Group actions of prime order on local normal rings

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In the theory of singularities, an important class of singularities is built by the famous Hirzebruch-Jung singularities. They arise by dividing out a finite cyclic group action on a smooth surface. The resolution of these singularities is well understood and have nice arithmetic properties related to continued fractions; cf. [H] and [J].

One can also look at such group actions from a purely algebraic point of view. So let \( B \) be a regular local ring and \( G \) a finite cyclic group of order \( n \) acting faithfully on \( B \) by local automorphisms. In the tame case; i.e. the order of \( G \) is prime to the characteristic of the residue field \( k \) of \( B \), there is a central result of J.P. Serre [S1] saying that the action is given by multiplying a suitable system of parameters \( (y_1, \ldots, y_d) \) by roots of unity \( \zeta^n \cdot y_i \) for \( i = 1, \ldots, d \) where \( \zeta \) is a primitive \( n \)-th root of unity. Moreover, the ring of invariants \( A := B^G \) is regular if and only if \( n_i \equiv 0 \mod n \) for \( d - 1 \) of the parameters. The latter is equivalent to the fact that \( \text{rk}((\sigma - \text{id})|T) \leq 1 \) for the action of \( \sigma \in G \) on the tangent space \( T := \mathfrak{m}_B/\mathfrak{m}_B^2 \). For more details see [B, Chap. 5, ex. 7].

Only very little is known in the case of a wild group action; i.e., \( \gcd(n, \text{char } k) > 1 \). In this paper we will restrict ourselves to the case of \( p \)-cyclic group actions; i.e. \( n = p \) is a prime number. We will present a sufficient condition for the fact that the ring of invariants \( A \) is regular. Our result is also valid in the tame case; i.e. where \( n \) is a prime different from \( \text{char } k \). As the method of Serre depends on an intrinsic formula for writing down the action explicitly, we provide also an explicit formula for presenting \( B \) as a free \( A \)-module if our condition is fulfilled.

The interest in our problem stems from the investigation on the relationship between the regular and the stable \( R \)-model of a smooth projective curve \( X_K \) over the field of fractions \( K \) of a discrete valuation ring \( R \). In general, the curve \( X_K \) admits a stable model \( X' \) over a finite Galois extension \( R \to R' \). Then the Galois group \( G = G(R'/R) \) acts on \( X' \). Our result provides a means to construct a regular model over \( R \) starting from the stable model \( X' \). We intend to work this out in a further article.

Finally, we want to mention that S. Wewers obtained partial results of ours by different methods cf. [W].

In this paper we will study only local actions of a cyclic group \( G \) of prime order \( p \) on a normal local ring \( B \). We fix a generator \( \sigma \) of \( G \) and obtain the augmentation map

\[
I := I_\sigma := \sigma - \text{id} : B \to B ; \ b \mapsto \sigma(b) - b .
\]

\footnotesize
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We introduce the $B$-ideal \[ I_G := (I(b) : b \in B) \subset B \]
which is generated by the image $I(B)$. This ideal is called augmentation ideal. If this ideal is generated by an element $I(y)$, we call $y$ an augmentation generator. Note that this ideal does not depend on the chosen generator $\sigma$ of $G$. Moreover, if $y$ is an augmentation generator with respect to a generator $\sigma$ of $G$, then $y$ is also an augmentation generator for any other generator of $G$. Since $B$ is local, the ideal $I_G$ is generated by an augmentation generator if $I_G$ is principal. Namely, $I_G/m_B I_G$ is a vector space over the residue field $k_B = B/m_B$ of $B$ of dimension 1. So it is generated by the residue class of $I(y)$ for some $y \in B$ and, hence, due to Nakayama’s Lemma, $I_G$ is generated by $I(y)$.

**Definition 1** An action of a group $G$ on a regular local ring $B$ by local automorphisms is called a pseudo-reflection if there exists a system of parameters $(y_1, \ldots, y_d)$ of $B$ such that $y_2, \ldots, y_d$ are invariant under $G$.

**Theorem 2** Let $B$ be a normal local ring with residue field $k_B := B/m_B$. Let $p$ be a prime number and $G$ a $p$-cyclic group of local automorphisms of $B$. Let $I_G$ be the augmentation ideal. Let $A$ be the ring of $G$-invariants of $B$. Consider the following conditions:

1. $I_G := B \cdot I(B)$ is principal.
2. $B$ is a monogenous $A$-algebra.
3. $B$ is a free $A$-module.

