A Novel Approach Based on Sine Trigonometric Picture Fuzzy Aggregation Operators and Their Application in Decision Support System

Saleem Abdullah, Saifullah Khan, Muhammad Qiyas, and Ronnason Chinram

1Department of Mathematics, Abdul Wali Khan University, Mardan, Pakistan
2Algebra and Applications Research Unit, Division of Computational Science, Faculty of Science, Prince of Songkla University, Hat Yai, Songkhla 90110, Thailand

Correspondence should be addressed to Ronnason Chinram; ronnason.c@psu.ac.th

Received 31 August 2020; Revised 28 November 2020; Accepted 12 January 2021; Published 8 February 2021

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Picture fuzzy sets (PFSs) are one of the fundamental concepts for addressing uncertainties in decision problems, and they can address more uncertainties compared to the existing structures of fuzzy sets; thus, their implementation was more substantial. The well-known sine trigonometric function maintains the periodicity and symmetry of the origin in nature and thus satisfies the expectations of the decision-maker over the multiple parameters. Taking this feature and the significances of the PFSs into consideration, the main objective of the article is to describe some reliable sine trigonometric laws (STLs) for PFSs. Associated with these laws, we develop new average and geometric aggregation operators to aggregate the picture fuzzy numbers. Also, we characterized the desirable properties of the proposed operators. Then, we presented a group decision-making strategy to address the multiple attribute group decision-making (MAGDM) problem using the developed aggregation operators and demonstrated this with a practical example. To show the superiority and the validity of the proposed aggregation operations, we compared them with the existing methods and concluded from the comparison and sensitivity analysis that our proposed technique is more effective and reliable.

1. Introduction

Multiple attribute group decision-making (MAGDM) method is one of the most relevant and evolving topics explaining how to choose the finest alternative with community of decision-makers (DMs) with some attributes. There are two relevant tasks in this system. The first is to define the context in which the values of the various parameters are effectively calculated, while the second is to summarize the information described. Traditionally, the information describing the objects is taken mostly to be deterministic or crisp in nature. With the increasing complexity of a system on a daily basis, however, it is difficult to aggregate the data, from the logbook, resources, and experts, in the crisp form. Therefore, [1] developed the core concept of fuzzy set (FS) and also [2] worked on it and further developed a new idea of intuitionistic fuzzy set (IFS). [3] developed the Pythagorean fuzzy sets (PyFSs), and [4] defined the idea of hesitant fuzzy sets, which are used by scholars to communicate the information clearly. In IFS, it is observed that each object has two membership grades, positive \( \hat{E} \) and negative \( \hat{Z} \), which satisfy the condition \( 0 \leq \hat{E} + \hat{Z} \leq 1 \), and, for all \( \hat{E}, \hat{Z} \) is lying in the closed interval 0 and 1. However, in the Pythagorean fuzzy sets, this constraint is relaxed from \( \hat{E} + \hat{Z} \leq 1 \) to \( \hat{E} + \hat{Z} \leq 1 \) for \( \hat{E}, \hat{Z} \in [0,1] \). Using this concept, many researchers have successfully addressed the above two critical tasks and discretion of the techniques under the different aspects. Verma and Sharma [5] proposed a new measure of inaccuracy with its application to multicriteria decision-making.
under intuitionistic fuzzy environment. Some of the basic results of IFSs and Pythagorean fuzzy sets are the operational laws [6, 7], some exponential operational laws [8], some distance or similarity measures [9, 10], and some information entropy [11]. Many researchers [12–17], under IFS, defined some basic aggregation operators (AOs), such as average and geometric, interactive, and Hamacher AOs. Meanwhile, for Pythagorean fuzzy sets, some basic operators are proposed by Peng and Yang [18]. To solve the MAGDM problems, Garg [19, 20] presented some basic concept of Einstein aggregation operators. Some extended aggregation operators are dependent on intuitionistic and Pythagorean fuzzy information, including the TOPSIS technique based on IF [21] and Pythagorean fuzzy set [22], partitioned Bonferroni mean [23], and Maclaurin symmetric mean [24, 25]. Apart from this, Yager et al. [26] intuitively developed the idea of q-rung orthopair fuzzy sets (q-ROFSs). Gao et al. [27] developed the basic idea of the continuities and differential of q-ROFSs. Peng et al. [28] presented the exponential and logarithm operational laws for q-ROFNs. Liu and Wang [29] developed weighted average and geometric aggregation operators for q-ROFNs.

Meanwhile, the ideas of IFSs and Pythagorean FSs are widely studied and implemented in various fields. But their ability to express the information is still limited. Thus, it was still difficult for the decision-makers (DMs) and their corresponding information to convey the information in such sets. To overcome this information, the notion of the picture fuzzy sets (PFs) was defined by Cuong and Kreinovich [30]. Thus, it was clearly noticed that the PFS is the extended form of the IFS to accommodate some more ambiguities. In picture fuzzy sets, each object was observed by defining three grades of the member named membership $\tilde{E}$, neutral $\tilde{R}$, and non-membership $\tilde{Z}$ with the constraint that $\tilde{E} + \tilde{R} + \tilde{Z} \leq 1$, for $\tilde{E}, \tilde{R}, \tilde{Z} \in [0, 1]$. The definition of the PFS will convey the opinions of experts like “yes”, “abstain”, “no,” and “refusal” while avoiding missing evaluation details and encouraging the reliability of the acquired data with the actual environment for decision-making. Although the concept of PFs is widely studied and applied in different fields and their extension focuses on the basic operational laws, which is the important aspect of the PFS as well as aggregation operators (AOs), which are an effective tool by the help of these AOs, we obtain raking of the alternatives by providing the comprehensive values to the alternatives. Wei [31] developed some operations of the PFS. Son [32] developed measuring analogousness in PFs. Apart from these, several other kinds of the AOs of the PFs have been developed such as logarithmic PF aggregation operators, which were presented by Khan et al. [33], Wang et al. [34] presented PF normalized projection based VIKOR method, and Wang et al. [35] developed PF Muirhead mean operators. Wei et al. [36] defined the idea of some q-ROF Maclaurin symmetric mean operators. Wang et al. [37] introduced a similarity measure of q-ROFSs. Wei et al. [38] developed bidirectional projection method for PFs. Ashraf et al. [39–41] developed the idea of different approaches to MAGDM problems, picture fuzzy linguistic sets and exponential Jensen PF divergence measure, respectively. Khan et al. [42] presented PF aggregation based on Einstein operation. Qiyas et al. [43] presented linguistic PF Dombi aggregation operators.

Among the above aspects, it is very clear that operational laws are a main role model for any aggregation process. In that direction, recently, Khan et al. [33] defined the new concept about logarithmic operation laws for PFs. Besides these mathematical logarithmic functions, another important feature is the sine trigonometry feature, which plays a main role during the fusion of the information. In this way, taking into consideration the advantages and usefulness of the sine trigonometric function, some new sine trigonometric operational laws need to be developed for PFs and their behavior needs to be studied. Consequently, the paper’s purpose is to develop some new operation laws for PFs and also introduce the MAGDM algorithm for managing the information for PFNs evaluation, as well as describing several more sophisticated operational laws for PFs in addition to a novel entropy to remove the weight of the attributes to prevent subjective and objective aspects. Some more generalized functional aggregation operators are presented with the help of the defined sine trigonometric operational laws (STOLs) for PFNs, and many basic relations between the developed AOs are discussed; also, a novel MAGDM technique depending on the developed operators to solve the group decision-making problems is presented. Finally, the proposed approach is compared with the existing methods. So, the goals and the motivations of this paper are as follows:

1. The paper presents some more advanced operational laws for PFNs by combining the features of the ST and PFNs.

2. A novel entropy is presented to extract the attributes’ weight for avoiding the influence of subjective and objective aspects.

3. Some more generalized functional AOs are presented with the help of the defined STOLs for PFNs. Also, the several fundamental relations between the proposed AOs are derived to show their significance.

4. A novel MAGDM method based on the proposed operators to solve the group decision-making problems is presented. The consistency of the proposed method is confirmed through these examples, and their evaluations are carried out in detail.

In Section 2 of the article, we can define some ideas related to PFNs. In Section 3, we define the new PF operation laws based on sine trigonometric functions and their properties. In Section 4, we present a series of AOs along with their required properties, based on sine trigonometric operational laws. Section 5 provides the basic connection between the developed AOs. In Section 6, using the new aggregation operators, we introduce a new MAGDM approach and give detailed steps. Examples are given in Section 7 to validate the new method and comparative analysis is carried out by the current method. Finally, the work is concluded in Section 8.
2. Preliminaries

Some fundamental ideas about picture fuzzy set (PFS) on the universal set $U$ are discussed in this portion.

$\mathcal{S}(\tilde{I}_1) > \mathcal{S}(\tilde{I}_2)$, then $\tilde{I}_1 > \tilde{I}_2$, and if the score function, that is,

$\mathcal{S}(\tilde{I}_1) = \mathcal{S}(\tilde{I}_2)$, and $\mathcal{H}(\tilde{I}_1) > \mathcal{H}(\tilde{I}_2)$, then $\tilde{I}_1 > \tilde{I}_2$; if $\mathcal{H}(\tilde{I}_1) = \mathcal{H}(\tilde{I}_2)$, then $\tilde{I}_1 = \tilde{I}_2$.

