Hopf algebras for matroids over hyperfields

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Abstract. Recently, M. Baker and N. Bowler introduced the notion of matroids over hyperfields as a unifying theory of various generalizations of matroids. In this paper we generalize the notion of minors and direct sums from ordinary matroids to matroids over hyperfields. Using this we generalize the classical construction of matroid-minor Hopf algebras to the case of matroids over hyperfields.

1. Introduction

A basic result in algebraic geometry is that the category $\text{Aff}_k$ of affine schemes over a field $k$ is equivalent to the opposite category $\text{Alg}_k^{op}$ of the category of commutative $k$-algebras. When one enhances $\text{Aff}_k$ to affine group schemes over $k$, one obtains Hopf algebras as an enrichment of commutative $k$-algebras. In fact, Hopf algebras naturally appear not only in algebraic geometry, but also in various fields of mathematics including (but not limited to) algebraic topology, representation theory, quantum field theory, and combinatorics. For a brief historical background for Hopf algebras, we refer the readers to [4]. For a comprehensive introduction to Hopf algebras in combinatorics, we refer the readers to [12]. In this paper, our main interest is in Hopf algebras arising in combinatorics, namely those obtained from matroids (or more generally matroids over hyperfields).

Matroids are combinatorial objects arising in two main ways: (1) as a model of cycle structures in graphs and (2) as combinatorial abstractions of linear independence properties in vector spaces. While matroids have their own charms, it is their rich interplay with other areas of mathematics which makes them truly interesting. For instance, N. Mnëv’s universality theorem [16] roughly states that any semi-algebraic set in $\mathbb{R}^n$ is the moduli space of realizations of an oriented matroid (up to homotopy equivalence). There is an analogue in algebraic geometry known as Murphy’s Law by R. Vakil [21] for ordinary matroids. Valuated matroids (a generalization of matroids by giving certain “weights” to bases) are analogous to “linear spaces” in the setting of tropical geometry. Indeed a moduli space of valuated matroids (a tropical analogue of a Grassmannian), called a Dressian, has received much attention.

Hopf algebras arising in combinatorics are usually created to encode the basic operations of an interesting class of combinatorial objects. Matroids have several basic operations, the most basic of which are deletion, contraction, and direct sum; an iterated sequence of deletions and contractions on a matroid results in a minor of the matroid. The Hopf algebra associated to a set of isomorphism classes of matroids closed under taking minors and direct sums is called a matroid-minor Hopf algebra. In this paper, we generalize the construction of the matroid-minor Hopf algebra to the setting of matroids over hyperfields, first introduced by M. Baker and N. Bowler in [5].
Remark 1.1. In fact, many combinatorial objects posses notions of “deletion” and “contraction” and hence one can associate Hopf algebras (or bialgebras in the disconnected case). In [10], C. Dupont, A. Fink, and L. Moci associate a universal Tutte character to such combinatorial objects specializing to Tutte polynomials in the case of matroids and graphs (among others) generalizing the work [14] of T. Krajewski, I. Moffatt, and A. Tanasa. See §6.3 in connection with our work.

Hyperfields were first introduced by M. Krasner in his work [15] on an approximation of a local field of positive characteristic by using local fields of characteristic zero. Krasner’s motivation was to impose, for a given multiplicative subgroup $G$ of a commutative ring $A$, “ring-like” structure on the set of equivalence classes $A/G$ ($G$ acts on $A$ by left multiplication). Krasner abstracted algebraic properties of $A/G$ and defined hyperrings, in particular, hyperfields. Roughly speaking, hyperfields are fields with multi-valued addition. For instance, when $A = k$ is a field and $G = k - \{0\}$, one has $k/G = \{[0], [1]\}$, where $[0]$ (resp. $[1]$) is the equivalence of 0 (resp. 1). Then one defines $[1] + [1] = \{[0], [1]\}$. This structure is called the Krasner hyperfield (see, Example 2.12). After Krasner’s work, hyperfields (or hyperrings, in general) have been studied mainly in applied mathematics. Recently several authors (including the second author of the current paper) began to investigate hyperstructures in the context of algebraic geometry and number theory. Furthermore, very recently M. Baker (later with N. Bowler) employ hyperfields in combinatorics: Baker and Bowler found a beautiful framework which simultaneously generalizes the notion of linear subspaces, matroids, oriented matroids, and valuated matroids. In light of various fruitful applications of Hopf algebra methods in combinatorics, one might naturally ask the following:

Question. Can we generalize matroid-minor Hopf algebras to the case of matroids over hyperfields?

We address this question in this paper. We first define minors for matroids over hyperfields which generalize the definition of minors for ordinary matroids:

Theorem A (§3). Let $H$ be a hyperfield. There are two cryptomorphic definitions (circuits and Grassmann-Plücker functions) of minors of matroids over $H$. Furthermore, if $M$ is a weak (resp. strong) matroid over $H$, then all minors of $M$ are weak (resp. strong).

Next, we introduce the notion of direct sums of matroids over hyperfields and prove that direct sums preserve the type (weak or strong) of matroids over hyperfields in the following sense:

Theorem B (§3). Let $H$ be a hyperfield. There are two cryptomorphic definitions (circuits and Grassmann-Plücker functions) of direct sums of matroids over $H$. Furthermore, if $M_1$ and $M_2$ are (weak or strong) matroids over $H$, then the direct sum $M = M_1 \oplus M_2$ is always a weak matroid over $H$ and $M$ is strong if and only if both $M_1$ and $M_2$ are strong.

Remark 1.2. We note that our definition of direct sum is a natural generalization of the direct sum of ordinary matroids. It also meshes nicely with the definition of direct sums for matroids over fuzzy rings in [9].

By appealing to the above results, we define Hopf algebras for matroids over hyperfields in §5. Finally, in §6 we explain how our current work can be thought in views of matroids over fuzzy rings as in Dress-Wenzel theory [9], matroids over partial hyperfields [6], and universal Tutte characters [10] as well as Tutte polynomials of Hopf algebras [14].
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2. Preliminaries

In this section, we recall the definitions of the following for readers who are not familiar with these key notions:

- Matroids.
- Hyperfields and matroids over hyperfields.
- Hopf algebras and matroid-minor Hopf algebras.

In what follows, we will only consider matroids on a finite set unless otherwise stated. For infinite matroids in the context of the current paper, we refer readers to [8].

Notice that we let $\mathbb{N} = \mathbb{Z}_{\geq 0}$.

2.1 Matroids. This section is intended as a brief refresher on the basic notions and operations in matroid theory. Readers familiar with matroid theory may skip this section. We refer readers to [17] and [22] for further details and proofs of the facts from this section.

A matroid is a combinatorial model for the properties of linear independence in a (finite dimensional) vector space and for the properties of cycles in combinatorial graphs. It is well-known that there are several “cryptomorphic” definitions for matroids. Chief among these are the notions of bases and circuits.

Let $E$ be a finite set (the ground set of a matroid). A nonempty collection $B \subseteq \mathcal{P}(E)$, where $\mathcal{P}(E)$ is the power set of $E$, is a set of bases of a matroid when $B$ satisfies the basis exchange axiom, given below:

1. For all $X,Y \in B$ and all $x \in X \setminus Y$ there is an element $y \in Y \setminus X$ such that $(X \setminus \{x\}) \cup \{y\} \in B$.

In the context of (finite dimensional) vector spaces, any pair of bases $B_1, B_2$ of a given subspace satisfies this property by the Steinitz Exchange Lemma. In the context of finite graphs, this is a corollary of the Tree Exchange Property satisfied by the edge sets of spanning forests in the graph.

A collection $C \subseteq \mathcal{P}(E)$ is a set of circuits of a matroid on $E$ when $C$ satisfies the following three axioms:

1. (Nondegeneracy) $\emptyset \notin C$.
2. (Incomparability) If $X,Y \in C$ and $X \subseteq Y$, then $X = Y$.
3. (Circuit elimination) For all $X,Y \in C$ and all $e \in X \cap Y$, there is a $Z \in C$ such that $Z \subseteq (X \cup Y) \setminus \{e\}$.

In the context of (finite) graphs, circuits are precisely the edge sets of cycles in the graph. In the context of (finite dimensional) vector spaces, circuits correspond with minimal dependence relations on a finite set of vectors.

Remark 2.1. There is a natural bijection between sets of circuits of a matroid on $E$ and sets of bases of a matroid on $E$. Given a set $C$ of circuits of a matroid, we define $B_C$ to be the set of maximal subsets of $E$ not containing any element of $C$. Likewise, given a set $B$ of bases of a matroid, we define $C_B$ to be the set of minimal nonempty subsets of $E$ which are not contained in any element of $B$. It is a standard exercise to show that (1) $B_C$ is a set of bases of a matroid, (2) $C_B$ is a set of circuits of a matroid, and (3) both $B_{C_B} = B$ and $C_{B_C} = C$. 

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In this sense \( \mathcal{B} \) and \( \mathcal{C}_\mathcal{B} \) carry the same information as \( \mathcal{C}_\mathcal{M} \) and \( \mathcal{B}_\mathcal{C} \) respectively; these are thus said to determine the same matroid on \( E \) “cryptomorphically.”

**Example 2.2.** The motivating examples of matroids (hinted at above) are given as follows:

1. Let \( V \) be a finite dimensional vector space and \( E \subseteq V \) a spanning set of vectors. The bases of \( V \) contain in \( E \) form the bases of a matroid on \( E \), and the minimal dependent subsets of \( E \) form the circuits of a matroid on \( E \). Furthermore, these are the same matroid.

2. Let \( \Gamma \) be a finite, undirected graph with edge set \( E \) (loops and parallel edges are allowed). The sets of edges of spanning forests in \( \Gamma \) form the bases of a matroid on \( E \), and the sets of edges of cycles form the circuits of a matroid on \( E \). Furthermore, these are the same matroid (called the *graphic matroid* of \( \Gamma \)).

One can define the notion of isomorphisms of matroids as follows.

**Definition 2.3.** Let \( M_1 \) (resp. \( M_2 \)) be a matroid on \( E_1 \) (resp. \( E_2 \)) defined by a set \( \mathcal{B}_1 \) (resp. \( \mathcal{B}_2 \)) of bases. We say that \( M_1 \) is isomorphic to \( M_2 \) if there exists a bijection \( f : E_1 \rightarrow E_2 \) such that \( f(B) \in \mathcal{B}_2 \) if and only if \( B \in \mathcal{B}_1 \). In this case, \( f \) is said to be an isomorphism.

**Example 2.4.** Let \( \Gamma_1 \) and \( \Gamma_2 \) be finite graphs and \( M_1 \) and \( M_2 \) be the corresponding graphic matroids. Every graph isomorphism between \( \Gamma_1 \) and \( \Gamma_2 \) gives rise to a matroid isomorphism between \( M_1 \) and \( M_2 \), but the converse need not hold.

Recall that given any base \( B \in \mathcal{B}(M) \) and any element \( e \in E \setminus B \), there is a unique circuit (fundamental circuit) \( C_{B,e} \) of \( e \) with respect to \( B \) such that \( C_{B,e} \subseteq B \cup \{e\} \).

One can construct new matroids from given matroids as follows:

**Definition 2.5 (Direct sum of matroids).** Let \( M_1 \) and \( M_2 \) be matroids on \( E_1 \) and \( E_2 \) given by bases \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \) respectively. The direct sum \( M_1 \oplus M_2 \) is the matroid on \( E_1 \sqcup E_2 \) given by the bases \( \mathcal{B} = \{B_1 \sqcup B_2 \mid B_i \in \mathcal{B}_i \text{ for } i = 1, 2\} \).

**Remark 2.6.** One can easily check that \( M_1 \oplus M_2 \) is indeed a matroid on \( E_1 \sqcup E_2 \).

**Definition 2.7 (Dual, Restriction, Deletion, and Contraction).** Let \( M \) be a matroid on a finite set \( E_M \) with the set \( \mathcal{B}_M \) of bases and the set \( \mathcal{C}_M \) of circuits. Let \( S \) be a subset of \( E_M \).

