Dimensionality Reduction for Categorical Data

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Abstract—Categorical attributes are those that can take a discrete set of values, e.g., colours. This work is about compressing vectors over categorical attributes to low-dimension discrete vectors. The current hash-based methods compressing vectors over categorical attributes to low-dimension discrete vectors do not provide any guarantee on the Hamming distances between the compressed representations. Here we present FSketch to create sketches for sparse categorical data and an estimator to estimate the pairwise Hamming distances among the uncompressed data only from their sketches. We claim that these sketches can be used in the usual data mining tasks in place of the original data without compromising the quality of the task. For that, we ensure that the sketches also are categorical, sparse, and the Hamming distance estimates are reasonably precise. Both the sketch construction and the Hamming distance estimation algorithms require just a single-pass; furthermore, changes to a data point can be incorporated into its sketch in an efficient manner. The compressibility depends upon how sparse the data is and is independent of the original dimension – making our algorithm attractive for many real-life scenarios. Our claims are backed by rigorous theoretical analysis of the properties of FSketch and supplemented by extensive comparative evaluations with related algorithms on some real-world datasets. We show that FSketch is significantly faster, and the accuracy obtained by using its sketches are among the top for the standard unsupervised tasks of RMSE, clustering and similarity search.

Index Terms—Dimensionality Reduction, Sketching, Feature Hashing, Clustering, Classification, Similarity Search.

1 INTRODUCTION

Of the many types of digital data that are getting recorded every second, most can be ordered – they belong to the ordinal type (e.g., age, citation count, etc.), and a good proportion can be represented as strings but cannot be ordered — they belong to the nominal type (e.g., hair colour, country, publication venue, etc.). The latter datatype is also known as categorical which is our focus in this work. Categorical attributes are commonly present in survey responses, and have been used earlier to model problems in bio-informatics [1], [2], market-basket transactions [3], [4], [5], web-traffic [6], images [7], and recommendation systems [8]. The first challenge practitioners encounter with such data is how to process them using standard tools most of which are designed for numeric data, that too often are real-valued.

Two important operations are often performed before running statistical data analysis tools and machine learning algorithms on such datasets. The first is encoding the data points using numbers, and the second is dimensionality reduction; many approaches combine the two, with the final objective being numeric vectors of fewer dimensions. To the best of our knowledge, the approaches usually followed are ad-hoc adaptations of those employed for vectors in the real space, and suffer from computational inefficiency and/or unproven heuristics [9]. The motivation of this work is to provide a solution that is efficient in practice and has proven theoretical guarantees.

For the first operation, we use the standard method of label encoding in this paper. In this a feature with $c$ categories is represented by an integer from $\{0, 1, 2, \ldots c\}$ where 0 indicates a missing category and $i \in \{1, 2, \ldots, c\}$ indicates the $i$-th category. Hence, an $n$-dimensional data point, where each feature can take at most $c$ values, can be represented by a vector from $\{0, 1, 2 \ldots c\}^n$ — we call such a vector as a categorical vector. Another approach is one-hot encoding (OHE) which is more popular since it avoids the implicit ordering among the feature values imposed by label-encoding. One-hot encoding of a feature with $c$ possible values is a $c$-dimensional binary vector in which the $i$-th bit is set to 1 to represent the $i$-th feature value. Naturally, one-hot encoding of an $n$-dimensional vector will be $nc$ dimensional — which can be very large if $c$ is large (e.g., for features representing countries, etc.). Not only label encoding avoids this problem, but is essential for the crucial second step – that of dimensionality reduction.

Dimensionality reduction is important when data points lie in a high-dimensional space, e.g., when encoded using one-hot encoding or when described using tens of thousands of categorical attributes. High-dimensional data vectors not only increase storage and processing cost, but they suffer from the “curse of dimensionality” that points to the decrease in performance after the dimension of the data points crosses a peak. Hence it is suggested that the high-dimensional categorical vectors be compressed to smaller vectors, essentially retaining the information only from the useful features. Baraniuk et al. [10] characterised a good dimensionality reduction in the Euclidean space as a compression algorithm that satisfies the following two conditions for any two vectors $x$ and $y$.

1) Information preserving: For any two distinct vectors $x$ and $y$, $R(x) \neq R(y)$. 

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2) \( \epsilon \)-Stability: (Euclidean) distances between all the points are approximately preserved (with \( \epsilon \) inaccuracy).

We call these two conditions the "well-designed" conditions. To obtain their mathematically precise versions, we need to narrow down upon a distance measure for categorical vectors. A natural measure for categorical vectors is an extension of the binary Hamming distance. For two \( n \)-dimensional categorical data points \( x \) and \( y \), the Hamming distance between them is defined as the number of features with different attributes in \( x \) and \( y \), i.e.,

\[
HD(x, y) = \sum_{i=1}^{n} \text{dist}(x[i], y[i]),
\]

\[
\text{dist}(x[i], y[i]) = \begin{cases} 
1, & \text{if } x[i] \neq y[i], \\
0, & \text{otherwise}.
\end{cases}
\]

**Problem statement:** The specific problem that we address is how to design a dimensionality reduction algorithm that can compress high-dimensional sparse label-encoded categorical vectors to low-dimensional categorical vectors so that (a) compressions of distinct vectors are distinct, and (b) the Hamming distance between two uncompressed vectors can be efficiently approximated from their compressed forms. These conditions, in turn, guarantee both information-preserving and stability. Furthermore, we would like to take advantage of the sparse nature of many real-world datasets. The most important requirement is the compressed vectors should be categorical as well, specifically not over real numbers and preferably not binary; this is to allow the statistical tests and machine learning tools for categorical datasets, e.g., k-mode, to run on the compressed datasets.

### 1.1 Challenges in the existing approaches

Dimensionality reduction is a well-studied problem \([11]\) (also see Table 8 in Appendix) but Hamming space does not allow the usual approaches applicable in the Euclidean spaces. Methods that work for continuous-valued data or even ordinal data (such as integers) do not perform satisfactorily for unordered categorical data. Among those that specifically consume categorical data, techniques via feature selection have been well studied. For example, in the case of labelled data \( \chi^2 \) \([12]\) and Mutual Information \([13]\) based methods select features based on their correlation with the label. This limits their applicability to only the classification tasks. Further, Kendall rank correlation coefficient \([14]\) "learns" the important features based on the correlation among them. Learning approaches tend to be computationally heavy and do not work reliably with small training samples. So what about task-agnostic approaches that do not involve learning? PCA-based methods, e.g., MCA is popular among the practitioners of biology \([11]\), however, we consider them merely a better-than-nothing approach since PCA is fundamentally designed for continuous data.

A quick search among internet forums, tutorials and Q&A websites revealed that the more favourable approach to perform machine learning tasks on categorical datasets is to convert categorical feature vectors to binary vectors using one-hot encoding \([15]\) see DictVectorizer — a widely-viewed tutorial on Kaggle calls it "The Standard Approach for Categorical Data" \([16]\). The biggest problem with OHE is that it is impractical for large \( n \) or large \( c \) followed by a technical annoyance that some OHE implementations do not preserve the Hamming distances for sparse vectors (see illustration in Figure 1). Hence, this encoding is used in conjunction with problem-specific feature selection or followed by dimensionality reduction from binary to binary vectors \([17]\), \([18]\), \([19]\). The latter is a viable heuristic that we wanted to improve upon by allowing non-binary compressed vectors (see Appendix A for a quick analysis of OHE followed by a state-of-the-art binary compression).

Another popular alternative, especially when \( n \times c \) is large, is feature hashing \([20]\) that is now part of most libraries, e.g., scikit-learn \([15]\) see FeatureHasher]. Feature hashing and other forms of hash-based approaches, also known as sketching algorithms, both encode and compress categorical feature vectors into integer vectors (sometimes signed) of a lower dimension, and furthermore, provide theoretical guarantees like stability, in some metric space. The currently known results for feature hashing apply only to the Euclidean space, however, Euclidean distance and Hamming distance are not monotonic for categorical vectors. It is neither known nor straightforward to ascertain whether feature hashing and its derivatives can be extended to the Hamming space which lacks the continuity that is crucial to their theoretical bounds. Other hash-based approaches either come with no guarantees and are used merely because of their compressibility or come with stability-like guarantees in a different space, e.g., cosine similarity by Simhash \([21]\). Our solution is a hashing approach that we prove to be stable in the Hamming space.

### 1.2 Overview of results

The commonly followed practices in dealing with categorical vectors, especially those with high dimensions and not involving supervised learning or training data, appear to be either feature hashing or one-hot encoding followed by dimensionality reduction of binary vectors \([22]\) Chapter 5. We provide a contender to these in the form of the FS sketch sketching algorithm to construct lower-dimensional categorical vectors from high-dimensional ones.

The lower-dimensional vectors, sketches, produced by FS sketch (we shall call these vectors as FS sketch too) have the desired theoretical guarantees and perform well on real-world datasets vis-à-vis related algorithms. Now we summarise the important features of FS sketch; in the summarisation, \( p \) is a constant that is typically chosen to be a prime number between 5-50.

**Lightweight and unsupervised:** First and foremost, FS sketch is an unsupervised process, and in fact, quite
lightweight making a single pass over an input vector and taking \(O(poly(\log p))\) steps per non-missing feature. The FSketch-\(es\) retain the sparsity of the input vectors and their size and dimension do not depend at all on \(c\). To make our sketches applicable out-of-the-box for modern applications where data keeps changing, we present an extremely lightweight algorithm to incorporate any change in a feature vector into its sketch in \(O(poly(\log p))\)-steps per modified feature. It should be noted that FSketch supports change of an attribute, deletion of an attribute and insertion of a previously missing attribute unlike some state-of-the-art sketches; for example, BinSketch [17] does not support deletion of an attribute.

Estimator for Hamming distance: We want to advocate the use of FSketch-\(es\) for data analytic tasks like clustering, etc. that use Hamming distance for the (dis)similarity metric. We present an estimator that can approximate the Hamming distance between two points by making a single pass over their sketches. The estimator follows a tight concentration bound and has the ability to estimate the Hamming distance from very low-dimensional sketches. In the theoretical bounds, the dimensions could go as low as \(4\sigma\) or even \(\sqrt{\sigma}\) (and independent of the dimension of the data) where \(\sigma\) indicates the sparsity (maximum number of non-zero attributes) of the input vectors; however, we later show that a much smaller dimension suffices in practice. Our sketch generation and the Hamming distance estimation algorithms combined meet the two conditions of “well-designed" dimensionality reduction.

