ON DELIGNE’S CONJECTURE FOR SYMMETRIC FIFTH
L-FUNCTIONS OF MODULAR FORMS

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Abstract. We prove Deligne’s conjecture for symmetric fifth L-functions of elliptic newforms of weight greater than 5. As a consequence, we establish period relations between motivic periods associated to an elliptic newform and the Betti–Whittaker periods of its symmetric cube functorial lift to GL_4.

1. Introduction

In [Del79], Deligne proposed a conjecture on the algebraicity of values of motivic L-functions at critical points in terms of motivic periods. A special class of examples are the symmetric power L-functions associated to modular forms. Given a normalized elliptic newform \( f \), we can define the twisted symmetric \( n \)-th L-function \( L(s, \text{Sym}^n(f) \otimes \chi) \) for each integer \( n \geq 1 \) and Dirichlet character \( \chi \). Deligne’s conjecture for \( L(s, \text{Sym}^n(f) \otimes \chi) \) at critical points was considered by various authors when \( n = 1, 2, 3, 4, 6 \) listed as follows:

- \( n = 1 \): Shimura [Shi76, Shi77].
- \( n = 2 \): Sturm [Stu80, Stu89].
- \( n = 3 \): Garrett–Harris [GH93, Che21a].
- \( n = 4, 6 \): Morimoto [Mor21, Che21b, Che21c].

In this paper, we consider the case when \( n = 5 \).

1.1. Main results. Let

\[
f(\tau) = \sum_{n=1}^{\infty} a_f(n)q^n \in S_\kappa(N, \omega), \quad q = e^{2\pi i \tau}\]

be a normalized elliptic newform of weight \( \kappa \geq 2 \), level \( N \), and nebentypus \( \omega \). For each prime number \( p \nmid N \), let \( \alpha_p, \beta_p \) be roots of the polynomial

\[
X^2 - a_f(p)X + p^{\kappa-1}\omega(p).
\]

For a Dirichlet character \( \chi : (\mathbb{Z}/M\mathbb{Z})^\times \to \mathbb{C} \), we define the twisted symmetric fifth L-function \( L(s, \text{Sym}^5(f) \otimes \chi) \) by an absolutely convergent Euler product

\[
L(s, \text{Sym}^5(f) \otimes \chi) = \prod_p L_p(s, \text{Sym}^5(f) \otimes \chi)
\]

for \( \text{Re}(s) > 1 + \frac{5(\kappa-1)}{2} \). Here the Euler factors are given by

\[
L_p(s, \text{Sym}^5(f) \otimes \chi) = \prod_{i=0}^{5} (1 - \alpha_p^{-i} \beta_p^i \cdot \chi(p)p^{-s})^{-1}
\]

if \( p \nmid NM \). We omit the definition of the Euler factors at \( p \mid NM \) but mentioning that as far as algebraicity concerns, it suffices to consider partial L-function. By the result of Barnet-Lamb–Geraghty–Harris–Taylor [BLGHT11, Theorem B] (see also [HSBT10]), the twisted
symmetric fifth $L$-function admits meromorphic continuation to $s \in \mathbb{C}$ and satisfies the expected functional equation relating $L(s, \text{Sym}^5(f) \otimes \chi)$ to $L(5(\kappa - 1) + 1 - s, \text{Sym}^5(f^\vee) \otimes \chi^{-1})$. Here $f^\vee \in S_\kappa(N, \omega^{-1})$ is the normalized newform dual to $f$. The Deligne’s periods for $\text{Sym}^5(f)$ are given by (cf. [Del79, Proposition 7.7])

\[ c^\pm(\text{Sym}^5(f)) = (2\pi \sqrt{-1})^{6-3\kappa} \cdot (\sqrt{-1})^\kappa \cdot G(\omega)^6 \cdot c^\pm(f)^3 \cdot \langle f, f \rangle^3. \]

Here $G(\omega)$ is the Gauss sum of the primitive Dirichlet character associated to $\omega$, $c^\pm(f) \in \mathbb{C}$ are the periods of $f$ as in [Shi77], and $\langle f, f \rangle$ is the Petersson norm of $f$ defined by

\[ \langle f, f \rangle = \text{vol}(\Gamma_0(N) \backslash \mathfrak{H})^{-1} \int_{\Gamma_0(N) \backslash \mathfrak{H}} |f(\tau)|^2 y^{\kappa-2} d\tau. \]

The set of critical points for $\text{Sym}^5(f)$ is equal to

\[ \{ m \in \mathbb{Z} \mid 2\kappa - 1 \leq m \leq 3\kappa - 3 \}. \]

We have the following special case of Deligne’s conjecture [Del79] on the algebraicity of the values of $L(s, \text{Sym}^5(f) \otimes \chi)$ at critical points. For $\sigma \in \text{Aut}(\mathbb{C})$, let $\sigma f \in S_\kappa(N, \sigma \omega)$ be the normalized newform defined so that $a_{\sigma f}(n) = \sigma(a_f(n))$ for $n \geq 1$.

**Conjecture 1.1** (Deligne). Let $\chi$ be a Dirichlet character and $m \in \mathbb{Z}$ a critical point for $\text{Sym}^5(f)$. For $\sigma \in \text{Aut}(\mathbb{C})$, we have

\[ \sigma \left( \frac{L(m, \text{Sym}^5(f) \otimes \chi)}{(2\pi \sqrt{-1})^{3m} \cdot G(\chi)^3 \cdot c^\pm(\text{Sym}^5(f))} \right) = \frac{L(m, \text{Sym}^5(\sigma f) \otimes \sigma \chi)}{(2\pi \sqrt{-1})^{3m} \cdot G(\sigma \chi)^3 \cdot c^\pm(\text{Sym}^5(\sigma f))}. \]

Here $\pm = (-1)^m \chi(-1)$.

Following is the main result of this paper.

**Theorem 1.2.** If $\kappa \geq 6$, then Conjecture 1.1 holds.

As a consequence of Theorem 1.2 we prove the following period relations between the periods associated to $f$ and the Betti–Whittaker periods $p(\text{Sym}^3(f), \pm) \in \mathbb{C}$ (cf. [Rag10]) of the symmetric cube lifting $\text{Sym}^3(f)$.

**Theorem 1.3.** Assume $\kappa \geq 6$. There exists a constant $C_\infty \in \mathbb{C}$, depending only on $\kappa$, such that for all $\sigma \in \text{Aut}(\mathbb{C})$, we have

\[ \sigma \left( \frac{p(\text{Sym}^3(f), \pm)}{C_\infty \cdot G(\omega)^8 \cdot c^\pm(f)^2 \cdot \langle f, f \rangle^4} \right) = \frac{p(\text{Sym}^3(\sigma f), \pm)}{C_\infty \cdot G(\sigma \omega)^8 \cdot c^\pm(\sigma f)^2 \cdot \langle \sigma f, \sigma f \rangle^4}. \]

1.2. **Notation.** Let $\mathbb{A}$ be the ring of adeles of $\mathbb{Q}$ and $\mathbb{A}_f$ its finite part. Let $\psi = \bigotimes_v \psi_v : \mathbb{Q} \backslash \mathbb{A} \to \mathbb{C}$ be the additive character defined so that

\[ \psi_p(x) = e^{-2\pi \sqrt{-1} x} \text{ for } x \in \mathbb{Z}[p^{-1}], \]

\[ \psi_x(x) = e^{2\pi \sqrt{-1} x} \text{ for } x \in \mathbb{R}. \]

Denote by $\text{GL}_n$ the general linear group over $\mathbb{Q}$. Let $U_n$ be the standard maximal unipotent subgroup of $\text{GL}_n$ consisting of upper unipotent matrices. Let $\psi_{U_n} : U_n(\mathbb{Q}) \backslash U_n(\mathbb{A}) \to \mathbb{C}$ be the additive character defined by

\[ \psi_{U_n}(u) = \psi(u_{12} + u_{23} + \cdots + u_{n-1,n}), \quad u = (u_{ij}) \in U_n(\mathbb{A}). \]

Let $K_n^\circ$ and $K_n$ be closed subgroups of $\text{GL}_n(\mathbb{R})$ defined by

\[ K_n^\circ = \mathbb{R}_+ \times \text{SO}(n), \quad K_n = \mathbb{R}_+ \times \text{O}(n). \]
In this way, we obtain a character of $\mathbb{P}$ automorphic representation of $GL_n$ respectively. For $W$ component $\phi$ exists an irreducible algebraic representation $M$ $\hat{\nu}$ is non-zero. Here the spaces of Whittaker functions of $p$ and $SO_t$ here we regard the set $\mathbb{R}_+$ of positive real numbers as the topological connected component of the center of $GL_n(\mathbb{R})$. We denote by $\mathfrak{g}_n$, $\mathfrak{k}_n$, and $so(n)$ the Lie algebras of $GL_n(\mathbb{R})$, $K_n$, and $SO(n)$, respectively.

Let $\sigma \in Aut(\mathbb{C})$. For a complex representation $\Pi$ of a group $G$ on the space $V_\Pi$ of $\Pi$, let $^\sigma\Pi$ be the representation of $G$ defined

\begin{equation}
^\sigma\Pi(g) = t \circ \Pi(g) \circ t^{-1},
\end{equation}

where $t : V_\Pi \to V_\Pi$ is a $\sigma$-linear isomorphism. Note that the isomorphism class of $^\sigma\Pi$ is independent of the choice of $t$.

2. Algebraicity of the Rankin–Selberg $L$-functions for $GL_n \times GL_{n-1}$

In this section, we recall the algebraicity of critical values of Rankin–Selberg $L$-functions for $GL_n(\mathbb{A}) \times GL_{n-1}(\mathbb{A})$ in terms of Betti–Whittaker periods following [Rag10], [Gro18], [GL21], [LLS21].

2.1. Tamely isobaric automorphic representations. Let $P = N_PM_P$ be the standard parabolic subgroup of $GL_n$ with Levi factor $M_P \cong GL_{n_1} \times \cdots \times GL_{n_k}$. For $1 \leq i \leq k$, let $\Pi_i$ be an irreducible cuspidal automorphic representation of $GL_{n_i}(\mathbb{A})$. Consider the isobaric automorphic representation of $GL_n(\mathbb{A})$:

$$\Pi = \Pi_1 \oplus \cdots \oplus \Pi_k.$$ 

Following [LLS21] §4.3, we say $\Pi$ is tamely isobaric if there exists $s \in \mathbb{R}$ such that $\Pi_i \otimes |_\mathbb{A}$ is unitary for all $1 \leq i \leq k$. In this case, $\Pi$ is fully induced, that is,

$$\Pi \cong Ind_{P(\mathbb{A})}^{GL_n(\mathbb{A})}(\Pi_1 \otimes \cdots \otimes \Pi_k).$$

We then realize $\Pi$ in the space $A(GL_n)$ of automorphic forms on $GL_n(\mathbb{A})$ using Eisenstein series as explained in [GL21] §1.4.3. Moreover, $\Pi$ is globally generic, that is, for non-zero automorphic form $\varphi \in \Pi$, the Whittaker function

$$W_\varphi(g) = \int_{U_n(\mathbb{Q}) \backslash U_n(\mathbb{A})} \varphi(ug) \overline{\psi_{U_n}(u)} du_T, \quad g \in GL_n(\mathbb{A})$$

is non-zero. Here $du_T$ is the Tamagawa measure on $U_n(\mathbb{A})$. Let $W(\Pi_\infty)$ and $W(\Pi_f)$ be the spaces of Whittaker functions of $\Pi_\infty$ and $\Pi_f = \bigotimes_p \Pi_p$ with respect to $\psi_{U_n,\infty}$ and $\psi_{U_n,f}$, respectively. For $W_\infty \in W(\Pi_\infty)$ and $W_f \in W(\Pi_f)$, there exists a unique automorphic form $\varphi \in \Pi$ such that

$$W_\varphi = W_\infty \cdot W_f.$$ 

In this way, we obtain a $((\mathfrak{g}_n, K_n) \times GL_n(\mathbb{A}_f))$-equivariant isomorphism

\begin{equation}
W(\Pi_\infty) \otimes W(\Pi_f) \longrightarrow \Pi. 
\end{equation}