1. The dual \( M^* \) of \( M \) is a matroid on \( E_M \) given by bases
   \[ \mathcal{B}_{M^*} := \{E_M - B \mid B \in \mathcal{B}_M\} \, . \]

2. The restriction \( M|S \) of \( M \) to \( S \) is a matroid on \( S \) given by circuits
   \[ \mathcal{C}_{M|S} = \{D \subseteq S \mid D \in \mathcal{C}_M\} \, . \]

3. The deletion \( M \setminus S \) of \( S \) is the matroid \( M \setminus S := M|(E \setminus S) \).

4. The contraction of \( M \) by \( S \) is \( M/S := (M^* \setminus S)^* \).

It is easy to show that \( M|S, M^*, \) and \( M/S \) are indeed matroids. A minor of a matroid \( M \) is any matroid obtained from \( M \) by a series of deletions and/or contractions. Basic properties of these operations are given below:

**Proposition 2.8.** Let \( M \) be a matroid on \( E \). We have the following for all disjoint subsets \( S \) and \( T \) of \( E \):

1. \( M/\emptyset = M = M \setminus \emptyset \)
2. \( (M \setminus S) \setminus T = M \setminus (S \cup T) \)
3. \( (M/S)/T = M/(S \cup T) \)
4. \( (M \setminus S)/T = (M/T) \setminus S \)

In particular, the minors of a matroid are in one-to-one correspondence with the ordered pairs of disjoint subsets of the ground set by deleting the first and contracting the second.
2.2 Matroids over hyperfields. In this section, we review basic definitions and properties for matroids over hyperfields first introduced by Baker and Bowler in [5]. Let’s first recall the definition of a hyperfield. By a hyperaddition on a nonempty set \( H \), we mean a function of \( H \times H \rightarrow \mathcal{P}^+(H) \) such that \((a,b) = +(b,a)\) for all \( a, b \in H \), where \( \mathcal{P}^+(H) \) is the set of nonempty subsets of \( H \). We will simply write \( a + b \) for \((a,b)\). A hyperaddition \( + \) on \( H \) is associative if the following condition holds: for all \( a, b, c \in H \),

\[
a + (b + c) = (a + b) + c.
\]  

(1)

Note that in general for subsets \( A \) and \( B \) of \( H \), we write \( A + B := \bigcup_{a \in A \in B} a + b \) and hence the notation in (1) makes sense. Also, we will always write a singleton \( \{a\} \) as \( a \).

Definition 2.9. Let \( H \) be a nonempty set with an associative hyperaddition \( + \). We say that \((H, +)\) is a canonical hypergroup when the following conditions hold:

- \( \exists! 0 \in H \) such that \( a + 0 = a \) for all \( a \in H \); existence of identity.
- \( \forall a \in H, \exists b (=: -a) \in H \) such that \( 0 \in a + b \); existence of inverses.
- \( \forall a, b, c \in H \), if \( a \in b + c \), then \( c \in a + (-b) \); ‘hyper-subtraction’ or reversibility.

We will write \( a - b \) instead of \( a + (-b) \) for brevity of notation.

Definition 2.10. By a hyperring, we mean a nonempty set \( H \) with a binary operation \( \cdot \) and hyperaddition \( + \) such that \((H, +, 0)\) is a canonical hypergroup and \((H, \cdot, 1)\) is a commutative monoid satisfying the following conditions: for all \( a, b, c \in H \),

\[
a \cdot (b + c) = a \cdot b + a \cdot c, \quad 0 \cdot a = 0, \quad \text{and} \quad 1 \neq 0.
\]

When \((H - \{0\}, \cdot, 1)\) is a group, we call \( H \) a hyperfield.

Definition 2.11. Let \( H_1 \) and \( H_2 \) be hyperrings. A homomorphism of hyperrings from \( H_1 \) to \( H_2 \) is a function \( f : H_1 \rightarrow H_2 \) such that \( f \) is a monoid morphism with respect to multiplication satisfying the following conditions:

\[
f(0) = 0 \quad \text{and} \quad f(a + b) \subseteq f(a) + f(b), \quad \forall a, b \in H_1.
\]

The following are some typical examples of hyperfields found in the literature:

Example 2.12 (\( \mathbb{K}; \) Krasner hyperfield). Let \( \mathbb{K} := \{0, 1\} \) and impose the usual multiplication \( 0 \cdot 0 = 0, \ 1 \cdot 1 = 1 \), and \( 0 \cdot 1 = 0 \). Hyperaddition is defined as \( 0 + 1 = 1, \ 0 + 0 = 0, \) and \( 1 + 1 = \mathbb{K} \). The structure \( \mathbb{K} \) is the Krasner hyperfield.

Example 2.13 (\( \mathbb{S}; \) hyperfield of signs). Let \( \mathbb{S} := \{-1, 0, 1\} \) and impose multiplication in a usual way following the rule of signs; \( 1 \cdot 1 = 1, \ (-1) \cdot 1 = (-1), \ (-1) \cdot (-1) = 1, \) and \( 1 \cdot 0 = (-1) \cdot 0 = 0 \cdot 0 = 0 \). Hyperaddition also follows the rule of signs as follows:

\[
1 + 1 = 1, \ (-1) + (-1) = (-1), \ 1 + 0 = 1, \ (-1) + 0 = (-1), \ 0 + 0 = 0, \ 1 + (-1) = \mathbb{S}.
\]

The structure \( \mathbb{S} \) is the hyperfield of signs.

Example 2.14 (\( \mathbb{P}; \) phase hyperfield). Let \( \mathbb{P} := S^1 \cup \{0\} \), where \( S^1 \) is the unit circle in the complex plane. The multiplication on \( \mathbb{P} \) is the usual multiplication of complex numbers. Hyperaddition is defined by:

\[
a + b = \begin{cases} 
\{-a, 0, a\} & \text{if } a = -b \text{ (} -b \text{ as a complex number)} \\
\text{the shorter open arc connecting } a \text{ and } b & \text{if } a \neq -b,
\end{cases}
\]

The structure \( \mathbb{P} \) is the phase hyperfield.
Example 2.15 \( (\mathbb{T}; \text{tropical hyperfield}) \). Let \( G \) be a (multiplicative) totally ordered abelian group. Then one can enrich the structure of \( G \) to define a hyperfield. To be precise, let \( G_{hyp} := G \cup \{ -\infty \} \) and define multiplication via the multiplication of \( G \) together with the rule \( g \cdot (-\infty) = -\infty \) for all \( g \in G \). Hyperaddition is defined as follows:

\[
a + b = \begin{cases} 
\max\{a, b\} & \text{if } a \neq b \\
-\infty, a & \text{if } a = b,
\end{cases}
\]

where \([-\infty, a] := \{ g \in G_{hyp} \mid g \leq a \} \) with \(-\infty\) the smallest element. Then one can easily see that \( G_{hyp} \) is a hyperfield. When \( G = \mathbb{R} \), the set of real numbers (considered as a totally ordered abelian group with respect to the usual addition), we let \( \mathbb{T} := \mathbb{R}_{hyp} \). The structure \( \mathbb{T} \) is the tropical hyperfield.

Remark 2.16. One easily observes that for any hyperfield \( H \), there exists a unique homomorphism \( \varphi : H \to \mathbb{K} \) sending every nonzero element to 1 and 0 to 0. In other words, \( \mathbb{K} \) is the final object in the category of hyperfields.

In what follows, let \((H, \boxplus, \odot)\) be a hyperfield, \( H^\times = H - \{0_H\} \), \( r \) a positive integer, \([r] = \{1, ..., r\} \), \( x \) an element of \( E^r \) such that \( x(i) \in E \) is the \( i \)th coordinate of \( x \) unless otherwise stated. Now we recall the two notions (weak and strong) of matroids over hyperfields introduced by Baker and Bowler. These notions are given cryptomorphically by structures analogous to the bases and circuits of ordinary matroids. Their definition simultaneously generalizes several existing theories of “matroids with extra structure,” evidenced by the following examples:

Example 2.17. Matroids over the following hyperfields have been studied in the past:

- A (strong or weak) matroid over a field \( K \) is a linear subspace.
- A (strong or weak) matroid over the Krasner hyperfield \( \mathbb{K} \) is an ordinary matroid.
- A (strong or weak) matroid over the hyperfield of signs \( \mathbb{S} \) is an oriented matroid.
- A (strong or weak) matroid over the tropical hyperfield \( \mathbb{T} \) is a valuaded matroid.

We first recall the generalization of bases to the setting of matroids over hyperfields. This is done via Grassmann-Plücker functions.

Definition 2.18. Let \( H \) be a hyperfield, \( E \) a finite set, \( r \) a nonnegative integer, and \( \Sigma_r \) the symmetric group on \( r \) letters with a canonical action on \( E^r \) (acting on indices).

1. A function \( \varphi : E^r \to H \) is a nontrivial \( H \)-alternating function when:
   (G1) The function \( \varphi \) is not identically zero.
   (G2) For all \( x \in E^r \) and all \( \sigma \in \Sigma_r \) we have \( \varphi(\sigma \cdot x) = \text{sgn}(\sigma)\varphi(x) \).
   (G3) If \( x \in E^r \) has \( x(i) = x(j) \) for some \( i < j \), then \( \varphi(x) = 0_H \).
2. A nontrivial \( H \)-alternating function \( \varphi : E^r \to H \) is a weak-type Grassmann-Plücker function over \( H \) when:
   (WG) For all \( a, b, c, d \in E \) and all \( x \in E^{r-2} \) we have
   \[
   0_H \in \varphi(a, b, x)\varphi(c, d, x) - \varphi(a, c, x)\varphi(b, d, x) + \varphi(b, c, x)\varphi(a, d, x).
   \]
3. A nontrivial \( H \)-alternating function \( \varphi : E^r \to H \) is a strong-type Grassmann-Plücker function over \( H \) when:
   (SG) For all \( x \in E^{r+1} \) and all \( y \in E^{r-1} \) we have
   \[
   0_H \in \sum_{k=1}^{r+1} (-1)^k \varphi(x|_{[r+1]\setminus\{k\}})\varphi(x(k), y).
   \]
4. The rank of a Grassmann-Plücker function \( \varphi : E^r \to H \) is \( r \).
5. Two Grassmann-Plücker functions \( \varphi, \psi : E^r \to H \) are equivalent when there is an element \( a \in H^\times \) with \( \psi = a \odot \varphi \).
A matroid over $H$ is an $H^\times$-equivalence class $[\varphi]$ of a Grassmann-Plücker function $\varphi$.

Before presenting circuits of matroids over hyperfields, we need the following technical definition.

**Definition 2.19.** Let $S$ be a collection of inclusion-incomparable subsets of a set $E$. A modular pair in $S$ is a pair of distinct elements $X,Y \in S$ such that for all $A,B \in S$, if $A \cup B \subseteq X \cup Y$, then $A \cup B = X \cup Y$.

Having Definition 2.19 we can now give definitions of collections of circuits for matroids over hyperfields. In what follows, we will simply write $\sum$ instead of $\boxplus$ if the context is clear.

**Definition 2.20.** Let $E$ be a finite set, $(H,\boxplus,\circ)$ a hyperfield, $H^E$ the set of functions from $E$ to $H$, and for any $X \in H^E$, we define $\text{supp}(X) := \{a \in E \mid X(a) \neq 0_H\}$.

1. A collection $C \subseteq H^E$ is a family of pre-circuits over $H$ when it satisfies the following axioms:
   (C1) $0 \notin C$
   (C2) $H^\times \circ C = C$
   (C3) For all $X,Y \in C$, if $\text{supp}(X) \subseteq \text{supp}(Y)$, then $Y = a \circ X$ for some $a \in H^\times$.

2. A pre-circuit set $C$ over $H$ is a weak-type circuit set when it satisfies the following additional axiom:
   (WC) $\forall X,Y \in C$ such that $\{\text{supp}(X),\text{supp}(Y)\}$ forms a modular pair in $\text{supp}(C) := \{\text{supp}(X) \mid X \in C\}$ and for all $e \in \text{supp}(X) \cap \text{supp}(Y)$, there is a $Z \in C$ such that
   \[Z(e) = 0_H \text{ and } Z \in X(e) \circ Y - Y(e) \circ X,\]
   i.e., for all $f \in E$, $Z(f) \in (X(e) \circ Y(f)) - (Y(e) \circ X(f))$.

3. A pre-circuit set $C$ over $H$ is a strong-type circuit set when it satisfies the following additional axioms:
   (SC1) The set $\text{supp}(C)$ is the set of circuits of an ordinary matroid $M_C$.
   (SC2) For all bases $B \in B_C$ and all $X \in C$ we have
   $X \in \sum_{e \in E \setminus B} X(e) \circ Y_{B,e},$
   where $Y_{B,e}$ is the (unique) element of $C$ with $Y_{B,e}(e) = 1$ and $\text{supp}(Y_{B,e})$ is the fundamental circuit of $e$ with respect to $B$.

