DECOUPLING OF TRANSLATIONAL AND ROTATIONAL MODES FOR A QUANTUM SOLITON

A.Dubikovsky and K.Sveshnikov

Department of Physics, Moscow State University, Moscow 119899, Russia

A set of integral relations for rotational and translational zero modes in the vicinity of the soliton solution are derived from the particle-like properties of the latter and verified for a number of models (solitons in 1+1-dimensions, skyrmions in 2+1- and 3+1-dimensions, non-abelian monopoles). It is shown, that by consistent quantization within the framework of collective coordinates these relations ensure the correct diagonal expressions for the kinetic and centrifugal terms in the Hamiltonian in the lowest orders of the perturbation expansion. The connection between these properties and virial relations is also determined.

Motivated by success of the skyrmion baryon models, there was recently much interest in quantization of hedgehog-type configurations including translational and rotational degrees of freedom by means of suitable collective coordinates. Generally, in this approach the corresponding quantum Hamiltonian contains a full bilinear form in conjugated momenta with nontrivial couplings between different collective variables. However, for a particle-like classical solution one should expect additive diagonal contributions of kinetic and centrifugal terms to the Hamiltonian, at least to the lowest orders in the appropriate weak coupling expansion. Actually, it is a part of the general problem of decoupling upon quantization of various soliton degrees of freedom, which takes place for any type of field models with classical solutions. In this paper we’ll present a consistent general analysis of this problem for translational and rotational variables, based on the particle-like properties of the classical solution combined with Lorentz covariance and virial relations.

Let us consider a field theory in $d+1$ space-time dimensions described by the Lagrangean density $\mathcal{L}(\varphi)$, which possesses a classical particle-like solution $\varphi_c(x)$. For brevity the Lorentz and internal symmetry group indices are suppressed. It is generally accepted, that if in the rest frame $\varphi_c$ is static with finite and localized energy density, then in quantum version of the theory such configuration describes an extended particle. Now we’ll

\[ \text{E-mail: costa@bog.msu.su} \]
show, that there exists a set of nontrivial integral relations, fulfilled by \( \varphi_c(x) \), which provide the validity of these assumptions.

For a given static solution \( \varphi_c(\vec{x}) \) the moving one is constructed via Lorentz boost, what results in the replacement

\[
x^i \rightarrow \Lambda^{-1}_\nu^i x^\nu
\]

in the arguments of \( \varphi_c \), where \( \Lambda^\mu_\nu \) is the corresponding Lorentz matrix. The momentum of the moving solution is

\[
P^\mu = \int T^{\mu 0}(\varphi_c(\vec{x}, x^0)) \, d\vec{x},
\]

where \( T^{\mu \nu}(\varphi_c) \) is the energy–momentum tensor. Transforming the r.h.s. in (2) to the rest frame, one gets

\[
P^\mu = \Lambda^\mu_\nu \Lambda^0_\sigma \int T^{\nu \sigma}'(\varphi_c(\vec{\xi})) \, J \, d\vec{\xi},
\]

where \( J = \Lambda^\nu_0^{-1} \) is the Jacobian of transition from \( d\vec{x} \) to the rest frame spatial variable. On the other hand, the l.h.s. of (2) should be the momentum of a particle with the mass \( M \), that is

\[
P^\mu = \Lambda^\mu_0 M.
\]

Then it follows from eqs.(3) and (4) that

\[
\int T^{\mu \nu}(\varphi_c(\vec{x})) \, d\vec{x} = \delta^\mu_0 \delta^\nu_0 M.
\]

For \( \nu = 0 \) we get \( P^0 = M \), \( P^i = 0 \), just that we should expect for a static solution. However, for \( \mu = i \), \( \nu = j \) we obtain

\[
\int \frac{\partial L(\varphi_c)}{\partial \partial^i \varphi_c(\vec{\xi})} \partial_i \varphi_c(\vec{\xi}) \, d\vec{\xi} = M \delta_{ij}.
\]

(Henceforth the derivative \( \partial_i \) acts on the indicated argument of the function.) So we get the first set of conditions (6), which holds for a particle-like classical configuration \( \varphi_c(\vec{\xi}) \) in the rest frame.

