On relative autocommutativity degree of a subgroup of a finite group

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Abstract

In this paper, we consider the probability that a randomly chosen automorphism of a finite group fixes a randomly chosen element of a subgroup of that group. We obtain several new results as well as generalizations and improvements of some existing results on this probability.

Key words: Automorphism group, Autocommuting probability, Autoisoclinism.

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1 Introduction

Let $H$ be a subgroup of a finite group $G$ and $\text{Aut}(G)$ be the automorphism group of $G$. The relative autocommutativity degree of $H$ denoted by $Pr(H, \text{Aut}(G))$ is the probability that a randomly chosen automorphism of $G$ fixes a randomly chosen element of $H$. In other words

$$Pr(H, \text{Aut}(G)) = \frac{|\{(x, \alpha) \in H \times \text{Aut}(G) : \alpha(x) = x\}|}{|H||\text{Aut}(G)|}. \quad (1.1)$$

The notion of $Pr(H, \text{Aut}(G))$ was introduced in [5] and studied in [5, 9]. Note that $Pr(G, \text{Aut}(G))$ is the probability that an automorphism of $G$ fixes an element of it. The ratio $Pr(G, \text{Aut}(G))$ is also known as autocommutativity degree of $G$. It is worth mentioning that the study of autocommutativity degree of a finite group was initiated by Sherman [10], in the year 1975. In this paper, we obtain several new results on $Pr(H, \text{Aut}(G))$ including some generalizations and improvements of existing results.

For any element $x \in G$ and $\alpha \in \text{Aut}(G)$ we write $[x, \alpha] := x^{-1}\alpha(x)$, the autocommutator of $x$ and $\alpha$. We also write $S(H, \text{Aut}(G)) := \{[x, \alpha] : x \in H$ and $\alpha \in \text{Aut}(G)\}$, $L(H, \text{Aut}(G)) := \{x \in H : \alpha(x) = x$ for all $\alpha \in \text{Aut}(G)\}$ and $[H, \text{Aut}(G)] := \langle S(H, \text{Aut}(G)) \rangle$. Note that $L(H, \text{Aut}(G))$ is a normal subgroup of $H$ contained in

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Hence, the result follows.

Also Aut(G) is a subgroup of L(G), the absolute centre of G (see [4]). Let C_{Aut(G)}(x) := \{ \alpha \in Aut(G) : \alpha(x) = x \} for x \in H and C_{Aut(G)}(H) := \{ \alpha \in Aut(G) : \alpha(x) = x \text{ for all } x \in H \}. Then C_{Aut(G)}(x) is a subgroup of Aut(G) and C_{Aut(G)}(H) = \bigcap_{x \in H} C_{Aut(G)}(x).

It is easy to see that \{(x, \alpha) \in H \times Aut(G) : \alpha(x) = x \} = \bigsqcup_{x \in H} (\{x\} \times C_{Aut(G)}(x)) = \bigsqcup_{\alpha \in Aut(G)} \{CH(\alpha) \times \{\alpha\}\},

where \bigsqcup stands for disjoint union of sets. Hence

$$|H||Aut(G)|Pr(H, Aut(G)) = \sum_{x \in H} |C_{Aut(G)}(x)| = \sum_{\alpha \in Aut(G)} |CH(\alpha)|. \tag{1.2}$$

Also Aut(G) acts on G by the action (\alpha, x) \mapsto \alpha(x) for \alpha \in Aut(G) and x \in G. Let orb(x) := \{\alpha(x) : \alpha \in Aut(G)\} be the orbit of x \in G. Then by orbit-stabilizer theorem, we have |orb(x)| = |Aut(G)|/|C_{Aut(G)}(x)| and hence, [12] gives the following generalization of [1] Proposition 2]

$$Pr(H, Aut(G)) = \frac{1}{|H|} \sum_{x \in H} \frac{1}{|orb(x)|} = \frac{|orb(H)|}{|H|} \tag{1.3}$$

where orb(H) = \{orb(x) : x \in H\}.

Note that Pr(H, Aut(G)) = 1 if and only if H = L(H, Aut(G)). Therefore, we consider H to be a subgroup of G such that H \neq L(H, Aut(G)) throughout the paper.

