MULTIVARIATE POLYNOMIAL GRAPH INVARIANTS: DUALITIES AND CRITICAL PROPERTIES

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ABSTRACT. We explore several types of functional relations on the family of multivariate Tutte polynomials: the Biggs duality and the star-triangle transformation at the critical point \( n = 2 \). We deduce the Matiyasevich theorem and its inverse from the Biggs duality, apply the duality argument to construct the recursion on the parameter \( n \). We provide two different proofs of the Zamolodchikov tetrahedron equation satisfied by the star-triangle transformation in the case of \( n = 2 \) multivariate Tutte polynomial, extend the latter to the case of valency 2 points and show that the Biggs duality and the star-triangle transformation commute.

CONTENTS

1. Introduction 1
Acknowledgements 2
1.1. Organization of the paper 2
2. Biggs interaction models 3
2.1. Ising–Potts Models and Tutte polynomial 3
2.2. \( n \)-Potts models and Matiyasevich theorem 5
2.3. Shifting the order in the Potts models 9
3. Star–triangle equation for Ising and Potts models 11
3.1. The case \( n > 2 \). 15
4. Tetrahedron equation 15
4.1. Local Yang-Baxter equation 16
4.2. Tetrahedron equation, first proof 17
4.3. Tetrahedron equation, second proof 19
5. Star-triangle transformation, Biggs duality and conclusion 24
Appendix A. The proof of Lemma 4.3 26
References 28

1. INTRODUCTION

The theory of polynomial invariants of graphs in its current state includes many methods and tools of integrable statistical mechanics. This phenomenon demonstrates the inherent intrusion of mathematical physics methods into topology and combinatorics. In this paper, the main subject of research is functional relations in the family of polynomial invariants for framed graphs, in particular for multivariate Tutte polynomials [15], their specializations for Potts models, multivariate chromatic and flow polynomials.

The flow generating function is closely related to the problems of electrical networks on a graph over a finite field. Each flow defines a discrete harmonic function, and non-zero flows can be interpreted as harmonic functions with a completely non-zero gradient. We specially discuss the full flow polynomial which is a linearization of the
flow polynomial and, in particular, corresponds to the point of the compactification of
the parameter space for the Biggs model.

One of the central tools of the paper is the Biggs formula (Lemma 2.11), which con-
nects Ising models for different parameter values as a convolution with some weight
over all edge subgraphs. In particular, we offer a new proof of the Matiyasevich The-
orem 2.16 on the connection of a flow and chromatic polynomial, as a special case of
the Biggs formula. This interpretation allows us to construct an inverse statement of
the Matiyasevich theorem. Moreover using the connection between the flow and the
complete flow polynomial we obtain a shifting parameters formulas in the Potts models
(Theorem 2.24).

The fundamental type of correspondences on the space of aforementioned invariants
is the star-triangle type relations and the related deletion-contraction relations. In a
sense, the kinship between these relations is analogous to the role of the tetrahedron
equation in the determining of the local Yang-Baxter equation. Despite the fact that the
invariance of the Ising model with respect to the star-triangle transformation known very
well [1], we have not found in the literature a full proof of the fact that the action of this
transformation on the weights of an anisotropic system is a solution of the tetrahedron
equation that corresponds to the orthogonal solution of the local Yang-Baxter equation
(parts of this statement were mentioned in [9, 11, 13]). We offer here two new proofs
of this fact. We find them instructive due to their anticipated relation to the theory of
positive orthogonal grassmannians [8].

Identification of the Potts model and the multivariate Tutte polynomial allows us
to assert the existence of a critical point for the parameter $n$ in the family of Tutte
polynomials. Namely, for $n = 2$, this model has a groupoid symmetry generated by a
family of transformations defined by the trigonometric solution of the Zamolodchikov
tetrahedron equation. We extend the star-triangle transformation for the graphs of lower
valency. In this way we obtain the 12-term correspondence. Such extension commute
with the Biggs duality. We should mention the relation of this subject with the theory of
cluster algebras. We suppose that the multivariate Tutte polynomial on standard graphs
at the critical point $n = 2$ corresponds to the orthogonal version of the Lusztig variety
[3] in the case of the unipotent group and the electrical variety [7] for the symplectic
group.

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1.1. Organization of the paper. In section 2 we concentrate our attention on the
Biggs formalism in the Ising-Potts type models. We define the main recurrence relations
and duality property. We also identify the Tutte polynomial with the Potts model. Then
we apply the Biggs duality to the proof of the Matiyasevich theorem and propose its
dual version. We examine in details the recursion of the Potts model with respect to
the parameter $n$.

In section 3 we show that the the $n = 2$ Potts model is invariant with respect to the
star-triangle transformation given by the orthogonal solution for the local Yang-Baxter
equation and the corresponding solution for the Zamolodchikov tetrahedron equation.
In section 4 we provide two different proofs for this fact. Both of them are interesting in
the applied technique of cluster variables on the space of Ising models. The first proof
operates with the space of boundary measurement matrices and the second with the
matrix of boundary partition function.
In section 5 we show that the Biggs duality considered as a correspondence on the set of multivariate Tutte polynomials commute with the star-triangle transformation.

2. Biggs interaction models

2.1. Ising–Potts Models and Tutte polynomial. We define the anisotropic Biggs model (interaction model) on an undirected graph \( G \) with the set of edges \( E \) and the set of vertices \( V \) (a graph can have multiple edges and loops) as follows:

- a state \( \sigma \) is a map \( \sigma : V \to A \), here \( A \) – some commutative ring with the unit;
- the energy of the state \( \sigma \) is defined by the formula
  \[
  H(\sigma) := \prod_{e \in E} i_e(\delta(e)),
  \]
  here \( \delta(e) = \sigma(v) - \sigma(w) \); the edge \( vw \) connects the vertices \( v \) and \( w \), the functions \( i_e : A \to \mathbb{C} \) are even: \( \forall b \in A : i_e(b) = i_e(-b) \);
- the partition function \( Z(G) \) of a model is the following sum:
  \[
  Z(G) = \sum_\sigma H(\sigma),
  \]
where the summation is taken over all possible states \( \sigma \).

Let us consider the most simple interaction models:

**Definition 2.1.** Let \( A \cong \mathbb{Z}_n \) and

\[
\begin{align*}
  i_e(0) &= \alpha_e, \\
  i_e(a) &= \beta_e, \forall a \neq 0 \in A.
\end{align*}
\]

We call this models \( M(G, i_e) \) the anisotropic \( n \)-Potts model with the set of parameters \( \alpha_e \) and \( \beta_e \). If the maps \( i_e = i \) do not depend on edges then we will call this model \( M(G, i) \) the isotropic \( n \)-Potts model (or just \( n \)-Potts model) with the parameters \( \alpha \) and \( \beta \). Also we will use the denotation \( M(G, \alpha, \beta) \) for the isotropic \( n \)-Potts models with the parameters \( \alpha \) and \( \beta \).

**Remark 2.2.** In the case \( A \cong \mathbb{Z}_2 \), \( i(0) = \exp \left( \frac{J}{kT} \right) \) and \( i(1) = \exp \left( -\frac{J}{kT} \right) \) this model can be identified with the isotropic Ising model [1].

**Remark 2.3.** For the empty graph, we define the partition function of any \( n \)-Potts model to be equal to 1, and for a disjoint set of \( m \) points to be equal to \( n^m \).

Now we will consider the combinatorial properties of the anisotropic \( n \)-Potts models (compare with [2] Theorem 3.2):

**Theorem 2.4.** Denote the partition function of the anisotropic \( n \)-Potts model on the graph \( G \) as \( Z_n(G) \).

- Let a graph \( G \) be the disjoint union of graphs \( G_1 \) and \( G_2 \), then
  \[
  Z_n(G) = Z_n(G_1)Z_n(G_2).
  \]

- Let a graph \( G \) be the joining of graphs \( G_1 \) and \( G_2 \) by the vertex \( v \), then
  \[
  nZ_n(G) = Z_n(G_1)Z_n(G_2).
  \]

- Consider a graph \( G \) and its edge \( e \), where \( e \) is neither a bridge nor a loop. Consider the graph \( G/e \) obtained by contraction of \( e \), and the graph \( G\backslash e \) obtained by deletion of \( e \). Then the following formula holds:
  \[
  Z_n(G) = (\alpha_e - \beta_e)Z_n(G/e) + \beta_eZ_n(G\backslash e).
  \]
Figure 1. The joining of two graphs by the vertex $v$

Proof.
1. The statement directly follows from Definition 2.1.
2. Let us rewrite the partition function $Z_n(G)$:

$$Z_n(G) = \sum_{k \in \{1, \ldots, n\}} \sum_{\sigma : \sigma(v) = k} H(\sigma).$$

Notice that $i(\sigma(v) - \sigma(w)) = i(\sigma(v) + 1 - \sigma(w) - 1)$, therefore for any $i \neq j$ we have the following identity:

$$\sum_{\sigma : \sigma(v) = i} H(\sigma) = \sum_{\sigma : \sigma(v) = j} H(\sigma).$$

Hence we obtain

$$Z_n(G) = n \sum_{\sigma : \sigma(v) = i} H(\sigma), \ \forall \ i \in \{0, \ldots, n - 1\}.$$ Let us introduce the partial partition functions $X_k := \sum_{\sigma \in \Sigma_k} H_G^1(\sigma)$ and $Y_k := \sum_{\sigma \in \Sigma_k} H_G^2(\sigma)$, then we could rewrite:

$$\begin{align*}
Z_n(G_1)Z_n(G_2) &= \left(\sum_{k} \sum_{\sigma : \sigma(v) = k} H_G^1(\sigma)\right) \left(\sum_{k} \sum_{\sigma : \sigma(v) = k} H_G^2(\sigma)\right) \\
&= (X_0 + X_1 + \ldots + X_{n-1})(Y_0 + Y_1 + \ldots Y_{n-1}) = n^2X_0Y_0 = \tag{5} \\
&= n(X_0Y_0 + X_1Y_1 + \ldots + X_{n-1}Y_{n-1}) = n \sum_{k} \sum_{\sigma : \sigma(v) = k} H_G(\sigma) = nZ_n(G).
\end{align*}$$

3. Let the edge $e$ is neither a bridge nor a loop and denote by $X$ the income in the partition function of all states such that the values of the ends of $e$ coincide and by $Y$ another part of the partition function (that of the distinct values of the ends of $e$), then

$$Z_n(G) = \alpha_e X + \beta_e Y, \ \ Z_n(G\setminus e) = X + Y, \ \ Z_n(G/e) = X$$

and we get the statement. \qed

Now let us recall the definition of the Tutte polynomial of a graph $G$.

