PICTURE GROUPS AND MAXIMAL GREEN SEQUENCES

KIYOSHI IGUSA AND GORDANA TODOROV

Abstract. We show that picture groups are directly related to maximal green sequences for valued Dynkin quivers of finite type. Namely, there is a bijection between maximal green sequences and positive expressions (words in the generators without inverses) for the Coxeter element of the picture group. We actually prove the theorem for the more general set up of “vertically and horizontally ordered” sets of positive real Schur roots for any hereditary algebra (not necessarily finite type).

Furthermore, we show that every picture for such a set of positive roots is a linear combination of “atoms” and we give a precise description of atoms as special semi-invariant pictures.

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1. Introduction

In this section we give the statements of the main theorems and an outline of the proofs. Although the results here are mainly in finite type, we have made the assumptions more general for possible applications to quivers of infinite type. We also believe that, using \( \tau \)-tilting theory, analogous statements can be obtained for any finite dimensional algebra. However, in this paper, all quivers will be valued quivers without oriented cycles.

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1.1. Basic definitions. We assume that $Q$ is a valued acyclic quiver and we always consider subsets $S$ of the set of positive real Schur roots of $Q$. The positive real Schur roots are precisely the dimension vectors of the exceptional modules over any modulated quiver with underlying valued quiver $Q$ \[12\]. We need to order the roots, in these subsets $S$, in two different ways which we call “lateral” and “vertical” ordering.

Notation 1.1.1. Let $Q$ be a modulated quiver and $\beta$ a positive real Schur root.

1. We will denote by $M_\beta$ the unique indecomposable exceptional module with the dimension vector equal to $\beta$.
2. A positive real Schur root $\beta'$ will be called a subroot of $\beta$ if the indecomposable exceptional module $M_{\beta'}$ is isomorphic to a submodule of $M_\beta$. This is denoted by $\beta' \subset \beta$.
3. A positive real Schur root $\beta''$ will be called a quotient root of $\beta$ if the indecomposable exceptional module $M_{\beta''}$ is isomorphic to a quotient of $M_\beta$. This is denoted by $\beta \twoheadrightarrow \beta''$.

Definition 1.1.2. By a lateral ordering $\leq$ on a set of real Schur roots $S$ we mean a total ordering on $S$ satisfying the following for any $\alpha, \beta \in S$.

1. If $\text{hom}(\alpha, \beta) \neq 0$ then $\alpha \leq \beta$, where $\text{hom}(\alpha, \beta) = \dim_K \text{Hom}_\Lambda(M_\alpha, M_\beta)$.
2. If $\text{ext}(\alpha, \beta) \neq 0$ then $\alpha > \beta$, where $\text{ext}(\alpha, \beta) = \dim_K \text{Ext}_\Lambda(M_\alpha, M_\beta)$.

Remark 1.1.3. We now state several basic facts and some examples of $S$ with lateral ordering and some $S$ which do not admit such ordering.

1. If $S$ has lateral ordering then for all $\alpha, \beta \in S$, either $\text{hom}(\alpha, \beta) = 0$ or $\text{ext}(\alpha, \beta) = 0$.
2. The left-to-right order of preprojective roots as they occur in the Auslander-Reiten quiver, together with any ordering on the summands of the middle term of each almost split sequence, is a lateral ordering.
3. The set of all regular roots does not admit a lateral ordering.
4. The simple roots can always be laterally ordered by taking $\alpha_i < \alpha_j$ whenever there is an arrow $j \rightarrow i$ in the quiver.
5. If $\omega$ is a rightmost root in $S$ in lateral order then $\text{ext}(\beta, \omega) = 0$ for all $\beta \in S$ and $\text{hom}(\omega, \beta') = 0$ for all $\beta' \neq \omega \in S$.
6. Any subset of a laterally ordered set of roots is laterally ordered with the same ordering.

Definition 1.1.4. A sequence of real Schur roots $S = (\beta_1, \cdots, \beta_m)$ is said to be vertically ordered if the following two conditions are satisfied for each $\beta_k \in S$.

1. Let $\beta' \subset \beta_k$ be any (positive real Schur) subroot of $\beta_k$. Then $\beta' = \beta_j$ for some $j \leq k$.
2. Let $\beta'' \twoheadrightarrow \beta_k$ be a (positive real Schur) quotient-root of $\beta_k$. Then $\beta'' = \beta_j$ for some $j \leq k$.

The sequence $S$ is weakly vertically ordered if at least one of the above two conditions is satisfied for each $\beta_k \in S$.

Remark 1.1.5. A finite set of positive real Schur roots closed under subroots and quotient roots is vertically ordered if the roots are ordered by length, and the roots of the same length are ordered in any way.

Definition 1.1.6. Let $S$ be a finite set of positive real Schur roots.

1. $S$ is called admissible if it has a lateral and a vertical ordering
2. $S$ is called weakly admissible if it has a lateral ordering and a weakly vertical ordering
3. an admissible sequence is an admissible set listed in its vertical ordering
4. a weakly admissible sequence is a weakly admissible set listed in its weakly vertical ordering
Remark 1.1.7. Let \((\beta_1, \ldots, \beta_m)\) be an admissible sequence (of positive real Schur roots). The following will be crucial for the induction steps in the proofs.

(1) If one of the \(\beta_i\)'s is removed, the sequence \((\beta_1, \ldots, \hat{\beta_i}, \ldots, \beta_m)\) is a weakly admissible sequence (not necessarily admissible sequence).

(2) If the last element \(\beta_m\) is removed then the resulting sequence will still be admissible sequence.

The notion of “picture groups” for hereditary artin algebras of finite representation type was introduced in [13] as groups associated to the “semi-invariant pictures”. These groups were defined using all positive roots (the algebras were of finite representation type). We now give a more general definition of “picture groups”, using admissible subsets of positive real Schur roots for all finite dimensional hereditary algebras.

Definition 1.1.8. Let \(S\) be an admissible set of (positive real Schur) roots. We will call a subset \(R \subseteq S\) relatively closed if \(R\) is closed under extensions in \(S\).

Relatively closed subsets \(R \subseteq S\) of admissible sets, have “picture groups”, which we now define.

Definition 1.1.9. For any relatively closed subset \(R \subseteq S\) of an admissible set of roots \(S\), we define the picture group \(G(R)\) as follows. There is one generator \(x(\beta)\) for each \(\beta \in R\). There is the following relation for each pair \(\beta_i, \beta_j\) of hom-orthogonal roots with \(\text{ext}(\beta_i, \beta_j) = 0\):

\[
x(\beta_i)x(\beta_j) = \prod x(\gamma_k)
\]

where \(\gamma_k\) runs over all roots in \(R\) which are linear combinations \(\gamma_k = a_k\beta_i + b_k\beta_j, a_k, b_k \in \mathbb{N}\) in increasing order of the ratio \(\frac{a_k}{b_k}\) (going from 0/1 where \(\gamma_1 = \beta_j\) to 1/0 where \(\gamma_k = \beta_i\)). For any \(g \in G(R)\), we define a positive expression for \(g\) to be any word in the generators \(x(\beta)\) (with no \(x(\beta)^{-1}\) terms) whose product is \(g\).

Remark 1.1.10. (a) Note that \(G(R)\) is independent of the choice of \(S\). However, the existence of an admissible \(S\) containing \(R\) is important. Also, by the well-known Theorem 4.2.4 each \(\gamma_k = a_k\beta_i + b_k\beta_j\) has \(\beta_i\) as a subroot and \(\beta_j\) as a quotient root if \(\beta_i, \beta_j\) are hom-orthogonal with \(\text{ext}(\beta_i, \beta_j) = 0\).

(b) Whenever \(R \subset R'\) are relatively closed subsets of an admissible set \(S\) we get a homomorphism of groups \(G(R) \to G(R')\) since any relation among the generators of \(G(R)\) is also a relation among the corresponding generators of \(G(R')\).

(c) Definition 1.1.9 is a generalization of the notion of “picture groups” for hereditary artin algebras of finite representation type as defined in [13]. Indeed, the picture group \(G(\Lambda)\) for such an algebra is, by definition, equal to the picture group \(G(\Phi^+(\Lambda))\) for the set \(\Phi^+(\Lambda)\) of all positive roots of \(\Lambda\). These roots are vertically ordered by dimension and laterally ordered by their position in the Auslander-Reiten quiver of \(\Lambda\).

Definition 1.1.11. Given \(S\) admissible, we define the Coxeter element \(c_S\) of \(S\) to be the product of the generators \(x(\alpha_i)\) for all simple roots \(\alpha_i \in S\) in lateral order, i.e., so that \(\alpha_i < \alpha_j\) whenever there is an arrow \(i \leftarrow j\) in the quiver of the algebra.

Remark 1.1.12. As an element of the picture group \(G(S)\), this product \(c_S = \prod x(\alpha_i)\) is independent of the choice of the lateral ordering. This is because one can pass from any lateral ordering to any other by transposing consecutive generators \(x(\alpha_i), x(\alpha_j)\) when there is no arrow between them in the quiver. But in that case, \(x(\alpha_i), x(\alpha_j)\) commute. So, the product remains invariant.
Example 1.1.13. Consider the quiver of type $A_3$ with straight orientation: $1 \leftarrow 2 \leftarrow 3$. The Auslander-Reiten quiver, with modules on the left and corresponding roots on the right is:

![Diagram of Auslander-Reiten quiver]

The set $S = (\alpha_1, \alpha_2, \alpha_4, \alpha_3)$ is vertically ordered since the subroot $\alpha_1$ and quotient root $\alpha_2$ of $\alpha_4$ come before it. The set $S$ is admissible since it also has a lateral ordering $\alpha_1 < \alpha_4 < \alpha_2 < \alpha_3$. The subsequence $S' = (\alpha_1, \alpha_4, \alpha_3)$ is weakly admissible. Also, $S'$ is relatively closed in $S$ since the missing element is simple.

The picture group $G(S)$ has four generators $x(\alpha_1), x(\alpha_2), x(\alpha_3), x(\alpha_4)$ and four relations given by the four pairs of hom-orthogonal roots:

1. $x(\alpha_1)x(\alpha_2) = x(\alpha_2)x(\alpha_4)x(\alpha_1)$ from the extension $\alpha_1 \hookrightarrow \alpha_4 \twoheadrightarrow \alpha_2$.
2. $x(\alpha_2)x(\alpha_3) = x(\alpha_3)x(\alpha_2)$ since the extension $\alpha_5$ of $\alpha_2$ by $\alpha_3$ is not in $S$.
3. $x(\alpha_1)x(\alpha_3) = x(\alpha_3)x(\alpha_1)$ since $\alpha_1, \alpha_3$ do not extend each other.
4. $x(\alpha_4)x(\alpha_3) = x(\alpha_3)x(\alpha_4)$ since $\alpha_6 \notin S$.

Thus $x(\alpha_4)$ is central. (This follows from the fact that $\alpha_3$ is last in both vertical and lateral orderings.) The picture group $G(S')$ has generators $x(\alpha_1), x(\alpha_3), x(\alpha_4)$ modulo the relation that $x(\alpha_3)$ is central. The Coxeter element of $G(S)$ is 

$$c_S = x(\alpha_1)x(\alpha_2)x(\alpha_3).$$

Remark 1.1.14. If $\beta_i, \beta_j$ are hom-orthogonal and $\beta_i < \beta_j$ in lateral order then 

$$x(\beta_i)x(\beta_j) = x(\beta_j)w$$

where $w$ is a positive expression in letters $\gamma$ where $\beta_i \leq \gamma < \beta_j$ in lateral order since $\text{hom}(\beta_i, \gamma) \neq 0$ and $\text{hom}(\gamma, \beta_j) \neq 0$ when $\gamma \neq \beta_i$.

An important case is when $j = m$, the size of $S$. For $\beta_i, \beta_m$ hom-orthogonal we get 

$$x(\beta_i)x(\beta_m) = x(\beta_m)x(\beta_i).$$

since the other roots $\gamma_k$ in the formula above would come after $\beta_m$ so do not lie in $S$.

We recall that, for all roots $\beta$, there is a unique exceptional module $M_\beta$ with dimension vector $\beta$. The subset $D(\beta) \subseteq \mathbb{R}^n$ is given by 

$$D(\beta) = \{x \in \mathbb{R}^n : \langle x, \beta \rangle = 0 \text{ and } \langle x, \beta' \rangle \leq 0 \forall \beta' \subset \beta\}$$

where $\beta' \subset \beta$ means that $M_\beta$ contains an exceptional submodule isomorphic to $M_{\beta'}$. The inner product $\langle x, \beta \rangle$ is the weighted dot product $\langle x, \beta \rangle = \sum x_i b_i f_i$ where $x_i, b_i$ are the $i$th coordinates of $x, \beta$ and $f_i = \text{dim}_K \text{End}(S_i)$ where $S_i$ is the $i$th module. So, $D(\beta)$ does not contain points in $\mathbb{R}^n$ all of whose coordinates are positive (or negative). For more details see Appendix II.

Theorem 4.1.1 in the Appendix also proves that $D(\beta)$ has the following equivalent description.

$$D(\beta) = \{x \in \mathbb{R}^n : \langle x, \beta \rangle = 0 \text{ and } \langle x, \beta'' \rangle \geq 0 \forall \beta'' \subset \beta\}$$

where $\beta'' \subset \beta$ means that $M_\beta$ has an exceptional quotient module isomorphic to $M_{\beta''}$.

Given $S$ a weakly admissible sequence of roots, let $CL(S) \subseteq \mathbb{R}^n$ denote the union of $D(\beta)$ for all $\beta \in S$. Since this set is invariant under scaling in the sense that $\lambda CL(S) = CL(S)$ for all $\lambda >
0, we usually consider just the intersection \( L(S) := CL(S) \cap S^{n-1} \). The semi-invariant picture for \( G(S) \) is defined to be this set \( L(S) \subseteq S^{n-1} \) together with the labels of its walls by positive roots and the normal orientation of each wall \( D(\beta) \) telling on which side the vector \( \beta \) lies. When \( n = 3 \), we draw the stereographic projection of this set onto the plane. (Projecting away from the negative octant. See Figures 1 and 2.)

**Definition 1.1.15.** Given \( S = (\beta_1, \ldots, \beta_m) \) weakly admissible and \( \epsilon = (\epsilon_1, \ldots, \epsilon_m) \in \{0, +, -\}^m \).

1. Define \( \mathcal{U}_\epsilon \) to be the convex open set given by
   \[ \mathcal{U}_\epsilon = \{ x \in \mathbb{R}^n : \langle x, \beta_i \rangle > 0 \text{ if } \epsilon_i = + \text{ and } \langle x, \beta_j \rangle < 0 \text{ if } \epsilon_j = - \} . \]
2. \( \epsilon \) will be called admissible (with respect to \( S \)) if for all \( 1 \leq k \leq m \) we have:
   \[ \epsilon_k = 0 \iff D(\beta_k) \cap \mathcal{U}_{\epsilon_1, \ldots, \epsilon_{k-1}, +} = \emptyset . \]
3. When \( \epsilon \) is admissible the open set \( \mathcal{U}_\epsilon \) will be called an \( S \)-compartment. See Fig. 1, 2.

