Gauged Floreanini-Jackiw type chiral boson and its BRST quantization

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The gauged model of Siegel type chiral boson is considered. It has been shown that the action of gauged model of Floreanini-Jackiw (FJ) type chiral boson is contained in it in an interesting manner. A BRST invariant action corresponding to the action of gauged FJ type chiral boson has been formulated using Batalin, Fradkin and Vilkovisky based improved Fujiwara, Igarishi and Kubo (FIK) formalism. An alternative quantization of the gauge symmetric action has been made with a Lorentz gauge and an attempt has been made to establish the equivalence between the gauge symmetric version of the extended phase space and original gauge non-invariant version of the usual phase space.

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I. INTRODUCTION

The self-dual field in (1 + 1) which is also known as chiral boson is the basic ingredient of heterotic string theory [1–4]. This very chiral boson plays a crucial role in the study of quantum hall effect too [5, 6]. Seigel initiated the study of chiral boson in his seminal work [7]. Another description of chiral boson came from the work of Srivastava [8]. In these two descriptions [7, 8], the lagrangian of chiral boson were constituted with the second order time derivative of the field. In the description of Seigel chiral constraint was in a quadratic form whereas in the description of Srivastava it was in a linear form. One more ingenious description of chiral boson came from the description of Floreanini and Jackiw [9]. In this description the lagrangian of chiral boson was constituted with first order time derivative of the field. In Ref [10], we find an interesting description towards quantization of that free FJ type chiral boson.

In a very recent work [11], we find an application of augmented super field approach to derive the off-shell nilpotent and absolutely anti-commuting (anti-)BRST and (anti-)co-BRST symmetry transformations for the BRST invariant Lagrangian density of a free chiral boson. Another recent important development towards the BFV quantization of the free chiral boson along with study of Hodge decomposition theorem in the context of conserved charges has came in [12].

The obvious generalization of free chiral boson is to take into account of the interaction of gauge field with that and this interacting field theoretical model is known as gauged model of chiral boson. The interacting theory of chiral boson was first described by Bellucci, Golterman and Petcher [13] with Seigel like kinetic term for chiral boson. So naturally the theory of interacting chiral boson with FJ type kinetic was wanted for as free FJ type chiral boson became available in [9] and that was successfully met up by Harada [14]. After the work of Harada [14], interacting chiral boson based on FJ type kinetic term attracted considerable attention [15–20] in spite of the fact that this theory of interacting chiral boson was not derived from the iterating theory of chiral boson as developed in [13]. Harada obtained it from Jackiw-Rajaraman (JR) version of chiral Schwinger model with an ingenious insertion of a chiral constraint in the phase space this theory [21]. So there is a missing link between the two types of interacting gauged chiral boson. An attempt towards search for a link is, therefore, a natural extension which we would like to explore. In fact, we want to show whether the gauged model of FJ type chiral boson is contained within the gauged chiral boson of Seigel type chiral boson which is available in [13] The study of this model may be benefical from another another point of view indeed; where anomaly is the central issue of investigation [14, 17, 21–25, 27], since it is known from Ref. [13] that the model took birth from the JR version of chiral Schwinger model and it is known that chiral generation of Schwinger model [28] due to Hagen [26] gets secured from unitarity problem when anomaly was taken into consideration in it by Jackiw and Rajaraman [21]. In this respect, the recent chiral generation of Thirring-Wess model is of worth-mentioning [24, 50]. So when the issue of searching of the desired link gets settled

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down a natural extension that comes automatically in mind is to study the symmetry underlying in the model and perform the quantization of the model. BRST quantization in this context scores over other. BRST formalism provide a natural framework of covariant quantization of field theoretical models and is interesting in its own right since it ensures unitarity and renormalizability of the theory \[31–33\]. Therefore, BRST quantization of the gauged chiral boson would certainly be of interest. So we apply the Batalin, Fradkin and Vilkovisky (BFV) \[34–37\] formalism in order to get a BRST invariant reformulation of the said model. In fact, we will use here the improved version due to FIK \[38\] in our work since it helps to get the Wess-Zumino \[39\] term in a transparent way which was found lacking in the work \[20\]. The Wess-Zumino term for the free chiral boson obtain in \[20\], though agrees with the conventional Wess-Zumino term that can be inherited from \[40\], the term which was demanded by the author as the Wess-Zumino term for the gauged model of chiral boson fails to do so. Surprisingly, however, the final BRST invariant effective action for gauged chiral boson presented in \[20\] shows on shell BRST symmetry. So a natural question arises whether or not FIK formalism fails to produce the appropriate Wess-Zumino term for the gauged model of FJ type chiral boson since it was found to be instrumental to get the BRST invariant reformulation for several physical sensible field theoretical models with the appropriate Wess-Zumino term \[41–49\]. To explore the above fact, we are in fact, driven towards the reinvestigation of the BRST invariant reformulation of the gauged model of FJ type chiral boson.

Gauged model of chiral boson with the Wess-Zumino term would be a gauge invariant theory in the extended phase space. So if our attempt gets a positive shape towards BRST quantization with the appearance of appropriate Wess-Zumino term then a natural extension would be to proceed towards the alternative quantization of the gauge invariant part of the theory and the next task would certainly be to show the equivalence between the physical content of the actual gauge non-invariant theory and the gauge invariant theory of the extended phase space which we would also like to address within this work. Note that this type of investigation is not possible without the appropriate Wess-Zumino term which was found lacking in \[20\].

