Hoops, Coops and the Algebraic Semantics of Continuous Logic

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Abstract

Büchi and Owen studied algebraic structures called hoops. Hoops provide a natural algebraic semantics for a class of substructural logics that we think of as intuitionistic analogues of the widely studied Lukasiewicz logics. Ben Yaacov extended Lukasiewicz logic to get what is called continuous logic by adding a halving operator. In this paper, we define the notion of continuous hoop, or coop for short, and show that coops provide a natural algebraic semantics for continuous logic. We characterise the simple and subdirectly irreducible coops and investigate the decision problem for various theories of coops. In passing, we give a new proof that hoops form a variety by giving an algorithm that converts a proof in intuitionistic Lukasiewicz logic into a chain of equations.

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1 Introduction

Around 1930, Łukasiewicz and Tarski [12] instigated the study of logics admitting models in which the truth values are real numbers drawn from some subset $T$ of the interval $[0, 1]$. In these models, conjunction is represented by capped addition $\land A \land B := \inf\{A + B, 1\}$ and negation is represented by inversion: $\neg A := 1 - A$. The set $T$ is required to contain 1 and to be closed under these operations. One then finds that $T$ is the intersection $G \cap [0, 1]$ where $G$ is some additive subgroup of $\mathbb{R}$ with $\mathbb{Z} \subseteq G$ and that $T$ is also closed under disjunction and implication defined by $A \lor B := \sup\{A + B - 1, 0\}$ and $A \Rightarrow B := \sup\{B - A, 0\}$. These logics are classical in that $\neg \neg A$ and $A$ are equivalent. Moreover the law of the excluded middle holds, but $A \Rightarrow A \land A$ only holds in the special case of Boolean logic for which $T = \{0, 1\}$, so apart from this special case, the logics are substructural.

These Łukasiewicz logics have been widely studied, e.g., as instances of fuzzy logics [9]. More recently ben Yaacov has used them as a building block in what is called continuous logic [2]. Continuous logic unifies work of Henson and others [10] that aims to overcome shortfalls of classical first-order model theory when applied to continuous structures such as metric spaces and Banach spaces. The language of continuous logic extends that of the usual propositional logic by adding a halving operator, written $A/2$. In the standard numerical model of continuous logic the set $T$ of truth values is the interval $[0, 1]$ and $A/2 := \frac{1}{2} A$.

Many basic facts about both Łukasiewicz and continuous logics depend on work of Rose and Rosser [14] and Chang [6, 5] who proved that the following axiom schemas together with the rule of modus ponens are complete for the propositional Łukawiecz logics:

\begin{align*}
A \Rightarrow (B \Rightarrow A) & \quad (A1) \\
(A \Rightarrow B) \Rightarrow (B \Rightarrow C) \Rightarrow (A \Rightarrow C) & \quad (A2) \\
((A \Rightarrow B) \Rightarrow B) \Rightarrow ((B \Rightarrow A) \Rightarrow A) & \quad (A3) \\
(\neg A \Rightarrow \neg B) \Rightarrow (B \Rightarrow A) & \quad (A4)
\end{align*}

This had been a long-standing conjecture of Łukasiewicz. Ben Yaacov [1] added the following axiom schemata for the halving operator:

\begin{align*}
(A/2 \Rightarrow A) & \Rightarrow A/2 \quad (A5) \\
A/2 & \Rightarrow (A/2 \Rightarrow A) \quad (A6)
\end{align*}

\footnote{We here follow the convention of the literature on continuous logic in ordering the truth values by increasing logical strength so that 0 represents truth and 1 falsehood.}
and showed that $A_1$–$A_6$ together with modus ponens are complete for the standard numerical model of continuous logic.

The goal of the present paper is to cast some light onto these axiomatizations by developing propositional Lukasiewicz logic and continuous logic as a series of extensions of intuitionistic affine logic. A similar approach for Lukasiewicz logic was developed in [7] where classical affine logic was taken as the starting point. The more restricted setting of intuitionistic affine logic will allow us to better calibrate the amount of contraction that needs to be added to affine logic to obtain Lukasiewicz logics. In particular, we obtain an intuitionistic counter-part of Lukasiewicz logic.

Our work began with the observation that Ben-Yaacov’s continuous logic, which we call $\text{CL}_c$, is an extension of a primitive intuitionistic substructural logic $\text{AL}_i$. We now consider an even more primitive logic $\text{AL}_u$ and develop $\text{CL}_c$ as depicted in Figure 1, which also shows how the Brouwer-Heyting intuitionistic propositional logic $\text{IL}$ and Boolean logic $\text{BL}$ relate to this development.

The structure of the rest of this paper is as follows:

Section 2 gives the definitions of the logical languages we deal with and of each of the twelve logics shown in Figure 1.

Section 3 gives sound and complete algebraic semantics for the logics in terms of certain classes of pocrims and hoops, algebraic structures that have been quite widely studied in connection with $\text{AL}_i$ and related...
logics. We introduce the notion of a continuous hoop or coop to give the algebraic semantics of continuous logic.

Section 4 considers coops from the perspective of universal algebra. We characterize the simple and subdirectly irreducible coops and use the results to begin an investigation of the decision problem for theories of coops.

Section 5 outlines further work, particularly concerning decidability.

2 The Logics

We work in a language $\mathcal{L}$ (or $\mathcal{L}_{1\frac{1}{2}}$ for emphasis) whose atomic formulas are the propositional constants 0 (truth) and 1 (falsehood) and propositional variables drawn from the set $\text{Var} = \{P, Q, \ldots\}$. If $A$ and $B$ are formulas of $\mathcal{L}$ then so are $A \otimes B$ (conjunction), $A \rightarrow B$ (implication) and $A/2$ (halving). We define $\mathcal{L}_1$ and $\mathcal{L}_{1\frac{1}{2}}$ to be the sublanguages of $\mathcal{L}$ that disallow halving and 1 respectively and we define $\mathcal{L}_0$ to be the intersection of $\mathcal{L}_1$ and $\mathcal{L}_{1\frac{1}{2}}$. We write $A^\bot$ as an abbreviation for $A \rightarrow 1$. The judgments of the logics we consider are sequents of the form $\Gamma \vdash A$, where the succedent $A$ is a formula and the antecedent $\Gamma$ is a multiset of formulas.

The inference rules for all our logics are the introduction and elimination rules for the two connectives\(^2\) shown in Figure 2. The various logics we deal with are distinguished only the by axioms we define for them. We define the axioms such that if $\Gamma \vdash B$ is an axiom, then so is $\Gamma, A \vdash B$ for any formula $A$. The way the antecedents of sequents are handled in the inference rules

\(^2\)Omitting disjunction from the logic greatly simplifies the algebraic semantics. While it may be unsatisfactory from the point of view of intuitionistic philosophy, disjunction defined using de Morgan’s law is adequate for our purposes.
then implies that we have the following derived rule of weakening:

\[
\Gamma \vdash B \\
\Gamma, A \vdash B \quad \text{[WK]}
\]

since any instance of this rule in a proof tree may be moved up the proof tree until it is just beneath an axiom and then the conclusion of the rule will already be an axiom.

It is easily proved for any logic with the inference rules of Figure 2 that a form of the deduction theorem holds in the sense that if one of the following three sequents is provable then so are the other two:

\[
A_1, \ldots, A_m \vdash B, \\
\vdash A_1 \leftarrow \cdots \leftarrow A_m \leftarrow B, \\
\vdash A_1 \otimes \cdots \otimes A_m \leftarrow B.
\]

The axiom schemata for our logics are selected from those shown in Figure 3. These are the axiom of assumption [ASM], ex-falso-quodlibet [EFQ], double negation elimination [DNE], commutative weak conjunction [CWC], commutative strong disjunction [CSD], the axiom of contraction [CON], and two axioms for the halving operator: one for a lower-bound [HLB] and one for an upper bound [HUB].

The logics we deal with are discussed in the next few paragraphs and the axioms for each logic are summarised in Table 1. In the logics that do not have the axiom schemata [EFQ], 1 plays no special role and may be omitted from the language and similarly halving may be omitted from the language in the logics that do not have the axioms schemata [HLB] and [HUB]. Alternatively, the full language \( \mathcal{L}_{1} \) may be used in all cases with 1 and \( \lambda / 2 \) effectively acting as variables in formulas that involve the m.

Unbounded\(^3\) intuitionistic affine logic, \( \mathbf{ALu} \), has for its axiom schemata [ASM] alone. All our other logics include \( \mathbf{ALu} \). Since the contexts \( \Gamma, \Delta \) are multisets, an assumption in the rules of Figure 2 can be used at most once. \( \mathbf{ALu} \) serves as a prototype for substructural logics with this property. It corresponds under the Curry-Howard correspondence to a \( \lambda \)-calculus with pairing (i.e., \( \lambda \)-abstractions of the form \( \lambda(x,y)\bullet t, \lambda((x,y),z)\bullet t, \lambda(x,(y,z))\bullet t \) etc.) in which no variable is used twice. Intuitionistic affine logic, \( \mathbf{ALi} \), extends \( \mathbf{ALu} \) with the axiom schemata [EFQ].

\(^3\)We use the term “unbounded” here for logics in which 1 has no special meaning and need not be an upper bound for the lattice of truth values.
Figure 3: Axiom Schemata

Classical affine logic, $\mathbf{AL}_c$, extends $\mathbf{AL}_i$ with the axiom schema $[\text{DNE}]$. It can also be viewed as the extension of the so-called multiplicative fragment of Girard’s linear logic [8] by allowing weakening and the axiom schema $[\text{EFQ}]$. We do not define an “unbounded” version of $\mathbf{AL}_c$ or its extensions, since $[\text{EFQ}]$ is derivable from $[\text{DNE}]$ in the presence of weakening $[\text{WK}]$.

Unbounded intuitionistic Lukasiewicz logic, $\mathbf{LL}_u$, extends $\mathbf{AL}_u$ with the axiom schema $[\text{CWC}]$. In $\mathbf{AL}_u$, $A \otimes (A \rightarrow B)$ can be viewed as a weak conjunction of $A$ and $B$. In $\mathbf{LL}_u$, we have commutativity of this weak conjunction. Intuitionistic Lukasiewicz logic, $\mathbf{LL}_i$, extends $\mathbf{LL}_u$ with the axiom schema $[\text{EFQ}]$.

Classical Lukasiewicz logic, $\mathbf{LL}_c$, extends $\mathbf{AL}_i$ with the axiom schema $[\text{CSD}]$. In $\mathbf{AL}_i$, $(A \rightarrow B) \rightarrow B$ can be viewed as a form of disjunction, stronger than that defined by $(A^+ \otimes B^\perp)^\perp$. In $\mathbf{LL}_c$ we have commutativity of this strong disjunction. This gives us the widely-studied multi-valued logic of Lukasiewicz.

