The Weierstrass semigroups on double covers of genus two curves

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We show that three numerical semigroups $\langle 5, 6, 7, 8 \rangle$, $\langle 3, 7, 8 \rangle$ and $\langle 3, 5 \rangle$ are of double covering type, i.e., the Weierstrass semigroups of ramification points on double covers of curves. Combining the result with [5] and [2] we can determine the Weierstrass semigroups of the ramification points on double covers of genus two curves.

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1 Introduction

Let $C$ be a complete nonsingular irreducible curve over an algebraically closed field $k$ of characteristic 0, which is called a curve in this paper. For a point $P$ of $C$, we set

$$H(P) = \{ \alpha \in \mathbb{N}_0 | \exists \text{ a rational function } f \text{ on } C \text{ with } (f)_{\infty} = \alpha P \},$$

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which is called the Weierstrass semigroup of \( P \) where \( \mathbb{N}_0 \) denotes the additive monoid of non-negative integers. A submonoid \( H \) of \( \mathbb{N}_0 \) is called a numerical semigroup if its complement \( \mathbb{N}_0 \setminus H \) is a finite set. The cardinality of \( \mathbb{N}_0 \setminus H \) is called the genus of \( H \), which is denoted by \( g(H) \). It is known that the Weierstrass semigroup of a point on a curve of genus \( g \) is a numerical semigroup of genus \( g \). For a numerical semigroup \( H \) we denote by \( d_2(H) \) the set of consisting of the elements \( h/2 \) with even \( h \in H \), which becomes a numerical semigroup. A numerical semigroup \( \hat{H} \) is said to be of double covering type if there exists a double covering \( \pi : \tilde{C} \rightarrow C \) of a curve with a ramification point \( \tilde{P} \) over \( P \) satisfying \( H(\tilde{P}) = \hat{H} \). In this case we have \( d_2(H(\tilde{P})) = H(P) \).

We are interested in numerical semigroups of double covering type. Let \( \hat{H} \) be a numerical semigroup of genus \( \hat{g} \) with \( d_2(\hat{H}) = \mathbb{N}_0 \) whose genus is 0. Then the semigroup \( \hat{H} \) is \( \langle 2, 2\hat{g} + 1 \rangle \), where for any positive integers \( a_1, a_2, \ldots, a_n \) we denote by \( \langle a_1, a_2, \ldots, a_n \rangle \) the additive monoid generated by \( a_1, a_2, \ldots, a_n \). In this case \( \hat{H} \) is the Weierstrass semigroup of a ramification point \( \tilde{P} \) on a double cover of the projective line which is of genus \( \hat{g} \). Hence, \( \hat{H} \) is of double covering type.

Let \( \hat{H} \) be a numerical semigroup of genus \( \hat{g} \) with \( d_2(\hat{H}) = \langle 2, 3 \rangle \) which is the only one numerical semigroup of genus 1. Then the semigroup \( \hat{H} \) is either \( \langle 3, 4, 5 \rangle \) or \( \langle 3, 4 \rangle \) or \( \langle 4, 5, 6, 7 \rangle \) or \( \langle 4, 6, 2\hat{g} - 3 \rangle \) with \( \hat{g} \geq 4 \) or \( \langle 4, 6, 2\hat{g} - 1, 2\hat{g} + 1 \rangle \) with \( \hat{g} \geq 4 \). We can show that there is a double covering of an elliptic curve with a ramification point whose Weierstrass semigroup is any semigroup in the above ones (for example, see [1], [2]).

Oliveira and Pimentel [5] studied the semigroup \( \hat{H} = \langle 6, 8, 10, n \rangle \) with an odd number \( n \geq 11 \). They showed that the semigroup \( \hat{H} \) is of double covering type. In this case we have \( d_2(\hat{H}) = \langle 3, 4, 5 \rangle \), which is of genus 2. Moreover, in [2] we proved that any numerical semigroup \( \tilde{H} \) with \( d_2(\hat{H}) = \langle 3, 4, 5 \rangle \) except \( \langle 5, 6, 7, 8 \rangle \), \( \langle 3, 7, 8 \rangle \), \( \langle 3, 5 \rangle \) and \( \langle 3, 5, 7 \rangle \) is of double covering type. In view of the fact that \( g(\langle 3, 5, 7 \rangle) = 3 < 2 \cdot 2 \) the semigroup \( \langle 3, 5, 7 \rangle \) is not of double covering type. There is another numerical semigroup of genus 2, which is \( \langle 2, 5 \rangle \). Using the result of Main Theorem in [4] every numerical semigroup \( \tilde{H} \) with \( d_2(\tilde{H}) = \langle 2, 5 \rangle \) is of double covering type. In this paper we will study the remaining three numerical semigroups. Namely we prove the following:

**Theorem 1** The three numerical semigroups \( \langle 5, 6, 7, 8 \rangle \), \( \langle 3, 7, 8 \rangle \) and \( \langle 3, 5 \rangle \) are of double covering type.