In Proposition 2.0.1 below we show that, for \( S \) weakly admissible, each compartment \( \mathcal{U}_\epsilon \) is open and convex and these regions form the components of the complement of \( CL(S) \) in \( \mathbb{R}^n \).

**Figure 1.** On the left is the semi-invariant picture \( L(S_0) \) for the admissible subsequences \( S_0 = (\alpha_1, \alpha_2, \alpha_4) \) of \( S \) from Example 1.1.13. \( L(S_0) \) is a subset of \( S^2 \subseteq \mathbb{R}^3 \). Thus, e.g., \( D(\alpha_i) \) are actually coordinate hyperplanes. The \( S_0 \)-compartments are the components of the complement of \( L(S_0) \). For example, \( \mathcal{U}_{++0} = \mathcal{U}_{++} \) is the region on the positive side of the two hyperplanes \( D(\alpha_1), D(\alpha_2) \). \( \mathcal{U}_{-++} \) is the set of point in \( \mathcal{U}_{--} \) on the positive side of \( D(\alpha_4) \). On the right, the wall \( D(\alpha_3) \) cuts all five \( S_0 \)-compartments in half giving the semi-invariant picture for \( S = (\alpha_1, \alpha_2, \alpha_4, \alpha_3) \) with ten compartments.

**Definition 1.1.16.** For any weakly admissible \( S \), we define a maximal \( S \)-green sequence \( (\epsilon_1, \ldots, \epsilon_s) \) to be a sequence of \( S \)-compartments \( \mathcal{U}_{\epsilon(0)}, \ldots, \mathcal{U}_{\epsilon(s)} \) satisfying the following.

1. Every pair of consecutive compartments \( \mathcal{U}_{\epsilon(i-1)}, \mathcal{U}_{\epsilon(i)} \) is separated by a wall \( D(\beta_k) \) so that \( \epsilon(i-1)_{k_j} = - \) and \( \epsilon(i)_{k_j} = + \) and \( \epsilon(i-1)_{j} = \epsilon(i)_{j} \) for all \( j < k_i \).
2. \( \mathcal{U}_{\epsilon(0)} \) is the compartment containing vectors all of whose coordinates are negative.
3. \( \mathcal{U}_{\epsilon(s)} \) is the compartment containing vectors all of whose coordinates are positive.
We say that \((U_t)\) is an \(S\)-green sequence if only the first condition is satisfied. We define an \(S\)-green path representing the \(S\)-green sequence \((U_t)\) to be a continuous path, \(\gamma : \mathbb{R} \to \mathbb{R}^n\), so that, for some \(t_1 < t_2 < \cdots < t_s\) we have the following

1. \(\gamma(t) \in U_{t(0)}\) when \(t < t_1\)
2. \(\gamma(t) \in U_{t(s)}\) when \(t > t_s\)
3. \(\gamma(t) \in U_{t(i)}\) for \(0 < i < s\) whenever \(t_i < t < t_{i+1}\)
4. For \(1 \leq i \leq s\), \(\gamma(t)\) goes from the negative side to the positive side of \(D(\beta_{k_i})\) for some \(\beta_{k_i} \in S\) when \(t\) crosses the value \(t_i\).

The word “maximal” may be misleading. (See Figure 2)

Remark 1.1.17. The green arrow in Figure 2 is an example of a “Coxeter path” which is given more generally as follows. Let \(S\) be an admissible set of roots. Let \(\alpha_1, \cdots, \alpha_k\) be the simple roots in \(S\) in any lateral order. In other words, any arrow between the corresponding vertices in the quiver go from \(\alpha_j\) to \(\alpha_i\) only when \(i < j\). Also, all roots in \(S\) have support at these vertices by definition of an admissible set. The corresponding Coxeter path is defined to be the linear path \(\gamma : \mathbb{R} \to \mathbb{R}^n\)

\[\gamma(t) = (t, t, \cdots, t) - \sum j \alpha_j.\]

This path crosses the hyperplanes \(D(\alpha_i)\) at time \(t = i\) (in the order \(\alpha_1, \alpha_2, \cdots\)) and it passes from the negative to the positive side of each hyperplane.

Also, this path is disjoint from all other walls \(D(\beta)\) for \(\beta \in S\) not simple. To see this, suppose \(\gamma(t_0) \in D(\beta)\). Then

\[\langle \gamma(t_0), \beta \rangle = 0 = \sum (t_0 - j) f_j b_j.\]
So, some of the coefficients $t_0 - j$ are positive and some are negative with the positive ones coming first, say $t_0 - 1, t_0 - 2, \ldots, t_0 - p$ positive and the rest negative. In that case $\beta' = b_1 \alpha_1 + b_2 \alpha_2 + \cdots + b_p \alpha_p$ is a sum of subroots of $\beta$, but $\langle \gamma(t_0), \beta' \rangle > 0$ which contradicts the assumption that $\gamma(t_0) \in D(\beta)$. So, the Coxeter path does not meet any $D(\beta)$ for $\beta \in S$ not simple. Thus, the Coxeter path is an $S$-green path. Since the coordinates of $\gamma(t)$ are all negative for $t << 0$ and all positive for $t >> 0$, this green path gives a maximal $S$-green sequence. The product of the group labels on the walls crossed by this green sequence form the Coxeter element $c_S = x(\alpha_1)x(\alpha_2) \cdots x(\alpha_k) \in G(S)$.

1.2. Statement of the main results. The main property of $U \subset \mathbb{R}^n$ is that it is convex and nonempty when $S$ is weakly admissible and $\epsilon$ is admissible with respect to $S$ (Proposition 2.0.1). Furthermore, when $S$ is admissible, the complement of the union of these regions forms a “picture” for the picture group $G(S)$. The precise statement is as follows.

**Theorem 1.2.1.** When $S$ is admissible, each $S$-compartment $U_\epsilon$ can be labelled with an element of the picture group $g(\epsilon) \in G(S)$ so that, if $U_\epsilon$ and $U_{\epsilon'}$ are separated by a wall $D(\beta)$, $\beta \in S$, with $U_{\epsilon'}$ on the positive side of $D(\beta)$, then

$$g(\epsilon)x(\beta) = g(\epsilon').$$

Note that, given any system of compartment labels $g(\epsilon)$ satisfying (1.2), left multiplication of all labels by a fixed element of $G(S)$ will preserve the condition. Therefore, we may, without loss of generality, assume that $g(\epsilon) = 1$ on the negative $S$-compartment $U_\epsilon$ where all $\epsilon_i$ are negative or zero. Theorem 1.2.1 follows from the following lemma.

**Lemma 1.2.2.** For $S$ weakly admissible, every $S$-compartment $U_\epsilon$ lies in a maximal $S$-green sequence given by an $S$-green path.

**Proof.** Given any $S$-compartment $U_\epsilon$, choose a general point $v \in U_\epsilon$ and consider the straight line $f(t) = v + (t, t, \cdots, t), t \in \mathbb{R}$. This line passes through walls $D(\beta)$ only in the positive direction since

$$\langle (1, 1, \cdots, 1), \beta \rangle > 0$$

for all positive roots $\beta$. Thus $f(t)$ is an $S$-green path giving an $S$-green sequence. For $t >> 0$, the coordinates of $f(t)$ are all positive. For $t << 0$, they are all negative. Therefore $f(t)$ gives a maximal $S$-green sequence passing through the $S$-compartment $U_\epsilon$ at $t = 0$. \qed

**Proof of Theorem 1.2.1.** Given an $S$-compartment $U_\epsilon$ choose an $S$-green path through $U_\epsilon$ as in the lemma above. Let $g(\epsilon)$ be the product of labels $x(\beta_i)$ for the walls crossed by this path on the way to $U_\epsilon$. Condition (1.2) will be satisfied. We only need to show that $g(\epsilon)$ is well defined. To do this suppose we have two $S$-green paths $\gamma, \gamma'$ from the negative compartment to $U_\epsilon$. Since $\mathbb{R}^n$ is contractible these paths are homotopic. The homotopy gives a mapping of $h : [0, 1]^2 \rightarrow \mathbb{R}^n$. Make this a smooth mapping transverse to $CL(S)$.

Since each wall $D(\beta)$ is contained in the hyperplane $H(\beta)$, $CL(S)$ is contained in the union of these hyperplanes. The intersection of two hyperplanes has codimension 2. Since $S$ is finite, there are only finitely many such subspaces. We ignore the other intersections which have higher codimension. By transversality, the homotopy $h$ will only meet these codimension 2 subspaces at a finite number of points. Let $x_0 \in \mathbb{R}^n$ be one these points and let $B$ be the set of all $\beta \in S$ so that $x_0 \in D(\beta)$. Let $A$ be the set of minimal elements of $B$, i.e., the set of all $\alpha \in B$ so that no subroot of $\alpha$ lies in $B$.

Then $A$ has at most two elements since, otherwise, by Proposition 4.1.3, the intersection of $D(\alpha)$ for $\alpha \in A$ has codimension $\geq 3$. If $A$ has only one element then $A = B$. In that case, the
wall crossing sequence is unchanged when the path is deformed past $x_0$. The remaining case is when $\mathcal{A}$ has two elements: $\mathcal{A} = \{\alpha_1, \alpha_2\}$. By Corollary 4.2.5, the other elements of $\mathcal{B}$ are positive linear combinations $\beta = x\alpha_1 + y\alpha_2$ and $D(\beta)$ lies on the negative side of $D(\alpha_1)$ and the positive side of $D(\alpha_2)$ since $\alpha_1 \subset \beta$ implies $\langle v, \alpha_1 \rangle \leq 0$ for all $v \in D(\beta)$. This means that, on one side of $x_0$, the $\mathcal{S}$-green path goes through $D(\alpha_1)$ followed by $D(\alpha_2)$ and on the other side, it goes through $D(\alpha_2)$, then, being on the positive side of $D(\alpha_2)$ and on the negative side of $D(\alpha_1)$ it goes through $D(\beta)$ for $\beta \in \mathcal{B}$. (See Figure 3.) This sequence of wall crossings gives the same element of the picture group. So, the group label $g(\epsilon)$ is independent of the path. This proves the theorem. □

![Figure 3](image-url)

**Figure 3.** A typical intersection of two walls $D(\alpha_1)$ and $D(\alpha_2)$ producing walls $D(\beta_i)$. In this drawing there is only $\beta = \alpha_1 + \alpha_2$. The green path $\gamma$ crosses $D(\alpha_1), D(\alpha_2)$ and $\gamma'$ crosses $D(\alpha_2), D(\beta), D(\alpha_1)$. The homotopy $h : \gamma \simeq \gamma'$ passes through $x_0$.

**Lemma 1.2.3.** Take any maximal $\mathcal{S}$-green sequence for $\mathcal{S}$ admissible and consider the sequence of walls $D(\beta_{k_1}), \cdots, D(\beta_{k_s})$ which are crossed by the sequence. Then the product of the corresponding generators $x(\beta_{k_i}) \in G(\mathcal{S})$ is equal to the Coxeter element $c_\mathcal{S} \in G(\mathcal{S})$:

$$x(\beta_{k_1}) \cdots x(\beta_{k_s}) = \prod x(\alpha_i).$$

**Proof.** Use Theorem 1.2.1 with the group element $g(\epsilon) = 1$ on the negative $\mathcal{S}$-compartment. The group label on the positive $\mathcal{S}$-compartment is equal to the product of the positive expression associated to any maximal $\mathcal{S}$-green sequence. By Remark 1.1.17 any Coxeter path gives the Coxeter element. Therefore, all maximal $\mathcal{S}$-green sequences give a positive expression for the Coxeter element of $G(\mathcal{S})$. □

Lemma 1.2.3 can be rephrased as follows. Any maximal $\mathcal{S}$-green sequence gives a positive expression for $c_\mathcal{S}$ by reading the labels of the walls which are crossed by the sequence. The main theorem of this paper is the following theorem and its corollary.

**Theorem A.** Suppose that $\mathcal{S}$ is an admissible set of roots. Then, the operation described above gives a bijection:

$$\{\text{maximal } \mathcal{S}\text{-green sequence}\} \cong \{\text{positive expression for } c_\mathcal{S} \text{ in } G(\mathcal{S})\}$$
It is clear that distinct maximal $S$-green sequences give distinct positive expressions. Therefore, it suffices to show that every positive expression for $c_S$ can be realized as a maximal $S$-green sequence.

Recall from Remark 1.1.10(c) that, for $\Lambda$ a hereditary artin algebra of finite representation type, the set $\Phi^+(\Lambda)$ of positive roots of $\Lambda$ forms an admissible set and that the picture group of $\Lambda$ is equal to the picture group of $\Phi^+(\Lambda)$. This leads to the following corollary.

**Corollary B.** For $\Lambda$ any hereditary artin algebra of finite representation type, there is a bijection between the set of maximal green sequences for $\Lambda$ and the set of positive expressions for the Coxeter element $c_{\Phi} = x(\alpha_1) \cdots x(\alpha_n)$ in $G(\Lambda) = G(\Phi^+(\Lambda))$. \hfill \Box

### 1.3. Outline of proof of Theorem A

The proof is by induction on $m$, the size of the finite set $S$. If $m = 1$, the root $\beta_1$ must be simple. So, the group $G(S)$ is infinite cyclic with generator $x(\beta_1)$ which is equal to $c_S$. There are two compartments $U_1, U_{-1}$ separated by the single hyperplane $D(\beta_1) = H(\beta_1)$. And $U_-, U_+$ is the unique $S$-green sequence. The associated positive expression is $x(\beta_1)$ which is the unique positive expression for $c_S$. So, the result holds for $m = 1$. Thus, we may assume that $m \geq 2$ and the theorem holds for the admissible sequence of roots $S_0 = (\beta_1, \cdots, \beta_{m-1})$.

**Remark 1.3.1.** One key property of the last element $\beta_m$ in an admissible sequence $S$ is that, for $\beta \neq \beta_m$ in $S$, $x(\beta)$ commutes with $x(\beta_m)$ if and only if $\beta$ is hom-orthogonal to $\beta_m$. The reason is that there is a formula for the commutator of two roots if and only if they are hom-orthogonal and, in that case, the commutator is a product of extensions of these roots. But any extension comes afterwards in admissible (vertical) order, so any extension of $\beta_m$ will not be in the set $S$.

**Lemma 1.3.2.** There is a surjective group homomorphism

$$\pi : G(S) \rightarrow G(S_0)$$

given by sending each $x(\beta_i) \in G(S)$ to the generator in $G(S_0)$ with the same name when $i < m$ and sending $x(\beta_m)$ to $1$.

