Multiplicative duality, $q$-triplet and $(\mu, \nu, q)$-relation

derived from the one-to-one correspondence

between the $(\mu, \nu)$-multinomial coefficient and Tsallis entropy $S_q$

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Abstract

We derive the multiplicative duality “$q \leftrightarrow 1/q$” and other typical mathematical structures as the special cases of the $(\mu, \nu, q)$-relation behind Tsallis statistics by means of the $(\mu, \nu)$-multinomial coefficient. Recently the additive duality “$q \leftrightarrow 2 - q$” in Tsallis statistics is derived in the form of the one-to-one correspondence between the $q$-multinomial coefficient and Tsallis entropy. A slight generalization of this correspondence for the multiplicative duality requires the $(\mu, \nu)$-multinomial coefficient as a generalization of the $q$-multinomial coefficient. This combinatorial formalism provides us with the one-to-one correspondence between the $(\mu, \nu)$-multinomial coefficient and Tsallis entropy $S_q$, which determines a concrete relation among three parameters $\mu, \nu$ and $q$, i.e., $\nu (1 - \mu) + 1 = q$ which is called “$(\mu, \nu, q)$-relation” in this paper. As special cases of the $(\mu, \nu, q)$-relation, the additive duality and the multiplicative duality are recovered when $\nu = 1$ and $\nu = q$, respectively. As other special cases, when $\nu = 2 - q$, a set of three parameters $(\mu, \nu, q)$ is identified with the $q$-triplet $(q_{\text{sen}}, q_{\text{rel}}, q_{\text{stat}})$ recently conjectured by Tsallis. Moreover, when $\nu = 1/q$, the relation $1/(1 - q_{\text{sen}}) = 1/\alpha_{\text{min}} - 1/\alpha_{\text{max}}$ in the multifractal singularity spectrum $f(\alpha)$ is recovered by means of the $(\mu, \nu, q)$-relation.

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I. INTRODUCTION

In the last two decades the so-called Tsallis statistics or \( q \)-statistics has been introduced \cite{1} and studied as a generalization of Boltzmann-Gibbs statistics with many applications to complex systems \cite{2,3}, whose information measure is given by

\[
S_q(p_1, \ldots, p_n) = \frac{1 - \sum_{i=1}^{n} p_i^q}{q - 1} \tag{1}
\]

where \( p_i \) is a probability of \( i \)th state and \( q \) is a real parameter. This generalized entropy \( S_q \) is nowadays called Tsallis entropy which recovers Boltzmann-Gibbs-Shannon entropy \( S_1 \) when \( q \to 1 \). The above entropic form (1) was first given in \cite{4} and \cite{5} from a mathematical motivation, but in 1988 \cite{1} Tsallis first applied the above form (1) to a generalization of Boltzmann-Gibbs statistics for nonequilibrium systems through the maximum entropy principle (MaxEnt for short) along the lines of Jaynes approach \cite{6}. Since then, many applications of (1) to the studies of complex systems with power-law behaviors have been presented using the MaxEnt as a main approach \cite{7}. In fact, the \( q \)-exponential function appeared in the MaxEnt plays a crucial role in the formalism and applications \cite{2,3}.

For all many applications of the MaxEnt for Tsallis entropy (1), there have been missing a combinatorial consideration in Tsallis statistics until recently \cite{8}, whose ideas originate from Boltzmann’s pioneering work \cite{9} (See \cite{10} for the comprehensive review). By means of the \( q \)-product uniquely determined by the \( q \)-exponential function \cite{11,12} as the \( q \)-exponential law, the one-to-one correspondence between the \( q \)-multinomial coefficient and Tsallis entropy is obtained as follows \cite{8}: for \( n = \sum_{i=1}^{k} n_i \) and \( n_i \in \mathbb{N} \) if \( q \neq 2 \),

\[
\ln_q \left[ \frac{n}{n_1 \cdots n_k} \right] \simeq \frac{n^{2-q}}{2-q} \cdot S_{2-q} \left( \frac{n_1}{n}, \cdots, \frac{n_k}{n} \right) \tag{2}
\]

where \( \left[ \frac{n}{n_1 \cdots n_k} \right]_q \) is the \( q \)-multinomial coefficient and \( \ln_q \) is the \( q \)-logarithm. The above correspondence (2) obviously recovers the well-known correspondence:

\[
\ln \left[ \frac{n}{n_1 \cdots n_k} \right] \simeq n \cdot S_1 \left( \frac{n_1}{n}, \cdots, \frac{n_k}{n} \right) \tag{3}
\]

when \( q \to 1 \). Moreover, the additive duality “\( q \leftrightarrow 2 - q \)” in Tsallis statistics is revealed in (2). In the MaxEnt formalism for Tsallis entropy, two kinds of dualities “\( q \leftrightarrow 2 - q \)” and
“$q \leftrightarrow 1/q$” have been observed and discussed [7][13][14][15][16], but in the combinatorial formalism the multiplicative duality “$q \leftrightarrow 1/q$” is still missing.

In this paper, we derive the multiplicative duality “$q \leftrightarrow 1/q$” along the lines of the above correspondence (2), which introduces the $(\mu, \nu)$-factorial as a generalization of the $q$-factorial. We apply the $(\mu, \nu)$-factorial to the formulation of the $(\mu, \nu)$-multinomial coefficient and the $(\mu, \nu)$-Stirling’s formula, which results in the following correspondence: for $n = \sum_{i=1}^{k} n_i$ and $n_i \in \mathbb{N}$ if $q, \nu \neq 0$,

$$\frac{1}{\nu} \ln_{\mu} \begin{bmatrix} n \\ n_1 \cdots n_k \end{bmatrix}_{(\mu,\nu)} \simeq \frac{n^q}{q} \cdot S_q \left( \frac{n_1}{n}, \cdots, \frac{n_k}{n} \right)$$

(4)

where $\begin{bmatrix} n \\ n_1 \cdots n_k \end{bmatrix}_{(\mu,\nu)}$ is the $(\mu, \nu)$-multinomial coefficient and three parameters $\mu, \nu, q$ satisfy the relation:

$$\nu (1 - \mu) + 1 = q$$

(5)

which is called “$(\mu, \nu, q)$-relation” throughout the paper.

