Facilitating adoption of network services with externalities via cost subsidization

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Abstract—This paper investigates the adoption level of a network service where the net utility perceived by each user incorporates three key features, namely, user service affinity heterogeneity, a network externality, and a subscription cost. Services with network externality face a “chicken and egg” adoption problem in that the service requires customers in order to attract customers. In this paper we study cost subsidization as a means to “reach the knee” and thereby change the equilibrium adoption level from zero to one. By focusing on a simple subsidy structure and a simple model for user heterogeneity, we can derive explicit expressions for quantities of natural interest, such as the minimum subsidy required, the minimum subsidy duration, and the aggregate cost of the subsidy to the service provider. We show that small or large subsidies are inefficient, but that there is a Pareto efficient frontier for “intermediate” subsidies wherein subsidy duration and aggregate cost are in tension with one another.

Index Terms—Economic Networks; Social Networks; Diffusion Adoption; Subsidization.

I. INTRODUCTION

With the Internet fueling the rise of a “network society” [2], many services and technologies realize their value only after reaching a certain level of adoption. In other words, they exhibit positive externalities, e.g., Metcalfe’s Law. Externalities are well-known [3], [4] to affect adoption, and in particular to create a “chicken-and-egg” problem (i.e., a service requires customers in order to attract customers) that can often stymie the success of new services. This is because, when a new service is offered, most potential adopters see a cost that varies; e.g., IPv6 [6], [7]. Understanding how to overcome this problem is vital to the success of future (network) services. Towards this end, the Internet Architecture Board (IAB) held a workshop on Internet Technology Adoption and Transition (ITAT) in December, 2013 to “develop protocol deployment strategies that enable new features to rapidly gain a foothold and ultimately realize broad adoption. Such strategies must be informed by both operational and economic factors.”

In our prior work [8], [9] we investigated service bundling as a means of overcoming initial adoption inertia. In this work, we analyze the service adoption dynamics (AD) under a standard diffusion model when the service provider employs cost subsidization. Our model incorporates three key assumptions: i) users are heterogeneous, i.e., their affinity for the service varies; ii) services exhibit positive network externalities, i.e., the utility perceived by a user is an increasing function of the service adoption level; and iii) services have a subscription cost, i.e., a user pays a fixed amount per unit time to participate in the service. There are no additional costs to initially join or leave the service, nor any contractual requirements that prevent a user from leaving the service at any time. Costs are assumed non-discriminatory, i.e., identical across users, and fixed (exogenous). This is in contrast to studies that endogenize costs to maximize an objective, e.g., revenue (1).

Subsidization is a natural solution for such services because it incentivizes adoption among initial adopters (“innovators” [10]), thereby allowing the adoption level to build up to the “knee”, i.e., the point at which the strength of the externality will incentivize the later adopters (“imitators”), and the subsidy will no longer be needed to sustain the service. Subsidization may take many forms; we provide a (necessarily) selective and brief review of this large topic in [1]. In this paper we restrict our attention to perhaps the most natural and simple subsidy, namely, the constant level subsidy (CLS), wherein the service provider subsidizes the cost for each adopter at a constant level (per adopter) over a finite duration. Specifically, an $(s,t)$ CLS starting at time $t_0$ for a service with cost (per unit time) of $c$ means that any adopter will pay at rate $c - s$ at any time $t \in [t_0, t_0 + T]$, and will pay at rate $c$ for any time $t > t_0 + T$. It is natural that the subsidy duration $T$ be selected so that the subsidy stops once adoption reaches some target level. A service provider employing a CLS will be interested in simultaneously minimizing two key performance metrics: i) the aggregate cost of the subsidy and ii) the required duration of the subsidy.

Our main result is Prop. [5], where we suppose the subsidy duration $T$ to be chosen to guide the AD to the boundary of the domain of attraction of the full adoption equilibrium. We give the minimum subsidy amount required to actually change the equilibrium from zero to full, the AD as a function of the subsidy amount, and both the required subsidy duration and the aggregate cost of the subsidy to the service provider as a function of the subsidy amount. We show that the required subsidy duration is nonincreasing in the subsidy amount, and that the aggregate subsidy cost is initially decreasing in the subsidy amount, but then will eventually (for large enough subsidies) be increasing in the subsidy amount. These results illustrate that the two key performance metrics may or may not be in tension with one another, depending upon the specific

1 For conciseness we will use the term services to refer to both.

2 http://www.iab.org/activities/workshops/itat/
choice of the model parameters.

The rest of the paper is organized as follows. [II] discusses related work and [III] presents the mathematical model. [IV] addresses AD for services without a network externality, while [V] and [VI] address AD for services with network externalities with and without cost subsidization, respectively. [VII] offers a brief conclusion. Some proofs are found in the appendices.

II. RELATED WORK

There is a long-standing awareness of the role of subsidies in realizing more efficient outcomes in “markets” that exhibit positive externalities i.e., by demonstrating the benefits of Pigouvian subsidies [II]. For example, [I2] examines the impact of early investments on a firm’s growth rate in the telecommunication industry. It identifies that early investments can facilitate the creation of an initial user base, and lead to greater overall market share. This awareness not withstanding, most of the focus to-date has been on case studies, e.g., see [I3] for a recent review.

There have been some recent efforts on the modeling front, stemming in part from interest in viral marketing in online (social) networks [I4]–[I7]. These works are closely related to studies of adoption dynamics in social networks [I8] Chapter 24, but with a focus on maximizing revenue rather than adoption. The optimal marketing strategy in a symmetric network, i.e., a product utility grows in proportion to its number of adopters, is investigated in [I5] by formulating it as the solution of a dynamic program. A general network setting is considered in [I4] with the important difference of considering a divisible good, so that consumption maximization is now the target.

Like [IO], we focus on product adoption among heterogeneous users in the presence of an externality, but our work differs in that [IO] studies two classes with no adoption costs, and no subsidization. Like [I3], we focus on subsidies (sponsorship in their paper) with network externalities, but our work differs in that [I3] looks at equilibrium pricing, whereas our interest is on adoption dynamics. Like [I5], we address optimizing over subsidies, but our work differs in that [I5] considers buyer-specific subsidies and externalities.

III. MATHEMATICAL MODEL

A. Without cost subsidization

The basic model captures AD in a large population of potential users of a network service exhibiting the three assumptions in [I]. Let \( x(t) = x(t) \) denote the fraction of the population that has adopted the service at each time \( t \geq t_0 \) subject to the initial condition \( x(t_0) = x_0 \).

Assumption 1: The net utility, \( V = V(x) \), perceived by a randomly selected user when the adoption level is \( x \), the cost is \( c \), and the externality parameter \( e \) is the random variable

\[
V(x) = U + ex - c. \tag{1}
\]

The net utility, and each of the three terms comprising it, should be thought of as values or costs per unit time. Each of the three terms in (1) reflect one of the key assumptions in [I].

Assumption 2: User service affinity heterogeneity is captured by the random variable \( U \) with a continuous complementary cumulative distribution function (CCDF) \( F_U \), denoted \( U \sim F_U \). Affinities are independent and identically distributed (iid).

Assumption 3: The network service externality is captured by the utility term \( ex \), where \( e \geq 0 \) is the externality parameter. The assumption that each user’s perceived utility is linear in the adoption level is consistent with Metcalfe’s Law [I].

Assumption 4: The cost of adoption is a constant \( c \geq 0 \) in the net utility. Although one can define an equivalent model with \( U' = U - c \), capturing each user’s “net affinity”, we retain \( c \) to facilitate investigation of cost subsidization.

