Mass formulas, mixing angles, and decay constants of the pseudoscalar $\pi^0$, $\eta$, and $\eta'$ mesons have been obtained in the theory with four-quark interactions with an accuracy to $O(1/N_c^2)$ terms inclusively. Unlike the standard Nambu–Jona-Lasinio model, the meson Lagrangian has been obtained using the Volterra series, which allows a more detailed description of effects caused by the inequality of quark masses. The results have been compared to similar calculations in the $1/N_c$ chiral perturbation theory. The first corrections to the interaction violating the Zweig formulas have been calculated, and it has been shown that they are important for the description of the mass spectrum of the $\eta-\eta'$ mesons.

**1. INTRODUCTION**

To study the properties of the pseudoscalar nonet of mesons, Leutwyler [1, 2] used the Lagrangian whose effective vertices are classified in terms of the powers of momenta, masses of light quarks, and $1/N_c$, where $N_c$ is the number of color degrees of freedom. The method called the $1/N_c$ chiral perturbation theory [3] made it possible to calculate the first correction to the main result of the current algebra for mass formulas of the charged pseudoscalar mesons with nonzero strangeness, to show that the correction is small (at that time, contradictions existed in the description of the $\eta \to 3\pi$ decay), and to obtain constraints on the masses of light quarks. Later, the method was extended to spin-1 states [4]. Among recently results obtained with this effective Lagrangian, we mention the analysis of the two-photon decays of the $\pi^0$, $\eta$, and $\eta'$ mesons including the first- and second-order corrections in $1/N_c$ [5].

I have recently shown [6] that the theory with four-quark interactions of the Nambu–Jona-Lasinio type [7, 8] gives the same mass formulas for the pseudoscalar $\pi^\pm$, $K^\pm$, $K^0$, and $\bar{K}^0$ mesons as in [1]. This theory allows one to relate the parameters of the effective Leutwyler theory to the parameters of the dynamic Nambu–Jona-Lasinio model. This is achieved by expanding the effective action of the Nambu–Jona-Lasinio model in the Volterra series in inverse powers of the masses of the constituent quarks rather than in the Taylor series in powers of the proper time [9–11]. In addition, the Leutwyler hypothesis on the $m_i = O(1/N_c)$ behavior of the masses of light quarks $m_i$ ($i = u, d, s$) in the limit of large $N_c$ values is used.

The aim of this work is to study the physical characteristics of the remaining terms of the nonet—$\pi^0$, $\eta$, and $\eta'$ mesons—for which the masses and mixing angles are calculated with an accuracy to the first correction in $1/N_c$ inclusively and the constant of the anomalous violation of the $U(1)_A$ symmetry, as well as the degree of violation of the Okubo–Zweig–Iizuka rule, is estimated. The latter estimates are of particular interest in the context of the current active study of the gluon structure of the $\eta$ and $\eta'$ mesons from different points of view (see, e.g., [12], where the dispersion approach was applied to study the axial anomaly). The formalism used in this work, as well as the numerical parameters of the model, is the same as in [6].

**2. MODIFIED NAMBU–JONA-LASINIO MODEL**

The initial four-quark Lagrangian used in this work is the same as in the standard quark version of the Nambu–Jona-Lasinio model [13–16]

$$\mathcal{L} = \bar{q}i\gamma^\mu \partial_\mu q - m q + \mathcal{L}_{\text{int}}. \quad (1)$$

Here, $\gamma^\mu$ are the Dirac matrices, $q$ are the quark fields, and $m = \text{diag}(m_u, m_d, m_s)$ is the diagonal matrix consisting of the current masses. The Lagrangian density has the form $\mathcal{L}_{\text{ph}} = \mathcal{L}^{(0)} + \mathcal{L}^{(1)}$, where the first and second terms are $U(3)_L \times U(3)_R$ chirally symmetric combinations describing four-quark interactions with spins zero and unity, respectively, given by the expressions

$$\mathcal{L}^{(0)} = \frac{G_F}{2} \left[ (\bar{q} \gamma_\mu q)^2 + (\bar{q} i \gamma_\mu \gamma_5 q)^2 \right], \quad (2)$$


\[ \mathcal{L}^{(1)} = -\frac{G_A}{2} \left[ (\overline{q} \gamma^\mu \lambda_q q)^2 + (\overline{q} \gamma^\mu \gamma_5 \lambda_q q)^2 \right]. \tag{3} \]