**Definition 1** (see [31]). Let $\tilde{U}$ be the nonempty fixed sets. Then, the set

$$\tilde{I} = \left( \tilde{u}, \tilde{E}_i (\tilde{u}), \tilde{R}_i (\tilde{u}), \tilde{Z}_i (\tilde{u}) \right)_{\tilde{u} \in \tilde{U}}$$

is said to be a picture fuzzy set (PFS), where $\tilde{E}_i (\tilde{u}), \tilde{R}_i (\tilde{u}), \tilde{Z}_i (\tilde{u}) \in [0, 1]$ are called the grade of membership, positive, neutral, and negative, of the elements $\tilde{u} \in \tilde{U}$ to the set $\tilde{I}$, respectively, where the following constraint has been fulfilled by $\tilde{E}(\tilde{u}), \tilde{R}(\tilde{u}), \tilde{I}(\tilde{u})$ for all $\tilde{u} \in \tilde{U}$:

$$0 \leq \tilde{E}(\tilde{u}) + \tilde{R}(\tilde{u}) + \tilde{Z}(\tilde{u}) \leq 1.$$ (2)

**Definition 2** (see [31]). Let three PFNs be $\tilde{I} = (\tilde{E}_i (\tilde{u}), \tilde{R}_i (\tilde{u}), \tilde{Z}_i (\tilde{u})) \tilde{I}_1 = (\tilde{E}_i (\tilde{u}), \tilde{R}_i (\tilde{u}), \tilde{Z}_i (\tilde{u}))$, and $\tilde{I}_2 = (\tilde{E}_i (\tilde{u}), \tilde{R}_i (\tilde{u}), \tilde{Z}_i (\tilde{u}))$. Also $\tilde{0} > 0$ is any scalar. Then,

$$\tilde{I} = \left[ \tilde{E}_i (\tilde{u}), \tilde{R}_i (\tilde{u}), \tilde{Z}_i (\tilde{u}) \right]$$

$\tilde{I} = \left[ \tilde{E}_i (\tilde{u}), \tilde{R}_i (\tilde{u}), \tilde{Z}_i (\tilde{u}) \right]$,

$$\tilde{I}_{1 \wedge} \tilde{I}_2 = \left[ \min \left( \tilde{E}_i (\tilde{u}), \tilde{E}_i (\tilde{u}) \right), \max \left( \tilde{R}_i (\tilde{u}), \tilde{R}_i (\tilde{u}) \right), \max \left( \tilde{Z}_i (\tilde{u}), \tilde{Z}_i (\tilde{u}) \right) \right],$$

$$\tilde{I}_{1 \lor} \tilde{I}_2 = \left[ \max \left( \tilde{E}_i (\tilde{u}), \tilde{E}_i (\tilde{u}) \right), \min \left( \tilde{R}_i (\tilde{u}), \tilde{R}_i (\tilde{u}) \right), \min \left( \tilde{Z}_i (\tilde{u}), \tilde{Z}_i (\tilde{u}) \right) \right],$$

$$\tilde{I}_{1 \oplus} \tilde{I}_2 = \left[ \tilde{E}_i (\tilde{u}) \oplus \tilde{E}_i (\tilde{u}), \tilde{R}_i (\tilde{u}) \oplus \tilde{R}_i (\tilde{u}), \tilde{Z}_i (\tilde{u}) \oplus \tilde{Z}_i (\tilde{u}) \right],$$

$$\tilde{I} \otimes \tilde{I}_2 = \left[ \tilde{E}_i (\tilde{u}) \otimes \tilde{E}_i (\tilde{u}), \tilde{R}_i (\tilde{u}) \otimes \tilde{R}_i (\tilde{u}), \tilde{Z}_i (\tilde{u}) \otimes \tilde{Z}_i (\tilde{u}) \right],$$

$$\tilde{I} = \left[ \tilde{E}_i (\tilde{u}), \tilde{R}_i (\tilde{u}), \tilde{Z}_i (\tilde{u}) \right]$$

**Definition 3** (see [44]). Let all the PFNs $\tilde{I} = (\tilde{E}_i (\tilde{u}), \tilde{R}_i (\tilde{u}), \tilde{Z}_i (\tilde{u}))$. The score and accuracy functions are then described as follows:

$$\mathcal{S} (\tilde{I}) = \tilde{E}_i (\tilde{u}) - \tilde{R}_i (\tilde{u}) - \tilde{Z}_i (\tilde{u}), \quad \mathcal{S} (\tilde{I}) \in [-1, 1],$$

$$\mathcal{H} (\tilde{I}) = \tilde{E}_i (\tilde{u}) + \tilde{R}_i (\tilde{u}) + \tilde{Z}_i (\tilde{u}), \quad \mathcal{H} (\tilde{I}) \in [0, 1].$$

**Definition 4** (see [44]). Let two PFNs be $\tilde{I}_1 = (\tilde{E}_i (\tilde{u}), \tilde{R}_i (\tilde{u}), \tilde{Z}_i (\tilde{u}))$ and $\tilde{I}_2 = (\tilde{E}_i (\tilde{u}), \tilde{R}_i (\tilde{u}), \tilde{Z}_i (\tilde{u}))$. Then, the rules for comparison can be defined as follows: if the score function, that is,

$$\mathcal{S} (\tilde{I}) = \tilde{E}_i (\tilde{u}) - \tilde{R}_i (\tilde{u}) - \tilde{Z}_i (\tilde{u}), \quad \mathcal{S} (\tilde{I}) \in [-1, 1],$$

$$\mathcal{H} (\tilde{I}) = \tilde{E}_i (\tilde{u}) + \tilde{R}_i (\tilde{u}) + \tilde{Z}_i (\tilde{u}), \quad \mathcal{H} (\tilde{I}) \in [0, 1].$$

3. New Sine Trigonometric Operational Laws (STOLs) for PFSs

We will define some operational laws for PFNs in this portion. First, the sine trigonometric PFSs are defined.

$$\sin \tilde{I} = \left( \sin \left( \frac{\pi}{2} \tilde{E}_i (\tilde{u}) \right), 2 \sin^2 \left( \frac{\pi}{4} \tilde{R}_i (\tilde{u}) \right), 2 \sin^2 \left( \frac{\pi}{4} \tilde{Z}_i (\tilde{u}) \right) \right), \quad 0 \leq \tilde{E}_i \leq 1.$$ (5)

From the above definition, it is clear that $\sin \tilde{I}$ is also a PFS and also satisfied the following conditions of the PFS as the membership, neutral, and nonmembership degrees of PFS are defined, respectively:

- $\sin \left( \frac{\pi}{2} \tilde{E}_i (\tilde{u}) \right): \tilde{U} \rightarrow [0, 1], \quad$ such that $0 \leq \sin \left( \frac{\pi}{2} \tilde{E}_i (\tilde{u}) \right) \leq 1,$
- $2 \sin^2 \left( \frac{\pi}{4} \tilde{R}_i (\tilde{u}) \right): \tilde{U} \rightarrow [0, 1], \quad$ such that $0 \leq 2 \sin^2 \left( \frac{\pi}{4} \tilde{R}_i (\tilde{u}) \right) \leq 1,$
- $2 \sin^2 \left( \frac{\pi}{4} \tilde{Z}_i (\tilde{u}) \right): \tilde{U} \rightarrow [0, 1], \quad$ such that $0 \leq 2 \sin^2 \left( \frac{\pi}{4} \tilde{Z}_i (\tilde{u}) \right) \leq 1.$ (6)

Therefore,
\[ \sin \tilde{I} = \left( \sin \left( \frac{\pi}{2} (E_1) \right) , 2 \sin^2 \left( \frac{\pi}{4} (R_1) \right) , 2 \sin^2 \left( \frac{\pi}{4} (Z_1) \right) \right) \]  \hspace{1cm} (7) 

is a PFS.

**Definition 6.** Let \( \tilde{I} = (\tilde{E}_1, \tilde{R}_1, \tilde{Z}_1) \) be a PFN, if
\[
\sin \tilde{I} = \left( \sin \left( \frac{\pi}{2} (E_1) \right) , 2 \sin^2 \left( \frac{\pi}{4} (R_1) \right) , 2 \sin^2 \left( \frac{\pi}{4} (Z_1) \right) \right), \quad 0 < \tilde{E}_1 \leq 1, 
\]  \hspace{1cm} (8)
is known as sine trigonometric (ST) operator and its value is known as sine trigonometric PFN.

**Definition 7.** Let the collection of PFNs be \( \tilde{I} = (\tilde{E}_1, \tilde{R}_1, \tilde{Z}_1) \), \( \tilde{I}_1 = (\tilde{E}_1, \tilde{R}_1, \tilde{Z}_1) \), and \( \tilde{I}_2 = (\tilde{E}_2, \tilde{R}_2, \tilde{Z}_2) \). Then, we define the following operational laws where \( \tilde{\omega} > 0 \) is any scalar:

\[
\sin \tilde{I}_1 \oplus \sin \tilde{I}_2 = \left[ 1 - \left( 1 - \sin \left( \frac{\pi}{2} \tilde{E}_1 \right) \right) \left( 1 - \sin \left( \frac{\pi}{2} \tilde{E}_2 \right) \right), \left( 2 \sin^2 \left( \frac{\pi}{4} \tilde{R}_1 \right) \right), \left( 2 \sin^2 \left( \frac{\pi}{4} \tilde{R}_2 \right) \right), \left( 2 \sin^2 \left( \frac{\pi}{4} \tilde{Z}_1 \right) \right), \left( 2 \sin^2 \left( \frac{\pi}{4} \tilde{Z}_2 \right) \right) \right], 
\]

\[
\sin \tilde{I}_1 \otimes \sin \tilde{I}_2 = \left[ \left( \sin \left( \frac{\pi}{2} \tilde{E}_1 \right) \right), \left( \sin \left( \frac{\pi}{2} \tilde{E}_2 \right) \right), \left( 1 - \left( 2 \sin^2 \left( \frac{\pi}{4} \tilde{R}_1 \right) \right), \left( 2 \sin^2 \left( \frac{\pi}{4} \tilde{R}_2 \right) \right), \left( 1 - \left( 2 \sin^2 \left( \frac{\pi}{4} \tilde{Z}_1 \right) \right), \left( 2 \sin^2 \left( \frac{\pi}{4} \tilde{Z}_2 \right) \right) \right], 
\]

\[
\tilde{\omega} \cdot \sin \tilde{I} = \left[ 1 - \left( 1 - \sin \left( \frac{\pi}{2} \tilde{E} \right) \right) \tilde{\omega}, \left( 2 \sin^2 \left( \frac{\pi}{4} \tilde{R} \right) \right) \tilde{\omega}, \left( 2 \sin^2 \left( \frac{\pi}{4} \tilde{Z} \right) \right) \tilde{\omega} \right], 
\]

\[
\left( \sin \tilde{I}_1 \oplus \sin \tilde{I}_2 \right) \oplus \sin \tilde{I}_3 = \sin \tilde{I}_1 \oplus \left( \sin \tilde{I}_2 \oplus \sin \tilde{I}_3 \right), 
\]

\[
\left( \sin \tilde{I}_1 \oplus \sin \tilde{I}_2 \right) \otimes \sin \tilde{I}_3 = \sin \tilde{I}_1 \otimes \left( \sin \tilde{I}_2 \oplus \sin \tilde{I}_3 \right). 
\]

