LOCAL GROWTH OF PLURI-SUBHARMONIC FUNCTIONS

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ABSTRACT. We obtain two-bound estimates for the local growth of pluri-subharmonic functions in terms of Siciak and relative extremal functions. As applications, we give simple new proofs of "Bernstein doubling inequality" and the main result in [Alexander Brudnyi, Local inequalities for pluri-subharmonic functions, Annals Math. 149 (1999), No. 2, pp. 511–533]. We propose a conjecture similar to the comparison theorem in [H. Alexander and B. A. Taylor, Comparison of two capacities in \( \mathbb{C}^n \), Math. Z. 186 (1984), 407–417], whose validity allows to obtain bounds for the local growth of pluri-subharmonic functions solely in term of the Siciak extremal functions.

1. Introduction

Let \( \Omega \) be an open subset of \( \mathbb{C}^n \). The set of pluri-subharmonic functions on \( \Omega \) is denoted as usual by \( PSH(\Omega) \). We are interested in obtaining bounds for the local growth of functions in \( PSH(\Omega) \). Given two non-pluripolar sets \( A, E \subset \subset \Omega \), we define the function:

\[
1.1 \quad h_E(z) := \sup_{f \in PSH(\Omega), \sup \Omega f \leq 0, \sup A f \geq -1} \{ f(z) - \sup_E f : f \in PSH(\Omega) \},
\]

where \( z \in \Omega \). The problem is to obtain good estimates of the function \( h_E(z) \) in terms of some intrinsic quantities of the set \( E \), such as (Lebesgue or Hausdorff) measures, or (logarithmic or relative) capacities. In this paper we will give some bounds of the function \( h_E(z) \) by the later quantities, via the Siciak and relative extremal functions. Let us recall the definitions of these extremal functions. The Siciak extremal function \( V_E \) is defined as follows: For \( z \in \mathbb{C}^n \)

\[
V_E(z) = \sup \{ f(z) : f \in \mathcal{L}(\mathbb{C}^n), f|_E \leq 0 \},
\]

where \( \mathcal{L}(\mathbb{C}^n) \) is the Lelong class

\[
\mathcal{L}(\mathbb{C}^n) = \{ f \in PSH(\mathbb{C}^n) : f(z) \leq \log^+ |z| + O(1) \}.
\]

The relative extremal function \( u_{E, \Omega} \) is defined as

\[
u_{E, \Omega}(z) = \sup \{ f(z) : f \in PSH(\Omega), f \leq 0, \sup_E f \leq -1 \},
\]

where \( z \in \Omega \).

Our first result is
Corollary 1. i) We have

\[
V_E(z) \leq \frac{h_E(z)}{\sup_{|z|} V_A} \leq \frac{u_{E,\Omega}(z) + 1}{|\sup_A u_{E,\Omega}|}.
\]

ii) If \( E \) is such that \( u_{E,\Omega} \) is a continuous function then

\[
h_E(z) = \frac{u_{E,\Omega}(z) + 1}{|\sup_A u_{E,\Omega}|}.
\]

As some applications of Lemma 1 we will give simple new proofs to the main result in [5] and to the "Berstein doubling inequality". The notation \( B(x, \rho) \) (respectively \( B_c(x, \rho) \)) denotes the Euclidean ball with center \( x \) and radius \( \rho \) in \( \mathbb{R}^n \) (respectively \( \mathbb{C}^n \)). Let \( r > 1 \) be a constant. Define \( \mathcal{F}_r \) to be the set of functions \( f \in \text{PSH}(B_c(0, r)) \) satisfying

\[
\sup_{B_c(0, r)} f \leq 0, \quad \sup_{B_c(0, 1)} f \geq -1.
\]

Theorem 1. (Theorem 1.2 in [5]) Let the ball \( B(x, t) \) satisfy \( B(x, t) \subset B_c(x, at) \subset B_c(0, 1) \), where \( a > 1 \) is a fixed constant. There are constants \( c = c(a, r) \), \( d = d(n) \) such that the inequality

\[
\sup_{B(x, t)} f \leq c \log \frac{|B(x, t)|}{|E|} + \sup_E f,
\]

holds for every \( f \in \mathcal{F}_r \), and every measurable set \( E \subset B(x, t) \). (Here \( |B(x, t)| \) and \( |E| \) mean the Lebesgue measures of \( B(x, t) \) and \( E \), respectively, as subsets of \( \mathbb{R}^n \).)

