CAN OUR NUMBER SYSTEM BE IMPROVED?

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Abstract

Our number system is a magnificent tool. But it is far from perfect. Can it be improved? In this paper some possibilities are discussed, including the use of a different base or directed (negative as well as positive) numerals. We also put forward some suggestions for further research.

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1 Introduction

We are so accustomed to our number system that we hardly notice it. It is not possible however to lay attention on this tool and not to feel captivated by its elegance, power and usefulness. If there is art in Mathematics there is no doubt that our number system is one of its best pieces. But if our number system has art in it, it is immensely useful and practical too.

The system we use to write number is usually called the 'decimal' system (Fig. 1). But the fact that it uses ten symbols is not its main or, even less so, defining characteristic. A more fundamental feature is the use it makes of the place value of the numerals. This place-value notation, which is easier to use than to explain - but we will come back to it later - implies, for instance, that 12 is not the same as 21.

Our number system is surprisingly modern. Its elements appeared in different historic moments in China and India and propagate to the West through the muslim world. But the system was not commonly used (by the educated elites) in the West until the dawn of the Renaissance. There must be no doubt that the introduction of this number system, so convenient, have
something to do with the quick developments in Mathematics, the sciences and technology from the Renaissance onwards. It is difficult to imagine that the calculus, for instance, would have been developed had Mathematicians have to wrestle with a system so awkward as the Roman.

![The number line in our number system](image)

Figure 1: The number line in our number system

It is surprising that the place value principle were introduced so late to write numbers considering that this same principle plays a principal role in language too. In fact, it is thanks to this principle that with a handful of letters and another symbols we are able to express an immense complexity of meanings. Thanks to place value the word 'top', for instance, can mean something completely different from the word 'pot' [1]. This principle of order also plays a role (minor perhaps in most languages) at a grammatical level. A different word order may produce a different meaning for the phrase. For example, 'You are tired' is an affirmation whereas 'Are you tired?' is a question [2].

## 2 Is base 6 better?

Our number system is so useful, so convenient, so practical for the function it is meant to fulfil that it is not possible to think about it and not to feel fascinated.

Not everything is perfect about this tool however. To start with children needs (many) years to learn the rudiments of the times table. And most of them reach adulthood without completely mastering it. I myself, in spite of having spent decades doing Mathematics at all levels, need to use all sorts of tricks (or a calculator) to find, for instance, what $7 \times 8$ is (exactly).

This difficulty in being mastered shed a shadow of doubt on the perfection of a tool which the modern man needs to use almost in a daily basis.

And this difficulty is not all. Our number system has several other problems and limitations. The fact is, for instance, that with our system we can write a 'few' numbers only. For instance, no irrational number can be written with this system. To denote irrational quantities we have to use other symbols, e.g. $\pi$, $e$, $\sqrt{2}$, and so on.

On the other hand, there exist the general impression that decimal numbers are more or less the same as fractions, i.e. the rational numbers. The truth however is that most of the common fractions, i.e. the fractions we use
on a daily basis - such as \(\frac{1}{3}, \frac{1}{6}\) and so on - cannot be written as a decimal number [3]. In fact, only fraction whose denominator is of the form \(2^n \times 5^m\), for some natural numbers \(n\) and \(m\), can be written as decimal numbers. This may not be a serious problem but it is a nuisance nonetheless.

If we want a nice decimal expression for \(\frac{1}{3}\) we must use a system whose basis is a multiple of 3, e.g. 3, 6 and so on.

A system with base 6 (Fig. 2) would be very handy indeed. And since this system has a very simple times table it is very convenient too and would be welcomed in every school.

| \(-\forall\) | \(-H\) | \(-\Pi\) | \(-\Gamma\) | 0 | \(\Gamma\) | \(\Delta\) | \(H\) | \(\forall\) | \(\Gamma 0\) |
|---|---|---|---|---|---|---|---|---|---|

Figure 2: Number line for the system \(N_{(0,5)}\)

In fact, all the times tables in this system are either trivial - the one for 0 and \(\Gamma\) - or very easy - the one for \(\Pi\), \(\Delta\) and \(\forall\) (which play the role of the 9 in our number system) with the exception of the table for \(H\). There is therefore a single difficult product, \(H \times H\). But we can easily move to this point of the times table from other points, for instance, by making \(H \times H = \forall \times \Delta + \Gamma\). Therefore with a system with basis 6, learning the times table would really be child’s play.

3 Directed numerals and the joy of rounding

Our figures can be misleading too. For example, 29 is far closer to 30 than it is to 20 but its first numeral is 2. This problems has psychological implication so that many shops use prices like 199.99 because they surely believe (with good reason, no doubt) that this figure conveys the impression of 'about one hundred pounds' while it is obvious that the actual price is two hundred pounds.

This problem is related to rounding. Rounding should be a natural things to do with numbers. But we see that our number system is not well behaved in this area.

