The bold/timorous walker on the trek from home

Maurizio Serva

Dipartimento di Ingegneria e Scienze dell’Informazione e Matematica, Università dell’Aquila, 67010 L’Aquila, Italy and
Departamento de Biofísica e Farmacologia, Universidade Federal do Rio Grande do Norte, 59072-970 Natal-RN, Brazil

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We study a one-dimensional random walk with memory. The behavior of the walker is modified with respect to the simple symmetric random walk (SSRW) only when he is at the maximum distance ever reached from his starting point (home). In this case, having the choice to move farther or to move closer, he decides with different probabilities. If the probability of a forward step is higher then the probability of a backward step, the walker is bold, otherwise he is timorous. We investigate the asymptotic properties of this bold/timorous random walk (BTRW) showing that the scaling behavior vary continuously from sub-diffusive (timorous) to super-diffusive (bold). The scaling exponents are fully determined with a new mathematical approach.

I. INTRODUCTION

In 1827, the botanist Robert Brown noticed that pollen grains in water perform a peculiar erratic movement. Many decades later, in 1905, Albert Einstein [1] gave an explanation of the pollen random walk in terms of collisions with the water molecules relating the diffusion coefficient to observable quantities. Indeed, Einstein was scooped by Louis Bachelier which five years before, in his 1900 doctoral thesis [2] and in a following paper [3], arrived to similar conclusions. Actually, Bachelier was interested in the motion of prices on French stock market, but (log-)prices move like pollen in water and their random walk can be treated mathematically on the same ground.

This twofold origin of random walk as a probabilistic tool is illuminating, in fact, this utensil can be applied everywhere "a walker" (a particle, a cell, an individual, a price, a language,...) moves erratically in such a way that its square displacement $x^2(t)$ increases in average according to $(x^2(t)) \sim t$.

In its simpler version, the path of a random walk is the output of a succession of independent random steps. In this case, the scaling relation $(x^2(t)) \sim t$ is immediate. Nevertheless, in most cases, this relation also holds if memory effect in size and direction of the steps are present, the requirement is that memory is short ranged and steps have not diverging length.

The scaling relation $(x^2(t)) \sim t$ is traditionally associated to the appellative normal diffusion, while anomalous diffusion corresponds to a scaling $(x^2(t)) \sim t^{2\nu}$ with $\nu \neq 1/2$. In particular sub-diffusive behavior corresponds to $\nu < 1/2$ and super-diffusive behavior to $\nu > 1/2$.

There is a very large number of phenomena which exhibit anomalous diffusion as well a variety of models which have been used to describe them, we refer to [4-8] for a review of both.

Broadly, anomalous diffusion may arise via diverging steps length, as in Lévy flights or via long-range memory effects as in fractional Brownian motion and in self avoiding random walks. Diverging steps length and long-range memory are two different ways of violating the necessary conditions for the central limit theorem when applied to random walks.

Anomalous diffusion (super-diffusion) in Lévy flights is the simple consequence of the fact that the length of the steps has a heavy-tailed probability distribution. This does not mean that the problem is trivial, see for example [9] where the authors consider the interesting case in which diffusion is strongly anomalous $(\langle x^q(t) \rangle \sim t^{q\nu}$ with $\nu$ depending on $q$).

Anomalous diffusion induced by long-range memory is the non self-evident output of the self-interaction of the walker position at different times; the most celebrate example probably being the "true" self avoiding random walk introduced quite a long time ago [10] and later rigorously studied (see [12] [15] and references therein). In this model, the exponent $\nu$ depends only on dimensionality.

In some case, the mechanism which gives origin to anomalous scaling can be different for example special deterministic or random environments (see for example [14] [15]) or multi-particle interactions [16].

Since exact solution of non-trivial models with memory are quite difficult to obtain, some effort has been made in this direction. For example, in the elephant model [17], the walker decides the direction of his step depending on his previous decisions. Unfortunately, this model, given the direction of the first step, can be exactly mapped in a Markovian model, without necessity of enlarging the phase space, and, more importantly, the anomalous scaling is not a consequence of an anomalous diffusion but of the movement of the center of mass of the probability distribution of the position. Some generalization of this model has been proposed (see [18]) which are genuinely non-Markovian but which show the same problem concerning the origin of the anomalous scaling.

