Spherically Symmetric Solutions of the $SU(N)$ Skyrme Models

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Abstract

Recently we have presented in [1] an ansatz which allows us to construct skyrmion fields from the harmonic maps of $S^2$ to $CP^{N-1}$. In this paper we examine this construction in detail and use it to construct, in an explicit form, new static spherically symmetric solutions of the $SU(N)$ Skyrme models. We also discuss some properties of these solutions.

1 Introduction

The Skyrme model presents an opportunity to understand nuclear physics as a low energy limit of quantum chromodynamics (QCD). The model was initially proposed as a theory of strong interactions of hadrons [2], but recently, it was shown to be the low energy limit of QCD in the large $N_c$ limit [3]. Since then further work has suggested that topologically nontrivial solutions of this model, known as skyrmions, can be identified with classical ground states of light nuclei. However, a thorough understanding of the structure and dynamics of multi-skyrmion configurations is required before a more qualitative assessment of the validity of this application of the model can be made.

The $SU(N)$ Skyrme model involves fields which take values in $SU(N)$; i.e. are described by $SU(N)$ valued functions of $\vec{x}$ and $t$. Its static solutions correspond to field configurations describing multi-skyrmions. In this paper new solutions have been obtained for fields whose energy density is spherically symmetric.

Multi-skyrmions are stationary points (maxima or saddle points) of the static energy functional, which is given in topological charge units by

$$E = \frac{1}{12\pi^2} \int_{R^3} \left\{ -\frac{1}{2} \text{tr} \left( \partial_i U U^{-1} \right)^2 - \frac{1}{16} \text{tr} \left[ \partial_i U U^{-1}, \partial_j U U^{-1} \right]^2 \right\} d^3x, \quad (1)$$

where $U(\vec{x}) \in SU(N)$.

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In this case multi-skyrmions are solutions of the equation
\[ \partial_i \left( \partial_i U U^{-1} - \frac{1}{4} [\partial_j U U^{-1}, [\partial_j U U^{-1}, \partial_i U U^{-1}]] \right) = 0. \] (2)

We have, for simplicity, set the mass terms to zero. This has been done for convenience, since the conventional mass terms introduce only small changes and, as we will see later, affect only the profile functions. Therefore, all our discussion can be easily generalised to include such mass terms.

Finiteness of the energy functional requires that \( U(x) \) approaches a constant matrix at spatial infinity, which can be chosen to be the identity matrix by a global \( SU(N) \) transformation. So, without any loss of generality, we can impose the following boundary condition on \( U \): \( U \to I \) as \( |\vec{x}| \to \infty \).

Since \( U \to I \) as \( |\vec{x}| \to \infty \) is a mapping from \( S^3 \to SU(N) \), it can be classified by the third homotopy group \( \pi_3(SU(N)) \equiv \mathbb{Z} \) or, equivalently, by the integer valued winding number
\[ B = \frac{1}{24\pi^2} \int_{\mathbb{R}^3} \varepsilon_{ijk} \text{tr} \left( \partial_i U U^{-1} \partial_j U U^{-1} \partial_k U U^{-1} \right) d^3\vec{x}, \] (3)
which is a topological invariant. This winding number classifies the solitonic sectors in the model, and as Skyrme has argued \[2\], \( B(U) \) may be identified with the baryon number of the field configuration.

Up to now most of the studies involving the Skyrme model have concentrated on the \( SU(2) \) version of the model and its embeddings into \( SU(N) \). The simplest nontrivial classical solution involves a single skyrmion \( (B = 1) \) and has already been discussed by Skyrme \[2\]. The energy density of this solution is radially symmetric and, as a result, using the so-called hedgehog ansatz one can reduce (2) to an ordinary differential equation, which then has to be solved numerically. Many solutions with \( B > 1 \) of the \( SU(2) \) model have also been computed numerically and, in all cases, the solutions are very symmetrical (cf. Battye et al. \[4\] and references therein). However, since the model is not integrable, with few exceptions, explicit solutions (even) for spherical symmetric \( SU(N) \) skyrmions are not known.

The first example of a non-embedded solution for a higher group was the \( SO(3) \) soliton, corresponding to a bound system of two skyrmions, which was found by Balachandran et al. \[6\]. Another solution, with a large \( SU(3) \) strangeness content, was found by Kopeliovich et al. \[7\]. However, all other known multi-skyrmion configurations seem to be the embeddings of the solutions of the \( SU(2) \) model.

Recently, we have showed in \[1\] how to construct low energy states of the \( SU(N) \) model by using \( CP^{N-1} \) harmonic maps. Our discussion involved only one projector. In this paper, we extend our method to more projectors. We show that, for the \( SU(N) \) model, when we take \( N - 1 \) projectors which lead to spherically symmetric energy densities, the full equations of the model separate and the problem of finding exact solutions is reduced to having to solve \( N - 1 \) coupled nonlinear ODE’s for \( N - 1 \) profile functions. This way we obtain a whole family of new spherical symmetric multi-baryon solutions of the \( SU(N) \) models. Our solutions include the \( SU(3) \) dibaryon configuration of Balachandran et al. \[6\] and the non-topological \( SU(3) \) four baryon configuration of \[1\].
2 Harmonic Maps

In [1] we generalised the $SU(2)$ ansatz of Houghton et al. [3] to $SU(N)$. This generalisation involved re-writing the expression of Houghton et al. as a projector from $S^2$ to $CP^{N-1}$. It gave us a new way of interpreting old results and of deriving expressions for the low energy $SU(N)$ field configurations which are not simple embeddings of $SU(2)$ fields. In particular, the energy distributions exhibit very different symmetries from those of the embeddings. The method also gave us a new solution of the $SU(3)$ model, which lies in the topologically trivial sector of the model (ie it has zero baryon number) and so, obviously, is not stable.

