Nous introduisons la notion de multi-motif, une abstraction combinatoire des phrases musicales à plusieurs voix. L’intérêt de cette approche réside dans le fait qu’il devient possible de composer deux multi-motifs pour en produire un plus long. Ceci s’inscrit dans un contexte algébrique puisque l’ensemble des multi-motifs possède une structure dite d’opérade ; les opérades étant des structures offrant une formalisation de la notion d’opérateur et de leurs compositions. Cette vision des phrases musicales comme des opérateurs permet de réaliser ainsi des calculs sur ces dernières et admet des applications en musique générative : étant donné un ensemble de courts motifs, nous proposons divers algorithmes pour produire de manière aléatoire une nouvelle phrase plus longue inspirée des motifs initiaux.

—

We introduce the notion of multi-pattern, a combinatorial abstraction of polyphonic musical phrases. The interest of this approach lies in the fact that this offers a way to compose two multi-patterns in order to produce a longer one. This dives musical phrases into an algebraic context since the set of multi-patterns has the structure of an operad; operads being structures offering a formalization of the notion of operators and their compositions. Seeing musical phrases as operators allows us to perform computations on phrases and admits applications in generative music: given a set of short patterns, we propose various algorithms to randomly generate a new and longer phrase inspired by the inputted patterns.

Introduction

Generative music is a subfield of computational musicology in which the focus lies on the automatic creation of musical material. This creation is based on algorithms accepting inputs to influence the result obtained, and having a randomized behavior in the sense that two executions of the algorithm with the same inputs produce different results. Several very different approaches exist. For instance, some of them use Markov chains, others genetic algorithms [10], still others neural networks [2], or even formal grammars [7,8]. The way in which such algorithms represent and manipulate musical data is crucial. Indeed, the data structures used to represent musical phrases orient the nature of the operations we can define of them. Considering operations producing new phrases from old ones is important to specify algorithms to randomly generate music. A possible way for this purpose consists to give at input some musical phrases and the algorithm creates a new one by blending them through operations. Therefore, the willingness to endow the infinite set of all musical phrases with operations in order to obtain suitable algebraic structures is a promising approach. Such interactions between music and algebra is a fruitful field of investigation [11].

In this work, we propose to use tools coming from combinatorics and algebraic combinatorics to represent musical phrases and operations on them, in order to introduce generative music algorithms close to the family of those based on formal grammars. More precisely, we introduce the music box model, a very simple model to represent polyphonic phrases, called multi-patterns. The infinite set of all these objects admits the structure of an operad. Such structures originate from algebraic topology and are used nowadays also in algebraic combinatorics and in computer science [4,11]. Roughly speaking, in these algebraic structures, the elements are operations with several inputs and the composition law is the usual composition of operators. Since the set of multi-patterns forms an operad, one can regard each pattern as an operation. The fallout of this is that each pattern is, at the same time, a musical phrase and an operation acting on musical phrases. In this way, our music box model and its associated operad provide an algebraic and combinatorial framework to perform computations on musical phrases.

All this admits direct applications to design random generation algorithms since, as introduced by the author in [5], given an operad there exist algorithms to generate some of its elements. These algorithms are based upon bud generating systems, which are general formal grammars based on colored operads [12]. In the present work, we propose three different variations of these algorithms to produce new musical phrases from old ones. More precisely, our algorithm works as follows. It takes as input a finite set of multi-patterns and an integer value to influence the size of the output. It works iteratively by choosing patterns from the initial collection in order to alter the current one by performing a composition using the operad structure. As we shall explain, the initial patterns can be colored in order to forbid some compositions and avoid in this way some musical intervals for instance. These generation algorithms are not intended to write complete

RéSUMÉ

Nous introduisons la notion de multi-motif, une abstraction combinatoire des phrases musicales à plusieurs voix. L’intérêt de cette approche réside dans le fait qu’il devient possible de composer deux multi-motifs pour en produire un plus long. Ceci s’inscrit dans un contexte algébrique puisque l’ensemble des multi-motifs possède une structure dite d’opérade ; les opérades étant des structures offrant une formalisation de la notion d’opérateur et de leurs compositions. Cette vision des phrases musicales comme des opérateurs permet de réaliser ainsi des calculs sur ces dernières et admet des applications en musique générative : étant donné un ensemble de courts motifs, nous proposons divers algorithmes pour produire de manière aléatoire une nouvelle phrase plus longue inspirée des motifs initiaux.
musical pieces; they are for obtaining, from short old patterns, a similar but longer one, presenting possibly new ideas to the human composer.

This text is organized as follows. Section 1 is devoted to setting our context and notations about music theory and to introduce the music box model. In Section 2 we begin by presenting a brief overview of operad theory and we build step by step the music box operad. For this, we introduce first an operad on sequences of scale degrees, an operad on rhythm patterns, and then an operad on monophonic patterns to end with the operad of multi-patterns. Three random generation algorithms for multi-patterns are introduced in Section 3. Finally, Section 4 provides some concrete applications of the previous algorithms. We focus here on random variations of a monophonic musical phrase as input leading to random changes of rhythm, harmonizations, and arpeggiations.

In this version of this work, most of the proofs of the announced results are omitted due to lack of space. A computer implementation of all the presented algorithms is, as well as its source code and concrete examples, available at [6].

General notations and conventions
For any integer \( n \), \( \lfloor n \rfloor \) denotes the set \( \{1, \ldots, n\} \). If \( a \) is a letter and \( n \) is a nonnegative integer, \( a^n \) is the word consisting in \( n \) occurrences of \( a \). In particular, \( a^0 \) is the empty word \( \epsilon \).

1. THE MUSIC BOX MODEL

The purpose of this section is to set some definitions and some conventions about music theory, and introduce multi-patterns that are abstractions of musical phrases.

1.1. Notes and scales
We fit into the context of an \( \eta \) tone equal temperament, also written as \( \eta \)

TET, where \( \eta \) is any nonnegative integer. An \( \eta \)-note is a pair \( (k, n) \) where \( 0 \leq k \leq \eta - 1 \) and \( n \in \mathbb{Z} \). We shall write \( k_n \) instead of \( (k, n) \). The integer \( n \) is the octave index and \( k \) is the step index of \( k_n \). The set of all \( \eta \)-notes is denoted by \( \mathcal{N}(\eta) \). Despite this level of generality, and even if all the concepts developed in the sequel work for any \( \eta \), in most applications and examples we shall consider that \( \eta = 12 \). Therefore, under this convention, we simply call note any 12-note and write \( \mathcal{N} \) for \( \mathcal{N}(12) \). We set in this context of 12-TET the “middle C” as the note \( 0_4 \), which is the first step of the octave of index 4. An \( \eta \)-scale is an integer composition \( \lambda \) of \( \eta \), that is a sequence \( (\lambda_1, \ldots, \lambda_\ell) \) of nonnegative integers satisfying \( \lambda_1 + \cdots + \lambda_\ell = \eta \). The length of \( \lambda \) is the number \( \ell(\lambda) := \ell \) of its elements. We simply call scale any 12-scale. For instance, \( (2, 2, 1, 2, 2, 1) \) is the major natural scale, \( (2, 1, 2, 1, 3, 1) \) is the harmonic minor scale, and \( (2, 1, 4, 1, 4) \) is the Hirajoshi scale. This encoding of a scale by an integer composition is also known under the terminology of interval pattern.

