Testing extra dimensions with the binary pulsar

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Abstract. In this paper we calculate the emission of gravity waves by the binary pulsar in the framework of five dimensional spacetime. We consider only spacetimes with one compact extra-dimension. We show that the presence of additional degrees of freedom, especially the 'gravi-scalar' leads to a modification of Einstein’s quadrupole formula. We compute the induced change for the binary pulsar PSR 1913+16 in the simple example of a 5d Minkowski background. In the example of a cylindrical braneworld it amounts to about 20% which is by far excluded by present experimental data.

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1. Introduction

During the last couple of years, the possibility that our 3+1 dimensional Universe might be a hypersurface in a higher dimensional spacetime has been investigated. This can be consistent with relatively 'large' extra-dimension if only gravitational interactions are propagating in the full spacetime, while gauge interactions are confined to the 3+1 dimensional 'brane' \[1, 2\]. The motivation for these 'braneworlds' is twofold. First of all, super string theory, which is consistently formulated only in 10 dimensions, predicts such branes as endpoints of strings, 'D-branes' \[3\]. Secondly, allowing for \( n \) extra dimensions of size \( L \), the observed, four dimensional Planck scale, \( M_4 \), relates to the fundamental Planck scale of the underlying string theory, \( M_P \), via \( M_4 = M_P (L M_P)^{n/2} \). Hence for a sufficiently large length \( L \) a small fundamental Planck scale, \( M_P \sim \text{TeV} \) can lead to the observed effective Planck scale, \( M_4 \equiv 1/\sqrt{4\pi G_4} \sim 3.4 \times 10^{18} \text{GeV} \), where \( G_4 \) denotes Newton’s constant \[1, 2\]. A short calculation shows that for \( n \geq 2 \) the required size of the extra dimensions is less than about 1mm at which scale Newton’s law has not been extensively tested so far. Here and in the following we use natural units to that \( \hbar = c = 1 \).

Motivated by this new attempt to solve the hierarchy problem, people have started to study the modified gravitational theory obtained on the brane, when starting from Einstein’s equation in the higher dimensional ‘bulk’. Many astrophysical and laboratory consequences of this idea have been investigated and found to be consistent with present bounds \[1\]. For simplicity, most of the work has concentrated on 1 single extra dimension for simplicity. We also do so in this work. In Refs. \[5, 6\] the modifications of cosmological solutions are derived. Following that, a lot of work has been devoted to cosmological perturbation theory in the context of brane worlds (see Refs. \[7\] to \[23\] and references therein). To do this, the authors have simply linearized the 5-dimensional Einstein equations around a cosmological background solution. In this paper we show that, at least for compact braneworlds, which allow a zero–mode like it is known from Kaluza—Klein theories, the five dimensional linearly perturbed Einstein equations are in conflict with observations.

For this we will make use of the following very generic issue: In \( d + 1 \) dimensions, the little group of the momentum state \((p, 0, \cdots, 0, p)\) of a massless particle is \( SO(d-1) \) (see e.g. \[24\]). For \( d = 3 \), the irreducible representations of \( SO(2) \) together with parity lead to the usual two helicity states of massless particles, independent of their spin. In \( d = 4 \) dimensions however, a massless particle with spin 2 like the graviton has the 5 spin states coming from the five dimensional tensor representation of \( SO(3) \). Projected onto a 3–dimensional hypersurface, these five states become the usual graviton, a massless particle with spin 1 usually termed ‘gravi-photon’ and a massless particle with spin 0, the ‘gravi-scalar’. These particles couple to the energy momentum tensor and should therefore be emitted by a time dependent mass/energy configuration like a binary system of heavenly bodies. This problem has not been discussed in the literature before. In
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analogy to the four dimensional quadrupole formula for graviton emission,

\[- \dot{E} = \frac{G}{4c^5} \ddot{Q}_{kl} \ddot{Q}_{kl}, \quad (1)\]

one might naively expect a formula for dipole emission of for the gravi-photon, like

\[- \dot{E} \propto \frac{G}{c^3} \dddot{S}_l \dddot{S}_l \quad \text{and} \quad (2)\]

\[- \dot{E} \propto \frac{G}{c} \dot{M} \dot{M} \quad (3)\]

for the monopole emission due to the gravi-scalar respectively. Here $S$ is the center of inertia,

\[S_j = \int d^3x \rho(x) x_j\]

so that $\dot{S} = P$ and $M$ is the mass of the system. Contrary to $\dddot{Q}$, the momentum, $\dot{S} = P$ and the mass are conserved to lowest order in $G$. Therefore, the contributions (2) and (3) vanish and the quadrupole formula, which is so well tested with the binary pulsar PSR 1913+16 that Hulse and Taylor have obtained the Nobel price of 1993, is maintained to sufficient accuracy.

