Nonuniversal finite-size scaling in anisotropic systems

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(Dated: October 25, 2004)

We study the bulk and finite-size critical behavior of the $O(n)$ symmetric $\varphi^4$ theory with spatially anisotropic interactions of non-cubic symmetry in $d < 4$ dimensions. In such systems of a given $(d,n)$ universality class, two-scale factor universality is absent in bulk correlation functions, and finite-size scaling functions including the Privman-Fisher scaling form of the free energy, the Binder cumulant ratio and the Casimir amplitude are shown to be nonuniversal. In particular it is shown that, for anisotropic confined systems, isotropy cannot be restored by an anisotropic scale transformation.

PACS numbers: 05.70.Jk, 64.60.-i, 75.40.-s

A basic tenet in the physics of critical phenomena is the notion of a universality class. It is characterized by the dimensionality $d$ of the system and by the number $n$ of the components of the order parameter. (See, e.g., the review article [1].) Within a certain $(d,n)$ universality class, the universal quantities (critical exponents, amplitude ratios and scaling functions) are independent of microscopic details, such as the particular type of (finite-range or van der Waals type) interactions or the lattice structure [2]. This implies that a given universality class includes both spatially isotropic and anisotropic systems.

Once the universal quantities of a universality class are known the asymptotic critical behavior of very different systems (e.g., fluids and magnets) is believed to be known completely provided that only two nonuniversal amplitudes $A_1$ and $A_2$ are given. This property is known as two-scale factor universality or hyperuniversality [3,4]. In terms of the singular part of the reduced bulk free energy density $F_s/V k_B T \equiv f_s(t,h)$ above $T_c$,

$$f_s(t,h) = A_1 t^{d-4} W (A_2 t^{-\beta})$$

with $W(0) = 1$ and $t = (T - T_c)/T_c \ll 1$, this property can be stated as [3,4]

$$\lim_{t \to 0^+} f_s(t,0) \xi^d = Q(d,n) = \text{universal} .$$

Thus the amplitude $\xi_0 = (Q/A_1)^{1/d}$ of the correlation length $\xi = \xi_0 t^{-\nu}$ at zero ordering field $h$ is not an independent amplitude but is universally related to $A_1$. The validity of two-scale factor universality has been established by the renormalization-group (RG) theory on the basis of an isotropic Hamiltonian with short-range interactions below the upper critical dimension $d^* = 4$ [3] but no general proof has been given for the anisotropic case.

In this paper we study the critical behavior of systems with a spatial anisotropy of non-cubic symmetry within a given $(d,n)$ universality class. An example is an Ising ferromagnet with an isotropic nearest-neighbor (NN) coupling $J > 0$ and an anisotropic next-nearest-neighbor (ANNN) coupling $J'$ on a simple-cubic lattice. In some range of $J'/J$ this model has the same type of critical behavior as the ordinary ($J' = 0$) Ising model. We shall show that for such systems Eq. (2) must be generalized to

$$\lim_{t \to 0^+} f_s(t,0) \prod_{i=1}^{d} \xi_{(i)} = Q(d,n) = \text{universal}$$

where $\xi_{(i)} = \xi_{(i)}^{(0)} t^{-\nu}$ are the correlation lengths associated with the principal directions of the anisotropic system and where $Q(d,n)$ is the same universal quantity for both isotropic and anisotropic systems. (For $d = 2, n = 1$, this is already known for the Ising model with anisotropic NN (ANN) interactions $J_x \neq J_y$.) There are, in general, $d+1$ nonuniversal bulk amplitudes $\xi_{(0)}^{(1)}, \ldots, \xi_{(d)}^{(d)}, A_2$ whose ratios are also nonuniversal. Note that there still exists a unique critical exponent $\nu(d,n)$ that is identical for isotropic and anisotropic systems within the same $(d,n)$ universality class [4,7,8,9,10].

A different type of critical behavior exists in the so-called strongly anisotropic systems [11,12,13,14] where not only amplitudes depend on the spatial directions but also the critical exponents (e.g., $\nu_1$ and $\nu_{1\perp}$) depend on the direction. These systems do not belong to the $(d,n)$ universality class of ordinary critical points and our analysis will not include such types of anisotropy.

While Eq. (3) is a natural generalization of Eq. (2) we shall call attention to the intriguing problem of finite-size effects in anisotropic systems. For simplicity we shall confine ourselves to the case of periodic boundary conditions in rectangular $L_1 \times L_2 \times \ldots \times L_d$ block geometries (including $L_d$ cubic geometry and $\infty \times \infty \times L$ film geometry). There have been several studies of this problem in the past [13,14,16,17,18]. We shall only briefly comment on the more complicated case of anisotropic confined systems with non-periodic boundary conditions [17,18,20,21,22,23].

