LIOUVILLE PROPERTIES OF PLURISUBHARMONIC FUNCTIONS

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§0 Introduction.

In this paper we will prove a Liouville theorem on smooth plurisubharmonic functions on a complete noncompact Kähler manifold with nonnegative bisectional curvature. Using this Liouville theorem we prove a splitting theorem for such manifolds as well as a gap theorem in terms of the curvature decay of such a manifold.

In [N], the first author raised the following question:

*On a complete noncompact Kähler manifold with nonnegative Ricci curvature, is a plurisubharmonic function of sub-logarithmic growth a constant?*

It is well-known that for the complex Euclidean space \(\mathbb{C}^n\), the answer is positive. An affirmative answer to the above question is also a natural analogue, for plurisubharmonic functions, of Yau’s Liouville theorem [Y] for positive harmonic functions on Riemannian manifolds with nonnegative Ricci curvature. In this paper we shall first prove the following result as a supporting evidence of the positive solution to the above mentioned question.

**Theorem 1.** Let \(M\) be a complete Kähler manifold with nonnegative holomorphic bisectional curvature. Let \(u\) be a plurisubharmonic function on \(M\) satisfying \(\Delta u \leq \exp(C(r^2(x) + 1))\) for some \(C > 0\). Suppose

\[
\limsup_{x \to \infty} \frac{u(x)}{\log r(x)} = 0,
\]

then \(u\) must be a constant.

Even the result holds only for the manifolds with nonnegative bisectional curvature it has interesting applications in studying the geometry of such complete Kähler manifolds. The first application is the following splitting result.

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Theorem 2. Let $M^m$ be a complete noncompact Kähler manifold with bounded nonnegative holomorphic bisectional curvature. Suppose $f$ is a nonconstant harmonic function on $M$ such that

$$\limsup_{x \to \infty} \frac{|f(x)|}{r^{1+\epsilon}(x)} = 0,$$

for any $\epsilon > 0$, where $r(x)$ is the distance of $x$ from a fixed point. Then $f$ must be of linear growth and $M$ can be split isometrically as $\tilde{M} \times \mathbb{R}$. Moreover the universal cover $\tilde{M}$ of $M$ can be split isometrically and holomorphically as $\tilde{M}' \times \mathbb{C}$, where $\tilde{M}'$ is a complete Kähler manifold with nonnegative holomorphic bisectional curvature. Suppose that there exists a holomorphic function $f$ on $M$ of growth satisfying (0.1). Then $M$ itself splits as $\tilde{M} \times \mathbb{C}$.

A consequence of Theorem 2 is the following corollary.

Corollary. Let $M^m$ be a complete noncompact Kähler manifold with nonnegative holomorphic bisectional curvature whose Ricci curvature is positive at some point. Then every harmonic function defined on $M$ satisfying (0.1) must be constant.

In [L1], Li proved that if $M^m$ is a complete noncompact Kähler manifold with complex dimension $m$ with nonnegative Ricci curvature and $M$ supports $n + 1$ linearly independent harmonic functions of linear growth over $\mathbb{R}$, then $M$ is holomorphically isometric to $\mathbb{C}^m$. Here $n = 2m$ is the real dimension of $M$. This result was later generalized to the real case by Cheeger-Colding-Minicozzi in [C-C-M]. They proved that Li’s result is still true for Riemannian manifolds with nonnegative Ricci curvature. In this case, the conclusion is that the manifold is isometric to the Euclidean space. In fact, they proved that if $M^n$ is a complete noncompact Riemannian manifold with nonnegative Ricci curvature which supports a non-constant harmonic function of linear growth, then the tangent cone at infinity splits a factor of $\mathbb{R}$. Theorem 2 shows that on a complete Kähler manifold with bounded nonnegative holomorphic bisectional curvature, the existence of a nonconstant linear growth harmonic function would split the manifold itself. One can also think this as a function-theoretic version of Cheeger-Gromoll’s splitting theorem.

On the other hand, in [Y] and [C-Y], Cheng and Yau proved that on a complete noncompact Riemannian manifold with nonnegative Ricci curvature, then any sublinear growth harmonic function must be constant. That is to say, if the growth rate of a harmonic function is ‘close’ to that of constant functions, then the harmonic function must be constant. It is an interesting question to locate the ‘next gap’. Namely, what is the minimum growth rate beyond the linear growth. An easy consequence of the theorem is that if a harmonic function $f(x)$ is of $O(r(x)(\log r(x))^a)$, for some $a > 0$, then it is of linear growth. On the other hand, for any $\delta > 0$, the ‘round off’ cones with metrics $dr^2 + r^2 ds^2_{S^1(1+\delta)}$, where $S^1(\frac{1}{\sqrt{1+\delta}})$ is the circle with radius $\frac{1}{\sqrt{1+\delta}}$, support harmonic functions of growth $r^{1+\delta}(x)$. Therefore, Theorem 1 provides the best ‘next gap’, at least for the Kähler manifolds with bounded nonnegative bisectional curvature. Whether the similar ‘gap’ exists for the manifolds with nonnegative Ricci curvature and whether the same splitting result remains true for the Riemannian manifolds with nonnegative sectional curvature remain to be interesting open questions.
Theorem 1 also shows its potential in the study of the structure of complete Kähler manifolds with nonnegative curvature by proving the following gap theorem.

**Theorem 3.** Let $M$ be a complete noncompact Kähler manifold with nonnegative bisectional curvature. Assume that $\mathcal{R}(x) \leq C(r^2(x) + 1)$ for some $C > 0$ and

$$
\int_0^r s \left( \int_{B_0(s)} \mathcal{R}(y) \, dy \right) \, ds = o(\log r)
$$

where $\mathcal{R}(x)$ is the scalar curvature function, $\int_{B_0(r)} \mathcal{R}(y) \, dy$ is the average of $\mathcal{R}$ over $B_0(r)$. Then $M$ must be flat. In particular, the universal cover of $M$ must be isometric to $\mathbb{C}^m$.

Theorem 3 is the best gap type theorem proved so far for the Kähler manifolds with nonnegative bisectional curvature. The first result of this sort was proved by Mok-Siu-Yau in [M-S-Y] through solving the Poincaré-Lelong equation. It was later generalized in [N] for non-parabolic manifolds. The best result to date is in [C-Z], where Chen and Zhu used W.-X. Shi’s argument in [Sh3] and proved that:

Let $M^m$ be a complete noncompact Kähler manifold with nonnegative bisectional curvature. Assume that $\mathcal{R}(x)$ is bounded and $\int_{B_x(r)} \mathcal{R}(y) \, dy \leq k(r)$ for all $x$ where $k(r)$ is a nonincreasing function satisfying $k(r) = o(r^{-2})$. Then $M$ must be flat.

In the proof of [C-Z], the long time existence on the Kähler-Ricci flow in [Sh3] was used together with the volume element estimate in [Sh3] and a Li-Yau-Hamilton inequality on Kähler-Ricci flow of H.-D. Cao [Co1]. Along this line, the authors of the current paper improved the above result slightly in [N-T2], after their simpler derivation of W.-X. Shi’s volume element estimate.

We prove Theorem 3 using a simpler and much more direct method. Note that in Theorem 3 we do not require the uniform decay as in the above mentioned result in [C-Z] (See also [N-T2]). Another advantage of this method is that we do not necessarily require the boundedness of the curvature tensor, which has to be assumed due to the current status of the existence theory on the Kähler-Ricci flow over complete noncompact manifolds. The connection between the Liouville theorem and the gap theorem is provided by the solution to the Poincaré-Lelong equation, especially the one constructed in Theorem 5.1 of [N-S-T1]. This connection was also illustrated in earlier papers [N-T1-2]. We should also remark that Theorem 3 is not true if one only assume that the manifold has nonnegative Ricci curvature. In fact, there are many examples of Kähler manifolds with curvature decay satisfying Theorem 3 and with maximum volume growth. But they are not flat. (To our knowledge that all such examples are Ricci flat. Whether this is generally true or not remains an interesting question.)