Unbounded intuitionistic propositional logic, $\mathbf{IL}_u$, extends $\mathbf{AL}_u$ with the axiom schema $[\text{CON}]$, which is equivalent to a contraction rule allowing $\Gamma, A \vdash B$ to be derived from $\Gamma, A, A \vdash B$. Intuitionistic propositional logic, $\mathbf{IL}$, extends $\mathbf{IL}_u$ with the axioms schemata $[\text{EFQ}]$. $\mathbf{IL}$ is the
Table 1: The logics and their models

| Logic  | Axioms               | Models            |
|--------|----------------------|-------------------|
| $AL_u$ | $[ASM]$              | pocrims           |
| $AL_i$ | $AL_u + [EFQ]$       | bounded pocrims   |
| $AL_c$ | $AL_i + [DNE]$       | involutive pocrims|
| $LL_u$ | $AL_u + [CWC]$       | hoops             |
| $LL_i$ | $LL_u + [EFQ]$       | bounded hoops     |
| $LL_c$ | $AL_i + [CSD]$       | bounded involutive hoops |
| $IL_u$ | $AL_u + [CON]$       | idempotent pocrims|
| $IL_i$ | $IL + [DNE]$         | involutive idempotent pocrims |
| $IL_c$ | $IL_i + [CSD]$       | coops             |
| $CL_u$ | $LL_u + [HLB] + [HUB]$ | bounded coops   |
| $CL_i$ | $LL_i + [HLB] + [HUB]$ | involutive coops |

conjunction-implication fragment of the well-known Brouwer-Heyting intuitionistic propositional logic.

Boolean logic, $BL$, extends $IL$ with the axiom schema $[DNE]$. This is the familiar two-valued logic of truth tables.

Unbounded intuitionistic continuous logic, $CL_u$, allows the halving operator and extends $LL_u$ with the axiom schemas $[HLB]$ and $[HUB]$, which effectively give lower and upper bounds on the logical strength of $A/2$. The two axioms can also be read as saying that $A/2$ is equivalent to $A/2 \rightarrow A$. Intuitionistic continuous logic, $CL_i$, extends $CL_u$ with the axiom schemata $[EFQ]$. $CL_i$ is an intuitionistic version of the continuous logic of ben Yaacov [2].

Classical Continuous logic, $CL_c$ extends $CL_i$ with the axiom schema $[DNE]$. This gives ben Yaacov’s continuous logic. The motivating model takes truth values to be real numbers between 0 and 1 with conjunction defined as capped addition.

Our initial goal was to understand the relationships amongst $AL_i$, $LL_c$ and $CL_c$. The other logics came into focus when we tried to decompose the somewhat intractable axiom $[CSD]$ into a combination of $[DNE]$ and an intuitionistic component. We will see that the twelve logics are related as shown in Figure 1. In the figure, an arrow from $T_1$ to $T_2$ means that $T_2$ extends $T_1$, i.e., the set of provable sequents of $T_2$ contains that of $T_1$. In each square, the north-east logic is the least extension of the south-west
logic that contains the other two. For human beings, at least, the proof of this fact is quite tricky for the $\mathbf{AL}_{I}-\mathbf{LL}_{C}$ square, see [9, chapters 2 and 3].

The routes in Figure 1 from $\mathbf{AL}_{u}$ to $\mathbf{IL}$ and $\mathbf{BL}$ have been quite extensively studied [3, 13]. We are not aware of any work on $\mathbf{CL}_{u}$ and $\mathbf{CL}_{i}$, but these are clearly natural objects of study in connection with ben Yaacov’s continuous logic. It should be noted that $\mathbf{IL}_{u}$ and $\mathbf{CL}_{u}$ are incompatible: given $[\text{CON}]$, $A/2$ and $A/2 \otimes A/2$ are equivalent, so that from $[\text{HLB}]$ and $[\text{HUB}]$ one finds that $A/2 \to A$ and $A/2$ are both provable; which proves $A$, for arbitrary formulas $A$.

3 Algebraic Semantics

We give an algebraic semantics to our logics using pocrims: partially ordered, commutative, residuated, integral monoids. A pocrim$^4$, is a structure for the signature $(0, +, \rightarrow)$ of type $(0, 2, 2)$ satisfying the following laws:

\[
\begin{align*}
(x + y) + z &= x + (y + z) & [m_1] \\
x + y &= y + x & [m_2] \\
x + 0 &= x & [m_3] \\
x \geq x & & [o_1] \\
x \geq y \land y \geq z & \Rightarrow x \geq z & [o_2] \\
x \geq y \land y \geq x & \Rightarrow x = y & [o_3] \\
x \geq y & \Rightarrow x + z \geq y + z & [o_4] \\
x \geq 0 & [le] \\
x + y \geq z & \Leftrightarrow x \geq y \to z & [r]
\end{align*}
\]

where $x \geq y$ is an abbreviation for $x \to y = 0$.

When working in a pocrim, we adopt the convention that $\to$ associates to the right and has lower precedence than $+$. So, for example, the brackets in $(a + b) \to (c \to (d + f))$ are all redundant, while those in $((a \to b) \to c) + d$ are all required.

Let $VM = (M, 0, +, \to)$ be a pocrim. The laws $[m_1]$, $[o_3]$ and $[le]$ say that $(M, 0, +; \geq)$ is a partially ordered commutative monoid with the identity 0 as least element. In particular, $+$ is monotonic in both its arguments. Law $[r]$,

\footnote{Strictly speaking, this is a dual pocrim, since we order it by increasing logical strength and write it additively, whereas in much of the literature the opposite order and multiplicative notation is used (so halves would be square roots). We follow the ordering convention of the continuous logic literature.}
the residuation property, says that for any \( x \) and \( z \) the set \( \{ y \mid x + y \geq z \} \) is non-empty and has \( x \rightarrow z \) as least element. As is easily verified, \( \rightarrow \) is antimonotonic in its first argument and monotonic in its second argument.

Let \( \alpha : \text{Var} \rightarrow M \) be an interpretation of logical variables as elements of \( M \) and extend \( \alpha \) to a function \( v_\alpha : L_0 \rightarrow M \) by interpreting 0, \( \otimes \) and \( \rightarrow \) as 0, + and \( \rightarrow \) respectively. We say that \( \alpha \) satisfies the sequent \( C_1, \ldots, C_n \vdash A \) iff \( v_\alpha(C_1) + \ldots + v_\alpha(C_n) \geq v_\alpha(A) \). We say that \( \Gamma \vdash A \) is valid in \( M \) if it is satisfied by every assignment \( \alpha : \text{Var} \rightarrow M \), in which case we say \( M \) is a model of \( \Gamma \vdash A \). If \( \mathcal{C} \) is a class of pocrim, we say \( \Gamma \vdash A \) is valid in \( \mathcal{C} \) if it is valid in every member of \( \mathcal{C} \). We say that a logic \( L \) whose language is \( L_0 \) is sound for a class of pocrim \( \mathcal{C} \) if every sequent over \( L_0 \) that is provable in \( L \) is valid in \( \mathcal{C} \). We say that \( L \) is complete for \( \mathcal{C} \) if the converse holds. We then have:

**Theorem 1** \( \text{AL}_u \) is sound and complete for the class of all pocrim.

**Proof:** This is standard. Soundness is a routine exercise. For the completeness, one defines an equivalence relation \( \simeq \) on formulas such that \( A \simeq B \) holds iff both \( A \vdash B \) and \( B \vdash A \) are provable in the logic. Writing \( [A] \) for the equivalence class of a formula \( A \), one then shows that the set of equivalence classes \( T \) is the carrier set of a pocrim \( T = (T; 0, +, \rightarrow) \), where \( 0 = [0] \) and the operators + and \( \rightarrow \) are defined so that \( [A] + [B] = [A \otimes B] \) \( [A] \rightarrow [B] = [A \rightarrow B] \). In \( T \), the term model of the logic, \( C_1, \ldots, C_n \vdash A \) is valid, i.e., \( [C_1] + \ldots + [C_n] \rightarrow [A] = 0 \) holds, iff \( C_1, \ldots, C_n \vdash A \) is provable. Completeness follows, since a sequent that is valid in all pocrim must be valid in the pocrim \( T \) and hence must be provable.

The above theorem says that a sequent is provable in \( \text{AL}_u \) iff it has every pocrim as a model. In the sequel we will often use the theorem to derive laws that hold in all pocrim. For example, it is easy to find a proof in \( \text{AL}_u \) of the sequent \( P, Q \rightarrow P \vdash Q \rightarrow (P \otimes P) \), from which we may conclude that the law \( x + (y \rightarrow x) \geq y \rightarrow x + x \) holds in any pocrim.

A hoop is a pocrim that is naturally ordered, i.e., whenever \( x \geq y \), there is \( z \) such that \( x = y + z \). It is a nice exercise in the use of the residuation property to show that a pocrim is a hoop iff it satisfies the identity

\[
x + (x \rightarrow y) = y + (y \rightarrow x)
\]

From this it follows that the logic \( \text{LL}_u \) is sound and complete for the class of all hoops. See [3] for more information on hoops.
We say a pocrim is *idempotent* if it is idempotent as a monoid, i.e., it satisfies \( x + x = x \). Note that this condition implies condition \([\text{cwc}]\), since it implies \( x + y \geq x + (x \rightarrow y) = x + (x + (x \rightarrow y)) \geq x + y \), whence, \( x + (x \rightarrow y) = x + y = y + x = y + (y \rightarrow x) \). Using this, we find that \( \text{IL}_u \) is sound and complete for the class of all idempotent pocrims.

To complete our treatment of the bottom layer in Figure 1, we need to prove a lemma about hoops that will help us with the algebraic semantics of the halving operator. The hoop axiom \([\text{cwc}]\) is surprisingly powerful but often requires considerable ingenuity to apply. We have been greatly assisted in our work by using the late Bill McCune’s Prover9 and Mace4 programs to prove algebraic facts and to find counter-examples. Readers who enjoy a challenge may like to look for their own proof of the following lemma before reading ours, which is a fairly direct translation of that found after a few minutes by Prover9.

**Theorem 2 (Prover9)** The following hold in any hoop:

(i) if \( a = a \rightarrow b \) and \( c = c \rightarrow b \), then \( a = c \)

(ii) if \( a \geq a \rightarrow b \) and \( c = c \rightarrow b \), then \( a \geq c \)

(iii) if \( a \leq a \rightarrow b \) and \( c = c \rightarrow b \), then \( a \leq c \).