Here, \( \lambda_q = \sqrt{2}/3 \), \( \lambda_q \) are the Gell–Mann matrices, and the constants \( G_S \) and \( G_V \) in the limit \( N_c \to \infty \) have the order \( \mathcal{O}(1/N_c) \); their values determined in [6] are \( G_S = 6.6 \text{ GeV}^{-2} \) and \( G_V = 6.8 \text{ GeV}^{-2} \).

The functional integral method allows the transformation of the Lagrangian density (1) to the form

\[ \mathcal{L}' = \overline{Q}(i \gamma^\mu \partial_\mu - M + \sigma)\mathcal{Q} + \frac{1}{4G_V} \text{tr}(V_\mu^2 + A_\mu^2) \]

\[ - \frac{1}{4G_S} \text{tr} \left( \sigma^2 - (\sigma, M) + (\sigma - M)\Sigma \right) \tag{4} \]

Here, the functional freedom of choice of dynamic variables was used in favor of the nonlinear realization of chiral symmetry; \( V_\mu = V_\mu^a \lambda_a \), \( A_\mu = A_\mu^a \lambda_a \), \( \sigma = \sigma \lambda_a \), and \( \phi = \phi \lambda_a \) are the Hermitian matrices describing the vector, axial vector, scalar, and pseudoscalar fields, respectively; \( Q = (\xi P_R + \xi' P_L)q \), where \( P_L = (1 - \gamma_5)/2 \) and \( P_R = (1 + \gamma_5)/2 \), are the quark fields belonging to the fundamental representation:

\[ d_\mu = \partial_\mu - i \left[ \xi_\mu^{(+)} + V_\mu + \gamma_5 (\xi_\mu^{(-)} + A_\mu) \right] \tag{5} \]

\[ \xi_\mu^{(\pm)} = \frac{i}{2} \left( \xi \partial_\mu \xi \pm \xi \partial_\mu \xi \right) \tag{6} \]

\[ \Sigma = \xi m \xi + \xi^\dagger m \xi^\dagger, \quad \xi = \exp \left( \frac{i \phi}{2} \right) \tag{7} \]

The pseudoscalar field \( \phi \) is dimensionless and acquires the necessary mass dimension at the transition to the field functions of physical states. The matrix \( M = \text{diag}(M_u, M_d, M_s) \) is the diagonal matrix consisting of the masses of constituent quarks \( Q \). These masses appear through the dynamical breaking of symmetry and are related to the masses of light quarks by the gap equation

\[ M_i \left( 1 - \frac{N_c G_S}{2\pi} J_0(M_i) \right) = m_i \quad (i = u, d, s), \tag{8} \]

where

\[ J_0(M_i) = M_i^2 - M_i^2 \ln \left( 1 + \frac{M_i^2}{M_i^2} \right) \tag{9} \]

Here, \( \Lambda \) is the cutoff parameter characterizing the scale at which the effective theory under consideration is studied. In this case, this is the scale of hadron masses, which is \( \Lambda \approx 1.1 \text{ GeV} \).

The gap equation is a mathematical minimum condition for the effective potential; this potential and kinetic part of the effective action are obtained after integration over the quark fields \( Q \). This integration gives the quark determinant with the local part described by the first terms of its asymptotic expansion in powers of the proper time. The difference between the quark masses leads to a problem concerning the inclusion of \( M_i - M_j \) difference effects. To solve this problem, an asymptotic expansion in the inverse powers of large masses was proposed in [11] on the basis of the Volterra series. This expansion distinguishes our calculations from the standard approach involving the Taylor series. This difference is significant because the Volterra series contains numerous finite (in the limit \( \Lambda \to \infty \)) vertices that vanish in the limit of equal quark masses. These vertices carry important additional information on violations of the isospin and flavor symmetries that is absent in the standard meson Lagrangian of the Nambu–Jona-Lasinio model.