The plan of the paper is as follows. In Sec. II we are intended to find the missing link between the two types of mutually exclusive developments of gauged chiral boson. Sec. III will be devoted toward the BRST invariant reformulation of the gauged model of chiral boson which is based on FJ type kinetic term for chiral boson. In Sec. IV, we quantize the gauge invariant part of the lagrangian obtained during the process of BRST quantization in Sec. III with the Lorentz gauge. In Sec. V, an equivalence is made between the actual gauge non-invariant theory with the gauge invariant transmuted form obtained in Sec. III.

II. A GAUGED MODEL OF CHIRAL BOSON WITH THE SIEGEL TYPE KINETIC TERM

The gauged model of chiral boson with the Siegel type of kinetic term is described by the lagrangian density \[13\]

\[
\mathcal{L}_B = \frac{1}{2}(\dot{\phi}^2 - \phi'^2) + e(\phi' + \dot{\phi})(A_0 - A_1) + \frac{\lambda}{2}[(\dot{\phi} - \phi') + e(A_0 - A_1)]^2 + \frac{1}{2}(\dot{A}_1 - A_0')^2 + \frac{1}{2}ae^2(A_0^2 - A_1^2).
\]

(1)

Here over dot and over prime represent the time and space derivative respectively. Here \(m^2\) is written as \(ae^2\) for later convenience. The symbol \(e\) indicates the coupling constant which has one mass dimension. The momenta corresponding to the field \(A_0, A_1, \phi\) and \(\lambda\) respectively are

\[
\frac{\partial \mathcal{L}_B}{\partial \dot{A}_0} = \pi_0 = 0
\]

(2)

\[
\frac{\partial \mathcal{L}_B}{\partial A_1} = \pi_1 = \dot{A}_1 - A_0'
\]

(3)

\[
\frac{\partial \mathcal{L}_B}{\partial \phi} = \pi_\phi = (1 + \lambda)\dot{\phi} - \lambda\phi' + e(1 + \lambda)(A_0 - A_1)
\]

(4)

\[
\frac{\partial \mathcal{L}_B}{\partial \lambda} = \pi_\lambda = 0
\]

(5)

The canonical Hamiltonian density of the system is obtained through a Legendre transformation:

\[
\mathcal{H}_c = \pi_\phi \dot{\phi} + \pi_1 \dot{A}_1 - L.
\]

(6)
Using equations (2), (3), (11) and (5), we find that $\mathcal{H}_c$ takes the following form

$$
\mathcal{H}_c = \frac{\pi_1^2}{2} + \pi_1 A_0' + \pi_0 \phi' + \frac{1}{2} e^2 (A_1 - A_0)^2 - e (\pi_0 + \phi')(A_0 - A_1) - \frac{ae^2}{2} (A_0^2 - A_1^2) + \frac{1}{2(1 + \lambda)} (\pi_0 - \phi')^2 + u \pi_0 + v \pi_0. 
$$

(7)

In equation (7), $u$ and $v$ are the two lagrange multipliers. The following two equations

$$
\Omega_1 = \pi_0 \approx 0, 
$$

(8)

$$
\Omega_2 = \pi_\lambda \approx 0, 
$$

(9)

are identified as primary constraints of this system since these two do not contain the time derivative of the fields. The preservation of the constraints (8) and (9) leads to the following two constraints:

$$
\Omega_3 = \pi_1' + e (\pi_0 + \phi') + e^2 [(a - 1) A_0 + A_1] \approx 0, 
$$

(10)

$$
\Omega_4 = \pi_0 - \phi' \approx 0. 
$$

(11)

In order to single out the physical degrees of freedom we proceed to quantize the theory with the following gauge fixing condition.

$$
\Omega_5 = \lambda - f \approx 0. 
$$

(12)

The generating functional of this system now reads

$$
Z = \int dA_0 dA_1 d\pi_1 d\phi d\pi_0 d\lambda e^{i \int d^2 x [\pi_0 \phi + \pi_1 A_0 - H]} \delta (\Omega_1) \delta (\Omega_2) \delta (\Omega_3) \delta (\Omega_4) \delta (\Omega_5). 
$$

(13)

After integrating out of the momenta of the fields we get the generating functional $Z$ in the following form

$$
Z = \int dA_0 dA_1 d\phi d\pi_0 d\lambda e^{i \int d^2 x L_{GCB}} 
$$

(14)

where

$$
L_{GCB} = \dot{\phi} \phi' - \dot{\phi}^2 + 2 e \phi' (A_0 - A_1) - \frac{1}{2} e^2 (A_0 - A_1)^2 + \frac{1}{2} ae^2 (A_0^2 - A_1^2) + \frac{1}{2} (A_1 - A_0').^2. 
$$

(15)

This is the gauged model of chiral boson with FJ type kinetic term. Note that $L_{GCB}$ represents a lagrangian density that has generated from $L_B$ and it agrees with the lagrangian found in [14]. So we find that the gauged model of chiral boson with FJ type kinetic term is contained within the gauged version of Siegel like chiral boson [13].