**Proof:** (i): this is immediate from parts (ii) and (iii).

(ii): by the hypothesis on \( a \), \( a + a \geq b \), and so, using the hypothesis on \( c \) and the fact that \( x + (y \rightarrow x) \geq y \rightarrow x + x \) discussed in the remarks following the proof of Theorem 1, we find:

\[
a + (c \rightarrow a) \geq c \rightarrow a + a \geq c \rightarrow b = c
\]

so that \( c \rightarrow a \geq a \rightarrow c \). Using the fact that if \( x \geq y \), then \( x + (y \rightarrow z) \geq z \), we have: \( (c \rightarrow a) + ((a \rightarrow c) \rightarrow c) \geq c \). As the hypothesis on \( c \) implies \( c + c \geq b \), this gives:

\[
c + (c \rightarrow a) + ((a \rightarrow c) \rightarrow c) \geq b.
\]

Using \([\text{cwc}]\) twice and the fact that \( x \rightarrow y \rightarrow x = 0 \), we find:

\[
b \leq c + (c \rightarrow a) + ((a \rightarrow c) \rightarrow c)
\]

\[
= a + (a \rightarrow c) + ((a \rightarrow c) \rightarrow c)
\]

\[
= a + c + (c \rightarrow a \rightarrow c)
\]

\[
= a + c.
\]

I.e., \( a + c \geq b \), so that \( a \geq c \rightarrow b = c \) as required.
(iii): by the hypothesis on \( c \) and using the fact that \((x\to y) + x \geq y\) twice, we have:
\[
c + (a \to c) + a = (c \to b) + (a \to c) + a \geq b.
\]
Using the hypothesis on \( a \), we have:
\[
c + (a \to c) \geq a \to b \geq a
\]
So \( c \geq (a \to c) \to a \), implying:
\[
c \to (a \to c) \to a = 0.
\]
Using \([\text{cwc}]\) and the facts that \( x \to y \leq (z \to x) \to y \) and \( (x \to y) + x \geq y \), we have:
\[
c = c + (c \to (a \to c) \to a)
\]
\[
= ((a \to c) \to a) + (((a \to c) \to a) \to c)
\]
\[
\geq ((a \to c) \to a) + (a \to c)
\]
\[
\geq a.
\]

We define a coop to be a structure for the signature \((0,+\to,/)\) of type \((0,2,2,1)\) whose \((0,+\to)-\)reduct is a hoop and such that for every \( x \) we have:
\[
x/2 = x/2 \to x \quad \text{[h]}
\]
From \([h]\), one clearly has \( x \geq x/2 \to x = x/2 \), i.e., \( x \to x/2 = 0 \) and so using also \([\text{cwc}]\) one finds \( x/2 + x/2 = x/2 + (x/2 \to x) = x + (x \to x/2) = x \) justifying the choice of notation. (Our convention is that \(/2\) binds tighter than the infix operators, so the brackets are needed in \((x+y)/2\) but not in \(x \to (x/2)\).

The following very useful theorem shows that the halving operator is uniquely defined by the condition \( x/2 = x/2 \to x \) (so we could have defined a coop to be a hoop that satisfies the axiom \( \forall x \exists y : y = y \to x \) and taken the halving operator to be defined on such a hoop by equation \([h]\)).

**Theorem 3** Let \( a \) and \( b \) be elements of a coop. Then the following hold:

(i) \( a = b/2 \iff a = a \to b \)

(ii) \( a \geq b/2 \iff a \geq a \to b \)

(iii) \( a \leq b/2 \iff a \leq a \to b \)

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**Proof:** $\Rightarrow$: let $R \in \{=, \geq, \leq\}$, then using the definition of a coop and the fact that $\to$ is antimonotonic in its left argument, we have:

\[
a R b/2 = b/2 \to b R a \to b
\]

$\Leftarrow$: immediate from Theorem 2 and the definition of a coop.

**Corollary 4** Let $a$ and $b$ be elements of a coop. Then the following hold:

(i) $a = b \iff a/2 = b/2$

(ii) $a \geq b \iff a/2 \geq b/2$

(iii) $a/2 = a \iff a = 0$

(iv) $a/2 + b/2 \geq (a + b)/2$

(v) $a/2 \to b/2 = (a \to b)/2$

**Proof:** (i): immediate from (ii).

(ii)$\Rightarrow$: if $a \geq b$, then as $a = a/2 + a/2$, we have $a/2 \geq a/2 \to b$ and then, by the theorem, $a/2 \geq b/2$.

(ii)$\Leftarrow$: if $a/2 \geq b/2$, then $a = a/2 + a/2 \geq b/2 + b/2 = b$.

(iii): if $a/2 = a$, then, by the theorem, $a/2 = a/2 \to a = a \to a = 0$.

(iv): We have

\[
a/2 + b/2 \to a/2 + b/2 \to a + b = a/2 + b/2 + a/2 + b/2 \to a + b = a + b \to a + b = 0
\]

I.e., $a/2 + b/2 \geq a/2 + b/2 \to a + b$, so, by the theorem, $a/2 + b/2 \geq (a + b)/2$

(v): I claim that $(a \to b)/2 \geq a/2 \to b/2$ and $(a \to b)/2 \leq a/2 \to b/2$, from which the result follows. For the first part of the claim, we have $(a \to b)/2 \geq a/2 \to b/2$ iff $a/2 + (a \to b)/2 \geq b/2$ and, by the theorem, this holds iff $a/2 + (a \to b)/2 \geq a/2 + (a \to b)/2 \to b$, i.e. iff $b \leq (a/2 + (a \to b)/2) + (a/2 + (a \to b)/2) = a + (a \to b)$ which is true. For the second part of the claim, we have:

\[
(a/2 \to b/2) \to a \to b = (a/2 \to b/2) + a \to b = a/2 + a/2 + (a/2 \to b/2) \to b = a/2 + b/2 + (a/2 \to b/2) \to b = a/2 + (b/2 \to a/2) \to b/2 \to b = a/2 + (b/2 \to a/2) \to b/2 \leq a/2 \to b/2
\]
where the inequality follows from the fact that \( \rightarrow \) is antimonotonic in its first argument. So, by the theorem, \( (a \rightarrow b)/2 \leq a/2 \rightarrow b/2 \) as required.

**Corollary 5** There are no non-trivial finite coops.

**Proof:** If \( a \) is a non-zero element of a coop, parts (ii) and (iii) of the corollary imply that the sequence \( a, a/2, (a/2)/2, \ldots \) is strictly decreasing. Hence a finite coop has no non-zero elements.

Given an interpretation, \( \alpha : \text{Var} \rightarrow M \) with values in a coop, we extend the function \( v_{\alpha} : \mathcal{L}_0 \rightarrow M \) to \( \mathcal{L}_{\frac{1}{2}} \) in such a way that \( v_{\alpha}(A/2) = (v_{\alpha}(A))/2 \) and extend the notions of satisfaction, etc. accordingly. The proof of Theorem 1 is easily extended to show that the logic \( \text{CL}_u \) is sound and complete for the class of coops (using Theorem 3 to show that the halving operation on the term model is well-defined).

We now have a sound and complete algebraic semantics for each of the logics in the bottom layer of Figure 1. Moving to the middle layer, let us say that a pocrim, hoop or coop is **bounded** if it has a (necessarily unique) annihilator, i.e., an element \( 1 \) such that for every \( x \) we have:

\[
x + 1 = 1 \quad \text{[ann]}
\]

Assume the pocrim \( M \) is bounded. Then \( 0 \leq x \leq x + 1 = 1 \) for any \( x \) and \( (M; \leq) \) is indeed a bounded ordered set. Given an interpretation, \( \alpha : \text{Var} \rightarrow M \) with values in a bounded pocrim, we extend the function \( v_{\alpha} : \mathcal{L}_0 \rightarrow M \) to \( \mathcal{L}_1 \) so that \( v_{\alpha}(1) = 1 \) and extend the notions of satisfaction etc. accordingly. Yet again the proof of Theorem 1 is easily extended to show that the logic \( \text{AL}_i \) is sound and complete for the class of bounded pocrims.

We then find that the logics \( \text{LL}_i, \text{CL}_i \) and \( \text{IL} \) are sound and complete for bounded hoops, bounded coops and idempotent bounded hoops respectively. Idempotent bounded hoops are also known as Brouwerian algebras and are known to be the conjunction-implication reducts of Heyting algebras (see [11] and the works cited therein).

Finally, for the top layer of Figure 1, we say a pocrim is **involutive** if it is bounded and satisfies \( \neg \neg x = x \), where we write \( \neg x \) as an abbreviation for \( x \rightarrow 1 \), Idempotent involutive hoops are easily seen to be the conjunction-implication reducts of Boolean algebras. We find that \( \text{AL}_c, \text{LL}_c, \text{CL}_c \) and \( \text{BL} \) are sound and complete for involutive pocrims, involutive hoops, involutive coops and idempotent involutive hoops respectively. This completes the proof of the following theorem:
Theorem 6 The logics $\text{ALi}_1, \ldots, \text{CLc}$, $\text{ILu}$, $\text{IL}$ and $\text{BL}$ of Figure 1 are sound and complete for the corresponding classes of pocrims, hoops and coops listed in Table 1.

A Wajsberg hoop is a hoop satisfying the identity
\[(x \to y) \to y = (y \to x) \to x \quad [\text{csd}]\]

It can be shown that Wajsberg hoops are the same as bounded involutive hoops.

The classes of pocrims associated with the logics in the left-hand column in Figure 1 are very general: any partial order can be embedded in an involutive pocrim. To see this, let $X$ be any partially ordered set. Take a disjoint copy $X^\perp$ of $X$ (say $X^\perp = X \times \{1\}$) and write $x^\perp$ for the image in $X^\perp$ of $x \in X$. Choose objects 0, 1, $r$ and $s$ distinct from each other and from the elements of $X \cup X^\perp$ and order the disjoint union $P_X = \{0, r\} \cup X \cup X^\perp \cup \{s, 1\}$ so that, (i), $0 < r < X < X^\perp < s < 1$, (ii), the subset $X$ has the given ordering and, (iii), $X^\perp$ has the opposite ordering. Extend the mapping $(\cdot)^\perp : X \to X^\perp$ to all of $P_X$ so that $0^\perp = 1$, $r^\perp = s$ and $a^\perp \perp = a$ for all $a$. Then $^\perp$ is an order-reversing mapping of $P_X$ onto itself and there is a unique commutative binary operation $+$ on $P_X$ with the following properties:
\[
\begin{align*}
    a + 0 &= a, & \text{for every } a; \\
    a + b &= s, & \text{for every } a, b \geq r \text{ such that } a \not\geq b^\perp; \\
    a + b &= 1, & \text{for every } a, b \geq r \text{ such that } a \geq b^\perp.
\end{align*}
\]

Now let $P_X = (P_X, 0, +, \to)$ where $\to$ is defined using de Morgan’s law: $a \to b = (a + b^\perp)^\perp$. Then one finds that $a \to b = 0$ iff $a \geq b$ in $P_X$ with respect to the order defined above and the laws for an involutive pocrim other than associativity of $+$ are then easily verified for $P_X$. For the associativity of $+$, first note that if $0 \in \{a, b, c\}$, $(a + b) + c = a + (b + c)$ is trivial. If $a, b, c \geq r$ then $a + b, b + c \geq r^\perp$ and we have:
\[
    1 \geq (a + b) + c \geq r^\perp + r = 1 = r + r^\perp \leq a + (b + c) \leq 1.
\]
so that $a + (b + c) = 1 = (a + b) + c$. Thus $P_X$ is indeed an involutive pocrim.

