Generalized Landen Transformation Formulas for Jacobi Elliptic Functions

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Abstract

Landen transformation formulas, which connect Jacobi elliptic functions with different modulus parameters, were first obtained over two hundred years ago by changing integration variables in elliptic integrals. We rediscover known results as well as obtain more generalized Landen formulas from a very different perspective, by making use of the recently obtained periodic solutions of physically interesting nonlinear differential equations and numerous remarkable new cyclic identities involving Jacobi elliptic functions. We find that several of our Landen transformations have a rather different and substantially more elegant appearance compared to the forms usually found in the literature. Further, by making use of the cyclic identities discovered recently, we also obtain some entirely new sets of Landen transformations. This paper is an expanded and revised version of our previous paper math-ph/0204054.

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1 Introduction

Jacobi elliptic functions \(dn(x, m), cn(x, m)\) and \(sn(x, m)\), with elliptic modulus parameter \(m \equiv k^2 \ (0 \leq m \leq 1)\) play an important role in describing periodic solutions of many linear and nonlinear differential equations of interest in diverse branches of engineering, physics and mathematics [1]. The Jacobi elliptic functions are often defined with the help of the elliptic integral

\[
\int \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}.
\]

(1)

Over two centuries ago, John Landen [2] studied the consequences of making a change to a different integration variable

\[
t = \frac{(1 + k') z \sqrt{1-z^2}}{\sqrt{1-k'^2z^2}}, \quad k' \equiv \sqrt{1-k^2} = \sqrt{1-m}.
\]

(2)

This transformation yields another elliptic integral

\[
\int \frac{dt}{(1+k')(\sqrt{1-t^2}(1-t^2z^2))}, \quad l \equiv \frac{1-k'}{1+k'}.
\]

(3)

It readily follows that [3]

\[
\begin{align*}
\text{dn} \left[ (1+k')u, \left( \frac{1-k'}{1+k'} \right)^2 \right] &= \frac{1-(1-k') \sn^2(u, m)}{\dn(u, m)}, \\
\text{cn} \left[ (1+k')u, \left( \frac{1-k'}{1+k'} \right)^2 \right] &= \frac{1-(1+k') \sn^2(u, m)}{\dn(u, m)}, \\
\text{sn} \left[ (1+k')u, \left( \frac{1-k'}{1+k'} \right)^2 \right] &= \frac{(1+k') \sn(u, m) \cn(u, m)}{\dn(u, m)}.
\end{align*}
\]

(4)

(5)

(6)

These celebrated relations are known as the quadratic Landen transformation formulas, or more simply, Landen transformations. They have the special property of providing a non-trivial connection between Jacobi elliptic functions involving two different unequal elliptic modulus parameters \(m\) and \(\tilde{m}\), where the transformed modulus parameter is \(\tilde{m} \equiv (1 - \sqrt{1-m})^2/(1 + \sqrt{1-m})^2\). It may be noted that in Eqs. (4) to (6), for \(0 < m < 1\), \(\tilde{m}\) is always less than \(m\) and also lies in the range \(0 \leq \tilde{m} \leq 1\).

There are also similar formulas, known as the quadratic Gauss transformations [4, 5], where the transformed parameter \(\tilde{m}\) is always greater than \(m\). These are obtained by a different change of integration variables

\[
t = \frac{(1+k) z}{1+kz^2}.
\]

(7)
This transformation yields the elliptic integral
\[ \int \frac{dt}{(1 + k)\sqrt{(1 - t^2)(1 - l^2t^2)}}, \quad l \equiv \frac{2\sqrt{K}}{1 + k}, \]  
(8)
from which it follows that
\[ \text{dn}\left[(1 + k)u, \frac{4k}{(1 + k)^2}\right] = \frac{1 - k\text{sn}^2(u, m)}{1 + k\text{sn}^2(u, m)}, \]  
(9)
\[ \text{cn}\left[(1 + k)u, \frac{4k}{(1 + k)^2}\right] = \frac{\text{cn}(u, m)\text{dn}(u, m)}{1 + k\text{sn}^2(u, m)}, \]  
(10)
\[ \text{sn}\left[(1 + k)u, \frac{4k}{(1 + k)^2}\right] = \frac{(1 + k)\text{sn}(u, m)}{1 + k\text{sn}^2(u, m)}. \]  
(11)
In these Gauss transformation formulas, note that for any choice of \( m \) in the range \( 0 < m < 1 \), the transformed modulus parameter \( \tilde{m} = 4\sqrt{m}/(1 + \sqrt{m})^2 \) is always greater than \( m \) but lies in the range \( 0 < \tilde{m} < 1 \).

The above described Landen and Gauss transformation formulas are of order two. Subsequently, generalizations of these formulas to arbitrary order \( p \) have been studied \([6]\). The purpose of this paper is to obtain these generalized transformations by an entirely different method. In particular, we have recently shown \([7, 8]\) that a kind of superposition principle works for several nonlinear problems like \( \lambda \phi^4 \) theory and for numerous nonlinear equations of physical interest [Korteweg deVries equation, modified Korteweg deVries equation, nonlinear Schrödinger equation, sine-Gordon equation]. Using the idea of superposition, we have obtained seemingly new periodic solutions of these nonlinear problems in terms of Jacobi elliptic functions. The purpose of this paper is to argue on mathematical as well as physical grounds that these solutions cannot really be new, but must be re-expressions of known periodic solutions \([9]\). In the process of proving this, we discover the generalized Landen and Gauss transformations.

As an illustration, let us focus on Eq. (4) first. Using the identity \( \text{dn}^2(u, m) = 1 - m\text{sn}^2(u, m) \), and changing variables to \( x = (1 + k')u \), one can re-write the order two “dn” Landen formula (4) in the alternative form
\[ \text{dn}\left[x, \left(\frac{1 - k'}{1 + k'}\right)^2\right] = \alpha \{\text{dn}[\alpha x, m] + \text{dn}[\alpha x + K(m), m]\}, \quad \alpha = \frac{1}{(1 + k')} \]  
(12)
Here, the right hand side contains the sum of two terms with arguments separated by \( K(m) \equiv \int_0^{\pi/2} d\theta [1 - m \sin^2 \theta]^{-1/2} \), the complete elliptic integral of the first kind \([11, 5]\). Our generalized Landen
formulas will have not two but \( p \) terms on the right hand side. We will show that the generalization of Eq. (4) [or equivalently Eq. (12)], valid for any integer \( p \) is given by

\[
dn(x, \tilde{m}) = \alpha \sum_{j=1}^{p} \dn\left[\alpha x + 2(j - 1)K(m) / p, m\right],
\]

where

\[
\alpha \equiv \left\{ \sum_{j=1}^{p} \dn\left[2(j - 1)K(m) / p, m\right] \right\}^{-1},
\]

and

\[
\tilde{m} = (m - 2)\alpha^2 + 2\alpha^3 \sum_{j=1}^{p} \dn\left[2(j - 1)K(m) / p, m\right].
\]

However, if one starts from the “cn” or “sn” Landen formulas of order 2 as given by Eqs. (5) and (6), the generalization to arbitrary order \( p \) is different depending on whether \( p \) is even or odd. The main results are:

\[
\begin{aligned}
\text{cn}(x, \tilde{m}) &\propto \sum_{j=1}^{p} \text{cn}(\tilde{x}_j, m), & p \text{ odd,} \\
&\propto \sum_{j=1}^{p} (-1)^{j-1} \dn(x_j, m), & p \text{ even,}
\end{aligned}
\]

\[
\begin{aligned}
\text{sn}(x, \tilde{m}) &\propto \sum_{j=1}^{p} \text{sn}(\tilde{x}_j, m), & p \text{ odd,} \\
&\propto \prod_{j=1}^{p} \text{sn}(x_j, m), & p \text{ even,}
\end{aligned}
\]

where \( \tilde{m} \) is as given in Eq. (15), and we have used the notation

\[
x_j \equiv \alpha x + 2(j - 1)K(m) / p, \quad \tilde{x}_j \equiv \alpha x + 4(j - 1)K(m) / p.
\]

Note that all the above formulas have the same non-trivial scaling factor \( \alpha \) of the argument \( x \), as well as real shifts which are fractions of the periods of Jacobi elliptic functions. The richness of the generalized results is noteworthy and reflects the many different forms of periodic solutions for nonlinear equations which we have recently obtained \([7, 8]\). Some formulas involve the sum of \( p \) terms, the even \( p \) “cn” formula involves alternating + and − signs, and the even \( p \) “sn” formula has a product of \( p \) terms. In fact, there are also several interesting additional alternative forms for the above results which follow from use of several identities involving Jacobi elliptic functions which we have recently discovered.
For instance, we will later show that for odd $p$, the Landen formulas for $dn$, $cn$, $sn$ which have been written above as the sum of $p$ terms, can also be written as the product of the same $p$ terms!

