REALIZATIONS OF ABSTRACT REGULAR POLYTOPES
FROM A REPRESENTATION THEORETIC VIEW

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Abstract. Peter McMullen has developed a theory of realizations of abstract
regular polytopes, and has shown that the realizations up to congruence form a
pointed convex cone which is the direct product of certain irreducible subcones.
We show that each of these subcones is isomorphic to a set of positive semi-definite
hermitian matrices of dimension $m$ over either the real numbers, the complex
numbers or the quaternions. In particular, we correct an erroneous computation
of the dimension of these subcones by McMullen and Monson. We show that
the automorphism group of an abstract regular polytope can have an irreducible
character $\chi$ with $\chi \neq \overline{\chi}$ and with arbitrarily large essential Wythoff
dimension. This gives counterexamples to a result of Herman and Monson, which was derived
from the erroneous computation mentioned before.

We also discuss a relation between cosine vectors of certain pure realizations
and the spherical functions appearing in the theory of Gelfand pairs.

1. Introduction

These notes are the result of an attempt to understand realizations of abstract
regular polytopes, as introduced by Peter McMullen [9, 13, 11, 12], from a repre-
sentation theoretic viewpoint, thereby showing that the theory actually generalizes
to a theory of “realizations of transitive $G$-sets”. That the theory applies in this
wider context was already pointed out by McMullen [12, Remark 2.1]. In particular,
we will derive the exact structure of McMullen’s realization cone using arguments
from basic representation theory and linear algebra.

To explain this in more detail, and to state our main theorem, we have to
introduce some notation. Let $G$ be a finite group and $\Omega$ a transitive $G$-set. (In
the original theory, $\Omega$ is the vertex set of an abstract regular polytope and $G$
the automorphism group of the polytope. But this assumption is in fact unnecessary
for a large part of the theory.) In this situation, one can define a closed pointed
convex cone called the realization cone which describes realizations of the transitive
$G$-set $\Omega$ up to congruence. (We will recall the exact definitions below.)

Let us write $\text{Irr}_{\mathbb{R}} G$ for the set of characters of irreducible representations over
the real numbers $\mathbb{R}$. McMullen [9] has shown that the realization cone is the direct
product of subcones, each subcone corresponding to some $\sigma \in \text{Irr}_{\mathbb{R}} G$ (or, what
is the same, to a similarity class of irreducible representations of $G$). We write $RC_{\sigma}(\Omega)$
for the subcone corresponding to $\sigma \in \text{Irr}_{\mathbb{R}} G$. The main new result of this
note concerns the structure of such a subcone.

1991 Mathematics Subject Classification. Primary 52B15, Secondary 20C15, 20B25.
Key words and phrases. Real representations of finite groups, abstract regular polytope, real-
ization cone, C-string group.

Author supported by the DFG (Project: SCHU 1503/6-1).
To state this result, we need some more notation. Let $\pi = \pi_\Omega$ be the permutation character corresponding to the $G$-set $\Omega$. We can write $\pi$ as a sum of irreducible real characters:

$$\pi = \sum_{\sigma \in \text{Irr}_R G} m_\sigma \chi_\sigma.$$

The multiplicities $m_\sigma$ are uniquely determined by this equation, and equal the essential Wythoff dimension defined by McMullen and Monson [13]. Moreover, to each $\sigma \in \text{Irr}_R G$ belongs a division ring $D_\sigma$ (the centralizer ring of a representation affording $\sigma$), which is isomorphic to either the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$ or the Hamiltonian quaternions $\mathbb{H}$.

We write $M_{m}(D)$ for the ring of $m \times m$-matrices over $D$, and if $A \in M_{m}(D)$, then $A^*$ denotes the (complex/quaternion) conjugate transpose of $A$ when $D = \mathbb{C}$ or $\mathbb{H}$, and the transpose of $A$ when $D = \mathbb{R}$. With this notation, we have:

**Main Theorem.** The realization cone of $\Omega$ is the direct product of subcones $\mathcal{RC}_\sigma(\Omega)$ corresponding to $\sigma \in \text{Irr}_R G$, where each $\mathcal{RC}_\sigma(\Omega)$ is isomorphic to the set of matrices

$$\{AA^* \mid A \in M_{m_\sigma}(D_\sigma)\}.$$

In other words, the subcone $\mathcal{RC}_\sigma(\Omega)$ is isomorphic to the set of hermitian positive semi-definite $m_\sigma \times m_\sigma$-matrices with entries in $D_\sigma$, with appropriate meaning of “hermitian” (depending on whether $D_\sigma = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$).

From the main theorem, one can immediately derive the dimension of $\mathcal{RC}_\sigma(\Omega)$ (see Corollary 3.7). This dimension has been computed by McMullen and Monson [13, Theorem 4.6] (using different notation). Unfortunately, the result of McMullen and Monson only matches with our description when $m_\sigma \leq 1$ or when $D_\sigma = \mathbb{R}$. If the computation of McMullen and Monson [13, Theorem 4.6] were correct in the original situation, where $G$ is the automorphism group of an abstract regular polytope with vertex set $\Omega$, then it would follow that we always have $m_\sigma \leq 1$ or $D_\sigma = \mathbb{R}$ for such $G$. And indeed, this is the main result of a paper by Herman and Monson [4]. They derive this from [13, Theorem 4.6] in a different way.

But unfortunately, the main result of Herman and Monson [4] is wrong: We show in Section 4 that we can have $D_\sigma = \mathbb{C}$ and $m_\sigma$ arbitrarily large even when $G$ is the automorphism group of an abstract regular polytope with vertex set $\Omega$. (See Example 4.1 for a concrete case where $m_\sigma = 2$. It seems to be unknown whether there are any abstract regular polytopes with $D_\sigma \cong \mathbb{H}$ for some $\sigma$.) These examples show that the computation of McMullen and Monson must be wrong even in the original setting. At the end of Section 3, we briefly discuss where we see the flaw in McMullen’s and Monson’s proof.

A later result of McMullen [12, Theorem 5.2] can be interpreted as saying that the subcone $\mathcal{RC}_\sigma(\Omega)$ is isomorphic to the symmetric positive semi-definite matrices of size $m_\sigma \times m_\sigma$, with entries in the reals. This is in general not correct, the correct statement is the main theorem stated above.

Another consequence of the mistake in [13] is that the $\Lambda$-orthogonal basis described in [12] is in general too small. In Section 5, we briefly discuss the relation between McMullen’s $\Lambda$-inner product and some other natural inner products, and indicate how to construct a complete orthogonal basis.
In Section 6, we discuss some relations between McMullen’s cosine vectors and the spherical functions appearing in the theory of Gelfand pairs. It turns out that when \((G, H)\) is a Gelfand pair (where \(H\) is the stabilizer of a vertex), then the cosine vectors are in principle the same as the spherical functions. (This applies to all classical regular polytopes in euclidean space, except the 120-cell.) We show that the values of cosine vectors are algebraic numbers, when the essential Wythoff dimension is 1. This was conjectured by McMullen [12, Remark 9.4]. Indeed, multiplied with the size of the corresponding layer, we get an algebraic integer.

Finally, in Section 7 we propose an explanation of an observation of McMullen [11, Remark 9.3] about the cosine vectors of the 600-cell.

2. Realizations as \(G\)-homomorphisms

Let \(G\) be a finite group. For convenience, we use the following terminology: An euclidian \(G\)-space is an euclidean vector space \((V, \langle \cdot, \cdot \rangle)\) on which the group \(G\) acts by orthogonal transformations. The action is denoted by \((v, g) \mapsto vg\). Equivalently, we are given an orthogonal representation \(D: G \to O(V)\), so that \(D(g)\) is the map \(v \mapsto vg = vD(g)\).

Let \(\Omega\) be a transitive (right) \(G\)-set. A realization of \((G, \Omega)\) is a map \(A: \Omega \to V\) into an euclidean \(G\)-space \(V\) such that \(\langle \omega gA, \omega A \rangle = \langle \omega A, \omega A \rangle g\) for all \(\omega \in \Omega\) and \(g \in G\). This definition agrees with McMullen’s definition [9, 11, 12] in the case where \(G\) is the automorphism group of an abstract regular polytope with vertex set \(\Omega\). We emphasize that in this paper, \(G\) is just some finite group and \(\Omega\) a transitive \(G\)-set. For example, we could take \(\Omega = G\), on which \(G\) acts by right multiplication.

Two realizations \(A_1: \Omega \to V_1\) and \(A_2: \Omega \to V_2\) are called congruent, if there is a linear isometry \(\sigma\) from the linear span of \(\{\omega A_1 \mid \omega \in \Omega\}\) into \(V_2\) such that \(A_1\sigma = A_2\). (A peculiarity of this definition is that the realization \(\Omega \to \mathbb{R}\) sending every \(\omega \in \Omega\) to 0 is not congruent to the realization sending every \(\omega \in \Omega\) to 1. It turns out to be useful to distinguish these.) The following is easy to see:

2.1. Lemma. Two realizations \(A_1: \Omega \to (V_1, \langle \cdot, \cdot \rangle_1)\) and \(A_2: \Omega \to (V_2, \langle \cdot, \cdot \rangle_2)\) are congruent if and only if \(\langle \xi A_1, \eta A_1 \rangle_1 = \langle \xi A_2, \eta A_2 \rangle_2\) for all \(\xi, \eta \in \Omega\).

Thus a realization \(A: \Omega \to V\) is determined up to congruence by the \(\Omega \times \Omega\) matrix \(Q = Q(A)\) with entries \(q_{\xi, \eta} = \langle \xi A, \eta A \rangle\). We call \(Q\) the inner product matrix of the realization \(A\). It is a symmetric positive semi-definite matrix and \(G\)-invariant in the sense that \(q_{g\xi, g\eta} = q_{\xi, \eta}\).

2.2. Remark. McMullen [11] uses inner product vectors instead of inner product matrices. The relation is as follows: A diagonal class is an orbit of \(G\) on the set of unordered pairs on \(\Omega\). Since the inner product matrix \(Q = (q_{\xi, \eta})\) is symmetric and \(G\)-invariant, the map \(\{\xi, \eta\} \mapsto q_{\xi, \eta}\) is well defined and constant along diagonal classes. Thus it is determined by its values on a set of representatives of the diagonal classes.

