Gravitational collapse of Vaidya spacetime in Rastall theory of gravity

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(Dated: December 20, 2019)

A Vaidya spacetime is considered for gravitational collapse of a type II fluid, in the context of Rastall theory of gravity. By considering a linear equation of state (EoS) for the fluid profiles, it is examined the conditions under which the dynamical evolution of the collapse can give rise to a naked singularity formation. It is found that depending on the model parameters, strong curvature naked singularities would arise as exact solutions to the Rastall’s field equations. The allowed values of these parameters are subject to fulfillment of the certain conditions on the physical reliability, nakedness and the curvature strength of the singularity.

PACS numbers: 04.20.Dw, 04.50.Kd, 04.20.Jb, 04.70.Bw

I. INTRODUCTION

Gravitational collapse and its final outcome has been a remarkable area of gravity theory and relativistic astrophysics since the formulation of singularity theorems by Hawking and Penrose \cite{1} (see also \cite{2}). As these theorems do not say much about the detailed features of the singularity, one would like to figure out whether, and under what conditions, the collapse results in formation of a black hole or a naked singularity. If the later occurs as the collapse end state, it would be regarded as counterexample to cosmic censorship conjecture (CCC) \cite{3}, which states that curvature singularities in asymptotically flat spacetimes are always dressed by event horizons of black holes so that, the singular region cannot be causally connected to external region through an observer at infinity (see also \cite{4,5} for reviews on the conjecture).

Despite several attempts made by many researchers, neither a conclusive proof or disproof nor a precise and firm mathematical formulation of CCC has been presented up until now and this conjecture has remained as one of the most outstanding unresolved issues in general relativity (GR). On the contrary, much efforts have been devoted in the past decades to find and develop exact spacetimes, as solutions to GR, which admit naked singularities as the collapse final state. This implies a situation for spacetime events in which the apparent horizons are failed to form during the collapse to cover the singular region. Consequently, causal curves terminating in the past at the singularity have chance to reach the faraway observers, thus exposing the super dense region in the past at the singularity have chance to reach the faraway observers, thus exposing the super dense region in the past at the singularity have chance to reach the faraway observers, thus exposing the super dense region in the past at the singularity have chance to reach the faraway observers, thus exposing the super dense region in the past at the singularity have chance to reach the faraway observers, thus exposing the super dense region in the past at the singularity have chance to reach the faraway observers, thus exposing the super dense region in the past at the singularity have chance to reach the faraway observers, thus exposing the super dense region in the past at the singularity have chance to reach the faraway observers, thus exposing the super dense region in the past at the singularity have chance to reach the faraway 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vides a suitable setting to study the gravitational lensing effects [34].

Following the discussion provided in above paragraphs, we are motivated now to study the process of gravitational collapse of a Vaidya spacetime and examine the CCC in the Rastall theory of gravity. In particular, we will consider the matter field of the collapsing system to be of a type II fluid and then we will check under what conditions on model parameters, a naked singularity may form. We organize our paper as follows. In Sec. II we will present the field equations for the Vaidya spacetime in the presence of a null fluid, within the context of Rastall gravity. In Sec. III we will establish the physical reasonably conditions for the dynamical evolution of the collapse within our model. Our main concern in this section will be to investigate the required circumstances for the formation of naked singularities as collapse outcome. Finally, in Sec. IV we will provide the conclusion and discussions of our work.

II. GRAVITATIONAL COLLAPSE IN RASTALL GRAVITY

According to the original idea of Rastall [26], vanishing of covariant divergence of the matter EMT is no longer valid and this vector field is proportional to the covariant derivative of the Ricci curvature scalar as

\[ \nabla_\mu T^\mu_\nu = \lambda \nabla_\nu R, \]

where \( \lambda \) is the Rastall parameter. The Rastall field equations are then given by [26, 30]

\[ G_{\mu\nu} + \gamma g_{\mu\nu} R = \kappa T_{\mu\nu}, \]

where \( \gamma = \kappa \lambda \) is the Rastall dimensionless parameter and \( \kappa \) being the Rastall gravitational coupling constant. By introducing an effective EMT, \( T^{\text{eff}}_{\mu\nu} \),

\[ T^{\text{eff}}_{\mu\nu} := T_{\mu\nu} - \frac{\gamma T}{4\gamma - \lambda} g_{\mu\nu}, \]

the above field equations can be rewritten in an equivalent form as

\[ G_{\mu\nu} = \kappa T^{\text{eff}}_{\mu\nu}. \]

The Newtonian limit of the Rastall theory implies that [35]

\[ \kappa = \frac{4\gamma - \lambda}{6\gamma - 1} 8\pi G, \]

where \( G \) is the universal gravitational constant.

For a spherically symmetric collapse of a null fluid we consider the Vaidya metric in Eddington-Finkelstein coordinates as

\[ ds^2 = - \left( 1 - \frac{2m(r, v)}{r} \right) dv^2 + 2\epsilon dr dv + r^2 d\Omega^2, \]

where \( m(r, v) \) is the mass function related to the gravitational energy confined within the radius \( r \) and \( d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2 \) is the standard metric on a unit two-sphere. For \( \epsilon = +1 \), the null coordinate \( v \) is the Eddington advanced time for which the \( r \) coordinate is decreasing towards the future along a ray with \( v = \text{const.} \) and represents a congruence of ingoing light rays. Likewise, the case \( \epsilon = -1 \) represents a congruence of outgoing light rays.

