On the twisted $G/H$ topological models

O. Aharony, O. Ganor, J. Sonnenschein and S. Yankielowicz

School of Physics and Astronomy
Beverly and Raymond Sackler
Faculty of Exact Sciences
Tel Aviv University
Ramat Aviv, Tel-Aviv, 69978, Israel

ABSTRACT

The twisted $G/H$ models are constructed as twisted supersymmetric gauged WZW models. We analyze the case of $G = SU(N), H = SU(N_1) \times \ldots \times SU(N_n) \times U(1)^r$ with $\text{rank } G = \text{rank } H$, and discuss possible generalizations. We introduce a non-abelian bosonization of the $(1,0)$ ghost system in the adjoint of $H$ and in $G/H$. By computing chiral anomalies in the latter picture we write the quantum action as a decoupled sum of “matter”, gauge and ghost sectors. The action is also derived in the unbosonized version. We invoke a free field parametrization and extract the space of physical states by computing the cohomology of $Q$, the sum of the BRST gauge-fixing charge and the twisted supersymmetry charge. For a given $G$ we briefly discuss the relation between the various $G/H$ models corresponding to different choices of $H$. The choice $H = G$ corresponds to the topological $G/G$ theory.

† Work supported in part by the US-Israel Binational Science Foundation and the Israel Academy of Sciences.
1. Introduction

The \( N = 2 \) super-conformal field theories (SCFT’s) have been investigated intensively during recent years. It is these theories that form the building blocks in the construction of four dimensional superstring theories. Moreover, some of these theories admit Landau-Ginzburg (LG) description. Upon twisting the \( N = 2 \) theories by the U(1) current in the \( N = 2 \) algebra one obtains a topological theory \cite{1,2,3}. It is still an open question whether all topological conformal theories are related to \( N = 2 \) super-conformal theories. The physical states in the topological theory are defined to be in the cohomology of the BRST charge \( Q \). In the corresponding \( N = 2 \) models, these are precisely the conditions defining the chiral primary fields.\cite{4} Since the LG potential describes the chiral fields it also gives, whenever it exists, a solution of the corresponding topological theory.\cite{5} The interplay between the \( N = 2 \) SCFT and its topological counterpart implies that the topological theory encodes important information which pertains to the chiral primary fields. In fact, correlators of these fields can be calculated in the corresponding topological theory.

Recently we have investigated the \( G/H \) topological theories\cite{6,7}. We have shown that there are intriguing similarities between the structure of \( G/H \) theories and the structure of 2d gravity theories coupled to \( c \leq 1 \) matter. As a matter of fact this correspondence goes much further. It has been demonstrated\cite{6,8} that upon twisting the \( G/H \) by a BRST exact current \( J^{(tot)}_0 \), one recovers the 2d gravity coupled to \((p,q)\) minimal models for the case \( G=SL(2,R) \) at level \( k = \frac{p}{q} - 2 \). The generalization to \( G=SL(N,R) \) at level \( k = \frac{p}{q} - N \) gives the \((p,q)\) minimal \( W_N \) matter coupled to \( W_N \) gravity.\cite{7} Among the \( N = 2 \) SCFT’s the so called Kazama-Suzuki theories\cite{9} have attracted much attention. They are based on the coset \( G/H \times SO(dimG/H) \) where \( G/H \) is a symmetric space and the \( SO(dimG/H) \) part is associated with the fermionic degrees of freedom which actually reside in the coset space \( G/H \). We shall refer to the topological version obtained upon twisting as the \( G/H \) topological model. In the present work we will restrict ourselves to the case \( G=SU(N) \) and \( H \) a
subgroup of \( G \) such that \( rank(H) = rank(G) \) and investigate the corresponding topological theories. This last requirement guarantees that \( \frac{G}{H} \) is a Kahler manifold which is known to admit \( N = 2 \) supersymmetry. Hence we may take \( H = U(1)^{(N-1)}, SU(2) \times U(1)^{(N-2)}, \ldots, SU(N-1) \times U(1) \) (Actually we will allow \( H = SU(N_1) \times \ldots \times SU(N_n) \times U(1)^r \) with \( r = N - 1 - \sum_{I=1}^{n} N_I + n \)). Note that the last case corresponds to the \( CP^{(N-1)} \) symmetric space \( SU(N)/SU(N-1) \times U(1) \) which is known to have a LG description\(^{10-14}\). The twist involved in the passage from the \( N = 2 \) theory to the topological theory turns the coset fermions into a \((1,0)\) ghost system. If we continue the series of models above one step further we will have \( H = SU(N) \) with no coset "fermions". This theory is nothing but the \( \frac{G}{H} \) topological theory. Thus, we expect that there exists a strong relationship between the whole series of topological models depicted above and the \( \frac{G}{H} \) theory. In the present work we shall elaborate and analyze this relationship. Some work in this direction has been done very recently\(^{12}\) for the case \( H = U(1)^{(N-1)} \). Related aspects in the symmetric space case \( H = SU(N-1) \times U(1) \) have been investigated in \([11]\). At this stage it is instructive to mention yet another place where this kind of theories is relevant. This is the black hole case\(^{18}\) which corresponds to the CFT coset \( SL(2,R)/U(1) \). It has been shown\(^{19}\) that restricting ourselves to the singularity region the theory becomes essentially a topological theory of the form \( U(1)/U(1) \). (At the vicinity of the singularity the symmetry currents become abelian). In ref. \([19]\) it has been shown that yet another description of the singularity is in terms of the twisted Kazama-Suzuki topological theory \( SL(2,R)/U(1) \). The association of space-time singularities with TFT’s of this kind is another motivation for investigating these theories.

It is a well known fact that \( \frac{G}{H} \) coset models can be realized as H-gauged G-WZW models\(^{13,14,15,16}\). Recently, Witten has demonstrated that the \( N = 2 \) Kazama-Suzuki \( \frac{G}{H} \) models can be formulated as supersymmetric WZW theories based on the group \( G \) where the \( H \) subgroup is gauged \([17]\). This will be the starting point of our work. The paper is organized as follows: In section 2 we discuss the non-abelian bosonization of \((1,0)\) ghost systems which appear in the construction
of $G/H$ topological theories. The approach would be to start with the corresponding $(1/2,1/2)$ fermionic system of the $N = 2$ theory, bosonizing it and identifying the $U(1)$ current which twists the $N = 2$ theory into the topological theory. This twist will transform the fermions into the required $(1,0)$ system. In particular, we will have to deal with fermions (ghosts) in the adjoint representation of $H$ and in $G/H$. Once we have learned how to bosonize the $(1,0)$ ghost systems we can go ahead and easily perform chiral rotations which enable us to decouple the gauge fields from the ghosts. This is a crucial step in the construction of the $G/H$ topological quantum action. This is done, following essentially the same steps as in the $G$ case,$^{14,20}$ in section 3. The outcome is that, as in the $G$ case, the action will be composed of three parts. The matter part is associated with G-WZW at level $k$. The gauge part is associated with $H^{(I)}$ WZW action at level $-(k + C_G + C_{H^{(I)}})$ for each nonabelian gauge factor $H^{(I)}$ and a free scalar action with background charge for each abelian $U(1)$ factor in $H$. The third part is a set of $(1,0)$ ghost systems. $C_G$ and $C_{H^{(I)}}$ are the second Casimir invariants of $G$ and $H^{(I)}$ respectively. The ghost part includes the ghosts arising from the $H$-gauge fixing as well as the coset ghosts obtained by twisting the $N = 2$ fermions. We repeat the derivation of the quantum action also in the “fermionic language”. We use a non-abelian generalization of Schwinger and Johnson’s method to compute the chiral anomalies. This results in an action which is identical to the one derived in the bosonization approach. Section 5 is devoted to the discussion of the algebraic structure of these theories and its relation to the algebraic structure of the $G$ theory. We establish directly that $c^{(tot)} = 0$ and identify the zero level Kac-Moody $J^{(tot)}_a$ currents associated with the $H$ subgroup. In particular, we have such a current per each member of the $G$ Cartan-subalgebra (recall $rank(H) = rank(G)$). As we will indicate, the algebraic structure of $G$ is not precisely the one inherited by twisting the simple $N = 2$ algebra. For the $G/H$ case the relevant BRST operator will be the sum of the $H$-fixing BRST operator and the supersymmetry generator. The resulting algebraic structure will be a combination of the one inherited from $H/H$ and the one obtained by twisting the $N = 2$ theory. The BRST cohomology and physical states
will be discussed in section 6. The cohomology of the $G/H$ topological theories will turn out to be isomorphic to that of the corresponding $\frac{G}{G}$ theory. The point is that on the Fock space the only relations that are crucial for determining the cohomology are $L_0|\text{phys} >= 0$ and $J_i^{(tot)}|\text{phys} >= 0$, where $J_i^{(tot)}$ are the currents in the $G$ Cartan algebra. Those relations continue to hold in the $G/H$ topological case. Moreover, the result will continue to hold on the space of matter irreducible representations since the matter corresponds to $G$-WZW of level $k$ in all cases. (Recall that we do not employ Felder’s procedure [21] in the $H$- gauge sector [6,7]). So, all of these theories will have the same number of primary fields and the partition functions will turn out to be the same. The obvious question which arises is to what extent all these models are equivalent. We will address this question in the last section, give a summary of our results and make some conjectures. We discuss the currents needed for the twisting in both the fermionic and bosonic languages in an appendix.

