Renormalizability of a generalized gauge fixing interpolating among the Coulomb, Landau and maximal Abelian gauges

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Abstract

A detailed discussion of the renormalization properties of a class of gauges which interpolates among the Landau, Coulomb and Maximal Abelian gauges is provided in the framework of the algebraic renormalization in Euclidean Yang-Mills theories in four dimensions.

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1 Introduction

Interpolating gauges in Yang-Mills theories have been used for renormalizability purposes and in order to understand the behavior of gauge invariant operators [1, 2, 3, 4]. In [1], a gauge fixing interpolating between the Coulomb gauge and the Landau gauge has been discussed and its renormalizability established. Further, a gauge which interpolates between the Landau and the maximal Abelian gauge (MAG) was constructed in [2]. In this work this interpolating gauge was used in order to study the vacuum energy in the MAG. It has been shown in [1, 2] that those two types of interpolating gauges are renormalizable to all orders in perturbation theory. A generalization which connects these two class of gauges was proposed and analyzed at the classical level in [4], providing thus a gauge which interpolates between the Coulomb, the Landau and the maximal Abelian gauges (the CLM gauge).

We point out that these three gauges (Coulomb, Landau, MAG) have been used to understand specific aspects of the nonperturbative infrared region of Yang-Mills theories, from theoretical as well as from lattice numerical simulations and phenomenological point of views. Therefore, a generalized gauge fixing interpolating among all these three gauges might be helpful in order to achieve a unifying picture of the behavior of gauge invariant quantities like, for instance, the vacuum energy.

Remarkably, the three gauges discussed here can be obtained through the minimization of a suitable functional, a feature which allows to construct a lattice formulation of these gauge conditions. Not surprisingly, the gauge fixing which interpolates among those three gauges turns out, in a suitable limit, to be defined as a minimization of an interpolating functional, making possible the implementation of a lattice formulation, see Appendix A.

In this work we discuss the quantum aspects of the interpolating gauge, our main result being that of establishing the all orders multiplicative renormalizability of this generalized gauge. In section 2 we review the construction of the CLM interpolating gauge introduced in [4]. In section 3 we explain how to control the breaking of the Lorentz invariance due to the gauge fixing. The proof of the multiplicative renormalizability is given in section 4. Finally, in section 5 we provide our conclusions.

2 Interpolating gauge fixing

In order to introduce the Coulomb-Landau-Maximal Abelian (CLM) interpolating gauge let us briefly fix the notation. According to the notation used in [2], we decompose the gauge field $A^A_{\mu}$, $A \in \{1, \ldots, N^2 - 1\}$, into off-diagonal and diagonal components, namely

$$A^A_{\mu} T^A = A^i_{\mu} T^i + A^a_{\mu} T^a,$$

where $T^A$ are the anti-hermitian, $T^\dagger = -T$, generators of the gauge group $SU(N)$, $[T^A, T^B] = f^{ABC} T^C$. The indices $\{i, j\}$ label the $N - 1$ diagonal generators of the Cartan subalgebra. The remaining $N(N - 1)$ off-diagonal generators will be labelled by the indices $a, b, c, \ldots$. 

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For the nilpotent BRST transformations of the fields, we have

\[ s_g A^a_{\mu} = -(D^a_{\mu} b + gf^{abc} A^b_{\mu} c + gf^{abi} A^b_{\mu} c^i) , \]

\[ s_g A^i_{\mu} = - (\partial_{\mu} c^i + gf^{abi} A^a_{\mu} c^i) , \]

\[ s_g c^a = gf^{abi} c^b c^i + \frac{g}{2} f^{abc} c^b c^c , \]

\[ s_g c^i = \frac{g}{2} f^{abi} c^a c^b , \]

where \((c^i, \bar{c}^i), (c^a, \bar{c}^a)\) stand for the diagonal and off-diagonal Faddeev-Popov ghosts, while \((b^i, b^a)\) denote the diagonal and off-diagonal Lagrange multipliers. The covariant derivative \(D^a_{\mu}\) in eq.(2) is defined as

\[ D^a_{\mu} = \delta^{ab} \partial_{\mu} - gf^{abi} A^b_{\mu} c^i . \]

Concerning the field strength \(F^A_{\mu \nu} = (F^i_{\mu \nu}, F^a_{\mu \nu})\), we have

\[ F^a_{\mu \nu} = D^a_{\mu} A^b_{\nu} - D^a_{\nu} A^b_{\mu} + gf^{abc} A^b_{\mu} A^c_{\nu} , \]

\[ F^i_{\mu \nu} = \partial_{\mu} A^i_{\nu} - \partial_{\nu} A^i_{\mu} + gf^{abi} A^a_{\mu} A^b_{\nu} . \]

Thus, for the Yang-Mills action one has

\[ S_{YM} = \frac{1}{4} \int d^4 x F^A_{\mu \nu} F^A_{\mu \nu} = \frac{1}{4} \int d^4 x \left( F^a_{\mu \nu} F^a_{\mu \nu} + F^i_{\mu \nu} F^i_{\mu \nu} \right) . \]