**Proof.** By Remark 1.3.1 there are only two kinds of relations in $G(S)$ involving $x(\beta_m)$:

1. Commutation relations: $[x(\beta_m), x(\beta_j)] = 1$ when $\beta_m, \beta_j$ are hom-orthogonal.
2. Relations in which $x(\beta_m)$ occurs only once:

$$x(\beta_1)x(\beta_2) \cdots x(\beta_m) \cdots x(\beta_k).$$

In both cases, when $x(\beta_m)$ is deleted, the relation in $G(S)$ reduces to a relation in $G(S_0)$ (or to the trivial relation $x(\beta_j) = x(\beta_j)$ in Case 1). Thus, $G(S_0)$ is given by $G(S)$ modulo the relation $x(\beta_m) = 1$. \hfill \Box

Suppose $m \geq 2$ and $\beta_m$ is simple, say $\beta_m = \alpha_k$ the $k$th simple root. Then, since $S$ is admissible, all previous roots $\beta_j, j < m$ have support disjoint from $\alpha_k$. Then $x(\beta_m)$ is central and $G(S)$ is the product $G(S) = G(S_0) \times \mathbb{Z}$ where the $\mathbb{Z}$ factor is generated by $x(\beta_m)$. Thus a positive expression for $c_S$ is given by any positive expression for $x_{S_0}$ with the letter $x(\beta_m)$ inserted at any point.

Each $S_0$-compartment $U_i$ is the inverse image in $\mathbb{R}^n$ of a compartment for $S_0$ in $\mathbb{R}^{n-1}$. Thus, any $S$-maximal green sequence will pass through these walls giving a maximal $S_0$-green sequence and must, at some point, pass from the negative side of the hyperplane $D(\beta_m)$ to its positive side. (See Figure 1 for an example.) By induction on $m$, this $S_0$-MGS is any positive
word for $c_S$ and the crossing of $D(\beta_m)$ inserts $x(\beta_m)$ at any point. This describes all words for $c_S$. So, the theorem holds in this case.

Now suppose $\beta_m$ is not simple. Then $S, S_0$ have the same set of simple roots. So, $\pi(c_S) = c_S$. Suppose that $w$ is a positive expression for $c_S$ in $G(S)$. Let $\pi(w) = w_0$ be the positive expression for $c_S$ in $G(S_0)$ given by deleting every instance of the generator $x(\beta_m)$ from $w$. By induction on $m$, there exists a unique maximal $S_0$-green sequence $U_{(0)}, \ldots, U_{(s)}$ which realizes the positive expression $w_0$. These fall into two classes.

Class 1. Each $S_0$-compartment $U_{(i)}$ in the maximal $S_0$-green sequence is disjoint from $D(\beta_m)$.

For maximal green sequences in this class, each $U_{(i)} = U_{(i)}'$ where $\epsilon(i) = (\epsilon_1, \ldots, \epsilon_m)$ with $\epsilon_m = 0$ and $\epsilon(i) = (\epsilon_1, \ldots, \epsilon_{m-1})$. Therefore, the maximal $S_0$-green sequence $U_{(i)}'$ is also a maximal $S$-green sequence and $w_0 = w$ by the following lemma proved in subsection 3.5. So, the positive expression $w$ is realized by a maximal $S$-green sequence.

Lemma C. Let $w, w'$ be two positive expressions for the same element of the group $G(S)$. Suppose that $w = \pi(w')$, i.e., the two expressions are identical modulo the generator $x(\beta_m)$. Then $x(\beta_m)$ occurs the same number of times in $w, w'$. In particular, $x(\beta_m) \neq 1$ in $G(S)$.

In the case at hand, $w' = w_0$ does not contain the letter $x(\beta_m)$. So, neither does $w$ and we must have $w = w_0$ as claimed. So, by Lemma C, the theorem holds when $w_0 = \pi(w)$ corresponds to a maximal $S_0$-green sequence of Class 1.

Class 2. At least one $S_0$-compartment in the $S_0$-green sequence meets $D(\beta_m)$.

For green sequences in this class, the $S_0$-compartments which intersect $D(\beta_m)$ are consecutive:

Lemma D. Let $U_{(0)}, \ldots, U_{(s)}$ be a maximal $S_0$-green sequence. Then

1. The $S_0$-compartments $U_{(i)}$ which meet $D(\beta_m)$ are consecutive, say $U_{(p)}, \ldots, U_{(q)}$.
2. Let $D(\beta_k)$ be the wall between $U_{(i-1)}$ and $U_{(i)}$ so that $w_0 = x(\beta_k) \cdots x(\beta_k)$. Then $\beta_m$ is hom-orthogonal to $\beta_k$ for $p < i \leq q$ but not hom-orthogonal to $\beta_{k_p}, \beta_{k_{q+1}}$.
3. For $p < i \leq q$ and $\delta \in \{+, -\}$, $D(\beta_k)$ is also the wall separating $U_{(i-1), \delta}$ and $U_{(i), \delta}$.

Lemma D tells us: (1) The $S_0$-compartments $U_{(r)}$ for $p \leq r \leq q$ are divided into two $S$-compartments by the wall $D(\beta_m)$. (3) The wall separating consecutive $S_0$-compartments $U_{(r)}, U_{(r+1)}$ for $p \leq r < q$ also separate the pairs of $S$-compartments $U_{(r), -}, U_{(r+1), -}$ and $U_{(r), +}, U_{(r+1), +}$. (See Figure 3.)

So, we can refine the maximal $S_0$-green sequence to a maximal $S$-green sequence, by staying on the negative side of $D(\beta_m)$ until we reach the $S$-compartment $U_{(r), -}$ for some $p \leq r \leq q$, then cross through $D(\beta_m)$ into $U_{(r), +}$ and continue in the given $S_0$-compartments but on the positive side of $D(\beta_m)$. This gives the maximal $S$-green sequence

$$U_{(0), 0}, \ldots, U_{(p-1), 0}, U_{(p), -}, \ldots, U_{(r), -}, U_{(r), +}, \ldots, U_{(q), +}, U_{(q+1), 0}, \ldots, U_{(s), 0}$$

of length $s + 1$ giving the positive expression

$$w_r = x(\beta_{k_1}) \cdots x(\beta_{k_p}) x(\beta_m) x(\beta_{k_{p+1}}) \cdots x(\beta_{k_{q+1}}).$$

By the defining relations in the group $G(S)$, the generators $x(\beta)$ and $x(\beta_m)$ commute if $\beta$ is hom-orthogonal to $\beta_m$. By (3) in the lemma this implies that $w_r$ is a positive expression for $c_S$ if $p \leq r \leq q$. We have just shown that each such $w_r$ is realizable by a maximal $S$-green sequence. So, it remains to show that the positive expression $w$ that we started with is equal to one of these $w_r$.\]
By Lemma C, \( x(\beta_0) \) occurs exactly once in the expression \( w \). We need to show that, if the generator \( x(\beta_m) \) occurs in the “wrong place” then \( w \) is not a positive expression for \( c_S \), in other words, the product of the elements of \( w \) is not equal to \( c_S \). This follows from the following lemma proved in subsection 3.5.

**Lemma E.** Let

\[
R(\beta_m) = \{ \beta_i \in S_0 : \text{hom}(\beta_i, \beta_m) = 0 = \text{hom}(\beta_m, \beta_i) \}.
\]

Let \( \beta_{j_1}, \ldots, \beta_{j_k} \) be elements of \( S_0 \) which do not all lie in \( R(\beta_m) \). Then \( x(\beta_m), \prod x(\beta_{j_i}) \) do not commute in the group \( G(S) \).

By part (2) of Lemma D, \( \beta_{kr} \in R(\beta_m) \) if \( p < r \leq q \) and \( \beta_{kp}, \beta_{kp+1} \notin R(\beta_m) \). So, this lemma implies that \( w_r \) is a positive expression for \( c_S \) if and only if \( p \leq r \leq q \). So, we must have \( w = w_r \), for one such \( r \) and \( w \) is realizable. This concludes the outline of the proof of the main theorem. It remains only to prove the three lemmas C, D, E invoked in the proof.

2. **Properties of compartments \( U_e \)**

We derive the basic properties of the compartments \( U_e \) and prove Lemma D. The basic property is the following.

**Proposition 2.0.1.** For all weakly admissible \( S \) and all admissible \( \epsilon \) the \( S \)-compartment \( U_e \) is convex and nonempty. When \( \epsilon_m \neq 0 \), or equivalently, when \( D(\beta_m) \cap U_{\epsilon_1, \ldots, \epsilon_{m-1}} \) is nonempty, the boundary of \( D(\beta_m) \) does not meet \( U_{\epsilon_1, \ldots, \epsilon_{m-1}} \). Equivalently,

\[
D(\beta_m) \cap U_{\epsilon_1, \ldots, \epsilon_{m-1}} = H(\beta_m) \cap U_{\epsilon_1, \ldots, \epsilon_{m-1}}.
\]

Consequently, the \( S \)-compartments for the components of the complement of \( CL(S) \) in \( \mathbb{R}^n \).

**Proof.** When \( m = 1 \), \( \beta_1 \) is simple and \( D(\beta_1) = H(\beta_1) \) is a hyperplane whose complement has two convex components \( U_+, U_- \). So, the proposition holds for \( m = 1 \). Now, suppose \( m \geq 2 \) and all statements hold for \( m - 1 \). Let \( S_0 = S \setminus \beta_m \). This a weakly admissible sequence of roots. So, the \( S_0 \)-components \( U_e \) are convex and open and their union is the complement of \( CL(S_0) \).

Since \( S \) is weakly admissible, it either contains all subroots of \( \beta_m \) or it contains all quotient roots of \( \beta_m \). By symmetry we assume the first condition. Let \( \epsilon = (\epsilon_1, \ldots, \epsilon_{m-1}) \) be admissible of length \( m - 1 \). If \( (\epsilon, 0) \) is admissible for \( S \) then \( U_{\epsilon, 0} = U_e \). Otherwise, \( U_e \) meets \( D(\beta_m) \). In that case \( U_e \cap \partial D(\beta_m) \) must be empty since, any element \( x_0 \in \partial D(\beta_m) \) must be an element of \( D(\beta') \) for some proper subroot \( \beta' \subsetneq \beta_m \). By assumption, \( \beta' \in S_0 \). So, \( x_0 \in CL(S_0) \). This gives a contradiction since \( U_e \) is disjoint from \( CL(S_0) \) by induction on \( m \). Therefore, \( U_e \) is divided into two convex open sets \( U_{e,+} \) and \( U_{e,-} \) separated by \( D(\beta_m) \). So the \( S \)-compartments fill up the complement of \( CL(S_0) \cup D(\beta_m) = CL(S) \). \( \square \)

2.1. **Inescapable regions.** For \( S_0 \) weakly admissible, let \( V \) be the closure of the union of some set of \( S_0 \)-compartments \( U_e \). Then \( V \) has internal and external walls. The internal walls of \( V \) are the ones between two of the compartments \( U_e, U_{e'} \) in \( V \). \( V \) has points on both sides of the internal walls. The external walls of \( V \) are the ones which separate \( V \) from its complement. The region \( V \) will be called inescapable if it is on the positive side of all of its external walls. I.e., they are all red on the inside. Once an \( S_0 \)-green sequence enters such a region, it can never leave. Since \( V \) is closed, it contains all of its internal and external walls. We also consider open regions \( W \) which are inescapable regions minus their external walls. Then \( W \) is the complement of the closure of the union of all compartments not in \( W \).
Given an admissible sequence $S$ with last object $\beta_m$ which we assume to be nonsimple, let $S_0 = (\beta_1, \ldots, \beta_{m-1})$. Recall that this is also admissible. We will construct two inescapable regions $W(\beta_m), V(\beta_m)$ where the first is open and the second is closed. All maximal $S_0$-green sequences start outside both regions, end inside both regions and fall into two classes: those that enter $W(\beta_m)$ before they enter $V(\beta_m)$ and those that enter $V(\beta_m)$ before they enter $W(\beta_m)$. And these coincide with the two classes of maximal $S_0$-green sequences discussed in the outline of the main theorem (Corollary 2.2.1 below).

The first inescapable region is the open set 

$$W(\beta_m) := \{x \in \mathbb{R}^n : \langle x, \alpha \rangle > 0 \text{ for some } \alpha \subset \beta_m\}.$$ 

For example, on the left side of Figure 1, $m = 3$ and $W(\beta_3)$ is the interior of $D(\alpha_1)$.

**Proposition 2.1.1.** The complement of $W(\beta_m)$ in $\mathbb{R}^n$ is closed and convex. Furthermore:

$$W(\beta_m) \cap H(\beta_m) = H(\beta_m) - D(\beta_m).$$

**Proof.** The complement of $W(\beta_m)$ is

$$\mathbb{R}^n \setminus W(\beta_m) = \{x \in \mathbb{R}^n : \langle x, \alpha \rangle \leq 0 \text{ for all } \alpha \subset \beta_m\}$$

which is closed and convex since it is given by closed convex conditions $\langle x, \alpha \rangle \leq 0$.

For the second statement, suppose that $v \in H(\beta_m)$. Then $\langle v, \beta_m \rangle = 0$. By the stability conditions which we are using to define $D(\beta_m)$, $v \in D(\beta_m)$ if and only if $\langle v, \alpha \rangle \leq 0$ for all $\alpha \subset \beta_m$, in other words,

$$D(\beta_m) = H(\beta_m) \cap (\mathbb{R}^n \setminus W(\beta_m))$$

which is equivalent to (2.1). □

**Proposition 2.1.2.** The region $W(\beta_m)$ is inescapable. I.e., all external walls are red. Furthermore, each external walls of $W(\beta_m)$ has the form $D(\alpha)$ for some $\alpha \subset \beta_m$. Consequently, every $S_0$-compartment is contained either in $W(\beta_m)$ or in its complement.

**Proof.** Take any external wall $D(\alpha)$ of $W(\beta_m)$. Let $v_t$ be a continuous path which goes through that wall from inside to outside. In other words, $v_t \in W(\beta_m)$ for $t < 0$ and $v_t \notin W(\beta_m)$ for $t \geq 0$. By definition of $W(\beta_m)$ this means that there is some $\alpha' \subset \beta_m$ so that $\langle v_t, \beta \rangle$ changes sign from positive to nonpositive at $t$ goes from negative to nonnegative.

By choosing $v_t$ in general position, $v_0$ will not lie in $H(\alpha')$ for any $\alpha' \neq \alpha$. So, we must have $\alpha \subset \beta_m$. And $\langle v_t, \alpha \rangle > 0$ for $t < 0$ and $\langle v_t, \alpha \rangle < 0$ for $t > 0$. Therefore, $W(\beta_m)$ is on the positive (red) side of the external wall $D(\alpha)$. So, $W(\beta_m)$ is inescapable.