If the quarks are massive, pseudoscalar fields are mixed with axial vector fields. To remove this mixing, it is necessary to redefine axial vector fields as [17]

\[ A_\mu = A_\mu' - \kappa_A \gamma^5 \xi^{(-)} \xi, \tag{10} \]

where \( \kappa_A \) is a matrix and \( \gamma \) stands for the Hadamard product of matrices [18] defined as \((A \circ B)_{ij} = A_{ij}B_{ij}\) without summation over repeated indices.

Only the diagonal elements of the matrix \( \kappa_A \) that are required here have the form

\[ (\kappa_A)^{-1} = 1 + \frac{\pi^2}{N_c G_V M_i^2 J_i(M_i)}, \tag{11} \]

where

\[ J_i(M_i) = \ln \left( 1 + \frac{\Lambda^2}{M_i^2} \right) - \frac{\Lambda^2}{\Lambda^2 + M_i^2}. \tag{12} \]

The quark determinant includes the kinetic terms of the Lagrangian of free meson fields. They have the canonical form after the corresponding redefinition of the variables

\[ \phi_i = f_i^{-1} \phi_i^R \quad (i = u, d, s), \tag{13} \]

where the relation of the components \( \phi_{u,d,s} \) to the components \( \phi_{0,3,8} \), as well as the relation between the components \( \phi_{u,d,s}^R \) and \( \phi_{0,3,8}^R \), is standard. As easily seen, the new field variables marked by the superscript \( R \) have the dimension of mass because the constants \( f_i \sim \mathcal{O}(\sqrt{N_c}) \) are given by the expression

\[ f_i = \sqrt{\frac{(\kappa_A)^{ii}}{4G_V}}. \tag{14} \]

As a result,

\[ \mathcal{L}_{\text{kin}} = \frac{1}{4} \sum_{i=u,d,s} \left( \partial_i \phi_i^R \right)^2 = \frac{1}{2} \sum_{a=0,3,8} \left( \partial_a \phi_a^R \right)^2. \tag{15} \]

The mass part of the Lagrangian (4) is diagonal in the flavor basis

\[ \mathcal{L}_m = -\frac{G_V}{G_S} \sum_{i=u,d,s} \frac{M_i m_i}{(\kappa_A)i} \left( \phi_i^R \right)^2. \tag{16} \]
However, the physical $\pi^0$, $\eta$, and $\eta'$ mesons are not pure flavor states. For this reason, it is convenient to pass to the singlet–octet components 0, 3, 8. In this basis, the off-diagonal elements of the mass matrix are zero only in the case of the exact $SU(3)_c$ symmetry. Thus, we obtain

$$\mathcal{L}_c = -\frac{1}{2} \sum_{a=0,3,8} \phi^R_a \Phi^R_a,$$

(17)

where $m^2_a$ is the symmetric matrix with the elements

$$m^2_a = \frac{4G_V}{3G_S} \sum_{i,a,d} \frac{M_m}{\kappa_A} \phi^R_a \phi^R_i,$$

$$m^2_{08} = \frac{2G_V}{3G_S} \left( \frac{M_m}{\kappa_A} \phi^R_a \phi^R_i \right),$$

$$m^2_{33} = \frac{2G_V}{3G_S} \left( \frac{M_m}{\kappa_A} \phi^R_a \phi^R_i \right),$$

$$m^2_{08} = \frac{2\sqrt{2}G_V}{3G_S} \left( \frac{M_m}{\kappa_A} \phi^R_a \phi^R_i \right),$$

$$m^2_{03} = \frac{2\sqrt{2}G_V}{3G_S} \left( \frac{M_m}{\kappa_A} \phi^R_a \phi^R_i \right),$$

$$m^2_{38} = \frac{1}{\sqrt{2}} m^2_{03}.$$

(18)

Here, it is necessary to take into account two important circumstances—$U(1)_A$ anomaly and violation of the Okubo–Zweig–Iizuka rule—both explained within the 1/$N_c$ expansion [19–21]. Their leading term has the order $O(1/N_c)$ coinciding with the order of the leading term in Eqs. (18), where $m^2_{0} \sim O(1/N_c)$. The Lagrangians corresponding to these processes have the form of the product of two traces. At the quark–gluon level, such a contribution comes from diagrams with quark loops coupled through the gluon exchange.