Proposition 1. (Proposition 2.5 in [5]) Let \( f \in \mathcal{F}_r \) and \( s \in [1, a] \), \( a > 1 \). Suppose that \( B_c(x, t) \subset B_c(x, at) \subset B_c(0, 1) \). Then there is a constant \( c = c(r) \) such that

\[
\sup_{B_c(x, st)} f \leq c \log s + \sup_{B_c(x, t)} f.
\]

Let us remark that already in [5], it was proved that when \( n = 1 \), in the RHS of (1.3) we can replace \( |E| \) by the Siciak capacity \( C(E) \) of \( E \). This suggests that for general \( n \), we may obtain a similar result. We propose the following conjecture, whose validity allows such an extension of Theorem 1 to the general cases when \( E \) needs not to have positive Lebesgue measure.

Conjecture 2. Let \( A = B_c(0, 1) \) and \( \Omega = B_c(0, a) \). There exists a constant \( C_{a,n} > 0 \) such that for all compact non-pluripolar set \( E \subset A \) we have

\[
|\sup_A u_{E,\Omega}| \sup_{\Omega} V_E \geq C_{a,n}.
\]

Let \( \gamma = C(E) \) be the Siciak capacity of \( E \), i.e.

\[
\lim_{s \to \infty} \sup_{B_s(0, s)} V_E - \log s = -\log \gamma.
\]

The following is a corollary of conjecture 2.

Corollary 1. If conjecture 2 is true, and if \( \Omega = B_c(0, a) \), \( A = B_c(0, 1) \) then there exists \( C_{a,n} > 0 \) such that for all compact non-pluripolar set \( E \subset B_c(0, 1) \) we have:

\[
\frac{1}{C_{a,n}} \log \frac{1}{\gamma} \leq \sup_A h_E \leq C_{a,n} \log \frac{1}{\gamma}.
\]
By Proposition 1, as argued in [5] (see also the proof of Theorem 1 in this paper), we can reduce proving (1.3) to estimating

\[
\sup_{B(0,1)} f - \sup_E f,
\]

where \( f \in PSH(B_c(0,a)) \), \( \sup_{B_c(0,a)} f \leq 0 \), \( \sup_{B_c(0,1)} f \geq -1 \). Since the middle term of (1.5) is an upper bound for the quantity in (1.6), Corollary 1 may be viewed as an extension of Theorem 1. Here the set \( E \) needs not to be a subset of \( \mathbb{R}^n \) or to have positive (\( \mathbb{R}^n \) or \( \mathbb{C}^n \)) Lebesgue measure.

Remark that conjecture 2 is similar to the comparison theorem of Alexander-Taylor [1]: There exists constants \( c_n > 0 \), \( c_a > 0 \) (here \( c_n \) depends only on \( n \) and \( c_a \) depends only on \( a \)) such that for all non-pluripolar set \( E \subset A \) we have

\[
\frac{c_n \text{cap}(E; \Omega)^{1/n}}{\sup_A V_E^*} \leq \text{sup}_A V_E^* \leq \frac{c_a \text{cap}(E; \Omega)}{\text{cap}(E; \Omega)},
\]

where \( \text{cap}(E; \Omega) \) is the relative capacity (for the definition, see for example [1]). Note that the exponents of \( \text{cap}(E; \Omega) \) in (1.7) can not be improved. As explained in [1], the exponent \( 1/n \) in the LHS of (1.7) occurs when \( E \) is a ball, while the exponent 1 in the RHS of (1.7) occurs when \( E \) is a small polydisk. More generally, if \( E = E_1 \times \ldots \times E_n \) where \( E_j \subset \mathbb{C} \), then in general the exponent may be any number between \( 1/n \) and 1. As will be shown later, in all these cases, conjecture 2 holds. It is interesting to observe that if \( E \) is a ball of center 0, then the LHS of (1.4) is the constant \( \log a \).