Analogous to the generalized Landen transformations mentioned above, we also obtain generalized Gauss transformations in which the parameter $\tilde{m}$ is greater than $m$. In particular, it is shown that Gauss transformations can be systematically obtained from the Landen transformations of arbitrary order $p$ by considering shifts in the arguments by pure imaginary amounts and making use of the fact that the Jacobi elliptic functions are doubly periodic.

By starting from the generalized Landen transformations discussed above, we show how to obtain Landen transformation formulas for other combinations of Jacobi elliptic functions like $\text{sn}(x, \tilde{m})\text{cn}(x, \tilde{m})$, $\text{sn}(x, \tilde{m})\text{dn}(x, \tilde{m})$, $\text{cn}(x, \tilde{m})\text{dn}(x, \tilde{m})$, $\text{cn}(x, \tilde{m})\text{dn}(x, \tilde{m})\text{sn}(x, \tilde{m})$, etc. Besides, by integrating the Landen transformation formula for $\text{dn}^2(x, \tilde{m})$ we also obtain Landen transformations for the Jacobi zeta function $Z(x, \tilde{m})$ as well as for the complete elliptic integral of the second kind $E(\tilde{m})$. Finally, by combining the recently discovered cyclic identities [10, 11, 12] with the Landen transformation results, we obtain new types of Landen transformations. For example, a simple cyclic identity is [12]

$$\sum_{j=1}^{p} \text{dn}^2(x_j, m)[\text{dn}(x_{j+r}, m) + \text{dn}(x_{j-r}, m)] = 2[\text{ds}(a, m) - \text{ns}(a, m)] \sum_{j=1}^{p} \text{dn}(x_j, m),$$

(19)

where $a = 2rK(m)/p$, $x_j$ is as given by Eq. (18) and $r$ is any integer less than $p$. Combining this identity with Eq. (13), we immediately obtain a novel Landen formula relating $\text{dn}(x, \tilde{m})$ with the left hand side of identity (19).

The plan of this paper is as follows. In Sec. 2 we obtain solutions of the sine-Gordon equation in two different ways and by requiring these solutions to be the same, thereby obtain the Landen transformations for the basic Jacobi elliptic functions $dn$, $cn$, $sn$ for both odd and even $p$. In Sec. 3 we show that these Landen transformations can be expressed in an alternative form. Further, we also show that for a given $p$, the relationship between $\tilde{m}$ and $m$ is the same, regardless of the type of Landen formula that one is considering. Using these results, in Sec. 4 we obtain Landen transformations for other combinations of Jacobi elliptic functions like $\text{sn} \text{cn}$, $\text{sn} \text{dn}$, etc. both when $p$ is an odd or an even integer. In Sec. 5 we obtain Gauss transformations of arbitrary order in which the parameter $\tilde{m}$ is always greater than $m$. This is done by starting from the Landen formulas and considering shifts by pure imaginary amounts instead of real shifts. In Sec. 6 we combine the recently obtained cyclic identities with the Landen formulas, thereby obtaining some unusual new Landen transformations. Finally, some concluding remarks are given in Sec. 7.
2 Generalized Landen Formulas

Given the diversity of the generalized Landen formulas for Jacobi elliptic functions, it is necessary to establish them one at a time.

2.1 “dn” Landen Formulas

To get an idea of our general approach, let us first focus on the proof of Eq. (13) in detail. Consider the periodic solutions of the static sine-Gordon field theory in one space and one time dimension, that is, consider the periodic solutions of the second order differential equation

$$\phi_{xx} = \sin \phi .$$

(20)

Note that the time dependent solutions are easily obtained from the static ones by Lorentz boosting. One of the simplest periodic solutions of Eq. (20) is given by

$$\sin[\phi(x)/2] = \text{dn}(x + x_0, \tilde{m}) ,$$

(21)

where \(x_0\) is an arbitrary constant and this is immediately verifiable by direct substitution. It was shown in refs. [7, 8] that a kind of linear superposition principle works even for such nonlinear equations as a consequence of several highly nontrivial, new identities satisfied by Jacobi elliptic functions [10, 11, 12]. In particular, one can show [8] that for any integer \(p\), one has static periodic solutions of Eq. (20) given by

$$\sin[\phi(x)/2] = \alpha \sum_{j=1}^{p} \text{dn}[\alpha x + 2(j - 1)K(m)/p, m] ,$$

(22)

where \(\alpha\) is as given by Eq. (14).

The question one would like to address here is whether solution (22) is completely new, or if it can be re-expressed in terms of simpler solutions like (21), but where \(m\) and \(\tilde{m}\) need not be the same. To that end, consider Eq. (20) more generally. We note that on integrating once, we obtain

$$\phi^2_x = C - 2 \cos \phi ,$$

(23)

where \(C\) is a constant of integration. Integrating again, one gets

$$\int \frac{d\phi}{\sqrt{C - 2 \cos \phi}} = x + x_0 ,$$

(24)
where $x_0$ is a second constant of integration, which we put equal to zero without loss of generality since it corresponds to a choice of the origin of coordinates. On substituting $\sin(\phi/2) = \psi$, equation (24) takes the form

$$\int \frac{d\psi}{\sqrt{1 - \psi^2} \sqrt{\frac{4}{c-2} + \psi^2}} = x. \quad (25)$$

Now the important point to note is that if we perform the integral for different values of $C$ then we will get all the solutions of Eq. (20). Further, if two solutions have the same value of $C$, then they must necessarily be the same. As far as the integral (25) is concerned, it is easily checked that the three simplest solutions covering the entire allowed range of $C$ are

$$\psi = \text{sech} \ x, \quad C = 2, \quad (26)$$

$$\psi = \text{dn}(x, \tilde{m}), \quad C = 4\tilde{m} - 2, \quad (27)$$

$$\psi = \text{cn}(x/\sqrt{\tilde{m}}, \tilde{m}), \quad C = \frac{4}{\tilde{m}} - 2, \quad (28)$$

where $0 \leq \tilde{m} \leq 1$. Note that the constant $C$ has been computed here by using Eq. (23), which in terms of $\psi(x)$ takes the form

$$C = 2 - 4\psi^2 + \frac{4\psi_x^2}{1 - \psi^2}. \quad (29)$$

Thus, whereas for the solution (27), $C$ lies in the range $-2 \leq C \leq 2$, for the solution (28), $C$ lies between 2 and $\infty$. Note that for $C < -2$, there is no real solution to Eq. (25).

Now the strategy is clear. We will take the solution (22) and compute $C$ for it and thereby try to relate it to one of the basic solutions as given by Eqs. (26) to (28). One simple way of obtaining the constant $C$ from Eq. (24) is to evaluate it at a convenient value of $x$, say $x = 0$. In this way, we find that for the solution (22), $C$ is given by

$$C = -2 + 4\alpha^2(m - 2) + 8\alpha^3 \sum_{j=1}^p \text{dn}^3(2(j - 1)K(m)/p, m). \quad (30)$$

Now, as $m \to 0$, $\alpha = 1/p$, $\text{dn}(x, m = 0) = 1$ and hence $C = -2$. On the other hand, as $m \to 1$, $K(m = 1) = \infty$, $\text{dn}(x, m = 1) = \text{sech} \ x$ and hence $\alpha = 1$ so that $C = 2$. Thus for solution (22), as $m$ varies in the range $0 \leq m \leq 1$, the value of $C$ varies in the range $-2 \leq C \leq 2$. Hence it is clear that the solutions (22) and (27) must be same. On equating the two $C$ values as given by Eqs. (27) and (30), we find that the two solutions are identical provided $m$ and $\tilde{m}$ are related by Eq. (15) and hence the appropriate Landen transformation valid for any integer $p$ is given by Eq. (13).
that when $p = 2$, one recovers the Landen formula $12$, since $\text{dn}(K(m), m) = k'$, and $\tilde{m}$ simplifies to $(1 - k')^2/(1 + k')^2$.

### 2.2 “cn” Landen Formulas

Unlike the $\text{dn}$ case, it turns out that in this case the Landen transformation formulas for odd and even $p$ have very different forms. We first derive the form for the odd $p$ case and then consider the even $p$ case.