Now fix some “initial” vertex \(\alpha \in \Omega\). A layer is the set of all elements \(\omega \in \Omega\) such that \(\{\alpha, \omega\}\) belongs to the same diagonal class. Choose a set of representatives \(\xi_0 = \alpha, \xi_1, \ldots, \xi_r\) of the layers in \(\Omega\). Then the unordered pairs \(\{\alpha, \xi_i\}\) represent all diagonal classes (as \(\Omega\) is a transitive \(G\)-set). The vector of length \(r + 1\) with values
\(q_{\alpha,\xi} = \langle \alpha A, \xi_i A \rangle\) as entries is the **inner product vector** of the realization \([11]\). It is clear that the inner product matrix is determined by the inner product vector. For the purposes of this paper, we find it more convenient to use the inner product matrix itself.

The set of all inner product matrices of realizations of \(\Omega\) is called the **realization cone** of \(\Omega\), and denoted by \(\mathcal{RC}(\Omega)\) or \(\mathcal{RC}(G, \Omega)\) (in the first variant, the group \(G\) is understood to be implicit in \(\Omega\)). It is in bijection to the set of all congruence classes of realizations.

The following operations on realizations show that the realization cone is indeed a cone: First, if \(A_1: \Omega \to V_1\) and \(A_2: \Omega \to V_2\) are two realizations, then their **blend** is the realization \(A_1 \oplus A_2: \Omega \to V_1 \oplus V_2\) sending \(\omega \in \Omega\) to \(\langle \omega A_1, \omega A_2 \rangle\) in the (outer) orthogonal sum of the two euclidean spaces \(V_1\) and \(V_2\). (McMullen denotes the blend by \(A_1 \neq A_2\).) For the corresponding inner product matrices, we have \(Q(A_1 \oplus A_2) = Q(A_1) + Q(A_2)\).

Second, we can scale realizations: for \(A: \Omega \to V\) and \(\lambda \in \mathbb{R}\), \(\lambda A: \Omega \to V\) is defined by \(\omega(\lambda A) = \lambda(\omega A)\). Obviously, \(Q(\lambda A) = \lambda^2 Q(A)\).

For completeness, we mention a third operation, the **tensor product** \(A_1 \otimes A_2: \Omega \to V_1 \otimes V_2\) of two realizations \(A_i: \Omega \to V_i\), defined on \(\Omega\) by \(\omega(A_1 \otimes A_2) := (\omega A_1) \otimes (\omega A_2)\). The inner product matrix \(Q(A_1 \otimes A_2)\) is the entry-wise (Hadamard) product of \(Q(A_1)\) and \(Q(A_2)\).

It follows from blending and scaling that \(\mathcal{RC}(\Omega)\) is a convex cone. It is also clear that \(\mathcal{RC}(\Omega)\) has an apex at 0.

A realization \(A: \Omega \to V\) is called **normalized**, if \(\|\omega A\|^2 := \langle \omega A, \omega A \rangle = 1\) for some (and hence for all) \(\omega \in \Omega\). If \(\omega A \neq 0\), then we may scale the realization by \(1/\|\omega A\|\), so that it becomes normalized. The inner product matrix of the normalized realization \((1/\|\omega A\|)A\) is called its **cosine matrix**, for obvious reasons. The set of cosine matrices of realizations forms a compact convex set.

### 2.3. Remark

As in Remark 2.2, a cosine matrix corresponds to a **cosine vector**, which contains the values \(\langle \alpha A, \xi_i A \rangle/\langle \alpha A, \alpha A \rangle\), where \(\xi_i\) runs over a set of representatives of the layers. We have to caution the reader that McMullen \([12]\) uses the term **cosine matrix** with a different meaning: In \([12]\), this is a square matrix whose rows are cosine vectors of different realizations (and maybe certain mixed cosine vectors), and such that the rows are orthogonal with respect to a certain inner product (\(\Lambda\)-orthogonality, see Section 5 below). This matrix is similar to the character table of a finite group, and thus we find the name “cosine table” more appropriate for this object.

An especially important realization is the **simplex realization** which we now define. Recall that the permutation module \(\mathbb{R}\Omega\) over \(\mathbb{R}\) belonging to the \(G\)-set \(\Omega\) is the set of formal sums

\[\mathbb{R}\Omega := \{ \sum_{\omega \in \Omega} r_{\omega} \omega \mid r_{\omega} \in \mathbb{R} \},\]

on which \(G\) acts by \((\sum r_{\omega} \omega)g = \sum r_{\omega}(\omega g)\). Also we think of \(\mathbb{R}\Omega\) as equipped with the standard scalar product

\[\langle \sum_{\omega} r_{\omega} \omega, \sum_{\omega} s_{\omega} \omega \rangle = \sum_{\omega} r_{\omega} s_{\omega} \omega.\]
This makes $\mathbb{R}\Omega$ into an euclidean $G$-space. The natural map $\Omega \to \mathbb{R}\Omega$ is a realization, called the *simplex realization*. (We usually identify its image, the canonical basis of $\mathbb{R}\Omega$, with the set $\Omega$.)

The next observation is obvious, but crucial for our proof of the structure theorems in the next section. Recall that a linear map $\hat{A}: U \to V$ between two $G$-modules is a $G$-module homomorphism if $ug\hat{A} = u\hat{A}g$ for all $u \in U$ and $g \in G$. Since $\Omega$ is a basis of $\mathbb{R}\Omega$, we have the following:

2.4. **Observation.** Realizations $A: \Omega \to V$ correspond to $G$-module homomorphisms $\hat{A}: \mathbb{R}\Omega \to V$.

From now on, we identify a realization $A: \Omega \to V$ with the corresponding linear map $\mathbb{R}\Omega \to V$, and use the same letter $A$ for both. We also identify a linear map $A: \mathbb{R}\Omega \to V$ with its matrix $A$ with respect to the canonical basis $\Omega$ and some fixed orthonormal basis of $V$. The inner product matrix of the realization $A$ is then $Q = AA^t$, and does not depend on the choice of basis of $V$.

We also write $A^t: V \to \mathbb{R}\Omega$ for the *adjoint map* of $A: \mathbb{R}\Omega \to V$ with respect to the inner products on $\mathbb{R}\Omega$ and $V$; if $A$ is a $G$-module homomorphism, then so is $A^t$.

From this viewpoint, $Q = AA^t$ is a $G$-module endomorphism of $\mathbb{R}\Omega$.

2.5. **Theorem.** Let $\Omega$ be a transitive $G$-set. Then

$$\mathcal{RC}(\Omega) = \{AA^t \mid A \in M_\Omega(\mathbb{R}) \text{ is } G\text{-invariant}\},$$

and this equals the set of $G$-invariant, symmetric positive semi-definite matrices.

This is the special case $U = \mathbb{R}\Omega$ of the following general observation:

2.6. **Lemma.** Let $U$ be an euclidean $G$-space and $Q \in \text{End}_\mathbb{R}(U)$. The following are equivalent:

(i) There is an euclidean $G$-space $V$ and a $G$-homomorphism $A: U \to V$ such that $Q = AA^t$.

(ii) $Q$ is symmetric positive semi-definite and commutes with $G$.

(iii) There is $A \in \text{End}_\mathbb{R}G(U)$ such that $Q = AA^t$.

**Proof.** Obviously, (iii) is a special case of (i), and (i) implies (ii).

It remains to show that (ii) implies (iii), so assume $Q$ is symmetric positive semi-definite and commutes with $G$. Then $U$ is the orthogonal sum of the eigenspaces of $Q$, and the eigenvalues of $Q$ are non-negative real numbers. For each eigenvalue $\lambda$ of $Q$, let $p_\lambda: U \to U$ be the orthogonal projection onto the corresponding eigenspace of $Q$. Since $Q$ commutes with $G$, it follows that the eigenspaces are $G$-invariant and thus the $p_\lambda$’s commute with $G$.

Since $U$ is the orthogonal sum of the eigenspaces, we have $\text{id}_U = \sum_\lambda p_\lambda$. For $u \in U$, it follows

$$uQ = \sum_\lambda up_\lambda Q = \sum_\lambda \lambda(up_\lambda) = u \sum_\lambda \lambda p_\lambda.$$

Since $p_\lambda p_\mu = \delta_{\lambda,\mu}p_\lambda$ for eigenvalues $\lambda$, $\mu$ of $Q$, and since all $\lambda \geq 0$, we get

$$Q = \sum_\lambda \lambda p_\lambda = \left(\sum_\lambda \sqrt{\lambda}p_\lambda\right)^2.$$
Set $A = \sum_\chi \sqrt{\lambda} p_\chi$, an element commuting with $G$. Then $A = A^t$, since orthogonal projections are self-adjoint, and thus $Q = A^2 = AA^t$ as required. \hfill \Box

3. The structure of the realization cone

In this section, we determine the structure of the realization cone. The general idea is the following: We can write the module $\mathbb{R}\Omega$ as an orthogonal sum of simple modules, say

$$\mathbb{R}\Omega \cong m_1S_1 \oplus \cdots \oplus m_kS_k,$$

with natural numbers $m_i$, and where the different $S_i$'s are non-isomorphic. It is well known that then

$$\text{End}_{RG}(\mathbb{R}\Omega) \cong M_{m_1}(\text{End}_{RG}(S_1)) \times \cdots \times M_{m_k}(\text{End}_{RG}(S_k)),$$

where for each $i$ the endomorphism ring $\mathbb{D}_i := \text{End}_{RG}(S_i)$ is a division ring by Schur's lemma, and thus either $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. The aim of this section is to fill in the details and to show that the above isomorphism, when restricted to the realization cone $\mathcal{RC}(\Omega)$ as a subset of $\text{End}_{RG}(\mathbb{R}\Omega)$, yields a similar decomposition into subcones of the form $\{AA^* | A \in M_{m_i}(\text{End}_{RG}(S_i))\}$.

