}_{\pm} \left( \varphi + e^{2A} \dot{\psi}_{w}(0) \psi_{w} + 2e^A \tau_{\pm(Y)} \right), $$

where $\tau_{\pm(Y)}$ and $\phi_{w}$ are doubly gauge invariant variables constructed from matter on $\Sigma_{\pm}$ and perturbation of the position of $\Sigma_{\pm}$, respectively, and satisfy the following equations derived from the perturbed Israel’s junction condition $[8]$.  

\[ ^2\text{For a mode with } k_{\mu} = 0 \text{ we do not have the second equation of (28) since there is no tensor harmonics of the type } (LL) \text{ for } k_{\mu} = 0. \text{ See appendix [A] for definition and properties of harmonics.} \]
\[ 2\tilde{\tau}_\pm(Y) = 3\eta^{\mu\nu}k_\mu k_\nu \tilde{\tau}_\pm(LL), \]
\[ \phi_{\pm w} = \mp \kappa_5^2 e^{-A - \alpha_\pm(0)} \tilde{\tau}_\pm(LL). \]

Here, \( \alpha_\pm^{(0)} = \alpha(\Psi^{(0)})|_{w = w_\pm}, \alpha_\pm''^{(0)} = \alpha''(\Psi^{(0)})|_{w = w_\pm} \) and \( \tilde{\tau}_\pm(LL) \) is another doubly gauge invariant variable constructed from matter on \( \Sigma_\pm \). The first of (28) is nothing but the conservation equation of matter stress energy tensor \( \tilde{S}_{\pm\mu\nu} \) on each brane. The second equation is the \( (LL) \)-component of the Israel’s junction condition and relates the perturbation of the brane position and the matter perturbation. The boundary condition (27) at \( w = w_\pm \), respectively, can be rewritten to the following form by eliminating \( \varphi, \tau(\pm Y) \) and \( \phi_{\pm w} \), and using the wave equation (24).

\[ C_\pm \left[ \dot{F} \mp 2\kappa_5^2 e^{-A - \alpha_\pm(0)} (\dot{A} + \dot{\bar{A}}) \tilde{\tau}_\pm(LL) \right] + \eta^{\mu\nu}k_\mu k_\nu \bar{\Psi}^{(0)} \left[ F \mp \kappa_5^2 \alpha_\pm''(0) e^{-A - \alpha_\pm(0)} \tilde{\Psi}^{(0)} \tilde{\tau}_\pm(LL) \right] = 0, \]  

where

\[ C_\pm = \bar{\Psi}^{(0)} + \dot{A} \bar{\Psi}^{(0)} \mp \frac{1}{2} e^{-A} \bar{\Psi}^{(0)} V''(\bar{\Psi}^{(0)}) \bigg|_{w = w_\pm}. \]  

Finally, the doubly gauge invariant perturbation of the physical metric \( \tilde{q}_{\pm\mu\nu} \) is expressed as

\[ e^{\alpha_\pm(0)} \tilde{f}_\pm = F \mp 2\kappa_5^2 \dot{A} e^{-A - \alpha_\pm(0)} \bar{\tau}_\pm(LL) - \eta^{\mu\nu}k_\mu k_\nu \frac{3\alpha_\pm''(0)}{2\kappa_5^2 C_\pm} \left[ F \mp \kappa_5^2 \alpha_\pm''(0) e^{-A - \alpha_\pm(0)} \bar{\Psi}^{(0)} \bar{\tau}_\pm(LL) \right] \bigg|_{w = w_\pm}. \]  

In the next section we shall assume that \( \bar{\Psi}^{(0)}(w_\pm)C_\pm \neq 0 \) to show the recovery of higher derivative gravity on a brane. In this case, the expression of \( \tilde{f}_\pm \) can be rewritten to the following form by eliminating \( \varphi, \tilde{\tau}_\pm \) and \( \phi_{\pm w} \), and using the boundary condition (24).

\[ e^{\alpha_\pm} \tilde{f}_\pm = F \mp 2\kappa_5^2 \dot{A} e^{-A - \alpha_\pm} \tilde{\tau}_\pm(LL) - \eta^{\mu\nu}k_\mu k_\nu \frac{3\alpha_\pm''(0)}{2\kappa_5^2 C_\pm} \left[ F \mp \kappa_5^2 \alpha_\pm''(0) e^{-A - \alpha_\pm} \bar{\Psi}^{(0)} \tilde{\tau}_\pm(LL) \right] \bigg|_{w = w_\pm}. \]

Hereafter, we consider \( \Sigma_+ \) as our brane and \( \Sigma_- \) as the hidden brane, and assume that there is no matter excitations on the hidden brane. Hence, we put \( \tilde{\tau}_{-(LL,Y)} = 0 \).

**D. Tensor perturbations**

For tensor perturbations we have only one 5-gauge invariant variable constructed from metric perturbation in the bulk: \( F(T) \). The Einstein equation leads to the wave equation

\[ e^A \left[ e^{-3A}(e^{2A} F(T)) \right]' - \eta^{\mu\nu}k_\mu k_\nu F(T) = 0, \]  

where \( k^\mu \) is the 4-dimensional momentum in the coordinate \( y^\mu \).

The Israel junction condition leads to the following boundary condition at \( w = w_\pm \), respectively.

\[ (e^{2A} F(T))' = \pm \kappa_5^2 e^{A - \alpha_\pm(0)} \tilde{\tau}_\pm(T), \]  

where \( \tilde{\tau}_\pm(T) \) is a doubly gauge invariant variable constructed from matter on \( \Sigma_\pm \).

Finally, the doubly gauge invariant perturbation of the physical metric \( \tilde{q}_{\pm\mu\nu} \) is expressed as

\[ \tilde{f}_\pm(T) = e^{-\alpha_\pm(0)} F(T) \bigg|_{w = w_\pm}. \]  

Hereafter, since we assumed that there is no matter excitations on the hidden brane, we put \( \tilde{\tau}_{-(T)} = 0 \).
III. LOW ENERGY EXPANSION

As already stated in the third-to-the-last paragraph of subsection III, our aim in this paper is to seek doubly-gauge invariant equations governing the physical metric perturbations and matter perturbations on our brane and to compare the resulting equations with the corresponding equations in 4-dimensional higher derivative gravity. Since we have only two gauge-invariant metric perturbations $\bar{f}_+^{(1)}$ and $\bar{f}_+^{(2)}$, and essentially two gauge-invariant matter perturbations $\bar{T}_{+\mu\nu}^{(LL)}$ and $\bar{T}_{+\mu\nu}^{(T)}$ on our brane, we shall seek the following form of the equations on the brane.

\[ C_{(s)} \bar{f}_+ = \bar{T}_{+\mu\nu}^{(LL)}, \]
\[ \bar{q}_+^{(0)\mu\nu} k_\mu k_\nu C_{(T)} \bar{f}_+^{(T)} = \bar{T}_{+\mu\nu}^{(T)}, \]

where $C_{(s)}$ and $C_{(T)}$ are functions of $\bar{q}_+^{(0)\mu\nu} k_\mu k_\nu$ and $\bar{q}_+^{(0)\mu\nu} = \Omega_+^{-2} \eta^{\mu\nu}$ is the inverse of the unperturbed physical metric $\eta^{\mu\nu}$. The reason why we expect the linear dependence of metric perturbations on matter perturbations is that boundary conditions summarized in the previous section are linear in the matter perturbations. The reason why $\bar{q}_+^{(0)\mu\nu} k_\mu k_\nu$ was put in front of $\bar{f}_+^{(T)}$ is that we expect 4-dimensional gravitons on our brane (non-vanishing $\bar{f}_+^{(T)}$ with $\bar{T}_{+\mu\nu}^{(T)} = 0$ and $\bar{T}_{+\mu\nu}^{(LL)} = 0$). What is important here is that the functions $C_{(s,T)}$ completely characterize the effective theory of weak gravity on our brane.

