HARMONIC ANALYSIS ON DOUBLE COSET SPACES

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Abstract. We prove the existence and uniqueness of quasi-invariant measure on double coset space $K\backslash G/H$ and study the Fourier and Fourier-Stieltjes algebras of these spaces

1. Introduction

Let $G$ be a locally compact group and $H,K$ be closed subgroups of $G$. The double coset space of $G$ by $H$ and $K$ is

$$K\backslash G/H = \{KxH : x \in G\}.$$

When $K = \{1\}$, this is the coset space $G/H$ and when $K = H$, it is the double coset space $G//H$ of $G$ by $H$. Although these are not groups when $H$ and $K$ are not normal, but a major part of the Harmonic analysis on $G$ carries over these spaces. Among these the coset space $G/H$ is quite well studied by several authors (see for instance [Fl], [RS], [Fo], [Sk] and references therein). Some part of these works has been carried over the double coset spaces [Li]. One important point, which is partly overlooked, is the fact that the double coset space $G//H$ is a hypergroup (convol) and the coset space $G/H$ is a semi hypergroup (semi convol) [Je].

In this paper we study double coset spaces. In particular we are interested in studying the function algebras on these spaces. One major motivation for this study is the fact that for a general hypergroup, the set of positive definite functions is not closed under pointwise multiplication [Vr]. In particular the Fourier and Fourier-Stieltjes spaces fail to be Banach algebras in general. Since there is no involution on $G/H$, the concept of positive definiteness could not be defined directly on this semi hypergroup (there are some alternative definitions, see for instance [Ar], [Fo]). The double coset space has neither of these pathologies. It has an involution and positive definite functions could be defined on it and they have the natural relation with their counterparts on $G$ [Je]. In section 4 we study the Fourier and Fourier-Stieltjes spaces over $G//H$. Also along the line of [Fo], we consider the corresponding subalgebras of the Fourier and Fourier-Stieltjes algebras over $G$ consisting of functions which are constant on double cosets of $H$.

The paper is organized as follows. In section 2 we study the (quasi) invariant measures on $K\backslash G/H$ and the analog of the Weil’s formula. The results of this section are mainly contained in [Li], but as our results are slightly more general and our approach is different at some points, and also because we later use some of the technical lemmas in this section which are not included in [Li], we preferred to present the details of proofs. Section 3 is a brief account of the theory of induced
representations (Mackey machine) for $K\backslash G/H$. The first part includes new results along the line of the classical theory for $G/H$. The second part considers the induction procedure from the subhypergroups of $G//H$. The results of this part are taken from [HKK] and [He], where these results are given in a more general setting. We only quote those results which are used in the next section. In section 4 we study function spaces on $K\backslash G/H$ and $G//H$. In particular we are interested in different versions of Banach spaces defined using positive definite functions. The proofs in this section partly follow the same line of reasoning as their coset space counterparts in [Fo]. But there are technical difficulties which force us to present the details of the proofs and show the spots where the arguments break down. There is however a major difference. In contrast with the case of $G/H$, we have an involution structure on $G//H$ and so we are able to define the function spaces directly on $G//H$, whereas in [Fo], the function (and operator) algebras on $G/H$ are defined indirectly using the corresponding appropriate subalgebras of the same algebras on $G$. Therefore from the hypergroup point of view, our study gives more direct information about the structure of $G//H$.

Our basic references for the harmonic analysis on $G$ are [Fl] and [Ey] and we use their now classical, notations and results without reference.

2. **Quasi-invariant measures**

Let $G$ be locally compact group and $H,K$ be closed subgroups of $G$ and $dx, dh,$ and $dk$ denote the Haar measures on them, respectively. If either of these groups are compact we assume that the corresponding Haar measure is normalized so that the measure of the group is one. Let $\Delta_G$, $\Delta_H$, and $\Delta_K$ denote the corresponding modular functions. The double quotient space of $G$ by $H$ and $K$ is denoted by $K\backslash G/H$ and is defined by

$$K\backslash G/H = \{KxH : x \in G\}.$$

Consider the map $q : G \to K\backslash G/H$, defined by $q(x) = KxH = : \bar{x}$ ($x \in G$), then the quotient topology of $K\backslash G/H$ is the weakest topology on $K\backslash G/H$ which makes $q$ continuous. This makes $K\backslash G/H$ a locally compact space on which $G$ acts by translations. For $f \in C_c(G)$ define

$$Qf(KxH) = \int_K \int_H f(kxh)\,dkdh,$$

then $Q : C_c(G) \to C_c(K\backslash G/H)$ and $\text{supp}(Qf) \subseteq q(\text{supp}(f))$ ($f \in C_c(G)$). Note that, by Fubini’s theorem, the order of integration in above formula is not important. The following lemma is trivial.

**Lemma 2.1.** For each $f \in C_c(G)$ and $\varphi \in C_c(K\backslash G/H)$

$$Q((\varphi \circ q).f) = \varphi.Qf.$$  \hfill \Box

**Lemma 2.2.** If $E \subseteq K\backslash G/H$ is compact then there is a compact set $F \subseteq G$ with $q(F) = E$.

**Proof** Let $V$ be a neighbourhood of identity in $G$ with compact closure and cover $E$ by the sets $q(xV)$, $x \in G$. Choose a subcover $\{q(x_jV)\}_{j=1}^n$ and put $F = q^{-1}(E) \cap (\cup_{j=1}^n x_jV)$. \hfill \Box
Lemma 2.3. If \( F \subseteq K \backslash G/H \) is compact then there is \( f \in C_c(G)_+ \) such that \( Qf = 1 \) on \( F \).

