Separating k-Player from t-Player One-Way Communication, with Applications to Data Streams

Elbert Du   Michael Mitzenmacher*   David Woodruff   Guang Yang

December 1, 2021

Abstract: In a k-party communication problem, the k players with inputs $x_1, x_2, \ldots, x_k$, respectively, want to evaluate a function $f(x_1, x_2, \ldots, x_k)$ using as little communication as possible. We consider the message-passing model, in which the inputs are partitioned in an arbitrary, possibly worst-case manner, among a smaller number $t$ of players ($t < k$). The $t$-player communication cost of computing $f$ can only be smaller than the $k$-player communication cost, since the $t$ players can trivially simulate the $k$-player protocol. But how much smaller can it be? We study deterministic and randomized protocols in the one-way model, and provide separations for product input distributions, which are optimal for low error probability protocols. We also provide much stronger separations when the input distribution is non-product.

A key application of our results is in proving lower bounds for data stream algorithms. In particular, we give an optimal $\Omega(\varepsilon^{-2}\log(N)\log\log(mM))$ bits of space lower bound for

*Supported in part by NSF grants CCF-2101140, CNS-2107078, and DMS-2023528, and by a gift to the Center for Research on Computation and Society at Harvard University.

ACM Classification: F.1.3, F.2.3, F.2.1

AMS Classification: 68Q17, 68W25, 68W20

Key words and phrases: streaming, space complexity, hamming norm, approximation, algorithms with predictions, direct sum
the fundamental problem of $(1 \pm \varepsilon)$-approximating the number $\|x\|_0$ of non-zero entries of an $n$-dimensional vector $x$ after $m$ integer updates each of magnitude at most $M$, and with success probability $\geq 2/3$, in a strict turnstile stream. We additionally prove the matching $\Omega(\varepsilon^{-2} \log(N) \log(\log(T)))$ space lower bound for the problem when we have access to a heavy hitters oracle with threshold $T$. Our results match the best known upper bounds when $\varepsilon \geq 1/\text{polylog}(mM)$ and when $T = 2^{\text{poly}(1/\varepsilon)}$ respectively. It also improves on the prior $\Omega(\varepsilon^{-2} \log(mM))$ lower bound and separates the complexity of approximating $L_0$ from approximating the $p$-norm $L_p$ for $p$ bounded away from 0, since the latter has an $O(\varepsilon^{-2} \log(mM))$ bit upper bound.

1 Introduction

Consider a $k$-party communication problem, in which the players have inputs $x_1, x_2, \ldots, x_k$ respectively, and want to compute a function $f(x_1, x_2, \ldots, x_k)$ of their inputs using as little communication as possible. We consider the message-passing model, in which the inputs are partitioned in an arbitrary, possibly non-contiguously, consisting of say $m$ subsets $S_1, S_2, \ldots, S_k$ such that $\bigcup_{i=1}^{k} S_i = \{1, 2, \ldots, k\}$ and $S_i \cap S_j = \emptyset$ for every $1 \leq i < j \leq t$, and let the $i$-th player $P_i$ hold the sequence of inputs $y_i := (x_{i1}, x_{i2}, \ldots, x_{i|S_i|})$. We are still interested in computing the original function $f$. The total communication required must be smaller than in the original $k$-player setting, since the $t$ players can simulate the protocol involving the original $k$ players. A natural question is: how much smaller can the communication be?

There are many communication models that are possible, but our main motivation for looking at this question comes from applications to data streams, see below, and so we are primarily interested in the one-way number-in-hand model. In this model, each of the $t$ players can only see its own input. The first player composes a message $m_1$ based on its input $y_1$ and sends $m_1$ to the second player. The second player takes $m_1$ and its input $y_2$ to compute a message $m_2$ for the third player, and so on. The $t$-th (also the last) player, upon receiving the message $m_{t-1}$ from the $(t-1)$-st player, computes the output of the protocol based on $m_{t-1}$ and its own input $y_t$. We sometimes abuse notation and refer to the output as $m_t$. The total communication cost is the maximum of $\Sigma_{i=1}^{t} |m_i|$, where $|m_i|$ denotes the length of the $i$-th message and the maximum is taken over all possible inputs $y_1, \ldots, y_t$ (which is a partition of $\{x_1, \ldots, x_k\}$) and all random coin tosses of the players. For streaming applications we are especially interested in $\max_{i \in \{1, \ldots, t\}} |m_i|$. 

To explain the connection to data streams, almost all known lower bound arguments on the memory required of a data stream algorithm are proven via communication complexity, or at least can be reformulated using communication complexity. The basic idea is to partition the elements of an input stream contiguously, consisting of say $k$ elements, into a possibly smaller number $t$ of players. Then one argues that if there is a data stream algorithm solving the problem, then the communication problem can be solved by passing the memory contents as messages from player to player. Note that this naturally gives rise to the one-way number-in-hand model. Since the total communication cost is $t \cdot S$, where $S$ is the size of the memory of the streaming algorithm, if the randomized $t$-player communication complexity of the function $f$ is $CC_t$, we must have $S \geq CC_t/t$. Many lower bounds in data streams are proven already with
two players. However, it is known that for some functions more players are needed to obtain stronger lower bounds, such as for estimating the frequency moments in insertion only streams (see, e.g., [3, 24] and references therein).

One cannot help but ask how powerful is communication complexity for proving data stream lower bounds? Another natural question is: for a given function \( f \), which number \( t \) of players should one partition the stream into? Yet another question is regarding the input distribution – should it be a product distribution for which the inputs to the players are chosen independently, or should the inputs be drawn from a non-product distribution to obtain the best space lower bounds? Since we are interested in the limits of using \( t \) players for establishing lower bounds for data stream algorithms, we allow the original \( k \) inputs (which correspond to the \( k \) elements in a stream) to be partitioned in the worst possible way for a \( t \)-player communication protocol, as this will give the strongest possible lower bound.

### 1.1 Our Results

In this paper we study these communication questions and their connections to data streams.

We first make the simple observation that for non-product input distributions, the communication complexity can be arbitrarily smaller if we partition the \( k \) inputs into \( t < k \) players. Indeed, consider the \( k \)-player set disjointness problem in which the \( i \)-th player, \( 1 \leq i \leq k \), has a set \( S_i \subseteq [n] \), where for notational simplicity we define \([n] := \{1, 2, \ldots, n\}\) for \( n \in \mathbb{N} \). The input distribution satisfies the promise that either (1) \( S_i \cap S_j = \emptyset \) for every \( 1 \leq i < j \leq k \), or (2) there is a unique item \( a \in [n] \) such that \( a \in S_i \) for all \( i \in [k] \), and for any other \( a' \neq a \), there is at most one \( i \in [k] \) for which \( a' \in S_i \). It is well-known that the randomized communication complexity of this problem is \( \Omega(n/k) \) \([3, 9, 12]\), and that the bound holds even for multiple rounds of communication and when players share a common blackboard. However, if we look at \( t < k \) players and an arbitrary, even if the worst-case mapping of the input sets \( S_1, \ldots, S_k \) to the \( t \) players, then by the pigeonhole principle there exists a player who gets two input sets \( S_i, S_j \) with \( i \neq j \). Now this player can locally determine the output of the function by checking if \( S_i \cap S_j = \emptyset \). Thus with \( t < k \) players the problem is solvable using \( O(1) \) bits per player. This simple argument shows that for non-product distributions, there can be an arbitrarily large gap between the \( k \)-player and the \( t \)-player worst-case-partitioned randomized communication complexities. Note that this example applies to a symmetric problem, meaning that the \( k \)-player set disjointness problem is invariant under any one-to-one assignment of \( x_1, \ldots, x_k \) to the \( k \) players.

Perhaps surprisingly, and this is one of the main messages of our work: for symmetric functions and product input distributions,

we show that for any \( t < k \), for deterministic one-way communication complexity or randomized one-way communication complexity with error probability \( 1/\text{poly}(k) \), that is, the gap between the \( k \)-player and \( t \)-player communication complexities is at most a multiplicative \( O(1) \) factor in maximum message length, or the maximum communication from a single player, and \( \text{O}(k) \) in total communication. Further, this gap is tight, as there are problems for which the input distribution is a product distribution, and the \( t \)-player communication with \( 1/\text{poly}(k) \) error probability is \( \text{O}(\log k) \) for constant \( t = \text{O}(1) \), while the \( k \)-player communication with \( 1/\text{poly}(k) \) error probability is \( \Omega(k \log k) \).

Thus, the answer for product input distributions is significantly different than what we saw for non-product distributions, even for symmetric functions.
We also show that for protocols with constant error and under product input distributions, the gap is at most a multiplicative $O(\log k)$ factor in message length and $O(k \log k)$ in total communication. Further, we show that there exists a symmetric function and input distribution which is product on any $k$ inputs, for which this gap is best possible. We leave open the question of the existence of a symmetric function and product input distribution (on all $k$ inputs rather than $k - 1$ out of $k$) which realizes this gap for constant error protocols.

One takeaway message from our results is that when showing space lower bounds for data stream algorithms computing symmetric functions on product distributions, by looking at 2-player communication complexity (which is by far the most common communication setup), there is only an $O(1)$ factor loss for error probability $1/\poly(k)$ protocols, and an $O(\log k)$ factor loss for constant error protocols.

However, for non-product distributions, which are often needed to show hardness of approximation in data streams (such as for the frequency moments [3]), one may need to use as many as $k$ players in order to obtain a non-trivial lower bound from communication complexity.

### 1.1.1 Data Stream Lower Bounds:

As a key application of our lower bound techniques, we provide a space lower bound for $(1 \pm \varepsilon)$-approximating the *Hamming norm* in the strict turnstile model. This problem, which is also known as the $L_0$ norm estimation and denoted by $T_\varepsilon$, requires estimating $||x||_0 := |\{i \mid x_i \neq 0\}|$ of a vector $x = (x_1, \ldots, x_N)$ and outputting an estimate $\tilde{F}$ for which $(1 - \varepsilon)||x||_0 \leq \tilde{F} \leq (1 + \varepsilon)||x||_0$ with constant probability. The vector $x$ is initialized to all zeros and undergoes a sequence of $m$ updates each of the form $(i, v) \in [N] \times \{\pm M\}$, where $\{\pm M\} := \{0, \pm 1, \ldots, \pm M\}$ and each update $(i, v)$ causes $x_i := x_i + v$. In the strict turnstile model, $x_i \geq 0$ holds for all $i$ and at all points in the stream. We obtain an $\Omega(\varepsilon^{-2} \log(N) \log \log(mM))$ bits of space lower bound for $(1 \pm \varepsilon)$-approximating the Hamming norm. This lower bound matches the best known upper bound $O(\varepsilon^{-2} \log(N) (\log(1/\varepsilon) + \log \log(mM)))$ [16] for any $\varepsilon \geq 1/\polylog(mM)$. Note that $\varepsilon \geq 1/\polylog(mM)$ is required in order to obtain polylogarithmic space, and so is the most common setting of parameters.

Perhaps surprisingly, there is an upper bound of $O(\varepsilon^{-2} \log(mM))$ bits of space for $(1 \pm \varepsilon)$-approximating $L_p$ for $p > 0$ [15] (improving an earlier $O(\log^2 N)$ bound of [11]; see also a time-efficient version in [14]), and thus we provide a strict separation in the complexities for $p = 0$ and $p > 0$.

The Hamming norm has many applications, as it corresponds to estimating the number of distinct values, and can be used to estimate set union and intersection sizes (see [8] where it was introduced).

**Lower Bounds in the Learning Augmented Setting** Recently, there has been a growing interest in using machine learning to infer information about the stream that would be useful for solving certain problems in the streaming setting. In this learning augmented setting, we have access an oracle (which in practice would have some degree of error and could be implemented with machine learning). Learned oracles have been used to develop improved algorithms for various problems, including frequency estimation [10], caching [19], scheduling [20], frequency moments [13], and more. A fairly comprehensive survey of learning augmented algorithms can be found here: [21]

In our setting, the oracle provides an additional operation: we can give the oracle a coordinate, and the oracle will tell us whether the frequency of this coordinate at the end of the stream is at least $T$ for a
threshold $T$. We refer to this oracle as the heavy hitters oracle. Approximate heavy hitter oracles have been used for frequency estimation [10].

We derive a novel method to prove space lower bounds even with a perfect heavy hitters oracle. We use this method to prove a lower bound of $\Omega\left(\varepsilon^{-2} \log(N) \log\log(T)\right)$ for approximating the $L_0$ norm, which is optimal when $T = 2^{\text{poly}(1/\varepsilon)}$ as it matches the upper bound in [13].

### 1.2 Technical Overview

We first illustrate the idea behind showing there is no gap between $k$-player and 2-player deterministic one-round communication complexity. The first player $P_1$ of the $k$-player protocol pretends to be Alice, the first player of the 2-player protocol, to create the message $m_1$ as Alice would do and sends it to the second player $P_2$ of the $k$-player protocol. Having received this message $m_1$, $P_2$ enumerates over all possible inputs of $P_1$ until finding one which would cause $P_1$ to send $m_1$. Since the protocol is deterministic and it evaluates a function defined on a product domain, meaning that it is a total function on a domain of the form $S_1 \times S_2 \times \cdots \times S_k$, the function value must be the same as long as $P_1$’s input results in the same message $m_1$ to be sent. So $P_2$ can arbitrarily pick one of those inputs as his guess for $P_1$. Now $P_2$ has a guess $x$ for $P_1$’s input together with his own input $y$, and $P_2$ can simulate Alice in the 2-player protocol. This is feasible because the 2-player protocol works under any partitioning of the inputs. Then $P_2$ sends to the third player $P_3$ the message that Alice would send to Bob in the 2-player protocol, given that Alice had input $(x, y)$. In case when every player $P_i$ cannot figure out how many input items have been processed from his own input and the received message $m_{i-1}$, which is important for his simulation of the 2-player protocol, an additional logarithmic-many-bits index carrying this piece of information should be passed together with the simulated messages. In this way, the entire $k$-player protocol can be simulated and the per player communication equals to the communication of the 2-player protocol between Alice and Bob, sometimes plus the additional logarithmic many bits for the index. Moreover, both protocols are deterministic.

For the randomized case with a product input distribution, we first consider 2-player protocols with error probability $1/\text{poly}(k)$.

We would like to run the same simulation as for deterministic protocols, except now it is unclear how the second player $P_2$ can reconstruct a valid input $x$ for the first player $P_1$ from the first message $m$. A natural thing would be for $P_2$ to choose the input $x = x_m$ to $P_1$ for which the probability of sending $m$, given that $P_1$’s input is $x_m$, is greatest. This is not correct though, since the overall probability of $P_1$ holding $x_m$ and sending $m$ may be less than the $1/\text{poly}(k)$ error bound and the protocol could afford to be always wrong on such a combination of $x_m$ and $m$. Thus we need some balancing between two probabilities: i) the first player $P_1$ sends $m$ on input $x$; and ii) the protocol output is correct given that $P_1$ has input $x$ and sends $m$.

The above naturally suggests that we should impose an input product distribution $\mu$. Then it must be that for a good fraction of $x$, weighted according to $\mu$, the $k$-player protocol is correct when the first player has input $x$ and sends message $m$. Thus we can sample $x$ from the conditional distribution on $\mu$ given that message $m$ is sent. Here, for correctness, it is crucial that $\mu$ is a product distribution; this ensures for most settings of remaining player’s inputs (weighted according to $\mu$), for most choices of $x$ (weighted according to $\mu$) giving rise to $m$, the function evaluated on the inputs is the same, and $x$ can be
Another natural approach is to use the fact that if a problem has a corruption bound, then one immediately
problem, where each rectangle, restricted to the first
k
and not for our setting.

Again though, this is only for two players or the
problem, which improves upon a bound in [6]. However, for
give an information cost lower bound on private coin protocols, though one can fix it for two players using
information bound, and then apply standard direct sum theorems for information. This approach does not
are much weaker. A natural route would be to take Viola’s
corruption bound
the technique of
r
fraction of
Ω
for his problem, showing its communication complexity increases to
probability.

It is an open question to give an optimal separation for product input distributions for constant error
probability (since we measure error with respect to an input distribution, equality has an
upper bound in the public coin model which comes from running an equality test with constant
error probability (since we measure error with respect to an input distribution, equality has an
O(1) upper bound with constant error).