Then the following implications are true:

\[(a) \iff (b) \implies (c)\]

Assume, in addition, that $B$ is regular. Consider the following conditions:

1. $A$ is regular.
2. $G$ acts as a pseudo-reflection.

Then the condition (c) implies (d).

Moreover if, in addition, the canonical map $k_A \to k_B$ is an isomorphism. Then condition (a) is equivalent to condition (e).

We start the proof of the theorem by several preparations.

**Remark 3** For $b_1, b_2, b \in B$, the following relations are true:

1. $I(b_1 \cdot b_2) = I(b_1) \cdot \sigma(b_2) + b_1 \cdot I(b_2)$
2. $I(b^n) = \left( \sum_{i=1}^{n} \sigma(b_1^{i-1}b_2^{n-i}) \right) \cdot I(b)$
3. $I\left( \frac{b_1}{b_2} \right) = \frac{I(b_1)b_2 - b_1I(b_2)}{b_2\sigma(b_2)}$ if $b_2 \neq 0$. 

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Proof. (i) follows by a direct calculation and (ii) by induction from (i).
(iii) The formula (i) holds for elements in the field of fractions as well. Therefore it holds
\[ I(b_1) = I \left( \frac{b_1 b_2}{b_2} \right) = I \left( \frac{b_1}{b_2} \right) \sigma(b_2) + \frac{b_1}{b_2} I(b_2) \]
and the formula follows. \( \square \)

For the implication (a) \( \rightarrow \) (b) we need a technical lemma.

**Lemma 4** Let \( y \in B \) be an augmentation generator. Then set, inductively,
\[
\begin{align*}
y_i^{(0)} &= y \quad \text{for } i = 0, \ldots, p - 1 \\
y_i^{(1)} &= \frac{I(y_i^{(0)})}{I(y_1^{(0)})} \quad \text{for } i = 1, \ldots, p - 1 \\
y_i^{(n+1)} &= \frac{I(y_i^{(n)})}{I(y_{n+1}^{(n)})} \quad \text{for } i = n + 1, \ldots, p - 1.
\end{align*}
\]
Then
\[
y_i^{(n)} = \sum_{0 \leq k_1 \leq \cdots \leq k_{i-n} \leq n} \prod_{j=1}^{i-n} \sigma^{k_j}(y) \quad \text{for } i = n, \ldots, p - 1
\]
and, in particular,
\[
\begin{align*}
y_n^{(n)} &= 1 \\
y_n^{(n+1)} &= \sum_{j=1}^{n+1} \sigma^{j-1}(y) \\
I(y_{n+1}^{(n)}) &= \sigma^{n+1}(y) - y
\end{align*}
\]
Furthermore, \( y_{n+1}^{(n)} \) is again an augmentation generator for \( n = 0, \ldots, p - 2 \).

**Proof.** We proceed by induction on \( n \). For \( n = 0 \) the formulae are obviously correct. For the convenience of the reader we also display the formulae for \( n = 1 \). Due to Remark 3 one has
\[
y_i^{(1)} = \frac{I(y_i^{(0)})}{I(y_1^{(0)})} = \frac{I(y)}{I(y)} = \sum_{j=1}^{i} \sigma(y)^{j-1} y^{j-j} = \sum_{0 \leq k_1 \leq \cdots \leq k_{i-1} \leq 1} \prod_{\nu=1}^{i-1} \sigma^{k_\nu}(y)
\]
since the last sum can be viewed as a sum over an index \( j \) where \( i - j \) is the number of the\( k_\nu = 0 \). In particular, the formulae are correct for \( y_1^{(1)} \) and \( y_2^{(1)} \). Moreover
\[
I\left(y_2^{(1)}\right) = I(\sigma(y) - y) = \sigma^2(y) - y.
\]
Since $\sigma^2$ is generator of $G$ for $2 < p$, the element $y_2^{(1)}$ is an augmentation generator as well.

Now assume that the formulae are correct for $n$. Since $y_n^{(n)}$ is an augmentation generator, $I(y_{n+1}^{(n)})$ divides $I(y_i^{(n)})$ for $i = n + 1, \ldots, p - 1$. Then it remains to show

$$I(\sigma^{n+1}(y) - y) \cdot y_i^{(n+1)}$$

for $i = n + 1, \ldots, p - 1$.