Proof Let \( E \) be a compact neighbourhood of \( F \) in \( K \backslash G/H \) and choose by above lemma a compact set \( F_1 \) in \( G \) such that \( q(F_1) = E \). Choose \( g \in C_c(G)_+ \) such that \( g > 0 \) on \( F_1 \) and using Uryshon’s Lemma, choose \( \varphi \in C_c(K \backslash G/H) \) with \( \text{supp}(\varphi) \subseteq E \) and \( \varphi = 1 \) on \( F \). Put \( f = \frac{g}{QgQg} \), then since \( Qg > 0 \) on \( \text{supp}(\varphi) \), we have \( f \in C_c(G) \) and clearly \( \text{supp}(f) \subseteq \text{supp}(g) \). Also by Lemma 2.1, \( Qf = Q(\varphi \circ q) \cdot \frac{g}{QgQg} = \varphi. \)

Proposition 2.1. If \( \varphi \in C_c(K \backslash G/H) \) then there is \( f \in C_c(G) \) such that \( Qf = \varphi \) and \( q(\text{supp}(f)) \subseteq \text{supp}(\varphi) \). Moreover if \( \varphi \geq 0 \) we may choose \( f \geq 0 \).

Proof Given \( \varphi \in C_c(K \backslash G/H) \), there is \( g \in C_c(G)_+ \) such that \( Qg = 1 \) on \( \text{supp}(\varphi) \). Put \( f = (\varphi \circ q)g \) and use Lemma 2.1. \( \square \)

Proposition 2.2. If \( \nu \) is a measure on \( G \) satisfying

\[
\int_G f(k^{-1}xh^{-1})d\nu(x) = \Delta_K(k)\Delta_H(h)\int_G f(x)d\nu(x) \quad (k \in K, h \in H, f \in C_c(G)),
\]

then there is a unique measure \( \mu = \hat{\nu} \) on \( K \backslash G/H \) such that

\[
\int_G f(x)d\nu(x) = \int_{K \backslash G/H} Qf(\varphi(x))d\hat{\nu}(\varphi(x)) = \int_{K \backslash G/H} \int_H f(kxh)d\hat{\mu}(kxh) \quad (f \in C_c(G)).
\]

Proof Fix \( f \in C_c(G) \), then for \( g \in C_c(G) \) we have

\[
\int_G f(x)g(\hat{x})d\nu(x) = \int_G f(x)\int_K \int_H g(kxh)d\hat{\nu}(kxh)d\nu(x) = \int_K \int_H \int_G f(x)g(kxh)d\hat{\nu}(x)d\nu(x) = \int_K \int_H \Delta_K(k^{-1})\Delta_H(h^{-1})\int_G f(k^{-1}xh^{-1})g(x)d\nu(x)d\nu(x) = \int_G g(x)\int_K \int_H f(kxh)d\hat{\nu}(x)d\nu(x) = \int_G g(x)Qf(\hat{x})d\nu(x).
\]

Now using Lemma 2.3, choose \( g \) such that \( Qg = 1 \) on \( q(\text{supp}(f)) \), then if \( Qf = 0 \) we have

\[
\int_G f(x)d\nu(x) = \int_G f(x)Qg(\hat{x})d\nu(x) = \int_G g(x)Qf(\hat{x})d\nu(x) = 0,
\]

so the map \( \hat{\nu}(Qf) = \int_G f d\nu \) is well defined on \( C_c(K \backslash G/H) \). Given compact subset \( E \) of \( K \backslash G/H \), by Lemma 2.2 choose a compact subset \( F \) of \( G \) such that \( q(F) = E \) and then, by Proposition 2.1, choose \( h \in C_c(G) \) supported in \( F \) such that \( Qh = 1 \) on \( E \). Put \( f' = (Qf \circ q)h \), then \( f' \) is supported in \( F \), \( Qf' = Qf \), and

\[
|\hat{\nu}(Qf)| = |\hat{\nu}(Qf')| = |\int_F f' d\nu| = |\int_F f' d\nu| \leq \nu(F)\|f'\|_{\infty} \leq \nu(F)\|h\|_{\infty}\|Qf\|_{\infty}.
\]
so \( \check{\nu} \) is a bounded linear functional on \( C_c(K \backslash G/H) \) and the corresponding measure \( \mu \) has the desired property. The uniqueness follows from Proposition 2.1.

The above condition leads us to the concept of generalized Bruhat function \( \rho \) as described in the next proposition. We would deal with the problem of existence of such functions later.

**Proposition 2.3.** If \( \rho : G \to \mathbb{C} \) is a continuous function, then \( d\nu(x) = \rho(x)dx \) satisfies the condition of above proposition if and only if

\[
\rho(kxh) = \rho(x) \frac{\Delta_K(k) \Delta_H(h)}{\Delta_G(h)} \quad (k \in K, h \in H, x \in G, f \in C_c(G)).
\]

**Proof** The condition of the above proposition for above \( \nu \) could be written as

\[
\int_G f(k^{-1}xh^{-1}) \rho(x) dx = \Delta_K(k) \Delta_H(h) \int_G f(x) \rho(x) dx,
\]

so the result follows from the following calculation

\[
\int_G f(k^{-1}xh^{-1}) \rho(x) dx = \Delta_G(h) \int_K \int_H \int_K \int_H f(kxh) dkdhd\mu(\check{x}) \quad (f \in C_c(G)).
\]

Now we are ready to prove a version of Weil’s formula for double coset spaces.

