Two Groups in a Curie–Weiss Model with Heterogeneous Coupling

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Abstract
We discuss a Curie–Weiss model with two groups with different coupling constants within and between groups. For the total magnetisations in each group, we show bivariate laws of large numbers and a central limit theorem which is valid in the high-temperature regime. In the critical regime, the total magnetisation normalised by $N^{3/4}$ converges to a non-trivial distribution which is not Gaussian, just as in the single-group Curie–Weiss model. Finally, we prove a kind of a ‘law of large numbers’ in the low-temperature regime, more precisely we prove that the empirical magnetisation converges in distribution to a mixture of two Dirac measures.

Keywords Curie–Weiss · Central limit theorems · Multi-population models

Mathematics Subject Classification 60F05 · 82B20

1 Introduction
The Curie–Weiss model is probably the easiest model of magnetism which shows a phase transition between a paramagnetic and a ferromagnetic phase. In this model, the spins can take values in $\{-1, 1\}$ (or up/down), each spin interacts with all the others in the same way. More precisely, for finitely many spins $X := (X_1, X_2, \ldots, X_N) \in \{-1, 1\}$ the energy of the spins is given by

$$H = H(X_1, \ldots, X_N) := -\frac{1}{2N} (\sum_{j=1}^{N} X_j)^2. \quad (1)$$
The ‘Gibbs measure’ or ‘canonical ensemble’ with coupling constant ($\approx$ inverse temperature) $J_0 \geq 0$ the probability of a spin configuration is given by

$$
P(X_1 = x_1, \ldots, X_N = x_N) := Z^{-1} e^{-J_0 H(x_1, \ldots, x_N)}$$

(2)

where $x_i \in \{-1, 1\}$ and $Z$ is a normalisation constant which depends on $N$ and $J_0$.

The quantity

$$S_N = \sum_{j=1}^{N} X_j$$

(3)

is called the (total) magnetisation. It is well known (see e.g. Ellis [5] or [14]) that the Curie–Weiss model has a phase transition at $J_0 = 1$ in the following sense

$$\frac{1}{N} S_N \rightarrow \frac{1}{2} (\delta_{m(J_0)} + \delta_{m(J_0)})$$

(4)

where $\Rightarrow$ denotes convergence in distribution, $\delta_x$ the Dirac measure in $x$.

For $J_0 \leq 1$, we have $m(J_0) = 0$ which is the unique solution of

$$\tanh(J_0 x) = x$$

(5)

for this case.

If $J_0 > 1$ Eq. (5) has exactly three solutions and $m(J_0)$ is the unique positive one. Equation (4) is a substitute for the law of large numbers for i.i.d. random variables.

Moreover, for $J_0 < 1$ there is a central limit theorem, i.e.

$$\frac{1}{\sqrt{N}} S_N \Rightarrow N\left(0, \frac{1}{1-J_0}\right)$$

(6)

There is a huge amount of literature on the Curie–Weiss model. We can just mention a few papers here. The Curie–Weiss model is also called the Husimi–Temperley model. It was first introduced by Husimi [11] and Temperley [20]. Subsequently, it was discussed by Kac [13], Ellis–Newman [6,7]. It was treated in the textbook Thompson [21], and Ellis [5].

More recently, the Curie–Weiss model has been used in the context of social and political interactions. See e.g. [4,15], and [10].

In this paper, we consider two groups of Curie–Weiss spins $X = (X_1, \ldots, X_{N_1})$ and $Y = (Y_1, \ldots, Y_{N_2})$ with $N := N_1 + N_2$. The spins $X$ and $Y$ are Curie–Weiss spins with coupling constant $J_1$ and $J_2$ respectively, in addition there is a Curie–Weiss-type interaction between the $X_i$ and the $Y_j$ with coupling constant $\bar{J}$.

We set

$$J := \begin{pmatrix} J_1 & \bar{J} \\ \bar{J} & J_2 \end{pmatrix}$$

(7)
and assume that $J_1, J_2, \bar{J} > 0$ and
\[
\Delta := J_1 J_2 - \bar{J}^2 > 0,
\]
so that the matrix $J$ is positive definite. Loosely speaking, this condition ensures that the interaction within groups dominates the interaction between the groups.

The energy function is given by
\[
H = H_J(X, Y) := -\frac{1}{2N}\left[ J_1\left(\sum_{j=1}^{N_1} X_j\right)^2 + J_2\left(\sum_{j=1}^{N_2} Y_j\right)^2 + 2J\sum_{i=1}^{N_1}\sum_{j=1}^{N_2} X_i Y_j \right].
\]

We denote the Gibbs measure associated with $H_J(X, Y)$ by $\mathbb{P}_J$ (sometimes abbreviated by $\mathbb{P}$) defined by
\[
\mathbb{P}_J(A) := Z^{-1} \sum_{(X,Y) \in A} e^{-H_J(X,Y)}
\]
where $Z$ is a normalising constant which makes $\mathbb{P}_J$ a probability measure. The corresponding expectation is called $\mathbb{E}_J$, sometimes abbreviated $\mathbb{E}$.

By sending $N$ to infinity, we mean that both $N_1$ and $N_2$ tend to infinity. We set
\[
\alpha_1 := \lim_{N \to \infty} \frac{N_1}{N}, \quad \alpha_2 := \lim_{N \to \infty} \frac{N_2}{N} = 1 - \alpha_1
\]
and assume that these limits exist and $0 < \alpha_1 < 1$.

In this paper, we consider the asymptotic behaviour of the two-dimensional random variables
\[
\left( \frac{1}{N_1 \gamma} \sum_{i=1}^{N_1} X_i, \frac{1}{N_2 \gamma} \sum_{j=1}^{N_2} Y_j \right)
\]
where $\gamma = 1, \frac{1}{2}$ or $\frac{3}{4}$ depending on the parameters of the model, namely $J_1, J_2, \bar{J}$ and $\alpha_1, \alpha_2$.

Our assumptions on $J_1, J_2$ and $\bar{J}$ exclude a few ‘borderline’ cases. If $\bar{J} = 0$, the two groups are independent of each other and can therefore be treated as independent single-group Curie–Weiss spins. This is also the case if $J_1 = 0$ or $J_2 = 0$ as this implies $\bar{J} = 0$ by assumption (8).

Condition 8 also excludes the (interesting) case $J = \left( \begin{smallmatrix} J_0 & 0 \\ 0 & J_0 \end{smallmatrix} \right)$. This case which we call the homogeneous one requires a somewhat different technique. It is treated in [16].

Another borderline case is given by $\alpha_1 = 0$ or $\alpha_2 = 0$ (assuming still that both $N_1$ and $N_2$ tend to infinity). We may even consider the following extension of our model: The groups may consist of $\tilde{N}_i \approx \rho_i N$, but the averages in (12) are taken over $N_i \approx \alpha_i N \leq \tilde{N}_i$. 
These cases can be treated by the techniques of this paper as well. With the obvious changes, the results and their proofs remain valid for these extensions. In order to avoid a notational overkill, we stick to the stronger assumptions made.

### 1.1 High-Temperature Regime

The parameter space of this model is

$$\Phi := \{(J_1, J_2, \bar{J}, \alpha_1, \alpha_2) \in (0, \infty)^5 | J_1 J_2 > \bar{J}^2, \alpha_1 + \alpha_2 = 1\}.$$

For the single-group model, the high-temperature regime is quite simply expressed by the single condition $J_0 < 1$. For two groups with a heterogeneous coupling matrix, we have a very different situation: each within-group coupling constant $J_\nu$ has to be small in relation to the reciprocal of the group’s size. Once the within-group couplings have been chosen, the between-groups coupling has to be small, too. How small depends on how close the other two couplings are to the reciprocals of the group sizes. If the within-group couplings are very small, that leaves more leeway for the between-groups coupling to be larger.

We shall assume that the interactions satisfy

$$J_1 < \frac{1}{\alpha_1},$$

$$J_2 < \frac{1}{\alpha_2},$$

$$\bar{J}^2 < \left(\frac{1}{\alpha_1} - J_1\right) \left(\frac{1}{\alpha_2} - J_2\right).$$

and refer to these conditions as the ‘high-temperature regime’, and we shall also refer to the subset $\Phi_h$ of $\Phi$ where these conditions hold by the same name. Note that if we use the symbol $\alpha$ for the diagonal $2 \times 2$ matrix with entries $\alpha_1$ and $\alpha_2$, we can formulate these conditions equivalently in matrix form: the matrix

$$J^{-1} - \alpha$$

is positive definite if and only if we are in the high-temperature regime (see Proposition 11).