We begin by recalling some general representation theory. As usual, we write $\operatorname{Irr} G$ for the set of irreducible complex characters of a group $G$. Furthermore, $\operatorname{Irr}_\mathbb{R} G$ denotes the set of characters of simple $\mathbb{R}G$-modules (equivalently, of irreducible representations $G \to \text{GL}(d, \mathbb{R})$). For class functions $\alpha$, $\beta$: $G \to \mathbb{C}$,

$$[\alpha, \beta] := \frac{1}{|G|} \sum_{g \in G} \alpha(g)\overline{\beta(g)}$$

denotes the usual inner product of class functions. It is well known that $\operatorname{Irr} G$ is an orthonormal basis of the space of class functions with respect to this inner product. For $\sigma \in \operatorname{Irr}_\mathbb{R} G$, we have the following possibilities [16, III.5A][7, Ch. 4]:

3.1. Lemma. Let $S$ be a simple $\mathbb{R}G$-module with character $\sigma \in \operatorname{Irr}_\mathbb{R} G$. Then one of the following three cases occurs:

(i) $[\sigma, \sigma] = 1$, $\sigma \in \operatorname{Irr} G$ and $\text{End}_{RG}(S) \cong \mathbb{R}$,

(ii) $[\sigma, \sigma] = 2$, $\sigma = \chi + \overline{\chi}$ with $\chi \neq \overline{\chi} \in \operatorname{Irr} G$ and $\text{End}_{RG}(S) \cong \mathbb{C}$,

(iii) $[\sigma, \sigma] = 4$, $\sigma = 2\chi$ with $\chi = \overline{\chi} \in \operatorname{Irr} G$ and $\text{End}_{RG}(S) \cong \mathbb{H}$.

We call $S$ and $\sigma$ of real, complex or quaternion type, respectively.

Let $S$ be a simple $\mathbb{R}G$-module with character $\sigma$. For any $\mathbb{R}G$-module $V$, let $V_{\sigma} = V_S$ be the sum of all submodules of $V$ isomorphic to $S$. The submodule $V_{\sigma}$ is called the $\sigma$-homogeneous component of $V$. Every module $V$ is the direct sum of the $V_{\sigma}$, as $\sigma$ runs over $\operatorname{Irr}_\mathbb{R} G$. This sum is orthogonal with respect to any $G$-invariant inner product defined on $V$. The orthogonal projection $V \to V_{\sigma}$ is given by the action of

$$e_\sigma = \frac{\sigma(1)}{[\sigma, \sigma]|G|} \sum_{g \in G} \sigma(g^{-1})g \in Z(\mathbb{R}G)$$

on $V$. (The formula for the idempotent $e_\sigma$ follows from the analogous one in the complex case [7, Theorem 2.12][16, III.7] together with Lemma 3.1.) We have

$$1 = \sum_{\sigma \in \operatorname{Irr}_\mathbb{R} G} e_\sigma, \quad \text{and} \quad e_\sigma e_\tau = \delta_{\sigma, \tau} e_\sigma \quad \text{for all } \sigma, \tau \in \operatorname{Irr}_\mathbb{R} G.$$
Notice that since $e_\sigma \in \mathbb{Z}(\mathbb{R} G)$, the action of $e_\sigma$ on modules commutes with both the action of $G$ and the action of $G$-module homomorphisms.

For each $\sigma \in \text{Irr}_R G$, define $\mathcal{R}C_\sigma(\Omega)$ to be the set of all inner product matrices which arise from a realization $A: \Omega \to V$ such that $V = V_\sigma$, so $V$ has character $k\sigma$ for some $k \in \mathbb{N}$. Equivalently, if $S$ is an irreducible module affording $\sigma$, then $V$ is isomorphic to a direct sum of copies of $S$. (The subcone $\mathcal{R}C_\sigma(\Omega)$ is denoted by $P_D$ in [9, 13], where $D$ is an irreducible representation of $G$ affording $\sigma$.)

In the next result, we view both the inner product matrix and the idempotent $e_\sigma$ as operators on the permutation module $\mathbb{R} \Omega$.

3.2. Theorem. (cf. [9, Theorem 16], [13, Theorem 4.1]) $\mathcal{R}C_\sigma(\Omega)$ is a closed subcone of $\mathcal{R}C(\Omega)$ and $\mathcal{R}C(\Omega)$ is the direct sum of the $\mathcal{R}C_\sigma(\Omega)$, where $\sigma \in \text{Irr}_R G$. More precisely, for $Q \in \mathcal{R}C(\Omega)$, we have

$$Q = \sum_{\sigma \in \text{Irr}_R G} Q_\sigma, \quad \text{where} \quad Q_\sigma = e_\sigma Q = Q e_\sigma \in \mathcal{R}C_\sigma(\Omega).$$

(In particular, $Q \in \mathcal{R}C_\sigma(\Omega)$ if and only if $e_\sigma Q = Q$, if and only if $Q = Q e_\sigma$.)

This means that if the inner product matrix $Q$ of a realization has entries $q_{\xi,\eta}$, then the inner product matrix $Q_\sigma = e_\sigma Q$ of the $\sigma$-homogeneous component of the realization has entries

$$s_{\xi,\eta} := \frac{\sigma(1)}{[\sigma,\sigma]|G|} \sum_{g \in G} \sigma(g^{-1})q_{\xi g,\eta}. \quad \text{for all } \xi, \eta \in \Omega.$$

Proof of Theorem 3.2. Suppose $A: \mathbb{R} \Omega \to V$ is a realization with inner product matrix $Q = AA' \in \mathcal{R}C(\Omega)$. Then $e_\sigma A = Ae_\sigma$ is a realization $\mathbb{R} \Omega \to Ve_\sigma$ with inner product matrix $(e_\sigma A)(e_\sigma A)' = e_\sigma Q e_\sigma = e_\sigma Q$, since $e_\sigma' = e_\sigma = e_\sigma^2$. Thus $e_\sigma Q$ is an inner product matrix in $\mathcal{R}C_\sigma(\Omega)$. Conversely, if $Q \in \mathcal{R}C_\sigma(\Omega)$, then $Q = AA'$ for some realization $A$ with $A = Ae_\sigma$, and thus $Q = e_\sigma Q$.

Since $Q = \sum_\sigma e_\sigma Q$ for any inner product matrix, the result follows. \hfill \Box

(That $\mathcal{R}C_\sigma(\Omega)$ is a subcone and that $\mathcal{R}C(\Omega)$ is the sum of these subcones is also immediate from the equation $Q(A_1 \oplus A_2) = Q(A_1) + Q(A_2)$ and the fact that every $\mathbb{R} G$-module can be written as an orthogonal sum of simple modules.)

Next we determine the structure of $\mathcal{R}C_\sigma(\Omega)$, for $\sigma \in \text{Irr}_R G$. Let $S$ be a simple $\mathbb{R} G$-module affording $\sigma$. We can write $(\mathbb{R} \Omega)_\sigma$ as the orthogonal sum of $m = m_\sigma = m_S$ copies of $S$, that is, $(\mathbb{R} \Omega)_\sigma \cong m S$. The non-negative integer $m$ is called the multiplicity of $S$ in $\mathbb{R} \Omega$ and of $\sigma$ in the character $\pi = (1_H)^G$ of $\mathbb{R} \Omega$. In other words, we have

$$\pi = (1_H)^G = \sum_{\sigma \in \text{Irr}_R G} m_\sigma \sigma,$$

and this equation determines the $m_\sigma$’s. (Here $H = G_\alpha$, the stabilizer of a vertex $\alpha$.)

Recall that the Wythoff space $W_S$ associated to $S$ (and $\alpha \in \Omega$) is the fixed space of $H$ on $S$. McMullen and Monson [13] defined the essential Wythoff dimension as the dimension of $W_S$ over the centralizer ring $D = \text{End}_{\mathbb{R} G}(S)$.

3.3. Lemma. The multiplicity $m_S = m_\sigma$ equals the essential Wythoff dimension.
Proof. Let \( \pi \) be the character of \( \mathbb{R} \Omega \). Then \( [\pi, \sigma]_G = m_\sigma [\sigma, \sigma]_G = m_\sigma \dim_{\mathbb{R}}(\mathbb{D}) \). On the other hand, \( \pi = (1_H)^G \) and \( [\pi, \sigma]_G = [1_H, \sigma]_H = \dim_{\mathbb{R}} W_S \) by Frobenius reciprocity. The result follows. \( \square \)

Before we give our structure theorem for \( \mathcal{RC}_\sigma(\Omega) \), we digress to reprove Theorems 4.4 and 4.5 of the McMullen-Monson paper [13], since, as we argue below, McMullen’s and Monson’s proofs of these theorems are not correct.

We recall that a realization \( A: \Omega \to V \) and the corresponding polytope are called pure, when the image \( A(\mathbb{R} \Omega) \) is simple as module over \( G \). The following contains Theorems 4.4 and 4.5 from the paper of McMullen and Monson [13].

### 3.4. Theorem

Every polytope in \( \mathcal{RC}_\sigma(\Omega) \) is the blend of at most \( m_\sigma \) pure polytopes, and has dimension at most \( m_\sigma \sigma(1) \), where \( m_\sigma \sigma(1) \) is possible.

**Proof.** Let \( A: \Omega \to V \) be a realization, which we identify as usual with a \( G \)-homomorphism \( \mathbb{R} \Omega \to V \). Without loss of generality, we can assume that \( V = (\mathbb{R} \Omega) A \), that is, \( V \) is the linear span of \( \{ \omega A \mid \omega \in \Omega \} \). The orthogonal complement of \( \ker A \) in \( \mathbb{R} \Omega \) is a \( G \)-invariant subspace isomorphic to \( V \). In particular, if \( V \cong kS \), where \( S \) affords \( \sigma \), it follows from the uniqueness of the decomposition of \( \mathbb{R} \Omega \) into irreducible summands that \( k \leq m_\sigma \). Then \( A \) is the blend of \( k \) pure realizations, and the polytope spanned by \( \{ \omega A \mid \omega \in \Omega \} \) has dimension \( k \sigma(1) \leq m_\sigma \sigma(1) \). Finally, \( e_\sigma \) viewed as realization \( \mathbb{R} \Omega \to U = \mathbb{R} \Omega e_\sigma \) yields a polytope of dimension \( \dim U = m_\sigma \sigma(1) \). \( \square \)

In the description of \( \mathcal{RC}_\sigma(\Omega) \), we use the following notation: for a matrix \( B \) over the complex numbers or the quaternions, \( B^\ast \) denotes the transposed conjugate. If \( B \) has real entries, then \( B^\ast = B^t \), the transposed matrix.