As shown in ref. 8, another periodic solution of the static sine-Gordon equation (20) is

$$
\sin(\phi(x)/2) = \alpha_1 \sum_{j=1}^{p} \text{cn} \left[ \frac{\alpha_1 x}{\sqrt{m}} + 4(j - 1)K(m)/p, m \right], \quad p \text{ odd},
$$

with $\alpha_1$ being given by

$$
\alpha_1 \equiv \left\{ \sum_{j=1}^{p} \text{cn}[4(j - 1)K(m)/p, m] \right\}^{-1}.
$$

Using Eq. 29, we can now compute the corresponding value of the constant $C$. We obtain

$$
C = -2 + \frac{4\alpha_1^2(1 - 2m)}{m} + 8\alpha_1^3 \sum_{j=1}^{p} \text{cn}^3[4(j - 1)K(m)/p, m].
$$

It is easily checked that since $0 \leq m \leq 1$, $C$ varies from 2 to $\infty$, and hence the solutions 28 and 31 must be identical. On equating the two values of $C$ as given by Eqs. 28 and 33, we then find that for odd $p$, the Landen transformation is

$$
\text{cn}(x, \tilde{m}_1) = \alpha_1 \sum_{j=1}^{p} \text{cn} \left[ \frac{\alpha_1 \sqrt{\tilde{m}_1} x}{\sqrt{m}} + 4(j - 1)K(m)/p, m \right],
$$

where $\tilde{m}_1$ is given by

$$
\tilde{m}_1 = m \alpha_1^{-2} \left\{ (1 - 2m) + 2m\alpha_1 \sum_{j=1}^{p} \text{cn}^3[4(j - 1)K(m)/p, m] \right\}^{-1}.
$$

What happens if $p$ is an even integer? As shown in 8, in that case another periodic solution of the static sine-Gordon Eq. 20 is

$$
\sin(\phi(x)/2) = \alpha_2 \sum_{j=1}^{p} (-1)^{j-1}\text{dn}[\alpha_2 x + 2(j - 1)K(m)/p, m], \quad p \text{ even},
$$

where $\alpha_2$ is given by

$$
\alpha_2 \equiv \left\{ \sum_{j=1}^{p} (-1)^{j-1}\text{dn}[2(j - 1)K(m)/p, m] \right\}^{-1}.
$$
Using Eq. (29), the value of $C$ for this solution is easily computed and we find that $2 \leq C \leq \infty$. On comparing with solution (28) we find that in this case the Landen formula is

$$
\text{cn}(x, \tilde{m}_2) = \alpha_2 \sum_{j=1}^{p} (-1)^{j-1} \text{dn}[\alpha_2 \sqrt{\tilde{m}_2} \ x + 2(j-1)K(m)/p, m],
$$

where $\tilde{m}_2$ is given by

$$
\tilde{m}_2 = \left\{ (m - 2) + 2\alpha_2 \sum_{j=1}^{p} (-1)^{j-1} \text{dn}^{3}[2(j-1)K(m)/p, m] \right\}^{-1}.
$$

As expected, in the special case of $p = 2$, we immediately recover the Landen formula (3).

2.3 “sn” Landen Formulas

As in the cn case, here too the Landen formulas for even and odd values of $p$ have very different forms and so we consider them separately.

We start from the sine-Gordon field equation

$$
\phi_{xx} - \phi_{tt} = \sin \phi,
$$

and look for time-dependent, traveling wave solutions with velocity $v > 1$ (which are called optical soliton solutions in the context of condensed matter physics). In terms of the variable

$$
\eta \equiv \frac{x - vt}{\sqrt{v^2 - 1}},
$$

Eq. (40) takes the simpler form

$$
\phi_{\eta \eta} = -\sin \phi.
$$

Note that the only change from Eq. (20) is an additional negative sign on the right hand side. On integrating this equation once, we obtain

$$
\phi_\eta^2 = C + 2 \cos \phi.
$$

On integrating further, we get

$$
\int \frac{d\phi}{\sqrt{C + 2 \cos \phi}} = \eta + \eta_0,
$$

where $\eta_0$ is a constant of integration which we put equal to zero without loss of generality. Substituting $\sin(\phi/2) = \psi$, yields

$$
\int \frac{d\psi}{\sqrt{1 - \psi^2 \sqrt{\frac{C+2}{4} - \psi^2}}} = \eta.
$$
If we now perform the integral for different values of $C$, then we get all the solutions. It is easily checked that the three simplest solutions of Eq. (45) covering the entire allowed range of $C$ are

$$\psi = \tanh \eta, \quad C = 2,$$

$$\psi = \sqrt{m} \text{sn} (\eta, \tilde{m}), \quad C = 4 \tilde{m} - 2,$$

$$\psi = \text{sn} \left( \frac{\eta}{\sqrt{\tilde{m}}}, \tilde{m} \right), \quad C = \frac{4}{\tilde{m}} - 2,$$

where $0 \leq \tilde{m} \leq 1$. Note that the constant $C$ has been computed here using Eq. (43), which in terms of $\psi$ takes the form

$$C = -2 + 4 \psi^2 + \frac{4\psi^2}{1 - \psi^2}.$$

Thus, for solution (47), $C$ is in the range $-2 \leq C \leq 2$, whereas for solution (48), $C$ lies between 2 and $\infty$. Note that for $C < -2$, there is no real solution to Eq. (45).

Using appropriate linear superposition, it was shown in ref. [8] that for odd $p$ one of the solutions of Eq. (42) is given by

$$\sin(\phi(\eta)/2) = \sqrt{ma} \sum_{j=1}^{p} \text{sn}[\alpha \eta + 4(j - 1)K(m)/p, m], \quad p \text{ odd},$$

with $\alpha$ being given by Eq. (14). Using Eq. (49), we can now compute the corresponding value of $C$. We find

$$C = -2 + 4\alpha_2^2 \frac{\alpha_2^2}{\alpha_1^2},$$

where $\alpha, \alpha_1$ are given by Eqs. (14) and (32) respectively. It is easily checked that since $0 \leq m \leq 1$, $C$ has values between -2 and 2. Hence the solutions (47) and (50) must be identical. On equating the two values of $C$ as given by Eqs. (47) and (51), we then find that the “sn” Landen transformation formula for odd $p$ is given by

$$\text{sn}(x, \tilde{m}_3) = \alpha_1 \sum_{j=1}^{p} \text{sn}[\alpha x + 4(j - 1)K(m)/p, m],$$

with $\tilde{m}_3$ and $m$ being related by

$$\tilde{m}_3 = m \frac{\alpha^2}{\alpha_1^2}.$$

Finally, we turn to the “sn” Landen transformation formula for the case when $p$ is an even integer. One can show [8] that in this case, a solution to Eq. (42) is given by

$$\sin(\phi(\eta)/2) = mp^{p/2} \alpha A_0 \prod_{j=1}^{p} \text{sn}[\alpha \eta + 2(j - 1)K(m)/p, m], \quad p \text{ even},$$

with $m$ being related by

$$m = \frac{\alpha^2}{\alpha_1^2}.$$
with \( \alpha \) being given by Eq. (14) and \( A_0 \) defined by

\[
A_0 = \prod_{j=1}^{p-1} \sin(2jK(m)/p,m) .
\] (55)

Using Eq. (49), we can now compute the corresponding value of \( C \). We obtain

\[
C = -2 + 4m^{p} \alpha^4 A_0^4 .
\] (56)

It is easily checked that since \( 0 \leq m \leq 1 \), the value of \( C \) varies between -2 and 2 and hence the solutions (47) and (54) must be identical. On equating the two values of \( C \) as given by Eqs. (47) and (56), we find that for even \( p \), the Landen transformation formula is

\[
A_0 \alpha \sin(x, \tilde{m}_4) = \prod_{j=1}^{p} \sin[\alpha x + 2(j-1)K(m)/p,m] ,
\] (57)

with \( \tilde{m}_4 \) given by

\[
\tilde{m}_4 = m^{p} \alpha^4 A_0^4 .
\] (58)

Not surprisingly, for \( p = 2 \) we recover the Landen transformation formula (6). It is amusing to notice that as \( m \to 0 \),

\[
A_0(p, m = 0) = \prod_{j=1}^{p-1} \sin(j\pi/p) = \frac{p}{2^{p-1}} .
\] (59)

At this point, we have generalized all three of the celebrated two hundred year old \( p = 2 \) Landen formulas [Eqs. (4), (5), (6)] to arbitrary values of \( p \), the generalization being different depending on whether \( p \) is an even or odd integer. In the next section, we re-cast the formulas in even simpler form. Although several of these Landen formulas are already known [6], to our knowledge, they have never been derived via the novel approach of this article which makes use of solutions of nonlinear field equations.