Since each part of the boundary lies in $D(\alpha)$ for some $\alpha \in S_0$, the boundary of $W(\beta_m)$ is contained in the union of the boundaries of the $S_0$-compartments. So, all such compartments are either entirely insider or entirely outside $W(\beta_m)$. □

The second inescapable region is the closed set

$$V(\beta_m) = \{y \in \mathbb{R}^n : \langle y, \gamma \rangle \geq 0 \text{ for all quotient roots } \gamma \text{ of } \beta_m\}.$$ 

For example, on the left side of Figure 1, $m = 3$ and $V(\beta_3)$ is the closure of the interior of $D(\alpha_2)$. In Figure 4, $V(\beta_m)$ is the region enclosed by the large oval. By arguments analogous to the ones above, we get the following.
Figure 4. The green path $\gamma_1$ is in Class 1 since it is disjoint from $D(\beta_m)$. The green path $\gamma_2$ is in Class 2 and passes through three $S_0$-compartments $U_{(p)}, U_{(r)}, U_{(q)}$ in $V_0 = \text{int}(V(\beta_m) \setminus W(\beta_m))$. Each of these is divided into two $S$-compartments by the wall $D(\beta_m)$ and $\gamma_2$ passes through four of these $S$-compartments in $V_0$. $D(\beta_m)$ is the part of the hyperplane $H(\beta_m)$ inside the oval region $V(\beta_m)$ and outside of $W(\beta_m)$.

Proposition 2.1.3. $V(\beta_m)$ is a closed convex inescapable region whose external walls all have the form $D(\gamma)$ where $\gamma$ is a quotient root of $\beta_m$. So, every $S_0$-compartment is contained in $V(\beta_m)$ or its complement. Furthermore,

$$V(\beta_m) \cap H(\beta_m) = D(\beta_m).$$

2.2. Class 1 and Class 2 maximal $S_0$-green sequences. Recall that a maximal $S_0$-green sequence with $S_0 = (\beta_1, \ldots, \beta_{m-1})$ is in:

1. Class 1 if each $S_0$-compartment $U_{(i)}$ in the green sequence is disjoint from $D(\beta_m)$.
2. Class 2 if at least one $S_0$-compartment, say $U_{(i)}$, in the $S_0$-green sequence meets $D(\beta_m)$.

So, $U_{(i)}$ is divided into two $S$-compartments $U_{(i),-}$ and $U_{(i),+}$. See Figure 4.

Corollary 2.2.1. A maximal $S_0$-green sequence is in Class 1 if and only if it passes through $V(\beta_m) \setminus W(\beta_m)$. It is in Class 2 if and only if it contains a compartment in

$$V(\beta_m) \setminus W(\beta_m) = \{x \in \mathbb{R}^n : \langle x, \alpha \rangle \leq 0 \text{ for all } \alpha \subset \beta_m \text{ and } \langle x, \gamma \rangle \geq 0 \text{ for all } \beta_m \to \gamma \}.$$

Proof. Every maximal green sequence starts on the negative side of the hyperplane $H(\beta_m)$ and ends on its positive side. Therefore the maximal $S_0$-green sequence must cross the hyperplane at some point. Since $\beta_\ell \notin S_0$, none of the $S_0$-compartments has $H(\beta_m)$ as a wall. So, there must be one compartment in the $S_0$-green sequences which meets the hyperplane $H(\beta_m)$. Let $U_\ell$ be the first such compartment. Then, either $U_\ell \cap D(\beta_m)$ is empty or nonempty. In the first case, $U_\ell$ is in $V(\beta_m)$ and it is outside $V(\beta_m)$. Since $W(\beta_m)$ is inescapable and does not meet $D(\beta_m)$, the green sequence is in Class 1. In the second case, $U_\ell$ is in $V(\beta_m)$ and not in $W(\beta_m)$ and the green sequence is in Class 2. So, these two cases correspond to Class 1 and Class 2 proving the corollary.

Recall that $\mathcal{R}(\beta_m)$ is the set of all $\alpha \in S_0$ which are hom-orthogonal to $\beta_m$. Let $V_0$ be the interior of the closed region $V(\beta_m) \setminus W(\beta_m)$. Thus

$$V_0 := \text{int}(V(\beta_m) \setminus W(\beta_m))$$
\[ \{ x \in \mathbb{R}^n : \forall \alpha \subseteq \beta_m \langle x, \alpha \rangle < 0 \text{ and } \langle x, \gamma \rangle > 0 \forall \beta_m \rightarrow \gamma, \gamma \neq \beta_m \}. \]

**Proposition 2.2.2.** For all \( \alpha \in S_0, \) \( \alpha \in \mathcal{R}(\beta_m) \) if and only if \( D(\alpha) \cap V_0 \neq \emptyset. \)

**Proof.** Suppose that \( x \in D(\alpha) \cap V_0 \) and \( \text{hom}(\beta_m, \alpha) \neq 0. \) Then there is a subroot \( \alpha' \) of \( \alpha \) which is also a quotient root of \( \beta_m: \beta_m \rightarrow \alpha' \subset \alpha. \) Since \( \alpha \in S_0 \) we cannot have \( \beta_m \subset \alpha. \) Therefore \( \alpha' \) is a proper quotient of \( \beta_m. \) Then \( \langle x, \alpha' \rangle > 0 \) since \( x \in V_0 \) and \( \langle x, \alpha' \rangle \leq 0 \) since \( x \in D(\alpha) \) and \( \alpha' \subset \alpha. \) This is a contradiction. So, \( \text{hom}(\beta_m, \alpha) = 0. \) A similar argument shows that \( \text{hom}(\alpha, \beta_m) = 0. \) So, \( \alpha \in \mathcal{R}(\beta_m). \)

Conversely, if \( \alpha \in \mathcal{R}(\beta_m) \) then \( \alpha, \beta_m \) span a rank 2 wide subcategory \( \mathcal{A}(\alpha, \beta_m) \) of \( \text{mod-}\Lambda. \) Choose any tilting object \( T \) in the left perpendicular category \( \perp \mathcal{A}(\alpha, \beta_m) \) (for example the sum of the projective objects). Then the \( g \)-vector \( g(\dim T) \) lies in the interior of both \( D(\alpha) \) and \( D(\beta_m) \) by Proposition 4.2.2 since \( \mathcal{M}_\alpha, \mathcal{M}_{\beta_m} \) are the minimal objects in \( T^\perp = \mathcal{A}(\alpha, \beta_m). \) So, \( g(\dim T) \in V_0. \) So, \( D(\alpha) \) meets \( V_0. \)

**Corollary 2.2.3.** The open region \( V_0 \) contains no vertices of the semi-invariant picture \( L(S_0). \)

**Proof.** Suppose that \( x_0 \in V_0 \) is a vertex of \( L(S_0). \) By Theorem 4.2.1 we have a wide subcategory \( \mathcal{W}(x_0) \) of all modules \( V \) so that \( x_0 \in D(V). \) Since \( x_0 \) is a vertex of \( L(S_0), \) the wide subcategory \( \mathcal{W}(x_0) \) must have rank \( n - 1 \) and its minimal objects must lie in \( S_0, \) i.e., \( \mathcal{W}(x_0) = \mathcal{A}(\alpha_1, \ldots, \alpha_{n-1}) \) where \( \alpha_i \in S_0. \) By Proposition 2.2.2 each \( \alpha_i \) is hom-orthogonal to \( \beta_m. \) This implies that \( \alpha_1, \ldots, \alpha_{n-1} \) together with \( \beta_m \) form the minimal roots of a wide subcategory of rank \( n. \) By Theorem 4.2.3 this must be all of \( \text{mod-}\Lambda. \) So, \( \beta_m \) must be a simple root contrary to our initial assumption. Therefore \( V_0 \) contains no vertices of \( L(S_0). \)

**Corollary 2.2.4.** Let \( \alpha_1, \ldots, \alpha_k \) be pairwise hom-orthogonal elements of \( \mathcal{R}(\beta_m) \) then the intersection \( D(\alpha_1) \cap \cdots \cap D(\alpha_k) \cap D(\beta_m) \cap V_0 \) is nonempty.

**Proof.** More precisely, let \( \mathcal{A}(\alpha_1, \ldots, \alpha_k, \beta_m) \) be the rank \( k + 1 \) wide subcategory of \( \text{mod-}\Lambda \) with simple objects \( M_{\alpha_1}, M_{\beta_m}. \) Let \( T = T_1 \oplus \cdots \oplus T_{n-k-1} \) be any cluster tilting object of the cluster category of \( \perp \mathcal{A}(\alpha_1, \ldots, \alpha_k, \beta_m). \) Then the \( g \)-vector \( g(\dim T) \) is a point in \( D(\alpha_1) \cap \cdots \cap D(\alpha_k) \cap D(\beta_m) \) which lies in the interior of \( D(\beta_m). \) This can be proved by induction on \( k \) using the argument in the proof of Proposition 2.2.2.

2.3. **Proof of Lemma [D]** We will show that maximal \( S_0 \)-green sequences satisfy the three properties listed in Lemma [D]

**Proposition 2.3.1.** An \( S_0 \)-compartment \( U_e \) meets \( D(\beta_m) \) if and only if \( U_e \subseteq V_0. \)

Before proving this we show that this implies the first property in Lemma [D]. Recall that this states:

**D(1)** In every maximal \( S_0 \)-green sequence in Class 2, the compartments which meet \( D(\beta_m) \) are consecutive.

**Proof of D(1).** Let \( \mathcal{U}_i \) be a maximal \( S_0 \)-green sequence. Let \( p, q \) be minimal so that \( \mathcal{U}_i \subseteq \mathcal{V}(\beta_m) \) and \( \mathcal{U}_q \subseteq \mathcal{W}(\beta_m). \) When the green sequence is in Class 2, \( p < q. \) Since \( \mathcal{V}(\beta_m) \) is inescapable, \( \mathcal{U}_i \subseteq \mathcal{V}(\beta_m) \) iff \( p \leq i. \) Since \( \mathcal{W}(\beta_m) \) is inescapable, \( \mathcal{U}_i \subseteq V_0 \) iff \( p \leq i < q. \) So, the compartments of the green sequence which lie in \( V_0 \) are consecutive. By the proposition these are the compartments which meet \( D(\beta_m). \)
Proof of Proposition 2.3.1. Let \( \mathcal{U}_x \) be an \( S_0 \)-compartment in \( V_0 \). Let \( x \in \mathcal{U}_x \). If \( \langle x, \beta_m \rangle = 0 \) then \( x \in H(\beta_m) \cap V_0 \subseteq D(\beta_m) \) and we are done. So, suppose \( \langle x, \beta_m \rangle \neq 0 \). Pick a point \( y \in D(\beta_m) \cap V_0 \) and take the straight line from \( x \) to \( y \). Since \( V_0 \) is convex, this line is entirely contained in \( V_0 \). If the line is not in \( \mathcal{U}_x \) then it must meet an internal wall \( D(\alpha) \) on the boundary of \( \mathcal{U}_x \). By Proposition 2.2.2, \( \alpha \in \mathcal{R}(\beta_m) \).

Let \( k \) be maximal so that the closure of \( \mathcal{U}_x \) contains a point \( z \in D(\alpha_x) = D(\alpha_1) \cap \cdots \cap D(\alpha_k) \) where \( \alpha_1, \ldots, \alpha_k \in \mathcal{R}(\beta_m) \) are pairwise hom-orthogonal. Then, by Corollary 2.2.4, \( D(\alpha_x) \cap D(\beta_m) \cap V_0 \) is nonempty. Let \( w \) be an element. Since \( D(\alpha_x) \) and \( V_0 \) are both convex, \( D(\alpha_x) \cap V_0 \) contains the straight line \( \gamma(t) = (1 - t)z + tw, 0 \leq t \leq 1 \).

Let \( \delta \) be a very small vector so that \( \langle \delta, \beta_m \rangle = 0 \) and \( z + \delta \in \mathcal{U}_x \). Consider the line \( \gamma(t) + \delta \). This is in \( \mathcal{U}_x \) for \( t = 0 \) and lies in \( D(\beta_m) \) when \( t = 1 \). This proves the proposition if \( \gamma(t) + \delta \in \mathcal{U}_x \) for all \( 0 \leq t \leq 1 \). So, suppose not. Let \( t_0 \) be minimal so that this open condition fails. Then the line \( \gamma(t) \) meets another wall at \( t = t_0 \) and \( \gamma(t_0) \) will be a point in the closure of \( \mathcal{U}_x \) which meets a codimension \( k + 1 \) set \( D(\alpha_0) \cap D(\alpha_1) \cap \cdots \cap D(\alpha_k) \) where \( \alpha_0 \in S_0 \) is hom-orthogonal to the other roots \( \alpha_i \). (Take \( \alpha_0 \) of minimal length among the new roots so that \( \gamma(t_0) \in D(\alpha_0) \).) This contradicts the maximality of \( k \). So, there is no point \( t_0 \) and \( \gamma(1) + \delta \in \mathcal{U}_x \cap D(\beta_m) \) as claimed.

We have already shown property (2) in Lemma D. Any maximal \( S_0 \)-sequence of Class 2 crosses a wall \( D(\gamma) \) at some point to enter region \( V_0 \), passes through several internal walls of \( V_0 \), then exists \( V_0 \) by a wall \( D(\alpha) \) of \( W(\beta_m) \). By Propositions 2.1.3, 2.1.2, \( \gamma \) is a quotient root of \( \beta_m \) and \( \alpha \) is a subroot of \( \beta_m \), both not hom-orthogonal to \( \beta_m \). By Proposition 2.2.2, the internal walls of \( V_0 \) are \( D(\beta) \) where \( \beta \in \mathcal{R}(\beta_m) \). So, property (2) in Lemma D holds.

The last property we need to verify in Lemma D is the following.

D(3) Suppose that the two \( S_0 \)-compartments \( \mathcal{U}_{(1)} \) and \( \mathcal{U}_{(2)} \) meet along a common internal wall \( D(\beta_j) \). Then the \( S \)-compartments \( \mathcal{U}_{(1),+}, \mathcal{U}_{(2),+} \) meet along the common internal wall \( D(\beta_j) \) and the \( S \)-compartments \( \mathcal{U}_{(1),-}, \mathcal{U}_{(2),-} \) also meet along \( D(\beta_j) \).

Proof. Let \( S', S'_0 \) be \( S, S_0 \) with \( \beta_j \) deleted. Then \( S', S'_0 \) are weakly admissible. Since \( \beta_j \notin S'_0 \), the two \( S_0 \)-compartments \( \mathcal{U}_{(1)} \) and \( \mathcal{U}_{(2)} \) merge to form one \( S'_0 \)-compartment \( \mathcal{U} \). This compartment meets \( D(\beta_m) \) so it breaks up into two \( S' \)-compartments \( \mathcal{U}_{+,+} \) and \( \mathcal{U}_{-,-} \). We know that \( D(\beta_j) \) must divide these two \( S' \)-compartments into four \( S \)-compartments since \( \mathcal{U}_{(1)}, \mathcal{U}_{(2)} \) are both divided into two parts by \( D(\beta_m) \). Since \( S' \)-compartments are convex by Proposition 2.0.1, this can happen only if \( D(\beta_j) \) meets both \( S' \)-compartments and forms the common wall separating the two halves of each.