### 3.5. Theorem

Let \( S \) be a simple module affording \( \sigma \in \text{Irr}_{\mathbb{R}} G \), let \( m = m_\sigma \) be its multiplicity in \( \mathbb{R} \Omega \) and set \( \mathbb{D} = \text{End}_{\mathbb{R} G}(S) \). Then

\[
\mathcal{RC}_\sigma(\Omega) \cong \{ BB^\ast \mid B \in M_m(\mathbb{D}) \}.
\]

### 3.6. Example

Let \( \Omega \) be the vertex set of the 120-cell (of size 600) and \( G \) its symmetry group. Using the computer algebra system GAP [3], one can compute the multiplicities of the irreducible characters in the permutation character. There are 15 characters occurring with multiplicity 1, three characters occurring with multiplicity 2 (of degrees 16, 16 and 48), and two characters occurring with multiplicity 3 (of degrees 25 and 36). All characters are of real type. The realization cone of the 120-cell is thus a direct product of 15 copies of \( \mathbb{R} > 0 \), of three copies of the cone of symmetric positive semidefinite \( 2 \times 2 \)-matrices, and two copies of the cone of symmetric positive semidefinite \( 3 \times 3 \)-matrices. The 120-cell is the only classical regular polytope for which the realization cone is not polyhedral.

A corollary of the theorem is the correct version of [13, Theorem 4.6].
3.7. Corollary. We have
\[
\dim \mathcal{RC}_\sigma(\Omega) = m + \frac{m(m-1)}{2} [\sigma, \sigma] = \begin{cases} 
\frac{m(m+1)}{2} & \text{for } \mathbb{D} \cong \mathbb{R}, \\
m^2 & \text{for } \mathbb{D} \cong \mathbb{C}, \\
m(2m-1) & \text{for } \mathbb{D} \cong \mathbb{H}.
\end{cases}
\]

Proof. It follows from Theorem 3.5 that the linear span of $\mathcal{RC}_\sigma(\Omega)$ is isomorphic to the $m \times m$ self-adjoint matrices over $\mathbb{D}$. Since $[\sigma, \sigma] = \dim_{\mathbb{R}}(\mathbb{D})$, the result follows.

In the proof of Theorem 3.5, and also later, we need the following simple observation:

3.8. Lemma. Let $S$ be an irreducible euclidean $G$-space and let $\mathbb{D} = \text{End}_{\mathbb{R}G}(S)$. Then for $d \in \mathbb{D}$ we have $d^0 = \overline{d}$ (that is, the adjoint map with respect to the scalar product on $S$ equals the complex/quaternion conjugate).

Proof. We have $d^0 \in \mathbb{D}$ again and thus $dd^0 \in \mathbb{D}$. The eigenspaces of $dd^0$ on $S$ are $G$-invariant, and thus $dd^0 = \lambda \text{id}_S$ with $\lambda \in \mathbb{R}_{\geq 0}$. This means that $\langle vd, vd \rangle = \lambda \langle v, v \rangle$ for all $v \in S$. For $d = i$ (or $d \in \{i, j, k\}$ when $D = \mathbb{H}$), it follows $\lambda = 1$ (because $\lambda^2 \langle v, v \rangle = \langle vd^2, vd^2 \rangle = \langle -v, -v \rangle$), and thus $d^0 = \overline{d}$ in this case. The general case follows from this.

Proof of Theorem 3.5. First, observe that it follows from Theorem 2.5 together with Theorem 3.2 that
\[
\mathcal{RC}_\sigma(\Omega) = \{AA^t \mid A \in \text{End}_{\mathbb{R}G}(\mathbb{R}\Omega), Ae_\sigma = A\}.
\]
Fix a $G$-invariant inner product $\langle \cdot, \cdot \rangle_S$ on the simple module $S$ affording $\sigma$. Suppose that $\mu: S \to \mathbb{R}\Omega$ is an isomorphism from $S$ onto some simple submodule of $\mathbb{R}\Omega$ (necessarily, $S\mu \subseteq \mathbb{R}\Omega e_\sigma$). After eventually scaling $\mu$, we may assume that $\langle v, w \rangle_S = \langle v\mu, w\mu \rangle_{\mathbb{R}\Omega}$. Then with $\pi = \mu^t: \mathbb{R}\Omega \to S$, we have $\mu\pi = \text{id}_S$ and $\pi\mu$ is the orthogonal projection from $\mathbb{R}\Omega$ onto $S\mu$. We know that $\mathbb{R}\Omega e_\sigma$ is isomorphic to a sum of $m$ copies of $S$. Thus we can find $G$-module homomorphisms $\mu_i: S \to \mathbb{R}\Omega$ and $\pi_i: \mathbb{R}\Omega \to S$, $i = 1, \ldots, m$, such that
\[
\pi_i = \mu_i^t, \quad \mu_i\pi_j = \delta_{ij} \text{id}_S, \quad \text{and} \quad e_\sigma = \sum_{i=1}^m \pi_i\mu_i.
\]
Using these maps, we can describe the algebra isomorphism between
\[
\{A \in \text{End}_{\mathbb{R}G}(\mathbb{R}\Omega) \mid Ae_\sigma = A\} \quad \text{and} \quad \text{M}_m(\mathbb{D}),
\]
where $\mathbb{D} = \text{End}_{\mathbb{R}G}(S)$: Send $A \in \text{End}_{\mathbb{R}G}(\mathbb{R}\Omega)$ to the matrix $(\mu_i A \pi_j) \in \text{M}_m(\mathbb{D})$. Conversely, map a matrix $(b_{ij})$ to $\sum_{i,j} \pi_i b_{ij}\mu_j$.
This isomorphism sends the adjoint map $A^t$ to the matrix $(\mu_i A^t \pi_j) = (\pi_j^t A^t \mu_i^t) = (\mu_j A\pi_i^t) = (\mu_j A\pi_i)^t$, where the last equality follows from Lemma 3.8. Thus it sends a inner product matrix $AA^t$ to a matrix $BB^*$ as claimed.

Finally, Theorem 4.7(b) of McMullen and Monson [13] has to be modified accordingly.
3.9. Corollary. Let $r + 1$ be the number of layers. Then

$$r + 1 = \sum_{\sigma \in \text{Irr}_G} m_\sigma + \sum_{\sigma \in \text{Irr}_G} \frac{m_\sigma(m_\sigma - 1)}{2} [\sigma, \sigma].$$

We can rewrite the right hand side of the above formula in terms of the irreducible complex characters. Recall that $m_\sigma = \langle (1_H)^G, \sigma \rangle / [\sigma, \sigma]$. Thus if $\sigma = \chi \in \text{Irr}_G$ or $\sigma = \chi + \chi$ with $\chi \neq 0$, then $m_\sigma = m_\chi = \langle (1_H)^G, \chi \rangle$, and if $\sigma = 2\chi$ with $\chi = \chi \in \text{Irr}_G$, then $m_\sigma = m_\chi / 2$. Also recall the Frobenius-Schur indicator $\nu_2(\chi) = (1 / |G|) \sum g \chi(g^2)$, which is 1, 0, and $-1$, respectively, in the three mentioned cases. Using all this, one can derive the following equation:

$$r + 1 = \frac{1}{2} \sum_{\chi \in \text{Irr}_G} m_\chi(m_\chi + \nu_2(\chi)).$$

Herman and Monson [4] derived this equation from Frame’s formula for the number of symmetric cosets. Conversely, we can derive Frame’s formula from the last equation.

We conclude this section with a discussion about what is actually wrong in McMullen’s and Monson’s proof [13]. The mistake is that the essential Wythoff space defined before Theorem 4.4 has not all the properties the authors assume (implicitly). It is in general not true that a traverse of the action of the unit complex numbers (or the unit quaternions) can be chosen as a subspace. For example, if the Wythoff space $W$ has dimension 4 over the reals and if the centralizer ring is the field $\mathbb{C}$ of complex numbers, then $W \cong \mathbb{C}^2$. Clearly, not every element of $\mathbb{C}^2$ can be written as $v \cdot z$ with $v \in \mathbb{R}^2$, $z \in \mathbb{C}$ and $|z| = 1$, for example, $(1, i)$ is not of this form. On the other hand, in the $\mathbb{R}$-linear hull of $\mathbb{R}^2 \cup \{(1, i)\}$ we have the vector $-(1, 0) + (1, i) = (0, i) = (0, 1)i$, so this is no longer a traverse for the unit complex numbers.

Of course, we can always choose a $\mathbb{D}$-basis of $W$ and then let $W^*$ be the $\mathbb{R}$-linear hull of this basis. This is what is essentially done in the proof of Theorem 4.4 in [13]. But then the sentence “The general pure polytope in $P_G$ arises from a point $\alpha_1 p_1 + \ldots + \alpha_\nu p_\nu \in W^*$” is no longer true. We should allow coefficients $\alpha_i \in \mathbb{D}$, but then different points in the Wythoff space yield congruent realizations. So the proof must be modified somehow.

This flaw in the arguments also bears upon results in the later paper [12]. Namely, in Theorem 5.2 there and the remarks before, the definition of the matrix $A$ has to be modified, allowing for entries in the centralizer ring. We may view Theorem 3.5 above as the correct version of [12, Theorem 5.2]. The $\Lambda$-orthogonal basis described in Sections 5 and 6 of [12] does not generate the full space of cosine vectors, if there is $\sigma$ with $m_\sigma > 1$ and $\mathbb{D}_\sigma \neq \mathbb{R}$, and has to be modified accordingly. (We will consider this below in Section 5.)

