On the spectrum of the normalized graph Laplacian

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February 1, 2008

Abstract

The spectrum of the normalized graph Laplacian yields a very comprehensive set of invariants of a graph. In order to understand the information contained in those invariants better, we systematically investigate the behavior of this spectrum under local and global operations like motif doubling, graph joining or splitting. The eigenvalue 1 plays a particular role, and we therefore emphasize those constructions that change its multiplicity in a controlled manner, like the iterated duplication of nodes.

Let Γ be a finite and connected graph with $N$ vertices. Two vertices $i, j \in \Gamma$ are called neighbors, $i \sim j$, when they are connected by an edge of Γ. For a vertex $i \in \Gamma$, let $n_i$ be its degree, that is, the number of its neighbors. For functions $v$ from the vertices of Γ to $\mathbb{R}$, we define the (normalized) Laplacian as

$$\Delta v(i) := v(i) - \frac{1}{n_i} \sum_{j \sim i} v(j). \quad (1)$$

This is different from the operator $Lv(i) := n_i v(i) - \sum_{j \sim i} v(j)$ usually studied in the graph theoretical literature as the (algebraic) graph Laplacian, see e.g. [3, 7, 10, 11, 2], but equivalent to the Laplacian investigated in [4]. This normalized Laplacian is, for example, the operator underlying random walks on graphs, and in contrast to the algebraic Laplacian, it naturally incorporates a conservation law.

We are interested in the spectrum of this operator as yielding important invariants of the underlying graph Γ and incorporating its qualitative properties. As in the case of the algebraic Laplacian, one can essentially recover the graph from its spectrum, up to isospectral graphs. The latter are known to exist, but are relatively rare and qualitatively quite similar in most respects (see e.g. [12] for a systematic discussion). For a heuristic algorithm for the algebraic Laplacian which can be easily modified for the normalized Laplacian, see [8].

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We now recall some elementary properties, see e.g. [4, 9]. The normalized Laplacian, henceforth simply called the Laplacian, is symmetric for the product 
\[(u, v) := \sum_{i \in V} n_i u(i)v(i)\] (2)
for real valued functions \(u, v\) on the vertices of \(\Gamma\). \(\Delta\) is nonnegative in the sense that \((\Delta u, u) \geq 0\) for all \(u\).
From these properties, we conclude that the eigenvalues of \(\Delta\) are real and non-negative, where the eigenvalue equation is
\[\Delta u - \lambda u = 0.\] (3)
A nonzero solution \(u\) is called an eigenfunction for the eigenvalue \(\lambda\).
The smallest eigenvalue is \(\lambda_0 = 0\), with a constant eigenfunction. Since we assume that \(\Gamma\) is connected, this eigenvalue is simple, that is
\[\lambda_k > 0\] (4)
for \(k > 0\) where we order the eigenvalues as
\[\lambda_0 = 0 < \lambda_1 \leq \ldots \leq \lambda_{N-1}.\]
with equality iff the graph is bipartite. The latter is also equivalent to the fact that whenever \(\lambda\) is an eigenvalue, then so is \(2 - \lambda\).
For a complete graph of \(N\) vertices, we have
\[\lambda_1 = \ldots = \lambda_{N-1} = \frac{N}{N-1},\] (6)
that is, the eigenvalue \(\frac{N}{N-1}\) occurs with multiplicity \(N - 1\). Among all graphs with \(N\) vertices, this is the largest possible value for \(\lambda_1\) and the smallest possible value for \(\lambda_{N-1}\).
The eigenvalue equation (3) is
\[\frac{1}{n_i} \sum_{j \sim i} u(j) = (1 - \lambda)u(i)\text{ for all }i.\] (7)
In particular, when the eigenfunction \(u\) vanishes at \(i\), then also \(\sum_{j \sim i} u(j) = 0\), and conversely (except for \(\lambda = 1\)). This observation will be useful for us below.