The Lagrangian breaking the $U(1)_A$ symmetry was obtained in [22] (see also [23]). Using this result, we set

$$\mathcal{L}_V = \frac{\lambda_V}{48} \left[ \text{tr} \left( \ln \xi \xi - \ln \xi \xi^1 \right) \right] = -\frac{\lambda_V}{2F^2} \phi^R \phi^R.$$

(19)

The dimensional constant $\lambda_V$ is fixed from the physical masses of the pseudoscalar mesons, and it does not change in the limit $N_c \to \infty$, having the order $\lambda_V = O(N_c^0)$.

The Lagrangian violating the Okubo–Zweig–Iizuka rule has the form [1]

$$\mathcal{L}_Z = \frac{i \lambda_Z}{\sqrt{6}} \text{tr} \left[ \chi \xi \xi - \xi \xi^1 \right],$$

(20)

where $\lambda_Z = O(N_c^1)$ is the dimensional constant and $\chi = 2BM$ is the matrix having the form

$$\chi = \frac{4G_V}{G_S} \text{diag} \left( \frac{M_m}{\kappa_A} \phi^R_a \phi^R_i \right).$$

(21)

The quadratic part of the Lagrangian (20) leads to mixing

$$\mathcal{L}_Z \to \lambda_Z \frac{G_V}{G_S} \phi_0 \sum_{i,a,d} \frac{M_m}{\kappa_A} \phi^R_i.$$  

(22)

Nonphysical fields $\phi_0$ and $\phi_i$ should be transformed to physical ones according to Eq. (13). In particular,

$$\phi_0 = \frac{\phi_0^R}{3} \left( \frac{1}{f_a} + \frac{1}{f_d} + \frac{1}{f_s} \right) + \phi_i^R \left( \frac{1}{f_a} \frac{1}{f_d} \frac{1}{f_s} \right) + O \left( \frac{N_c^0}{f} \right).$$

(23)

Consequently, beyond the leading approximation in Eq. (22), additional mixing between the neutral components is induced by breakings of the isospin and $SU(3)_c$ symmetries.

Further, we consider the first terms of the 1/$N_c$ expansions of the resulting formulas. Here, we do only the first two steps, i.e., represent the elements of the mass matrix of the physical states in the form of the sum of the leading order (LO) term having the order $O(1/N_c)$ and the next-to-leading order (NLO) term with the order $O(1/N_c^2)$, which is the first correction to the LO term. The calculation of the second correction (NNLO term) requires additional consideration of meson one-loop contributions, which is beyond the scope of this work.

3. 1/$N_c$ EXPANSION

The six model parameters $\Lambda$, $G_S$, $G_V$, and $m_i$ have the orders $\Lambda \sim O(1)$ and $G_S, G_V, i \sim O(1/N_c)$. The first parameter specifies the characteristic energy scale and the other parameters are much smaller than the first parameter. This property allows the systematic 1/$N_c$ expansion of the effective theory.

Let us seek the solution of the gap equation (8) in the form

$$M_i (m_i) = M_0 + M^* (0) m_i + O(m_i^2),$$

(24)

where $M_0$ is the solution of the equation at large $N_c$ values. We find

$$M^* (0) = \frac{\pi^2}{N_c G_S M^2_0} = \frac{G_E}{G_S} (\kappa^{-1}_{d0} - 1) \equiv a.$$  

(25)

Here and below, the index 0 means that this function of the quark masses $m_i$ is calculated in the limit $m_i$ → 0. In particular, $J^0 = J_i (M_0)$ and $\kappa^{-1}_{i0} = \lim_{m_i \to 0} (\kappa_A)^{-1}_{ii}$. 