4. Counterexamples to a result of Herman and Monson

The main case of interest of the preceding theory is when $\Omega$ is the vertex set of an abstract regular polytope $P$ and $G$ is the automorphism group of $P$. Equivalently, $G = \langle s_0, s_1, \ldots, s_{n-1} \rangle$ is a string $C$-group and $H = \langle s_1, \ldots, s_{n-1} \rangle$ is the stabilizer
of some element of $\Omega$. By definition, this means that the generators $s_0, s_1, \ldots$ are involutions, that the *intersection property*

$$\langle s_i \mid i \in I \rangle \cap \langle s_j \mid j \in J \rangle = \langle s_k \mid k \in I \cap J \rangle$$

holds for all subsets $I, J \subseteq \{0, 1, \ldots, n-1\}$, and that $s_is_j = s_js_i$ for $|i-j| \geq 2$. Since the polytope can be recovered from the group $G$ and the distinguished generators $s_0, s_1, \ldots, s_{n-1}$ [14, Section 2E], we do not need to recall here what an abstract regular polytope actually *is*. The concepts of abstract regular polytopes and string C-groups are, in a certain sense, equivalent, and we work solely with the latter.

We now give an example which shows that we can have $m_\sigma > 1$ for $\sigma$ of complex type, even when $\Omega$ is the vertex set of an abstract regular polytope. This shows that Theorem 2 in [4] is wrong. The example is a special case of a more general construction which we will consider afterwards.

### 4.1. Example

Consider the matrices

$$S_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad S_1 = \begin{pmatrix} 0 & y \\ -y^{-1} & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \in \text{SL}(2, 19).$$

It is not difficult to see that their images $s_0, s_1$ and $s_2$ in $G := \text{PSL}(2, 19)$ generate $G$ and that $G$ is a string C-group with respect to these involutions (see Lemma 4.2 below). The element $s_1s_2$ has order 3 and thus $H = \langle s_1, s_2 \rangle \cong S_3$ has order 6. Now $G$ has an irreducible character $\chi$ of degree 9 with $\chi \not\cong \overline{\chi}$. We have $[1_H]^G \chi|_G = [1_H, \chi_H] = 2 > 1$. Thus the corresponding irreducible module over the reals has a Wythoff space of dimension 4 and essential Wythoff dimension (=multiplicity) 2. (The corresponding abstract regular polytope has Schläfli type $\{9, 3\}$.)

We are now going to show that there are in fact string C-groups with irreducible representations of complex type and arbitrary large essential Wythoff dimension. The following is probably well known:

### 4.2. Lemma

Let $F$ be a field. Let

$$S_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad S_1 = \begin{pmatrix} 0 & y \\ -y^{-1} & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \in \text{SL}(2, F),$$

where $y \neq 0, \pm 1$, $a^2 + b^2 = -1$ and $a \neq 0$. Then

$$G = \langle S_0, S_1, S_2 \rangle/\{\pm 1\} \leqslant \text{PSL}(2, F)$$

is a string C-group.

*Proof.* Let $s_i$ be the image of $S_i$ in $\text{PSL}(2, F)$. It is easily checked that $s_0, s_1$ and $s_2$ are mutually distinct involutions and that $s_0s_2 = s_2s_0$.

It remains to check the intersection property. For this, it suffices to show that

$$\langle s_0, s_1 \rangle \cap \langle s_1, s_2 \rangle = \langle s_1 \rangle = \{1, s_1\},$$

the other equalities then follow [14, Proposition 2E16]. We have

$$\langle s_0, s_1 \rangle \cap \langle s_1, s_2 \rangle = \langle s_1 \rangle C \quad \text{where} \quad C = \langle s_0s_1 \rangle \cap \langle s_1s_2 \rangle,$$

and we want to show that $C = \{1\}$. As

$$S_0S_1 = \begin{pmatrix} -y^{-1} & 0 \\ 0 & -y \end{pmatrix}, \quad y \neq y^{-1},$$
the matrix $S_0 S_1$ and its powers have eigenvectors $(1, 0)$ and $(0, 1)$. Since
\[ S_1 S_2 = \begin{pmatrix} yb & -ya \\ -ya & y^{-1} a \end{pmatrix}, \quad ya \neq 0, \]
the vectors $(1, 0)$ and $(0, 1)$ are not eigenvectors of $S_1 S_2$, but $S_1 S_2$ has an eigenvector,
possibly over an algebraic extension $\mathbb{E}$ of $\mathbb{F}$. Thus the elements of $C$ fix three different
lines in $\mathbb{E}^2$, and thus come from scalar matrices as claimed. \hfill \Box

The matrices in the last lemma have been used by Cherkassoff and Sjerve [2] to
generate $\text{PSL}(2,q)$ for $q \equiv -1 \mod 4$, $q \geq 19$. In fact, their argument shows the
following, which is sufficient for our purposes:

4.3. Lemma. In Lemma 4.2, let $\mathbb{F}$ be a field with $p$ elements, where $p$ is a prime
and $p \equiv -1 \mod 4$, and let $s_1$ be the image of $S_1$ in $\text{PSL}(2,p)$. If the order of $s_0 s_1$
or $s_1 s_2$ is $\geq 6$, then $\langle s_0, s_1, s_2 \rangle = \text{PSL}(2,p)$.

Proof. We use Dickson’s classification of the subgroups of $\text{PSL}(2,p)$ [17, Chapter 3,
Theorem 6.25]. By this classification, each proper subgroup of $\text{PSL}(2,p)$ is a subgroup
of a dihedral group, a group of affine type, which means that it is isomorphic
to a subgroup of
\[ \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid a \in \mathbb{F}^*, b \in \mathbb{F} \right\}/\{\pm 1\}, \]
or it is isomorphic to one of the groups $A_4$, $S_4$ or $A_5$.

Since $p \equiv -1 \mod 4$ and $y \neq \pm 1$, we see that $s_1$ does not commute with any
of $s_0$, $s_2$ and $s_0 s_2$. It follows (as in [2]) that $G = \langle s_0, s_1, s_2 \rangle$ is not a subgroup of a
dihedral group, since in such a group we would have $\langle s_0, s_2 \rangle \cap \mathbb{Z}(G) \neq \{1\}$.

Since $C_2 \times C_2 \cong \langle s_0, s_2 \rangle \leq G$, the group can not be of affine type, either. Since $G$
contains an element of order $\geq 6$, the exceptional cases $G \cong A_4$, $S_4$ or $A_5$ are
ruled out, too. Thus $G = \text{PSL}(2,p)$, as claimed. \hfill \Box

4.4. Lemma. If $p \equiv -1 \mod 4$, then there is $\chi \in \text{Irr}(\text{PSL}(2,p))$ such that
\[ \chi(1) = \frac{p-1}{2}, \quad \chi(g) \in \mathbb{C} \setminus \mathbb{R} \quad \text{if} \quad \text{ord}(g) = p, \]
and $\chi(g) \in \{-1, 0, 1\}$ else.

In particular, $\overline{\chi} \neq \chi$.

Proof. We show this by using the Weil representation of $\text{SL}(2,p)$, which equals the
symplectic group in dimension 2. The character $\psi$ of the Weil representation
has the property $|\psi(g)|^2 = |\text{Ker}(g-1)|$ for all $g \in \text{SL}(2,p)$, and decomposes into two
irreducible characters $\psi = \psi_+ + \psi_-$ [6, Theorem 4.8]. (See also [5] and [15] for an
elementary approach to the Weil representation.) Here $\psi_+(1) = \psi_+(1)$, so that the
kernel of $\psi_+$ contains $\{\pm 1\} = \mathbb{Z}(\text{SL}(2,p))$ and we can view $\chi = \psi_+$ as character of
$\text{PSL}(2,p)$. On the other hand, the constituent $\psi_-$ is defined by $\psi_-(1) = -\psi_-(1)$. Thus we have
$\psi(g) = \psi_+(g) + \psi_-(g)$ and $\psi(-g) = \psi_+(g) - \psi_-(g)$. It follows that
\[ \chi(g) = \psi_+(g) = \frac{1}{2}(\psi(g) + \psi(-g)). \]
In particular, $\chi(1) = (p \pm 1)/2$. For our application this is actually all we need
to know, but for completeness, let us mention that for $p \equiv -1 \mod 4$ we have
\[ \psi(-1) = -1, \text{ so } \chi(1) = (p-1)/2. \] (This follows from the known formulas for \( \psi \) \cite{18}, but is easiest seen from remarking that \( \psi(-1) \) must be even because \(-1\) is in the kernel of the determinant of \( \psi \).)

If \( g \in \text{SL}(2,p) \) has order \( p \), then \( \psi(g) = \pm \sqrt{-p} \) \cite[Corollary 6.2]{18}, and \( \psi(-g) = -1. \) (Again, we only need to know that \( |\psi(-g)| = 1 \).) Therefore, \( \chi(g) = (\pm \sqrt{-p} - 1)/2, \) and thus \( \chi(g) \neq \chi(-g) \).

If neither \( g \in \text{SL}(2,p) \) nor \(-g\) has order \( p \), then the order of \( g \) is not divisible by \( p \). In this case, \( \psi(g) \) is rational \cite[Proposition 2]{5}. Also, we have \( \text{Ker}(g-1) = \text{Ker}(g+1) = \{0\} \), except when \( g = \pm 1 \). It follows that \( \psi(g), \psi(-g) \in \{\pm 1\} \). Thus \( \chi(g) = (1/2)(\pm 1 \pm 1) \in \{-1,0,1\} \).

\[ \square \]

4.5. **Theorem.** There are abstract regular polytopes which have a pure realization of complex type with arbitrary large essential Wythoff dimension.

**Proof.** Let \( p \) be a prime such that \( p \equiv -1 \mod 4 \) and \( p \equiv 1 \mod 7 \). Choose \( y \in \mathbb{F}_p \) in Lemma 4.2 of multiplicative order 7, and let \( S_t \) and \( s_t \) be as in Lemmas 4.2 and 4.3. Then \( s_0 s_1 \) has order 7. By these lemmas, \( G = \text{PSL}(2,p) \) is a string C-group with respect to \( s_0, s_1 \) and \( s_2 \). Thus there is an abstract regular polytope with vertex set the right cosets of \( H = \langle s_0, s_1 \rangle \). (Compared with Example 4.1, the rôles of \( s_0 \) and \( s_2 \) are now interchanged.) Notice that \( H \) is a dihedral group of order \( 2 \cdot 7 = 14 \).