2. Non-abelian Bosonization of the $\frac{G}{H}$ ghost systems.

The ghost systems of the twisted $\frac{G}{H}$ models, as will be shown explicitly in the next section, can be decomposed into two sectors. The first one is an anti-commuting ghost system of dimensions $(1, 0)$ in the adjoint representation of $H$. The second sector comprises of dimension one anti-ghosts which correspond to the positive roots of the $\frac{G}{H}$ coset and dimension zero ghosts associated with the negative roots. These ghosts are coupled to non-abelian gauge fields, denoted by $A$, which take their values in the algebra of $H$. The ghost action, thus, takes the form:

$$S_{gh} = -i \int d^2 z Tr_H [\rho D\chi + \bar{\rho} D\bar{\chi}] - i \int d^2 z \sum_{\alpha \in \frac{G}{H}} [\rho^{+\alpha} (D\chi)^{-\alpha} + \bar{\rho}^{+\alpha} (D\bar{\chi})^{-\alpha}]$$ (2.1)

where $Tr_H$ denotes tracing in the group $H$ and $D\chi = \partial \chi - i[A, \chi]$. The group $H$ is not necessarily simple. In fact we are interested in the general case where the
non-abelian group $H$ can be decomposed into $H = \prod_I H^{(I)} \times U^r(1)$. Here $H^{(I)}$ stands for a non-abelian simple group factor and $r = \text{rank } H - \sum_I \text{rank } H^{(I)}$. $Tr_H$ therefore means $\sum Tr_{H^{(I)}}$ plus a sum over the $U(1)$ factors. The ghosts that correspond to the abelian parts, are neutral under $H$ and therefore they are not coupled to any gauge fields. The free ghost system is obviously invariant under the transformations

\[
\begin{align*}
\delta \rho^a &= f^a_{bc}(z) \rho^c \\
\delta \chi^a &= f^a_{bc}(z) \chi^c \\
\delta \rho^{+\alpha} &= f^{+\alpha}_{a+\beta} \epsilon^a(z) \rho^{+\beta} \\
\delta \chi^{-\alpha} &= f^{-\alpha}_{a-\beta} \epsilon^a(z) \chi^{-\beta},
\end{align*}
\]

where $f^a_{bc}$ are the structure constants of $H$, which are generated by the following currents

\[
J^a = f^a_{bc} \rho^b \chi^c + f^a_{+\alpha-\beta} \rho^{+\alpha} \chi^{-\beta}.
\]

The indices $a, b, c$ run over the adjoint of $H$ while $\alpha, \beta, \gamma$ denote positive roots of $G_H$. These currents which obey the $H$ Kac-Moody algebra of level

\[
k = 2C_H + (C_G - C_H) = C_H + C_G
\]

are coupled to the $H$ gauge field. Obviously there are anti-holomorphic currents which play a similar role. The energy momentum tensor of the corresponding free ghost system is $T(z) = Tr_H[\rho \partial \chi] + \sum_{\alpha \in G_H} [\rho^{+\alpha} \partial \chi^{-\alpha}]$ and is associated with a Virasoro algebra having a central charge

\[
c = -2d_H - 2\frac{(d_G - d_H)}{2} = -(d_G + d_H).
\]

Before presenting the bosonized ghost system it is useful to express the $(1, 0)$ ghost system in terms of a system of Dirac fermions $(\psi^\dagger, \psi)$ which transform under $H$ and $G_H$ in the same way as $(\rho, \chi)$. The Kac-Moody level of the corresponding fermionic currents is identical to that of the ghost system whereas the Virasoro anomaly is $c = \frac{1}{2}(d_G + d_H)$. Introducing now the twist $T = T_f + \frac{1}{2} \partial J_f^\#$, where the fermion number current $J_f^\# = \psi^\dagger a \psi_a + \psi^{+\alpha} \psi_{-\alpha}$, it is easy to check that the Dirac fermions turn into a $(1, 0)$ ghost system identical to the one described in eqns. [(2.1)-(2.5)].
We bosonize first the system of Dirac fermions in the adjoint representation of \( H \). A bosonization is required to produce a bosonic system with the same representation of the Kac-Moody and Virasoro algebras as those of the fermionic system. In addition, one has to check that the sets of primary fields of the two formulations are identical. For our purpose of determining the chiral anomaly it is enough to fulfill the condition on the Kac-Moody and Virasoro algebras as well as to identify the currents that couple to the gauge fields and the twisting currents.

From here on we will restrict our discussion to the case of \( G = SU(N) \) and \( H = SU(N_1) \times \ldots \times SU(N_n) \times U(1)^r \) with \( r = N - 1 - \sum_{I=1}^{n} N_I + n \). The bosonized action takes now the form

\[
S^b = \sum_I [S_{N_I}(l_1^{(I)}, A^{(I)}) + S_{N_I}(l_2^{(I)}, A^{(I)})] + \frac{1}{2\pi} \int d^2 z \sum_{s=1}^{r} \partial \phi^s \bar{\partial} \phi^s \quad (2.6)
\]

where \( l_1^{(I)}, l_2^{(I)} \in SU(N_I) \). The Kac-Moody currents associated with \( SU(N_I) \) corresponds to the sum of the two WZW currents appearing in (2.6). These currents obviously have level \( 2N_I \). The total Virasoro anomaly is (for \( A = 0 \))

\[
c = 2 \sum_I \frac{C_{H^{(I)}d_{H^{(I)}}}}{2C_{H^{(I)}}} + r = \sum_I d_{H^{(I)}} + r = \sum_I (N_I^2 - 1) + r \quad (2.7)
\]

which is identical to that of the fermions in the adjoint of \( H \). The operator \( Tr[T^a(u_1(z) + u_1^{-1}(z)) \] where \( u_1(z) \) is defined \([23]\) in the appendix, has conformal dimension \( \frac{1}{2} \). It is the bosonic operator analog to \( \psi^a_1 \), where \( \psi^a = \psi^a_1 + i\psi^a_2 \) and \( \psi_1 \) and \( \psi_2 \) are Majorana fermions. Similar relations hold obviously for \( \psi^a_2 \) and \( u_2(z) \). Thus we can write down the bosonic version of the Dirac field \( \psi^a \) as \( Tr[T^a(u_1(z) + u_1^{-1}(z) + i(u_2(z) + u_2^{-1}(z)))] \). To get to the bosonic version of the \((1,0)\) ghosts one has to twist \( T_b \rightarrow T_b + \frac{1}{2} \partial J^\#_b \). The identification of \( J^b \) as a generator for a mixed symmetry, namely a symmetry that involves the two WZW factors, is given in the appendix. For each \( SU(N_I) \) group factor the current

\[
J^\#_b = i Tr[u_1^{-1}(z)u_2(z) - u_2^{-1}(z)u_1(z)], \quad (2.8)
\]

is the bosonic counter-part of \( J^\#_f \). The holomorphic dimension of the fields which
correspond to $\psi^a$ and $\psi^a$ are shifted from $\frac{1}{2}$ to 0 and 1 respectively upon twisting $T$ with $J_b^\#$. The Virasoro central charge is shifted by $-3(N_f^2 - 1)$ thus leading to $c = -2(N_f^2 - 1)$ which is equal to that of the ghost system. The $U(1)$ currents $J_s = \partial \phi_s$ do a similar job on the abelian parts of $H$. The action of the twisted bosonic system, thus, reads

$$S_{(gh)}^b = S^b - \int d^2z \left[ \frac{1}{2} \sum_I J_{\bar{b}}^I \bar{\partial} \sigma - \frac{1}{2} \sum_s \phi_s \partial \bar{\partial} \bar{\sigma} \right]$$  \hspace{1cm} (2.9)

where the metric is taken to be $h_{zz} = e^\sigma$ so that $\sqrt{hR} = \partial \bar{\partial} \bar{\sigma}$.