In this work we show that Eq. (2) is indeed verified and there is no emission of gravi-photons to lowest order in $G$. However, Eq. (3) is not. In formulating it we omitted another scalar quantity, the trace of the second moment on the mass distribution,

\[I = \int \rho x^i x^j d^3x.\]

As we shall derive here, this term leads to an emission of gravi-scalars which induce an energy loss of the same order as the usual quadrupole term. We will see that for the binary pulsar PSR 1913+16 this modifies the prediction from Einstein’s quadrupole formula by about 20% in blatant contradiction with the observations which confirm the formula within an error of about 0.5%.

In the next section we derive the result announced above. In Section III we discuss its consequences for models with one extra–dimension.

Throughout we use capital Latin letters for five dimensional indices 0, 1, 2, 3, 4, Greek letters for four dimensional ones and lower case Latin letters for three dimensional spatial indices. The spatial Laplacian is denoted by, $\delta^i \delta_i \partial_j \equiv \Delta$, and spatial vectors are indicated in bold face.

2. The modified quadrupole formula for spacetimes with one extra–dimension

In order to be specific and for simplicity, we consider a 4 + 1 dimensional cylinder, with a rolled up fourth spatial dimension of length $L$. Since this length which is of the order of micrometers or smaller, is much smaller that the scale of the 3 + 1 dimensional system which is emitting gravity waves, we shall only consider states which are zero-modes with respect to the fourth spatial dimension.
We choose a gauge such that the perturbed five dimensional line element is given by

\[ ds^2 = -(1 + 2\Psi)dt^2 - 2\Sigma_i dt dx^i - 2B dt dy + 2\mathcal{E}_i dx^i dy + [(1 - 2\Phi)\delta_{ij} + 2H_{ij}] dx^i dx^j + (1 + 2\mathcal{C}) dy^2 \]

\[ = (\eta_{AB} + h_{AB}) dx^A dx^B. \]  

(4)

Here \(\Sigma_i\) and \(\mathcal{E}_i\) are divergence free vectors, \(H_{ij}\) is a traceless, divergence free tensor and \(\eta_{AB}\) is the five dimensional Minkowski metric. In the appendix we show, that the variables defined above are gauge-invariant combinations of the most general metric perturbations. There we also see that the gauge choice made in Eq. (4) fixes the gauge completely. The extra-dimension is parameterized by the co-ordinate \(y\), \(0 \leq y \leq L\) for our cylindric spacetime. We are working within linear perturbation theory and therefore require \(|h_{AB}| \ll 1\). The source free linearized Einstein equations for this geometry can be reduced to three wave equations,

\[ (\partial_t^2 - \partial_y^2 - \Delta)\Phi = 0, \]  

(5)

\[ (\partial_t^2 - \partial_y^2 - \Delta)\Sigma_i = 0, \]  

(6)

\[ (\partial_t^2 - \partial_y^2 - \Delta)H_{ij} = 0. \]  

(7)

The other variables are then determined by constraint equations,

\[ \Delta \mathcal{C} = (2\Delta + 3\partial_y^2)\Phi \]  

(8)

\[ \Psi = \Phi - \mathcal{C} \]  

(9)

\[ \Delta \mathcal{B} = -6\partial_t \partial_y \Phi \]  

(10)

\[ \partial_y \mathcal{E}_i = -\partial_t \Sigma_i \]  

(11)

(for more details see the Appendix and [26]). The important point is that the above wave equations describe simply the five propagating modes which we also expect from our group theoretical argument above. We now consider a four dimensional, Newtonian source given by

\[ T_{\mu 4} = T_{44} = 0 \]  

(12)

\[ T_{00} = \rho(x, t)\delta(y) \]  

(13)

\[ T_{0i} = (\partial_i v + v_i)(x, t)\delta(y) \]  