Using the additive duality “$q \leftrightarrow 2 - q$” in (2), (2) is rewritten by

$$\ln_{2-q} \begin{bmatrix} n \\ n_1 \cdots n_k \end{bmatrix}_{2-q} \simeq \frac{n^q}{q} \cdot S_q \left( \frac{n_1}{n}, \cdots, \frac{n_k}{n} \right).$$

(6)

Hence the above generalized correspondence (4) is found to recover (6) when $\mu = 2 - q$ and $\nu = 1$. As will be shown later,

$$\begin{bmatrix} n \\ n_1 \cdots n_k \end{bmatrix}_{(\mu,1)} = \begin{bmatrix} n \\ n_1 \cdots n_k \end{bmatrix}_\mu.$$

(7)

The $(\mu, \nu, q)$-relation (5) among three parameters $\mu, \nu, q$ yields the additive duality “$q \leftrightarrow 2 - q$” when $\nu = 1$ and the multiplicative duality “$q \leftrightarrow 1/q$” when $\nu = q$, respectively. As other special cases of the $(\mu, \nu, q)$-relation, when $\nu = 2 - q$, it is shown that the $q$-triplet $(q_{\text{sen}}, q_{\text{rel}}, q_{\text{stat}})$ recently conjectured by Tsallis [17] is identified with the $(\mu, \nu, q)$-relation (5) in the following sense:

$$\mu = \frac{1}{q_{\text{sen}}}, \quad \nu = \frac{1}{q_{\text{rel}}}, \quad q = q_{\text{stat}}.$$

(8)

Moreover, when $\nu = 1/q$, the relation [18]:

$$\frac{1}{1 - q_{\text{sen}}} = \frac{1}{\alpha_{\text{min}}} - \frac{1}{\alpha_{\text{max}}}$$

(9)
in the multifractal singularity spectrum \( f(\alpha) \) is recovered by means of the \((\mu, \nu, q)\)-relation in the following sense:

\[
\mu = q_{\text{sen}}, \quad \nu = \frac{1}{\alpha_{\text{max}}}, \quad q = \alpha_{\text{max}}
\]  

with \( \alpha_{\text{max}} - \alpha_{\text{min}} = 1 \). The above new results are derived in detail in the following sections.

This paper consists of the 5 sections including this introduction. In the next section, we briefly review the fundamental formulas such as the \(q\)-product, the \(q\)-factorial, the \(q\)-multinomial coefficient and the \(q\)-Stirling’s formula which are applied to the derivation of (2). In Section III, the correspondence (2) is modified to derive the multiplicative duality “\( q \leftrightarrow 1/q \)” in our combinatorial formalism. In this derivation, a slight generalization of the \(q\)-factorial is required, which is called “\((\mu, \nu)\)-factorial”. As similarly as Section II, we formulate the \((\mu, \nu)\)-multinomial coefficient and the \((\mu, \nu)\)-Stirling’s formula based on the \((\mu, \nu)\)-factorial, and apply them to finding the generalized correspondence (4). In Section IV, we derive the additive duality and the multiplicative duality as special cases of (4). Moreover, when \( \nu = 2 - q \) and \( \nu = 1/q \), each interpretation of the \((\mu, \nu, q)\)-relation shown in (8) and (10) is respectively presented. The final section is devoted to our conclusion.

II. ADDITIVE DUALITY DERIVED FROM THE \(q\)-MULTINOMIAL COEFFICIENT

The MaxEnt for Boltzmann-Gibbs-Shannon entropy \( S_1 \) yields the exponential function \( \exp(x) \) which is well known to be characterized by the linear differential function \( dy/dx = y \). In parallel with this, the MaxEnt for Tsallis entropy \( S_q \) yields a generalization of the exponential function \( \exp_q(x) \) which is characterized by the nonlinear differential function \( dy/dx = y^q \). In Tsallis statistics, the fundamental functions are the \(q\)-logarithm \( \ln_q x \) and the \(q\)-exponential \( \exp_q(x) \), respectively defined as follows:

**Definition 1** (\( q \)-logarithm, \( q \)-exponential) The \( q \)-logarithm \( \ln_q x : \mathbb{R}^+ \to \mathbb{R} \) and the \( q \)-exponential \( \exp_q(x) : \mathbb{R} \to \mathbb{R} \) are defined by

\[
\ln_q x := \frac{x^{1/q} - 1}{1 - q},
\]

\[
\exp_q(x) := \begin{cases} 
[1 + (1 - q) x]^{1/q} & \text{if } 1 + (1 - q) x > 0, \\
0 & \text{otherwise}.
\end{cases}
\]
Then a new product $\otimes_q$ to satisfy the following identities as the $q$-exponential law is introduced.

\[
\ln_q (x \otimes_q y) = \ln_q x + \ln_q y, \quad (13)
\]
\[
\exp_q (x) \otimes_q \exp_q (y) = \exp_q (x + y). \quad (14)
\]

For this purpose, the new multiplication operation $\otimes_q$ is introduced in [11][12]. The concrete forms of the $q$-logarithm and $q$-exponential are given in (11) and (12), so that the above requirement (13) or (14) as the $q$-exponential law leads to the definition of $\otimes_q$ between two positive numbers.

**Definition 2 (q-product)** For $x, y \in \mathbb{R}^+$, the $q$-product $\otimes_q$ is defined by

\[
x \otimes_q y := \begin{cases} 
[x^{1-q} + y^{1-q} - 1]^{\frac{1}{1-q}}, & \text{if } x > 0, y > 0, x^{1-q} + y^{1-q} - 1 > 0, \\
0, & \text{otherwise}.
\end{cases} \quad (15)
\]

The $q$-product recovers the usual product such that $\lim_{q \to 1} (x \otimes_q y) = xy$. The fundamental properties of the $q$-product $\otimes_q$ are almost the same as the usual product, but

\[
a (x \otimes_q y) \neq (ax) \otimes_q y \quad (a, x, y \in \mathbb{R}). \quad (16)
\]

The other properties of the $q$-product are available in [11][12].