In order to interpolate between the Coulomb, the Landau and the maximal Abelian gauges, we introduce the following, power counting renormalizable, gauge fixing term

\[ S_{gf} = s_g \int d^4 x \left[ c^a \left( a_{\mu \nu} \partial_{\mu} A^a_{\nu} + gK_{\mu \nu} f^{abi} A^b_{\mu} A^c_{\nu} + \frac{\alpha}{2} b^a - \frac{\alpha}{2} g f^{abi} c^b c^i - \frac{\beta}{4} g f^{abc} c^b c^c \right) + h_{\mu \nu} c^i \partial_{\mu} A^i_{\nu} \right] , \]

where \(a_{\mu \nu}, h_{\mu \nu}, k_{\mu \nu}\) and \(K_{\mu \nu}\) are diagonal constant matrices defined as

\[ a_{\mu \nu} \equiv \text{diag}(1,1,1,1) , \]

\[ h_{\mu \nu} \equiv \text{diag}(1,1,1,1) , \]

\[ k_{\mu \nu} \equiv \text{diag}(k_s, k_s, k_s, k_t) , \]

\[ K_{\mu \nu} = (k_{\mu \rho} - \delta_{\mu \rho}) a_{\sigma \nu} \equiv \text{diag}(k_s - 1, k_s - 1, k_s - 1, a(k_t - 1)) , \]

and, \(\{a, h, k_s, k_t, \alpha, \beta\}\) are gauge parameters.
The explicit expression for $S_{gf}$ reads

$$
S_{gf} = \int d^4x \left\{ b^a \left[ a_{\mu\nu} \partial_\mu A_\nu^a + gK_{\mu\nu} f^{abi} A_\mu^b A_\nu^i + \frac{\alpha}{2} b^a - \alpha g f^{abc} c^b c^c - \frac{\beta}{2} g f^{abc} c^b c^c \right] + h_{\mu\nu} b^\dagger \partial_\mu A_\nu^a \right. \\
+ \left. a_{\mu\nu} c^a \partial_\mu D^b_\nu c^b + h_{\mu\nu} c^a \partial_\mu \left( \partial_\nu c^3 + g f^{abi} A_\nu^b \right) \right. \\
+ \left. gK_{\mu\nu} f^{abi} A_\mu^b \partial_\nu c^b + gK_{\mu\nu} f^{abi} c^a A_\nu^i \partial_\nu c^i + g^2 K_{\mu\nu} f^{abi} f^{cde} c^d A_\nu^b A_\mu^c \\
+ \left. g^2 K_{\mu\nu} f^{abi} f^{bcj} A_\mu^b A_\nu^c c^j + g^2 K_{\mu\nu} f^{abi} f^{bcd} A_\mu^b A_\nu^c c^d + \frac{\alpha}{4} g^2 f^{abi} f^{cde} c^b c^c c^d \\
- \left. \frac{\beta}{4} g^2 f^{abi} f^{acj} c^b c^d c^i \right. \\
- \left. \frac{\beta}{8} g^2 f^{abc} f^{ade} c^b c^c c^d c^e \right\} . \tag{8}
$$

The gauge fixed action is then

$$
S = SYM + S_{gf} \tag{9}
$$

where $SYM$ and $S_{gf}$ are given by (5) and (8), respectively.

The various gauges are attained in the following way.

- The Landau gauge is achieved by setting

$$
a_{\mu\nu} = h_{\mu\nu} = \delta_{\mu\nu} , \\
K_{\mu\nu} = \alpha = \beta = 0 . \tag{10}
$$

namely

$$
a = h = k_t = k_s = 1 , \\
\alpha = \beta = 0 . \tag{11}
$$

Substitution of the values (11) in the action (8) provides the Landau gauge fixing,

$$
S_L = sg \int d^4x \left( \bar{c}^a \partial_\mu A_\mu^a + \bar{c}^i \partial_\mu A_\mu^i \right) = sg \int d^4x \left( \bar{c}^A \partial_\mu A_\mu^A \right) , \tag{12}
$$

- The Coulomb gauge is achieved by setting

$$
a_{\mu\nu} = h_{\mu\nu} \equiv \text{diag}(1, 1, 1, 0) , \\
K_{\mu\nu} = \alpha = \beta = 0 , \tag{13}
$$

i.e.

$$
a = h = \alpha = \beta = 0 \\
k_t = k_s = 1 . \tag{14}
$$

The substitution of (14) in the action (8) gives the Coulomb gauge fixing,

$$
S_C = sg \int d^4x \left( \bar{c}^a \partial_\mu A_\mu^a + \bar{c}^i \partial_\mu A_\mu^i \right) = sg \int d^4x \bar{c}^A \partial_\mu A_\mu^A . \tag{15}
$$
The Maximal Abelian gauge is achieved by setting
\[ a_{\mu \nu} = h_{\mu \nu} = K_{\mu \nu} = \delta_{\mu \nu}, \beta = \alpha, \]
which gives
\[ a = h = 1, \quad k_t = k_s = 0, \quad \beta = \alpha. \] (17)