The EMT of the collapsing ball is assumed to be given by a two-fluids system to be

\[ T^{\mu\nu} = T^{(n)}_{\mu\nu} + T^{(m)}_{\mu\nu}, \]

where

\[ T^{(n)}_{\mu\nu} = \sigma n_\mu n_\nu, \]

\[ T^{(m)}_{\mu\nu} = (\rho + p)(n_\mu \ell_\nu + n_\nu \ell_\mu) + pg_{\mu\nu}. \]

The first relation, \( T^{(n)}_{\mu\nu} \), denotes the EMT of a null fluid which corresponds to the component of the matter field moving along the null hypersurfaces \( v = \text{const.} \), while the second equation, \( T^{(m)}_{\mu\nu} \), is the EMT of an ordinary matter. By considering a congruence of ingoing light rays (i.e., by setting \( \epsilon = +1 \)), we define the vectors \( n_\mu \) and \( \ell_\nu \) to be a two null vector fields

\[ n_\mu = \delta^0_\mu, \quad \ell_\nu = \frac{1}{2} \left[ 1 - \frac{2m(r, v)}{r} \right] \delta^0_\nu + \delta^1_\nu, \]

such that

\[ n_\lambda \ell^\lambda = 0, \quad n_\lambda \ell_\lambda = -1. \]

Then, by replacing Eq. (2.7) into the Eq. (2.3) the matrix form of the (Rastall induced) effective EMT becomes

\[ T^{\text{eff}}_{\mu\nu} = \begin{pmatrix} -\bar{\rho} & 0 & 0 & 0 \\ \bar{\sigma} & -\bar{\rho} & 0 & 0 \\ 0 & 0 & \bar{\rho} & 0 \\ 0 & 0 & 0 & \bar{\rho} \end{pmatrix}, \]

whose (effective) components \( \bar{\rho}, \bar{\sigma}, \bar{\rho} \) are defined in terms of the matter energy density \( \rho \), and pressure \( p \), and the energy density \( \sigma \) of the Vaidya null fluid as

\[ \bar{\rho}(r, v) = \frac{2\gamma - \lambda}{4\gamma - 1} \rho(r, v) + \frac{2\gamma}{4\gamma - 1} p(r, v), \]

\[ \bar{\sigma}(r, v) = \sigma(r, v), \]

\[ \bar{\rho}(r, v) = \frac{2\gamma - \lambda}{4\gamma - 1} \rho(r, v) + \frac{2\gamma}{4\gamma - 1} \rho(r, v). \]

In terms of these components, now the field equations (2.4) become

\[ \sigma(r, v) = \frac{2}{\kappa^2} \dot{m}, \]

\[ \rho(r, v) = -\frac{2}{\kappa^2} \left[ (2\gamma - 1)m' + r\gamma m'' \right], \]

\[ p(r, v) = \frac{1}{\kappa^2} \left[ r(2\gamma - 1)m'' + 4\gamma m' \right]. \]
in which, a prime and an over dot denote, respectively, a derivative with respect to r and v. Let us assume that the EoS of the two-fluids system is of the form $p = \omega \rho$. Then, by replacing this into Eqs. (2.17) and (2.18), we find a second order differential equation for $m(r, v)$. By solving the resulting equation, we get the following solutions for the mass function:

$$m(r, v) = \begin{cases} \alpha F_1(v) r^\beta + F_2(v), & \text{if } \gamma \neq \frac{2w-1}{2(1+w)}, \\ F_1(v) \ln(r) + F_2(v), & \text{if } \gamma = \frac{2w-1}{2(1+w)}, \end{cases}$$

(2.19)

in which, $F_1(v)$ and $F_2(v)$ are two arbitrary functions, while $\beta$ and $\alpha$ were defined by

$$\beta(\gamma, w) = -1 - \frac{2(1-w)}{2\gamma(1+w) - 1} = \frac{1}{\alpha(\gamma, w)}.$$  

(2.20)

By projecting the EMT (2.12) into the orthonormal basis:

$$e_{(\alpha)\mu} = \begin{pmatrix} \sqrt{2} & (1 - \frac{w}{\gamma}) & -\frac{1}{\sqrt{2}} & 0 & 0 \\
\sqrt{2} & \frac{1}{\gamma} & 0 & 0 \\
0 & 0 & r & 0 & 0 \\
0 & 0 & 0 & \sin \theta & 0 \end{pmatrix},$$

(2.21)

we find that $T^{\text{eff}(a)b)} \equiv e_{(\alpha)\mu} e^{(b)v} T^{\text{eff}}_{\mu\nu}$ takes the form

$$T^{\text{eff}(a)b)} = \begin{pmatrix} \tilde{\sigma} + \tilde{\rho} & \tilde{\sigma} & 0 & 0 \\
\frac{\tilde{\rho}}{\gamma} & 0 & \tilde{\rho} & 0 \\
0 & 0 & \tilde{\rho} & 0 \\
0 & 0 & 0 & \tilde{\rho} \end{pmatrix},$$

