A theory of finite structures

Daniel Leivant
SICE (Indiana University) and IRIF (Paris-Diderot)
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Abstract

We develop a novel formal theory of finite structures, based on a view of finite structures as a fundamental artifact of computing and programming, forming a common platform for computing both within particular finite structures, and in the aggregate for computing over infinite data-types construed as families of finite structures. A finite structure is here a finite collection of finite partial-functions, over a common universe of atoms. The theory is second-order, as it uses quantification over finite functions.

Our formal theory FS uses a small number of fundamental axiom-schemas, with finiteness enforced by a schema of induction on finite partial-functions. We show that computability is definable in the theory by existential formulas, generalizing Kleene’s Theorem on the $\Sigma_1$-definability of RE sets, and use that result to prove that FS is mutually interpretable with Peano Arithmetic.

1 Introduction

We develop a formal theory of finite structures, motivated in part by an imperative programming language for transforming such structures.

Our point of departure is to posit finite structures as a fundamental artifact of computing and programming. It is a truism that a database is a finite structure. But elements of infinite data-types, such as the natural numbers or binary strings, are also finite-structures that obey certain requirements. Computing over such a data-type can thus be construed as a transformation process driven by those structures’ internal making. Viewed from that angle, a function computed by a transducer-program is perceived as a mapping over the space of finite-structures, rather than a function within a particular infinite structure. Thus, finite partial-structures form a common platform for computing both within particular finite structures, and in the aggregate for computing over infinite data-types, as long as their elements are not infinite themselves (e.g. streams).

Our theory is second-order, in that we quantify over structures (via a quantification over finite functions). The well-known second-order nature of inductive data is thus manifested in the computation objects being themselves second-order, albeit finite ones. The atoms, from which our finite structures are built, are the first-order elements. They are nameless and structure-less, and there must be an unlimited supply of them.
to permit unhindered structure extension during computation. One intends to identify structures that differ only by the choice of atoms, i.e. ones that are isomorphic to each other, though it turns out that we can often convey that intent implicitly, without complicating matters with permanent references to equivalence classes.

Since referencing objects by constant and function identifiers is central to imperative programming, we choose to base our structures on partial-functions, rather than sets or relations. That is, each one of our structures is a finite set of finite partial-functions. The Tarskian notion of an explicit structure universe is superfluous here: the only atoms that matter are the ones that appear as inputs and/or outputs of a structure’s functions, a set we shall call the structure’s scope.

Finally, as we base our finite structures on finite partial-functions, the basic operations on structures must be function-updates of some form. This is analogous to the operation of adjoining an element to a set, which underlies several existing theories of finite sets (see below).

In fact, the idea that inductive data-objects, such as the natural numbers, can be construed as being composed of underlying units, goes back all the way to Euclid, who defined a number as “a multitude composed of units” [4, 7th book, definition 2]. The logicist project of Frege, Dedekind and Russell, which attempted to reduce mathematics to logic’s first principles, included this very same reduction of natural numbers to finite sets. The current predominant view of natural numbers as irreducible primal objects was advocated by opponents of the logicist project, notably Poincaré [5] and Kronecker, whose critique was subsequently advanced by the emergence of Tarskian semantics and by Gödel’s incompleteness theorems.

Philosophical considerations aside, the kinship between natural numbers and finite sets raises interesting questions about the formalization of finite set theory, and the mutual interpretation of such a theory with formal theories for arithmetic. Several formal theories proposed for finite set theory are based on $\mathbf{ZF}$ with the Axiom of Infinity replaced by its negation, which we denote here by $\mathbf{ZF}_f$. [10] shows that $\mathbf{ZF}_f$ can be interpreted in $\mathbf{PA}$ (Theorem 3.1), and that $\mathbf{PA}$ can be interpreted in $\mathbf{ZF}_f$ (Theorem 4.5), but not by the inverse of the former. That inverse would work for the interpretation of $\mathbf{PA}$ provided $\mathbf{ZF}_f$ is augmented with a Transitive Containment axiom, asserting that every set is contained in a transitive set (Theorem 6.5). A result analogous to the latter was proved by Wang already in 1953 [22], where he uses an axiom (ALG) analogous to Transitive Containment, stating the enumerability of the collection of all finite sets.

It seems, though, that $\mathbf{ZF}_f$ is somewhat of an oxymoron, in that it embraces $\mathbf{ZF}$, a theory designed specifically to reason about infinity, only to eviscerate it off the bat by excluding infinity. An approach more germane to finiteness was proposed already by Zermelo [24, Theorem 3], Whitehead and Russell [23, *120.23], Sierpiński [18, p.106], Kuratowski [11], and Tarski [20]. It enforces finiteness by an induction principle for sets: if $\emptyset$ satisfies a property $\varphi$ and whenever $X$ satisfies $\varphi$ then so does $X \cup \{y\}$ for each $y$, then all sets satisfy $\varphi$. This induction principle corresponds to the inductive

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1Wang removes from $\mathbf{ZF}$ Infinity without replacing it with its negation, since he does not interpret finite set theory in $\mathbf{PA}$.

2The subtext is, of course, the conviction that $\mathbf{ZFC}$ is the canonical formal framework for all of mathematics.
definition of finite sets as the objects generated by successively joining elements to \(\emptyset\).

Mayberry [13] defines an arithmetic based on induction on sets, which he deftly dubs Euclidean Arithmetic. He considers a number of induction principles derived from the set-induction principle above, but allows for neither set quantifiers nor unbounded atomic quantifiers in the induction formula. Consequently, his system is mutually interpretable with a weak number theory, namely \(I\Delta_0\) extended with an exponentiation function (detailed proof in [9]). Pettigrew [15, 16] shows that these two theories are also equipollent with a bounded version of \(ZF_f\).

Our approach differs from the ones above in several respects. Our theory \(FS\) of finite structures is based on finite structures rather than finite sets, and is formally second-order, in contrast to theories based on \(ZF\), which on the one hand are first-order, and on the other hand refer to a cumulative hierarchy of sets. At the same time, \(FS\)'s second-order nature is weak, in that its induction principle implies that its second-order object are finite. As a result, our theory of finite structures, although based on a variant of set induction (and not \(ZF\)) is equipollent to full Peano Arithmetic.

Principally, our motivation is to develop new natural tools for the analysis and verification of computing and programming. We are particularly interested in novel forms of calibrating computational resources by syntactic methods. In [12] we already show that, by using finite structures as focal concept, one obtains an abstract characterization of primitive recursive mathematics based on the concept of loop variant, familiar from program verification.

The remaining of the paper is organized as follows. In §2 we lay out our fundamental concepts and notations. Our theory \(FS\) of finite structures is presented in §3, followed by its programming language counterpart \(ST\) in §4. We then prove in §5 an abstract generalization of Kleene’s classical theorem about the existential definability of computability, and use it in §6 to show that \(FS\) is mutually interpretable in Peano Arithmetic.

## 2 Finite structures

### 2.1 Finite functions

Our basic notion is the finite partial structure, in which function-identifiers are interpreted as partial-functions. For example, we construe binary strings as structures over the vocabulary with a constant \(e\) and unary function identifiers 0 and 1. E.g., \(011\) is taken to be the four element structure

\[
e \circ 0 \rightarrow \circ \circ 1 \circ 1 \circ
\]

Here 0 is interpreted as the partial-function defined only for the leftmost atom, and 1 as the partial-function defined only for the second and third. As in Gurevich’s ASM, we posit our finite structures to live within a denumerable set \(A\) of atoms, i.e. unspecified and unstructured objects. To accommodate non-denoting terms we extend \(A\) to a flat domain \(A_\perp\), where in addition to the atoms we posit a fresh object \(\perp\), intended to denote “undefined.” The elements of \(A\) are the standard elements of \(A_\perp\)\(^3\).