It has been hypothesized [5] that two-scale factor universality holds not only for bulk systems but also for confined systems, except that the finite-size scaling functions depend on the geometry and on the boundary conditions. For example, for a system in a cube of volume $L^d$ with
periodic boundary conditions, the singular part of the reduced free energy density \( f_{s}(t, h, L) \) near bulk \( T_{c} \) was predicted to have the asymptotic scaling form for large \( L \)

\[
f_{s}(t, h, L) = L^{-d}Y_{cube}(C_{1}tL^{1/\nu}, C_{2}hL^{\delta/\nu})
\]

where the function \( Y_{cube}(x, y) \) is universal and where \( C_{1} \) and \( C_{2} \) are the only nonuniversal parameters. A similar ansatz was made for the correlation length \( \xi(t) \) in a \( L^{d-1} \times \infty \) cylinder \([9]\). As a consequence, the amplitude \( Y_{cube}(0, 0) \) and the Binder cumulant ratio \([1, 24, 27]\)

\[
U = \frac{1}{3} \left[ \left( \frac{\partial^{4} Y_{cube}/\partial y^{4}}{\partial^{2} Y_{cube}/\partial y^{2}} \right)^{2} \right]_{y=0, x=0}
\]

are predicted to be universal. (For example, they should be independent of the ratio \( J'/J \).) The scaling form \([4]\), if extended to realistic geometries and boundary conditions, has far-reaching consequences for measurable quantities \([1, 24, 27]\). In particular the prediction of a universal character of the Casimir amplitude

\[
\Delta = (d-1)Y_{film}(0, 0)
\]

is of interest, e.g., for fluid \([27]\), superfluid \([28]\), and superconducting \([29]\) films.

The universality of the scaling functions \( Y \) of Eqs. \([11\) or \([13]\) was supposed to be valid for all systems in a given universality class \([1, 3]\) including anisotropic lattice systems provided that an appropriate rescaling of the lattice spacings (or length \( L \)) is performed \([3]\). This appears to be consistent with existing studies of finite-size effects in anisotropic systems where it was stated that isotropy can be restored asymptotically by an anisotropic scale transformation \([11, 13, 16, 18, 21, 23, 30]\).

We have found that this picture of finite-size effects in anisotropic systems, though valid in special cases, is, in general, not correct. In the present paper we show, for periodic boundary conditions, that Eqs. \([11\) - \([13]\), though valid for isotropic systems and for systems with cubic symmetry in the range \( L/\xi \lesssim O(1)\) \([31]\), are not universally valid for the anisotropic systems of the type described above (e.g. spin models with NN and ANN interactions on simple-cubic lattices) although these systems belong to the same universality class as isotropic systems. In such anisotropic systems the finite-size scaling functions depend, in general, on additional nonuniversal parameters (apart from \( C_{1} \) and \( C_{2} \)), even after a rescaling of the lattice spacing or of the length \( L \). Thus, in general, two-scale factor universality and isotropy cannot be restored and the notion of a universality class is only of restricted relevance for the scaling functions of confined systems.

We shall prove our claims within the \( O(n) \) symmetric \( \varphi^{4} \) field theory with the spatially anisotropic Hamiltonian

\[
(\text{at } h = 0)
\]

\[
H(r_{0}, u_{0}; \Lambda; A; V; \varphi) = \int_{V'} d^{d}x' \left[ \frac{r_{0}}{2} \varphi^{2} + \frac{d}{\Lambda} \frac{\partial}{\partial x_{\alpha}} \frac{\partial}{\partial x_{\beta}} \varphi + u_{0}(\varphi^{2})^{2} \right]
\]

for the \( n \)-component field \( \varphi(x) \). The sum runs over the components \( x_{\alpha} \) of the spatial coordinates \( x, \alpha = 1, \ldots, d \). The \( d \times d \) anisotropy matrix \( \mathbf{A} \equiv (A_{\alpha\beta}) \) is assumed to be real, symmetric and positive definite. This model has a critical point at some value \( r_{0} = r_{0}(A; u_{0}, \Lambda) \) where \( \Lambda \) is a (sharp or smooth) cutoff in \( k \) space. In addition to the three parameters \( r_{0}, u_{0} \) and \( \Lambda \) of the standard isotropic \( (\mathbf{A} = 1) \) model, our model has \( d(d + 1)/2 \) nonuniversal parameters contained in the matrix \( \mathbf{A} \). Below we shall argue that the non-diagonality of the anisotropy matrix \( \mathbf{A} \) is a generic case of real anisotropic systems. For simplicity we assume a cubic volume, \( V = L^{d}, 0 \leq x_{\alpha} \leq L \), with periodic boundary conditions.

First we prove that the model defined by Eq. \([7]\) belongs to the same bulk universality class as the standard isotropic Landau-Ginzburg-Wilson model with \( \mathbf{A} = 1 \).