It is beneficial to compute the Dirac brackets for completeness of the analysis since it is a constrained theory and ordinary Poisson brackets become inadequate for the theories endowed with constraint. The Dirac bracket [50] for the two field variables $A$ and $B$ is defined by

$$
[A(x), B(y)]^* = [A(x), B(y)] - \int [A(x) \omega_i(\eta)] C^{-1}_{ij}(\eta, z) [\omega_j(z), B(y)] d\eta dz, 
$$

(16)

where $C^{-1}_{ij}(x, y)$ is defined by

$$
\int C^{-1}_{ij}(x, z)[\Omega_i(z), \Omega_j(y)] dz = 1. 
$$

(17)

Here $\Omega_i$'s stands for the standing second class constraints embedded in the phase space of the theory. Therefore, to compute Dirac brackets we need to construct the matrix constituted with the Poisson brackets of the constraints [53], [9], [10], [11] and [12]. The required matrix is

$$
C_{ij} = \begin{pmatrix}
0 & 0 & -e^2(a - 1) & 0 & 0 \\
0 & 0 & 0 & 0 & -1 \\
e^2(a - 1) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -2\theta & 0 \\
0 & 1 & 0 & 0 & 0
\end{pmatrix} \delta (x - y). 
$$

(18)
The matrix $C_{ij}$ is nonsingular. So inverse of it exists which is found out to be

$$C^{-1}_{ij} = \begin{pmatrix} 0 & 0 & \frac{1}{e^2(a-1)}\delta(x-y) & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$  \hspace{1cm} (19)

Here $\epsilon(x)$ is the sign function, $\epsilon(x) = +1$ for $x > 0$ and $\epsilon(x) = -1$ for $x < 0$. $\frac{d}{dx}\epsilon(x) = 2\delta(x)$ Using the definition (16), straightforward calculations renders the following Dirac brackets between the field variables.

$$[A_0(x), A_1(y)]^* = \frac{1}{e^2(a-1)}\delta'(x-y),$$  \hspace{1cm} (20)

$$[\phi(x), \phi(y)]^* = -\frac{1}{4}\epsilon(x-y),$$  \hspace{1cm} (21)

$$[A_0(x), \phi(y)]^* = \frac{1}{e(a-1)}\delta(x-y),$$  \hspace{1cm} (22)

$$[A_0(x), \pi_1(y)]^* = -\frac{1}{a-1}\delta(x-y),$$  \hspace{1cm} (23)

$$[A_0(x), \pi_\phi(y)]^* = -\frac{1}{e(a-1)}\delta'(x-y),$$  \hspace{1cm} (24)

$$[A_1(x), \pi_1(y)]^* = \delta(x-y),$$  \hspace{1cm} (25)

$$[\phi(x), \pi_\phi(y)]^* = \delta(x-y).$$  \hspace{1cm} (26)

Here (*) indicate the Dirac bracket. Here we end up the description of this Sec. and in the following section we proceed towards BRST quantization.

III. BRST QUANTIZATION OF THE GAUGED MODEL OF CHIRAL BOSON WITH FJ TYPE KINETIC TERM

In this section we are intendant to carry out the BRST quantization of the gauged model of chiral boson with FJ type kinetic term using the BFV based improved version of FIK since we are familiar with the several successful attempts with this improved version towards the generation of the appropriate Wess-Zumino term during the process of BRST quantization [41–49].

According to this formalism $H_m$ is usually known as the minimal Hamiltonian which is defined by

$$H_m = H_c + \bar{P}_a V^a_b C^b,$$  \hspace{1cm} (27)

and the BRST charge $Q$ and the gauge fixing function $G$ have the following expressions respectively:

$$Q = C^a \omega_a - \frac{1}{2} C^b c^c U^a_{cb} \bar{P}_a + P^a \pi_a,$$  \hspace{1cm} (28)

$$G = \bar{C}_a \chi^a + \bar{P}_a \lambda^a.$$  \hspace{1cm} (29)

The structure coefficients $U^c_{ab}$ and $V^a_b$ come from the Poisson brackets among the constraints $\Omega$’s themselves of the theory and the Poisson’s brackets with the canonical Hamiltonian $H_c(q^i, p_i)$:

$$[\Omega_a, \Omega_b] = i\Omega_c U^c_{ab}, [H_c, \Omega_a] = iw_b V^a_b,$$  \hspace{1cm} (30)
where $w_a$ (a = 1, 2,.......,N) represents the $N$ number of constraints embedded in the phase space of the theory defined by the Hamiltonian $H_c(q^i, p_i)$. In order to single out the physical degrees of freedom $N$ number of additional conditions $\phi^a = 0$, are are required to impose within the phase space. The constraints $\phi^a = 0$ and $\Omega_a = 0$ together with the Hamiltonian equations may be obtained from the action

\[ S = \int [p_i q^i - H(p_i, q^i)] - \lambda^a \Omega_a + \pi_a \phi^a] dt, \quad (31) \]

where $\lambda^a$ and $\pi_a$ are lagrange multiplier having Poisson’s bracket $[\lambda^a, \pi^a] = i \delta^a_b$ and $\lambda^a$ is contained within the gauge fixing conditions in the form $\phi^a = \lambda^a + \chi^a$. The variables $(C^1 \bar{P}_1)$ and $(P^i, \bar{C}_i)$ are the two sets of canonically conjugate anti-commuting ghost coordinates and momenta having the algebra $[C_i, \bar{P}_i] = i \delta(x - y)$ and $[P^i, \bar{C}_i] = i \delta(x - y)$. The lagrange multiplier $\lambda^a$ and $\pi_a$ have the Poisson bracket $[\lambda^a, \pi^a] = i \delta^a_b$. The quantum theory, therefore, can be described by the generating functional

\[ Z_G = \int dq^i dp_i d\lambda^a d\pi_a dC_a d\bar{C}_a dP^i d\bar{P}_a e^{iS_G}, \quad (32) \]

where the action $S_G$ is

\[ S_G = \int [p_i q^i + \bar{P}_i C_i + \bar{C}_a \bar{P}_a - H_m + \lambda^a \pi_a + i(Q, G)] dt. \quad (33) \]