Assumption 5: The adoption level follows standard diffusion dynamics [I9], with time-scale parameter \( \gamma > 0 \):

\[
\dot{x}(t) = \gamma \left( P(V(x(t)) > 0) - x(t) \right) = \gamma \left( F_U(c-ex(t)) - x(t) \right). \tag{2}
\]

The dynamics in (2) assert the rate of change of the adoption level is proportional to the difference between the fraction of the population that would adopt at adoption level \( x(t) \) (in light of (1)), and the fraction of the population that has adopted, i.e., \( x(t) \). The initial (\( t = t_0 \)) adoption level is denoted \( x(t_0) \equiv x_0 \). Equilibria and stability are defined in the natural way:

Definition 1: A level of adoption \( \tilde{x} \in [0, 1] \) is an equilibrium if \( \dot{x}(t) \equiv 0 \), i.e., \( P(V(\tilde{x}) > 0) = \tilde{x} \). The set of equilibria is denoted \( \mathcal{X} \). An equilibrium \( \tilde{x} \) is stable if \( \dot{x}(t) \equiv 0 \), i.e., \( P(V(\tilde{x}) > 0) < 1 \). The set of stable equilibria is \( \mathcal{X} \subset \mathcal{X} \).

B. With cost subsidization

A subsidy \( s > 0 \) is a reduction of the cost \( c \) so that the net utility (1) under the subsidy is \( V = U + ex - (c-s) \). It is natural to consider subsidies that depend upon time \( (s(t)) \), the adoption level \( (s(x)) \), or both \( (s(t,x)) \).

Definition 2: The (normalized) cost of the subsidy to the service provider is

\[
S = \int_{t_0}^{\infty} s(t,x(t))x(t)dt, \tag{3}
\]

where the AD \( x(t) \) are affected by the subsidy \( s(t,x(t)) \).

In this paper we focus on a simple but natural subsidy.

Definition 3: The constant level subsidy (CLS) with parameters \( (s,T) \in [0,c] \times \mathbb{R}_+ \) is

\[
s(t) = \left\{ \begin{array}{ll}
s, & t \in [t_0,t_0+T] \\
0, & \text{else}
\end{array} \right. \tag{4}
\]

AD under CLS are denoted \( y(t) = y(t,t_0,y_0) \) to distinguish from unsubsidized AD \( x(t) \). The subsidized net utility is

\[
V(t,y) = \left\{ \begin{array}{ll}
U + ey - (c-s), & t \in [t_0,t_0+T] \\
U + ey - c, & \text{else}
\end{array} \right. \tag{5}
\]

and the subsidized AD (with initial condition \( y(t_0) \equiv y_0 \)) are:

\[
y(t) = \left\{ \begin{array}{ll}
\gamma \left( F_U(c-s-ex(t)) - y(t) \right), & t \in [t_0,t_0+T] \\
\gamma \left( F_U(c-ex(t)) - y(t) \right), & \text{else}
\end{array} \right. \tag{6}
\]

The subsidy cost (3) under CLS is

\[
S(s,T) = s \int_{t_0}^{t_0+T} y(t,t_0,y_0)dt = s \int_0^T y(t,0,y_0)dt, \tag{7}
\]

where \( y(t) \) is the solution to (6). Table I lists notation.

3Other externality terms can be argued to be more suitable [I9].
TABLE I

| Notation |
|----------|
| $t_0$    | initial time |
| $x_0 = y_0$ | initial adoption level without and with subsidization |
| $x(t|t_0,x_0)$ | adoption level without subsidization |
| $y(t|t_0,y_0)$ | adoption level with subsidization |
| $V(x), V(t, y)$ | net utility with and without subsidization |
| $U \sim F_U$ | random user service affinity |
| $e \geq 0$ | network externality coefficient |
| $c \geq 0$ | adoption cost (per unit time) |
| $\gamma$ | time-scale parameter for AD |
| $X(X, s, T)$ | set of (stable) adoption equilibria |
| $S(s, T)$ | total cost of subsidy to provider |
| $\text{Unif}(u_m, u_M)$ | uniformly distributed affinities |
| $x^*(c)$ | unique equilibrium in $(0, 1)$ under uniform affinities |

IV. ADOPTION DYNAMICS AND EQUILIBRIA WITHOUT EXTERNALITIES

In this section we study the impact of subsidization in the absence of a network externality ($e = 0$). In contrast with [4], [5], and [11], where $F_U$ is assumed uniform, in this section we make no assumption about the affinity distribution $F_U$ beyond the continuity in Assumption [2].

A. Without subsidization

Without any externality and no subsidization ($s = 0$), the net utility [1] is $V(x) = U - c$ and the AD [2] are $\dot{x}(t) = \gamma(\bar{F}_U(c, t) - x(t))$ (with $x(t_0) = x_0$), which has solution

$$x(t|t_0, x_0) = \bar{F}_U(c) - (\bar{F}_U(c) - x_0)e^{-\gamma(t-t_0)}, \quad t \geq t_0. \quad (8)$$

There is a unique (stable) equilibrium $\bar{X} = \{\bar{F}_U(c)\}$, and $x(t) \to \bar{F}_U(c)$ as $t \to \infty$ for all $(t_0, x_0)$.

B. With constant level subsidy (CLS)

With no externality, if the subsidy level $s(t)$ only depends upon time $t$ and not on the adoption level $y$, then the AD becomes [4]

$$\dot{y}(t|t_0, y_0) = y_0 e^{-\gamma(t-t_0)} + \int_{t_0}^{t} e^{-\gamma(t-\tau)} \bar{F}_U(c(\tau)) d\tau. \quad (9)$$

If we assume the subsidy $s(t)$ is aCLS then more can be said.

**Proposition 1:** Assume no externality ($e = 0$) and fix $s(t)$ to be aCLS with parameters $(s, T)$. The following seven statements hold.

i) The AD [6] are [7]

$$\begin{cases} \bar{F}_U(c - s) - (\bar{F}_U(c) - y_0)e^{-\gamma(t-t_0)}, & t - t_0 \leq T, \\ \bar{F}_U(c) - (\bar{F}_U(c) - y(t_0 + T))e^{-\gamma(t-t_0-T)}, & t - t_0 > T. \end{cases} \quad (10)$$

ii) The cost of subsidization [8] is $S(s, T) = \bar{F}_U(c - s)T - \frac{1}{\gamma}(\bar{F}_U(c - s) - y_0)(1 - e^{-\gamma T}). \quad (11)$

iii) The subsidy duration $T(s, y)$ required to reach a target adoption level $y$ with subsidy level $s$ is

$$T(s, y) = \frac{1}{\gamma} \log \left( \frac{\bar{F}_U(c - s) - y_0}{\bar{F}_U(c - s) - y} \right). \quad (12)$$

which is positive provided $y_0 < y < \bar{F}_U(c - s)$ or $\bar{F}_U(c - s) < y < y_0$. iv) The cost $S(s, T(s, y))$ for a subsidy with duration $T(s, y)$ follows by substitution of [12] into [11]:

$$S(s, T(s, y)) = s \left( \frac{\bar{F}_U(c - s)T(s, y) - y - y_0}{\gamma} \right). \quad (13)$$

v) For any $s \in [0, c]$ and $T < \infty$, the unique stable equilibrium adoption level is $\bar{X} = \{\bar{F}_U(c)\}$. vi) For any target $y > y_0$, the subsidy duration $T(s, y)$ is nonincreasing in the subsidy amount $s$, i.e., $\frac{d}{ds}T(s, y) \leq 0$. vii) For any target $y$,

$$\bar{F}_U(c - s) - sf_U(c - s) \Rightarrow \frac{d}{ds}S(s, T(s, y)) < 0, \quad (14)$$

i.e., the LHS is a sufficient condition for the subsidy cost $S(s, T(s, y))$ to be decreasing in the subsidy amount $s$.