We note that the
k
-player protocol has communication
Ω(k \log k)
for constant error protocols, which
gives the
Ω(k \log k)
factor gap we claimed. The only downside is that the
Ω(k \log k)
lower bound holds for an input distribution which is product on
k − 1 out of
k
players, rather than all
k
players. We leave it as an open question to give an optimal separation for product input distributions for constant error
probability.

Given the importance of Viola’s problem in showing separations, we next show a direct sum theorem
for his problem, showing its communication complexity increases to
Ω(\sum_{i=1}^{\log k} p^i)
for solving a constant fraction of
r
independent copies.

To show the direct sum theorem for Viola’s problem, one issue is that, unlike for two players where
the technique of information complexity often provides direct sum theorems, for
k
-players the analogues
are much weaker. A natural route would be to take Viola’s corruption bound, argue it implies a high
information bound, and then apply standard direct sum theorems for information. This approach does not
give an information cost lower bound on private coin protocols, though one can fix it for two players using
[5], which improves upon a bound in [6]. However, for
k
players similarly strong bounds are unknown. Another natural approach is to use the fact that if a problem has a corruption bound, then one immediately
has a direct sum for it [4]. Again though, this is only for two players or the number on forehead model, and not for our setting.

Instead, our proof is inspired by Viola’s rectangle argument for a single copy of the SUM-EQUAL
problem, where each rectangle, restricted to the first
k − 1
players, is a product distribution on which the
protocol generates a message to the $k$-th player. We use a rectangle argument on multiple copies where the output is now a binary vector instead of a single bit. The main obstacle is that we must consider the Hamming distance between the protocol output and the correct answer in a vector space, which is much more involved than studying the error probability for a single instance. The intuition of our proof is that for every large rectangle, there must be linearly many copies that appear (almost) uniformly random in the last player’s view. The above argument is fairly intricate, and involves several levels of conversion: i) a large rectangle implies large conditional entropy in many players’ inputs; ii) the large entropy of all copies implies we have min-entropy at least 1 on many copies; iii) a random variable of min-entropy at least 1 can always be decomposed into a convex combination of uniform distributions over two elements; iv) the summation of sufficiently many independent random variables that are each drawn from a uniform-over-two-element distribution turns out to be nearly uniform, and hence many SUM-EQUAL copies look uniform to the last player.

Thus, the last player can hardly outperform a random guess. Note that it is insufficient to prove uniformity for many copies individually (which is not too hard using the same idea as in Viola’s proof), since such a situation could be simulated with a much smaller rectangle with very small error. We instead perform our rectangle argument inductively to show most copies appear almost uniform, even if conditioned on previous copies.

This direct sum technique has further applications. One application is to proving a lower bound for approximating the Hamming norm in a strict turnstile stream. Using a result of [2], to show lower bounds for streaming algorithms in the strict turnstile model, it suffices to show lower bounds in the simultaneous communication model, where each player simultaneously sends a linear sketch to a referee who outputs the answer. To get the desired direct sum property, we have a chain of reductions leading to the SUM-EQUAL problem which we compute the information complexity of.

Specifically, we consider a composition of the Gap-Orthogonality problem on top of the SUM-EQUAL instances as well as an augmented index version of the composed problem. When we compose these problems, each coordinate of the Gap-Orthogonality problem becomes a SUM-EQUAL instance, and we show that in order to solve Gap Orthogonality, we must solve most of the SUM-EQUAL instances. Thus, we can use a direct sum to bound the information cost of the composed problem in a similar manner as in [24]. We then prove that approximating the Hamming norm reduces to the augmented index version of this, which allows us to bound its communication complexity and accordingly its streaming complexity.

In the augmented problem we additionally give a referee an index $i$ and the answers to all copies $j$, with $j > i$. Similar augmentation has been studied for $L_p$-norms [15]. This allows us to reduce our communication problem to Hamming norm approximation, and ultimately prove our data stream lower bound.

2 Preliminaries

A function $f : \Sigma^k \rightarrow \Gamma$ is called a $k$-party symmetric function if for every $(x_1, x_2, \ldots, x_k) \in \Sigma^k$ and for every permutation $\sigma$ over $\{1, 2, \ldots, k\}$, there is $f(x_1, \ldots, x_k) = f(x_{\sigma(1)}, \ldots, x_{\sigma(k)})$.

A $k$-dimensional vector space $S$ is called a product space if it can be represented as $S = S_1 \times S_2 \times \cdots \times S_k$. A distribution $\mu$ is called a product distribution if it is obtained by taking the product of $k$ independent distributions, i.e., $\mu = \mu_1 \times \mu_2 \times \cdots \times \mu_k$. 

Theory of Computing
In the $t$-player communication complexity model, there are $t$ computationally unbounded players, e.g., $P_1, \ldots, P_t$, required to compute a function $f : X_1 \times \cdots \times X_t \to Y$, where $f$ is usually a $t$-party symmetric function. Each player $P_i$ is given a private input $x_i \in X_i$ and follows a fixed protocol to exchange messages. For every input $(x_1, \ldots, x_t)$, the message transcript is denoted by $\Pi_t(x_1, \ldots, x_t)$ when all players follow the protocol $\Pi_t$ (when $\Pi_t$ is randomized, $\Pi_t(x_1, \ldots, x_t)$ is a random variable taking probabilities over players’ random coins). A deterministic protocol $\Pi_t$ computes $f$ if there is a function $\Pi_{out}$ such that $\Pi_{out} (\Pi_t^{(i)}(x_1, \ldots, x_t)) = f$, where $\Pi_t^{(i)}(x_1, \ldots, x_t)$ denotes $P_i$’s view under the execution of $\Pi_t$ on input $(x_1, \ldots, x_t)$ and for simplicity we let $\Pi_{out} (x_1, \ldots, x_t) := \Pi_{out} (\Pi_t^{(i)}(x_1, \ldots, x_t), x_t)$. A $\delta$-error randomized protocol $\Pi_t$ for $f$ requires the existence of $\Pi_{out}$ such that for all inputs $(x_1, \ldots, x_t)$, $\Pr [\Pi_{out} (x_1, \ldots, x_t) = f(x_1, \ldots, x_t)] \geq 1 - \delta$. The communication cost of $\Pi_t$ is the maximum size of $\Pi_t(x_1, \ldots, x_t)$ over all $x_1, \ldots, x_t$ and all random coins. The $t$-player deterministic communication complexity (resp. $t$-player $\delta$-error randomized communication complexity), denoted by $\text{DCC}_t(f)$ (resp. $\text{RCC}_{t, \delta}(f)$), is the cost of the best $t$-player deterministic (resp. $\delta$-error randomized) protocol $\Pi_t$ for $f$.

Given a $k$-party function $f : X_1 \times \cdots \times X_k \to Y$ and $t < k$, we define $\text{DCC}_t(f)$ and $\text{RCC}_{t, \delta}(f)$ under a worst-case partition of inputs. That is, let $f_i(z_1, \ldots, z_i) = f(x_1, \ldots, x_k)$ be defined for every partition $i_0 = 0 \leq i_1 \leq \cdots \leq i_t = k$ and $z_j := (x_{j-1+1}, \ldots, x_j)$, and the $t$-player communication complexity of $f$ is defined with respect to the worst choice of $f_i$, i.e., $\text{DCC}_t(f) := \max_{f_i} \text{DCC}_t(f_i)$ and $\text{RCC}_{t, \delta}(f) := \max_{f_i} \text{RCC}_{t, \delta}(f_i)$.

Given a $t$-party function $f$ and its input distribution $\mu$, we let $\text{DCC}_{t, \mu}^\delta(f)$ denote the communication cost of the best $t$-player deterministic protocol $\Pi_t$ computing $f$ such that $\Pr_{x \sim \mu} [\Pi_{out}(x) \neq f(x)] \leq \delta$. Similarly we define $\text{RCC}_{t, \mu}^\delta(f)$ for randomized protocols.

In the restricted one-way communication model [22, 1, 17], the $i$-th player sends exactly one message to the $(i+1)$-st player for $i \in [t-1]$ following $\Pi_t$, and then $P_t$ announces the output of $\Pi_t$ as specified by $\Pi_{out}$. Note that in this setting there are only $k-1$ messages sent by $P_1, \ldots, P_{k-1}$, and we do not count the final output announced by $P_t$ in the communication in order to best correspond to streaming algorithms. This is also known as a sententious protocol in previous work, e.g., [23]. We denote the $t$-player one-way communication complexities of $f$ by $\overrightarrow{\text{DCC}}_{t, \mu}(f)$ and $\overrightarrow{\text{RCC}}_{t, \delta}(f)$, respectively.

In the common reference string model (aka CRS model), there is a sequence of public random coins, which is by default a uniformly random binary string, accessible to all players. The obvious advantage of communication in the CRS model is that players have access to the same random string and thus save the cost of synchronizing their private coins.

A streaming algorithm is an algorithm that scans the input $(x_1, \ldots, x_m) \in \Sigma^m$ as $m$ stream input items in sequence, updates its internal memory of size $s = o(m \log |\Sigma|)$ (i.e., a streaming automaton with $2^s$ states, where the space cost of updating the internal memory is not accounted for), and finally outputs a function $f(x_1, \ldots, x_m)$ evaluated on all input items. If the best deterministic (resp. $\delta$-error randomized) streaming algorithm computes $f$ with $s$ bits of memory and $t$ passes over the data stream, then we say the deterministic (resp. $\delta$-error) streaming complexity of $f$ is $st$, denoted by $\text{DSC}(f) = st$ (resp. $\text{RSC}_\delta(f) = st$). In a popular and standard setting, a streaming algorithm scans the input stream in a single pass and only processes every input item once. The necessary amount of memory required by such single-pass algorithms is called the single-pass deterministic/$\delta$-error streaming complexity and denoted by $\overrightarrow{\text{DSC}}(f)$ and $\overrightarrow{\text{RSC}}_\delta(f)$ respectively.
Note that every streaming algorithm can be naturally interpreted as a communication protocol where
each party holds some (possibly an empty set of) input items on the stream and the messages capture the
memory updates. The connection between streaming complexity and communication complexity trivially
follows in the following lemma.

**Lemma 2.1.** For every function $f$ and error tolerance $\delta$, for every $k \in \mathbb{N}$, it holds that:

$$DSC(f) \geq \frac{1}{k} \cdot DCC_k(f), \quad RSC_\delta(f) \geq \frac{1}{k} \cdot RCC_k, \delta(f)$$

Furthermore, similar relations hold for single-pass streaming complexities
versus $k$-player one-way communication complexities:

$$\tilde{DSC}(f) \geq \frac{1}{k-1} \cdot \tilde{DCC}_k(f), \quad \tilde{RSC}_\delta(f) \geq \frac{1}{k-1} \cdot \tilde{RCC}_k, \delta(f)$$

Additionally, we let $D_{k,\delta,\mu}(f)$ denote the communication complexity of $f$ with $k$ players and $\delta$ error
under input distribution $\mu$ and $IC_{k,\delta}(f)$ denote the information complexity of $f$ with $k$ players and $\delta$ error.
We extend the notion of information complexity from [7] to this setting by summing the information costs
over all of the players and allowing some probability of returning an incorrect answer. The following
lemma from relates $IC_{k,\delta}(f)$ to $RCC_{k,\delta}(f)$:

**Lemma 2.2.** For any function $f$,

$$IC_{k,\delta}(f) \leq RCC_{k,\delta}(f)$$

This follows from the fact that the mutual information of the message $M$ that a player sends with their
input must be smaller than the number of bits in the message.

Additionally, $IC(f)$ is well-behaved in the sense that it satisfies the direct sum property:

**Theorem 2.3.** For any function $f$ and any positive integer $m$,

$$IC_{k,\delta}(f^m) \geq m \cdot IC_{k,\delta}(f)$$

where a $\delta$ probability of failure for $f^m$ is defined to mean a $\delta$ probability of failure on each instance.

This follows from the direct sum theorem on two players and no error by grouping all but player $i$ into
the referee for each $i$ and summing over the information complexities of the protocols for each $i$. Then, to
deal with the $\delta$ probability of error, we simply force the protocols to be deterministic and consider the
function only on the values for which it is correct.

Finally, we introduce the linear sketch model of communication. In this setting, we have $n$ players
and the only protocols allowed are of the following form:

There is some matrix $A$ such that if player $i$ receives input $x_i$, they compute $Ax_i$ and send the result to
the referee. The referee then computes $\sum_{i=1}^n Ax_i$ and uses the result to compute the answer. We denote the
randomized communication complexity of a function $f$ in this model by $RCC_{k,\delta}(f)$. 

We first show that 2-player one-way communication complexity is equivalent to the streaming complexity of a function \( f \). Theorem 4.1. For every symmetric function \( f : D \to \{0,1\} \) defined on \( D \subseteq \{0,1\}^n = \{(0,1)^{n/4}\} \) such that for every error tolerance \( \epsilon < 1/4 \), \( \text{DCC}^{n/4}_{t-1}(f) \leq t - 1 \) but \( \text{RCC}_{t,\delta}(f) = \Omega(n/t) \). In particular, as long as \( t = O(1) \) is a constant, we have \( \text{DCC}^{1/2}_{t-1}(f) = O(1) \) and \( \text{RSC}_\delta(f) \geq 1/t \cdot \text{RCC}_{t,\delta}(f) = \Omega(n) \).

Proof. Consider the \( t \)-party set disjointness problem \( \text{Disj}_{n/t} \) defined as follows: there are \( t \) players \( P_1, \ldots, P_t \) such that every player \( P_i \) holds a private indicator vector \( x_i \in \{0,1\}^{n/4} \) which represents a subset of \( [n/t] \), i.e., \( \text{Disj}_{n/t}(x_1, \ldots, x_t) = \bigvee_{j=1}^{n/4} (\bigwedge_{i=1}^{t} x_{i,j}) \), where \( x_{i,j} \) denotes the \( j \)-th coordinate of \( x_i \). We consider the domain \( D \) such that the vectors \( x_1, \ldots, x_t \in \{0,1\}^{n/4} \) are either (1) pairwise disjoint, or (2) sharing a unique element \( j \in [n/t] \). Let \( f \) be the function that computes \( \text{Disj}_{n/t} \) on domain \( D \).

On the one hand, it is easy to verify that \( \text{DCC}^{1/2}_{t-1}(f) \leq t - 1 \). Indeed, at least one of the \( t - 1 \) players obtains two distinct indicator vectors and hence can itself decide the output of \( f \). The communication is 1 bit per player to pass the result, and hence the total communication is bounded by \( t - 1 \) since there are \( t - 1 \) players.

On the other hand, the \( \Omega(n/t) \) lower bound for \( \text{RCC}_{t,\delta}(f) \) follows from the known lower bound for multi-player set disjointness (see [3], which was improved to optimal in [9, 12]). The lower bound for \( \text{RSC}_\delta(f) \) immediately follows by theorem 2.1.

4 Deterministic Communication and Streaming Complexity

We first show that 2-player one-way communication complexity is equivalent to the streaming complexity of single-pass streaming algorithms in the deterministic setting. In the following theorem, we assume for convenience that \( m \) is known to both players.

Theorem 4.1. For every symmetric function \( f : \Sigma^m \to \Gamma \), \( \text{DSC}^\Gamma_2(f) \leq \text{DCC}^\Gamma_2(f) \leq \text{DCC}^\Gamma_2(f) + \log m \).

Proof. Obviously, \( \text{DSC}^\Gamma_2(f) \geq \text{DCC}^\Gamma_2(f) \) since a 2-player communication protocol simulates a streaming algorithm. It remains to prove \( \text{DSC}^\Gamma_2(f) \leq \text{DCC}^\Gamma_2(f) + \log m \).

