ON VOLUMES OF COMPLEMENTS OF PERIODIC GEODESICS

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Abstract. Every closed geodesic $\gamma$ on a surface has a canonically associated knot $\hat{\gamma}$ in the projective unit tangent bundle. We study, for $\gamma$ filling, the volume of the associated knot complement with respect to its unique complete hyperbolic metric.

1. Introduction

Let $\Sigma$ be a complete, orientable hyperbolic surface of finite area. Every closed geodesic $\gamma$ in $\Sigma$ has a canonical lift $\hat{\gamma}$ in the projective unit tangent bundle $PT^1 \Sigma$, namely the image under the map $T^1 \Sigma \to PT^1 \Sigma$ of the corresponding periodic orbit of the geodesic flow. Such canonical lift $\hat{\gamma}$ is an embedding of $S^1$ into $PT^1 \Sigma$ and can then be considered as a knot in $PT^1 \Sigma$.

More generally, given a closed geodesic in a hyperbolic 2–orbifold one can consider its canonical lift as a knot in the corresponding unit tangent bundle. This is maybe particularly interesting in the case of the modular surface $\Sigma_{\text{mod}} = \mathbb{H}^2 / \text{PSL}_2(\mathbb{Z})$, because its unit tangent bundle is homeomorphic to the complement of the trefoil knot in $S^3$. Therefore, canonical lifts of closed geodesics on the modular surface are knots in $S^3$. In fact, in [14] Ghys observed that the obtained knots are Lorentz knots. For facts about such knots [5, 4, 10].

The aim of this paper is to study $M_{\hat{\gamma}}$, the complement of a normal neighborhood of the canonical lift $\hat{\gamma}$ in $PT^1 \Sigma$, for a general hyperbolic surface $\Sigma$. Foulon and Hasselblatt [13] proved that $M_{\hat{\gamma}}$ admits a complete hyperbolic metric of finite volume as soon as $\gamma$ fills the surface. It is unique by the Mostow’s Rigidity Theorem, meaning that any geometric invariant is a topological invariant. We will be interested in estimating the volume of $M_{\hat{\gamma}}$, when $\gamma$ is a filling closed geodesic.

Our main result is to give a lower bound on the volume of the complement $M_{\hat{\gamma}}$ (Theorem 1.5). We start however commenting on upper bounds for the volume of $M_{\hat{\gamma}}$. Bergeron, Pinsky and Silberman have already studied in [3] the problem of finding an upper bound, by giving one which is linear in the length of the geodesic. Nevertheless, it is easy to construct sequences of closed geodesics with length approaching to infinity but whose associated canonical lift complements have uniformly bounded volume. For example, suppose that $\gamma$ is filling and $\phi$ is a diffeomorphism of the surface representing an infinite order element in the mapping class group: the length of the geodesics in the sequence $\{\phi^n(\gamma)\}_{n \in \mathbb{N}}$ tend to infinity as $n$ grows, but every $M_{\phi^n(\gamma)}$ is homeomorphic to $M_{\hat{\gamma}}$. There are however more interesting examples:

**Theorem 1.1.** Given a hyperbolic surface $\Sigma$, there exist a constant $V_0 > 0$ and a sequence $\{\gamma_n\}$ of filling closed geodesics on $\Sigma$, with $M_{\gamma_n} \not\cong M_{\gamma_k}$ for every $k \neq n$, such that $\text{Vol}(M_{\gamma_n}) < V_0$ for every $n \in \mathbb{N}$. Moreover, for any sequence $\{X_n\}$ of hyperbolic metrics on $\Sigma$, we have that $\ell_{X_n}(\gamma_n) \nearrow \infty$. 

Bergeron, Pinsky and Silberman gave already in [3], Example 3.1 examples satisfying Theorem 1.1 in the case of the modular surface. Their examples and the ones in the proof of Theorem 1.1 are different. To explain why this is the case we recall that in the modular surface ([3], Sec. 3) they found an upper bound which is proportional to the period plus the sum of the logarithms of the coefficients corresponding to the geodesic’s continued fraction expansion. The examples provided in ([3], Example 3.1) have unbounded length but bounded period. On the other hand, the examples used to prove Theorem 1.1 have unbounded period. This yields:

**Corollary 1.2.** For the modular surface \( \Sigma_{\text{mod}} \), there exist a constant \( V_0 > 0 \) and a sequence \( \{ \gamma_k \} \) of filling closed geodesics on \( \Sigma_{\text{mod}} \), with \( M_{\gamma_n} \not\sim M_{\gamma_k} \) for every \( k \neq n \), such that \( \text{Vol}(M_{\gamma_k}) < V_0 \) for every \( k \in \mathbb{N} \) and the period of the continued fraction expansion of \( \gamma_k \) tends to infinity.

It is perhaps worth mentioning that there is a numerical example for Corollary 1.2 in ([6], Example 5.2). Even more interesting is that in their paper the authors presented numerical evidence of geodesics on the modular for which the volume’s growth is linear in the geometric length of the geodesics. We show that, up to a logarithmic factor, such geodesics do actually exist in any hyperbolic surface:

**Theorem 1.3.** Given a hyperbolic metric \( X \) on a surface \( \Sigma \), there exist a sequence \( \{ \gamma_n \} \) of filling closed geodesics on \( \Sigma \) with \( \ell_X(\gamma_n) \nearrow \infty \) such that,

\[
\text{Vol}(M_{\gamma_n}) \geq c_X \frac{\ell_X(\gamma_n)}{\ln(\ell_X(\gamma_n))},
\]

where \( c_X \) depends on the hyperbolic metric \( X \).

Returning to the case of the modular surface, we can reformulate Theorem 1.3 in terms of the period of the geodesic’s continued fraction expansion, as follows:

**Theorem 1.4.** For the modular surface \( \Sigma_{\text{mod}} \), there exist a sequence \( \{ \gamma_k \} \) of filling closed geodesics on \( \Sigma_{\text{mod}} \) with \( n_{\gamma_k} \nearrow \infty \) such that,

\[
\text{Vol}(M_{\gamma_k}) \geq v_3 \frac{n_{\gamma_k}}{12},
\]

where \( n_{\gamma_k} \) is half the period of the continued fraction expansion of \( \gamma_k \) and \( v_3 \) is the volume of a regular ideal tetrahedra.

The reader might wonder why we insist in the period of the geodesics on the modular surface. The reason for this is that in a further research we intend to pursue an upper bound for the volume of the canonical lift complement which is linear in the period of the geodesic’s continued fraction expansion.

We point out that the main ingredient to achieve Theorem 1.3 and Theorem 1.4, is our next result which estimates a lower bound for the volume of the canonical lift complement in terms of combinatorial data between the geodesic and a given pants decomposition on \( \Sigma \).

**Theorem 1.5.** Given a pants decomposition \( \Pi := \{ P_i \}_{i=1}^{\chi(\Sigma)} \) on a hyperbolic surface \( \Sigma \), and \( \gamma \) a filling closed geodesic, we have that:

\[
\text{Vol}(M_{\gamma}) \geq \frac{v_3}{2} \sum_{i=1}^{\chi(\Sigma)} \sharp\{ \text{homotopy classes of } \gamma\text{-arcs in } P_i \},
\]
where \( v_3 \) is the volume of a regular ideal tetrahedra.

It is well known in 3–dimensional topology that giving a lower bound for the volume of a complete hyperbolic 3–manifolds is a more difficult problem than the upper bound, so it will be not surprising that Theorem 1.5 uses a more profound result ([2], Theorem 9.1) due to Agol, Storm and Thurston. This result allows us to give a lower bound in terms of the simplicial volume of the manifold constructed by doubling the pieces of \( M_\gamma \) that result from cutting it along the pre-image under the map \( PT^1\Sigma \to \Sigma \) of the pants curves.

Before presenting the structure of our paper, let us go back to the case of the modular surface. It will be interesting to calculate, by using Theorem 1.5, a lower bound for the geodesics in \([6]\), which come from the ideal class group of the fields \( \mathbb{Q}(\sqrt{d}) \) with \( d \) a square-free positive integer bigger than 1. The interest of these closed geodesics is that they are uniformly distributed on \( PT^1\Sigma_{mod} \).

The paper is organized as follows. After the introduction, we begin section 2 by presenting some well known facts on transversal homotopies, one of the main technical tools of this note that allows us to deform the geodesics without changing the topological type of their canonical lift complement. We also recall some 3–manifold properties of \( M_\gamma \), such as its JSJ-decomposition, which relies on the result of the hyperbolicity of \( M_\gamma \) ([13], Theorem 1.12). The only new result found in section 2 is the following on the topological type of \( M_\gamma \), which maybe is interesting in its own right:

**Theorem 1.6.** Let \( \gamma \) and \( \eta \) be two closed geodesics, where the class of \( \gamma \) in \( H_1(\Sigma; \mathbb{R}) \) is not trivial. Then \( M_\gamma \) is homeomorphic to \( M_\eta \) if and only if there is a diffeomorphism \( \phi \) of \( \Sigma \) such that \( \phi(\gamma) \) homotopic to \( \eta \).

Section 3 is devoted to construct sequences of geodesic whose canonical lift complements are not homeomorphic and their volumes are universally bounded (Theorem 1.1). This sequence of canonical lifts is obtained by performing annular Dehn fillings on a link in \( PT^1\Sigma \). It follows by using transversal homotopies that the length of the geodesic in the sequence is unbounded for any sequence of hyperbolic metrics. Specifically, one shows that the self-intersection number of the sequence is unbounded.

Later, in section 4 we prove our main result (Theorem 1.5) that gives a combinatorial lower bound on the volume of \( M_\gamma \). This allows to construct sequences of geodesics whose volume grows as the geodesic complexity increases, in terms of the length (Theorem 1.3) or in terms of the period (Corollary 1.4).

We conclude in section 5 by discussing what happens to the volume’s lower bound on the lift complement if one changes the canonical lift to a non-canonical lift in \( PT^1\Sigma \) over the same filling closed geodesic in the surface.

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2. Topology and geometry of $M_\gamma$

In this section we start noting that the canonical lift can be defined for a larger set of closed curves on surfaces and recall a special type of homotopies between them which preserve the topological type of their canonical lift complement. Later, in Theorem 1.6 we prove that for any non null-homologous closed geodesic $\gamma$, the topological type of $M_\gamma$ is determined by its mapping class group orbit. Finally, we set some preliminaries on 3-dimensional manifolds in order to investigate the topological and geometrical features of $M_\gamma$, for example we recall why the filling condition on $\gamma$ implies that $M_\gamma$ admits a unique hyperbolic structure.

2.1. Transversal homotopies. Any immersed closed curve $\gamma$ on $\Sigma$, has a canonical lift into $PT^1\Sigma$ by considering:

$$\hat{\gamma}(s) := \left( \gamma(s), \frac{\dot{\gamma}(s)}{\|\dot{\gamma}(s)\|} \right).$$

It happens to be an embedding if $\gamma$ is self-transverse, that is, an immersion without self-tangency points. Furthermore, we say that an homotopy $h : I \times S^1 \to \Sigma$, between two self-transverse closed curves $h_0$ and $h_1$, is transversal, if $h_t$ is self-transverse for all $t \in I$.