The method of [1] can be generalised further, to involve more projectors. In fact, we can exploit here some ideas taken from the theory of harmonic maps of $S^2 \to CP^{N-1}$ [8, 9]; since they play an important role in our construction.

Recall (cf. [9]) that in $N$-dimensional space there is a “natural” set of projectors: $S^2 \to CP^{N-1}$ maps, which are constructed as follows:

Write each projector $P$ as

$$P(V) = \frac{V \otimes V^\dagger}{|V|^2},$$

(4)

where $V$ is a $N$-component complex vector of two variables $\xi$ and $\bar{\xi}$ which locally parametrise $S^2$. In terms of the more familiar $\theta$ and $\varphi$, they are given by $\xi = \tan(\theta/2)e^{i\varphi}$. The first projector is obtained by taking $V = f(\xi)$, ie an analytic vector of $\xi$; while the other projectors are obtained from the original $V$ by differentiation and Gramm-Schmidt orthogonalisation. If we define an operator $P_+$ by its action on any vector $v \in C^N$ [9] as

$$P_+v = \partial_\xi v - v^\dagger \partial_\bar{\xi} v / |v|^2,$$

(5)

then the further vectors $P^k v$ can be defined by induction: $P_+^k v = P_+(P_+^{k-1} v)$.

Therefore, in general, we can consider projectors $P_k$ of the form [4] corresponding to the family of vectors $V \equiv V_k = P_+^k f$ (for $f = f(\xi)$) as

$$P_k = P(P_+^k f), \quad k = 0, \ldots, N - 1,$$

(6)

where, due to the orthogonality of the projectors, we have $\sum_{k=0}^{N-1} P_k = 1$.

The orthogonality properties of our projectors follow from the following properties of vectors $P_+^k f$ which hold when $f$ is holomorphic:

$$(P_+^k f)^\dagger P_+^l f = 0, \quad k \neq l,$$

(7)

$$\partial_\xi (P_+^k f) = -P_+^{k-1} f |P_+^{k-1} f|^2 / |P_+^{k-1} f|^2, \quad \partial_\bar{\xi} \left( P_+^{k-1} f / |P_+^{k-1} f|^2 \right) = P_+^{k-1} f / |P_+^{k-1} f|^2.$$

(8)

Note that, for $SU(N)$, the last projector $P_{N-1}$ in the sequence corresponds to an anti-analytic vector; (ie the components of $V_{N-1} = P_{N-1}^{-1} f$, up to an irrelevant overall factor which cancels in the projector, are functions of only $\bar{\xi}$).

Our new $SU(N)$ generalisation of [4] involves the introduction of $N - 1$ projectors, ie

$$U = \exp\{ig_0(P_0 - I / N) + ig_1(P_1 - I / N) - \ldots + ig_{N-2}(P_{N-2} - I / N)\}. $$


\[ e^{-ig_0/N}(I + A_0 P_0) e^{-ig_1/N}(I + A_1 P_1) \ldots e^{-ig_{N-2}/N}(I + A_{N-1} P_{N-2}), \quad (9) \]

where \( g_k = g_k(r), \) for \( k = 0, \ldots, N - 2, \) are the profile functions and \( A_k = e^{ig_k} - 1. \) Note that the projector \( P_{N-1} \) is not included in the above formula since it is the linear combination of the others. [Our previous ansatz given in [1] corresponds to putting all the profile functions, but the first one, equal to zero.]

The spherically symmetric maps into \( CP^{N-1} \) are given by

\[ f = (f_0, f_1, \ldots, f_{N-1})^t, \quad \text{where} \quad f_k = \xi^k \sqrt{C_{k+1}^{N-1}}, \quad (10) \]

where \( C_{k+1}^{N-1} \) denote the binomial coefficients. Furthermore, as we prove in the appendix, the modulus of the corresponding vector \( P_k f \) for \( f \) of the above form is

\[ |P_k f|^2 = \alpha (1 + |\xi|^2)^{N-2k-1}, \quad (11) \]

where \( \alpha \) depends on \( N \) and \( k. \)

### 3 Constructing the Skyrmion Solutions

In this section we construct a family of exact spherical symmetric solutions of the \( SU(N) \) Skyrme models. In fact, we show that for each \( SU(N) \) model the Skyrme field involving \( N - 1 \) projectors leads to an exact solution involving \( N - 1 \) profile functions.