It is known that the class of involutive pocrims is not a variety i.e., it cannot be characterised by equational laws. Since involutive pocrims are characterised over bounded pocrims and over pocrims by equational laws, it follows that the class of pocrims and the class of bounded pocrims are also not varieties. See [13] and the works cited therein for these results and their
Bosbach [4] gave a direct proof of an equational axiomatization of the class of hoops. Using Theorem 6, we can give an alternative proof that shows how a proof of a sequent ⊢ A may be translated into an equational proof that α = 0, where α is a translation into the language of pocrims of the formula A.

**Theorem 7** A structure $H = (H; 0, +, \to)$ is a hoop iff $(H; 0, +)$ is a commutative monoid and the following equations hold in $H$:

1. $x \to x = 0$
2. $x \to 0 = 0$
3. $0 \to x = x$
4. $x + y \to z = x \to y \to z$
5. $x + (x \to y) = y + (y \to x)$

**Proof:** It follows easily from the definitions (or from Theorem 6) that the equations hold in any hoop. For the converse, Theorem 6 implies that it is sufficient to show that if there is proof of ⊢ A in LLu then $[A]$ (the element of the term model of LLu represented by A) can be reduced to 0 using the commutative monoid laws and equations 1 to 5. More generally, if $B_1, \ldots, B_m$ and A are formulas, with $\gamma = [B_1] + \ldots + [B_m]$ and $a = [A]$, we will show how to translate a proof of $B_1, \ldots, B_m \vdash A$ into a sequence of equations $\gamma \to a = a_1 = \ldots = a_n = 0$ where each equation $a_i = a_{i+1}$ is obtained by applying one of the equations 1 to 5 to a subterm of $a_i$ or $a_{i+1}$ and then simplifying or rearranging as necessary using the commutative monoid laws. We have base cases for the axioms [ASM] and [CWC] of Figure 3 and inductive steps for the rules of Figure 2.

[ASM]: we want $\gamma + a \to a = 0$ for arbitrary $\gamma$ and $a$:

\[
\begin{align*}
\gamma + a \to a &= \quad \text{(eq. 4)} \\
\gamma \to a \to a &= \quad \text{(eq. 1)} \\
\gamma \to 0 &= 0 \quad \text{(eq. 2)}
\end{align*}
\]
[CWC]: we want $\gamma + a + (a \rightarrow b) \rightarrow b + (b \rightarrow a) = 0$ for arbitrary $\gamma$, $a$ and $b$:

\[
\begin{align*}
\gamma + a + (a \rightarrow b) \rightarrow b + (b \rightarrow a) &= \quad \text{(eq. 5)} \\
\gamma + b + (b \rightarrow a) \rightarrow b + (b \rightarrow a) &= \quad \text{(eq. 4)} \\
\gamma \rightarrow b + (b \rightarrow a) \rightarrow b + (b \rightarrow a) &= \quad \text{(eq. 1)} \\
\gamma \rightarrow 0 &= 0 \quad \text{(eq. 2)}
\end{align*}
\]

[→I]: we are given $\gamma + a \rightarrow b = 0$ and we want $\gamma \rightarrow a \rightarrow b = 0$:

\[
\begin{align*}
\gamma \rightarrow a \rightarrow b &= \quad \text{(eq. 4)} \\
\gamma + a \rightarrow b &= \quad \text{(hyp.)}
\end{align*}
\]

[→E]: we are given $\gamma \rightarrow a = 0$ and $\delta \rightarrow a \rightarrow b = 0$ and we want $\gamma + \delta \rightarrow b = 0$:

\[
\begin{align*}
\gamma + \delta \rightarrow b &= \quad \text{(hyp.)} \\
\gamma + (\gamma \rightarrow a) + \delta \rightarrow b &= \quad \text{(eq. 5)} \\
a + (a \rightarrow \gamma) + \delta \rightarrow b &= \\
(a \rightarrow \gamma) + \delta + a \rightarrow b &= \quad \text{(hyp.)} \\
(a \rightarrow \gamma) + \delta + a + (\delta \rightarrow a \rightarrow b) \rightarrow b &= \quad \text{(eq. 4)} \\
(a \rightarrow \gamma) + \delta + a + (\delta + a \rightarrow b) \rightarrow b &= \quad \text{(eq. 5)} \\
(a \rightarrow \gamma) + b + (b \rightarrow \delta + a) \rightarrow b &= \\
(a \rightarrow \gamma) + (b \rightarrow \delta + a) + b \rightarrow b &= \quad \text{(eq. 4)} \\
(a \rightarrow \gamma) + (b \rightarrow \delta + a) \rightarrow b \rightarrow b &= \quad \text{(eq. 1)} \\
(a \rightarrow \gamma) + (b \rightarrow \delta + a) \rightarrow 0 &= 0 \quad \text{(eq. 2)}
\end{align*}
\]

[⊗I]: we are given $\gamma \rightarrow a = 0$ and $\delta \rightarrow b = 0$ and we want $\gamma + \delta \rightarrow a + b = 0$.

\[
\begin{align*}
\gamma + \delta \rightarrow a + b &= \quad \text{(hyp.)} \\
\gamma + (\gamma \rightarrow a) + \delta + (\delta \rightarrow b) \rightarrow a + b &= \quad \text{(eq. 5)} \\
a + (a \rightarrow \gamma) + b + (b \rightarrow \delta) \rightarrow a + b &= \\
(a \rightarrow \gamma) + (b \rightarrow \delta) + a + b \rightarrow a + b &= \quad \text{(eq. 4)} \\
(a \rightarrow \gamma) + (b \rightarrow \delta) \rightarrow a + b \rightarrow a + b &= \quad \text{(eq. 1)} \\
(a \rightarrow \gamma) + (b \rightarrow \delta) \rightarrow 0 &= 0 \quad \text{(eq. 2)}
\end{align*}
\]

[⊗E]: we are given $\gamma \rightarrow a + b = 0$ and $\delta + a + b \rightarrow c = 0$ and we want
\( \gamma + \delta \rightarrow c = 0: \)

\[
\begin{align*}
\gamma + \delta & \rightarrow c = \\
\gamma + (\gamma \rightarrow a + b) + \delta & \rightarrow c = \\
a + b + (a + b \rightarrow \gamma) + \delta & \rightarrow c = \\
(a + b \rightarrow \gamma) + \delta + a + b & \rightarrow c = \\
(a + b \rightarrow \gamma) & \rightarrow \delta + a + b & \rightarrow c = \\
(a + b \rightarrow \gamma) & \rightarrow 0 = 0
\end{align*}
\] (hyp.)

(eq. 5)

(eq. 4)

(hyp.)

(eq. 2)

This completes the induction.

The axiomatization in the statement of Theorem 7 is natural and convenient but by no means minimal. See [4] for more concise axiomatizations.

4 Algebra of Coops

Blok and Ferreirim [3] have studied hoops from the perspective of universal algebra. Here we undertake an analogous study of coops. Our goal is to obtain decision problems for useful theories of coops. This will require various facts about hoops, most of which may be found in [3], but in the dual (multiplicative) notation. We begin by looking at some special classes of coops, for which certain facts that hold for involutive hoops can be obtained rather efficiently by dint of the halving operator.

4.1 Some Special Classes of Coops

We say a hoop is cancellative if its underlying monoid is a cancellation monoid \((x + y = x + z \text{ implies } y = z)\). Let us say a hoop is semi-cancellative if \(x + y = x + z \text{ and } y \neq z \text{ implies } x + y \text{ is an annihilator} \) (i.e., the hoop is bounded with \(x + y = 1\)). Thus a hoop that is semi-cancellative and not bounded is cancellative. In a linearly ordered hoop, the semi-cancellative property is easily seen to be equivalent to the condition that \(x + y = x \text{ implies that either } y = 0 \text{ or } x \text{ is an annihilator.} \)

Semi-cancellative coops enjoy the property that halving is almost a homomorphism, or, indeed, a real homomorphism if the coop is cancellative:

\textbf{Lemma 8} Let \(C\) be a semi-cancellative coop, then, for any \(x, y \in C\), either \((x + y)/2 = x/2 + y/2\) or \(x + y = 1\).
Proof: Since $x/2 + x/2 = x$ and $y/2 + y/2 = y$, we have $x + y = x/2 + y/2 + x/2 + y/2$. On the other hand, since $x + y \geq x/2 + y/2$, [cwc] implies that $x + y = x/2 + y/2 + (x/2 + y/2 \to x + y)$. By the semi-cancellative property, either $x + y = 1$ or $x/2 + y/2 = x/2 + y/2 \to x + y$. In the latter case, Theorem 3 (i) tells us that $x/2 + y/2 = (x + y)/2$. ■

We now prove a very useful theorem that will let us transfer some important results about bounded coops to unbounded coops. This corresponds to Chang’s construction of the enveloping group of an MV-algebra but the proof involves much less tricky algebra.

Before stating the theorem, we introduce some notation and terminology that will be used throughout the sequel. Let $G = (G_0; +, \geq)$ be a 2-divisible linearly ordered commutative group. Writing $G_{\geq 0}$ for the set of non-negative elements of $G$, we then have a coop $G_{\geq 0} = (G_{\geq 0}; 0, +, \to, /2)$ where $x \to y := \sup \{0, y - x\}$ and $x/2$ is that element of $G$ such that $x/2 + x/2 = x$ (this is unique because $G$ is linearly ordered and hence torsion-free).

If $L$ is any coop and $a$ is any non-zero element of $L$, we have a bounded coop $L_a = (\{x \in L \mid x \leq a\}, +, \to, /2)$ where $x + a y := \inf \{a, x + y\}$. We say $L_a$ is $L$ capped at $a$. We will just write $x + y$ for $x + a y$ in contexts where it is clear that we are working in $L_a$. If $L = G_{\geq 0}$ for some 2-divisible linearly ordered commutative group $G$, we write $G_{[0,a]}$ for $L_a$. Note that $G_{[0,a]}$ is an involutive coop: with $-x = a - x$, we clearly have $-\neg x = x$.

As an example, take $G$ to be the additive group $\mathbb{D}$ of dyadic rationals $\mathbb{D} = (\{\frac{n}{2^m} \mid i \in \mathbb{Z}, n \in \mathbb{N}\}; 0, +, \geq)$. We then have an unbounded coop $\mathbb{D}_{\geq 0}$ and from $\mathbb{D}_{\geq 0}$, we obtain the bounded coops $\mathbb{D}_{[0,a]} = ([0,a] \cap \mathbb{D}, 0, +, \to, /2)$ for $a$ any positive dyadic rational. Note that the isomorphism type of $\mathbb{D}_{[0,a]}$ depends on $a$: $\mathbb{D}_{[0,1]}$ contains no $x$ such that $3x$ is the annihilator but $\mathbb{D}_{[0,3]}$ does.

**Theorem 9** Let $C$ be a semi-cancellative bounded coop. Then there exist a cancellative unbounded coop $\hat{C}$, an element $\hat{1} \in \hat{C}$ and an isomorphism $\alpha : C \to \hat{C}_1$. Every element of $C$ has the form $2^m\alpha(a)$ for some $a \in C$ and $m \in \mathbb{N}$. If $C$ is linearly ordered then so is $\hat{C}$.

