Thickness conditions and Littlewood–Paley sets

by

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Abstract. We consider sets in the real line that have Littlewood–Paley properties LP\(p\) or LP and study the following question: How thick can these sets be?

1. Introduction. Let \(E\) be a closed Lebesgue measure zero set in the real line \(\mathbb{R}\) and let \(I_k, k = 1, 2, \ldots,\) be the intervals complementary to \(E\), i.e., the connected components of the complement \(\mathbb{R} \setminus E\). Let \(S_k\) be the operator defined by
\[
\hat{S_k}f = 1_{I_k} \cdot \hat{f}, \quad f \in L^2 \cap L^p(\mathbb{R}),
\]
where \(1_{I_k}\) is the characteristic function of \(I_k\), and \(\hat{\cdot}\) stands for the Fourier transform. Consider the corresponding quadratic Littlewood–Paley function:
\[
S(f) = \left( \sum_k |S_k f|^2 \right)^{1/2}.
\]

Following [12] we say that \(E\) has property LP\(p\) \((1 < p < \infty)\) if for all \(f \in L^p(\mathbb{R})\) we have
\[
c_1 \|f\|_{L^p(\mathbb{R})} \leq \|S(f)\|_{L^p(\mathbb{R})} \leq c_2 \|f\|_{L^p(\mathbb{R})},
\]
where \(c_1, c_2\) are positive constants independent of \(f\). When a set has property LP\(p\) for all \(p, 1 < p < \infty\), we say that it has property LP.

The role of such sets in harmonic analysis and particularly in multiplier theory is well-known. We recall that if \(G\) is a locally compact Abelian group and \(\Gamma\) is the group dual to \(G\), then a function \(m \in L^\infty(\Gamma)\) is called an \(L^p\)-Fourier multiplier, \(1 \leq p \leq \infty\), if the operator \(Q\) given by
\[
\hat{Q}f = m \cdot \hat{f}, \quad f \in L^p \cap L^2(G),
\]
is bounded from \(L^p(G)\) to itself (here \(\hat{\cdot}\) is the Fourier transform on \(G\)). The space of all such multipliers is denoted by \(M_p(\Gamma)\). Provided with the norm
\[
\|m\|_{M_p(\Gamma)} = \|Q\|_{L^p(G) \to L^p(G)},
\]

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the space $M_p(\Gamma)$ is a Banach algebra (with the usual multiplication of functions). For basic facts on multipliers in the cases when $\Gamma = \mathbb{R}$, $\mathbb{Z}$, $\mathbb{T}$, where $\mathbb{Z}$ is the group of integers and $\mathbb{T} = \mathbb{R}/2\pi \mathbb{Z}$ is the circle, see [1], [13] Chap. IV, [7].

A classical example of an infinite set that has property LP is the set $E = \{\pm 2^k : k \in \mathbb{Z}\} \cup \{0\}$ (see, e.g., [13] Chap. IV, Sec. 5). From the arithmetic and combinatorial point of view, sets that have property LP($p$) or LP were studied extensively: see, e.g., [1–3], [12]. With the exception of [12] these works deal with countable sets, particularly, with subsets of $\mathbb{Z}$. At the same time there exist uncountable sets that have property LP. This was first established by Hare and Klemes [3]; see also [8] and [9, Sec. 4].

In this paper we study the following question: How thick can a set $E \subseteq \mathbb{R}$ that has property LP($p$) ($p \neq 2$) or property LP be? In Theorems 1 and 2 we show that such a set cannot be metrically very thick, namely it is porous and the measure of the $\delta$-neighbourhood of any portion of it tends to zero quite rapidly (as $\delta \to +0$). As a consequence we obtain (see Corollary) an estimate for the Hausdorff dimension of these sets. An immediate consequence of our estimate is that if a set has property LP, then its Hausdorff dimension is zero. In Theorem 3 we show that there exist sets which are thin in several senses simultaneously but have property LP($p$) for no $p \neq 2$. In Theorem 4 we show that a set can be quite thick but at the same time have property LP. In part our arguments are close to those used by other authors to study subsets of $\mathbb{Z}$, but the mere fact of existence of uncountable (i.e. thick in the sense of cardinality) sets that have property LP brings some specific details to the subject.

It is well-known that a set has property LP($p$) if and only if it has property LP($q$), where $1/p + 1/q = 1$ (see, e.g., [12]). Thus, it suffices to consider the case when $1 < p < 2$.

