On the Ornstein-Zernike behaviour for the supercritical Random-Cluster model on $\mathbb{Z}^d, d \geq 3$.

M. Campanino#,†, M. Gianfelice°,†
#Dipartimento di Matematica
Università degli Studi di Bologna
P.zza di Porta San Donato, 5 I-40127
massimo.campanino@unibo.it

°Dipartimento di Matematica e Informatica
Università della Calabria
Campus di Arcavacata
Ponte P. Bucci - cubo 30B
I-87036 Arcavacata di Rende
gianfelice@mat.unical.it

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Abstract

We prove Ornstein-Zernike behaviour in every direction for finite connection functions of the random cluster model on $\mathbb{Z}^d, d \geq 3$, for $q \geq 1$, when occupation probabilities of the bonds are close to 1. Moreover, we prove that equi-decay surfaces are locally analytic, strictly convex, with positive Gaussian curvature.

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1 Introduction and results

Ornstein-Zernike behaviour of correlation functions for Gibbs random fields and of connection functions for percolation models gives an exact power law correction outside critical points. Apart from its intrinsic interest, in the two-dimensional case it is related to the behaviour of fluctuations of interfaces and therefore to the study of phases of two dimensional systems ([Ga], [CCC], [CI], [CIL], [CIV2], [CD-CIV]).

Initially Ornstein-Zernike behaviour has been rigorously proved in the high temperature/low probability region (see e. g. [BF]). In the last few decades these results have been extended to subcritical percolation models and to high temperature finite-range Ising models up to their critical points ([CCC], [CI], [CIV1], [CIV2]).

Above the critical probability connection functions converge to a positive constant as the distance of the sites tend to infinity. One is then led to study the asymptotic behaviour of finite connection functions, i.e. the probabilities that two sites belong to a common finite open cluster. These correspond for Gibbs random fields to truncated correlation functions. In [BF] Bricmont and Fröhlich proved Ornstein-Zernike behaviour for truncated correlation functions of Ising model in the direction of axes in dimension $d \geq 3$ at low temperatur. In the same paper arguments are given, suggested by their proof, in favour of a different asymptotic behaviour in the two-dimensional case. A rigorous proof of this in the case of finite connection functions of two-dimensional Bernoulli percolation above critical probability is given in [CIL].

The analysis of the asymptotic behaviour of finite connection functions in dimension $d \geq 3$ has been carried on for Bernoulli percolation with the parameter close to 1 in [BPS] for connection functions along Cartesian axes and then in [CG] for connections in all directions. [BPS] uses cluster expansions, whereas [CG] exploits the methods developed in [CI] and [CIV1], [CIV2], together with specific techniques built up to deal with probabilities of non-monotone events such as finite connections. Here we extend the results of [CG] to FK random cluster models, with $q \geq 1$, when the probability parameter $p$ is close to 1. The exponential decay of finite connection functions of FK...
random clusters can be established by using an inequality proved in [BHK].

In the rest of this section we present the main results of the paper and the notation that we will use. In the next section we prove the existence of the finite correlation length for translation invariant Random Cluster measures and show that, for \( p \) sufficiently close to 1, finite supercritical clusters, up to a negligible probability, have a one-dimensional structure. This will allow us to reduce the analysis of the exact asymptotics of the finite two-point connection function to the proof of a local limit theorem result for an effective stationary random walk via thermodynamic formalism.

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1.1 Notation

Given a set \( A \subset \mathbb{R}^d, d \geq 1 \), let us denote by \( A^c \) its complement. We also set \( \mathcal{P}(A) \) to be the collection of all subsets of \( A \), \( \mathcal{P}_n(A) := \{ A \in \mathcal{P}(A) : |A| = n \} \) and \( \mathcal{P}_f(A) := \bigcup_{n \geq 1} \mathcal{P}_n(A) \), where \( |A| \) is the cardinality of \( A \). Moreover, we denote by \( \mathcal{A}, \overline{\mathcal{A}} \) respectively the interior of \( A \) and the closure of \( A \) and set \( \partial A := \overline{\mathcal{A}} \setminus \mathcal{A} \) the boundary of \( A \) in the Euclidean topology. Furthermore, for \( B \subset \mathbb{R}^d \), we set

\[
\mathcal{B} + A := \bigcup_{x \in \mathcal{B}} (x + A),
\]

where, given \( x \in \mathbb{R}^d \),

\[
x + A := \{ y \in \mathbb{R}^d : y - x \in A \}.
\]

Moreover, we denote by \( |x| := \sum_{i=1}^{d} |x_i| \), by \( \langle \cdot, \cdot \rangle \) the scalar product in \( \mathbb{R}^d \) and by \( \| \cdot \| := \sqrt{\langle \cdot, \cdot \rangle} \) the associated Euclidean norm. We then set, for \( x \neq 0 \), \( \hat{x} := \frac{x}{\|x\|}, S_{d-1} := \{ z \in \mathbb{R}^d : \|z\| = 1 \} \) and, denoting by \( B \) the closed unit ball in \( \mathbb{R}^d \), for \( r > 0 \), we let \( rB := \{ x \in \mathbb{R}^d : \|x\| \leq r \} \) and \( B_r(x) := x + rB \).

For any \( t \in \mathbb{R}^d \) we define

\[
\mathcal{H}_t^i := \{ x \in \mathbb{R}^d : \langle t, x \rangle = 0 \}
\]

(3)

to be the \((d - 1)\)-dimensional hyperplane in \( \mathbb{R}^d \) orthogonal to the vector \( t \) passing through the origin and the corresponding half-spaces

\[
\mathcal{H}_t^{i-} := \{ x \in \mathbb{R}^d : \langle t, x \rangle \leq 0 \},
\]

(4)

\[
\mathcal{H}_t^{i+} := \{ x \in \mathbb{R}^d : \langle t, x \rangle \geq 0 \},
\]

(5)

so that, setting for \( t \in \mathbb{R}^d, \mathcal{H}^i_t := x + \mathcal{H}_t^i \), we denote by \( S_{xy}^i := H_x^{i+} \cap H_y^{i-} \).

We also denote by \( \text{dist}(\mathcal{A}, \mathcal{B}) \) the Euclidean distance between two subset \( \mathcal{A}, \mathcal{B} \) of \( \mathbb{R}^d \).
1.1.1 Graphs

To make the paper self-contained, we will now introduce those notions of graph theory which are going to be used in the sequel and refer the reader to \[\text{Bo}\] for an account on this subject.

Let \( G = (V, E) \) be a graph whose set of vertices and set of edges are given respectively by a finite or denumerable set \( V \) and \( E \subseteq \mathcal{P}_2(V) \). \( G' = (V', E') \) such that \( V' \subseteq V \) and \( E' \subseteq \mathcal{P}_2(V') \cap E \) is said to be a subgraph of \( G \) and this property is denoted by \( G' \subseteq G \). If \( G' \subseteq G \), we denote by \( V(G') \) and \( E(G') \) respectively the set of vertices and the collection of the edges of \( G' \). \( |V(G')| \) is called the order of \( G' \) while \( |E(G')| \) is called its size. Given \( G_1, G_2 \subseteq G \), we denote by \( G_1 \cup G_2 := (V(G_1) \cup V(G_2), E(G_1) \cup E(G_2)) \subseteq G \) the graph union of \( G_1 \) and \( G_2 \). Moreover, we say that \( G_1, G_2 \subseteq G \) are disjoint if \( V(G_1) \cap V(G_2) = \emptyset \). A path in \( G \) is a subgraph \( \gamma \) of \( G \) such that there is a bijection \( \{0, ..., |E(\gamma)|\} \ni i \mapsto v(i) := x_i \in V(\gamma) \) with the property that any \( e \in E(\gamma) \) can be represented as \( \{x_{i-1}, x_i\} \) for \( i = 1, ..., |E(\gamma)| \). A walk in \( G \) of length \( l \geq 1 \) is an alternating sequence \( x_0, e_1, x_1, ..., e_l, x_l \) of vertices and edges of \( G \) such that \( e_i = \{x_{i-1}, x_i\} \) \( i = 1, ..., l \). Therefore, paths can be associated to walks having distinct vertices. Two distinct vertices \( x, y \) of \( G \) are said to be connected if there exists a path \( \gamma \subseteq G \) such that \( x_0 = x, x_{|E(\gamma)|} = y \). A graph \( G \) is said to be connected if any two distinct elements of \( V(G) \) are connected. The maximal connected subgraphs of \( G \) are called components of \( G \) and their number is denoted by \( \kappa(G) \). Moreover, to denote that \( \gamma \subseteq G \) is a component of \( G \) we write \( \gamma \subset G \). Given \( E' \subseteq E \), we denote by \( G(E') := (V, E') \) the spanning graph of \( E' \). We also define

\[
V(E') := \left( \bigcup_{e \in E'} e \right) \subseteq V.
\]

Given \( V' \subseteq V \), we set

\[
E(V') := \{ e \in E : e \subseteq V' \}
\]

and denote by \( G[V'] := (V', E(V')) \) that is called the subgraph of \( G \) induced or spanned by \( V' \). Moreover, if \( G' \subset G \), we denote by \( G \setminus G' \) the graph \( G[V \setminus V(G')] \subseteq G \) and define the boundary of \( G' \) as the set

\[
\partial G' := \{ e \in E \setminus E(G') : |e \cap V(G')| = 1 \} \subseteq E.
\]

1.1.2 The Random Cluster measure

Let \( \mathbb{L}^d \) denote the graph associated to \( (\mathbb{Z}^d, \mathbb{E}^d) \), with

\[
\mathbb{E}^d := \{ (x, y) \in \mathcal{P}_2(\mathbb{Z}^d) : |x - y| = 1 \}.
\]

Let \( \mathcal{L}_0 \) be the collection of subgraphs of \( \mathbb{L}^d \) of finite order. If \( G \in \mathcal{L}_0 \), we denote by \( \overline{G} \) the graph induced by the union of \( V(G) \) with the the sets of vertices of the components
of the $L^d \setminus G$ of finite size. We define the external boundary of $G$ to be $\partial G := \partial \overline{G}$. We remark that, given $G_i := (V_i, E_i)$, $i = 1, 2$ two connected subgraphs of $L^d$ of finite size, by $(\mathbb{R})$, $\partial (G_1 \cup G_2) \subseteq \partial G_1 \cup \partial G_2$. Moreover,

$$
\partial (G_1 \cup G_2) = \partial (G_1 \cup G_2) \subseteq \partial \overline{G_1} \cup \partial \overline{G_2}.
$$

(10)

We then set $G_0 := \{ G \in \mathcal{L}_0 : \partial G = \overline{\partial G} \}$ and denote by $G_c$ the collection of connected elements of $G_0$.