The characteristic properties of the matrix \( \mathbf{A} \) are described in terms of the \( d \) eigenvalues \( \lambda_{i} > 0 \) and eigenvectors \( \mathbf{e}_{i} \) defined by \( \mathbf{Ae}_{i} = \lambda_{i} \mathbf{e}_{i} \). A rotation by the orthogonal matrix \( \mathbf{U} \) yields the diagonal matrix \( \mathbf{UAU}^{-1} = \mathbf{\Lambda} \) with diagonal elements \( \lambda_{i} \). After the transformation of the spatial coordinates

\[
x' = \lambda^{-1/2} \mathbf{U} x
\]

and of the field

\[
\varphi'(x') = (\text{det } \mathbf{A})^{1/4} \varphi(\mathbf{U}^{-1/2} \mathbf{\Lambda}^{1/2} x),
\]

\[
\text{det } \mathbf{A} = \prod_{i=1}^{d} \lambda_{i} > 0,
\]

the Hamiltonian \([7]\) becomes

\[
H(r_{0}, u_{0}; \Lambda; A; V; \varphi) = H'(r_{0}, u_{0}', A'; V'; \varphi') = \int_{V'} d^{d}x' \left[ \frac{r_{0}'}{2} \varphi'(x')^{2} + \frac{1}{2} \left( \nabla' \varphi' \right)^{2} + u_{0}'(\varphi'^{2})^{2} \right]
\]

with the unchanged four-point coupling

\[
u_{0}' = (\text{det } \mathbf{A})^{-1/2} u_{0},
\]

with the changed (non-cubic) volume

\[
V' = \prod_{i=1}^{d} L_{i}' = (\text{det } \mathbf{A})^{-1/2} V,
\]
\[ L_i' = L \lambda_i^{-1/2}, \]  
(15)

with a transformed cutoff \( \Lambda' \) in \( \mathbf{k}' \) space, \( \mathbf{k}' = \lambda^{1/2} \mathbf{U} \mathbf{k} \), and with a critical point at

\[ r_{oc}'(u_0', \Lambda') = r_{oc}(A; u_0, \Lambda). \]  
(16)

The temperature variable \( r_0 - r_{oc} = r_0 - r_{oc}' = a_d \) remains invariant under the transformation (5) and (9).

According to Eq. (12), the bulk critical behavior of the anisotropic model \( \mathcal{H} \) with the coupling \( u_0 \), Eq. (7), can be calculated within the minimally renormalized isotropic bulk theory (\( \mathcal{V}' \to \infty, \lambda' \to \infty \)) for \( \mathcal{H}' \) with the coupling \( u_0' \) in \( 2 < d < 4 \) dimensions (8), provided that \( u_0' > 0 \). Specifically, the renormalized quantities of the Hamiltonian \( \mathcal{H}' \) are defined as

\[ u' = \mu^{-\delta} Z_{\varphi'}^{-1/2} Z_{\varphi'} A_d u_0' \]  
(17)

\[ \varphi'_R = Z_{\varphi'}^{-1/2} \varphi' \]  
(18)

\[ r = Z_{\lambda}^{-1} (r_0 - r_{oc}') \]  
(19)

with \( A_d = \Gamma(3 - d/2)^2 / \pi^{d/2} (d - 2)^{-1} \) and

\[ r_{oc}' = (u_0')^{2/3} S(\varepsilon), \]  
(20)

\( \varepsilon = 4 - d \), where \( S(\varepsilon) \) and the \( Z \) factors \( Z_{\varphi}(u', \varepsilon) \) depend on \( \varepsilon \) and \( u' \) in the same way as they depend on \( \varepsilon \) and \( u \) in the standard (\( A = 1, V \to \infty, \lambda \to \infty \)) theory (8), with an identical fixed point value \( u^* = u^* \). This statement applies also to the field-theoretic functions \( \zeta_r(u') \) and \( \zeta_{\varphi'}(u') \) which determine the critical exponents \( \nu \) and \( \eta \). This proves that the critical behavior of \( \mathcal{H} \) and \( \mathcal{H}' \) belongs to the same universality class in the whole range of \( A \) where \( \det A > 0 \).

Our model, Eq. (7) with \( A \neq c_0 \mathbf{1} \), can be considered as the continuum version of a \( \varphi^4 \) lattice Hamiltonian \( \mathcal{H}_{\text{lattice}} \) with short-range interactions \( J_{ij} \) (see, e.g., Eq. (50) below) for a lattice model with a single lattice constant \( \tilde{a} \). Non-cubic anisotropies may arise either from a non-cubic lattice structure or from non-cubic interactions on a cubic lattice (as an example see Eq. (51) below) or from both types of anisotropies. In some range of \( A \) near \( A \approx c_0 \mathbf{1} \) with \( c_0 > 0 \), \( \mathcal{H}_{\text{lattice}} \) and \( \mathcal{H} \) belong to the same universality class. Note, however, that in general \( r_{oc, \text{lattice}}(J_{ij}; u_0, \tilde{a}) \neq r_{oc}(A; u_0, \Lambda) \).