\(^3\)We refer neither to boolean values not to “background structures” \(\Box\) on top of atoms.
By an \textit{A-function} we mean a finite \(k\)-ary partial-function over \(A\), where \(k \geq 0\); thus, the nullary \(A\)-functions are the atoms. We identify an \(A\)-function \(F\) with the total function \(\tilde{F} : A_k^k \rightarrow A_{\perp}\) satisfying

\[
\tilde{F}(a_1, \ldots, a_k) = \begin{cases} 
\min(F(a_1, \ldots, a_k), \perp) & \text{if } F(a_1, \ldots, a_k) \in A \\
\perp & \text{else}
\end{cases}
\]

Note that \(\tilde{F}\) is necessarily strict, i.e. its output is \(\perp\) whenever one of its inputs is. For each \(k\) we let \(\emptyset_k\) be the empty \(k\)-ary \(A\)-function, i.e. the one that returns \(\perp\) for every input. When in no danger of confusion we write \(\emptyset\) for \(\emptyset_k\). The \textit{scope} of \(F\) is the set of atoms occurring in its entries.

Function partiality provides a natural representation of finite relations over \(A\) by partial functions, without recourse to booleans. We represent a finite \(k\)-ary relation \(R\) over \(A\) by \(k\)-ary \(A\)-functions whose domain is \(R\), notably the function \(\xi_R(a_1, \ldots, a_k) = \begin{cases} 
R(a_1, \ldots, a_k) & \text{if } R(a_1, \ldots, a_k) \in A \\
\perp & \text{else}
\end{cases}\)

### 2.2 \(A\)-structures over a vocabulary

A \textit{vocabulary} is a finite set \(V\) of function-identifiers, with each \(f \in V\) assigned an \textit{arity} \(r(f) \geq 0\). We refer to nullary function-identifiers as \textit{tokens} and to unary ones as \textit{pointers}. For the moment we might think of these identifiers as reserved names, but they can be construed just the same as (permanently free) variables.

An \textit{entry} of an \(A\)-function \(F\) is a tuple \((a_1 \ldots a_k, b)\) where \(b = F(a_1, \ldots, a_k) \neq \perp\). A \(\sigma\) is an \(A\)-structure over \(V\) if it is a mapping that to each \(f^k \in V\), assigns a \(k\)-ary \(A\)-function \(\sigma(f)\), said to be a \textit{component} of \(\sigma\). The \textit{scope} of \(\sigma\) is the union of the scopes of its components.

If \(\sigma\) is an \(A\)-structure over \(V\), and \(\tau\) an \(A\)-structure over \(W \supseteq V\), then \(\tau\) is an \textit{expansion} of \(\sigma\) (to \(W\)), and \(\sigma\) a \textit{reduct} of \(\tau\) (to \(V\)), if the two structures have the same interpretation for identifiers in \(V\) (the scope of \(\tau\) may be strictly larger than that of \(\sigma\)). For \(A\)-structures \(\sigma\) and \(\tau\) over the same vocabulary, we say that \(\sigma\) is a \textit{substructure} of \(\tau\) if every entry of \(\sigma\) is an entry of \(\tau\).

A finite collection of structures is itself a structure: if \(\sigma_i\) is a structure over \(V_i\) \((i \in I)\), then the collection is the union \(\sigma = \bigcup_{i \in I} \sigma_i\) over the disjoint union of the vocabularies.

Given a vocabulary \(V\), the set \(\text{Tm}_V\) of \(V\)-\textit{terms} is generated by

- \(\omega \in \text{Tm}_V\).
- If \(f^k \in V\), and \(t_1, \ldots, t_k \in \text{Tm}_V\) then \(ft_1 \cdots t_k \in \text{Tm}_V\)

Note that we write function application in formal terms without parentheses and commas, as in \(fxy\) or \(f\vec{x}\). We optionally superscript function identifiers by their arity, and implicitly posit that the arity of a function matches the number of arguments displayed; thus writing \(f^k\vec{a}\) assumes that \(\vec{a}\) is a vector of length \(k\), and \(f\vec{a}\) (with no superscript) that the vector \(\vec{a}\) is as long as \(f\)'s arity. A term is \textit{standard} if \(\omega\) does not occur in it.
Given an $A$-structure $\sigma$ over $V$, the value of a $V$-term $t$ in $\sigma$, denoted $\sigma(t)$, is obtained by recurrence on $t$:

1. $\sigma(\omega) = \bot$
2. For $f^k \in V$, $\sigma(ft_1 \cdots t_k) = \sigma(f)(\sigma(t_1), \ldots, \sigma(t_k))$

An atom $a \in A$ is $V$-accessible in $\sigma$ if $a = \sigma(t)$ for some $t \in Tm_V$. An A-structure $\sigma$ over $V$ is accessible if every atom in the scope of $\sigma$ is $V$-accessible. For example, if $V$ is without tokens then no atom of a $V$-structure can be accessible. If every atom in the scope of an accessible structure $\sigma$ is the value of a unique $V$-term we say that $\sigma$ is free. Note that an accessible A-structure $\sigma$ can fail to be free even if all its components are injective:

However, we have the straightforward observation:

**Lemma 1** An accessible A-structure $\sigma$ over $V$ is free iff there is a finite set $T$ of $V$-terms, closed under taking sub-terms, such that the function $\sigma : T \to A$ is injective.

If $q$ is a standard $V$-term, and $T$ consists of the sub-terms of $q$, then we write $T(q)$ for the resulting free A-structure. It is often convenient to fix a reserved token, say $\bullet$, to denote in each structure $T(q)$ its root, i.e. the term $q$ as a whole. For example, for nullary $z$ and $u$, unary $s$, and binary $f$, the structures $T(szs)$, $T(fzu)$, $T(fszsz)$, $T(fsz)$ are, respectively,

![Diagram]

### 2.3 A second-order language

We wish to present a formal theory of finite structures, that deals not only with one structure at a time, but with structure transformation, in particular by suitable imperative programs. To that end, we generalize our discussion from a single vocabulary $V$ to a vocabulary “on demand.” We posit for each $k \geq 0$ a denumerable sets of $k$-ary variables, intended to range over the $k$-ary $A$-functions. The variables of arity $k = 0$ and $k > 0$ are dubbed atomic and functional, respectively. When a particular vocabulary $V = f_1^1 \ldots f_k^k$ is of interest (with the exhibited ordering of its identifiers), we write
\( g^V \) (with the vocabulary’s name as superscript) for a vector of variables \( g_1 \ldots g_k \), with \( r(g_i) = r_j \). We update our definitions of the set \( Tm \) of terms, and of their semantics in a structure \( \sigma \), by referring to function variables rather than identifiers of a fixed vocabulary.

An equation is a phrase of the form \( t \doteq q \), where \( t, q \in Tm \). The set \( Fm \) of formulas is generated inductively by

1. Every equation is a formula;
2. \( Fm \) is closed under the propositional operators \((\neg, \land, \lor, \rightarrow)\).
3. If \( \varphi \) is a formula, then so are \( \forall f^k \varphi \) and \( \exists f^k \varphi \) (\( r \geq 0 \)).