We prove a ‘law of large numbers’.

**Theorem 1** In the high-temperature regime, we have

$$\left(\frac{1}{N_1} \sum_{i=1}^{N_1} X_i, \frac{1}{N_2} \sum_{j=1}^{N_2} Y_j\right) \quad \overset{N \to \infty}{\Longrightarrow} \quad (0, 0).$$

Above ‘$\overset{\rightarrow}{\Longrightarrow}$’ denotes convergence in distribution of the 2-dimensional random variable on the left hand side.
We also have a ‘central limit theorem’. Using \( \alpha = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix} \) we define the matrix
\[
C = 1 + \left( \alpha^{-1/2} J^{-1} \alpha^{-1/2} - 1 \right)^{-1}
\]  
(16)
where 1 denotes the identity matrix.

**Theorem 2** *In the high-temperature regime, we have*
\[
\left( \frac{1}{\sqrt{N_1}} \sum_{i=1}^{N_1} X_i, \frac{1}{\sqrt{N_2}} \sum_{j=1}^{N_2} Y_j \right) \overset{N \to \infty}{\Rightarrow} \mathcal{N}((0, 0), C),
\]  
(17)

*The covariance matrix C (as in (16)) is given by*
\[
C = \frac{1}{(1 - \alpha_1 J_1)(1 - \alpha_2 J_2) - \alpha_1 \alpha_2 \bar{J}} \left[ 1 - \alpha_2 J_2 \sqrt{\alpha_1 \alpha_2 \bar{J}} 1 - \alpha_1 J_1 \right].
\]  
(18)

**Remark 3** *Theorem 2 implies that also expressions like \( \frac{1}{\sqrt{N_1}} \sum_{i=1}^{N_1} X_i \pm \frac{1}{\sqrt{N_2}} \sum_{j=1}^{N_2} Y_j \) are asymptotically Gaussian distributed.*

### 1.2 Critical Regime

The critical regime is where \( (\sum_{i=1}^{N_1} X_i, \sum_{j=1}^{N_2} Y_j) \) abruptly changes behaviour. In the single-group model, this occurs at \( J_0 = 1 \). For two groups with a heterogeneous coupling matrix, in the critical regime, each within-group coupling constant \( J_i \) has to be small in relation to the reciprocal of the group’s size. Once the within-group couplings have been chosen, the between-groups coupling has to have an exact magnitude, which is larger than in the high-temperature regime:
\[
J_1 < \frac{1}{\alpha_1},
\]  
(19)
\[
J_2 < \frac{1}{\alpha_2},
\]  
(20)
\[
\bar{J}^2 = \left( \frac{1}{\alpha_1} - J_1 \right) \left( \frac{1}{\alpha_2} - J_2 \right).
\]  
(21)

We shall call the subset of \( \Phi \) where these conditions hold \( \Phi_c \) and we can also formulate these conditions equivalently in matrix form: the matrix
\[
J^{-1} - \alpha
\]

is singular and has positive diagonal entries if and only if we are in the critical regime.

We also note that if \( J_1 = \frac{1}{\alpha_1} \) or \( J_2 = \frac{1}{\alpha_2} \) then (21) implies \( \bar{J} = 0 \); hence, the two groups are independent of each other and can be treated as in the single-group case.
In the critical regime, we consider here (i.e. for (19)–(21)) the law of large numbers, Theorem 1, still holds, but the central limit theorem, Theorem 2, has to be replaced by a theorem describing the asymptotic behaviour of

\[ T_N = \left( \frac{1}{N_1^{3/4}} S_{N_1}^{(1)}, \frac{1}{N_2^{3/4}} S_{N_2}^{(2)} \right). \] (22)

This sequence \( T_N \) converges in distribution but not to a normal distribution. We state the moments of the limiting measure in Theorem 5.

The critical regime results are:

**Theorem 4** *In the critical regime, we have*

\[ \left( \frac{1}{N_1} \sum_{i=1}^{N_1} X_i, \frac{1}{N_2} \sum_{j=1}^{N_2} Y_j \right) \xrightarrow{N \to \infty} \delta_{(0,0)}. \]

If we choose as normalising factors \( N_0^{3/4} \) instead of \( N_0 \), then we obtain

**Theorem 5** *In the critical regime, the random variables*

\[ \left( \frac{1}{N_1^{3/4}} \sum_{i=1}^{N_1} X_i, \frac{1}{N_2^{3/4}} \sum_{j=1}^{N_2} Y_j \right) \]

*converge in distribution to a measure \( \mu \) (on \( \mathbb{R}^2 \)) with moments*

\[
m_{K,Q} := \int x^K y^Q d\mu(x, y) = \left[ \alpha_1 (L_2 - \alpha_2)^2 + \alpha_2 (L_1 - \alpha_1)^2 \right]^{\frac{K+Q}{4}} (L_1 - \alpha_1)^{\frac{Q}{2}} (L_2 - \alpha_2)^{\frac{K}{2}} \cdot \frac{\Gamma \left( \frac{K+Q+1}{4} \right)}{\Gamma \left( \frac{1}{4} \right)} \alpha_1^\frac{K}{4} \alpha_2^\frac{Q}{4}. \tag{23}
\]

where the matrix \( L \) is defined by \( L = \begin{pmatrix} L_1 & -\tilde{L} \\ -\tilde{L} & L_2 \end{pmatrix} = J^{-1} \)

### 1.3 Low-Temperature Regime

In the single-group model, the low-temperature regime is characterised by the inequality \( J_0 > 1 \). The magnetisation \( \frac{1}{N} \sum X_i \) converges in distribution to the measure \( \frac{1}{2} (\delta_{m^*} + \delta_{-m^*}) \) where \( m^* \) is the unique positive solution of the equation \( m = \tanh(J_0 m) \). We regard this fact as a (substitute for the) law of large numbers.

In the case of two groups, we define the low-temperature regime to be the complement \( \Phi \setminus (\Phi_h \cup \Phi_c) \) in the parameter space. We have a similar ‘law of large numbers’ in this case.
Theorem 6 In the low-temperature regime, there are exactly two nonzero solutions $m^* = (m^*_1, m^*_2)$ and $-m^*$ of the system

$$m_1 = \tanh(J_1 \alpha_1 m_1 - \tilde{J} \alpha_2 m_2)$$  \hspace{1cm} (24)

and $m_2 = \tanh(J_2 \alpha_2 m_2 - \tilde{J} \alpha_1 m_1).$  \hspace{1cm} (25)

We have

$$\left( \frac{1}{N_1} \sum_{i=1}^{N_1} X_i, \frac{1}{N_2} \sum_{j=1}^{N_2} Y_j \right) \Rightarrow_{N \to \infty} \frac{1}{2} (\delta_{m^*} + \delta_{-m^*}).$$  \hspace{1cm} (26)

Moreover, we may assume $m^*_1 > 0$ and $m^*_2 > 0.$

Obviously, in the low-temperature case, there can be no central limit theorem in the sense that $\left( \frac{1}{N_1} \sum_{i=1}^{N_1} X_i, \frac{1}{N_2} \sum_{j=1}^{N_2} Y_j \right)$ converges to a non-trivial limit measure. However, we conjecture that there is a ‘conditional version’ of a central limit theorem.

2 A Rough Sketch of the Proofs

Our proofs are based on the method of moments, the basis of which is the following well-known Theorem (see e. g. [1]).

Theorem 7 Suppose that $\mu_N$ and $\mu$ are probability measure on $\mathbb{R}^d$ for which all moments are finite and assume that $\mu$ is determined by its moments $m_k(\mu).$ If $m_k(\mu_N) \to m_k(\mu)$ for all $k \in \mathbb{N}^d$ as $N \to \infty$ then the measures $\mu_N$ converge weakly to $\mu.$

It is also known that all (multidimensional) normal distributions are determined by their moments (see e. g. [17]).

