Supporting Information S1 for “The effects of spatially heterogeneous prey distributions on detection patterns in foraging seabirds”

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1 Landscape properties of the Fractal Local Density Model

1.1 Calculation of the fractal dimension

In this model, prey are distributed on a plane that is randomly subdivided into non-overlapping patches of heterogeneous diameters (see Materials and Methods, main article). The patch diameter probability distribution function (PDF), \( \psi(R) \), is given by the power-law

\[
\psi(R) = (\nu - 1) R_0^{\nu - 1} R^{-\nu}, \quad R \geq R_0, \quad \nu > 1,
\]

and \( \psi(R) = 0 \) for \( R < R_0 \), where \( R_0 \) is the minimum patch diameter. \( \int_0^\infty \psi(R)dR = 1. \) A patch of size \( R \) contains on average \( n_p(R) \) prey uniformly and randomly distributed inside the patch. We assume \( n_p(R) = kR^\epsilon \) with \( k \) a constant and \( \epsilon \) a scaling exponent. Let \( N_0(>1) \) be the average number of prey that a patch of smallest size \( R_0 \) contains. Thus,

\[
k = \frac{N_0}{R_0}. \tag{2}
\]

If \( \epsilon = 2 \), the number of prey per unit area, \( \rho(R) = n_p(R)/R^2 = kR^{\epsilon-2} \), is independent of \( R \) and the medium is uniform. We show below that the prey system can be fractal if \( \epsilon < 2 \), when \( \rho(R) \to 0 \) for large patches, and calculate its fractal dimension \( D_F \) using the standard box-counting method [1]. For a fractal medium, we expect a relation of the form

\[
\mathcal{N}(\sigma) \sim \sigma^{-D_F}, \tag{3}
\]

where \( \mathcal{N}(\sigma) \) is the average number (per patch) of non-overlapping square boxes of length \( \sigma \) that cover all the prey (\( \sigma > R_0 \) in the following). Two cases are distinguished below: \( 0 \leq \epsilon < 2 \) (big patches have more prey) and \( \epsilon < 0 \) (big patches have fewer prey). Although fits to the albatross data suggest that the prey system has \( \epsilon < 0 \) (see main article), we also present the results of the other case for completeness.

a) Case \( 0 \leq \epsilon < 2 \).

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If $\epsilon > 0$, there are always more than one prey per patch. As the prey density $\rho(R) = kR^{\nu - 2}$ is uniform inside the patch, the typical distance between neighboring prey is $d(R) = 1/\rho(R)^{1/2}$, which is $< R$. Consider square boxes of size $\sigma$, larger than the smallest patch diameter $R_0$. If $d(R)$ does not exceeds $\sigma$, the patch can be completely covered by $(R/\sigma)^2$ boxes on average [this number is $< 1$ if $R < \sigma$, meaning that one box covers more than one patch on average]. If on the other hand $d(R) > \sigma$, $n_p(R) = kR^\epsilon$ boxes are needed in average to cover all the prey inside the patch. For a fixed $\sigma$, there is a patch size $R^*$ such that $d(R^*) = \sigma$. Using notation (2),

$$R^* = R_0 N_0^{1/(2-\epsilon)} (\sigma/R_0)^{2/(2-\epsilon)}. \quad (4)$$

From the considerations above, one can write

$$N(\sigma) = \int_{R_0}^{R^*} dR \psi(R) \left( \frac{R}{\sigma} \right)^2 + \int_{R^*}^{\infty} dR \psi(R) kR^\epsilon \equiv N_1(\sigma) + N_2(\sigma). \quad (5)$$

Several cases are to be distinguished.

If the patch exponent $\nu$ is $> 3$, one finds

$$N_1(\sigma) \simeq \frac{\nu - 1}{\nu - 3} \left( \frac{\sigma}{R_0} \right)^{-2}, \quad \sigma \gg R_0 \quad (6)$$

$$N_2(\sigma) = \frac{\nu - 1}{\nu - \epsilon - 1} N_0^{\frac{3-\nu}{2}} \left( \frac{\sigma}{R_0} \right)^{-\frac{2(\nu - \epsilon - 1)}{2 - \epsilon}}. \quad (7)$$

Since $N_0 > 1$ and $2(\nu - \epsilon - 1)/(2 - \epsilon) > 2$, then $N_1 \gg N_2$ when $\sigma \gg R_0$. Comparing with the definition (3), one concludes:

$$D_F = 2. \quad (8)$$

Hence, if $\nu > 3$, the prey system is bidimensional, although with a non-uniform density.

If $\epsilon + 1 < \nu < 3$, one finds, for $\sigma \gg R_0$:

$$N(\sigma) = \left[ \frac{1}{3 - \nu} + \frac{1}{\nu - \epsilon - 1} \right] (\nu - 1)N_0^{\frac{3-\nu}{2}} \left( \frac{\sigma}{R_0} \right)^{-\frac{2(\nu - \epsilon - 1)}{2 - \epsilon}}. \quad (9)$$

Therefore,

$$D_F = 2 \frac{\nu - \epsilon - 1}{2 - \epsilon}, \quad (10)$$

which is lower than 2.

If $\nu < 1 + \epsilon$, the average prey number per patch is infinite and the medium can not be described as a fractal. This case will not be considered here.

b) Case $\epsilon < 0$. 

