Quantum braided Poincaré group

Jakub Rembieliński
Department of Theoretical Physics
University of Łódź
ul. Pomorska 149/153
90–236 Łódź, Poland

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Abstract
A new deformation of the of the Poincaré group and of the Minkowski space-time is given. From the mathematical point of view this deformation is rather quantum-braided group. Global and local structure of this quantum-braided Poincaré group is investigated. A kind of “quantum metrics” is introduced in the $q$-Minkowski space.

1 Introduction

The idea of quantization of space-time and of the Poincaré group by means of quantum group theory methods was examined recently in a number of papers [1, 2, 3, 4, 5, 6, 7, 8]. In particular two-dimensional deformed space-time groups was considered in [1, 2, 3], while the four-dimensional ones by Podleś & Woronowicz [4], Lukierski, Nowicki, Ruegg & Tolstoy [5, 6], Schmidke, Wess & Zumino [7].

In this paper we introduce a new quantization of the four-dimensional Minkowski space-time and Poincaré group. This deformation has many nice properties, like isotropy, existence of a kind of “quantum geometry” and relatively clear physical interpretation. From the mathematical point of view the resulting group is rather a quantum-braided group—a hybryde of the both quantum and braided groups.

Quantum braided group structure was initiated in [9, 10].

The main assumption we made is the following:

**Assumption** The $q$-space-time and $q$-Poincaré algebras are isotropic, i.e. the rotation group is an automorphism group of the above algebras. Strictly speaking we assume that there are no distinguished direction in the space sector of the space-time.

It seems that this assumption is very strongly supported by experimental data. Moreover, as was shown by Bacry and Levy-Leblond [11], rotational symmetry very strongly restricts possible space-time groups. Notice that isotropy condition is not fulfilled in the approach presented in the [7].

Under the above assumption the space-time generators $x^\mu$ satisfy

\[
\begin{align*}
  x^i x^j &= g(x^0, x^2) x^j x^i, \\
  x^0 x &= x f(x^0, x^2).
\end{align*}
\] (1)
Taking into account the permutational symmetry we immediately obtain that the function \( g(x^0, x^2) = 1 \).

In the following we restrict ourselves to an inhomogenous Manin’s reordering rules for every space-time plane, i.e. we accept the Ansatz

\[
    f(x^0, x^2) = ax^0 + b. \tag{2}
\]

Now, by means of the hermicity of coordinates

\[
    x^\mu \ast = x^\mu, \quad (\ast \text{ denotes an antilinear anti-involution in our algebra, we deduce that eq. } (1) \text{ takes the form}) \tag{3}
\]

\[
    \begin{align*}
        x^i x^j & = x^j x^i, \\
        x^0 x & = qx^0 x + i\kappa q^2 x
    \end{align*} \tag{4}
\]

with \(|q| = 1, \kappa = \kappa_0\).

Notice that the variant with \( q = 1 \) is a dual form of the Lukierski-Nowicki-Ruegg \( \kappa \) deformation \([5, 6, 12]\). In this paper we consider complementary case of an arbitrary phase \( q, |q| = 1 \) and \( \kappa = 0 \). Our choice follows naturally from the requirement of the invariance under geometrical time reflection; namely \([4]\) is invariant under \( x^0 \rightarrow -x^0, x \rightarrow x \) only for \( \kappa = 0 \). (We know that \( T \)-invariance is broken only for a restricted class of interactions, so it should be connected with dynamics rather than with the space-time algebra).

Summarizing, our reordering rules for space-time generators have the form

\[
    \begin{align*}
        x^i x^j & = x^j x^i, \\
        x^0 x & = qx^0 x
    \end{align*} \tag{5}
\]

with \(|q| = 1\).