A rooted scale is a pair \( (\lambda, r) \) where \( \lambda \) is a scale and \( r \) is a note. This rooted scale describes a subset \( \mathcal{N}(\lambda, r) \) of \( \mathcal{N} \) consisting in the notes reachable from \( r \) by following the steps prescribed by the values \( \lambda_1, \lambda_2, \ldots, \lambda_\ell(\lambda) \) of \( \lambda \).

If \( \lambda \) is the major natural scale, then

\[
\mathcal{N}(\lambda, 0_4) = \{ \ldots, 7_3, 8_3, 0_4, 2_4, 3_4, 7_4, 8_4, 0_5, \ldots \}. \tag{1}
\]

If \( \lambda \) is the major natural scale, then

\[
\mathcal{N}(\lambda, 2_4) = \{ \ldots, 1_4, 2_4, 4_4, 6_4, 7_4, 9_4, 11_4, 1_5, 2_5, \ldots \}. \tag{2}
\]

1.2. Patterns
We now introduce degree patterns, rhythm patterns, patterns, and finally multi-patterns.

A degree \( d \) is any element of \( \mathbb{Z} \). Negative degrees are denoted by putting a bar above their absolute value. For instance, \( -3 \) is denoted by \( \bar{3} \). A degree pattern \( d \) is a finite word \( d_1 \ldots d_\ell \) of degrees. The arity \( |d| \), also denoted by \( |d| \), is the number \( \ell \) of its elements.

Given a rooted scale \( \lambda \), a degree pattern \( d \) specifies a sequence of notes by assigning to the degree 0 the note \( r \), to the degree 1 the following higher note in \( \mathcal{N}(\lambda_1, r) \) next to \( r \), to the degree 1 the lower note in \( \mathcal{N}(\lambda_1, r) \) next to \( r \), and so on. For instance, the degree pattern \( 10\bar{2}\bar{3}5\bar{0}7 \) specifies, in the context of the rooted scale \( \lambda \), where \( \lambda \) is the major natural scale, the sequence of notes

\[
2_4, 0_4, 9_3, 7_3, 0_4, 2_4, 0_5.
\]

A rhythm pattern \( r \) is a finite word \( r_1 \ldots r_\ell \) on the alphabet \( \{\Box, \blacksquare\} \). The symbol \( \Box \) is a rest and the symbol \( \blacksquare \) is a beat. The length of \( r \) is \( \ell \) and the arity \( |r| \) of \( r \) is its number of occurrences of beats. The duration sequence of a rhythm pattern \( r \) is the unique sequence \( (\alpha_0, \alpha_1, \ldots, \alpha_{|r|}) \) of nonnegative integers such that

\[
r = \Box^{\alpha_0} \blacksquare^{\alpha_1} \cdots \blacksquare^{\alpha_{|r|}}.
\]

The rhythm pattern \( r \) specifies a rhythm wherein each beat has a relative duration: the rhythm begins with a silence of \( \alpha_0 \) units of time, followed by a first beat sustained \( 1 + \alpha_1 \) units of time, and so on, and finishing by a last beat sustained \( 1 + \alpha_{|r|} \) units of time. We adopt here the convention that each rest and beat last each the same amount of time of one eighth of the duration of a whole note. Therefore, given a tempo specifying how many there are rests and beats by minute, any rhythm pattern encodes a rhythm.

For instance, let us consider the rhythm pattern

\[
r := \Box\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare.
\]

The duration sequence of \( r \) is \( (1, 0, 1, 3, 0, 1, 0) \) so that \( r \) specifies the rhythm consisting in an eighth rest, an eighth note, a quarter note, a half note, an eighth note, a quarter note, and finally an eighth note.

A pattern is a pair \( p := (d, r) \) such that \( |d| = |r| \). The arity \( |p| \) of \( p \) is the arity of both \( d \) and \( r \), and the length \( \ell(p) \) of \( d \) is the length \( \ell(r) \) of \( r \).
In order to handle concise notations, we shall write any pattern \((d, r)\) as a word \(p\) on the alphabet \([\square] \cup \mathbb{Z}\) where
the subword of \(p\) obtained by removing all occurrences of \(\square\) is the degree pattern \(d\), and the word obtained by
replacing in \(p\) each integer by \(\blacksquare\) is the rhythm pattern \(r\). For instance,
\[
1\square2\square12
\]
is the concise notation for the pattern
\[
(1212, \blacksquare\square\square\square\square\square). \tag{5}
\]
For this reason, thereafter, we shall see and treat any pattern \(p\) as a word \(p_1 \ldots p_r\) on the alphabet \([\square] \cup \mathbb{Z}\).
Remark that the length of \(p\) is \(\ell\) and that its arity is the number of letters of \(\mathbb{Z}\) it has.

Given a rooted scale \((\lambda, r)\) and a tempo, a pattern \(p := (d, r)\) specifies a musical phrase, that is a sequence of
notes arranged into a rhythm. The notes of the musical phrase are the ones specified by the degree pattern \(d\) and
their relative durations are specified by the rhythm pattern \(r\). For instance, consider the pattern
\[
p := 00\square12\square012\square100\square\square. \tag{7}
\]
By choosing the rooted scale \((\lambda, 9_3)\) where \(\lambda\) is the harmonic minor scale, and by setting 128 as tempo, one obtains the musical phrase
\[
J = 128
\]
For any positive integer \(m\), an \(m\)-multi-pattern is an \(m\)-tuple \(m := (m^{(1)}, \ldots, m^{(m)})\) of patterns such that all
\(m^{(i)}\) have the same arity and the same length. The arity \(|m|\) of \(m\) is the common arity of all the \(m^{(i)}\), and the
length \(\ell(m)\) of \(m\) is the common length of all the \(m^{(i)}\).
An \(m\)-multi-pattern \(m\) is denoted through a matrix of dimension \(m \times \ell(m)\), where the \(i\)-th row contains the pattern
\(m^{(i)}\) for any \(i \in [m]\). For instance,
\[
m := \begin{bmatrix}
0 & \square & 1 & \square & 1 \\
\square & 2 & 3 & \square & 0
\end{bmatrix} \tag{8}
\]
is a 2-multi-pattern having arity 3 and length 5. The fact that all patterns of an \(m\)-multi-pattern must have the same length
ensures that they last the same amount of units of time. This is important since an \(m\)-multi-pattern is used to handle musical sequences consisting in \(m\) stacked voices.
The condition about the arities of the patterns, and hence, about the number of degrees appearing in these, is a particularity of our model and comes from algebraic reasons.
This will be clarified later in the article.

Given a rooted scale \((\lambda, r)\) and a tempo, an \(m\)-multi-pattern \(m\) specifies a musical phrase obtained by considering
the musical phrases specified by each \(m^{(i)}\), \(i \in [m]\), each forming a voice. For instance, consider the 2-multi-pattern
\[
m := \begin{bmatrix}
0 & 4 & \square & 4 & 0 & 0 \\
7 & 7 & 0 & \square & 3 & 3
\end{bmatrix}. \tag{9}
\]
By choosing the rooted scale \((\lambda, 9_3)\) where \(\lambda\) is the minor natural scale and by setting 128 as tempo, one obtains the musical phrase
\[
J = 128
\]
Due to the fact that \(m\)-multi-patterns evoke paper tapes of a programmable music box, we call music box model
the model just described to represent musical phrases by \(m\)-multi-patterns within the context of a rooted scale and a tempo.