We use the following notation. For a set $F \subseteq \mathbb{R}$ we denote its open $\delta$-neighbourhood ($\delta > 0$) by $(F)_\delta$. If $F$ is measurable, then $|F|$ means its Lebesgue measure. A portion of a set $F \subseteq \mathbb{R}$ is a set of the form $F \cap I$, where $I$ is a bounded interval. By dim $F$ we denote the Hausdorff dimension of $F$. For basic properties of the Hausdorff dimension we refer the reader to [11]. For a set $F \subseteq \mathbb{R}$ and a point $t \in \mathbb{R}$ we put $F + t = \{x + t : x \in F\}$. By card $A$ we denote the number of elements of a finite set $A$. By an arithmetic progression of length $N$ we mean a set of the form $\{a + kd : k = 1, \ldots, N\}$, where $a, d \in \mathbb{R}$ and $d \neq 0$. We use $c, c(p), c(p, E), \ldots$ to denote various positive constants which may depend only on $p$ and the set $E$.

2. Results. We recall that a set $F \subseteq \mathbb{R}$ is said to be porous if there exists a constant $c > 0$ such that every bounded interval $I \subseteq \mathbb{R}$ contains a subinterval $J$ with $|J| \geq c|I|$ and $J \cap F = \emptyset$. 
THEOREM 1. Let \( E \subseteq \mathbb{R} \) be a closed set of measure zero. Suppose that \( E \) has property LP\((p)\) for some \( p, p \neq 2 \). Then \( E \) is porous.

Earlier Hare and Klemes showed that if a set in \( \mathbb{Z} \) has property LP then it is porous [2, Theorem 3.7].

To prove Theorem 1 we need certain lemmas.

**Lemma 1.** Let \( 1 < p < \infty \). Let \( \varphi : \mathbb{R}^n \to \mathbb{R} \) be a nonconstant affine mapping. Suppose that a function \( m \in M_p(\mathbb{R}) \) is continuous at each point of the set \( \varphi(\mathbb{Z}^n) \). Then the restriction \( m \circ \varphi|_{\mathbb{Z}^n} \) of the superposition \( m \circ \varphi \) to \( \mathbb{Z}^n \) belongs to \( M_p(\mathbb{Z}^n) \), and \( \|m \circ \varphi|_{\mathbb{Z}^n}\|_{M_p(\mathbb{Z}^n)} \leq c\|m\|_{M_p(\mathbb{R})} \), where \( c = c(p) > 0 \) is independent of \( \varphi \), \( m \) and the dimension \( n \).

**Proof.** The proof is a trivial combination of two well-known assertions on multipliers. The first one is the theorem on superpositions with affine mappings [1 Chap. I, Sec. 1.3], which implies that for every \( m \in M_p(\mathbb{R}) \) we have \( m \circ \varphi \in M_p(\mathbb{R}^n) \) and \( \|m \circ \varphi\|_{M_p(\mathbb{R}^n)} = \|m\|_{M_p(\mathbb{R})} \). The second one is the de Leeuw theorem [10] (see also [5]) on restrictions to \( \mathbb{Z}^n \), according to which if a function \( g \in M_p(\mathbb{R}^n) \) is continuous at all points of \( \mathbb{Z}^n \), then \( g|_{\mathbb{Z}^n} \in M_p(\mathbb{Z}^n) \) and \( \|g|_{\mathbb{Z}^n}\|_{M_p(\mathbb{Z}^n)} \leq c(p)\|g\|_{M_p(\mathbb{R}^n)} \).

**Lemma 2.** Let \( E \subseteq \mathbb{R} \) be a nowhere dense set and let \( F \subseteq \mathbb{R} \) be a finite or countable set. Then for each \( \delta > 0 \) there exists \( \xi \in \mathbb{R} \) such that \( |\xi| < \delta \) and \( (F + \xi) \cap E = \emptyset \).

**Proof.** The set

\[
\bigcup_{t \in F} (E - t),
\]

being a union of at most countable family of nowhere dense sets, cannot contain the whole interval \((-\delta, \delta)\), hence there exists \( \xi \in (-\delta, \delta) \) that does not belong to the union.

We say that a (finite or countable) set \( F \subseteq \mathbb{R} \) splits a closed set \( E \subseteq \mathbb{R} \) if \( F \subseteq \mathbb{R} \setminus E \) and no two distinct points of \( F \) are contained in the same interval complementary to \( E \).