The rest of this paper is organized as follows. In Section 2, we prove Lemma 1 and Proposition 1. In Section 3, we verify conjecture 2 in some cases, and prove Corollary 1.

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## 2. Proofs of Lemma 1, Proposition 1 and Theorem 1

**Proof of Lemma 1**

Proof. i) Let \( f \in PSH(\Omega) \) be such that \( f \leq 0 \), \( \sup_A f \geq -1 \). Define

\[
\alpha := \sup_E f.
\]

Then by the definition of \( u_{E,\Omega} \) we have

\[
f(x) \leq |\alpha| u_{E,\Omega}(x) = |\alpha|(u_{E,\Omega}(x) + 1) + \alpha.
\]

Hence

\[
f(x) - \sup_E f = f(x) - \alpha \leq |\alpha|(u_{E,\Omega}(x) + 1).
\]

Now we estimate \( |\alpha| \). We have

\[
0 \geq |\alpha| \sup_A u_{E,\Omega} \geq \sup_A f \geq -1.
\]

Hence

\[
|\alpha| \leq \frac{1}{\sup_A u_{E,\Omega}}.
\]
Combining these inequalities we obtain
\[ f(x) - \sup_E f \leq \frac{u_{E, \Omega}(x) + 1}{|\sup_A u_{E, \Omega}|}. \]

Take supremum on over all such \( f \), we obtain the RHS inequality of (1.2).

Now we prove the LHS of (1.2). Let \( f \in L(\mathbb{C}^n) \) be not a constant function with \( \sup_E f = 0 \). Consider the function
\[ g(z) = \frac{f(z) - \sup_{\Omega} f}{\sup_{\Omega} f - \sup_A f}. \]

Then \( g \in PSH(\Omega) \), \( \sup_{\Omega} g \leq 0 \) and \( \sup_A g = -1 \). Hence by definition of Siciak extremal function, we have
\[ \frac{f(z)}{\sup_{\Omega} V_A} \leq \frac{f(z)}{\sup_{\Omega} f - \sup_A f} = g(z) - \sup_{\Omega} g \leq h_E(z). \]

If we take supremum of the above inequality on over all such \( f \) we obtain the LHS inequality of (1.2).

ii) If \( E \) is such that \( u_{E, \Omega} \) is a continuous function then \( u_{E, \Omega} \) itself is plurisubharmonic in \( \Omega \). Consider the function
\[ g(z) = \frac{u_{E, \Omega}(z)}{|\sup_A u_{E, \Omega}|}, \]
where \( z \in \Omega \). Then \( g \in PSH(\Omega) \), \( \sup_{\Omega} g \leq 0 \) and \( \sup_A g = -1 \). Thus by definition of the \( h_E \) we have
\[ \frac{u_{E, \Omega}(z)}{|\sup_A u_{E, \Omega}|} = g(z) - \sup_E g \leq h_E(z). \]

\[ \Box \]

Proof of Proposition 1

Proof. In this case \( \Omega = B_c(0, r) \), \( A = B_c(0, 1) \) and \( E = B_c(x, t) \).

By Lemma 1 we have
\[ \sup_{B_c(x, t)} f \leq \frac{\sup_{B_c(x, t)} u_{B_c(x, t), B_c(0, r)} + 1}{|\sup_{B_c(0, 1)} u_{B_c(x, t), B_c(0, r)}|} + \sup_{B_c(x, t)} f. \]

By Proposition 5.3.3 in [8] we have
\[ \sup_{B_c(x, t)} u_{B_c(x, t), B_c(0, r)} + 1 \leq \frac{\sup_{B_c(x, t)} V_{B_c(x, t)}}{\inf_{\partial B_c(0, r)} V_{B_c(x, t)}}. \]