Notice that a transversal homotopy $h$ on $\Sigma$, induces an isotopy between the respective canonical lifts $\hat{h}_0$ and $\hat{h}_1$,

$$\hat{h} : I \times S^1 \to PT^1\Sigma \text{ where } \hat{h}(t,s) = \left( h_t(s), \frac{\dot{h}_t(s)}{\|\dot{h}_t(s)\|} \right).$$

By the isotopy extension lemma, there is an ambient isotopy $H : I \times PT^1\Sigma \to PT^1\Sigma$ between $\hat{h}_0$ and $\hat{h}_1$, which extends the isotopy $\hat{h}$, that satisfies:

$$H(t, \hat{h}_0(s)) = \hat{h}_t(s).$$

It follows that the topological type of the canonical lift complement of a self-transverse closed curve is invariant under transversal homotopies. Recall now that in [18] Hass and Scott proved that two homotopic self-transverse closed curves with minimal self-intersection, are transversally homotopic. Therefore, for any self-transverse closed curve with minimal self-intersection, its canonical lift complement is homeomorphic to the canonical lift complement of the unique closed geodesic associated to it.

Putting all this together we get the following fact, which we record here for later use:

**Lemma.** The homeomorphism type of $M_\gamma$ is independent of the chosen hyperbolic metric on the hyperbolic surface. \hfill $\square$

In the constructions of examples around this article, it will be important to characterize when a closed curve has minimal self-intersection. In ([17], Theorem 2) Hass and Scott proved the following result which will help us to decide when a closed curve on a surface has this property:

**Theorem 2.1** (Hass-Scott). Let $\alpha$ be a closed curve on a surface which has excess self-intersection. Then there is a singular 1–gon or 2–gon on the surface bounded by part of the image of $\alpha$.

A third Reidemeister move on a closed curve is a local move which corresponds to pushing a branch of a curve across a self-intersection, as shown in Figure 1 that is a special type
of transversal homotopy. In ([18], Theorem 2.1, and [15], Theorem 1) Hass and Scott, and later De Graaf and Schrijver, proved that if two closed curves with minimal self-intersection are on the same homotopy class, then there is a sequence of third Reidemeister moves and an ambient isotopy on the surface between them. Consequently, we have the following result which might exist in the literature of curves on surfaces, but we were not able to find it in this form useful for our study.

**Corollary.** Let \( \gamma \) and \( \eta \) be two homotopic self-transverse closed curves with minimal self-intersection on \( \Sigma \), then there is a homeomorphism between the non simply connected components of \( \Sigma \setminus \gamma \) and \( \Sigma \setminus \eta \). \( \square \)

A closed curve \( \alpha \) with minimal self-intersection on a surface \( \Sigma \) is said to be **filling** if \( \Sigma \setminus \alpha \) is a collection of disks or once-punctured disks; equivalently, if its geometric intersection with respect any essential simple closed curve is not zero. By the previous result, we conclude that this property is preserved by transversal isotopies.

**Corollary 2.2.** Given a self-transverse closed curve with minimal self-intersection, then the property of being filling over a surface is preserved by transversal homotopies. \( \square \)

### 2.2. Topological type of \( M_\gamma \)

In this subsection we prove that the canonical lift complement of a non null-homologous closed geodesic is determined by its mapping class group orbit.

**Theorem 1.6.** Let \( \gamma \) and \( \eta \) be two closed geodesics, where the class of \( \gamma \) in \( H_1(\Sigma; \mathbb{R}) \) is not trivial. Then \( M_\gamma \) is homeomorphic to \( M_\eta \) if and only if there is a diffeomorphism \( \phi \) of \( \Sigma \) such that \( \phi(\gamma) \) homotopic to \( \eta \).

**Proof.** By the discussion at the beginning of subsection 2.1 and the fact that diffeomorphisms preserve self-intersection number, we imply that closed geodesics in the same mapping class group orbit have homeomorphic canonical lift complements. This reduces the proof to one implication.

The first step is proving that every homeomorphism \( f : M_\gamma \to M_\eta \), can be extended it to a self-homeomorphism \( \hat{f} \) of \( PT^1 \Sigma \) with \( \hat{f}(\hat{\gamma}) = \hat{\eta} \). To show that this is the case, it is enough that \( f \) maps \( m_\gamma \), the meridian of \( \hat{\gamma} \), to the meridian of \( \hat{\eta} \). As the morphism in homology induced by the restriction of \( f \) on the boundary component corresponding to \( \hat{\gamma} \) is an isomorphism, then we just need to prove the following claim:

**Claim 2.3.** If the class of \( \gamma \) in \( H_1(\Sigma; \mathbb{R}) \) is not trivial, then, for the canonical lift \( \hat{\gamma} \) in \( PT^1 \Sigma \) we have:

\[
\text{Ker}(H_1(\partial N_\hat{\gamma}; \mathbb{R}) \xrightarrow{i_*} H_1(M_\hat{\gamma}; \mathbb{R})) = \text{Span}_\mathbb{R}\{[m_\hat{\gamma}]\},
\]

where \( N_\hat{\gamma} \) is a normal neighborhood of \( \hat{\gamma} \).
Assuming the Claim 2.3, we conclude the proof of Theorem 1.4 by showing that \( \hat{f} \) induces a diffeomorphism on the surface \( \Sigma \). Using the fact that the kernel of the morphism \( \pi_1(PT^1 \Sigma) \to \pi_1(\Sigma) \) is characteristic, we have the following morphism,

\[ \text{Out}(\pi_1(PT^1 \Sigma)) \to \text{Out}(\pi_1(\Sigma)). \]

By the Baer-Dehn-Nielsen Theorem, there exist \( \phi \in \text{Diff}(\Sigma) \) associated to \( \hat{f} \) such that,

\[ \phi_* \circ \pi_* = \pi_* \circ \hat{f}_*. \]

Then \( \phi_*(\gamma) = \phi_* \circ \pi_*(\gamma) = \pi_* \circ \hat{f}_*(\gamma) = \pi_*(\eta) = [\eta], \) meaning that \( \phi(\gamma) \) is homotopic to \( \eta \).

\[ \square \]

It remains to prove Claim 2.3:

**Proof of Claim 2.3.** Notice that the image of the homology class of a longitude in \( \partial N_\gamma \) under \( i_* \) is not trivial in \( H_1(M_\gamma; \mathbb{R}) \) because \( \hat{\gamma} \) is not trivial in \( H_1(PT^1 \Sigma; \mathbb{R}) \) by hypothesis. Then it is enough to show that there is a non trivial element of \( \text{Span}_\mathbb{R}\{[m_\gamma]\} \) whose image under \( i_* \) vanishes in \( H_1(M_\gamma; \mathbb{R}) \). To prove this, consider the Mayer-Vietoris sequence in homology with real coefficients,

\[ \cdots \to H_2(PT^1 \Sigma) \xrightarrow{\delta} H_1(\partial N_\gamma) \xrightarrow{(i_*j_*)} H_1(M_\gamma) \oplus H_1(\gamma) \to H_1(PT^1 \Sigma) \to \cdots \]

Then by the exactness of the previous sequence, the last condition reduces to finding a surface on \( PT^1 \Sigma \) whose image under \( \hat{\delta} \), the connecting morphism of the sequence, is a non trivial element of \( \text{Span}_\mathbb{R}\{[m_\gamma]\} \).

Since the class of \( \gamma \) is not trivial in \( H_1(\Sigma; \mathbb{Z}) \), there is a non separating simple closed curve \( \alpha \) such that \( n[\alpha] = [\gamma] \) for some non zero integer \( n \) ([12], Proposition 6.2). Choose \( [\beta] \) a primitive element in \( H_1(\Sigma; \mathbb{Z}) \) such that its algebraic intersection with \( [\alpha] \) is one. Consider \( T_\beta \) the pre-image of \( \beta \), a simple closed geodesic, under the map \( PT^1 \Sigma \to \Sigma \), which is a torus because \( PT^1 \Sigma \) is orientable. Then

\[ \hat{\delta}([T_\beta]) = [\partial(T_\beta \cap N_\gamma)] = ([\beta], [\gamma])[m_\gamma] = n[m_\gamma], \text{ with } n \neq 0. \]

\[ \square \]

**Remark.** By the discussion in section 2.1, if \( \gamma \) and \( \eta \) are two self-transverse closed curves with minimal self-intersection on \( \Sigma \), where the class of \( \gamma \) in \( H_1(\Sigma; \mathbb{R}) \) is not trivial, then \( M_\gamma \) is homeomorphic to \( M_\eta \) if and only if there is a diffeomorphism \( \phi \) of \( \Sigma \) such that \( \phi(\gamma) \) is homotopic to \( \eta \).

We conclude this subsection by giving a second proof of the fact that two closed geodesics in the same mapping class group orbit have homeomorphic canonical lift complements. This alternative proof relies on an action of the mapping class group of \( \Sigma \) into its unit tangent bundle, which leaves invariant the foliation given by the geodesic flow.

Let \( \phi \) be in \( \text{Diff}(\Sigma) \) and consider \( \tilde{\phi} : \mathbb{H}^2 \to \mathbb{H}^2 \) a lift of \( \phi \). In that case we can give rise to a \( \phi_* \)-equivariant homeomorphism of the boundary at infinity of the hyperbolic space to itself,

\[ \partial \tilde{\phi} : \partial \mathbb{H}^2 \to \partial \mathbb{H}^2. \]

Using the fact that \( T^1 \mathbb{H}^2 \) is diffeomorphic to

\[ \{(a_1, a_2, a_3) \in \partial \mathbb{H}^2 \times \partial \mathbb{H}^2 \times \partial \mathbb{H}^2 \mid a_i = a_j \forall i \neq j \}, \]
quotiented via the fix-point free involution,

\[(a_1, a_2, a_3) \mapsto (a_2, a_1, a_3).\]

This diffeomorphism maps \((a_1, a_2, a_3)\) to the unique unit tangent vector \(v\) normal to the geodesic in \(\mathbb{H}^2\) with endpoints \(a_1, a_2\) and pointing to \(a_3\).

From this point of view, we induce a \(\phi^*\)-equivariant map,

\[\Phi: T^1\mathbb{H}^2 \to T^1\mathbb{H}^2\]

such that \(\Phi(x, y, z) = (\partial\phi(x), \partial\phi(y), \partial\phi(z))\).

The \(\phi^*\)-equivariance property implies that the map \(\Phi\) descends to \(\hat{\Phi}: \hat{\pi}T^1\Sigma \to \hat{\pi}T^1\Sigma\), sending the orbits of the geodesic flow to themselves.

### 2.3. Hyperbolicity of \(M_\hat{\gamma}\)

In this subsection we study some 3-manifold properties of \(M_\hat{\gamma}\). In particular we describe the J.S.J. decomposition of \(M_\hat{\gamma}\), this is mainly due to Foulon and Hasselblatt ([13], Theorem 1.12), which states that the canonical lift complement over a filling closed geodesic is hyperbolic. This result is establish by verifying the hypotheses of the Hyperbolization Theorem [29]:

**Theorem (Hyperbolization Theorem).** The interior of a closed 3-manifold \(N\) with torus boundary admits a hyperbolic metric with finite volume if and only if it is irreducible, (homotopically) atoroidal, with infinite fundamental group and not homeomorphic to \(S^1 \times D^2\), \(T^2 \times [0, 1]\) or \(K \times [0, 1]\) (the twisted interval bundle over the Klein bottle).

