Entropic trade–off relations for quantum operations

Wojciech Roga\textsuperscript{1,2}, Zbigniew Puchała\textsuperscript{3}, Łukasz Rudnicki\textsuperscript{4}, Karol Życzkowski\textsuperscript{2,4}

\textsuperscript{1}Università degli Studi di Salerno, Via Ponte don Melillo, I-84084 Fisciano (SA), Italy
\textsuperscript{2}Institute of Physics, Jagiellonian University, ul. Reymonta 4, 30-059 Kraków, Poland
\textsuperscript{3}Institute of Theoretical and Applied Informatics, Polish Academy of Sciences, Bałtycka 5, 44-100 Gliwice, Poland
\textsuperscript{4}Center for Theoretical Physics, Polish Academy of Sciences, al. Lotników 32/46, 02-668 Warszawa, Poland

(Dated: 05–02–2013, ver. 19.2)

Spectral properties of an arbitrary matrix can be characterized by the entropy of its rescaled singular values. Any quantum operation can be described by the associated dynamical matrix or by the corresponding superoperator. The entropy of the dynamical matrix describes the degree of decoherence introduced by the map, while the entropy of the superoperator characterizes the a priori knowledge of the receiver of the outcome of a quantum channel. We prove that for any map acting on a \(N\)–dimensional quantum system the sum of both entropies is not smaller than \(\ln N\). For any bistochastic map this lower bound reads \(2 \ln N\). We investigate also the corresponding Rényi entropies, providing an upper bound for their sum and analyze entanglement of the bi–partite quantum state associated with the channel.

PACS numbers: 03.67.Hk 02.10.Ud 03.65.Aa

\section{I. INTRODUCTION}

From the early days of quantum mechanics the uncertainty principle was one of its the most significant features, as it shows in what respect the quantum theory differs from its classical counterpart. It was manifested that the variances of the two noncommuting observables cannot be simultaneously arbitrarily small. Therefore, if we prepare the quantum state as an eigenstate of one observable we get the perfect knowledge about the corresponding physical quantity, however, we loose ability to predict the effect of measurement of the second, noncommuting observable. Preparing a state one shall always consider some trade–off regarding the observables which will be specified in the experiment. Limits for such a trade–off have been formulated as different uncertainty relations \[1,2\]. However, we shall point out that these uncertainty relations are not necessarily related to non–commuting observables, but may describe the trade–off originating from different descriptions of the same quantum state. As an example consider the entropic uncertainty relation \[3\] derived for two probability distributions related to the same quantum state in position and momentum representations.

The original formulation of Heisenberg \[3\] of the uncertainty principle concerns the product of variances of two non–commuting observables. Assume that a physical system is described by a quantum state \(|\psi\rangle\), and several copies of this state are available. Measuring an observable \(A\) in each copy of this state results with the standard deviation \(\Delta_\psi A = \sqrt{\langle \psi | A^2 |\psi\rangle - \langle \psi | A |\psi\rangle^2}\), while \(\Delta_\psi B\) denotes an analogous expression for another observable \(B\). According to the approach of Robertson \[2\] the product of these deviations is bounded from below,

\begin{equation}
\Delta_\psi A \Delta_\psi B \geq \frac{1}{2} |\langle \psi | [A,B] |\psi\rangle|,
\end{equation}

where \([A,B] = AB - BA\) denotes the commutator. If the operators \(A\) and \(B\) do not commute it is thus impossible to specify simultaneously precise values of both observables.

Uncertainty relations can also be formulated for other quantities characterizing the distributions of the measurement outcomes. One possible choice is to use entropy which leads to entropic uncertainty relations of Bialynicki–Birula and Mycielski \[3\]. This formulation can be considered as a generalization of the standard approach as it implies the relations of Heisenberg.

In the case of a finite dimensional Hilbert space the uncertainty relation can be formulated in terms of the Shannon entropy. Consider a non–degenerate observable \(A\), the eigenstates \(|a_i\rangle\) of which determine an orthonormal basis. The probability that this observable measured in a pure state \(|\psi\rangle\) gives the outcome reads \(a_i = \langle |a_i| |\psi\rangle\). The non-negative numbers \(a_i\) satisfy \(\sum_{i=1}^{N} a_i = 1\), so this distribution can be characterized by the Shannon entropy, \(H(A) = - \sum_{i=1}^{N} a_i \ln a_i\). Let \(H(B)\) denotes the Shannon entropy corresponding the the probability vector \(b_i = \langle |b_i| |\psi\rangle\) associated with an observable \(B\). If both observables do not commute the sum of both entropies is bounded from below, as shown by Deutsch \[4\]. His result was improved by Maassen and Uffink \[5\], who proved that

\begin{equation}
H(A) + H(B) \geq -2 \ln c,
\end{equation}

where \(c^2 = \max_{j,k} \langle a_j | b_k \rangle^2\) denotes the maximal overlap between the eigenstates of both observables. Note that
this bound depends solely on the choice of the observables and not on the state $|\psi\rangle$. Recent reviews on entropic uncertainty relations can be found in [6, 7], while a link to the stabilizer formalism was discussed in [8]. Certain generalizations of uncertainty relations for more than two spaces can be based on the strong subadditivity of entropy [9].

Relation (2) describes a bound for the information which can be obtained in two non-complementary projective measurements. Entropic uncertainty relations formulated for a pair of arbitrary measurements, described by positive operator valued measures (POVM), were obtained by Krishna and Parthasarathy [11]. A more general class of inequalities was derived later by Rastegin [11, 12]. Related recent results [13–16] concerned sum of two conditional entropies characterizing two quantum measurements described in terms of their POVM operators.

The so-called collapse of wave function during the measurement is often considered as another characteristic feature of quantum mechanics. This postulate of quantum theory implies that the measurement disturbs the quantum state subjected to the process of quantum measurement. In general, describing a quantum operation performed on an arbitrary state one can consider a kind of trade–off relations between the efficiency of the measurement and the disturbance introduced to the measured states.

Even though the trade–off relations were investigated from the beginnings of quantum mechanics, this field became a subject of a considerable scientific interest in the recent decade [19, 21]. The notion of disturbance of a state introduced by Maccone [17, 18] can be related to the average fidelity between an initial state of the system and the state after the measurement [17, 19]. Another version of disturbance can be defined as a difference between the initial entropy of a quantum state and the coherent information between the system and the measuring apparatus [18].

In this work we will investigate a single measurement process described by a quantum operation: a complete positive, trace preserving linear map which acts on an input state of size $N$. We attempt to compare the information loss introduced by the map (disturbance) with the information the receiver knows about the outgoing state before the measurement (the information gained by the apparatus). The former quantity can be characterized [22] by the entropy of a map $S_{\text{map}}(\Phi)$, equal to the von Neumann entropy of the quantum state which corresponds to the considered map by the Jamiołkowski isomorphism [23, 24]. The latter quantity will be described by the singular quantum entropy $S_{\text{sing}}(\Phi)$ of Jumarie [23], given by the Shannon entropy of the normalized vector of singular values of the superoperator matrix. We are going to show that the sum of these two entropies is bounded from below by $\ln N$. Note that our approach concerns a given quantum map $\Phi$, but it does not depend on the particular choice of the Kraus operators (or POVM operators) used to represent the quantum operation. We also derive an upper bound for the sum of entropies $S_{\text{map}}(\Phi)$ and $S_{\text{sing}}(\Phi)$ and analyze entanglement properties of the corresponding Jamiołkowski-Choi state.

Our paper is organized as follows. In Section II we review basic concepts on quantum maps, define entropies investigated and present a connection with trade–off relations for quantum measurements. A motivation for our study stems from investigations of the one–qubit maps presented in Section III. General entropic inequalities for arbitrarily reordered matrices are formulated in Section IV. Main results of the work are contained in Section V, in which the trade–off relations for quantum channels are derived and entanglement of the corresponding states is analyzed. Discussion of some other properties of the dynamical matrix and a bound for the entropy of a map are relegated to Appendices.