3 Alternative Forms for Landen Transformations

3.1 Relation between the transformed modulus parameters \( \tilde{m}_i \) and \( m \)

So far, we have obtained several seemingly different relationships between the transformed modulus parameters \( \tilde{m}, \tilde{m}_1, \tilde{m}_2, \tilde{m}_3, \tilde{m}_4 \) and \( m \). Let us recall that while relation (13) for \( \tilde{m} \) is valid for all \( p \), relations (34) and (52) for \( \tilde{m}_1 \) and \( \tilde{m}_3 \) are only valid for odd \( p \) and relations (38) and (57) for \( \tilde{m}_2 \) and \( \tilde{m}_4 \) are only valid for even \( p \). One would like to know if for any given \( p \), the different Landen
transformations give the same relationship between \( \tilde{m}_i \) and \( m \) or not. For \( p = 2 \), it is known that
\[ \tilde{m} = \tilde{m}_2 = \tilde{m}_4 \] [see Eqs. (1) to (3)]. Similarly, with a little algebraic manipulation, one can show that for \( p = 3 \) the corresponding relations are
\[ \tilde{m} = \tilde{m}_1 = \tilde{m}_3 = m \left( \frac{1-q}{1+q} \right)^2 \left( \frac{1+2q}{1+q} \right)^2 , \] (60)
where \( q = \text{dn}(2K(m)/3, m) \). Note that while deriving this result, use has been made of the fact that \( \text{cn}(4K(m)/3, m) = -\frac{q}{1+q} \) and that \( q \) satisfies the identity \( q^4 + 2q^3 - 2(1-m)q - (1-m) = 0 \). Similarly, using the relations \( \text{dn}(K(m)/2, m) = \text{dn}(3K(m)/2, m) = (1-m)^{1/4} \equiv t \) and \( \text{dn}(K(m), m) = t^2 \), it is easily proved that for \( p = 4 \), the relations are \( \tilde{m} = \tilde{m}_2 = \tilde{m}_4 = (1-t)^4/(1+t)^4 \).

As \( p \) increases, the algebra becomes messier and it is not easy to write the corresponding relations between \( \tilde{m}_i \) and \( m \) in a neat closed form. However, we can still establish the equivalence of all \( \tilde{m}_i \) for any given \( p \). For this purpose we equate the periods of the left and right hand sides of the various Landen transformation relations \([13, 34, 58, 52, 57] \). We get:

\[ K(\tilde{m}) = \frac{K(m)}{p\alpha} , \] (61)
\[ K(\tilde{m}_1) = \frac{K(m)\sqrt{m}}{p\alpha_1 \sqrt{\tilde{m}_1}} , \] (62)
\[ K(\tilde{m}_2) = \frac{K(m)}{p\alpha_2 \sqrt{\tilde{m}_2}} , \] (63)
\[ K(\tilde{m}_3) = \frac{K(m)}{p\alpha} , \] (64)
\[ K(\tilde{m}_4) = \frac{K(m)}{p\alpha} . \] (65)

From Eqs. (61), (62) and (65) above, one immediately sees that \( \tilde{m} = \tilde{m}_3 \) for odd \( p \) and \( \tilde{m} = \tilde{m}_4 \) for even \( p \). In order to establish that \( \tilde{m}_1 = \tilde{m}_3 \) for odd \( p \), we suitably rescale, square and add the Landen transformations \([34, 52] \). This yields
\[ \text{sn}^2(x, \tilde{m}) + \text{cn}^2\left(\frac{\alpha \sqrt{mx}}{\alpha_1 \sqrt{m_1}}, \tilde{m}_1\right) = C , \] (66)
where the constant \( C \) on the right hand side comes from making use of the cyclic identities \([10, 11, 12] \) given in eqs. (105) and (106). Since Eq. (66) is valid for all \( x \), and in particular for \( x = 0 \), it follows that \( C = 1 \), and hence one must have
\[ \text{sn}^2(x, \tilde{m}) = \text{sn}^2\left(\frac{\alpha \sqrt{mx}}{\alpha_1 \sqrt{m_1}}, \tilde{m}_1\right) . \] (67)
This implies that \( \frac{\alpha \sqrt{m}}{\alpha_1 \sqrt{m_1}} = 1 \) and hence \( \tilde{m}_1 = \tilde{m} \) for odd \( p \). Similar reasoning yields \( \frac{\alpha}{\alpha_2 \sqrt{m_2}} = 1 \) and \( \tilde{m}_2 = \tilde{m} \) for even \( p \).

Consequently, for odd \( p \), the three Landen transformations are

\[
\text{dn}(x, \tilde{m}) = \alpha \sum_{j=1}^{p} \text{dn}(x_j, m),
\]

(68)

\[
\text{cn}(x, \tilde{m}) = \alpha_1 \sum_{j=1}^{p} \text{cn}(\tilde{x}_j, m),
\]

(69)

\[
\text{sn}(x, \tilde{m}) = \alpha_1 \sum_{j=1}^{p} \text{sn}(\tilde{x}_j, m),
\]

(70)

where \( \alpha, \alpha_1 \) are given by Eqs. (14) and (32), \( x_j, \tilde{x}_j \) are as given by Eq. (18) while \( \tilde{m} \) and \( m \) are related by Eq. (15).

Likewise, for even \( p \), the three Landen formulas are

\[
\text{dn}(x, \tilde{m}) = \alpha \sum_{j=1}^{p} \text{dn}(x_j, m),
\]

(71)

\[
\text{cn}(x, \tilde{m}) = \alpha_2 \sum_{j=1}^{p} (-1)^{j-1} \text{dn}(x_j, m),
\]

(72)

\[
A_0 \alpha \text{sn}(x, \tilde{m}) = \prod_{j=1}^{p} \text{sn}(x_j, m),
\]

(73)

where \( \alpha, \alpha_2, A_0 \) are given by Eqs. (14), (37) and (55) respectively while \( \tilde{m} \) and \( m \) are related by Eq. (15).

The numerical results for \( \tilde{m} \) as a function of \( m \) for various values of \( p \) ranging from 2 to 7 are shown in Table 1. Note that for any fixed value of \( p \), as \( m \) increases from 0 to 1, \( \tilde{m} \) also increases monotonically from 0 to 1 (but is always less than \( m \)). Also, for any given fixed value of \( m \), \( \tilde{m} \) decreases monotonically as \( p \) increases and is always less than \( m \). For this reason, Landen transformations are sometimes referred to as ascending Landen transformations [4].

### 3.2 Alternative Forms of Landen Transformation Formulas

Recently we [10, 11, 12] have obtained several new identities for Jacobi elliptic functions. Three of these, valid for odd \( p \), are

\[
\prod_{j=1}^{p} \text{dn}(x_j, m) = \prod_{n=1}^{(p-1)/2} \text{cs}^2 \left( \frac{2Kn}{p}, m \right) \sum_{j=1}^{p} \text{dn}(x_j, m),
\]

(74)
Here \( ds(x, m) \), etc. are defined by \( ds(x, m) = \frac{dn(x, m)}{sn(x, m)} \). Hence, for odd \( p \), the three Landen transformation formulas can also be written as products (rather than sums) of \( p \) terms. In particular, for odd \( p \) the three Landen transformation formulas (68) to (70) can also be written in the form

\[
\prod_{j=1}^{p} sn(\bar{x}_j, m) = (-1/m)^{(p-1)/2} \prod_{n=1}^{(p-1)/2} ns^2\left(\frac{4Kn}{p}, m\right) \sum_{j=1}^{p} sn(\bar{x}_j, m), \tag{75}
\]

\[
\prod_{j=1}^{p} cn(\bar{x}_j, m) = (1/m)^{(p-1)/2} \prod_{n=1}^{(p-1)/2} ds^2\left(\frac{4Kn}{p}, m\right) \sum_{j=1}^{p} cn(\bar{x}_j, m). \tag{76}
\]

Similarly, for even \( p \), on making use of the identity \[10, 11\] we can rewrite the Landen formula (73) for \( sn \) in the form

\[
\prod_{j=1}^{p} sn(\bar{x}_j, m) = \frac{\prod_{j=1}^{p} sn(\bar{x}_j, m)}{\prod_{n=1}^{(p-1)/2} sn(2nK(m)/p, m)} \sum_{j=1}^{p} (-1)^{j-1} Z(x_j, m), \tag{80}
\]

where \( Z(u, m) \) is the Jacobi zeta function, we can rewrite the Landen formula (73) for \( sn \) in the form

\[
\prod_{j=1}^{p} sn(\bar{x}_j, m) = \prod_{n=1}^{(p-1)/2} sn(2nK(m)/p, m). \tag{81}
\]

We have not seen this particular form for the \( sn \) Landen transformation in the mathematics literature. Here the coefficient multiplying the right hand side of Eq. (81) has been fixed by demanding consistency. From now onward, we shall mostly be using this form of the Landen formula rather than the one given by Eq. (73).