By means of the $q$-product (15), the $q$-factorial is naturally defined in the following form.

**Definition 3 (q-factorial)** For a natural number $n \in \mathbb{N}$ and $q \in \mathbb{R}^+$, the $q$-factorial $n!_q$ is defined by

\[
n!_q := 1 \otimes_q \cdots \otimes_q n. \quad (17)
\]

Thus, we concretely compute the $q$-Stirling’s formula.

**Theorem 4 (q-Stirling’s formula)** Let $n!_q$ be the $q$-factorial defined by (17). The rough $q$-Stirling’s formula $\ln_q (n!_q)$ is computed as follows:

\[
\ln_q (n!_q) = \begin{cases} 
\frac{n \ln_q n - n}{2 - q} + O (\ln_q n) & \text{if } q \neq 2, \\
n - \ln n + O (1) & \text{if } q = 2.
\end{cases} \quad (18)
\]
The above rough \( q \)-Stirling’s formula is obtained by the approximation:

\[
\ln_q (n!_q) = \sum_{k=1}^{n} \ln_q k \simeq \int_1^n \ln_q x \, dx. \tag{19}
\]

The rigorous derivation of the \( q \)-Stirling’s formula is given in [8].

Similarly as for the \( q \)-product, \( q \)-ratio is introduced from the requirements:

\[
\ln_q \left( \frac{x}{y} \right) = \ln_q x - \ln_q y, \tag{20}
\]

\[
\exp_q \left( \frac{x}{y} \right) = \exp_q(x) \cdot \exp_q(y) = \exp_q(x - y). \tag{21}
\]

Then we define the \( q \)-ratio as follows.

**Definition 5** \((q\text{-ratio})\) For \( x, y \in \mathbb{R}^+ \), the inverse operation to the \( q \)-product is defined by

\[
x \cdot_q y := \begin{cases} 
[x^{1-q} - y^{1-q} + 1]^{\frac{1}{1-q}}, & \text{if } x > 0, y > 0, x^{1-q} - y^{1-q} + 1 > 0, \\
0, & \text{otherwise}
\end{cases} \tag{22}
\]

which is called \( q \)-ratio in [12].

The \( q \)-product, \( q \)-factorial and \( q \)-ratio are applied to the definition of the \( q \)-multinomial coefficient [8].

**Definition 6** \((q\text{-multinomial coefficient})\) For \( n = \sum_{i=1}^{k} n_i \) and \( n_i \in \mathbb{N} \) \((i = 1, \cdots, k)\), the \( q \)-multinomial coefficient is defined by

\[
\begin{bmatrix} n \\ n_1 & \cdots & n_k \end{bmatrix}_q := (n!_q) \cdot_q \left( (n_1!_q) \cdot_q \cdots \cdot_q (n_k!_q) \right). \tag{23}
\]

From the definition (23), it is clear that

\[
\lim_{q \to 1} \begin{bmatrix} n \\ n_1 & \cdots & n_k \end{bmatrix}_q = \begin{bmatrix} n \\ n_1 & \cdots & n_k \end{bmatrix} = \frac{n!}{n_1! \cdots n_k!}. \tag{24}
\]

Throughout the present paper, we consider the \( q \)-logarithm of the \( q \)-multinomial coefficient to be given by

\[
\ln_q \begin{bmatrix} n \\ n_1 & \cdots & n_k \end{bmatrix}_q = \ln_q (n!_q) - \ln_q (n_1!_q) \cdots - \ln_q (n_k!_q). \tag{25}
\]

Based on these fundamental formulas, we obtain the one-to-one correspondence (2) between the \( q \)-multinomial coefficient and Tsallis entropy as follows [8].
Theorem 7 When $n \in \mathbb{N}$ is sufficiently large, the $q$-logarithm of the $q$-multinomial coefficient coincides with Tsallis entropy [1] in the following correspondence:

$$
\ln_q \left[ \begin{array}{c} n \\ n_1 \cdots n_k \end{array} \right]_q \simeq \begin{cases} \frac{n^2 - q}{2 - q} \cdot S_{2-q} \left( \frac{n_1}{n}, \ldots, \frac{n_k}{n} \right) & \text{if } q > 0, \ q \neq 2 \\ -S_1 (n) + \sum_{i=1}^{k} S_1 (n_i) & \text{if } q = 2 \end{cases} \quad (26)
$$

where $S_q$ is Tsallis entropy (1) and $S_1 (n) := \ln n$.

Straightforward computation of the left side of (26) by means of the $q$-Stirling’s formula (18) yields the above result (26). (See [8] for the proof.)

Clearly the additive duality “$q \leftrightarrow 2 - q$” is appeared in the above one-to-one correspondence (26). In the following sections these fundamental formulas are generalized for the derivation of the multiplicative duality “$q \leftrightarrow 1/q$” in the similar correspondence as (26).