Considering the realization of $L^d$ as a geometric graph embedded in $\mathbb{R}^d$, which, with abuse of notation, we still denote by $L^d$, we can look at it as a cell complex, i.e. as the union of $\mathbb{Z}^d$ and $\mathbb{E}^d$ representing respectively the collection of 0-cells and of 1-cells, we denote by $(\mathbb{Z}^d)^*$ the collection of $d$-cells dual to 0-cells in $L^d$, that is the collection of Voronoi cells of $L^d$, and by $(\mathbb{E}^d)^*$ the collection of $(d-1)$-cells dual 1-cells in $L^d$, usually called plaquettes in the physics literature.

We also define

$$
\mathcal{C} := \left\{ \{e_1^*, e_2^*\} \in \mathcal{P}_2 \left( (\mathbb{E}^d)^* \right) : \text{codim} (\partial e_1^* \cap \partial e_2^*) = 2 \right\}
$$

(11)

and consider the graph $\mathcal{G} := ((\mathbb{E}^d)^*, \mathcal{C})$.

We remark that since duality is an involution: if $E^* \subset (\mathbb{E}^d)^*$, $E^{**} = E \subset \mathbb{E}^d$.

A bond percolation configuration on $L^d$ is a map $\mathbb{E}^d \ni e \longmapsto \omega_e \in \{0, 1\}$. Setting $\Omega := \{0, 1\}^{\mathbb{E}^d}$, we define

$$
\Omega \ni \omega \longmapsto \mathbf{E}(\omega) := \{ e \in \mathbb{E}^d : \omega_e = 1 \} \in \mathcal{P}(\mathbb{E}^d),
$$

(12)

Denoting by $\mathcal{G} := \{ G \subset L^d : G = G(E), \ E \in \mathcal{P}(\mathbb{E}^d) \}$ the collection of spanning subgraphs of $L^d$, we define the random graph

$$
\Omega \ni \omega \longmapsto \mathcal{G}(\omega) := G(\mathbf{E}(\omega)) \in \mathcal{G}
$$

(13)

and by $\kappa(\omega)$ the number of its components. Then, given $l \geq 1$, $x_1, \ldots, x_l \in \mathbb{Z}^d$, we denote by

$$
\Omega \ni \omega \longmapsto \mathcal{C}_{\{x_1, \ldots, x_l\}}(\omega) \in \mathcal{P}(\mathbb{Z}^d)
$$

(14)

the common open cluster of the points $x_1, \ldots, x_l \in \mathbb{Z}^d$, that is the set of vertices of the component of the random graph $\mathcal{G}$ to which these points belong, provided it exists, and define, in the case $\mathcal{C}_{\{x_1, \ldots, x_l\}}$ is finite, the random set $\partial \mathcal{C}_{\{x_1, \ldots, x_l\}}$ to be equal to $\partial G$ if $G$ is the component of $\mathcal{G}$ whose set of vertices is $\mathcal{C}_{\{x_1, \ldots, x_l\}}$ and the random set

$$
\mathcal{S}_{\{x_1, \ldots, x_l\}} := (\partial \mathcal{C}_{\{x_1, \ldots, x_l\}})^*.
$$

(15)

Let $\mathcal{F}$ be the $\sigma$-algebra generated by the cylinder events of $\Omega$. If $\Lambda \subset \subset \mathbb{Z}^d$, let $\mathbb{E}^\Lambda$ be the subset of $\mathbb{E}^d$ such that $V(\mathbb{E}^\Lambda) = \Lambda$ and denote by $\Omega_\Lambda := \{0, 1\}^{\mathbb{E}^\Lambda}$, by
$\mathcal{F}_\Lambda$ the corresponding product $\sigma$-algebra and by $\mathcal{T}_\Lambda$ the $\sigma$-algebra generated by the cylinder events $\{\omega \in \Omega : \omega_\Lambda \in A\}$, where $\Delta \subset \Lambda^c, A \in \mathcal{F}_\Delta$. The Random Cluster (RC) measures on $\mathbb{Z}^d$ (see [FK], [ES]) with parameters $q \geq 1$ and $p := \{p_e\}_{e \in \mathbb{Z}^d}$, where $\mathbb{E}^d \ni e \mapsto p_e \in [0, 1]$, are the dependent bond percolation probability measures $\mathbb{P}$ on $(\Omega, \mathcal{F})$ specified by

$$
\mathbb{P} (A | \mathcal{T}_\Lambda) = \mathbb{P}_{\Lambda,q,p} (A) \quad \mathbb{P} - a.s., \quad A \in \mathcal{F},
$$

where, setting for any $\pi \in \Omega^\Lambda, \Omega^\Lambda := \{\omega \in \Omega : \omega_e = \pi_e, e \in \mathbb{E}^d \setminus \mathbb{E}^\Lambda\}, \mathbb{P}^\pi_{\Lambda,q,p}$ is the probability measure on $(\Omega, \mathcal{F})$ with density

$$
\mathbb{P}^\pi_{\Lambda,q,p} (\omega) := \frac{1}{Z^\pi_{\Lambda} (q,p)} \prod_{e \in \mathbb{E}^\Lambda} p^\omega_e (1 - p^\omega) (1 - \omega_e) \kappa^\Lambda (\omega) 1_{\Omega^\Lambda} (\omega),
$$

where $\kappa^\Lambda (\omega)$ is the number of the components of $G (\omega)$ intersecting $\Lambda$.

Random Cluster measures satisfy the FKG inequality, that is, for any couple $f, g$ of r.v.’s increasing w.r.t. the natural partial order defined on $\Omega$, $\mathbb{P} (f g) \geq \mathbb{P} (f) \mathbb{P} (g)$. Moreover, the partial order of $\Omega$ induces a stochastic ordering on the elements of the collection of probability measures defined by (17); namely, for any increasing r.v. $f, \mathbb{P}^\pi_{\Lambda,q,p} (f) \leq \mathbb{P}^\pi_{\Lambda',q,p} (f)$ if $\pi_1 \leq \pi_2$. Hence, denoting by $\preceq$ such ordering, $\forall \pi \in \Omega^\Lambda, \mathbb{P}^\pi_{\Lambda,q,p} \leq \mathbb{P}^\pi_{\Lambda',q,p} \leq \mathbb{P}^\pi_{\Lambda',q,p}$, where $\mathbb{P}^\pi_{\Lambda,q,p}$ and $\mathbb{P}^\pi_{\Lambda',q,p}$ stand for respectively the probability measure with density (17) corresponding to the free ($\pi = 0$) and to the wired ($\pi \equiv 1$) boundary conditions. Since, for $\# = f, w$, the (weak) limit of the sequence $\{\mathbb{P}^\pi_{\Lambda,q,p}\}_{\pi \in \Omega^\Lambda}$ along any exhaustion $\{\Lambda\} \uparrow \mathbb{Z}^d$ exists (see e.g. [Gr] Theorem 4.19) and is the Random Cluster measure which we denote by $\mathbb{P}^\#_{q,p}$, the ordering $\preceq$ extends as well to Random Cluster measures and $\mathbb{P}^\#_{q,p} \preceq \mathbb{P} \preceq \mathbb{P}^w_{q,p}$.

Furthermore, denoting by $\mathbb{P}^p' := \mathbb{P}_{1,p'}$ the independent Bernoulli bond percolation probability measures on $\mathbb{Z}^d$ with parameter set $p' = \{p'_e\}_{e \in \mathbb{Z}^d}$, by Theorem (3.21) p.43 of [Gr], we obtain the following stochastic domination inequalities

$$
\mathbb{P} (p'_{q,p}) \leq \mathbb{P}^\#_{q,p} \leq \mathbb{P}^w_{q,p} \leq \mathbb{P}_p,
$$

where $\forall e \in \mathbb{E}^d, p_e (q) := \frac{p_e}{p_e + q (1 - p_e)}$.

In the following, we assume the Random Cluster random field specification defined in (16) to be translation invariant; therefore we set, $\forall e \in \mathbb{E}^d, p_e = p$. Moreover, we assume the Random Cluster measure $\mathbb{P}_{q,p}$ to be translation invariant.

### 1.2 Results

**Theorem 1** For any $d \geq 3$ and any $q \geq 1$, there exists $p_0 = p_0 (q, d)$ such that, $\forall p > p_0$, uniformly in $x \in \mathbb{Z}^d$ as $\|x\| \to \infty$,

$$
\mathbb{P}_{q,p} \{0 \leftrightarrow x, |C_{\{0,x\}}| < \infty\} = \frac{\Phi_{q,p} (\hat{x})}{\sqrt{(2\pi \|x\|)^d}} e^{-\gamma_{q,p}(x)} (1 + o(1)),
$$

(19)
where \( \Phi_{q,p} \) is a positive real analytic function on \( S^{d-1} \) and \( \tau_{q,p} \) an equivalent norm in \( \mathbb{R}^d \).

As a by-product of the proof of the previous theorem we also obtain the following result.

**Theorem 2** For any \( d \geq 3 \) and any \( q \geq 1 \), there exists \( p_0 = p_0 (q, d) \) such that, \( \forall p > p_0 \), the equi-decay set of the two-point finite connectivity function is locally analytic and strictly convex. Moreover, the Gaussian curvature of the equi-decay set is uniformly positive.

