Ascoli’s theorem for pseudocompact spaces

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Abstract
A Tychonoff space \( X \) is called (sequentially) Ascoli if every compact subset (resp. convergent sequence) of \( C_k(X) \) is equicontinuous, where \( C_k(X) \) denotes the space of all real-valued continuous functions on \( X \) endowed with the compact-open topology. The classical Ascoli theorem states that each compact space is Ascoli. We show that a pseudocompact space \( X \) is Ascoli iff it is sequentially Ascoli iff it is selectively \( \omega \)-bounded. The class of selectively \( \omega \)-bounded spaces is studied.

Keywords \( C_k(X) \) · Ascoli · Sequentially Ascoli · Selectively \( \omega \)-bounded · Pseudocompact · Compact-covering map

Mathematics Subject Classification 54A05 · 54B05 · 54C35 · 54D30

1 Introduction

All topological spaces in the article are assumed to be Tychonoff. We denote by \( C_k(X) \) the space \( C(X) \) of all continuous real-valued functions on a space \( X \) endowed with the compact-open topology. One of the basic theorems in Analysis is the Ascoli theorem which states that if \( X \) is a \( k \)-space, then every compact subset of \( C_k(X) \) is equicontinuous. For the proof of the Ascoli theorem and various its applications see for example the classical books [8,9] or [23]. The Ascoli theorem motivates us in [4] to introduce and study the class of Ascoli spaces. A space \( X \) is called Ascoli if every compact subset of \( C_k(X) \) is equicontinuous. Recall that a subset \( H \) of \( C(X) \) is equicontinuous if for every \( x \in X \) and each \( \varepsilon > 0 \) there is an open neighborhood \( U \) of \( x \) such that \( |f(y) - f(x)| < \varepsilon \) for all \( y \in U \) and \( f \in H \). Being motivated by the classical notion of \( c_0 \)-barrelled locally convex spaces and the fact that in many highly important cases in analysis only convergent sequences are considered (as in the Lebesgue dominated convergence theorem), we defined in [13] a space \( X \) to be sequentially Ascoli if every convergent sequence in \( C_k(X) \) is equicontinuous. Clearly, every Ascoli space is sequentially Ascoli, but the converse is not true in general (every non-discrete \( P \)-space is sequentially Ascoli but not Ascoli, see [13]). Ascoli and sequentially Ascoli spaces in
various classes of topological, function and locally convex spaces are thoroughly studied in [2,4,11,12,14–17,24].

By the Ascoli theorem every compact space is Ascoli. Although the compact spaces form the most important class of topological spaces, there are other classes of compact-type topological spaces (as sequentially compact or countably compact spaces etc.) which play a considerable role both in analysis and general topology, see for example [9,18,23] or the articles [22,26]. The most general class of compact-type spaces is the class of pseudocompact spaces. Recall that a space \( X \) is called *pseudocompact* if every continuous function on \( X \) is bounded. So the following question arises naturally: Which pseudocompact spaces \( X \) are (sequentially) Ascoli? A partial answer to this question was obtained in [14] where showed that totally countably compact spaces and near sequentially compact spaces are sequentially Ascoli, however, there are countably compact spaces which are not sequentially Ascoli (for definitions see Sect. 2).

Let \( X \) be a pseudocompact space. We denote by \( \beta X \) the Stone-Čech compactification of \( X \), and let \( \beta : X \to \beta X \) be the canonical embedding. Then the adjoint (or restriction) map \( \beta^* : C(\beta X) \to C_k(X) \), \( \beta^*(f) = f \circ \beta \), is a continuous linear isomorphism from the Banach space \( C(\beta X) \) onto \( C_k(X) \). One of the most important properties of continuous functions is the property of being compact-covering. A continuous function \( f : X \to Y \) between topological space \( X \) and \( Y \) is called *compact-covering* if for every compact subset \( K \) of \( Y \) there is a compact subset \( C \) of \( X \) such that \( f(C) = K \). It is well known that perfect mappings are compact-covering [9, Theorem 3.7.2], and compact-covering functions are important for the study of functions spaces as \( C_k(X) \), see [21]. Therefore one can ask: For which pseudocompact spaces \( X \) the adjoint map \( \beta^* : C(\beta X) \to C_k(X) \) is compact-covering?

In [10], Frolík defined the class \( \mathcal{P}^* \) consisting of spaces with the following property: each infinite collection of disjoint open sets has an infinite subcollection each of which meets some fixed compact set. We shall say that a space \( X \) is *selectively* \( \omega \)-*bounded* if \( X \) belongs to the class \( \mathcal{P}^* \). Our terminology is explained by the possibility to “select” special (sub)sequences and the classical notion of \( \omega \)-bounded spaces as the following proposition shows (recall that a space \( X \) is \( \omega \)-bounded if every sequence in \( X \) has compact closure).