**Theorem 1.** Let a collection of PFNs be \( \tilde{I}_j = (\tilde{E}_j, \tilde{R}_j, \tilde{Z}_j) \), where \( j = 1, 2, 3 \). Then,

\[
\sin \tilde{I}_1 \oplus \sin \tilde{I}_2 = \sin \tilde{I}_1 \oplus \sin \tilde{I}_1, 
\]

\[
\sin \tilde{I}_1 \otimes \sin \tilde{I}_2 = \sin \tilde{I}_2 \otimes \sin \tilde{I}_1, 
\]

\[
\text{Proof.} \text{ Here, we solve the first two parts using the STOLs (sine trigonometric operation laws) defined in Definition 7, and the proof of the other two parts is similar to the first parts, so we omit it here; we get}
\]

\[
\sin \tilde{I}_1 \oplus \sin \tilde{I}_2 = \left[ \sin \left( \frac{\pi}{2} \tilde{E}_1 \right), \left( 2 \sin^2 \left( \frac{\pi}{4} \tilde{R}_1 \right) \right), \left( 2 \sin^2 \left( \frac{\pi}{4} \tilde{Z}_1 \right) \right) \right] \oplus \left[ \sin \left( \frac{\pi}{2} \tilde{E}_2 \right), \left( 2 \sin^2 \left( \frac{\pi}{4} \tilde{R}_2 \right) \right), \left( 2 \sin^2 \left( \frac{\pi}{4} \tilde{Z}_2 \right) \right) \right], 
\]

\[
= \left[ 1 - \left( 1 - \sin \left( \frac{\pi}{2} \tilde{E}_1 \right) \right), \left( 1 - \sin \left( \frac{\pi}{2} \tilde{E}_2 \right) \right), \left( 2 \sin^2 \left( \frac{\pi}{4} \tilde{R}_1 \right) \right), \left( 2 \sin^2 \left( \frac{\pi}{4} \tilde{R}_2 \right) \right), \left( 2 \sin^2 \left( \frac{\pi}{4} \tilde{Z}_1 \right) \right), \left( 2 \sin^2 \left( \frac{\pi}{4} \tilde{Z}_2 \right) \right) \right], 
\]

\[
= \left[ 1 - \left( 1 - \sin \left( \frac{\pi}{2} \tilde{E}_2 \right) \right), \left( 1 - \sin \left( \frac{\pi}{2} \tilde{E}_1 \right) \right), \left( 2 \sin^2 \left( \frac{\pi}{4} \tilde{R}_1 \right) \right), \left( 2 \sin^2 \left( \frac{\pi}{4} \tilde{R}_2 \right) \right), \left( 2 \sin^2 \left( \frac{\pi}{4} \tilde{Z}_1 \right) \right), \left( 2 \sin^2 \left( \frac{\pi}{4} \tilde{Z}_2 \right) \right) \right], 
\]

\[
= \left[ \sin \left( \frac{\pi}{2} \tilde{E}_2 \right), \left( 2 \sin^2 \left( \frac{\pi}{4} \tilde{R}_2 \right) \right), \left( 2 \sin^2 \left( \frac{\pi}{4} \tilde{Z}_2 \right) \right) \right] \oplus \left[ \sin \left( \frac{\pi}{2} \tilde{E}_1 \right), \left( 2 \sin^2 \left( \frac{\pi}{4} \tilde{R}_1 \right) \right), \left( 2 \sin^2 \left( \frac{\pi}{4} \tilde{Z}_1 \right) \right) \right], 
\]

\[
= \sin \tilde{I}_2 \oplus \sin \tilde{I}_1. 
\]
Therefore, from the above solution,
\[
\sin \tilde{I}_1 \circ \sin \tilde{I}_2 = \sin \tilde{I}_2 \circ \sin \tilde{I}_1.
\]
\hspace{1cm} (13)

**Theorem 2.** Let a collection of PFNs be \( \tilde{I} = (\tilde{E}, \tilde{R}, \tilde{Z}) \) and \( \tilde{I}_1 = (\tilde{E}_1, \tilde{R}_1, \tilde{Z}_1), \) where \( \tilde{I} = 1, 2. \) Also let \( \bar{N}, \bar{N}_1, \bar{N}_2 > 0 \) be the real number; then
\[
\bar{N} \cdot \left( \sin \tilde{I}_1 \circ \sin \tilde{I}_2 \right) = \bar{N} \cdot \sin \tilde{I}_1 \circ \bar{N} \cdot \sin \tilde{I}_2,
\]
\[
\left( \sin \tilde{I}_1 \circ \sin \tilde{I}_2 \right) = \left( \sin \tilde{I}_1 \right) \circ \left( \sin \tilde{I}_2 \right),
\]
\[
\bar{N}_1 \cdot \sin \tilde{I}_1 \circ \bar{N}_2 \cdot \sin \tilde{I}_2 = \bar{N}_1 \cdot \sin \tilde{I}_1 \circ \bar{N}_2 \cdot \sin \tilde{I}_2,
\]
\[
\left( \sin \tilde{I}_1 \right) \circ \left( \sin \tilde{I}_2 \right) = \left( \sin \tilde{I}_1 \right) \circ \left( \sin \tilde{I}_2 \right),
\]
\hspace{1cm} (14)

**Proof.** Here, we will prove the first part of the above theorem only by using the STOLs defined in Definition 7, while the rest can be proven similarly. But, \[
\sin \tilde{I}_1 = \left( \sin \left( \frac{\pi}{2} E_1 \right), 2 \sin^2 \left( \frac{\pi}{4} R_1 \right), 2 \sin^2 \left( \frac{\pi}{4} Z_1 \right) \right),
\]
\[
\sin \tilde{I}_2 = \left( \sin \left( \frac{\pi}{2} E_2 \right), 2 \sin^2 \left( \frac{\pi}{4} R_2 \right), 2 \sin^2 \left( \frac{\pi}{4} Z_2 \right) \right),
\]
\hspace{1cm} (15)

and, by using the STOLs, we have
\[
\sin \tilde{I}_1 \circ \sin \tilde{I}_2 = \sin \tilde{I}_2 \circ \sin \tilde{I}_1.
\]
\hspace{1cm} (16)
Corollary 1. Let a collection of two PFNs be \( \tilde{I}_1 = (E_1, R_1, Z_1) \) and \( \tilde{I}_2 = (E_2, R_2, Z_2) \), where \( J = 1, 2 \), such that \( E_1 \geq E_2, R_1 \leq R_2, \) and \( Z_1 \leq Z_2 \). Then show that \( \sin \tilde{I}_1 \geq \sin \tilde{I}_2 \).

Proof. Let \( \tilde{I}_1 = (E_1, R_1, Z_1) \) and \( \tilde{I}_2 = (E_2, R_2, Z_2) \) be the PFNs with condition \( E_1 \geq E_2 \), since in the closed interval \([0, \pi/2] \) sine is an increasing function; thus, we have \( \sin((\pi/2)E_1) \geq \sin((\pi/2)E_2) \). But also, given that \( R_1 \leq R_2 \) which implies that \( (1 - R_1) \geq (1 - R_2) \), since in closed interval \([0, \pi/2] \) sine is an increasing function, we have \( \sin(((\pi/2)(1 - R_1)) \geq \sin(((\pi/2)(1 - R_2)) \), which implies that \( 2 \sin^2((\pi/4)R_1) \leq 2 \sin^2((\pi/4)R_2) \). Similarly, \( Z_1 \leq Z_1 \), which implies that \( (1 - Z_1) \geq (1 - Z_2) \), since in closed interval \([0, \pi/2] \) sine is an increasing function; thus, we have \( 2 \sin^2((\pi/4)Z_1) \leq 2 \sin^2((\pi/4)Z_2) \); hence, we get

\[
\geq \left( \sin\left(\frac{\pi}{2}E_1\right), 2 \sin^2\left(\frac{\pi}{4}R_1\right), 2 \sin^2\left(\frac{\pi}{4}Z_1\right) \right) \tag{18}
\]

and, therefore, we get the required result by using Definition 7:

\[
\sin \tilde{I}_1 \geq \sin \tilde{I}_2. \tag{19}
\]

4. Sine Trigonometric Aggregation Operators

We have described a number of aggregation operators in this portion of the article on the basis of sine trigonometric operational laws (STOLs).

Definition 8. Let a collection of PFNs be \( \tilde{I}_J = (E_J, R_J, Z_J) \), where \( J = 1, \ldots, n \). Then, the mapping \( ST - \text{PFWA} : \Psi^n \rightarrow \Psi \) is known as the sine trigonometric picture fuzzy average (ST - PFWA) operator, if

\[
ST - \text{PFWA}\left( \tilde{I}_1, \ldots, \tilde{I}_n \right) = \tilde{\omega}_1 \cdot \sin \tilde{I}_1 \oplus \cdots \oplus \tilde{\omega}_n \cdot \sin \tilde{I}_n, \tag{20}
\]

where \( \tilde{\omega}_j \) are the weighted vectors of \( \tilde{I}_j (J = 1, \ldots, n) \) which fulfilled the criteria \( \tilde{\omega}_j > 0 \) and \( \sum_{j=1}^{n} \tilde{\omega}_j = 1 \).

Theorem 3. Let a collection of PFNs be \( \tilde{I}_J = (E_J, R_J, Z_J) \), where \( J = 1, \ldots, n \). Then, the aggregated value is also a PFN by utilizing the ST - PFWA operator and is given by

\[
\text{ST - PFWA}\left( \tilde{I}_1, \ldots, \tilde{I}_n \right) = \left[ 1 - \prod_{j=1}^{n} \left( 1 - \sin\left(\frac{\pi}{2}E_j\right) \right) \right]^{\tilde{\omega}_1} \prod_{j=1}^{n} \left( 2 \sin^2\left(\frac{\pi}{4}R_j\right) \right)^{\tilde{\omega}_j}, \tag{21}
\]

Proof. By using the process of mathematical induction, we prove the said theorem. Because \( \tilde{I}_J = (E_J, R_J, Z_J) \) is a PFN for each \( J \), which implies that \( E_J + R_J + Z_J \in [0, 1] \) and also \( E_J + R_J + Z_J \leq 1 \), the following mathematical induction steps were then performed.