2 Some upper bounds

In this section we obtain several upper bounds for Pr(H, Aut(G)). We begin with the following result.

**Proposition 2.1.** Let H and K be two subgroups of a finite group G such that H \subseteq K. Then

$$Pr(H, Aut(G)) \leq |K : H| Pr(K, Aut(G)).$$

The equality holds if and only if H = K.

**Proof.** By [12], we have

$$|H||Aut(G)|Pr(H, Aut(G)) = \sum_{x \in H} |C_{Aut(G)}(x)| \leq \sum_{x \in K} |C_{Aut(G)}(x)| = |K||Aut(G)| Pr(K, Aut(G)).$$

Hence, the result follows.

As a corollary, we have the following result.

**Corollary 2.2.** Let H be a subgroup of a finite group G. Then

$$Pr(H, Aut(G)) \leq |G : H| Pr(G, Aut(G))$$

with equality if and only if H = G.
Theorem 2.3. Let $H$ be a subgroup of a finite group $G$ and $p$ the smallest prime dividing $|\Aut(G)|$. Then

$$\Pr(H, \Aut(G)) \leq \frac{(p-1)|L(H, \Aut(G))| + |H| - |X_H|(|\Aut(G)| - p)}{p|H||\Aut(G)|},$$

where $X_H = \{x \in H : C_{\Aut(G)}(x) = \{I\}\}$ and $I$ is the identity automorphism of $G$.

Proof. We have $X_H \cap L(H, \Aut(G)) = \emptyset$. Therefore

$$\sum_{x \in H} |C_{\Aut(G)}(x)| = |X_H| + |\Aut(G)||L(H, \Aut(G))|$$

$$+ \sum_{x \in H \setminus (X_H \cup L(H, \Aut(G)))} |C_{\Aut(G)}(x)|.$$ 

For $x \in H \setminus (X_H \cup L(H, \Aut(G)))$ we have $C_{\Aut(G)}(x) \leq \Aut(G)$ which implies $|C_{\Aut(G)}(x)| \leq \frac{|\Aut(G)|}{p}$. Therefore

$$\sum_{x \in H} |C_{\Aut(G)}(x)| \leq |X_H| + |\Aut(G)||L(H, \Aut(G))|$$

$$+ \frac{|\Aut(G)||H| - |X_H| - |L(H, \Aut(G))|}{p}. \quad (2.1)$$

Hence, the result follows from (1.2) and (2.1). \hfill \square

We would like to mention here that the upper bound obtained in Theorem 2.3 is better than the upper bound obtained in [5, Theorem 2.3 (i)]. We also have the following improvement of [5, Corollary 2.2].

Corollary 2.4. Let $H$ be a subgroup of a finite group $G$. If $p$ and $q$ are the smallest primes dividing $|\Aut(G)|$ and $|H|$ respectively then

$$\Pr(H, \Aut(G)) \leq \frac{p + q - 1}{pq}.$$ 

In particular, if $q \geq p$ then $\Pr(H, \Aut(G)) \leq \frac{2p - 1}{pq} \leq \frac{4}{q}.$

Proof. Since $H \neq L(H, \Aut(G))$ we have $|H : L(H, \Aut(G))| \geq q$. Therefore, by Theorem 2.3 we have

$$\Pr(H, \Aut(G)) \leq \frac{1}{p} \left( \frac{p - 1}{|H : L(H, \Aut(G))|} + 1 \right) \leq \frac{p + q - 1}{pq}.$$ 

Further, if $H$ is a non-abelian subgroup of $G$ then we have the following result.

Corollary 2.5. Let $H$ be a non-abelian subgroup of a finite group $G$. If $p$ and $q$ are the smallest primes dividing $|\Aut(G)|$ and $|H|$ respectively then

$$\Pr(H, \Aut(G)) \leq \frac{q^2 + p - 1}{pq^2}.$$ 

In particular, if $q \geq p$ then $\Pr(H, \Aut(G)) \leq \frac{p^2 + p - 1}{p^2} \leq \frac{6}{5}$. 

