Non-uniqueness of weak solutions to hyperviscous Navier–Stokes equations: on sharpness of J.-L. Lions exponent

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Abstract
Using the convex integration technique for the three-dimensional Navier–Stokes equations introduced by Buckmaster and Vicol, it is shown the existence of non-unique weak solutions for the 3D Navier–Stokes equations with fractional hyperviscosity \((-\Delta)^{\theta}\), whenever the exponent \(\theta\) is less than Lions’ exponent 5/4, i.e., when \(\theta < 5/4\).

Mathematics Subject Classification 35Q30

1 Introduction

In this paper we consider the question of non-uniqueness of weak solutions to the 3D Navier–Stokes equations with fractional viscosity (FVNSE) on \(\mathbb{T}^3\)
\[
\begin{cases}
\partial_t v + \nabla \cdot (v \otimes v) + \nabla p + v(-\Delta)^\theta v = 0, \\
\nabla \cdot v = 0,
\end{cases}
\]
(1)
where \(\theta \in \mathbb{R}\) is a fixed constant, and for \(u \in C^\infty(\mathbb{T}^3)\) with \(\int_{\mathbb{T}^3} u(x)dx = 0\), the fractional Laplacian is defined via the Fourier transform as
\[
\mathcal{F}((-\Delta)^{\theta} u)(\xi) = |\xi|^{2\theta} \mathcal{F}(u)(\xi), \quad \xi \in \mathbb{Z}^3.
\]
**Definition** (weak solutions) A vector field \( v \in C^0_{weak}(\mathbb{R}; L^2(\mathbb{T}^3)) \) is called a weak solution to the FVNSE if it solves (1) in the sense of distribution.

When \( \theta = 1 \), FVNSE (1) is the standard Navier–Stokes equations. Lions first considered FVNSE (1) in [20], and showed the existence and uniqueness of weak solutions to the initial value problem, which also satisfied the energy equality, for \( \theta \in [5/4, \infty) \) in [21]. Moreover, an analogue of the Caffarelli–Kohn–Nirenberg [6] result was established in [18] for the FVNSE system (1), showing that the Hausdorff dimension of the singular set, in space and time, is bounded by \( 5 - 4\theta \) for \( \theta \in (1, 5/4) \). The existence, uniqueness, regularity and stability of solutions to the FVNSE have been studied in [17, 26, 28, 29] and references therein. Very recently, using the method of convex integration introduced in [12], Colombo et al. [8] showed the non-uniqueness of Leray weak solutions to FVNSE (1) for \( \theta \in (0, 1/5) \) and for \( \theta \in (0, 1/3) \) in [13].

In the recent breakthrough work [5], Buckmaster and Vicol obtained non-uniqueness of weak solutions to the three-dimensional Navier–Stokes equations. They developed a new convex integration scheme in Sobolev spaces using intermittent Beltrami flows which combined concentrations and oscillations. Later, the idea of using intermittent flows was used to study non-uniqueness for transport equations in [23–25] employing scaled Mikado waves, and for stationary Navier–Stokes equations in [7, 22] employing viscous eddies.

The schemes in [5, 24] are based on the convex integration framework in Hölder spaces for the Euler equations, introduced by De Lellis and Székelyhidi [12], subsequently refined in [2, 3, 10, 15], and culminated in the proof of the second half of the Onsager conjecture by Isett in [16]; also see [4] for a shorter proof. For the first half of the Onsager conjecture, see, e.g., [1, 9], and the references therein.

The main contribution of this note is to show that the results in Buckmaster–Vicol’s paper hold for FVNSE (1) for \( \theta < 5/4 \):

**Theorem 1** Assume that \( \theta \in [1, 5/4) \). Suppose \( u \) is a smooth divergence-free vector field, define on \( \mathbb{R}_+ \times \mathbb{T}^3 \), with compact support in time and satisfies the condition

\[
\int_{\mathbb{T}^3} u(t, x) dx \equiv 0.
\]

Then for any given \( \varepsilon_0 > 0 \), there exists a weak solution \( v \) to the FVNSE (1), with compact support in time, satisfying

\[
\|v - u\|_{L^\infty_t W^{2\theta-1, 1}_x} < \varepsilon_0.
\]

As a consequence there are infinitely many weak solutions of the FVNSE (1) which are compactly supported in time; in particular, there are infinitely many weak solutions with initial values zero.

**Remark 1** In the above theorem we assume that \( \theta \in [1, 5/4) \). However, using the constructions in [5] with a slightly different choice of parameters, one can actually show that Theorem 1.2 and Theorem 1.3 in [5] hold for the 3D FVNSE, i.e., there exist non-unique weak solutions \( v \in C^0_t W^{\beta, 2}_x \), with a different \( \beta > 0 \), depending on \( \theta \). However, in this paper we choose to prove a weaker result, Theorem 1, in order to simplify the presentation while retaining the main idea.

**Remark 2** For the case \( \theta \in (-\infty, 1) \), the same construction also yields weak solutions \( v \in C^0_t L^2_x \cap C^0_t W^{1, 1}_x \) with a suitable choice of parameters.
We now make some comments on the analysis in this paper. Using the technique in [5], we adapt a convex integration scheme with intermittent Beltrami flows as the building blocks. The main difficulty in a convex integration scheme for (FVNSE), is the error induced by the frictional viscosity $\nu(\frac{-\Delta}{\Delta^1})^\theta v$, which is greater for a larger exponent $\theta$. This error is controlled by making full use of the concentration effect of intermittent flows introduced in [5]. As it is shown in the crucial estimate (36), the error is controllable only for $\theta < \frac{5}{4}$. Compared with [5], since our goal is to construct weak solutions $v \in C^0_t L^2_x, weak \cap L^\infty_t W^{2\theta-1,1}_x$, we adapt a slightly simpler cut-off function and prove only estimates that are sufficient for this purpose.

2 Outline

2.1 Iteration lemma

Following [5], we consider the approximate system

$$\begin{cases}
\partial_t v + \nabla \cdot (v \otimes v) + \nabla p + (\Delta)^\theta v = \nabla \cdot R,

\nabla \cdot v = 0,
\end{cases}$$

(2)

where $R$ is a symmetric $3 \times 3$ matrix.

Lemma 1 (Iteration Lemma for $L^2$ weak solutions) Let $\theta \in (-\infty, 5/4)$. Assume $(v_q, R_q)$ is a smooth solution to (2) with

$$\|R_q\|_{L^\infty_t L^1_x} \leq \delta_{q+1},$$

(3)

for some $\delta_{q+1} > 0$. Then for any given $\delta_{q+2} > 0$, there exists a smooth solution $(v_{q+1}, R_{q+1})$ of (2) with

$$\|R_{q+1}\|_{L^\infty_t L^1_x} \leq \delta_{q+2},$$

(4)

and $\text{supp}_t v_{q+1} \cup \text{supp}_t R_{q+1} \subset \text{supp}_t v_q \cup \text{supp}_t R_q$.