Suppose the input stream is \( (x_1, \ldots, x_m) \in \Sigma^m \), and for every partition into \( (x_1, \ldots, x_i) \) and \( (x_{i+1}, \ldots, x_m) \) there is a deterministic 2-player one-way protocol \( \Pi_2 \) computing \( f \). We design the deterministic single-pass streaming algorithm \( A \) for \( f \) by simulating 2-player one-way communication protocols under different partitions. The memory usage of \( A \) is therefore bounded by the maximum communication cost of the simulated 2-player protocols plus an index in \( [m] \) recording the number of processed items.
Notice that when processing the item \( x_{i+1} \), \( A \) has already processed \( x_1, \ldots, x_i \) and has \((m_i,i)\) in memory. \( A \) can thus reconstruct a compatible guess of \( x_1'', \ldots, x_i'' \) that would induce exactly the message \( m_i \) as in \( \Pi_2^i \), and then sets the memory to be \((m_{i+1},i+1)\) where \( m_{i+1} \) is the message sent in \( \Pi_2^{i+1} \) when \( P_1 \) has \((x_1'', \ldots, x_i'', x_{i+1})\) and \( P_2 \) has \((x_{i+2}, \ldots, x_m)\). \( A \) repeats this process for every \( i = 1, \ldots, m-1 \) and at the end it outputs \( f(x_1, \ldots, x_m) \).

Therefore, we complete the proof with \( \overrightarrow{\text{DCC}}_2(f) \leq \overrightarrow{\text{DSC}}(f) \leq \overrightarrow{\text{DCC}}_2(f) + \log m. \)

Note that the additional index \( i \) in the above simulation, which results in the additive \( \log m \) term in the upper bound, indicates which 2-player protocol should be simulated in the reconstruction, and it is implicitly shared in the 2-player communication case when \( m \) is common knowledge.

When \( m \) is not known, the memory used for the index follows any previously agreed upon encoding, which uses \( O(\log m) \) space. For functions that are well-defined for an arbitrary number of input items, e.g. the parity function, this index can be saved, and hence \( \overrightarrow{\text{DSC}}(f) = \overrightarrow{\text{DCC}}_2(f) \).

For communication complexity among more players, we establish the following corollary.

**Corollary 4.2.** For every \( k \)-party symmetric function \( f \),

\[
(k-1) \cdot \overrightarrow{\text{DCC}}_2(f) \leq \overrightarrow{\text{DCC}}_k(f) \leq (k-1) \cdot \left( \overrightarrow{\text{DCC}}_2(f) + \log k \right)
\]

**Proof.** Combining theorems 2.1 and 4.1, it follows that

\[
\overrightarrow{\text{DCC}}_k(f) \leq (k-1) \cdot \overrightarrow{\text{DSC}}(f) \leq (k-1) \cdot \left( \overrightarrow{\text{DCC}}_2(f) + \log k \right)
\]

The other direction \( \overrightarrow{\text{DCC}}_k(f) \geq (k-1) \cdot \overrightarrow{\text{DCC}}_2(f) \) holds by giving \( z_j = 0 \) to every player \( j \in \{2, \ldots, k-1\} \) in the \( k \)-player case, when the problem degenerates to 2-player communication but the same message has to be passed \( k-1 \) times. \( \square \)

Such a linear separation naturally extends to the communication complexity of \( t \)-player versus \( k \)-player protocols, as long as \( 2 \leq t < k \). Thus, the deterministic communication complexity grows linearly in the number of parties.

We remark that if every player must get a non-trivial input, i.e., at least one input element to the function, the linear growth remains for some but not all problems. For example, the communication complexity of the parity of \( k \) bits is linear in the number of players. However, to decide whether \( k \) elements in \( [k] \) are distinct, the 2-player protocol requires communication \( \log \binom{k}{2} \approx k - \log \sqrt{k} \), whereas the \( k \)-player worst-case communication grows sublinearly, i.e. for \( k \) players the communication is no more than \( \sum_{i=1}^{k-1} \log \binom{k}{i} \ll (k-1) \cdot \log \binom{k}{k/2} \).

## 5 Communication Complexity for Functions on a Product Space

### 5.1 Separations for Randomized Communication Complexity

In this section, we consider the communication cost of randomized multi-player protocols defined on product input distributions and present a \( k \log k \) versus \( t \log t \) separation between \( k \)-player and \( t \)-player communication complexity.

First we introduce the \textsc{Sum-Equal} problem (as used in Viola’s work [23]).
Definition 5.1. The $k$-player SUM-EQUAL over integers, denoted by $\text{SUM-EQUAL}_k$, requires deciding whether $\sum_{i=1}^k x_i = 0$, where each player $P_i$ is given an integer $x_i$ as his private input together with the integer $k$ as public input shared by all players. In the CRS model, an additional public random string is also known to all players. The $k$-player SUM-EQUAL over $\mathbb{Z}_m$, denoted by $\text{SUM-EQUAL}_{k,m}$, is defined similarly as $\text{SUM-EQUAL}_k$, except that the input items are drawn from $\mathbb{Z}_m$ and the summation is over $\mathbb{Z}_m$, for a publicly known $m$.

Lemma 5.2 ([23], Theorem 15 and Theorem 29). For every $k \in \mathbb{N}$, $0 \leq \delta \leq 1/3$, and in the CRS model, the $k$-player $\delta$-error communication complexity of SUM-EQUAL satisfies:

(a) For every $m \in \mathbb{N}$, $\text{REC}_{k,\delta}(\text{SUM-EQUAL}_{k,m}) = O(k \log(1/\delta))$.

(b) For every prime $p \in (k^{1/4}, 2k^{1/4})$, $\text{REC}_{k,\delta}(\text{SUM-EQUAL}_{k,p}) = \Omega(k \log k)$.

In particular, $\text{REC}_{k,\delta}(\text{SUM-EQUAL}_{k,p}) = \Theta(k \log k)$ in the CRS model if $\delta = \Omega(1/poly(k))$.

We remark that Viola’s lower bound for $\text{SUM-EQUAL}_{k,p}$ is proved for a non-product distribution $\mu_H$ whose support covers exactly a $2/p$ fraction of the whole (product) input space. Thus if a $k$-player protocol solves $\text{SUM-EQUAL}_{k,p}$ with error $\delta \leq 1/k$ on a uniform distribution $\mu$ over the whole input space, then its error with respect to $\mu_H$ is bounded by $1/k^3 < k^{-3/4}$. Notice that the two player version of $\text{SUM-EQUAL}_{k,p}$ degenerates to testing equality over $\mathbb{Z}_p$ whose upper bound is $O((\log(1/\delta) + \log \log k)$, see more details in appendix A. By theorem 5.2, the $\Omega(k)$ separation in theorem 5.3 naturally follows.

Corollary 5.3. For every prime $p \in (k^{1/4}, 2k^{1/4})$ and $\delta \leq 1/poly(k)$, there is a product distribution $\mu$ such that $\text{REC}_{k,\delta}^\mu(\text{SUM-EQUAL}_{k,p}) = \Omega(k \log k)$, $\text{REC}_{2,\delta}^{\mu,\star}(\text{SUM-EQUAL}_{k,p}) = O(\log k)$.

For a larger error tolerance, say $\delta$ is a constant, we have a stronger separation between $k$-party communication and $t$-party communication. However, the hard distribution is slightly non-product, that is, it is a product distribution on any $k-1$ out of the $k$ players.

Corollary 5.4. For every $k \in \mathbb{N}$, there is a $k$-party symmetric function $f$ such that

(a) For any product distribution $\mu$, for every $2 \leq t \leq k$ and $0 \leq \delta \leq 1/3$, $\text{REC}_{t,\delta}^\mu(f) = O(t \log(t/\delta))$.

In particular, $\text{REC}_{2,\delta}^\mu(f) = O(\log(1/\delta))$.

(b) There exists a distribution $\mu_H$, which is product on any $k-1$ out of $k$ players, for which $\text{REC}_{k,\delta}^\mu(f) = \Omega(k \log k)$ as long as $\delta \leq 1/3$.

For $\delta \geq 1/poly(t)$, the gap between $\text{REC}_{k,\delta}^\mu(f)$ and $\text{REC}_{t,\delta}^\mu(f)$ is bounded as below:

$$\text{REC}_{k,\delta}^\mu(f) / \text{REC}_{t,\delta}^\mu(f) = \Omega\left(\frac{k \log k}{t \log t}\right)$$

The outline of the proof of theorem 5.4 was given in Section 1. That is, the upper bound in part (a) follows from applying $k = t$ in the first part of Lemma 5.2, while the lower bound in part (b) follows from the second part of Lemma 5.2.

---

1 Viola’s states the lower bound for constant $\delta$, but it naturally holds for smaller $\delta$ (sometimes not tight).
5.2 Tightness of the Communication Complexity Separation

The following theorem and corollary show tightness of our separations.

**Theorem 5.5.** For every $k$-party function $f : \Sigma^k \to \Gamma$, product distribution $\mu$ over $\Sigma^k$, and error tolerance $\delta < 1/3$,
then the following holds:
\[
\overline{\text{RCC}}_{k,\delta}^\mu(f) = \begin{cases} 
O(k \log k \cdot \overline{\text{RCC}}_{2,\delta}^\chi(f)) & \text{if } \delta = \Omega(1) \\
O(k \cdot \overline{\text{RCC}}_{2,\delta}^\chi(f) + O(k \log k)) & \text{if } \delta \leq 1/k^{\Omega(1)}
\end{cases}
\]

When $\delta > 0$, we have that $\overline{\text{RCC}}_{2,\delta}^\chi(f) = \Omega(\log k)$ and thus the following holds:
\[
\overline{\text{RCC}}_{k,\delta}^\mu(f) / \overline{\text{RCC}}_{2,\delta}^\chi(f) \leq O\left(k \cdot \left(1 + \frac{\log k}{\log(1/\delta)}\right)\right) = \begin{cases} 
O(k \log k) & \text{if } \delta = \Omega(1) \\
O(k) & \text{if } \delta = 1/k^{\Omega(1)}
\end{cases}
\]

**Proof.** First we let $\Pi_0$ be the optimal $\delta$-error 2-player one-way protocol $\Pi_0$ that computes $f$ with communication $C = \overline{\text{RCC}}_{2,\delta}^\chi(f)$, and construct a new protocol $\Pi_2$ by taking the majority of $M$ independent parallel copies of $\Pi_0$ such that $\Pi_2$ has error $\varepsilon = \delta^2/(16k^2)$ and communication $CM$. Recall that $\Pi_0$ has $\delta < 1/3$, it suffices to let $t$ and $M$ be defined as in Lemma 5.6 below:
\[
t = \left\lceil \log \left(\frac{\delta/(16k^2)}{\log(4\delta(1-\delta))}\right) \right\rceil \tag{5.1}
\]
\[
M = 1 + 2t = 1 + 2\left[\frac{\log(1/\delta) + 2\log k + 4}{\log(1/\delta) + \log(1/(1-\delta)) - 2}\right] = \Theta\left(1 + \frac{\log k}{\log(1/\delta)}\right) \tag{5.2}
\]

**Lemma 5.6.** Let $t \in \mathbb{N}$ and $X_1, X_2, \ldots, X_{2t+1}$ be i.i.d. binary random variable such that $\Pr[X_i = 1] = \delta < 1/2$ for every $i \in [t]$, and let $Y = \text{Majority}\{X_1, \ldots, X_{2t+1}\}$ be the majority of all $X_i$'s. Then $\Pr[Y = 1] \leq \varepsilon$ as long as $t \geq \log(\varepsilon/\delta)/\log(4\delta(1-\delta))$.

**Proof.** For $0 < \delta < 1/2$ and $t \geq \log(\varepsilon/\delta)/\log(4\delta(1-\delta))$, we have
\[
\Pr[Y = 1] = \Pr[|\{i \mid X_i = 1\}| \geq t + 1]
= \sum_{j=t+1}^{2t+1} \binom{2t+1}{j} \delta^j (1-\delta)^{2t+1-j}
\leq \sum_{j=t+1}^{2t+1} \binom{2t+1}{j} \delta^{t+1} (1-\delta)^j
= \frac{2^{2t+1}}{2} \cdot \delta^{t+1} (1-\delta)^j = (4\delta(1-\delta))^{t+1} \cdot \delta
\leq \frac{\varepsilon}{\delta} \cdot \delta = \varepsilon
\]

The first inequality holds because $\delta < 1/2$ and hence $\delta^j (1-\delta)^{2t+1-j} \leq \delta^{t+1} (1-\delta)^j$ for $j \geq t + 1$.

The second inequality holds because $4\delta(1-\delta) < 1$ for $\delta < 1/2$, and $(4\delta(1-\delta))^{t+1} \leq (4\delta(1-\delta))^{\log(\varepsilon/\delta)/\log(4\delta(1-\delta))} = \varepsilon/\delta$. Thus, we have proved that $\Pr[Y = 1] \leq \varepsilon$ for $t \geq \log(\varepsilon/\delta)/\log(4\delta(1-\delta))$. \qed
We next turn to our direct sum theorem for Viola’s problem, which is a crucial building block for our streaming application. Note that the theorem is proved for \( \delta < 1/3 \), but lower bounds for large error tolerance such as \( \delta = 1/3 \) can be obtained using a standard error amplification argument.

**Theorem 6.1.** Let \( F : (\mathbb{Z}_p^m)^k \to \{0,1\}^m \) be the \( k \)-party function computing \( m \) independent copies of \( \text{SUM-EQUAL}_{k,p} \), where \( p \) is a prime between \( k^{1/4} \) and \( 2k^{1/4} \). For every error tolerance \( \delta \in (0,1/9) \), we say a protocol \( \Pi \) is correct with probability \( 1 - \delta \) if there is a reconstruction function \( G \) such that for every fixed \( i \in [m] \) and input \( x \in (\mathbb{Z}_p^m)^k \), \( G(i, \Pi_{out}(x)) \) equals the output of the \( i \)-th instance of \( \text{SUM-EQUAL}_{k,p} \) with probability at least \( 1 - \delta \), over the internal randomness of \( \Pi \). Then the communication cost of any \( \Pi \) which is correct with probability \( 1 - \delta \), is \( \Omega(mk \log k) \).

**Proof.** For simplicity of notation in the proof, we flip the output of \( F \), so that it outputs 0 if the input to the corresponding \( \text{SUM-EQUAL}_{k,p} \) instance sums to 0 in \( \mathbb{Z}_p \), and \( F \) outputs 1 on instances with summation other than 0.
Let $\Pi$ be an $\delta$-error randomized protocol for $F$, and let $\Pi_{\text{out}}(x)$ denote the output of $\Pi$ on input $x$. Here by “the $\delta$-error protocol” we mean that the expected error rate of $\Pi$ is bounded by $\delta$, since both $\Pi_{\text{out}}(x)$ and $F(x)$ are binary vectors in $\{0,1\}^m$. Therefore,

$$\Pr_{i\in[m]} [\Pi_{\text{out}}(x)_i \neq F_i(x)] \leq \delta$$

where the input to $F$ is partitioned as $x = (x^{(1)}, x^{(2)}, \ldots, x^{(m)}) \in \mathbb{Z}_p^{m \times k}$ such that $F_i(x) := \text{SUM-EQUAL}_{k,p}(x^{(i)})$ computes the $i$-th instance of $\text{SUM-EQUAL}_{k,p}$ for each $i \in [m]$.

We abuse notation a little in this proof and let $\| \cdot \|$ denote the Hamming weight of a not necessarily binary vector, which measures the number of non-zero coordinates of the vector. Then,

$$E[|\Pi_{\text{out}}(x) - F(x)|] \leq \delta m$$

To prove that $\text{RCC}_{k,\delta}(F) = \max_{x} |\Pi(X)| = \Omega(mk \log k)$ for the optimal $\delta$-error protocol $\Pi$, we will deduce a contradiction if $\Pi$ uses $c < \gamma m k \log k$ bits of communication, for a constant $\gamma = (1 - 9\delta)/135 > 0$ and sufficiently large $k$. Thus, we can conclude a communication lower bound of $c \geq \gamma m k \log k = \Omega(mk \log k)$.

For the purposes of a contradiction, we first convert the randomized protocol $\Pi$ into a deterministic protocol $\Pi'$ that has small error with respect to a specific distribution $\mathcal{H}$. The deterministic protocol $\Pi'$ is obtained by fixing all internal random coins of $\Pi$ so that $\Pi'$ has error rate at most $\delta$ for inputs drawn from $\mathcal{H}$.

$$E_{X \sim \mathcal{H}}[|\Pi'_{\text{out}}(X) - F(X)|] \leq \delta m$$

Since $\Pi'$ can never generate a transcript larger than the communication that $\Pi$ uses in the worst case, i.e., $|\Pi'(X)| \leq \max_{x} |\Pi(x)| = c$, it suffices to prove a communication lower bound for $\Pi'$ on inputs drawn from $\mathcal{H}$.