On comparing Eqs. (78), (80) and (81) we obtain an interesting identity

\[
m^{p/2} \prod_{j=1}^{p} sn(x_j, m) = \left[ \prod_{n=1}^{\frac{p-1}{2}} ns^2(2nK(m)/p, m) \right] \sum_{j=1}^{p} (-1)^{j-1} Z(x_j, m), \tag{82}
\]

In concluding this section, we note that many of the above derived Landen transformation formulas lead to interesting known trigonometric relations by taking the limiting case \( m = \bar{m} = 0 \). For instance, Eqs. (78), (79) and (80) lead to

\[
\sin px = 2^{p-1} \prod_{j=1}^{p} \sin[x + (j - 1)\pi/p], \quad p \text{ even,} \tag{83}
\]
\[
\cos px = 2^{p-1} \prod_{j=1}^{p} \cos[x + 2(j-1)\pi/p], \quad p \text{ odd},
\]

\[
\sin px = (-4)^{\frac{p-1}{2}} \prod_{j=1}^{p} \sin[x + 2(j-1)\pi/p], \quad p \text{ odd}.
\]

4 Landen Transformations for Products of Jacobi Elliptic Functions

We shall now show that starting from the Landen formulas for the three basic Jacobi elliptic functions \(sn, \ cn, \ dn\), we can obtain Landen formulas for their products and various other combinations.

4.1 Any integer \(p\)

We start from the basic Landen formula, Eq. (68) valid for any integer \(p\). Differentiating it gives the Landen formula:

\[
\text{sn}(x, \tilde{m})\text{cn}(x, \tilde{m}) = \frac{m\alpha^2}{\tilde{m}} \sum_{j=1}^{p} \text{sn}(x_j, m)\text{cn}(x_j, m),
\]

For odd values of \(p\), this can also be proved by multiplying the two Landen formulas [Eqs. (69) and (70)] and using the cyclic identity

\[
\sum_{j=1}^{p} \text{sn}(x_j, m)[\text{cn}(x_{j+r}, m) + \text{cn}(x_{j-r}, m)] = 0,
\]

obtained in refs. [10, 11]. Here \(r = 1, 2, \ldots, (p - 1)/2\). On the other hand, for even \(p\), multiplying relations (72) and (81) and comparing with identity (86) yields the remarkable identity

\[
m \sum_{j=1}^{p} \text{sn}(x_j, m)\text{cn}(x_j, m) = \left[ \sum_{j=1}^{p} (-1)^{j-1}Z(x_j, m) \right] \left[ \sum_{j=1}^{p} (-1)^{j-1}\text{dn}(x_j, m) \right].
\]

Nontrivial results also follow from squaring any of the Landen formulas. For example, squaring the Landen formula (68) and using the cyclic identity

\[
\sum_{j=1}^{p} \text{dn}(x_j, m)\text{dn}(x_{j+r}, m) = p[\text{dn}(a, m) - \text{cs}(a, m)Z(a, m)], \quad a \equiv 2rK(m)/p,
\]

yields the Landen formula for \(\text{dn}^2\), i.e. we get

\[
\text{dn}^2(x, \tilde{m}) = \alpha^2 \left[ \sum_{j=1}^{p} \text{dn}^2(x_j, m) + 2A_d \right],
\]
where

\[ A_d \equiv \sum_{i<j=1}^{p} \text{dn}(x_i, m)\text{dn}(x_j, m) \]

\[ = p \sum_{j=1}^{(p-1)/2} [\text{dn}(2jK(m)/p, m) - \text{cs}(2jK(m)/p, m)Z(2jK(m)/p)] , \quad p \text{ odd ,} \]

\[ = (p/2)\sqrt{1-m} + p \sum_{j=1}^{(p-2)/2} [\text{dn}(2jK(m)/p, m) - \text{cs}(2jK(m)/p, m)Z(2jK(m)/p)] , \quad p \text{ even.}(91) \]

It is worth noting that the linearly superposed solutions of KdV equation \[7\] discovered recently, essentially correspond to this Landen transformation.

Differentiating both sides of Eq. (90) yields the Landen transformation for the product \(\text{sn \ cn \ dn}\) of the three basic Jacobi elliptic functions. We obtain

\[ \text{dn}(x, \tilde{m})\text{sn}(x, \tilde{m})\text{cn}(x, \tilde{m}) = \frac{m\alpha^2}{\tilde{m}} \sum_{j=1}^{p} \text{dn}(x_j, m)\text{sn}(x_j, m)\text{cn}(x_j, m). \quad (92) \]

On the other hand, if we integrate both sides of (90) with respect to \(x\), then we simultaneously obtain Landen transformations for both the Jacobi Zeta function as well as the complete elliptic integral of the second kind \(E(m)\). In particular, on using the well known formula \[4, 5\]

\[ \int \text{dn}^2(x, m) dx = Z(x, m) + \frac{E(m)}{K(m)} x , \quad (93) \]

integration of Eq. (90) yields

\[ Z(x, \tilde{m}) + \frac{E(\tilde{m})}{K(\tilde{m})} x = \alpha^2 x \left[ 2A_d + \frac{E(m)}{K(m)} \right] + \alpha \sum_{j=1}^{p} Z(x_j, m) . \quad (94) \]

Here the constant on the right hand side has been fixed by considering the equation at \(x = 0\) and using the fact that \(Z(x = 0, m) = 0\), that it is an odd function of its argument \(x\) and that it is a periodic function with period \(2K(m)\). It is now obvious that terms proportional to \(x\) on the left and right hand sides must cancel among themselves, since the remaining terms are oscillatory and do not grow. This immediately yields two remarkable Landen transformations:

\[ Z(x, \tilde{m}) = \alpha \sum_{j=1}^{p} Z(x_j, m) , \quad (95) \]

\[ E(\tilde{m}) = \alpha \left[ E(m) + \frac{2A_dK(m)}{p} \right] , \quad (96) \]
where use has been made of relation (61). As expected, for \( p = 2 \) the Landen transformation agrees with the well known relation given in [4]. However, it seems that the generalized Landen transformation for Jacobi zeta functions is a new result.

Landen formulas for higher order combinations like say \( \text{dn}^n(x, \tilde{m}) \) can be obtained recursively by differentiating the lower order Landen formulas [86] and [92]. For example, differentiating relation (86) yields the Landen formula for \( \text{dn}^3(x, \tilde{m}) \)

\[
\text{dn}^3(x, \tilde{m}) = \alpha^3 \sum_{j=1}^{p} \text{dn}^3(x_j, m) + \alpha [2 - \tilde{m} - (2 - m)\alpha^2] \sum_{j=1}^{p} \text{dn}(x_j, m). \tag{97}
\]

### 4.2 Odd Integer Case

We start from the two basic Landen formulas, Eqs. (69) and (70) valid for any odd integer \( p \). Differentiating and using the relation between \( \tilde{m} \) and \( m \), gives the following Landen formulas:

\[
\text{sn}(x, \tilde{m})\text{dn}(x, \tilde{m}) = \alpha \alpha^1 \sum_{j=1}^{p} \text{sn}(\tilde{x}_j, m)\text{dn}(\tilde{x}_j, m), \tag{98}
\]
\[
\text{cn}(x, \tilde{m})\text{dn}(x, \tilde{m}) = \alpha \alpha^1 \sum_{j=1}^{p} \text{cn}(\tilde{x}_j, m)\text{dn}(\tilde{x}_j, m). \tag{99}
\]

It may be noted that these relations can also be proved by multiplying appropriate Landen formulas [Eqs. (68) to (70)] and using the cyclic identities analogous to (87) obtained in refs. [10, 11].

Landen formulas for higher order combinations like \( \text{sn}^{2n+1}(x, \tilde{m}) \) or \( \text{cn}^{2n+1}(x, \tilde{m}) \) can be obtained recursively from here by differentiating the lower order Landen formulas. For example, differentiation of relations (98) and (99) yield the following Landen formulas for \( \text{sn}^3(x, \tilde{m}) \) and \( \text{cn}^3(x, \tilde{m}) \):

\[
\text{cn}^3(x, \tilde{m}) = \alpha^3 \left[ \sum_{j=1}^{p} \text{cn}^3(\tilde{x}_j, m) + \left( \frac{1 - \alpha^2}{\alpha^4} - \frac{1 - \alpha^2}{2m\alpha^2} \right) \sum_{j=1}^{p} \text{cn}(\tilde{x}_j, m) \right], \tag{100}
\]
\[
\text{sn}^3(x, \tilde{m}) = \alpha^3 \left[ \sum_{j=1}^{p} \text{sn}^3(\tilde{x}_j, m) + \left( \frac{1 - \alpha^2}{2m\alpha^2} + \frac{1 - \alpha^2}{2\alpha^4} \right) \sum_{j=1}^{p} \text{sn}(\tilde{x}_j, m) \right]. \tag{101}
\]

Before concluding the discussion about the Landen formulas for odd \( p \), we want to point out the consistency conditions which we obtain by demanding \( \text{dn}^2(x, \tilde{m}) + \tilde{m}\text{sn}^2(x, \tilde{m}) = 1 \), and \( \text{sn}^2(x, \tilde{m}) + \text{cn}^2(x, \tilde{m}) = 1 \). In particular, using relations (68) to (70) and demanding these constraints, yields

\[
\frac{1}{\alpha^2} = p + 2(A_d + A_s), \quad \frac{m}{\alpha^4} = mp + 2(A_s + A_c), \tag{102}
\]
where in view of the cyclic identities derived in \([10, 11, 12]\) we have
\[
A_s \equiv m \sum_{i<j=1}^{p} \text{sn}(x_i, m)\text{sn}(x_j, m) = p \sum_{j=1}^{(p-1)/2} \frac{Z(2jK(m)/p, m)}{\text{sn}(2jK(m)/p)},
\]
(103)
\[
A_c \equiv m \sum_{i<j=1}^{p} \text{cn}(x_i, m)\text{cn}(x_j, m) = p \sum_{j=1}^{(p-1)/2} \left[ \text{cn}(2jK(m)/p) - \frac{\text{dn}(2jK(m)/p, m)Z(2jK(m)/p, m)}{\text{sn}(2jK(m)/p)} \right],
\]
(104)
and \(A_d\) is as given by Eq. \([91]\).