\section{Operad Structures}

The purpose of this section is to introduce an operad structure on multi-patterns, called music box operad. The main
interest of endowing the set of multi-patterns with the structure of an operad is that this leads to an algebraic framework
to perform computations on patterns.

\subsection{A primer on operads}

We set here the elementary notions of operad theory used in the sequel. Most of them come from \cite{4}.

A graded set is a set \(O\) decomposing as a disjoint union
\[
O := \bigcup_{n \in \mathbb{N}} O(n), \tag{10}
\]
where the \(O(n), n \in \mathbb{N}\), are sets. For any \(x \in O\), there is by definition a unique \(n \in \mathbb{N}\) such that \(x \in O(n)\) called
arity of \(x\) and denoted by \(|x|\).

A nonsymmetric operad, or an operad for short, is a triple \((O, o_1, 1)\) such that \(O\) is a graded set, \(o_i\) is a map
\[
o_i : O(n) \times O(m) \to O(n + m - 1), \quad i \in [n], \tag{11}
\]
called partial composition map, and \(1\) is a distinguished element of \(O(1)\), called unit. This data has to satisfy, for any \(x, y, z \in O\), the three relations
\[
(x o_i y) o_{i+1} z = x o_i (y o_j z), \quad i \in |x|, \quad j \in |y|, \tag{12}
\]
\[
(x o_i y) o_{j+|y|-1} z = (x o_j z) o_i y, \quad 1 \leq i < j \leq |x|, \tag{13}
\]
\[
o_1 x = x = o_1 1, \quad i \in |x|. \tag{14}
\]
Intuitively, an operad is an algebraic structure wherein each element can be seen as an operator having \(|x|\) inputs and one output. Such an operator is depicted as
\[
\begin{array}{c}
\xymatrix{\circlearrowleft_x} \\
\ar_{1} \ldots \ar_{|x|}
\end{array}
\]
where the inputs are at the bottom and the output at the top. Given two operations \(x\) and \(y\) of \(O\), the partial composition \(x o_i y\) is a new operator obtained by composing \(y\)
into $x$ onto its $i$-th input. Pictorially, this partial composition expresses as

$$
\begin{align*}
\begin{array}{c}
\text{x} \\
1 \\
\vdots \\
\vdots \\
1 \\
\cdots \\
|y| \\
\vdots \\
\vdots \\
\vdots \\
1 \\
\cdots \\
|x| \\
\end{array}
\end{align*}
\begin{array}{c}
o_i \\
\text{y} \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\cdots \\
\vdots \\
\vdots \\
\vdots \\
1 \\
\cdots \\
|y| \\
\vdots \\
\vdots \\
\vdots \\
1 \\
\cdots \\
|x| \\
\end{array}
\begin{array}{c}
\circ_i \\
= \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\cdots \\
\vdots \\
\vdots \\
\vdots \\
1 \\
\cdots \\
|y| \\
\vdots \\
\vdots \\
\vdots \\
1 \\
\cdots \\
|x| \\
\end{array}
\begin{array}{c}
1 \\
\cdots \\
|y| \\
\vdots \\
\vdots \\
\vdots \\
\cdots \\
\vdots \\
\vdots \\
\vdots \\
1 \\
\cdots \\
|x| \\
\end{array}
\end{align*}
\tag{16}
$$

Relations (12), (13), and (14) become clear when they are interpreted into this context of abstract operators and rooted trees.

Let $(O, o_2, 1)$ be an operad. The full composition map of $O$ is the map

$$
o: O(n) \times O(m_1) \times \cdots \times O(m_n) \to O(m_1 + \cdots + m_n),
\tag{17}
$$
defined, for any $x \in O(n)$ and $y_1, \ldots, y_n \in O$ by

$$
x \circ [y_1, \ldots, y_n] := (\ldots ((x \circ o_n y_n) \circ_{n-1} y_{n-1}) \ldots) \circ_1 y_1.
\tag{18}
$$

Intuitively, $x \circ [y_1, \ldots, y_n]$ is obtained by grafting simultaneously the outputs of all the $y_i$ onto the $i$-th inputs of $x$.

Let $(O', o'_1, 1')$ be a second operad. A map $\phi: O \to O'$ is an operad morphism if for any $x \in O(n)$, $o_i(x) \in O'(n)$, $\phi(1) = 1'$, and for any $x, y \in O$ and $i \in [x]$, $\phi(xo_i y) = \phi(x) o'_i \phi(y)$.

If instead holds by replacing the second occurrence of $i$ by $|x| + 1 - i$, then $\phi$ is an operad antimorphism. We say that $O'$ is a suboperad of $O$ if for any $n \in \mathbb{N}$, $O'(n)$ is a subset of $O(n)$, $1 = 1'$, and for any $x, y \in O'$ and $i \in [x]$, $x o_i y = x o'_i y$. For any subset $\mathcal{G}$ of $O$, the operad generated by $\mathcal{G}$ is the smallest suboperad $O^{\mathcal{G}}$ of $O$ containing $\mathcal{G}$.

The Hadamard product of $O$ and $O'$ is the operad $O \boxtimes O'$ defined, for any $n \in \mathbb{N}$, by $(O \boxtimes O')(n) := O(n) \times O'(n)$, endowed with the partial composition map $o_i^{\boxtimes}$ defined, for any $(x, x'), (y, y') \in O \boxtimes O'$ and $i \in [x + x']$, by

$$
(x \circ x') o_i^{\boxtimes} (y \circ y') := (x o_i y, x' o'_i y'),
\tag{20}
$$
and having $(1, 1')$ as unit.

2.2. The music box operad

We build an operad on multi-patterns step by step by introducing an operad on degree patterns and an operad on rhythm patterns. The operad of patterns is constructed as the Hadamard product of the two previous ones. Finally, the operad of multi-patterns if a suboperad of an iterated Hadamard product of the operad of patterns with itself.

Let $DP$ be the graded collection of all degree patterns, wherein for any $n \in \mathbb{N}$, $DP(n)$ is the set of all degree patterns of arity $n$. Let us define on $DP$ the partial composition $o_i$ wherein, for any degree patterns $d$ and $d'$, and any integer $i \in [|d|]$, $d o_i d' := d_1 \cdots d_{i-1} (d_i + d'_i) \cdots (d_i + d'_{|d'|}) d_{i+1} \cdots d_{|d|}$.

For instance,

$$
01234 o_2 \bar{1} = 0021234.
\tag{21}
$$

We denote by $\epsilon$ the empty degree pattern. This element is the only one of $DP(0)$.

**Proposition 2.2.1.** The triple $(DP, o_\epsilon, 0)$ is an operad.

**Proof.** This is the consequence of the fact that $(DP, o_\epsilon, 0)$ is the image of the monoid $(\mathbb{Z}, +, 0)$ by the construction $T$ defined in [1]. Since this construction associates an operad with any monoid, the result follows.

We call $DP$ the degree pattern operad.

2.2.2. The operad $DP$ admits $\{\epsilon, \bar{1}, 1, 00\}$ and $\{\epsilon, 11\}$ as minimal generating sets.

Let $RP$ be the graded collection of all rhythm patterns, wherein for any $n \in \mathbb{N}$, $RP(n)$ is the set of all rhythm patterns of arity $n$. Let us define on $RP$ the partial composition $o_i$ wherein, for any rhythm patterns $r$ and $r'$, and any integer $i \in [|r|]$, $r o_i r'$ is obtained by replacing the $i$-th occurrence of $\bar{0}$ in $r$ by $r'$. For instance,

$$\bar{0} \bar{1} \bar{0} \bar{1} o_3 \bar{0} \bar{1} = \bar{0} \bar{1} \bar{0} \bar{1} \bar{0} \bar{1}. \tag{23}$$

We denote by $\epsilon$ the empty rhythm pattern. This element is not the only one of $RP(0)$ since $RP(0) = \{\bar{0}^\alpha : \alpha \in \mathbb{N}\}$.