**Remark 2.21.** The definition of strong-type circuit sets given above is not the original definition; rather this is equivalent to the original (much less intuitive) definition by [5, Theorem 3.8, Remark 3.9]. For our purposes, we shall use the definition given above.

The following result is proved in [5]:

**Proposition 2.22.** Let $H$ be a hyperfield. The $H^\times$-orbits of Grassmann-Plücker functions over $H$ are in natural one-to-one correspondence with the $H$-circuits of a matroid, preserving both ranks and types (weak and strong).

The correspondence is described as follows:

Given a Grassmann-Plücker function $\varphi$ over $H$, one first shows that the collection of subsets $B \subseteq E$ for which an ordering $B$ has $\varphi(B) \neq 0_H$ forms a set of bases for an ordinary matroid $M_\varphi$. Next, one can define a set of $H$-circuits by defining for all ordered bases $B$ of $M_\varphi$ and all $e \in E \setminus B$ a function $X = X_{B,e}$ supported on the fundamental circuit for $e$ by $B$ via the equality
\[X(B(i))X(e)^{-1} = (-1)^i\varphi(e,B_{[\sigma] \setminus \{i\}})\varphi(B)^{-1}\]
This equality uniquely determines $X : E \to H$, up to the multiplicative action of $H^\times$. The collection of all such $X$ is a collection of $H$-circuits of the same type as $\varphi$.

Constructing a Grassmann-Plücker function from circuits is more difficult to describe, and requires the additional notion of dual pairs. An explicit description of this construction is unnecessary for our purposes; the interested reader is referred to [5].

We now describe the duality operation for matroids over hyperfields in terms of Grassmann-Plücker functions and subsequently in terms of circuits. It should be noted that the duality described in [5] incorporates a notion of conjugation generalizing the complex conjugation. This changes the duality operation, but the change is equally well described by another operation (called “pushforward through a morphism”) as noted in a footnote in [5, §6]. Our treatment will also assume that the conjugation is trivial.

Fix a total ordering $\leq$ of $E$. A dual of a Grassmann-Plücker function $\varphi$ over $H$ is defined by the equation $\varphi^*(B) := \text{sgn}_{\leq}(B, E \setminus B)\varphi(E \setminus B)$ for all cobases $B$ of the underlying matroid of $\varphi$, using the convention that $S$ denotes the ordered tuple with coordinates the elements of $S$ arranged according to our fixed total ordering on $E$ and $\text{sgn}_{\leq}(B, E \setminus B)$ denotes the sign of the permutation given by the word $(B, E \setminus B)$ with respect to the ordering $\leq$. The definition can be uniquely extended to the set $E^{#E-r}$ by alternation and the degeneracy conditions for Grassmann-Plücker functions over $H$. It is relatively easy to see that if $\varphi$ is a Grassmann-Plücker function, then $\varphi^*$ is a Grassmann-Plücker function of the same type. Notice that this duality is well-defined up to the chosen ordering; a different ordering will induce a Grassmann-Plücker function which is multiplied by the sign of the permutation used to translate between the two orderings. In particular, this notion of duality is constant on the level of $H^\times$-orbits of Grassmann-Plücker functions, and thus sends an $H$-matroid $M$ to an $H$-matroid $M^*$ of the same type despite the fact that there is no canonical dual to the original Grassmann-Plücker function.

The dual of a circuit set requires some more care to define; it is here that the contrast between weak and strong $H$-matroids is most stark.

**Definition 2.23.** Let $H$ be a hyperfield and $E$ a finite set.

1. The dot product of two functions $X, Y : E \to H$ is the following subset of $H$:
   
   $X \cdot Y := \sum_{e \in E} X(e)Y(e)$.

2. Two functions $X$ and $Y$ are strong orthogonal, denoted $X \perp_s Y$, when $0_H \in X \cdot Y$.

3. Two functions $X$ and $Y$ are weak orthogonal, denoted $X \perp_w Y$, when either $X \perp_s Y$ or the following condition holds:
   
   $\#(\text{supp}(X) \cap \text{supp}(Y)) > 3$.

4. Let $C$ be a set of strong $H$-circuits on $E$, we define the following subset of $H^E$:
   
   $C^\perp := \{ X : E \to H \mid X \perp_s Y \text{ for all } Y \in C \}$.

5. Let $C$ be a set of weak $H$-circuits, we define the following subset of $H^E$:
   
   $C^\perp := \{ X : E \to H \mid X \perp_w Y \text{ for all } Y \in C \}$.

For ease of notation the symbol $\perp$ is to be understood in context as either $\perp_s$ or $\perp_w$.

**Definition 2.24.** Let $H$ be a hyperfield and $M$ be a weak (resp. strong) $H$-matroid with the set $C$ of weak type (resp. strong type) $H$-circuits. The cocircuits of $C$, denoted by $C^*$, are the elements of the perpendicular set $C^\perp$ (resp. $C^\perp$) with minimal support.
Remark 2.25. In [5, §6.6], Baker and Bowler show that this determines an $H$-matroid with the properties $\text{supp}(C^*) = (\text{supp}(C))^*$ and $C^{**} = C$; in other words, the underlying matroid of the dual is the dual of the underlying matroid and the double dual is identical to the original $H$-matroid.

Remark 2.26. From an algebraic geometric viewpoint, Baker and Bowler’s definition of matroids over hyperfields can be considered as points of a Grassmannian over a hyperfield $H$. Motivated by this observation, in [13], the second author proves that certain topological spaces (the underlying spaces of a scheme, Berkovich analytification of schemes, real schemes) are homeomorphic to sets of rational points of a scheme over a hyperfield. Also, recently L. Anderson and J. Davis defined and investigated hyperfield Grassmannians in connection to the MacPhersonian (from oriented matroid theory) in [2].

2.3 Matroid-Minor Hopf algebras. In this subsection, we recall the definition of matroid-minor Hopf algebras. First we briefly recall the definition of Hopf algebras; interested readers are referred to [7] for more details.

Definition 2.27. Let $k$ be a field. A commutative $k$-algebra $A$ is a Hopf algebra if $A$ is equipped with maps

1. (Comultiplication) $\Delta : A \rightarrow A \otimes_k A$,
2. (Counit) $\varepsilon : A \rightarrow k$,
3. (Antipode) $S : A \rightarrow A$

such that the following diagrams commute:

\[
\begin{array}{ccc}
A \otimes_k A & \xrightarrow{\Delta \otimes \text{id}} & A \otimes_k A \\
\text{id} \otimes \Delta & \downarrow & \Delta \\
A & \xrightarrow{\Delta} & A \otimes_k A,
\end{array}
\quad
\begin{array}{ccc}
A \otimes_k A & \xrightarrow{\varepsilon \otimes \text{id}} & k \otimes_k A \\
\Delta & \downarrow \sim & \Delta \\
A & \xrightarrow{\text{id}} & A,
\end{array}
\quad
\begin{array}{ccc}
A \otimes_k A & \xrightarrow{\mu \circ (S \otimes \text{id})} & A \\
\Delta & \downarrow & \varepsilon \\
A & \xrightarrow{\text{id}} & k,
\end{array}
\]

where $\mu : A \otimes_k A \rightarrow A$ is the multiplication of $A$. If $A$ is only equipped with $\Delta$ and $\varepsilon$ satisfying the first two commutative diagrams, then $A$ is a bialgebra.

Definition 2.28. Let $(A, \mu, \Delta, \eta, \varepsilon)$ be a bialgebra over a field $k$.

1. $A$ is graded if there is a grading $A = \bigoplus_{i \in \mathbb{N}} A_i$ which is compatible with the bialgebra structure of $A$, i.e., $\mu$, $\Delta$, $\eta$, and $\varepsilon$ are graded $k$-linear maps.
2. $A$ is connected if $A$ is graded and $A_0 = k$.

Definition 2.29. Let $A_1$ and $A_2$ be Hopf algebras over a field $k$. A homomorphism of Hopf algebras is a $k$-bialgebra map $\varphi : A_1 \rightarrow A_2$ which preserves the antipodes, i.e., $S_{A_2} \varphi = \varphi S_{A_1}$.

The following theorem shows that indeed there is no difference between bialgebra maps and Hopf algebra maps.

Theorem 2.30. [7, Proposition 4.2.5.] Let $A_1$ and $A_2$ be Hopf algebras over a field $k$. Let $\varphi : A_1 \rightarrow A_2$ be a morphism of $k$-bialgebras. Then $\varphi$ is indeed a homomorphism of Hopf algebras.

We also introduce the following notation:

Definition 2.31. Let $A$ be a Hopf algebra over a field $k$ and $i \in \mathbb{Z}_{\geq 1}$.

1. (Iterated multiplication): $\mu^i : A^{\otimes(i+1)} \rightarrow A$ is defined inductively as $\mu^i := \mu \circ (\text{id} \otimes \mu^{(i-1)})$.
2. (Iterated comultiplication): $\Delta^i : A \rightarrow A^{\otimes(i+1)}$ is defined inductively as $\Delta^i := (\text{id} \otimes \Delta^{(i-1)}) \circ \Delta$. 

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Now, let's recall the definition of matroid-minor Hopf algebras, first introduced by W. R. Schmitt in [18]. Let $\mathcal{M}$ be a collection of matroids which is closed under taking minors and direct sums. Let $\mathcal{M}_{\text{iso}}$ be the set of isomorphism classes of matroids in $\mathcal{M}$. For a matroid $M$ in $\mathcal{M}$, we write $[M]$ for the isomorphism class of $M$ in $\mathcal{M}_{\text{iso}}$. One easily see that $\mathcal{M}_{\text{iso}}$ can be enriched to a commutative monoid with the direct sum:

$$[M_1] \cdot [M_2] := [M_1 \oplus M_2]$$

and the identity $[\emptyset]$, the equivalence class of the empty matroid (considered as the unique matroid associated to the empty ground set). Let $A$ be the monoid algebra $k[\mathcal{M}_{\text{iso}}]$ over a field $k$.

For any matroid $M$, let $E_M$ denote the ground set of $M$. Consider the following maps:

- (Comultiplication)
  \[ \Delta : k[\mathcal{M}_{\text{iso}}] \to k[\mathcal{M}_{\text{iso}}] \otimes_k k[\mathcal{M}_{\text{iso}}], \quad [M] \mapsto \sum_{S \subseteq E_M} [M|_S] \otimes [M/S]. \]

- (Counit)
  \[ \varepsilon : k[\mathcal{M}_{\text{iso}}] \to k, \quad [M] \mapsto \begin{cases} 1 & \text{if } E_M = \emptyset \\ 0 & \text{if } E_M \neq \emptyset. \end{cases} \]

Under the above maps, $k[\mathcal{M}_{\text{iso}}]$ becomes a connected bialgebra; $k[\mathcal{M}_{\text{iso}}]$ is graded by cardinalities of ground sets. It follows from the result of M. Takeuchi [20] that $k[\mathcal{M}_{\text{iso}}]$ has a unique Hopf algebra structure with a unique antipode $S$ given by:

\[ S = \sum_{i \in \mathbb{N}} (-1)^i \mu^{-1} \circ \pi^* \circ \Delta^{-1}, \tag{3} \]

where $\mu^{-1}$ is a canonical map from $k$ to $k[\mathcal{M}_{\text{iso}}]$, $\Delta^{-1} := \varepsilon$, and $\pi : k[\mathcal{M}_{\text{iso}}] \to k[\mathcal{M}_{\text{iso}}]$ is the projection map defined by

\[ \pi|_{A_n} \begin{cases} \text{id} & \text{if } n \geq 1 \\ 0 & \text{if } n = 0, \end{cases} \]

and extended linearly to $k[\mathcal{M}_{\text{iso}}]$, where $A_n$ is the $n$th graded piece of $A$.

### 3. Minors and sums of matroids over hyperfields

In this section we explicitly write out the constructions of restriction, deletion, contraction, and direct sums for matroids over hyperfields. We do this cryptomorphically via both circuits and Grassmann-Plücker functions in both the weak and strong cases. Primarily, we define the restriction, and subsequently use our characterization to derive the other cryptomorphic descriptions of minors. It should be noted that formulas for deletion and contraction in the case of phirotopes are given in [3] for phased matroids and in [5] for general Grassmann-Plücker function without proof. For completeness, we give full proofs and expand the previous work by giving formulas for the circuits of these objects as well.

