Abundance of fast growth of the number of periodic points in 2-dimensional area-preserving dynamics

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Abstract

We prove that there exists an open subset of the set of real-analytic Hamiltonian diffeomorphisms of a closed surface in which diffeomorphisms exhibiting fast growth of the number of periodic points are dense. We also prove that there exists an open subset of the set of smooth area-preserving diffeomorphisms of a closed surface in which typical diffeomorphisms exhibit fast growth of the number of periodic points.

1 Introduction

In this article, we give an application of the KAM theory to a classical problem on the growth rate of the number of periodic points of a dynamical system. We say that a continuous self-map \( f \) on a topological space \( X \) satisfies the condition (Exp) if the number of periodic points of \( f \) grows at most exponentially, i.e., there exists \( \lambda > 1 \) such that

\[
\limsup_{n \to \infty} \frac{1}{\lambda^n} \# \{ x \in X \mid f^n(x) = x \} = 0,
\]

where \( \# S \) denotes the cardinality of a set \( S \). In 1965, Artin and Mazur [2] proved that \( C^r \) maps satisfying (Exp) are dense in the space of \( C^r \)-maps on a closed manifold (see also [12]). For one-dimensional dynamics, Theorem B' in [21] by Martens, de Melo, and van Strien implies that a \( C^r \) map on a compact interval with \( r \geq 3 \) satisfies (Exp) if it does not admit any \((r - 1)\)-flat critical points (see also [17, Corollary 1]). In particular, the condition (Exp) is open and dense in the space of \( C^r \) maps on a compact interval for any \( 3 \leq r \leq \infty \). If we allow the existence of flat critical points, super-exponential growth is possible for interval maps. For any given sequence of positive integers and any \( 1 \leq r < \infty \), Kaloshin and Kozlovski [17] constructed an example of \( C^r \) unimodal map on a compact interval whose number of periodic points grows faster than the sequence. \(^1\)

\(^1\) In the appendix of this paper, we give a simple example for \( C^\infty \) case.
For higher-dimensional dynamics, it turned out that super-exponential growth is abundant. In [13], Kaloshin proved that super-exponential growth is generic in the Newhouse domain for $C^r$ surface diffeomorphisms with $2 \leq r \leq \infty$. Kaloshin and Saprykina [13], Bonatti, Díaz, and Fisher [7], and the author, Shinohara, and Turaev [3] also gave open sets of maps in which generic maps exhibit super-exponential growth in higher dimensions or in other settings.

It is natural to ask whether any real-analytic map satisfies the condition (Exp) or not. Any real-analytic map on a compact interval satisfies (Exp) since it has no flat-critical point. In higher dimension, as far as the author’s knowledge, there was no known real-analytic example which exhibits super-exponential growth of the number of periodic points.

The first main result of this paper is the density of maps with super-exponential growth in an open subset of the set of real-analytic Hamiltonian diffeomorphisms on the two-dimensional torus. Let $\mathbb{T}^2$ be the two-dimensional torus $\mathbb{R}^2/\mathbb{Z}^2$. A diffeomorphism of $\mathbb{T}^2$ is called a Hamiltonian diffeomorphism if it is a composition of finite number of the time-one maps of the Hamiltonian flows associated with real-analytic time-dependent Hamiltonian functions and the standard volume form $dx \wedge dy$ on $\mathbb{T}^2$. Let $\text{Ham}^\omega(\mathbb{T}^2)$ be the space of real-analytic Hamiltonian diffeomorphisms of $\mathbb{T}^2$ with the $C^\omega$-topology (see Section 3.1 for the definition of the $C^\omega$ topology). For a diffeomorphism $f$ of a surface and $n \geq 1$, let $\text{Per}_h(f,n)$ (resp., $\text{Per}_e(f,n)$) be the the set of hyperbolic (resp., elliptic) periodic points of $f$ whose least period is $n$. We call a periodic point non-degenerate if it is hyperbolic or elliptic.

**Theorem 1.1.** There exists an open subset $U_*$ of $\text{Ham}^\omega(\mathbb{T}^2)$ such that maps satisfying the following conditions are dense in $U_*$ for any given sequence $(\gamma_n)_{n \geq 1}$ of positive integers:

1. All periodic point of $f$ are non-degenerate.
2. $\limsup_{n \to \infty} \min_{\gamma_n} \{ \# \text{Per}_h(f,n), \# \text{Per}_e(f,n) \} = \infty$.

For $\gamma_n = 2^{n^2}$, any map in the above dense set does not satisfy (Exp). Remark that genericity is almost non-sense in the space $\text{Ham}^\omega(\mathbb{T}^2)$ with our $C^\omega$ topology since it does not satisfies the Baire property. In particular, a residual subset of $\text{Ham}^\omega(\mathbb{T}^2)$ may be empty in general.

We also apply the idea of the proof of Theorem 1.1 to show that fast growth of the number of periodic points is $C^\infty$-typical in the sense of Kolmogorov-Arnold in an open subset of the set of $C^\infty$ area-preserving diffeomorphisms of a closed surface. Let $M$ be a $C^r$ manifold with $1 \leq r \leq \infty$. We denote the space of $C^r$ diffeomorphisms with the $C^r$-topology by $\text{Diff}^r(M)$. For a finite-dimensional closed unit disk $\Delta$ and a subset $D$ of $\text{Diff}^r(M)$, a family $(f_\mu)_{\mu \in \Delta}$ is called a $C^r$ family in $D$ with parameter set $\Delta$ if each $f_\mu$ is an element of $D$ and the map $F(x, \mu) = f_\mu(x)$ is a $C^r$ map from $M \times \Delta$ to $M$. The identification

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2. Kaloshin stated his result for finite $r$. However, his proof works for the $C^\infty$ case.
3. A related example is given by Kozlovski [19].
of the family \((f_\mu)_{\mu \in \Delta}\) with the \(C^r\) map \(F\) induces the \(C^r\)-topology to the set of \(C^r\) family in \(D\) with parameter set \(\Delta\). We say that a \(C^r\)-typical map in \(D\) satisfies a given property if for any finite dimensional closed unit disk \(\Delta\) and any generic \(C^r\) family \((f_\mu)_{\mu \in \Delta}\) of maps in \(D\), the set of parameters \(\mu \in \Delta\) for which the map \(f_\mu\) satisfies the property contains a Lebesgue full measure subset of \(\Delta\). See [10, Section 3.1] for more information on typicality and related concepts.

In many cases, it is harder to show that a dynamical property is typical than to show that it is generic. For example, Newhouse [23] proved that finiteness of periodic attractors is not generic for \(C^r\) diffeomorphisms on a closed surface with \(r \geq 2\) in 1970’s. But, \(C^r\)-typicality of finiteness of periodic attractors had been a long-standing open problems (see e.g., [24] Section 2.7) until Berger [1] recently proved that this property is not \(C^r\) typical for \(d\)-dimensional diffeomorphisms with \(d \geq 3\) and \(d \leq r \leq \infty\). Typicality of the condition (Exp) has been also an open problem (see e.g., [1] Problems 1992-13, 1994-47). The second main result of this paper shows that super-exponential growth is \(C^\infty\)-typical in an open subset of the set of \(C^\infty\) area-preserving diffeomorphisms on a closed surface. Let \(\Sigma\) be a \(C^\infty\) two-dimensional closed orientable manifold and \(\Omega\) a \(C^\infty\) volume form on \(\Sigma\). We denote the set of \(C^\infty\) diffeomorphisms of \(\Sigma\) which preserve the volume \(\Omega\) by \(\text{Diff}_{C^\infty}^\infty(\Omega, \Sigma)\). The set \(\text{Diff}_{C^\infty}^\infty(\Omega, \Sigma)\) is endowed with the \(C^\infty\)-topology.

**Theorem 1.2.** There exists an open subset \(U\) of \(\text{Diff}_{C^\infty}^\infty(\Sigma)\) such that fast growth of the number of periodic points is \(C^\infty\)-typical in \(U\). More precisely, for any given sequence \((\gamma_n)_{n \geq 1}\) of positive integers, a \(C^\infty\)-typical map \(f\) in \(U\) satisfies

\[
\limsup_{n \to \infty} \frac{\min\{\#\text{Per}_h(f,n), \#\text{Per}_e(f,n)\}}{\gamma_n} = \infty \quad \text{4}
\]

After the first version of this article was submitted to arXiv, Berger [5] proved that the property (Exp) is not \(C^r\)-typical in \(\text{Diff}^r(M)\) for \(1 \leq r < \infty\) and \(\text{dim} M \geq 3\). Remark that Kaloshin and Hunt proved a result on the growth rate of the number of periodic points which contrasts with the above results on typical fast growth. In [14] [15] [16], they proved that at most stretched exponential growth of the number of periodic points is \(C^r\)-prevalent in the sense of Hunt-Sauer-Yorke (see [9] [28]) for \(1 \leq r \leq \infty\).