By Markov’s inequality, we have that for every positive constant $\epsilon > 0$,

$$\Pr_{X \sim \mathcal{H}} [ |\Pi'_{\text{out}}(X) - F(X)| > \epsilon m ] \leq \frac{E_{X \sim \mathcal{H}}[|\Pi'_{\text{out}}(X) - F(X)|]}{\epsilon m} \leq \frac{\delta}{\epsilon} \quad (6.1)$$

Now we specify the distribution $\mathcal{H}$. Let $G, B$ be defined as

$$\begin{align*}
G &:= (G_1, \ldots, G_{k-1}, -\sum_{j=1}^{k-1} G_j) \\
B &:= (B_1, \ldots, B_{k-1}, 1 - \sum_{j=1}^{k-1} B_j)
\end{align*}$$

for uniform and independent $G_j, B_j \in \mathbb{Z}_p$ for every $j \in [k-1]$. Note that: a) $\text{SUM-EQUAL}_{k,p}(G) = 1$, $\text{SUM-EQUAL}_{k,p}(B) = 0$ and hence $F_i(G) = 0, F_i(B) = 1$; b) the first $k - 1$ elements of $G$ and $B$, denoted by $G_{-k}$ and $B_{-k}$, follow the same distribution, i.e., the uniform distribution over $\mathbb{Z}_p^{k-1}$. For convenience we can write $B = (G_{-k}, 1 + G_k)$.

Let $H := G/2 + B/2$ be a mixture of $G$ and $B$ and let $\mathcal{H}$ be $m$ independent copies of $H$ as below:

$$\mathcal{H} := H^m = (G/2 + B/2)^m$$

**Theory of Computing**
Since $B = (G_{-k}, 1 + G_k)$ and $H = G/2 + B/2$, we note that

$$\mathcal{H} = \sum_{v \in \{0, 1\}^m} \frac{1}{2m}(G_{-k}^m, v + G_k^m) = (G_{-k}^m, V + G_k^m),$$

where $G_{-k}^m$ is uniformly distributed over $\mathbb{Z}_p^{m \times (k-1)}$, $G_k^m$ is a vector in $\mathbb{Z}_p^m$ such that $G_k^m = -\sum_{j=1}^{k-1} G_j^m$, and $V$ is a random variable that is uniform over $\{0, 1\}^m$, that we will think of as an element in $\mathbb{Z}_p^m$. With the above notation of $\mathcal{H}, V$, we have

$$F(\mathcal{H}) = F(G_{-k}^m, V + G_k^m) = V$$

To prove the communication lower bound of a deterministic protocol $\Pi'$ that has error probability $\leq \delta$ w.r.t. $\mathcal{H}$, we recall the following protocol decomposition by monochromatic rectangles, c.f. Claim 24 in [23] or Lemma 1.16 in [18].

**Claim 6.2** ([23], Claim 24). A $k$-player (number-in-hand) deterministic protocol using communication $\leq c$ partitions the inputs into $C \leq 2^c$ sets of inputs $R^1, R^2, \ldots, R^C$ such that

- the protocol outputs the same value on inputs in the same set, and
- the sets are rectangles: each $R^i$ can be written as $R^i = R^i_1 \times R^i_2 \times \ldots \times R^i_k$ where $R^i_j$ is a subset of the inputs of Player $j$.

For every $i \in [C]$ and rectangle $R^i$, we use the notation $R^i_{-j} := R^i_1 \times R^i_2 \times \ldots \times R^i_{j-1} \times R^i_{j+1} \times \ldots \times R^i_k$ to denote the projection of $R^i$ on to the $k - 1$ coordinates except the $j$-th one, for every $j \in [k]$. In particular, $R^i_{-k} := R^i_1 \times R^i_2 \times \ldots \times R^i_{k-1}$ denotes the first $k - 1$ coordinates. Sometimes the index $i$ of rectangle $R^i$ is clear from context, for which we simply write $R$ instead of $R^i$.

In what follows we show a contradiction when $\Pi'$ has communication $c < \gamma mk \log k$ and hence there are $C \leq 2^c < k^\gamma mk$ rectangles. The argument depends on the following lemma, which essentially guarantees that for every large rectangle, $\Pi'$ is likely to make mistakes on more than $\varepsilon m$ coordinates.

**Lemma 6.3.** For every rectangle $R$ satisfying $\Pr[\mathcal{H}_{-k} \in R_{-k}] \geq \frac{1}{\alpha C} > \frac{1}{\alpha \gamma mk}$ for which $\alpha = p^{O(1)}$, there must be a set $L \subseteq [m]$ such that $|L| = (1 - 135\gamma)m$ and $G_{-k}^L \mid G_{-k}^m \in R_{-k}$ is $\frac{|L|}{p}$-close to uniform over $\mathbb{Z}_p^{|L|}$.

Theorem 6.3 implies the following claim:

**Claim 6.4.** For every rectangle $R$ on which $\Pi'$ outputs $w \in \{0, 1\}^m$, if $\Pr[\mathcal{H}_{-k} \in R_{-k}] \geq \frac{1}{\alpha C}$, then for every $u \in R_k$ and for $\gamma, \varepsilon$ satisfying $1 - 135\gamma \geq 3\varepsilon$,

$$\Pr \left[ \mathcal{H} \in R, |F(\mathcal{H}) - w| \leq \varepsilon m \right] \leq \frac{1}{2} \Pr \left[ \mathcal{H} \in R \right] \quad (6.2)$$

For compactness of the proof of Theorem 6.1 we defer the proofs of Claim 6.4 and Lemma 6.3 to the end of this section.

Let $\mathcal{R}$ be the set of the $C$ rectangles and $\mathcal{R} \subseteq \mathcal{R}$ be the set of all large rectangles satisfying $\Pr[\mathcal{H}_{-k} \in R_{-k}] \geq \frac{1}{\alpha C} > \frac{1}{\alpha \gamma mk}$. Then for every rectangle $R \in \mathcal{R} \setminus \mathcal{R}$,

$$\Pr[\mathcal{H} \in R] \leq \Pr[\mathcal{H}_{-k} \in R_{-k}] < \frac{1}{\alpha C} \leq \frac{1}{\alpha |\mathcal{R} \setminus \mathcal{R}|}$$
Using Claim 6.4, we have
\[
\Pr_{X \sim \mathcal{D}} \left[ \left| \Pi'_{out}(X) - F(X) \right| \leq \varepsilon m \right] = \sum_{R \in \mathcal{R}} \Pr \left[ \mathcal{H} \in R, \left| F(\mathcal{H}) - \Pi'_w(R) \right| \leq \varepsilon m \right] \\
\leq \sum_{R \in \mathcal{R}} \Pr \left[ \mathcal{H} \in R, \left| F(\mathcal{H}) - \Pi'_w(R) \right| \leq \varepsilon m \right] + \sum_{R \in \mathcal{R}} \Pr[\mathcal{H} \in R] \\
\leq \frac{1}{2} \sum_{R \in \mathcal{R}} \Pr \left[ \mathcal{H} \in R \right] + \frac{1}{2} \sum_{R \in \mathcal{R}} \left| \mathcal{R} \right| - \frac{1}{2} \leq \frac{1}{2} + \frac{1}{2\alpha} \\
\]
Combining it with (6.1), we have
\[
1 - \frac{\delta}{\varepsilon} \leq \Pr_{X \sim \mathcal{D}} \left[ \left| \Pi'_w(X) - F(X) \right| \leq \varepsilon m \right] \leq \frac{1}{2} + \frac{1}{2\alpha} \implies 1 - \frac{2\delta}{\varepsilon} \leq \frac{1}{\alpha}
\]
However, the above inequality cannot be true if we set $\varepsilon = 3\delta$ and pick a constant $\alpha > 3$. Let $\gamma := (1 - 9\delta)/135$ be the constant for which we want to show $c \geq \gamma mk \log k = \Omega(mk \log k)$. Then $1 - 135\gamma = 9\delta \geq 3\varepsilon$ satisfies the condition in theorem 6.4 and $\alpha = O(1)$ satisfies the requirement in theorem 6.3.

Thus we finish the contradiction argument and complete the proof of theorem 6.1 with $RCC_{k,\delta}(F) \geq \gamma mk \log k = \Omega(mk \log k)$. \qed

Proof of Claim 6.4. Recall that $G' := G_{k}^{(L)} | G'_{k} \in R_{-k}$, $G'$ is $|L|/p$ close to the uniform distribution by Lemma 6.3. Therefore for every fixed $u \in \mathbb{Z}_p^{[L]}$,
\[
\sum_{v \in \{0,1\}^{[L]}: |v| \leq \varepsilon m} \Pr \left[ G' = u - v \right] \\
= \frac{1}{2} \left( \sum_{v: |v| \leq \varepsilon m} \Pr \left[ G' = u - v \right] + \sum_{v: |v| \geq |L| - \varepsilon m} \Pr \left[ G' = u - v \right] \right) \\
\leq \frac{1}{2} \left( \sum_{v: |v| \leq \varepsilon m} \Pr \left[ G' = u - v \right] + \sum_{v: |v| \geq |L| - \varepsilon m} \Pr \left[ G' = u - v \right] + \frac{2|L|}{p} \right) \\
\leq \frac{1}{2} \sum_{v \in \{0,1\}^{[L]}} \Pr \left[ G' = u - v \right]
\]
where the first inequality follows Lemma 6.3, and the last inequality holds since as long as $G'$ is close to the uniform distribution and $|L| = (1 - 135\gamma)m \geq 3\varepsilon m$, there is
\[
\sum_{v: |v| < |L| - \varepsilon m} \Pr \left[ G' = u - v \right] = \Omega \left( \frac{|L|}{p} \right)
\]
Recall that $u_L$ and $v_L$ denote $u$ and $v$ restricted to coordinates in the set $L$, $u_{-L}$ and $v_{-L}$ denote $u$ and $v$ restricted to coordinates not in $L$, and $G_k^{(-L)}$ denotes $G_k$ restricted to coordinates not in $L$. We then apply the above inequality and get the following bound relating probabilities on a single coordinate conditional on the rest of the coordinates being contained in the rectangle $R$:

$$
\sum_{v \in \{0,1\}^m : |v-w| \leq \varepsilon m} \Pr \left[ G^m_k = u - v \mid G^m_{-k} \in R_{-k} \right] \\
\leq \sum_{v \in \{0,1\}^m : |v_L-w_L| \leq \varepsilon m} \Pr \left[ G^m_k = u - v \mid G^m_{-k} \in R_{-k} \right] \\
= \sum_{v_L \in \{0,1\}^{|L|} : |v_L-w_L| \leq \varepsilon m} \Pr \left[ G_k^{(L)} = u_L - v_L \mid G^m_{-k} \in R_{-k} \right] \\
\cdot \sum_{v_{-L} \in \{0,1\}^{|L|}} \Pr \left[ G_k^{(-L)} = u_{-L} - v_{-L} \mid G^m_{-k} \in R_{-k}, G_k^{(L)} = u_L - v_L \right] \\
< \frac{1}{2} \sum_{v_L \in \{0,1\}^{|L|}} \Pr \left[ G_k^{(L)} = u_L - v_L \mid G^m_{-k} \in R_{-k} \right] \\
\cdot \sum_{v_{-L} \in \{0,1\}^{|L|}} \Pr \left[ G_k^{(-L)} = u_{-L} - v_{-L} \mid G^m_{-k} \in R_{-k}, G_k^{(L)} = u_L - v_L \right] \\
= \frac{1}{2} \sum_{v \in \{0,1\}^m} \Pr \left[ G^m_k = u - v \mid G^m_{-k} \in R_{-k} \right]
$$

The above inequality (6.3) implies (6.2) since:

$$
\Pr \left[ \exists \mathcal{F} \subseteq R, |F(\mathcal{F}) - w| \leq \varepsilon m \right] \\
= \Pr \left[ \exists \mathcal{F}_{-k} \subseteq R_{-k}, \exists \mathcal{F}_k \subseteq R_k, |\mathcal{F}(\mathcal{F}) - w| \leq \varepsilon m \right] \\
= \Pr \left[ G^m_{-k} \in R_{-k} \right] \cdot \sum_{v \in \{0,1\}^m} \frac{1}{2m} \Pr \left[ \mathcal{F}_k \subseteq R_k, |F(\mathcal{F}) - w| \leq \varepsilon m \mid G^m_{-k} \in R_{-k}, F(\mathcal{F}) = v \right] \\
= \Pr \left[ G^m_{-k} \in R_{-k} \right] \cdot \sum_{v \in \{0,1\}^m} \frac{1}{2m} \Pr \left[ v + G^m_k \in R_k, |v-w| \leq \varepsilon m \right] \\
= \Pr \left[ G^m_{-k} \in R_{-k} \right] \cdot \sum_{u \in R_k} \frac{1}{2m} \sum_{v \in \{0,1\}^m : |v-w| \leq \varepsilon m} \Pr \left[ v + G^m_k = u \mid G^m_{-k} \in R_{-k} \right] \\
< \frac{1}{2m} \Pr \left[ G^m_{-k} \in R_{-k} \right] \cdot \sum_{u \in R_k} \frac{1}{2} \sum_{v \in \{0,1\}^m : |v-w| \leq \varepsilon m} \Pr \left[ v + G^m_k = u \mid G^m_{-k} \in R_{-k} \right] \\
= \frac{1}{2} \Pr \left[ \mathcal{F} \subseteq R \right]
$$

Thus we complete the proof of Claim 6.4

\[ \square \]
**Proof of Lemma 6.3.** We prove this lemma inductively for the indices in $L$. In what follows, let $\delta_i := \frac{i}{p}$ for every $i \in [\ell]$. Given that $\left(G^{(1)}_k, \ldots, G^{(\ell-1)}_k\right) \mid G^m_{-k} \in R_{-k}$ is $\delta_{\ell-1}$-close to the uniform distribution over $\mathbb{Z}_p^{\ell-1}$, we will show that there exists another instance which, w.l.o.g., we label as $G^{(\ell)}_k$, for which $\left(G^{(1)}_k, \ldots, G^{(\ell-1)}_k, G^{(\ell)}_k\right) \mid G^m_{-k} \in R_{-k}$ is $\delta_\ell$-close to uniform distribution over $\mathbb{Z}_p^{\ell}$.

The base case for $\ell = 0$ is trivial. In what follows we suppose that the conditional distribution $\left(G^{(1)}_k, \ldots, G^{(\ell)}_k\right) \mid G^m_{-k} \in R_{-k}$ is already $\delta_{\ell-1}$-uniform and we do our induction for $G^{(\ell)}_k$.

First we fix $x \in \mathbb{Z}_p^{\ell-1} \times (k-1)$ for which $\Pr\left[\left(G^{(1)}_{-k}, \ldots, G^{(\ell-1)}_{-k}\right) = x \mid G^m_{-k} \in R_{-k}\right] \geq \frac{1}{\eta p^{(\ell-1)(k-1)}}$, and let $E_x$ denote the event $\left(G^{(1)}_{-k}, \ldots, G^{(\ell-1)}_{-k}\right) = x$. Then we discuss the conditional distribution of the remaining instances given $E_x$.

Let $J_x := \left\{ j \in [k-1] \mid \Pr\left[ G^m_j \in R_j \mid E_x \right] \geq \frac{1}{\beta k^{2m}} \right\}$.