We will begin by reviewing our list of definitions. For technical details see [19] and [21].

Suppose \(N\) is a compact, orientable 3-manifold with boundary. A embedded surface in \(N\) is said to be **compressible** if there is an essential circle (that is one that bounds no disk in the surface) on it that bounds a disk in \(N\). In the case where the embedded surface is not compressible, is said to be **incompressible**.

An incompressible torus is **boundary-parallel** if there is an isotopy from it to \(\partial N\). That is, there is an embedded \(T^2 \times [0, 1]\) such that \(T^2 \times \{0\}\) parametrizes the given torus and \(T^2 \times \{1\}\) a boundary component.

The 3-manifold \(N\) is said to be **irreducible** if every embedded 2-sphere bounds a ball. In this case we also say that \(N\) is **homotopically atoroidal** if every \(\pi_1\)-injective map from the torus to \(N\) is homotopic to a map into \(\partial N\). Being homotopically atoroidal is a stronger property than being **atoroidal** (here, atoroidal means that every incompressible torus in \(N\) is boundary-parallel). Nevertheless, the two notions agree except for some Seifert fibered manifolds (see [21], Lemma IV.2.6), which will not appear in this paper.

Let us investigate for a given closed geodesic \(\gamma\) on \(\Sigma\), which of properties on the Hyperbolization Theorem are satisfied by its canonical lift complement:

- \(M_\hat{\gamma}\) is irreducible, because if \(S^2\) is embedded in \(M_\hat{\gamma}\) and hence into \(\hat{\pi}T^1\Sigma\), then it bounds a ball \(B\) in \(\hat{\pi}T^1\Sigma\), by irreducibility of \(\hat{\pi}T^1\Sigma\). As \(S^2 \cap \hat{\gamma} = \emptyset\) we conclude that \(\hat{\gamma}\) is embedded outside \(B\), because if \(\hat{\gamma}\) is inside \(B\), then its projection to \(\Sigma\) would be nullhomotopic.

- \(\pi_1(M_\hat{\gamma})\) is infinite, because the inclusion map \(i: M_\hat{\gamma} \to \hat{\pi}T^1\Sigma\) induces, by transversality, a \(\pi_1\)-surjective map and \(\pi_1(\hat{\pi}T^1\Sigma)\) is infinite.
Definition 2.4. Let \( \eta \) be a simple closed geodesic on \( \Sigma \), then we define the following embedded surfaces on \( M_\gamma \) :

\[
T_\eta \text{ the pre-image of } \eta \text{ under the map } PT^1\Sigma \to \Sigma, \text{ if } i(\eta, \gamma) = 0,
\]
and

\[(T_\eta)_\gamma := (T_\eta) \setminus \mathcal{N}_\gamma, \text{ if } i(\eta, \gamma) \neq 0.\]

- For the atoroidality of \( M_\gamma \), notice that if a simple closed geodesic \( \eta \) does not intersect \( \gamma \) in \( \Sigma \), then \( T_\eta \) is an embedded incompressible torus in \( M_\gamma \), \([\text{Theorem B.8}]\) which is not boundary-parallel. Therefore, a necessary condition for \( M_\gamma \) to be atoroidal is that \( \gamma \) fills \( \Sigma \).

- Conversely, Foulon and Hasselblatt proved in \([\text{Theorem 1.12}]\) that the filling condition on \( \gamma \) is sufficient to provide atoroidality to \( M_\gamma \), and consequently a unique hyperbolic structure with finite volume.

**Theorem** (Foulon-Hasselblatt). If \( \gamma \) is a filling self-transverse closed curve with minimal self-intersection number on a hyperbolic surface \( \Sigma \), and \( \tilde{\gamma} \) any continuous lift of \( \gamma \). Then \( M_{\tilde{\gamma}} \) is a hyperbolic manifold of finite volume.

In the case where \( \gamma \) is just a non simple closed geodesic, that is \( M_\gamma \) is an irreducible 3–manifold with torus boundary, then by JSJ-decomposition Theorem \([\text{20, 22}]\), we can canonically split \( M_\gamma \) along a finite (maybe empty) collection of disjoint and non-parallel nor boundary-parallel incompressible embedded tori into connected components which are either atoroidal or Seifert fibered:

**Theorem** (JSJ-decomposition). Let \( N \) be a compact, irreducible 3–manifold with boundary. In the interior of \( N \), there exists a family \( C = \{T_1, ..., T_r\} \) of disjoint tori that are incompressible and not boundary parallel, with the following properties:

1. each connected component of \( N \setminus C \) is either a Seifert manifold or is atoroidal;
2. the family \( C \) is minimal among those satisfying (1).

Moreover, such a family \( C \) is unique up to ambient isotopy.

We recall that the characteristic submanifold of \( N \) is the union of all Seifert fibered JSJ-components of \( N \).

**Corollary.** Let \( \gamma \) be a non simple closed geodesic in \( \Sigma \). Denote by \( \Sigma_1 \) be the essential subsurface filled by \( \gamma \), \( \Sigma_2 \) the closure in \( \Sigma \) of \( \Sigma \setminus \Sigma_1 \) and \( M_\gamma := PT^1\Sigma_1 \setminus \mathcal{N}_\gamma \). Then the JSJ-decomposition of \( M_\gamma \) in its atoroidal piece and characteristic submanifold is:

\[
M_\gamma \cup PT^1\Sigma_2, \text{ with the respective splitting incompressible tori } T_{(\partial \Sigma_1 \setminus \partial \Sigma)}. \quad \square
\]

Lastly, we point out that the embedded surfaces \( (T_\eta)_\gamma \) on Definition 2.4 are also incompressible on \( M_\gamma \). This surfaces, will be important for our main result in section 4.

**Lemma 2.5.** For every pair of intersecting closed geodesic \( \eta \) and \( \gamma \) in \( \Sigma \), where \( \eta \) is simple then the embedded surface \( (T_\eta)_\gamma \) in \( M_\gamma \) is \( \pi_1 \)-injective.

**Proof.** We consider for the sake of concreteness, the case where the surface \( (T_\eta)_\gamma \) is separating and leave the non separating case to the reader. Let us split \( M_\gamma \) by \((T_\eta)_\gamma\), into pieces \( N_1 \) and \( N_2 \). By Van Kampen it is enough to show the \( \pi_1 \)-injectivity for the surface \((T_\eta)_\gamma\) in \( N_i \).
If the surface \((T_{\eta})_{\hat{\gamma}}\) is not \(\pi_1\)-injective, then by the Loop Theorem, there is an embedded disk \(D_0\) in \(M_{\hat{\gamma}}\) whose boundary is an essential simple closed curve \(\alpha\) in \((T_{\eta})_{\hat{\gamma}}\).

As \(T_{\eta}\) is an incompressible surface in \(PT^1\Sigma\) ([13], Theorem B.8), then \(\alpha\) bounds a disk \(D_1\) in \(T_{\eta}\) that intersects \(\hat{\gamma}\). So by the irreducibility of \(PT^1\Sigma\), the embedded sphere formed by \(D_0 \cup D_1\) would bound a ball \(B\) in \(PT^1\Sigma\).

By the periodicity of \(\hat{\gamma}\), there is at least one arc of \(\hat{\gamma}\) inside \(B\) with endpoints at \(D_1\). This arc is homotopic relative to the boundary to an arc in \(B\) that union with a simple arc in \(D_1\) with the same endpoints, bounds a disk in \(N_{\hat{\gamma}}\). Such homotopy inside \(PT^1\Sigma\), induces an homotopy on \(\Sigma\) that reduces the intersection number between \(\gamma\) and \(\eta\), contradicting the fact that \(\gamma\) and \(\eta\) are in minimal position. \(\square\)

3. Sequences of closed geodesics with uniformly bounded volume complement

In this section we prove one of our main results, Theorem 1.1, which states a way of constructing sequences of closed geodesics of increasing length, whose canonical lift complement are not homeomorphic and their volume is universally bounded by the volume of a link complement on \(PT^1\Sigma\). Although this kind of examples already existed (see [2], Subsection 3.2 or [6], Example 5.2), we point out that our method generalizes the previous examples.

**Theorem 1.1.** Given a hyperbolic surface \(\Sigma\), there exist a constant \(V_0 > 0\) and a sequence \(\{\gamma_n\}\) of filling closed geodesics on \(\Sigma\), with \(M_{\gamma_n} \neq M_{\gamma_k}\) for every \(k \neq n\), such that \(\text{Vol}(M_{\gamma_n}) < V_0\) for every \(n \in \mathbb{N}\). Moreover, for any sequence \(\{X_n\}\) of hyperbolic metrics on \(\Sigma\), we have that \(\ell_{X_n}(\gamma_n) \nearrow \infty\).

**Proof.** Choose \(\gamma_0\) and \(\eta\) closed geodesics on \(\Sigma\) such that the first one is filling, the second has a fixed orientation and \(i(\gamma_0, \eta) > 1\).

For each intersection point between \(\gamma_0\) and \(\eta\), we measure the angle of intersection at that point by using the orientation of \(\eta\) and the right side on \(\Sigma\) with respect \(\eta\). Let \(p\) be a point of intersection of \(\gamma_0\) with \(\eta\) that minimizes the angle of intersection.

Define \(\gamma_n\) the unique closed geodesic homotopic to the closed curve that starts at \(p\), travels \(n\) times about \(\eta\) with its same orientation and then one time about \(\gamma_0\), that is

\[\eta^n \ast_p \gamma_0\] (denoted as \(\gamma_{n,0}\)).

Notice that \(\gamma_{n,0}\) is not self-transverse, so we will slightly modify it to obtain one that is, in following manner:

Choose a \(\delta\) neighborhood of \(\eta\) such that for all self-intersections of \(\gamma_0\) that lie outside the geodesic \(\eta\), also lie outside the \(\delta\) neighborhood of \(\eta\).

Erase the geodesic arc of \(\gamma_0\) that has \(p\) and is contained in the \(\delta\) neighborhood of \(\eta\). Link the extremal points by the geodesic segment \(\eta_n\) in the \(\delta\) neighborhood of \(\eta\) that winds \(n\) times in the orientation of \(\eta\). Let us denote this piecewise geodesic as \(\gamma_{n,1}\).

Using elementary hyperbolic geometry, one easily shows that \(\gamma_{n,1}\) has not self-tangency points, and the angle in the intersection points between \(\gamma_{n,1} \setminus \eta_n\) and \(\eta_n\) are smaller than the angles in the gluing points.

Notice that by construction,

\[n^2i(\eta, \eta) + n(i(\gamma_0, \eta) - 1) = \sharp(\gamma_{n,1} \cap \gamma_{n,1}).\]
Claim 3.1. \( \gamma_{n,1} \) has the minimal number of self-intersection.

**Proof.** By Theorem 2.1, the proof reduces to show that \( \gamma_{n,1} \) has no 1-gons nor 2-gons. Consider the lifts of \( \gamma_{n,1} \) in the universal cover of \( \Sigma \), which are piecewise geodesic lines in \( \mathbb{H}^2 \). It is enough to show that every pair of them intersects at most once. Indeed, if there were two consecutive intersections they can not happen in segments of the same geodesic segment lifts, so in between there must be at least one gluing point of two distinct geodesic segments.