### 4.3 Even Integer Case

We start from the two Landen formulas for even \(p\) given by Eqs. \([72]\) and \([81]\). Differentiating them and using the relation for \(\tilde{m}\) leads to the following Landen formulas:

\[
\text{sn}(x, \tilde{m})\text{dn}(x, \tilde{m}) = m\alpha_2 \sum_{j=1}^{p} (-1)^{j-1}\text{sn}(x_j, m)\text{cn}(x_j, m),
\]
(105)
\[
\text{cn}(x, \tilde{m})\text{dn}(x, \tilde{m}) = \alpha_2 \sum_{j=1}^{p} (-1)^{j-1}\text{dn}^2(x_j, m).
\]
(106)

It may be noted that the relation \([104]\) can also be derived by multiplying the Landen formulas \([71]\) and \([72]\) and using the cyclic identity
\[
\left[ \sum_{j=1}^{p} \text{dn}(x_j, m) \right] \sum_{j=1}^{p} (-1)^{j-1}\text{dn}(x_j, m) = \sum_{j=1}^{p} (-1)^{j-1}\text{dn}^2(x_j, m).
\]
(107)

On the other hand, multiplying relations \([71]\) and \([81]\) and comparing with identity \([105]\) yields a remarkable identity
\[
m \sum_{j=1}^{p} (-1)^{j-1}\text{sn}(x_j, m)\text{cn}(x_j, m) = \left[ \sum_{j=1}^{p} (-1)^{j-1}\text{Z}(x_j, m) \right] \sum_{j=1}^{p} \text{dn}(x_j, m).
\]
(108)

Landen formulas for higher powers like say \(\text{sn}^{2n+1}(x, \tilde{m})\) can be obtained recursively from here by differentiating the lower order Landen formulas. For example, differentiation of the relations \([105]\) and \([106]\) gives rise to the following Landen formulas for \(\text{cn}^3(x, \tilde{m})\) and \(\text{sn}^3(x, \tilde{m})\):

\[
\text{cn}^3(x, \tilde{m}) = \alpha_2^3 \sum_{j=1}^{p} (-1)^{j-1}\text{dn}^3(x_j, m) + \left( \frac{2\alpha^2 - \alpha_2^2}{2\alpha^2\alpha_2^2} + \frac{m-2}{2} \right) \sum_{j=1}^{p} (-1)^{j-1}\text{dn}(x_j, m),
\]
(109)
\[
\text{sn}^3(x, \tilde{m}) = \alpha_2 \left( \frac{\alpha^2 + \alpha_2^2}{2\alpha^2} \right) \sum_{j=1}^{p} (-1)^{j-1}\text{Z}(x_j, m) - m\alpha_2^3 \sum_{j=1}^{p} (-1)^{j-1}\text{sn}(x_j, m)\text{cn}(x_j, m)\text{dn}(x_j, m).
\]
(110)
Before finishing the discussion about the Landen formulas for even \( p \), we want to point out the consistency conditions which we obtain by demanding \( \text{dn}^2(x, \tilde{m}) + \tilde{m}\text{sn}^2(x, \tilde{m}) = 1 \), and \( \text{sn}^2(x, \tilde{m}) + \text{cn}^2(x, \tilde{m}) = 1 \). In particular, using relations (71), (72) and (81) and demanding these constraints, yields

\[
\frac{1}{\alpha^2} = \text{dn}^2(x_j, m) + 2A_d + \left[ \sum_{j=1}^{p} (-1)^{j-1}Z(x_j, m) \right]^2, \tag{111}
\]

\[
\frac{1}{\alpha^2} = \left[ \sum_{j=1}^{p} (-1)^{j-1}\text{dn}(x_j, m) \right]^2 + \left[ \sum_{j=1}^{p} (-1)^{j-1}Z(x_j, m) \right]^2, \tag{112}
\]

where \( A_d \) is as given by Eq. (91).

## 5 Gauss Transformation Formulas

For \( p = 2 \), one has the Gauss quadratic transformation formulas \[4\] as given by Eqs. (9) to (11) in which \( \tilde{m} \) is greater than \( m \). Gauss transformations are sometimes referred to as descending Landen transformations \[4\]. It is natural to ask whether Gauss transformation formulas can be generalized to arbitrary \( p \). In this context it may be noted that the Landen transformation generalizations which we have established so far are in terms of shifts involving the period \( K(m) \) on the real axis. We now show that the generalization of the Gauss transformation formulas to arbitrary order result from shifts involving the period \( iK'(m) \) on the imaginary axis.

The procedure consists of starting with any Landen transformation formula, using the standard results \[3, 4\]

\[
\text{dn}(x, m') = \text{dc}(ix, m), \quad \text{cn}(x, m') = \text{nc}(ix, m), \quad \text{sn}(x, m') = -i\text{sc}(ix, m), \tag{113}
\]

and then redefining \( ix = u \). Note that \( m' = 1 - m \) while \( \text{dc}(x, m) \equiv \frac{\text{dn}(x, m)}{\text{cn}(x, m)} \), etc. In this way, we find that for odd \( p \), the three Landen transformation formulas as given by Eqs. (68) to (70) take the form

\[
\text{dc}(x, \tilde{m}) = \beta \sum_{j=1}^{p} \text{dc}[\beta x + 2i(j - 1)K'(m)/p, m], \tag{114}
\]

\[
\text{nc}(x, \tilde{m}) = \beta_1 \sum_{j=1}^{p} \text{nc}[\beta x + 4i(j - 1)K'(m)/p, m], \tag{115}
\]

\[
\text{sc}(x, \tilde{m}) = \beta_1 \sum_{j=1}^{p} \text{sc}[\beta x + 4i(j - 1)K'(m)/p, m], \tag{116}
\]
where
\[
\beta \equiv \beta(m) = \alpha(m' = 1 - m) = \left\{ \frac{p}{j=1} \text{dn}[2(j - 1)K'(m)/p, m'] \right\}^{-1},
\]
(117)
\[
\beta_1 \equiv \beta_1(m) = \alpha_1(m' = 1 - m) = \left\{ \frac{p}{j=1} \text{cn}[4(j - 1)K'(m)/p, m'] \right\}^{-1}.
\]
(118)

It might be noted here that the Landen formula (114) is in fact valid for both even and odd \(p\). On the other hand, for even \(p\), instead of relations (115) and (116) we have the Landen formulas
\[
\text{nc}(x, \tilde{m}) = \beta_2 \sum_{j=1}^{p} (-1)^{j-1} \text{dc}[\beta x + 2i(j - 1)K'(m)/p, m],
\]
(119)
\[
\text{sc}(\beta x, \tilde{m}) \beta B_0 = (-i)^{p-1} \prod_{j=1}^{p} \text{sc}[\beta x + 2i(j - 1)K'(m)/p, m],
\]
(120)
where \(\beta_2\) and \(B_0\) are given by
\[
\beta_2 \equiv \beta_2(m) = \alpha_2(m' = 1 - m) = \left\{ \frac{p}{j=1} (-1)^{j-1} \text{dn}[2(j - 1)K'(m)/p, m'] \right\}^{-1},
\]
(121)
\[
B_0 \equiv B_0(m) = A_0(m') = \prod_{j=1}^{p-1} \text{sn}[2jK'(m)/p, m'].
\]
(122)

For odd \(p\), \(\tilde{m}\) is now given by
\[
1 - \tilde{m} = (1 - m) \frac{\beta^2}{\beta_1^2},
\]
(123)
while for even \(p\) it is given by
\[
1 - \tilde{m} = \frac{\beta^2}{\beta_2^2},
\]
(124)
where \(\beta, \beta_1, \beta_2\) are as given by Eqs. (117), (118) and (121) respectively.

