Luttinger model for current and spin-Hall conductivities

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Abstract. We evaluate, based on a dimensional analysis, the ratio of the spin-Hall conductivity to current conductivity in a sample where the electric current is dominated by the Luttinger model. The ratio is composed of a size-dependent factor multiplied by the corresponding ratio in the Drude model.

1. Introduction
The spin Hall effect (SHE), which was first predicted by D’yakonov and Perel [1] in doped semiconductors, has attracted much attention after a series of experiments in semiconductors [2–5] and in metals [6]. Not a few microscopic models have been proposed for extrinsic (doped) SHE (doped) [1, 7, 8] and for intrinsic (impurity free) SHE [9, 10]. Recently, the universal relation between the spin-Hall conductivity $\sigma_s$ and current conductivity $\sigma_c$ has been derived from extending the Drude model as $\sigma_s \propto \sigma_c^2$ (with the proportional constant being independent of the sample size), which is independent of the details of the scattering mechanism. The crucial point of this relation is that $\sigma_s \propto \tau^2$ while $\sigma_c \propto \tau$, where $\tau$ represents the mean free time.

However, the relation of $\sigma_s \propto \sigma_c^2$ holds for the sample size $L$ being sufficiently larger than the mean free path $\ell$, where the Drude model can be applied. When $L$ is comparable to $\ell$, the Drude model cannot be applied. If the electric current is dominated by the carriers near the Fermi surface, as in metals and doped semiconductors, the effective action can be described by a massless fermion. Instead of dealing with a 4D massless fermion, it may be practical to proceed to the following steps. First, we regard that the sample can be described by a 2D massless fermion (where the Luttinger model can be applied) in that the component of the electric current parallel to the applied electric field is large enough, compared to that perpendicular to it. Then, we evaluate the spin-Hall current by a dimensional analysis.

In this paper, we first reexamine the relation of $\sigma_s \propto \sigma_c^2$ for the Drude model by using a dimensional analysis. Then we derive from a similar analysis the relation between $\sigma_s$ and $\sigma_c$ for the Luttinger model.

2. Dimensional analysis
The nonrelativistic Dirac Hamiltonian $\mathcal{H}$ (with mass $m$, charge $e$, and spin $\frac{\hbar}{2}\sigma$) under an electric potential $U(r)$ is given by

$$\mathcal{H} = \frac{1}{2m} \left( \mathbf{p} - \frac{e}{c} \mathbf{A} \right)^2 + U(r) \quad \text{(with } \mathbf{A} = \frac{\lambda_\sigma}{2} \mathbf{\sigma} \times \mathbf{E})$$
where $\lambda_c = h/(mc)$ and $E = -\nabla U(r)$, representing the electron Compton length and the electric field, respectively. From the Heisenberg equation of motion: $i\hbar \dot{\mathcal{O}} = \{\mathcal{O}, \mathcal{H}\}$ (for $\mathcal{O} = \mathbf{r}, \mathbf{p}$) with $[r_i, p_j] = i\hbar \delta_{ij}$, we obtain up to order $e^{-2}$ the following equation for $\mathbf{r}$ by eliminating $\mathbf{p}$ and $\dot{\mathbf{p}}$:

$$m\ddot{\mathbf{r}} = \mathbf{F}_E + \mathbf{F}_L \quad \text{[with \ } \mathbf{F}_E = e\mathbf{E}, \ \mathbf{F}_L = (e/c)\dot{\mathbf{r}} \times (\nabla \times \mathbf{A})\text{]}.$$  

(1)

For a cubic sample, the current conductivity $\sigma_c$ and the spin-Hall conductivity $\sigma_s$ are given by

$$\mathbf{j} = \sigma_c \mathbf{E} + \sigma_s (\mathbf{\sigma} \times \mathbf{E}) + O(\mathbf{E}^2),$$

(2)

where $\mathbf{j} (= \mathbf{en}\dot{\mathbf{r}})$ represents the electric current density. Comparing Eqs. (1) and (2), we find that $\sigma_c$ and $\sigma_s$ are attributed to the Coulomb force $\mathbf{F}_E$ and the Lorentz force $\mathbf{F}_L$, respectively.

