Stability of shortest paths in complex networks with random edge weights

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We study shortest paths and spanning trees of complex networks with random edge weights. Edges which do not belong to the spanning tree are inactive in a transport process within the network. The introduction of quenched disorder modifies the spanning tree such that some edges are activated and the network diameter is increased. With analytic random-walk mappings and numerical analysis, we find that the spanning tree is unstable to the introduction of disorder and displays a phase-transition-like behavior at zero disorder strength $\varepsilon = 0$. In the infinite network-size limit $(N \to \infty)$, we obtain a continuous transition with the density of activated edges $\Phi$ growing like $\Phi \sim \varepsilon^2$ and with the diameter-expansion coefficient $\Upsilon$ growing like $\Upsilon \sim \varepsilon^2$ in the regular network, and first-order transitions with discontinuous jumps in $\Phi$ and $\Upsilon$ at $\varepsilon = 0$ for the small-world (SW) network and the Barabási-Albert scale-free (SF) network. The asymptotic scaling behavior sets in when $N \gg N_c$, where the crossover size scales as $N_c \sim \varepsilon^{-2}$ for the regular network, $N_c \sim \exp[\alpha^{-2}]$ for the SW network, and $N_c \sim \exp[\alpha \ln \varepsilon^{-2}]$ for the SF network. In a transient regime with $N \ll N_c$, there is an infinite-order transition with $\Phi \sim \Upsilon \sim \exp[-\alpha/(\varepsilon^2 \ln N)]$ for the SW network and $\sim \exp[-\alpha/(\varepsilon^2 \ln N/\ln \ln N)]$ for the SF network. It shows that the transport pattern is practically most stable in the SF network.

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I. INTRODUCTION

A network is a new paradigm to study complex systems in many disciplines in science [1, 2]. A complex system consists of a large number of interacting units, and the nature of the interaction determines equilibrium and dynamical properties of the system. Frequently the simplifying assumption is made that the units are arranged to form a simple pattern like a regular lattice or to interact with all others as in a mean-field theory. Recent studies, however, have revealed that the structure of complex systems is much richer [3, 4, 5]. In general this structure is captured by a network which consists of vertices representing the units and edges connecting interacting vertex pairs.

Complex networks exhibit so-called small-world phenomena: vertices are highly clustered and the average separation between vertices grows slowly with the total number of vertices. Watts and Strogatz [6] introduced a small-world (SW) network as a model for these phenomena. It is obtained from a regular lattice with edges randomly rewired with probability $p_r$. Later it was found that some complex networks have a power-law distribution $P_{\text{deg}}(z) \sim z^{-\gamma}$ of the degree $z$. The degree of a vertex is the number edges incident upon it. The class of networks with a power-law degree distribution is called the class of scale-free (SF) networks and is found in many areas including physics, computer science, biology, sociology, etc (we refer readers to Ref. [7] for examples). The Barabási-Albert model [8] generate a SF network ($\gamma = 3$) growing via a preferential attachment rule [9]. Initially one starts with $Z_0$ vertices and introduces a new vertex at each step [growing]. It is then attached to $Z$ existing vertices, which are selected with probability linearly proportional to their degree [preferential attachment].

The discovery of the new classes of networks triggers extensive research. Order-disorder phase transitions [10, 11, 12] and nonequilibrium phase transitions [13] have been studied. Interestingly, the critical behavior is described well by mean-field theories and strongly depends on the degree distribution [14]. Stability of complex networks has also been studied against a strong disorder such as a vertex dilution [14, 15, 16]. When the fraction of diluted vertices increases, a network may disintegrate into finite clusters undergoing a percolation-type transition. In this paper, we study the effect of weak disorder on the transport properties of networks.

The shortest path plays an important role for the transport within a network [3, 4, 17]. A path denotes a sequence of vertices, successive pairs of which are connected via edges. In general there exist many paths connecting two given vertices. The shortest path is the one with minimum path length among all the paths. The minimum path length is called a separation between the two vertices. Suppose one needs to send, e.g., a data packet from one computer to the other through the Internet. The shortest path provides an optimal path way since one would achieve a fastest transfer and save system resources. The shortest path is also important in studying an internal structure of a network [18]. The separation can be used as a measure of intimacy between vertices. The number of shortest paths that pass through a vertex is called the “betweenness” or “load” [19, 20, 21]. It reflects the importance of a vertex in mutual relationship or in transport. The load follows universal power-law distributions in scale-free networks [22, 23, 24].

Consider shortest paths from a vertex $s$, called the source, to all other vertices in a network. In an unweighted network the length of a path is just the number of edges that it contains and shortest paths can simply be found using the “breadth-first search algorithm” [7].
A sub-network consisting of all the shortest paths from $s$ is the *spanning tree* $T_s$, which characterizes the optimal transport pattern. Figure 1 shows an example of a network and its spanning tree. It is convenient to represent the spanning tree by a diagram in which vertices are arranged hierarchically in the ascending order of their separation from the source. Then, the shortest path to each vertex is given by a directed path on $T_s$. In general, the spanning tree does not have a tree structure. If there are degenerate shortest paths, $T_s$ contains a loop. Some edges do not belong to $T_s$. They do not contribute to any flow from/to $s$.