Therefore, for the Maximal Abelian gauge fixing\(^*\) we get
\[ S_{MAG} = s g \int d^4 x \left[ \bar{c}^i (D_{\mu} A^i_{\mu} + \frac{\alpha}{2} b^a - \frac{\alpha}{2} g f^{abc} \bar{c}^b \vec{e}^c - \frac{\alpha}{4} g f^{abc} \bar{c}^b \bar{c}^c) + \bar{c}^i \partial_{\mu} A^i_{\mu} \right]. \] (18)

We point out that the gauge fixing (8) is slightly more general than the one reported in [4], which was in fact limited only to tree level aspects. As it will be shown in the rest of this article, the gauge fixing (8) turns out to be suitable in order to establish the multiplicative renormalizability of the model.

Let us now proceed by discussing the Lorentz breaking induced by the gauge (8) and the way to control it.

3 BRST quantization and the breaking of Lorentz invariance

A problem to be faced in Yang-Mills theories quantized in the gauge (8) is that of the breaking of the Lorentz invariance. This problem was successfully treated in [1] for the case of the Coulomb-Landau interpolating gauge. Here, we will use the same technique in order to control the Lorentz breaking. We refer thus to [1] and references therein for all details. This section is then devoted to the treatment of the Lorentz breaking in the specific case of the CLM gauges.

3.1 BRST quantization method

Let us start by recalling the main steps of the BRST quantization method [6, 7]. For that we consider a general gauge model with classical action \( S(A) \) and coupling \( g \), where \( A \) is the gauge field. The action \( S(A) \) is invariant under transformations of a certain Lie group \( G \) with elements
\[ u = e^{\omega} \in G \mid \omega = \omega^{A} \lambda^{A}, \] (19)
where \( \lambda^{A} \) are the group generators. The index \( A \) is used here as a general index, with arbitrary dimension. The algebra of the generators is, typically, given by
\[ [\lambda^{A}, \lambda^{B}] = f^{ABC} \lambda^{C}, \] (20)

\(^*\)As is known, the gauge parameter \( \alpha \) has to be introduced for renormalization purposes. The real MAG condition, namely \( D_{\mu} A^b_{\mu} = 0 \), is attained in the limit \( \alpha \to 0 \), which has to be taken after renormalization [7].
where $f^{ABC}$ are the structure constants of the group. The BRST method of quantization amounts to introduce a set of Lie algebra valued anticommuting fields, $C = C^A \lambda^A$, for each generator of the symmetry, the so called ghost fields or Faddeev-Popov ghosts. Together with the ghost fields, a set of nilpotent transformations is obtained, giving rise to the BRST transformations. For the transformation of the ghost fields we have\(^1\)

\[
 sC^A = \frac{g}{2} f^{ABC} C^B C^C ,
\]

which is just the Maurer-Cartan structure equation of the Lie group. The operator $s$ is the BRST operator. For the gauge field the BRST transformation is, in fact, obtained by replacing the group element parameter $\omega$ by the ghost field $C$, namely

\[
 sA^A_\mu = - (\partial_\mu C^A + g f^{ABC} A^B_\mu C^C) ,
\]

ensuring thus the nilpotency of the BRST operator, $s^2 = 0$.

From the gauge invariance of the action $S(A)$, it follows that

\[
 sS(A) = 0 .
\]

The quantization of the theory is achieved by introducing a gauge fixing in a BRST invariant way. Suppose that the constraint expressing the gauge condition is given by the equation

\[
 f^A(A) = 0 .
\]

The gauge fixed action then reads

\[
 S = S(A) + s\Delta^{-1}_{gf}(f, C, \bar{C}, b) ,
\]

where $\Delta^{-1}_{gf}$ is a local polynomial with ghost number $-1$ and dimension four,

\[
 \Delta^{-1}_{gf}(f, C, \bar{C}, b) = \int d^4x \left\{ \bar{C}^A \left[ f^A(A) + \frac{\alpha}{2} b^A \right] + \delta^{-1}(\bar{C}, C) \right\} .
\]

Here $b$ and $\bar{C}$ are respectively the Lautrup-Nakanishi and anti-ghost fields, transforming as a BRST doublet, i.e.

\[
 s\bar{C} = b ,

 s_b = 0 .
\]

The parameter $\alpha$ is a gauge parameter, and the polynomial function $\delta^{-1}$ might be necessary for renormalizability purposes. For example, nonlinear gauges need a quartic ghost self-interaction generated, for instance, by

\[
 \delta^{-1} = f^{ABC} \bar{C}^A C^B C^C .
\]

\(^1\)Notice that the appearance of the coupling constant $g$ in eq. (21) is just a matter of convention. Here we adopt a different convention than that employed in [I].
3.2 Controlling the Lorentz breaking