#### 3.1 Circuits of $H$-Matroid Restrictions

Let $H$ be a hyperfield, $E$ be a finite set, $\mathcal{C}$ be a set of (either weak-type or strong-type) $H$-circuits on $E$, and $S \subseteq E$. Recall that $H^S$ is the set of functions from $S$ to $H$. We define the following notation:

\[ \mathcal{C} \mid S := \{ X|_S \in H^S \mid X \in \mathcal{C} \text{ and } \text{supp}(X) \subseteq S \}. \tag{4} \]

We have the following:

---

1There is an error in [3]; in particular, the authors make the false assumption that the Axiom (WG) implies Axiom (SG). Thus they fail to handle the weak case separately from the strong case. While [5] fixes this issue, the authors merely state this result without presenting details.
Proposition 3.1. Let $C$ be a set of weak-type (resp. strong-type) $H$-circuits of a matroid $M$ over $H$ on a ground set $E$.

1. $\forall S \subseteq E$, the set $C|S$ is a set of weak-type (resp. strong-type) $H$-circuits on $S$.
2. The underlying matroid of the $H$-matroid $M$ determined by $C|S$ is precisely the restriction of the underlying matroid $\text{supp}(M)|S$. In other words, the restriction commutes with the push-forward operation to the Krasner hyperfield $\mathbb{K}$.

Proof. One can easily see that if $C$ is a set of circuits of an $H$-matroid, then

$$\text{supp}(C|S) = \text{supp}(C)|S$$

and hence $\text{supp}(C|S)$ is the set of circuits of the restriction of the ordinary matroid; in particular, the second statement follows immediately from the first statement.

Now, we prove the first statement. Suppose that $X, Y \in C$ have $\text{supp}(X)$, $\text{supp}(Y) \subseteq S$. In this case, we have that

$$\text{supp}(X|S) = \text{supp}(X) \text{ and } \text{supp}(Y|S) = \text{supp}(Y).$$

We first claim that if $C$ is a set of pre-circuits over $H$ on $E$, then $C|S$ is also a set of pre-circuits over $H$ on $S$. Indeed, since $\text{supp}(X) \subseteq S$ and $X \neq \emptyset$, we have that $X|S \neq \emptyset$ and

$$(a \circ X|S) \in C|S, \quad \forall a \in H^\times.$$

Finally, if $\text{supp}(X|S) \subseteq \text{supp}(Y|S)$, then

$$\text{supp}(X) = \text{supp}(X|S) \subseteq \text{supp}(Y|S) = \text{supp}(Y)$$

yields $Y = a \circ X$ for some $a \in H^\times$ and hence $Y|S = a \circ X|S$ as desired. This proves that $C|S$ is a set of pre-circuits over $H$ on $S$.

Next we prove that if $C$ is a set of weak-type $H$-circuits on $E$, then $C|S$ is also a set of weak-type $H$-circuits on $S$. In fact, if $X|S$ and $Y|S$ form a modular pair in $C|S$, then (5) implies immediately that $X$ and $Y$ are a modular pair as well in $C$. More precisely, in this case, the condition

$$A \cup B \subseteq \text{supp}(X) \cup \text{supp}(Y) \subseteq S, \quad A, B \subset \text{supp}(C)$$

implies that $A, B \subseteq S$. Thus, $\forall e \in \text{supp}(X) \cap \text{supp}(Y)$, there exists $Z \in C$ such that

$$Z(e) = 0 \text{ and } Z \in X(e)Y - Y(e)X.$$ 

On the other hand, if $a \notin \text{supp}(X) \cup \text{supp}(Y)$, then

$$X(e)Y(a) - Y(e)X(a) = \{0\}. $$

Thus, for $A \in C$, $A \in X(e)Y - Y(e)X$ implies that

$$\text{supp}(A) \subseteq \text{supp}(X) \cup \text{supp}(Y) \subseteq S.$$

Hence, $\text{supp}(Z) \subseteq S$ and $Z|S \in C|S$. One can easily see that $C|S$ satisfies the Axiom (WC). This proves that $C|S$ is a set of weak-type $H$-circuits on $S$.

Finally, we show that if $C$ is a set of strong-type $H$-circuits on $E$, then $C|S$ is also a set of strong-type $H$-circuits on $S$. As we mentioned before, $\text{supp}(C|S) = \text{supp}(C)|S$ is a set of circuits of a matroid as these are given by the same formula and $\text{supp}C$ is a set of circuits of an ordinary matroid; in particular $C|S$ satisfies axiom (SC1). Let $B_{C|S}$ (resp. $B_C$) be the set of bases of an underlying matroid $M_C|S$ (resp. $M_C$) given by the set $\text{supp}(C|S)$ (resp. $\text{supp}(C)$) of circuits. If $B \in B_{C|S}$, then we have that

$$B = \tilde{B} \cap S \text{ for some } \tilde{B} \in B_C.$$ 

It follows from (SC2), applied to $C$ with $\tilde{B}$ and $X$, that

$$X \in E \setminus \tilde{B} \subseteq Y_{\tilde{B}, e}.$$
Now $Y_{B,e}|S = Y_{B,e}$ by incomparability of circuits in ordinary matroids, and thus we see (7) implies that

$$X|S \in \sum_{e \in E \setminus B} X|S(e) \odot Y_{B,e}.$$  

It follows that the axiom (SC2) holds for $C|S$ and hence $C|S$ is a strong-type $H$-circuit set, as claimed. \hfill \square

Now, thanks to Proposition 3.3, the following definition makes sense.

**Definition 3.2.** Let $M$ be a matroid over hyperfield $H$ on a ground set $E$ given by weak (resp. strong) $H$-circuits $C$, and let $S$ be a subset of $E$. The restriction of matroid $M$ to $S$ is the matroid $M|S$ over $H$ given by weak (resp. strong) $H$-circuits $C|S$.

### 3.2 Grassmann-Plücker Functions of $H$-Matroid Restrictions

We now describe restriction of $H$-matroids via Grassmann-Plücker functions. Let $H$ be a hyperfield, $E$ a finite set, $r$ a positive integer, and $\varphi$ a (weak-type or strong-type) Grassmann-Plücker function over $H$ on $E$ of rank $r$. Let $M_\varphi$ denote the underlying matroid of $\varphi$ given by bases:

$$B_\varphi = \{\{b_1, \ldots, b_r\} \subseteq E \mid \varphi(b_1, \ldots, b_r) \neq 0\}.$$ 

Recall that for any ordered basis $B = \{b_1, b_2, \ldots, b_k\}$ of $M_\varphi/(E \setminus S)$, we let

$$B = (b_1, b_2, \ldots, b_k) \in E^k.$$ 

Now, for any subset $S \subseteq E$ and any (ordered) basis $B = \{b_1, b_2, \ldots, b_k\}$ of $M_\varphi/(E \setminus S)$, we define

$$\varphi^B : S^{r-k} \to H, \quad A \mapsto \varphi(A, B). \quad (8)$$

**Proposition 3.3.** Let $\varphi$ be a weak-type (resp. strong-type) Grassmann-Plücker function over $H$ on $E$ of rank $r$ and let $S \subseteq E$. For all ordered bases $B$ of $M_\varphi/(E \setminus S)$, the function $\varphi^B$ is a weak-type (resp. strong-type) Grassmann-Plücker function. Moreover, all such $\varphi^B$ determine the $H$-circuits $C|S$ of $M|S$.

**Proof.** For the notational convenience, we let $[n] = \{1, 2, \ldots, n\}$ and we regard $B$ as a function $B : [k] \to E$. First, one can observe that $\varphi^B$ is a nontrivial $H$-alternating function as $\varphi^B$ is a restriction of a nontrivial $H$-alternating function to a subset containing a base of $M_\varphi$. We claim that if $\varphi$ is a weak-type Grassmann-Plücker function over $H$, then $\varphi^B$ is also a weak-type Grassmann-Plücker function over $H$. To see this, let $a, b, c, d \in E$ and $Y : [r-k-1] \to E$ be given. Applying Axiom (WG) to $a, b, c, d \in E$ and $x = (Y, B) \in E^{r-2}$, we obtain the following:

$$0_H \in \varphi(a, b, Y, B)\varphi(c, d, Y, B) - \varphi(a, c, Y, B)\varphi(b, d, Y, B) + \varphi(a, d, Y, B)\varphi(b, c, Y, B)$$

$$= \varphi^B(a, b, Y)\varphi^B(c, d, Y) - \varphi^B(a, c, Y)\varphi^B(b, d, Y) + \varphi^B(a, d, Y)\varphi^B(b, c, Y).$$

This shows that Axiom (WG) holds for $\varphi^B$ and hence $\varphi^B$ is a weak-type Grassmann-Plücker function over $H$.

We next show that if $\varphi$ is a strong-type Grassmann-Plücker function over $H$, then $\varphi^B$ is also a strong-type Grassmann-Plücker function over $H$. Indeed, let $X : [r-k+1] \to S$ and $Y : [r-k-1] \to S$ be given. Applying Axiom (SG) to $x := (X, B)$ and $y := (Y, B)$, we
obtain the following:

\[ 0_H \in \sum_{j \in [r-k+1]} (-1)^j \varphi(\mathbf{X}_{[r-k+1] \setminus (j)}, \mathbf{B}) \varphi(\mathbf{X}(j), \mathbf{Y}, \mathbf{B}) + \sum_{j \in [k]} (-1)^{r-k+1+j} \varphi(\mathbf{X}, \mathbf{B}_{[k] \setminus (j)}) \varphi(\mathbf{B}(j), \mathbf{Y}, \mathbf{B}) \]

\[ = \sum_{j \in [r-k+1]} (-1)^j \varphi(\mathbf{X}_{[r-k+1] \setminus (j)}, \mathbf{B}) \varphi(\mathbf{X}(j), \mathbf{Y}, \mathbf{B}) \]

\[ = \sum_{j \in [r-k+1]} (-1)^j \varphi^B(\mathbf{X}_{[r-k+1] \setminus (j)}) \varphi^B(\mathbf{X}(j), \mathbf{Y}). \]

This shows that Axiom (SG) holds for \( \varphi^B \) and hence \( \varphi^B \) is a strong-type Grassmann-Plücker function over \( H \).

Finally, we show that \( \varphi^{B'} \) determines the same set of circuits as \( \varphi^B \) for all ordered bases \( \mathbf{B} \) and \( \mathbf{B}' \) of \( M_\varphi/(E \setminus S) \). Indeed, we show that the circuits determined by \( \varphi^B \) are precisely \( C | S \).

Fix an ordered base \( \mathbf{A} \) of \( M_\varphi | S \). Now, \( \mathbf{y} := (\mathbf{A}, \mathbf{B}) \) is an ordered base of \( M_\varphi \).

Moreover, for all \( e \in S \setminus A \), the fundamental \( H \)-circuit \( X = X_{A \cup B,e} \) satisfies

\[ X|_S(A(i))X|_S(e)^{-1} = X(y(i))X(e)^{-1} \]

\[ = (-1)^i \varphi(e, A_{[r-k] \setminus (i)}, B) \varphi(A, B)^{-1} \]

\[ = (-1)^i \varphi^B(e, A_{[r] \setminus (i)}) \varphi^B(A)^{-1}. \]

for all \( i \in [r-k] \) by the cryptomorphism relating \( C \) and \( \varphi \). On the other hand, \( X|_S = X_{A \cup B,e} | S \) is the fundamental \( H \)-circuit for \( e \) by the basis \( A \) in \( C | S \). Hence \( C | S \) is the set of \( H \)-circuits determined by the Grassmann-Plücker function \( \varphi^B \) for all ordered bases \( \mathbf{B} \) of \( M_\varphi/(E \setminus S) \). \( \square \)

We summarize our results from this section as follows:

**Proposition 3.4.** The restriction of an \( H \)-matroid to a subset is well-defined, and admits cryptomorphic description in terms of Grassmann-Plücker functions over \( H \) and \( H \)-circuits. Furthermore, this correspondence preserves types and all such restrictions have underlying matroid the ordinary restriction. Finally, we have the following:

1. The restriction \( M|S \) is given by \( H \)-circuits

\[ C | S = \{ X|_S \mid X \in C \text{ and } \text{supp}(X) \subseteq S \}. \]

2. The restriction \( M|S \) is obtained by fixing any base \( \mathbf{B} = (b_1, b_2, \cdots, b_k) \) of \( M_\varphi/(E \setminus S) \) and defining:

\[ \varphi^B : S^{r-k} \rightarrow H, \quad x \mapsto \varphi(x, \mathbf{B}). \]

In particular, the \( H \)-matroid \( M|S \) is determined by the \( H\times\)-class \([\varphi^B] \) of any such \( \mathbf{B} \).