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4It seems possible to prove that \(C^r\)-typical diffeomorphisms in \(\text{Diff}_{C^\infty}^\infty(\Sigma)\) has no degenerate periodic points by an argument analogous to the proof of \(C^r\)-typicality of Kupka-Smale diffeomorphisms in [11].
2 Smooth case

Let \( \Sigma \) be a \( C^\infty \) two-dimensional orientable closed manifold and \( \Omega \) a \( C^\infty \) volume on \( \Sigma \). For an elliptic fixed point \( p \) of a diffeomorphism \( f \) in \( \text{Diff}_{\Omega}^\infty(\Sigma) \), the Birkhoff normal form of \( f \) at \( p \) up to the third order coincides with

\[
\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + O((x^2 + y^2)^2),
\]

where \( \alpha(x, y) = \alpha_0 + \alpha_1(x^2 + y^2) \) with real numbers \( \alpha_0, \alpha_1 \) (see e.g. [22, Theorem 2.11]). We say that the elliptic fixed point \( p \) satisfies the Moser stability condition if \( \alpha_1 > 0 \). The following theorem implies Theorem 1.2.

**Theorem 2.1.** Suppose that a diffeomorphism \( f_0 \in \text{Diff}_{\Omega}^\infty(\Sigma) \) admits an elliptic fixed point with the Moser stability condition. Then, there exists an open neighborhood \( U \) of \( f_0 \) in \( \text{Diff}_{\Omega}^\infty(\Sigma) \) such that for any finite dimensional closed unit disk \( \Delta \) and any sequence \( (\gamma_n)_{n \geq 1} \) of positive integers, a generic family \( (f_\mu)_{\mu \in \Delta} \) in \( U \) satisfies that

\[
\limsup_{n \to \infty} \frac{1}{\gamma_n} \min \{ \# \text{Per}_h(f_\mu, n), \# \text{Per}_e(f_\mu, n) \} = \infty
\]

for any \( \mu \in \Delta \). In particular, the condition (1) is \( C^r \)-typical in \( U \).

The rest of this section is devoted to the proof of Theorem 2.1. Fix a finite dimensional closed unit disk \( \Delta \). Let \( (f_\mu)_{\mu \in \Delta} \) be a \( C^\infty \) family in \( \text{Diff}_{\Omega}^\infty(\Sigma) \) and \( \theta \) a real number. We call a family \( (\psi_\mu)_{\mu \in \Delta} \) a family of KAM circles with rotation number \( \theta \) if \( (\psi_\mu)_{\mu \in \Delta} \) is a \( C^\infty \) family of embeddings from \( S^1 \times [-1, 1] \) to \( \Sigma \) such that

\[
f_\mu \circ \psi_\mu(x, 0) = \psi_\mu(x + \theta, 0)
\]

and

\[
\left( \frac{\partial}{\partial y} [P_x \circ (\psi^{-1}_\mu \circ f_\mu \circ \psi_\mu)] \right)(x, 0) > 0
\]

for any \( \mu \in \Delta \) and \( x \in S^1 \), where \( P_x : S^1 \times [-1, 1] \to S^1 \) is the projection to the first coordinate. The former condition means that \( f_\mu \) is \( C^\infty \) conjugate to the rigid rotation of angle \( \theta \) on the invariant circle \( \psi_\mu(S^1 \times \{0\}) \). The latter means that \( f_\mu \) ‘twists’ a neighborhood of the circle. The following is an easy consequence of a classical result by Moser (see e.g., [22, Theorem 2.11]).

**Proposition 2.2.** Let \( f_0 \) be a diffeomorphism in \( \text{Diff}_{\Omega}^\infty(\Sigma) \) and \( p \) be an elliptic fixed point of \( f_0 \) with the Moser stability condition. Then, there exists a neighborhood \( U \) of \( f_0 \) in \( \text{Diff}_{\Omega}^\infty(\Sigma) \) such that any \( C^\infty \) family \( (f_\mu)_{\mu \in \Delta} \) of maps in \( U \) admits a family of KAM circles.

The following proposition implies Theorem 2.1 combining with Proposition 2.2.
Proposition 2.3. Let \((f_\mu)_{\mu \in \Delta}\) be a \(C^\infty\) family of maps in \(\text{Diff}^r_\Omega(\Sigma)\) which admits a family of KAM circles. For any neighborhood \(V\) of \((f_\mu)_{\mu \in \Delta}\) in the space of \(C^\infty\) families of maps in \(\text{Diff}^r_\Omega(\Sigma)\), any sequence \((\gamma_n)_{n \geq 1}\) of positive integers, and any \(n_0 \geq 1\), there exists a family \((\hat{f}_\mu)_{\mu \in \Delta}\) in \(V\) and an integer \(\hat{n} \geq n_0\) such that

\[
\min\{\#\text{Per}_h(\hat{f}_\mu, \hat{n}), \#\text{Per}_e(\hat{f}_\mu, \hat{n})\} \geq \hat{n} \cdot \gamma_{\hat{n}}
\]

for any \(\mu \in \Delta\).

Proof of Theorem 2.4 from Proposition 2.3 Let \(U\) be an open neighborhood of \(f_0\) in \(\text{Diff}^\infty_\Omega(\Sigma)\) given by Proposition 2.2. Fix a sequence \((\gamma_n)_{n \geq 1}\) of positive integers. Let \(V\) be the set of \(C^\infty\) families \((f_\mu)_{\mu \in \Delta}\) of maps in \(U\). Put

\[
V_n = \{(f_\mu)_{\mu \in \Delta} \in V \mid \min\{\#\text{Per}_h(f_\mu, n), \#\text{Per}_e(f_\mu, n)\} \geq \gamma_n \text{ for any } \mu \in \Delta\}.
\]

By the persistence of hyperbolic or elliptic fixed points, the set \(V_n\) is open for any \(n \geq 1\). By Proposition 2.3, the set \(\bigcup_{n \geq n_0} V_n\) is dense in \(V\) for any \(n_0 \geq 1\). The set \(R = \bigcap_{n_0 \geq 0} \bigcup_{n \geq n_0} V_n\) is a residual subset of \(V\) and any family \((f_\mu)_{\mu \in \Delta}\) in \(R\) satisfies (1).

In order to prove Proposition 2.3, we produce arbitrary many periodic points by a series of perturbations on KAM circles.

Lemma 2.4. Let \((f_\mu)_{\mu \in \Delta}\) of maps in \(\text{Diff}^\infty_\Omega(\Sigma)\) and \((\psi_\mu)_{\mu \in \Delta}\) a family of KAM circles. There exists a family \((\Phi_\mu)_{\mu \in \Delta}\) of \(\Omega\)-preserving flows on \(\Sigma\) such that the map \((p, t, \mu) \mapsto \Phi_\mu(t, p)\) is of class \(C^\infty\) and \(\Phi_\mu(x, 0) = x\) for any \(\mu \in \Delta\), \(x \in S^1\), and \(t \in R\).

Proof. Let \(\beta_\mu : S^1 \times [-1, 1] \to R\) be the function given by \(\beta_\mu(x, y) = (\psi_\mu)^* \Omega\). Take a \(C^\infty\) function \(H\) on \(\Sigma \times \Delta\) such that \(H(\psi_\mu(x, y), \mu) = y \cdot \beta_\mu(x, y)\) on \(S^1 \times [-1, 1] \times \Delta\) and put \(H_\mu(p) = H(p, \mu)\) for \(p \in \Sigma\) and \(\mu \in \Delta\). Let \(\Phi_\mu\) be the Hamiltonian flow associated with the Hamiltonian function \(H_\mu\) and the symplectic form \(\Omega\). Then, the map \((p, t, \mu) \mapsto \Phi_\mu(t, p)\) is of class \(C^\infty\). The vector field \(D\psi_\mu^{-1}(X_\mu)\) on \(S^1 \times [-1, 1]\) is the Hamiltonian vector field associated with the Hamiltonian function \(\beta_\mu(x, y)\) and the symplectic form \(\beta_\mu dx \wedge dy\). This vector field satisfies that

\[
D\psi_\mu^{-1}(X_\mu)(x, 0) = \frac{\partial}{\partial x}.
\]

Hence, we have \(\Phi_\mu(\psi_\mu(x, 0)) = \psi_\mu(x + t, 0)\).

Lemma 2.5. Let \(f : S^1 \times [-1, 1] \to S^1 \times R\) be a \(C^1\) embedding homotopic to the inclusion map and \(g : S^1 \to [-1, 1]\) be a \(C^1\) map such that \(\Gamma_g = \{(x, g(x)) \mid x \in S^1\}\) is \(f\)-invariant and \(\frac{\partial}{\partial t}[P_x \circ f(t)] > 0\) on \(\Gamma_g\), where \(P_x : S^1 \times [-1, 1] \to S^1\) is the projection to the first coordinate. Then, \(\frac{\partial}{\partial t}[P_x \circ f^n(t)] > 0\) on \(\Gamma_g\) for any \(n \geq 1\).
Proof. Define a (half) cone \( C(x, g(x)) \) in \( T_{(x,g(x))}(S^1 \times \mathbb{R}) \) for \((x, g(x)) \in \Gamma_g\) by
\[
C(x, g(x)) = \left\{ s \frac{\partial}{\partial x} + t \frac{\partial}{\partial y} \left| s > 0, t > g'(x)s \right. \right\}.
\]
Since the restriction of \( f \) to \( \Gamma_g \) is an orientation preserving diffeomorphism and \((\partial/\partial y)[P_x \circ f] > 0 \) on \( \Gamma_g \), we have \( Df_p(\partial/\partial y) \in C(f(p)) \) and \( Df_p(C(p)) \subset C(f(n\partial/\partial y)) \) for any \( p \in \Gamma_g \). By induction, \( Df^n_p(\partial/\partial y) \in C(f^n(p)) \) for any \( p \in \Gamma_g \) and \( n \geq 1 \). This means that \((\partial/\partial y)[P_x \circ f^n] > 0 \) on \( \Gamma_g \), \( \square \)

The following lemma is probably well-known for experts. However, we give a proof because we use an analogous argument for the real-analytic case.