Since we are dealing with gauge-invariant variables only, there is no ambiguity of gauge freedom when we compare the result with the corresponding equations in 4-dimensional higher derivative gravity. Namely, we only have to compare functions $C_{(s,T)}$ of $\bar{q}_+^{(0)\mu\nu} k_\mu k_\nu$ with the corresponding functions of momentum squared in the Fourier transformed, linearized 4-dimensional higher derivative gravity.

In this section we expand the basic equations summarized in the previous section by the parameter $\mu = l^2 \eta^{\mu\nu} k_\mu k_\nu$ and solve them iteratively, where $\mu$ has the characteristic length scale of the model which we shall determine by comparing the results of order $O(1)$ and $O(\mu)$. The purpose of the $\mu$-expansion is to analyze the behavior of the functions $C_{(s)}$ and $C_{(T)}$ near $\mu = 0$. Namely, we shall seek first few coefficients $C_{(s,T)}^{[i]}$ (i = 0, 1, · · ·) of the expansion

\[ C_{(s,T)} = \sum_{i=0}^{\infty} \mu^i C_{(s,T)}^{[i]}, \]

Since the 4-dimensional physical energy scale $m_+$ on $\Sigma_+$ is given by $m_+^2 = -\bar{q}_+^{(0)\nu\nu} k_\nu = -\mu l^{-2} e^\alpha S + 2A_+$, the expansion in $\mu$ is nothing but the low energy expansion. Hence, the first few coefficients $C_{(s,T)}^{[i]}$ (i = 0, 1, · · ·) of the expansion determine the low energy behavior of weak gravity on our brane. We expect that $C_{(s,T)}^{[0]}$ give 4-dimensional Einstein gravity, that $C_{(s,T)}^{[1]}$ give curvature-squared corrections to the Einstein gravity, and so on. The length scale $l$ gives the energy scale $l^{-1} e^\alpha S / 2A_+$ below which we can trust the $\mu$-expansion. Nonetheless, we can defer the determination of $l$ until we obtain the results of order $O(\mu)$ since $l$ can be eliminated from all formal results in each order of the $\mu$-expansion. Meanwhile, we shall keep it in intermediate calculations in order to make the expansion parameter dimensionless.

First, by expanding $F$ and $F(T)$ as

\[ F(u) = \sum_{i=0}^{\infty} \mu^i F^{[i]}(u), \]
\[ F(T)(u) = \sum_{i=0}^{\infty} \mu^i F^{[T]}(u), \]

we can solve the wave equations order by order. The result is

\[ F^{[i]}(u) = \sum_{i=0}^{\infty} \mu^i F^{[i]}(u), \]

If one likes, one can restore gauge fixed equations for any gauge choices from the functions $C_{(s,T)}^{[i]}$ only. For example, see [21].
\[ F^{[i]}(w) = \tilde{A}(w)e^{\tilde{A}(w)} \left[ C_{1}^{[i]} + C_{2}^{[i]} \int_{w_{-}}^{w} dw' \frac{\ddot{A}(w') + \dot{A}(w')^{2}}{A(w')^{2}} e^{-3A(w')} \right. \\
\left. + l^{-2} \int_{w_{-}}^{w} dw' \frac{\dot{A}(w') + A(w')^{2}}{A(w')^{2}} e^{-3A(w')} \int_{w_{-}}^{w} dw'' \frac{\dot{A}(w'') e^{2A(w'')}}{A(w'') + A(w'')^{2}} F^{[i-1]}(w'') \right], \]

\[ F_{(T)}^{[i]}(w) = e^{-2A(w)} \left[ D_{1}^{[i]} \int_{w_{-}}^{w} dw' e^{3A(w')} + D_{2}^{[i]} \int_{w_{-}}^{w} dw' e^{-A(w')} F^{[i-1]}(w') \right], \]

(39)

with \( F^{[i-1]} = F^{[i]}_{(T)} = 0 \), where \( C_{1,2}^{[i]} \) and \( D_{1,2}^{[i]} \) are constants.

Next, let us analyze the boundary condition. For this purpose, we expand \( \tilde{\tau}_{(LL,T)} \) as

\[
\tilde{\tau}_{+(LL)} = \sum_{i=0}^{\infty} \mu_{\tau}^{[i]}_{+(LL)}, \\
\tilde{\tau}_{+(T)} = \sum_{i=0}^{\infty} \mu_{\tau}^{[i]}_{+(T)},
\]

(40)

and put \( \tilde{\tau}_{-(LL)} = \tilde{\tau}_{-(T)} = 0 \) since we assumed that there is no matter excitation on the hidden brane. The boundary condition (29) for scalar perturbations can be rewritten as

\[
C_{1}^{[i]} + B_{+} C_{2}^{[i]} = -l^{-2} X_{+}^{[i-1]} - 2\kappa_{5}^{2} e^{-2A_{+}-\alpha_{+}^{(0)}} \tilde{\tau}_{+(LL)}^{[i]}, \\
C_{1}^{[i]} + B_{-} C_{2}^{[i]} = -l^{-2} X_{-}^{[i-1]},
\]

(41)

where

\[
B_{+} = \int_{w_{-}}^{w_{+}} dw' \frac{\ddot{A}(w) + \dot{A}(w)e^{-3A(w)} + e^{-3A_{+}}}{A(w)} , \\
B_{-} = e^{-3A_{-}} \frac{A_{-}}, \\
X_{+}^{[i]} = \int_{w_{-}}^{w_{+}} dw' \frac{\dot{A}(w') + A(w')^{2}}{A(w')^{2}} e^{-3A(w')} \int_{w_{-}}^{w} dw'' \frac{\dot{A}(w'') e^{2A(w'')}}{A(w'') + A(w'')^{2}} F^{[i]}(w') \\
+ \frac{e^{-3A_{+}}}{A_{+}} \int_{w_{-}}^{w_{+}} dw' \frac{\dot{A}(w') e^{2A(w')}}{A(w') + A(w')^{2}} F^{[i]}(w') \\
+ \frac{3 e^{-A_{+}}}{\kappa_{5}^{2} \Psi^{(0)}_{+}} \left[ F^{[i]}(w_{+}) \right. - \frac{2 \kappa_{5}^{2} e^{-2A_{+}}}{\kappa_{5}^{2} \Psi^{(0)}_{+} - A_{+} + \alpha_{+}^{(0)}} \left. \Psi^{(0)}_{+} \tilde{\tau}_{+(LL)}^{[i]} \right] , \\
X_{-}^{[i]} = \frac{3 e^{-A_{-}}}{\kappa_{5}^{2} \Psi^{(0)}_{-}} F^{[i]}(w_{-}) ,
\]