(14)

\[ T_{ij} = \left[ P\delta_{ij} + (\partial_i \partial_j - \frac{1}{3}\delta_{ij}\Delta)\Pi + \frac{1}{2}(\partial_i \Pi_j + \partial_j \Pi_i) + \Pi_{ij}\right](x, t)\delta(y) \]  

(15)

where \(v_i\) and \(\Pi_i\) are divergence free vector fields and \(\Pi_{ij}\) is a traceless divergence free tensor in three spatial dimensions. Eqs. (12) to (15) represent the most general energy momentum tensor on the brane with vanishing energy flux into the bulk. However, being interested in the binary pulsar, we shall restrict our attention to Newtonian sources, i.e. sources with \(\rho = |T_{00}| \gg |T_{0i}| \sim \rho V\), \(|T_{00}| \gg |T_{ij}| \sim \rho V^2\), where \(V \ll 1\) is a typical velocity of the system.
Since we are only interested in the zero-mode with respect to the fourth spatial dimension, we may integrate over the fifth dimension. The four dimensional linearized Einstein equations with source term then become (see Appendix)

\begin{align}
\triangle (2\Phi - \mathcal{C}) &= 8\pi G_4 \rho \quad (16) \\
\partial_t \mathcal{C} - 2\partial_t \Phi &= 8\pi G_4 v \quad (17) \\
\partial_t^2 (-\mathcal{C} + 2\Phi) &= 8\pi G_4 \left( P + \frac{2}{3} \triangle \Pi \right) \quad (18) \\
\Phi - \Psi - \mathcal{C} &= 8\pi G_4 \Pi \quad (19) \\
\triangle \mathcal{B} &= 0 \quad (20) \\
\partial_t^2 (2\Phi - \Psi) + 3\partial_t^2 \Phi &= 0 \quad (21)
\end{align}

vector perturbations:

\begin{align}
\frac{1}{2} \triangle \Sigma_i &= 8\pi G_4 v_i \quad (22) \\
\partial_t \Sigma_i &= 8\pi G_4 \Pi_i \quad (23) \\
\triangle \mathcal{E}_i - \partial_t^2 \mathcal{E}_i &= 0 \quad (24)
\end{align}

tensor perturbations:

\begin{align}
-\triangle H_{ij} + \partial_t^2 H_{ij} &= 8\pi G_4 \Pi_{ij} \quad (25)
\end{align}

where \( G_4 = (\pi/2) L^{-1} G_5 \) is the four dimensional Newton constant, \( M_4 = (2\pi^2 G_5 / L)^{-1/2} \). Here we have used the relation \( G_d^{-1} = M_d^{d-2} \text{vol}(S^{d-2}) = M_d^{d-2} 2\pi^{d-2} \Gamma(\frac{d-1}{2}) \) between the \( d \)-dimensional Newton constant and Planck mass. This relation ensures that Gauss’ law is true in any dimension. Einstein’s equations in an arbitrary number of dimensions are \( G_{AB} = 2M_d^{-d} T_{AB} \). Note that the equations for the zero-mode are very similar to the Kaluza-Klein approach. Both \( \mathcal{B} \) and \( \mathcal{E}_i \) are completely decoupled. In Kaluza-Klein theories they play the role of the electromagnetic field which couples to the current \( J_\mu \propto T_{\mu4} \) which we have set to zero in this work. In a coordinate system where the brane is fixed (no brane bending) the only difference for the zero-mode, in our ’test-brane’ analysis is the relation between the four- and five-dimensional gravitational constant. From Eqs. (16) to (25) we can derive three wave equations with source term:

\begin{align}
(\partial_t^2 - \triangle) \Phi &= -8\pi G_4 (\rho - \triangle \Pi) \quad (26) \\
(\partial_t^2 - \triangle) \Sigma_i &= 8\pi G_4 (\dot{\Pi}_i + 2v_i) \quad (27) \\
\partial_t \Sigma_i &= 8\pi G_4 \Pi_i \quad (28) \\
(\partial_t^2 - \triangle) H_{ij} &= 8\pi G_4 \Pi_{ij} \quad (29)
\end{align}

The other variables are again determined by constraint equations.