With this input, we may proceed towards the BRST quantization of theory under consideration. The lagrangian density of gauged model of FJ type chiral boson is given by

\[ L = \dot{\phi} \dot{\phi}' - \phi'^2 + 2e \phi' (A_0 - A_1) - \frac{1}{2} e^2 (A_0 - A_1)^2 + \frac{1}{2} a e^2 (A_0^2 - A_1^2) + \frac{1}{2} (\dot{A}_1 - A_0')^2. \quad (34) \]

For this lagrangian density $L_{CB}$ the canonical momenta corresponding to the field $A_0$, $A_1$ and $\phi$ respectively are

\[ \frac{\partial L_{CB}}{\partial A_0} = \pi_0 = 0, \quad (35) \]

\[ \frac{\partial L_{CB}}{\partial A_1} = \pi_1 = \dot{A}_1 - A_0', \quad (36) \]

\[ \frac{\partial L_{CB}}{\partial \phi} = \pi_\phi = \phi'. \quad (37) \]

The canonical hamiltonian can be calculated using equations (35), (36) and (37) through a Legendre transformation as has been done earlier:

\[ H_c = \int dx \left[ \frac{1}{2} \pi_1^2 + \pi_1 A'_0 + \phi'^2 - 2e \phi' (A_0 - A_1) + \frac{1}{2} e^2 (A_0 - A_1)^2 - \frac{1}{2} a e^2 (A_0^2 - A_1^2) \right] \quad (38) \]

Equation (35) and (37), do not contain any time derivative of the fields. So these two are the primary constraint of the theory.

\[ \omega_1 = \pi_\phi - \phi' \approx 0, \quad (39) \]

\[ \omega_2 = \pi_0 \approx 0. \quad (40) \]

Therefore, the Hamiltonian reads

\[ H_p = \int dx \left[ \frac{1}{2} \pi_1^2 + \pi_1 A'_0 + \phi'^2 2 e \phi' (A_0 - A_1) + \frac{1}{2} e^2 (A_0 - A_1)^2 - \frac{1}{2} a e^2 (A_0^2 - A_1^2) + u(\pi_\phi - \phi') + v \pi_0 \right]. \quad (41) \]

The preservation of $\omega_2$ renders the following new constraint

\[ \omega_3 = \pi_1' + 2e \phi' + e^2 (a - 1) A_0 + e^2 A_1 \approx \pi_1' + e \phi' + e \pi_\phi + e^2 (a - 1) A_0 + e^2 A_1 \approx 0. \quad (42) \]
The preservation of $\omega_1$ and $\omega_3$ however do not give rise to any new constraint. These two conditions fix the velocities $u$ and $v$ respectively:

$$u = A'_1 - \frac{1}{(a-1)} \pi_1,$$

and

$$v = \phi' - e(A_0 - A_1).$$

Note that the constraints labelled by $\Omega'$ in Sec. II differs in number with the constraints labelled by $\omega'$s in this section because of the presence lagrange multiplier $\lambda$ in that section. However careful look reveals that $\Omega_1 \approx \omega_2, \Omega_3 \approx \omega_3$ and $\Omega_4 \approx \omega_1$. Now imposing the expression of $u$ and $v$ in (41) the Hamiltonian turns into

$$H = \int dx \left[ \frac{1}{2} \pi_1^2 + \pi_1 A'_0 + \pi_\phi (\phi' - e(A_0 - A_1)) - e\phi'(A_0 - A_1) + \frac{1}{2} e^2(A_0 - A_1)^2 - \frac{1}{2} a e^2(A_0 - A_1^2) + \pi_0 (A'_1 - \frac{1}{(a-1)} \pi_1) + (\pi_\phi - \phi')(\phi' - e(A_0 - A_1)) \right]$$

The constraints of the theory satisfy the following Poisson brackets between themselves

$$[\omega_1, \omega_1] = -2i \delta'(x - y),$$

$$[\omega_1, \omega_3] = 0,$$

$$[\omega_2, \omega_2] = 0,$$

$$[\omega_2, \omega_3] = -ie^2(a-1) \delta(x - y).$$

The involution relation between the Hamiltonian and the constraints $\omega_1, \omega_2$ and $\omega_3$ are

$$-i[\omega_1, H_P] = \omega'_1,$$

$$-i[\omega_2, H_P] = \omega_3,$$

$$-i[\omega_3, H_P] = \omega'_2 - \frac{e^2}{(a-1)} \omega_2.$$
For consistency, the time evaluation of these first class set must be identical to the (50), (51) and (52). Precisely these are the following.

\[ -i[\bar{\omega}_1, \bar{H}] = \bar{\omega}'_1, \]  
\[ -i[\bar{\omega}_2, \bar{H}] = \bar{\omega}_3, \]  
\[ -i[\bar{\omega}_3, \bar{H}] = \bar{\omega}''_2 - \frac{e^2}{(a - 1)} \bar{\omega}_2. \]  

(58) \hspace{1cm} (59) \hspace{1cm} (60)