(42)

and \( X_{-}^{[i-1]} = 0 \). We have assumed that \( \Psi_{\pm}^{(0)} C_{\pm} \neq 0 \) and that \( \dot{A}_{\pm} \neq 0 \). Here, in order to rewrite the last term of \( X_{+}^{[i]} \) and \( X_{-}^{[i]} \), we have used the background equation (14). We have adopted the following abbreviation: \( A_{\pm} = A(w_{\pm}) \), \( \dot{A}_{\pm} = \dot{A}(w_{\pm}) \), \( \ddot{A}_{\pm} = \ddot{A}(w_{\pm}) \) and \( \Psi_{\pm}^{(0)} = \Psi^{(0)}(w_{\pm}) \). The boundary condition (34) for tensor perturbations can be easily solved to give

\[
\tilde{\tau}_{+(T)}^{[i]} = 0, \\
D_{1}^{[i]} = 0, \\
D_{2}^{[i]} = \frac{1}{\int_{w_{-}}^{w} dw e^{-3A(w)}} \left[ \kappa_{5}^{2}^{2} e^{-2A_{+}-\alpha_{+}^{(0)}} \tilde{\tau}_{+(T)}^{[i+1]} - l^{-2} \chi_{+(T)}^{(i-1)} \right],
\]

(43)
where

\[
Y^{[i]} = \int_{w_-}^w dw e^{-3A(w)} \int_{w_-}^w dw' e^{3A(w')} \int_{w_-}^{w'} dw'' e^{-A(w'')} F^{[i]}_{(T)}(w''),
\]

(44)

and \(Y^{[-1]} = 0\).

Thirdly, the doubly gauge invariant variables \(\tilde{f}_+\) and \(\tilde{f}_+(T)\) corresponding to perturbation of the physical metric \(\bar{q}_{\mu\nu}\) on \(\Sigma_+\) are expanded as

\[
\tilde{f}_+ = \sum_{i=0}^{\infty} \mu^i \tilde{f}_+^{[i]},
\]

\[
\tilde{f}_+(T) = \sum_{i=0}^{\infty} \mu^i \tilde{f}_+^{[i]}(T),
\]

(45)

where the expansion coefficients are given by

\[
e^{\alpha^0} \tilde{f}_+^{[i]} = F^{[i]}(w_+) + 2\kappa_5^2 A e^{-A_+ - \alpha^0} \tilde{\tau}_+(LL),
\]

\[
e^{\alpha^0} \tilde{f}_+^{[i]}(T) = F^{[i]}_{(T)}(w_+),
\]

with \(F^{[-1]} = \tilde{\tau}_+^{[-1]} = 0\).

Now let us summarize first few terms in the \(\mu\)-expansion.

\[
\tilde{f}_+^{[0]} = -16\pi G_N + \Omega_+^2 \tilde{\tau}_+(LL),
\]

\[
\tilde{f}_+^{[1]} = -16\pi G_N + \left[ \Omega_+^2 \tilde{\tau}_+(LL) - l^2 \tau_{S_+}^{[1]}(LL) \right],
\]

\[
\tilde{\tau}_+(T) = 0,
\]

\[
\tilde{f}_+^{[0]}(T) = 16\pi G_N + \Omega_+^2 \tilde{\tau}_+^{[0]}(T),
\]

\[
\tilde{f}_+^{[1]}(T) = 16\pi G_N + \left[ \Omega_+^2 \tilde{\tau}_+^{[2]}(T) + l^2 \tau_{T_+}^{[1]}(T) \right],
\]

(47)

where

\[
\Omega_+^2 = e^{-2A_+ - \alpha^0},
\]

\[
16\pi G_N = \frac{\int_{w_-}^w \int_{w_-}^w dw e^{-3A(w)} \left( \frac{w'}{A(w)} \right)^2}{\int_{w_-}^w dw e^{-3A(w)}}.
\]

(48)

From these results, it is easy to show that

\[
\left[ 1 + l^2 S_+ q_+^{(0)\mu\nu} k_{\mu} k_{\nu} + O(\mu^2) \right] \tilde{f}_+ = -16\pi G_N + \Omega_+^2 \tilde{\tau}_+(LL),
\]

\[
q_+^{(0)\mu\nu} k_{\mu} k_{\nu} \left[ 1 - l^2 T_+ q_+^{(0)\mu\nu} k_{\mu} k_{\nu} + O(\mu^2) \right] \tilde{f}_+(T) = 16\pi G_N \Omega_+^2 \tilde{\tau}_+(T),
\]

(49)
where \( \tilde{q}_+^{(0)} = \Omega_+^{-2} \eta^{\mu\nu} \) is the inverse of the unperturbed physical metric, \( \tilde{q}_+^{(0)}_{\mu\nu} = \Omega_+^{-2} \eta_{\mu\nu} \), on \( \Sigma_+ \).

As promised, the length scale \( l \) does not appear in the results \([19]\). However, we have obtained two lengths \( |l_{S^+}| \) and \( |l_{T^+}| \). It is evident that we cannot trust the \( \mu \)-expansion, or the low energy expansion, at energies above \( \min(|l_{S^+}^{-1}|, |l_{T^+}^{-1}|) \).

Hence, we should choose \( l = \Omega_+^{-1} \max(|l_{S^+}|, |l_{T^+}|) \).

The result \([14]\) is of the expected form \([13]\) with the expansion \([17]\) of the functions \( C_{(S,T)} \). We have determined first two coefficients of the expansions:

\[
C_{(S)}^{[0]} = \frac{-\Omega_+^{-2}}{16\pi G N_+},
C_{(T)}^{[0]} = \frac{1}{16\pi G N_+},
C_{(S)}^{[1]} = \left( \frac{l_{S^+}}{\Omega_+ l} \right)^2 C_{(S)}^{[0]},
C_{(T)}^{[1]} = -\left( \frac{l_{T^+}}{\Omega_+ l} \right)^2 C_{(T)}^{[0]},
\]

\( IV. \) **Higher derivative gravity in four-dimensions**

In this section, for the purpose of comparison, we review the linear perturbations in 4-dimensional higher derivative gravity. In particular, we consider a theory whose action includes curvature-squared terms:

\[
I = \frac{1}{16\pi G_N} \int d^4 x \sqrt{-g} \left[ R + \tilde{a}_1 R^2 + \tilde{a}_2 R_\mu^2 R^\mu + \tilde{a}_3 R_\mu^\nu R^\mu R_\nu^\sigma R^\sigma_{\mu\nu} \right], \quad (51)
\]

where \( \tilde{a}_1, \tilde{a}_2 \) and \( \tilde{a}_3 \) are constants. The variation of this action plus matter action with respect to the metric gives the following equation of motion.