#### 3.1 Skyrme Equations

The Skyrme equations \([2]\), when re-written in spherical coordinates, take the form:

\[
\partial_r \left[ r^2 R_r + \frac{1}{4} \left( A_{\theta \theta} + \frac{1}{\sin^2 \theta} A_{\phi \phi} \right) \right] + \frac{1}{\sin \theta} \partial_\theta \left[ \sin \theta \left( R_\theta + \frac{1}{4} \left( A_{\theta \theta} + \frac{1}{\sin^2 \theta} A_{\phi \phi} \right) \right) \right] + \frac{1}{\sin^2 \theta} \partial_\phi \left[ R_\phi + \frac{1}{4} \left( A_{\phi \phi} + \frac{1}{\sin^2 \theta} A_{\theta \theta} \right) \right] = 0, \quad (12)
\]

where \( R_i = U^{-1} U_i \) and \( A_{\alpha \beta \gamma} \equiv [R_\alpha, [R_\beta, R_\gamma]]. \)

It is easy to see, using \([3]\), that

\[
R_r = i \sum_{j=0}^{N-2} \dot{g}_j \left( P_j - \frac{I}{N} \right), \quad (13)
\]

where \( \dot{g}_j(r) \) denotes the derivative of \( g_j(r) \) with respect to its argument; and that, in terms of the holomorphic variables \( \xi \) and \( \bar{\xi}, \)

\[
R_\xi = e^{-i \sum_{k=0}^{N-2} g_k P_k} \partial_\xi \left[ e^{i \sum_{i=0}^{N-2} g_i P_i} \right] = \left[ 1 + \sum_{k=0}^{N-2} (e^{-ig_k} - 1) P_k \right] \left[ \sum_{i=0}^{N-2} (e^{ig_i} - 1) P_\xi \right] = \sum_{i=1}^{N-1} \left[ e^{i(g_i - g_{i-1})} - 1 \right] \frac{V_i V_{i-1}^*}{|V_{i-1}|^2}, \quad (14)
\]
where the last line follows from the identity \( e^{-i \sum_{k=0}^{N-2} g_k P_k} = 1 + \sum_{k=0}^{N-2} (e^{-ig_k} - 1) P_k \). Here, \( g_{N-1} = 0 \) and \( R_\xi = -(R_\xi)^\dagger \).

Next we note that

\[
\partial_\theta = \frac{1 + |\xi|^2}{2 \sqrt{|\xi|^2}} \left( \xi \partial_\xi + \bar{\xi} \partial_{\bar{\xi}} \right), \quad \partial_\varphi = i \left( \xi \partial_\xi - \bar{\xi} \partial_{\bar{\xi}} \right), \tag{15}
\]

and re-write all the terms in (12) in terms of \( R_\xi, R_{\bar{\xi}} \) and \( R_r \) and their commutators, ie

\[
\partial_\theta (\sin \theta R_\theta) + \frac{1}{\sin \theta} \partial_\varphi R_\varphi = (1 + |\xi|^2) \sqrt{|\xi|^2} \left( (R_\xi)_{\bar{\xi}} + (R_{\bar{\xi}})_{\xi} \right), \tag{16}
\]

\[
A_{\theta \varphi \theta} + \frac{1}{\sin^2 \theta} A_{\varphi r \varphi} = \frac{(1 + |\xi|^2)^2}{2} \left\{ [R_\xi, [R_r, R_\xi]] + [R_{\bar{\xi}}, [R_r, R_{\bar{\xi}}]] \right\}, \tag{17}
\]

\[
\sin \theta \partial_\theta (\sin \theta A_{r \theta r}) + \partial_\varphi (A_{r \varphi r}) = 2|\xi|^2 \left( [R_r, [R_\xi, R_r]]_{\bar{\xi}} + [R_r, [R_{\bar{\xi}}, R_r]]_{\xi} \right), \tag{18}
\]

\[
\partial_\theta \left( \frac{A_{\varphi r \varphi}}{\sin \theta} \right) + \frac{1}{\sin \theta} \partial_\varphi (A_{\theta \varphi \theta}) = \frac{(1 + |\xi|^2)^2 \sqrt{|\xi|^2}}{2} \left[ \partial_{\bar{\xi}} \left( (1 + |\xi|^2)^2 [R_\xi, [R_r, R_\xi]] \right) - \partial_{\xi} \left( (1 + |\xi|^2)^2 [R_{\bar{\xi}}, [R_r, R_{\bar{\xi}}]] \right) \right]. \tag{19}
\]

Thus equation (12), when re-written in the holomorphic variables, becomes

\[
\partial_r \left[ r^2 R_r + \frac{(1 + |\xi|^2)^2}{8} \left( [R_\xi, [R_r, R_\xi]] + [R_{\bar{\xi}}, [R_r, R_{\bar{\xi}}]] \right) \right] + \frac{(1 + |\xi|^2)^2}{2} \left( (R_\xi)_{\bar{\xi}} + (R_{\bar{\xi}})_{\xi} \right) + \frac{(1 + |\xi|^2)^2}{8 \nu^2} \left( \xi [R_\xi, [R_r, R_\xi]] - \xi [R_{\bar{\xi}}, [R_r, R_{\bar{\xi}}]] \right) + \frac{(1 + |\xi|^2)^4}{16 \nu^2} \left( [R_\xi, [R_r, R_\xi]]_{\bar{\xi}} - [R_{\bar{\xi}}, [R_r, R_{\bar{\xi}}]]_{\xi} \right) + \frac{(1 + |\xi|^2)^2}{8} \left( [R_r, [R_\xi, R_r]]_{\bar{\xi}} + [R_r, [R_{\bar{\xi}}, R_r]]_{\xi} \right) = 0. \tag{20}
\]

