IDONEAL GENERA AND K3 SURFACES
COVERING AN ENRIQUES SURFACE

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ABSTRACT. Idoneal genera are a generalization of Euler’s idoneal numbers. We enumerate all idoneal genera by means of the Smith–Minkowski–Siegel mass formula. As an application, we classify transcendental lattices of K3 surfaces covering an Enriques surface.

1. INTRODUCTION

An idoneal (or suitable or convenient) number is a positive integer \( n \in \mathbb{N} \) with the following property: every odd number \( m \) prime to \( n \) is a prime number whenever \( m \) can be written in the form \( x^2 + ny^2 \) with \( x, y \) relatively prime, and the equation \( m = x^2 + ny^2 \) has only one solution with \( x, y \geq 0 \). This terminology goes back to Euler \([4]\).

Equivalently (see e.g. \([8]\), Theorem 6), \( n \in \mathbb{N} \) is an idoneal number if and only if the lattice \([1] \oplus [n]\) is unique in its genus (our conventions on integral lattices are explained in §1.2). An idoneal genus is a positive definite genus \( g \) with the property that each lattice \( L \in g \) contains a vector of square 1 or, equivalently, \( L \cong [1] \oplus L' \) for some lattice \( L' \).

Remark 1.1. If \( L \) belongs to an idoneal genus \( g \) of rank 2 and determinant \( n \), then necessarily \( L \cong [1] \oplus [n] \); in particular, \( L \) is unique in \( g \). Thus, a genus \( g \) of rank 2 is idoneal if and only if \( g = \{ [1] \oplus [n] \} \), with \( n \) an idoneal number.

The problem of enumerating all idoneal genera was posed by Kani \([8]\), Problem 45). In this paper, we completely solve this problem under the assumption of the generalized Riemann hypothesis (see Remark 1.5).

Theorem 1.2 (see §2.7). For each \( r \in \mathbb{N} \), there exist \( I_r \) idoneal genera of rank \( r \), with \( I_r \) given in Table 1 for \( r \leq 13 \) and \( I_r = 0 \) otherwise. There exist no other idoneal genera of rank \( r \neq 2 \) and there exist at most two more of rank 2 if the generalized Riemann hypothesis does not hold. The list of all 577 known idoneal genera can be found in the ancillary file idoneal.genera.txt on arXiv.

The main idea is to compare the mass of an idoneal genus with the masses of its twigs (Definition 2.2), leading to a certain mass condition (Proposition 2.6). Through the Smith–Minkowski–Siegel mass formula, we translate this condition into an explicit
bound on the rank and the determinant of idoneal genera. By means of computer algebra (Remark 2.20), we examine all genera satisfying the given bounds.

In this paper, we also present an application of the notion of idoneal genus to the problem of characterizing complex K3 surfaces covering an Enriques surface.

Recall that a smooth proper algebraic surface $X$ defined over $\mathbb{C}$ such that $H^1(X, \mathcal{O}) = 0$ is called a K3 surface if its canonical bundle $\mathcal{K}$ is trivial and it is called an Enriques surface if $\mathcal{K}$ is not trivial, but $\mathcal{K}^{\otimes 2}$ is. The cohomology group $H^2(X, \mathbb{Z})$ of a K3 surface $X$, together with the Poincaré pairing, is a unimodular lattice of rank 22. It contains the Néron–Severi lattice, defined as the image $S$ of the map $H^1(X, \mathcal{O}^*) \rightarrow H^2(X, \mathbb{Z})$ coming from the exponential sheaf sequence. The transcendental lattice of $X$ is the orthogonal complement of $S$ in $H^2(X, \mathbb{Z})$, and it has signature $(2, \lambda - 2)$. We set

$$\Lambda^- := U \oplus U(2) \oplus E_8(2).$$

Our starting point is the following criterion proved by Keum (under an additional assumption which is actually superfluous, see [17]).

**Theorem 1.3** (Keum’s criterion [9]). A K3 surface $X$ with transcendental lattice $T$ covers an Enriques surface if and only if there exists a primitive embedding $T \hookrightarrow \Lambda^-$ such that there exists no vector $v \in T^\perp$ with $v^2 = -2$.

Our aim is to restate Keum’s criterion in a way that makes the condition on $T$ more explicit and easy to be checked, completing the work started by Sertöz [21] and Ohashi [17], and partially continued by Lee [11] and Yörük [28].

By Keum’s criterion, there are two reasons why a K3 surface $X$ may not cover any Enriques surface: either $T$ does not embed primitively into $\Lambda^-$, or $T$ does embed primitively into $\Lambda^-$, but for each primitive embedding $T \hookrightarrow \Lambda^-$ there exists $v \in T^\perp$ with $v^2 = -2$. In the latter case, we call $T$ a co-idoneal lattice.

The technical core result of this paper is the enumeration of all co-idoneal lattices. Note that a co-idoneal lattice is always of the form $T = T'(2)$ for some odd lattice $T'$, called the half of $T$ (Corollary 3.13).

**Theorem 1.4** (see §3.5). For each $\lambda \in \mathbb{N}$ there exist $E_\lambda$ co-idoneal lattices of signature $(2, \lambda - 2)$, with $E_\lambda$ given in Table 1 if $2 \leq \lambda \leq 11$ and $E_\lambda = 0$ otherwise. There exist no other co-idoneal lattices of rank $\lambda \neq 10$, and there exist at most two more of rank 10 if the generalized Riemann hypothesis does not hold. The list of all halves of the known 550 co-idoneal lattices is contained in Table 5 and in the ancillary file half.co-idoneal.lat.txt on arXiv.

The key observation is the connection between idoneal genera and co-idoneal lattices (Proposition 3.14): the orthogonal complement in $\Lambda^-$ of a co-idoneal lattice is always of the form $L(-2)$, where $L$ is a lattice belonging to a uniquely determined idoneal genus.

**Remark 1.5.** There are exactly 65 idoneal numbers known, the highest one being 1848 (sequence A000926 in the OEIS [23]). Weinberger [27] proved that the list is complete if the generalized Riemann hypothesis holds. If it does not hold, then there could exist two more idoneal numbers (see [8, Corollary 23]), hence two more idoneal genera of rank 2 and two more co-idoneal lattices of rank 10. All other statements in this paper are valid also without the assumption of the generalized Riemann hypothesis.
By means of Nikulin’s theory of discriminant forms, we obtain the following theorem, where \( \tilde{E} \) denotes the lattice given in shorthand form by

\[
[2, 0, 2, 0, 0, 2, 0, 0, 0, 1, 1, 1, 1, 4, 0, 0, 0, 0, 0, 1, 2, 1, 1, 1, 1, 1, -1, 4, 1, 1, 1, 1, -1, 2, 4][-1).
\]

(By Theorem 3.6, \( \tilde{E} \) can be equivalently defined as a negative definite lattice of rank 8 and discriminant form \( 3u_{1} \), where \( u_{1} \) is the discriminant form of \( U(2) \).

**Theorem 1.6** (see §3.4). If \( X \) is a K3 surface with transcendental lattice \( T \) of rank \( \lambda \), then \( X \) covers an Enriques surface if and only if \( T \) is not a co-idoneal lattice and one of the following conditions holds:

(i) \( 2 \leq \lambda \leq 6 \) and \( T \) admits a Gram matrix of the form

\[
\begin{pmatrix}
2a_{11} & a_{12} & \ldots & a_{1\lambda} \\
 a_{12} & 2a_{22} & \ldots & \vdots \\
 \vdots & \vdots & \ddots & \vdots \\
a_{1\lambda} & \ldots & \ldots & 2a_{\lambda\lambda}
\end{pmatrix}
\]

such that \( a_{ij} \) is even for each \( 2 \leq i, j \leq \lambda \),

(ii) \( \lambda = 7 \) and there exists an even lattice \( T' \) with \( T \cong U \oplus T'(2) \),

(iii) \( \lambda = 7 \) and there exists a lattice \( T' \) with \( T \cong U \oplus T'(2) \),

(iv) \( \lambda = 8 \) and there exists an even lattice \( T' \) with \( T \cong U \oplus U(2) \oplus T'(2) \),

(v) \( \lambda = 8 \) and there exists a lattice \( T' \) with \( T \cong U \oplus T'(2) \),

(vi) \( \lambda = 9 \) and there exists an even lattice \( T' \) with \( U(2) \oplus T \cong U \oplus T'(2) \),

(vii) \( \lambda = 9 \) and there exists a lattice \( T' \) with \( U \oplus T \cong U \oplus T'(2) \),

(viii) \( \lambda = 10 \) and there exists an even lattice \( T' \) with \( T \cong U \oplus T'(2) \),

(ix) \( \lambda = 10 \) and there exists a lattice \( T' \) with \( T \cong E_8(2) \oplus T'(2) \),

(x) \( \lambda = 11 \) and there exists \( n > 0 \) with \( T \cong U \oplus E_8(2) \oplus [4n] \),

(xi) \( \lambda = 11 \) and there exists \( n > 0 \) with \( T \cong U(2) \oplus E_8(2) \oplus [2n] \),

(xii) \( \lambda = 12 \) and \( T \cong \Lambda^{-} \).

A characterization analogous to Theorem 1.6 was obtained by Morrison [14] for Kummer surfaces. Morrison restated Nikulin’s criterion [15] that a K3 surface \( X \) with transcendental lattice \( T \) is a Kummer surface if and only if there exists a primitive embedding \( T \hookrightarrow 3U(2) \) (see [14, Corollary 4.4]).

According to Ohashi [18], the number of isomorphism classes of Enriques surfaces covered by a fixed K3 surface is finite. Theorem 1.6 classifies K3 surfaces for which this number is nonzero. For the related problem of computing this number explicitly, see...
[17, 18, 22]. For an application of our results to the case of one-dimensional families of K3 surfaces, see [5].

1.1. Contents of the paper. The paper is divided into two sections. In §2 the relevant facts about the Smith–Minkowski–Siegel mass formula are summarized. The section is then devoted to the classification of idoneal genera and contains the proof of Theorem 1.2. In §3, after recalling some results on finite discriminant forms, we first determine which transcendental lattices embed into $\Lambda^-$, proving Theorem 1.6. We conclude the paper with the list all co-idoneal lattices.

1.2. Conventions on lattices. In this paper, an (integral) lattice of rank $r$ is a finitely generated free $\mathbb{Z}$-module $L \cong \mathbb{Z}^r$ endowed with a nondegenerate symmetric bilinear pairing $b: L \times L \to \mathbb{Z}$. A morphism $L \to L'$ of $\mathbb{Z}$-modules is an isomorphism of lattices if it is an isomorphism of $\mathbb{Z}$-modules respecting the bilinear pairings. If it exists, we write $L \cong L'$. We denote the group of automorphisms of $L$ by $\text{Aut}(L)$.

If $e_1, \ldots, e_r \in L$ is a system of generators, the associated Gram matrix is the square matrix with entries $b_{ij} = b(e_i, e_j)$. A lattice $L$ is denoted by any of its Gram matrices, using the following shorthand notation:

$$[b_{11}, b_{12}, b_{22}, \ldots, b_{rr}] := \begin{pmatrix} b_{11} & b_{12} & \ldots & b_{1r} \\ b_{12} & b_{22} & & b_{2r} \\ \vdots & \ddots & \ddots & \vdots \\ b_{1r} & \cdots & b_{rr} \end{pmatrix}.$$ 

The determinant $\det L$ is the determinant of any such matrix. A lattice $L$ is called even if $e^2 = b(e, e) \in 2\mathbb{Z}$ for each $e \in L$, otherwise it is called odd.

An embedding $L \hookrightarrow L'$ is primitive if $L'/L$ is free, and $L'$ is an overlattice of $L$ of index $m$ if rank $L' = \text{rank } L$ and $m = |L'/L|$. We write $L(n)$ for the lattice with the pairing defined by the composition $L \times L \to \mathbb{Z} \to \mathbb{Z}$ and we put $nL := L \oplus \ldots \oplus L$ ($n$ times).

The abelian group $A = L^\vee/L$, where

$$L^\vee := \{ e \in L \otimes \mathbb{Q} \mid b(e, f) \in \mathbb{Z} \text{ for all } f \in L \},$$

has order $\det L$. If $L$ is even, the finite quadratic form $q(L): A \to \mathbb{Q}/2\mathbb{Z}$ induced by the linear extension of $b$ to $\mathbb{Q}$ is called the discriminant (quadratic) form of $L$. Our conventions on finite forms are explained in §3.1.

The negative definite ADE lattices are denoted $A_n, D_n, E_n$ and the hyperbolic plane is denoted $U$.

In this paper, a genus is a complete set of isomorphism classes of lattices which are equivalent over $\mathbb{R}$ and over $\mathbb{Z}_p$ for each prime $p$ to a given lattice. Each genus is a finite set (see [10, Kapitel VII, Satz (21.3)]). The parity, rank, signature or determinant of a genus $\mathfrak{g}$ are by definition the parity, rank, signature or determinant of any lattice $L \in \mathfrak{g}$.

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2. Classification of idoneal genera

Trivially, the only idoneal genus of rank 1 is $g = \{[1]\}$. We already observed that there exists a bijective correspondence between idoneal genera of rank 2 and Euler’s idoneal numbers (Remark 1.1). This section is dedicated to the classification of idoneal genera of rank $\geq 3$.

We first fix the notation concerning the Smith–Minkowski Siegel mass formula (§2.2–§2.4). The main idea is explained in §2.5: on account of Proposition 2.6, we can classify idoneal genera by searching for slender genera, which are genera satisfying a certain condition on the mass (Definition 2.5). We are led to compare the mass of $g$ with the mass of a related genus $\tilde{g}$, which is done in §2.6. Finally, §2.7 contains the proof of Theorem 1.2.

We assume that the reader is familiar with Conway–Sloane’s paper [2], to which we refer for further details.

2.1. Zeta functions. The gamma function is denoted by $\Gamma$, and the Riemann zeta function by $\zeta$. For $D \in \mathbb{Z}$, we introduce the following Dirichlet character modulo $4D$:

$$
\chi_D(m) := \begin{cases} 
0 & \text{if } (m, 2D) \neq 1, \\
(D \mod m) & \text{if } (m, 2D) = 1,
\end{cases}
$$

where $(D \mod m)$ denotes the Jacobi symbol. In [2, §7], the zeta function $\zeta_D(s)$ is defined as the Dirichlet $L$-series with respect to the character $\chi_D$.

$$
\zeta_D(s) := \sum_{m=1,3,5,...} \left( \frac{D}{m} \right) m^{-s} = \prod_p \frac{1}{1 - \chi_D(p)p^{-s}}.
$$

For a genus $g$ of rank $n = 2s$ or $n = 2s - 1$ and determinant $d$, we put

$$
D := (-1)^s d, \quad \varepsilon_p(g) := \chi_D(p).
$$

2.2. Jordan decomposition. Each lattice $L$ admits a Jordan decomposition over the $p$-adic integers

$$
L = \ldots \oplus \frac{1}{p} J_p^{-1}(L) \oplus 1J_p^0(L) \oplus pJ_p^1(L) \oplus p^2 J_p^2(L) \oplus \ldots
$$

Since the pairing of $L$ takes values in $\mathbb{Z}$, each Jordan constituent $J_p^i(L)$ with $i < 0$ is a lattice of dimension 0. Nonetheless, Conway–Sloane’s formalism takes the Jordan constituent $J_p^{-1}(L)$ into account to compute the mass of $L$.

We write $J_p^i(g)$ for the $i$th $p$-adic Jordan constituent of any $L \in g$. Denoting by $\nu_p$ the $p$-adic valuation, we have

$$
\sum_i i \dim J_p^i(g) = \nu_p(\det g).
$$

2.3. Mass and $p$-mass. The mass of a genus $g$ is defined as

$$
m(g) := \sum_{L \in g} \frac{1}{|\text{Aut}(L)|}.
$$
The $p$-mass of $\mathfrak{g}$ is defined by the formula
\begin{equation}
 m_p(\mathfrak{g}) := \Delta_p(\mathfrak{g}) \cdot \chi_p(\mathfrak{g}) \cdot \text{type}_p(\mathfrak{g}),
\end{equation}
where $\text{type}_p(\mathfrak{g}) := 1$ if $p \neq 2$ and the other factors are defined as follows:
\begin{align*}
\Delta_p(\mathfrak{g}) &:= \prod_{i} M^i_p(\mathfrak{g}) \quad \text{(diagonal product)}, \\
\chi_p(\mathfrak{g}) &:= \prod_{i<j} p^{\frac{1}{2}(j-i)} \dim J^i_p(\mathfrak{g}) \dim J^j_p(\mathfrak{g}) \quad \text{(cross product)}, \\
\text{type}_2(\mathfrak{g}) &:= 2^{n_{I,2}(\mathfrak{g}) - n_{I}(\mathfrak{g})} \quad \text{(type factor)}.
\end{align*}
Here, $n_{II}(\mathfrak{g})$ is the sum of the dimensions of all Jordan constituents that have type II, $n_{I,II}$ is the total number of pairs of adjacent constituents $J^i_2(\mathfrak{g}), J^{i+1}_2(\mathfrak{g})$ that are both of type I, and $M^i_p(\mathfrak{g})$ is the diagonal factor associated to $J^i_p(\mathfrak{g})$. How to compute the diagonal factor is explained in [2, §5].