## 2 Quantum-braided inhomogenous linear group

\((QBIGL(4R))\)

Now, the (braided) co-module action of the real inhomogenous linear quantum group is defined as

\[
    \left( \begin{array}{c}
        x^0' \\
        x'
    \end{array} \right) = \left( \begin{array}{cc}
        \Lambda & a^0 \\
        0 & 1
    \end{array} \right) \left( \begin{array}{c}
        x^0 \\
        x
    \end{array} \right) \tag{6}
\]

with

\[
    \Lambda = [A^\mu, a] = \begin{pmatrix}
        \beta & q^{\frac{i}{2}} w \\
        q^{\frac{i}{2}} w^\dagger & B
    \end{pmatrix}. \tag{7}
\]

Here "\( \dagger \)" denotes matrix transposition accompanied with \( * \) conjugation and \( B = [B_{ij}] \). The generators \( A^\mu \) and \( a^\mu \) are real, i.e.

\[
    (A^\mu)^* = A^\mu, \quad (a^\mu)^* = a^\mu. \tag{8}
\]

As was announced we admit a braiding in our quantum group action. This means that \( x^i \)'s do not commute with the group generators \( A^\mu, a^\mu \) in general. Moreover generators of the next group action \( A'^\mu \) and \( a'^\mu \) do not commute with \( A^\mu \) and \( a^\mu \) in general. We will assume the group composition (co-product) in the form

\[
    \left( \begin{array}{cc}
        A'' & a'' \\
        0 & 1
    \end{array} \right) = \left( \begin{array}{cc}
        A' & a' \\
        0 & 1
    \end{array} \right) \left( \begin{array}{cc}
        \Lambda & a \\
        0 & 1
    \end{array} \right). \tag{9}
\]
In the formulas (10) and (11) we do not use the tensor product notation because of braiding and explicit distinguishing between $\Lambda, a$ and $\Lambda', a'$. Now, because in principle we can apply firstly $\Lambda', a'$ and next $\Lambda, a$ in the eq. (10) (with a different result $(\Lambda', a', a')$) then $(\Lambda'_{\nu}, a'_{\mu})$ should satisfy the same reordering rules with $x^\mu$ as $(\Lambda'_{\nu}, a'_{\mu})$. Moreover the reordering rules between generators $(\Lambda'_{\nu}, a'_{\mu})$ and $(\Lambda'_{\nu}, a'_{\mu})$ should be symmetric under intertwining of primed and unprimed generators. Taking into account the above remarks and by means of the isotropy condition we obtain (under the generalized Bethe Ansatz assumption) the following algebra for generators

$$A^\mu_{\nu}A^\sigma_{\lambda} = A^\sigma_{\lambda}A^\mu_{\nu},$$

(10)

$$
\begin{align*}
a^0_0 \beta &= \beta a^0, & a^\beta &= \beta a, \\
a^0_0 B^i_j &= B^i_j a^0, & a B^i_j &= B^i_j a, \\
a^0_0 w &= qw a^0, & a w^i &= q w^i a, \\
a^0_0 u &= q^{-1} u a^0, & a u^i &= q^{-1} u^i a,
\end{align*}
$$

(11)

$$a^0_0 a = q a a^0, \quad a^i a^j = a^j a^i,$$

(12)

The co-module braiding rules takes the form

$$
\begin{align*}
x^0_0 \beta &= \beta x^0, & x \beta &= \beta x, \\
x^0_0 B^i_j &= B^i_j x^0, & x B^i_j &= B^i_j x, \\
x^0_0 w &= qw x^0, & x w^i &= qw^i x, \\
x^0_0 u &= q^{-1} u x^0, & x u^i &= q^{-1} u^i x, \\
x^0_0 a &= a^0 x^0, & x a^0 &= q^{-1} a^0 x, \\
x^0_0 a &= q a x^0, & x a^i &= a^i x.
\end{align*}
$$

(13)