**Proposition 1.1** [25, Lemma 3.3] A space \( X \) is selectively \( \omega \)-bounded if and only if for any sequence \( \{U_n\}_{n \in \omega} \) of nonempty open subsets of \( X \) there exists a sequence \( \{x_n\}_{n \in \omega} \in \prod_{n \in \omega} U_n \) containing a subsequence \( \{x_{n_k}\}_{k \in \omega} \) with compact closure.

Clearly, \( \omega \)-bounded (in particular, compact) spaces and sequentially compact spaces are selectively \( \omega \)-bounded, and every selectively \( \omega \)-bounded space is pseudocompact. In Sect. 2 we establish basic properties of selectively \( \omega \)-bounded spaces.

The next theorem answers the aforementioned questions and is the main result of the paper.

**Theorem 1.2** For a pseudocompact space \( X \) the following assertions are equivalent:

(i) \( X \) is selectively \( \omega \)-bounded;
(ii) the adjoint map \( \beta^* : C(\beta X) \to C_k(X) \) is compact-covering;
(iii) \( X \) is an Asoli space;
(iv) \( X \) is a sequentially Asoli space.

We prove Theorem 1.2 in Sect. 3. Recall (see [3]) that a topological space \( Z \) is called *\( k \)-metrizable* if there exists a bijective, compact-covering continuous function \( \Phi \) from a metrizable space \( M \) onto \( Z \). By Theorem 1.2, if a space \( X \) is selectively \( \omega \)-bounded, then the space \( C_k(X) \) is \( k \)-metrizable. In the last Sect. 4 we show that the converse assertion is not true in general.
Proposition 1.3 There is a countably compact, non-selectively $\omega$-bounded space $X$ such that the space $C_k(X)$ is $k$-metrizable.

2 Basic properties of selectively $\omega$-bounded spaces

In this section we collect some basic properties of the class of selectively $\omega$-bounded spaces and show relationships of this class with other important classes of pseudocompact spaces. In what follows we shall use Proposition 1.1 without mentioning.

Proposition 2.1 Let $f : X \to Y$ be a continuous function from a space $X$ onto a space $Y$. If $X$ is selectively $\omega$-bounded, then so is $Y$.

Proof Let $\{V_n\}_{n \in \omega}$ be a sequence of nonempty open subsets of $Y$. Then $\{f^{-1}(V_n)\}_{n \in \omega}$ is a sequence of nonempty open subsets of $X$. Since $X$ is selectively $\omega$-bounded, there exists a sequence $(x_n)_{n \in \omega} \subseteq \prod_{n \in \omega} f^{-1}(V_n)$ containing a subsequence $(x_{n_k})_{k \in \omega}$ with compact closure. Then $(f(x_{n_k}))_{k \in \omega}$ is a subsequence of the sequence $(f(x_n))_{n \in \omega} \subseteq \prod_{n \in \omega} V_n$ such that $\{f(x_{n_k}) : k \in \omega\} = f([x_{n_k} : k \in \omega])$ is compact in $Y$. Thus $Y$ is selectively $\omega$-bounded. $\square$

Proposition 2.2 If a dense subspace $Y$ of a space $X$ is selectively $\omega$-bounded, then so is $X$.

Proof Let $\{U_n\}_{n \in \omega}$ be a sequence of nonempty open subsets of $X$. Then $\{Y \cap U_n\}_{n \in \omega}$ is a sequence of nonempty open subsets of $Y$. Since $Y$ is selectively $\omega$-bounded, there exists a sequence $(x_n)_{n \in \omega} \subseteq \prod_{n \in \omega} Y \cap U_n$ containing a subsequence $(x_{n_k})_{k \in \omega}$ with compact closure (in $Y$). It is clear that $(x_{n_k})_{k \in \omega} \subseteq \prod_{n \in \omega} U_n$ and $(x_{n_k})_{k \in \omega}$ has compact closure in $X$. Thus $X$ is selectively $\omega$-bounded. $\square$

Let $\mathcal{A}$ and $\mathcal{B}$ be family of subsets of a set $\Omega$. We say that the family $\mathcal{B}$ swallows the family $\mathcal{A}$ if for every $A \in \mathcal{A}$ there is $B \in \mathcal{B}$ such that $A \subseteq B$.

Proposition 2.3 Let $X = \bigcup_{n \in \omega} X_n$, where $\{X_n\}_{n \in \omega}$ is an increasing sequence of subspaces of $X$ swallowing the compact sets of $X$. If $X$ is selectively $\omega$-bounded, then there is an $m \in \omega$ such that $X_m$ is dense in $X$. Consequently, if all $X_n$ are selectively $\omega$-bounded, then $X$ is selectively $\omega$-bounded if and only if $X_m$ is dense in $X$ for some $m \in \omega$.