**II. QUANTUM OPERATIONS AND ENTROPY**

A quantum state is described by a density matrix – a Hermitian, positive semi-definite matrix of trace one. A density matrix of dimension $N$ represents the operator acting on $\mathcal{H}_N$. The set of density matrices of dimension $N$ is denoted as:

$$\mathcal{M}_N = \{\rho : \rho = \rho^\dagger, \rho \geq 0, \text{Tr}\rho = 1\}. \quad (3)$$

A quantum operation $\Phi$, also called a quantum channel, is defined as a completely positive (CP) and trace preserving (TP) quantum map which acts on the set of density matrices:

$$\Phi : \mathcal{M}_N \rightarrow \mathcal{M}_N. \quad (4)$$

Complete positivity means that any extended map acting on an enlarged quantum system

$$\Phi \otimes 1_d : \mathcal{M}_{Nd} \rightarrow \mathcal{M}_{Nd} \quad (5)$$

transforms positive matrices into positive matrices for any extension of dimension $d$. Due to the Choi theorem, see e.g. [21], to verify whether a given quantum map $\Phi^A$ acting on a quantum $N$–level system $A$ is completely positive it is necessary and sufficient that the following operator on the Hilbert space $\mathcal{H}_N^A \otimes \mathcal{H}_B^B$ of a composed subsystems $A$ and $B$

$$D_{\Phi^A} := N(\Phi^A \otimes 1^B)(|\phi_+^{AB}\rangle \langle \phi_+^{AB}|) \geq 0, \quad (6)$$

transforms positive matrices into positive matrices for any extension of dimension $d$. Due to the Choi theorem, see e.g. [21], to verify whether a given quantum map $\Phi^A$ acting on a quantum $N$–level system $A$ is completely positive it is necessary and sufficient that the following operator on the Hilbert space $\mathcal{H}_N^A \otimes \mathcal{H}_B^B$ of a composed subsystems $A$ and $B$: 

$$D_{\Phi^A} := N(\Phi^A \otimes 1^B)(|\phi_+^{AB}\rangle \langle \phi_+^{AB}|) \geq 0, \quad (6)$$

transforms positive matrices into positive matrices for any extension of dimension $d$. Due to the Choi theorem, see e.g. [21], to verify whether a given quantum map $\Phi^A$ acting on a quantum $N$–level system $A$ is completely positive it is necessary and sufficient that the following operator on the Hilbert space $\mathcal{H}_N^A \otimes \mathcal{H}_B^B$ of a composed subsystems $A$ and $B$
is non-negative. Here \( \phi^{AB} = \frac{1}{N} \sum_{i=1}^{N} |i^A \rangle \otimes |i^B \rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \) denotes the maximally entangled state in the extended space. The above relation, called the Jamiołkowski isomorphism, implies a correspondence between quantum maps \( \Phi \) and quantum states \( \omega_{\Phi} = \frac{1}{N} D_{\Phi} \).

The operator \( \omega_{\Phi} \) is called the Jamiołkowski–Choi state, whereas the matrix \( D_{\Phi} \) is called the dynamical matrix associated with the map \( \Phi \).

Any quantum channel acting on a quantum system \( A \) can be represented by a unitary transformation \( U^{AB} \) acting on an enlarged system and followed by the partial trace over the ancillary subsystem \( B \):

\[
\Phi(\rho^A) = \text{Tr}_B \left[ U^{AB}(\rho^A \otimes |1^B \rangle \langle 1^B|)(U^{AB})^\dagger \right].
\]

This formula is called the environmental representation of a quantum channel. Another useful representation of a quantum channel is given by a set of operators \( K_i \) satisfying an identity resolution, \( \sum_i K_i^\dagger K_i = 1 \), which implies the trace preserving property. The Kraus operators \( K_i \) define the Kraus representation of the map \( \Phi \),

\[
\Phi(\rho) = \sum_i K_i \rho K_i^\dagger.
\]

Since \( \Phi : \rho \rightarrow \rho' \) acts on an operator \( \rho \), it is sometimes called a superoperator. If we reshape a density matrix into a vector of its entries \( \vec{\rho} \), the superoperator \( \Phi \) is represented by a matrix of size \( N^2 \). It is often convenient to write the discrete dynamics \( \vec{\rho}' = \Phi \vec{\rho} \) using the four–index notation

\[
\rho'_{kl} = \Phi_{mn} \rho_{mn},
\]

where the sum over repeating indices is implied and

\[
\Phi_{mn} = (k \mid \Phi \mid m n).
\]

The dynamical matrix \( D = D_{\Phi} \) is related to the superoperator matrix \( \Phi \) by reshuffling its entries,

\[
D_{lm} = \Phi_{mn} \delta_{in}
\]

written \( D_{\Phi} = \Phi^R \) or \( \Phi = D_{\Phi}^R \).

The von Neumann entropy of the Jamiołkowski-Choi state was studied in \([26-30]\) and it is also investigated in this work. We are going to compare the spectral properties of the Jamiołkowski-Choi state \( \omega_{\Phi} \) and the spectral properties of the corresponding superoperator matrix \( \Phi \).

### A. Entropy of a map

Entropy \( S_{\text{map}}(\Phi) \) is defined \([22]\) as the von Neumann entropy of the corresponding Jamiołkowski-Choi state \( \omega_{\Phi} = \frac{1}{N} D_{\Phi} \),

\[
S_{\text{map}}(\Phi) := -\text{Tr} \omega_{\Phi} \ln \omega_{\Phi}.
\]

This quantity can be interpreted as the special case of the exchange entropy \([31]\)

\[
S_{\text{exchange}}(\rho^A, \rho^A) \equiv S \left( \Phi^A \otimes \mathbb{1} (|\phi^{AB}_\rho \rangle \langle \phi^{AB}_\rho|) \right),
\]

where \( |\phi^{AB}_\rho \rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \) is a purification of \( \rho^A \), that is such a pure state of an enlarged system which has the partial trace given by \( \text{Tr}_B |\phi^{AB}_\rho \rangle \langle \phi^{AB}_\rho| = \rho^A \). The exchange entropy characterizes the information exchanged during a quantum operation between a principal quantum system \( A \) and an environment \( B \), assumed to be initially in a pure state. Under the condition that an initial state of the quantum system \( A \) is maximally mixed, \( \rho^A = \frac{1}{N} \mathbb{1} \), the exchange entropy \( S_{\text{exchange}}(\Phi^A, \rho^A) \) is equal to the entropy of a channel \( S_{\text{map}}(\Phi^A) \).

We will treat the entropy of a map as a measure of disturbance caused by a measurement performed on the quantum system. The work \([18]\) contains a list of the properties expected from a good measure of disturbance. Among them there is the requirement that the disturbance measure should be equal to zero if and only if the measuring process is invertible. For unitary transformations of the quantum state the dynamical matrix given in Eq. \([10]\) has rank one, so the related entropy of the map is equal to zero as expected. Moreover, if the map preserves identity, the entropy of a map is equivalent to the state independent disturbance analyzed in \([18]\).

It is useful to generalize the von Neumann entropy and to introduce the family of the Rényi entropies

\[
S_q(\rho) = \frac{1}{1-q} \ln \text{Tr} \rho^q,
\]

as they allow to formulate a more general class of uncertainty relations \([35]\). Here \( q \geq 0 \) is a free parameter and in the limit \( q \rightarrow 1 \) the generalized entropy tends to the von Neumann entropy, \( S_1(\rho) \equiv S(\rho) \). For any classical probability vector and any quantum state \( \rho \) the Rényi entropy \( S_q(\rho) \) is a monotonously decreasing function of the Rényi parameter \( q \) \([36]\). The generalized entropy \([12]\) of a quantum map \( \Phi \), obtained by applying the above form of Rényi to the state \( \omega_{\Phi} \) will be denoted by \( S_{q,\text{map}}(\Phi) \).

#### 1. Connection with the uncertainty principle for measurements

Let \( \Phi \) be a CP TP map with Kraus operators \( \{A_i\} \), we define, \( P_i = A_i^\dagger A_i \) and note that the operators \( \{P_i\} \) form a POVM \([24]\), i.e. are positive semidefinite and

\[
\sum_i P_i = \mathbb{1}.
\]
If $\rho$ is a state of a given system, the probability of the outcome associated with a measurement of the operator $P_i$ reads

$$p_i = \text{Tr} P_i \rho. \quad (16)$$

The uncertainty involved in a described measurement can be quantified by the entropy

$$H_q(P, \rho) = S_q(p). \quad (17)$$

Let us consider an uncertainty involved in the measurement when the state of a given system is maximally mixed, i.e.

$$p_i = \text{Tr} \left( P_i \frac{1}{N} \mathbb{1} \right) = \frac{1}{N} \text{Tr} P_i. \quad (18)$$

We have the following corollary

**Corollary 1.** If the state of a given system is maximally mixed, then

$$\min_p H_q \left( P, \frac{1}{N} \mathbb{1} \right) = S_q^{\text{map}}(\Phi), \quad (19)$$

where the minimum is taken over all possible POVM’s such that

$$P_i = A_i^\dagger A_i \quad (20)$$

and $A_i$ are Kraus operators of the quantum channel $\Phi$.

