1 Introduction

The following text is extracted from a longer article in the making. Its aim is to show how one can calculate the production rate, decay rate and angular distribution of the decay products of a hypothetical "heavy neutrino" having the couplings of an ordinary neutrino to standard model particles, up to a (very small) mixing matrix element. It is aimed at experimental groups wishing to evaluate the sensitivity of their apparatus to such heavy neutrino production and decay through simulation. Masses have to be taken into account at every stage of the production/decay process since, for example, the well known helicity suppression of $\nu_e$ production in two-body $0^-$ mesons decays no longer works when the neutrino is hypothetized to have a mass of a few MeV. Also, polarization of the neutrino must be taken into account because it bears on the angular distribution of its decay products and therefore on the acceptance of the experimental set-up to a given combination of mass and mode. We do not claim to give here any new result. Most of them can probably be found in the litterature but they are scattered among many experimental or theoretical papers, and this is why we think an article...
like the one in the making might be useful. Moreover, some of these papers contain errors [1]. The calculations involved—evidently at lowest order—are possibly too straightforward to be of interest to theoreticians but too involved (or boring) for many an experimentalist.

We will therefore give fairly complete derivations so as to allow anyone with a minimal literacy in Dirac algebra to check our results.

The present extract is devoted to "heavy neutrino" production by decaying charged $0^-$ mesons and is restricted to two-body decays of said heavy neutrino.

2 Generalities

Neutrino states related to charged leptons through weak charged currents are thought to be linear combinations of mass eigenstates. The mechanism giving rise to these combinations is not known, but given the successes of "standard" physics, we assume that the interaction lagrangian is that of the Standard Model, namely:

$$L_{int} = eA^\alpha J_{em}^\alpha + \frac{g}{\cos \theta_w} Z^\alpha J_{neut}^\alpha + \frac{g}{\sqrt{2}} (W^\alpha\dagger J_{ch}^\alpha + W^\alpha J_{ch}^\alpha\dagger)$$

where:

- $J_{ch}^\alpha = \sum_{\beta=e,\mu,\tau} \nu_\beta \gamma_\alpha P_L l_\beta +$ quark currents
- $J_{neut}^\alpha = \sum_f \bar{\gamma}_\alpha (P_L T^3_w - \sin^2 \theta_w Q) f$
- $J_{em}^\alpha = \sum_f \gamma_\alpha Q f$

- $f$ is any elementary fermion field, $\nu_\beta$ and $l_\beta$ stand for the neutrino and charged lepton fields of "flavour" $\beta (= e, \mu, \tau)$.

- $T^3_w$ and $Q$ are the third weak isospin component and electric charge operators.

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1 We apologize in advance to the many people who published such or such formula for not quoting them. It is a task beyond our capacity and anyway, quite useless in a work of this kind.

2 Einstein’s summation convention is used thoroughly for space-time indices
\[ P_L = \frac{1}{2}(1 - \gamma^5) \] is the left-handed projector.

\[ g = \frac{e}{\sin \theta_w} \]

\( \nu_\beta \)'s are assumed to be linear superpositions of fields corresponding to definite mass quanta which can be either Dirac or Majorana. The notation will be as follows:

\[ \nu_\beta = \sum_h U_{\beta h} N_h \]

where \( N_h \) represents the field of a neutrino of mass \( \mu_h \). Greek indices will be used for leptonic "flavours" and latin indices for definite mass fields. It is known that there must exist three different light masses, but in the following, we will assume that there is at least an extra "heavy neutrino" (Heavy neutral lepton or HNL henceforth). \( U \) is therefore a rectangular extension of the PMNS mixing matrix.

The processes of interest are at low energies and will always involve virtual \( W \) and \( Z \)'s. Therefore, they will be at least second order in \( \mathcal{L}_{\text{int}} \). Neglecting \( q^2 \) w.r.t. \( m^2 \) in the bosons propagators written in momentum space, one finds the effective lagrangian:

\[ \mathcal{L}_{\text{eff}} = 4G_F\sqrt{2}\left(J_{\text{neut},\alpha}^\dagger J_{\text{neut},\alpha} + J_{\text{ch},\alpha}^\dagger J_{\text{ch},\alpha}\right) \] (1)

with \( G_F\sqrt{2} = \frac{g^2}{8M_W^2} \) in tree approximation.

In the following, we shall note \( j \) for a leptonic current and \( J \) for a hadronic current.

