Abstract

A joint mix is a random vector with a constant component-wise sum. It is known to represent the minimizing dependence structure of some common objectives, and it is usually regarded as a concept of extremal negative dependence. In this paper, we explore the connection between the joint mix structure and one of the most popular notions of negative dependence in statistics, called negative orthant dependence (NOD). We show that a joint mix does not always have NOD, but some natural classes of joint mixes have. In particular, the Gaussian class is characterized as the only elliptical class which supports NOD joint mixes of arbitrary dimension. For Gaussian margins, we also derive a necessary and sufficient condition for the existence of an NOD joint mix. Finally, for identical marginal distributions, we show that an NOD Gaussian joint mix solves a multi-marginal optimal transport problem under uncertainty on the number of components. Analysis of this optimal transport problem with heterogeneous marginals reveals a trade-off between NOD and the joint mix structure.

Keywords: Complete mixability; concordance order; optimal transport; extreme dependence; negative dependence.

1 Introduction

Dependence among multiple sources of randomness has always been an active topic in operations research, statistics, transport theory, economics, and finance; see Denuit et al. (2005), Joe (2014), Rüschendorf (2013), McNeil et al. (2015) and Galichon (2016) for standard textbook treatment in different fields, and the recent work Blanchet et al. (2020) for relevant examples in operations research.

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In the literature, much attention has been paid to positive dependence. In particular, comonotonicity, the commonly accepted notion of extremal positive dependence, has been an active topic of research over the past few decades, giving rise to many applications in decision theory (Yaari, 1987; Schmeidler, 1989), game theory (Carlier et al., 2012), quantitative finance (Kusuoka, 2001), economic development (Kremer, 1993), and actuarial science (De neberg, 1994; Dhaene et al., 2002, 2006). Other notions of positive dependence are also useful and extensively studied in statistics (e.g., Lehmann, 1966; Benjamini and Yekutieli, 2001). In contrast to positive dependence, considerably less studies are found on negative dependence, partially due to its more complicated mathematical nature. In particular, it is well known that the natural counter-part of comonotonicity, called counter-monotonicity, is only well-defined in dimension 2. For a review and historical account on extremal positive and negative dependence concepts, we refer to Puccetti and Wang (2015).

In the past decade, the notion of joint mixability proposed by Wang et al. (2013), which generalizes complete mixability (Wang and Wang, 2011), has been shown useful for solving many optimization problems involving the dependence of multiple risks. In particular, joint mixability is essential to worst-case bounds on Value-at-Risk and other risk measures under dependence uncertainty (Puccetti and Rüschendorf, 2013; Embrechts et al., 2013; Bernard et al., 2014), as well as bottleneck assignment and scheduling problems (Coffman and Yannakakis, 1984; Hsu, 1984; Haus, 2015; Bernard et al., 2018).

Joint mixability concerns, for given marginal distributions, the existence of a random vector which has a constant component-wise sum. Such a random vector is called a joint mix supported by the given marginal distributions, and it represents a very simple concept of dependence. A joint mix is commonly regarded as a notion of extremal negative dependence; see the review of Puccetti and Wang (2015). The reason why a joint mix represents negative dependence is that it minimizes many objectives which are maximized by comonotonicity. For instance, for fixed marginal distributions of the risks, comonotonicity maximizes the variance, the stop-loss premium, and the Expected Shortfall (ES) of the sum of the risks, whereas a joint mix, if it exists, minimizes these quantities; see e.g., Rüschendorf (2013). As such, joint mixability is seen as the safest dependence structure, as long as risk aggregation is concerned (Embrechts et al., 2014).

Although a joint mix has been treated as a concept of negative dependence, it remains unclear whether it is consistent with classic notions of negative dependence in statistics. The most popular notion of negative dependence is the negative orthant dependence proposed by...
Block et al. (1982) which is further based on Lehmann (1966). The connection between joint mixes and negative orthant dependence (we will omit “orthant” below) is the main object that we address in this paper. We will answer four main questions on this matter.