Note that it is customary to write $m(L)$, $m_p(L)$ and so on, but we preferred to stress the dependence on the genus and not on the chosen representative.

**Remark 2.1** (cf. [2, §7]). If $p \nmid 2d$, then the Jordan decomposition of any $L \in \mathfrak{g}$ is concentrated in degree 0. Hence, $m_p(\mathfrak{g}) = \Delta_p(\mathfrak{g}) = M^0_p(\mathfrak{g})$ takes on the so-called standard value
\begin{equation}
 \text{std}_p(\mathfrak{g}) := \frac{1}{2(1 - p^{-2})(1 - p^{-4}) \cdots (1 - p^{2n-2})}. \tag{3}
\end{equation}

2.4. Smith–Minkowski–Siegel mass formula. The following formula, known as the Smith–Minkowski–Siegel mass formula, relates the mass of $\mathfrak{g}$ to its $p$-masses $m_p(\mathfrak{g})$.

\begin{equation}
 m(\mathfrak{g}) = 2\pi^{-\frac{1}{2}n(n+1)} \cdot \prod_{j=1}^{n} \Gamma \left(\frac{1}{4}j\right) \cdot \prod_{p} (2m_p(\mathfrak{g})).
\end{equation}

2.5. Twigs and slender genera. If $L'$, $L''$ are two lattices belonging to the same genus $\mathfrak{g}$, then also $[1] \oplus L'$ and $[1] \oplus L''$ belong to the same genus, denoted by $\tilde{\mathfrak{g}}$.

**Definition 2.2.** We say that $\mathfrak{g}$ is a twig of a genus $\mathfrak{f}$ if $\mathfrak{f} = \tilde{\mathfrak{g}}$.

**Lemma 2.3** (see [2, Lemma 3]). A genus $\mathfrak{f}$ can have at most two twigs, and if it has two, then one twig is odd and the other one is even. \hfill $\square$

**Lemma 2.4.** If $\mathfrak{T}$ is the set of twigs of an idoneal genus $\mathfrak{f}$, then
\begin{equation}
 2 \sum_{\mathfrak{g} \in \mathfrak{T}} m(\mathfrak{g}) \leq \sum_{\mathfrak{g} \in \mathfrak{T}} m(\mathfrak{g}). \tag{4}
\end{equation}

**Proof.** Any lattice in $\mathfrak{f}$ is of the form $[1] \oplus L$ with $L \in \mathfrak{g}, \mathfrak{g} \in \mathfrak{T}$, If $L$ and $L'$ are not isomorphic, then $[1] \oplus L$ and $[1] \oplus L'$ are also not isomorphic, by the uniqueness of the decomposition of positive definite lattices into irreducible lattices (see [10, Satz 27.2]). Therefore, since $|\text{Aut}([1] \oplus L)| \geq 2|\text{Aut}(L)|$, we have
\begin{equation}
 m(\mathfrak{f}) = \sum_{\mathfrak{g} \in \mathfrak{T}} \sum_{L \in \mathfrak{g}} \frac{1}{|\text{Aut}([1] \oplus L)|} \leq \frac{1}{2} \sum_{\mathfrak{g} \in \mathfrak{T}} m(\mathfrak{g}). \hfill \square
\end{equation}
Lemma 2.3.

Proposition 2.6. Any idoneal genus has at least one slender twig.

Proof. If \( f \) is an idoneal genus and \( \mathfrak{T} \) is its set of twigs, then \( \mathfrak{T} \) has at most 2 elements by Lemma 2.3. If each \( g \in \mathfrak{T} \) is not slender, i.e. \( m(f) > m(g) \), then \( 2m(f) > \sum_{g \in \mathfrak{T}} m(g) \), contradicting Lemma 2.4.

Remark 2.7. If an idoneal genus has two twigs, not both of them need to be slender. For instance, the two twigs of the idoneal genus \( f = \{9[1], [1] \oplus E_8\} \) are \( g = \{8[1]\} \), which is slender, and \( h = \{E_8\} \), which is not. (Here \( E_8 \) is meant to be positive definite.)

2.6. Comparison of masses. We set out to compare the mass of \( \tilde{g} \) with the mass of \( g \). The Smith–Minkowski–Siegel mass formula \([3]\) implies

\[
\frac{m(\tilde{g})}{m(g)} = \frac{1}{2} \frac{\Delta_2(\tilde{g})}{\Delta_2(g)} \prod_{p \in \mathbb{P}} \frac{\Delta_p(\tilde{g})}{\Delta_p(g)} \prod_{I \neq 2} \frac{\chi_p(\tilde{g})}{\chi_p(g)} \prod_{E \neq 2} \frac{\text{type}_2(\tilde{g})}{\text{type}_2(g)}.
\]

We proceed to estimate the factors \( A, \ldots, E \). To begin with, we observe that the Jordan constituents behave in the following way:

\[
J^i_p(\tilde{g}) = \begin{cases} 
J^i_p(g) & \text{for } i \neq 0, \\
[1] \oplus J^0_p(g) & \text{for } i = 0.
\end{cases}
\]

Proposition 2.8 (factor A). For a genus \( g \),

\[
\frac{\Delta_2(\tilde{g})}{\Delta_2(g)} \geq \begin{cases} 
\frac{1}{2} & \text{if } g \text{ is odd}, \\
\frac{1}{4} & \text{if } g \text{ is even}.
\end{cases}
\]

Proof. We refer to [2] for an explanation of the terms used in this proof.

If \( J^0_2(g) \) is of type \( \Pi_2 \) or \( \Pi_{2t+1} \) or \( \Pi_{2t+2} \), then \( J^0_2(\tilde{g}) \) is of type \( \Pi_{2t+1} \) or \( \Pi_{2t+2} \) or \( \Pi_{2(t+2)+1} \). Thus, by [2, Table 1] the type factor of \( J^0_2(g) \) is \( 2t\pm \) or \( 2t+1 \) and that of \( J^0_2(\tilde{g}) \) is \( 2t\pm, 2(t+1)\), \( 2t+1 \) or \( 2(t+1) + 1 \). However, if \( J^0_2(\tilde{g}) \) is of type \( 2t+ \), then \( J^0_2(g) \) cannot be of type \( 2t- \), since the octane value of \( J^0_2(g) \) increases by 1 when passing to \( J^0_2(\tilde{g}) \).

Using [2, Table 2], for \( t \neq 0 \) we have the following inequalities:

\[
M_2(2(t+1)-) \leq M_2(2t+1) \leq M_2(2t+),
\]

\[
M_2(2t\pm) \leq M_2(2(t+1)\pm),
\]

\[
M_2(2t+1) \leq M_2(2(t+1) + 1).
\]

This leads us to the two cases for \( t > 0 \). If \( J^0_2(g) \) has type \( 2t+ \), then

\[
\frac{M^0_2(\tilde{g})}{M^0_2(g)} \geq \frac{M_2(2(t+1)-)}{M_2(2t+)} = \frac{1}{(1+2^{-t})(1+2^{t-1})} \geq \frac{8}{15} \geq \frac{1}{2}.
\]
else

\[ \frac{M_2^0(\tilde{g})}{M_2^0(g)} \geq \frac{M_2(2t)}{M_2(2t+1)} = (1 - 2^{-t}) \geq \frac{1}{2}. \]

For \( t = 0 \) we reach the lower bound \( 1/2 \).

Suppose first that \( g \) is odd, so that \( J_2^{-1}(g) \) is of type I. Then both \( J_2^{-1}(g) \) and \( J_2^1(g) \) are bound, so their contributions do not vary when passing to \( \tilde{g} \); using the estimates above, we obtain

\[ \frac{\Delta_2(\tilde{g})}{\Delta_2(g)} = \frac{M_2^0(\tilde{g})}{M_2^0(g)} \geq \frac{1}{2}. \]

Assume now that \( g \) is even, so that \( J_2^{0}(g) \) is of type II. As \( J_2^{0}(\tilde{g}) \) is of type I, the status of \( J_2^{1}(g) \) and \( J_2^{1}(g) \) might change from free to bound, so we must take into account their contributions. Arguing as above,

\[ \frac{\Delta_2(\tilde{g})}{\Delta_2(g)} = \frac{M_2^{-1}(\tilde{g})}{M_2^{-1}(g)} \cdot \frac{M_2^0(\tilde{g})}{M_2^0(g)} \cdot \frac{M_2^1(\tilde{g})}{M_2^1(g)} \geq \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}. \]

Proposition 2.9 (factor B). Define

\[ \xi(n, d) := \prod_{p|d, p \neq 2} \left( 1 + p^{-\max(0, \left\lceil \frac{n - \nu_p(d)}{2} \right\rceil)} \right)^{-1}. \]

If \( g \) is a genus of rank \( n \) and determinant \( d \), then

\[ \prod_{p|d, p \neq 2} \frac{\Delta_p(\tilde{g})}{\Delta_p(g)} \geq \xi(n, d). \]

Proof. Indeed, since \( \text{dim} \, J_p^0(g) \geq n - \nu_p(d) \) by (1), and since the contribution of \( J_p^0(g) \) does not vary when passing to \( \tilde{g} \) for \( i \geq 1 \), we have

\[ \frac{\Delta_p(\tilde{g})}{\Delta_p(g)} = \frac{M_p^0(\tilde{g})}{M_p^0(g)} \geq \left( 1 + p^{-\max(0, \left\lceil \frac{n - \nu_p(d)}{2} \right\rceil)} \right)^{-1}. \]

We obtain the result by multiplying over all odd prime divisors of \( d \). \qed

Lemma 2.10 (cf. [12, Lemma 5.1]). If \( s > 1 \), then

\[ (1 + 2^{-s}) \frac{\zeta(2s)}{\zeta(s)} \leq \zeta_D(s) \leq (1 - 2^{-s}) \zeta(s). \]

Proof. On the one hand,

\[ \zeta_D(s) \geq \prod_{p \neq 2} \frac{1}{1 + p^{-s}} = (1 + 2^{-s}) \prod_p \frac{1}{1 + p^{-s}} = (1 + 2^{-s}) \frac{\zeta(2s)}{\zeta(s)}. \]

On the other hand,

\[ \zeta_D(s) \leq \prod_{p \neq 2} \frac{1}{1 - p^{-s}} = (1 - 2^{-s}) \zeta(s). \]

\[ \square \]

\[ \]
Proposition 2.11 (factor C). For a genus $g$ of determinant $d$ and rank $n$,

$$
\prod_{p \nmid 2d} \frac{\Delta_p(\tilde{g})}{\Delta_p(g)} = \begin{cases} 
\zeta_D(s) & \text{if } n = 2s - 1, \\
\prod_{p \nmid 2d} (1 - p^{-2s}) \zeta(2s) / \zeta_D(s) & \text{if } n = 2s.
\end{cases}
$$

Proof. By Remark 2.1, for $p \nmid 2d$ it holds that

$$
\frac{\Delta_p(\tilde{g})}{\Delta_p(g)} = \frac{\text{std}_p(\tilde{g})}{\text{std}_p(g)}.
$$

If $g$ is of rank $n = 2s - 1$, then $\tilde{g}$ is of rank $2s$, hence

$$
\frac{\text{std}_p(\tilde{g})}{\text{std}_p(g)} = \frac{1 - \varepsilon_p(g)p^{-s}}{1 - (D/p)p^{-s}},
$$

whereas if $g$ is of rank $n = 2s$, then $\tilde{g}$ is of rank $2s + 1 = 2(s + 1) - 1$, hence

$$
\frac{\text{std}_p(\tilde{g})}{\text{std}_p(g)} = \frac{1 - \varepsilon_p(g)p^{-s}}{1 - p^{-2s}} = \frac{1 - (D/p)p^{-s}}{1 - p^{-2s}}.
$$

The result is obtained by multiplying over all primes $p \nmid 2d$. \hfill \Box

Corollary 2.12 (factor C, $n \geq 3$). For a genus $g$ of determinant $d$ and rank $n = 2s$ or $n = 2s - 1$, with $s > 1$,

$$
(8) \prod_{p \nmid 2d} \frac{\Delta_p(\tilde{g})}{\Delta_p(g)} \geq (1 + 2^{-s}) \frac{\zeta(2s)}{\zeta(s)}.
$$

Proof. This follows from Lemma 2.10 and Proposition 2.11. \hfill \Box

Proposition 2.13 (factor D). For a genus $g$ of determinant $d$,

$$
\prod_p \frac{\chi_p(\tilde{g})}{\chi_p(g)} = \sqrt{d}.
$$

Proof. From (1) and (7) it follows that

$$
\chi_p(\tilde{g}) = \prod_{1 < j} p^{\frac{1}{2}j(1 + \dim J_j^0(\tilde{g}))} \dim J_j^0(g) \prod_{0 < i < j} p^{\frac{1}{2}(j - i) \dim J_i^0(\tilde{g}) \dim J_j^0(g)}
$$

$$
= \prod_j p^{\frac{1}{2} \dim J_j^0(g)} \prod_{0 \leq i < j} p^{\frac{1}{2}(j - i) \dim J_i^0(g) \dim J_j^0(g)}
$$

$$
= p^{\frac{1}{2} \nu_p(d)} \chi_p(g).
$$

The result is obtained by multiplying over all primes. \hfill \Box

Proposition 2.14 (factor E). For a genus $g$ of determinant $d$,

$$
\frac{\text{type}_2(\tilde{g})}{\text{type}_2(g)} = \begin{cases} 
1 & \text{if } g \text{ is odd}, \\
2^{\max(0, n - \nu_2(d))} & \text{if } g \text{ is even}.
\end{cases}
$$
Proof. If \( g \) is odd, then \( J_2^0(g) \) is of type I. In this case, both \( n_{II} \) and \( n_{II} \) do not vary when passing to \( \tilde{g} \). (Equality holds.) On the other hand, if \( g \) is even, then \( J_2^0(g) \) is even, so \( n_{II}(\tilde{g}) \geq n_{II}(g) \) and \( n_{II}(\tilde{g}) = n_{II}(g) - \dim J_2^0(g) \). We conclude observing that \( \dim J_2^0(g) \geq \max(0, n - \nu_2(d)) \) by (1).  

We can now draw all estimates together in order to obtain a lower bound on \( m(\tilde{g})/m(g) \) for genera \( g \) of rank > 2 (Theorem 2.15). The case of rank 2 is more delicate and is treated in Theorem 2.17.

**Theorem 2.15.** Define \( \xi \) as in Proposition 2.9 and

\[
F_1(n, d) := \frac{1}{2} \xi(n, d) \sqrt{d}, \\
F_\Pi(n, d) := \frac{1}{8} \xi(n, d) \sqrt{d} \cdot 2^{\max(0, n - \nu_2(d))}.
\]

For \( s > 1 \), put

\[
c_n := \pi^{-\frac{1}{2}(n+1)} \Gamma \left( \frac{1}{2}(n + 1) \right) (1 + 2^{-s}) \frac{\zeta(2s)}{\zeta(s)}.
\]

Then, for a genus \( g \) of rank \( n = 2s \) or \( n = 2s - 1 \) and determinant \( d \),

\[
\frac{m(\tilde{g})}{m(g)} \geq \begin{cases} 
  c_n F_1(n, d) & \text{if } g \text{ is odd,} \\
  c_n F_\Pi(n, d) & \text{if } g \text{ is even.}
\end{cases}
\]

Proof. The statement follows from (5) and (6), by comparing the factors \( A, \ldots, E \) using respectively Proposition 2.8, Proposition 2.9, Corollary 2.12 (here, \( s > 1 \) is crucial), Proposition 2.13 and Proposition 2.14.

We turn to the case \( s = 1 \).

**Lemma 2.16.** Put \( \kappa_1 = 3/2 \), \( \kappa_2 = 3 \). Then, for every \( d > 0 \),

\[
\zeta_D(1) \leq \kappa_1 \log 4d + \kappa_2.
\]

Proof. Put \( D = -d \) and let \( e \equiv 0, 1 \mod 4 \), \( e \neq 0 \). Define \( \psi_e(m) \) as the Kronecker symbol \( \left( \frac{e}{m} \right) \). The associated L-series is \( L(\psi_e, s) = \sum_{m=1}^{\infty} \psi_e(m) m^{-s} \). By [7, §12.14, Theorem 14.3] we have \( L(\chi_e, 1) \leq 2 + \log |e| \).