\[
G_{\mu\nu} + \tilde{a}_1 E_{1\mu\nu} + \tilde{a}_2 E_{2\mu\nu} + \tilde{a}_3 E_{3\mu\nu} = 8\pi G_N T_{\mu\nu}, \quad (52)
\]

where

\[
E_{1\mu\nu} = -2 \nabla_\mu \nabla_\nu R + 2 \nabla^2 R g_{\mu\nu} + 2 R R_{\mu\nu} - \frac{1}{2} R^2 g_{\mu\nu},
E_{2\mu\nu} = -\nabla_\mu \nabla^\rho R_{\rho\nu} - \nabla_\nu \nabla^\rho R_{\rho\mu} + \nabla^2 R_{\mu\nu} + \nabla^\rho \nabla_\rho R_{\mu\nu} g_{\rho\sigma} + 2 R_{\mu\rho} R_{\nu}^\rho - \frac{1}{2} R^\rho_{\rho\sigma} R^\sigma_{\mu\nu} g_{\rho\mu},
E_{3\mu\nu} = 2(\nabla^\rho \nabla_\sigma + \nabla_\rho \nabla_\sigma) R_{\mu\rho\sigma\alpha} + 2 R_{\mu\rho\sigma\alpha} R_{\nu}^\rho R_{\lambda\sigma}^\alpha - \frac{1}{2} R_{\rho\sigma}^\alpha R_{\mu\nu}^\rho R_{\alpha\beta}^\sigma g_{\mu\nu}. \quad (53)
\]

It is known that these three tensors \( E_{i\mu\nu} \) (\( i = 1, 2, 3 \)) are not independent. Actually, it can be shown that \( E_{1\mu\nu} - 4E_{2\mu\nu} + E_{3\mu\nu} = 0 \) in general. The easiest way to see this relation is to note that the choice \( (a_1, a_2, a_3) = (a, -4a, a) \) leads to a combination called Gauss-Bonnet term. Hence, the relation \( E_{1\mu\nu} - 4E_{2\mu\nu} + E_{3\mu\nu} = 0 \) follows from the fact that the Gauss-Bonnet term is topological. Because of the linear relation among \( E_{i\mu\nu} \) (\( i = 1, 2, 3 \)) it seems convenient to reparameterize the action \((51)\). For later convenience we adopt the following reparameterization by taking another combination \( 3R_{\mu}^\rho R_{\mu}^\rho - R^2 \).

\[
I = \frac{1}{16\pi G_N} \int d^4 x \sqrt{-g} \left[ R + a_1 R^2 + a_2 (3R_{\mu}^\rho R_{\mu}^\rho - R^2) + a_3 (R_{\mu}^\rho R_{\rho\sigma}^\mu R_{\mu}^\sigma - 4R_{\mu}^\rho R_{\mu}^\nu + R^2) \right], \quad (54)
\]

where \( a_1 = \tilde{a}_1 - \tilde{a}_2 + \tilde{a}_3, a_2 = 3\tilde{a}_2 - 4\tilde{a}_3 \) and \( a_3 = \tilde{a}_3 \). The equation of motion \((52)\) becomes

\[
G_{\mu\nu} + a_1 E_{1\mu\nu} + a_2 (3E_{2\mu\nu} - E_{1\mu\nu}) = 8\pi G_N T_{\mu\nu}, \quad (55)
\]

which is explicitly independent of \( a_3 \).

We consider general perturbations around the Minkowski spacetime. Namely, we consider the metric of the form

\[
ds_4^2 = (\hat{q}_{\mu\nu}^{(0)} + \delta q_{\mu\nu}) dy^\mu dy^\nu, \quad (56)
\]
where
\[ \tilde{q}^{(0)}_{\mu \nu} = \Omega^2 \eta_{\mu \nu}, \]

\( \Omega \) is a non-zero constant, and
\[ \delta \tilde{q}_{\mu \nu} = \tilde{\sigma}(T)T_{(T)\mu \nu} + \tilde{\sigma}(LT)T_{(LT)\mu \nu} + \tilde{\sigma}(LL)T_{(LL)\mu \nu} + \tilde{\sigma}(Y)T_{(Y)\mu \nu}. \]

Here, the coefficients \( \tilde{\sigma}(T), \tilde{\sigma}(LT), \tilde{\sigma}(LL), \tilde{\sigma}(Y) \) are constants. For the definition of the harmonics \( T_{(T,LT,LL,Y)\mu \nu} \), see appendix A. Similarly to (24), we can construct gauge-invariant variables \( f(T) \) and \( \bar{f} \).

\[ f(T) = \tilde{\sigma}(T), \]
\[ \bar{f} = \tilde{\sigma}(Y) + \frac{1}{2} \eta^{\mu \nu} k_\mu k_\nu \tilde{\sigma}(LL). \]

As for the stress energy tensor \( \bar{S}_{\mu \nu} \), we consider it as a first order quantity, and expand it as follows.
\[ \bar{S}_{\mu \nu} = \bar{\tau}(T)T_{(T)\mu \nu} + \bar{\tau}(LT)T_{(LT)\mu \nu} + \bar{\tau}(LL)T_{(LL)\mu \nu} + \bar{\tau}(Y)T_{(Y)\mu \nu}, \]

where coefficients \( \bar{\tau}(T,LT,LL,Y) \) are constants. In the Minkowski background these coefficients are gauge-invariant by themselves.

We can expand the equation of motion up to the first order in the perturbations and express it in terms of the above gauge-invariant variables. The equation of motion is
\[ 2\bar{\tau}(Y) = 3\eta^{\mu \nu} k_\mu k_\nu \bar{\tau}(LL), \]
\[ \left[ 1 + 6a_1 \bar{q}^{(0)\mu \nu} k_\mu k_\nu \right] \bar{f} = -16\pi G_N \Omega^2 \bar{\tau}(LL) \]
for scalar perturbations,
\[ \bar{\tau}(LT) = 0 \]
for vector perturbations, and
\[ \bar{q}^{(0)\mu \nu} k_\mu k_\nu \left[ 1 - 3a_2 \bar{q}^{(0)\mu \nu} k_\mu k_\nu \right] \bar{f}(T) = 16\pi G_N \bar{\tau}(T) \]
for tensor perturbations.

Therefore, the weak gravity equations (49) (and the first of (28)) on our brane is indistinguishable from the linearized gravitational equations (51) and (53) in the higher derivative gravity, provided that the following correspondence is understood.
\[ G_N \leftrightarrow G_{N+}, \]
\[ 6a_1 \leftrightarrow l_{S+}^2, \]
\[ 3a_2 \leftrightarrow l_{T+}^2 \]
where \( G_{N+} \), \( l_{S+}^2 \) and \( l_{T+}^2 \) are given by (48).

