PLANETARY TORQUES AS THE VISCOSITY OF PROTOPLANETARY DISKS

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ABSTRACT

We revisit the idea that density wave wakes of planets drive accretion in protostellar disks. The effects of many small planets can be represented as a viscosity if the wakes damp locally but the viscosity is proportional to the damping length. Damping occurs mainly because of shocks even for Earth-mass planets. The excitation of the wake follows from standard linear theory including the torque cutoff. We use this as input to an approximate but quantitative nonlinear theory based on Burger's equation for the subsequent propagation and shock. Shock damping is indeed local, but weakly so. If all metals in a minimum-mass solar nebula are invested in planets of a few Earth masses each, dimensionless viscosities \( \alpha \) of the order of \(-4\) dex to \(-3\) dex result. We compare this with observational constraints. Such small planets would have escaped detection in radial velocity surveys and could be ubiquitous. If so, then the similarity of the observed lifetime of T Tauri disks to the theoretical timescale for assembling a rocky planet may be fate rather than coincidence.

Subject headings: planetary systems — planets and satellites: general — solar system: formation

1. INTRODUCTION

The disks of young low-mass protostars ("protostellar disks") are observed to accrete at rates \( \sim 10^{-8} M_\odot \text{ yr}^{-1} \) (Hartmann et al. 1998). A comparable average accretion rate is implied by the disk masses inferred from dust emission \( \left( \sim 10^{-2.1} M_\odot \right) \) (Osterloh & Beckwith 1995) when combined with the maximum ages of T Tauri stars that show evidence for disks \( \left( \sim 10^6-10^7 \text{ yr} \right) \) (Strom et al. 1989) or with radio-chemical estimates for the lifetime of the protostellar nebula (Podosek & Cassen 1994). Indeed, the evidence that young protestors are surrounded by viscous accretion disks has only strengthened in the quarter-century since Lynden-Bell & Pringle (1974) proposed it.

While the effective viscosity of the hotter accretion disks surrounding degenerate stars and black holes is probably due to magnetohydrodynamic (MHD) turbulence (Balbus & Hawley 1998), protoplanetary disks may be too cold, too weakly ionized, and too resistive to sustain MHD turbulence (Blaes & Balbus 1994; Jin 1996; Hawley & Stone 1998) except perhaps at small radii, in surface layers (Gammie 1996), or during FU Orionis-type outbursts (Hartmann, Kenyon, & Hartigan 1993). In view of several physical uncertainties in the ionization balance (Gammie 1996) and the possibly subtle nature of the conductivity itself (Wardle 1999), this conclusion is only tentative, but it is probably fair to say that the best reason for supposing that the viscosity of protoplanetary disks is magnetic remains the lack of a theoretically attractive alternative. Convection (Lin & Papaloizou 1980) and nonlinear shear instability (Zahn 1991; Dubrulle 1992) have been proposed, but these mechanisms appear to fail because of the very strongly stabilizing influence of a positive radial gradient in specific angular momentum (Ryu & Goodman 1992; Balbus, Hawley, & Stone 1996; Stone & Balbus 1996).

Larson (1989) suggested that density waves might provide the effective viscosity mechanism in protostellar disks. As he made clear, wave transport is rather different from viscous transport. In the standard theory of thin accretion disks, viscosity \( \nu \) plays two roles. First, it provides an outward angular momentum flux \( \nu \) by the part of the disk interior to a given radius on the part exterior. Second, it heats the disk by dissipating the differential rotation. In steady state, \( F_j \) balances the inward flux of angular momentum carried by the accreting matter, while the dissipation balances the loss of its orbital energy. Even if the disk is not steady, provided only that it is truly viscous,

\[
F_j = -v_2 \pi \Sigma^3 \frac{d\Omega}{dr},
\]

where \( \Sigma \) is the surface density and \( \Omega \) the angular velocity. Also, the dissipation rate per unit area is related to \( F_j \) by

\[
\dot{E} = -F_j \frac{d\Omega}{dr}. \tag{1}
\]

Any dissipative mechanism such that \( F_j \propto d\Omega/dr \) and equation (1) hold can be described by an effective viscosity \( \nu_{\text{eff}} \); one expects this to be the case for dissipative mechanisms that are sufficiently local.

Since density waves are not local, one may deduce a different value of \( \nu_{\text{eff}} \) from \( F_j \) than from \( \dot{E} \). In particular, in inviscid linear theory (Goldreich & Tremaine 1980, hereafter GT80) predicts that a planet embedded in a gas disk excites density waves at the inner and outer Lindblad resonances of its (circular) orbit; these waves carry \( F_j > 0 \), but \( \dot{E} = 0 \) by assumption, so that \( v_k = 0 \) but \( v_j > 0 \). Actually, the theory implicitly assumes some damping at a large distance from the resonances or else the waves would reflect from the edges of the disk and produce zero net flux. If one is concerned with the torques on the disk only, then the damping length is unimportant as long as it is large, but without dissipation the waves cause no secular changes to the orbits of the gas even at the Lindblad resonances (Goldreich & Nicholson 1989).

By demanding that equation (1) be satisfied, Larson (1990) found that steady spiral shocks produce

\[
\alpha \sim 0.013 \left( \frac{c}{V} \right)^3 + 0.08 \left( \frac{c}{V} \right)^2 \right)^{1/2} \tag{2}
\]

independently of the source of excitation of the waves. Here \( c \) is the sound speed and \( V = r \Omega \) the orbital velocity. For \( c/V = 0.03 \), perhaps characteristic of the later phases of the protosolar nebula at 1 AU, this gives \( \alpha \approx 10^{-4} \). Larson's...
result depends on a number of additional assumptions, some inspired by earlier work by Spruit (1987) on self-similar shocks: the waves were taken to form two spiral arms separated by 180° in azimuth, and because they would be very tightly wrapped since \( c \ll V \), Larson (1990) approximated the wave profile by analyzing a train of perfectly axisymmetric nonlinear waves. Most importantly, at least in contrast to our own work reported below, no source of excitation for the waves exists within the range of radii where equation (2) applies. Larson argued, however, that a bisymmetric spiral shock of the required strength could be launched from the 2:1 resonance of a Jovian-mass planet, and he deduced an accretion timescale \( \sim 10^6-10^7 \) yr in accordance with observational requirements.