Proof: Let $D = C^\mathbb{N}$ be the product of countably many copies of $C$. Thus elements of $D$ are sequences $x = \langle x_0, x_1, \ldots \rangle$ of elements of $C$ and the coop operations are defined pointwise: $(x + y)_i = x_i + y_i$, $(x \to y)_i = x_i \to y_i$ and $(x/2)_i = x_i/2$. For this proof, let us say $x \in D$ is regular if $x_{i+1} = x_i/2$ for all but finitely many $i$. Using Corollary 4 and Lemma 8 as appropriate, it is easy to see that if $x$ and $y$ are regular then so are $x \to y$, $x + y$ and $x/2$. Thus
the regular elements comprise a subcoop \( R \) of \( D \). Define a relation \( \sim \) on \( R \) by \( x \sim y \) iff \( x_i = y_i \) for all but finitely many \( i \). It is a routine exercise to verify that \( \sim \) is a congruence. Let \( \hat{\mathcal{C}} \) be \( R/\sim \) and, for \( a \in \mathcal{C} \), let \( \alpha(a) \) be given by

\[
(\alpha(a))_i = \frac{1}{2}a \quad \text{and let } \hat{1} = \alpha(1).
\]

By Corollary 4, \( \alpha(a \rightarrow b) = \alpha(a) \rightarrow \alpha(b) \) for any \( a, b \in \mathcal{C} \), and, by Lemma 8, if \( a + b < 1 \), \( \alpha(a + b) = \alpha(a) + \alpha(b) \). It is easy to verify that \( \alpha \) is an injection and that \( \alpha(C) = \{ a \in \hat{\mathcal{C}} | \hat{1} \geq a \} \), from which it follows that \( \alpha \) is an isomorphism between \( \mathcal{C} \) and \( \hat{\mathcal{C}}_1 \). If \( x \in R \), there is \( a \in \mathcal{C} \) and \( m \in \mathbb{N} \) such that for all \( i \in \mathbb{N} \), \( x_{m+i} = \frac{1}{2^i}a \) and then \( [x] = 2^m \alpha(a) \). Hence, for any \( s \in \hat{\mathcal{C}} \), \( \frac{1}{2^n}s \in \alpha(C) \) for all but finitely many \( i \) and from this it follows that, \( \mathcal{C} \) is semi-cancellative and hence cancellative and that, if \( \mathcal{C} \) is linearly ordered, then so is \( \hat{\mathcal{C}} \).

\textbf{Theorem 10} Let \( \mathcal{C} \) be a linearly ordered cancellative unbounded coop. Then there exist a 2-divisible linearly ordered group \( \overline{\mathcal{C}} \) and an isomorphism \( \beta : \mathcal{C} \rightarrow \overline{\mathcal{C}}_{\geq 0} \).

\textbf{Proof:} Define \( \overline{\mathcal{C}} \) to be the group of differences of \( \mathcal{C} \) and let \( \beta : \mathcal{C} \rightarrow \overline{\mathcal{C}} \) be the natural homomorphism. Every element of \( \overline{\mathcal{C}} \) has the form \( \beta(a) - \beta(b) \) for \( a, b \in \mathcal{C} \). \( \beta(a) - \beta(b) = \beta(c) - \beta(d) \) iff there are \( x, y \in \mathcal{C} \), such that \( a + x = c + y \) and \( b + x = d + y \). We have \( (\beta(a/2) - \beta(b/2)) + (\beta(a/2) - \beta(b/2)) = \beta(a) - \beta(b) \), so \( \overline{\mathcal{C}} \) is 2-divisible. As \( \mathcal{C} \) is linearly ordered, given \( a, b \in \mathcal{C} \), either (i) \( a \geq b \), in which case, \( \beta(a) - \beta(b) = \beta(b \rightarrow a) \), since \( a + 0 = (b \rightarrow a) + b \) and \( b + 0 = 0 + b \), or (ii) \( b \geq a \), in which case, \( \beta(a) - \beta(b) = -\beta(a \rightarrow b) \), since \( a + 0 = 0 + a \) and \( b + 0 = (a \rightarrow b) + a \). Thus for any \( s \in \overline{\mathcal{C}} \), either \( s \in \beta(C) \) or \( s \in -\beta(C) \). Moreover if \( s \in \beta(C) \cap -\beta(C) \), we have \( s = \beta(a) = -\beta(b) \) whence for some \( x, y \in \mathcal{C} \) we have \( a + x = y \) and \( x = b + y \), whence \( a + b + y = y \) implying \( a = b = 0 \), thus \( \beta(C) \cap -\beta(C) = \{ 0 \} \). Since \( \beta(C) + \beta(C) = \beta(C) \), it follows that \( \beta(C) \) is the non-negative cone of a linear order on \( \overline{\mathcal{C}} \) and that \( \beta \) is an isomorphism of \( \mathcal{C} \) with \( \overline{\mathcal{C}}_{\geq 0} \).

\textbf{Theorem 11} The first order theories of the following classes of coops are decidable: (i) linearly ordered cancellative coops (ii) linearly ordered bounded semi-cancellative coops, (iii) linearly ordered semi-cancellative coops.

\textbf{Proof:} Using Theorems 9 and 10, one can find primitive recursive reductions of the theory of linearly ordered bounded semi-cancellative coops to that of linearly ordered cancellative coops and of the latter theory to the theory of linearly ordered 2-divisible groups. The theory of linearly ordered groups is decidable by a well-known result of Gurevich, and hence so is the theory of 2-divisible linearly ordered groups (since the latter is a finitely
axiomatisable extension of the former). Hence, (i) and (ii) hold. As for
(iii), a general linearly ordered semi-cancellative coop is either cancellative
or bounded, so the theory in (iii) is the intersection of the theories in (i)
and (ii).

4.2 Homomorphisms and Ideals

Let $H$ be a hoop. An ideal $I$ of $H$, is a downwards-closed submonoid:

\begin{align*}
0 \in I & \subseteq H \\
I + I & \subseteq I \\
I & \downarrow \subseteq I
\end{align*}

where, for any $X, Y \subseteq H$, $X + Y = \{x + y \mid x \in X, y \in Y\}$ and $X_\downarrow = \{y \in H \mid \exists x \in X \cdot x \geq y\}$. For example, if $X \subseteq H$, the ideal generated by $X$, $l(X)$, is the set comprising all $y \in H$, such that for some $x_1, \ldots, x_n \in X$, $y \leq x_1 + \ldots + x_n$. $l(X)$ is easily seen to be an ideal and is clearly the smallest ideal containing $X$. As a special case, the ideal $l(x)$ generated by $x \in H$, comprises all elements $y$ such that $y \leq nx$ for some $n \in \mathbb{N}$. We say an ideal $I$ is proper if $0 \neq I \neq H$.

If $I$ is an ideal, then $I$ is actually the carrier set of a subhoop, since, we
have $I \rightarrow I \subseteq H \rightarrow I \subseteq I$ (since $I$ is downwards-closed and $x \rightarrow y \leq y$ for any $x$ and $y$). If $K$ is also a hoop and $f : H \rightarrow K$ is a homomorphism of hoops, we define the kernel of $f$, $\ker(f)$, as follows:

$$\ker(f) := \{x : H \mid f(x) = 0\}.$$ 

$\ker(f)$ is clearly a submonoid of $H$. Moreover, if $y \in \ker(f)$ and $x \leq y$, then, by definition, $f(y) = 0$ and $y \rightarrow x = 0$, and then $f(x) = f(y) \rightarrow f(x) = f(y \rightarrow x) = f(0) = 0$, so $x \in \ker(f)$. Thus $\ker(f)$ is an ideal of $H$. Conversely, if $I$ is an ideal of $H$, define a relation $\theta \subseteq H \times H$, by $x \theta y \iff x \rightarrow y \in I \wedge y \rightarrow x \in I$. It is then routine to verify that $\theta$ is a hoop congruence on $H$ and that, with $p_\theta : H \rightarrow H/\theta$, the natural projection onto the quotient hoop, we have $\ker(p_\theta) = I$. It follows that the lattice of congruences on $H$ is isomorphic to its lattice of ideals. In particular, a hoop is simple (i.e., it admits no non-trivial congruences) iff it has no proper ideals (so that $l(x) = H$ for every non-zero $x \in H$).

**Theorem 12** If $C$ and $D$ are coops then a mapping $f : C \rightarrow D$ is a homomorphism of coops iff it is a homomorphism of the underlying hoops of $C$ and $D$. 

20
Proof: Necessity is trivial. For sufficiency, assume \( f : C \rightarrow D \) is a homomorphism of hoops. By definition, \( f(x \rightarrow y) = f(x) \rightarrow f(y) \) for any \( x, y \in C \). So for any \( x \in C \), we have:

\[
f(\frac{x}{2}) \rightarrow f(x) = f(\frac{x}{2} \rightarrow x) = f(\frac{x}{2})
\]
whence by Theorem 3 we must have \( f(\frac{x}{2}) = f(x)/2 \). It follows that \( f \) is a homomorphism of coops.

Thus we need no new notion for the kernels of coop homomorphisms: the lattice of congruences on a coop is isomorphic to its lattice of ideals in the sense defined above. We have the following immediate corollary:

**Corollary 13** A coop is simple iff its \((0, +, \rightarrow)\)-reduct is a simple hoop.

In categorical language, the forgetful functor from the category of coops to the category of hoops provides an isomorphism between the category of coops and the full subcategory of the category of hoops comprising the objects satisfying the axiom \( \forall x \cdot \exists y \cdot x = y \rightarrow x \). In fact, there is a functor that maps a hoop to an enveloping coop. This is adjoint to the forgetful functor from coops to hoops. The forgetful functor is faithful (as they always are) and the above says that it is full as well.

### 4.3 Simple Coops

The hoop \( H \) is said to be archimedean iff, for any non-zero \( x \in H \) and any \( y \in H \), there is \( m \in \mathbb{N} \), such that \( y \leq mx \). We then have:

**Theorem 14** A hoop is simple iff it is archimedean.

**Proof:** Immediate from the definition of \( l(x) \) and the fact that \( H \) is simple iff \( l(x) = H \) for every non-zero \( x \in H \).

**Theorem 15** A coop is simple iff it is archimedean.

**Proof:** Immediate from Corollary 13 and Theorem 14.

We will need an interesting property of hoops due to Bosbach [4]. From a logical perspective, this says that \( \text{LL}_1 \) enjoys the principle that to prove an implication one may assume the converse implication.

**Lemma 16** Let \( H \) be a hoop, \( x, y \in H \). Then

\[
(x \rightarrow y) \rightarrow (y \rightarrow x) = y \rightarrow x
\]
Proof: Clearly $y \to x \geq (x \to y) \to y \to x$, so it is enough to prove that
$((x \to y) \to y \to x) \geq y \to x$, or equivalently that $y + ((x \to y) \to y \to x) \geq x$,
but we have:

\[
y + ((x \to y) \to y \to x) = \\
y + (y \to (x \to y) \to x) = \\
((x \to y) \to x) + ((x \to y) \to y) \geq \\
((x \to y) \to x) + (x \to y) \geq x
\]

where the penultimate inequality holds since $\to$ is antimonotonic in its first argument and $(x \to y) \to x \leq x$.