Given an \( A \)-structure \( \sigma \) for a set \( W \) of variables, a variable \( f^k \), and \( k \)-ary \( A \)-function \( F \), we write \( \{ f := F \} \sigma \) for the structure \( \sigma' \) which is identical to \( \sigma \), except that \( \sigma'(f) = F \). Given a formula \( \varphi \) and a structure \( \sigma \), where all variables free in \( \varphi \) are in the domain of \( \sigma \), the truth of \( \varphi \) in \( \sigma \), denoted \( \sigma \models \varphi \), is defined by recurrence on formulas, as follows.

\[
\begin{align*}
\sigma & \models t \doteq q \iff \sigma(t) = \sigma(q) \\
\sigma & \models \neg \varphi \iff \sigma \not\models \varphi
\end{align*}
\]

Similarly for other connectives

\[
\begin{align*}
\sigma & \models \forall f^k \varphi \iff \{ f \leftarrow F \} \sigma \models \varphi \text{ for all finite } F : A^k \rightarrow A \\
\sigma & \models \exists f^k \varphi \iff \{ f \leftarrow F \} \sigma \models \varphi \text{ for some finite } F : A^k \rightarrow A
\end{align*}
\]

When \( \sigma \models \varphi \) we also say that \( \sigma \) verifies \( \varphi \). Clearly, \( \sigma \models \varphi \) depends only on \( \sigma(f) \) for variables occurring free in \( \varphi \).

Recall from \( \mathbb{1} \) that we identify a finite \( F : A^k \rightarrow A \) with its strict extension \( \tilde{F} : A^{k+1}_\bot \rightarrow A_\bot \), so the quantifiers range over all strict \( \bot \)-valued functions, which for the nullary case \( k = 0 \) means that atomic variables may take any value in \( A_\bot \). This definition departs from Tarski’s semantics in that quantification ranges over all \( A \)-functions, regardless of the scope of \( \sigma \).

### 2.4 First-order formulas

We call quantification over atomic variables first-order, and quantification over functional variables second-order. A formula is first-order if all its quantifiers are first-order.

We use notation conventions for some important first-order formulas:

- \( \vec{t} \in f^k \) for \( f \vec{t} \neq \omega \). We overload this convention, and write \( \vec{t} \in f^1 \) to mean \( t_1, \ldots, t_k \in f \), where \( \vec{t} = t_1 \ldots t_k \).
  
  Dually, \( t \in f^k \) abbreviates \( \bigvee_{i+j=k-1} \exists \vec{v}, \vec{w} \vec{v} \vec{t} \vec{w} \in f \)

- \( \text{Scope}_V[x] = \bigvee_{f \in V} (x \in f) \lor (\exists \vec{v} x \doteq f \vec{v}) \)
states that the atom denoted by $x$ is in the scope of $\sigma$. Thus

$$\sigma \models \forall x \text{ Scope}_V[x] \rightarrow \varphi[x]$$

means that $\varphi$ is true “within” $\sigma$, which is not the same as $\forall x \varphi[x]$. For example, $\forall x x \neq \omega$ is true for all $A$-structures, whereas $\forall x \text{ Scope}_V[x] \rightarrow x \neq \omega$ is false for all $A$-structures.

- Let $V = \{z^0, s^1\}$. The $V$-structures that model the following formula are precisely the structures $T(s^2z)$ for the natural numbers, described in §2.2.  

$$\nu \equiv ( \forall x, y \ s x \hat{=} s y \land x \hat{=} \omega \land y \hat{=} \omega \rightarrow x \hat{=} y ) \land ( \forall x \ s x \hat{=} z \lor \exists y x \hat{=} s y )$$  

(4)

The following observation is a variant of a basic result of Finite Model Theory. We shall not use it in this paper.

**Proposition 2** Let $\varphi$ be a first-order formula. The following problem is decidable in polynomial time (in the size of the input structure presentation). We assume that $A$-structure are given as tables.

- Given an $A$-structure $\sigma$, is $\varphi$ true for $\sigma$.

**Proof.** The proof is by induction on $\varphi$. Absent second-order quantifiers, the only case of interest is when the main operator of $\varphi$ is an atomic quantifier, say $\forall u$. The input’s format provides direct scanning of all entries that are in the scope of $\sigma$, and the denotations of the $A$-functions all yield $\bot$ for all other entries. From this the calculation of the truth in $\sigma$ of $\forall u \varphi$ is immediate.  

\[ \square \]

### 3 A theory of finite structures

#### 3.1 On axiomatizing finiteness

Since our $A$-structure are built of functions, we refer to a generative process that extends functions rather than sets or relations. The set $\mathfrak{F}^k$ of $k$-ary $A$-functions is generated by:

- The empty $k$-ary function $\emptyset_k$ is in $\mathfrak{F}^k$;

- If $F \in \mathfrak{F}^k$, $v \in \bar{u} \in A$, and $F(\bar{u}) = \bot$, then extending $F$ with an entry $F\bar{u} = v$ yields an $A$-function in $\mathfrak{F}_k$.

\[ \text{We disregard here the token } \bullet \]
From this inductive definition we obtain an Induction Schema for $A$-functions. Using the abbreviations

$$\varphi[\emptyset] \equiv \forall g (\forall \vec{u} \, g\vec{u} = \omega) \rightarrow \varphi[g]$$

and

$$\varphi[\{\bar{u} \mapsto v\} f] \equiv \forall g (g\bar{u} \approx v \land \forall \vec{w} \neq \bar{u} \, g\vec{w} \approx f\vec{w}) \rightarrow \varphi[g]$$

Induction for a formula $\varphi[f^k]$ (with a distinguished function-variable $f$) reads

$$(\forall f^k, \bar{u}, v \, \varphi[f] \land (f\bar{u} \approx \omega) \rightarrow \varphi[\{\bar{u} \mapsto v\} f]) \rightarrow \varphi[\emptyset] \rightarrow \forall f \varphi[f]$$

Since the components of $A$-structures are $A$-functions, without constraints that relate them, there is no need to articulate a separate induction principle for $A$-structures. Indeed, every $A$-structure $\sigma$ for a vocabulary $V = \{f_1, \ldots, f_k\}$ is obtained by first generating the entries of $\sigma(f_1)$, then those for $\sigma(f_2)$, and so on. Note that this would no longer be the case for an inductive definition of the class $\mathfrak{A}$ of accessible structures, whose components must be generated in tandem.  

- If $\sigma = (F_1 \ldots F_k)$ is in $\mathfrak{A}$, $\bar{u}$ is in the scope of $\sigma$, $(\sigma f)(\bar{u}) = \perp$, and $b \in A$, then extending $\sigma$ with an entry $F_i\bar{u} = b$ yields a structure in $\mathfrak{A}$.

3.2 The theory FS

Our axiomatic theory $FS$ for $A$-functions has free and bound variables for $atoms$ and free and bound variables for $functions$ of arbitrary positive arity. We use $u, v, \ldots$ as syntactic parameters for atomic variables, and $f, g, h, \ldots$ as syntactic parameters for functional variables, optionally superscripted with their arity when convenient.