We proceed now to bosonize the coset fermions. It is easy to describe the $\frac{G}{H}$ fermions and those in the adjoint of $H$ by considering an $N \times N$ matrix. The latter consists of $n$ blocks of sizes $N_I \times N_I$ and $N - \sum_I N_I$ unit blocks along the diagonal. The coset fermions fill the rest of the upper triangle with $\frac{n(n-1)}{2}$ blocks of sizes $N_I \times N_J$, $r + 1 - n$ blocks of size $1 \times N_I$ for all $N_I$ and $\frac{(r-n)(r+1-n)}{2}$ unit blocks. The fermions in these off-diagonal blocks furnish the following representations under the non-abelian parts of $H$: $(1,\ldots,N_I,\ldots,\bar{N}_J,\ldots,1)$, $(1,\ldots,N_I,\ldots,1)$ and $(1,\ldots,1)$ respectively, where the numbers inside each (...) denote the representation under the non-abelian factors. The contribution of each block to $c$ is obviously equal to the dimension of the corresponding group representation. The fermions in the $N_I \times N_J$ block lead to $SU(N_f)$ Kac-Moody currents of level $N_f$, $SU(N_J)$ currents of level $N_I$, and a fermion number current of level $N_I N_J$. A similar current structure appears for the one column and unit blocks. The bosonized action of such a block is

$$S_{N_I N_J}^b = S_{N_I}(l_I) + S_{N_J}(l_I) + \frac{1}{2\pi} \int d^2 z \partial \phi_{IJ} \bar{\partial} \bar{\phi}_{IJ}$$  \hspace{1cm} (2.10)

with $l_I \in SU(N_f)$. (In the case of one or two abelian factors one can set $N_I = 1$ or $N_f = 1$, $N_J = 1$ with $S_1 = 0$). Obviously this action results in the same Kac-Moody currents as the fermionic ones ($\sqrt{N_IN_J} \partial \phi_{IJ}$ corresponds to $J^\#$). It is easy to check that $c = N_I N_J$, as required. The bosonic operator which corresponds to
a fermion in the $IJ$ block can be written in terms of $u_I(z)u_J(z)e^{i\phi_{IJ}(z)}$. It is tempting to write the bosonic analog of the mass bilinear in terms of $l_I$ and $l_J$ as $\psi_{IJ}(z)\bar{\psi}_{IJ}(\bar{z}) \simeq (l_I)^{ij}(z,\bar{z})(l_J)^{ji}(\bar{z},z)e^{i\sqrt{N_IN_J}\phi_{IJ}(z,\bar{z})}$. These operators have the same conformal dimensions and group properties as those of free Dirac fermions and the corresponding mass bilinear. However, as was shown in ref.[24] these operators do not lead to correlators which are identical to those in the fermionic version. It is plausible that one has to simply use the bosonization of $\psi$ in terms of $u_I$ and $u_J$. In any case this is not relevant for our present purpose of finding the twisting current which relates only to the abelian part $e^{i\sqrt{N_IN_J}\phi_{IJ}(z)}$. To transmute a fermionic block to that of $(1,0)$ ghosts we twist $T$ in the usual way $T \to T + \frac{1}{2}\partial J^\#$. Changing now the abelian part of $S_{N_I,N_J}^b$ to

$$\frac{1}{2\pi} \int d^2z [\partial \phi_{IJ} \bar{\partial} \phi_{IJ} + \sqrt{\frac{N_IN_J}{2}} \phi_{IJ} R] \quad (2.11)$$

one finds the following contribution to $c$, $c = 1 - 3N_IN_J$, so that altogether $c = -2N_IN_J$ as that of $(1,0)$ ghosts in the $N_I \times N_J$ block. The bosonic version of the ghost system coupled to $H$ gauge fields is easily achieved by gauging the bosonic actions constructed above.

3. The $\frac{GH}{H}$ quantum action using bosonization of the ghosts.

Let us start with a derivation of the quantum action of the $\frac{GH}{H}$ twisted Kazama-Suzuki model. The classical action of this model\footnote{17} is that of level $k$ twisted supersymmetric $G$-WZW model coupled to gauge fields in the algebra of $H \in G$. In other words it is the usual $\frac{G}{H}$ model with an extra set of $(1,0)$ anti-commuting ghosts where the dimension one fields take their values in the positive roots of $\frac{G}{H}$ and the dimension zero fields in the negative ones. The action of the model
\[
S_{(tKS)} = S_k(g, A, \bar{A}) + S_{gh}^k \\
S_k(g, A, \bar{A}) = S_k(g) - \frac{k}{2\pi} \int d^2z Tr_G[g^{-1}\partial g \bar{A}z + g\partial g^{-1}A - \bar{A}g^{-1}Ag + A\bar{A}] \\
S_{gh}^k = -i \int d^2z \sum_{\alpha \in \hat{G}} [\rho^+\alpha(\bar{D}\chi)^{-\alpha} + \bar{\rho}^+\alpha(D\bar{\chi})^{-\alpha}]
\]

(3.1)

where \(g \in G\) and \(S_k(g)\) is the WZW action at level \(k\). In the case that \(H = G\) the model coincides with the \(\frac{G}{G}\) model. In fact, we shall follow closely the derivation of the quantum action of the latter. In the case that \(\Sigma\) is topologically trivial the gauge fields can be parametrized as follows \(A = ih^{-1}\partial h, \bar{A} = i\bar{h}\partial\bar{h}^{-1}\) where \(h(z), \bar{h}(z) \in H^c\). The WZW part of the action then\(^{[14,15]}\) takes the form

\[
S_k(g, A) = S_k(hg\bar{h}) - S_k(h\bar{h})
\]

(3.2)

The Jacobian of the change of variables introduces a dimension (1, 0) system of anticommuting ghosts \(\chi\) and \(\rho\) in the adjoint representation of \(H\). The WZW action thus becomes

\[
S_k(g, A) = S_k(hg\bar{h}) - S_k(h\bar{h}) - i \int d^2z Tr_H[\rho\bar{D}\chi + \bar{\rho}D\bar{\chi}]
\]

(3.3)

where \(D\chi = \partial\chi - i[A, \chi]\). One then fixes the gauge by setting \(h^* = 1\) which implies \(\bar{A} = 0\) and redefining \(hg \to g\). For our case where \(G = SU(N)\) and \(H = SU(N_1) \times \ldots \times SU(N_n) \times U(1)^r\) with \(r = N - 1 - \sum_{I=1}^n N_I + n\), the gauge fields \(A\) take the form \(A = i \sum_{I=1}^n h^{(I)}(-1)\partial h^{(I)} + i \sum_{s=1}^r \partial\mathcal{H}_s\) and the twisted Kazama-Suzuki action is given by

\[
S_{(tKS)} = S_k(g) - \sum_{I=1}^n S_k(h^{(I)}) - \frac{k}{4\pi} \int d^2z \sum_{s=1}^r \partial\mathcal{H}_s \bar{\partial}\mathcal{H}_s \\
- i \int d^2z Tr_H[\rho\bar{D}\chi + \bar{\rho}D\bar{\chi}] - i \int d^2z \sum_{\alpha \in \hat{G}} [\rho^+\alpha(\bar{D}\chi)^{-\alpha} + \bar{\rho}^+\alpha(D\bar{\chi})^{-\alpha}]
\]