The stage is now set to introduce the two pairs of ghost and anti-ghost fields \((C^i, \bar{P}_i)\) and \((P^i, \bar{C}_i)\). We also need to introduce a pair of multiplier fields \((N_i, B_i)\) The multipliers and the ghost anti-ghost pairs satisfy the following canonical Poisson’s Brackets:

\[ [C^i, \bar{P}_j] = [P^i, \bar{C}_j] = [N^i, B_j] = i\delta^i_j \delta(x - y), \]  

where \(i = 1, 2, 3\). According to the definition

\[ H_{BRST} = H_P + H_{BF} + \int dx \bar{P}_a V^a_b C^b, \]  

(61)

and

\[ H_U = H_{BRST} + i \int dx [Q, G]. \]  

(62)

In this situation BRST charge \(Q\) and the fermionic gauge fixing function \(G\) can be written down as

\[ Q = \int dx (C^i \bar{\omega}_i + P^i B_i), \]  

(63)

\[ G = \int dx (\bar{C}_i \chi^i + \bar{P}_i N^i). \]  

(64)

We are now in a position to fix up the gauge condition which is very crucial for achieving appropriate Wess-Zumino term. It is found that the following gauge fixing conditions render the required service towards that end.

\[ \chi_1 = \pi_\phi - \phi', \]  

(65)

\[ \chi_2 = N^2 + A_0, \]  

(66)

\[ \chi_3 = -A_1' + \frac{\alpha}{2} B_3. \]  

(67)

Let us now calculate the commutation relation between the BRST charge and the gauge fixing function:

\[ [Q, G] = B_1 \chi^1 + P_1 P^1 - C^3 \bar{C}_3'' + \bar{\omega}_3 N^1 - C^2 \bar{C}_2 - 2 C^1 \bar{C}_1. \]  

(68)

Generating functional for this system can be written down as

\[ Z = \int [D\mu] \exp^{iS}. \]  

(69)

where \([D\mu]\) is the Liouville measure in the extended phase space.

\[ d\mu = [d\phi][d\pi_\phi]\sum_{i=0}^{1} [dA_i][d\pi_i][d\eta][d\pi_\eta][d\theta][d\pi_\theta] \sum_{k=1}^{3} [dN^k][dB_k][dC^k], [d\bar{C}_k][d\bar{P}^k], [d\bar{P}_k], \]  

(70)

and the action \(S\) is explicitly given by

\[ S = \int d^2 x [\dot{\phi} \pi_\phi + \dot{A}_1 \pi_1 + \dot{A}_0 \pi_0 + \dot{\eta} \pi_\eta + \dot{\theta} \pi_\theta + \dot{\bar{N}}_i B_i + \dot{\bar{P}}_i \bar{C}_i + \dot{\bar{C}}_i \bar{P}^i - H_U]. \]  

(71)
The above formulation allows the following simplification:

$$\int dx (B_t N^1 + C_t \dot{P}^1) = i[Q, \int dx \tilde{C} N^1].$$

(72)

Exploiting the above simplification (72) we obtain the effective action in the following form.

$$S_{\text{eff}} = \int d^2x [\dot{\phi} \pi_\phi + \dot{A}_1 \pi_A + \dot{A}_0 \pi_0 + \dot{\eta} \pi_\eta + \theta \pi_\theta + \dot{N}^2 B_2 + \dot{N}^3 B_3 + \dot{P}_1 \dot{C}_1 + \dot{P}_2 \dot{C}_2$$

$$+ \dot{P}_3 \dot{C}_3 + \dot{C}_2 \dot{P}^2 + \dot{C}_3 \dot{P}^3 - P_1 \dot{P}^1 - P_2 \dot{P}^2 - P_3 \dot{P}^3 - \{\pi_\phi (\phi' - e(A_0 - A_1))$$

$$+ \frac{1}{2} \pi_1^2 + \pi_1 A_0' - e\phi'(A_0 - A_1) + \frac{1}{2} e^2 (A_0 - A_1)^2 - \frac{1}{2} ae^2 (A_0^2 - A_1^2)$$

$$+ \pi_0 (A_1' \frac{1}{(a-1)} - \frac{1}{4} \pi_1 + \frac{1}{4} (\pi_0 + \theta')^2 + \frac{1}{2} e^2 (a-1)^2 \eta^2 + \frac{1}{2 (a-1)^2} \pi_0'' + \frac{1}{2 (a-1)^2} \pi_0')$$

$$+ (\pi_0 - \phi' + \pi_\eta + \dot{\eta}) N^1 + (\pi_0 - \pi_\eta) N^2 + (\pi_1 + e \phi' + e \pi_\eta + e^2 (a-1) A_0 + e^2 A_1 + e^2 (a-1) \eta) N^3$$

$$- B_1 \chi^1 - B_2 \chi^2 - B_3 \chi^3 - \dot{P}_1 C_1' + \dot{P}_3 C_2' - \dot{P}_2 C_3 - \frac{1}{(a-1)} e^2 \dot{P}_1 C_3 + C^2 \dot{C}_2 + C^3 \dot{C}_3'' + 2 C^1 \dot{C}_1].$$

(73)

We are in a state to integrate out of the fields $\pi_1, \pi_0, \eta, B_1, B_2, N^1, N^2, C_1', \dot{P}_1, \dot{P}_3, \dot{P}_2$, one by one to have the action in a desired shape. After integrating out of the said fields and choosing $N_3 = A_0$, the action reduces to

$$S_{\text{eff}} = \int d^2x [\phi \phi' - \phi^2 + 2 e \phi'(A_0 - A_1) - \frac{1}{2} e^2 (A_0 - A_1)^2 + \frac{1}{2} e^2 (A_0 - A_1)^2$$

$$+ (\pi''_1 A_1 - \pi''_0 A_0 + \frac{1}{2 e^2 (a-1)} (\pi''_1 - \pi''_0) + B_3 \dot{A}_0 - B_3 A_1' + \frac{\alpha}{2} B^2 + \partial_\mu C^3 \partial^\mu \dot{C}_3].$$