III. ONE-TO-ONE CORRESPONDENCE BETWEEN THE $(\mu, \nu)$-MULTINOMIAL COEFFICIENT AND TSALLIS ENTROPY

In this section the correspondence (26) is generalized for the multiplicative duality “$q \leftrightarrow 1/q$”. For this purpose, replace $q$ in (26) by $1/q$ at first. Then we obtain

$$
\ln_{1/q} \left[ \begin{array}{c} n \\ n_1 \cdots n_k \end{array} \right]_{1/q} \simeq \frac{n^2 - 1}{2 - 1/q} \cdot S_{2-1/q} \left( \frac{n_1}{n}, \ldots, \frac{n_k}{n} \right) \quad (27)
$$

where we consider the case $q > 0$ and $q \neq 1/2$ only. The left side of (27) is computed as

$$
\ln_{1/q} \left[ \begin{array}{c} n \\ n_1 \cdots n_k \end{array} \right]_{1/q} = \ln_{1/q} \left( n!_{1/q} \right) - \ln_{1/q} \left( n_1!_{1/q} \right) \cdots - \ln_{1/q} \left( n_k!_{1/q} \right). \quad (28)
$$

Using this formula (28), we will represent the left side of (27) by means of the forms such as $\ln_q$ or $\ln_{2-q}$ to find the multiplicative duality. The important relation for this purpose is the following identity:

$$
\ln_{1/q} \left( \frac{1}{x^q} \right) = -q \ln_q x. \quad (29)
$$

Each term $\ln_{1/q} \left( n!_{1/q} \right)$ on the right side of (28) is equal to

$$
\ln_{1/q} \left( n!_{1/q} \right) = \ln_{1/q} \left( \frac{1}{\left( n!_{1/q} \right)^{-1}} \right). \quad (30)
$$
\((n!_q)^{-1}\) is expanded in accordance with the definition of the \(q\)-product \((15)\).

\[
(n!_q)^{-1} = (1 \otimes q^1 \cdots \otimes q^n)^{-1} = \left[1^{1-q} + 2^{1-q} + \cdots + n^{1-q} - (n-1)\right]^\frac{1}{q}
\]

\[
= \left[\left((\frac{1}{1})\right)^{1-q} + \left((\frac{1}{2})\right)^{1-q} + \cdots + \left((\frac{1}{n})\right)^{1-q}\right] - (n-1)\right]^\frac{1}{q}
\]

\[
= \left[\left((\frac{1}{1})\right)^{\frac{1}{q}} \otimes_q \left((\frac{1}{2})\right)^{\frac{1}{q}} \otimes_q \cdots \otimes_q \left((\frac{1}{n})\right)^{\frac{1}{q}}\right]^q
\]

Then \(\ln_q (n!_q)\) is given by

\[
\ln_q (n!_q) = \ln_q \left(\frac{1}{\left[\left((\frac{1}{1})\right)^{\frac{1}{q}} \otimes_q \left((\frac{1}{2})\right)^{\frac{1}{q}} \otimes_q \cdots \otimes_q \left((\frac{1}{n})\right)^{\frac{1}{q}}\right]^q}\right) \quad (\because (30) \text{ and } (33))
\]

\[
= -q \ln_q \left[\left((\frac{1}{1})\right)^{\frac{1}{q}} \otimes_q \left((\frac{1}{2})\right)^{\frac{1}{q}} \otimes_q \cdots \otimes_q \left((\frac{1}{n})\right)^{\frac{1}{q}}\right] \quad (\because (29))
\]

\[
= -q \sum_{j=1}^{n} \ln_q j^{-\frac{1}{q}}. \quad (36)
\]

Thus, substitution of \((36)\) to \((28)\) yields

\[
\ln_q \left[\begin{array}{c}
\frac{n}{n_1} \\
\vdots \\
\frac{n}{n_k}
\end{array}\right] = q \left(- \sum_{j=1}^{n} \ln_q j^{-\frac{1}{q}} + \sum_{j_1=1}^{n_1} \ln_q j_1^{-\frac{1}{q}} + \cdots + \sum_{j_k=1}^{n_k} \ln_q j_k^{-\frac{1}{q}}\right). \quad (37)
\]

Applying the general formula:

\[
- \ln_q j^{-\frac{1}{q}} = \ln_q j^{\frac{1}{q}} \quad (38)
\]

to the above result \((37)\), we have

\[
\frac{1}{q} \ln_q \left[\begin{array}{c}
\frac{n}{n_1} \\
\vdots \\
\frac{n}{n_k}
\end{array}\right] = \sum_{j=1}^{n} \ln_{q^{-2}} j^{-\frac{1}{q}} - \sum_{j_1=1}^{n_1} \ln_{q^{-2}} j_1^{-\frac{1}{q}} - \cdots - \sum_{j_k=1}^{n_k} \ln_{q^{-2}} j_k^{-\frac{1}{q}} \quad (39)
\]

\[
= \ln_{q^{-2}} \left[\left(1^{\frac{1}{q}} \otimes_{q^{-2}} 2^{\frac{1}{q}} \otimes_{q^{-2}} \cdots \otimes_{q^{-2}} n_{1}^{\frac{1}{q}}\right) \otimes_{q^{-2}} \left(1^{\frac{1}{q}} \otimes_{q^{-2}} 2^{\frac{1}{q}} \otimes_{q^{-2}} \cdots \otimes_{q^{-2}} n_{1}^{\frac{1}{q}}\right) \otimes_{q^{-2}} \cdots \otimes_{q^{-2}} \left(1^{\frac{1}{q}} \otimes_{q^{-2}} 2^{\frac{1}{q}} \otimes_{q^{-2}} \cdots \otimes_{q^{-2}} n_{k}^{\frac{1}{q}}\right)\right]. \quad (40)
\]
On the other hand, from (25) the $q$-logarithm of the $q$-multinomial coefficient is given by

$$
\ln_q \left[ \begin{array}{c} n \\ n_1 \cdots n_k \end{array} \right] = \sum_{j=1}^{n} \ln_q j - \sum_{j=1}^{n_1} \ln_q j_1 - \cdots - \sum_{j=1}^{n_k} \ln_q j_k
$$

where

$$
\ln_q [(1 \otimes_q 2 \otimes_q \cdots \otimes_q n)]
$$

$$
\otimes_q (1 \otimes_q 2 \otimes_q \cdots \otimes_q n_1)
$$

$$
\cdots
$$

$$
\otimes_q (1 \otimes_q 2 \otimes_q \cdots \otimes_q n_k)].
$$

Comparing the argument of $\ln_{2-q}$ on the right side of (40) with that of $\ln_q$ on (41), a generalization of the $q$-factorial (17) is found to be required for our purpose.