(3.4)
The next step taken is to use the results of the previous section and introduce the bosonized actions of the $H$ and $G_H$ ghost sectors. The ghost part in the previous equation is thus replaced by

$$S^b_{gh} = S^b_H(gh) + \sum_{1 \leq I < J \leq r+1} S^b_{N_I N_J}(gh) \quad (3.5)$$

where $S^b_H$ and $S^b_{N_I N_J}$ are given by eqns. (2.9) and (2.10) (after twisting) respectively. Each of the non-abelian terms in eqn. (3.5) has the form

$$S_{k'}(\tilde{l}, A) = S_{k'}(\tilde{l}) - \frac{k'}{2\pi} \int d^2 z Tr[\tilde{l}^{-1} h^{-1} \partial h]$$

$$= S_{k'}(\tilde{l}h) - S_{k'}(h) = S_{k'}(l) - S_{k'}(h) \quad (3.6)$$

where we have used the Polyakov-Wiegmann relation and $l = \tilde{l}h$. The level of $S(h_I)$ in eqn. (3.4) is thus shifted

$$-k \rightarrow -k - 2N_I - (N - N_I) = -k - (N + N_I) \quad (3.7)$$

where the first term ($-2N_I$) is due to $S^b_H(gh)$ while the second one [-(N - N_I)] comes from $S^b_G$. Similarly, it is expected that for arbitrary groups $G$ and $H_I$ the level is shifted to $-k -(C_G + C_{H_I})$. Now let us examine the coupling to gauge fields of the abelian parts of $S^b_{gh}$. As was stated above the abelian parts of $S^b_H(gh)$ are not coupled to the gauge fields, so we have to discuss only the coupling of the currents $\bar{\partial} \phi_{IJ}$ of each $N_I \times N_J$ block. These currents couple to a linear combination of the $U(1)$ gauge fields. Choosing a convenient basis, which fits the Wakimoto free field parametrization which we will later use for the non-abelian parts of $H$ (see eqn. (6.4)), the latter can be written as $(\bar{\alpha}_{IJ} \cdot \partial \tilde{H}) \bar{\partial} \phi_{IJ}$ where $\bar{\alpha}_{IJ} = \sum \bar{\alpha}$ and the sum is over the roots which correspond to the block. Hence one has to add to $S_{\phi_{IJ}}$ a term of the form $-\sqrt{2/N_I N_J} (\bar{\alpha}_{IJ} \cdot \partial \tilde{H}) \bar{\partial} \phi_{IJ}$. The coefficient is determined by the charge of the fermions in the block with respect to this abelian gauge field. To
diagonalize this new action we shift \( \phi_{IJ} \rightarrow \phi_{IJ} + \sqrt{\frac{1}{2N_IN_J}} \vec{\alpha}_{IJ} \cdot \vec{\mathcal{H}} \). Under this shift the action is

\[
S_{\phi_{IJ}} \rightarrow S_{\phi_{IJ}} + \frac{1}{2\pi} \int d^2z \left[ \frac{1}{2} (\vec{\alpha}_{IJ} \cdot \vec{\mathcal{H}}) R - \frac{1}{2N_IN_J} (\vec{\alpha}_{IJ} \cdot \partial \vec{\mathcal{H}})(\vec{\alpha}_{IJ} \cdot \partial \vec{\mathcal{H}}) \right] \quad (3.8)
\]

Collecting now all the various terms one finds that the total quantum action is

\[
S_k = S_k(g) + \sum_{I=1}^n S_{-(k+C_G+C_H(I))}(h^{(I)}) + \frac{1}{2\pi} \int d^2z \left[ \sum_{s=1}^r \partial \mathcal{H}_s \bar{\partial} \mathcal{H}_s + i \sqrt{\frac{2}{k+C_G}} (\bar{\rho}_G - \bar{\rho}_H) \cdot \vec{\mathcal{H}} R \right] - i \int d^2z \text{Tr}_H [\bar{\rho} \partial \chi + \rho \bar{\partial} \chi] - i \int d^2z \sum_{\alpha \in \frac{G}{H}} [\rho^{+\alpha} (\partial \chi)^{-\alpha} + \bar{\rho}^{+\alpha} (\partial \bar{\chi})^{-\alpha}] \quad (3.9)
\]

where we have normalized the \( \vec{\mathcal{H}} \) fields to be free bosons, and \( \bar{\rho}_G \) and \( \bar{\rho}_H \) are half the sums of the positive roots of \( G \) and \( H \) respectively. The action is composed of three decoupled sectors: the matter sector, the gauge sector and the ghost sector involving ghosts in \( H \) and \( \frac{G}{H} \). This action describes a TCFT model as is shown in section 5.

4. The Quantum action using Schwinger’s method

The derivation of the quantum action of the twisted Kazama-Suzuki model as a sum of the matter sector, the gauge sector and the ghost sector which are decoupled from each other involved a chiral rotation in the ghost sector. In the previous section a bosonized version of the model was invoked to perform this transformation which amounts in fact to calculating the chiral anomaly of the model. Here we repeat this computation in the “fermionic language” for the \( \frac{G}{H} \) ghosts. A similar derivation applies also to the ghosts in the adjoint of \( H \).

Let us define

\[
e^{iI(h,\bar{h})} = \int D\rho D\bar{\rho} D\chi D\bar{\chi} e^{\frac{i}{2\pi} \int d^2z \sum_{\alpha \in \frac{G}{H}} [\rho^{+\alpha} (\partial \chi)^{-\alpha} + \bar{\rho}^{+\alpha} (\partial \bar{\chi})^{-\alpha}]} \quad (4.1)
\]

where again we parametrize \( A = ih^{-1} \partial h, \bar{A} = i\bar{h} \bar{\partial} h^{-1} \) where \( h(z) \in H^c \). Under
the infinitesimal chiral transformations \( \delta h = \epsilon h, \delta \bar{h} = \bar{\epsilon} h \) we have

\[
\frac{\delta e^{iI(h,\bar{h})}}{e^{iI(h,\bar{h})}} = - \frac{1}{2\pi} \int d^2 z \sum_{\alpha \in \mathbb{Z}} \left[ \rho^\alpha [\delta \bar{A}, \chi]^{-\alpha} + \bar{\rho}^\alpha [\delta A, \bar{\chi}]^{-\alpha} \right] > h \bar{h}
\]

\[
= - \frac{1}{2\pi} \int d^2 z \sum_{\alpha \in \mathbb{Z}} f_{\alpha\alpha,-\beta} < \rho^\alpha \chi^{-\beta} >_{h,\bar{h}} \delta \bar{A}^a + < \bar{\rho}^\alpha \bar{\chi}^{-\beta} >_{h,\bar{h}} \delta A^a
\]

From here on we consider a chiral transformation only of \( \bar{h} \), namely \( \epsilon = 0, \bar{\epsilon} \neq 0 \). This last relation is meaningful only provided that we regularize in a gauge invariant way the propagator \(< \rho^\alpha (z, \bar{z}) \chi^{-\beta} (z, \bar{z}) >\). This can be achieved by generalizing a method proposed originally by Schwinger and Johnson \([26]\) for the abelian case namely:

\[
< \rho^\alpha (z, \bar{z}) \chi^{-\beta} (z, \bar{z}) >_{h \bar{h}} = \lim_{z' \to z} G^\alpha, -\beta (z', \bar{z}) = \lim_{z' \to z} \left[ \bar{h} (z') \right]_{\alpha} \left[ G^0_{\alpha, -\beta} (z', z) \right]_{\alpha} \left[ \bar{h} (z) \right]_{\beta}
\]

where the Green's function \( G^\alpha, -\beta \) is a solution of the equation

\[
\bar{D}_{z'} G^\alpha, -\beta = \bar{\partial}_{z'} G^\alpha, -\beta - i f^\alpha_{a\gamma} A^a (z) G^{\gamma, -\beta} = \pi \delta^{\alpha \beta} \delta (2) (z' - z).
\]

Some clarifications are in order: (i) \( z', z \) are on the plane \( C - \{0\} \) and we use the equal radius limit \( |z'| = |z| \) (which may be viewed as an equal time limit on a cylinder). (ii) The path ordered integral is not uniquely defined since the path is not defined. However, the ambiguity is of order \( O(|z' - z|^2) \) and, therefore, is negligible since \( G \sim \frac{1}{z' - z} \). (iii) For any \( b \in H \) \([b]^\beta_{\alpha} \equiv Tr[T^{-\gamma} b^{-1} T^{\beta} b]g_{\alpha, -\gamma}\). It is straightforward to prove that eqn. (4.3) is indeed gauge invariant. Now the solution of equation (4.4) is given by

\[
G^\alpha, -\beta (z', z) = \left[ \bar{h} (z') \right]_{\alpha} G^0_{\alpha, -\beta} (z', z) \left[ \bar{h} (z) \right]_{-\beta}
\]

where \( G^0_{\alpha, -\beta} = \frac{g_{\alpha, -\beta}}{2\pi} \) is the free propagator. We take now the limit in eqn. (4.4)
and find

\[ <\rho^\alpha(z,\bar{z})\chi^{-\beta}(z,\bar{z})>_{h}\bar{h}=<\rho^\alpha(z,\bar{z})\chi^{-\beta}(z,\bar{z})>^{0}+f^{\alpha\alpha,-\beta}[(\partial\bar{h}\bar{h}^{-1})_{a}-(h^{-1}\partial h)_{a}] \]