Now we apply the BRST quantization method to our gauge fixing. First, one has to identify the symmetries broken by the gauge fixing term. In our case these are: the local $SU(N)$ gauge symmetry, the global color invariance due to the Abelian decomposition, eq. (1), and the Lorentz symmetry. Here, the Lorentz symmetry coincides in fact with the rotation group $O(4)$, because we are dealing with a four dimensional Euclidean space-time. Thus, we shall make use of a BRST symmetry associated to the large symmetry group $SU(N) \otimes O(4)$, this will give rise to the introduction of a set of ghost fields which will enable us to control the gauge breaking together with the Lorentz breaking in a powerful and simple fashion. The algebra obeyed by the generators of this group is

\[
\begin{align*}
[T^a, T^b] &= f^{abc} T^c + f^{abi} T^i, \\
[T^a, T^i] &= f^{abi} T^b, \\
[T^i, T^j] &= 0, \\
[\Sigma_{\mu\nu}, \Sigma_{\gamma\delta}] &= f_{\mu\nu\gamma\delta} \Sigma_{\alpha\beta}, \\
[\Sigma_{\mu\nu}, T^a] &= f^{ab\mu\nu} T^b, \\
[\Sigma_{\mu\nu}, T^i] &= f^{ij\mu\nu} T^j, \\
\end{align*}
\]  

(29)

where $T^a$ and $T^i$ are, respectively, the generators of the non-Abelian and Abelian part of the gauge group $SU(N)$, and $\Sigma_{\mu\nu}$ are the generators of the rotation group $O(4)$. As already explained in Section 2, the indices $(a, b, c)$ refer to the $N(N-1)$ off-diagonal generators, while the indices $(i, j)$ label the $(N-1)$ diagonal generators of the Cartan subgroup of $SU(N)$. The structure constants of $O(4)$ are given by

\[
f_{\mu\nu\rho\sigma\alpha\beta} = -\frac{1}{2} \left[ (\delta_{\mu\rho} \delta_{\sigma\alpha} - \delta_{\nu\rho} \delta_{\sigma\beta}) \delta_{\mu\beta} + (\delta_{\nu\rho} \delta_{\sigma\alpha} - \delta_{\nu\sigma} \delta_{\rho\alpha}) \delta_{\nu\beta} \right],
\]

(30)

while the mixed structure constants are found to be

\[
\begin{align*}
f^{ab}_{\mu\nu} &= -\frac{1}{2} \delta^{ab} (x_\mu \partial_\nu - x_\nu \partial_\mu), \\
f^{ij}_{\mu\nu} &= -\frac{1}{2} \delta^{ij} (x_\mu \partial_\nu - x_\nu \partial_\mu).
\end{align*}
\]

(31)

In addition of the Faddeev-Popov ghost, $\{c^a, c^i\}$, corresponding to the gauge symmetry $SU(N)$ we have to include further global ghosts, $\{V_{\mu\nu}\}$, associated with the global $O(4)$ symmetry breaking. These global ghosts, being space-time independent, will behave as external sources.

According to the group structure and to equation (21), it is straightforward to deduce the BRST transformations of all ghosts

\[
\begin{align*}
sc^a &= g f^{abi} T^b c^i + \frac{g}{2} f^{abc} c^b c^c - g V_{\mu\nu} x_\mu \partial_\nu c^a, \\
sc^i &= \frac{g}{2} f^{abi} c^b - g V_{\mu\nu} x_\mu \partial_\nu c^i, \\
sV_{\mu\nu} &= -g V_{\mu\gamma} V_{\gamma\nu}.
\end{align*}
\]

(32)
Also, for the BRST doublets one has

\[
\begin{align*}
sc^a & = b^a , \\
sc^i & = b^i , \\
sb^a & = 0 , \\
sb^i & = 0 .
\end{align*}
\] (33)

The BRST transformations of the gauge fields are those corresponding to infinitesimal gauge transformations of the large group \( SU(N) \otimes O(4) \), maintaining the nilpotence of \( s \),

\[
\begin{align*}
sA^a_\mu & = -D^a_\mu c^b - gf^{abc} A^b_\mu c^c - gV_{\mu\nu} A^a_\nu - gV_{\gamma\nu} x_\gamma \partial_\nu A^a_\mu , \\
sA^i_\mu & = -\partial_\mu c^i - gf^{abi} A^b_\mu c^b - gV_{\mu\nu} A^i_\nu - gV_{\gamma\nu} x_\gamma \partial_\nu A^i_\mu .
\end{align*}
\] (34)

Such extended BRST symmetry \( 32-34 \) encodes the Lorentz rotations. As such, it is a generalization of the previous one, eq. (2).