Lemma 17 Let $H$ be a hoop such that for all $x, y \in H$, if $y = x \to y$, then $x = 0$ or $y = 0$. Then $H$ is linearly ordered.

Proof: By Lemma 16, $(a \to b) \to (b \to a) = b \to a$ and then by assumption, either $a \to b = 0$ or $b \to a = 0$, i.e., either $a \geq b$ or $b \geq a$.

Lemma 18 If $H$ is a simple hoop and $x, y \in H$ are such that $y = x \to y$, then $x = 0$ or $y = 0$.

Proof: If $y = x \to y$, it is easy to see by induction that $y = nx \to y$, for every $n \in \mathbb{N}$. But by Theorem 14, $H$ is archimedean, so either $x = 0$ or, for some $n$, $y = nx \to y = 0$.

Theorem 19 Simple hoops and simple coops are linearly ordered.

Proof: For hoops, this is immediate from Lemmas 18 and 17. The statement for coops follows using Corollary 13.

We will see later that simple coops are also Wajsberg hoops.

Lemma 20 Let $C$ be a coop such that $C \subseteq \mathbb{R}$ and let $G$ be the subgroup of the additive group $\mathbb{R}$ generated by $C$. Then $G$ is 2-divisible and:

(i) if $C$ is a subcoop of $\mathbb{R}_{\geq 0}$, then $G = C \cup -C$ and $C = G_{\geq 0}$;
(ii) if $C$ is a subcoop of $\mathbb{R}_{[0,1]}$ and $1 \in C$, then $G = \bigcup_{n \in \mathbb{Z}} (n + C)$ and $C = G_{[0,1]}$.

Proof: If $g \in G$, $g$ can be written as $i_1x_1 + \ldots i_mx_m$ where $x_j \in C$ and $i_j \in \mathbb{Z}$. But then $g/2 = i_1y_1 + \ldots i_my_m$, where $y_j = x_j/2 \in C$. So $G$ is indeed 2-divisible.
(i): It is enough to prove that \( G = C \cup -C \), for then \( C = G \cap \mathbb{R}_{\geq 0} \) and so \( C = G_{\geq 0} \). Since clearly \( C \cup -C \subseteq G \), we have only to show \( C \cup -C \) is closed under negation and addition. Closure under negation is clear. To show closure under addition, we have to show that if \( x, y \in C \), then \( x + y, -x + -y \) and \( x - y \) are in \( C \cup -C \). This is clear for \( x + y \) and \( -x + -y \), since \( C \) is closed under addition. As for \( x - y \), if \( x \geq y \), then, by definition, \( y \to x = x - y \in C \), while, if \( x < y \), \( x \to y = y - x \in C \) and so \( x - y \in -C \).

(ii): It is enough to prove that \( G = \bigcup_{n \in \mathbb{Z}} (n + C) \), for then \( C = G \cap [0,1] \) and so \( C = G_{[0,1]} \). Clearly \( \bigcup_{n \in \mathbb{Z}} (n + C) \subseteq G \), so we have only to show that \( \bigcup_{n \in \mathbb{Z}} (n + C) \) is closed under negation and addition. So let \( x, y \in C \) and \( j, k \in \mathbb{Z} \) be given. We have:

\[
-(j + x) = -(j + 1) + 1 - x = -(j + 1) + (x \to 1) \in -(j + 1) + C
\]
giving closure under negation. If \( x + y \leq 1 \) (in \( G \), not \( C \)), then we have:

\[
(j + x) + (k + y) = (j + k) + (x + y) \in (j + k) + C,
\]
while if \( 1 < x + y < 2 \), we can find \( i, n \in \mathbb{N} \) with \( i \leq 2^n \), such that \( x > \frac{i}{2^n} \) and \( y > \frac{2^n - i}{2^n} \) and then we have:

\[
(j + x) + (k + y) = (j + k + 1) + (x - \frac{i}{2^n}) + (y - \frac{2^n - i}{2^n})
\]

\[
= (j + k + 1) + (\frac{i}{2^n} \to x) + (\frac{2^n - i}{2^n} \to y)
\]

\[
\in (j + k + 1) + C
\]

since \( 1 \in C \), so that \( \frac{i}{2^n}, \frac{2^n - i}{2^n} \in C \), since \( C \) is closed under halving and coop addition (which agrees with the group addition when the sum in the group is at most 1). Finally if \( x + y = 2 \), we have:

\[
(j + x) + (k + y) = (j + k + 2) + 0 \in (j + k + 2) + C.
\]

In all cases, \( (j + x) + (k + y) \in \bigcup_{n \in \mathbb{Z}} (n + C) \) and so \( \bigcup_{n \in \mathbb{Z}} (n + C) \) is closed under addition, as claimed.

Dyadic rational numbers will play an important rôle in the sequel as they did in the above proof. We will now generalise the halving operator on a coop to multiplication by arbitrary non-negative dyadic rationals. So, let \( C = (C; 0,+,\to,/2) \) be any coop and define a function \( \phi : \mathbb{N}_{>0} \times \mathbb{N} \times C \to C \) such that:

\[
\phi(1,0,x) = x
\]
\[
\phi(1,n+1,x) = \phi(1,n,x)/2
\]
\[
\phi(i,n,x) = i\phi(1,n,x)
\]

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Using the fact that $x/2 + x/2 = x$, we find that the following holds for any $i, n \in \mathbb{N}$ and $x \in \mathbb{C}$.

$$\phi(2i, n + 1, x) = \phi(i, n, x)$$

Thus, if $\frac{i}{2^n} = \frac{j}{2^m}$ (in $\mathbb{Q}$), $\phi(i, n, x) = \phi(j, m, x)$ for any $x$, and so $\phi$ induces a function $\mathbb{D}_{\geq 0} \times \mathbb{C} \to \mathbb{C}$ which we write multiplicatively: $(p, x) \mapsto px$. (Here, as with $\mathbb{N}, \mathbb{Z}$, etc., we abuse notation by writing $\mathbb{D}_{\geq 0}$ and $\mathbb{D}_{[0,a]}$ both for the structures and for their carrier sets.) So for example $\frac{3}{4}x = \left(\frac{x}{2}/2 + \frac{x}{2}/2 + \frac{x}{2}/2\right)$.

Clearly we have $(p + q)x = px + qx$, so, for fixed $x$, $p \mapsto px$ defines a homomorphism of monoids from $\mathbb{D}_{\geq 0}$ to $\mathbb{C}$. Also, we have $p(x+y) = px + py$, so that for fixed $p, x \mapsto px$ is a homomorphism of monoids from $\mathbb{C}$ to itself.

If $p, q \in \mathbb{D}$ with $0 \leq p, q \leq 1$, we have $p(qx) = (pq)x$, so we have an action on $\mathbb{C}$ qua monoid of the multiplicative monoid of dyadic rationals in the interval $[0,1]$. However, if $p > 1$ or $q > 1$, $p(qx) \neq (pq)x$ in general; e.g. with $M = \mathbb{D}_{[0,1]}$ and $x = 1$, one has $2x = x$, so that $\frac{1}{2}(2x) = \frac{1}{2}x = \frac{1}{2}$, while $(\frac{1}{2}2)x = 1x = 1$.

**Lemma 21** Let $x \neq 0$ be an element of a coop, $\mathbb{C}$, and $0 \leq i < j \leq 2^n$. Then (i) $\frac{i}{2^n}x < \frac{j}{2^n}x$ and (ii) $\frac{i}{2^n}x \rightarrow \frac{j}{2^n}x = \frac{i-j}{2^n}x$.

**Proof**: We prove (ii) first. Note that since $\frac{i}{2^n}x + \frac{j-i}{2^n}x = \frac{j}{2^n}x$, we have $\frac{i-j}{2^n}x \geq \frac{j}{2^n}x \rightarrow \frac{j}{2^n}x$ by the residuation property. Thus as $a \rightarrow b \geq a+c \rightarrow b+c$, it is enough to prove (ii) in the special case when $j = 2^n$, for then for $j < 2^n$ we have:

$$\frac{i}{2^n}x \rightarrow \frac{j}{2^n}x \geq \frac{i + 2^n - j}{2^n}x \rightarrow \frac{2^n}{2^n}x = \frac{2^n - (i + 2^n - j)}{2^n}x = \frac{j - i}{2^n}x.$$

So taking $j = 2^n$, let us prove (ii) by induction on $n$. The statement is trivial when $n = 0$. So given $n \geq 0$ assume that $\frac{1}{2^n}x \rightarrow x = \frac{2^n}{2^{n+1}}x$ holds for any $x$ and $i$ with $0 \leq i < 2^n$. Let $x$ and $i$ with $0 \leq i < 2^{n+1}$ be given. If $i = 2^n$, then $\frac{i}{2^{n+1}}x = \frac{1}{2^n}x$ and we have $\frac{1}{2^n}x \rightarrow x = \frac{1}{2^n}x$ by the coop laws. If
\( i < 2^n \), we have (using the inductive hypothesis on the line marked \((\ast)\)):

\[
\frac{2^{n+1} - i}{2^{n+1}} x = \frac{2^n - i}{2^{n+1}} x + \frac{1}{2^n} x
\]

\[
= \frac{2^n - i}{2^{n+1}} x + \left( \frac{1}{2} x \to x \right)
\]

\[
= \frac{2^n - i}{2^{n+1}} x + \left( \frac{2^n - i}{2^{n+1}} x + \frac{i}{2^{n+1}} x \to x \right)
\]

\[
= \frac{2^n - i}{2^{n+1}} x + \left( \frac{2^n - i}{2^{n+1}} x \to \frac{i}{2^{n+1}} x \to x \right)
\]

\[
= \left( \frac{i}{2^{n+1}} x \to x \right) + \left[ \left( \frac{i}{2^{n+1}} x \to x \right) \to \frac{2^n - i}{2^{n+1}} x \right]
\]

\[
= \left( \frac{i}{2^{n+1}} x \to x \right) + \left[ \left( \frac{i}{2^{n+1}} x \to x \right) \to \frac{2^n - i}{2^n} x \right]
\]

\[
= \left( \frac{i}{2^{n+1}} x \to x \right) + \left[ \left( \frac{i}{2^{n+1}} x \to x \right) \to \left( \frac{i}{2^n} x \to \frac{1}{2^n} x \right) \right]
\]

\[
= \left( \frac{i}{2^{n+1}} x \to x \right) + \left[ \left( \frac{i}{2^{n+1}} x \to x \right) \to \left( \frac{i}{2^n} x \to \frac{1}{2^n} x \right) \right]
\]

\[
= \frac{i}{2^{n+1}} x \to x.
\]

If \( 2^{n+1} > i > 2^n \), then we have:

\[
\frac{i}{2^{n+1}} x \to x = \frac{i - 2^n}{2^{n+1}} x \to \frac{1}{2^n} x \to x
\]

\[
= \frac{i - 2^n}{2^{n+1}} x \to \frac{1}{2^n} x
\]

\[
= \frac{i - 2^n}{2^n} x \to \frac{1}{2^n} x
\]

\[
= \frac{2^n - (i - 2^n)}{2^n} x
\]

\[
= \frac{2^{n+1} - i}{2^{n+1}} x.
\]

This completes the proof of part \((ii)\). Part \((i)\) follows since, by part \((ii)\), we have \( \frac{i}{2^n} x \to \frac{i + 1}{2^n} x = \frac{1}{2^n} x \neq 0 \), whence \( \frac{i}{2^n} x < \frac{i + 1}{2^n} x \leq \frac{j}{2^n} x \).