In an other analytically treatable model it is considered the case of a semi-Markovian sub-diffusive processes in which the waiting time for a step is given by a probability distribution with a diverging mean value [19].

Random walks with memory have been also employed to model the spreading of an infection in a medium with a history-dependent susceptibility [20, 21], the focus, in
this case, is the time scaling of the survival probability (a trap is collocated somewhere) and not the scaling of diffusion. Moreover, random walks with memory have been used in finance as, for example, in [22]. In this paper it is described the strategy of a prudent investor which tries to maximize the invested capital while never decreasing his standard of life. In [20, 21] and in [22], as in the model presented in this paper, the behavior of the walker is modified only when it is at the maximum distance from the origin and Markovianity is recovered only when the phase space is properly enlarged.

Motivated by the scarcity of exact solutions, we present in this paper a model which is treatable, one-dimensional, genuinely non-Markovian and which shows anomalous scaling ranging from sub-diffusion to super diffusion according to a single continuous parameter.

The paper is organized as follows: in Section II we present the model and we expose our results; in Section III we describe the decomposition of the dynamics which is at the basis of our mathematical approach; the asymptotic behavior is computed in Section IV; in section V we write and exactly numerically solve the associated forward Kolmogorov equation; Section VI contains our conclusions. Some of the calculations whose result is used in Section III are postponed in a final Appendix.

II. MODEL AND RESULTS

The model presented in this paper is one-dimensional, steps all have the same unitary length, time is discrete and the walker can only move left or right at any time step. The behavior of the random walker is modified with respect to the simple symmetric random walk (SSRW) only when he is at the maximum distance ever reached from his starting point (home). In this case, he decides with different probabilities to make a step forward (going farther from home) or a step backward (going closer to home).

More precisely, the model is the following: the walker starts from home \(x(0) = 0\), then, at any time he can make a (unitary length) step to the right or to the left

\[
x(t+1) = x(t) + \sigma(t)
\]

with \(\sigma(t) = \pm 1\). We define

\[
y(t) = \max_{0 \leq s \leq t} |x(s)|
\]

which is the maximum distance from home he ever attained which obviously implies \(-y(t) \leq x(t) \leq y(t)\). Then we assume

- \(\sigma(0) = \pm 1\) with equal probability, i.e. the walker choses with equal probability the direction of the first step,
- \(\sigma(t) = \pm 1\) with equal probability if the walker is not at is maximum distance from home, i.e., \(|x(t)| < y(t)|

\[
\sigma(t) = \text{sign}(x(t)) \text{ with probability } p(y(t)) \text{ and } \sigma(t) = -\text{sign}(x(t)) \text{ with probability } 1 - p(y(t)) \text{ if } |x(t)| = y(t),
\]

- the probability \(p(y)\) depends on \(y\) according to \(p(y) = y^\gamma/(1 + y^\gamma)\).

Therefore, simple symmetric random walk (SSRW) holds when \(|x(t)| < y(t)|\) but when the walker is at the maximum distance from home \(|x(t)| = y(t)|\), he boldly prefers to move farther when \(\gamma > 0\) or timorously prefers to move closer if \(\gamma < 0\).

Our goal is to find the asymptotic behavior of \(y(t)|\) and \(x(t)|\). We preliminarily observe that in case \(\gamma = 0\) one has \(p(y) = 1/2\) which implies SSRW holds everywhere, also if the walker is at maximum distance. In this case, ordinary scaling applies: \(\langle y^\alpha(t) \rangle \sim \langle |x(t)|^\alpha \rangle \sim t^{\alpha/2}\) for any real positive \(\alpha\) (the sign \(\sim\) indicates that the ratio of the two sides asymptotically tends to a strictly positive constant. We use the sign \(\simeq\) for the stronger statement that the ratio tends to 1).

Results of this paper can be summarized as follows:

- \(\langle y^\alpha(t) \rangle \simeq \langle y(t) \rangle^\alpha \simeq (t/2\nu)^{\alpha\nu}\) with \(\nu = 1/(2 - \gamma)\) for \(-\infty < \gamma < 0\),
- \(\langle y^\alpha(t) \rangle \sim t^{\alpha/\nu}\) with \(\nu = 1/(2 - 2\gamma)\) for \(0 \leq \gamma \leq 1/2\),
- \(\langle y^\alpha(t) \rangle \simeq \langle y(t) \rangle^\alpha \simeq t^\alpha\) for \(1/2 < \gamma < \infty\).