Thus, we will consider suitably normalised moments of the form

$$M_{K, Q} := \mathbb{E}_J \left( \left( \sum_{i=1}^{N_1} X_i \right)^K \left( \sum_{j=1}^{N_2} Y_j \right)^Q \right)$$

$$= \sum_{i_1, j_2, \ldots, i_K, j_1, j_2, \ldots, j_Q} \mathbb{E}_J \left( X_{i_1} \cdots X_{i_K} \cdot Y_{j_1} \cdots Y_{j_Q} \right)$$

$$= \sum_{i \in \mathcal{N}_1^K} \sum_{j \in \mathcal{N}_2^Q} \mathbb{E}_J \left( X(i) \cdot Y(j) \right)$$  \hspace{1cm} (27)

where $i = (i_1, \ldots, i_K), \ j = (j_1, \ldots, j_Q), \ \mathcal{N}_1 = \{1, 2, \ldots, N_1\}, \ \mathcal{N}_2 = \{1, 2, \ldots, N_2\}$ and $X(i) = \prod_{v=1}^{K} X_{i_v}.$
Since \( X_i^2 = Y_j^2 = 1 \) and due to exchangeability we have

\[
\mathbb{E}_J\left(X_{i_1} \cdot \ldots \cdot X_{i_{\tilde{K}}} \cdot Y_{j_1} \cdot \ldots \cdot Y_{j_{\tilde{Q}}}ight) = \mathbb{E}_J\left(X_1 \cdot \ldots \cdot X_{\tilde{K}} \cdot Y_1 \cdot \ldots \cdot Y_{\tilde{Q}}\right)
\]

(28)

where \( \tilde{K} \) (resp. \( \tilde{Q} \)) is the number of \( i_k \) (resp. \( j_\ell \)) which occur an odd number of times.

In order to evaluate the moments in (27), we need to estimate correlations as in (28).

In Sect. 3, we will prove asymptotic estimates for the correlations. For example in Sect. 3.3, we show that in the high-temperature regime

\[
\left| \mathbb{E}_J\left(X_1 \cdot \ldots \cdot X_{K} \cdot Y_1 \cdot \ldots \cdot Y_{Q}\right) \right| \leq c_{K,Q} \frac{1}{N^{(K+Q)/2}}
\]

(29)

where \( c_{K,Q} \) depends on the matrix \( J \) and the numbers \( \alpha_1 \) and \( \alpha_2 \) but not on \( N \). Note that in the case of independent random variables, i.e. if \( J = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \) correlation (29) is zero unless \( K = Q = 0 \).

In a second step of the proof, we need a ‘bookkeeping’ method, to keep track of the variety of terms in sum (27). We have to count the number of simple, double, triple, etc., occurrences of the \( X_i \) and \( Y_j \) in (27). We start the discussion of this topic with Lemma 22. A more precise discussion is given in Sect. 5.1. In Sect. 4, we prove the laws of large numbers, Theorems 1 and 4. The central limit theorem is proved in 5.3 combining the results on the correlations and the bookkeeping method.

### 3 Computing Expectations

In this section, we compute expectations (= correlations) of the form

\[
\mathbb{E}_J\left(X_1 \cdot X_2 \cdot \ldots \cdot X_K \cdot Y_1 \cdot Y_2 \cdot \ldots \cdot Y_Q\right)
\]

(30)

asymptotically for the three regimes of \( J \).

#### 3.1 A Two-Dimensional Hubbard–Stratonovich Transform

For any configuration of the spins

\[
(X, Y) = (X_1, X_2, \ldots, X_{N_1}, Y_1, Y_2, \ldots, Y_{N_2})
\]

(31)

we set

\[
S_1 = \sum_{i=1}^{N_1} X_i \quad S_2 = \sum_{j=1}^{N_2} Y_j
\]

(32)
and \( S = \begin{pmatrix} S_1 \\ S_2 \end{pmatrix} \) \( (33) \)

and define the function

\[
h(S_1, S_2) = \frac{1}{2N} (S_1, S_2) J \begin{pmatrix} S_1 \\ S_2 \end{pmatrix} = \frac{1}{2N} S' JS,
\]

For a symmetric positive definite \( 2 \times 2 \) matrix \( A \) and a point \( x_0 \in \mathbb{R}^2 \), we can use the following equality to express a value of the exponential function as an integral:

\[
e^{-\frac{x'Ax}{2}} = \frac{\sqrt{\det A}}{2\pi} \int_{\mathbb{R}^2} e^{-\frac{x'Ax}{2}} e^{-x'Ax_0} dx.
\]

According to this equality,

\[
e^{h(S_1, S_2)} = e^{\frac{S'JS}{2N}} = c \int_{\mathbb{R}^2} e^{-\frac{1}{2N} J^x x} e^{-\frac{1}{N} x'JS} dx,
\]

where \( c = \frac{\sqrt{\det J}}{2\pi} \).

We define the inverse matrix

\[
L = \begin{bmatrix} L_1 & -\bar{L} \\ -\bar{L} & L_2 \end{bmatrix} = \frac{1}{J_1 J_2 - \bar{J}^2} \begin{bmatrix} J_2 & -\bar{J} \\ -\bar{J} & J_1 \end{bmatrix} = J^{-1}.
\]

(34)

Switching variables \( y = \frac{1}{N} Jx \), we obtain

\[
e^{h(S_1, S_2)} = c' \int_{\mathbb{R}^2} e^{-\frac{N}{2} y' Ly} e^{S'y} d^2y,
\]

where \( c' \) is a term that depends on the matrix \( L \) and on \( N \). Equation (35) is our two-dimensional version of the Hubbard–Stratonovich transform.

Summing over all \((X, Y) \in \{-1, +1\}^N\), we obtain

\[
\sum_{X, Y} e^{S'y} = (e^{+y_1} + e^{-y_1})^{N_1} \cdot (e^{+y_2} + e^{-y_2})^{N_2}
\]

\[
= 2^N \cosh^{N_1}(y_1) \cdot \cosh^{N_2}(y_2)
\]

(36)
\[
\sum_{X,Y} X_1 \ldots X_K \cdot Y_1 \ldots Y_Q \ e^{S_{Y}} = \frac{(e^{+y_1} - e^{-y_1})^K}{(e^{+y_1} + e^{-y_1})^K} \cdot \frac{(e^{+y_2} - e^{-y_2})^Q}{(e^{+y_2} + e^{-y_2})^Q} \ e^{+y_2 + e^{-y_2}} \ N_2 \\
= 2^N \ \tanh^K (y_1) \ \tanh^Q (y_2) \ \cosh^{N_1}(y_1) \ \cosh^{N_2}(y_2)
\]

Consequently, we have
\[
\sum_{X,Y} X_1 \ldots X_K \cdot Y_1 \ldots Y_Q \ e^{-H_J(X,Y)}
\]
\[
= c \int e^{-N(1/2 \cdot L \cdot y - N_1/N \ln \cosh y_1 - N_2/N \ln \cosh y_2)} \ \tanh^K y_1 \ \tanh^Q y_2 \ d^2y
\]
\[
= c \int e^{-N F_J(y)} \ \tanh^K y_1 \ \tanh^Q y_2 \ d^2y.
\]

where
\[
F_J(y) := \frac{1}{2} L_1 y_1^2 + \frac{1}{2} L_2 y_2^2 - \bar{L} y_1 y_2 - \alpha_1 \ln \cosh y_1 - \alpha_2 \ln \cosh y_2.
\]

Let us define
\[
Z_J(K, Q) := \int e^{-N F_J(y)} \ \tanh^K y_1 \ \tanh^Q y_2 \ d^2y.
\]

then
\[
\mathbb{E}_J(X_1 \cdot X_2 \ldots \cdot X_K \cdot Y_1 \cdot Y_2 \ldots \cdot Y_Q) = \frac{Z_J(K, Q)}{Z_J(0, 0)}
\]

Thus, if we can compute \(Z_J(K, Q)\) asymptotically we will be able to compute correlations (30).

3.2 Extrema of the Function \(F\)

3.2.1 High-Temperature Regime

We are going to apply the Laplace method to evaluate the quantities \(Z_J(K, Q)\). In order to do so, we need to determine the minima of the function
\[
F(y_1, y_2) = \frac{1}{2} L_1 y_1^2 + \frac{1}{2} L_2 y_2^2 - \bar{L} y_1 y_2 - \alpha_1 \ln \cosh y_1 - \alpha_2 \ln \cosh y_2.
\]
Proposition 8 If

\[ L_1 > \alpha_1, \quad (L_1 - \alpha_1)(L_2 - \alpha_2) > \bar{L}^2, \]

then the function \( F \) has a unique minimum at \((0, 0)\).