2.2. Betti–Whittaker periods for $GL_n$. Let $\Pi = \Pi_1 \oplus \cdots \oplus \Pi_k$ be a tamely isobaric automorphic representation of $GL_n(\mathbb{A})$ as in [2.1]. Let $\rho_P$ be the square-root of the modulus character of $P(\mathbb{A})$. For $1 \leq i \leq k$, let $\rho_i$ be the restriction of $\rho_P$ to the factor $GL_{n_i}$ of the Levi component $M_P \cong GL_{n_1} \times \cdots \times GL_{n_k}$. Assume further that $\Pi$ is cohomological, that is, there exists an irreducible algebraic representation $M$ of $GL_n$ such that the $(\mathfrak{g}_n, K_n)$-cohomology

$$H^*(\mathfrak{g}_n, K_n, \Pi_\infty \otimes M_C) \neq 0.$$
Here $M_\mathbb{C} = M \otimes_{\mathbb{Q}} \mathbb{C}$. Note that $M$ is uniquely determined by $\Pi_\infty$ and $\bullet = b_n = \lfloor \frac{n^2}{4} \rfloor$ is the least integer such that the $(\mathfrak{g}_n, K_n^\circ)$-cohomology is non-vanishing (cf. [Clo90, §3]). Moreover, we have

$$\dim_\mathbb{C} H^{b_n}(\mathfrak{g}_n, K_n^\circ; \Pi_\infty \otimes M_\mathbb{C}) = \begin{cases} 1 & \text{if $n$ is odd,} \\ 2 & \text{if $n$ is even.} \end{cases}$$

When $n$ is odd, we fix a generator

$$(2.2) \quad [\Pi_\infty] \in H^{b_n}(\mathfrak{g}_n, K_n^\circ; \Pi_\infty \otimes M_\mathbb{C}).$$

When $n$ is even, the $\pm 1$-eigenspace $H^{b_n}(\mathfrak{g}_n, K_n^\circ; \Pi_\infty \otimes M_\mathbb{C})[\pm]$ under the action of $K_n/K_n^\circ$ is 1-dimensional. We fix a generator

$$(2.3) \quad [\Pi_\infty]_\pm \in H^{b_n}(\mathfrak{g}_n, K_n^\circ; \Pi_\infty \otimes M_\mathbb{C})[\pm].$$

2.2.1. **Rational structure via Whittaker model.** Let $\sigma \in \text{Aut}(\mathbb{C})$. For $1 \leq i \leq k$, let $\sigma(\Pi_i \otimes \rho_i^{-1})$ be the irreducible admissible $((\mathfrak{g}_n, K_n) \times \text{GL}_n(\mathbb{A}_f))$-module defined by

$$\sigma(\Pi_i \otimes \rho_i^{-1}) = (\Pi_i, \text{GL}_n(\mathbb{A}_f)) \otimes \sigma(\Pi_i \otimes \rho_i^{-1}).$$

By [BW00, III, Theorem 3.3], $\Pi_i \otimes \rho_i^{-1}$ is cohomological. In particular, $\sigma(\Pi_i \otimes \rho_i^{-1})$ is cohomological and cuspidal by the result of Clozel [Clo90, Théorème 3.19]. Let $\sigma \Pi$ be the isobaric automorphic representation defined by

$$\sigma \Pi = \bigoplus_{i=1}^k \sigma(\Pi_i \otimes \rho_i^{-1}) \otimes \rho_i.$$ 

It is clear that $\sigma \Pi$ is cohomological and tamely isobaric. Moreover, we have $(\sigma \Pi)_f = \sigma(\Pi_f)$ (cf. [Gro18, Lemma 1.2]). We write $\sigma \Pi_f = (\sigma \Pi)_f$. Let $\mathbb{Q}(\Pi)$ be the rationality field of $\Pi$, which is the fixed field of $\{\sigma \in \text{Aut}(\mathbb{C}) \mid \sigma \Pi = \Pi\}$. Note that $\mathbb{Q}(\Pi)$ is equal to the composite of the rationality fields of $\mathbb{Q}(\Pi_i)$ for $1 \leq i \leq k$. In particular, $\mathbb{Q}(\Pi)$ is a number field. Let $t_\sigma : W(\Pi_f) \to W(\sigma \Pi_f)$ be the $\sigma$-linear $\text{GL}_n(\mathbb{A}_f)$-equivariant isomorphism defined by

$$t_\sigma W(g) = \sigma(W(\text{diag}(u_{\sigma}^{-n+1}, u_{\sigma}^{-n+2}, \ldots, 1)g)), \quad g \in \text{GL}_n(\mathbb{A}_f).$$

Here $u_{\sigma}$ is a $\hat{\mathbb{Z}}^\times$ is the unique element such that $\sigma(\psi(x)) = \psi(u_{\sigma} x)$ for all $x \in \mathbb{A}_f$. We thus obtain a $\mathbb{Q}(\Pi)$-rational structure on $W(\Pi_f)$ given by taking the Galois invariants:

$$(2.4) \quad W(\Pi_f)^{\text{Aut}(\mathbb{C}/\mathbb{Q}(\Pi))} = \{W \in W(\Pi_f) \mid t_\sigma W = M \text{ for } \sigma \in \text{Aut}(\mathbb{C}/\mathbb{Q}(\Pi))\}.$$ 

2.2.2. **Rational structure via sheaf cohomology.** Consider the orbifold

$$X_n = \text{GL}_n(\mathbb{Q}) \backslash \text{GL}_n(\mathbb{A})/K_n^\circ.$$ 

The algebraic representation $M_\mathbb{C}$ defines a locally constant sheaf $\mathcal{M}_\mathbb{C}$ of $\mathbb{C}$-vector spaces on $X_n$ (cf. [HR20, §2.2.7]). We denote by

$$H^*(X_n, \mathcal{M}_\mathbb{C})$$

the sheaf cohomology of $\mathcal{M}_\mathbb{C}$. For $\sigma \in \text{Aut}(\mathbb{C})$, the canonical $\sigma$-linear isomorphism $M_\mathbb{C} \to M_\mathbb{C}$ naturally induces a $\sigma$-linear isomorphism

$$\sigma^* : H^*(X_n, \mathcal{M}_\mathbb{C}) \to H^*(X_n, \mathcal{M}_\mathbb{C}).$$

Assume $n$ is odd. Following the proof of [Gro18, Propositions 1.6 and 1.7] (cf. [GR14a, Theorem 7.23] and [LLS21, Proposition 4.3]), there exists a $\text{GL}_n(\mathbb{A}_f)$-equivariant injection

$$\Psi_\Pi : H^{b_n}(\mathfrak{g}_n, K_n^\circ; \Pi \otimes M_\mathbb{C}) \to H^{b_n}(X_n, \mathcal{M}_\mathbb{C})$$
such that the image of $\sigma^* \circ \Psi_\Pi$ is the image of $\Psi_{\sigma \Pi}$. This induces a $\sigma$-linear $GL_n(\mathbb{A}_f)$-equivariant isomorphism

$$\sigma^* : H^{b_n}(\mathfrak{g}_n, K_n^\circ, \Pi \otimes M_{\mathbb{C}}) \longrightarrow H^{b_n}(\mathfrak{g}_n, K_n^\circ, \sigma \Pi \otimes M_{\mathbb{C}}).$$

We thus obtain a $\mathbb{Q}(\Pi)$-rational structure on $H^{b_n}(\mathfrak{g}_n, K_n^\circ, \Pi \otimes M_{\mathbb{C}})$ given by taking the Galois invariants:

$$H^{b_n}(\mathfrak{g}_n, K_n^\circ, \Pi \otimes M_{\mathbb{C}})^{\text{Aut}(\mathbb{C}/\mathbb{Q}(\Pi))} = \{ c \in H^{b_n}(\mathfrak{g}_n, K_n^\circ, \Pi \otimes M_{\mathbb{C}}) \mid \sigma^* c = c \text{ for } \sigma \in \text{Aut}(\mathbb{C}/\mathbb{Q}(\Pi)) \}.$$

Let

$$\Phi_\Pi : \mathcal{W}(\Pi_f) \longrightarrow H^{b_n}(\mathfrak{g}_n, K_n^\circ, \Pi \otimes M_{\mathbb{C}})$$

be the $GL_n(\mathbb{A}_f)$-equivariant isomorphism defined as follows: For $W \in \mathcal{W}(\Pi_f)$, we have

$$[\Pi_\infty] \otimes W \in H^{b_n}(\mathfrak{g}_n, K_n^\circ, \mathcal{W}(\Pi_\infty) \otimes \mathcal{W}(\Pi_f) \otimes M_{\mathbb{C}}).$$

Then $\Phi_\Pi(W)$ is the image of $[\Pi_\infty] \otimes W$ under the $GL_n(\mathbb{A}_f)$-equivariant isomorphism induced by the isomorphism $(2.1)$. When $n$ is even, similarly we have a $\mathbb{Q}(\Pi)$-rational structure $H^{b_n}(\mathfrak{g}_n, K_n^\circ, \Pi \otimes M_{\mathbb{C}})[\pm]^{\text{Aut}(\mathbb{C}/\mathbb{Q}(\Pi))}$ on $H^{b_n}(\mathfrak{g}_n, K_n^\circ, \Pi \otimes M_{\mathbb{C}})[\pm]$ and a $GL_n(\mathbb{A}_f)$-equivariant isomorphism

$$\Phi_{\Pi, \pm} : \mathcal{W}(\Pi_f) \longrightarrow H^{b_n}(\mathfrak{g}_n, K_n^\circ, \Pi \otimes M_{\mathbb{C}})[\pm].$$

Comparing the $\mathbb{Q}(\Pi)$-rational structures given by $(2.4)$ and $(2.5)$, we have the following lemma/definition of the Betti–Whittaker periods of $\Pi$.

**Lemma 2.1.** There exists $p(\Pi) \in \mathbb{C}^\times$ if $n$ is odd and $p(\Pi, \pm) \in \mathbb{C}^\times$ if $n$ is even, unique up to $\mathbb{Q}(\Pi)^\times$, such that

$$\frac{\Phi_\Pi(\mathcal{W}(\Pi_f)^{\text{Aut}(\mathbb{C}/\mathbb{Q}(\Pi))})}{p(\Pi)} = H^{b_n}(\mathfrak{g}_n, K_n^\circ, \Pi \otimes M_{\mathbb{C}})^{\text{Aut}(\mathbb{C}/\mathbb{Q}(\Pi))}$$

if $n$ is odd and

$$\frac{\Phi_{\Pi, \pm}(\mathcal{W}(\Pi_f)^{\text{Aut}(\mathbb{C}/\mathbb{Q}(\Pi))})}{p(\Pi, \pm)} = H^{b_n}(\mathfrak{g}_n, K_n^\circ, \Pi \otimes M_{\mathbb{C}})[\pm]^{\text{Aut}(\mathbb{C}/\mathbb{Q}(\Pi))}$$

if $n$ is even. Moreover, we can normalize the periods so that

$$\sigma^* \left( \frac{\Phi_\Pi(W)}{p(\Pi)} \right) = \frac{\Phi_{\sigma \Pi}(t_\sigma W)}{p(\sigma \Pi)}$$

if $n$ is odd and

$$\sigma^* \left( \frac{\Phi_{\Pi, \pm}(W)}{p(\Pi, \pm)} \right) = \frac{\Phi_{\Pi, \pm}(t_\sigma W)}{p(\sigma \Pi, \pm)}$$

if $n$ is even for all $W \in \mathcal{W}(\Pi_f)$ and $\sigma \in \text{Aut}(\mathbb{C})$. 