**Definition 2.5.** Let us define the Tutte polynomial $T_G(x, y)$ by the deletion-contraction recurrent relation

(1) If an edge $e$ is neither a bridge nor a loop, then $T_G(x, y) = T_{G\setminus e}(x, y) + T_{G/e}(x, y)$.

(2) If the graph $G$ consists of $i$ bridges and $j$ loops, then $T_G(x, y) = x^iy^j$. 

Now we are going to connect the partition function of the isotropic $n$-Potts model $Z_n(G)$ of the graph $G$ with the Tutte polynomial $T_G(x, y)$ of the same graph $G$ using the well-known trick (for instance see [2]). Let us consider the weighted partition function:

$$\hat{Z}_n(G) := \frac{Z_n(G)}{n^{k(G)}},$$

where $k(G)$ is the number of connected components in the graph $G$.

The significance of this definition is revealed by the following theorem:

**Theorem 2.6 ([17],[5]).** Let the function $F(G)$ of a graph $G$ satisfies the following conditions:

- $F(G) = 1$, if $G$ consists of only one vertex.
- $F(G) = aF(G \setminus e) + bF(G/e)$, if an edge $e$ is not a bridge neither a loop.
- $F(G) = F(G_1)F(G_2)$, if either $G = G_1 \cup G_2$ or the intersection $G_1 \cap G_2$ consists of only one vertex.

Then

$$F(G) = a^{c(G)}b^{r(G)}T_G\left(\frac{F(K_2)}{b}, \frac{F(L)}{a}\right),$$

where $K_2$ is a complete graph on two vertices, $L$ is a loop, $r(G) = v(G) - k(G)$ is a rank of $G$ and $c(G) = c(G) - r(G)$ is a corank.

Let us note that the weighted partition function $\hat{Z}_n(G)$ satisfies Theorem 2.6, therefore the following theorem holds:

**Theorem 2.7** (Compare with [2] Corollary 3.3). The partition function $Z_n(G)$ of the $n$-Potts model $M(G, \alpha, \beta)$ coincides with the Tutte polynomial of a graph $G$ up to a multiplicative factor:

$$Z_n(G) = n^{k(G)}\beta^{c(G)}(\alpha - \beta)^{r(G)}T_G\left(\frac{\alpha + (n - 1)\beta}{\alpha - \beta}, \frac{\alpha}{\beta}\right).$$

**Example 2.8.** (The bad coloring polynomial) Consider a graph $G$ and all possible colorings of $V(G)$ in $n$ colors. Define the bad coloring polynomial as:

$$B_G(n, t) = \sum_j b_j(G, n)t^j,$$

here $b_j(G, n)$ is the number of such colorings that each of them has exactly $j$ edges which ends have different colors. So, easy to see that $B_G(n, t) = Z_n(G)$, here $Z_n(G)$ is the partition function of the $n$-Potts model $M(G, t, 1)$. Hence, using the Theorem 2.7 we immediately obtain:

$$B_G(n, t + 1) = n^{k(G)}t^{r(G)}T_G\left(\frac{t + n}{t}, t + 1\right).$$

2.2. $n$-Potts models and Matiyasevich theorem. The connection between the $n$-Potts models and Tutte polynomials allows us to give a simple proof of the Matiyasevich theorem about the chromatic and flow polynomials.

**Definition 2.9.** A graph $A$ is called a spanning subgraph of a graph $G$, if graphs $G$ and $A$ share the same set of vertices: $V(G) = V(A)$, and the set of edges $E(A)$ is the subset of the set of edges $E(G)$.

**Definition 2.10.** A graph $A$ is called an edge induced subgraph (Fig. 2) of a graph $G$, if $A$ is induced by a subset of the set $E(G)$. Every edge induced subgraph $A$ of a graph $G$ could be completed to the spanning subgraph $A'$ by adding all the vertices of $G$ which is not contained in the subgraph $A$. 

We will start with the following lemma, which is a generalization of the high temperature formula for the Ising model:

**Lemma 2.11** (Biggs formula [4]). Let us consider two $n$-Potts models $M_1(G, i_1)$ with the parameters $\alpha_1, \beta_1$ and $M_2(G, i_2)$ with the parameters $\alpha_2, \beta_2$ then the partition function $Z_n^1(G)$ of the first model could be expressed in terms of the partition functions of the models of all edge induced subgraphs of the second model:

\[
(6) \quad \tilde{Z}_n^1(G) = q^{e(G)} \sum_{A \subseteq G} \left( \frac{p}{q} \right)^{e(A)} \tilde{Z}_n^2(A),
\]

here $p = \frac{\alpha_1 - \beta_1}{\alpha_2 - \beta_2}$, $q = \frac{\alpha_2 \beta_1 - \alpha_1 \beta_2}{\alpha_2 - \beta_2}$ and $\tilde{Z}_n^1(G) = \frac{Z_n^1(G)}{n^\nu(G)}$ (we assume that $\tilde{Z}_n^1(\emptyset) = 1$).

**Proof.** Let us notice that $i_1 = p \cdot i_2 + q$, therefore

\[
\begin{aligned}
Z_n^1(G) &= \sum_{\sigma:V(G)\rightarrow\mathbb{Z}_n} \prod_e i_1(\delta(e)) = \sum_{\sigma:V(G)\rightarrow\mathbb{Z}_n} \prod_e (p i_2(\delta(e)) + q) \\
&= \sum_{\sigma:V(G)\rightarrow\mathbb{Z}_n} \sum_{A \subseteq G} p^{e(A)} q^{e(G)-e(A)} \prod_{e \in E(A)} i_2(\delta(e))
\end{aligned}
\]

In order to complete the proof we consider the following term for a fixed $A$:

\[
\begin{aligned}
&\sum_{\sigma:V(G)\rightarrow\mathbb{Z}_n} p^{e(A)} q^{e(G)-e(A)} \prod_{e \in E(A)} i_2(\delta(e)) = q^{e(G)} \left( \frac{p}{q} \right)^{e(A)} \sum_{\sigma:V(G)\rightarrow\mathbb{Z}_n} \prod_{e \in E(A)} i_2(\delta(e)) \\
&= q^{e(G)} \left( \frac{p}{q} \right)^{e(A)} n^{\nu(G)-\nu(A)} \sum_{\sigma:V(A)\rightarrow\mathbb{Z}_n} \prod_{e \in E(A)} i_2(\delta(e)) = n^{\nu(G)} q^{e(G)} \left( \frac{p}{q} \right)^{e(A)} \tilde{Z}_n^2(G).
\end{aligned}
\]

\[\blacksquare\]

**Proposition 2.12.** Consider two anisotropic $n$-Potts models $M_1(G, i_1^k)$ and $M_2(G, i_2^k)$. By the same fashion we can obtain:

\[
(7) \quad \tilde{Z}_n^1(G) = \prod_{e \in G} q_e \sum_{\sigma:V(G)\rightarrow\mathbb{Z}_n} \prod_{e \in A} p_e \tilde{Z}_n^2(A),
\]

where $p_e = \frac{\alpha_1^e - \beta_1^e}{\alpha_2^e - \beta_2^e}$, $q_e = \frac{\alpha_2^e \beta_1^e - \alpha_1^e \beta_2^e}{\alpha_2^e - \beta_2^e}$.

We consider further the chromatic and flow polynomials, first of all remain some well-known definitions.
Theorem 2.16 (Matiyasevich theorem [12]). Let us consider two Potts models with the special parameters: the model with the parameters \( \alpha_1 = 0, \beta_1 = 1 \) and the model with the parameters \( \alpha_2 = 1 - n, \beta_2 = 1 \). By Theorem 2.7 we could express the partition function of the first model in terms of the chromatic polynomial:

\[
\chi_G(n) = (-1)^{\nu(G)-k(G)} n^{k(G)} T_G(1-n,0) = \frac{(-1)^{\nu(G)-k(G)} n^{k(G)}}{n^{k(G)}} Z^1_n(G).
\]

So we have:

\[
Z^1_n(G) = \frac{\chi_G(n)}{n^{\nu(G)}}.
\]

Analogously, we express the partition function of the second model in terms of the flow polynomial:

\[
C_G(n) = (-1)^{\nu(G)+v(G)+k(G)} T_G(0,1-n) = \frac{(-1)^{\nu(G)+v(G)+k(G)} Z^2_n(G)}{n^{k(G)} v^{v(G)} - k(G)} = (-1)^{\nu(G)} \tilde{Z}^2_n(G).
\]

So we have:

\[
\tilde{Z}^2_n(G) = (-1)^{\nu(G)} C_G(n).
\]
Then by Lemma 2.11 after the substitutions (10) and (11) we obtain:

\[
\chi_G(n) = \frac{(n - 1)^{e(G)}}{n^{e(G)}} \sum_{A \subseteq G} \frac{C_A(n)}{(1 - n)^{e(A)}}
\]

(12)

Where the summation goes through all edge induced subgraphs \(A'\).