(2.22)

which is the EMT of a type II fluid as defined in Ref. [2]. Despite the arbitrariness of the functions $F_1(v)$ and $F_2(v)$, they should be chosen carefully so that the mass function $m(r, v)$ provides a physically reasonable EMT (2.22), satisfying relevant energy conditions. In particular, for such a fluid the weak, strong and dominant energy conditions (ECs) read [2]:

$$\tilde{\sigma} \geq 0, \quad \tilde{\rho} \geq 0, \quad \tilde{\rho} \geq 0; \quad \text{weak & strong ECs},$$

$$\tilde{\sigma} \geq 0, \quad \tilde{\rho} \geq \tilde{\rho} \geq 0; \quad \text{dominant EC}.$$  

(2.23)

Therefore, the mass function $m(r, v)$ should suitably be chosen to satisfy all the conditions (2.23). This would constrain the statements for $F_1(v), F_2(v)$, as well as the parameters $\gamma$ and $w$. Using the expressions (2.13)–(2.15) along with the Eqs. (2.16)–(2.18) and the mass function (2.19) we can rewrite the effective profiles as

$$\tilde{\sigma}(v, r) = \frac{2(6\gamma - 1)}{(4\gamma - 1)r^2} \left( \dot{F}_1(v) + \alpha \nu \beta \dot{F}_1(v) \right),$$  

(2.24)

$$\tilde{\rho}(v, r) = \frac{2(6\gamma - 1)}{4\gamma - 1} F_1(v) r^{-\beta - 3},$$

(2.25)

$$\dot{\rho}(v, r) = \frac{(1 - \beta)(6\gamma - 1)}{4\gamma - 1} F_1(v) r^{-\beta - 3},$$

(2.26)

in which $\gamma \neq \frac{1}{2}$ and $\gamma \neq \frac{1}{4}$. Moreover, we have set the units in which $8\pi G = 1$. As we shall consider in Subsec. IIIA the mass function (2.19) is a growing function due to the inward flow of null fluid. We therefore take the functions $F_1(v)$ and $F_2(v)$ to be positive and increment in $v$. Then, to satisfy the ECs, the coefficients of these functions and their derivatives are required to be positive. Now, following the above arguments together with the conditions (2.23) the pair $\{\gamma, w\}$ are constrained to the following sets of ranges:

$$\{\gamma, w\} \in \bigcup_{i=1}^{4} S_i,$$

(2.27)

where, $S_i$’s are given by

$$S_1 = \{ \gamma < \frac{1}{6}, \quad w_\gamma \leq w < \tilde{w}_\gamma \},$$

$$S_2 = \{ \frac{1}{4} < \gamma < \frac{1}{2}, \quad \tilde{w}_\gamma < w \leq w_\gamma \},$$

$$S_3 = \{ \frac{1}{2} \leq \gamma < 1, \quad w > \tilde{w}_\gamma \},$$

$$S_4 = \{ \frac{1}{2} \leq \gamma < 1, \quad w \leq \tilde{w}_\gamma \},$$

(2.28)

in which, for any given value of $\gamma$, the parameters $w_\gamma$ and $\tilde{w}_\gamma$ are constants and are defined by

$$w_\gamma := -\frac{2\gamma}{2\gamma - 1}, \quad \tilde{w}_\gamma := -\frac{1 + 2\gamma}{2(\gamma - 1)}.$$  

(2.29)

III. THE STATUS OF FINAL SINGULARITY

The formation of naked singularity or black hole and the conditions under which any of these two outcomes would occur are of great significance in the study of gravitational collapse. The occurrence of these outcomes can be examined via investigating the behavior of outgoing light rays in the neighborhood of the singular region. In particular, if there exist families of null geodesics terminating at the singularity in the past, whose tangent vectors are positive-definite, the singularity can be revealed to the exterior observer due to arrival of such curves. Otherwise, the singularity would be covered through a black hole horizon. Nevertheless, existence of such rays do not guarantee that the singularity will be certainly naked. Indeed, to reach the exterior universe, it should be insured that these outgoing rays will not be trapped within their journey due to formation of apparent horizons. This requires a careful analysis of the evolution of trapped surfaces prior to formation of the singularity.

Motivated from above paragraph, our aim in this section will be studying the conditions for the nakedness of the final singularity through the gravitational collapse of the Vaidya spacetime with a type II fluid. To be more concrete, the required properties for our collapsing model to be qualified in the sequel are:
(i) The effective EMT of the two-fluids system, as the collapse matter field, should satisfy the energy conditions such that its associated parameters \( \gamma, w \) should hold the ranges \( (2.27) \).

(ii) There should exist at least one non-negative, real root for the null geodesic equation \( (3.23) \); in fact, such roots are tangents to the null geodesics originating from singularity (cf. Subsec. [III.A])

(iii) The tangent vectors to the radial null geodesics, \((dv/dr)_{r,v=0}\), should be smaller than the slope of trapping surfaces, \((dv/dr)_{AH}\), (cf. Eq. (3.29)) in the vicinity of the singularity. This ensures that the outgoing null geodesics will not be trapped due to formation of apparent horizons (see Subsec. [III.B])

(iv) And finally, the emergent singularity at the collapse end state should be gravitationally strong so that the extension of spacetime through it cannot be possible (see Subsec. [III.C]).

Once the above conditions are fulfilled, we will provide in Subsec. [IV.A], a numerical analysis of the physically relevant solutions for the herein collapse model and study the situations in which a naked singularity can form as the collapse end state.