For the left hand side one computes

$$LHS = I \left( \sum_{0 \leq k_1 \leq \ldots \leq k_{i-n} \leq n} \prod_{j=1}^{i-n} \sigma^{k_j}(y) \right) = \sum_{0 \leq k_1 \leq \ldots \leq k_{i-n} \leq n} I \left( \prod_{j=1}^{i-n} \sigma^{k_j}(y) \right)$$

$$= \sum_{0 \leq k_1 \leq \ldots \leq k_{i-n} \leq n} \prod_{j=1}^{i-n} \sigma^{k_j}(y) - \sum_{0 \leq k_1 \leq \ldots \leq k_{i-n} \leq n+1} \prod_{j=1}^{i-n} \sigma^{k_j}(y).$$

Now all terms occurring in both sums cancel. These are the terms with $k_{i-n} \leq n$ in the first sum and $1 \leq k_1$ in the second sum. For the right hand side one computes

$$RHS = (\sigma^{n+1}(y) - y) \cdot \sum_{0 \leq k_1 \leq \ldots \leq k_{i-n-1} \leq n+1} \prod_{j=1}^{i-n-1} \sigma^{k_j}(y)$$

$$= \sum_{0 \leq k_1 \leq \ldots \leq k_{i-n} = n+1} \prod_{j=1}^{i-n} \sigma^{k_j}(y) - \sum_{0 \leq k_1 \leq \ldots \leq k_{i-n} \leq n+1} \prod_{j=1}^{i-n} \sigma^{k_j}(y).$$

Comparing both sides one obtains $LHS = RHS$. In particular we have

$$y_{n+1}^{(n+1)} = 1$$

$$y_{n+2}^{(n+1)} = \sum_{0 \leq k_1 \leq n+1} \prod_{j=1}^{n+2} \sigma^{k_j}(y) = \sum_{j=1}^{n+2} \sigma^{j-1}(y)$$

$$I(y_{n+2}^{(n+1)}) = \sigma^{n+2}(y) - y.$$

So $y_{n+2}^{(n+1)}$ is an augmentation generator for $n + 2 < p$, since $\sigma^{n+2}$ generates $G$. This concludes the technical part.

**Proposition 5** Assume that the augmentation ideal $I_G$ is principal and let $y \in B$ be an augmentation generator. Then $B$ decomposes into the direct sum

$$B = A \cdot y^0 \oplus A \cdot y^1 \oplus \ldots \oplus A \cdot y^{p-1}.$$

**Proof.** Since $I(y) \neq 0$, the element $y$ generates the field of fractions $Q(B)$ over $Q(A)$. Therefore

$$Q(B) = Q(A) \cdot y^0 \oplus Q(A) \cdot y^1 \oplus \ldots \oplus Q(A) \cdot y^{p-1}.$$
Then it suffices to show the following claim:

Let \( a, a_0, \ldots, a_{p-1} \in A \). Assume that \( a \) divides

\[
    b = a_0 \cdot y^0 + a_1 \cdot y^1 + \ldots + a_{p-1} \cdot y^{p-1}.
\]

Then \( a \) divides \( a_0, a_1, \ldots, a_{p-1} \).

If \( b = a \cdot \beta \), then \( I(b) = a \cdot I(\beta) \). Since \( I(\beta) = \beta_1 \cdot I(y) \), we get \( I(b) = a\beta_1 \cdot I(y) \). So we see that \( a \) divides \( I(b)/I(y) \in B \). Using the notations of Lemma 3 set

\[
    b^{(0)} := b = a_0 \cdot y^0 + a_1 \cdot y^1 + \ldots + a_{p-1} \cdot y^{p-1}.
\]

\[
    b^{(1)} := \frac{I(b^{(0)})}{I(y)} = a_1 + a_2 \frac{I(y^2)}{I(y)} + \ldots + a_{p-1} \frac{I(y^{p-1})}{I(y)} = a_1 \cdot y_1 + a_2 \cdot y_2 + \ldots + a_{p-1} \cdot y_{p-1}.
\]

\[
    b^{(n)} := \frac{I(b^{(n-1)})}{I(y^{n-1})} = a_n \cdot y_n + a_{n+1} \cdot y_{n+1} + \ldots + a_{p-1} \cdot y_{p-1}.
\]

