A physical model and a Monte Carlo estimate for the spatial derivative of the specific intensity

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Abstract

Starting from the radiative transfer equation and its usual boundary conditions, the objective of the present article is to design a Monte Carlo algorithm estimating the spatial derivative of the specific intensity. There are two common ways to address this question. The first consists in using two independent Monte Carlo estimates for the specific intensity at two locations and using a finite difference to approximate the spatial derivative; the associated uncertainties are difficult to handle. The second consists in considering any Monte Carlo algorithm for the specific intensity, writing down its associated integral formulation, spatially differentiating this integral, and reformulating it so that it defines a new Monte Carlo algorithm directly estimating the spatial derivative of the specific intensity; the corresponding formal developments are very demanding [1]. We here explore an alternative approach in which we differentiate both the radiative transfer equation and its boundary conditions to set up a physical model for the spatial derivative of the specific intensity. Then a standard path integral translation is made to design a Monte Carlo algorithm solving this model. The only subtlety at this stage is that the model for the spatial derivative is coupled to the model for the specific intensity itself. The paths associated to the spatial derivative of the specific intensity give birth to paths associated to specific intensity (standard radiative transfer paths). When designing a Monte Carlo algorithm for the coupled problem a double randomization approach is therefore required.

1 Introduction

We address the question of modeling and numerically simulating the spatial derivative \( \partial_\vec{\gamma} I \equiv \partial_{\vec{\gamma}} I(\vec{x}, \vec{\omega}) \) of the specific intensity \( I \equiv I(\vec{x}, \vec{\omega}) \) at location \( \vec{x} \) in the transport direction \( \vec{\omega} \). This spatial derivative is made along a given direction, namely along a unit vector \( \vec{\gamma} \), which means that

\[
\partial_{\vec{\gamma}} I = \vec{\gamma} \cdot \nabla I
\]

(1)

Intensity \( I \) has two independent variables \( (\vec{x}, \vec{\omega}) \); the spatial derivative \( \partial_{\vec{\gamma}} I \) has three independent variables \( (\vec{x}, \vec{\omega}, \vec{\gamma}) \). As two of these variables are directions (vectors in the unit sphere), they will be distinguished by specifying the transport direction for \( \vec{\omega} \) and the differentiation direction for \( \vec{\gamma} \) (see Figure 1).

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The reason why we address $\partial_{1,\xi} I$, a scalar quantity, instead of the vector $\nabla I$ as a whole, is the attempt to make explicit connections between the modeling of spatial derivatives and standard radiative transfer modeling. Starting from the available transport physics for $I$, our main objective is to introduce a new, very similar transport physics for $\partial_{1,\xi} I$. Then all the standard practice of analysing and numerically simulating $I$ can be directly translated into new tools for analysing and numerically simulating spatial derivatives.

Standard radiative transfer physics can be gathered into two equations: the partial differential equation governing $I$ at any location inside the field $G$ (the radiative transfer equation) and an integral constraint at the boundary $\partial G$ (the incoming radiation equation), relating $I$ in any direction toward the field to $I$ in all the directions exiting the field. Recognizing, in the writing of these equations, the processes of volume emission/absorption/scattering and surface emission/absorption/reflection, translating them into path statistics, is quite straightforward. We will do the same with $\partial_{1,\xi} I$:

- Two equations will be constructed for $\partial_{1,\xi} I$ by differentiating the radiative transfer equation and the incoming radiation equation (differentiating the equations of the $I$ model).
- The resulting equations will be physically interpreted using transport physics processes, defining volume emission/absorption/scattering and surface emission/absorption/reflection processes for the spatial derivative. A particular attention will be devoted to the identifications of the sources of the spatial derivative.
- Statistical paths will then be defined for $\partial_{1,\xi} I$, from the sources to the location and direction of observation.

Numerically estimating $\partial_{1,\xi} I$ will then be simply achieved using a Monte Carlo approach, i.e. sampling large numbers of paths. We will display the observed variance of the resulting Monte Carlo estimate but no attempt will be made to optimize convergence in the frame of the present article. Configurations for which $\partial_{1,\xi} I$ is known analytically will be used both to validate the formal developments and to illustrate the physical meaning of each of the identified processes of emission, absorption, scattering and reflection as far as spatial derivatives are concerned.

Even if the presentation of the mathematical developments remains strictly formal, we will try to stick to the spirit of radiative transfer: trying to write down the physics of spatial derivatives by maintaining a parallel, as strict as possible, with the physics of photon transport. This parallel will not be complete. Beer-Lambert and phase functions will be entirely recovered, BRDFs also but with a significant new feature: reflection changes the differentiation direction (note that a far parallel can be made with surface reflection modifying the polarization state).

The text is essentially a short note with three sections:

- Section 2 provides the model in its differential form for boundary surfaces without any discontinuities.
- Section 3 deals with the specific case of discontinuities at the junction between two plane surfaces.
- Section 4 provides the associated statistical paths and illustrates how a standard Monte Carlo approach can be used to estimate $\partial_{1,\xi} I$ (or any radiative transfer observable defined as an integral of $\partial_{1,\xi} I$).

## 2 Convex domain with differentiable boundaries

Noting $\mathcal{C}$ the collision operator, the stationary monochromatic radiative transfer equation is

\[ \nabla I(\vec{x}, \Omega) = \mathcal{C}[I] + S \quad \vec{x} \in G \]  

(2)

with

\[ \mathcal{C}[I(\vec{x}, \Omega)] = -k_a(\vec{x}) I(\vec{x}, \Omega) - k_s(\vec{x}) I(\vec{x}, \Omega) + k_s(\vec{x}) \int_{4\pi} p_{\Omega\Omega'}(-\Omega'|\vec{x}, -\Omega)d\Omega' I(\vec{x}, \Omega') \]  

(3)

where $k_a$ is the absorption coefficient, $k_s$ the scattering coefficient and $p_{\Omega\Omega'}(-\Omega'|\vec{x}, -\Omega)$ is the probability density that the scattering direction is $-\Omega'$ for a photon scattered at $\vec{x}$ coming from direction $-\Omega$ (the single scattering phase function, see Figure 2 for a single collision and Figure 3 for a multiple-scattering photon trajectory).

$S \equiv S(\vec{x}, \Omega)$ is the volumic source. When this source is due to thermal emission, under the assumption that the matter is in a state of local thermal equilibrium, then it is isotropic and $S = k_v I^{eq}(T)$ where $T$ is the local temperature and $I^{eq}$ is the specific intensity at equilibrium (following Planck function).

We make the very same choice in $\partial_{1,\xi} I$ as far as angular derivatives are concerned (considering only one rotation around a given axis).
At the boundary, noting $\mathcal{C}_b$ the reflection operator, the incoming radiation equation is

$$I = \mathcal{C}_b[I] + S_b \quad \bar{x} \in \partial G \ ; \ \bar{\omega}, \bar{n} > 0$$

with

$$\mathcal{C}_b[I] = \rho(\bar{x}, -\bar{\omega}) \int_{\mathcal{M}'} p_{\mathcal{YT}, b}(-\bar{\omega}'|\bar{x}, -\bar{\omega}) d\omega' I(\bar{x}, \bar{\omega}')$$

where $\bar{n}$ is the normal to the boundary at $\bar{x}$, oriented toward the inside, $\bar{\omega}$ is a direction within the inside hemisphere $\mathcal{H}'$, $\bar{\omega}'$ is any direction within the outside hemisphere $\mathcal{H}'$, $\rho(\bar{x}, -\bar{\omega})$ is the surface reflectivity for a photon reflected at $\bar{x}$ coming from direction $-\bar{\omega}$ (the product $p_{\mathcal{YT}, b}$ is the bidirectional reflectivity density function, see Figure 4 collision at the boundary and Figure 5 for a multiple-reflection photon trajectory). When the surfacic source $S_b \equiv S_b(\bar{x}, \bar{\omega})$ is due to the thermal emission of an opaque surface, under the assumption that the matter at this surface is in a state of local thermal equilibrium, then $S_b = (1 - \rho(\bar{x}, -\bar{\omega})) I^T(T_b)$ where $T_b$ is the local surface temperature.

Using the linearity of the collision operator, spatially differentiating equations 2 provides a transport model for $\partial_1,_{\bar{\omega}} I$

$$\nabla \cdot (\partial_1,_{\bar{\omega}} I) \cdot \bar{\omega} = \mathcal{C}'[\partial_1,_{\bar{\omega}} I] + S_{\bar{\omega}}[I]$$

with $S_{\bar{\omega}}[I] = \partial_1,_{\bar{\omega}} \mathcal{C}'[I] + \partial_1,_{\bar{\omega}} S$, leading to

$$S_{\bar{\omega}}[I] = -\partial_1,_{\bar{\omega}} k_s I - \partial_1,_{\bar{\omega}} k_s I$$

$$+ \partial_1,_{\bar{\omega}} k_s \int_{4\pi} p_{\mathcal{YT}}(-\bar{\omega}'|\bar{x}, -\bar{\omega}) d\omega' I(\bar{x}, \bar{\omega}')$$

$$+ k_s \int_{4\pi} \partial_1,_{\bar{\omega}} p_{\mathcal{YT}}(-\bar{\omega}'|\bar{x}, -\bar{\omega}) d\omega' I(\bar{x}, \bar{\omega}')$$

$$+ \partial_1,_{\bar{\omega}} S$$

Establishing the boundary condition for equation 6 is less straightforward because the boundary properties are attached to the boundary and spatially differentiating $I$ in any direction a implies a differential step that is not parallel to the boundary. We retained the following approach that we believe is an essential argument when attempting to read the physics of $\partial_1,_{\bar{\omega}} I$ in pure transport terms:

— $\bar{\omega}$ is decomposed as the sum of two vectors, one parallel to the direction of sight $\bar{\omega}$, the other parallel to a direction $\bar{u}$ parallel to the boundary (see figure 6 and Appendix A), i.e.

$$\bar{\omega} = \alpha \bar{\omega} + \beta \bar{u}$$

with

$$\alpha = \frac{\bar{\omega} \cdot \bar{n}}{\bar{n} \cdot \bar{n}}; \quad \beta = \|\bar{\omega} - \alpha \bar{\omega}\|; \quad \bar{u} = \frac{\bar{\omega} - \alpha \bar{\omega}}{\beta} \quad \text{or} \quad \beta \bar{u} = \frac{(\bar{\omega} \wedge \bar{\gamma}) \wedge \bar{n}}{\bar{\omega} \cdot \bar{n}}$$

The spatial derivative in direction $\bar{\omega}$ can then be addressed by successively considering the spatial derivative in direction $\bar{\omega}$ and the spatial derivative in direction $\bar{u}$:

$$\partial_1,_{\bar{\omega}} I = \alpha \partial_1,_{\bar{\omega}} I + \beta \partial_1,_{\bar{u}} I$$