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Theorem 2.6. Let $H$ be a subgroup of a finite group $G$. If $p$ and $q$ are two primes such that $\Pr(H, \Aut(G)) = \frac{q^2 + p - 1}{pq}$, then $pq$ divides $|H||\Aut(G)|$. Further, if $p$ and $q$ are the smallest primes dividing $|\Aut(G)|$ and $|H|$ respectively, then

$$\frac{H}{L(H, \Aut(G))} \cong \mathbb{Z}_q.$$  

Proof. By (1.1), we have $(p + q - 1)|H||\Aut(G)| = pq\{(x, \alpha) \in H \times \Aut(G) : \alpha(x) = x\}$. Therefore, $pq$ divides $|H||\Aut(G)|$.

If $p$ and $q$ are the smallest primes dividing $|\Aut(G)|$ and $|H|$ respectively, then, by Theorem 2.3 we have

$$\frac{p + q - 1}{pq} \leq \frac{1}{p} \left( \frac{p - 1}{|H : L(H, \Aut(G))|} + 1 \right)$$

which gives $|H : L(H, \Aut(G))| \leq q$. Hence, $\frac{H}{L(H, \Aut(G))} \cong \mathbb{Z}_q$. $lacksquare$

It is worth mentioning here that Theorem 2.6 is a generalization of [5, Theorem 2.4].

Theorem 2.7. Let $H$ be a non-abelian subgroup of a finite group $G$. If $p$ and $q$ are two primes such that $\Pr(H, \Aut(G)) = \frac{q^2 + p - 1}{pq}$, then $pq$ divides $|H||\Aut(G)|$. Further, if $p$ and $q$ are the smallest primes dividing $|\Aut(G)|$ and $|H|$ respectively, then

$$\frac{H}{L(H, \Aut(G))} \cong \mathbb{Z}_q \times \mathbb{Z}_q.$$  

In particular, if $H$ and $\Aut(G)$ are of even order and $\Pr(H, \Aut(G)) = \frac{q}{8}$, then $\frac{H}{L(H, \Aut(G))} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof. By (1.1), we have $(q^2 + p - 1)|H||\Aut(G)| = pq^2\{(x, \alpha) \in H \times \Aut(G) : \alpha(x) = x\}$. Therefore, $pq$ divides $|H||\Aut(G)|$.

If $p$ and $q$ are the smallest primes dividing $|\Aut(G)|$ and $|H|$ respectively, then, by Theorem 2.3 we have

$$\frac{q^2 + p - 1}{pq^2} \leq \frac{1}{p} \left( \frac{p - 1}{|H : L(H, \Aut(G))|} + 1 \right)$$

which gives $|H : L(H, \Aut(G))| \leq q^2$. Since $H$ is non-abelian, $|H : L(H, \Aut(G))| \neq 1, q$. Hence, $\frac{H}{L(H, \Aut(G))} \cong \mathbb{Z}_q \times \mathbb{Z}_q$. $lacksquare$

The following result gives partial converses of Theorems 2.6 and 2.7 respectively.
Theorem 3.1. Let $H$ be a subgroup of a finite group $G$ and $p$ the smallest prime dividing $|\text{Aut}(G)|$. Then

$$\text{Pr}(H, \text{Aut}(G)) \geq \frac{|L(H, \text{Aut}(G))|}{|H|} + p\left(\frac{|H| - |L(H, \text{Aut}(G))|}{|H||\text{Aut}(G)|}\right) + |X_H|,$$

where $X_H = \{x \in H : C_{\text{Aut}(G)}(x) = \{I\}\}$ and $I$ is the identity automorphism of $G$. 

**Proposition 2.8.** Let $H$ be a subgroup of a finite group $G$. Let $p, q$ be the smallest prime divisors of $|\text{Aut}(G)|$, $|H|$ respectively and $|\text{Aut}(G) : C_{\text{Aut}(G)}(x)| = p$ for all $x \in H \setminus L(H, \text{Aut}(G))$.

(a) If $\frac{H}{L(H, \text{Aut}(G))} \cong \mathbb{Z}_q$ then $\text{Pr}(H, \text{Aut}(G)) = \frac{p+q-1}{pq}$.