Lemma 2.6. Let \((f_\mu)_{\mu \in \Delta} \) be a \( C^\infty \) family of maps in \( \text{Diff}_r^2(\Sigma) \), \((\psi_\mu)_{\mu \in \Delta} \) a family of KAM circles with a rational rotation number \( \theta \), and \( N \) the denominator of \( \theta \). For any neighborhood \( \mathcal{V} \) of the family \((f_\mu)_{\mu \in \Delta} \) and \( \gamma \geq 1 \), there exists a family \((\hat{f}_\mu)_{\mu \in \Delta} \) in \( \mathcal{V} \) such that
\[
\min\{\#\text{Per}_{h}(\hat{f}_\mu, N), \#\text{Per}_{e}(\hat{f}_\mu, N)\} \geq N\gamma
\]
for any \( \mu \in \Delta \).

Proof. Since \( f_\mu \) preserves the circle \( \psi_\mu(S^1 \times \{0\}) \), we can take \( \delta > 0 \) such that the map \( F_\mu = \psi^{-1}_\mu \circ f_\mu \circ \psi_\mu \) is well-defined on \( S^1 \times [-\delta, \delta] \). Remark that \( F_\mu(x,0) = (x+\theta,0) \) and \((\partial/\partial y)[P_x \circ f_\mu](x,0) > 0 \) for any \( x \in S^1 \). Put \( x_{i,j} = \frac{2i\pi}{2\gamma} + j\theta \in S^1 \) and \( p_{i,j} = (x_{i,j},0) \) for \( i = 0, \ldots , 2\gamma - 1 \) and \( j = 0, \ldots , N \). Then, \( F_\mu(p_{i,j}) = p_{i,j+1} \) and \( p_{i,N} = p_{i,0} \). In particular, \( \{p_{i,0}, \ldots , p_{i,N-1}\} \) is an \( N \)-periodic orbit of \( F_\mu \).

Take a \( C^\infty \) function \( h \) on \( S^1 \) such that the support of \( h \) is contained in \( S^1 \times (-\delta, \delta) \), \( h''(x_{i,j}) = 0 \) and
\[
h''(x_{i,j}) = \begin{cases} (-1)^j & (j = 0) \\ 0 & (j = 1, \ldots , N-1) \end{cases}
\]
for any \( i = 0, \ldots , 2\gamma - 1 \) and \( j = 0, \ldots , N - 1 \). Put \( A_\mu = \psi_\mu(S^1 \times (-\delta, \delta)) \) for \( \mu \in \Delta \) and define a \( C^\infty \) function \( H \) on \( \Sigma \times \Delta \) by \( H(\cdot, \mu) = 0 \) on \( \Sigma \setminus A_\mu \) and
\[
H(\psi_\mu(x,y), \mu) = -h(x)
\]
for any \( (x,y) \in S^1 \times [-\delta/2, \delta/2] \) and any \( \mu \in \Delta \). Let \( \Psi_\mu \) be the Hamiltonian flow on \( \Sigma \) associated with the smooth Hamiltonian function \( H(\cdot, \mu) \). Since the support of the flow \( \Psi_\mu \) is contained in \( A_\mu \), \( G^t_\mu(x,y) = \psi_\mu^{-1} \circ \Psi^t_\mu \circ \psi_\mu(x,y) \) is well-defined for any \( \mu \in \Delta \), \( (x,y) \in S^1 \times (-\delta, \delta) \), and \( t \in \mathbb{R} \).

It is sufficient to show that
\[
\min\{\#\text{Per}_{h}(F_\mu \circ G^t_\mu, N), \#\text{Per}_{e}((F_\mu \circ G^t_\mu), N)\} \geq N\gamma.
\]
for any sufficiently small $t > 0$. By construction of the Hamiltonian function $H(\cdot, \mu)$, we have $G^t_\mu(x, y) = (x, y + h'(x)t)$ and hence,

$$ (DG^t_\mu)(x, y) = \begin{pmatrix} 1 & 0 \\ h''(x)t & 1 \end{pmatrix} $$

for any $(x, y) \in S^1 \times (-\delta/2, \delta/2)$ and any $t$ with $|y + h'(x)t| \leq \delta/2$. Since $h'(x_i, j) = 0$, each $p_{i,j}$ is a fixed point of $G^t_\mu$, and hence,

$$(F_\mu \circ G^t_\mu)(p_{i,j}) = F_\mu(p_{i,j}) = p_{i,j+1}$$

for any $t \geq 0$. In particular, $\{p_{i,0}, \ldots, p_{i,N-1}\}$ is an $N$-periodic orbit of $F_\mu \circ G^t_\mu$ for each $i = 0, \ldots, 2\gamma - 1$. By the condition on $h''$, we also have

$$D(F_\mu \circ G^t_\mu)_{p_{i,j}} = \begin{cases} (DF^N_\mu)_{p_{i,0}} \begin{pmatrix} 1 & 0 \\ (-1)^jt & 1 \end{pmatrix} & (j = 0) \\ (DF^N_\mu)_{p_{i,j}} & (j = 1, \ldots, N - 1). \end{cases}$$

To determine the hyperbolicity or ellipticity of the periodic orbit $\{p_{i,0}, \ldots, p_{i,N-1}\}$, it is sufficient to compute the trace of $D(F_\mu \circ G^t_\mu)^N_{p_{i,0}}$. By Equations (2) and (3),

$$\text{tr} D(F_\mu \circ G^t_\mu)^N_{p_{i,0}} = \text{tr} \left[ D(F_\mu \circ G^t_\mu)_{p_{i,N-1}} \cdots D(F_\mu \circ G^t_\mu)_{p_{i,0}} \right]$$

$$= \text{tr} \left[ (DF^N_\mu)_{p_{i,0}} \begin{pmatrix} 1 & 0 \\ (-1)^jt & 1 \end{pmatrix} \right]$$

$$= \text{tr}(DF^N_\mu)_{p_{i,0}} + (-1)^jt \cdot \frac{\partial}{\partial y} [P_\mu \circ F^N_\mu](p_{i,0}).$$

Since $F^N_\mu(x, 0) = (x + N\theta, 0) = (x, 0)$ for any $x \in S^1$, the derivative $(DF^N_\mu(x, 0))$ has the form

$$\begin{pmatrix} 1 & \ast \\ 0 & \ast \end{pmatrix}.$$

Since $F_\mu$ preserves a volume form, we have $\det(DF^N_\mu(x, 0)) = 1$, and hence, $\text{tr}(DF^N_\mu(x, 0)) = 1$. By Lemma 2.5, we also have $\frac{\partial}{\partial y} [P_\mu \circ F^N_\mu](x, 0) > 0$. Hence, by Equation (4), the $N$-periodic orbit $\{p_{i,0}, \ldots, p_{i,N-1}\}$ of $(F_\mu \circ G^t_\mu)$ is hyperbolic if $j = 0, \ldots, 2\gamma - 1$ is even and elliptic if $j$ is odd for any sufficiently small $t > 0$. This implies that

$$\min\{\# \text{Per}_h((F_\mu \circ G^t_\mu), N), \# \text{Per}_e((F_\mu \circ G^t_\mu), N)\} \geq N\gamma.$$

For any sufficiently small $t > 0$, the family $(F_\mu \circ G^t_\mu)_{\mu \in \Delta}$ is contained in $\mathcal{V}$.

Now, we prove Proposition 2.3. Fix an open neighborhood $\mathcal{V}$ of the family $(f_\mu)_{\mu \in \Delta}$, a sequence $(\gamma_n)_{n \geq 1}$ of positive integers, and $n_0 \geq 1$. By Lemma 2.4, there exist a rational number $\theta$ and a family $(\hat{f}_\mu)_{\mu \in \Delta}$ in $\mathcal{V}$ such that the denominator $N$ of $\theta$ is greater than $n_0$ and the family $(\hat{f}_\mu)_{\mu \in \Delta}$ admits a KAM
circle with rotation number $\theta$. By Lemma 2.6 there exists a family $(\hat{f}_\mu)_{\mu \in \Delta}$ in $\mathcal{V}$ such that
\[
\min\{\#\text{Per}_h(\hat{f}_\mu, N), \#\text{Per}_c(\hat{f}_\mu, N)\} \geq N \cdot \gamma_N.
\]
for any $\mu \in \Delta$.