Step 1. Now, for \( n = 2 \), we get \( \text{ST - PFWA}(\tilde{I}_1, \tilde{I}_2) = \tilde{\omega}_1 \cdot \sin \tilde{I}_1 \oplus \tilde{\omega}_2 \cdot \sin \tilde{I}_2 \), where

\[
\tilde{\omega}_1 \cdot \sin \tilde{I}_1 = \left( 1 - (1 - \sin\left(\frac{\pi}{2}E_1\right) \right)^{\tilde{\omega}_1} \left( 2 \sin^2\left(\frac{\pi}{4}R_1\right) \right)^{\tilde{\omega}_1}, \tag{22}
\]

\[
\tilde{\omega}_2 \cdot \sin \tilde{I}_2 = \left( 1 - (1 - \sin\left(\frac{\pi}{2}E_2\right) \right)^{\tilde{\omega}_2} \left( 2 \sin^2\left(\frac{\pi}{4}R_2\right) \right)^{\tilde{\omega}_2}, \tag{23}
\]

and hence, by using the definition \([7]\), we get

\[
\tilde{\omega}_1 \cdot \sin \tilde{I}_1 \oplus \tilde{\omega}_2 \cdot \sin \tilde{I}_2 = \left[ 1 - \prod_{j=1}^{2} \left( 1 - \sin\left(\frac{\pi}{2}E_j\right) \right) \right]^{\tilde{\omega}_1} \prod_{j=1}^{2} \left( 2 \sin^2\left(\frac{\pi}{4}R_j\right) \right)^{\tilde{\omega}_j}, \tag{24}
\]
Step 2. Now say it is true for \( n = k \).

\[
\text{ST} - \text{PFWA}(\tilde{I}_1, \ldots, \tilde{I}_n) = \left[ 1 - \prod_{j=1}^{k} \left( 1 - \sin \left( \frac{\pi \tilde{E}_j}{2} \right) \right) \right] \hat{\omega}_1 \prod_{j=1}^{k} \left( 2 \sin^2 \left( \frac{\pi \tilde{R}_j}{4} \right) \right) \hat{\omega}_1 \prod_{j=1}^{k} \left( 2 \sin^2 \left( \frac{\pi \tilde{Z}_j}{4} \right) \right),
\]

(24)

Step 3. Now, we prove that this is true for \( n = k + 1 \):

\[
\text{ST} - \text{PFWA}(\tilde{I}_1, \ldots, \tilde{I}_{k+1}) = \hat{\omega}_1 \sin \tilde{I}_1 \Phi \cdots \Phi \hat{\omega}_n \sin \tilde{I}_n \Phi \hat{\omega}_{k+1} \sin \tilde{I}_{k+1}
\]

\[
= \left[ 1 - \prod_{j=1}^{k+1} \left( 1 - \sin \left( \frac{\pi \tilde{E}_j}{2} \right) \right) \right] \hat{\omega}_{k+1} \prod_{j=1}^{k+1} \left( 2 \sin^2 \left( \frac{\pi \tilde{R}_j}{4} \right) \right) \hat{\omega}_{k+1} \prod_{j=1}^{k+1} \left( 2 \sin^2 \left( \frac{\pi \tilde{Z}_j}{4} \right) \right),
\]

(25)

and, again, by using Definition 7, we obtain

\[
\text{ST} - \text{PFWA}(\tilde{I}_1, \ldots, \tilde{I}_{k+1}) = \left[ 1 - \prod_{j=1}^{k+1} \left( 1 - \sin \left( \frac{\pi \tilde{E}_j}{2} \right) \right) \right] \hat{\omega}_{k+1} \prod_{j=1}^{k+1} \left( 2 \sin^2 \left( \frac{\pi \tilde{R}_j}{4} \right) \right) \hat{\omega}_{k+1} \prod_{j=1}^{k+1} \left( 2 \sin^2 \left( \frac{\pi \tilde{Z}_j}{4} \right) \right),
\]

(26)

Hence, \( n = k + 1 \) holds. Then, the statement is valid for all \( n \) through the principal of mathematical induction. \( \square \)

Property 1. If all collection of PFNs \( \tilde{I}_j = \tilde{I} \), where \( \tilde{I} \) is another PFN \((\tilde{I} = 1, \ldots, n)\), then

\[
\text{Proof. Let} \, \tilde{I} = (\tilde{E}, \tilde{R}, \tilde{Z}) \, \text{be a PFN, such that} \, \tilde{I}_j = \tilde{I} \, \text{. Then, by using Theorem 4, we get}
\]

\[
\text{ST} - \text{PFWA}(\tilde{I}_1, \ldots, \tilde{I}_n) = \sin \tilde{I}.
\]

(27)

\[
\text{ST} - \text{PFWA}(\tilde{I}_1, \ldots, \tilde{I}_n) = \left[ 1 - \prod_{j=1}^{n} \left( 1 - \sin \left( \frac{\pi \tilde{E}_j}{2} \right) \right) \right] \hat{\omega}_n \prod_{j=1}^{n} \left( 2 \sin^2 \left( \frac{\pi \tilde{R}_j}{4} \right) \right) \hat{\omega}_n \prod_{j=1}^{n} \left( 2 \sin^2 \left( \frac{\pi \tilde{Z}_j}{4} \right) \right),
\]

(28)

\[
\text{ST} - \text{PFWA}(\tilde{I}_1, \ldots, \tilde{I}_n) = \sum_{j=1}^{n} n \hat{\omega}_j \sum_{j=1}^{n} n \hat{\omega}_j \sum_{j=1}^{n} n \hat{\omega}_j,
\]

(28)

\[
\text{ST} - \text{PFWA}(\tilde{I}_1, \ldots, \tilde{I}_n) = \left[ 1 - \left( 1 - \sin \left( \frac{\pi \tilde{E}}{2} \right) \right) \right] \left( 2 \sin^2 \left( \frac{\pi \tilde{R}}{4} \right) \right) \left( 2 \sin^2 \left( \frac{\pi \tilde{Z}}{4} \right) \right).
\]

(28)
Property 2. If $\tilde{I} = (\tilde{E}_1, \tilde{R}_1, \tilde{Z}_1)$, where we let $\tilde{j} = 1, \ldots, n$, $\tilde{I}^- = (\min_{\tilde{j}}\{\tilde{E}_{\tilde{j}}\}, \max_{\tilde{j}}\{\tilde{R}_{\tilde{j}}\}, \max_{\tilde{j}}\{\tilde{Z}_{\tilde{j}}\}$, and $\tilde{I}^+ = (\max_{\tilde{j}}\{\tilde{E}_{\tilde{j}}\}, \min_{\tilde{j}}\{\tilde{R}_{\tilde{j}}\}, \min_{\tilde{j}}\{\min_{\tilde{j}}\{\tilde{Z}_{\tilde{j}}\}\}$ be PFNs, then

$$\sin \tilde{I}^- \leq \text{ST – PFWA}(\tilde{I}_1, \tilde{I}_2, \ldots, \tilde{I}_n) \leq \sin \tilde{I}^+. \quad (29)$$

Proof. Since, for any $\tilde{j}$, $\min_{\tilde{j}}\{\tilde{E}_{\tilde{j}}\} \leq \tilde{E}_j \leq \max_{\tilde{j}}\{\tilde{E}_{\tilde{j}}\}$, $\min_{\tilde{j}}\{\tilde{R}_{\tilde{j}}\} \leq \tilde{R}_j \leq \max_{\tilde{j}}\{\tilde{R}_{\tilde{j}}\}$, $\min_{\tilde{j}}\{\min_{\tilde{j}}\{\tilde{Z}_{\tilde{j}}\}\} \leq \tilde{Z}_j \leq \max_{\tilde{j}}\{\tilde{Z}_{\tilde{j}}\}$, this implies that $\tilde{I}^- \leq \tilde{I}_j \leq \tilde{I}^+$.

Assume that $\text{ST} – \text{PFWA} (\tilde{I}_1, \tilde{I}_2, \ldots, \tilde{I}_n) = \sin \tilde{I} = (\tilde{E}_1, \tilde{R}_1, \tilde{Z}_1)$, $\sin \tilde{I}^+ = (\tilde{E}_1, \tilde{R}_1, \tilde{Z}_1)$, and $\sin \tilde{I}^- = (\tilde{E}_1, \tilde{R}_1, \tilde{Z}_1)$. Then, by the monotonicity of the sine trigonometric function, we have

$$\tilde{E}_j = 1 - \frac{1}{n} \left(1 - \sin \left(\frac{\pi}{2} \min_{\tilde{j}} \{\tilde{E}_{\tilde{j}}\} \right)\right) \geq 1 - \prod_{j=1}^{n} \left(1 - \sin \left(\frac{\pi}{2} \min_{\tilde{j}} \{\tilde{E}_{\tilde{j}}\} \right)\right)^{\frac{1}{\tilde{j}}}$$

$$= 1 - \left(1 - \sin \left(\frac{\pi}{2} \min_{\tilde{j}} \{\tilde{E}_{\tilde{j}}\} \right)\right)^{\sum_{j=1}^{n} \frac{1}{\tilde{j}}}$$

$$= \sin \left(\frac{\pi}{2} \min_{\tilde{j}} \{\tilde{E}_{\tilde{j}}\} \right)$$

$$= \tilde{E}_j^+,$$

$$\tilde{R}_j = \prod_{j=1}^{n} \left(2 \sin^2 \left(\frac{\pi}{4} \min_{\tilde{j}} \{\tilde{R}_{\tilde{j}}\} \right)\right) \geq \prod_{j=1}^{n} \left(2 \sin^2 \left(\frac{\pi}{4} \min_{\tilde{j}} \{\tilde{R}_{\tilde{j}}\} \right)\right)^{\frac{1}{\tilde{j}}}$$