The same relations hold for $A^\mu_{\nu}$ and $a^\mu$. The braiding rules reads

$$A^\mu_{\nu}A^\sigma_{\lambda} = A^\sigma_{\lambda}A^\mu_{\nu},$$

(14)

$$
\begin{align*}
a^0_0 \beta &= \beta a^0, & a^\beta &= \beta a, \\
a^0_0 B^i_j &= B^i_j a^0, & a B^i_j &= B^i_j a, \\
a^0_0 w &= qw a^0, & a w^i &= q w^i a, \\
a^0_0 u &= q^{-1} u a^0, & a u^i &= q^{-1} u^i a,
\end{align*}
$$

(15)

$$
\begin{align*}
a^0_0 \beta &= \beta a^0, & a^\beta &= \beta a, \\
a^0_0 B^i_j &= B^i_j a^0, & a B^i_j &= B^i_j a, \\
a^0_0 w &= qw a^0, & a w^i &= q w^i a, \\
a^0_0 u &= q^{-1} u a^0, & a u^i &= q^{-1} u^i a,
\end{align*}
$$

(16)

$$
\begin{align*}
a^0_0 a &= a^0 a^0, & a a^0 &= q^{-1} a^0 a, \\
a^0_0 a &= q a a^0, & a^i a^j &= a^j a^i,
\end{align*}
$$

(17)
Co-unity is defined standardly as

\[
\begin{align*}
\epsilon(\Lambda) &= \epsilon(\Lambda') = I, \\
\epsilon(a^\mu) &= \epsilon(a'^\mu) = 0,
\end{align*}
\]

(18)

where \( I \) is the \( 4 \times 4 \) unit matrix. The antipode has the form

\[
\begin{pmatrix}
\Lambda^{-1} & -\Lambda^{-1}a \\
0 & 1
\end{pmatrix}
\]

(19)

where \( \Lambda^{-1} \) is understood in the usual sense as a matrix inverse because \( \{A^\mu_\nu\} \) is a commutative subalgebra of our quantum braided group (see eqs. (10), (14)). In particular \( \text{det} \Lambda \) is of the standard form and belongs to the center of the above algebra.

3 Quantum Minkowski space-time and the quantum braided Poincaré group

To select in \( QBIGL(4R) \) an appropriate Poincaré subgroup it is necessary to define a substitute of geometry in our space-time algebra. To do this let us firstly introduce a covariant differential calculus in that algebra. Taking into account our isotropy assumption, the algebra (3) and the group action (6), we obtain, with help of the classification [13] of the differential calculi, the following covariant reordering rules for differentials

\[
\begin{align*}
\dot{x}^0 \, dx^0 &= dx^0 \, \dot{x}^0, \\
x^i \, dx^k &= dx^k \, x^i, \\
x^0 \, dx &= q \, dx \, x^0, \\
x \, dx^0 &= q^{-1} \, dx^0 \, x, \\
(dx^0)^2 &= 0, \\
dx^i \, dx^j &= -dx^j \, dx^i, \\
dx^0 \, dx &= -q \, dx \, dx^0.
\end{align*}
\]

(20)

Now, a trajectory in the bundle of the space-time algebras is given by a parametric dependence of \( x^\mu \) on an affine parameter, say \( \tau \), i.e.

\[
x^\mu = x^\mu(\tau),
\]

(22)

so

\[
dx^\mu = \dot{x}^\mu(\tau) \, d\tau,
\]

(23)

where the “four-velocity” \( \dot{x}^\mu \) satisfy, according to the eq. (21) and the covariance condition, the following first order differential calculus rules

\[
\begin{align*}
\dot{x}^0 \dot{x}^0 &= \dot{x}^0 \dot{x}^0, \\
\dot{x}^i \dot{x}^k &= \dot{x}^k \dot{x}^i, \\
\dot{x}^0 \dot{x}^k &= q \dot{x}^k \dot{x}^0, \\
\dot{x}^k \dot{x}^0 &= q^{-1} \dot{x}^k \dot{x}^0, \\
\dot{x}^i \dot{x}^j &= \dot{x}^j \dot{x}^i, \\
\dot{x}^0 \dot{x}^i &= q \dot{x}^i \dot{x}^0.
\end{align*}
\]