3 Real analytic case

3.1 The $C^\omega$ topology

In this subsection, we describe the $C^\omega$-topology of the set of real-analytic Hamiltonian diffeomorphisms of $T^2$ and show maps whose all $n$ periodic points are non-degenerate are open and dense with respect to the topology.

Let $M_1$ and $M_2$ be complex manifolds and $d_{M_2}$ a distance on $M_2$ compatible with the topology of $M_2$. By $\text{Hol}(U, M_2)$, we denote the set of holomorphic maps from $M_1$ to $M_2$. For a compact subset $K$ of $M_1$, we define a pseudo-distance $d_K$ on $\text{Hol}(M_1, M_2)$ by
\[
d_K(f, g) = \sup_{z \in K} d_{M_2}(f(z), g(z)).
\]
The space $\text{Hol}(M_1, M_2)$ is endowed with the topology defined by the family of pseudo-distances $d_K$, where $K$ runs over all compact subsets of $M_1$.

Recall that $\text{Ham}^\omega(T^2)$ is the set of real-analytic Hamiltonian diffeomorphisms on $T^2$. For an open neighborhood $U$ of $T^2$ in $C^2/Z^2$, let $\text{Ham}^\omega_U(T^2)$ be the set of maps in $\text{Ham}^\omega(T^2)$ which extend to holomorphic maps from $U$ to $C^2/Z^2$. Remark that the holomorphic extension to $U$ is unique for each map in $\text{Ham}^\omega_U(T^2)$. In the rest of this article, we identify a map $f$ in $\text{Ham}^\omega_U(T^2)$ and its holomorphic extension to $U$. The space $\text{Ham}^\omega(T^2)$ is the union of $\text{Ham}^\omega_U(T^2)$’s, where $U$ runs over all open neighborhoods of $T^2$ in $C^2/Z^2$. Each $\text{Ham}^\omega_U(T^2)$ is endowed with the topology as a subset of $\text{Hol}(U, C^2/Z^2)$. The $C^\omega$-topology of $\text{Ham}^\omega(T^2)$ is the direct limit topology associated with the inclusions $\text{Ham}^\omega_U(T^2) \hookrightarrow \text{Ham}^\omega(T^2)$. More precisely, a subset $\mathcal{U}$ of $\text{Ham}^\omega(T^2)$ is open if and only if $\mathcal{U} \cap \text{Ham}^\omega_U(T^2)$ is an open subset of $\text{Ham}^\omega_U(T^2)$ for any open neighborhood $U$ of $T^2$ in $C^2/Z^2$. Remark that this topology is weaker than another topology on the space of real-analytic maps defined by Takens [27] (see also [6]). Takens’ topology has the Baire property, but our topology does not.

For $n \geq 1$, let $\mathcal{ND}_n$ be the subset of $\text{Ham}^\omega(T^2)$ consisting of maps whose all periodic points of period less than $n + 1$ are non-degenerate.

Proposition 3.1. Let $U$ be an openneighborhood of $T^2$ in $C^2/Z^2$ whose closure is compact. Then, the subset $\mathcal{ND}_n \cap \text{Ham}^\omega_U(T^2)$ is open and dense in $\text{Ham}^\omega_U(T^2)$ for any $n \geq 1$.

\footnote{Takens’ topology is too strong for our purpose. For example, let $\Phi = (\Phi^t)_{t \in \mathbb{R}}$ be a real-analytic flow on $T^2$ generated by a real-analytic vector field $X$. If $X$ does not extends to $C^2/Z^2$, then the map $t \mapsto \Phi^t \in \text{Diff}^\omega(T^2)$ is not continuous with respect to Takens’ $C^\omega$ topology while it is continuous with respect to our $C^\omega$ topology.}
Theorem 3.2. The following is a version of the KAM theorem proved in [26, Section 32, p.230–]

To show density, we follow the argument in [25 Section 11.3] (see also [3]). Let $Y_1, \ldots, Y_4$ be the Hamiltonian vector fields associated to Hamiltonian functions $\cos(2\pi x), \sin(2\pi x), \cos(2\pi y)$ and $\sin(2\pi y)$. Then, for any $(x, y) \in \mathbb{T}^2$, the tangent space $T_{(x,y)}\mathbb{T}^2$ is spanned by \{ $Y_1(x,y), \ldots, Y_4(x,y)$ \}. Let $\Psi_t$ be the Hamiltonian flow generated by the vector field $Y_i$. Fix $f_0 \in \text{Ham}_{\mathbb{C}}^\omega(\mathbb{T}^2)$ and define a $C^\omega$ map $F_n : \mathbb{T}^2 \times \mathbb{R}^4 \to \mathbb{T}^2 \times \mathbb{T}^2$ by $F_n(t_1, \ldots, t_4, (x, y)) = ((x, y), (\Psi_{t_1} \circ \cdots \circ \Psi_{t_4} \circ f_0)(x, y))$. Since $Y_i$ extends to $\mathbb{C}^2/\mathbb{Z}^2$ and the closure of $U$ is compact, $\Psi_t$ extends to $U$ for any sufficiently small $t > 0$. By the choice of $Y_1, \ldots, Y_4$, the map $F_n$ is transverse to the diagonal $\{(p, p) \mid p \in \mathbb{T}^2\}$ in $\mathbb{T}^2 \times \mathbb{T}^2$ if $f$ belongs to $\mathcal{N}\mathcal{D}_{\alpha-1}$. Using this map $F_n$, we can show that the density of $\mathcal{N}\mathcal{D}_{\alpha}$ in $\text{Ham}_{\mathbb{C}}^\omega(\mathbb{T}^2)$ by the same argument as in [25 Section 11.3].

\[ \text{3.2 Persistence of KAM curves} \]

Put $A(r) = \{ z \in \mathbb{C}/\mathbb{Z} \mid |\text{Im } z| < r \}$ for $r > 0$. By $P_x$ and $P_y$, we denote the natural projections from $\mathbb{T}^2$ to the first and second coordinates. For an open neighborhood $U$ of $\mathbb{T}^2$ in $\mathbb{C}^2/\mathbb{Z}^2$, a map $f \in \text{Ham}_{\mathbb{C}}^\omega(\mathbb{T}^2)$, and a real number $\theta$, we call a holomorphic map $\varphi : A(r) \to \mathbb{C}^2/\mathbb{Z}^2$ a graph KAM curve with rotation number $\theta$ if $f \circ \varphi(z) = \varphi(z + \theta)$ for any $z \in A(r)$, $P_x \circ \varphi$ is a diffeomorphism between $A(r)$ onto its image, and the twist condition

\[ \frac{\partial}{\partial y}(P_x \circ f) > 0 \]

holds on $\varphi(S^1)$. For a graph KAM curve $\varphi$ and $0 < r' \leq r$, we put

\[ U_{\varphi,r'} = \{ (z_1, z_2) \in U \mid z_1 \in P_x \circ \varphi(A(r')) \}. \]

We say that a real number $\theta$ is Diophantine if there exists $\tau > 0$ and $c > 0$ such that

\[ |\theta - \frac{p}{q}| \geq \frac{c}{|q|^{2+\tau}} \]

for any rational number $p/q$. We say that a map $f : S^1 \times (a,b) \to S^1 \times \mathbb{R}$ has the intersection property if for any continuous map $g : S^1 \to (a,b)$ the curve $C_g = \{ (x, g(x)) \mid x \in S^1 \}$ intersects with $f(C_g)$. It is known that if $f_H$ is a Hamiltonian diffeomorphism of $\mathbb{T}^2$ and $\varphi : S^1 \times (a,b) \to \mathbb{T}^2$ be an embedding, then $\varphi^{-1} \circ f_H \circ \varphi$ satisfies the intersection property. For $r > 0$ and $\delta > 0$, put

\[ V(r, \delta) = \{ (z_1, z_2) \in (\mathbb{C}/\mathbb{Z}) \times \mathbb{C} \mid |\text{Im } z_1| < r, |z_2| < \delta \}. \]

The following is a version of the KAM theorem proved in [26] Section 32, p.230–235.

**Theorem 3.2.** For any given Diophantine real number $\theta$, real numbers $\alpha > 0$, $0 < \tilde{r} < r$, and $\eta > 0$, there exists $0 < \delta_* < 1$ such that for any holomorphic map $F : V(r, \delta) \to (\mathbb{C}/\mathbb{Z}) \times \mathbb{C}$ and $0 < \delta < \delta_*$ satisfying that
1. \( F(S^1 \times (-\delta, \delta)) \subset S^1 \times \mathbb{R} \).
2. the restriction of \( F \) on \( S^1 \times (-\delta, \delta) \) has the intersection property, and
3. \( \sup_{(z_1, z_2) \in V(r, \delta)} |F(z_1, z_2) - (z_1 + \theta + \alpha z_2, z_2)| < \delta \).

we can find a holomorphic map \( \psi : A(\hat{r}) \rightarrow V(r, \delta) \) such that \( F \circ \psi(z) = \psi(z + \theta) \) for any \( z \in A(\hat{r}) \) and \( \sup_{z \in A(\hat{r})} \| \psi(z) - (z, 0) \| < \eta \).

The following lemma corresponds to Proposition 2.2 in real-analytic case.