Inserting this result into eqn. (4.2) we get

\[ \delta I(h,\bar{h}) = \frac{i}{2\pi} \int d^2 z \sum_{\alpha\in\hat{\mathcal{H}}} f^{\alpha\alpha,-\beta}[<\rho^\alpha\chi^{-\beta}>^{0} + if^{\beta\alpha,-\beta}((\partial\bar{h}\bar{h}^{-1})_{b}+(h^{-1}\partial h)_{b})(\bar{h}\partial)_{b}e\bar{h}^{-1})_{a}^{a} \]

Using the non-conservation of the fermionic current due to the gravitational anomaly, namely \( \bar{\partial}(\rho^\alpha\chi^{-\beta}) = \frac{1}{2}\delta^{\alpha\beta}R \) one finds that

\[ I(h,\bar{h}) = S_{(C_{G}-C_{H})}(h\bar{h}) + \frac{1}{2\pi} \int i\sqrt{\frac{2}{k+C_{G}}}(\bar{\rho}_{G}-\rho_{H})\cdot\bar{H}R \]

where we have used \( \sum_{\alpha,-\beta\in\hat{\mathcal{H}}} f_{i}^{\alpha,-\beta}\delta^{\alpha\beta} = \rho_{G}^{i}-\rho_{H}^{i} \) and \( f_{i}^{\alpha,-\beta}f_{j}^{\alpha,-\beta} = (C_{G}\delta_{j}^{i}-C_{H}^{i}) \)
and where \( C_{H}^{i} = C_{H_{I}}\delta_{i}^{I} \) if both \( \alpha \) and \( \beta \) are in \( H_{I} \) and otherwise zero. Thus the fermionic derivation of the chiral transformation of the quantum action is identical to that derived using the bosonization method.

5. The algebraic structure of the twisted Kazama-Suzuki model

The next step in analyzing the twisted \( \frac{G}{H} \) models is the derivation of their algebraic structure. Twisted \( N = 2 \) super conformal models, as well as the \( \frac{G}{G} \) models, obey the TCFT algebra.\(^{[27]}\) As one would expect the twisted Kazama-Suzuki models share the same algebra. We start with the Kac-Moody algebra associated with the group \( H \). We define the currents \( J^{(tot)\alpha}_{I} \) for each non-abelian group factor \( H^{(I)} \),
and for each $U(1)$ group as

\[ J^{(\text{tot})a}_I = J^a + I^a + i f^a_{bc} \rho^b \chi^c + i \sum_{\alpha \beta \in G/H} f^a_{\alpha \beta} \rho^\alpha \chi^-\beta. \]  

(5.1)

where $J^a$, $I^a$ are the contributions of the $g$ and $h$ sectors respectively. The contribution of the ghost currents consists of both the $H$ and $G/H$ parts to be denoted by $J^a_H$ and $J^a_{G/H}$ respectively. The level of these currents vanishes

\[ k^{(\text{tot})}_i = k - (k + C_G + C_{H_i}) + 2C_{H_i} + (C_G - C_{H_i}) = 0. \]  

(5.2)

For the abelian case there is a similar expression now with $C_{H_i} = 0$.

The energy momentum tensor $T$ can be decomposed, in a way which will be found later to be natural, into $T = T^H + T^G/H$ as follows

\[
T(z) = \frac{1}{2(k + c_G)} g_{\tilde{a}\tilde{b}} : J^{\tilde{a}} \cdot J^{\tilde{b}} : - \frac{1}{2(k + c_G)} g_{ab} : I^a \cdot I^b : + g_{ab} \rho^a \partial \chi^b \\
- \frac{\sqrt{2}}{k + C_G} (\vec{\rho}_G - \vec{\rho}_H) \cdot \vec{J} + \sum_{\alpha \in G/H} \rho^{+\alpha} (\partial \chi)^{-\alpha} \\
T^H(z) = \frac{1}{2(k + c_G)} g_{ab} : (J^a + J^a_{G/H}) (J^b + J^b_{G/H}) : - \frac{1}{2(k + c_G)} g_{ab} : I^a \cdot I^b : \\
- \frac{\sqrt{2}}{k + C_G} (\vec{\rho}_G - \vec{\rho}_H) \cdot \partial (\vec{J} + \vec{J}_{G/H}) + g_{ab} \rho^a \partial \chi^b \\
T^G/H(z) = \frac{1}{2(k + c_G)} g_{\tilde{a}\tilde{b}} : J^{\tilde{a}} \cdot J^{\tilde{b}} : - \frac{1}{2(k + c_G)} g_{ab} : (J^a + J^a_{G/H}) (J^b + J^b_{G/H}) : \\
+ \frac{\sqrt{2}}{k + C_G} (\vec{\rho}_G - \vec{\rho}_H) \cdot \partial (\vec{J} + \vec{J}_{G/H}) + \sum_{\alpha \in G/H} \rho^{+\alpha} (\partial \chi)^{-\alpha}
\]

(5.3)

where $\tilde{a}$ and $\tilde{b}$ go over the adjoint of $G$. $\vec{J}$, $\vec{I}$ and $\vec{J}_{G/H}$ are the Cartan-subalgebra currents given in the basis in which $[J^i_n, J^j_m] = kn \delta^{ij} \delta_{m+n}$. The total Virasoro central charge vanishes since
\begin{equation}
\frac{c}{k + C} = \frac{kd_G}{k + C} + \sum_{l=1}^{n} \left( k + C_G + C_{H_l} \right) d_{H_l} + r - 2d_H - (d_G - d_H) + 6 \sqrt{\frac{2}{k + C_G}} \left( \tilde{\rho}_G - \tilde{\rho}_H \right) \right] = 0
\end{equation}

where we have used, assuming simply laced groups, the relations \( 12 \rho_G^2 = d_G C_G \), and \( \tilde{\rho}_H \cdot (\tilde{\rho}_G - \tilde{\rho}_H) = 0 \)

In addition to the commuting holomorphic (and anti-holomorphic) symmetry generators there are also anti-commuting ones. Upon gauge fixing, the gauge invariance is transformed into a BRST symmetry generated by a dimension one current \( J^{(BRST)} \). This current has a dimension two partner \( G \). These two anti-commuting currents are given by

\[ J^{(BRST)} = g_{ab} \chi^a [J^b + t^b + i{\frac{1}{2}} J^{(gh)}_H] = g_{ab} \chi^a [J^b + t^b + i{\frac{1}{2}} f_{\alpha \gamma, -\beta} \rho^\gamma \chi^{-\beta}] \]

\[ G^H = \frac{g_{ab}}{2(k + C_G)} \rho^a [J^b - t^b + i{\frac{1}{2}} f_{\alpha \gamma, -\beta} \rho^\gamma \chi^{-\beta}] - \frac{\sqrt{2}}{k + C_G} (\tilde{\rho}_G - \tilde{\rho}_H) \cdot \partial \tilde{\rho} \]

It is straightforward to realize that \( T^H(z) \) is BRST exact

\[ T^H(z) = \{ Q^{(BRST)}, G^H(z) \} \]

where \( Q^{(BRST)} = \oint dz J^{(BRST)} \). As was shown in ref. [17] the addition of the coset ghosts turned the model into a twisted \( N = 2 \) model. The twisted \( N = 2 \) algebra is generated by \( Q^\pi \) and \( G^\pi \) given by their \( N = 2 \) counterparts \([9]\) with the fermions replaced with ghosts.

\[ Q^\pi = \sum_{\alpha \beta \gamma \in \pi} \chi^{-\alpha} (J^{\alpha} + i{\frac{1}{2}} f^{\alpha \gamma, -\beta} \rho^\gamma \chi^{-\beta}) \]

\[ G^\pi = \frac{1}{k + C_G} \sum_{\alpha \beta \gamma \in \pi} \rho^\alpha (J^{-\alpha} + i{\frac{1}{2}} f^{-\alpha \gamma, -\beta} \rho^\gamma \chi^{-\beta}) \].