Proof Suppose for a contradiction that $X_n \neq X$ for all $n \in \omega$. Then $X \setminus \bigcup_{n \in \omega} X_n$ is a sequence of nonempty open subsets of $X$. As $X$ is selectively $\omega$-bounded, there exists a sequence $(x_n)_{n \in \omega} \subseteq \prod_{n \in \omega} X \setminus \bigcup_{n \in \omega} X_n$ containing a subsequence $(x_{n_k})_{k \in \omega}$ with compact closure $K$ in $X$. Observe that $x_{n_k+1} \notin X_k$ for every $k \in \omega$. Since $\{X_n\}_{n \in \omega}$ swallows the compact sets of $X$, there is an $m \in \omega$ such that $K \subseteq X_m$. But then $x_{n_{m+1}} \in X_m$, a contradiction. The second assertion follows from the first one and Proposition 2.2. $\square$

Proposition 2.4 If every countable subset of a space $X$ is contained in a selectively $\omega$-bounded subspace of $X$, then $X$ is selectively $\omega$-bounded.

Proof Let $\{U_n\}_{n \in \omega}$ be a sequence of nonempty open subsets of $X$. For every $n \in \omega$, take a point $y_n \in U_n$. By our hypothesis, the sequence $\{y_n\}_{n \in \omega}$ is contained in a selectively $\omega$-bounded subspace $Y$ of $X$. Therefore there is a sequence $(x_n)_{n \in \omega} \subseteq \prod_{n \in \omega} Y \cap U_n$ containing a subsequence $(x_{n_k})_{k \in \omega}$ with compact closure in $Y$ and hence in $X$. Thus $X$ is selectively $\omega$-bounded. $\square$
Proposition 2.5 A clopen subspace $X$ of a selectively $\omega$-bounded space $Y$ is selectively $\omega$-bounded.

Proof Let $\{U_n\}_{n \in \omega}$ be a sequence of nonempty open subsets of $X$. Since $X$ is open in $Y$, $\{U_n\}_{n \in \omega}$ is also a sequence of nonempty open subsets in $Y$. Therefore there is a sequence $(x_n)_{n \in \omega} \in \prod_{n \in \omega} U_n$ containing a subsequence $(x_{n_k})_{k \in \omega}$ with compact closure $K := \{x_{n_k} : k \in \omega\}$ in $Y$. As $X$ is also closed in $Y$ we have $K \subseteq Y$, and hence $K$ is compact also in $X$. Thus $X$ is selectively $\omega$-bounded. \(\square\)

Note that closed subspaces (as well as open subspaces) of selectively $\omega$-bounded spaces can be even not pseudocompact, see Example 2.9 below.

To show that the property of being a selectively $\omega$-bounded space is productive we need the following lemma.

Lemma 2.6 [25, Theorem 3.4] A countable product of selectively $\omega$-bounded spaces is selectively $\omega$-bounded.

Let $\{X_i : i \in I\}$ be a family of sets, and let $p$ be a point of the product $X = \prod_{i \in I} X_i$. Then the subset

$$\Sigma(p, X) := \{x \in X : |\{i \in I : x(i) \neq p(i)\}| \leq \omega\}$$

of $X$ is called the $\Sigma$-product of $\{X_i : i \in I\}$ with the base point $p \in X$.

The following proposition extends Theorem 3.4 of [25] which states that the class of selectively $\omega$-bounded spaces is closed under arbitrary products.

Proposition 2.7 Let $\{X_i : i \in I\}$ be a family of topological spaces, and let $X = \prod_{i \in I} X_i$. Then the following assertions are equivalent:

(i) all $X_i$ are selectively $\omega$-bounded;
(ii) for every point $p \in X$, the space $\Sigma(p, X)$ is selectively $\omega$-bounded;
(iii) $X$ is selectively $\omega$-bounded.

Proof (i) $\Rightarrow$ (ii) Fix an arbitrary point $p \in X$. To apply Proposition 2.4 we have to show that each countable subset $A$ of the space $\Sigma(p, X)$ is contained in a selectively $\omega$-bounded subspace of $X$. The definition of $\Sigma(p, X)$ and the countability of $A$ imply that the set

$$I_0 := \{i \in I : a(i) \neq p(i) \text{ for some } a \in A\}$$

is countable. It is clear that the set $A$ is contained in the subspace

$$Y := \{x \in \Sigma(p, X) : \{i \in I : x(i) \neq p(i)\} \subseteq I_0\}$$

of $\Sigma(p, X)$ which is homeomorphic to the product $\prod_{i \in I_0} X_i$. Now Lemma 2.6 implies that $Y$ is selectively $\omega$-bounded.