**Lemma 3.3.** Let \( U \) be an open neighborhood of \( T^2 \) in \( \mathbb{C}^2 / \mathbb{Z}^2 \), \( f \) a diffeomorphism in \( \text{Ham}_0^r (T^2) \), \( r > \hat{r} \) positive real numbers, \( \varphi : A(r) \rightarrow U \) a graph KAM curve with a Diophantine rotation number \( \theta \), and \( K \) a compact neighborhood of \( T^2 \) in \( \mathbb{C}^2 / \mathbb{Z}^2 \) such that \( K \subset U_{\varphi, \hat{r}} \). Then, there exists a neighborhood \( \mathcal{U} \) of \( f \) in \( \text{Ham}_0^r (T^2) \) such that any \( f \in \mathcal{U} \) admits a graph KAM curve \( \hat{\varphi} : A(\hat{r}) \rightarrow \mathcal{U} \) with rotation number \( \theta \) such that \( K \subset U_{\hat{\varphi}, \hat{r}} \).

**Proof.** Define a map \( \psi_1 : A(r) \times \mathbb{C} \rightarrow \mathbb{C}^2 / \mathbb{Z}^2 \) by

\[
\psi_1(z_1, z_2) = \left( P_x \circ \varphi(z_1), \frac{1}{(P_x \circ \varphi)'(z_1)} \cdot z_2 + P_y \circ \varphi(z_1) \right).
\]

Then, \( \psi_1(z, 0) = \varphi(z) \) and \( f \circ \psi_1(z, 0) = \psi_1(z + \theta, 0) \). Hence, \( \psi_1(A(r') \times \{0\}) \) is an \( f \)-invariant subset of \( U_{\varphi, r} \) for any \( 0 < r' \leq r \). Take \( r_1 \in (\hat{r}, r) \) and \( \delta_1 > 0 \) such that \( \psi_1(V(r_1, \delta_1)) \subset U_{\varphi, 1} \) and \( F_1 = \psi_1^{-1} \circ f \circ \psi_1 \) is well-defined on \( V(r_1, \delta_1) \). Remark that \( F_1(z, 0) = (z + \theta, 0) \) for any \( x \in A(r_1) \) and the restriction of \( F_1 \) to \( S^1 \times (-\delta_1, \delta_1) \) preserves the standard volume form \( dx \wedge dy \).

Put \( \alpha(z) = (\partial / \partial y) [P_x \circ F_1](z, 0) \) for \( z \in A(r_1) \). Then, for any \( x \in S^1 \), we have \( \alpha(x) > 0 \) and

\[
(DF_1)(x, 0) = \begin{pmatrix} 1 & \alpha(x) \\ 0 & 1 \end{pmatrix}.
\]

Since \( \theta \) is Diophantine, the equation

\[
\alpha(x) = \beta(x + \theta) - \beta(x) + \bar{\alpha}
\]

has a solution \( \beta : S^1 \rightarrow \mathbb{R} \) and \( \alpha_\ast \in \mathbb{R} \) and the function \( \beta \) extends to a holomorphic function on \( A(r_1) \). Since \( \alpha(x) > 0 \) for any \( x \in S^1 \), we have \( \alpha_\ast > 0 \).

Define a holomorphic embedding \( \psi_2 : A(r_1) \times \mathbb{C} \rightarrow (\mathbb{C} / \mathbb{Z}) \times \mathbb{C} \) by

\[
\psi_2(z_1, z_2) = (z_1 + \beta(z_1) z_2, z_2).
\]

Since \( \psi_2(z, 0) = (z, 0) \), there exist \( r_2 \in (\hat{r}, r_1) \) and \( \delta_2 \in (0, \delta_1) \) such that \( F_2 = \psi_2^{-1} \circ F_1 \circ \psi_2 \) is well-defined on \( V(r_2, \delta_2) \). The map \( F_2 \) satisfies that \( F_2(x, 0) = (x + \theta, 0) \) and

\[
(DF_2)(x, 0) = \begin{pmatrix} 1 & \beta(x + \theta) \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & \alpha(x) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \beta(x) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \alpha_\ast \\ 0 & 1 \end{pmatrix}.
\]

\(^6\) In [26] Section 32, p.230–235, the proof is given for the case \( \hat{r} = r/2 \). But, it is easy to show the theorem in our form by a modification of constants. The constants \( \delta \) and \( \delta_\ast \) in the theorem correspond to \( s_0 \) and \( d_0 \) in [26].
This implies that there exist $C > 0$, $r_3 \in (\hat{r}, r_2)$, and $\delta_3 \in (0, \delta_2)$ such that
\[
\sup_{(z_1, z_2) \in \mathcal{V}(r_3, \delta_3)} \| F_2(z_1, z_2) - (z_1 + \theta + \alpha_2 z_2, z_2) \| \leq C \delta^2 \tag{5}
\]
for any $\delta \in (0, \delta_3)$. Put $\psi = \psi_1 \circ \psi_2$. Then, $\psi(V(r_2, \delta_2)) \subset U_{\varphi, r}$, $\psi(z, 0) = \varphi(z)$, and $F_2 = \psi^{-1} \circ f \circ \psi$ has the intersection property. Fix $r_4 \in (\hat{r}, r_3)$. By Cauchy’s integral formula, there exists $\eta > 0$ such that for any holomorphic map $\Phi : A(r_4) \to \mathbb{C}/\mathbb{Z} \times \mathbb{C}$ with $\sup_{z \in A(r_4)} \| \Phi(z) - (z, 0) \| < \eta$, the the restriction of $P_\varphi (\psi \circ \Phi)$ to $A(\hat{r})$ is a holomorphic diffeomorphism onto its image and $K \subset U_{\psi \circ \Phi, \hat{r}}$.

Let $\delta_e \in (0, \delta_3)$ be the constant in Theorem 3.2 for $\theta, \alpha_2, r_3 > r_4$ and $\eta$. Take $\hat{\delta} \in (0, \delta_e)$ such that $C \hat{\delta}^2 < \hat{\delta}^2/2$. Since $F_2 = \psi^{-1} \circ f \circ \psi$ is well-defined on $V(r_2, \delta_2)$, there exists a neighborhood $\mathcal{U}$ of $f$ in $\text{Ham}_{\mathbb{C}}^0(T^2)$ such that for any $\hat{f} \in \mathcal{U}$, $\psi^{-1} \circ f \circ \psi$ is well-defined on $V(r_3, \delta_3)$ and
\[
\sup_{(z_1, z_2) \in \mathcal{V}(r_3, \delta_3)} \| \psi^{-1} \circ \hat{f} \circ \psi(z_1, z_2) - F_2(z_1, z_2) \| < \frac{\delta^2}{2}.
\]
With (5), we have
\[
\sup_{(z_1, z_2) \in \mathcal{V}(r_3, \delta_3)} \| \psi^{-1} \circ \hat{f} \circ \psi(z_1, z_2) - (z_1 + \theta + \alpha_2 z_2, z_2) \| < \delta^2
\]
for any $\hat{f} \in \mathcal{U}$. By Theorem 3.2 for $\hat{f} \in \mathcal{U}$ there exists a holomorphic map $\Phi : A(r_4) \to V(r_3, \delta_3)$ such that $(\psi^{-1} \circ \hat{f} \circ \psi) \circ \Phi(z) = \Phi(z + \theta)$ and $\sup_{z \in A(r_4)} \| \psi(z) - (z, 0) \| < \eta$. Put $\hat{\varphi} = \psi \circ \Phi$. Then, $\hat{f} \circ \hat{\varphi}(z) = \hat{\varphi}(z + \theta)$, $P_\varphi \circ \hat{\varphi} : A(\hat{r}) \to \mathbb{C}/\mathbb{Z}$ is a holomorphic diffeomorphism onto its image, and $K \subset U_{\hat{\varphi}, \hat{r}}$. This means that $\hat{\varphi}$ is a graph KAM curve of $\hat{f}$ such that $K \subset U_{\hat{\varphi}, \hat{r}}$. \hfill \Box

### 3.3 Proof for the real-analytic case

The following proposition corresponds to Proposition 2.3 in real-analytic case.