Using (10) we observe that

\[
[R_\xi, R_\xi] = 2 P_0 \frac{|V_1|^2}{|V_0|^2} (1 - \cos(g_1 - g_0)) - 2 P_{N-1} \frac{|V_{N-1}|^2}{|V_{N-2}|^2} (1 - \cos(g_{N-2})) + 2 \sum_{i=1}^{N-2} P_i \left[ \frac{|V_{i+1}|^2}{|V_i|^2} (1 - \cos(g_{i+1} - g_i)) - \frac{|V_i|^2}{|V_{i-1}|^2} (1 - \cos(g_i - g_{i-1})) \right], \tag{21}
\]

\[
[R_\xi, [R_\xi, R_\xi]] = \sum_{i=1}^{N-1} \left( \mu_i \frac{|V_{i+2}|^2}{|V_{i+1}|^2} + \nu_i \frac{|V_{i+1}|^2}{|V_{i}|^2} + \rho_i \frac{|V_i|^2}{|V_{i-1}|^2} \right) \frac{V_{i} V_{i+1}^\dagger}{|V_{i-1}|^2}, \tag{22}
\]

\[
[R_r, [R_\xi, R_r]] = \sum_{i=1}^{N-1} s_i \frac{V_{i-1} V_{i}^\dagger}{|V_{i-1}|^2}, \tag{23}
\]

where \( \mu, \nu \) and \( \rho \) are functions of \( g_k(r) \), only; while \( s_i \) are functions of \( g_k(r) \) and their derivatives.

Since \( V_k = P_k f \), one can show that \( \frac{|V_i|^2}{|V_{i-1}|^2} \propto (1 + |\xi|^2)^{-2} \); while \( \partial_\xi (1 + |\xi|^2)^{-2} = -2\bar{\xi}(1 + |\xi|^2)^{-3} \) and thus, the derivative terms involving \( [R_\xi, [R_r, R_\xi]] \) in (20) cancel leaving us with derivatives of \( \frac{|V_i|^2}{|V_{i-1}|^2} \) - which are proportional to \( \sum_{i=1}^{N-1} (1 + |\xi|^2)^{-2} (P_i - P_{i-1}) h_i \), where \( h_i \) involve functions of \( g_k(r) \) (due to (8)), multiplied by terms of the form \( \frac{|V_i|^2}{|V_{i-1}|^2} \).
So the factors $(1 + |\xi|^2)^{-4}$ in (20) cancel – leaving us with a sum of differences of two successive projectors multiplied by functions dependent only on $r$.

Following the above argument and using the properties of $R_r$, etc one can show that the terms $[R_r, [R_{\xi}, R_{\bar{\xi}}]]\bar{\xi}$ in (20), are proportional to $\sum_{i=1}^{N-1} S_i (1 + |\xi|^2)^{-2}(P_i - P_{i-1})$, where $S_i$ are functions of $g_k(r)$ and their derivatives – leaving us, once again, with a sum of differences of two successive projectors multiplied by functions dependent only on $r$.

Finally, the contribution of the terms $(1 + |\xi|^2)^{-2}$ is given by $\sum_{i=1}^{N-1} (P_i - P_{i-1}) H_i$, where $H_i$ are only functions of $g_k$, while the commutators in (17) are equal to a sum of projectors multiplied by $(1 + |\xi|^2)^{-2}$, which cancel out in (20). In addition, $\partial_r (r^2 R_r) = i \sum_{i=0}^{N-2} \left( P_i - \frac{i}{N} \right) (2r \dot{g}_i + r^2 \ddot{g}_i)$.

We note that, for our choice of the vectors $V_k$, all the dependence on $\xi$ and $\bar{\xi}$ in (20) resides only in the projectors (the rest of it cancels out). The terms involving $\partial_r (r^2 R_r)$ give us expressions involving $\frac{1}{N} - P_i$ while all the other terms give us expressions involving $P_i - P_{i-1}$. Although $N$ projectors arise in (20), the projector $P_{N-1}$ can be re-expressed in terms of the previous ones – giving $N - 1$ factors involving the harmonic maps $P_i - \frac{1}{N}$ (for $i = 0, \ldots, N - 2$). To satisfy (20) the coefficients of such factors have to vanish leaving us with $N - 1$ equations for the $N - 1$ profile functions $g_i$. Hence, if these equations have solutions then they correspond to exact solutions of the $SU(N)$ Skyrme models. Notice that (14) implies that these solutions have a covariant axial symmetry, ie any rotation by an angle $\alpha$ around the $z$-axis is equivalent to the gauge transformation $U \to AU^\dagger$ where $A = \text{diag}(1, e^{i\alpha}, e^{2i\alpha}, \ldots, e^{(N-1)i\alpha})$. On the other hand, as will be shown below, the energy density for these solution is radially symmetric.

The $N - 1$ equations for the profile functions can be obtained either from (20) – which is a hard task; or from the variation of the energy (1) – using (9) and integrating out $\xi$ and $\bar{\xi}$ variables. Clearly, the two methods give the same equations.