Since \( V_{B_c(x, t)}(z) = \log^+(|z - x|/t) \), we obtain
\[ \sup_{B_c(x, t)} u_{B_c(x, t), B_c(0, r)} + 1 \leq \frac{\log s}{\log((r - 1 + t)/t)}. \]

Now we estimate \( |\sup_{B_c(0, 1)} u_{B_c(x, t), B_c(0, r)}| \). Fix \( z_0 \in \partial B_c(0, 1) \). We choose \( l_{z_0} \) to be the complex line containing both points \( x \) and \( z_0 \). Then
\[ |u_{B_c(x, t), B_c(0, r)}(z_0)| \geq |u_{B_c(x, t) \cap l_{z_0}, B_c(0, r) \cap l_{z_0}}(z_0)| \geq |\sup_{B_c(0, 1) \cap l_{z_0}} u_{B_c(x, t) \cap l_{z_0}, B_c(0, r) \cap l_{z_0}}|. \]
Now by the 1-dimensional case of conjecture [2] which is known to be true (see for example [1] or [5], see also Section 4 in this paper), since $B_c(x,t) \cap I_{z_0}$ is a 1-dimensional ball of radius $t$, there is a constant $C = C(r)$ depending only on $r$ such that

$$\sup_{B_c(0,1)^c \cap I_{z_0}} u_{B_c(x,t) \cap I_{z_0}} \geq C/\sup_{B_c(0,1)^c \cap I_{z_0}} V_{B_c(x,t) \cap I_{z_0}} \geq C/\log((r+1-t)/t).$$

Since $t \in [0,1]$, substituting all these inequalities into (2.1) we obtain

$$\sup_{B_c(x,t)} f \leq \frac{1}{C} \log((r + 1 - t)/t) \log s + \sup_{B_c(x,t)} f \leq C_1 \log s + \sup_{B_c(x,t)} f,$$

where $C_1 > 0$ is a constant depending only on $r$. □

Proof of Theorem [3]

Proof. Using the "Bernstein doubling inequality" (Proposition [1], as observed in [5] (page 523) it suffices to prove the following equivalent statement. Let $a > 1$ be a constant. Let $R_a$ be the family of pluri-subharmonic functions on $B_c(0,a)$ satisfying the conditions

$$\sup_{B_c(0,a)} f \leq 0, \quad \sup_{B_c(0,1)} f \geq -1.$$

Then for every measurable subset $E \subset B(0,1)$ of positive measure and every $f \in R_a$

$$\sup_{B(0,1)} f \leq c \log \frac{d|B(0,1)|}{|E|} + \sup_E f. \quad (2.3)$$

Here $d = d(n)$ and $c = c(a)$.

In this case $\Omega = B_c(0,a)$ and $A = B_c(0,1)$. Apply Lemma [4] we get

$$\sup_{B(0,1)} f \leq \frac{\sup_{B(0,1)} u_{E,B_c(0,a)}}{|\sup_{B(0,1)} u_{E,B_c(0,a)}|} + \sup_E f. \quad (2.4)$$

We divide the estimation of the first term in the RHS of (2.4) into several steps:

Step 1:

$$|\sup_{B(0,1)} u_{E,B_c(0,a)}| \geq |\sup_{B(0,1)} u_{B(0,1),B_c(0,a)}| \sup_{B(0,1)} u_{E,B_c(0,a)}|.$$

Proof: Let $f$ be any function in $PSH(B_c(0,a))$ with $\sup_{B_c(0,a)} f \leq 0$ and $\sup_E f \leq -1$. Define the function

$$g(z) = \frac{f(z)}{|\sup_{B(0,1)} u_{E,B_c(0,a)}|}.$$

Then $g \in PSH(B_c(0,a))$, $\sup_{B_c(0,a)} g \leq 0$, and since $f(z) \leq u_{E,B_c(0,a)}(z)$ we have also $\sup_{B(0,1)} g \leq -1$. Thus by definition of the relative extremal function

$$\frac{f(z)}{|\sup_{B(0,1)} u_{E,B_c(0,a)}|} = g(z) \leq u_{B(0,1),B_c(0,a)}(z),$$

for all $z \in \Omega$. Take supremum of the above inequality on over all such functions $f$, we obtain

$$\frac{u_{E,B_c(0,a)}(z)}{|\sup_{B(0,1)} u_{E,B_c(0,a)}|} \leq u_{B(0,1),B_c(0,a)}(z).$$