**V. PHYSICAL IMPLICATIONS**

The result of the previous sections up to the order \( O(\mu) \) can be summarized by the following 4-dimensional effective gravitational action on our brane.
\[ I = \frac{1}{16\pi G_{N\pm}} \int d^4 y \sqrt{-q_{\pm}} \left[ \tilde{R}_{\pm} + \frac{1}{6} \tilde{l}_{S_{\pm}}^2 \tilde{R}_{\pm}^2 + \frac{1}{3} \tilde{l}_{S_{\pm}}^2 (3 \tilde{R}_{\pm}^{\mu \nu} \tilde{R}_{\pm}^{\rho \sigma} - \tilde{R}_{\pm}^{\rho \sigma}) + 4 \tilde{R}_{\pm}^{\mu \nu} \tilde{R}_{\pm}^{\rho \sigma} + \tilde{R}_{\pm}^{\mu \nu} \tilde{R}_{\pm}^{\rho \sigma} - a_{\pm} (\tilde{R}_{\pm}^{\mu \nu} \tilde{R}_{\pm}^{\rho \sigma} - 4 \tilde{R}_{\pm}^{\mu \nu} \tilde{R}_{\pm}^{\rho \sigma} + \tilde{R}_+^2) \right], \]
where \( \tilde{R}_{\pm}, \tilde{R}_{\pm}^{\mu \nu} \) and \( \tilde{R}_{\pm}^{\mu \nu} \) are the Ricci scalar, Ricci tensor and Riemann tensor of the minimally coupled physical metric \( q_{\pm \mu \nu} \) on the brane \( \Sigma_{\pm} \), and constants \( G_{N\pm}, l_{S\pm}^2 \) and \( l_{T\pm}^2 \) are given by (48) (and corresponding expressions for quantities with the subscript “-“). The expression with the undetermined coefficient \( a_{\pm} \) is the Gauss-Bonnet term.
and does not contribute to the equations of motion at all. Hence 4-dimensional Einstein gravity is restored only at distances much longer than $\max(|l_{S+}|, |l_{T+}|)$ or at energies much lower than $\min(|l_{S+}^2|, |l_{T+}^2|)$. Otherwise, 4-dimensional Einstein gravity is not valid on the brane $\Sigma_{\pm}$. What governs weak gravity on the brane at higher energies is the higher derivative gravity. Note that the energy scale $\min(|l_{S+}^2|, |l_{T+}^2|)$ can be lower than both 5-dimensional and 4-dimensional Planck energies.

The restoration of the higher derivative gravity on a brane was almost insensitive to the form of potentials of the scalar field and conformal transformation to a frame in which matter on the branes is minimally coupled to the metric. Only condition which we had to impose is that there is a background solution with 4-dimensional Poincare symmetry, scalar field and conformal transformation to a frame in which matter on the branes is minimally coupled to the metric.

Only condition which we had to impose is that there is a background solution with 4-dimensional Poincare symmetry, that $\dot{\Psi}_{\pm}^{(0)} C_{\pm} \neq 0$ and that $\dot{A}_{\pm} \neq 0$. The condition $\dot{\Psi}_{\pm}^{(0)} C_{\pm} \neq 0$ can be rewritten as follows

$$V_{\pm}^{(0)'(0)} \left( U_{\pm}^{(0)} + \frac{\kappa_5^2}{3} V_{\pm}^{(0)} V_{\pm}^{(0)'} - \frac{1}{4} V_{\pm}^{(0)} V_{\pm}^{(0)''} \right) \neq 0.$$  \hspace{1cm} (66)

This condition is equivalent to the condition that the background bulk scalar field is changing near the branes and that the radion is stabilized. Since $\dot{\Psi}_{\pm}^{(0)}$ is related to $\dot{\Psi}^{(0)}$ by $\dot{\Psi}^{(0)} = \frac{1}{2} \dot{\Psi}_{\pm}^{(0)}$ on branes, finally the condition (66) is equivalent to the condition that the background bulk scalar field is changing near the branes and that the radion is stabilized. The final condition $\dot{A}_{\pm} \neq 0$ can be restated that the bulk geometry should be warped near the branes.

A higher derivative theory with the action (64) is known to be unstable if the coefficient $l_{S\pm}^2$ is negative [23]. One can see this instability in the first equation of (49) at the linearized level: if $l_{S\pm}^2$ is negative then the equation has a tachyonic solution $(q_{\pm}^{(0)\mu\nu} k_{\mu} k_{\nu} = |l_{S\pm}|^{-2})$ with vanishing matter on the brane ($\bar{\tau}_{\pm}(LL) = 0$). Since the coefficient $l_{S\pm}^2$ depends on the background solution as shown in (63), one can perhaps conclude that the stability condition $l_{S\pm}^2 \geq 0$ should be imposed as a constraint on models of the brane-world. However, while we adopted the expansion in the parameter $\mu$, the tachyonic solution corresponds to $\mu$ of order unity. Hence, this solution is outside the domain of validity of the expansion in $\mu$. In other words, terms of more than the second power of $\mu$ can alter the stability/instability significantly. Further investigation is necessary to understand the stability of the brane world.

One could also derive another stability condition at the linearized level from the second equation of (64). The stability condition at the linearized level would be $l_{T\pm}^2 \leq 0$. Surprisingly, this condition is always violated since $l_{T\pm}^2$ is positive as shown in (63). Hence, the second equation of (64) has a tachyonic solution $(q_{\pm}^{(0)\mu\nu} k_{\mu} k_{\nu} = l_{T\pm}^{-2})$ with vanishing matter on the brane ($\bar{\tau}_{\pm}(T) = 0$). The appearance of this tachyonic solution may be considered as the brane-world version of Horowitz instability [24]: noting that Horowitz instability is caused by the Weyl-squared term $C_{\mu\nu} C_{\mu\nu}$ in a semiclassical effective action and that $C_{\mu\nu} C_{\mu\nu} = (2/3)(3 R_{\mu\nu} R_{\mu\nu} - R^2) + (the \ Gauss-Bonnet \ term)$, one can expect that the term $(3 R_{\mu\nu} R_{\mu\nu} - R^2)$ in the action (63) may cause an analogue of Horowitz instability. However, the tachyonic solution in this case is also outside the domain of validity of the expansion in $\mu$. Hence, terms of more than the second power of $\mu$ can alter the instability significantly. From this point of view we expect that the tachyonic solution is just a spurious solution and does not indicate any physically harmful instability of the brane world. Actually, it can be shown that $\eta^{\mu\nu} k_{\mu} k_{\nu} \int_{w_-}^{w_+} d\omega e^A F_{(T)}^2 = - \int_{w_-}^{w_+} d\omega e^{-3A} \left[(e^{2A} F_{(T)})^2 \right] \leq 0.$  \hspace{1cm} (69)

For a similar statement without branes, see ref. [10].
Therefore, we conclude that the brane-world version of Horowitz instability is spurious and does not indicate any physically harmful instability of the brane world.

Hence, we have to remove the spurious tachyonic solution for tensor perturbations whenever we deal with the low energy equation \[ \text{(13)} \] or the low energy effective action \[ \text{(35)}. \] One of the possible ways to remove spurious solutions is the method of “self-consistent reduction of order”. This method has been used in many areas of physics including the radiation reaction equation \[ \text{(28)} \], the post-Newtonian equations of motion in classical relativity \[ \text{(29)} \], higher derivative gravity \[ \text{(30)} \], and the semiclassical gravity \[ \text{(31)} \] (see also \[ \text{(32)} \]).