Then

$$\Pr\left[ G^m_{-k} \in R_{-k} \mid E_x \right] = \prod_{j \in [k-1]} \Pr\left[ G^m_j \in R_j \mid E_x \right] \leq \prod_{j \in (k-1) \setminus J_x} \frac{1}{\beta k^{2m}} = \left( \frac{1}{\beta k^{2m}} \right)^{k-1-|J_x|} \quad \text{(6.4)}$$

On the other hand, recalling that $\left(G^{(1)}_{-k}, \ldots, G^{(\ell-1)}_{-k}\right)$ is uniformly distributed and hence $\Pr[E_x] = \frac{1}{p^{(\ell-1)(k-1)}}$, we have

$$\Pr\left[ G^m_{-k} \in R_{-k} \mid E_x \right] = \frac{\Pr\left[ G^m_{-k} \in R_{-k} \mid E_x \right]}{\Pr[E_x]}$$

$$= \Pr\left[ E_x \mid G^m_{-k} \in R_{-k} \right] \cdot \Pr\left[ G^m_{-k} \in R_{-k} \right] / \Pr[E_x]$$

$$\geq \frac{1}{\eta p^{(\ell-1)(k-1)}} \cdot \left( \frac{1}{\alpha k^{2m}} \right) \left( \frac{1}{p^{(\ell-1)(k-1)}} \right) = \frac{1}{\eta \alpha k^{2m}} \quad \text{(6.5)}$$

Combining eqs. (6.4) and (6.5) and letting $\beta \geq (\eta \alpha)^{2\gamma/k}$, we can conclude

$$\left( \frac{1}{\beta k^{2m}} \right)^{k-1-|J_x|} \geq \frac{1}{\eta \alpha k^{2m}} \implies |J_x| \geq k - 1 - \frac{\gamma m \log k + \log \eta \alpha}{2 \gamma m \log k + \log \beta} \geq \left( 1 - \frac{\gamma}{2\gamma} \right) k - 1$$

Thus the size of $J_x$ is at least $|J_x| \geq \left( 1 - \frac{\gamma}{2\gamma} \right) k - 1 = \Omega(k)$.

For every $j \in J_x$, we have $\Pr\left[ G^m_j \in R_j \mid E_x \right] \geq 1/\beta k^{2m}$ by definition of $J_x$ and hence

$$\Pr\left[ G^m_j \in R_j \mid G^m_{-k} \in R_{-k}, E_x \right] \geq \log \left( \frac{p^{m-\ell} / \beta k^{2m}}{p} \right) = (m - \ell) \log p - 2m \log k - \log \beta \quad \text{(6.6)}$$

Note that for every $i \in [m]$, $G^{(i)}$ is uniform over $\mathbb{Z}_p$ as long as $j \in [k-1]$. Thus conditioned on $E_x$ and $G^m_{-k} \in R_{-k}$, if $\exists a \in \mathbb{Z}_p$, $\Pr[G^{(i)}_j = a \mid G^m_j \in R_j, E_x] = p_a > \frac{1}{2}$ then we have an upper bound for the conditional
Then \( x \in -m \) the size of \( \text{entropy of } G_j \):

\[
H[G_j^{(i)} \mid G_j^m \in R_j, E_x] \leq p_a \log \frac{1}{p_a} + (1 - p_a) \log (p - 1) < (1 + \log (p - 1))/2
\]

Let \( I_{j,x} \) be defined as

\[
I_{j,x} := \left\{ i \in [m] \mid H_\infty \left[ G_j^{(i)} \mid G_j^m \in R_j, E_x \right] \geq 1 \right\} = \left\{ i \mid \forall a, \Pr \left[ G_j^{(i)} = a \mid G_j^m \in R_j, E_x \right] \leq \frac{1}{2} \right\}
\]

Then \( \forall i \in I_{j,x} := (\lfloor m/\ell - 1 \rfloor \mid I_{j,x}), \) \( H[G_j^{(i)} \mid G_j^m \in R_j, E_x] < (1 + \log (p - 1))/2 \), and in particular for \( i \in [\ell - 1], H[G_j^{(i)} \mid G_j^m \in R_j, E_x] = 0 \) since \( G_j^{(i)} \) is already fixed in \( E_x \).

\[
H \left[ G_j^m \mid G_j^m \in R_j, E_x \right] \leq \sum_{i=1}^m H[G_j^{(i)} \mid G_j^m \in R_j, E_x]
\]

\[
= \sum_{i \in I_{j,x}} H[G_j^{(i)} \mid G_j^m \in R_j, E_x] + \sum_{i \notin I_{j,x}} H[G_j^{(i)} \mid G_j^m \in R_j, E_x] + \sum_{i \in I_{j,x}} H[G_j^{(i)} \mid G_j^m \in R_j, E_x]
\]

\[
\leq |I_{j,x}| \cdot \log p + (m - \ell + 1 - |I_{j,x}|) (1 + \log (p - 1))/2
\]

Combining the above with the lower bound for \( H \left[ G_j^m \mid G_j^m \in R_j, E_x \right] \) in (6.6),

\[
(m - \ell) \log p - 2\gamma m \log k - \log \beta \leq |I_{j,x}| \cdot \log p + (m - \ell + 1 - |I_{j,x}|) (1 + \log (p))/2
\]

\[
\Rightarrow \left( \frac{\log p - 1}{2} \right) |I_{j,x}| \geq (m - \ell) \left( \frac{\log p - 1}{2} \right) - 2\gamma m \log k - \frac{1 + \log p}{2} - \log \beta
\]

Therefore, recalling that \( p > k^{1/4} \) and for \( \log \beta = o(\log p) = o(\log k) \), we have

\[
|I_{j,x}| \geq m - \ell - \frac{4\gamma m \log k}{\log p - 1} - O \left( \frac{\log \beta}{\log p} \right) > m - \ell - \frac{4\gamma m \log k}{\log p - 1} - o(1) > m - \ell - 18\gamma m + 1
\]

Therefore, for every \( x \in \mathbb{Z}_p^{(l-1) \times (k-1)} \) for which

\[
\Pr \left[ \left( G_{j,k}, \ldots, G_{j-k}^{(l-1)} \right) = x \mid G_{j-k}^m \in R_{j-k} \right] \geq \frac{1}{\eta_{p^{(l-1)(k-1)}}}
\]

the size of \( |J_{j,k}| \geq \left( 1 - \frac{2}{\sqrt{p}} \right) k - 1 = \Omega(k) \); and for every \( j \in J_x, |I_{j,x}| > m - \ell - 18\gamma m + 1 \) and \( |I_{j,x}| = m - \ell + 1 - |I_{j,x}| < 18\gamma m \).

That is, these three bounds hold with probability at least \( 1 - \frac{1}{\eta} \) by taking a union bound over all \( x \in \mathbb{Z}_p^{(l-1) \times (k-1)} \) where

\[
\Pr \left[ \left( G_{j-k}^{(i)} \right) = x \mid G_{j-k}^m \in R_{j-k} \right] < \frac{1}{\eta_{p^{(l-1)(k-1)}}}
\]
for \( x \sim \left( G^{(1)}_{-k}, \ldots, G^{(\ell-1)}_{-k} \right) \) \( G^m_{-k} \in R_{-k} \). In what follows we abuse notation a little by assuming \( X := \left( G^{(1)}_{-k}, \ldots, G^{(\ell-1)}_{-k} \right) \) a distribution over \( \mathbb{Z}_p^{\ell(k-1)} \) for which \( X \) satisfies all the above statements of \( J_x \) and \( I_{j,x} \). This causes at most an additional loss of \( \frac{1}{n} \) in the error probability.

Notice that the conditional distribution \( \left( G^{(1)}_{-k}, \ldots, G^{(\ell-1)}_{-k} \right) \mid G^m_{-k} \in R_{-k} \) is indeed a product distribution since \( R \) is a rectangle. That is, letting \( x = (x_1, \ldots, x_{k-1}) \) where \( x_j \in \mathbb{Z}_p^{\ell-1} \) for \( j \in [k-1] \), then \( \mathcal{E}_x \) can be decomposed into \( k-1 \) independent events \( \mathcal{E}_{x,j} \), where each \( \mathcal{E}_{x,j} \) denotes the event \( \left( G^{(1)}_{j}, \ldots, G^{(\ell-1)}_{j} \right) = x_j \) and \( \mathcal{E}_x = \cap_{j=1}^{k-1} \mathcal{E}_{x,j} \). Therefore the conditional distribution \( G^m_{-k} \mid \mathcal{E}_x \) is identical to \( G^m_{-k} \mid \mathcal{E}_{x,j} \) since the distribution of \( G^m_{-k} \) is independent from inputs of the remaining \( k-2 \) players (among the first \( k-1 \) players) in the product distribution. As a result, we have \( \Pr \left[ G^m_{j} \in R_j \mid \mathcal{E}_x \right] = \Pr \left[ G^m_{j} \in R_j \mid \mathcal{E}_{x,j} \right] \) so that \( \mathcal{E}_{x,j} \) and \( x_j \) fully determines whether \( j \in J_x \) following the definition of \( J_x \). Similarly we have \( G^m_{j} \mid \left\{ G^m_{i} \in R_i, \mathcal{E}_x \right\} \) identical to \( G^m_{j} \mid \left\{ G^m_{i} \in R_i, \mathcal{E}_{x,i} \right\} \), so that \( I_{j,x} \) is also fully determined by \( x_j \) and \( \mathcal{E}_{x,j} \).

Next we fix \( j \in [k-1] \) and pick \( x_j \in \mathbb{Z}_p^{\ell-1} \) for which \( j \in J_x \) for \( x \) extended from \( x_j \). Now we have \( \mathcal{E}_{x,j} \) and \( I_{j,x} := I_{j,x} \) containing all but a fraction of \( \leq \frac{18 \gamma m}{m-\ell+1} \) coordinates, since \( |T_{j,x,j}| < 18 \gamma m \) out of the \( m-\ell+1 \) unfixed coordinates in total. Then for \( X_j \sim \mathcal{U}_{i \in \mathbb{Z}_p} \) and \( \mathcal{J}(\cdot) \) denoting the indicator function,

\[
\sum_{i=\ell}^{m} \mathcal{J} \left( \mathbb{P}_{X_j} [i \in T_{j,x,j} \mid j \in J_x] \geq \frac{1}{3} \right) \\
\leq \sum_{i=\ell}^{m} \mathbb{P}_{X_j} [i \in T_{j,x,j} \mid j \in J_x] \\
= \sum_{i=\ell}^{m} \mathbb{P}_{X_j} [X_j = x_j \cap j \in J_x] \\
= \sum_{i=\ell}^{m} \mathbb{P}_{X_j} [X_j = x_j \mid j \in J_x] \cdot \mathbb{P}_{X_j} [X_j = x_j \cap j \in J_x] \\
= \sum_{i=\ell}^{m} \mathbb{P}_{X_j} [X_j = x_j \mid j \in J_x] \cdot \mathbb{P}_{X_j} [X_j = x_j \mid j \in J_x] < 54 \gamma m
\]

That is, for every fixed \( j \in [k-1] \), there are at least \( m-\ell+1-54 \gamma m \) coordinates \( i \in [m] \) satisfying \( \mathbb{P}_{X_j} [i \in T_{j,x,j} \mid j \in J_x] > \frac{2}{3} \), i.e., with probability \( \frac{2}{3} \), \( G^{(i)}_{j} \) satisfies \( H_{\infty} \left( G^{(i)}_{j} \mid G^m_{-k} \in R_j, \mathcal{E}_x \right) \geq 1 \) for a randomly selected \( x_j \) conditioned on that \( j \in J_x \) specifies a big component in the rectangle. This is exactly the probability that the \( i \)-th coordinate \( G^{(i)}_{j} \) of \( G^m_{-k} \) can be decomposed into a convex combination of a uniform distribution over 2 elements.

Now we have at least \((m-\ell+1-54 \gamma m)(k-1) \) pairs of \((i,j)\) \( \in \{\ell, \ell+1, \ldots, m\} \times [k-1] \) satisfying the above condition \( \mathbb{P}_{X_j} [i \in T_{j,x,j} \mid j \in J_x] > \frac{2}{3} \), which means at least one fixed \( i \) must appear in \( \frac{(m-\ell+1-54 \gamma m)(k-1)}{m-\ell+1} = \left( \frac{54 \gamma m}{m-\ell+1} \right)(k-1) \) pairs for different \( j \in [k-1] \) by a standard averaging.
argument. Without loss of generality we may assume \( i = \ell \), and let \( G'' := (G''_1, \ldots, G''_k) \) denote the conditional distribution of \( G^{(i)} \), i.e., each \( G''_j := G''_j \bigm/ \{ G''_j \in R, \mathcal{E}_x \} \) denotes the conditional distribution of \( G^{(i)}_j \). Recalling that \( |J_x| \geq \left( 1 - \frac{\gamma}{2\gamma} \right) k - 1 \), the number of elements in \( |J_x| \) hit by those pairs containing \( \ell \) is at least

\[
\left( 1 - \frac{\gamma}{2\gamma} \right) k - 1 + \left( 1 - \frac{54\gamma m}{m - \ell + 1} \right) (k - 1) - (k - 1) \geq \left( 1 - \frac{\gamma}{2\gamma} - \frac{54\gamma m}{m - \ell + 1} \right) k - 1 = \Omega(k)
\]

We say the pair \((i, j)\) is good for \( x \) if \( j \in J_x \) and \( i \in I_{j,x} \). Then recalling that \( |J_x| \geq \left( 1 - \frac{\gamma}{2\gamma} \right) k - 1 \), the expected number of good \((\ell, j)\) over \( x \sim X \) is lower bounded as follows.

\[
\mathbb{E}_x \left[ \sum_{j=1}^{k-1} I((\ell, j) \text{ is good for } x) \right] \geq \mathbb{E}_x \left[ \sum_{j=1}^{k-1} \mathbb{E}_x [I((\ell, j) \text{ is good for } x)] \right] = \mathbb{E}_x \left[ \sum_{j=1}^{k-1} \mathbb{E}_x [\mathbb{P}(\ell \in I_{j,x}, j \in J_x)] \right] \geq \mathbb{E}_x \left[ \sum_{j \in J_x} \mathbb{P}((\ell \in I_{j,x}, j \in J_x)] \right] \geq \frac{2}{3} \left( 1 - \frac{\gamma}{2\gamma} - \frac{54\gamma m}{m - \ell + 1} \right) k - 1
\]

By a Chernoff bound it implies

\[
\mathbb{P}_x \left[ \left( \sum_{j \in [k-1]} I((\ell, j) \text{ is good for } x) \right) \leq \frac{1}{3} \left( 1 - \frac{\gamma}{2\gamma} - \frac{54\gamma m}{m - \ell + 1} \right) k \right] \leq \exp \left( -\Omega \left( 1 - \frac{\gamma}{2\gamma} - \frac{54\gamma m}{m - \ell + 1} \right) k \right)
\]

Let \( \delta' = \exp \left( -\Omega \left( 1 - \frac{\gamma}{2\gamma} - \frac{54\gamma m}{m - \ell + 1} \right) k \right) \) be an upper bound of this error probability. Then with probability at least \( 1 - \delta' \), the conditional distribution \( G''_j \) can be decomposed into a convex combination of uniform distributions over two distinct elements for at least \( \frac{1}{3} \left( 1 - \frac{\gamma}{2\gamma} - \frac{54\gamma m}{m - \ell + 1} \right) k \) indices \( j \in [k-1] \).

Next we show that conditioned on the above decomposition, which happens with probability \( \geq 1 - \delta' \), the conditional distribution \( G''_k \) is close to uniform by the following claim.

**Claim 6.5** (Claim 31 in [23]). Let \( p \) be a prime number. Let \( X \) be the sum of \( t \) independent random variables each uniform over \( \{a_i, b_i\} \subseteq \mathbb{Z}_p \) for \( a_i \neq b_i \). Then \( X \) modulo \( p \) is \( \delta \leq 0.5\sqrt{p}\exp \left( -\Omega \left( t/p^2 \right) \right) \) close to uniform.