**Theorem 2.1.** Assume that \( K \) is unimodular. There is a \( G \)-invariant Radon measure \( \mu \) on \( K \backslash G/H \) if and only if \( \Delta_G \upharpoonright H = \Delta_H \). In this case \( \mu \) is unique up to constant factors and if suitably chosen then

\[
\int_G f(x) dx = \int_{K \backslash G/H} \int_K \int_H f(kxh) dkd\mu(\check{x}) \quad (f \in C_c(G)).
\]

**Proof** If \( \mu \) exists then the uniqueness and above relation follow from the fact that the (left) Haar measure on \( G \) (up to constant factors) and the fact that \( f \mapsto \int_{K \backslash G/H} Qf d\mu \) is clearly a left invariant positive linear functional on \( C_c(G) \). Now again if \( \mu \) exists, then for each \( k_0 \in K, h_0 \in H \) and \( f \in C_c(G) \) we have

\[
\Delta_G(h_0) \int_G f(x) dx = \int_G f(k_0^{-1}xh_0^{-1}) dx = \int_{K \backslash G/H} \int_K \int_H f(k_0^{-1}xkh_0^{-1}) dkd\mu(\check{x}) = \Delta_H(h_0) \int_{K \backslash G/H} \int_K \int_H f(xh) dkd\mu(\check{x}) = \Delta_H(h_0) \int_G f(x) dx.
\]

Therefore, choosing \( f \) appropriately, we get \( \Delta_G(h_0) = \Delta_H(h_0) \). Conversely suppose that \( \Delta_G \upharpoonright H = \Delta_H \), then the result follows from Propositions 2.2 (with \( \rho = 1 \)) and 2.1.

**Corollary 2.1.** If \( K \) and \( H \) are compact then there is a \( G \)-invariant Radon measure on \( K \backslash G/H \).
When \( K = \{1\} \), the above theorem gives the well-known Wiel’s formula [Fl, 2.49]. When \( K = H \) we get the following result.

**Corollary 2.2.** If \( H \) be a compact subgroup of \( G \), then there is a \( G \)-invariant Radon measure \( \mu \) on \( G//H \). In this case \( \mu \) is unique up to constant factors and if suitably chosen then

\[
\int_G f(x)dx = \int_{G//H} \int_H \int_H f(kxh)dkdhd\mu(x) \quad (f \in C_c(G)). \tag*{\Box}
\]

3. **Mackey Machine**

There is a well-developed theory of induced representations from a closed subgroup \( H \) to \( G \) using the "intermediate space" \( G/H \), usually referred to as the Mackey Machine (see for instance [Fl]). Here we add one more ingredient to this construction, namely a compact subgroup \( K \). Then what we get is induced representations from \( H \) to \( N_G(K) \), the normalizer of \( K \) in \( G \), with the intermediate space \( K\backslash G/H \). When \( K = 1 \), this reduces to the usual Mackey machine, so it is reasonable to call it a generalized Mackey machine. To get the induced representation we follow the same construction as in the classical case [Fl].

The induction process related to \( K\backslash G/H \) could be studied from another point of view. This is when we consider \( K\backslash G/H \) as a semi convo and try to induce a representation on \( K\backslash G/H \) from a sub semi convo. When \( K = H \), this has been studied in a more general framework [He], [HKK]. We briefly mention their results at the end of this section, as we need some of them in the next section.

Let \( K \) and \( H \) be compact and closed subgroups of \( G \), respectively. Let \( q : G \to K\backslash G/H \) be the quotient map. Consider a unitary representation \( \{\sigma, H_{\sigma}\} \) of \( H \). Let \( \mathfrak{F}_0 \) be the set of all elements \( f \in C(G, \mathcal{S}_{\sigma}) \) such that \( q(\text{supp}(f)) \) is compact in \( K\backslash G/H \) and

\[
f(kxh) = \sigma(h^{-1})(f(x)) \quad (k \in K, h \in H, x \in G).
\]

**Proposition 3.1.** If \( \alpha \in C_c(G, \mathcal{S}_{\sigma}) \) then

\[
f_\alpha(x) = \int_K \int_H \sigma(h)\alpha(kxh)dkdh \quad (x \in G)
\]

is uniformly continuous on \( G \) and \( \alpha \mapsto f_\alpha \) maps \( C_c(G, \mathcal{S}_{\sigma}) \) onto \( \mathfrak{F}_0 \).