Now, if \( D \equiv 0 \mod 4 \), then \( \zeta_D(s) = L(\psi_D, s) \). If \( D \equiv 2, 3 \mod 4 \), then \( \zeta_D(s) = \zeta_{4D}(s) = L(\psi_{4D}, s) \). Finally, if \( D \equiv 1 \mod 4 \), then

\[
(1 - \psi_D(2)2^{-s})L(\psi_D, s) = \sum_{m=1}^{\infty} \psi_D(m) m^{-s} - \sum_{m=1}^{\infty} \psi_D(2m) (2m)^{-s} = \sum_{m=1,3,5,...}^{\infty} \chi_D(m) m^{-s} = \zeta_D(s).
\]

Note that \( (1 - \psi_D(2)2^{-1}) \leq 3/2 \). In any case, we conclude that

\[
\zeta_D(1) \leq 3 + \frac{3}{2} \log(4d).
\]
Theorem 2.17. Put \( c_2 := \pi^{-\frac{3}{2}} \Gamma \left( \frac{3}{2} \right) = \frac{1}{2\pi}, \ k_1 := 3/2, k_2 := 3 \) and
\[
\tilde{F}_P(d) := \frac{F_P(2, d)}{\kappa_1 \log 4d + \kappa_2}
\]
for \( P = I, II \), where \( F_I, F_{II} \) are defined as in Theorem 2.15. Then, for a genus \( g \) of rank 2 and determinant \( d \),
\[
\frac{m(\tilde{g})}{m(\tilde{g})} > \begin{cases} 
\left( c_2 \tilde{F}_I(d) \right) & \text{if } \tilde{g} \text{ is odd} \\
\left( c_2' \tilde{F}_{II}(d) \right) & \text{if } \tilde{g} \text{ is even.}
\end{cases}
\]

Proof. The statement follows from (5) and (6), by comparing the factors \( A, \ldots, E \) using respectively Proposition 2.8, Proposition 2.9, Proposition 2.11 together with Lemma 2.16, Proposition 2.13 and Proposition 2.14. \( \square \)

2.7. Proof of Theorem 1.2. Let us first fix \( r \geq 4 \) and let \( \tilde{f} \) be an idoneal genus of rank \( r \) and determinant \( d \). Then, by Proposition 2.6, \( \tilde{f} \) has at least one slender twig \( \tilde{g} \) of rank \( n = r - 1 \) (Definition 2.5). On account of Theorem 2.15, since \( d \) is also the determinant of \( \tilde{g} \), \( d \) belongs to one of the following sets, depending on the parity of \( \tilde{g} \):
\[
D^I_n := \{ d \in \mathbb{Z}_{>0} : c_n F_I(n, d) \leq 1 \},
\]
\[
D^{II}_n := \{ d \in \mathbb{Z}_{>0} : c_{II}(n, d) \leq 1 \}.
\]
Note that \( \sqrt{p}/(1 + p^{-n}) > 1 \) unless \((n, p) = (0, 2), (1, 2) \) or \((0, 3) \). Hence
\[
F_I(n, d) < \tilde{F}_I(n, d')
\]
whenever \( d' \) is a multiple of \( d \). This property implies the following preliminary bound on the rank \( n \).

Lemma 2.18. For \( n \geq 21 \), the sets \( D^I_n \) and \( D^{II}_n \) are empty.
Proof. Indeed, we have \( F_P(n, d) \geq F_P(n, 1) \geq \frac{1}{8} \) for \( n \geq 2 \) and \( P = I, II \). Moreover, \( 1 + 2^{-s} > 1 \) and the functions \( \zeta(2s)/\zeta(s) \) and \( \pi^{-\frac{1}{2}(n+1)} \Gamma \left( \frac{1}{2}(n + 1) \right) \) are increasing for \( s \geq 2 \) and \( n \geq 6 \). Therefore, for \( n \geq 21 \) we have
\[
c_n \geq c_{21} > 8.
\]

In order to determine the set \( D^I_n \), we set \( D_0 = \{1\} \) and compute inductively
\[
D_{i+1} := \{pd \in \mathbb{Z}_{>0} : d \in D_i, p \text{ prime}, pd \in D^I_n \},
\]
so that \( D^I_n = \bigcup_i D_i \). The set \( D^{II}_n \) can be computed analogously, by first fixing \( \nu_2(d) \). Thus, the following strategy enumerates all idoneal genera of rank \( r \).

(i) Set \( n = r - 1 \).
(ii) Compute the set \( D_n = D^I_n \cup D^{II}_n \).
(iii) For each \( d \in D_n \), enumerate all sets of twigs of genera of rank \( r \) and determinant \( d \).
(iv) For each set of twigs, determine if it satisfies condition (4).
(v) For each set satisfying (4), determine if it is the set of twigs of an idoneal genus.
Table 2. The number $S_n$ of slender genera of rank $n \geq 2$ and the corresponding maximal determinant $d_n$ (see Addendum 2.19).

| $n$ | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  | 11  | 12  | 13  | 14  | 15  | 16  |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $S_n$ | 534 | 977 | 1546 | 2134 | 2396 | 2296 | 1808 | 1230 | 712 | 361 | 157 | 61  | 19  | 4   | 1   |
| $d_n$ | 10800 | 6912 | 4096 | 4096 | 1152 | 1024 | 1024 | 320  | 128  | 68  | 36  | 17  | 8   | 3   | 1   |

For rank $r = 3$ we argue similarly, using Theorem 2.17 instead of Theorem 2.15 and

\[
\tilde{D}_1^I := \{ d \in \mathbb{Z}_{>0} : c_2 \tilde{F}_1(d) \leq 1 \}, \\
\tilde{D}_2^I := \{ d \in \mathbb{Z}_{>0} : c_2 \tilde{F}_1(d) \leq 1 \},
\]

instead of $D_n^I, D_n^\Pi$. The sets $\tilde{D}_1^I, \tilde{D}_2^I$ can also be computed inductively, by observing that $\tilde{F}_1(d) < \tilde{F}_1(pd)$ for every prime $p \neq 3$, or $p = 3$ and $d > 39$. Similarly, $\tilde{F}_2(d) < \tilde{F}_2(pd)$ for $p > 5$, or $p = 2$ and $4 \mid d$, or $p = 3$ and $d > 39$. Since our bound on $\zeta_D(1)$ is far from optimal, a further step (ii-b) speeds up the computation:

(ii-b) Sieve the results of step (ii) using $F_P(2, d)/\zeta_D(1)$ instead of $\tilde{F}_P(d)$.

As a byproduct of this strategy, using Steps (i)–(iii), one can classify all slender genera of rank $n \geq 2$.

Addendum 2.19. For $n \in \mathbb{N}, n \geq 2$, there exist exactly $S_n$ slender genera of rank $n$ and maximal determinant $d_n$, with $S_n$ and $d_n$ given in Table 2 for $n \leq 16$ and $S_n = 0$ otherwise. \hfill $\square$

Remark 2.20. The computations were carried out using sageMath [24] or Magma [1]. Step (iii) can be carried out using the sageMath [24] function genera found in sage. quadratic_forms.genera.genus and implemented by the first author. Step (iv) involves only computations of the mass and is not difficult for a computer to carry out. We used the code in sageMath written by John Hanke for this purpose [6]. Step (v) involves computing all representatives of a genus (stopping whenever a representative which does not represent 1 is found). It is the most time-consuming step. For this purpose, we used Kneser’s neighbour method (see [10, §28] and [20] for algorithmic considerations). The most important subtasks are isometry testing and the computation of the orthogonal groups. We used the algorithm of [19] shipped with PARI [25].

Remark 2.21. In its general formulation, the Smith–Minkowski–Siegel mass formula computes the weighted average of representations of a lattice by a genus. If $L$ is a lattice of rank $l$ and $g$ a positive definite genus of rank $m$, then

\[
\frac{1}{m(g)} \sum_{M \in g} a(L, M) = \frac{\gamma(m - l)}{\gamma(m)} \prod p \alpha_p(L, g)
\]

where $a(L, M)$ is the number of representations of $L$ by $M$, $\alpha_p(L, g)$ is the local representation density of $L$ by the genus $g$ and $\gamma$ is known [10, Kapitel X, Satz (33.6)]. If $g$ is an idoneal genus, then $a([1], M) \geq 1$ for every $M \in g$, so that (9) yields

\[
\frac{\gamma(m - 1)}{\gamma(m)} \prod p \alpha_p([1], g) \geq 1.
\]
This gives an alternative way to bound the determinant of an idoneal genus. However, the local representation densities \( \alpha_p([1], \mathfrak{g}) \) need to be estimated.

3. K3 surfaces covering an Enriques surface

Theorem 1.6 is proven in §3.4. First, some elementary properties of finite quadratic forms and Nikulin’s theory of discriminant forms are reviewed (§3.1 and §3.2). Then, the possible shapes of the discriminant form of lattices of signature \((2, \lambda - 2)\) embedding in \( \mathbf{A}^- \) are determined in §3.3 and listed in Table 4. Finally, we find all co-idoneal lattices \( \Lambda \).

16 §3.5

§3.3

3

1

§2

1

§1

1

§0

1

3.1. Finite quadratic forms. Let \( A \) be a finite commutative group. The bilinear form associated to a finite quadratic form \( q: A \to \mathbb{Q}/2\mathbb{Z} \) is given by

\[
q^b: A \times A \to \mathbb{Q}/2\mathbb{Z}, \quad q^b(\alpha, \beta) := (q(\alpha + \beta) - q(\alpha) - q(\beta))/2.
\]

A torsion quadratic form \( q \) is nondegenerate if the homomorphism of groups \( A \to \text{Hom}(A, \mathbb{Q}/2\mathbb{Z}) \) given by \( \alpha \mapsto (\beta \mapsto q^b(\alpha, \beta)) \) is an isomorphism. If \( H \subset A \) is a subgroup, then \( q/H \) denotes the restriction of \( q \) to \( H \).

We adopt Miranda–Morrison’s notation [13] for the elementary finite quadratic forms \( u_k, v_k \) and \( w_{p,k}^\varepsilon \). Moreover, the degenerate quadratic form \( q: A \to \mathbb{Z}/2\mathbb{Z} \) with \( A \cong \mathbb{Z}/2\mathbb{Z} \) taking the values \( 0 \mod 2\mathbb{Z} \) (resp. \( 1 \mod 2\mathbb{Z} \)) on the nontrivial element is denoted by \( (0) \) (resp. \( (1) \)). If \( A \) is 2-elementary, then the short exact sequence \( 0 \to q^+ \to q \to q/q^+ \to 0 \) splits. Hence, \( q \) can be written as the direct sum of copies of \( u_1, v_1, w_{1,1}^1, w_{2,1}^1, (0), (1) \). A (possibly degenerate) finite quadratic form \( q \) is odd if \( q \cong w_{2,1}^\varepsilon \oplus q' \) for some \( \varepsilon \) and some finite quadratic form \( q' \), otherwise \( q \) is even.

From now on, unless explicitly stated, we assume a quadratic form to be nondegenerate.

The signature of a finite quadratic form \( q \) is defined as \( \text{sign } q = s_+ - s_- \mod 8 \), where \((s_+, s_-)\) is the signature of any lattice \( L \) such that \( q(L) \cong q \). Explicit formulas for the signature of elementary quadratic forms were found by Wall [26] (and are also reproduced in [16, Proposition 1.11.2]). Given a prime number \( p \), \( q_p \) denotes the \( p \)-torsion part of \( q \).

We put \( \ell_p(q) = \ell(q_p) \), where \( \ell \) denotes the length of a finite abelian group.

Lemma 3.1. For any finite quadratic form \( q \), \( \ell_2(q) \equiv \text{sign } q \mod 2 \).

Proof. This can be checked for all \( u_k, v_k, w_{p,k}^\varepsilon \). The claim follows by linearity, as each finite quadratic form is isomorphic to the direct sum of elementary forms [3]. \( \square \)

Lemma 3.2. Any finite quadratic form \( q \) contains a 2-elementary subgroup \( H \) of length \( \ell_2(q) - 1 \) such that \( q/H \) is even. A finite quadratic form \( q \) contains a 2-elementary subgroup \( H \) of length \( \ell_2(q) \) such that \( q/H \) is even if and only if \( q \) is even.

Proof. Let \( G \) be the underlying group of \( q \). The subgroup \( H = \{ \alpha \in G : 2\alpha = 0 \} \) is 2-elementary of length \( \ell_2(q) \). The (possibly degenerate) quadratic form \( q/H \) is even if and only if \( q \) is.

Write \( q/H \) as the direct sum of copies of \( u_1, v_1, w_{2,1}^1, w_{2,1}^2, (0), (1) \). By [16, Proposition 1.8.2], we can suppose that there are at most two copies of \( w_{2,1}^\varepsilon \). But \( w_{2,1}^1 \oplus w_{2,1}^2 \) and \( w_{2,1}^1 \oplus w_{2,1}^2 \) contain a copy of \( (1) \), while \( w_{2,1}^1 \oplus w_{2,1}^2 \) contains a copy of \( (0) \), so we conclude. \( \square \)
The following lemma uses similar ideas as the previous ones. We leave the details to the reader.

**Lemma 3.3.** Let \( q = m\mathbf{u}_1 \). If \( H \) is a subgroup of \( q \) and \( m = \max\{0, \ell_2(H) - n\} \), then there exists a (possibly degenerate) finite quadratic form \( q' \) such that
\[
q|H \cong m\mathbf{u}_1 \oplus q'.
\]

3.2. Nikulin’s theory of discriminant forms. If \( L \) is an even lattice, \( q(L) \) denotes its discriminant form.

**Theorem 3.4** (Nikulin [16, Theorem 1.9.1]). For each finite quadratic form \( q \) and prime number \( p \) there exists a unique \( p \)-adic lattice \( K_p(q) \) of rank \( \ell_p(q) \) whose discriminant form is isomorphic to \( q_p \), except in the case when \( p = 2 \) and \( q \) is odd.

We introduce the following conditions (depending on \( s, s' \in \mathbb{Z} \)) on a finite quadratic form \( q \).

\[A(s): \quad \text{sign } q \equiv s \mod 8.\]

\[B(s, s'): \quad \text{for all primes } p \neq 2, \ell_p(q) \leq s + s'; \text{ moreover, } |q| \equiv (-1)^{s'} \text{ discr } K_p(q) \mod (\mathbb{Z}_p^\times)^2 \text{ if } \ell_p(q) = s + s'.\]

\[C(s): \quad \ell_2(q) \leq s; \text{ moreover, } |q| \equiv \pm \text{ discr } K_2(q) \mod (\mathbb{Z}_2^\times)^2 \text{ if } \ell_2(q) = s \text{ and } q \text{ is even.}\]

**Theorem 3.5** (Nikulin [16, Theorem 1.10.1]). An even lattice of signature \((t_+, t_-), t_+, t_- \in \mathbb{Z}_{\geq 0}\), and discriminant quadratic form \( q \) exists if and only if \( q \) satisfies conditions \( A(t_+ - t_-), B(t_+, t_-) \) and \( C(t_+ + t_-) \).

**Theorem 3.6** (Nikulin [16, Theorem 1.14.2]). If \( T \) is an even, indefinite lattice satisfying the following conditions:

(a) \( \text{rank} T \geq \ell(q(T)) + 2 \) for all \( p \neq 2 \),

(b) if \( \text{rank} T = \ell_2(q(T)) \), then \( q(T) \cong \mathbf{u}_1 \oplus q' \) or \( q(T) \cong \mathbf{v}_1 \oplus q' \),

then the genus of \( T \) contains only one class.

Given a pair of nonnegative integers \((m_+, m_-)\) and a finite quadratic form \( q \), Nikulin [16, Proposition 1.15.1] establishes a useful way to enumerate the set of primitive embeddings of a fixed lattice \( T \) into any even lattice belonging to the genus \( \mathfrak{g} \) of signature \((m_+, m_-)\) and discriminant form \( q \). The following proposition is a simplified version of Nikulin’s proposition in the case that the genus \( \mathfrak{g} \) contains only one class.

**Proposition 3.7.** Let \( T \) be an even lattice of signature \((t_+, t_-)\) and \( \Lambda \) be an even lattice of signature \((m_+, m_-)\) which is unique in its genus. Then, for each lattice \( S \), there exists a primitive embedding \( T \to \Lambda \) with \( T^\perp \cong S \) if and only if \( \text{sign } S = (m_+ - t_+, m_- - t_-) \) and there exist subgroups \( H \subset \langle \Lambda \rangle \) and \( K \subset \langle T \rangle \), and an isomorphism of quadratic forms \( \gamma: q(\Lambda)|H \to q(T)|K \), whose graph we denote by \( \Gamma \), such that
\[
q(S) \cong q(\Lambda) \oplus (-q(T))|\Gamma^\perp / \Gamma.
\]
3.3. Transcendental lattices embedding in $\Lambda^-$. Recall that we defined $\Lambda^- = \mathbb{U} \oplus \mathbb{U}(2) \oplus E_8(2)$. It holds $\text{sign } \Lambda^- = (2, 10)$ and $q(\Lambda^-) \cong 5\mathfrak{u}_1$.

In this section the possible discriminant forms of even lattices of signature $(2, \lambda - 2)$ which embed primitively into $\Lambda^-$ are determined. Necessarily, $2 \leq \lambda \leq 12$.