The axiom schemas are the universal closures of the following templates, for all arities $i, k$, terms $t$, and formulas $\varphi$. For a function variable $f$ and variables $\bar{u}$ we write $\bar{u} \in f$ for $f\bar{u} \neq \omega$.

\[\text{[Strictness]} \quad f_{u_1} \cdots u_k \approx \omega \quad \rightarrow \quad \bigwedge_i u_i \approx \omega\]

\[\text{[Infinity]} \quad \exists \bar{u} \, f^k\bar{u} \approx \omega\]

Note that this states that $A$ is infinite (unbounded), and has no bearing on the finiteness of $A$-functions.

\[\text{[Empty-function]} \quad \exists g^k \forall \bar{u}. \, g\bar{u} \approx \omega.\]

\[\text{[Extension]} \quad \exists g \, g\bar{u} \approx v \land \forall \vec{w} \neq \bar{u} \, g\vec{w} \approx f\vec{w}\]

\[\text{[Explicit-definition]} \quad \exists g \forall \bar{u} \, (\bar{u} \in f \land g\bar{u} \approx t) \lor (\bar{u} \notin f \land g\bar{u} \approx \omega)\]

This schema combines Zermelo’s Separation Schema with an Explicit Definition principle: $g$ is defined by the term $t$, for arguments in the domain of $f$.

\[\text{[Explicit-definition]} \quad \exists g \forall \bar{u} \, (\bar{u} \in f \land g\bar{u} \approx t) \lor (\bar{u} \notin f \land g\bar{u} \approx \omega)\]

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This schema combines Zermelo’s Separation Schema with an Explicit Definition principle: $g$ is defined by the term $t$, for arguments in the domain of $f$.
Recalling (6) above:

\[(\forall f, \vec{u}, v \ \varphi[f] \land (f \vec{u} = \omega) \rightarrow \varphi[[\vec{u} \mapsto v] f]) \rightarrow \varphi[\emptyset] \rightarrow \forall f \ \varphi[f]\]

For each of the schemas [Empty-function], [Function-extension], and [Explicit-definition], the function \(g\) asserted to exist is trivially unique, and so adding identifiers for these functions is conservative over \(FS\). We write \(\emptyset\), \{\(\vec{u} \mapsto v\)\} \(f\) and \(\lambda \vec{x} \in f \ \vec{x} = t\), respectively, for these functions (one for every arity \(k \geq 1\)).

The relevant axiom-schemas above can be rephrased using these three constructs as primitives:

[Empty-function] \(\emptyset_{k}u_{1} \cdots u_{k} = \omega\)

[Extension] \(\{\(\vec{u} \mapsto v\)\} f = v \land \forall \vec{w} \neq \vec{u} \ (\{\(\vec{u} \mapsto v\)\} \vec{w} = f \vec{w}\)

[Explicit-definition] \(\forall \vec{u} \ (\vec{u} \in f \land (\lambda \vec{u} \in f) \vec{u} = t\) \land (\vec{u} \notin f \land (\lambda \vec{u} \in f) \vec{u} = \omega)\)

Also note the contra-positive form of \(f\)-Induction:

\[(\forall f, \vec{u}, v \ \psi[[\vec{u} \mapsto v] f] \rightarrow \psi[f]) \rightarrow (\psi[h] \rightarrow \psi[\emptyset]) \quad (7)\]

### 3.3 Some derived schemas

[Union] \(\exists g \ \forall \vec{u} \ \vec{u} \in g \leftrightarrow (\vec{u} \in f_{1} \lor \vec{u} \in f_{2})\).

The proof is by \(f\)-Induction on \(f_{2}\) and Extension.

[Composition] \(\exists h^{k} \ \forall \vec{x} \ h\vec{x} \equiv f_{1}(g_{1}\vec{x}) \cdots (g_{i}\vec{x})\) \(\quad (8)\)

This follows from Explicit-definition and Union.

[Branching] \(\exists h^{k} \ \forall \vec{x} \ (f\vec{x} \neq \omega \land h\vec{x} \equiv f\vec{x}) \lor (f\vec{x} \equiv \omega \land h\vec{x} \equiv g\vec{x})\)

The universal closure with respect to \(f, g\) is proved by inductions on \(f\) and \(g\), using Extension.

[Contraction] \(\exists g \ g\vec{u} \equiv \omega \land \forall \vec{w} \neq \vec{u} \ g\vec{w} \equiv f\vec{w}\) \(\quad (9)\)

This is the dual of Extension. The proof of (refeq:contraction) is by induction on \(f\) for the universal closure of (9) with respect to \(\vec{u}\).

[Function pairing] \(\forall f^{k}, g^{k} \exists h^{k+1} \ \exists a, b \ \forall \vec{x} \ f\vec{x} \equiv ha\vec{x} \land g\vec{x} \equiv hb\vec{x}\) \(\quad (10)\)

Much of the expressive and proof theoretic power of arithmetic is due to the representation of finite sequences and finite sets of numbers by numbers. The Function-pairing schema provides a representation of two \(k\)-ary \(A\)-functions by a single \((k+1)\)-ary \(A\)-function. Namely, \(f, g\) are “tagged” withing \(h\) by the tags \(a, b\) respectively. In other words, writing \(h_{a}\) for \(\lambda \vec{x} hu\vec{x}\), we have \(f = h_{a}\) and \(g = h_{b}\). (We might require, in addition, that \(hu\vec{x} = \bot\) for all atoms \(u \neq a, b\), but this is inessential if we include \(a, b\) explicitly in the representation.)
[Atomic-choice] A more interesting form of tagging is provided by the following
principles of Choice.

\[(\forall x \in f \exists y \varphi[x, y]) \rightarrow \exists g \forall x \in f \varphi[x, g] \] (11)

This is analogous to [14, Lemma 2(e)], and is straightforward by induction on \(f\). Suppose the schema holds for \(f\), yielding the function \(g\). To show the schema true for \(f' = \{\bar{u} \mapsto v\}f\) suppose it satisfies the premise

\[\forall \bar{x} \in f \exists y \varphi[\bar{x}, y] \] (12)

Then \(f\) satisfies (12) as well, yielding a \(g\) for the conclusion. Also, by (12) there is an atom \(y\) such that \(\varphi[\bar{u}, y]\), and so the conclusion is satisfied by \(\{\bar{u} \mapsto y\}g\) in place of \(g\).

[Function-choice] \[(\forall \bar{x} \in f \exists g \varphi[\bar{x}, g]) \rightarrow \exists h \forall \bar{x} \in f \varphi[\bar{x}, h] \]

where \(\varphi[\bar{x}, h] \) abbreviates \(\forall j \left( (\forall y \ jy \equiv h\bar{x}y) \rightarrow \varphi[\bar{x}, j] \right)\). Note that Atomic-choice is a special case of Function-choice, with \(g\) nullary.

Note that the bounding condition in the choice schemas above is essential: even the simplest case

\[(\forall x^0 \exists y^0 \varphi[x, y]) \rightarrow \exists f^1 \forall x \varphi[x, fx] \] (13)

is false already for \(\varphi \equiv x \equiv y\), since the identity function over \(A\) is not finite.

4 Imperative programs over \(A\)-structures

We define a variant of Gurevich’s abstract state machines (ASMs) [2] [6] [8], for the transformation of \(A\)-structures. Transducer-programs define mappings between structures, which are akin to the mappings underlying Fraenkel’s Replacement Axiom, and more generally the proper classes of Bernays-Gödel set theory. Namely, these mappings are not \(A\)-functions, and are referred to in the theory \(FS\) via the formulas that define them.