(74)

If we now define $\pi_\eta = e(a-1) \eta$, $C_3 = C$, and $B_3 = B$ we get the desired BRST invariant effective action:

$$S_{\text{BRST}} = \int d^2x [\phi \phi' - \phi^2 + 2 e \phi'(A_0 - A_1) - \frac{1}{2} e^2 (A_0 - A_1)^2 + \frac{1}{2} e^2 (A_0 - A_1)^2$$

$$+ \frac{1}{2} e^2 (A_0 - A_1)^2 + \frac{1}{2} e^2 (A_0 - A_1)^2 + \frac{1}{2} (a-1)(\eta^2 - \eta'^2)$$

$$+ e(A_0 \eta' - A_1 \dot{\eta}) + e(a-1)(A_1 \eta' - A_0 \dot{\eta}) + \partial_\mu C \partial^\mu \dot{C} + B \partial_\mu A^\mu + \frac{\alpha B^2}{2}$$.  

(75)

The action (75) is now found to remain invariant if the fields transform as follows.

$$\delta \phi = e \lambda C,$$

(76)

$$\delta A_0 = - \lambda \dot{C},$$

(77)

$$\delta A_1 = - \lambda C',$$

(78)

$$\delta \eta = - \lambda e C,$$

(79)

$$\delta C = 0,$$

(80)

$$\delta \dot{C} = - \lambda B.$$  

(81)

The above transformations are the very BRST transformation generated from the BRST charge (63). The Wess-Zumino term for the theory under consideration can easily be identified as

$$S_{\text{WESS}} = \int d^2x \left[ \frac{1}{2} (a-1)(\eta^2 - \eta'^2) + e(A_0 \eta' - A_1 \dot{\eta}) + e(a-1)(A_1 \eta' - A_0 \dot{\eta}) \right].$$

(82)

This very action (82) contains the appropriate Wess-Zumino term corresponding to the theory of our present consideration and it agrees with the Ref. [40]. We would like to reiterate that in [21] it was lacking for. In fact, in [20], the term which was demanded by the author as the Wess-Zumino term though do not agree with Ref. [40], nevertheless, the author finds on shell BRST invariance with that Wess-Zumino term. The term standing in equation (82) however establishes the off-shell BRST invariance. To achieve the appropriate Wess-Zumino term for this theory which agrees with [40], is a novel aspect of this reinvestigation.
IV. AN ALTERNATIVE QUANTIZATION OF THE GAUGE INVARIANT VERSION OF THE THEORY

The quantization of gauged model of FJ type chiral boson was available in [14]. It was attempted there to quantize it in a gauge non-invariant manner. The gauge invariant version certainly can be quantized. We refer the works [49, 51], where the authors made alternative quantization of chiral Schwinger model with the Faddeevian anomaly and generalized version of QED where vector and axial vector interaction gets mixed up with different weight respectively. Some gauge fixing is needed in this situation indeed. We choose the Lorentz gauge and proceed to quantize the gauge symmetric version of the gauged model of FJ chiral boson. The gauge symmetric version of the said theory with Lorentz gauge is described by the lagrangian density.

\[
\mathcal{L} = \dot{\phi}\phi' - \phi'^2 + 2e\phi'(A_0 - A_1) - \frac{1}{2}e^2(A_0 - A_1)^2 + \frac{1}{2}ac^2(A_0^2 - A_1^2) + \frac{1}{2}(A_1 - A_0')^2 + \frac{1}{2}(a-1)(\dot{\eta}^2 - \eta'^2)
+ e(A_0\eta' - A_1\dot{\eta}) + e(a - 1)(A_1\eta' - A_0\dot{\eta}) + B\partial_\mu A^\mu + \frac{\alpha B^2}{2}.
\] (83)

Gauge fixing is needed in order to single out the real physical degrees of freedom from the gauge symmetric version of the extended phase space. The Euler-Lagrange equations of motion corresponding to the fields \(\phi, A_0, A_1, B\) and \(\eta\) that follow from the lagrangian density \((83)\) respectively are

\[
\dot{\phi}' - \phi'' + e(A_0' - A_1') = 0,
\] (84)

\[
A_0'' - A_1' + e^2(1 - a)A_0 - e^2A_1 + e(a - 1)\dot{\eta} - e\eta' - 2e\phi' - \dot{B} = 0,
\] (85)

\[
A_1' - A_0 + ae^2A_1 + e^2A_0 - e(a - 1)\eta' + e\dot{\eta} + 2e\phi' + B' = 0,
\] (86)

\[
\partial_\mu A^\mu + \alpha B = 0,
\] (87)

\[
(a - 1)\ddot{\eta} - (a - 1)\eta'' - e(a - 1)\dot{A}_0 + e(a - 1)A_1' + eA_0' - eA_1 = 0.
\] (88)