Finally, we summarize some earlier work on Liouville properties of the plurisubharmonic functions and explain the methods we use to put our theorem into the right perspective. The first attempt to the question raised at the beginning on the plurisubharmonic functions was made in [N], where the first author proved that if $M$ is quasi-projective and $u$ is bounded then $u$ is a constant. In general case, it was also proved there that $u$ satisfies a homogenous Monge-Amperé equation. This fact turns out to be very useful in our
proof here. In [N-T1], using the Kähler-Ricci flow and a linear trace Li-Yau-Hamilton inequality (which is also called Harnack inequality in [H2], [Co1] and [C-H]) established there, the authors answered the question raised at the beginning affirmatively under the assumptions that $M$ has bounded bisectional curvature, the average of the scalar curvature of $M$ has quadratic decay and the Laplacian of the plurisubharmonic function grows at most exponentially (cf. [Theorem 3.2, N-T1] for a more precise statement).

The proof here is complete different, simpler and is based on the fact that on a complete Kähler manifold with nonnegative Ricci curvature, a plurisubharmonic function of sub-logarithmic growth must satisfies a homogeneous Monge-Amperé equation (see Lemma 2.3 below). Namely, at each point, at least one of the eigenvalue of the complex Hessian of the plurisubharmonic function is zero. Hence a natural way to prove the theorem is to use the induction. However, the foliation defined by the Monge-Amperé equation might be singular. We overcome this difficulty by deforming the plurisubharmonic function through heat equation. It turns out that, under the condition that the manifold has nonnegative holomorphic bisectional curvature, the deformed function is still plurisubharmonic for $t > 0$ and satisfies the homogeneous Monge-Amperé equation. Then the manifold, or its universal cover if not simply-connected, can be splitted with a factor whose tangent space corresponds to the kernel of the complex Hessian of the function at each point at some time $t > 0$. (Namely, the foliation at $t > 0$ becomes a product.) Therefore, we can indeed use the induction to conclude the result.

We should point out that the upper-bound assumption on the Laplacian is believed not necessary. However, due to the lack of direct method to the problem and the heat equation method in our proof, this assumption is necessary to obtain a maximum principle which holds for tensors satisfying a heat equation. The classical uniqueness for the solution to the heat equation on Euclidean space requires a similar necessary assumption on the solution.

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§1 Estimates on solutions to the heat equation.

In this section, we derive some basic estimates on the solution of the heat equation with plurisubharmonic initial data on a complete noncompact Kähler manifold with nonnegative bisectional curvature. First, we need the following:

**Lemma 1.1.** Let $M^n$ be a complete noncompact Riemannian manifold with nonnegative Ricci curvature. Let $u_0$ be a smooth function on $M$ such that

$$
\exp(a(r^2(x) + 1)) \leq u_0(x) \leq \exp(b(r^2(x) + 1))
$$

for some constants $b > a > 0$, where $r(x)$ is the distance of $x$ from a fixed point $o \in M$. Then there exists a $T > 0$ depending only on $b$ such that the Cauchy problem

$$
\left( \frac{\partial}{\partial t} - \Delta \right) u = 0, \quad u(x, 0) = u_0(x)
$$

holds.
has a solution $u$ on $M \times [0, T]$. Moreover, there exist constants $C_1, C_2 > 0$ such that

$$C_1 \exp \left( \frac{a}{4} r^2(x) \right) \leq u(x, t) \leq C_2 \exp \left( 3br^2(x) \right)$$

on $M \times [0, T]$.

**Proof.** To prove the existence of the solution, it is sufficient to show that for some $T > 0$,

$$u(x, t) = \int_M H(x, y, t)u_0(y)dy$$

is uniformly bounded on $B_o(r) \times [0, T]$ for all $r$. Let $x \in M$ and let $r(x) = 2R$, using the upper bound of the heat kernel of Li-Yau [L-Y], we have that

$$\int_M H(x, y, t)u_0(y)dy = \int_{B_x(R)} H(x, y, t)u_0(y)dy + \int_{M \setminus B_x(R)} H(x, y, t)u_0(y)dy$$

$$\leq \exp \left( b(9R^2 + 1) \right) + C_3 \int_{M \setminus B_x(R)} V_x^{-1}(\sqrt{t}) \exp \left( \frac{-r^2(x, y)}{5t} + 9br^2(x, y) \right) dy$$

for some constant $C_3$ depending only on $n$ and $b$, where we have used the fact that $\int_M H(x, y, t)dy = 1$ and the fact that $r(y) \leq 3r(x, y)$ outside $B_x(R)$. Here $r(x, y)$ is the distance between $x$ and $y$. If we choose $T = \min\{1, \frac{1}{10br}\}$, then for $0 < t \leq T$, we have that

$$\int_M H(x, y, t)u_0(y)dy \leq \exp \left( b(9R^2 + 1) \right) + \frac{C_4 R^n}{V_o(1)} \int_R t^{-\frac{n}{2}} \exp \left( -\frac{r^2}{10t} \right) r^{n-1} dr$$

$$\leq \exp \left( b(9R^2 + 1) \right) + C_5 R^n$$

$$\leq C_6 \exp \left( b(9R^2 + 1) \right)$$

for some constants $C_4 - C_6$ depending only on $n$, $b$ and $V_o(1)$. Here we have used the volume comparison so that $V_x(1) \geq \frac{1}{(1+2R)^n} V_o(1)$ and $A(\partial B_x(r)) \leq C(n)r^{n-1}$. From this it is easy to see that (1.1) has a solution $u(x, t)$ so that the second inequality of (1.2) is true. To prove the lower bound of $u(x, t)$, if $2R = r(x) > 2$, then for $0 < t \leq T$,

$$u(x, t) \geq \int_{B_x(R)} H(x, y, t)u_0(y)dy$$

$$\geq C_7 \exp \left( aR^2 \right) \int_{B_x(\sqrt{t})} V_x^{-1}(\sqrt{t}) \exp \left( -\frac{r^2(x, y)}{3t} \right) dy$$

$$\geq C_8 \exp \left( aR^2 \right)$$

for some positive constants $C_7$ and $C_8$ depending only on $n$ and $a$, where we have used the lower estimate of the heat kernel in [L-Y, page 182]. From this it is easy to see the first inequality in (1.2) is also true.

Similarly, one can prove that:
Lemma 1.2. Let $M^n$ be a complete noncompact manifold with nonnegative Ricci curvature and let $u_0(x)$ be a smooth function on $M$ such that $|u_0(x)| \leq \exp \left( b(r^2(x) + 1) \right)$, then the Cauchy problem (1.1) has a solution $u(x,t)$ on $M \times [0,T]$ for some $T > 0$, such that $|u(x,t)| \leq C \exp \left( 3br^2(x) \right)$ for all $(x,t)$. If in addition,

\begin{equation}
\limsup_{x \to \infty} \frac{u_0(x)}{\log r(x)} = 0
\end{equation}

then for any $T \geq t > 0$,

\begin{equation}
\limsup_{x \to \infty} \frac{u(x,t)}{\log r(x)} = 0.
\end{equation}

In the next lemma, we give sufficient conditions for a function $u(x)$ to satisfy $|u(x)| \leq \exp \left( C(r^2(x) + 1) \right)$.