— The spatial derivative in the direction of the line of sight is simply the transport term of the radiative transfer equation 2. It can therefore be replaced by field collisions and sources:

$$\partial_1,_{\bar{\omega}} I = \mathcal{C}[I] + S$$

— The spatial derivative in a direction tangent to the boundary can finally be obtained by a straightforward differentiation of the incoming radiation equation 4:

$$\partial_1,_{\bar{u}} I = \mathcal{C}_b[\partial_1,_{\bar{u}} I] + \partial_1,_{\bar{u}} \mathcal{C}_b[I] + \partial_1,_{\bar{u}} S_b$$

Altogether, the boundary condition of the transport model for $\partial_1,_{\bar{\omega}} I$ is

$$\partial_1,_{\bar{\omega}} I = \beta \mathcal{C}_b[\partial_1,_{\bar{u}} I] + S_b,_{\bar{\omega}} I$$

$$\bar{x} \in \partial G \ ; \ \bar{\omega}, \bar{n} > 0$$
Figure 1 – The spatial derivative $\partial_{1,\vec{\gamma}} I$ pictured as an elementary displacement in the differentiation direction $\vec{\gamma}$ according to $\partial_{1,\vec{\gamma}} I(\vec{x}, \vec{\omega}) = \vec{\gamma} \cdot \nabla_{\vec{x}} I = \lim_{\tau \to 0} \frac{I(\vec{x} + \tau \vec{\gamma}, \vec{\omega}) - I(\vec{x}, \vec{\omega})}{\tau}$. When picturing photon transport, we need to draw the location $\vec{x}$ and the line of sight, i.e. the transport direction $\vec{\omega}$. When picturing the physics of spatial derivatives, we will need to draw the location $\vec{x}$ and two vectors: $\vec{\omega}$ for the transport direction and $\vec{\gamma}$ for the differentiation direction.

Figure 2 – Sources (emission) and collisions (absorption and scattering) within the volume. The formulation of Eq. 3 favors a reciprocal/adjoint interpretation thanks to the micro-reversibility relation $p_{\Omega'}(-\vec{\omega}'|\vec{x}, -\vec{\omega}) = p_{\Omega'}(\vec{\omega}|\vec{x}, \vec{\omega}')$. The physical picture then becomes that of a photon initially in direction $-\vec{\omega}$ scattered in direction $-\vec{\omega}'$. 

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Figure 3 – Left: a multiple-scattering photon trajectory leading to location $\vec{x}$ and transport direction $\vec{\omega}$. Right: its correspondence for spatial derivatives (differentiation direction $\vec{\gamma}$). Nothing changes. The differentiation direction is preserved at each scattering event.

Figure 4 – Sources (emission) and collisions (absorption and reflection) at the boundary. The formulation of Eq. 4 favors a reciprocal/adjoint interpretation thanks to the micro-reversibility relation $(\vec{\omega}.\vec{n})\rho(\vec{x}, -\vec{\omega})p_{\text{TV}, b}(\vec{\omega}'|\vec{x}, -\vec{\omega}) = -(\vec{\omega}''.\vec{n})\rho(\vec{x}, \vec{\omega}')p_{\text{TV}, b}(\vec{\omega}|\vec{x}, \vec{\omega}')$. The physical picture then becomes that of a photon initially in direction $-\vec{\omega}$ reflected in direction $-\vec{\omega}'$. 
Figure 5 – Left: a multiple-reflection photon trajectory leading to location $\vec{x}$ and transport direction $\vec{\omega}$. Right: its correspondance for spatial derivatives (differentiation direction $\vec{\gamma}$). The characteristics of surface reflection are unchanged, but the differentiation direction is modified at each reflection event. Note that once again we favor a reciprocal reading of this transport physics: $\vec{\gamma}$ is transformed into $\vec{u}_1$ at the first reflection backward along the line of sight, then $\vec{u}_1$ is transformed into $\vec{u}_2$ at the second reflection, etc.

Figure 6 – At the boundary, the differentiation direction $\vec{\gamma}$ is decomposed by projection along the transport direction $\vec{\omega}$ and along a unit vector $\vec{u}$ tangent to the boundary: $\vec{\gamma} = \alpha \vec{\omega} + \beta \vec{u}$ with $\alpha$ that can be positive or negative and $\beta$ always positive. Four configurations are illustrated. The bottom right configuration illustrates that when the transport direction is nearly tangent to the surface, then the coefficient $\beta$ can take very large values. This will be an important point when discussing convergence issues for Monte Carlo simulations. $\beta$ appears indeed as a factor in front of the collision operator, which is translated by the Monte Carlo weight being multiplied by $\beta$ at each reflection, possibly leading to very large weight values.
with \( S_{b,\gamma}[I] = \alpha (\mathcal{C}[I] + S) + \beta (\partial_{1,\alpha}\mathcal{C}_b[I] + \partial_{1,\alpha}S_b) \), leading to

\[
S_{b,\gamma}[I] = -\alpha k_a I - \alpha k_s I + \alpha k_s \int_{4\pi} p_{\gamma'}(-\bar{\omega}',\bar{\omega})d\bar{\omega}' I(\bar{x},\bar{\omega}') + \alpha S + \beta \partial_{1,\alpha} \rho(\vec{x},-\bar{\omega}) \int_{\mathbb{S}^2} p_{\gamma',\beta}(-\bar{\omega}',\bar{\omega})d\bar{\omega}' I(\vec{x},\bar{\omega}') + \beta \rho(\vec{x},-\bar{\omega}) \int_{\mathbb{S}^2} \partial_{1,\alpha} p_{\gamma',\beta}(-\bar{\omega}',\bar{\omega})d\bar{\omega}' I(\vec{x},\bar{\omega}') + \beta \partial_{1,\alpha} S_b
\]

The model for \( I \) was (see Eq. 2 and Eq. 4)

\[
\begin{cases}
\tilde{\mathcal{C}} I = \mathcal{C}[I] + S & \vec{x} \in G \\
I = \mathcal{C}_b[I] + S_b & \vec{x} \in \partial G ; \quad \vec{x} \n \end{cases}
\]

The model for \( \partial_{1,\gamma} I \) is (see Eq. 6 and Eq. 13)

\[
\begin{cases}
\tilde{\mathcal{C}} (\partial_{1,\gamma} I) = \mathcal{C}[\partial_{1,\gamma} I] + S_{\gamma}[I] & \vec{x} \in G \\
\partial_{1,\gamma} I = \beta \mathcal{C}_b[\partial_{1,\gamma} I] + S_{b,\gamma}[I] & \vec{x} \in \partial G ; \quad \vec{x} \n
\end{cases}
\]

The main differences are the following:

- At the boundary, the collision operator is multiplied by \( \beta \), a pure geometrical quantity, function of \( \bar{\omega} \), \( \hat{\gamma} \) and \( \vec{n} \), that is always positive but is not framed inside the unit interval. It can take large values when the transport direction is close to surface tangent (see Figure 6). At least this \( \beta \) factor cannot be interpreted as a simple modification of the surface reflectivity: at each reflection we will have to account for this multiplication factor as an additional amplification or attenuation mechanism.

- Again at the boundary, the collision operator is applied to another spatial derivative, \( \partial_{1,\hat{\gamma}} I \) instead of \( \partial_{1,\gamma} I \), i.e. a spatial derivative along a direction tangent to the boundary. In physical terms, there is still a surface reflection mechanism, with the same reflection properties, but the direction of the spatial derivative changes at each reflection (see Figure 5).

- In the standard radiative transfer model, the sources \( S \) and \( S_b \) are given quantities (functions of the volume and surface properties), but in the model for \( \partial_{1,\gamma} I \), the sources \( S_{\gamma}[I] \) and \( S_{b,\gamma}[I] \) depend on \( I \).

In pure mathematical terms, they are sources in the model for \( \partial_{1,\gamma} I \) only if this model is decoupled from the radiative transfer model. But the complete physics implies that the models are coupled: \( S_{\gamma}[I] \) and \( S_{b,\gamma}[I] \) express this coupling.

The sources \( S_{\gamma}[I] \) and \( S_{b,\gamma}[I] \) can be reformulated, depending on the configuration and the addressed question, in order to highlight a chosen set of features of spatial derivatives. Hereafter, as an example, we put forward the fact that when reaching a state of radiative equilibrium, intensity is uniform and therefore \( \partial_{1,\gamma} I \) is null whatever the deviation direction \( \hat{\gamma} \): there must be no sources for \( \partial_{1,\gamma} I \). Equations 7 and 14 can be transformed the following way to help picturing this equilibrium limit:

\[
S_{\gamma}[I] = \partial_{1,\gamma} S - \partial_{1,\gamma} k_a I + \partial_{1,\gamma} k_s \int_{4\pi} p_{\gamma'}(-\bar{\omega}',\bar{\omega})d\bar{\omega}' (I(\vec{x},\bar{\omega}') - I)
\]

and

\[
S_{b,\gamma}[I] = \alpha (S - k_a I) + \alpha k_s \int_{4\pi} p_{\gamma'}(-\bar{\omega}',\bar{\omega})d\bar{\omega}' (I(\vec{x},\bar{\omega}') - I) + \beta \int_{\mathbb{S}^2} p_{\gamma',\beta}(-\bar{\omega}',\bar{\omega})d\bar{\omega}' (\partial_{1,\hat{\gamma}} \rho(\vec{x},\bar{\omega}) I(\vec{x},\bar{\omega}') + \partial_{1,\hat{\gamma}} S_{b,\gamma})
\]

This leaves us with three terms for \( S_{\gamma}[I] \) and four terms for \( S_{b,\gamma}[I] \):

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— The first term of $S_{\gamma}[I]$ expresses the fact that when moving along the differentiation direction $\gamma$, if the absorption coefficient changes ($k_a$ non-uniform) then extinction by volume absorption changes, and also if the source changes ($S$ non-uniform) then amplification by volume sources changes. When the physical problem is compatible with equilibrium, then $S = k_a I^{eq}(T)$ and this first term of $S_{\gamma}[I]$ becomes

$$k_a \partial_{\gamma} S_{\gamma}[I] = \partial_{\gamma} \left( I^{eq}(T) \right) + \partial_{\gamma} k_a (I^{eq}(T) - I)$$

Its physical meaning is the following: i) $k_a \partial_{\gamma} I^{eq}(T)$ means that even for $k_a$ uniform, the source may change spatially if the volume is non-isothermal; ii) as $k_a$ is in factor of both extinction by absorption and amplification by emission, the source associated to a non-uniform absorption coefficient is proportional to the difference $I^{eq}(T) - I$. Obviously both mechanisms vanish at equilibrium: $\partial_{\gamma} I^{eq}(T) = 0$ because $T$ is uniform and $I^{eq}(T) - I = 0$ because $I = I^{eq}(T)$. As expected, this first term (competition of volume emission and volume absorption) is null at equilibrium.