1. Is a joint mix always negatively dependent? We will illustrate by examples that the answer is negative for dimension larger than 2, even if we further require that the joint mix is exchangeable. In addition, some completely mixable distributions may not support any negatively dependent joint mixes. Nevertheless, we are able to show that, a joint mix is never positively dependent, although it is not necessarily negatively dependent. Therefore, the joint mix can be seen as a notion of negative dependence, although in a sense different from that of Block et al. (1982). This is the topic of Section 3.

2. Which joint mixes are negatively dependent? Since not all joint mixes are negatively dependent as discussed in the first question, one needs to understand which ones are. We will show that there are some natural classes of joint mixes that are indeed negatively dependent. Moreover, any joint mixes with tail dependence cannot be negatively dependent, and among all elliptical classes, only the Gaussian class supports negatively dependent joint mixes of any dimension, leading to a new characterization of Gaussian distributions among elliptical ones. This is the topic of Sections 4 and 5.

3. Which marginal distributions support a negatively dependent joint mix? As discussed in the first question, not all marginal distributions, assumed jointly mixable, support a negatively dependent joint mix. However, for commonly used jointly mixable distributions in risk management and statistics, we show that there exists a way to construct negatively dependent joint mixes supported by these marginal distributions. In particular, we derive a necessary and sufficient condition for a tuple of Gaussian distributions to support a negatively dependent joint mix. This is the topic of Section 6.

4. Are there any special properties of joint mixes which are negatively dependent, useful in applications? In the context of a multi-marginal optimal transport problem, we establish the optimality of a special class of negatively dependent Gaussian joint mixes under a novel setting of uncertainty. This optimizer is further shown to be unique among Gaussian vectors, and the results can be extended to other marginal distributions, showing an interesting interplay between joint mixes and negative orthant dependence. This is the topic of Section 7.
Other notions of extremal negative dependence than a joint mix exist, and they all require some conditions. A generalization of counter-monotonicity to higher dimension is called mutual exclusivity in the actuarial literature; see Dhaene et al. (1999) and Cheung and Lo (2014). Mutual exclusivity is the safest dependence structure for more risk management problems than the joint mix, but with this, its existence requires very restrictive conditions on the marginal distributions, in particular ruling out any continuous distributions, making it incompatible with studies on copulas. Another generalization of counter-monotonicity is studied by Lee and Ahn (2014) which are joint mixes after marginal transforms.

The study of joint mixability was originally motivated by questions in risk management and operations research, and it has a strong connection to the theory of multi-marginal optimal transport (Santambrogio, 2015) and variance reduction in random sampling (Craiu and Meng, 2001, 2005); see also our Section 7. Recently, there is a growing spectrum of applications of joint mixability outside the above fields, including multiple hypothesis testing (Vovk et al., 2022), wireless communications (Besser and Jorswieck, 2020), labor market matching (Boerma et al., 2021), and resource allocation games (Perchet et al., 2022). Results in this paper connect the two topics of joint mixability and negative dependence, allowing us to bring tools from one area to the other.

2 Preliminaries

We first recall the classic notions of negative dependence proposed by Block et al. (1982). The random vector $X = (X_1, \ldots, X_n)$ or its distribution is said to be negatively upper orthant dependent (NUOD) if for every $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$,

$$\mathbb{P}(X > x) \leq \prod_{i=1}^{n} \mathbb{P}(X_i > x_i).$$

It is said to be negatively lower orthant dependent (NLOD) if for every $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$,

$$\mathbb{P}(X \leq x) \leq \prod_{i=1}^{n} \mathbb{P}(X_i \leq x_i).$$

The word “negative” would be “positive” if the inequalities are reversed, as originally studied by Lehmann (1966); this leads to the terms PLOD and PUOD. If a random vector is both NLOD and NUOD, we say that it is negatively orthant dependent (NOD). These abbreviations are also used as nouns to represent orthant dependence.