Due to the observation above, we see by induction that \( a \) divides \( b^{(0)}, b^{(1)}, \ldots, b^{(p-1)} \), since \( y_{n+1}^{(n)} \) is an augmentation generator for \( n = 1, \ldots, p-2 \). So we obtain

\[
    a \mid b^{(p-1)} = a_{p-1} \cdot y_{p-1}^{(p-1)} = a_{p-1}.
\]

Now proceeding downwards, one obtains

\[
    a \mid b^{(p-2)} = a_{p-2} + a_{p-1} \cdot y_{p-1}^{(p-2)} \quad \text{and, hence,} \quad a \mid a_{p-2}
\]

\[
    a \mid b^{(n)} = a_n + a_{n+1} \cdot y_{n+1}^{(n)} + \ldots + a_{p-1} \cdot y_{p-1}^{(n)} \quad \text{and, hence,} \quad a \mid a_n
\]

for \( n = p-2, p-3, \ldots, 0 \). \( \square \)

**Proof of the first part of Theorem 2**

(a) \( \rightarrow \) (b): This follows from Proposition 3.

(b) \( \rightarrow \) (a): If \( B = A[y] \) is monogenous, then \( I_G = B \cdot I(y) \) is principal.

(b) \( \rightarrow \) (c) is clear. Namely, if \( B = A[y] \), the minimal polynomial of \( y \) over the field of fraction is of degree \( p \) and the coefficients of this polynomial belong to \( A \). Then \( B \) has \( y^0, y^1, \ldots, y^{p-1} \) as an \( A \)-basis.

Next we do some preparations for proving the second part of the theorem where \( B \) is assumed to be regular.

**Lemma 6** Let \( R \) be a local subring of \( B \) which is invariant under \( G \) such that the canonical map \( R/m_R \rightarrowtail B/m_B \) is an isomorphism. Let \( (y_1, \ldots, y_d) \) be a generating system of the maximal ideal \( m_B \). Then \( I_G \) is generated by \( I(y_1), \ldots, I(y_d) \).

**Proof.** Due to the assumption, we have \( B = R + m_B \) and, hence, \( I(B) = I(m_B) \). Furthermore, we have

\[
    m_B = m_B^2 + \sum_{i=1}^{d} R \cdot y_i.
\]

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Since $I$ is $R$-linear, we get

$$I(m_B) = I(m_B^2) + \sum_{i=1}^{d} R \cdot I(y_i).$$

Due to Remark 3, one knows $I(m_B^2) \subset m_B \cdot I(m_B)$. So one obtains

$$I(m_B) \subset m_B \cdot I(m_B) + \sum_{i=1}^{d} R \cdot I(y_i).$$

Since $B$ is local, Nakayama’s Lemma yields

$$I_G = B \cdot I(B) = B \cdot I(m_B) = \sum_{i=1}^{d} B \cdot I(y_i).$$

Thus the assertion is proved. □

**Proposition 7** Keep the assumption of the second part of the theorem; namely that $B$ is regular and that the canonical morphism $k_A \rightarrow k_B$ is an isomorphism. Let $(y_1,\ldots,y_d)$ be a generating system of the maximal ideal $m_B$. Then the following assertions are true:

(i) $I_G = B \cdot I(y_1) + \ldots + B \cdot I(y_d)$

(ii) If the ideal $I_G = B \cdot I(B)$ is principal, then there exists an index $i \in \{1,\ldots,d\}$ with $I_G = B \cdot I(y_i)$.

**Proof.** Let $\hat{B}$ be the $m_B$-adic completion of $B$. Recall that a regular ring is noetherian by definition. Therefore the extension $B \rightarrow \hat{B}$ is faithfully flat and $m_B \hat{B} = \hat{m}_B$; cf. [AM 10.14 & 10.15]. Since $G$ acts by local morphism, any $\sigma \in G$ extends to a local automorphism $\hat{\sigma}$ of $\hat{B}$. Due to the assumption that the canonical morphism $k_A \rightarrow k_B$ is an isomorphism, any $b \in B$ can be written as $B = a + m$ where $a \in A$ is invariant under $G$ and $m \in m_B$ and, hence, $I(b) = I(m) \in I(m_B)$. Therefore $I_G$ is generated by $I(m_B)$.

(i) Since $B \rightarrow \hat{B}$ is faithfully flat and $\hat{B} \cdot m_B = \hat{m}_B$, it suffices to prove the assertion for the completion $\hat{B}$. For complete local rings there exists a $G$-stable lift $R$ of the residue field $k$. Namely, in the case of mixed characteristic $(0,p)$, one can choose the ring of Witt vectors $W(k) \subset A$ as $R$ and, in the equal characteristic case, the residue field $k$ lifts into $\hat{A}$; cf. [C].