— The second term expresses the volume source associated to the competition between extinction by outgoing scattering and amplification by incoming scattering in the case of a non-uniform scattering coefficient. This expression is obtained by noting that

$$\int_{4\pi} p_{TV}(-\omega'|\vec{x},-\omega)d\omega' I = \int_{4\pi} p_{TV}(-\omega'|\vec{x},-\omega)d\omega' = I$$

Again, this second term is null at equilibrium because intensity is isotropic and $I(\vec{x},\omega') = I$ for all $\omega'$.

— The third term is strictly similar for non-uniform phase functions. It is obtained by observing that $\int_{4\pi} p_{TV}(-\omega'|\vec{x},-\omega)d\omega' = 1$ at all locations, therefore $\partial_{\gamma} \int_{4\pi} p_{TV}(-\omega'|\vec{x},-\omega)d\omega' = 0$, or $\int_{4\pi} \partial_{\gamma} p_{TV}(-\omega'|\vec{x},-\omega)d\omega' = 0$, leading to

$$\int_{4\pi} \partial_{\gamma} p_{TV}(-\omega'|\vec{x},-\omega)d\omega' I = \int_{4\pi} \partial_{\gamma} p_{TV}(-\omega'|\vec{x},-\omega)d\omega' = 0$$

— The first term of $S_{b,\gamma}[I]$ expresses the fact that except when $\gamma$ is strictly parallel to the surface, when moving along the differentiation direction $\gamma$ the distance to the surface inceases (if $\gamma, \vec{n} > 0$, i.e. $\alpha > 0$), which creates a new volume of emitting and absorbing medium between the current location and the surface, or the distance to the surface decreases (if $\gamma, \vec{n} < 0$, i.e. $\alpha < 0$), which supresses some amount of emitting and absorbing medium. When the adressed radiative transfer problem is compatible with equilibrium, $S = k_a I^{eq}(T)$ and this first term of $S_{b,\gamma}[I]$ becomes

$$\alpha k_a (I^{eq}(T) - I)$$

which is obviously null at the equilibrium state.

— The second term of $S_{b,\gamma}[I]$ expresses the very same phenomenon, but as far as scattering is concerned: increase or decrease of the amount of participating medium between the current location and the surface when moving along the differentiation direction, therefore increasing or reducing the extinction by outgoing scattering as well as the amplification by incoming scattering. This second term is obtained by noting that

$$\int_{4\pi} p_{TV}(-\omega'|\vec{x},-\omega)d\omega' I = \int_{4\pi} p_{TV}(-\omega'|\vec{x},-\omega)d\omega' = I$$

— The third term accounts for surfaces with a non-homogeneous reflectivity and/or a non-homogeneous surface emission. A displacement along the differentiation direction $\gamma$ is associated to a displacement along the projected direction $\vec{\alpha}$ of the location where the line of sight intersects the surface. $\rho$ and $S_b$ are therefore spatially differentiated along $\vec{\alpha}$. When the adressed radiative transfer problem is compatible with equilibrium, $S_b = (1 - \rho(\vec{x}, -\omega)) I^{eq}(T_b)$ and this third term of $S_{b,\gamma}[I]$ becomes

$$\beta \rho(\vec{x}, -\omega) \partial_{\gamma} S_{b}[I] + \beta \partial_{\gamma} \rho(\vec{x}, -\omega) \int_{4\pi} p_{TV, b}(-\omega'|\vec{x}, -\omega)d\omega' (I(\vec{x}, \omega') - I)$$

Its first part accounts for non-isothermal surfaces, even for uniform reflectivities. The second term is null: $T_b$ is uniform along the surface and intensity is isotropic ($I(\vec{x}, \omega') = I$ for all $\omega'$).

— The last term of $S_{b,\gamma}[I]$ deals similarly with $p_{TV, b}$ non-uniform. Its expression is obtained by observing that $\int_{4\pi} p_{TV, b}(-\omega'|\vec{x}, -\omega)d\omega' = 1$ at all locations along the surface, therefore $\partial_{\gamma} \int_{4\pi} p_{TV, b}(-\omega'|\vec{x}, -\omega)d\omega' = 0$, or $\int_{4\pi} \partial_{\gamma} p_{TV, b}(-\omega'|\vec{x}, -\omega)d\omega' = 0$, leading to

$$\int_{4\pi} \partial_{\gamma} p_{TV, b}(-\omega'|\vec{x}, -\omega)d\omega' I = \int_{4\pi} \partial_{\gamma} p_{TV, b}(-\omega'|\vec{x}, -\omega)d\omega' = 0$$
3 Boundary discontinuities at the junction of two plane surfaces

We have set up a transport model for $\partial_{1,2}I$. The corresponding source terms define the emission, in the elementary solid angle $d\mathbf{\Omega}$ around $\mathbf{\Omega}$ (see Figure 7),

- of any elementary volume $dv \equiv d\mathbf{x}$ around $x \in \mathcal{G}$:
  \[
  \text{Volumic emission : } S_v[I] \ dv \ d\mathbf{\Omega}
  \]

- of any elementary surface $d\sigma \equiv d\mathbf{\ell}$, of normal $\mathbf{n}$, around $x \in \partial \mathcal{G}$
  \[
  \text{Surfacic emission : } S_s[I](\mathbf{n} \cdot \mathbf{\ell}) \ d\sigma \ d\mathbf{\Omega}
  \]

When the boundary is discretized as an ensemble of plane surfaces, typically an ensemble of triangles, then new lineic emissions appear along the edge $\mathcal{L}_{12}$ between adjacent plane surfaces ($\mathcal{F}_1, \mathcal{F}_2$). These emissions are nothing more than the extension of our preceding developments to discontinuous intensities fields. The intensity in a given direction becomes discontinuous at the edge, either because the intensity sources are different on the two plane surfaces (discontinuous surface temperatures for thermal emission), or because reflection properties are different, or simply with the same reflection properties because the normals are different. In all such cases, the outgoing intensity is discontinuous when crossing the edge and this creates localised sources that require a Dirac formulation. When these Dirac sources are integrated over the surface, only the integral over the edge remains and an emission is associated to each elementary length $dl \equiv d\mathbf{x}$ around $x \in \mathcal{L}_{12}$ (see Appendix B and Figure 7):

\[
\text{Lineic emission : } (\mathbf{\Omega} \wedge \mathbf{\gamma}) \cdot \mathbf{t} \ (I_1 - I_2) \ d\ell \ d\mathbf{\Omega}
\]

where $I_1$ and $I_2$ are the two intensity values at the discontinuity and $\mathbf{t}$ is a unit tangent to the edge. In this expression, the indexes $1$ and $2$ for the two adjacent surfaces $\mathcal{F}_1$ and $\mathcal{F}_2$, of unit normals $\mathbf{n}_1$ and $\mathbf{n}_2$, are chosen so that $\mathbf{n}_1 = \mathbf{t} \wedge \mathbf{\Gamma}_1$ is oriented toward the inside of $\mathcal{F}_1$, and $\mathbf{n}_2 = -\mathbf{t} \wedge \mathbf{\Gamma}_2$ is oriented toward the inside of $\mathcal{F}_2$ (see Figure 8). As for the volumic and surfacic sources of the preceeding section, this lineic source of spatial derivative is a function of intensity via its dependance on $I_1$ and $I_2$: via its sources, the transport physics of the spatial derivative of intensity is coupled to the physics of intensity itself. Evaluating $I_1$ (or $I_2$) raises different questions depending on the sign of $\mathbf{\Omega} \cdot \mathbf{n}_1$ (or $\mathbf{\Omega} \cdot \mathbf{n}_2$). If $\mathbf{\Omega} \cdot \mathbf{n}_1 > 0$, then $I_1$ is the sum of surface emission $S_b$ and surface reflection $S_b[I] = \rho(x, -\mathbf{\Omega}) \int_{\mathcal{F}_1} p(\mathbf{\Omega}, b(-\mathbf{\Omega} \cdot \mathbf{x}, -\mathbf{\Omega})) d\mathbf{\Omega} I(x, \mathbf{\Omega})$, using the physical properties of $\mathcal{F}_1$:

\[
I_1 = \lim_{\epsilon \to 0} I(x + \epsilon \mathbf{\Gamma}_1, \mathbf{\Omega}) \quad \text{for} \quad \mathbf{\Omega} \cdot \mathbf{n}_1 > 0
\]

\[
= \lim_{\epsilon \to 0} S_b(x + \epsilon \mathbf{\Gamma}_1, \mathbf{\Omega}) + \rho(x + \epsilon \mathbf{\Gamma}_1, -\mathbf{\Omega}) \int_{\mathcal{F}_1} p(\mathbf{\Omega}, b(-\mathbf{\Omega} \cdot \mathbf{x} + \epsilon \mathbf{\Gamma}_1, -\mathbf{\Omega})) d\mathbf{\Omega} I(x + \epsilon \mathbf{\Gamma}_1, \mathbf{\Omega})
\]

If $\mathbf{\Omega} \cdot \mathbf{n}_1 < 0$, then $I_1$ is not exiting $\mathcal{F}_1$ and cannot be expressed using surface emission and surface reflection: it corresponds to radiation tangenting the edge, coming from the part of the system facing $\mathcal{F}_1$ (see Figure 9):

\[
I_1 = \lim_{\epsilon \to 0} I(x - \epsilon \mathbf{\Gamma}_1, \mathbf{\Omega}) \quad \text{for} \quad \mathbf{\Omega} \cdot \mathbf{n}_1 < 0
\]

Anticipating Monte Carlo discussions, we need to emphasize that these lineic emissions are the result of surface integrations over the boundary of a Dirac sources. This implies that when a Dirac source at $x$ is viewed from a point $x_{obs}$ at distance $r$, i.e. $x_{obs} = x + r\mathbf{\Omega}$, the surface integration comes from the angular integration (see Figure 10). A typical formulation is therefore the following. At $x_{obs}$, let us consider a solid angle $\Omega$ under which a subpart of the boundary $\partial G$ is viewed, noted $\partial G^\Omega$, including a subpart of the edge $\mathcal{L}_{12}$, noted $\mathcal{L}_{12}^\Omega$. If we adress the integration over $\Omega$ of the surfacic sources as they are viewed from $x_{obs}$ (temporarily ignoring extinction by absorption and scattering), each elementary solid angle $d\Omega$ defines an elementary surface $d\sigma$ at the boundary according to $d\sigma = (\mathbf{\Omega} \wedge \mathbf{\gamma}) r^2$ and the angular integration becomes

\[
\int_{\partial \mathcal{L}_{12}} \partial_{1,2} I(x_{obs}, \mathbf{\Omega}) d\Omega = \int_{\partial G^\Omega} \frac{(\mathbf{\Omega} \wedge \mathbf{\gamma})}{r^2} (\beta \rho_b(\partial_1, aI) + S_b(\mathbf{\Omega}) [I]) d\sigma + \int_{\mathcal{L}_{12}^\Omega} \frac{(\mathbf{\Omega} \wedge \mathbf{\gamma}) \cdot \mathbf{t}}{r^2} (I_1 - I_2) d\ell
\]
Figure 7 – Volumic, surfacic and lineic emissions of spatial derivatives.