Let us take NLOD as the primary object. Clearly, NLOD of $X$ implies, for each $i < j$ and
For two $n$-dimensional random vectors $X$ and $Y$, $X$ is said to be less than $Y$ in

(i) lower concordance order (denoted by $X \leq_{cL} Y$) if $P(X \leq t) \leq P(Y \leq t)$ for all $t \in \mathbb{R}^n$,

(ii) upper concordance order (denoted by $X \leq_{cU} Y$) if $P(X > t) \leq P(Y > t)$ for all $t \in \mathbb{R}^n$,

(iii) concordance order (denoted by $X \leq_c Y$) if $X \leq_{cL} Y$ and $X \leq_{cU} Y$, and

(iv) supermodular order (denoted by $X \leq_{sm} Y$) if $E[\psi(X)] \leq E[\psi(Y)]$ for all supermodular functions $\psi$ such that the expectations exist.

Here a function $\psi : \mathbb{R}^n \to \mathbb{R}$ is called supermodular if $\psi(x \land y) + \psi(x \lor y) \geq f(x) + \psi(y)$, where $x \land y = (\min(x_1, y_1), \ldots, \min(x_n, y_n))$ and $x \lor y = (\max(x_1, y_1), \ldots, \max(x_n, y_n))$.

For an $n$-dimensional random vector $X$, denote by $X^\perp$ a random vector with the same marginals as $X$ but with independent components. We say that $X$ is negatively supermodular dependent (NSD) if $X \leq_{sm} X^\perp$. Moreover, negative upper orthant dependence, negative lower orthant dependence, and negative orthant dependence of $X$ are equivalently written as $X \leq_{cU} X^\perp$, $X \leq_{cL} X^\perp$ and $X \leq_c X^\perp$, respectively. All of these relations are weaker than $X \leq_{sm} X^\perp$ since the functions $x \mapsto 1_{\{x \leq t\}}$ and $x \mapsto 1_{\{x > t\}}$ are supermodular for all $t \in \mathbb{R}^n$.

Next, we define joint mixes and joint mixability. A joint mix (Wang and Wang, 2016) is a random vector $X = (X_1, \ldots, X_n)$ satisfying $X_1 + \cdots + X_n = c$ almost surely (a.s.) for some constant $c$, called a center of the joint mix. Equivalently, $1_n^\top X$ is a constant, where $1_n$ is the $n$-vector with all components being 1; the vector $0_n$ is defined analogously. Both the random vector and its dependence structure will be referred to as the joint mix, and no confusion should arise. An $n$-tuple $(F_1, \ldots, F_n)$ of distributions on $\mathbb{R}$ is called jointly mixable if there exists a joint mix with these marginal distributions. A distribution $F$ is called $n$-completely mixable if the $n$-tuple $(F, \ldots, F)$ is jointly mixable.

Let $\Pi_n$ be the set of $n$-permutations. A random vector $X = (X_1, \ldots, X_n)$ is exchangeable if $X \overset{d}{=} X^\pi$ for all $\pi \in \Pi_n$, where $X^\pi = (X_{\pi(1)}, \ldots, X_{\pi(n)})$. Many examples of joint mixes in the literature are indeed exchangeable. Throughout, denote by $[n] = \{1, \ldots, n\}$.

As mentioned in the Introduction, it is often mentioned in the literature that a joint mix has a negative dependence structure, for the reason that it minimizes a large class of objectives.
which are maximized by comonotonicity. The rest of the paper is to formally explore the relationship between the joint mix and NLOD or NUOD.

3 Motivating examples and observations

We start from the simplest case. Our first observation is that a joint mix in dimension \( n = 2 \) is always NLOD and NUOD, as joint mix is counter-monotonic in case \( n = 2 \). In what follows, we focus on the case \( n \geq 3 \).

The next observation, drawn from the following examples, is that a joint mix is not necessarily NLOD or NUOD in general, even if some restrictions are put on the marginal distributions.