We finish the proof by noticing that the edge induced subgraph differs from the spanning subgraph by the set of disjoint union of vertices. Therefore we can complete each edge induced subgraph to its corresponding spanning subgraph and then replace the summation over all edge induced subgraph by the summation over all spanning subgraph, because the value of the each flow polynomial \(C_A\) remains the same and finally we obtain:

\[
\chi_G(n) = \frac{(n - 1)^{e(G)}}{n^{e(G)} - v(G)} \sum_{A \subseteq G} \frac{C_A(n)}{(1 - n)^{e(A)}}
\]

(13)

\[
\chi_G(n) = \frac{(n - 1)^{e(G)}}{n^{e(G)} - v(G)} \sum_{A \subseteq G} \frac{C_A(n)}{(1 - n)^{e(A)}}
\]

\[\Box\]

Note that we could produce series of statements that look like Theorem 2.16:

**Theorem 2.17.** Let us consider a graph \(G\), then we can obtain the following formulas:

\[
n^{k(G)} \beta_1^{e(G)} (\alpha_1 - \beta_1)^{r(G)} T_G \left( \frac{\alpha_1 + (n - 1) \beta_1}{\alpha_1 - \beta_1}, \frac{\alpha_1}{\beta_1} \right) = q^{e(G)} \sum_{A \subseteq G} \left( \frac{p}{q} \right)^{e(A)} \chi_A(n),
\]

(13)

\[
n^{k(G)} \beta_1^{e(G)} (\alpha_1 - \beta_1)^{r(G)} T_G \left( \frac{\alpha_1 + (n - 1) \beta_1}{\alpha_1 - \beta_1}, \frac{\alpha_1}{\beta_1} \right) = q^{e(G)} \sum_{A \subseteq G} \left( \frac{p}{q} \right)^{e(A)} \chi_A(n),
\]

(14)

\[
n^{k(G)} \beta_1^{e(G)} (\alpha_1 - \beta_1)^{r(G)} T_G \left( \frac{\alpha_1 + (n - 1) \beta_1}{\alpha_1 - \beta_1}, \frac{\alpha_1}{\beta_1} \right) = q^{e(G)} \sum_{A \subseteq G} \left( \frac{p}{q} \right)^{e(A)} \chi_A(n),
\]

(15)

\[
\chi_G(n) = (n - 1)^{e(G)} \sum_{A \subseteq G} n^{e(A) - v(G)} (1 - n)^{e(A)} \chi_A(n);
\]

(16)

\[
\chi_G(n) = (n - 1)^{e(G)} \sum_{A \subseteq G} n^{e(A) - v(G)} (1 - n)^{e(A)} \chi_A(n);
\]

\[\Box\]

**Proof.** We give only the sketch of the proof, because the full proof almost literally repeat the reasoning of the Theorem 2.16.

First of all, we consider the two \(n\)-Potts models:

- The model \(M_1(G, i_1)\) with the parameters \(\alpha_1, \beta_1\) and \(M_2(G, i_2)\) with the parameters \(\alpha_2 = 0, \beta_2 = 1\) (for the proof of the formula (13)).
- The specification of the first case – \(M_1(G, i_1)\) with \(\alpha_1 = 1 - n, \beta_1 = 1\) and the same \(M_2(G, i_2)\) with \(\alpha_2 = 0, \beta_2 = 1\) (for the proof of the formula (14)).
- The model \(M_1(G, 1 - n, 1)\) and \(M_2(G, i_2)\) with the parameters \(\alpha_1, \beta_1\) (for the proof of the formula (15)).
- Oppositely, \(M_1(G, i_1)\) with the parameters \(\alpha_1, \beta_1\) and \(M_2(G, 1 - n, 1)\) (for the proof of the formula (16)).
Then, using the Theorem 2.7 we might write down the formulas (we call them (∗) and (**)) which represent the link between the partition function \( Z^i_n(G) \), \( i = 1, 2 \) and the corresponding Tutte polynomial, this step is similar to obtaining the formulas (8) and (9). The next step consists in obtaining the formula (∗), it can be done by the substitution of formulas (∗) and (∗∗) in Lemma 2.11, this reasoning is similar to obtaining the formula (12). Finally, we need to rewrite the formula (∗) in term of spanning subgraphs.

For this we complete each edge induced subgraph \( A' \) to its corresponding spanning subgraph \( A \) and compute how to change each term of the right part of (∗) after replacing \( A' \) to \( A \). For instance, we show in detail how to do computation in case of the formula (13). Consider polynomials \( X_{A'}(n) \) and \( X_A(n) \), easy to see that the following formula holds:

\[
\tilde{Z}^2_n(A') = \frac{X_{A'}(n)}{n^{v(A')}} = \frac{X_A(n)n^{-v(G)+v(A')}}{n^{v(A')}} = \frac{X_A(n)}{n^{v(G)}}. 
\]

\[\Box\]

Remark 2.18. We notice that the formula (14) naturally can be considered as “inversion” of Theorem 2.16.

2.3. Shifting the order in the Potts models. Biggs Lemma 2.11 allows us to relate the values of the partition functions of the \( n \)-Potts models with fixed \( n \), but different values of parameters \( \alpha \) and \( \beta \). The goal of the current subsection is to present a method how to associate the partition functions of the \( n \)-Potts models of different \( n \). We will call it shifting order formulas.

The first method is based on the multiplicativity property of the complete flow polynomial.

**Definition 2.19.** Let \( G \) be a graph with the edge set \( E \) and the vertex set \( V \). The function \( f : E \times V \times V \to \mathbb{Z}_n \) is called an \( n \)-flow (or just flow) if the following conditions hold:

- \( f(e, v, w) = 0 \), if \( e \neq vw \)
- \( f(e, v, w) = -f(e, w, v) \)
- \( \forall u \in V : \sum_{e \in E, V \setminus \{u\}} f(e, u, w) = 0 \)

we will denote the number of all \( n \)-flows by \( FC_G(n) \)

Let us formulate few well known results concerning flow and complete flow polynomials, the proofs could be found for example in [15].

**Proposition 2.20.** The following identity holds:

\[
FC_G(n) = \sum_{A \subseteq G} C_A(n),
\]

where the summation goes through all spanning subgraphs \( A \) of the graph \( G \).

**Proposition 2.21.**

\[
FC_G(n) = n^{e(G)-v(G)+k(G)},
\]

where \( e(G), v(G), k(G) \) are numbers of edges, vertices and connected components in the graph \( G \) correspondingly.

**Proposition 2.22.** The flow polynomial \( C_G(n) \) of a graph \( G \) could be expressed in terms of the complete flow polynomials of its spanning subgraphs by the following identity:

\[
C_G(n) = \sum_{A \subseteq G} (-1)^{e(G)-e(A)} FC_A(n).
\]
Remark 2.23. We could interpret the complete flow polynomial \( FC_G(n) \) as a weighted partition function of the \( n \)-Potts model with the parameters \( \alpha = \beta \). Therefore we could understand formulas (17) as a particular case of the limit in the Biggs formula (6).

The complete flow polynomial \( FC_G(n) \) is a multiplicative invariant: \( FC_G(n_1n_2) = FC_G(n_1)FC_G(n_2) \), therefore we are ready to formulate the following theorem:

**Theorem 2.24.** The partition function \( Z_{n_1n_2}(G) \) of the \( n_1n_2 \)-Potts model \( M(G, \alpha_1, \beta_1) \) could be expressed in terms of the partition functions \( Z_{n_1}(A) \) and \( Z_{n_2}(A) \) of the \( n_1 \)-Potts model \( M_1(A, \alpha_1, \beta_1) \) and \( n_2 \)-Potts model \( M_2(A, \alpha_1, \beta_1) \) of all spanning subgraphs \( A \) of the graph \( G \) correspondingly.

**Proof.** Indeed, by Theorem 2.7 and the formula (15) we have:

\[
Z_{n_1n_2}(G) = \gamma_G T_G \left( \frac{\alpha_1 + (n - 1)\beta_1}{\alpha_1 - \beta_1}, \frac{\alpha_1}{\beta_1} \right) = \sum_{A \subseteq G} \lambda_A C_A(n_1n_2)
\]

By Proposition 2.22 we obtain:

\[
\sum_{A \subseteq G} \lambda_A C_A(n_1n_2) = \sum_{A \subseteq G} \lambda_A \left( \sum_{A' \subseteq A} (-1)^{e(A')-e(A')} FC_{A'}(n_1n_2) \right) = \sum_{A \subseteq G} \omega_A FC_A(n_1n_2) = \sum_{A \subseteq G} \omega_A FC_A(n_1)FC_A(n_2)
\]

The Proposition 2.20 implies:

\[
\sum_{A \subseteq G} \omega_A FC_A(n_1)FC_A(n_2) = \sum_{A \subseteq G} \omega_A \left( \sum_{A' \subseteq A} C_{A'}(n_1) \right) \left( \sum_{A'' \subseteq A''} C_{A''}(n_2) \right) = \sum_{A \subseteq A'} \sum_{A'' \subseteq A''} \mu_{A'A''} C_A(n_1)C_{A''}(n_2).
\]

Finally with the help of formula (16) and Theorem 2.7 we get:

\[
\sum_{A \subseteq A'} \sum_{A'' \subseteq A''} \mu_{A'A''}(\sum_{B \subseteq A'} \delta_B Z_{n_1}(B))(\sum_{C \subseteq A''} \delta_C Z_{n_2}(C)) = \sum_{A \subseteq A'} \sum_{A'' \subseteq A''} \eta_{A'A''} Z_{n_1}(A')Z_{n_2}(A''),
\]

where \( \eta_{A'A''} \) — some constants, appeared after the resummations.

**Remark 2.25.** (Convolution formula [10]) It seems extremely interesting and fruitful to compare Lemma 2.11 and Theorem 2.24 with the convolution formula:

\[
T_G(x, y) = \sum_{A \subseteq E(G)} T_{G|A}(0, y)T_{G/A}(x, 0),
\]

here the summation is over all possible subsets of \( E(G) \), for the definitions of \( G|A \) and \( G/A \) we refer to [5].

Our second method is based on the Tutte identity for the chromatic polynomial:

**Theorem 2.26** ([5]). Consider a graph \( G \) with the set of edge \( V(G) \) then the following formula holds:

\[
\chi_G(n_1 + n_2) = \sum_{B \subseteq V(G)} \chi_{G|B}(n_1)\chi_{G|B^c}(n_2),
\]

where \( G|B \) (\( G|B^c \)) is the restriction of \( G \) on the vertex subset \( B \in V(G) \) (\( B^c \in V(G) \), where \( B^c = V(G) \setminus B \)). We will call \( G|B \) — \( B \)-vertex inducted subgraph.

Using this fact we can formulate the following theorem:
Theorem 2.27. The partition function $Z_{n_1+n_2}(G)$ of the $n_1+n_2$-Potts model $M(G, \alpha, \beta)$ could be expressed in terms of the partition functions $Z_{n_1}(A)$ and $Z_{n_2}(A)$ of the $n_1$-Potts model $M_1(A, \alpha, \beta)$ and $n_2$-Potts model $M_2(A, \alpha, \beta)$ of all spanning subgraphs $A$ of the graph $G$ correspondingly.