2.3. Algebraicity for \( \text{GL}_n \times \text{GL}_{m-1} \). For isobaric automorphic representations \( \Pi_n \) and \( \Pi_m \) of \( \text{GL}_n(\mathbb{A}) \) and \( \text{GL}_m(\mathbb{A}) \), respectively, let

\[ L(s, \Pi_n \times \Pi_m) \]

be the Rankin–Selberg \( L \)-function for \( \Pi_n \times \Pi_m \). We denote by \( L^{(\infty)}(s, \Pi_n \times \Pi_m) \) the \( L \)-function obtained by excluding the archimedean \( L \)-factor.

Let \( \Pi_{n,\infty} \) and \( \Pi_{n-1,\infty} \) be irreducible admissible generic essentially unitary \((\mathfrak{g}_n, K_n^0)\) and \((\mathfrak{g}_{n-1}, K_{n-1}^0)\) modules, respectively. Let \( M_n \) and \( M_{n-1} \) be the irreducible algebraic representations of \( \text{GL}_n \) and \( \text{GL}_{n-1} \) such that

\[ H^{bn}(\mathfrak{g}_n, K_n^0; \Pi_{n,\infty} \otimes M_{n,\mathbb{C}}) \neq 0, \quad H^{bn-1}(\mathfrak{g}_{n-1}, K_{n-1}^0; \Pi_{n-1,\infty} \otimes M_{n-1,\mathbb{C}}) \neq 0. \]

Here \( M_{n,\mathbb{C}} = M_n \otimes_{\mathbb{Q}} \mathbb{C} \) and \( M_{n-1,\mathbb{C}} = M_{n-1} \otimes_{\mathbb{Q}} \mathbb{C} \). Note that \( M_n \) and \( M_{n-1} \) are uniquely determined by \( \Pi_{n,\infty} \) and \( \Pi_{n-1,\infty} \) \((\text{cf. } [\text{ClO90}] \text{ §3})\). We say \((\Pi_{n,\infty}, \Pi_{n-1,\infty})\) is balanced if there exists \( m \in \mathbb{Z} \) such that

\[ \text{Hom}_{\text{GL}_{n-1}}(M_n \otimes M_{n-1}, \text{det}^m) \neq 0. \]

In this case, \( m' \in \mathbb{Z} \) satisfying \((2.6)\) if and only if \( L(s, \Pi_{n,\infty} \times \Pi_{n-1,\infty}) \) and \( L(1-s, \Pi_{n,\infty} \times \Pi_{n-1,\infty}) \) are holomorphic at \( s = m' + \frac{1}{2} \) \((\text{cf. } [\text{Rag16}] \text{ Theorem 2.21})\). Fix \( m \in \mathbb{Z} \) satisfying \((2.6)\). We define a quantity \( Z(m, \Pi_{n,\infty}, \Pi_{n-1,\infty}) \in \mathbb{C} \) as follows: Let \( \iota : \text{GL}_{n-1} \to \text{GL}_n \) be the embedding defined by

\[ \iota(g) = \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}. \]

- Fix a non-zero \( \xi_m \in \text{Hom}_{\text{GL}_{n-1}}(M_n \otimes M_{n-1}, \text{det}^m) \).
- For \( 1 \leq i, j \leq n-1 \), let \( e_{ij} \in \mathfrak{g}_{n-1} \) be the \( n-1 \) by \( n-1 \) matrix with 1 in the \((i, j)\)-entry and zeros otherwise. By abuse of notation, the image of \( e_{ij} \in \mathfrak{g}_{n-1} \) in \( \mathfrak{g}_{n-1}/\mathfrak{so}(n-1) \) is denoted by the same symbol \( e_{ij} \). Then the generator

\[ (\wedge_{i=1}^{n-1} e_{ii}^*) \wedge (\wedge_{1 \leq i < j \leq n-1} e_{ij}^*) \in \wedge^{n(n-1)/2}(\mathfrak{g}_{n-1,\mathbb{C}}/\mathfrak{so}(n-1,\mathbb{C}))^* \]

defines an invariant measure on \( \text{GL}_{n-1}(\mathbb{R})/\text{SO}(n-1) \). By pull-back, this then defines a Haar measure on \( \text{GL}_{n-1}(\mathbb{R}) \) by requiring \( \text{vol}(\text{SO}(n-1)) = 1 \). For \( X \in \wedge^{bn}(\mathfrak{g}_{n,\mathbb{C}}/\mathfrak{t}_{n,\mathbb{C}})^* \) and \( Y = \wedge^{bn-1}(\mathfrak{g}_{n-1,\mathbb{C}}/\mathfrak{t}_{n-1,\mathbb{C}})^* \), let \( s(X, Y) \in \mathbb{C} \) defined so that

\[ \iota^*(X) \wedge \text{pr}(Y) = s(X, Y) \cdot (\wedge_{i=1}^{n-1} e_{ii}^*) \wedge (\wedge_{1 \leq i < j \leq n-1} e_{ij}^*). \]

Here \( \text{pr} : \wedge^{bn}(\mathfrak{g}_{n,\mathbb{C}}/\mathfrak{t}_{n,\mathbb{C}})^* \to \wedge^{bn-1}(\mathfrak{g}_{n-1,\mathbb{C}}/\mathfrak{so}(n-1,\mathbb{C}))^* \) is the natural surjection and \( \iota^* : \wedge^{bn}(\mathfrak{g}_{n,\mathbb{C}}/\mathfrak{t}_{n,\mathbb{C}})^* \to \wedge^{bn}(\mathfrak{g}_{n-1,\mathbb{C}}/\mathfrak{so}(n-1,\mathbb{C}))^* \) is induced by the embedding \( \iota \).

- For \( W_n \in \mathcal{W}(\Pi_{n,\infty}) \) and \( W_{n-1} \in \mathcal{W}(\Pi_{n-1,\infty}) \), we define local zeta integral

\[ Z(s, W_n, W_{n-1}) = \int_{U_{n-1}(\mathbb{R}) \backslash \text{GL}_{n-1}(\mathbb{R})} W_n(\iota(g)) W_{n-1}(g) \det g|^{s-1/2} \, dg. \]

Here \( dg \) is the quotient measure induced by the Lebesgue measure on \( U_{n-1}(\mathbb{R}) \cong \mathbb{R}^{(n-1)(n-2)/2} \) and the Haar measure on \( \text{GL}_{n-1}(\mathbb{R}) \) described above. The integral converges absolutely for \( \text{Re}(s) \) sufficiently large and admits meromorphic continuation to \( s \in \mathbb{C} \). Moreover, the ratio

\[ \frac{Z(s, W_n, W_{n-1})}{L(s, \Pi_{n,\infty} \times \Pi_{n-1,\infty})} \]

is entire \((\text{cf. } [\text{JS90b}, \text{Jac09}]\)\). In particular, \( Z(s, W_n, W_{n-1}) \) is holomorphic at \( s = m + \frac{1}{2} \).
We then define the $K_{n-1}^\circ$-equivariant bilinear pairing $\langle \cdot, \cdot \rangle_m$ on

$$\langle W_n \otimes X \otimes \mathbf{v}, W_{n-1} \otimes Y \otimes \mathbf{v} \rangle_m = Z(m + \frac{1}{2}, W_n, W_{n-1}) \cdot s(X, Y) \cdot \xi_m(\mathbf{v} \otimes \mathbf{v}).$$

By restriction, we obtain a bilinear pairing $\langle \cdot, \cdot \rangle_m$ on

$$H^{bn}(\mathfrak{g}_n, K_n^\circ) \otimes M_n, \mathfrak{c}) \times H^{bn-1}(\mathfrak{g}_{n-1}, K_{n-1}^\circ) \otimes M_{n-1}, \mathfrak{c})$$

Let $\mathfrak{w}(\Pi_{n, x})$ and $\mathfrak{w}(\Pi_{n-1, x})$ be the integers such that

$$|\omega_{\Pi_{n, x}}| = |\mathfrak{w}(\Pi_{n, x})/2|, \quad |\omega_{\Pi_{n-1, x}}| = |(n-1)\mathfrak{w}(\Pi_{n-1, x})/2|.$$

When $n$ is odd (resp. even), $\mathfrak{w}(\Pi_{n, x})$ (resp. $\mathfrak{w}(\Pi_{n-1, x})$) must be even and we put

$$\varepsilon(\Pi_{n, x}) = (-1)^\mathfrak{w}(\Pi_{n, x})/2\omega_{\Pi_{n, x}}(-1) \quad (\text{resp. } \varepsilon(\Pi_{n-1, x}) = (-1)^\mathfrak{w}(\Pi_{n-1, x})/2\omega_{\Pi_{n-1, x}}(-1)).$$

We fix generators in the $(\mathfrak{g}_n, K_n^\circ)$ and $(\mathfrak{g}_{n-1}, K_{n-1}^\circ)$ cohomologies as in (2.2 and 2.3). Define $Z(m, \Pi_{n, x}, \Pi_{n-1, x}) \in \mathbb{C}$ by

$$Z(m, \Pi_{n, x}, \Pi_{n-1, x}) = \left\{ \begin{array}{ll} \langle [\Pi_{n, x}], [\Pi_{n-1, x}] \rangle & \text{if } n \text{ is odd,} \\ \langle [\Pi_{n, x}], [\Pi_{n-1, x}] \rangle & \text{if } n \text{ is even.} \end{array} \right.$$
As an immediate consequence of the theorem, we have the following corollary on the algebraicity of ratios of critical values of the Rankin–Selberg $L$-functions.

**Corollary 2.6.** Let $\Pi_n$ and $\Pi'_n$ (resp. $\Pi_{n-1}$ and $\Pi'_{n-1}$) be cohomological irreducible cuspidal (resp. tamely isobaric) automorphic representations of $\GL_n(\A)$ (resp. $\GL_{n-1}(\A)$) satisfying the following conditions:

- $(\Pi_{n,\infty}, \Pi'_{n,\infty})$ is balanced.
- $\Pi_{n,\infty} = \Pi'_{n,\infty}$ and $\Pi_{n-1,\infty} = \Pi'_{n-1,\infty}$.