**Lemma 3.8.** Let $f: F \to \mathbb{Q}/2\mathbb{Z}$ and $g: G \to \mathbb{Q}/2\mathbb{Z}$ be finite quadratic forms with $f \cong nu_1$. Let $H \subset F$ and $K \subset G$ be subgroups and $\gamma: f|H \to g|K$ an isometry. Let $\Gamma$ be the graph of $\gamma$ in $F \oplus G$. Then

$$\ell_2(H)u_1 \oplus \left( f \oplus (-g) \right) \cong f \oplus (-g).$$

In particular,

$$\ell_2(f) + \ell_2(g) = \ell_2(\Gamma^+ / \Gamma) + 2\ell_2(H).$$

**Proof.** Recall that $f^\oplus$ and $g^\oplus$ denote the bilinear forms induced by the quadratic forms $f$ and $g$, respectively. Let $C = \ker(f^\oplus|H)$. The exact sequence

$$0 \to C^\perp \to F \to \text{Hom}(C, \mathbb{Q}/\mathbb{Z}) \to 0$$

(induced by $f^\oplus$) splits, because $F$ is 2-elementary by assumption.

Let $s: \text{Hom}(K, \mathbb{Q}/\mathbb{Z}) \to F$ be a section and $C_s^\perp$ its image. Then, since $C$ is a totally isotropic subspace with respect to $f^\oplus$, we infer that $f^\oplus|(C \oplus C_s^\perp) \cong \ell u_1^\perp$, with $\ell = \ell_2(C)$. By modifying the section $s$, we may assume that $f^\oplus|C_s^\perp = 0$. Consider the subgroups $F^\perp_s = (H \oplus C_s^\perp)^\perp \subset F$, $D_s = (C \oplus C_s^\perp)^\perp \subset H \oplus C_s^\perp$ and $G_s^\perp = \gamma(D_s)^\perp \subset G$. Putting $f^\perp = f|F^\perp_s$, $g^\perp = g|G_s^\perp$ and $d = f|D_s$, we obtain

$$f^\perp \cong f^\perp|D_s \oplus (C \oplus C_s^\perp) \oplus F^\perp_s \cong d^\perp \oplus \ell u_1^\perp \oplus (f^\perp)^\perp,$$

$$g^\perp \cong g^\perp|\gamma(D_s) \oplus G_s^\perp \cong d^\perp \oplus (g^\perp)^\perp.$$ 

Let $\varphi: G_s^\perp \to C_s^\perp$ be defined by $f^\perp(\varphi(\beta), \alpha) = g^\perp(\beta, \gamma(\alpha))$ for all $\alpha \in C \subset H$ and $\beta \in G_s^\perp$. Define $\psi: G_s^\perp \to \Gamma^\perp$ by $\psi(\beta) = \varphi(\beta) + \beta$. Since $f^\perp|C_s^\perp = 0$, we have

$$f^\perp + (-g^\perp)(\varphi(\beta), \psi(\beta')) = f^\perp(\varphi(\beta), \varphi(\beta')) - g^\perp(\beta, \beta') = -g^\perp(\beta, \beta').$$

It follows that $f^\perp + (-g^\perp)$ restricted to $F^\perp_s \oplus \psi(G_s^\perp) \subset \Gamma^\perp$ is nondegenerate. Since the orders of $\Gamma^\perp / \Gamma$ and $F^\perp_s \oplus \psi(G_s^\perp)$ coincide, this shows that

$$(f^\perp + (-g^\perp))^\perp \cong (f^\perp + (-g^\perp))^\perp.$$

We claim that we can choose the section $s$ in such a way that the quadratic forms coincide. Indeed, if $f|C_s^\perp = 0$, then (10) also holds at the level of quadratic forms, so we can replace $f^\perp$, $g^\perp$ by $f$, $g$, respectively, and we are done (note that $2d \cong (\ell_2(H) - \ell)u_1$).

If $C \oplus C_s^\perp = F$, the section $s$ can be modified so that $f|C_s^\perp = 0$, because $f \cong nu_1$. Otherwise, as $f|\pi(C \oplus C_s^\perp)^\perp$ is then even, nondegenerate and nonzero, there exists $\alpha \in (C \oplus C_s^\perp)^\perp \subset \Gamma^\perp$ with $f(\alpha) = 1$. Let $\delta_1, \ldots, \delta_\ell$ be a basis of $\text{Hom}(C, \mathbb{Q}/\mathbb{Z})$, so that $C_s^\perp = \langle s(\delta_1), \ldots, s(\delta_\ell) \rangle$. Define $s'$ by $s'(\delta_i) = s(\delta_i)$ if $x(s(\delta_i)) = 0$ and $s'(\delta_i) = s(\delta_i) + \alpha$ else. By replacing $s$ with $s'$ we are again in the situation where $x|C_s^\perp = 0$ and we conclude.

$\square$
Table 3. The three cases appearing in the proof of Proposition 3.9.

| $\lambda_{\text{case}}$ | $\ell_2(q(T))$ | $\ell_2(H)$ | $\ell_2(q(T^\perp))$ | Parity of $q(T)$ |
|--------------------------|----------------|-------------|------------------------|------------------|
| $\lambda_3$             | $\lambda - 2$  | $\lambda - 2$ | $12 - \lambda$        | Even             |
| $\lambda_6$             | $\lambda$      | $\lambda - 1$ | $12 - \lambda$        | Even or odd      |
| $\lambda_c$             | $\lambda$      | $10 - \lambda$ | Even                  |                  |

Proposition 3.9. An even lattice $T$ of signature $(2, \lambda - 2)$ embeds primitively into $\Lambda^-$ if and only if $q(T)$ is of the form given in Table 4 for some nondegenerate finite quadratic form $q$ satisfying the given conditions. In that case, $q(T^\perp)$ is isomorphic to the form given in the corresponding column of the table.

Proof. If $\lambda = 12$, then rank $T = \text{rank} \Lambda^-$, so the claim is trivial. For the rest of the proof we will suppose that $2 \leq \lambda \leq 11$.

Assume first that $T$ embeds in $\Lambda^-$ and let $H \subset q(T)$ be the subset given by Proposition 3.7. By the last equation in Lemma 3.8 we see that

$$\ell_2(q(T)) + 10 = \ell_2(q(T^\perp)) + 2\ell_2(H).$$

Using $\ell_2(q(T^\perp)) \leq \text{rank}(T^\perp) = 12 - \lambda$ and $\ell_2(H) \leq \ell_2(q(T))$, we see that

$$\lambda - 2 = 10 - \text{rank}(T^\perp) \leq 2\ell_2(H) - \ell_2(q(T)) \leq \ell_2(q(T)) \leq \text{rank}(T) = \lambda.$$

By Lemma 3.1, $\ell_2(q(T))$ can only assume two values, namely $\lambda - 2$ or $\lambda$.

We also infer that if $\ell_2(q(T)) = \lambda - 2$, then $\ell_2(H) = \lambda - 2$, whereas if $\ell_2(q(T)) = \lambda$, then either $\ell_2(H) = \lambda - 1$ or $\ell_2(H) = \lambda$ (if $\lambda = 11$ only $\ell_2(H) = \lambda - 1$ is possible, as $\ell_2(H) \leq \ell_2(q(\Lambda^-)) = 10$). Moreover, by Lemma 3.2, $q$ must be even whenever $\ell_2(H) = \ell_2(q(T))$.

Summarizing, for each $\lambda \in \{2, \ldots, 11\}$ we have three cases, described in Table 3, except for $\lambda = 11$ where the last case does not occur.

Let $m = \max\{0, \ell_2(H) - 5\}$. Lemma 3.3 ensures the existence of a subgroup of $H$ (hence of $q(T)$) isometric to $m\mathbf{u}_1$; therefore, $q(T) \cong m\mathbf{u}_1 \oplus q$. The form of $q(T^\perp)$ is then given by Lemma 3.8 and we can apply Theorem 3.5 to find all necessary conditions on $q$. We only write those that are also sufficient: for instance, condition $A(12 - \lambda)$ for $q(T^\perp)$ is always equivalent to $A(\lambda)$ for $q(T)$; if $2 \leq \lambda \leq 6$, condition $B(2, \lambda - 2)$ for $q(T)$ automatically implies $B(0, 12 - \lambda)$ for $q(T^\perp)$.

Conversely, if $T$ is given as in one of the rows of Table 4, then the conditions on $q$, together with Proposition 3.7 and Lemma 3.2, ensure the existence of a primitive embedding $T \hookrightarrow \Lambda^-$ with $q(T^\perp)$ of the given form. \hfill $\Box$

3.4. Proof of Theorem 1.6. We start with an elementary but useful lemma, of which we omit the proof.

Lemma 3.10 (cf. [16, end of p. 130]). A lattice $L$ satisfies $\ell_2(q(L)) = \text{rank} L$ if and only if there exists a lattice $L'$ with $L = L'(2)$. In this case, $L'$ is even if and only if $q(L)$ is even. \hfill $\Box$
Table 4. Discriminant forms of lattices $T$ of signature $(2, \lambda - 2)$ embedding primitively into $\Lambda^-$ (see Proposition 3.9).

| $\lambda_{\text{case}}$ | $q(T)$ | $q(T^\perp)$ | $\ell_2(q)$ | conditions on $q$ |
|--------------------------|--------|--------------|-------------|-------------------|
| $2_a$                    | $q$    | $5u_1 \oplus q(-1)$ | 0           | $C(0)$           |
| $2_b$                    | $q$    | $4u_1 \oplus q(-1)$ | 2           | –                |
| $2_c$                    | $q$    | $3u_1 \oplus q(-1)$ | 2           | even            |
| $3_a$                    | $q$    | $4u_1 \oplus q(-1)$ | 1           | $C(1)$, even    |
| $3_b$                    | $q$    | $3u_1 \oplus q(-1)$ | 3           | –                |
| $3_c$                    | $q$    | $2u_1 \oplus q(-1)$ | 3           | even            |
| $4_a$                    | $q$    | $3u_1 \oplus q(-1)$ | 2           | $C(2)$, even    |
| $4_b$                    | $q$    | $2u_1 \oplus q(-1)$ | 4           | –                |
| $4_c$                    | $q$    | $u_1 \oplus q(-1)$  | 4           | even            |
| $5_a$                    | $q$    | $2u_1 \oplus q(-1)$ | 3           | $C(3)$, even    |
| $5_b$                    | $q$    | $u_1 \oplus q(-1)$  | 5           | –                |
| $5_c$                    | $q$    | $(q(-1))$          | 5           | even            |
| $6_a$                    | $q$    | $u_1 \oplus q(-1)$  | 4           | $C(4)$, even    |
| $6_b$                    | $q$    | $(q(-1))$          | 6           | –                |
| $6_c$                    | $u_1 \oplus q$ | $(q(-1))$ | 4           | even            |
| $7_a$                    | $q$    | $(q(-1))$          | 5           | $B(5,0)$, $C(5)$, even |
| $7_b$                    | $u_1 \oplus q$ | $(q(-1))$ | 5           | $B(5,0)$        |
| $7_c$                    | $2u_1 \oplus q$ | $(q(-1))$ | 3           | $B(5,0)$, even  |
| $8_a$                    | $u_1 \oplus q$ | $(q(-1))$ | 4           | $B(4,0)$, $C(4)$, even |
| $8_b$                    | $2u_1 \oplus q$ | $(q(-1))$ | 4           | $B(4,0)$        |
| $8_c$                    | $3u_1 \oplus q$ | $(q(-1))$ | 2           | $B(4,0)$, even  |
| $9_a$                    | $2u_1 \oplus q$ | $(q(-1))$ | 3           | $B(3,0)$, $C(3)$, even |
| $9_b$                    | $3u_1 \oplus q$ | $(q(-1))$ | 3           | $B(3,0)$        |
| $9_c$                    | $4u_1 \oplus q$ | $(q(-1))$ | 1           | $B(3,0)$, even  |
| $10_a$                   | $3u_1 \oplus q$ | $(q(-1))$ | 2           | $B(2,0)$, $C(2)$, even |
| $10_b$                   | $4u_1 \oplus q$ | $(q(-1))$ | 2           | $B(2,0)$        |
| $10_c$                   | $5u_1 \oplus q$ | $(q(-1))$ | 0           | $B(2,0)$        |
| $11_a$                   | $4u_1 \oplus q$ | $(q(-1))$ | 1           | $B(1,0)$, $C(1)$, even |
| $11_b$                   | $5u_1 \oplus q$ | $(q(-1))$ | 1           | $B(1,0)$        |
| $12$                     | $5u_1$ | –             | –           | –                |

Let $X$ be a K3 surface with transcendental lattice $T$ of rank $\lambda$. By Keum’s criterion (Theorem 1.3), if $T$ is not a co-idoneal lattice, then $X$ covers an Enriques surface if and only if there exists a primitive embedding $T \hookrightarrow \Lambda^-$. Theorem 1.6 follows from Theorem 1.4 once we prove that the conditions given by Proposition 3.9 are equivalent to conditions (i)–(xii). Therefore, we need to analyze all cases of Table 4.
Let us first consider the case $2 \leq \lambda \leq 6$. We want to prove that (i) holds if and only if one of the following holds

(a) $\ell_2(q(T)) = \lambda - 2$, $q(T)$ is even and satisfies condition $C(\lambda - 2)$ (case $\lambda_a$);

(b) $\ell_2(q(T)) = \lambda$ (case $\lambda_b$ or $\lambda_c$).

Let $e_1, \ldots, e_\lambda$ be a system of generators of $T$. Suppose first that the corresponding Gram matrix satisfies (i).

If $a_{ij}$ is even for $2 \leq j \leq \lambda$, then $\ell_2(q(T)) = \text{rank}T = \lambda$ by Lemma 3.10, hence (b) holds.

If this is not true, we can suppose $a_{12}$ to be odd and $a_{1j}$ to be even for $3 \leq j \leq \lambda$, up to relabelling and substituting $e_j$ with $e_j + e_2$. Let $T'$ be the sublattice generated by $e'_1 = 2e_1, e_2, \ldots, e_\lambda$. Then $q(T') \cong u_1 \oplus q(T)$, where the copy of $u_1$ is generated by $e'_1/2$ and $e_2/2$. Since $T' \cong T''(2)$ with $T''$ even, $q(T')$ is even and $\ell_2(q(T')) = \lambda$, by Lemma 3.10. Moreover, $q(T')$ satisfies condition $C(\lambda)$ by Theorem 3.5. This implies (a).

Conversely, if (b) holds, then Lemma 3.10 implies that $T \cong T'(2)$ for some lattice $T'$. If $T'$ is even, then (i) holds. If $T'$ is odd, then up to relabelling we can suppose that $e'_1 \equiv 2 \mod 4$. Then, up to substituting $e_j$ with $e_j + e_1$, we can suppose that $e'_j \equiv 0 \mod 4$. Hence, (i) holds.

Finally, suppose that (a) holds. Since $T$ exists, $q$ satisfies also conditions $A(4 - \lambda)$ and $B(2, \lambda - 2)$. Therefore, $u_1 \oplus q$ satisfies conditions $A(4 - \lambda)$, $B(2, \lambda - 2)$ and $C(\lambda)$, so the genus $g$ of even lattices of signature $(2, \lambda - 2)$ and discriminant form $u_1 \oplus q$ is nonempty. All lattices $T'$ in $g$ are of the form $T' \cong T''(2)$ for some even lattice $T''$. By Proposition 1.4.1 in [16], $T$ must be an overlattice of such a lattice $T'$. The fact that $\det T' = 4 \det T$ implies that $T'$ has index 2 in $T$. Therefore we can find a basis $e_1, \ldots, e_\lambda$ with $e_1 \notin T'$ and $e_j \in T'$ for $j = 2, \ldots, \lambda$, whose corresponding Gram matrix satisfies (i).

We now turn to $\lambda \geq 7$. The case $\lambda = 12$ follows immediately from Keum’s criterion. The arguments for $\lambda \in \{7, \ldots, 11\}$ are very similar, so we illustrate here only the case $\lambda = 10$.

Suppose case $\lambda_a$ holds, i.e. $\lambda = 10$ and $q(T) \cong 3u_1 \oplus q$, with $\ell_2(q) = 2$, $q$ even and satisfying conditions $B(2, 0)$, $C(2)$. Since $T$ exists, $q$ satisfies also $A(2)$, by Theorem 3.5. Hence, using Theorem 3.5 again and Lemma 3.10, we infer that there exists an even lattice $T'$ of signature $(2, 0)$ such that $T'(2)$ has discriminant form $q$. Since $T$ is unique in its genus (Theorem 3.6), $T \cong \tilde{E} \oplus T'(2)$, so (viii) holds.

Suppose case $\lambda_b$ or $\lambda_c$ holds, i.e. $\lambda = 10$ and $q(T) \cong 4u_1 \oplus q$, with $\ell_2(q) = 2$, and $q$ satisfying condition $B(2, 0)$. Since $T$ exists, $q$ satisfies also $A(2)$ and $C(2)$. Hence, there exists a lattice $T'$ of signature $(2, 0)$ such that $T'(2)$ has discriminant form $q$. Again by uniqueness, $T \cong E_8(2) \oplus T'(2)$, so (ix) holds.