Jovian-mass planets have been discovered during the past few years by exquisitely sensitive radial velocity surveys (Marcy, Cochran, & Mayor 2000). Most of these planets have orbital periods shorter than that of Mercury. They are detected around 4%-5% of the stars surveyed, mostly G, F, and K dwarfs. There are indications that stars of higher-than-solar metallicity are preferred (Gonzalez et al. 2001). Because of observational selection effects, it is probably too early to derive the incidence of giant planets in long-period orbits (\( \sim 10 \) yr). If such planets are as rare as those on short-period orbits, however, then we may seek another explanation for the observed accretion rates and inferred lifetimes of most protostellar disks. Nevertheless, it remains an intriguing notion that planet formation is not just constrained by disk lifetimes but may actually determine them.

Our paper therefore inquires as to whether accretion could be driven by planets of roughly terrestrial mass, which radial velocity surveys would not have detected. Other things being equal, the torque exerted on the disk by a planet is proportional to the square of the planet’s mass \( M_p \), so the influence of \( M_p \sim 10^{-5} M_\odot \) might be thought negligible compared with that of \( M_p \sim 10^{-3} M_\odot \). Just because its local effects are so strong, however, the larger planet will almost certainly open a gap in the disk, whereas the smaller one will not; since the torque per unit radius (smoothed over resonances) decreases rapidly with distance from the planet (GT80), the shorter range of its interaction may partly compensate for the terrestrial planet’s smaller mass. At solar abundance there could be tens of Earth-mass planets in the disk, each too small to have captured a significantly massive atmosphere. We assume that the planets are distributed through the disk in proportion to the local surface density of the gas, neglecting orbital migration and radial changes in the timescale of planet formation. Unlike Larson, therefore, we have a distributed source of excitation for density waves, so that equation (2) need not apply (although in fact we obtain a comparable value of \( \alpha \) in the minimum-mass solar nebula [MMSN]).

As discussed, no sensible effective viscosity can be derived from density waves without considering their damping. Section 2 shows that for a generic damping mechanism in a thin disk, if the waves are absorbed over a small distance compared to the disk radius, but possibly large compared to its thickness, and if the local damping rate depends mainly on distance from the planet, then equation (1) is satisfied so that a consistent value of \( v_{\text{eff}} \) or \( \alpha \) can in fact be defined. However, the viscosity is directly proportional to the damping length. After briefly considering other mechanisms in § 3, we conclude that shock damping is probably most important even at \( M_p \sim M_\odot \). In § 4, we calculate the excitation of the wake from linear theory, which is valid for sufficiently small \( M_p \) (eq. [31]). In § 5 and the Appendix, we show that the subsequent nonlinear propagation of the wake away from the planet can be reduced to Burger’s equation using some transformations and some approximations that should be accurate for sufficiently low-mass planets in sufficiently thin disks. We describe quantitatively the formation of the shock and its subsequent decay with increasing distance from the planet. Most importantly, we obtain formulae for the shock-damping length, and hence for \( \alpha_{\text{eff}} \), as functions of \( M_p \) and local disk parameters. Section 6 summarizes our main results and discusses their implications for disks resembling a standard MMSN.

2. EFFECTIVE \( \alpha \)-PARAMETER.

Let \( f^{(0)}(M_p, r_p) \) be the angular momentum flux excited by a planet of small mass \( M_p \) and orbital radius \( r_p \), embedded in a disk of local surface density \( \Sigma \), sound speed \( c \), and angular velocity \( \Omega \sim (GM_p/r_p)^{1/2} \), where \( M_\odot \) is the mass of the central star. According to GT80,

\[
f^{(0)}(M_p, r_p) = (GM_p)^2 \frac{\Sigma \Omega}{c^3} \left\{ \frac{4}{3} \left( \frac{r_p}{\mu_{\text{max}}(Q)} \right)^3 \left[ 2K_0 \left( \frac{2}{3} \right) + K_1 \left( \frac{2}{3} \right) \right] \right\} .
\]

The dimensionless quantity \( \mu \) is defined in such a way that \( \mu \Omega c/\Omega \gg 1 \) is roughly the largest azimuthal harmonic contributing to the flux (the “torque cutoff”); it is a function of the gravitational stability parameter \( Q = c\Omega/\pi G \Sigma \) (Toomre 1964). Self-gravity is probably very weak in the later evolutionary phases of protostellar disks, so it is appropriate to take \( Q \to \infty \). In this limit, it follows from the results of GT80 that the quantity in the brackets \( \approx 0.93 \). As acknowledged by GT80 and elaborated on by later authors, equation (3) is valid only to leading order in \( c/\Omega r \), the ratio of disk thickness to radius. At the next order, there is a slight imbalance between the interior \( (r < r_p) \) and exterior \( (r > r_p) \) fluxes; the difference between the two torques represents a net torque on the planet, whose orbit therefore “migrates,” probably toward smaller \( r_p \) (Ward 1997 and references therein). The effective viscosity we calculate depends on the sum of the interior and exterior fluxes, so we neglect the difference between them.

The flux rises quickly to the value (eq. [3]) over a radial distance from the planet of the order of \( \Omega l \approx c/|r d\Omega/dr|^{-1} \), the length over which the rotation velocity changes by Mach 1. The strongest Lindblad resonances giving rise to the wake lie within this distance. Absent damping, the angular momentum flux is radially constant at \( |r - r_p| \gg l \). We can characterize the strength of damping by the reciprocal of the damping length \( l_d \) at which \( f_j \) falls to \( \sim e^{-1} \) of equation (3).

The larger the damping length, the greater the dissipation of orbital energy by the density wave wake, for the following reason: If a quantity \( \Delta \Omega \) of angular momentum is added to the disk in the form of waves having angular pattern speed \( \Omega_p \) (in steady state, the wake has the pattern speed of the planet), the work required to create the waves is \( \Omega_p \Delta \Omega \). When these waves damp at radius \( r \), the work done on the mean flow there is \( \Omega(r) \Delta \Omega \). The difference \( [\Omega_p - \Omega(r)] \Delta \Omega \)
represents the energy available to be dissipated, assuming that this is positive. It can be shown that \(\Delta J\) is positive (negative) for waves excited exterior (interior) to the planet in the approximation that \(\Omega_p = \Omega(r_p)\). From these considerations, it follows that the total mechanical energy dissipated by the wake of the one planet is

\[
\dot{e}(M_p, r_p) = \int_0^r \left[ \Omega(r) - \Omega_p \right] \frac{df}{dr} \, dr \, .
\]  

(4)

Clearly \(\dot{e} \propto l_d\), assuming that \(l_d \ll r_p\). Suppose that the background properties of the disk are approximately constant over radial distances of the order of \(l_d\). Write the angular momentum flux in terms of its ideal asymptotic value (eq. [3]) and a dimensionless distribution function, \(f_J(r) = f_J^0(\phi(r - r_p))\). Our notation presumes a damping mechanism such that \(\phi\) depends much more rapidly on \(r - r_p\) than \((r + r_p)/2\), the latter dependence being significant only on the scale of itself. The integral (eq. [4]) can now be approximated by

\[
\dot{e}(M_p, r_p) = f_J^0(M_p, r_p) \left( \frac{d\Omega}{dr} \right)_{r_p} \int_{-\infty}^{\infty} \frac{x \, d\phi}{dx} \, dx \, .
\]  

(5)

where \(x \equiv r - r_p\). The lower limit of integration has been extended to \(-\infty\) for convenience since \(\phi(x) \approx 0\) unless \(|x| \ll r_p\).