(24)
Now, we try to find a substitute of the relativistic line element

\[ ds^2 = (\dot{x}^0 - \dot{x}^2) \, d\tau^2. \]  

(25)

We see that \( ds^2 \) in the above form does not belong to the center of our algebra. Therefore we have difficulties with interpretation of \( ds^2 \) as a line parameter and consequently with a formulation of the action principle, definition of geodesics, free motion, etc. To omit this difficulty let us introduce a “quantum metric” \( g_{\mu\nu} \) satisfying reality and isotropy condition; namely we define

\[ ds_q^2 = \dot{x}^\mu g_{\mu\nu} \dot{x}^\nu \, d\tau^2 = (\dot{x}^1 g \dot{x}) \, d\tau^2, \]

(26)

with covariant, “quantum flat” metric tensor

\[ g = [g_{\mu\nu}] = \begin{pmatrix} \gamma & 0 & 0 & 0 \\ 0 & -\Gamma & 0 & 0 \\ 0 & 0 & -\Gamma & 0 \\ 0 & 0 & 0 & -\Gamma \end{pmatrix} \]  

(27)

where the new generators are hermitean i.e. \( \gamma^* = \gamma, \Gamma^* = \Gamma \) and \( \dot{\gamma} = \dot{\Gamma} = 0 \). The square of the line element \( ds_q^2 \) belongs to the center if the space-time algebra is completed by

\[ \begin{align*}
\gamma \Gamma &= q^4 \Gamma \gamma, \\
x^0 \gamma &= \gamma x^0, \\
x^0 \Gamma &= q^{-2} \Gamma, \\
x\gamma &= q^2 \gamma x, \\
x\Gamma &= \Gamma x.
\end{align*} \]

(28)

The algebra generated by \( x^\mu \) and equipped with the quantum metrics \( ds_q^2 \) will be called \textit{quantum Minkowski space-time}. Now, we are able to select in \( QBIGL(4\mathbb{R}) \) group the quantum braided Poincaré group \( (QB\mathbb{P}) \). It is defined in analogy to the usual Poincaré group as an invariance group of the line element \( (26) \). Consequently the generators \( \Lambda^\mu_{\nu} \) should satisfy the matrix equation

\[ \Lambda^\dagger g \Lambda = g \]

(29)

with \( g \) defined in the eqs. (27, 28). By universality, \( \Lambda' \) satisfy also eq. (29) so \( \Lambda' \Lambda \) belongs to the same category.

It is necessary to complete the algebra \( (28) \) by reordering rules consistent with the eq. (23); we will use the parametrization \( (7) \) of \( \Lambda \):

\[ \begin{align*}
\beta \gamma &= \gamma \beta, \\
\beta \Gamma &= \Gamma \beta, \\
B^i_{\ j} \gamma &= \gamma B^i_{\ j}, \\
B^i_{\ j} \Gamma &= \Gamma B^i_{\ j}, \\
B^i_{\ j} \gamma &= q^{-2} \gamma u, \\
B^i_{\ j} \Gamma &= q^{-2} \Gamma u, \\
\beta \gamma &= q \gamma a, \\
\beta \Gamma &= \Gamma a.
\end{align*} \]

(30)

The same relations hold for primed generators \( \Lambda'^\mu_{\nu} \).

Now, we can solve the constraints \( (23) \) by means of the “polar decomposition” of \( \Lambda \):

\[ \Lambda = \begin{pmatrix} \beta & q^{-\frac{1}{2}} G w \\ q^{\frac{1}{2}} w & I + \frac{1}{\beta + 1} w \times w \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix} = L(w)R. \]

(31)
Here
\[
R^i_k = R^i_k, \quad R^T R = RR^T = I
\]
and \(R^i_k\)'s belong to the center of the whole algebra. Thus \(R^i_k\) generate the \(O(3)\) group.