1 The eigenvalue 1
For the eigenvalue \(\lambda = 1\), (7) becomes simply
\[\sum_{j \sim i} u(j) = 0\text{ for all }i,\] (8)
that is, the average of the neighboring values vanishes for each \( i \). We call a solution \( u \) of (8) balanced. The multiplicity \( m_1 \) of the eigenvalue 1 then equals the number of linearly independent balanced functions on \( \Gamma \).

There is an equivalent algebraic formulation: Let \( A = (a_{ij}) \) be the adjacency matrix of \( \Gamma \); \( a_{ij} = 1 \) if \( i \) and \( j \) are connected by an edge and =0 else. Then simply means

\[
Au = \sum_j a_{ij}u(j) = 0,
\]

that is, the vector \( u(j)_{j \in \Gamma} \) is in the kernel of the adjacency matrix. Thus,

\[
m_1 = \dim \ker A.
\]

We are interested in the question of estimating the multiplicity of the eigenvalue 1 on a graph. An obvious method for this is to determine restrictions on corresponding eigenfunctions \( f_1 \). We shall do that by graph theoretical considerations, and in this sense, this constitutes a geometric approach to the algebraic question of determining or estimating the kernel of a symmetric 0-1 matrix with vanishing diagonal. [1] systematically investigated the effect of the addition of a single vertex on \( m_1 \). Here, we are also interested in the effect of more global graph operations.

We start with the following simple observation

**Lemma 1.1.** Let \( q \) be a vertex of degree 1 in \( \Gamma \) (such a \( q \) is called a pending vertex). Then any eigenfunction \( f_1 \) for the eigenvalue 1 vanishes at the unique neighbor of \( \Gamma \).

2 Motif doubling, graph splitting and joining

Let \( \Sigma \) be a connected subgraph of \( \Gamma \) with vertices \( p_1, \ldots, p_m \), containing all of \( \Gamma \)'s edges between those vertices. We call such a \( \Sigma \) a motif. The situation we have in mind is where \( N \), the number of vertices of \( \Gamma \), is large while \( m \), the number of vertices of \( \Sigma \), is small.

Let 1 be an eigenvalue of \( \Sigma \) with eigenfunction \( f_\Sigma^1 \). \( f_\Sigma^1 \) when extended by 0 outside \( \Sigma \) to all of \( \Gamma \) need not be an eigenfunction of \( \Gamma \), and 1 need not even be an eigenvalue of \( \Gamma \). We can, however, enlarge \( \Gamma \) by doubling the motif \( \Sigma \) so that the enlarged graph also possesses the eigenvalue 1, with a localized eigenfunction:

**Theorem 2.1.** Let \( \Gamma^\Sigma \) be obtained from \( \Gamma \) by adding a copy of the motif \( \Sigma \) consisting of the vertices \( q_1, \ldots, q_m \) and the corresponding connections between them, and connecting each \( q_\alpha \) with all \( p \notin \Sigma \) that are neighbors of \( p_\alpha \). Then \( \Gamma^\Sigma \) possesses the eigenvalue 1, with a localized eigenfunction that is nonzero only at the \( p_\alpha \) and the \( q_\alpha \).

**Proof.** A corresponding eigenfunction is obtained as

\[
f_1^{\Gamma^\Sigma}(p) = \begin{cases} f_\Sigma^1(p_\alpha) & \text{if } p = p_\alpha \in \Sigma \\ -f_\Sigma^1(p_\alpha) & \text{if } p = q_\alpha \\ 0 & \text{else.} \end{cases}
\]
The theorem also holds for the case where \( \Sigma \) is a single vertex \( p_1 \) (even though such a motif does not possess the eigenvalue 1 itself). Thus, we can always produce the eigenvalue by **vertex doubling**. This is a reformulation of a result of [6].

Thus, if we wish to produce a high multiplicity for the eigenvalue 1, we can perform many vertex doublings. We could either duplicate different vertices, or we could duplicate one vertex repeatedly. In fact, the repeated doubling of one vertex leaves a characteristic trace in the number of certain small motifs in the graph. Let \( p_1 \) be a vertex and \( q_1 \) its double. We consider any motif \( \Sigma \) consisting of a certain collection \( p, p', p'', \ldots \) of neighbors of \( p_1 \) together with their connections to both \( p_1 \) and \( q_1 \) and possibly some connections among them.