Let \( \chi \) be the character of Lemma 4.4 and \( S \) an irreducible module over \( \mathbb{R}G \) with character \( \chi + \overline{\chi} \). Then the essential Wythoff dimension of \( S \) is

\[
[(1_H)^G, \chi]_G = [1_H, \chi]_H \geq \frac{1}{14} \left( \frac{p-1}{2} - 13 \right) = \frac{p-1}{28} - \frac{13}{14}.
\]

Since there are infinitely primes \( p \) with \( p \equiv -1 \mod 4 \) and \( p \equiv 1 \mod 7 \) by Dirichlet’s theorem, we can make this lower bound as large as we wish. \( \square \)

The condition \( p \equiv 1 \mod 7 \) in the proof was chosen only for convenience. It is clear from the preceding lemmas that for “big” primes \( p \), us usual get a lot of possibilities of representing \( \text{PSL}(2,p) \) as a string C-group of type \( \{k,l\} \), with one or both of \( k, l \) “small”.

Checking small primes suggests that every \( \text{PSL}(2,p) \), \( 19 \leq p \equiv -1 \mod 4 \), is even a string C-group with respect to some generating set \( \{s_0, s_1, s_2\} \) such that \( s_0 s_1 \) has order 3.

In \cite[Remark 5.4]{12}, McMullen says that he has “not as yet encountered any instances with [essential Wythoff dimension] \( w > 2 \).” Of course, the examples of Theorem 4.5 are such instances. However, another example is the 120-cell. As we mentioned in Example 3.6, there are two pure realizations of the 120-cell having Wythoff space of essential dimension 3.

Even another example are the duals of the polytopes \( \mathcal{L}_p^3 \) with group \( \text{PGL}(2,p) \) \cite{10,11}. The stabilizer of a facet of \( \mathcal{L}_p^3 \) has order 6, this is the stabilizer of a vertex of the dual polytope. Since \( \text{PGL}(2,p) \) is 2-transitive on the \( p+1 \) lines of \( \mathbb{F}_p \) (in fact, sharply 3-transitive), the corresponding permutation character contains an irreducible character of degree \( p \), which has values in \( \{-1,0,1\} \) on the non-identity elements of \( \text{PGL}(2,p) \). The corresponding Wythoff space has dimension at least \( (p-5)/6 \).
5. Orthogonality

On the set of matrices $M_\Omega(\mathbb{R})$, the standard inner product is defined by

$$\langle A, B \rangle = \text{tr}(AB^t).$$

Now assume that $A = (a_{\xi\eta})$ and $B = (b_{\xi\eta})$ are $G$-invariant matrices, and fix some $\alpha \in \Omega$. Then for $\xi = \alpha g$ (say) we have

$$\sum_{\eta \in \Omega} a_{\xi\eta} b_{\xi\eta} = \sum_{\eta \in \Omega} a_{\alpha\eta} b_{\alpha\eta}. $$

Thus

$$\text{tr}(AB^t) = \sum_{\xi, \eta \in \Omega} a_{\xi\eta} b_{\xi\eta} = |\Omega| \sum_{\eta \in \Omega} a_{\alpha\eta} b_{\alpha\eta}. $$

If additionally $A$ and $B$ are symmetric (for example, $A$ and $B$ are inner product matrices of realizations of $\Omega$), then $\eta \mapsto a_{\alpha\eta} b_{\alpha\eta}$ is constant on the layers of $\Omega$. Let $\xi_0 = \alpha$, $\xi_1$, $\ldots$, $\xi_r$ be representatives of the layers and define vectors $a_i, b_i \in \mathbb{R}^{r+1}$ by $a_i = a_{\alpha\xi_i}$, $b_i = b_{\alpha\xi_i}$. Let $\ell_i$ be the size of the layer containing $\xi_i$. Then

$$\text{tr}(AB^t) = |\Omega| \sum_{\eta \in \Omega} a_{\alpha\eta} b_{\alpha\eta} = |\Omega| \sum_{i=0}^{r} \ell_i a_i b_i = |\Omega|^2 \langle a, b \rangle_\Lambda,$$

where $\langle a, b \rangle_\Lambda$ is the $\Lambda$-inner product defined by McMullen [12] for inner product vectors. So the correspondence between inner product vectors and inner product matrices identifies the $\Lambda$-inner product of McMullen with the standard inner product on matrices, up to a scalar. To maintain consistency with McMullen’s notation, we write

$$\langle A, B \rangle_\Lambda = \frac{1}{|\Omega|^2} \text{tr}(AB^t)$$

for $G$-invariant, symmetric matrices $A$ and $B$.

5.1. Theorem. If the simplex realization is written as the blend of realizations $A_1 \oplus \cdots \oplus A_s$, $A_i : \mathbb{R} \Omega \to V_i$, with inner product matrices $Q_i$, then

$$\langle Q_i, Q_j \rangle_\Lambda = \delta_{ij} \frac{\text{dim}(V_i)}{|\Omega|^2}.$$

Proof. The simplex realization is simply the identity $\text{id}: \mathbb{R} \Omega \to \mathbb{R} \Omega$. The $A_i$ are then simply the orthogonal projections onto $V_i$, as are the $Q_i = A_i A_i^t = A_i^2 = A_i$. It follows $Q_i Q_j = 0$ for $i \neq j$, and $\text{tr}(Q_i^2) = \text{tr}(Q_i) = \text{dim} V_i$. \hfill $\square$

Notice that the $A_i$'s are not normalized realizations. To normalize $A_i$, we have to scale $A_i$ by a factor $\sqrt{|\Omega|/\text{dim}(V_i)}$. So for the cosine matrices $C_i = |\Omega|/\text{dim}(V_i)$ of the $A_i$, we get $\langle C_i, C_i \rangle_\Lambda = 1/\text{dim}(V_i)$. This is in accordance with [12, Theorem 4.5].

The $\Lambda$-orthogonal basis of the realization cone which McMullen constructs in [12] is in general too small, due to the mistake in [13]. We now indicate how to repair this. We need to find orthogonal bases of the subcones $\mathcal{R}C_\sigma(\Omega)$, for each $\sigma \in \text{Irr}_R G$. For this, we have to see what the isomorphism of Theorem 3.5 does to the scalar
product. Suppose that $A$ and $B \in \text{End}_{\mathbb{R}\Omega}(\mathbb{R}\Omega)$ are such that $e_{\sigma}A = A$ and $e_{\sigma}B = B$. Choose $\mu_i$ and $\pi_i$ as in the proof of Theorem 3.5, and let $U = \mathbb{R}\Omega e_\sigma$. Then

$$\text{tr}_{\mathbb{R}\Omega}(AB^t) = \text{tr}_U(AB^t) = \text{tr}_U(\sum_i \pi_i \mu_i AB^t \sum_j \pi_j \mu_j) = \sum_i \text{tr}_S(\mu_i AB^t \pi_i) = \text{tr}_S(\sum_{i,j} a_{ij} \overline{b_{ij}}),$$

where $a_{ij} = \mu_i A \pi_j \in \mathbb{D}$ and $\overline{b_{ij}} = (b_{ij})^t = (\mu_i B \pi_j)^t = \mu_j B^t \pi_i$. Let $d = \sum_{i,j} a_{ij} \overline{b_{ij}} = \text{tr}((a_{ij})(b_{ij})^*)$. Then $\text{tr}_S(d) = (\dim_{\mathbb{R}} S)(d + \overline{d})/2$.

Thus the isomorphism of Theorem 3.5 respects the canonical inner products on the involved spaces, up to a scaling. It is now clear how to choose an orthogonal basis in the linear span of $RC_\sigma(\Omega)$. For example, if $m = 2$ and $\mathbb{D} = \mathbb{C}$, we choose matrices corresponding to

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

under the isomorphism of Theorem 3.5. Notice that the last two matrices do not correspond to realizations (they are not positive semi-definite). Also, if $m > 1$, the isomorphism of Theorem 3.5 is by no means canonical, and thus we do not get a uniquely defined basis.

### 6. Cosine vectors and spherical functions

In this section, we explain the relation between cosine vectors and spherical functions, and use it to show that the entries of a cosine vector of a realization with essential Wythoff dimension 1 are algebraic numbers. We continue to assume that $G$ is a finite group, $\Omega$ is a transitive $G$-set and $H = G_\alpha$ is the stabilizer of some fixed initial vertex $\alpha$. In the following, we set

$$e_H := e_{1_H} = \frac{1}{|H|} \sum_{h \in H} h.$$  

#### 6.1. Theorem

Let $S$ be a simple euclidean $G$-space with character $\sigma$ and with centralizer ring $\mathbb{D} = \text{End}_{\mathbb{R}G}(S)$. Let $W = \text{Fix}_S(H)$ be the Wythoff space in $S$ and let $w_1, \ldots, w_m$ be a basis of $W$ over $\mathbb{D}$ such that the following hold: We have $\langle w_i, w_i \rangle = 1$, and whenever $i \neq j$ and $d_1, d_2 \in \mathbb{D}$, then $\langle w_i d_1, w_j d_2 \rangle = 0$. Then for all $g \in G$ we have

$$\sigma(e_H g) = [\sigma, \sigma] \sum_{i=1}^m \langle w_i g, w_i \rangle.$$  

Before beginning with the proof, let us show how to construct a basis as in the theorem: Begin with some $w_1 \in W$ such that $\langle w_1, w_1 \rangle = 1$. The orthogonal complement $U$ of $w_1 \mathbb{D}$ is closed under multiplication with $\mathbb{D}$, since $\langle u d, w_1 \rangle = \langle u, w_1 \overline{d} \rangle = 0$ for $u \in U$ and $d \in \mathbb{D}$. By induction on the dimension, we find a basis in $U$ with the required properties, and thus one in $W$.

The case $m = 1$ of the theorem is worth mentioning as a separate corollary:
Corollary. Let $S$ be a simple euclidean $G$-space with character $\sigma$ and essential Wythoff dimension $m = 1$. Then for any $w \in W = \text{Fix}_S(H)$ with $\langle w, w \rangle = 1$ we have

$$\langle wg, w \rangle = \frac{\sigma(e_H g)}{[\sigma, \sigma]}.$$ 

Thus the cosine matrix of the corresponding pure realization can be expressed in terms of the character of the corresponding irreducible representation.