For some purposes it is useful to rewrite the gauge-invariant equations \[ \text{(47)} \] in a gauge-fixed form. We adopt a gauge (the harmonic gauge) such that \( \partial_\pm \hat{h}_\mu^\nu = 0 \), where \( \hat{h}_\mu^\nu \equiv \delta h^\mu_\nu - \delta q^\mu_\nu/2 \) and \( \delta q^\mu_\nu = \delta \phi^\mu / \mp \). Hereafter, indices are raised by \( q^{(0)\mu\nu} = \Omega^2 q^{\mu\nu} \) and lowered by \( h^{(0)\mu\nu} = \Omega^2 h^{\mu\nu} \). In this gauge the scalar-type gauge invariant variable \( f_\pm \) is reduced to \( f_\pm = (2/3) \tilde{\sigma}_\pm(Y) \), and the first equation of \[ \text{(49)} \] becomes

\[
\ddot{q}_\pm^{(0)\mu\nu} k_\mu k_\nu + O(\mu^2) \tilde{\sigma}_\pm(Y) = -16\pi G_N \tilde{\sigma}_\pm(Y) \tag{70}
\]

Hence, combining this equation with the second equation of \[ \text{(49)} \] and the conservation equation (the first equation of \[ \text{(28)} \]) we obtain

\[
\Box \hat{h}_\mu^\nu + l_2^2 \Box^2 \hat{h}_\mu^\nu + \frac{1}{3} (l_2^2 + l_2^2) (\partial^\nu \Box \partial_\nu - \delta^\nu_\nu \Box^2) \hat{h} = -16\pi G_N \tilde{S}_{\pm}^\mu_\nu, \tag{71}
\]

where \( \tilde{S}_{\pm}^\mu_\nu \) is the surface energy momentum tensor of matter on the brane \( \Sigma_\pm \) in the minimally coupled physical frame, \( \Box = \partial^\nu \partial_\nu \), and \( \hat{h} = \hat{h}_\mu^\nu + \delta \tilde{\sigma}_\mu / \mp \). Alternatively, the equation \[ \text{(71)} \] can be derived from the action \[ \text{(65)} \] by using the well-known formula \( \tilde{R}^\mu_\mu = -\Box (\hat{h}_\mu^\nu - \hat{h} \delta^\mu_\nu / 2 + O(\hat{h}^2) \) and \[ \text{(55)} \] with \( a_1 = l_2^2 / 6 \) and \( a_2 = l_2^2 / 3 \). In the low energy limit the gauge-fixed equation \[ \text{(71)} \], of course, reduces to the linearized Einstein equation \( \Box \hat{h}_\mu^\nu = -16\pi G_N \tilde{S}_{\pm}^\mu_\nu \). Again, we have to remove the spurious tachyonic solution for tensor perturbations whenever we deal with the low energy gauge-fixed equation \[ \text{(71)} \] by eg. the method of “self-consistent reduction of order”.

Given the high energy correction to 4-dimensional Einstein gravity, it is possible to derive corrections to the Newtonian potential. Actually, it is known that the spherically symmetric, static solutions of the linearized field equations in theories with curvature-squared terms are combinations of Newtonian and Yukawa potentials \[ \text{[13]} \]. Hence, the inclusion of the higher derivative terms indeed changes the short distance behavior of gravity, while it does not change the long distance behavior. For this reason, the correction presented in this paper is relevant for laboratory experiment on the validity of the Newtonian force. On the other hand, the long distance correction obtained by Garriga and Tanaka \[ \text{[11]} \] is relevant for astronomical tests although they considered a different model (the Randall-Sundrum infinite bulk model without a scalar field). Calculation of long distance corrections to the Newtonian potential in the model with a bulk scalar field between two branes is a worthwhile task as a future work. Analogy between calculations of long distance quantum corrections \[ \text{[19]} \] and the brane world calculation of long distance corrections is also an interesting subject.

Equipped with the result up to the order \( O(\mu) \), we conjecture that in the order \( O(\mu^N) \), weak gravity on the brane should be still indistinguishable from a higher derivative gravity whose action includes up to \( (N+1) \)-th power of curvature tensors. The coefficients of higher derivative terms can be in principle calculated by using the iterative results in section \[ \text{[11]} \] as we have done in this paper up to the order \( O(\mu) \). Since the expansion in \( \mu \) is in principle an infinite series, gravity in the brane world becomes non-local at high energies even at the linearized level. Actually, in this case the left hand side of \[ \text{(71)} \] will have up to \( 2(N+1) \)-th derivatives \( (N \to \infty) \) and, thus, we need to specify the 0-th to \( (2N+1) \)-th time-derivatives \( (N \to \infty) \) of the field \( \tilde{h}_{\mu\nu} \) on a spacelike 3-surface in order to predict the future evolution. In other words, we need to specify the whole (past) history of \( \tilde{h}_{\mu\nu} \) to predict its future evolution, provided that \( \tilde{h}_{\mu\nu} \) can be Taylor expanded with respect to the time. This explains how the 4-dimensional local description breaks down at high energies. Of course, at low energies below \( \min(l_{S\pm}^{-1}, l_{T\pm}^{-1}) \) we expect that the non-local behavior is suppressed.

It is known that if a gravitational action includes a metric tensor (without derivatives) and its Ricci tensor only, then the system is equivalent to another system described by Einstein gravity and additional massive spin-2 and spin-0 fields \[ \text{[20]} \]. Evidently, this observation is applicable to the effective action \[ \text{(55)} \]. Hence, we can expect appearance of effective spin-2 and spin-0 fields on the branes. The appearance of these fields is consistent with the observation that the Yukawa potential appears in the correction to Newtonian potential by curvature-squared terms as a result of exchanges of effective massive spin-2 and spin-0 fields \[ \text{[13]} \]. In the brane world context, these spin-2 and spin-0 fields should be composite fields due to superpositions of bulk gravitational and scalar fields, respectively.

The appearance of the composite spin-0 field may suggest a possibility of brane-world inflation without an inflaton on the brane. Actually, in 4-dimensional higher derivative gravitational theories it is known that inflation can be
driven by higher derivative terms. Such an inflationary model is called Starobinsky model [22, 24]. Hence, there is a possibility that the brane-world version of Starobinsky model may be caused by the higher derivative terms found in this paper. In this case, the composite spin-0 field is the brane-world version of the so-called scalaron. This possibility of brane-world inflation due to higher derivative terms seems to be related to an interesting brane inflation scenario proposed by Himemoto and Sasaki [21]. It seems interesting to explore relations between their model and the brane-world version of Starobinsky model. It may be sufficient to suppose that brane inflation is driven by the composite spin-0 field, or the brane-world version of the scalaron. It is expected that fluctuations of the spin-2 composite field as well as the spin-0 field (scalaron) play important roles in the generation of cosmological perturbations in this case.

Although the composite spin-2 and spin-0 fields are generally expected to appear on the brane, it is not the end of the story. As far as the author knows, there is no known equivalence between a higher derivative gravity theory and Einstein theory with additional fields if the higher derivative terms in the action cannot be expressed in terms of Ricci tensor only. In other words, if the higher derivative terms depend on the Weyl tensor explicitly then the composite spin-2 and spin-0 fields are not enough to describe the whole system. Actually, the non-locality discussed above seems to require that the appearance of these composite fields is not the whole story since these fields have 4-dimensional local actions.

The expected non-locality should be due to gravitational and scalar waves in the bulk. Hence, the infinite series of higher derivative terms is one description of the non-locality pointed out in ref. [33] in the context of brane world cosmology [34] (see also [35]). Another description was given as an integro-differential equation in ref. [36], where a complete set of four equations governing scalar-type cosmological perturbations was derived by using the doubly gauge invariant formalism developed in refs. [38, 35]. One of the four equations is an integro-differential equation, which describes non-local effects due to gravitational and scalar waves propagating in the bulk. Further investigation of the relation between the two different descriptions may be an interesting future subject.