## 2 Analysis of connectivities

Given \( x, y \in \mathbb{Z}^d \), we set
\[
\varphi (x, y) := \min \left\{ |S_{\{x,y\}} (\omega)| : \omega \in \{ |C_{\{x,y\}}| > 0 \} \right\} \quad x \neq y ,
\]
\[
\varphi (x, y) = \varphi (x - y) . \quad (20)
\]
\( \varphi \) is symmetric and translation invariant, therefore in the sequel we will write
\[
\varphi (x, y) \quad (21)
\]

For any \( x \in \mathbb{Z}^d \) and \( k \geq \varphi (x) \), let us set \( A_k (x) := \{ |S_{\{0,x\}} (\omega)| = k \} \) and \( A^k (x) := \bigvee_{l \geq k} A_l (x) \). We define
\[
\psi_k (x) := \min \{ |E (C_{\{0,x\}} (\omega))| : \omega \in A_k (x) \} ,
\]
\[
\psi (x) := \psi_{\varphi (x)} .
\]

By Lemma 6 in [CG] it follows that that there exists \( c_2 = c_2 (d) > 1 \) such that, for any \( x \in \mathbb{Z}^d \),
\[
c_2^{-1} \leq \frac{\varphi (x)}{\psi (x)} \leq c_2 . \quad (23)
\]

**Proposition 3** There exists a constant \( c_3 = c_3 (d) > 1 \) such that, for any \( p \in (p^*, 1) \),

\[
p^* = p^* (q, d) := \frac{q \left( 1 - \frac{1}{c_3} \right)}{\frac{1}{c_3} + q \left( 1 - \frac{1}{c_3} \right)} , \quad (24)
\]
and any $\delta > \delta^*$, with

$$\delta^* = \delta^* (p, q, d) := \log \frac{c^2(d)q(p+q(1-p))\delta^2}{p^2(1-\delta^2)}$$

(25)

$$\mathbb{P}_{q,p} (\{|S_{[0,x]}| \geq (1+\delta) \varphi (x)\} \cup \{0 < |C_{[0,x]}| < \infty \}) \leq \frac{1}{1-c_3 \text{ } (\frac{q(1-p)}{p+q(1-p)})_{\delta + 1} (1-p)^{\delta}}$$

(26)

**Proof.** For any $k \geq 2d$, we define the (possibly empty) collection of subgraphs of $\mathcal{G}$

$$\mathcal{G}_k := \{ \mathcal{G} \subset \mathcal{G} : \mathcal{G} = G' \left[ (\partial G')^* \right] , \mathcal{G}' \in \mathcal{G}_c^d : |V (\mathcal{G})| = k \}$$

(27)

We have

$$\{0 < |C_{[0,x]}| < \infty \} = \bigvee_{k \geq \varphi (x)} A_k (x)$$

(28)

and, for any $E \in \{ E' \subset E^d : E' = E (\omega) , \omega \in A_k (x) \}$, denoting by

$$A_k (E; x) := \{ \omega \in A_k (x) : E (\omega) = E \}$$

(29)

we get

$$\mathbb{P}_{q,p} (A_k (x)) = \mathbb{P}_{q,p} (\{ \omega \in \Omega : |S_{[0,x]} (\omega)| = k \} | A_k (E; x) \} \mathbb{P}_{q,p} (A_k (E; x))$$

(30)

$$\leq \mathbb{P}_{q,p} (\{ \omega \in \Omega : |S_{[0,x]} (\omega)| = k \} | A_k (E; x) \}$$

Moreover, because $\{ |S_{[0,x]}| = k \}$ is a decreasing event, it holds, since (18) is also valid for $\mathbb{P}_{q,p} (A_k (E; x))$ (see [Gr] Theorem (3.1) p.37), that

$$\mathbb{P}_{q,p} (\{ |S_{[0,x]}| = k \} | A_k (E; x) \}) \leq \mathbb{P}_{p+q(1-p)} (|S_{[0,x]}| = k | A_k (E; x) \})$$

(31)

$$\leq \left( \frac{q (1-p)}{p+q(1-p)} \right)^k \sum_{G \in \mathcal{G}_k} \mathbb{P}_{p+q(1-p)} (G | S_{[0,x]} = G) \leq |\mathcal{G}_k| \left( \frac{q (1-p)}{p+q(1-p)} \right)^k$$

We can choose for each $\mathcal{G} \in \mathcal{G}_k$ a minimal spanning tree $T_\mathcal{G}$ and consider the collection of graphs

$$\mathcal{T}_k := \{ T_\mathcal{G} : \mathcal{G} \in \mathcal{G}_k \}$$

(32)
Since given a connected tree there is a walk passing only twice through any edge of the
graph, there exists a constant \( c_3 = c_3(d) > 1 \) such that \(|\mathcal{G}_k| = c_3^k\). Therefore,
\[
P_{q,p} \{ |S_{0,x}| \geq (1 + \delta) \varphi(x), 0 < |C_{0,x}| < \infty \} \leq P_{q,p} \{ |S_{0,x}| \geq (1 + \delta) \varphi(x) \}
\leq \sum_{k \geq (1 + \delta) \varphi(x)} P_{q,p}(A_k(x)) \leq \sum_{k \geq (1 + \delta) \varphi(x)} c_3^k \left( \frac{q(1-p)}{p+q(1-p)} \right)^k.
\]
Since
\[
P_{q,p} \{ 0 < |C_{0,x}| < \infty \} = \sum_{k \geq \varphi(x)} P_{q,p}(A_k(x)) \geq P_{q,p}(A(x))
\]
for any \( E \in \{ E' \subset \mathbb{E}^d : |E'| = \psi(x), E' = E(\omega), \omega \in A(x) \} \), denoting by \( A(E;x) := \{ \omega \in A(x) : E(\omega) = E \} \), by (18) and (23),
\[
P_{q,p}(A(x)) = P_{q,p}(A(x) | A(E;x)) P_{q,p}(A(E;x)) \geq P_{p} \left( \{ |S_{0,x}| = \varphi(x) \} | A(E;x) \right) \frac{p}{p+q(1-p)} \psi(x)
\geq \left( \frac{p}{p+q(1-p)} \right) \psi(x) (1-p)^{\varphi(x)}
\geq \left\{ (\frac{p}{p+q(1-p)})^{c_2} (1-p) \right\} ^{\varphi(x)}.
\]
Therefore, \( \forall p \in \left( \frac{q(1-p)}{c_3 q(1-p)}, 1 \right) \), choosing \( \delta^* \) as in (25), \( \forall \delta > \delta^* \), we have
\[
P_{q,p} \left( \{ |S_{0,x}| \geq (1 + \delta) \varphi(x) \} \mid \{ 0 < |C_{0,x}| < \infty \} \right) \leq \frac{1}{1-c_3} \left[ \left( \frac{q(1-p)}{p+q(1-p)} \right)^{(1 + \delta) \varphi(x)} \left( \frac{p}{p+q(1-p)} \right)^{c_2} (1-p) \right] ^{\varphi(x)}
= \frac{1}{1-c_3} \left[ \left( \frac{q(1-p)}{p+q(1-p)} \right)^{1 + \delta} \left( \frac{p}{p+q(1-p)} \right)^{c_2} (1-p) \right] ^{\varphi(x)}.
\]

**Proposition 4** Given \( q \geq 1 \) and \( p \in (0,1) \) let \( P_{q,p} \) be a translation invariant Random Cluster measure on \( \mathbb{L}^d \) with parameters \( q \) and \( p \). Then, for any \( x \in \mathbb{R}^d \),
\[
\tau_{q,p}(x) := \lim_{n \to \infty} \frac{1}{n} \log P_{q,p} \{ 0 \leftrightarrow [nx] \mid |C_{(0,|nx|)}| < \infty \}
\]
exists and is a convex and homogeneous-of-order-one function on \( \mathbb{R}^d \).
Proof. For any $\Delta \subseteq \mathbb{Z}^d$, let us denote by $E_\Delta := \bigcup_{x \in \Delta} E \{C_x\} \subseteq \mathbb{R}^d$ the set of edges belonging to open paths starting at the vertices of $\Delta$.

Let now $\Lambda$ be a finite subset of $\mathbb{Z}^d$ such that $\Lambda \ni 0$. For any two distinct lattice points $x, y \in \Lambda$, looking at $I_{0 \leftarrow x, 0 \leftrightarrow \Lambda^c}$, $I_{x \leftarrow y, y \leftrightarrow \Lambda^c}$ as functions of $(E_x, E_{\Lambda^c})$, they are both nondecreasing on $E_x$ and nonincreasing on $E_{\Lambda^c}$. Therefore, by Theorem 2.1 in [BHK],

\[
\mathbb{P}_{q,p} \{0 \leftarrow x, 0 \leftrightarrow \Lambda^c\} \cap \{x \leftarrow y, y \leftrightarrow \Lambda^c\} \geq \mathbb{P}_{q,p} \{0 \leftarrow x, x \leftrightarrow \Lambda^c\} \times \mathbb{P}_{q,p} \{x \leftarrow y, x \leftrightarrow \Lambda^c\},
\]

that is

\[
\mathbb{P}_{q,p} \{x \leftrightarrow \Lambda^c\} \mathbb{P}_{q,p} \{0 \leftarrow x, x \leftrightarrow y, x \leftrightarrow \Lambda^c\} \geq \mathbb{P}_{q,p} \{0 \leftarrow x, x \leftrightarrow \Lambda^c\} \times \mathbb{P}_{q,p} \{x \leftarrow y, x \leftrightarrow \Lambda^c\},
\]

which implies

\[
\mathbb{P}_{q,p} \{0 \leftarrow x, x \leftrightarrow y, C_{(0,x)} \cap \Lambda^c = \emptyset\} \geq \mathbb{P}_{q,p} \{0 \leftarrow x, C_{(0,x)} \cap \Lambda^c = \emptyset\} \times \mathbb{P}_{q,p} \{x \leftarrow y, C_{(x,y)} \cap \Lambda^c = \emptyset\},
\]

But

\[
\mathbb{P}_{q,p} \{0 \leftarrow y, C_{(0,y)} \cap \Lambda^c = \emptyset\} \geq \mathbb{P}_{q,p} \{0 \leftarrow x, x \leftrightarrow y, C_{(0,x,y)} \cap \Lambda^c = \emptyset\},
\]

hence

\[
\mathbb{P}_{q,p} \{0 \leftarrow y, C_{(0,y)} \cap \Lambda^c = \emptyset\} \geq \mathbb{P}_{q,p} \{0 \leftarrow x, C_{(0,x)} \cap \Lambda^c = \emptyset\} \times \mathbb{P}_{q,p} \{x \leftarrow y, C_{(x,y)} \cap \Lambda^c = \emptyset\}.
\]

Taking the limit $\Lambda \uparrow \mathbb{Z}^d$ we have

\[
\mathbb{P}_{q,p} \{0 \leftarrow y, |C_{(0,y)}| < \infty\} \geq \mathbb{P}_{q,p} \{0 \leftarrow x, |C_{(0,x)}| < \infty\} \times \mathbb{P}_{q,p} \{x \leftarrow y, |C_{(x,y)}| < \infty\}.
\]

Proceeding as in the proof of Proposition 15 in [CG] we obtain the thesis. ■

2.1 Effective structure of connectivities

2.1.1 Definitions

Let $t \in \mathbb{R}^d$. Given two points $x, y \in \mathbb{Z}^d$ such that $\langle x, t \rangle \leq \langle y, t \rangle$, we denote by $C_{x,y}^t$ the cluster of $x$ and $y$ inside the strip $S_{x,y}^t$, if they are connected in the restriction of the configuration to $S_{x,y}^t$.