Now suppose that there are many planets of the same mass distributed through the disk in numbers \(N(r)\) per unit radius and that \(rN(r) \gg 1\). Then the total flux at radius \(r\) is (on average)

\[
F_J(r) = \int_0^r N(r_p) f_J^0(M_p, r_p) \phi(r - r_p) \, dr_p \approx N(r) f_J^0(M_p, r) \int_{-\infty}^{\infty} \phi(x) dx \, ,
\]  

(6)

assuming that \(N(r)\) and \(f_J^0(M_p, r)\) vary only slowly with radius. Similarly, the dissipation rate per unit radius at \(r\) is

\[
\dot{E}(r) = \int_0^r \left[ \Omega(r) - \Omega_p \right] N(r_p) f_J^0(M_p, r_p) \frac{d\phi}{dx} \, (r - r_p) \, dr_p \approx \frac{d\Omega}{dr} \int_{-\infty}^{\infty} \frac{x \, d\phi}{dx} \, dx = -\frac{d\Omega}{dr} F_J(r) \, .
\]  

(7)

where the last step has used integration by parts. So, as promised in § 1, equation (1) is satisfied by density wave wakes under the assumption of many planets and local damping.

In the standard theory of viscous accretion disks (Pringle 1981), the viscous energy dissipation rate per unit radius is related to the Shakura \& Sunyaev (1973) viscosity parameter \(\alpha\) by (assuming \(\Omega \propto r^{-3/2}\))

\[
\dot{E}_{\text{visc}} = 9\pi \frac{\alpha}{2} \rho \Omega c^2 \, .
\]  

(8)

Let the planets have average mass \(\langle M_p \rangle\), and let them account for a fraction \(Z_p \ll 1\) of the total nebular mass; we expect that \(Z_p\) is comparable to the metallicity of the gas if planet formation is efficient but the planets are too small to have captured massive gaseous envelopes (Mizuno, Nakazawa, \& Hiyashi 1978; Podolak, Hubbard, \& Pollack 1993; Ikoma, Nakazawa, \& Emori 2000). Then the average number of planets per unit radius is

\[
N = \frac{2\pi Z_p}{\langle M_p \rangle} \, .
\]  

(9)

The total dissipation rate per unit radius due to planetary wakes is \(N\langle \dot{e} \rangle\), so by comparison with equation (8),

\[
\dot{e}_{\text{eff}} = \frac{4}{9} \frac{Z_p}{\Omega c^2} \langle \dot{e} \rangle \, .
\]  

(10)

Despite the relation equations (3) and (5), we are not yet in a position to evaluate \(\dot{e}_{\text{eff}}\) numerically because we have not determined the width of the distribution \(\phi\) or, equivalently, the damping length.

In the rest of the paper we assume an MMSN (cf. Hayashi 1981),

\[
\Sigma(r) = 1700 \, \text{g cm}^{-3} \, r_{\text{AU}}^{-3/2} \, ,
\]  

(11)

\[
c(r) = 1.2 \, \text{km s}^{-1} \, r_{\text{AU}}^{1/4} \, ,
\]  

(12)

where \(r_{\text{AU}}\) is the radius \(r\) expressed in astronomical units. For these parameters,

\[
Q = 67 r_{\text{AU}}^{1/4} \, ,
\]  

(13)

which is effectively infinite for our purposes, so that we completely neglect self-gravity of the disk. The MMSN plausibly corresponds to the state of protostellar disks as observed around T Tauri stars, but disks are likely to have been more massive at earlier evolutionary phases when the star was enshrouded by infalling dust and gas. Self-gravity could then have been important to the effective viscosity and accretion rate (Gammie 2001).

Another important assumption is that the sound speed is independent of height above the midplane. This is appropriate for the later evolutionary phases of protostellar disks, when the local thermal structure is probably dominated by passive reprocessing of radiation from the central star (Chiang \& Goldreich 1997), and it is in these phases that planets are most likely to be dynamically important. Vertical isothermality allows us to treat even short-wavelength acoustic disturbances two-dimensionally, i.e., in radius and azimuth. When the disk is predominantly self-luminous, as is likely in earlier phases and during FU Orionis outbursts, the temperature may decrease away from the midplane, in which case acoustic waves will concentrate toward the surface and perhaps damp more quickly than we estimate here (Lubow \& Ogilvie 1998; Ogilvie \& Lubow 1999). The thermal stratification of an active disk is very uncertain, however, since the distribution of dissipation over height is poorly understood; in some models, dissipation occurs mainly near the surface, which would also tend to make the disk vertically isothermal (Gammie 1996).

### 3. Linear Damping

Linear damping mechanisms are those for which the damping length is independent of amplitude and which do not couple different azimuthal harmonics of the wake. One possibility is a background viscosity \(\eta_0\) that acts on density waves as well as the mean shear. Takeuchi, Miyama, \& Lin (1996) showed that in this case the wave’s angular momentum decays with distance as (for definiteness, consider \(r > r_p\))

\[
f_{Jm} \propto \exp \left[ -2 \int \Delta k_v \, dr \right] ,
\]  

where \(f_{Jm}\) is the angular momentum flux carried by the \(m\)th azimuthal harmonic, and \(\Delta k_v\) is the contribution to the imaginary part of the
The most interesting feature of this expression is that now compared to shock damping, which, as will be seen, leads for larger. Therefore, these mechanisms are unimportant
morphisms produce damping lengths as large as the radius itself,
Numerical estimates show, however, that the damping
function at the disk photosphere. Because they take an iso-
evaluating the adiabatic (lossless) density wave eigen-
turbation at the photosphere is as large as it is at the disk
warmed by the central star, the Ðrst-order temperature per-
order in (but vital to the net torque on the planet;
becomes and a gap may open in the disk (GT80), at least in a non-
become a minimum radial separation at which density
Ñux can be accurately calculated by linear theory. It is well
known that in the absence of linear damping (viscosity, thermal conductivity), a sound wave of small but Ñnite amplitude propagating into still air eventually shocks; the
distance to the shock is proportional to the wavelength and inversely proportional to the initial amplitude (Landau &
Whitham 1974). In a disk the shock length is shorter and the dependence on initial amplitude is weaker because differential rotation compresses the radial width of the wake as it propagates.