To put Corollary 6.2 in perspective, we recall the notions of Gelfand pairs and spherical functions. (See [8, VII.1] or [1] for more on Gelfand pairs and spherical functions.) Let $\pi$ be the permutation character of $G$ on $\Omega$ (we can think of $\Omega$ as the set of right cosets of $H$ in $G$ here). The pair $(G, H)$ is called a Gelfand pair, if $\pi$ is multiplicity free (as $G$-module over $\mathbb{C}$), that is, if $[\pi, \chi] \leq 1$ for all $\chi \in \text{Irr}(G)$.

(In our terminology, this is equivalent to all essential Wythoff dimensions being 1, and the Wythoff dimensions itself are 1 or 2.) If $[\pi, \chi] = 1$, then the corresponding spherical function $s_\chi$ is defined by

$$s_\chi(g) = \chi(e_H g) = \frac{1}{|H|} \sum_{h \in H} \chi(hg).$$

Thus Corollary 6.2 says that if $S$ is of real type, then the entries of the corresponding cosine vector are values of the spherical function $s_\chi$, and if $S$ is of complex type, then the values of the cosine vector are the real parts of the spherical function. It is well known that spherical functions can be expressed using a $G$-invariant inner product [8, VII (1.6)].

For example, it is a remarkable fact that the irreducible representations of all finite Coxeter groups are of real type, and it is another remarkable fact that the automorphism group of almost every classical regular polytope acts multiplicity freely on the vertices of the polytope; the only exception is the 120-cell. In the other cases, the cosine vectors of the pure realizations are thus the spherical functions. These cosine vectors have been computed by McMullen [9, 11, 12].

Notice that when $\pi = (1_H)^G$ has a constituent $\sigma$ of quaternion type, then $(G, H)$ can not be a Gelfand pair, since then $\sigma = 2\chi$ and $[(1_H)^G, \chi]$ is a multiple of 2. We may say that $(G, H)$ is a Gelfand pair over $\mathbb{R}$, if $m_\sigma \in \{0, 1\}$ for $\sigma \in \text{Irr}_\mathbb{R} G$, that is, all essential Wythoff dimensions are 0 or 1.

Proof of Theorem 6.1. Suppose $d = -d$ for $d \in \mathbb{D}$. Then $\langle vd, v \rangle = \langle v, vd \rangle = -\langle v, vd \rangle = -\langle vd, v \rangle$ and thus $\langle vd, v \rangle = 0$. We now choose a basis $B$ of $\mathbb{D}$ over $\mathbb{R}$. If $\mathbb{D} = \mathbb{R}$, we choose $B = \{1\}$, if $\mathbb{D} = \mathbb{C}$, we choose $B = \{1, i\}$, and if $\mathbb{D} = \mathbb{H}$, we choose $B = \{1, i, j, k\}$. In each case, it follows that $\langle vb, vc \rangle = 0$ for $b \neq c \in B$ and $\langle vb, wb \rangle = \langle v, w \rangle$. Thus $\{w_i b \mid i = 1, \ldots, m, b \in B\}$ is an orthonormal basis of $W$ over $\mathbb{R}$. Extend this basis by some set $X$ (say) to an orthonormal basis of the whole space $S$. For any $\mathbb{R}$-linear map $\alpha : S \to S$ we have

$$\text{tr}(\alpha) = \sum_{i,b} \langle w_i b \alpha, w_i b \rangle + \sum_{x \in X} \langle x \alpha, x \rangle.$$
We apply this to the map induced by $e_H g$. Since $xe_H = 0$ for $x \not\in W$ and $we_H = w$ for $w \in W$, we get

$$\sigma(e_H g) = \text{tr}(e_H g) = \sum_{i=1}^{m} \sum_{b \in B} \langle w_i b e_H g, w_i b \rangle = \sum_{i=1}^{m} \sum_{b \in B} \langle w_i g b, w_i b \rangle$$

$$= \sum_{i=1}^{m} \sum_{b \in B} \langle w_i g, w_i \rangle$$

$$= \abs{B} \sum_{i=1}^{m} \langle w_i g, w_i \rangle$$

$$= \abs{\sigma} \sum_{i=1}^{m} \langle w_i g, w_i \rangle,$$

as claimed. \qed

It follows from Corollary 6.2 that the values of the cosine vector are algebraic numbers, if $m = 1$. This confirms a “guess” of McMullen \[12, \text{Remark 9.4}\]. We can say somewhat more: It is known \[8, \text{VII(1.10)}\] that

$$\frac{|HgH \cup Hg^{-1}H|}{|H|} \langle wg, w \rangle$$

is an algebraic integer.

Notice that $|HgH \cup Hg^{-1}H|/|H|$ is the size of the corresponding layer. Another formulation of the corollary is thus: the component-wise product of a cosine vector of a pure realization of essential Wythoff dimension 1 with the layer vector has algebraic integers as entries.

**Proof.** For each double coset $K = HgH$, let

$$e_K = \frac{1}{|H|} \sum_{x \in K} x \in \mathbb{R}G.$$

It is known \[8, \text{remarks before VII(1.10)}\] that the product of two such elements is a $\mathbb{Z}$-linear combination of these elements. Thus $\mathbb{Z}[e_K | K \in H \setminus G/H]$ is a ring which is finitely generated as $\mathbb{Z}$-module, so its elements are integral.

Let $W = S e_H \cong \mathbb{D} = \text{End}_{\mathbb{R}G}(S)$ be the Wythoff space. Then $e_K = e_{HgH}$ acts as some $\mathbb{D}$-linear map on $W$, and can thus be identified with some $d \in \mathbb{D}$. Then $e_K + e_{K^{-1}} = e_{HgH} + e_{Hg^{-1}H}$ acts as the scalar $\lambda = d + d$ on $W$. Since $e_K$ is integral over $\mathbb{Z}$, it follows that $d$ and $\lambda$ are integral over $\mathbb{Z}$. In the case where $d \in \mathbb{R}$ we have

$$d = \frac{\sigma(e_K)}{[\sigma, \sigma]} = \frac{\sigma(e_{HgH})}{[\sigma, \sigma]} = \frac{1}{|H|} \sum_{x \in K} \langle wx, w \rangle = \frac{|K|}{|H|} \langle wg, w \rangle,$$

and in any case we have

$$\lambda = \frac{\sigma(e_K + e_{K^{-1}})}{[\sigma, \sigma]} = \frac{2 \sigma(e_K)}{[\sigma, \sigma]} = 2 \frac{|K|}{|H|} \langle wg, w \rangle.$$
7. On the realizations of the 600-cell

In this section we explain two observations of McMullen [11, Remark 9.3] about the pure realizations of the 600-cell. Namely, we have the following:

7.1. Theorem. There is a “natural” bijection between the irreducible characters of the finite group $\text{SL}(2, 5)$ and the pure realizations of the 600-cell. If $\varphi \in \text{Irr}(\text{SL}(2, 5))$, then the corresponding pure realization has dimension $\varphi(1)^2$, and the entries of its cosine vector are of the form $\varphi(u)/\varphi(1)$, where $u$ runs through $\text{SL}(2, 5)$. (More precisely, we also have a natural bijection between the conjugacy classes of $\text{SL}(2, 5)$ and the layers of the 600-cell, and $\varphi(u)/\varphi(1)$ is the value at the layer corresponding to the conjugacy class of $u$.)

This “explains” that the dimension of each pure realization is a square $q^2$, and that its cosine vector has entries of the form $a/q$, where $a$ is an algebraic integer (in fact, $a \in \mathbb{Z}[\tau]$ with $\tau = (-1 + \sqrt{5})/2$).

We have to warn the reader that the proof of Theorem 7.1, while not difficult, is rather long, in particular longer than working out the cosine vectors directly. On the other hand, we work out the realization cone of a class of G-sets, of which the 600-cell is an example.

We will use that the automorphism group of the 600-cell, the reflection group of type $H_4$, is the factor group of a certain wreath product: Let $U$ be a group. The cyclic group $C_2 = \{1, t\}$ of order 2 acts on the direct product $U \times U$ by exchanging components, that is $(u, v)^t = (v, u)$. The corresponding semidirect product of $C_2$ and $U \times U$ is the wreath product, denoted by $U \wr C_2$. The following lemma is of course known, but for completeness, we work out a large part of the proof:

7.2. Lemma. Set $U = \text{SL}(2, 5)$ and $\hat{G} = U \wr C_2$, and let $\hat{H}$ be the subgroup of $\hat{G}$ generated by the pairs $\{(u, u) \mid u \in U\}$ and by $C_2$. (Notice that $\hat{H} \cong C_2 \wr U$.) The automorphism group of the 600-cell is isomorphic to the factor group $\hat{G}/\mathbb{Z}(\hat{G})$ in such a way that the stabilizer of a vertex is identified with $\hat{H}/\mathbb{Z}(\hat{G})$.

Proof. We can express the automorphism group of the 600-cell as a group of transformations on the quaternions $\mathbb{H}$. For $u \in \mathbb{H}$, let $\lambda_u : \mathbb{H} \to \mathbb{H}$ and $\rho_u : \mathbb{H} \to \mathbb{H}$ be the maps defined by

$$ x\lambda_u = ux \quad \text{and} \quad x\rho_u = xu \quad (x \in \mathbb{H}). $$

Let $\sigma : \mathbb{H} \to \mathbb{H}$ be conjugation.

Let $U$ be a (finite) subgroup of the multiplicative group $\mathbb{H}^*$. Mapping $t$ to $\sigma$ and $(u, v) \in U \times U$ to $\lambda_u\rho_v$ defines a group homomorphism from $U \wr C_2$ into $\text{GL}_\mathbb{R}(\mathbb{H}) \cong \text{GL}(4, \mathbb{R})$. The kernel is $\langle (-1, -1) \rangle \subseteq U \times U$.

The reflection group of type $H_4$ can be realized as the image of such a homomorphism: Let

$$ \alpha_1 = j, \quad \alpha_2 = \frac{1}{2}(ai + bj - k), \quad \alpha_3 = k, \quad \alpha_4 = \frac{1}{2}(a + bi - k), $$

where $a = 2 \cos(2\pi/5) = (-1 + \sqrt{5})/2$ and $b = 2 \cos(4\pi/5) = (-1 - \sqrt{5})/2$. Then $\alpha_1, \ldots, \alpha_4$ form a simple root system of type $H_4$.\hfill \square
Let $s_1, \ldots, s_4$ be the reflections corresponding to $\alpha_1, \ldots, \alpha_4$. These generate the automorphism group $G$ of the 600-cell, and the stabilizer of a vertex is $H = \langle s_1, s_2, s_3 \rangle$. (The vertices are all points in the orbit of $1 = 1_{\mathbb{H}}$.)