Lemma 1.3. Let $M^n$ be a complete noncompact manifold with nonnegative Ricci curvature. Let $u$ be a smooth function. Suppose there exists a constant $b > 0$ such that

(i) $u(x) \leq \exp \left( b(r^2(x) + 1) \right)$; and

(ii) $0 \leq \Delta u(x) \leq \exp \left( b(r^2(x) + 1) \right)$.

Then

\begin{equation}
u(x) \geq -C \exp \left( 5b(r^2(x) + 1) \right)
\end{equation}

for some constant $C > 0$ for all $x$.

Proof. Consider $\tilde{M} = M \times \mathbb{R}^4$, then $\tilde{M}$ has positive Green’s function $G(x,y)$ which satisfies:

\begin{equation}
G(\tilde{x},\tilde{y}) \leq \frac{C\tilde{r}^2(\tilde{x},\tilde{y})}{V_x(\tilde{x},\tilde{y})}
\end{equation}

for some constant $C$ depending only on $n$, see [Sh3, p.162]. If we define $\tilde{u}(\tilde{x}) = u(x)$, where $\tilde{x} = (x,x') \in \tilde{M} = M \times \mathbb{R}^4$, then $\tilde{u}$ also satisfies conditions similar to (i) and (ii). Suppose we can prove that (1.5) is true for $\tilde{u}(\tilde{x})$ on $\tilde{M}$, then it is easy to see that $u$ also satisfies (1.5). Hence we may assume that $M$ has a positive Green’s function which satisfies condition similar to (1.6) on $\tilde{M}$.

Now for any $R > 0$, let $f(x) = \Delta u(x)$ and let

$v_R(x) = -\int_{B_\sigma(R)} G_R(x,y)f(y)dy.$

Then for $x \in B_\sigma\left(\frac{R}{2}\right)$,

\begin{equation}|v_R(x)| \leq \exp \left( b(R^2 + 1) \right) \int_{B_\sigma(R)} G(x,y)dy
\end{equation}

\begin{align*}
&\leq \exp \left( b(R^2 + 1) \right) \int_{B_\sigma(2R)} G(x,y)dy \\
&\leq C_1 \exp \left( b(R^2 + 1) \right) \int_0^{2R} \frac{r^2 A_x(r)}{V_x(r)} dr \\
&\leq C_2 R^2 \exp \left( b(R^2 + 1) \right)
\end{align*}
for some constants $C_1 - C_2$ independent of $x$ and $R$. Here we have used condition (ii), (1.6) and volume comparison. Since $v_R$ satisfies $\Delta v_R = f$ on $B_o(R)$ with zero boundary data, we conclude that $v_R + \exp \left(b(R^2 + 1)\right) \geq u$ in $B_o(R)$ by condition (i) and the maximum principle. By the Harnack inequality derived from the gradient estimate for positive harmonic functions of Cheng-Yau [C-Y], we have

$$\sup_{B_o(\frac{R}{2})} \left(v_R + \exp \left(b(R^2 + 1)\right) - u\right) \leq C_3 \left(v_R(o) + \exp \left(b(R^2 + 1)\right) - u(o)\right)$$

for some constant $C_3$ depending only on $n$. Hence for $x \in B_o(\frac{R}{2})$,

$$-u(x) \leq -v_R(x) + C_3 \left(\exp \left(b(R^2 + 1)\right) - u(o)\right) \leq C_2 R^2 \exp \left(b(R^2 + 1)\right) + C_3 \left(\exp \left(b(R^2 + 1)\right) - u(o)\right)$$

where we have used (1.7) and the fact that $v_R \leq 0$. From this, it is easy to see that (1.5) is true.

Let $M^m$ be a complete Kähler manifold and let $u(x, t)$ be a solution to the heat equation on $M$. Namely,

$$\left(\frac{\partial}{\partial t} - \Delta\right) u(x, t) = 0.$$

By the computation in Lemma 2.1 in [N-T1], we have:

**Lemma 1.4.** Let $u(x, t)$ be a solution to the heat equation. Then the complex Hessian $u_{\alpha\beta}(x, t)$ of $u(x, t)$ satisfies the complex Lichnerowicz equation:

$$\left(\frac{\partial}{\partial t} - \Delta\right) u_{\gamma\overline{\delta}} = R_{\beta\overline{\alpha}\gamma\overline{\delta}} u_{\alpha\beta} - \frac{1}{2} \left(R_{\gamma\overline{\delta}} u_{\beta\overline{\delta}} + R_{\beta\overline{\delta}} u_{\gamma\overline{\delta}}\right).$$

Here $R_{\beta\overline{\alpha}\gamma\overline{\delta}}$ and $R_{\gamma\overline{\delta}}$ are the curvature tensor and the Ricci tensor of $M$.

**Lemma 1.5.** Let $M^m$ be a complete noncompact Kähler manifold with nonnegative bisectional curvature. Let $u(x, t)$ be a solution to the heat equation. Then

$$\left(\frac{\partial}{\partial t} - \Delta\right) \|u_{\alpha\overline{\beta}}\|^2 \leq -\|\nabla_\gamma u_{\alpha\overline{\beta}}\|^2 - \|\nabla_\overline{\gamma} u_{\alpha\overline{\beta}}\|^2.$$

**Proof.** Choose a normal coordinate. Using Lemma 1.4, the direct calculation shows that:

$$\Delta \|u_{\alpha\overline{\beta}}\|^2 = \|\nabla_\gamma u_{\alpha\overline{\beta}}\|^2 + \|\nabla_\overline{\gamma} u_{\alpha\overline{\beta}}\|^2 + u_{\alpha\overline{\beta}} u_{\overline{\beta}\overline{s}\overline{s}} R_{\gamma\overline{\delta}} + u_{\alpha\overline{\beta}} u_{\gamma\overline{\delta}} R_{\overline{s}\overline{s}\overline{\delta}}$$

$$- u_{\alpha\overline{\beta}} u_{\overline{s}\overline{s}} R_{\overline{\alpha}\overline{\beta}\overline{s}\overline{s}} - u_{\alpha\overline{\beta}} u_{\overline{\delta}} R_{\overline{\alpha}\overline{\beta}\overline{\delta}} + u_{\alpha\overline{\beta}} (u_t)_{\alpha\overline{\beta}} + u_{\alpha\overline{\beta}} (u_t)_{\alpha\overline{\beta}}.$$

Therefore,

$$\left(\frac{\partial}{\partial t} - \Delta\right) \|u_{\alpha\overline{\beta}}\|^2 = -\|\nabla_\gamma u_{\alpha\overline{\beta}}\|^2 - \|\nabla_\overline{\gamma} u_{\alpha\overline{\beta}}\|^2 - \sum_{\alpha\beta} R_{\alpha\overline{\alpha}\beta\overline{\beta}} (\lambda_\alpha - \lambda_\beta)^2,$$

where $\lambda_\alpha$ are eigenvalues of $u_{\alpha\overline{\beta}}$. Here we calculate under a normal coordinate around a fixed point such that $u_{\alpha\overline{\beta}}$ is diagonalized. By the nonnegativity of the bisectional curvature the lemma follows.
Lemma 1.6. Let $M^m$ be a complete noncompact Kähler manifold with nonnegative bisectional curvature. Let $u_0$ be a plurisubharmonic function on $M$ such that

\begin{equation}
 u_0(x) \leq \exp \left( b(r^2(x) + 1) \right),
\end{equation}

and

\begin{equation}
 0 \leq \Delta u_0(x) \leq \exp \left( b(r^2(x) + 1) \right)
\end{equation}

for some constant $b > 0$. Then there exists $T > 0$ such that the Cauchy problem (1.1) has a solution $u(x, t)$ on $M \times [0, T]$ such that for some constant $b^* > 0$ (which might depend on $T$),

\begin{equation}
 ||u_{\alpha\beta}||(x, t) \leq \exp \left( b^*(r^2(x) + 1) \right)
\end{equation}

for all $(x, t)$.