**Definition 8** ($(\mu, \nu)$-factorial) For a natural number $n \in \mathbb{N}$ and $\mu, \nu \in \mathbb{R}$, the $(\mu, \nu)$-factorial $n!_{(\mu, \nu)}$ is defined by

$$
n!_{(\mu, \nu)} := 1^\nu \otimes_\mu 2^\nu \otimes_\mu \cdots \otimes_\mu n^\nu.
$$

where $\nu \neq 0$.

Clearly when $\mu = q, \nu = 1$ the $q$-factorial (17) is recovered.

$$
n!_{q} = n!_{(q, 1)}
$$

Moreover, when $\mu = 1$, the $(\mu, \nu)$-factorial $n!_{(\mu, \nu)}$ is equal to $(n!)^\nu$ because the $\mu$-product recovers the usual product.

$$
n!_{(1, \nu)} = 1^\nu 2^\nu \cdots n^\nu = (n!)^\nu
$$

Thus, throughout the paper we consider the case $\mu \neq 1$ only.

Using the $(\mu, \nu)$-factorial, we have

$$
\frac{1}{q} \ln_{\frac{1}{q}} \left[ \begin{array}{c} n \\ n_1 \cdots n_k \end{array} \right] = \ln_{2-q} \left( n!(2-q, \frac{1}{q}) \otimes_{2-q} n_1!(2-q, \frac{1}{q}) \cdots \otimes_{2-q} n_k!(2-q, \frac{1}{q}) \right)
$$

Then we define the form of the argument of $\ln_{2-q}$ on the right side of (46) by the $(\mu, \nu)$-multinomial coefficient as a generalization of the $q$-multinomial coefficient (23).
Definition 9 \((\mu, \nu)\)-multinomial coefficient) For \(n = \sum_{i=1}^{k} n_i\) and \(n_i \in \mathbb{N}\) \((i = 1, \cdots, k)\), the \((\mu, \nu)\)-multinomial coefficient is defined by

\[
\begin{bmatrix}
n \\
n_1 & \cdots & n_k
\end{bmatrix}_{(\mu, \nu)} := \left(n!_{(\mu, \nu)} \right) \otimes_{\mu} \left(\left(m_1!_{(\mu, \nu)} \right) \otimes_{\mu} \cdots \otimes_{\mu} \left(n_k!_{(\mu, \nu)} \right) \right).
\]

(47)

where \(n!_{(\mu, \nu)}\) is the \((\mu, \nu)\)-factorial defined in (42).

Clearly when \(\mu = q\) and \(\nu = 1\) the \(q\)-multinomial coefficient (23) is recovered.

Using the \((\mu, \nu)\)-multinomial coefficient, (46) becomes

\[
\frac{1}{q} \ln_q \left[ \begin{bmatrix}
n \\
n_1 & \cdots & n_k
\end{bmatrix}_{(\mu, \nu)} \right] = \ln_{2-q} \left[ \begin{bmatrix}
n \\
n_1 & \cdots & n_k
\end{bmatrix}_{(q,1)} \right].
\]

(49)

Moreover, the \((\mu, \nu)\)-Stirling’s formula is computed as the following form:

Theorem 10 \((\mu, \nu)\)-Stirling’s formula) Let \(n!_{(\mu, \nu)}\) be the \((\mu, \nu)\)-factorial defined by (42). The \((\mu, \nu)\)-Stirling’s formula \(\ln_{\mu} \left(n!_{(\mu, \nu)}\right)\) is computed as follows:

\[
\ln_{\mu} \left(n!_{(\mu, \nu)}\right) = \begin{cases} 
\frac{n \ln_{\mu} n^\nu - \nu n}{\nu (1 - \mu) + 1} + O \left(\ln_{\mu} n \right) & \text{if } \nu (1 - \mu) + 1 \neq 0, \\
\nu (n - \ln n) + O \left(1 \right) & \text{if } \nu (1 - \mu) + 1 = 0.
\end{cases}
\]

(50)

This formula is computed by the approximation:

\[
\ln_q \left(n!_{(\mu, \nu)}\right) = \sum_{k=1}^{n} \ln_{\mu} k^\nu \simeq \int_{1}^{n} \ln_{\mu} x^\nu dx.
\]

(51)

Based on these results, we obtain the one-to-one correspondence between the \((\mu, \nu)\)-multinomial coefficient and Tsallis entropy as follows.

Theorem 11 When \(n\) is sufficiently large, the \(\mu\)-logarithm of the \((\mu, \nu)\)-multinomial coefficient coincides with Tsallis entropy (7) as follows:

\[
\frac{1}{\nu} \ln_{\mu} \left[ \begin{bmatrix}
n \\
n_1 & \cdots & n_k
\end{bmatrix}_{(\mu, \nu)} \right] \simeq \begin{cases} 
n^q/q \cdot S_q \left(\frac{n_1}{n}, \cdots, \frac{n_k}{n}\right) & \text{if } q \neq 0 \\
-S_1 \left(n\right) + \sum_{i=1}^{k} S_1 \left(n_i\right) & \text{if } q = 0
\end{cases}
\]

(52)
where \( \nu \neq 0 \),

\[
\nu (1 - \mu) + 1 = q, \tag{53}
\]

\( S_q \) is Tsallis entropy \((1)\) and \( S_1 (n) := \ln n \).

The proof is given in the appendix A.

The generalized correspondence \((52)\) between the \((\mu, \nu)\)-multinomial coefficient and Tsallis entropy includes some typical mathematical structures such as the two kinds of dualities and the \( q \)-triplet as the special cases, shown in the next section.

IV. MULTIPLICATIVE DUALITY AND OTHER TYPICAL MATHEMATICAL STRUCTURES DERIVED FROM THE COMBINATORIAL FORMALISM

We consider the case \( q \neq 0 \) only. Then, the generalized correspondence \((52)\) is given by

\[
\frac{1}{\nu} \ln_{\mu} \left[ \begin{array}{c} n \\ n_1 \cdots n_k \end{array} \right]_{(\mu, \nu)} \simeq \frac{n^q}{q} \cdot S_q \left( \frac{n_1}{n}, \cdots, \frac{n_k}{n} \right). \tag{54}
\]

In this paper, we call the above relation \((53)\) “\((\mu, \nu, q)\)-relation” which provides us interesting features in Tsallis statistics. In particular, we consider the following four cases.