Moreover, \(\langle |x(t)|^\alpha \rangle \sim \langle y^\alpha(t) \rangle\) for all \(\gamma\).

In both regions \(-\infty < \gamma < 0\) and \(1/2 < \gamma < \infty\) relations are indicated with \(\simeq\), i.e. the ratio of the two sides tends to 1 in the limit \(t \to \infty\) providing both the scaling exponent and the scaling factor for \(\langle y^\alpha(t) \rangle\). Furthermore, \(\langle y^\alpha(t) \rangle \simeq \langle y(t) \rangle^\alpha\) which implies that the variable \(y(t)\) scales deterministically as its average.

In particular, in region \(-\infty < \gamma < 0\) the behaviour is sub-diffusive, as a consequence of the propensity the walker has to step in the home direction when at maximum distance, while, in the region \(1/2 < \gamma < \infty\), behavior is ballistic, with coefficient 1, as a consequence of the strong propensity to step away from home when at maximum distance.

Finally, in the intermediate region \(0 \leq \gamma \leq 1/2\) only the scaling exponent of \(\langle y^\alpha(t) \rangle\) is determined. If \(\gamma = 1/2\), ordinary diffusion holds (with standard coefficients), while in the region \(0 < \gamma \leq 1/2\), we have non-ballistic super-diffusive behaviour as a consequence of the (not too strong) propensity the walker has to step away from home when at maximum distance.

The behavior of the anomalous scaling exponent \(\nu\) with respect to the control parameter \(\gamma\) in region \(\gamma < 1/2\) is depicted in Fig. 1. In region \(\gamma \geq 1/2\) the exponent \(\nu\) equals 1, i.e. behavior is ballistic (notice that, by construction, it cannot be super-ballistic).

Next three sections are devoted to the validation of the results here presented.
Here we outline the decomposition of the dynamics which is at the basis of our new mathematical approach.

Trajectories are decomposed in active journeys and lazy journeys. The lazy journey starts at the time \( t \) when the walker leaves the maximum and it ends when he reaches it again at time \( t + m + 1 \), i.e. \( |x(t)| = y(t) = y \), \( |x(t + s)| < y \) for \( 1 \leq s \leq m \) and \( |x(t + m + 1)| = y \). The total number of steps of this journey is \( 1 + m \) since the first step is for leaving the maximum and \( m \) is the random number of steps necessary to reach it again starting from a position \( |x| = y - 1 \). During all steps of the lazy journey the maximum remains the same. The minimum duration of the lazy journey is two time steps (1 + 2) when the walker immediately steps back to the maximum after having left it.

The active journey starts at the time \( t + m + 1 \) when the walker arrives on a maximum and it ends when he leaves it at time \( t + m + n + 1 \), i.e. \( |x(t + m)| = y - 1 \), \( |x(t + m + s)| = y + s \) for \( 0 \leq s \leq n \) and \( |x(t + m + n + 2)| = y + n - 1 \) (the first step of a new lazy journey). The total (random) number of time steps of this journey is \( n \) with a minimum duration of zero steps (\( n = 0 \) when the walker immediately leaves the maximum after being arrived). During the active journey the maximum increases from \( y \) to \( y + n \).

A cycle journey is composed by a lazy journey followed by an active journey, its duration is \( 1 + m + n \) and the maximum increases of \( n \).

Notice that both \( n = n(y) \) and \( m = m(y) \) are random variables whose distribution only depends on \( y \). In fact \( m(y) \) is the SSRW first hitting time of one of the barriers \( y \) or \( -y \) starting from position \( x = y - 1 \) or \( x = -y + 1 \), while the statistics of \( n(y) \) is determined by \( y \) through \( p(y) \).

We start by evaluating the probability \( \pi(n|y) \) that the walker makes at least \( n \) steps during the active journey, i.e. \( \pi(n|y) = \text{prob} (n(y) \geq n) \). Straightforwardly:

\[
\pi(n|y) = \prod_{s=0}^{n-1} p(y + s) \tag{3}
\]

where \( p(y + s) = (y + s)\gamma/(1 + (y + s)\gamma) \).