Proof. By Lemmas 1.2 and 1.3, we conclude that for some $T > 0$, (1.1) has a solution $u(x, t)$ in $M \times [0, T]$ such that

\begin{equation}
 |u(x, t)| \leq C_1 \exp \left( 15br^2(x) \right).
\end{equation}

It remains to prove (1.12). By (1.11) and (1.13), one can easily prove that

\begin{equation}
 \int_{B_o(r)} |\nabla u_0|^2 dx \leq \exp \left( b_1(r^2 + 1) \right)
\end{equation}

for some constant $b_1$ for all $r$. Since

\begin{equation*}
 \left( \frac{\partial}{\partial t} - \Delta \right) u^2 = -2|\nabla u|^2,
\end{equation*}

we can multiply the equation by $\varphi^2(x)$ and integrate by parts, where $\varphi(x)$ is a smooth function such that $0 \leq \varphi \leq 1$, $\varphi = 1$ on $B_o(R)$, $\varphi = 0$ outside $B_o(2R)$ and $|\nabla \varphi| \leq C/R$ for some constant $C$ independent of $R$. We have

\begin{align*}
 2 \int_0^T \int_M \varphi^2 |\nabla u|^2 dx dt &= -\int_0^T \int_M \varphi^2 \left( \frac{\partial}{\partial t} - \Delta \right) u^2 \\
 &\leq \int_M \varphi^2 u_0^2(x) dx + 4 \int_0^T \int_M \varphi u |\nabla \varphi| |\nabla u| dx dt \\
 &\leq \int_M \varphi^2 u_0^2(x) dx + 4 \int_0^T \int_M |\nabla \varphi|^2 u_0^2 dx dt + \int_0^T \int_M \varphi^2 |\nabla u|^2 dx dt.
\end{align*}

Hence by (1.13), we have

\begin{equation}
 \int_0^T \int_M \exp \left( -b_2(r^2(x) + 1) \right) |\nabla u|^2 dx dt < \infty
\end{equation}
for some $b_2 > 0$. As in [N-T2, Lemma 1.1], using the fact that the Ricci curvature is nonnegative, we have

$$
\left( \frac{\partial}{\partial t} - \Delta \right) |\nabla u|^2 \leq -||u_{\alpha\bar{\beta}}||^2 - ||u_{\alpha\beta}||^2.
$$

Multiplying this by $\varphi^2$ and integrating by parts, using (1.14) and (1.15), one can repeat the above argument and show that

$$
(1.16) \quad \int_0^T \int_M \exp \left(-b_3(r^2(x) + 1)\right) ||u_{\alpha\bar{\beta}}||^2(x) dx dt < \infty
$$

for some constant $b_3 > 0$.

To conclude the proof of (1.12), let $\Phi = ||u_{\alpha\bar{\beta}}||^2$, by Lemma 1.5 we have

$$
(1.17) \quad \left( \frac{\partial}{\partial t} - \Delta \right) \Phi(x,t) \leq 0.
$$

By the mean value inequality [Theorem 1.2, L-T], we have, for $\frac{1}{4}r^2 \geq T$,

$$
\exp(-C_1r^2t)(1+\Phi)(x,t) \leq C_2 \left[ \frac{1}{r^{2m+2}} \int_0^T \int_{B_r(\frac{t}{4})} \exp(-C_1r^2s)\Phi(y,s)dyds + \sup_{B_r(\frac{t}{4})} \Phi(y,0) \right].
$$

Combining this with (1.11) and (1.16), we can conclude that (1.12) is true.

§2 Proof of the Liouville theorem.

The proof of Theorem 1 is based on the following three lemmas. Let $M^m$ and $u_0$ as in Lemma 1.6, and let $u(x,t)$ be the solution of (1.1) on $M \times [0, T]$ for some $T > 0$ constructed in the lemma. In the following, the eigenvalues of a Hermitian form are arranged in ascending order. Hence the first eigenvalue is the smallest one.

**Lemma 2.1.** With the above notations and assumptions, we have the following:

(a) $u_{\alpha\bar{\beta}}(x,t) \geq 0$ for all $(x, t)$.

(b) For any $T > t' \geq 0$, suppose there is a point $x'$ in $M^m$ such that the first $k$ eigenvalues $\lambda_1, \ldots, \lambda_k$ of $u_{\alpha\bar{\beta}}(x',t')$ satisfy $\lambda_1 + \cdots + \lambda_k > 0$, then for all $t > t'$ and for all $x \in M$, the sum of the first $k$ eigenvalues of $u_{\alpha\bar{\beta}}(x,t)$ is also positive.

**Proof.** The proofs of (a) and (b) are similar. Let us first prove (b) for the case that $t' = 0$. By Lemma 1.6, there is a constant $b_1 > 0$ such that on $M \times [0, T]$

$$
(2.1) \quad ||u_{\alpha\bar{\beta}}||(x,t) \leq \exp \left(b_1(r^2(x)+1)\right).
$$

It is easy to see that there exists a smooth function $h_0(x)$ such that

$$
\exp \left(8b_1(r^2(x)+1)\right) \leq h_0(x) \leq \exp \left(b_2(r^2(x)+1)\right)
$$
for some $b_2 > 8b_1$. By Lemma 1.1, we can find a solution $h(x,t)$ of the heat equation on $M \times [0, T_1]$ where $T_1 = \min\{1, \frac{1}{100b_2}\}$ such that

\begin{equation}
(2.2) \quad h(x,t) \geq C_1 \exp(2b_1r^2(x))
\end{equation}

for some $C_1 > 0$. Let $\phi(x,t) = e^t h(x,t)$, then

\begin{equation}
(2.3) \quad \left(\frac{\partial}{\partial t} - \Delta\right) \phi = \phi.
\end{equation}

Assume at $t = 0$, there is $x_0 \in M$, such that the sum of the first $k$ eigenvalues of $u_{\alpha\bar{\beta}}(x_0,0)$ is positive. Then we can find a smooth nonnegative function $f$ with $f(x_0) > 0$ and with support in a neighborhood of $x_0$, such that the sum of the first $k$ eigenvalues of

$$u_{\alpha\bar{\beta}} = fg_{\alpha\bar{\beta}}$$

is still nonnegative at $t = 0$, where we have used the fact that $u_{\alpha\bar{\beta}} \geq 0$ at $t = 0$. As in [B], solve the scalar heat equation

\begin{equation}
(2.4) \quad \left(\frac{\partial}{\partial t} - \Delta\right) f = -f
\end{equation}

such that $f(x,0) = f(x)$. The solution can simply be written as $e^{-t} \cdot \int_M H(x,y,t)f(y) \, dy$. Then by the strong maximum principle, $f > 0$ for $t > 0$ and $f$ is bounded.

Let $\epsilon > 0$, define $\psi = -f + \epsilon \phi$, and let $\eta_{\alpha\bar{\beta}} = u_{\alpha\bar{\beta}} + \psi g_{\alpha\bar{\beta}}$, where $g_{\alpha\bar{\beta}}$ is the metric tensor of $M$. Then at $t = 0$, at each point the sum of the first $k$ eigenvalues of $\eta$ is positive. We want to prove that for any $T_1 \geq t > 0$ and any $x \in M$, the sum of the first $k$ eigenvalues of $\eta$ is positive. Otherwise, because of (2.1), (2.2), the definition of $\phi$ and the fact that $f$ is bounded, it is easy to see that there is a first $T_1 \geq t_1 > 0$ and a point $x_1 \in M$, such that the sum of the first $k$ eigenvalues of $\eta$ at $x_1$ at time $t_1$ is zero.