### 2.1. Drude model

Due to the impurity scattering and/or thermal fluctuation, the drift velocity tends to be saturated for sufficiently large $t$. Phenomenologically, the saturation of the drift velocity can be realized by introducing a drag force $-(m/\tau)\dot{\mathbf{r}}$ into the right-hand side of Eq. (1), where $\tau$ represents the mean free time. For a cubic sample, the conductivity $\sigma_c$ and $\sigma_s$ are related to $[11]

$$\sigma_s/\sigma_c \approx \gamma \cdot \sigma_c \tau_e \quad \text{(with } \gamma = Zn_0/n),$$

(3)

where $\tau_e = \lambda_e/c$, representing the electron Compton time; $Z, n_0$, and $n$ are the valence of the ion, the ion concentration, and the carrier concentration, respectively.

Although the relation of Eq. (3) can be legitimately derived from Eqs. (1) and (2), here we estimate the ratio of $\sigma_s/\sigma_c$ by a dimensional analysis. First of all, $\sigma_s$ may depend on all the parameters available, together with the functional form of $U(r)$, so that we can write $\sigma_s$ as $\sigma_s = f(h, m, c, e, n; \tau)[U]$, where $f$ is a certain function to be determined. Based on the Drude model, the mean free time $\tau$ is proportional to $\sigma_c$ as $\sigma_c = e^2n\tau/m$, so that it may be convenient to eliminate $\tau$ in $f$ by using $\sigma_s$ as $\sigma_s = f_1(h, m, c, e, n; \sigma_c)[U]$, where $f_1$ takes the same value of $f$ but has a different functional form. Recall that $\sigma_s$ is attributed to the Lorentz force $\mathbf{F}_L$, so that the $U$-dependence of $\sigma_s$ can be obtained through the relation of $\sigma_s \propto \nabla \times \mathbf{A} \propto \nabla^2 U(r) \propto Zn_0$, where use has been made of the Gauss’ law: $\nabla \cdot \mathbf{E} \propto Zn_0$. Thus $\sigma_s/\sigma_c$ can be rewritten as

$$\sigma_s/\sigma_c = \gamma \cdot \tilde{f}(\lambda_e, \tau_e, m, \alpha; \sigma_c), \quad \tilde{f}(\lambda_e, ..., n; \sigma_c) = (n/\sigma_c)f_1(h, ..., n; \sigma_c),$$

where we have changed the set of parameters $(h, m, c, e) \rightarrow (\lambda_e, \tau_e, m, \alpha)$, with $\alpha \equiv e^2/(hc) \approx 1/137$. Since $\tilde{f}$ is dimensionless, $f$ should be a function of a dimensionless parameters, so that $\tilde{f}$ can be rewritten as $f \rightarrow F(\tau_e, \sigma_c, n\lambda_e^3, \alpha)$, where use has been made of $[\sigma_c] = T^{-1}$ (in Gauss units) and $[n] = L^{-3}$.

The next thing is evaluate the $(m, n, h)$-dependence of $\sigma_s/\sigma_c$. Recall that $\sigma_s/\sigma_c \propto F_L/F_C$. Assuming that $\tau$ does not depend on $m, n, h$ explicitly, we find from $\dot{\mathbf{r}} = (eE/m)\tau$ (by $e\mathbf{E} = (m/\tau)\dot{\mathbf{r}}$) that $F_L \propto \dot{\tau}_c \propto m^{-2}n^0h^0$ (while $F_C \propto m^0n^0h^0$), so that $\sigma_s/\sigma_c \propto m^{-2}n^0h^0$ (the $n$-independence of $\sigma_s/\sigma_c$ is also found in the microscopic model [12, 13]), meaning that $F \propto m^{-2}n^0h$. This, together with $\tau_e, \sigma_c \propto m^{-2}n^0h^0$, $n\lambda_e^3 \propto m^{-3}n^1h^0$, and $\alpha \propto m^0n^0h^{-1}$ (they are linearly independent with respect to $m, n, h$), implies that $F \propto \tau_e/\sigma_c$. If the proportional constant is of order unity, we finally obtain the relation of Eq. (3).