In order to elucidate the effect of the non-diagonality of the anisotropy matrix \( A \) we first discuss the bulk order-parameter correlation function for \( T \geq T_c \)

\[ G(\mathbf{x}; A, u_0) \equiv \varphi(\mathbf{x})\varphi(0) > H \]  
(21)

where \( \langle ... \rangle_H \) means an average with the exponential weight \( e^{-H} \). Equations (5), (10) and (12) imply

\[ G(\mathbf{x}; A, u_0) = (\det A)^{-1/2} G'(|\mathbf{x}'|; u_0') \]  
(22)

where

\[ G'(|\mathbf{x}'|; u_0') \equiv \varphi'(|\mathbf{x}'|)\varphi'(0) > H'. \]  
(23)

The second-moment bulk correlation length \( \xi'(u_0') \) associated with \( H' \) is defined by

\[ \xi'(u_0') = \frac{1}{2d} \lim_{v' \to \infty} \int d^d x' \xi^2 G'(|\mathbf{x}'|; u_0') \right)^{1/2}. \]  
(24)

For \( T \geq T_c \) and \( |\mathbf{x}'|/\xi' < O(1) \) the asymptotic scaling form of \( G'(|\mathbf{x}'|; u_0') \) reads \( \xi' \) for \( |\mathbf{x}'| \gg \xi'^{-1}, \xi' \gg \lambda'^{-1} \)

\[ G'(|\mathbf{x}'|; u_0') = A_G |\mathbf{x}'|^{-d+2-\eta} \Phi(|\mathbf{x}'|/\xi') \]  
(25)

with a universal scaling function \( \Phi \), a nonuniversal amplitude \( A_G(u_0', \lambda') \), and with \( \xi' = \xi_0'(u_0') t^{-\nu} \), apart from corrections to scaling. Eqs. (8), (22) and (25) imply asymptotically

\[ G(\mathbf{x}; A, u_0) = A_G |\mathbf{x}|^{-1/2} U_{\mathbf{x}} |^{-d+2-\eta} \Phi(|\lambda'^{-1/2} U_{\mathbf{x}} | / \xi'), \]  
(26)

with \( A_G' = A_G(\det A)^{-1/2} \). Thus the anisotropy does not change the universal structure of the scaling function \( \Phi \) but makes the scaling argument of \( \Phi \) and the spatial behavior of \( G \) anisotropic, even right at \( T_c(A) \) (see also [12,13]).

Choosing \( \mathbf{x} = x_i \mathbf{e}_i \) along the principal direction \( i, i = 1, ..., d \) defined by the eigenvector \( \mathbf{e}_i \), we have \( (U_{\mathbf{x}})_j = x_i \delta_{ij} \) and

\[ G(x_i \mathbf{e}_i; A, u_0) = A' G(|x_i|/\lambda_{i}^{1/2})^{-d+2-\eta} \Phi(|x_i| / \xi^{(i)}), \]  
(27)

where

\[ \xi^{(i)}(A, u_0) = \xi_0^{(i)} t^{-\nu} \]  
(28)

are the principal correlation lengths of the anisotropic system with the nonuniversal amplitudes

\[ \xi_0^{(i)}(A, u_0) = \lambda_{i}^{1/2} \xi_0'(u_0'). \]  
(29)

(The amplitudes \( \xi_0' \) and \( \xi_0^{(i)} \) may depend, in general, also on the cutoff.) Their product

\[ V_{\text{corr}}(A) = \prod_{i=1}^d \xi^{(i)} \]  
(30)

constitutes the appropriate measure of the correlation volume whose shape is ellipsoidal rather than spherical. This is seen by determining the singular part \( F_s(t; A, u_0)/V k_B T = \int f_s(t; A, u_0) \) of the bulk free energy density \( f = -\lim_{\kappa \to \infty} V^{-1} \int d^d x e^{-H} \) of the
anisotropic system. Using Eqs. (10), (11), (13) and (15) we obtain
\[ f_s(t; A, u_0) = (\det A)^{-1/2} f'_s(t; u'_0) \] (31)
where \( f'_s(t; u'_0) \) is the singular part of the bulk free energy density \( f' = -\lim_{r' \to \infty} V^{-1} \ln \int D\varphi e^{-H} \) associated with \( H' \), Eq. (24). Together with Eq. (22) for the isotropic system, Eqs. (10), (13) and (25) - (31) lead to
\[ \lim_{t \to 0^+} f_s(t; A, u_0)V_{\text{corr}}(A) = Q(d, n) = \text{universal} \] (32)
which is identical with Eq. (3).