It has been assumed that all edges are equivalent having the same cost. However a real network would be described better with weighted edges. A weight of an edge may represent an access cost, a physical length, or an intimacy between vertices \cite{Barrat,Boguna}. For example, edges between scientists in scientific collaboration networks may have weights which depends on the number of coauthored papers \cite{Barrat}. In a weighted network, the path with the minimum number of edges is not necessarily an optimal one. In this work, we study disordered networks with randomly-weighted edges and investigate the effect of the disorder on the transport pattern. The regular network, the SW network \cite{Barabasi}, and the Barabási-Albert SF network \cite{Barabasi} are considered. This paper is organized as follows: In Sec. II, we define the shortest path and the spanning tree in the disordered network. The disorder modifies the shape of the spanning tree. The response is described for the regular and SW networks in Sec. III and for the SF network in Sec. IV. We conclude in Sec. V.

II. SHORTEST PATHS AND THE SPANNING TREE OF DISORDERED NETWORKS

Consider a disordered undirected network. An edge $e$ between two vertices $u$ and $v$ will be denoted as $e = [u; v] = [v; u]$. To each edge $e$ a non-negative weight $c(e)$ is assigned which is called the *edge cost* of $e$. Here we neglect all system-dependent details and assume that

$$c(e) = 1 + \eta(e),$$ (1)

where $\eta(e)$’s are random variables distributed independently with distribution $\mathcal{F}(\eta)$ ($\eta(e) > -1$).

In a disordered network, a *minimum-cost path* plays the role of the shortest path in a pure ($\eta = 0$) network. For given vertices $u$ and $v$, the minimum-cost path is given by the one with minimum path cost. The path cost is defined as the sum of all edge costs in the path. Without disorder ($\eta(e) = 0$ for all $e$), the path cost is equivalent to the path length and the minimum-cost path is the same as the shortest path. Hereafter, the minimum-cost path will be called the shortest path, and the minimum cost is denoted as a *distance*. The path length of the shortest path is called the *separation* of the two vertices connected by it.

The spanning tree $T_s$ of a disordered network can be found using the “Dijkstra algorithm” \cite{Dijkstra}: Divide all vertices into two sets $S$ and its complement $\bar{S}$. Initially $S = \{s\}$ and the source is assigned to a distance label $d(s) = 0$ and a separation label $l(s) = 0$. At each iteration, one selects an optimal edge $e^* = [u^*; v^*]$ that has a minimum value of $d(u) + c(u; v)$ among all edges $e = [u; v]$ with $u \in S$ and $v \in \bar{S}$. Then, the vertex $v^*$ gets the labels $d(v^*) = d(u^*) + c(u^*; v^*)$ and $l(v^*) = l(u^*) + 1$ and a predecessor label $\text{pred}(v^*) = u^*$, and is shifted from $\bar{S}$ to $S$. The iteration terminates when the set $\bar{S}$ is empty. The shortest path to each vertex is then found by tracing the predecessor iteratively back to $s$. The distance and separation from $s$ to each vertex $v$ are given by $d(v)$ and $l(v)$, respectively. The average separation $D_s = \frac{1}{N-1} \sum_{v \neq s} l(v)$ will be called a *diameter*.

In a homogeneous (e.g. regular) network and a weakly disordered network (e.g. SW network), all vertices are equivalent after an average over all disorder realizations. The diameter is independent of the source, $D_s = D$. In such cases we select the source $s$ arbitrarily. On the other hand, the SF network has a highly inhomogeneous structure. For the SF network, we select the *hub* which has the largest degree as the source $s$ since it plays the most important role in the transport \cite{Barrat}.

The spanning tree $T_s$ of a disordered network will be different from $T^0_s$ of the same network without disorder. If the disorder has a continuous distribution, with probability one all shortest paths are uniquely determined. Therefore, $T_s$ has a tree structure, whereas $T^0_s$ may have loops. Without loops, $T_s$ consists of only $(N-1)$ edges. Moreover, a vertex may have a shortest path that cannot be found in $T^0_s$. For example, a path $(s, v_1, v_9, v_8, v_6)$ in a network in Fig. 1 may be a shortest path to $v_6$, so that $T_s$ includes the edge $[v_6; v_9]$ which does not belong to $T^0_s$. Such an edge of $T_s$ that does not belong to $T^0_s$ will be called an *activated* edge. The disorder activates it to play a role in the transport. The activated edge results in a drastic change in the shape of the spanning tree and increases the network diameter. We quantify the change by the density of disorder-induced activated edges $\Phi$, which is given by the number of activated edges in $T_s$ divided by $(N-1)$, and the diameter-expansion coefficient $\Upsilon = (D_s - D^0_s)/D^0_s$ with $D_s$ ($D^0_s$) the diameter...
with (without) disorder. The activated edge emerges as a result of competition between all paths connecting a vertex to the source. So networks with different structures respond differently. In the next section we will study the evolution of the spanning trees of the regular network and the small-world network \( b \).

III. SMALL-WORLD NETWORK

Consider a regular network consisting of \( N \) vertices on a one-dimensional ring, each of which is connected up to \( Z \)th nearest neighbors with undirected edges. The small-world (SW) network is obtained by rewiring each edge with probability \( p_r \) (see Ref. [3] for a detailed procedure). Except for extreme cases with \( p_r = 0 \) (regular network) and \( p_r = 1 \) (random network), the SW network displays the small-world phenomena \( Z \). We introduce a quenched disorder in edge costs as in Eq. (1) with the disorder distribution

\[
\mathcal{F}(\eta) = \begin{cases} 
\frac{1}{2\varepsilon} & \text{for } -\varepsilon \leq \eta < \varepsilon \\
0 & \text{otherwise.}
\end{cases}
\]

Disorder strength is controlled by the parameter \( \varepsilon < 1 \). First we focus on the regular network \( (p_r = 0) \) with \( Z = 2 \), which gives us a lot of insights.