$$= \frac{1}{2} \sin \left(\frac{\pi}{4} \min_{\tilde{j}} \{\tilde{R}_{\tilde{j}}\} \right)^{\sum_{j=1}^{n} \frac{1}{\tilde{j}}}$$

$$= \tilde{R}_j^-,$$

$$\tilde{Z}_j = \prod_{j=1}^{n} \left(2 \sin^2 \left(\frac{\pi}{4} \min_{\tilde{j}} \{\tilde{Z}_{\tilde{j}}\} \right)\right) \geq \prod_{j=1}^{n} \left(2 \sin^2 \left(\frac{\pi}{4} \min_{\tilde{j}} \{\tilde{Z}_{\tilde{j}}\} \right)\right)^{\frac{1}{\tilde{j}}}$$

$$= \frac{1}{2} \sin \left(\frac{\pi}{4} \min_{\tilde{j}} \{\tilde{Z}_{\tilde{j}}\} \right)^{\sum_{j=1}^{n} \frac{1}{\tilde{j}}}$$

$$= \tilde{Z}_j^-.$$
and also

\[
\tilde{E}_j = 1 - \prod_{j=1}^n \left( 1 - \sin \left( \frac{\pi}{2} \hat{E}_j \right) \right) \phi_i \geq 1 - \prod_{j=1}^n \left( 1 - \sin \left( \frac{\pi}{2} \max \left\{ \tilde{E}_j \right\} \right) \right) \phi_j
\]

\[
= 1 - \left( 1 - \sin \left( \frac{\pi}{2} \max \left\{ \tilde{E}_j \right\} \right) \right) \sum_{j=1}^n \tilde{E}_j
\]

\[
= \sin \left( \frac{\pi}{2} \max \left\{ \tilde{E}_j \right\} \right)
\]

\[
= \tilde{E}_j^*,
\]

\[
\tilde{R}_j = \prod_{j=1}^n \left( 2 \sin^2 \left( \frac{\pi}{4} \hat{R}_j \right) \right) \sum_{j=1}^n \tilde{R}_j
\]

\[
= \left( 2 \sin^2 \left( \frac{\pi}{4} \max \left\{ \tilde{R}_j \right\} \right) \right) \sum_{j=1}^n \tilde{R}_j
\]

\[
= \tilde{R}_j^*,
\]

\[
\tilde{Z}_j = \prod_{j=1}^n \left( 2 \sin^2 \left( \frac{\pi}{4} \hat{Z}_j \right) \right) \sum_{j=1}^n \tilde{Z}_j
\]

\[
= \left( 2 \sin^2 \left( \frac{\pi}{4} \max \left\{ \tilde{Z}_j \right\} \right) \right) \sum_{j=1}^n \tilde{Z}_j
\]

\[
= \tilde{Z}_j^*.
\]

Based on score function in Definition 3, we get

\[
S(\tilde{T}) = E_1 - R_1 - Z_1 \leq E_1 - R_1 - Z_1 = S(\tilde{T}^+),
\]

\[
S(\tilde{T}) = E_1 - R_1 - Z_1 \geq E_1 - R_1 - Z_1 = S(\tilde{T}^+).
\]

(32)

Hence, \(S(\tilde{T}^-) \leq S(\tilde{T}) \leq S(\tilde{T}^+)\). Now, we explain three cases:

If \(S(\tilde{T}^-) \leq S(\tilde{T}) \leq S(\tilde{T}^+)\), then the result holds.

If \(S(\tilde{T}^-) = S(\tilde{T})\), then \(E_1 - R_1 - Z_1 = E_1 - R_1 - Z_1\), which implies that \(E_1 = E_1, R_1 = R_1\), and \(Z_1 = Z_1\) and \(H(\tilde{T}^-) = H(\tilde{T})\).

If \(S(\tilde{T}^-) = S(\tilde{T})\), then \(E_1 - R_1 - Z_1 = E_1 - R_1 - Z_1\), which implies that \(E_1 = E_1, R_1 = R_1\), and \(Z_1 = Z_1\) and \(H(\tilde{T}^-) = H(\tilde{T})\); therefore, by combining all these cases, we get

\[
\sin(\tilde{T}^-) \leq ST - \text{PFWA}(\tilde{T}_1, \tilde{T}_2, \ldots, \tilde{T}_n) \leq \sin(\tilde{T}^+).
\]

\\(\square\\)

Property 3. Let the collection of PFNs be \(\tilde{T}_j = (E_j, R_j, Z_j)\) and \(\tilde{T}_j^* = (E_j^*, R_j^*, Z_j^*)\), where \(j = 1, \ldots, n\). If \(E_j \leq E_j^*, R_j \geq R_j^*, Z_j \geq Z_j^*\), then

\[
ST - \text{PFWA}(\tilde{T}_1, \tilde{T}_2, \ldots, \tilde{T}_n) \leq ST - \text{PFWA}(\tilde{T}_1^*, \tilde{T}_2^*, \ldots, \tilde{T}_n^*).
\]

(34)

Proof. It follows from the above, so we omit it here. \(\square\\)

Definition 9. A sine trigonometric PF ordered weighted average (ST - PFOWA) operator is a mapping ST - PFOWA: \(\Psi^n \rightarrow \Psi^*\) such that weighted vector \(\vec{\tilde{o}} = (\tilde{o}_1, \tilde{o}_2, \ldots, \tilde{o}_n)^T\), which fulfilled the criteria of \(\tilde{o}_j > 0\) and \(\sum_{j=1}^n \tilde{o}_j = 1\).
\[ ST - PFOWA = \tilde{\alpha}_1 \cdot \sin \tilde{I}_{\tilde{\sigma}(1)} \oplus \tilde{\alpha}_2 \cdot \sin \tilde{I}_{\tilde{\sigma}(2)} + \cdots + \tilde{\alpha}_n \cdot \sin \tilde{I}_{\tilde{\sigma}(n)}, \]

where \((1, \ldots, n)\) is the permutation of \(\tilde{\sigma}\), such that \(I_{\tilde{\sigma}(j-1)} \geq I_{\tilde{\sigma}(j)}\) for any \(\tilde{j}\).

\[ ST - PFHA(\tilde{I}_1, \ldots, \tilde{I}_n) = \left[ 1 - \prod_{j=1}^{n} \left( 1 - \sin \left( \frac{\pi}{2} \frac{\tilde{E}_{\tilde{\sigma}(j)}}{\tilde{I}_{\tilde{\sigma}(j)}} \right) \right) \right]^{\tilde{\phi}_1} \cdot \prod_{j=1}^{n} \left( 2 \sin \left( \frac{\pi}{4} \frac{\tilde{R}_{\tilde{\sigma}(j)}}{\tilde{I}_{\tilde{\sigma}(j)}} \right) \right)^{\tilde{\phi}_j}, \]

\[ ST - PWFG(\tilde{I}_1, \ldots, \tilde{I}_n) = \left[ \prod_{j=1}^{n} \left( \sin \left( \frac{\pi}{2} \frac{\tilde{E}_{\tilde{\sigma}(j)}}{\tilde{I}_{\tilde{\sigma}(j)}} \right) \right) \right]^{\tilde{\phi}_1} \cdot \prod_{j=1}^{n} \left( 2 \sin \left( \frac{\pi}{4} \frac{\tilde{Z}_{\tilde{\sigma}(j)}}{\tilde{I}_{\tilde{\sigma}(j)}} \right) \right)^{\tilde{\phi}_j}. \]

**Theorem 4.** Let a collection of PFNs be \(\tilde{I}_j = (\tilde{E}_j, \tilde{R}_j, \tilde{Z}_j)\), where \(\tilde{j} = 1, \ldots, n\). Then, by utilizing the operator, that is, \(ST - PFOWA\), the aggregated value is also a PFN and is given by

\[ \[ ] \]

**Theorem 5.** Let a collection of PFNs be \(\tilde{I}_j = (\tilde{E}_j, \tilde{R}_j, \tilde{Z}_j)\), where \(\tilde{j} = 1, \ldots, n\). Then, the aggregated value is also a PFN by utilizing the operator \(ST - PWFG\) and is given by

\[ \[ ] \]

**Definition 10.** A sine trigonometric PF hybrid average (ST - PFHA) operator is a mapping \(ST - PFHA: \Psi^n \rightarrow \Psi\) such that the associated vectors \(\xi = (\xi_1, \xi_2, \ldots, \xi_n)^T\) which fulfilled the criteria of \(\xi_{\tilde{j}} > 0\) and \(\sum_{j=1}^{n} \xi_{\tilde{j}} = 1\).

**Definition 11.** Let a collection of PFNs be \(\tilde{I}_j = (\tilde{E}_j, \tilde{R}_j, \tilde{Z}_j)\), where \(\tilde{j} = 1, \ldots, n\). Then the mapping \(ST - PFWG: \Psi^n \rightarrow \Psi\) is known as the sine trigonometric picture fuzzy weighted geometric (ST - PFWG) operator, if

\[ \[ ] \]

**Definition 12.** A ST - PFWWG is a mapping from \(\Psi^n\) to \(\Psi\) such that the weighted vectors \(\tilde{\alpha} = (\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n)^T\) which fulfilled the criteria of \(\tilde{\alpha}_{\tilde{j}} > 0\) and \(\sum_{j=1}^{n} \tilde{\alpha}_{\tilde{j}} = 1\).
Theorem 7. Let a family of PFNs be \( \tilde{I}_j = (\tilde{E}_j, \tilde{R}_j, \tilde{Z}_j) \), where \( j = 1, \ldots, n \). Then, the aggregated value is also a PFN by using the ST – PFOWG operator and is given by

\[
\text{ST – PFOWG}(\tilde{I}_1, \ldots, \tilde{I}_n) = \left[ \prod_{j=1}^{n} \left( \sin \left( \frac{\pi}{4} \tilde{E}_{\sigma(j)} \right) \right)^{\tilde{\omega}} \right]^{-1} \left[ \prod_{j=1}^{n} \left( 2 \sin^{2} \left( \frac{\pi}{3} \tilde{R}_{\sigma(j)} \right) \right)^{\tilde{\omega}} \right]^{-1} \left[ \prod_{j=1}^{n} \left( 2 \sin^{2} \left( \frac{\pi}{3} \tilde{Z}_{\sigma(j)} \right) \right)^{\tilde{\omega}} \right]^{-1}. \tag{42}
\]

Proof. The proof is similar to that of Theorem 4.