Let $u$ be the first of the unit vectors in the direction of the coordinate axes $u_1, \ldots, u_d$ such that $\langle t, u \rangle$ is maximal.
Definition 5 Given \( t \in \mathbb{R}^d \), let \( x, y \in \mathbb{Z}^d \) such that \( \langle x, t \rangle \leq \langle y, t \rangle \) be connected inside \( S_{x,y}^t \). The points \( b \in C_{\{x,y\}}^t \) such that:

1. \( \langle t, x + u \rangle \leq \langle t, b \rangle \leq \langle t, y - u \rangle ; \)
2. \( C_{\{x,y\}}^t \cap S_{b-u,b+u}^t = \{b - u, b, b + u\} ; \)

are said to be \( t \)-break points of \( C_{\{x,y\}} \). The collection of such points, which we remark is a totally ordered set with respect to the scalar product with \( t \), will be denoted by \( B^t(x,y) \).

Definition 6 Given \( t \in \mathbb{R}^d \), let \( x, y \in \mathbb{Z}^d \) such that \( \langle x, t \rangle \leq \langle y, t \rangle \) be connected inside \( S_{x,y}^t \). An edge \( \{b, b + u\} \) such that \( b, b + u \in B^t(x,y) \) is called a \( t \)-bond of \( C_{\{x,y\}} \). The collection of such edges will be denoted by \( E^t(x,y) \), while \( B_{\{x,y\}}^t(x,y) \subset B^t(x,y) \) will denote the subcollection of \( t \)-break points \( b \) of \( C_{\{x,y\}}^t \) such that the edge \( \{b, b + u\} \in E^t(x,y) \).

For any \( t \in \mathbb{R}^d \) and \( \varepsilon \in (0,1) \), let

\[
C_{\varepsilon}(t) := \{x \in \mathbb{R}^d : \langle \hat{t}, x \rangle \geq (1 - \varepsilon) \parallel x \parallel \} .
\]

(44)

Definition 7 Given \( t \in \mathbb{R}^d \), let \( x, y \in \mathbb{Z}^d \) such that \( \langle x, t \rangle \leq \langle y, t \rangle \) be connected inside \( S_{x,y}^t \). Then, for any \( \varepsilon \in (0,1) \):

1. \( x \) is said to be a \( (t,\varepsilon) \)-forward cone point if \( C_{\{x,x+u\}}^t = \{x, x + u\} \) and \( C_{\{x,y\}}^t \cap H_{x,+}^{t,t} < x + C_{\varepsilon}(t) \);
2. \( y \) is said to be a \( (t,\varepsilon) \)-backward cone point if \( C_{\{y-u,y\}}^t = \{y - u, y\} \) and \( C_{\{x,y\}}^t \cap H_{x,-}^{t,t} < y - C_{\varepsilon}(t) \);
3. \( z \) is said to be a \( (t,\varepsilon) \)-cone point if \( z \in B^t(x,y) \) and \( C_{\{z,y\}}^t \subset C_{\{x,z\}}^t \cap z - C_{\varepsilon}(t) \). The collection of \( (t,\varepsilon) \)-cone points is denoted by \( K_{\varepsilon}^t(x,y) \).

Definition 8 Given \( t \in \mathbb{R}^d \), let \( x, y \in \mathbb{Z}^d \) such that \( \langle x, t \rangle \leq \langle y, t \rangle \) be connected inside \( S_{x,y}^t \). Then, for any \( \varepsilon \in (0,1) \):

1. \( C_{\{x,y\}}^t \) is said to be \( (t,\varepsilon) \)-forward irreducible if \( x \) is a \( (t,\varepsilon) \)-forward cone point and \( K_{\varepsilon}^t(x+u,y) = \emptyset \);
2. \( C_{\{x,y\}}^t \) is said to be \( (t,\varepsilon) \)-backward irreducible if \( y \) is a \( (t,\varepsilon) \)-backward cone point and \( K_{\varepsilon}^t(x,y-u) = \emptyset \);
3. \( C_{\{x,y\}}^t \) is said to be \( (t,\varepsilon) \)-irreducible if \( x, y \in K_{\varepsilon}^t(x,y) \) and \( K_{\varepsilon}^t(x+u,y-u) = \emptyset \).
Notice that by definition, if \( x \) is a \((t, \varepsilon)\)-forward cone point, then is also a \((t, \varepsilon')\)-forward cone point for any \( \varepsilon' \in (\varepsilon, 1) \). The same remark also applies to \((t, \varepsilon)\)-backward cone points and therefore to \((t, \varepsilon)\)-cone points implying \( K_t^+ (x,y) \subseteq K_{t,\varepsilon'} (x,y) \). Hence, if for \( t \in \mathbb{R}^d \) and \( x,y \in \mathbb{Z}^d \) as in Definition \( \text{8} \) there exists \( \varepsilon \in (0,1) \) such that \( C \) of \( \gamma \) satisfies either condition 1 or 2 or 3 of that definition, then \( C \) is said to be respectively \( t \)-forward irreducible, \( t \)-backward irreducible, \( t \)-irreducible and we denote by \( K_t (x,y) := \bigcup_{\varepsilon \in (0,1)} K_t^+ (x,y) \) the collection of \( t \)-cone points of \( C \) as well as \( E_t (x,y) := \{ e \in E_t^+ (x,y) : e \subseteq K_t^+ (x,y) \} \).

**Definition 9** Given \( t \in \mathbb{R}^d \), let \( x,y \in \mathbb{Z}^d \) such that \( (x,t) \leq (y,t) \) be connected. Two subclusters \( \gamma_1 \) and \( \gamma_2 \) of \( C \) are said to be compatible, which condition we denote by \( \gamma_1 \bigcap \gamma_2 \), if they are connected and there exists \( b \in K_t (x,y) \) such that \( \gamma_1 \) is a subcluster of \( C \) containing \( b \) and \( \gamma_2 \) is a subcluster of \( C \) containing \( b + u \).

Therefore, two subsets \( s_1, s_2 \) of \( S \) will be called compatible, and we will still denote this condition by \( s_1 \bigcap s_2 \), if there exist two compatible subclusters \( \gamma_1, \gamma_2 \) of \( C \), such that \( s_i = (\overline{\gamma_i})^+ \cap S \), \( i = 1,2 \).

### 2.1.2 Renormalization

In Lemma 4 and Proposition 5 in [CG] we proved that \( \varphi \) is subadditive and the sequence \( \{ \bar{\varphi}_n \}_{n \in \mathbb{N}} \), such that \( \forall n \in \mathbb{N}, \mathbb{R}^d \ni x \longmapsto \bar{\varphi}_n (x) := \mathcal{A}(nx) \in \mathbb{R}^+ \), converges pointwise on \( \mathbb{R}^d \), and uniformly on \( \mathbb{S}^{d-1} \), to a convex, homogeneous-of-order-one function \( \bar{\varphi} \). As in [CG] we also define

\[
\mathcal{W} := \bigcap_{x \in \mathbb{S}^{d-1}} \left\{ w \in \mathbb{R}^d : \langle w, \hat{x} \rangle \leq \bar{\varphi} (\hat{x}) \right\} .
\]

(45)

Given \( x \in \mathbb{Z}^d \), let \( \mathcal{W} (x) := \{ w \in \mathcal{W} : \langle w, x \rangle = \bar{\varphi} (x) \} \).

For \( N \in \mathbb{N} \) larger than 1, let us set \( t_N (x) := \left\lfloor \frac{\langle x, \hat{x} \rangle}{N} \right\rfloor - 1 \) and

\[
\begin{align*}
y_i & := \left\lfloor \frac{i}{N} \right\rfloor ; \quad \mathcal{H}^t_{y_i} := \mathcal{H}_{y_i}^t ; \quad \mathcal{H}^{t+}_{y_i} := \mathcal{H}_{y_i}^{t+} ; \quad \mathcal{H}^{t-}_{y_i} := \mathcal{H}_{y_i}^{t-}, \quad i = 0, \ldots, t_N; \\
y_{N+1} & := x ; \quad \mathcal{H}^t_{y_{N+1}} := \mathcal{H}_x^t ; \quad \mathcal{H}^{t+}_{y_{N+1}} := \mathcal{H}^t_{x} ; \quad \mathcal{H}^{t-}_{y_{N+1}} := \mathcal{H}_x^{t-}; \\
S_i^t & := \mathcal{H}^{t+}_{i} \cap \mathcal{H}^{t-}_{i+1}. \quad (46) \end{align*}
\]

With a slight notational abuse we still denote by \( S_{0,x} \) its representation as a hypersurface in \( \mathbb{R}^d \) and define

\[
C_i^t := C_{0,x} \cap S_i^t ; \quad S_i^t := S_{0,x} \cap S_i^t .
\]

(49)

Hence, \( C_{0,x} = \bigcup_{i=0}^{t_N} C_i^t \) and \( S_{0,x} \cap S_i^t \subseteq \bigcup_{i=0}^{t_N} S_i^t \).
We call crossing any connected component $s$ of $S^i_t$ such that, denoting by $K(s)$ the compact subset of $S^i_t$ whose boundary is $s$, there exist $y \in H^{i-1}_t \cap \mathbb{Z}^d$ and $y' \in H^{i+1}_t \cap \mathbb{Z}^d$, both belonging to $C_{(0,x)}$, which are connected by an open path in $\mathbb{L}^d \cap K(s)$.

We remark that since $C_{(0,x)}$ is connected, the existence of two crossings in $S^i_t$ implies the existence of two disjoint paths connecting $H^i_t$ and $H^{i+1}_t$ while the converse does not hold true in general.

We say that a slab $S^i_t$ is good if $S^i_t$ is connected and made by just a single crossing of size smaller than twice the minimal one, otherwise we call it bad.