Conversely, if (viii) holds, then $q(T) \cong 3u_1 \oplus q$, with $q = q(T')$ being an even finite quadratic form satisfying conditions $A(2)$, $B(2, 0)$ and $C(2)$ by Theorem 3.5. Hence, by Theorem 3.5 and Proposition 3.7, there exists a primitive embedding $T \hookrightarrow A^{-}$. An analogous argument works for (ix).

**Remark 3.11.** For the equivalence between (vi) and case $9_a$, one uses the following fact: if $q, q'$ are two torsion quadratic forms, then $u_1 \oplus q \cong u_1 \oplus q'$ if and only if $q \cong q'$ (see [10, Kapitel I, Satz (4.3)]).
3.5. Co-idoneal lattices. Recall that a co-idoneal lattice is a lattice $T$ of signature $(2, \lambda - 2)$ which embeds primitively into $\mathbf{A}^-$, and such that for each primitive embedding $T \hookrightarrow \mathbf{A}^-$ there exists $v \in T^\perp$ with $v^2 = -2$. Here we list all co-idoneal lattices.

**Lemma 3.12.** If $T$ is a co-idoneal lattice, of rank $\lambda$, then case $\lambda_b$ of Table 3 holds and $q(T)$ is odd.

**Proof.** Consider a primitive embedding $T \hookrightarrow \mathbf{A}^-$. By inspection of Table 4, one sees that $q(T)$ is even if and only if $q(T^\perp)$ is even. In case $\lambda_b$, $\ell_2(q(T^\perp)) = \text{rank } T^\perp$ and $q(T)$ is even, so $T^\perp \cong T'(2)$ for some even lattice $T'$, by Lemma 3.10. Thus, $T^\perp$ does not contain a vector of square $-2$ and $T$ cannot be co-idoneal. If an embedding as in case $\lambda_c$ exists, then $q(T)$ is even and an embedding of $T$ as in case $\lambda_b$ exists: it suffices to choose a smaller subgroup $H$ in Proposition 3.7. Indeed, this changes the type $(\lambda_c, q)$ in Table 4 to $(\lambda_b, u \oplus q)$ which does not affect condition $B(12 - \lambda_b, 0)$. Moreover, if an embedding as in case $\lambda_b$ exists and $q(T)$ is even, then we can argue as before and $T$ cannot be co-idoneal.

Since in case $\lambda_b$ it holds that $\ell_2(q(T)) = \lambda = \text{rank } T$, Lemma 3.10 implies the following corollary.

**Corollary 3.13.** If $T$ is a co-idoneal lattice, then $T \cong T'(2)$ for some odd lattice $T'$.

The following proposition explains the connection between co-idoneal lattices and idoneal genera.

**Proposition 3.14.** For each co-idoneal lattice $T$ there exists a unique idoneal genus $g$ with the following property: for each primitive embedding $T \hookrightarrow \mathbf{A}^-$, there exists $L \in g$ with $T^\perp \cong L(-2)$.

**Proof.** Let $\lambda = \text{rank } T$ and consider a primitive embedding $T \hookrightarrow \mathbf{A}^-$. It follows from Lemma 3.12 that $\ell_2(q(T^\perp)) = 12 - \lambda = \text{rank } T^\perp$ and $q(T)$ is odd. By Lemma 3.10, there exists an odd lattice $L$ with $T^\perp \cong L(-2)$. Let $g$ be the genus of $L$.

The discriminant form of $T^\perp$ is determined by the discriminant form of $T$ according to Table 4. According to [16, Corollary 1.16.3], the genus of a lattice is determined by its signature, parity and discriminant bilinear form. Hence, each lattice $L$ such that $T^\perp \cong L(-2)$ belongs to the same genus $g$.

Conversely, as $\mathbf{A}^-$ is unique in its genus, each lattice $L \in g$ satisfies $L(-2) \cong T^\perp$ for some embedding $T \hookrightarrow \mathbf{A}^-$, by Proposition 3.7. Since $T^\perp$ always contains a vector of square $-2$, each lattice $L \in g$ contains a vector of square $1$, i.e. $g$ is idoneal.

**Proof of Theorem 1.4.** Thanks to Proposition 3.14 and Theorem 1.2, one can proceed in the following way.

(i) For each $g$ of rank $r = 12 - \lambda$ in the list we pick any $L \in g$ and compute $x = q(L(-2))$.

(ii) If $x$ is one of the suitable forms $q(T^\perp)$ in Table 4, then we compute the corresponding $y = q(T)$.

(iii) We compute all lattices $T = T'(2)$ of signature $(2, \lambda - 2)$ and discriminant form $y$.

(iv) We add $T'$ to Table 5.

In this way we enumerate all co-idoneal lattices.
The following turns out to be true.

**Addendum 3.15.** All co-idoneal lattices are unique in their genus. □

**Table 5.** Halves of co-idoneal lattices (see Theorem 1.4), listed by increasing rank $\lambda = \text{rank } T'$ and determinant.

| $\lambda$ | No. | $\text{det } T'$ | $T'$ |
|-----------|-----|----------------|------|
| 2         | 1   | 1              | 2[1] |
| 2         | 2   | 2              | [1]  ⊕ [2] |
| 2         | 3   | 4              | [1]  ⊕ [4] |
| 3         | 1   | −1             | [1]  ⊕ U |
| 3         | 2   | −2             | [1]  ⊕ [2] ⊕ [−1] |
| 3         | 3   | −3             | 2[1]  ⊕ [−3] |
| 3         | 4   | −4             | [1]  ⊕ [2] ⊕ [−2] |
| 3         | 5   | −4             | [1]  ⊕ U(2) |
| 3         | 6   | −5             | [1]  ⊕ [2, 1, −2] |
| 3         | 7   | −8             | [2]  ⊕ [0, 2, −1] |
| 3         | 8   | −9             | [−1]  ⊕ U(2) |
| 4         | 1   | 1              | 2[1]  ⊕ [2, −1] |
| 4         | 2   | 2              | [1]  ⊕ [−2] ⊕ U |
| 4         | 3   | 3              | [−1]  ⊕ U(3) ⊕ U |
| 4         | 4   | 4              | [1]  ⊕ [−1, 1, 1, −1, 1, 1](−1) |
| 4         | 5   | 4              | [1]  ⊕ [−1] ⊕ U(2) |
| 4         | 6   | 4              | [1]  ⊕ [−1] ⊕ [0, 2, −1] |
| 4         | 7   | 4              | [1]  ⊕ [−1] ⊕ [0, 2, 1] |
| 4         | 8   | 5              | [−1]  ⊕ [1] ⊕ [2, 1, −2] |
| 4         | 9   | 6              | [1]  ⊕ [1, 2, −1, 0, 1, 1](−1) |
| 4         | 10  | 8              | [1]  ⊕ [2] ⊕ U(2) |
| 4         | 11  | 8              | [1]  ⊕ [−2, 1, 2, 0, 1, 2](−1) |
| 4         | 12  | 9              | [3]  ⊕ [−3] ⊕ U |
| 4         | 13  | 9              | [0, 3, 2, 0, 2, 0, 0, −5, 1, 1] |
| 4         | 14  | 12             | [3]  ⊕ [0, 2, −2, 0, 1, 1](−1) |
| 4         | 15  | 16             | [−4]  ⊕ [0, 2, 1, 0, 1, 1] |
| 4         | 16  | 16             | [4]  ⊕ [0, 2, 1, 0, 1, 1](−1) |
| 4         | 17  | 16             | [1]  ⊕ [−1] ⊕ U(4) |
| 5         | 1   | −1             | [−1]  ⊕ 2U |
| 5         | 2   | −2             | [1]  ⊕ [−1] ⊕ [−2] ⊕ U |
| 5         | 3   | −3             | 2[−1]  ⊕ [3] ⊕ U |
| 5         | 4   | −3             | [1]  ⊕ [−1] ⊕ [−3] ⊕ U |
| 5         | 5   | −4             | [1]  ⊕ [2][−1] ⊕ [2] ⊕ [−2] |
| 5         | 6   | −4             | [−1]  ⊕ U ⊕ U(2) |

Continued on next page.
Table 5, continued from previous page.

| $\lambda$ | No. | $\det T'$ | $T'$ |
|----------|-----|-----------|-------|
| 5 | 7 | -4 | $[1] \oplus [-1] \oplus [-4] \oplus U$ |
| 5 | 8 | -4 | $[-1] \oplus U \oplus [-1, 1, 3]$ |
| 5 | 9 | -5 | $2[-1] \oplus [5] \oplus U$ |
| 5 | 10 | -6 | $[1] \oplus [-1] \oplus [-6] \oplus U$ |
| 5 | 11 | -7 | $[-1] \oplus U \oplus [2, 1, -3]$ |
| 5 | 12 | -7 | $2[-1] \oplus [7] \oplus U$ |
| 5 | 13 | -8 | $[1] \oplus [-1] \oplus [-2] \oplus U(2)$ |
| 5 | 14 | -8 | $[-1] \oplus [-2] \oplus [4] \oplus U$ |
| 5 | 15 | -8 | $2[-1] \oplus [8] \oplus U$ |
| 5 | 16 | -8 | $[-1] \oplus [1] \oplus [0, 2, -2, 0, 1, 2](-1)$ |
| 5 | 17 | -8 | $[1] \oplus [-1] \oplus [-8] \oplus U$ |
| 5 | 18 | -9 | $U \oplus [0, 3, -2, 0, 1, 1](-1)$ |
| 5 | 19 | -9 | $[0, 3, 2, 0, 1, 0, 0, 0, 1, 0, 1, 0, 0, 0](-1)$ |
| 5 | 20 | -10 | $[1] \oplus [-1] \oplus [-10] \oplus U$ |
| 5 | 21 | -12 | $[1] \oplus [-1] \oplus [-3] \oplus U(2)$ |
| 5 | 22 | -12 | $2[-1] \oplus [3] \oplus U(2)$ |
| 5 | 23 | -12 | $2[-1] \oplus [1] \oplus [2, 1, -1](2)$ |
| 5 | 24 | -12 | $[-1] \oplus [-3] \oplus [4] \oplus U$ |
| 5 | 25 | -13 | $2[-1] \oplus [13] \oplus U$ |
| 5 | 26 | -15 | $[3] \oplus U \oplus [2, 1, 3](-1)$ |
| 5 | 27 | -16 | $[-1] \oplus 2U(2)$ |
| 5 | 28 | -16 | $[1] \oplus [-1] \oplus [-4] \oplus U(2)$ |
| 5 | 29 | -16 | $[1] \oplus [2-4] \oplus U$ |
| 5 | 30 | -16 | $[0, 2, -2, 0, 1, 1, 0, 1, 0, -2, 0, 1, 0, 2](-1)$ |
| 5 | 31 | -16 | $[-4] \oplus U \oplus [-1, 1, 3]$ |
| 5 | 32 | -16 | $[1] \oplus [-1] \oplus [-16] \oplus U$ |
| 5 | 33 | -18 | $[-1] \oplus [-2] \oplus [9] \oplus U$ |
| 5 | 34 | -20 | $2[-1] \oplus [5] \oplus U(2)$ |
| 5 | 35 | -21 | $[-1] \oplus [-3] \oplus [7] \oplus U$ |
| 5 | 36 | -24 | $[1] \oplus [-1] \oplus [-24] \oplus U$ |
| 5 | 37 | -27 | $[1] \oplus [-1] \oplus [3] \oplus 2[-3]$ |
| 5 | 38 | -32 | $[1] \oplus [-1] \oplus [-2] \oplus U(4)$ |
| 5 | 39 | -32 | $[1] \oplus [-1] \oplus [-8] \oplus U(2)$ |
| 5 | 40 | -48 | $U(2) \oplus [0, 2, -1, 0, 1, 3](-1)$ |
| 5 | 41 | -64 | $[1] \oplus [2-4] \oplus U(2)$ |

6 1 1 $2[-1] \oplus 2U$
6 2 2 $2[-1] \oplus [1] \oplus [-2] \oplus U$
6 3 3 $[-1] \oplus [-3] \oplus 2U$
6 4 3 $[1] \oplus [-1] \oplus U \oplus A_2$
6 5 4 $[1] \oplus [-1] \oplus [2-2] \oplus U$

Continued on next page.
Table 5, continued from previous page.

| $\lambda$ | No. | $\det T''$ | $T''$ |
|-----------|-----|-------------|--------|
| 6  | 6  | 4  | $2[-1] \oplus U \oplus U(2)$ |
| 6  | 7  | 4  | $2[-1] \oplus U \oplus [0,2,-1]$ |
| 6  | 8  | 4  | $[1] \oplus [-1] \oplus [-4] \oplus 2U$ |
| 6  | 9  | 5  | $3[-1] \oplus [5] \oplus U$ |
| 6  | 10 | 5  | $2U \oplus [2,1,3](-1)$ |
| 6  | 11 | 6  | $2[-1] \oplus [-2] \oplus [3] \oplus U$ |
| 6  | 12 | 6  | $[1] \oplus [-1] \oplus [-2] \oplus [-3] \oplus U$ |
| 6  | 13 | 7  | $[1] \oplus [2,-1] \oplus [3,-7] \oplus U$ |
| 6  | 14 | 8  | $[1] \oplus [2,-2] \oplus U(2) \oplus U$ |
| 6  | 15 | 8  | $2[-1] \oplus [-2] \oplus [4] \oplus U$ |
| 6  | 16 | 8  | $3[-1] \oplus [8] \oplus U$ |
| 6  | 17 | 8  | $[1] \oplus [1] \oplus [3,1,3](-1)$ |
| 6  | 18 | 8  | $[1] \oplus [-8] \oplus 2U$ |
| 6  | 19 | 9  | $[1] \oplus [1] \oplus [-3] \oplus U$ |
| 6  | 20 | 9  | $2[-1] \oplus U \oplus [-2,1,4]$ |
| 6  | 21 | 9  | $3[-1] \oplus [9] \oplus U$ |
| 6  | 22 | 10 | $2[-1] \oplus [-2] \oplus [5] \oplus U$ |
| 6  | 23 | 11 | $[1] \oplus [1] \oplus U \oplus [3,1,4](-1)$ |
| 6  | 24 | 11 | $[1] \oplus [-1] \oplus [-2,1,2,0,1,2,0,1,1,2](-1)$ |
| 6  | 25 | 12 | $[1] \oplus [-1] \oplus U(2) \oplus A_2$ |
| 6  | 26 | 12 | $[1] \oplus [-1] \oplus [-2] \oplus [-6] \oplus U$ |
| 6  | 27 | 12 | $[1] \oplus [-1] \oplus [-3] \oplus U(2)$ |
| 6  | 28 | 12 | $[1] \oplus [-1] \oplus [-3] \oplus [-4] \oplus U$ |
| 6  | 29 | 12 | $[1] \oplus [-4] \oplus U \oplus A_2$ |
| 6  | 30 | 12 | $[1] \oplus [2,-1] \oplus [-12] \oplus U$ |
| 6  | 31 | 14 | $[-1] \oplus [-2] \oplus U \oplus [2,1,-3]$ |
| 6  | 32 | 14 | $2[-1] \oplus [-2] \oplus [7] \oplus U$ |
| 6  | 33 | 15 | $[1] \oplus [2,-1] \oplus [-15] \oplus U$ |
| 6  | 34 | 15 | $[-1] \oplus [-3] \oplus U \oplus [1,2,-1]$ |
| 6  | 35 | 16 | $[1] \oplus [-1] \oplus [2,-2] \oplus U(2)$ |
| 6  | 36 | 16 | $[2] \oplus U \oplus [0,2,-1,0,1,1](-1)$ |
| 6  | 37 | 16 | $[-1] \oplus [-4] \oplus U \oplus U(2)$ |
| 6  | 38 | 16 | $[-1] \oplus U(2) \oplus [0,2,-2,0,1,1](-1)$ |
| 6  | 39 | 16 | $2[-1] \oplus [2] \oplus [-8] \oplus U$ |
| 6  | 40 | 16 | $[1] \oplus U \oplus [3,1,3,0,1,1,1,2](-1)$ |
| 6  | 41 | 16 | $[1] \oplus [-1] \oplus [2,-4] \oplus U$ |
| 6  | 42 | 16 | $2[-1] \oplus U \oplus U(4)$ |
| 6  | 43 | 16 | $2U \oplus [4,2,5](-1)$ |
| 6  | 44 | 16 | $2[-1] \oplus U \oplus [-2,1,8]$ |
| 6  | 45 | 18 | $2[-1] \oplus [1] \oplus [-2] \oplus [3] \oplus [-3]$ |

Continued on next page.
Table 5, continued from previous page.