Let us stress that our procedure hinges on having $N - 1$ profile functions and on the very special form of our vectors $V_k$. Had we taken other vectors $V_k$, we would have got some $\xi$ and $\bar{\xi}$ dependence outside the projectors; while had we taken less than $N - 1$ profile functions and projectors we would have got too many equations for our functions. It is only in the case of $N - 1$ projectors that we get the right number of equations.

### 3.2 Energy Dependence on Profile Functions

The energy (1), when written in the holomorphic variables, becomes

$$E = -\frac{i}{12\pi^2} \int r^2 dr \, d\xi d\bar{\xi} \text{tr} \left( \frac{1}{(1 + |\xi|^2)^2} R_r^2 + \frac{1}{r^2} |R_{\xi}|^2 + \frac{1}{4r^2} [R_r, R_{\xi}]^2 - \frac{(1 + |\xi|^2)^2}{16r^4} [R_{\bar{\xi}}, R_{\xi}]^2 \right).$$

Using (13) and (14) we find that

$$\text{tr} R_r^2 = \frac{1}{N} \left( \sum_{i=0}^{N-2} \dot{g}_i \right)^2 - \sum_{i=0}^{N-2} \ddot{g}_i^2,$$

$$\text{tr} |R_{\xi}|^2 = -2 \sum_{i=1}^{N-1} B_i,$$
\[ \text{tr}[R_r, R_\xi][R_r, R_\xi] = -2 \sum_{k=1}^{N-1} B_k (\dot{g}_k - \dot{g}_{k-1})^2, \]  
\[ \text{tr}[R_\xi, R_\xi]^2 = 4 \left( B_1^2 + \sum_{i=1}^{N-2} (B_i - B_{i+1})^2 + B_{N-1}^2 \right), \]

where \( B_i = \frac{|V_i|^2}{|V_{i-1}|^2} (1 - \cos(g_i - g_{i-1})) \) and \([R_\xi, R_\xi] = 2 \sum_{i=1}^{N-1} (P_{i-1} - P_i) B_i.\)

Since \( \frac{|V_i|^2}{|V_{i-1}|^2} = k(N - k)(1 + |\xi|^2)^{-2} \) (see appendix) all terms in (24) have a factor \((1 + |\xi|^2)^{-2}\) and the integration over \( \xi \) and \( \bar{\xi} \) is a topological constant, \( \text{i.e., if } d\xi d\bar{\xi}(1 + |\xi|^2)^{-2} = 2\pi. \) Thus we get

\[ E = \frac{1}{6\pi} \int r^2 dr \left\{ -\frac{1}{N} \left( \sum_{i=0}^{N-2} \dot{g}_i \right)^2 + \sum_{i=0}^{N-2} \dot{g}_i^2 + \frac{1}{2r^2} \sum_{k=1}^{N-1} (\dot{g}_k - \dot{g}_{k-1})^2 D_k + \frac{2}{r^2} \sum_{k=1}^{N-1} D_k + \frac{1}{4r^4} \left( D_1^2 + \sum_{k=1}^{N-2} (D_k - D_{k+1})^2 + D_{N-1}^2 \right) \right\}, \]

where \( D_k = k(N - k)(1 - \cos(g_k - g_{k-1})). \)

Let us, for simplicity, take \( F_k = g_k - g_{k+1}, \) \( (k = 0, \ldots, N - 2) \) with \( F_{N-2} = g_{N-2}. \) Then, the variation of the integrand of the energy \( E \) with respect to the functions \( \dot{F}_l \) (for \( l = 0, \ldots, N - 2 \)) is

\[ \frac{\partial \tilde{E}}{\partial \dot{F}_l} = \left[ -\frac{2(l + 1)}{N} \sum_{i=0}^{N-2} (i + 1) \ddot{F}_i + 2 \sum_{i=0}^{N-2} \ddot{F}_i \right] + 2 \left( \sum_{i=0}^{N-2} \ddot{F}_i \right) + \frac{1}{r^2} \dot{F}_l D_{l+1} \]

where \( D_k = k(N - k)(1 - \cos F_{k-1}). \)

Therefore, the equations of motion for the functions \( F_i, \) and thus for the profile functions, are

\[ -\frac{2(l + 1)}{N} \sum_{i=0}^{N-2} (i + 1) \dddot{F}_i + 2 \sum_{i=0}^{N-2} \dddot{F}_i + \frac{1}{r^2} \dddot{F}_l (l + 1)(N - l - 1)(1 - \cos F_l) + \frac{1}{2r^2} \dddot{F}_l^2 (l + 1)(N - l - 1) \sin F_l + \frac{2}{r} \left( -\frac{2(l + 1)}{N} \sum_{i=0}^{N-2} (i + 1) \dddot{F}_i + 2 \sum_{i=0}^{N-2} \dddot{F}_i \right) \]

\[ -\frac{2}{r^2} (l + 1)(N - l - 1) \sin F_l - \frac{1}{r^4} (l + 1)^2 (N - l - 1)^2 (1 - \cos F_l) \sin F_l + \frac{1}{2r^4} (l + 1)(N - l - 1) \sin F_l \]

These equations can be solved numerically by imposing the appropriate boundary conditions on the profile functions. To do this we have to specialise to a particular model, \( \text{i.e., for specific } N \) and diagonalise the terms involving the second derivatives. The simplest cases: \( N = 2, \) \( N = 3 \) and \( N = 4 \) involve 1, 2 or 3 functions and will be discussed in the next sections.