From Eqs. (20) - (29) we see that a complete knowledge of the asymptotic behavior of the correlation function \( G \) requires the knowledge of the \( d+1 \) nonuniversal amplitudes \( \xi_0^{(i)} \) and of the \( d(d-1)/2 \) nonuniversal parameters characterizing the directions of the \( d \) eigenvectors \( e_i \). For real magnetic materials these quantities are unknown as they depend on all microscopic details. Furthermore, real magnetic materials may have lattice structures and anisotropic interactions (e.g., ANN, ANNN and third ANN interactions) corresponding to a nondiagonal matrix \( A \). It is because of the non-diagonality of \( A \) that both a scale transformation and a rotation is necessary and that a simple rescaling of \( d \) amplitudes is not sufficient. Clearly two-scale factor universality is absent in the bulk correlation functions of such anisotropic systems (e.g., metamagnets) although they belong to the same universality class as isotropic systems (e.g., fluids).

While the anisotropy does not destroy the universality of the scaling function \( \Phi \) of the bulk correlation function \( G \) (in the non-exponential regime \( r/\xi \lesssim O(1) \), a fundamental complication arises for confined systems since, in general, the principal directions \( e_i \) of the intrinsic anisotropy are totally unrelated to the orientation of the surfaces of the confining geometry (e.g., \( L_1 \times L_2 \times \ldots \times L_d \) rectangular geometry). This introduces a source of non-universality that cannot be absorbed only by a transformation of the lengths \( L_i \) of the confining geometry or of the scaling argument. Within our model (7), a complete information of this source of non-universality requires, at \( h = 0 \), the knowledge of \( d+d(d-1)/2 = d(d+1)/2 \) nonuniversal parameters (rather than \( d \) parameters). Within this model we shall show that this implies not only the absence of two-scale factor universality but the absence of universality itself for all finite-size scaling functions and finite-size amplitude ratios of anisotropic systems with non-cubic symmetry. In particular, two-scale factor universality and isotropy cannot be restored by an anisotropic scale transformation for confined systems in rectangular geometries with a non-diagonal anisotropy matrix \( A \). This is the central general result of this paper to be demonstrated in the following on the basis of exact results in the large-\( n \) limit and of one-loop RG results for \( n = 1, 2, 3 \).

First we consider the susceptibility \( \chi \) (per component) of the field-theoretic model (7) above \( T_c \) in a finite cube with periodic boundary conditions. In the limit \( n \to \infty \) at fixed \( u_0 n \) it is determined by
\[ \chi^{-1} = r_0 + 4u_0 n L^{-d} \sum_k (\chi^{-1} + k \cdot A k)^{-1} \] (33)
with \( k \cdot A k = \sum_{\alpha \beta} A_{\alpha \beta} k_\alpha k_\beta \). The sum \( \sum_k \) runs over \( k \) vectors with components \( k_\alpha = 2\pi n \alpha / L, n_\alpha = 0, \pm 1, \ldots \) up to some cutoff \( \Lambda \). For \( 2 < d < 4 \) the asymptotic form of the correlation length \( \xi' \) defined by Eq. (24) is \( \xi' = \xi'_0 t^{1/(d-2)} \).
\[ \xi'_0 = (4u_0 n A_d^{-1/2})^{1/(d-2)}. \] (34)
For large \( L \gg \Lambda^{-1} \) and small \( 0 < t < 1 \) we find the asymptotic scaling form for \( L'/\xi' \lesssim O(1) \)
\[ \chi(t, L; A) = L'^{\gamma / \nu} g_{\text{cube}}(L'/\xi'; \bar{A}), \gamma / \nu = 2 \] (35)
with the rescaled length
\[ L' = L (\det A)^{-1/2d} \] (36)
and the normalized anisotropy matrix
\[ \bar{A} = A / (\det A)^{1/d} \] (37)
where \( g_{\text{cube}}(x; \bar{A}) \) is determined implicitly by
\[ x^{d-2} - g_{\text{cube}}^{(2-d)/2} = (4 - d) A_d^{-1} I_1 (g_{\text{cube}}^{-1} \bar{A}), \] (38)
\[ I_j(z; \bar{A}) = \int_0^\infty ds (4\pi^2)^{-1} s^{j-1} P(s, \bar{A}) e^{-zs/4\pi^2}, \] (39)
with
\[ P(s, \bar{A}) = (\pi / s)^{d/2} - \sum_m e^{-m \bar{A} m}. \] (40)
The sum \( \sum_m \) runs over \( m = (m_1, \ldots, m_d) \) with all integers \( m_\alpha = 0, \pm 1, \ldots \). For \( \bar{A} = 1, g_{\text{cube}}(x; 1) \equiv g_{\text{cube, iso}}(x) \) is the known scaling function of the isotropic case (34). For \( \bar{A} \neq 1 \), however, \( g_{\text{cube}}(x; \bar{A}) \) is nonuniversal and depends on \( \bar{A} \) in a highly complicated way via the inhomogeneous \( m \neq 0 \) modes, even after having introduced the rescaled length \( L' \), Eq. (36). The effect of these modes depends on the orientation of the eigenvectors \( e_i \) relative to the shape of the confining geometry. In general this anisotropy effect cannot be inferred from the knowledge of finite-size scaling functions of isotropic systems of the same universality class and cannot be described by a transformation of the argument \( x \) of \( g_{\text{cube, iso}}(x) \) (unlike the case for the scaling function \( \Phi \) of the bulk correlation function \( G \)) or by a rescaling of \( L \).