Now let us consider the orbital part of the 4-rotation tensor (without the spin term)

\[
L^{\mu \nu} = \int (x^\nu T^{\mu 0}(\varphi_c(\vec{x}, x^0))) - x^\mu T^{\nu 0}(\varphi_c(\vec{x}, x^0))) \, d\vec{x}.
\]

Returning to the rest frame, we can write similarly

\[
L^{\mu \nu} = \Lambda^\mu_\mu' \Lambda^\nu_\nu' \Lambda^0_\sigma \int (\xi^\sigma T^{\mu \sigma}(\varphi_{c}(\vec{\xi}))) - \xi^\sigma T^{\nu \sigma}(\varphi_{c}(\vec{\xi}))) \, J \, d\vec{\xi}.
\]
On the other hand, for a static particle-like solution in the rest frame one obviously has $L^{ij} = 0$, while $L^{0i}$ coincide with the center-of-mass coordinates and therefore can be made vanish by a spatial translation. But since $L^{\mu\nu}$ is a tensor, then it must vanish in any other Lorentz system. Then it follows from eq. (8) that

$$0 = \int \left( \xi^\nu T^{\mu\sigma}(\varphi_c(\xi)) - \xi^\mu T^{\nu\sigma}(\varphi_c(\xi)) \right) \, d\xi. \quad (9)$$

For $\sigma = 0$ these relations mean, that in the rest frame $L^{\mu\nu} = 0$, just that we expected to have. When $\mu = 0$ or $\nu = 0$ but $\sigma \neq 0$, then due to eq. (8) one obtains from (9) that $\int \xi^i T^{0j}(\varphi_c) \, d\xi = 0$, what gives an identity provided by symmetry of $T^{\mu\nu}$. The latter statement is valid even for theories with Chern–Simons terms, since these terms do not contribute to $T^{\mu\nu}$ [6]. However, for $\mu = i$, $\nu = j$, $\sigma = k$ we get from (9) the following relations (for definiteness, we take $d = 3$)

$$\int \varepsilon_{lij} \xi_i \partial_j \varphi_c(\xi) \frac{\partial \mathcal{L}(\varphi_c)}{\partial \partial^k \varphi_c(\xi)} \, d\xi = \int \varepsilon_{lik} \xi_i \mathcal{L}(\varphi_c) \, d\xi = 0, \quad (10)$$

since $L^{0k}$ vanish in the rest frame by assumption. This is the second set of relations on $\varphi_c(\xi)$, following from the Lorentz covariance and particle-likeness of the classical solution.

So each particle-like solution should be subject of conditions (6) and (10). It should be noted, that the relation (4) for $\mu = 0$ reproduces nothing else but the relativistic mass-energy relation. For the moving $\varphi^4$-kink solution this relation has been explicitly verified in [6], and for the moving skyrmion — in (7) by direct calculations. However, the eqs. (9) are more general and, moreover, the eqs. (10) also take place. Note also, that these relations, being consistent with the field equations and conservation laws, do not be the direct consequences of the latters, and should be considered separately.

As a direct result of these relations we get the orthogonality of the zero-frequency eigenfunctions in the neighborhood of the classical particle-like solution cite8. Let us discuss the theory of a nonlinear scalar field in 3 spatial dimensions, described by the Lagrangean density

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \varphi)^2 - U(\varphi), \quad (11)$$

which possesses a classical static solution $\varphi_c(x) = u(\vec{x})$. According to the virial theorem such solutions are unstable in more then one spatial dimension, but for our purposes this is not so important compared to simplicity.
of presentation. In the general case the non-spherical configuration \( u(\vec{x}) \) yields 6 zero-frequency modes — three translational ones

\[
\psi_i(\vec{x}) = \partial_i u(\vec{x}),
\]

(12)

and three rotational

\[
f_i = \varepsilon_{ijk} x_j \partial_k u(\vec{x}).
\]

(13)