Plugging our parameters into the above claim and following exactly the same argument as in [23] \( (G''_k) \) is \( \delta'' \)-close to uniform if every component in the above convex decomposition of \( G''_k \) is \( \delta'' \)-close.
to uniform), the statistical distance between $G_k'' = -\sum_{j=1}^{T-1} G_j''$ and the uniform distribution over $\mathbb{Z}_p$ is bounded by

$$\delta'' \leq 0.5 \sqrt{p} \exp \left( -\Omega \left( \frac{1}{3} \left( 1 - \frac{\gamma}{2\gamma} - \frac{54\gamma m}{m - \ell + 1} \right) k/p^2 \right) \right)$$

$$= \exp \left( -\Omega \left( \left( 1 - \frac{\gamma}{2\gamma} - \frac{54\gamma m}{m - \ell + 1} \right) \sqrt{k} \right) \right)$$

Putting it all together, we conclude that $G_k^{(\ell)} \mid \{G^{(1)}_{m}, G^{(1)}, \ldots, G^{(\ell-1)}_{k}\}$ is close to uniform, which implies $\left( G^{(1)}_{m}, \ldots, G^{(\ell-1)}_{k}, G^{(\ell)}_{k}\right) \mid G_{m:k} \in R_{-k}$ is also close to uniform. Moreover, its statistical distance to uniform is bounded by

$$\delta_\ell \leq \delta_{\ell-1} + \frac{1}{\eta} + \delta' + \delta''$$

Let $\eta = 2p$ and $\beta = 2 \geq (\eta \alpha)^{2\gamma/(\gamma^k)} = 2^{O(\eta \alpha)}$ for $\alpha = p^{O(1)}$. Then for sufficiently large $km$ the above induction argument goes through for $\ell \leq (1 - 135\gamma)m$, with error $\delta', \delta''$ bounded by

$$\delta' = \exp(-\Omega(k)), \delta'' = \exp(-\Omega(\sqrt{k})) \iff 1 - \frac{\gamma}{2\gamma} - \frac{54\gamma m}{m - \ell + 1} \geq 0.1 \iff \ell \leq (1 - 135\gamma)m + 1$$

Therefore the conditional distribution $\left( G^{(1)}_{k}, \ldots, G^{(\ell-1)}_{k}, G^{(\ell)}_{k}\right) \mid G_{m:k} \in R_{-k}$ is $\delta_\ell$-close to uniform for $\delta_\ell$ bounded by $\frac{\ell}{p}$ as follows:

$$\delta_\ell \leq \delta_{\ell-1} + \frac{1}{\eta} + \delta' + \delta'' \leq \frac{\ell - 1}{p} + \frac{1}{2p} + \exp \left( -\Omega \left( \sqrt{k} \right) \right) \leq \frac{\ell}{p}$$

Thus we have proved the induction hypothesis for every $\ell \leq (1 - 135\gamma)m$. Let $L$ be the first $(1 - 135\gamma)m$ indices as in the induction hypothesis, we complete the proof of Lemma 6.3 for $|L| = (1 - 135\gamma)m$ and statistical distance $\frac{|L|}{p}$. \qed

### 7 Lower bound for Hamming Norm Estimation

In this section we present a space lower bound for single-pass streaming algorithms for $(1 \pm \varepsilon)$-approximating the Hamming norm $L_0$ in the strict turnstile model, which is denoted by $T_{\varepsilon}$ as in section 1.1.1.

Formally, in the Hamming norm estimation problem there is an underlying vector $(x_1, \ldots, x_N)$ which starts from the all zero vector and processes up to $m$ updates each of the form $(i, v) \in [N] \times \{\pm M\}$. The update $(i, v)$ means one should add $v$ to the $i$-th coordinate $x_i$ in the vector $x$. After processing all $m$ updates, we have $\|x\|_0 = |\{i \mid x_i \neq 0\}|$ and we want to output a number within $(1 \pm \varepsilon)\|x\|_0$ with probability $\geq 2/3$. We additionally assume all players have access to a heavy hitters oracle, which tells them whether the frequency of a given coordinate is greater than $T$. This is a generalization of the case without a heavy
hitters oracle, where we simply let $T = mM$ and we know that all frequencies are guaranteed to be smaller. The strict turnstile model guarantees that $x_i \geq 0$ for all $i \in [N]$ at all positions in the stream, in which case it suffices to prove the space lower bound in the simultaneous communication model following the reduction in Theorem 4.1 of [2]. Furthermore, it is also guaranteed that for every $i \in [N]$, $x_i \leq \text{poly}(n)$ at the end of the stream. In this setting, the algorithm of [16] approximates $\|x\|_0$ up to a $(1 \pm \varepsilon)$ factor with $O\left(\varepsilon^{-2} \log(N) \left(\log(1/\varepsilon) + \log \log(T)\right)\right)$ bits of space\(^2\), as long as $\varepsilon > 0$.

We first note that solving distinct elements with a heavy hitters oracle reduces to solving distinct elements given a threshold on the frequency of the coordinates. As such, we will solve the complexity question of space complexity given a threshold $T$ for the frequency.

**Theorem 7.1.** The space complexity of $(1 \pm \varepsilon)$ approximating $L_0$ with probability at least $2/3$ in a strict turnstile stream with access to a heavy hitters oracle with a threshold of $T > 1$ is $\Omega(\varepsilon^{-2} \log n \log \log T)$.

We note that the assumption $T > 1$ is necessary for this bound to be well defined. When $T = 1$, the heavy hitters oracle tells us exactly whether or not the frequency of a coordinate is $0$ at the end of the stream, so the complexity is $\Theta(\log n)$. This lower bound follows as we need to write down the answer and the upper bound follows as we can directly count the elements with nonzero frequency.

To prove this theorem, we first prove the following lemma:

**Lemma 7.2.** The space complexity of $(1 \pm \varepsilon)$ approximating $L_0$ with probability at least $2/3$ in a strict turnstile stream with access to a heavy hitters oracle with a threshold of $T > 1$ is at least $RSC^{T/2/3}(T)$.

**Proof.** Suppose we have an algorithm $A$ which gives us a $(1 \pm \varepsilon)$ approximation of $L_0$ in a strict turnstile stream with access to a heavy hitters oracle with a threshold of $T$.

Now, if we are given an input where the maximum frequency of any element is at most $T$, then we can go through our input and do exactly what $A$ would do for everything other than calls to the heavy hitters oracle. If $A$ would make a call to a heavy hitters oracle, we just treat the answer as $0$ without making this query and proceed as $A$ would.

Since we assumed the input has a maximum frequency of $T$, the heavy hitters oracle would return $0$ for every element, so this would give us the same answer as $A$, and by correctness of $A$, it’s a $(1 \pm \varepsilon)$ approximation.

Now, we will state and prove our main theorem:

**Theorem 7.3.** For error tolerance $\varepsilon < 1/3$ and $\varepsilon = \max\left\{\Omega\left(\sqrt{\frac{\log k}{k}}\right), \frac{1}{N0^{\varepsilon}}\right\}$, any single-pass streaming algorithm solving $T_\varepsilon$ with probability $\geq 2/3$ in the strict turnstile model must use $\Omega\left(\varepsilon^{-2} \log(N) \log \log(T)\right)$ bits of space.

First we introduce GAP-ORT and GAP-ORT-SUM-EQUAL:

---

\(^2\)Indeed, their algorithm stores $O(\varepsilon^{-2} \log N)$ counters modulo primes that are each $O(\log(1/\varepsilon) + \log \log(T))$ bits in magnitude, and it does not matter how large the values of $x_i$ are at intermediate positions in the stream.
Definition 7.4. In the $c$-Gap-ORT$_n$ problem, we have two players Alice and Bob. They each have as input a vector in $\{0,1\}^n$ and we wish to compute

$$c\text{-Gap-ORT}_n(x,y) = \begin{cases} 1, & \left| \sum_{i \in n} \text{XOR}(x_i, y_i) \right| \leq \frac{n}{2} \geq 2c\sqrt{n}, \\ 0, & \left| \sum_{i \in n} \text{XOR}(x_i, y_i) \right| \leq \frac{n}{2} \leq c\sqrt{n}, \end{cases}$$

and otherwise, it can return anything.

Definition 7.5. In the $c$-Gap-ORT-SUM-EQUAL$_n$ problem, we have two players Alice and Bob. They each have as input a vector in $\mathbb{Z}^n$ and we wish to compute

$$c\text{-Gap-ORT-SUM-EQUAL}_n(x,y) = \begin{cases} 1, & \left| \sum_{i \in n} \delta(x_i + y_i) \right| \leq \frac{n}{2} \geq 2c\sqrt{n}, \\ 0, & \left| \sum_{i \in n} \delta(x_i + y_i) \right| \leq \frac{n}{2} \leq c\sqrt{n}, \end{cases}$$

and otherwise, it can return anything. We let $\delta(x_i + y_i)$ denote the indicator function which is 1 if $x_i + y_i = 0$ and 0 otherwise.

Additionally, we will let $\text{SUM-EQUAL}^{m,a}_{k,\delta}$ denote $m$ independent instances of $\text{SUM-EQUAL}_{k,\delta}$, and our protocol needs to be able to solve at least $am$ of these instances correctly with probability at least $1 - \delta$.

Lemma 7.6. for every $k \in \mathbb{N}$, $0 \leq \delta \leq 1/2$, and $n \geq c^2/100\varepsilon^2 = n'$,

$$\text{RCC}_{k,\delta}^{\text{LIN,T}}(10\varepsilon\sqrt{n}\text{-Gap-ORT-SUM-EQUAL}_n) \geq \text{RCC}_{k,\delta}^{\text{LIN,T}}(c\text{-Gap-ORT-SUM-EQUAL}_{n'})$$

Proof of theorem 7.6. Given $n' = c^2/100\varepsilon^2$ and an input instance of $c$-Gap-ORT-SUM-EQUAL$_n$ with underlying SUM-EQUAL problems outputting $x' \in \{0,1\}^n$, we create the new input to $10\varepsilon\sqrt{n}$-Gap-ORT-SUM-EQUAL$_n$ by taking $100\varepsilon^2 n/c^2$ copies of each coordinate, with results of underlying problems being $x \in \{\pm 1\}^n$.

As a result, $\sum_{j=1}^{n'} x_j = \frac{100\varepsilon^2 n}{c^2} \cdot \sum_{j=1}^{n} x'_j$.

If $|\sum_j x'_j| \leq c\sqrt{n'}$, then $|\sum_j x_j| \leq 10\varepsilon n$, and on the other hand $|\sum_j x'_j| \geq 2c\sqrt{n'}$ implies $|\sum_j x_j| \geq 20\varepsilon n$.

Thus, any $k$-player $\delta$-error simultaneous communication protocol for $10\varepsilon\sqrt{n}$-Gap-ORT-SUM-EQUAL$_n$ immediately implies a $k$-player $\delta$-error simultaneous communication protocol for $c$-Gap-ORT-SUM-EQUAL$_{n'}$.

Since all we are doing is copying coordinates, this does not change the threshold.

Theorem 7.7. Given some simultaneous communication protocol $\Pi$ with two players that solves 1-Gap-ORT-SUM-EQUAL$_n$ when each SUM-EQUAL instance has input drawn from its hard distribution $\mu$, there exists a protocol $\Pi'$ such that $\text{RCC}_{2,\delta}^{\text{LIN}}(\Pi') \leq O(\text{RCC}_{2,\delta}^{\text{LIN}}(\Pi))$ which solves $\Omega(n)$ of the individual sum-equal instances with probability at least $\frac{1}{2} + \beta$ for some constant $\beta > 0$.

Proof. Suppose $\Pi$ is a protocol that solves 1-Gap-ORT-SUM-EQUAL$_n$. Now, if Alice has input $X = (x_1,x_2,\ldots,x_n)$ and Bob has input $Y = (y_1,y_2,\ldots,y_n)$ to 1-Gap-ORT-SUM-EQUAL$_n$, we define a corresponding instance of 1-Gap-ORT$_n$ where Alice gets input $X' = (x'_1,x'_2,\ldots,x'_n)$ and Bob gets input $Y' = (y'_1,y'_2,\ldots,y'_n)$ where $y'_i = 0$ with probability 1 if $y_i < M/2$, probability $\frac{1}{2}$ if $y_i = M$, and probability 0 otherwise where $M$ is defined as in the proof of theorem 7.9, and $x'_i = 1 - y'_i$ iff $x_i + y_i = 0$.
Thus, we get that $p_1 - G$ must have $p_0 < 1$ such that there are at least $\alpha n$ indices $j$ such that $I(x_j' ; M, Y) \geq \alpha$ for some constant $\alpha > 0$.

Now, if we let $M$ be the message sent by Alice to Bob in protocol $\Pi$, then

$$I(X' ; M, Y) \geq IC(1\text{-Gap-Ort}_n) = \Omega(n)$$

since Bob can solve 1-Gap-Ort$_n$ where Alice has input $X'$ and Bob has input $Y'$ when he has access to $(M, Y)$ by returning the answer to 1-Gap-Ort-Sum-Equal$_n$ using protocol $\Pi$ with input $Y$ after being sent the message $M$.

We now note that $X'$ is $n$ iid uniformly random bits. As such,

$$I(X' ; M, Y) = \sum_{i=1}^{n} I(x_i' ; M, Y | x_1, x_2, \ldots, x_{i-1}) \geq \sum_{i=1}^{n} I(x_i' ; M, Y).$$

Each of these terms is upper bounded by 1, so in order for the sum to be $\Omega(n)$, there exists some constant $c > 0$ such that there are at least $cn$ indices $j$ such that $I(x_j' ; M, Y) \geq \alpha$ for some constant $\alpha > 0$.

Now, let

$$J = \{ j \ | I(x'_j ; M, Y) \geq \alpha \}.$$

We claim that the transcript of $\Pi$ must contain the solution to the $j^{th}$ Sum-Equal instance with probability at least $\frac{1}{2} + \beta$ for a constant $\beta > 0$ for each $j$. To see this, we note that Bob has as input $Y$ for 1-Gap-Ort-Sum-Equal$_n$ so he can compute $y'_j$. Then, we note that

$$H(x_j' | M, Y) = H(x_j') - I(x_j' ; M, Y) \leq 1 - \alpha$$

Since $x_j' \in \{0, 1\}$, let $Pr[x_j' = 0 | M, Y] = p$. Then, if $p = 0$, the entropy is 0 so this is satisfied for any $0 < \alpha \leq 1$. If $p > 0$, we have

$$-(p \log p + (1-p) \log (1-p)) \leq 1 - \alpha.$$

Since this is symmetric about $p = \frac{1}{2}$ and cannot be satisfied by $p = \frac{1}{2}$ since $\alpha > 0$, we assume WLOG that $p < \frac{1}{2}$, in which case the entropy monotonically decreases as $p$ decreases. Now, we claim that we must have $p < \frac{1}{2} - \frac{\alpha}{2}$. It suffices to show that

$$-(\left(\frac{1-\alpha}{2}\right) \log \left(\frac{1-\alpha}{2}\right) + \left(\frac{1+\alpha}{2}\right) \log \left(\frac{1+\alpha}{2}\right)) \geq 1 - \alpha$$

Simplifying this expression yields the solution

$$0 < \alpha < 1.$$  

Thus, for $0 < \alpha < 1$, we must have $p < \frac{1-\alpha}{2}$. If $\alpha = 1$, then the entropy is 0 so we must have $p = 0$. Thus, we get that $p \leq \frac{1-\alpha}{2}$.
By symmetry, we thus have that either
\[ \Pr[x_j = 0 \mid M, Y] \leq \frac{1 - \alpha}{2} \]

or
\[ \Pr[x_j = 0 \mid M, Y] \geq \frac{1 + \alpha}{2}. \]

In the former case, Bob lets \( \hat{x}_j = 1 \) and in the latter case, Bob lets \( \hat{x}_j = 0 \). Bob then computes \( y_j' \) from \( y_j \). Then, if \( y_j' = y_j \), Bob concludes that \( x_j + y_j \neq 0 \) and if \( \hat{x}_j = 1 - y_j' \), Bob concludes that \( x_j + y_j = 0 \). By construction, this succeeds with probability at least \( \frac{1 + \alpha}{2} \), and all we did was run \( \Pi \) and compute the value from the transcript.

**Corollary 7.8.** When \( \alpha \) and \( c \) are the constants from the proof of theorem 7.7,
\[
D_{2, \delta, \mu'}^{LIN,T} (1\text{-GAP-ORT-SUM-EQUAL}_{\varepsilon^{2}/100}) \geq D_{2,(1+\alpha)/2, \mu}^{LIN,T} (\text{SUM-EQUAL}_{\varepsilon^{2}/100,c})
\]

**Proof.** This follows directly from theorem 7.7. Each instance of Sum-Equal corresponds to a single coordinate from 1-GAP-ORT-SUM-EQUALso their frequencies must all be bounded by \( T \) as well.

**Theorem 7.9.** When \( \delta < \frac{1}{2} \) and \( a \) is some constant fraction,
\[
IC_{k, \delta}^{T}(\text{SUM-EQUAL}_{k}^{n', a}) \geq \Omega(n' \log \log T)
\]

(7.1)

where \( \text{SUM-EQUAL}_{k}^{n', a} \) is the problem where we are given \( n' \) independent instances of Sum-Equal and we are asked to solve \( an' \) of them with probability \( 1 - \delta \) each.