Only in the special cases where \( A = \lambda \) is diagonal at the outset and where the eigenvectors \( e_i \) happen to be
parallel to the edges of the confining cube, the finite-
size scaling function of the anisotropic system in a cu-
bic geometry can be reexpressed in terms of the scaling
function of the isotropic system in a \( L' \times \ldots \times L'_d \) block
geometry, \( L'_i = L_i^{-1/2} \). Such special cases with a diago-
nal matrix \( \Lambda \) are \( d = 2 \) or \( d = 3 \) spin models on sc cubic
lattices with only NN couplings \( J_x \neq J_y \) [3, 13, 28] or
\( J_x \neq J_y \neq J_z \), respectively.

We note that a conclusive answer about the appropri-
ate way of rescaling the length \( L \) cannot be inferred only
on the basis of the result of \( \chi(0, L; A) \) at \( T_c \), without
further knowledge. The same statement applies to the
anisotropy effects being formally absorbed in the length
scaling function of the \( \xi \parallel (0, L; A) \) in a \( L^{d-1} \times \infty \) cylinder. It
would always be possible to rewrite \( \chi \) at \( T_c \) in the form

\[
\chi(0, L; A) = \hat{L}^{\nu'/\nu} g_{\text{cube,iso}}(0)
\]

(41)

with the amplitude \( g_{\text{cube,iso}}(0) \) of the isotropic system
if all anisotropy effects are formally absorbed in the length

\[
\hat{L} = L' \left[ g_{\text{cube}}(0; \Lambda)/g_{\text{cube,iso}}(0) \right]^{\nu'/\nu}.
\]

(42)

But after the calculation of a different physical quantity
at \( T_c \) it becomes obvious that this length \( \hat{L} \) is inappro-
priate as will be demonstrated in the following.

Next we present the anisotropy effect on the finite-size
scaling function of the singular part of the reduced free
energy density per component in the large-\( n \) limit for
cubic geometry and periodic boundary conditions. For
\( L \gg \Lambda^{-1}, 0 \leq t < 1, L' / \xi' \lesssim O(1) \) we find

\[
f_s(t, L; A) = L^{-d} Y_{\text{cube}}(L' / \xi' ; \Lambda),
\]

(43)

\[
Y_{\text{cube}}(x; \Lambda) = -\ln \frac{2}{2} + \frac{(d-2)A_d}{2d(4-d)} g(x; \Lambda)^{-d/2}
\]

\[+
\frac{1}{8\pi^2} \int_0^\infty ds \left[ \frac{4\pi^2}{s} + \frac{1}{g(x; \Lambda)} \right] P(s, \Lambda) e^{-s/(4\pi^2)}
\]

(44)

where \( g(x; \Lambda) \equiv g_{\text{cube}}(x; \Lambda) \) is determined implicitly by Eqs. [33] - [40]. The scaling function \( Y_{\text{cube}}(x; \Lambda) \), in-
cluding the amplitude \( Y_{\text{cube}}(0; \Lambda) \), is nonuniversal. Only
on the level of a lowest-mode \( (k = 0) \) approximation in
Eq. [33] the explicit dependence of \( Y_{\text{cube}} \) on \( \Lambda \) dis-
appears. The effect of the \( m \neq 0 \) modes cannot be de-
scribed simply by a transformation of the scaling variable
\( x \) of \( Y_{\text{cube}}(x; 1) \equiv Y_{\text{cube,iso}}(x) \) of the isotropic case and it
depends on \( d(d+1)/2 - 1 \) nonuniversal parameters con-
tained in \( \Lambda \). Equivalent parameters appear already in
\( G \), Eq. [20]. This holds, of course, also for the rele-
vant case of general finite \( n < \infty \) as can be shown [37]
within a one-loop RG calculation for the model [7]. The
exact scaling function \( Y_{\text{cube}}(x; \Lambda) \) for general \( n \) remains
unknown even if the exact scaling function \( Y_{\text{cube,iso}}(x) \)
were given for general \( n \) and if the exact matrix \( \Lambda \) were
given for a special anisotropic system.

We note that the same rescaled length \( L' \), Eq. [36], is employed in the scaling argument \( L'/\xi' \) of \( Y_{\text{cube}} \) as in
g_{cube} but not in the leading \( L^{-d} \) power law of Eq. [36].
At \( T = T_c \), it would of course be possible to rewrite \( f_s \)
in the form

\[
f_s(t, L; A) = \hat{L}^{-d} Y_{\text{cube,iso}}(0)
\]

(45)

with the amplitude \( Y_{\text{cube,iso}}(0) \) of the isotropic system if
the anisotropy effect is formally absorbed in the length

\[
\hat{L} = L \left[ Y_{\text{cube}}(0; \Lambda)/Y_{\text{cube,iso}}(0) \right]^{-1/d}.
\]

(46)

This length \( \hat{L} \) differs, however, from the length \( \hat{L}, \)
introduced formally for the susceptibility \( \chi(0, L; A) \).