**Lemma 3.** Let \( 1 < p < 2 \). Let \( E \subseteq \mathbb{R} \) have property LP\((p)\). Suppose that \( F \) is a subset of an arithmetic progression of length \( N \), and \( F \) splits \( E \). Then \( \text{card } F \leq c(p, E)N^{2/q} \), where \( 1/p + 1/q = 1 \).

**Proof.** This lemma can be deduced from Theorems 1.2 and 1.3 of [12]. We give an independent simple proof based on a quite standard argument. Consider an arithmetic progression \( \{a + kd : k = 1, \ldots, N\} \). We can assume that \( d > 0 \). Suppose that a set \( F = \{a + kjd : j = 1, \ldots, \nu\} \), where \( 1 \leq k_j \leq N \), splits \( E \). For \( j = 1, \ldots, \nu \) let \( \Delta_j \) be the interval of length \( \delta \) centered at \( a + k_jd \), where \( \delta > 0 \) is so small that \( \delta < d \) and \( \Delta_j \cap E = \emptyset, j = 1, \ldots, \nu \).
We put
\[ m_\theta = \sum_{j=1}^{\nu} r_j(\theta) \cdot 1_{\Delta_j}, \]
where \( r_j(\theta) = \text{sign} \sin 2^j \pi \theta, \theta \in [0, 1], j = 1, 2, \ldots, \) are the Rademacher functions.

It is well-known that if a set \( E \) has property \( LP(p) \), then it has the Marcinkiewicz property \( \text{Mar}(p) \), namely \(^{(1)}\) for each function \( m \in L_\infty(\mathbb{R}) \) whose variations \( \text{Var}_{I_k} m \) on the intervals \( I_k \) complementary to \( E \) are uniformly bounded, we have \( m \in M_p(\mathbb{R}) \) and
\[
\| m \|_{M_p(\mathbb{R})} \leq c(p, E) \left( \| m \|_{L_\infty(\mathbb{R})} + \sup_k \text{Var}_{I_k} m \right).
\]
Thus we have \( \| m_\theta \|_{M_p(\mathbb{R})} \leq c \), where \( c > 0 \) is independent of \( N \) and \( \theta \).

Consider the affine mapping \( \phi(x) = a + dx, x \in \mathbb{R} \). Using Lemma 1 for \( n = 1 \), we see that
\[
\| m_\theta \circ \phi \|_{Z_{M_p(\mathbb{R})}} \leq c(p) \| m_\theta \|_{M_p(\mathbb{R})} \leq c_1(p).
\]
Thus
\[
\left\| \sum_k m_\theta(a + kd)c_k e^{ikx} \right\|_{L_p(\mathbb{T})} \leq c_1(p) \left\| \sum_k c_k e^{ikx} \right\|_{L_p(\mathbb{T})}
\]
for every trigonometric polynomial \( \sum_k c_k e^{ikx} \). In particular,
\[
\left\| \sum_{k=1}^{N} m_\theta(a + kd)e^{ikx} \right\|_{L_p(\mathbb{T})} \leq c_1(p) \left\| \sum_{k=1}^{N} e^{ikx} \right\|_{L_p(\mathbb{T})}.
\]
Hence,
\[
\left\| \sum_{j=1}^{\nu} r_j(\theta)e^{ik_j x} \right\|_{L_p(\mathbb{T})} \leq c_1(p) \left\| \sum_{k=1}^{N} e^{ikx} \right\|_{L_p(\mathbb{T})}.
\]
It is easy to verify that
\[
\left\| \sum_{k=1}^{N} e^{ikx} \right\|_{L_p(\mathbb{T})} \leq c(p) N^{1/q},
\]
so \(^{(2)}\) yields
\[
\int_{T} \left| \sum_{j=1}^{\nu} r_j(\theta)e^{ik_j x} \right|^p dx \leq c_2(p) N^{p/q}.
\]

\(^{(1)}\) Actually \( LP(p) \) and \( \text{Mar}(p) \) are equivalent: see, e.g., \(^{[12]} \) Theorem 1.1.
By integrating this inequality with respect to $\theta \in [0,1]$ and using the Khintchine inequality
\[
\left( \int_0^1 \left| \sum_j c_j r_j(\theta) \right|^p d\theta \right)^{1/p} \geq c \left( \sum_j |c_j|^2 \right)^{1/2}, \quad 1 \leq p < 2,
\]
(see, e.g., [14, Chap. V, Sec. 8]), we obtain $\nu^{p/2} \leq c_3(p) N^{p/q}$.