**Theorem 2.2.** Let the graph \( \bar{\Gamma} \) be obtained from \( \Gamma \) by \( n \) successive doublings of the vertex \( p_1 \), and let \( \Sigma \) be any motif of the type just described. Then \( \bar{\Gamma} \) contains at least \( \binom{n}{2} \) instances of the motif \( \Sigma \).

**Proof.** An instance of the motif \( \Sigma \) is obtained by taking any two copies of \( p_1 \) and the vertices \( p, p', p'', \ldots \) together with the connections defining \( \Sigma \). There exist \( \binom{n}{2} \) such pairs of copies of \( p_1 \) in \( \bar{\Gamma} \).

Theorem 2.1 however, does not apply to eigenvalues other than 1 because for \( \lambda \neq 1 \), the vertex degrees \( n_i \) in (7) are important, and this is affected by embedding the motif \( \Sigma \) into another graph \( \Gamma \). However, we have the following variant in the general case.

**Theorem 2.3.** Let \( \Sigma \) be a motif in \( \Gamma \). Suppose \( f \) satisfies

\[
\frac{1}{n_i} \sum_{j \in \Sigma, j \sim i} f(j) = (1 - \lambda) f(i) \text{ for all } i \in \Sigma \text{ and some } \lambda. \tag{12}
\]

Then the motif doubling of Theorem 2.1 produces a graph \( \Gamma^\Sigma \) with eigenvalue \( \lambda \) and an eigenfunction \( f^{\Gamma_\Sigma} \) agreeing with \( f \) on \( \Sigma \), with \(-f\) on the double of \( \Sigma \), and being 0 on the rest of \( \Gamma^\Sigma \).

**Proof.** (12) implies that \( f \) satisfies the eigenvalue equation on \( \Sigma \), and therefore \(-f\) satisfies it on its double. As before, the doubling has the effect that for all other vertices \( j \in \Gamma^\Sigma \),

\[
\frac{1}{n_j} \sum_{\ell \sim j} f^{\Gamma_\Sigma}(\ell) = 0. \tag{13}
\]

The simplest motif is an edge connecting two vertices \( p_1, p_2 \). The corresponding relations (12) then are

\[
\frac{1}{n_{p_1}} f(p_2) = (1 - \lambda) f(p_1), \quad \frac{1}{n_{p_2}} f(p_1) = (1 - \lambda) f(p_2) \tag{14}
\]
which admit the solutions
\[ \lambda = 1 \pm \frac{1}{\sqrt{n p_1 n p_2}}. \] (15)

Thus edge doubling leads to those eigenvalues which when \( p_1 \) or \( p_2 \) has a large degree become close to 1. In any case, the two values are symmetric about 1.

We can also double the entire graph:

**Theorem 2.4.** Let \( \Gamma_1 \) and \( \Gamma_2 \) be isomorphic graphs with vertices \( p_1, \ldots, p_n \) and \( q_1, \ldots, q_n \) respectively, where \( p_i \) corresponds to \( q_i \) for \( i = 1, \ldots, n \). We then construct a graph \( \Gamma_0 \) by connecting \( p_i \) with \( q_j \) whenever \( p_j \sim p_i \). If \( \lambda_1, \ldots, \lambda_n \) are the eigenvalues of \( \Gamma_1 \) and \( \Gamma_2 \), then \( \Gamma_0 \) has these same eigenvalues, and the eigenvalue 1 with multiplicity \( n \).

**Proof.** Since the degree of every vertex \( p \) in \( \Gamma_0 \) is \( 2 n p \) where \( n_p \) is its original degree in \( \Gamma_1 \), we have for an eigenfunction \( f_\lambda \) of \( \Gamma_1 \) (which then is also an eigenfunction on \( \Gamma_2 \)),
\[
\frac{1}{2 n p} \sum_{s \in \Gamma_0, s \sim p} f_\lambda(s) = \frac{1}{n p} \sum_{s \in \Gamma_1, s \sim p} f_\lambda(s) = (1 - \lambda) f_\lambda(p).
\] (16)

Thus, by (7), it is an eigenfunction on \( \Gamma_0 \).