Proof. The proof is very similar to the proof of Theorem 2.24. Again by Theorem 2.7 and the formula (13) we have:

$$Z_{n_1+n_2}(G) = \gamma_G T_G \left( \frac{\alpha_1 + (n-1)\beta_1}{\alpha_1 - \beta_1}, \frac{\alpha_1}{\beta_1} \right) = \sum_{A \subseteq G} \lambda_A \chi_A(n_1 + n_2)$$

By Theorem 2.26 we obtain:

$$\sum_{A \subseteq G} \lambda_A \chi_A(n_1 + n_2) = \sum_{A \subseteq G} \lambda_A \left( \sum_{B \subseteq V(A)} \chi_{A|B}(n_1) \chi_{A|B^c}(n_2) \right) =$$

$$= \sum_{A \subseteq G} \lambda_A \left( \sum_{B \subseteq V(A)} \sum_{A_1 \subseteq A|B} \omega_{A_1} Z_{n_1}(A_1) \sum_{A_2 \subseteq G|B^c} \omega_{A_2} Z_{n_2}(A_2) \right) =$$

$$= \sum_{A \subseteq G} \sum_{B \subseteq V(A)} \sum_{A_1 \subseteq A|B} \sum_{A_2 \subseteq A|B^c} \mu_{A_1, A_2} Z_{n_1}(A_1) Z_{n_2}(A_2)$$

Let us complete each subgraph $A'$ (each $A''$) to the corresponding spanning subgraph of $G$ by adding isolating vertices:

$$\sum_{A \subseteq G} \sum_{B \subseteq V(A)} \sum_{A_1 \subseteq A|B} \sum_{A_2 \subseteq A|B^c} \mu_{A_1, A_2} Z_{n_1}(A_1) Z_{n_2}(A_2) = \sum_{A \subseteq A'} \sum_{A \subseteq A''} \eta_{A', A''} Z_{n_1}(A') Z_{n_2}(A''),$$

where $\eta_{A', A''}$ — again some constants, appeared after the resumptions. □

3. Star–triangle equation for Ising and Potts models

We will use the following formalism of Ising–Potts models which is just the slight reformulation of Definition (2.1):

Definition 3.1. ([15]) Let $G$ be a graph (with possible loops and multiple edges) with the set of vertices $V$ and the set of edges $E$. Let us define a partition function on $G$ by the following:

$$Z(G) = \prod_{\sigma} \prod_{e_i \in E} (\beta_{e_i} + (\alpha_{e_i} - \beta_{e_i}) \delta(\sigma_{e_i})) = \prod_{e_i \in E} \beta_{e_i} \sum_{e_i \in E} \prod_{\sigma} (1 + (t_{e_i} - 1) \delta(\sigma_{e_i})),$$

here $\alpha_{e_i}$ and $\beta_{e_i}$ are the weights of the edge $e_i$, depending on the state of the boundary vertices. $t_{e_i} = \frac{\alpha_{e_i}}{\beta_{e_i}}$ is a reduced weight of the edge $e_i$, $\sigma : V \rightarrow \mathbb{Z}_q$ and $\delta(\sigma_{e_i})$ is a value of standard Kronecker delta function of the values of $\sigma$ on the boundary vertices of the edge $e_i$.

Also we define the boundary partition function of the Ising–Potts models:

Definition 3.2. Let $G$ be a graph (also with possible loops and multiple edges) with the set of vertices $V$, the set of edges $E$ and the boundary subset $S \subseteq V$ of enumerated
vertices: \( S = \{v_1, v_2, \ldots, v_k\} \). The boundary partition function on \( G \) is defined by the following expression:

\[
Z_{S(\overline{\alpha})}(G) = \sum_{\sigma} \prod_{e_i \in E} (\beta_{e_i} + (\alpha_{e_i} - \beta_{e_i}) \delta(\sigma_{e_i})),
\]

here \( \overline{\alpha} = \{a_1, a_2, \ldots, a_k\}, \forall i : a_i \in \mathbb{Z}_q \) is the set of fixed values, and the summation is over such states \( \sigma_{a} \) that \( \sigma_{a}(v_i) = a_i \).

The next Lemma connects these two definitions:

**Lemma 3.3.** Consider two graphs \( G_1 = (V_1, E_1) \) and \( G_2 = (V_2, E_2) \) with the only common vertices in the boundary subset \( S = \{v_1, v_2, \ldots, v_n\} \) in \( V_1 \) and \( V_2 \). We can glue these graphs and obtain the third graph \( G = (V, E) \), where \( E = E_1 \sqcup E_2 \), \( V = V_1 \cup S \cup V_2 \).
Then the following identity holds:

\[
Z(G) = \sum_{\overline{\alpha}} Z_{S(\overline{\alpha})}(G_1)Z_{S(\overline{\alpha})}(G_2),
\]

here the summation is over all possible sets \( \overline{\alpha} \).

**Proof.** The formula is obtained directly from the Definitions 3.1 and 3.2. Indeed, by the Definition 3.2 we can write down \( Z(G) = \sum_{\overline{\alpha}} Z_{S(\overline{\alpha})}(G) \), but also \( Z_{S(\overline{\alpha})}(G) = Z_{S_1(\overline{\alpha})}(G_1)Z_{S_2(\overline{\alpha})}(G_2) \).

**Remark 3.4.** The topological quantum field theory due to Atiah formalism [16] is a functor

\[
TQFT : \text{Cob} \rightarrow \text{Vect}
\]

from the cathegory of cobordisms to the cathegory fo vector spaces. The considered property of the partition function in statistical mechanics in lemma 3.3 is similar to that in the TQFT.

In what follows we will consider the case \( n = 2 \). Our first goal is to find such conditions that the partition function (22) is invariant under the star–triangle transformation which changes the subgraph \( \Omega \) to the subgraph \( \Omega' \).

We derive these conditions with the use of the boundary partition functions: consider a graph \( G \) with the subgraph \( \Omega \), then using Lemma 3.3 for graphs \( \Omega \) and \( G - \Omega \) we

![Figure 3. G is obtained by merging of S = {v_1, v_2, v_3, v_4}](image)
obtain the following identity:

\[
Z(G) = \sum_\pi Z_{S(\pi)}(\Omega) Z_{S(\pi)}(G - \Omega),
\]

where \( S = \{ v_1, v_2, v_3 \} \) (Fig. 4). After the star–triangle transformation we will obtain a graph \( G' \) with the following partition function:

\[
Z(G') = \sum_\pi Z_{S(\pi)}(\Omega') Z_{S(\pi)}(G' - \Omega')
\]

Due to the fact that the star–triangle transformation does not change edges of the graph \( G - \Omega \), we deduce that \( \forall \pi : Z_{S(\pi)}(G - \Omega) = Z_{S(\pi)}(G' - \Omega') \). Therefore, the sufficient (but not necessary, see the Remark 3.7) conditions for the invariance of the partition function are the following ones:

\[
\forall \pi : Z_{S(\pi)}(\Omega) = Z_{S(\pi)}(\Omega')
\]

We write them down in details. Let us note that these conditions do not depend on the states of the vertices (see on the picture), but depend on the number and the positions of the vertices with equal states. Therefore we have the following possibilities:

- two states in the triangle are the same, then the central vertex either has the same state or has the different state, then \( \alpha_1 \beta_2 \beta_3 + \beta_1 \alpha_2 \alpha_3 \mapsto \alpha'_1 \beta'_2 \beta'_3 \) and two more maps after changing indexes;
- all states are the same, then \( \alpha_1 \alpha_2 \alpha_3 + \beta_1 \beta_2 \beta_3 \mapsto \alpha'_1 \alpha'_2 \alpha'_3 \).
In this way we get the following equations:

\[
\begin{align*}
\alpha_1 \beta_2 \beta_3 + \beta_1 \alpha_2 \alpha_3 &= \alpha_1' \beta_2' \beta_3' \\
\alpha_2 \beta_2 \beta_3 + \beta_2 \alpha_1 \alpha_3 &= \alpha_2' \beta_2' \beta_3' \\
\alpha_3 \beta_1 \beta_2 + \beta_3 \alpha_1 \alpha_2 &= \alpha_3' \beta_1' \beta_2' \\
\alpha_1 \alpha_2 \alpha_3 + \beta_1 \beta_2 \beta_3 &= \alpha_1' \alpha_2' \alpha_3'
\end{align*}
\]

(27)

After the substitution

\[
t_i = \frac{\alpha_i}{\beta_i}
\]

we rewrite (27) as

\[
\begin{align*}
\beta_1 \beta_2 \beta_3 (t_1 + t_2 t_3) &= \beta_1' \beta_2' \beta_3' t_1' \\
\beta_1 \beta_2 \beta_3 (t_2 + t_1 t_3) &= \beta_1' \beta_2' \beta_3' t_2' \\
\beta_1 \beta_2 \beta_3 (t_3 + t_1 t_2) &= \beta_1' \beta_2' \beta_3' t_3' \\
\beta_1 \beta_2 \beta_3 (t_1 t_2 t_3 + 1) &= \beta_1' \beta_2' \beta_3' t_1' t_2' t_3'
\end{align*}
\]

This set of equations defines a correspondence which preserves the Ising model partition function if we mutate the graph \( G \) to \( G' \). Let us denote the product \( \beta_1 \beta_2 \beta_3 \) by \( \beta \) and the product \( \beta_1' \beta_2' \beta_3' \) by \( \beta' \) then we obtain the following map from the \((t, \beta)\)-variables to the \((t', \beta')\) variables, we will call it \( \tilde{F} \).

\[
\tilde{F}(t_1, t_2, t_3, \beta) = (t_1', t_2', t_3', \beta') :
\]

\[
\begin{align*}
t_1' &= \sqrt{\frac{(t_1 + t_2 t_3)(t_1 t_2 t_3 + 1)}{(t_2 + t_1 t_3)(t_3 + t_1 t_2)}}, \\
t_2' &= \sqrt{\frac{(t_2 + t_1 t_3)(t_1 t_2 t_3 + 1)}{(t_1 + t_2 t_3)(t_3 + t_1 t_2)}}, \\
t_3' &= \sqrt{\frac{(t_3 + t_1 t_2)(t_1 t_2 t_3 + 1)}{(t_1 + t_2 t_3)(t_2 + t_1 t_3)}}, \\
\beta' &= \beta \sqrt{\frac{(t_1 + t_2 t_3)(t_3 + t_1 t_2)(t_2 + t_1 t_3)}{(t_1 t_2 t_3 + 1)}}.
\end{align*}
\]

(29)

Remark 3.5. Formally speaking to define a map on the space of edge weight adopted to the star-triangle transformation we have to resolve the map \( \tilde{F} \) somehow for the parameters \( \beta_i \). For example one can take the following one:

\[
\beta'_i = \beta_i (\beta' / \beta)^{1/3}.
\]

Actually the choice of a resolution is not important in what follows.