A. \( \nu = 1 \)

In this case, from the \((\mu, \nu, q)\)-relation \((53)\) \( \mu \) is given by

\[
\mu = 2 - q. \tag{55}
\]

Then the generalized correspondence \((54)\) becomes

\[
\ln_{2-q} \left[ \begin{array}{c} n \\ n_1 \cdots n_k \end{array} \right]_{2-q} \simeq \frac{n^q}{q} \cdot S_q \left( \frac{n_1}{n}, \cdots, \frac{n_k}{n} \right) \tag{56}
\]

which is equivalent to \((6)\) or \((26)\) revealing the additive duality “\( q \leftrightarrow 2 - q \)”.

B. \( \nu = q \)

In this case, from the \((\mu, \nu, q)\)-relation \((53)\) \( \mu \) is determined as

\[
\mu = \frac{1}{q}. \tag{57}
\]
Then the generalized correspondence (54) becomes
\[
\ln_{\frac{1}{q}} \left[ \frac{n}{n_1 \cdots n_k} \right] \simeq n^q \cdot S_q \left( \frac{n_1}{n}, \cdots, \frac{n_k}{n} \right)
\]
which reveals the multiplicative duality “\( q \leftrightarrow \frac{1}{q} \).

Aside from the above representation (58), the multiplicative duality in Tsallis statistics is easily derived from the definition of the escort distribution. See the appendix B for the detail.

C. \( \nu = 2 - q \)

In this case, from the \((\mu, \nu, q)\)-relation (53) \( \mu \) is obtained as
\[
\mu = \frac{3 - 2q}{2 - q}.
\]
Then the generalized correspondence (54) becomes
\[
\frac{1}{2 - q} \ln_{\frac{3 - 2q}{2 - q}} \left[ \frac{n}{n_1 \cdots n_k} \right] \simeq \frac{n^q}{q} \cdot S_q \left( \frac{n_1}{n}, \cdots, \frac{n_k}{n} \right)
\]
where the \((\mu, \nu, q)\)-relation for this case is equivalent to the \(q\)-triplet \((q_{\text{sen}}, q_{\text{rel}}, q_{\text{stat}})\) recently conjectured by Tsallis [17][24] in the following sense. In [17], Tsallis first conjectured the three entropic \(q\)-indices \((q_{\text{sen}}, q_{\text{rel}}, q_{\text{stat}})\), respectively for \(q\)-exponential sensitivity to the initial conditions, \(q\)-exponential relaxation of macroscopic quantities to thermal equilibrium and \(q\)-exponential distribution describing a stationary state. More concretely, based on his recent results in [23], he conjectured the concrete \(q\)-triplet \((q_{\text{sen}}, q_{\text{rel}}, q_{\text{stat}})\) satisfying the following relation [24]:
\[
q_{\text{rel}} + \frac{1}{q_{\text{sen}}} = 2, \quad q_{\text{stat}} + \frac{1}{q_{\text{rel}}} = 2.
\]
From this relation, we immediately obtain
\[
\frac{1}{q_{\text{sen}}} = \frac{3 - 2q_{\text{stat}}}{2 - q_{\text{stat}}}
\]
which is the same form as \( \mu \) obtained in (59). Therefore, when \( \nu = 2 - q \), the present \((\mu, \nu, q)\)-relation is identified with the \(q\)-triplet \((q_{\text{sen}}, q_{\text{rel}}, q_{\text{stat}})\) in the following sense:
\[
\mu = \frac{1}{q_{\text{sen}}}, \quad \nu = \frac{1}{q_{\text{rel}}}, \quad q = q_{\text{stat}}.
\]
As shown in this paper, the above identification (63) is derived from the mathematical discussion only. Besides our analytical derivation, the $q$-triplet $(q_{\text{sen}}, q_{\text{rel}}, q_{\text{stat}})$ has been already confirmed in the experimental observations in [25]. Therefore, Tsallis’ conjecture on the $q$-triplet $(q_{\text{sen}}, q_{\text{rel}}, q_{\text{stat}})$ in [24] is correct in both theoretical and experimental aspects.

Note that in our theoretical derivation of (63) we never use the definition of the three entropic $q$-indices $q_{\text{sen}}, q_{\text{rel}}, q_{\text{stat}}$, which may be remained as a future work from the theoretical points of view in this case $\nu = 2 - q$. However, the present identification (63) is just an interpretation of the $(\mu, \nu, q)$-relation. In fact, in our formulation the additive duality, the multiplicative duality and the $q$-triplet are derived as special cases of the $(\mu, \nu, q)$-relation (53). For example, we present other possible interpretation of the present $(\mu, \nu, q)$-relation (53) for the case $\nu = \frac{1}{q}$, shown in the next subsection.

D. $\nu = \frac{1}{q}$

As an other special case of the $(\mu, \nu, q)$-relation, we consider the case $\nu = \frac{1}{q}$. For this case, we obtain

$$\frac{1}{1-\mu} = \frac{1}{q-1} - \frac{1}{q}. \quad (64)$$

This identity reminds us of the following relationship [18]:

$$\frac{1}{1-q_{\text{sen}}} = \frac{1}{\alpha_{\text{min}}} - \frac{1}{\alpha_{\text{max}}} \quad (65)$$

where $q_{\text{sen}}$ is the same entropic $q$-index as the case C for the $q$-exponential sensitivity to the initial conditions, $\alpha_{\text{min}}$ and $\alpha_{\text{max}}$ are the values of $\alpha$ at which the multifractal singularity spectrum $f(\alpha)$ vanishes (with $\alpha_{\text{min}} < \alpha_{\text{max}}$). These $\alpha_{\text{min}}$ and $\alpha_{\text{max}}$ are given by

$$\alpha_{\text{min}} = \frac{\ln b}{z \ln \alpha_F}, \quad \alpha_{\text{max}} = \frac{\ln b}{\ln \alpha_F} \quad (66)$$

where $b$ stands for a natural scale for the partitions, $\alpha_F$ is the Feigenbaum universal scaling factor and $z$ represents the nonlinearity of the map at the vicinity of its extremal point [18].