### 2.2. Luttinger model

Under an electromagnetic field $A^\mu (\mu = 0, 1)$, the effective Lagrangian for a 2D massless fermion is given (after bosonization) by the Luttinger model as [14]

$$\mathcal{L} = \frac{\hbar}{4K} \left[ (\partial_{\mu} \phi)^2 - (\partial_{x} \phi)^2 \right] - e \sum_{\mu, \nu} e^{\mu \nu} A_\mu \partial_\nu \phi,$$
where \( v \) and \( K \) \([= K(x)]\) represent the carrier velocity and the interaction-dependent parameter, respectively; the Levi-Civita tensor \( \epsilon^{\mu \nu} \) is chosen as \( \epsilon^{10} = 1 \) [with the metric \((+,−,−,−)\)], and the scalar field \( \phi \) is normalized as such that the current \( j^\mu \) \([= (\rho, j)]\) is given by \( j^\mu = \epsilon^{\mu \nu} \partial_\nu \phi \). Then in the stationary current regime \( \partial_t j = 0 \), the equation for \( \phi \) indicates that

\[
\partial_x (v \rho / K) = 2eE / h, \tag{4}
\]

where \( E \) \( (= −\partial_x A_0 + \partial_t A_1) \) represents the electric field. Suppose that the sample is attached to metal contacts at \( x = 0, L \) (with their length \( \Delta L \)). Under the boundary condition that the carrier propagates rightward for \( x > L + \Delta L \) or leftward for \( x < −\Delta L \), the electric current \( I \) (per unit spin orientation) can be written as \([15]\)

\[
I = \frac{e^2 K_L}{h} \int_\Omega E(x) \, dx, \tag{5}
\]

where \( K_L = K(x = L + \Delta L) = K(x = −\Delta L) \) and \( \Omega = [−\Delta L, L + \Delta L] \). The relation of Eq. (5), which is the same form of the Laudauer formula (with \( K_L \) being identified with the transition probability \( T \) for the carrier to penetrate from the left to right reservoirs), can be applied to a purely one-dimensional sample, where there is only one mode for the electronic state in the \( x \)-axis direction. As the section \( S \) of the sample increases, the number of the model tends to increase. For \( N \) modes, the electric current may be simply given by multiplying \( I \) in Eq. (5) by \( N \). In this case, the current conductivity \( \sigma_c \) can be written as \( \sigma_c = (e^2 K_L / h)(L/S)N \), where we have replaced \( E \) by \( \frac{1}{L} \int_\Omega E(x) \, dx \) for the condition of \( \Delta L \ll L \).

Now we estimate the ratio of \( \sigma_s / \sigma_c \) by a dimensional analysis. To begin with, \( \sigma_s \) depends on all the parameters available, together with the functional form of \( U(r) \), so that \( \sigma_s \) may be written as \( \sigma_s = g(h, m, c, e; n; L, S)[U] \), where \( n = N/(L S) \), and we have neglected the \( K_L \)-dependence of \( \sigma_s \) (because \( K_L \), which can be interpreted as the transition probability \( T \), can be chosen as unity in an ideal case.). The \( U \)-dependence of \( \sigma_s \) and the null \( n \)-dependence of \( \sigma_s / \sigma_c \) is the same as in the Drude model, so that \( \sigma_s / \sigma_c = \gamma N \cdot \tilde{g}(\lambda_e, \tau_e, m, \alpha; L, S) \), where we have again changed the set of parameters as \( (h, m, c, e) \longrightarrow (\lambda_e, \tau_e, m, \alpha) \). Since \( \tilde{g} \) is a dimensionless quantity, it should be a function of dimensionless variables, so that \( \tilde{g} \) can be rewritten as \( \tilde{g} \longrightarrow G(\lambda_e / L, \lambda_e^2 / S, \alpha) \).

The next thing is to take account of the \((m, h)\)-dependence of \( G \) (or \( \sigma_s / \sigma_c \)). Different from the \((m, h)\)-dependence of the drift velocity \( v \) in the Drude model as \( \dot{r} \propto m^{-1} h^0 \), the \((m, h)\)-dependence of the carrier velocity \( v \) in the Luttinger model is found from Eq. (4) (and \( \partial_t E \propto \rho \)) to be given by \( v \propto m^0 h^{-1} \) (for doped semiconductors, the proportional constant is explicitly given by Ref. [16]). Thus we obtain \( \sigma_s / \sigma_c \propto F_L / F_C \propto v \lambda_e \propto m^{-1} h^0 \). This, along with \( \lambda_e / L \propto m^{-1} h, \lambda_e^2 / S \propto m^{-2} h^2 \), and \( \alpha \propto m^0 h^{-1} \), indicates that \( G \) can be rewritten as \( G \propto \alpha(\lambda_e / L)^{\nu}(\lambda_e^2 / S)^{\delta} \) with \( \nu + \delta = 1 \), so that we obtain

\[
\sigma_s / \sigma_c \propto Z n_0 \alpha \lambda_e L^2 S^{1-\delta} \sim \gamma(\tau_e \lambda_e)(S/L^2)^{1-\delta}, \tag{6}
\]

where the second relation holds for \( K_L \approx 1 \).