Then from eqs. (6) and (10) one immediately obtains

\[
\int d\vec{\xi} \psi_i(\vec{\xi}) \psi_j(\vec{\xi}) = M \delta_{ij},
\]

(14)

\[
\int d\vec{\xi} f_i(\vec{\xi}) \psi_j(\vec{\xi}) = 0.
\]

(15)

Further, by spatial rotations one can always achieve that

\[
\int d\vec{\xi} f_i(\vec{\xi}) f_j(\vec{\xi}) = \Omega_{ij} = \Omega_i \delta_{ij},
\]

(16)

where \( \Omega_i \) are the moments of inertia of the classical configuration. Obviously, the relations (14) and (15) remain unchanged. As a result, the normalized set of translational and rotational zero-modes can always be written as \( \{ \psi_i(\vec{x})/\sqrt{M} , f_i(\vec{x})/\sqrt{\Omega_i} \} \).

So the particle-likeness of the classical solution results in the diagonality of the zero-frequency scalar product matrix. This diagonality plays an essential role in the procedure of quantization in the vicinity of a classical soliton solution by means of collective coordinates [3, 4, 5]. Following the conventional procedure [4], let us consider the field \( \varphi(\vec{x}) \) in the Schroedinger picture in the vicinity of the solution \( u(\vec{x}) \). The substitution, introducing translational and rotational collective coordinates, reads

\[
\varphi(\vec{x}) = u \left( R^{-1}(\vec{c})(\vec{x} - \vec{q}) \right) + \Phi \left( R^{-1}(\vec{c})(\vec{x} - \vec{q}) \right),
\]

(17)

where \( \Phi \) is the meson field, \( R(\vec{c}) \) is the rotation matrix, \( \vec{q} \) and \( \vec{c} \) are the translational and rotational collective coordinates correspondingly.

For our purposes the parametrization of the rotation group by means of the vector–parameter [9] is the most convenient. In this parametrization the rotation matrix \( R(\vec{c}) \) is taken in the form

\[
R(\vec{c}) = 1 + 2 \frac{c^x + c^2}{1 + c^2} = \frac{1 - c^2 + 2c^x + 2c \cdot c}{1 + c^2},
\]

(18)

where \( c^x_{ab} = \varepsilon_{adc} c_d, (c \cdot c)^{ab} = c_a c_b, c^2 = \vec{c} \cdot \vec{c} \). The composition law for vector–parameters, corresponding to product of rotations \( R(\vec{a}) \) \( R(\vec{b}) = R(\vec{c}) \), is given by

\[
\vec{c} = \langle \vec{a}, \vec{b} \rangle = \frac{\vec{a} + \vec{b} + \vec{a} \times \vec{b}}{\vec{a} \cdot \vec{a}}.
\]

(19)
The generators of infinitesimal rotations are

\[ \vec{S} = -\frac{i}{2} (1 + c \cdot c + c^\times) \frac{\partial}{\partial \vec{c}}, \tag{20} \]

while the finite rotations \( U(\vec{a}) \), defined so that \( U^+(\vec{a}) \vec{b} U(\vec{a}) = \langle \vec{a}, \vec{b} \rangle \), take the form

\[ U(\vec{a}) = \exp \left\{ -2i\vec{a}\vec{S} \right\}. \tag{21} \]

Returning to the decomposition (17), one finds that the total momentum of the field is now represented as

\[ \vec{P} = -i \frac{\partial}{\partial \vec{q}}, \tag{22} \]

and the total angular momentum is equal to

\[ \vec{J} = \vec{L} + \vec{S}, \tag{23} \]

where \( \vec{L} = \vec{q} \times \vec{P} \) is the orbital angular momentum, and the spin \( \vec{S} \) is defined by relation (20).