If there are more than one gluing points in between the consecutive intersections, we would have geodesic side quadrilaterals whose sum of interior angles is bigger or equal to \( 2\pi \), which is a contradiction in the hyperbolic plane.

If there are exactly one gluing point cannot happen thanks to the choice of \( p \) as the intersection points between \( \gamma_0 \) and \( \eta \) with minimal angle. \( \square \)

In order to make \( \gamma_{n,1} \) smooth, just round the two corners of \( \gamma_{n,1} \), and denote it by the same name. This gives us a self-transverse closed curve homotopically transversal to \( \gamma_0 \), which means that \( M_{\gamma_n} \not\cong M_{\gamma_{n,1}} \). By Theorem 1.6, \( M_{\gamma_n} \not\cong M_{\gamma_k} \) for every \( k \neq n \), because their self-intersection number, which is an invariant under the mapping class group action, is different for each element of the sequence \( \{ \gamma_n \} \). Moreover, for any sequence \( \{ X_n \} \) of
hyperbolic metrics on $\Sigma$, we have that $\ell_{X_n}(\gamma_n) \nearrow \infty$, because the self-intersection number is increasing too.

The hyperbolicity of $M_{\hat{\gamma}_n}$ comes from the following fact:

**Claim 3.2.** $\gamma_n$ is filling for every $n \in \mathbb{N}$.

![Figure 4. The images on the left are two examples of partitions of $\Sigma$ via $\eta \cup \gamma_0$, on the center $\gamma_1$, and on right for $\gamma_2$. a) The partition around $p$. b) The partition where the arcs does not contain $p$.]

**Proof.** Take the partition of $\Sigma$ by $\gamma_0$ and draw $\eta$ over this partition. Firstly will be interested in comparing the partitions between $\Sigma \setminus \gamma_0$ and $\Sigma \setminus \gamma_{1,1}$.

Consider the four sides of the graph induced by $\gamma_0 \cup \eta$ that have $p$ as vertex and cut them at $p$ and glue them in a particular order that depends on the orientation of $\eta$. Next, superpose the pair of new arcs to geodesic segments on $\eta_1$.

In the case were the boundary discs of the partition of $\Sigma$ by $\gamma_0$ does not contain $p$ we simply consider the partition of the discs given by $\gamma_0 \cup \eta_1$.

Thus if $\gamma_0$ is filling then $\gamma_{1,1}$ is also filling. Moreover, as this property of being filling is preserved by transversal homotopy (Corollary 2.2), implies that $\gamma_1$ is also filling.

Notice that, $\Sigma \setminus \gamma_{n,1}$ is a subdivision of the partition given by $\gamma_{1,1}$, then $\gamma_n$ is also filling. $\square$

Finally, in order to find an upper bound to the volume of all the hyperbolic 3–manifolds $M_{\hat{\gamma}_n}$ we will construct $L_0$ and $L_1$, two knots inside $M_{\gamma_0}$ and show that each $M_{\hat{\gamma}_n}$ is obtained by making Dehn surgery along $L_0$ and $L_1$. As Dehn filling decreases volume ([28], Theorem 6.5.6), then:

$$Vol(M_{\hat{\gamma}_n}) < Vol(M_{\gamma_0} \setminus \{L_0, L_1\}), \text{ for every } k \in \mathbb{N}.$$  

For the construction of $L_0$ and $L_1$, let $\theta$ the angle of intersection of $\gamma_0$ and $\eta$ at $p$. As the number of possible angles in the intersection of $\gamma_0$ and $\eta$ is finite, there exist a small $\varepsilon$ such that the following annulus:

$$A := \{(\eta(t), \hat{\eta}(t)e^{2i\pi s}) \mid t \in S^1, \ s \in [0, \theta + \varepsilon]\} \subset PT^1\Sigma,$$

is intersected only once by $\hat{\gamma}_0$ at the point whose projection on $\Sigma$ is $p$. Denote the boundary component of $A$ associated to $\hat{\eta}$ as $L_0$ and let $L_1$ be the other boundary component of
A. Moreover, \( M_{\tilde{\gamma}} \setminus \{L_0, L_1\} \) is atoroidal because \( M_{\tilde{\gamma} \cup L_1} \) and \( M_{\tilde{\gamma} \cup L_0} \) are atoroidal ([13], Theorem 1.12). So if \( M_{\tilde{\gamma}} \setminus \{L_0, L_1\} \) is not atoroidal, there would be an incompressible torus in \( M_{\tilde{\gamma}} \setminus \{L_0, L_1\} \) which is not boundary-parallel and contains \( L_0 \cup L_1 \), consequently also \( \Lambda \), contradicting that \( \tilde{\gamma} \) intersects \( A \).

Take \( V \) a solid torus inside the preimage the \( \delta \) neighborhood of \( \eta \) under the map \( \Sigma \to PT^1\Sigma \) which contains \( A \) and does not intersect other arcs of \( \tilde{\gamma} \) different from the one that intersects \( A \).

Let \( V' \) be the resulting 3–manifold obtained by performing a \( \frac{1}{n} \) Dehn surgery on one component of \( \partial A \) and a \( \frac{1}{n} \) Dehn surgery on the other component of \( \partial A \) inside \( V \). By ([24], Theorem 2.1) there exist a diffeomorphism \( \phi_n : V' \to V \) such that \( \phi_n |_{\partial V} = id \). Moreover, each homotopy class relative to the boundary of an arc transversal to the annulus is send to the homotopy class of the initial arc concatenated, at the point of intersection with the annulus, with an arc that winds along the soul of \( V \) \( n \) times.

In conclusion, the diffeomorphism \( \phi_n \) extends to a self-homeomorphism of \( PT^1\Sigma \) that sends the isotopy class of \( \tilde{\gamma} \) to \( \tilde{\gamma}_n \).

\[ \square \]

**Remark.** From Theorem [1.1] it is natural to research for bounds of the volume of \( M_\alpha \) more intrinsic than the length or the self-intersection number, whose complexity is invariant under the mapping class group action.

We can also apply our method in Theorem [1.1] to explain why the volume of the sequence of geodesics found in (Example 5.2, [6]) is bounded. We recall that every closed geodesics on the modular surface is represented as a positive word in the code for a closed geodesic \( \gamma \) containing both symbols. Conversely, any such word encodes a unique periodic geodesic \( x \) on the modular surface is represented as a positive word containing \( x \) and \( y \), whose representing geodesic intersects \( \gamma \).

\[ \frac{1}{2} \leq \frac{1}{2} \leq \frac{1}{2} \]

**Corollary 1.2.** For the modular surface \( \Sigma_{\text{mod}} \), there exist a constant \( V_0 > 0 \) and a sequence \( \{\gamma_k\} \) of filling closed geodesics on \( \Sigma_{\text{mod}} \), with \( M_{\tilde{\gamma}_0} \neq M_{\tilde{\gamma}_k} \) for every \( k \neq n \), such that \( Vol(M_{\tilde{\gamma}_k}) < V_0 \) for every \( k \in \mathbb{N} \) and the period of the continued fraction expansion of \( \gamma_k \) tends to infinity.

**Proof.** Consider the closed geodesics \( \gamma_k \) whose code is of the form \( \{x^2y\alpha^k\} \), where \( \alpha \) is any positive word containing \( x \) and \( y \) that ends with \( y \), whose representing geodesic intersects the geodesic encoded by \( x^2y \) more than once. Proceeding as in Theorem [1.1] one shows that the volume associated to \( x^2y\alpha^k \) is bounded by the volume of

\[ PT^1\Sigma_{\text{mod}} \setminus (\hat{x}^2 \hat{y} \cup \hat{\alpha} \cup \hat{\alpha}_\theta), \]

where \( \hat{\alpha}_\theta \) is a knot obtain by translating \( \hat{\alpha} \) by some small angel \( \theta \) in the fiber direction of \( PT^1\Sigma_{\text{mod}} \).

\[ \square \]

**Remark 3.3.** For the sequence of geodesics in Theorem [1.1] we observe that if we fix a suitable pants decomposition on \( \Sigma \), then the homotopy classes of arcs in each pair of pants is bounded. The only thing that changes is the number of arcs in the fixed homotopy classes.

This remak is the motivation for our next main result, that we present in the following section.
4. LOWER BOUND ON THE VOLUME OF $M_\gamma$

In this section we prove a lower bound for the volume of the canonical lift complement (Theorem 1.5). The lower bound is obtained in terms of combinatorial data coming from the geodesic and a pants decomposition of the surface. We apply this result to find a sequence of geodesics in the modular surface where the lower bound can be written in terms of the period of the continued fraction extension of the geodesics (Corollary 1.4). We conclude by finding a sequence of geodesics on surfaces, whose lower bound is written in terms of their length (Theorem 1.3).

Theorem 1.5. Given a pants decomposition $\Pi := \{P_i\}_{i=1}^{-\chi(\Sigma)}$ on a hyperbolic surface $\Sigma$, and $\gamma$ a filling geodesic, we have that:

$$\text{Vol}(M_\gamma) \geq \frac{v_3}{2} \sum_{i=1}^{\chi(\Sigma)} \sharp\{\text{homotopy classes of } \gamma\text{-arcs in } P_i\},$$

where $v_3$ is the volume of a regular ideal tetrahedra.

Given a pair of pants $P$, we say that two arcs $\alpha, \beta : [0, 1] \to P$ with $\alpha([0, 1]) \cup \beta([0, 1]) \subset \partial P$ are in the same homotopy class in $P$, if there exist an homotopy $h : [0, 1] \times [0, 1] \to P$ such that:

$h_0(t_2) = \alpha(t_2), \ h_1(t_2) = \beta(t_2)$ and $h([0, 1] \times \{0, 1\}) \subset \partial P$.

Before stating the main result to prove Theorem 1.5 we recall some definitions.

If $N$ is a hyperbolic 3–manifold and $S \subset N$ is an embedded incompressible surface, we will use $N \setminus S$ to denote the manifold that is obtained by cutting along $S$; it is homeomorphic to the complement in $N$ of an open regular neighborhood of $S$. If one takes two copies of $N \setminus S$, and glues them along their boundary by using the identity diffeomorphism, one obtains the double of $N \setminus S$, which is denoted by $D(N \setminus S)$.