**Proposition 2.2.3.** The triple $(RP, o_\epsilon, \bar{0})$ is an operad.

We call $RP$ the rhythm pattern operad.

2.2.4. The operad $RP$ admits $\{\epsilon, \bar{0}, 00\}$ as minimal generating set.

Let $P$ be the operad defined as

$$P := DP \boxtimes RP. \tag{24}$$

Since a pattern is a pair $(d, r)$ where $d$ is a degree pattern and $r$ is a rhythm pattern of the same arity, for any $n \in \mathbb{N}$, $P(n)$ is in fact the set of all patterns of arity $n$. For this reason, $P$ is the graded set of all patterns. We call $P$ the pattern operad. For instance, by using the concise notation for patterns,

$$\bar{0} \bar{2} \bar{1} \bar{0} \bar{2} \bar{0} \bar{1} = \bar{0} \bar{2} \bar{1} \bar{0} \bar{1}. \tag{25}$$

We denote by $\epsilon$ the empty pattern.

**Proposition 2.2.5.** The operad $P$ admits $\{\epsilon, \bar{0}, 00\}$ and $\{\epsilon, \bar{0}, 11\}$ as minimal generating sets.

A consequence of Proposition 2.2.5 is that any pattern $p$ expresses as a tree on the internal nodes in $\{\epsilon, \bar{0}, 11\}$.
Let also the map \( \text{mir} : P_m \to P_m \) be the map such that, for any \( m \in P_m \), \( \text{mir}(m) \) is the \( m \)-multi-pattern obtained by reading the \( m \) from right to left. For instance,

\[
\text{mir} \left( \begin{array}{ccc} 1 & 2 & 2 \\ 3 & 1 & 3 \\ \end{array} \right) = \begin{array}{ccc} 2 & 1 & 1 \\ 3 & 1 & 1 \\ \end{array} .
\] (31)

**Proposition 2.2.8.** For any positive integer \( m \), the map \( \text{mir} \) sending any \( m \)-multi-pattern to its mirror is an operad anti-automorphism of \( P_m \).

Due to the \( m \)-music box operad and more precisely, to the operad structure on \( m \)-multi-patterns, we can see any \( m \)-multi-pattern as an operator. Therefore, we can build \( m \)-multi-patterns and then musical sequences by considering some compositions of small building blocks \( m \)-multi-patterns. For instance, by considering the small 2-multi-patterns

\[
m_1 := \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad m_2 := \begin{bmatrix} 1 & 0 \\ 7 & 0 \end{bmatrix}, \quad m_3 := \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \end{bmatrix},
\] (32)

one can build a new 2-multi-pattern by composing them as specified by the tree

\[
\begin{tikzpicture}
\node (m1) at (0,0) [circle,fill] {m1};
\node (m2) at (1,0) [circle,fill] {m2};
\node (m3) at (0,-1) [circle,fill] {m3};
\draw (m1) -- (m2) node[midway, above] {m1};
\draw (m1) -- (m3) node[midway, below] {m1};
\draw (m2) -- (m3) node[midway, above] {m2};
\end{tikzpicture}
\]

(33)

This produces the new 2-multi-pattern

\[
\begin{bmatrix} 1 & 1 & 0 & 1 & 2 & 3 & 1 \\ 7 & 7 & 0 & 0 & 1 & 0 & 1 \\ \end{bmatrix} .
\] (34)

Besides, by Proposition 2.2.7, the image of \( \text{mir} \) through the map, for instance, \( \phi((-1,2),3) \) is the same as the 2-multi-pattern obtained from \( (33) \) by replacing each 2-multi-pattern appearing in it by its image by \( \phi((-1,2),3) \).

### 3. GENERATION AND RANDOM GENERATION

We exploit now the music box operad to design three random generation algorithms devoted to generate new musical phrases from a finite set of multi-patterns. This relies on colored operads and bud generating systems, a sort of formal grammars introduced in [5].

#### 3.1. Colored operads and bud operads

We provide here the elementary notions about colored operads [12]. We also explain how to build colored operads from an operads.

A set of colors is any nonempty finite set \( \mathcal{C} := \{ b_1, \ldots, b_k \} \) wherein elements are called colors. A \( \mathcal{C} \)-colored set is a set \( C \) decomposing as a disjoint union

\[
C := \bigsqcup_{a \in \mathcal{C}} C(a, u),
\] (35)
where $\mathcal{C}^*$ is the set of all finite sequences of elements of $\mathcal{C}$, and the $\mathcal{C}(a, u)$ are sets. For any $x \in \mathcal{C}$, there is by definition a unique pair $(a, u) \in \mathcal{C} \times \mathcal{C}^*$ such that $x \in \mathcal{C}(a, u)$. The *arity* $|x|$ of $x$ is the length $|u|$ of $u$ as a word, the *output color* $\text{out}(x)$ of $x$ is $a$, and for any $i \in [[|x|]]$, the *i-th input color* $\text{in}_i(x)$ of $x$ is the $i$-th letter $u_i$ of $u$. We also denote, for any $n \in \mathbb{N}$, by $\mathcal{C}(n)$ the set of all elements of $\mathcal{C}$ of arity $n$. Therefore, a colored graded set is in particular a graded set.

A *$\mathcal{C}$-colored operad* is a triple $(\mathcal{C}, \circ_i, 1)$ such that $\mathcal{C}$ is a $\mathcal{C}$-colored set, $\circ_i$ is a map
\[
\circ_i : \mathcal{C}(a, u) \times \mathcal{C}(u_i, v) \to \mathcal{C}(a, u \circ_i v), \quad i \in [[|u|]],
\]
called *partial composition* map, where $u \circ_i v$ is the word on $\mathcal{C}$ obtained by replacing the $i$-th letter of $v$ by $u$, and $1$ is a map
\[
1 : \mathcal{C} \to \mathcal{C}(a, a),
\]
called *colored unit* map. This data has to satisfy Relations (12) and (13) when their left and right members are both well-defined, and, for any $x \in \mathcal{C}$, the relation
\[
1(\text{out}(x)) \circ_i x = x = x \circ_i 1(\text{in}_i(x)), \quad i \in [[|x|]].
\] (38)

Intuitively, an element $x$ of a colored operad having $a$ as output color and $u_i$ as $i$-th input color for any $i \in [[|x|]]$ can be seen as an abstract operator wherein colors are assigned to its output and to each of its inputs. Such an operator is depicted as
\[
\begin{array}{c}
\xymatrix{
\circ_i & x \ar@{.}[r] & y \\
1 \ar[u] & \vdots \ar[u] & \vdots \ar[u]
}
\end{array}
\] (39)

where the colors of the output and inputs are put on the corresponding edges. The partial composition of two elements $x$ and $y$ in a colored operad expresses pictorially as
\[
\begin{array}{c}
\xymatrix{
\circ_i & u_i \ar@{.}[r] & v_i \\
1 \ar[u] & \vdots \ar[u] & \vdots \ar[u]
}
\end{array}
\begin{array}{c}
\xymatrix{
\circ_i & x \ar@{.}[r] & y \\
1 \ar[u] & \vdots \ar[u] & \vdots \ar[u]
}
\end{array} = \begin{array}{c}
\xymatrix{
\circ_i & u_i \ar@{.}[r] & v_i \\
1 \ar[u] & \vdots \ar[u] & \vdots \ar[u]
}
\end{array}
\begin{array}{c}
\xymatrix{
\circ_i & x \ar@{.}[r] & y \\
1 \ar[u] & \vdots \ar[u] & \vdots \ar[u]
}
\end{array}
\] (40)

Besides, most of the definitions about operads recalled in Section 2 generalize straightforwardly to colored operads. In particular, one can consider the full composition map of a colored operad defined by (18) when its right member is well-defined.