### 3.3 Deletion and Contraction.

As noted previously, deletion and contraction for \( H \)-matroids were defined by Baker and Bowler in [5] by using Grassmann-Plücker functions. In this section, we also provide a cryptomorphic definition for deletion and contraction via \( H \)-circuits by appealing to the definitions of dual \( H \)-matroids and restrictions. Throughout let \( H \) be a hyperfield, \( E \) a finite set, \( r \) a positive integer, and \( M \) be a matroid over \( H \) on ground set \( E \) of rank \( r \) with circuits \( C \) and a Grassmann-Plücker function \( \varphi \).

**Definition 3.5.** Let \( S \) be a subset of \( E \).

1. The deletion \( M \setminus S \) of \( S \) from \( M \) is the \( H \)-matroid \( M|(E \setminus S) \).
2. The contraction \( M/S \) of \( S \) from \( M \) is the \( H \)-matroid \((M^* \setminus S)^* \).

**Remark 3.6.** It follows from Definition 3.5 that if \( M \) is a weak-type (resp. strong-type), then the deletion \( M \setminus S \) and the contraction \( M/S \) are also weak-type (resp. strong-type).

**Proposition 3.7.** Let \( S \) be a subset of \( E \).
(1) The deletion $M \setminus S$ is given by $H$-circuits
\begin{equation}
\mathcal{C}|(E \setminus S) = \{X|_{E \setminus S} \mid X \in \mathcal{C} \text{ and } S \cap \text{supp}(X) = \emptyset\}.
\end{equation}

(2) The deletion $M \setminus S$ is obtained by fixing base $B = (b_1, b_2, \ldots, b_{r-k})$ of $M_\varphi/S$ and letting $\varphi^B : (E \setminus S)^k \to H : \mathbf{x} \mapsto \varphi(\mathbf{x}, B)$. The $H$-matroid $M \setminus S$ is determined by the $H^\times$-class $[\varphi^B]$ for any such $B$.

Proof. The first statement is immediate from the definition of the deletion and the second statement directly follows from Proposition 3.4.

A description of contractions is a bit more complicated.

**Proposition 3.8.** Let $S$ be a subset of $E$.

1. The contraction $M/S$ is given by $H$-circuits $\mathcal{C}'' = (\mathcal{C}^*|(E \setminus S))^\perp$. More explicitly
\begin{equation}
\mathcal{C}'' = \min \left\{ Z \in H^{E \setminus S} \setminus \{0\} \mid \begin{array}{l}
Z \perp Y \forall Y \in \mathcal{C} \\
\text{with supp}(X) \cap S = \emptyset \text{ and } X \perp Y \text{ for all } Y \in \mathcal{C}
\end{array} \right\}.
\end{equation}

2. The contraction $M/S$ is given by the class of Grassmann-Plücker functions $((\varphi^*)_{E \setminus S})^\perp$. More explicitly, let $B = (b_1, \ldots, b_k)$ be an ordered basis of $M_\varphi|S$. A representative of the $H^\times$-orbit of Grassmann-Plücker functions determining $M/S$ is given by
\begin{equation}
\varphi'' : (E \setminus S)^{r-k} \to H, \ x \mapsto \varphi(B, x).
\end{equation}

Proof. We note that the formula for $\varphi''$ in (11) is given in [5], with proof deferred to [3]; we give a new proof here.

Proof of (1): Since $M/S := (M^* \setminus S)^\perp$, the formula $\mathcal{C}'' = (\mathcal{C}^*|(E \setminus S))^\perp$ follows from the duality cryptomorphism and the restriction constructions of Propositions 3.4 and 3.7. Recall that
\[
\mathcal{C}^* = \min \left\{ X \in H^E \setminus \{0\} \mid X \perp Y \text{ for all } Y \in \mathcal{C} \right\},
\]
where “min” means minimal support. Now, the following shows (10):
\[
(\mathcal{C}^*|(E \setminus S))^\perp = \left( \min \left\{ X \in H^E \setminus \{0\} \mid X \perp Y \text{ for all } Y \in \mathcal{C} \right\} \right)^\perp
\]
\[
= \min \left\{ X|_{E \setminus S} \in H^{E \setminus S} \mid \begin{array}{l}
X \in H^E \setminus \{0\} \text{ and supp}(X) \cap S = \emptyset \\
\text{and } X \perp Y \text{ for all } Y \in \mathcal{C}
\end{array} \right\}^\perp
\]
\[
= \min \left\{ Z \in H^{E \setminus S} \setminus \{0\} \mid \begin{array}{l}
Z \perp Y \forall Y \in \mathcal{C} \\
\text{with supp}(X) \cap S = \emptyset \text{ and } X \perp Y \text{ for all } Y \in \mathcal{C}
\end{array} \right\}
\]

Proof of (2): We prove that $\varphi''$ is the Grassmann-Plücker function determined by $\mathcal{C}''$. Let $B$ be a base of the ordinary matroid $M_\varphi|S$. Then, for any base $A$ of the ordinary matroid $M_\varphi/S$, $A \cup B$ is a base of $M_\varphi$. Let $e \in (E \setminus S) \setminus A$ be given, and let $\mathcal{C}_{A \cup B, e}$ be the fundamental circuit of $e$ with respect to $A \cup B$ in $M$ (see (2) and the paragraph before it). Now suppose $X \in H^E \setminus \{0\}$ satisfies the conditions that:
\[
\text{supp}(X) \cap S = \emptyset, \quad X \perp Y \forall Y \in \mathcal{C}.
\]
In particular, $X \perp \mathcal{C}_{A \cup B, e}$ since $\mathcal{C}_{A \cup B, e} \in \mathcal{C}$. On the other hand, as $X(s) = \{0\} \forall s \in S$, we have that
\[
X|_{E \setminus S} \perp \mathcal{C}_{A \cup B, e}|_{E \setminus S}.
\]
Hence $\mathcal{C}_{A \cup B, e}|_{E \setminus S} = C_{A, e}$ is the fundamental $H$-circuit of $e \in E \setminus S$ with respect to $A$ in $M/S$ by incomparability of supports of elements in $\mathcal{C}''$ and the fact that $\text{supp}(\mathcal{C}_{A \cup B, e}|_{E \setminus S})$
is precisely the fundamental circuit of $e$ by $A$ in the underlying matroid of the contraction. What remains is a computation:

$$(-1)^i \varphi''(e, A|_{[r-k]\setminus\{i\}})\varphi''(A)^{-1} = (-1)^i \varphi(B, e, A|_{[r-k]\setminus\{i\}})\varphi(B, A)^{-1}$$

$$= \tilde{C}_{A\cup B, e}(A(i))\tilde{C}_{A\cup B, e}(e)^{-1}$$

$$= C_{A, e}(A(i))C_{A, e}(e)^{-1}.$$  

\[ \Box \]

### 3.4 Elementary Properties of Minors

We summarize the constructions of the preceding sections below for easy reference:

**Proposition 3.9.** Let $H$ be a hyperfield, $E$ a finite set, $r$ a positive integer, and $M$ be a matroid over $H$ of rank $r$ with circuits $C$ and a Grassmann-Plücker function $\varphi$. Let $S$ be a subset of $E$.

1. The restriction $M|S$ is given by $H$-circuits:

$$\mathcal{C}_S = \{ X|_S \mid X \in \mathcal{C} \text{ and } \text{supp}(X) \subseteq S \}. $$

2. The restriction $M|S$ is obtained by fixing an ordered base $B = (b_1, b_2, \ldots, b_k)$ of the underlying matroid $M_{\varphi}/(E \setminus S)$ and defining:

$$\varphi^B : S^{r-k} \rightarrow H, \ x \mapsto \varphi(x, B).$$

In particular, the $H$-matroid $M|S$ is determined by the $H^\times$-class $[\varphi^B]$ of any such $B$.

3. The deletion $M \setminus S$ is given by $H$-circuits:

$$\mathcal{C} |(E \setminus S) = \{ X|_{E \setminus S} \mid X \in \mathcal{C} \text{ and } S \cap \text{supp}(X) = \emptyset \}. $$

4. The deletion $M \setminus S$ is obtained by fixing an ordered base $B = (b_1, b_2, \ldots, b_{r-k})$ of $M_{\varphi}/S$ and defining:

$$\varphi^B : (E \setminus S)^k \rightarrow H, \ x \mapsto \varphi(x, B).$$

In particular, the $H$-matroid $M \setminus S$ is determined by the $H^\times$-class $[\varphi^B]$ of any $B$.

5. The contraction $M/S$ is given by $H$-circuits $\mathcal{C} = (\mathcal{C}^* |(E \setminus S))^*$. More explicitly,

$$\mathcal{C}'' = \left\{ Z \in H^E \setminus \{0\} \mid \begin{array}{l} Z \perp X|_{E \setminus S} \text{ for all } X \in H^E \setminus \{0\} \text{ with } \text{supp}(X) \cap S = \emptyset \text{ and } X \perp Y \text{ for all } Y \in \mathcal{C} \end{array} \right\}$$

6. The contraction $M/S$ is given by the class of Grassmann-Plücker functions $([\varphi^*]_{E \setminus S})^*$. More explicitly, let $B = (b_1, \ldots, b_k)$ be an ordered basis of $M_{\varphi}/S$. A representative of the $H^\times$-orbit of Grassmann-Plücker functions determining $M/S$ is given by

$$\varphi'' : (E \setminus S)^{r-k} \rightarrow H, \ x \mapsto \varphi(B, x).$$

Using the constructions above, one can easily verify that the following properties hold (cf. the properties of minors in ordinary matroids):

**Corollary 3.10.** Let $M$ be an $H$-matroid on $E$ with $S, T \subseteq E$ disjoint. We have

1. $M/\emptyset = M = M \setminus \emptyset$
2. $(M \setminus S) \setminus T = M \setminus (S \cup T)$
3. $(M/S)/T = M/(S \cup T)$
4. $(M \setminus S)/T = (M/T) \setminus S$

**Proof.** The proof of parts (1)-(3) is clear by choosing an appropriate Grassmann-Plücker function to represent both sides of the equalities using our characterization in Proposition 3.9. To see (4), one can calculate the $H$-circuits of these and obtain that they are equal.  

Finally, we have the following corollary stating that restriction, deletion, and contraction commute with pushforwards:
Example 3.12. Recall that the Krasner hyperfield $\mathbb{K}$ is the final object in the category of hyperfields. Let $H$ be a hyperfield and $f : H \rightarrow \mathbb{K}$ be the canonical map. For any $H$-matorid $M$, the pushforward $f_*M$ is just the underlying matroid of $M$. In this case, one can clearly see Corollary 3.11.

3.5 Direct sums of Matroids over Hyperfields. For matroids over hyperfields, we provide two cryptomorphic definitions (Grassmann-Plücker functions and circuits) for direct sum. Let $M$ and $N$ be $H$-matroids. To state a precise formula for a sum of Grassmann-Plücker functions, we will need some additional notation. Fix a total order $\leq$ on $E_M \sqcup E_N$ such that $x < y$ whenever $x \in E_M$ and $y \in E_N$. Now for every $x \in (E_M \sqcup E_N)^{r_M+r_N}$ either $x$ has exactly $r_M$ components in $E_M$ or not. If so, we let $\sigma_x$ denote the unique permutation of $[r_M + r_N]$ such that $\sigma_x \cdot x$ is monotone increasing with respect to $\leq$. Now, we have the following:

**Proposition 3.13.** Let $H$ be a hyperfield and $M$ (resp. $N$) be $H$-matroids of rank $r_M$ (resp. $r_N$) given by a Grassmann-Plücker function $\varphi_M$ (resp. $\varphi_N$). The function $\varphi_M \oplus \varphi_N$, defined by the following formula (\[\begin{array}{ll}
\varphi_M \oplus \varphi_N : (E_M \sqcup E_N)^{r_M+r_N} \rightarrow H, \\
x \mapsto \begin{cases} 
0, & \text{if } x \text{ does not have precisely } r_M \text{ components in } E_M \\
\text{sgn}(\sigma_x)\varphi_M(\langle \sigma_x \cdot x \rangle_{[r_M]} )\varphi_N(\langle \sigma_x \cdot x \rangle_{[r_M+r_N \setminus [r_M]}), & \text{otherwise.}
\end{cases}
\end{array}\]) is a weak-type Grassmann-Plücker function on $(E_M \sqcup E_N)^{r_M+r_N}$.