(5)

Here for a given set $A \subset \mathbb{R}$, the $\delta$-neighborhood of $A$ is denoted by

$$N_\delta(A) = \{y \in \mathbb{R} : \exists y' \in A, |y - y'| < \delta\}.$$

Furthermore, the increment $w_{q+1} = v_{q+1} - v_q$ satisfies the estimates

$$\|w_{q+1}\|_{L^\infty_t L^2_x} \leq C\delta_{q+1}^{1/2},$$

(6)

$$\|w_{q+1}\|_{L^\infty_t W^{2\theta-1,1}_x} \leq \delta_{q+2},$$

(7)

where the positive constant $C$ depends only on $\theta$.

Proof of Theorem 1 Assume Lemma 1 is valid. Let $v_0 = u$. Then

$$\int_{T^3} \partial_t v_0(t, x)dx = \frac{d}{dt} \int_{T^3} v_0(t, x)dx \equiv 0.$$

Let

$$R_0 = \mathcal{R}(\partial_t v_0 + (\Delta)^\theta v_0) + v_0 \otimes v_0 + p_0 I, \quad p_0 = -\frac{1}{3}|v_0|^2,$$

where $\mathcal{R}$ is the symmetric anti-divergence operator established in Lemma 5, below. Clearly $(v_0, R_0)$ solves (2). Set
\[ \delta_1 = \| R_0 \|_{L_t^\infty L_x^1}, \]
\[ \delta_{q+1} = 2^{-q} \varepsilon_0, \quad \text{for } q \geq 1. \]

Apply Lemma 1 iteratively to obtain smooth solution \((v_q, R_q)\) to (2). It follows from (6) that
\[ \sum \| v_{q+1} - v_q \|_{L_t^\infty L_x^2} = \sum \| w_{q+1} \|_{L_t^\infty L_x^2} \leq C \sum \delta_{q+1}^{1/2} < \infty. \]

Thus \(v_q\) converge strongly to some \(v \in C_t^0 L_x^2\). Since \(\| R_{q+1} \|_{L_t^\infty L_x^1} \to 0\), as \(q \to \infty\), \(v\) is a weak solution to the FVNSE (1). Estimate (7) leads to
\[ \| v - v_0 \|_{L_t^\infty W^{2q-1,1}} \leq \sum_{q=1}^\infty \| w_q \|_{L_t^\infty W^{2q-1,1}} \leq \sum_{q=1}^\infty \delta_{q+1} \leq \varepsilon_0. \]

Furthermore, it follows from (5) that
\[ \text{supp}_t v \subset \bigcup_{q \geq 0} \text{supp}_q v \subset N \sum_{q \geq 0} \delta_{q+1} (\text{supp}_t u) \subset N \delta_{1+\varepsilon_0} (\text{supp}_t u). \]

Now we show the existence of infinitely many weak solutions with initial values zero. Let \(u(t, x) = \varphi(t) \sum_{|k| \leq N} a_k e^{i k \cdot x}\) with \(a_k \neq 0, a_k \cdot k = 0, a_{-k} = a_k^*\) for all \(|k| \leq N\), and \(\varphi \in C_c^\infty(\mathbb{R}_+).\) Thus \(\nabla \cdot u = 0\) satisfies the conditions of the theorem. Hence there exists a weak solution \(v\) to (1) close enough to \(u\) so that \(v \neq 0.\)

\[\square\]

3 Iteration scheme

3.1 Notations and parameters

For a complex number \(\zeta \in \mathbb{C}\), we denote by \(\zeta^*\) its complex conjugate. Let us normalize the volume
\[ |\mathbb{T}^3| = 1. \]

For smooth functions \(u \in C_c^\infty(\mathbb{T}^3)\) with \(\int_{\mathbb{T}^3} u(x) dx = 0\) and \(s \in \mathbb{R}\), we define
\[ \mathcal{F}(\nabla^s u)(\xi) = |\xi|^s \mathcal{F}(u)(\xi), \quad \xi \in \mathbb{Z}^3. \]

For \(M, N \in [0, +\infty]\), denote the Fourier projection of \(u\) by
\[ \mathcal{F}(\mathbb{P}_{(M,N)} u) = \begin{cases} u(\xi), & M \leq |\xi| < N, \xi \in \mathbb{Z}^3, \\ 0, & \text{otherwise.} \end{cases} \]

We also denote \(\mathbb{P}_{\leq k} = \mathbb{P}_{[0,k)}\) and \(\mathbb{P}_{\geq k} = \mathbb{P}_{[k, +\infty)}\) for \(k > 0.\)

Following the notation in [5], we introduce here several parameters \(\sigma, r, \lambda,\) with
\[ 0 < \sigma < 1 < r < \lambda < \mu < \lambda^2, \quad \sigma r < 1, \quad (8) \]

where \(\lambda = \lambda_{q+1} \in 5\mathbb{N}\) is the ‘frequency’ parameter; \(\sigma\) with \(1/\sigma \in \mathbb{N}\) is a small parameter such that \(\lambda \sigma \in \mathbb{N}\) parameterizes the spacing between frequencies; \(r \in \mathbb{N}\) denotes the number of frequencies along edges of a cube; \(\mu\) measures the amount of temporal oscillation.

Later \(\sigma, r, \mu\) will be chosen to be suitable powers of \(\lambda_{q+1}.\) We also fix a constant \(p > 1\) which will be chosen later to be close to 1. The constants implicitly in the notation ‘\(\lesssim\)’ may depend on \(p\) but are independent of the parameters \(\sigma, r, \lambda.\)
3.2 Intermittent Beltrami flows

We use intermittent Beltrami flows introduced in [5] as the building blocks. Recall some basic facts of Beltrami waves.