| \( \lambda \) | No. | \( \det T' \) | \( T' \) |
|--------------|-----|--------------|--------|
| 6            | 46  | 2[-1] \( \oplus \) [2] \( \oplus \) [-10] \( \oplus \) U | |
| 6            | 47  | 2[-1] \( \oplus \) [-1] \( \oplus \) U \( \oplus \) [3, 1, 7](-1) | |
| 6            | 48  | 2[-1] \( \oplus \) [-4] \( \oplus \) [5] \( \oplus \) U | |
| 6            | 49  | 2[-1] \( \oplus \) [2] \( \oplus \) [-23] \( \oplus \) U | |
| 6            | 50  | 2[-1] \( \oplus \) [-2] \( \oplus \) [2, 1, -1](2) | |
| 6            | 51  | 2[-1] \( \oplus \) [-1] \( \oplus \) U \( \oplus \) [4, 2, 7](-1) | |
| 6            | 52  | 2[-1] \( \oplus \) [-3] \( \oplus \) [8] \( \oplus \) U | |
| 6            | 53  | 2[-1] \( \oplus \) [-2] \( \oplus \) [13] \( \oplus \) U | |
| 6            | 54  | 2[-1] \( \oplus \) [-3] \( \oplus \) U \( \oplus \) [-2, 1, 4] | |
| 6            | 55  | 2[-1] \( \oplus \) [-1] \( \oplus \) [3](-3) | |
| 6            | 56  | 2[-1] \( \oplus \) [0, 3, 2] \( \oplus \) A_2 | |
| 6            | 57  | [-1] \( \oplus \) U \( \oplus \) [-1, 3, -1, 2, 0, 4](-1) | |
| 6            | 58  | [2] \( \oplus \) [4] \( \oplus \) U \( \oplus \) [3, 1, 3](-1) | |
| 6            | 59  | [-1] \( \oplus \) [4] \( \oplus \) U \( \oplus \) [3, 1, 3](-1) | |
| 6            | 60  | [-1] \( \oplus \) [4] \( \oplus \) U \( \oplus \) [3, 1, 3](-1) | |
| 6            | 61  | 2[-1] \( \oplus \) U \( \oplus \) [2, -16] \( \oplus \) U | |
| 6            | 62  | 2[-1] \( \oplus \) [-2] \( \oplus \) 2U(2) | |
| 6            | 63  | [1] \( \oplus \) U \( \oplus \) [4, 2, 4, 1, -1, 4](-1) | |
| 6            | 64  | [-1] \( \oplus \) U(2) \( \oplus \) [0, 2, -1, 0, 1, 3](-1) | |
| 6            | 65  | [1] \( \oplus \) [-3] \( \oplus \) [4] \( \oplus \) U(2) | |
| 6            | 66  | [1] \( \oplus \) [-1] \( \oplus \) U \( \oplus \) A_2(4) | |
| 6            | 67  | [1] \( \oplus \) U(2) \( \oplus \) [3, 1, 3, -1, 1, 3](-1) | |
| 6            | 68  | [1] \( \oplus \) [-1] \( \oplus \) [2, -1] \( \oplus \) U(4) | |
| 6            | 69  | [1] \( \oplus \) [-1] \( \oplus \) [-2] \( \oplus \) U(2) | |
| 6            | 70  | [1] \( \oplus \) [-3] \( \oplus \) U | |
| 6            | 71  | [-4] \( \oplus \) U(2) \( \oplus \) [0, 2, -1, 0, 1, 1](-1) | |
| 6            | 72  | 2U(2) \( \oplus \) [3, 1, 3](-1) | |
| 6            | 73  | 2U(2) \( \oplus \) [0, 2, -1, 0, 1, 3](-1) | |
| 6            | 74  | [1] \( \oplus \) [3] \( \oplus \) [4](-3) | |
| 6            | 75  | [1] \( \oplus \) U(4) \( \oplus \) [3, 1, 3, -1, 1, 3](-1) | |
| 6            | 76  | 2U(2) \( \oplus \) [3, 1, 3](-1) | |
| 6            | 77  | 2U(2) \( \oplus \) [0, 2, -1, 0, 1, 1](-1) | |
| 7            | 1   | 3[-1] \( \oplus \) 2U | |
| 7            | 2   | 2[-1] \( \oplus \) [-2] \( \oplus \) 2U | |
| 7            | 3   | 2[-1] \( \oplus \) [-3] \( \oplus \) 2U | |
| 7            | 4   | 2[-1] \( \oplus \) [1] \( \oplus \) A_2 \( \oplus \) U | |
| 7            | 5   | 2[-1] \( \oplus \) [2] \( \oplus \) [-2] \( \oplus \) U | |
| 7            | 6   | 3[-1] \( \oplus \) U(2) \( \oplus \) U | |
| 7            | 7   | 3[-1] \( \oplus \) U(2) \( \oplus \) U | |
| 7            | 8   | 2[-1] \( \oplus \) [-4] \( \oplus \) 2U | |
| 7            | 9   | 2[-1] \( \oplus \) [-5] \( \oplus \) 2U | |

Continued on next page.
Table 5, continued from previous page.

| λ | No. | \( \det T' \) | \( T' \) |
|---|-----|-----------------|---------|
| 7 | 10  | -5              | \([-1] \oplus 2U \oplus [2, 1, 3](-1)\) |
| 7 | 11  | -6              | \([-1] \oplus [-2] \oplus [-3] \oplus 2U\) |
| 7 | 12  | -6              | \([1] \oplus [-1] \oplus [-2] \oplus A_2 \oplus U\) |
| 7 | 13  | -7              | \(2[-1] \oplus U \oplus [1, 1, -2, 0, 1, 2](-1)\) |
| 7 | 14  | -7              | \([-1] \oplus 2U \oplus [2, 1, 4](-1)\) |
| 7 | 15  | -8              | \(2[-1] \oplus [-2] \oplus U(2) \oplus U\) |
| 7 | 16  | -8              | \([-1] \oplus [-2] \oplus [-4] \oplus 2U\) |
| 7 | 17  | -8              | \(2U \oplus [2, 1, 3, 0, 1, 2](-1)\) |
| 7 | 18  | -8              | \([-1] \oplus 2U \oplus [3, 1, 3](-1)\) |
| 7 | 19  | -8              | \(2[-1] \oplus [-8] \oplus 2U\) |
| 7 | 20  | -9              | \([1] \oplus 2A_2 \oplus U\) |
| 7 | 21  | -9              | \(3[-1] \oplus U(3) \oplus U\) |
| 7 | 22  | -9              | \(3[-1] \oplus [2, 1, -4] \oplus U\) |
| 7 | 23  | -10             | \([-2] \oplus 2U \oplus [2, 1, 3](-1)\) |
| 7 | 24  | -10             | \([-1] \oplus [-2] \oplus [-5] \oplus 2U\) |
| 7 | 25  | -11             | \(2[-1] \oplus [-11] \oplus 2U\) |
| 7 | 26  | -12             | \(2[-1] \oplus [-3] \oplus U(2) \oplus U\) |
| 7 | 27  | -12             | \([1] \oplus [-1] \oplus 2[-2] \oplus [-3] \oplus U\) |
| 7 | 28  | -12             | \(2[-1] \oplus [1] \oplus [-2] \oplus [-6] \oplus U\) |
| 7 | 29  | -12             | \([-1] \oplus U \oplus [1, 1, -3, 0, 2, 1, 0, 0, 1, 2](-1)\) |
| 7 | 30  | -12             | \([1] \oplus [-1] \oplus U \oplus [2, 1, 4, 0, 1, 2](-1)\) |
| 7 | 31  | -12             | \([-1] \oplus [1] \oplus U \oplus [2, 1, 3, 1, 1, 3](-1)\) |
| 7 | 32  | -13             | \(2[-1] \oplus [-13] \oplus 2U\) |
| 7 | 33  | -13             | \(3[-1] \oplus [-2, 1, 6] \oplus U\) |
| 7 | 34  | -14             | \([1] \oplus [-1] \oplus [-2] \oplus [2, 1, 4](-1) \oplus U\) |
| 7 | 35  | -15             | \([-3] \oplus 2U \oplus [2, 1, 3](-1)\) |
| 7 | 36  | -15             | \([-1] \oplus [-3] \oplus [-5] \oplus 2U\) |
| 7 | 37  | -16             | \([-1] \oplus 2[-2] \oplus [0, 2, 1] \oplus U\) |
| 7 | 38  | -16             | \([1] \oplus [-1] \oplus [-2] \oplus [3, 1, 3](-1) \oplus U\) |
| 7 | 39  | -16             | \(U \oplus [0, 2, -1, 0, 1, 2, 0, 0, 1, 1, 2](-1)\) |
| 7 | 40  | -16             | \([-1] \oplus 2[-4] \oplus 2U\) |
| 7 | 41  | -16             | \([-1] \oplus [-4] \oplus U \oplus [0, 2, -2, 0, 1, 1](-1)\) |
| 7 | 42  | -16             | \(2U \oplus [3, 1, 3, -1, 1, 3](-1)\) |
| 7 | 43  | -16             | \([1] \oplus [-1] \oplus U \oplus [2, 1, 2, 1, 1, 6](-1)\) |
| 7 | 44  | -16             | \(2[-1] \oplus [-16] \oplus 2U\) |
| 7 | 45  | -16             | \([-1] \oplus 2[-2] \oplus U(2) \oplus U\) |
| 7 | 46  | -17             | \([-1] \oplus 2U \oplus [3, 1, 6](-1)\) |
| 7 | 47  | -18             | \([-2] \oplus U \oplus [0, 3, 1, 0, 1, 0, 1, 0, 1](-1)\) |
| 7 | 48  | -18             | \([-2] \oplus 2[-3] \oplus 2U\) |
| 7 | 49  | -19             | \(2[-1] \oplus [-19] \oplus 2U\) |

Continued on next page.
| \( \lambda \) | No. | \( \det T' \) | \( T' \)         |
|--------|-----|-------------|----------------|
| 7      | 50  | \(-20\)     | \([-1] \oplus 2U \oplus [2, 1, 3](−2)\) |
| 7      | 51  | \(-20\)     | \(2U \oplus [3, 1, 3, 1, 1, 3](-1)\) |
| 7      | 52  | \(-20\)     | \([2, 1, 3](−1) \oplus U \oplus [−2, 1, 1, 1, 0, 1](−1)\) |
| 7      | 53  | \(-21\)     | \(2[−1] \oplus [3] \oplus [2, 1, 4](−1) \oplus U\) |
| 7      | 54  | \(-21\)     | \([-1] \oplus [1] \oplus U \oplus [3, 1, 3, 1, 3](−1)\) |
| 7      | 55  | \(-22\)     | \([1] \oplus [−1] \oplus U \oplus [2, 1, 2, 1, 0, 8](−1)\) |
| 7      | 56  | \(-24\)     | \([-1] \oplus [−6] \oplus U(2) \oplus U\) |
| 7      | 57  | \(-24\)     | \([-1] \oplus [−2] \oplus [−12] \oplus 2U\) |
| 7      | 58  | \(-24\)     | \([1] \oplus [−1] \oplus U \oplus [2, 1, 4, 1, 0, 4](−1)\) |
| 7      | 59  | \(-24\)     | \(U \oplus [−1, 1, 2, 1, 0, 2, 0, 0, 0, 1, 1, 3](−1)\) |
| 7      | 60  | \(-24\)     | \([−1] \oplus [−3] \oplus [−8] \oplus 2U\) |
| 7      | 61  | \(-25\)     | \(U \oplus [1, 1, −1, 0, 1, 0, 1, 3, 0, −1, 0, 2, 3](−1)\) |
| 7      | 62  | \(-27\)     | \([-1] \oplus [−3] \oplus [2, 1, −4] \oplus U\) |
| 7      | 63  | \(-27\)     | \([-1] \oplus [1] \oplus [3] \oplus U\) |
| 7      | 64  | \(-27\)     | \([-9] \oplus 2U \oplus A_{2}\) |
| 7      | 65  | \(-28\)     | \(2[−1] \oplus U \oplus [3, 1, 3, 0, 2, −2](−1)\) |
| 7      | 66  | \(-28\)     | \([1] \oplus [−1] \oplus [−4] \oplus [2, 1, 4](−1) \oplus U\) |
| 7      | 67  | \(-30\)     | \([−2] \oplus [2, 1, −1] \oplus [2, 1, 3](−1) \oplus U\) |
| 7      | 68  | \(-32\)     | \(U(2) \oplus U \oplus [2, 1, 3, 0, 1, 2](−1)\) |
| 7      | 69  | \(-32\)     | \(2[−1] \oplus [−2] \oplus U(4) \oplus U\) |
| 7      | 70  | \(-32\)     | \([-1] \oplus U \oplus [0, 4, 1, 0, 1, 1, 0, 2, 0, 2](−1)\) |
| 7      | 71  | \(-32\)     | \([-1] \oplus U(2) \oplus [3, 1, 3](−1) \oplus U\) |
| 7      | 72  | \(-32\)     | \([−2] \oplus 2U \oplus [4, 2, 5](−1)\) |
| 7      | 73  | \(-32\)     | \([−4] \oplus 2U \oplus [3, 1, 3](−1)\) |
| 7      | 74  | \(-32\)     | \([-1] \oplus [−2] \oplus [−4] \oplus U(2) \oplus U\) |
| 7      | 75  | \(-33\)     | \([-1] \oplus [−3] \oplus [−11] \oplus 2U\) |
| 7      | 76  | \(-36\)     | \([1] \oplus U \oplus [2, 1, 4, 1, 0, 4, 0, −1, 1, 2](−1)\) |
| 7      | 77  | \(-36\)     | \([1] \oplus [−1] \oplus [−12] \oplus A_{2} \oplus U\) |
| 7      | 78  | \(-40\)     | \([-1] \oplus 2U \oplus [7, 3, 7](−1)\) |
| 7      | 79  | \(-40\)     | \([-8] \oplus 2U \oplus [2, 1, 3](−1)\) |
| 7      | 80  | \(-45\)     | \([-15] \oplus 2U \oplus A_{2}\) |
| 7      | 81  | \(-48\)     | \([1] \oplus [−1] \oplus [−4] \oplus A_{2}(2) \oplus U\) |
| 7      | 82  | \(-48\)     | \([-1] \oplus [1] \oplus U \oplus [3, 1, 3, 1, −1, 7](−1)\) |
| 7      | 83  | \(-48\)     | \(U \oplus [0, 2, −1, 0, 1, 2, 0, 1, 2, 0, 1, −1, 1, 4](−1)\) |
| 7      | 84  | \(-48\)     | \([-1] \oplus [2] \oplus [−2] \oplus A_{2}(2) \oplus U\) |
| 7      | 85  | \(-49\)     | \([-7] \oplus 2U \oplus [2, 1, 4](−1)\) |
| 7      | 86  | \(-54\)     | \([-1] \oplus [−2] \oplus [9] \oplus A_{2} \oplus U\) |
| 7      | 87  | \(-64\)     | \([0, 2, 1] \oplus U \oplus [3, 1, 3, −1, 1, 3](−1)\) |
| 7      | 88  | \(-64\)     | \([1] \oplus U \oplus [3, 1, 3, −1, 3, 1, −1] \oplus U(−1)\) |
| 7      | 89  | \(-64\)     | \([1] \oplus [−1] \oplus [3] \oplus U\) |

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Table 5, continued from previous page.

| $\lambda$ | No. | $\det T''$ | $T''$ |
|-----------|-----|-------------|-------|
| 7         | 90  | $-64$       | $U \oplus [2, 1, 2, 1, 1, 2, 1, 1, 0, 3, 0, -1, 1, 3, 2](-1)$ |
| 7         | 91  | $-64$       | $2[-1] \oplus U \oplus [-3, 1, 5, 1, -3, 5](-1)$ |
| 7         | 92  | $-64$       | $[-1] \oplus [2] \oplus [-2] \oplus [2][-4] \oplus U$ |
| 7         | 93  | $-64$       | $U(2) \oplus U \oplus [3, 1, 3, -1, 1, 3](-1)$ |
| 7         | 94  | $-80$       | $2U \oplus [4, 2, 7, 0, 2, 4](-1)$ |
| 7         | 95  | $-80$       | $U(2) \oplus U \oplus [3, 1, 3, 1, 1, 3](-1)$ |
| 7         | 96  | $-81$       | $[-3] \oplus U(3) \oplus A_2 \oplus U$ |
| 7         | 97  | $-96$       | $U(2) \oplus U \oplus [2, 1, 7, 0, 1, 2](-1)$ |
| 7         | 98  | $-128$      | $U(4) \oplus U \oplus [2, 1, 3, 0, 1, 2](-1)$ |
| 7         | 99  | $-128$      | $[-4] \oplus U(2) \oplus [3, 1, 3](-1) \oplus U$ |
| 7         | 100 | $-144$      | $[1] \oplus 2A_2(2) \oplus U$ |
| 7         | 101 | $-162$      | $[-2] \oplus 2[-3] \oplus U(3) \oplus U$ |
| 7         | 102 | $-192$      | $[-4] \oplus [0, 2, 1] \oplus A_2(2) \oplus U$ |
| 7         | 103 | $-243$      | $[-9] \oplus U \oplus [-1, 1, 2, -1, 1, 2, 1, -1, 1, 2](-1)$ |
| 7         | 104 | $-256$      | $[1] \oplus [-16] \oplus U \oplus [3, 1, 3, -1, 1, 3](-1)$ |
| 7         | 105 | $-256$      | $U \oplus [1, 1, -1, 0, 2, 4, 0, 2, 0, 4, 0, 2, 2, 6](-1)$ |
| 7         | 106 | $-256$      | $[-4] \oplus U \oplus [3, 1, 3, -1, 1, -1, 1, -1, 1, 3](-1)$ |
| 7         | 107 | $-1024$     | $U \oplus [4, 2, -1, -2, 1, 7, 0, 2, 2, 4, 0, 2, 2, 0, 4](-1)$ |