Now we can apply Lemma 5 and obtain the assertion.

(ii) Since $I_G$ is principal, $I_G/m_B I_G$ is generated by one of the $I(y_i)$ and, hence, again by Nakayama’s Lemma $I_G = B \cdot I(y_i)$ for a suitable $i \in \{1,\ldots,d\}$.

□

**Proof of the second part of Theorem 2**

(c) → (d) follows from [M Theorem 51]. Namely, $B$ is noetherian due to the definition of a regular ring. Since $A \rightarrow B$ is faithfully flat, so $A$ is noetherian. Then one can apply loc.cit.

(a) → (e) We assume that the canonical map $k_A \rightarrow k_B$ of the residue fields is an isomorphism. If $I_G$ is principal, one can choose an augmentation generator $y \in m_B$ which is part of a system of parameters $(y,y_2,\ldots,y_d)$ due to Proposition 7. Due to Proposition 5 we know that $B$ decomposes into the direct sum

$$B = A \cdot y^0 \oplus A \cdot y^1 \oplus \ldots \oplus A \cdot y^p.$$
Now we can represent

\[ y_j = \sum_{i=0}^{p-1} a_{i,j} \cdot y^i \text{ for } j = 2, \ldots, d . \]

Then set

\[ \tilde{y}_j := y_j - \sum_{i=1}^{p-1} a_{i,j} y^i = a_{0,j} \in A \cap m_B = m_A \text{ for } j = 2, \ldots, d . \]

So \((y, \tilde{y}_2, \ldots, \tilde{y}_d)\) is a system of parameters of \(B\) as well. Thus \(G\) acts by a pseudo-reflection.

(e) \(\rightarrow\) (a): If \(G\) is a pseudo-reflection, \(I_G\) is generated by \(I(y)\) due to Proposition 7 where \(y, x_2, \ldots, x_p\) is a system of parameters with \(x_i \in m_A\) for \(i = 2, \ldots, p\) if \(k_A = k_B\). \(\square\)

If \(k_A \rightarrow k_B\) is not an isomorphism, the implication (e) \(\rightarrow\) (a) is false as the following shows.

**Example 8** Let \(k\) be a field of positive characteristic \(p\) and look at the polynomial ring

\[ R := k[Z, Y, X_1, X_2] \]

over \(k\). We define a \(p\)-cyclic action of \(G = \langle \sigma \rangle\) on \(R\) by

\[ \sigma[k] := \text{id}_k, \sigma(Z) = Z + X_1, \sigma(Y) = Y + X_2, \sigma(X_i) = X_i \text{ for } i = 1, 2 . \]

This is a well-defined action of order \(p\), since \(p \cdot X_i = 0\) for \(i = 1, 2\), and it leaves the ideal \(I := (Y, X_1, X_2)\) invariant. Furthermore, for any \(g \in k[Z] - \{0\}\) the image is given by \(\sigma(g) = g + I(g)\) with \(I(g) \in X_1 \cdot k[Z, X_1]\).

Then consider the polynomial ring

\[ S := k(Z)[Y, X_1, X_2] \]

over the field of fractions \(k(Z)\) of the polynomial ring \(k[Z]\). Then \(S\) has the maximal ideal \(m = (Y, X_1, X_2)\). Then set

\[ B := S_m = k(Z)[Y, X_1, X_2](Y, X_1, X_2) . \]

We can regard all these rings as subrings of the field of fractions of \(R\)

\[ R \subset S \subset B \subset k(Z, Y, X_1, X_2) . \]

Clearly, \(\sigma\) acts on \(R\) and, hence, it induces an action on its field of fractions; denote this action by \(\sigma\) as well. Then we claim that the restriction of \(\sigma\) to \(B\) induces an action on \(B\) by local automorphisms. For this, it suffices to show that for any \(g \in R - I\) the image \(\sigma(g)\) does not belong to \(I\). The latter is true, since

\[ \sigma(g) = g + I(g) \text{ with } I(g) \in I . \]

The augmentation ideal \(I_G = B \cdot X_1 + B \cdot X_2\) is not principal although \(G\) acts through a pseudo-reflection.

\[ \square \]

**Remark 9** In the tame case \(p \neq \text{char}(k_B)\), the converse (d) \(\rightarrow\) (a) is also true due to the theorem of Serre as explained in the introduction.
In the case of a wild group action; i.e. \( p = \text{char}(k_B) \), it is not known whether the converse is true, but we would conjecture it.