Combining this theorem with the results in [5] and [2], we have the following conclusion:

**Theorem 2** Let \( \hat{H} \) be a numerical semigroup with \( g(d_2(\hat{H})) = 2 \). If \( \hat{H} \neq \langle 3, 5, 7 \rangle \), then it is of double covering type.
2 The proof of Theorem

To prove that the three numerical semigroups are of double covering type we use the following remark which is stated in Theorem 2.2 of [3].

Remark 1. Let \( \tilde{H} \) be a numerical semigroup. We set

\[
n = \min \{ \tilde{h} \in \tilde{H} \mid \tilde{h} \text{ is odd} \} \quad \text{and} \quad g(\tilde{H}) = 2g(d_2(\tilde{H})) + \frac{n-1}{2} - r
\]

with some non-negative integer \( r \). Assume that \( H = d_2(\tilde{H}) \) is Weierstrass. Take a pointed curve \( (C, P) \) with \( H(P) = H \). Let \( Q_1, \ldots, Q_r \) be points of \( C \) different from \( P \) with \( h^0(Q_1 + \cdots + Q_r) = 1 \). Moreover, assume that \( H \) has an expression

\[
\tilde{H} = 2H + \langle n, n + 2l_1, \ldots, n + 2l_s \rangle
\]

of generators with positive integers \( l_1, \ldots, l_s \) such that

\[
h^0(l_iP + Q_1 + \cdots + Q_r) = h^0((l_i - 1)P + Q_1 + \cdots + Q_r) + 1
\]

for all \( i \). If the divisor \( nP - 2Q_1 - \cdots - 2Q_r \) is linearly equivalent to some reduced divisor not containing \( P \), then there is a double covering \( \pi : \tilde{C} \longrightarrow C \) with a ramification point \( \tilde{P} \) over \( P \) satisfying \( H(\tilde{P}) = \tilde{H} \), hence \( \tilde{H} \) is of double covering type.

By seeing the proof of Theorem 2.2 in [3] we may replace the assumption in Theorem 2.2 in [3] that the complete linear system \( |nP - 2Q_1 - \cdots - 2Q_r| \) is base point free by the above assumption that the divisor \( nP - 2Q_1 - \cdots - 2Q_r \) is linearly equivalent to some reduced divisor not containing \( P \).

Case 1. Let \( \tilde{H} = \langle 5, 6, 7, 8 \rangle \). Then we have \( H = d_2(\tilde{H}) = \langle 3, 4, 5 \rangle \) and \( g(\tilde{H}) = 5 = 2 \cdot 2 + \frac{5-1}{2} - 1 \). Moreover, we have \( \tilde{H} = 2H + \langle 5, 5 + 2 \cdot 1 \rangle \). Let \( C \) be a curve of genus 2 and \( \iota \) the hyperelliptic involution on \( C \). Let us take a point \( P \) of \( C \) with \( H(P) = \langle 3, 4, 5 \rangle \) and \( 3(P - \iota(P)) \neq 0 \). Then we get \( h^0(P + \iota(P)) = 2 = h^0(\iota(P)) + 1 \). Moreover, we have \( R \neq P \) if the complete linear system \( |5P - 2\iota(P)| \) has a base point \( R \). Indeed, we assume that \( R = P \). Then we have

\[
h^0(5P - 2\iota(P) - P) = h^0(5P - 2\iota(P)) = 3 + 1 - 2 = 2,
\]

which implies that

\[
4P - 2\iota(P) \sim g_2^1 \sim P + \iota(P).
\]

Hence, we get \( 3(P - \iota(P)) \sim 0 \). This is a contradiction.