**Proposition 3.4.** Let $U_0$ be an open neighborhood of $T^2$ in $C^2/\mathbb{Z}^2$, $f_0$ a diffeomorphism in $\text{Ham}_{\mathbb{C}}^0(T^2)$, $r_0 > r' > r_1$ positive real numbers, $\varphi_0 : A(r_0) \to U_0$ a graph KAM curve of $f_0$, and $K$ a compact subset of $(U_0)_{\varphi_0, r_1}$. Then, there exists an open neighborhood $U_1$ of $T^2$ which satisfies the following conditions:

1. The closure of $U_1$ is compact.
2. $\varphi_0(A(r')) \subset U_1$ and $K \subset (U_1)_{\varphi_0, r_1}$.
3. For any sequence $(\gamma_n)_{n \geq 1}$ of positive integers, any $n_0 \geq 1$, and any neighborhood $\mathcal{U}$ of $f_0$ in $\text{Ham}_{\mathbb{C}}^0(T^2)$, there exist $f_1 \in \mathcal{U}$ and $n_1 > n_0$ such that
   \[
   \min\{\# \text{Per}_h(f_1, n_1), \# \text{Per}_r(f_1, n_1)\} \geq n_1 \cdot \gamma_{n_1}.
   \]
Proof. Proof is analogous to Proposition 2.3

Since $K \cup \varphi_0(A(r'))$ is a compact subset of $(U_0)_{\varphi_0,r_0}$, we can take a compact subsets $K'$ and $K_1$ of $(U_0)_{\varphi_0,r_0}$ such that $K \cup \varphi_0(A(r_1)) \subset \text{Int} \ K_1$ and $K_1 \subset \text{Int} \ K'$. Let $U_1$ be interior of $K_1$. Then, the closure of $U_1$ is compact, $K \subset (U_1)_{\varphi_0,r_1}$, and $\varphi_0(A(r_1)) \subset U_1$. Put $\varphi_x = P_x \circ \varphi_0$ and $g = (P_y \circ \varphi_0) \circ \varphi_x^{-1}$. Define a holomorphic function $H_1$ on $(U_0)_{\varphi_0,r_0}$ by

$$H_1(z_1,z_2) = \frac{1}{2\pi i}(\varphi_x)'(z_1) \cdot \sin[2\pi(z_2 - g(z_1))].$$

Let $X_1$ be the Hamiltonian vector field associated with the Hamiltonian function $H_1$ and $\Phi_1^t$ the (local) flow generated by $X_1$. The vector field $X_1$ extends to a holomorphic vector field on $(U_0)_{\varphi_0,r_0}$, and hence, there exists $T_1 > 0$ such that $\Phi_1^t$ extends to $K'$ for any $t \in [0,T_1]$. The vector field $X_1$ satisfies that

$$X_1(\varphi_0(z)) = X_1(\varphi_x(z),g(\varphi_x(z)))$$

$$= \varphi_x'(z) \left( \frac{\partial}{\partial x} + g'(\varphi_x(z)) \frac{\partial}{\partial y} \right)$$

$$= (D\varphi_0)_z \left( \frac{\partial}{\partial z} \right).$$

This implies that $\Phi_1^t(\varphi_0(z)) = \varphi_0(z + t)$ for any $z \in A(r_1)$ and $t \in [0,T_1]$.

Fix a sequence $(\gamma_n)_{n \geq 1}$ of positive integers, $n_0 \geq 1$, and an open neighborhood $U$ of $f_0$ in $\text{Ham}^0(T^2)$. Let $\theta_0$ be the rotation number of the KAM curve $\varphi_0$. Take $t_1 \in (0,T_1)$ such that $\theta_1 = \theta_0 + t_1$ is a rational number with denominator $n_1 > n_0$ and $f_0 \circ \Phi_1^t$ is contained in $U$. Put $\hat{f} = f_0 \circ \Phi_1^t$, $\gamma = \gamma_{n_1}$, $p_{i,j} = \varphi_0(\frac{1}{2\gamma_{n_1}} + j\theta_1)$, and $x_{i,j} = P_x(p_{i,j})$ for $i = 0,\ldots,2\gamma - 1$ and $j \geq 0$. Then,

$$\hat{f}(x_{i,j},g(x_{i,j})) = \hat{f}(p_{i,j}) = p_{i,j+1} = (x_{i,j+1},g(x_{i,j+1})).$$

Using trigonometric functions, we can construct a real-analytic function $h_2$ on $S^1$ which extends to $\mathbb{C}/\mathbb{Z}$ and satisfies that $h_2'(x_{i,j}) = 0$, $h_2''(x_{i,0}) = (-1)^i$, and $h_2''(x_{i,1}) = \cdots = h_2''(x_{i,n_1-1}) = 0$ for any $i = 0,\ldots,2\gamma - 1$ and $j = 0,\ldots,n_1 - 1$. Put $H_2(x,y) = -h_2(x)$ for $(x,y) \in T^2$ and let $\Phi_2$ be the Hamiltonian flow associated with the Hamiltonian function $H_2$. The flow $\Phi_2$ satisfies that $\Phi_2^t(x,y) = (x,y + h_2(x)t)$ for any $(x,y) \in T^2$ and $t \geq 0$. Since $H_2$ extends to $\mathbb{C}^2/\mathbb{Z}^2$, there exists $T_2 > 0$ such that $\Phi_2^t$ extends to $T_2$ for any $t \in [0,T_2]$.

Since $\hat{f}^{n_1} \circ \varphi_0(x) = \varphi_0(x + n_1\theta_1) = \varphi_0(x)$ for any $x \in S^1$ and $\hat{f}$ preserves the standard volume form $dx \wedge dy$ of $T^2$, we have $\text{tr} D\hat{f}^{n_1}_{x_{i,j}} = 1$. By the same argument as in Lemma 2.3, we can show that $\{p_{i,0},\ldots,p_{i,n_1-1}\}$ is a periodic orbit of $\hat{f} \circ \Phi_2^t$ which is hyperbolic if $i$ is even and elliptic if $i$ is odd for any $t \in (0,T_2)$. Take a small $T \in (0,T_2)$ such that $\hat{f} \circ \Phi_2^T$ is contained in $U$. Then, $f_1 = \hat{f} \circ \Phi_2^T$ satisfies that

$$\min\{\# \text{Per}_h(f_1,n_1),\# \text{Per}_e(f_1,n_1)\} \geq n_1\gamma.$$
With Lemma 3.3 and Proposition 3.4 we show that any Hamiltonian diffeomorphism with a graph KAM curve can be approximated by another Hamiltonian diffeomorphism which admits a graph KAM curve and many periodic points of high period. Recall that \( \mathcal{ND}_n \) is the subset of \( \text{Ham}^\omega(T^2) \) consisting of maps such that all periodic points of period less than \( n + 1 \) are non-degenerate.

**Lemma 3.5.** Let \((\gamma_n)_{n \geq 1}\) be a sequence of positive integers, \( K \) a compact subset of \( \mathbb{C}^2/\mathbb{Z}^2 \), and \( \theta \) a Diophantine real number. Take an open neighborhood \( U_0 \) of \( T^2 \) in \( \mathbb{C}^2/\mathbb{Z}^2 \), a diffeomorphism \( f_0 \) in \( \text{Ham}_{\mathbb{C}^2}^\omega(T^2) \), positive numbers \( r_0 > r_1 \), and a graph KAM curve \( \varphi_0 : A(r_0) \rightarrow U_0 \) with rotation number \( \theta \) such that \( K \subset (U_0)_{\varphi_0,r_1} \). For any given \( n_0 \geq 1 \) and \( \epsilon > 0 \), there exists \( n_1 > n_0 \), an open neighborhood \( U_1 \) of \( T^2 \) in \( \mathbb{C}^2/\mathbb{Z}^2 \), a diffeomorphism \( f_1 \) in \( \text{Ham}_{U_1}^\omega(T^2) \cap \mathcal{ND}_{n_1} \), and \( \varphi_1 : A(r_1) \rightarrow U_1 \) a graph KAM curve with rotation number \( \theta \) such that \( K \subset (U_1)_{\varphi_1,r_1} \), \( d_K(f_1,f_0) < \epsilon \), and

\[
\min\{\#\text{Per}_h(f_1,n_1),\#\text{Per}_e(f_1,n_1)\} \geq n_1 \cdot \gamma_{n_1}.
\]

**Proof.** Take \( r' \in (r_1,r_0) \). By Proposition 3.4 we can find an open neighborhood \( U_1 \) of \( T^2 \) satisfying the following conditions:

1. The closure of \( U_1 \) is compact.
2. \( \varphi_0(A(r')) \subset U_1 \) and \( K \subset (U_1)_{\varphi_0,r_1} \).
3. For any neighborhood \( \mathcal{U} \) of \( f_0 \) in \( \text{Ham}_{U_1}^\omega(T^2) \), there exists \( f_\mathcal{U} \in \text{Ham}_{U_1}^\omega(T^2) \) and \( n_\mathcal{U} > n_0 \) such that

\[
\min\{\#\text{Per}_h(f_\mathcal{U},n_\mathcal{U}),\#\text{Per}_e(f_\mathcal{U},n_\mathcal{U})\} \geq n_\mathcal{U} \cdot \gamma_{n_\mathcal{U}}.
\]

The set \( \mathcal{ND}_{n_1} \cap \text{Ham}_{U_1}^\omega(T^2) \) is dense in \( \text{Ham}_{U_1}^\omega(T^2) \) by Proposition 3.3. Since the numbers \( \#\text{Per}_h(f,n) \) and \( \#\text{Per}_e(f,n) \) are lower semi-continuous with respect to \( f \), we may assume that \( f_\mathcal{U} \) is contained in \( \mathcal{ND}_{n_\mathcal{U}} \) by approximating \( f_\mathcal{U} \) by a map in \( \mathcal{ND}_{n_\mathcal{U}} \) if it is necessary. Applying Lemma 3.3 to \( U_1, f_0, r_1 < r', \theta \), and \( \varphi_0 : A(r') \rightarrow U_1 \), we can find a neighborhood \( \mathcal{U}_1 \) of \( f_0 \) in \( \text{Ham}_{U_1}^\omega(T^2) \) such that any \( f \in \mathcal{U}_1 \) admits a graph KAM curve \( \varphi_f : A(r_1) \rightarrow U_1 \) with rotation number \( \theta \) such that \( K \subset (U_1)_{\varphi_f,r_1} \). By shrinking \( \mathcal{U}_1 \) if it is necessary, we may assume that \( \mathcal{U}_1 \subset \{\hat{f} \in \text{Ham}_{U_1}^\omega(T^2) \mid d_K(\hat{f},f_0) < \epsilon\} \). Put \( n_1 = n_{\mathcal{U}_1}, f_1 = f_{\mathcal{U}_1}, \) and \( \varphi_1 = \varphi_{f_1} \). Then, the triple \((n_1,f_1,\varphi_1)\) satisfies the required conditions. \( \square \)

Next, we show that any Hamiltonian diffeomorphism with a graph KAM curve can be approximated by another one which exhibits fast growth of the number of periodic points. Put \( \mathcal{ND}_\infty = \bigcap_{n \geq 1} \mathcal{ND}_n \).