Eq. (29) leads over well-known steps to the quadrupole formula for Newtonian sources. The retarded solutions of the wave Eqs. (26) and (27) determine \( \Sigma_i \) and \( \Phi \) far away from the source. It is easy to see that the a priori dominant terms \( \rho \) in (26) and \( v_i \) in (27) lead exactly to the terms predicted in Eqs. (3) and (2). Hence their contribution to the energy emission vanishes (to lowest order) by mass and momentum
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conservation. We therefore concentrate on the terms coming from the stress tensor. Energy momentum conservation $(T^\mu{}_{\nu}, = 0)$ gives

\[ \Delta(\Delta\Pi + \frac{3}{2}P) = \frac{3}{2} \partial_i \partial_j T^{ij} = \frac{3}{2} \partial_i^2 \rho, \]  

so that $\Delta\Pi = \frac{3}{2}(\dot{U} - P)$, where

\[ U(t, x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\rho(t', x')}{|x - x'|} d^3 x'. \]  

Taking the divergence of (15) and using the above expression for $\Delta\Pi$ we find

\[ \Delta\Pi_j = 2\partial_i(T^i_j - \delta^i_j \dot{U}), \]  

so that, to order $1/r$, each component of the vector potential $\Pi_i$ is the integral over a divergence which vanishes for a confined source. Hence $\Pi_i \propto r^{-2}$; together with (28) this implies that also $\Sigma_i$ is of order $r^{-2}$. The energy flux from $\Sigma_i$ is expected to be proportional to $t_{0j} \propto (\partial_i \Sigma_i)(\partial_0 \Sigma_j) \propto 1/r^4$ (the bracket $[..]$ indicates anti-symmetrization) which shows that we have no energy emission from the gravi-photon far away from the source.

The situation is different for the gravi-scalar, $\Phi$. There the energy flux is proportional to $t_0i \propto \partial_i \Phi \partial_0 \Phi$. To order $1/r = 1/|x|$, the scalar field $\Phi$ is given by

\[ \Phi(t, x) = \frac{G_4}{r} \int_{\mathbb{R}^3} \left[ 3(\dot{U}(t', x') - P(t', x')) - 2\rho(t', x') \right] d^3 x'. \]  

Here $t' = t - |x - x'|$ is retarded time. The last term does not contribute to $t_{0i}$ since $\partial_i \int \rho = M = 0$. Furthermore, $\dot{\rho} = \partial_t T^{0i}$ so that

\[ 4\pi \dot{U}(t, x) = \int_{\mathbb{R}^3} \frac{\partial^j T^{0i}(t, x)}{|x - x'|} d^3 x' = -\int_{\mathbb{R}^3} T^{0i}(t, x') \partial_i \frac{1}{|x - x'|} d^3 x' \]  

\[ = -\partial_i \int_{\mathbb{R}^3} T^{0i}(t, x') d^3 x'. \]  

Here $\partial'_i$ denotes differentiation w.r.t. $x'$ while $\partial_i$ is differentiation w.r.t. $x$. In other words, $\dot{U}$ is a divergence and therefore does not contribute in the integral (33).

Finally, we use the identity (see [27])

\[ 3 \int_{\mathbb{R}^3} P = \int_{\mathbb{R}^3} \delta^{ij} T_{ij} = \frac{1}{2} \int_{\mathbb{R}^3} \delta^{ij} \dot{\rho} x'_i x'_j \]  

which is a simple consequence of energy momentum conservation and Gauss’ law. All this leads to

\[ \dot{\Phi}(t, x) = \frac{G_4}{2r} \int \frac{\dot{\rho}(t', x') x'^2}{|x - x'|} d^3 x' + O\left(\frac{1}{r^2}\right) \]  

and

\[ \partial_i \Phi(t, x) = \frac{G_4 n_i}{2r} \int \frac{\rho(t', x') x'^2}{|x - x'|} d^3 x' + O\left(\frac{1}{r^2}\right). \]  

Here $n^i = n_i = x^i/r$. The energy emission from the gravi-scalar is then given by

\[ \left. -\left(\frac{dE}{dt}\right)\right|_{\text{scalar}} = \lim_{r \to \infty} \int_{S^2} r^2 d\Omega n^i t_{0i}(r n) \]  

\[ \propto \lim_{r \to \infty} \int_{S^2} r^2 d\Omega n^i \dot{\Phi} \partial_i \Phi = \pi G_4^2 \left[ \frac{d^3}{dt^3} \int \rho x^2 d^3 x \right]^2. \]  

(38)
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For the last equality sign we have used the results \[36\] and \[37\] and integrated over the sphere.