Example 1. Since NLOD requires \( \text{Cov}(X_i, X_j) \leq 0 \), generally a joint mix may not satisfy this. For instance, take \( X = (U, U, -2U) \) where \( U \) is a standard uniform random variable. Clearly, \( \text{Cov}(X_1, X_2) = 1 \) although \( X_1 + X_2 + X_3 = 0 \). Thus, we cannot say that a joint mix is NLOD, unless \( n = 2 \). Moreover, it is easy to see that for the marginal distributions in this example, the distribution of \( (U, U, -2U) \) is the only possible distribution of a joint mix, and thus no NLOD vector can be found for a joint mix with these marginal distributions.

Example 2. This example shows that NLOD cannot be expected even if the joint mix is exchangeable. Take \( p \in (0, 1/3) \). Let \( X = (X_1, X_2, X_3) \) be constructed by \( \mathbb{P}(X = (0, 0, 0)) = 1 - 3p \), and

\[
\mathbb{P}(X = (-1, -1, 2)) = \mathbb{P}(X = (-1, 2, -1)) = \mathbb{P}(X = (2, -1, -1)) = p.
\]

Clearly, \( X \) is exchangeable and its marginal distributions are identical and supported on \( \{-1, 0, 2\} \).

1. Take \( x = (x_1, x_2, x_3) = (-1, -1, 2) \). Note that \( \mathbb{P}(X \leq x) = p \) and \( \prod_{i=1}^3 \mathbb{P}(X_i \leq x_i) = 4p^2 \).

Since \( 4p^2 < p \) if and only if \( p < 1/4 \), \( X \) is not NLOD if \( p < 1/4 \).

2. Take \( x = (x_1, x_2, x_3) = (0, 0, 0) \). Note that \( \mathbb{P}(X > x) = 1 - 3p \) and \( \prod_{i=1}^3 \mathbb{P}(X_i > x_i) = (1 - 2p)^3 \). Clearly, \( X \) is not NUOD for some \( p \), such as \( p = 1/5 \).

Moreover, \( X \) has the only possible distribution for a joint mix with the given marginal distributions. Therefore, there exists a completely mixable tuple of distributions such that none of its joint mixes are NLOD or NUOD.

Remark 1. Lee and Ahn (2014) defined a notion of \( n \)-dimensional counter-monotonicity for a random vector \((X_1, \ldots, X_n)\), which means that there exists strictly increasing continuous functions \( f_1, \ldots, f_n \) such that \( \mathbb{P}(\sum_{i=1}^n f_i(X_i) = 1) = 1 \) that is, \((f_1(X_1), \ldots, f_n(X_n))\) is a joint
mix. Obviously, this property is weaker than being a joint mix. Hence, the above examples also illustrate that \( n \)-dimensional counter-monotonicity does not imply NLOD or NUOD unless \( n = 2 \).

We have seen from the above examples that a joint mix is not always NOD. Therefore, a joint mix does not always satisfy some concepts of negative dependence stronger than NOD, such as NSD and negative association (NA); see Amini et al. (2013) for relationships among various concepts of negative dependence. Nevertheless, we show below that a non-degenerate joint mix cannot be PLOD or PUOD. Hence, although a joint mix may not always fit the definition of negative dependence, it never fits the definition of a positive dependence; this reassures that a joint mix may be referred to as negative dependence (in a different sense than NOD) in the literature.

To show this statement, we first note that a joint mix \( \mathbf{X} \) cannot be PLOD if its components have finite second moment, since PLOD of \( \mathbf{X} \) implies \( \text{Cov}(X_i, X_j) \geq 0 \) for all \( i, j \), and thus \( \text{Var}(X_1 + \cdots + X_n) > 0 \). In case second moment is not assumed, we need a more detailed analysis.