It is found that the following expression of \(A_\mu, \phi\) and \(\eta\) represents the exact solution of the equations \((84), (85), (86)\) and \((87)\) and \((88)\)

\[
A_\mu = \frac{1}{ac^2}\left[\frac{(a - 1)}{a} \partial_\mu F + \partial_\mu B - e\partial_\mu h - ea\partial_\mu \zeta\right],
\] (89)

\[
\phi = -\frac{(a - 1)}{ea^2}F - \frac{h}{a} + \zeta,
\] (90)

\[
\eta = -\frac{F}{ea^2} - \zeta - \frac{h}{a(a - 1)}.
\] (91)

if the following free field equations

\[
(\partial_0 - \partial_1)h = 0,
\] (92)

\[
(\partial_0 - \partial_1)B = 0,
\] (93)

\[
\Box \zeta = \alpha eB,
\] (94)

\[
[\Box + m^2]F = 0,
\] (95)

\[
m^2 = \frac{a^2e^2}{(a - 1)}.
\] (96)
are maintained. Therefore, the free fields in terms of which the system is completely described are
\begin{equation}
h = -(a - 1)(\phi + \eta + \frac{1}{ea}F),
\end{equation}
\begin{equation}
\zeta = \frac{1}{a} \phi - \frac{(a - 1)}{a} \eta,
\end{equation}
\begin{equation}
F = \pi_1,
\end{equation}
\begin{equation}
B = \pi_0.
\end{equation}
The equal time commutation relations corresponding to the free fields are found out to be
\begin{equation}
[F, \dot{F}] = im^2 \delta(x - y),
\end{equation}
\begin{equation}
[\zeta, \dot{\zeta}] = i \frac{1}{a} \delta(x - y),
\end{equation}
\begin{equation}
[h, \dot{h}] = i \delta(x - y),
\end{equation}
\begin{equation}
[B, \dot{\zeta}] = ie \delta(x - y).
\end{equation}
Note that $F = \pi_1$ represents a massive field with mass $m$ and $h$ represents a massless chiral boson. These two are the replica of the spectrum as obtained in [14]. The equations involving $B$ appear because of the presence of the auxiliary field in the Lorentz gauge fixing. Note that $B$ has the vanishing commutation relation with the physical field $B$ and $h$. The field $\zeta$ represents the zero mass dipole field playing the role of gauge degrees of freedom that can be eliminated by operator gauge transformation. So the theoretical spectrum agrees in an exact manner with the theoretical spectrum obtained in [14].

V. TO SHOW THE EQUIVALENCE BETWEEN THE GAUGE INVARIANT AND GAUGE VARIANT VERSION OF THE MODEL

In this section an attempt is made to show the equivalence between the gauge invariant version of the extended phase space and the gauge variant version of the usual phase space of the gauged model of FJ chiral boson. It is important because to make the model gauge invariant phase space was needed to extend introducing the Wess-Zumino fields. So, what service does the Wess-Zumino fields actually render is a matter of utter curiosity.

To meet it let us start with the lagrangian of the gauged FJ type chiral boson with the appropriate Wess-Zumino term as is obtained from our investigation. The said lagrangian density reads
\begin{equation}
\mathcal{L} = \dot{\phi} \phi' - \phi'^2 + 2e\phi'(A_0 - A_1) - \frac{1}{2} \epsilon^2 (A_0 - A_1)^2 + \frac{1}{2} a e^2 (A_0^2 - A_1^2)
+ \frac{1}{2} (A_1 - \dot{A}_0)^2 + \frac{1}{2} (a - 1) (\dot{\eta}^2 - \eta'^2)
+ e(A_0 \eta' - A_1 \dot{\eta}) + e(a - 1)(A_1 \eta' - A_0 \dot{\eta}).
\end{equation}
To show the equivalence between the gauge invariant and the gauge variant version of this model we proceed with computation of the canonical momenta corresponding to the fields $\phi, A_0, A_1$ and $\eta$:
\begin{equation}
\frac{\partial L}{\partial \dot{\phi}} = \pi_\phi = \phi',
\end{equation}
\begin{equation}
\frac{\partial L}{\partial \dot{A}_0} = \pi_0 = 0.
\end{equation}
Under insertion of the conditions (114) and (115), by using the equations (106), (107), (108) and (109), a Legendre transformation leads to the canonical Hamiltonian 

\[
\frac{\partial L}{\partial A_1} = \pi_1 = \dot{A}_1 - A'_1.
\]

\[
\frac{\partial L}{\partial \dot{\eta}} = \pi_\eta = (a - 1)\dot{\eta} - eA_1 - e(a - 1)A_0.
\]

The equations (106) and (107) are independent of velocity so these two represent the two primary constraints. Explicitly these two are

\[
T_1 = \pi_0 \approx 0,
\]

\[
T_2 = \pi_\phi - \phi' \approx 0.
\]

Using the equations (106), (107), (108) and (109), a Legendre transformation leads to the canonical Hamiltonian \( H_c \) corresponding to the lagrangian density (105):

\[
H_c = \int dx \left[ \frac{1}{2}\pi_1^2 + \pi_1 A'_0 + \phi'^2 - 2e\phi'(A_0 - A_1) + \frac{1}{2}e^2(A_0 - A_1)^2 - \frac{1}{2}ae^2(A'_0 - A'_1)^2 + \frac{1}{2}e^2((a - 1)A_0 + A_1)^2 \right].
\]

The preservation of the constraint of \( T_1 \) leads to a new constraint

\[
T_3 = \pi'_1 + e\pi_\phi + e\phi' + e\pi_\eta \approx 0.
\]

The ref. [52], suggests that we have to choose appropriate gauge fixing at this stage to meet our need and we find that gauge fixing conditions those which have been found suitable for this system are the following:

\[
T_4 = e\pi_\eta \approx 0,
\]

\[
T_5 = e^2(a - 1)A_0 + e^2A_1 + e\pi_\eta \approx 0.
\]