4.1 Structure revisions

Our structure transformation programming language, \(ST\), is designed to be a Turing-complete computing system for the transformation of finite partial-structures, using the simplest possible building blocks while maintaining expressiveness. An \(ST\)-program takes an \(A\)-structure as input, and successively applies basic structure revisions to it. The process may terminate with a final \(A\)-structure when no further revision is called for. For example, addition over \(\mathbb{N}\) might be computed by a program that takes as input a structure representing two natural numbers \(n_1\) and \(n_2\), and grafting the second on top of the first. Doubling a number might be performed by copying the input before grafting the copy over the input, or alternatively generating a new structure by repeatedly extending it with two atoms while depleting the input.

We start by defining the basic operations of \(ST\), to handle entries. Each such operation maps a structure \(\sigma\) into a structure \(\sigma'\) which is identical to \(\sigma\) with the exceptions noted.
1. An *extension* is a phrase $f t_1 \cdots t_k := q$ where the $t_i$'s are standard terms and $q$ is a term. The intent is that $\sigma'$ is identical to $\sigma$, except that if $\sigma(f t_1 \cdots t_k) = \bot$ then $\sigma'(f t_1 \cdots t_k) = \sigma(q)$.

2. An *inception* is a phrase of the form $c \downarrow$, where $c$ is a token. A common alternative notation is $c := \text{new}$. The intent is that $\sigma'$ is identical to $\sigma$, except that if $\sigma(c) = \bot$, then $\sigma'(c)$ is an atom not in the scope of $\sigma$. A more general form of inception, with a fresh atom assigned to a term $t$ is obtained as the composition

$$c \downarrow; \; f \alpha_1 \cdots \alpha_k := c; \; c := \omega$$

where $c$ is a reserved token.

We allow extensions and inceptions to refer to an identifier $f$ not in the vocabulary of $\sigma$, in which case the vocabulary of $\sigma'$ extends that of $\sigma$, and we posit that $\sigma(f) = \emptyset$.

3. A *contraction*, the dual of an extension, is a phrase of the form $f \alpha_1 \cdots \alpha_k \uparrow$. The intent is that $\sigma'$ is identical to $\sigma$, except that $\sigma'(f \alpha_1 \cdots \alpha_k) = \bot$.

4. A *deletion*, the dual of an inception, is a phrase of the form $c \uparrow$, where $c$ is a token. The intent is that $\sigma'$ is obtained by removing $\sigma(c)$. That is, $\sigma'$ is identical to $\sigma$, except that for all $A$-functions $f^k$ present and all $a_1 \ldots a_k \in A$, if $\sigma(f) \vec{a} = \sigma(c)$, then $\sigma'(f) \vec{a} = \bot$.

Note that a deletion cannot be obtained via the composition of contractions, because $\sigma(c)$ might be reached by $A$-functions of $\sigma$ from atoms that are not accessible, in which case the atom $\sigma(c)$ cannot be eliminated from the scope of $\sigma'$ by contractions alone.

We refer to extensions, contractions, inceptions and deletions as *revisions*. Extensions and inceptions are then *constructive revisions*, whereas contractions and deletions are the *destructive revisions*.

An extension and a contraction can be combined into an *assignment*, i.e. a phrase of the form $f: = q$. This can be viewed as an abbreviation, with $b$ a fresh token, for the composition of four revisions:

$$b \downarrow q; \; f \uparrow; \; f \downarrow b; \; b \uparrow$$

(Note that the atom denoted by $q$ (when defined) is being memorized by $b$, since $q$ may become inaccessible due to the contraction $f \uparrow$.) Although assignments are common and useful, we prefer contractions and extensions as basic constructs, for two reasons. First, these constructs are truly elemental. More concretely, the distinction between constructive and destructive revisions plays a role in implicit characterizations of computational complexity classes, as for example in [12].

### 4.2 ST programs

Our programming language **ST** for structure transformation consists of guarded iterative programs using revisions as basic operations. Define a *guard* to be a quantifier-free
The programs of ST are generated inductively as follows.

1. A structure-revision is a program.

2. If \( P \) and \( Q \) are programs and \( G \) is a guard, then \( P; Q, \text{if}[G] \{ P \} \{ Q \} \) and \( \text{do}[G] \{ P \} \) are programs.

For a program \( P \) we define the binary yield relation \( \Rightarrow \) between structures by recurrence on \( P \). For \( P \) a revision the definition follows the intended semantics described informally above. The cases for composition, branching, and iteration, are straightforward as usual.

Let \( \Phi : \mathcal{C} \rightarrow \mathcal{C}' \) be a partial-mapping from a class \( \mathcal{C} \) of \( A \)-structures to a class \( \mathcal{C}' \) of \( A \)-structures. A program \( P \) computes \( \Phi \) if for every \( \sigma \in \mathcal{C} \), \( \sigma \Rightarrow_P \tau \) for some expansion \( \tau \) of \( \Phi(\sigma) \). Note that the vocabulary of the output structure need not be related to the input vocabulary.

4.3 Turing completeness

Guarded iterative programs are well known to be sound and complete for Turing computability, and proofs of the Turing completeness of abstract state machines have been given before (see for example [12, §3.1]). To dispel any concern that those proofs need more than finite structures and our simple revision operations, we outline a proof here.

**THEOREM 3** Let \( M \) be a Turing transducer computing a partial-function \( f_M : \Sigma^* \rightarrow \Sigma^* \). There is an ST-program \( P_M \) that, for every \( w \in \Sigma^* \), transforms \( T(w) \) to \( T(f_M(w)) \).

**PROOF.** Suppose \( M \) uses an extended alphabet \( \Gamma \supseteq \Sigma \), set of states \( Q \), start state \( s \), print state \( p \), and transition function \( \delta \). Recall that for \( w = \gamma_1 \cdots \gamma_k \in \Gamma^* \) we write \( T(w) \) for the structure \( \circ \gamma_1 \rightarrow \circ \cdots \circ \gamma_k \rightarrow \circ \).

Define \( V_M \) to be the vocabulary with \( e \), \( c \) and each state in \( Q \) as tokens; and with \( r \) and each symbol in \( \Gamma \) as pointers. The intent is that a configuration \( (q, \sigma_1 \cdots \sigma_i \cdots \sigma_k) \) (i.e. with \( \sigma_i \) cursored) be represented by the \( V_M \)-structure

\[
\begin{align*}
\circ & \xrightarrow{\sigma_1} \circ \cdots \circ \xrightarrow{\sigma_i} \circ \cdots \circ \xrightarrow{\sigma_k} \circ \\
e, q & \quad c
\end{align*}
\]

All remaining tokens are undefined.

We define the program \( P_M \) to implement the following phases:

1. Convert the input structure into the structure for the initial configuration, and initialize \( c \) to the initial input element. Use a loop to initialize a fresh pointer \( r \) to be the destructor function for the input string, to be used for backwards movements of the cursor.

\(^6\)Taking for guards arbitrary first-order formulas would not make a difference anywhere.
2. Main loop: configurations are revised as called for by \( \delta \). The loop’s guard is \( p \) (the “print” state) being undefined.

3. Convert the final configuration into the output.

\[\square\]

5  Computability implies existential definability

5.1 Expressing relation iteration

Suppose \( \varphi[\vec{f}^V, \vec{g}^V] \) is a formula, with variable-vectors \( \vec{f}, \vec{g} \), both for the vocabulary \( V \). Define the relation \( \varphi^*[\vec{f}^V, \vec{g}^V] \) as the least fixpoint of the closure conditions:

- \( \varphi^*[\vec{f}^V, \vec{f}^V] \) for all \( \vec{f}^V \).
- If \( \varphi[\vec{f}^V, \vec{g}^V] \) and \( \varphi^*[\vec{g}^V, \vec{h}^V] \) then \( \varphi^*[\vec{g}^V, \vec{h}^V] \).