Let us introduce another operation, specific to colored operads. Let $(\mathcal{C}, \circ_i, 1)$ be a colored operad. The *colored composition* map of $\mathcal{C}$ is the map
\[
\odot : \mathcal{C}(a, u) \times \mathcal{C}(b, v) \to \mathcal{C}, \quad a, b \in \mathcal{C}, \quad u, v \in \mathcal{C}^*,
\] (41)
defined, for any $x \in \mathcal{C}(a, u)$ and $y \in \mathcal{C}(b, v)$, by using the full composition map, by
\[
x \circ y := x \circ \left[ y^{(1)}, \ldots, y^{(|x|)} \right],
\] (42)
where for any $i \in [[|x|]],$
\[
y^{(i)} := \begin{cases} y & \text{if } \text{in}_i(x) = \text{out}(y), \\ 1(\text{in}_i(x)) & \text{otherwise}. \end{cases}
\] (43)

Intuitively, $x \circ y$ is obtained by grafting simultaneously the outputs of copies of $y$ into all the inputs of $x$ having the same color as the output color of $y$.

Let us describe a general construction building a colored operad from a noncolored one introduced in [5]. Given a noncolored operad $(\mathcal{O}, \circ_i, 1)$ and a set of colors $\mathcal{C}$, the *$\mathcal{C}$-bud operad* of $\mathcal{O}$ is the $\mathcal{C}$-colored operad $\mathcal{B}_\mathcal{C}(\mathcal{O})$ defined in the following way. First, $\mathcal{B}_\mathcal{C}(\mathcal{O})$ is the $\mathcal{C}$-colored set defined, for any $a \in \mathcal{C}$ and $u \in \mathcal{C}^*$, by
\[
\mathcal{B}_\mathcal{C}(\mathcal{O})(a, u) := \{(a, x, u) : x \in \mathcal{O}(|u|)\}.
\] (44)

Second, the partial composition maps $\circ_i$ of $\mathcal{B}_\mathcal{C}(\mathcal{O})$ are defined, for any $(a, x, u), (u_i, y, v) \in \mathcal{B}_\mathcal{C}(\mathcal{O})$ and $i \in [[|u|]]$, by
\[
(a, x, u) \circ_i (u_i, y, v) := (a, x \circ_i y, u_i \circ_i v)
\] (45)
where the first occurrence of $\circ_i$ in the right member of (45) is the partial composition map of $\mathcal{O}$ and the second one is a substitution of words: $u \circ_i v$ is the word obtained by replacing in $u$ the $i$-th letter of $u$ by $v$. Finally, the colored unit map $1$ of $\mathcal{B}_\mathcal{C}(\mathcal{O})$ is defined by $1(a) := (a, 1, a)$ for any $a \in \mathcal{C}$, where $1$ is the unit of $\mathcal{O}$. The *pruning* $\text{pr}((a, x, u))$ of an element $(a, x, u)$ of $\mathcal{B}_\mathcal{C}(\mathcal{O})$ is the element $x$ of $\mathcal{O}$.

Intuitively, this construction consists in forming a colored operad $\mathcal{B}_\mathcal{C}(\mathcal{O})$ out of $\mathcal{O}$ by surrounding its elements with an output color and input colors coming from $\mathcal{C}$ in all possible ways.

We apply this construction to the $m$-music box operad by setting, for any set $\mathcal{C}$ of colors,
\[
\mathcal{B}_\mathcal{C}^m := \mathcal{B}_\mathcal{C}(P_m).
\] (46)

We call $\mathcal{B}_\mathcal{C}^m$ the *$\mathcal{C}$-bud $m$-music box operad*. The elements of $\mathcal{B}_\mathcal{C}^m$ are called *$\mathcal{C}$-colored $m$-multi-patterns*. For instance, for $\mathcal{C} := \{b_1, b_2, b_3\}$,
\[
\left( b_1, \left[ \begin{array}{c} 1 \begin{array}{c} 0 \begin{array}{c} 0 \begin{array}{c} 1 \begin{array}{c} 1 \begin{array}{c} 0 \begin{array}{c} 1 \begin{array}{c} 2 \begin{array}{c} 1 \begin{array}{c} 2 \begin{array}{c} b_2 b_2 b_1 \end{array} \end{array} \end{array} \end{array} \end{array} \end{array} \end{array} \end{array} \end{array} \end{array} \end{array} \end{array} \begin{array}{c} b_3 b_2 b_3 b_2 \end{array} \right) \right)
\] (47)
is a $\mathcal{C}$-colored 2-multi-pattern. Moreover, in the colored operad $\mathcal{B}_{\mathcal{C}}^2$, one has
\[
\left( b_3, \left[ \begin{array}{c} 0 \begin{array}{c} 1 \begin{array}{c} 0 \begin{array}{c} 1 \begin{array}{c} 2 \begin{array}{c} 1 \begin{array}{c} 1 \begin{array}{c} 0 \begin{array}{c} 1 \begin{array}{c} 2 \begin{array}{c} 1 \begin{array}{c} 2 \begin{array}{c} b_2 b_2 b_1 \end{array} \end{array} \end{array} \end{array} \end{array} \end{array} \end{array} \end{array} \end{array} \end{array} \end{array} \end{array} \end{array} \begin{array}{c} b_3 b_2 b_3 b_2 \end{array} \right) \right)
\] (48)

The intuition that justifies the introduction of these colored versions of patterns and of the $m$-music box operad...
is that colors restrict the right to perform the composition of two given patterns. In this way, one can for instance forbid some intervals in the musical phrases specified by the patterns of a suboperad of $\text{BP}_m$ generated by a given set of $\mathcal{C}$-colored $m$-multi-patterns. Moreover, given a set $\mathcal{G}$ of $\mathcal{C}$-colored $m$-multi-patterns, the elements of the suboperad $\text{BP}_m^\mathcal{G}$ of $\text{BP}_m$ generated by $\mathcal{G}$ are obtained by composing elements of $\mathcal{G}$. Therefore, in some sense, these elements inherit from properties of the patterns $\mathcal{G}$.

The next section uses these ideas to propose random generation algorithms outputting new patterns from existing ones in a controlled way.

### 3.2. Bud generating systems and random generation

We describe here a sort of generating systems using operads introduced in [5]. Slight variations are considered in this present work. We also design three random generation algorithms to produce musical phrases.

A *bud generating system* [5] is a tuple $(\mathcal{O}, \mathcal{C}, \mathcal{R}, b)$ where

1. $(\mathcal{O}, c_0, 1)$ is a noncolored operad, called *ground operad*;
2. $\mathcal{C}$ is a finite set of colors;
3. $\mathcal{R}$ is a finite subset of $\mathcal{B}_c(\mathcal{O})$, called *set of rules*;
4. $b$ is a color of $\mathcal{C}$, called *initial color*.