Definition 13. A sine trigonometric picture fuzzy hybrid geometric (ST – PFHG) operator is a mapping ST – PFHG: \( \Psi_n \rightarrow \Psi_1 \), such that the associated vectors are \( \tilde{\xi} = (\tilde{\xi}_1, \tilde{\xi}_2, \ldots, \tilde{\xi}_n)^T \), which fulfilled the conditions of \( \sum_{j=1}^{n} \tilde{\xi}_j = 1 \).

\[
\text{ST – PFHG}(\tilde{I}_1, \ldots, \tilde{I}_n) = \left[ \prod_{j=1}^{n} \left( \sin \tilde{I}_{\sigma(j)} \right)^{\xi_j} \right]^{-1} \left[ \prod_{j=1}^{n} \left( 2 \sin^{2} \tilde{I}_{\sigma(j)} \right)^{\xi_j} \right]^{-1} \left[ \prod_{j=1}^{n} \left( 2 \sin^{2} \tilde{I}_{\sigma(j)} \right)^{\xi_j} \right]^{-1}. \tag{43}
\]

Proof. The proof is the same as that of Theorem 4, so it is omitted here.

Similar to ST – PFWA operator, ST – PFOWA, ST – PFHA, ST – PFWG, ST – PFOWG, and ST – PFHG operators satisfy some properties such as blondeness and monotonicity.

5. Fundamental Properties of the Proposed Aggregation Operators

In this section of the paper, we discuss many relations between the proposed aggregation operators and also discuss their fundamental properties.

\[
\sin \tilde{I}_1 \oplus \sin \tilde{I}_2 = \left[ 1 - \left( 1 - \sin \left( \frac{\pi}{2} \tilde{E}_1 \right) \right) \left( 1 - \sin \left( \frac{\pi}{2} \tilde{E}_2 \right) \right) \left( 2 \sin^{2} \left( \frac{\pi}{3} \tilde{R}_1 \right) \right) \left( 2 \sin^{2} \left( \frac{\pi}{3} \tilde{Z}_1 \right) \right) \right]^{-1} \left[ 1 - \left( 1 - \sin \left( \frac{\pi}{2} \tilde{E}_2 \right) \right) \left( 1 - \sin \left( \frac{\pi}{2} \tilde{E}_1 \right) \right) \left( 2 \sin^{2} \left( \frac{\pi}{3} \tilde{R}_2 \right) \right) \left( 2 \sin^{2} \left( \frac{\pi}{3} \tilde{Z}_2 \right) \right) \right]^{-1} \left[ 1 - \left( 1 - \sin \left( \frac{\pi}{2} \tilde{E}_1 \right) \right) \left( 1 - \sin \left( \frac{\pi}{2} \tilde{E}_2 \right) \right) \left( 2 \sin^{2} \left( \frac{\pi}{3} \tilde{R}_1 \right) \right) \left( 2 \sin^{2} \left( \frac{\pi}{3} \tilde{Z}_1 \right) \right) \right]^{-1} \left[ 1 - \left( 1 - \sin \left( \frac{\pi}{2} \tilde{E}_2 \right) \right) \left( 1 - \sin \left( \frac{\pi}{2} \tilde{E}_1 \right) \right) \left( 2 \sin^{2} \left( \frac{\pi}{3} \tilde{R}_2 \right) \right) \left( 2 \sin^{2} \left( \frac{\pi}{3} \tilde{Z}_2 \right) \right) \right]^{-1}. \tag{45}
\]
Theorem 10. For any PFN, that is, $0 \leq a, b \leq 1$, and also $0 \leq x \leq 1$; then $0 \leq ax + b(1-x) \leq 1$.

Proof. The proof is the same as that of Theorem 9. □

Lemma 1. For $a_j \geq 0$ and $b_j \geq 0$, we have $\prod_{j=1}^{n} a_j \leq \sum_{j=1}^{n} b_j$.

Lemma 2. Let $0 \leq a, b \leq 1$, and $0 \leq x \leq 1$; then $0 \leq ax + b(1-x) \leq 1$.

Theorem 11. For PFNs $\tilde{I}_j = (\tilde{E}_j, \tilde{R}_j, \tilde{Z}_j)$, the operators $ST - \text{PFWA}$ and $ST - \text{PFWG}$ satisfy the inequality

\[ ST - \text{PFWA}(\tilde{I}_1, \ldots, \tilde{I}_n) \geq ST - \text{PFWG}(\tilde{I}_1, \ldots, \tilde{I}_n), \]

where the equality holds if $\tilde{I}_1 = \tilde{I}_2 = \cdots = \tilde{I}_n$.

Proof. For $n$, PFNs $\tilde{I}_j = (\tilde{E}_j, \tilde{R}_j, \tilde{Z}_j)$ and normalized weight vector $\tilde{\omega}_j > 0$; we have

\[ ST - \text{PFWA}(\tilde{I}_1, \ldots, \tilde{I}_n) = \left[ 1 - \prod_{j=1}^{n} \left( 1 - \sin \left( \frac{\pi}{2} \tilde{E}_j \right) \right) \tilde{\omega}_j, \prod_{j=1}^{n} \left( 2 \sin^2 \left( \frac{\pi}{4} \tilde{R}_j \right) \right) \tilde{\omega}_j, \prod_{j=1}^{n} \left( 2 \sin^2 \left( \frac{\pi}{4} \tilde{Z}_j \right) \right) \tilde{\omega}_j \right], \]

\[ ST - \text{PFWG}(\tilde{I}_1, \ldots, \tilde{I}_n) = \left[ \prod_{j=1}^{n} \left( \sin \left( \frac{\pi}{2} \tilde{E}_j \right) \right) \tilde{\omega}_j, 1 - \prod_{j=1}^{n} \left( 2 \sin^2 \left( \frac{\pi}{4} \tilde{R}_j \right) \right) \tilde{\omega}_j, 1 - \prod_{j=1}^{n} \left( 2 \sin^2 \left( \frac{\pi}{4} \tilde{Z}_j \right) \right) \tilde{\omega}_j \right]. \]

For $\tilde{\omega}_j > 0$, $0 \leq \left( \frac{\pi}{2} \tilde{E}_j \right) \in [0, 1]$, and, by Lemma 3, we get

\[ 1 - \prod_{j=1}^{n} \left( 1 - \sin \left( \frac{\pi}{2} \tilde{E}_j \right) \right) \tilde{\omega}_j \geq 1 - \sum_{j=1}^{n} \tilde{\omega}_j \cdot \left( 1 - \sin \left( \frac{\pi}{2} \tilde{E}_j \right) \right) \]

\[ \geq 1 - 1 + \sum_{j=1}^{n} \tilde{\omega}_j \left( \sin \left( \frac{\pi}{2} \tilde{E}_j \right) \right) \geq \prod_{j=1}^{n} \left( \sin \left( \frac{\pi}{2} \tilde{E}_j \right) \right) \tilde{\omega}_j, \]
which implies that
\[
1 - \prod_{j=1}^{n} \left( 1 - \sin \left( \frac{\pi}{2} E_j \right) \right) \geq \prod_{j=1}^{n} \left( \sin \left( \frac{\pi}{2} E_j \right) \right), \quad (51)
\]

For neutral and negative membership components, we have
\[
\prod_{j=1}^{n} \left( 2 \sin^{2} \left( \frac{\pi}{4} R_j \right) \right) \geq \sum_{j=1}^{n} \omega_j \left( 2 \sin^{2} \left( \frac{\pi}{4} R_j \right) \right),
\]
\[
\leq 1 - \sum_{j=1}^{n} \omega_j \left( 2 \sin^{2} \left( \frac{\pi}{4} R_j \right) \right), \quad (52)
\]
\[
\leq 1 - \prod_{j=1}^{n} \left( 2 \sin^{2} \left( \frac{\pi}{4} R_j \right) \right),
\]
which implies that
\[
\prod_{j=1}^{n} \left( 2 \sin^{2} \left( \frac{\pi}{4} R_j \right) \right) \geq 1 - \prod_{j=1}^{n} \left( 2 \sin^{2} \left( \frac{\pi}{4} R_j \right) \right), \quad (53)
\]
and, similarly, the negative grade is
\[
\prod_{j=1}^{n} \left( 2 \sin^{2} \left( \frac{\pi}{4} Z_j \right) \right) \geq 1 - \prod_{j=1}^{n} \left( 2 \sin^{2} \left( \frac{\pi}{4} Z_j \right) \right),
\]
\[
\leq 1 - \sum_{j=1}^{n} \omega_j \left( 2 \sin^{2} \left( \frac{\pi}{4} Z_j \right) \right), \quad (54)
\]
\[
\leq 1 - \prod_{j=1}^{n} \left( 2 \sin^{2} \left( \frac{\pi}{4} Z_j \right) \right),
\]
which implies that
\[
\prod_{j=1}^{n} \left( 2 \sin^{2} \left( \frac{\pi}{4} Z_j \right) \right) \leq 1 - \prod_{j=1}^{n} \left( 2 \sin^{2} \left( \frac{\pi}{4} Z_j \right) \right), \quad (55)
\]
Hence, from all the above equations, we get
\[
ST - PFWA \left( \bar{I}_1, \ldots, \bar{I}_n \right) \geq ST - PFWG \left( \bar{I}_1, \ldots, \bar{I}_n \right). \quad (56)
\]