As seen from our results for \( \chi(t, L; A) \) and \( f_s(t, L; A) \), a possible ambiguity of defining a rescaled length dis-
appears after calculating the complete temperature depen-
dence of the finite-size scaling functions of the anisotropic
system. At the same time such results clarify whether or
not isotropy can be restored by a scale transformation.
The exact analytic form of our results [35] and [36] for
\( T \geq T_c \) unambiguously answers this question for cubic ge-
ometry and periodic boundary conditions. An extension of
our results to rectangular \( L_1 \times L_2 \times \ldots \times L_d \) block ge-
ometry [35] confirms our findings, i.e., even after a rescaling
of the lengths \( L_i \) the finite-size scaling functions remain
nonuniversal for systems with a nondiagonal matrix \( A \).

We conclude that, for rectangular geometry and pe-
riodic boundary conditions, finite-size scaling functions
are, in general, not universally determined only by the
bulk universality class but do depend on nonuniversal
parameters in a highly complicated way if the system is
anisotropic in the sense specified above. In particular,
within our model [7], if the matrix \( A \) is nondiagonal, isotropy cannot be restored by a rescaling of lengths [36].

We expect that this conclusion holds also for nonperi-
odic boundary conditions and for non-rectangular geomet-
ries. For example, we expect that the universality of the
amplitude \( u \) of the ”corner” term \( u L^{-d} \) in \( L \) of the \( d = 2 \)
and \( d = 3 \) free energy density for free boundary condi-
tions at \( T_c \) [37, 38] is not generally valid for anisotropic
systems. The universality of \( u \) was proven in [37] only for
isotropic \( (d = 2) \) systems whereas in [38] it was supposed
to be valid ”within a given RG universality class”. Fur-
thermore, there have been calculations [11, 20, 21, 22, 23]
of edge exponents of anisotropic spin models in wedge
geometries with free boundary conditions. It was found
that the anisotropy enters explicitly into the exponents
and that it was possible to rescale \( L_i \) and \( L_j \) anisotropi-
cally to bring the problem into an isotropic form. We ex-
pect, however, that this is, in general, not possible for
the temperature-dependent finite-size scaling functions of
lattice systems with edges whose continuum limit yields
an effective Hamiltonian of the form of Eq. [7] with a
nondiagonal matrix \( A \).
We briefly extend our analysis to $\infty^{d-1} \times L$ film geometry with periodic boundary conditions, with $L$ being the thickness in the $d^{th}$ direction. In the large-$n$ limit we find

$$f_{s,film}(t; L; \mathbf{A}) = L^{-d} \left[ (\mathbf{\bar{A}}^{-1})_{dd} \right]^{-d/2} Y_{film,iso}(\bar{x}),$$

where $Y_{film,iso}$ is the scaling function for the isotropic system, with a transformed argument $\bar{x} = \bar{L}/\xi'$,

$$\bar{L} = \left[ (A^{-1})_{dd} \right]^{1/2} L$$

and where $(\mathbf{A}^{-1})_{dd}$ and $(\mathbf{\bar{A}}^{-1})_{dd}$ are the $d^{th}$ diagonal elements of the inverse of $\mathbf{A}$ and $\mathbf{\bar{A}}$, respectively. In contrast with $\mathbf{41}$, a nonuniversal amplitude appears at bulk $T_c$ and the Casimir amplitude

$$\Delta = (d-1) \left[ (\mathbf{\bar{A}}^{-1})_{dd} \right]^{-d/2} Y_{film,iso}(0)$$

is nonuniversal. The simplicity of this anisotropy effect is due to the one-loop structure of diagrams contributing to the large-$n$ limit. From finite-size theory at order $u_0^2 \mathbf{10}$ we infer a highly complicated $\mathbf{\bar{A}}$ dependence of $f_{s,film}$ for finite $n$. Furthermore we expect that the amplitudes $\mathbf{41}$ and scaling functions $\mathbf{52}$ of density profiles in film geometry are nonuniversal for anisotropic systems with non-cubic symmetry. More generally, our results suggest that the feature of universality in the theory of boundary critical phenomena $\mathbf{43}$ $\mathbf{44}$ $\mathbf{45}$ as well as the notion of a "surface universality class" and of "(2+1)-scale factor universality" $\mathbf{42}$ need to be reconsidered for the case of anisotropic systems.

It would also be interesting to interpret finite-size effects in percolation problems of anisotropic systems $\mathbf{17}$ $\mathbf{22}$ in the light of the results of the present paper.