**Proposition 1** [5, Proposition 3.1] Given \( \xi \in S^2 \cap Q^3 \), let \( A_\xi \in S^2 \cap Q^3 \) be such that

\[
A_\xi : \xi = 0, \quad |A_\xi| = 1, \quad A_{-\xi} = A_\xi.
\]

Let \( \Lambda \) be a given finite subset of \( S^2 \) such that \( -\Lambda = \Lambda \), and \( \lambda \in \mathbb{Z} \) be such that \( \lambda \Lambda \subset \mathbb{Z}^3 \). Then for any choice of coefficients \( a_\xi \in \mathbb{C} \) with \( a_\xi^* = a_{-\xi} \) the vector field

\[
W(x) = \sum_{\xi \in \Lambda} a_\xi B_\xi e^{i\lambda \cdot \xi} \cdot x, \quad \text{with} \quad B_\xi = \frac{1}{\sqrt{2}} \left( A_\xi + i\xi \times A_\xi \right),
\]

is real-valued, divergence-free and satisfies

\[
\nabla \times W = \lambda W, \quad \nabla \cdot (W \otimes W) = \nabla |W|^2 / 2.
\]

Furthermore,

\[
(W \otimes W) := \int_{T^3} W \otimes W dx = \sum_{\xi \in \Lambda} \frac{1}{2} |a_\xi|^2 (\text{Id} - \xi \otimes \xi).
\]

Let \( \Lambda, \Lambda^+, \Lambda^- \subset S^2 \cap Q^3 \) be defined by

\[
\Lambda^+ = \left\{ \frac{1}{5} (3e_1 \pm 4e_2), \frac{1}{5} (3e_2 \pm 4e_3), \frac{1}{5} (3e_3 \pm 4e_1) \right\},
\]

\[
\Lambda^- = -\Lambda^+, \quad \Lambda = \Lambda^+ \cup \Lambda^-.
\]

Clearly we have

\[
5\Lambda \in \mathbb{Z}^3, \quad \text{and} \quad \min_{\xi, \xi' \in \Lambda, \xi + \xi' \neq 0} |\xi' + \xi| \geq \frac{1}{5}.
\] (9)

Also it is direct to check that

\[
\frac{1}{8} \sum_{\xi \in \Lambda} (\text{Id} - \xi \otimes \xi) = \text{Id}.
\]

In fact, representations of this form exist for symmetric matrices close to the identity. We have the following simple variant of [5, Proposition 3.2].

**Proposition 2** Let \( B_\varepsilon (\text{Id}) \) denote the ball of symmetric matrices, centered at the identity, of radius \( \varepsilon \). Then there exist a constant \( \varepsilon_\gamma > 0 \) and smooth positive functions \( \gamma_{(\xi)} \in C^\infty (B_{\varepsilon_\gamma} (\text{Id})) \), such that

1. \( \gamma_{(\xi)} = \gamma_{(-\xi)} \);
2. for each \( R \in B_{\varepsilon_\gamma} (\text{Id}) \) we have the identity

\[
R = \frac{1}{2} \sum_{\xi \in \Lambda} \left( \gamma_{(\xi)} (R) \right)^2 (\text{Id} - \xi \otimes \xi).
\]
Define the Dirichlet kernel
\[ D_r(x) = \frac{1}{(2r + 1)^{3/2}} \sum_{\xi \in \Omega_r} e^{i\xi \cdot x}, \quad \Omega_r = \{(j, k, l) : j, k, l \in \{-r, \ldots, r\}\}. \]
It has the property that, for \(1 < p \leq \infty\),
\[ \|D_r\|_{L^p} \lesssim r^{3/2 - 3/p}, \quad \|D_r\|_{L^2} = (2\pi)^3. \]
Following [5], for \(\xi \in \Lambda^+\), define a directed and rescaled Dirichlet kernel by
\[ \eta(\xi)(t, x) = \eta(\lambda, \sigma, r, \mu)(t, x) = D_r(\lambda \sigma (\xi \cdot x + \mu t), A_\xi \cdot x, (\xi \times A_\xi) \cdot x)), \quad \xi \in \Lambda^+. \]
and for \(\xi \in \Lambda^-\), define
\[ \eta(\xi)(t, x) = \eta(-\xi)(t, x). \]
Note the important identity
\[ \frac{1}{\mu} \partial_t \eta(\xi)(t, x) = \pm (\xi \cdot \nabla) \eta(\xi)(t, x), \quad \xi \in \Lambda^\pm. \]
Since the map \(x \mapsto \lambda \sigma (\xi \cdot x + \mu t), A_\xi \cdot x, (\xi \times A_\xi) \cdot x)\) is the composition of a rotation by a rational orthogonal matrix mapping \(\{e_1, e_2, e_3\} \to \{\xi, A_\xi, -\xi \times A_\xi\}\), a translation, and a rescaling by integers, for \(1 < p \leq \infty\), we have
\[ \int_{\mathbb{R}^3} \eta(\xi)(t, x)^2(t, x) dx = 1, \quad \|\eta(\xi)\|_{L^\infty_t L^p_\xi(\mathbb{R}^3)} \lesssim r^{3/2 - 3/p}. \]
Let \(W(\xi)\) be the Beltrami plane wave at frequency \(\lambda\),
\[ W(\xi) = W(\xi, \lambda)(x) = B_\xi e^{i\lambda \xi \cdot x}. \]
Define the intermittent Beltrami wave \(\mathbb{W}(\xi)\) as
\[ \mathbb{W}(\xi)(t, x) := \mathbb{W}(\xi, \lambda, r, \mu)(t, x) = \eta(\xi)(t, x)W(\xi)(x). \]
It follows from the definitions and (9) that
\[ \mathbb{P}_L^{(\frac{1}{2}, 2\lambda)} \mathbb{W}(\xi) = \mathbb{W}(\xi), \quad \mathbb{P}_L^{(\frac{1}{2}, 4\lambda)} \left(\mathbb{W}(\xi) \otimes \mathbb{W}(\xi')\right) = \mathbb{W}(\xi) \otimes \mathbb{W}(\xi'), \quad \xi' \neq -\xi. \]
The following properties are immediate from the definitions.

**Proposition 3** [5, Proposition 3.4] Let \(a_\xi \in \mathbb{C}\) be constants with \(a_\xi^* = a_{-\xi}\). Let
\[ W(x) = \sum_{\xi \in \Lambda} a_\xi \mathbb{W}(\xi)(x). \]
Then \(W(x)\) is real valued. Moreover, for each \(R \in B_{\mathbb{R}^3}(\text{Id})\) we have
\[ \sum_{\xi \in \Lambda} \left(\gamma(\xi)(R)\right)^2 \int_{\mathbb{R}^3} \mathbb{W}(\xi) \otimes \mathbb{W}(-\xi) = \sum_{\xi \in \Lambda} \left(\gamma(\xi)(R)\right)^2 B_\xi \otimes B_{-\xi} = R. \]

**Proposition 4** [5, Proposition 3.5] For any \(1 < p \leq \infty, N \geq 0, K \geq 0\):
\[ \left\| \nabla^N \partial_t^K \mathbb{W}(\xi) \right\|_{L^\infty_t L^p_\xi} \lesssim \lambda^N (\lambda \sigma r \mu)^K r^{3/2 - 3/p}, \quad (15) \]
\[ \left\| \nabla^N \partial_t^K \eta(\xi) \right\|_{L^\infty_t L^p_\xi} \lesssim (\lambda \sigma r)^N (\lambda \sigma r \mu)^K r^{3/2 - 3/p}. \quad (16) \]
3.3 Perturbations