**Theorem 1.** Let \(x\) and \(y\) be distinct categorical vectors, and \(\phi(x)\) and \(\phi(y)\) be their d-dimensional compressions.

1) \(\phi(x)\) and \(\phi(y)\) are distinct with probability \(\approx HD(x,y)/d\).
2) Let \(HD'(x, y)\) denote the approximation to the Hamming distance between \(x\) and \(y\) computed from \(\phi(x)\) and \(\phi(y)\). If \(d\) is set to \(4\sigma\), then with probability at least \(1 - \delta\) (for any \(\delta\) of choice),

\[
|HD(x, y) - HD'(x, y)| = O \left(\sqrt{\sigma \ln \frac{2}{\delta}}\right).
\]

The proof of (1) follows from Lemma 3 and the proof of (2) follows from Lemma 8 for which we used McDiarmid’s inequality. The theorem allows us to use compressed forms of the vectors in place of their original forms for data analytic and statistical tools that depend largely on their pairwise Hamming distances.

**Practical performance:** All of the above claims are proved rigorously but one may wonder how do they perform in practice. For this, we design an elaborate array of experiments on real-life datasets involving many common approaches for categorical vectors. The experiments demonstrate these facts.

- Some of the baselines do not output categorical vectors (see Section 4). Our FSketch algorithm is super-fast among those that do and offer comparable accuracy.
- When used for typical data analytic tasks like clustering, similarity search, etc. low-dimension FSketch-\(es\) bring immense speedup \textit{vis-a-vis} using the original (uncompressed) vectors, yet achieving very high accuracy. The NYTimes dataset saw 140x speedup upon compression to 0.1%.
- Even though highly compressed, the results of clustering, etc. on FSketch-\(es\) are close to what could be obtained from the uncompressed vectors and are comparable with the best alternatives. For example, we were able to compress the Brain cell dataset of dimensionality 1306127 to 1000 dimensions in a few seconds, yet retaining the ability to correctly approximating the pairwise Hamming distances from the compressed vectors. This is despite many other baselines giving either an out-of-memory error, not stopping even after running for a sufficiently long time, or producing significantly worse estimates of pairwise Hamming distances.
- The parameter \(p\) can be used to fine-tune the quality of results and the storage of the sketches.

We claim that FSketch is the best method today to compress categorical datasets for data analytic tasks that require pairwise Hamming distances with respect to both theoretical guarantee and practical performance.

### 1.3 Organisation of the paper

The rest of the paper is organised as follows. We discuss several related works in Section 2. In Section 3 we present our algorithm FSketch and derive its theoretical bounds. In Section 4 we empirically compare the performance of FSketch on several end tasks with state-of-the-art algorithms. We conclude our presentation in Section 5. The proofs of the theoretical claims and the results of additional experiments are included in Appendix.

### 2 Related work

**Dimensionality reduction:** Dimensionality reduction has been studied in-depth for real-valued vectors, and to some extent, also for discrete vectors. We categorise them into these broad categories — (a) random projection, (b) spectral projection, (c) locality sensitive hashing (LSH), (d) other hashing approaches, and (e) learning-based algorithms. All of them compress high-dimensional input vectors to low-dimensional ones that explicitly or implicitly preserve some measure of similarity between the input vectors.

The seminal result by Johnson and Lindenstrauss [23] is probably the most well known random projection-based algorithm for dimensionality reduction. This algorithm compresses real-valued vectors to low-dimensional real-valued vectors such that the Euclidean distances between the pairs of vectors are approximately preserved, but in such a manner that the compressed dimension does not depend upon the original dimension. The algorithm involves projecting a data matrix onto a random matrix whose each entry is sampled from a Gaussian distribution. This result has seen lots of enhancements, particularly with respect to generating the random matrix without affecting the accuracy \([24, 25]\). However, it is not clear whether any of those ideas can be made to work for categorical data and that too, for approximating Hamming distances.

Principal component analysis (PCA) is a spectral projection-based technique for reducing the dimensionality of high dimensional datasets by creating new uncorrelated variables that successively maximise variance. There are extensions of PCA that employ kernel methods that try to
capture non-linear relationships [27]. Multiple Correspondence Analysis (MCA) [28] does the analogous job for the categorical datasets. However, these methods perform dimensionality reduction by creating un-correlated features in a low-dimensional space whereas our aim is to preserve the pairwise Hamming distances in a low-dimensional space.

Another line of dimensionality reduction techniques builds upon the “Locality Sensitive Hashing (LSH)” algorithms. LSH algorithms have been proposed for different data types and similarity measures, e.g., real-valued vectors and the Euclidean distance [29], real-valued vectors and the cosine similarity [30], binary vectors and the Jaccard similarity [31]. However, generally speaking, the objective of an LSH is to group items so that similar items are grouped together and dissimilar items are not; unlike FSketch they do not provide explicit estimators of any similarity metric.

There are quite a few learning-based dimensionality reduction algorithms available such as Latent Semantic Analysis (LSA) [32], Latent Dirichlet Allocation (LDA) [33], Non-negative Matrix Factorisation (NNMF) [34], Generalized feature embedding learning (GEL) [35] all of which strive to learn a low-dimensional representation of a dataset while preserving some inherent properties of the full-dimensional dataset. They are rather slow due to the optimization step involved during learning. T-distributed Stochastic Neighbour Embedding (t-SNE) [36] is a faster non-linear dimensionality reduction technique that is widely used for the visualisation of high-dimensional datasets. However, the low-dimensional representation obtained from t-SNE is not recommended for use for other end tasks such as clustering, classification, anomaly detection as it does not necessarily preserve densities or pairwise distances. An autoencoder [37] is another learning-based non-linear dimension reduction algorithm. It basically consists of two parts: An encoder which aims to learn a low-dimensional representation of the input and a decoder which tries to reconstruct the original input from the output of the encoder. However, these approaches involve optimising a learning objective function and are usually slow and CPU-intensive.

The other hashing approaches randomly assign each feature (dimension) to one of several bins, and then compute a summary value for each bin by aggregating all the feature values assigned to it. A list of such summaries can be viewed as a low-dimensional sketch of the input. Such techniques have been designed for real-valued vectors approximating inner product (e.g., feature hashing [20]), binary vectors allowing estimation of several similarity measures such as Hamming distance, Inner product, Cosine, and Jaccard similarity (e.g., BinSketch [17]), etc. This work is similar to these approaches but for categorical vectors and only aiming to estimate the Hamming distances.

Another approach in this direction could be to encode categorical vectors to binary and then apply dimensionality reduction for binary vectors; unfortunately, the popular encodings, e.g. OHE, do not preserve Hamming distance for vectors with missing features. Nevertheless, it is possible to encode using OHE and then reduce its dimension. However, our theoretical analysis led to a worse accuracy compared to that of FSketch (see Appendix A for the analysis) and this approach turned out to be one of the worst performers in our experiments (see Section 4).

While our motivation was to design an end-task agnostic dimensionality reduction algorithm, there exist several that are designed for specific tasks, e.g., for clustering [38], for regression and discriminant analysis of labelled data [39], and for estimating covariance matrix [40]. Deep learning has gained mainstream importance and several researchers have proposed a dimensionality reduction “layer” inside a neural network [41]; this layer is intricately interwoven with the other layers and cannot be separated out as a standalone technique that outputs compressed vectors.

Feature selection is a limited form of dimensionality reduction whose task is to identify a set of good features, and maybe learn their relative importance too. Banerjee and Pal [42] recently proposed an unsupervised technique that identifies redundant features and selects those with bounded correlation, but only for real-valued vectors. For our experiments we chose the Kendall-Tau rank correlation approach that is applicable to discrete-valued vectors.

**Sketching algorithm:** The use of “sketches” for computing Hamming distance has been explicitly studied in the streaming algorithm framework. The first well-known solution was proposed by Cormode et al. [43] where they showed how to estimate a Hamming distance with high accuracy and low error. There have been several improvements to this result, in particular, by Kane et al. [44] where a sketch with the optimal size was proposed. However, we neither found any implementation nor an empirical evaluation of those approaches (the algorithms themselves appear fairly involved). Further, their objective was to minimise the space usage in the asymptotic sense in a streaming setting, whereas, our objective is to design a solution that can be readily used for data analysis. This motivated us to compress categorical vectors onto low-dimensional categorical vectors, unlike the real-valued vectors that the theoretical results proposed. A downside of our solution is that it heavily relies on the sparsity of a dataset unlike the sketches output by the streaming algorithms.

### Table 1

| Notations                                                                 | Description                                                                 |
|---------------------------------------------------------------------------|-----------------------------------------------------------------------------|
| categorical data vectors                                                 | $x, y$                                                                      |
| their Hamming distance                                                   | $h$                                                                         |
| compressed categorical vectors (sketches)                                | $\phi(x), \phi(y)$                                                         |
| $j$-th bit of a sketch $\phi(x)$                                        | $\phi_j(x)$                                                                |
| observed Hamming distance between sketches                               | $f$                                                                        |
| expected Hamming distance between sketches                                | $f^*$                                                                      |
| estimated Hamming distance between data vectors                           | $h$                                                                        |

### 3 Category sketching and Hamming distance estimation

Our technical objective is to design an effective algorithm to compress high-dimensional vectors over $\{0, 1, \ldots, c\}$ to integer vectors of a low dimension, aka sketches; $c$ can even be set to an upper bound on the largest number of categories among all the features. The number of attributes in the input vectors is denoted $n$ and the dimension of the compressed vector is denoted $d$. We will later show how to choose $d$ depending on the sparsity of a dataset that we denote $\sigma$. 
Fig. 2. An example illustrating how to compress a data point with categorical features using $\text{FSketch}$ to a 3-dimensional integer vector. The data point has 10 feature values, each of which is a categorical variable (the corresponding label encoded values are present inside the brackets). $c$ is chosen as 195 since the fifth, sixth, seventh, and eighth features have 195 categories which is the largest. $p$, $p'$ and $R$ are internal variables of $\text{FSketch}$.

The commonly used notations in this section are listed in Table 1.