**Proof.** Let $A_i$ be a Kraus representation of the channel $\Phi$ and denote by $|\text{res}(A_i)\rangle$ a vector obtained from the matrix $A_i$ by putting its elements in the lexicographical order i.e. rows follow one after another. We introduce

$$\kappa_i = \langle \text{res}(A_i) | \text{res}(A_i) \rangle = \text{Tr} A_i^\dagger A_i = \text{Tr} P_i,$$

$$|a_i\rangle = \frac{1}{\sqrt{\kappa_i}} |\text{res}(A_i)\rangle. \quad (21)$$

Assume that we put the $\kappa_i$ coefficients in a decreasing order such that $\kappa_1$ has the largest value. We have [24]

$$D_\Phi = \sum_{i=1}^l |\text{res}(A_i)\rangle \langle \text{res}(A_i)|, \quad (22)$$

where $l \leq N^2$. Using the variational characterization of eigenvalues, for the Hermitian matrix $D_\Phi$, we get for each $k \leq N^2$ the following expression for the sum of $k$ largest eigenvalues

$$\sum_{i=1}^k \lambda_i(D_\Phi) = \max_{U_k} \text{Tr} \left( U_k^\dagger D_\Phi U_k \right) \quad (23)$$

$$= \max_{\tilde{U}_k} \sum_{i=1}^l \kappa_i \text{Tr} \left( \tilde{U}_k^\dagger |a_i\rangle \langle a_i| \tilde{U}_k \right),$$

where $U_k$ is a matrix of size $N^2 \times k$ fulfilling the relation $U_k^\dagger U_k = \mathbb{1}_k$, and $\mathbb{1}_k$ denotes the $k \times k$ identity. For a specific choice $\tilde{U}_k$ of the matrix $U_k$, such that the vectors $|\text{res}(A_1)\rangle, ..., |\text{res}(A_k)\rangle$ belong to the subspace spanned by all $k$ columns of $\tilde{U}_k$, we have $\text{Tr} \left( \tilde{U}_k^\dagger |a_i\rangle \langle a_i| \tilde{U}_k \right) = 1$, for $i = 1, ..., k$. This property implies

$$\sum_{i=1}^k \lambda_i(D_\Phi) = \max_{\tilde{U}_k} \sum_{i=1}^l \kappa_i \text{Tr} \left( \tilde{U}_k^\dagger |a_i\rangle \langle a_i| \tilde{U}_k \right) \geq \sum_{i=1}^k \kappa_i \text{Tr} \left( \tilde{U}_k^\dagger |a_i\rangle \langle a_i| \tilde{U}_k \right) \quad (24)$$

In the last inequality we neglected the remaining non-negative terms labeled by $i > k$. The above set of inequalities imply the majorization relation, $\kappa \prec \lambda(D_\Phi)$.

Using the fact that Rényi entropies are Schur–concave, we arrive at the desired inequality,

$$H_q \left( P, \frac{1}{N} \mathbb{1} \right) = S_q \left( \left\{ \text{Tr} P_i \frac{1}{N} \mathbb{1} \right\} \right) = S_q(\kappa/N), \quad (25)$$

$$\geq S_q(\lambda(D_\Phi)/N) = S_q^{\text{map}}(\Phi).$$

**□**

Using the monotonicity of the Rényi entropies, we get

$$S_q^{\text{map}}(\Phi) \geq S_q^{\text{map}}(\Phi) = -\log(\lambda_1(D_\Phi)/N). \quad (26)$$

For $q = 1$ this inequality combined with [20] resembles the uncertainty principle for a single quantum measurement [10, 13, 14], since for an optimal POVM the lower bound obtained in these papers depends on $c = \max_j \text{Tr} P_j = \lambda_1/N$.

**B. Receiver entropy**

Since a superoperator matrix $\Phi$ is in general not Hermitian, we characterize this matrix by means of the entropy of the normalized vector of its singular values

$$S^{\text{rec}}(\Phi) = -\sum_i \mu_i \ln \mu_i. \quad (27)$$

Here $\mu_i = \frac{\sigma_i^2}{\sum_k \sigma_k}$ and $\sigma_i$ denote the singular values of $\Phi$, so that $\sigma_i^2$ are eigenvalues of the positive matrix $\Phi \Phi^\dagger$. This quantity characterizes an arbitrary matrix $\Phi$ and depends only on its singular values, so it was called singular quantum entropy by Jumarie [25].
In general, the receiver entropy is bounded by the logarithm of the rank of the superoperator characterizing the channel $S_{rec}(\Phi) \leq \ln \text{rank}(\Phi)$. Every quantum state can be represented by a real vector in the basis of generalized Pauli matrices (see for instance [32]). Therefore, in this basis the superoperator is a real matrix and its rank characterizes the dimensionality of the vector space accessible for the outcomes from the channel.

Consider any orthonormal basis $\{K_i\}$ with respect to the Hilbert-Schmidt scalar product which includes rescaled identity. Such a set of matrices satisfies normalization condition $\sum_i K_i \dagger K_i = 1$, therefore can define the POVM measurement. During the measurements of a quantum state $\rho$ the outcomes $K_i \rho K_i \dagger / (\text{Tr} K_i \rho K_i \dagger)$ are observed with probabilities $p_i = \text{Tr} K_i \rho K_i \dagger$. The receiver entropy is related to the probability distribution characterizing frequency of different outcomes of the measurement apparatus. If the entropy is low the receiver may expect that only a small amount of outcomes of the measuring apparatus will occur. High values of the entropy imply that several different results of the measurement will appear. Hence the receiver entropy $S_{rec}$ characterizes the number of measurement operators needed to obtain a complete information about the measured state.

### III. ONE QUBIT EXAMPLES

To analyze discrete dynamics of a one-qubit system let us define two matrices of order four:

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} G_1 \\ G_2 \\ G_3 \\ G_4 \end{pmatrix},$$

(28)

and

$$C = G^R = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(29)

Note that the first row of $C$ is obtained by reshaping the block $G_1$ into a vector, the second row of $C$ contains the reshaped block $G_2$, etc. Such transformation of a matrix is related to the fact that in any linear, one-qubit map, $\rho^\prime = \Phi \rho$, the $2 \times 2$ matrix $\rho$ is treated as a vector of length 4.

Normalizing the spectra of both matrices to unity we get the entropies $S(G) = \ln 4$ and $S(C) = 0$. Making use of the above notation we can represent the identity map $\Phi_1 = G$ and the corresponding dynamical matrix $D_1 = \Phi_1^R = C$. Moreover, the completely depolarizing channel which maps any state $\rho$ into the maximally mixed state $\Phi_* : \rho \mapsto \frac{1}{2} 1$ can be written as $\Phi_* = \frac{1}{2} C$ while $D_* = \Phi_1^R = \frac{1}{2} G$. In both cases the sum of the entropy of a dynamical matrix $S^{\text{map}} = S(\frac{1}{2} D)$ and the entropy of normalized singular values of superoperator.
\( S_{\text{rec}} \equiv S \left( \frac{\Phi_B}{M_N} \right) \) reads \( S_{\text{map}} + S_{\text{rec}} = 2 \ln 2 \). Thus, both maps \( \Phi_1 \) and \( \Phi_2 \) are in a sense distinguished, as they occupy extreme positions at both entropy axes. It is easy to see that the above reasoning can be generalized for an arbitrary dimension \( N \). For the identity map acting on \( M_N \) and the maximally depolarizing channel \( \Phi_2 \), one obtains \( S_{\text{map}} + S_{\text{rec}} = 2 \ln N \). For these two maps the above relation holds also for the Rényi entropies, \( S_{\text{map}}^{(q)} + S_{\text{rec}}^{(q)} = 2 \ln N \).

Investigations of one–qubit quantum operations enabled us to specify the set of admissible values of the channel entropy \( S_{\text{map}}(\Phi) \) and the receiver entropy \( S_{\text{rec}}(\Phi) \). We analyzed the images of the set of one–qubit quantum maps on to the plane \( (S_{\text{map}}, S_{\text{rec}}) \). This problem was first analyzed numerically by constructing random one-qubit maps \([38]\) and marking their position on the plane. A special care was paid to the case of bistochastic maps, i.e. maps preserving the identity, which form a tetrahedron spanned by the identity \( \sigma_0 \) and the three Pauli matrices \( \sigma_i \) (see e.g. \([24]\))

\[
\Phi_{\text{bist}}(\rho) = \sum_{i=0}^{3} p_i \sigma_i \rho \sigma_i. \quad (30)
\]

Fig. 2 can thus be interpreted as a non–linear projection of the set of all one–qubit channels onto the plane \( (S_{\text{map}}, S_{\text{rec}}) \), in which bistochastic maps correspond to the dark stripped region.

The distinguished points of the allowed region in the plane \( (S_{\text{map}}, S_{\text{rec}}) \) correspond to:

- a) completely depolarizing channel: \( \Phi_* : \rho \rightarrow \rho_* = \frac{1}{2} I_2 \),
- b) identity channel \( \Phi_1 = I \),
- c) coarse graining channel \( \Phi_{CG} \), which sets all off–diagonal elements of a density matrix to zero, and preserves the diagonal populations unaltered,

\[
\Phi_{CG} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad (31)
\]
- d) spontaneous emission channel sending any state into a certain pure state (e.g. the ground state of the system), \( \Phi_{SE} : \rho \rightarrow |0\rangle\langle 0| \).