Let us fix the notations. Suppose $v_1, \ldots, v_m$ are unit eigenvectors of $\eta$ at $x_1$ at time $t_1$, with eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m$. We may choose normal coordinates at $x_1$ such that $v_j = \frac{\partial}{\partial z_j}$ at $x_1$. In particular, if we write $v_j = v_j^\alpha \frac{\partial}{\partial z^\alpha}$, we have $v_j^\alpha = \delta_{\alpha j}$ at $x_1$. Note that the sum of the first $k$ eigenvalues of a Hermitian form is the infimum of the traces of the form restricted to $k$-dimensional subspaces. Therefore $\sum_{\alpha,\beta=1}^k \left(g^{\alpha\bar{\beta}} \eta_{\alpha\bar{\beta}}\right) \geq 0$ for all $(x,t)$ with $t \leq t_1$ and equals to zero at $(x_1, t_1)$.

Hence at $(x_1, t_1)$, we have

\begin{equation}
(2.5) \quad 0 \geq \left(\frac{\partial}{\partial t} - \Delta\right) \left(\sum_{\alpha,\beta=1}^k \eta_{\alpha\bar{\beta}} g^{\alpha\bar{\beta}}\right).
\end{equation}

From now on repeated indices mean summation from 1 to $m$ if there is no specification. Now

\begin{equation}
(2.6) \quad \frac{\partial}{\partial t} \left(\sum_{\alpha,\beta=1}^k \eta_{\alpha\bar{\beta}} g^{\alpha\bar{\beta}}\right) = \sum_{\alpha,\beta=1}^k \left(\frac{\partial}{\partial t} \eta_{\alpha\bar{\beta}}\right) g^{\alpha\bar{\beta}}.
\end{equation}
Also at \((x_1, t_1)\), we have
\[
\Delta \left( \sum_{\alpha, \beta=1}^{k} \eta_{\alpha\beta} g^{\alpha\bar{\beta}} \right) = \sum_{\alpha, \beta=1}^{k} (\Delta \eta_{\alpha\beta}) g^{\alpha\bar{\beta}}.
\]

Combining this with (2.5), (2.6) and (1.8) in Lemma 1.4, at \((x_1, t_1)\) we have,
\[
0 \geq \sum_{\alpha, \beta=1}^{k} \left[ R_{\delta\bar{\gamma}alpha} (u_{\gamma\bar{\delta}} + \psi g_{\gamma\bar{\delta}}) - \frac{1}{2} R_{alpha\bar{p}} (u_{p\bar{\delta}} + \psi g_{p\bar{\delta}}) - \frac{1}{2} R_{p\bar{\delta}} (u_{alpha\bar{p}} + \psi g_{alpha\bar{p}}) \right] g^{\alpha\bar{\beta}}
\]
\[
+ \sum_{\alpha, \beta=1}^{k} \left[ \left( \frac{\partial}{\partial t} - \Delta \right) \psi \right] g^{\alpha\bar{\beta}} - R_{\delta\bar{\gamma}alpha\bar{\beta}} \psi g_{\gamma\bar{\delta}} + \frac{1}{2} \psi R_{alpha\bar{p}} g_{p\bar{\delta}} + \frac{1}{2} \psi R_{alpha\bar{p}} g_{p\bar{\delta}} \right] g^{\alpha\bar{\beta}}.
\]

Since at \((x_1, t_1)\), \(\eta\) has eigenvectors \(v_p = \frac{\partial}{\partial z_p}\), for \(1 \leq p \leq m\), with eigenvalue \(\lambda_p\)
\[
0 \geq k \sum_{\alpha, \beta=1}^{k} \left[ R_{\delta\bar{\gamma}alpha\bar{\beta}} (u_{\gamma\bar{\delta}} + \psi g_{\gamma\bar{\delta}}) - \frac{1}{2} R_{alpha\bar{p}} (u_{p\bar{\delta}} + \psi g_{p\bar{\delta}}) - \frac{1}{2} R_{p\bar{\delta}} (u_{alpha\bar{p}} + \psi g_{alpha\bar{p}}) \right] g^{\alpha\bar{\beta}}
\]
\[
= k \sum_{j=1}^{k} \sum_{\gamma=1}^{m} R_{\gamma\bar{\gamma}j\bar{j}} \lambda_\gamma - k \sum_{j=1}^{k} R_{j\bar{j}} \lambda_j
\]
\[
= \sum_{j=1}^{k} \sum_{\gamma=1}^{m} R_{\gamma\bar{\gamma}j\bar{j}} \lambda_\gamma - \sum_{j=1}^{k} \sum_{\gamma=1}^{m} R_{\gamma\bar{\gamma}j\bar{j}} \lambda_j
\]
\[
= \sum_{j=1}^{k} \sum_{\gamma=1}^{m} \lambda_\gamma R_{\gamma\bar{\gamma}j\bar{j}} - \sum_{j=1}^{k} \sum_{\gamma=1}^{m} R_{\gamma\bar{\gamma}j\bar{j}} \lambda_j
\]
\[
= \sum_{j=1}^{k} \sum_{\gamma=1}^{m} R_{\gamma\bar{\gamma}j\bar{j}} (\lambda_\gamma - \lambda_j)
\]
\[
\geq 0
\]

where we have used that fact that \(M\) has nonnegative bisectional curvature, and \(\lambda_\gamma \geq \lambda_j\) for \(\gamma \geq j\). Also by (2.3) and (2.4)
\[
\left[ \left( \frac{\partial}{\partial t} - \Delta \right) \psi \right] = k(f + \epsilon \phi) > 0.
\]

Moreover
\[
-R_{\delta\bar{\gamma}alpha\bar{\beta}} \psi g_{\gamma\bar{\delta}} + \frac{1}{2} \psi R_{alpha\bar{p}} g_{p\bar{\delta}} + \frac{1}{2} \psi R_{alpha\bar{p}} g_{p\bar{\delta}} \right] g^{\alpha\bar{\beta}} = 0.
\]
From (2.7)–(2.10), we have a contradiction. Hence the sum of the first \( k \) eigenvalues of \( \eta \) is nonnegative for all \( (x,t) \in M \times (0,T_1) \). Let \( \epsilon \to 0 \), we conclude that the sum of the first \( k \) eigenvalues of \( u_{\alpha\bar{\beta}}(x,t) - f(x,t)g_{\alpha\bar{\beta}}(x,t) \) is nonnegative on \( M \times [0,T_1] \). Since \( f \) is positive for \( t > 0 \), the sum of the first \( k \) eigenvalues of \( u_{\alpha\bar{\beta}}(x,t) \) must be positive for \( 0 < t \leq T_1 \).

Take \( f \equiv 0 \) in the above, one can prove similarly that \( u_{\alpha\bar{\beta}} \geq 0 \) on \( M \times [0,T_1] \). One can then apply the same arguments as above to prove that (b) is true on \([0,T_1]\). The results then follow by iteration, because \( T_1 > 0 \) is a fixed number.

**Lemma 2.2.** Let \( M, u_0(x) \) and \( u(x,t) \) be as in Lemma 2.1. Let

\[
\mathcal{K}(x,t) = \{ v \in T_{x,t}^1(M) | u_{\alpha\bar{\beta}}(x,t)v^\alpha = 0, \text{ for all } \beta \}
\]

be the null space of \( u_{\alpha\bar{\beta}}(x,t) \). Then there exists \( 0 < T_1 < T \) such that for any \( 0 < t < T_1 \), \( \mathcal{K}(x,t) \) is distribution on \( M \). Moreover the distribution is invariant under parallel translation.