Let \( \psi(t) \) be a smooth cut-off function such that
\[
\psi(t) = 1 \text{ on } \text{supp}_t R_q, \quad \text{supp} \psi(t) \subset N_{\delta_{q+1}}(\text{supp}_t R_q), \quad |\psi'(t)| \leq 2\delta_{q+1}^{-1}. \tag{17}
\]
Take a smooth increasing function \( \chi \) such that
\[
\chi(s) = \begin{cases} 
1, & 0 \leq s < 1 \\
1 - s, & s \geq 2
\end{cases},
\]
and set
\[
\rho(t, x) = \varepsilon^{-1}_q \delta_{q+1} \chi \left( \delta_{q+1}^{-1} |R_q(t, x)| \right) \psi^2(t).
\]
where \( \varepsilon_q \) is the constant in Proposition 2. Then clearly
\[
\text{supp}_t \rho \subset N_{\delta_{q+1}}(\text{supp}_t R_q). \tag{18}
\]
It follows from the above definition that
\[
|R_q|/\rho = \varepsilon_q \frac{|R_q|}{\delta_{q+1} \chi \left( \delta_{q+1}^{-1} |R_q(t, x)| \right) \psi^2} \leq \varepsilon_q \implies \text{Id} - R_q/\rho \in B_{\varepsilon_q}(\text{Id}) \text{ on supp } R_q.
\]
Therefore, the amplitude functions
\[
a_{(\xi)}(t, x) := \rho^{1/2}(t, x) \gamma_{(\xi)}(\text{Id} - \rho(t, x)^{-1} R_q(t, x))
\]
are well-defined and smooth. Define the velocity perturbation to be \( w = w_{q+1} \):
\[
w = w^{(p)} + w^{(c)} + w^{(t)},
\]
\[
w^{(p)} = \sum_{\xi \in \Lambda} a_{(\xi)} W_{(\xi)} = \sum_{\xi \in \Lambda} a_{(\xi)}(t, x) \eta_{(\xi)}(t, x) B_\xi e^{i\lambda \xi \cdot x},
\]
\[
w^{(c)} = \frac{1}{\lambda_{q+1}} \sum_{\xi \in \Lambda} \nabla \left( a_{(\xi)} \eta_{(\xi)} \right) \times W_{(\xi)},
\]
\[
w^{(t)} = \frac{1}{\mu} \sum_{\xi \in \Lambda^*} P_{LH} \neq 0 \left( a_{(\xi)}^2 \eta_{(\xi)}^2 \xi \right),
\]
where \( P_{LH} = \text{Id} - \nabla \Delta^{-1} \text{div} \) is the Leray-Helmholtz projection into divergence-free vector field, and \( P_{LH} \neq 0 f = f - \int_{\mathbb{T}^3} f dx \). It is well-known that \( P_{LH} \) is bounded on \( L^p, 1 < p < \infty \) (see, e.g., [14]). It follows from Proposition 3 that
\[
\sum_{\xi \in \Lambda} a_{(\xi)}^2 \int_{\mathbb{T}^3} W_{(\xi)} \otimes W_{(-\xi)} dx = \rho \text{Id} - R_q. \tag{19}
\]

3.4 Estimates for perturbations

Lemma 2 The following bounds hold:
\[
\|\rho\|_{L^\infty_t L^1_x} \leq C \delta_{q+1}, \tag{20}
\]
\[
\|\rho^{-1}\|_{C^0(\text{supp } R_q)} \lesssim \delta_{q+1}^{-1}. \tag{21}
\]
\[ \|\rho\|_{C^{N}_{t,x}} \leq C(\delta_{q+1}, \|R_q\|_{C^{N}}), \quad (22) \]

\[ \|a_{(\xi)}\|_{L^\infty_t L^2_x} \lesssim \|\rho\|_{L^\infty_t L^1_x}^{1/2} \leq \delta_{q+1}^{1/2}, \quad (23) \]

\[ \|a_{(\xi)}\|_{C^{N}_{t,x}} \leq C(\delta_{q+1}, \|R_q\|_{C^{N}}). \quad (24) \]

**Proof** It follows from (3) that

\[ \|\rho(t, \cdot)\|_{L^1_t} = \int_{|R_q| \leq \delta_{q+1}} \rho + \int_{|R_q| > \delta_{q+1}} \rho \lesssim \delta_{q+1} + \int_{|R_q| > \delta_{q+1}} |R_q| \]
\[ \leq C \delta_{q+1}. \]

It is direct to verify (21) and (23), while (22) and (24) follow from (17) and (21).

Now we can estimate the time support of \( w_{q+1} \):

\[ \text{supp}_t w_{q+1} \subset \text{supp}_t \rho \subset \text{supp}_t \psi \subset N_{\delta_{q+1}}(\text{supp}_t R_q). \quad (25) \]

We need the following Lemma, which is a variant of [5, Lemma 3.6].

**Lemma 3** ([24, Lemma 2.1]) Let \( f, g \in C^\infty(\mathbb{T}^3) \), and \( g \) is \((\mathbb{T}/N)^3\) periodic, \( N \in \mathbb{N} \). Then for \( 1 \leq p \leq \infty \),

\[ \|fg\|_{L^p} \leq \|f\|_{L^p} \|g\|_{L^p} + C_p N^{-1/p} \|f\|_{C^1} \|g\|_{L^p}. \]

Let us denote

\[ C_N = C \left( \sup_{\xi \in \Lambda} \|a_{(\xi)}\|_{C^{N}_{t,x}} \right) \]

(26) to be some polynomials depending on \( \sup_{\xi \in \Lambda} \|a_{(\xi)}\|_{C^{N}_{t,x}} \).

**Lemma 4** Suppose the parameters satisfy (8) and

\[ \rho^{3/2} \leq \mu. \quad (27) \]

Then the following estimates for the perturbations hold:

\[ \|u_{(q+1)}^{(p)}\|_{L^\infty_t L^2_x} \lesssim \delta_{q+1}^{1/2} + (\lambda_{q+1}\sigma)^{-1/2} C_1, \]
\[ \|w_{q+1}\|_{L^\infty_t L^1_x} \lesssim \rho^{3/2-3/p} C_1, \]
\[ \|u_{(q+1)}^{(c)}\|_{L^\infty_t L^2_x} + \|u_{(q+1)}^{(f)}\|_{L^\infty_t L^2_x} \lesssim (\sigma r + \mu^{-1} r^{3/2}) \rho^{3/2-3/p} C_1, \]
\[ \|\partial_t w_{q+1}\|_{L^\infty_t L^1_x} + \|\partial_t u_{(q+1)}^{(c)}\|_{L^\infty_t L^1_x} \lesssim \lambda_{q+1} \mu^{5/2-3/p} C_2, \]
\[ \|\nabla^N w_{q+1}\|_{L^\infty_t L^1_x} \lesssim \rho^{3/2-3/p} \lambda_{q+1}^N C_{N+1}, \quad (32) \]

for \( 1 < p < \infty, N \geq 1 \).