Let $M$ be a connected, orientable 3-manifold with boundary and let $S(M; \mathbb{R})$ be the singular chain complex of $M$. More concretely, $S_k(M; \mathbb{R})$ is the set of formal linear combination of $k$–simplices, and we set as usual $S_k(M, \partial M; \mathbb{R}) = S_k(M; \mathbb{R})/S_k(\partial M; \mathbb{R})$. We denote by $\|c\|$ the $l_1$–norm of the $k$–chain $c$. If $\alpha$ is a homology class in $H^\text{sing}_k(M, \partial M; \mathbb{R})$, the Gromov norm of $\alpha$ is defined as:

$$\|\alpha\| = \inf_{\|c\| = \alpha} \{\|c\| = \sum_{\sigma} |r_\sigma| \text{ such that } c = \sum_{\sigma} r_\sigma \sigma\}.$$  

The simplicial volume of $M$ is the Gromov norm of the fundamental class of $(M, \partial M)$ in $H^\text{sing}_3(M, \partial M; \mathbb{R})$ and is denoted by $\|M\|$. The key ingredient to prove Theorem 1.5 is the following result due to Agol, Storm and Thurston ([2], Theorem 9.1):

Theorem (Agol-Storm-Thurston). Let $N$ be a compact manifold with interior a hyperbolic 3–manifold of finite volume. Let $S$ be an embedded incompressible surface in $N$. Then

$$\text{Vol}(N) \geq \frac{v_3}{2} \|D(N \setminus S)\|.$$  

We now prove the lower bound for the volume of the canonical lift complement:
Proof of Theorem 1.5. Let \( \{ \eta_i \}_{i=1}^{3g-3} \) be the simple closed geodesics on the boundary of the pants decomposition \( \Pi \). Consider the incompressible surface \( S := \bigsqcup_{i=1}^{3g-3} (T_\eta) \) in \( M_\gamma \) (Lemma 2.5). From ([2], Theorem 9.1) we deduce that:

\[
\text{Vol}(M_\gamma) \geq \frac{v_3}{2} \|D(M_\gamma \setminus S)\| = \frac{v_3}{2} \sum_{i=1}^{\chi(\Sigma)} \|D((P_\gamma)_{\gamma})\|.
\]

For each pair of pants \( P \) we have:

\[
v_3 \|D(P_\gamma)\| = v_3 \|D(P_\gamma)^{hyp}\| \geq \text{Vol}(D(P_\gamma)^{hyp}) \geq v_3 \#\{\text{cusps of } D(P_\gamma)^{hyp}\},
\]

where \( D(P_\gamma)^{hyp} \) is the atoroidal piece of \( D(P_\gamma) \), complement of the characteristic submanifold, with respect to its JSJ-decomposition. The first and third inequality come form [16] and \([1]\) respectively.

In Lemma 4.1 we show that there is a injection between the homotopy classes of \( \gamma \)-arcs in \( P \) and the cusps of \( D(P_\gamma)^{hyp} \), which completes the proof.

Since two isotopic \( \tilde{\gamma} \)-arcs in \( T^1 P \) induce a homotopy between their projections in \( P \), the number of homotopy classes of \( \gamma \)-arcs in \( P \) is less or equal than the number of isotopy classes of \( \gamma \)-arc in \( T^1 P \). Therefore, it is enough to prove the following fact:

Lemma 4.1. The number of isotopy classes of \( \tilde{\gamma} \)-arc in \( T^1 P \), is less or equal than the number of cusps of \( D(P_\gamma)^{hyp} \).

Proof. Firstly we will define the following function:

\[
\{ \tilde{\gamma} \text{-arcs in } T^1 P \} \varphi \rightarrow \{ \text{cusps of } D(P_\gamma)^{hyp} \} = \left\{ \begin{array}{c} \text{splitting tori of the} \\ \text{JSJ-decomposition of} \\ D(P_\gamma) \end{array} \right\} \Pi \left\{ \begin{array}{c} \text{tori contained in} \\ \partial D(P_\gamma) \cap D(P_\gamma)^{hyp} \end{array} \right\}.
\]

If the boundary component on \( D(P_\gamma) \) induced by the \( \tilde{\gamma} \)-arc in \( T^1 P \) belongs to the characteristic submanifold of \( D(P_\gamma) \), \( \varphi \) sends it to a splitting tori corresponding to the connected component of the characteristic submanifold where it is contained. Otherwise, \( \varphi \) sends it to itself.

If there are more isotopy classes of \( \tilde{\gamma} \)-arcs in \( T^1 P \) than the number of cusps of \( D(P_\gamma)^{hyp} \), then there are two non isotopic \( \tilde{\gamma} \)-arcs in \( T^1 P \), that belong to the same connected component of the characteristic submanifold. Meaning that, they are isotopic in the corresponding component and so in \( T^1 P \), but this contradicts the fact that they were not isotopic.

By Lemma 4.1 and proceeding as in Theorem 1.5, we give a lower bound for the simplicial volume of any continuous lift complement over a closed geodesic on a hyperbolic surface:

Corollary 4.2. Given a pants decomposition \( \Pi := \{ P_i \}_{i=1}^{\chi(\Sigma)} \) on a hyperbolic surface \( \Sigma \) and \( \gamma \) a closed geodesic, we have that for any continuous lift \( \tilde{\gamma} \) we have:

\[
\|M_\gamma\| \geq \frac{1}{2} \sum_{i=1}^{\chi(\Sigma)} \#\{ \text{isotopy classes of } \tilde{\gamma} \text{-arcs in } T^1 P_i \}.
\]

In the case of the canonical lift, we can prove a correspondance between the homotopy classes of \( \gamma \)-arcs in a pair of pants \( P \) and the isotopic classes of \( \tilde{\gamma} \)-arcs in \( T^1 P \).
Remark 4.3. If $\omega_1$ and $\omega_2$ is pair of homotopic $\gamma-$arcs on $P$ then their respective canonical lifts $\tilde{\omega}_1$ and $\tilde{\omega}_2$ are isotopic $\hat{\gamma}-$arcs in $T^1P$. Indeed, choose $\tilde{\omega}_1$ and $\tilde{\omega}_2$ be lifts on the universal cover starting in the same fundamental domain. Take an homotopy of geodesics that varies $T$ that the geodesic homotopy induces an isotopy in $P$. This will give us a homotopy $h$ of geodesics arcs $h_t$ that starts in $\omega_1$ and ends in $\omega_2$. Notice that the image of the geodesic homotopy does not intersects other $\hat{\gamma}-$arcs because of local unicity of the geodesics. Then we have that the geodesic homotopy induces an isotopy in $T^1P$.

Remark. Notice that if $\Omega$ is the set of all $\gamma-$arcs on $P$ minus one arc for each homotopy class, we have that:

$$D(P_{\tilde{\gamma}})^{hyp} \cong D(P_{\gamma \backslash \Omega})^{hyp}.$$  

This result implies that there exist geodesics $\gamma$ on $\Sigma$ such that the $Vol(M_\gamma)$ can be as big as we want. Let us fix a pants decomposition on $\Sigma$, then for any $N \in \mathbb{N}$ there exist a closed geodesic with at least $N$ homotopy classes of geodesic arcs in one pair of pants. This is constructed by taking $N$ non homotopic geodesic arcs in a pair of pants and linking them to form a filling close geodesic on $\Sigma$.

The lower bound of the volume of $M_\gamma$ obtained in Theorem 1.5 does not have control on the length of the geodesic, even if each homotopy class of $\gamma-$arcs contributes to the length of $\gamma$. So a natural question is how big can the volume of $M_\gamma$ be when the length of the filling close geodesic $\gamma$ is bounded. In subsection 4.2 we try to understand this relation.

4.1. Coding filling geodesics on surfaces by splitting along a simple closed geodesic. This subsection gives a method for describing closed geodesics on hyperbolic surfaces that intersect a given simple closed geodesic. This will be useful for the results in the subsequent sections because it also allows us identifying the homotopy classes of geodesic arcs generated by splitting along the simple closed geodesic.

Let $\Sigma$ be a hyperbolic surface and fix a simple closed geodesic $\beta$. By applying either HNN extensions or amalgamated products (see [27]), more concretely van Kampen’s Theorem, we can code closed geodesics on $\Sigma$ as a product only involving certain sequences of elements. This elements come from the fundamental group of the connected components of the surface minus the simple closed geodesic $\beta$.

Let $\beta$ be a separating (non-separating) simple closed geodesic on $\Sigma$. Denote by $\Sigma_1$ the union of $\beta$ and one of the connected components of $\Sigma \setminus \beta$ and by $\Sigma_2$ the union of $\beta$ with the other connected component (resp. let $\Sigma_1$ be the complement of $N_\beta$ maximal normal embedded neighborhood of $\beta$ in $\Sigma$ whose boundary is a closed geodesic). Choose a point $p \in \beta$ to be the base point of the fundamental group of $\Sigma$ (resp. $\Sigma_1$ in the self intersection of the closed geodesic which is the boundary of $N_\beta$). As a consequence of van Kampen’s Theorem, when $\beta$ is separating, $\pi_1(\Sigma,p)$ is isomorphic to the free product of $\pi_1(\Sigma_1,p)$ with $\pi_1(\Sigma_2,p)$ amalgamated by the subgroup $\pi_1(\beta,p)$. If $\beta$ is non-separating, $\pi_1(\Sigma,p)$ is isomorphic to the HNN extension of $\pi_1(\Sigma_1,p)$ relative to an isomorphism between two simple curves in the boundary of $N_\beta$. Notice also that the simple closed geodesic $\tau$ inside $N_\beta$ perpendicular to $\beta$, passing thought $p$, represents the stable element, denoted by $t$.

Definition. Let $\beta$ be a separating (non-separating) simple closed geodesic on $\Sigma$, and $n$ be a positive integer. A finite sequence

$$(g_1, g_2, \ldots, g_n) \quad \text{(resp.} \quad (g_0, t^{e_1}, g_1, t^{e_2}, \ldots, g_{n-1}, t^{e_n})\text{)}$$  

of elements of \(\pi_1(\Sigma_1, p) \ast_\beta \pi_1(\Sigma_2, p)\) (resp. \(\pi_1(\Sigma_1, p) \ast_\beta\)) is \(\beta\)-cyclically reduced if the following conditions hold:

1. each \(g_i\) is in one factor \(\pi_1(\Sigma_1, p)\) or \(\pi_1(\Sigma_2, p)\) (resp. \(\varepsilon_i \in \{1, -1\}\) and \(g_i \in \pi_1(\Sigma_1, p)\));
2. for each \(i \in \{1, \ldots, n-1\}\), \(g_i\) and \(g_{i+1}\) are not in the same factor, (resp. there is no consecutive subsequence of the form \((t^\varepsilon, g_i, t^{-\varepsilon})\) whose product is reducible);
3. if \(n = 1\), then \(g_1\) is not the identity (resp. for \(n=0\));
4. any cyclic permutation of the sequence satisfies (1) – (3).

The following statement allows, by fixing a minimal system of generators in \(\pi_1(\Sigma_i, p)\), to associate to each oriented closed geodesic in \(\Sigma\) a cyclically reduced sequence. The interested reader can find this material in [8].

**Theorem.**

1. Let \(s\) be a conjugacy class of \(\pi_1(\Sigma_1, p) \ast_\beta \pi_1(\Sigma_2, p)\) (resp. \(\pi_1(\Sigma_1, p) \ast_\beta\)). Then there is a representative of \(s\) which can be written as a product:

\[
g_1g_2\ldots g_n \quad \text{(resp. } g_0t^{\varepsilon_1}g_2t^{\varepsilon_2}\ldots g_{n-1}t^{\varepsilon_n})\text{,}
\]

where \((g_1, g_2, \ldots, g_n)\) (resp. \((g_0, t^{\varepsilon_1}, g_1, t^{\varepsilon_2}, \ldots, g_{n-1}, t^{\varepsilon_n})\)) is a \(\beta\)-cyclically reduced sequence.

2. If \(n\) is a positive integer and \((g_1, g_2, \ldots, g_n)\) (resp. \((g_0, t^{\varepsilon_1}, g_1, t^{\varepsilon_2}, \ldots, g_{n-1}, t^{\varepsilon_n})\)) is \(\beta\)-cyclically reduced sequence then the product \(g_1g_2\ldots g_n\) (resp. \(g_0t^{\varepsilon_1}g_2t^{\varepsilon_2}\ldots g_{n}t^{\varepsilon_n}\)) is not the identity.