We illustrate our theory by the example of the Binder cumulant ratio $U$ for $L \to \infty$ at $T_c$, Eq. $\mathbf{13}$. We consider a $\varphi^4$ lattice model

$$H_{lattice} = \tilde{\alpha}^d \left\{ \sum_{i} \left[ \frac{T_0}{2} \varphi_i^2 + u_0 (\varphi_i^2)^2 \right] + \sum_{i,j} \frac{J_{ij}}{2\tilde{\alpha}^2} (\varphi_i - \varphi_j)^2 \right\}$$

with an isotropic ferromagnetic interaction $J_{ij} = J > 0$ between nearest neighbors but an anisotropic interaction $J_{ij} = J'$ with only 6 (rather than 12) next-nearest neighbors in the $\pm(1,1,0)$, $\pm(1,0,1)$, and $\pm(0,1,1)$ directions on a simple-cubic lattice with a lattice constant $\tilde{\alpha}$ in a cube with periodic boundary conditions. It is expected that a ferromagnetic critical point exists not only for $J > 0, J' \geq 0$ but also for $J > 0, J' < 0$. In the continuum limit ($\tilde{\alpha} \to 0$) this model is reduced to Eq. $\mathbf{16}$ with

$$\mathbf{A} = c_0 \left( \begin{array}{ccc} 1 & w & w \\ w & 1 & w \\ w & w & 1 \end{array} \right)$$

FIG. 1: Cumulant ratio $1 - U(w)/U(0)$ vs coupling ratio $w = J'/(J + 2J')$ of the field-theoretic model, Eq. $\mathbf{7}$, in three dimensions for $n = 1, 2, 3$ (solid, dotted, dashed lines) according to Eqs. $\mathbf{53}$ - $\mathbf{57}$.

One of the eigenvalues of $\mathbf{A}$ vanishes at $w_c = -\frac{1}{2}$, i.e., $J' = -J/4$ (the two other eigenvalues vanish at $w = 1, J' = -J$). Thus $w$ may vary in the range $-\frac{1}{2} < w < \frac{1}{2}$ corresponding to a line of ferromagnetic critical points $T_c(w)$ terminating at a Lifshitz point $T_c(w_c)$ of the $\varphi^4$ continuum model $\mathbf{41}$ (but not necessarily of the $\varphi^4$ lattice model $\mathbf{40}$ whose line of critical points $T_c(w)$ may end at a value of $w$ different from $-\frac{1}{2}$).

From a RG treatment of the model $\mathbf{41}$ within the minimal renormalization scheme in three dimensions $\mathbf{32}$ parallel to previous work $\mathbf{40}$ we obtain $U(w)$ for $L \to \infty$ at $T_c(w)$ in one-loop order for $n = 1$ as

$$U(w) = 1 - \frac{1}{3} \partial_1(\bar{Y}) \left[ \partial_2(\bar{Y}) \right]^{-2}$$

where

$$\partial_m(\bar{Y}) = \int_0^\infty ds \exp\left( \frac{1}{2} \bar{Y} s^2 - s^4 \right)$$

\int_0^\infty ds \exp\left( \frac{1}{2} \bar{Y} s^2 - s^4 \right).
Here the quantity $\tilde{Y}$ depends on $w$ through $\tilde{A}(w)$,

$$\tilde{Y} = -b \left\{ \frac{4\pi}{l} [\tilde{l}^2 + I_1(\tilde{l}^2; \tilde{A})] + \frac{1}{2} + 4\pi \tilde{l}^4 + I_2(\tilde{l}^2; \tilde{A}) \right\},$$

with

$$b = 144 u^* \vartheta_2(0),$$

and $u^* = u^* = 0.0412$ where $I_j(z; \tilde{A})$ is given by Eq. (39). Clearly there is no way of eliminating the complicated internal dependence on the anisotropy matrix $\tilde{A}$ in Eq. (55), thus isotropy cannot be restored by means of a scale transformation.

We have also extended this result to general $n$. While the $w$ dependence is weak for $-0.4 \lesssim w \leq \frac{1}{2}$ it becomes appreciable upon approaching $w_c = -\frac{1}{2}$, as shown in Fig. 1 for $n = 1, 2, 3$. This proves the nonuniversality of $U(w)$. Similarly one can derive a $w$ dependence of the Casimir amplitude $\Delta(w)$ and of other scaling functions. Note, however, that because of the nonuniversal character of $U(w)$ and $\Delta(w)$, these quantities may, in principle, differ, e.g., for the $(d = 3, n = 1)$ field-theoretic model [Eq. (7) with the matrix (51)], and the $(d = 3, n = 1)$ Ising model (with NN and ANNN couplings) even if the geometries and the boundary conditions are the same in both models.

This kind of nonuniversal finite-size effect should exist near critical points of real systems and should be detectable in Monte Carlo simulations of $d = 2$ and $d = 3$ spin models. Support by NNSFC, by Max-Planck-Gesellschaft, by DLR and by NASA is acknowledged.

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