A. Regular network \( (p_r = 0) \) with \( Z = 2 \)

A regular network with \( Z = 2 \) and \( N = 18 \) is shown in Fig. 2(a). We consider the shortest path from the source \( s \). Without disorder \( (\varepsilon = 0) \), the shortest path to \( b_n \) is unique. On the other hand, there are \( n \) degenerate shortest paths to \( a_n \). So the spanning tree \( T_s^n \) has a ladder shape with diagonal rungs from \( b_n \) to \( a_{n+1} \) as shown in Fig. 2(b) \( b \). All edges \([a_n, b_n]\) are missing in \( T_s^n \).

With infinitesimal disorder, all loops break up in the spanning tree. So either \([a_{n-1}, a_n]\) or \([b_{n-1}, a_n]\) should be removed from \( T_s^n \). Consequently, \( T_s^n \) has a tree shape with a single branch for \( b_n \)'s and with sub-branches for \( a_n \)'s, cf., Fig. 2(c). This shape will be denoted as a type-B tree. The branching points are determined from recursion relations for the distances \( d(a_n) \) and \( d(b_n) \):

\[
\begin{align*}
d(a_n) &= \min\{d(a_{n-1}) + c([a_{n-1}, a_n]), \\
d(b_{n-1}) + c([b_{n-1}, a_n])\} \\
d(b_n) &= d(b_{n-1}) + c([b_{n-1}, b_n]) .
\end{align*}
\]

If \( d(a_{n-1}) + c([a_{n-1}, a_n]) < d(b_{n-1}) + c([b_{n-1}, a_n]) \), the predecessor of \( a_n \) is \( a_{n-1} \), otherwise the predecessor is \( b_{n-1} \). The recursion relation \( \mathcal{F}(\eta) \) holds if \( T_s^n \) has the type-B structure, which is valid as long as

\[
d(a_n) + c([a_n, b_n]) > d(b_n) .
\]

When it is violated, the predecessor of \( b_n \) is \( a_n, [a_n, b_n] \) becomes activated, and \( T_s^n \) changes its shape.

We introduce now a random walk interpretation of the recursion relation. Define \( X_A(n) \equiv d(a_n) - n \) and \( X_B \equiv d(b_n) - n \) and insert it into Eq. (2). Using Eq. (2), one obtains \( X_B(n) = X_B(n-1) + \eta([b_{n-1}, b_n]) \). \( X_B(n) \) can be interpreted as the coordinate of a one-dimensional random walker (walker \( B \)) after \( n \) jumps. \( X_A(n) \) is given by the minimum of \( X_A(n-1) + \eta([a_{n-1}, a_n]) \) and \( X_B(n-1) + \eta([b_{n-1}, a_n]) \). The first term also suggests that \( X_A(n) \) can be interpreted as the coordinate of another random walker (walker \( A \)) after \( n \) jumps. But its motion is constrained: After each jump, one has to compare the current position of \( A \) [the first term] with the position of \( B \) [the second term \( \equiv X_B(n) + \mathcal{O}(\varepsilon) \)], and take the minimum as \( X_A(n) \). Therefore, one may assume a hard-core repulsion between two walkers (the interaction range fluctuates by an amount of \( \mathcal{O}(\varepsilon) \)). The inequality in Eq. (2) can be rewritten as \( X_B(n) - X_A(n) < c([a_n, b_n]) = 1 + \mathcal{O}(\varepsilon) \), which imposes a constraint on the random-walk motion.

It is convenient to introduce \( X_R(n) \equiv X_B(n) - X_A(n) \). It can be interpreted as the coordinate of a one-dimensional random walker \( R \) in the presence of a fluctuating reflecting wall at \( X_R = \mathcal{O}(\varepsilon) \) and a fluctuating absorbing wall at \( X_R = 1 + \mathcal{O}(\varepsilon) \). At each time step, the walker \( R \) performs a jump of size \( \eta' \) obeying the distribution \( \mathcal{F}_R(\eta') \equiv \int d\eta_1 \int d\eta_2 \mathcal{F}(\eta_1)\mathcal{F}(\eta_2)\delta(\eta' - \eta_1 + \eta_2) \). Hereafter the boundary walls are assumed to be fixed at \( X_R = 0 \) and \( X_R = 1 \). The fluctuations do not modify the scaling behavior of \( \Phi \) and \( \Upsilon \) with a possible change in the coefficient of the leading order term.