**Proof:** i) Consider first $t_0 \leq t \leq t_0 + T$, where under CLS $s(t)$ in [4], the net utility [1] for each user is $V(t) = U - (c - s)$, and thus the AD [2] are $\dot{y}(t) = \gamma(\bar{F}_U(c - s) - y(t))$ (with $y(t_0) = y_0$), with solution in [8]. Consider next $t > t_0 + T$, where the net utility [1] for each user is $V(t) = U - c$, and thus the AD [2] are $\dot{y}(t) = \gamma(\bar{F}_U(c) - y(t))$ (with $y(t_0 + T) = \bar{F}_U(c - s) - (\bar{F}_U(c - s) - y_0)e^{-\gamma T}$), with solution in [8]. ii) The subsidization cost [11] follows from substituting $c(t) = c$ and $y(t)$ in [10] into [3]. iii) The subsidization duration [12] follows by equating [10] with $y$ and solving for $T$. iv) The subsidy cost [13] follows by substituting $T(s, y)$ in (7) and simplifying. v) The (stable) equilibrium property is immediate from Def. [4] and the dynamics [10]. vi) That $\frac{d}{ds}T(s, y) < 0$ follows from

$$\frac{d}{ds}S(s, T(s, y)) = -\frac{f_U(c - s)(y - y_0)}{\gamma(\bar{F}_U(c) - s) - y_0(\bar{F}_U(c - s) - y)} \leq 0. \quad (15)$$

vii) The proof of [14] is in App. [A].

If affinities are uniformly distributed ($U \sim \text{Unif}(u_m, u_M)$), then the results in Prop. [4] may be extended as follows. Recall $T(s, y)$ [12] is the subsidy duration required to reach a target adoption level $y$ with subsidy level $s$.

**Corollary 1:** Under the assumptions of Prop. [4] and uniform affinities, the following hold.

i) $T(s, y)$ is finite if $s > (c - u_m) + (u_m - u_M)y$, ii) $T(s, y)$ depends upon $s$ if $c - u_m \leq s \leq c - u_m$.

The sufficient condition in [14] simplifies to $u_M < c \Rightarrow \frac{1}{\gamma}S(s, T(s, y)) < 0$.

Combining i) and ii) above, $T(s, y)$ is finite and depends upon $s$ provided $s \in [(c - u_m) + (u_m - u_M)y, c - u_m]$.

**Example 1:** To illustrate Cor. [4] fix $e = 0$, $t_0 = y_0 = 0$, $\gamma = 1$. Fig. [1] (top) shows the AD $y(t)$ vs. $t$ under CLS for $s = c = 1/2$ for $u_m = 0, u_2 = 1, T \in [0, 1, 2, 8]$. After the subsidy ends, the AD converge to the equilibrium, here, $\bar{F}_U(c) = 1/2$. Fig. [1] (bottom) shows $T(s, y)$ and $S(s, T(s, y))$ vs. $s$ for $c = 3, y = 1/2, u_m = 1$ and $u_2 = 0$. By Cor. [4] point i), we have $T(s, y) < \infty$ if $s > 1/2$. By Cor. [4] point ii), we have $T(s, y)$ depends upon $s$ if $3 < s < 2$. By Cor. [4] point iii), if $6 < 3$ then $\frac{d}{ds}S(s, T(s, y)) < 0$. All three points are born out by the plot. Regarding the last point, in fact the subsidy cost $S(s, T(s, y))$ is increasing in $s$. Increasing the subsidy over $s \in [0, 2]$ trades off decreasing duration $T(s, y)$ with increasing cost $S(s, T(s, y))$, while any subsidy $s > 2$ is cost-inefficient relative to $s = 2$. 


In the absence of an externality, the resulting unique (stable) equilibrium \( \tilde{X} = \{ F_U(c) \} \) means that any finite-time subsidy cannot alter the final equilibrium level. In the presence of a network externality, however, the final equilibrium level may (in some cases) be alterable by a finite-term subsidy, which suggests subsidies are of mathematical interest primarily for services with externalities. With this justification, the rest of the paper investigate AD with network externalities — first without (IV-B) and then with (VII) cost subsidization.

V. ADOPTION DYNAMICS AND EQUILIBRIA WITHOUT COST SUBSIDIZATION

We now turn to our main focus in this paper – services exhibiting a network externality. In this section, we seek to i) characterize the set of equilibria \( \tilde{X} \) and stable equilibria \( \tilde{X} \), and ii) explicitly solve the AD \( x(t) \) in (2) in the absence of any cost subsidization. In the interest of providing explicit expressions, we hereafter assume the following.

Assumption 6: The i.i.d. user service affinities are uniformly distributed, \( U \sim \text{Unif}[u_m, u_M] \), for \( u_m < u_M \).

The following notation will be employed. First, let

\[
x^o(c) \equiv \frac{u_M - c}{u_M - (u_m + e)}
\]

denote the unique equilibrium in \((0, 1)\) of (2) under uniform affinities (see Prop. 2). Next, let

\[
\hat{x}(t_0, x_0, c) \equiv x^o(c) + (x_0 - x^o(c))e^{\gamma \frac{u_m - u_M}{u_M - u_m}(t - t_0)}
\]

and

\[
\hat{i}(x|t_0, x_0, c) \equiv t_0 + \frac{1}{\gamma e + u_m - u_M} \log \left( \frac{x - x^o(c)}{x_0 - x^o(c)} \right)
\]

represent the solution of (2) with \( x(t_0) = x_0 \) under uniform affinities, for \( (c - u_M)/e < x \leq (c - u_m)/e \) (cf., Lem. 1 and Cor. 2 in App. B). Finally, define

\[
\hat{T}_M(x_0|c) \equiv \hat{t} \left( \frac{c - u_M}{e}, 0, x_0, c \right)
\]

\[
\hat{T}_m(x_0|c) \equiv \hat{t} \left( \frac{c - u_m}{e}, 0, x_0, c \right)
\]

as the time durations required for the dynamics in (18) to reach \( (c - u_M)/e \) and \( (c - u_m)/e \), respectively, starting from \( x_0 \) at \( t_0 \) (assuming such times are finite).

Proposition 2: Suppose \( U \sim \text{Unif}(u_m, u_M) \) (Ass. 6). The four possible equilibria sets \( \tilde{X} \) are:

\[
\tilde{X} = \begin{cases} 1) \{ 0 \}, & \max \{ u_M, u_m + e \} \leq c \\ 2) \{ x^o(c) \}, & u_m + e \leq c \leq u_M \\ 3) \{ 0, x^o(c), 1 \}, \quad u_M \leq c \leq u_m + e \\ 4) \{ 1 \}, & c \leq \min \{ u_M, u_m + e \} \end{cases}
\]

All equilibria are stable, aside from \( x^o(c) \) when \( u_m \leq c \leq u_m + e \) (case 3). The AD, denoted \( x(t_0, x_0) \), are given by (66) through (69), where the subscripts 1 through 4 correspond to the four equilibrium cases in (21).

The proof is found in App. B (Fig. 2 shows (21). The four cases each imply a range for \( x^o(c) \), \( (c - u_M)/e \), \( (c - u_m)/e \):

| case | ordering | \( x^o(c) \) | \( (c - u_M)/e \) | \( (c - u_m)/e \) |
|------|----------|----------------|-----------------|-----------------|
| 1a   | \( u_m < u_m + e < c \) | > 1 | > 1 | > 0 |
| 1b   | \( u_m + e < u_M < c \) | < 0 | < 0 | < 0 |
| 2    | \( u_m + e < c < u_m + e \) | (0, 1) | (x^o, 1) | (0, x^o) |
| 3    | \( u_M < c < u_m + e \) | \( x^o, 1 \) | \( 0, x^o \) |
| 4    | \( c < u_M < u_m + e \) | < 0 | < 0 | < 0 |
| 4b   | \( c < u_m + e < u_M \) | > 1 | > 1 | < 0 |

Remark 1: The unstable equilibrium \( x^o(c) \) in case 3 is the boundary between the domains of attraction of the two stable equilibria, 0 and 1, i.e., \( x_0 < (> \) \( x^o(c) \) ensures \( x(t) \to 0 \) (1) as \( t \to \infty \).