The reflection corresponding to an element $\alpha \in \mathbb{H}$ of norm 1 is the map

$$x \mapsto -\alpha x \alpha = x \sigma \lambda_{-\pi} \rho_\alpha,$$

as is easily checked (it sends $\alpha$ to $-\alpha$ and fixes $i\alpha$, $j\alpha$ and $k\alpha$). It follows that

$$\langle s_1, s_2, s_3, s_4 \rangle \subseteq \{ \text{id}_H, \sigma \} \{ \lambda_u \rho_v \mid u, v \in U \},$$

where $U$ is the group generated by $\alpha_1, \ldots, \alpha_4$ and $-1$.

Since $\alpha_4^2 = -1$ and $\alpha_4 = (\alpha_1 \alpha_2)^2$, we see that $U = \langle \alpha_1, \alpha_2, \alpha_3 \rangle$. We see that the reflections $s_1, s_2, s_3$ generate the subgroup

$$H = \{ \text{id}_H, \sigma \} \{ \lambda_u \rho_v \mid u, v \in U \}.$$

Then it is also not difficult to see that

$$\langle s_1, s_2, s_3, s_4 \rangle = \{ \text{id}_H, \sigma \} \{ \lambda_u \rho_v \mid u, v \in U \}.$$

We leave out the proof that $U \cong \mathrm{SL}(2, 5)$. Apart from this, the lemma is proved. \qed

We now slightly change notation. Let $U$ be an arbitrary finite group, let $G$ be the wreath product $U \wr C_2$ and let $H \leq G$ be the subgroup

$$H = \{1, t\} \{(u, u) \mid u \in U\} \cong C_2 \times U.$$

We will describe the realization cone of the $G$-set $[G : H]$ (the cosets of $H$ in $G$) for such $G$ and $H$.

Set $N = U \times U$, a normal subgroup of $G$ of index 2. The irreducible characters of $N$ are of the form $\varphi \times \vartheta$ with $\varphi, \vartheta \in \text{Irr} U$ [16, Theorem III.9.1].

**Lemma.**

(i) If $\varphi \neq \vartheta \in \text{Irr} U$, then $\langle \varphi \times \vartheta \rangle^G \in \text{Irr} G$.

(ii) For $\varphi \in \text{Irr} U$, the character $\varphi \times \varphi$ has exactly two extensions to a character of $G$, namely

$$\chi(t(u, v)) = \varphi(uv) \quad \text{and} \quad \chi(t(u, v)) = -\varphi(uv).$$

**Proof.** The first point is clear from Clifford theory ($\langle \varphi \times \vartheta \rangle^G$ denotes the Frobenius induced character).

It is also known that $\varphi \times \varphi$ has two different extensions to $G$ [16, III.11]. Here, we can describe these extensions explicitly. Let $X$ be a $CU$-module affording the character $\varphi$. We may define an action of $t$ on $X \otimes X$ by $(x \otimes y)t = y \otimes x$ or $(x \otimes y)t = -y \otimes x$. These are the two extensions to a representation of $G$.

We treat the first case. Then

$$(x \otimes y)t(u, v) = yu \otimes xv.$$

Suppose that $\{e_i\}$ is a basis of $X$ and $e_iu = \sum d_{ij}(u)e_j$. Then $\{e_i \otimes e_j\}$ is a basis of $X \otimes X$, and we get for the trace of $t(u, v)$ on $X \otimes X$:

$$\chi(t(u, v)) = \sum_{i,j} d_{ji}(u)d_{ij}(v) = \sum_j d_{jj}(uv) = \varphi(uv). \quad \Box$$

Let $U, G, H$ and $N$ be as defined before the last lemma.
7.4. Lemma. If $\chi \in \text{Irr } G$ with $[\chi_H, 1] \neq 0$, then either $\chi = (\varphi \times \overline{\varphi})^G$ with $\varphi \neq \overline{\varphi} \in \text{Irr } U$, or $\chi_N = \varphi \times \varphi$ with $\varphi = \overline{\varphi} \in \text{Irr } U$ and $\chi(\sigma(u, v)) = \nu_2(\varphi)\varphi(uv)$. In both cases, $[\chi_H, 1] = 1$.

(Here $\nu_2(\varphi)$ denotes the Frobenius-Schur indicator of $\varphi$. Recall that for $\varphi \in \text{Irr } U$,

$$\nu_2(\varphi) = \frac{1}{2|U|} \sum_{u \in U} \varphi(u^2) \in \{0, \pm 1\},$$

and $\nu_2(\varphi) \neq 0$ if and only if $\varphi = \overline{\varphi}$ [16, Theorem III.5.1].)

Lemma 7.4 explains the first part of Theorem 7.1. Since $U = \text{SL}(2, 5)$ has only real-valued characters, every pure realization corresponds to a $\varphi \in \text{Irr } U$ and has dimension $\varphi(1)^2$.

In the general case, notice that the realizations correspond to $\text{Irr}_\mathbb{R} U$. The Wythoff dimension is 1 for all pure realizations. (In particular, the corresponding irreducible representations are of real type.) Thus the realization cone is polyhedral, in fact a direct product of copies of $\mathbb{R}_{\geq 0}$ by Theorem 3.5.

Proof of Lemma 7.4. First, suppose that $\chi = (\varphi \times \vartheta)^G$ with $\varphi \neq \vartheta \in \text{Irr } U$. Then

$$[\chi_H, 1_H] = \left[\left((\varphi \times \vartheta)^G\right)_H, 1_H\right] = \left[\left((\varphi \times \vartheta)_{H \cap N}\right)^H, 1_H\right] = \left[(\varphi \times \vartheta)_{H \cap N}, 1_{H \cap N}\right] = \frac{1}{|U|} \sum_{u \in U} \varphi(u)\vartheta(u) = [\varphi, \overline{\vartheta}] = \delta_{\varphi, \overline{\vartheta}}.$$

Here the second equality follows from $G = HN$ and Mackey’s formula, and the third equality follows from Frobenius reciprocity. Thus $\vartheta = \overline{\varphi}$ when $[\chi_H, 1_H] \neq 0$.

Second, suppose that $\chi$ extends $\varphi \times \varphi$, and that $\chi(t(u, v)) = \varepsilon \varphi(uv)$. Then

$$[\chi_H, 1_H] = \frac{1}{2|U|} \sum_{u \in U} \left(\chi((u, u)) + \chi(t(u, u))\right) = \frac{1}{2|U|} \left(\sum_{u \in U} \varphi(u)^2 + \sum_{u \in U} \varepsilon \varphi(u^2)\right) = \frac{1}{2}(\langle \varphi, \overline{\varphi} \rangle + \varepsilon \nu_2(\varphi)).$$

The last expression is non-zero only when $\varphi = \overline{\varphi}$ and $\varepsilon = \nu_2(\varphi)$, and in this case $[\chi_H, 1] = 1$. 

The next result finishes the proof of Theorem 7.1. As in the last results, we only assume that $G = U \wr C_2$ for some finite group $U$, and that $H = C_2\{ (u, u) \mid u \in U \}$. We notice in passing that in this situation,

$$Ht(x, y)H = H(x, y)H \leftrightarrow (x^{-1}y)^U \cup (y^{-1}x)^U$$

defines a bijection between double cosets of $H$ and “symmetrized” conjugacy classes of $U$. The double cosets of $H$ in turn correspond to the layers. (If $U = \text{SL}(2, 5)$, then all conjugacy classes of $U$ are real, that is, $u$ and $u^{-1}$ are always conjugate.) The following lemma describes an arbitrary entry of a cosine vector of a pure realization.
7.5. Lemma. Let $V$ be an irreducible euclidean $G$-space and suppose the non-zero element $w \in V$ is fixed by $H$. Then the character $\chi$ of $V$ is irreducible. Let $\varphi \in \text{Irr} \ U$ be the character defined in Lemma 7.4. Let $n = (x, y) \in N = U \times U$. Then

$$\langle wn, w \rangle = \varphi(x^{-1}y) / \varphi(1).$$

Proof. Since $w \neq 0$ is fixed by $H$, we have $[\chi_H, 1_H] \neq 0$. It follows from Lemma 7.4 that $[\chi_H, 1_H] = 1$, and $\chi$ is as in that lemma. We may assume that $\langle w, w \rangle = 1$ and apply Corollary 6.2. We only treat the case that $\chi_N = \varphi \times \varphi$. (The case $\chi = (\varphi \times \overline{\varphi})^G$ is similar, but in fact simpler.) We get

$$\langle wn, w \rangle = \chi(e_{hn}) = \frac{1}{2|U|} \left( \sum_{u \in U} \chi(ux, uy) + \sum_{u \in U} \chi(tx, uy) \right)$$

$$= \frac{1}{2|U|} \left( \sum_{u \in U} \varphi(ux) \varphi(uy) + \sum_{u \in U} \nu_2(\varphi) \varphi(uxuy) \right).$$

The first sum equals $|U|\varphi(x^{-1}y) / \varphi(1)$ by the generalized orthogonality relation [7, Theorem 2.13] and the fact that $\varphi(uy) = \varphi(uy) = \varphi(y^{-1}u^{-1})$. For the second sum, we get

$$\frac{1}{|U|} \sum_{u \in U} \varphi(uxuy) = \frac{1}{|U|} \varphi \left( \sum_{u \in U} v^2 x^{-1}y \right) = \varphi(zx^{-1}y),$$

where $z = (1/|U|) \sum_{u \in U} v^2$ is a central element in the group algebra and is mapped to a scalar matrix by any irreducible representation. Thus $\varphi(zx^{-1}y) = (\varphi(z)/\varphi(1)) \varphi(x^{-1}y)$. But clearly, $\varphi(z) = \nu_2(\varphi)$. Plugging in above, we get that $\chi(e_{hn}) = \varphi(x^{-1}y)/\varphi(1)$ as claimed. \qed

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