By the following lemma, simple coops are semi-cancellative.

**Lemma 22** Let \( C \) be a coop and let \( x, y \in C \) be such that \( x + y = x \), then either \( x = mx \) for all \( m \in \mathbb{N} \) or \( y \leq \frac{1}{2^n} x \) for all \( n \in \mathbb{N} \). In particular, if \( C \) is simple, and hence archimedean, either \( x \) is an annihilator or \( y = 0 \).
Proof: By an easy induction, we have \( x + my = x \) for all \( m \in \mathbb{N} \). If \( y > \frac{1}{2^n} x \) for some \( n \in \mathbb{N} \), then we have
\[
x = x + 2^ny \geq x + 2^n(\frac{1}{2^n}x) = 2x
\]
and then by another easy induction we have \( x = mx \) for all \( m \in \mathbb{N} \). 

If \( H \) is a hoop and \( 0 \neq x \in H \), define the depth of \( x \) to be the smallest \( d \in \mathbb{N} \) such that \( (d+1)x = dx \), or to be \( \infty \) if no such \( d \) exists. Lemma 21 implies that if \( x \) is a non-zero element of a coop, then the depth of \( \frac{1}{2^n}x \) is at least \( 2^n \).

Lemma 23 Let \( H \) be a simple hoop. Then either (i) every non-zero element has infinite depth or (ii) \( H \) is bounded and every non-zero element has finite depth.

Proof: Assume (i) does not hold, so there is a non-zero \( x \in H \) with finite depth \( d \), so \( dx = (d+1)x \). By induction, for any \( n > d \), we have \( dx = nx \). Let \( a = dx \). Then \( na = ndx = dx = a \) for any \( n > 0 \), so that, as \( H \) is simple, \( H = 1(a) = a \downarrow \). Now if \( y \) is any non-zero element, \( 1(y) = H \), so \( a \leq ny \) for some \( n \) and we have \( ny \geq a \geq (n+1)y \) so that \( ny = (n+1)y \) and \( y \) has finite depth.

Theorem 24 Let \( C \) be a simple coop. Then there is 2-divisible subgroup \( G \) of the additive group \( \mathbb{R}^+ \), such that either (i) \( C \) is isomorphic to \( G_{\geq 0} \), or (ii) \( C \) is isomorphic to \( G_{[0,1]} \).

Proof: By Theorems 15 and 19, \( C \) is archimedean and linearly ordered. We use these properties without further comment in the rest of the proof.

If \( C \) is not bounded, then, by Lemma 23, there is a non-zero \( e \in C \) with infinite depth, so that \( ne < (n+1)e \) for every \( n \in \mathbb{N} \). We will show that case (i) holds. To see this, define \( f : C \to \mathbb{R}_{\geq 0} \) by:
\[
f(x) = \sup \{ \frac{i}{2^n} | i, n \in \mathbb{N}, \frac{i}{2^n} e \leq x \}.
\]
For every \( x \), we have \( 0e \leq x \leq ne \) for all large enough \( n \). Thus the set whose supremum is used in the definition of \( f \) is always non-empty and bounded above, so \( f \) is well-defined. Clearly, \( f \) is at least weakly monotonic, i.e., if \( x \leq y \), then \( f(x) \leq f(y) \). We claim that \( f \) is a homomorphism from \( C \) to the coop \( \mathbb{R}_{\geq 0} \). By Lemma 22, the \((0,+)-\)reduct of \( C \) is a cancellation
monoid, so that as \( 2e = e + e = e + (e \rightarrow 2e) \), we have \( e = e \rightarrow 2e \), whence \( e = (2e)/2 \). By induction, \( e = \frac{1}{2^n}(2^n e) \) for any \( n \in \mathbb{N} \). Hence, for any \( m \), taking \( x = 2^m e \) in Theorem 21 we find that for any \( a, b \) of the form \( \frac{i}{2^n}x \), with \( 0 \leq i \leq 2^n \) we have \( f(a + b) = f(a) + f(b) \) and \( f(a \rightarrow b) = f(a) \rightarrow f(b) \).

Letting \( m \) tend to infinity, these equations hold for any \( a, b \in D \), where \( D \) is the set \( \{ \frac{i}{2^n}e \mid i, n \in \mathbb{N} \} \) of all dyadic rational multiples of \( e \). Thus \( D \) is a subcoop of \( C \) isomorphic to \( \mathbb{D}_{\geq 0} \). By Theorem 21, given \( x, y \in C \) and any \( n \in \mathbb{N} \), there are \( p_n, q_n, r_n, s_n \in D \) such that \( p_n \leq x \leq q_n \), \( r_n \leq y \leq s_n \), \( f(q_n) - f(p_n) = \frac{1}{2^{n+1}} \) and \( f(s_n) - f(r_n) = \frac{1}{2^{n+1}} \). But then as \( f|_D \) is a coop-homorphism and \( f \) is weakly monotonic, we have:

\[
\begin{align*}
\ f(p_n + r_n) &= f(p_n) + f(r_n) \\
\ f(q_n + s_n) &= f(q_n) + f(s_n) \\
\ f(p_n + r_n) &\leq f(x + y) \leq f(q_n + s_n) \\
\ f(p_n) + f(r_n) &\leq f(x) + f(y) \leq f(q_n) + f(s_n) \\
\ f(q_n + s_n) - f(p_n + r_n) &\leq \frac{1}{2^n}
\end{align*}
\]

Letting \( n \) tend to infinity, we must have that \( f(x + y) = f(x) + f(y) \) and \( f(x \rightarrow y) = f(x) \rightarrow f(y) \) and \( f \) is indeed a homomorphism from \( C \) to \( \mathbb{R}_{\geq 0} \) as claimed. But \( C \) is simple, hence \( f \) is either identically zero or is one-to-one, but clearly \( f(e) = 1 \neq 0 \), so \( f \) embeds \( C \) as a subcoop of \( \mathbb{R}_{\geq 0} \). (i) follows immediately using Lemma 20.

Now assume \( C \) is bounded, with annihilator \( a \), say. So \( a \geq x \) for every \( x \in C \). To see that that case (ii) holds, define \( g : C \rightarrow \mathbb{R}_{[0,1]} \) by:

\[
g(x) = \sup\{ \frac{i}{2^n} \mid i, n \in \mathbb{N}, i \leq 2^n, \frac{i}{2^n}a \leq x \}.
\]

Then by an argument very similar to the one used above in the unbounded case, \( C \) has a dense subcoop \( D_1 \) such that \( g|_{D_1} \) is an isomorphism from \( D_1 \) to \( \mathbb{D}_{[0,1]} \). Then, approximating \( x + y \) and \( x \rightarrow y \) by elements of \( D_1 \) just as we did above, we find that \( g \) is a homomorphism embedding \( C \) as a subcoop of \( \mathbb{R}_{[0,1]} \), from which (ii) follows using Lemma 20. \[ \square \]
4.4 Subdirectly Irreducible Coops

Recall that an algebra $A$ is subdirectly irreducible iff the intersection $\mu$ of its non-identity congruences is not the identity congruence, in which case $\mu$ is called the monolith. Thus a hoop or a coop is subdirectly irreducible iff the intersection of all its non-zero ideals is non-zero. In this section, we determine the structure of subdirectly irreducible coops.

If $C$ and $D$ are subcoops of a coop $E$, we say $E$ is the ordinal sum of $C$ and $D$ and write $E = C \nabla D$ iff $C \cap D = \{0\}$, $C \cup D = E$ and whenever $c \in C$ and $0 \neq d \in D$, $c + d = d$. It is easy to see that, if $E = C \nabla D$ and $c \in C$ and $0 \neq d \in D$, then $d > c$ and $c \rightarrow d = d$. Thus $C$ is an ideal and $E/C \cong D$. We will find that any subdirectly irreducible coop is $S \nabla F$ where $S$ is totally ordered and subdirectly irreducible and $F$ can be any coop. This could also be establishing using the analogous result for subdirectly irreducible hoops proved in [3], but the extra structure in a coop admits a slightly more efficient presentation.

**Theorem 25** Hoops and coops have the congruence extension property.

**Proof:** By Theorem 12 and the discussion of ideals that precedes it, it suffices to show that if $C$ is a subhoop of a hoop $D$, then for any ideal $I \subseteq C$, there is an ideal $J \subseteq D$, such that $I = J \cap C$. But, if $I \subseteq C$ is an ideal, it is easily verified from the definitions that $I = J \cap C$ where $J$ is the ideal of $D$ generated by $I$.

Let $C$ be a subdirectly irreducible coop, so that the set of all non-zero ideals of $C$ intersect in a non-zero ideal $M$, which we call the monolithic ideal. Since coops have the congruence extension property, $M$ viewed as a coop in its own right can have no non-trivial ideals, so $M$ is a simple coop, and so by Theorems 15 and 19, $M$ is archimedean and linearly ordered.

If $x \in C$, we define the implicative stabilizer $IS(x)$ as follows:

$$ IS(x) := \{ s \in C \mid s \rightarrow x = x \}. $$

It is easily verified that $IS(x)$ is an ideal. So, for any $x$, either $IS(x) = \{0\}$ or $IS(x) \supseteq M$. If $X \subseteq C$, we write $IS(X)$ for $\bigcap_{x \in X} IS(x)$.