Form this we obtain the claim of Step 1.
Step 2: Apply Step 1 to (2.4), for any $f \in \mathcal{R}_a$ we have
\[
\sup_{B(0,1)} f \leq C_1 \frac{\sup_{B(0,1)} u_{E,B,c}(0,a) + 1}{|\sup_{B(0,1)} u_{E,B,c}(0,a)|} + \sup_f,
\]
where
\[
C_1 = \frac{1}{|\sup_{B(0,1)} u_{B(0,1),B,c}(0,a)|},
\]
depends only on $a$.

Step 3: Let $x_0$ be any point in $B(0,1)$. Then by Lemma 3 of [6], there exists a ray $l_0$ such that
\[
\frac{\text{mes}_1(B(0,1) \cap l_0)}{\text{mes}_1(E \cap l_0)} \leq \frac{n|B(0,1)|}{|E|}.
\]
Let $l'_0$ be the one-dimensional affine complex line containing $l_0$. Using the properties of extremal functions in one-dimensional and (2.6), we obtain
\[
\sup_{B(0,1)} u_{E,B,c}(0,a) + 1 \leq \sup_{z_0 \in B(0,1)} u_{E,B,c}(z_0) + 1 \leq \sup_{z_0 \in B(0,1)} u_{E,B,c}(z_0) + 1 \leq \inf_{\partial(B(0,1) \cap l_0)} V_{E,B,c}(z_0) \leq C_2 \log \frac{4 \text{mes}_1(B(0,1) \cap l_0)}{\text{mes}_1(E \cap l_0)} \leq C_2 \log \frac{4n|B(0,1)|}{|E|},
\]
for some constant $C_2 > 0$ depending only on $a$. This inequality together with (2.5) complete the proof of Theorem 1. \hfill \Box

3. Verification of conjecture 2 in some cases

Throughout this section $\Omega = B_c(0,a), A = B_c(0,1)$ and $E$ is a compact subset of $A$.

We need the following results

Claim 1:
\[
(3.1) \quad \log \frac{1}{\gamma} \leq \sup_A V_E \leq 2e^2 n \log \frac{n}{\gamma}.
\]

Proof. The LHS of (3.1) follows easily from the following two facts:

i) If $s \geq t > 0$ then
\[
\sup_{B_c(0,s)} V_E - \log s \leq \sup_{B_c(0,t)} V_E - \log t.
\]

ii) \[
\limsup_{s \to \infty} \sup_{B_c(0,s)} V_E - \log s = -\log \gamma.
\]

The proof of the RHS of (3.1) is similar to the proof of formula (1.2) in [10]: we use Taylor’s inequality (see [9]) applied to estimate the integration of $V_E^*$ on the sphere $|z| = n$, and the Harnack inequality for positive PSH functions. \hfill \Box
Claim 2: \[
\sup_A u_{E, \Omega} + 1 \leq \frac{2}{\sup_{\Omega} V_E} \sup_A V_E.
\]

Proof. Define \(M = \sup_{P} V_E\). For a function \(u\), let \(u^*\) be the upper-semicontinuous regularization of \(u\). Then it is well-known that the function \(V_E^*\) is in the Lelong class \(L(C^n)\). Consider the following function
\[
V(z) = (\sup \{ M(f(z) + 1), V(z) \} | z \in \Omega \})^*.
\]
Then \(V(z)\) is also in the Lelong class \(L(C^n)\).

Fix a function \(f \in PSH(\Omega)\) with \(\sup_{\Omega} f \leq 0\), \(\sup_E f \leq -1\). Define \(u(z) = \max\{ M(f(z) + 1), V(z) \} \)\( | z \in \Omega \}\)\(, z \in C^n \setminus \Omega\).