Remark 2: The AD admit a natural interpretation:

Case 1) \( \tilde{X} = \{ 0 \} \). If \( c \geq u_M + e \) then for all \( x_0 \) the convergence rate to 0 depends solely on \( \gamma \). The same is true if \( c \leq u_m + e \) but \( x_0 \leq (c - u_m)/e \). If instead \( x_0 \geq (c - u_m)/e \) then the dynamics consist of two parts: first \( x(t|t_0, x_0) \) decays like \( \hat{x}(t_0, x_0, c) \) for \( t \in [t_0, t_0 + \hat{T}_M(x_0|c)] \), then \( x(t|t_0, x_0) \) again converges to 0 at a rate that depends solely on \( \gamma \) for \( t \in [t_0 + \hat{T}_M(x_0|c), \infty) \).

Case 2) \( \tilde{X} = \{ x^o(c) \} \). Here, \( x(t|t_0, x_0) \) converges to \( x^o(c) \) according to \( \hat{x}(t|t_0, x_0, c) \) for all \( t \).

Case 2a) \( \tilde{X} = \{ x^o(c), 1 \} \). There are two possibilities: i) if \( x(0) \leq x^o(c) \) then \( x(t|t_0, x_0) \) decays to 0, or ii) if \( x(0) \geq x^o(c) \) then \( x(t|t_0, x_0) \) rises to 1. In either case, \( x(t|t_0, x_0) \) initially evolves according to \( \hat{x}(t_0, x_0, c) \) until either time \( t_0 + \hat{T}_M(x_0|c) \) if \( x_0 \leq x^o(c) \) or \( t_0 + \hat{T}_M(x_0|c) \) (if \( x_0 \geq x^o(c) \)). After this time, \( x(t|t_0, x_0, c) \) converges to 0 or 1, respectively, at a rate that depends solely on \( \gamma \).
\[
x_1(t|t_0, x_0) = \begin{cases} 
  x_0 e^{-\gamma(t-t_0)} & x_0 \leq \frac{c-u_m}{e} \\
  \frac{c-u_m}{e} e^{-\gamma(t-t_0-T_M(x_0|c))} & x_0 \geq 1 \quad \text{and} \quad t-t_0 \leq T_M(x_0|c)
\end{cases}
\]

\[
x_2(t|t_0, x_0, c) = \begin{cases} 
  x_0 e^{-\gamma(t-t_0)} & x_0 \leq \frac{c-u_m}{e} \\
  \frac{c-u_m}{e} e^{-\gamma(t-t_0-T_M(x_0|c))} & x_0 \geq 1 \quad \text{and} \quad t-t_0 \geq T_M(x_0|c)
\end{cases}
\]

\[
x_3(t|t_0, x_0) = \begin{cases} 
  x_0 e^{-\gamma(t-t_0)} & x_0 \leq \frac{c-u_m}{e} \\
  \frac{c-u_m}{e} e^{-\gamma(t-t_0-T_M(x_0|c))} & x_0 \geq 1 \quad \text{and} \quad t-t_0 \leq T_M(x_0|c)
\end{cases}
\]

\[
x_4(t|t_0, x_0) = \begin{cases} 
  x_0 e^{-\gamma(t-t_0)} & x_0 \leq \frac{c-u_m}{e} \\
  \frac{c-u_m}{e} e^{-\gamma(t-t_0-T_M(x_0|c))} & x_0 \geq 1 \quad \text{and} \quad t-t_0 \geq T_M(x_0|c)
\end{cases}
\]

Case 4): \( X = \{1\} \). If \( c \leq u_m \) then for all \( x_0 \) the convergence rate to 1 depends solely on \( \gamma \). The same is true if \( c \geq u_m \) but \( x_0 \geq (c-u_m)/e \). If instead \( x_0 \leq (c-u_m)/e \) then the dynamics consist of two parts: first \( x(t|t_0, x_0) \) rises like \( \hat{x}(t|t_0, x_0, c) \) for \( t \in [t_0, t_0 + T_{M(x_0|c)}] \), then \( x(t|t_0, x_0) \) again converges to 1 at a rate that depends solely on \( \gamma \) for \( t \in [t_0 + T_{m(x_0|c)}, \infty) \).

Example 2: To illustrate Prop. 2 consider these four cases:

| case | \( u_m \) | \( u_M \) | \( c \) | \( e \) | \( x^0(c) \) | \( \frac{c-u_m}{e} \) | \( \frac{c-u_m}{e} \) |
|------|------|------|------|------|----------|----------|----------|
| 1    | 1    | 2    | 5    | 2    | 3        | 2        | -2       |
| 2    | 1    | 2    | 7/2  | 1    | 1        | 1/2      | -1/2     |
| 3    | 1    | 2    | 3/2  | 2    | 1        | 3/4      | -1/4     |
| 4    | 1    | 2    | 1    | 1/2  | 2        | 0        | -2       |

each with \( t_0 = 0 \) and one of four initial conditions: \( x_0 \in \{1/10, 1/3, 2/3, 9/10\} \). These four cases are in one-to-one correspondence with the four cases for \( X \) and \( x(t) \) in Prop. 2. The equilibria and AD are illustrated in Fig. 2.

Significantly, as cases 1, 2, and 4 have only one equilibrium, there is no possibility for a finite-duration subsidy to change the equilibrium adoption level. By contrast, such a change is possible under case 3 (where \( X = \{0, 1\} \)) provided the initial adoption level \( x_0 \) lies below the boundary \( x^0(c) \) between the two domains of attraction. As such, in the following section we focus on this case.

VI. ADOPTION DYNAMICS WITH COST SUBSIDIZATION

We now study AD under cost subsidization in the presence of a network externality when the service affinity distribution is uniform (Ass. 6). Consistent with the remarks at the end of Ass. 6 we focus on the case where subsidies may be used to positive effect. Recall the subsidized AD (6) are denoted \( y(t) \) with \( x_0 = x(t_0) = y_0 = y(t_0) \).

Assumption 7: The parameters \( (u_m, u_M, c, e) \) are such that there are multiple stable equilibria, namely, \( X = \{0, x^0(c), 1\} \); equivalently, \( u_M \leq c \leq u_m + e \) (case 3 in Prop. 2).

Assumption 8: The initial adoption level \( y_0 = y(t_0) \) is such that the stable equilibrium without subsidization is zero, i.e., \( 0 \leq y_0 < x^0(c) \leq 1 \).

The AD under CLS (6) may be specialized under the assumption of uniformly distributed affinities.

Proposition 3: Under CLS with \( U \sim \text{Uni}(u_m, u_M) \)
\[
y(t|t_0, y_0) = \begin{cases} 
  x(t|t_0, y_0)e^{-c-s} & t-t_0 \leq T \\
  x(t|t_0 + T, y(t_0 + T)) & t-t_0 > T
\end{cases}
\]

where \( x(t|t_0, y_0)e^{-c} \) denotes substitution of the cost \( c \) with...
the subsidized cost $c - s$ in the unsubsidized AD from Prop. 2 and the initial adoption level at the end of the subsidy is $y(t_0 + T) = x(t_0 + T|t_0, y_0) = c - s$.