The random walker \( R \) determines the shape of the spanning tree. If the walker does not touch either wall at \( X_R = 0 \) or 1 at a moment \( n \), then the predecessor of \( a_n \) \( (b_n) \) is \( a_{n-1} \) \( (b_{n-1}) \). When it bounces at the reflecting wall in step \( n, a_n \) has \( b_{n-1} \) as its predecessor, and the spanning tree has a new sub-branch, cf. \( a_3, a_4, \) and

![Diagram](image-url)
as in Fig. 3(c). If it collides with the absorbing wall at step \( n_0 \), the inequality (3) is violated and \( b_{n_0} \) has \( a_{n_0} \) as predecessor instead of \( b_{n_0-1} \). The same random-walk mapping can be established after an edge \([a_{n_0}, b_{n_0}]\) is activated. If one interprets \( s' = a_{n_0} \) as a new source and re-defines \( a'_{n} = a_{n+n_0} \) and \( b'_{n} = b_{n+n_0-1} \), the distances from \( s' \) to \( a'_{n} \) and \( b'_{n} \) satisfy the same recursion relation as in Eqs. (3) and (4) and the same constraint as in Eq. (5) with \( a \) (b) replaced by \( b' \) (a'). Thus, the spanning tree consists of a single branch for \( a' \)'s and sub-branches for \( b' \)'s. This shape will be denoted as a type A, see Fig. 3(d). The creation of sub-branches and the switch into type-A tree are described by the same random-walk mapping used for the type-B segment.

Combining the mappings for type-A and type-B segments, the shape of the whole spanning tree can be determined by the the random walker \( R \) in the presence of two hard-core walls at \( X_R = 0 \) and 1. Initially \( X_R(0) = 0 \), and the wall at \( X_R = 0 \) (1) is reflecting (absorbing). When the absorbing wall is at \( X_R = 1 \) (0), the spanning tree has the type-B (type-A) shape. The sub-branch emerges when the random walker collides with the reflecting wall. When it collides with the absorbing wall, the role of the two walls are exchanged and the spanning tree switches its shape. It is interesting to note that the random walk with two types of boundary walls were used to find the exact ground states of one-dimensional random-field Ising-spin chain [20].

The shape of \( T_{\phi} \) is characterized by two length scales \( \xi_1 \) and \( \xi_2 \), see Fig. 3. The former characterizes the length of the sub-branch, and the latter the length of each type-A or type-B segment. They are given by the mean time scales between successive collisions with the reflecting wall and the absorbing wall, respectively. Then, \( \xi_1 \) can be approximated as the life time of the random walker \( R \), being at the origin initially, in the presence of two absorbing walls at \( X = 0 \) and \( X = 1 \). And \( \xi_2 \) is given by the mean life time of the random walker \( R \), being at the origin initially, in the presence of two absorbing walls at \( X = \pm 1 \). Such time scales are calculated in time-continuum limit in Appendix. Using the results in Eqs. (A10) and (A11) with \( a = \sigma \) and \( \sigma^2 \equiv \int d\eta' \mathcal{F}_R(\eta')\eta'^2 = \frac{<\varepsilon^2>}{<\varepsilon>^2} \), we obtain that \( \xi_1 \simeq \sqrt{3/2} \varepsilon^{-1} \) and \( \xi_2 \equiv (3/2)\varepsilon^{-2} \). Note that the \( \mathcal{O}(\varepsilon) \) fluctuation of the walls does not change the scaling exponents, but may modify the coefficient of \( \xi_1 \).

One edge among \( Z \) edges in a row is activated when a sub-branch appears only in the type-A segment (see Fig. 3(d)). So, the activated-edge density is inversely proportional to \( \xi_1 \):

\[
\Phi_{\text{REG}}(\varepsilon) = \frac{1}{Z} - \frac{1}{Z} \xi_1^{-1} \simeq \frac{1}{\sqrt{24}} \varepsilon, \quad (6)
\]

where \( Z = 2 \) and \((Z - 1)/Z\) is the fraction of the type-A segments. When the spanning tree changes its shape at a certain vertex (cf. \( a_5 \) in Fig. 3(d)), its all descendant vertices increases their separation from \( s \) by one. So, the
approximately equal to those for the two-random-walker problem. As in the case with an activated edge \([a_1; a_0]\) appears when this inequality is violated in the \((n_0 + 1)\)th step. Then, we can use the same random-walk mapping after a cyclic permutation \((A_1, \ldots, A_Z) \rightarrow (A_2, \ldots, A_Z, A_1)\), which continues repeatedly. As in the case with \(Z = 2\), the shape of the spanning tree is characterized by the length scale \(\xi_1\), the mean time scale for a collision between adjacent walkers, and \(\xi_2\), the mean time scale for violating the constraint \(|X_1 - X_Z| \lesssim 1\). The time scales are approximately equal to those for the two-random-walker problem with a constraint \(|X_1 - X_Z| \lesssim 1/Z\). With this approximation, one gets \(\xi_1 \sim (Z\varepsilon)^{-1}\) and \(\xi_2 \sim (Z\varepsilon)^{-2}\). Using \(\Phi_{\text{REG}} = Z\varepsilon^{-1}\) and \(\Upsilon_{\text{REG}} = Z\varepsilon^{-2}\), we finally obtain \(\Phi_{\text{REG}} \propto \chi_\Phi \varepsilon^1\) and \(\Upsilon_{\text{REG}} \propto \chi_\Upsilon \varepsilon^2\) with \(Z\)-dependent coefficients \(\chi_\Phi \propto (1 - 1/Z)^\gamma\) and \(\chi_\Upsilon \propto Z\). The scaling exponents are universal for all \(Z\). We determine the coefficients of \(\varepsilon\) and \(\varepsilon^2\) for \(\Phi_{\text{REG}}\) and \(\Upsilon_{\text{REG}}\), respectively, numerically and plot them as a function of \(Z\) in the insets of Fig. 3. One sees that \(\chi_\Phi \propto (1 - 1/Z)\) and \(\chi_\Upsilon \propto Z\), as estimated above.

