On the Secure Domination of Benes Networks

Angel D¹, Mary Jeya Jothi R¹, Revathi R¹ and A. Raja

¹Department of Mathematics, Sathyabama Institute of Science and Technology, Chennai
²Department of Electronics and Communication Engineering, Saveetha School of Engineering, Chennai.

*angel.zara1001@gmail.com, jeyajothi31@gmail.com, revathirangan75@gmail.com, rajaa.sse@saveetha.com

Abstract. This paper focuses on the fascinating and highly active area of research on domination in graphs. In this article, the secure domination problem in graph theory is solved for Benes networks. The results obtained can be used in finding minimum secure dominating sets for an interconnection network whose vertices are the routing servers while it becomes necessary to obtain an optimal solution for designing the network defence strategy.

1. Introduction

The modern study of graph protection was initiated in the late twentieth century and has its historical roots in the time of the ancient Roman Empire that referred to the military strategy of Emperor Constantine [4]. Currently while studying domination, emperor Constantine’s strategy is termed as Roman domination. Dominating sets in a graph provide numerous applications for protecting networks which involves placing a set of guards at designated nodes of a network.

In the defending technique using dominating sets, some guards in designated vertices may not necessarily be adjacent to any other guards and when a guard moves along an edge to deal with an attack at a vertex without a guard, the resulting placement of guards may leave the system unprotected. To overcome this deficiency recently, a novel strategy for placing guards in order to protect a graph had been introduced which necessitates the definition of secure domination [4]. Secure domination is a defence strategy that can be used when it is not possible or desirable to station two defence units at the same location. The objective in this strategy is to evaluate or determine the minimum number of guards needed to defend a graph. Secure domination had been studied in [1, 3, 8]. There are various problems in the literature related to domination. This work deals with secure domination problem on Benes networks.

Our motivation for studying secure domination comes from the potential usefulness in this learning. There are so many security issues in the use of AI and complex interconnected networks and just as quickly as some issues are solved, more challenges are coming up. Because of these constant changes, it is important to keep studying. This is where secure domination comes into play. With this graph theoretical approach, we hope to provide a tool that can be used for network monitoring.
2. Secure Domination in graphs and Bene Networks

A graph with vertex set \( V \) and edge set \( E \) is denoted by \( G(V, E) \). The cardinality of the vertex set is denoted by \( |V| = n \). For other definitions concerning graph theory not mentioned here, one can refer [5, 7, 9]. A dominating set for a graph \( G = (V, E) \) is a subset \( D \) of \( V \) such that every vertex not in \( D \) is adjacent to at least one member of \( D \). The domination number \( \gamma(G) \) is the number of vertices in a smallest dominating set for \( G \). A secure dominating set \( X \) of a graph \( G \) is a dominating set with the property that each vertex \( u \in V - X \) is adjacent to a vertex \( v \in X \) such that \( (X - \{v\}) \cup \{u\} \) is dominating. The minimum cardinality of such a set is called the secure domination number, denoted by \( \gamma_s(G) \) [6]. Figure 1 shows a graph with only two vertices connected by an edge, and the secure dominating set is either \( \{v_1\} \) or \( \{v_2\} \) and hence the secure domination number is one.

An \( r \)-dimensional Benes network denoted by \( B(r) \) has \( 2r+1 \) levels, with \( (2r+1) \cdot 2^r \) vertices. Figure 2 shows a \( B(2) \) network. The Benes networks are bipartite and are invertible graphs.

![Fig. 1: \( \gamma_s(G) = 1 \)](image1)

![Fig. 2: A 2-dimensional Benes network](image2)

3. Our Results

A communication network is an undirected graph where the nodes in the network represent processors and edges represent communication channels. The \( d \)-dimension Benes network is one of the most versatile and powerful interconnection networks. For all values of \( d \), Benes are bipartite. Calculating secure dominating set for Benes is a significant problem because this facilitates message passing to all the other processors securely. In this section, the exact value of the parameter which gives the
minimum number of guards required to protect the networks Benes is obtained. These findings can be applied for secure message passing in interconnected networks.