**Proof** Clearly \( q(\text{supp}(f_\alpha)) \) is included in \( q(\text{supp}(\alpha)) \), and so it is compact. Also for each \( h_0 \in H \) we have

\[
f_\alpha(k_0xh_0) = \int_K \int_H \sigma(h)\alpha(kk_0xh_0h)dkdh
\]

\[
= \int_K \int_H \sigma(h_0^{-1}h)\alpha(kxh)dkdh
\]

\[
= \sigma(h_0^{-1})\int_K \int_H \sigma(h)\alpha(kxh)dkdh
\]

\[
= \sigma(h_0^{-1})f_\alpha(x).
\]

Also since \( K \) is compact, the function

\[
\alpha'(x) = \int_K \alpha(kx)dk \quad (x \in G)
\]
is continuous of compact support, and so by [Fl, 6.1],
\[ f_\alpha(x) = \int_H \sigma(h)\alpha'(xh)dh \]
is uniformly continuous on \( G \). Finally if \( f \in \mathfrak{G}_0 \), then by Lemma 2.1 there is \( \psi \in C_c(G) \) with \( Q(\psi) = 1 \) on \( \text{supp}(f) \), and if we put \( \alpha = \psi f \), then
\[
\begin{align*}
f_\alpha(x) &= \int_K \int_H \psi(kxh)\sigma(h)f(kxh)dkdh \\
&= \int_K \int_H \psi(kxh)\sigma(h)\sigma(h^{-1})f(x)dkdh \\
&= f(x)Q(\psi)(x) = f(x). \quad \square
\end{align*}
\]

Now let \( N = N_G(K) = \{ x \in G : x^{-1}Kx = K \} \) be the normalizer of \( K \) in \( G \), then \( N \) acts on \( \mathfrak{G}_0 \) by left translations.

**Lemma 3.1.** For each \( x \in N \) and \( f \in \mathfrak{G}_0 \), we have \( L_x(f) \in \mathfrak{G}_0 \) and \( f \mapsto L_x(f) \) is a bijective linear map.

**Proof** Given \( f \in \mathfrak{G}_0 \), the set \( q(\text{supp}(L_x(f))) = L_q(x)(\text{supp}(f)) \) is compact, since \( L_q(x) \) is continuous. Also for each \( k \in K, h \in H, y \in N, x \in G \) we have
\[
L_y(f)(kxh) = f(y^{-1}kxh) = f((y^{-1}ky)y^{-1}xh) = \sigma(h^{-1})f(y^{-1}x) = \sigma(h^{-1})L_y(f)(x),
\]
so \( \mathfrak{G}_0 \) is stable under left translation by \( G \). The other assertions are trivial. \( \square \)

Now let us assume that \( \Delta_G \mid_H = \Delta_H \), so that there is a \( G \)-invariant measure \( \mu \) on \( K \backslash G / H \). Let \( f, g \in \mathfrak{G}_0 \), consider \( h(x) = \langle f(x), g(x) \rangle_\sigma \quad (x \in G) \), then for each \( k \in K, h \in H, x \in G \)
\[
h(kxh) = \langle f(kxh), g(kxh) \rangle_\sigma = \langle \sigma(h^{-1})f(x), \sigma(h^{-1})g(x) \rangle_\sigma = \langle f(x), g(x) \rangle_\sigma,
\]
so we may regard \( h \) as a continuous function of compact support on \( K \backslash G / H \), and thereby we may define
\[
\langle f, g \rangle = \int_{K \backslash G / H} \langle f(x), g(x) \rangle_\sigma d\mu(x),
\]
which is an inner product on \( \mathfrak{G}_0 \). Also we have \( \langle L_x(f), L_x(g) \rangle = \langle f, g \rangle \), for each \( x \in N \). Let \( \mathfrak{H} \) be the Hilbert space completion of \( \mathfrak{G}_0 \). Then for each \( x \in N \), the translation operator \( L_x \) extends to a unitary operator on \( \mathfrak{H} \), and \( x \mapsto L_x(f) \) is a continuous map from \( N \) to \( \mathfrak{H} \), for each \( f \in \mathfrak{G}_0 \). Now as operators \( L_x \) are uniformly bounded, they are strongly continuous on all of \( \mathfrak{G}_0 \), and so \( x \mapsto L_x(f) \) is a continuous unitary representation of \( N \) in \( \mathfrak{H} \), which is denoted by \( \text{ind}^{N_G(K)}_H \sigma \) and is called the induced representation of \( \sigma \) from \( H \) to \( N_G(K) \).

**Remark 3.1.** There is a more general framework introduced by L. Pavel, which could be used to get almost the same construction [Pl]. To show that \( K \backslash G / H \) fits into Pavel’s framework one may use Theorem 3.1 of [Li] (one should put \( H, G, \) and \( X \) of the definition in page 434 of [Pa] equal \( H, N_G(K), \) and \( G \) in our notation).

In the second part of this section we briefly study those representations of \( G // H \), as a hypergroup, which are canonically induced by representations of its subhypergroups. We quote some of the results in [HKK],[He]. As before \( H \) is a closed subgroup of \( G \) with the left Haar measure \( \sigma \) (normalized in the compact case).
Proposition 3.2. Let \( \{ \pi, \mathcal{S}_\pi \} \) be a representation of \( G \), then
(i) If \( \pi(\sigma) \neq 0 \), then \( \pi \) gives rise to a representation \( \hat{\pi} \) of \( G//H \) given by \( \hat{\pi} = \pi(\sigma) \mathcal{S}_\pi(\cdot) \pi(\sigma) \) on \( \pi(\sigma)(\mathcal{S}_\pi) \). If \( \pi \) is irreducible, then so is \( \hat{\pi} \).
(ii) \( \pi(\sigma) \neq 0 \) if and only if the trivial representation \( 1_H \) of \( H \) is a subrepresentation of \( \pi |_H \).

If (i) holds we say that \( \pi \) is an extension of \( \hat{\pi} \) to \( G \).