We conclude that quenched disorder is a relevant perturbation to the spanning tree of the regular network. Using the random-walk mapping, we have shown that a finite fraction of edges in the spanning tree are modified at nonzero disorder strength. This fraction is linearly proportional to the disorder strength, \(\Phi_{\text{REG}} \propto \varepsilon\). We have also shown that the disorder exponent coefficient is proportional to the square of the disorder strength, \(\Upsilon_{\text{REG}} \propto \varepsilon^2\).

**B. Small-world network**

Next we study the effect of the quenched disorder in the SW network. Figure 4 shows an example of a SW network and its spanning tree \(T_0^s\) without disorder \((\varepsilon = 0)\). The random rewiring of edges randomizes \(T_0^s\), too, which is not suited for an exact description. Therefore we study \(\Phi_{\text{SW}}\) and \(\Upsilon_{\text{SW}}\) first numerically. We calculate \(\Phi_{\text{SW}}\) in SW networks with \(Z = 4\) and \(p_r = 0.2\) and compare it with \(\Phi_{\text{REG}}\) in Fig. 2. \(\Phi_{\text{SW}}\) shows two noticeable features: (i) At small \(\varepsilon\), \(\Phi_{\text{SW}}\) appears to display a threshold behavior at a nonzero value of \(\varepsilon\). (ii) \(\Phi_{\text{SW}}\) has a strong size dependence. \(\Phi_{\text{SW}}\) does not approach an asymptotic saturation at any value of \(\varepsilon\) and \(N\) considered. The same feature is commonly observed at other values of \(Z\) and \(p_r\), and also for \(\Upsilon_{\text{SW}}\). In what follows we describe the response of the spanning tree with an effective random walk process and we will show that the origin of the apparent threshold behavior is the presence of a well-defined \(\varepsilon\)-dependent crossover size in the network.

All edges that connect vertices at the same hierarchy level in \(T_0^s\), as represented by dashed lines in the example in Fig. 4, are candidates for an activated edge. Focus on a pair of vertices \(a\) and \(b\) with \([a; b] \notin T_0^s\) in Fig. 4. They are descendants of a common ancestor \(s'\). The edge \([a; b]\) belongs to \(T_s\) if a difference between costs of the two paths (denoted by thick lines) from \(s'\) to \(a\) and to \(b\) is larger than the edge cost \(c([a; b]) = 1 + \eta([a; b]) = 1 + O(\varepsilon)\), where \(O(\varepsilon)\) term can be neglected. The probability with which this happens will be denoted as \(P_{\text{act.}}([a; b])\). Let \(l(a, b)\) be the separation of \(a\) and \(b\) to their common ancestor \(s'\) in \(T_0^s\). Using Eq. (4), the path costs are given by a sum of \(l(a, b)\) independent random variables \(\eta's\) (plus \(l(a, b)\)). So, with the common term \(l(a, b)\) discarded, they can be interpreted as coordinates of two one-dimensional random walkers after \(l(a, b)\) jumps. Each jump follows the distribution \(F(\eta)\). \(P_{\text{act.}}([a; b])\) is then given by the probability that the distance between two random walkers is larger than 1 after \(l\) jumps. Or equivalently, it is given by the probability that a random walker deviates from a starting position by a distance larger than
1 after \( l \) jumps, where each jump follows a distribution \( F_R(\eta') = \int d\eta \int d\eta_2 F(\eta_1) F(\eta_2) \delta(\eta' - \eta_1 + \eta_2) \). The probability distribution of the random walker after \( l \) steps is given by \( P_{RW}(x, l) = (2\pi a^2 l)^{-1/2} e^{-x^2/(2a^2 l)} \) with \( a^2 = \int d\eta \eta^2 F_R(\eta') = 2e^2/3 \). Therefore, one obtains

\[
P_{\text{act.}}([a; b]) = \text{erfc} \left( \frac{\sqrt{3}}{2\varepsilon \sqrt{l(a, b)}} \right),
\]

where \( \text{erfc}(x) \equiv \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} \, dt \) is the complementary error function.

Now we make a mean-field-type approximation that each edge \( e \notin T_0^s \) may be an activated edge independently. That is only true for the edges \( \{e_i = [a_i; b_i] \notin T_0^s \} \) if the paths from \( a_i \) and \( b_i \) to their common ancestor do not overlap for different \( e_i \)'s. Otherwise, two probabilities \( P_{\text{act.}}(e) \) and \( P_{\text{act.}}(e') \) are not independent. If \( e \) is activated, it modifies the shape of the spanning tree and hence \( P_{\text{act.}}(e') \), and vice versa. The approximation would give the correct scaling behavior if the spanning tree would be random and self-averaging. In the mean-field scheme, the activated edge density is proportional to the probability in Eq. (10) with \( l(a, b) \) replaced by the average value, i.e., the diameter of the network \( D_s \sim \ln N \): 

\[
\Phi_{SW}(N, \varepsilon) \simeq \alpha_1 \text{erfc} \left( \frac{\alpha_2}{\varepsilon \sqrt{\ln N}} \right) \tag{11}
\]

with constants \( \alpha_{1,2} \) being independent of \( \varepsilon \) and \( N \).