**Proof of Theorem 1.** We can assume that $1 < p < 2$. For a bounded interval $I \subseteq \mathbb{R}$ let
\[
d(I) = \sup\{|J| : J \text{ is an interval, } J \subseteq I, J \cap E = \emptyset\}.
\]
Suppose that $E$ is not porous. Then, for each positive integer $N$ we can find a (bounded) interval $I$ such that $0 < d(I) < |I|/3N$. Let $d = 2d(I)$. Consider an arithmetic progression $t_k = a + kd, k = 1, \ldots, N$, that lies in the interior of $I$. Using Lemma 2, we can find $\xi$ such that $t_k + \xi \notin E, k = 1, \ldots, N$, and $\xi$ is so small that $\{t_k + \xi : k = 1, \ldots, N\} \subseteq I$. Note that since $d = 2d(I)$, no two distinct points of the progression $\{t_k + \xi : k = 1, \ldots, N\}$ lie in the same interval complementary to $E$. Thus this progression splits $E$. By Lemma 3 this is impossible if $N$ is sufficiently large.

**Theorem 2.** Let $1 < p < 2$. Let $E \subseteq \mathbb{R}$ be a closed set of measure zero. Suppose that $E$ has property $\text{LP}(p)$. Then each portion $E \cap I$ of $E$ satisfies
\[
|(E \cap I)_\delta| \leq c |I|^{2/q} \delta^{1-2/q},
\]
where $1/p + 1/q = 1$ and the constant $c = c(p, E) > 0$ is independent of $I$ and $\delta$.

Theorem 2 immediately implies an estimate for the Hausdorff dimension of sets that have property $\text{LP}(p)$:

**Corollary.** If $1 < p < 2$ and a set $E \subseteq \mathbb{R}$ has property $\text{LP}(p)$, then $\dim E \leq 2/q$, where $1/p + 1/q = 1$. Thus, if $E$ has property $\text{LP}$, then $\dim E = 0$.

**Proof of Theorem 2.** Consider an arbitrary portion $E \cap I$ of $E$. Let $J$ be the interval concentric with $I$ and of twice its length. Denote the left endpoint of $J$ by $a$. Fix a positive integer $N$ and consider the progression $a + kd, k = 1, \ldots, N$, where $d = |J|/N$. By Lemma 2 one can find $\xi$ such that no element of $\{a + kd + \xi : k = 1, \ldots, N\}$ is in $E$ and $I \subseteq J + \xi = (a + \xi, a + N d + \xi)$.

We define intervals $J_k$ by
\[
J_k = (a + (k - 1) d + \xi, a + kd + \xi), \quad k = 1, \ldots, N.
\]
Consider the intervals $J_{kj}$ such that $J_{kj} \cap E \neq \emptyset$. Obviously their right endpoints split $E$, so, by Lemma 3, their number is at most $c(p) N^{2/q}$. Thus $E \cap I$ is covered by at most $c(p) N^{2/q}$ intervals of length $d = 2|I|/N$ each.
Let $\delta > 0$. We can assume that $\delta < |I|$ (otherwise the assertion of the theorem is trivial). Choosing a positive integer $N$ so that

$$\frac{2|I|}{N} \leq \frac{\delta}{3} < \frac{4|I|}{N},$$

we see that $E \cap I$ can be covered by at most $c(p)(12|I|/\delta)^{2/q}$ intervals of length $\delta/3$ each. It remains to replace each of these intervals with the corresponding concentric interval of nine times its length. This proves the theorem. The corollary follows.

We note now that a set can be quite thin and at the same time have property LP($p$) for no $p \neq 2$. Consider the set

$$(3) \quad F = \left\{ \sum_{k=1}^{\infty} \epsilon_k l_k : \epsilon_k = 0 \text{ or } 1 \right\},$$

where $l_k$, $k = 1, 2, \ldots$, are positive numbers with $l_{k+1} < l_k/2$. It was shown by Sjögren and Sjölin [12] that such sets have property LP($p$) for no $p$, $p \neq 2$. (In particular, the Cantor triadic set does not have property LP($p$) for $p \neq 2$.) Taking a rapidly decreasing sequence $\{l_k\}$ one can obtain a set $F$ of the form (3) that is porous and has the property that the measure of its $\delta$-neighbourhood rapidly tends to zero. Still, in a sense, any set of the form (3) is thick: it is uncountable and all its points are its accumulation points. Theorem 3 below shows that a set can be thin in several senses simultaneously, and at the same time have property LP($p$) for no $p$, $p \neq 2$.