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In turn, the constants given by Eq. (14) have the form
\[
f_i = F \left[ 1 + \frac{m_i}{2M_0} (a - \delta_M) \right], \quad F = \frac{\sqrt{\kappa_40}}{4G_V},
\]
where
\[
\delta_M = a \left[ 1 - 2(1 - \kappa_{ab}) \left[ 1 - \frac{\Lambda^4}{J_i(\Lambda^2 + \bar{\Omega}_i^2)} \right] \right].
\]

The elements of the mass matrix \( m_{ab}^2 \) including all above contributions can be represented in the form
\[
m_{ab}^2 \rightarrow M_{ab}^2 = M_{ab}^0 + \Delta M_{ab}^2,
\]
where the first term is the leading contribution and the second term is the first correction to it. The leading contribution has the order \( \mathcal{O}(1/N) \) and is described by the formulas
\[
M_{00}^2 = \frac{2}{3} B_0 (m_u + m_d + m_s)(1 - 2\Delta_N) + \lambda_\eta^2,
\]
\[
M_{08}^2 = \frac{1}{3} B_0 (m_u + m_d + 4m_s),
\]
\[
M_{08}^2 = \frac{\sqrt{2}}{3} B_0 (m_u + m_d - 2m_s)(1 - \Delta_N),
\]
\[
M_{03}^2 = \frac{\sqrt{2}}{3} B_0 (m_u - m_d)(1 - \Delta_N),
\]
\[
M_{08}^2 = \frac{1}{\sqrt{3}} B_0 (m_u - m_d),
\]
\[
M_{33}^2 = B_0 (m_u + m_d),
\]
where
\[
\Delta_N = 2\sqrt{2} \frac{\Lambda_\eta}{F^2}, \quad \lambda_\eta^2 = \frac{\Delta_\nu}{F^2},
\]
\[
B_0 = \frac{(\bar{q}d)_0}{F^2} = \frac{2G_V M_0}{G_S \kappa_{40}} = M_0 \frac{M_0}{2G_S F^2}.
\]

\( \phi_0^R = (\epsilon^* \cos \theta - \epsilon \sin \theta) \pi^0 - \sin \theta \eta + \cos \theta \eta \).

This orthogonal transformation diagonalizes the mass matrix \( M_{ab}^2 \) if the mixing angles satisfy the conditions
\[
\tan 2\theta = -\frac{2M_{08}}{M_{00}^2 - M_{88}^2}, \quad \epsilon = \frac{\sin \theta M_{03} - \cos \theta M_{38}}{m_\eta - m_\pi^0},
\]
\[
\epsilon^* = -\frac{\cos \theta M_{03} + \sin \theta M_{38}^2}{m_\eta^2 - m_\pi^0},
\]

where
\[
m_{\eta,\eta'}^2 = \frac{1}{2} \left[ M_{00}^2 + M_{88}^2 \pm \sqrt{(M_{00}^2 - M_{88}^2)^2 + 4M_{38}^4} \right],
\]
\[
m_{\pi^0}^2 = M_{33}^2.
\]

Since Eqs. (28) contain only two unknown parameters \( \Delta_N \) and \( \lambda_\eta^2 \), they can be determined from the known masses of the \( \eta \) and \( \eta' \) mesons. To this end, we first determine the masses of the quarks using the physical masses of the \( \pi^+ \), \( K^+ \), and \( K^0 \) mesons, which in this approximation are given by the current algebra formulas \( \pi^+_1 = B_0 (m_u + m_d) \), \( \pi^+_K = B_0 (m_u + m_s) \), and \( \pi^0 = B_0 (m_d + m_s) \). Here, the overline means that these expressions were obtained disregarding the electromagnetic interaction. The inclusion of this interaction increases the masses of the charged states:
\[
\mu^2_\pi^+ = \mu^2_\pi^0 + \lambda_\pi^0, \quad \mu^2_\pi^+ = \mu^2_\pi^0 + \lambda_\pi^0, \quad \mu^2_\pi^0 = \mu^2_\pi^0.
\]