**Proposition 1.** A non-degenerate joint mix cannot be PLOD or PUOD.

**Proof.** Let \( c \in \mathbb{R} \) be the center of \( \mathbf{X} \) and \( x_1, \ldots, x_n \) be the essential infimums of \( X_1, \ldots, X_n \), respectively. If \( x_1 + \cdots + x_n \geq c \), then \( \mathbb{P}(X_i = x_i) = 1 \) for each \( i \in [n] \), and thus \( \mathbf{X} \) is degenerate. Hence, \( x_1 + \cdots + x_n < c \), and there exists \( \varepsilon > 0 \) such that \( x_1 + \cdots + x_n + n\varepsilon < c \). We can calculate

\[
\begin{align*}
\mathbb{P}(X_1 \leq x_1 + \varepsilon, \ldots, X_n \leq x_n + \varepsilon) &\leq \mathbb{P}(X_1 + \cdots + X_n \leq x_1 + \cdots + x_n + n\varepsilon) \\
&\leq \mathbb{P}(X_1 + \cdots + X_n < c) = 0,
\end{align*}
\]

and

\[
\prod_{i=1}^{n} \mathbb{P}(X_i \leq x_i + \varepsilon) > 0;
\]

thus \( \mathbf{X} \) cannot be PLOD. The case of PUOD is symmetric. \( \square \)

### 4 Properties of NOD joint mixes

In this section, we study some useful properties of NOD joint mixes. Recall that NOD of a random vector is defined through a concordance relation with another independent random vector. Supermodular order and concordance order are closed with respect to various operations as
shown in Theorem 3.8.7 of Muller and Stoyan (2002). Since joint mixability is also closed with respect to some of these operations, NOD joint mixes enjoy some closedness properties summarized in the following proposition. These properties can be checked using standard techniques, and a proof is put in Appendix A.

**Proposition 2.** The following properties hold.

(i) (Location-scale transforms) Let \( X \) be an NOD \( n \)-joint mix. Then so is \( a + bX \) for any \( a \in \mathbb{R}^n \) and \( b > 0 \).

(ii) (Permutation) Let \( X \) be an NOD \( n \)-joint mix. Then \( X^\pi \) is an NOD \( n \)-joint mix for any permutation \( \pi \in \Pi_n \).

(iii) (Concatenation) Let \( X \) and \( Y \) be independent NOD joint mixes. Then \( (X, Y) \) is an NOD joint mix.

(iv) (Convolution) Let \( X \) and \( Y \) be independent NOD \( n \)-joint mixes. Then \( X + Y \) is an NOD \( n \)-joint mix.

(v) (Weak convergence) Let \( X^{(m)}, m \in \mathbb{N}, \) be a sequence of NOD \( n \)-joint mixes converging weakly to a random vector \( X \). Then \( X \) is an NOD \( n \)-joint mix.

The properties (i)–(v) hold true if NOD is replaced by NUOD or NLOD.

**Remark 2.** All statements of Proposition 2 hold true if all joint mixes are replaced by random vectors, following the same proof. Proposition 2 is stated for joint mixes as they are the main object of our paper.

For any given marginal distribution which is \( n \)-completely mixable, there exists an exchangeable joint mix with marginals \( F \) (e.g., Proposition 2.1 of Puccetti et al., 2019). Such an exchangeable joint mix can be easily obtained from any joint mix by randomly permuting all the indices. More specifically, if \( X \) is any joint mix, then \( X^\pi \) is an exchangeable joint mix, where \( \pi \) follows a uniform distribution on \( \Pi_n \) and is independent of \( X \). Obviously, if \( X \) has identical marginal distributions, then \( X^\pi \) has the same marginal distributions as \( X \). The next proposition states that such an exchangeable joint mix \( X^\pi \) is also NOD if \( X \) is NOD.

We say that a distribution \( F \) (or, analogously, a tuple of distributions) supports an NOD \( n \)-joint mix if there exists a joint mix \((X_1, \ldots, X_n)\) which is NOD and satisfies \( X_i \sim F \) for each \( i \in [n] \).
Proposition 3. Suppose that a univariate distribution function $F$ supports an NOD $n$-joint mix. Then $F$ supports an exchangeable NOD $n$-joint mix. The statement holds true if NOD is replaced by NUOD or NLOD.