For large \( y \), we have to distinguish three different ranges of \( \gamma \):

\begin{enumerate}
  \item \( -\infty < \gamma < 0 \), in this case \( \pi(n|y) \leq y^{n\gamma} \) where the approximate equality \( \pi(n|y) \approx y^{n\gamma} \) holds for \( n \) small with respect to \( y \),
  \item \( 0 < \gamma < 1 \), in this case \( \pi(n = \beta y^{\gamma}|y) \approx e^{-\beta} \), which means that \( n(y) \approx \xi y^{\gamma} \) where \( \xi \) is a random variable distributed according to an unitary exponential probability,
  \item \( \gamma > 1 \), in this case \( \pi(n|y) \approx e^{-\psi(y,n)} \) where \( \psi(y,n) = \sum_{s=0}^{n-1} 1/(y + s)^{\gamma} \), noticeably, \( \pi(\infty|y) \) is finite which implies that \( n(y) \) is infinite with finite probability.
\end{enumerate}

The above relations are derived in the Appendix.

In both cases \( i \) and \( ii \) the average is \( \langle n(y) \rangle \approx y^{\gamma} \), while in case \( iii \) it diverges, moreover, the standard deviation in case \( i \) is \( \sigma(n(y)) \approx y^{\gamma/2} \) while in case \( ii \) is \( \sigma(n(y)) \approx y^{\gamma} \).

The statistical properties of \( m(y) \) are well known in fact, as already stressed, \( m(y) \) is simply the SSRW time for hitting one of the frontiers of the interval \([-y,y]\) starting from position \( y - 1 \) (or \(-y + 1\)). Using standard martingale approach, it is easy to compute the average \( \langle m(y) \rangle \approx 2y \) and the standard deviation \( \sigma(m(y)) \approx (8/3)^{1/2}y^{3/2} \) where the approximations hold for large \( y \) (see the Appendix).

We underline that standard deviation is larger than average and that both diverge when \( y \to \infty \), nevertheless, \( m(y) \) has a finite probability to be of order 1 (consider that \( m(y) = 1 \) with probability 1/2 independently on \( y \)). This is reflected in the fact that the averages \( \langle m(y)^{\beta} \rangle \) are of order 1 when \( \beta \) is negative. For \( m \) not very small, \( \theta(m(y)) \approx m^{-1/2} \) with a cutoff at \( m_c \sim y^2 \) where \( \theta(m(y)) \) drops to 0 (\( m_c \) is the typical time the walker can "see" the far barrier). This implies that \( \theta(m|y) \) is approximately a truncated Lévy distribution.

### IV. ASYMPTOTIC ANALYSIS

We have seen in previous section that in a cycle journey starting from a maximum \( y \), time increases of is \( 1 + m(y) + n(y) \) and the maximum increases of \( n(y) \).

Let us indicate with \( k \) (to be not confused with time \( t \)) the progressive number identifying cycle journeys, each composed by a lazy journey followed by an active journey. Also, let us indicate with \( y(k) \) the value of the maximum when the cycle journey number \( k \) starts.
The time $t$ is linked to the progressive number $k$ by the stochastic relation
\[ t(k + 1) = t(k) + 1 + m(y(k)) + n(y(k)) \tag{4} \]
while the value of the maximum by
\[ y(k + 1) = y(k) + n(y(k)) \tag{5} \]
where $m(y(k))$ and $n(y(k))$ are all independent random variables whose statistical properties we have already described.

In principle one should simply solve the two equations and, by substitution, obtain the scaling behavior of $y(t)$. Obviously, this asks for some work.

We start our asymptotic analysis by considering the region $-\infty < \gamma < 1$. Let us consider first equation (5). In the region of $\gamma$ we are considering, the variables have average $\langle n(y(k)) \rangle \simeq y(k)^\gamma$, then, from equation (5), one has $\langle y(k + 1)^{1-\gamma} \rangle \simeq (y(k)^{1-\gamma} + (1-\gamma))$. The omitted terms are of lower order in $y(k)$ since the standard deviation of the $n(y(k))$ can be $\sigma_n(y(k)) \simeq y(k)^{\gamma/2}$ (for $-\infty < \gamma < 0$) or $\sigma_n(y(k)) \simeq y(k)^{\gamma}$ (for $0 < \gamma < 1$). By integration we obtain $\langle y(k)^{1-\gamma} \rangle \simeq (1-\gamma)k^\gamma$ and by iteration $\langle y(k)^{l(1-\gamma)} \rangle \simeq (1-\gamma)^l k^l$ where $l$ is a positive integer number. Finally, by analytical continuation we have $\langle y(k)^\alpha \rangle \simeq (1-\gamma)^{\alpha/(1-\gamma)} k^{\alpha/(1-\gamma)} \simeq (y(k)^\alpha)$ for any real $\alpha$. We have thus proven the relation
\[ y(k) \simeq (1-\gamma)^{1/(1-\gamma)} k^{1/(1-\gamma)} \tag{6} \]
which holds deterministically, i.e. fluctuations are comparatively negligible in the large $y(k)$ limit.