The new hermitean generator \(G\) in the eq. (31) is defined by
\[
G = q^2 \gamma^{-1} \Gamma
\]
so it satisfy
\[
A^\nu_{\mu} G = GA^\nu_{\mu}, \quad a^\mu G = q^{-2} G a^\mu, \quad x^\mu G = q^{-2} G x^\mu.
\]
Furthermore \(\beta\) and \(w\) fulfill the constraint
\[
\beta^2 - Gw^2 = 1.
\]

Notice that appearance of \(G\) in the above relations reflects the fact that the generators \(A^\nu_{\mu}\) have mixed (co- and contra-variant) tensor nature and the “metric tensor” is given by (27). Notice also that inverse of the boost \(L^{-1}(w) = L(-w)\) as in the classical case.

Summarising, QBP group contains translation \(a^\mu\) and \(q\)-Minkowski rotations \(A^\nu_{\mu}\) satisfying (29) (or (35) in the parametrization (31)) and the corresponding multiplication and braiding rules hold (eqs. (10–17)).

It is interesting to note that contrary to the scheme proposed in [5, 6, 7] the QBP group has Lorentz group and translations group as subgroups.

### 4 Local structure of QBP group

An important feature of QBIGL group is that the diagonal generators \(\beta\) and \(B^{ij}\) belong to the center of the whole group and the space-time algebra, while the off-diagonal generators \(w, u, a^\mu\) satisfy homogenous reordering rules. This enables us to apply infinitesimal-like methods in that case by replacing \(\beta \rightarrow 1 + \delta \beta, \quad B^{ij} \rightarrow \delta^{ij} + \delta B^{ij}, \quad w \rightarrow \delta w, \quad u \rightarrow \delta u, \quad a^\mu \rightarrow \delta a^\mu\) and under the same algebraic rules. This is equivalent with use of the corresponding Cartan-Maurer forms and appropriate definitions of commutators according to the reordering rules for Cartan-Maurer forms. In the case of QBP group both methods gives us the following algebra for “infinitesimal generators” of rotations \((J_i)\), boosts \((K_i)\) and translations \((P_{\mu})\):

\[
\begin{align*}
[J_i, J_j] &= i \varepsilon_{ijk} J_k, \\
[K_i, K_j] &= -i G \varepsilon_{ijk} J_k, \\
[J_i, k_j] &= i \varepsilon_{ijk} K_j, \\
[J_i, P_j] &= i \varepsilon_{ijk} P_k, \\
[J_i, P_0] &= 0, \\
[K_i, P_j] &= 0, \\
[J_i, P_0] &= 0, \\
[P_0, K_i] &= -i q^2 P_i, \\
[P_i, K_j] &= -i q^2 G \delta_{ij} P_0,
\end{align*}
\]

where \([A, B]_q = AB - qBA, \quad [A, B] = [A, B]_1\). The detailed identification of Lie generators and the algebra of Cartan-Maurer forms is given in the Appendix.
Now, in a unitary realization of $QBP$, $J$, $K$ and $P_\mu$ are hermitean and by consistency
\begin{align*}
J\gamma &= \gamma J, & K\gamma &= q^{-2}\gamma K, \\
J\Gamma &= \Gamma J, & K\Gamma &= q^{-2}\Gamma K, \\
P_0\gamma &= \gamma P_0, & P_k\gamma &= q^{-2}\gamma P_k, \\
P_0\Gamma &= q^2\Gamma P_0, & P_k\Gamma &= \Gamma P_k,
\end{align*}
(37)
as well as
\begin{itemize}
\item $J$ and $K$ commute with $R^i_j$, $w$, $\beta$ and $G$, i.e. $[J_k, R^i_j] = [J_k, w] = [J_k, \beta] = [J_k, G] = 0$, $[K_k, R^i_j] = [K_k, w] = [K_k, \beta] = [K_k, G] = 0$;
\item $J$ commute with $a^\mu$: $[J_k, a^\mu] = 0$;
\item $K$ $q$-commute with $a^\mu$, i.e. $[K_i, a^\mu]_q = 0$;
\item $P_\mu$ commute with $R^i_j$, $\beta$.
\end{itemize}
and
\begin{align*}
[w, P_\mu]_q &= 0, & [P_\mu, R^i_j] &= [P_\mu, \beta] = 0, \\
a^0P_0 &= P_0a^0, & aP_0 &= qP_0a, \\
a^kP_i &= P_ia^k, & a^0P_i &= q^{-1}P_ia^0.
\end{align*}
(38)
Quadratic Casimir $P^2 = P_\mu g_{\mu\nu}P_\nu = P_\mu P^\mu$.