Linear excitation has been studied by Goldreich & Tre-
maine (1978, hereafter GT78); GT80; Artymowicz (1993);
Ward (1997); and others. The relative phases of the azi-
muthal Fourier harmonics of the wake are normally dis-
carded because they are not relevant to the torque, but the
onset of the shock depends on the local slope of the wake
front, which in turn depends on these phases. Therefore, we
have repeated the linear calculation using the methods of
GT78 and GT80.

As usual, \( x = r - r_p \) and \( y = r_p (\theta - \theta_p) \) denote pseudocartesian radial and azimuthal coordinates, respectively, in a corotating system centered on the planet, and all properties of the background ßow are expanded to lowest order in \( x/r_p \). The background surface density \( \Sigma \) and sound speed \( c \) are made constant, as are the rotation rate \( = \text{Coriolis parameter} \) \( \Omega \), shear rate \( 2A \equiv \partial \Sigma/\partial r \), and vorticity \( 2B \equiv 2\Omega + A \). The background ßow with respect to the planet is \( 2Ax \epsilon \). No distinction is made between \( \Omega \) and the angular velocity of the planet since this is not important for the angular momentum ßux carried by the wake to leading order in \( h/r_p \) (but vital to the net torque on the planet; Ward 1997). Although we are interested in Keplerian disks, for which \( 2A/\Omega = -3/2 \) and \( 2B/\Omega = +1/3 \), the physics is clariÐed by retaining \( A \) and \( B \) in basic formulae. We use the Mach 1 distance \( l \equiv c/(2A) \) rather than \( h = c/\Omega = (3/2)l \) as a reference length. A convenient unit for the planetary mass is

\[ M_1 \equiv \frac{c^3}{|2A|G}, \]

which reduces to \( 2c^3/30G \) in a Keplerian disk. In linear
theory, the amplitude of the wake is \( \propto M_p \), so it is sufficient to calculate as if \( M_p = M_1 \) and scale the results accordingly.

Note that \( M_p \gtrsim M_1 \) is also the point at which the linear approximation begins to fail; the Roche lobe of the planet becomes \( \gtrsim l \), and a gap may open in the disk (§ 6).

The wake is steady in the planet frame, so that its pertur-
bations in radial and azimuthal velocity, \( u, v \ll c \), and the
\( \propto M_p^{2/5} \) but is still only a modest multiple of the disk
thickness, even if \( M_p \) is as small as \( M_\odot \).

3. LINEAR EXCITATION OF A PLANETARY WAKE

Even if the planet does not open a gap in the disk, there is
a minimum radial separation \( |r - r_p| = l \) at which density
waves can be excited because at smaller distances the velocity
of the gas in the corotating frame of the planet is sub-
sonic, and a stationary perturber does not excite acoustic
waves in a subsonic flow (Landau & Lifshitz 1959). This
explains the torque cutoff (GT80), at least in a non-self-
gravitating disk where the density waves are basically
acoustic. For a planet of sufÞciently low mass (eq. [31]), the
forces exerted at this minimum distance are weak enough so
that the excitation of the wake and its angular momentum
ßux can be accurately calculated by linear theory. It is well
known that in the absence of linear damping (viscosity, thermal conductivity), a sound wave of small but Ñnite amplitude propagating into still air eventually shocks; the
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relative perturbation in surface density, $\sigma = \delta \Sigma / \Sigma \ll 1$, are functions of $(x, y)$ only. Their spatial Fourier transforms $\hat{u}, \hat{v},$ and $\hat{\sigma}(k_x, k_y)$ satisfy (GT78; GT80)

$$
\frac{d^2 \hat{v}}{d\tau^2} + \left[ c^2 k(x)^2 + x^2 \right] \hat{v} = -i k_y \frac{d \hat{\phi}}{d\tau} + 2i k_x(\tau) B \hat{\phi},$$

$$
\hat{u} = -\frac{1}{c^2 k_y^2 + 4B^2} \left( 2B \frac{d \hat{\phi}}{d\tau} - c^2 k_x k_y \hat{v} + 2Bk_y \hat{\phi} \right),$$

$$
\hat{\sigma} = \frac{i}{c^2 k_y^2 + 4B^2} \left( k_y \frac{d \hat{\phi}}{d\tau} + 2Bk_x \hat{v} + ik_x^2 \hat{\phi} \right).
$$

(20)

The timelike variable $\tau$ is related to the pitch angle of the wave:

$$
\tau = -\frac{k_x}{2Ak_y};
$$

the notation is standard and will not be confused with optical depth. An individual Fourier component evolves at constant $k_x$ but varying $k_y$ because of the differential rotation. Since $A < 0$, all Fourier information flows along the characteristics $dk_y/d\tau = -2Ak_y$ from leading ($\tau < 0$) to trailing ($\tau > 0$) waves, and equation (20) describes the evolution of the amplitudes along those characteristics. The quantity $k$ is the instantaneous wavenumber ($k_x^2 + k_y^2)^{1/2}$, and $\hat{\phi} = -2\pi G M_p / k$ is the Fourier transform of the point mass planetary potential. Appropriate initial conditions are $\hat{v} = 0$ at $\tau = -\infty$. Unlike GT80, we have neglected the self-gravity of the disk. Unless otherwise noted, the rest of this section assumes units $c = -2A = 1$, so that $l = 1$, and the wake amplitude is scaled to $M_p = M_1$.

We solved equation (20) numerically to obtain $(\hat{u}, \hat{v}, \hat{\sigma})$ on a grid of $N_x \times N_y = 4096 \times 8192$ points in $(k_x, k_y)$. The very large grid proved necessary to minimize numerical artifacts of the discrete Fourier transform (DFT). We chose the maximum (Nyquist) value of $k_y = 8$, well beyond the torque cutoff. The corresponding azimuthal grid resolution and periodicity length are $\Delta y = (\pi/8)$ and $N_y \Delta y \approx 3217$, respectively. Since the locus of the wake is approximately $y = -x^2 \text{ sgn} x \tau/2$, we chose $L_x = 2(\sigma L_y)^{1/2} \approx 113$, implying a radial resolution $\Delta x = L_x/N_x = 0.028$.