Notice that the extended BRST operator \( s \) can be decomposed into an ordinary BRST operator \( s_g \) and a Lorentz BRST operator \( s_L \),

\[
s = s_g + s_L ,
\] (35)

obeying

\[
s^2 = s_g^2 = s_L^2 = \{ s_g , s_L \} = 0 .
\] (36)

The operator \( s_g \) corresponds to the pure gauge sector and is given in eq. (2), together with

\[
s_g V_{\mu\nu} = 0 .
\] (37)

The Lorentz BRST operator acts on the fields like

\[
\begin{align*}
s_L A^a_\mu & = -gV_{\mu\nu} A^a_\nu - gV_{\alpha\beta} x_\alpha \partial_\beta A^a_\mu , \\
s_L c^a_\mu & = -gV_{\alpha\beta} x_\alpha \partial_\beta c^a_\mu , \\
s_L \bar{c}^a_\mu & = 0 , \\
s_L b^a_\mu & = 0 , \\
s_L V_{\mu\nu} & = -gV_{\gamma\nu} V_{\gamma\mu} .
\end{align*}
\] (38)

4 Renormalizability of the CLM gauges

Having defined the structure of the BRST operator, we can write down the gauge fixing term which has to be added to the Yang-Mills action \( 33 \). This is done by replacing the ordinary BRST operator \( s_g \), in expression \( 32 \), by the extended BRST operator \( s \) defined in \( 32-34 \). Thus,

\[
S_{gf} = s \int d^4 x \left[ \bar{c}^a \left( a_{\mu\nu} \partial_\mu A^a_\nu + gK_{\mu\nu} f^{abc} A^b_\mu A^c_\nu + \frac{\alpha}{2} b^a - \frac{\alpha}{2} gf^{abi} c^b c^i - \frac{\beta}{4} gf^{abc} c^b c^c \right) + h_{\mu\nu} \bar{c}^a \partial_\mu A^a_\nu \right] ,
\] (39)
where transformations (5). Thus, the complete action we will work with is fixed action (9), one has to introduce external sources \( \Omega, L, M \).

In order to establish the set of Ward identities describing the symmetries displayed by the gauge fields / sources is a constant field, and so is \( S_{gf} \). The Slavnov-Taylor identity

\[
S_{gf} = \int d^4 x \left\{ \frac{1}{2} g_{\mu\nu} \left[ A^\mu_{\nu} + \Omega^\mu_{\nu} + L^\mu_{\nu} + M^\mu_{\nu} \right] + \frac{1}{2} g \left[ E_{\mu}^2 + \Omega^{\mu}_{\nu} \right] \right\}
\]

The complete action (41) obeys the following set of Ward identities:

\[
\Sigma = S_{YM} + S_{gf} + S_{ext}
\]

4.1 Ward identities

In order to establish the set of Ward identities describing the symmetries displayed by the gauge fixed action (39), one has to introduce external sources \( \Omega, L, M \) coupled to the nonlinear BRST transformations (5). Thus, the complete action we will work with is

\[
\Sigma = S_{YM} + S_{gf} + S_{ext}
\]

where

\[
S_{ext} = \int d^4 x \left\{ -\Omega^\mu_{\nu} A^\mu_{\nu} - \Omega^i_{\mu} A^i_{\mu} + L^\mu_{\nu} + M^\mu_{\nu} \right\}
\]

Notice that \( V \) is a constant field, and so is \( M \). The total set of fields and sources, with their corresponding ghost numbers and dimensions, is displayed in Table 1.

The complete action (41) obeys the following set of Ward identities:

- The Slavnov-Taylor identity

\[
S(\Sigma) = 0
\]
with
\[
S(\Sigma) = \int d^4x \left( \frac{\delta \Sigma}{\delta \Omega_\mu^a} \frac{\delta \Sigma}{\delta A^b_\mu} + \frac{\delta \Sigma}{\delta \Omega_\mu^a} \frac{\delta \Sigma}{\delta A^b_\mu} + \frac{\delta \Sigma}{\delta L^a} \frac{\delta \Sigma}{\delta c^a} + \frac{\delta \Sigma}{\delta L^a} \frac{\delta \Sigma}{\delta c^a} + b^a \frac{\delta \Sigma}{\delta \Omega_\mu^a} + b^a \frac{\delta \Sigma}{\delta \Omega_\mu^a} \right) + \frac{\delta \Sigma}{\delta M_{\mu\nu}} \frac{\delta \Sigma}{\delta V_{\mu\nu}} .
\] (44)

- The diagonal gauge fixing and diagonal antighost equations
\[
\frac{\delta \Sigma}{\delta b^i} = h_{\mu\nu} \partial_\mu A^b_\nu , \quad \frac{\delta \Sigma}{\delta \bar{c}^i} + h_{\mu\nu} \partial_\mu \frac{\delta \Sigma}{\delta \Omega_\nu^a} = 0 .
\] (45)

- The linearly broken integrated diagonal ghost Ward identity
\[
\int d^4x \left( \frac{\delta \Sigma}{\delta \bar{c}^i} + g f^{abi} \frac{\delta \Sigma}{\delta b^i} \right) = \int d^4x \left( \frac{\delta \Sigma}{\delta \bar{c}^i} + g f^{abi} \frac{\delta \Sigma}{\delta b^i} \right).
\] (46)

- The $M$ equation
\[
\frac{\delta \Sigma}{\delta M_{\mu\nu}} = -g V_{\mu\gamma} V_{\nu\gamma} .
\] (47)

The last Ward identity is possible only because $V$ is a global ghost field. In addition, we also have a residual global $U(1)^{N-1}$ symmetry and a residual global $O(3)$ symmetry, corresponding to the three-space rotations. As shown in [8], the $U(1)^{N-1}$ global symmetry follows by anticommuting the diagonal ghost equation (46) with the Slavnov-Taylor identities (44). The $O(3)$ symmetry will be tacitly assumed in the construction of the counterterm.