(b) If $\frac{H}{L(H, \text{Aut}(G))} \cong \mathbb{Z}_q \times \mathbb{Z}_q$ then $\text{Pr}(H, \text{Aut}(G)) = \frac{q^2 + p - 1}{pq}$.

**Proof.** Since $|\text{Aut}(G) : C_{\text{Aut}(G)}(x)| = p$ for all $x \in H \setminus L(H, \text{Aut}(G))$ we have $|C_{\text{Aut}(G)}(x)| = \frac{\text{Aut}(G)}{p}$ for all $x \in H \setminus L(H, \text{Aut}(G))$. Therefore, by (1.2), we have

$$\text{Pr}(H, \text{Aut}(G)) = \frac{|L(H, \text{Aut}(G))|}{|H|} + \frac{1}{|H||\text{Aut}(G)|} \sum_{x \in H \setminus L(H, \text{Aut}(G))} |C_{\text{Aut}(G)}(x)|$$

$$= \frac{|L(H, \text{Aut}(G))|}{|H|} + \frac{|H| - |L(H, \text{Aut}(G))|}{p|H|}.$$

Thus

$$\text{Pr}(H, \text{Aut}(G)) = \frac{1}{p} \left(\frac{p-1}{|H : L(H, \text{Aut}(G))|} + 1\right). \quad (2.2)$$

Hence, the results follow from (2.2). 

Note that if we replace $\text{Aut}(G)$ by $\text{Inn}(G)$, the inner automorphism group of $G$, then from (1.1), we have $\text{Pr}(H, \text{Inn}(G)) = \text{Pr}(H, G)$ where

$$\text{Pr}(H, G) = \frac{|\{(x, y) \in H \times G : xy = yx\}|}{|H||G|}.$$

Various properties of the ratio $\text{Pr}(H, G)$ are studied in [2] and [8]. We conclude this section showing that $\text{Pr}(H, \text{Aut}(G))$ is bounded by $\text{Pr}(H, G)$.

**Proposition 2.9.** Let $H$ be a subgroup of a finite group $G$. Then

$$\text{Pr}(H, \text{Aut}(G)) \leq \text{Pr}(H, G).$$

**Proof.** By [8] Lemma 1, we have

$$\text{Pr}(H, G) = \frac{1}{|H|} \sum_{x \in H} \frac{1}{|c_G(x)|} \quad (2.3)$$

where $c_G(x) = \{\alpha(x) : \alpha \in \text{Inn}(G)\}$. Since $c_G(x) \subseteq \text{orb}(x)$ for all $x \in H$, the result follows from (1.3) and (2.3). 

### 3 Some lower bounds

We begin with the following lower bound.

**Theorem 3.1.** Let $H$ be a subgroup of a finite group $G$ and $p$ the smallest prime dividing $|\text{Aut}(G)|$. Then

$$\text{Pr}(H, \text{Aut}(G)) \geq \frac{|L(H, \text{Aut}(G))|}{|H|} + p\left(\frac{|H| - |L(H, \text{Aut}(G))|}{|H||\text{Aut}(G)|}\right) + |X_H|,$$

where $X_H = \{x \in H : C_{\text{Aut}(G)}(x) = \{I\}\}$ and $I$ is the identity automorphism of $G$. 

Proof. We have $X_H \cap L(H, \text{Aut}(G)) = \phi$. Therefore
\[
\sum_{x \in H} |C_{\text{Aut}(G)}(x)| = |X_H| + |\text{Aut}(G)||L(H, \text{Aut}(G))|
\]
\[
\quad + \sum_{x \in H \setminus (X_H \cup L(H, \text{Aut}(G)))} |C_{\text{Aut}(G)}(x)|.
\]

For $x \in H \setminus (X_H \cup L(H, \text{Aut}(G)))$ we have $\{I\} \leq C_{\text{Aut}(G)}(x)$ which implies $|C_{\text{Aut}(G)}(x)| \geq p$. Therefore
\[
\sum_{x \in H \setminus (X_H \cup L(H, \text{Aut}(G)))} |C_{\text{Aut}(G)}(x)| \geq |X_H| + |\text{Aut}(G)||L(H, \text{Aut}(G))|
\]
\[
\quad + p(|H| - |X_H| - |L(H, \text{Aut}(G))|).
\]
(3.1)

Hence, the result follows from (1.2) and (3.1).