(ii) $\Rightarrow$ (iii) For every $p \in X$, the $\Sigma(p, X)$ is dense in $X$ and, by hypothesis, is selectively $\omega$-bounded. Now Proposition 2.2 implies that $X$ is selectively $\omega$-bounded.

(iii) $\Rightarrow$ (i) Assume that $X$ is selectively $\omega$-bounded. Then all $X_i$, being projections of $X$, are selectively $\omega$-bounded by Proposition 2.1. \(\square\)

Corollary 2.8 A product of locally compact pseudocompact spaces is selectively $\omega$-bounded.

Proof Let $\{X_i : i \in I\}$ be a family of locally compact pseudocompact spaces. Since any locally compact space is Ascoli, Theorem 1.2 implies that all spaces $X_i$ are selectively $\omega$-bounded and Proposition 2.7 applies. \(\square\)
For example, if \( \kappa \) is an ordinal of uncountable cofinality, then the ordered space \([0, \kappa)\) is locally compact and countably compact, for details see [9]. Therefore, by Corollary 2.8, the space \([0, \kappa)^\lambda\) is selectively \( \omega \)-bounded for every cardinal \( \lambda \).

Now we compare the class of selectively \( \omega \)-bounded spaces with other important classes of pseudocompact spaces. We recall that a space \( X \) is called

- **sequentially compact** if every sequence in \( X \) has a convergent subsequence;
- **totally countably compact** if every sequence in \( X \) has a subsequence with compact closure;
- **near sequentially compact** if for any sequence \( \{U_n\}_{n \in \omega} \) of open subsets of \( X \) there exists a sequence \( (x_n)_{n \in \omega} \in \prod_{n \in \omega} U_n \) containing a convergent subsequence \( (x_{n_k})_{k \in \omega} \);
- **countably compact** if every sequence in \( X \) has a cluster point.

Near sequentially compact spaces were introduced and studied by Dorantes-Aldama and Shakhmatov [7], who called them selectively sequentially pseudocompact spaces. Later those spaces were applied in [5] to the study of the Josefson–Nissenzweig property in the realm of locally convex spaces. Totally countably compact spaces, introduced by Frolík, were intensively studied by Vaughan in [28]. Evidently, totally countably compact spaces and near sequentially compact spaces are selectively \( \omega \)-bounded.

**Example 2.9** Let \( \Psi(A) \) be the Mrówka–Isbell space associated with a maximal almost disjoint family \( A \) on the discrete space \( \omega \), see [9, 3.6.I]. The space \( \Psi(A) \) is locally compact (hence Ascoli) and pseudocompact. Moreover, by Example 2.6 of [7], \( \Psi(A) \) is near sequentially compact. From the definition of the space \( \Psi(A) \) it follows that the subspace \( \Psi(A) \setminus \omega \) of \( \Psi(A) \) is closed, discrete and has cardinality \( \omega \) (in particular, \( \Psi(A) \) is not sequentially compact).

Therefore closed subspaces of near sequentially compact spaces and hence selectively \( \omega \)-bounded spaces can be even not pseudocompact. Observe also that, by Corollary 4.4 of [7], the power \( \Psi(A)^\lambda \) is near sequentially compact for every cardinal \( \lambda \). □

**Example 2.10** It is easy to see that every infinite near sequentially compact space has non-trivial convergent sequences. Therefore, the Stone-Čech compactification \( \beta\omega \) of the discrete space \( \omega \) is not near sequentially compact. □

Examples 3.2, 2.9 and 2.10 and Examples 2.11 and 2.14 from [28] show that none of the implications in the following diagram (which summarizes the relationships between the above-defined notions) is in general reversible

\[
\text{\( \omega \)-bounded} \quad \Rightarrow \quad \text{totally countably compact} \quad \Rightarrow \quad \text{countably compact} \quad \Rightarrow \quad \text{pseudocompact}
\]

\[
\text{sequentially compact} \quad \Rightarrow \quad \text{near sequentially compact} \quad \Rightarrow \quad \text{selectively \( \omega \)-bounded} \quad \Rightarrow \quad \text{Th.1.2 (sequentially) Ascoli pseudocompact}
\]

It is well known that the product of a sequentially compact space and a pseudocompact space is pseudocompact (see [9, Theorem 3.10.37]), and by [9, Theorem 3.10.26], the product of a pseudocompact \( k \)-space and a pseudocompact space is pseudocompact. The next assertion generalizes these results.

**Proposition 2.11** [10, Theorem 3.5] The product \( X \times Y \) of a selectively \( \omega \)-bounded (=sequentially Ascoli pseudocompact) space \( X \) and a pseudocompact space \( Y \) is pseudocompact.