The third thing is to estimate the proportional constant in Eq. (6). Recall that Eq. (3) can be applied to a cubic sample with its side \( r \gg \ell \), where \( \ell \) represents the mean free path. As we diminish the side \( r \) down to on the order of \( \ell \), it is expected that the relation of Eq. (3) may still hold in a rough way. On the other hand, the Luttinger model can be applied to the regime of \( L \sim \ell \). In order that the sample can be compared to the Drude model, the sample should be chosen as cubic \((S = L^2)\). From the (nealy) equality of \( \sigma_s / \sigma_c \) between the Drude and Luttinger models for \( r \sim \ell \sim L \) \((= \sqrt{S})\), it is found that the proportional constant in Eq. (6) should be of order unity.
Finally, we obtain from the $L$-dependence of $\sigma_s$ the restriction on $\delta$ as $\delta > -1/2$. Consider the scaling of $L \rightarrow \lambda L$ with $E$ fixed. Then it follows from Eq. (5) that $\sigma_c \rightarrow \lambda \sigma_c$. This is due to the enhancement of the charge density $\rho$ in the $x$-axis direction, which is also found from Eq. (4). When the spin-Hall current is taken into account, part of the electric current propagates in the direction perpendicular to the $x$-axis direction. This implies that if the charge density in the $x$-axis direction is multiplied by $\lambda$, then the electric current in the direction perpendicular to the $x$-axis must increase (or decrease) for $\lambda > 1$ (or $\lambda < 1$), so that the scaling of $\sigma_s$ may be written as in the form $\sigma_s \rightarrow \lambda^?\sigma_s$ ($\beta > 0$) under $L \rightarrow \lambda L$. On the other hand, the $L$-dependence of $\sigma_s$ is given by $\sigma_s \propto L^{1+2\delta}$ [because $\sigma_s/\sigma_c \propto L^{2\delta}$ by Eq. (6), with $\sigma_c \propto L$ by Eq. (5) (recall that $E$, in this case, is fixed)], so that $\beta = 1 + 2\delta$. Hence the condition of $\beta > 0$ leads to $1 + 2\delta > 0$.

To summarize, we obtain for $K_L \simeq 1$

$$\frac{(\sigma_s/\sigma_c)_{\text{Luttin}}}{(\sigma_s/\sigma_c)_{\text{Drude}}} \approx \left(\frac{S}{L^2}\right)^{1-\delta} \quad (\text{with } \delta > -1/2),$$

(7)

where $(\sigma_s/\sigma_c)_{\text{Drude}}$ represents the right-hand side of Eq. (3). Rough estimate of $\delta$ is as follows. By comparing the scaling of $\sigma_c \rightarrow \lambda \sigma_c$ and $\sigma_s \rightarrow \lambda^?\sigma_s$ under $L \rightarrow \lambda L$, it seems reasonable to expect that $\beta = O(1)$, so that $-\frac{1}{2} < \delta = \frac{1}{2}(\beta - 1) \lesssim O(1)$.

3. Summary

We have evaluated, based on a dimensional analysis, the ratio of $\sigma_s/\sigma_c$ in a sample where the electric current is dominated by the Luttinger model. The result is given by Eq. (7). The crucial point of the simple introduction of the size-dependent factor of $(S/L^2)^{1-\delta}$ is due to the $(m, h)$-dependence of the carrier (drift) velocity $v$ as $v \propto m^0 h^{-1}$. Suppose that $v \propto m^{-1} h^0$, as in the Drude model, the left-hand side of Eq. (7) cannot be given by a function of $S/L^2$ only; another length scale, such as $\lambda_c$, is introduced. There still remains arbitrariness of the value of $\delta$; it cannot be determined by a dimensional analysis. To obtain the precise value of $\delta$, we should start with a model Lagrangian for a 4D massless fermion. Different from a 2D fermion, the bosonization of a 4D fermion does not hold in a strict sense of the word. However, the bosonization technique, in itself, can be applied (under certain condition) to interacting fermions in arbitrary dimensions (see, for example, [17]). This technique may resolve the arbitrariness of the value of $\delta$.

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