In [CG] we proved that, for any $S_{(0,x)}$, the number of plaquettes of $S$ since $t$ triples of consecutive slabs, the last being smaller than $\frac{\delta}{2}$, is at most the number of such bad slabs with a single crossing. Since any crossing is composed by at least $\delta \frac{\|x\|}{N}$ plaquettes, we have

$$\frac{\|x\|}{N} \left( 2 \delta \phi_t ([N\hat{x}]) + (1 - \eta) \phi_t ([N\hat{x}]) \right) = \frac{\|x\|}{N} (1 + \eta) \phi_t ([N\hat{x}]) < (1 + \delta) \phi_t (x).$$

Moreover, given $\epsilon > 0$, there exists $R_\epsilon > 0$ such that, for any $x \in \mathbb{Z}^d \cap (R_\epsilon B)^c$, $\phi_t (x) \leq \phi_t (x) (1 + \epsilon)$. Hence, choosing $N$ sufficiently large such that $\phi_t ([N\hat{x}]) \leq \phi_t (N\hat{x}) (1 + \epsilon)$, since $t \in d\mathcal{W}(x)$, $\phi_t (N\hat{x}) = \phi_t (N\hat{x})$ and, by the previous inequality, we get $\eta < \delta$. Furthermore, since the number of plaquettes of $S_{(0,x)}$ exceeding $\phi_t (x)$ is at most $\delta \phi_t (x)$, if $S_{(0,x)}$ does not give rise to multiple crossings and $S^i_t$ is a bad slab, the components of $S^i_t$ which are not crossings must be connected either to $S^i_{t-1}$ or to $S^i_{t+1}$. Therefore, the number of bad slabs with a single crossing is at most the same as the number of such triples of consecutive slabs, the last being smaller than $\delta \frac{\|x\|}{N}$. Hence, the total number of bad slabs can be at most equal to $2 \delta \frac{\|x\|}{N}$.

Let $t \in d\mathcal{W}(x)$. Denoting by $\{v_i\}_{i=1}^d$ an orthonormal basis of $\mathbb{R}^d$ where $v_1 = \hat{t}$ and $\{v_j\}_{j=2}^d$ is any orthonormal basis of $H^i_t$, we define, for $i \in \{0, \ldots, t_N\}$ and $n := (n_2, \ldots, n_d) \in \mathbb{Z}^{d-1}$,

$$Q_N (i,n) := S^i_t \cap \bigcap_{j=2, \ldots, d} \{ z \in \mathbb{R}^d : \langle v_j, z \rangle \in [n_j N, (n_j + 1) N] \}$$

(53)
which we call $N$-blocks.

For any $i = 0, \ldots, t_N$, we define the $N$-sets $D^i_N (i)$ to be convex hull in $\mathbb{R}^d$ of the $N$-blocks $Q_N (i, n)$ $n \in \mathbb{Z}^{d-1}$ intersecting $S^i_t$. Denoting by $\{S^i_t\}_{i=1}^{g_N}$ the set of good slabs, the corresponding $N$-sets $D^i_N (i)$, will be called good while the remaining $N$-sets bad, while the set

$$\mathcal{C}_N := \bigcup_{i=0}^{t_N} D^i_N (i)$$

will be called $N$-renormalized cluster.

**Definition 10** Given $\varepsilon \in (0, 1)$ and $l \in \{1, \ldots, g_N\}$, a point of $z \in \mathcal{C}_N^l$ is called a $(t, \varepsilon)$-correct point and the collection of these points is denoted by $\mathbb{K}^l_\varepsilon (0, x)$, if

$$\bigcup_{j=0}^{i-1} D^i_N (j) \subset z - C_\varepsilon (t) , \quad \bigcup_{j=i+1}^{t_N} D^i_N (j) \subset z + C_\varepsilon (t) .$$

Thus, setting $t_N := |\mathbb{K}^l_\varepsilon (0, x)|, \mathbb{K}^l_\varepsilon (0, x) = \{z_1, \ldots, z_{t_N}\}$. Let $\{S^i_t\}_{k=1}^{t_N} \subseteq \{S^i_t\}_{i=0}^{t_N}$ such that, for any $k = 1, \ldots, t_N$, $S^i_t \supseteq z_k$ and define

$$D^i_k (k) := ((z_k + C_\varepsilon (t)) \cap (z_{k+1} - C_\varepsilon (t))) , k = 1, \ldots, t_N - 1 .$$

We select among these compact subsets of $\mathbb{R}^d$ those containing $\bigcup_{j=i_k}^{t_N} D^i_N (j)$ and denote their collection by $\{D^i_k (k_m)\}_{m=1}^{c_N-1}$.

**Lemma 11** There exist $\varepsilon \in (0, 1)$ sufficiently large and a positive constant $c_5 = c_5 (\delta, \varepsilon), c_5 (\delta, \varepsilon)$, such that $c_N \geq c_5 |x|$.

**Proof.** Let $M$ a positive constant to be chosen later. If $\{S^i_t\}_{i=1}^{g_N}$, we set $m^+_1 := i_1$ and

$$r^+_1 := \min \left\{ k \in \{i_1, \ldots, t_N + 1\} : \sum_{j=i_1}^{k} |S^i_j| > MN (k - i_1) \right\} ;$$

$$m^+_i := \min \left\{ r^+_1 + 1, \ldots, t_N + 1 \right\} \cap \{i_1, \ldots, i_{g_N}\} ;$$

$$r^+_{i+1} := \min \left\{ k \in \{m^+_i, \ldots, t_N + 1\} : \sum_{j=m^+_i}^{k} |S^i_j| > MN (k - m^+_i) \right\} .$$

Analogously, we define $m^-_1 := i_{g_N}$ and

$$r^-_1 := \max \left\{ k \in \{0, \ldots, i_{g_N}\} : \sum_{j=k}^{g_N} |S^i_j| > MN (i_{g_N} - k) \right\} ;$$

$$m^-_i := \max \{0, \ldots, r^-_1 - 1\} \cap \{i_1, \ldots, i_{g_N}\} ;$$

$$r^-_{i+1} := \max \left\{ k \in \{0, \ldots, m^-_i\} : \sum_{j=k}^{m^-_i} |S^i_j| > MN (m^-_i - k) \right\} .$$
there exist two positive constants $c_+$ such that $\phi(x) \leq c_+ \|x\|$, we get

$$(1 + \delta) c_+ \|x\| \geq \sum_{i \geq 1} \sum_{j=r_i^+} m_i^{+1} |S_i| \geq \sum_{i \geq 1} (m_i^{+1} - 1 - r_i^+) MN \quad (63)$$

as well as

$$\sum_{i \geq 1} (r_i^+ - m_i^{+1} - 1) \leq \frac{(1 + \delta) c_+}{MN} \|x\| . \quad (64)$$

Hence, if we denote by $\tau_N^+$ the number of the slabs labelled by the $r_i^+$'s and $m_N$ the number of the (good) slabs labelled by the $m_i^+$'s, we obtain $\tau_N^+ \leq \frac{(1+\delta)c_+}{MN} \|x\|$ that is, choosing $M > 2 (1 + \delta) c_+$,

$$c_N = \left| \{ m_i^+ \}_{i=1}^{m_N} \cap \{ m_i^- \}_{i=1}^{m_N} \right| \geq \left( 1 - \frac{(1 + \delta) c_+}{M} \right) \frac{\|x\|}{N} . \quad (65)$$

This implies that we can find $\varepsilon = \varepsilon (\delta, d)$ and, setting $\{ m_i^+ \}_{i=1}^{m_N} \cap \{ m_i^- \}_{i=1}^{m_N} =: \{ m_1, \ldots, m_N \}$, in each $S_{m_i}, i = 1, \ldots, c_N$, a $(t, \varepsilon)$-correct point $z_i$ such that $D_N^N (k) \supset \cup_{j=1}^{m_N} D_N^j (j)$. 

By construction, for any $k = 1, \ldots, c_N$, each $z_k$ belongs to a given $D_N (i_k), l = 1, \ldots, g_N$. Therefore, if also $z_k + u = z_{k+1} \in D_N (i_k)$, by the finite-energy property of $\mathbb{P}_{q,p}$, the probability that $\{ z_k, z_k + u \} \in E^t (0, x)$ is bounded below by $\beta = \beta (M, N) > 0$ regardless of the percolation configuration outside the $N$-set $D_N^N (i_k)$. Hence, for a fixed $N$-renormalized cluster $\mathcal{C}_N$ containing a subset $\{ z_1, \ldots, z_{2L} \}$ of $t$-correct points such that, $\forall i = 1, \ldots, L, z_{2i} = z_{2i-1} + u$ and any pair $\{ z_{2i-1}, z_{2i-1} + u \}$ belong to distinct $N$-sets $D_N^N (i_l), l \in \{ 1, \ldots, g_N \}$, the conditional distribution of $|E^t (0, x)|$ given $\mathcal{C}_N$ stochastically dominates the binomial distribution of parameters $2L$ and $\beta$. Since the number of $t$-correct points $\tau_N \geq c_N \frac{\|x\|}{N}$, we have:

**Proposition 12** For any $p \in (p^*, 1)$ sufficiently close to $1$ and $\delta > \delta^*$ sufficiently small, with $p^*$ and $\delta^*$ given in respectively (22) and (23), uniformly in $x$ and in $t \in W (x)$, there exist two positive constants $c_6 = c_6 (\delta, \varepsilon), c_7 = c_7 (N, M, p, q, d)$ such that

$$P_{q,p} \left( |E^t (0, x)| \leq c_6 \frac{\|x\|}{N} \right) \left\{ 0 < \left| C_{(0,x)} \right| < \infty \right\} \leq e^{-c_7 \|x\|} . \quad (66)$$

2.1.3 Reduction to a one-dimensional thermodynamics

Since by Definition 7 $K^t (0, x) \subseteq B^t (0, x)$, it is a totally ordered set with respect to the scalar product with $t$, we can relabel the elements of $K^t (0, x)$ in increasing order and consider $K^t_e (0, x) := \bigcup_{i \geq 1} \{ b_i, b_{i+u} \} \subseteq K^t (0, x)$, where $\forall i \geq 1, (b_{i+1}, t) > (b_i, t), \{ b_i, b_{i+u} \} \in$
\( \mathcal{E}'(0, x) \), which implies that \( \mathbf{C}_{(b_{i+u,b_{i+1}})} \) is a \( t \)-irreducible subcluster of \( \mathbf{C}_{(0,x)} \). Therefore, we have proven that, with probability larger than \( 1 - e^{-c N \| x \|} \), there exists \( \varepsilon = \varepsilon (\delta) \in (0, 1) \) such that, as in the subcritical case [CIV2], \( \mathbf{C}_{(0,x)} \) can be decomposed as a concatenation of \((t, \varepsilon)\)-irreducible compatible subclusters, that is \( \mathbf{C}_{(0,x)} = \gamma^b \prod \gamma_1 \prod \gamma_n \prod \gamma^f \), for some \( n \in \mathbb{N} \), \( n \geq \frac{c}{N} \| x \| \), where \( \gamma^b = \mathbf{C}_{(0,b_1)} \cap \mathcal{H}^{t, -}_{b_1} \) is \((t, \varepsilon)\)-backward irreducible, \( \gamma^f = \mathbf{C}_{(b_{n+1}, x)} \cap \mathcal{H}^{t, +}_{b_n} \) is \((t, \varepsilon)\)-forward irreducible and, for \( i = 1, \ldots, n \), \( \gamma_i = \mathbf{C}_{(b_{i+u}, b_{i+1})} \) is \((t, \varepsilon)\)-irreducible. From this follows, by Definition 9, that \( \mathbf{S}_{(0,x)} \) can be decomposed as a concatenation of compatible subclusters, namely

\[
\mathbf{S}_{(0,x)} = s^b \prod s_1 \prod \ldots \prod s_n \prod s^f,
\]

with \( s^b = (\partial \gamma)^* \cap \mathbf{S}_{(0,x)}, s^f = (\partial \gamma)^* \cap \mathbf{S}_{(0,x)} \) and for \( i = 1, \ldots, n \), \( s_i = (\partial \gamma_i)^* \cap \mathbf{S}_{(0,x)} \).