**Theorem 4** For every formula \( \varphi[\vec{f}^V, \vec{g}^V] \) there is a formula \( \varphi^*[\vec{f}^V, \vec{g}^V] \) that defines the relation \( \varphi^*[\vec{f}^V, \vec{g}^V] \).

**Proof.** By definition, \( \varphi^*[\vec{f}^V, \vec{g}^V] \) just in case there are \( \vec{h}^V_0, \ldots, \vec{h}^V_k \) \( (k \geq 0) \) such that

\[ \vec{h}^V_0 = \vec{f}, \quad \varphi[\vec{h}^V_i, \vec{h}^V_{i+1}] \quad (i < k), \quad \text{and} \quad \vec{h}^V_k = \vec{g} \]

The function-vectors \( \vec{h}_i \) can be bundled jointly into a single vector \( \vec{h} \), using a vocabulary \( \hat{V} \) with a fresh variable \( \bar{f}_k \) for each \( f \) in \( V \). We also refer to two fresh auxiliary variables \( z^0 \) and \( s^1 \), and let the bundled vector \( \vec{h} \) be defined by

\[ \vec{h}^{\hat{V}}(s^0[z]) = h_i(x) \]

For a term \( t \) let \( \vec{h}_t = \vec{f} \) abbreviate \( \forall \vec{u} \vec{t} \vec{u} = \vec{h} \).

Now define

\[ \varphi^*[\vec{f}^V, \vec{g}^V] \equiv \begin{align*}
\exists z^0, s^1(z \neq \vec{\omega} & \land \forall x^0, y^0 \quad (sx \neq z) \land (sx = sy \rightarrow x = y) \\
& \land \exists \vec{h}^{\hat{V}} \quad \vec{h}_z = \vec{f} \\
& \land \forall x \quad (sx \neq \vec{\omega} \rightarrow (\forall \vec{f}^{\hat{V}} m^V \vec{\ell} = \vec{h}_x \land \vec{m} = \vec{h}_{sx} \rightarrow \varphi[\vec{\ell}, \vec{m}]) \\
& \land (sx \neq \vec{\omega} \land x \neq \vec{\omega} \rightarrow \vec{h}_x = \vec{g}))
\end{align*} \]  

\[ (14) \]

\[ \square \]
Example. Let $V$ be a vocabulary with identifiers of arity $\leq k$, and $\vec{x}, \vec{y}$ etc. be vectors of $k$ atomic variables. Take for $\varphi[\vec{g}, \vec{h}]$ the following formula (where $g$ and $h$ are construed as sets).

$$\forall x \ (hx = gx \lor \bigvee_{f \in V} \exists y_1 \ldots y_r \in g \ f\vec{y} \in h \land f\vec{y} \notin g)$$

That is, $h$ is identical to $g$ except that it may contain additional atoms, all obtained by applying some $f \in V$ to elements of $g$. Then the existential formula

$$A_V[u] \equiv \exists y^1 \varphi^*[\emptyset, g] \land u \in g$$

defines the set of $V$-accessible atoms. Consequently,

$$\forall u \ (\bigvee_{f \in V} u \in f) \rightarrow A_V[u]$$
is true for a $V$-structure $\sigma$ iff $\sigma$ is $V$-accessible.

5.2 Existential definability of computable relations between $A$-structures

Theorem 5 For every $\text{ST}$-program $P$ over $V$-structures, there is an existential formula $\varphi_P[\vec{f}^V, \vec{g}^V]$ that holds iff $\vec{f} \Rightarrow_P \vec{g}$.

Proof. By induction on $P$.

- If $P$ is a revision then $\varphi_P$ is in fact first-order, and trivially defined.
- If $P$ is $Q; R$, then by IH there are existential formulas $\varphi_Q[\vec{f}, \vec{h}]$ and $\varphi_R[\vec{h}, \vec{f}]$ that hold just in case $\vec{f} \Rightarrow_Q \vec{h}$ and $\vec{h} \Rightarrow_Q \vec{g}$. We thus define

$$\varphi_P[\vec{f}, \vec{g}] \equiv \exists \vec{h} \ \varphi_Q[\vec{f}, \vec{h}] \land \varphi_R[\vec{h}, \vec{f}]$$

which is existential if $\varphi_Q$ and $\varphi_R$ are.
- If $P$ is $\text{if}[G]\{Q\}\{R\}$ we define

$$\varphi_P[\vec{f}, \vec{g}] \equiv \ (G[\vec{f}] \land \varphi_Q[\vec{f}, \vec{g}] ) \lor ( \neg G[\vec{f}] \land \varphi_R[\vec{f}, \vec{g}] )$$

which is existential if $\varphi_Q$ and $\varphi_R$ are. Note that $G$ here is first-order, so the negation is harmless.
- If $P$ is $\text{do}[G]\{Q\}$, let

$$\psi[\vec{h}, \vec{j}] \equiv \ G[\vec{h}] \land \varphi_Q[\vec{h}, \vec{j}]$$

and define

$$\varphi_P[\vec{f}, \vec{g}] \equiv \ \psi^*[\vec{f}, \vec{g}] \land \neg G[\vec{g}]$$

Here $\psi^*$ is the existential formula defined in (14).
Theorem 5 defines a binary relation between the initial and final configurations of an ST-program. Given a convention on which program variables are to be considered inputs and which output, the formula $\varphi_P$ can be modified to express the input-output relation computed by the program. For instance, if $P$ uses variables $f_1, \ldots, f_k$, of which $f_1$ and $f_2$ are used for the inputs, and $f_2$ and $f_3$ are used for the outputs, then the I/O relation defined by $P$ is

$$
IO_P[g_1, g_2; g'_2, g'_3] \equiv \exists g'_1, g'_4 \ldots g'_k \varphi_P[g_1, g_2, \vec{0}; g'_1, \ldots, g'_k] \quad (15)
$$

6 Equipollence of FS and PA

6.1 Interpretation of arithmetic formulas in finite structures

The intended model of FS has no “universe” in the traditional, Tarskian, sense. Using the structures $T(s^{(0)}z)$ as the target “elements” falls short of the natural embedding, where natural numbers are interpreted as equivalence classes of such structures. Note that we cannot take one representative from each equivalence class because there is no way to formally identify such representatives. Consequently, we depart from the traditional definition of interpretations between languages and between formal theories, (see e.g. [3, \S 2.7]), relax the requirement that the source universe be interpreted by a definable subset of the target universe, and interpret the natural numbers instead by equivalence classes of structures satisfying $\nu$ (as defined in (4)).

The representation of natural numbers by equivalence classes of structures can now be formalized as follows. Take PA to be based on logic without equality, that is with equality considered a binary relation identifier rather than a logical constant, which for PA happens to be interpreted as identity. The point is that our interpretation of PA in FS must now include a definition of the interpretation of equality, though not as the identity relation between the atoms of FS, but rather as structure isomorphism. The property of a unary function $f$ being an isomorphism between structures over the vocabulary $z^0, s^1$ can be defined by

$$
\text{Isom}[f; z, s; z', s'] \equiv fz = z' \\
\land \forall a \ sa \neq \omega \rightarrow (fsa = s'fa \neq \omega) \quad (16)
$$

Given that $\nu$ (as defined in (10)) is assumed true for $(z, s)$ and $(z', s')$, and that $f$ must be strict, the condition Isom$[f; z, s; z', s']$ above implies that the function $f$ must be a bijection. We now define the interpretation of the equality relation of PA as the relation

$$
\text{Iso}[z, s; z', s'] \equiv \exists f^1 \text{ Isom}[z, s; z's']
$$

between structures representing natural numbers.