\( T^\pi \) defined above is exact with respect to \( Q^\pi \)

\[ T^\pi = \{ Q^\pi, G^\pi \} \].

16
The various $Q$'s and $G$'s obey the following anti-commutation relations:

$$\{Q^\pi, Q^{(BRST)}\} = \{Q^\pi, G^H\} = \{Q^{(BRST)}, G^\pi\} = 0$$

$$\{Q^{(BRST)}, Q^{(BRST)}\} = \{Q^\pi, Q^\pi\} = \{G^\pi, G^\pi\} = 0$$

$$\{G^H, G^H\} = \frac{1}{4(k + C_G)^2} \sum_{abc} f_{abc} \bar{\rho}^a \rho^b J^{(tot)c}$$

The fact that $G^H$ is not nilpotent is shared by several other TCFT's, in particular the $G$ models. Defining now the combined generators

$$Q = Q^{(BRST)} + Q^\pi \quad G = G^H + G^\pi$$

we find the following relations which are common to all TCFT algebras:

$$T(z) = \{Q, G(z)\}$$

$$J^{(BRST)} = \{Q, j^\#(z)\}$$

$$J^{(tot)a} = \{Q, \rho^a\}$$

where $J^{(tot)a}$ denotes a current in the algebra of $H$ and $j^\#(z) = g_{ab} \rho^a \chi^b + \sum_{\alpha} \chi^\alpha \chi^{-\alpha}$. Notice that $G$ is not nilpotent. Following eqn.(5.9) it is clear that the algebra given in eqn.(5.11) is not closed. A complete analysis of the algebraic structure is under current investigation.

6. The $Q$ cohomology of the twisted $G_H$ model

Next we proceed to extract the space of physical states of the model. We take as our definition of a physical state a state in the cohomology of $Q = Q^{(BRST)} + Q^\pi$, namely, $|\text{phys}> \in H^*(Q)$. In the case that the spectral sequence of the double complex of $Q^{(BRST)}$ and $Q^\pi$ degenerates at the $E_2$ term, this is the same as taking one cohomology and then the other. It is not clear to us whether this
always happens in our case. Expanding the various currents in modes, $Q$ takes the form

$$Q = \sum_{n,m=-\infty}^{\infty} \left[ g_{ab} \chi_n^a (j_n^b + f_n^b) - \frac{1}{2} f_{abc} \chi_n^a \chi_m^b \rho_{n+m}^c \right]$$

$$+ \sum_{\alpha\beta\gamma \in \mathbb{G}_n} \sum_{n,m=-\infty}^{\infty} \left[ g_{ab} f_{\alpha,-\beta}^b : \chi_n^a \rho_n^\alpha \chi_m^\beta : + \chi_n^- (J_n^\alpha + \frac{1}{2} f_{\alpha,-\gamma}^\alpha : \rho_n^\beta \chi_m^- : \right]$$

(6.1)

The extraction of the physical states from here on follows the same lines as in the $G$ models described in details in refs. [6,7]. Therefore, here we only briefly summarize the procedure. The physical states obey

$$L_0|\text{phys}\rangle = 0 \quad J^{(\text{tot})}_i|\text{phys}\rangle = 0 \quad (i = 1, \ldots, \text{rank } H) \quad (6.2)$$

The generalized BRST charge is decomposed into

$$Q = \chi_0^i J^{(\text{tot})}_i^0 + M^i \rho_0^i + \hat{Q}$$

$$M^i = -\frac{1}{2} f_{bc}^i \sum_n : \chi_n^b \chi_n^c :$$

(6.3)

where the sum over $i$ is over the Cartan subalgebra, so that on the sub-space of states annihilated by $\rho_0^i$, $Q = \hat{Q}$. The cohomology on this subspace is called the relative cohomology. Let us start to compute this cohomology. The states of $H^\ast(\hat{Q})$ are built on a highest weight vacuum $|J,I\rangle$ defined in ref. [6,7] by applying creation operators which correspond to a free field parametrization of the $J$ and $I$ sectors. (The root denoted in ref. [7] by $(ij)$ correspond to $\alpha_i + \ldots + \alpha_j$). Notice that the bosonization of the $I$ sector is opposite to the one used in the $J$ sector. In both cases there are bosons which correspond to the Cartan sub-algebra denoted by $\mathcal{H}_i$ and $\bar{\mathcal{H}}_i$ for the $J$ and $I$ sectors respectively. In addition there is a pair of dimension $(1,0)$ ghost for each root. The dimension one fields denoted by $\beta$ relate to the positive roots in the $J$ sector, while their analogs $\bar{\beta}$ are associated with the negative roots in the $I$ sector. For instance in this parametrization the
components of the $I$ currents which are associated with the Cartan sub-algebra of the $H_I = SU(N_I)$ group factor take the form \[ ( \text{in a basis which satisfies } I^i_n I^j_m = -(k + N + N_I)ng^{ij}\delta_{m+n} \]

\[ I^i_n = -\sum_{j=1}^{N_I-1} g_{H_I}^{ij} [i\sqrt{k + N}\bar{\alpha}_j \cdot \bar{\mathcal{H}}_n + 2 \sum_m :\gamma_m^{(jj)} \beta_{n-m}^{(jj)} : - \sum_{m} \sum_{k=1}^{j-1} (:\gamma_m^{(k,j-1)} \beta_{n-m}^{(k,j-1)} : - :\gamma_m^{(k,j)} \beta_{n-m}^{(k,j)} :)] (6.4) \]

\[ - \sum_{m} \sum_{k=j+1}^{N_I-1} (:\gamma_m^{(j+1,k)} \beta_{n-m}^{(j+1,k)} : - :\gamma_m^{(jk)} \beta_{n-m}^{(jk)} :)] \]

Here $\bar{\mathcal{H}}_i$, $i = 1, \ldots, N_I - 1$ are scalar fields with $\bar{\mathcal{H}}_i(z)\bar{\mathcal{H}}_j(\omega) = -\delta_{ij}\log(z - \omega)$ and $\beta^{(ij)}$, $\gamma^{(ij)}$ ($i \leq j$) are the commuting $(1, 0)$ ghost systems obeying $\gamma^{(ij)}(z)\beta^{(kl)}(\omega) = \delta^{ik}\delta^{jl}(z - \omega)^{-1}$. Defining degrees and using a spectral sequence decomposition for $\hat{Q}$, it turns out \[ that the required cohomology is isomorphic to that of the zero degree component of $Q$. The contribution of the excitations of various fields to $L_0$ are exact under the latter operator and hence there are no excitations in $H^*(Q)$. Finally one finds \[ that the relative cohomology contains only a single state given by \]

\[ H^{rel}(Q) = \{ \prod_{\alpha \in H, \alpha > 0} \chi_0^\alpha | \bar{J}, \bar{I} >; \quad \bar{J} + \bar{I} + 2\bar{\rho}_H = 0 \}. (6.5) \]

Since the $J$ sector has a background charge of $i\sqrt{k + C_G} \vec{\rho}_H \cdot \partial^2 \vec{\phi}$ while in the $I$ sector has a background charge of $i\sqrt{k + C_G} [(2\vec{\rho}_H - \vec{\rho}_G) \cdot \partial^2 \vec{H}$, we find that the weight of this state is $L_0 = \frac{1}{k + C_G} [\bar{J} \cdot (\bar{J} + 2\vec{\rho}_G) - \bar{I} \cdot (\bar{I} + 2(2\vec{\rho}_H - \vec{\rho}_G))] = 0$ for $\bar{J} + \bar{I} + 2\bar{\rho}_H = 0$. This state corresponds to the “tachyon” state of the $W_N$ models based on $H = G = SU(N)$ \[. The absolute cohomology (without the restriction $\rho_0^i = 0$) is \]

\[ H^{abs}(Q) \simeq H^{rel}(Q) \oplus \sum_{\{k_1, \ldots, k_l\}} \chi_0^{k_1} \ldots \chi_0^{k_l} H^{rel}(Q) (6.6) \]

where the sum is over $\{k_1, \ldots, k_l\}$ which are all possible subsets of the set $1, \ldots, N-1$. Thus, each state in the relative cohomology gives rise to $2^{N-1}$ states in the absolute
cohomology. In the $G$ models the physical states were deduced after translating the cohomology on the Fock space into the space of irreducible representations of the Kac-Moody algebra associated with the $G$-WZW matter sector. Following the same steps in the present case one finds for each maximal weight $J$ of $G_k$ a rank $G$ dimensional vector of states. This implies also that there is an $N - 2$ dimensional lattice of states for each ghost number and $J$. The latter situation follows from the $N - 1$ dimensional lattice of Fock spaces which are derived by Weyl reflections as well as shifts by linear combinations of roots. A correspondence between the field content (up to “topological sectors”) and partition functions of the $(p, q) W_N$ strings and $G$ models for $A^{(1)}_{N-1}$ at level $k = \frac{p}{q} - N$ (the case of $N = 2$ corresponds to the minimal models coupled to gravity) was derived by further twisting the models according to

$$T(z) \rightarrow \tilde{T}(z) = T(z) + \sum_{i=1}^{N-1} \partial J^{(tot)}(z).$$

(6.7)

It is easy to realize that the torus partition sum in the present case is equal to the one of the $A^{(1)}_{N-1}$ and thus shares the same relation with the $W_N$ string models.