Furthermore, $\varphi_M \oplus \varphi_N$ is of strong-type precisely when both $\varphi_M$ and $\varphi_N$ are of strong-type. Moreover, the $H^X$-class of $\varphi_M \oplus \varphi_N$ depends only on $M$ and $N$.

**Proof.** Let $\varphi := \varphi_M \oplus \varphi_N$. We first show that $\varphi$ is a nondegenerate $H$-alternating function. Indeed, the function $\varphi$ is clearly nontrivial. To see that $\varphi_M \oplus \varphi_N$ is $H$-alternating, we let $\tau$ be an arbitrary permutation of $[r_M + r_N]$; note that $x$ does not have precisely $r_M$ components in $E_M$ if and only if $\tau \cdot x$ does not have precisely $r_M$ components in $E_M$. If $x$ does have precisely $r_M$ components in $E_M$, then $\sigma_{\tau \cdot x} = \sigma_x \tau^{-1}$ and so the following completes the proof of our claim:

$$(\varphi_M \oplus \varphi_N)(\tau \cdot x) = \text{sgn}(\sigma_{\tau \cdot x} )\varphi_M(\langle \sigma_{\tau \cdot x} \cdot \tau \cdot x \rangle_{[r_M]} )\varphi_N(\langle \sigma_{\tau \cdot x} \cdot \tau \cdot x \rangle_{[r_M+r_N \setminus [r_M]} )$$

$$= \text{sgn}(\sigma_x \tau^{-1} )\varphi_M(\langle \sigma_x \cdot \tau^{-1} \cdot \tau \cdot x \rangle_{[r_M]} )\varphi_N(\langle \sigma_x \cdot \tau^{-1} \cdot \tau \cdot x \rangle_{[r_M+r_N \setminus [r_M]} )$$

$$= \text{sgn}(\tau )\text{sgn}(\sigma_x )\varphi_M(\langle \sigma_x \cdot x \rangle_{[r_M]} )\varphi_N(\langle \sigma_x \cdot x \rangle_{[r_M+r_N \setminus [r_M]} )$$

$$= \text{sgn}(\tau )\varphi_M \oplus \varphi_N(x).$$

Next, we prove that if $\varphi_M$ and $\varphi_N$ are weak-type Grassmann-Plücker functions, then $\varphi$ is also a weak-type Grassmann-Plücker function; we should show that the 3-term Grassmann-Plücker relation (WG) holds. In other words, we have to show that $\forall a, b, c, d \in E_M \sqcup E_N$ and $y \in (E_M \sqcup E_N)^{r_M+r_N}$,

$$0_H \in \varphi(a, b, y)\varphi(c, d, y) - \varphi(a, c, y)\varphi(b, d, y) + \varphi(a, d, y)\varphi(b, c, y).$$

(WG)

Before proceeding, notice that we may assume $a < b < c < d$ and $y$ is strictly increasing with respect to the ordering $\leq$ by alternation and degeneration conditions. We may further
assume that none of \(a, b, c, d\) are coordinates of \(y\) by degeneracy.

**Case 1:** Suppose all of \(a, b, c, d\) belong to the same part of \(E_M \cup E_N\) (either \(E_M\) or \(E_N\)). In this case, we may assume that \(a, b, c, d \in E_M\). If \(y\) does not have exactly \(r_M - 2\) components in \(E_M\), then the relation follows trivially as all terms are zero. Otherwise, notice that \(\sigma_x\) is identity on the components with elements from \(E_N\) and \(\varphi_N\) contributes the same constant to the relation in each term (namely \(\varphi_N(y_{[r_M+r_N-2]\setminus[r_M-2]^2})\)). Thus we may reduce to a consideration of the terms contributed by \(\varphi_M\), namely

\[
0_H \in \varphi_M(\rho_{a,b} \cdot (a, b, y'))\varphi_M(\rho_{c,d} \cdot (c, d, y'))
\]

\[
-\varphi_M(\rho_{a,c} \cdot (a, c, y'))\varphi_M(\rho_{b,d} \cdot (b, d, y')) + \varphi_M(\rho_{a,d} \cdot (a, d, y'))\varphi_M(\rho_{b,c} \cdot (b, c, y'))
\]

where \(\rho_{p,q} = \sigma_{(p, q, y)}|_{[r_M]}\) and \(y' = y_{[r_M]}\). For each \(p \in \{a, b, c, d\}\) let

\[
\delta_p := \# \{ i \in [r_M] \mid y'(i) < p \}.
\]

Now one permutes the coordinates in the expression in the following manner. First permute \(d\) to the front of all terms which contain it; this results in a global change of sign \((-1)^{\delta_d+1}\) as \(d\) must pass over \(\delta_d\) coordinates of \(y'\) and the coordinates \(c\) in the first term, \(b\) in the second term, and \(a\) in the third term. Next permute \(c\) to the front of all terms which contain it; in each term the sign changes by \((-1)^{\delta_c+1}\) as \(c\) must pass over \(\delta_c\) coordinates of \(y'\) and the coordinates \(d\) in the first, \(a\) in the second, and \(b\) in the third. Next permute \(b\) to the front of all terms which contain it; in each term the sign changes by \((-1)^{\delta_b+1}\) as \(b\) must pass over \(\delta_b\) coordinates of \(y'\) and the coordinates \(a\) in the first, \(d\) in the second, and \(c\) in the third. Finally permute \(a\) to the front of all terms which contain it; in each term the sign changes by \((-1)^{\delta_a+1}\) as \(a\) must pass over \(\delta_a\) coordinates of \(y'\) and the coordinates \(b\) in the first, \(c\) in the second, and \(d\) in the third. Hence permuting coordinates in this way we arrive at the relation \([\text{WG}]\) for \(\varphi_M\) up to a global sign change of \((-1)^{\delta_a+\delta_b+\delta_c+\delta_d+4}\). Hence Axiom \([\text{WG}]\) holds in this case.

**Case 2:** Suppose not all of \(a, b, c, d\) belong to the same part of \(E_M \cup E_N\). By our arrangement of \(a < b < c < d\) and our choice of order \(\leq\) as above we see that \(a \in E_M\) and \(d \in E_N\). If \(y\) does not have precisely \(r_M - 1\) components in \(E_M\), then the relation holds trivially as all terms are zero. Thus, we may further assume that \(y\) has precisely \(r_M - 1\) components in \(E_M\) (and thus precisely \(r_N - 1\) components in \(E_N\)). Now if \(b\) and \(c\) belong to the same part of \(E_M \cup E_N\), again we see that the relation trivially holds as all terms are zero. Thus we can reduce to the case that \(b \in E_M\) and \(c \in E_N\). We must see the following to conclude our desired result:

\[
\varphi(a, c, y)\varphi(b, d, y) = \varphi(a, d, y)\varphi(b, c, y). \tag{12}
\]

We now obtain the relation by a similar trick as in the first case, “walking” each of \(d, c, b, a\) back to the first coordinate in that order to obtain the relation by corresponding relations on \(\varphi_M\) and \(\varphi_N\).

Next, we prove that \(\varphi\) is strong-type only if \(\varphi_M\) and \(\varphi_N\) are strong-type. Suppose \(\varphi_P\) is weak-type but not strong-type for either \(P = M\) or \(P = N\). Then there is an \((r_P + 1)\)-tuple \(x\) and \((r - 1)\)-tuple \(y\) for which (SG) is violated. Pick any ordered basis \(z\) of the other \(H\)-matroid \(P'\); trivially \(\varphi_M \oplus \varphi_N\) fails (SG) for \((x, z)\) and \((y, z)\), as this reduces to the \(\varphi_P(z)\)-multiple of the failing relation for \(\varphi_P\); this yields that \(\varphi_M \oplus \varphi_N\) is weak-type but not strong-type. On the other hand, if \(\varphi_M\) and \(\varphi_N\) are both strong-type, then the relations required by (SG) for \(\varphi_M \oplus \varphi_N\) can be rewritten as a constant times an (SG)-relation for \(M\) plus a constant times an (SG)-relation for \(N\). This immediately implies that \(\varphi_M \oplus \varphi_N\) is strong-type.

Finally, invariance of the resulting \(H^x\)-class is immediate from the following:

\[
\alpha \varphi_M \oplus \beta \varphi_N = \alpha \beta (\varphi_M \oplus \varphi_N), \quad \forall \alpha, \beta \in H^x.
\]
Proposition 3.14. Let $M$ and $N$ be $H$-matroids of rank $r_M$ and $r_N$ on disjoint ground sets $E_M$ and $E_N$ given by $H$-circuits $C_M$ and $C_N$ respectively. Define

$$C_M \oplus C_N = \left\{ X : E_M \sqcup E_N \to H \middle| \begin{array}{l}
\text{either both } X|_{E_M} \in C_M \text{ and } X|_{E_N} = 0 \\
\text{or both } X|_{E_M} = 0 \text{ and } X|_{E_N} \in C_N
\end{array} \right\}$$

Then, $C_M \oplus C_N$ is a set of $H$-circuits. Furthermore, $C_M \oplus C_N$ is of strong-type exactly when both $C_M$ and $C_N$ are of strong-type.

Proof. That $C_M \oplus C_N$ is a set of pre-circuits over $H$ follows trivially from its definition. Moreover, one can see easily see that

$$\text{supp}(C_M \oplus C_N) = \text{supp}(C_M) \sqcup \text{supp}(C_N)$$

and hence the underlying matroid of the $H$-matroid determined thereby is the direct sum of the underlying matroids of the summands. It follows that every modular pair in $C_M \oplus C_N$ reduces to two modular pairs, one in $C_M$ and one in $C_N$. Thus (WC) holds by noting that any modular pair with nontrivial intersection is either a modular pair in $C_M$ or a modular pair in $C_N$. If $C_M$ and $C_N$ are both strong, then by (13), (SC1) holds. Moreover (SC2) holds by noting that the computation reduces to a computation in precisely one of $C_M$ or $C_N$. \qed

The next result shows that the direct sum of $H$-matroids admits the cryptomorphic descriptions given in this section.

Proposition 3.15. If $M$ is an $H$-matroid given by $H$-circuits $C_M$ and Grassmann-Plücker function $\varphi_M$ on $E_M$ and $N$ is an $H$-matroid given by $H$-circuits $C_N$ and Grassmann-Plücker function $\varphi_N$ on $E_N$ such that $E_M \cap E_N = \emptyset$. Then, $\varphi_M \oplus \varphi_N$ and $C_M \oplus C_N$ both determine the same $H$-matroid under cryptomorphism. Furthermore, this matroid has underlying matroid the direct sum of the underlying matroids of $M$ and $N$.

Proof. We must verify that $C_M \oplus C_N$ is cryptomorphically determined by $\varphi_M \oplus \varphi_N$. Let $B_M$ and $B_N$ be any bases of the underlying matroids of $M$ and $N$, respectively. Notice that for all $e \in (E_M \sqcup E_N) \setminus (B_M \sqcup B_N)$, the fundamental circuit $X_{(B_M \sqcup B_N \setminus \{e\})}$ has support contained in $E_M$ or in $E_N$. Thus, the cryptomorphism relation required reduces to the relation on the fundamental circuit the part containing $e \in E_M \sqcup E_N$. Hence $\varphi_M \oplus \varphi_N$ and $C_M \oplus C_N$ determine the same $H$-matroid as desired. \qed

Corollary 3.16. The pushforward of a direct sum of matroids is the direct sum of the pushforwards. In other words, direct sum commutes with pushforwards.

Proof. This is trivially verified on Grassmann-Plücker functions. \qed

Example 3.17. The case when we pushforward to the Krasner hyperfield $K$, i.e., taking underlying matroids, is directly proven in terms in $H$-circuits in Proposition 3.15.