\( F \) has strictly positive definite Hessian

\[ H = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \]

and is therefore strictly convex.

We used \( F_{ij} \) to denote the partial derivative of \( F \) with respect to \( y_i \) and \( y_j \).

Remark 9 Conditions (43) and (44) are equivalent to the high-temperature regime, as we shall show in Proposition 11.

Proof We take derivatives with respect to both variables

\[ F_1(y_1, y_2) = L_1 y_1 - \bar{L} y_2 - \alpha_1 \tanh y_1, \]
\[ F_2(y_1, y_2) = L_2 y_2 - \bar{L} y_1 - \alpha_2 \tanh y_2, \]
\[ F_{11}(y_1, y_2) = L_1 - \frac{\alpha_1}{\cosh^2 y_1}, \]
\[ F_{22}(y_1, y_2) = L_2 - \frac{\alpha_2}{\cosh^2 y_2}. \] (46)

The Hessian matrix of \( F \) is

\[ H = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} = \begin{bmatrix} L_1 - \frac{\alpha_1}{\cosh^2 y_1} & -\bar{L} \\ -\bar{L} & L_2 - \frac{\alpha_2}{\cosh^2 y_2} \end{bmatrix}. \]

One solution to the first-order conditions (45) and (46) is \( y_1 = y_2 = 0 \).

The matrix \( H \) is positive definite at the origin if and only if (43) and (44) hold. Hence, there is a local minimum at the origin. If the Hessian matrix at the origin is positive definite, it is also positive definite at any other point due to \( \cosh |s| > \cosh |t| \) for all \( |s| > |t| \). Thus, \( F \) is strictly convex and it follows that the minimum is unique and global.

Lemma 10

\[ \text{sgn} \left( \det \left( \frac{1}{\alpha_1} - J_1 \frac{\tilde{J}}{\tilde{J}} \frac{1}{\alpha_2} - J_2 \right) \right) = \text{sgn} \left( \det \left( L_1 - \frac{\alpha_1}{\cosh^2 y_1} - \bar{L} \\ L_2 - \frac{\alpha_2}{\cosh^2 y_2} \right) \right) \] (47)
We used the notation

\[
\text{sgn} (x) = \begin{cases} 
1, & x > 0; \\
0, & x = 0; \\
-1, & x < 0.
\end{cases}
\]

**Proof** We write

\[
\alpha^{-1} - J = J(J^{-1} - \alpha)\alpha^{-1}
\]

Since \( \det(J), \det(\alpha) > 0 \) the assertion follows. \( \square \)

**Proposition 11** The conditions on the Hessian matrix \( H \) given in Proposition 8 are equivalent to the following conditions on the coupling matrix \( J \):

\[
J_1 < \frac{1}{\alpha_1}, \tag{48}
\]

\[
J_2 < \frac{1}{\alpha_2}, \tag{49}
\]

\[
\bar{J}^2 < \left( \frac{1}{\alpha_1} - J_1 \right) \left( \frac{1}{\alpha_2} - J_2 \right). \tag{50}
\]

**Proof** The equivalence of (44) and (50) is the contents of Lemma 10.

From either (44) or (50), it follows that \( L_1 - \alpha_1 \) and \( L_2 - \alpha_2 \) have the same sign and also that \( \frac{1}{\alpha_1} - J_1 \) and \( \frac{1}{\alpha_2} - J_2 \) have the same sign.

A straightforward calculation shows

\[
L_1 > \alpha_1 \iff J_1 - \frac{1}{\alpha_1} < \frac{\bar{J}^2}{J_1} \tag{51}
\]

and

\[
L_2 > \alpha_2 \iff J_2 - \frac{1}{\alpha_2} < \frac{\bar{J}^2}{J_1}. \tag{52}
\]

Now suppose (43) and (44) hold. If \( J_1 - \frac{1}{\alpha_1} > 0 \) holds, then also \( J_2 - \frac{1}{\alpha_2} > 0 \). Consequently, (51) and (52) imply that

\[
\left( J_1 - \frac{1}{\alpha_1} \right) \left( J_2 - \frac{1}{\alpha_2} \right) < \frac{\bar{J}^4}{J_1 J_2} < \bar{J}^2, \tag{53}
\]

but this contradicts (50). Thus, \( \frac{1}{\alpha_1} - J_1 > 0 \).

If, on the other hand, \( \frac{1}{\alpha_1} - J_1 > 0 \) then (51) implies \( L_1 > \alpha_1 \). \( \square \)
3.2.2 Critical Regime

We turn to the critical regime:

**Proposition 12** If \( L_\nu - \alpha_\nu > 0 \) for both groups and \((L_1 - \alpha_1)(L_2 - \alpha_2) = \bar{L}^2\), then the function \( F \) defined in (42) has a unique global minimum at the origin.

**Remark 13** The conditions stated in the proposition are equivalent to the critical regime. This is shown in analogous fashion to the proof of Proposition 11.

**Proof** We take derivatives of \( F \) with respect to both variables

\[
F_1(y_1, y_2) = L_1y_1 - \bar{L}y_2 - \alpha_1 \tanh y_1 = 0, \\
F_2(y_1, y_2) = L_2y_2 - \bar{L}y_1 - \alpha_2 \tanh y_2 = 0.
\]

One solution to this system of equations is \( y_1 = y_2 = 0 \). We proceed to show that this solution is unique. We rewrite the function \( F \):

\[
F(tx_0, ty_0) = \frac{1}{2}L_1t^2x_0^2 + \frac{1}{2}L_2t^2y_0^2 - \bar{L}x_0y_0t^2 - \alpha_1 \ln \cosh tx_0 - \alpha_2 \ln \cosh ty_0,
\]

where \((x_0, y_0)\) indicates the direction, \(x_0^2 + y_0^2 = 1\), and \( t \) is the distance from the origin. The first derivative of \( F \) with respect to \( t \) is 0 at the origin, independently of the direction \((x_0, y_0)\).

We show that the second derivative \( \frac{d^2F(tx_0, ty_0)}{dt^2} \) is positive in all directions, except for two.

\[
\frac{d^2F(tx_0, ty_0)}{dt^2} = L_1x_0^2 + L_2y_0^2 - 2\bar{L}x_0y_0 - \frac{\alpha_1x_0^2}{\cosh^2 tx_0} - \frac{\alpha_2y_0^2}{\cosh^2 ty_0}.
\]

Therefore, we have

\[
\left. \frac{d^2F(tx_0, ty_0)}{dt^2} \right|_{t=0} \geq 0
\]

with equality if and only if both \( t = 0 \) and

\[
\sqrt{L_1 - \alpha_1x_0} - \sqrt{L_2 - \alpha_2y_0} = 0
\]

hold.

Hence, there are two directions \((x_0, y_0)\), one pointing into quadrant one, the other into quadrant three, in which the second derivative is 0 at the origin. In all other directions, the second derivative is strictly positive. For any direction, the second derivative is strictly positive for all \( t > 0 \).

This concludes the proof that the minimum at the origin is unique and global. \( \square \)
3.2.3 Low-Temperature Regime

Assume we are in the low-temperature regime, i.e. at least one of the following conditions holds:

\[ J_1 > \frac{1}{\alpha_1}, \]  
\[ J_2 > \frac{1}{\alpha_2}, \]  
\[ J^2 > \left( \frac{1}{\alpha_1} - J_1 \right) \left( \frac{1}{\alpha_2} - J_2 \right). \]  

In terms of the inverse matrix \( L = J^{-1} \), at least one of the following inequalities has to hold:

\[ L_1 \leq \alpha_1, \]  
\[ L_2 \leq \alpha_2, \]  
\[ (L_1 - \alpha_1)(L_2 - \alpha_2) < \tilde{L}^2. \]  

In order to apply Laplace’s method, we need to determine the minima of the function

\[ F(x, y) = \frac{1}{2} L_1 x^2 + \frac{1}{2} L_2 y^2 - \tilde{L} x y - \alpha_1 \ln \cosh x - \alpha_2 \ln \cosh y. \]  

The first-order conditions are

\[ F_1(x, y) = L_1 x - \tilde{L} y - \alpha_1 \tanh x = 0, \]  
\[ F_2(x, y) = L_2 y - \tilde{L} x - \alpha_2 \tanh y = 0. \]  

These equations always have a solution \((x, y) = (0, 0)\). We define functions \( X : [0, \infty) \rightarrow [0, \infty) \) and \( Y : [0, \infty) \rightarrow [0, \infty) \) by setting \( X(y) \) equal to the largest solution \( x \) of Eq. (61) given a value \( y \geq 0 \). Similarly, \( Y(x) \) is defined as the largest solution \( y \) of (62) given \( x \geq 0 \).