Can this problem be solved? A simple solution - and one that, as a byproduct, improves other areas of our number system too - involves the introduction of the substraction operation in the way we write numbers. The idea is to use numerals that instead of adding, take away. Let, for instance, \(\alpha\) be the symbol for taking away one unit. Then we have \(3\alpha = 30 - 1 = 29\).

Introducing these symbols may appear an artificial thing to do since now we would have two different ways of writing the same number. However,
it is easy to see that we can write any number by using just five negative numerals: \(\alpha, \beta, \delta, \gamma\) and \(\epsilon\), and four positive ones: 1, 2, 3, and 4 (Fig. 3). (The negative numerals and the positive numerals can be collectively referred to as 'directed numerals'.)

\[
\begin{array}{ccccccccccc}
\alpha & 3 & \alpha & 4 & \epsilon & \gamma & \delta & \beta & \alpha & 0 & 1 & 2 & 3 & 4 & 1\epsilon & 1\gamma & 1\delta \\
\end{array}
\]

Figure 3: Number line for the system \(N_{(5,4)}\)

For example, we have now 284 = 3\(\beta\)4. If we now round the last numeral we get 3\(\beta\)0. If we round now to hundreds we get 300. So we see that this number system is better behaved under rounding than the current one.

Moreover, this number system brings with it unexpected gains, a saving of symbols, for example. The sign '-' for indicating negative numbers is no longer needed. With the new notation a number will be negative if the first significant digit is negative and positive if the first significant digit is positive. For instance, \(\delta\)3 is negative (= -27).

A disadvantage of the new system with respect to the standard one is its bad behavior under 'changing sign'. With the standard number system this is done by simple adding (or removing) a '-' sign. With the new system this is considerably more complicated. For instance, the opposite of 2\(\epsilon\) is \(\alpha\epsilon\).

This complication can be traced back to the lack of symmetry between negative numerals and positive ones. Whereas there are five negative numerals there are only four positive ones [4]. This problem would not exist if a system with an odd base were used. For example with a base 7 made of three positive numerals, three negative ones and zero.

4 More generalizations

The best way of understanding a structure is exploring ways in which it could be changed while preserving its principal characteristics. We have seen above that in addition to different bases, number systems can be considered that make use of directed numerals. This whole family of number systems can be characterized by a pair of natural numbers \((a, b)\) where \(a\) corresponds to the number of negative numerals and \(b\) to the number of positive ones. In this way our number system would be \(N_{(0,9)}\) and the one described in the previous section would be \(N_{(5,4)}\). A symmetrical system - and well behaved therefore under rounding - with base 7 would be \(N_{(3,3)}\).

Are there more radical generalization available? The following definitions offer a wide family of possibilities.
Definition 1. A normal pre-numeration system is a triplet \( N = \langle I, i, (a) \rangle \) where \( I \) is a finite or countable set; \( i \) is an injective application from \( I \) unto the set of integer numbers \( \mathbb{Z} \); and \( (a) \) is a sequence valued in \( \mathbb{Z} \).

The numerical value of a string (figure) \( c_n \ldots c_2 c_1 \) (with \( c_n, \ldots, c_1 \in I \)) is given by

\[
V_N(c_n \ldots c_2 c_1) = i(c_1) \times a_1 + i(c_2) \times a_2 + \ldots + i(c_n) \times a_n
\]  

(1)

It is clear that in this definition the elements of \( I \) play the role of the symbols (numerals) of the number system; the function \( i \) fixes the value of each numeral; and the sequence \( (a) \) gives a value to the numerals’ place in the figure.

For our numeration system \( I = \{0, 1, \ldots, 9\} \); the application \( i \) assigns the value ‘nil’ to 0, the value ‘unity’ to 1, and so on; and the sequence \( (a) \) is given by \( a_n = 10^n \).

Definition 2. A normal pre-numeration system \( N = \langle I, i, (a) \rangle \) is said to be

- \textit{Finite} iff \( I \) is finite.
- \textit{Perfectly ordered} iff the sequence \( (a) \) fulfil the condition that \( a_n = a_m \iff n = m \)
- \textit{Perfectly disordered} iff the sequence \( (a) \) is constant [5].

Def. 1 allows us to easily consider ‘exotic’ number systems, where positive and negative numerals are taken at random from the number line. Other even more exotic possibilities - but which may be more convenient to our psychology than the standard system - include using sequences such as

\[
a_n = n! \quad \text{so that, for instance, } 23 \text{ means } 2 \times 2! + 3 \times 1
\]  

(2)

or also

\[
a_n = n^n \quad \text{so that } 57 \text{ means } 5 \times 2^2 + 7 \times 1
\]  

(3)

But, how many of these pre-numeration systems are well behaved?

Definition 3. A normal pre-numeration system \( N \) is said to be a \textit{normal number system} if every integer number \( n \) can be written within \( N \), i.e. if for all integer \( n \), there exist a string of numerals \( c_n \ldots c_1 \) such that \( V_N(c_n \ldots c_1) = n \) [6].