Algorithm 1 Constructing $d$-dimensional $\text{FSketch}$ of $n$-dimensional vector $\vec{x}$

1: procedure $\text{INITIALIZE}$
2: Choose random mapping $\rho : \{1, \ldots, n\} \rightarrow \{1, \ldots, d\}$
3: Choose some prime $p$
4: Choose $n$ random numbers $R = r_1, \ldots, r_n$ with each $r_i \in \{0, \ldots, p - 1\}$
5: end procedure

1: procedure $\text{CREATESKETCH}(x \in \{0, 1, \ldots, c\}^n)$
2: Create empty sketch $\phi(x) = 0^d$
3: for $i = 1 \ldots n$
4: $j = \rho(i)$
5: $\phi_j(x) = (\phi_j(x) + x_i \cdot r_i) \mod p$
6: end for
7: return $\phi(x)$
8: end procedure

3.1 $\text{FSketch}$ construction

Our primary tool for sketching categorical data is a randomised sketching algorithm named $\text{FSketch}$ that is described in Algorithm 1; see Figure 2 for an example.

Let $x \in \{0, 1, \ldots, c\}^n$ denote the input vector, and the $i$-th feature or co-ordinate of $x$ is denoted by $x_i$. The sketch of input vector $x$ will be denoted $\phi(x) \in \{0, 1, \ldots, p - 1\}^d$ whose coordinates will be denoted $\phi_1(x), \phi_2(x), \ldots, \phi_d(x)$. Note that the initialisation step of $\text{FSketch}$ needs to run only once for a dataset. We are going to use the following characterisation of the sketches in the rest of this section; a careful reader may observe the similarity to Freivald’s algorithm for verifying matrix multiplication [45].

Observation 2. It is obvious from Algorithm 1 that the sketches created by $\text{FSketch}$ satisfy $\phi_j(x) = (\sum_{i \in \rho^{-1}(j)} x_i \cdot r_i) \mod p$.

3.2 Hamming distance estimation

Here we explain how the Hamming distance between $x$ and $y$ denoted $HD(x, y)$, percolates to their sketches as well. The objective is derive an estimator for $HD(x, y)$ from the Hamming distance between $\phi(x)$ and $\phi(y)$.

The sparsity of a set of vectors denoted $\sigma$, is the maximum number of non-zero coordinates in them. For the theoretical analysis, we assume that we know the sparsity of the dataset, or at least an upper bound of the same. Note that, for a pair of sparse vectors $x, y \in \{0, 1, \ldots, c\}^n$, the Hamming distance between them can vary from 0 (when they are same) to $2\sigma$ (when they are completely different).

We first prove case (a) of Theorem 1 which states that sketches of different vectors are rarely the same.

Lemma 3. Let $h$ denote $HD(x, y)$ for two input vectors $x, y$ to $\text{FSketch}$. Then

$$\Pr_{\rho,R}[^{\phi}(x) \neq {\phi}(y)] = (1 - \frac{1}{p})(1 - (1 - \frac{1}{d})^h).$$

Proof. Fix a mapping $\rho$ and then define $F_j(x)$ as the vector \(x_{i_1}, x_{i_2}, \ldots : i_k \in \{1, \ldots, n\}\) of values of $x$ that are mapped to $j$ in $\phi(x)$ in the increasing order of their coordinates, i.e., $\rho(i_k) = j$ and $i_1 < \ldots < i_k < i_{k+1}$. Since $\rho$ is fixed, $F_j(x)$ is also a vector of the same length. The key observation is that if $F_j(x) = F_j(y)$ then $\phi_j(x) = \phi_j(y)$ but the converse is not always true. Therefore we separately analyse both the conditions (a) $F_j(x) \neq F_j(y)$ and (b) $F_j(x) = F_j(y)$.

It is given that $x$ and $y$ differ at $h$ coordinates. Therefore, $F_j(x) \neq F_j(y)$ iff any of those coordinates are matched to $j$ by $\rho$. Thus,

$$\Pr_{\rho,R}[\phi_j(x) = {\phi}_j(y) | F_j(x) \neq F_j(y)] = \frac{1}{p}. \quad (2)$$

Due to Equations (1)-(2) we have

$$\Pr_{\rho,R}[\phi_j(x) \neq {\phi}_j(y)] = \Pr_{\rho,R}[\phi_j(x) \neq {\phi}_j(y) | F_j(x) \neq F_j(y)] \cdot \Pr_{\rho,R}[F_j(x) \neq F_j(y)]$$

$$+ \Pr_{\rho,R}[\phi_j(x) \neq {\phi}_j(y) | F_j(x) = F_j(y)] \cdot \Pr_{\rho,R}[F_j(x) = F_j(y)]$$

$$= (1 - \frac{1}{p})(1 - (1 - \frac{1}{d})^h).$$

The right-hand side of the expression in the statement of the lemma can be approximated as $(1 - \frac{1}{p})^\frac{h}{2}$ which is stated as case (a) of Theorem 1. The lemma also allows us to relate the Hamming distance of the sketches to the Hamming distance of the vectors which is our main tool to define an estimator.

Lemma 4. Let $h$ denote $HD(x, y)$ for two input vectors $x, y$ to $\text{FSketch}$, $f$ denote $HD(\phi(x), \phi(y))$ and $f^*$ denote $\mathbb{E}[HD(\phi(x), \phi(y))]$. Then

$$f^* = \mathbb{E}[f] = d \left(1 - \frac{1}{p}\right) \left(1 - (1 - \frac{1}{d})^h\right).$$

Note that the initialisation step of $\text{FSketch}$ is denoted by $\phi_1(x), \phi_2(x), \ldots, \phi_d(x)$.
The lemma is easily proved using Lemma 3 by applying the linearity of expectation on the number of coordinates \( j \) such that \( \phi_j(x) \neq \phi_j(y) \). We are now ready to define an estimator for the Hamming distance.

Using \( D = (1 - \frac{1}{2}) \) and \( P = (1 - \frac{1}{2}) \), we can write

\[
f^* = dP(1 - D^h) \quad \text{and} \quad h = \ln \left( 1 - \frac{f^*}{dP} \right) / \ln D. \tag{3}
\]

Our proposal to estimate \( h \) is to obtain a tight approximation of \( f^* \) and then use the above expression.

**Definition 5 (Estimator of Hamming distance).** Given sketches \( \phi(x) \) and \( \phi(y) \) of data points \( x \) and \( y \), suppose \( f \) represents \( HD(\phi(x), \phi(y)) \). We define the estimator of \( HD(x, y) \) as \( \hat{h} = \ln \left( 1 - \frac{f^*}{dP} \right) / \ln D \) if \( f < dP \) and \( 2 \sigma \) otherwise.

Observe that \( \hat{h} \) is set to \( 2 \sigma \) if \( f \geq dP \). However, we shall show in the next section that this occurs very rarely.

### 3.3 Analysis of Estimator

\( \hat{h} \) is pretty reliable when the actual Hamming distance is 0; in that case \( \phi(x) = \phi(y) \) and thus, \( f = 0 \) and so is \( \hat{h} \). However, in general, \( \hat{h} \) could be different from \( h \). The main result of this section is that their difference can be upper bounded when we set the dimension of \( \text{FSketch} \) to \( d = 4\sigma \).

The results of this subsection rely on the following lemma that proves that an observed value of \( f \) is concentrated around its expected value \( f^* \).

**Lemma 6.** Let \( \alpha \) denote a desired additive accuracy. Then, for any \( x, y \) with sparsity \( \sigma \),

\[
\Pr \left[ |f - f^*| \geq \alpha \right] \leq 2 \exp \left( -\frac{\alpha^2}{dP} \right).
\]

The proof of the lemma employs martingales and McDiarmid’s inequality and is available in Appendix B. The lemma allows us to upper bound the probability of \( f \geq dP \).

**Lemma 7.** \( \Pr[f \geq dP] \leq 2 \exp(-P^2 \sigma) \).

The right-hand side is a very small number, e.g., it is of the order of \( 10^{-278} \) for \( p = 5 \) and \( \sigma = 1000 \). The proof is a straightforward application of Lemma 3 and is explained in Appendix B. Now we are ready to show that the estimator \( \hat{h} \), which uses \( f \) instead of \( f^* \) (refer to Equation 3) is almost equal to the actual Hamming distance.

**Lemma 8.** Choose \( d = 4\sigma \) as the dimension of \( \text{FSketch} \) and choose a prime \( p \) and an error parameter \( \delta \in (0, 1) \) (ensure that \( 1 - \frac{1}{p} \geq \frac{1}{\sqrt{2}} \ln \frac{2}{\delta} \)). Then the estimator defined in Definition 5 is close to the Hamming distance between \( x \) and \( y \) with high probability, i.e.,

\[
\Pr \left[ |\hat{h} - h| \geq \frac{32}{1-1/p} \sqrt{\sigma \ln \frac{2}{\delta}} \right] \leq \delta.
\]

If the data vectors are not too dissimilar which is somewhat evident from Figure 3, then a better compression is possible which is stated in the next lemma. The proofs of both these lemmas are fairly algebraic and use standard inequalities; they are included in Appendix B.

**Lemma 9.** Suppose we know that \( h \leq \sqrt{\sigma} \) and choose \( d = 16\sqrt{\sigma \ln \frac{2}{\delta}} \) as the dimension for \( \text{FSketch} \). Then (a) also \( f < dP \) with high probability and moreover we get a better estimator. That is, (b) \( \Pr \left[ |\hat{h} - h| \right] \geq \frac{8}{1-1/p} \sqrt{\sigma \ln \frac{2}{\delta}} \leq \delta \).

The last two results prove case (b) of Theorem 1 which states that the estimated Hamming distances are almost always fairly close to the actual Hamming distances. We want to emphasise that the above claims on \( d \) and accuracy are only theoretical bounds obtained by worst-case analysis. We show in our empirical evaluations that an even smaller \( d \) leads to better accuracy in practice for real-life instances.

There is a way to improve the accuracy even further by generating multiple \( \text{FSketch} \) using several independently generated internal variables and combining the estimates obtained from each. We observed that the median of the estimates can serve as a good statistic, both theoretically and empirically. We discuss this in detail in Appendix C.