**Proof.** By Lemma 2.1, it is easy to see that there exists \( 0 < T_1 < T \) such that \( \dim \mathcal{K}(x,t) \) is constant on \( M \times (0,T_1) \). Hence for each \( 0 < t < T_1 \), \( \mathcal{K}(x,t) \) is a smooth distribution on \( M \). It remains to prove that the distribution is parallel for fixed \( t \). We can proceed as in [H1, Lemma 8.2].

Fix \( 0 < t_0 < T_1 \), let \( x_0 \in M \) and let \( w_0 \in \mathcal{K}(x_0,t_0) \). Let \( \gamma(\tau) \) be a smooth curve from \( x_0 \) and let \( w(\tau) \) be the vector field obtained by parallel translation along \( \gamma \). We want to prove that \( w(\tau) \) is also in the null space \( \mathcal{K}(\gamma(\tau),t_0) \) at \( \gamma(\tau) \). First extend \( w \) to be a vector field in a neighborhood of \( \gamma(\tau) \), and then extend \( w \) to be a vector field independent of time \( t \). Now, projecting \( w \) onto \( \mathcal{K}(x,t) \), we have a vector field \( v \) such that \( v \) is in \( \mathcal{K}(x,t) \) for all \( x \) in a neighborhood of \( \gamma \) and for all \( t \). The following computations are performed in a neighborhood of \( \gamma \).

Since

\[
(2.10) \quad \quad u_{\alpha\bar{\beta}}v^\alpha = 0
\]

for all \( \beta \), we have

\[
(2.11) \quad 0 = \frac{\partial}{\partial t} \left( u_{\alpha\bar{\beta}}v^\alpha \bar{v}^\beta \right)
= \left( \frac{\partial}{\partial t} u_{\alpha\bar{\beta}} \right) v^\alpha \bar{v}^\beta + u_{\alpha\bar{\beta}} \frac{\partial v^\alpha}{\partial t} \bar{v}^\beta + u_{\alpha\bar{\beta}} v^\alpha \frac{\partial \bar{v}^\beta}{\partial t}
= \left( \frac{\partial}{\partial t} u_{\alpha\bar{\beta}} \right) v^\alpha \bar{v}^\beta
\]

where we have used (2.10). Choosing a unitary frame \( e_s \) at a point \( \gamma(\tau) \), we have

\[
(2.12) \quad 0 = \Delta \left( u_{\alpha\bar{\beta}}v^\alpha \bar{v}^\beta \right)
= \frac{1}{2} (\nabla_s \nabla_{\bar{s}} + \nabla_{\bar{s}} \nabla_s) \left( u_{\alpha\bar{\beta}}v^\alpha \bar{v}^\beta \right)
= (\Delta u_{\alpha\bar{\beta}}) v^\alpha \bar{v}^\beta - u_{\alpha\bar{\beta}} \nabla_{\bar{s}}v^\alpha \nabla_s \bar{v}^\beta - u_{\alpha\bar{\beta}} \nabla_s v^\alpha \nabla_{\bar{s}} \bar{v}^\beta
\]
where we have used (2.10) so that

\((\nabla_s u)_{\alpha\overline{\beta}} v^\alpha = -u_{\alpha\overline{\beta}} \nabla_s v^\alpha, \ (\nabla_{\overline{s}} u)_{\alpha\overline{\beta}} v^\alpha = -u_{\alpha\overline{\beta}} \nabla_{\overline{s}} v^\alpha\)

and their complex conjugates.

Combining with (1.8), (2.11), (2.12), we have

\[(2.13) \ 0 = R_{t\overline{s} \alpha \overline{\beta}} u_{st} v^\alpha \overline{v^\beta} + 2u_{\alpha\overline{\beta}} \nabla_s v^\alpha \nabla_{\overline{s}} v^\beta + 2u_{\alpha\overline{\beta}} \nabla_{\overline{s}} v^\alpha \nabla_s \overline{v^\beta} .\]

We may choose \(e_s\) so that at a point \(u_{st} = a_s \delta_{st}\). Then

\[R_{t\overline{s} \alpha \overline{\beta}} u_{st} v^\alpha \overline{v^\beta} = R_{s\overline{s} \alpha \overline{\beta}} a_s v^\alpha \overline{v^\beta} = a_s R_{s\overline{s} v^\alpha \overline{v^\beta}} \geq 0\]

because \(a_s \geq 0\) and \(M\) has nonnegative bisectional curvature. Hence (2.13) implies that \(\nabla_s v\) and \(\nabla_{\overline{s}} v\) are in the null space \(K(\gamma(\tau), t_0)\).

Since \(w(\tau)\) is parallel along \(\gamma(\tau)\), and \(w = v + w^\perp\), where \(w^\perp\) is perpendicular to \(K(\gamma(\tau), t_0)\), we have

\[0 = \frac{D}{d\tau} \frac{w}{t} = \frac{D}{d\tau} v + \frac{D}{d\tau} w^\perp.\]

Hence

\[\frac{D}{d\tau} w^\perp = -\frac{D}{d\tau} v\]

which is in \(K\).

Now

\[\frac{d}{d\tau} \langle w^\perp, w^\perp \rangle = \langle \frac{D}{d\tau} w^\perp, w^\perp \rangle + \langle w^\perp, \frac{D}{d\tau} w^\perp \rangle = 0\]

because \(\frac{D}{d\tau} w^\perp\) is in \(K\) and \(w^\perp\) is perpendicular to \(K\). At \(\gamma(0) = x_0, w = v_0\) and so \(w^\perp = 0\) at \(\gamma(0)\). Hence \(w^\perp = 0\) for all \(\tau\) and so \(w\) is in \(K\).

**Lemma 2.3.** ([N, Proposition 4.1]) Let \(M^m\) be a complete noncompact Kähler manifold of complex dimension \(m\), with nonnegative Ricci curvature. Let \(u(x)\) be a plurisubharmonic function on \(M\) satisfying:

\[(2.14) \ \lim_{x \to \infty} \sup \frac{u(x)}{\log r(x)} = 0.\]

Then \((\partial \overline{\partial} u)^m = 0\)

Proposition 4.1 stated in [N] is under the assumption that \(M\) is nonparabolic. However, the proof without any changes also works for general complete Kähler manifolds with nonnegative Ricci curvature.

We are ready to prove Theorem 1.

**Proof of Theorem 1.** Let \(M\) and \(u\) satisfy the conditions in Theorem 1. Let \(\tilde{M}\) be the universal cover of \(M\), then the distance function in \(\tilde{M}\) dominates the distance function in \(M\). Hence \(\tilde{M}\) and the lift \(\tilde{u}\) of \(u\) also satisfy the conditions in the theorem. Therefore, we may assume that \(M\) is simply connected.
By Lemmas 1.6, 2.1 and 2.2, there exists $T > 0$ such that the Cauchy problem (1.1) has a solution $u(x, t)$ with $u(x, 0) = u(x)$. Moreover, let $0 < t_0 < T$ be fixed, we have $u_{\alpha \bar{\beta}}(x, t_0) \geq 0$ and the null space $K(x, t_0)$ is a parallel distribution on $M$. Using the De Rham decomposition (Cf. Theorem 8.1, page 172 of [K-N]) we know that $M = M_1 \times M_2$ isometrically and holomorphically such that $u_{\alpha \bar{\beta}}$ is zero when restricted on $M_1$ and $u_{\alpha \bar{\beta}}$ is positive everywhere when restricted on $M_2$. By Lemma 1.2, we still have

$$\limsup_{x \to \infty} \frac{u(x, t_0)}{\log r(x)} = 0. \tag{2.15}$$

Hence when restricted on $M_2$, (2.15) is still true. This contradicts Lemma 2.3 and the fact that $u_{\alpha \bar{\beta}}$ is positive when restricted on $M_2$, unless $M = M_1$. Hence $u_{\alpha \bar{\beta}}(x, t_0) \equiv 0$ on $M$ for all $0 < t_0 < T$. From this it is easy to see that $u_{\alpha \bar{\beta}}(x) \equiv 0$. By applying the gradient estimate of Cheng-Yau [C-Y] and the fact that $u$ satisfies (2.14), we know that $u$ is a constant.

