Bumpless Pipe Dreams and Alternating Sign Matrices

Anna Weigandt

University of Michigan

weigandt@umich.edu

September 21st, 2018
Bumpless Pipe Dreams
A **bumpless pipe dream** is a tiling of the $n \times n$ grid with the six tiles pictured above so that there are $n$ pipes which

1. start at the right edge of the grid,
2. end at the bottom edge, and
3. pairwise cross at most one time.

Lam, T., Lee, S. J., and Shimozono, M. (2018). *Back stable Schubert calculus*. *arXiv:1806.11233*
Example

Not an Example
Write Pipes($w$) for the set of bumpless pipe dreams which trace out the permutation $w$. 
Why Bumpless?

Ordinary pipe dreams use the tiles:

Example:

Fomin, S. and Kirillov, A. N. (1996). The Yang-Baxter equation, symmetric functions, and Schubert polynomials. *Discrete Mathematics*, 153(1):123–143

Bergeron, N. and Billey, S. (1993). RC-graphs and Schubert polynomials. *Experimental Mathematics*, 2(4):257–269

Knutson, A. and Miller, E. (2005). Gröbner geometry of Schubert polynomials. *Annals of Mathematics*, pages 1245–1318
The Weight of a Bumpless Pipe Dream

If a blank tile sits in row $i$ and column $j$, assign it the weight $(x_i - y_j)$. The weight of a bumpless pipe dream is the product of the weights of its blank tiles.

$$\text{wt}(\mathcal{P}) = (x_1 - y_1)(x_1 - y_2)(x_3 - y_2)$$
Theorem (Lam-Lee-Shimozono, 2018)

The double Schubert polynomial $\mathcal{G}_w(x; y)$ is the weighted sum

$$\mathcal{G}_w(x; y) = \sum_{P \in \text{Pipes}(w)} \text{wt}(P).$$
Example: $w = 2143$

$\mathcal{S}_w = (x_1 - y_1)(x_3 - y_3) + (x_1 - y_1)(x_2 - y_1) + (x_1 - y_1)(x_1 - y_2)$.

\begin{figure}[ht]
\centering
\includegraphics[width=\textwidth]{example1.png}
\end{figure}

$\mathcal{S}_w = (x_1 - y_1)(x_3 - y_1) + (x_1 - y_1)(x_2 - y_2) + (x_1 - y_1)(x_1 - y_3)$.

\begin{figure}[ht]
\centering
\includegraphics[width=\textwidth]{example2.png}
\end{figure}
**Upshot:** There is **not** a weight preserving bijection from bumpless pipe dreams to ordinary pipe dreams for $w$.

**Problem:** Specializing the $y_i$’s to 0, find a weight preserving bijection between bumpless and ordinary pipe dreams for $w$. 

![Diagram](image-url)
Alternating Sign Matrices
A matrix $A$ is an alternating sign matrix (ASM) if:

- $A$ has entries in $\{-1, 0, 1\}$
- Rows and columns sum to 1
- Non-zero entries alternate in sign along rows and columns

\[
\begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 1 & -1 & 1 \\
1 & -1 & 1 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}
\]

Let $\text{ASM}(n)$ be the set of $n \times n$ ASMs.
Plot the 1’s of $A$ as black dots and the $-1$’s as white dots:

Write $D(A)$ for the positions of boxes in the diagram and $N(A)$ for the positions of the negative entries in $A$. 

Lascoux, A. Chern and Yang through ice
Defining Segments are Pipes
A Quick Sanity Check

The formula which enumerates ASMs was famously conjectured by Mills-Robbins-Rumsey in 1983. It was proved independently, first by Zeilberger and then Kuperberg.

| $n$ | $\#ASM(n)$ | $\prod_{k=0}^{n-1} \frac{(3k + 1)!}{(n + k)!}$ | $\sum_{w \in S_n} \mathcal{G}_w(1; 0)$ |
|-----|-------------|---------------------------------------------|-----------------------------------|
| 1   | 1           | 1                                           | 1                                 |
| 2   | 2           | 2                                           | 2                                 |
| 3   | 7           | 7                                           | 7                                 |
| 4   | 42          | 41                                          | 41                                |
| 5   | 429         | 393                                         | 393                               |

Anna Weigandt
Bumpless Pipe Dreams and ASMs
This site is supported by donations to The OEIS Foundation.

THE ON-LINE ENCYCLOPEDIA OF INTEGER SEQUENCES®

founded in 1964 by N. J. A. Sloane

1, 2, 7, 41, 393, 6080, 150371, 5903710

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

Search: seq:1,2,7,41,393,6080,150371,5903710

Sorry, but the terms do not match anything in the table.

If your sequence is of general interest, please submit it using the form provided and it will (probably) be added to the OEIS! Include a brief description and if possible enough terms to fill 3 lines on the screen. We need at least 4 terms.

Search completed in 0.007 seconds
**Upshot:** We can extend the definition of bumpless pipe dreams to allow multiple crossings. Say a bumpless pipe dream is **reduced** if each pair of pipes crosses at most one time.

Write $\text{BPD}(n)$ for the set of all $n \times n$ bumpless pipe dreams.
A Bijection Through Ice
Theorem (W-, 2018+)

ASM(n) is in bijection with BPD(n).

Call an ASM **reduced** if its corresponding bumpless pipe dream is reduced.
The Six Vertex Model
Anna Weigandt
Bumpless Pipe Dreams and ASMs
Six Vertices and Six Tiles
Lascoux’s Formula for Grothendieck Polynomials
Write \( z_{ij} = x_i + y_j - x_i y_j \).

Fix \( A \in \text{ASM}(n) \).