### 3.4 Complexity analysis

The results in the previous section show that the accuracy of the estimator \( h \) can be tightened, or a smaller probability of error can be achieved, by choosing large values of \( p \) which has a downside of a larger storage requirement. In this section, we discuss these dependencies and other factors that affect the complexity of our proposal.

The USP of \( \text{FSketch} \) is its efficiency. There are two major operations with respect to \( \text{FSketch} \) — construction of sketches and estimation of Hamming distance from two sketches. Their time and space requirements are given in the following table and explained in detail in Appendix C.
Space savings offered by \textit{FSketch} on an example scenario with $2^{20}$ data points, each of $2^{10}$ dimensions but having only $2^7$ non-zero entries where non-zero entry belongs to one of $2^3$ categories. \textit{FSketch} dimension is $2^9$ (as prescribed theoretically) and its parameter $p$ is close to $2^3$. (*)

| Construction          | Uncompressed | Compressed |
|-----------------------|--------------|------------|
| Naive                 | \( \mathcal{O}(n) \) | \( \mathcal{O}(d \log p) \) |
| Sparse vector format  | \( \mathcal{O}(n \log p) \) | \( \mathcal{O}(d \log p) \) |

We are aware of efficient representations of sparse data vectors, but for the sake of simplicity we assume full-size arrays to store vectors in this table; similarly, we assume simple dictionaries for storing the internal variables \( p, R \) and \( p \). While it may be possible to reduce the number of random bits by employing \( k \)-wise independent bits and mappings, we left it out of the scope of this work.

Both the operations are quite fast compared to the matrix-based and learning-based methods. There is very little space overhead too; we explain the space requirement with the help of an example in Table 2 — one should keep in mind that a sparse representation of a vector has to store the non-zero entries as well as their positions in it.

Apart from the efficiency in both time and space measures, \textit{FSketch} provides additional benefits. Recall that each entry of an \textit{FSketch} is an integral value from 0 to \( p - 1 \). Even though 0 does not necessarily indicate a missing feature in a compressed vector, we show below that 0 has a predominant presence in the sketches. The sketches can therefore be treated as sparse vectors that further facilitates their efficient storage.

\textbf{Lemma 10.} If \( d = 4\sigma \) (as required by Lemma 8), then the expected number of non-zero entries of \( \phi(x) \) is upper bounded by \( \frac{d}{2} \). Further, at least 50% of \( \phi(x) \) will be zero with probability at least \( \frac{1}{2} \).

The lemma can be proved using a balls-and-bins type analysis (see Appendix D for the entire proof).

### 3.5 Sketch updating

Imagine a situation where the categories of attributes can change dynamically, and they can both “increase”, “decrease” or even “vanish”. We present Algorithm 2 to incorporate such changes without recomputing the sketch afresh. The algorithm simply uses the formula for a sketch entry as given in Observation 2.

Most hashing-based sketching and dimensionality reduction algorithms that we have encountered either require complete regeneration of \( \phi(x) \) when some attributes of \( x \) change or are able to handle addition of previously missing attributes but not their removal.

### 4 Experiments

We performed our experiments on a machine having Intel(R) Xeon(R) CPU E5-2650 v3 @ 2.30GHz, 94 GB RAM, and running a Ubuntu 64-bits OS.

We first study the effect of the internal parameters of our proposed solution on its performance. We start with the effect of the prime number \( p \); then we compare \textit{FSketch} with the appropriate baselines for several unsupervised data-analytic tasks (see Table 3) and objectively establish these advantages of \textit{FSketch} over the others.

(a) Significant speed-up in the dimensionality reduction time,
(b) considerable savings in the time for the end-tasks (e.g., clustering) which now runs on the low-dimensional sketches,
(c) but with comparable accuracy of the end-tasks (e.g., clustering).

Several baselines threw out-of-memory errors or did not stop on certain datasets. We discuss the errors separately in Section 6 in Appendix.

### 4.1 Dataset description

The efficacy of our solution is best described for high-dimensional datasets. Publicly available categorical datasets being mostly low-dimensional, we treated several integer-valued freely available real-world datasets as categorical. Our empirical evaluation was done on the following seven such datasets with dimensions between 5000 and 1.3 million, and sparsity from 0.07% to 30%.

- **Gisette Data Set** [47], [48]: This dataset consists of integer feature vectors corresponding to images of handwritten digits and was constructed from the MNIST data. Each image, of \( 28 \times 28 \) pixels, has been pre-processed (to retain the pixels necessary to disambiguate the digit 4 from 9) and then projected onto a higher-dimensional feature space represented to construct a 5000-dimension integer vector.
- **BoW (Bag-of-words)** [47], [49]: We consider the following five corpus – NIPS full papers, KOS blog entries, Enron Emails, NYTimes news articles, and tagged web pages from the social bookmarking site delicious.com. These datasets are “BoW”(Bag-of-words) representations of the corresponding text cor-
neurons reduction technique that preserves Hamming distance. We which is the state-of-the-art binary-to-binary dimensionality use it is by further compressing the one-hot encoded coding the NYTimes dataset is OHE actually increases the dimension to very high levels categorical data to a numeric vector and can approximate which is one of the most common methods to convert 4.2 Baselines categories, and the sparsity of these datasets in the Table 3. We summarise the dimensionality, the number of dimension and the earlier ones due to their popularity in dimensionality-reduction experiments. We consider all the data points for KOS, Enron, Gisette, DeliciousMIL, a 10,000 sized sample for NYTimes, and a 2000 sized samples for BrainCell. We summarise the dimensionality, the number of categories, and the sparsity of these datasets in the Table 3.

4.3 Choice of \( p \)
We discussed in Section 3 that a larger value of \( p \) (a prime number) leads to a tighter estimation of Hamming distance but degrades sketch sparsity, which negatively affects performance at multiple fronts, and moreover, demands more space to store a sketch. We conducted an experiment to study this trade-off, where we ran our proposal with different values of \( p \), and computed the corresponding RMSE values. The RMSE is defined as the square-root of the average error, among all pairs of data points, between their actual Hamming distances and the corresponding estimate obtained via FS sketch. Note that a lower RMSE indicates that the sketch correctly estimates the underlying pairwise Hamming distance. We also note the corresponding space overhead which is defined as the ratio of the space used by uncompressed vector and its sketch obtained from FS sketch. We consider storing a data point in a typical sparse vector format – a list of non-zero entries affecting performance at multiple fronts, and moreover, demands more space to store a sketch. We conducted an experiment to study this trade-off, where we ran our proposal with different values of \( p \), and computed the corresponding RMSE values. The RMSE is defined as the square-root of the average error, among all pairs of data points, between their actual Hamming distances and the corresponding estimate obtained via FS sketch. Note that a lower RMSE indicates that the sketch correctly estimates the underlying pairwise Hamming distance. We also note the corresponding space overhead which is defined as the ratio of the space used by uncompressed vector and its sketch obtained from FS sketch. We consider storing a data point in a typical sparse vector format – a list of non-zero entries and their positions (see Table 2). We summarise our results in Figures 4 and 5 respectively. We observe that a large value of \( p \) leads to a lower RMSE (in Figure 4), however simultaneously it leads to a smaller space compression (Figure 5). As a heuristic, we decided to set \( p \) as the next prime after \( c \) as shown in this table.

4.4 Variance of \( F \) sketch
In Section 3.3 we explained that the bias of our estimator is upper bounded with a high likelihood. However, there re-

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**TABLE 3**

| Dataset                | Categories | Dimension | Sparsity | No. of points |
|------------------------|------------|-----------|----------|---------------|
| Enron                  | 150        | 28,102    | 2021     | 39,861        |
| DeliciousMIL           | 56         | 8,519     | 200      | 12,234        |
| NYTimes articles       | 114        | 10,266    | 871      | 10,000        |
| NIPS full papers       | 132        | 12,419    | 914      | 1,500         |
| KOS blog entries       | 42         | 6,006     | 457      | 3,430         |
| Million Brain Cells E18 Mice | 2036 | 13,061,277 | 1,051 | 2,000 |

**TABLE 4**

| Baseline     | Description                                      |
|--------------|--------------------------------------------------|
| 1. SSD       | Sketch via Stable Distribution                   |
| 2. OHE       | One Hot Encoding (BinSketch)                     |
| 3. FH        | Feature Hashing (FH)                             |
| 4. SH        | Signed-random projection/SimHash                 |
| 5. KT        | Kendall rank correlation coefficient              |
| 6. LSA       | Latent Semantic Analysis                         |
| 7. LDA       | Latent Dirichlet Allocation                      |
| 8. MCA       | Multiple Correspondence Analysis                 |
| 9. NNMF      | Non-neg. Matrix Factorization                    |
| 10. PCA      | Vanilla Principal component analysis             |
| 11. VAE      | Variational autoencoder                          |
| 12. CATPCA   | Categorical PCA                                  |
| 13. HCA      | Hierarchical Cluster Analysis                    |

**TABLE 5**

| Dataset | Categories | Dimension | Sparsity | No. of points |
|---------|------------|-----------|----------|---------------|
| KOS     | 43         | 59        | 151      | 1,500         |
| DeliciousMIL | 59   | 59        | 1,009    | 1,000         |
| NIPS    | 137        |           |          |               |

---

- 1.3 Million Brain Cell Dataset [50]: This dataset contains the result of a single cell RNA-seq (scRNA-seq) of 1.3 million cells captured and sequenced from an E18.5 mouse brain. Each gene represents a data point and for every gene, the dataset stores the read-count of that gene corresponding to each cell – these read-counts form our features. We chose the last dataset due to its very high dimension and the earlier ones due to their popularity in dimensionality-reduction experiments. We consider all the data points for KOS, Enron, Gisette, DeliciousMIL, a 10,000 sized sample for NYTimes, and a 2000 sized samples for BrainCell. We summarise the dimensionality, the number of categories, and the sparsity of these datasets in the Table 3.

**TABLE 4**

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| 5. KT   | Kendall rank correlation coefficient              |
| 6. LSA  | Latent Semantic Analysis                         |
| 7. LDA  | Latent Dirichlet Allocation                      |
| 8. MCA  | Multiple Correspondence Analysis                 |
| 9. NNMF | Non-neg. Matrix Factorization                    |
| 10. PCA | Vanilla Principal component analysis             |
| 11. VAE | Variational autoencoder                          |
| 12. CATPCA | Categorical PCA                                    |
| 13. HCA | Hierarchical Cluster Analysis                    |

In all these datasets, the attribute takes integer values which we consider as categories.