Finally, similarly to the proof of Theorem 2.1, we obtain the eigenvalue 1 with multiplicity \( n \): for each \( p \in \Gamma_1 \), we construct an eigenfunction with value 1 at \( p \), \(-1\) at its double in \( \Gamma_2 \), and 0 elsewhere.

We now turn to a different operation. Let \( \Gamma \) be a graph with an eigenfunction \( f_1 \). We arbitrarily divide \( \Gamma \) into subgraphs \( \Sigma_0, \Sigma_1, \Sigma_2 \) such that there is no edge between an element of \( \Sigma_1 \) and an element of \( \Sigma_2 \). We then take the graphs \( \Gamma_1 = \Sigma_1 \cup \Sigma_0 \) and \( \Gamma_2 = \Sigma_2 \cup \Sigma_0 \), in such a manner that each edge between two elements of \( \Sigma_0 \) is contained in either \( \Gamma_1 \) or \( \Gamma_2 \), but not in both of them, and form a connected graph \( \Gamma_0 \) by taking an additional vertices \( w \) for each vertex \( q \in \Sigma_0 \) and connect it with the two copies of \( q \) in \( \Gamma_1 \) and \( \Gamma_2 \).

**Theorem 2.5.** \( \Gamma_0 \) possesses the eigenvalue 1 with an eigenfunction that agrees with \( f_1 \) on \( \Gamma_1 \).

**Proof.** We put
\[
f_1^{\Gamma_0}(p) = \begin{cases} 
  f_1(p) & \text{for } p \in \Gamma_1 \\
  -f_1(p) & \text{for } p \in \Gamma_2 \\
  -\sum_{s \in \Gamma_1, s \sim q} f_1(s) & \text{when } p = w \text{ is one of the added vertices connected to } q \in \Gamma_1
\end{cases}
\] (17)

This works out because \( \sum_{s \in \Gamma_1, s \sim q} f_1(s) + \sum_{s \in \Gamma_2, s \sim q} f_1(s) = \sum_{s \in \Gamma, s \sim q} f_1(s) = 0 \) since \( f_1 \) is an eigenfunction on \( \Gamma \).
A simple and special case consists in taking a node \( p \) and joining a chain of length 2 to it, that is, connect \( p \) with a new node \( p_1 \) and that node in turn with another new node \( p_2 \) and put the value 0 at \( p_1 \) and the value \(-f_1(p)\) at \( p_2 \). This case was obtained in [1].

The next operation, graph joining, works for any eigenvalue, not just 1:

**Theorem 2.6.** Let \( \Gamma_1, \Gamma_2 \) be graphs with the same eigenvalue \( \lambda \) and corresponding eigenfunctions \( f_1, f_2 \). Assume that \( f_1(p_1) = 0 \) and \( f_2(p_2) = 0 \) for some \( p_1 \in \Gamma_1, p_2 \in \Gamma_2 \). Then the graph \( \Gamma \) obtained by joining \( \Gamma_1 \) and \( \Gamma_2 \) via identifying \( p_1 \) with \( p_2 \) also has the eigenvalue \( \lambda \) with an eigenfunction given by \( f_1 \) on \( \Gamma_1 \), \( f_2 \) on \( \Gamma_2 \).

**Proof.** We observe from (7) that for an eigenfunction \( f_\lambda \) whenever \( f_\lambda(q) = 0 \) at some \( q \), then also \( \sum_{s \sim q} f_\lambda(s) = 0 \). This applies to \( p_1 \) and \( p_2 \), and therefore, we can also join the eigenfunctions on the two components.

This includes the case where either \( f_1 \) or \( f_2 \) is identically 0.