Basing on the numerical analysis the following curves are recognized as the limits of the region

- the curve \( ab \) given by the depolarizing channels \( \Phi_a = \alpha I + (1 - \alpha) \Phi_* \), for \( 0 \leq \alpha \leq 1 \). This family of states provides the upper bound for the entire region of \( (S_{\text{map}}, S_{\text{rec}}) \) available for the one-qubit quantum maps. Because of the importance of this curve we provide its parametric expression

\[
S_{\text{map}} = -\frac{3}{4}(1 - \alpha) \ln \left[ \frac{1}{4}(1 - \alpha) \right] - (1 + 3 \alpha) \ln(1 + 3 \alpha),
\]

\[
S_{\text{rec}} = \ln(1 + 3 \alpha) - \frac{3 \alpha \ln \alpha}{(1 + 3 \alpha)}.
\]

- the curve \( bc \) which represents the combination of identity and the coarse graining,
- the interval \( ad \) which represents completely contracting channels: linear combinations of the completely depolarizing channel and the spontaneous emission,
- the interval \( cd \) which includes the maps of the form:

\[
\Phi_{cd} = \begin{pmatrix}
\alpha & 0 & 0 & 0 \\
0 & \beta & \sqrt{\beta(1 - \beta)} e^{i \phi_2} & \sqrt{\beta(1 - \beta)} e^{-i \phi_2} \\
0 & 0 & 1 - \alpha & 0 \\
0 & 0 & 0 & 1 - \beta
\end{pmatrix} \quad (32)
\]

with \( \alpha, \beta \in (0, 1) \) and two arbitrary phases \( \phi_1 \) and \( \phi_2 \).
The above maps belong to a broader family of one-qubit operations:

\[
\Phi_I = \begin{pmatrix}
\alpha & 0 & 0 \\
\gamma_1 & 0 & \gamma_2 \\
1-\alpha & 0 & 1-\beta
\end{pmatrix},
\]

with complex numbers \(\gamma_1\) and \(\gamma_2\), such that the first column reshaped into a matrix of order two forms a positive state \(\rho_1\), while the reshaped last column corresponds to a state \(\rho_2\).

These operations can be called *interval channels*, as they transform the entire Bloch ball into an interval given by the convex combination of the states \(\rho_1\) and \(\rho_2\). The dynamical matrix corresponding to an interval map can be transformed by permutations into a block diagonal form.

Fig. 2 representing all one-qubit channels distinguishes two regions. Bistochastic quantum operations correspond to the dark region. As the set of one–qubit bistochastic maps forms a tetrahedron (a convex set given in Eq. (30)), to justify this observation that the bistochastic maps form a tetrahedron (a convex set given in Fig. 2) it is sufficient to analyze the images of the edges of the antisymmetric part of the tetrahedron onto the plane \((S^\text{map}, S^\text{rec})\). The white striped region \(acd\) contains for instance interval maps, which will be shown in Proposition 3. Note that there exist several maps which correspond to a given point in Fig. 2.

A further insight into the interpretation of the receiver entropy is due to the fact that for any completely contractive channel (interval \(ad\) in the plot), which sends any initial state into a concrete, selected state, \(\Phi_{\xi}: \rho \rightarrow \xi\), the receiver entropy is equal to zero. This is implied by the fact that the dynamical matrix of such an operation reads \(D_{\Phi_{\xi}} = \xi \otimes \mathbb{1}\). After reshuffling of this matrix we obtain the superoperator matrix of rank one, since all non-zero column are the same, therefore it has only one nonzero singular value. Normalization of the vector of singular values sets this number to unity so that \(S^\text{rec}(\Phi_{\xi}) = 0\). This observation supports an interpretation of \(S^\text{rec}\) as the amount of information missing to the receiver of the output \(\rho'\) of a quantum channel, who knows the operation \(\Phi\), but does not know the input state \(\rho\).

### IV. ENTROPIC INEQUALITIES FOR REORDERED MATRICES

Before we establish several trade-off relations for quantum channels we shall introduce a framework concerning matrices (in general non–hermitian) together with their reordered counterparts. An arbitrary \(d \times d\) matrix \(X\) has \(d^2\) independent matrix elements. A matrix \(Y_x\) can be called a reordering of \(X\) if \(Y_x = X^\pi\), where \(\pi\) denotes some permutation of matrix entries. Thus, for each matrix \(X\) we can consider \((d^2)!\) reordered matrices \(Y_\pi\).

Denote by \(x_i\) the singular values of the matrix \(X\) and introduce the following \(q\)-norms:

\[
\|X\|_q = \left(\text{Tr} \left[XX^\dagger\right]^{q/2}\right)^{1/q} = \left(\sum_i x_i^q\right)^{1/q}.
\]

Moreover, by \(x_1 \equiv \|X\|_\infty\) denote the greatest singular value of the matrix \(X\) and by

\[
\Lambda_x = \|X\|_1 = \sum_i x_i,
\]

the trace norm of \(X\), i.e. the sum of all singular values \(x_i\). Finally, define the Rényi entropy

\[
S_q(X) = \frac{1}{1-q} \ln \sum_i \left(\frac{x_i}{\|X\|_1}\right)^q.
\]

The first result holds in general.

**Lemma 1.** For an arbitrary matrix \(X\) and \(1 \leq q < \infty\) we have

\[
\ln \left(\frac{\Lambda_x}{x_1}\right) \leq S_q(X) \leq \frac{q}{q-1} \ln \left(\frac{\Lambda_x}{x_1}\right),
\]

The second inequality relates matrices \(X\) and \(Y_\pi\).

**Lemma 2.** If \(Y_x = X^\pi\) where the transformation \(\pi\) is an arbitrary permutation of matrix entries, then we have for \(1 \leq q < \infty\)

\[
F_{\min} \ln \left(\frac{\Lambda_y}{\sqrt{x_1\Lambda_x}}\right) \leq S_q(Y_\pi) \leq F_{\max} \ln \left(\frac{\Lambda_y}{x_1}\right),
\]

where \(F_{\min} = \min \left(\frac{q}{q-1}; 2\right)\) and \(F_{\max} = \max \left(\frac{q}{q-1}; 2\right)\).

The symbol \(\Lambda_y\) inside Lemma 2 denotes the trace norm of \(Y_\pi\). Both lemmas are proven in Appendix A.

### V. TRADE–OFF RELATIONS FOR QUANTUM CHANNELS

The structure of the set of allowed values of both entropies \(S^\text{map}\) and \(S^\text{rec}\) describing all one–qubit stochastic maps shown in Fig. 2 suggests that their sum is bounded from below. For the smaller class of bistochastic maps the bound looks to be more tight. Indeed we are going to prove the following trade–off relation for the sum of two von Neumann entropies

\[
S^\text{map}(\Phi) + S^\text{rec}(\Phi) \geq \ln N,
\]

(39a)
and a sharper inequality
\[ S_{\text{map}}(\Phi) + S_{\text{rec}}(\Phi) \geq 2\ln N, \]  
(39b)
which holds for any bistochastic map acting on an \( N \) dimensional system. Note that the second expression can be interpreted as a kind of entropic trade–off relation for unital quantum channels: if the map entropy \( S_{\text{map}}(\Phi) \), which quantifies the interaction with the environment during the operation or the degree of disturbance of a quantum state, is small, the receiver entropy \( S_{\text{rec}}(\Phi) \) cannot be small as well. This implies that the results of the measurement could be very diverse. Conversely, a small value of the receiver entropy implies that the map \( \Phi \) is strongly contracting, so the map entropy is sufficiently large and a lot of information escapes from the system to an environment and the disturbance of the initial state is strong.

Instead of proving directly the bounds \((39a)\) and \((39b)\) for the von Neumann entropy \( S \equiv S_1 \) we are going to prove a more general inequalities formulated for the Rényi entropies \( S_q \) with \( q \in [1, \infty] \). All bounds in the limiting case \( q = 2 \), related to the Hilbert–Schmidt norm of a matrix, are shown in Fig. 3 for one-qubit quantum operations.

In the case of the \( N^2 \times N^2 \) matrices \( \Phi \) and \( D_\Phi \), let us denote by \( \sigma_1 \) the greatest singular value of \( \Phi \), by \( d_1 \) the greatest eigenvalue of \( D_\Phi \), and by \( \Lambda_\Phi = \| \Phi \|_1 \) the sum of all singular values of \( \Phi \). Since the Jamiołkowski–Choi state \( \omega_\Phi \) is normalized we have \( \| D_\Phi \|_1 = N \).