The Fourier data were filtered in pitch angle to avoid aliasing. When $k_x > k_y$ and $k$, the homogeneous solutions to equation (20) change phase by $k_x \Delta \tau = \tau \Delta k_x$ from $k_x$ to $k_x + \Delta k_x$, where $\Delta k_x = 2\pi/L_x$ is the resolution in Fourier space and $k_x = \tau k_y$ in the present units. This phase change should be less than $\pi$, in other words $\tau_{\text{max}} = L_x/2 \approx 57$, or else the phase will be aliased and the inverse DFT will assign the wave to the wrong x-position. We multiplied the Fourier components by the pitch-angle filter,

$$
W(k_x, k_y) = \begin{cases} 
1 & \text{if } \tau < \tau_{\text{max}}/2, \\
2(1 - \tau/\tau_{\text{max}}) & \text{if } \tau_{\text{max}}/2 \leq \tau < \tau_{\text{max}}, \\
0 & \text{if } \tau \geq \tau_{\text{max}}.
\end{cases}
$$

(21)

Because the pitch angle increases linearly with $|x|$ in coordinate space, this filter attenuates the wake at $|x| \gtrsim L_x/4$. We also used a similar filter in the $k_y$-direction, tapering linearly from unity at $|k_y| = k_y_{\text{Nyquist}}/2$ to zero at $|k_y| = k_y_{\text{Nyquist}}$.

We checked our calculation against the angular momentum flux reported by GT80 in two different ways. One can show that as $L_x, L_y \to \infty$,

$$
f_j = \lim_{\tau \to -\infty} \frac{\int_0^{|x|} \langle \bar{u}(k_x, k_y) \bar{v}(k_x, k_y) \rangle_x |k_y| \, dk_y}{4\pi^2},
$$

(22)

the average being taken over oscillations in $\tau$ at fixed $k_y$. We evaluated a numerical approximation to this formula in Fourier space (before applying the windows discussed above) and found agreement with the flux (eq. [3]) reported by GT80 for $Q = \infty$ to the accuracy they quoted, viz. two significant digits. We also evaluated the flux as a function of $x$ in coordinate space using equation (32). The errors are larger because the coordinate-space wake is softened by the pitch-angle and $k_y$ filters, but nevertheless $f_j(x)$ is nearly constant and correct within $\pm 3\%$ between $|x| = 6$ and $|x| = 28$.

Figure 1 shows that beyond a few Mach lengths (0) from the planet, the linearized wake profile is approximately constant as a function of $y + x^2 \text{ sgn} x/2$, as expected if the source terms are negligible at $|x| \gtrsim 1$ and the wake propagates as a tightly wrapped acoustic disturbance (see below). To the extent that this is true, the profiles can be regarded equally well as azimuthal cuts across the wake at fixed radius or as radial cuts at fixed azimuth but with $x^2/2$ rather than $x$ as radial coordinate. If plotted against $x$, the profile would shrink in width $\propto |x|^{-1}$ for $|x| \gg 1$. The profile is about as compact as possible given that the torque cut off (the integrand of eq. [21]) peaks at $k_y \approx 0.36 l^{-1}$, showing that the phases of the Fourier components of the wake are closely aligned.

5. NONLINEAR PROPAGATION

The fully nonlinear steady state fluid equations in $xy$ are

$$
(2Ax + v)\partial_x u + u \partial_x u - 2\omega v + \frac{c^2}{\Sigma} \partial_x \Sigma = 0,
$$

$$
(2Ax + v)\partial_y v + u \partial_y v + 2Bu + \frac{c^2}{\Sigma} \partial_y \Sigma = 0,
$$

$$
(2Ax + v)\partial_y \Sigma + u \partial_y \Sigma + \Sigma \partial_x u + \Sigma \partial_y v = 0.
$$

(22)
Henceforth $\Sigma$ denotes the full surface density and $\Sigma_0$ its unperturbed value. The sound speed follows the adiabatic law $c^2(\Sigma) = c^2_0(\Sigma/\Sigma_0)^{5/3}$, a good approximation for weak shocks. We have dropped the planetary source terms, since at distances $\gg l$ they are unimportant.

Equation (22) is hyperbolic, and in the Appendix, we demonstrate that for $|x| \gg l$, it can be reduced to a single first-order nonlinear equation:

$$\frac{\partial}{\partial t} \chi - \chi \frac{\partial}{\partial x} \chi = 0,$$

(23)

which is the inviscid Burger’s equation (Whitham 1974). The dimensionless variables appearing here are related to radius, azimuth, and density contrast as follows:

$$t = \frac{4}{5} \frac{y}{l} \left( \frac{5/4}{3/4} \right) \frac{x}{l}^{3/2},$$

(24)

$$\eta = \frac{y}{\sqrt{\frac{2}{3}}} \frac{x^2}{2c_0} \text{sgn} \chi,$$

(25)

$$\chi = \frac{y}{l} \left( \frac{\gamma + 1}{\gamma - 1} \right) \frac{c - c_0}{c_0} \approx \frac{y}{l}^{1/4} \frac{\gamma + 1}{2} \frac{\Sigma - \Sigma_0}{\Sigma_0}.$$  

(26)

5.1. The Shock

For almost any smooth choice of initial conditions in Burger’s equation (eq. 23), the solution will eventually threaten to become double-valued, so that one must fit in a shock (Whitham 1974). The identity $\left( \frac{\partial}{\partial x} \chi \right)/(\partial_t \chi) = - (\partial_x \eta) \chi$ converts equation (23) to the linear equation

$$\left( \frac{\partial}{\partial t} \right) \chi = - \chi,$$

(27)

which says that each point on the wave profile moves forward at a “speed” determined by the value of $\chi$ at that point. A shock develops when higher (larger $\chi$) parts of the profile overtake lower ones. The first shock develops where the initial profile is steepest, after a delay

$$t_{\text{shock}} = t_0 \left[ \max \left( \frac{\partial_x \chi}{\partial_t \text{sgn} \chi} \right) \right]^{-1}.$$  

(28)

If the initial profile changes sign, as it does in Figure 1, there will be at least two shocks eventually: one moving to the left and one to the right in this reference frame, which is moving radially away from the planet at $c_0$.