For $\sigma \in \Aut(\C)$ and $m + \frac{1}{2}$ a critical point such that $L(m + \frac{1}{2}, \Pi_n \times \Pi_{n-1}) \cdot L(m + \frac{1}{2}, \Pi'_n \times \Pi'_{n-1})$ is non-zero, we have

\[
\frac{L^{(\sigma)}(m + \frac{1}{2}, \Pi_n \times \Pi'_{n-1}) \cdot L^{(\sigma)}(m + \frac{1}{2}, \Pi'_n \times \Pi_{n-1})}{L^{(\sigma)}(m + \frac{1}{2}, \Pi_n \times \Pi_{n-1}) \cdot L^{(\sigma)}(m + \frac{1}{2}, \Pi'_n \times \Pi'_{n-1})} = \frac{L^{(\sigma \Pi \otimes \sigma \Pi')}(m + \frac{1}{2}, \Pi_n \times \Pi_{n-1}) \cdot L^{(\sigma \Pi' \otimes \sigma \Pi)}(m + \frac{1}{2}, \Pi'_n \times \Pi'_{n-1})}{L^{(\sigma \Pi \otimes \sigma \Pi)}(m + \frac{1}{2}, \Pi'_n \times \Pi'_{n-1}) \cdot L^{(\sigma \Pi' \otimes \sigma \Pi)}(m + \frac{1}{2}, \Pi_n \times \Pi_{n-1})}.
\]

2.4. Case $n = 2$. Let $\Pi$ be a cohomological irreducible cuspidal automorphic representation of $\GL_2(\A)$ with central character $\omega_\Pi$. Then there exist $\kappa \geq 2$ and $w \in \Z$ such that $\kappa \equiv w \pmod{2}$ and

\[
\Pi_{\infty} = D_\kappa \otimes |w/2|.
\]

Let $M_\mu$ be the irreducible algebraic representation of $\GL_2$ of highest weight

\[
\mu = \left( \frac{\kappa - 2 - w}{2}, \frac{-\kappa + 2 - w}{2} \right).
\]

Following the normalization in [RT11] (3.6) and (3.7), we define the generators

\[
[\Pi_{\infty}]_{\pm} \in H^1(\g_2, K_0^2; \W(\Pi_{\infty}) \otimes M_{N, C})[\pm].
\]

In the following theorem, we recall a refinement of Theorem 2.4 when $n = 2$ due to Raghuram–Tanabe [RT11] and a period relation. Let $f_{\Pi} \in \Pi$ be the normalized newform of $\Pi$ in the sense of Casselman [Cas73]. Let $\|f_{\Pi}\|$ be the Petersson norm of $f_{\Pi}$ defined by

\[
\|f_{\Pi}\| = \int_{A^\infty \setminus \GL_2(\Q) \setminus \GL_2(\A)} \|f_{\Pi}(g)\|^2 \det g^{-w} \, dg^{\Tam}.
\]

Here $dg^{\Tam}$ is the Tamagawa measure on $A^\infty \setminus \GL_2(\A)$.

**Theorem 2.7.**

1. (Raghuram–Tanabe) Let $\chi$ be a finite order Hecke character of $A^\infty$ and $m + \frac{1}{2}$ a critical point of $L(s, \Pi \otimes \chi)$. For $\sigma \in \Aut(\C)$, we have

\[
\sigma \left( \frac{L^{(\chi)}(m + \frac{1}{2}, \Pi \otimes \chi)}{(2\pi \sqrt{-1})^{m+(\kappa + w)/2} \cdot G(\chi) \cdot p(\Pi, -1)^m \cdot \chi_{\infty}(-1)} \right) = \frac{L^{(\sigma \Pi \otimes \sigma \chi)}(m + \frac{1}{2}, \chi_{\infty}(-1))}{(2\pi \sqrt{-1})^{m+(\kappa + w)/2} \cdot G(\sigma \chi) \cdot p^{(\sigma \Pi, -1)^m} \cdot \chi_{\infty}(-1)}.
\]

2. For $\sigma \in \Aut(\C)$, we have

\[
\sigma \left( \frac{p(\Pi, +) \cdot p(\Pi, -)}{(2\pi \sqrt{-1}) \cdot (\sqrt{-1})^w \cdot G(\Pi)} \|f_{\Pi}\| \right) = \frac{p^{(\sigma \Pi, +)} \cdot p^{(\sigma \Pi, -)}}{(2\pi \sqrt{-1}) \cdot (\sqrt{-1})^w \cdot G(\sigma \omega_\Pi) \cdot \|f_{\Pi}\|}.
\]

Proof. We refer to [Che22] Lemma 4.1 for the proof of the second assertion.

\[\square\]
3. Proof of main results

3.1. Some algebraicity results. Firstly we recall some algebraicity results in the literature which will be used in the proof of our main result.

**Theorem 3.1.** For \( i = 1, 2, 3 \), let \( \Pi_i \) be a cohomological irreducible cuspidal automorphic representation of \( \text{GL}_2(A) \) with \( \Pi_i \mid \mid \omega \mid \mid^{|w_i/2} \) for some \( \kappa_i \equiv w_i \pmod{2} \).

(1) (Shimura [Shi76, Theorem 3]) Assume \( \kappa_1 > \kappa_2 \). For \( \sigma \in \text{Aut}(\mathbb{C}) \) and \( m \in \mathbb{Z} \) critical for \( L(s, \Pi_1 \times \Pi_2) \), we have

\[
\sigma \left( \frac{L^{(\infty)}(m, \Pi_1 \times \Pi_2)}{(2\pi \sqrt{-1})^{2m+\kappa_1+w_1+w_2} \cdot (\sqrt{-1})^{w_1} \cdot G(\omega_{\Pi_1} \omega_{\Pi_2}) \cdot \|f_{\Pi_1}\|} \right) = \frac{L^{(\infty)}(m, \sigma \Pi_1 \times \sigma \Pi_2)}{(2\pi \sqrt{-1})^{2m+\kappa_1+w_1+w_2} \cdot (\sqrt{-1})^{w_1} \cdot G(\sigma \omega_{\Pi_1} \sigma \omega_{\Pi_2}) \cdot \|f_{\sigma \Pi_1}\|}.
\]

(2) (Garrett–Harris [GH93, Theorem 4.6], C.- [Che21a, Theorem 1.2]) Assume

\[
\kappa_1 + \kappa_2 + \kappa_3 \geq 2 \max\{\kappa_1, \kappa_2, \kappa_3\} + 2.
\]

For \( \sigma \in \text{Aut}(\mathbb{C}) \) and \( m + \frac{1}{2} \in \mathbb{Z} + \frac{1}{2} \) critical for the triple product L-function \( L(s, \Pi_1 \times \Pi_2 \times \Pi_3) \), we have

\[
\sigma \left( \frac{L^{(\infty)}(m, \Pi_1 \times \Pi_2 \times \Pi_3)}{(2\pi \sqrt{-1})^{4m+\kappa_1+\kappa_2+\kappa_3+2w_1+2w_2} \cdot (\sqrt{-1})^{w_1+w_2} \cdot G(\omega')^2 \cdot \|f_{\Pi_1}\| \|f_{\Pi_2}\| \|f_{\Pi_3}\|} \right) = \frac{L^{(\infty)}(m, \sigma \Pi_1 \times \sigma \Pi_2 \times \sigma \Pi_3)}{(2\pi \sqrt{-1})^{4m+\kappa_1+\kappa_2+\kappa_3+2w_1+2w_2} \cdot (\sqrt{-1})^{w_1+w_2} \cdot G(\sigma \omega')^2 \cdot \|f_{\sigma \Pi_1}\| \|f_{\sigma \Pi_2}\| \|f_{\sigma \Pi_3}\|}.
\]

Here \( w' = w_1 + w_2 + w_3 \) and \( \omega' = \omega_{\Pi_1} \omega_{\Pi_2} \omega_{\Pi_3} \).

(3) Assume

\[
3\kappa_1 - 7 \geq \kappa_2 \geq \kappa_1 + 5.
\]

For \( \sigma \in \text{Aut}(\mathbb{C}) \) and \( m \in \mathbb{Z} \) critical for \( L(s, \text{Sym}^3 \Pi_1 \times \Pi_2) \), we have

\[
\sigma \left( \frac{L^{(\infty)}(m, \text{Sym}^3 \Pi_1 \times \Pi_2)}{(2\pi \sqrt{-1})^{4m+3\kappa_1+\kappa_2+6w_1+2w_2} \cdot (\sqrt{-1})^{w_1+w_2} \cdot G(\omega_{\Pi_1}^3 \omega_{\Pi_2})^2 \cdot \|f_{\Pi_1}\|^3 \|f_{\Pi_2}\|} \right) = \frac{L^{(\infty)}(m, \text{Sym}^3 \sigma \Pi_1 \times \sigma \Pi_2)}{(2\pi \sqrt{-1})^{4m+3\kappa_1+\kappa_2+6w_1+2w_2} \cdot (\sqrt{-1})^{w_1+w_2} \cdot G(\sigma \omega_{\Pi_1}^3 \sigma \omega_{\Pi_2})^2 \cdot \|f_{\sigma \Pi_1}\|^3 \|f_{\sigma \Pi_2}\|}.
\]

**Proof.** We prove the third assertion. We assume \( \Pi_1 \) is non-CM. The case when \( \Pi_1 \) is of CM-type follows from the algebraicity of critical values for \( \text{GL}_2 \times \text{GL}_2 \) due to Shimura [Shi76]. First we recall a result of Liu [Lin21]: Let \( \Psi \) be a cohomological irreducible cuspidal automorphic representation of \( \text{GSp}_4(A) \) such that \( \Psi_{\infty} \) is a holomorphic discrete series representation of \( \text{GSp}_4(\mathbb{R}) \). Then the minimal \( U(2) \)-type of \( \Psi_{\infty} \) is equal to \( (k_1, k_2) \) for some \( k_1 \geq k_2 \geq 3 \). Let \( u \in \mathbb{Z} \) be the integer such that \( |\omega_{\Psi_{\infty}}| = |\omega|^u \). Consider the standard L-function \( L(s, \Psi, \text{std}) \) of \( \Sigma \). Liu proved that there exists a sequence of complex numbers \( (P^i(\sigma \Psi))_{\sigma \in \text{Aut}(\mathbb{C})} \) such that

\[
(3.1) \quad \sigma \left( \frac{L^{(\infty)}(m, \Psi, \text{std})}{(2\pi \sqrt{-1})^{3m} \cdot (\sqrt{-1})^u \cdot P(\Psi)} \right) = \frac{L^{(\infty)}(m, \sigma \Psi, \text{std})}{(2\pi \sqrt{-1})^{3m} \cdot (\sqrt{-1})^u \cdot P(\sigma \Psi)}
\]
for all critical point $m \geq 1$ of $L(s, \Psi, \text{std})$ and $\sigma \in \text{Aut}(\mathbb{C})$. On the other hand, Morimoto [Mor18] proved the following result on algebraicity of critical values for $\text{GSp}_4 \times \text{GL}_2$: Let $\Sigma$ be as above. Consider the Rankin–Selberg $L$-function $L(s, \Psi \times \Pi_2)$. Assume

\[
\kappa_1 + \kappa_2 - 7 \geq \kappa_2 \geq \kappa_1 - \kappa_2 + 7.
\]

Then

\[
\begin{align*}
\sigma \left( \frac{L^{(\infty)}(m, \Psi \times \Pi_2)}{(2\pi \sqrt{-1})^{4m+\kappa_2+2\kappa_1+2w_2} \cdot (\sqrt{-1})^{u+w_2} \cdot G(\omega_{\Psi} \omega_{\Pi_2})^2 \cdot P(\Psi) \cdot \|f_{\Pi_2}\|} \right) \\
= \frac{L^{(\infty)}(m, \sigma \Psi \times \Pi_2)}{(2\pi \sqrt{-1})^{4m+\kappa_2+2\kappa_1+2w_2} \cdot (\sqrt{-1})^{u+w_2} \cdot G(\sigma \omega_{\Psi} \omega_{\Pi_2})^2 \cdot P(\sigma \Psi) \cdot \|f_{\Pi_2}\|}
\end{align*}
\]  