As a corollary we obtain the following non-trivial result of Glicksberg [19, Theorem 4(a)].
Corollary 2.12 Let \( \{X_i\}_{i \in I} \) be a family of pseudocompact spaces such that all but one \( X_i \) is locally compact. Then the product \( X = \prod_{i \in I} X_i \) is pseudocompact.

**Proof** If all spaces \( X_i \) are locally compact, then, by Corollary 2.8, \( X \) is even selectively \( \omega \)-bounded. Assume now that there is (unique) \( i_0 \in I \) such that \( X_{i_0} \) is not locally compact. Then the product \( T := \prod_{i \in I \setminus \{i_0\}} X_i \) is selectively \( \omega \)-bounded by Corollary 2.8. Thus the product \( X = T \times X_{i_0} \) is pseudocompact by Proposition 2.11.

\[ \square \]

3 Proof of Theorem 1.2

Let \( X \) be a (Tychonoff) space. Then the sets of the form

\[ [K; \varepsilon] := \{ f \in C(X) : |f(x)| < \varepsilon \text{ for all } x \in K \}, \]

where \( K \subseteq X \) is compact and \( \varepsilon > 0 \), form a base of the compact-open topology on \( C(X) \).

The space \( C(X) \) endowed with the pointwise topology is denoted by \( C_p(X) \).

Recall that a continuous function \( f : X \to Y \) between topological spaces \( X \) and \( Y \) is called sequence-covering if for every convergent sequence \( S \) in \( Y \) with the limit point there is a convergent sequence \( C \subseteq X \) such that \( f(C) = K \).

**Proposition 3.1** Let \( X \) be a subspace of a space \( Y \) such that the adjoint map \( i^* : C_k(Y) \to C_k(X) \) of the identical embedding \( i : X \hookrightarrow Y \) is surjective. If \( i^* \) is compact (sequence) covering and \( Y \) is a (resp. sequentially) Ascoli space, then so is \( X \).

**Proof** Let \( K \) be a compact subset (or a convergent sequence) in \( C_k(X) \). We have to show that \( K \) is equicontinuous. Fix a point \( x_0 \in X \) and \( \varepsilon > 0 \). Choose a compact subset (or a convergent sequence) \( C \) in \( C_k(Y) \) such that \( i^*(C) = K \). Since \( Y \) is (sequentially) Ascoli, there is an open neighborhood \( U \) of \( x_0 \) in \( Y \) such that

\[ |g(y) - g(x_0)| < \varepsilon \text{ for all } y \in U \text{ and } g \in C. \]  \hspace{1cm} (1)

For every \( x \in U \cap X \) and each \( f \in C \), take \( g \in C \) such that \( f = g \circ i \) and then (1) implies

\[ |f(x) - f(x_0)| = |g(i(x)) - g(i(x_0))| < \varepsilon. \] Thus \( K \) is equicontinuous. \( \square \)

Now we are able to prove our main result.

**Proof of Theorem 1.2** (i) \( \Rightarrow \) (ii) Assume that \( X \) is selectively \( \omega \)-bounded, and let \( K \) be a compact subset of \( C_k(X) \). We will show that the closed subset \( C := (\beta^*)^{-1}(K) \) of the Banach space \( C(\beta X) \) is compact. Suppose for a contradiction that \( C \) is not compact. Since \( C(\beta X) \) is complete and \( C \) is closed, it follows that \( C \) is not precompact in \( C(\beta X) \). Therefore, by [6, Theorem 5], there exist a sequence \( \{f_n\}_{n \in \omega} \subseteq C \) and \( \varepsilon > 0 \) such that

\[ \|f_n - f_m\|_{\infty} > \varepsilon \text{ for all distinct } n, m \in \omega, \]  \hspace{1cm} (2)

where \( \|f\|_{\infty} \) denotes the sup-norm of \( f \in C(\beta X) \).

It is clear that \( K \) is compact also in the space \( C_p(X) \). Since \( X \) is pseudocompact, Theorem III.4.22 of [1] implies that \( K \) is an Eberlein compact, and hence \( K \) is Fréchet–Urysohn by Theorem III.3.6 of [1]. Therefore, passing to a subsequence if needed, we can assume that the sequence \( \{f_n|_X\}_{n \in \omega} \) converges in \( C_k(X) \) to some function \( g \in K \). Replacing \( K \) by \( K - g \), we can also suppose that \( g = 0 \) is the zero function.

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Since $X$ is dense in $\beta X$, (2) implies that for every $n \in \omega$, the open set
\[ U_n := \{ x \in X : |f_n(x) - f_{n+1}(x)| > \varepsilon \} \]
is not empty. As $X$ is selectively $\omega$-bounded, there exists a sequence $(x_n)_{n \in \omega} \in \prod_{n \in \omega} U_n$ containing a subsequence $(x_{n_k})_{k \in \omega}$ whose closure $S := \{ x_{n_k} : k \in \omega \}$ is a compact subset of $X$.