The shock jump condition depends on what is to be conserved. In our case, $\chi$ is approximately proportional to the density contrast $(\Sigma - \Sigma_0)/\Sigma_0$, provided that the latter is small, so mass conservation demands

$$(\chi_1 - U_x)\chi_1 = (\chi_2 - U_x)\chi_2 \iff U_x = \frac{\chi_1 + \chi_2}{2},$$

(29)

where $\chi_1$ and $\chi_2$ are the values of $\chi$ before and after the shock, and $U_x \equiv \left( d\chi/dt \right)_{\text{shock}}$ is the “velocity” of the shock in these variables. In physical variables, the radial velocity of the shock is the mean of the sound velocities on either side of the jump, as usual for weak shocks.

For an initial pulse that changes sign and has a finite range, the final asymptotic behavior is an “N-wave,” called such because the profile resembles that letter of the alphabet (Fig. 2). The unperturbed gas flows from left to right in these variables, so it first encounters the left-hand shock, followed by a rarefaction wave in which the pressure falls below its unperturbed value and finally a jump back to the unperturbed conditions at the right-hand shock. The point where $\chi = 0$ (on the rarefaction wave) does not move, and the flux vanishes there, so mass conservation implies that the areas in the left-hand and right-hand lobes of the N-wave are both constant. From this and the jump condition (eq. [29]), it follows that the height and width of the N-wave scale as $t^{-1/2}$ and $t^{+1/2}$, respectively (cf. Whitham 1974; Landau & Lifshitz 1959).

We choose $x_0 = -2l$ as the matching point between the linear and nonlinear calculations. This is not far enough from the planet to be completely outside the excitation region: the flux in the linear wake is about 7% smaller than its asymptotic value. For $M_p \ll M_1$, one should probably use a larger value of $|x_0|$. In practice, the interesting values of $M_p$ will be not much smaller than $M_1$, and it is then necessary to match at a rather small $|x_0|$ in order to remain within the range where the linear calculations are valid.

Applying the criterion (eq. [28]) to the linear wake profile at $|x_0| = 2l$ (see Fig. 1), whose amplitude is $\propto M_p$, we find that the shock develops at $t_{\text{shock}} - t_0 \approx 0.79(M_1/M_p)$. The constant $t_0 \approx 1.89$ is $\propto |x_0|/l^{3/2}$, independent of $M_p$. In order that subsequent formulae have a simple power-law scaling with $M_p$, we shall neglect $t_0$, but it becomes important as $M_p$ approaches $M_1$. Using the relation equation (24) and $h \equiv c_0/\Omega = (2/3)l$, $t_{\text{shock}}$ translates to

$$|x|_{\text{shock}} \approx 0.93 \left( \frac{\gamma + 1}{12/5 M_1} \right)^{2/5} l.$$  

(30)
This result indicates that if $M_p \geq M_1$, then the shock begins immediately and cannot be separated from the excitation region. Note that for $M_\ast = M_\odot$, 
\[
M_1 = \frac{2c^3}{3GM} \approx 8.0 \left( \frac{c}{1 \text{ km s}^{-1}} \right)^3 \left( \frac{r_p}{1 \text{ AU}} \right)^{3/4} M_\odot . \tag{31}
\]

5.2. Angular Momentum Flux and Energy Dissipation

By summing over the contribution of each azimuthal harmonic as given by GT80, one can show that the angular momentum flux carried by the wake in a non-self-gravitating disk is
\[
f_j(x) = \frac{c_0^3 r_p \Sigma_0}{2A\Sigma_0} \int_{\nu_p}^\infty \left( \Sigma - \Sigma_0 \right)^2 dy . \tag{32}
\]
In terms of our dimensionless variables,
\[
f_j(x) = \frac{2r_p c_0^3 \Sigma_0}{(\gamma + 1)^2 2A} \Phi(t) , \text{ where } \Phi(t) \equiv \int x^2(\eta, t) d\eta . \tag{33}
\]

We have used the fact that the wake depends on radius and azimuth most rapidly in the combination $\eta$ and that its slowly varying amplitude can be thought of as a function of radius or azimuth according to the correspondence $|\eta| \approx (2\eta')^{1/2} = 2^{1/2}(5/4)^{1/2}l$. Without shocks, the dimensionless flux $\Phi$ is strictly conserved; in fact, the integral over $\eta$ of any function of $\chi$ is conserved.

Once the shock develops, $f_j$ decays with distance as the angular momentum carried by the wake is transferred to the mean flow. For the scalings of the N-wave discussed above, it follows that $\Phi(t) \propto t^{-1/2}$ at large $t$; hence
\[
f_j(x) \propto |x|^{-5/4} \left( |x| \gg |x|_{\text{shock}} \right) , \tag{34}
\]
which was also confirmed by direct analytical N-wave expansion of the original set of equation (22). The full evolution of $\Phi(t)$ is shown in Figure 3.

The total energy dissipation rate associated with the absorption of the wake is (cf. eq. [4])
\[
\dot{\varepsilon} = 2 \int_0^\infty [\Omega(r) - \Omega(r_p)] \frac{df_j}{dr} dx
\]
\[
\approx \left( \frac{4}{5} \right)^{3/5} \frac{2}{\gamma + 1} c^3 \Sigma \int_0^\infty \Phi(t) t^{-3/5} dt ,
\]
where we have used integration by parts to transfer the radial derivative from $f_j$ to $\Omega(r)$. Since $t^{-3/5}\Phi(t) \propto t^{-11/10}$ at large $t$, the integral converges, but slowly; its value is $\approx 12.3$ when $M_p = M_1$.