It is known that the difference between the masses of the charged and neutral pions is due primarily to the electromagnetic interaction. The contribution of the strong interaction is proportional to \((m_d - m_s)^2\) and is thereby negligibly small. Using the Dashen theorem \( \Delta_{\eta'} = \Delta_{\eta} \) [24], we obtain \( m_\pi = 2.6 \text{ MeV}, m_\eta = 4.7 \text{ MeV}, \) and \( m_\eta = 95 \text{ MeV} \) from the above mass formulas. Then, the parameters \( \lambda_\eta^2 = 0.891 \text{ GeV}^2 \) and \( \Delta_N = 0.48 \) are determined from the system of two equations for the masses of the \( \eta \) and \( \eta' \) mesons. The mixing angles are \( \epsilon = 0.014, \epsilon^* = 0.0037, \) and \( \theta = -10.5^\circ \).

The estimates presented above show that the Zweig rule is strongly violated in the leading approximation, as also mentioned in [1, 25]. Only paying this price can one satisfactorily describe the mass spectrum of the \( \eta - \eta' \) mesons. In the absence of the Lagrangian (20), the mass of the \( \eta \) meson is much smaller than its phenomenological value and the angle \( \theta = -18.3^\circ \). The interaction violating the Zweig rule allows one to reproduce the mass spectrum but reduces the absolute value of the angle \( \theta \). The result for the angle \( \theta \) is close to \( \theta = -12.3^\circ \) [26] and exactly coincides with the
result obtained in [27] with the inclusion of the first correction in $1/N_c$ chiral perturbation theory.

Now, we do the next step and calculate the first correction $\Delta M_{ab}$ to the leading term. This correction includes the contributions from Eqs. (18) and (22). As a result,

$$
\Delta M_{ab}^2 = \frac{2B_0}{3M_0} \left[ (m_u^2 + m_d^2 + m_s^2) [a - 3\delta_M] \Delta_N + \delta_M \right]
+ \frac{1}{3} (m_u + m_d + m_s) (a - \delta_M),
$$

$$
\Delta M_{ss}^2 = \frac{B_0}{3M_0} \left[ (m_u^2 + m_s^2 + 4m_s^2) \delta_M
+ \frac{1}{3} (2m_s - m_u - m_d)^2 \Delta_N \right],
$$

$$
\Delta M_{dst}^2 = \frac{\sqrt{2}B_0}{3M_0} \left[ (m_u^2 - m_d^2 - m_s^2) \frac{3\delta_M - a}{2} \Delta_N - \delta_M \right]
+ \frac{1}{3} \Delta_N (2m_s - m_u - m_d) (m_u + m_d + m_s) (\delta_M - a),
$$

$$
\Delta M_{ds}^2 = \frac{1}{\sqrt{3}M_0} \left[ (m_d - m_s) \frac{3\delta_M - a}{2} \Delta_N - \delta_M \right]
+ \frac{1}{3} \Delta_N (m_d - m_s) (m_u + m_d + m_s) (\delta_M - a),
$$

$$
\Delta M_{ss}^2 = \frac{B_0}{M_0} \left[ (m_u - m_s) \frac{a - \delta_M}{3} \Delta_N
- (m_d + m_s) \delta_M \right].
$$

Corrections caused by Eq. (18) with an accuracy to a common factor coincide with the result of $1/N_c$ chiral perturbation theory [27]. The correspondence between factors has the form

$$
\frac{\delta_M}{M_0} \to 16 \frac{B_0}{F_0} (2L_8 - L_3).
$$

Corrections $\sim \Delta_N$ were not considered in [27].

As before, the second-order terms in the breaking of isospin symmetry were neglected when deriving Eq. (35). We recall that the mass matrix $M_{ab}^2$ in this approximation is still diagonalized by the orthogonal transformation (31).