---

The experiments reveal that, at least for the datasets in the above experiments, setting \( p \) to be at least \( c/4 \) may be practically sufficient, since there does not appear to be much advantage in using a larger \( p \).
Fig. 4. Comparison of RMSE measure obtained from FSketch algorithm on various choices of $p$. Values of $c$ for NIPS, Enron, NYTimes, and GISTE are 132, 150, 114, and 999, respectively.

Fig. 5. Space overhead of uncompressed vectors stored as a list of non-zero entries and their positions. $Y$-axis represents the ratio of the space used by uncompressed vector to that obtained from FSketch.

Fig. 6. Comparison of avg. error in estimating Hamming distance of a pair of points from the Enron dataset.

Fig. 7. Comparison among the baselines on the dimensionality reduction time. See Appendix G for results on the other datasets which show a similar trend and Section F for the errors encountered by some baselines.
mains the question of its variance. To decide the worthiness of our method, we compared the variance of the estimates of the Hamming distance obtained from FSketch and from the other randomised sketching algorithms with integer-valued sketches (KT was not included as it is a deterministic algorithm, and hence, has zero variance).

Figure 6 shows the Hamming error (estimation error) for a randomly chosen pair of points from the Enron dataset, averaged over 100 iterations. We make two observations.

First is that the estimate using FSketch is closer to the actual Hamming distance even at a smaller reduced dimension; in fact, as the reduced dimension is increased, the variance becomes smaller and the Hamming error converges to zero. Secondly, FSketch causes a smaller error compared to the other baselines. On the other hand, feature hashing highly underestimates the actual Hamming distance, but has low variance, and tends to have negligible Hamming error with an increase of the reduced dimension. The behaviour of SimHash is counter-intuitive as on lower reduced dimensions it closely estimates the actual Hamming distances, but on larger dimensions it starts to highly underestimate the actual Hamming distances. This creates an ambiguity on the choice of a dimension for generating a low-dimensional sketch of a dataset. Similar to FSketch, the sketches produced by SSD, though real-valued, allow estimation of pairwise Hamming distances. However the estimation error increases with the reduced dimension. Lastly, OHE seems to be highly underestimating pairwise Hamming distances.

4.5 Speedup in dimensionality reduction

We compress the datasets to several dimensions using FSketch and the baselines and report their running times in Figure 7. We notice that FSketch has a comparable speed w.r.t. Feature hashing and SimHash, and is significantly faster than the other baselines. However, both feature
hashing and SimHash are not able to accurately estimate the Hamming distance between data points and hence perform poorly on RMSE measure (Subsection 4.6) and the other tasks. Many baselines such as OHE, KT, NNMF, MCA, CATPCA, HCA give “out-of-memory” (OOM) error, and also didn’t stop (DNS) even after running for a sufficiently long time (~ 10 hrs) on high dimensional datasets such as BrainCell and NYTtimes. On other moderate dimensional datasets such as Enron and KOS, our speedup w.r.t. these baselines are of the order of a few thousand. We report the numerical speedups that we observed in Table 6.

4.6 Performance on root-mean-squared-error (RMSE)

How good are the sketches for estimating Hamming distances between the uncompressed points in practice? To answer this, we compare FSketch with integer-valued sketching algorithms, namely, feature hashing, SimHash, Kendall correlation coefficient and OHE+BinSketch. Note that feature hashing and SimHash are known to approximate inner product and cosine similarity, respectively. However, we consider them in our comparison nonetheless as they output discrete sketches and Hamming distance can be computed on their sketch. We also include SSD for comparison which outputs real-valued sketches and estimates original pairwise Hamming distance. For each of the methods we compute its RMSE as the square-root of the average error, among all pairs of data points, between their actual Hamming distances and their corresponding estimates (for FSketch the estimate was obtained using Definition 5). Figure 8 compares these values of RMSE for different dimensions; note that a lower RMSE is an indication of better performance. It is immediately clear that the RMSE of FSketch is the lowest among all; furthermore, it falls to zero rapidly with increase in reduced dimension. This demonstrates that our proposal FSketch estimates the underlying pairwise Hamming distance better than the others.

4.7 Performance on clustering

We compare the performance of FSketch with baselines on the task of clustering and similarity search, and present the results for the first task in this section. The objective of the clustering experiment was to test if the data points in the reduced dimension maintain the original clustering structure. If they do, then it will be immensely helpful for those techniques that use a clustering, e.g., spam filtering. We used the purity index to measure the quality of k-mode and k-means clusters on the reduced datasets obtained through the compression algorithms; the ground truth was obtained using k-mode on the uncompressed data (for more details refer to Appendix E.2).

We summarise our findings on quality in Figure 9. The compressed versions of the NIPS, Enron, and KOS datasets that were obtained from FSketch yielded the best purity index as compared to those obtained from the other baselines; for the other datasets the compressed versions from FSketch are among the top. Even though it appears that KT offers comparable performance on the KOS, DeliciousMIL, and Gisette datasets w.r.t. FSketch, the downside of using KT is that its compression time is much higher than that of FSketch (see Table 6) on those datasets, and moreover it gives OOM/DNS error on the remaining datasets. Performance of FH also remains in the top few. However, its performance degrades on the NIPS dataset.

We tabulate the speedup of clustering of FSketch-compressed data over uncompressed data in Table 7 where we observe significant speedup in the clustering time, e.g., 139x when run on a 1000 dimensional FSketch. Recall that the dimensionality reduction time of our proposal is among the fastest among all the baselines which further reduces the total time to perform clustering by speeding up the dimensionality reduction phase. Thus the overall observation is that FSketch appears to be the most suitable method for clustering among the current alternatives, especially, for high-dimensional datasets on which clustering would take a long time.

4.8 Performance on similarity search

We take up another unsupervised task – that of similarity search. The objective here is to show that after dimensionality reduction the similarities of points with respect to some query points are maintained. To do so, we randomly split the dataset in two parts 5% and 95% – the smaller partition is referred to as the query partition and each point of this partition is called a query vector; we call the larger partition as training partition. For each query vector, we find top-k similar points in the training partition. We then perform dimensionality reduction using all the methods (for various values of reduced dimensions). Next, we process the compressed dataset where, for each query point, we compute the top-k similar points in the corresponding low-dimensional version of the training points, by maintaining the same split. For each query point, we compute the accuracy of baselines by taking the Jaccard ratio between the set of top-k similar points obtained in full dimensional data with the top-k similar points obtained in reduced dimensional dataset. We repeat this for all the points in the querying partition, compute the average, and report this as accuracy.

We summarise our findings in Figure 10. Note that PCA, MCA and LSA can reduce the data dimension up to the minimum of the number of data points and the original data dimension. Therefore their reduced dimension is at most 2000 for Brain cell dataset.

The top few methods appear to be feature hashing (FH), Kendall-Tau (KT), HCA along with FSketch. However, KT gives OOM and DNS on the Brain cell, NYTtimes and Enron datasets, and HCA gives DNS error on BrainCell and NYTtimes datasets. Further, their dimensionality reduction time are much worse than FSketch (see Table 6). FSketch outperforms FH on the BrainCell and the Enron datasets; however, on the remaining datasets, both
of them appear neck to neck for similarity search despite the fact that there is no known theoretical understanding of FH for Hamming distance — in fact, it was included in the baselines as a heuristic because it offers discrete-valued sketches on which Hamming distance can be calculated. Here want to point out that FH was not a consistent top-performer for clustering and similarity search.

The two other methods that are designed for Hamming distance, namely SSD and OHE, perform significantly worse than FSketch; in fact, the accuracy of OHE lies almost to the bottom on all the four datasets.

We also summarise the speedup of FSketch-compressed data over uncompressed data, on similarity search task, in Table 7. We observe a significant speedup — e.g., 1231.6× speedup on the BrainCell dataset when run on a 1000 dimensional FSketch.

To summarise, FSketch is one of the best approaches towards similarity search for high-dimensional datasets and the best if we also require theoretical guarantees or applicability towards other data analytic tasks.

5 Conclusion

In this paper, we proposed a sketching algorithm named FSketch for sparse categorical data such that the Hamming distances estimated from the sketches closely approximate the original pairwise Hamming distances. The low-dimensional data obtained by FSketch are discrete-valued, and therefore, enjoy the flexibility of running the data analytics tasks suitable for categorical data. The sketches allow tasks like clustering, similarity search to run which might not be possible on a high-dimensional dataset.

Our method does not require learning from the dataset and instead, exploits randomization to bring forth large speedup and high-quality output for standard data analytic tasks. We empirically validated the performance of our algorithm on several metric and end tasks such as RMSE, clustering, similarity search, and observed comparable performance while simultaneously getting significant speedup in dimensionality reduction and end-task with respect to several baselines. A common practice to analyse high-dimensional datasets is to partition them into smaller datasets. Given the simplicity, efficiency, and effectiveness of our proposal, we hope that FSketch will allow such analysis to be done on the full datasets and on general-purpose hardware.

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APPENDIX A

ANALYSIS OF ONE-HOT ENCODING + BINARY COMPRESSION

Let x and y be two n-dimensional categorical vectors with sparsity at most σ; c will denote the maximum number of values any attribute can take. Let $x'$ and $y'$ be the one-hot encodings of x and y, respectively. Further, let $x''$ and $y''$ denote the compression of $x'$ and $y'$, respectively, using BinSketch [17], which is the state-of-the-art dimensionality reduction for binary vectors using Hamming distance.

Observe that the sparsity of $x'$ is the same as that of x and a similar claim holds for $y'$ and y. However, $HD(x', y')$ does not hold a monotonic relationship with $HD(x, y)$. It is easy to show that $HD(x, y) \leq HD(x', y') \leq 2HD(x, y)$. Therefore,

$$|HD(x, y) - HD(x', y')| \leq 2\sigma. \tag{4}$$

We need the following lemma that was used to analyse BinSketch [17, Lemma 12, Appendix A].