3. For any pair of \(\beta\)-cyclically reduced sequences representing the same conjugacy class in \(\pi_1(\Sigma_1, p) \ast_\beta \pi_1(\Sigma_2, p)\) (resp. \(\pi_1(\Sigma_1, p) \ast_\beta\)). Then the number of terms of the two sequences is the same.

Consider \(\beta\) any essential simple closed geodesic in \(\Sigma_{1,1}\) and the HNN extension of \(\pi_1(\Sigma_{1,1}, p)\) obtained by splitting \(\Sigma_{1,1}\) along \(\beta\):

\[
\langle a, b, t \mid tat^{-1} = b \rangle,
\]

where \(ab^{-1}\) represents the conjugacy class of a simple closed curve parallel to the boundary, \(t\) the stable element representing the class of \(\tau\) and \(b\) representing the class of \(\beta\).

**Corollary 4.4.** Let \(\gamma\) be an oriented closed geodesic in \(\Sigma_{1,1}\) intersecting \(\beta\), with its \(\beta\)-cyclically reduced sequence of the form \((g_0, t, g_1, t, \ldots, g_{n-1}, t)\) such that each \(g_i\) starts with \(b^{\pm 1}\) and ends with \(a^{\pm 1}\). Then the sequence of \(\gamma\)-arcs \((\gamma \mid I_1, \gamma \mid I_2, \ldots, \gamma \mid I_n)\) that result from splitting \(\gamma\) along \(\beta\) satisfy the following properties:

1. the \(\gamma\)-arc \(\gamma \mid I_i\) is representative of \((t, g_i, t)\);
2. the inclusion \(\gamma \mid I_i \subset \Sigma_{1,1} \setminus \beta\) holds;
3. if \(\gamma \mid I_i\) and \(\gamma \mid I_j\) are freely homotopic relative to \(\beta\), then \(g_i = g_j\).

**Proof.** By Lemma 5.2 in [8] there exist sequence of curves \((\gamma_0, \gamma_1, \ldots, \gamma_{n-1})\) such that the curve \(\gamma_i \subset \Sigma_1\) is a representative of \(g_i\).

Denote by \(\tilde{\gamma}\) the curve \(\gamma_0\tau\gamma_1\tau\ldots\gamma_{n-1}\tau\), which clearly is a representative of \(\gamma\) and consider the \(\gamma\)-arc \(\gamma \mid I_i \subset \Sigma_{1,1}\) which is homotopic relative to the boundary to the concatenation of arcs \(\tau_0, \gamma_i\) and \(\tau_1\), where \(\tau_0\) and \(\tau_1\) are the two \(\tau\)-arcs from the intersection point of \(\tau\) with \(\beta\) to \(p\).

To prove 3) it is enough to show that there is and isomorphism between \(\pi_1(\Sigma_1, p)\) and the group of free homotopy classes of arcs in \(\Sigma_{1,1} \setminus \beta\) relative to the boundary, which start in one component and end in the other. This isomorphism is given as follows, given \(\alpha\) a
loop based at \( p \) construct the arc \( \tau_0 * p * \alpha * p \tau_1 \). The inverse of the isomorphism consist on collapsing the endpoints of a given arcs to \( p \) along \( \tau_0 \) and \( \tau_1 \).

**Remark 4.5.** The \( \gamma \)-arcs \( \gamma \mid \iota_i \) of Corollary 4.4 can be encoded by using two oriented seam geodesic arcs between \( \beta \) and the boundary of \( \Sigma_{1,1} \). The coding comes from replacing each letter in \( g_i \) with an arc connecting one seam edge to another following the direction prescribed by \( g_i \) (see Figure 5).

In the case when \( \beta \) is a separating simple closed geodesic on a hyperbolic surface \( \Sigma \), by \( [8] \), Lemma 3.2 we have the following result:

**Corollary 4.6.** Let \( \gamma \) be an oriented closed geodesic in \( \Sigma \) intersecting the separating geodesic \( \beta \). Consider its \( \beta \)-cyclically reduced sequence of the form \( (g_1, g_2, \ldots, g_n) \). Then the sequence of \( \gamma \)-arcs \( \gamma \mid \iota_1, \gamma \mid \iota_2, \ldots, \gamma \mid \iota_n \) that result from splitting \( \gamma \) along \( \beta \) satisfy that if we identify the endpoint of an arc \( \gamma \mid \iota_i \) along \( \beta \), then it is representative of the conjugacy class of \( g_i \).

The following result is a criteria to verify when the unique \( \beta \)-cyclically reduced sequence associated to \( \gamma \) in Corollary 4.4 is a filling geodesic.

**Lemma 4.7.** Let \( \gamma \) be an oriented closed geodesic as in Corollary 4.4. If there exist a term \( g_i \), in the \( \beta \)-cyclically reduced sequence of \( \gamma \), with a sub word of the form \( a \pm 1b \pm 1 \) or \( b \pm 1a \pm 1 \). Then \( \gamma \) is filling.

**Proof.** It is enough to show that there exist two simple loops formed by sub arcs \( \alpha_1 \) and \( \alpha_2 \) of \( \gamma \) such that \( i(\alpha_1, \alpha_2) \neq 0 \).

Since \( \gamma \) intersects \( \beta \) and there exist \( g_i \notin \langle a \rangle \cup \langle b \rangle \). Then \( \gamma \) is not a simple closed curve in \( \Sigma_{1,1} \) and there is a simple loop formed by a sub arc \( \alpha_1 \) of \( \gamma \) which is transversal to \( \beta \) (\( [8] \), Theorem 5.3).

By the Remark 4.5, the fact that there exist \( g_i \) with a sub word of the form \( a \pm 1b \pm 1 \) or \( b \pm 1a \pm 1 \) implies that there is a \( \gamma \)-arcs \( \gamma \mid \iota_i \) corresponding to \( g_i \) (Corollary 4.4) that contains a sub arc \( \alpha_2 \) which induces loop homotopic to \( \beta \). □

A first application to the previous Lemma 4.7 and Theorem 1.5, is the following result.

**Theorem 1.4.** For the modular surface \( \Sigma_{mod} \), there exist a sequence \( \{\gamma_k\} \) of filling closed geodesics on \( \Sigma_{mod} \) with \( n_{\gamma_k} \not\to \infty \) such that,

\[
Vol(M_{\gamma_k}) \geq v_3 \frac{n_{\gamma_k}}{12},
\]
where \( n_{\gamma_k} \) is half the period of the continued fraction expansion of \( \gamma_k \) and \( v_3 \) is the volume of a regular ideal tetrahedron.

**Proof.** Consider the family of closed geodesics \( \{ \gamma_k \} \) on the modular surface \( \Sigma_{mod} \) with the following coding

\[
\omega(\gamma_k) := x^6 y^x x^{13} y^{19} x^{25} y \ldots x^{6k+1}(xy^2 x^2 y).
\]

Let's lift the family \( \{ \gamma_k \} \) to \( \{ \tilde{\gamma}_k \} \) into the once-punctured modular torus cover, whose fundamental group is generated by \( \{ t := yx, a := y^{-1}x^{-1} \} \) and has the following presentation \( \langle a, b, t \mid tat^{-1} = b \rangle \). Notice that if we split the once-punctured modular torus along the simple closed geodesic associated to \( \gamma_1 \), then we rewrite the coding of the geodesics \( \{ \gamma_k \} \), and obtain the \( a \)-cyclic reduced words:

\[
\omega_{1,1}(\tilde{\gamma}_k) := ab^{-1}t(ab^{-1})^2t(ab^{-1})^3t \ldots t(ab^{-1})^ka^{-1}b^{-1}t,
\]

whose \( a \)-cyclic reduced word in the form of Corollary 4.4 is:

\[
tb^{-1}(ab^{-1})ab^{-1}(ab^{-1})^2at \ldots tb^{-1}(ab^{-1})^{k-1}a^{-1}t.
\]

The closed geodesics \( \tilde{\gamma}_k \) are filling by Lemma 4.7 and by Corollary 4.4, the number of homotopy classes after splitting by the simple closed geodesic associated to \( a \) is \( k+1 \). So we have a lower bound of the volume of the canonical lift complement of \( \tilde{\gamma}_k \) given by \( v_3 \frac{k+1}{12} \).

**Remark.** For the filling closed geodesics \( \{ \gamma_k \} \) on Corollary 1.4 we can also rewrite the combinatorial upper bound found by Bergeron, Pinsky and Silberman ([3], Theorem 3.5) for the volume of the canonical lift complement of \( \tilde{\gamma}_k \) in terms of \( n_{\gamma_k} \), as

\[
Cn_{\gamma_k} \ln(n_{\gamma_k}),
\]

where \( n_{\gamma_k} = k+1 \) and \( C \) a positive constant that depends on \( \Sigma_{mod} \).

**4.2. Lower bound of the volume of \( M_{\gamma} \) in terms of the length of \( \gamma \).** Given \( P \) a pair of pants in a pants decomposition of a hyperbolic surface, we construct all possible non homotopic geodesic arcs on \( P \) whose length is bounded by some constant and then induce a filling geodesic which contains all this arcs. In this case, we have rewritten the lower bound for the volume of the canonical lift complement given by Theorem 1.3 in terms of the length of the geodesic.

**Theorem 4.8.** Given a hyperbolic metric \( X \) on the surface \( \Sigma_{1,1} \), there is a sequence \( \{ \gamma_n \} \) of filling closed geodesics on \( \Sigma \) with \( \ell_X(\gamma_n) \nearrow \infty \) such that,

\[
Vol(M_{\gamma_n}) \geq k_X \frac{\ell_X(\gamma_n)}{\ln(\ell_X(\gamma_n))},
\]

where \( k_X \) depends on the hyperbolic metric \( X \).

**Proof.** For the sake of concreteness, we will start proving the result for a particular hyperbolic metric on \( \Sigma_{1,1} \). Fix the following representation \( \rho: \pi_1(\Sigma_{1,1}) := \langle a, t \rangle \to PSL_2(\mathbb{C}) \) such that

\[
\rho(a) = A = \begin{pmatrix} \sqrt{2} & 1 + \sqrt{2} \\ -1 + \sqrt{2} & \sqrt{2} \end{pmatrix} \quad \text{and} \quad \rho(t) = T = \begin{pmatrix} -1 + \sqrt{2} & 0 \\ 0 & 1 + \sqrt{2} \end{pmatrix}.
\]

Consider the splitting of \( \Sigma_{1,1} \) along a simple close geodesic associated to \( A \), then we have the HNN extension of \( \pi_1(\Sigma_{1,1}, p) = \langle a, b, t \mid tat^{-1} = b \rangle \), with:

\[
\rho(b) = TAT^{-1} = \begin{pmatrix} \sqrt{2} & -1 + \sqrt{2} \\ 1 + \sqrt{2} & \sqrt{2} \end{pmatrix}.
\]
Notice that $ab^{-1}$ is freely homotopic to a simple loop that is retractable to the boundary of $\Sigma_{1,1}$.

For each $n \in \mathbb{N}$ we define the following $A-$cyclically reduced sequence:

$$(g_1,t,g_2,t, \ldots, t, g_{12}, (3^n-1), t)$$

where $g_i \in \langle a, b \rangle$.