3. Planar pictures and group theory

In this section we will use planar pictures to prove the two properties of the group \( G(S) \) that we are using: Lemmas 3 and 4. The key tool will be the “sliding lemma” (Lemma 3.4.3) which comes from the first author’s PhD thesis [7]. Unless otherwise stated, all pictures in this section will be planar. We begin with a review of the topological definition of a (planar) picture with special language coming from the fact that all relations in our group \( G(S) \) are commutator relations. Since this section uses only planar diagrams, we feel that theorems can be proven using diagrams and topological arguments. Algebraic versions of these arguments using HNN extensions, geometric realizations of categories and cubical \( \text{CAT}(0) \) categories can be found in other papers which prove similar results for pictures of arbitrary dimension [15, 10, 13].
3.1. **Planar pictures.** Suppose that the group $G$ has a presentation $G = \langle \mathcal{X} \mid \mathcal{Y} \rangle$. This means there is an exact sequence

$$R_\mathcal{Y} \hookrightarrow F_\mathcal{X} \twoheadrightarrow G$$

where $F_\mathcal{X}$ is the free group generated by the set $\mathcal{X}$ and $R_\mathcal{Y} \subseteq F_\mathcal{X}$ is the normal subgroup generated by the subset $\mathcal{Y} \subseteq F_\mathcal{X}$. Then $G$ is the fundamental group of a 2-dimensional CW-complex $X^2$ given as follows. Let $X^1$ denote the 1-dimensional CW-complex having a single 0-cell $e^0$, one 1-cell $e^1(x)$ for every generator $x \in \mathcal{X}$ attached on $e^0$. Then $\pi_1 X^1 = F_\mathcal{X}$ and any $f \in F_\mathcal{X}$ gives a continuous mapping $\eta_f : S^1 \to X^1$ given by composing the loops corresponding to each letter in the unique reduced expression for $f$. Here $S^1 = \{ z \in \mathbb{C} : ||z|| = 1 \}$, $1 \in S^1$ is the basepoint and $S^1$ is oriented counterclockwise.

Let $X^2$ denote the 2-dimensional CW-complex given by attaching one 2-cell $e^2(r)$ for every relations $r \in \mathcal{Y}$ using an attaching map $\eta_r : S^1 \to X^1$ homotopic to the one described above. We choose each mapping $\eta_r$ so that it is transverse to the centers of the 1-cells of $X^1$. So, the inverse images of these center points are fixed finite subsets of $S^1$. The relation $r$ is given by the union of these finite sets, call it $E_r \subset S^1$, together with a mapping $\lambda : E_r \to \mathcal{X} \cup \mathcal{X}^{-1}$ indicating which 1-cell the point goes to and in which direction the image of $\eta_r$ traverses that 1-cell. Then we have:

$$r = \prod_{x \in E_r} \lambda(x) \in F_\mathcal{X}.$$

The circle $S^1$ is the boundary of the unit disk $D^2 = \{ x \in \mathbb{C} : ||x|| \leq 1 \}$. Let $CE_r \subset D^2$ denote the cone of the set $E_r$:

$$CE_r := \bigcup_{x \in E_r} \{ ax \in D^2 : 0 \leq a \leq 1 \}.$$

This is the union of the straight lines from all $x \in E_r$ to $0 \in D^2$.

Example:

$$r = x^{-1}yxz$$

![Diagram](image)

**Figure 5.** The cone of $E_r$ in $D^2$ is the part inside the circle $S^1$. The asterisks * indicates the position of the basepoint $1 \in S^1$. The labels are drawn on the negative side of each edge.

A picture is a geometric representation of a continuous pointed mapping $\theta : S^2 \to X^2$ where pointed means preserving the base point. A (pointed) deformation of a picture represents a homotopy of such a mapping. Deformation classes of pictures form a module over the group ring $\mathbb{Z}G$.

**Definition 3.1.1.** Given a group $G$ with presentation $G = \langle \mathcal{X} \mid \mathcal{Y} \rangle$ and fixed choices of $E_r \subset S^1$, $\lambda : E^1 \to \mathcal{X} \cup \mathcal{X}^{-1}$, a picture for $G$ is defined to be a graph $L$ embedded in the plane $\mathbb{R}^2$ with circular edges allowed, together with:

1. a label $x \in \mathcal{X}$ for every edge in $L$,
(2) a normal orientation for each edge in \( L \),
(3) a label \( r \in \mathcal{Y} \cup \mathcal{Y}^{-1} \) for each vertex in \( L \),
(4) for each vertex \( v \), a smooth \((C^\infty)\) embedding \( \theta_v : D^2 \to \mathbb{R}^2 \) sending 0 to \( v \)
satisfying the following where \( E(x) \) denotes the union of edges labeled \( x \).

(a) Each \( E(x) \) is a smoothly embedded 1-manifold in \( \mathbb{R}^2 \) except possibly at the vertices.
(b) For each vertex \( v \in L \), \( \theta_v^{-1}(E(x)) \subset CE_r \) is equal to the cone of \( \lambda^{-1}\{x, x^{-1}\} \subset E_r \).

The image of \( 1 \in S^1 \) under \( \theta_v : D^2 \to \mathbb{R}^2 \) will be called the basepoint direction of \( v \) and will be indicated with \(*\) when necessary.

The embedding \( \theta_v \) has positive, negative orientation when \( r \in \mathcal{Y}, r \in \mathcal{Y}^{-1} \), respectively.

One easy consequence of this definition is the following.

**Proposition 3.1.2.** Given a picture \( L \) for \( G \), there is a unique label \( g(U) \in G \) for each component \( U \) of the complement of \( L \) in \( \mathbb{R}^2 \) having the following properties.

1. \( g(U_\infty) = 1 \) for the unique unbounded component \( U_\infty \).
2. \( g(V) = g(U)x \) if the regions \( U, V \) are separated by an edge labeled \( x \) and oriented towards \( V \).

**Proof.** For any region \( U \), choose a smooth path from \( \infty \) to any point in \( U \). Make the path transverse to all edge sets. Then let \( g(U) = x_1^{e_1} \cdots x_m^{e_m} \) if the path crosses \( m \) edges labeled \( x_1, \ldots, x_m \) with orientations given by \( e_i \). This is well defined since any deformation of the path which fixes the endpoints and which pushes it through a vertex will not change the product \( g(U) \) since the paths on either side of the vertex have edge labels giving a relation in the group and therefore give the same product of labels in the group \( G \). \( \Box \)

**Remark 3.1.3.** Any particular smooth path \( \gamma \) from \( U_\infty \) to \( U \) gives a lifting \( f_\gamma(U) \) of \( g(U) \) to the free group \( F_X \).

It is well-known that the set of deformation classes of pictures for any group \( G \) is a \( \mathbb{Z}G \)-module \( P(G) \). (See Theorem 3.1.5 and Corollary 3.1.7 below.)

The action of the group \( G \) is very easy to describe. Given any picture \( L \) and any generator \( x \in X \), the pictures \( xL, x^{-1}L \) are given by enclosing the set \( L \) with a large circle, labeling the circle with \( x \) and orienting it inward or outward, respectively. Addition of pictures is given by disjoint union of translates of the pictures.

To define the equivalence relation which we call “deformation equivalence” of pictures, it is helpful to associate to each picture \( L \) an element \( \psi(L) \in \mathbb{Z}G(\mathcal{Y}) \) where \( \mathbb{Z}G(\mathcal{Y}) \) is the free \( \mathbb{Z}G \) module generated by the set of relations \( \mathcal{Y} \). This is given by

\[
\psi(L) = \sum_{v_i} g(v_i) \langle r_i \rangle
\]

where the sum is over all vertices \( v_i \) of \( L \), \( r_i \in \mathcal{Y} \cup \mathcal{Y}^{-1} \) is the relation at \( v_i \), \( g(v_i) \in G \) is the group label at the basepoint direction of \( v_i \) and \( \langle r^{-1} \rangle = -\langle r \rangle \) by definition.

**Definition 3.1.4.** A deformation \( L_0 \simeq L_1 \) of pictures for \( G \) is defined to be a sequence of allowable moves given as follows.

1. **Isotopy.** \( L_0 \simeq L_1 \) if there is an orientation preserving diffeomorphism \( \varphi : \mathbb{R}^2 \to \mathbb{R}^2 \) so that \( L_1 = \varphi(L_0) \) with corresponding labels. By isotopy we can make the images of the embeddings \( \theta_v : D^2 \to \mathbb{R}^2 \) disjoint and arbitrarily small.
2. **Smooth concordance of edge sets.** There are two concordance moves:
(a) If $L_0$ contains a circular edge $E$ with no vertices and $L_0$ does not have any point in the region enclosed by $E$ then $L_0 \simeq L_1$ where $L_1$ is obtained from $L_0$ by deleting $E$.

(b) If $U$ is a connected component of $\mathbb{R}^2 - L_0$ and two of the walls of $U$ have the same label $x$ oriented in the same way (inward towards $U$ or outward) then, choose a path $\gamma$ in $U$ connecting points on these two edges then perform the following modification of $L_0$ in a neighborhood of $\gamma$ to obtain $L_1 \simeq L_0$.

(3) Cancellation of vertices. Suppose that two vertices $v_0, v_1$ of $L_0$ have inverse labels $r, r^{-1}$. Suppose that there is a path $\gamma$ disjoint from $L_0$ connecting the basepoint directions of $v_0, v_1$. Let $V$ be the union of the $\theta_{v_0}(D^2), \theta_{v_1}(D^2)$ and a small neighborhood of the path $\gamma$. We can choose $V$ to be diffeomorphic to $D^2$. Then $L_0 \simeq L_1$ if $L_0, L_1$ are identical outside of the region $V$ and $L_1$ has no vertices in $V$. (The two vertices in $V \cap L_0$ cancel.)

Concordance means $L_0, L_1$ have the same vertex sets and are equal in a neighborhood of each vertex and that $f_{\gamma_i} \in F_X$ are equal for $L_0, L_1$ for some (and thus every) choice of paths $\gamma_i$ disjoint from vertices from $\infty$ to the basepoint direction of each vertex of $L_0$. The same paths work for $L_1$ since $L_0, L_1$ have the same vertex set.

**Theorem 3.1.5.** [8][16][9] Prop 7.4 $L_0, L_1$ are deformation equivalent if and only if $\psi(L_0) = \psi(L_1)$. Furthermore, the set of possible values of $\psi(L)$ for all pictures $L$ is equal to the kernel of the mapping

$$\mathbb{Z}G\langle\mathcal{Y}\rangle \xrightarrow{d_2} \mathbb{Z}G\langle\mathcal{X}\rangle$$

where $d_2 \langle r \rangle = \sum \partial_x r\langle x \rangle$, where $\partial_x$ is the Fox derivative of $r$ with respect to $x$.

The Fox derivative of $w \in F_X$ is given recursively on the reduced length of $w$ by

1. $\partial_x(x) = 1, \partial_x(x^{-1}) = -x^{-1}$.
2. $\partial_x(y) = 0$ if $y \in \mathcal{X} \cup \mathcal{X}^{-1}$ is not equal to $x, x^{-1}$.
3. $\partial_x(ab) = \partial_xa + a\partial_xb$ for any $a, b \in F_X$.

**Definition 3.1.6.** The group of pictures $P(G)$ is defined to be the group of deformation classes of pictures for $G$.

**Corollary 3.1.7.** There is an exact sequence of $\mathbb{Z}G$-modules

$$0 \to P(G) \to \mathbb{Z}G\langle\mathcal{Y}\rangle \xrightarrow{d_2} \mathbb{Z}G\langle\mathcal{X}\rangle \xrightarrow{d_1} \mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

where $d_1 \sum a_i \langle x_i \rangle = \sum a_i(x_i - 1), \epsilon : \mathbb{Z}G \to \mathbb{Z}$ is the augmentation map and $d_2$ is as above.
Remark 3.1.8. The chain complex $\mathbb{Z}G\langle Y \rangle \xrightarrow{d_2} \mathbb{Z}G\langle X \rangle \xrightarrow{d_1} \mathbb{Z}G$ is the cellular chain complex of the universal covering $\tilde{X}^2$ of the 2-dimensional CW complex $X^2$ constructed above. Since $\tilde{X}^2$ is simply connected, we have

$$P(G) = H_2(\tilde{X}^2) = \pi_2(\tilde{X}^2) = \pi_2(X^2).$$

Therefore, $P(G) = \pi_2(X^2)$ as claimed at the beginning of this subsection.

We also use “partial pictures”. These are given by putting a picture in half using a straight line transverse to the picture.

Definition 3.1.9. Let $w$ be a word in $X \cup X^{-1}$ given by a finite subset $W$ of the $x$-axis together with a mapping $W \to X \cup X^{-1}$. A partial picture with boundary $w$ is defined to be a closed subset $L$ of the upper half plane so that the intersection of $L$ with the $x$-axis is equal to $W$ together with labels on $L$ so that the union of $L$ and its mirror image $L_-$ in the lower half plane is a picture for $G$ and so that the labels on the edges which cross the $x$-axis agree with the given mapping $W \to X \cup X^{-1}$. We call $L \cup L_-$ the double of $L$.

A deformation of a partial picture $L$ is defined to be any deformation of its double which is, at all times, transverse to the $x$-axis. It is clear that deformation of partial pictures preserves its boundary $\partial L = w$ and that $w$ lies in the relation group $R_Y \subseteq F_X$. The main theorem about partial pictures is the following.

Theorem 3.1.10. [8] The set of deformation classes of partial pictures forms a (nonabelian) group $Q(G)$ given by generators and relations as follows.

1. The generators of $Q(G)$ are pairs $(f, r)$ where $f \in F_X$ and $r \in Y$.
2. The relations in $Q(G)$ are given by

$$((f, r)(f', r'))(f, r)^{-1} = (frf^{-1}f', r').$$

Note that there is a well defined group homomorphism

$$\varphi : Q(G) \to F_X$$

given by $\varphi(f, r) = frf^{-1}$. The image is $R_Y$, the normal subgroup generated by all $r \in Y$.

3.2. Pictures with good commutator relations. If the same letter, say $x$, occurs more than twice in a relation $r$, then, at the vertex $v$, the edge set $E(x)$ cannot be a manifold. (For example, if $G = \langle x \mid x^3 \rangle$ then $E(x)$ will not be a manifold.) However, this does not happen in our case because our relations are “good”.

We define a [good commutator relation](#) to be a relation of the form

$$r(a, b) := ab(bc_1, \ldots, c_k a)^{-1}$$

where $a, b, c_1, \ldots, c_k$ are distinct elements of $X$ and $k \geq 0$. The letters $a, b$ will be called $X$-letters and the letters $c_j$ will be called $Y$-letters in the relation. In the picture, the two $X$-letters in any commutator relation form the shape of the letter “X” since the lines labeled with these letters go all the way through the vertex. Call these $X$-edges at the vertex. The edges labeled with the $Y$-letters go only half way and stop at the vertex. Call these $Y$-edges at the vertex. (See Figure 6.) In the definition of a picture we can choose the sets $E_r \subseteq S^1$ so that the points labeled $a, a^{-1}$ (and $b, b^{-1}$) are negatives of each other. Then the edge sets $E(a), E(b)$ will be manifolds. (Since $a, b, c_j$ are all distinct there are no other coincidences of labels at the vertices.)
Proposition 3.2.1. Suppose that $G = \langle X \mid Y \rangle$ is a group having only good commutator relations. Then, given any label $x$, the edge set $E(x)$ in $L$ is a disjoint union of smooth simple closed curves and smooth paths. At both endpoints of each path, $x$ occurs as a $Y$-letter. It occurs as $x$ at one end and $x^{-1}$ at the other.