The scaling function behaves as \( \text{erfc}(x) \simeq 1 - \frac{2}{\sqrt{\pi}} x e^{-x^2} \) for small \( x \). So, in the asymptotic limit where \( N \gg N_c \simeq e^{-\alpha_2^2 \varepsilon^{-2}} \), the activated edge density is a constant \( [\Phi_{SW} \simeq \alpha_1] \) in the infinite \( (N \to \infty) \) network for all values of \( \varepsilon \neq 0 \). Therefore, the spanning tree of the SW network undergoes a discontinuous transition at \( \varepsilon = \varepsilon_c = 0 \) in the asymptotic limit, which is contrasted to the continuous transition in the regular network.

The asymptotic behavior sets in when the network size is bigger than the crossover size \( N_c \simeq e^{-\alpha_2^2 \varepsilon^{-2}} \). It grows very fast as \( \varepsilon \) goes to zero. For instance, when \( \varepsilon = 0.1 \) with \( \alpha_2 \simeq 1 \), only SW networks with \( N \gg N_c \simeq 10^{43} \) are in the asymptotic region, which is improbable in any real networks. So, the behavior in the non-asymptotic regime is more important in practice. Using \( \text{erfc}(x) \simeq 1 - \frac{2}{\sqrt{\pi}} x e^{-x^2} \) for \( x \gg 1 \), we obtain that

\[
\Phi_{SW} \simeq \frac{\alpha_1 \varepsilon \sqrt{\ln N}}{\alpha_2 \sqrt{\pi}} \exp \left[ -\frac{\alpha_2^2}{\varepsilon^2 \ln N} \right] \tag{12}
\]

for \( N \ll N_c \). It has an essential singularity at \( \varepsilon = 0 \) and increases continuously and extremely slowly. Therefore, the spanning tree of finite SW networks undergoes an infinite-order transition at \( \varepsilon = 0 \).

Numerical data are in good agreement with the mean-field results. In Fig. 5 we plot \( \Phi_{SW} \) as a function of the scaling variable \( \varepsilon \sqrt{\ln N} \). Data at different values of \( N \) collapse onto a single curve, which supports the scaling form in Eq. (12). In the inset, we plot \( \Phi_{SW} \) at given values of \( \varepsilon \) against \( 1/\ln N \) for \( N = 100, \ldots, 800000 \) in the semilog scale. We observe that the data align along straight lines, which support the result in Eq. (12). Extrapolating the straight lines to the limit \( \ln N \to \infty \), we obtained that \( \Phi_{SW} \)'s converge to \( \alpha_1 \simeq 0.25 \).

In the regular network, an activated edge may or may not increase the separation of vertices from the source, which leads to the different scaling behaviors of \( \Phi_{\text{REG}} \sim \varepsilon \) and \( \Upsilon_{\text{REG}} \sim \varepsilon^2 \). However, in the SW network, all activated edges considered above increase the separation. So \( \Upsilon_{SW} \) shows the same scaling behavior as \( \Phi_{SW} \), which we also confirmed numerically.

**IV. BARABÁSI-ALBERT SCALE-FREE NETWORK**

In this section we study the response of the spanning tree of a SF network to the quenched disorder. We consider the Barabási-Albert (BA) network with the degree-distribution exponent \( \gamma = 3 \). Among all vertices, the hub which has the largest degree plays an important role in the SF network [13]. So we consider the spanning tree of the BA network with the hub as the source. \( \Phi_{SF} \) is measured for the BA network with \( Z_0 = Z = 4 \), and compared with \( \Phi_{\text{REG}} \) and \( \Phi_{SW} \) in Fig. 6. \( \Phi_{SF} \) behaves similarly to \( \Phi_{SW} \).

Though the SF network is different from the SW network in many aspects, its spanning tree also has a random structure. Thus, we expect that the scaling behavior of \( \Phi_{SF} \) can be explained with the same mean-field-type approximation as \( \Phi_{SW} \). Under the approximation, each edge that does not belong to \( T_0^s \) may be activated independently. Then, the mean density of the activated edges is obtained by replacing \( l(a, b) \) in Eq. (11) with the
diameter of the network. Recently, it was found that the scale-free network is ultra small in diameter \(22\). In particular, for \(\gamma = 3\) (\(\gamma\): the degree-distribution exponent) the diameter scales as \(D \sim \ln N / \ln \ln N\), i.e., with a double logarithmic correction to the SW scaling \(D \sim \ln N\). It leads us to conclude that

\[
\Phi_{SF} = \alpha_1' \, \text{erfc} \left( \frac{\alpha_2'}{\varepsilon \sqrt{\ln N / \ln \ln N}} \right)
\]

with constants \(\alpha_1'\) and \(\alpha_2'\).

This scaling form is indeed supported by our numerical data. In Fig. 7 we plot \(\Phi_{SF}\) for the BA network with \(Z = Z_0 = 2\) as a function of the scaling variable \(\varepsilon \sqrt{\ln N / \ln \ln N}\), and obtain a good data collapse. Though the double-logarithmic correction is very weak, the data do not collapse at all without it. We note that the data also scale well with a scaling variable \(\varepsilon (\ln N)^{1/4}\), however, we presume that this is accidental since the ratio between \((\ln N / \ln \ln N)\) and \((\ln N)^{1/4}\) remains almost constant up to \(N = 800000\). The inset in Fig. 7 shows that \(\ln \Phi_{SF}\) is linear in \(1/(\varepsilon^2 \ln N / \ln \ln N)\). It implies that the scaling function has the essential singularity as in Eq. (12). The same scaling behavior is observed universally for other values of \(Z = Z_0 = 3\) and 4, and also for \(\Upsilon_{SF}\).