Proposition 3.3. Let \( \{ \hat{\pi}, \zeta, \tilde{\mathcal{S}} \} \) be a cyclic representation of \( G//H \) and
\[
\psi(x) = \langle \hat{\pi}(x)\zeta, \zeta \rangle \quad (x \in G),
\]
then the following are equivalent:
(i) There is a representation \( \pi \) of \( G \) such that \( \hat{\pi} = \pi \).
(ii) \( \psi \in P(G) \).

Now let \( G_1 \) be a closed subgroup of \( G \) containing \( H \). We want to consider those representations \( \theta \) of \( G_1//H \) which induce a representation \( \theta' = \text{ind}^{G_1//H}_{G//H} \theta \) of \( G//H \). These are so called inducible representations. Let \( P : C_c(G//H) \to C_c(G_1//H) \) be defined by
\[
P(f) = \left( \frac{\Delta_{G_1}}{\Delta_G} \right)^{1/2} f \mid_{G_1//H} \quad (f \in C_c(G//H)).
\]

Definition 3.1. With the above notations, \( \theta \) is called inducible if for each \( \zeta \in \mathcal{S}_\theta, f, g \in C_c(G//H) \), we have
(i) \( \langle \theta(P(f^* \ast f))\zeta, \zeta \rangle \geq 0 \),
(ii) \( \langle \theta(P(g^* \ast f^* \ast f \ast g))\zeta, \zeta \rangle \leq \|f\|^2 \langle \theta(P(g^* \ast g))\zeta, \zeta \rangle \).

In this case the induced representation \( \theta' = \text{ind}^{G_1//H}_{G//H} \theta \) of \( G//H \) is defined as follows: Take \( V = C_c(G//H) \otimes \mathcal{S}_\theta \) and define
\[
\langle f \otimes \zeta, g \otimes \eta \rangle := \langle \theta(P(g^* \ast f))\zeta, \eta \rangle,
\]
where the right hand side is the inner product in \( \mathcal{S}_\theta \). Let \( N = \{ v \in V : \langle v, v \rangle = 0 \} \) and identify elements \( u, v \in V \) if \( u - v \in N \) (take the quotient), then take the completion of this quotient with respect to the norm defined by the above inner product to get the Hilbert space \( \mathcal{S}_{\theta'} \) and define
\[
\theta'(f)(g \otimes \zeta) = (f \ast g) \otimes \zeta \quad (f, g \in C_c(G//H, \zeta \in \mathcal{S}_\theta)).
\]
This uniquely extends to a bounded linear operator on \( \mathcal{S}_{\theta'} \), giving a representation of \( C_c(G//H) \), which by (ii) extends to a nondegenerate representation of \( L^1(G//H) \), denoted by \( \theta' = \text{ind}^{G_1//H}_{G//H} \theta \).

Proposition 3.4. If \( H \) is compact and \( \hat{\pi} \) is a representation of \( G_1//H \) admitting an extension \( \pi \) to \( G_1 \) which is inducible to a representation \( \pi' \) of \( G \), then \( \pi \) is inducible to a representation \( \pi' \) of \( G//H \) and there is an isometric embedding of \( \mathcal{S}_{\pi'} \) into \( \mathcal{S}_{\pi} \). If moreover \( \pi(G_1)(\pi(\sigma)(\mathcal{S}_\pi)) \) is total in \( \mathcal{S}_\pi \), then \( \pi' \) extends \( \hat{\pi}' \) to \( G \).

Proposition 3.5. If \( \lambda_{G//H} \) is the left regular representation of \( G//H \), then \( \lambda_{G//H} = \text{ind}^{G//H}_1 1_H \) and \( \text{ind}^{G//H}_H 1_H \) is the extension of \( \lambda_{G//H} \) to a representation of \( G \).
4. Algebras of functions on $K \backslash G / H$

In this section we study the algebras of functions on the double coset space $K \backslash G / H$, where $H$ and $K$ are closed subgroups of $G$. In particular, following [Fr] we study the Fourier and Fourier-Stieltjes algebras of $K \backslash G / H$. The case $K = H$ is of particular interest. Recall that $G//H$ is a hypergroup with the left Haar measure $m = \int_G \delta_{xH} dx$ and convolution $\delta_{q(x)} * \delta_{q(y)} = \int_H \delta_{q(xy^{-1})} dh$ (x, y ∈ G), and identity $\delta_e$ and involution $q(x)^{-1} = q(x^{-1})$, where $q : G \rightarrow G//H$ is the quotient map [Je, 8.3]. Moreover, the hypergroup $G^H$ of orbits of the any action of $H$ on $G$ is canonically isomorphic to $(G \times H)//(1 \times H)$ [Je, 8.3]. Therefore $M(G//H)$ is a Banach $*$-algebra. In general $K \backslash G / H$ is only a semi-hypergroup (semi-convo in [Je] terminology) and $M(K \backslash G / H)$ is only a Banach algebra.

Let $\sigma = dh$ and $\tau = dk$ be the left Haar measures on $H$ and $K$, respectively (normalised if $H$ or $K$ are compact), then put $M(K : G : H) = \{ \mu \in M(G) : \sigma * \mu * \tau = \mu \}$. The following result follows exactly like [Je, 14.2].