**Lemma 11.** Suppose we compress two n'-dimensional binary vectors $x'$ and $y'$ with sparsity at most σ to g-dimensional binary sketches, denotes $x''$ and $y''$ respectively, by following an algorithm proposed in the BinSketch work. If $g$ is set to $\sigma \cdot \frac{\ln 6}{\delta}$ for any $\delta \in (0, 1)$, then the following holds with probability at least $1 - \delta$.

$$|HD(x', y') - HD(x'', y'')| \leq 6\sqrt{\frac{\sigma}{\delta} \ln \frac{6}{\delta}}. \tag{5}$$

Combining the above inequality with that in Equation (4) gives us

$$|HD(x', y') - HD(x'', y'')| \leq 2\sigma + 6\sqrt{\frac{\sigma}{\delta} \ln \frac{6}{\delta}} \leq 2\sigma \sqrt{\frac{\ln 2}{\delta}}$$

if we set the reduced dimension to $\sigma \cdot \frac{\ln 6}{\delta}$.

This bound is worse compared to that of BinSketch where we can prove an accuracy of $O(\sqrt{\ln \frac{1}{\delta}})$ using reduced dimension value of 4σ (see Lemma 8).

APPENDIX B

PROOFS FROM SECTION 3.3

**Lemma 6.** Let α denote a desired additive accuracy. Then, for any x, y with sparsity σ,

$$\Pr \left[ |f - f^*| \geq \alpha \right] \leq 2\exp \left( -\frac{\alpha^2}{4\sigma} \right).$$

**Proof.** Fix any $R$ and x, y; the rest of the proof applies to any $R$, and therefore, holds for a random $R$ as well. Define a vector $z \in \{0, \pm 1, \ldots, \pm \epsilon\}^n$ in which $z_i = (x_i - y_i)$; the number of non-zero entries of $z$ are at least 2σ since the number of non-zero entries of x and y are at most σ. Let $J_0$ be the set of coordinates from {1, ..., n} at which $z = 0$, and let $J_1$ be the set of the rest of the coordinates; from above, $J_1 \leq 2\sigma$.

Define the event $E_j$ as "$\phi_j(x) \neq \phi_j(y)$". Note that $f$ can be written as a sum of indicator random variables, $\sum_j f(E_j)$, and we would like to prove that $f$ is almost always close to $f^* = E[f]$. Observe that $\phi_j(x) = \phi_j(y)$ iff $\sum_{i \in R^{-1}(j)} z_i \cdot r_i = 0 \mod p$ iff $\sum_{i \in R^{-1}(j) \cap J_1} z_i \cdot r_i = 0 \mod p$. In other words, $\rho(i)$ could be set to anything for $i \in J_0$ without any effect on the event $E_j$; hence, we will assume that the mapping $\rho$ is defined as a random mapping only for $i \in J_1$, and further for the ease of analysis, we will denote them as $\rho(i_1), \rho(i_2), \ldots, \rho(i_{2\sigma})$ (if $|J_1| < 2\sigma$ then move a few coordinates from $J_0$ to $J_1$ without any loss of correctness).

To prove the concentration bound we will employ martingales. Consider the sequence of these random variables $\rho^j = \rho(i_1), \rho(i_2), \ldots, \rho(i_{2\sigma})$—these are independent. Define a function $g(\rho^j)$ of these random variables as a sum of indicator random variables as stated below (note that $R$ and $\rho(i)$, for $i \in J_0$, are fixed at this point)

$$g(\rho(i_1), \rho(i_2), \ldots, \rho(i_{2\sigma}))$$

$$= \sum_j I \left( \sum_{i \in R^{-1}(j) \cap J_1} z_i \cdot r_i \neq 0 \mod p \right)$$

Now consider an arbitrary $t \in \{1, \ldots, 2\sigma\}$ and let $q = \rho(i_t)$; observe that $z_{i_t}$ influences only $E_q$. Choose an arbitrary value $q' \in \{1, \ldots, d\}$ that is different from q. Observe that, if $\rho$ is modified only by setting $\rho(i_t) = q'$ then we claim that "bounded difference holds".

**Proposition 12.** $|g(\rho(i_1), \ldots, \rho(i_{t-1}), q, \ldots, \rho(i_{2\sigma})) - g(\rho(i_1), \ldots, \rho(i_{t-1}), q', \ldots, \rho(i_{2\sigma}))| \leq 2$.

The proposition holds since the only effects of the change of $\rho(i_t)$ from q to q' are seen in $E_q$ and $E_{q'}$ (earlier $E_q$ depended upon $z_{i_t}$ that now changes to $z_{i_t}''$ being depended upon $z_{i_t}$). Since $g()$ obeys bounded difference, therefore, we can apply McDiarmid’s inequality [46, Ch 17, 54].

**Theorem 13 (McDiarmid’s inequality).** Consider independent random variables $X_1, \ldots, X_m \in \mathcal{X}$, and a mapping $f : \mathcal{X}^m \rightarrow \mathbb{R}$ which for all i and for all $x_1, \ldots, x_m, x'_i$ satisfies the property:

$$|f(x_1, \ldots, x_i, \ldots, x_m) - f(x_1, \ldots, x_i', \ldots, x_m)| \leq c_i,$$

where $x_1, \ldots, x_m, x'_i$ are possible values for the input variables of the function f. Then,

$$\Pr \left[ |E[f(X_1, \ldots, X_m)] - f(X_1, \ldots, X_m)| \geq \epsilon \right] \leq 2\exp \left( -\frac{2\epsilon^2}{\sum_{i=1}^m c_i^2} \right).$$

This inequality implies that, for every $x, y, R$,

$$\Pr_\rho \left[ |E[f] - f| \geq \alpha \right] \leq 2\exp \left( -\frac{\alpha^2}{(2\sigma)^2} \right) = \exp \left( -\frac{\alpha^2}{4\sigma} \right).$$

Hence, the lemma is proved.

**Lemma 7.** If $|f \geq dP| \leq 2\exp(-P^2\sigma)$. Proof. Since $f^* = dP(1 - D^h) = dP - dPD^h$, if $f \geq dP$ then $|f - f^*| \geq dPD^h$.

$$\Pr[f \geq dP] \leq \Pr[|f - f^*| \geq dPD^h]$$

$$\leq 2\exp(-\frac{d^2 P^2 D^h}{4\sigma}) \tag{using Lemma 6}$$

$$= 2\exp(-\frac{P^2}{4\sigma} (1 - \frac{1}{d})^{2h})$$

$$\leq 2\exp(-\frac{P^2}{4\sigma} (d - h)^2) : (1 - \frac{1}{d}) \geq 1 - \frac{\sigma}{4h}$$

$$\leq 2\exp(-P^2\sigma).$$
Here we have used the fact that $h \leq 2\sigma$ which, along with the setting $d = 4\sigma$, implies that $(d - h) \geq 2\sigma$.

**Lemma 8.** Choose $d = 4\sigma$ as the dimension of $\mathbb{F}^{\text{Sketch}}$ and choose a prime $p$ and an error parameter $\delta \in (0, 1)$ (ensure that $1 - \frac{1}{p} \geq \frac{1}{2\sqrt{\ln \frac{2}{\delta}}}$ — see the proof for discussion). Then the estimator defined in Definition 5 is close to the Hamming distance between $x$ and $y$ with high probability, i.e.,

$$\Pr \left[ |\hat{h} - h| \geq \frac{\sigma}{\sqrt{\ln \frac{2}{\delta}}} \right] \leq \delta.$$ 

**Proof.** Denote $|\hat{h} - h|$ by $\Delta h$ and let $\alpha = \sqrt{d \ln \frac{2}{\delta}}$. We will prove that $\Delta h < \frac{32}{\sqrt{\ln(2/\delta)}}$ for the case $|f - f^*| \leq \alpha$ which, by Lemma 3, happens with probability at least $(1 - 2\exp(-\frac{\delta^2}{2})) = 1 - \delta$.

First we make a few technical observations all of which are based on standard inequalities of binomial series and logarithmic functions. It will be helpful to remember that $D = 1 - 1/d \in (0, 1)$.

**Observation 14.** For reasonable values of $\sigma$ and reasonable values of $\delta$, almost all primes satisfy the bound $P \geq \frac{1}{\sqrt{\ln(2/\delta)}}$. We will assume this inequality to hold without loss of generality.

For example, $p = 2$ is sufficient for $\sigma \approx 1000$ and $\delta \approx 0.001$ (remember that $P = 1 - \frac{1}{p}$). Furthermore, observe that $P$ is an increasing function of $p$, and the right hand side is a decreasing function of $\sigma$, increasing with decreasing delta but at an extremely slow logarithmic rate.

**Observation 15.** $\frac{dP}{\alpha} > 4$ can be assumed without loss of generality. This holds since the left hand side is $\frac{dP}{\sqrt{\ln(2/\delta)}} = \frac{P\sqrt{\alpha}}{\sqrt{\ln(2/\delta)}}$ by Observation 14 and the last value is $\alpha \sqrt{2}$ which is at least 4.

**Observation 16.** Based on the above assumptions, $f < dP$.

**Proof of Observation.** We will prove that $\sqrt{d \ln \frac{2}{\delta}} < dP \Delta h$. Since $|f - f^*| \leq \sqrt{d \ln \frac{2}{\delta}}$ and $f^* = dP(1 - D^h)$, it follows that $f \leq f^* + \sqrt{d \ln \frac{2}{\delta}} < dP$.

$$\sqrt{dP} \Delta h = \frac{dP \Delta h}{\sqrt{d}} \geq \frac{d P}{\sqrt{d}} d(1 - \frac{1}{d}) \geq \frac{P}{\sqrt{d}} d(1 - \frac{1}{d^2}) = \frac{P}{\sqrt{d}} (d - h) \geq \frac{P \sqrt{\alpha}}{\sqrt{\ln(2/\delta)}} \geq 4 \sqrt{\ln \frac{2}{\delta}}$$

which proves the claim stated at the beginning of the proof.

Based on this observation, $\hat{h}$ is calculated as $\ln \left(\frac{1 - \frac{1}{dP}}{dP}\right) / \ln D$ (see Definition 5). Thus, we get $D^h = 1 - \frac{1}{dP}$. Further, from Equation 5 we get $D^h = 1 - \frac{1}{dP}$.

**Observation 17.** $D^h \geq D^{2\sigma} \geq \frac{9}{16}$. This is since $h \leq 2\sigma$ and $D^\sigma = (1 - \frac{1}{d})^{2\sigma} \geq 1 - \frac{1}{d^2} = 1 - \frac{3}{4}$.