A. Existence of outgoing radial null geodesics

The situation being considered here is that of a Vaidya spacetime \( (2.6) \) with a radially injected flow of radiation in it, which initiated from an empty region of Minkowski spacetime \[6, 7\]. The radiation would emerge from a central singularity at \( r = 0, v = 0 \) in the past, with a growing Vaidya mass \( m(r,v) \). Hence, the Vaidya mass \( m(v,r) \) is an arbitrary non-negative increment function of the radial coordinate and the advanced time.

Let us define \( \zeta^\mu = d\xi^\mu/d\eta \) as the tangent null geodesics with \( \eta \) being an affine parameter. In terms of \( \zeta^\mu \) the null geodesic equation is given by \( \zeta^\mu \zeta_\mu = 0 \). The Lagrangian associated to the spacetime metric \( (2.6) \) is thus given by

\[
\mathcal{L} = \frac{1}{2}g_{\mu\nu}\zeta^\mu \zeta^\nu = -\frac{1}{2} \left( 1 - \frac{2m}{r} \right) (\zeta^v)^2 + \zeta^v \zeta^r + \frac{r^2}{2} \left[ (\zeta^\theta)^2 + \sin^2 \theta (\zeta^\phi)^2 \right].
\]  

(3.1)

The evolution equation of the geodesic curves is determined by the Euler-Lagrange equation:

\[
\frac{\partial \mathcal{L}}{\partial x^\sigma} - \frac{d}{d\eta} \frac{\partial \mathcal{L}}{\partial (dx^\sigma/\partial \eta)} = 0,
\]

(3.2)

where \( \partial_\eta \equiv \partial/\partial \eta \) and \( x^\sigma = (v, r, \theta, \phi) \). For a given mass function \( m(r,v) \) of the collapsing matter field, the above Euler-Lagrange equation leads to four equations for the four unknown components of the tangent vector \( [\zeta^\mu] = (\zeta^v, \zeta^\phi, \zeta^\theta, \zeta^r) \). They are given as \[16\]:

\[
\begin{align*}
\frac{d\zeta^v}{d\eta} &= \left[ \frac{m' - m}{v - \frac{m}{r}} \right] (\zeta^v)^2 \\
&\quad + r \left[ (\zeta^\theta)^2 + (\zeta^\phi)^2 \sin^2 \theta \right], \quad (3.3) \\
\frac{d\zeta^r}{d\eta} &= \left[ \frac{2m^2}{v - \frac{m}{r}} \right] \left( 1 + 2m' + \frac{m'}{r} - \frac{m}{r} \right) \\
&\quad + (r - 2m) \left[ (\zeta^\theta)^2 + (\zeta^\phi)^2 \sin^2 \theta \right] \\
&\quad + 2\zeta^v \zeta^r \left( \frac{m}{r^2} - \frac{m'}{r} \right), \quad (3.4) \\
\frac{d\zeta^\phi}{d\eta} &= - \frac{2\zeta^\phi}{r} \left[ \zeta^r + r \zeta^\phi \cot \theta \right], \quad (3.5) \\
\frac{d\zeta^\theta}{d\eta} &= (\zeta^\phi)^2 \sin \theta \cos \theta - \frac{2}{r} \zeta^\phi \zeta^r. \quad (3.6)
\end{align*}
\]

By solving equations (3.5) and (3.6) we get

\[
\zeta^\phi = \frac{C_0}{r^2 \sin^2 \theta}, \quad \zeta^\theta = \frac{1}{r^2} \left( C_1 - \frac{C_0^2}{\sin^2 \theta} \right)^{1/2}, \quad (3.7)
\]

where \( C_0 \) and \( C_1 \) are constants of integration. By setting \( C_1 \equiv b^2 \) and \( C_0 \equiv b \cos \theta \), Eq. (3.7) becomes

\[
\zeta^\phi = \frac{b \cos \theta}{r^2 \sin^2 \theta}, \quad \zeta^\theta = \frac{b}{r^2 \sin \theta \cos \theta} \left( \sin^2 \theta - \cos^2 \theta \right)^{1/2}, \quad (3.8)
\]

where \( b \) and \( \theta \) are impact and isotropy parameters, respectively \[6, 16\]. The relations above can be rewritten as a single equation

\[
(\zeta^\phi)^2 + (\zeta^\theta)^2 \sin^2 \theta = \frac{b^2}{r^4}. \quad (3.9)
\]

This together by utilizing the null condition, \( \zeta^\mu \zeta_\mu = 0 \), lead to

\[
\left( 1 - \frac{2m}{r} \right) (\zeta^r)^2 - 2\zeta^v \zeta^r = \frac{b^2}{r^2}. \quad (3.10)
\]

Let us now define \( \zeta^v = Q(r,v) \), following \[16, 21\], for an arbitrary function \( Q(r,v) \). By replacing this definition for \( \zeta^v \) into the Eq. (3.10) we find a relation for \( \zeta^v \) as

\[
\zeta^v = \left( 1 - \frac{2m}{r} \right) \frac{Q}{2r} - \frac{b^2}{2rQ}. \quad (3.11)
\]

Moreover, since \( \zeta^v \) holds the equation \( (3.4) \) we obtain a constraint on the function \( Q(r,v) \) due to the following differential equation:

\[
\frac{dQ}{d\eta} - \frac{Q^2}{2r^2} \left( 1 - \frac{4m}{r} \right) - \frac{b^2 Q}{2r^2} - \frac{Q^2}{r^2} m' = 0. \quad (3.12)
\]