Figure 8 – The unit vectors attached to \( S_1 \) and \( S_2 \) at the edge \( L_{12} \). They form two direct orthonormal basis : \((\vec{m}_1, \vec{t}, \vec{n}_1)\) and \((\vec{m}_2, -\vec{t}, \vec{n}_2)\).
Figure 9 – The two limit values of intensity, $I_1$ and $I_2$, at the edge $\mathcal{L}_{12}$ between $\mathcal{S}_1$ and $\mathcal{S}_2$. Top: $\vec{\omega} \cdot \vec{n}_1 > 0$ and $\vec{\omega} \cdot \vec{n}_2 > 0$; both $I_1$ and $I_2$ are the limits of the intensity exiting the corresponding surface when reaching the edge: $I_1 = \lim_{\epsilon \to 0} I(\vec{x} + \epsilon \vec{m}_1, \vec{\omega})$ and $I_2 = \lim_{\epsilon \to 0} I(\vec{x} + \epsilon \vec{m}_2, \vec{\omega})$. Bottom: $\vec{\omega} \cdot \vec{n}_1 < 0$ and $\vec{\omega} \cdot \vec{n}_2 > 0$; $I_2$ is the limits of the intensity exiting $\mathcal{S}_2$ when reaching the edge, but $I_1$ corresponds to the intensity within the volume, tangenting the edge: $I_1 = \lim_{\epsilon \to 0} I(\vec{x} - \epsilon \vec{m}_1, \vec{\omega})$ and $I_2 = \lim_{\epsilon \to 0} I(\vec{x} + \epsilon \vec{m}_2, \vec{\omega})$.

Figure 10 – Surfacic and lineic sources viewed from a distant point $\vec{x}_{obs}$ within a solid angle $\Omega$. 
4 Path statistics and Monte Carlo

Notice : This is a preliminary version of the final paper, consequently the reader might find some missing parts, especially in the results section where some of the pseudo-algorithms and results tables are not included in this current version.

Our main point in this text is that the model of the spatial derivative of intensity resembles so much the model of intensity (the radiative transfer model) that the whole radiative transfer literature about path statistics and Monte Carlo simulation can be reinvested in a straightforward manner to numerically estimate spatial derivatives. In this last section, we illustrate the practical meaning of this statement. The technical steps that we will highlight with some specificity are the following:

— As already mentioned, at each reflection event the projection factor \( \beta \) needs to be stored and the differentiation direction is changed (see Figure 5). Such a state change at reflection events leads to algorithmic steps that are very similar to those of the Monte Carlo algorithms designed for polarized radiation (note that here nothing similar occurs at scattering events).

— Via the volumic, surfacic and lineic sources of spatial derivatives, that depend on intensity, the model of spatial derivatives is coupled to the radiative transfer model. This coupling can be handled using the very same Monte Carlo techniques as those recently developed for the coupling of radiative transfer with other heat-transfer modes[2, 3, 4, 5], or the coupling of radiative transfer with electromagnetism and photosynthesis[6, 7, 8, 9]. In both cases, the main idea is double randomisation: in standard Monte Carlo algorithms for pure radiative transfer, when a volumic source or a surfacic source is required it is known (typically the temperature is known for infrared radiative transfer); if it is not known but a Monte Carlo algorithm is available to numerically estimate the source as an average of a large number of sampled Monte Carlo weights, then in the coupled problem the source can be replaced by only one sample. The resulting coupled algorithm is rigorously unbiased thanks to the law of expectation (“the expectation of an expectation is an expectation”). In practice, this means that the Monte Carlo algorithms estimating spatial derivatives can be designed as if the sources were known, and when a source is required that depends on \( I(\vec{x}, \vec{\omega}) \) then one single radiative path is sampled as if estimating the intensity \( I(\vec{x}, \vec{\omega}) \) with any available Monte Carlo algorithm.

— The lineic sources need a specific treatment otherwise they would be missed by the standard algorithms integrating over surfaces or solid angles. This can be achieved using the techniques developed to handle collimated Dirac sources for solar/laser applications[10, 11, 12, 13, 14] or satellite observation: at each reflection or scattering event, the directions of the Dirac sources are first sampled, specifically, before continuing the path in another sampled reflected or scattered direction.

We provide hereafter some examples of algorithms that illustrate these three points. They estimate either \( \partial_{\vec{r}} I \) at a location \( \vec{x} \) in a direction \( \vec{\omega} \), or the spatial derivative of the incident flux density \( \varphi \) at a location \( \vec{x} \) on a surface of unit normal \( \vec{n} \), i.e. \( \partial_{\vec{r}} \varphi = \int_{2\pi} (\vec{\omega} \cdot \vec{n}) \partial_{\vec{r}} I(\vec{x}, -\vec{\omega}) \, d\vec{\omega} \). Each example is implemented and tested against exact solutions (see Figure 11):

— Solution 1 : the solution provided by Chandrasekhar for a uniform flux in a stratified heterogeneous scattering atmosphere[15] (see Appendix C). This one-dimensional solution is cut by a three-dimensional closed boundary (a sphere or a cube) and the boundary conditions are adjusted to ensure that Chandrasekhar’s solution is still satisfied. In Chandrasekhar’s solution, there is no volume absorption; when we need to add volume absorption, we compensate it by introducing an adjusted volume emission ensuring that Chandrasekhar’s solution is again still satisfied.

— Solution 2 : a transparent slab between a black isothermal surface at \( T_{hot} \) and an emitting/reflecting diffuse surface of temperature \( T_{cold} \) everywhere except for a square subsurface where the temperature is \( T_{hot} \).

These algorithms sample Monte Carlo weights noted \( w_{z} \) for each quantity \( Z \), meaning that \( N \) samples \( w_{1}, w_{2}, \ldots, w_{N} \) are required to estimate \( \hat{Z} \) as \( \hat{Z} = \frac{1}{N} \sum_{i=1}^{N} w_{z,i} \),

— \( w_{I} \) for the intensity \( I \) when referring to a standard Monte Carlo algorithm estimating the solution of the radiative transfer equation;

— \( w_{\partial_{\vec{r}} I} \) for the spatial derivative of intensity;

— \( w_{\partial_{\vec{r}} \varphi} \) for the spatial derivative of the incident flux density.
Figure 11 – The two configurations used for illustration. Top: the solution provided by Chandrasekhar for a uniform flux in a stratified heterogeneous scattering atmosphere cut by a three-dimension closed boundary (a sphere of radius $a$ or a cube of side $a$). Bottom: a transparent slab of thickness $c$ between a black isothermal surface at $T_{hot}$ and an emitting/reflecting diffuse surface of temperature $T_{cold}$ everywhere except for a square subsurface of side $a$ where the temperature is $T_{hot}$. The emissivity $\epsilon$ of the emitting/reflecting diffuse surface is uniform.
4.1 Emissive surfaces, no reflection, uniform scattering, no volume absorption, no volume emission

Convex domain with differentiable boundaries The intensity \( I \) and its spatial derivative \( \partial_1 \varphi \) are estimated at location \( \vec{x}_{\text{obs}} \) and direction \( \vec{\omega}_{\text{obs}} \) as described in Fig. 11. The geometrical configuration is a sphere, with center located at \( \vec{x}, \vec{e}_3 = 0 \), inserted into an infinite scattering medium. The scattering coefficient \( k_s \) lead to the optical thickness \( \tau = k_s ||\vec{x}, \vec{e}_1 || \). In this example there is no volume absorption (\( k_a = 0 \)) and no volume emission. The radiative configuration is built so that Chandrasekhar’s analytical solution \( L \) (Appendix. C) apply at any \((\vec{x}, \vec{\omega})\). Therefore, the sphere volume has the same properties than the rest of the infinite medium and the sphere surface is considered as a blackbody with boundary conditions set in Eq. 32 as Chandrasekhar’s solution \( L \) for each position on the sphere boundary \( \vec{x} \in \partial G \) and each \( \vec{\omega}.\vec{n} > 0 \).

\[
I = S_b = L(\vec{x}, \vec{\omega}) \quad \vec{x} \in \partial G; \vec{\omega}.\vec{n} > 0
\]  

The intensity Monte-Carlo weight \( w_I \) sampling is detailed in Algorithm 1 and the resulting intensity estimated by Monte-Carlo at \((\vec{x}_{\text{obs}}, \vec{\omega}_{\text{obs}})\) is compared to the analytical solution \( L(\vec{x}_{\text{obs}}, \vec{\omega}_{\text{obs}}) \) in Fig. 12 and table (the table is not included in the current state of the paper). The spatial derivative \( \partial_1 \varphi \) weight sampling is detailed in Algorithm 2 and the resulting spatial derivative estimated by Monte-Carlo at \((\vec{x}_{\text{obs}}, \vec{\omega}_{\text{obs}})\) is compared to the analytical solution \( \partial_1 \varphi L(\vec{x}_{\text{obs}}, \vec{\omega}_{\text{obs}}) \) in Fig. 12 and table (the table is not included in the current state of the paper).

Supplementary informations on the algorithms and the spatial derivative boundary conditions will be found in Appendix. E.1.

Boundary discontinuities The density flux \( \phi \) and its spatial derivative \( \partial_1 \varphi \) are estimated at location \( \vec{x}_{\text{obs}} \) as described in Fig. 11. The geometric configuration is composed by two parallel planes, the lower plane \((\partial G_{\text{bottom}})\) is modelled as a black body at temperature \( T_{\text{cold}} \) and the upper plane \( \partial G_{\text{top}} \) as a blackbody at temperature \( T_{\text{hot}} \) in a square surface \( \mathcal{J}_{\text{hot}} \) and \( T_{\text{cold}} \) outside the square surface. The observation position is located on the lower plane so that we aim to estimate the flux density outgoing the lower plane. The analytical solution of the flux density in this configuration is stated in Appendix. D and will be compared with the Monte-Carlo estimations of the flux density and its spatial gradient.

The flux density is solved by sampling \( w_\phi \) (see Algorithm 3) and results are compared to analytical solution in Fig. 13 and table (the table is not included in the current state of the paper). The spatial derivative of the flux density is solved by sampling \( w_{\partial_1 \varphi} \) (see Algorithm 6) and results are presented in Fig. 13 and compared with the analytical solution.

Supplementary informations on the algorithms and the spatial derivative boundary conditions will be found in Appendix. E.1.