**Theorem 12.** Let \( \bar{I}_j = \left( E_j, \tilde{R}_j, \tilde{Z}_j \right) (j = 1, \ldots, n) \) and \( \bar{I} = \left( E, \tilde{R}, \tilde{Z} \right) \) be PFNs; then
\[
ST - PFWA \left( \bar{I}_1 \otimes \cdots \otimes \bar{I}_n \otimes \bar{I} \right) \geq ST - PFWA \left( \bar{I}_1 \otimes \cdots \otimes \bar{I}_n \otimes \bar{I} \right),
\]
\[
ST - PFWG \left( \bar{I}_1 \otimes \cdots \otimes \bar{I}_n \otimes \bar{I} \right) \geq ST - PFWG \left( \bar{I}_1 \otimes \cdots \otimes \bar{I}_n \otimes \bar{I} \right),
\]
\[
ST - PFWA \left( \bar{I}_1, \ldots, \bar{I}_n \right) \otimes \sin \bar{I} \geq ST - PFWA \left( \bar{I}_1, \ldots, \bar{I}_n \right) \otimes \sin \bar{I},
\]
\[
ST - PFWA \left( \bar{I}_1, \ldots, \bar{I}_n \right) \otimes \sin \bar{I} \geq ST - PFWG \left( \bar{I}_1, \ldots, \bar{I}_n \right) \otimes \sin \bar{I}.
\]

**Proof.** Here, we prove only the first part, while the other parts can be deduced similarly; for this, let \( I_j = \left( E_j, \tilde{R}_j, \tilde{Z}_j \right) \) and \( I = \left( E, \tilde{R}, \tilde{Z} \right) \), since both \( I_j \) and \( I \) are PFNs.

\[
ST - PFWA \left( I_1 \oplus \cdots \oplus I_n \oplus \bar{I} \right) = \left[ 1 - \prod_{j=1}^{n} \left( 1 - \sin \left( \frac{\pi}{2} \left( 1 - \bar{E} \right) \right) \left( 1 - E \right) \right) \right]^{\bar{\omega}} \prod_{j=1}^{n} \left( 2 \sin^{2} \left( \frac{\pi}{4} R \right) \right)^{\bar{\omega}},
\]
\[
ST - PFWA \left( I_1 \otimes \cdots \otimes I_n \otimes \bar{I} \right) = \left[ 1 - \prod_{j=1}^{n} \left( 1 - \sin \left( \frac{\pi}{2} \tilde{E} \right) \right) \left( 1 - E \right) \right]^{\bar{\omega}} \prod_{j=1}^{n} \left( 2 \sin^{2} \left( \frac{\pi}{4} \tilde{R} \right) \right)^{\bar{\omega}},
\]
\[
\bar{E}, \tilde{E} \in \left[ 0, 1 \right] \text{ and from Lemma 3, we have } 1 - (1 - E_j)(1 - E) > E_j, E. \text{ Since “sine” is an increasing function, we get } \sin \left( \frac{\pi}{2} (1 - (1 - \bar{E}) (1 - E)) \right) \geq \sin \left( \frac{\pi}{2} \right), (\bar{E}, \tilde{E}) \text{, which gives that}
\]

\[
\sin\left(\frac{\pi}{2} \left(1 - \left(1 - E_j\right)(1 - \bar{E})\right)\right) \geq \sin\left(\frac{\pi E_j}{2}\right) \cdot \bar{E} 
\]
\[
\Rightarrow 1 - \sin\left(\frac{\pi}{2} \left(1 - \left(1 - E_j\right)(1 - \bar{E})\right)\right) \leq 1 - \sin\left(\frac{\pi E_j}{2}\right) \cdot \bar{E} 
\]
\[
\Rightarrow \prod_{j=1}^{n} \left(1 - \sin\left(\frac{\pi}{2} \left(1 - \left(1 - E_j\right)(1 - \bar{E})\right)\right)\right)^{\frac{1}{n}} \leq \prod_{j=1}^{n} \left(1 - \sin\left(\frac{\pi E_j}{2}\right) \cdot \bar{E}\right)^{\frac{1}{n}}. 
\]

Similarly, for the neutral and negative grades, we get
\[
\prod_{j=1}^{n} \left(2 \sin^2\left(\frac{\pi}{4} R_{ij}\right)\right)^{\frac{1}{n}} \leq \prod_{j=1}^{n} \left(2 \sin^2\left(\frac{\pi}{4} \bar{R}_{ij}\right) \cdot \bar{R}_{ij}\right)^{\frac{1}{n}}. 
\]  
(60)

Therefore, from the above equation, we get
\[
\text{ST} = \text{PFWA}\left(\bar{I}_1 \oplus \cdots \oplus \bar{I}_n \oplus \bar{I}\right) 
\geq \text{ST} = \text{PFWA}\left(\bar{I}_1 \oplus \cdots \oplus \bar{I}_n \oplus \bar{I}\right). 
\]  
(61)

6. Decision-Making Approach

This section provides a strategy, preceded by an illustrative example, to solve the decision-making problem. For this reason, assume \( m \) alternative \((\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_m)\) that is evaluated by a group of DMs under the \( n \) different criteria \((\bar{G}_1, \bar{G}_2, \ldots, \bar{G}_n)\). That decision-maker tests \( \bar{y}_i \) and \( \bar{G}_j \) and gives their preferences in terms of PFNs \( a_i^{(x)} = (E_{ij}, R_{ij}, Z_{ij}) \), where \( i = 1(1)m \); \( j = 1(1)n \); \( x = 1(1)\bar{D} \). Then, the rating of each alternative \( \bar{y}_i \) under \( \bar{G}_j \) is expressed as
\[
\bar{y}_i = \left(\bar{G}_1, a_{i1}\right), \left(\bar{G}_2, a_{i2}\right), \ldots, \left(\bar{G}_n, a_{in}\right). 
\]  
(62)

and let \( \bar{a}_j > 0 \) be the normalized weight vector of criteria \( \bar{G}_j \). The following steps are taken to calculate the best choice:

Step 4: if the weights of the attributes are known as before, then use them. Otherwise, we measure these by using the entropy principle. For this, the information entropy of criteria \( \bar{G}_j \) is computed as
\[
E_j = \frac{1}{(\sqrt{2} - 1)m} \sum_{j=1}^{m} \sin\left(\frac{\pi}{4} \left(1 + \bar{E}_{ij} - R_{ij} - Z_{ij}\right)\right) + \sin\left(\frac{\pi}{4} \left(1 - \bar{E}_{ij} + R_{ij} + Z_{ij}\right)\right) - 1, 
\]  
(64)

where \( 1/(\sqrt{2} - 1)m \) is a constant for assuring \( 0 \leq E_j \leq 1 \).

Based on it, the weights of the attributes are computed as \( \omega = (\omega_1, \omega_2, \ldots, \omega_n) \), where
\[
\omega_j = \frac{1 - E_j}{n - \sum_{j=1}^{n} E_j}. 
\]  
(65)

Step 5: with weight vector \( \omega \) and the proposed averaging or geometric PF aggregation operators, the collective values are obtained as \( r_i = (\bar{E}_i, \bar{R}_i, \bar{Z}_i) \) for each alternative \( \bar{y}_i \).

Step 6: find the score values of \( r_i = (\bar{E}_i, \bar{R}_i, \bar{Z}_i) (i = 1, \ldots, m) \).

Step 7: grade all the possible alternatives \( \bar{y}_i (i = 1, \ldots, m) \) and select the most desirable alternative(s).

7. Illustrative Example

In this portion, we discuss with an example the result of the defined MAGDM approach and compare its results with the existing approaches [38].

7.1. Application of the Proposed MAGDM Method

Assume that the five companies \( \bar{y}_1, \bar{y}_2, \bar{y}_3, \bar{y}_4, \) and \( \bar{y}_5 \) were assessed by three decision-makers \( \text{DM}_1(1), \text{DM}_2(2), \) and \( \text{DM}_3(3) \) for funding focused on four criteria, which are given as follows:

(1) \( \bar{G}_1 \) denotes the enterprise level of the management
(2) \( \bar{G}_2 \) denotes the growth ability of the business
(3) \( \bar{G}_3 \) denotes the economic benefit
Step 1: the evaluations of all decision-makers are summarized in Tables 1–3. The aim of this issue is to choose the best company to invest.

Step 2: by taking the weight of the experts, that is, \( \omega = (0.37, 0.41, 0.22) \), and then utilizing the ST-PFWA operator to achieve the collective data on each alternative, the results are shown in Table 4.

Step 3: almost all of the four attributes are just to be the benefit types; then normalization is not needed.

Step 4: we used the idea of the entropy in this step to obtain the values:

\[
\begin{align*}
E_1 &= 0.728313575, \\
E_2 &= 0.697921077, \\
E_3 &= 0.653517061, \\
E_4 &= 0.664506725.
\end{align*}
\]

By the help of this, we find the attributes \( \omega = (0.216355366, 0.240558194, 0.275918987, 0.267167453) \).

Step 5: based on \( \omega = (0.216355366, 0.240558194, 0.275918987, 0.267167453) \) and utilizing the ST-PFWA operator, the collective values of each alternative are gained as

\[
\begin{align*}
\gamma_1 &= (0.905540446, 1.26702E-07, 3.5098E-08), \\
\gamma_2 &= (0.976969493, 3.90763E-08, 1.68931E-10), \\
\gamma_3 &= (0.904107853, 6.55832E-08, 1.24568E-07), \\
\gamma_4 &= (0.935101452, 4.68173E-08, 4.68173E-08), \\
\gamma_5 &= (0.790485389, 6.43519E-08, 7.16207E-06).
\end{align*}
\]

Step 6: we can get the scores of each by using the definition

\[
\begin{align*}
\mathfrak{S}(\gamma_1) &= 0.905540284, \\
\mathfrak{S}(\gamma_2) &= 0.976969454, \\
\mathfrak{S}(\gamma_3) &= 0.904107663, \\
\mathfrak{S}(\gamma_4) &= 0.935101326, \\
\mathfrak{S}(\gamma_5) &= 0.790478163.
\end{align*}
\]

Step 7: according to \( \mathfrak{S}(\gamma_2) > \mathfrak{S}(\gamma_4) > \mathfrak{S}(\gamma_1) > \mathfrak{S}(\gamma_3) > \mathfrak{S}(\gamma_5) \), the ranking order is \( \tilde{\gamma}_2 > \tilde{\gamma}_4 > \tilde{\gamma}_1 > \tilde{\gamma}_3 > \tilde{\gamma}_5 \). Hence, \( \tilde{\gamma}_2 \) is the best alternative.