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The calculation above must be modified for negative values of $\epsilon$. The average prey number is $<1$ in patches larger than

$$R_c = R_0 N_0^{-1/\epsilon}.$$  

(11)

One can figure these low density patches as containing one prey with probability $kR^\epsilon$ and zero prey with probability $1 - kR^\epsilon$. Hence, if $R > R_c$, $d(R)$ does not represent the distance between neighboring prey as in the previous case. Let us restrict our analysis to boxes of size $\sigma > R_c$ in the following. A patch with $R < R_c$ is completely covered (and therefore all its prey) by $(R/\sigma)^2 (1/R_0)$ boxes. A patch with $R_c < R < \sigma$ is covered by $(R/\sigma)^2$ boxes, too, and has a probability $kR^\epsilon (1/R_0)$ of containing one prey. A patch with $R > \sigma$ has a probability $kR^\epsilon$ of containing one prey and therefore requires $kR^\epsilon$ boxes in average to be covered. Therefore:

$$N(\sigma) = \int_{R_0}^{R_c} dR \psi(R) \left( \frac{R}{\sigma} \right)^2$$

$$+ \int_{R_c}^{\sigma} dR \psi(R) \left( \frac{R}{\sigma} \right)^2 kN^\epsilon$$

$$+ \int_{\sigma}^{\infty} dR \psi(R) kR^\epsilon$$

$$\equiv N_1(\sigma) + N_2(\sigma) + N_3(\sigma).$$

Performing the integrals, one finds ($\sigma \gg R_0$):

$$N_1(\sigma) \simeq \begin{cases} 
\frac{\nu-1}{\nu-3} \left( \frac{\sigma}{R_0} \right)^{-2}, & \nu > 3 \\
\frac{\nu-1}{\nu-3} N_0^{(\nu-3)/\epsilon} \left( \frac{\sigma}{R_0} \right)^{-2}, & \nu < 3,
\end{cases}$$

(13)

$$N_2(\sigma) \simeq \begin{cases} 
\frac{\nu-1}{\nu-3-\epsilon} N_0^{(\nu-3)/\epsilon} \left( \frac{\sigma}{R_0} \right)^{-2}, & \nu > 3 + \epsilon \\
\frac{\nu-1}{\nu-3-\epsilon} N_0 \left( \frac{\sigma}{R_0} \right)^{-(\nu-\epsilon-1)}, & \nu < 3 + \epsilon,
\end{cases}$$

(14)

$$N_3(\sigma) = \begin{cases} 
\frac{\nu-1}{\nu - \epsilon - 1} N_0 \left( \frac{\sigma}{R_0} \right)^{-(\nu-\epsilon-1)}, & \forall \nu > 1.
\end{cases}$$

(15)

For a fixed patch exponent $\nu$, the fractal dimension is given by the leading term(s) in eqs.(13)-(15), i.e., the term(s) with the slowest power-law decay as a function of $\sigma/R_0$. One obtains:

$$D_F = 2 \quad \text{for } \nu > 3 + \epsilon$$

$$D_F = \nu - \epsilon - 1 < 2 \quad \text{for } \nu < 3 + \epsilon.$$  

(16)

1.2 Distribution of the local density

In this subsection, we show that in the Fractal Patch model with patch size distribution of the form $\psi(R) \sim R^{-\nu}$ and $\epsilon < 2$, the probability distribution
function $f(\rho)$ of the local density $\rho$ is also a power-law, $f(\rho) \sim \rho^{-\alpha_\rho}$, with exponent:

$$\alpha_\rho = \frac{5 - \epsilon - \nu}{2 - \epsilon}. \quad (18)$$

(See equation (4), main article). A motivation for calculating $\alpha_\rho$ is that recent acoustic measurements have shown that the prey of some top marine predators have densities that are power-law distributed in space [2].

The probability that a small region of the plane has a density larger than a value $\rho$ is equal to the fraction area occupied by patches smaller than $R$, with $\rho = kR^{\epsilon-2}$. Namely,

$$\int_\rho^\infty f(x)dx = \int_0^R x^2 \psi(x)dx, \quad (19)$$

with $\rho_0 = \rho(R_0)$ the largest density, that is found in the smallest patches. Taking the derivative of eq.(19) with respect to $\rho$, one obtains

$$f(\rho) \sim \left|\frac{dR}{d\rho}\right| R^{2-\nu}. \quad (20)$$

Using the relation $R \sim \rho^{1/(\epsilon-2)}$ yields to the above mentioned result.

If $\nu < 3$, then $\alpha_\rho > 1$ from eq.(18): The presence of very large patches of low densities produce a sharp increase of $f(\rho)$ as $\rho \to 0$. This is akin to the situation encountered in [2], where $\alpha_\rho \approx 1.7$ was observed for krill densities. A realistic description of prey in the sea requires that the patch size distribution $\psi(R)$ is exponentially cut-off at large scales, with few patch exceeding a value $R_m$ (see eq.(1), main article). This puts a lower bound on $\rho$ and ensures that $f(\rho)$ is integrable in the small $\rho$ region.

If $\nu > 3$, then $\alpha_\rho < 1$. The medium is more homogeneous than in the previous case due to the relative scarcity of large patches. Even if the patch size distribution has no large size cut-off (i.e. $\psi(R)$ is a pure power-law), $f(\rho)$ remains integrable in $\rho = 0$. At large densities, on the other hand, the distribution $f(\rho)$ is very “flat”. It drops to zero, however, once the highest density in the system, $\rho_0$, is reached.

References

[1] B. B. Mandelbrot. The Fractal Geometry of Nature. W. H. Freeman (1983).

[2] D. W. Sims et al. Scaling laws of marine predator search behaviour. Nature 451, 1098–1102 (2008).