Corollary 3.2.2. Suppose that $G$ has only good commutator relations. Then, for any picture $L$ for $G$ and any label $x$, the number of vertices of $L$ having $x$ as $Y$-letter is equal to the number of vertices of $L$ having $x^{-1}$ as $Y$-letter.

3.3. Atoms. Let $S = (\beta_1, \ldots, \beta_m)$ be an admissible sequence of real Schur roots for a hereditary algebra $\Lambda$. Then $G(S)$ has only good commutator relations. We need the Atomic Deformation Theorem which says that every picture in $G(S)$ is a linear combination of “atoms”. In other words atoms generate $P(G)$. The definition comes from [13] and [15] but is based on [14] where similar generators of $P(G)$ are constructed for a torsion-free nilpotent group $G$.

Suppose that $S$ is admissible and $\alpha_s = (\alpha_1, \alpha_2, \alpha_3)$ is a sequence of three hom-orthogonal roots in $S$ ordered in such a way that $\text{ext}(\alpha_i, \alpha_j) = 0$ for $i < j$. Let $A(\alpha_s)$ be the rank 3 wide subcategory of $\text{mod-}\Lambda$ with simple objects $\alpha_i$. One easy way to describe this category is

$$A(\alpha_s) = (\perp M_{\alpha_s})^\perp$$

where $M_{\alpha_s} = M_{\alpha_1} \oplus M_{\alpha_2} \oplus M_{\alpha_3}$. In other words, $A(\alpha_s)$ is the full subcategory of $\text{mod-}\Lambda$ of all modules $X$ having the property that $\text{Hom}(X, Y) = 0 = \text{Ext}(X, Y)$ for all $Y$ having the property that $\text{Hom}(M_{\alpha_s}, Y) = 0 = \text{Ext}(M_{\alpha_s}, Y)$. The objects of $A(\alpha_s)$ are modules $M$ having filtrations where the subquotients are $M_{\alpha_i}$. Since $\text{ext}(\alpha_i, \alpha_j) = 0$ for $i < j$, the modules $M_{\alpha_1}$ occur at the bottom of the filtration and $M_{\alpha_3}$ occurs at the top of the filtration. Let $\text{wide}(\alpha_s)$ denote the set of all dimension vectors of the objects of $A(\alpha_s)$. The elements of $\text{wide}(\alpha_s)$ are all nonnegative integer linear combinations of the roots $\alpha_i$. These are elements of the 3-dimensional vector space $\mathbb{R}\alpha_s$ spanned by the roots $\alpha_s$.

Let $L(\alpha_s) \subseteq S^2$ be the semi-invariant picture for the category $A(\alpha_s)$. We recall ([13], [15], [10]) that $L(\alpha_s)$ is the intersection with the unit sphere $S^2 \subseteq \mathbb{R}\alpha_s \cong \mathbb{R}^3$ with the union of the 2-dimensional subset $D(\beta)$ of $\mathbb{R}\alpha_s$ where $\beta \in \text{wide}(\alpha_s)$ given by the stability conditions:

$$D(\beta) := \{ x \in \mathbb{R}\alpha_s : \langle x, \beta \rangle = 0, \langle x, \beta' \rangle \geq 0 \text{ for all } \beta' \subset \beta, \beta' \in \text{wide}(\alpha_s) \}$$

When we stereographically project $L(\alpha_s) \subset S^2$ into the plane $\mathbb{R}^2$ we get a planar picture for the group $G(\text{wide}(\alpha_s))$ according to the definitions in this section.

Definition 3.3.1. Let $S, \alpha_s$ be as above. Then the atom $A_S(\alpha_s) \subset \mathbb{R}^2$ is defined to be the picture for $G(S)$ given by taking the semi-invariant picture $L(\alpha_s) \subset S^2$, stereographically projecting it away from the point $-\sum \dim P_i \in \mathbb{R}\alpha_s$ where $P_i$ are the projective objects of $A(\alpha_s)$ and deleting all edges having labels $x(\gamma)$ where $\gamma \notin S$.

![Figure 6. The X-letters a, b have edge sets which are smooth at the vertex. The basepoint direction is on the negative side of both X-edges E(a), E(b).](image)
Figure 7 gives an example of an atom. We need to prove that certain aspects of the shape are universal.

**Proposition 3.3.2.** Any atom $A_S(\alpha_1, \alpha_2, \alpha_3)$ has three circles $E(\alpha_i) = D(\alpha_i)$ with labels $x(\alpha_i) \in G$ and all other edge sets have two endpoints. There is exactly one vertex $v$ outside the $\alpha_3$ circle. This vertex has the relation $r(\alpha_1, \alpha_2)$. Dually, there is exactly one vertex inside the $\alpha_1$ circle with relation $r(\alpha_2, \alpha_3)^{-1}$.

We use the notation $r(\alpha, \beta)$ for $r(x(\alpha), x(\beta))$. For example, the blue lines in Figure 6 meet at two vertices giving the relations

$$r(\alpha, \beta) = x(\alpha)x(\beta)(x(\beta)x(\gamma_1)x(\gamma_2)x(\alpha))^{-1}$$

at the top and $r(\alpha, \beta)^{-1}$ in the middle of the brown $x(\omega)$ circle.

**Proof.** The only objects of $A(\alpha_4)$ which do not map onto $M_{\alpha_3}$ are the objects of $A(\alpha_1, \alpha_2)$ which are the objects $M_{\alpha_1}, M_{\alpha_2}$ and their extensions $M_{\gamma_j}$. These give the terms in the commutator relation $r(\alpha_1, \alpha_2)$ and these lines meet at only two vertices in the atom. All other edges of the atom have at least one abutting edge with a label $\gamma$ where $\gamma \rightarrow \alpha_3$. By the stability condition defining $D(\gamma)$, these points must be inside or on the $\alpha_3$ circle as claimed. \hfill \square

**3.4. Sliding Lemma and Atomic Deformation Theorem.** We will prove the Sliding Lemma 3.4.3 and derive some consequences such as the Atomic Deformation Theorem 3.4.4 which says that every picture for $G(S)$ is a linear combination of atoms. First, some terminology. We say that $L'$ is an atomic deformation of $L$ if $L'$ is a deformation of $L$ plus a linear combination of atoms. Thus the Atomic Deformation Theorem states that every picture has an atomic deformation to the empty picture.

Suppose that $S$ is an admissible set of roots with a fixed lateral ordering and let $\omega \in S$. Recall that $S_-(\omega)$ is the set of all $\beta \leq \omega$ in lateral order in $S$. In particular, either $\beta = \omega$ or
\( \text{hom}(\omega, \beta) = 0 \) and \( \text{ext}(\beta, \omega) = 0 \). Also, \( R_-(\omega) \) is the set of all \( \beta \in S_-(\omega) \) which are hom-orthogonal to \( \omega \). Since these are relatively closed subsets of \( S \), the picture groups \( G(S_-(\omega)) \) and \( G(R_-(\omega)) \) are defined. (See Remark [1.1.10])

\[
S_-(\omega) := \{ \beta \in S : \beta \leq \omega \text{ in lateral order} \}
\]

\[
R_-(\omega) := \{ \beta \in S_-(\omega) : \text{hom}(\beta, \omega) = 0 \}
\]

for any \( \omega \in S \).

**Lemma 3.4.1** (Monomorphism Lemma). The homomorphism \( G(R_-(\omega)) \to G(S_-(\omega)) \) induced by the inclusion \( R_-(\omega) \hookrightarrow S_-(\omega) \) has a retraction \( \rho \) given on generators by

\[
\rho(x(\beta)) = \begin{cases} 
  x(\beta) & \text{if } \beta \in R_-(\omega) \\
  1 & \text{otherwise}
\end{cases}
\]

Furthermore, \( \rho \) takes pictures and partial pictures \( L \) for \( G(S_-(\omega)) \) and gives a picture or partial picture \( \rho(L) \) for \( G(R_-(\omega)) \) by simply deleting all edges with labels \( x(\beta) \) where \( \beta \notin R_-(\omega) \).

Figure 9 gives an example of how this lemma is used. The proof is analogous to the proof of the dual statement which goes as follows. Recall that, for any \( \alpha \) in an admissible set of roots \( S \), \( S_+ (\alpha) \) is the set of all \( \beta \geq \alpha \) in \( S \) and \( R_+(\alpha) \) is the set of all \( \beta \in S_+ (\alpha) \) which are hom-orthogonal to \( \alpha \). As in the case of \( S_-(\omega) \), \( R_-(\omega) \) these are relatively closed subsets of \( S \).

**Lemma 3.4.2.** The homomorphism \( G(R_+(\alpha)) \to G(S_+(\alpha)) \) induced by the inclusion \( R_+(\alpha) \hookrightarrow S_+(\alpha) \) has a retraction \( \rho \) given on generators by

\[
\rho(x(\beta)) = \begin{cases} 
  x(\beta) & \text{if } \beta \in R_+(\alpha) \\
  1 & \text{otherwise}
\end{cases}
\]

Furthermore, \( \rho \) takes pictures and partial pictures \( L \) for \( G(S_+(\alpha)) \) and gives a picture or partial picture \( \rho(L) \) for \( G(R_+(\alpha)) \) by simply deleting all edges with labels \( x(\beta) \) where \( \beta \notin R_+(\alpha) \).

**Proof.** The key is that \( R_+(\alpha) \) is given by a linear condition. Since \( \text{ext}(\alpha, \beta) = 0 \) for all \( \beta \in S_+(\alpha) \) (and \( \text{hom}(\beta, \alpha) = 0 \) for all \( \beta \neq \alpha \) in \( S_+(\alpha) \)) we have:

\[
R_+(\alpha) = \{ \beta \in S_+(\alpha) : \langle g(\alpha), \beta \rangle = \text{hom}(\alpha, \beta) - \text{ext}(\alpha, \beta) = 0 \}.
\]

Since any two letters in any relation in are linearly independent, if two letters in any relation in \( G(S_+(\alpha)) \) lie in \( R_+(\alpha) \) then all the letters in the relation lie in \( R_+(\alpha) \). Thus, if only part of the relation survives under the retraction it must be a single letter. This letter, say \( \gamma \), cannot be a Y-letter: If it were and \( \gamma_1, \gamma_2 \) are the X-letters in that relation then \( \text{hom}(\alpha, \gamma_1) \) and \( \text{hom}(\alpha, \gamma_2) \) would both be nonzero. Since one of these is a subroot of \( \gamma \), this would also make \( \text{hom}(\alpha, \gamma) \neq 0 \) and \( \gamma \notin R_+(\alpha) \). So, none of the letters in such a relation will lie in \( R_+(\alpha) \). Therefore, the retraction \( S_+(\alpha) \to R_+(\alpha) \) sends relations to relations and induces a retraction of groups \( \rho : G(S_+(\alpha)) \to G(R_+(\alpha)) \).

Given any picture or partial picture \( L \) for \( G(S_+(\alpha)) \), each vertex has a relation \( r \) which has the property that either \( \rho(r) = r \) or \( \rho(r) \) is an unreduced relation of the form \( xx^{-1} \) or \( r(r) \) is empty. In the second case \( \rho(r) = xx^{-1} \) we consider the vertex as part of the smooth curve \( E(x) \). Removal of all edges with labels not in \( R_+(\alpha) \) therefore keeps \( L \) looking locally like a picture for \( R_+(\alpha) \). But pictures and partial pictures are defined by local conditions. \( \square \)

Using the Monomorphism Lemma [3.4.1] we can now state and prove the key lemma about pictures for \( G(S) \). Recall that \( E(\omega) \) is the union of the set of edges with label \( x(\omega) \) and that, for any root \( \beta \in R_-(\omega) \), any vertex with relation \( r(\beta, \omega) \) or \( r(\beta, \omega)^{-1} \) has Y-edges on the
positive side of the X-line $E(\omega)$. (For example, in Figure 7, $\alpha, \beta$ and all letters $\gamma_i$ in $r(\alpha, \beta)$ lie in $R_-(\omega)$). So the edges corresponding to the commutator relations $r(\gamma_i, \omega)$ for all letters $\gamma_i$ in $r(\alpha, \beta)$ lie in the interior of the brown circle $E(\omega) = D(\omega)$. Since Figure 7 is an atom, the edges are curved in the positive direction.) We also note that the base point direction is on the negative side of both X-lines at each crossing.

**Lemma 3.4.3 (Sliding Lemma).** Suppose that $L$ is a picture for $G(S)$ so that $E(\omega)$ is a disjoint union of simple closed curves. Let $U$ be one of the components of the complement of $E(\omega)$ and let $\Sigma = \overline{U} \cap E(\omega)$ be the boundary of the closure $\overline{U}$ of $U$. Suppose that $U$ is on the negative side of $\Sigma$ and that all edges in $L \cap U$ have labels $x(\beta)$ for $\beta \in R_-(\omega)$. Then there is an atomic deformation $L \sim L'$ which alters $L$ only in an arbitrarily small neighborhood $V$ of $\overline{U}$ so that $L' \cap V$ contains no edges with labels $\geq \omega$ in lateral order.

**Proof.** By assumption, every edge which crosses $\Sigma$ has a label $x(\beta)$ where $\beta \in R_-(\omega)$. This implies that all Y-edges at all vertices on $\Sigma$ lie outside the region $U$. So, at each vertex of $\Sigma$, only one edge $E(\beta)$ goes into the region $U$. Also, all basepoint directions of all vertices on $\Sigma$ lie inside $U$.

The proof of the lemma is by induction on the number of vertices in the region $V$ containing $\Sigma$. Suppose first that this number is zero. Then $\Sigma$ has no vertices and $L \cap U$ is a union of disjoint simple closed curves (scc) which can be eliminated by concordance one at a times starting with the innermost scc. This includes $\Sigma$. The result has no edges with labels $\geq \omega$.

Suppose next that $L$ has vertices on the set $\Sigma$ but no vertices in the region $U$ enclosed by $\Sigma$. Then every edge of $L$ in $U$ is an arc connecting two vertices on $\Sigma$ and the negative side of each arc has a path connecting the two basepoint directions at these two vertices. So, we can cancel all pair of vertices and we will be left with no vertices in $V$. As before, we can then eliminate all closed curves in $V$ including $\Sigma$ which has now become a union of scc’s.

Finally, suppose that $U$ contains a vertex $v$ having relation $r(\alpha, \beta)^\pm$. So, $v$ contributes $\pm g \langle r(\alpha, \beta) \rangle$ to the algebraic expression for $L$. Then $\alpha, \beta \in R_-(\omega)$ by assumption. Now add the atom $\mp A(\alpha, \beta, \omega)$ (which resembles Figure 7) in the region containing the basepoint direction of $v$. (See the left side of Figure 8.) This adds $\mp g A(\alpha, \beta, \omega)$ to the algebraic expression for $L$. The atom has a circle labeled $x(\omega)$ oriented inward with exactly one vertex outside this circle with relation $r(\alpha, \beta)^\mp$ (the mirror image of the relation at $v$) by Proposition 3.3.2. The new vertex cancels $v$. (See the right side of Figure 8.) Repeating this process eliminates all vertices in the new region $U'$.