### 4.2 General invariant counterterm

Let us face now the construction of the most general counterterm consistent with the symmetries of the action (41). Following the algebraic renormalization theory [5], this can be performed by adding a generic local field functional, $\Sigma^c$, to the classical action (41)
\[
\Sigma^c = \Sigma + \epsilon \Sigma^c ,
\] (48)
where $\epsilon$ is a small expansion parameter. Imposing now the Ward identities (43, 45-47), one obtains that $\Sigma^c$ has to fulfill the following set of constraints
\[
B_{\Sigma} \Sigma^c = 0 , \quad \frac{\delta \Sigma^c}{\delta b^i} = 0 , \quad \frac{\delta \Sigma^c}{\delta \bar{c}^i} + h_{\mu\nu} \partial_\mu \frac{\delta \Sigma^c}{\delta \Omega_\nu^a} = 0 ,
\]
\[
\int d^4x \left( \frac{\delta \Sigma^c}{\delta \bar{c}^i} + g f^{abi} \frac{\delta \Sigma^c}{\delta b^i} \right) = 0 , \quad \frac{\delta \Sigma^c}{\delta M_{\mu\nu}} = 0 ,
\] (49)
where $B_\Sigma$ is the nilpotent, $B_\Sigma B_\Sigma = 0$, linearized Slavnov-Taylor operator,

$$B_\Sigma \equiv \int d^4 x \left( \frac{\delta \Sigma}{\delta \phi^a} \frac{\delta}{\delta A^a_\mu} + \frac{\delta \Sigma}{\delta A^a_\mu} \frac{\delta}{\delta \phi^a} + \frac{\delta \Sigma}{\delta \phi^a} \frac{\delta}{\delta A^a_\mu} + \frac{\delta \Sigma}{\delta A^a_\mu} \frac{\delta}{\delta \phi^a} \right) + \frac{\delta \Sigma}{\delta \phi^a} \frac{\delta}{\delta A^a_\mu} + \frac{\delta \Sigma}{\delta A^a_\mu} \frac{\delta}{\delta \phi^a} \right) + \frac{\delta \Sigma}{\delta A^a_\mu} \frac{\delta}{\delta \phi^a} \right) + \frac{\delta \Sigma}{\delta A^a_\mu} \frac{\delta}{\delta \phi^a} \right) + \frac{\delta \Sigma}{\delta \phi^a} \frac{\delta}{\delta A^a_\mu} + \frac{\delta \Sigma}{\delta A^a_\mu} \frac{\delta}{\delta \phi^a} \right)$$

From general cohomological arguments [5], it follows that the first condition of eqs. (19) implies that $\Sigma^c$ can be written as

$$\Sigma^c = a_0 S_Y + B_\Sigma \Delta^{-1},$$

where $\Delta^{-1}$ is the most general local polynomial in the fields with dimension 4 and ghost number -1. Furthermore, from the constraints (49), it turns out that

$$\Delta^{-1} = \int d^4 x \left\{ a_1 c^a \left( b^a - g f^{abc} c^b c^c \right) + a_5 g f^{abc} c^b c^c + a_3 L^a c^a + a_\mu d_{\mu\nu} \right\}
+ a_\nu d_{\nu\sigma} \left( \frac{1}{2} \Omega^{ab}_{\gamma} \right) - d_{\nu\gamma} \left( 2 g f^{abc} c^b c^c \right) \right\} + d_{\mu\nu} \left( \Omega^{ab}_{\nu} A^a_{\mu} A^b_{\nu} \right) + d_{\mu\nu} \left( \Omega^{ab}_{\nu} A^a_{\mu} A^b_{\nu} \right)$$

where

$$d_{\mu\nu} \equiv \text{diag} \left( a_j, a_j, a_j, a_i \right).$$

Explicitly, (52) reads

$$\Delta^{-1} = \int d^4 x \left\{ a_1 c^a \left( b^a - g f^{abc} c^b c^c \right) + a_5 g f^{abc} c^b c^c + a_3 L^a c^a + a_8 a^a c^a D^{ab}_{\mu} A^b_{\mu} + a_9 c^a D^{ab}_{\mu} A^b_{\mu} + a_10 \left( \frac{1}{2} \Omega^{ab}_{\nu} \right) A^a_{\nu} \right\}$$

The parameters $(a_0, a_1, a_3, a_5, a_8, a_9, a_{10}, a_{11}, a_{12}, a_{13}, a_{14}, a_{15})$ are free independent constants. Thus, in order to establish the renormalizability of our model, we have to show that these twelve independent parameters can be reabsorbed in the action, through a renormalization of the fields, parameters and external sources. This will be the task of the next section.