**Proof** Since \( W(\xi) \) is \((\mathbb{T}/\lambda\sigma)^3\) periodic, it follows from (15), (23), and Lemma 3 that

\[ \|w_{(q+1)}^{(p)}\|_{L^\infty_t L^2_x} \lesssim \sum_{\xi \in \Lambda} \left( \|a_{(\xi)}\|_{L^\infty_t L^1_x} + (\lambda_{q+1}\sigma)^{-1/2} \right) C_1 \|W(\xi)\|_{L^\infty_t L^2_x} \]
\[ \lesssim \delta_{q+1}^{1/2} + (\lambda_{q+1}\sigma)^{-1/2} C_1. \]

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In view of (8), (15) and (16) yield that
\[
\|w_q^{(p)}\|_{L^\infty_t L^p_x} \lesssim \sum_{\xi \in \Lambda} \|a(\xi)\|_{C^0} \|\mathcal{W}(\xi)\|_{L^\infty_t L^p_x} \lesssim r^{3/2-3/p} C_0,
\]
\[
\|w_q^{(c)}\|_{L^\infty_t L^p_x} \lesssim \lambda_{q+1}^{-1} \sum_{\xi \in \Lambda} \left( \|\eta(\xi)\|_{L^\infty_t L^p_x} + \|\nabla \eta(\xi)\|_{L^\infty_t L^p_x} \right) \|a(\xi)\|_{C^0} \|\mathcal{W}(\xi)\|_{L^\infty_t L^p_x}
\lesssim (\sigma r)^{3/2-3/p} C_1,
\]
\[
\|w_q^{(t)}\|_{L^\infty_t L^p_x} \lesssim \mu^{-1} \sum_{\xi \in \Lambda^*} \|a(\xi)\|_{C^0} \|\eta(\xi)\|_{L^\infty_t L^p_x}^2 \lesssim \mu^{-1} \sum_{\xi \in \Lambda^*} a(\xi)^2 \|\eta(\xi)\|_{L^\infty_t L^p_x}^2
\lesssim \mu^{-1} r^{3/2-3/p} C_0,
\]
where the boundedness of \(P_{L^H}\) and \(P_{\neq 0}\) on \(L^p\), for \(1 < p < \infty\), is used in the first inequality of the estimate for \(\|u_q^{(t)}\|_{L^\infty_t L^p_x}\). In the same way, we can estimate
\[
\|\partial_t w_q^{(p)}\|_{L^\infty_t L^p_x} \lesssim \sum_{\xi \in \Lambda} \|\partial_t a(\xi)\|_{C^0} \|\mathcal{W}(\xi)\|_{L^\infty_t L^p_x} + \|a(\xi)\|_{C^0} \|\partial_t \mathcal{W}(\xi)\|_{L^\infty_t L^p_x}
\lesssim \lambda_{q+1} \sigma \mu r^{3/2-3/p} C_1,
\]
\[
\|\partial_t w_q^{(c)}\|_{L^\infty_t L^p_x} \lesssim \lambda_{q+1}^{-1} \sum_{\xi \in \Lambda} \left( \|\eta(\xi)\|_{L^\infty_t L^p_x} + \|\nabla \eta(\xi)\|_{L^\infty_t L^p_x} + \|\partial_t \eta(\xi)\|_{L^\infty_t L^p_x} \right)
\lesssim \sigma r \lambda_{q+1} \sigma \mu r^{3/2-3/p} C_2 \lesssim \lambda_{q+1} \sigma \mu r^{3/2-3/p} C_2.
\]
For \(N \geq 1\), using (15) and (16), we obtain that
\[
\|\nabla^N w_q^{(p)}\|_{L^\infty_t L^p_x} \lesssim \sum_{\xi \in \Lambda} \sum_{k=0}^N \|\nabla^k a(\xi)\|_{C^0} \|\nabla^{N-k} \mathcal{W}(\xi)\|_{L^\infty_t L^p_x}
\lesssim \lambda_{q+1}^N r^{3/2-3/p} C_N,
\]
\[
\|\nabla^N w_q^{(c)}\|_{L^\infty_t L^p_x} \lesssim \lambda_{q+1}^{-1} \sum_{\xi \in \Lambda} \sum_{m=0}^N \sum_{k=0}^m \lambda_{q+1}^{N-m} \|\nabla^k a(\xi)\|_{C^0} \|\nabla^{m-k} \eta(\xi)\|_{L^\infty_t L^p_x}
\lesssim \lambda_{q+1}^N r^{3/2-3/p} C_{N+1},
\]
\[
\|\nabla^N w_q^{(t)}\|_{L^\infty_t L^p_x} \lesssim \mu^{-1} \sum_{\xi \in \Lambda} \sum_{m=0}^N \|\nabla^{N-m} \eta(\xi)\|_{L^\infty_t L^p_x} \|\nabla^m \eta(\xi)\|_{L^\infty_t L^p_x} \|\nabla^{m-k} \eta(\xi)\|_{L^\infty_t L^p_x}
\lesssim \lambda_{q+1}^N r^{3/2-3/p} \frac{(\sigma r)^{3/2}}{\mu} C_N \lesssim \lambda_{q+1}^N r^{3/2-3/p} C_N,
\]
where we use (8) and (27).
3.5 Estimates for the stress

Let us recall the following operator in [12].

Lemma 5 (symmetric anti-divergence) There exists a linear operator \( \mathcal{R} \), of order \(-1\), mapping vector fields to symmetric matrices such that

\[
\nabla \cdot \mathcal{R}(u) = u - \int_{\mathbb{T}^3} u,
\]

with standard Calderon–Zygmund estimates, for \( 1 < p < \infty \),

\[
\| \mathcal{R} \|_{L^p \to W^{1,p}} \lesssim 1, \quad \| \mathcal{R} \|_{C^0 \to C^0} \lesssim 1, \quad \| \mathcal{R} P_{\neq 0} u \|_{L^p} \lesssim \| \nabla | -1 P_{\neq 0} u \|_{L^p}.
\]

Proof Suppose \( u \in C^\infty(\mathbb{T}^3, \mathbb{R}^3) \) is a smooth vector field. Define

\[
\mathcal{R}(u) = \frac{1}{4} \left( \nabla P_{LH} v + (\nabla P_{LH} v)^T \right) + \frac{3}{4} \left( \nabla v + (\nabla v)^T \right) - \frac{1}{2} (\nabla \cdot v) \text{Id}
\]

where \( v \in C^\infty(\mathbb{T}^3, \mathbb{R}^3) \) is the unique solution to \( \Delta v = u - f_{\mathbb{T}^3} u \) with \( f_{\mathbb{T}^3} v = 0 \).