Now, since $f_n \to 0$ in $C_k(X)$, there is an $m \in \omega$ such that $f_n \in [S; \frac{\varepsilon}{3}]$ for all $n \geq m$. In particular, we have
\[ |f_{n_k}(x_{n_k}) - f_{n_k+1}(x_{n_k})| < \frac{2\varepsilon}{3} \quad (3) \]
for all sufficiently large $k \in \omega$. But since $x_{n_k} \in U_{n_k}$ for all $k \in \omega$, (3) contradicts the choice of the open sets $U_n$. This contradiction shows that $C$ is compact in $C(\beta X)$, and hence the map $\beta^*$ is compact-covering.

The implication (ii) $\Rightarrow$ (iii) follows from Proposition 3.1 applied to $X$ and $Y = \beta X$, and the implication (iii) $\Rightarrow$ (iv) is trivial.

(iv) $\Rightarrow$ (i) Assume that $X$ is a sequentially Ascoli space. We have to show that $X$ is selectively $\omega$-bounded. Suppose for a contradiction that $X$ is not a selectively $\omega$-bounded space. Then there exists a sequence $(U_n)_{n \in \omega}$ of nonempty open subsets of $X$ such that for every sequence $(z_n)_{n \in \omega} \in \prod_{n \in \omega} U_n$ there is no subsequence $(z_{n_k})_{k \in \omega}$ whose closure is compact.

For every $n \in \omega$, choose a point $x_n \in U_n$ and a continuous function $f_n : X \to [0, 1]$ such that $f_n(x_n) = 1$ and $f_n(X \setminus U_n) \subseteq [0]$. We claim that $f_n \to 0$ in $C_k(X)$. Indeed, fix a compact subset $K$ of $X$ and $\varepsilon > 0$. Then the choice of the sequence $(U_n)_{n \in \omega}$ implies that the set $A := \{ n \in \omega : U_n \cap K \neq \emptyset \}$ is finite (indeed, otherwise, we could choose a point $z_n \in U_n \cap K$ for every $n \in A$ and an arbitrary point $z_n \in U_n$ for every $n \in \omega \setminus A$, and then the closure of the subsequence $(z_n : n \in A)$ of $(z_n)_{n \in \omega}$ would be compact that contradicts the choice of the sequence $(U_n)_{n \in \omega}$). This means that $f_n \in [K; \varepsilon]$ for every $n \in \omega \setminus A$. Thus $f_n \to 0$. Set $S := \{ f_n \}_{n \in \omega} \cup \{ 0 \}$, so $S$ is a convergent sequence in $C_k(X)$.

For every $n \in \omega$, set $V_n := \{ x \in X : f_n(x) > \frac{1}{3} \}$; so $x_n \in V_n \subseteq U_n$. Since $X$ is pseudocompact, the family $(V_n)_{n \in \omega}$ is not locally finite (see [9, Theorem 3.10.22]), and therefore there is a point $z \in X$ such that for every neighborhood $W$ of $z$, the set $\{ n \in \omega : V_n \cap W \neq \emptyset \}$ is infinite.

Finally, to get a desired contradiction we show that the sequence $S$ is not equicontinuous. Since $X$ is sequentially Ascoli, Theorem 2.7 of [14] states that $S$ is equicontinuous if and only if $S$ is evenly continuous, i.e. the evaluation map $\psi : S \times X \to \mathbb{R}$, $\psi(f, x) := f(x)$, is continuous (see also Lemma 2.1 of [11]). Therefore it is sufficient to show that the map $\psi$ is not continuous at the point $(0, z)$. To this end, fix a $k \in \omega$ and an open neighborhood $W$ of the point $z$. Since the set $\{ n \in \omega : V_n \cap W \neq \emptyset \}$ is infinite, there is an $m > k$ such that $V_m \cap W$ contains some point $t_m$. By the definition of $V_m$ we obtain $|\psi(f_m, t_m) - \psi(0, z)| = f_m(t_m) > \frac{1}{2}$. Thus $\psi$ is not continuous at $(0, z)$. \qed

Let $X$ be a selectively $\omega$-bounded space. Then, by Theorem 1.2, every compact subset of $C_k(X)$ is metrizable. So, it is natural to ask whether the converse is true. Assume that $X$ is a pseudocompact space such that all compact subsets of $C_k(X)$ are metrizable. Is it true that $X$ is selectively $\omega$-bounded? The answer to this question is negative as the following example shows.