In order to keep the number of degrees of freedom we impose on the field \( \Phi(\vec{y}) \) 6 subsidiary conditions, which in the theory of a weak coupling are usually taken as linear combinations

\[ \int d\vec{y} N^{(\alpha)}(\vec{y}) \Phi(\vec{y}) = 0, \quad \alpha = 1, \ldots, 6. \tag{24} \]

The set \( \{N^{(\alpha)}(\vec{y})\} \) should ensure the condition of orthogonality of the meson field \( \Phi(\vec{y}) \) to zero-frequency modes and is chosen in the following way. Let us denote \( M^{(\alpha)}(\vec{y}) = \{\psi_i(\vec{y}), f_i(\vec{y})\} \). In the general case [4] \( N^{(\alpha)}(\vec{y}) \) are given by linear combinations of \( M^{(\beta)}(\vec{y}) \) subject of relations

\[ \int d\vec{y} N^{(\alpha)}(\vec{y}) M^{(\beta)}(\vec{y}) = \delta_{\alpha\beta}. \tag{25} \]

In our case the system of zero-frequency modes is orthogonal, so one immediately gets

\[ N^{(\alpha)}(\vec{y}) = \{\psi_i(\vec{y})/M, f_i(\vec{y})/\Omega_i\}. \tag{26} \]

It is the relation (26), that ensures the additive form of the collective coordinate part of the Hamiltonian within the weak coupling expansion in powers of the meson fields. Let us consider the condition (24) as relation, defining \( \vec{q} \) and \( \vec{c} \) as functionals of \( \varphi(\vec{x}) \). A straightforward calculation gives

\[ N^{(\alpha)} \left( R^{-1}(\vec{c})(\vec{x} - \vec{q}) \right) + \]

\[ + R_{ji} \frac{\partial R_{jk}}{\partial c_i} \frac{\partial c_l}{\partial \varphi(\vec{x})} \int d\vec{y} N^{(\alpha)}(\vec{y}) y_k \partial_i (u(\vec{y}) + \Phi(\vec{y})) + \]

\[ + R_{ij} \frac{\partial q_l}{\partial q_j} \int d\vec{y} N^{(\alpha)}(\vec{y}) \partial_i (u(\vec{y}) + \Phi(\vec{y})) = 0. \tag{27} \]
By means of eqs. (18)–(21) one can easily verify that

\[ R_{ji} \frac{\partial R_{jk}}{\partial c_l} = -\frac{2}{1 + c^2} \varepsilon_{ikm}(1 - c^\times)_{ml}, \]

(28)

\[ (1 - c^\times)^{-1} = \frac{1 + c \cdot c + c^\times}{1 + c^2}. \]

The simplicity of these relations demonstrates the convenience of vector parametrization (18)–(21) for such type of problems [8]. Then from eqs. (27), (28) we immediately get the following lowest-order expressions for \( \partial \vec{q}/\partial \varphi(\vec{x}) \) and \( \partial \vec{c}/\partial \varphi(\vec{x}) \)

\[ -\bar{\psi} \left( R^{-1}(\vec{c})(\vec{x} - \vec{q}) \right) = MR^{-1} \frac{\partial \vec{q}}{\partial \varphi(\vec{x})}, \]

(29)

\[ -\bar{f} \left( R^{-1}(\vec{c})(\vec{x} - \vec{q}) \right) = 2 \Omega \frac{1 - c^\times}{1 + c^2} \frac{\partial \vec{c}}{\partial \varphi(\vec{x})}. \]

Calculating the conjugate momentum \( \pi(\vec{x}) = -i \delta / \delta \varphi(\vec{x}) \) as a composite derivative

\[
\pi(\vec{x}) = -i \frac{\delta}{\delta \varphi(\vec{x})} = \frac{\partial \vec{c}}{\partial \varphi(\vec{x})} \left( -i \frac{\partial}{\partial \vec{c}} \right) + \\
+ \frac{\partial \vec{q}}{\partial \varphi(\vec{x})} \left( -i \frac{\partial}{\partial \vec{q}} \right) + \int d\vec{y} \frac{\delta \Phi(\vec{y})}{\delta \varphi(\vec{x})} \left( -i \frac{\delta}{\delta \Phi(\vec{y})} \right),
\]