**Proof:** For $t \in [t_0, t_0 + T]$ the subsidized AD follow the unsubsidized AD in Prop. 2 with subsidized cost $c - s$. For $t > t_0 + T$, the subsidized AD follow the unsubsidized AD in Prop. 2 with unsubsidized cost $c$.

Prop. 3 shows the AD under a CLS are (trivially) expressed in terms of the unsubsidized AD, but is otherwise not insightful. We consider two specific cases to gain more insight into CLS. First, VI-A studies full cost subsidization, i.e., $s = c$, as a function of the duration $T$. Second, VI-B studies minimum duration subsidization, i.e., $T = T(s)$ (72), where the duration is selected to bring the adoption level to the boundary $x^o(c)$ between the domains of attraction of the stable equilibria (Remark 1), with the subsidy level $s$ a free parameter.

### A. Full cost subsidization

Under CLS with full cost subsidization $(s = c)$, the subsidized AD $(t \in [t_0, t_0 + T])$ are $y(t) = \gamma(1 - y(t))$ (since each user has a positive net utility) with solution

$$y(t|t_0, y_0) = 1 - (1 - y_0)e^{-\gamma(t-t_0)}, \ t - t_0 \leq T.$$  \hspace{1cm} (25)

Define

$$\hat{T}_o(y_0) = \frac{1}{\gamma} \log \left( \frac{1 - y_0}{1 - x^o(c)} \right) = \frac{1}{\gamma} \log \left( 1 - y_0 \frac{u_m + c - u_M}{u_m + c - e} \right)$$  \hspace{1cm} (26)

as the duration $T$ such that $y(t_0 + T|t_0, y_0) = x^o(c)$, and

$$\hat{T}_m(y_0) = \frac{1}{\gamma} \log \left( \frac{1 - y_0}{u_m + c - e} \right)$$  \hspace{1cm} (27)

$$\hat{T}_M(y_0) = \frac{1}{\gamma} \log \left( \frac{1 - y_0}{u_M + c - e} \right)$$  \hspace{1cm} (28)

as the durations such that $y(t_0 + T|t_0, y_0) = \frac{c - u_m}{e}$ and $y(t_0 + T|t_0, y_0) = \frac{c - u_M}{e}$, respectively. These boundary adoption values $\frac{c - u_m}{e}, \frac{c - u_M}{e}$ are boundary values in $x_3(t)$ in (68). Observe Ass. 7 implies $0 \leq \frac{c - u_m}{e} \leq x^i(c) \leq \frac{c - u_M}{e} \leq 1$, which in turn implies $\min\{\hat{T}_M(y_0), 0\} < \hat{T}_o(y_0) < \hat{T}_m(y_0)$. This ordering defines four time duration intervals. The post-subsidy dynamics and final equilibrium are determined by which of these four intervals contain the given subsidy duration $T$.

**Proposition 4:** Let Assumptions 6 (uniform affinities), 7 (multiple stable equilibria), and 8 (initial adoption level) hold. The following three statements hold under a CLS $s(t)$ with duration $T > 0$ and full subsidy $s = c$.

i) The subsidized $(t_0 \leq t \leq t_0 + T)$ AD are given by (25), for $\hat{T}_M = \hat{T}_M(y(t_0 + T)|c)$ in (19), $\hat{T}_m = \hat{T}_m(y(t_0 + T)|c)$ in (20), and $x$ in (17).

ii) The cost of the subsidy (7) is $S(T)$

$$S(T) = c \left( T - \frac{1}{\gamma}(1 - y_0)(1 - e^{-\gamma T}) \right)$$  \hspace{1cm} (29)

iii) If $T < (>) \hat{T}_o(y_0)$ (26) then $y(t) \to 0 (1)$ as $t \to \infty$.

In (70), the three columns are i) the four interval durations for $T$, ii) the associated ordering of the adoption level $y(t_0 + T)$ at the end of the subsidy relative to $x^o(c)$, $\frac{c - u_m}{e}, \frac{c - u_M}{e}$, and iii) the post-subsidy $(t > t_0 + T)$ AD $y(t)$.

**Proof:** i) The unsubsidized dynamics (70) follow from case 3 in Prop. 2. ii) The subsidy cost $S(T) = 29$ follows from substituting (25) into (7). iii) The convergence to 0 (1), respectively follows from Remark 1.

**Example 3:** As an illustration of Prop. 4 suppose $t_0 = 0, y_0 = \frac{1}{4}, u_m = 1, u_M = 2, c = 3, e = 3, \gamma = \frac{1}{3}$. (30)

Note $u_M \leq c \leq u_m + e$ and as such the unsubsidized dynamics follow case 3) in Prop. 2. Furthermore, $(c - u_M)/e = 1/3$, $\frac{c - u_m}{e} = 1/2$, and $(c - u_m)/e = 2/3$, and $y_0$ satisfies $0 \leq x^o(c)$, meaning without subsidization the adoption level will converge to 0. For the given parameters, the fully subsidized dynamics $y(t)$ reach the adoption thresholds $(c - u_M)/e, x^o(c), (c - u_m)/e$ at times $\hat{T}_o = \hat{T}_M(y_0) = \log(9/8) \approx 0.353, \hat{T}_o = \hat{T}_M(y_0) = 3\log(3/2) \approx 1.216, \hat{T}_m = \hat{T}_m(y_0) = 3\log(9/4) \approx 2.433$, respectively. We consider seven different subsidy durations $T$,

$$0, \frac{T_0}{2}, \frac{T_0}{2}, 19\frac{3}{20}, 21\frac{3}{20}, 20\frac{3}{20}, 2, 3\frac{T_0}{2} - \frac{T_0}{2},$$  \hspace{1cm} (31)

which are “evenly spaced” around thresholds $\hat{T}_o, \hat{T}_o, \hat{T}_m$.

$$\begin{align*}
T_1 &= 0.177, \\
T_2 &= 0.785, \\
T_3 &= 1.156, \\
T_4 &= 1.277, \\
T_5 &= 1.824, \\
T_6 &= 3.041
\end{align*}$$  \hspace{1cm} (32)

Fig. 3 shows the AD for these seven durations $T$. Subsidies with durations below (above) $\hat{T}_o(y_0)$ converge to 0 (1), respectively.

### B. Minimum duration subsidization

We now consider the case of a general subsidy level $s$ with a subsidy duration $T = T(s)$ (72) chosen to ensure that the adoption level at the end of the subsidy is the *minimum required* for the unsubsidized dynamics to converge to 1, i.e.,
\[ T < \hat{T}_M(y_0) \quad y(t_0 + T) \leq \frac{c - u_M}{e} x^0(c) \quad y(t) = y(t_0 + T) e^{-\gamma(t - t_0 - T)} \]

\[ \hat{T}_M(y_0) < T < \hat{T}_o(y_0) \quad \frac{c - u_M}{e} \leq y(t_0 + T) \leq x^0(c) \quad y(t) = \begin{cases} \hat{x}(t_0 + T, y(t_0 + T), c), & T \leq t - t_0 \leq T + \hat{T}_M \\ \frac{c - u_M}{e} e^{-\gamma(t - t_0 - (T + \hat{T}_M))}, & T + \hat{T}_M \leq t - t_0 \end{cases} \]

\[ T > \hat{T}_o(y_0) \quad x^0(c) < y(t_0 + T) < \frac{c - u_m}{e} \quad y(t) = \begin{cases} \hat{x}(t_0 + T, y(t_0 + T), c), & T \leq t - t_0 \leq T + \hat{T}_M \\ 1 - (1 - \frac{c - u_m}{e}) e^{-\gamma(t - t_0 - (T + \hat{T}_m))}, & T + \hat{T}_m \leq t - t_0 \end{cases} \]

(70)

\[ y(t_0 + \hat{T}(s)) = x^0(c) \] (Remark 1). In what follows, we refer to the ratio \( s/e \) as the normalized subsidy.