We interpret the remaining non-logical constants of PA, namely 0, successor, addition and multiplication, via basic equations for them. For each variable $x$ of PA let $z^0_x$ and $s^1_x$ be FS-variables. In any given context only finitely many $\mathbb{N}$-variables will
be present, so the collection of all corresponding FS-variables will be finite as well. Every equation of \( \text{PA} \) is equivalent to a formula involving only equation of one of the following five forms:

\[
\begin{align*}
  x_i &= 0, \quad x_i = x_j, \quad x_i = sx_j, \quad x_i = x_j + x_k, \quad \text{and} \quad x_i = x_j \cdot x_k
\end{align*}
\]

(17)

We define the following interpretation for such equations.

- \((x_i = 0)^{\Delta} \equiv (z_i \neq \omega \land s_i = \emptyset)\),
- \((x_i = x_j)^{\Delta} \equiv \exists f^1 \text{Iso}[f; z_i, s_i; z_j, s_j]\) where \(\text{Iso} \) is as in (16),
- \((x_i = sx_j)^{\Delta} \equiv \exists t^1 \text{Suc}[s_j; t] \land \text{Iso}[z_j, t; z_i, s_i]\) where \(\text{Suc}[s,t] \) is
  \[
  \forall a \ a \neq \omega \rightarrow (sa \neq \omega \land ta = sa) \lor (sa = \omega \land ta \neq \omega \land tta = \omega)
  \]

(18)
- \((x_i = x_j + x_k)^{\Delta} \equiv \exists z^0, t^1 \varphi_A[z_j, s_j, z_k, s_k; z, t] \land \text{Iso}[z, t; z_i, s_i]\) where \(\varphi_A \) is the existential formula defined by (16) for the ST-program \(A\) computing addition,
- \((x_i = x_j \times x_k)^{\Delta} \) is defined similarly, referring to the ST-program for multiplication.
- Combining the previous cases, we easily define (by discourse-level induction on terms) an interpretation \((t = q)^{\Delta}\) for all terms \(t, q\).

We let the mapping \(\Delta\) commute with the connectives: \((\varphi \land \psi)^{\Delta} \equiv \varphi^{\Delta} \land \psi^{\Delta}\), etc. For the quantifiers we let

\[
\begin{align*}
  (\forall x_i \psi)^{\Delta} & \equiv \forall z_i, s_i \ \nu[z_i, s_i] \rightarrow \psi^{\Delta} \\
  (\exists x_i \psi)^{\Delta} & \equiv \exists z_i, s_i \ \nu[z_i, s_i] \land \psi^{\Delta}
\end{align*}
\]

(19)

Let \(N_\times\) be the standard model of \(\text{PA}\), with zero, successor, addition, and multiplication as functions.

**Theorem 6** The interpretation \(\Delta\) is semantically sound and complete: for every closed formula \(\varphi\) of \(\text{PA}\), \(\varphi\) is true in \(N_\times\) iff \(\varphi^{\Delta}\) is true for all \(A\)-structures.

**Proof.** Consider the following fixed interpretation of the natural numbers as \(A\)-structures. Let \(z_0\) be an atom (to interpret 0), and \(s\) an injective unary partial-function \(z_0 \rightarrow a_1 \rightarrow a_2 \rightarrow \cdots\). That is, \(z_0\) and \(s\) form a copy of \(\mathbb{N}\) in \(A\). Define the canonical interpretation of \(n \in \mathbb{N}\) to be the \(A\)-structure \((z_0, s_n)\) where \(s_n\) is \(s\) truncated to its first \(n\) steps. Thus every \(A\)-structure satisfying \(\nu\) is isomorphic to \((z_0, s_n)\) for some \(n\).

We prove that for each formula \(\varphi[x_1, \ldots, x_k]\) of \(\text{PA}\), the following conditions are equivalent.
1. $N \times, \ [x_1 \leftarrow n_1, \ldots, x_k \leftarrow n_k] \models \varphi$

2. $\sigma \models \varphi^\Delta[z_1, s_1 \ldots z_k, s_k]$ where $\sigma$ assigns to $(z_i, s_i)$ the canonical interpretation of $n_i$ ($i = 1\ldots k$).

3. $\sigma \models \varphi^\Delta[z_1, s_1 \ldots z_k, s_k]$ for every $\sigma$ that assigns to each $(z_i, s_i)$ an interpretation satisfying $\nu$ and of size $n_i$.

We use induction on $\varphi$. For the induction basis, (1) and (2) are equivalent by the definition of $\Delta$, (2) implies (3) since the structures satisfying $\nu$ and of size $n$ are all isomorphic to the canonical structure for $n$, and (3) implies (2) trivially. The cases for propositional connectives are all immediate. Finally, for quantifiers we have that $N \times, \ [x_1 \leftarrow n_1, \ldots, x_k \leftarrow n_k] | = \forall x_{k+1} \varphi$ iff $N \times, \ [x_1 \leftarrow n_1, \ldots, x_k \leftarrow n_k, x_{k+1} \leftarrow n] | = \varphi$ for all $n \in \mathbb{N}$, which by IH is equivalent to $\varphi^\Delta[z_1, s_1 \ldots z_k, s_k, z_{k+1}, s_{k+1}]$ being true where $(z_i, s_i)$ is the canonical interpretation of $n_i$ ($i \leq k$) and $(z_{k+1}, s_{k+1})$ is any structure of size $n$ that satisfies $\nu$, i.e. $\forall z_{k+1}, s_{k+1} \nu[z_{k+1}, s_{k+1}] \rightarrow \varphi^\Delta$ being true, i.e. $(\forall x_{k+1} \varphi)^\Delta$. This proves the equivalence of (1) and (3). The equivalence of (1) and (2) is similar. $\blacksquare$

6.2 Interpreting PA in FS

We proceed to show that the interpretation $\Delta$ above from the language of PA to that of FS is not only semantically sound and complete, but is also a proof-theoretic interpretation of Peano Arithmetic in the theory FS.

**Theorem 7** For every closed formula of PA, if $PA \models \varphi$, then $FS \models \varphi^\Delta$.

**Proof.** We verify that for every axiom $\psi$ of PA, the formula

$$\psi^\circ \equiv \bigwedge \{ \nu[z_i, s_i] \mid x_i \text{ free in } \psi \} \rightarrow \psi^\Delta$$

is provable in FS.

- For Peano's Third Axiom, $\psi \equiv sx_i \neq 0$, we have

  $$\psi^\circ \equiv \nu[z_i, s_i] \rightarrow (\exists t \ Suc[s_i; t] \rightarrow \neg \text{Iso}[z_i, t; z_i, \emptyset])$$

  Reasoning within FS, Iso$[z_i, t; z_i, \emptyset]$ implies $t = \emptyset$, which contradicts Suc$[s_i; t]$.