4. Isomorphisms of matroids over hyperfields

In this section, we introduce a notion of isomorphisms of matroids over hyperfields which generalizes the definition of isomorphisms of ordinary matroids. We will subsequently use this definition to construct matroid-minor Hopf algebras for matroids over hyperfields in §5.

Definition 4.1 (Isomorphism via Grassmann-Plücker function). Let $E_1$ and $E_2$ be finite sets, $r$ be a positive integer, and $H$ be a hyperfield. Let $M_1$ (resp. $M_2$) be a matroid on $E_1$ (resp. $E_2$) of rank $r$ over $H$ which is represented by a Grassmann-Plücker function $\varphi_1$
(resp. \( \varphi_2 \)). We say that \( M_1 \) and \( M_2 \) are isomorphic if there is a bijection \( f : E_1 \to E_2 \) and an element \( \alpha \in H^\times \) such that the following diagram commutes:

\[
\begin{array}{ccc}
E_1^r & \xrightarrow{\varphi_1} & H \\
\downarrow{f^r} & & \downarrow{\circ \alpha} \\
E_2^r & \xrightarrow{\varphi_2} & H
\end{array}
\]  \hspace{1cm} (14)

**Proposition 4.2.** Definition [4.4] is well-defined.

**Proof.** Let \( \varphi_1 \) and \( \varphi_2 \) be different representatives of \( M_1 \) and \( M_2 \). In other words, there exist \( \beta, \gamma \in H^\times \) such that \( \varphi_1 = \beta \circ \varphi_1 \) and \( \varphi_2 = \gamma \circ \varphi_2 \). In this case, we have that

\[
\gamma^{-1} \circ \varphi_2 \circ f^r = \varphi_2 \circ f^r = \alpha \circ \varphi_1 = (\alpha \circ \beta^{-1}) \circ \varphi_1
\]

It follows that \( \varphi_2 \circ f^r = (\gamma \circ \alpha \circ \beta^{-1}) \circ \varphi_1 \) and hence Definition [4.4] is well-defined. \( \Box \)

**Proposition 4.3.** Let \( H \) and \( K \) be hyperfields and \( g : H \to K \) be a morphism of hyperfields. If \( M_1 \) and \( M_2 \) are matroids over \( H \) which are isomorphic, then the pushforwards \( g^*M_1 \) and \( g^*M_2 \) are isomorphic as well.

**Proof.** Let \( M_1 \) (resp. \( M_2 \)) be represented by a Grassmann-Plücker function \( \varphi_1 \) (resp. \( \varphi_2 \)). Since \( M_1 \) and \( M_2 \) are isomorphic, there exist \( a \in H^\times \) and a bijection \( f : E_1 \to E_2 \) such that \( \varphi_2 \circ f^r = a \circ \varphi_1 \). Notice that the pushforward \( g^*M_1 \) (resp. \( g^*M_2 \)) is represented by the Grassmann-Plücker function \( g \circ \varphi_1 \) (resp. \( g \circ \varphi_2 \)), we obtain

\[
(g \circ \varphi_2) \circ f^r = g \circ (\varphi_2 \circ f^r) = g \circ (a \circ \varphi_1) = g(a) \circ (g \circ \varphi_1).
\]

One notes that in the special case \( K = \mathbb{K} \), the underlying matroids of two isomorphic matroids are isomorphic in the classical sense. Therefore, our definition of isomorphisms generalizes the definition of isomorphisms of ordinary matroids.

**Proposition 4.4.** If \( M \) and \( M' \) (resp. \( N \) and \( N' \)) are isomorphic \( H \)-matroids, then \( M \oplus N \) and \( M' \oplus N' \) are isomorphic \( H \)-matroids.

**Proof.** Consider Grassmann-Plücker functions \( \varphi_M, \varphi_{M'}, \varphi_N, \) and \( \varphi_{N'} \). By assumption there are bijections \( f_M : E_M \to E_{M'} \) and \( f_N : E_N \to E_N' \) and constants \( \alpha_M, \alpha_N \in H^\times \) such that \( \alpha_M \circ \varphi_M = \varphi_{M'} \circ f_M^r \) and \( \alpha_N \circ \varphi_N = \varphi_{N'} \circ f_N^r \). Let \( f_M \sqcup f_N : E_M \sqcup E_N \to E_{M'} \sqcup E_{N'} \) denote the obvious bijection. Then, we have

\[
\alpha_M \alpha_N \circ (\varphi_M \oplus \varphi_N) = (\alpha_M \circ \varphi_M) \oplus (\alpha_N \circ \varphi_N)
\]

\[
= (\varphi_{M'} \circ f_M^r) \oplus (\varphi_{N'} \circ f_N^r)
\]

\[
= (\varphi_{M'} \oplus \varphi_{N'}) \circ (f_M \sqcup f_N)^{TM+FN}.
\]

**Remark 4.5.** Although we stick with Definition [4.4] in this paper, any \( \oplus \)-congruence relation could be used in place of “isomorphism” for matroids over hyperfields; indeed, all that we will need from our notion of isomorphism is that \( M \sim M' \) and \( N \sim N' \) implies \( M \oplus N \sim M' \oplus N' \) to define Hopf algebras for matroids over hyperfields.

**Remark 4.6.** Our initial definition for isomorphism was as follows: \( M_1 \) and \( M_2 \) are isomorphic if there is a bijection \( f : E_1 \to E_2 \) and an automorphism \( g : H \to H \) such that the following diagram commutes:

\[
\begin{array}{ccc}
E_1^r & \xrightarrow{\varphi_1} & H \\
\downarrow{f^r} & & \downarrow{g} \\
E_2^r & \xrightarrow{\varphi_2} & H
\end{array}
\]
This definition was inspired by the classical notion of \textit{semilinear maps}, i.e., linear maps up to “twist” of scalars by the automorphisms of a ground field. Unfortunately, this is not well-defined on the level of $H^\times$-equivalence classes and hence we use the current definition. Although we do not pursue this line of thought in this paper, it seems really interesting to investigate a notion of general linear groups over hyperfields. For instance, a proper notion of general linear groups over hyperfields is needed to study matroid bundles (a combinatorial analogue of vector bundles, as in [1]) for matroids over hyperfields.

5. The matroid-minor Hopf algebra associated to a matroid over a hyperfield

In this section, by appealing to Definition 4.1, Propositions 3.13 and 3.14, we generalize the classical construction of matroid-minor Hopf algebras to the case of matroids over hyperfields. Let $H$ be a hyperfield. Let $\mathcal{M}$ be a set of matroids over $H$ which is closed under taking direct sums and minors. Let $\mathcal{M}_{\text{iso}}$ be the set of isomorphism classes of elements in $\mathcal{M}$, where the isomorphism class is defined by Definition 4.1. Then, $\mathcal{M}_{\text{iso}}$ has a canonical monoid structure as follows:

\[
\cdot : \mathcal{M}_{\text{iso}} \times \mathcal{M}_{\text{iso}} \to \mathcal{M}_{\text{iso}}, \quad ([M_1], [M_2]) \mapsto [M_1 \oplus M_2].
\]

(15)

Note that (15) is well-defined thanks to Proposition 4.4 and the isomorphism class of the empty matroid $[\emptyset]$ becomes the identity element. Let $k$ be a field. Then we have the monoid algebra $k[\mathcal{M}_{\text{iso}}]$ over $k$ with the unit map $\eta : k \to k[\mathcal{M}_{\text{iso}}]$ sending 1 to $[\emptyset]$ and the multiplication:

\[
\mu : k[\mathcal{M}_{\text{iso}}] \otimes_k k[\mathcal{M}_{\text{iso}}] \to k[\mathcal{M}_{\text{iso}}], \quad \text{generated by } [M_1] \otimes [M_2] \mapsto [M_1 \oplus M_2].
\]

Proposition 5.1. Let $k$ be a field and $H$ be a hyperfield. Let $(\mathcal{M}_{\text{iso}}, \cdot)$ be the monoid and $k[\mathcal{M}_{\text{iso}}]$ be the monoid algebra over $k$ as above. Then $\mathcal{H} := k[\mathcal{M}_{\text{iso}}]$ is a bialgebra with the following maps:

- (Comultiplication)

\[
\Delta : \mathcal{H} \longrightarrow \mathcal{H} \otimes_k \mathcal{H}, \quad [M] \mapsto \sum_{A \subseteq E} [M |_A] \otimes_k [M/A].
\]

(16)

- (Counit)

\[
\varepsilon : \mathcal{H} \longrightarrow k, \quad [M] \mapsto \begin{cases} 1 & \text{if } E_M = \emptyset \\ 0 & \text{if } E_M \neq \emptyset. \end{cases}
\]

(17)

Furthermore, $\mathcal{H}$ is graded and connected and hence has a unique Hopf algebra structure.

Proof. There is a canonical grading on $\mathcal{H}$ via the cardinality of the underlying set of each element $[M]$ and this is clearly compatible with the bialgebra structure of $\mathcal{H}$. In this case, $[\emptyset]$ has a degree 0 and hence $\mathcal{H}$ is connected. The last assertion simply follows from the result of [20].

Example 5.2. Let $H$ be the hyperfield which is obtained from $\mathbb{Q}$ as in Example 2.15 (considered as a totally ordered abelian group) and $k$ be a field. Let $M = U^1_1$ (the uniform rank-1 matroid on one element) and $\mathcal{M}_{\text{iso}}$ be the free monoid generated by the isomorphism class $[M]$ of $M$. Then one can easily see that the Hopf algebra $k[\mathcal{M}_{\text{iso}}]$ is just $k[T]$, where $T$ is the isomorphism class of $U^1_1$.

Let $X$ be the set of matroids over $H$ whose pushforward is $U^1_1$. Let $[M_1], [M_2] \in X$. Then $M_1$ and $M_2$ are isomorphic if and only if there exists $q \in H^\times = \mathbb{Q}$ such that $a \odot \varphi_1 = \varphi_2$, where $\varphi_1$ (resp. $\varphi_2$) is a Grassmann-Plücker function for $M_1$ (resp. $M_2$). It follows that the free monoid $\mathcal{M}_{\text{iso}}^H$, which is generated by the isomorphisms classes of $X$, is as follows:

\[
\mathcal{M}_{\text{iso}}^H = \{T_{q_1}^{n_1}T_{q_2}^{n_2} \cdots T_{q_j}^{n_j} \mid n_i, j \in \mathbb{N}, q_i \in \mathbb{Q} \}.
\]
Hence, the Hopf algebra $k[\mathcal{M}^H_{\text{iso}}]$ is just $k[T_q]_{q \in \mathbb{Q}}$. One then has the following surjection:

$$
\pi : k[\mathcal{M}^H_{\text{iso}}] = k[T_q]_{q \in \mathbb{Q}} \twoheadrightarrow k[\mathcal{M}_{\text{iso}}] = k[T], \quad T_q \mapsto T.
$$

Note that the map $\pi$ is a surjection since any matroid in $\mathcal{M}_{\text{iso}}$ is realizable over $H$ by some matroid in $\mathcal{M}^H_{\text{iso}}$. One can easily see that $\text{Ker}(\pi)$ is generated by elements of the form $T_{q_1} - T_{q_2}$ for $q_i \in \mathbb{Q}$.

6. Relations to other generalizations

6.1 Relation to matroids over fuzzy rings. In this section, we investigate the results in previous sections in a view of matroids over fuzzy rings, introduced by A. Dress in [8] (and later with W. Wenzel in [9]). We will employ the functor, which is constructed by the second author together with J. Giansiracusa and O. Lorscheid, from the category of hyperfields to the category of fuzzy rings for this purpose. We note that the most recent work of Baker and Bowler [6] generalizes matroids over hyperfields and matroids over fuzzy rings at the same time. For the brief overview of this approach in connection to our previous work, see §6.2.

We first review the definition of matroids over fuzzy rings. In what follows, we let $E$ be a finite set, $K$ a fuzzy ring, and $K^\times$ the group of multiplicatively invertible elements of $K$, unless otherwise stated. Roughly speaking a fuzzy ring $K$ is a set, equipped with two binary operations $+$, $\cdot$ such that $(K, +, 0_K)$ and $(K, \cdot, 1_K)$ are commutative monoids (but not assuming that two binary operations are compatible), together with a distinguished subset $K_0$ and a distinguished element $\varepsilon$, satisfying certain list of axioms. The element $\varepsilon$ of $K$ plays the role of the additive inverse of 1 and $K_0$ is “the set of zeros”; this is where the term “fuzzy” came from. For the precise definition of fuzzy rings, we refer the readers to [11, §2.3.].