After that, all edges in $\overline{U}'$, the closure of $U'$ can be eliminated. This eliminates all edges with label $\omega$ from $V$. However, it also introduces new edge sets (the interior of the $\omega$ oval in the atom). However, these all have labels $< \omega$. So, we are done.

Since the entire process was a sequence of picture deformations and addition of $ZG(S)$ multiples of atoms, it is an atomic deformation.

**Theorem 3.4.4 (Atomic Deformation Theorem).** Suppose that $S$ is an admissible set of real Schur roots. Then any picture for $G(S)$ has a null atomic deformation. I.e., it is deformation equivalent to a $ZG$ linear combination of atoms. Equivalently, the $ZG(S)$-module $P(G(S))$ is generated by atoms.

This theorem follows from the Sliding Lemma and we will see that it implies Lemma 3.4.3.

**Proof.** Let $S = (\beta_1, \cdots, \beta_m)$ be an admissible set of roots. Let $\beta^1, \cdots, \beta^m$ be the same set rearranged in lateral order. Let $R^k$ be the set of all elements of $S$ which are $\leq \beta^k$ in lateral
If $\Sigma$ is a disjoint union of $E(\omega)$ closed curves which encloses a region $U = \Sigma \cup U$. All Y-edges for vertices on $\Sigma$ lie outside $U$. The atom $A(\alpha, \beta, \omega)$ in the proof has already been added on the left. The new region $U'$ is the complement of the new $\omega$ oval in $U$. The vertex $v$ has been cancelled with the vertex in the atom on the right.

order. Thus, $R^k = S_-(\beta^k)$. Take $k$ minimal so that the labels which occurs in $L$ all lie in $R^k$. If $k = 1$ then $L$ has no vertices and is a disjoint union of simple closed curves (scc’s) which are null homotopic. By induction, it suffices to eliminate $\omega = \beta^k$ as a label from the picture $L$ by picture deformations and addition of atoms without introducing labels $\beta^j$ for $j > k$.

Since $\omega$ is a rightmost element in the set $R^k$, $x(\omega)$ does not occur as a Y-letter at any vertex of $L$. Therefore the edge set $E(\omega)$ is a disjoint union of simple closed curves. Let $\Sigma$ be innermost such curve and $\Sigma'$ be a curve parallel to $\Sigma$ on the negative side. (See Figure 9) Then $\Sigma'$ crosses on those edges $E(\beta)$ where $\beta \in S_-(\omega)$ are hom-orthogonal to $\omega$. In other words, $\beta \in R_-(\omega)$.

Let $L_0'$ be the mirror image of $L_0$ through $\Sigma'$. Then $L_0 \cup L_0'$ is null deformable, i.e., $L_0 + L_0' = 0$ in the group of partial pictures $Q(R^k)$. Since $\Sigma'$ meets only edges with labels in $R_-(\omega)$, we can apply the retraction $\rho$ from the Monomorphism Lemma 3.4.1 to just one side of $\Sigma'$ and still have a well-defined picture. This construction gives us two pictures: $L' = \rho(L_0) \cup L_1$ and $L'' = L_0 \cup \rho(L_0')$.

Claim $L$ is deformation equivalent to $L' \bigsqcup L''$, i.e., $L = L' + L''$ in the group $P(R^k)$.

Pf. The group of pictures $P(R^k)$ is a a subgroup of the group of partial pictures $Q(R^k)$ and in that group we have:

$$L = L_0 + L_1 = L_0 + \rho(L_0) + \rho(L_0) + L_1 = L'' + L'$$

since $\rho(L_0') + \rho(L_0) = \rho(L_0 + L_0') = \rho(0) = 0$.

The scc $\Sigma$ lies either in $L'$ or $L''$. If $\Sigma \subset L'$ then $\Sigma$ can be removed by $L'$ by an atomic deformation by the Sliding Lemma 3.4.3 since the edges inside $\Sigma$ are in $R_-(\omega)$, bing in $\rho(L_0)$. If $\Sigma \subset L''$ (as drawn in Figure 9), the region outside $\Sigma$ has all labels in $R_-(\omega)$. So, it can be removed by Lemma 3.4.3. In both cases, the number of $E(\omega)$ components in $L' \bigsqcup L''$ (the same as the number of components in $L$) has been reduced by one by an atomic deformation without introducing any new labels $\geq \omega$. By induction on the number of components of $E(\omega)$, this set can be removed and $k$ can be reduced by one. So, by induction on $k$, we are done. The entire picture can be deformed into nothing by atomic deformation. \hfill $\square$

3.5. Proofs of the lemmas. The proofs of Lemmas $C$ and $E$ are very similar.
Figure 9. Illustrating proof of Atomic Deformation Theorem 3.4.4. \(\Sigma'\) (in red) is on the negative side of an innermost \(E(\omega)\) curve \(\Sigma\) (in blue). The picture \(L = L_0 \cup L_1\), on the left, is deformation equivalent to the disjoint union of two pictures: \(L'' = L_0 \cup \rho(L_0')\), in the middle, and \(L' = \rho(L_0) \cup L_1\) on the right. The \(E(\omega)\) component \(\Sigma\) lie either in \(L'\) or \(L''\). (Here it is in \(L''\) in the middle.) In either case, it can be removed by the Sliding Lemma 3.4.3.

Proof of Lemma 3.4.4. Suppose that \(w, w'\) are expressions for the same element of \(G(\mathcal{S})\) and \(\pi(w), \pi(w')\) are equal as words in the generators of \(G(\mathcal{S}_0)\). This means that \(\pi(w^{-1}w')\) reduces to the trivial (empty) word in \(G(\mathcal{S}_0)\).

Let \(L\) be a partial picture giving the proof that \(w^{-1}w'\) is trivial in \(G(\mathcal{S})\). Then \(\pi(L)\) can be completed to a true picture \(L_0\) for the group \(G(\mathcal{S}_0)\) by joining together cancelling letters in \(\pi(w^{-1}w')\). By the Atomic Deformation Theorem 3.4.4, \(L_0\) is equivalent to a sum of atoms. However, each atom \(A\) for \(G(\mathcal{S}_0)\) can be lifted to an atom \(\hat{A}\) for \(G(\mathcal{S})\) by definition of the atoms. Therefore, up to deformation equivalence, \(L\) can be lifted to a picture \(\hat{L}\) for \(G(\mathcal{S})\). By Corollary 3.2.2 the number of vertices of \(\hat{L}\) having \(x(\beta_m)\) as \(Y\)-letter is equal to the number of vertices having \(x(\beta_m)^{-1}\) as \(Y\)-letter. This implies that the number of vertices in \(L_0\) lifting to ones in \(\hat{L}\) having \(x(\beta_m)^{-1}\) as \(Y\)-letter is equal to the number of vertices in \(L_0\) lifting to ones in \(\hat{L}\) having \(x(\beta_m)^{-1}\) as \(Y\)-letter are equal. So, the number of times \(x(\beta_m), x(\beta_m)^{-1}\) occur as \(Y\)-letters in \(L\) are equal. So, the number of times that \(x(\beta_m), x(\beta_m)^{-1}\) occur in the word \(w^{-1}w'\) are equal. So, \(x(\beta_m)\) occurs the same number of times in the words \(w, w'\) as claimed.

Proof of Lemma 3.4.5. Recall that \(\beta_m\) is the last element of an admissible set \(\mathcal{S}\). Lemma E says that if \(w_0\) is a positive expression for some element of \(G(\mathcal{S})\) which commutes with \(x(\beta_m)\) then every letter of \(w_0\) commutes with \(\beta_m\). To prove this, suppose not and let \(w_0\) be a minimal length positive expression in the letters \(\mathcal{S}\) satisfying the following.

1. As an element of \(G(\mathcal{S})\), \(w_0\) commutes with \(x(\beta_m)\).
2. One of the letters of \(w_0\), say \(x(\beta)\), does not commute with \(x(\beta_m)\). Equivalently, \(\beta, \beta_m\) are not hom-orthogonal (Remark 1.3.1).

Clearly, \(w_0\) has at least 2 letters and the first and last letter of \(w_0\) do not commute with \(x(\beta_m)\).

In the group \(G(\mathcal{S})\) we have the relation

\[ W = w_0 x(\beta_m) w_0^{-1} x(\beta_m)^{-1} = 1. \]

A proof of the relation \(W = 1\) gives a partial picture \(L\) for \(G(\mathcal{S})\) having the word \(W\) as its boundary. Let \(\beta^1, \ldots, \beta^m\) be the letters in \(\mathcal{S}\) in lateral order. Then \(\beta_m = \beta^k\) for some \(k\). Let \(\beta^i, \beta^j\) be the letters which occurs in the partial picture \(L\) with \(i\) minimal and \(j\) maximal. Then
\(i < j\) and \(i \leq k \leq j\). In particular, either \(i < k\) or \(k < j\). By symmetry we may assume that \(k < j\). Then we will use the Monomorphism Lemma \([3.4.1]\) for \(\omega = \beta^j \neq \beta_m\). (For \(k = j\) the argument is the same using the dual lemma \([3.4.2]\) with \(\alpha = \beta^i\).)

There are two cases. Either \(\lambda = \beta^j\) is a letter in \(W\) or not.

**Case 1** \(\lambda\) is not a letter in \(W\). Then the edge set \(E(\lambda)\) is a disjoint union of simple closed curves. We claim that these can all be eliminated by Lemmas \([3.4.1, 3.4.3]\). Let \(\Sigma\) be any component of \(E(\lambda)\). Let \(\Sigma'\) be a parallel curve on the negatives side of \(\Sigma\). Then \(\Sigma'\) crosses only edges \(E(\beta)\) where \(\beta\) is hom-orthogonal to \(\lambda\). Therefore, we can apply the retraction \(\rho : G(S_-(\lambda)) \to G(R_-(\lambda))\) to the region enclosed by \(\Sigma'\) to eliminate all edges in that region which are not hom-orthogonal to \(\lambda\). By the Sliding Lemma \([3.4.3]\) we can then eliminate \(\Sigma\) if it is still there. Repeating this process produces a new partial picture \(L'\) with boundary \(W\) so that the laterally rightmost letter in \(L'\) is a letter in \(W\), i.e., we are reduced to Case 2.

**Case 2** \(\lambda = \beta^j\) is a letter in \(W\). Since \(j > k\), \(\lambda\) is then a letter in \(w_0\). The generator \(x(\lambda)\) may occur several times in \(w_0\) and \(x(\lambda)^{-1}\) occurs in \(w_0^{-1}\). Taking the first occurrence of \(x(\lambda)\) in \(w_0\) we can write \(w_0 = w_1x(\lambda)w_2\) there \(x(\lambda)\) is not a letter in \(w_1\). Then

\[W = w_1x(\lambda)w_2x(\beta_m)w_2^{-1}x(\lambda)^{-1}w_1^{-1}x(\beta_m)^{-1}\]

is the boundary of \(L\) which is a partial picture for \(G(S_-(\lambda))\). Since \(\lambda\) is rightmost in later order, \(x(\lambda)\) does not occur as a \(Y\)-letter at any of the vertices of \(L\). Therefore, the edge set \(E(\lambda)\) is a disjoint union of simple closed curves and disjoint arcs connecting the \(x(\lambda)\) in \(w_0\) to the \(x(\lambda)^{-1}\) in \(w_0^{-1}\). Since these arc are disjoint, the outermost such arc \(\Sigma\) connects the first occurrence of \(x(\lambda)\) in \(w_0\) to the last occurrence of \(x(\lambda)^{-1}\) in \(w_0^{-1}\). Let \(\Sigma'\) be an arc parallel to \(\Sigma\) on its negative side. Thus \(L = L_0 \cup L_1\) where \(L_0\) is the portion of \(L\) enclosed by \(\Sigma'\). Since \(x(\lambda)\) is to the left of \(x(\lambda)^{-1}\), \(\Sigma \subset L_0\). (See the left side of Figure 10)

**Figure 10.** (Proof of Lemma E) The partial picture \(L\) for \(G(S_-(\lambda))\) is divided into two parts \(L = L_0 \cup L_1\) by \(\Sigma'\). Applying \(\rho : G(S_-(\lambda)) \to G(R_-(\lambda))\) to \(L_0\) eliminates \(x(\lambda)\) from the word \(w_0 = w_1x(\lambda)x_2\) but does not eliminate \(x(\beta_m)\). Then \(w_1\rho(w_2)\) commutes with \(x(\beta_m)\) contradicting the minimality of \(w_0\).

Using the Monomorphism Lemma \([3.4.1]\) we apply the retraction \(\rho\) to \(L_0\). This will eliminate \(\Sigma\) and all occurrences of the letter \(x(\lambda)\) in \(W\) giving a new relation:

\[w_1\rho(w_2)\rho(x(\beta_m))\rho(w_2)^{-1}w_1^{-1}x(\beta_m)^{-1} = 1\]

or, equivalently, \(w_1\rho(w_2)\rho(x(\beta_m)) = x(\beta_m)w_1\rho(w_2)\). By Lemma C proved above, \(x(\beta_m)\) occurs the same number of times in these two expressions. So, \(\rho(x(\beta_m)) = x(\beta_m)\). In particular, \(\lambda\) is hom-orthogonal to \(\beta_m\). Equivalently \(x(\lambda)\) commutes with \(x(\beta_m)\). So, \(x(\lambda)\) is not the first letter of \(w_0\) which means \(w_1\) is a nontrivial word.
This gives a new word $w'_0 = w_1\rho(w_2)$ which is shorter than $w_0$, commutes with $x(\beta_m)$ and has at least one letter (the first letter of $w_1$) which does not commute with $x(\beta_m)$. This contradicts the minimality of $w_0$ and completes the proof of Lemma E.

4. Appendix

This Appendix contains basic background material for this paper. Details can be found in [12] and [11]

4.1. Exceptional representations of modulated quivers. We assume throughout the paper that $Q$ is a quiver without loops, oriented cycles or multiple edges $i \to j$ (since multiplicity of edges is included in the valuation). We recall briefly that a valuation on a quiver $Q$ is given by assigning positive integers $f_i$ to each vertex $i$ and pairs of positive integers $(d_{ij}, d_{ji})$ to every arrow $i \to j$ in $Q$ having the property that $f_id_{ij} = f_jd_{ji}$. For example, the Kronecker quiver is $\bullet \xrightarrow{(2,2)} \bullet$. A $K$-modulation of a valued quiver is given by assigning a division algebra $F_i$ of dimension $f_i$ at each vertex and an $F_i$-$F_j$-bimodule $M_{ij}$ on each arrow $i \to j$ with $\dim_K M_{ij} = f_id_{ij} = f_jd_{ji}$. A representation of a modulated quiver consists of a right $F_i$-vector space $V_i$ at each vertex and an $F_j$-linear map $V_i \otimes M_{ij} \to V_j$ on each arrow $i \to j$. A representation $V$ is called a brick if its endomorphism ring is a division algebra. An exceptional module is a brick having no self-extensions. For hereditary algebras of finite type, all bricks are exceptional.