**Theorem 3.** Let $\psi$ be a positive function on an interval $(0, \delta_0)$, $\delta_0 > 0$, with $\lim_{\delta \to +0} \psi(\delta)/\delta = +\infty$. There exists a strictly increasing bounded sequence $a_1 < a_2 < \cdots$ such that the set $E = \{a_k\}_{k=1}^{\infty} \cup \{\lim_{k \to \infty} a_k\}$ satisfies the following conditions: 1) $E$ is porous; 2) $|(E)_{\delta}| \leq \psi(\delta)$ for all sufficiently small $\delta > 0$; 3) $E$ has property LP($p$) for no $p$, $p \neq 2$.

**Proof.** Given (real) numbers $a$ and $l_1, \ldots, l_n$ consider the set of all points $a + \sum_{j=1}^{n} \epsilon_j l_j$, where $\epsilon_j = 0$ or 1. Assume that the cardinality of this set is $2^n$. Following [6] we call such a set an $n$-chain $\square$

We shall need the following refinement of the Sjögren and Sjölin result on the sets (3). This refinement also provides a partial extension of Proposition 3.4 of [2], that treats subsets of integers, to the general case of closed measure zero sets in the line.

**Lemma 4.** Let $E \subseteq \mathbb{R}$ be a closed set of measure zero. Suppose that $E$ contains $n$-chains with arbitrarily large $n$. Then $E$ has property LP($p$) for no $p \neq 2$.

$\square$ An $n$-chain is a particular case of what is called a parallelepiped of dimension $n$, that is, of a set of cardinality $2^n$, obtained as the Minkowski sum of $n$ two-element sets.
Proof. Suppose that, contrary to the assertion, $E$ has property $\text{LP}(p)$ for some $p$, $p \neq 2$. We can assume that $1 < p < 2$.

Let $n$ be such that $E$ contains an $n$-chain

$$a + \sum_{j=1}^{n} \varepsilon_j l_j, \quad (\varepsilon_1, \ldots, \varepsilon_n) \in \{0, 1\}^n. \quad (4)$$

Consider the set

$$B = \left\{ a + \sum_{j=1}^{n} k_j l_j : (k_1, \ldots, k_n) \in \mathbb{Z}^n \right\}.$$

By Lemma 2 there exists an arbitrarily small $\xi$ such that

$$(B + \xi) \cap E = \emptyset. \quad (5)$$

Clearly, if $\xi$ is small enough, then no two distinct points of the chain obtained by the same shift $\xi$ of the chain (4) can lie in the same interval complementary to $E$. Thus, there exists $\xi$ such that (5) holds and the $n$-chain

$$a + \xi + \sum_{j=1}^{n} \varepsilon_j l_j, \quad (\varepsilon_1, \ldots, \varepsilon_n) \in \{0, 1\}^n,$$

splits $E$.

For each $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \{0, 1\}^n$ let $I_\varepsilon$ denote the interval complementary to $E$ that contains the point $a + \xi + \sum_{j=1}^{n} \varepsilon_j l_j$. For an arbitrary choice of signs $\pm$ consider the function

$$m = \sum_{\varepsilon \in \{0, 1\}^n} \pm 1_{I_\varepsilon}.$$  

We have (see (1))

$$\|m\|_{M_p(\mathbb{R})} \leq c, \quad (6)$$

where $c > 0$ is independent of $n$ and the choice of signs.

Consider the following affine mapping $\varphi$:

$$\varphi(x) = a + \xi + \sum_{j=1}^{n} x_j l_j, \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

Note that condition (5) implies that the function $m$ is continuous at each point of $\varphi(\mathbb{Z}^n)$. Using Lemma 1 we obtain (see (6)) $m \circ \varphi|_{\mathbb{Z}^n} \in M_p(\mathbb{Z}^n)$ and

$$\|m \circ \varphi|_{\mathbb{Z}^n}\|_{M_p(\mathbb{Z}^n)} \leq c,$$

where the constant $c > 0$ is independent of $n$ and the choice of signs.