During Step 5 of the established method, the complete analysis of changing aggregation operators is analyzed, and their results are shown in Table 5.

We can therefore conclude from all the abovementioned computational process that the alternative \( \tilde{\gamma}_2 \) is really the best option among the other options and therefore it is strongly recommended to choose the appropriate option. In Figure 1, we draw the graphical representation of all the alternatives ranked based on the score values and show that the alternative \( \tilde{\gamma}_2 \) is the best one.

### 8. Comparative Analysis

In this section, we give some brief discussion on the comparison of the proposed method with some well-known related methods [33, 38, 45, 46].

#### 8.1. Comparison with [38]

In the existing method, the bidirectional project methods for MAGDM problems with PFNs are discussed, but, in the proposed method, we defined the sine trigonometric entropy aggregation operators for MAGDM problem. The results of the MAGDM approach are listed in Table 6. It is concluded that the best alternative remains the same. Therefore, the suggested approach is more rational than the current one [38].
Table 3: DM\((\gamma)\).

| \(G_1\) | \(G_2\) | \(G_3\) | \(G_4\) |
|---|---|---|---|
| \(\tilde{y}_1\) | (0.53, 0.21, 0.26) | (0.51, 0.11, 0.38) | (0.55, 0.23, 0.22) | (0.34, 0.25, 0.41) |
| \(\tilde{y}_2\) | (0.61, 0.38, 0.01) | (0.54, 0.17, 0.29) | (0.65, 0.20, 0.15) | (0.77, 0.10, 0.13) |
| \(\tilde{y}_3\) | (0.58, 0.15, 0.27) | (0.19, 0.13, 0.68) | (0.61, 0.11, 0.28) | (0.25, 0.18, 0.57) |
| \(\tilde{y}_4\) | (0.42, 0.31, 0.27) | (0.58, 0.20, 0.22) | (0.81, 0.10, 0.09) | (0.52, 0.15, 0.33) |
| \(\tilde{y}_5\) | (0.26, 0.24, 0.50) | (0.27, 0.29, 0.44) | (0.34, 0.39, 0.27) | (0.52, 0.14, 0.34) |

Table 4: Aggregated values of experts by using the ST-PFWA operator.

| \(G_1\) | \(G_2\) | \(G_3\) | \(G_4\) |
|---|---|---|---|
| \(\tilde{y}_1\) | (0.684, 0.025, 0.138) | (0.615, 0.081, 0.063) | (0.857, 0.059, 0.016) | (0.656, 0.069, 0.099) |
| \(\tilde{y}_2\) | (0.875, 0.027, 0.011) | (0.823, 0.044, 0.037) | (0.892, 0.041, 0.014) | (0.849, 0.064, 0.015) |
| \(\tilde{y}_3\) | (0.665, 0.072, 0.101) | (0.638, 0.047, 0.139) | (0.724, 0.072, 0.049) | (0.801, 0.037, 0.054) |
| \(\tilde{y}_4\) | (0.674, 0.087, 0.065) | (0.813, 0.042, 0.049) | (0.748, 0.039, 0.094) | (0.807, 0.056, 0.032) |
| \(\tilde{y}_5\) | (0.546, 0.047, 0.170) | (0.635, 0.101, 0.075) | (0.643, 0.074, 0.071) | (0.180, 0.032, 0.289) |

Table 5: Score value and ranking of the different operators.

| Operators | \(\tilde{y}_1\) | \(\tilde{y}_2\) | \(\tilde{y}_3\) | \(\tilde{y}_4\) | \(\tilde{y}_5\) | Ranking |
|---|---|---|---|---|---|---|
| ST-PFWA | 0.9055 | 0.9769 | 0.9041 | 0.9351 | 0.7905 | \(\tilde{y}_2 > \tilde{y}_4 > \tilde{y}_1 > \tilde{y}_3 > \tilde{y}_5\) |
| ST-PFOWA | 0.8948 | 0.9760 | 0.8972 | 0.9314 | 0.7846 | \(\tilde{y}_2 > \tilde{y}_4 > \tilde{y}_1 > \tilde{y}_3 > \tilde{y}_5\) |
| ST-PFWA | 0.9829 | 0.9991 | 0.9849 | 0.9943 | 0.9368 | \(\tilde{y}_2 > \tilde{y}_4 > \tilde{y}_1 > \tilde{y}_3 > \tilde{y}_5\) |
| ST-PFWG | 0.8607 | 0.9719 | 0.8791 | 0.9186 | 0.7327 | \(\tilde{y}_2 > \tilde{y}_4 > \tilde{y}_1 > \tilde{y}_3 > \tilde{y}_5\) |
| ST-PFOWG | 0.8525 | 0.9710 | 0.8715 | 0.9138 | 0.7260 | \(\tilde{y}_2 > \tilde{y}_4 > \tilde{y}_1 > \tilde{y}_3 > \tilde{y}_5\) |
| ST-PFHG | 0.9705 | 0.9988 | 0.9763 | 0.9887 | 0.9245 | \(\tilde{y}_2 > \tilde{y}_4 > \tilde{y}_1 > \tilde{y}_3 > \tilde{y}_5\) |

Figure 1: Graphical representation of the obtained results using different proposed operators.
Furthermore, we compare our proposed aggregation operators with some other existing approaches, which are proposed by [33, 45, 46], to deal with picture fuzzy quantities. FT_hen, the calculating results are the same in ranking alternatives and the best alternative is also the same. FT_hus, these four methods with PFNs are conducted to further illustrate the advantages of the new approach.

We can therefore conclude from all the abovementioned comparative studies that the alternative $\frac{c_2}{c_2}$ is the best among the other options. In Figure 2, we draw the graphical representation of all the alternatives ranked based on the score values by using the proposed operators and existing operators and show that the alternative $\frac{c_2}{c_2}$ is the best one.

| Proposed operators | $\hat{y}_1$ | $\hat{y}_2$ | $\hat{y}_3$ | $\hat{y}_4$ | $\hat{y}_5$ | Ranking |
|--------------------|-------------|-------------|-------------|-------------|-------------|---------|
| ST-PFWA            | 0.9055      | 0.9769      | 0.9041      | 0.9351      | 0.7905      | $\hat{y}_2 > \hat{y}_4 > \hat{y}_1 > \hat{y}_3 > \hat{y}_5$ |
| ST-PFOWA           | 0.8948      | 0.9760      | 0.8972      | 0.9314      | 0.7846      | $\hat{y}_2 > \hat{y}_4 > \hat{y}_1 > \hat{y}_3 > \hat{y}_5$ |
| ST-PFHA            | 0.9829      | 0.9991      | 0.9849      | 0.9943      | 0.9368      | $\hat{y}_2 > \hat{y}_4 > \hat{y}_1 > \hat{y}_3 > \hat{y}_5$ |
| ST-PFWG            | 0.8607      | 0.9719      | 0.8791      | 0.9186      | 0.7327      | $\hat{y}_2 > \hat{y}_4 > \hat{y}_1 > \hat{y}_3 > \hat{y}_5$ |
| ST-PFOWG           | 0.8525      | 0.9710      | 0.8715      | 0.9138      | 0.7260      | $\hat{y}_2 > \hat{y}_4 > \hat{y}_1 > \hat{y}_3 > \hat{y}_5$ |
| ST-PFHG            | 0.9705      | 0.9988      | 0.9763      | 0.9887      | 0.9245      | $\hat{y}_2 > \hat{y}_4 > \hat{y}_1 > \hat{y}_3 > \hat{y}_5$ |
| Existing method [38]| 0.8681      | 0.8837      | 0.8690      | 0.8754      | 0.8600      | $\hat{y}_2 > \hat{y}_4 > \hat{y}_1 > \hat{y}_3 > \hat{y}_5$ |
| Existing method [33]| 0.7570      | 0.7726      | 0.7580      | 0.7350      | 0.7500      | $\hat{y}_2 > \hat{y}_4 > \hat{y}_1 > \hat{y}_3 > \hat{y}_5$ |
| Existing method [45]| 0.6480      | 0.8037      | 0.6790      | 0.7600      | 0.6300      | $\hat{y}_2 > \hat{y}_4 > \hat{y}_1 > \hat{y}_3 > \hat{y}_5$ |
| Existing method [46]| 0.7382      | 0.9217      | 0.9060      | 0.8761      | 0.7702      | $\hat{y}_2 > \hat{y}_4 > \hat{y}_1 > \hat{y}_3 > \hat{y}_5$ |

Figure 2: Graphical representation of the obtained result utilizing proposed operators and other existing methods.

9. Conclusion

A research related to aggregation operators was investigated in this study by establishing some new sine trigonometric operation laws for PFSs. During decision-making problems, the well-defined operational laws play a major role. On the other hand, the sine trigonometric function has the features of periodicity as well as being symmetric about the origin and hence is more likely to satisfy the decision-maker’s preference over the multiple time periods. We therefore describe some sine trigonometric operational laws for PFNs and study their properties in order to take these advantages and make a smoother and more important decision. We have defined various averaging and geometric aggregation operators on the basis of these operators to club decision maker’s preference. FT_he different elementary relations between the aggregation operators are studied and explained in detail. We developed a new MAGDM algorithm for group decision-making problems, in which goals are classified in terms of PFNs to enforce the proposed laws on decision-making problems. Further, we compute the weight of the attribute by combining the subjective and objective data in terms of the measure. The functionality of the proposed method is applied to an example, and superiority and
feasibility of the approach are investigated in detail. A comparative study is often carried out with current works to verify its performance.

In the future, we will use the framework built on new multiattribute assessment models to tackle fuzziness and ambiguity in a variety of decision-making parameters, for example, advanced study of the similarity measures of intuitionistic fuzzy sets based on the set pair analysis theory, generalized intuitionistic fuzzy entropy-based approach for solving MADM problems with unknown attribute weights; intuitionistic fuzzy Hamacher aggregation operators with entropy weight and their applications to MCDM problems, and linguistic picture fuzzy Dombi aggregation operators and their application in a MAGDM problem.

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that there are no conflicts of interest.

**Acknowledgments**

This research was supported by Algebra and Applications Research Unit, Department of Mathematics and Statistics, Faculty of Science, Prince of Songkla University, Thailand.

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