**Proposition 14** The functions \( X \) and \( Y \) are strictly increasing and strictly concave.

**Proof** We show the properties for \( X \). By the implicit function theorem, we can calculate the first derivative of the function \( X \) by dividing the partial derivative of the function

\[ G(x, y) := L_1 x - \tilde{L} y - \alpha_1 \tanh x \]

with respect to \( x \) by the partial derivative of \( G \) with respect to \( y \). That yields

\[ X'(y) = \frac{\tilde{L}}{L_1 - \frac{\alpha_1}{\cosh^2 X(y)}}. \]
We show that $X'$ is always positive. We define two auxiliary functions

$$f, g : [0, \infty) \rightarrow [0, \infty),$$

$$f(x) := L_1 x - \bar{L} y,$$

$$g(x) := \alpha_1 \tanh x.$$

We are looking for the intersections of the functions $f$ and $g$ given a value of $y \geq 0$, the largest of which is precisely the value $X(y)$. If $y = 0$ and $L_1 - \alpha_1 \geq 0$, $f$ and $g$ only intersect at 0, so $X(0) = 0$ in this case. If $L_1 - \alpha_1 < 0$, or $y > 0$ holds, then $f(0) = -\bar{L} y \leq 0$, $g(0) = 0$, and $f'(0) < g'(0)$, so at the origin $f(0) \leq g(0)$, and for small values of $x$ $f(x) < g(x)$. However, whereas $f'(x) = L_1 > 0$ is constant, $g'(x) = \frac{-\alpha_1}{\cosh^2 x} > 0$ is strictly decreasing in $x$ and $\lim_{x \to \infty} g'(x) = 0$. Therefore, there is exactly one value $x_1 > 0$ such that $f(x_1) = g(x_1)$. Since for $x < x_1$ $f(x) < g(x)$, it must be that $f'(x_1) > g'(x_1)$. This $x_1$ is $X(y)$. Hence, we have

$$L_1 > \frac{\alpha_1}{\cosh^2 X(y)},$$

and $X'(y) > 0$ has been shown.

The second derivative of $X$ is

$$X''(y) = -\frac{2\alpha_1 \bar{L} X'(y)}{\left(L_1 - \frac{\alpha_1}{\cosh^2 X(y)}\right)^2 \cosh^3 X(y)} < 0,$$

and so $X$ is strictly concave.

\[ \square \]

**Proposition 15** The limits of the first derivatives of $X$ and $Y$ are

$$\lim_{y \to \infty} X'(y) = \frac{\bar{L}}{L_1} > 0,$$

$$\lim_{x \to \infty} Y'(x) = \frac{\bar{L}}{L_2} > 0.$$

**Proof** We have

$$\lim_{y \to \infty} X'(y) = \frac{\bar{L}}{L_1 - \frac{\alpha_1}{\cosh^2 X(y)}} = \frac{\bar{L}}{L_1}$$

due to $\frac{\alpha_1}{\cosh^2 X(y)} \to 0$ as $X(y) \to \infty$. On the other hand, as $y$ goes to infinity, the solution $X(y)$ of (61) has to go to infinity due to the boundedness of the term $\alpha_1 \tanh x$. 

\[ \square \]
Corollary 16

\[
\lim_{y \to \infty} X'(y) \lim_{x \to \infty} Y'(x) = \frac{\bar{L}^2}{L_1 L_2} < 1
\]  

(63)

holds.

Proof By assumption, \(J\), and therefore \(L = J^{-1}\), are positive definite. In particular, the determinant of \(L\) must be positive. Hence

\[L_1 L_2 - \bar{L}^2 > 0.\]

\(\square\)

We define the curves \(\gamma_1, \gamma_2 : [0, \infty) \to \mathbb{R}^2\) by setting

\[
\gamma_1 (y) := (X(y), y),
\]

\[
\gamma_2 (x) := (x, Y(x)).
\]

These curves originate at a certain point in the first quadrant that depends on the parameters \(L_1, L_2, \alpha_1, \alpha_2\):

\[
\gamma_1 (0) := \begin{cases} 
(0, 0), \quad L_1 - \alpha_1 \geq 0, \\
(X(0), 0), \quad L_1 - \alpha_1 < 0,
\end{cases}
\]

\[
\gamma_2 (x) := \begin{cases} 
(0, 0), \quad L_2 - \alpha_2 \geq 0, \\
(0, Y(0)), \quad L_2 - \alpha_2 < 0.
\end{cases}
\]

So curve \(\gamma_1\) starts at the origin if and only if \(L_1 - \alpha_1 \geq 0\). Similarly, \(\gamma_2\) starts at the origin if and only if \(L_1 - \alpha_1 \geq 0\).

Let us first assume the two curves do not meet at the origin. Then, they start at points apart, but due to (63), we have

\[
\lim_{x \to \infty} Y'(x) < \frac{1}{\lim_{y \to \infty} X'(y)}.
\]

This implies the curves have to meet at some point in the interior of the first quadrant. Call this point \((x_1, y_1)\). Once they have met, the strict concavity of both \(X\) and \(Y\) drives them apart and they do not intersect again. Hence, the point \((x_1, y_1)\) is uniquely determined.

If the two curves do meet at the origin, we have \(L_1 - \alpha_1 \geq 0\) and \(L_2 - \alpha_2 \geq 0\). We distinguish the two cases

1. \(L_1 - \alpha_1 > 0\) and \(L_2 - \alpha_2 > 0\),
2. \(L_1 - \alpha_1 = 0\) or \(L_2 - \alpha_2 = 0\).
In the first case, since we are in the low-temperature regime,
\[(L_1 - \alpha_1)(L_2 - \alpha_2) < \tilde{L}^2\]
must hold, and we have
\[X'(0)Y'(0) = \frac{\tilde{L}}{L_1 - \alpha_1} \frac{\tilde{L}}{L_2 - \alpha_2} > 1.\]

So
\[Y'(0) > \frac{1}{X'(0)},\]
which means that the two curves starting at \((0, 0)\) initially move apart. So for some points close to the origin, the curves are apart, and the previous reasoning for the existence of a unique point of intersection inside the first quadrant applies.

In the second case, if \(L_1 - \alpha_1 = 0\), then the derivative of function \(X\) is infinite at the origin, meaning \(\gamma_1\) moves parallel to the \(x\)-axis. The function \(Y\) on the other hand either has positive or infinite derivative at the origin, so \(\gamma_2\) either moves into the interior of the first quadrant or it moves parallel to the \(y\)-axis. In any case, the two curves move apart after leaving the origin.

We summarise:

**Theorem 17** In the low-temperature regime, the function \(F\) has exactly two minima \(\mu^* = (\mu_1^*, \mu_2^*)\) and \(-\mu^*\) and we may suppose that \(\mu_1^*, \mu_2^* > 0\).

### 3.3 Correlations for the High-Temperature Regime

In this section, we use the Laplace method to evaluate the expression \(Z_J(K, Q)\) and thus correlation (30) asymptotically in the high-temperature regime.

Let \(H = J^{-1} - \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}\) be the Hessian of \(F\) at 0, \(\mathcal{N}(0, H^{-1})\) the two-dimensional normal distribution with covariance matrix \(H^{-1}\) and let
\[\nu_{K, Q} = \nu_{K, Q}(0, H^{-1}) = \sqrt{\text{det} H} \left( x_1 x_2 ight)_{x_1 K x_2 Q \text{d}x_1 \text{d}x_2} 2\pi \int_{\mathbb{R}^2} e^{-\frac{1}{2} \left( x_1 x_2 \right)_{x_1 K x_2 Q \text{d}x_1 \text{d}x_2}} \]
be the moments of \(\mathcal{N}(0, H^{-1})\).