The elements of such a decomposition of \( \mathbf{S}_{(0,x)} \) will also be called \( t \)-irreducible.

If \( s \) is a realization of the random element \( s_i, i = 1, \ldots, n \), part of the just described decomposition of \( \mathbf{S}_{(0,x)} \), considering the just given representation of \( \mathbf{K}^t_e (0, x) \), we define

\[
i_+ (s) := \max \left\{ i \geq 1 : s \subset \mathcal{H}^{t, +}_{b_i} ; b_i \in \mathbf{K}^t_e (0, x) \right\},
\]

\[
i_- (s) := \min \left\{ i \geq i_- (s) + 1 : s \subset \mathcal{H}^{t, -}_{b_i} ; b_i \in \mathbf{K}^t_e (0, x) \right\}.
\]

Moreover, if \( s \) is a realization of \( s_b \), we define

\[
i_+ (s) := \min \left\{ i \geq 1 : s \subset \mathcal{H}^{t, -}_{b_i} ; b_i \in \mathbf{K}^t_e (0, x) \right\},
\]

and set \( b_- (s) := b_-(s), b_+ (s) := b_+(s) \). Clearly, by definition, \( e_- (s) := \left\{ b_- (s) , b_- (s) + u \right\} \) and \( e_+ (s) := \left\{ b_+ (s) - u, b_+ (s) \right\} \) belong to \( \mathcal{E}'(0, x) \). We also set \( \Lambda_s \) to be the subset \( V \) of \( \mathbb{Z}^d \) such that \((\partial G [V \setminus \mathcal{E}'(0, x)]^*) = s \). Hence, defining, for any realization \( s \) of \( s_i, i = 1, \ldots, n \),

\[
\Omega_s := \left\{ \omega \in \Omega : \exists \gamma \sqsubset G (\omega) \text{ s.t. } (\partial \gamma)^* \supset s ; \ \omega_e = 1, \right\}
\]

\[
\forall e \in \left\{ e_- (s), e_- (s) + u \right\} \bigcup \left\{ e_+ (s) - u, e_+ (s) \right\},
\]

for any realization \( s_b \) of \( s_b \)

\[
\Omega_{s_b} := \left\{ \omega \in \Omega : \exists \gamma \sqsubset G (\omega) \text{ s.t. } (\partial \gamma)^* \supset s_b ; \ \omega_e = 1, \right\}
\]

\[
\forall e \in \left\{ e_+ (s_b) - u, e_+ (s_b) \right\}
\]

and for any realization \( s_f \) of \( s_f \)

\[
\Omega_{s_f} := \left\{ \omega \in \Omega : \exists \gamma \sqsubset G (\omega) \text{ s.t. } (\partial \gamma)^* \supset s_f ; \ \omega_e = 1, \right\}
\]

\[
\forall e \in \left\{ e_- (s_f), e_- (s_f) + u \right\},
\]
up to factors of order $1 + o \left( e^{-c r [x]} \right)$,
\[
\mathbb{P}_{q,p} \{ 0 \leftarrow x , \ |C_{0,x}| < \infty \} = \sum_{s_b \not\in 0} \sum_{f \not\in x} \sum_{n \geq 1} \sum_{(s_1, \ldots, s_n)} \mathbb{P}_{q,p} \{ 0 \leftarrow x , \ S_{0,x} = s_b \prod s_1 \prod s_n \prod s_f \} = \sum_{s_b \not\in 0} \sum_{f \not\in x} \sum_{n \geq 1} \sum_{(s_1, \ldots, s_n)} \mathbb{P}_{q,p} (\Omega_b \cap (\bigcap_{i=1}^n \Omega_i) \cap \Omega_f) ,
\]
where $\Omega_{\#} := \Omega_{s_{\#}}$, $\# = b, f, 1, \ldots, n$, and $\sum_{s_b \not\in 0} \sum_{f \not\in x}$ stand respectively for the sum over the elements of
\[
\{ s \subset (E)^* : s = (\overline{\gamma})^* \cap S_{0,x} \text{ s.t. } \gamma \ni 0 \text{ and is } t\text{-backward irreducible} \} ,
\]
\[
\{ s \subset (E)^* : s = (\overline{\gamma})^* \cap S_{0,x} \text{ s.t. } \gamma \ni x \text{ and is } t\text{-forward irreducible} \} ,
\]
while the last sum is over all the realizations $(s_1, \ldots, s_n)$ of the strings $(s_1, \ldots, s_n)$ of $t$-irreducible compatible subsets of $S_{0,x}$.

**Decomposition of probabilities** Setting, for any $n \geq 1$,
\[
\{ s_{\#} \} := \{ \omega \in \Omega : s_{\#}(\omega) = s_{\#} \} , \# = b, f, 1, \ldots, n
\]
and
\[
\mathbb{P}^f_{A_{s_b} \cup \{ b - (s_b) \}; q,p} =: \mathbb{P}^f_{q,p;s_b^*} ; \mathbb{P}^f_{\{ b - (s_f) \} \cup A_{s_f}; q,p} =: \mathbb{P}^f_{q,p;s_f^*} ,
\]
\[
\mathbb{P}^f_{\{ b - (s_i) \} \cup A_{s_i}; \cup \{ b - (s_i) \}; q,p} =: \mathbb{P}^f_{q,p;s_i^*} ,
\]
\[
\mathbb{P}^f_{A_{s_b} \cup \left( \bigcup_{i=1}^n A_{s_i} \right); A_{s_f}; q,p} =: \mathbb{P}^f_{q,p; (s_b \prod s_1 \prod s_n \prod s_f)} ,
\]
we have
\[
\mathbb{P}_{q,p} (\Omega_b \cap (\bigcap_{i=1}^n \Omega_i) \cap \Omega_f) \{ s_b \} \cap (\bigcap_{i=1}^n \{ s_i \}) \cap \{ s_f \} (\bigcap_{i=1}^n \Omega_i) \cap \Omega_f )
\]
\[
= \mathbb{P}^f_{q,p; s_b^*} (\Omega_b) \prod_{i=1}^n \mathbb{P}^f_{q,p; s_i^*} (\Omega_i)
\]
and
\[
\mathbb{P}_{q,p} (\Omega_{\#}) = \mathbb{P}_{q,p} (\Omega_{\#} \{ s_{\#} \}) \mathbb{P}_{q,p} \{ s_{\#} \} = \mathbb{P}^f_{q,p; s_{\#}^*} (\Omega_{\#}) \mathbb{P}_{q,p} \{ s_{\#} \} , \# = b, f .
\]
Therefore,

\[ P_{q,p} (\Omega_b \cap (\bigcap_{i=1}^n \Omega_i) \cap \Omega_f) = P_{q,p} (\Omega_b) \prod_{i=1}^n P_{q,p,s_i}^f (\Omega_i) \times \]
\[ \times \frac{P_{q,p} (\{s_b\} \cap (\bigcap_{i=1}^n \{s_i\}) \cap \{s_f\})}{P_{q,p} \{s_b\} P_{q,p} \{s_f\}}. \]  

(84)

Furthermore, the last term in the r.h.s. of the previous formula admits the equivalent decompositions

\[ \frac{P_{q,p} (\{s_b\} \cap (\bigcap_{i=1}^n \{s_i\}) \cap \{s_f\})}{P_{q,p} \{s_b\} P_{q,p} \{s_f\}} = \frac{P_{q,p} (\{s_b\} | (\bigcap_{i=1}^n \{s_i\}) \cap \{s_f\})}{P_{q,p} \{s_b\}} \times \]
\[ \times \prod_{j=1}^{n-1} P_{q,p} (\{s_j\} | (\bigcap_{i=j+1}^n \{s_i\}) \cap \{s_f\}) P_{q,p} (\{s_n\} | \{s_f\}) \]
\[ = \frac{P_{q,p} (\{s_f\} | (\bigcap_{i=1}^n \{s_i\}) \cap \{s_b\})}{P_{q,p} \{s_f\}} \times \]
\[ \times \prod_{j=0}^{n-2} P_{q,p} (\{s_{n-j}\} | (\bigcap_{i=j+1}^{n-1} \{s_i\}) \cap \{s_b\}) P_{q,p} (\{s_1\} | \{s_b\}). \]  

(85)

Once we have fixed \( s_b \) and \( s_f \), we choose one of the just defined representations, say the first, and, for any \( n \in \mathbb{N} \), denoting by \( \mathcal{I}_t^n \) the collection of strings \( (s_1, ..., s_n) \) of \( t \)-irreducible compatible subsets of \( S_{(0,x)} \), we set

\[ \mathcal{I}_t^n \ni (s_1, ..., s_n) \mapsto g (s_1, ..., s_n; s_b, s_f) := \frac{P_{q,p} (\{s_b\} | (\bigcap_{i=1}^n \{s_i\}) \cap \{s_f\})}{P_{q,p} \{s_b\}} \in [0, +\infty), \]

(87)

\[ \mathcal{I}_t^n \ni (s_1, ..., s_n) \mapsto \Xi (s_1, ..., s_n; s_f) := \log \frac{P_{q,p,s_i}^f (\Omega_1) P_{q,p} (\{s_1\} | (\bigcap_{j=2}^n \{s_j\}) \cap \{s_f\})}{P_{q,p} \{s_f\}} \in (-\infty, 0]. \]  

(88)