- Assign each \((i, j) \in D(A)\) the weight \(-z_{ij}\).
- Assign each \((i, j) \in N(A)\) the weight \(1 - z_{ij}\).
- All other entries are given the weight 1.

Define \( \text{wt}(A) \) to be the product of the weights over all entries in \( A \).
The weight of an ASM

Example:

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$D(A) = \begin{array}{c}
\end{array}$$

$$\text{wt}(A) = -z_{11}z_{12}z_{21}(1 - z_{33})$$
Lascoux’s Formula

**Theorem (Lascoux)**

Fix $w \in S_n$. The double Grothendieck polynomial $G_w(x; y)$ is a weighted sum over ASMs:

$$G_w(x; y) = (-1)^{\ell(w)} \sum_{A \in \text{asm}(w)} \text{wt}(A).$$

Here, $\text{asm}(w)$ is the set of ASMs whose key is $w$. 
If $A_{ij} = 1$, we say it is a **neighbor** of $A_{rs}$ if

1. $(i, j)$ sits weakly northwest of $(r, s)$,
2. $(i, j) \neq (r, s)$, and
3. there are no other nonzero entries in $A$ which sit weakly between $(i, j)$ and $(r, s)$.
A $-1$ in $A \in \text{ASM}(n)$ is **removable** if there are no negative entries weakly northwest of its position in $A$. The region enclosed by a removable entry and its neighbors forms a (reverse) partition shape.

The **key** of an ASM is the permutation matrix obtained by applying inflation iteratively until no removable entries remain.
Example: $w = 2143$

\[ G_w(x; y) = (-1)^{\ell(w)} \sum_{A \in \text{asm}(w)} \text{wt}(A). \]

\[ G_w = z_{11}z_{33} + z_{11}z_{21}(1 - z_{33}) + z_{11}z_{12}(1 - z_{33}) - z_{11}z_{12}z_{21}(1 - z_{33}). \]
Lemma

$A \in \text{ASM}(n)$ is reduced if and only if $\#D(A) = \ell(\text{key}(A))$. In particular, if $A$ is not reduced, then $\#D(A) > \ell(\text{key}(A))$.

To get $G_w(x; y)$ from $G_w(x; y)$, replace each $y_i$ with $-y_i$ and then take the lowest degree terms. This agrees with the weight on bumpless pipe dreams given in [Lam et al., 2018].
Example: $w = 2143$

\[ z_{ij} = x_i + y_j - x_i y_j \quad \mapsto \quad (x_i - y_j) \]

\[ \mathcal{G}_w = z_{11}z_{33} + z_{11}z_{21}(1 - z_{33}) + z_{11}z_{12}(1 - z_{33}) - z_{11}z_{12}z_{21}(1 - z_{33}) \]

\[ \mathcal{G}_w = (x_1 - y_1)(x_3 - y_3) + (x_1 - y_1)(x_2 - y_1) + (x_1 - y_1)(x_1 - y_2). \]
Transition and Alternating Sign Matrices
Take \( w \in S_n \) and let \((r, s)\) be the position of the last box in the last row of its diagram. Let \( \nu \) be the (unique) permutation whose diagram is obtained by removing this box.

Let \( i_1 < i_2 < \cdots < i_k \) be the list of rows of the neighbors of \((r, s)\).
Transition for Double Grothendieck Polynomials

Theorem (Lascoux)

\[ G_w = G_v - (1 - z_{rs})G_v \cdot (1 - t_{i_1r}) \cdots (1 - t_{i_kr}). \]

Here \( G_v \cdot u = G_{vu} \).

Lascoux proved the ASM formula for Grothendieck polynomials by showing the weight on ASMs is compatible with transition.
Example: $w = 2143$

\[
\mathcal{G}_w = \mathcal{G}_v - (1 - z_{33})(\mathcal{G}_v - \mathcal{G}_{vt_{23}} - \mathcal{G}_{vt_{13}} + \mathcal{G}_{vt_{13}t_{23}})
\]

\[
= z_{33}\mathcal{G}_v + (1 - z_{33})(\mathcal{G}_{vt_{23}} + \mathcal{G}_{vt_{13}} - \mathcal{G}_{vt_{13}t_{23}}).
\]
Keys from Pipes
Given $\mathcal{P} \in \text{BPD}(n)$, we can read off a permutation $\delta(\mathcal{P})$ by replacing any crossing tiles with bumping tiles whenever two pipes have previously crossed.
Theorem (W-, 2018+)

If $A \in \text{ASM}(n)$ corresponds to the bumpless pipe dream $\mathcal{P}$ then $\text{key}(A) = \delta(\mathcal{P})$. 
References I

Bergeron, N. and Billey, S. (1993). RC-graphs and Schubert polynomials. *Experimental Mathematics, 2*(4):257–269.

Fomin, S. and Kirillov, A. N. (1996). The Yang-Baxter equation, symmetric functions, and Schubert polynomials. *Discrete Mathematics, 153*(1):123–143.

Knutson, A. and Miller, E. (2005). Gröbner geometry of Schubert polynomials. *Annals of Mathematics*, pages 1245–1318.

Kuperberg, G. (1996). Another proof of the alternating-sign matrix conjecture. *International Mathematics Research Notices, 1996*(3):139–150.

Lam, T., Lee, S. J., and Shimozono, M. (2018). Back stable Schubert calculus. *arXiv:1806.11233*.

Lascoux, A. Chern and Yang through ice.
Mills, W. H., Robbins, D. P., and Rumsey, H. (1983). Alternating sign matrices and descending plane partitions. *Journal of Combinatorial Theory, Series A*, 34(3):340–359.

Zeilberger, D. (1996). Proof of the alternating sign matrix conjecture. *Electron. J. Combin*, 3(2):R13.
Thank you!