Now we obtain two lower bounds analogous to the lower bounds obtained in [8, Theorem A] and [7, Theorem 1].

**Theorem 3.2.** Let $H$ be a subgroup of a finite group $G$. Then
\[
\Pr(H, \text{Aut}(G)) \geq \frac{1}{|S(H, \text{Aut}(G))|} \left( 1 + \frac{|S(H, \text{Aut}(G))| - 1}{|H : L(H, \text{Aut}(G))|} \right).
\]
The equality holds if and only if $\text{orb}(x) = xS(H, \text{Aut}(G)) \forall x \in H \setminus L(H, \text{Aut}(G))$.

Proof. For all $x \in H \setminus L(H, \text{Aut}(G))$ we have $\alpha(x) = x[x, \alpha] \in xS(H, \text{Aut}(G))$. Therefore $\text{orb}(x) \subseteq xS(H, \text{Aut}(G))$ and so $|\text{orb}(x)| \leq |S(H, \text{Aut}(G))|$ for all $x \in H \setminus L(H, \text{Aut}(G))$. Now, by (3.1), we have
\[
\Pr(H, \text{Aut}(G)) = \frac{1}{|H|} \left( \sum_{x \in L(H, \text{Aut}(G))} \frac{1}{|\text{orb}(x)|} + \sum_{x \in H \setminus L(H, \text{Aut}(G))} \frac{1}{|\text{orb}(x)|} \right)
\]
\[
\geq \frac{|L(H, \text{Aut}(G))|}{|H|} + \frac{1}{|H|} \sum_{x \in H \setminus L(H, \text{Aut}(G))} \frac{1}{|S(H, \text{Aut}(G))|}.
\]
Hence, the result follows.

**Corollary 3.3.** Let $H$ be a subgroup of a finite group $G$. Then
\[
\Pr(H, \text{Aut}(G)) \geq \frac{1}{|[H, \text{Aut}(G)]|} \left( 1 + \frac{|[H, \text{Aut}(G)]| - 1}{|H : L(H, \text{Aut}(G))|} \right).
\]
Proof. For any two integers $m \geq n$, we have
\[
\frac{1}{n} \left( 1 + \frac{n - 1}{|H : L(H, \text{Aut}(G))|} \right) \geq \frac{1}{m} \left( 1 + \frac{m - 1}{|H : L(H, \text{Aut}(G))|} \right).
\]
(3.2)
Now, the result follows from Theorem 3.2 and (3.2) noting that $|[H, \text{Aut}(G)]| \geq |S(H, \text{Aut}(G))|$. 

\[\square\]
Suppose that (a) holds. Then, by (1.3), we have

First note that for all \( f \) between two groups. Recall that two groups 1940. After many years, in 2013, Moghaddam et al. [6] have introduced autoisoclinism 4 Autoisoclinism between pairs of groups

Note that Corollary 3.3 is a generalization of [1, Equation (3)]. Also (d) follows. Since if there exist isomorphisms \( \psi \) of \( \{ \)

Harmonic, Corollary 3.3 gives better lower bound than the lower bound obtained in [5, Theorem 2.3 (i)]. We conclude this section with the following generalization of [1, Proposition 3] which gives several equivalent conditions for equality in Corollary 3.3.

Proposition 3.4. If \( H \) is a subgroup of a finite group \( G \) then the following statements are equivalent.