The first step in manifestation of the singular region is the existence of outgoing non-spacelike geodesics originating from the singularity. The basic classification of
the families of such curves is best given in terms of the
limiting values of \( Z \equiv \xi^v/\zeta^v \) near the singularity. Let us
consider radial null geodesics, i.e. curves with zero tan-
ent components \( \xi^0 \) and \( \zeta^0 \) in Eq. (3.8), for which \( b = 0 \).
By setting this in Eq. (3.10) along with the Eq. (3.11) we get
\[
Z = \frac{dv}{dr} = \frac{2r}{1 - 2m}.
\] (3.13)
This quantity is indeed the tangent vector to the radial
null geodesic. For any family of non-spacelike curves
meeting the singularity, the tangents to the curves are
definite [24]. This means that the parameter \( Z \) given in
the limit \( r = 0, \ v = 0 \):
\[
Z_0 := \lim_{v, r \to 0} Z = \left( \frac{dv}{dr} \right)_{v, r = 0},
\] (3.14)
is well-defined at the singularity. For the given metric
(2.6), Eq. (3.14) becomes
\[
Z_0 = \frac{2}{(1 - 2m_0) - 2\hat{m}Z_0},
\] (3.15)
where we have defined
\[
\hat{m}_0 := \left( \frac{\partial m}{\partial v} \right)_{v, r = 0} \quad \text{and} \quad m_0' := \left( \frac{\partial m}{\partial r} \right)_{v, r = 0}.
\] (3.16)
The status of final singularity is determined by the
characteristic (limiting) parameter \( Z \) on the singular
geodesics. This is given by solving the quadratic equation
(3.15) for \( Z_0 \):
\[
2\hat{m}_0 Z_0^2 - (1 - 2m_0)Z_0 + 2 = 0.
\] (3.17)
We thus obtain
\[
Z_0 = 1 - 2m_0 \pm \left[ (1 - 2m_0)^2 - 16\hat{m}_0 \right]^{1/2}
\] (4\hat{m}_0).
(3.18)
To determine, now, the limiting parameter \( Z_0 \) we should
specify the mass function \( m(r, v) \) in equation above.
A suitable choice for the mass function \( m(r, v) \) and
its partial derivatives ensures the existence of the well-
defined solutions for the tangents \( Z_0 \) to the null geodesics
in the vicinity of the singularity. To be more precise,
positive-definiteness of the \( Z_0 \) in Eq. (3.18) implies that
the partial derivatives of the mass function should ex-
ist and be continuous on the entire spacetime of the
collapse. Moreover, they should hold the conditions
\( (1 - 2m_0)^2 - 16\hat{m}_0 \geq 0 \) and \( \hat{m}_0 > 0 \) (provided by the
requirement that \( \hat{m} \neq 0 \) and the weak energy condition,
\( \sigma(r, v) > 0 \), is satisfied) at the central singularity. Having
the solutions for \( Z_0 \) with the above properties then guar-
antees the existence of families of future directed non-
spacelike trajectories that can reach faraway observers in
spacetime.
In order to determine the tangents \( Z_0 \) we then proceed
with computing the mass function \( m(r, v) \) in the limit
when the singularity is reached. In particular, we follow
Ref. [3] and consider an influx of null fluid collapses to the
singularity, where the first shell arrives at \( r = 0 \) at time
\( v = 0 \) while the last shell arrives at \( v = v_0 \). We further
consider a situation in which the radial null fluid starts
its evolution, for \( v < 0 \), from an initially empty region of
the Minkowski spacetime where \( m(r, v) = 0 \). Then, for
\( v > v_0 \) we would have a Schwarzschild spacetime with a
constant mass \( m(r, v) = M_0 \). Then, the suitable expres-
sions for the arbitrary functions \( F_1(v) \) and \( F_2(v) \) can be
given by
\[
F_1(v) := \begin{cases} 0, & \text{if } v < 0; \\ \xi v^{1-\beta}, & \text{if } 0 \leq v \leq v_0; \\ 0, & \text{if } v > v_0, \end{cases}
\] (3.19)
and
\[
F_2(v) := \begin{cases} 0, & \text{if } v < 0; \\ \lambda v, & \text{if } 0 \leq v \leq v_0; \\ M_0, & \text{if } v > v_0, \end{cases}
\] (3.20)
where, \( \lambda \) and \( \xi \) are some constants. Using this choice in
Eq. (2.19) gives the mass function as
\[
m(r, v) = \begin{cases} 0, & \text{if } v < 0; \\ \frac{\xi v^{1-\beta}}{2} + \lambda v, & \text{if } 0 \leq v \leq v_0; \\ M_0, & \text{if } v > v_0. \end{cases}
\] (3.21)
The solution for \( m(r, v) \) when \( 0 \leq v \leq v_0 \) is known as the
self-similar Vaidya spacetime [2].
The limiting values for the partial derivatives of the
mass function (3.21) in the vicinity of the singularity now
read
\[
m_0' = \xi Z_0^{1-\beta} \quad \text{and} \quad \hat{m}_0 = \lambda + \frac{1-\beta}{\beta} \xi Z_0^{-\beta}.
\] (3.22)
By replacing \( m_0' \) and \( \hat{m}_0 \) from equation above into the
Eq. (3.18) we get the root equation as
\[
\frac{2\xi}{\beta} Z_0^2 + 2\lambda Z_0^2 - Z_0 + 2 = 0,
\] (3.23)
where
\[
n(\gamma, w) := 2 - \beta(\gamma, w).
\] (3.24)
Eq. (3.23) is indeed an algebraic equation whose solutions
represent the behaviour of the outgoing null rays responsi-
ble for revelation of the central singularity to the exter-
ior universe. To find the desired solutions to this equa-
tion we first need to determine the exponent \( n(\gamma, w) \) due to
the physically reasonable ranges of parameter \( \{ \gamma, w \} \).
This follows from qualification of the remaining conditions
(i.e., items [iii] and [iv]) listed at the beginning of
this section.
B. Equation of trapped surfaces and conditions for the nakedness of singularity