A choice of the nonlinearity $z$ to satisfy

$$\left(\frac{b}{\alpha_F}\right)^z = b \quad (67)$$

implies $\alpha_{\text{max}} - \alpha_{\text{min}} = 1$. In other words, (67) means a rescaling of $\alpha_{\text{max}} - \alpha_{\text{min}}$ to be 1. Therefore, if the nonlinearity $z$ is determined by the above requirement (67), we have the
following identification:
\[\mu = q_{\text{sen}}, \quad \nu = \frac{1}{\alpha_{\text{max}}}, \quad q = \alpha_{\text{max}}\] (68)
which is one of the interpretations of the \((\mu, \nu, q)\)-relation. Note that the above identification (68) implies (67).

All results in cases A-D mean that the \((\mu, \nu, q)\)-relation is found to be a more general nature in Tsallis statistics to recover these specific mathematical structures.

V. CONCLUSION

We present the one-to-one correspondence between the \((\mu, \nu)\)-multinomial coefficient and Tsallis entropy \(S_q\) to represent both the additive duality “\(q \leftrightarrow 2 - q\)” and the multiplicative duality “\(q \leftrightarrow 1/q\)” in one unified formula (54). In this derivation, \((\mu, \nu)\)-factorial, \((\mu, \nu)\)-multinomial coefficient and \((\mu, \nu)\)-Stirling’s formula are concretely formulated as a generalization of \(q\)-factorial, \(q\)-multinomial coefficient and \(q\)-Stirling’s formula, respectively. In the present one-to-one correspondence (54), when \(\nu = 2 - q\), the \((\mu, \nu, q)\)-relation among three parameters \(\mu, \nu, q\) is shown to be identified with the \(q\)-triplet \((q_{\text{sen}}, q_{\text{rel}}, q_{\text{stat}})\) in the sense of (63). In addition, as other interpretation of the \((\mu, \nu, q)\)-relation, the multifractal structure \(1/(1 - q_{\text{sen}}) = 1/\alpha_{\text{min}} - 1/\alpha_{\text{max}}\) is recovered.

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APPENDIX A: PROOF OF THEOREM 11

When \( \mu = 1 \), \( \mu \)-product recovers the usual product regardless of \( \nu \). Thus, we consider the case \( \mu \neq 1 \) only.

If \( \nu (1 - \mu) + 1 \neq 0 \),

\[
\ln_{\mu} \left[ \frac{n!}{n_1! \cdots n_k!} \right]_{(\mu, \nu)} = \ln_{\mu} n_1!_{(\mu, \nu)} - \ln_{\mu} n_1!_{(\mu, \nu)} - \cdots - \ln_{\mu} n_k!_{(\mu, \nu)} \tag{A1}
\]

\[
\approx \frac{n \ln_{\mu} n^\nu - \nu n}{\nu (1 - \mu) + 1} - \frac{n_1 \ln_{\mu} n_1^\nu - \nu n_1}{\nu (1 - \mu) + 1} - \cdots - \frac{n_k \ln_{\mu} n_k^\nu - \nu n_k}{\nu (1 - \mu) + 1} \quad \left( \because \begin{align*}
\end{align*} \right) \tag{A2}
\]

\[
= \frac{n \ln_{\mu} n^\nu}{\nu (1 - \mu) + 1} - \frac{n_1 \ln_{\mu} n_1^\nu}{\nu (1 - \mu) + 1} - \cdots - \frac{n_k \ln_{\mu} n_k^\nu}{\nu (1 - \mu) + 1} \quad \left( \because n = \sum_{i=1}^{k} n_i \right) \tag{A3}
\]

\[
= \frac{n \left( n^\nu (1 - \mu) - 1 \right)}{(1 - \mu) (\nu (1 - \mu) + 1)} - \frac{n_1 \left( n_1^\nu (1 - \mu) - 1 \right)}{(1 - \mu) (\nu (1 - \mu) + 1)} - \cdots - \frac{n_k \left( n_k^\nu (1 - \mu) - 1 \right)}{(1 - \mu) (\nu (1 - \mu) + 1)} \tag{A4}
\]

\[
= \frac{n^\nu (1 - \mu) + 1}{(1 - \mu) (\nu (1 - \mu) + 1)} - \frac{n_1^\nu (1 - \mu) + 1}{(1 - \mu) (\nu (1 - \mu) + 1)} - \cdots - \frac{n_k^\nu (1 - \mu) + 1}{(1 - \mu) (\nu (1 - \mu) + 1)} \tag{A5}
\]

\[
= \frac{n^\nu (1 - \mu) + 1}{(1 - \mu) (\nu (1 - \mu) + 1)} \left( 1 - \left( \frac{n_1}{n} \right)^{\nu (1 - \mu) + 1} - \cdots - \left( \frac{n_k}{n} \right)^{\nu (1 - \mu) + 1} \right) \tag{A6}
\]

\[
= \frac{n^\nu (1 - \mu) + 1}{\nu (1 - \mu) + 1} \left( 1 - \sum_{i=1}^{k} \left( \frac{n_i}{n} \right)^{\nu (1 - \mu) + 1} \right) \tag{A7}
\]

\[
= \frac{\nu n^\nu (1 - \mu) + 1}{\nu (1 - \mu) + 1} \left( 1 - \sum_{i=1}^{k} \left( \frac{n_i}{n} \right)^{\nu (1 - \mu) + 1} \right) \tag{A8}
\]