Let \( \mathcal{S}_t := \bigcup_{n \in \mathbb{N}} \mathcal{I}_t^n \), where \( \mathcal{I}_t^n \) is the set of infinite sequences \( \omega := (s_1, ...) \) such that the string composed by the first \( n \) symbols appearing in \( \omega \) label the elements of \( \mathcal{I}_t^n \), while the remaining symbols are fixed to be the empty set. Setting, for any \( \omega, \omega' \in \mathcal{S}_t \) such that \( \omega \neq \omega' \), \( i (\omega, \omega') := \min \{ k \geq 1 : s_k \neq s'_k \} \) and, for any complex-valued function \( f \) on \( \mathcal{S}_t \), \( \var{f} := \sup \{ |f (\omega) - f (\omega')| : i (\omega, \omega') \geq k \}, \) let \( \mathcal{H}_\theta \) be the Banach space of real bounded continuous functions on \( \mathcal{S}_t \) which are also uniformly Hölder continuous for a given exponent \( \theta < 1 \) endowed with the norm \( \| \cdot \|_\theta := \| \cdot \|_\infty + \sup_{k \geq 2} \frac{\var{f} (\omega)}{\var{f} (\omega')}. \)

In the next subsection we will prove that \( g (\cdot; s_b, s_f) \) and \( \Xi (\cdot; s_f) \) admit a unique extension on \( \mathcal{H}_\theta \) for some \( \theta < 1 \) denoted respectively by \( g_{s_b,s_f} \) and \( \Xi_{s_f} \). This will allow us to define the Ruelle’s operator

\[ L_{s_f} f (\omega) := \sum_{\omega \in \mathcal{I}_t} e^{\Xi_{s_f} (s, \omega)} f (s, \omega), \quad f \in \mathcal{H}_\theta, \]

(89)
where \( I_t := I_t^t \), whose largest isolated eigenvalue is finite and has multiplicity 1, since by Proposition 12
\[
\sup_{t \in \mathcal{S}_t} \sum_{s \in I_t} e^{x_{sf}(s,t)} < \infty.
\]
Therefore, up to factors of order \( 1 + o\left(e^{-cr\|x\|}\right) \), by (74),
\[
\mathbb{P}_{q,p} \left\{ 0 \leftrightarrow x , \ |C_{[0,x]}| < \infty \right\} = \sum_{n_{\mathcal{B}} \geq q, s \in \mathcal{S}_t} \mathbb{P}_{q,p} (\Omega_b) \mathbb{P}_{q,p} (\Omega_f) \left[ L_{sf} \right] g_{n_{\mathcal{B}},s_f} (\emptyset),
\]
where \( \emptyset \) stands for the sequence \((\emptyset, ...) \in \mathcal{S}_t \).

Let
\[
R_{q,p} := \bigcap_{y \in \mathbb{R}^{d-1}} \left\{ w \in \mathbb{R}^d : \langle w, \hat{y} \rangle \leq \tau_{q,p} (\hat{y}) \right\}
\]
the convex body polar with respect to \( U_{q,p} := \left\{ y \in \mathbb{R}^d : \tau_{q,p} (y) \leq 1 \right\} \). Since \( \tau_{q,p} \) and \( \tilde{\varphi} \) are equivalent norms in \( \mathbb{R}^d \), if \( v \in \partial R_{q,p} \) is polar to \( x \) (i.e. \( \langle v, x \rangle = \tau_{q,p} (x) \)), we can choose \( t = t(v) \) as one of the elements of \( \partial W(x) \) maximizing its scalar product with \( v \). Notice that, by translation invariance of the RC random field, we can consider any realization of \( S_{[0,x]} \) as a collection \( s_b, (s_1, ..., s_n) \), \( s_f \) of realizations of its \( t \)-irreducible components modulo \( \mathbb{Z}^d \)-shift patched together. Then, for any element \( s_i, i \geq 1 \) of the \( t \)-irreducible decomposition of \( S_{[0,x]} \), we define
\[
X (s_i) := b_{i+1} - b_i.
\]
Thus, up to factors of order \( 1 + o\left(e^{-cr\|x\|}\right) \), we can write
\[
e^{\tau_{q,p}(x)} \mathbb{P}_{q,p} \left\{ 0 \leftrightarrow x , \ |C_{[0,x]}| < \infty \right\} = \sum_{y \in \mathcal{H}_{q}^{0,+} \cap \mathcal{H}_{p}^{1,+}} \sum_{z \in \mathcal{H}_{q}^{0,-} \cap \mathcal{H}_{p}^{1,-}} \sum_{s_b \in \mathcal{B}} \sum_{s_f \in \mathcal{F}} \mathbb{P}_{q,p} (\Omega_b) \mathbb{P}_{q,p} (\Omega_f) e^{\langle v,x-(z-y) \rangle} \times \sum_{n \geq 1} \left( \sum_{s_1, ..., s_n : \sum_{i=1}^n x(s_i) = z-y} \sum_{s_{sf}} \mathbb{P}_{q,p} (\Omega_b) \mathbb{P}_{q,p} (\Omega_f) e^{\langle v,x-(z-y) \rangle} \left[ L_{sf} \right] \right) g_{n_{\mathcal{B}},s_f} (\emptyset),
\]
where, assuming the shifts of \( t \)-backward and \( t \)-forward irreducible clusters are normalised in such a way that \( b_1 = b_{n+1} = 0, \sum_{s_b \in \mathcal{B}} \sum_{s_f \in \mathcal{F}} \) now stand respectively for the sum over the elements of
\[
\left\{ s \in (E)^* : s = (\partial \gamma)^* \cap S_{[0,x]} \text{ s.t. } \gamma \ni -y \text{ and is } t \text{-backward irreducible} \right\},
\]
\[
\left\{ s \in (E)^* : s = (\partial \gamma)^* \cap S_{[0,x]} \text{ s.t. } \gamma \ni x - z \text{ and is } t \text{-forward irreducible} \right\}.
\]
where we set $\Psi$ for any $q > 0$. However, in order to prove Proposition 14 below, instead of adapting to our purpose the formalism developed in that work, we find it more convenient to perform the expansion in a form closer to the one presented in [KP].

We can look at the elements of the collection of the connected subgraphs of finite order of $\mathfrak{G} = (\mathfrak{G}^d, \mathfrak{C})$, where $\mathfrak{C}$ is defined in (11), as a set of polymers which we denote by $\mathfrak{S}$. Two polymers $s, s' \in \mathfrak{S}$ are said to be compatible, and we write $s \sim s'$, if they are not connected (as subgraphs of $\mathfrak{G}$), otherwise are said to be incompatible and we write $s \not\sim s'$. Given $S \subset \mathfrak{S}$, we denote by $\mathfrak{P}(S)$ the collection of the subsets of $S$ consisting of mutually compatible polymers and call contours the elements of $\mathfrak{P}_0(S) := \{\sigma \in \mathfrak{P}(S) : |\sigma| < \infty\}$. We also set $\mathfrak{P} := \mathfrak{P}(\emptyset), \mathfrak{P}_0 := \mathfrak{P}_0(\emptyset)$. Given $S \in \mathcal{P}_f(\mathfrak{S}), s \in \mathfrak{S}$ we write $S \sim s$ if there exists $s' \in S$ such that $s' \sim s$. Moreover, we call $S$ a polymer cluster if it cannot be decomposed as a union of $S_1, S_2 \in \mathcal{P}_f(\mathfrak{S})$ such that every pair $s_1 \in S_1, s_2 \in S_2$ is compatible. We denote by $\mathfrak{C}(S)$ the collection of polymer clusters in $S$ and let $\mathfrak{C}$ be the collection of polymer clusters in $\mathfrak{S}$.

Given $\Lambda \subset \subset \mathbb{Z}^d$, we denote by $\mathfrak{S}_\Lambda$ the subset of $\mathfrak{S}$ such that, for any $s \in \mathfrak{S}_\Lambda, s \subset G[\mathfrak{G}^\Lambda]$ is the subgraph of $\mathfrak{G}$ induced by $\mathfrak{G}^\Lambda$. We also set $\mathcal{E}_S := V(\bigcup_{s \in \mathfrak{S}} s)$. Then, we define $\kappa_w(S)$ to be the number of the components of $(\mathcal{E}^d, \mathbb{E}^d \setminus \mathcal{E}_S)$ and $\kappa_t(S)$ to be the number of the components of $(\Lambda, \mathbb{E}^\Lambda \setminus \mathcal{E}_S)$. Moreover, for any $s \in \mathfrak{S}_\Lambda$, we set $||s||_\# := \kappa_\#(s) - 1, \# = f,w$.

Let $\Psi_\Lambda := \mathfrak{P}(\mathfrak{S}_\Lambda)$. We remark that, given $s \in \mathfrak{S}_\Lambda$, for any $\sigma \in \Psi_\Lambda$ such that $\sigma \supset s, ||s||_\# = \kappa_\#(\sigma) - \kappa_\#(\sigma \setminus s)$. The function

$$\Psi_\Lambda \supset \sigma \mapsto \Psi_\#(\sigma) := \prod_{s \in \sigma} \left( \frac{1-p}{p} \right)^{|s|} q^{||s||_\#} \in \mathbb{R}^+, \quad \# = f,w,$$

where we set $\Psi_\#(\emptyset) := 1$, is called activity of the contour $\sigma$. Since, $||s||_\# \leq |s|$, we get

$$\Psi_\#(\sigma) \leq \prod_{s \in \sigma} \left( \frac{1-p}{p} q \right)^{|s|}.$$
We then define, for any \( S \subseteq S_\Lambda \),

\[
\mathfrak{3}_{q,p}^\# (S) := \sum_{\sigma \in \psi(S)} \Psi_\# (\sigma) = \sum_{\sigma \in \psi(S)} \prod_{s \in \sigma} \left( \frac{1-p}{p} \right)^{|s|} q^{|s|}_\# , \quad \# = f,w. \tag{100}
\]

Considering for each \( s \in S_\Lambda \) a minimal spanning tree and bounding their number as in Proposition 3, we obtain that we can choose \( c_8 > 0 \) such that, for \( p \in (p_0, 1) \), with \( p_0 = p_0 (q,d) := \frac{1}{1-e^{c_8 q c_3}} + \frac{c_8 c_3}{1-c_8 c_3} \),

\[
\sum_{s' \in S_\Lambda : s' \sim s} e^{c_8 |s'|} \left( \frac{1-p}{p} q \right)^{|s'|} \leq |s| \frac{c_3 e^{c_8} \left( \frac{1-p}{p} q \right)}{1-c_3 e^{c_8} \left( \frac{1-p}{p} q \right)} \leq \frac{c_8}{2} |s| . \tag{101}
\]