The proof of this theorem can be found in appendix B.

**Corollary 7.10.** For the input distribution \( \mu \) defined in the proof of theorem 7.9, \( \delta < \frac{1}{2} \), and \( 0 < a < 1 \),
\[
D_{2, \delta, \mu}^{LIN,T} (\text{SUM-EQUAL}_{2}^{n', a}) \geq \Omega(n' \log \log T)
\]

**Proof.** If we plug \( k = 2 \) into (7.1), we get
\[
D_{2, \delta, \mu}^{LIN,T} (\text{SUM-EQUAL}_{2}^{n', a}) \geq IC_{2, \delta}^{T}(\text{SUM-EQUAL}_{2}^{n', a}) \geq \Omega(n' \log \log T)
\]
since by definition \( \mu \) is the hard distribution that we got the information complexity bound from.

**Definition 7.11.** The Aug-Index-GoSe\( _{n,k}^{t} \) problem consists of \( t \) independent instances of \( \varepsilon \sqrt{n} \)-Gap-Ort-Sum-Equal\( _{a} \), denoted \( g_1, g_2, \ldots, g_t \), with \( k \) players and \( n \) coordinates each. In this problem, the referee is asked to estimate \( g_i \) based on an index \( i \in [t] \) together with the auxiliary information of \( f_{i+1}, \ldots, f_t \), where for convenience we let \( f_i \in [\pm n] \) denote the bias of the number of underlying Sum-Equal\( _{k} \) instances outputting 1 in \( g_i \).

**Theorem 7.12.** \( \text{RCC}_{k, 1/3}^{LIN,T}(T_{\varepsilon}) \geq \text{RCC}_{k, 0.4}^{LIN,T}(\text{AUG-INDEX-GoSe}) \)
Proof. We let the $i$-th $\varepsilon\sqrt{n}$-GAP-ORT-SUM-EQUAL$_n$ instance $g_i$ in the AUG-INDEX-GOSE$^{T}_{n,k}$ problem have frequency $100^{i-1}$, i.e., each element in $g_i$ is counted $100^{i-1}$ times (as that many distinct elements). Thus the universe contains $N := n + 100^{0} \cdot n + \cdots + 100^{1-1} \cdot n \leq 100^{0} \cdot n/99$ distinct elements in total, which is $N \leq n^{1.01}$ for sufficiently small $t$ (and hence $1/n^{0.49} > 1/\sqrt{n}$). The final Hamming norm is a weighted sum $F' := \sum_{i=1}^{t} 100^{i-1} f_i$. The advantage of $F'$ is hence $F := 2F' - N = \sum_{i=1}^{t} 100^{i-1} f_i$.

Then we invoke the simultaneous communication protocol for $T_{\varepsilon}$ to estimate $F'$, which returns a value $\tilde{F}'$ satisfying $(1 - \varepsilon)F' \leq \tilde{F}' \leq (1 + \varepsilon)F'$. Translating to the advantage we get $|F - \tilde{F}'| \leq 2\varepsilon F' \leq 2\varepsilon N$.

From this approximated value $\tilde{F}$, together with the index $i$ and auxiliary information $f_{i+1}, \ldots, f_t$, we need to determine the output value of $g_i$. Since the influence of $f_j$ with $j > i$ can be precisely removed from $F$ before getting the approximated norm $\tilde{F}$, in what follows it suffices to consider the estimation of $g_i$ when the index is indeed $i = t$. Recall that $F = 100^{i-1} f_i + \sum_{i=1}^{t-1} 100^{i-1} f_i$, and thus $\tilde{F}$ is also an approximation of $100^{i-1} f_i$ as long as the additive error $\sum_{i=1}^{t-1} 100^{i-1} f_i$ is bounded.

Let the input distribution to every $f_i$ be padded from the 1-GAP-ORT-SUM-EQAUL$_{t-2}$ distribution $\mu'$ as in theorem 7.7, where the coordinates are iid bits drawn uniformly from $\{0, 1\}$. Thus, each $f_i$ has expectation 0 and variance $25\varepsilon^2 n^2$. It immediately follows by Cheby’s inequality that $\Pr[|f_i| \geq 50\varepsilon n] \leq 1/100$. Similarly, $\Pr[|f_i| \geq 50\varepsilon n] \leq 1/100$. Therefore,

$$\Pr \left[ \sum_{i=1}^{t-1} 100^{i-1} f_i > 100^{i-1} \varepsilon n \right] \leq \sum_{i=1}^{t-1} \Pr[|f_{i-j}| > 50i \varepsilon n] \leq \sum_{i=1}^{t-1} \frac{1}{100^i} \leq \frac{1}{99} \tag{7.2}$$

where the first inequality holds because if $|f_{i-j}| \leq 50\varepsilon n$ for every $i$, then $|\sum_{i=1}^{t-1} 100^{i-1} f_i| \leq \sum_{i=1}^{t-1} 100^{i-1} \times 50\varepsilon n \leq \frac{50\varepsilon n}{100^i} \sum_{i=1}^{t-1} 2^i < 100^{i-1} \varepsilon n$.

Notice that as long as $\tilde{F}$ is a $(1 \pm \varepsilon)$-approximation of $F$, we must have $|\tilde{F} - F| \leq 2\varepsilon N$. Furthermore suppose that we return 0 if $\tilde{F} < 15 \cdot 100^{i-1} \varepsilon n$ and 1 if $\tilde{F} \geq 15 \cdot 100^{i-1} \varepsilon n$. Since we know that $N \leq 100^n/99$, we have $2\varepsilon N \leq 2\varepsilon 100^n/99 < 3 \cdot 100^{i-1} \varepsilon n$.

So in particular, if $T_{\varepsilon}$ succeeds, if $g_i = 0$, we have $|f_i| \leq 10 \cdot 100^{i-1} \varepsilon n$, so $|F| = |\sum_{i=1}^{t-1} 100^{i-1} f_i| \leq 11 \cdot 100^{i-1} \varepsilon n$ with probability at least $\frac{98}{99}$. Then, $|\tilde{F}| < 14 \cdot 100^{i-1} \varepsilon n$ and our algorithm succeeds.

Similarly, if $g_i = 1$, we have $|f_i| \geq 20 \cdot 100^{i-1} \varepsilon n$. Thus, with probability at least $\frac{98}{99}$, $|F| = |\sum_{i=1}^{t-1} 100^{i-1} f_i| \geq 19 \cdot 100^{i-1} \varepsilon n$, so $|\tilde{F}| > 16 \cdot 100^{i-1} \varepsilon n$ and our algorithm succeeds.

Thus, if $T_{\varepsilon}$ succeeds with probability $\frac{2}{3}$, the above algorithm succeeds with probability $\frac{2}{3} \cdot \frac{98}{99} > 0.6$. Thus we can determine the value of $g_i$ with probability $\geq 0.6$. The thresholds stay the same because all we did to change the input was copy coordinates, which does not change the frequencies. Hence,

$$\text{RCC}^{\text{LIN.}}_{k, 1/3}(T_{\varepsilon}) \geq \text{RCC}^{\text{LIN.}}_{k, 0.4}(\text{AUG-INDEX-GOSE}^T_{n,k}).$$

Finally, to bound the complexity $\text{RSC}^{T}_{k, 0.4}(T_{\varepsilon})$, we conclude as follows:
When we solve AUG-INDEX-GOSE, we claim that in order to solve AUG-INDEX-GOSE, with probability at least 0.6 on every input, we must solve every instance of GAP-ORT-SUM-EQUAL with probability at least 0.6.

To see this, suppose we have a protocol that solves AUG-INDEX-GOSE with probability at least 0.6. Let the index corresponding to this index be \( j \). Then, we can choose our input such that the index is \( j \), and the probability of success of our protocol is the same as the probability that the protocol solves instance \( j \) correctly. By assumption, our protocol succeeds with probability at least 0.6, so it solves the \( j \)'th instance of GAP-ORT-SUM-EQUAL with probability at least 0.6. This holds for every \( j \), so our protocol must solve every instance of GAP-ORT-SUM-EQUAL with probability at least 0.6.

Thus,

\[
\begin{align*}
RCC_{k,2/3}^{LIN,T}(T_\epsilon) & \geq RCC_{k,0.4}^{LIN,T} \left( \text{AUG-INDEX-GOSE}_i \right) \\
& \geq RCC_{k,0.4}^{LIN,T} \left( 10\sqrt{n} \cdot \text{GAP-ORT-SUM-EQUAL}_i \right) \\
& \geq RCC_{k,0.4}^{LIN,T} \left( 1-\text{GAP-ORT-SUM-EQUAL}_i \right) \\
& \geq \frac{1}{k} \sum_{i=1}^{T} \left( \text{SUM-EQUAL}_i \right) \\
& \geq \Omega \left( ktn \log \log T \right) = \Omega \left( \epsilon^{-2} k \log n \log \log T \right)
\end{align*}
\]

so

\[
RSC_{k,0.4}(T_\epsilon) \geq \frac{1}{k} \sum_{i=1}^{T} \text{RCC}_{k,0.4}^{LIN,T}(T_\epsilon) \geq \Omega \left( \epsilon^{-2} \log n \log \log T \right)
\]

\( \square \)

### A Communication Upper Bound for EQUALITY

The standard \( \delta \)-error protocol solving the EQUALITY problem starts by sending and comparing the digest under a random hash function \( h : [p] \to [q] \) where \( q = O \left( \delta^{-1} \log p \right) \). For example, let \( q \) be a random prime drawn from the interval \([\delta^{-2} \log^2 p, 3\delta^{-2} \log^2 p]\) and let \( h \) compute a number modulo \( q \). By the prime number theorem there are at least \( 2\sqrt{N} \) primes in the interval \([N, 2N]\), which implies the existence of \( 2\delta^{-1} \log(p) \) distinct primes in that range. For any two distinct numbers \( x, y \in \mathbb{Z}_p \), since \( z = x - y \) has no more than \( \log |z| \leq \log p \) prime factors, the error probability of the protocol is bounded by the collision probability of \( h \) as follows:

\[
Pr_q \left[ h(x) = h(y) \right] = Pr_q \left[ x \equiv y \mod q \right] = Pr_q \left[ q \mid (x-y) \right] \leq \frac{\log p}{2\delta^{-1} \log p} < \delta
\]

The communication is a message of the form \((h, h(x))\) (indeed \((q, x \mod q)\) in the above example), whose length is at most \( 2 \log q = O \left( \log(1/\delta) + \log \log p \right) = O \left( \log(1/\delta) + \log \log k \right) \) bits. In particular
this is an upper bound for one-way communication protocols computing \textsc{Equality}. Recalling that 
\( p = \Theta(k^{1/4}) \), we can conclude
\[
\text{RCC}_{2,\delta}(f) \leq \overline{\text{RCC}}_{2,\delta}(f) = O((\log(1/\delta)) + \log \log k)
\]

We note that the \( 1/\delta \) factor in \( q \) is unavoidable, since otherwise more than an \( \delta \) fraction of numbers would share the same message and hence the collision probability, as well as the error probability, would exceed \( \delta \).

**B The lower bound for \textsc{Sum-Equal}_{k}^{m,a} over integers**

**Theorem 7.9 (restated).** Let \( \Pi \) be the \( \delta \)-error simultaneous \( k \)-player protocol for solving \( m \) independent instances of the \textsc{Sum-Equal}_{k}^{m,a'} \) problem, where \( m \leq \frac{k \log T}{20 \log k} \) and the error tolerance \( \delta \in (0,1/6) \). The simultaneous communication complexity of \( \Pi \) is \( \text{RCC}_{k,\delta}^{\text{Lin,T}}(\Pi) = \Omega(mk \log \log T) \).

**Proof.** To prove the \( \Omega(mk \log \log T) \) lower bound we will deduce a contradiction if \( \Pi \) uses \( c < \gamma mk \log \log T \) bits of communication, for a sufficiently small constant \( \gamma \). By decreasing \( \gamma \) we may assume that \( k \) is arbitrarily large.

For the hard distribution we first introduce a magnitude bound \( a \) defined to be the largest integer such that \( a! \leq T \). We define \( M = a! \). We note that \( M \leq T \) and \( a = O(\log M) \) so \( a = O(\log T) \). Let \( \alpha = \gamma \log T \).

Now we specify the distribution \( \mathcal{H} \) for the \textsc{Sum-Equal}_{k} instances. \( \mathcal{H} := \left(G/2 + B/2\right)^{\text{m}} \) consists of \( m \) independent copies of \( G/2 + B/2 \), for \( G, B \) defined as follows:
\[
\begin{align*}
G &:= (G_1, \ldots, G_{k-1}, -\sum_{j=1}^{k-1} G_j) \\
B &:= (B_1, \ldots, B_{k-1}, M - \sum_{j=1}^{k-1} B_j)
\end{align*}
\]
where \( G_j, B_j \) are uniformly and independently chosen from \( [a] \) for every \( j \in [k-1] \). Note that: a) \textsc{Sum-Equal}_{k}(G) = 0, \textsc{Sum-Equal}_{k}(B) = 1 \); b) the first \( k-1 \) elements of \( G \) and \( B \), denoted by \( G_{-k} \) and \( B_{-k} \), are the same uniform distribution over \( [a]^{k-1} \). Thus we can write \( B = (G_{-k}, M + G_k) \); c) for \( j \in [k-1] \), the \( j \)-th player’s input \( \mathcal{H}_j \) is uniform over \( [a]^m \) and independent from other players’ input.

Besides \( \mathcal{H}_k \), the referee gets in addition an index \( n \) uniformly drawn from \([m]\) together with the answers \( Y^{(j)} = \textsc{Sum-Equal}_{k}(X^{(j)}) \) for \( j = n+1, \ldots, m \). Let \( \mathcal{H}_n' := (\mathcal{H}, Y^{(n+1)}, \ldots, Y^{(m)}) \) and the hard input distribution is defined as \( \mathcal{H}_n' := \sum_{n=1}^{m} \frac{1}{m} \cdot \mathcal{H}_n' \).

Now we derandomize the protocol \( \Pi \) by fixing the randomness and thus get an \( \delta \)-error deterministic protocol \( \Pi' \) with respect to the above input distribution. That is, \( \Pi' \) outputs \( \text{SUM-Equal}_{k}^{(n)} = \text{SUM-Equal}_{k}(X^{(n)}) \) with probability \( \geq 1 - \delta \).

By averaging, for at least \( m/2 \) choices of the index \( n \in [m] \) and the restricted distribution \( \mathcal{H}_n' \), the error of \( \Pi' \) is bounded by \( 2 \delta \).

\[
\Pr_{(X,Y) \sim \mathcal{H}_n'} \left[ \Pi'_n(X,Y) \neq \text{SUM-Equal}_{k}(X^{(n)}) \right] \leq 2\delta \quad \text{(B.1)}
\]

Then we introduce theorem B.1 that lower bounds \( I \left( X^{(n)}; M_1, \ldots, M_{k-1} \right) \geq 0.1 k \log a \) for protocols with small error. For compactness the proof of theorem B.1 is deferred to the end of this section.
Lemma B.1. For every \( n \) such that \( \Pi' \) err\$ with probability \( \leq 1/3 \) on input \( (X, Y) \sim \mathcal{H}_n' \), on at least \( d'm \) of the SUM-EQUAL instances, the mutual information between \( X^{(n)} \) and \( \Pi'(X, Y) \) must be \( I\left( X^{(n)} ; M_1, \ldots, M_{k-1} \right) \geq 0.1 k \log a \).

Using theorem B.1, it immediately follows that for \( \delta \leq 1/6 \) the protocol \( \Pi' \) must use \( \Omega \left( mk \log a \right) \) bits of communication. Since

\[
RCC_{k, \delta}^{(n)} (\Pi') \geq I\left( X^{(n)} ; M_1, \ldots, M_{k-1} \right)
= \sum_{i=1}^{m} I \left( X^{(i)} ; M_1, \ldots, M_{k-1} | X^{(i)}_{i-k}, \ldots, X^{(i)}_{k-1} \right)
\geq \sum_{i=1}^{m} I \left( X^{(i)} ; M_1, \ldots, M_{k-1} \right)
\geq \frac{a'}{2} m \cdot 0.1 k \log a = \Omega \left( mk \log a \right)
\]

since \( a' \) is some constant between 0 and 1.