It is easy to construct a realization of the algebra (36) in the space of functions over the space-time algebra:
\begin{align*}
J_k &= -i\varepsilon_{kij}(x^i\partial_j - x^j\partial_i), \\
K_i &= -i(q^2Gx^i\partial_0 + q^2x^0\partial_i), \\
P_\mu &= -i\partial_\mu.
\end{align*}
(39)
Here $\partial_\mu$ are the corresponding Jackson derivatives defined via $df = dx^\mu \partial_\mu f$.

5 Kinematics

Let us define the (contravariant) four-momentum $p^\mu$ of a free particle by the standard formula
\begin{equation}
p^\mu = m\dot{x}^\mu.
\end{equation}
(40)
Now, a consistency of the above definition with the geometric interpretation of $p_\mu = g_{\mu\nu}p^\nu$ as translation generators (see eq. (36)) demands commutativity of the inertial mass $m$, i.e. $m$ belongs to the center of the space-time (and group) algebra. Moreover, $m$ is hermitean (i.e. $m^* = m$). By means of the egs. (24) and (28) we obtain
\begin{align*}
p^0p^0 &= qpp^0, & x^0p^0 &= qpx^0, & x^0p^0 &= p^0x^0, \\
p^ip^i &= p^ip^i, & x^ip^i &= p^ix^i, & x^0p^0 &= qpx^0
\end{align*}
(41)
and
\begin{align*}
p^0\gamma &= \gamma p^0, & p\gamma &= q^2\gamma p, \\
p^0\Gamma &= q^{-2}\Gamma p^0, & p\Gamma &= \Gamma p.
\end{align*}
(42)
The braid relations between $p^\mu$ and $(\Lambda^\mu_\nu, \omega^\mu)$ can be obtained by means of the eqs. (13).

Notice that

\[ p^2 = p^\mu g_{\mu\nu} p^\nu = m^2 c^2 \]  

(43)

with the square of the light velocity

\[ c^2 = \dot{x}^\mu g_{\mu\nu} \dot{x}^\nu, \]  

(44)

where $\dot{c} = 0$ by means the free motion $\ddot{x}^\mu = 0$ equation. Therefore the light velocity is a constant of motion.

Free particle trajectories (geodesics) can be obtained via the least action principle

\[ \delta \int ds = 0, \]  

(45)

where $ds = \sqrt{\dot{x}^\mu g_{\mu\nu} \dot{x}^\nu} d\tau$.

If we identify two affine parameters $s$ and $\tau$ via

\[ ds = c \, d\tau \]  

(46)

then the least action principle implies

\[ \ddot{x}^\mu = 0 \]  

(47)

for geodesics. This confirms our claim about constancy of the light velocity.

6 Conclusions

In this paper we introduce a quantum-braided variant of the deformed Poincaré group under isotropy assumption. In its structure $QBP$ group is more close to the standard Poincaré group than other deformations. In particular it admits a kind of quantum Minkowski geometry. Also infinitesimal-like methods can be applied; it is important from the point of view of identification of physical observables and construction of representations.

It is interesting an analogy of $QBP$ group with supersymmetric groups; namely the braiding rules in the supersymmetric case lies in the anticommutativity of group parameters and coordinates.