Remark 3.6. We choose the positive branch of the root function for real positive values of variables \( t_i \) for purposes emphasized further. This is relevant to the almost positive version of the orthogonal grassmanian.

Remark 3.7. As we mentioned above the conditions (27) are not necessary: let us consider a graph \( G \) such that each black edge of \( G \) has the same weights equal to \( \alpha, \beta \).

Let us note that \( Z_{S(\pi_1)}(G - \Omega) = Z_{S(\pi_2)}(G - \Omega) = Z_{S(\pi_3)}(G - \Omega) \), here \( \pi_1 = \{a_2, a_2, a_1\}, \pi_2 = \{a_2, a_1, a_2\}, \pi_3 = \{a_1, a_2, a_2\} \) and \( \pi_3 = \{a_1, a_1, a_1\} \). This is a consequence of the symmetry of \( G - \Omega \).
Figure 6. The graph $G$

Hence, to prove the invariance of the partition function of the graph $G$ (Pic. 6) under the star–triangle transformation it is sufficient that the following conditions hold:

\[
\begin{align*}
Z_{S(\pi)}(\Omega) + Z_{S(\pi')}\left(\alpha_1\right) + Z_{S(\pi')}\left(\alpha_2\right) + Z_{S(\pi')}\left(\alpha_3\right) &= Z_{S(\pi)}\left(\alpha_1'\right) + Z_{S(\pi)}\left(\alpha_2'\right) + Z_{S(\pi)}\left(\alpha_3'\right) \\
Z_{S(\pi)}(\Omega) &= Z_{S(\pi)}(\Omega')
\end{align*}
\]

Or equivalently:

\[
\begin{align*}
\beta_1 \beta_2 \beta_3 (t_1 + t_2 + t_3 + t_2 t_3 + t_1 t_2) &= \beta_1' \beta_2' \beta_3' (t_1' + t_2' + t_3') \\
\beta_1 \beta_2 \beta_3 (t_1 t_2 t_3 + 1) &= \beta_1' \beta_2' \beta_3' t_1' t_2' t_3'.
\end{align*}
\]

3.1. The case $n > 2$. Let us demonstrate how the method, described above, works for the star–triangle transformation in the case $n \geq 3$. Using the same ideas as in the previous subsection we could obtain the following conditions:

\[
\begin{align*}
\beta_1 \beta_2 \beta_3 (t_1 + t_2 t_3 + n - 2) &= \beta_1' \beta_2' \beta_3' t_1' \\
\beta_1 \beta_2 \beta_3 (t_2 + t_1 t_3 + n - 2) &= \beta_1' \beta_2' \beta_3' t_2' \\
\beta_1 \beta_2 \beta_3 (t_3 + t_1 t_2 + n - 2) &= \beta_1' \beta_2' \beta_3' t_3' \\
\beta_1 \beta_2 \beta_3 (t_1 t_2 t_3 + n - 1) &= \beta_1' \beta_2' \beta_3' t_1' t_2' t_3' \\
\beta_1 \beta_2 \beta_3 (t_1 + t_2 + t_3 + n - 3) &= \beta_1' \beta_2' \beta_3'.
\end{align*}
\]

Here the last equation follows from the extra case in which all states are different.

Figure 7. The extra case

In general the system (32) does not have a solution.

4. Tetrahedron equation

Tetrahedron equation firstly was considered by A. Zamolodchikov [18] who has constructed it’s first solution in S-form. We consider the following form of the equation:

\[
T_{123}T_{145}T_{246}T_{356} = T_{356}T_{246}T_{145}T_{123},
\]

where $T_{ijk}$ is an operator acting nontrivially in the tensor product of three vector spaces $V_i, V_j, V_k$, indexed by $i, j$ and $k$. For the complete introduction to the topic see for example [14]. In this section we present two proofs of the main theorem of the paper:

Theorem 4.1. The change of variables (29) defines the solution of the tetrahedron equation (33).
These two proofs have a lot of common points and ideas, but have the crucial differences in the last stages. For us these differences are quite important due to the intention to understand the case of \( n > 2 \). Also it is interesting to compare these two proofs for the purpose of combining arguments of boundary partition functions and the technique of correlation functions in the Ising-Potts models.

At first in the next subsection we prove that the change of variables (29) is a trigonometric solution of a local Yang-Baxter equation.

### 4.1. Local Yang-Baxter equation

Let us recall that the following change of variables \((t_1, t_2, t_3) \mapsto (t'_1, t'_2, t'_3)\) gives an invariance of the Ising model (22) under the star-triangle transformation (29):

\[
\begin{align*}
\frac{t'}{t_1} &= \sqrt{(t_1 + t_2 t_3)(t_1 t_2 t_3 + 1)} \\
\frac{t'}{t_2} &= \sqrt{(t_2 + t_1 t_3)(t_1 t_2 t_3 + 1)} \\
\frac{t'}{t_3} &= \sqrt{(t_3 + t_1 t_2)(t_1 t_2 t_3 + 1)} \\
\end{align*}
\]

(34)

\[
\frac{t'_1 t'_2}{t_1 t_2} = \frac{t_1 t_2 t_3 + 1}{t_3 + t_1 t_2} \\
\frac{t'_2 t'_3}{t_1 t_2} = \frac{t_1 t_2 t_3 + 1}{t_1 + t_2 t_3} \\
\frac{t'_3 t'_1}{t_2 t_3} = \frac{t_1 t_2 t_3 + 1}{t_2 + t_1 t_3} \\
\]

(35)

Following [11] we construct orthogonal hyperbolic \(3 \times 3\) matrices \(R_{ij}\) which solve the local Yang-Baxter equation:

\[
R_{12}(t_3)R_{13}(S(t_2))R_{23}(t_1) = R_{23}(S(t'_1))R_{13}(t'_2)R_{12}(S(t'_3)),
\]

(36)

where \(S(x)\) is the following involution:

\[
S(t) = \frac{t - 1}{t + 1}.
\]

(37)

In the left hand side of (36) we have

\[
R_{12}(t_3) = \begin{pmatrix}
i \sinh(\log(t_3)) & \cosh(\log(t_3)) & 0 \\
\cosh(\log(t_3)) & -i \sinh(\log(t_3)) & 0 \\
0 & 0 & 1
\end{pmatrix};
\]

(38)

\[
R_{13}(t_2) = \begin{pmatrix}
i \sinh(\log(S(t_2))) & 0 & \cosh(\log(S(t_2))) \\
0 & 1 & 0 \\
\cosh(\log(S(t_2))) & 0 & -i \sinh(\log(S(t_2)))
\end{pmatrix};
\]

(39)

\[
R_{23}(t_1) = \begin{pmatrix}
1 & 0 & 0 \\
0 & i \sinh(\log(t_1)) & \cosh(\log(t_1)) \\
0 & \cosh(\log(t_3)) & -i \sinh(\log(t_3))
\end{pmatrix}.
\]

(40)
**Theorem 4.2.** Matrices (38), (39), (40) together with the rules (29), (37) give a solution of (36).

**Proof.** The proof is a straightforward computation. For example let us write down the result of the product in the left hand side:

\[
\begin{pmatrix}
  t_2(t_3^2 - 1) & it_2(t_1^2 t_3^2 - t_1^2 - t_2^2 + t_3^2) & it_2(t_1^2 t_3^2 - t_1^2 + t_2^2 - t_3^2) \\
  t_3(t_2^2 - 1) & it_2(t_1 t_3(t_2^2 - 1)) & -it_2(t_1^2 t_3^2 - t_1^2 + t_2^2 - t_3^2) \\
  t_2(t_3^2 + 1) & it_2(t_1^2 t_3^2 + t_1^2 + t_2^2 + t_3^2) & it_2(t_1^2 t_3^2 + t_1^2 - t_2^2 - t_3^2) \\
  t_3(t_2 - 1) & it_2(t_1^2 - 2) & t_2(t_1^2 - 2)
\end{pmatrix}
\]

At the first glance the product in the right hand side looks much more cumbersome, but occasionally all terms are simplified and the matrix in the right hand side coincides with (41).

4.2. **Tetrahedron equation, first proof.** Let us encode the tetrahedron equation by the Fig. 8.

The standard graph encodes $R$-matrices in the following way: in each inner vertex numbered by $k$, which is the intersection of strands $i$ and $j$ we put the matrix $R_{ij}(t_k)$ which is the $2 \times 2$ matrix in the 4-dimensional space with basis vectors indexed by $a, b, c, d$. For instance,

\[
R_{ac}(t_5) = \begin{pmatrix}
\sinh(\log(t_5)) & 0 & \cosh(\log(t_5)) & 0 \\
0 & 1 & 0 & 0 \\
\cosh(\log(t_5)) & 0 & -\sinh(\log(t_5)) & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

Let us orient each strand from the left to the right and multiply $R$-matrices in order of the orientation, for instance for the picture 8 we have the following product of $R$-matrices:

\[
R_{cd}(t_1)R_{bd}(S(t_2))R_{bc}(t_3)R_{ad}(t_4)R_{ac}(t_5)R_{ab}(t_6).
\]

Let us note, that the orientation defines the product (43) uniquely.

Then let us apply four local Yang–Baxter equations consequently to the inner triangles with vertices numbered $(1, 2, 3)$, $(1, 4, 5)$, $(2, 4, 6)$ and $(3, 5, 6)$ as on the figure 9. As a result we have one and the same standard graph as on the figure 8 rotated by $\pi$. 

\[\text{Figure 8.} \quad \text{Encoding the tetrahedron equation by the standard graph}\]
At the same time we could apply local Yang–Baxter equations in the opposite direction: firstly to the triangle \((3, 5, 6), (2, 4, 6), (1, 4, 5)\) and \((1, 2, 3)\). Eventually in this case we will have again the same standard graph.