Proposition 5: Let Assumptions 6 (uniform affinities), 7 (multiple stable equilibria), and 8 (initial adoption level) hold. The following seven statements hold under a CLS \( s(t) \) with \( s \in [0, c] \) and minimum required duration \( T = \hat{T}(s) \).

i) The minimum normalized subsidy \( s/e \) required to change the equilibrium from 0 to 1 is

\[ \hat{s} = \frac{1}{e} \left( 1 - \frac{u_M - u_m}{c} \right) \left( x^0(c) - y_0 \right) \leq \frac{c - u_M}{e} \].

(ii) For any \( \frac{c - u_M}{e} \leq y_0 \leq x^0(c - s) \), the AD under the subsidy for case 3) in Prop. 2 but follow case 4) for \( \frac{c - u_M}{e} \leq \frac{c - u_M}{e} \leq x^0(c) \), for \( \frac{c - u_M}{e} \) in (33) and (recalling \( t \) in (18))

\[ \hat{T}_m(s) = \hat{t} \left( \frac{c - s - u_m}{e} \right) 0, y_0, c - s \). \]

(iii) The required subsidy duration \( \hat{T}(s) \) is given in (72).

v) The duration \( T(s) \) is nonincreasing in the subsidy \( s \):

\[ \frac{d}{ds} T(s) = \begin{cases} < 0, & \frac{c - u_M}{e} - y_0 \leq 0, \\ 0, & \frac{c - u_M}{e} - y_0 \leq \frac{c - u_M}{e} \leq \frac{c - u_M}{e} \end{cases} \]

(iv) The cost of the subsidy \( S(s) \) is given in (73).

vii) The cost of the subsidy \( S(s) \) is initially increasing in \( s \) for \( \frac{c - u_M}{e} \leq s \leq \frac{c - u_M}{e} \), then decreasing for \( \frac{c - u_M}{e} \leq s \leq \frac{c - u_M}{e} - x^0(c) \), and ultimately increasing in \( s \) for \( \frac{c - u_M}{e} - x^0(c) \leq y_0 \leq \frac{c - u_M}{e} \), i.e.,

\[ \frac{dS(s)}{ds} \begin{cases} > 0, & 0 < s \leq \frac{c - u_M}{e} - y_0 \\ \leq 0, & \frac{c - u_M}{e} - y_0 \leq \frac{c - u_M}{e} \leq \frac{c - u_M}{e} - x^0(c) \\ \leq 0, & \frac{c - u_M}{e} - x^0(c) \leq \frac{c - u_M}{e} - y_0 \leq \frac{c - u_M}{e} \leq \frac{c - u_M}{e} - x^0(c) \\ > 0, & \frac{c - u_M}{e} - y_0 \leq \frac{c - u_M}{e} \leq \frac{c - u_M}{e} - x^0(c) \end{cases} \]

(36)

where the \( \leq \) in case 4 denotes there exists some \( \frac{c - u_M}{e} \leq x^0(c) \) such that \( \frac{dS(s)}{ds} < (\) 0 for \( \frac{c - u_M}{e} \leq x^0(c) \), \( \frac{c - u_m}{e} - y_0 \) and \( \frac{c - u_m}{e} - x^0(c) \), \( \frac{c - u_m}{e} - y_0 \), respectively.

Proof:

i) and ii) Assumption 7, i.e., \( u_M \leq c \leq u_m + e \) in Prop. 2, means that any subsidized cost \( c - s \) will follow either case 3 or case 4, since \( c - s \leq u_m + e \) precludes cases 1 and 2. If \( c - s \leq u_M \) then the subsidized AD follow case 4, while if \( u_M \leq c - s \) then they follow case 3. In the case \( s \leq c - u_M \) the equilibria under the subsidy are \( \chi = \{0, x^0(c - s), 1\} \), and the adoption level will increase only if \( y_0 \geq x^0(c - s) \), as in (33).

iii) Suppose \( \frac{c - u_M}{e} \leq x^0(c - s) \leq y_0 \leq x^0(c) \) \leq 1. There are three possibilities for \( \frac{c - u_M}{e} \):

\[ \frac{c - u_m}{e} - x^0(c) \leq \frac{c - u_M}{e} \leq \frac{c - u_m}{e} - y_0 \] (37)

which imply the following orderings, respectively:

\[ \frac{c - u_m}{e} - x^0(c) \leq \frac{c - u_m}{e} - y_0 \] (38)

In the first case the subsidized adoption level rises from \( y_0 \) to the target \( x^0(c) \) without crossing the threshold \( (c - s - u_m)/e \).

In the second case the subsidized adoption level rises to the threshold \( (c - s - u_m)/e \) after a time duration \( \hat{T}_m(s) \), then rises to the target \( x^0(c) \). In the third case the subsidized adoption level starts above the threshold \( (c - s - u_m)/e \) and rises to the target \( x^0(c) \).

iv) Recalling Remark 1, \( \hat{T}(s) \) in (72) follows from

\[ \hat{T}(s) = \min \{ T : y(t_0 + T, t_0, y_0) = x^0(c) \} \].

The derivation is straightforward and is omitted.

v) The analysis of \( \frac{d}{ds} \hat{T}(s) \) is in App. C.

vi) The analysis of \( S(s) \) is in App. D.

vii) The analysis of \( \frac{d}{ds} S(s) \) is in App. E.

Example 4: To illustrate Prop. 5 let \( y_0 \in \{0, 1/8\} \) and fix

\[ u_m = 1, u_M = 2, e = 3, c = 5/2, \gamma = 1, t_0 = 0 \].

First note \( u_M \leq c \leq u_m + e \), so the unsubsidized AD \( x(t)[t_0, y_0] \) will follow case 3 in Prop. 2. For \( y_0 < x^0(c) = 1/4 \), the unsubsidized AD will converge to 0. The minimum normalized subsidies \( s/e \) to change the equilibrium from 0 to 1 are \( s/e = 1/6 \) for \( y_0 = 0 \) and \( s/e = 1/12 \) for \( y_0 = 1/8 \). The thresholds on the subsidized AD are \( (c - u_m)/e - x^0(c) = 1/4, (c - u_m)/e - y_0 \leq 1/2, 3/8 \), and \( c/e = 5/6 \). The performance metrics \( S(s), \hat{T}(s) \) are shown in Fig. 4. On the plot of \( \hat{T}(s) \) and \( S(s) \) vs. \( s \) the vertical lines show the critical subsidy levels. Observe \( \hat{T}(s) \) is strictly decreasing for \( s/e \leq (c - u_m)/e - y_0 \), and then \( \hat{T}(s) \) is independent of \( s \) for \( s > 3/2 \). Further, observe \( \hat{T}(s) \) grows without bound as \( s \) decreases towards the minimum subsidy threshold \( \hat{s} \). Next, observe the aggregate subsidy cost \( S(s) \) is
strictly decreasing for $\hat{s}/\hat{e} \leq s/e \leq (c - u_m)/e - x^0(c)$, and then increases linearly for $s/e \geq (c - u_m)/e - y_0$. The cost $S(s)$ for $s/e \in [(c - u_m)/e - x^0(c), (c - u_m)/e - y_0]$ are computed numerically. Finally, observe the tradeoff between $S(s), \hat{T}(s)$ when plotted parametrically. In particular, for large $s$ the cost increases without decreasing the duration, hence these points are inefficient, while for small $s$ the cost again increases as does the duration, and thus such points are also inefficient. There is a critical interval for the subsidy $s$ within which $S(s)$ is decreasing while $\hat{T}(s)$ is increasing only marginally from its minimum value; this interval represents the efficient frontier for the subsidy.