Given any module $X$ we denote by $X^\perp$ the full subcategory of $\text{mod-}\Lambda$ with all objects $Y$ so that

$$\text{Hom}_\Lambda(X, Y) = \text{Ext}_\Lambda(X, Y)$$

Similarly, $\perp X$ is the category of all $\Lambda$-modules $Y$ so that $X \in Y^\perp$. An exceptional sequence of length $k$ is defined to be a sequence of exceptional modules $E_1, E_2, \ldots, E_k$ so that $E_i \in E_j^\perp$ for all $i \leq j$.

The dimension vector $\dim V$ of a representation of a modulated quiver is defined to be $(d_1, d_2, \ldots, d_n)$ where $d_i$ is the dimension of $V_i$ as a vector space over $F_i$. A real Schur root of the valued quiver $Q$ is defined to be the dimension vector of an exceptional module for any modulation of $Q$. This concept is known to be independent of the choice of modulation. See [12] for details. In this paper we assume a modulation is given.

The semi-stability set $D(V)$ of any module $V$ is defined by

$$D(V) := \{ x \in \mathbb{R}^n : \langle x, \dim V \rangle = 0 \text{ and } \langle x, \dim V' \rangle \leq 0 \text{ for all submodules } V' \subseteq V \}$$

where we use the bilinear pairing:

$$\langle x, y \rangle = \sum x_iy_if_i.$$

For any real Schur root $\beta$ let $D(\beta) = D(M_\beta)$ where $M_\beta$ is the unique exceptional module with dimension vector $\beta$. In this paper we use the following refinement of the definition of $D(\beta)$ which is essentially proved in [12].

**Theorem 4.1.1.** For $\beta$ a real Schur root and $x \in \mathbb{R}^n$ so that $\langle x, \beta \rangle = 0$, the following are equivalent.

1. $\langle x, \beta' \rangle \leq 0$ for all real Schur subroots $\beta'$ of $\beta$.
2. $\langle x, \dim V' \rangle \leq 0$ for all submodules $V \subseteq M_\beta$.
3. $\langle x, \dim V'' \rangle \geq 0$ for all quotient modules $V''$ of $M_\beta$.
4. $\langle x, \beta'' \rangle \geq 0$ for all real Schur quotient roots of $\beta$. 
Proof. It is shown in [12] that (1) is equivalent to (2) for \( x \in \mathbb{Z}^n \). This easily implies that (1) and (2) are equivalent for \( x \in \mathbb{Q}^n \). Taking the closure we get that (1) and (2) are equivalent for all \( x \in \mathbb{R}^n \).

The equivalence (2) \( \iff \) (3) is obvious. The equivalence (3) \( \iff \) (4) follows from the equivalence (1) \( \iff \) (2). Indeed, applying the duality functor \( D = \text{Hom}(\cdot, K) \), the exceptional \( \Lambda \)-module \( M_\beta \) and quotient module \( M_\beta'' \) become exceptional \( D\Lambda \) modules with the same dimension vectors, but \( DM_\beta'' \subset DM_\beta \). So, \( x \in \mathbb{R}^n \) satisfies (4) for \( \Lambda \) if and only if \( \langle -x, \beta'' \rangle \leq 0 \) for \( \beta'' \subset \beta \) (as \( D\Lambda \)-roots). Equivalently, \( x \in D\Lambda(\beta) \) using the criteria (1),(2) if and only if \( -x \in D\Lambda(\beta) \) using the quotient root criteria (4),(3) respectively. So (3) \( \iff \) (4). \( \square \)

Following [11], we use \( g \)-vectors and modified dot product in this paper instead of the Euler product used in [12]. and we define the \( g \)-vector of a module \( X \) to be

\[
g(X) := \dim P_0/\text{rad}P_0 - \dim P_1/\text{rad}P_1
\]

where

\[
0 \to P_1 \to P_0 \to X \to 0
\]

is the minimal projective presentation of \( X \). Equivalently, \( g(X) = C_\Lambda^{-1} \dim X \) where \( C_\Lambda \) is the Cartan matrix of \( \Lambda \).

Lemma 4.1.2. The \( g \)-vector of \( X \) satisfies the following for any representation \( V \).

\[
\langle g(X), \dim V \rangle = \dim K \text{Hom}_\Lambda(X, V) - \dim K \text{Ext}_\Lambda(X, V).
\]

In particular, \( \langle g(X), \dim V \rangle = 0 \) when \( X \in \perp V \).

Proof. This follows from the exact sequence:

\[
0 \to \text{Hom}_\Lambda(X, V) \to \text{Hom}_\Lambda(P_0, V) \to \text{Hom}_\Lambda(P_1, V) \to \text{Ext}_\Lambda(X, V) \to 0
\]

and the evident fact that \( \dim K \text{Hom}_\Lambda(P, V) = \langle g(P), \dim V \rangle \). \( \square \)

This immediately gives the following.

Proposition 4.1.3. The dimension vectors of modules in an exceptional sequence are linearly independent.

Proof. Suppose that \( E_1, \ldots, E_k \) is an exceptional sequence. Lemma 4.1.2 implies

\[
\langle g(E_j), \dim E_i \rangle = 0
\]

for all \( i < j \). But \( \langle g(E_j), \dim E_j \rangle = \dim K \text{End}_\Lambda(E_j) \neq 0 \). So, \( \dim E_j \) cannot be a linear combination of \( \dim E_i \) for \( i < j \). \( \square \)

The \( g \)-vector of a shifted projective module \( P[1] \) is define by \( g(P[1]) := -g(P) \).

We have the following “Virtual Stability Theorem” from [12].

Theorem 4.1.4. If \( X \in \perp M_\beta \) then \( g(X) \in D(\beta) \). If \( P \in \perp M_\beta \) is projective then \( g(P[1]) = -g(P) \in D(\beta) \). Conversely, for any \( x \in D(\beta) \cap \mathbb{Z}^n \) there is a module \( X \) and a projective module \( P \) so that

1. \( x = g(X \oplus P[1]) = g(X) - g(P) \).
2. \( X, P \in \perp M_\beta \), i.e., \( \text{Hom}(X \oplus P, M_\beta) = 0 = \text{Ext}(X, M_\beta) \).
4.2. Wide subcategories. Recall that a full subcategory \( \mathcal{W} \) of an abelian category \( \mathcal{A} \) is wide if it is closed under extension and kernels and cokernels of morphism between objects. This implies in particular that \( \mathcal{W} \) is closed under taking direct summands.

Returning to the case of \( \text{mod-} \Lambda \) for a hereditary algebra \( \Lambda \), we note that \( X^\perp \) is a wide subcategory for any object \( X \). To see this, look at the following six term exact sequence for any short exact sequence \( 0 \to A \to B \to C \to 0 \).

\[
0 \to \text{Hom}(X, A) \to \text{Hom}(X, B) \to \text{Hom}(X, C) \to \text{Ext}(X, A) \to \text{Ext}(X, B) \to \text{Ext}(X, C) \to 0
\]

If \( A, C \in X^\perp \) then we see that \( B \in X^\perp \). If \( B \in X^\perp \) then \( \text{Hom}(X, A) = 0 = \text{Ext}(X, C) \). So, any object which is both a subobject and quotient object of an object of \( X^\perp \) is also in \( X^\perp \). So, \( X^\perp \) is a wide subcategory of \( \text{mod-} \Lambda \). Similarly, \( \perp X \) is a wide subcategory.

Closely related to this example is the following well-known fact. (See [11] for a short proof.)

**Theorem 4.2.1.** Let \( R \) be any subset of \( \mathbb{R}^n \). Then the set \( \mathcal{W}(R) \) of all representation \( V \) so that \( R \subset \text{D}(V) \) is a wide subcategory of \( \text{mod-} \Lambda \).

Consider the case when \( R = \{ x_0 \} \) is a single point \( x_0 \neq 0 \in \mathbb{R}^n \). Suppose that \( \mathcal{S} \) is an admissible set of real Schur roots. Recall our notation that \( D(\beta) = D(M_\beta) \) where \( M_\beta \) is the unique exceptional module with dimension vector \( \beta \).

What can we say about the set of \( \beta \in \mathcal{S} \) so that \( x_0 \in D(\beta) \)?

**Proposition 4.2.2.** Let \( \alpha \in \mathcal{S} \). Then \( x_0 \) is in the interior of \( D(\alpha) \) if and only if \( M_\alpha \) is a minimal object of the wide subcategory \( \mathcal{W}(x_0) \).

**Proof.** If \( x_0 \) lies in the interior of \( D(\alpha) \), \( \langle x_0, \gamma \rangle < 0 \) for all subroots \( \gamma \subset \alpha \). So, \( x_0 \notin D(\gamma) \). So, \( \alpha \) is minimal. The converse follows in the same way. \( \square \)

A wide subcategory \( \mathcal{W} \subset \text{mod-} \Lambda \) has rank \( k \) if it is isomorphic to the module category of an hereditary algebra with \( k \) simple modules. More concretely, such a wide subcategory contains \( k \) Hom-orthogonal exceptional modules forming an exceptional sequence: \( X_1, X_2, \cdots, X_k \). In other words, \( \text{Ext}(X_j, X_i) = 0 \) for \( j \geq i \). And all other objects of \( \mathcal{W} \) are iterated extensions of the \( X_i \) with each other. From this description we see that the \( X_i \) are objects of \( \mathcal{W} \) of minimal length, i.e., proper subobjects and proper quotient objects of the \( X_i \) do not lie in \( \mathcal{W} \). In particular, the \( X_i \) are uniquely determined by \( \mathcal{W} \).

One special case which we need in this paper is the case \( k = n \).

**Theorem 4.2.3.** Let \( (E_1, \cdots, E_n) \) be an exceptional sequence of Hom-orthogonal objects in \( \text{mod-} \Lambda \). Then all \( E_i \) are simple. In particular, \( \text{mod-} \Lambda \) is the only wide subcategory of rank \( n \).

**Proof.** This follows from the theory of exceptional sequences. By [2] and [4], the action of the braid group on \( n \) strands acts transitively on the set of exceptional sequences of length \( n \). However, by definition, braid moves keep objects in the same wide subcategory which is the category of all objects which are iterated extensions of the \( E_i \) with each other. By the theorem of [2] and [4], this includes all exceptional sequences. But the sequence of simple modules of \( \text{mod-} \Lambda \) form an exceptional sequence. So, every simple \( \Lambda \)-module is in our wide subcategory. So, the wide subcategory is all of \( \text{mod-} \Lambda \). Since the \( E_i \) are minimal objects, they must all be simple. \( \square \)

Let \( \alpha_1, \cdots, \alpha_k \) be real Schur roots so that \( (M_{\alpha_1}, \cdots, M_{\alpha_k}) \) is a sequence of Hom-orthogonal sequence of modules forming an exceptional sequence. Then we denote by \( \mathcal{A}(\alpha_1, \cdots, \alpha_k) \), or \( \mathcal{A}(\alpha_\ast) \) for short, the wide subcategory of \( \text{mod-} \Lambda \) generated by the modules \( M_{\alpha_i} \). As remarked
above, this is a rank \( k \) wide subcategory whose objects have a filtration with subquotients \( M_{\alpha_i} \).

Another description is:

\[
\mathcal{A}(\alpha_1, \cdots, \alpha_k) = \frac{1}{\gamma} \left( (M_{\alpha_1} \oplus \cdots \oplus M_{\alpha_k}) \right)
\]

In other words, \( \mathcal{A}(\alpha_s) = \frac{1}{\gamma}(E_1 \oplus \cdots \oplus E_{n-k}) \) for any choice of a complete exceptional sequence \( (E_1, \cdots, E_{n-k}, M_{\alpha_1}, \cdots, M_{\alpha_k}) \) ending in the \( M_{\alpha_i} \).

Here is another well-known fact that we need.

**Theorem 4.2.4.** The wide subcategory \( \mathcal{W} = \mathcal{A}(\alpha_1, \cdots, \alpha_k) \) described above contains the exceptional module \( M_\beta \) if and only if \( \beta \) is a nonnegative linear combination of the \( \alpha_i \).

**Proof.** Necessity of this condition is clear since all objects of \( \mathcal{W} \) are iterated extensions of the modules \( M_{\alpha_i} \). For the converse, we choose an extension of this sequence to a complete exceptional sequence \( (E_1, \cdots, E_{n-k}, M_{\alpha_1}, \cdots, M_{\alpha_k}) \). Then \( \mathcal{W} = \frac{1}{\gamma}(E_1 \oplus \cdots \oplus E_{n-k}) \). By Theorem 4.1.4, an exceptional module \( M_\beta \) lies in \( \mathcal{W} \) if and only if \( g(\beta) \in \bigcap_j D(E_j) \). But this is a convex set. Since this condition holds for the roots \( \alpha_i \), it holds for any nonnegative linear combination of the \( \alpha_i \). \( \square \)

For an admissible set of roots \( S \), this theorem and Proposition 4.2.2 imply the following.

**Corollary 4.2.5.** For \( x_0 \neq 0 \in \mathbb{R}^n \), let \( \alpha_1, \cdots, \alpha_k \) be the elements of \( S \) for which \( M_{\alpha_i} \) is minimal in \( \mathcal{W}(x_0) = \{M : x_0 \in D(M)\} \). Then, \( S \cap \mathcal{W}(x_0) \) is the set of elements of \( S \) which are sums of these roots (\( \beta = \sum n_i \alpha_i \) for \( n_i \geq 0 \)).

**Proof.** Let \( \beta \in S \cap \mathcal{W}(x_0) \). So, \( x_0 \in D(\beta) \). If \( \beta \) is not one of the \( \alpha_i \) then, by Proposition 4.2.2, \( x_0 \in \partial D(\beta) \). This implies that \( x_0 \in D(\gamma) \) for a subroot \( \gamma \subseteq \beta \). It follows that \( x_0 \in D(\gamma') \) for all components \( \gamma' \) of the quotient root \( \beta - \gamma \). These subroots and quotient roots of \( \beta \) all lie in \( S \) since \( S \) is admissible. By induction on the length of \( \beta \) we conclude that each \( \gamma, \gamma' \) is a nonnegative linear combination of the \( \alpha_i \). So, the same holds for their sum \( \beta \).

Conversely, suppose \( \beta \in S \) has the form \( \beta = \sum n_i \alpha_i \) for \( n_i \geq 0 \). Since \( S \) is admissible, the modules \( M_{\alpha_1}, \cdots, M_{\alpha_k} \) are Hom-orthogonal and form an exceptional sequence (being in lateral order). By Theorem 4.2.4, \( M_\beta \) lies in the wide subcategory \( \mathcal{W}(x_0) \) as claimed. \( \square \)

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Department of Mathematics, Brandeis University, Waltham, MA 02454

*E-mail address*: igusa@brandeis.edu

Department of Mathematics, Northeastern University, Boston, MA 02115

*E-mail address*: g.todorov@northeastern.edu