**Proposition 3.6.** Let \( U_0 \) be an open neighborhood of \( T^2 \) in \( \mathbb{C}^2/\mathbb{Z}^2 \), \( f_0 \) a diffeomorphism in \( \text{Ham}_{U_0}^\omega(T^2) \), \( \varphi : A(r_0) \rightarrow U_0 \) a graph KAM curve with a Diophantine rotation number, \( K \) a compact subset of \( (U_0)_{\varphi_0,r_0} \), and \((\gamma_n)_{n \geq 1}\) a sequence of positive integers. Then, we can find an open subset \( U_\infty \) of \( U_0 \) such that
1. $K \subset U_\infty$ and

2. for any neighborhood $U$ of $f_0$ in $\text{Ham}^\omega_{U_\infty}(T^2)$, there exists $f_\infty \in U \cap \mathcal{N} \mathcal{D}_\infty$ such that 

$$\limsup_{n \to \infty} \frac{1}{\gamma_n} \min \{ \# \text{Per}_h(f_\infty, n), \# \text{Per}_e(f_\infty, n) \} = \infty.$$  \hspace{1cm} (6)

**Proof.** Take a compact subset $K_\infty$ of $(U_0)_{\varphi_0,r_0}$ such that $K \subset \text{Int} K_\infty$. Put $U_\infty = \text{Int} K_\infty$ and fix a neighborhood $U$ of $f_0$ in $\text{Ham}^\omega_{U_\infty}(T^2)$. Take $\epsilon_0 > 0$ and a compact set $K'_\infty \subset U_\infty$ such that the set 

$$\{ f \in \text{Ham}^\omega_{U_\infty}(T^2) \mid d_{K'_\infty}(f, f_0) < \epsilon_0 \}$$

is contained in $U$. Put $n_0 = 1$. We claim that there exist a sequence $(U_k, f_k, r_k, \varphi_k, n_k, \epsilon_k)_{k \geq 1}$ of sextuplets such that

1. $U_k$ is an open neighborhood of $T^2$ in $C^2/\mathbb{Z}^2$, $f_k \in \text{Ham}^\omega_{U_k}(T^2)$,
2. $r_k$ is a positive number and $\varphi_k : A(r_k) \to U_k$ is a graph KAM curve of $f_k$ with a Diophantine rotation number such that $K_\infty \subset (U_k)_{\varphi_k,r_k}$,
3. $n_k > n_{k-1}$ and $0 < \epsilon_k < \epsilon_{k-1}/3$,
4. $d_{K_\infty}(f_k, f_{k-1}) \leq \epsilon_{k-1}/3$, and
5. any $f \in \text{Ham}^\omega_{U_k}(T^2)$ with $d_{K_\infty}(f, f_k) < \epsilon_k$ is contained in $\mathcal{N} \mathcal{D}_{n_k}$ and satisfies that 

$$\min \{ \# \text{Per}_h(f, n_k), \# \text{Per}_e(f, n_k) \} \geq n_k \cdot \gamma_{n_k}.$$ 

for any $k \geq 1$. By assumption, the quadruplet $(U_0, f_0, r_0, \varphi_0)$ satisfies the first and second items in the above claim for $k = 0$. We show that if $(U_{k-1}, f_{k-1}, r_{k-1}, \varphi_{k-1})$ satisfies the first and second items in the above claim for some $k \geq 1$, then there exists $(U_k, f_k, r_k, \varphi_k, n_k, \epsilon_k)$ which satisfies all the items. Then, the claim will be proved by induction.

Take $r_k \in (0, r_{k-1})$ such that $K \subset (U_{k-1})_{\varphi_{k-1},r_k}$. By Lemma 5.3, there exist an open neighborhood $U_k$ of $T^2$ in $C^2/\mathbb{Z}^2$, $f_k \in \text{Ham}^\omega_{U_k}(T^2) \cap \mathcal{N} \mathcal{D}_{n_k}$, a graph KAM curve $\varphi_k : A(r_k) \to U_k$ of $f_k$ with a Diophantine rotation number such that $K_\infty \subset (U_k)_{\varphi_k,r_k}$, $d_{K_\infty}(f_k, f_{k-1}) < \epsilon_{k-1}/3$, and 

$$\min \{ \# \text{Per}_h(f_k, n_k), \# \text{Per}_e(f_k, n_k) \} \geq n_k \cdot \gamma_{n_k}.$$ 

Since $\mathcal{N} \mathcal{D}_{n_k}$ is an open subset of $\text{Ham}^\omega(T^2)$, the set $\text{Ham}^\omega_{\text{Int} K_\infty}(T^2) \cap \mathcal{N} \mathcal{D}_{n_k}$ is an open subset of $\text{Ham}^\omega_{\text{Int} K_\infty}(T^2)$. In $\mathcal{N} \mathcal{D}_{n_k}$, the numbers of hyperbolic (resp. elliptic) periodic points of period $n_k$ is locally constant. Hence, we can take $\epsilon_k \in (0, \epsilon_{k-1}/3)$ such that any $f \in \text{Ham}^\omega_{U_k}(T^2)$ with $d_{K_\infty}(f, f_k) < \epsilon_k$ is contained in $\mathcal{N} \mathcal{D}_{n_k}$ and satisfies that 

$$\min \{ \# \text{Per}_h(f, n_k), \# \text{Per}_e(f, n_k) \} \geq n_k \cdot \gamma_{n_k}.$$ 

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Therefore, \((U_k, f_k, r_k, \varphi_k, n_k, \epsilon_k)\) satisfies all the items in the claim for \(k\).

Since \(d_{K_\infty}(f_k, f_{k-1}) < \epsilon_{k-1}/3\) and \(0 < \epsilon_k < \epsilon_{k-1}/3\) for any \(k \geq 1\), we have \(\epsilon_k < 3^{-k} \epsilon_0\) and \(d_{K_\infty}(f_l, f_k) < 2\epsilon_k/3\) for any \(l \geq k \geq 0\). This implies that the limit \(f_\infty = \lim_{k \to \infty} f_k\) exists and it is a continuous map on \(K_\infty\) such that \(d_{K_\infty}(f_\infty, f_k) < \epsilon_k\) for any \(k \geq 0\). Since \(d_{K_\infty}(f_\infty, f_0) < \epsilon_0\), the map \(f_\infty\) is an element of \(U\). By the choice of \(\epsilon_k\), the map \(f_\infty\) is contained in \(\mathcal{N}_D\) and

\[
\min\{\# \text{Per}_h(f_\infty, n_k), \# \text{Per}_e(f_\infty, n_k)\} \geq n_k \cdot \gamma_{n_k}
\]

for any \(k \geq 1\).

Now, we prove Theorem 1.1. Let \(U_*\) be the subset of \(\text{Ham}^\omega(\mathbb{T}^2)\) consisting of diffeomorphisms admitting a graph KAM curve with a Diophantine rotation number. It is easy to see that the map \(f(x, y) = (x + \sin(2\pi y), y)\) is contained in \(U_*\), and hence, \(U_*\) is non-empty. By Lemma 3.3 for any open neighborhood \(U\) of \(\mathbb{T}^2\) and any \(f \in \text{Ham}^\omega_U(\mathbb{T}^2)\), there exists a neighborhood \(U_f\) of \(f\) in \(\text{Ham}^\omega_U(\mathbb{T}^2)\) such that any \(\tilde{f} \in U_f\) admits a graph KAM curve. This implies that \(U_* \cap \text{Ham}^\omega_U(\mathbb{T}^2)\) is an open subset of \(\text{Ham}^\omega_U(\mathbb{T}^2)\) for any \(U\). Hence, \(U_*\) is an open subset of \(\text{Ham}^\omega(\mathbb{T}^2)\).