7. Summary and discussion

In this paper we have analyzed the twisted $G/H$ models. We focused on the case of $G = SU(N)$, $H = SU(N_1) \times \ldots \times SU(N_n) \times U(1)^r$ with $\text{rank } G = \text{rank } H$, and discussed possible generalizations to other groups. The quantum action was written as a decoupled sum of “matter”, gauge and ghost sectors. This was achieved by computing chiral anomalies both by introducing a non-abelian bosonization of the $(1, 0)$ ghost system in the adjoint of $H$ and in $G/H$, as well as in the unbosonized version. The algebraic structure of the models was presented emphasizing its relations with the TCFT algebra. We invoke a free field parametrization and extract the space of physical states by computing the cohomology of $Q$, the sum of the BRST gauge-fixing charge and the twisted supersymmetry charge.
The exact relations between models based on the same $G$ but different $H$ can be understood only after computing all possible correlators for the various models. Even though this stage of understanding the models is still ahead of us, we can make the following remarks.

(i) The field content of the various models is very similar. Two such twisted $G\sslash H$ models differ in quartets of fields $\chi^{\alpha}, \rho^{-\alpha}, \bar{\beta}^{-\alpha}, \bar{\gamma}^{\alpha}$, corresponding to positive roots $\alpha$ which are contained in one $H$ subgroup and not in the other. Those quartets contribute nothing to the Virasoro anomaly.

(ii) The cohomologies of the various cosets are very similar. When an inverted bosonization was used for the gauge sector (as was done in section 6), the Fock-space cohomology was found to consist of one state for every eigen-value $\vec{J}$ of the Cartan subalgebra. The $\vec{I}$ eigenvalue was connected to that of $\vec{J}$ via $\vec{I} + \vec{J} + 2\rho_H = 0$ and the ghost number of the state was equal to the number of positive roots of $H$. It is thus clear that there is a simple isomorphism between the cohomologies of the different coset models which preserves the $\vec{J}$ and $L_0$ eigenvalues of the states.

(iii) A similar relation holds when we work out the cohomology of the physical states in the space of irreducible representation of the Kac-Moody algebra of the matter sector. Since the matter sector is identical for all the various choices of $H$ we find, after applying the method of ref. [21], that the states of the different models are connected by a constant shift of $I$ and of the ghost number.

(iv) As an immediate outcome of the previous points, it is clear that the torus partition sum is the same for all subgroups $H$.

(v) Non-trivial correlation functions on the sphere are those that obey the condition that the sum of the ghost number of the operators is equal to the ghost number anomaly. The latter is proportional to the rank of the group and thus is the same for the various cosets. (vi) The LG approach, whenever it exists, gives a description of chiral primary fields of the $N = 2$ theory and, therefore, of the primary fields of the corresponding twisted topological theory. Thus the LG theory gives a solution of the corresponding topological theory. We conjecture
that all the twisted $G$ models, including the $G$ model, are connected by smooth perturbations. In the LG approach this means that the leading power in all the corresponding potentials is the same. Thus, the different twisted $G$ models would correspond to different points in the space of parameters associated with the LG potentials, at which the model is both conformal and topological. This conjecture is supported by some specific examples. As an illustration take the case of $A_k$ minimal $N = 2$ models based on $SU(2) / U(1)$ coset. The corresponding TCFT\textsuperscript{[2,3,27]} are described by the potential $W(x) = \frac{x^{k+2}}{k+2}$ and the primary fields are $\phi_i \sim x^i$ with $i = 0, \ldots, k$. They satisfy the chiral ring relation $\phi_i \cdot \phi_j = \phi_{i+j} \mod(k)$ associated with the ring of polynomials $R = \frac{C[x]}{dW}$. One can perturb this potential by turning on a set of couplings $\{t_n\}$ to the operators $\phi_n$. There is a well defined iterative way to construct the perturbed LG potential. By this procedure one obtains a multi-parameter family of potentials $W(x, \{t_n\})$ which in this example takes the form $W(x, \{t_n\}) = \frac{x^{k+2}}{k+2} - \sum_{i=0}^{k} g_i(t)x^i$, with well defined coefficients $g_i(t)$ (e.g. $g_i(t) = t_i + \ldots$). The corresponding primary fields are given by $\phi_j(x, \{t_n\}) = -\frac{\partial W(x, \{t_n\})}{\partial t_j}$ and they satisfy the ring structure $\phi_i(x, \{t_n\}) \cdot \phi_j(x, \{t_n\}) = \sum_{k} c_{ij}^{k} \phi_k(x, \{t_n\})$. At the origin of the $\{t_n\}$ parameter space we have the twisted $SU(2) / U(1)$ theory\textsuperscript{[2,3,31]} at level $k$. Note that these two points are connected along the line $t_i = t\delta_{i,k}, 0 \leq t \leq 1$. At the point $t = 1$, $\phi_i(x, t = 1) = P_i(x)$ where the $P_i$'s are the modified Chebyshev polynomials. These polynomials subject to the constraint $W'(x, t = 1) \equiv P_{k+1}(x) = 0$ are known to satisfy the fusion ring structure\textsuperscript{[31,10]} i.e. $C_{ij}^k(t = 1) = N_{ijk}$. Other specific examples that support our conjecture can be worked out given the LG potentials given in ref. [11]. It would be very interesting to prove our conjecture, identify and provide characterization to all the points in the LG parameter space that correspond to TCFT’s. Note that upon twisting the latter one can get the LG description of the corresponding $N = 2$ theories which for most of the cases under consideration are not known yet.

Acknowledgements: We would like to thank T. Eguchi for a stimulating discussion. One of us J.S would like to thank A. Schwimmer for a useful conversation.

22
REFERENCES

1. E. Witten, *Comm. Math. Phys.* **117** (1988) 353, **118** (1988) 411.
2. T. Eguchi and N. Yang, *J. Mod. Phys.* A(1990) 1635
3. K. K. Li *Nucl. Phys.* B354 (1991) 711.
4. W. Lerche, C. Vafa and N. p. Warner *Nucl. Phys.* B324 (1989) 427.
5. C. Vafa, *Mod. Phys. Lett.* A6 (1991) 337.
6. O. Aharony, O. Ganor N. Sochen J. Sonnenschein and S. Yankielowicz, “Physical states in the $G$ models and two dimensional gravity”, TAUP-1947-92 April 1992.
7. O. Aharony, J. Sonnenschein and S. Yankielowicz, “$G$ models and $W_N$ strings”, TAUP-1977-92 June 1992 (to be published in *Phys. Lett. B*).
8. L. H. Hu and M. Yu, “On BRST cohomology of SL(2)/SL(2) gauged WZWN models” Academia Sinica preprint AS-ITP-92-32.
9. Y. Kazama and H. Suzuki, *Nucl. Phys.* B321 (1989).
10. D. Gepner, *Phys. Lett.* 222B (1989) 207; *Nucl. Phys.* B322 (1989) 65.
11. D. Nemeschansky and N. P. Warner “Topological Matter, Integrable Models and Fusion Rings” USC-91/031.
12. T. Nakatsu and Y. Sugawara “Topological gauged WZW models and 2D gravity” Tokyo preprint UT-598 “BRST fixed points and Topological Conformal Symmetry” UT-599.
13. K. Bardacki, E. Rabinovici, and B. Serin *Nucl. Phys.* B299 (1988) 151.
14. K. Gawedzki and A. Kupianen, *Phys. Lett.* 215B (1988) 119, *Nucl. Phys.* B320 (1989)649.
15. D. Karabali and H. J. Schnitzer, *Nucl. Phys.* B329 (1990) 649.
16. E. Witten, “On Holomorphic Factorization of WZW and Coset Models” IASSNS-91-25.
17. E. Witten, “The N matrix model and the gauged WZW model” IASSNS-91-26.