To obtain the proportionality factor which is still missing in Eq.\(38\), we write the five dimensional action up to second order in the scalar metric perturbations

\[S = \frac{1}{8\pi G_5} \int_0^L dy \int_{\mathbb{R}^4} d^4x \sqrt{|g|} R\]

\[= \frac{1}{16\pi G_4} \int_{\mathbb{R}^4} d^4x (1 + \frac{1}{2} h_A^A) (R^{(1)} + R^{(2)})\]

\[= \int_{\mathbb{R}^4} d^4x \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi + \text{ a total derivative} . \]  

(39)

Here \(h_A^A = 2(\Psi - 3\Phi + \mathcal{C})\) is the trace of the metric perturbations and \(R^{(1)}\) and \(R^{(2)}\) are the first and second order (scalar) perturbations of the five dimensional Riemann scalar. The last equation is just the ansatz for a usual, canonically normalized scalar field action. The source free Einstein equations for the zero-mode can be obtained by varying this action. Therefore, the energy flux of the mode is given by \(t_{0i} = \partial_0 \varphi \partial_i \varphi\). A straightforward but somewhat tedious calculation of \(R^{(1)}\) and \(R^{(2)}\), in which we use the homogeneous constraint equations to express all scalar perturbations variables in terms of \(\Phi\), gives finally

\[\varphi = \sqrt{\frac{21}{8\pi G_4}} \Phi \] 

so that \(t_{0i} = \frac{21}{8\pi G_4} \dot{\Phi} \partial_i \Phi\).

Inserting this proportionality factor in Eq. \(38\) leads to

\[-\left( \frac{dE}{dt} \right)_{\text{scalar}} = \frac{21 G_4}{8} \left[ \frac{d^3}{dt^3} \int_{\mathbb{R}^3} \rho x^2 \right]^2 . \]  

(40)

A textbook calculation for a binary system in Keplerian orbit, like it is presented e.g. in Ref. \[27\], now gives for the energy loss averaged over one period of the system

\[-\left( \frac{dE}{dt} \right) = \frac{21 G_4 M_1^2 M_2^2 (M_1 + M_2) g(e)}{4a^5 (1 - e^2)^{7/2}} \]  

(41)

with \(g(e) = e^2(1 + e^2)^2\).  

(42)

Here \(M_{1,2}\) are the two masses of the system, \(a\) is the semi-major axis and \(e\) is the eccentricity of the orbit. Comparing this with the quadrupole formula for a Keplerian binary system which can be found e.g. in \[24\], we see that \(41\) agrees with it upon replacing \(\frac{21g(e)}{4}\) by \(\frac{32f(e)}{5}\) with \(f = 1 + (73/24)e^2 + (37/96)e^4\). Inserting the value \(e = 0.617\) for PSR 1913+16 given in \[23\], one finds that this new contribution amounts to 19.9% of the ordinary quadrupole prediction. Therefore, the slowdown of the period of PSR 1913+16 should be

\[\dot{T}|_{\text{tens+scal}} = -2.88 \times 10^{-12} \]  

instead of

\[\dot{T}|_{\text{tens}} = -2.4024 \times 10^{-12} . \]  

(43)

(44)

But the experimental value \[25\] is in perfect agreement with the quadrupole formula \[44\] and contradicts such a huge correction:

\[\dot{T}|_{\text{exp}} = -(2.408 \pm 0.01) \times 10^{-12} . \]  

(45)
Our result is not entirely self-consistent in that we assume that the masses $M_1$ and $M_2$ inferred for the two neutron stars are the same as in ordinary four dimensional gravity. This is not necessarily the case and probably also depends on the method by which they are determined. In principle we would have to specify two additional measurements and check whether different masses $M'_1$ and $M'_2$ can be found so that all the experiments agree with the results from four dimensional gravity for the masses $M_1$ and $M_2$. This procedure has been undertaken in the work on general scalar-tensor theories of gravity in Ref. [28]. Clearly, it would be a very surprising coincidence if such masses could be found.

Finally, we want to mention that other tests, like for example light deflection around the sun most probably also lead to deviations from the four dimensional result.