It is direct to verify that \( \mathcal{R}(u) \) is a symmetric matrix field depending linearly on \( u \) and satisfies (33). Note that \( \mathcal{R} \) is a constant coefficient elliptic operator of order \(-1\). We refer to [14] for the Calderon-Zygmund estimates \( \| \mathcal{R} \|_{L^p \to W^{1,p}} \lesssim 1 \) and \( \| \mathcal{R} P_{\neq 0} u \|_{L^p} \lesssim \| \nabla | -1 P_{\neq 0} u \|_{L^p} \). Combining these with Sobolev embeddings, we have \( \| \mathcal{R} u \|_{C^\alpha} \lesssim \| \mathcal{R} u \|_{W^{1,4}} \lesssim \| u \|_{L^4} \lesssim \| u \|_{C^0}, \) with \( \alpha = 1/4 \).

We have the following variant of [5, Lemma B.1] in [5].

Lemma 6 Let \( a \in C^2(\mathbb{T}^3) \). For \( 1 < p < \infty \), and any smooth function \( f \in L^p(\mathbb{T}^3) \), we have

\[
\| \nabla | -1 P_{\neq 0} (a P_{\geq k} f) \|_{L^p(\mathbb{T}^3)} \lesssim k^{-1} \| \nabla^2 a \|_{L^\infty(\mathbb{T}^3)} \| f \|_{L^p(\mathbb{T}^3)}.
\]

Proof of Lemma 6 We follow the proof in [5]. Note that

\[
\| \nabla | -1 P_{\neq 0} (a P_{\geq k} f) \|_{L^p(\mathbb{T}^3)} = \| \nabla | -1 P_{\geq k/2} (P_{\leq k/2} a P_{\geq k} f) + \| \nabla | -1 P_{\neq 0} (P_{\geq k/2} a P_{\geq k} f).
\]

As direct consequences of the Littlewood–Paley decomposition and Schauder estimates we have the bounds for \( 1 < p < \infty \) (see, for example, [14])

\[
\| P_{\leq k/2} \|_{L^p \to L^p} \lesssim 1, \quad \| \nabla | -1 P_{\geq k/2} \|_{L^p \to L^p} \lesssim k^{-1}, \quad \| \nabla | -1 P_{\neq 0} \|_{L^p \to L^p} \lesssim 1.
\]

Combining these bounds with Hölder’s inequality and the embedding \( W^{1,4}(\mathbb{T}^3) \subset L^\infty(\mathbb{T}^3), \) we obtain

\[
\| \nabla | -1 P_{\neq 0} (a P_{\geq k} f) \|_{L^p} \lesssim k^{-1} \| P_{\leq k/2} a P_{\geq k} f \|_{L^p} + \| P_{\geq k/2} a P_{\geq k} f \|_{L^p}
\]

\[
\lesssim k^{-1} (\| P_{\leq k/2} a \|_{L^\infty} + k \| P_{\geq k/2} a \|_{L^\infty}) \| f \|_{L^p}
\]

\[
\lesssim k^{-1} (\| \nabla P_{\leq k/2} a \|_{L^4} + k \| \nabla P_{\geq k/2} a \|_{L^4}) \| f \|_{L^p}
\]

\[
\lesssim k^{-1} (\| \nabla a \|_{L^4} + \| \nabla^2 P_{\geq k/2} a \|_{L^4}) \| f \|_{L^p}
\]

\[
\lesssim k^{-1} \| \nabla a \|_{L^4} \| f \|_{L^p} \lesssim k^{-1} \| \nabla^2 a \|_{L^4} \| f \|_{L^p}.
\]

\[\square\]
It follows from the definition of $w_{q+1}$ that
\[
\int_{\mathbb{T}^3} w_{q+1} dx = \int_{\mathbb{T}^3} \frac{1}{\lambda_{q+1}} \sum_{\xi \in \Lambda} \nabla \left( a(\tilde{\xi}) \eta(\tilde{\xi}) W(\tilde{\xi}) \right) dx + \int_{\mathbb{T}^3} \frac{1}{\mu} \sum_{\xi \in \Lambda^+} P_{LH} \mathbb{P} \neq 0 \left( a^2(\tilde{\xi}) \eta^2(\tilde{\xi}) \right) dx = 0.
\]

Hence $\int_{\mathbb{T}^3} \nu(-\Delta)^q w_{q+1} dx = 0$ and $\frac{d}{dt} \int_{\mathbb{T}^3} w_{q+1} dx = 0$. We obtain $R_{q+1}$ by plugging $v_{q+1} = v + w_{q+1}$ in (2), using (33) and the assumption that $(v_q, R_q)$ solves (2):
\[
\nabla \cdot R_{q+1} = \nabla \cdot \left[ R \left( v(-\Delta)^q w_{q+1} + \partial_t w^{(p)} + \partial_t w^{(c)} + v_q \otimes w_{q+1} + w_{q+1} \otimes v_q \right) + \nabla \cdot \left[ (u_{q+1}^{(c)} + u_{q+1}^{(t)}) \otimes w_{q+1} + w^{(p)} \otimes (w_{q+1}^{(c)} + u_{q+1}^{(t)}) \right] \right] \\
\quad \times \left[ \nabla \cdot (w^{(p)} \otimes w^{(p)} - R_q) + \partial_t w^{(p)} \right] + \nabla (p_{q+1} - p_q) \\
:= \nabla \cdot (\tilde{R}_{\text{linear}} + \tilde{R}_{\text{corrector}} + \tilde{R}_{\text{oscillation}}) + \nabla (p_{q+1} - p_q).
\]

It follows from Lemma 4 that
\[
\| \tilde{R}_{\text{corrector}} \|_{L_t^\infty L_x^p} \lesssim \left( \| w_{q+1}^{(c)} \|_{L_t^\infty L_x^{2p}} + \| w_{q+1}^{(t)} \|_{L_t^\infty L_x^{2p}} \right) \left( \| w_{q+1} \|_{L_t^\infty L_x^{2p}} + \| w_{q+1}^{(p)} \|_{L_t^\infty L_x^{2p}} \right) \\
\quad \lesssim (\sigma r + \mu^{-1/2}) r^{3-3/p} C_1.
\]