VI. SUMMARY

In summary we have extended the analysis of linearized gravity in brane world models with a bulk scalar field between two branes to higher energies by investigating the next relevant order in the expansion in the parameter \( \mu = l^2 \eta^{\mu \nu} k_\mu k_\nu \), where \( l \) is the characteristic length scale of the model given by \( l = \Omega_{N+1}^{-1} \max(|l_{S\pm}|, l_{T\pm}) \). Since the 4-dimensional physical energy scale \( m_+ \) on our brane \( \Sigma^+ \) is given by \( m_+^2 = -\lambda_+^2 l^2 \mu \), the expansion in \( \mu \) is nothing but the low energy expansion. For the investigation we used the formalism developed in ref. [16], in which all quantities and equations including the surface energy momentum tensor and the junction condition are completely Fourier transformed with respect to the 4-dimensional coordinates so that the problem is essentially reduced to a set of 1-dimensional problems.

We compared the result with the so-called higher derivative gravity. It has been shown that in the order \( O(\mu) \), gravity on the brane is indistinguishable from the higher derivative gravity whose action includes the Einstein term and curvature-squared terms, provided that the inter-brane distance (radion) is stabilized, that the background scalar field is changing near the branes and that the background bulk geometry near the branes is warped. This result holds for a general conformal transformation to a frame in which matter on the branes is minimally coupled to the metric. The obtained indistinguishability between brane gravity and higher derivative gravity agrees with ref. [38] in which the non-relativistic limit of the theories with non-warped extra dimension was considered. In the present paper we considered weak gravity (including the non-relativistic and relativistic limits) in the brane world model with a scalar field in a warped bulk geometry. Newton’s constant and coefficients of curvature-squared terms except for the Gauss-Bonnet topological term have been determined as in (14). (The coefficient of the 4-dimensional Gauss-Bonnet term cannot and does not need to be determined as far as classical dynamics is concerned.) The result is summarized by the 4-dimensional effective action (15). In other words, we have provided the brane-world realization of the so-called \( R^2 \)-model.

Equipped with the result up to the order \( O(\mu) \), we conjectured that in the order \( O(\mu^N) \), weak gravity on the brane is still indistinguishable from a higher derivative gravity whose action includes up to \( (N+1) \)-th power of curvature tensors. We discussed the appearance of composite spin-2 and spin-0 fields in addition to the graviton on the brane and pointed out a possibility that the spin-0 field may play the role of an effective inflaton to drive brane-world inflation. We also showed that the brane-world version of Horowitz instability is spurious. Finally, we conjectured that the sequence of higher derivative terms is an infinite series and, thus, indicates non-locality in the brane world scenarios.
ACKNOWLEDGMENTS

The author would like to thank Lev Kofman, Takahiro Tanaka and Alexander Vilenkin for useful discussions and valuable comments. He would be grateful to Werner Israel for continuing encouragement and careful reading of the manuscript. This work is supported by JSPS Postdoctoral Fellowship for Research Abroad.

APPENDIX A: HARMONICS IN MINKOWSKI SPACETIME

In this appendix we give definitions of scalar, vector and tensor harmonics in an $n$-dimensional Minkowski spacetime. Throughout this appendix, $n$-dimensional coordinates are $x^\mu$ ($\mu = 0, 1, \cdots, n-1$), $\eta_{\mu\nu}$ is the Minkowski metric, and all indices are raised and lowered by the Minkowski metric and its inverse $\eta^{\mu\nu}$.

1. Scalar harmonics

The scalar harmonics are given by

$$Y = \exp(-ik_\rho x^\rho),$$

(A1)

by which any function $f$ can be expanded as

$$f = \int dk cY,$$

(A2)

where $c$ is a constant depending on $k$. Hereafter, $k$ and $dk$ are abbreviations of $\{k^\mu\}$ ($\mu = 0, 1, \cdots, n-1$) and $\prod_{\mu=0}^{n-1} dk^\mu$, respectively. We omit $k$ in most cases.

2. Vector harmonics

In general, any vector field $v_\mu$ can be decomposed as

$$v_\mu = v_{(T)\mu} + \partial_\mu f,$$

(A3)

where $f$ is a function and $v_{(T)\mu}$ is a transverse vector field:

$$\partial^\mu v_{(T)\mu} = 0.$$  

(A4)

Thus, the vector field $v_\mu$ can be expanded by using the scalar harmonics $Y$ and transverse vector harmonics $V_{(T)\mu}$ as

$$v_\mu = \int dk \left[ c_{(T)} V_{(T)\mu} + c_{(L)} \partial_\mu Y \right].$$

(A5)

Here, $c_{(T)}$ and $c_{(L)}$ are constants depending on $k$, and the transverse vector harmonics $V_{(T)\mu}$ are given by

$$V_{(T)\mu} = u_\mu \exp(-ik_\rho x^\rho),$$

(A6)

where the constant vector $u_\mu$ satisfies the following condition.

$$k^\mu u_\mu = 0$$

(A7)

for $k^\mu k_\mu \neq 0$, and

$$k^\mu u_\mu = 0,$$

$$\tau^\mu u_\mu = 0$$

(A8)

for non-vanishing $k_\mu$ satisfying $k^\mu k_\mu = 0$, where $\tau^\mu$ is an arbitrary constant timelike vector. For $k_\mu = 0$, the constant vector $u^\mu$ does not need to satisfy any of the above conditions. For the special case $k^\mu k_\mu = 0$, the second condition in
can be imposed by redefinition of $c(L)$. Actually this condition is necessary to eliminate redundancy. Note that the number of independent vectors satisfying the above condition is $n - 1$ for $k^\mu k_\mu \neq 0$ and $n - 2$ for $k^\mu k_\mu = 0$ and that these numbers are equal to the numbers of physical degrees of freedom for massive and massless spin-1 fields in $n$-dimensions, respectively.

Because of the expansion (A15), it is convenient to define longitudinal vector harmonics $V(L)_{\mu}$ by

$$V(L)_{\mu} \equiv \partial_\mu Y = -i k_\mu Y.$$ \hspace{1cm} (A9)

### 3. Tensor harmonics

In general, a symmetric second-rank tensor field $t_{\mu\nu}$ can be decomposed as

$$t_{\mu\nu} = t(T)_{\mu\nu} + \partial_\mu v_\nu + \partial_\nu v_\mu + f_{\mu\nu},$$ \hspace{1cm} (A10)

where $f$ is a function, $v_\mu$ is a vector field and $t(T)_{\mu\nu}$ is a transverse traceless symmetric tensor field:

$$t^{(T)}_{\mu\nu} = 0,$$

$$\partial^\rho t^{(T)}_{\mu\nu} = 0. \hspace{1cm} (A11)$$

Thus, the tensor field $t_{\mu\nu}$ can be expanded by using the scalar harmonics $Y$, the vector harmonics $V(T)$ and $V(L)$, and transverse traceless tensor harmonics $T(T)$ as

$$t_{\mu\nu} = \int dk \left[ c(T)T_{(T)\mu\nu} + c(L)T_{(LT)\mu\nu} + c(L)\partial_\mu V(T)_{\nu} + \partial_\nu V(T)_{\mu} \right] + c(L)(\partial_\mu V(L)_{\nu} + \partial_\nu V(L)_{\mu}) + \tilde{c}(Y)Y_{\mu\nu}. \hspace{1cm} (A12)$$