Proof of Lemma B.1. Suppose by contradiction that \( I\left( X^{(n)} ; M_1, \ldots, M_{k-1} \right) < 0.1 k \log a \) and recall that \( m \leq \frac{k \log \log T}{20 \log k} \leq \frac{0.1 k \log a}{\log (ka)} \) for \( a = \gamma' \log T \) and sufficiently large \( T \),

\[
I\left( X^{(n)} ; M_1, \ldots, M_{k-1}, X_k, Y^{(n+1)}, \ldots, Y^{(m)} \right) < 0.1 k \log a + m \log (ka) < 0.2 k \log a
\]

Therefore, recalling that \( I(A; B, C) = I(A; B \mid C) \) when \( A \) is independent from \( C \) and that \( X_j^{(n)} \) is independent from \( X_1^{(n)}, \ldots, X_{j-1}^{(n)} \),

\[
\sum_{j=1}^{k-1} I\left( X_j^{(n)} ; M_1, \ldots, M_{k-1}, X_k, Y^{(n+1)}, \ldots, Y^{(m)} \right)
\leq \sum_{j=1}^{k-1} I\left( X_j^{(n)} ; M_1, \ldots, M_{k-1}, X_k, Y^{(n+1)}, \ldots, Y^{(m)}, X_1^{(n)}, \ldots, X_{j-1}^{(n)} \right)
= \sum_{j=1}^{k-1} I\left( X_j^{(n)} ; M_1, \ldots, M_{k-1}, X_k, Y^{(n+1)}, \ldots, Y^{(m)} | X_1^{(n)}, \ldots, X_{j-1}^{(n)} \right)
\leq I\left( X^{(n)} ; M_1, \ldots, M_{k-1}, X_k, Y^{(n+1)}, \ldots, Y^{(m)} \right) < 0.2 k \log a
\]

As a result, there is \( J \subseteq [k-1] \) and \( |J| > k/2 \) such that for every \( j \in [k-1] \), it holds that \( I\left( X_j^{(n)} ; M_1, \ldots, M_{k-1}, X_k, Y^{(n+1)}, \ldots, Y^{(m)} \right) < 0.2 k \log a \)
\[-1 + 0.5 \log a, \text{ and hence}\]
\[
H \left[ X_j^{(n)} \mid M_1, \ldots, M_{k-1}, X_k, Y^{(n+1)}, \ldots, Y^{(m)} \right] = H \left[ X_j^{(n)} \right] - I \left( X_j^{(n)} ; M_1, \ldots, M_{k-1}, X_k, Y^{(n+1)}, \ldots, Y^{(m)} \right) > \log a - (-1 + 0.5 \log a) = 1 + 0.5 \log a \tag{B.2}
\]

Note that \( H_{\infty} \left[ X_j^{(n)} \mid M_1, \ldots, M_{k-1}, X_k, Y^{(n+1)}, \ldots, Y^{(m)} \right] < 1 \) implies the existence of \( x \in [a] \) such that \( \Pr \left[ X_j^{(n)} = x \mid M_1, \ldots, M_{k-1}, X_k, Y^{(n+1)}, \ldots, Y^{(m)} \right] > \frac{1}{2} \), and hence it follows that
\[
H \left[ X_j^{(n)} \mid M_1, \ldots, M_{k-1}, X_k, Y^{(n+1)}, \ldots, Y^{(m)} \right] = \sum_{i \in [a]} p_i \log \frac{1}{p_i} \\
\leq p_x \log \frac{1}{p_x} + (1 - p_x) \log \frac{a - 1}{1 - p_x} \\
< 1 + 0.5 (a - 1) \tag{B.3}
\]

Thus, (B.2) ensures that \( H_{\infty} \left[ X_j^{(n)} \mid M_1, \ldots, M_{k-1}, X_k, Y^{(n+1)}, \ldots, Y^{(m)} \right] \geq 1 \) for every \( j \in J \). In what follows, we prove that if \( H_{\infty} \left[ X_j^{(n)} \mid M_1, \ldots, M_{k-1}, X_k, Y^{(n+1)}, \ldots, Y^{(m)} \right] \geq 1 \) for every \( j \in J \) and \(|J| > k/2\), then the conditional distribution \( B_j' := G_j' + M \) and \( G_j' := - \sum_{j=1}^{k-1} X_j^{(n)} \mid \{ M_1, \ldots, M_{k-1}, X_k, Y^{(n+1)}, \ldots, Y^{(m)} \} \) have statistical distance \( \leq k^{-1/8} \).

Notice that for \( j \in J \) and \( H_{\infty} \left[ X_j^{(n)} \mid M_1, \ldots, M_{k-1}, X_k, Y^{(n+1)}, \ldots, Y^{(m)} \right] \geq 1 \), the conditional distribution \( G_j' := X_j^{(n)} \mid \{ M_1, \ldots, M_{k-1}, X_k, Y^{(n+1)}, \ldots, Y^{(m)} \} \) is a convex combination of distributions uniform over two values. More specifically, \( G_j' = \sum_{\nu} \alpha_{\nu} \cdot G^{[\nu]} \), where \( \alpha_{\nu} \in (0, 1) \) and each \( G^{[\nu]} \) is a random variable uniform over two values. For \( j \notin J \), \( G_j' = \sum_{\nu} \alpha_{\nu} \cdot G^{[\nu]} \) where \( G^{[\nu]} \) is fixed, i.e., a random variable that equals one value with probability 1. For \( \nu = (v_1, \ldots, v_{k-1}) \), let \( \alpha_{\nu} = \prod_{j=1}^{k-1} \alpha_{v_j} \) and \( G^{[\nu]} = \left( G^{[v_1]}, \ldots, G^{[v_{k-1}]} - \sum_{j=1}^{k-1} G^{[v_j]} \right) \), then \( G' \) can be decomposed as \( G' = \sum_{\nu} \alpha_{\nu} \cdot G^{[\nu]} \).

Now for every \( j \in J \) and \( G^{[\nu]} \) uniform over \( \{ a_j, b_j \} \subseteq [a] \), we can assume w.l.o.g., \( a_j < b_j \) and write \( G^{[\nu]} = a_j + \left( b_j - a_j \right) Z_j \) where \( Z_j \) is uniform over \( \{ 0, 1 \} \). Since \( b_j - a_j \in [a] \), among the \( > k/2 \) indices \( j \in J \) for which \( G^{[\nu]} \) takes two values, we must have \( t \geq |J|/a > k/\Theta(\log k) > \sqrt{k} \) indices \( J' \) such that for any \( j \in J' \) the value \( b_j - a_j \) is the same value \( M' \).

Thus \( G^{[\nu]} \) can be further decomposed into a convex combination of \( G^{[u]} \) where, among the indices in \( J \), only those in \( J' \) are not fixed. Fix any \( u \) and denote \( G^{[u]} \) by \( G'' \). Let \( S = \sum_{j \in J'} Z_j \) denote the sum of \( t \) uniform i.i.d. \( 0/1 \) random variables. Then we can write
\[
G''_j = b + M' \cdot S \\
B'_j = b + M' \cdot S + M
\]

Since \( 1 \leq M' < a, M' \) divides \( M \) and hence \( M = M' q \) for \( q \in \mathbb{Z} \) and \( q \leq M \leq k^{1/8} \). Now we can apply \( q \) times the shift-invariance of the binomial distribution, which is stated as follows:
Claim B.2 (Claim 39 in [23]). Let $S$ be the sum of $t$ uniform, i.i.d. Boolean random variables. Then $S$ and $S + 1$ have statistical distance $\leq O\left(1/\sqrt{t}\right)$.

This yields that $G_k^{''}$ and $B_k^{''}$ have statistical distance

$$\text{SD}(G_k^{''}, B_k^{''}) = \text{SD}(M' \cdot S, M' \cdot (q + S)) \leq q \cdot O\left(1/\sqrt{k}\right) \leq k^{1/8}/k^{1/4} = k^{-1/8}$$

Recalling that $G'$ is just a convex combination of $G''$, the statistical distance between $G_k'$ and $B_k' = G_k' + M$ is also bounded by $k^{-1/8}$. However, by definition of $G_k'$ and $B_k'$ we conclude that the referee cannot distinguish the two cases of $X^{(n)} \sim G$ and $X^{(n)} \sim B$ with advantage greater than $k^{-1/8} < 1/6$, which contradicts the condition that $\Pi'$ has error probability $< 1/3$.

Therefore, $I\left(X^{(n)}_{-k}; M_1, \ldots, M_{k-1}\right) \geq 0.1k \log a = \Omega(k \log a)$. 

References

[1] Farid Abayev: Lower bounds for one-way probabilistic communication complexity and their application to space complexity. *Theoretical Computer Science*, 157(2):139–159, 1996.

[2] Yuqing Ai, Wei Hu, Yi Li, and David P. Woodruff: New characterizations in turnstile streams with applications. In *31st Conference on Computational Complexity, CCC 2016, May 29 to June 1, 2016, Tokyo, Japan*, pp. 20:1–20:22, 2016.

[3] Ziv Bar-Yossef, T. S. Jayram, Ravi Kumar, and D. Sivakumar: An information statistics approach to data stream and communication complexity. *Journal of Computer and System Sciences*, 68(4):702–732, 2004.

[4] Paul Beame, Toniann Pitassi, Nathan Segerlind, and Avi Wigderson: A direct sum theorem for corruption and the multiparty NOF communication complexity of set disjointness. In *20th Annual IEEE Conference on Computational Complexity, CCC 2005, 11-15 June 2005, San Jose, CA, USA*, pp. 52–66, 2005.

[5] Mark Braverman and Ankit Garg: Public vs private coin in bounded-round information. In *Automata, Languages, and Programming - 41st International Colloquium, ICALP 2014, Copenhagen, Denmark, July 8-11, 2014, Proceedings, Part I*, pp. 502–513, 2014.

[6] Joshua Brody, Harry Buhrman, Michal Koucký, Bruno Loff, Florian Speelman, and Nikolay K. Vereshchagin: Towards a reverse Newman’s theorem in interactive information complexity. *Algorithmica*, 76(3):749–781, 2016 (also CCC 2013).

[7] A. Chakrabarti, Yaojun Shi, A. Wirth, and A. Yao: Informational complexity and the direct sum problem for simultaneous message complexity. In *Proceedings 42nd IEEE Symposium on Foundations of Computer Science*, pp. 270–278, 2001. [doi:10.1109/SFCS.2001.959901]

[8] Graham Cormode, Mayur Datar, Piotr Indyk, and S. Muthukrishnan: Comparing data streams using hamming norms (how to zero in). *IEEE Trans. Knowl. Data Eng.*, 15(3):529–540, 2003.
[9] ANDRÉ GRONEMEIER: Asymptotically optimal lower bounds on the nih-multi-party information complexity of the and-function and disjointness. In 26th International Symposium on Theoretical Aspects of Computer Science, STACS 2009, February 26-28, 2009, Freiburg, Germany, Proceedings, pp. 505–516, 2009. 3, 10

[10] CHEN-YU HSU, PIOTR Indyk, DINA KATABI, AND ALI VAKILIAN: Learning-based frequency estimation algorithms. In International Conference on Learning Representations, 2019. 4, 5

[11] PIOTR Indyk: Stable distributions, pseudorandom generators, embeddings, and data stream computation. J. ACM, 53(3):307–323, 2006. 4

[12] T. S. Jayram: Hellinger strikes back: A note on the multi-party information complexity of and. Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques. APPROX ’09 / RANDOM ’09, Berkeley, CA, USA, August 21 - 23, 2009, pp. 562–573, 2009. 3, 10

[13] Tanqiu Jiang, Yi Li, Honghao Lin, Yisong Ruan, and David P. Woodruff: Learning-augmented data stream algorithms. In International Conference on Learning Representations, 2020. 4, 5

[14] Daniel M. Kane, Jelani Nelson, Ely Porat, and David P. Woodruff: Fast moment estimation in data streams in optimal space. In Proceedings of the 43rd ACM Symposium on Theory of Computing, STOC 2011, San Jose, CA, USA, 6-8 June 2011, pp. 745–754, 2011. 4

[15] Daniel M. Kane, Jelani Nelson, and David P. Woodruff: On the exact space complexity of sketching and streaming small norms. In Proceedings of the Twenty-First Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2010, Austin, Texas, USA, January 17-19, 2010, pp. 1161–1178, 2010. 4, 7

[16] Daniel M. Kane, Jelani Nelson, and David P. Woodruff: An optimal algorithm for the distinct elements problem. In Proceedings of the twenty-ninth ACM SIGMOD-SIGACT-SIGART symposium on Principles of database systems, pp. 41–52, 2010. 4, 24

[17] Ilan Kremer, Noam Nisan, and Dana Ron: On randomized one-round communication complexity. Computational Complexity, 8(1):21–49, 1999. 8

[18] Eyal Kushilevitz: Communication complexity. Advances in Computers, 44:331–360, 1997. 16

[19] Thodoris Lykouris and Sergei Vassilvitskii: Competitive caching with machine learned advice. In International Conference on Machine Learning, pp. 3296–3305. PMLR, 2018. 4

[20] Michael Mitzenmacher: Scheduling with predictions and the price of misprediction. In Thomas Vidick, editor, 11th Innovations in Theoretical Computer Science Conference (ITCS 2020), volume 151 of Leibniz International Proceedings in Informatics (LIPIcs), Dagstuhl, Germany, January 2020. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik. [doi:10.4230/LIPICS.ITCS.2020.14] 4

[21] Michael Mitzenmacher and Sergei Vassilvitskii: Algorithms with predictions. In Tim Roughgarden, editor, Beyond the Worst-Case Analysis of Algorithms, pp. 646–662. Cambridge University Press, 2020. [doi:10.1017/9781108637435.037] 4

[22] Christos H. Papadimitriou and Michael Sipser: Communication complexity. Journal of Computer and System Sciences, 28(2):260–269, 1984. 8

[23] Emanuele Viola: The communication complexity of addition. Combinatorica, 35(6):703–747, 2015 (also SODA 2013). 6, 8, 11, 12, 16, 22, 33
[24] DAVID P. WOODRUFF AND QIN ZHANG: Tight bounds for distributed functional monitoring. In Proceedings of the 44th Symposium on Theory of Computing Conference, STOC 2012, New York, NY, USA, May 19 - 22, 2012, pp. 941–960, 2012. 3, 7

AUTHORS
Elbert Du
4th year Undergraduate
Department of Computer Science and
Department of Mathematics
Harvard University
Cambridge, MA, USA
du@college.harvard.edu

Michael Mitzenmacher
Professor of Computer Science
Department of Computer Science
Harvard University
Cambridge, MA, USA
michaelm@eecs.harvard.edu

David Woodruff
Associate Professor of Computer Science
Department of Computer Science
Carnegie Mellon University
Pittsburgh, PA, USA
dwoodruf@andrew.cmu.edu

Guang Yang
Research Director
Tree-Graph Blockchain Innovation Center of Shanghai and
Tree-Graph Blockchain Innovation Center of Xiang River Hunan
Conflux Foundation
Shanghai, China
guang.research@gmail.com

https://sites.google.com/site/guangyangresearch/home
ABOUT THE AUTHORS

ELBERT DU is currently a senior at Harvard College, concentrating in mathematics and getting a fourth year masters with the AB/SM program in Computer Science. He was first introduced to the world of academic mathematics when he was in fifth grade, and he began attending the late Professor Paul Sally Jr.’s Young Scholars’ Program at the University of Chicago. Elbert is now interested in studying complexity, differential privacy, and adaptive data analysis. In his spare time, Elbert enjoys reading, solving chess puzzles, and playing video games.

MICHAEL MITZENMACHER is a Professor of Computer Science at Harvard University. He is the co-author of a well-known textbook on randomized algorithms and probabilistic techniques in computer science with Eli Upfal. He is an ACM and IEEE Fellow.

DAVID WOODRUFF is an associate professor in the Computer Science Department at Carnegie Mellon University. He works on the foundations of data science, specifically in data streams, machine learning, randomized numerical linear algebra, sketching and sparse recovery.

GUANG YANG is currently the research director at Conflux, a startup blockchain project initiated by Fan Long and Andrew Yao. Before joining Conflux, he was an assistant professor at Institute of Computing Technology (ICT), Chinese Academy of Sciences.