Under insertion of the conditions of (114) and (115), \( T_3 \) and \( H_c \), turns into \( \tilde{T}_3 \) and \( \tilde{H}_c \) those which are explicitly given by

\[
\tilde{T}_3 = \pi'_1 + e\pi_\phi + e\phi' + e^2(a - 1)A_0 + e^2A_1 \approx 0.
\]

and

\[
\tilde{H}_c = \int dx \left[ \frac{1}{2}\pi_1^2 + \pi_1 A'_0 + \phi'^2 - 2e\phi'(A_0 - A_1) + \frac{1}{2}e^2(A_0 - A_1)^2 - \frac{1}{2}ae^2(A'_0 - A'_1)^2 \right],
\]

respectively. Note that with the gauge fixing conditions (114) and (115) push back the constraint \( T_3 \) into \( \tilde{T}_3 \) which was the constraint of the usual phase space and as a result \( H_c \) lands onto \( \tilde{H}_c \) which was the hamiltonian of the usual phase space. It has therefore become evident that physical contents remains the same in the gauge symmetric version of the theory in the extended phase space. The extra fields therefore renders there incredible service towards bring back of the symmetry without disturbing the physical sector.

For completeness of the analysis we compute the Dirac brackets of the physical fields using the definition (10). The matrix \( C_{ij} \) in this situation is

\[
C_{ij} = \begin{pmatrix}
0 & 0 & 0 & 0 & -e^2(a - 1) \\
0 & -2\theta & 0 & 0 & 0 \\
0 & 0 & -e^2\theta & 0 & 0 \\
0 & 0 & -e^2\theta & 0 & e^2\theta \\
e^2(a - 1) & 0 & 0 & e^2\theta & 0
\end{pmatrix} \delta(x - y),
\]

(118)
and the inverse of it is the following

\[
C_{ij}^{-1} = \begin{pmatrix}
0 & 0 & \frac{1}{e^2(a-1)} \delta(x-y) & 0 & \frac{1}{e^2(a-1)} \delta(x-y) \\
0 & -\frac{e}{2e^2(a-1)} \delta(x-y) & 0 & 0 & 0 \\
-\frac{1}{e^2(a-1)} \delta(x-y) & 0 & 0 & -\frac{1}{2e^2} \delta(x-y) & 0 \\
0 & -\frac{1}{e^2(a-1)} \delta(x-y) & 0 & 0 & 0 \\
-\frac{1}{e^2(a-1)} \delta(x-y) & 0 & 0 & 0 & 0
\end{pmatrix}
\]  \quad (119)

Using the definition (18), it is straightforward to compute the Dirac brackets between the field variables:

\[
[A_0(x), A_1(y)]^* = \frac{1}{e^2(a-1)} \delta'(x-y),
\]  \quad (120)

\[
[\phi(x), \phi(y)]^* = -\frac{1}{4} \epsilon(x-y),
\]  \quad (121)

\[
[A_0(x), \phi(y)]^* = \frac{1}{\epsilon(a-1)} \delta(x-y),
\]  \quad (122)

\[
[A_0(x), \pi_1(y)]^* = -\frac{1}{\epsilon(a-1)} \delta(x-y),
\]  \quad (123)

\[
[A_0(x), \pi_\phi(y)]^* = -\frac{1}{\epsilon(a-1)} \delta'(x-y),
\]  \quad (124)

\[
[A_1(x), \pi_1(y)]^* = \delta(x-y),
\]  \quad (125)

\[
[\phi(x), \pi_\phi(y)]^* = \delta(x-y),
\]  \quad (126)

Here also (*) stands to symbolize the Dirac bracket. Note that the Dirac brackets between the fields computed here are identical with the set of Dirac brackets computed in Sec. II. It is indeed the expected result.

VI. CONCLUSION

We have started our investigation with the gauged version of the Siegel type chiral boson. From this action we have landed onto the gauged version of FJ type chiral boson. Harada in [14] showed that this action can be derived from JR version of Chiral Schwinger model imposing a chiral constraint into the phase space of the theory. Our investigation however reveals that the gauged version of FJ type chiral boson is contained within the Seigel action in an interesting way. In fact, it is a successful endeavor of obtaining the gauged version of chiral boson in a different line of approach.

An extension towards the BRST invariant reformulation of the gauged version of the FJ type chiral boson has been made using BFV based improved FIK formalism. Though in [20], an attempt was made towards BRST quantization of the same model. However, in that work the part of the effective action which was demanded as the Wess-Zumino term did not agree with the Ref. [40]. In spite of that, with that Wess-Zumino term the author established the on-shell BRST invariance.

The way we have made the BRST invariant reformulation leads to the appropriate Wess-Zumino term and this does agree with the Ref. [40]. It is interesting that the appropriate Wess-Zumino term has automatically appeared during the process of BRST quantization and with Wess-Zumino term, we observe the off shell BRST invariance.

An alternative quantization has found possible due the appearance of the appropriate Wess-Zumino term. From alternative quantization we have seen that the theoretical spectrum agrees with the spectrum obtained in the quantization of the gauge non-invariant version of this model. It is indeed the expected result.

An equivalence between the gauge invariant version of the gauge model of FJ type chiral boson of the extended phase space has been established with the gauge non-invariant version of the usual phase space following the same line of approach as it was available from the work [52]. It is worth-mentioning that the gauge fixing plays an important role to establish this equivalence.
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