![Diagram](image)

**Figure 9.** Local Yang–Baxter equations applied to the standard graph

As the reader already guessed every local Yang–Baxter equation applied to the triangle \(A, B, C\) defines the factor \(T_{ABC}\) in the tetrahedron equation (33). For example we get

\[
T_{123} : (t_1, S(t_2), t_3, t_4, t_5, t_6) \mapsto (S(t_1'), t_2', S(t_3'), t_4, t_5, t_6).
\]

By the Theorem 4.2 the product (43) preserves at each step (encoded on the Fig. 9). As a result of two sequences of four Local Yang–Baxter equations we have an equality of two products of six \(4 \times 4\) \(R\)-matrices:

\[
R_{cd}(u_1)R_{bd}(u_2)R_{bc}(u_3)R_{ad}(u_4)R_{ac}(u_5)R_{ab}(u_6) = \\
R_{cd}(v_1)R_{bd}(v_2)R_{bc}(v_3)R_{ad}(v_4)R_{ac}(v_5)R_{ab}(v_6),
\]

where the parameters \(u_i, v_j, i, j = 1, \ldots, 6\) depends somehow on the initial variables \(t_i\), on the mapping (29) and on the involution (37).

Let us consider this equation element-wise, and note that we could uniquely express parameters in the right hand side in terms of the parameters in the left hand side.

\[
U_{1,4} = b(t_4) \\
U_{2,4} = -a(t_4)b(t_2); U_{1,3} = a(t_4)b(t_5) \\
U_{1,2} = a(t_4)a(t_5)b(t_6); U_{3,4} = a(t_2)a(t_4)b(t_1) \\
U_{2,3} = b(t_2)b(t_4)b(t_5) - a(t_2)a(t_5)b(t_3)
\]

Here \(U\) is a matrix in the left hand side and \(a, b\) are some invertible functions, come from (38),(39),(40). So we could uniquely determine \(t_4\) from the first equation, \(t_2\) and \(t_5\) from the second and the third, then \(t_1\) and \(t_6\), and finally \(t_3\) from the element \(U_{2,3}\).
Let us note that this algebraic proof could be formulated in terms of the paths on the standard graph (Fig. 8) with orientation. So the equation (45) provides coincidence of the parameters in the vertices given by the two sides of the tetrahedron equation. This finishes the proof.

4.3. Tetrahedron equation, second proof.

4.3.1. Involution Lemma. Let us consider the map $F(t_1, t_2, t_3) = (t_1', t_2', t_3')$, where prime variables defines by the (29). First of all, we formulate one technical Lemma:

**Lemma 4.3.** The following identity holds for all $t_1, t_2, t_3$:

$$S \times S \times S \circ F \circ S \times S \times S = F^{-1},$$

where $S(t) = \frac{t-1}{t}$. We postpone the proof to the Appendix A.

![Figure 10. The graphical representation of the left and right parts of (51)](image)

4.3.2. Towards the local tetrahedron equation. Let us consider any graph $G$ with a subgraph $\Gamma_1$ which coincides with the leftmost graph on the Fig. 10. We can transform the graph $G$ to the graph $G'$ with the subgraph $\Gamma_2$ which is coincides with the rightmost graph on the Fig. 10. We could make this mutation by two different chains of star–triangle transformations: $F_{356}^{-1}F_{246}F_{145}^{-1}F_{123}$ and $F_{123}^{-1}F_{145}F_{246}^{-1}F_{356}$. Both are figured out on the Fig. 9. This observation turns us to the following hypothesis:

$$F_{356}^{-1}F_{246}F_{145}^{-1}F_{123} = F_{123}^{-1}F_{145}F_{246}^{-1}F_{356}. \quad (50)$$

This equality is equivalent to the Zamolodchikov equation

$$\Phi_{356}\Phi_{246}\Phi_{145}\Phi_{123} = \Phi_{123}\Phi_{145}\Phi_{246}\Phi_{356}, \quad (51)$$

where $\Phi_{ijk} = S_{i}S_{k}F_{ijk}S_{j}$. Indeed, using Lemma 4.3 and the simple observation that $S_{l}F_{ijk} = F_{ijk}S_{l}$, $l \neq \{i, j, k\}$ we can write down:

$$F_{356}^{-1}F_{246}F_{145}^{-1}F_{123} = S_{3}S_{5}S_{6}F_{356}S_{3}S_{5}S_{6}F_{246}S_{1}S_{4}S_{6}F_{145}S_{1}S_{4}S_{5}S_{123},$$

$$= S_{2}S_{5}(S_{3}S_{6}F_{356}S_{5})(S_{2}S_{6}F_{246}S_{4})(S_{1}S_{5}F_{145}S_{4})(S_{1}S_{3}F_{123}S_{2})S_{2}S_{5},$$

and

$$F_{123}^{-1}F_{145}F_{246}^{-1}F_{356} = S_{1}S_{2}S_{3}F_{123}S_{1}S_{2}S_{3}F_{145}S_{2}S_{4}S_{6}F_{246}S_{2}S_{4}S_{6}F_{356},$$

$$= S_{2}S_{5}(S_{1}S_{3}F_{123}S_{2})(S_{1}S_{5}F_{145}S_{4})(S_{2}S_{6}F_{246}S_{4})(S_{1}S_{3}F_{123}S_{2})S_{2}S_{5}. \quad (50)$$

Conjugating both sides of (50) by $S_{2}S_{5}$ we get the Zamolodchikov tetrahedron equation.
4.3.3. Local tetrahedron equation for the Ising model.

**Proposition 4.4.** The functions

- \( \frac{\partial \ln(Z(G))}{\partial t_{e_i}}, \) where \( e_i \) — any edge belonging to \( G - \Omega \) and
- \( \frac{Z_{S(\pi)}(G)}{Z(G)} \), where \( Z_{S(\pi)}(G) \) is the boundary partition function

are invariant with respect to the star-triangle transformation those inner point does not an element of \( S \). Moreover, these functions do not depend on variables \( \beta_{e_k} \).

**Remark 4.5.** The function \( \frac{Z_{S(\pi)}(G)}{Z(G)} \) can be interpreted as a probability of the fixed values \( \pi \) of spins in \( S \), related to the boundary partition function \( Z_{S(\pi)}(G) \).

**Proof.** The crucial point in the demonstration of the first part of the statement is the fact that the derivative \( \frac{\partial \ln(Z(G))}{\partial t_{e_i}} \) does not depend on parameters \( \beta \). Indeed, this follows from the explicit form of the partition function

\[
Z(G, \beta, t) = \prod_{e \in E} \beta_e \sum_{\sigma} \prod_{e \in E} \left( 1 + (t_e - 1)\delta(e) \right).
\]

The proof of the second part of the statement is straightforward. It follows from the Definition 3.2 and the condition (26). \( \square \)

We will prove the Zamolodchikov equation in the equivalent form:

\[
(52) \quad F_{356}^{-1} F_{246}^{-1} F_{145}^{-1} F_{123} = F_{123}^{-1} F_{145} F_{246}^{-1} F_{356}.
\]

Let us notice that due to the local nature of the star-triangle transformation and the convolution property of the boundary partition function we have a choice to take some suitable graph to prove equation (52). So let us take the graph \( \Gamma_1 \) from the Fig. 11 with the following choice of boundary set \( S_0 \):

\[
S_0 := \{ v_1, v_2, v_3, v_4 \}.
\]

**Figure 11.** The graphs \( \Gamma_1 \) and \( \Gamma_2 \)

We will prove that the values of the second-type invariant functions which are preserved by both sides of the equation (52) allows us to uniquely reconstruct weights of all edges. Explain this idea in detail, let us consider the left hand side of the equation (52) and the map \( F_{123} \), then for any \( \bar{a} = \{a_1, \ldots, a_4\} \) the following identity holds:

\[
(53) \quad \frac{Z_{S(\bar{a})}(\Gamma_1)}{Z(\Gamma_1)} = \frac{Z_{S(\bar{a})}(\Gamma_1)}{Z(\Gamma_1)} + \frac{Z_{S(\bar{a})}(\Gamma_1)}{Z(\Gamma_1)}.
\]
here $S_1 = \{v_1, v_2, v_3, v_4, v_5\}$, (see Fig. 12) $\overline{a_1} = \{a_1, \ldots, a_4, 0\}$, $\overline{a_2} = \{a_1, \ldots, a_4, 1\}$. The Proposition 4.4 provides:

$$Z_{S_1(\pi)}(\Gamma_1) \frac{Z_{S\overline{1}(\pi)}(\Gamma)}{Z(\Gamma')} = Z_{S\overline{1}(\pi)}(\Gamma') \frac{Z_{S_1(\pi)}(\Gamma)}{Z(\Gamma)},$$

where $\Gamma'$ is obtained from $\Gamma_1$ by the star–triangle transformation (see Fig.12). And therefore we deduce that

$$Z_{S_1(\pi)}(\Gamma_1) = Z_{S\overline{1}(\pi)}(\Gamma').$$

Repeating these arguments for the remaining maps $F_{ijk}$ from the left hand side of (52) we obtain:

$$Z_{S_1(\pi)}(\Gamma_1) = Z_{S\overline{1}(\pi)}(\Gamma_2).$$

By the same fashion if we consider the right hand side of the equation (52), we similarly obtain that

$$Z_{S_1(\pi)}(\Gamma_1) = Z_{S\overline{1}(\pi)}(\Gamma_2).$$

Hence, in order to prove the equation (52) it is sufficient to prove that we can reconstruct the parameters $t_i$, $i = 1, \ldots, 6$ from the values $Z_{S_0(\pi)}(\Gamma_2)/Z(\Gamma_2)$ for different values of $\pi$ in a unique way.
We understand the identity (55) as a system of $2^4$ linear equations with unknowns $Z_{S_0(\bar{\sigma})}(\Gamma_2)$ of the following type: $Z_{S_0(\bar{\sigma})}(\Gamma_2)$:
\[
\frac{Z_{S_0(\bar{\sigma})}(\Gamma_2)}{Z(\Gamma_2)} = \alpha(\bar{\sigma}), \quad \forall \bar{\sigma} = (a_1, \ldots, a_4)
\]
which is equivalent to
\[
\sum_{\bar{\sigma}} Z_{S_0(\bar{\sigma})}(\Gamma_2) = Z_{S_0(\bar{\sigma})}(\Gamma_2)/\alpha(\bar{\sigma}).
\]
The rank of the system is equal to 15. Indeed, the rank is $\geq 15$ and we know that there is a nontrivial solution coming from the boundary partition functions for the graph $\Gamma_2$.

Hence any solution has the form
\[
Z_{S_0(\bar{\sigma})}(\Gamma_2) = C \cdot \alpha_0(a_1, a_2, a_3, a_4),
\]
where $C$ is some constant and $a_i$ are the states. Now we will prove that the parameters $t_1, \ldots, t_6$ are reconstructed uniquely from the equation (57).