Therefore, given \( s \in S_\Lambda \), if \( \ell (s) \) denotes the diameter of \( V(s) \) considered as a subset of \( \mathbb{R}^d \), since \( \ell (s) \leq |s| \),

\[
\sum_{s' \in S_\Lambda : s' \sim s} e^{c_8 |s'|} \left( \frac{1-p}{p} q \right)^{|s'|} |s|^{\ell (s')} q^{|s'|}_\# \leq \frac{c_8}{2} |s| . \tag{102}
\]

Thus, by the theorem in \([KP]\), for any \( S \subseteq S_\Lambda \),

\[
\log \mathfrak{3}_{q,p}^\# (S) = \sum_{S' \in C(S)} \vartheta_\# (S) , \quad \# = f,w \tag{103}
\]

where, setting \( C_\Lambda := C (S_\Lambda ) \),

\[
C_\Lambda \ni S \mapsto \vartheta_\# (S) := \sum_{S' \in P(S)} (-1)^{|S|-|S'|} \log \mathfrak{3}_{q,p}^\# (S') \tag{104}
\]

is such that, \( \forall s \in S_\Lambda \),

\[
\sum_{S \in C_\Lambda : S \sim s} e^{\vartheta_\# \sum_{s' \in S \ell (s')} |\vartheta_\# (S)|} \leq \frac{c_8}{2} |s| . \tag{105}
\]

Condition \([15]\) provides the existence of thermodynamics for the polymer model with partition function \( \mathfrak{3}_\Lambda^\# (q,p) := \sum_{\sigma \in \psi_\Lambda} \Psi_\# (\sigma) \), i.e. the existence of the limit

\[
\lim_{\Lambda \uparrow \mathbb{Z}^d} \log \mathfrak{3}_\Lambda^\# (q,p) \text{ along any cofinal sequence (see [Ge])} \{ \Lambda \uparrow \mathbb{Z}^d \} \quad [KP], \text{ this limit being independent of the boundary conditions.}
\]

Considering the realization of the elements of the decomposition of \( S_{(0,x)} \) given in \([67]\) as elements of \( S \) we have
Lemma 13 Let $s_1, s, s_f, s'_f$ be realizations of respectively $s_1, s, s_f$. Then, by (78) and (85) there exists $c_{10} > 0$ such that
\[
\frac{\P_{q,p} \left( \{s_1\} \mid \{s\} \cap \{s'_f\} \right)}{\P_{q,p} \left( \{s_1\} \mid \{s\} \cap \{s'_f\} \right)} \leq \exp e^{-c_{10}\text{dist}(s, s_f \triangle s'_f)}.
\] (106)

Proof. By (117), (100), and (103), we obtain
\[
\frac{\P_{q,p} \left( \{s_1\} \mid \{s\} \cap \{s'_f\} \right)}{\P_{q,p} \left( \{s_1\} \mid \{s\} \cap \{s'_f\} \right)} = \frac{\P_{q,p} \left( \{s_1\} \mid \{s\} \cap \{s'_f\} \right)}{\P_{q,p} \left( \{s_1\} \mid \{s\} \cap \{s'_f\} \right)}
\] (107)
\[
= \exp \left[ \sum_{S \in C : S = s_1 \perp \perp s_f} \vartheta_w(S) - \sum_{S \in C : S = s \perp \perp s_f} \vartheta_w(S) \right] \\
+ \sum_{S \in C : S = s \perp \perp s'_f} \vartheta_w(S) - \sum_{S \in C : S = s_1 \perp \perp s'_f} \vartheta_w(S)
\]
\[
= \exp \left[ \sum_{S \in C : S = s_1, S = s \perp \perp s'_f} \vartheta_w(S) - \sum_{S \in C : S = s_1, S = s \perp \perp s'_f} \vartheta_w(S) \right]
\]
\[
= \exp \sum_{S \in C : S = s_1, S = s \perp \perp s'_f} \vartheta_w(S).
\]

Since, by definition of $s_1, s_f$ and $s'_f$, there exists $\varepsilon > 0$ and $b_+ (s_1), b_- (s_f), b_- (s'_f) \in \mathbb{Z}^d$

such that $s_1 \subset b_+ (s_1) - C_\varepsilon (t), s_f \subset b_- (s_f) + C_\varepsilon (t)$ and $s'_f \subset b_- (s'_f) + C_\varepsilon (t)$

for any $s \in S$ such that $S \in C$ and $S \sim s_1 \perp \perp s_f, \ell (s) \geq \|b_+ (s_1) - b_- (s_f)\|$.

Hence, given $\zeta \in (0, 1)$, let us define, for any $l \geq 0$, $S^{l}_{i\ell} \cup \mathbb{N}^i b_2 := H^{l\perp}_{b_2+i\perp\mathbb{N}^i} \cap H^{l\perp}_{b_2+i+l(1+i)\perp\mathbb{N}^i}$ and

$S^{l}_{i\ell} \cup \mathbb{N}^i b_1 := H^{l\perp}_{b_1+i\perp\mathbb{N}^i} \cap H^{l\perp}_{b_1+i+l(1+i)\perp\mathbb{N}^i}$; where $b_2$ is the element of the set $\{b_+ (s_f), b_- (s'_f)\}$

closer to $b_1 := b_+ (s_1)$ w.r.t. the Euclidean distance. Setting $s_1^{(l)} := s_1 \cap S^{l}_{i\ell} \cup \mathbb{N}^i b_1, s_f^{(k)} := (s_f \triangle s'_f) \cap S^{l}_{i\ell} \cup \mathbb{N}^i b_1$, there exists $c_{9} = c_{9} (\varepsilon) > 0$ such that, by (105), we have

\[
\sum_{S \in C : S = s_1, S = s \perp \perp s'_f} \sum_{k,l \geq 0} |\vartheta_w(S)| = \sum_{k,l \geq 0} \sum_{S \in C : S = s \perp \perp s'_f} |\vartheta_w(S)|
\]
\[
\leq \sum_{k,l \geq 0} e^{-c_{9} \text{dist}(s_1^{(k)}, s_f^{(l)})} \sum_{S \in C : S = s \perp \perp s'_f} |\vartheta_w(S)| e^{c_{9} \sum_{S \in C : S = s \perp \perp s'_f} |\vartheta_w(S)|}
\]
\[
\leq e^{-c_{9} \|b_1-b_2\|} \left( c_{9} \zeta N \int_{0}^{\infty} dr e^{-c_{9} r d-1} \right)^{2}.
\]

A straightforward consequence of this result is the following
Proposition 14 There exists $\theta = \theta (p, q, d) \in (0, 1)$ and two positive constants $c_{11}, c_{12}$ such that uniformly in $v \in \mathcal{D}_{p,q}, t$-irreducible subsets $s_1, s_b$, strings of $t$-irreducible subsets $s, s'$, and pairs of $t$-irreducible subsets $s_f, s'_f$:

\[ c_{11} \leq g (s | s_b, s_f) \leq \frac{1}{c_{11}}, \quad (109) \]
\[ |g (s_1, s | s_b, s_f) - g (s_1, s' | s_b, s'_f)| \leq c_{12} \theta (s | s') \quad (110) \]
\[ |\Xi_v (s_1, s | s_b, s_f) - \Xi_v (s_1, s' | s_b, s'_f)| \leq c_{12} \theta (s | s'). \quad (111) \]

2.2 Exact asymptotics of finite connections

We refer to [CIV1] section 5 for the derivation of local limit type results associated with Ruelle's operators on countable alphabets.

Let $p \in (p_0 \vee p^*, 1)$. For any $n \in \mathbb{N}$, the measure on $\mathcal{I}_1^n$,

\[ \nu_n^\nu (s_1, \ldots, s_n | s_b, s_f) := e^{\sum_{i=1}^n \Xi_v (s_i, \ldots, s_n | s_b, s_f)} g (s_1, \ldots, s_n | s_b, s_f) \quad (112) \]

allow us to represent (94) as

\[ e^{\tau_{q,p} (x)} \mathbb{P}_{q,p} \{ 0 \rightarrow x, |C_{0,z}| < \infty \} = \sum_{y \in \mathcal{H}_v^{0,1} \cap \mathcal{H}_b^{0,1}} \sum_{z \in \mathcal{H}_v^{0,1} \cap \mathcal{H}_b^{0,1}} \sum_{s_b \not\parallel y} \sum_{s_f \not\parallel x-z} \mathbb{P}_{q,p} (\Omega_b) \mathbb{P}_{q,p} (\Omega_f) e^{(v,x-(z-y))} \times \]
\[ \times \sum_{n \geq 1} \nu_n^\nu \left( \sum_{i=1}^n X (s_i) = z - y | s_b, s_f \right) \quad (113) \]

Because Proposition [12] implies

\[ \sum_{s_b \not\parallel u} \mathbb{P}_{q,p} (\Omega_b) e^{(v,u)} \leq e^{-c_7 ||u||} \quad ; \sum_{s_f \not\parallel u} \mathbb{P}_{q,p} (\Omega_f) e^{(v,u)} \leq e^{-c_7 ||u||} \quad (114) \]

uniformly in $u \in \mathbb{Z}^d$, the main contribution of the r.h.s. of (94) comes from the last sum in (113) when $z - y$ is close to $x$ and $n$ is close to the optimal value. Therefore, proceeding as in section 4.1 of [CIV2] we have

\[ \sum_{n \geq 1} \nu_n^\nu \left( \sum_{i=1}^n X (s_i) = z - y | s_b, s_f \right) = \frac{\Theta_{q,p} (\hat{x})}{\sqrt{2\pi ||x||^{d-1}}} F (s_b) F (s_f) (1 + o (1)), \quad (115) \]

where $\Theta$ is a locally analytic positive function defined on a neighborhood of $\hat{x}$ in $\mathbb{S}^{d-1}$ and $F$ is a function on the set of all the possible realizations of $t$-backward and $t$-forward subsets $s_b$ and $s_f$ which is bounded above and below.
This proves Theorem 1 with

\[ \Phi_{q,p}(\hat{x}) = \Theta_{q,p}(\hat{x}) \left( \sum_{u \in \mathcal{H}_0^{n,+} \cap \mathcal{H}_0^{s,+}} \sum_{s_f \ni u} \mathbb{P}_{q,p}(\Omega_f) e^{(n,u)} F(s_f) \right)^2. \]  

(116)

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