Many questions are open. For example an open question is physical identification of the deformation parameter $q$. Furthermore it is clear that (like in general relativity) we should distinguish between dynamical time and coordinate time. The first one can be identified with the invariant parameter $\tau$ ($s$) and permit us to causal ordering of events. For the other hand $x^0$ should be interpreted as a coordinate time. We have no direct relation between $\tau$ and $x^0$ even in the rest frame ($\tau$ cannot be interpreted as the proper time). It seems that from the point of view of the measurement process, the directly measured observable are $x^\mu$'s so $x^0$ is measured directly rather than $\tau$.

A more detailed discussion of these rather delicate questions will be given in the forthcoming paper.

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The Cartan-Maurer differential form is defined as

$$\Theta = \left( \begin{array}{c|c} \Lambda^{-1} & -\Lambda^{-1}a \\ \hline 0 & 1 \end{array} \right) d \left( \begin{array}{c|c} \Lambda & a \\ \hline 0 & 1 \end{array} \right) = \left( \begin{array}{c|c} \Lambda^{-1} d\Lambda & -\Lambda^{-1} da \\ \hline 0 & 0 \end{array} \right)$$  (A.1)

Now the Lie generators are identified via

$$\Omega \equiv \Lambda^{-1} d\Lambda = i\Omega_{\mu\nu} J_{\mu}^{\nu}$$  (A.2)

$$\Lambda^{-1} da = i\rho_{\mu} P_{\mu}.$$  (A.3)

Putting

$$\phi^k = \varepsilon^{kij} \Omega_j^i,$$  (A.4)

$$\kappa^k = q^{-1} \Omega_k^0$$  (A.5)

we can show that

$$\phi^* = \phi, \quad \kappa^* = \kappa, \quad \rho^{\mu*} = \rho^\mu$$  (A.6)

A bi-covariant differential calculus for QBP is defined by the following reordering rules

$$\begin{align*}
\phi^i \phi^k &= -\phi^k \phi^i, \\
\phi^i \rho^\mu &= -\rho^\mu \phi^i, \\
\phi^i \kappa^k &= -\kappa^k \phi^i, \\
\kappa^i \kappa^k &= -\kappa^k \kappa^i, \\
\kappa^i \rho^\mu &= -q^{-1} \rho^\mu \kappa^i, \\
\rho^0 \rho &= -q \rho^0 \rho, \\
\rho^i \rho^k &= -\rho^k \rho^i,
\end{align*}$$  (A.7)

$$\begin{align*}
\phi^k A_{\mu}^{\nu} &= A_{\nu}^{\mu} \phi^k, \\
\phi^k a_{\mu} &= a_{\nu}^{\mu} \phi^k, \\
\kappa^i A_{\mu}^{\nu} &= A_{\nu}^{\mu} \kappa^i, \\
\kappa^i a_{\mu} &= q^{-1} a_{\nu}^{\mu} \kappa^i, \\
\rho^0 a^0 &= a^0 \rho^0, \\
\rho^0 a^i &= qa^i \rho^0, \\
\rho^i a^k &= a^k \rho^i, \\
\rho^0 a^0 &= q^{-1} a^0 \rho^i.
\end{align*}$$  (A.8)

Furthermore

$$\begin{align*}
\phi^i \gamma &= \gamma \phi^i, \\
\phi^i \Gamma &= \Gamma \phi^i, \\
\kappa^i \gamma &= q^2 \gamma \kappa^i, \\
\kappa^i \Gamma &= q^2 \Gamma \kappa^i, \\
\rho^i \gamma &= q^2 \gamma \rho^i, \\
\rho^i \Gamma &= \Gamma \rho^i, \\
\rho^0 \gamma &= \gamma \rho^0, \\
\rho^0 \Gamma &= q^{-2} \Gamma \rho^0.
\end{align*}$$  (A.9)

Finally, the generators $J_k$ and $K_i$ are related to $J_m u^\nu$ by

$$\begin{align*}
J_k &= \varepsilon_{kij} J_j^i, \\
K_k &= J_k^0 + GJ_0^k.
\end{align*}$$
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