In the following proposition as in the whole paper by \(a_N \approx b_N\) we mean \(\lim_{N \to \infty} \frac{a_N}{b_N} = 1\).
Proposition 18 Let \( J, \alpha_1, \alpha_2 \) satisfy (13)–(15) (high-temperature regime), then

\[
Z_J(K, Q) = \int e^{-NF_J(y)} \tanh^K y_1 \tanh^Q y_2 \, d^2y \approx \frac{2\pi}{\sqrt{\det H}} v_{K, Q}(0, H^{-1}) N^{-\frac{K+Q}{2} - 1} \quad \text{as } N \to \infty. \tag{65}
\]

**Proof** We use the Laplace method to evaluate \( Z_J(K, Q) \). We only sketch the main idea. For the details, in particular the remainder estimates, we refer to [19] or [14]. In the integral, we replace both \( F \) and the \( \tanh \) terms by the leading terms in their Taylor expansion around 0. This gives

\[
Z_J(K, Q) \approx \int e^{-\frac{1}{2}(y'Hy)} \frac{K}{2} y_2^Q \, dy_1 dy_2 = \frac{2\pi}{\sqrt{\det H}} v_{K, Q}(0, H^{-1}) N^{-\frac{K+Q}{2} - 1}
\]

where we changed variable \( x = \sqrt{N} y \). \( \Box \)

Proposition 18 immediately gives:

Theorem 19 Let \( J, \alpha_1, \alpha_2 \) be in the high-temperature regime ((13)–(15) then

\[
\mathbb{E}(X_1 \cdot X_2 \cdot \ldots \cdot X_K \cdot Y_1 \cdot Y_2 \cdot \ldots \cdot Y_Q) \approx v_{K, Q}(0, H^{-1}) N^{-(K+Q)/2} \tag{66}
\]

### 3.4 Correlations for the Critical Regime

Expanding again \( F \) to leading order gives in the critical regime

\[
F(y_1, y_2) \approx \frac{1}{2} \left( (L_1 - \alpha_1)y_1^2 + (L_2 - \alpha_2)y_2^2 - 2(L_1 - \alpha_1)y_1 - 2L_2 y_2 + \frac{2\alpha_1}{12} y_1^4 + \frac{2\alpha_2}{12} y_2^4 \right)
\]

\[
= \frac{1}{2} \left( \left( \sqrt{L_1 - \alpha_1} y_1 - \sqrt{L_2 - \alpha_2} y_2 \right)^2 + \frac{\alpha_1}{6} y_1^4 + \frac{\alpha_2}{6} y_2^4 \right)
\]

Thus, we have

\[
Z_J(K, Q) \approx \int e^{-N/2 \left( \left( \sqrt{L_1 - \alpha_1} y_1 - \sqrt{L_2 - \alpha_2} y_2 \right)^2 + \frac{\alpha_1}{6} y_1^4 + \frac{\alpha_2}{6} y_2^4 \right)} \frac{K}{2} y_2^Q \, dy_1 dy_2
\]

We substitute

\[
u = N^{1/4}(\sqrt{L_1 - \alpha_1} y_1 - \sqrt{L_2 - \alpha_2} y_2)
\]

\[
u = N^{1/4}(\sqrt{L_1 - \alpha_1} y_1 + \sqrt{L_2 - \alpha_2} y_2)
\]
which gives

\[
\int_{\mathbb{R}^2} e^{-\frac{1}{2} \left[ u'^2 + \frac{\alpha_1}{253(L_1 - \alpha_1)^2} \left( \frac{u'}{N^{1/4}} + v' \right)^4 + \frac{\alpha_2}{253(L_2 - \alpha_2)^2} \left( v' - \frac{u'}{N^{1/4}} \right)^4 \right]} \left( \frac{u'}{N^{1/2}} + \frac{v'}{N^{1/4}} \right)^K \left( \frac{v'}{N^{1/4}} - \frac{u'}{N^{1/2}} \right)^L \, du' \, dv'
\]

times a constant equal to

\[
\frac{1}{2^{K+L+1} (L_1 - \alpha_1) \frac{K+1}{2} (L_2 - \alpha_2) \frac{L+1}{2} N^{\frac{3}{4}}}
\]

Since we are interested merely in the ratio \( \frac{Z_{J}(K,Q)}{Z_{J}(0,0)} \), we may (and will) neglect multiplicative constants in the evaluation of \( Z_{J}(K,Q) \) as long as these constants are independent of \( K \) and \( Q \). To shorten notation, we define

\[
a_N(K, Q) \sim b_N(K, Q) \quad \text{if} \quad \lim_{N \to \infty} \frac{a_N}{b_N} \to c
\]

for a constant \( 0 < c < \infty \) which is independent of \( K \) and \( Q \). With this notation, we have

\[
Z_{J}(K, Q) \sim \int_{\mathbb{R}^2} e^{-\frac{1}{2}a^2} e^{-1/(3-26) \left( \frac{\alpha_1}{(L_1 - \alpha_1)^2} + \frac{\alpha_2}{(L_2 - \alpha_2)^2} \right) v^4} v^{K+Q} \, du \, dv
\]

\[
\sim \int e^{-1/(3-26) \left( \frac{\alpha_1}{(L_1 - \alpha_1)^2} + \frac{\alpha_2}{(L_2 - \alpha_2)^2} \right) v^4} v^{K+Q} \, dv
\]

for constants \( c, c' \) independent of \( K \) and \( Q \). We note that

\[
\int_0^\infty e^{-a \cdot x^4} x^m \, dx = \frac{1}{4a^{m+1}} \Gamma \left( \frac{m + 1}{4} \right)
\]

Summing up, we obtain

**Theorem 20** Let \( J, \alpha_1, \alpha_2 \) be in the critical regime ((19)–(20)) then

\[
\mathbb{E}(X_1 \cdots X_K \cdot Y_1 \cdots Y_Q) \approx \left[ \frac{12}{\alpha_1 (L_2 - \alpha_2)^2 + \alpha_2 (L_1 - \alpha_1)^2} \right]^{\frac{K+Q}{4}} \cdot (L_1 - \alpha_1)^{Q/2} (L_2 - \alpha_2)^{K/2} \frac{\Gamma \left( \frac{K+Q+1}{4} \right)}{\Gamma \left( \frac{1}{4} \right)} \cdot N^{-\frac{K+Q}{4}}
\]

\( \odot \) Springer
3.5 Correlations for the Low-Temperature Regime

Using again Laplace’s method to evaluate the expressions for $Z_J(K, Q)$ for the low-temperature regime, we immediately get:

**Theorem 21** In the low-temperature regime, we have

$$E\left(X_1 \cdots X_K \cdot Y_1 \cdots Y_Q\right) \approx \begin{cases} \tanh^K(\mu_1) \tanh^Q(\mu_2), & \text{if } K + Q \text{ is even;} \\ 0, & \text{otherwise.} \end{cases}$$

(69)

where $\mu_\ast = (\mu_{1\ast}, \mu_{2\ast})$ is given in Theorem 17.

4 Laws of Large Numbers

In this section, we prove Theorems 1, 4 and 6. We set

$$W_{K,N_1} := \{1, 2, \ldots, N_1\}^K.$$  

(70)

We also denote by $W_{K,N_1}(r)$ the set of all multiindices $\mathbf{i} = (i_1, i_2, \ldots, i_K) \in W_{K,N_1}$ for which exactly $r$ indices occur only once and by $w_{K,N_1}(r)$ the cardinality of $W_{K,N_1}(r)$.

We have

**Lemma 22**

$$w_{K,N_1}(r) \leq K! \frac{N_1^{K+r}}{(K+r)!}$$

(71)

**Proof** The multiindices in $W_{K,N_1}(r)$ contain at most $r + \frac{K-r}{2} = \frac{K+r}{2}$ different indices. There are at most $N_1^{K+r}$ ways to choose them and at most $K!$ ways to order them. $\square$

**Theorem 23** If (13)–(15) (high-temperature regime) hold, then for all $K, Q \in \mathbb{N}, K, Q > 0$

$$E\left(\left(\frac{1}{N_1} \sum_{i=1}^{N_1} X_i\right)^K \left(\frac{1}{N_2} \sum_{j=1}^{N_2} Y_j\right)^Q\right) \rightarrow 0$$

(72)

(72) is also true if (19)–(21) (critical regime) hold.