Moreover $\{g_i\}_{i=1}^{12(3^n-1)}$ is the set of different reduced words in $\langle a, b \rangle$ starting with $b$ or $b^{-1}$, and ending with $a$ or $a^{-1}$, which have word length at most $n$ and at least $4$.

By Corollary 4.4, the number of homotopy classes of arcs in each $n^{th}$-sequence is equal to $\sharp \{g_i\}_{i=1}^{12(3^n-1)} = 12 \cdot (3^n - 1)$. Let $\gamma_n$ be the unique geodesic associated to the product of the $n^{th}$-sequence. Each one of it belongs to different mapping class group orbits, because in this case the self-intersection number of $\gamma_n$ is bounded from below by the number of terms of the sequence, which grows exponentially. Therefore, by Lemma 4.7, $\{\gamma_n\}$ is a sequence of filling closed geodesics on $\Sigma_{1,1}$ with $\ell_{X_0}(\gamma_n) \not\to \infty$.

By Theorem 1.5, we have that:

$$v_3 \cdot 6 \cdot (3^n - 1) = \frac{v_3}{2} \sharp \{\text{homotopy classes of } \gamma_n\text{-arcs} \} \leq Vol(M_{\gamma_n}).$$

The last part of the proof consist in rewriting the combinatorial lower bound in terms of the length of $\gamma_n$. To do so, for each $\gamma_n$, let $L_n$ be such that:

$$n = \frac{\ln(\ln(3)L_n) - \ln(\ln(3)L_n)) - \ln(3)}{\ln(3)}.$$

Using the trace of the matrices we can calculate the length of the geodesics associated to $ab$ and $t$. In this way we give the following upper bound of the length of $\gamma_n$ by using the word length with respect the generating set $\{a, b, t\}$:

$$\ell_{X_0}(\gamma_n) \leq 1.15 \cdot |g_1tg_2t \ldots t g_{12}(3^n-1)t| \leq 1.15 \cdot 12 \cdot 3^n(n + 3) \leq 14 \cdot L_n.$$

Then,

$$6 \cdot (3^n - 1) \geq \frac{\ln(3)L_n}{\ln(\ln(3)L_n)}.$$

Finally, by inequality (4.1) we have that:

$$v_3 \frac{\ell_{X_0}(\gamma_n)}{\ln(\ell_{X_0}(\gamma_n))} \leq Vol(M_{\gamma_n}).$$

The proof of this result for any hyperbolic metric, follows from the fact that any pair of hyperbolic metrics on a hyperbolic surface are bi-Lipschitz ([3], Lemma 4.1).

To generalize Theorem 4.8 to any hyperbolic surface $\Sigma$, consider the following method for extending a given closed geodesic that fills a subsurface of $\Sigma$ to a filling closed geodesic on $\Sigma$. Let $\beta$ be a separating oriented simple closed geodesic on $\Sigma$ and denote $\Sigma \setminus \beta$ by $\Sigma_1 \coprod \Sigma_2$.

Given two closed geodesics $\alpha_1$ and $\alpha_2$ on $\Sigma_1$ and $\Sigma_2$ respectively, choose a minimal length common perpendicular segment $\eta$ between $\alpha_1$ and $\alpha_2$, with extremal points $p_1$ and $p_2$. Let $\alpha_1 \ast \alpha_2$ be the closed geodesic in the homotopy class of the piecewise closed geodesic $\tilde{\alpha}_{1,2}$ that travels from $p_1$ along $\alpha_1$ on the direction given by $\beta$, then follows $\eta$, continues from $p_2$ along $\alpha_2$ on the same direction given by $\beta$ and finishes by $\eta^{-1}$.

As $\tilde{\alpha}_{1,2}$ is not self-transverse, we will slightly modify it to obtain one that does.
Figure 6. From $\bar{\alpha}_{1,2}$ to $\alpha_1 \star \alpha_2$.

Take a $\delta$ neighborhood of $\eta$ such that it does not contains self-intersection of $\alpha_1$ and $\alpha_2$. Erase the segment of $\bar{\alpha}_{1,2}$ that is contained the $\delta$ neighborhood of the $\eta$ geodesic segment and link the consecutive extremal points in the same $\delta$ neighborhood by a pair of intersecting minimal length geodesic segments linking the remaining segments of $\bar{\alpha}_{1,2}$. This piecewise geodesic, denoted by $\alpha_1 \star \alpha_2$, has not self-tangency points.

Figure 7. a) A lift of $\bar{\alpha}_{1,2}$ in $\mathbb{H}^2$. b) The modification from $\tilde{\bar{\alpha}}_{1,2}$ to $\tilde{\alpha}_1 \star \tilde{\alpha}_2$ by erasing the segments $\tilde{\eta}$ and re-gluing. c) Possible bigons given by the intersection of two lifts of $\alpha_1 \star \alpha_2$.

Claim 4.9. $\alpha_1 \star \alpha_2$ has minimal self-intersection.

Proof. By Theorem 2.1, we just need to show that $\alpha_1 \star \alpha_2$ has no 1–gons nor 2–gons. Indeed, take the lifts of $\alpha_1 \star \alpha_2$ in the universal cover of $\Sigma$, which are piecewise geodesic lines. It is enough to show that that every pair of them intersects at most once. Certainly, if there were two consecutive intersections they cannot happen in segments of the same geodesic segment, so in-between there must be at least one gluing point of two distinct geodesic segments. Moreover, from the fact that the angle on the gluing points is obtuse, there must be at least two gluing points in each 2–gon side. If that is the case, we would have geodesic side hexagons whose sum of interior angles is bigger or equal to $4\pi$, which is a contradiction in the hyperbolic plane.

By rounding the four corners of $\alpha_1 \star \alpha_2$, we get a self-transverse closed curve homotopically transversal to $\alpha_1 \star \alpha_2$, in particular this means that $M_{\alpha_1 \star \alpha_2} \cong M_{\alpha_1 \star \alpha_2}$. 

Proof. Notice that \( \Sigma \setminus (\alpha_1 \cup \alpha_2) \) differs from \( \Sigma \setminus (\alpha_1 \ast \alpha_2) \) only in the regions whose sides contains \( p_1 \) and \( p_2 \). The later splits the ring around \( \beta \) in the original partition and deforms the other two adjacent regions, preserving their homeomorphism type. Moreover, as the property of being filling is preserved by transversal homotopy (Corollary \ref{cor:transversal_homotopy}), then \( \alpha_1 \ast \alpha_2 \) is also filling. \( \square \)

**Theorem 1.3.** Given a hyperbolic metric \( X \) on a surface \( \Sigma \), there exist a sequence \( \{\gamma_n\} \) of filling closed geodesics on \( \Sigma \) with \( \ell_X(\gamma_n) \not\to \infty \) such that,

\[
\text{Vol}(M_{\gamma_n}) \geq c_X \frac{\ell_X(\gamma_n)}{\ln(\ell_X(\gamma_n))},
\]

where \( c_X \) depends on the hyperbolic metric \( X \).

**Proof.** If the surface \( \Sigma \) contains a once-punctured torus, choose a separating closed geodesic \( \beta \) such that \( \Sigma \setminus \beta := \Sigma_1 \coprod \Sigma_2 \), where \( \Sigma_1 \) is homeomorphic to \( \Sigma_{1,1} \). Choose a pair of simple closed geodesics \( \alpha \) and \( \tau \) on \( \Sigma_1 \) such that \( i(\alpha, \tau) = 1 \). Consider the closed geodesics \( \alpha_n \) on \( \Sigma_1 \) homotopic to \( \alpha \) under the group \( \pi_1(\Sigma) \) associated to:

\[
\alpha_n \ast \eta_0.
\]

By Theorem \ref{thm:geodesic_bound} we have that:

\[
(4.2) \quad v_3 \cdot 6 \cdot (3^n - 1) = v_3 \sum_{i=1}^{\chi(\Sigma)} 2 \{\text{homotopy classes of } \gamma\text{-arcs in } P_i\} \leq \text{Vol}(M_{\gamma_n}).
\]

The last part of the proof consist in rewriting the combinatorial lower bound in terms of the length of \( \gamma_n \). By using the word length with respect the generating set \( \{\alpha, \tau\} \) we can find constants \( c_X \) and \( \varepsilon_X \) that only depends on the metric such that:

\[
c_X \frac{\ell_X(\gamma_n)}{\ln(\ell_X(\gamma_n))} \leq k_X \frac{\ell_X(\gamma_n) - \varepsilon_X}{\ln(\ell_X(\gamma_n))} \leq k_X \frac{\ell_X(\alpha_n)}{\ln(\ell_X(\alpha_n))} \leq v_3 \cdot 6 \cdot (3^n - 1).
\]

Finally, by the inequality (4.2) we have that:

\[
c_X \frac{\ell_X(\gamma_n)}{\ln(\ell_X(\gamma_n))} \leq \text{Vol}(M_{\gamma_n}).
\]

In the case where \( \Sigma \) is a \( n \)-punctured sphere, we consider \( \pi : \Sigma' \to \Sigma \) a finite cover where \( \Sigma' \) contains a once-punctured torus, and the induced covering \( \tilde{\pi} \) from \( PT^1\Sigma' \) to \( PT^1\Sigma \). Consider the previous geodesics \( \{\gamma_n\} \) on \( \Sigma' \) and \( \{\pi(\gamma_n)\} \) in \( \Sigma \). Then the volume of the complement of \( \pi(\gamma_n) \) on \( PT^1\Sigma \) is a constant (given by the degree of the covering) times the volume of the complement of the canonical lifts of all the translates of \( \gamma_n \) under the group of deck transformations on \( PT^1\Sigma' \), which is grater than the volume corresponding to the complement of \( \gamma_n \) on \( PT^1\Sigma' \). So the filling closed geodesics \( \{\pi(\gamma_n)\} \) in \( \Sigma \) is the sequence wanted. \( \square \)
5. Further comments

As noticed by Foulon and Hasselblatt in [13], Theorem 1.12) any continuous lift complement of a filling closed geodesic $γ$ is a hyperbolic 3-manifold of finite volume. This volume is bounded from below by the isotopy classes of $γ$-arcs in the unit tangent bundle over a given pants decomposition of the surface (Corollary 4.2). This gives rise to new questions, for example, how the volume complement varies from the canonical lift to other lifts in $PT^1Σ$ over the same closed geodesic.

We start by showing that given a hyperbolic metric on hyperbolic surface we can construct a sequence of closed geodesics and their respective non-canonical lifts, whose volume complement has a lower bound which is linear in the length of the geodesic.

**Proposition 5.1.** For any hyperbolic metric $X$ on $Σ$, there exist a sequence of $\{γ_n\}$ filling closed geodesics and respective lifts $\{\hat{γ}_n\}$ in $PT^1Σ$ with $ℓ_X(γ_n) \searrow \infty$, such that,

$$k_Xℓ_X(γ_n) \leq Vol(M_{\hat{γ}_n}),$$

where $k_X$ is a positive constant that depends on the metric $X$. Moreover, there exists a constant $V_0 > 0$ such that $Vol(M_{\hat{γ}_n}) < V_0$ for every $n ∈ \mathbb{N}$.