(30)

and using the relations (29), we obtain to the leading order the following result

\[
\pi(\vec{x}) = \Pi \left( R^{-1}(\vec{c})(\vec{x} - \vec{y}) \right) \\
- \frac{1}{M} \bar{\psi} \left( R^{-1}(\vec{c})(\vec{x} - \vec{q}) \right) \left( \bar{K} + \int d\vec{y} (\partial \Phi(\vec{y}) \Pi(\vec{y})) \right) \\
- \bar{f} \left( R^{-1}(\vec{c})(\vec{x} - \vec{q}) \right) \Omega^{-1} \left( \bar{T} + \int d\vec{y} (\partial \Phi(\vec{y}) \Pi(\vec{y})) \right).
\]

(31)

In eq. (31) \( \bar{K} = R^{-1}(\vec{c})\bar{P} \) and \( \bar{T} = R^{-1}(\vec{c})\bar{S} \) are the momentum and the spin of the field, corresponding to the rotating frame, and the meson field momentum \( \Pi(\vec{y}) \) is defined as

\[
\Pi(\vec{y}) = \int d\vec{z} A(\vec{z}, \vec{y}) \left( -i \frac{\delta}{\delta \Phi(\vec{z})} \right),
\]

(32)

where \( A(\vec{z}, \vec{y}) \) is the projection matrix on the subspace, orthogonal to zero-frequency modes

\[
A(\vec{x}, \vec{y}) = \delta(\vec{x} - \vec{y}) - \sum M^{(a)}(\vec{x}) N^{(a)}(\vec{y})
\]

(33)
Inserting eqs. (17) and (31) in the Hamiltonian

\[ H = \int d\vec{x} \left\{ \frac{1}{2} \pi^2(\vec{x}) + \frac{1}{2}(\vec{\partial} \varphi)^2 + U(\varphi) \right\}, \]  

we obtain finally the following lowest-order expression

\[ H = M + \int d\vec{y} \left\{ \frac{1}{2} \Pi^2 + \frac{1}{2}(\vec{\partial} \Phi)^2 + \frac{1}{2}U''(u(\vec{y}))\Phi^2 \right\}(\vec{y}) \]

\[ + \frac{1}{2M} \left( \vec{K} + \int d\vec{y} (\vec{\partial} \Phi) \Pi(\vec{y}) \right)^2 \]

\[ + \frac{1}{2} \sum_i \left( I_i + \int d\vec{y} ([\vec{y} \times \vec{\partial}]_i \Phi) \Pi(\vec{y}) \right)^2 / \Omega_i. \]

It is indeed such form of the Hamiltonian, that provides to interpret the resulting ground state as non-relativistic particle with the mass \(M\) and moments of inertia \(\Omega_i\). So the correct form of the Hamiltonian with additive kinetic and centrifugal terms, that means the absence of correlations between translational and rotational degrees of freedom, is ensured by the diagonality of zero–frequency scalar product matrix (14)–(16). In turn, this is a direct consequence of relations (6) and (10). Note also, that this result will be actually valid for any field model in the neighborhood of the suitable soliton solution.

These general considerations can be easily illustrated by concrete models. Firstly, we consider the theory of a scalar field in 1+1-dimensions, described by the Lagrangean density (11). In this case we have only one relation (6)

\[ \int dx (\varphi'(x))^2 = M, \]  

where the mass \(M\) is given by

\[ M = \int dx \frac{1}{2}(\varphi'(x))^2 + \int dx U(\varphi(x)). \]  

Performing the dilatation \(\varphi(x) \rightarrow \varphi(\lambda x)\) and demanding for the solution at \(\lambda = 1\), i.e. \(\left(\frac{dM(\lambda)}{d\lambda}\right)_{\lambda=1} = 0\), we find the well-known Hobart–Derrick virial relation [10]

\[ \frac{1}{2} \int dx (\varphi'(x))^2 = \int dx U(\varphi(x)), \]  

owing to which the "particle-likeness condition" (36) is fulfilled automatically.