For any color $a \in \mathcal{C}$, we shall denote by $\mathcal{R}_a$ the set of all rules of $\mathcal{R}$ having $a$ as output color.

Bud generating systems are devices similar to context-free formal grammars wherein colors play the role of nonterminal symbols. These last devices are designed to generate sets of words. Bud generating systems are designed to generate more general combinatorial objects (here, $m$-multi-patterns). More precisely, a bud generating system $(\mathcal{O}, \mathcal{C}, \mathcal{R}, b)$ allows us to build elements of $\mathcal{O}$ by following three different operating modes. We describe in the next sections the three corresponding random generation algorithms. These algorithms are in particular intended to work with $\text{BP}_m$ as ground operad in order to generate $m$-multi-patterns.

Hereafter, we shall provide some examples based upon the bud generating system

$$B := (P_2, \{b_1, b_2, b_3\}, \{c_1, c_2, c_3, c_4, c_5\}, b_1)$$  \hspace{1cm} (49)$$

where

$$\begin{align*}
c_1 &:= \left( b_1, \begin{array}{c} 0 \\ 5 \end{array}, \begin{array}{c} 0 \end{array}, \begin{array}{c} 0 \end{array}, \begin{array}{c} 0 \end{array}, \begin{array}{c} 0 \end{array}, \begin{array}{c} 4 \end{array}, b_3 b_2 b_1 b_3 \right), \\
c_2 &:= \left( b_1, \begin{array}{c} 1 \end{array}, \begin{array}{c} 4 \\ 1 \end{array}, \begin{array}{c} 0 \end{array}, \begin{array}{c} 0 \end{array}, \begin{array}{c} 0 \end{array}, \begin{array}{c} 0 \end{array}, \begin{array}{c} 1 \end{array}, b_1 \right), \\
c_3 &:= \left( b_2, \begin{array}{c} 1 \\ 1 \end{array}, \begin{array}{c} 1 \end{array}, b_1 \right), \\
c_4 &:= \left( b_3, \begin{array}{c} 0 \end{array}, \begin{array}{c} 0 \\ 0 \end{array}, \begin{array}{c} 0 \end{array}, b_1 b_1 \right), \\
c_5 &:= \left( b_3, \begin{array}{c} 0 \\ 0 \end{array}, \begin{array}{c} 0 \end{array}, b_3 \right) .
\end{align*}$$  \hspace{1cm} (50)$$

Moreover, to interpret the generated multi-patterns, we choose to consider a tempo of 128 and the rooted scale $(\lambda, 9_3)$ where $\lambda$ is the Hirajoshi scale.

Let $B := (\mathcal{O}, \mathcal{C}, \mathcal{R}, b)$ be a bud generating system. Let $\Rightarrow$ be the binary relation on $\mathcal{B}_c(\mathcal{O})$ such that

$$\begin{align*}
(a, x, u) \Rightarrow (a, y, v)
\end{align*}$$  \hspace{1cm} (53)$$

if there is a rule $r \in \mathcal{R}$ and $i \in [[u]]$ such that

$$\begin{align*}
(a, y, v) = (a, x, u) \circ_i r.
\end{align*}$$  \hspace{1cm} (54)$$

An element $x$ of $\mathcal{O}$ is partially generated by $B$ if there is an element $(b, x, u)$ such that $(b, 1, b)$ is in relation with $(b, x, u)$ w.r.t. the reflexive and transitive closure of $\Rightarrow$.

For instance, by considering the bud generating system [49], since

$$\begin{align*}
\Rightarrow &\left( \left( b_1, \begin{array}{c} 0 \\ 0 \end{array}, b_1 \right), \left( b_1, \begin{array}{c} 1 \end{array}, \begin{array}{c} 0 \end{array}, \begin{array}{c} 0 \end{array}, \begin{array}{c} 4 \\ 1 \end{array}, \begin{array}{c} 0 \end{array}, b_3 b_2 b_1 b_3 \right) \right), \\
\Rightarrow &\left( \left( b_1, \begin{array}{c} 0 \\ 4 \end{array}, \begin{array}{c} 0 \end{array}, \begin{array}{c} 0 \end{array}, \begin{array}{c} 0 \end{array}, \begin{array}{c} 1 \end{array}, b_1 b_3 b_1 b_1 b_3 \right), \left( b_1, \begin{array}{c} 1 \end{array}, \begin{array}{c} 0 \end{array}, \begin{array}{c} 2 \end{array}, \begin{array}{c} 2 \end{array}, b_3 b_2 b_1 b_3 \right) \right), \\
\Rightarrow &\left( \left( b_1, \begin{array}{c} 0 \\ 0 \end{array}, \begin{array}{c} 0 \end{array}, \begin{array}{c} 1 \end{array}, b_1 b_1 b_3 b_1 b_1 b_3 \right), \left( b_1, \begin{array}{c} 1 \end{array}, \begin{array}{c} 0 \end{array}, \begin{array}{c} 0 \end{array}, \begin{array}{c} 4 \end{array}, b_3 b_2 b_1 b_3 \right) \right),
\end{align*}$$  \hspace{1cm} (55)$$

the 2-multi-pattern

$$\begin{align*}
\left[ \begin{array}{c} 1 \end{array}, \begin{array}{c} 0 \end{array}, \begin{array}{c} 2 \end{array}, \begin{array}{c} 2 \end{array}, \begin{array}{c} 1 \end{array}, \begin{array}{c} 0 \end{array}, \begin{array}{c} 4 \end{array}, \begin{array}{c} 1 \end{array}, \begin{array}{c} 1 \end{array}, \begin{array}{c} 1 \end{array} \end{array} \right]
\end{align*}$$  \hspace{1cm} (56)$$

is partially generated by $B$.

The *partial random generation algorithm* is the algorithm defined as follows:

- **Inputs:**
  1. A bud generating system $B := (\mathcal{O}, \mathcal{C}, \mathcal{R}, b)$;
  2. An integer $k \geq 0$.

- **Output:** an element of $\mathcal{O}$.

1. Set $x$ as the element $(b, 1, b)$;
2. Repeat $k$ times:
   - (a) Pick a position $i \in [[x]]$ at random;
   - (b) If $\mathcal{R}_{\text{in}(x)} \neq 0$:
     i. Pick a rule $r \in \mathcal{R}_{\text{in}(x)}$ at random;
     ii. Set $x := x \circ_i r$;
3. Returns $\text{pr}(x)$.

This algorithm returns an element partially generated by $B$ obtained by applying at most $k$ rules to the initial element $(b, 1, b)$. The execution of the algorithm builds a composition tree of elements of $\mathcal{R}$ with at most $k$ internal nodes.