## 3 Production of massive neutrinos

### 3.1 Production through 2-body 0\(^-\) charged mesons decay

The relevant part of the effective lagrangian is here:

\[ \frac{4G_F}{\sqrt{2}}(j_{\text{ch},\alpha}^\dagger J_{\alpha}^{\text{ch}} + h.c.) \]

From now on, we assume a \( M^+ \) (momentum \( P \), mass \( M \)) decaying to HNL \( N_h \) (momentum \( p_N \), mass \( \mu \)), and antilepton \( l^+ \) of "flavour" \( \beta \) (momentum \( p \)).
\( p_l \) mass \( m \). \( M^+ \) being spinless, the only vector available to parametrize the hadronic current matrix element is its 4-momentum \( P^\alpha \). Introducing \( M^+ \) ’decay constant’ \( f_M \) and using Lorentz invariance one makes the usual ansatz for the current matrix element:

\[
< O | A^{ch\alpha\dagger}(x) | M^+ > = i f_M V_\alpha e^{-iP\cdot x} P^\alpha
\]

where \( V_\alpha \) is the relevant CKM matrix element for \( M^+ \rightarrow W^+ \).

The leptonic current matrix element for producing a state \( N_h \) of definite mass \( \mu \) and 4-momentum \( p_N \) together with a charged antilepton \( \beta^+ \) of flavour \( \beta \) mass \( m \) and 4-momentum \( p_l \) is:

\[
< \beta^+ N_h | \sum_{k,\delta} U_{\delta k}^* N_k \gamma_\alpha \mathcal{P} L_{\delta}(x) | O > = U_{\beta h}^* \bar{u}(N_h) \gamma_\alpha \mathcal{P} L v(\beta) e^{i(p_l + p_N)\cdot x}
\]

so that the transition matrix element for \( M^+ \rightarrow N_h \beta^+ \) will be:

\[
-i \sqrt{2} G_F f_M U_{\beta h}^* \bar{v}(N_h) \mathcal{P} (1 - \gamma^5) v(\beta)
\]

This result is obviously independent of the Dirac or Majorana nature of the \( N_h \) field.

As said in the introduction, we will give here a complete derivation. We only assume that the reader knows how to calculate traces of products of Dirac algebra matrices. Our way of calculating the HNL polarization vector and using it in the second decay is inspired by [2]

1. using \( P = p_N + p_l \) and the Dirac equations:

\[
\bar{u}_N = \mu \bar{u} \text{ and } \bar{p}_l v = -mv
\]

simplify the matrix element to: \(-i\kappa \bar{u}(\alpha - \gamma^5 \beta) v\)

where \( \kappa = \sqrt{2} G_F f_M U_{\beta h}^* \mathcal{P}, \alpha = \mu - m, \beta = \mu + m \)

2. multiply the m.e. by its complex conjugate:

\[\text{Further notice that only the axial part of the hadronic current can have a non zero matrix element between a pseudoscalar state and the hadronic vacuum.}\]

\[W^+ \text{ is an off-shell } W^+\]

\[\mathcal{P} \text{ stands for } P^* \gamma_\alpha \text{ (Feynman’s notation.)}\]
\[ \kappa^2 \bar{u}(\alpha - \gamma^5 \beta)v\bar{v}(\alpha + \gamma^5 \beta)u \]

\[ = \kappa^2 \text{Tr}(u\bar{u}(\alpha - \gamma^5 \beta)v\bar{v}(\alpha + \gamma^5 \beta)) \]

3. Sum over antilepton polarizations, which amounts to the replacement:

\[ v\bar{v} \rightarrow (p_l - m). \]

4. In order to calculate the HNL polarization, keep its full density matrix for both momentum and spin:

\[ u\bar{u} \rightarrow (\varphi_N + \mu)\frac{1}{2}(1 + \gamma^5 \gamma) \]

where \( s \) is the HNL polarization 4-vector which reduces, in the rest frame, to \((0, P)\) with \( P \) the usual polarization 3-vector for spin 1/2, i.e. twice the spin expectation value.

5. The squared m.e. thus becomes:

\[ \kappa^2 \text{Tr} (\varphi_N^2 + \mu^2)(\alpha - \gamma^5 \beta)(\alpha + \gamma^5 \beta) \]

6. Calculate the trace. Using again 4-momentum conservation, this yields:

\[ \frac{1}{4} \text{Tr} = M^2(m^2 + \mu^2) - (m^2 - \mu^2)^2 + 2\mu(m^2 - \mu^2)s \cdot p_l \quad (2) \]

\[ \bullet \text{ To calculate the rate, sum over HNL spin states by replacing } s \rightarrow 0 \text{ and multiplying by 2.} \]