2. If the reader is wondering why we are not proving this fact, it may be observed that this relationship does not hold for small values of $\sigma$, e.g., $\sigma = 16, \delta = 0.01$.

**Observation 18.** $D^h \geq \frac{9}{16}$.

This is not so straightforward as Observation 17 since $\hat{h}$ is calculated using a formula and is not guaranteed, ab initio, to be upper bounded by $2\sigma$.

**Proof of Observation.** We will prove that $\frac{dP}{\alpha} > \frac{11}{16}$ which will imply that $D^h = 1 - \frac{1}{dP} > \frac{9}{16}$.

For the proof of the lemma we have considered the case that $f \leq f^* + \alpha$. Therefore, $\frac{dP}{\alpha} \leq \frac{dP}{\alpha} + \frac{dP}{\alpha}$. Substituting the value of $f^* = dP(1 - D^h)$ from Equation 3 and using Observation 17 we get the bound $D^h = 1 - \frac{1}{dP}$. We can further simplify the bound using Observation 15.

$$\frac{dP}{\alpha} \cdot \frac{dP}{\alpha} \leq \frac{7}{16} + \frac{7}{16} \leq \frac{1}{16} + \frac{1}{16},$$ validating the observation.

Now we get into the main proof which proceeds by considering two possible cases.

**Case $\hat{h} \geq h$, i.e., $\Delta h = \hat{h} - h$** We start with the identity $D^h - D^\hat{h} = D^\hat{h}(1 - D^h) > \frac{9}{16}(1 - D^h)$ where we have used Observation 17.

Combining these facts we get $\frac{dP}{\alpha} > \frac{9}{16}(1 - D^h)$. .$\Delta h \ln D \geq \ln \left(1 - \frac{16}{5dP} \right) \geq -\frac{16\alpha}{5dP} / \left(1 - \frac{16\alpha}{5dP}\right) = -\frac{16\alpha}{5dP - 16\alpha}$

(using the inequality $\ln(1 + x) \geq \frac{1}{1 + x}$ for $x > -1$)

\[ \Delta h \leq \frac{\ln \frac{1}{dP}}{\ln \frac{5dP - 16\alpha}{16\alpha} \leq \frac{16d\alpha}{5dP - 16\alpha} \leq \frac{16d\alpha}{5dP - 16\alpha} \]

(it is easy to show that $\ln \frac{1}{dP} = \ln \frac{1}{1 - 1/d} \geq 1/d$)

$$= \frac{dP}{\alpha} - \frac{16\alpha}{dP} 
\leq \frac{16\alpha}{dP}$$

(2) using Observation 15.

$$= \frac{16\alpha}{dP} \geq \frac{16\alpha}{dP} \geq \frac{16\alpha}{dP}$$

(3) we get a better estimator. That is, (b) $\Pr [|\hat{h} - h|] \geq \frac{8}{1 - dP} \leq \delta$.

**Proof of Observation.** We will prove that $f \geq dP$ with high probability. Following the steps of the proof of Lemma 7.

$$\Pr [f \geq dP] \leq 2 \exp(-\frac{dP^2 \Delta h^2}{4\sigma}) \leq 2 \exp(-\frac{P^2 (d - h)^2}{4\sigma})$$
Let $L$ denote $\sqrt{\ln \frac{2}{\delta}}$; note that $L > 1$. Now, $d = 16L\sqrt{\sigma}$ and $h \leq \sqrt{\sigma}$. So, $d - h \geq 15L\sqrt{\sigma} > 15\sqrt{\sigma}$ and, therefore, $(d - h)^2 > 225$. Using this bound in the equation above, we can upper bound the right-hand side as $2 \exp(-225(1 - \frac{1}{8})^2/4)$ which is a decreasing function of $p$, the lowest (for $p = 2$) being $2 \exp(-225/4 + 4) \approx 10^{-6}$.

Proof of (b) a better estimator of $h$. The proof is almost exactly same as that of Lemma 5, with only a few differences. We set $\alpha = d/8$ where $d = 16\sqrt{\sigma \ln \frac{2}{\delta}}$. Incidentally, the value of $\alpha$ remains the same in terms of $\sigma$ ($\alpha = \sqrt{4\sigma \ln \frac{2}{\delta}}$). Thus, the probability of error remains same as before:

$$2 \exp \left( -\frac{d^2}{64\sigma} \right) = \delta.$$

Observation 14 is true without any doubt. $\frac{dp}{d} = 8P$, which is greater than 4 for any prime number; so Observation 15 is true in this scenario.

Observation 16 requires a new proof. Following the steps of the above proof of Observation 16, it suffices to prove that $PD^h > \frac{d}{8}$.

$$PD^h = P(1 - \frac{h}{d})^h \geq P(1 - \frac{h}{d})^d = P(\frac{d - h}{d})^d \geq P\left(\frac{15L\sqrt{\sigma}}{16L\sqrt{\sigma}}\right)^{15L\sqrt{\sigma} \geq \frac{15L\sqrt{\sigma}}{16L\sqrt{\sigma}} > \frac{15}{16} > \frac{1}{8}.$$

Observation 17 is now tighter since $D^h \geq D^{\sqrt{\sigma}} = (1 - \frac{1}{d})^{\sqrt{\sigma}} \geq 1 - \frac{1}{\sqrt{16\ln 2/d}} > \frac{3}{4}$ for reasonable values of $\delta$. Similarly Observation 18 is also tighter (it relies on only the above observations) since $\frac{1}{d^2} = 1 - D^h \leq 1 - \frac{3}{4}$ and $\frac{d^2}{d^2} > \frac{1}{4}$; we get $D^h > \frac{1}{4}$.

Following similar steps as above, for the case $\hat{h} \geq h$, we get $\frac{\alpha}{d^2} > \frac{3}{4}(1 - D^h\Delta)$ and for the case $\hat{h} < h$, we get $\frac{\alpha}{d^2} > \frac{1}{4}(1 - D^h\Delta)$ leading to the common condition that $\Delta h \geq \frac{1}{4}(1 - D^h\Delta)$.

The final thing to calculate is the bound on $\Delta h$.

$$\Delta h \ln D \geq \ln (1 - \frac{2\alpha}{d^2}) \geq -\frac{2\alpha}{d^2}(1 - \frac{2\alpha}{d^2}) = -\frac{2\alpha^2}{d^2} = \frac{2\alpha^2}{d^2 - 2\alpha^2}$$

(using the inequality $\ln (1 + x) \geq \frac{x^2}{1+x}$ for $x < -1$)

$$\Delta h \leq \frac{1}{d^2 - 2\alpha^2} \frac{2\alpha}{dP - 2\alpha} \leq \frac{2\alpha}{dP - 2\alpha}$$

(it is easy to show that $\ln \frac{1}{d} = \ln \frac{1}{1/d} \geq 1/d$)

$$= \frac{2d}{P} - 2$$

$$< \frac{2d}{\frac{2\alpha}{P}}(\text{using Observation 15})$$

$$= \frac{4\alpha}{P} \cdot \frac{2\sigma}{d^2} \geq \frac{8P\sqrt{4\sigma \ln \frac{2}{\delta}}}{P\sqrt{\sigma \ln \frac{2}{\delta}}}$$

\[\Box\]

**APPENDIX D**

**PROOFS FROM SECTION 3.4**

**Lemma 10.** If $d = 4\sigma$ (as required by Lemma 3), then the expected number of non-zero entries of $\phi(x)$ is upper bounded by $\frac{d}{4}$. Further, at least 50% of $\phi(x)$ will be zero with probability at least $\frac{1}{2}$.

*Proof.* The lemma can be proved by treating it as a balls-and-bins problem. Imagine throwing $\sigma$ balls (treat them as the non-zero attributes of $x$) into $d$ bins (treat them as the sketch cells) independently and uniformly at random. If the $j$th-bin remains empty then $\phi_j(x)$ must be zero (the converse is not true). Therefore, the expected number of non-zero cells in the sketch is upper bounded by the expected number of empty bins, which can be easily shown to be $d[1 - (1 - \frac{1}{d})^\sigma]$. Using the stated value of $d$, this expression can further be upper bounded.

$$d[1 - (1 - \frac{1}{d})^\sigma] \leq d[1 - (1 - \frac{1}{4})] = \frac{d}{4}$$

Furthermore, let $NZ$ denote the number of non-zero entries in $\phi(x)$. We derived above $E[NZ] \leq \frac{d}{4}$. Markov inequality can help in upper bounding the probability that $\phi(x)$ contains many non-zero entries.

$$\Pr[NZ \geq \frac{d}{4}] \leq \frac{E[NZ]}{\frac{d}{4}} \leq \frac{1}{2}$$

\[\Box\]

**APPENDIX E**

**REPRODUCIBILITY DETAILS**

**E.1 Baseline implementations**

1) We implemented the feature hashing (FH) [20], SimHash (SH) [21], Sketching via Stable Distribution.
We first generated the ground truth clustering results on the compressed datasets. On the other hand, Latent Semantic Analysis (LSA) [32], Latent Dirichlet Allocation (LDA) [33], Non-negative Matrix Factorisation (NNMF) [34], and vanilla Principal component analysis (PCA), we used their implementations available in the `sklearn.decomposition` library.

4) For Multiple Correspondence Analysis (MCA) [28], we used a Python library.

5) For HCA [53], we performed hierarchical clustering over the features in which we set the number of clusters to the value of reduced dimension. We then randomly sampled one feature from each of the clusters, and considered the data points restricted to the sampled features.

6) For CATPCA [53], we used an R package.

It should be noted that PCA, MCA and LSA cannot reduce the dimension beyond the number of data points.

E.2 Reproducibility details for clustering task

We first generated the ground truth clustering results on the datasets using k-mode [55] (we used a Python library).

We then compressed the datasets using the baselines. Of them, feature hashing [20], SimHash [21], and Kendall rank correlation coefficient [14] generate integer/discrete valued sketches on which we can define a Hamming distance. Therefore we use the k-mode algorithm on compressed datasets. On the other hand, Latent Semantic Analysis (LSA) [32], Latent Dirichlet Allocation (LDA) [33], Non-negative Matrix Factorisation (NNMF) [34], Principal component analysis (PCA), and Multiple Correspondence Analysis (MCA) [28] generate real-valued sketches. For these we used the k-means algorithm (available in the `sklearn` library) on the compressed datasets. We set random_state = 42 for both k-mode and k-means.