If we apply Lemma 10 to both matrices we obtain the following bounds:
\[
\ln \left( \frac{\Lambda_\Phi}{\sigma_1} \right) \leq S_q^{\text{rec}}(\Phi) \leq \frac{q}{q - 1} \ln \left( \frac{\Lambda_\Phi}{\sigma_1} \right),
\]
(40a)
\[
\ln \left( \frac{N}{d_1} \right) \leq S_q^{\text{map}}(\Phi) \leq \frac{q}{q - 1} \ln \left( \frac{N}{d_1} \right).
\]
(40b)

Since \( \Phi = D_\Phi^R \) and \( D_\Phi = \Phi^R \), where the reshuffling operation \( R \) is a particular example of reordering we have two additional bounds originating from Lemma 2
\[
F_{\text{min}} \ln \left( \frac{\Lambda_\Phi}{\sqrt{N d_1}} \right) \leq S_q^{\text{rec}}(\Phi) \leq F_{\text{max}} \ln \left( \frac{\Lambda_\Phi}{d_1} \right),
\]
(40c)
\[
F_{\text{min}} \ln \left( \frac{N}{\sqrt{\sigma_1 \Lambda_\Phi}} \right) \leq S_q^{\text{map}}(\Phi) \leq F_{\text{max}} \ln \left( \frac{N}{\sigma_1} \right).
\]
(40d)

The bounds \((40c)\) and \((40d)\) are in fact implied by the equality of Hilbert–Schmidt norms \( \| \Phi \|_2 = \| D_\Phi \|_2 \), what is a consequence of the reshuffling relation \( D_\Phi = \Phi^R \).

Inequalities \((40a)\) and \((40b)\) provide individual limitations for ranges of the entropies \( S^{\text{rec}}_q \) and \( S^{\text{map}}_q \). However, if we consider a particular inequality we can always recover a full range \([0, 2\ln N]\). The above inequalities can be combined in four different ways: \((40a)\) with \((40b)\), \((40c)\) with \((40d)\), \((40a)\) with \((40d)\) and \((40b)\) with \((40c)\) in order to obtain upper and lower bounds for the sum \( S_q^{\text{map}} + S_q^{\text{rec}} \). These bounds shall depend on the three parameters: \( \sigma_1 \), \( d_1 \) and \( \Lambda_\Phi \), thus, without an additional knowledge about these parameters, they do not lead to a trade–off relation. In particular, for \( d_1 = \sigma_1 = \Lambda_\Phi = N \) we find from \((40a)\) \((40b)\) that \( S_q^{\text{rec}} = 0 \) and \( S_q^{\text{map}} = 0 \). This case would correspond to a pure, separable Jamiołkowski–Choi state \( \omega_\Phi \).

In order to show that the above example cannot be realized by a CP TP map we shall prove the following theorem which provides an upper bound on the greatest singular value \( \sigma_1 \).

**Theorem 1.** Let \( \Phi \) be a CP TP channel acting on a set of density operators of size \( N \). Its superoperator \( \Phi \) is a \( N^2 \times N^2 \) matrix. The greatest singular value \( \sigma_1 \) is:

1. given by the expression
\[
\sigma_1(\Phi) = \max_{\rho \in \mathcal{M}_N} \sqrt{\frac{\text{Tr}(\Phi(\rho)^2)}{\text{Tr}\rho^2}},
\]
(41)

2. bounded
\[
\sigma_1(\Phi) \leq \sqrt{N \tau_1} \leq \sqrt{N},
\]
(42)

where \( \tau_1 \) denotes the greatest eigenvalue of the density matrix \( \Phi \left( \frac{1}{N} \mathbb{1} \right) \in \mathcal{M}_N \).

The bound \( \sigma_1 \leq \sqrt{N} \) is saturated for quantum channels which transform the maximally mixed state onto a pure state (only in that case \( \tau_1 = 1 \)). In the case of a bistochastic map all eigenvalues of \( \Phi \left( \frac{1}{N} \mathbb{1} \right) \) are equal to \( 1/N \) and therefore \( \sigma_1(\Phi) \leq 1 \). The proof of Theorem 1 is presented in Appendix 13. Some other bounds on singular values of reshuffled density matrices have been studied in \( [22] \). In particular, there was shown that for \( \rho \in \mathcal{M}_N \) the largest singular value of the matrix \( \rho^R \) is greater than \( N^{-1} \). Because \( \Phi = N \omega_\Phi^R \) we immediately find that \( \sigma_1 \geq 1 \). Thus, for bistochastic maps we have the equality \( \sigma_1 = 1 \).

We are now prepared to prove the following theorem which establishes the entropic trade–off relations between \( S_q^{\text{map}} \) and \( S_q^{\text{rec}} \).

**Theorem 2.** For a CP TP map \( \Phi \) acting on a system of an arbitrary dimension \( N \) the following relations hold:

1. For an arbitrary map \( \Phi \)
\[
S_q^{\text{map}}(\Phi) + S_q^{\text{rec}}(\Phi) \geq \frac{F_{\text{min}}}{2} \ln N,
\]
(43)
2. If the quantum channel $\Phi$ is bistochastic

$$S_q^{\text{map}}(\Phi) + S_q^{\text{rec}}(\Phi) \geq F_{\min} \ln N. \quad (44)$$

Since for $q = 1$ the coefficient $F_{\min} = 2$, from Theorem 2 we recover the particular bounds \cite{33a,39b} for the von Neumann entropies.

**Proof of Theorem 2.** In a first step we shall add two lower bounds present in \cite{10a} and \cite{10d} to obtain

$$S_q^{\text{map}}(\Phi) + S_q^{\text{rec}}(\Phi) \geq F_{\min} \ln \left( \frac{N}{\sigma_1} \right) + \left( 1 - \frac{F_{\min}}{2} \right) \ln \left( \frac{\Lambda_\Phi}{\sigma_1} \right). \quad (45)$$

Since $F_{\min} \leq 2$ and the greatest singular value $\sigma_1$ is less than the sum $\Lambda_\Phi$ of all singular values, the second term is always nonnegative. Thus, due to the upper bound \cite{42} we have

$$S_q^{\text{map}}(\Phi) + S_q^{\text{rec}}(\Phi) \geq \frac{F_{\min}}{2} \ln \left( \frac{N}{\sigma_1} \right). \quad (46)$$

The first statement of Theorem 2 follows immediately, when instead of $\tau_1$ we put its maximal value 1 into the inequality \cite{40a}. The second statement is related to the fact that bistochastic quantum channels preserve the identity i.e. $\Phi \left( \frac{1}{N} \mathbb{1} \right) = \frac{1}{N} \mathbb{1}$. The greatest eigenvalue $\tau_1$ is in this case equal to $\frac{1}{N}$, thus the value of $N^2$ appears inside the logarithm and cancels the factor of 2 in the denominator.

In fact, the inequality (46) quantifies the deviation from the set of bistochastic maps, with the greatest eigenvalue of $\Phi \left( \frac{1}{N} \mathbb{1} \right)$ playing the role of the interpolation parameter.

A. Additional upper bounds

The receiver entropy $S_q^{\text{rec}}(\Phi)$ is upper bounded due to the relations \cite{40d} and \cite{40e}. However, these bounds diverge in the limit $q \to 1$. Since the greatest singular value $\sigma_1$ is not less than 1 we can derive another upper bound which gives a nontrivial limitation valid for all values of $q$.

**Theorem 3.** For a CP TP map $\Phi$ acting on a system of an arbitrary dimension $N$ the following relation holds:

$$S_q^{\text{rec}}(\Phi) \leq \frac{1}{1-q} \ln \left( \Lambda_\Phi^{-q} + \frac{(\Lambda_\Phi - 1)^q}{\Lambda_\Phi^q (N^2 - 1)^{q-1}} \right). \quad (47)$$

**Proof.** Since the map $\Phi$ is CP TP the greatest singular value $\sigma_1 \geq 1$. Thus, the vector $\sigma$ of the singular values of the $N^2 \times N^2$ matrix $\Phi$ majorizes ($\sigma \succ \sigma_0$) the vector:

$$\sigma_0 = \left( 1, \frac{\Lambda_\Phi - 1}{N^2 - 1}, \frac{\Lambda_\Phi - 1}{N^2 - 1}, \ldots, \frac{\Lambda_\Phi - 1}{N^2 - 1} \right). \quad (48)$$

Since $S_q^{\text{rec}}(\Phi) = S_q(\sigma/\Lambda_\Phi)$ and the Rényi entropy is Schur concave we obtain the inequality $S_q^{\text{rec}}(\Phi) \leq S_q(\sigma_0/\Lambda_\Phi)$ which is equivalent to (47).

As a limiting case of Theorem 3 we have the corollary

**Corollary 2.** The von Neumann entropy $S_q^{\text{rec}}(\Phi)$ is bounded

$$S_q^{\text{rec}}(\Phi) \leq \frac{\Lambda_\Phi - 1}{\Lambda_\Phi} \ln \left( \frac{N^2 - 1}{\Lambda_\Phi - 1} \right) + \ln \Lambda_\Phi \leq 2 \ln N. \quad (49)$$

![Figure 3: (Color online) Stripped region represents the set of one-qubit operations projected into the plane spanned by the linear entropy of the map, $S_2^{\text{map}}(\Phi)$, and the linear receiver entropy $S_2^{\text{rec}}(\Phi)$, i.e. the Rényi entropies of order $q = 2$. Dark region represents the bistochastic quantum operations. Dashed antidiagonal line represents the lower bound (44) which holds for bistochastic operations, while solid antidiagonal line denotes the weaker bound (43) which holds for all quantum operations. Dotted antidiagonal line represents the upper bound (50) applied for $N = 2$.](image-url)

The relation between the matrices $\Phi$ and $D_\Phi$ allows us to derive an upper bound for the sum of the Rényi entropies $S_2^{\text{map}}(\Phi) + S_2^{\text{rec}}(\Phi)$.