By using techniques found in [9], maybe this result could be improved by constructing a sequence of non-canonical lifts, whose volume complement has a lower bound which is quadratic in the length of the geodesic. Motivated by this, we conjecture that the volume of the canonical lift complement minimizes the volume of the complement over all the possible lifts of a fixed filling closed geodesic on the surface.

**Conjecture 5.2.** Given $γ$ a filling closed geodesic on a hyperbolic surface, then

$$Vol(M_γ) = \min_{\hat{γ}}\{Vol(M_\hat{γ})\}.$$

**Proof of Proposition 5.1.** We will start proving the statement for the case $Σ = Σ_{1.2}$. We start by splitting $Σ_{1.2}$ along a simple closed separating geodesic into pieces homeomorphic to $Σ_{1.1}$ and $Σ_{0.3}$. Choose a base point $p$ and we write $π_1(Σ_{1.2}, p)$ as the amalgamated product along the previous separating simple geodesic, then $π_1(Σ_{1.1}, p) * \mathbb{Z} π_1(Σ_{0.3}, p) = \langle a_1, t_1 \rangle *_{\mathbb{Z}} \langle a_2, b_2 \rangle$.

The first step consists on constructing the family of closed filling geodesics. For each $n ∈ \mathbb{N}$ we define the following cyclically reduced sequence:

$$(t_1^n a_1 t_1 a_1 t_1^{-1}, b_2, a_1, b_2, ..., a_1, b_2)$$

which has length $2[\ln(n)]$, let $γ_n$ be the unique geodesic associated to the product of this sequence.

By Corollary 4.6 consider the sequence of subintervals $I_t, \{I_{a_i}\}_{i=1}^{[\ln(n)]−1}$ and $\{I_{b_i}\}_{i=1}^{[\ln(n)]}$ of $[0, ℓ_X(γ_n)]$, such that $γ_n |_{I_t}$, $γ_n |_{I_{a_i}}$, and $γ_n |_{I_{b_i}}$ are respectively homotopic to $t_1^n a_1 t_1 a_1 t_1^{-1}$, $a_1$ and $b_2$ relative to the boundary in $Σ_{1.1}$. Since the closed geodesics associated to $a_1$ and $t_1$ intersect each other, this implies that each $γ_n |_{I_{a_i}}$ intersects $γ_n |_{I_{b_j}}$ $n + 1$ times. Notice that $γ_n$ is filling because by Lemma 4.7 $γ_n |_{I_t}$ is filling in $Σ_{1.1}$ and the piece corresponding to $Σ_{0.3}$ is split by $γ_n |_{I_{b_j}}$. We now construct the non-canonical lifts $\hat{γ}_n$ by modifying the canonical lift $γ_n$ in the pre-image of $γ_n |_{I_t}$ under the map $PT^1Σ → Σ$. We start by splitting the sub arc $γ_n |_{I_t}$ by the closed simple geodesic associated to $a_1$ in $\{γ_n |_{I_{t_j}}\}_{j=0}^n$ sub arcs. As there is an injection between the number of sub arcs $\{γ_n |_{I_{t_j}}\}_{j=0}^n$ and the set $\{0, 1\}^{[\ln(n)]}$, then to each sub arc we can associate a unique sequence.
Let us take the pre-image of \( \gamma_n \mid_{I_{t_j}} \) in \( PT^1 \Sigma_{1,2} \) under the projection map \( PT^1 \Sigma \to \Sigma \). This is an annulus that contains \( \gamma_n \mid_{I_{t_j}} \) and \( \ln(n) \) punctures corresponding to the intersection of \( \gamma_n \mid_{I_{t_i}} \) with \( \hat{\gamma}_n \mid_{I_{t_j}} \). If the sequence associated to \( \gamma_n \mid_{I_{t_j}} \) has 0 in the \( k \)-coordinate we do not modify the lift \( \hat{\gamma}_n \mid_{I_{t_j}} \) in the fiber corresponding to the intersection with the \( \gamma_n \mid_{I_{a_k}} \). If it has 1 in the \( k \)-coordinate then we modify \( \hat{\gamma}_n \mid_{I_{t_j}} \) in such way that the new lift \( \tilde{\gamma}_n \mid_{I_{t_j}} \) goes around the puncture generated by \( \hat{\gamma}_n \mid_{I_{a_k}} \).

![Figure 8](image)

Figure 8. a) The lift \( \hat{\gamma}_n \mid_{I_{t_j}} \) in the pre-image of \( \gamma_n \mid_{I_{t_j}} \), and the \( \gamma_n \mid_{I_{a_k}} \)’s punctures. b) The new lift \( \tilde{\gamma}_n \mid_{I_{t_j}} \), in the pre-image of \( \gamma_n \mid_{I_{t_j}} \), associated to the sequence \( (0, 1, 0, 1) \).

**Claim 5.3.** Any pair of distinct sub arcs \( \gamma_n \mid_{I_{t_j}} \) and \( \gamma_n \mid_{I_{t_i}} \) are not ambient isotopic in 

\[
(X, \partial X) := (T^1 P \setminus \{ \gamma_n \mid_{I_{a_k}} \}_{k=1}^{\ln(n)-1}, \partial(T^1 P \setminus \{ \gamma_n \mid_{I_{a_k}} \}_{k=1}^{\ln(n)-1})),
\]

where \( P \) is the pair of pants after splitting the piece \( \Sigma_{1,1} \) by the simple closed geodesic associated to \( a_1 \).

**Proof.** Take \( \gamma_n \mid_{I_{t_i}} \) and \( \gamma_n \mid_{I_{t_j}} \) with \( i \neq j \), and their respective sequence in \( \{0, 1\}^{\ln(n)} \), then there exist the first coordinate \( k \) where the two sequence do not coincide. Consider an embedded surface \( S_k \) in \( T^1 P \setminus \{ \gamma_n \mid_{I_{a_k}} \}_{k=1}^{\ln(n)} \) bounding the arcs \( \gamma_n \mid_{I_{a_k}} \) and \( \hat{\gamma}_n \mid_{I_{a_{k+1}}} \).

By the cup product in \( H^*(X; \mathbb{Z}) \) and Lefschetz duality we have the following paring:

\[
H_2(X, \partial X; \mathbb{Z}) \times H_1(X, \partial X; \mathbb{Z}) \to H_0(X, \partial X; \mathbb{Z}) \cong \mathbb{Z}
\]

such that \( ([S_k], [\gamma_n \mid_{I_{t_i}}]) \neq ([S_k], [\gamma_n \mid_{I_{t_j}}]) \).

Then \( [\gamma_n \mid_{I_{t_i}}] \neq [\gamma_n \mid_{I_{t_j}}] \), meaning that \( \gamma_n \mid_{I_{t_i}} \) is not ambient isotopic to \( \gamma_n \mid_{I_{t_j}} \) in \( (X, \partial X) \).

The previous fact implies that the isotopy classes of \( \hat{\gamma} \)-arcs in \( T^1 P \) corresponding to \( \{\gamma_n \mid_{I_{t_j}}\}_{j=1}^{n} \) are different. So by Corollary 4.2 we conclude that:

\[
\frac{v_3}{2} n < Vol(M_{\hat{\gamma}_n}).
\]

Moreover, by Theorem 4.1 if we take the canonical lifts of the sequence \( \{\gamma_n\} \), then there exists a constant \( V_0 > 0 \) such that \( Vol(M_{\gamma_n}) < V_0 \) for every \( n \in \mathbb{N} \).
To finish the case $\Sigma = \Sigma_{1,2}$, it is enough to estimate the length of $\gamma_n$ by comparing it to its word length,

$$\ell_X(\gamma_n) \leq (4 + n + 2 \ln(n))\ell_{\text{max}} \leq 5n\ell_{\text{max}},$$

where $\ell_{\text{max}}$ is the maximal length of three simple closed curves representing $a_1$, $t_1$ and $b_2$ with the same base point.

We have proved the Proposition 5.1 for the case $\Sigma = \Sigma_{1,2}$. Suppose now that $\Sigma$ contains a subsurface of topological type as $\Sigma_{1,2}$. Proceeding as in Theorem 1.3 we get the wanted sequence of filling closed geodesics on $\Sigma$, and their respective canonical lifts are constructed in the same way as in the previous case. Furthermore, if $\Sigma$ does not contain a subsurface of topological type as $\Sigma_{1,2}$, we consider $\pi : \Sigma' \to \Sigma$ a finite cover where $\Sigma'$ contains a subsurface $\Sigma_{1,2}$, and the induced covering $\tilde{\pi}$ from $PT^1\Sigma'$ to $PT^1\Sigma$. Consider the previous geodesics $\{\gamma_n\}$ on $\Sigma'$ and $\{\pi(\gamma_n)\}$ in $\Sigma$. Then the volume of the complement of $\tilde{\pi}(\tilde{\gamma}_n)$ on $PT^1\Sigma$ is a constant (given by the degree of the covering) times the volume of the complement of the non-canonical lifts of all the translates of $\gamma_n$ under the group of deck transformations on $PT^1\Sigma'$, which is greater than the volume corresponding to the complement of $\tilde{\gamma}_n$ on $PT^1\Sigma'$. So the filling closed geodesics $\{\pi(\gamma_n)\}$ in $\Sigma$ and non-canonical lifts $\tilde{\pi}(\tilde{\gamma}_n)$ on $PT^1\Sigma$ are the sequences wanted.

One of the question that arrises from the fact that the topology of $M_{\tilde{\gamma}_n}$ does not depend on the metric given on $\Sigma$ is to find a metric for each element in the sequence $\{\gamma_n\}$ of Proposition 5.1 (see also Theorem 1.3) which minimizes its length $\ell(\gamma_n)$. This implies that the volume lower bounds (Corollary 4.2) in terms of the length with respect this particular metric will be bigger. For example, in the following result, we choose for each $\gamma_n$ on Proposition 5.1 a different hyperbolic metric on $\Sigma_{1,2}$ which can be interpreted as an approximation of the minimal metric associated to each $\gamma_n$, and rewrite the lower bound in terms of this new metrics.

**Proposition 5.4.** There exist $X_n$ hyperbolic metrics on $\Sigma_{1,2}$ corresponding to the closed geodesics $\gamma_n$ of Proposition 5.1 with $\ell_X(\gamma_n) \uparrow \infty$, such that:

$$\frac{v_3}{2\sqrt{2}} e^{\frac{\ell_X(\gamma_n)}{2}} \leq Vol(M_{\tilde{\gamma}_n}),$$

where $v_3$ is the volume of a regular ideal tetrahedra and $\tilde{\gamma}_n$ are the sequence of lifts in Proposition 3.1.

**Proof.** For each $n \in \mathbb{N}$ we construct a hyperbolic metric $X_n$ on $\Sigma_{1,2}$ by using the pants decomposition $\Pi$ of Proposition 5.1. Let us fix the length of the boundary components of the pair of pants corresponding to the piece $\Sigma_{1,1}$ by $4\ln(\sqrt{2}n)$, this implies that the length of the seam arcs joining the boundary components is approximately $\frac{1}{n}$. Also fix the length of the boundary components of $\Sigma_{1,2}$ by a constant.

The conclusion follows from the following inequality, which results from the upper estimation of the geodesic length with respect the hexagonal decomposition induced by $\Pi$ and the seam edges:

$$\ell_{X_n}(\gamma_n) \leq 4\ln(\sqrt{2}n)^2.$$
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