Let us now consider equation (4), by simple sum we get
\[ t(k) = y(k) + k + M((k)) \tag{7} \]
where $y(k)$ is given by (6) and $M(k) = \sum_{i=0}^{k-1} m(y(i))$ which is a sum of independent variables distributed according to truncated Lévy distributions with $(m(y(i))) \simeq 2y(i)$ and $\sigma_{m(y(i))} \simeq (8/3)^{1/2} y(i)^{3/2}$. According to (6) we obtain the average
\[ \langle M(k) \rangle \simeq \frac{2}{2-\gamma} [(1-\gamma)k^{(2-\gamma)/(1-\gamma)}] \tag{8} \]
and the standard deviation $\sigma_M(k) \sim k^{(2-\gamma)/(2(1-\gamma))}$.

If $\gamma < 0$, the standard deviation $\sigma_M(k)$ is asymptotically negligible with respect to the average $\langle M(k) \rangle$ and we can thus replace $M(k)$ with its average $\langle M(k) \rangle$ in (7). Furthermore we can neglect the smaller terms $y(k)$ and $k$ and obtain $t(k) \simeq \langle M(k) \rangle$ which solved with respect to $k$ and substituted in (7) finally gives the relation
\[ y(t) \simeq (1-\gamma/2)^{1/(2-\gamma)} t^{1/(2-\gamma)} \tag{9} \]
which holds deterministically in the region $-\infty < \gamma < 0$.

On the contrary, if $0 < \gamma < 1$ the standard deviation of $M(k)$ is larger than its average. In this case, it is necessary to determine the behavior of its probability distribution. This can be done considering that all the independent $m(y(i))$ in the sum which defines $M(k)$ are distributed according to a truncated Lévy. Then, according to the generalized central limit for leptokurtic variables [9],
\[ L(k) = \frac{1}{k^2} \sum_{i=1}^{k} m(y(i)) = \frac{1}{k^2} M(k) \tag{10} \]
is also a truncated leptokurtic variable, notice, in fact, that the denominator equals the power two of the number of the summed variables (Lévy is $\alpha = 1/2$ stable). Also notice that $L(k)$ has average $\sim k^{\gamma/(1-\gamma)}$ and variance $\sim k^{(3/2)\gamma/(1-\gamma)}$ which both diverge in the large $k$ limit (truncation disappears). Accordingly, $L(k)$ is of order 1 with probability 1 and all the averages $\langle L(k)^\beta \rangle$ with negative $\beta$ are of order 1. This property is true for any $k$ and also holds in the limit $k \rightarrow \infty$ where $L(k) \rightarrow L$.

Then, in the region $0 < \gamma < 1$, equation (7) can be rewritten as
\[ t(k) \simeq y(k) + k^2 L \tag{11} \]
where the term $k$ has been dropped since it is smaller both with respect to $y(k)$ and $k^2 L$.

The region $0 < \gamma < 1$ splits into two subregions, when $1/2 < \gamma < 1$, the term $y(k)$ is larger than $k^2 L$, therefore we can assume $t(k) \simeq y(k)$ and therefore $y(t) \simeq t$ holds deterministically.

When $0 < \gamma < 1/2$, on the contrary, $k^2 L$ is larger than $y(k)$ so that $t(k) \simeq k^2 L$. This implies $k = (t(k)/L)^{1/2}$ which substituted in (10) finally gives the relation
\[ \langle y^\alpha(t) \rangle \simeq \langle L^{-\alpha\nu} \rangle (1-\gamma)^{\alpha/(1-\gamma)} t^{\alpha\nu} \tag{12} \]
with $\nu = 1/(2 - 2\gamma)$. The average $\langle L^{-\alpha\nu} \rangle$ is of order 1 since the exponent is negative, but we are unable to determine its exact value in terms of $\gamma$ and $\alpha$. So we simply conclude that $\langle y^\alpha(t) \rangle \sim t^{\alpha\nu}$.