**Proposition 3.** The following relation holds:

$$S_2^{\text{map}}(\Phi) + S_2^{\text{rec}}(\Phi) \leq 2 \ln \left( \frac{N(N + 1)}{2} \right). \quad (50)$$
Proof. Since \( \| \Phi \|_2 = \| D_{\Phi} \|_2 \) we have an easy relation between both entropies:
\[
S^\text{map}_2(\Phi) = S^\text{rec}_2(\Phi) + 2 \ln N - 2 \ln \Lambda_{\Phi}.
\] (51)
According to (47) we are able to estimate
\[
S^\text{map}_2(\Phi) + S^\text{rec}_2(\Phi) \leq 2 \ln(N \Lambda_{\Phi}) - 2 \ln \left(1 + \frac{(\Lambda_{\Phi} - 1)^2}{N^2 - 1}\right).
\] (52)
In order to complete the proof of Proposition 4 we shall perform the maximization of the above upper bound over the parameter \( \Lambda_{\Phi} \in [1, N^2] \).

The bound presented in Proposition 4 can be saturated by a quantum channel, which is a mixture of the identity channel and the maximally depolarizing channel, i.e.
\[
\Phi = \frac{1}{N + 1} \mathbb{I} + \frac{N}{N + 1} \Phi^*.
\] (53)
In fact, we are able to generalize the relation (51) to the case of all \( 1 \leq q \leq \infty \).

**Proposition 4.** The following relation holds:
\[
S^q_{\text{map}}(\Phi) \geq F_{\text{min}} \ln \frac{N}{\Lambda_{\Phi}} + G_{\text{min}} S^q_{\text{rec}}(\Phi),
\] (54)
where \( G_{\text{min}} = \min \left\{ \frac{q}{2(q-1)}, \frac{2(q-1)}{q} \right\} \).

Proof. Assume that \( q \leq 2 \). In that case we have the following monotonicity properties for the Rényi entropy:
\[
S_q \geq S_2 \geq 2 \left( \frac{q-1}{q} \right) S_q.
\] These relations together with Eq. (51) provide a chain of inequalities:
\[
S^q_{\text{map}}(\Phi) \geq S^2_{\text{map}}(\Phi) = 2 \ln \left( \frac{N}{\Lambda_{\Phi}} \right) + S^2_{\text{rec}}(\Phi)
\geq 2 \ln \left( \frac{N}{\Lambda_{\Phi}} \right) + 2 \left( \frac{q-1}{q} \right) S^q_{\text{rec}}(\Phi),
\] (55)
The same method applied for \( q \geq 2 \) with associated monotonicity relations \( 2 \left( \frac{q-1}{q} \right) S_q \geq S_2 \geq S_q \) completes the proof of inequality (54).

We can also show in which region of the plot \((S^\text{map}, S^\text{rec})\) the interval maps are located. Notice that the classical maps, which transform the set of \( N \)-point probability vectors into itself, also satisfy these inequalities.

**Proposition 5.** The interval maps satisfy the following inequalities \( S^\text{rec}(\Phi) \leq \ln N \leq S^\text{map}(\Phi) \).

Proof. The left inequality concerning the receiver entropy follows from the fact that the entire set of states is mapped into an interval. To show the right inequality observe that the dynamical matrix corresponding to an interval channel is block diagonal or can be transformed to this form by a permutation. Due to the trace preserving condition every block of the normalized dynamical matrix can be interpreted as \( \frac{1}{N} \rho_i \) where \( \rho_i \) is some density matrix. Therefore, up to a permutation \( P \) the normalized dynamical matrix has the structure
\[
\omega = \frac{1}{N} P^\dagger D_{\Phi} P = \sum_{i=1}^N \frac{1}{N} \rho_i \otimes |i\rangle \langle i|.
\] (56)
Hence the entropy of the normalized dynamical matrix reads
\[
S(\omega) = S \left( \frac{D_{\Phi}}{N} \right) = - \sum_i \text{Tr} \frac{1}{N} \rho_i \ln \frac{1}{N} \rho_i
= \ln N + \sum_i \frac{1}{N} S(\rho_i).
\] (57)
This implies the desired inequality for the entropy of a map, \( S^\text{map}(\Phi) \).

The last string of equations exemplifies the Shannon rule known as the grouping principle \([39, 40]\) that the information of expanded probability distribution should be the sum of a reduced distribution and weighted entropy of expansions. Notice that the grouping rule does not hold for all dynamical matrices corresponding to generic quantum operations. As an example take a maximally entangled state which is a purification of the maximally mixed state.

**B. Super–positive maps and separability of the Jamiołkowski–Choi state**

The aim of this part is to answer the question: How the separability (entanglement) of the state \( \omega_{\Phi} \) can be described in terms of the entropies \( S^\text{map}_q \) and \( S^\text{rec}_q \)? In other words we wish to identify the class of superpositive maps (also called entanglement breaking channels – see \([24]\)), for which \( \omega_{\Phi} \) is separable on the plane \((S^\text{map}_q, S^\text{rec}_q)\). Furthermore, we will determine the region on this plane where no such maps can be found. The method to answer these questions is based on the previously given uncertainty relations and the realignment separability criteria \([41, 42]\). These criteria state that if \( \omega_{\Phi} \) is separable then the sum of all singular values of the matrix \( \frac{1}{N} \Phi \) cannot be greater than 1, what straightforwardly implies \( \Lambda_{\Phi} \leq N \). We shall prove the following proposition

**Proposition 6.** If \( \omega_{\Phi} \) is separable, then:
In order to prove the statements 1–3 we apply the following bounds given in Proposition 5 concerning the separability of the dynamical matrix. Region A contains no superpositive channels, while region B contains both classes of the maps. Region C (determined by PPT criteria) contains only superpositive channels (all corresponding states are separable). Lower and upper bounds imply that there are no one–qubit quantum operations projected into region D. Diagonal of the figure contains the reshuffling–invariant channels, in particular, the coarse graining channel (c) and the transition depolarizing channel \( \Phi_{1/3} = \frac{1}{3} \mathbb{I} + \frac{2}{3} \Phi_s \), located at the boundary of the set of superpositive maps (e).

1. \( S_{\text{map}}^{\text{map}} (\Phi) \geq \frac{E_{\min}}{4} \ln N, \) and
2. \( S_{\text{map}}^{\text{rec}} (\Phi) \leq \frac{1}{1-q} \ln \left( \frac{(N+1)^3 + N^2 - 1}{N^q (N+1)^2} \right), \) and
3. \( S_{\text{map}}^{\text{map}} \geq G_{\min} S_{\text{map}}^{\text{rec}} (\Phi). \)

Proof. In order to prove the statements 1–3 we apply the separability criteria \( \Lambda_0 \leq N \) directly to the inequalities 40, 47 and 54 respectively. In the case 1 we also include the bound \( \sigma_1 \leq \sqrt{N} \).

The above result leads immediately to the separability criteria. If at least one inequality from Proposition 6 is violated, then the state \( \omega_\Phi \) is entangled, so the map \( \Phi \) is not superpositive – see Fig. 1.

The last inequality in Proposition 6 is saturated for the channels for which \( S_{\text{map}}^{\text{map}} = S_{\text{map}}^{\text{rec}} \). They are located at the diagonal of Fig. 1. This class contains maps with dynamical matrix symmetric with respect to the reshuffling, \( D = D^R = \Phi \). This condition implies that the superoperator \( \Phi \) is hermitian so its spectrum is real. The following proposition characterizes the set of one–qubit channels invariant with respect to reshuffling.

**Proposition 7.** The following one–qubit bistochastic channels \( \Phi_{R–inv} \) are reshuffling–invariant

\[
\Phi_{R–inv} = \Phi_U \Phi_{\eta_1, \eta_2} \Phi_{U^*},
\]

where

\[
\Phi_{\eta_1, \eta_2} = \frac{1}{2} \begin{pmatrix}
1 + \eta_3 & 0 & 0 & 1 - \eta_3 \\
0 & \eta_1 + \eta_2 & \eta_1 - \eta_2 & 0 \\
0 & \eta_1 - \eta_2 & \eta_1 + \eta_2 & 0 \\
1 - \eta_3 & 0 & 0 & 1 - \eta_3
\end{pmatrix}
\]

The map \( \Phi_U = U \otimes \bar{U} \) describes an arbitrary unitary channel, as \( U \) is a unitary matrix of order two and \( \bar{U} \) denotes its complex conjugation.