**Remarks.** 1. There is a result by Cao [Co2] related to the splitting phenomena in the above proof. Cao has told the second author that using the Kähler-Ricci flow he has proved that if $M$ is a complete noncompact simply connected Kähler manifold of bounded and nonnegative holomorphic bisectional curvature, then $M$ splits holomorphically isometrically into $C^k \times M'$ with respect to the metric at time $t > 0$, where $M'$ is a complete simply connected Kähler manifold of nonnegative bisectional curvature and positive Ricci curvature. For the compact cases, there are results of this type in [H-S-W], see also [B] and [M1, p.64].

2. For the ALE Kähler manifolds, a Liouville theorem on plurisubharmonic functions was proved earlier in [N-S-T2].

§3 Proof of Theorem 2 and 3.

In order to prove Theorem 2 we need a result in [L1, Corollary 5]. For the sake of completeness, we will sketch the proof. It seems that in the proof of this result, we need to assume that the holomorphic bisectional curvature is nonnegative.

**Lemma 3.1.** Let $M$ be a complete noncompact Kähler manifold with nonnegative holomorphic bisectional curvature. If $f$ is a harmonic function with sub-quadratic growth defined on $M$, then $f$ is pluri-harmonic.

**Proof.** Let $h = ||f_{\alpha \bar{\beta}}||^2 = g^{\alpha \bar{\delta}} g^{\bar{\gamma} \bar{\beta}} f_{\alpha \beta} f_{\bar{\gamma} \bar{\delta}}$, where $g_{\alpha \bar{\beta}}$ is the metric of $M$ and $g^{\gamma \alpha \bar{\beta}}$ is its inverse. Since $f$ is harmonic, by (1.8)

$$\Delta f_{\bar{\gamma} \bar{\delta}} = -R_{\beta \bar{\alpha} \gamma \bar{\delta}} f_{\alpha \beta} + \frac{1}{2} (R_{\gamma \bar{p} \bar{\beta} \bar{p}} f_{\bar{p} \bar{\beta}} + R_{\gamma \bar{p} \bar{\beta} \bar{p}} f_{\bar{p} \bar{\beta}}).$$

Hence in normal coordinates so that at a point $x$, $f_{\alpha \bar{\beta}} = \lambda_{\alpha} \delta_{\alpha \beta}$, we have

$$\Delta h = 2f_{\gamma \bar{p} \bar{q} \delta} f_{\bar{p} \bar{q} \delta} + ||f_{\alpha \bar{\beta} \gamma}||^2 + ||f_{\alpha \bar{\beta} \bar{\gamma}}||^2$$

$$= -2R_{\beta \alpha \gamma \bar{\delta}} f_{\alpha \beta} f_{\bar{\delta} \bar{\gamma}} + (R_{\gamma \bar{p} \bar{q} \bar{p}} f_{\bar{p} \bar{q} \delta} + R_{\gamma \bar{p} \bar{q} \bar{p}} f_{\bar{p} \bar{q} \delta}) f_{\bar{p} \bar{q} \delta} + ||f_{\alpha \bar{\beta} \gamma}||^2 + ||f_{\alpha \bar{\beta} \bar{\gamma}}||^2$$

$$= -2R_{\alpha \bar{\alpha} \gamma \bar{\gamma}} \lambda_{\alpha} \lambda_{\gamma} + 2R_{\gamma \bar{\gamma} \bar{\gamma} \gamma} \lambda_{\gamma}^2 + ||f_{\alpha \bar{\beta} \gamma}||^2 + ||f_{\alpha \bar{\beta} \bar{\gamma}}||^2$$

$$\geq 0,$$
where we have used the fact that $M$ has nonnegative holomorphic bisectional curvature. Since $|f(x)| = o\left(r^2(x)\right)$ where $r(x)$ is the distance from a fixed point $o \in M$, as in [L1, p.90-91], we have

$$\frac{1}{V_o(R)} \int_{B_o(R)} h \leq \frac{C}{R^{-2}V_o(R)} \int_{B_o(R)} |\nabla f|^2 = o(1),$$

as $R \to \infty$. Here $C$ is a constant independent of $R$ and we has used the gradient estimate in [C-Y]. Since $h$ is subharmonic, $h \equiv 0$ by the mean value inequality in [L-S]. Hence $f$ is pluri-harmonic.

**Lemma 3.2.** Let $M$ be a complete noncompact Kähler manifold with nonnegative holomorphic bisectional curvature. Let $f$ be a pluri-harmonic function. Then $\log(1 + |\nabla f|^2)$ is pluri-subharmonic.

**Proof.** We adapt the complex notation. Let $h = |\nabla f|^2 = \sum_{\alpha,\beta} g^{\alpha\beta} f_\alpha f_\beta$. Here $g^{\alpha\beta}$ is the Kähler metric and $(g^{\alpha\beta})$ is the inverse of $(g_{\alpha\beta})$. To prove that $\log(1 + h)$ is pluri-subharmonic, it is sufficient to show that $[\log(1 + h)]_{\gamma\bar{\gamma}} \geq 0$ in normal coordinates. Direct calculation shows that:

$$h_{\gamma\bar{\gamma}} = \left(\sum_{\alpha,\beta} g^{\alpha\beta} f_\alpha f_\beta\right)_{\gamma\bar{\gamma}}$$

$$= \sum_{\alpha,\beta} g^{\alpha\beta} \left[f_{\alpha\gamma} f_{\beta\bar{\gamma}} + f_{\alpha\bar{\gamma}} f_{\beta\gamma} + f_{\alpha\gamma} f_{\beta\bar{\gamma}} + f_{\alpha\bar{\gamma}} f_{\beta\gamma}\right]$$

$$= \sum_{\alpha} f_{\alpha\gamma} f_{\bar{\alpha}\bar{\gamma}} + \sum_{\alpha, \bar{s}} R_{\gamma\alpha\bar{s}} f_\alpha f_{\bar{s}}$$

where we have used the fact that $f$ is pluri-harmonic. Hence

$$[\log(1 + h)]_{\gamma\bar{\gamma}} = \frac{1}{(1 + h)^2} [(1 + h) h_{\gamma\bar{\gamma}} - h_{\gamma} h_{\bar{\gamma}}]$$

$$= \frac{1}{(1 + h)^2} \left[(1 + h) \left(\sum_{\alpha} f_{\alpha\gamma} f_{\bar{\alpha}\bar{\gamma}} + \sum_{\alpha, \bar{s}} R_{\gamma\alpha\bar{s}} f_\alpha f_{\bar{s}}\right) - \sum_{\alpha} f_{\alpha\gamma} f_{\bar{\alpha}} \sum_{\alpha} f_{\alpha} f_{\bar{\alpha}\bar{\gamma}}\right]$$

$$\geq \frac{1}{(1 + h)^2} \left(\sum_{\alpha} f_{\alpha\gamma} f_{\bar{\alpha}\bar{\gamma}} + \sum_{\alpha, \bar{s}} R_{\gamma\alpha\bar{s}} f_\alpha f_{\bar{s}}\right)$$

where we have used the fact that $f$ is pluri-harmonic. From (3.2), the fact that $M$ has nonnegative holomorphic bisectional curvature, it is easy to see that $\log(1 + h)$ is pluri-subharmonic.
Lemma 3.3. Let $M$ be a complete noncompact Kähler manifold with nonnegative holomorphic bisectional curvature such that $\|Rm\|$ and $\|\nabla Rm\|$ are bounded. Let $f$ be a harmonic function on $M$ satisfying (0.1). Then

\begin{equation}
(3.3) \quad ||f_{\alpha\beta}||(x) \leq C \left(1 + r(x)\right)^{3/2}
\end{equation}

for some constant $C$, where $r(x)$ is the distance from $x$ to a fixed point $o \in M$.