In the special case of \(p = 2\), it is easily checked that \(\tilde{m} = 4\sqrt{m}/(1 + \sqrt{m})^2\) and as expected the relations (114), (119) and (120) reduce to the well known \[4\] Gauss transformation formulas as given by Eqs. (9) to (11). Note that there is an obvious, interesting relationship between the transformed modulus parameters \(\tilde{m}_G(m)\) and \(\tilde{m}_L(m)\) in Gauss and Landen transformations respectively. For any choice of \(p\), the Gauss transformation undoes the effects of the Landen transformation and vice versa. This means that \(\tilde{m}_G[\tilde{m}_L(m)] = m\) and \(\tilde{m}_L[\tilde{m}_G(m)] = m\). For \(p = 2\), these relations are easily checked from Eqs. (11) and (9).

Before ending this section, it is worth remarking that just as we have considered Landen formulas where the shifts are in units of \(K(m)\) or \(iK'(m)\) on the real or imaginary axis respectively, we can also
consider Landen formulas corresponding to shifts in units of $K(m) + iK'(m)$ in the complex plane. In this context it is worth noting that by starting from the Landen formulas for $p = 2$ as given by Eqs. (4) to (6), changing $m$ to $1/m$, $x$ to $kx$, and using the formulas
\[ \text{dn}(kx, \frac{1}{m}) = \text{cn}(x, m), \quad \text{cn}(kx, \frac{1}{m}) = \text{dn}(x, m), \quad \text{sn}(kx, \frac{1}{m}) = k \text{sn}(x, m), \] (125)
one readily obtains the Landen transformations \[ [3]
\[ \text{dn} \left[ (k + ik')u, \left( \frac{k - ik'}{k + ik'} \right)^2 \right] = \frac{1 - k(k - ik') \text{sn}^2(u, m)}{\text{cn}(u, m)}, \] (126)
\[ \text{cn} \left[ (k + ik')u, \left( \frac{k - ik'}{k + ik'} \right)^2 \right] = \frac{1 - k(k + ik') \text{sn}^2(u, m)}{\text{cn}(u, m)}, \] (127)
\[ \text{sn} \left[ (k + ik')u, \left( \frac{k - ik'}{k + ik'} \right)^2 \right] = \frac{(k + ik') \text{sn}(u, m) \text{dn}(u, m)}{\text{cn}(u, m)}. \] (128)

The generalizations of these Landen transformations to arbitrary $p$ is immediate. In particular, by changing $m(\tilde{m})$ to $1/m(1/\tilde{m})$, changing $x$ to $kx$ and using formulas (125) in the Landen formulas as given by Eqs. (68) to (73), we obtain the corresponding Landen formulas for shifts by complex amounts in units of $K(m) + iK'(m)$. For example, for any integer $p$, we get
\[ \text{cn}(x, \tilde{m}) = \delta \sum_{j=1}^{p} \text{cn}[\delta_1 x + 2(j - 1)(K(m) + iK'(m))/p, m], \] (129)
while for odd $p$ we have
\[ \text{dn}(x, \tilde{m}) = \delta_2 \sum_{j=1}^{p} \text{dn}[\delta_2 x + 4(j - 1)(K(m) + iK'(m))/p, m], \] (130)
\[ \text{sn}(x, \tilde{m}) = \delta_1 \sum_{j=1}^{p} \text{sn}[\delta_2 x + 4(j - 1)(K(m) + iK'(m))/p, m], \] (131)
with $\tilde{m}$ and $m$ being related by $\tilde{m} = \delta_1^2/\delta_2^2$.

On the other hand, the corresponding Landen formulas for even $p$ are
\[ \text{dn}(x, \tilde{m}) = \delta_2 \sum_{j=1}^{p} (-1)^{j-1} \text{cn}[\delta_2 x/\sqrt{\tilde{m}} + 2(j - 1)(K(m) + iK'(m))/p, m], \] (132)
\[ \delta_2 D_0 \text{sn}(x, \tilde{m}) = \sqrt{\tilde{m}} \prod_{j=1}^{p} \text{sn}[\delta_2 x/\sqrt{\tilde{m}} + 2(j - 1)(K(m) + iK'(m))/p, m], \] (133)
where $\tilde{m}$ is given by $\tilde{m} = \delta_2^2/\delta_1^2$. Here, $\delta, \delta_1, \delta_2$ are given by
\[ \delta \sum_{j=1}^{p} \text{cn}[2(j - 1)(K(m) + iK'(m))/p, m] = 1, \] (134)
\[
\delta_1 \sum_{j=1}^{p} \text{dn}[4(j-1)(K(m) + iK'(m))/p, m] = 1 , \quad (135)
\]
\[
\delta_2 \sum_{j=1}^{p} (-1)^{j-1} \text{cn}[2(j-1)(K(m) + iK'(m))/p, m] = 1 , \quad (136)
\]
while \(D_0\) is given by
\[
D_0 = \prod_{j=1}^{p-1} \text{sn}[2(jK(m) + iK'(m))/p, m] . \quad (137)
\]
It is easily checked that for \(p = 2\) the Landen formulas \(129\), \(132\) and \(133\) reduce to the well known “complex” Landen formulas \(127\), \(126\) and \(128\) respectively.

## 6 Landen Transformation Formulas and Cyclic Identities

Recently, we \[10, 11\] have obtained a large number of cyclic identities where combinations of Jacobi elliptic functions at different points are expressed in terms of sums like \(\sum_{j=1}^{p} \text{dn}(2(j-1)K(m)/p, m)\), \(\sum_{j=1}^{p} (-1)^{j-1} \text{dn}(2(j-1)K(m)/p, m)\), etc. Now the remarkable thing is that it is precisely these sums for which Landen and others have obtained the famous transformation formulas mentioned above. Thus by combining the cyclic identities and the Landen formulas given above, we can obtain a wide class of generalized Landen transformations for many combinations of Jacobi elliptic functions.

For example, a simple cyclic identity valid for both even and odd \(p\) is
\[
\sum_{j=1}^{p} \text{dn}^2(x_j, m)[\text{dn}(x_{j+r}, m) + \text{dn}(x_{j-r}, m)] = 2[\text{ds}(a, m)\text{ns}(a, m) - \text{cs}^2(a, m)] \sum_{j=1}^{p} \text{dn}(x_j, m) , \quad (138)
\]
where \(x_j\) is given by Eq. \(15\) and \(a = 2rK(m)/p\). On combining with relation \(68\), we obtain the Landen formula for the left hand side of \(138\) given by
\[
\sum_{j=1}^{p} \text{dn}^2(x_j, m)[\text{dn}(x_{j+r}, m) + \text{dn}(x_{j-r}, m)] = (2/\alpha)[\text{ds}(a, m)\text{ns}(a, m) - \text{cs}^2(a, m)]\text{dn}(x, \tilde{m}) , \quad (139)
\]
where \(\tilde{m}\) and \(m\) are related by Eq. \(15\).

Proceeding in this way, using the many identities obtained by us \[10, 11, 12\], we obtain a huge class of new Landen transformation formulas. As illustrations, we give four examples:
\[
m \sum_{j=1}^{p} \text{cn}(x_j, m)[\text{sn}(x_{j+r}, m) - \text{sn}(x_{j-r}, m)] = (2/\alpha)[\text{ns}(a, m) - \text{ds}(a, m)]\text{dn}(x, \tilde{m}) ; \quad (140)
\]
\[
\sum_{j=1}^{p} \text{dn}^2(x_j, m)[\text{dn}(x_{j+r}, m) - \text{dn}(x_{j-r}, m)] = -(2m/A)\text{cs}(a, m)\text{cn}(x, \tilde{m})\text{sn}(x, \tilde{m}) , \quad (141)
\]
where $A$ is $\alpha_1^2$ or $\alpha_2^2$ respectively depending on whether $p$ is an odd or an even integer, and $\alpha, \alpha_1, \alpha_2$ are given by Eqs. (14), (32) and (37) respectively;

$$
\sum_{j=1}^{p} \cot^2(x_j, m)\cot^2(x_{j+r}, m) = -(2/\alpha^2)\cot^2(a, m)\cot^2(x, m)
+ 4A_d\cot^2(a, m) + p[\cot^2(a, m) + \text{ds}^2(a, m) - 2\cot(a, m)\text{ds}(a, m)\text{ns}(a, m)\text{Z}(a, m)],
$$

where $A_d$ is given by Eq. (91);

$$
\sum_{j=1}^{p} \cot^3(x_j, m)[\cot(x_{j+r}, m) - \cot(x_{j-r}, m)] = -(2m/A)\cot(b, m)\text{cs}(b, m)\cot(x, m)\cot(x, m)\cot(x, m),
$$

where $A$ is $\alpha_2a^2$ for $p$ odd and $ma\alpha_2^2$ for $p$ even.