**Proof.** To justify the above statement we use the following general property of the reshuffling operation, which can be easily verified by checking the matrix entries of both sides

\[
\left[ (X_n^1 \otimes X_n^2) Y_{n^2} (X_n^3 \otimes X_n^4) \right]^R = \left[ (X_n^1 \otimes (X_n^2)^T) Y_{n^2}^R ((X_n^3)^T \otimes X_n^4) \right],
\]

where lower indices denote the dimensionalities of square matrices. Since 50 is a reshuffling–invariant matrix, using 60 we see that 55 is preserved after reshuffling.

Two extreme examples of the reshuffling–invariant maps are distinguished in Fig. 1: the coarse graining channel (c) for which \( \eta_1 = \eta_2 = 0 \) and \( \eta_3 = 1 \), and the transition depolarizing channel (e) at the boundary of super–positivity, \( \Phi_{1/3} = \frac{1}{3} \mathbb{I} + \frac{2}{3} \Phi_s \), for which \( \eta_1 = \eta_2 = \eta_3 = 1/3 \).

**VI. CONCLUDING REMARKS**

In this work an entropic trade–off relation analogue of the entropic uncertainty relation characterizing a given quantum operation 13 was established. We have shown that for any stochastic quantum map the sum of the map entropy, characterizing the decoherence introduced to the system by the measurement process, and the receiver entropy, which describes the knowledge on the output state without any information on the input, is bounded from below. The more one knows a priori concerning the outcome state, the more information was exchanged between the principal subsystem and the environment due to the quantum operation. A stronger bound 14 is obtained for the class of bistochastic maps, for which the maximally mixed state is preserved. Entanglement properties...
of a Jamiołkowski–Choi state were investigated in terms of the entropies $S_q^\text{map}$ and $S_q^\text{rec}$.

Dynamical entropic trade–off relations were obtained also for the Rényi entropies of an arbitrary order $q$. From a mathematical perspective this result is based on inequalities relating the spectrum of a positive hermitian matrix $X = X^\dagger$ and the singular values of the non–hermitian reshuffled matrix $X^R$. Related algebraic results were recently obtained in [37] and applied to the separability problem. It is tempting to believe that further algebraic investigations on the spectral properties of a reshuffled matrix will lead to other results applicable to physical problems motivated by the quantum theory.

Acknowledgments

The authors would like to thank P. Gawron for his help with the preparation of the figures. We are grateful to M. Zwołak and A.E. Rastegin for helpful correspondence and appreciate encouraging discussions with I. Białynicki–Birula, J. Korbicz and R. Horodecki. W.R. acknowledges financial support from the EU STREP Projects HIP, Grant Agreement No. 221889. Z.P. was supported by MNiSW under the project number IP2011 044271, for years 2012-2014. K.Z. acknowledges financial support by the Polish NCN research grant, decision number DEC-2011/02/A/ST2/00305.

Appendix A: Algebraic lemmas

In order to prove Lemma 3 we need the following norm inequality:

**Lemma 3.** For an arbitrary vector $x_i$ with non-negative coefficients $x_i$, and for $1 \leq q < \infty$ we have

$$\|x\|_q \leq \|x\|_1^{1/q} \|x\|_\infty^{(q-1)/q}. \quad (A1)$$

**Proof.** For $1/r + 1/s = 1$ we shall write $x_i^q = x_i^{q/r} x_i^{q/s}$ and next use the Hölder inequality for $1/\alpha + 1/\beta = 1$

$$\|x\|_q \leq \left( \sum_i x_i^{qa/r} \right)^{1/(qr)} \left( \sum_i x_i^{q\beta/s} \right)^{1/(qs)}. \quad (A2)$$

When we choose $\alpha = \infty$, $\beta = 1$, $s = q$ and $r = q/(q-1)$ we obtain the desired result $\|x\|_q \leq \|x\|_1^{1/q} \|x\|_\infty^{(q-1)/q}$. \hfill $\square$

**Proof of Lemma 4**. Lemma 3 together with the fact that the $q$-norms [38] are decreasing functions of the $q$ parameter provide a chain of norm inequalities

$$x_1 = \|X\|_\infty \leq \|X\|_q \leq \|X\|_1^{1/q} \|X\|_\infty^{(q-1)/q} = \Lambda_x^{1/q} x_1^{(q-1)/q}. \quad (A3)$$

We shall divide $\|x\|_q$ by $\|X\|_1 \equiv \Lambda_x$ to obtain

$$x_1 \Lambda_x \leq \|X\|_q \leq \left( \frac{x_1}{\Lambda_x} \right)^{(q-1)/q}. \quad (A4)$$

When we take the logarithm of the above inequality and then multiply by $q/(1-q)$, we boil down to the result [37]. \hfill $\square$

**Proof of Lemma 5**. Reordering operations do not change the matrix entries, thus also do not change the Hilbert–Schmidt norm $\|\|_\text{HS} \equiv \|\|_2$ which is a sum of squares of moduli of all matrix entries. This implies the equality $\|X\|_2 = \|Y\|_2$.

First we shall prove the right hand side of (A6). For $1 \leq q \leq 2$ we have

$$\|Y\|_q \geq \|X\|_2 \geq x_1, \quad (A5)$$

what by the same steps as before transforms into

$$S_q (Y_x) \leq \frac{q}{q-1} \ln \left( \frac{\Lambda_y}{x_1} \right). \quad (A6)$$

For $q \geq 2$ we extend the above inequality using the monotonicity property of the Rényi entropy $S_q \leq S_2$. Finally, we introduce the function $F_{\text{max}} = \max \left( \frac{q}{q-1}; 2 \right)$ to describe properly the transition from $1 \leq q \leq 2$ to $q \geq 2$.

In the case of the lower bound [38] we have for $q \geq 2$

$$\|Y\|_q \leq \|X\|_2 \leq \sqrt{\|X\|_1 \|X\|_\infty} = \sqrt{x_1 \Lambda_x}, \quad (A7)$$

what gives

$$S_q (Y_x) \geq \frac{q}{q-1} \ln \left( \frac{\Lambda_y}{x_1 \Lambda_x} \right). \quad (A8)$$

For $1 \leq q \leq 2$ we have $S_q \geq S_2$ what extends the above result providing the function $F_{\text{min}} = \min \left( \frac{q}{q-1}; 2 \right)$. \hfill $\square$

Appendix B: The greatest singular value of $\Phi$

Before the proof of Theorem 1 we state the lemma.

**Lemma 4.** For any matrix $M$ with $\|M\|_{\text{HS}} = \text{Tr}(M^\dagger M) = 1$ there exist a positive semi–definite matrix $P_M$ with $\|P_M\|_{\text{HS}} = 1$ such that

$$\|\Phi(M)\|_{\text{HS}} \leq \|\Phi(P_M)\|_{\text{HS}}. \quad (B1)$$

**Proof.** First we will show, that one can choose hermitian matrix $H_M$, such that

$$\|\Phi(M)\|_{\text{HS}} \leq \|\Phi(H_M)\|_{\text{HS}}. \quad (B2)$$
If we consider a decomposition of $M = H + iL$, where $H, L$ are hermitian matrices, we obtain that
\[ 1 = \|M\|_{HS}^2 = \|H\|_{HS}^2 + \|L\|_{HS}^2. \] (B3)

Let $(0 \leq p \leq 1)$
\[ \|H\|_{HS}^2 = p, \quad \|L\|_{HS}^2 = 1 - p, \] (B4)
and define normalized hermitian matrices
\[ H_0 = H/\sqrt{p}, \quad L_0 = L/\sqrt{1-p}. \] (B5)

Now we write
\[
\|\Phi(M)\|_{HS}^2 = \|\Phi(H) + i\Phi(L)\|_{HS}^2 \\
= \|\Phi(H)\|_{HS}^2 + \|\Phi(L)\|_{HS}^2 \\
= p\|\Phi(H_0)\|_{HS}^2 + (1-p)\|\Phi(L_0)\|_{HS}^2.
\] (B6)

Since $\|\Phi(M)\|_{HS}^2$ is a convex combination of $\|\Phi(H_0)\|_{HS}^2$ and $\|\Phi(L_0)\|_{HS}^2$, therefore $\|\Phi(H_0)\|_{HS}^2 \geq \|\Phi(M)\|_{HS}^2$ or $\|\Phi(L_0)\|_{HS}^2 \geq \|\Phi(M)\|_{HS}^2$ and this shows, that for any matrix $M$, with $\|M\|_{HS} = 1$ there exist a hermitian matrix $H_M$, which satisfies (B2).