In more spatial dimensions the situation with the model (11) is more complicated. Namely, it is a trivial task to verify by the same arguments, that for each \(i\)

\[ \int (\partial_i \varphi)^2 \, d\vec{x} = M. \]
However, the orthogonality conditions between different spatial derivatives, predicted by eqs. (14) and (15), cannot be derived by such simple considerations. So here the additional arguments, used by derivation of relations (8) and (10), are crucial.

In 2+1-dimensions, the solitons in $CP_N$-models are interesting examples with such particle-like properties. As it is well-known, for $N = 1$ the $CP_N$-model is reduced to $O(3)$-model [11], described by

$$\mathcal{L} = \frac{1}{2} \partial_\mu \varphi^a \partial^\mu \varphi^a$$

with subsidiary condition

$$\varphi^a \varphi^a = 1.$$  \hspace{1cm} (41)

This theory is a planar analog of the Skyrme model [12]. The standard one-particle solution of the model is given by [13]

$$\varphi^1 = \phi(r) \cos n\vartheta, \quad \varphi^2 = \phi(r) \sin n\vartheta, \quad \varphi^3 = (1 - \phi^2)^{1/2},$$  \hspace{1cm} (42)

where $r, \vartheta$ are polar coordinates and

$$\phi(r) = \frac{4r^n}{r^{2n} + 4},$$  \hspace{1cm} (43)

and describes the ”baby-skyrmeon” configuration with the topological charge $Q = n$ and the mass $M = 4\pi Q$. The direct insertion of expression (42) into conditions (8) and (10) yields

$$\int \partial_i \varphi^a \partial_j \varphi^a d^2x = 4\pi n \delta_{ij} = M \delta_{ij},$$  \hspace{1cm} (44)

and

$$\int \varepsilon_{ij} x_i \partial_j \varphi^a \partial_k \varphi^a d^2x = 0,$$  \hspace{1cm} (45)

that means the particle-likeness of the solution (42) in the way described above.

As a more nontrivial example, we consider the $SU(2)$-Skyrme model in 3+1-dimensions [1, 14], including the break–symmetry pion mass term

$$\mathcal{L} = -\frac{f^2_\pi}{4} \text{tr} L^2_\mu + \frac{1}{32g^2} \text{tr} [L_\mu L_\nu]^2 + \frac{m^2_\pi}{4} \text{tr} (U + U^+ - 2)$$

$$= \mathcal{L}^{(2)} + \mathcal{L}^{(4)} + \mathcal{L}_B,$$  \hspace{1cm} (46)

where, as usually, $L_\mu = U^{-1} \partial_\mu U$ is the left chiral current and $U = \sigma + i\tau^a \pi^a$ is the quaternion field. In the quaternion representation one has

$$\mathcal{L}^{(2)} = \frac{f^2_\pi}{2} (\partial_\mu \sigma)^2 + \frac{f^2_\pi}{2} (\partial_\mu \pi^a)^2.$$
\[ \mathcal{L}^{(4)} = -\frac{1}{4g^2}((\partial_\mu \pi^a)^4 - (\partial_\mu \pi^a \partial_\nu \pi^a)^2 ) + 2((\partial_\mu \pi^a)^2 (\partial_\nu \pi^a)^2 - \partial_\mu \pi^a \partial_\nu \pi^a \partial_\mu \pi^a \partial_\nu \pi^a)), \]

\[ \mathcal{L}_B = m^2_\pi (\sigma - 1). \]

Supposing the conventional "hedgehog" Ansatz

\[ \sigma = \cos \phi(r), \quad \pi^a = \frac{r^a}{r} \sin \phi(r) \]

we find for the mass of the skyrmion

\[ M = M^{(2)} + M^{(4)} + M_B, \]

\[ M^{(2)} = 4\pi f^2 \int r^2 dr \frac{1}{2} \left( \phi'^2 + \frac{2}{r^2} \sin^2 \phi \right), \]

\[ M^{(4)} = \frac{4\pi}{g^2} \int r^2 dr \frac{\sin^2 \phi}{2r^2} \left( \frac{\sin^2 \phi}{r^2} + 2\phi'^2 \right), \]

\[ M_B = 4\pi m^2_\pi \int r^2 dr \sin^2 \frac{\phi}{2}. \]