4 Topological Properties and Symmetries

Before we discuss special cases, let us first investigate the general topological properties of our fields.
The topological charge (3), which in many applications of the Skyrme model is identified with the baryon number, is given by

\[ B = \frac{1}{8\pi^2} \int dr d\xi d\bar{\xi} \text{tr} \left( R_r [R_\xi, R_{\bar{\xi}}] \right), \]  

when written in the complex coordinates.

Due to (21) and (13) the terms involving \( \dot{g}_i/N \) in \( R_r \) after taking the trace cancel and the expression for the baryon number simplifies to

\[ B = -i \frac{4}{\pi^2} \int dr d\xi d\bar{\xi} \sum_{i=0}^{N-2} \dot{F}_i (1 - \cos F_i) \frac{|V_{i+1}|^2}{|V_i|^2} \]

\[ = \frac{1}{2\pi} \int dr \sum_{i=0}^{N-2} \dot{F}_i (1 - \cos F_i) (i + 1) (N - i - 1) \]

\[ = \frac{1}{2\pi} \sum_{i=0}^{N-2} (i + 1) (N - i - 1) \left( F_i - \sin F_i \right) r_{r=\infty}. \]  

(33)

As \( g_i(\infty) = 0 \) (required for the finiteness of the energy) the only contributions to the topological charge come from \( F_i(0) \).

The \( N - 1 \) equations for the profile functions and their differences given in (31) have many symmetries. These symmetries can be used to derive special skyrmion solutions which involve a smaller number of profile functions and projectors.

The main symmetry of our expressions, are the independent interchanges

\[ F_k \leftrightarrow F_{N-k-2}, \quad \text{for} \quad k = 0, \ldots, N - 2. \]  

(34)

This symmetry follows from the fact the terms \( D_k = k(N - k)(1 - \cos F_{k-1}) \) which appear in the energy are symmetric under the interchange: \( D_k \leftrightarrow D_{N-k} \) when \( F_{k-1} \leftrightarrow F_{N-k-1} \). In addition, all the other terms in the energy also exhibit this symmetry since they are combinations of \( F_i \) and their derivatives.

5 Spherical Skyrmions

The simplest case corresponds to the \( SU(2) \) spherically symmetric skyrmion. This is the solution which was found thirty years ago by Skyrme and is usually presented in terms of the well-known hedgehog ansatz.

5.1 \( SU(3) \) Skyrme Model

In this case \( N = 3 \) and we have two functions: \( F_0 \) and \( F_1 \). The energy density \( \mathcal{E} \), such that \( E = (6\pi)^{-1} \int \mathcal{E} r^2 dr \), is given by

\[ \mathcal{E} = \frac{2}{3} (\dot{F}_0^2 + \dot{F}_1^2 + \dot{F}_0 \dot{F}_1) + \frac{1}{r^2} \left[ (\dot{F}_0^2 + 4)(1 - \cos F_0) + (\dot{F}_1^2 + 4)(1 - \cos F_1) \right] + \]

\[ \frac{2}{r^3} \left[ (1 - \cos F_0)^2 - (1 - \cos F_0)(1 - \cos F_1) + (1 - \cos F_1)^2 \right], \]  

(35)
and the equations for the profile functions are

\[
\ddot{F}_0 \left( 1 + \frac{3}{2r^2} (1 - \cos F_0) \right) + \frac{\ddot{F}_1}{2} + 2 \dot{F}_0 + \frac{\dot{F}_1}{r} + 3 \sin F_0 \left[ \dot{F}_0^2 - 4 - \frac{4(1 - \cos F_0)}{r^2} + \frac{2(1 - \cos F_1)}{r^2} \right] = 0,
\]

\[
\ddot{F}_1 \left( 1 + \frac{3}{2r^2} (1 - \cos F_1) \right) + \frac{\ddot{F}_0}{2} + 2 \dot{F}_1 + \frac{\dot{F}_0}{r} + 3 \sin F_1 \left[ \dot{F}_1^2 - 4 - \frac{4(1 - \cos F_1)}{r^2} + \frac{2(1 - \cos F_0)}{r^2} \right] = 0.
\]

(36)

These equations can be solved numerically when the right boundary conditions have been imposed.

However, by letting \( F_0 = F_1 = F \) (ie using the symmetry) they reduce to

\[
\ddot{F} \left( 1 + \frac{1 - \cos F}{r^2} \right) + \frac{\ddot{F}}{r} + \frac{\sin F}{2r^2} \left[ \dot{F}^2 - 4 - \frac{2(1 - \cos F)}{r^2} \right] = 0.
\]

(37)

This equation coincides with the equation for the profile function of a single \( SU(2) \) skyrmion. Here we note that as \( F_0(0) = F_1(0) = 2\pi \) the topological charge of our solution is four. Thus the energy of this configuration, which corresponds to four skyrmions is \( E_{B=4} = 4 \times 1.232 \), ie is exactly four times the energy of one skyrmion. We see that we have a static solution corresponding to four non-interacting skyrmions, placed on top of each other in such a way that their energy density is spherically symmetric.