3. Conclusion

Our calculations show clearly that a $4+1$ dimensional cylindrical spacetime which satisfies the five dimensional Einstein equations contradicts observations. This finding is independent of the size of the extra dimension. Note that we did only take into account the Kaluza-Klein zero-mode and hence the size $L$ of the extra dimension just enters in the relation between the five and four dimensional gravitational constants, $G_4 = G_5 \pi / 2L$. It is important to note at this point that the zero-mode in general of course depends on the background geometry and thus on the warp factor (it is a solution to the five dimensional d’Alambertian, $\Box_5$ with respect to the background geometry). In our case, the background geometry is trivial. If the warp factor depends on the size of the extra-dimension this will of course also imprint itself on the zero-mode.

In our calculation we have considered a simple cylindrical spacetime. But our result would not alter substantially if we would consider a more realistic, warped spacetime which corresponds to an expanding universe. The mayor exception are warped models with non-compact extra-dimensions like the Randall Sundrum II model [29], where the scalar gravity mode is not normalizable and therefore our discussion does not apply. But in all models with compact extra-dimensions, for example in all models with two branes, like Randall Sundrum I [30] or the ekpyrotic universe [31] the mode discussed here is normalizable. Its coupling is determined by the five dimensional Einstein equations. We therefore expect that (besides all the other problems of scalar–tensor theories of gravity [28]) this mode be excited by a system like the binary pulsar and lead to deviations from Einstein’s quadrupole formula. Even though the exact amount of change will depend on the details of the model, generically we can expect it to be of the same order of magnitude. The expansion of the Universe, which determines the jump of the extrinsic curvature between the two sides of the brane is not relevant on scales of a binary pulsar and so we expect a realistic braneworld model to lead to very similar results. The possibly more complicated geometry of the extra-dimension may somewhat
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affect the numerical result but as long as the zero-mode discussed here exists (i.e. as long as it is normalizable) the problem will remain.

In order to find consistent four dimensional Einstein gravity on the brane, the five dimensional Einstein equations have to be modified somehow at very low energy. One possibility might be that the gravi-scalar $\varphi$ acquires a small mass which is larger than the frequency of the binary pulsar, $m_\varphi > 1/T \simeq 3.6 \times 10^{-5}\text{Hz} \simeq 2.4 \times 10^{-20}\text{eV}$ (e.g. via the Goldberger–Wise mechanism \[32\]). Such a tiny mass would already suffice to inhibit the field $\varphi$ to be emitted by the binary pulsar PSR 1916+13. To interpret the gravi-scalar physically, it is useful to notice that in source free space, for the zero modes we have $2\Phi = C$. But $C$ determines the size of the extra dimension, $L + \delta L = \int_0^L (1 + C)dy$. Therefore, if one wants to stabilize the size of the extra dimensions one needs a positively curved potential, hence a mass, for $C$ and thus for $\Phi$. This argument is well known in string inspired Kaluza-Klein theories. There, the size of the extra dimensions is given by the string scale, $\ell$ and T-duality, the duality of string theory under the transformation $r \rightarrow \ell^2/r$, indicates that a potential for $\delta L$ and hence for $C$ should have a minimum at $L = \ell$, $C = 0$, in other words a positive mass term, see e.g. \[33\]. Some non-gravitational (stringy) correction must provide this mass.

But even if we set $C = 0$, the scalar perturbation equations (16) to (21) do not reduce to the usual four dimensional perturbation equations: using (18) and the Laplacian of (19) to eliminate $\partial^2_t \Phi$ and $\Box(\Phi - \Psi)$ in (21), we obtain

$$\Box \Phi = 8\pi G_4 \frac{3}{2} \rho P \quad (46)$$

Comparing this with Eq. (16) we find that these equations are only compatible when $\rho = 3P$, in other words, if the matter is relativistic. This shows that the problem discussed in this paper goes beyond the radion problem: even if the radion $C$ is fixed, we do not recover the four dimensional Einstein equations.

Moreover, if non-gravitational corrections influence the evolution equation of the gravi-scalar, they may very well also introduce modifications to the interactions on the brane which are usually inferred from the five dimensional Einstein equations, like the Israel junction conditions \[34\], or certainly the gravitational perturbation equations as derived in Refs. \[7\] to \[23\].