A straightforward calculation gives

\[ \int \frac{\partial \mathcal{L}}{\partial \partial_i u^A} \partial_i u^A d^3x = \frac{2}{3} \delta_{ij}(M^{(2)} + 2M^{(4)}), \]

here \( u^A = \{\sigma, \pi^a\}. \) Inserting eqs. (49) and (50) into (48), we obtain the first "particle-likeness condition" for the skyrmion

\[ M^{(2)} - M^{(4)} + 3M_B = 0. \]

On the other hand, the scaling \( u^A(x) \rightarrow u^A(\lambda x) \) yields

\[ M(\lambda) = \frac{1}{\lambda} M^{(2)} + \lambda M^{(4)} + \frac{1}{\lambda^3} M_B. \]

It is easy to verify, that the requirement of \( \left( \delta \frac{M(\lambda)}{d\lambda} \right)_{\lambda=1} = 0 \) coincides with the eq.(51) for the skyrmion. So the particle-likeness condition (51) for the skyrmion is fulfilled due to virial relation. Further, inserting the substitution (48) into eqs. (10) we find in the same way, that the second set of relations for the skyrmion is provided by the symmetry properties. So we'll get upon quantization, that the full bilinear form considered in [2], automatically simplifies up to a diagonal construction similar to eq.(35), and therefore the quantized skyrmion describes an extended non-relativistic particle.
Finally, we consider the 't Hooft–Polyakov monopole for the \( SU(2) \)-Yang–Mills–Higgs theory [3, 15], described by the Lagrangean

\[
\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 + \frac{1}{2}(D_\mu\phi)^2 - V(\phi),
\]

where

\[
V(\phi) = \frac{\lambda}{4}(\phi^a\phi^a - \eta^2)^2,
\]

\[
F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g\varepsilon^{abc} A_\mu^b A_\nu^c,
\]

\[
D_\mu\phi^a = \partial_\mu\phi^a + g\varepsilon^{abc} A_\mu^b \phi^c.
\]

The monopole solution is given by

\[
\phi^a = \frac{1}{g}\frac{r^a}{r^2} H(r), \quad A_i^a = \frac{1}{g}\varepsilon_{aij} \frac{r_j}{r^2}(1 - K(r)), \quad A_0^a = 0.
\]

The mass of the monopole is equal to

\[
M = \frac{4\pi\eta}{g} \int \frac{d\xi}{\xi^2} \left[ \xi^2 K'^2 + \frac{1}{2}(1 - K^2)^2 + \frac{1}{2}(H - \xi H')^2 + K^2 H^2 + \frac{\lambda}{4g^2}(H^2 - \xi^2)^2 \right],
\]

where the integration variable \( \xi \) is connected with the radial coordinate \( r \) via \( \xi = g\eta r \). The l.h.s of condition (56) reads

\[
\int \frac{\partial \mathcal{L}}{\partial \partial^\mu u^A} \partial_\mu u^A d^3x = 4\pi\frac{\eta}{g} \cdot \frac{2}{3} \delta_{ij} \int \frac{d\xi}{\xi^2} \left[ \xi^2 K'^2 + 2(1 - K)^2 + (1 - K)^3 + \frac{1}{2}(H - \xi H')^2 + H^2 K \right],
\]

here \( u^A = \{ A_\mu^a, \phi^a \} \). Now we take into account of the virial considerations. Performing the scaling \( u^A(\bar{x}) \to u^A(\lambda \bar{x}) \), we find

\[
M(\lambda) = \frac{1}{\lambda} \lambda^2 M_2 + \frac{1}{\lambda^2} M_1 + \frac{1}{\lambda^3} M_0 + \frac{1}{\lambda^4} M_V,
\]

\[
M_2 = \int d^3x \frac{1}{2} \left[ (\partial_\mu A_\mu^a)^2 - \partial_\mu A_\nu^a \partial_\nu A_\mu^a - (\partial_\mu \phi^a)^2 \right] = 4\pi\frac{\eta}{g} \int \frac{d\xi}{\xi^2} \left[ \xi^2 K'^2 + 2(1 - K)^2 + \frac{1}{2}(H - \xi H')^2 + H^2 \right],
\]