4.2 Emissive and reflective surfaces, uniform scattering, no volume absorption, no volume emission

Convex domain with differentiable surfaces The intensity \( I \) and its spatial derivative \( \partial_1 \varphi \) are estimated at location \( \vec{x}_{\text{obs}} \) and direction \( \vec{\omega}_{\text{obs}} \) as described in Fig. 11. The geometrical configuration is the same as in Sec. 4.1: a sphere, with center located at \( \vec{x}, \vec{e}_3 = 0 \), inserted into an infinite scattering medium. Again the radiative configuration is built so that Chandrasekhar’s analytical solution \( L \) apply at any \((\vec{x}, \vec{\omega})\). The only difference with Sec. 4.1 is that this time the sphere boundaries are looked at as emissive and reflective (diffuse) surfaces with reflection coefficient \( \rho = 0.6 \) and reflection probability density function \( p_{H, b}(-\vec{\omega}'|\vec{x}, -\vec{\omega}) = \frac{\vec{\omega}.\vec{n}}{2} \). To ensure that Chandrasekhar’s analytical solution still apply in this configuration the reflection term of the boundary condition will be compensated by the emission (surfacic source \( S_b \)) part. Intensity boundary conditions are stated in Eq. 4 with \( \mathcal{C}_b \) the collisional operator of the radiative boundary conditions \((\vec{x} \in \partial G; \vec{\omega}.\vec{n} > 0)\):

\[
\mathcal{C}_b[I] = \rho \int_{\mathcal{E}} p_{H, b}(-\vec{\omega}' | -\vec{\omega}) d\vec{\omega}' I(\vec{x}, \vec{\omega}')
\]  

(33) and \( S_b \) the surfacic source

\[
S_b = L(\vec{x}, \vec{\omega}) - \mathcal{C}_b[L(\vec{x}, \vec{\omega})]
\]  

(34) that will account for the intensity coming out of the sphere surface (Changrasekhar’s solution \( L \)) and the compensation term \( \mathcal{C}_b[L] \). Note that the surfacic source \( S_b \) is supposed to be known and is a function of the
analytical solution $\mathcal{L}$ whereas the reflected part of the boundary condition is considered as a function of the unknown incoming intensity.

The intensity spatial derivative is solved by sampling the Monte-Carlo weight $w_{\partial \mathcal{L} I, r}$ (the pseudo-algorithm is not included in the paper at this stage). The results obtained for the spatial derivative of the intensity at $(\vec{x}_{\text{obs}}, \vec{\omega}_{\text{obs}})$ are presented in Fig. 14 and compared with the analytical solution.

Supplementary informations on the algorithms and the spatial derivative boundary conditions will be found in Appendix. E.2.

**Boundary discontinuities**  The flux density $\varphi$ and its spatial derivative $\partial_1 \varphi$ are estimated at location $\vec{x}_{\text{obs}}$ as described in Fig. 11. The geometric configuration is identical to Sec. 4.1 slab configuration. The medium between the two parallel planes is still transparent and the bottom plane surface is still a blackbody at cold temperature. However, this time the top plane is an emissive and reflective (diffuse) surface at temperature $T_{\text{hot}}$ in a square surface $\mathcal{S}_{\text{hot}}$ and $T_{\text{cold}}$ outside the square surface. The reflection coefficient and the reflection probability density function are homogenous along the plane and stated as $\rho = 0.6$ and $\rho_{\varphi, \mathcal{S}_{\text{hot}}} (-\vec{\omega} | \vec{x}, -\vec{\omega}) = \vec{\omega}_{\mathcal{S}_{\text{hot}}} \cdot \vec{n}_{\text{top}} / \pi$.

The flux density is solved by sampling $w_{\varphi, r}$ (the algorithm is not included in the paper at this stage) and results are compared to analytical solution in Fig. 15 and table (the table is not included in the current state of the paper). The flux density spatial derivative is solved by sampling $w_{\partial_1 \varphi, r}$ (pseudo-algorithm is not included in the current state of the paper). Results of the flux density (and its spatial derivative) estimations are presented in Fig. 15 and compared with the analytical solution.

Supplementary informations on the algorithms and the spatial derivative boundary conditions will be found in Appendix. E.2.

**4.3 Emissive surfaces, no reflection, non-uniform scattering, non-uniform volume absorption, non-uniform volume emission**

Convex domain with differentiable boundaries  The intensity $I$ and its derivative $\partial_1 \varphi I$ are estimated at location $\vec{x}_{\text{obs}}$ and direction $\vec{\omega}_{\text{obs}}$ as described in Fig. 11. The geometrical and radiative configurations are identical to Sec. 4.1 : a sphere which surface is considered as a blackbody with the Chandrasekhar solution $\mathcal{L}$ as emitted intensity. The only difference here is that the scattering properties of the infinite medium and sphere volume change : the scattering coefficient field is now non-uniform and is stated as $k_s = k_0 \exp(k_1 \vec{x} \cdot \vec{e}_3)$.

The optical thickness is then $\tau = \frac{k_0}{k_1} (\exp(k_1 \vec{x} \cdot \vec{e}_3) - 1)$. Volume emission and absorption are not considered in the current state of the paper. With this example we illustrate how non-uniform scattering will impact the Mont-Carlo algorithm used to solve the spatial derivative.

The results of spatial gradient estimation at $(\vec{x}_{\text{obs}}, \vec{\omega}_{\text{obs}})$ are presented in Fig. 16 and compared with the analytical solution.

Supplementary informations on the algorithms and the spatial derivative boundary conditions will be found in Appendix. E.3.

**Boundary discontinuities**  In the present state of the paper this configuration is not described.
Algorithm 1 : Sample the Monte Carlo weight \( w_I \) for domains of any shape, with emissive surfaces, no reflection, uniform scattering, no volume absorption and no volume emission.

Initialise \( \vec{x} \) and \( \vec{\omega} \);
Reverses the direction : \( \vec{\omega} \leftarrow -\vec{\omega} \);
Set \( \text{intersection} \) to \( \text{False} \);
while \( \text{intersection} = \text{False} \) do
    Sample a scattering free path \( \ell \) according to \( p(\ell) = k_s \exp(-k_s \ell) \);
    Find the distance \( \ell_b \) to the boundary from \( \vec{x} \) in direction \( \vec{\omega} \);
    if \( \ell < \ell_b \) then
        \( \vec{x} \leftarrow \vec{x} + \ell \vec{\omega} \);
        Sample \( \vec{\omega}_s \) according to \( p_\Omega(\vec{\omega}_s | \vec{x}, \vec{\omega}) \);
        \( \vec{\omega} \leftarrow \vec{\omega}_s \);
    else
        \( \vec{x} \leftarrow \vec{x} + \ell_b \vec{\omega} \);
        \( \text{intersection} \leftarrow \text{True} \);
        \( w_I \leftarrow S_b(\vec{x}, -\vec{\omega}) \);
    end
end

Algorithm 2 : Sample the Monte Carlo weight \( w_{\partial_1, \vec{\gamma}} \) for a convex domain with differentiable boundaries, emissive surfaces, no reflection, uniform scattering, no volume absorption and no volume emission.

Initialise \( \vec{x} \) and \( \vec{\omega} \);
Reverse the direction : \( \vec{\omega} \leftarrow -\vec{\omega} \);
Set \( \text{intersection} \) to \( \text{False} \);
while \( \text{intersection} = \text{False} \) do
    Sample a scattering free path \( \ell \) according to \( p(\ell) = k_s \exp(-k_s \ell) \);
    Find the distance \( \ell_b \) to the boundary from \( \vec{x} \) in direction \( \vec{\omega} \);
    if \( \ell < \ell_b \) then
        \( \vec{x} \leftarrow \vec{x} + \ell \vec{\omega} \);
        Sample \( \vec{\omega}_s \) according to \( p_\Omega(\vec{\omega}_s | \vec{x}, \vec{\omega}) \);
        \( \vec{\omega} \leftarrow \vec{\omega}_s \);
    else
        \( \vec{x} \leftarrow \vec{x} + \ell_b \vec{\omega} \);
        \( \text{intersection} \leftarrow \text{True} \);
        Compute \( \alpha, \beta \) and \( \vec{u} \) for the transport direction \( -\vec{\omega} \);
        Sample \( \vec{\omega}_s \) according to \( p_\Omega(\vec{\omega}_s | \vec{x}, \vec{\omega}) \);
        if \( \vec{\omega}_s \cdot \vec{n} < 0 \) then
            \( \vec{\omega}_{\partial_1, \vec{\gamma}} \leftarrow \alpha(S_b(\vec{x}, -\vec{\omega}_s) - S_b(\vec{x}, -\vec{\omega})) + \beta \partial_1 S_b(\vec{x}, -\vec{\omega}) \);
        else
            Sample \( w_I \) for \( \vec{x} \) and \( -\vec{\omega}_s \) using Algorithm 1;
            \( w_{\partial_1, \vec{\gamma}} \leftarrow \alpha(\vec{w} - S_b(\vec{x}, -\vec{\omega})) + \beta \partial_1 S_b(\vec{x}, -\vec{\omega}) \);
        end
    end
end

Algorithm 3 : Sample the Monte Carlo weight \( w_\phi \) for a convex domain with differentiable boundaries, emissive surfaces, no reflection, uniform scattering, no volume absorption and no volume emission.

Initialise \( \vec{x} \) and \( \vec{n} \);
Sample \( \vec{\omega} \) according to a Lambert distribution around \( \vec{n} \) (i.e. \( p(\vec{\omega}) = \frac{\vec{\omega} \cdot \vec{n}}{\pi} \));
Sample \( w_I \) for \( \vec{x} \) and \( -\vec{\omega} \) using Algorithm 1;
\( w_\phi \leftarrow \pi w_I \);
Algorithm 4: Sample the Monte Carlo weight $w_\partial \gamma \phi$ for a convex domain with differentiable boundaries, emissive surfaces, no reflection, uniform scattering, no volume absorption and no volume emission.

Initialise $\vec{x}$ and $\vec{n}$;
sample $\vec{\omega}$ according to a Lambert distribution around $\vec{n}$ (i.e. $p(\vec{\omega}) = \frac{\vec{\omega} \cdot \vec{n}}{\pi}$);
Sample $w_\partial \gamma I$ for $\vec{x}$ and $-\vec{\omega}$ using Algorithm 2;
$w_\partial \gamma \phi \leftarrow \pi w_\partial \gamma I$;

Algorithm 5: Sample the Monte Carlo weight $w_\partial \gamma I$ for a closed cavity composed of adjacent plane surfaces with emissive surfaces, no reflection, uniform scattering, no volume absorption and no volume emission.