As mentioned above, the situation is mitigated if the geometry is warped, like in the Randall-Sundrum model \[29\]. In such models, the fifth dimension may be non-compact and the gravi-scalar zero-mode can be non-normalizable in the sense that the integral $\int \sqrt{|g|} \varphi^2 dy = \infty$. Then the gravi-scalar cannot be excited as it has infinite energy. Within the language of classical relativity, we may simply say that the 'gravi-scalar mode' turns out to be large for some values of $y$ and linear perturbation theory is therefore not applicable.

Let us finally summarize our main conclusion: If a model with extra dimensions is such that the gravi-scalar coming from Einstein’s equations in five dimensions is normalizable, which is always the case if the extra-dimensions are compact, non-gravitational interactions need to intervene in order to make the model compatible.
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with observations. They have to give the gravi-scalar a mass. In such models therefore, we cannot simply use the five dimensional Einstein equations to infer the four dimensional equations of motion. Especially, we cannot apply five dimensional gravitational perturbation theory to, e.g., a cosmological solution in order to learn about four dimensional cosmological structure formation. The perturbation theory derived in Refs. [7] to [23] can at best be applied to models where the gravi-scalar is not normalizable.

Clearly, the situation does not improve if there are more than one extra dimensions. In this case there are several gravi-scalars which may all contribute a similar amount to the emission of gravity waves.

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Appendix A.
Gauge–invariant perturbation equations for a 5d Minkowski braneworld

We consider a five dimensional Minkowski bulk with background metric \((\eta_{AB}) = \text{diag}(-1,1,1,1,1)\). The most general perturbed metric is then of the form
\[
ds^2 = -(1 + 2A)dt^2 - 2S_i dt dx^i + (\delta_{ij} + h_{ij})dx^i dx^j - 2Bdydt + 2E_i dy dx^i + (1 + 2C)dy^2.
\]

(A.1)

The spatial vectors \(S_i\) and \(E_i\) can be decomposed into scalar (spin zero) and vector (spin one) components and the spatial tensor \(h_{ij}\) can be decomposed into scalar, vector and tensor (spin two) components;
\[
S_i = \nabla_i S + \overline{S}_i, \quad E_i = \nabla_i E + \overline{E}_i,
\]

(A.2)

\[
h_{ij} = 2H\gamma_{ij} + 2F_{ij}, \quad F_{ij} = \nabla_i F_j + H_{ij}, \quad F_i = \nabla_i F + \overline{F}_i.
\]

(A.3)

Here \(\overline{E}_i, \overline{F}_i\) and \(\overline{S}_i\) are divergence free vectors and \(H_{ij}\) is a divergence free, traceless symmetric tensor. To first order, scalar vector and tensor perturbations evolve independently. Spatial indices can be raised and lowered with the background metric \(\delta_{ij}\).

Let us now study the transformation of the defined perturbation variables under infinitesimal coordinate transformations (gauge transformations) in the bulk. We consider an infinitesimal coordinate transformation
\[x^A \rightarrow x^A + \xi^A,\]

where we set
\[
(\xi^A) = (T, L^i, L^4), \quad L^i = \nabla^i L + \overline{E}^i.
\]

(A.5)
The three scalar \((T, L, L^4)\) and one vector type \((\mathbf{L})\) gauge-transformation allow us to
gauge to zero three scalar and one vector variable. Instead of choosing a specific gauge
(like in the main text of this paper) we shall define gauge invariant combinations and
express the first order Einstein equations in terms of these. This is always possible as a
consequence of the Steward-Walker Lemma [35].

Under the above coordinate change, the geometrical perturbations transform
according to the Lie derivative of the background metric,

\[
A \rightarrow A + \partial_t T, \quad \mathcal{S}_i \rightarrow \mathcal{S}_i - \partial_t \mathbf{L}_i, \quad (A.6)
\]
\[
H \rightarrow H, \quad \mathcal{T}_i \rightarrow \mathcal{T}_i + \mathbf{L}_i, \quad (A.7)
\]
\[
H_{ij} \rightarrow H_{ij}, \quad B \rightarrow B - \partial_i L_4 + \partial_j T, \quad (A.8)
\]
\[
\bar{E}_i \rightarrow \bar{E}_i + \partial_y \mathbf{L}_i, \quad C \rightarrow C + \partial_y L_4, \quad (A.9)
\]
\[
S \rightarrow S + T - \partial_t L, \quad F \rightarrow F + L, \quad (A.10)
\]
\[
E \rightarrow E + \partial_y L + L_4. \quad (A.11)
\]