**Theorem 26** Let $C$ be a subdirectly irreducible coop with monolithic ideal $M$ and let $F, S \subseteq C$ be defined as follows:

$$ F := \{ f \in C \mid M \subseteq IS(f) \} $$

$$ S := IS(F) $$

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Then:

(i) \( \forall x \in C \setminus \{0\} \cdot \exists a \in M \setminus \{0\} \cdot x \geq a; \)
(ii) \( \forall f \in F \setminus \{0\}, a \in M \cdot f \geq a; \)
(iii) \( \forall a \in M, f \in F \setminus \{0\} \cdot a + f = f; \)
(iv) \( \forall f \in F \setminus \{0\}, x \in C \cdot x \geq f \Rightarrow x \in F; \)
(v) \( \forall x \in C, f \in F, x \cdot f \Rightarrow f \in F; \)
(vi) \( \forall f \in F \setminus \{0\}, x \in C \cap F \cdot f > x; \)
(vii) \( F \) is the carrier set of a subcoop \( F \) of \( C; \)
(viii) \( S \) is a linearly ordered ideal of \( C, \) and \( S \cap F = \{0\}; \)
(ix) Writing \( S \) for the subcoop with carrier set \( S, \) \( S \) is semi-cancellable;
(x) \( C = S \setminus F. \)

**Proof:** First note that if \( x \in C \) and \( a \rightarrow x = x \) for some \( a \in C \setminus \{0\}, \) then \( IS(x) \neq \{0\}, \) hence \( M \subseteq IS(x) \) so that \( x \in F. \)
(i): if \( 0 \neq x \in C, \) then \( \{0\} \neq M \subseteq l(x), \) there is \( a \in M \) and \( n \in \mathbb{N}, \) with \( 2^n x \geq a \neq 0, \) but then \( 0 \neq \frac{1}{2^n} a \in M \) and \( x > 2^n x \geq a \) (where the first inequality follows by induction using part (iv) of Corollary 4).
(ii): since \( a \in M \subseteq l(f), \) \( m f \geq a \) for some \( m \in \mathbb{N} \). By Lemma 16, \( f \rightarrow a = (a \rightarrow f) \rightarrow f \rightarrow a = f \rightarrow a = 2f \rightarrow a, \) since \( f \in F. \) By induction, \( f \rightarrow a = n f \rightarrow a \) for every \( n \in \mathbb{N}. \) In particular, \( f \rightarrow a = mf \rightarrow a = 0. \)
(iii): by part (ii), \( f \geq a, \) i.e. \( f \rightarrow a = 0. \) Hence, using \( [cwc], \) \( a + f = a + (a \rightarrow f) = f + (f \rightarrow a) = f. \)
(iv) assume \( f \in F, x \in C \) and \( x \geq f \neq 0. \) We need to show that if \( a \in M, a \rightarrow x = x. \) But given \( a \in M, \) we have \( a \rightarrow x \geq a \rightarrow f = f, \) i.e., \( ((a \rightarrow x) \rightarrow f) = 0. \) Hence:

\[
\begin{align*}
    a \rightarrow x &= (a \rightarrow x) + ((a \rightarrow x) \rightarrow f) \\
    &= f + (f \rightarrow a \rightarrow x) \quad [cwc] \\
    &= f + (a \rightarrow f \rightarrow x) \\
    &= f + a + (a \rightarrow f \rightarrow x) \quad (iii) \\
    &= f + (f \rightarrow x) + ((f \rightarrow x) \rightarrow a) \quad [cwc] \\
    &\geq x.
\end{align*}
\]

So \( x \geq a \rightarrow x \geq x \) giving \( x = a \rightarrow x \) as required.
(v): if \( a \in M \) and \( f \in F, a \rightarrow f = f \) by the definition of \( F. \) So, for any \( x \in C, a \rightarrow x \rightarrow f = x \rightarrow a \rightarrow f = x \rightarrow f, \) whence \( x \rightarrow f \in F. \)
(vi): Let \( f \in F \) and \( x \in C \setminus F. \) if \( a \in M, \) we have \( a \rightarrow f \rightarrow x = a + f \rightarrow x =
\( f \to x \), by part (iii), so \( x \geq f \to x \in F \) and by part (iv) we can only have \( f \to x = 0 \), i.e., \( f \geq x \), and the inequality must be strict, since \( x \notin F \).

(vii): Clearly \( 0 \in F \). Given \( f, g \in F \), we must show that \( f + g, f \to g \) and \( f/2 \) all belong to \( F \). As \( f + g \geq f \), \( f + g \in F \) follows from part (iv). That \( f \to g \in F \) follows from part (v). Finally \( f/2 \in F \) follows from part (v) together with part (v) of Corollary 4, since given \( 0 \neq a \in M \) and \( a \to f = f \), then we have \( 0 \neq a/2 \in M \) and \( a/2 \to f/2 = (a \to f)/2 = f/2 \).

(viii): As the intersection of a set of ideals, \( S \) is itself an ideal. If \( x \in S \cap F \), then \( x = x \to x = 0 \), so \( S \cap F = \{0\} \). If \( s, t \in S \) and \( s \leftrightarrow t = t \), I claim that either \( s = 0 \) or \( t = 0 \), whence \( S \) is linearly ordered by Lemma 17. To prove the claim, if \( s \leftrightarrow t = t \) and \( s \neq 0 \), then, by part (i), there is \( a \in M \) such that \( s \geq a > 0 \), but then \( t \geq a \to t \geq s \to t \geq t \), so \( a \to t = t \) and \( t \in F \), so \( t \in S \cap F \) and therefore \( t = 0 \).

(ix): Let \( s, t \in S \) with \( t \neq 0 \) and \( s + t = s \). We must prove that \( s \) annihilates \( S \), i.e., \( S = s \cdot \). We may choose an \( a \in M \) such that \( t \geq a \neq 0 \) and then \( s = s + t \geq s + a \geq s \), whence \( s + a = s \). If \( u \in S \), then we have \( a \leftrightarrow s \leftrightarrow u = a + s \leftrightarrow u = s \leftrightarrow u \), so \( s \leftrightarrow u \in S \cap F = \{0\} \) by part (viii). Hence \( s \leftrightarrow u = 0 \), i.e., \( s \geq u \).

(\( x/2 \): by part (viii) \( S \cap F = \{0\} \). I claim that, if \( x \in C \setminus F \) and \( 0 \neq f \in F \), then \( x \in S \) and \( x \to f = f \). Given this, we must have that \( C = S \cup F \) and \( S \cap F = \{0\} \) and so \( C = S \to F \). So assume \( x \in C \setminus F \) and \( 0 \neq f \in F \). We must prove that \( f = x \to f = x + f \). By part (v), \( x \to f \to f \in F \), but \( x \geq (x \to f) \to f \) and \( x \notin F \), so by part (iv), we must have \( (x \to f) \to f = 0 \), i.e., \( x \to f \geq f \) implying \( f = x \to f \). Using \([\text{cwc}]\) and part (vi), we have \( f = f + (x \to f) = x + (x \to f) = x + f \) and the claim is true. \( \blacksquare \)

We refer to the subcoops \( F \) and \( S \) of the theorem as the fixed subcoop and the support subcoop respectively. Since the support is linearly ordered and semi-cancellative the following theorem applies to it.

**Theorem 27** Let \( L \) be a linearly ordered semi-cancellative coop. Then

(i) If \( L \) is bounded, it is involutive;

(ii) \( L \) is a Wajsberg coop, i.e., for any \( s, t \in L \), \( (t \to s) \to s = (s \to t) \to t \).

**Proof:** (i): As usual write 1 for the annihilator of \( L \) and \( \neg x \) for \( x \to 1 \). Note that by Corollary 4, \((-s)/2 = s/2 \to 1/2 \). We must show that \( \neg s = s \) for any \( s \in L \). We claim that \( \neg(\cdot) : L \to L \) is injective. To see this, assume \( s, t \in L \) with \( \neg s = \neg t \). Then \( s/2 \to 1/2 = (-s)/2 = (-t)/2 = t/2 \to 1/2 \). As \( 1/2 \geq s/2 \) and \( 1/2 \geq t/2 \), we have \( 1/2 = (-s)/2 + s/2 = (-t)/2 + t/2 \). But \( 1/2 \) is not an annihilator, so by part (i), this implies \( s/2 = t/2 \), whence \( s = t \),
completing the proof that \( \neg \) is injective. But for any \( s \in L \), \( \neg \neg s = \neg s \), so if \( \neg \) is injective, \( \neg s = s \).

(ii): We claim that for any \( s, t \in L \), \( (t \to s) \to s = \min\{s, t\} \), which is well-defined because \( L \) is linearly ordered. Assuming the claim, we have:

\[
(t \to s) \to s = \min\{s, t\} = \min\{t, s\} = (s \to t) \to t
\]

so the claim implies the required identity. To prove the claim, note that if \( t \geq s \), we have:

\[
(t \to s) \to s = 0 \to s = s = \min\{s, t\}
\]

while if \( s \geq t \), we have

\[
s = (t \to s) + t = (t \to s) + ((t \to s) \to s).
\]

If \( L \) is unbounded or if \( L \) is bounded but \( s \) is not the annihilator, then the semi-cancellative property gives us

\[
(t \to s) \to s = t = \min\{s, t\}.
\]

If \( L \) is bounded, let us write \( 1 \) for its annihilator and \( \neg x \) for \( x \to 1 \) as we did in the proof of part (i). Then if \( s = 1 \), we have:

\[
(t \to s) \to s = \neg t = t = \min\{s, t\}
\]

by part (i). In all cases, the claim holds and the proof is complete.

Note that a non-trivial ordinal sum is never Wajsberg: if \( s \in S \) and \( f \in F \), then in \( S \dashv F \), we have:

\[
(s \to f) \to f = f \to f = 0
\]

\[
(f \to s) \to s = 0 \to s = s
\]

So \( (s \to f) \to f = (f \to s) \to s \) iff \( 0 \in \{s, f\} \).

**Theorem 28** The universal theory of Wajsberg coops is decidable. (I.e., the set of purely universal formulas in the language of a coop that are valid in all coops is decidable).

**Proof:** We claim that any Wajsberg coop is isomorphic to a subcoop of a product of linearly ordered semi-cancellative coops. Given part (ii) of Theorem 27, such a product is itself a Wajsberg hoop, hence, given the claim, the universal theory of Wajsberg coops reduces to that of linearly ordered
semi-cancellative coops and by Theorem 11 the full first order theory of linearly ordered semi-cancellative coops is decidable.

As for the claim, let $W$ be a Wajsberg coop. By Birkhoff’s theorem, $W$ embeds in a product $\prod_i C_i$, where each $C_i$ is a subdirectly irreducible homomorphic image of $W$. By the remarks above, when we write $C_i$ as the ordinal sum of its support and fixed part, $S_i \cong F_i$, $F_i = \{0\}$, so $C_i$ is isomorphic to $S_i$. The claim follows from parts (viii) and (ix) of Theorem 26 and Theorem 27.

5 Future Work

An important goal of our work is to understand the decision problem for useful classes of coop, and we have presented some results on Wajsberg coops, in particular, in the present paper. We already have some more results about general coops, but the proofs are not yet in a very satisfactory form. Blok and Ferreirim have shown that the quasi-equational theory of hoops is decidable. Using their results on subdirectly irreducible hoops one can show that any hoop embeds in a coop and from this conclude that the quasi-equational theory of coops is decidable (since this implies that a horn clause in the language of coops, can be translated into an equisatisfiable Horn clause in the language of hoops\(^5\) However, this approach doesn’t yield a practically feasible algorithm and gives no information about the complexity of the decision problem. We hope to improve on this position in future work.

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\(^5\) To do this, replace subterms of the form $t/2$ by $v_t$ where $v_t$ is a fresh variable and add hypotheses $v_t = v_t \rightarrow t$. 32
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