In addition, there is a further symmetry which allows us to set \( F_0 = -F_1 = G \). In this case the equations reduce to

\[
\ddot{G} \left( \frac{1}{2} + \frac{3}{2r^2} (1 - \cos G) \right) + \frac{\ddot{G}}{r} + \frac{3 \sin G}{4r^2} \left[ \dot{G}^2 - 4 - \frac{2(1 - \cos G)}{r^2} \right] = 0.
\]

(38)

Let us note that, since \( F_0 = g_0 - g_1 \) and \( F_1 = g_1 \), this case corresponds to \( g_0 = 0 \) and thus, the field (3) involves only one projector, namely \( P_1 \). This solution is the topologically trivial solution discussed in [1] and its energy is 3.861.

Finally, Balachandran et al. skyrmion solution can be obtained from (36) by imposing the following boundary conditions: \( g_0(0) = 2\pi \), \( g_1(0) = 0 \) and \( g_0(\infty) = 0 \), \( g_1(\infty) = 0 \); its energy is \( E_{B=2} = 2.3764 \).

5.2 \( SU(4) \) Skyrme Model

In this case the energy density becomes

\[
\mathcal{E} = \frac{1}{4} \left( 3\ddot{F}_0^2 + 4\dot{F}_1^2 + 3\dot{F}_2^2 + 4\dot{F}_0\dot{F}_1 + 4\dot{F}_1\dot{F}_2 + 2\dot{F}_0\dot{F}_2 \right) +
\frac{1}{2r^2} \left[ 3(\ddot{F}_0^2 + 4)(1 - \cos F_0) + 4(\dot{F}_1^2 + 4)(1 - \cos F_1) + 3(\dot{F}_2^2 + 4)(1 - \cos F_2) \right] +
\frac{1}{2r^4} \left\{ 9(1 - \cos F_0)^2 + 16(1 - \cos F_1)^2 + 9(1 - \cos F_2)^2 -
12(1 - \cos F_0)(1 - \cos F_1) - 12(1 - \cos F_1)(1 - \cos F_2) \right\},
\]

(39)

while the equations for \( F_0 \), \( F_1 \) and \( F_2 \) are more complicated:

\[
\ddot{F}_0 \left( 1 + \frac{2(1 - \cos F_0)}{r^2} \right) + \frac{2\dot{F}_1 + \dot{F}_2}{3} + 3\dot{F}_0 + 4\dot{F}_1 + 2\dot{F}_2 + \frac{\sin F_0}{r^2} \left[ \dot{F}_0^2 - 4\frac{(1 - \cos F_0)}{r^2} + \frac{4(1 - \cos F_1)}{r^2} \right] = 0,
\]

\[
\ddot{F}_1 \left( 1 + \frac{3}{2r^2} (1 - \cos F_1) \right) + \frac{2\dot{F}_0 + \dot{F}_2}{3} + 3\dot{F}_0 + 4\dot{F}_1 + 2\dot{F}_2 + \frac{\sin F_1}{r^2} \left[ \dot{F}_1^2 - 4\frac{(1 - \cos F_1)}{r^2} + \frac{2(1 - \cos F_0)}{r^2} \right] = 0,
\]
These equations have the previously mentioned symmetry $F_k \leftrightarrow F_{N-k-2}$ which allows us to set $F_0 = F_2$ by keeping $F_1$ arbitrary.

In addition, letting $F_0 = F_1 = F_2 = F$ the above system reduces to equation (37) and therefore, the configuration which consists of ten skyrmions (as $B = \frac{3E_0(0) + 4F_1(0) + 3F_2(0)}{2\pi} = 10$) is $E_{B=10} = 10 \times 1.232$, i.e. exactly ten times the energy of one skyrmion. Once again we have a solution describing many skyrmions, which are non-interacting and whose energy density is spherically symmetric.

In addition, letting $F_0 = -F_2 = G$ we spot that when $F_1 = 0$, we have a solution of the form

$$
\dot{G} \left(1 + \frac{3(1 - \cos G)}{r^2}\right) + \frac{2\dot{G}}{r} + \frac{3\sin G}{2r^2} \left[\dot{G}^2 - 4 - \frac{6(1 - \cos G)}{r^2}\right] = 0,
$$

which corresponds to a non-topological solution, i.e. its baryon number is zero; however the corresponding configuration consists of three skyrmions and three anti-skyrmions. [Recall, that the profile functions are $g_0 = 0$ and $g_1 = g_2$, i.e. the field (9) involves only one projector of rank two – namely $P_1 + P_2$.]

In general, however, the solutions depend on more functions. We can always assume that the functions $F_i$ go to zero at infinity, so the topological charge of a solution is determined, using (33), by the boundary value of each $F_i$ at the origin. When each of these values is positive the solution is a mixture of skyrmions. When some $F_i$’s take positive and some $F_i$’s take negative values at the origin the solution corresponds to a mixture of skyrmions and anti-skyrmions. This is very similar to what happens in the two-dimensional Grassmannian sigma model [5].