- Peano's Fourth Axiom,

  $$\psi \equiv sx_i = sx_j \rightarrow x_i = x_j$$

  is rendered by

  $$\psi^\circ \equiv \nu[z_i, s_i] \land \nu[z_j, s_j]$$

  $$\rightarrow \text{Suc}[s_i, s_i'] \land \text{Suc}[s_j, s_j']$$

  $$\rightarrow \text{Iso}[z_i, s_i'; z_j, s_j'] \rightarrow \text{Iso}[z_i, s_i; z_j, s_j]$$

  which is easily provable from the definitions.
• The defining equations of addition and multiplication are treated similarly.

• Finally, consider an instance of PA’s Induction Schema,

\[ \psi \equiv (\forall x_i \varphi[x_i] \rightarrow \varphi[b x_i]) \rightarrow \varphi[0] \rightarrow \varphi[x_j] \]

with free variables, say a single variable \( x_m \). We have

\[ \psi^\circ \equiv \nu[z_m, s_m] \]

\[ \rightarrow (\forall z_i, s_i \nu[z_i, s_i] \land \varphi^\Delta[z_i, s_i] \land \forall t, z_k, s_k \text{Suc}[s_i, t] \land \text{Iso}[z_i, t; z_k, s_k] \rightarrow \varphi^\Delta[z_k, s_k]) \]

\[ \rightarrow \nu[z_j, s_j] \rightarrow \varphi^\Delta[z_j, s_j] \]

(20)

We shall use the provability in FS of the following simple observations:

1. \( \nu[z, s] \land \text{Iso}[z, s; z', s'] \rightarrow \nu[z', s'] \)

2. \( \nu[z, s] \land \text{Suc}[s; t] \rightarrow \nu[z, t] \)

3. \( \nu[z, \{a \leftrightarrow b\} s] \rightarrow \nu[z, s] \)

4. \( \varphi[z, s] \land \text{Iso}[z, s; z', s'] \rightarrow \varphi[z', s'] \)

5. \( a, b \not\in f, \bar{g} \rightarrow \xi[\{\bar{u} \mapsto a\} f, \bar{g}] \rightarrow \xi[\{\bar{u} \mapsto b\} f, \bar{g}] \)

To prove \( \psi^\circ \) within FS assume

6. \( \nu[z_m, s_m] \)

7. \( \varphi^\Delta[z_j, \emptyset] \)

8. \( \forall z_i, s_i \nu[z_i, s_i] \land \varphi^\Delta[z_i, s_i] \land \forall t, z_k, s_k \text{Suc}[s_i, t] \land \text{Iso}[z_i, t; z_k, s_k] \rightarrow \varphi^\Delta[z_k, s_k]) \)

9. \( \nu[z_j, s_j] \)

We use f-Induction for the formula

\[ \zeta[s] \equiv \nu[z_k, s] \rightarrow \varphi^\Delta[z_k, s] \]

By (7) and (1) we have \( \zeta[\emptyset] \). Assume \( \zeta[s] \), and consider \( \zeta[s'] \) for \( s' = \{a \mapsto b\} s \). If \( \nu[z_k, s'], \) then \( \nu[z_k, s] \), by (3), and so \( \varphi^\Delta[z_k, s] \), from \( \zeta[s] \). This implies, by (8),

\[ \forall t, z_k, s_k \text{Suc}[s, t] \land \text{Iso}[z, t; z_k, s_k] \rightarrow \varphi^\Delta[z_k, s_k]) \]

But we have \( \text{Suc}[s; s'] \), so taking \( s' \) for both \( s_k \) and \( t \), and \( z \) for \( z_k \), we get \( \varphi^\Delta[z, s'] \). We have thus completed both the basis and the step of the f-induction for \( \zeta[s] \), and conclude \( \psi^\circ \).

In summary, we have shown that for every formula \( \varphi \) of PA we have

\[ \text{PA} \vdash \varphi \quad \text{implies} \quad \text{FS} \vdash \varphi^\Delta \] (21)

We shall show in Theorem 10 below that the inverse of (21) also hold, i.e. the interpretation \( \varphi \mapsto \varphi^\circ \) is faithful.
6.3 Interpretation of FS in PA

Developing finite mathematics (and even much of mathematical analysis [19, Part 2] in PA is nowadays a routine exercise. It would still be useful to articulate a particular interpretation. To simplify, we refer to the extension \( \overline{PA} \) of PA with identifiers for all primitive recursive functions and with their defining equations as axioms. \( \overline{PA} \) is well-known to be conservative over PA [21].

Our interpretation \( \psi \mapsto \psi^\triangle \) of FS into PA represents the atoms by the natural numbers: posit an enumeration \( a_0, a_1 \ldots \) of the atoms, and represent \( a_n \) by \( n + 1 \). \( \bot \) is represented by 0. We code a finite set \( S = \{m_1, \ldots, m_k\} \) of natural numbers, displayed in increasing order, by the natural number \( \#S \) whose binary numeral is \( 1d_0\ldots d_m \), where \( d_i \) is 1 iff \( i \in S \). Thus \( \emptyset \) is coded by 1, \{0\} by 3 (binary 11), and \{0, 2\} by 13 (binary 1101). We represent an entry \( e = (a_{n_1}, \ldots, a_{n_k}, a_m) \) of a \( k \)-ary function \( (k > 0) \) by the number \( \#e \) whose binary numeral is \( 10^n10^{n_1} \ldots 10^{n_k} \). An \( A \)-function \( f \) is the finite set of its entries, so we represent it as \( \#f = \#\{\#e \mid e \text{ an entry of } f\} \).

Clearly we have for each \( k \geq 0 \) a primitive-recursive (p.r.) predicate \( G_k \) that identifies the codes of strict \( k \)-ary \( A \)-functions. We use these predicates to refer to the syntax of FS, and to interpret formulas \( \psi \) of FS as formulas \( \psi^\triangle \) of (extended) PA, with quantifiers bounded to their intended range.

**Proposition 8** For every formula \( \psi \) of FS, \( \psi \) is valid iff \( \psi^\triangle \) is true in the structure \( N_{PR} \) of the natural numbers with the primitive recursive functions.

**Proof.** By straightforward induction on \( \psi \). \( \square \)

The semantic correctness of the interpretation \( \psi \mapsto \psi^\triangle \) extends to a proof theoretic interpretation:

**Proposition 9** For every formula \( \psi \) of FS, if \( FS \vdash \psi \) then \( PA \vdash \psi^\triangle \).

**Proof.** Other than f-Induction, the interpretations of the axioms of FS are all trivially provable in PA. As for an instance \( \psi \) of f-Induction (with fixed arity), \( \psi^\triangle \) is provable in PA by induction on the size of the A-function \( f \). \( \square \)

From Proposition 9 we infer:

**Theorem 10** The interpretation \( \varphi \mapsto \varphi^\triangle \) is faithful. That is, for every closed formula \( \varphi \) of PA,

\[
FS \vdash \varphi^\triangle \quad \text{implies} \quad PA \vdash \varphi \quad (22)
\]

**Proof.**

Proposition 9 implies that for a PA-formula \( \varphi \),

\[
FS \vdash \varphi^\triangle \quad \text{implies} \quad PA \vdash (\varphi^\triangle)^\varphi \quad (23)
\]

It remains to observe that for every PA-formula \( \varphi \),

\[
PA \vdash (\varphi^\triangle)^\varphi \rightarrow \varphi \quad (24)
\]
This is proved by induction on $\varphi$, starting with the five forms of \[17\]. The details are straightforward.

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