So far we have established, due to Eq. (3.23), a situation for the existence of outgoing null rays, through positive-definite tangent \( Z_0 \), as a functor for revelation of singularity. However, this does not guarantee the nakedness of the final singularity. Indeed, if apparent horizons form early enough prior to singularity formation, the outgoing rays will be trapped and the singularity would be covered by a black hole horizon. To avoid such situations, our task would be now to set up an evolution equation for the trapping horizons and examine circumstances for the null geodesics to stand outside the trapped region.

To this aim we first write the expansion parameters along the null vector fields \( n_\nu, \ell_\mu \) as [36] [37]

\[
\Theta_\ell(r,v) = \frac{r - 2m(r,v)}{r^2}, \quad \Theta_n(r,v) = -\frac{2}{r}.
\]  (3.25)

For spheres in \( r < 2m(r,v) \) region, we get both expansions to be negative. Such spheres are known as trapped surfaces whose union form a trapped region. Thus, apparent horizon is the boundary of the trapped region and is defined by the conditions

\[
\Theta_\ell = 0 \quad \text{and} \quad \Theta_n < 0.
\]  (3.26)

From this, the equation for the apparent horizon can be written in the form

\[
2m(r,v) = r.
\]  (3.27)

Now, we can calculate the slope of apparent horizon in the limit where approaching the singularity. By differentiating the Eq. (3.27) we obtain \( 2dm(r,v) = dr \), which can be written as

\[
m'(r,v) \left( \frac{dv}{dr} \right)_{AH} + m'(r,v) = \frac{1}{2}.
\]  (3.28)

Here, the term \( (dv/dr)_{AH} \) represents the tangent to the trapping surfaces. Therefore, by solving the above equation for the slope of apparent horizon, \( X_0 \equiv (dv/dr)_{AH} \), in the limit where \( v, r \to 0 \) (i.e., as the singularity is approached), we get

\[
X_0 = \frac{1 - 2\xi Z_0^{1-\beta}}{2\lambda + \frac{2}{\beta}(1 - \beta)Z_0^{-\beta}}.
\]  (3.29)

In derivation of equation above, we have used Eq. (3.22) for the values of \( m_0 \) and \( m'_0 \) in the vicinity of the singularity. Therefore, once a positive solution of \( Z_0 \) is given due to Eq. (3.23), the slope of the apparent horizon, \( X_0 \), would be determined. If the the tangents to the outgoing null geodesics are less than the slope of the apparent horizon in the vicinity of the central singularity, i.e. \( Z_0 < X_0 \), then the outgoing rays will lie outside the trapped region and the final singularity would be naked.

Depending on the qualified values of parameters \( \{\gamma, w, \xi, \lambda\} \), in Subsec. III D, we will provide analyses to examine the existence of our favorite solutions satisfying the condition \( Z_0 < X_0 \), for the nakedness of the final singularity.

C. Strength of singularity

According to Tippler [38], an important physical consequence of the existence of spacetime singularity is the strength of the singularity. Indeed, if singularity is gravitationally weak, then extension of the spacetime through it may be possible. On the contrary, when a strong curvature singularity forms, the gravitational tidal forces associated with it are so strong that any object trying to cross it gets destroyed. Therefore, as argued by [39], the extension of spacetime becomes meaningless for such a strong singularity for which all the objects terminating at it crush to zero size. In order to estimate the curvature strength of the singularity we consider a congruence of null geodesics parameterized by the affine parameter \( \eta \) that terminate at the singularity. Then, the singularity would be strong if the following condition holds [10]:

\[
\psi = \lim_{\eta \to 0} \eta^2 R_{\mu \nu} \xi^\mu \xi^\nu > 0.
\]  (3.30)

To compute the above quantity we consider the required components for the Ricci tensor as

\[
R_{\nu \nu} = \left( 2m - r \right) \frac{m''}{r^2} + \frac{2m'}{r^2},
\]

\[
R_{\nu \mu} = \frac{m''}{r}. \tag{3.31}
\]

Then, using this together with the tangent vector field

\[
\zeta^\mu = \frac{Q}{r} \left( 1, \frac{r - 2m}{2r}, 0, 0 \right), \tag{3.32}
\]

through Eq. (3.30) we get

\[
\psi = \lim_{\eta \to 0} 2 \tilde{m}(r,v) \left( \frac{Q \eta}{r^2} \right)^2
\]

\[
= 2 \tilde{m}(r,v) \lim_{\eta \to 0} \left( \frac{dv}{d\eta} \right)^2 \frac{\eta^2}{r^2}. \tag{3.33}
\]