Definition 4. A normal number system is said to be \textit{univocal} if every integer \( n \) can be expressed in a single way within \( N \).
A exciting line of research would be to find out what kind of normal
pre-number systems are normal number systems and among these which are
univocal.

It is clear that generalizations that go beyond normal number systems
can also be considered.

There are three basic things that can be done with two numbers: 1) compare them; 2) add them; and 3) multiply them. A number system is
a way of giving names to the numbers. The natural demand to be made
of a number system is that these names are well behaved under these three
operations. It turns out, however, that while our number system is good with
respect to the first operation (comparing numbers) it is very badly behaved
with respect to the other two (adding and multiplying numbers). For there
is nothing in the pair of symbols 49, for instance, that indicates that this
number is $7 \times 7$. Or there is nothing in 18 that indicates that it is worth
$3 \times 6$. Or for that matter there is nothing in 9 that indicates that it denotes
‘nine’.

On the other hand, the most important property of a number system
is that the representation of a number conveys clearly the actual size of
the number. We have seem that not even in this area is our current system
specially good. On the contrary, it can be misleading and is not well behaved
under rounding.

A notation is good when it is used unnoticeably. This is clearly not the
case with our number system. Instead of facilitating our calculations it is an
obstacle that we have to climb to carry them out.

From these consideration it follows that the shape of the numerals(and the
figures) deserves attention. Perhaps it would be possible to devise numerals
(and figures) that would multiply (and add) naturally.

Ideally, it should be possible to graphically combine two figures to obtain
the figure representing their total or their product.

5 Conclusions and perspectives

We have discussed some of the positive aspects and some of the negative
aspects of our number system and have proposed ways of overcoming the
latter.

We have found that once generalizations are considered it becomes ap-
parent that our number system is not as magnificent at it appears on the
surface. Far from being perfect, our number system has many problems and
limitations.

An important problem of our number system is the complexity of its
times table. Many children start loathing Mathematics as soon as they come across the times table. If the first thing our children find of Mathematics is so unattractive, should we not take into serious consideration the possibility of changing it?

We have suggested that a number system with base 6 is advantageous in a number of ways. Its times table in particular is much easier to learn. This system has the additional advantage that a system in base 12 (dozens) is already used in some realms, e.g. in the hours of the day, minutes in the hour, the months of the year, and so on. Perhaps it would be advantageous to give more protagonism to a system with base 6, which could be increasingly used in everyday life and in the first years of Mathematics education.

We have also discussed other, more 'exotic' generalization of our number system. Among these we have found several that solve some of the problems of our number system. We have not found however any which represent a significant improvement.

It is perfectly possible, however, that among the generalizations yet to be analyzed a number system can be found that is really advantageous. Perhaps one of these systems allows us to seamlessly calculate $4,349 \times 5,673$ or $12.7\%$ of $15,642$ and it is perfect for everyday Mathematics. Perhaps another is most suitable for advanced Mathematics, e.g. number theory.

It would not be inconceivable that alongside the current number system another one (or several others) more appropriate for the task at hand could be used.

It is clear that more research is needed in this area which has so clear practical applications and so exciting theoretical implications. The objective is devising a way of denoting the numbers which is well behaved under addition and multiplication. Ideally the figure denoting the product (or addition) of two numbers should be obtainable by combining graphically (in an easy way) the figures representing the factors. It is possible that this can be done within the current decimal system by simply using numerals with better, more suitable shapes.

Will our grand-grandchildren wonder in the future why despite all our technological advances we have failed to perfect our inconvenient number system for several centuries?

6 Notes

[1] In a sense a number system is just a way (more or less convenient) of labelling (or naming) numbers. Thanks to the place-value principle we can name $10^n$ numbers using figures with no more than $n$ numerals. But numbers
can also be denoted with their name, in English, for instance.
[2] In fact, an interesting exercise would be to study the possibility (theoretical at least) of constructing a perfectly-ordered language, i.e. a language such that the order of the words plays a fundamental role in conveying meaning and such that a meaning could be conveyed with one order only.
[3] Sometimes an arc over the recurring digits is used. But in purity this, in addition to being inelegant, implies actually going beyond the place value principle.
[4] This asymmetry implies that the new number system is not completely well behaved under rounding. For instance, 2εε(= 145) is rounded (hundreds) to 100 despite its first digit being 2.
[5] Originally the order of the symbols was completely irrelevant in the Roman number system and VI, for instance, and IV denoted the same number, namely 'six'. The original Roman number system was therefore completely disordered. The (minor) role of the order was introduced in the Middle Ages (and it is not clear at all that it represents an improvement).
[6] A subtlety may complicate this definition. In fact it is only necessary that the figures cover half the integers - as long as the other half can be reached by using a negative '−' sign, for instance. In this definition we have not paid attention either to the decimal numbers.
[7] A base 6 appears natural too from a Mathematical point of view in that 6 is the product of the first two prime numbers. According to this criterion the possible bases would be 2, 6 = 2 × 3, 30 = 2 × 3 × 5, 210... and so on.