3.1 Theorem: If \( G \) is a d-dimensional Benes network with \( d > 1 \) then the secure dominating set of \( G \) is half the number of the vertices of \( G \).

Proof. Let \( G \) be a d-dimensional Benes network. Such graphs exist for \( d > 1 \). Let \( S \) be a secure dominating set of \( G \). Since \( G \) is bipartite, the vertex set \( V \) can be partitioned as two disjoint subsets of \( V \) (say \( A \) and \( B \)) such that every edge starts in \( A \) and ends in \( B \). The odd levels in Benes are \( 2r+1 \). For proving the result, we consider the rows and levels of \( G \). For example if \( d = 2 \), then there are 4 rows and 5 levels in a Benes network BF(2). Since the graph \( G \) is bipartite there will be no odd cycles in \( G \). Therefore, \( S \) will contain all those vertices in alternate levels. That is, starting from level 0, level 2, level 4,…From fig 2, it is clear that from level 0, all the 4 vertices are put in set \( S \), from level 2, again 4 vertices and once again from level 4 another 4 vertices and all together the secure dominating set will contain 12 vertices. Therefore, \((2r + 1).2^r\) divided by 2 is a secure dominating number. We claim that the set \( S \) is minimum. If not, adding or removing vertices from \( S \) will spoil the domination property and hence we conclude that the secure domination number is half the number of its vertices.

The next theorem yields the relation between covering number of a graph \( G \), that is, covering edges by vertices, and the secure domination number of a graph \( G \).

3.2 Theorem: If \( G \) be a d-dimensional Benes network with \( d > 1 \), then the minimum number of vertices covering the edges of \( G \) is equal to the secure domination number of the \( G \).

Proof. Let \( G \) be a d-dimensional Benes network where \( d > 1 \). Since \( G \) is bipartite and there are even number of vertices in \( G \), there exists a perfect matching in \( G \), such that the ends of the perfect matching form a covering set. Since the number of maximum perfect matchings are half the number of vertices of \( G \), using theorem 3.1, we have the result that, the minimum number of edges for covering the vertices of \( G \) is equal to the secure domination number of the graph \( G \).

4. Conclusion

In this paper, secure domination for Benes networks is solved and the minimum secure dominating sets are obtained for all Benes networks of dimension \( d \). The results obtained be used to model many physical systems. This problem is open for various other networks.

References

[1] Araki, Toru. Miyazaki and Hiroka 2018 Secure domination in proper interval graphs Discrete Applied Mathematics DOI: 10.1016/j.dam.2018.03.040.
[2] Athas, W. C. and C. L. Seitz 1988 Multicomputers: Message-Passing Concurrent Computers IEEE Computer 21 pp. 9–24
[3] Castillano Elmer C. Ugbinada. Rose Ann L. and Canoy. Sergio R. 2014 Secure domination In the joins of graphs Applied Mathematical Sciences DOI:10.12988/ams.2014.47519
[4] Cockayne E. J. Grobler P. J.P. Grundlingh W. R. Munganga, J. and Van Vuuren. 2005 Protection of a Graph Utilitas Mathematica 67 pp.1-14
[5] Harary F. 1993 Graph theory Narosa Publishers pp. 21–23
[6] Klostermeyer William F. Mynhardt and Christina M. 2008 Secure domination and secure total domination in graphs Discussiones Mathematicae Graph Theory 28 pp. 267-284
[7] Mary Jeya Jothi R and Revathi R 2019 SSP structure on the cartesian product \( Sm+1 \times Pn \) International Journal of Innovative Technology and Exploring Engineering 8 pp.1102-1104
[8] Parimalazhagan R. Sulochana V. and Vimal Kumar S. 2018 Secure domination in 4-regular planar and non-planar graphs with girth 3 Utilitas Mathematica pp. 1-12

[9] R.Revathi and R. Mary Jeya Jothi 2020 Even Arbitrary Supersubdivision of Corona Related MMD Graphs Journal Combinatorial Mathematics and Combinatorial Computing 112 pp. 147-152

[10] Roushini Leely Pushpam P and Chitra Suseendran. 2019 Secure domination in middle graphs Bulletin 9 pp.25-38