**Example:** A triangle, that is, a complete graph of 3 vertices \( i_1, i_2, i_3 \), possesses the eigenvalue 3/2 with multiplicity 2. An eigenfunction \( f_{3/2} \) vanishes at one of the vertices, say \( f_{3/2}(i_1) = 0 \) and takes the values +1 and -1, resp., at the two other ones. Thus, when a triangle is joined at one vertex to another graph, the eigenvalue 3/2 is kept. For instance (see [4]), the petal graph, that is, a graph where \( m \) triangles are joined at a single vertex, has the eigenvalue 3/2 with multiplicity \( m + 1 \) (here, \( m \) of these eigenvalues are obtained via the described construction, and the remaining eigenfunction has the value \(-2\) at the central vertex where all the triangles are joined and 1 at all other ones).

Also, when the condition of Theorem 2.6 is satisfied at several pairs of vertices, we can form more bonds by vertex identifications between the two graphs. For the eigenvalue 1, the situation is even better: We need not require \( f_1(p_1) = 0 \) and \( f_2(p_2) = 0 \), but only \( f_1(p_1) = f_2(p_2) \) to make the joining construction work.

### 3 Examples

A chain of \( m \) vertices (that is, where we have an edge between \( p_j \) and \( p_{j+1} \) for \( j = 1, \ldots, m - 1 \)), by the lemma and node doubling, possesses the eigenvalue 1 (with multiplicity 1) iff \( m \) is odd, with eigenfunction \( f_1(p_1) = 1, f_1(p_2) = 0, f_1(p_3) = -1, f_1(p_4) = 0, \ldots \). Similarly, a closed chain (that is, where we add an edge between \( p_0 \) and \( p_1 \) ) possesses the eigenvalue 1 (with multiplicity 2) iff \( m \) is a multiple of 4.

Local operations like adding an edge may increase or decrease \( m \) or leave it invariant. Adding a pending vertex to a chain of length 2 increases \( m \) from 0 to 1, adding a pending vertex to closed chain of length 3, a triangle, leaves \( m = 0 \), adding a pending vertex to a closed chain of length 4, a quadrangle,
reduces $m_1$ from 2 to 1 (see [1] for general results in this direction). Similarly, closing a chain by adding an edge between the first and last vertex may increase, decrease or leave $m_1$ the same. In any case, the question of the eigenvalue 1 is not a local one. Take closed chains of lengths $4k - 1$ and $4\ell + 1$. Neither of them supports the eigenvalue 1, but if we join them at a single point (that is, we take a point $p_0$ in the first and a point $q_0$ in the second graph and form a new graph by identifying $p_0$ and $q_0$), the resulting graph has 1 as an eigenvalue. An eigenfunction has the value 1 at the joined node, and the values $\pm 1$ occurring always in neighboring pairs in the rest of the chains, where the two neighbors of $p_0$ in the first chain both get the value $-1$, and the ones in the second chain the value 1.

4 Construction of graphs with eigenvalue 1 from given data

Let $f$ be an integer valued function on the vertices of the graph $\Gamma$. We define the excess of $p \in \Gamma$ as

$$e(p) := \sum_{q \sim p} f(q).$$

Thus, $f$ is an eigenfunction for the eigenvalue iff $e(p) = 0$ for all $p$.

We are going to show that we can construct graphs $\Gamma$ and functions $f$ with the property that $e(p) = 0$ except for one single vertex $p_0$ where the pair $(f(p_0), e(p_0))$ assumes any prescribed integer values $(n, m)$. These will be assembled from elementary building blocks.

1. A triangle with a function $f$ that takes the value $-1$ at two vertices and the value 1 at the third vertex, our $p_0$, realizes the pair $(1, -2)$.

2. The same triangle, with a pending vertex, our new $p_0$, connected to the vertex with value 1, and given the value 2, realizes $(2, 1)$.

3. Joining instead $\ell$ triangles at a single vertex, our $p_0$, with value 1, assigning $-1$ to all the other vertices as before, yields $(1, -2\ell)$.

4. A pentagon, i.e., a closed chain of 5 vertices, with value $-1$ at two adjacent vertices and 1 at the remaining three, the middle one of which is our $p_0$, realizes $(1, 2)$.