(3.2)

for all critical point $m \in \mathbb{Z}$ of $L(s, \Psi \times \Pi_2)$ and $\sigma \in \text{Aut}(\mathbb{C})$. Note that the above algebraicity result was proved in [Mor18] under the assumption that $u$ is even, and we extend the result to arbitrary $u$ in [Che21b §4.5]. Now we take $\Psi$ be the descent of $\text{Sym}^3 \Pi_1$ from $\text{GL}_4(\mathbb{A})$ to $\text{GSp}_4(\mathbb{A})$ with respect to the spin representation of $\text{GSp}_4(\mathbb{C})$. The existence of the descent is a consequence of Arthur’s multiplicity formula for $\text{GSp}_4(\mathbb{A})$ established by Gee and Taïbi [GT19 Theorem 7.4.1]. Note that in this case,

\[(k_1, k_2) = (2\kappa_1 - 1, \kappa_1 + 1), \quad u = 3w_1, \quad \omega_{\Psi} = \omega_{\Pi_2}^3\]

By the functoriality, we have

\[
L(s, \Psi, \text{std}) = L(s, \text{Sym}^4 \Pi_1 \otimes \omega_{\Pi_2}^{-2}).
\]

It follows from Theorem 3.1 and our previous result [Che21b] on Deligne’s conjecture for critical values of $L(s, \text{Sym}^4 \Pi_1 \otimes \omega_{\Pi_2}^{-2})$ (see also [Mor21]), we obtain the period relation

\[
\begin{align*}
\sigma \left( \frac{P(\Psi)}{(2\pi \sqrt{-1})^{3\kappa_1} \cdot \|f_{\Pi_2}\|^3} \right) = \frac{P(\sigma \Psi)}{(2\pi \sqrt{-1})^{3\kappa_1} \cdot \|f_{\Pi_2}\|^3}
\end{align*}
\]

for all $\sigma \in \text{Aut}(\mathbb{C})$. Combining this period relation with (3.2), we obtain assertion (3). This completes the proof. \hfill \Box

3.2. Proof of Theorem 1.2. Now we begin the proof of Theorem 1.2. Let

\[
f(\tau) = \sum_{n=1}^{\infty} a_f(n)q^n \in S_\kappa(N, \omega), \quad q = e^{2\pi \sqrt{-1} \tau}
\]

be a normalized elliptic newform. We assume $f$ is not a CM-form. Fix $w \in \mathbb{Z}$ such that $\kappa \equiv w \pmod{2}$. Let $\Pi$ be the cohomological irreducible cuspidal automorphic representation of $\text{GL}_2(\mathbb{A})$ characterized by the following conditions:

- For each prime number $p \nmid N$, $\Pi_p$ is unramified and its Satake parameters are roots of the polynomial

\[
X^2 - p^{(1-\kappa-w)/2}a_f(p)X + p^{-w}\omega(p).
\]

- $\Pi_{\infty} = D_\kappa \otimes |w|^{w/2}$.

Let $f_{\Pi} \in \Pi$ be the normalized newform of $\Pi$ and $\|f_{\Pi}\|$ its Petersson norm in (2.7). Then we have

\[
\|f_{\Pi}\| = 2 \cdot \langle f, f \rangle.
\]
Recall $c^\pm(f) \in \mathbb{C}^\times$ is the periods of $f$ in $\text{Shi77}$. Let $p(\Pi, \pm)$ be the Betti-Whittaker periods of $\Pi$ with respect to the generator $[\Pi_c]_\pm$ following the normalization in $\text{RT11}$ (3.6) and (3.7)]. We have the following period relation (cf. $\text{RT11}$ § 4.6):

$$
\sigma \left( \frac{c^\pm(f)}{p(\Pi, \pm (-1)^{(\kappa + w)/2})} \right) = \frac{c^\pm(\sigma f)}{p(\sigma \Pi, \pm (-1)^{(\kappa + w)/2})}
$$

for all $\sigma \in \text{Aut}(\mathbb{C})$. For $n \geq 1$, let $\text{Sym}^n \Pi$ be the functorial lift of $\Pi$ to $\text{GL}_{n+1}(\mathbb{A})$ with respect to the symmetric $n$-th power representation of $\text{GL}_2(\mathbb{C})$. The existence of the lifts was proved by Newton–Thorne $\text{NT21}$ (see also $\text{GJ72}$, $\text{KS02}$, $\text{Kim03}$, $\text{CT15}$, $\text{CT17}$ for $n \leq 8$). Note that $\text{Sym}^n \Pi$ is a cohomological irreducible cuspidal automorphic representation of $\text{GL}_{n+1}(\mathbb{A})$ (cf. $\text{Rag10}$ Theorem 5.3]). For any primitive Dirichlet character $\chi$, we have

$$
L^{(\infty)}(s + \frac{1}{2}, \text{Sym}^5 \Pi \otimes \chi_\mathbb{A}) = L(s + \frac{5(\kappa + w)}{2} - 2, \text{Sym}^5(f) \otimes \chi).
$$

Here $\chi_\mathbb{A}$ is the finite order Hecke character of $\mathbb{A}^\times$ such that $\chi_\mathbb{A}, p$ is unramified and $\chi_\mathbb{A}, p(p) = \chi(p)$ for all prime numbers $p$ coprime to the conductor of $\chi$.

We fix an auxiliary cohomological irreducible cuspidal automorphic representation $\Pi'$ of $\text{GL}_2(\mathbb{A})$ satisfying the following conditions:

- $\Pi'$ is non-CM.
- $\Pi'_\infty = D_{2\kappa - 1} \otimes |^{1/2+w}$.

We also fix a finite order Hecke character $\eta$ of $\mathbb{A}^\times$ such that $\eta_\mathbb{A}(-1) = (-1)^{1+w}$.

Let $\Pi \boxtimes \Pi'$ be the functorial lift of the Rankin–Selberg convolution of $\Pi$ and $\Pi'$ to $\text{GL}_4(\mathbb{A})$. The existence of the lift was proved by Ramakrishnan in $\text{Ram00}$. Since $\Pi$ and $\Pi'$ are both non-CM and the weights of $\Pi_\infty$ and $\Pi'_\infty$ are different, we see that $\Pi \boxtimes \Pi'$ is cuspidal automorphic by the cuspidality criterion in $\text{Ram00}$ Theorem M]. Let $\Sigma_3$ and $\Sigma_4$ be tamely isobaric automorphic representations of $\text{GL}_3(\mathbb{A})$ and $\text{GL}_4(\mathbb{A})$, respectively, defined by

$$
\Sigma_3 = (\Pi' \otimes |_{\mathbb{A}^\times} |^{1/2}) \boxtimes \eta |_{\mathbb{A}^\times}, \quad \Sigma_4 = (\Pi \boxtimes \Pi') \otimes |_{\mathbb{A}^\times}^{1/2}.
$$

It is easy to verify that $\Sigma_3$ and $\Sigma_4$ are cohomological. Indeed, we have

$$
\Sigma_{3,\infty} = \text{Sym}^2 \Pi_\infty, \quad \Sigma_{4,\infty} = \text{Sym}^3 \Pi_\infty.
$$

Also note that $(\Sigma_{3,\infty}, \Sigma_{4,\infty})$ is balanced (cf. $\text{Rag10}$ Theorem 5.3]). Let $\chi$ be a finite order Hecke character of $\mathbb{A}^\times$. We write

$$
\Sigma_3' = \text{Sym}^2 \Pi, \quad \Sigma_4' = \text{Sym}^3 \Pi
$$

and consider four Rankin–Selberg $L$-functions for $\text{GL}_4 \times \text{GL}_3$:

$$
L(s, \Sigma_4 \times \Sigma_3' \otimes \chi), \quad L(s, \Sigma_4 \times \Sigma_3 \otimes \chi),
$$

$$
L(s, \Sigma_4' \times \Sigma_3' \otimes \chi), \quad L(s, \Sigma_4' \times \Sigma_3 \otimes \chi).
$$

Note that the sets of critical points for these $L$-functions are the same and is given by

$$\{m + \frac{1}{2} \in \mathbb{Z} + \frac{1}{2} | \frac{-\kappa - 5w}{2} + 1 \leq m \leq \frac{\kappa - 5w}{2} - 1\}.$$

From now on, we assume $\kappa \geq 6$. 
3.2.1. Case \( m + \frac{1}{2} \neq \frac{1-5w}{2} \). Let \( m + \frac{1}{2} \neq \frac{1-5w}{2} \) be a non-central critical point. In particular, since \( m + \frac{1}{2} \) is non-central, the Rankin–Selberg \( L \)-functions in (3.6) are non-vanishing at \( s = m + \frac{1}{2} \) by the results of Jacquet–Shalika [JS81, Theorem 5.3] and Shahidi [Sha81, Theorem 5.2]. We have the following factorizations of \( L \)-functions:

\[
L(s, \Sigma_4 \times \Sigma' \otimes \chi) = L(s - \frac{1}{2}, \text{Sym}^3 \Pi \times \Pi' \otimes \omega \Pi \chi), \\
L(s, \Sigma_4 \times \Sigma_3 \otimes \chi) = L(s - 1, \Pi \times \Pi' \otimes \chi) \cdot L(s - \frac{1}{2} + w, \Pi \times \Pi' \otimes \eta \chi).
\]

By Theorem 3.1 for \( \sigma \in \text{Aut}(\mathbb{C}) \) we have

\[
\sigma \left( \frac{L^{(x)}(m + \frac{1}{2}, \Sigma_4 \times \Sigma_3 \otimes \chi) \cdot L^{(x)}(m + \frac{1}{2}, \Sigma_4 \times \Sigma_3 \otimes \chi)^{-1}}{(2\pi \sqrt{-1}) \cdot (\sqrt{-1}) \cdot G(\omega_\Pi^2 \omega_\Pi^{-1} \eta^{-1})^2 \cdot \|f_\Pi\|^2 \cdot \|f_\Pi\|^{-1}} \right)
\]

\[
= \frac{L^{(x)}(m + \frac{1}{2}, \Sigma_4 \times \Sigma_3 \otimes \chi) \cdot L^{(x)}(m + \frac{1}{2}, \Sigma_4 \times \Sigma_3 \otimes \chi)^{-1}}{(2\pi \sqrt{-1}) \cdot (\sqrt{-1}) \cdot G(\omega_\Pi^2 \omega_\Pi^{-1} \eta^{-1})^2 \cdot \|f_\Pi\|^2 \cdot \|f_\Pi\|^{-1}}.
\]

Note that the assumption \( \kappa \geq 6 \) is needed in order to apply Theorem 3.1-(3). We also have the following factorizations of \( L \)-functions:

\[
L(s, \Sigma'_4 \times \Sigma_3 \otimes \chi) = L(s - \frac{1}{2}, \text{Sym}^3 \Pi \times \Pi' \otimes \omega \Pi \chi), \\
L(s, \Sigma'_4 \times \Sigma_3 \otimes \chi) = L(s, \text{Sym}^3 \Pi \otimes \chi) \cdot L(s, \text{Sym}^3 \Pi \otimes \omega \Pi \chi) \cdot L(s, \Pi \otimes \omega_\Pi^2 \chi).
\]

By [GH93, Theorem 6.2] and [Che21a, Theorem 1.6] on Deligne’s conjecture for symmetric cube \( L \)-functions, for any finite order Hecke character \( \xi \) of \( \mathbb{A}_F^\times \) and \( \sigma \in \text{Aut}(\mathbb{C}) \), we have

\[
\sigma \left( \frac{L^{(x)}(m + w + \frac{1}{2}, \text{Sym}^3 \Pi \otimes \xi)}{(2\pi \sqrt{-1})^{2m+2n+5w} \cdot (\sqrt{-1})^w \cdot G(\omega_\Pi^2 \eta_\Pi^2 \chi)^2 \cdot p(\Pi, (-1)^m \chi \xi (-1))^2 \cdot \|f_\Pi\|} \right)
\]

\[
= \frac{L^{(x)}(m + w + \frac{1}{2}, \text{Sym}^3 \Pi \otimes \xi)}{(2\pi \sqrt{-1})^{2m+2n+5w} \cdot (\sqrt{-1})^w \cdot G(\omega_\Pi^2 \eta_\Pi^2 \chi)^2 \cdot p(\Pi, (-1)^m \chi \xi (-1))^2 \cdot \|f_\Pi\|^3}.
\]