A écrire;
Initialise $\vec{x}$ and $\vec{\omega}$;
Reverse the direction: $\vec{\omega} \leftarrow -\vec{\omega}$;
Set intersection to False;
Set $w_\partial \gamma I$ to 0;
while intersection = False do
    Sample a scattering free path $\ell$ according to $p(\ell) = k_s \exp(-k_s \ell)$;
    Find the distance $\ell_b$ to the boundary from $\vec{x}$ in direction $\vec{\omega}$;
    if $\ell < \ell_b$ then
        $\vec{x} \leftarrow \vec{x} + \ell \vec{\omega}$;
        Find the edges potentially visible from $\vec{x}$ and compute their total length $L_{\text{edges}}$;
        Uniformly sample a location $\vec{y}$ on the edges;
        Trace a ray from $\vec{x}$ to $\vec{y}$ and check if there is an intermediate surface is intersected;
        if No intermediate surface is detected then
            Compute the unit vector $\vec{t}$ tangent to the edge at $\vec{y}$ (its orientation defines which of the two adjacent surfaces is labeled $\mathcal{I}_1$ and $\mathcal{I}_2$);
            Compute the unit vector $\vec{\omega}_s$ from $\vec{x}$ to $\vec{y}$;
            Compute the distance $r$ from $\vec{x}$ to $\vec{y}$;
            Get $S_b(\vec{y}, -\vec{\omega}_s)$ for each of the two adjacent surfaces ($S_{b,1}(\vec{y}, -\vec{\omega}_s)$ for $\mathcal{I}_1$ and $S_{b,2}(\vec{y}, -\vec{\omega}_s)$ for $\mathcal{I}_2$);
            $w_\partial \gamma I \leftarrow w_\partial \gamma I + L_{\text{edges}} \frac{p_\Omega(\vec{\omega}_s|x,\vec{\omega})}{\ell} [S_{b,1}(\vec{y}, -\vec{\omega}_s) - S_{b,2}(\vec{y}, -\vec{\omega}_s)]$;
        end
        Sample $\vec{\omega}_s$ according to $p_\Omega(\vec{\omega}_s|x,\vec{\omega})$;
        $\vec{\omega} \leftarrow \vec{\omega}_s$;
    else
        $\vec{x} \leftarrow \vec{x} + \ell_b \vec{\omega}$;
        intersection $\leftarrow$ True;
        Compute $\alpha$, $\beta$ and $\vec{u}$ for the transport direction $-\vec{\omega}$;
        Sample $\vec{\omega}_s$ according to $p_\Omega(\vec{\omega}_s|x,\vec{\omega})$;
        if $\vec{\omega}_s \cdot \vec{n} < 0$ then
            $w_\partial \gamma I \leftarrow \alpha(S_b(\vec{x}, -\vec{\omega}_s) - S_b(\vec{x}, -\vec{\omega})) + \beta \partial_1 q S_b(\vec{x}, -\vec{\omega})$;
        else
            Sample $w_I$ for $\vec{x}$ and $-\vec{\omega}_s$ using Algorithm 1;
            $w_\partial \gamma I \leftarrow w_\partial \gamma I + \alpha(w_I - S_b(\vec{x}, -\vec{\omega})) + \beta \partial_1 q S_b(\vec{x}, -\vec{\omega})$;
        end
    end
end
Algorithm 6: Sample the Monte Carlo weight $w_{\partial_{x} \varphi}$ for a closed cavity composed of adjacent plane surfaces with emissive surfaces, no reflection, uniform scattering, no volume absorption and no volume emission.

À écrire;  
Initialise $\vec{x}$ and $\vec{n}$;  
set $w_{\partial_{x} \varphi}$ to 0;  
Find the edges potentially visible from $\vec{x}$ and compute their total length $L_{\text{edges}}$;  
Uniformly sample a location $\vec{y}$ on the edges;  
Trace a ray from $\vec{x}$ to $\vec{y}$ and check if there is an intermediate surface is intersected;  
if No intermediate surface is detected then  
  Compute the unit vector $\vec{t}$ tangent to the edge at $\vec{y}$ (its orientation defines which of the two adjacent surfaces is labeled $S_{1}$ and $S_{2}$);  
  Compute the unit vector $\vec{ω}_{s}$ from $\vec{x}$ to $\vec{y}$;  
  Compute the distance $r$ from $\vec{x}$ to $\vec{y}$;  
  Get $S_{b}((\vec{y}, -\vec{ω}_{s})$ for each of the two adjacent surfaces ($S_{b,1}(\vec{y}, -\vec{ω}_{s})$ for $S_{1}$ and $S_{b,2}(\vec{y}, -\vec{ω}_{s})$ for $S_{2}$);  
  $w_{\partial_{x} \varphi} \leftarrow w_{\partial_{x} \varphi} + \frac{1}{p_{\text{edge}}} (\vec{ω} \cdot \vec{n}) [S_{b,1}(\vec{y}, -\vec{ω}_{s}) - S_{b,2}(\vec{y}, -\vec{ω}_{s})]$;  
end  
sample $\vec{ω}$ according to a Lambert distribution around $\vec{n}$ (i.e. $p(\vec{ω}) = \frac{\vec{ω} \cdot \vec{n}}{\pi}$);  
Sample $w_{\partial_{x} \varphi}$ for $\vec{x}$ and $\vec{ω}$ using Algorithm 2;  
$w_{\partial_{x} \varphi} \leftarrow w_{\partial_{x} \varphi} + \pi w_{\partial_{x} \varphi}$;  

Algorithm 7: Sample the Monte Carlo weight $w_{I,r}$ for domains of any shape, with diffuse surfaces (emission and diffuse reflection), uniform scattering, no volume absorption and no volume emission.

Initialise $\vec{x}$ and $\vec{ω}$;  
Reverse the direction : $\vec{ω} \leftarrow -\vec{ω}$;  
Set stop criterion to 1;  
Set $w_{I,r}$ to 0;  
while stop criterion $> 0.1$ do  
  Set intersection to False;  
  while intersection = False do  
    Sample a scattering free path $\ell$ according to $p(\ell) = k_{s} \exp(-k_{s}\ell)$;  
    Find the distance $\ell_{b}$ to the boundary from $\vec{x}$ in direction $\vec{ω}$;  
    if $\ell < \ell_{b}$ then  
      $\vec{x} \leftarrow \vec{x} + \ell \vec{ω}$;  
      Sample $\vec{ω}_{s}$ according to $p_{\Omega}(\vec{ω}_{s}|\vec{x}, \vec{ω})$;  
      $\vec{ω} \leftarrow \vec{ω}_{s}$;  
    else  
      $\vec{x} \leftarrow \vec{x} + \ell_{b} \vec{ω}$;  
      intersection $\leftarrow$ True;  
      $w_{I,r} \leftarrow w_{I,r} + \text{stop criterion} \times S_{b}(\vec{x}, -\vec{ω})$;  
      stop criterion $\leftarrow$ stop criterion $\times \rho$;  
      Sample $\vec{ω}$ according to $p_{\Omega,b}(-\vec{ω}|\vec{x}, -\vec{ω}_{s})$;  
  end  
end  
end
Figure 12 – Monte-Carlo estimations and analytical solutions for the intensity (left) and its spatial derivative in the direction $\hat{\gamma} = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ (right). The corresponding configuration is described in Sec. 4.1: a convex domain with differentiable boundaries (the sphere in Fig. 11) and emissive surfaces. Intensity and its derivative are estimated at optical thickness $\tau = k_s \hat{x}_{\text{obs}} \cdot \hat{e}_1$ and in the direction $\hat{\omega}_{\text{obs}} = (0, 0, 1)$. Monte-Carlo number of sampling is $N = 10^8$, the sphere diameter $D_s = 1\text{m}$.

Figure 13 – Monte-Carlo estimations and analytical solutions for the radiative flux $j$ (left) and its spatial derivative $\partial_1 \hat{\gamma} j$ in the direction $\hat{\gamma} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$ (right). The corresponding configuration is described in Sec. 4.1: a convex domain with boundary discontinuity between emissive surfaces at different temperatures (top plane of the slab geometry described in Fig. 11). The radiative flux and its derivative are estimated at the position $\hat{x}_{\text{obs}}$ such as $\hat{x}_{\text{obs}} \cdot \hat{e}_1 = \hat{x}_{\text{obs}} \cdot \hat{e}_2$ and $\hat{x}_{\text{obs}} \cdot \hat{e}_3 = 0$. Monte-Carlo number of samplings is $N = 10^9$, the top hot square dimensions are $D \times D$ with $D = 2\text{m}$. 
Figure 14 – Monte-Carlo estimations and analytical solutions for the intensity (left) and its spatial derivative in the direction \( \vec{\gamma} = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}) \) (right). The corresponding configuration is described in Sec. 4.2: a convex domain with differentiable boundaries (the sphere described in Fig. 11) and diffuse and emissive surfaces. The intensity and its spatial derivative are estimated at optical thickness \( \tau = k_s \vec{x}_{\text{obs}} \cdot \vec{e}_1 \) and in the direction \( \vec{\omega}_{\text{obs}} = (0, 0, 1) \). Monte-Carlo number of samplings is \( N = 10^8 \), the sphere diameter \( D_s = 1 \text{m} \).

Figure 15 – Monte-Carlo estimations and analytical solutions for the radiative flux \( j \) (left) and its spatial derivative \( \partial_1 \vec{\gamma} j \) in the direction \( \vec{\gamma} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0) \) (right). The configuration is described in Sec. 4.2: a convex domain with boundary discontinuities between adjacent diffuse surfaces at different temperatures (top plane of the slab geometry described in Fig. 11). The radiative flux and its derivative are estimated at the position \( \vec{x}_{\text{obs}} \) such as \( \vec{x}_{\text{obs}} \cdot \vec{e}_1 = \frac{\vec{x}_{\text{obs}} \cdot \vec{e}_2}{D} \) and \( \vec{x}_{\text{obs}} \cdot \vec{e}_3 = 0 \). Monte-Carlo number of samplings is \( N = 10^9 \), the top hot square dimensions are \( D \times D \) with \( D = 2 \text{m} \).
Figure 16 – Monte-Carlo estimations and analytical solutions for the intensity (left) and its spatial derivative in the direction $\vec{\gamma} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ (right). The corresponding configuration is described in Sec. 4.3: a convex domain with differentiable boundaries (the sphere described in Fig. 11), emissive surfaces and non-homogeneous scattering coefficient. The intensity and its spatial derivative are estimated at optical thickness $\tau = \frac{k_0}{\sigma_s} (\exp (k_1 \vec{x}_{obs} \cdot \vec{e}_1) - 1)$ and in the direction $\hat{\omega}_{obs} = (0, 0, 1)$. Monte-Carlo number of samplings is $N = 10^8$, the sphere diameter $D_s = 1$ m.
A  Projection on the surface

Omitting the index 1, we make use of the same direct orthonormal basis \((\vec{m}, \vec{t}, \vec{n})\) as for \(\mathcal{S}_1\) in Figure 8. \(\vec{r}\) is decomposed using the non-orthogonal basis \((\vec{\omega}, \vec{m}, \vec{t})\):

\[
\vec{r} = \alpha \vec{\omega} + \zeta \vec{m} + \chi \vec{t}
\]

(35)

Taking the scalar product of \(\vec{r}\) with \(\vec{n}, \vec{\omega} \wedge \vec{t}\) and \(\vec{\omega} \wedge \vec{m}\) leads to

\[
\alpha = \frac{\vec{r} \cdot \vec{n}}{\vec{\omega} \cdot \vec{n}}
\]

\[
\zeta = \frac{\vec{r} \cdot (\vec{\omega} \wedge \vec{t})}{\vec{m} \cdot (\vec{\omega} \wedge \vec{t})}
\]