Therefore, for every trigonometric polynomial $\sum_{k \in \mathbb{Z}^n} c_k e^{i(k,t)}$ on the torus $\mathbb{T}^n$,

$$\left\| \sum_{k \in \mathbb{Z}^n} m \circ \varphi(k)c_k e^{i(k,t)} \right\|_{L_p(\mathbb{T}^n)} \leq c \left\| \sum_{k \in \mathbb{Z}^n} c_k e^{i(k,t)} \right\|_{L_p(\mathbb{T}^n)}.$$
(We use \((k, t)\) to denote the usual inner product of vectors \(k \in \mathbb{Z}^n\) and \(t \in \mathbb{T}^n\).) In particular, taking \(c_k = 1\) for \(k \in \{0, 1\}^n\) and \(c_k = 0\) for \(k \notin \{0, 1\}^n\), we obtain

\[
\left\| \sum_{\varepsilon \in \{0, 1\}^n} m(a + \xi + \sum_{j=1}^{n} \varepsilon_j l_j) e^{i(\varepsilon, t)} \right\|_{L^p(\mathbb{T}^n)} \leq c \left\| \sum_{\varepsilon \in \{0, 1\}^n} e^{i(\varepsilon, t)} \right\|_{L^p(\mathbb{T}^n)}.
\]

That is,

\[
\left\| \sum_{\varepsilon \in \{0, 1\}^n} \pm e^{i(\varepsilon, t)} \right\|_{L^p(\mathbb{T}^n)} \leq c \left\| \sum_{\varepsilon \in \{0, 1\}^n} e^{i(\varepsilon, t)} \right\|_{L^p(\mathbb{T}^n)}.
\]

Raising this inequality to the power \(p\) and averaging with respect to the signs \(\pm\) (i.e., using the Khintchine inequality), we obtain

\[
\left\| \sum_{\varepsilon \in \{0, 1\}^n} e^{i(\varepsilon, t)} \right\|_{L^2(\mathbb{T}^n)} \leq c \left\| \sum_{\varepsilon \in \{0, 1\}^n} e^{i(\varepsilon, t)} \right\|_{L^p(\mathbb{T}^n)}.
\]

Note that

\[
\sum_{\varepsilon \in \{0, 1\}^n} e^{i(\varepsilon, t)} = \prod_{j=1}^{n} (1 + e^{it_j}), \quad t = (t_1, \ldots, t_n) \in \mathbb{T}^n,
\]

so (7) yields

\[
\parallel 1 + e^{it} \parallel_{L^2(\mathbb{T})} \leq c \parallel 1 + e^{it} \parallel_{L^p(\mathbb{T})}.
\]

Since \(n\) can be arbitrarily large, relation (8) implies

\[
\parallel 1 + e^{it} \parallel_{L^2(\mathbb{T})} \leq \parallel 1 + e^{it} \parallel_{L^p(\mathbb{T})},
\]

which, as one can easily verify, is impossible for \(1 < p < 2\).

**Lemma 5.** Let \(l_k, k = 1, 2, \ldots\), be positive numbers satisfying \(l_{k+1} < l_k/2\). Then the set \(F\) defined by (3) contains a strictly increasing sequence \(S = \{a_k\}_{k=1}^\infty\) that contains an \(n\)-chain for every \(n\).

**Proof.** For \(n = 1, 2, \ldots\) let

\[
\alpha_n = \sum_{k=1}^{n^2} l_k, \quad \beta_n = \sum_{k=1}^{n^2+n} l_k.
\]

Clearly \(\alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots\), so the closed intervals \([\alpha_n, \beta_n], n = 1, 2, \ldots\), are pairwise disjoint.

Define sets \(F_n \subseteq F, n = 1, 2, \ldots,\) as follows:

\[
F_n = \left\{ l_1 + l_2 + \cdots + l_{n^2} + \sum_{k=n^2+1}^{n^2+n} \varepsilon_k l_k : \varepsilon_k = 0 \text{ or } 1 \right\}.
\]

Note that \(F_n \subseteq [\alpha_n, \beta_n]\) for all \(n = 1, 2, \ldots\).

It remains to put \(S = \bigcup_{n=1}^\infty F_n\).
We shall now complete the proof of the theorem. Replacing, if needed, the function \( \psi(\delta) \) with
\[
\tilde{\psi}(\delta) = \frac{\psi(t)}{\delta} \inf_{0 < t \leq \delta} \frac{\psi(t)}{t},
\]
we can assume that \( \psi(\delta)/\delta \not\to +\infty \) as \( \delta \searrow 0 \).