At this point only remains the region $1 < \gamma < \infty$, but for these values of $\gamma$, active walks of infinite length have a finite probability, therefore, after some excursions away from the maximum the walker decides once of all to follow the same direction remaining always on the maximum. Accordingly, the relation $y(t) \simeq |x(t)| \simeq t$ holds deterministically. Our analysis of the scaling behavior of $y(t)$ is thus concluded.

Since the walker spends part of the time on the maximum where $|x(t)| = y(t)$ and part of the time in ordinary random walk in the interval $-y(t) < x(t) < y(t)$ where in average $|x(t)| \geq y(t)/2$ (the lazy trek always starts form the frontier), we also conclude that $\langle |x(t)|^\alpha \rangle \sim \langle y^\alpha(t) \rangle$.

**V. FORWARD KOLMOGOROV EQUATION**

We would like to test our results against the output of the exact numerical solution of the forward Kolmogorov
equation. Indeed, the process is non-Markovian, but it can be rendered Markovian by enlarging the phase space including the variable $y(t)$, and, therefore, jointly considering the evolution of the variables $x(t)$ and $y(t)$. Accordingly, it is possible to write a forward Kolmogorov equation for $P(x, y, t)$ which is the probability that $x(t) = x$ and $y(t) = y$.

The initial condition ($t = 0$) is $P(0, 0, 0) = 1$ while all others $P(x, y, 0)$ equal zero. First of all, notice that $P(x, y, t)$ obviously vanishes when $|x| > y$ and when $y > t$. Moreover, the symmetry of both initial condition and dynamics implies $P(x, y, t) = P(-x, y, t)$ for all $x$, $y$ and $t$. Thus, let us write the forward equation only for $x \geq 0$.

At any time $t \geq 1$ one has $P(0, 0, t) = 0$, furthermore one has $P(1, 1, t) = 0$ at even times, $P(1, 1, t) = (1/2)^{t/2}$ at odd times, $P(0, 1, t) = 0$ at odd times and $P(0, 1, t) = (1/2)^{t/2}$ at even times.

Assuming that $y \geq 2$, the forward Kolmogorov equation is completed by

$$P(x, y, t+1) = \frac{1}{2} P(x+1, y, t) + \frac{1}{2} P(x-1, y, t)$$

which holds when $0 \leq x \leq y - 2$. In case $x = 0$, we can use the symmetry to replace $P(-1, y, t)$ with $P(1, y, t)$ in the right hand side of the equation. When $x = y - 1$ we have

$$P(y-1, y, t+1) = (1-p(y)) P(y, y, t) + \frac{1}{2} P(y-2, y, t)$$

where $p(y) = y^{\gamma}/(1 + y^{\gamma})$ and, finally, when $x = y$ we have

$$P(y, y, t+1) = p(y-1) P(y-1, y-1, t) + \frac{1}{2} P(y-1, y, t).$$

We have numerically exactly solved the Kolmogorov equation and computed $\langle y(t) \rangle$, $\langle |x(t)| \rangle$, $\langle y^{2}(t) \rangle$ and $\langle |x(t)|^{2} \rangle$. The scaling exponent can be obtained by the ratio $\log(\langle y(t) \rangle)/\log(t)$ and analogous expressions where $\langle y(t) \rangle$ is replaced by $\langle |x(t)| \rangle$, $\langle y^{2}(t) \rangle$ or $\langle |x(t)|^{2} \rangle$. Nevertheless, convergence is much faster if one computes the exponent from $\log(\langle y(t) \rangle)/\log(t)$ and analogous expressions since the scaling factor is wiped out. In Fig. 1 we plot our results for $t_{2} = 11000$ and $t_{1} = 10000$ against prevision. Independently of the use of $(y(t))$, $(|x(t)|)$, $(y^{2}(t))$ or $(|x(t)|^{2})$ we find excellent agreement. Notice that for $\gamma = 1/2$ the exponent $\nu$ equals 1 (ballistic behavior), for larger values of $\gamma$ necessarily it must have the same value, thus, without loss of information, Fig. 1 ends at $\gamma = 1/2$.