Combining with Theorems 2.7 and 3.1 for \( \sigma \in \text{Aut}(\mathbb{C}) \) we have

\[
\sigma \left( \frac{L^{(x)}(m + \frac{1}{2}, \text{Sym}^3 \Pi \otimes \chi) \cdot L^{(x)}(m + \frac{1}{2}, \Sigma_4' \times \Sigma_3 \otimes \chi) \cdot L^{(x)}(m + \frac{1}{2}, \Sigma_4' \times \Sigma_3 \otimes \chi)^{-1}}{(2\pi \sqrt{-1})^{3m-1+(9n+15w)/2} \cdot (\sqrt{-1})^{1+w} \cdot G(\omega_\Pi^2 \eta_\Pi^2 \chi)^2 \cdot p(\Pi, (-1)^m \chi \xi (-1))^3 \cdot \|f_\Pi\|^3} \right)
\]

\[
= \frac{L^{(x)}(m + \frac{1}{2}, \text{Sym}^3 \Pi \otimes \chi) \cdot L^{(x)}(m + \frac{1}{2}, \Sigma_4' \times \Sigma_3 \otimes \chi) \cdot L^{(x)}(m + \frac{1}{2}, \Sigma_4' \times \Sigma_3 \otimes \chi)^{-1}}{(2\pi \sqrt{-1})^{3m-1+(9n+15w)/2} \cdot (\sqrt{-1})^{1+w} \cdot G(\omega_\Pi^2 \eta_\Pi^2 \chi)^2 \cdot p(\Pi, (-1)^m \chi \xi (-1))^3 \cdot \|f_\Pi\|^3}.
\]

On the other hand, it follows from Corollary 2.6 that

\[
\sigma \left( \frac{L^{(x)}(m + \frac{1}{2}, \Sigma_4' \times \Sigma_3') \cdot L^{(x)}(m + \frac{1}{2}, \Sigma_4' \times \Sigma_3') \cdot L^{(x)}(m + \frac{1}{2}, \Sigma_4' \times \Sigma_3') \cdot L^{(x)}(m + \frac{1}{2}, \Sigma_4' \times \Sigma_3')}{L^{(x)}(m + \frac{1}{2}, \Sigma_4' \times \Sigma_3') \cdot L^{(x)}(m + \frac{1}{2}, \Sigma_4' \times \Sigma_3')} \right)
\]

\[
= \frac{L^{(x)}(m + \frac{1}{2}, \Sigma_4' \times \Sigma_3') \cdot L^{(x)}(m + \frac{1}{2}, \Sigma_4' \times \Sigma_3') \cdot L^{(x)}(m + \frac{1}{2}, \Sigma_4' \times \Sigma_3') \cdot L^{(x)}(m + \frac{1}{2}, \Sigma_4' \times \Sigma_3')}{L^{(x)}(m + \frac{1}{2}, \Sigma_4' \times \Sigma_3') \cdot L^{(x)}(m + \frac{1}{2}, \Sigma_4' \times \Sigma_3')}.
\]
for all $\sigma \in \text{Aut}(\mathbb{C})$. We thus conclude from (3.7) and (3.8) that
\[
\sigma \left( \frac{L(x)(m + \frac{1}{2}, \text{Sym}^5\Pi \otimes \chi)}{(2\pi\sqrt{-1})^{3m+(9\chi+15w)/2} \cdot \left(\sqrt{-1}\right)^w \cdot G(\omega^2_{\Pi} \chi)^3 \cdot p(\Pi, (1 - 1)^m \lambda_{\chi}(1)) \cdot \|f_{\Pi}\|^3} \right) 
\]
(3.9)
\[
= \frac{L(x)(m + \frac{1}{2}, \text{Sym}^5\Pi \otimes \sigma\chi)}{(2\pi\sqrt{-1})^{3m+(9\chi+15w)/2} \cdot \left(\sqrt{-1}\right)^w \cdot G(\omega^2_{\Pi} \sigma\chi)^3 \cdot p(\sigma\Pi, (1 - 1)^m \lambda_{\chi}(1)) \cdot \|f_{\Pi}\|^3}
\]
for all $\sigma \in \text{Aut}(\mathbb{C})$. Recall the identity (3.5) for $L$-functions and the period relations (3.3), (3.4). This completes the proof of Theorem 1.2 for non-central critical points.

3.2.2. Case $m + \frac{1}{2} = \frac{1-5w}{2}$. Now we consider the central critical point $s = \frac{1-5w}{2}$. Note that in this case $w$ must be even. We have a factorization of the twisted exterior square $L$-function of $\text{Sym}^5\Pi$:
\[
L(s, \text{Sym}^5\Pi, \lambda^2 \otimes \omega^{-5}) = L(s, \text{Sym}^8\Pi \otimes \omega^{-4}) \cdot L(s, \text{Sym}^4\Pi \otimes \omega^{-2}) \cdot \zeta_{\mathbb{F}}(s).
\]
In particular, $L(s, \text{Sym}^5\Pi, \lambda^2 \otimes \omega^{-5})$ has a pole at $s = 1$. In this case, for $\sigma \in \text{Aut}(\mathbb{C})$, we have the Betti–Shilika period $\omega(\text{Sym}^5\Pi, \pm) \in \mathbb{C}^\times$ of $\text{Sym}^5\Pi$ in [GR14b, Definition/Proposition 4.2.1] (recalled in (A.3)). Fix a finite order Hecke character $\eta$ of $\mathbb{A}^\times$ such that $\eta_{\chi}(1) = -\chi(\chi)$. By [GR14b] Theorem 7.1.2, we have
\[
\sigma \left( \frac{L(x)(m + \frac{1}{2}, \text{Sym}^5\Pi \otimes \chi) \cdot L(x)(m + \frac{1}{2}, \text{Sym}^5\Pi \otimes \eta)^{-1}}{\omega(\text{Sym}^5\Pi, (1 - 1)^m \lambda_{\chi}(1)) \cdot \omega(\text{Sym}^5\Pi, (1 - 1)^m \lambda_{\chi}(1))^{-1} \cdot G(\chi^{-1})^3} \right) 
\]
(3.10)
\[
= \frac{L(x)(m + \frac{1}{2}, \text{Sym}^5\Pi \otimes \sigma\chi) \cdot L(x)(m + \frac{1}{2}, \text{Sym}^5\Pi \otimes \sigma\eta)^{-1}}{\omega(\text{Sym}^5\Pi, (1 - 1)^m \lambda_{\chi}(1)) \cdot \omega(\text{Sym}^5\Pi, (1 - 1)^m \lambda_{\chi}(1))^{-1} \cdot G(\chi^{-1})^3}
\]
for all critical points $m + \frac{1}{2} \neq \frac{1-5w}{2}$ and $\sigma \in \text{Aut}(\mathbb{C})$. On the other hand, by [HR20] Theorem 7.21, we have
\[
\sigma \left( \frac{L(x)(m - \frac{1}{2}, \text{Sym}^5\Pi \otimes \chi)}{(2\pi\sqrt{-1})^{-3} \cdot \Omega(\text{Sym}^5\Pi, (1 - 1)^m \lambda_{\chi}(1)) \cdot L(x)(m + \frac{1}{2}, \text{Sym}^5\Pi \otimes \chi)} \right) 
\]
(3.11)
\[
= \frac{L(x)(m - \frac{1}{2}, \text{Sym}^5\Pi \otimes \sigma\chi)}{(2\pi\sqrt{-1})^{-3} \cdot \Omega(\text{Sym}^5\Pi, (1 - 1)^m \lambda_{\chi}(1)) \cdot L(x)(m + \frac{1}{2}, \text{Sym}^5\Pi \otimes \sigma\chi)}
\]
for all critical points $m + \frac{1}{2} \neq \frac{1-5w}{2}$ and $\sigma \in \text{Aut}(\mathbb{C})$, where $\Omega(\text{Sym}^5\Pi, \pm) \in \mathbb{C}^\times$ is the relative period of $\text{Sym}^5\Pi$ defined in [HR20, Definition 5.3] (recalled in (A.2)). In Theorem A.1 below, we show that
\[
\sigma \left( \frac{\Omega(\text{Sym}^5\Pi, \pm)}{\omega(\text{Sym}^5\Pi, \pm)} \right) = \Omega(\text{Sym}^5\Pi, \pm) \cdot \frac{\omega(\text{Sym}^5\Pi, \pm)}{\omega(\text{Sym}^5\Pi, \pm)}
\]
for all $\sigma \in \text{Aut}(\mathbb{C})$. We then conclude from (3.10) and (3.11) that
\[
\sigma \left( \frac{L(x)(m + \frac{1}{2}, \text{Sym}^5\Pi \otimes \chi)}{(2\pi\sqrt{-1})^{-3} \cdot G(\chi^{-1})^3 \cdot L(x)(\frac{1-5w}{2}, \text{Sym}^5\Pi \otimes \eta)} \right) 
\]
\[
= \frac{L(x)(m + \frac{1}{2}, \text{Sym}^5\Pi \otimes \sigma\chi)}{(2\pi\sqrt{-1})^{-3} \cdot G(\chi^{-1})^3 \cdot L(x)(\frac{1-5w}{2}, \text{Sym}^5\Pi \otimes \sigma\chi)}
\]
for all $\sigma \in \text{Aut}(\mathbb{C})$. Therefore, (3.9) for the central critical value $L(x)(\frac{1-5w}{2}, \text{Sym}^5\Pi \otimes \chi)$ follows from (3.9) for $L(x)(\frac{3-5w}{2}, \text{Sym}^5\Pi \otimes \eta)$. This completes the proof.
Remark 3.2. In the above proof of Conjecture 1.1, the argument works for \( \kappa \geq 3 \). It suffices to relax the assumption in Theorem 3.1-(3) in order to prove Conjecture 1.1 for \( \kappa \geq 3 \).

Remark 3.3. Since we assume \( \kappa \geq 6 \), \( m + \frac{1}{2} = \frac{5-5w}{2} \) is also a critical point when \( w \) is even. Then, by [HR20, Theorem 7.21] alone, we can deduce (3.9) for the central critical value from that for \( L^{(\infty)}(\frac{5-5w}{2}, \text{Sym}^{5} \Pi \otimes \chi) \). However, the above argument are necessary if we consider \( \kappa = 4 \). As in this case, there are no non-central critical points with the same parity as the central critical point.

3.3. Period relations. We keep the notation of the previous section. For \( n \geq 1 \), recall \( \text{Sym}^{n} \Pi \) is the symmetric \( n \)-th power lifting of \( \Pi \). Note that \( \text{Sym}^{n} \Pi \) is a cohomological tamely isobaric automorphic representation of \( \text{GL}_{n+1}(\mathbb{A}) \). Based on Deligne’s conjecture for critical values of the standard \( L \)-function of \( \text{Sym}^{n} \Pi \) in [Del79, Proposition 7.7], we propose the following conjectural period relations between the periods associated to \( \Pi \) and the Betti–Whittaker periods of \( \text{Sym}^{n} \Pi \).