Proof. Denote by $X = (X_1, \ldots, X_n)$ the NLOD $n$-joint mix with joint distribution $F_X$. Let $\bar{F} = \frac{1}{n!} \sum_{\pi \in \Pi_n} F_{X^\pi}$. Then $\bar{F}$ is an exchangeable $n$-joint mix of $F$. Moreover, $\bar{F}$ is NLOD since

$$\bar{F}(x) = \frac{1}{n!} \sum_{\pi \in \Pi_n} F_{X^\pi}(x) \leq \frac{1}{n!} \sum_{\pi \in \Pi_n} F(x_1) \cdots F(x_n) = F(x_1) \cdots F(x_n).$$

Similarly, $\bar{F}$ is NUOD if $F_X$ is, and thus we have the desired result. \hfill \Box

In the next proposition, we show that NLOD and NUOD conflict with tail dependence or intermediate tail dependence; see Hua and Joe (2011). For a bivariate distribution with continuous marginals and copula $C$, if

$$\lim_{t \downarrow 0} \frac{C(t, t)}{t^k} > 0,$$

for some $k \in [1, 2)$, then we say that the distribution has lower tail dependence ($k = 1$) or intermediate tail dependence ($1 < k < 2$).

Proposition 4. If $X$ has continuous marginal distributions, and there exists a two-dimensional projection of $X$ that has lower tail dependence or intermediate tail dependence, then $X$ is not NLOD.

Proof. Let $(X_i, X_j)$ be a two-dimensional projection of $X$ such that its copula $C_{ij}$ has lower tail dependence or intermediate tail dependence. Let $F_i$ and $F_j$ be the distributions of $X_i$ and $X_j$ respectively. It holds that, for some $k \in [1, 2)$,

$$\frac{\mathbb{P}(X_i \leq F_i^{-1}(t), X_j \leq F_j^{-1}(t))}{\mathbb{P}(X_i \leq F_i^{-1}(t)) \mathbb{P}(X_j \leq F_j^{-1}(t))} = \frac{C_{ij}(t, t)}{t^2} \frac{C_{ij}(t, t)}{t^k t^2 - k} \to \infty, \quad \text{as } t \downarrow 0. \quad (1)$$

Such an $X$ cannot be NLOD since

$$\frac{\mathbb{P}(X_i \leq F_i^{-1}(t), X_j \leq F_j^{-1}(t))}{\mathbb{P}(X_i \leq F_i^{-1}(t)) \mathbb{P}(X_j \leq F_j^{-1}(t))} \leq 1 \quad \text{for every } t \in (0, 1),$$

which contradicts (1). Therefore, joint mixes with lower tail dependence or intermediate tail dependence are not NLOD. \hfill \Box
Since upper tail dependence and intermediate tail dependence of \( X \) are defined as lower tail dependence and intermediate tail dependence of \(-X\), a joint mix cannot be NUOD if it has upper tail dependence or intermediate tail dependence.

**Example 3.** This example shows that \( t \)-distributed joint mixes cannot be NLOD or NUOD. Let \( X \) follow a multivariate \( t \) distribution (see also Section 5.2) with \( \nu > 0 \) degrees of freedom, location parameter \( \mu \in \mathbb{R}^n \) and \( n \times n \) positive semi-definite dispersion matrix \( \Sigma = (\sigma_{ij}) \); this distribution is denoted by \( t_n(\mu, \Sigma, \nu) \). If \( \mathbf{1}_n^\top \Sigma \mathbf{1}_n = 0 \), then \( X \) is a joint mix since \( \sum_{i=1}^n X_i \sim t_1(\mathbf{1}_n^\top \mu, \mathbf{1}_n^\top \Sigma \mathbf{1}_n, \nu) = t_1(\mathbf{1}_n^\top \mu, 0, \nu) \) which is degenerate. On the other hand, a bivariate projection \((X_1, X_j)\) of \( X \) has positive upper and lower tail dependence coefficient if and only if \( \sigma_{ij} > -1 \) (Demarta and McNeil, 2005). If \( n \geq 3 \), then we cannot have \( \sigma_{ij} = -1 \) for all \( i \neq j \) (otherwise \( \mathbf{1}_n^\top \Sigma \mathbf{1}_n < 0 \)). Hence, we know that \( X \) is neither NLOD nor NUOD. Note that for \( n = 2 \), the joint mix \( X \) is NOD since it is counter-monotonic. Similarly, other heavy-tailed elliptical distributions exhibit tail dependence, and cannot be NLOD or NUOD.