### VII. CONCLUSION

This paper investigates the use of cost subsidization as a means of increasing the adoption level of services exhibiting network externalities. Specializing the problem to the simple CLS subsidy and uniform user service affinities, we obtain explicit expressions for the AD, and two key performance indicators, namely, the aggregate cost of the subsidy to the service provider and the duration of the subsidy. We chose the subsidy duration to be as small as possible while still ensuring the subsidy will change the equilibrium from zero to full adoption. For this reason, we demonstrated that an efficient service provider will not choose “small” or “large” subsidies, but that there is a Pareto efficient frontier for “intermediate” subsidies wherein the two performance metrics are in tension with one another.

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In particular (with Application of (42) gives
The time $t$ at which $x(t) = x$ is given by $t_{a,b}(x|t_0,x_0)$ below.

$$x_{a,b}(t|t_0,x_0) = \frac{1}{a} \left( (ax_0 + b)e^{\gamma(t-t_0)} - b \right)$$

$$t_{a,b}(x|t_0,x_0) = t_0 + \frac{1}{\gamma} \log \left( \frac{x}{x_0} \right)$$

The proof is trivial. Cor. 2 is immediate from Lem. 1.

**Corollary 2:** The DEs below have the following solutions.

$$x_{-1,0}(t|t_0,x_0) = x_0 e^{-\gamma(t-t_0)}$$

$$t_{-1,0}(x|t_0,x_0) = t_0 - \frac{1}{\gamma} \log \left( \frac{x}{x_0} \right)$$

$$x_{-1,1}(t|t_0,x_0) = 1 - (1 - x_0)e^{-\gamma(t-t_0)}$$

$$t_{-1,1}(x|t_0,x_0) = t_0 - \frac{1}{\gamma} \log \left( \frac{1-x}{1-x_0} \right)$$

**iii)** For $\hat{x}(t) = \gamma \left( \frac{e + u_m - u_m}{u_M - u_m} - x \right)$, i.e.,

$$a = \frac{e + u_m - u_m}{u_M - u_m}, b = \frac{u_M - c}{u_M - u_m}$$

the solution is $x(t) = x_{\hat{a}\hat{b}}(t|t_0,x_0)$ in (17) and reaches state $x$ at time $t_{a,b}(x|t_0,x_0)$ in (18).

Consider Prop. 2. When $U \sim \text{Uni}(u_m, u_M)$

$$h(x) = \mathbb{P}(V(x) > 0) = \begin{cases} 0, & x < \frac{c - u_M}{e} \\ \frac{c - u_M}{u_M - u_m}, & \frac{c - u_M}{e} \leq x < \frac{c - u_m}{e} \\ 1, & \frac{c - u_m}{e} \leq x \\ \end{cases}$$

Characterization of $X$ and $X'$ is straightforward using $h(x)$ (49) and Cor. 2 and is omitted. Likewise, establishing the AD in (66) through (69) requires simply a careful case analysis and is omitted.

**Appendix C**

**Proof of (35), point iii) in Prop. 5**

There are four intervals of $s/e$ to consider. i) If $s/e \leq \hat{s}/e$ then $\hat{T}(s) = \infty$ by point i) in Prop. 5 ii) If $\hat{e} \in \left[ \frac{c}{e}, \frac{c - u_m}{e} \right]$, then by (72), $\hat{T}(s) = t(x^0(c)|0,y_0,c-s) < \infty$. For any $f(s)$,

$$\frac{d}{ds} \log \left( \frac{a - f(s)}{b - f(s)} \right) = \frac{a - b}{(a - f(s))(b - f(s))} f'(s).$$

Then $\frac{d}{ds} \hat{t}(x^0(c)|0,y_0,c-s) = \frac{1}{\gamma} \frac{u_m - u_m}{(u_m - u_M)^2} \frac{x^0(c) - y_0}{x^0(c) - x^0(c-s)}(y_0 - x^0(c-s))$, which is negative by inspection. iii) If $\hat{e} \in \left( \frac{c - u_m}{e}, \frac{c - u_m}{e} \right]$ then, by (72), $\hat{T}(s) = \hat{T}_m(s) + \frac{1}{\gamma} \log \left( \frac{1 - \frac{c - u_m}{1-x^0(c)}}{1-x^0(c)} \right)$.

Observe $\frac{d}{dt} S(s) < 0$ if $F \leq s f$, which establishes (14).

**Appendix A**

**Proof of (14) in Prop. 1**

We use the inequality $\log x \leq \frac{x^2 - 1}{2x}$, which for $x = a/b$ is

$$\log \left( \frac{a}{b} \right) \leq \frac{a^2 - b^2}{2ab} = \frac{(a + b)(a - b)}{2ab}.$$ (41)

In particular (with $f = f_U(c-s)$ and $\hat{F} = \hat{F}_U(c-s)$):

$$\log \left( \frac{\hat{F} - y_0}{\hat{F} - y} \right) \leq \frac{((\hat{F} - y_0) + (\hat{F} - y))((\hat{F} - y_0) - (\hat{F} - y))}{2(\hat{F} - y_0)(\hat{F} - y)}$$

$$= \frac{((\hat{F} - y_0) + (\hat{F} - y))(y_0 - y)}{2(\hat{F} - y_0)(\hat{F} - y)}$$

Application of (42) gives $\gamma \frac{d}{dt} S(s)$

$$= (sf + \hat{F}) \log \left( \frac{\hat{F} - y_0}{\hat{F} - y} \right) - sf \hat{F}(y_0 - y)/(\hat{F} - y) - (y_0 - y)$$

$$\leq \frac{(sf + \hat{F})(\hat{F} - y_0) + (\hat{F} - y))(y_0 - y)}{2(\hat{F} - y_0)(\hat{F} - y)}$$

$$- \frac{sf \hat{F}(y_0 - y) - (y_0 - y)}{\hat{F} - y_0}$$

$$= \frac{y_0 - y}{\hat{F} - y_0} \left[ \frac{1}{2} (\hat{F} - s f)(y_0 + y) - y_0 \right]$$

Observe $\frac{d}{ds} S(s) < 0$ if $\hat{F} \leq s f$, which establishes (14).

**Appendix B**

**Proof of Prop. 2**

**Lemma 1:** The differential equation (DE) $\dot{x}(t) = \gamma(ax + b)$ (with $x(t_0) = x_0$) with $a \neq 0$ solution $x_{a,b}(t|t_0,x_0)$ below.

$$x_{a,b}(t|t_0,x_0) = \frac{1}{a} \left( (ax_0 + b)e^{\gamma(t-t_0)} - b \right)$$

$$t_{a,b}(x|t_0,x_0) = t_0 + \frac{1}{\gamma} \log \left( \frac{x}{x_0} \right)$$
and \( \gamma \tilde{T}(s) = \frac{u_M - u_m}{e + u_m - u_M} \log \left( \frac{(u_M - u_m)(u_m + e - c) + (u_M - u_m)s}{(u_M + e)(u_m + e - c) + u_M (1 - y_0) - c + es} \right) \) 

so that, after some manipulation, we have 

\[
\frac{d}{ds} \tilde{T}(s) = -\frac{1}{\gamma (u_m + e - c + s)} \left( 1 - \frac{(u_M - u_m)(1 - y_0)}{\text{den}} \right) 
\]

for \( \text{den} \equiv (u_m + e)y_0 + u_M (1 - y_0) - c + s \). Then, 

\[
\frac{d}{ds} \tilde{T}(s) = \frac{1}{\gamma (u_m + e - c + s)} \left( 1 - \frac{(u_M - u_m)(1 - y_0)}{\text{den}} \right) 
\]

\[
\frac{u_M - u_m}{e + u_m - u_M} \log \left( \frac{(u_M - u_m)(u_m + e - c) + (u_M - u_m)s}{(u_M + e)(u_m + e - c) + u_M (1 - y_0) - c + es} \right) 
\]

Substituting for \( \tilde{T}(s) \) yields \( (73) \). Case iv): \( \frac{c - u_m}{e} - x^o(c) \leq \frac{\delta}{e} \leq \frac{c - u_m}{e} - y_0 \). Then: 

\[
\frac{\gamma}{s} S(s) = \gamma \int_{t_0}^{t_0 + \tilde{T}(s)} (1 - (1 - y_0)e^{-\gamma(t - t_0)}) dt
\]

Substituting for \( \tilde{T}(s) \) yields \( (73) \). 