Let us introduce some auxiliary variables and rewrite the partition function in the following way: we have 16 states of boundary vertices $S_0 = \{v_1, v_2, v_3, v_4\}$. Each expression $Z_{S_0(\bar{\sigma})}(\Gamma_2)$ is a sum of four terms corresponding to the states of internal vertices $v_5$ and $v_6$. We consider in details the case $S_0 = \{0, 0, 0, 0\}$. Let us denote the weights of the states of the square $\{v_1, v_0, v_5, v_4\}$ by $v, z, y$ and $x$ (Fig. 14) then we obtain the following equations:
\[
\begin{align*}
v &= t_3 t_5 B, \quad v_1 = t_5 t_6 B \\
z &= t_3 t_4 t_5 B, \quad z_1 = t_4 t_5 B \\
y &= t_6 t_3 B, \quad y_1 = B \\
x &= t_3 t_4 B, \quad x_1 = t_4 t_6 B.
\end{align*}
\]
where $B = \beta_3 \beta_4 \beta_5 \beta_6$ and
\[
\begin{align*}
v + t_1 t_2 z + t_2 y + t_1 x &= \frac{C}{B_1} \alpha_0(0, 0, 0, 0), \quad B_1 = \beta_1 \beta_2.
\end{align*}
\]
Similarly we get seven more equations (we omit eighth equation with $v_1 = 1$ due to the symmetry of the model with respect to the total involution of spins):
Figure 14. The auxiliary variables

\[ v_1 + t_1 t_2 z_1 + t_2 y_1 + t_1 x_1 = \frac{C}{B_1} b_1 = \frac{C}{B_1} \alpha' (0, 0, 0, 1) \]
\[ t_1 v_1 + t_2 z_1 + y_1 t_1 t_2 + x_1 = \frac{C}{B_1} b_2 = \frac{C}{B_1} \alpha' (0, 1, 0, 1) \]
\[ t_1 t_2 v_1 + z_1 + t_1 y_1 + t_2 x_1 = \frac{C}{B_1} b_3 = \frac{C}{B_1} \alpha' (0, 1, 1, 1) \]
\[ t_2 v_1 + t_1 z_1 + y_1 + t_1 t_2 x_1 = \frac{C}{B_1} b_4 = \frac{C}{B_1} \alpha' (0, 0, 1, 1) \]
\[ v + t_1 t_2 y + t_2 x + t_1 x = \frac{C}{B_1} a_1 = \frac{C}{B_1} \alpha' (0, 0, 0, 0) \]
\[ t_1 v + t_2 z + y t_1 t_2 + x = \frac{C}{B_1} a_2 = \frac{C}{B_1} \alpha' (0, 1, 0, 0) \]
\[ t_1 t_2 v + z + t_1 y + t_2 x = \frac{C}{B_1} a_3 = \frac{C}{B_1} \alpha' (0, 1, 1, 0) \]
\[ t_2 v + t_1 z + y + t_1 t_2 x = \frac{C}{B_1} a_4 = \frac{C}{B_1} \alpha' (0, 0, 1, 0) \]

By a straightforward calculation we get:

\begin{align*}
y_1 &= \frac{C}{B_1} \frac{-t_2 b_1 + t_1 t_2 b_2 + b_4 - t_1 b_3}{-t_2^2 + 1 + t_1^2 t_2^2 - t_1^2} \\
z_1 &= \frac{C}{B_1} \frac{t_1 t_2 b_1 - t_2 b_2 + b_3 - t_1 b_4}{-t_2^2 + 1 + t_1^2 t_2^2 - t_1^2} \\
x_1 &= \frac{C}{B_1} \frac{t_2 t_1 b_4 - t_1 b_1 - t_2 b_3 + b_2}{-t_2^2 + 1 + t_1^2 t_2^2 - t_1^2} \\
v_1 &= \frac{C}{B_1} \frac{b_1 + t_1 t_2 b_3 - t_2 b_4 - t_1 b_2}{-t_2^2 + 1 + t_1^2 t_2^2 - t_1^2}
\end{align*}
Using the auxiliary variables it is easy to see that

\[
\begin{align*}
y &= \frac{C - t_2a_1 + t_1t_2a_2 + a_4 - t_1a_3}{B_1 - t_2^2 + 1 + t_1^2t_2^2 - t_1^2} \\
z &= \frac{C t_1t_2a_1 - t_2a_2 + a_3 - t_1a_4}{B_1 - t_2^2 + 1 + t_1^2t_2^2 - t_1^2} \\
x &= \frac{C t_2t_1a_4 - t_1a_1 - t_2a_3 + a_2}{B_1 - t_2^2 + 1 + t_1^2t_2^2 - t_1^2} \\
v &= \frac{C a_1 + t_1t_2a_3 - t_2a_4 - t_1a_2}{B_1 - t_2^2 + 1 + t_1^2t_2^2 - t_1^2}
\end{align*}
\]

By the same fashion we obtain:

\[
\begin{align*}
\frac{z_1}{x_1} &= \frac{v}{y} = \frac{-t_2a_4 + t_1t_2a_3 - t_1a_2 + a_1}{t_2t_1a_2 - t_2a_1 - t_1a_3 + a_4} = \frac{t_1t_2b_1 - t_2b_2 + b_3 - t_1b_4}{t_2t_1b_4 - t_1b_1 - t_2b_3 + b_2}.
\end{align*}
\]

This completes the proof of the Zamolodchikov equation due to the fact that there is a unique way to choose positive weights for the edges of the model to provide the expected values of boundary partitions function for the graph \(\Gamma_2\).

5. STAR-TRIANGLE TRANSFORMATION, BIGGS DUALITY AND CONCLUSION

The main results of the paper are related to the functional relations on the space of multivariate Tutte polynomials. This problem is a step of the program of investigation of the framed graph structures and the related statistical models. We examined in details the Biggs duality and applied it to the multivariate case. We also provided a new proof of the Matiyasevich theorem as a partial case of such duality. The second principal result is the reveal of the tetrahedral symmetry of the multivariate Tutte polynomial at the point \(n = 2\). Therefore we have a connection between the multivariate Tutte polynomial, functions on Lustig cluster manifolds [3] and its electrical analogues [7, 19]. We would like to interpret this property as the critical point of the model described by the multivariate Tutte polynomial, and the tetrahedral symmetry as a longstanding analog of the conformal symmetry of the Ising model at the critical point [6].

Both correspondences are related by the following observation. Let \(G\) and \(G'\) be two graphs related by the star-triangle transformation. The partition function is invariant with respect to this transformation

\[Z_n(G') = Z_n(G).\]

By the other hand the star-triangle transformation provides a groupoid symmetry on a wide class of objects, in our case on the space of Ising models. The Biggs formula allows us to extend this action to the points of valency 1 and 2. And we can obtain
the 14-term relation (Theorem 5.1) by comparing the right-hand sides of Biggs duality formulas for $G$ and $G'$.

Complain this idea in detail, consider two pairs of $n$-Potts models: $M_1(G, i^1_e)$ and $M_1(G', i^1_e)$; $M_2(G, i^2_e)$ and $M_2(G', i^2_e)$. After multiplying both parts of the formula (7) for $M_1(G, i^1_e)$ and $M_2(G, i^2_e)$ by $n^{v(G)}$ (by $n^{v(G')}$ for $M_1(G', i^1_e)$ and $M_2(G', i^2_e)$) we can write:

$$Z^i_n(G) = \prod_{e \in G} q_e \sum_{A \subseteq G} \prod_{e \in A} \frac{p_e}{q_e} Z_n^2(A)$$

$$= Z^i_n(G') = \prod_{e \in G'} q'_e \sum_{A' \subseteq G'} \prod_{e \in A'} \frac{p'_e}{q'_e} Z_n^2(A'),$$

here in both cases we take the sum over the set of all spanning subgraphs.

Let us rewrite the first part of the formula (61) by separating two kinds of terms:

$$Z^1_n(G) = \prod_{e \in G} q_e \sum_{A_1 \subseteq G} \prod_{e \in A_1} \frac{p_e}{q_e} Z_n^2(A_1) + \prod_{e \in G} q_e \sum_{A_2 \subseteq G} \prod_{e \in A_2} \frac{p_e}{q_e} Z_n^2(A_2),$$

where each subgraph $A_1$ contains the full triangle and each $A_2$ contains only a part of the triangle.

After the star-triangle transformation of the $M_1$ and $M_2$ we obtain the following formula for the models $M'_1(G', i'^1_e)$ and $M'_2(G', i'^2_e)$:

$$Z^i'_n(G') = \prod_{e \in G'} q'_e \sum_{A'_1 \subseteq G'} \prod_{e \in A'_1} \frac{p'_e}{q'_e} Z_n^2(A'_1) + \prod_{e \in G'} q'_e \sum_{A'_2 \subseteq G'} \prod_{e \in A'_2} \frac{p'_e}{q'_e} Z_n^2(A'_2),$$

where each subgraph $A'_1$ contains the full star and each $A'_2$ contains only a part of the star. Then, we compare the terms of these formulas:

- We notice that due to the star-triangle transformation $Z^i_n(G) = Z^i_n(G')$ and $Z^i_n(A_1) = Z^i_n(A'_1)$ (here and below $A_1$ is different from $A'_1$ only by the star-triangle transformation).
- Also it is easy to see that $\prod_{e \in G} q_e \prod_{e \in A_1} \frac{1}{q_e} = \prod_{e \in G'} q'_e \prod_{e \in A'_1} \frac{1}{q'_e}$.
- If the model $M_1(G, i^1_e)$ is chosen such that $p_1 p_2 p_3 = p'_1 p'_2 p'_3$, we conclude that $\prod_{e \in A_1} p_e = \prod_{e \in A'_1} p'_e$.