We have solved numerically equations (40) taking all combinations, modulo the exchange of $F_0$ and $F_2$, of $0, 2\pi$ and $-2\pi$ for the value of $F_i$ at the origin. The results are summarised in the Table below.

| $F_0(0)$ | $F_1(0)$ | $F_2(0)$ | $B$ | Energy | E/baryon | SU(2) En |
|----------|----------|----------|-----|--------|----------|----------|
| 2\pi     | 0        | 0        | 3   | 3.518  | 1.173    | 3.429    |
| 0        | 2\pi     | 0        | 4   | 4.788  | 1.197    | 4.464    |
| 2\pi     | 0        | 2\pi     | 6   | 7.22553| 1.204    | 6.654    |
| 2\pi     | 2\pi     | 0        | 7   | 8.45219| 1.207    | 7.7693   |
| 2\pi     | 2\pi     | 2\pi     | 10  | 12.32  | 1.232    | -        |
| 2\pi     | -2\pi    | 2\pi     | 6-4 | 8.852  | 0.8852   | -        |
| 2\pi     | 2\pi     | -2\pi    | 7-3 | 9.896  | 0.9896   | -        |
| 2\pi     | 0        | -2\pi    | 3-3 | 6.63422| 1.106    | -        |
| -2\pi    | 2\pi     | 0        | 4-3 | 6.61478| 0.945    | -        |

The first five solutions are bound states of skyrmions with energies larger than the energies of the corresponding $SU(2)$ solutions [4]. Moreover, the excitation energy of the
first two solutions is very small. As mentioned above, the energy of the \( B = 10 \) solution is exactly ten times the energy of a single skyrmion solution. These solutions are all axially symmetric (but their energy densities are radially symmetric) and thus they are all more symmetrical than the corresponding \( SU(2) \) solutions.

The last four solutions are bound states of skyrmions and anti-skyrmions. Although their energies are very small, we know that these solutions must be unstable.

6 Conclusions

In this paper we have shown how to construct radially symmetric solutions of the \( SU(N) \) Skyrme models. In the general case these solutions depend on \( N - 1 \) profile functions which have to be determined numerically. In some cases we can exploit symmetries of our expressions and reduce the number of functions. Thus in the case of \( SU(3) \) we can recover the topologically trivial solution discussed in [1].

We have not discussed the derived solutions in much detail. Their properties and their relation to physics deserve further study and these topics are currently under investigation.

It is worth pointing out that there is a rather close connection between \( SU(N) \) BPS monopoles and skyrmions. Both monopole and skyrmion fields can be constructed in terms of harmonic maps between Riemann spheres. Thus, recently, it has been shown [10] that the monopoles fields can also be constructed using the projector ansatz. Its generalisation to multi-projectors and the construction in [10] provide explicit examples of spherically symmetric \( SU(N) \) monopoles with various symmetry breaking patterns. In the monopole case, the Bogomolny equation is the analogue of our Skyrme equation and the monopole number corresponds to our baryon number.

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Appendix

In this appendix we prove equation (11) for \( f \) given by (10). This result can be proved by induction. Note that, the modulus of the \( N \)-dimensional vector \( f \) is \(|f|^2 = (1 + |\xi|^2)^{N-1}\) and therefore, 

\[
|P_+ f|^2 f|^2 = |f|^2 |\partial_+ f|^2 - |f^\dagger \partial_+ f|^2 = (N-1)(1 + |\xi|^2)^{N-3}|f|^2,
\]

therefore, 

\[
|P_+ f|^2 = (N-1)(1 + |\xi|^2)^{N-3}.
\]

As

\[
P_{+1}^k f = \partial_\xi P_+^k f - P_+^k f \frac{(P_+^k f)^\dagger \partial_\xi P_+^k f}{|P_+^k f|^2},
\]

(A.1)

using (8) we get

\[
|P_{+1}^k f|^2 |P_+^k f|^2 = |\partial_\xi P_+^k f|^2 |P_+^k f|^2 - |\partial_\xi |P_+^k f|^2|^2,
\]

(A.2)
\[
\partial_\xi \partial_\bar{\xi} |P_k f|^2 = |\partial_\xi P_k f|^2 + (P_k f) \dagger \partial_\xi P_k f, \quad (A.3)
\]
\[
(P_k f) \dagger \partial_\bar{\xi} (P_k f) = -(P_k f) \dagger \partial_\xi (P_k f) \frac{|P_k f|^2}{|P_{k-1} f|^2}
\]
\[
= - \frac{|P_k f|^4}{|P_{k-1} f|^2}. \quad (A.4)
\]

Which finally, leads to

\[
|P_{k+1} f|^2 = \partial_\xi \partial_\bar{\xi} |P_k f|^2 + \frac{|P_k f|^4}{|P_{k-1} f|^2} - \frac{\partial_\xi |P_k f|^2 |P_k f|^2}{|P_k f|^2}.
\]

(A.5)

Therefore if \(|P_k f|^2 = \alpha (1 + |\xi|^2)^{N-2k-1}\) and \(|P_{k-1} f|^2 = \beta (1 + |\xi|^2)^{N-2k+1}\) then

\[
|P_{k+1} f|^2 = \gamma (1 + |\xi|^2)^{N-2k-3},
\]

where \(\gamma = \alpha (N - 2k - 1) + \alpha^2/\beta\). To find \(\gamma\) we again use induction, recalling that the coefficients of the two lowest terms in the sequence are 1 and \(N - 1\), respectively. Then it is easily seen that

\[
|P_{k+1} f|^2 = k!(N - 1)(N - 2) \cdots (N - k)(1 + |\xi|^2)^{N-2k-1},
\]

(A.6)
in which the last term in the sequence corresponds to \(k = N - 1\).

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