Fix a sequence \((\gamma_n)_{n \geq 1}\) of positive integers, \(f \in U_*\), and an open neighborhood \(U \subset U_*\) of \(f\). In order to prove Theorem 1.1 it is sufficient to show that there exists \(f_\infty \in U \cap \mathcal{N}_D\) such that

\[
\limsup_{n \to \infty} \frac{1}{\gamma_n} \min\{\# \text{Per}_h(f_\infty, n), \# \text{Per}_e(f_\infty, n)\} = \infty. \tag{7}
\]

Let \(U\) be an open neighborhood of \(\mathbb{T}^2\) such that \(f\) is an element of \(\text{Ham}^\omega_U(\mathbb{T}^2)\). The map \(f\) admits a graph KAM curve \(\varphi : A(r) \to U\) with a Diophantine rotation number. By Proposition 3.6 there exist an open subset \(U_\infty\) of \(U\) and \(f_\infty \in \text{Ham}^\omega_{U_\infty}(\mathbb{T}^2) \cap U \cap \mathcal{N}_D\) which satisfies the condition \(7\).

A Another example of smooth unimodal map with fast growth

In this appendix, we give a simple example of a \(C^\infty\) unimodal map on a compact interval which exhibits fast growth of the number of periodic points. Let \(\text{Per}(f, n)\) be the set of periodic point of a map \(f\) whose minimal period is \(n\).

**Theorem A.1.** Let \((\gamma_n)_{n \geq 1}\) be a sequence of positive integers. Then, there exists a \(C^\infty\) unimodal self-map \(f\) on \([-1, 1]\) such that all periodic points of \(f\) are hyperbolic and

\[
\# \text{Per}(f, n) \geq \gamma_n \tag{8}
\]

for any \(n \geq 2\).

The unique critical point of the above map must be flat as mentioned in Section 1. For any finite \(r \geq 1\), Kaloshin and Kozlovski [17] constructed a \(C^r\)
unimodal map \( f \) which is of class \( C^\infty \) except at the critical point and satisfies 
\[ \# \text{Per}(f, 3^k) \geq \gamma_3k \] for any \( k \geq 1 \). While their construction of the example 
starts from an infinitely many renormalizable map, our construction starts from a 
Misiurewicz map.

**Proof.** Our proof is inspired by de Melo’s construction of a \( C^\infty \) interval map 
with infinitely many periodic attractors [8].

Recall that the \( C^\infty \) topology of the space of \( C^\infty \) self-maps on \([-1, 1]\) is given 
by the metric 
\[
d_{C^\infty}(f, g) = \sum_{s=0}^{\infty} 2^{-s} \frac{\|f^{(s)} - g^{(s)}\|_{sup}}{1 + \|f^{(s)} - g^{(s)}\|_{sup}},
\]
where \( \|h\|_{sup} = \sup_{x \in [-1, 1]} |h(x)| \) and \( f^{(s)} \) is the \( s \)-th derivative of \( f \).

Take a \( C^\infty \) non-decreasing function \( \chi \) on \( \mathbb{R} \) such that \( \chi(t) = 0 \) if \( t \leq 0 \) and 
\( \chi(t) = 1 \) if \( t \geq 1 \). Fix \( 0 < \delta < 1/4 \) and define a \( C^\infty \) function \( g_k \) by 
\[ g_k(x) = 2^{-k}(1 + \chi((2k - 1)(2k\delta^{-1}x - 1))) (1 + x). \]

Put \( x_k = \delta/(2k + 1) \) and \( x'_k = \delta/(2k) \). Then, we have \( g_k(x) = g_{k+1}(x) = 2^{-k}(1 + x) \) \( x \in [x_k, x'_k] \) and \( \lim_{k \to \infty} d_{C^\infty}(g_k, 0) = 0 \). There exists a \( C^\infty \) 
self-map \( f \) on \([-1, 1]\) such that \( x = 0 \) is the unique critical point of \( f_0 \) and 
\[
f_0(x) = \begin{cases} 
1 + 2x & (x \in [-1, -1/3]), \\
1 - 2x & (x \in [1/3, 1]), \\
1 - g_k(x) & (x \in [x_k, x_k - 1], k \geq 1).
\end{cases}
\]

Since \( f_0(y - 1) = 2y - 1 \) for \( y \in [0, 2/3] \), we have 
\[ f_0^{k+1}(x) = f_0^k(1 - 2^{-k}(1 + x)) = f_0^{k-1}(2^{-(k-1)}(1 + x) - 1) = x \]
for any \( k \geq 1 \) and \( x \in [x_k, x'_k] \).

Since \( x = 0 \) is the unique critical point of \( f_0 \), we can take a continuous 
function \( \eta \) on \([0, 1]\) such that \( 0 < \eta(|x|) < |f_0(x)| \) for any \( x \in [-1, 1] \setminus \{0\} \). 
Let \( C^\infty_\eta([-1, 1]) \) be the set of \( C^\infty \) self-maps \( f \) on \([-1, 1]\) such that \( 0 < \eta(|x|) < |f'(x)| \) 
for any \( x \in [-1, 1] \setminus \{0\} \). For \( n \geq 1 \), let \( \mathcal{N}D_n^\eta \) be the subset of \( C^\infty_\eta([-1, 1]) \) 
consisting of map whose all periodic points of period less than \( n + 1 \) are hyperbolic. Remark that \( \mathcal{N}D_n^\eta \) is open and dense in \( C^\infty_\eta([-1, 1]) \) with respect to the 
\( C^\infty \) topology.

For \( m \geq 1 \), define a subset \( I_m \) of \([-1, 1]\) by 
\[
I_m = \bigcup_{k=m}^{\infty} \bigcup_{n=0}^{k} f^m_0([x_k, x'_k])
\]
Let \( \bar{I}_m \) be the closure of \( I_m \). For \( 0 \leq j \leq k - 2 \), we have 
\[
f_0^{k-j}([x_k, x_{k+1}]) = [2^{-(j+1)}(1 + x_k) - 1, 2^{-(j+1)}(1 + x'_k) - 1] \subset [2^{-(j+1)} - 1, 2^{-j} - 1].
\]
This implies that
\[ \tilde{I}_m \setminus I_m = \{-1, 0, 1\} \cup \{2^{-n} - 1 \mid n \geq 1\}. \]

Therefore, \( x = -1 \) is the unique periodic point of \( f_0 \) in \( \tilde{I}_m \) whose period is less than \( m + 1 \). Fix \( \epsilon_0 > 0 \). By a small perturbation on \([-1, 1] \setminus \tilde{I}_1\), we can take \( \tilde{f}_0 \in C^\infty([0, 1]) \cap \mathcal{ND}' \) such that \( dC^\infty(\tilde{f}_0, f_0) < \epsilon_0/3 \) and \( \text{supp}(\tilde{f}_0 - f_0) \cap \tilde{I}_1 = \emptyset \).

By another small perturbation on \([x_1, x_1']\), we also obtain \( f_1 \in C^\infty([0, 1]) \cap \mathcal{ND}' \) such that \( dC^\infty(f_1, f_0) < \epsilon_0/3 \), \( \text{supp}(f_1 - f_0) \cap \tilde{I}_2 = \emptyset \), and \( \text{Per}(f_1, 2) \cap (\tilde{x}_1, \tilde{x}_1') \) consists of at least \( \gamma_2 \) hyperbolic periodic points of period 2. Take \( \epsilon_1 \in (0, \epsilon_0/3) \) such that any \( f \in C^\infty([0, 1]) \) with \( dC^\infty(f, f_1) < \epsilon_1 \) is contained in \( \mathcal{ND}' \) and satisfies that \( \# \text{Per}(f, 2) \geq \gamma_2 \).

By inductive perturbations in the same way, we can obtain sequences \((f_k)_{k \geq 1}\) of maps in \( C^\infty([0, 1]) \) and \((\epsilon_k)_{k \geq 1}\) of positive real numbers which satisfy the following conditions for any \( k \geq 1\):

1. \( \epsilon_k \in (0, \epsilon_{k-1}/3) \).
2. \( dC^\infty(f_k, f_{k-1}) < \epsilon_{k-1}/3 \).
3. \( \text{supp}(f_k - f_{k-1}) \cap \tilde{I}_{k+1} = \emptyset \).
4. Any \( f \in C^\infty([0, 1]) \) with \( dC^\infty(f, f_k) < \epsilon_k \) is contained in \( \mathcal{ND}' \) and satisfies that \( \# \text{Per}(f, k + 1) \geq \gamma_{k+1} \).

Since \( \epsilon_k \in (0, \epsilon_{k-1}/3) \) and \( dC^\infty(f_k, f_{k-1}) < \epsilon_{k-1}/3 \), \( f_k \) converges to a \( C^\infty \) map \( f \) and it satisfies that \( dC^\infty(f, f_k) < \epsilon_k \) for any \( k \geq 1 \). By the choice of \( \epsilon_k \), all periodic points of \( f \) are hyperbolic and \( \# \text{Per}(f, k + 1) \geq \gamma_{k+1} \) for any \( k \geq 1 \). Since \( |f_k'(x)| > \eta(x) > 0 \) for any \( x \in [-1, 1] \setminus \{0\} \) and \( k \geq 1 \), we have \( |f'(x)| \geq \eta(x) > 0 \) for any \( x \neq 0 \). Therefore, \( x = 0 \) is the unique critical point of \( f \). \( \square \)

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