Conjecture 3.4. For \( n \geq 1 \), there exists a constant \( C_{n,\infty} \in \mathbb{C}^{\times} \) unique up to \( \mathbb{Q}^{\times} \), depending only on \( n \) and \( \Pi_{\infty} \), such that for all \( \sigma \in \text{Aut}(\mathbb{C}) \), we have

\[
\sigma \left( \frac{p(\text{Sym}^{2r+1} \Pi, \pm)}{C_{2r+1,\infty} \cdot G(\omega_{\Pi})^{2r(r+1)^2} \cdot p(\Pi, \pm)^{r+1} \cdot \|f_{\Pi}\|^{2r(r+1)(2r+1)/3}} \right) = \frac{p(\text{Sym}^{2r+1} \sigma \Pi, \pm)}{C_{2r+1,\infty} \cdot G(\omega_{\Pi})^{2r(r+1)^2} \cdot p(\sigma \Pi, \pm)^{r+1} \cdot \|f_{\Pi}\|^{2r(r+1)(2r+1)/3}}
\]

if \( n = 2r + 1 \), and

\[
\sigma \left( \frac{p(\text{Sym}^{2r} \Pi)}{C_{2r,\infty} \cdot G(\omega_{\Pi})^{2r(r+1)^2} \cdot \|f_{\Pi}\|^{r(r+1)(2r+1)/3}} \right) = \frac{p(\text{Sym}^{2r} \sigma \Pi)}{C_{2r,\infty} \cdot G(\omega_{\Pi})^{(r+1)^2} \cdot \|f_{\Pi}\|^{r(r+1)(2r+1)/3}}
\]

if \( n = 2r \).

Remark 3.5. In [Che22, Theorem 4.13], we prove the conjecture for \( n = 2 \) and explicitly determine the constant \( C_{2,\infty} \).

As a consequence of Theorem 1.2, we prove the conjecture for \( n = 3 \).

Theorem 3.6. If \( \kappa \geq 6 \), then Conjecture 3.4 holds for \( n = 3 \).

Remark 3.7. It is customary in the literature to choose \( w = -\kappa \). In this case, we let \( p(\text{Sym}^{3}(f, \pm)) = p(\text{Sym}^{3} \Pi, \pm) \). Then Theorem 3.6 is equivalent to Theorem 1.3.

Proof. Let \( \chi \) be any finite order Hecke character of \( \mathbb{A}^{\times} \). We have the following factorizations of \( L \)-functions:

\[
L(s, \text{Sym}^{3} \Pi \times \text{Sym}^{2} \Pi \otimes \chi) = L(s, \text{Sym}^{5} \Pi \otimes \chi) \cdot L(s, \text{Sym}^{3} \Pi \otimes \omega_{\Pi} \chi) \cdot L(s, \Pi \otimes \omega_{\Pi}^{2} \chi),
\]

\[
L(s, \text{Sym}^{2} \Pi \times \Pi \otimes \omega_{\Pi} \chi) = L(s, \text{Sym}^{3} \Pi \otimes \omega_{\Pi} \chi) \cdot L(s, \Pi \otimes \omega_{\Pi}^{2} \chi).
\]

Let \( m + \frac{1}{2} \) be any non-central critical point for \( \text{Sym}^{5} \Pi \). By Theorem 1.2, for all \( \sigma \in \text{Aut}(\mathbb{C}) \) we have

\[
\sigma \left( \frac{L^{(\infty)}(m + \frac{1}{2}, \text{Sym}^{3} \Pi \times \text{Sym}^{2} \Pi \otimes \chi)}{(2\pi \sqrt{-1})^{3m+9w+15w}/2 \cdot (\sqrt{-1})^{w} \cdot G(\omega_{\Pi}^{2} \chi)^{3} \cdot p(\Pi, (-1)^{m} \chi \mp (-1)^{3} \cdot \|f_{\Pi}\|^{3}} \right)
\]

\[
= \frac{L^{(\infty)}(m + \frac{1}{2}, \text{Sym}^{3} \sigma \Pi \times \text{Sym}^{2} \sigma \Pi \otimes \chi)}{(2\pi \sqrt{-1})^{3m+9w+15w}/2 \cdot (\sqrt{-1})^{w} \cdot G(\omega_{\Pi}^{2} \sigma \chi)^{3} \cdot p(\sigma \Pi, (-1)^{m} \chi \mp (-1)^{3} \cdot \|f_{\Pi}\|^{3}} \right).
\]
Let \( C_{3, \infty} = (2\pi \sqrt{-1})^{3n+1+(9\kappa+15\lambda)/2} \cdot \frac{Z(m, \text{Sym}^3 \Pi, \text{Sym}^2 \Pi)}{Z(m + \omega, \text{Sym}^2 \Pi, \Pi)} \).

By Theorem \ref{main_thm}, it is easy to verify that Conjecture \ref{main_conj} holds for this choice of \( C_{3, \infty} \). \( \square \)

**APPENDIX A. PERIOD RELATIONS BETWEEN RELATIVE PERIODS AND BETTI–SHALIKA PERIODS**

Let \( \Pi \) be a cohomological irreducible cuspidal automorphic representation of \( \text{GL}_n(\mathbb{A}) \). We assume \( n \) is even and fix a non-trivial representative \( \delta \) of \( K_n/K_n^0 \). In the appendix, we establish period relations between relative periods and ratios of Betti–Shalika periods of \( \Pi \).

First we recall the definition of the relative periods \( \Omega(\Pi, \pm) \) of \( \Pi \) (cf. \cite{HR20}, § 5.2). Recall \( b_n = \frac{n^2}{4} \). We put \( t_n = \frac{n^2}{4} + \frac{n}{2} - 1 \). Let \( M \) the irreducible algebraic representation of \( \text{GL}_n \) such that
\[
H^{b_n}(\mathfrak{g}_n, K_n^0; \Pi \otimes M_{\mathbb{C}}) \neq 0, \quad H^{t_n}(\mathfrak{g}_n, K_n^0; \Pi \otimes M_{\mathbb{C}}) \neq 0.
\]

Since \( n \) is even, the above cohomology groups are 2-dimensional. Moreover, the restriction to \( (\mathfrak{g}_n, K_n^0) \) of \( \Pi_{\infty} \) decomposes as \( \Pi_{\infty} = \Pi_{\infty}^+ \oplus \Pi_{\infty}^- \) such that \( \delta \) interchanges the two summands.

We fix generators
\[
c_b^{++} \in H^{b_n}(\mathfrak{g}_n, K_n^0; \Pi_{\infty}^+ \otimes M_{\mathbb{C}}), \quad c_t^{++} \in H^{t_n}(\mathfrak{g}_n, K_n^0; \Pi_{\infty}^+ \otimes M_{\mathbb{C}}).
\]

We then define generators
\[
[I_{\infty}]^b_{\pm} \in H^{b_n}(\mathfrak{g}_n, K_n^0; \Pi_{\infty} \otimes M_{\mathbb{C}})[\pm], \quad [I_{\infty}]^t_{\pm} \in H^{t_n}(\mathfrak{g}_n, K_n^0; \Pi_{\infty} \otimes M_{\mathbb{C}})[\pm]
\]
by
(A.1)
\[
[I_{\infty}]^b_{\pm} = c_b^{++} \pm \delta \cdot c_b^{++}, \quad [I_{\infty}]^t_{\pm} = c_t^{++} \pm \delta \cdot c_t^{++}.
\]

Let
\[
T_{\Pi}^\pm : H^{b_n}(\mathfrak{g}_n, K_n^0; \Pi_{\infty} \otimes M_{\mathbb{C}})[\pm] \to H^{b_n}(\mathfrak{g}_n, K_n^0; \Pi_{\infty} \otimes M_{\mathbb{C}})[\mp]
\]
be the \( \text{GL}_n(\mathbb{A}_f) \)-equivariant isomorphism defined as follows: Let \( \Phi : I_{\infty} \otimes I_f \to \Pi \) be a \(((\mathfrak{g}_n, K_n) \times \text{GL}_n(\mathbb{A}_f))\)-equivariant isomorphism. It induces a \((K_n \times \text{GL}_n(\mathbb{A}_f))\)-isomorphism
\[
\Phi : H^{b_n}(\mathfrak{g}_n, K_n^0; \Pi_{\infty} \otimes M_{\mathbb{C}}) \otimes I_f \to H^{b_n}(\mathfrak{g}_n, K_n^0; \Pi \otimes M_{\mathbb{C}}).
\]

Then \( T_{\Pi}^\pm \) is defined so that
\[
T_{\Pi}^\pm \circ \Phi([I_{\infty}]^b_{\pm} \otimes \mathbf{v}) = \Phi([I_{\infty}]^b_{\mp} \otimes \mathbf{v}), \quad \mathbf{v} \in I_f.
\]

It is clear that \( T_{\Pi}^\pm \) is independent of the choice of \( \Phi \). The relative period \( \Omega(\Pi, \pm) \) is the non-zero complex number, unique up to \( \mathbb{Q}(\Pi)^\times \), such that
\[
(A.2) \quad T_{\Pi}^\pm \left( H^{b_n}(\mathfrak{g}_n, K_n^0; \Pi \otimes M_{\mathbb{C}})[\pm]^{\text{Aut}(\mathbb{C}/\mathbb{Q}(\Pi))} \right) \cong \frac{H^{b_n}(\mathfrak{g}_n, K_n^0; \Pi \otimes M_{\mathbb{C}})[\mp]}{\Omega(\Pi, \pm)}^\times.
\]