8 1 1 4[-1] $\oplus 2U$
8 2 2 3[-1] $\oplus [-2] \oplus 2U$
8 3 3 3[-1] $\oplus [-3] \oplus 2U$
8 4 3 2[-1] $\oplus 2U \oplus A_2$
8 5 4 2[-1] $\oplus 2[-2] \oplus 2U$
8 6 4 4[-1] $\oplus 2U \oplus A_3$
8 7 4 3[-1] $\oplus [-4] \oplus 2U$
8 8 5 2[-1] $\oplus 2U \oplus [2, 1, 3](-1)$
8 9 5 3[-1] $\oplus [-5] \oplus 2U$
8 10 6 2[-1] $\oplus [-2] \oplus [-3] \oplus 2U$
8 11 6 2[-1] $\oplus [-2] \oplus 2U \oplus A_2$
8 12 7 2[-1] $\oplus 2U \oplus [2, 1, 2, 1, 0, 3](-1)$
8 13 7 2[-1] $\oplus 2U \oplus [2, 1, 4](-1)$
8 14 8 2[-1] $\oplus [-2] \oplus [-4] \oplus 2U$
8 15 8 2U $\oplus [2, 1, 2, 1, 2, 1, 0, 3](-1)$
8 16 8 2U $\oplus [2, 1, 3, 0, 1, 2](-1)$
8 17 8 2[-1] $\oplus 2U \oplus [3, 1, 3](-1)$
8 18 8 2[-1] $\oplus 3[-2] \oplus 2U$
8 19 9 2[-1] $\oplus [-3] \oplus 2U \oplus A_2$
8 20 9 3[-1] $\oplus [-9] \oplus 2U$
8 21 9 2[-1] $\oplus 2[-3] \oplus 2U$
8 22 10 2[-1] $\oplus [-2] \oplus 2U \oplus [2, 1, 3](-1)$

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| $\lambda$ | No. | $\det T'$ | $T'$ |
|----------|-----|----------|------|
| 8        | 23  | 10       | $2[-1] \oplus [-2] \oplus [-3] \oplus 2U$ |
| 8        | 24  | 11       | $2U \oplus [2, 1, 2, 1, 0, 3, 0, 0, 1, 2](-1)$ |
| 8        | 25  | 12       | $2U \oplus [2, 1, 3, 1, 2, 3, 0, 1, 1, 2](-1)$ |
| 8        | 26  | 12       | $2[-1] \oplus [-2] \oplus [-6] \oplus 2U$ |
| 8        | 27  | 12       | $[-1] \oplus [2, -2] \oplus [-3] \oplus 2U$ |
| 8        | 28  | 12       | $2[-1] \oplus [-3] \oplus [-4] \oplus 2U$ |
| 8        | 29  | 12       | $[-3] \oplus 2U \oplus A_3$ |
| 8        | 30  | 12       | $[-1] \oplus 2U \oplus [2, 1, 3, 1, 1, 3](-1)$ |
| 8        | 31  | 13       | $2[-1] \oplus 2U \oplus [2, 1, 2, 1, 7](-1)$ |
| 8        | 32  | 13       | $3[-1] \oplus [-13] \oplus 2U$ |
| 8        | 33  | 14       | $[-2] \oplus 2U \oplus [2, 1, 2, 1, 1, 3](-1)$ |
| 8        | 34  | 15       | $2U \oplus [2, 1, 2, 1, 0, 3, 1, 0, 3, 0](-1)$ |
| 8        | 35  | 15       | $2[-1] \oplus 2U \oplus [4, 1, 4](-1)$ |
| 8        | 36  | 16       | $2U \oplus [2, 1, 3, 1, 1, 3, 0, 1, 1, 2](-1)$ |
| 8        | 37  | 16       | $2[-1] \oplus [-2] \oplus [-8] \oplus 2U$ |
| 8        | 38  | 16       | $[-1] \oplus 2U \oplus [3, 1, 3, 1, -1, 3](-1)$ |
| 8        | 39  | 16       | $2U \oplus [2, 1, 2, 1, 1, 3, 1, 1, 1, 3](-1)$ |
| 8        | 40  | 16       | $2U \oplus [2, 1, 2, 1, 0, 2, 1, 1, 1, 5](-1)$ |
| 8        | 41  | 16       | $2[-1] \oplus [2, -2] \oplus [-4] \oplus 2U$ |
| 8        | 42  | 17       | $[-1] \oplus 2U \oplus [2, 1, 3, 1, 0, 4](-1)$ |
| 8        | 43  | 18       | $[-1] \oplus [-2] \oplus 2U \oplus [2, 1, 5](-1)$ |
| 8        | 44  | 18       | $[-1] \oplus [-6] \oplus 2U \oplus A_3$ |
| 8        | 45  | 19       | $2U \oplus [2, 1, 3, 1, 0, 3, 0, 1, 0, 2](-1)$ |
| 8        | 46  | 20       | $[-1] \oplus 2U \oplus [3, 1, 3, 1, 1, 3](-1)$ |
| 8        | 47  | 20       | $[-1] \oplus 2U \oplus [2, 1, 6, 0, 1, 2](-1)$ |
| 8        | 48  | 20       | $[-1] \oplus [-4] \oplus 2U \oplus [2, 1, 3](-1)$ |
| 8        | 49  | 20       | $[-1] \oplus [-4] \oplus 2U \oplus [2, 1, 3](-1)$ |
| 8        | 50  | 21       | $[-1] \oplus [-3] \oplus 2U \oplus [2, 1, 4](-1)$ |
| 8        | 51  | 21       | $3[-1] \oplus [-21] \oplus 2U$ |
| 8        | 52  | 22       | $2[-1] \oplus [-2] \oplus [-11] \oplus 2U$ |
| 8        | 53  | 24       | $[-2] \oplus 2U \oplus [2, 1, 3, 1, 1, 3](-1)$ |
| 8        | 54  | 24       | $2U \oplus [2, 1, 3, 1, 0, 3, 0, 1, 1, 3](-1)$ |
| 8        | 55  | 24       | $[-1] \oplus [-8] \oplus 2U \oplus A_3$ |
| 8        | 56  | 24       | $[-1] \oplus [-3] \oplus 2U \oplus [3, 1, 3](-1)$ |
| 8        | 57  | 24       | $[-1] \oplus [-2] \oplus 2U \oplus A_2(2)$ |
| 8        | 58  | 25       | $[-1] \oplus [-5] \oplus 2U \oplus [2, 1, 3](-1)$ |
| 8        | 59  | 27       | $2[-3] \oplus 2U \oplus A_2$ |
| 8        | 60  | 27       | $2U \oplus [2, 1, 3, 1, 1, 3, 0, 1, 3](-1)$ |
| 8        | 61  | 27       | $2U \oplus [2, 1, 5](-1) \oplus A_2$ |
| 8        | 62  | 28       | $[-1] \oplus [2, -2] \oplus [-7] \oplus 2U$ |

Continued on next page.
| $\lambda$ | No. | $\det T'$ | $T'$ |
|-------|-----|----------|-----|
| 8     | 63  | 28       | $[-1] \oplus 2U \oplus [2, 1, 3, 0, 1, 6](-1)$ |
| 8     | 64  | 30       | $2 [-1] \oplus [-3] \oplus [-10] \oplus 2U$ |
| 8     | 65  | 32       | $[-4] \oplus 2U \oplus [2, 1, 3, 0, 1, 2](-1)$ |
| 8     | 66  | 32       | $[-1] \oplus [-4] \oplus 2U \oplus [3, 1, 3](-1)$ |
| 8     | 67  | 32       | $2 [-1] \oplus 2U \oplus [3, 1, 3](-2)$ |
| 8     | 68  | 32       | $2 [-1] \oplus [-2] \oplus [-16] \oplus 2U$ |
| 8     | 69  | 32       | $[-1] \oplus [-2] \oplus 2 [-4] \oplus 2U$ |
| 8     | 70  | 32       | $2 [-2] \oplus 2U \oplus [3, 1, 3](-1)$ |
| 8     | 71  | 32       | $2U \oplus [3, 1, 3, -1, 1, 3, -1, 1, 3](-1)$ |
| 8     | 72  | 33       | $2U \oplus [2, 1, 2, 1, 2, 1, 1, 1, 9](-1)$ |
| 8     | 73  | 33       | $[-1] \oplus 2U \oplus [2, 1, 4, 0, 1, 5](-1)$ |
| 8     | 74  | 35       | $2U \oplus [2, 1, 2, 1, 1, 7, 0, 0, 1, 2](-1)$ |
| 8     | 75  | 36       | $[-1] \oplus [-3] \oplus 2U \oplus A_2(2)$ |
| 8     | 76  | 36       | $[-1] \oplus 2U \oplus [2, 1, 10, 0, 1, 2](-1)$ |
| 8     | 77  | 36       | $[-3] \oplus 2U \oplus [2, 1, 4, 0, 1, 2](-1)$ |
| 8     | 78  | 40       | $2U \oplus [2, 1, 3](-1) \oplus [3, 1, 3](-1)$ |
| 8     | 79  | 40       | $[-1] \oplus 2U \oplus [3, 1, 3, -1, 1, 6](-1)$ |
| 8     | 80  | 42       | $[-2] \oplus [-7] \oplus 2U \oplus A_2$ |
| 8     | 81  | 44       | $2U \oplus [2, 1, 3, 0, 1, 2, 0, 1, 0, 6](-1)$ |
| 8     | 82  | 45       | $[-1] \oplus [-5] \oplus 2U \oplus [2, 1, 5](-1)$ |
| 8     | 83  | 48       | $[-3] \oplus 2U \oplus [3, 1, 3, 1, -1, 3](-1)$ |
| 8     | 84  | 48       | $2U \oplus [3, 1, 3, 1, -1, 3, 1, 1, 1, 4](-1)$ |
| 8     | 85  | 48       | $2U \oplus [2, 1, 3, 1, 1, 7, 0, 1, 1, 2](-1)$ |
| 8     | 86  | 48       | $2U \oplus [2, 1, 2, 1, 1, 5, 1, 1, -1, 5](-1)$ |
| 8     | 87  | 48       | $[-1] \oplus [-2] \oplus [11] \oplus 2U$ |
| 8     | 88  | 48       | $2U \oplus [3, 1, 3, 1, 1, 3, 1, 1, 3, -1]$ |
| 8     | 89  | 51       | $2U \oplus [2, 3, 3, 0, 1, 6, 0, 0, 1, 2](-1)$ |
| 8     | 90  | 54       | $[2] \oplus 3 \oplus 3 \oplus 2U$ |
| 8     | 91  | 56       | $2U \oplus [2, 1, 3, 1, 1, 4, 0, 1, 4](-1)$ |
| 8     | 92  | 60       | $[-1] \oplus [-2] \oplus [-15] \oplus 2U$ |
| 8     | 93  | 64       | $[-8] \oplus 2U \oplus [2, 1, 3, 0, 1, 2](-1)$ |
| 8     | 94  | 64       | $2 [-2] \oplus 2U \oplus [4, 2, 5](-1)$ |
| 8     | 95  | 64       | $[-4] \oplus 2U \oplus [3, 1, 3, 1, -1, 3](-1)$ |
| 8     | 96  | 72       | $2U \oplus [2, 1, 6, 0, 1, 3, 0, 1, -1, 3](-1)$ |
| 8     | 97  | 80       | $2U \oplus [2, 1, 7, 0, 1, 2, 0, 2, 0, 4](-1)$ |
| 8     | 98  | 80       | $[-4] \oplus 2U \oplus [3, 1, 3, 1, 3](-1)$ |
| 8     | 99  | 81       | $[-3] \oplus [-9] \oplus 2U \oplus A_2$ |
| 8     | 100 | 81       | $2 [-3] \oplus 2U \oplus [2, 1, 5](-1)$ |
| 8     | 101 | 96       | $[-4] \oplus 2U \oplus [2, 1, 7, 0, 1, 2](-1)$ |
| 8     | 102 | 96       | $2U \oplus [3, 1, 3](-1) \oplus A_2(2)$ |

Continued on next page.
| $\lambda$ | No. | $\det T'$ | $T'$ |
|--------|-----|----------|------|
| 8      | 103 | 108      | $2U \oplus [2, 1, 7, 0, 1, 2, 0, 3, 0, 6](-1)$ |
| 8      | 104 | 108      | $[-2] \oplus [2, -3] \oplus [-6] \oplus 2U$ |
| 8      | 105 | 112      | $[-7] \oplus 2U \oplus [3, 1, 3, -1, 1, 3](-1)$ |
| 8      | 106 | 125      | $[-1] \oplus 3[-5] \oplus 2U$ |
| 8      | 107 | 128      | $2U \oplus [2, 1, 7, -1, 2, 7, 0, 1, 1, 2](-1)$ |
| 8      | 108 | 128      | $[-2] \oplus 2U \oplus [4, 2, 5, 2, 1, 5](-1)$ |
| 8      | 109 | 128      | $[-1] \oplus 2U \oplus [3, 1, 3, 1, -1, 3](-2)$ |
| 8      | 110 | 128      | $2[-4] \oplus 2U \oplus [3, 1, 3](-1)$ |
| 8      | 111 | 135      | $2U \oplus A_2 \oplus [2, 1, 3](-3)$ |
| 8      | 112 | 144      | $[3, -3] \oplus [-4] \oplus 2U \oplus A_2(2)$ |
| 8      | 113 | 162      | $[-6] \oplus [-9] \oplus 2U \oplus A_2$ |
| 8      | 114 | 192      | $[-1] \oplus 2U \oplus [2, 1, 7, 0, 1, 2](-2)$ |
| 8      | 115 | 192      | $[-3] \oplus 3[-4] \oplus 2U$ |
| 8      | 116 | 216      | $2U \oplus A_2 \oplus [3, 1, 3](-3)$ |
| 8      | 117 | 240      | $[-15] \oplus 2U \oplus [3, 1, 3, -1, 1, 3](-1)$ |
| 8      | 118 | 243      | $2U \oplus [2, 1, 5](-1) \oplus A_2(-3)$ |
| 8      | 119 | 256      | $2U \oplus [4, 2, 7, 0, 2, 4, 0, 2, 0, 4](-1)$ |
| 8      | 120 | 432      | $2[-3] \oplus 2U \oplus A_2(-4)$ |
| 8      | 121 | 512      | $2U \oplus [4, 2, 7, 2, 3, 7, 2, 3, -1, 7](-1)$ |
| 8      | 122 | 768      | $2U \oplus [4, 2, 15, 0, 2, 4, 0, 2, 0, 4](-1)$ |

| 9      | 1   | -1      | $5[-1] \oplus 2U$ |
| 9      | 2   | -2      | $4[-1] \oplus [-2] \oplus 2U$ |
| 9      | 3   | -3      | $4[-1] \oplus [-3] \oplus 2U$ |
| 9      | 4   | -3      | $3[-1] \oplus 2U \oplus A_3$ |
| 9      | 5   | -4      | $2[-1] \oplus 2U \oplus A_3$ |
| 9      | 6   | -4      | $3[-1] \oplus 2U \oplus 2U$ |
| 9      | 7   | -5      | $3[-1] \oplus 2U \oplus [2, 1, 3](-1)$ |
| 9      | 8   | -5      | $4[-1] \oplus [-5] \oplus 2U$ |
| 9      | 9   | -6      | $3[-1] \oplus [-2] \oplus 2U$ |
| 9      | 10  | -6      | $4[-1] \oplus [-6] \oplus 2U$ |
| 9      | 11  | -7      | $2[-1] \oplus 2U \oplus [2, 1, 2, 1, 0, 3](-1)$ |
| 9      | 12  | -8      | $[-1] \oplus 2U \oplus [2, 1, 2, 1, 0, 2, 1, 0, 3](-1)$ |
| 9      | 13  | -8      | $2[-1] \oplus 2U \oplus [2, 1, 3, 0, 1, 2](-1)$ |
| 9      | 14  | -8      | $[-1] \oplus [-2] \oplus 2U \oplus A_3$ |
| 9      | 15  | -9      | $4[-1] \oplus [-9] \oplus 2U$ |
| 9      | 16  | -9      | $2U \oplus [2, 1, 2, 1, 2, 1, 1, 1, 2, 1, 0, 0, 3](-1)$ |
| 9      | 17  | -9      | $[-1] \oplus 2U \oplus 2A_3$ |
| 9      | 18  | -10     | $2[-1] \oplus 2U \oplus [2, 1, 2, 1, 0, 4](-1)$ |
| 9      | 19  | -11     | $4[-1] \oplus [-11] \oplus 2U$ |
| 9      | 20  | -11     | $[-1] \oplus 2U \oplus [2, 1, 2, 1, 0, 3, 0, 0, 1, 2](-1)$ |