Then \(u(z)\) is in the Lelong class \(L(C^n)\). Hence
\[
u(z) \leq V_E(z) + \sup_E u.
\]
Now we estimate \(\sup_E u\). Since \(E \subset A = B_2(0,1)\), we have:
\[
\sup_E u = \sup_E V \leq \sup_A V = \sup_A V_E = \sup_A V_E.
\]
In particular
\[
M(f(z) + 1) \leq V_E(z) + \sup_A V_E.
\]
Take supremum on over all such \(f\), we obtain
\[
M(\sup_{\Delta} u_{E, \Omega} + 1) \leq V_E(z) + \sup_A V_E.
\]
Thus
\[
\sup_A u_{E, \Omega} + 1 \leq 2 \frac{\sup_A V_E}{\sup_{\Omega} V_E}.
\]

We verify conjecture \(2\) in the following four cases:

Case 1: \(n = 1\). In this case Conjecture \(2\) is just the Alexander-Taylor inequality \((1.7)\), using the equivalence between \(\text{cap}(E; \Omega)\) and \(\sup_A u_{E, \Omega}\) (see \[1\]).

Case 2: \(E = \bigcap_{j=1}^n D_j\) is a polydisk, where \(D_j\) is a disk in \(C\). In this case the Siciak capacity \(\gamma = \text{Cap}(E)\) of \(E\) is the smallest radius of the disks \(D_j\)’s. The same argument as that of the proof of Proposition \[1\] together with \((5.1)\), proves conjecture \(2\) in this case.

Case 3: \(E \subset B_2(z_0, \gamma^n)\) where \(\gamma = \text{Cap}(E)\) is the Siciak capacity of \(E\), and
\[
\tau_n = 1 - \frac{1}{8 e^2 n}.
\]
Without loss of generality (using the automorphism of \(\Omega\) translating \(z_0\) to the origin \(0 \in C^n\)), we may assume that \(z_0 = 0\). It suffices to prove Conjecture \(2\) when \(\gamma\) is small enough.

The proof of Claim 2 and \((5.1)\) gives
\[
\sup_{B_2(0, \gamma^n)} u_{E, \Omega} \leq 2 \frac{\sup_{B_2(0, \gamma^n)} V_E}{\sup_{\Omega} V_E} - 1 \leq 4 e^2 n (1 - \tau_n) \frac{- \log \gamma}{\log a - \log \gamma} - 1.
\]
Hence when \( \gamma \) is small enough we have

\[
\sup_{B_{c}(0,\gamma_{n})} u_{E,\Omega} \leq -\frac{1}{3}.
\]

Then it follows that

\[
|\sup_{A} u_{E,\Omega}| \geq \frac{1}{3} |\sup_{A} u_{B_{c}(0,\gamma_{n}),\Omega}|.
\]

This inequality, together with the LHS of (3.1) completes the proof of Conjecture 2 for Case 3.

Remark: A similar constraint was used in [10] (see Lemma 1 in [10]) when exploring sets non-thin at \( \infty \) in \( \mathbb{C}^{n} \).

Case 4: \( E = E_{1} \times \ldots \times E_{n} \), where \( E_{j} \subset \mathbb{C} \) are compact non-pluripolar, and \( a > \sqrt{n} \). In this case, there exists \( r > 1 \) such that \( A = B_{c}(0,1) \subset B = D(0,r) \times D(0,r) \ldots \times D(0,r) \subset \Omega = B_{c}(0,a) \), where \( D(0,r) \subset \mathbb{C} \) is the one-dimensional disk. Then

\[
|\sup_{A} u_{E,\Omega}| \geq |\sup_{A} u_{E,B}| = |\sup_{A} u_{E,B}^{*}|.
\]

We also have

\[
\sup_{A} V_{E} = \sup_{A} V_{E}^{*}.
\]

Using the product property of the function \( u_{E,B}^{*} \) and \( V_{E}^{*} \) (see for example [7] and [4]), Case 4 is reduced to Case 1 above.

Proof of Corollary 1: From Lemma 1 and the arguments above, Corollary 1 follows easily.

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