Then, by using of l’Hôpital’s rule in equation above, in the limit \( r = 0 \) near the central singularity, we get

\[
\psi = \tilde{m}_0 \lim_{\eta \to 0} \left( \frac{dv}{d\eta} \right)^2 = 2\tilde{m}_0 Z_0^2. \tag{3.34}
\]

Now, by setting \( \tilde{m}_0 \) from Eq. (3.22) into equation above, we obtain

\[
\psi = 2\lambda Z_0^2 + \frac{2\xi}{\beta} (1 - \beta) Z_0^{-\beta}. \tag{3.35}
\]

It therefore follows that depending on the model parameters, \( \psi \) can be positive and the strong curvature condition (3.30) is satisfied.
D. Numerical results and fate of the singularity

Once the conditions (i)–(iv) are met, the herein model for gravitational collapse of Vaidya spacetime with the (Rastall gravity induced) effective null fluid is qualified whose central singularity will be visible to the distant observer.

For the existence of the outgoing radial null geodesics reaching the faraway observers, Eq. (3.23) should admit real positive roots depending on the physically reasonable values of parameters \( \{ \gamma, w, \xi, \lambda \} \), generating a 4-dimensional space whose allowed regions are subject to fulfillment of the conditions (i)–(iii). In particular, to satisfy condition (i) we demand that the pair \( (\gamma, w) \) should fulfill the bounds given in Eq. (2.27). We therefore have a two dimensional slice, as shown in Fig. 1, which represents the domain of validity of energy conditions. For the red region, the second and third inequalities in the both lines of the Eq. (2.23) are satisfied, while the gray one stands for validity of the first inequalities of both lines. Therefore, the intersection of these two regions, as shown by the striped region, encompasses the condition Eq. (2.27).

In order to determine the degree \( n(\gamma, w) \) (which is associated to \( \beta(\gamma, w) \) through Eq. (3.24)) of the algebraic equation (3.23), we need to find suitable ranges of values for \( \{ \gamma, w \} \) following the numerical analysis presented in Fig. 1. The simplest choices for the exponent \( n(\gamma, w) \) are \( n = 1 \) and \( n = 2 \) for which the corresponding EoS parameters read, respectively

\[
w = \frac{2\gamma}{1 - 2\gamma}, \quad (n = 1), \tag{3.36}
\]
\[
w = \frac{1 + 2\gamma}{2(1 - \gamma)}, \quad (n = 2). \tag{3.37}
\]

The values of parameters \( \{ \gamma, w \} \) satisfying the above equations constitute two curves in Fig. 1 (the black solid curve for \( n = 1 \), and the blue curve for \( n = 2 \)) that lay on the border of the striped region. Then, to fulfill the condition (i), the allowed values of \( n(\gamma, w) \) are all those curves that are lied between the blue and the black solid curves. We note that for \( n = 2 \) (or identically \( \beta = 0 \)), the parameter \( \alpha(\gamma, w) \) (cf. Eq. (2.20)) is not defined, so we neglect this case\(^1\). We therefore require that \( n \) holds the range \( 1 < n(\gamma, w) < 2 \).

The case \( n = 1 \) (or identically \( \beta = 1 \)), corresponds to a vanishing effective pressure \( \tilde{p} \) for the fluid (cf. Eq. (2.26)). In this case, Eq. (3.23) reduces to a quadratic equation:

\[
2\lambda Z_0^2 + (2\xi - 1)Z_0 + 2 = 0, \tag{3.38}
\]

whose solutions are

\[
Z_0^\pm = \frac{1 - 2\xi \pm [1 - 16\lambda - 4\xi + 4\xi^2]^{1/2}}{4\lambda}. \tag{3.39}
\]

By substituting the above solutions into the Eq. (3.29) we find the slope of the apparent horizon as

\[
X_0 = \frac{1 - 2\xi}{2\lambda}. \tag{3.40}
\]

Then, the conditions (ii) and (iii) demand the following restrictions on the pair \( \{ \lambda, \xi \} \) as

\[
0 < \xi < \frac{1}{2}, \quad \text{and} \quad 0 < \lambda \leq \frac{1}{16} (4\xi^2 - 4\xi + 1). \tag{3.41}
\]

The solution (3.39) looks similar to the solution presented in Ref. [21] for a general relativistic model (being identical to our model with \( \gamma = 0 \)). However, unlike their fluid model with a zero pressure, in our model (i.e., for \( \gamma \neq 0 \)) only the effective pressure \( \tilde{p} \) vanishes while the fluid itself can contain non-vanishing pressures. This is a consequence of the fact spacetime and matter are non-minimally coupled due to the coupling term encoded in the Rastall parameter \( \gamma \).