**Proposition 4.1.** The Banach algebras $M(K \backslash G / H)$ and $M(K : G : H)$ are isometrically isomorphic. When $K = H$ these are also isomorphic as Banach $*$-algebras. \(
\)

Also let

$L^1(K : G : H) = \{ f \in L^1(G) : f(kxh) = f(x), \ a.a. x \in G, k \in K, h \in H \}$,

then we have

**Lemma 4.1.** If $K$ and $H$ are compact, then for each $f \in L^1(G)$, $f dx \in M(K : G : H)$ if and only if $f \in L^1(K : G : H)$.

**Proof** Given $f \in L^1(K : G : H)$, put $\mu_f = fdx$, then for each $h \in H, k \in K$ and each Borel subset $A$ of $G$ we have

$\mu_f(k^{-1}Ah^{-1}) = \int_{k^{-1}Ah^{-1}} f(x)dx = \int_A f(kxh)dx = \int_A f(x)dx = \mu_f(A)$,

hence by Fubini’s theorem

$\tau * \mu_f * \sigma(A) = \int_K \int_G \int_H \chi_{A}(kxh)dkd\mu_f(x)dh = \int_K \int_H \int_G \chi_{A}(kxh)dkd\mu_f(x)dh$

$= \int_K \int_H \mu_f(k^{-1}Ah^{-1})dkdh = \mu_f(A) \int_K \int_H dkdh = \mu_f(A),$

that is $\mu_f \in M(K : G : H)$. Conversely if $\mu_f$ satisfies $\tau * \mu_f * \sigma = \mu_f$, then for each $g \in C_c(G)$ we have

$\int_K \int_G \int_H g(kxh)dkd\mu_f(x)dh = \int_G g(x)d\mu_f(x),$

that is

$\int_K \int_H \int_G [f(k^{-1}xh^{-1}) - f(x)]g(x)dxdkh = 0.$

This implies that $f(k^{-1}xh^{-1}) - f(x) = 0$ for a.a $x, k, h$, that is $f \in L^1(K : G : H)$. \(\square\)
Corollary 4.1. If $K$ and $H$ are compact, then the Banach algebras $L^1(K \backslash G/H)$ and $L^1(K : G : H)$ are isometrically isomorphic. When $K = H$ these are also isomorphic as Banach $*$-algebras.

It is easy to see that the Banach $*$-algebra $C_b(K \backslash G/H)$ of bounded continuous functions on $K \backslash G/H$ is isometrically isomorphic to
\[ C_b(K : G : H) = \{ u \in C_b(G) : u(kxh) = u(x) \mid x \in G, k \in K, h \in H \} \]
Now let $K$ be unimodular and assume that a generalized Bruhat function $\rho$ exists on $G$. Let $d\nu = \rho dx$ and let $\mu = \nu$ be the corresponding quasi-invariant measure on $K \backslash G/H$. Define
\[ Q_\rho(f)(\check{x}) = \int_K \int_H \frac{f(kxh)}{\rho(kxh)} dk dh \quad (x \in G, f \in C_c(G)) . \]
Then we have the generalized Mackey-Bruhat formula
\[ \int_{K \backslash G/H} Q_\rho(f)(\check{x}) d\mu(\check{x}) = \int_G f(x) dx . \]
Let $J_\rho = \{ f \in C_c(G) : Q_\rho(f) = 0 \}$, then

Lemma 4.2. If $K$ is unimodular, there is an isometric linear isomorphism of Banach spaces $C_c(K \backslash G/H) \cong C_c(G)/J_\rho$ with respect to the $||\cdot||_1$-norms defined by $\mu$ and Haar measure, respectively.

Proof: It could be proved just as in the Proposition 2.1 that the map $Q_\rho : C_c(K \backslash G/H) \rightarrow C_c(G)$ is surjective. Also for each $f \in C_c(G)$ we have
\[ \|Q_\rho(f)\|_1 = \int_{K \backslash G/H} \left( \int_K \int_H \frac{|f(kxh)|}{\rho(kxh)} dk dh \right) d\mu(\check{x}) \leq \int_{K \backslash G/H} \left( \int_K \int_H \frac{|f(kxh)|}{\rho(kxh)} dk dh \right) d\mu(\check{x}) = \int_G |f(x)| dx = \|f\|_1 . \]
On the other hand, given $g \in C_c(K \backslash G/H)$, using the argument of Proposition 2.1, we can choose $f \in C_c(G)$ such that $Q_\rho(f) = g$ and $Q_\rho(|f|) = |g|$. Then
\[ \|f\|_1 = \int_G |f| dx = \int_{K \backslash G/H} Q_\rho(|f|) d\mu = \int_{K \backslash G/H} |g| d\mu = \|g\|_1 . \]
Therefore
\[ \|g\|_1 = \inf \{ \|f\|_1 : f \in C_c(G), Q_\rho(f) = g \} = \|f + J_\rho\|_1 . \]

Now the following result, which follows from the above lemma and [RS, Lemma 3.4.4], shows that the generalized Mackey-Bruhat formula is valid in $L^1(G)$ also.

Proposition 4.2. If $K$ is unimodular then there is an isometric linear isomorphism $L^1(K \backslash G/H, \mu) \cong L^1(G)/J_\rho$, where $J - \rho$ is the closure of $J_\rho$ in $L^1(G)$.