Now it is easy to notice, that by taking the absolute value of a hermitian matrix $H_M$ we do not decrease the norm of the channel output, i.e. let
\[ H_M = \sum \lambda_i |\phi_i\rangle \langle \phi_i|, \quad |H_M| = \sum |\lambda_i| |\phi_i\rangle \langle \phi_i|. \] (B7)

We have
\[
\|\Phi(H_M)\|_{HS}^2 = \sum \lambda_i \lambda_j \text{Tr}\Phi(|\phi_i\rangle \langle \phi_i|)\Phi(|\phi_j\rangle \langle \phi_j|) \\
\leq \sum |\lambda_i| \lambda_j |\text{Tr}\Phi(|\phi_i\rangle \langle \phi_i|)\Phi(|\phi_j\rangle \langle \phi_j|)| \\
= \|\Phi(|H_M|)\|_{HS}^2.
\] (B8)

Now we are in position to prove Theorem 1.

**Proof of Theorem 1.** The definition of the greatest singular value of the super–operator reads
\[ \sigma_1(\Phi) = \max_{\|M\|_{HS} = 1} \|\Phi(M)\|_{HS}. \] (B9)

According to Lemma 1 we can restrict the maximization to positive semi–definite matrices $H$ such that $\|H\|_{HS} = 1$. Such matrix can be written as $H = \frac{\rho}{\sqrt{\text{Tr}\rho^2}}$ for $\rho \in \mathcal{M}_N$.
\[
\sigma_1(\Phi) = \max_{\|M\|_{HS} = 1} \|\Phi(M)\|_{HS} \\
= \max_{\rho \in \mathcal{M}_N} \|\text{Tr}\Phi\left(\frac{\rho}{\sqrt{\text{Tr}\rho^2}}\right)\|_{HS} \\
= \max_{\rho \in \mathcal{M}_N} \sqrt{\frac{\text{Tr}\Phi(\rho)^2}{\text{Tr}\rho^2}}.
\] (B10)

This proves the first part of Theorem 1. In order to derive the second part we shall use the Kraus representation:
\[ \Phi : \rho \mapsto \Phi(\rho) = \sum_i A_i \rho A_i^\dagger, \quad \sum_i A_i^\dagger A_i = 1, \] (B11)
and write
\[ \text{Tr}\Phi(\rho)^2 = \sum_{i,j} \text{Tr} \left(A_i^\dagger A_i \rho^2 A_j^\dagger A_j\right). \] (B12)

where we also took an advantage from the trace invariance under cyclic permutations. Applying the matrix version of the Cauchy–Schwarz inequality $\text{Tr}XY \leq \sqrt{\text{Tr}X^2 \sqrt{\text{Tr}Y^2}}$ we obtain the bound
\[ \text{Tr}\Phi(\rho)^2 \leq \sum_{i,j} \sqrt{\text{Tr} \left(A_i^\dagger A_i \rho^2 A_j^\dagger A_j\right)} \sqrt{\text{Tr} \left(A_i^\dagger A_i \rho^2 A_j^\dagger A_j\right)}. \] (B13)

Obviously both families of matrices $A_i^\dagger A_i \rho^2 A_j^\dagger A_j$ and $A_i^\dagger A_i \rho^2 A_j^\dagger A_j$ are positive semi–definite. We shall apply to (B13) the usual Cauchy–Schwarz inequality to find that
\[ \text{Tr}\Phi(\rho)^2 \leq \text{Tr} \left(\sum_{i,j} A_i^\dagger A_i \rho^2 A_j^\dagger A_j\right). \] (B14)

Next, we shall rearrange the right hand side of (B14) to the form
\[ \text{Tr} \left(\sum_{i,j} A_i^\dagger A_i \rho^2 A_j^\dagger A_j\right) = N\text{Tr}\rho^2 \text{Tr} \left(\Phi\left(\frac{1}{N}\mathbbm{1}\right) \Phi(\tilde{\rho})\right), \] (B15)
where $\mathcal{M}_N \ni \tilde{\rho} = \frac{\rho^2}{\text{Tr}\rho^2}$. Finally, we use that expression to bound $\sigma_1$ given by the formula (B11):
\[ \sigma_1(\Phi) \leq \sqrt{N} \max_{\tilde{\rho} \in \mathcal{M}_N} \sqrt{\text{Tr} \left(\Phi\left(\frac{1}{N}\mathbbm{1}\right) \Phi(\tilde{\rho})\right)} \] (B16)
The term $\text{Tr} \left(\Phi\left(\frac{1}{N}\mathbbm{1}\right) \Phi(\tilde{\rho})\right)$ is bounded by the greatest eigenvalue of the matrix $\Phi\left(\frac{1}{N}\mathbbm{1}\right)$, which implies the desired result.

**Appendix C: Estimating channel entropy**

The following inequality allows us to estimate the entropy of a channel. A similar estimation was recently formulated in [43].

**Proposition 8.** For any quantum channel $\Phi$ acting on $\mathcal{M}_N$ the following inequality holds
\[ \ln N - S\left(\Phi\left(\frac{1}{N}\mathbbm{1}\right)\right) \leq S^{\text{map}}(\Phi) \leq \ln N + S\left(\Phi\left(\frac{1}{N}\mathbbm{1}\right)\right). \] (C1)
where \( S \) denotes the von Neumann entropy. Moreover, the right inequality is satisfied for the Rényi entropy \( S_q(\rho) = \frac{1}{1-q} \ln \text{Tr}\rho^q \) of an arbitrary order \( q \).

**Proof.** For any quantum operation \( \Phi \) the corresponding dynamical matrix \( D_\Phi \) obeys the following relations [24].

\[
\text{Tr}_1 \left( \frac{1}{N} D_\Phi \right) = \frac{1}{N} \mathbb{1}, \quad \text{(C2)}
\]

\[
\text{Tr}_2 \left( \frac{1}{N} D_\Phi \right) = \Phi \left( \frac{1}{N} \mathbb{1} \right), \quad \text{(C3)}
\]

Thus in the case of the von Neumann entropies the upper bound follows from subadditivity, while the lower bound is a consequence of Araki–Lieb triangle inequality [44].

In the case of the Rényi entropies the upper bound follows directly from the weak subadditivity [43]. Although the lower bound (C1) for the von Neumann entropy of a map can not be directly extended for Rényi entropies, we provide another generalized bound, which holds for any \( q \geq 0 \),

\[
\ln N - \ln \text{rank} \left( \Phi \left( \frac{1}{N} \mathbb{1} \right) \right) \leq S_q^{\text{map}}(\Phi). \quad \text{(C4)}
\]

This lower bound for the generalized entropy of a map \( S_q^{\text{map}} \) follows also from the weak subadditivity [43].

Notice that in the special case of complete contraction \( \Phi_\xi : \rho \to \xi \) when the dynamical matrix has a form \( D_{\Phi_\xi} = \xi \otimes \mathbb{1} \) the right hand side of inequality (C1) is saturated. The bounds established by Propositions [8] are illustrated in Figure 5.
probability and complex fractals: classical and quantum approach, (Springer, 2000).

[26] F. Verstraete, H. Verschelde, preprint arXiv:0202124.
[27] W. Roga, M. Fannes, K. Życzkowski, J. Phys. A 41, 035305 (2008).
[28] M. Ziman, Phys. Rev. A 78, 032118 (2008).
[29] W. Roga, M. Fannes, K. Życzkowski, Int. J. Quantum Information 9, 1031 (2011).
[30] A.E. Rastegin, preprint arXiv:1206.3056.
[31] B. Schumacher, Phys. Rev. A 54, 2614 (1996).
[32] A. Fujiwara and P. Algoet, Phys. Rev. A 59, 3290 (1999).
[33] M. B. Ruskai, S. Szarek, E. Werner, Linear Algebr. Appl. 347, 159 (2002).
[34] A. Pittenger, M. Rubin, Linear Algebr. Appl. 390, 255 (2004).
[35] I. Białynicki-Birula, Phys. Rev. A 74, 052101 (2006).
[36] C. Beck and F. Schlogl, Thermodynamics of chaotic systems, (Cambridge University Press, Cambridge, 1993).

[37] C.-K. Li, Y.-T. Poon and N.-S. Sze, J. Phys. A 44, 315304 (2011).
[38] W. Bruzda, V. Cappellini, H.-J. Sommers, K. Życzkowski, Phys. Lett. A 373, 320 (2009).
[39] C. Shannon, The Bell System Technical Journal 27, 379 and 623 (1948).
[40] M. J. W. Hall, Phys. Rev. A 59, 2602 (1999).
[41] K. Chen and L. A. Wu, Quantum Inf. Comput. 3, 193 (2003).
[42] M. Horodecki, P. Horodecki and R. Horodecki, Open Syst. Inf. Dyn. 13, 103 (2006).
[43] L. Zhang, preprint arXiv:1110.6321.
[44] H. Araki and E. H. Lieb, Commun. Math. Phys. 18, 160 (1970).
[45] W. Van Dam and P. Hayden, preprint arXiv:0204093.
[46] R. A. Horn and C. R. Johnson, Matrix Analysis, (Cambridge University Press, Cambridge, 1985).