Proof. By Lemma 3.1, $f$ is pluri-harmonic. Let $\Psi = ||f_{\alpha\beta}||^2$. Then in normal coordinates:

$$
\Delta \Psi = \sum_s \left( \sum_{\alpha,\beta,\gamma,\delta} g^{\alpha\gamma} g^{\beta\delta} f_{\alpha\beta} f_{\gamma\delta} \right)_{s\bar{s}}
$$

$$
\geq \sum_{\alpha,\beta,s} f_{\alpha\beta s\bar{s}} f_{\bar{\alpha}\bar{\beta}} + f_{\bar{\alpha}\beta s\bar{s}} f_{\alpha\beta}
$$

$$
\geq \sum_{\alpha,\beta,s} \left( R_{\alpha s\bar{\alpha},\beta} f_{s} f_{\bar{\beta}} + R_{s\bar{\alpha},\beta\bar{\alpha}} f_{s} f_{\alpha\beta} \right) + 2 \sum_{\alpha,\beta,s,t} R_{\alpha t\beta\bar{s}t} f_{s} f_{\bar{\alpha}\beta}
$$

where $R_{\alpha\beta}$ is the Ricci tensor of $M$. Hence

$$
\Delta \Psi \geq -C_1 (\Psi + h)
$$

for some constant $C_1 > 0$ depending only on $m$ and the bound of $\|Rm\| + \|\nabla Rm\|$. By (3.1), we also have

$$
\Delta h \geq \Psi.
$$

Let $S_R = \sup_{B_o(R)} h$, then

$$
\Delta (\Psi + C_1 h) \geq -C_1 S_{2R}
$$

on $B_o(2R)$. Hence for any $T > 0$, we have

$$
\left( \Delta - \frac{\partial}{\partial t} \right) (\Psi + C_1 h + C_1 S_{2R}(T-t)) \geq 0.
$$

Since $\Psi + C_1 h + C_1 S_{2R}(T-t) \geq 0$ for $0 \leq t \leq T$, by [L-T, Theorem 1.1], for any $R > 0$, if we let $T = \frac{1}{4} R^2$, we have

$$
\sup_{B_o(\frac{1}{4} R) \times [\frac{1}{8} R^2, \frac{1}{4} R^2]} (\Psi + C_1 h + C_1 S_{2R}(T-t)) \leq \frac{C_2}{R^2 V_o(R)} \int_{\frac{1}{4} R^2}^{\frac{1}{2} R^2} \int_{B_o(R)} (\Psi + C_1 h + C_1 S_{2R}(T-t)) dx dt
$$

$$
\leq C_3 \left( R^{-2} + S_{2R} R^2 \right)
$$
for some constants $C_2, C_3 > 0$ independent of $R$. Here we have used the fact that
\[ \frac{1}{V_0(R)} \int_{B_0(R)} \Psi \leq CR^{-2} \]
for some constant $C$ independent of $R$, see [L1, p.90-91]. Since $S_{2R} = o(R^{1/2})$ by the gradient estimate in [C-Y], we have
\[ \Psi(x) \leq C_4 (1 + r(x))^3. \]

Now we are ready to prove Theorem 2.

**Proof of Theorem 2.** Let $f$ be a nonconstant harmonic function on $M$ satisfying (0.1). Then $f$ is pluri-harmonic by Lemma 3.1. Since $M$ has bounded curvature, we can solve the Kähler-Ricci flow
\[ \frac{\partial \tilde{g}_{\alpha\bar{\beta}}}{\partial t} = -\tilde{R}_{\alpha\bar{\beta}} \]
where initial data $\tilde{g}_{\alpha\bar{\beta}}(\cdot, 0) = g_{\alpha\bar{\beta}}$. The equation has a short time solution so that for any fixed $t > 0$, $(M, \tilde{g}_{\alpha\bar{\beta}})$ still has nonnegative holomorphic bisectional curvature so that the curvature tensor and the covariant derivatives of the curvature tensor are bounded. Moreover, $\tilde{g}_{\alpha\bar{\beta}}$ is uniformly equivalent to $g_{\alpha\bar{\beta}}$. All these results are in [Sh2]. Fix $t > 0$, then $f$ is still a pluri-harmonic function on $(M, \tilde{g}_{\alpha\bar{\beta}})$ satisfying (0.1). By Lemma 3.2, the function $u = \log(1 + |\nabla_t f|^2)$ is pluri-subharmonic, where $\nabla_t$ is the gradient with respect to the metric $\tilde{g}_{\alpha\bar{\beta}}(\cdot, t)$. By the gradient estimates in [C-Y], $|u|(x) = o(\log r(x))$. By Lemma 3.3 and (3.2), we have
\[ \Delta^{(t)} u \leq C (1 + r(x))^3 \]
if $r(x) > 1$, where $\Delta^{(t)}$ is the Laplacian for the metric $\tilde{g}_{\alpha\bar{\beta}}(\cdot, t)$. By Theorem 1, we conclude that $|\nabla^{(t)} f|$ depends only on $t$, where $\nabla^{(t)}$ is the gradient with respect to $\tilde{g}_{\alpha\bar{\beta}}(\cdot, t)$. Let $t \to 0$, we conclude that $|\nabla f|$ is constant. Hence $f$ must be of linear growth. Moreover, by the Bochner formula, we conclude that $\nabla f$ must be parallel. Hence $J(\nabla f)$ is also parallel, where $J$ is the complex structure of $M$. From this it is easy to see that the universal cover of $M$ splits as $\tilde{M}' \times \mathbb{C}$ isometrically and holomorphically. At the same time by integrating along $\nabla f$, $M$ splits as $\tilde{M} \times \mathbb{R}$ isometrically, where $\tilde{M}$ can be taken to be the component of $f^{-1}(0)$. In this case that $M$ supports a nonconstant holomorphic function of growth rate (0.1), both the real and imaginary part will split a factor of $\mathbb{R}$ and clearly that they consist a complex plane $\mathbb{C}$.

**Proof of Theorem 3.** By the assumptions that
\[ (3.4) \quad \int_0^r s \left( \int_{B_\alpha(s)} R(y) dy \right) ds = o \left( \log r \right), \]
and that $R(x) \leq C \left( r^2(x) + 1 \right)$, it is easy to see that the conditions in Theorem 5.1 of [N-S-T1] are satisfied and so there exists a solution $u(x)$ to the Poincaré-Lelong equation $\sqrt{-1} \partial \bar{\partial} u = Ric_M$ such that
\[ \limsup_{x \to \infty} \frac{u(x)}{\log r(x)} = 0. \]
Obviously, $u$ is plurisubharmonic. By Theorem 1, $u$ must be constant and so $M$ must be flat.

Finally, we should point out that in order to solve the Poincaré-Lelong equation we only need (3.4) together with \( \lim \inf_{r \to \infty} \int_{B_r(x)} R^2(y) \, dy = 0 \) which is slightly more general than the assumptions on $R(x)$ in Theorem 3. Therefore, Theorem 3 holds under these more general assumptions.

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