**Example 3.2** In [27], Terasaka constructed a separable countably compact space $X$ whose square $X \times X$ is not pseudocompact. Therefore, by Proposition 2.7, $X$ is not selectively $\omega$-bounded. Let $D$ be a countable dense subspace of $X$. Then the restriction map $C_p(D) \to C_p(D)$ is continuous and injective. Since $C_p(D)$ is a metric space, it follows that all compact subsets of $C_p(X)$ and hence of $C_k(X)$ are metrizable. \qed
Recall that a topological space $X$ is called a $k_{R}$-space if a real-valued function $f$ on $X$ is continuous if and only if its restriction $f|_{K}$ to any compact subset $K$ of $X$ is continuous. In [24], Noble proved that each $k_{R}$-space is Ascoli. Therefore we have the following implications

\[
\text{k-space} \rightarrow \text{k}_{R}\text{-space} \rightarrow \text{Ascoli} \rightarrow \text{sequentially Ascoli}.
\]

None of these implications is reversible, see [4,9,13]. We complete this remark by the next example.

**Example 3.3** There is an Ascoli pseudocompact space which is not a $k_{R}$-space. Indeed, Kato constructed in [20] a space $X$ in the class $\mathfrak{P}^{*}$ which is not a $k_{R}$-space. It remains to note that, by Theorem 1.2, $X$ is an Ascoli space. $\square$

### 4 Proof of Proposition 1.3

Let $X = \bigcup_{n \in \omega} X_{n}$, where $\{X_{n}\}_{n \in \omega}$ is an increasing sequence of subspaces of the Tychonoff space $X$. Define

\[
p : C_{k}(X) \to \prod_{n \in \omega} C_{k}(X_{n}), \quad p(f) := (f|_{X_{n}}),
\]

\[
T : \prod_{n \in \omega} C(\beta X_{n}) \to \prod_{n \in \omega} C_{k}(X_{n}), \quad T(f_{n}) := (f_{n}|_{X_{n}}).
\]

It is clear that $p$ and $T$ are continuous injective linear maps. Assume additionally that all $X_{n}$ are pseudocompact. Then $T$ is an isomorphism and we can define a bijective linear map $\Phi$ from the linear subspace $(T^{-1} \circ p)(C_{k}(X))$ of the product $\prod_{n \in \omega} C(\beta X_{n})$ of Banach spaces $C(\beta X_{n})$ into the space $C_{k}(X)$ by

\[
\Phi(h_{n}) := p^{-1}(T(h_{n})), \quad \text{where } (h_{n})_{n \in \omega} \in (T^{-1} \circ p)(C_{k}(X)).
\]

**Proposition 4.1** Let $X = \bigcup_{n \in \omega} X_{n}$, where $\{X_{n}\}_{n \in \omega}$ is an increasing sequence of subspaces of $X$ swallowing the compact sets of $X$.

(i) The map $p$ is an embedding.

(ii) If all $X_{n}$ are selectively $\omega$-bounded (with the induced topology), then the bijective map $\Phi$ is continuous and compact-covering.

**Proof** (i) Since $p$ is continuous and injective, we have show only that $p$ is open onto $p(C_{k}(X))$. To this end, fix a standard basic neighborhood $[K; \varepsilon]$ of $0 \in C_{k}(X)$, where $K \subseteq X$ is compact and $\varepsilon > 0$. Choose an $m \in \omega$ such that $K \subseteq X_{m}$. Put $[K; \varepsilon]_{m} := \{g \in C_{k}(X_{m}) : g(K) \subseteq (-\varepsilon, \varepsilon)\}$ and

\[
U := \prod_{n \in \omega, n \neq m} C_{k}(X_{n}) \times [K; \varepsilon]_{m}.
\]

Then $U$ is an open neighborhood of the zero-function in $\prod_{n \in \omega} C_{k}(X_{n})$. It is easy to see that $p(f) \in U \cap p(C_{k}(X))$ if and only if $f \in [K; \varepsilon]$. Thus $p$ is an embedding.

(ii) For every $n \in \omega$, let $\pi_{n} : C_{k}(X) \to C_{k}(X_{n})$, $\pi_{n}(f) := f|_{X_{n}}$, be the restriction map. Since, by (i), $p$ is an embedding and $T$ is continuous, the map $\Phi$ is continuous. To show that $\Phi$ is compact-covering, let $\mathcal{K}$ be a compact subset of $C_{k}(X)$. Then for every $n \in \omega$, the set $\pi_{n}(\mathcal{K})$ is compact in $C_{k}(X_{n})$, and hence, by Theorem 1.2, the set $\mathcal{K}_{n} := (\beta^{*})^{-1}(\pi_{n}(\mathcal{K}))$ is compact in $C(\beta X_{n})$. Therefore, the set...
\[ \tilde{K} := \prod_{n \in \omega} K_n \] is compact in the Fréchet space \( \prod_{n \in \omega} C(\beta X_n) \). So the set \( \mathcal{K}_0 := T^{-1}(p(\mathcal{K})) \subseteq \tilde{K} \) is compact in \( \prod_{n \in \omega} C(\beta X_n) \). Finally, since \( \Phi(\mathcal{K}_0) = \mathcal{K} \) the map \( \Phi \) is compact-covering. \( \square \)

Now we are ready to prove Proposition 1.3.