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Table 5, continued from previous page.

| $\lambda$ | No. | det $T''$ | $T''$ |
|-----------|-----|-----------|-------|
| 9         | 21  | $-12$     | $2U \oplus [2,1,2,1,1,1,1,1,0,0,0,0,1,2](-1)$ |
| 9         | 22  | $-12$     | $2[-1] \oplus 2U \oplus [2,1,3,1,1,3](-1)$ |
| 9         | 23  | $-12$     | $[-1] \oplus 2U \oplus [2,1,3,1,1,3,0,1,1,-1,2](-1)$ |
| 9         | 24  | $-12$     | $2[-1] \oplus 2[-2] \oplus [-3] \oplus 2U$ |
| 9         | 25  | $-14$     | $2[-1] \oplus [-2] \oplus 2U \oplus [2,1,4](-1)$ |
| 9         | 26  | $-14$     | $[-1] \oplus [-2] \oplus 2U \oplus [2,1,2,1,0,3](-1)$ |
| 9         | 27  | $-15$     | $2[-1] \oplus [-5] \oplus 2U \oplus A_2$ |
| 9         | 28  | $-15$     | $2[-1] \oplus [-3] \oplus 2U \oplus [2,1,3](-1)$ |
| 9         | 29  | $-16$     | $2U \oplus [2,1,2,1,1,1,1,1,3,0,0,1,1,-1,2](-1)$ |
| 9         | 30  | $-16$     | $[-2] \oplus 2U \oplus [2,1,2,1,0,2,1,1,1,3](-1)$ |
| 9         | 31  | $-16$     | $[-1] \oplus 2U \oplus [2,1,3,1,1,3,0,1,1,2](-1)$ |
| 9         | 32  | $-17$     | $4[-1] \oplus [-17] \oplus 2U$ |
| 9         | 33  | $-18$     | $3[-1] \oplus [-3] \oplus [-6] \oplus 2U$ |
| 9         | 34  | $-20$     | $2[-1] \oplus 2U \oplus [2,1,6,0,1,2](-1)$ |
| 9         | 35  | $-20$     | $2[-1] \oplus [-4] \oplus 2U \oplus [2,1,3](-1)$ |
| 9         | 36  | $-20$     | $[-1] \oplus 2[-2] \oplus 2U \oplus [2,1,3](-1)$ |
| 9         | 37  | $-21$     | $2[-1] \oplus [-7] \oplus 2U \oplus A_2$ |
| 9         | 38  | $-23$     | $2[-1] \oplus 2U \oplus [2,1,3,0,1,5](-1)$ |
| 9         | 39  | $-24$     | $2U \oplus [2,1,2,1,1,0,3,1,0,1,0,4](-1)$ |
| 9         | 40  | $-24$     | $[-1] \oplus 2U \oplus [2,1,2,1,0,2,1,0,0,7](-1)$ |
| 9         | 41  | $-24$     | $2[-1] \oplus [-3] \oplus 2U \oplus [3,1,3](-1)$ |
| 9         | 42  | $-24$     | $2[-1] \oplus [-2] \oplus [-3] \oplus [-4] \oplus 2U$ |
| 9         | 43  | $-25$     | $3[-1] \oplus 2[-5] \oplus 2U$ |
| 9         | 44  | $-26$     | $2[-1] \oplus 2U \oplus [2,1,4,0,1,4](-1)$ |
| 9         | 45  | $-27$     | $3[-1] \oplus [-3] \oplus [-9] \oplus 2U$ |
| 9         | 46  | $-27$     | $[-1] \oplus 2U \oplus A_2 \oplus [2,1,5](-1)$ |
| 9         | 47  | $-28$     | $[-1] \oplus 2U \oplus [2,1,3,1,1,3,1,0,0,3](-1)$ |
| 9         | 48  | $-29$     | $4[-1] \oplus [-29] \oplus 2U$ |
| 9         | 49  | $-30$     | $2[-1] \oplus [-6] \oplus 2U \oplus [2,1,3](-1)$ |
| 9         | 50  | $-32$     | $[-2] \oplus 2U \oplus [2,1,2,1,0,3,1,0,1,3](-1)$ |
| 9         | 51  | $-32$     | $[-2] \oplus 2U \oplus [2,1,2,1,0,2,1,1,1,5](-1)$ |
| 9         | 52  | $-32$     | $2U \oplus [2,1,3,1,0,3,1,2,1,4,0,1,1,-1,2](-1)$ |
| 9         | 53  | $-35$     | $2[-1] \oplus [-5] \oplus 2U \oplus [2,1,4](-1)$ |
| 9         | 54  | $-36$     | $2U \oplus [2,1,2,1,0,3,1,0,1,4,0,0,1,0,2](-1)$ |
| 9         | 55  | $-36$     | $2[-1] \oplus 2[-3] \oplus [-4] \oplus 2U$ |
| 9         | 56  | $-36$     | $[-1] \oplus 2[-2] \oplus 2[-3] \oplus 2U$ |
| 9         | 57  | $-39$     | $2[-1] \oplus [-13] \oplus 2U \oplus A_2$ |
| 9         | 58  | $-40$     | $2[-1] \oplus 2U \oplus [2,1,2,1,0,14](-1)$ |
| 9         | 59  | $-41$     | $4[-1] \oplus [-41] \oplus 2U$ |
| 9         | 60  | $-44$     | $[-1] \oplus 2U \oplus [2,1,2,1,1,3,1,0,0,7](-1)$ |

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Table 5, continued from previous page.

| $\lambda$ | $\text{No.}$ | $\det T'$ | $T'$ |
|-----------|--------------|-----------|------|
| 9         | 61           | $-44$     | $2[-1] \oplus 2[-2] \oplus [-11] \oplus 2U$ |
| 9         | 62           | $-45$     | $3[-1] \oplus 2U \oplus [2, 1, 3][-3]$ |
| 9         | 63           | $-48$     | $2U \oplus [2, 1, 2, 1, 1, 3, 0, 0, 1, 2, 0, 0, 0, 1, 5][-1]$ |
| 9         | 64           | $-48$     | $[-1] \oplus [-3] \oplus 2U \oplus [3, 1, 3, -1, 1, 3][-1]$ |
| 9         | 65           | $-48$     | $[-1] \oplus [-4] \oplus 2U \oplus [2, 1, 3, 1, 1, 3][-1]$ |
| 9         | 66           | $-50$     | $2U \oplus [2, 1, 3][-1] \oplus [2, 1, 2, 1, 0, 4][-1]$ |
| 9         | 67           | $-54$     | $[-1] \oplus [-2] \oplus [-3] \oplus 2U \oplus [2, 1, 5][-1]$ |
| 9         | 68           | $-54$     | $3[-1] \oplus [-6] \oplus [-9] \oplus 2U$ |
| 9         | 69           | $-56$     | $2U \oplus [2, 1, 3, 1, 1, 3, 1, 0, 0, 3, 0, -1, 1, 0, 3][-1]$ |
| 9         | 70           | $-56$     | $2U \oplus [2, 1, 4][-1] \oplus [2, 1, 3, 0, 1, 2][-1]$ |
| 9         | 71           | $-60$     | $[-1] \oplus 2U \oplus [2, 1, 3][-1] \oplus A_2(2)$ |
| 9         | 72           | $-63$     | $[-7] \oplus 2U \oplus 2A_2$ |
| 9         | 73           | $-64$     | $2U \oplus [2, 1, 3, 1, 1, 4, 1, 1, 2, 4, 0, 1, 1, 1, 2][-1]$ |
| 9         | 74           | $-65$     | $3[-1] \oplus 2U \oplus [2, 1, 33][-1]$ |
| 9         | 75           | $-72$     | $2U \oplus [2, 1, 2, 1, 1, 3, 1, 0, 1, 3, 1, 1, 0, 1, 6][-1]$ |
| 9         | 76           | $-75$     | $[-5] \oplus 2U \oplus [2, 1, 3][-1] \oplus A_2$ |
| 9         | 77           | $-80$     | $[-2] \oplus 2U \oplus [2, 1, 4, 0, 1, 3, 0, -1, 1, 3][-1]$ |
| 9         | 78           | $-80$     | $[-1] \oplus 2U \oplus [2, 1, 3][-1]$ |
| 9         | 79           | $-81$     | $[-9] \oplus 2U \oplus [2, 1, 2, 1, 1, 2, 1, 1, 1, 3][-1]$ |
| 9         | 80           | $-84$     | $2U \oplus [2, 1, 3, 1, 1, 8, 0, 1, 0, 2, 0, 1, 1, 0, 2][-1]$ |
| 9         | 81           | $-90$     | $[-1] \oplus [-2] \oplus [-3] \oplus 2U \oplus [2, 1, 8][-1]$ |
| 9         | 82           | $-96$     | $[-1] \oplus [-3] \oplus 2U \oplus [3, 1, 3, 1, -1, 5][-1]$ |
| 9         | 83           | $-96$     | $[-2] \oplus 2U \oplus [2, 1, 4, 0, 1, 3, 0, 1, 1, 3][-1]$ |
| 9         | 84           | $-96$     | $2U \oplus [3, 1, 3][-1] \oplus [2, 1, 4, 0, 1, 2][-1]$ |
| 9         | 85           | $-104$    | $2U \oplus [2, 1, 4, 0, 1, 5, 0, 0, 1, 2, 0, 0, 0, 1, 0, 2][-1]$ |
| 9         | 86           | $-108$    | $2U \oplus [2, 1, 2, 1, 1, 2, 1, 1, 0, 7, 1, 1, 0, -2, 7][-1]$ |
| 9         | 87           | $-108$    | $2[-1] \oplus [-3] \oplus 2U \oplus [2, 1, 5][-2]$ |
| 9         | 88           | $-117$    | $[-1] \oplus 2U \oplus [2, 1, 20][-1] \oplus A_2$ |
| 9         | 89           | $-120$    | $[-1] \oplus 2U \oplus [2, 1, 3][-1] \oplus [4, 2, 7][-1]$ |
| 9         | 90           | $-125$    | $[-1] \oplus [-5] \oplus 2U \oplus [2, 1, 5, 0, 1, 3][-1]$ |
| 9         | 91           | $-128$    | $2U \oplus [2, 1, 3, 1, 1, 3, 1, 1, 4, 1, 0, 0, 1, 4][-1]$ |
| 9         | 92           | $-135$    | $2U \oplus A_2 \oplus [2, 1, 5, 1, 2, 6][-1]$ |
| 9         | 93           | $-140$    | $2[-2] \oplus [-5] \oplus 2U \oplus [2, 1, 4][-1]$ |
| 9         | 94           | $-144$    | $[-1] \oplus [-6] \oplus [-8] \oplus 2U \oplus A_2$ |
| 9         | 95           | $-144$    | $[-1] \oplus 2[-3] \oplus 2[-4] \oplus 2U$ |
| 9         | 96           | $-176$    | $[-1] \oplus [-11] \oplus 2U \oplus [3, 1, 3, -1, 1, 3][-1]$ |
| 9         | 97           | $-189$    | $[-7] \oplus 2U \oplus [2, 1, 5][-1] \oplus A_2$ |
| 9         | 98           | $-192$    | $2U \oplus [2, 1, 3, 1, 1, 4, -1, 0, 1, 4, 0, 0, 1, -1, 5][-1]$ |
| 9         | 99           | $-200$    | $2U \oplus [2, 1, 2, 1, 1, 7, 1, 1, 2, 7, 0, 0, 0, 1, 1, 2][-1]$ |
| 9         | 100         | $-216$    | $2U \oplus [2, 1, 6, 0, 1, 3, 0, 1, -1, 3, 0, 1, 1, 1, 4][-1]$ |

Continued on next page.
| $\lambda$ | No. | $\det T''$ | $T''$ |
|-------|-----|-------------|-------|
| 9     | 101 | $-216$     | $2U \oplus [2, 1, 2, 1, 0, 3, 1, 0, 0, 6, 1, 0, 1, -2, 8](-1)$ |
| 9     | 102 | $-300$     | $[-2] \oplus [-5] \oplus [-6] \oplus 2U \oplus [2, 1, 3](-1)$ |
| 9     | 103 | $-320$     | $2U \oplus [2, 1, 4, 1, 2, 4, 0, -1, 1, 5, 0, -1, 1, 1, 5](-1)$ |
| 9     | 104 | $-360$     | $[-3] \oplus 2U \oplus [2, 1, 8](-1) \oplus [3, 1, 3](-1)$ |
| 9     | 105 | $-384$     | $2U \oplus [3, 1, 3, -1, 1, 6, 0, 0, 1, 4, 0, 0, 2, 2, 4](-1)$ |
| 9     | 106 | $-432$     | $[-3] \oplus 2U \oplus [3, 1, 4, 1, 0, 4, 1, 0, 0, 4](-1)$ |
| 9     | 107 | $-441$     | $[-21] \oplus 2U \oplus [2, 1, 2, 1, 2, 1, 0, 0, 6](-1)$ |
| 9     | 108 | $-560$     | $2U \oplus [2, 1, 4](-1) \oplus [4, 2, 7, 0, 2, 4](-1)$ |
| 9     | 109 | $-576$     | $[-1] \oplus 2U \oplus [6, 3, 7, -3, 1, 7, 3, 0, 0, 7](-1)$ |
| 9     | 110 | $-1200$    | $2U \oplus [3, 1, 4, 1, 0, 4, 1, 2, 2, 7, 1, 2, 2, 8](-1)$ |

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| $\lambda$ | No. | det $T'$ | $T'$   |
|---------|-----|---------|--------|
| 10      | 31  | 70      | $[1] \oplus [70] \oplus E_8$ |
| 10      | 32  | 72      | $[1] \oplus [72] \oplus E_8$ |
| 10      | 33  | 78      | $[1] \oplus [78] \oplus E_8$ |
| 10      | 34  | 85      | $[1] \oplus [85] \oplus E_8$ |
| 10      | 35  | 88      | $[1] \oplus [88] \oplus E_8$ |
| 10      | 36  | 93      | $[1] \oplus [93] \oplus E_8$ |
| 10      | 37  | 102     | $[1] \oplus [102] \oplus E_8$ |
| 10      | 38  | 105     | $[1] \oplus [105] \oplus E_8$ |
| 10      | 39  | 112     | $[1] \oplus [112] \oplus E_8$ |
| 10      | 40  | 120     | $[1] \oplus [120] \oplus E_8$ |
| 10      | 41  | 130     | $[1] \oplus [130] \oplus E_8$ |
| 10      | 42  | 133     | $[1] \oplus [133] \oplus E_8$ |
| 10      | 43  | 165     | $[1] \oplus [165] \oplus E_8$ |
| 10      | 44  | 168     | $[1] \oplus [168] \oplus E_8$ |
| 10      | 45  | 177     | $[1] \oplus [177] \oplus E_8$ |
| 10      | 46  | 190     | $[1] \oplus [190] \oplus E_8$ |
| 10      | 47  | 210     | $[1] \oplus [210] \oplus E_8$ |
| 10      | 48  | 232     | $[1] \oplus [232] \oplus E_8$ |
| 10      | 49  | 240     | $[1] \oplus [240] \oplus E_8$ |
| 10      | 50  | 253     | $[1] \oplus [253] \oplus E_8$ |
| 10      | 51  | 273     | $[1] \oplus [273] \oplus E_8$ |
| 10      | 52  | 280     | $[1] \oplus [280] \oplus E_8$ |
| 10      | 53  | 312     | $[1] \oplus [312] \oplus E_8$ |
| 10      | 54  | 330     | $[1] \oplus [330] \oplus E_8$ |
| 10      | 55  | 345     | $[1] \oplus [345] \oplus E_8$ |
| 10      | 56  | 357     | $[1] \oplus [357] \oplus E_8$ |
| 10      | 57  | 385     | $[1] \oplus [385] \oplus E_8$ |
| 10      | 58  | 408     | $[1] \oplus [408] \oplus E_8$ |
| 10      | 59  | 462     | $[1] \oplus [462] \oplus E_8$ |
| 10      | 60  | 520     | $[1] \oplus [520] \oplus E_8$ |
| 10      | 61  | 760     | $[1] \oplus [760] \oplus E_8$ |
| 10      | 62  | 840     | $[1] \oplus [840] \oplus E_8$ |
| 10      | 63  | 1320    | $[1] \oplus [1320] \oplus E_8$ |
| 10      | 64  | 1365    | $[1] \oplus [1365] \oplus E_8$ |
| 10      | 65  | 1848    | $[1] \oplus [1848] \oplus E_8$ |

| 11      | 1   | $-1$    | $[1] \oplus U \oplus E_8$ |
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