(a) \( \text{Pr}(H, \text{Aut}(G)) = \frac{1}{|H, \text{Aut}(G)|} \left( 1 + \frac{|[H, \text{Aut}(G)]| - 1}{|H : L(H, \text{Aut}(G))|} \right) \).
(b) \( \text{orb}(x) = |H, \text{Aut}(G)| \) for all \( x \in H \setminus L(H, \text{Aut}(G)) \).
(c) \( \text{orb}(x) = x[H, \text{Aut}(G)] \) for all \( x \in H \setminus L(H, \text{Aut}(G)) \), and hence \( [H, \text{Aut}(G)] \subseteq L(H, \text{Aut}(G)) \).
(d) \( C_{\text{Aut}(G)}(x) \leq \text{Aut}(G) \) and \( z_{\text{Aut}(G)}(x) \) for all \( x \in H \setminus L(H, \text{Aut}(G)) \).
(e) \( [H, \text{Aut}(G)] = \{ x^{-1} \alpha(x) : \alpha \in \text{Aut}(G) \} \) for all \( x \in H \setminus L(H, \text{Aut}(G)) \).

Proof. First note that for all \( x \in H \)

\[
\text{orb}(x) \leq x[H, \text{Aut}(G)]. \tag{3.3}
\]

Suppose that (a) holds. Then, by (3.3), we have

\[
\sum_{x \in H \setminus L(H, \text{Aut}(G))} \left( \frac{1}{\text{orb}(x)} - \frac{1}{|[H, \text{Aut}(G)]|} \right) = 0.
\]

Now using, we get (b). Also, if (b) holds then from (a), we have (a). Thus (a) and (b) are equivalent.

Suppose that (b) holds. Then for all \( x \in H \setminus L(H, \text{Aut}(G)) \) we have \( \text{orb}(x) = |x[H, \text{Aut}(G)]| \). Hence, using [3.3], we get (c). If \( [H, \text{Aut}(G)] \nsubseteq L(H, \text{Aut}(G)) \) then there exist \( y \in [H, \text{Aut}(G)] \setminus L(H, \text{Aut}(G)) \). Therefore \( \text{orb}(y) = y[H, \text{Aut}(G)] = [H, \text{Aut}(G)] \), a contradiction. Hence \( [H, \text{Aut}(G)] \subseteq L(H, \text{Aut}(G)) \). It can be seen that the map \( f : \text{Aut}(G) \to [H, \text{Aut}(G)] \) given by \( \alpha \mapsto x^{-1} \alpha(x) \), where \( x \) is a fixed element of \( H \setminus L(H, \text{Aut}(G)) \), is a surjective homomorphism with kernel \( C_{\text{Aut}(G)}(x) \). Therefore (d) follows. Since \( |\text{Aut}(G)|/|C_{\text{Aut}(G)}(x)| = |\text{orb}(x)| \) for all \( x \in H \setminus L(H, \text{Aut}(G)) \) we have (b). Thus (b), (c) and (d) are equivalent.

The equivalence of (c) and (e) follows from the fact that \( \text{orb}(x) = x[H, \text{Aut}(G)] \) if and only if \( x^{-1} \text{orb}(x) = [H, \text{Aut}(G)] \) for all \( x \in H \setminus L(H, \text{Aut}(G)) \). This completes the proof.

4 Autoisoclinism between pairs of groups

The concept of isoclinism between two groups was introduced by Hall [3] in the year 1940. After many years, in 2013, Moghaddam et al. [6] have introduced autoisoclinism between two groups. Recall that two groups \( G_1 \) and \( G_2 \) are said to be autoisoclinic if there exist isomorphisms \( \psi : \frac{G_1}{Z(G_1)} \to \frac{G_2}{Z(G_2)} \), \( \gamma : \text{Aut}(G_1) \to \text{Aut}(G_2) \) and \( \beta : [G_1, \text{Aut}(G_1)] \to [G_2, \text{Aut}(G_2)] \) such that the following diagram commutes
if there exist isomorphisms \( \psi \) and \( \gamma \) such that

\[
\begin{array}{ccc}
\frac{G_1}{L(G_2)} \times \text{Aut}(G_1) & \xrightarrow{\psi \times \gamma} & \frac{G_2}{L(G_2)} \times \text{Aut}(G_2) \\
\downarrow a_{(G_1, \text{Aut}(G_1))} & & \downarrow a_{(G_2, \text{Aut}(G_2))} \\
[G_1, \text{Aut}(G_1)] & \xrightarrow{\beta} & [G_2, \text{Aut}(G_2)]
\end{array}
\]

where the maps \( a_{(G_i, \text{Aut}(G_i))} : \frac{G_i}{L(G_i)} \times \text{Aut}(G_i) \to [G_i, \text{Aut}(G_i)] \) for \( i = 1, 2 \) are given by

\[ a_{(G_i, \text{Aut}(G_i))}(x_i L(G_i), \alpha_i) = [x_i, \alpha_i]. \]

Such a triple \((\psi, \gamma, \beta)\) is called an autoisoclinism between \(G_1\) and \(G_2\). We generalize the notion of autoisoclinism between two groups in the following definition.