For instance, by considering the bud generating system [49], this algorithm called with $k := 5$ builds the tree of colored 2-multipatterns

$$\begin{align*}
\begin{array}{c}
| \\
\downarrow \\
| \\
\downarrow \\

\hat{c}_1
\end{array}
\begin{array}{c}
\downarrow \\
| \\
\downarrow \\
\hat{c}_4
\end{array} \\
\begin{array}{c}
\downarrow \\
\hat{c}_2 \begin{array}{c}
| \\
\downarrow \\
| \\
\downarrow \\
\hat{c}_3
\end{array}
\begin{array}{c}
\downarrow \\
\hat{c}_5
\end{array}
\end{array}
\end{align*}$$  \hspace{1cm} (57)$$
which produces the 2-multi-pattern
\[
\begin{array}{c}
0 & 1 & 2 & 1 & 3 & 2 & 1 & 5 & 0 & 4 \\
5 & 0 & 1 & 0 & 4 & 0 & 1 & 1 & 1 & 0
\end{array}
\] (58)
Together with the aforementioned interpretation, the generated musical phrase is
\[
\text{J=128}
\]
Let \( \rightarrow \) be the binary relation on \( B_c(O) \) such that
\[
(a, x, u) \rightarrow (a, y, v)
\] (59)
if there are rules \( r_1, \ldots, r_{|x|} \in R \) such that
\[
(a, y, v) = (a, x, u) \circ [r_1, \ldots, r_{|x|}].
\] (60)
An element \( x \) of \( O \) is fully generated by \( B \) if there is an element \((b, x, u)\) such that \((b, 1, b)\) is in relation with \((b, x, u)\) w.r.t. the reflexive and transitive closure of \( \circ \).
For instance, by considering the bud generating system (49), since
\[
\begin{aligned}
(b_1, [0], b_1) & \rightarrow (b_1, [0 2 1 0 0 0 0 0], b_3 b_1 b_1 b_3) \\
& \rightarrow (b_1, [0 1 2 1 0 2 1 0 0 0 0 0 0 0 0 0], b_3 b_1 b_1 b_1 b_3 b_1 b_1 b_3),
\end{aligned}
\] (61)
the 2-multi-pattern
\[
\begin{array}{c}
0 & 1 & 2 & 1 & 0 & 2 & 1 & 0 & 4 & 4 \\
5 & 0 & 1 & 0 & 1 & 5 & 0 & 0 & 0 & 0
\end{array}
\] (62)
is fully generated by \( B \).

The full random generation algorithm is the algorithm defined as follows:

- **Inputs:**
  1. A bud generating system \( B := (O, C, R, b) \);
  2. An integer \( k \geq 0 \).

- **Output:** an element of \( O \).

1. Set \( x \) as the element \((b, 1, b)\);
2. Repeat \( k \) times:
   (a) If all \( R_{\text{in}}(x_i) \), \( i \in [k] \), are nonempty:
      i. Let \( (r_1, \ldots, r_{|x|}) \) be a tuple of rules such that each \( r_i \) is picked at random in \( R_{\text{in}}(x_i) \);
      ii. Set \( x := x \circ [r_1, \ldots, r_{|x|}] \);
3. Returns \( pr(x) \).

This algorithm returns an element synchronously generated by \( B \) obtained by applying at most \( k \) rules to the initial element \((b, 1, b)\). The execution of the algorithm builds a composition tree of elements of \( R \) of height at most \( k + 1 \) wherein the leaves are all at the same distance from the root.

For instance, by considering the bud generating system (49), this algorithm called with \( k := 2 \) builds the tree of colored 2-multipatterns
\[
\begin{aligned}
&c_2 \\
&\downarrow \\
&c_3 c_1 c_2 c_1 c_3 c_3 c_2 c_1
\end{aligned}
\] (63)
which produces the 2-multi-pattern
\[
\begin{array}{c}
2 & 4 & 3 & 2 & 6 & 2 & 1 & 0 & 1 & 2 & 0 & 1 & 2 & 1 & 0 & 4 \\
5 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 4 & 0 & 0 & 1 & 2 & 1 & 2 & 1
\end{array}
\] (64)
Together with the aforementioned interpretation, the generated musical phrase is
\[
\text{J=128}
\]
Let \( \circ \) be the binary relation on \( B_c(O) \) such that
\[
(a, x, u) \circ (a, y, v)
\] (65)
if there is a rule \( r \in R \) such that
\[
(a, y, v) = (a, x, u) \circ r.
\] (66)
An element \( x \) of \( O \) is colorfully generated by \( B \) if there is an element \((b, x, u)\) such that \((b, 1, b)\) is in relation with \((b, x, u)\) w.r.t. the reflexive and transitive closure of \( \circ \).
For instance, by considering the bud generating system (49), since
\[
\begin{aligned}
(b_1, [0], b_1) & \circ (b_1, [0 2 1 0 0 0 0 0], b_3 b_2 b_1 b_3) \\
& \circ (b_1, [0 2 2 1 1 0 0 0 0 0 0 0 0 0 0 0], b_3 b_2 b_1 b_3 b_1 b_3),
\end{aligned}
\] (67)
the 2-multi-pattern
\[
\begin{array}{c}
0 & 1 & 2 & 1 & 1 & 1 & 0 & 4 \\
5 & 0 & 1 & 0 & 1 & 0 & 1 & 0
\end{array}
\] (68)
is colorfully generated by \( B \).

The colored random generation algorithm is the algorithm defined as follows:

- **Inputs:**
  1. A bud generating system \( B := (O, C, R, b) \);
  2. An integer \( k \geq 0 \).
• Output: an element of $O$.

1. Set $x$ as the element $(b, 1, b)$;

2. Repeat $k$ times:
   (a) Pick a rule $r \in R$ at random;
   (b) Set $x := x \circ r$;

3. Returns $pr(x)$.

This algorithm returns an element colorfully generated by $B$ obtained by applying at most $k$ rules to the initial element $(b, 1, b)$. The execution of the algorithm builds a composition tree of elements of height at most $k + 1$.

For instance, by considering the bud generating system $\{9\}$, this algorithm called with $k := 3$ builds the tree of colored 2-multipatterns

\[
\begin{align*}
&c_3 \quad c_3 \quad c_2 \quad c_1 \quad c_1 \quad c_1 \quad c_1 \\
&c_3 \quad c_3 \quad c_3 \quad c_3 \quad c_3 \quad c_3 \quad c_3
\end{align*}
\]

which produces the 2-multi-pattern

\[
\begin{bmatrix}
0 & 1 & 0 & 1 & 2 & 2 & 1 & 5 & 0 & 1 & 0 & 1 & 0 & 0 & 4 & 4 \\
5 & 0 & 0 & 1 & 5 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]

Together with the aforementioned interpretation, the generated musical phrase is

\[
J = 128
\]

4. APPLICATIONS: EXPLORING VARIATIONS OF PATTERNS

We construct here some particular bud generated systems devoted to work with the algorithms introduced in Section 3.2. They generate variations of a single 1-multipattern $p$ given at input, with possibly some auxiliary data. Each performs a precise musical transformation of $p$.

4.1. Random temporizations

Given a pattern $p$ and an integer $t \geq 1$, we define the temporizer bud generating system $B_{p,t}^{\text{temp}}$ of $p$ and $t$ by

\[
B_{p,t}^{\text{temp}} := (P_t, \mathcal{C}, \{c_1, c_2, c_3, \ldots, c_j\}, b_1)
\]

where $\mathcal{C}$ is the set of colors $\{b_1, b_2, b_3\}$ and $c_1, c_2, c_3, \ldots, c_j$ are the $\mathcal{C}$-colored 1-multi-patterns

\[
\begin{align*}
&c_1 := (b_1, p, b_2^{[p]}) \quad , \quad c_2 := (b_2, p, b_2^{[p]}) \quad , \\
&c_j := (b_2, [0 \; \square], b_3) \quad , \quad j \in [t]
\end{align*}
\]

The temporizator bud generating system of $p$ and $t$ generates a version of the pattern $p$ composed with itself where the durations of some beats have been increased by at most $t$. The colors, and in particular the color $b_3$, prevent multiple compositions of the colored patterns $c_j$, $j \in [t]$, in order to not overly increase the duration of some beats.