Here, $c(T)$, $c(LT)$, $c(LL)$, and $\tilde{c}(Y)$ are constants depending on $k$, and the transverse traceless tensor harmonics $T(T)$ are given by

$$T(T)_{\mu\nu} = s_{\mu\nu} \exp(-i k_\rho x^\rho),$$ \hspace{1cm} (A13)

where the constant symmetric second-rank tensor $s_{\mu\nu}$ satisfies the following condition.

$$k^\mu s_{\mu\nu} = 0,$$

$$s^\mu = 0.$$ \hspace{1cm} (A14)

for $k^\mu k_\mu \neq 0$, and

$$k^\mu s_{\mu\nu} = 0,$$

$$s^\mu = 0,$$

$$\tau^\mu s_{\mu\nu} = 0.$$ \hspace{1cm} (A15)

for non-vanishing $k_\mu$ satisfying $k^\mu k_\mu = 0$, where $\tau^\mu$ is an arbitrary constant timelike vector. For $k_\mu = 0$, the constant tensor $s_{\mu\nu}$ does not need to satisfy any of the above conditions. For the special case $k^\mu k_\mu = 0$, the last condition in (A15) can be imposed by redefinition of $c(LT)$, $c(LL)$ and $\tilde{c}(Y)$. Actually this condition is necessary to eliminate redundancy. Note that the number of independent symmetric second-rank tensors satisfying the above conditions is $(n + 1)(n - 2)/2$ for $k^\mu k_\mu \neq 0$ and $n(n - 3)/2$ for $k^\mu k_\mu = 0$ and that these numbers are equal to numbers of physical degrees of freedom for massive and massless spin-2 fields in $n$-dimensions, respectively.

Because of the expansion (A12), it is convenient to define tensor harmonics $T(LT)$, $T(LL)$, and $T(Y)$ by

$$T(LT)_{\mu\nu} \equiv \partial_\mu V(T)_{\nu} + \partial_\nu V(T)_{\mu},$$

$$T(LL)_{\mu\nu} \equiv \partial_\mu V(L)_{\nu} + \partial_\nu V(L)_{\mu} - \frac{2}{n} \eta_{\mu\nu} \partial^\rho V(L)_{\rho},$$

$$= \left( -2k_\mu k_\nu + \frac{2}{n} k^\rho k_\rho \eta_{\mu\nu} \right) Y,$$

$$T(Y)_{\mu\nu} \equiv \eta_{\mu\nu} Y. \hspace{1cm} (A16)$$
[1] J. Polchinski, *String Theory I & II* (Cambridge University Press, 1998).
[2] N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Rev. B429, 263 (1998); Phys. Rev. D59, 086004 (1999).
[3] L. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Lett. B436, 257 (1998).
[4] L. Randall and R. Sundrum, Phys. Rev. Lett. 83, 3370 (1999).
[5] L. Randall and R. Sundrum, Phys. Rev. Lett. 83, 4690 (1999).
[6] K. Akama, in Lect. Notes Phys. 176, Gauge Theory and Gravitation, Proceedings, Nara, 1982, edited by K. Kikkawa, N. Nakanishi and H. Nariai (Springer-Verlag, 1983) [hep-th/0001113]; V. A. Rubakov and M. E. Shaposhnikov, Phys. Lett. 152B, 136 (1983); M. Visser, Phys. Lett. B159, 22 (1985).
[7] P. Horava and E. Witten, Nucl. Phys. 460, 506 (1996).
[8] A. Lukas, B. A. Ovrut, K. S. Stelle and D. Waldram, Nucl. Phys. B552, 246 (1999).
[9] W. D. Goldberger and M. B. Wise, Phys. Rev. Lett. 83, 4922 (1999).
[10] O. DeWolfe, D. Z. Freedman, S. S. Gubser and A. Karch, Phys. Rev. D62, 046008 (2000).
[11] L. Randall and T. Tanaka, Phys. Rev. Lett. 84, 2778 (2000).
[12] M. Sasaki, T. Shiromizu and K. Maeda, Phys. Rev. D62, 024008 (2000).
[13] S. B. Giddings, E. Katz, and L. Randall, J. High Energy Phys. 03, 023 (2000).
[14] T. Tanaka and X. Montes, Nucl. Phys. B582, 259 (2000).
[15] H. Kudoh and T. Tanaka, Phys. Rev. D64, 084022 (2001); hep-th/0112013.
[16] S. Mukohyama and L. Kofman, [hep-th/0112115].
[17] S. Mukohyama, [gr-qc/0108048] to appear in Phys. Rev. D.
[18] K. S. Stelle, Gen. Rel. and Grav. 9, 353 (1978).
[19] J. F. Donoghue, Phys. Rev. Lett. 72, 2996 (1994).
[20] K. Maeda, Phys. Rev. D59, 3159 (1999); J. Koga and K. Maeda, Phys. Rev. D58, 064020 (1998).
[21] Y. Himemoto and M. Sasaki, Phys. Rev. D63, 044015 (2001); N. Sago, Y. Himemoto and Misao Sasaki, [gr-qc/0104033].
[22] Y. Himemoto, T. Tanaka and M. Sasaki, [gr-qc/0112027].
[23] A. A. Starobinsky, Phys. Lett. B91, 99 (1980).
[24] A. Vilenkin, Phys. Rev. D32, 2511 (1985).
[25] L. Kofman, V. Mukhanov and D. Pogosyan, Sov. Phys. JETP 66, 433 (1987).
[26] V. Müller and H.-J. Schmidt, Gen. Rel. and Grav. 17, 769 (1985); H.-J. Schmidt, Phys. Rev. D50, 5452 (1994).
[27] The author thanks Lev Kofman for pointing this out.
[28] For example, L. D. Landau and E. M. Lifshitz, *The classical Theory of Fields* (Pergamon, Oxford, 1962).
[29] For example, T. Damour, in *Gravitational Radiation*, edited by N. Deruelle and T. Piran (North-Holland, Amsterdam, 1983), p.214.
[30] L. Bel and H. S. Zia, Phys. Rev. D32, 3128 (1985).
[31] L. Parker and J. Z. Simon, Phys. Rev. D47, 1339 (1993).
[32] É. É. Flanagan and R. M. Wald, Phys. Rev. D54, 6233 (1996).
[33] S. Mukohyama, Phys. Rev. D62, 084015 (2000).
[34] J. M. Cline, C. Grojean and G. Servant, Phys. Rev. Lett. 83, 4245 (1999); E. E. Flanagan, S. H. H. Tye, I. Wasserman, Phys. Rev. D62, 044039 (2000); P. Binétruy, C. Defayet, U. Ellwanger and D. Langlois, Phys. Lett. B477, 285 (2000); S. Mukohyama, Phys. Lett. B473, 241 (2000); P. Kraus, J. High Energy Phys. 9912, 011 (1999); D. Ida, JHEP 0009, 014 (2000); S. Mukohyama, T. Shiromizu and K. Maeda, Phys. Rev. D62, 024028 (2000), Erratum-ibid. D63, 029901 (2001).
[35] E. Alvarez and F. D. Mazzitelli, Phys. Lett. B 505, 236 (2001).
[36] S. Mukohyama, Phys. Rev. D64, 064006 (2000).
[37] S. Mukohyama, Class. Quantum Grav. 17, 4777 (2000).
[38] B. Tekin, [hep-th/0106134].