### 4.3 Stability

It remains to show that the counterterm (51) can be in fact reabsorbed by means of a multiplicative renormalization of the fields and parameters, according to

$$\Sigma(\phi_0, J_0, \xi_0) = \Sigma(\phi, J, \xi) + \epsilon \Sigma^\prime(\phi, J, \xi) + O(\epsilon^2),$$

where

\[\Sigma^\prime(\phi, J, \xi) = \text{linearized Slavnov-Taylor operator}, \]

\[\epsilon \Sigma^\prime(\phi, J, \xi) + O(\epsilon^2).]
where

\begin{align*}
\phi_0^a &= Z^{1/2}_\phi \phi^a, \\
\phi_i^0 &= z^{1/2}_o \phi^i, \\
J^a &= Z_J J^a, \\
J^i &= z_J J^i, \\
\xi_0 &= Z_\xi \xi, \\
\end{align*}

and

\begin{align*}
\phi &\in \{ A, c, \bar{c}, V \} , \\
J &\in \{ \Omega, L, M \} , \\
\xi &\in \{ g, \alpha, \beta, a, h, k_t, k_s \} . \\
\end{align*}

Due to the use of a non covariant gauge fixing, we shall distinguish the renormalization of the fourth component of the gluon field from the remaining three components, according to

\begin{align*}
A_{04}^a &= \tilde{Z}_{A}^{1/2} A_4^a, \\
A_{0k}^a &= Z_{A}^{1/2} A_k^a, \\
A_{04}^i &= z_{A}^{1/2} A_4^i, \\
A_{0k}^i &= z_{A}^{1/2} A_k^i. \\
\end{align*}

The corresponding renormalization factors are given by

\begin{align*}
\tilde{Z}_{A}^{1/2} &= 1 + \epsilon \left( \frac{a_0}{2} + a_{14} \right), \\
Z_{A}^{1/2} &= 1 + \epsilon \left( \frac{a_0}{2} + a_{15} \right), \\
z_{A}^{1/2} &= 1 + \epsilon \left( \frac{a_0}{2} - a_{10} \right), \\
z_{A}^{1/2} &= 1 + \epsilon \left( \frac{a_0}{2} - a_{11} \right). \\
\end{align*}

For the renormalization of the coupling constant one has

\begin{align*}
Z_g &= 1 - \epsilon \frac{a_0}{2}. \\
\end{align*}

For the space-time gauge parameters $a$ and $h$ the renormalization factors reads

\begin{align*}
Z_a &= 1 + \epsilon (a_8 - a_9), \\
Z_h &= z_{A}^{1/2} z_{A}^{-1/2}. \\
\end{align*}
while for the gauge group parameters we have

\[
\begin{align*}
Z_\alpha &= 1 + \epsilon \left( a_0 + 2a_1 - 2a_9 \right), \\
Z_\beta &= 1 + \epsilon \left( a_0 - 4a_5 - 2a_9 \right), \\
Z_{k_1} &= 1 + \epsilon \left( -a_8 + a_{12} \right), \\
Z_{k_2} &= 1 + \epsilon \left( -a_9 + a_{13} \right).
\end{align*}
\] (62)

The Faddeev-Popov ghosts renormalize according to

\[
\begin{align*}
Z_{c_1}^{1/2} &= 1 + \epsilon \left( -a_3 + \frac{a_{11}}{2} \right), \\
Z_{\bar{c}}^{1/2} &= 1 + \epsilon \left( a_9 - \frac{a_{11}}{2} \right), \\
z_{c_1}^{1/2} &= z_{\bar{c}}^{1/2} = 1 + \epsilon \frac{a_{11}}{2} = Z_g^{-1/2} z_A^{-1/4}. \\
\end{align*}
\] (63)

For the Lagrange multipliers we obtain

\[
\begin{align*}
Z_{b_1}^{1/2} &= 1 + \epsilon \left( -a_0 + a_9 \right) = Z_g^{1/2} z_A^{-1/4} Z_{c_1}^{1/2}, \\
z_{b_1}^{1/2} &= z_A^{-1/2},
\end{align*}
\] (64)

and, for the external sources \( \Omega \) and \( L \)

\[
\begin{align*}
\tilde{Z}_\Omega &= 1 - \epsilon \left( \frac{a_{11}}{2} + a_{14} \right) = Z_g^{-1/2} z_A^{-1/4} \tilde{z}_A^{1/2}, \\
Z_\Omega &= 1 - \epsilon \left( \frac{a_{11}}{2} + a_{15} \right) = Z_g^{-1/2} z_A^{-1/4} \tilde{Z}_A^{1/2}, \\
\tilde{z}_\Omega &= 1 + \epsilon \left( a_{10} - \frac{a_{11}}{2} \right) = Z_g^{-1/2} z_A^{-1/4} \tilde{z}_A^{-1/2}, \\
z_\Omega &= 1 + \epsilon \frac{a_{11}}{2} = Z_g^{-1/2} z_A^{-1/4}, \\
Z_L &= Z_g^{-1} Z_{c_1}^{-1/2} z_A^{-1/2}, \\
z_L &= Z_g^{-1} z_A^{-1}.
\end{align*}
\] (65)

Finally, the Lorentz ghost and its associated BRST external source renormalize as

\[
\begin{align*}
Z_{V_1}^{1/2} &= z_{c_1}^{1/2}, \\
Z_M &= Z_g^{-1} z_A^{-1}.
\end{align*}
\] (66)

This ends the proof of the multiplicatively renormalizability of Yang-Mills theory quantized in the general interpolating gauge \( \mathbf{5} \).