We evaluated the clustering outputs using purity index. Let m be the number of data points and \( \Omega = \{\omega_1, \omega_2, \ldots, \omega_m\} \) be a set of clusters obtained on the original data. Further, let \( C = \{c_1, c_2, \ldots, c_k\} \) be a set of clusters obtained on reduced dimensional data. Then the purity index of the clusters \( C \) is defined as

\[
\text{purity index}(\Omega, C) = \frac{1}{m} \sum_{i=1}^{k} \max_{1 \leq j \leq k} |\omega_i \cap c_j|.
\]

APPENDIX F

Errors during dimensionality reduction experiments

Several baselines give out-of-memory error or their running time is quite high on some datasets. This makes it infeasible to include them in empirical comparison on RMSE and other end tasks.

We list these errors here. OHE gives out-of-memory error for Brain cell dataset. HCA gives DNS errors on NYTimes and BrainCell datasets. CATPCA could only on KOS and DeliciousMIL datasets that too upto only 300 reduced dimension. Other than that it gives a DNS error. VAE gives DNS errors on Enron datasets. KT gives out-of-memory error for NYTimes and Brain cell and on Enron it didn’t stop even after 10 hrs. MCA also gives out-of-memory error for NYTimes and Brain cell datasets. Further, the dimensionality reduction time for NNMF was quite high – on NYTimes it takes around 20 hrs to do the dimensionality reduction for 3000 dimension, and on the Brain cell dataset, NNMF didn’t stop even after 10 hrs. These errors prevented us from performing dimensionality reduction for all dimension using some of the algorithms.

APPENDIX G

Extended experimental results

This section contains the remaining comparative plots for the RMSE (Figure 11), clustering (Figure 12), similarity search experiments (Figure 13) and the dimensionality reduction time (Figure 14).

APPENDIX H

Median-FSketch: Combining multiple FSketch

We proved in Lemma 8 that our estimate \( \hat{h} \) is within an additive error of \( h \). A standard approach to improve the accuracy in such situations is to obtain several independent estimates and then compute a suitable statistic of the estimates. We were faced with a choice of mean, median and minimum of the estimates of which we decided to choose median after extensive empirical evaluation (see Section H.3) and obtaining theoretical justification (explained in Section H.2). We first explain our algorithms in the next subsection.

H.1 Algorithms for generating a sketch and estimating Hamming distance

Let \( k, d \) be some suitably chosen integer parameters. An arity-\( k \) dimension-\( d \) Median-FSketch for a categorical data, say \( x \), is an array of \( k \) sketches: \( \Phi(x) = (\phi^1(x), \phi^2(x), \ldots \phi^k(x)) \); the \( i \)-th entry of \( \Phi(x) \) is a \( d \)-dimensional FSketch. See Figure 15 for an illustration. Note that the internal parameters \( \rho, R, p \) required to run FSketch to obtain the \( i \)-entry are same across all data points; the parameters corresponding to different \( i \) are, however, chosen independently \( (p \) can be the same).

Our algorithm for Hamming distance estimation is inspired from the Count-Median sketch [56] and Count sketch [57]. It estimates the Hamming distances between the pairs of "rows" from \( \Phi(x) \) and \( \Phi(y) \) and returns the median
Fig. 11. Comparison of RMSE values. A lower value is an indication of better performance. The GISETTE dataset is of 5000 dimensions and hence, FSketch suffers from an increase in RMSE as the embedding dimension also reaches 5000.

Fig. 12. Comparing the quality of clusters on the compressed datasets.

Fig. 13. Comparing the performance of the similarity search task (estimating top-$k$ similar points with $k = 100$) achieved on the reduced dimensional data obtained from various baselines.

Fig. 14. Comparison of the dimensionality reduction times.
of the estimated distances. This procedure is followed in Algorithm 3.

Algorithm 3 Estimate Hamming distance between x and y from their Median-FSketch

Input: \( \Phi(x) = \langle \phi^1(x), \phi^2(x), \ldots, \phi^k(x) \rangle \), \( \Phi(y) = \langle \phi^1(y), \phi^2(y), \ldots, \phi^k(y) \rangle \)

1. for \( i = 1 \ldots k \) do
2. Compute \( f = \) Hamming distance between \( \phi^i(x) \) and \( \phi^i(y) \)
3. If \( f < dP \), \( \hat{h}^i = \ln \left(1 - \frac{1}{dP}\right) / \ln D \)
4. Else \( \hat{h}^i = 2\sigma \)
5. end for
6. return \( h = \min\{\hat{h}^1, \hat{h}^2, \ldots, \hat{h}^k\} \)

H.2 Theoretical justification

We now give a proof that our Median-FSketch estimator offers a better approximation. Recall that \( \sigma \) indicates the maximum number of non-zero attributes in any data vector, and is often much small compared to the their dimension, \( n \). Surprisingly, our results are independent of \( n \).

Lemma 19. Let \( h^m \) denote the median of the estimates of Hamming distances obtained from \( t \) independent FSketch vectors of dimension \( 4\sigma \) and let \( h \) denote the actual Hamming distance. Then,

\[
\Pr [ |h^m - h| \geq 18\sqrt{\sigma} ] \leq \delta
\]

for any desired \( \delta \in (0,1) \) if we use \( t \geq 48 \ln \frac{1}{\delta} \).

Proof. We start by using Lemma 2 with \( p = 3 \) and error (\( \delta \) in the lemma statement) = \( \frac{1}{4} \). Let \( \hat{h}^i \) denote the \( k \)-th estimate.

From the lemma we get that

\[
\Pr [ |\hat{h}^i - h| \geq 18\sqrt{\sigma} ] \leq \frac{1}{4}
\]

Define indicator random variables \( W_1 \ldots W_t \) as \( W_i = 1 \) iff \( |\hat{h}^i - h| \geq 18\sqrt{\sigma} \). We immediately have \( \Pr[W_i] \leq \frac{1}{4} \).

Notice that \( W_i = 1 \) can also be interpreted to indicate the event \( h - 18\sqrt{\sigma} \leq \hat{h}^i \leq h + 18\sqrt{\sigma} \). Now, \( h^m \) is the median of \( \{\hat{h}^1, \hat{h}^2, \ldots, \hat{h}^t\} \), and so, \( h^m \) falls outside the range \( [h - 18\sqrt{\sigma}, h + 18\sqrt{\sigma}] \) only if more than half of the estimates fall outside this range., i.e., if \( \sum_{i=1}^t W_i > t/2 \).

Since \( E[\sum_i W_i] \leq t/4 \), the probability of this event is easily bounded by \( \exp \left(-\frac{t^2}{4}/3\right) = e^{-t^2/48} \leq \delta \) using Chernoff’s bound.

H.3 Choice of statistics in Median-FSketch

We conducted an experiment to decide whether to take median, mean or minimum of \( k \) FSketch estimates in the Median-FSketch algorithm. We randomly sampled a pair of points and estimated the Hamming distance from its low-dimensional representation obtained from FSketch. We repeated this 10 times over different random mappings and computed the median, mean, and minimum of those 10 different estimates. We further repeat this experiment 10 times and generate a box-plot of the readings which is presented in Figure 16. We observe that median has the lowest variance and also closely estimates the actual Hamming distance between the pair of points.

APPENDIX I

DIMENSIONALITY REDUCTION ALGORITHMS
Fig. 16. Box plot for the median, mean, and minimum of the FSketch's estimate obtained from its 10 repetitions, then each experiment is repeated 10 times for computing the variance of these statistics. The black dotted line corresponds to the actual Hamming distance.

| S. No. | Data type of input vectors | Objective/Properties | Data type of sketch vectors | Result | Supervised or Unsupervised | Type of dimensionality reduction |
|--------|---------------------------|----------------------|-----------------------------|--------|-----------------------------|---------------------------------|
| 1      | Real-valued vectors       | Approximating pairwise Euclidean distance, inner product | Real-valued vectors          | JL-lemma [23] | Unsupervised | Linear |
| 2      | Real-valued vectors       | Approximating pairwise Euclidean distance, inner product | Real-valued vectors          | Feature Hashing [20] | Unsupervised | Linear |
| 3      | Real-valued vectors       | Approximating pairwise cosine or angular similarity | Binary vectors               | SimHash [21] | Unsupervised | Non-Linear |
| 4      | Real-valued vectors       | Approximating pairwise $\ell_p$ norm for $p \in (0, 2]$ | Real-valued vectors          | $p$-stable random projection (SSD) [58] | Unsupervised | Linear |
| 5      | Sets                      | Approximating pairwise Jaccard similarity | Integer valued vectors        | MinHash [30] | Unsupervised | Non-linear |
| 6      | Sparse binary vectors     | Approximating pairwise Hamming distance, Inner product, Jaccard and Cosine similarity | Binary vectors               | BinSketch [17] | Unsupervised | Non-linear |
| 7      | Real-valued vectors       | Minimize the variance in low dimension | Real-valued vectors          | Principal Component Analysis (PCA) | Unsupervised | Linear |
| 8      | Real-valued vectors (labelled input) | Maximizes class separability in the reduced dimensional space | Real-valued vectors          | Linear Discriminant Analysis [59] | Supervised | Linear |
| 9      | Real-valued vectors       | Embedding high-dimensional data for visualization in a low-dimensional space of two or three dimensions | Real-valued vectors          | t-SNE [36] | Unsupervised | Non-linear |
| 10     | Real-valued vectors       | Minimize the reconstruction error | Real-valued vectors          | Auto-encoder [60] | Unsupervised | Non-linear |
| 11     | Real-valued vectors       | Extracting nonlinear structures in low-dimension via Kernel function | Real-valued vectors          | Kernel-PCA [61] | Unsupervised | Non-linear |
| 12     | Real-valued vectors       | Factorize input matrix into two small size non-negative matrices | Real-valued vectors          | Non-negative matrix factorization (NNMF) [34] | Unsupervised | Linear |
| 13     | Real-valued vectors       | Compute a quasi-isometric low-dimensional embedding | Real-valued vectors          | Isomap [62] | Unsupervised | Non-linear |
| 14     | Real-valued vectors       | Preserves the topological structure of the data | Real-valued vectors          | Self-organizing map [63] | Unsupervised | Non-linear |