18. E. Witten *Phys. Rev.* D44 (1991) 314.

19. T. Eguchi *Mod. Phys. Lett.* A7 (1992) 85.

20. M. Spigelglas and S. Yankielowicz “$G_k$ Topological Field Theories by Coseting

   $G_k$; “Fusion Rules As Amplitudes in $G_k$ Theories,”

21. Bernard and G, Felder *Comm. Math. Phys.* (1991) 145.

22. P. Goddard, W. Nahm and D. Olive *Phys. Lett.* 160B (1985) 111.

23. M. Chu, P. Goddard, I. Haliday, D. Olive and A. Schwimmer *Phys. Lett.* 266B (1991) 71.

24. Y. Frishman and J. Sonnenschien *Nucl. Phys.* B294 (1987) 801, “Bosonization

   and QCD in two dimensions” TAUP-1981-92.

25. M. Wakimoto, *Comm. Math. Phys.* 104 (1989) 605.

26. K. Johnson *Phys. Lett.* 5 (1963) 253.

27. R. Dijkgraaf, E. Verlinde, and H. Verlinde, *Nucl. Phys.* B352 (1991) 59;

   “Notes On Topological String Theory And 2D Quantum Gravity,” Princeton

   preprint PUPT-1217 (1990).

28. R. Bott L. W. Tu “Differential Forms in Algebraic Topology”, Springer-Verlag

   NY 1982.

29. M. Bershadsky, W. Lerche, D. Nemeshansky and N. P. Warner “A BRST

   operator for non-critical W strings” CERN-TH 6582/92.

30. P. Bowknebt, J. Mc Carthy and K. Pilch Cern Preprint TH-6162/91.

31. M. Speigelglas, *Phys. Lett.* 274B (1992) 21.

32. A. Polyakov and P. B. Wigmann *Phys. Lett.* 131B (1983) 121.
APPENDIX

Mixed symmetries in non-abelian bosonizations Let us start with the simple case of abelian bosonization of two Dirac fermions. The fermionic action is

\[ S = \int d^2z \sum_{i=1}^{2} [\psi_i^\dagger \bar{\partial} \psi_i + \bar{\psi}_i^\dagger \partial \bar{\psi}_i] \quad (A.1) \]

Each sector is obviously invariant under holomorphic (and anti-holomorphic) transformations generated by \( J_f^i = \psi_i^\dagger \psi_i \) and \( T_f^i = \frac{1}{2} [\psi_i^\dagger \bar{\partial} \psi_i - \partial \bar{\psi}_i^\dagger \psi_i] \). In addition the current

\[ J^f = \psi_1^\dagger \psi_2 - \psi_2^\dagger \psi_1 \quad (A.2) \]

is also a holomorphic current. Its dimension is (1,0) and \( \bar{\partial} J_f = 0 \) since the fermions \( \psi, \bar{\psi}^\dagger \) are all holomorphic by the equation of motion. Under \( J_f \) one has the transformations:

\[ \delta \psi_1 = -\epsilon(z) \psi_2 \quad \delta \psi_2 = \epsilon(z) \psi_1 \quad (A.3) \]

which obviously leave the action invariant.

The corresponding bosonic system has the action

\[ S = \int d^2z \sum_{i=1}^{2} \partial \phi_i \bar{\partial} \phi_i \quad (A.4) \]

Again the symmetries of each sector are \( J_b^i = \partial \phi_i \) and \( T_b^i = \partial \phi_i \bar{\partial} \phi_i \) and in addition there are some mixed symmetries. We want to identify what is the symmetry which corresponds to \( J_f \). The transformations \( \delta \phi_1 = -\epsilon \phi_2 \quad \delta \phi_2 = -\epsilon \phi_1 \) leave the action invariant but the corresponding current

\[ J^b = \phi_1 \partial \phi_2 - \phi_2 \partial \phi_1, \quad \bar{\partial} J^b = \bar{\partial} \phi_1 \partial \phi_2 - \bar{\partial} \phi_2 \partial \phi_1 \neq 0 \quad (A.5) \]

is not holomorphically conserved. The corresponding vector current is conserved namely: \( \bar{\partial} J^b + \partial \bar{J}^b = 0 \) where \( \bar{J}^b = \phi_1 \bar{\partial} \phi_2 - \phi_2 \bar{\partial} \phi_1 \). The difference between the
chiral and vector conservations manifests itself in the nature of $\epsilon$ the parameter of transformation. The action is invariant under global transformations only and not with $\epsilon(z)$ or $\bar{\epsilon}(\bar{z})$. Notice that unlike the usual currents $J^b_i$ where the conservation of $J^V_\mu$ implies the conservation of $J^A_\mu = \epsilon_{\mu\nu}J^{V\nu}$, here this relation does not hold. This is of course equivalent to the non-invariance of the chiral current. It is thus obvious that this current cannot correspond to $J^f$. The bosonic version of $J^f$ can be obtained by bosonizing separately $\psi_1$ and $\psi_2$. The result is given by

$$J^b = :e^{-i\phi_1}::e^{i\phi_2}:+ :e^{-i\phi_2}::e^{i\phi_1}:= 2 :\cos(\phi_1 - \phi_2) : .$$

(A.6)

Notice that here $\phi = \phi(z)$. (This can be written in Mandelstam formulation).

Therefore, by construction $\bar{\partial}J = 0$. Using the transformations

$$\delta \partial \phi_1 = -\epsilon(z) : \sin(\phi_2 - \phi_1) : \quad \delta \partial \phi_2 = \epsilon(z) : \sin(\phi_2 - \phi_1) :$$

(A.7)

it is easy to verify that the action is indeed invariant: $\delta S^b = \int d^2z \partial \phi = \int d^2z \partial \phi = \int d^2z : \sin(\phi_2 - \phi_1) : = 0$. In addition the dimension of $J^b$ is $(1, 0)$. So we have identified the bosonized version of $J^f$.

Let us now pass to the non-abelian case. As discussed in section 2 the bosonized theory of Dirac fermions in the adjoint representation is the sum of two WZW models at level $N$ for $H = SU(N)$. Following the discussion above, in addition to the symmetries of each sector there are also some mixed symmetries. The current relevant to the twisting is

$$J^f_\# = Tr_H[\psi_1 \psi_2] = \psi_1^a \psi_2^a$$

(A.8)

For each index $a$ the transformations are those given in eqn.(A.3). The Kac-Moody level of this current is $N^2 - 1$. In analogy to the abelian bosonization we construct the operator

$$J^b_\# = iTr[ u_1^{-1}(z)u_2(z) - u_2^{-1}(z)u_1(z)]$$

(A.9)

The chiral fields $u(z), \bar{u}(\bar{z})$ are the non-abelian analogs of $e^{i\phi(z)}$ and $e^{i\bar{\phi}(\bar{z})}$. Classically, the solution of the equation of motion can be written as $g(z, \bar{z}) = \rho(z, \bar{z})$.
$u(z)\bar{u}(\bar{z})$ just like $\phi(z, \bar{z}) = \phi(z) + \bar{\phi}(\bar{z})$ for the abelian case. The quantization of these chiral group elements, including a careful treatment of the ambiguities involved in their definitions is discussed in ref. [23]. For $u_1, u_2$ in the adjoint representation of $SU(N)$ at level $N$, the dimension of $u(z)$ is $\Delta = \frac{C_G}{k+N} = \frac{N}{N+N} = \frac{1}{2}$. Thus, $J^b_H$ has dimension $(1, 0)$ (by construction $\bar{\partial} J^b_H = 0$). Notice that the Kac-Moody algebra of $J^b_H$ is of level $N^2 - 1$ so it matches that of $J^f_H$.

Twisting the energy momentum tensor with $\partial J^b_H$ produces a shift of $-3(N^2 - 1)$ as requested. So this twisted $T$ leads to the $(1, 0)$ system in the adjoint representation. To introduce an action that corresponds to this energy-momentum tensor, it seems that one has to write some non-local term. If $J^b_H = \partial \phi$ then we can add a term of the form $\phi R$, alternatively if $R = \partial \bar{\partial} \sigma$, then one can add the term $J^b_H \bar{\partial} \sigma$. 