\[
M_1 = \int d^3x g \left[ \varepsilon^{abc} \partial_\mu A_\nu^a A_\mu^b A_\nu^c - \varepsilon^{abc} \partial_\mu \phi^a A_\mu^b \phi^c \right] = 4\pi\frac{\eta}{g} \int \frac{d\xi}{\xi^2} \left[ -2(1 - K)^3 - 2H^2 (1 - K) \right],
\]
$$M_0 = \int d^3x \frac{1}{2} g^2 \left[ \frac{1}{2} \left( (A_\mu^a)^4 - (A_\mu^a A_\nu^a)^2 \right) - \left( (A_\mu^a)^2 (\phi^b)^2 - (A_\mu^a \phi^a)^2 \right) \right] =$$
$$= 4\pi \frac{\eta}{g} \int \frac{d\xi}{\xi^2} \left[ \frac{1}{2} (1 - K)^4 + H^2 (1 - K)^2 \right],$$
$$M_V = \int d^3x \ V(\phi(x)) = 4\pi \frac{\eta}{g} \int \frac{d\xi}{\xi^2} \frac{\lambda}{4g^2} (H^2 - \xi^2)^2.$$  

The requirement \( \left( \frac{dM(\lambda)}{d\lambda} \right)_{\lambda=1} = 0 \) yields

$$\int \frac{d\xi}{\xi^2} \left[ \xi^2 K'^2 + \frac{1}{2} (1 - K)^2 (3K^2 + 2K - 1) + \frac{1}{2} (H - \xi H')^2 \right] - 2H^2 K + 3H^2 K^2 + 3\frac{\lambda}{4g^2} (H^2 - \xi^2)^2 = 0. \quad (59)$$

Inserting eqs.\((\text{57})\) and \((\text{58})\) into \((\text{5})\), we see that the particle-likeness condition \((\text{6})\) coincides with eq.\((\text{59})\). The condition \((\text{10})\) is fulfilled due to symmetry properties of the expression \((\text{55})\), just as in the case of skyrmion.

So we have proved the validity of the "particle-likeness" conditions \((\text{6})\) and \((\text{10})\) for the most important soliton solutions. Note, that there is a close connection between the condition \((\text{6})\) and virial relations for the static configuration by homogeneous dilatations. The relations \((\text{10})\) are usually fulfilled on account of symmetry properties of classical solutions. Since the Lorentz group is six-parametric and there is only one parameter by homogeneous dilatations, then it looks naturally, that the relations \((\text{6})\) and \((\text{10})\) should be splitted in two parts:

1) the l.h.s. of relations \((\text{6})\) should be diagonal

$$\int \frac{\partial L(\tilde{\xi})}{\partial \partial^i \varphi_c(\tilde{\xi})} \partial_i \varphi_c(\tilde{\xi}) \ d\tilde{\xi} = I \delta_{ij}, \quad (60)$$

and l.h.s. of eqs.\((\text{10})\) should vanish

$$\int \varepsilon_{iij} \xi_i \partial_j \varphi_c(\tilde{\xi}) \frac{\partial L(\tilde{\xi})}{\partial \partial^k \varphi_c(\tilde{\xi})} \ d\tilde{\xi} = 0, \quad (61)$$

provided by symmetry properties of the static configuration;

2) the value of \(I\) coincides with the mass \(M\) of the solution

$$I = M \quad (62)$$

due to the virial arguments.

The last statement is verified for all the interesting classical solutions. However, the general proof of this connection is not found yet.
To conclude let us mention, that the present analysis can be easily extended to internal degrees of freedom. On the other hand, the relations (6) and (10), being automatically consistent with exact solutions of equations of motion, can play an essential role of additional constraints in approximate calculations. For example, they can be explored as a test for various sample functions, used in describing the shape of the skyrmeon [1, 16]. The results of this work will be reported separately.

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