Coming back to the case, $K = H$, following [Je], we denote the positive definite elements of $C_b(G//H)$ by $P(G//H)$ and its linear span by $B(G//H)$. Also, following [Fo], we consider
\[ B(G :: H) = \{ u \in B(G) : u(kxh) = u(x) \mid x \in G, k \in K, h \in H \} \]
where $B(G)$ is the Fourier-Stieltjes algebra of $G$, and denote the closure of the set of those $u \in B(G :: H)$ for which $q(supp(u)) \subseteq G//H$ is compact, by $A(G :: H)$. The following result is proved in [Je, 14.4].
Lemma 4.3. For each \( f \in P(G) \cap C_0(G :: H) \), there is \( g \in P(G/\!\!/H) \) such that \( f = g \circ q \).

Proposition 4.3. The vector spaces \( B(G/\!\!/H) \) and \( B(G :: H) \) are linearly isomorphic.

**Proof** By above lemma, The linear map \( g \to g \circ q \) from \( B(G/\!\!/H) \) to \( B(G :: H) \) is surjective. It is also injective, since \( q \) is surjective.

The following is proved in [AM] in a more general context.

**Proposition 4.4.** If \( H \) is compact then the above linear isomorphism is norm increasing. In particular it is an open map.

**Proposition 4.5.** If \( H \) is compact, then there is a surjective projection \( Q : B(G) \to B(G :: H) \) with \( \|Q\| \leq 1 \). Also the restriction of \( Q \) on \( A(G) \) is a projection onto \( A(G :: H) \).

**Proof** Put
\[
Q_1(u)(x) = \int_H u(kx)dk, \quad Q_2(u)(x) = \int_H u(xh)dh \quad (u \in B(G)),
\]
then for \( i = 1, 2 \), \( Q_i : B(G) \to B(G) \) is a continuous projection with \( \|Q_i\| \leq 1 \) [Fo]. Also clearly \( Q_1Q_2 = Q_2Q_1 \). Hence \( Q = Q_1Q_2 \) is a continuous projection on \( B(G) \) with \( \|Q\| \leq 1 \). Now \( \text{Im}(Q_2) = B(G :: H) \) [Fo, 3.4], and a similar left version holds for \( Q_1 \), hence \( \text{Im}(Q) = B(G :: H) \). The assertion about \( A(G) \) follows similarly from [Fo, 3.4].

**Corollary 4.2.** If \( H \) is compact, \( A(G :: H) \) is a complemented subspace of both \( A(G) \) and \( A(G :: H) \).

As another corollary we can prove the following extension property. But first we need the following analogue of a result of Herz [Hr].

**Lemma 4.4.** If \( H, \ K \) are compact and closed subgroups of \( G \) respectively, and \( H \subseteq K \). Then each \( u \in A(K :: H) \) extends to some \( v \in A(G) \) with the same norm.

**Corollary 4.3.** If \( H, \ K \) are compact and closed subgroups of \( G \) respectively, and \( H \subseteq K \). Then each \( u \in A(K :: H) \) extends to some \( v \in A(G) \) with the same norm.

**Proof** Extend \( u \) to \( v_1 \in A(G) \) with the same norm and put \( v = Q(v_1) \). Then \( v \) extends \( u \) so \( \|v\| \geq \|u\| \). Also \( \|v\| \leq \|v_1\| = \|u\| \), so the result.

**Remark 4.1.** One can show that there is a continuous surjection \( Q \) as above, even when \( H \) is closed but we don’t know if this is a projection. Indeed we can argue as follows: Recall that \( f \in C_0(G) \) is weakly almost periodic if the set \( O_L(f) \) of left translates of \( f \) is relatively weakly compact. This is equivalent to its right version. We denote the set of all such functions with \( WAP(G) \). This is a \( C^* \)-subalgebra of \( C_0(G) \). Now let \( \Psi \) be the unique invariant mean on \( WAP(G) \) and \( u \in B(G) \). Let \( \{x_\beta\} \) and \( \{y_\gamma\} \) be complete sets of left and right coset representatives of \( H \) in \( G \). Put
\[
\beta u(h) = u(x_\beta h), \quad u_\gamma(h) = u(hy_\gamma) \quad (h \in H),
\]
and set
\[ Q_1(u)(x) = \Psi_1(u), \quad Q_2(u)(x) = \Psi_2(u) \quad (x \in x_\beta H) \quad (x \in Hy), \]
then for \( i = 1, 2 \), \( Q_i : B(G) \to B(G) \) is a continuous projection with \( \|Q_i\| \leq 1 \). Then \( Q = Q_1Q_2 \) is the required map. We are not able to show that \( Q_1Q_2 = Q_2Q_1 \), ensuring that \( Q \) is indeed a projection.

The following is proved similar to [Fo, 3.5,3.6]

**Proposition 4.6.** If \( H_1 \) and \( H_2 \) are compact subgroups of \( G \), then \( B(G : H_1) = B(G : H_2) \) if and only if \( A(G : H_1) = A(G : H_2) \) if and only if \( H_1 = H_2 \).

Let \( H \) be a compact subgroup of \( G \). Consider \( Q : L^1(G) \to L^1(G) \) defined by
\[ Q(f)(x) = \int_{H/H} \int_H f(kxh)dkdh \quad (x \in G, f \in L^1(G)). \]

**Lemma 4.5.** \( \text{Im}(Q) = L^1(G : H) \) is a self-adjoint subalgebra of \( L^1(G) \).

Let \( C^*(G : H) \) and \( VN(G : H) \) denote the norm and weak* closure of \( L^1(G : H) \) in \( C^*(G) \) and \( VN(G) \), respectively. The following result is interesting as its analogue for one sided cosets fails (c.f. remarks before Theorem 4.1 in [Fo].