We can therefore define the following four scalar and two vector perturbation variables
which are gauge invariant,

\[
\Psi = A - \partial_t (S + \partial_t F), \quad (A.12)
\]
\[
\Phi = - H, \quad (A.13)
\]
\[
\mathcal{B} = B - \partial_y (S + \partial_t F) + \partial_t (E - \partial_y F), \quad (A.14)
\]
\[
\mathcal{C} = C - \partial_y (E - \partial_y F), \quad (A.15)
\]
\[
\Sigma_i = S_i + \partial_t \mathbf{F}_i, \quad (A.16)
\]
\[
\mathcal{E}_i = \mathbf{F}_i - \partial_y \mathbf{F}_i. \quad (A.17)
\]

The tensor variable \(H_{ij}\) is gauge invariant since there are no tensor type gauge
transformations.

The stress energy tensor defined in Eqs. (12) to (15) is a first order perturbation
with vanishing background component and it is therefore gauge invariant by itself. This
is a specialty of the Minkowski background. The much more complicated general gauge-
invariant variables for a generic evolving bulk can be found in Ref. [23].

From Eqs. (A.7) and (A.11) it is clear that the gauge is completely fixed by setting\(F = S = E = 0\) and \(\mathbf{F}_i = 0\). This ‘generalized longitudinal gauge’ is precisely the gauge
choice adopted in the main text of this paper.

We can now go on and compute the components of the curvature in terms of our
perturbation variables. In Ref. [23] this is done for the generic evolving case. Here we
repeat for convenience the expressions for the Christoffel symbols and for the Einstein
tensor in our simple case. In contrast to the generic situation, the Einstein tensor is
gauge invariant in for a Minkowski bulk.

**Christoffel symbols**

\[
\Gamma_{00}^0 = \partial_t A, \quad \Gamma_{0i}^0 = \partial_i A, \quad (A.18)
\]
Evolving bulk can be found in Ref. [23].

Over the Ricci tensor one can now compute also the Einstein tensor. This is a somewhat lengthy but straight forward calculation. Also here, the more complicated results for an evolving bulk can be found in Ref. [23].

Here $X(iY_j) = \frac{1}{2}(X_i Y_j + X_j Y_i)$ indicates symmetrization in the indices $i$ and $j$.

**Einstein tensor**

Over the Ricci tensor one can now compute also the Einstein tensor. This is a somewhat lengthy but straight forward calculation. Also here, the more complicated results for an evolving bulk can be found in Ref. [23].

\[
G_{00} = \triangle(-C + 2\Phi) + 3\partial_y^2\Phi, \\
G_{0i} = \partial_i[\partial_t(2\Phi - C) - \frac{1}{2}\partial_y B] + \frac{1}{2}\partial_y[\partial_y\Sigma_i + \partial_t\xi_i] + \frac{1}{2}\triangle\Sigma_i, \\
G_{ij} = (\partial_i\partial_j - \partial_j\partial_i)(\Psi - \Phi + C) + \delta_{ij}[2(\Psi - 2\Phi) - \partial^2_y(\Psi - 2\Phi) \\
- \partial_t\partial_y B] + \partial_t\partial_i\Sigma_j + \partial_y\partial_i\xi_j + (\partial^2_t - \triangle - \partial^2_y)H_{ij}, \\
G_{04} = \frac{1}{2}\triangle B + 3\partial_t\partial_y\Phi, \\
G_{i4} = \partial_i[\partial_y(2\Phi - \Psi) + \frac{1}{2}\partial_t B] + \frac{1}{2}(\partial_t\partial_y\Sigma_i + \partial^2_t\xi_i - \triangle\xi_i), \\
G_{44} = \triangle(\Psi - 2\Phi) + 3\partial^2_y\Phi.
\]

The source free Einstein equations (5) to (11) are now simply obtained by setting $G_{AB} = 0$. Integrating the five dimensional Einstein equation, $G_{AB} = 2M_5^{-3}T_{AB}$ over the extra dimension $y$, with the ansatz given in Eqs. (12) to (15) for the stress energy tensor one obtains the 4-dimensional Einstein equations given in Eqs. (16) to (25).

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