Let us now study other possible solutions to the equation (3.23) for the cases in which \( 1 < n(\gamma, w) < 2 \). As an

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\(^1\) In fact, this case should be studied separately within the second solution of Eq. (2.19) for the mass function. However, by using the chosen functions \( F_1(v) \) and \( F_2(v) \) different from those we have considered herein this work.
example, let us consider the case \( n = 5/4 \). The EoS then reads
\[
w = -\frac{1 + 14\gamma}{2(7\gamma - 4)}, \quad (3.42)
\]
In Fig. 1 we have plotted the EoS parameter as a function of Rastall parameter (see the black dashed curve in the striped region). We observe that some values of \( \{\gamma, w\} \) lay within the striped region for which the condition (ii) is satisfied. Now Eq. (3.23) can be re-expressed as
\[
8\xi Z_0^{5/4} + 6\lambda Z_0^2 - 3Z_0 + 6 = 0. \quad (3.43)
\]
The above equation cannot be solved analytically, thus, we proceed with finding the roots numerically. By doing so, we arrive at a three-dimensional parameter space constructed by \( (Z_0,\lambda,\xi) \). The left panel in Fig. 2 represents the numerical solutions for the Eq. (3.43) along with the expression \( (3.29) \) in terms of the parameter \( \gamma \). We observe that \( Z_0 < X_0 \) so the conditions (ii) and (iii) are satisfied which implies that the central singularity will be naked as collapse final state.

As another example, let us consider the case where \( n = 7/4 \). The EoS for this case reads
\[
w = -\frac{3 + 10\gamma}{2(5\gamma - 4)}, \quad (3.44)
\]
which corresponds to the black dot-dashed curve in Fig. 1. The Eq. (3.23) for \( n = 7/4 \) can be rewritten then:
\[
8\xi Z_0^{7/4} + 2\lambda Z_0^2 - Z_0 + 2 = 0. \quad (3.45)
\]
For a fixed value of \( \xi > 0 \), numerical solution to the above equation reveals that the tangent to the radial null geodesics (black curve) is positive for the ranges of parameter \( \lambda > 0 \) (cf. the right panel in Fig. 2). Moreover, this tangent is less than the slope of apparent horizon (red curve) in the limit of singular node, hence, the conditions (ii) and (iii) are fulfilled. We further note that for all the above studied cases we have \( \lambda > 0 \) and \( \xi > 0 \).

On the other hand, since \( 0 < \beta(\gamma, w) \leq 1 \) for the range of the exponent \( 1 < n(\gamma, w) < 2 \), the condition (iv) (cf. Eq. (3.30)) is satisfied as well, thus the central singularity is gravitationally strong, and hence we are led to a physically reasonable solutions for the naked singularity formation as the collapse final state in our model.

IV. CONCLUDING REMARKS

In the present work we studied gravitational collapse of a Vaidya spacetime with a type II fluid in the framework of Rastall gravity. By considering a particular choice for the Vaidya mass \( m(r, v) \), we observed that depending on the model parameters \( \{\gamma, w, \xi, \lambda\} \), strong curvature naked singularities could emerge as the collapse final states. The nakedness of these singularities were examined by pursuing the radial null geodesics terminating in the past at the central singularities with positive tangents to the geodesic curves, \( (dv/dr)_v=0 \) (cf. Eqs. (3.18) and (3.23)). However, the existence of such geodesic curves could not necessarily imply that the singularities are naked, as these curves might be turned back to the singularity from their starting points due to formation of apparent horizons. To deal with this issue, we computed the tangent to the apparent horizon, \( (dv/dr)_{AH} \) (cf. Eq. (3.29)), in the limit where the singularity is reached. It was observed that for the certain values of the model parameters, the tangents to the apparent horizons are greater than those of the geodesics, hence, the radial null geodesics could emerge from the singularities, standing outside the trapped regions so that by continuing to remain untrapped, they could arrive in the exterior universe.

By fixing the above conditions, the solutions generating naked singularity were classified through the parameter \( n(\gamma, w) \) whose values were subject to the fulfillment of the energy conditions. A particular solution (i.e. the case \( n = 1 \)) was found for which the collapse scenario
ends in a naked singularity whose null fluid matter field displayed an effective dust-like behaviour. Despite the vanishing effective pressure profile (induced by Rastall gravity modifications), the fluid itself had nonzero pressure. This is a consequence of the non-minimal coupling between geometry and matter through the Rastall parameter $\gamma \neq 0$.

As discussed in Sec. [III A] the particular Vaidya mass we considered in this work was an increment mass function of the coordinates $(v, r)$, initiated from an empty Minkowski spacetime (see Eq. 3.21). Its arbitrary functions were chosen as $F_1(v) = \xi v^{\gamma-\beta}$ and $F_2(v) = \nu v$ (cf. 3.19) and (3.20). If we chose instead two different free functions as $F_1(v) = a_0/2$ and $F_2(v) = 0$ with $a_0$ being a constant, then the mass function would become $m(r) = (a_0/2)r$, with the effective density profiles:

$$\tilde{\rho} = \frac{a_0}{\kappa r^2}, \quad \tilde{\sigma} = 0, \quad p = \frac{2\gamma}{1 - 2\gamma} \tilde{\rho} = \frac{a_0 \gamma}{\kappa r^2}.$$  (4.1)

This solution represents the gravitational field of a monopole [11]. The gravitational collapse of a monopole and the situation in which naked singularities can form has been studied in Refs. [6, 7].

We therefore conclude that, Rastall theory of gravity can provide a framework which leads to remarkable solutions for occurrence of strong curvature naked singularities, being counterexamples to the CCC. These solutions can arise for a general class of spacetimes, namely the generalized Vaidya metric, with a non-linear mass function, which describes spherically symmetric gravitational collapse of a type II fluid in an initially empty region of Minkowski spacetime.

Acknowledgments

This article is based upon work from European Cooperation in Science and Technology (COST) action CA18108–Quantum gravity phenomenology in the multimessenger approach–supported by COST.

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