**Proposition 4.7.** If \( H \) is compact then
(i) \( C^*(G : H) \) is a \( C^* \)-subalgebra of \( C^*(G) \) and \( Q \) extends to a continuous projection on \( C^*(G) \) with range \( C^*(G : H) \).
(ii) \( VN(G : H) \) is a von Neumann subalgebra of \( VN(G) \) and \( Q \) extends to a continuous projection on \( VN(G) \) with range \( VN(G : H) \).

**Proof** The first part of both assertions follow from above lemma. The second part could be proved from the analogous result in [Fo] and its left version using the method of Proposition 4.5.

**Theorem 4.1.** If \( H \) is compact, the Banach spaces \( C^*(G : H)^* \) and \( B(G : H) \) are isometrically isomorphic.

**Proof** We have
\[ C^*(G : H)^* = \text{Im}(Q)^* \cong (C^*(G)/\ker(Q))^* \cong (\ker(Q))^\perp, \]
so we only need to check that \( (\ker(Q))^\perp = B(G : H) \). Let \( u \in B(G : H) \) and \( f \in L^1(G) \) with \( Q(f) = 0 \). Then
\[
\langle u, f \rangle = \int_G u(x)f(x)dx = \int_{G/H} \int_H \int_H u(kxh)f(kxh)dkdhd\mu(\hat{x}) = u(x)\int_{G/H} \int_H \int_H f(kxh)dkdhd\mu(\hat{x}) = u(x)\int_{G/H} Q(f)(\hat{x})d\mu(\hat{x}) = 0.
\]
Conversely if \( u = 0 \) on \( \ker(Q) \), then for each \( k_0, h_0 \in H \) let \( g(x) = f(k_0^{-1}xh_0^{-1}) - f(x) \), then clearly \( g \in \ker(Q) \) so we have
\[
\int_G (u(k_0xh_0) - u(x))f(x)dx = \int_G u(x)g(x)dx = 0.
\]
This being true for each $f$, we have $u(k_0 x_0) - u(x) = 0$, for each $x \in G$, that is $u \in B(G :: H)$.

Similarly we have

**Theorem 4.2.** If $H$ is compact, the Banach spaces $VN(G :: H)_*$ and $A(G :: H)$ are isometrically isomorphic.

**Proposition 4.8.** Let $H$ be compact, then the following are equivalent.

(i) $G$ is amenable.

(ii) $A(G :: H)$ has a bounded approximate identity consisting of functions of compact support.

(iii) $A(G :: H)$ is weakly factorized.

**Proof**  (i) $\Rightarrow$ (ii) Given $E \subseteq G$ compact, let $F = HEH$, then $F \subseteq G$ is compact and so by Reiter’s condition, for each $\epsilon > 0$, there is an element $u = u_{E,\epsilon} \in A(G)$ with compact support such that $u = (1 + \epsilon)^{-1}$ on $F$ and $\|u\| \leq 1$. Let $v = v_{E,\epsilon} = Q(u) \in A(G :: H)$, then clearly $v = (1 + \epsilon)^{-1}$ on $E$ and supp$(v) \subseteq H$(supp$(u))H$, so $v$ is of compact support. Let $w \in A(G :: H)$ be any element of compact support, and let $E = supp(w)$, then for $v = v_{E,\epsilon}$ we have $vw = (1 + \epsilon)^{-1}w$, so $\|vw - w\| = \|w\|(1 + \epsilon)^{-1} - 1 < \epsilon \|w\|$. Since elements of $A(G :: H)$ with compact support are dense in $A(G :: H)$, this proves (ii).

(ii) $\Rightarrow$ (iii) Cohen’s factorization theorem.

(iii) $\Rightarrow$ (i) Let $F \subseteq G$ be compact, then by weak factorization property, there is an element $u = u_F \in A(G :: H)$ with $u > 1$ on $F$ and $\|u\| \leq M$, for some constant $M$ independent of $F$ [Fo]. Take any $f \in C_c(G)_+$ and let $F = supp(f)$ and observe that

$$< u_F, f > = \int_G u_F(x)f(x)dx \geq \int_G f(x)dx = \|f\|_1.$$

The rest of the proof is standard and could be finished as in [Fo,4.2].

**Theorem 4.3.** If $G$ is amenable and $H$ is compact, then the multiplier algebra $M(A(G :: H))$ is isometrically isomorphic to $B(G :: H)$.

**Proof** By above theorem, $A(G :: H)$ has a bounded approximate identity $\{u_\alpha\}$ (bounded by 1). Let $M = M(A(G :: H))$ and $\|\cdot\|_M$ denote the multiplier norm. It is clear that $B(G :: H) \subseteq M \subseteq C_b(G)$. Take $u \in M$, then $uu_\alpha \in A(G :: H)$ and $\|uu_\alpha\| \leq \|u\|_M \|u_\alpha\| \leq \|u\|_M$, for each $\alpha$. Now $\{uu_\alpha\}$ converges to $u$ in the norm of $A(G)$ and so pointwise. Hence $u \in B(G :: H)$ and $\|u\| \leq \|u\|_M$. The inequality in other direction now follows from $B(G :: H) \subseteq M$.

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