5. Similarly, adding a pending vertex, again our new $p_0$, connected to the former $p_0$ in the pentagon, and assigned the value $-2$, realizes $(-2, 1)$.

6. Likewise, joining $\ell$ such pentagons instead at $p_0$ yields $(1, 2\ell)$.

7. In general, connecting a pending vertex as the new $p_0$ to the former $p_0$ changes $(n, m)$ to $(-m, n)$. 

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In general, joining the $p_0$s from graphs with values $(n, m_1), \ldots (n, m_k)$ yields $(n, \sum_i^k m_i)$.

Thus, from the triangle and the pentagon, by adding pending vertices and graph joining, we can indeed realize all integer pairs $(n, m)$.

**Theorem 4.1.** Let $\Sigma$ be a graph, $f$ an integer valued function on its vertices. We can then construct a graph $\Gamma$ containing the motif $\Sigma$ with eigenvalue 1 and an eigenfunction coinciding with $f$ on $\Sigma$.

**Proof.** At each $p \in \Sigma$, we attach a graph realizing the pair $(f(p), -e(p))$. This ensures (7) at $p$.

The preceding constructions also tell us how $m_1$, the multiplicity of the eigenvalue 1, behaves when we modify a graph $\Gamma'$, consisting possibly of two disjoint components $\Gamma_1, \Gamma_2$, by either identifying vertices or by joining vertices by new edges. The graph resulting from these operations will be called $\Gamma$. We consider two cases:

1. We identify the vertex $p_j$ with $q_j$ for $j = 1, \ldots, m$, assuming that they do not have common neighbors. Then
   
   (a) We can generate an eigenfunction on $\Gamma$ whenever we find a function $g$ on $\Gamma'$ with vanishing excess except possibly at the joined points where we require
   
   \[ g(p_j) = g(q_j) \text{ and } e_g(p_j) = -e_g(q_j) \text{ for } j = 1, \ldots, m. \]  
   \hspace{1cm} (19)

   (b) As a special case of (19), an eigenfunction $f_1^{\Gamma'}$ produces an eigenfunction $f_1^\Gamma$ whenever
   
   \[ f_1^{\Gamma'}(p_j) = f_1^{\Gamma'}(q_j) \text{ for } j = 1, \ldots, m. \]  
   \hspace{1cm} (20)

   In the case where $\Gamma'$ consists of two disjoint components $\Gamma_1, \Gamma_2$, this includes the case where that value is 0 for all $j$ and $f_1^{\Gamma'}$ vanishes identically on one of the components. In other words, we can extend an eigenfunction from $\Gamma_1$, say, to the rest of the graph by 0 whenever that function vanishes at all joining points.

   Since in general, (20) cannot be satisfied for a basis of eigenfunctions, by this process, we can only expect to generate fewer than $m_1^{\Gamma'}$ linearly independent eigenfunctions on $\Gamma$.

   Whether $m_1^\Gamma$ is larger or smaller than $m_1^{\Gamma'}$ then depends on the balance between these two processes, that is, how many eigenfunctions satisfy (20) vs. how many new eigenfunctions can be produced by functions satisfying (19) with nonvanishing excess at some of the joined vertices.

2. We connect the vertex $p_j$ by an edge with $q_j$ for $j = 1, \ldots, m$. Then
(a) We can generate eigenfunctions on $\Gamma$ whenever we find a function $g$ on $\Gamma'$ with vanishing excess except possibly at the connected points where we require
\[ g(p_j) = -e_g(q_j) \text{ and } g(q_j) = -e_g(p_j) \text{ for } j = 1, \ldots, m. \] (21)

(b) Again, as a special case of (21), an eigenfunction $f_1^{\Gamma'}$ produces an eigenfunction $f_1^\Gamma$ whenever
\[ f_1^{\Gamma'}(p_j) = 0 = f_1^{\Gamma'}(q_j) \text{ for } j = 1, \ldots, m. \] (22)

This imposes a stronger constraint than in (20) on eigenfunctions to yield an eigenfunction on $\Gamma$.

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