Adding normalization and phase-space factors, one gets the width:

\[ \Gamma(M^+ \rightarrow \beta^+ N_h) = \frac{G^2 f_M^2 |V_{\beta h}|^2}{8\pi M^2} \left( m^2 + \mu^2 - \frac{(m^2 - \mu^2)^2}{M^2} \right)^{1/2} \lambda(M^2, m^2, \mu^2) \]

where \( M, m, \mu \) are the masses of \( M^+, \beta \) and \( N_h \) respectively and \( \lambda \) is the usual kinematical function

\[ \lambda(x, y, z) = x^2 + y^2 + z^2 - 2(xy + yz + zx) \]
By letting \( \mu \to 0 \) (and forgetting about \( U \)) one retrieves the ordinary well known formula for decay into an antilepton and a standard model massless neutrino.

\[
\Gamma = \frac{G_F^2 f^2 |V_{ef}|^2}{8\pi} M m^2 (1 - \frac{m^2}{M^2})^2 \tag{3}
\]

which, being proportional to \( m^2 \), explains the tiny ratio \( \Gamma(e^+\nu)/\Gamma(\mu^+\nu) \), due to helicity conservation by V and A vertices in the ultra-relativistic limit. For the domain envisioned here (\( \mu \geq \) a few MeV) the suppression no longer works and both modes acquire the same order of magnitude modulo the coefficients \( |U_{\beta h}| \).

- To find the HNL polarization:

The squared m.e. (cf. 2) is proportional to the probability of finding 4-polarization \( s \) and must therefore be equal to \( \text{Tr}(\rho \rho_f) \) (with \( \rho_f \) the true HNL polarization matrix) up to a factor.

In the rest frame, \( \rho \) reduces to \( 1 + \sigma \cdot P \) with \( \sigma \) the Pauli matrices and \( P \) the polarization 3-vector, so that the expression obtained is proportional to \( \text{Tr} (1 + \sigma \cdot P)(1 + \sigma \cdot P^f) \) or to \( 1 + P \cdot P^f \)

By expliciting the proportionality of this last expression with (2) written in the HNL rest frame, we find the following for the HNL polarization vector to be used when simulating its decay:

\[
P = \frac{(m^2 - \mu^2)\lambda^{1/2}(M^2, m^2, \mu^2)}{M^2(m^2 + \mu^2) - (m^2 - \mu^2)^2} \hat{n} = \mathbb{P} \hat{n} \tag{4}
\]

where \( \hat{n} \) is a unit vector in the direction of the parent meson or of the decay lepton in the \( N_h \) rest frame and the second equality defines \( \mathbb{P} \). Although the formula obtained by the authors of [1] is not given in their paper, it is readily seen graphically (compare with fig. 1) that it must coincide with our above result for the case where the initial particle is a charged kaon decaying into muon and HNL. In particular, it is seen from the graph (fig. 1) or formula (4), that if the HNL mass \( \mu \) coincides with the muon mass, its polarization vector is zero. The graph or formula (4) also show that when \( \mu \to 0 \), the coefficient of \( \hat{n} \to 1 \) that is, the massless neutrino will be pure \(-1\) helicity.
Figure 1: Polarization of HNL produced in $K^+ \rightarrow \text{HNL} + \mu^+$ as a function of HNL mass. It is seen that when the latter coincides with the muon mass, the polarization vanishes.

4 Decays of massive neutrinos

4.1 Two-body HNL decay into $0^-$ meson and lepton

These are crossed channels of those envisioned above for production. Amplitudes are trivial to write; in the 'charged' case, one finds e.g., for a decay into $\pi^+, l^-:$

(Here $k, q$ and $p$ are the 4-momenta of $N_h$ (mass $\mu$), $l^-$ (mass $m_l$ flavour $\gamma$) and $\pi^+$ (mass $m_{\pi}$) so that $k = p + q$)

$$A_{N_h \rightarrow \pi^+ l^-} = -i \frac{G_F}{\sqrt{2}} f_\pi V_{u,d} U_{h,\gamma} \bar{u}(l) \not{\!p}(1 - \gamma^5) u(N_h)$$

Using Dirac equation, one gets:

$$A_{N_h \rightarrow \pi^+ l^-} = -i \frac{G_F}{\sqrt{2}} f_\pi V_{u,d} U_{h,\gamma} \bar{u}(l)(\alpha + \beta \gamma^5) u(N_h)$$
with $\alpha = \mu - m_l$ and $\beta = \mu + m_l$

Squaring, summing over $l$ polarizations and introducing the HNL polarization matrix with polarization 4-vector $s$ one gets:

$$|A|^2 = \frac{G_F^2 f_\pi^2}{2} |U_{h,\gamma}|^2 |V_{u,d}|^2 \text{Tr}(q + m_l)(\alpha + \beta \gamma^5)(k + \mu)\frac{1}{2}(1 + \gamma^5 \sigma)(\alpha - \beta \gamma^5)$$