**Proof**

$$E\left(\left(\frac{1}{N_1} \sum_{i=1}^{N_1} X_i\right)^K \left(\frac{1}{N_2} \sum_{j=1}^{N_2} Y_j\right)^Q\right)$$
\[
\begin{align*}
&= \frac{1}{N_1^K N_2^Q} \sum_{i \in W_{K,N_1}} \sum_{j \in W_{Q,N_2}} \mathbb{E}\left( X_{i_1} \cdot X_{i_2} \cdot \ldots \cdot X_{i_K} Y_{j_1} \cdot Y_{j_2} \cdot \ldots \cdot Y_{j_Q} \right) \\
&= \frac{1}{N_1^K N_2^Q} \sum_{k=0}^{K} \sum_{q=0}^{Q} \sum_{i \in W_{K,N_1}(k)} \sum_{j \in W_{Q,N_2}(q)} \mathbb{E}\left( X_{i_1} \cdot X_{i_2} \cdot \ldots \cdot X_{i_k} Y_{j_1} \cdot Y_{j_2} \cdot \ldots \cdot Y_{j_Q} \right) \\
&\leq C \frac{1}{N_1^K N_2^Q} N_1^{K+k} N_2^{Q+q} N^{-(k+q)/2} \to 0 \quad (73)
\end{align*}
\]

where we used Theorem 19 and (72) in the final estimate. Note that the constant \( C \) depends on \( K \) and \( Q \), but not on \( N \).

Using estimate (68) instead, we obtain the result in the critical regime as well.

We also note that the estimates hold in the cases \( \alpha_1 = 0 \) or \( \alpha_2 = 0 \). \( \square \)

Theorems 1 and 4 follow immediately from 23 and 7.

Remark 24 We have actually proved that

\[
\mathbb{E}\left( \frac{1}{N_1} \sum_{i=1}^{N_1} X_i \right)^K \left( \frac{1}{N_2} \sum_{j=1}^{N_2} Y_j \right)^Q
\]

is bounded with \( \gamma = \frac{1}{2} \) in the high-temperature regime and \( \gamma = \frac{3}{4} \) in the critical regime. This is an indication that the limit Theorems 2 and 5 may hold.

We turn to the low-temperature regime. Analogous to (73), we obtain

\[
\begin{align*}
&\mathbb{E}\left( \frac{1}{N_1} \sum_{i=1}^{N_1} X_i \right)^K \left( \frac{1}{N_2} \sum_{j=1}^{N_2} Y_j \right)^Q \\
&= \frac{1}{N_1^K N_2^Q} \sum_{k=0}^{K} \sum_{q=0}^{Q} \sum_{i \in W_{K,N_1}(k)} \sum_{j \in W_{Q,N_2}(q)} \mathbb{E}\left( X_{i_1} \cdot X_{i_2} \cdot \ldots \cdot X_{i_k} Y_{j_1} \cdot Y_{j_2} \cdot \ldots \cdot Y_{j_Q} \right) \\
&\leq C \frac{1}{N_1^K N_2^Q} N_1^{K+k} N_2^{Q+q} N^{-(k+q)/2} \to 0 \quad (74)
\end{align*}
\]

The terms with \( k < K \) and \( q < Q \) are cancelled by the term in front of the sum. Thus, (74) is asymptotically given by

\[
\mathbb{E}\left( X_1 \cdot X_2 \cdot \ldots \cdot X_K Y_1 \cdot Y_2 \cdot \ldots \cdot Y_Q \right) \\
\approx \frac{1}{2} \left( 1 + (-1)^{K+Q} \right) \tanh^K (\mu_1^*) \cdot \tanh^Q (\mu_2^*) \quad (75)
\]

5 The Central Limit Theorem

5.1 Some Combinatorics

To prove the central limit Theorem 2, we need a more detailed analysis of Lemma 22.
Let us define $W_{K,N_1}^0(r)$ to be the set of all multiindices $\underline{i} = (i_1, i_2, \ldots, i_K) \in W_{K,N_1}(r)$ for which no index occurs more than twice. We also set

$$W_{K,N_1}^+(r) := W_{K,N_1}(r) \setminus W_{K,N_1}^0(r)$$

and denote by $w_{K,N_1}^+(r)$ and $w_{K,N_1}^0(r)$ their cardinalities.

**Lemma 25**

$$w_{K,N_1}^+(r) \leq K! N_1^{\frac{K+r}{2} - \frac{1}{2}}. \tag{76}$$

**Proof** If the $K$-tuple $\underline{i}$ contains $r$ indices with only one occurrence and at least one index with three or more occurrences, there are at most $r - 3$ places left for indices with (exactly) two occurrences. Therefore, a tuple in $w_{K,N_1}^+(r)$ contains at most $k + 1 + \frac{K-r-3}{2}$ different indices. Consequently, there are at most $K! N_1^{\frac{K+r}{2} - \frac{1}{2}}$ such tuples. \(\square\)

**Lemma 26**

$$w_{K,N_1}^0(r) = \begin{cases} \frac{N_1!}{(N_1 - \frac{K+r}{2})!} \cdot \frac{K!}{r! (\frac{K-r}{2})! 2^{\frac{K-r}{2}}} \cdot \frac{K!}{2^{\frac{K-r}{2}}} & \text{if } K - r \text{ is even;} \\ 0 & \text{else.} \end{cases} \tag{77}$$

**Proof** We choose an (ordered) $r$-tuple $\rho$ of $r$ indices to occur once and an ordered $(K - r)/2$-tuple $\lambda$ of indices to occur twice in $\underline{i}$. We have

$$\frac{N_1!}{(N_1 - \frac{K+r}{2})!}$$

ways to do so.

Then, we choose the $r$ positions for those indices which occur once. We can do this in

$$\binom{K}{r} = \frac{K!}{r! (K-r)!}$$

ways. We fill these positions in $\underline{i}$ with $\rho_1, \rho_2, \ldots, \rho_r$ starting with the left most open position.

Finally, we distribute the indices $\lambda_1, \ldots, \lambda_{(K-r)/2}$, twice each. The index $\lambda_1$ is put at the left most free place in $\underline{i}$ and in one of the remaining $K - r - 1$ positions, $\lambda_2$ is put at the then first free place in $\underline{i}$ and in one of the $K - r - 3$ remaining free places and so on.

This gives

$$(K - r - 1)!! = \frac{(K-r)!}{(\frac{K-r}{2})! 2^{\frac{K-r}{2}}} \tag{78}$$
possibilities.

We summarise the above considerations in the following Theorem.

**Theorem 27** In the high-temperature regime, we have

\[
\mathbb{E}\left( \left( \frac{1}{N_1^2} \sum_{i=1}^{N_1} X_i \right)^{2K} \left( \frac{1}{N_2^2} \sum_{j=1}^{N_2} Y_j \right)^{2Q} \right) \\
\approx \sum_{k=0}^{K} \sum_{q=0}^{Q} \frac{(2K)! \alpha_1^k}{(2k)! (K - k)! 2^{K-k}} \frac{(2Q)! \alpha_2^q}{(2q)! (Q - q)! 2^{Q-q}} v_{2k,2q}(0, H^{-1}) 
\]

and

\[
\mathbb{E}\left( \left( \frac{1}{N_1^2} \sum_{i=1}^{N_1} X_i \right)^{2K+1} \left( \frac{1}{N_2^2} \sum_{j=1}^{N_2} Y_j \right)^{2Q+1} \right) \\
\approx \sum_{k=0}^{K} \sum_{q=0}^{Q} \frac{(2K+1)! \alpha_1^{k+\frac{1}{2}}}{(2k+1)! (K - k)! 2^{K-k}} \frac{(2Q+1)! \alpha_2^{q+\frac{1}{2}}}{(2q+1)! (Q - q)! 2^{Q-q}} v_{2k+1,2q+1}(0, H^{-1}) 
\]

### 5.2 Moments of a 2D-Normal Distribution

Let \( \Sigma = \begin{pmatrix} \sigma_1 & \bar{\sigma} \\ \bar{\sigma} & \sigma_2 \end{pmatrix} \) be a covariance matrix, and let \( \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \sim \mathcal{N}(0, \Sigma) \) distributed.

We write \( \nu_{K,Q}(\Sigma) = \mathbb{E}(Z_1^K Z_2^Q) \) to denote the moment of \( \mathcal{N}(0, \Sigma) \) of order \( (K, Q) \).