In the region $-\infty < \gamma < 0$ we also have found the explicit scaling factor and the relation $(y^{\alpha}(t)) \simeq (y(t))^{\alpha}$. In order to confirm the latter we consider the residual difference $\sqrt{\langle y^{2}(t) \rangle}/(y(t)) - 1$ up to 30000 time steps. This difference converges to 0 according to a power law as shown by the log-log plot in Fig. 2 for two orders of magnitude of time. Moreover, the log-log plot of $(y(t))//(t/2\nu)^{\nu} - 1$ shows power law convergence to 0 (Fig. 2) proving that both scaling factor and scaling exponent are correct. A faster power law convergence (Fig. 2) can be obtained considering $2((y(t)+1)-(y(t)))/(t/2\nu)^{\nu} - 1$, since only the differential average at the largest time contributes. Plot in Fig. 2, corresponds to the case $\gamma = -1$ but we have verified the same power law behavior for various values in the region $-\infty < \gamma < 0$.

VI. CONCLUSIONS

Anomalous diffusion in this model is induced by long-range memory in a conceptually very simple manner, furthermore, the model is one-dimensional and it is controlled by a single parameter. In spite of this conceptual simplicity, the scaling behavior unfold all possibilities varying continuously from sub-diffusive to ballistic. More precisely, if the walker timidly prefers to go back when it is at the frontier of unexplored regions, it is sub-diffusive, on the contrary, if he boldly prefers to go where he never has gone before, it is super-diffusive.

The sub-diffusive region is below the threshold $\nu = 0$. Above the threshold $\nu = 1/2$ the process is ballistic, and the walker moves uniformly at constant velocity. Finally, in the region above the threshold $\nu = 0$ but below the threshold $\nu = 1/2$ the process is super-diffusive but sub-ballistic. This region is probably the most interesting since the walker has an intermittent behavior, with bursts of linear growth, followed by longer bursts of random motion. This behavior is typical of the transition from laminar to turbulent behavior in chaotic systems [24].

This reach phenomenology can be used, in principle, to model a variety of phenomena. We think, for example, to the problem of foraging strategies, with the walker (animal) changing his attitude when he is at the frontier of unexplored regions. The aim, in this case, is to evaluate the degree of success of the search in comparison with ordinary random walk search and Lévy search [25]. Also in epidemics, recent focus is on the effects of super-diffusive spreading of an infection, via heavy-tailed distributed jumps [26]. The present model could be al-
ternative, with super-diffusive (or sub-diffusive) spreading arising as an effect of memory of infection agents. Moreover, the orthography of languages performs a random walk on the discrete space of possible vocabularies \cite{27,28}. As in present model, the jump rates are different if changes are in the direction of a radical innovation or if they run on an already treaded territory. Finally, as we already mentioned, the (non-ballistic) super-diffusive region of the present model could represent a stochastic counterpart to chaotic systems with intermittent behavior (see also \cite{29}).

We conclude pointing out that the mathematical characterization of the BTRW in this paper is far to be complete, for example, all the scaling factors for the variable \( x(t) \) and the scaling factors for the variable \( y(t) \) in the region \( 0 < \gamma \leq 1/2 \) remain unknown as well all the correlations at different times among variables.

Finally, we would like to underline that this model could be successfully extended to higher dimensions.

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\section*{APPENDIX}

In the first part of this Appendix we prove the three relations i), ii) and iii) of Section III. In the region \( -\infty < \gamma < 0 \), one has \( p(y+s) = (y + s)^\gamma /(1 + (y + s)^\gamma) \leq y^\gamma \) which implies \( \pi(n|y) \leq y^{\gamma n} \).

If \( n \) is small with respect to \( y \), one also has \( p(y+s) = (y+s)^\gamma /(1 + (y+s)^\gamma) \approx y^\gamma \) which, using \( \gamma \), immediately gives the approximated equality \( \pi(n|y) \approx y^{\gamma n} \).

In the region \( 0 < \gamma < 1 \), we directly obtain from \( \gamma \),

\begin{equation}
[p(y)]^n \leq \pi(n|y) = \prod_{s=0}^{n-1} p(y+s) \leq [p(y+n)]^n,
\end{equation}

in fact, being \( \nu \) positive, \( p(y) \) is the smallest among the elements of the product and \( p(y + \beta y^\gamma) \) the largest.