## 5 Conclusions

In this work we have proven the renormalizability of a generalized gauge fixing which interpolates between the Coulomb, the Landau and the maximal Abelian gauges.
It is worth underlining that all these three gauges are extensively used in lattice numerical simulations. The introduction of such a generalized interpolating gauges seems thus appropriate, as it could be helpful in order to achieve a kind of unifying understanding of the physical operators in all these gauges.

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A Minimizing functional

In this Appendix we discuss the possibility of introducing a suitable minimizing functional for the interpolating gauge, a feature which could allow for a lattice implementation of this generalized gauge fixing. For that, we shall consider the interpolating gauge (8) when

\[ h_{\mu\nu} \rightarrow a_{\mu\nu} , \]
\[ k_t \rightarrow k , \]
\[ k_s \rightarrow k , \]
\[ \beta \rightarrow \alpha , \]

which gives

\[
S_{\text{class}} = \int d^4x \left\{ b^a \left[ a_{\mu\nu} \partial_\mu A^a_{\nu} + (k-1)g f^{abc} a_{\mu\nu} A^i_{\mu} A^b_{\nu} + \frac{\alpha}{2} b^a - \alpha g f^{abi} c^b c^i - \frac{\alpha}{2} g f^{abc} b^c c^c \right] + b^i a_{\mu\nu} \partial_\mu A^i_{\nu} + c^a a_{\mu\nu} D^{ab} D^{bc} c^c + \frac{\beta}{2} a_{\mu\nu} \partial_\mu \left( \partial_\nu c^i + g f^{abi} A^a_{\nu} c^i \right) + g c^a a_{\mu\nu} D^{ab} \left( f^{bcd} A^c_{\nu} c^d \right) - \frac{\alpha}{2} b^i a_{\mu\nu} \partial_\mu A^i_{\nu} A^b_{\mu} A^c_{\nu} + \frac{\alpha}{4} g^2 f^{abi} f^{cde} c^b c^c c^d + \frac{\alpha}{8} g^2 f^{abc} f^{ade} c^b c^c c^d \right\}. \tag{68}
\]

This expression coincides with the gauge fixing introduced in [4], which interpolates between the Coulomb, Landau and MAG as well. In order to achieve the real MAG condition, i.e.
\[ D^a_b A^b_\mu = 0, \] the limit \( \alpha \to 0 \) has to be taken, namely

\[
S_{\text{class}} \bigg|_{\alpha \to 0} = \int d^4x \left\{ b^a \left[ a_{\mu\nu} \partial_\mu A^a_\nu + (k - 1) g f^{abi} a_{\mu\nu} A^i_\mu A^b_\nu \right] + b^i a_{\mu\nu} \partial_\mu A^i_\nu + \bar{c}^a a_{\mu\nu} D^a_\mu D^b_\nu c^c + \epsilon^i a_{\mu\nu} \partial_\mu c^i + g f^{abi} a_{\mu\nu} A^i_\mu A^b_\nu \right\} + g \bar{c}^a f^{abi} a_{\mu\nu} A^i_\mu A^b_\nu c^c - g c^a a_{\mu\nu} D^a_\mu \left( f^{bcd} A^c_\nu c^d \right) - g^2 f^{abi} f^{cdi} c^d a_{\mu\nu} A^b_\nu A^c_\mu c^c + k g^2 f^{abi} f^{cdi} c^d a_{\mu\nu} A^b_\nu A^c_\mu c^c + k g^2 f^{abi} f^{cdi} c^d a_{\mu\nu} A^b_\nu A^c_\mu c^c \right\} .
\] (69)

The gauge fixing conditions which stem from expression (69) are now easily seen to be derived by requiring that the following field functional

\[
\mathcal{F} = \int d^4x \frac{1}{2} a_{\mu\nu} \left( A^a_\mu A^a_\nu + k A^i_\mu A^i_\nu \right)
\] (70)

is stationary under the action of infinitesimal gauge transformations. This requirement gives precisely the gauge fixing conditions corresponding to (69), \( \text{i.e.} \)

\[
a_{\mu\nu} \left( D^a_b A^b_\nu + k g f^{abi} A^b_\nu A^i_\mu \right) = 0 ,
\]

\[
a_{\mu\nu} \partial_\mu A^i_\nu = 0 .
\] (71)

One sees thus that a suitable minimizing functional can be associated to the interpolating gauge (69), providing thus a useful way to implement it on the lattice.

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