(36)

\[
\chi = \frac{\vec{r} \cdot (\vec{\omega} \wedge \vec{m})}{\vec{t} \cdot (\vec{\omega} \wedge \vec{m})}
\]

Replacing \(\vec{m}\) with \(\vec{t} \wedge \vec{n}\) and using standard algebra (line 2: circulation property of triple products; line 3: development of double vectorial products; line 4: \(\vec{t} \cdot \vec{n} = 0\) and \(\vec{t} \cdot \vec{t} = 1\),

\[
\vec{m} \cdot (\vec{\omega} \wedge \vec{t}) = (\vec{t} \wedge \vec{n}) \cdot (\vec{\omega} \wedge \vec{t})
\]

\[
= ((\vec{\omega} \wedge \vec{t}) \wedge \vec{n}) \cdot \vec{t}
\]

\[
= (-\vec{t} \cdot \vec{t}) \vec{\omega} + (\vec{\omega} \cdot \vec{t}) \vec{t} \cdot \vec{n}
\]

\[
= -\vec{\omega} \cdot \vec{n}
\]

(37)

Similarly \(\vec{t} \cdot (\vec{\omega} \wedge \vec{m}) = \vec{\omega} \cdot \vec{n}\) and we get

\[
\alpha = \frac{\vec{r} \cdot \vec{n}}{\vec{\omega} \cdot \vec{n}}
\]

\[
\zeta = \frac{\vec{r} \cdot (\vec{\omega} \wedge \vec{t})}{\vec{m} \cdot (\vec{\omega} \wedge \vec{t})}
\]

(38)

\[
\chi = \frac{\vec{r} \cdot (\vec{\omega} \wedge \vec{m})}{\vec{t} \cdot (\vec{\omega} \wedge \vec{m})}
\]

By definition, \(\beta \vec{u} = \zeta \vec{m} + \chi \vec{t}\) and observing (line 1: development of double vectorial products; line 2: replacement of \(\vec{r}\) with its development; line 3: \(\alpha = \frac{\vec{r} \cdot \vec{n}}{\vec{\omega} \cdot \vec{n}}\))

\[
(\vec{r} \wedge \vec{\omega}) \wedge \vec{n} = -(\vec{\omega} \cdot \vec{n}) \vec{r} + (\vec{r} \cdot \vec{n}) \vec{\omega}
\]

\[
= -\vec{\omega} \cdot \vec{n} \left( \alpha \vec{\omega} + \zeta \vec{m} + \chi \vec{t} \right) + (\vec{r} \cdot \vec{n}) \vec{\omega}
\]

\[
= -\vec{\omega} \cdot \vec{n} \left( \zeta \vec{m} + \chi \vec{t} \right)
\]

(39)

we get

\[
\beta \vec{u} = \frac{(\vec{\omega} \wedge \vec{r}) \wedge \vec{n}}{\vec{\omega} \cdot \vec{n}}
\]

(40)

B  Lineic emission

For any location \(\vec{x} \in \mathcal{S}_1\), we note \((y, \ell)\) the coordinates of \(\vec{x}\) in a two dimension cartesian system of basis \((\vec{m}_1, \vec{t})\). Therefore \(y\) is also the distance from \(\vec{x}\) to the edge \(\mathcal{L}_{12}\). Remembering that \(\beta \vec{u} = \zeta \vec{m}_1 + \chi \vec{t}\), when intensity is discontinuous at the edge \((I_1 \text{ on } \mathcal{S}_1, I_2 \text{ on } \mathcal{S}_2)\), the term \(\beta \partial_{1,\vec{u}} I = \beta \vec{u} \cdot \vec{\omega} \cdot \vec{e}_d(I)\) in Eq. 10 induces a Dirac in the coordinate \(y\) normal to the edge:

\[
\zeta \vec{m}_1 \cdot \vec{\omega} \cdot \vec{e}_d(I) = \zeta \delta(y)(I_1 - I_2)
\]

(41)

When this Dirac is multiplied by \(\vec{\omega} \cdot \vec{n}_1\) to get a flux density, and then integrated over the surface (including the edge), writting the differential surface \(d\sigma = dyd\ell\), the integral over \(y\) vanishes to give

\[
\int_{\mathcal{S}_1} (\vec{\omega} \cdot \vec{n}_1) \zeta \delta(y)(I_1 - I_2) \ d\sigma = \int_{\mathcal{L}_{12}} (\vec{\omega} \cdot \vec{n}_1) \zeta I (I_1 - I_2) \ d\ell
\]

(42)

Reporting the expression of \(\zeta\) and using the circulation property of triple products:

\[
\int_{\mathcal{S}_1} (\vec{\omega} \cdot \vec{n}_1) \zeta \delta(y)(I_1 - I_2) \ d\sigma = -\int_{\mathcal{L}_{12}} \vec{n} \cdot (\vec{\omega} \wedge \vec{t}) \ d\ell = \int_{\mathcal{L}_{12}} (\vec{\omega} \wedge \vec{r}) \cdot \vec{t} (I_1 - I_2) d\ell
\]

(43)

The lineic emission associated to each differential length \(d\ell\) is therefore \((\vec{\omega} \wedge \vec{r}) \cdot \vec{t} (I_1 - I_2) d\ell\).
C Chadrasekhar’s exact solution for heterogeneous multiple-scattering atmospheres

In a heterogeneous, purely scattering and infinite medium, with plane parallel stratified intensity field, the radiative transfer equation has an analytical solution $I(\tau, \mu)$ ([15]):

$$I(\tau, \mu) = \frac{\eta(0)}{4\pi} + \frac{3}{4\pi} j [(g - 1) \tau + \mu]$$

(44)

with $\eta(0)$ and $j$ being constants, $g$ is the asymmetric coefficient, $\tau$ is the optical thickness normal to the plane of stratification and $\mu$ the direction cosine, $\vec{e}_3$ being the plane normal unit vector and a vector of the cartesian coordinate system $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ we state the normal optical thickness as :

$$\tau = \int_0^{\vec{x}.\vec{e}_3} k_s(l)dl$$

(45)

with $\vec{x}$ the position in the infinite medium. The cosine $\mu = \vec{\omega}.\vec{e}_3$ with $\vec{\omega}$ the transport direction. We state the analytical intensity $\mathcal{L}$ as $\mathcal{L}(\vec{x}, \vec{\omega}) = I(\tau, \mu)$.

The analytical spatial derivative $\partial_1, \gamma \mathcal{L}$ is obtain by differentiating $I(\tau(\vec{x}), \mu)$:

$$\partial_1, \gamma \mathcal{L}(\vec{x}, \vec{\omega}) = \partial_1, \gamma I(\tau(\vec{x}), \mu) = \frac{3}{4\pi} j [(g - 1) \partial_1, \gamma \tau(\vec{x})$$

(46)

with

$$\partial_1, \gamma \tau(\vec{x}) = \int_0^{\vec{x}.\vec{e}_3} \partial_1, \gamma k_s(l)dl + (\vec{\gamma}.\vec{e}_3) k_s(\vec{x}.\vec{e}_3)$$

(47)

D The slab

The black surface is at $x_3 = 0$. Its temperature is $T_{hot}$. The emissive/reflective diffuse surface is at $x_3 = c$. Its temperature is $T_{hot}$ if $x_1 \in [0, a], x_2 \in [0, a]$ and $T_{cold}$ elsewhere. Its emissivity $\epsilon$ is uniform. The observation location is at $x_1 \in [0, a], x_2 \in [0, a]$ and $x_3 = 0$ (on the black surface, facing the square). The flux density is

$$\varphi = \pi \epsilon F \tau^q(T_{hot}) + \pi (1 - \epsilon) F \tau^q(T_{cold}) + \pi (1 - F) \tau^q(T_{cold})$$

(48)

with

$$F = \frac{\mathcal{B}_1}{2\pi \sqrt{1 + \mathcal{B}_1^2}} \tan^{-1} \frac{\mathcal{B}_2}{\sqrt{1 + \mathcal{B}_1^2}} + \frac{\mathcal{B}_2}{2\pi \sqrt{1 + \mathcal{B}_2^2}} \tan^{-1} \frac{\mathcal{B}_1}{\sqrt{1 + \mathcal{B}_2^2}}$$

$$+ \frac{\mathcal{B}_1}{2\pi \sqrt{1 + (\mathcal{B}_1 - \mathcal{B}_2)^2}} \tan^{-1} \frac{\mathcal{B}_2}{\sqrt{1 + (\mathcal{B}_1 - \mathcal{B}_2)^2}} + \frac{\mathcal{B}_2}{2\pi \sqrt{1 + (\mathcal{B}_2 - \mathcal{B}_1)^2}} \tan^{-1} \frac{\mathcal{B}_1}{\sqrt{1 + (\mathcal{B}_2 - \mathcal{B}_1)^2}}$$

$$+ \frac{\mathcal{B}_1}{2\pi \sqrt{1 + (\mathcal{B}_1 + \mathcal{B}_2)^2}} \tan^{-1} \frac{\mathcal{B}_2}{\sqrt{1 + (\mathcal{B}_1 + \mathcal{B}_2)^2}} + \frac{\mathcal{B}_2}{2\pi \sqrt{1 + (\mathcal{B}_2 + \mathcal{B}_1)^2}} \tan^{-1} \frac{\mathcal{B}_1}{\sqrt{1 + (\mathcal{B}_2 + \mathcal{B}_1)^2}}$$

$$\mathcal{B}_1 = x_1/c, \mathcal{B}_2 = x_2/c$$

(49)

E Examples supplementary information

E.1 Emissive surfaces, no reflection, uniform scattering, no volume absorption, no volume emission

Convex domain with differentiable boundaries Solving the intensity using a Monte-Carlo algorithm falls down to sample a scattered radiative path in the medium until it reaches the sphere boundary. The Monte-Carlo weight is then implemented with the sphere surface emission, that is $\mathcal{L}(\vec{x}, \vec{\omega})$.

The spatial derivative $\partial_1, \gamma I$ transport equation is identical to the radiative transfer equation in this configuration. In term of Monte-Carlo algorithm it implies that the spatial derivative scattering paths sampling will
be identical to radiative paths sampling until paths reach the sphere boundary. At the boundary the spatial derivative \( \partial_{\hat{n},\hat{\omega}} I \) is derived from Eq. 14 and Eq. 32:

\[
\partial_{\hat{n},\hat{\omega}} I = -\alpha k_{s} L(\vec{x}, \vec{\omega}) - \int_{4\pi} p_{\gamma}(\vec{x}', \vec{\omega}')d\vec{\omega}' I(\vec{x}, \vec{\omega}') + \beta \partial_{\hat{n},\hat{\omega}} L(\vec{x}, \vec{\omega}) \quad \vec{x} \in \partial G; \vec{\omega} \cdot \hat{n} > 0
\]

With \( L \) and \( \partial_{\hat{n},\hat{\omega}} L \) derived in Appendix. C. At the boundary the spatial derivative is coupled with the intensity \( I(\vec{x}, \vec{\omega}') \) so that sampling a spatial derivative path comes down to sample a scattering path until it reaches the sphere boundary, sample a direction \(-\vec{\omega}'\), and sample \( \vec{x} \) from \((\vec{x}, \vec{\omega}')\).