**Conjecture 10** Let \( B \) be a regular local ring and let \( G \) be a \( p \)-cyclic group acting on \( B \) by local automorphisms. Then the following conditions are *conjectured* to be equivalent:

1. \( I_G \) is principal.
2. \( A := B^G \) is regular.

The implication \((1) \implies (2)\) was shown in Theorem 2. Of course the converse is true if \( \dim A \leq 1 \).

In higher dimension, the converse \((2) \implies (1)\) is uncertain, but it holds for small primes \( p \leq 3 \) as we explain now. Since \( A \) is regular, the ring \( B \) is a free \( A \)-module of rank \( p \); cf. [S2, IV, Prop. 22]. So,

\[
B/Bm_A^n \text{ is a free } A/m_A^n \text{-module of rank } p \text{ for any } n \in \mathbb{N}.
\]

In the case \( p = 2 \) the rank of \( m_B/Bm_A \) is 0 or 1. In the first case, \( k_B \) is an extension of degree \( [k_B : k_A] = 2 \) over \( k_A \) and \( m_B = Bm_A \). So there exists an element \( \beta \in B \) such that \( B/Bm_A \) is generated by the residue classes of 1 and \( \beta \). Due to Nakayama’s Lemma \( B = A[\beta] \) is monogenous and, hence, \( I_G \) is principal. In the second case, where \( k_A \to k_B \) is an isomorphism, then there exists an element \( \beta \in m_B \) such that \( m_B = B\beta + Bm_A \). Then \( G \) acts as a pseudo-reflection and, hence, \( I_G \) is principal.

In the case \( p = 3 \) we claim that \( Bm_A \not\subseteq m_B^2 \).

If we assume the contrary \( Bm_A \subseteq m_B^2 \), then these ideals coincide; \( Bm_A = m_B^2 \). Namely, the rank of \( B/Bm_A \) as \( A/m_A \)-module is 3 and the rank of \( B/m_B^2 \) is at least 3 due to \( d := \dim B \geq 2 \), so \( Bm_A = m_B^2 \). Therefore the length of \( B/Bm_A^2 = B/m_B^3 \) is 3 times the length of \( A/m_A^2 \) which is \( 3 \cdot (\dim A + 1) \). On the other hand the rank of \( B/m_B^3 \) is equal to

\[
(1 + \dim m_B/m_B^2) + \dim m_B^2/m_B^3 + \dim m_B^3/m_B^4 = \sum_{n=0}^{3} \binom{d + n - 1}{d - 1}
\]

which larger than

\[
(1 + \dim A/m_A^2) + (1 + \dim m_A/m_A^2) + (1 + \dim m_A/m_A^2)
\]

since for \( d \geq 2 \) holds

\[
\binom{d + 1}{d - 1} = \frac{(d + 1)d}{2} \geq 1 + d = 1 + \dim m_A/m_A^2
\]

and

\[
\binom{d + 3 - 1}{d - 1} = \frac{(d + 2)(d + 1)d}{2 \cdot 3} > 1 + d
\]

Here we used the formula for the number \( \lambda_{n,d} \) of monomials \( T_1^{m_1} \cdots T_d^{m_d} \) in \( d \) variables of degree \( n = m_1 + \cdots + m_d \)

\[
\lambda_{n,d} = \binom{d + n - 1}{d - 1}.
\]
So, using only the condition \((\ast)\) and proceeding by induction on \(\dim(A)\), we see that here exists a system of parameters \(\alpha_1, \ldots, \alpha_d\) of \(A\) such that \(\alpha_2, \ldots, \alpha_d\) is part of a system of parameters of \(B\). In the case, where \(k_A \to k_B\) is an isomorphism, \(G\) acts as a pseudo-reflection and, hence, \(I_G\) is principal. If \(k_A \to k_B\) is not an isomorphism, then we must have \(\mathfrak{m}_B = B\mathfrak{m}_A\); otherwise the rank of \(B/\mathfrak{m}_B\) is at least 4. Since \([k_B : k_A] \leq 3\), the field extension \(k_A \to k_B\) is monogenous and, hence, \(A \to B\) is monogenous due to the Lemma of Nakayama. \(\square\)

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