By scaling our numerical results to other $M_p$ and $\gamma$, we find that the dissipation produced by a small planet is
\[
\dot{\varepsilon} \approx 11 \left( \frac{\gamma + 1}{7/5 + 1} \right)^{-2/5} c^3 \Sigma \frac{3\Omega GM_p}{2c^3} \left( \frac{M_p}{2e^n} \right)^{8/5} \tag{35}
\]
in a Keplerian disk. From equation (10), it now follows that effective viscosity parameter is
\[
\alpha_{\text{eff}} = 3.2 \left( \frac{\gamma + 1}{7/5 + 1} \right)^{-2/5} r_p^2 Z_p M_p \left( \frac{3\Omega GM_p}{2c^3} \right)^{8/5} \left( \frac{M_p}{M_\odot} \right)^{3/5}
\approx 1.2 \times 10^{-4} \left( \frac{r_p}{\text{AU}} \right)^{-1/5} \left( \frac{Z_p}{0.01} \right) \left( \frac{M_p}{M_\odot} \right)^{3/5} \tag{36}
\]

6. DISCUSSION

We have shown that density wave wakes excited by planets in a non-self-gravitating protostellar disk serve as an effective viscosity mechanism, provided that the wakes are damped over a distance smaller than the orbital radius of the planet. The effective viscosity is proportional to the damping length, which is most likely to be limited by shocks unless the mass per planet $M_p \ll M_\odot$. If a fixed total mass is available to form planets, then $\alpha_{\text{eff}} \propto M_p^{3/5}$. Equation (36) suggests that $\alpha_{\text{eff}} \sim 10^{-4}$ in the inner solar system where $M_p \sim M_\odot$, and $\alpha_{\text{eff}} \sim 3 \times 10^{-4}$ at $r \sim 10$ AU if $M_p \sim 10 M_\odot$, which is characteristic of the rock + ice cores of the giant planets, though very much less than the total mass of Jupiter or Saturn (Guillot 1999).

Our estimates for $\alpha_{\text{eff}}$ may seem optimistic because they assume that all of the rocks and ice are efficiently converted into planets. On the other hand, $\alpha_{\text{eff}}$ could be even larger if $M_p$ is augmented by gas. In that case, $\alpha_{\text{eff}} \propto M_p^{8/3}$ because the mass in planets is not limited to the metals. There are
two restrictions on the amount of gas we can allow for. The first is technical: we have calculated the excitation of the density waves from linear theory, which cannot be justified when \( M_p \) exceeds the limit of equation (31). This could be overcome by nonlinear two-dimensional numerical calculations (see, e.g., Korycansky & Papaloizou 1996). The other restriction is more substantial: once \( M_p \) is large enough to open a gap in the disk, the dissipation rate probably scales more slowly than \( M_p^{8.5} \).

Several estimates exist for the minimum mass required to open a gap, \( M_{gap} \). Lin & Papaloizou (1993) and Bryden et al. (1999) argue that in a completely inviscid disk, \( M_{gap} \approx M_1 \) as given by equation (31) because of radial gas pressure gradients; recall that at this mass, the Roche lobe (or Hill sphere) of the planet is \( \approx h \). Ward (1997) argues for \( M_{gap} \approx 0.14\pi \Sigma h r \), or \( M_{gap} \approx 0.8(r/AU)^{3/4} M_\odot \) in the fiducial MMSN (eqs. [11]–[12]). This results from comparing the time to open a gap with the time to migrate across it, assuming that the planetary wake damps immediately at the resonances where it is excited. Presumably \( M_{gap} \) should be increased to allow for the finite damping length that we have computed, but it would still be proportional to \( \Sigma \), unlike equation (31).

The mass \( M_{gap} \) is expected to be larger in a viscous disk than in an inviscid one because the torque exerted by the planet can be balanced by a viscous torque. Bryden et al. (1999) cite

\[
\frac{M_{gap}}{M_*} = 40v \frac{\Omega r^2}{\Sigma h} \approx 40\alpha \left( \frac{h}{r} \right)^2,
\]

which is again independent of \( \Sigma \) (if \( \alpha \) is). In the MMSN, this becomes

\[
M_{gap} \approx 2 \left( \frac{\alpha}{10^{-2}} \right) \left( \frac{r}{10 \text{ AU}} \right)^{1/2} M_\odot.
\]

In our case, it seems likely that the shocks created by one planet will resist the opening of a gap next to an adjacent planet, so it is perhaps reasonable to use the value in equation (36) for \( \alpha \) on the right-hand side above.

In the discussion to follow, we shall follow Lin & Papaloizou (1993) and adopt equation (31) as a provisional value for \( M_{gap} \), but clearly the question deserves further study. Then \( M_{gap} \approx 45 M_\odot \) at 10 AU using equation (12). If the planets at 10 AU accrete enough gas to reach this limit (Saturn exceeds it by a factor \( \sim 2 \)), then \( \alpha_{eff} \approx 3 \times 10^{-3} \).

In short, values for \( \alpha_{eff} \) in the range from \( 10^{-4} \) up to at least \( 10^{-3} \) seem plausible for the density wave mechanism in an MMSN. The predicted accretion timescale is

\[
t_{acc} = \alpha^{-1} \left( \frac{r}{h} \right)^2 \Omega^{-1} \approx 5 \times 10^6 \left( \frac{\alpha}{10^{-3}} \right)^{-1} \left( \frac{r}{30 \text{ AU}} \right) \text{ yr},
\]

in reasonable agreement with the disk lifetimes cited in § 1.

By comparing observed sizes, ages, and accretion rates of T Tauri disks with self-similar models, Hartmann et al. (1998) estimate \( \alpha \approx 10^{-2} \), which may be too large to be explained by planetary wakes, especially if one must rely on the rocky core masses only. The strongest lower bounds on \( \alpha \) assume that the disks started out much more compact than observed. In view of the wide scatter and uncertainties in the observationally derived parameters, the questionable relevance of self-similar models, and the theoretical uncertainties just discussed, it is unclear whether the result of Hartmann et al. is a major difficulty for us. Another concern is that planets are likely to migrate inward with respect to the gas (Ward 1997), perhaps enhancing the viscosity of the inner disk but diminishing that of the outer parts where most of the gas probably resides. Still another is that our mechanism operates only after planet growth, which standard models place late in the lifetime of the gaseous nebula, if not afterward (Podolak et al. 1993; Ikoma et al. 2000), whereas Hartmann et al. find larger accretion rates in younger disks on average.

All of these objections have merit. Nevertheless, the density wave mechanism is interesting from several points of view. First, it provides a minimum effective viscosity when other mechanisms are in abeyance. For example, MHD turbulence may dominate only during FU Orionis–like outbursts when the temperature and ionization of the disk rise (Gammie 1999); self-gravity may dominate in young, massive, cool disks (Lin & Pringle 1987), but neither is likely to be effective in the late, thermally passive phases when it is believed that planets form. Second, calculable models for the effective viscosity are so difficult to come by that it is worthwhile to examine any plausible candidate, even if others seem more likely to dominate (but cannot be calculated). Third, the formalism offered in § 5 and the Appendix for the development and damping of the shock, though only approximate, is expected to be reasonably accurate for weak shocks and simple enough to use that it may find other applications. Finally, there may be other nonaxisymmetric structures in protostellar disks of greater mass than planets, for example vortices (Adams & Watkins 1995; Godon & Livio 2000; Nauta 2000). Any such structure would create a density wave wake.