\[
= \nu \frac{n^\nu (1 - \mu) + 1}{\nu (1 - \mu) + 1} S_{\nu (1 - \mu) + 1} \left( \frac{n_1}{n}, \cdots, \frac{n_k}{n} \right) \tag{A9}
\]
if \( \nu (1 - \mu) + 1 = 0 \),

\[
\ln_{\mu} \left[ \begin{array}{c}
\frac{n}{n_1} \\
\vdots \\
\frac{n}{n_k}
\end{array} \right]_{(\mu, \nu)} = \ln_{\mu} n!_{(\mu, \nu)} - \ln_{\mu} n_1!_{(\mu, \nu)} - \cdots - \ln_{\mu} n_k!_{(\mu, \nu)} \quad (A10)
\]

\[
\simeq \nu (n - \ln n) - \nu (n_1 - \ln n_1) - \cdots - \nu (n_k - \ln n_k) \quad (\therefore \Box)
\]

\[
= \nu (- \ln n + \ln n_1 + \cdots + \ln n_k) \quad \quad (\therefore n = \sum_{i=1}^{k} n_i)
\]

\[
= \nu \left( -S_1 (n) + \sum_{i=1}^{k} S_1 (n_i) \right) \quad \quad (A13)
\]

**APPENDIX B: THE MULTIPLICATIVE DUALITY DERIVED FROM THE DEFINITION OF THE ESCORT DISTRIBUTION**

The escort distribution was first introduced in [26] to scan the structure of a given distribution by using similarities as thermodynamic equilibrium distribution.

**Definition 12** (escort distribution) For any given probability distribution \( \{p_i\} \), the escort distribution \( \{P_i\} \) is defined as

\[
P_i := \frac{p_i^q}{\sum_{j=1}^{n} p_j^q} \quad (q > 0) . \quad (B1)
\]

Note that a given distribution \( \{p_i\} \) in the above definition is not necessarily normalized, but in our formulations we require \( \{p_i\} \) to be a normalized distribution, that is, a probability distribution.

Until now, the multiplicative duality "\( q \leftrightarrow 1/q \)" in Tsallis statistics is based on the following property derived from the above definition of the escort distribution:

\[
p_i = \frac{P_i^q}{\sum_{j=1}^{n} P_j^q} . \quad (B2)
\]

However, the escort distribution \( \{P_i\} \) is originally associated with Tsallis entropy in the following sense:

**Theorem 13** For any given probability distribution \( \{p_i\} \) and its associated escort distribution \( \{P_i\} \), the next identity is satisfied.

\[
\exp_q (S_q (p_i)) = \exp_{\frac{1}{q}} \left( S_{\frac{1}{q}} (P_i) \right) \quad (B3)
\]
where $S_q(p_i)$ is Tsallis entropy \([7]\).

**Proof.** Using the definition of the escort distribution \([21]\), we have

$$
\sum_{i=1}^{n} P_i^{\frac{1}{q}} = \sum_{i=1}^{n} \left( \frac{p_i}{\sum_{j=1}^{n} p_j^q} \right)^{\frac{1}{q}} = \sum_{i=1}^{n} \frac{p_i}{\left( \sum_{j=1}^{n} p_j^q \right)^{\frac{1}{q}}} = \left( \sum_{j=1}^{n} p_j^q \right)^{\frac{1}{q}}.
$$

(B4)

Both sides to the power $\frac{1}{1-q} = -\frac{q}{1-q}$ is

$$
\left( \sum_{j=1}^{n} P_j^{\frac{1}{q}} \right)^{\frac{1}{1-q}} = \left( \sum_{i=1}^{n} p_i^q \right)^{\frac{1}{1-q}}
$$

(B5)

which is equivalent to

$$
\left( 1 + \left( 1 - \frac{1}{q} \right) \frac{1 - \sum_{j=1}^{n} P_j^{\frac{1}{q}}}{\frac{1}{q} - 1} \right)^{\frac{1}{1-q}} = \left( 1 + (1 - q) \frac{1 - \sum_{i=1}^{n} p_i^q}{q - 1} \right)^{\frac{1}{1-q}}.
$$

(B6)

Clearly, this is identical to the simple form

$$
\exp_{\frac{1}{q}} \left( S_{\frac{1}{q}} (P_j) \right) = \exp_q \left( S_q (p_i) \right).
$$

(B7)

Note that the above result \((B3)\) obviously reveals the multiplicative duality \(q \leftrightarrow \frac{1}{q}\) of Tsallis entropy, which is derived from the definition of the escort distribution only.

The above relation \((B3)\) provides us a key to unify several entropies such as Boltzmann-Gibbs-Shannon entropy, Rényi entropy \([27]\), Tsallis entropy \([1]\), Gaussian entropy \([28]\), Sharma-Mittal entropy \([29]\) and Supra-extensive entropy \([30]\). Among these entropies, Sharma-Mittal entropy and Supra-extensive entropy are the two-parameterized entropies including the other entropies as special cases. For these two entropies, the similar identities as above are satisfied in the forms:

$$
\exp_{r} \left( S_{q,r}^{\text{Sharma-Mittal}} \right) = \exp_q \left( S_q^{\text{Tsallis}} \right),
$$

(B8)

$$
\exp_q \left( S_{q,r}^{\text{Supra-extensive}} \right) = \exp_{r} \left( S_q^{\text{Rényi}} \right)
$$

(B9)
where

\[
S_{q,r}^{\text{Sharma-Mittal}} (p_i) := \frac{\left( \sum_{i=1}^{n} p_i^q \right)^{\frac{1-r}{1-q}} - 1}{1 - r}, \tag{B10}
\]

\[
S_{q,r}^{\text{Supra-extensive}} (p_i) := \frac{\left( 1 + \frac{1-r}{1-q} \log \sum_{i=1}^{n} p_i^q \right)^{\frac{1-r}{1-q}} - 1}{1 - q}. \tag{B11}
\]

The above identities (B8) and (B9) are respectively simple mathematical modifications of (2.9) and (2.10) in [30].

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