We assume that \( |5P - 2\iota(P)| \) has a base point \( R \). Then we get \( 5P - 2\iota(P) \sim R + E \), where \( E \) is an effective divisor of degree 2 with projective dimension 1. In this case the complete linear system \( |E| \) is base point free. Therefore, the divisor \( 5P - 2\iota(P) \) is linearly equivalent to some reduced divisor not containing \( P \). If
$|5P - 2\iota(P)|$ is base point free, then the divisor $5P - 2\iota(P)$ satisfies the above condition. By Remark 1 the semigroup $\tilde{H} = \langle 5, 6, 7, 8 \rangle$ is of double covering type.

**Case 2.** Let $\tilde{H} = \langle 3, 7, 8 \rangle$. Then we have $H = d_2(\tilde{H}) = \langle 3, 4, 5 \rangle$ and $g(\tilde{H}) = 4 = 2 \cdot 2 + \frac{3 - 1}{2} - 1$. Moreover, we have $\tilde{H} = 2H + \langle 3, 3 + 2 \cdot 2 \rangle$. Let $C$ be a curve of genus 2 and $\iota$ the hyperelliptic involution on $C$. We take a point $P$ of $C$ with $H(P) = \langle 3, 4, 5 \rangle$. Let $\varphi : C \to \mathbb{P}^1$ be a covering of degree 3 corresponding to the complete linear system $|3P|$. We may take the pointed curve $(C, P)$ such that $\varphi$ has a simple ramification point $Q$. Then there is another simple ramification point of $\varphi$ by Riemann-Hurwitz formula. Hence, we may assume that $\iota P \neq Q$, which implies that $P + Q \not\sim g_2^1$. Thus, we get $h^0(2P + Q) = 2 = h^0(P + Q) + 1$. Let $R$ be the point satisfying $2Q + R \sim 3P$. Then we have $R \neq P$ and $3P - 2Q \sim R$. By Remark 1 the semigroup $\tilde{H} = \langle 3, 7, 8 \rangle$ is of double covering type.

**Case 3.** Let $\tilde{H} = \langle 3, 5 \rangle$. Then we have $H = d_2(\tilde{H}) = \langle 3, 4, 5 \rangle$ and $g(\tilde{H}) = 4 = 2 \cdot 2 + \frac{3 - 1}{2} - 1$. Moreover, we have $\tilde{H} = 2H + \langle 3, 3 + 2 \cdot 1 \rangle$. Let $C$ be a curve whose function field is $k(x, y)$ with an equation $y^3 = (x - c_1)(x - c_2)(x - c_3)^2$, where $c_1, c_2$ and $c_3$ are distinct elements of $k$. Let $\pi : C \to \mathbb{P}^1$ be the morphism corresponding to the inclusion $k(x) \subset k(x, y)$. Then $C$ is of genus 2. Let $P = P_1$, $P_2$, $P_3$ and $P_4$ be the ramification points of $\pi$. Since $\pi$ is a cyclic covering, it induces an automorphism $\sigma$ of $C$ with $C/\langle \sigma \rangle \cong \mathbb{P}^1$. Let $\iota$ be the hyperelliptic involution on $C$. Then we have $\sigma \circ \iota = \iota \circ \sigma$. Indeed, we have

$$(\sigma \circ \iota \circ \sigma^{-1}) \circ (\sigma \circ \iota \circ \sigma^{-1}) = \sigma \circ \iota \circ \sigma = \sigma \circ \sigma^{-1} = \text{id}.$$  

Hence, the automorphism $\sigma \circ \iota \circ \sigma^{-1}$ is an involution. Moreover, we have a bijective correspondence between the sets $\text{Fix}(\iota)$ and $\text{Fix}(\sigma \circ \iota \circ \sigma^{-1})$ sending $Q$ to $\sigma(Q)$, where $\text{Fix}(\iota)$ and $\text{Fix}(\sigma \circ \iota \circ \sigma^{-1})$ are the sets of the fixed points by $\iota$ and $\sigma \circ \iota \circ \sigma^{-1}$ respectively. Hence, $\sigma \circ \iota \circ \sigma^{-1}$ is also the hyperelliptic involution. Thus, we have $\sigma \circ \iota \circ \sigma^{-1} = \iota$. Since $\sigma(\iota(P)) = \iota(\sigma(P)) = \iota(P)$, the point $\iota(P)$ is a fixed point of $\sigma$. Hence, we may assume that $\iota(P) = P_2$. Then we obtain $h^0(P + P_2) = 2 = h^0(P_2) + 1$. Moreover, we have

$$3P - 2P_2 \sim 3P_2 - 2P_2 = P_2.$$  

By Remark 1 the semigroup $\langle 3, 5 \rangle$ is of double covering type. \hfill \Box

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