Calculating the trace, one gets

$$\frac{1}{4} \text{Tr} = (\alpha^2 + \beta^2)q \cdot k + 2\mu \alpha \beta q \cdot s + m_l \mu (\alpha^2 - \beta^2)$$

$$= 2(\mu^2 + m_l^2)q \cdot k + 2\mu(\mu^2 - m_l^2)q \cdot s - 4m_l^2 \mu^2$$

$$= (\mu^2 - m_l^2)^2 - m_l^2(\mu^2 + m_l^2) + 2\mu(2\mu^2 - m_l^2)q \cdot s$$

$$= (\mu^2 - m_l^2)^2 - m_\pi^2(\mu^2 + m_l^2) - 2\mu(2\mu^2 - m_l^2)q \cdot P$$

in the $N_h$ rest frame, therefore:

$$|A|^2 = G_F^2 f_\pi^2 |U_{h,\gamma}|^2 |V_{u,d}|^2 [(\mu^2 - m_l^2)^2 - m_\pi^2(\mu^2 + m_l^2) - 2\mu(2\mu^2 - m_l^2)q \cdot P] \quad (5)$$

With phase space (integrated over the angles) equal to $\frac{|q|^4}{4\pi \mu}$ or $\frac{\lambda_{1/2}(\mu^2, m_l^2, m_\pi^2)}{8\pi \mu^2}$ we find for the rate:

$$\Gamma(N_h \to l^- \pi^+) = \frac{G_F^2 f_\pi^2}{16\pi \mu^3} [(\mu^2 - m_l^2)^2 - m_\pi^2(\mu^2 + m_l^2)] \lambda_{1/2}(\mu^2, m_l^2, m_\pi^2) |U_{h,\gamma}|^2 |V_{u,d}|^2$$

If $\gamma = e$, $m_l$ can be neglected and this becomes:

$$\frac{G_F^2 f_\pi^2}{16\pi \mu^3} \mu^3 \left(1 - \frac{m_\pi^2}{\mu^2}\right)^2 |U_{h,e}|^2 |V_{u,d}|^2$$

The angular distribution is non isotropic due to polarization (cf. (4)) as shown by (5).

Normalizing formula (5) in such a way that the constant term be equal to $1/2$ (so that the integral over $\cos(\hat{n}, \hat{q})$ equals 1 ) we get:

\[\text{We have used } 2q \cdot k = \mu^2 + m_l^2 - m_\pi^2\]
\[
\frac{dN}{d\cos \theta} = 1/2 - 1/2 \frac{\mu^2 - m_l^2}{(\mu^2 - m_\pi^2)^2 - m_\pi^2 (\mu^2 + m_\pi^2)} \lambda^{1/2}(\mu^2, m_l^2, m_\pi^2) P \cos \theta
\]

\(\theta = (\hat{n}, \hat{q})\) is the angle between the recoil lepton direction (\(\hat{n}\)) in the first decay \(M^+ \rightarrow \text{HNL} + \beta^+\) and the secondary lepton direction (\(\hat{q}\)) from HNL decay seen in the HNL rest frame. \(P\) has been defined in (4). It is clear that, contrary to formula (16) of ref.[1], the HNL decay is isotropic when its mass is equal to the mass of the lepton recoiling against it in the first decay. Said formula is incoherent on different other grounds, making, for example, no distinction between the c.o.m. momenta in the HNL-generating meson two-body decay and the HNL two-body decay itself.

5 A pedagogical conclusion

It is interesting to note that the heavy neutrino which is in general only partially polarized is NOT a quantum mechanical linear superposition of helicity 1 and helicity -1 states contrary to what is stated in many places (see e.g. [3]). Since its polarization vector modulus is not one, there is no direction in which a spin measurement will yield 1/2 with certainty and this system, which is in a mixed state, cannot be represented by a wave function. Although the spin 0 initial meson can be thought of as being in a pure state, the HNL, being but a subsystem of the -evolved- initial state can only be represented by a density matrix (see e.g. [4]).

References

[1] Formaggio J.A. et al., Phys.Rev. D57, 7037 (1998)
[2] Berestetskii, Lifschitz and Pitaevskii, Relativistic Quantum Theory, §§ 29 and 66 (Pergamon Press, 1971)
[3] K. Nakamura and S.T. Petcov in Review of Particle Physics, 14. Neutrino masses, mixing and oscillations (Particle Data Group 2017)
[4] Landau and Lifschitz, Quantum Mechanics, § 12 (Pergamon Press 1962)