Take a strictly increasing sequence of positive integers \( n_k, k = 1, 2, \ldots \), so that
\[
6 \cdot 2^k \leq \psi(3^{-n_k})/3^{-n_k}, \quad k = 1, 2, \ldots.
\]
Consider the set
\[
F = \left\{ \sum_{k=1}^{\infty} \varepsilon_k 3^{-n_k} : \varepsilon_k = 0 \text{ or } 1 \right\}.
\]
It is clear that \( F \) is porous (as a subset of the Cantor triadic set).

Assuming that \( \delta > 0 \) is sufficiently small, we can find \( k \) such that
\[
3^{-n_k+1} \leq \delta < 3^{-n_k}.
\]
Note that \( F \) can be covered by \( 2^{k+1} \) closed intervals of length \( 3^{-n_k+1} \) each. Consider the \( \delta \)-neighbourhood of each of these intervals. We infer that (see (10))
\[
|\{(F)_\delta\}| \leq 2^{k+1}3\delta.
\]
Hence, taking (9), (10) into account, we obtain
\[
|\{(F)_\delta\}| \leq \frac{\psi(3^{-n_k})}{3^{-n_k}} \delta \leq \psi(\delta).
\]

Using Lemma 5 we can find a strictly increasing sequence \( S = \{a_k\}_{k=1}^{\infty} \) contained in \( F \) such that for every \( n \) the sequence \( S \) contains an \( n \)-chain. Let \( E = S \cup \{a\} \), where \( a = \lim_{k \to \infty} a_k \). It remains to apply Lemma 4. ■

Our next goal is to construct a set that has property LP\((p)\) or property LP and at the same time is thick. Theorem 2 implies that if \( 1 < p < 2 \) and a bounded set \( E \) has property LP\((p)\), then \( |(E)_\delta| = O(\delta^{1-2/q}) \) as \( \delta \to +0 \). Hence, if a bounded set \( E \) has property LP, then \( |(E)_\delta| = O(\delta^{1-\varepsilon}) \) for all \( \varepsilon > 0 \). The author does not know if these estimates are sharp. A partial solution to this problem is given by Theorem 4 below. It is a simple consequence of the Hare and Klemes theorem [3 Theorem A], which provides a sufficient condition for a set to have property LP\((p)\). Stated for sets in \( \mathbb{Z} \), this theorem, as noted at the end of [3], easily transfers to sets in \( \mathbb{R} \) and allows one to construct perfect sets that have this property.

We shall use the version of the Hare and Klemes theorem stated in [9 Sec. 4]. According to this version, for each \( p, 1 < p < \infty \), there is a constant \( \tau_p \) (\( 0 < \tau_p < 1 \)) with the following property. Let \( E \) be a closed set of measure zero in the interval \([0, 1]\). Suppose that, under an appropriate numbering, the intervals \( I_k, k = 1, 2, \ldots \), complementary to \( E \) in \([0, 1]\) (i.e., the connected
components of \([0, 1] \setminus E\) satisfy
\begin{equation}
\delta_{k+1}/\delta_k \leq \tau_p, \quad k = 1, 2, \ldots,
\end{equation}
where \(\delta_k = |I_k|\). Then \(E\) has property \(\text{LP}(p)\). This in turn implies that if
\begin{equation}
\lim_{k \to \infty} \delta_{k+1}/\delta_k = 0,
\end{equation}
then \(E\) has property \(\text{LP}\).

**Theorem 4.**

(a) Let \(1 < p < \infty\). There exists a perfect set \(E \subseteq [0, 1]\) which has property \(\text{LP}(p)\) and at the same time satisfies \(|(E)_\delta| \geq c\delta \log(1/\delta)\) for all sufficiently small \(\delta > 0\).

(b) Let \(\gamma(\delta)\) be a positive nondecreasing function on \((0, \infty)\) with \(\lim_{\delta \to +0} \gamma(\delta) = 0\). There exists a perfect set \(E \subseteq [0, 1]\) which has property \(\text{LP}\) and at the same time satisfies \(|(E)_\delta| \geq c\gamma(\delta)\delta \log(1/\delta)\).