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APPENDIX A

REDUCTION TO BURGER’S EQUATION

At \(|x| \gg l\), equation (22) is hyperbolic, as usual for supersonic steady flows (Landau & Lifshitz 1959): that is, it describes an

---

1 Burger’s equation can be solved analytically given a closed form for the initial wave profile (Whitham 1974).
We approximate $c_u$ only those parts of that are in phase with $(\text{parts make a slight but negligible change to the characteristic velocity). The operator changes the phase by 90º, so from the equation above,}

$$\partial_y u + u\partial_x u + \frac{c^2(\Sigma)}{\Sigma} \partial_x \Sigma = 0$$

Equation (22) simplifies for $|x| \gg L$ where a linearized WKB treatment suggests that $\partial_y \ll \partial_x$ and $v \ll u$. Dropping $v$ entirely and $\partial_y$ except in the combination $2Ax\partial_x$, results in

$$\partial_x u + u\partial_x u + c^2(\Sigma)/\Sigma \partial_x \Sigma = 0$$

We have replaced $x$ with $\xi \equiv Ax^2$ so that the equations are autonomous; i.e., the independent variables do not appear explicitly.

The system of equation (A1) is formally identical to one-dimensional isentropic gas dynamics and can be solved by the same methods (Landau & Lifshitz 1959). It has characteristics $C_+$ and Riemann invariants $R_+$,

$$C_+ \left( \frac{d\xi}{dy} \right)_\pm = \pm c + u, \quad R_+ = u \pm \frac{2c}{\gamma - 1}.$$  

The $C_+$ characteristics propagate toward the planet from the unperturbed disk, where $u = 0$ and $c = c_0$. Therefore $R_+ = 2c_0/(\gamma - 1) \text{ everywhere}$, and $u = 2(c_0 - c)/(\gamma - 1)$. Equation (A1) then reduces to a single first-order equation,

$$0 = \partial_y \psi - (1 + \psi)c_0 \partial_x \psi,$$

$$\psi(y, \xi) \equiv \frac{\gamma + 1 - c - c_0}{\gamma - 1} \psi \approx \frac{\gamma + 1 - \Sigma_0 - \Sigma_0}{2 - \Sigma_0}$$

which is the well-known inviscid Burger's equation: the simplest nonlinear equation capable of displaying a shock (Whitham 1974).

Unfortunately, the analysis above is oversimple. In the limit of an infinitesimal disturbance, equation (A1) predicts that the wave propagates at constant amplitude. But a linear WKB treatment of equation (22) shows that the amplitude of such a wave grows in proportion to $|x|^{1/2} \sim |\xi|^{1/4}$ as a consequence of conservation of angular momentum flux. To recover this behavior, the larger subdominant terms involving $v$ that were omitted in passing from equation (22) to equation (A1) must be reinstated. The equations for the $C_\pm$ characteristics become

$$\partial_x R_\pm \approx \frac{1}{2Ax} \left( 2\Omega v + c\partial_x v \right) \equiv S_\pm .$$

where $R_\pm$ has the same meaning as before, and we have introduced the abbreviations $\partial_x \equiv \partial_x + (u + c)\partial_x$ for the derivatives along $C_\pm$. The source term $S_\pm$ is smaller than individual terms in $R_\pm$ by $O(\xi^{-1})$ but nonzero, so that the $R_\pm$ is not invariant. Yet $R_+ \text{ is still well approximated by its undisturbed value } 2c_0/(\gamma - 1)$: First, $R_+ \text{ is truly undisturbed until the incoming } C_+ \text{ characteristic encounters the leading edge of the wake, which propagates in the opposite direction on a bundle of } C_- \text{ characteristics of roughly constant width } \Delta \xi. \text{ Then, since the encounter lasts a "time" } \Delta y \sim \Delta \xi/2c, \text{ the maximum change in } R_+ \text{ relative to its undisturbed value is } O(\Delta \xi/\xi) \ll 1, \text{ which may be neglected at large } \xi.$

The source term $S_-$ cannot be neglected because it “travels with” the wake so that its effects accumulate. We solve for $S_-$ using the second line of equation (22), which, to an adequate approximation, reads

$$\partial_x v \approx \frac{1}{2Ax} \left( 2Bu + c^2 \Sigma \partial_x \Sigma \right).$$

Only those parts of $S_-$ that are in phase with $u$ or $u - \Sigma_0$ are important for the growth of the amplitude; the out-of-phase parts make a slight but negligible change to the characteristic velocity. The operator $\partial_y$ changes the phase by 90º, so from the equation above,

$$-\frac{1}{2Ax} [c\partial_y v + 2\Omega v] \text{ in phase} = -\frac{1}{2Ax} \left( 2Bu + 2\Omega v \frac{c^2 - c_0^2}{\gamma - 1} \right).$$

We approximate $c$ by $c_0$ except in the difference $c - c_0$ and invoke the approximate constancy of $R_+$ to write $u \approx 2(c_0 - c)/(\gamma - 1)$. Finally we write $\psi \equiv (\gamma + 1)(c - c_0)/(\gamma - 1)c_0$ as before, and so obtain the corrected $C_-$ equation in the form

$$\partial_y \psi - (1 + \psi)c_0 \partial_x \psi = -\frac{c_0}{4c_0} \psi .$$  

As expected, the general solution of the linearized form of equation (A4) is $\psi = |\xi|^{1/4}f(\xi + c_0 y)$, where $f$ is an arbitrary function. The full equation incorporates all three processes that dominate propagation of the wake: (1) increasing radial
wavenumber with distance from the planet ($\hat{c}_e = 2Ax\hat{c}_q$); (2) increasing amplitude due to conservation of angular momentum flux; and (3) nonlinear steepening.

Fortunately, equation (A4) can be transformed back into Burger’s equation by changing the independent and dependent variables. First let $\xi \to c_0(t \eta - y/l)$ (so $\eta$ is dimensionless):

$$l\hat{c}_q \psi - \psi \hat{c}_q \psi = \frac{1}{4y} \psi .$$

Since the wake is confined to $|\eta| \ll y/l$, we have approximated $y - \eta l$ by $y$ on the right-hand side. Next, let $\psi = (y/l)^{1/4} \chi$ to absorb the source term:

$$l\hat{c}_q \chi - (y/l)^{1/4} \chi_0 \chi = 0 .$$

Finally, introduce a new dimensionless “time” (or azimuth) variable $t$ as in equation (24), so that the dimensionless equation of motion becomes equation (23).

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