The SF network shares the same scaling behavior as the SW network with the scaling variable \(\varepsilon (\ln N / \ln \ln N)^{1/2}\). The spanning tree of the SF network undergoes a discontinuous transition in the asymptotic regime: \(\Phi_{SF}\) and \(\Upsilon_{SF}\) in the infinite SF networks jump from zero at \(\varepsilon = 0\) to finite values as the disorder turns on. The asymptotic behavior sets in only when \(N \gg N_c \sim \exp[\alpha \ln \varepsilon \varepsilon^{-2}]\) with a positive constant \(\alpha\). In the transient regime with \(N \ll N_c\), the spanning tree of the SF networks undergoes a continuous infinite-order transition with the essential singularity in \(\Phi_{SF}\) and \(\Upsilon_{SF}\) at \(\varepsilon = 0\). Note that the crossover size \(N_c\) of the SF network grows much faster than that of the SW network. Therefore we conclude that the scale-free network has the most stable transport pattern.

\[
\Phi = \Phi(\varepsilon, N) = \Phi(\varepsilon D^{1/2}) \quad \text{where} \quad D = \ln N / \ln \ln N
\]

for the SW and SF networks, we obtain the scaling form \(\Phi(\varepsilon, N) = \Phi(\varepsilon D^{1/2})\) where \(D\) is the network diameter. It scales as \(D \sim \ln N\) in the SW network and \(D \sim \ln \ln N\) in the BA network \(22\). The diameter-expansion-coefficient satisfies the same scaling form. The scaling function approaches a constant as its argument \(x = \varepsilon D^{1/2} \to \infty\). It has the essential singularity, \(\Phi(x) \sim \exp[-\alpha x^{-2}]\) with a positive constant \(\alpha\), as \(x \to 0\). Therefore, \(\Phi\) and \(\Upsilon\) have a discontinuous jump at \(\varepsilon = 0\) in infinite-size SW and SF networks. It shows that the SW network and the SF network are more affected by the quenched disorder than the regular network. However, the asymptotic scaling behavior emerges only when the network size is larger than the crossover size \(N_c\). It grows as \(N_c \sim \exp[\alpha \ln \varepsilon \varepsilon^{-2}]\) for the SW network and \(N_c \sim \exp[\alpha \ln \ln \varepsilon \varepsilon^{-2}]\) for the BA network with a positive constant \(\alpha\) as \(\varepsilon\) vanishes. Since the crossover size grows extremely fast as \(\varepsilon\) goes to zero, the behavior in the transient regime with \(N \ll N_c\) is very important in practice. In contrast to the discontinuous jump when \(N \gg N_c\), \(\Phi\) at \(N \ll N_c\) grows continuously and very slowly with the essential singularity at \(\varepsilon = 0\). Numerically we obtained that \(\Phi < 10^{-7}\) for \(\varepsilon \lesssim 0.1\) up to \(N = 800000\). Therefore, we conclude that the transport pattern in finite SW and SF networks is extremely robust against the quenched disorder. In particular, the SF network is most stable.

The interesting scaling behavior of the SW and SF networks can be explained within a mean-field picture. \(\Phi\) and \(\Upsilon\) are proportional to the probability that a displacement of a random walker with diffusion constant \(O(\varepsilon^2)\) is larger than 1 after \(D\), the network diameter, time steps. The crossover size is determined from the condition \(\varepsilon^2 D(N_c) \sim 1\), and the spanning tree is very stable against the disorder as long as \(N \ll N_c\).

V. CONCLUSION

In a network the shortest paths between vertices play an important role. The optimal transport pattern from/to a vertex \(s\) is characterized by the spanning tree \(T_s\), a set of all shortest paths. We have investigated the response of the spanning tree of complex networks to a quenched randomness in edge costs, and found that quenched disorder is a relevant perturbation. As the disorder turns on, the spanning tree evolves with a finite fraction of edges modified and the network diameter expands.

For the regular network, the shape of the spanning tree can be described exactly using the random-walk mapping. The spanning tree undergoes a continuous transition with \(\Phi_{REG} \sim \varepsilon\) and \(\Upsilon_{REG} \sim \varepsilon^2\) with \(\varepsilon\) the disorder strength. For the SW and SF networks, we obtain the scaling form \(\Phi(\varepsilon, N) = \Phi(\varepsilon D^{1/2})\) where \(D\) is the network diameter. It scales as \(D \sim \ln N\) in the SW network and \(D \sim \ln \ln N\) in the BA network \(22\). The diameter-expansion-coefficient satisfies the same scaling form. The scaling function approaches a constant as its argument \(x = \varepsilon D^{1/2} \to \infty\). It has the essential singularity, \(\Phi(x) \sim \exp[-\alpha x^{-2}]\) with a positive constant \(\alpha\), as \(x \to 0\). Therefore, \(\Phi\) and \(\Upsilon\) have a discontinuous jump at \(\varepsilon = 0\) in infinite-size SW and SF networks. It shows that the SW network and the SF network are more affected by the quenched disorder than the regular network. However, the asymptotic scaling behavior emerges only when the network size is larger than the crossover size \(N_c\). It grows as \(N_c \sim \exp[\alpha \ln \ln \varepsilon \varepsilon^{-2}]\) for the SW network and \(N_c \sim \exp[\alpha \ln \ln \varepsilon \varepsilon^{-2}]\) for the BA network with a positive constant \(\alpha\) as \(\varepsilon\) vanishes. Since the crossover size grows extremely fast as \(\varepsilon\) goes to zero, the behavior in the transient regime with \(N \ll N_c\) is very important in practice. In contrast to the discontinuous jump when \(N \gg N_c\), \(\Phi\) at \(N \ll N_c\) grows continuously and very slowly with the essential singularity at \(\varepsilon = 0\). Numerically we obtained that \(\Phi < 10^{-7}\) for \(\varepsilon \lesssim 0.1\) up to \(N = 800000\). Therefore, we conclude that the transport pattern in finite SW and SF networks is extremely robust against the quenched disorder. In particular, the SF network is most stable.
$D(N) \sim \ln N$ [21] and $D(N) \sim \ln N \ln N$ [22], for the small-world and the scale-free network, respectively. The diameter of the SF networks with $2 < \gamma < 3$ scales as $D(N) \sim \ln N$ [22] and within the mean-field picture one expects the crossover size of these networks to scale like $N_c \sim \exp[-\alpha \varepsilon^{-2}]$.