**Proof of Proposition 1.3** We shall show that there is a countably compact, non-selectively \( \omega \)-bounded space \( X \) such that the space \( C_k(X) \) is \( k \)-metrizable.

Let \( H \) be a countably compact subspace of \( \beta \omega \) containing \( \omega \) such that every compact subset of \( H \) is finite, see [28]. Set \( H_0 := H \setminus \omega = H \cap \omega^*, \) where \( \omega^* := \beta \omega \setminus \omega \) is the remainder of \( \omega \) in \( \beta \omega \). So \( \omega^* \) is a compact space and \( H_0 \) is a closed subspace of \( H \). Consider the following subspace of the compact space \( [0, \omega_1] \times \beta \omega \)

\[
X := ([0, \omega_1] \times \omega^*) \cup ([\omega_1] \times H) = ([0, \omega_1] \times \omega^*) \cup ([\omega_1] \times H_0) \cup ([\omega_1] \times \omega),
\]

and set \( Y := ([0, \omega_1] \times \omega^*) \cup ([\omega_1] \times H_0). \) Since \( X \) is the union of countably compact subspaces \( [0, \omega_1] \times \omega^* \) and \( [\omega_1] \times H \), we obtain that \( X \) is a countably compact space.

**Claim 1** \( X \) is not a selectively \( \omega \)-bounded space. Indeed, for every \( n \in \omega \), put

\[
U_n := X \cap ([0, \omega_1] \times \{n\}) = ([\omega_1, n]) \subseteq [\omega_1] \times H,
\]

so \( U_n \) is a nonempty open subset of \( X \). Then, by the choice of \( H \), there is no subsequence of \( \{([\omega_1, n])\}_{n \in \omega} \) with compact closure. Thus \( X \) is not a selectively \( \omega \)-bounded space. (Note that Claim 1 follows also from Claims 2 and 3 and Proposition 2.3.)

**Claim 2** \( Y \) is a closed selectively \( \omega \)-bounded subspace of \( X \). Indeed, since \( Y = X \setminus \bigcup_{n \in \omega} U_n \), it follows that \( Y \) is a closed subspace of \( X \). To show that \( Y \) is a selectively \( \omega \)-bounded space, let \( \{V_n\}_{n \in \omega} \) be a sequence of nonempty open subsets of \( Y \). Then for every \( n \in \omega \), the set \( V_n \) contains a point \( y_n = (t_n, \xi_n) \) with \( t_n < \omega_1 \) and \( \xi_n \in \omega^* \) (recall that \( H_0 \subseteq \omega^* \)). Choose \( t < \omega_1 \) such that \( t_n < t \) for all \( n \in \omega \). Then the closure of the sequence \( \{y_n\}_{n \in \omega} \) is contained in the compact subset \( [0, t] \times \omega^* \) of \( Y \). Thus \( Y \) is selectively \( \omega \)-bounded. The claim is proved.

For every \( n \in \omega \), set \( X_n := Y \cup \{(\omega_1, 0), \ldots, (\omega_1, n)\} \). Then, by Claim 2, \( X_n \) is a closed selectively \( \omega \)-bounded subspace of \( X \) and \( X = \bigcup_{n \in \omega} X_n \).

**Claim 3** The sequence \( \{X_n\}_{n \in \omega} \) swallows the compact sets of \( X \). Indeed, let \( K \) be a compact subset of \( X \). Then the inclusion \( K \cap ([\omega_1] \times \omega) \subseteq ([\omega_1] \times H) \) and the fact that all compact subsets of the closed subspace \( ([\omega_1] \times H) \) of \( X \) are finite imply that the set \( K \cap ([\omega_1] \times \omega) \) is finite. Therefore \( K \subseteq X_m \) where \( m = \max\{n \in \omega : (\omega_1, n) \in K\} \). Thus \( \{X_n\}_{n \in \omega} \) swallows the compact sets of \( X \), as desired.

Finally, Claim 3 and Proposition 4.1 imply that the space \( C_k(X) \) admits a stronger metrizable locally convex topology \( T \) induced from the Fréchet space \( \prod_{n \in \omega} C(\beta X_n) \) such that the identity map \( (C(X), T) \to C_k(X) \) is compact-covering. Thus the space \( C_k(X) \) is \( k \)-metrizable. \( \square \)

We finish the article with the following question.

**Problem 4.2** Characterize (pseudocompact) spaces \( X \) for which the space \( C_k(X) \) is \( k \)-metrizable.

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