**Proposition 28**

\[
\nu_{2K,2Q}(\Sigma) = \sum_{r=0}^{K \wedge Q} \frac{2K!}{(2K-2r)! 2^r} \frac{2Q!}{(2Q-2r)! (Q-r)!} \frac{(2K-2r)!}{2^{K-r}} \\
\left( \frac{(2Q-2r)!}{(Q-r)! 2^{Q-r} \sigma_1^{K-r} \bar{\sigma}^{2r} \sigma_2^{Q-r}} \right)
\]

**Proof** Suppose \( V_1, \ldots, V_{2n} \) are random variables. Denote by \( \mathcal{P}_2 = \mathcal{P}_2(2n) \) the set of pair partitions of \( 1, \ldots, 2n \). For a pair partition \( \pi = \{ \pi_1, \ldots, \pi_n \} \in \mathcal{P}_2 \), we set \( \prod_{\pi}(V_1, \ldots, V_{2n}) = \prod_{i=1}^{n} \mathbb{E}(V^{\pi_i}) \) where \( V^{[i,j]} = V_i \cdot V_j \).
By Isserlis’ theorem [12], we have

$$\nu_{2K,2Q}(\Sigma) = \sum_{\pi \in \mathcal{P}_2(2K+2Q)} \Pi(\pi) \left( \frac{Z_1, Z_1, \ldots, Z_1}{2K \text{ times}} \frac{Z_2, Z_2, \ldots, Z_2}{2Q \text{ times}} \right)$$

$$= \sum_{r=0}^{K \wedge Q} \rho_r \cdot \sigma_1^{2K-2r} \bar{\sigma}^{2r} \sigma_2^{2Q-2r}$$

where \( \rho_r = \#\{\pi | \pi \text{ contains exactly } 2r \text{ mixed pairs}\} \). (Mixed pairs are of the form \( \{i, j\} \) with \( i \leq 2K \) and \( j > 2K \).)

To compute \( \rho_r \), we first choose \( 2r \) ‘\( Z_1 \)’s’. This can be done in \( \binom{2K}{2r} \) ways. For these \( Z_1 \)’s, choose \( 2r \) \( Z_2 \)’s: \( \frac{(2L)!}{(2L-2r)!} \). This gives \( \binom{2K}{2r} \frac{(2L)!}{(2L-2r)!} \). The remaining terms come from the pair partitions of the \( 2K - 2r \) \( Z_1 \)’s and \( 2Q - 2r \) \( Z_2 \)’s.

\( \square \)

5.3 Proof of Theorem 2

We calculate the moments \( \mathbb{E} \left( \left( \frac{1}{N_1^{1/2}} \sum X_i \right)^{2K} \left( \frac{1}{N_2^{1/2}} \sum Y_j \right)^{2Q} \right) \), i.e. those with even exponents. The case of odd exponents is done in a similar way.

We write \( H^{-1} = \left( \begin{array}{cc} \sigma_1 & \bar{\sigma} \\ \bar{\sigma} & \sigma_2 \end{array} \right) \)

From Theorem 27 and Proposition 28, we know that the moments \( \mathbb{E} \left( \left( \frac{1}{N_1^{1/2}} \sum X_i \right)^{2K} \left( \frac{1}{N_2^{1/2}} \sum Y_j \right)^{2Q} \right) \) are approximately given by

$$\sum_{k=0}^{K} \sum_{l=0}^{Q} \frac{2K! \alpha_1^k}{2k! (K-k)!(2K-k)} \frac{2Q! \alpha_2^l}{2l! (Q-l)!(2Q-l)} \nu_{k,l}(H^{-1})$$

$$= \sum_{k=0}^{K} \sum_{l=0}^{Q} \frac{2K! \alpha_1^k}{2k! (K-k)!(2K-k)} \frac{2Q! \alpha_2^l}{2l! (Q-l)!(2Q-l)} \sum_{r=0}^{K \wedge L} \frac{2k!}{(2k-2r)!} \frac{2l!}{(2l-2r)!} \frac{(2k-2r)!}{(k-r)!(2k-2r)!(l-r)!} \sigma_1^{k-r} \bar{\sigma}^{2r} \sigma_2^{l-r}$$

$$= \sum_{r=0}^{K \wedge Q} \sum_{k=0}^{K} \sum_{l=0}^{Q} \frac{2K! \alpha_1^k}{(K-k)!2K-k} \frac{2Q! \alpha_2^l}{(Q-l)!2Q-l} \sum_{r=0}^{K \wedge L} \frac{2k!}{(2k-2r)!} \frac{2l!}{(2l-2r)!} \frac{(2k-2r)!}{(k-r)!(2k-2r)!(l-r)!} \sigma_1^{k-r} \bar{\sigma}^{2r} \sigma_2^{l-r}$$

$$= \sum_{r=0}^{K \wedge Q} \sum_{k=0}^{K} \sum_{l=0}^{Q} \frac{2K! \alpha_1^k}{(K-k)!2K-r} \frac{2Q! \alpha_2^l}{(Q-l)!2Q-r} \frac{1}{(k-r)!(2k-r)!(l-r)!} \sigma_1^{k-r} \bar{\sigma}^{2r} \sigma_2^{l-r}$$

Setting \( s = k - r, t = l - r \) (i.e. \( k = s + r, l = t + r \)) gives:

$$= \sum_{r=0}^{K \wedge Q} \sum_{s=0}^{K-r} \sum_{t=0}^{Q-r} \frac{2K! \alpha_1^{s+r}}{(K-r-s)!2K-r} \frac{2Q! \alpha_2^{t+r}}{(Q-r-s)!2Q-r} \frac{1}{2r!s!t!} \sigma_1^{s} \bar{\sigma}^{2r} \sigma_2^{t}$$
5.4 Proof of Theorem 5

Similar to the high-temperature regime, we evaluate

$$
\mathbb{E} \left( \left( \frac{1}{N_1^{\frac{3}{4}}} \sum_{i=1}^{N_1} X_i \right)^K \left( \frac{1}{N_2^{\frac{3}{4}}} \sum_{j=1}^{N_2} Y_j \right)^Q \right) = \frac{1}{N_1^{\frac{3}{4}} k N_2^{\frac{3}{4}} Q} \sum_{k=0}^{K} \sum_{q=0}^{Q} w^0_{K,N_1}(k) w^0_{Q,N_2}(q) \mathbb{E} \left( X_1 \ldots X_k \cdot Y_1 \ldots Y_q \right) \quad (81)
$$

By Theorem 20 and Lemma 26, we obtain

$$
\frac{1}{N_1^{\frac{3}{4}} k N_2^{\frac{3}{4}} Q} w^0_{K,N_1}(k) w^0_{Q,N_2}(q) \mathbb{E} \left( X_1 \ldots X_k \cdot Y_1 \ldots Y_q \right) \leq C N^{-\frac{K+k}{4} - \frac{Q-q}{4}}
$$

Consequently, only the term with \( k = K \) and \( q = Q \) in (81) does not vanish in the large-\( N \)-limit. Thus,

$$
\mathbb{E} \left( \left( \frac{1}{N_1^{\frac{3}{4}}} \sum_{i=1}^{N_1} X_i \right)^K \left( \frac{1}{N_2^{\frac{3}{4}}} \sum_{j=1}^{N_2} Y_j \right)^Q \right) \approx \alpha_1^K \alpha_2^Q N^{\frac{K+Q}{4}} \mathbb{E}(X_1 \cdot \ldots \cdot X_K \cdot Y_1 \cdot \ldots \cdot Y_Q)
$$

$$
\approx \alpha_1^K \alpha_2^Q \left[ \frac{12}{\alpha_1(L_2 - \alpha_2)^2 + \alpha_2(L_1 - \alpha_1)^2} \right]^{\frac{K+Q}{4}} \Gamma \left( \frac{K+Q+1}{4} \right) \Gamma \left( \frac{1}{4} \right)
$$

with \( C \) is defined in (16).

This finishes the proof of Theorem 2.
Acknowledgements While finishing this paper, we became aware of the papers [8,9] which contain the above results as special cases. The methods used by those authors are very different from ours. We are grateful to Francesca Collet for drawing our attention to the papers [8,9]. See also [2,3]. We would also like to thank Matthias Löwe and Kristina Schubert [18] as well as an unnamed referee for valuable comments which in our opinion improved this paper considerably.

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