For instance, by considering the pattern $p := 02\square\square\square04$ and the parameter $t := 2$, the partial random generation algorithm ran with the bud generating system $B_{p,t}^{\text{temp}}$ and $k := 16$ as inputs produces the pattern

\[
02\square\square\square04 \quad 02\square\square\square04 \quad 02\square\square\square04. \quad (73)
\]

Together with the interpretation consisting in a tempo of 128 and the rooted scale ($\lambda, \lambda_3$) where $\lambda$ is the Hirajoshi scale, the generated musical phrase is

\[
\begin{align*}
\hat{\text{J}} &= 128
\end{align*}
\]

4.2. Random rhythmic variations

Given a pattern $p$ and a rhythm pattern $r$, we define the rhythmic bud generating system $B_{p,r}^{\text{rhy}}$ of $p$ and $r$ by

\[
B_{p,r}^{\text{rhy}} := (P_r, \mathcal{C}, \{c_1, c_2, c_3\}, b_1)
\]

where $\mathcal{C}$ is the set of colors $\{b_1, b_2, b_3\}$ and $c_1, c_2, c_3$ are the three $\mathcal{C}$-colored 1-multi-patterns

\[
\begin{align*}
&c_1 := (b_1, p, b_2^{[p]}) \quad , \quad c_2 := (b_2, p, b_2^{[p]}) \quad , \\
&c_3 := (b_2, r', b_3^{[r']})
\end{align*}
\]

where $r'$ is the pattern ($0 \; [r'], r$). The rhythmic bud generating system of $p$ and $r$ generates a version of the pattern $p$ composed with itself where some beats are repeated accordingly to the rhythm pattern $r$. The colors, and in particular the color $b_3$, prevent multiple compositions of the colored pattern $c_3$. Observe that when $r = c$, each composition involving $c_3$ deletes a beat in the generated pattern.

For instance, by considering the pattern $p := 1\square011\square2$ and the rhythm pattern $r := \square\square\square\square\square\square$, the partial random generation algorithm ran with the bud generating system $B_{p,r}^{\text{rhy}}$ and $k := 8$ as inputs produces the pattern

\[
22\square\square\square122\square\square\square101\square22\square\square\square22
\]

Together with the interpretation consisting in a tempo of 128 and the rooted scale ($\lambda, \lambda_3$) where $\lambda$ is the minor natural scale, the generated musical phrase is

\[
\hat{\text{J}} = 128
\]

4.3. Random harmonizations

For any pattern $p$ and an integer $m \geq 1$, we denote by $[p]_m$ the $m$-multi-pattern $([p]_1^{[m]} \ldots , [p]_m^{[m]})$ satisfying $[p]_{i}^{[i]} = p$ for all $i \in [m]$.

Given a pattern $p$ and a degree pattern $d$ of arity $m \geq 1$, we define the harmonizator bud generating system $B_{p,d}^{\text{har}}$ of $p$ and $d$ by

\[
B_{p,d}^{\text{har}} := (P_m, \mathcal{C}, \{c_1, c_2, c_3\}, b_1)
\]
where \( \mathcal{C} \) is the set of colors \( \{b_1, b_2, b_3\} \) and \( e_1, e_2, \) and \( e_3 \) are the three \( \mathcal{C} \)-colored \( m \)-multi-patterns

\[
e_1 := (b_1, [p]^m, b_2^m), \quad e_2 := (b_2, [p]^m, b_2^m), \quad (87a)
\]

\[
e_3 := \left( b_2, \begin{bmatrix} d_1 & \cdots & \square \cdots & \square \cdots & \square \cdots & \square \cdots & \square \cdots \end{bmatrix}, b_3 \right). \quad (87b)
\]

The harmonizator bud generating system of \( p \) and \( d \) generates an arpeggiated version of the pattern \( p \) composed with itself, with chords controlled by \( d \). The colors, and in particular the color \( b_1 \), prevent multiple compositions of the colored pattern \( e_3 \).

For instance, by considering the pattern \( p := 210210\square0\square \) and the degree pattern \( d := 057 \), the partial random generation algorithm ran with the bud generating system \( \mathcal{B}_{p,d}^{arp} \) and \( k := 3 \) as inputs produces the 3-multi-pattern

\[
\begin{bmatrix}
2 & 1 & 0 & 2 & 0 & 1 & 0 & \square \\
2 & 6 & 5 & 2 & 0 & 1 & 0 & 0 \\
2 & 6 & 7 & 2 & 0 & 1 & 0 & 0
\end{bmatrix}.
\]

Together with the interpretation consisting in a tempo of 128 and the rooted scale \( (\lambda, b_3) \) where \( \lambda \) is the minor natural scale, the generated musical phrase is

\[
J_{\text{-128}}
\]

### 4.4. Random arpeggations

Given a pattern \( p \) and a degree pattern \( d \) of arity \( m \geq 1 \), we define the arpeggiator bud generating system \( \mathcal{B}_{p,d}^{arp} \) of \( p \) and \( d \) by

\[
\mathcal{B}_{p,d}^{arp} := (p, \mathcal{C}, \{e_1, e_2, e_3\}, b_1) \quad (88)
\]

where \( \mathcal{C} \) is the set of colors \( \{b_1, b_2, b_3\} \) and \( c_1, c_2, \) and \( c_3 \) are the three \( \mathcal{C} \)-colored \( m \)-multipatterns

\[
c_1 := (b_1, [p]^m, b_2^m), \quad c_2 := (b_2, [p]^m, b_2^m), \quad (89a)
\]

\[
c_3 := \left( b_2, \begin{bmatrix} d_1 & \square & \cdots & \square \cdots & \square \cdots & \square \cdots & \square \cdots \end{bmatrix}, b_3 \right). \quad (89b)
\]

The arpeggiator bud generating system of \( p \) and \( d \) generates an arpeggiated version of the pattern \( p \) composed with itself, where the arpeggio is controlled by \( d \). The colors, and in particular the color \( b_1 \), prevent multiple compositions of the colored pattern \( e_3 \). Observe in particular that when \( d = 0^m \), each composition involving \( e_3 \) creates a repetition of a same beat over the \( m \) stacked voices.

For instance, by considering the pattern \( p := 0213\square1 \) and the degree pattern \( d := 024 \), the partial random generation algorithm ran with the bud generating system \( \mathcal{B}_{p,d}^{arp} \) and \( k := 8 \) as inputs produces the 3-multi-pattern

\[
\begin{bmatrix}
0 & 0 & 2 & 1 & 0 & 3 & 3 & 0 & 1 & 2 & 0 & 0 & 3 & 3 & 0 & 1 \\
0 & 2 & 0 & 3 & 3 & 0 & 1 & 2 & 0 & 0 & 3 & 3 & 0 & 1 \\
0 & 2 & 0 & 3 & 0 & 5 & 3 & 0 & 1 & 2 & 0 & 0 & 3 & 3 & 0 & 1
\end{bmatrix}.
\]

Together with the interpretation consisting in a tempo of 128 and the rooted scale \( (\lambda, b_3) \) where \( \lambda \) is the major natural scale, the generated musical phrase is

\[
J_{\text{-128}}
\]

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