Then assume \( n = \beta y^\gamma \) (if \( \beta y^\gamma \) is not an integer then \( n = \beta y^\gamma + \epsilon \) where \( 0 < \epsilon < 1 \), one immediately gets

\begin{equation}
[p(y)]^{\beta y^\gamma + \epsilon} \leq \pi(n = \beta y^\gamma |y) \leq [p(y + \beta y^\gamma)]^{\beta y^\gamma + \epsilon}
\end{equation}

Then, using the definition of \( p(y) \) and taking into account that \( 0 < \gamma < 1 \), it is straightforward to verify that the limit for \( y \to \infty \) of both bounds is \( e^{-\beta} \) so that

\begin{equation}
\pi(n = \beta y^\gamma |y) \approx e^{-\beta}
\end{equation}

The above approximated equality means that \( n(y) \approx \xi y^\gamma \) where \( \xi \) is a random variable distributed according to an unitary exponential probability.

Finally consider the region \( \gamma > 1 \), we have

\begin{equation}
p(y+s) = 1/(1 + (y+s)^{-\gamma}) \approx e^{-1/(y+s)^{\gamma}}
\end{equation}

which implies \( \pi(n|y) \approx e^{-\psi(y,n)} \) where \( \psi(y,n) = \sum_{s=0}^{n-1} 1/(y+s)^{\gamma} \). Noticeably, \( \nu \geq 1 \) implies that \( n(y) \) is finite which, in turn, implies that \( n(y) \) is infinite with finite probability.

We compute now the average and standard deviation of \( m(y) \) which appear in Section III.

We preliminary remark that the process \( x(t) \) is a SSRW when it is not on the maximum, therefore \( m(y) \) is simply the random time necessary for hitting one of the frontiers of the interval \([-y, y]\) starting from position \( y - 1 \) (or \( -y + 1 \)).

Assume that at time \( t + 1 \) the walker is in \( y - 1 \), i.e. \( x(t+1) = y - 1 \) (the choice \( x(t+1) = y + 1 \) is symmetrical and leads to the same results). Also assume that the walker hits for the first time one of the barriers at time \( t + 1 + m(y) \), i.e. \( x(t+1 + m(y)) = y \) or \( x(t+1 + m(y)) = -y \). Strong martingale property implies that the average of \( x(t+1 + s) - x(t+1) \) equals zero at any non-negative \( s \) also if \( s \) is a random time, therefore

\begin{equation}
\langle x(t+1 + m(y)) - x(t+1) \rangle = 0
\end{equation}

Given that \( a \) is the probability of hitting the barrier \( y + 1 \) and \( 1 - a \) is the probability of hitting the barrier \( -y \), one has that \( \langle x(t+1 + m(y)) \rangle = ay + (1-a)(-y) \). Therefore, equality \( \gamma \) rewrites \( ay - y(1-a) - y = 0 \) which, in turn, implies \( a = 1 - 1/(2y) \).

Then we notice that \( [x(t+1+s) - x(t+1)]^2 - s \) is also a martingale, therefore

\begin{equation}
\langle [x(t+1 + m(y)) - x(t+1)]^2 - m(y) \rangle = 0
\end{equation}

where \( \langle [x(t+1 + m(y))]^2 \rangle = ay^2 + (1-a)y^2 = y^2 \) which implies \( m(y) = a + (1-a)(2y - 1)^2 \approx 2y \).

Moreover, \( [x(t+1+s) - x(t+1)]^4 - 3s^2 \) is as well a martingale and, therefore,

\begin{equation}
\langle [x(t+1 + m(y)) - x(t+1)]^4 - 3m^2(y) \rangle = 0
\end{equation}

where \( \langle [x(t+1 + m(y))]^2 \rangle = ay^4 + (1-a)y^4 = y^4 \) which implies \( m^2(y) = (a + (1-a)(2y - 1)^4)/3 \approx (8/3)y^4 \).

The standard deviation can be finally easily computed as \( \sigma_m(y) = \langle m^2(y) \rangle - \langle m(y) \rangle^2 \approx (8/3)^{1/2} y^{3/2} \).
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