Noting that $\nabla \times \frac{w_{q+1}^{(p)}}{\lambda_{q+1}} = w_{q+1}^{(p)} + w_{q+1}^{(c)}$, Lemma 4 and (34) yield that
\[
\| \tilde{R}_{\text{linear}} \|_{L_t^\infty L_x^p} \\
\quad \lesssim \lambda_{q+1}^{-1} \| \partial_t R \nabla \times (w_{q+1}^{(p)}) \|_{L_t^\infty L_x^{p}} + \| R \left( v(-\Delta)^q w_{q+1} \right) \|_{L_t^\infty L_x^{p}} \\
\quad + \| v_q \otimes w_{q+1} + w_{q+1} \otimes v_q \|_{L_t^\infty L_x^{p}} \\
\quad \lesssim \lambda_{q+1}^{-1} \| \partial_t w_{q+1}^{(p)} \|_{L_t^\infty L_x^{p}} + \| \nabla \|_{2 \theta - 1} \| w_{q+1} \|_{L_t^\infty L_x^{p}} + \| v_q \|_{C_0} \| w_{q+1} \|_{L_t^\infty L_x^{p}} \\
\quad \lesssim \sigma \mu r^{5/2 - 3/p} C_2 + r^{3/2 - 3/p} \left( \lambda_{q+1}^{-1} + \| v_q \|_{C_0} \right) C_3. \tag{36}
\]

This is the crucial estimate to control the fractional viscosity. If we assume that $p \sim 1$, $r \sim \lambda_{q+1}^{-1}$, we must have $\theta < 5/4$ in order that the second term in (36) is small for $\lambda_{q+1}$ sufficiently large.

It remains to estimate $\tilde{R}_{\text{oscillation}}$, which can be handled in the same way as in [5]. It follows from (19) that
\[
\nabla \cdot (w_{q+1}^{(p)} \otimes w_{q+1}^{(p)} - R_q) = \nabla \cdot \left( \sum_{\tilde{\xi}, \tilde{\xi} \in \Lambda} a_{(\tilde{\xi})} a_{(\tilde{\xi})} W_{(\tilde{\xi})} \otimes W_{(\tilde{\xi})} - R_q \right) \\
\quad = \nabla \cdot \left( \sum_{\tilde{\xi}, \tilde{\xi} \in \Lambda} a_{(\tilde{\xi})} a_{(\tilde{\xi})} \mathbb{P}_{\geq \lambda_{q+1} \sigma/2} W_{(\tilde{\xi})} \otimes W_{(\tilde{\xi})} \right) + \nabla \rho \\
:= \sum_{\tilde{\xi}, \tilde{\xi} \in \Lambda} E_{(\tilde{\xi}, \tilde{\xi})} + \nabla \rho.
\]
Since $E_{(ξ, ξ')}$, has zero mean, we can split it as

$$E_{(ξ, ξ')} + E_{(ξ', ξ)} = \mathbb{P} \neq 0 \left( \nabla \left( a_ξ a_{ξ'} \right) \cdot \left( \mathbb{P} \geq λ_{q+1}/10 \left( W_ξ \otimes W_{ξ'} + W_{ξ'} \otimes W_ξ \right) \right) \right)$$

Using (15), (34) and (35), we obtain

$$\mathcal{R} E_{(ξ, ξ'), 1} \leq \left\| \mathbb{P} \right\|_{L^p_t L^q_x} \left\| \nabla^{-1} \left( E_{(ξ, ξ'), 1} \right) \right\|_{L^p_t L^q_x}$$

$$\leq \left( λ_{q+1} σ \right)^{-1} a_ξ^2 a_{ξ'} \left\| W_ξ \otimes W_{ξ'} \right\|_{L^p_t L^q_x}$$

Using (15), (34) and (35), we obtain

$$\mathcal{R} E_{(ξ, ξ'), 1} \leq \left\| \mathbb{P} \right\|_{L^p_t L^q_x} \left\| \nabla^{-1} \left( E_{(ξ, ξ'), 1} \right) \right\|_{L^p_t L^q_x}$$

Recall the vector identity $A \cdot \nabla B + B \cdot \nabla A = \nabla (A \cdot B) - A \times (\nabla \times B) - B \times (\nabla \times A).$ For $ξ, ξ' \in Λ$, using the anti-symmetry of the cross product, we can write

$$\nabla \cdot \left( W_ξ \otimes W_{ξ'} + W_{ξ'} \otimes W_ξ \right) \nabla \left( \eta_ξ \eta_{ξ'} \right)$$

$$= \left( W_ξ \otimes W_{ξ'} + W_{ξ'} \otimes W_ξ \right) \nabla \left( \eta_ξ \eta_{ξ'} \right)$$

$$+ \eta_ξ \eta_{ξ'} \left( W_ξ \cdot \nabla W_{ξ'} + W_{ξ'} \cdot \nabla W_ξ \right)$$

For the term $E_{(ξ, ξ'), 2},$ first consider the case $ξ + ξ' \neq 0.$ It follows from the above identity and (14) that

$$a_ξ a_{ξ'} \nabla \cdot \left( W_ξ \otimes W_{ξ'} + W_{ξ'} \otimes W_ξ \right)$$

where the second term is a pressure, the third can be estimated analogously to $E_{(ξ, ξ'), 1}. $ Also note that the first and fourth term can estimated analogously. Using (16), (34) and (35), we obtain

$$\mathcal{R} \left( a_ξ a_{ξ'} \mathbb{P} \geq λ_{q+1}/10 \left( \nabla \left( \eta_ξ \eta_{ξ'} \right) \cdot \left( W_ξ \otimes W_{ξ'} + W_{ξ'} \otimes W_ξ \right) \right) \right) \left\| L^p_t L^q_x \right\|$$
\[ \lesssim \lambda_{q+1}^{-1} \| a_{(\xi)} a_{(\xi')} \|_{C^1} \left( \| \nabla \left( \eta_{(\xi)} \eta_{(\xi')} \right) \|_{L_t^\infty L_x^p} \right) \]
\[ \lesssim \sigma r^{4-3/p} C_3. \]