**Remark 6.1.** We restrict ourselves to the case that $E$ is a finite set to make an exposition simpler, although one interesting facet of Dress and Wenzel’s theory is that $E$ does not have to be finite.

**Definition 6.2.** Let $E$ be a finite set and $(K; +, \cdot; \varepsilon, K_0)$ a fuzzy ring.

1. The unit-support of a function $f : E \to K$ is defined by
   $$\text{usupp}(f) := f^{-1}(K^\times).$$

2. The inner product of two functions $f, g : E \to K$ is defined by
   $$\langle f, g \rangle := \sum_{e \in \text{supp}(f) \cap \text{supp}(g)} f(e) \cdot g(e).$$

3. Two functions $f, g : E \to K$ are orthogonal, denoted $f \perp g$, when $\langle f, g \rangle$ is an element of $K_0$.

4. The wedge of $f, g : E \to K$ is the function
   $$f \wedge g : E \times E \to K, \quad (x, y) \mapsto \begin{cases} 0 & \text{if } x = y \\ f(x) \cdot g(y) + \varepsilon f(y) \cdot g(x) & \text{otherwise} \end{cases} \quad (18)$$

Clearly, we have $\text{usupp}(f) \subseteq \text{supp}(f)$. The following lemma now directly follows from the definition:

**Lemma 6.3.** Let $f, g, h : E \to K$ be functions. Suppose that $f$ is orthogonal to both $g$ and $h$. If

$$\text{supp}(f) \cap (\text{supp}(g) \cup \text{supp}(h)) \subseteq \text{usupp}(f),$$

then for all $x \in E$ the function $(g \wedge h)|_{\{x\} \times E}$ is orthogonal to $f$. 

We further define for all $R \subseteq K^E$,
\[
\bigwedge R := \{ f_1 \land f_2 \land \cdots \land f_n \ | \ n \in \mathbb{N} \text{ and for all } i \in [n], \text{ we have } f_i \in R \}. \tag{19}
\]
For each $R \subseteq K^E$, we let
\[
[R] := \left\{ r \big| (x_1, x_2, \ldots, x_{n-1}) \times E \big| \ r = f_1 \land f_2 \land \cdots \land f_n \in \bigwedge R \text{ and } x_i \in E \text{ for } i \in [n] \right\}. \tag{20}
\]
For $S \subseteq E$ and $R \subseteq K^E$, we define
\[
R_S := \{ f \in R \ | \ \text{supp}(f) \cap S = \text{usupp}(f) \cap S \}. \]

**Definition 6.4.** A matroid over $K$ on $E$ is presented by $(X, R)$ if

1. $X \subseteq \mathcal{P}(E)$ is a set of bases of an ordinary matroid,
2. $R$ is a subset of $(K^E)$ satisfying the following: for any $n \in \mathbb{N}$, $f = f_1 \land f_2 \land \cdots \land f_n$ with $f_i \in R$ for $i = 1, \ldots, n$, and $(x_1, \ldots, x_n) \in E^n$ with $x_n \in Y$ for $Y \in X$ such that $
abla f(x_1, \ldots, x_n) \notin K_0$, there exists $g \in R|_Y$ such that
\[
x_n \in \text{supp}(g) \cap Y \subseteq \text{supp}(f).
\]

**Remark 6.5.** Two different data $(X, R)$ and $(X', R')$ on $E$ may present the same matroid over $K$ (see, [8]). Also, it is not difficult to see that the set of circuits of a matroid over $K$ (i.e. the set of support minimal elements of $R$) has a supports set which is the set of circuits of an ordinary matroid.

We now review the functors constructed in [11] to link matroids over hyperfields and matroids over fuzzy rings. Note that Dress and Wenzel also introduced the cryptomorphic description of matroids over fuzzy rings by using Grassmann-Plücker functions in [9]. There are two types of morphisms for fuzzy rings (called weak and strong morphisms in [11]). We let $\text{Hyperfields}$ be the category of hyperfields, $\text{FuzzyRings}_{wk}$ the category of fuzzy rings with weak morphisms, and $\text{FuzzyRings}_{str}$ the category of fuzzy rings with strong morphisms. Then, one has the following:

**Theorem 6.6.** [11, §3] There exists a fully faithful functor from $\text{Hyperfields}$ to $\text{FuzzyRings}_{wk}$.

The construction goes as follows. For a hyperfield $(H, \oplus, \odot)$, we let $K := \mathcal{P}^*(H)$ and impose two binary operations $+ \text{ and } \cdot$ as follows:

\[
A + B := \bigcup_{a \in A, b \in B} a \oplus b, \quad A \cdot B := \{ a \odot b \ | \ a \in A, b \in B \}.
\]

Then, $(K, +, \{0_H\})$ and $(K, \cdot, \{1_H\})$ become commutative monoids. It is shown in [11] that with $K_0 = \{ A \subseteq H \ | \ 0_H \in A \}$ and $\varepsilon = \{-1_H\}$, $(K, +, \cdot, \varepsilon, K_0)$ becomes a fuzzy ring. For any hyperfield $H$, we let $\mathcal{F}(H)$ be the fuzzy ring defined in this way. For morphisms and more details, we refer the readers to [11].

**Remark 6.7.** It is also shown in [11] that there exists a quasi-inverse $S$ of the functor $\mathcal{F}$.

Now, we employ the functor $\mathcal{F}$ to yield minors of matroids over fuzzy rings as in Dress and Wenzel from minors of matroids over hyperfields as in Baker and Bowler. In what follows, all matroids over hyperfields are assumed to be strong. One has the following:

**Theorem 6.8.** [11, §7.2] Let $E$ be a finite set, $H$ be a hyperfield, $K = \mathcal{F}(H)$ be the fuzzy ring obtained from $H$, and $r$ a positive integer. Then a function
\[
\varphi : E^r \rightarrow H^\times = \mathcal{F}(H)^\times
\]
is a Grassmann-Plücker function over the fuzzy ring $K = \mathcal{F}(H)$ in the sense of Dress and Wenzel in [9] if and only if $\varphi$ is a strong-type Grassmann-Plücker function over $H$. 

For a matroid $M$ over $H$ we abuse notation and let $\mathcal{F}(M)$ be the corresponding matroid over the fuzzy ring $\mathcal{F}(H)$. One has the following corollary:

**Corollary 6.9.** Let $E$ be a finite set, $H$ a hyperfield, $K = \mathcal{F}(H)$ the fuzzy ring obtained from $H$, $r$ a positive integer, and $S$ a subset of $E$. Then, we have

1. $\mathcal{F}(M|S) = \mathcal{F}(M)|S$.
2. $\mathcal{F}(M\setminus S) = \mathcal{F}(M)\setminus S$.
3. $\mathcal{F}(M/S) = \mathcal{F}(M)/S$.

In particular, if $N$ is a minor of $M$, then $\mathcal{F}(N)$ is a minor of $\mathcal{F}(M)$.

**Proof.** This directly follows from the Grassmann-Plücker function characterizations of minors for matroids over hyperfields in §3 and for matroids over fuzzy rings in [9, §5]. □

**Remark 6.10.** By using the quasi-inverse $\mathcal{G}$ constructed in [11], for field-like fuzzy rings (see, [11, §4] for the definition), one can also obtain minors of matroids over hyperfields from minors of matroids over fuzzy rings.

**Remark 6.11.** One can use our definition of direct sums of matroids over hyperfields in §3.5 to define direct sums for matroids over fuzzy rings and hence obtain matroid-minor Hopf algebras for matroids over fuzzy rings.

### 6.2 Relation to matroids over partial hyperfields

In this section, we review Baker and Bowler’s more generalized framework, namely matroids over partial hyperfields [6] and explain how our work can be generalized in this setting.

**Definition 6.12.** [6, §1] A tract is an abelian group $G$ together with a designated subset $N_G$ of the group semiring $\mathbb{N}[G]$ such that

1. $0_{[G]} \in N_G$ and $1_G \notin N_G$.
2. $\exists! \varepsilon \in G$ such that $1 + \varepsilon \in N_G$.
3. $G \cdot N_G = N_G$.

The idea is similar to fuzzy rings; $\varepsilon$ plays the role of $-1$ and $N_G$ encodes “non-trivial dependence” relations; in the case of fuzzy rings, one has a designated subset $K_0$ of “zeros”, however, by using $\varepsilon$ one can always change $K_0$ to $N_G$ as above.

For a hyperfield $(H, \boxplus, \odot)$, one can canonically associate a tract $(G, N_G)$; this is very similar to the functor from the category of hyperfield to the category of fuzzy rings in §3. To be precise, one sets $G = H^\times$, and lets $f = \sum a_i g_i \in \mathbb{N}[G]$ be in $N_G$ if and only if

$$0_H \in \boxplus(a_i \odot g_i) \quad \text{(as elements of } H).$$

Recall that partial fields are introduced by C. Semple and G. Whittle in [19] to study realizability of matroids. A partial field $(G \cup \{0_R\}, R)$ consists of a commutative ring $R$ and a multiplicative subgroup $G$ of $R^\times$ such that $-1 \in G$ and $G$ generates $R$. Inspired by this definition (along with hyperfields), Baker and Bowler define the following:

**Definition 6.13.** [6, §1] A partial hyperfield is a hyperdomain $R$ (a hyperring without zero divisors) together with a designated subgroup $G$ of $R^\times$.

One can naturally associate a tract to a partial hyperfield $(G, R)$ in a manner similar to the previous association of a tract to a hyperfield by stating that $\sum a_i g_i \in \mathbb{N}[G]$ if and only if (21) holds.

With tracts (or partial hyperfields), Baker and Bowler generalize their previous work on matroids over hyperfields. Their main idea is that in their proofs for matroids over hyperfields, one only needs the three conditions of tracts given in Definition 6.12. Therefore, although we only focus on the case of matroids over hyperfields, one can easily generalize our results to the case of matroids over partial hyperfields.
6.3 Tutte polynomials of Hopf algebras and Universal Tutte characters. The Tutte polynomial is one of the most interesting invariants of graphs and matroids. In [14], T. Krajewsky, I. Moffatt, and A. Tanasa introduced Tutte polynomials associated to Hopf algebras. More recently, C. Dupont, A. Fink, and L. Moci introduced universal Tutte characters generalizing [14]. In fact, both [10] and [14] consider the case when one has combinatorial objects which have notions of “deletion” and “contraction” (e.g. graphs and matroids). In the context of our work, the following is straightforward.

**Proposition 6.14.** Let $H$ be a hyperfield. A set $\mathcal{M}_{\text{iso}}$ of isomorphisms classes of matroids over $H$, which is stable under taking direct sums and minors, satisfies the axioms of a minor system in [14, Definition 2].

**Proof.** This directly follows from Corollary 3.10. □

The term a *minor system* is used in [10] to define universal Tutte characters. The following is an easy consequence of §3.

**Proposition 6.15.** Let $H$ be a hyperfield and $\text{Mat}_H$ be the set species such that $\text{Mat}_H(E)$ is the set of matroids over $H$ with an underlying set $E$. Then $\text{Mat}_H$ is a connected multiplicative minor system as in [10, Definition 2.6. and 2.8].

**Proof.** Let $S := \text{Mat}_H$. Clearly, $S$ is connected since the empty matroid over $H$ is the only object of $S[\emptyset]$. Multiplicative structure of $S$ comes from direct sums. The axioms (M1)-(M3), (M4’-M8’) can be easily checked as in the ordinary matroids case. □

**Remark 6.16.** It follows from the above observations that the construction in [14, §2] can be applied to define the Tutte polynomial for $k[\mathcal{M}_{\text{iso}}]$. Furthermore, one can also associate the universal Tutte characters in our setting.

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Department of Mathematical Sciences, Binghamton University, Binghamton, NY, 13902, USA

E-mail address: eppolito@math.binghamton.edu

Department of Mathematical Sciences, Binghamton University, Binghamton, NY, 13902, USA

E-mail address: jjun@math.binghamton.edu

Department of Mathematics and Statistics, Boston University, 111 Cummington Mall, Boston, USA

E-mail address: szczesny@math.bu.edu