**Proof.** Let \(\delta_k, k = 1, 2, \ldots,\) be a sequence of positive numbers with
\begin{equation}
\sum_k \delta_k = 1.
\end{equation}
Let \(E \subseteq [0, 1]\) be a closed set. Assume that, under an appropriate numbering, the intervals \(I_k, k = 1, 2, \ldots,\) complementary to \(E\) in \([0, 1]\) satisfy \(|I_k| = \delta_k, k = 1, 2, \ldots,\) In this case we say that \(E\) is generated by the sequence \(\{\delta_k\}\). (Certainly \(|E| = 0\).) Note that for each sequence \(\{\delta_k\}\) of positive numbers with (13) there exists a perfect set \(E \subseteq [0, 1]\) generated by \(\{\delta_k\}\).

It is easy to see that if \(E\) is generated by a positive sequence \(\{\delta_k\}\) satisfying (13), then for all \(\delta > 0\) we have
\begin{equation}
|(E)_\delta| \geq 2\delta \text{card}\{k : \delta_k > 2\delta\}.
\end{equation}
Indeed, if \(I_k = (a_k, b_k)\) is an arbitrary interval complementary to \(E\) in \([0, 1]\) such that \(|I_k| > 2\delta\), then the \(\delta\)-neighbourhood of \(E\) contains the intervals \((a_k, a_k + \delta)\) and \((b_k - \delta, b_k)\).

We now prove part (a) of the theorem. Fix \(p, 1 < p < \infty\). Let
\[\delta_k = ae^{-kb}, \quad k = 1, 2, \ldots,\]
where the positive constants \(a\) and \(b\) are chosen so that conditions (11), (13) hold. Consider a perfect set \(E \subseteq [0, 1]\) generated by \(\{\delta_k\}\). Using (14), we see that
\[|(E)_\delta| \geq 2\delta \left(\frac{1}{b} \log \frac{a}{2\delta} - 1\right),\]
which proves (a).

Now we prove (b). Without loss of generality we can assume that \(\gamma(1/e) = 1/4\). Let
\[b(x) = \frac{1}{\gamma(e^{-x})}, \quad x > 0.\]
The function \(b\) is nondecreasing, \(b(x) \to \infty\) as \(x \to \infty\), and \(b(1) = 4\).
Define
\[ \delta_k = a e^{-kb(k)}, \quad k = 1, 2, \ldots, \]
where \( a > 0 \) is chosen so that (13) holds. Note that
\[ \delta_{k+1}/\delta_k = e^{-(k+1)b(k+1) - kb(k)} \leq e^{-b(k)} \to 0 \quad \text{as} \quad k \to \infty, \]
and thus (12) holds.

Consider a perfect set \( E \subseteq [0, 1] \) generated by the sequence \( \{\delta_k\} \).

Let \( \delta > 0 \) be sufficiently small. Choose a positive integer \( k = k(\delta) \) so that
\[ \delta_{k+1} \leq 2\delta < \delta_k. \]
We have
\[ \text{card}\{k : \delta_k > 2\delta\} \geq k(\delta). \]
So (see (14))
\[ |(E)_\delta| \geq 2\delta k(\delta). \]

Note that (15) implies
\[ kb(k) < \log \frac{a}{2\delta} \leq (k + 1)b(k + 1). \]
Hence, for all sufficiently small \( \delta > 0 \) we have
\[ \frac{1}{2}kb(k) < \log \frac{1}{\delta} \leq 2(k + 1)b(k + 1). \]
The left-hand inequality in (17) yields (recall that \( b(1) = 4 \))
\[ 2k = \frac{1}{2}kb(1) \leq \frac{1}{2}kb(k) < \log \frac{1}{\delta}, \]
whence
\[ b(2k) \leq b\left( \log \frac{1}{\delta} \right) = \frac{1}{\gamma(\delta)}. \]
Combining this inequality and the right-hand inequality in (17), we see that
\[ \log \frac{1}{\delta} \leq 2(k + 1)b(k + 1) \leq 4kb(2k) \leq 4k \frac{1}{\gamma(\delta)}. \]
So,
\[ \frac{1}{4} \gamma(\delta) \log \frac{1}{\delta} \leq k = k(\delta). \]
Thus (see (16)),
\[ |(E)_\delta| \geq \frac{1}{2} \gamma(\delta) \delta \log \frac{1}{\delta}. \]

Remark. As far as the author knows, the problem of the existence of a set that has property LP(\( p \)) for some \( p, p \neq 2 \), but does not have property LP is open.
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