We now present four examples of novel Landen formulas which are only valid for odd $p$:

$$
m \sum_{j=1}^{p} \cot^2(x_j, m)[\cot(x_{j+r}, m) + \cot(x_{j-r}, m)] = -(2/\alpha_1)[\text{ds}(b, m)\cot(b, m) - \text{ns}^2(b, m)]\cot(x, m),
$$

$$
m^2 \sum_{j=1}^{p} \cot^3(x_j, m)[\cot^2(x_{j+r}, m) - \cot^2(x_{j-r}, m)]
= (2/\alpha_1)\text{ns}(b, m)[2\text{ds}(b, m)\cot(b, m) + \text{ns}^2(b, m)]\cot(x, m)\cot(x, m),
$$

$$
m \sum_{j=1}^{p} \cot^2(x_j, m)[\cot(x_{j+r}, m) + \cot(x_{j-r}, m)] = (2/\alpha_1)[\text{ns}(b, m)\cot(b, m) - \text{ds}^2(b, m)]\cot(x, m),
$$

$$
m \sum_{j=1}^{p} \cot(x_j, m)\text{cot}(x_{j+r}, m)\text{cot}(x_{j-r}, m)\text{cot}(x_{j+r}, m)\text{cot}(x_{j-r}, m)
- \text{cot}(x_{j-r}, m)\text{cot}(x_{j+r}, m)]
= (2/\alpha_2)\text{ds}(b, m)\text{cs}(b, m) + \text{ns}^2(b, m) + \text{cs}^2(b, m)\text{sn}(x, m)\text{dn}(x, m),
$$

where $x_j, \tilde{x}_j$ are given by Eq. (18) and $b = 4rK(m)/p$.

On the other hand, here are five examples of novel Landen formulas which are only valid for even $p$ and odd $r < p$:

$$
m \sum_{j=1}^{p} (-1)^{j-1}\cot(x_j, m)[\cot(x_{j+r}, m) - \cot(x_{j-r}, m)] = (2/\alpha_2)[\text{ns}(a, m) + \text{ds}(a, m)]\cot(x/\alpha, m),
$$

$$
m \sum_{j=1}^{p} (-1)^{j-1}\cot(x_j, m)\text{cot}(x_{j+r}, m)\text{cot}(x_{j-r}, m)\text{cot}(x_{j+r}, m)\text{cot}(x_{j-r}, m)
- \text{cot}(x_{j-r}, m)\text{cot}(x_{j+r}, m)]
= -(2/\alpha_2)\cot(a, m)[\text{ds}(a, m) - \text{ns}(a, m)]\cot(x/\alpha, m)\cot(x/\alpha, m),
$$

23
\[ \sum_{j=1}^{p} (-1)^{j-1} \text{dn}(x_j, m) \text{dn}(x_{j+r}, m) = -(2/\alpha_2) \text{cs}(a, m) \text{sn}(x/\alpha, \tilde{m}) ; \quad (150) \]

\[ \sum_{j=1}^{p} (-1)^{j-1} \text{dn}(x_j, m) \text{dn}(x_{j+r}, m) \text{dn}(x_{j+2r}, m) \text{dn}(x_{j+3r}, m) = (2/\alpha^2) \text{cs}(a, m) \text{cs}(2a, m) \text{cs}(3a, m) \text{sn}(x/\alpha, \tilde{m}) ; \quad (151) \]

\[ \sum_{j=1}^{p} (-1)^{j-1} \text{dn}^3(x_j, m) [\text{dn}(x_{j+r}, m) + \text{dn}(x_{j-r}, m)] = (2/\alpha^2) \text{ns}(a) \text{ds}(a) \text{cn}(x/\alpha, \tilde{m}) \text{dn}(x/\alpha, \tilde{m}) . \quad (152) \]

## 7 Conclusion

The results of this paper clarify the relationship between the well known periodic solutions of various nonlinear differential equations and those obtained recently by us using the idea of judicious linear superposition \[7, 8\]. In fact, we obtain a deep connection between the highly nonlinear Landen transformation formulas involving changes of the modulus parameter and certain linear superpositions of an arbitrary number of Jacobi elliptic functions. Further, we have found that using the identities with and without alternating signs along with Landen transformations, one can obtain novel Landen formulas for suitable combinations of Jacobi elliptic functions. In this context, it is worth recalling that recently we have also obtained cyclic identities with arbitrary weight factors and a large number of local identities \[12\]. It would be very interesting if these could also be profitably combined with the Landen transformations.

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References

[1] See, for example, P. G. Drazin and R. S. Johnson, *Solitons: an Introduction* (Cambridge, 1989).

[2] John Landen, Phil. Trans. LXV, 283 (1775).

[3] Harris Hancock, *Theory of Elliptic Functions* (Dover, 1958).

[4] For the properties of Jacobi elliptic functions, see, for example, M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions* (Dover, 1964).

[5] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products* (Academic Press, 2000).

[6] See, for example, D. F. Lawden, *Elliptic Functions and Applications*, Applied Math. Sc. Vol. 80 (Springer, 1989); A. Cayley, *An Elementary Treatise on Elliptic Functions*, (G. Bell, 1895).

[7] A. Khare and U. Sukhatme, *Linear Superposition in Nonlinear Equations*, Phys. Rev. Lett. **88** (2002) 244101.

[8] F. Cooper, A. Khare and U. Sukhatme, *Periodic Solutions of Nonlinear Equations Obtained by Linear Superposition*, J. Phys. A: Math. Gen. **35** (2002) 10085.

[9] W. P. Reinhardt, A. Khare, and U. P. Sukhatme, *Relating Linearly Superposed Periodic Solutions of Nonlinear Equations to One Soliton Solutions*, math-ph/0212069 (2002); M. Jaworski and M. Lakshmanan, *Comment on "Linear Superposition in Nonlinear Equations"*, Phys. Rev. Lett. 90 (2003) 239401 and A. Khare and U. Sukhatme, *Khare and Sukhatme Reply*, Phys. Rev. Lett. 90 (2003) 239402.

[10] A. Khare and U. Sukhatme, *Cyclic Identities Involving Jacobi Elliptic Functions*, J. Math. Phys. **43** (2002) 3798.

[11] A. Khare, A. Lakshminarayan and U. Sukhatme, *Cyclic Identities for Jacobi Elliptic and Related Functions*, J. Math. Phys. **44** (2002) 1841; also see, math-ph/0207019.

[12] A. Khare, A. Lakshminarayan and U. Sukhatme, *Local Identities Involving Jacobi Elliptic Functions*, math-ph/0306028 (2003).
Table 1: A table showing the transformed modulus parameter $\tilde{m}$ appearing in the generalized Landen transformation formulas as a function of the modulus parameter $m$ and the number of terms $p$ in the formula. The values of $\tilde{m}$ have been computed using Eq. (15). As discussed in the text, the same values of $\tilde{m}$ could equally well have been obtained from the equivalent expressions Eqs. (35) or (53) for odd integers $p$, and Eqs. (39) or (58) for even integers $p$. Note that the table also gives the change of modulus parameter for Gauss transformations if one interchanges the roles of $m$ and $\tilde{m}$.

| $m$  | $\tilde{m}(p = 2)$ | $\tilde{m}(p = 3)$ | $\tilde{m}(p = 4)$ | $\tilde{m}(p = 5)$ | $\tilde{m}(p = 6)$ | $\tilde{m}(p = 7)$ |
|------|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|
| 0    | 0                 | 0                 | 0                 | 0                 | 0                 | 0                 |
| 0.25 | .5155 x 10^{-2}   | .9288 x 10^{-4}   | .1669 x 10^{-5}   | .3000 x 10^{-7}   | .5392 x 10^{-9}   | .9693 x 10^{-11}  |
| 0.5  | .2944 x 10^{-1}   | .1290 x 10^{-2}   | .5580 x 10^{-4}   | .2411 x 10^{-5}   | .1042 x 10^{-6}   | .4503 x 10^{-8}   |
| 0.75 | .1111             | .1005 x 10^{-1}   | .8666 x 10^{-3}   | .7438 x 10^{-4}   | .6381 x 10^{-5}   | .5475 x 10^{-6}   |
| 0.9  | .2699             | .4311 x 10^{-1}   | .6158 x 10^{-2}   | .8655 x 10^{-3}   | .1213 x 10^{-3}   | .1701 x 10^{-4}   |
| 0.99 | .6694             | .2506             | .7283 x 10^{-1}   | .1963 x 10^{-1}   | .5185 x 10^{-2}   | .1362 x 10^{-2}   |
| 0.999| .8811             | .5292             | .2374             | .9312 x 10^{-1}   | .3464 x 10^{-1}   | .1264 x 10^{-1}   |
| 0.9999| .9608            | .7446             | .4481             | .2293             | .1080             | .4891 x 10^{-1}   |
| 0.99999| .9874           | .8721             | .6374             | .3973             | .2239             | .1193             |
| 1    | 1                 | 1                 | 1                 | 1                 | 1                 | 1                 |