Now consider \( E_{(\xi, -\xi, 2)} \). We can write
\[ \nabla \cdot \left( W_{(\xi)} \otimes W_{(-\xi)} + W_{(-\xi)} \otimes W_{(\xi)} \right) = \left( W_{(-\xi)} \cdot \nabla \eta_{(\xi)}^2 \right) W_{(\xi)} + \left( W_{(\xi)} \cdot \nabla \eta_{(\xi)}^2 \right) W_{(-\xi)} \]
\[ = \left( A_{\xi} \cdot \nabla \eta_{(\xi)}^2 \right) A_{\xi} + \left( (\xi \times A_{\xi}) \cdot \nabla \eta_{(\xi)}^2 \right) (\xi \times A_{\xi}) \]
\[ = \nabla \eta_{(\xi)}^2 - \left( \xi \cdot \nabla \eta_{(\xi)}^2 \right) \xi \]
\[ = \nabla \eta_{(\xi)}^2 - \frac{\xi}{\mu}, \]

where we use (11) and the fact that \( \{\xi, A_{\xi}, \xi \times A_{\xi}\} \) forms an orthonormal basis of \( \mathbb{R}^3 \). Therefore, we can write
\[ E_{(\xi, -\xi, 2)} = P \neq 0 \left( a_{(\xi)}^2 \nabla P_{\geq \lambda_{q+1}\sigma/2} \eta_{(\xi)}^2 - a_{(-\xi)}^2 \nabla \eta_{(\xi)}^2 \right) \]
\[ = \nabla \left( a_{(\xi)}^2 P_{\geq \lambda_{q+1}\sigma/2} \eta_{(\xi)}^2 \right) - P \neq 0 \left( P_{\geq \lambda_{q+1}\sigma/2} \nabla a_{(\xi)}^2 \right) \]
\[ - \mu^{-1} \partial_t P \neq 0 \left( a_{(\xi)}^2 \eta_{(\xi)}^2 \xi \right) + \mu^{-1} P \neq 0 \left( \partial_t a_{(\xi)}^2 \eta_{(\xi)}^2 \xi \right). \]

Using the identity \( \Id - P_{LH} = \nabla \Delta^{-1} \text{div} \), we obtain
\[ \sum_{\xi} E_{(\xi, -\xi, 2)} + \partial_t w_{q+1} = \nabla \sum_{\xi} \left( a_{(\xi)}^2 P_{\geq \lambda_{q+1}\sigma/2} \eta_{(\xi)}^2 \right) - \nabla \sum_{\xi} \mu^{-1} \Delta^{-1} \nabla \cdot \partial_t \left( a_{(\xi)}^2 \eta_{(\xi)}^2 \xi \right) \]
\[ - \sum_{\xi} P \neq 0 \left( P_{\geq \lambda_{q+1}\sigma/2} \nabla a_{(\xi)}^2 \right) + \mu^{-1} \sum_{\xi} P \neq 0 \left( \partial_t a_{(\xi)}^2 \eta_{(\xi)}^2 \xi \right), \]

where the first and second terms are pressure terms. Using (16), (34) and (35), we obtain
\[ \| \mathcal{R} P \neq 0 \left( P_{\geq \lambda_{q+1}\sigma/2} \nabla a_{(\xi)}^2 \right) \|_{L_t^\infty L_x^p} \lesssim (\lambda_{q+1}\sigma)^{-1} \| \eta_{(\xi)}^2 \|_{L_t^\infty L_x^p} C_3 \]
\[ \lesssim (\lambda_{q+1}\sigma)^{-1} r^{3-3/p} C_3. \]

It follows from (16) and (34) that
\[ \mu^{-1} \| \mathcal{R} P \neq 0 \left( \partial_t \left( a_{(\xi)}^2 \eta_{(\xi)}^2 \xi \right) \right) \|_{L_t^\infty L_x^p} \lesssim \mu^{-1} \| \partial_t \left( a_{(\xi)}^2 \eta_{(\xi)}^2 \xi \right) \|_{L_t^\infty L_x^p} \]
\[ \lesssim \mu^{-1} r^{3-3/p} C_1. \]

Let us now give the explicit definition of \( \tilde{R}_{\text{oscillation}} \):
\[ \tilde{R}_{\text{oscillation}} = \sum_{\xi, \xi' \in \Lambda} P \neq 0 \left( \nabla (a_{(\xi)} a_{(\xi')}) \cdot (P_{\geq \lambda_{q+1}\sigma/2} (W_{(\xi)} \otimes W_{(\xi')} + W_{(\xi')} \otimes W_{(\xi)})) \right) \]
\[ + \sum_{\xi, \xi' \in \Lambda, \xi \neq \xi'} a_{(\xi)} a_{(\xi')} P_{\geq \lambda_{q+1}/10} \left( \nabla (\eta_{(\xi)} \eta_{(\xi')} \eta_{(\xi')} \eta_{(\xi)}) \cdot \left( W_{(\xi)} \otimes W_{(\xi')} + W_{(\xi')} \otimes W_{(\xi)} \right) \right) \]
\[ - \sum_{\xi, \xi' \in \Lambda, \xi \neq \xi'} \nabla (a_{(\xi)} a_{(\xi')}) P_{\geq \lambda_{q+1}/10} \left( W_{(\xi)} \cdot W_{(\xi')} \right) \]
Finally, we estimate the time support of $R_{q+1}$. Using (25) we obtain

$$\text{supp}_t R_{q+1} \subset \text{supp}_t w_{q+1} \cup \text{supp}_t R_q \subset N_{\delta_{q+1}}(\text{supp}_t R_q).$$

Now we choose the parameters $r, \sigma, \mu$. Fix $\alpha$ so that

$$\max\left\{0, \frac{2}{3}(2\theta - 1)\right\} < \alpha < 1,$$

which is possible since $\theta \in (-\infty, 5/4)$. Fix

$$r = \lambda_{q+1}^\alpha, \quad \sigma = \lambda_{q+1}^{-(\alpha+1)/2}, \quad \mu = \lambda_{q+1}^{(5\alpha+1)/4}.$$

(37)

Clearly (27) is satisfied. Choose $p > 1$ sufficiently close to 1 so that

$$-\frac{\alpha + 1}{2} + \frac{5\alpha + 1}{4} + \left(\frac{5}{2} - \frac{3}{p}\right)\alpha < 0, \quad \left(\frac{3}{2} - \frac{3}{p}\right)\alpha + \max(0, 2\theta - 1) < 0,$$

$$-\frac{5\alpha + 1}{4} + \left(\frac{9}{2} - \frac{3}{p}\right)\alpha < 0, \quad -\frac{1 - \alpha}{2} + \left(3 - \frac{3}{p}\right)\alpha < 0.$$

Note that $C_N$ is independent of $\lambda_{q+1}$, due to (24). Combining the above estimates with Lemma 4, it is easy to check that, by taking $\lambda_{q+1}$ sufficiently large, we arrive at (4), (6) and (7). This completes the proof of Lemma 1.

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