To obtain numerical values, we first determine the masses of light quarks using the following mass formulas for the $\pi^\pm$, $K^\pm$, and $\eta'$ mesons, which also include the first correction to the current algebra result:

$$
m_{\pi^+}^2 = \mu_{\pi^+}^2 \left( 1 + \frac{m_u + m_d}{2M_0} \delta_M \right),
$$

$$
m_{K^+}^2 = \mu_{K^+}^2 \left( 1 + \frac{m_u + m_d}{2M_0} \delta_M \right),
$$

$$
m_{\eta'}^2 = \mu_{\eta'}^2 \left( 1 + \frac{m_u + m_d}{2M_0} \delta_M \right).
$$

Second, it is also necessary to take into account that the Dashen theorem is valid only in the leading approximation of the chiral expansion. Beyond this approximation, the possible deviation from the Dashen theorem should be taken into account; i.e., it is necessary to accept that $\Delta_{\pi^0}^2 \neq \Delta_{\eta'}^2 = m_{\pi^0}^2 - m_{\eta'}^2$. As a result, mass formulas (37)–(39) acquire an additional dependence on the parameter $\Delta_{\eta'}$, which varies in the range that can be determined from the observed $\eta \to \pi^+\pi^-\pi^0$ decay width [1]. This allows one to establish $\Delta_{\eta'} = (44.8 \pm 4.5)$ MeV and, as a result, to fix the quark masses $m_u = (2.65 \pm 0.07)$ MeV, $m_d = (4.63 \pm 0.07)$ MeV, and $m_s = (85.94 \pm 0.07)$ MeV. As above, using the phenomenological masses of the $\eta$ and $\eta'$ mesons, we now obtain $\theta = -20.4^\circ$,

$$
\epsilon = 0.013 \pm 0.001, \quad \epsilon' = 0.0039 \pm 0.0003,
$$

$$
\Delta_N = 0.38, \quad \lambda_{\eta'}^2 = (0.730 \pm 0.001) \text{ GeV}^2.
$$

It is noteworthy that the NLO corrections hardly affect the angles $\epsilon$ and $\epsilon'$, which are in agreement with the phenomenological estimates $\epsilon = 0.014$ and $\epsilon' = 0.0037$ [28], whereas these corrections noticeably increase the absolute value of the angle $\theta$, which is in agreement with the result of chiral perturbation theory $\theta = -20^\circ \pm 4^\circ$ [29] but is above the result of anomalous sum rules $\theta = -14.2^\circ \pm 0.7^\circ$ [30]. The parameter $\Delta_N$ characterizing the degree of violation of the Zweig rule decreases. The constant $\lambda_{\eta'}^2$ also decreases and almost coincides with the estimate $\lambda_{\eta'}^2 = 0.726 \text{ GeV}^2$ obtained in [20].

Formula (14) gives the constants $f_\pi = 92.7$ MeV, $f_\eta = 93.7$ MeV, and $f_{\eta'} = 132.8$ MeV. According to the model, the weak decay constant of the pion is $f_\pi = 93.2$ MeV. As a result, $f_\eta/f_{\eta'} = 1.42$, which is in agreement with the estimate $f_\eta/f_{\eta'} = 1.65 \pm 0.25$ obtained from anomalous sum rules [30] and is close to the phenomenological estimate $f_\eta/f_{\eta'} = 1.34 \pm 0.06$ presented in [31].

4. CONCLUSIONS

To summarize, the main characteristics of the $\pi^0$–$\eta$–$\eta'$ system in the model with four-quark interactions have been calculated. A new method for the low-energy expansion of the quark determinant with virtual particles with different masses has been used to bosonize the quark vertices. Assuming that the masses of light quarks vanish in the limit $N_c \to \infty$, we have
obtained mass formulas and decay constants of these pseudoscalar states in the form of the first two terms of their expansion in series in $1/N_c$. As a result, we have described the mass spectrum of the $\eta - \eta'$ mesons and have analyzed the degree of the violation of the Zweig rule. The estimate $\Delta_N = 0.38$ obtained in this work indicates the moderate violation of the Okubo–Zweig–Iizuka rule. We emphasize that the considered $\mathcal{O}(1/N^2_c)$ corrections to the interaction violating the Zweig rule have not yet been studied. In particular, at least the effective Lagrangian in [32] includes interactions violating the Zweig rule but only in the leading order in $1/N_c$.

CONFLICT OF INTEREST

The author declares that he has no conflicts of interest.

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