**Definition 4.1.** Let \(H_1\) and \(H_2\) be two subgroups of the groups \(G_1\) and \(G_2\) respectively. A pair of groups \((H_1, G_1)\) is said to be autoisoclinic to another pair of groups \((H_2, G_2)\) if there exist isomorphisms \(\psi : \frac{H_1}{L(H_1, \text{Aut}(G_1))} \to \frac{H_2}{L(H_2, \text{Aut}(G_2))}\), \(\gamma : \text{Aut}(G_1) \to \text{Aut}(G_2)\) and \(\beta : [H_1, \text{Aut}(G_1)] \to [H_2, \text{Aut}(G_2)]\) such that the following diagram commutes

\[
\begin{array}{ccc}
\frac{H_1}{L(H_1, \text{Aut}(G_1))} \times \text{Aut}(G_1) & \xrightarrow{\psi \times \gamma} & \frac{H_2}{L(H_2, \text{Aut}(G_2))} \times \text{Aut}(G_2) \\
\downarrow a_{(H_1, \text{Aut}(G_1))} & & \downarrow a_{(H_2, \text{Aut}(G_2))} \\
[H_1, \text{Aut}(G_1)] & \xrightarrow{\beta} & [H_2, \text{Aut}(G_2)]
\end{array}
\]

where the maps \( a_{(H_i, \text{Aut}(G_i))} : \frac{H_i}{L(H_i, \text{Aut}(G_i))} \times \text{Aut}(G_i) \to [H_i, \text{Aut}(G_i)] \) for \( i = 1, 2 \) are given by

\[ a_{(H_i, \text{Aut}(G_i))}(x_i L(H_i), \alpha_i) = [x_i, \alpha_i]. \]

Such a triple \((\psi, \gamma, \beta)\) is said to be an autoisoclinism between the pairs \((H_1, G_1)\) and \((H_2, G_2)\).

We conclude this section with the following generalization of [9, Lemma 2.5].

**Theorem 4.2.** Let \(G_1\) and \(G_2\) be two finite groups with subgroups \(H_1\) and \(H_2\) respectively. If \((\psi, \gamma, \beta)\) is an autoisoclinism between the pairs \((H_1, G_1)\) and \((H_2, G_2)\) then

\[ \text{Pr}(H_1, \text{Aut}(G_1)) = \text{Pr}(H_2, \text{Aut}(G_2)). \]

**Proof.** Consider the sets \(S = \{(x_1 L(H_1, \text{Aut}(G_1)), \alpha_1) \in \frac{H_1}{L(H_1, \text{Aut}(G_1))} \times \text{Aut}(G_1) : \alpha_1(x_1) = x_1\}\) and \(T = \{(x_2 L(H_2, \text{Aut}(G_2)), \alpha_2) \in \frac{H_2}{L(H_2, \text{Aut}(G_2))} \times \text{Aut}(G_2) : \alpha_2(x_2) = x_2\}\). Since \((H_1, G_1)\) is autoisoclinic to \((H_2, G_2)\) we have \(|S| = |T|\). Again, it is clear that

\[ |\{(x_1, \alpha_1) \in H_1 \times \text{Aut}(G_1) : \alpha_1(x_1) = x_1\}| = |L(H_1, \text{Aut}(G_1))||S| \quad (4.1) \]

and

\[ |\{(x_2, \alpha_2) \in H_2 \times \text{Aut}(G_2) : \alpha_2(x_2) = x_2\}| = |L(H_2, \text{Aut}(G_2))||T|. \quad (4.2) \]

Hence, the result follows from (1.1), (4.1) and (4.2).

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