For \( \sigma \in \text{Aut}(\mathbb{C}) \), similarly we can define \( T_{\sigma \Pi}^\pm \) and \( \Omega(\sigma \Pi, \pm) \). We normalize the relative periods so that the diagram
\[
\begin{array}{ccc}
H^{b_n}(\mathfrak{g}_n, K_n^0; \Pi \otimes M_{\mathbb{C}})[\pm] & \xrightarrow{\Omega(\Pi, \pm) \cdot T_{\Pi}^\pm} & H^{b_n}(\mathfrak{g}_n, K_n^0; \Pi \otimes M_{\mathbb{C}})[\mp] \\
\downarrow^{\sigma^*} & & \downarrow^{\sigma^*} \\
H^{b_n}(\mathfrak{g}_n, K_n^0; \sigma \Pi \otimes M_{\mathbb{C}})[\pm] & \xrightarrow{\Omega(\sigma \Pi, \pm) \cdot T_{\sigma \Pi}^\pm} & H^{b_n}(\mathfrak{g}_n, K_n^0; \sigma \Pi \otimes M_{\mathbb{C}})[\mp]
\end{array}
\]
commutes. In other words, we have
\[ \sigma^* \circ T_{\Pi}^c (\Omega(\Pi, \pm) \cdot c) = T_{\sigma n}^c (\Omega(\sigma \Pi, \pm) \cdot \sigma^* c) \]
for all \( c \in H^b_n(\mathfrak{g}_n, K_n^\circ, \Pi \otimes M_C)[\pm] \). Note that in the notation and normalization of [HR20, §5.2.3, Definition 5.3], we have
\[ \Omega(\sigma \Pi, \pm) = (\sqrt{-1})^{n/2} \cdot \Omega^\pm(\sigma \lambda, \sigma \Pi_f). \]
Now we recall the definition of Betti–Shalika periods \( \omega(\Pi, \pm) \) of \( \Pi \). Assume there exists an algebraic Hecke character \( \eta \) of \( \mathbb{A}^\times \) such that the twisted exterior square \( L \)-function
\[ L(s, \Pi, \lambda^2 \otimes \eta^{-1}) \]
has a pole at \( s = 1 \). By the result of Jacquet–Shalika [JS90a], the assumption is equivalent to saying that the \( (\eta, \psi) \)-Shalika functional is non-vanishing on \( \Pi \). For \( \varphi \in \Pi \), let \( S_\varphi \) be the \( (\eta, \psi) \)-Shalika function of \( \varphi \) (cf. [GR14b, §3.1]). Let \( S(\Pi_{\infty}) \) and \( S(\Pi_f) \) be the spaces of \( (\eta_{\infty}, \psi_{\infty}) \)-Shalika functions and \( (\eta_f, \psi_f) \)-Shalika functions of \( \Pi_{\infty} \) and \( \Pi_f \), respectively. For \( S_{\infty} \in S(\Pi_{\infty}) \) and \( S_f \in S(\Pi_f) \), there exists a unique cuspidal form \( \varphi \in \Pi \) such that
\[ S_\varphi = S_{\infty} \cdot S_f. \]
In this way, we obtain a \( ((\mathfrak{g}_n, K_n) \times \GL_n(\mathbb{A}_f)) \)-equivariant isomorphism
\[ (A.3) \quad S(\Pi_{\infty}) \otimes S(\Pi_f) \rightarrow \Pi. \]
Let
\[ (A.4) \quad \Phi^S_{\Pi, \pm} : S(\Pi_f) \rightarrow H^b_n(\mathfrak{g}_n, K_n^\circ; \Pi \otimes M_C)[\pm] \]
be the \( \GL_n(\mathbb{A}_f) \)-equivariant isomorphism defined as follows: For \( S \in S(\Pi_f) \), we have
\[ [\Pi_{\infty}]^t_+ \otimes S \in H^b_n(\mathfrak{g}_n, K_n^\circ; S(\Pi_{\infty}) \otimes S(\Pi_f) \otimes M_C)[\pm]. \]
Then \( \Phi^S_{\Pi, \pm}(S) \) is the image of \([\Pi_{\infty}]^t_+ \otimes S\) under the \( \GL_n(\mathbb{A}_f) \)-equivariant isomorphism induced by the isomorphism \[ (A.3) \]. Here \([\Pi_{\infty}]^t_+\) is the generator fixed in \[ (A.1) \]. For \( \sigma \in \Aut(\mathbb{C}) \), let \( t_\sigma : S(\Pi_f) \rightarrow S(\sigma \Pi_f) \) be the \( \sigma \)-linear \( \GL_n(\mathbb{A}_f) \)-equivariant isomorphism defined by
\[ t_\sigma S(g) = \sigma(S(\text{diag}(u_{\sigma^{-1}}, \ldots, u_{\sigma^{-1}}, 1, \ldots, 1)g)), \quad g \in \GL_n(\mathbb{A}_f). \]
Here \( 1 \) appears in the diagonal matrix \( n/2 \)-times and \( u_{\sigma} \in \hat{\mathbb{Z}}^\times \) is the unique element such that \( \sigma(\psi(x)) = \psi(u_{\sigma}x) \) for all \( x \in \mathbb{A}_f \). The Betti–Shalika period \( \omega(\Pi, \pm) \) is the non-zero complex number, unique up to \( \mathbb{Q}(\Pi, \eta)^\times \), such that
\[ \Phi^S_{\Pi, \pm} \left( \frac{S(\sigma \Pi_f)}{\omega(\Pi, \pm)} \right) = H^b_n(\mathfrak{g}_n, K_n^\circ; \Pi \otimes M_C)[\pm]^{\Aut(\mathbb{C}/\mathbb{Q}(\Pi, \eta))}. \]
For \( \sigma \in \Aut(\mathbb{C}) \), similarly we can define \( \Phi^S_{\sigma \Pi, \pm} \) and \( \omega(\sigma \Pi, \pm) \). We normalize the Betti–Shalika periods such that
\[ (A.5) \quad \sigma^* \left( \frac{\Phi^S_{\Pi, \pm}(S)}{\omega(\Pi, \pm)} \right) = \frac{\Phi^S_{\sigma \Pi, \pm}(t_\sigma S)}{\omega(\sigma \Pi, \pm)} \]
for all \( S \in S(\Pi_f) \). As the main result of the appendix, we have the following period relation.

**Theorem A.1.** For \( \sigma \in \Aut(\mathbb{C}) \), we have
\[ \sigma \left( \frac{\Omega(\Pi, \pm) \cdot \omega(\Pi, \pm)}{\omega(\Pi, \pm)} \right) = \Omega(\sigma \Pi, \pm) \cdot \frac{\omega(\sigma \Pi, \pm)}{\omega(\sigma \Pi, \pm)}. \]
Proof. Let

\[ \Phi_{\Pi, \pm}^H : W(\Pi_f) \longrightarrow H^1_n(\mathfrak{g}_n, K_n^\infty; \Pi \otimes M_\mathbb{C})[\pm] \]

be the GL$_n$(A$_f$)-equivariant isomorphism defined as above with respect to the generator $[\Pi_x]_\pm$ and the isomorphism (2.1). Similarly we define the (top degree) Betti–Whittaker period $q(\Pi, \pm)$ of $\Pi$ such that

\[ \sigma^* \left( \frac{\Phi_{\Pi, \pm}^H(W)}{q(\Pi, \pm)} \right) = \frac{\Phi_{\sigma \Pi, \pm}^H(t_\sigma W)}{q(\sigma \Pi, \pm)} \]

for all $W \in W(\Pi_f)$. By the period relations in [RS08 (4.6)] and [BR17 Corollary 3.3.14], we have

\[ \sigma \left( \frac{p(\Pi, \pm) \cdot q(\Pi, \mp)}{p(\Pi, \mp) \cdot q(\Pi, \pm)} \right) = \frac{p(\sigma \Pi, \pm) \cdot q(\sigma \Pi, \mp)}{p(\sigma \Pi, \mp) \cdot q(\sigma \Pi, \pm)} \]

for all $\sigma \in \text{Aut}(\mathbb{C})$. On the other hand, it is clear from the definition and normalization of the relative periods and (bottom degree) Betti–Whittaker periods in §2.2 that

\[ \sigma \left( \Omega(\Pi, \pm) \cdot \frac{p(\Pi, \mp)}{p(\Pi, \pm)} \right) = \Omega(\sigma \Pi, \pm) \cdot \frac{p(\sigma \Pi, \mp)}{p(\sigma \Pi, \pm)} \]

for all $\sigma \in \text{Aut}(\mathbb{C})$ (see the explanation in [HR20 §5.2.3]). Therefore, to prove the assertion, it suffices to establish the following period relation:

\[ \sigma \left( \frac{q(\Pi, \pm) \cdot \omega(\Pi, \mp)}{q(\Pi, \mp) \cdot \omega(\Pi, \pm)} \right) = \frac{q(\sigma \Pi, \pm) \cdot \omega(\sigma \Pi, \mp)}{q(\sigma \Pi, \mp) \cdot \omega(\sigma \Pi, \pm)} \]

for all $\sigma \in \text{Aut}(\mathbb{C})$. Let $\Pi^+$ be the $((\mathfrak{g}_n, K_n^\infty) \times \text{GL}_n(A_f))$-submodule of $\Pi$ consisting of cusp forms in $\Pi$ with archimedean components in $\Pi^+$. Let

\[ \iota_{\Pi}^S : S(\Pi_f) \longrightarrow H^1_n(\mathfrak{g}_n, K_n^\infty; \Pi^+ \otimes M_\mathbb{C}), \quad \iota_{\Pi}^W : W(\Pi_f) \longrightarrow H^1_n(\mathfrak{g}_n, K_n^\infty; \Pi^+ \otimes M_\mathbb{C}) \]

be the GL$_n$(A$_f$)-equivariant isomorphisms defined similar to (A.4) and (A.6) with $[\Pi_x]_\pm$ and $H^1_n(\mathfrak{g}_n, K_n^\infty; \Pi \otimes M_\mathbb{C})[[\pm]]$ replaced by $c_{\Pi}^{+\dagger}$ and $H^1_n(\mathfrak{g}_n, K_n^\infty; \Pi^+ \otimes M_\mathbb{C})$, respectively. Let $C_{\mathbb{C}W}(\Pi)$ be the non-zero complex number, unique up to $\mathbb{Q}(\Pi, \eta)^\times$, such that

\[ \frac{\iota_{\Pi}^S(W(\Pi_f)^{\text{Aut}(\mathbb{C}/\mathbb{Q}(\Pi, \eta))})}{C_{\mathbb{C}W}(\Pi)} = \frac{\iota_{\Pi}^S(S(\Pi_f)^{\text{Aut}(\mathbb{C}/\mathbb{Q}(\Pi, \eta))})}{C_{\mathbb{C}W}(\Pi)} \]

For $\sigma \in \text{Aut}(\mathbb{C})$, similarly we define $C_{\mathbb{C}W}(\sigma \Pi)$. We normalize these periods so that

\[ \frac{\iota_{\Pi}^S((\iota_{\Pi}^S)^{-1} \circ \iota_{\Pi}^W(W))}{C_{\mathbb{C}W}(\Pi)} = \frac{\iota_{\Pi}^S((\iota_{\Pi}^S)^{-1} \circ \iota_{\Pi}^W(t_\sigma W))}{C_{\mathbb{C}W}(\sigma \Pi)} \]

for all $W \in W(\Pi_f)$. On the other hand, by definition we have

\[ \Phi_{\Pi, \pm}^S = \iota_{\Pi}^S \pm \delta \cdot \iota_{\Pi}^S, \quad \Phi_{\Pi, \pm}^W = \iota_{\Pi}^W \pm \delta \cdot \iota_{\Pi}^W. \]
Therefore, for $\sigma \in \text{Aut}(\mathbb{C})$ and $W \in \mathcal{W}(\Pi_f)$, we have

$$
\sigma^* \left( \frac{\Phi^W_{\Pi, \pm}(W)}{C^S_{\mathcal{W}(\Pi)} \cdot \omega(\Pi, \pm)} \right) = \sigma^* \left( \frac{\Phi^S_{\mathcal{W}(\Pi), \pm}((t^S_{\mathcal{W}(\Pi)})^{-1} \circ \iota^W_{\Pi}(W))}{C^S_{\mathcal{W}(\Pi)} \cdot \omega(\Pi, \pm)} \right) \\
= \Phi^S_{\mathcal{W}(\Pi), \pm} \left( t_{\sigma} \left( (t^S_{\mathcal{W}(\Pi)})^{-1} \circ \iota^W_{\Pi}(W) \right) \right) \cdot \frac{1}{\omega(\sigma \Pi, \pm)} \\
= \Phi^S_{\mathcal{W}(\sigma \Pi), \pm} \left( (t^S_{\mathcal{W}(\sigma \Pi)})^{-1} \circ \iota^W_{\sigma \Pi}(t_{\sigma}W) \right) \\
= \frac{\Phi^W_{\mathcal{W}(\sigma \Pi), \pm}(t_{\sigma}W)}{C^S_{\mathcal{W}(\sigma \Pi)} \cdot \omega(\sigma \Pi, \pm)}.
$$

Here the second and third equalities follow from (A.5) and (A.9), respectively. Comparing with (A.7), we deduce that

$$
\sigma \left( \frac{q(\Pi, \pm)}{C^S_{\mathcal{W}(\Pi)} \cdot \omega(\Pi, \pm)} \right) = \frac{q(\sigma \Pi, \pm)}{C^S_{\mathcal{W}(\sigma \Pi)} \cdot \omega(\sigma \Pi, \pm)}
$$

for all $\sigma \in \text{Aut}(\mathbb{C})$. In particular, (A.8) follows immediately. This completes the proof. \hfill \Box

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