Another consequence of the disorder is that the shortest path from one vertex to the other becomes unique. One expects the crossover size of these networks to scale like $N_c \sim \exp[-\alpha \varepsilon^{-2}]$.

The diameter of the SF networks with $2 < \gamma < 3$ scales as $D(N) \sim \ln N$ [22] and within the mean-field picture one expects the crossover size of these networks to scale like $N_c \sim \exp[-\alpha \varepsilon^{-2}]$.

The survival probability $S(t)$ of the random walker is given by the spatial integral of the probability density:

$$S(t) = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\sin((2n+1)\pi a/L)}{(2n+1)^3}.$$  

Then the death probability during $t$ and $t+dt$ is given by $S(t) - S(t+dt) \sim D(t)dt$ with $D(t) = -\frac{dS}{dt}$. Therefore the mean life time of the walker is given as $\tau = \int_{0}^{\infty} dt D(t) = \int_{0}^{\infty} dt S(t)$. Using (A4), one obtains that

$$\tau = \frac{8L^2}{\sigma^2 \pi^2} \sum_{n=0}^{\infty} \frac{\sin((2n+1)\pi a/L)}{(2n+1)^3}.$$  

It becomes simpler in a special case. When $a = L/2$, $\tau = \frac{\pi^2}{4 \sigma^2} \beta(3)$ with $\beta(s) = \sum_{n=1}^{\infty} (1)(n+1)^{-s}$. Using $\beta(3) = \pi^4/32$ [23], one obtains that

$$\tau = \frac{L^2}{4 \sigma^2}.$$  

When $a \ll L$, one can use an expansion $\sin(n\pi a/L) \sim n\pi a/L$, which yields that $\tau = \frac{2 \pi a}{\sigma^2} \sum_{n=1}^{\infty} n^{-2}$. Using that $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ with Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ [23], one obtains

$$\tau = \frac{aL}{\sigma^2}.$$  

APPENDIX A: RANDOM WALK IN THE PRESENCE OF ABSORBING WALL

Consider a discrete-time random walk in one dimension in the presence of absorbing walls at $x = 0$ and $x = L$. The walk is killed upon a collision with the absorbing walls. Initially it is at $x = a$ and its position after $t$ steps is given by $x(t) = a + \sum_{i=1}^{t} x_i$ with random variables $x_i$’s with $\langle x_i \rangle = 0$ and $\langle x_i^2 \rangle = \sigma^2$. In the time-continuum limit, the probability density $P(x,t)$ satisfies a diffusion equation $\partial P/\partial t = D \partial^2 P/\partial x^2$ with a diffusion constant $D = \sigma^2/2$. The absorbing walls impose boundary conditions $P(0,t) = P(L,t) = 0$, and the initial condition implies $P(x,0) = \delta(x-a)$. In the Fourier series expansion $P(x,t) = \sum_{n=1}^{\infty} p_n(t) \sin \frac{n\pi x}{L}$, the diffusion equation becomes $\partial p_n/\partial t = -\left(\frac{n^2 \sigma^2}{L^2}\right) p_n$ with the solution $p_n(t) = p_n(0)e^{-\left(\frac{n^2 \sigma^2}{L^2}\right)t}$. The coefficients $p_n(0)$ are determined from the initial condition. Using the Fourier series expansion of the delta function, we obtain that

$$P(x,t) = \frac{2}{L} \sum_{n=1}^{\infty} \left(\frac{n\pi a}{L}\right) \left(\frac{n\pi x}{L}\right) e^{-\mu n^2 t}.$$  

The survival probability $S(t)$ of the random walker is given by the spatial integral of the probability density:

$$S(t) = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\sin((2n+1)\pi a/L)}{(2n+1)^3}.$$  

Then the death probability during $t$ and $t+dt$ is given by $S(t) - S(t+dt) \sim D(t)dt$ with $D(t) = -\frac{dS}{dt}$. Therefore the mean life time of the walker is given as $\tau = \int_{0}^{\infty} dt D(t) = \int_{0}^{\infty} dt S(t)$. Using (A4), one obtains that

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It becomes simpler in a special case. When $a = L/2$, $\tau = \frac{\pi^2}{4 \sigma^2} \beta(3)$ with $\beta(s) = \sum_{n=1}^{\infty} (1)(n+1)^{-s}$. Using $\beta(3) = \pi^4/32$ [23], one obtains that

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