Now, we are ready to formulate the following theorem:

**Figure 15.** The 14-term relation
Theorem 5.1. Consider two $n$-Potts models $M_2(G, i^2)$ and $M'_2(G', i'^2)$, which are different from each other by the star-triangle transformation. Then, the following formula holds:

\[
q_1q_2q_3\left( Z^2_n(G_0) + \frac{p_1}{q_1} Z^2_n(G_1) + \frac{p_2}{q_2} Z^2_n(G_2) + \frac{p_3}{q_3} Z^2_n(G_3) + \frac{p_1p_2}{q_1q_2} Z^2_n(G_{12}) + \frac{p_1p_3}{q_1q_3} Z^2_n(G_{13}) \right) + \frac{p_2p_3}{q_2q_3} Z^2_n(G_{23}) = q'_1q'_2q'_3\left( Z^2_n(G'_0) + \frac{p'_1}{q'_1} Z^2_n(G'_1) + \frac{p'_2}{q'_2} Z^2_n(G'_2) + \frac{p'_3}{q'_3} Z^2_n(G'_3) + \frac{p'_1p'_2}{q'_1q'_2} Z^2_n(G'_{12}) + \frac{p'_1p'_3}{q'_1q'_3} Z^2_n(G'_{13}) \right) + \frac{p'_2p'_3}{q'_2q'_3} Z^2_n(G'_{23}),
\]

where

\[
p_i = \frac{\alpha^1_i - \beta^1_i}{\alpha^2_i - \beta^2_i}, \quad q_i = \frac{\alpha^2_i - \alpha^1_i}{\alpha_i^2 - \beta_i^2}, \quad p'_i = \frac{\alpha^1_i - \beta^1_i}{\alpha'_i^2 - \beta'_i^2}, \quad q'_i = \frac{\alpha'_i^2 - \alpha'_i^1}{\alpha'_i^2 - \beta'_i^2}.
\]

and variables $\alpha^k_i, \beta^k_i$ and $\alpha'_k_i, \beta'_k_i$ are related by the star-triangle transformation with the condition $p_1p_2p_3 = p'_1p'_2p'_3$.

Proof. We will proof this theorem using induction on $ex(G) := e(G) - 3$.

We show that the base of the induction $k = 0$ is trivial. Hence, let us consider the $n$-Potts models $M_2(G, i^2)$ and $M'_2(G', i'^2)$ and the special models $M_1(G, i^1)$ and $M'_1(G', i'^1)$ such that $p_1p_2p_3 = p'_1p'_2p'_3$. Then, we write the formulas (62) and (63), after the comparison for each terms using the reasoning above we immediately obtain the result in the case $ex(G) = 0$.

Then, make the step of induction. Again, let us write down the formulas (62) and (63):

\[
Z^1_n(G) = \prod_{e \in G} q_e \sum_{A_1 \subseteq G} \prod_{e \in A_1} \frac{p_e}{q_e} Z^2_n(A_1) + \prod_{e \in G} q_e \sum_{A_2 \subseteq G} \prod_{e \in A_2} \frac{p_e}{q_e} Z^2_n(A_2) + \prod_{e \in \partial G} q_e S_1,
\]

where each $A_1$ contains the full triangle, each $A_2$ contains only a part of the triangle and such that $ex(A_2) \neq ex(G)$, and by $S_1$ we denoted the left hand side of (64).

\[
Z^1_n(G') = \prod_{e \in G'} q'_e \sum_{A'_1 \subseteq G'} \prod_{e \in A'_1} \frac{p'_e}{q'_e} Z^2_n(A'_1) + \prod_{e \in G'} q'_e \sum_{A'_2 \subseteq G'} \prod_{e \in A'_2} \frac{p'_e}{q'_e} Z^2_n(A'_2) + \prod_{e \in \partial G'} q'_e S_2,
\]

where each $A'_1$ contains the full star, each $A'_2$ contains only a part of the star and such that $ex(A'_2) \neq ex(G')$, and by $S_2$ we denoted the right hand side of (64).

Then the induction assumption ends the proof.

\[\Box\]

Appendix A. The proof of Lemma 4.3

Proof. We start by reformulating this statement in terms of equivalent rational identities. Let us introduce the $x$-variables by the following formula:

\[
F \circ S^3(t_1, t_2, t_3) = (x_1, x_2, x_3),
\]

\[
S^3(x_1, x_2, x_3) = (t'_1, t'_2, t'_3).
\]

Here $S^3(t_1, t_2, t_3) = (S \times S \times S)(t_1, t_2, t_3) = (S(t_1), S(t_2), S(t_3))$. Then the statement of the Lemma is equivalent to:

\[
(t'_1, t'_2, t'_3) = S^3(x_1, x_2, x_3) = F^{-1}(t_1, t_2, t_3).
\]
This identity is equivalent to three algebraic relations (we will write down only one of them, because the others differ just by replacing the indices):

\[
\begin{align*}
t_1t_2 &= \frac{t_1t_2t'_3 + 1}{t_3 + t_1t'_2} \\
&= \frac{(x_1 - 1)(x_2 - 1)(x_3 - 1)}{(x_1 + 1)(x_2 + 1)(x_3 + 1) + 1} / \frac{(x_3 - 1) + (x_1 - 1)(x_2 - 1)}{(x_3 + 1)(x_1 + 1)(x_2 + 1)} \\
&= \frac{x_1 + x_2 + x_3 + x_1x_2x_3}{-x_2 - x_1 + x_3 + x_1x_2x_3}.
\end{align*}
\]

(68)

Now let us introduce some additional variables:

\[
t_{12} = x_1x_2, \quad t_{23} = x_2x_3, \quad t_{13} = x_1x_3, \quad a_1 = x_1^2, \quad a_2 = x_2^2, \quad a_3 = x_3^2.
\]

We could rewrite (68) in the following way:

\[
t_{12} = \frac{a_1t_{23} + a_2t_{13} + a_3t_{12} + t_{12}t_{23}t_{13}}{-a_2t_{13} - a_1t_{23} + a_3t_{12} + t_{12}t_{23}t_{13}}
\]

The equations (35) and (65) with the identification \(y_i := S(t_i)\) provide the following system:

\[
\begin{align*}
t_{12} &= \frac{y_1y_2y_3 + 1}{y_3 + y_1y_2} = \frac{t_3 + t_2 + t_1 + t_1t_2t_3}{t_3 + t_2 - t_1 + t_1t_2t_3} \\
t_{13} &= \frac{y_1y_2y_3 + 1}{y_2 + y_3y_1} = \frac{t_3 + t_2 + t_1 + t_1t_2t_3}{-t_3 - t_1 + t_1t_2t_3 + t_2} \\
t_{23} &= \frac{y_1y_2y_3 + 1}{y_1 + y_3y_2} = \frac{t_3 + t_2 + t_1 + t_1t_2t_3}{-t_2 - t_3 + t_1t_2t_3 + t_1}
\end{align*}
\]

\[
\begin{align*}
a_1 &= \frac{(y_1y_2y_3 + 1)(y_1 + y_2y_3)}{(y_2 + y_1y_3)(y_3 + y_1y_2)} = \frac{(-t_2 - t_3 + t_1t_2t_3 + t_1)(t_3 + t_2 + t_1 + t_1t_2t_3)}{(t_3 - t_2 - t_1 + t_1t_2t_3)(-t_1 - t_3 + t_1t_2t_3 + t_2)} \\
a_2 &= \frac{(y_1y_2y_3 + 1)(y_2 + y_1y_3)}{(y_1 + y_2y_3)(y_3 + y_1y_2)} = \frac{(-t_1 - t_3 + t_1t_2t_3 + t_2)(t_3 + t_2 + t_1 + t_1t_2t_3)}{(t_3 - t_2 - t_1 + t_1t_2t_3)(t_3 + t_2 + t_1 + t_1t_2t_3 + t_1)} \\
a_3 &= \frac{(y_1y_2y_3 + 1)(y_3 + y_1y_2)}{(y_2 + y_1y_3)(y_1 + y_3y_2)} = \frac{(-t_2 - t_3 + t_1t_2t_3 + t_1)(-t_1 - t_3 + t_1t_2t_3 + t_2)}{(t_3 - t_2 - t_1 + t_1t_2t_3)(t_3 + t_2 + t_1 + t_1t_2t_3 + t_1)}
\end{align*}
\]

Using these expressions we can compute:

\[
\begin{align*}
t_{12}a_3 + t_{13}a_2 + a_1t_{23} + t_{12}t_{23}t_{13} &= \frac{4(t_3 + t_2 + t_1 + t_1t_2t_3)^2t_1t_2t_3}{(t_3 - t_2 - t_1 + t_1t_2t_3)(-t_1 - t_3 + t_1t_2t_3 + t_2)(-t_2 - t_3 + t_1t_2t_3 + t_1)}; \\
t_{12}(-a_2t_{13} - a_1t_{23} + a_3t_{12} + t_{12}t_{23}t_{13}) &= \frac{4(t_3 + t_2 + t_1 + t_1t_2t_3)^2}{(t_3 - t_2 - t_1 + t_1t_2t_3)(-t_1 - t_3 + t_1t_2t_3 + t_2)(-t_2 - t_3 + t_1t_2t_3 + t_1)}.
\end{align*}
\]

In this way we observe that

\[
t_{12}(-a_2t_{13} - a_1t_{23} + a_3t_{12} + t_{12}t_{23}t_{13}) = t_{12}a_3 + t_{13}a_2 + a_1t_{23} + t_{12}t_{23}t_{13}.
\]

This completes the proof.

Now we give the full formulas for \(t_1\) and \(t_2\):
(69)
\[
t_1 = (-b_3a_3 + a_2b_2 + a_1b_1 - b_4a_4 + (b_5^2a_3^2 - 2b_3a_3a_2b_2 - 2b_3a_3a_1b_1 - 2b_3a_5b_4a_4 + a_2^2b_2^2 - \\
-2a_2b_2a_1b_1 - 2a_2b_2b_1a_4 + a_1^2b_1^2 - 2a_1b_1a_4 + b_4^2a_4^2 + 4b_1a_3a_1b_2 + 4a_2b_1b_3a_4)^{1/2})/(2(-b_4a_3 + a_2b_1)),
\]

(70)
\[
t_2 = (a_2b_4 - b_3a_1 + b_3b_1 + (a_2^2b_4^2 - 2a_4b_4a_2b_2 - 2a_2b_4b_3a_1 - 2a_2b_1a_3b_4 + b_4^2a_4^2 - 2b_3a_4a_1b_2 - \\
- 2b_2a_4b_3a_1 + b_3^2a_1^2 - 2a_3b_3a_1b_1 + a_2^2b_1^2 + 4a_3b_1a_3b_2 + 4a_2b_1b_3a_4)^{1/2})/(2(a_3b_4 - b_3a_4)).
\]

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