**Appendix D**

**Proof of (73), Point vii) in Prop. 5**

There are five cases in (73). Case i): \( 0 \leq \frac{\delta}{e} \leq \frac{c - u_m}{e} - y_0 \). The second inequality may equivalently be expressed as \( 0 \leq y_0 \leq \frac{c - u_m}{c - \delta e} \), and \( \frac{\delta}{e} \leq \frac{\delta}{e} \) implies \( \tilde{T}(s) = \infty \). Thus, the AD follow the first sub-case in case 3 in Prop. 2 and so the cost, normalized by \( \frac{\gamma}{s} \), is 

\[
\frac{\gamma}{s} S(s) = \gamma \int_0^\infty y_0 e^{-\gamma t} dt = y_0.
\]

Case ii): \( \frac{c - u_m}{e} - y_0 \leq \frac{\delta}{e} \leq \frac{c - u_m}{e} - x^o(c) \). This inequality may equivalently be expressed as \( \frac{\delta}{e} \leq \frac{c - u_m}{e} - x^o(c) \), and again \( \frac{\delta}{e} \leq \frac{\delta}{e} \) implies \( \tilde{T}(s) = \infty \). Thus, the AD follow the second and third sub-cases in case 3 in Prop. 2 and so \( \frac{\gamma}{s} S(s) \)

\[
= \gamma \int_0^{\tilde{T}(s)} x(t) dt + \frac{c - s - u_M}{e} \frac{1}{\tilde{T}(s)} \int_0^{\tilde{T}(s)} e^{-\gamma(t - \tilde{T}(s)(c,s))} dt 
\]

\[
= \frac{y_0 - x^o(c - s) (u_M - u_m)}{e + u_m - u_M} \left( e^{\frac{c - s - u_M}{e} \tilde{T}(s)(c,s)} - 1 \right)
\]

Substituting for \( \tilde{T}(s)(y_0 | c - s) \) yields \( (73) \). Case iii): \( \frac{e}{c} \leq \frac{\delta}{e} \leq \frac{c - u_m}{e} - x^o(c) \). Then \( \frac{\gamma}{s} S(s) \)

\[
= \gamma \int_0^{\tilde{T}(s)} \left( x^o(c - s) + (y_0 - x^o(c - s)) e^{\frac{c - s - u_M}{e} \tilde{T}(s)(c,s)} \right) dt
\]

\[
= \frac{y_0 - x^o(c - s) (u_M - u_m)}{e + u_m - u_M} \left( e^{\frac{c - u_m}{e} \tilde{T}(s)} - 1 \right)
\]

\[
+ x^o(c - s) \gamma \tilde{T}(s)
\]

Substituting for \( \tilde{T}(s)(y_0 | c - s) \) yields \( (73) \). Case iv): \( \frac{c - u_m}{e} - x^o(c) \leq \frac{\delta}{e} \leq \frac{c - u_m}{e} - y_0 \). Then: 

\[
\frac{\gamma}{s} S(s) = \gamma \int_{t_0}^{t_0 + \tilde{T}(s)} (1 - (1 - y_0)e^{-\gamma(t - t_0)}) dt
\]

Substituting for \( \tilde{T}(s) \) yields \( (73) \). 

**Appendix E**

**Proof of (73), Point viii) in Prop. 5**

There are five intervals defined in (69). Case i): \( 0 \leq \frac{\delta}{e} \leq \frac{c - u_m}{e} - y_0 \). Here, \( \frac{d}{ds} S(s) = \frac{y_0}{e} > 0 \). Case ii): \( \frac{c - u_m}{e} - y_0 \leq \frac{\delta}{e} \leq \frac{c - u_m}{e} - x^o(c) \). We simplify notation as \( f = f(s) = x^o(c, s) \), \( g = y_0 \), \( h = x^o(s) \), and \( f' = f'(s) = \frac{d}{ds} x^o(c, s) \), where by assumption \( f \leq g \leq h \). Using (66) yields \( x^o(c - s) - s \frac{d}{ds} x^o(c - s) = x^o(c) \), which in the above notation asserts \( f - s f' \). Finally, denote \( \tilde{S}(s) = \gamma \frac{c - u_m}{e} - u_m S(s) \) so that 

\[
\tilde{S}(s) = s f \log \left( \frac{h - f}{g - f} \right) + h - g
\]

holds all dependence of \( S(s) \) upon \( s \). Its derivative is 

\[
\frac{d}{ds} \tilde{S}(s) = (f + s f') \log \left( \frac{h - f}{g - f} \right) + \frac{h - g}{(h - f)(g - f)} \left( (h - f)(g - f) + s f f' \right)
\]

We employ (41), which in this context gives 

\[
\log \left( \frac{h - f}{g - f} \right) \leq \frac{h - g}{(h - f)(g - f)} \frac{1}{2} \left( (h - f) + (g - f) \right)
\]

Substitution of this inequality gives \( \frac{d}{ds} \tilde{S}(s) \)

\[
< \frac{(f + s f')}{(h - f)(g - f)} \frac{1}{2} \left( (h - f) + (g - f) \right) + \frac{h - g}{(h - f)(g - f)} \left( (h - f)(g - f) + s f f' \right)
\]

\[
= \left( \frac{1}{2} (f + s f') \right) ((h - f) + (g - f)) + ((h - f)(g - f) + s f f')
\]

\[
\times \frac{h - g}{(h - f)(g - f)}
\]

Let \( \tilde{S}(s) \) be the first term. It suffices to show \( \frac{d}{ds} \tilde{S}(s) < 0 \):

\[
\frac{d}{ds} \tilde{S}(s) = \frac{1}{2} \left( (h - f) + (g - f) \right) + ((h - f)(g - f) + s f f')
\]

\[
= \frac{h - g}{2} \leq 0
\]

via \( f + s f' = h \) and \( f \leq g \leq h \). We’ve shown \( \gamma \frac{d}{ds} S(s) \)

\[
\frac{u_M - u_m}{e + u_m - u_M} \left( x^o(c)(x^o(c) - y_0)^2 + 2(x^o(c) - x^o(c - s))(y_0 - x^o(c - s)) \right) < 0
\]

Case iv): \( \frac{c - u_m}{e} - x^o(c) \leq \frac{\delta}{e} \leq \frac{c - u_m}{e} - y_0 \), is omitted due to space. Case v): \( \frac{c - u_m}{e} - y_0 \leq \frac{\delta}{e} \leq \frac{\delta}{e} \). Here, \( \gamma \frac{d}{ds} S(s) = \log \left( \frac{1 - y_0}{y_0} \right) \) and \( log p > 1 - 1/p \) yields 

\[
\frac{d}{ds} S(s) \geq \frac{y_0}{1 - y_0} (x^o(c) - y_0) > 0
\]