**Boundary discontinuities** The medium is transparent so that the radiative paths between the surfaces will only be strait lines whether it be for the intensity or for its spatial gradient. For the intensity boundary conditions we refer to Eq. 4 with:

\[
S_{b} = I^{c}(T_{\text{hot}})H(\bar{x} \in \mathcal{H}_{\text{hot}}) + I^{c}(T_{\text{cold}})H(\bar{x} \notin \mathcal{H}_{\text{hot}})\bar{x} \in \partial G_{\text{top}}
\]

The flux density is solved by sampling \( w_{s} \) (see Algorithm 3) and results are compared to analytical solution in Fig. 13 and table (the table is not included in the current state of the paper).

The flux density spatial gradient is estimated by solving:

\[
\partial_{\hat{n},\hat{\omega}} \varphi = \int_{\partial G_{\text{top}}} (\vec{\omega} \cdot \hat{n}) \partial_{\hat{n},\hat{\omega}} I d\vec{\omega} = \int_{\partial G_{\text{top}}} (\vec{\omega} \cdot \hat{n}) \frac{\vec{\omega} \cdot \hat{n}_{\text{top}}}{r^{2}} (\beta \mathcal{C}_{b}[\partial_{\hat{n},\hat{\omega}} I] + S_{b,\gamma}[I])dS_{\text{top}} + \int_{\omega_{\text{top}}} (\vec{\omega} \cdot \hat{n}) \left( \frac{\gamma}{r^{2}} (I_{1} - I_{2})d\ell_{\text{top}}
\]

which is the direct application of Eq. 31 for this configuration. Here:

\[
\mathcal{C}_{b}[\partial_{\hat{n},\hat{\omega}} I] = 0 \quad \text{and} \quad S_{b,\gamma}[I] = 0
\]

The intensities \( I_{1} \) and \( I_{2} \) are stated as \( I_{1} = S_{b,1} \) and \( I_{2} = S_{b,2} \) (see Eq. 29 and 30). According to \( \vec{n}_{1} \) and \( \vec{n}_{2} \) the sources \( S_{b,1} \) and \( S_{b,2} \) can take the values of \( I^{c}(T_{\text{hot}}) \) or \( I^{c}(T_{\text{cold}}) \). In the Monte-Carlo algorithm the discontinuity sources are sampled uniformly from \( p_{L_{\text{edges}}} \) along the square edges, \( I_{1} \) and \( I_{2} \) are evaluated and the Monte-Carlo weight \( (\vec{\omega} \cdot \vec{n}_{\text{bottom}})\frac{1}{p_{L_{\text{edges}}}} \frac{\hat{n}_{\text{top}}}{r^{2}} (I_{1} - I_{2}) \) is computed.

**E.2 Emissive and reflective surfaces, uniform scattering, no volume absorption, no volume emission**

**Convex domain with differentiable boundaries** Solving numerically the intensity at \((\bar{x}_{\text{obs}}, \vec{\omega}_{\text{obs}})\) comes down to sample a radiative path that will be scattered in the medium (according to the medium scattering properties in Eq. 2) and reflected at the boundary (according to the reflection properties in Eq. 33). As we usually do in the case of reflective surfaces, the Monte-Carlo weight will account for the source accumulation of each radiative path encounter with the boundary. The results obtained for the intensity at \((\bar{x}_{\text{obs}}, \vec{\omega}_{\text{obs}})\) are presented in Fig. 14 and compared with the analytical solution \( L(\bar{x}_{\text{obs}}, \vec{\omega}_{\text{obs}})\).

The spatial derivative \( \partial_{\hat{n},\hat{\omega}} I \) is estimated at the same observation location as the intensity and the spatial derivative transport equation (Eq. 6) is identical to the intensity radiative transfer equation. For the boundary conditions we refer to Eq. 13 with \( \rho \) and \( p_{\gamma,\nu,\beta} \) being constant along the sphere surface:

\[
\mathcal{C}_{b}[\partial_{\hat{n},\hat{\omega}} I] = \rho \int_{4\pi} p_{\gamma,\nu,\beta} (-\vec{\omega}'|\vec{x}, \vec{\omega}')d\vec{\omega}' \partial_{\hat{n},\hat{\omega}} I(\vec{x}, \vec{\omega}')
\]

\[
S_{b,\gamma}[I] = \alpha \mathcal{C}[I] + \beta \partial_{\hat{n},\hat{\omega}} L(\vec{x}, \vec{\omega}) - \beta \mathcal{C}_{b}[\partial_{\hat{n},\hat{\omega}} L(\vec{x}, \vec{\omega})]
\]

with

\[
\mathcal{C}[I] = -k_{s} L(\vec{x}, \vec{\omega}) + k_{s} \int_{4\pi} p_{\gamma} (-\vec{\omega}'|\vec{x}, \vec{\omega}')I(\vec{x}, \vec{\omega}')
\]

**Boundary discontinuities** The medium is transparent so that the radiative paths between the surfaces will only be strait lines whether it be for the intensity or for its spatial gradient. For the intensity boundary conditions we refer to Eq. 4 with:

\[
S_{b} = \epsilon I^{c}(T_{\text{hot}})H(\bar{x} \in \mathcal{H}_{\text{hot}}) + \epsilon I^{c}(T_{\text{cold}})H(\bar{x} \notin \mathcal{H}_{\text{hot}})\bar{x} \in \partial G_{\text{top}}
\]
The flux density spatial gradient is estimated by solving:

\[
\partial_1 \varphi = \int_{\partial G_{\text{top}}} \langle \vec{\omega}, \vec{n} \rangle \partial_1 \tilde{I} d\tilde{\omega} = \int_{\partial G_{\text{top}}} \langle \vec{\omega}, \vec{n} \rangle \frac{\tilde{\omega} \cdot \vec{n}_{\text{top}}}{r^2} (\beta C_b [\partial_1 \tilde{I}] + S_b, \vec{\gamma} [I]) dS_{\text{top}} + \int_{\Omega_{\text{top}}} \langle \vec{\omega}, \vec{n} \rangle \frac{\tilde{\omega} \wedge \vec{\gamma}}{r^2} (I_1 - I_2) d\epsilon_{\text{top}}
\]

which is the direct application of Eq. ref for this configuration. Here:

\[
C_b [\partial_1 \tilde{I}] = \rho \int_{\partial G_{\gamma}} p_{\Omega, b} (-\tilde{\omega} | \vec{x}, -\tilde{\omega}) d\tilde{\omega}^2 \partial_1 \tilde{I}(\vec{x}, \tilde{\omega}^\prime)
\]

and

\[
S_b, \vec{\gamma} [I] = 0
\]

A realization of the Monte-Carlo weight consist on sampling a direction \(\tilde{\omega}\) from Lambert distribution, sample a path in the \(-\tilde{\omega}\) direction. If the path reaches the hot square part of the top surface then \(C_b [\partial_1 \tilde{I}]\) is evaluated by sampling a direction \(\tilde{\omega}^\prime\) from \(p_{\Omega, b}\) and sampling a path in that direction. \(\partial_1 \tilde{I}(x, \tilde{\omega}^\prime)\) depends on the intensity gradient at the bottom surface since in a transparent medium a path leaving the top surface will only reach the bottom surface. If the boundary condition at the bottom is homogeneous, as it is the case here, then \(\partial_1 \tilde{I}_{\text{bottom}}\) is null and \(C_b [\partial_1 \tilde{I}]\) is null.

The intensities \(I_1\) and \(I_2\) are stated in Eq. 29 and 30. The discontinuities sources are sampled uniformly from \(p_{\Omega, \text{edges}}\), along the square edges, \(I_1\) and \(I_2\) are evaluated and the Monte-Carlo weight \((\tilde{\omega} \cdot \vec{n}_{\text{bottom}}) p_{\Omega, \text{edges}} \frac{2\gamma S}{\Omega} (I_1 - I_2)\) is computed. The only difference with example Sec. 4.1 at that stage is the evaluation of \(I_1\) and \(I_2\). In the previous example the top boundary was a blackbody so that only \(S_{b,1}\) and \(S_{b,2}\) were used. Here the top surface is diffuse so that the incoming intensity also has to be evaluated at the edges to evaluate \(I_1\) and \(I_2\).

### E.3 Emissive surfaces, no reflection, non-uniform scattering, non-uniform volume absorption, non-uniform volume emission

**Convex domain with differentiable boundaries** The radiative transfer model in the medium \(G\) is stated by Eq. 2 with the collisional operator \(C\) containing only the scattering terms, the volumic source being \(S = 0\). The intensity is solved by sampling the Monte-Carlo weight \(w_{I,k_s}\) (the pseudo-algorithm is not present in the paper at its current state). The only difference with Algorithm 1 is in the non-uniform scattering coefficient sampling. Other than that the algorithm remain identical.

The spatial derivative is estimated at the same observation location and its boundary conditions are stated in the first example Eq. 32. The model in the medium \(G\) is stated in Eq. 7 with \(S_\gamma\) being:

\[
S_\gamma = -\partial_1 \varphi \cdot k_s (\vec{x}, \tilde{\omega}) + \partial_1 \varphi k_s \int_{4\pi} p_{\Omega, b} (-\tilde{\omega} | \vec{x}, -\tilde{\omega}) d\tilde{\omega}^2 \tilde{I}(\vec{x}, \tilde{\omega}^\prime)
\]

with \(\partial_1 \varphi k_s = (\tilde{\gamma}, \vec{e}_s) (-k_1 k_0 \exp(-k_1 x))\). The source \(S_\gamma\) is regarded as a volumic source in the spatial gradient model. To estimate the spatial gradient numerically this source will be stored along the sampled path, as any usual Monte-Carlo algorithm that estimate the intensity in presence of thermal emission. The only difference with such algorithm is that the source \(S_\gamma\) is here a function of the intensity in the medium and will be estimated by sampling \(w_I\) (Algorithm 1) in the double-randomization process. The spatial gradient is then solved by sampling \(w_{\partial_1 \varphi k_s}\) (pseudo-algorithm is not present in the paper at its current state): it starts by sampling a scattering path until it reaches the sphere boundary and then sampling \(S_\gamma\) uniformaly along the path. To evaluate \(S_\gamma\) at \((\vec{x}, \tilde{\omega})\) a direction \(\tilde{\omega}^\prime\) is sampled from the phase function and the Monte-Carlo weight \(w_I\) is sampled twice: from \((\vec{x}, \tilde{\omega}^\prime)\) and \((\vec{x}, \tilde{\omega})\); then both contributions are added as in Eq. 61 and counted in the Monte-Carlo \(w_{\partial_1 \varphi k_s}\) as well as the boundary condition weight (from Eq. 50).

**Boundary discontinuities** In the present state of the paper this configuration is not described.

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