### 5 Classes of NOD joint mixes

In this section, we study classes of multivariate distributions which are NOD joint mixes. Despite various counterexamples in Section 3, one can find concrete examples of NOD joint mixes in the literature.

A simple way to construct NOD joint mixes can be obtained through conditional distribution on the sum of a random vector. This construction technique appears in the literature of negative dependence; see Block et al. (1982) and Hu (2000).

**Example 4.** Let \( Y = (Y_1, \ldots, Y_n) \) be an \( n \)-dimensional random vector and let \( c \) be a real number in the range of \( Y_1 + \cdots + Y_n \). Define the distribution of \( X = (X_1, \ldots, X_n) \) via

\[
X \overset{d}{=} Y \mid \{\mathbf{1}_n^\top Y = c\},
\]

where the conditional distribution is a version of the regular conditional distribution. By construction, \( X \) is a joint mix. If \( Y_1, \ldots, Y_n \) have logconcave pdfs or pmfs and are independent with each other, then \( X \) is known to be NSD and thus NOD; see Theorem 4.3 of Block et al. (1982) and Theorem 3.1 of Hu (2000). Consequently, multinomial distributions and Dirichlet distributions are NOD joint mixes since Poisson and gamma distributions have logconcave pmf and pdf, respectively; see Section 5 of Block et al. (1982).

In the rest of this section, we study exchangeable joint mixes and elliptical joint mixes that
are NOD.

5.1 Exchangeable joint mixes

We first focus on exchangeable joint mixes. For a vector $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n$, a discrete uniform (DU) distribution $F_{\mathbf{a}}$ on $n$ points takes the form $\frac{1}{n} \sum_{i=1}^{n} \delta_{a_i}$. As illustrated by Puccetti et al. (2012, Lemma 3.1), the simplest joint mix with marginals $F_{\mathbf{a}}$ can be constructed from the uniform distribution on the set

$$\Pi(\mathbf{a}) := \{ \mathbf{a}^\pi : \pi \in \Pi_n \},$$

and we denote this distribution by $U_{\mathbf{a}}$.

The next proposition follows from Theorem 3.2 of Hu (2000) based on induction. We give a different proof by direct calculation in Appendix A for the interested reader.

**Proposition 5.** For $\mathbf{a} \in \mathbb{R}^n$, the distribution $U_{\mathbf{a}}$ is both NLOD and NUOD.

Although the special distribution $U_{\mathbf{a}}$ is NOD, the following example shows that not all exchangeable joint mixes with marginal distributions $F_{\mathbf{a}}$ are NLOD or NUOD.

**Example 5.** Let $\mathbf{a} = (-2, -1, 1, 2)$, and $\mathbf{X} = (X_1, X_2, X_3, X_4)$ be uniformly distributed on the set

$$A = \Pi(-1, -1, 1, 1) \cup \Pi(-2, -2, 2, 2).$$

Note that $A$ is not $\Pi(\mathbf{a})$. We can see that $\mathbf{X}$ has marginal distributions $F_{\mathbf{a}}$, and $\mathbf{X}$ is a 4-dimensional exchangeable joint mix. However,

$$\mathbb{P}(\mathbf{X} \leq (1, 1, 1, 1)) = \frac{1}{2} > \left( \frac{3}{4} \right)^4 = \prod_{i=1}^{4} \mathbb{P}(X_i \leq 1).$$

Thus, $\mathbf{X}$ is not NLOD. Similarly,

$$\mathbb{P}(\mathbf{X} > (-2, -2, -2, -2)) = \frac{1}{2} > \left( \frac{3}{4} \right)^4 = \prod_{i=1}^{4} \mathbb{P}(X_i > -2),$$

and hence $\mathbf{X}$ is not NUOD either.

An immediate consequence of Proposition 5 shows that the DU distribution $F_{\mathbf{a}}$ always supports exchangeable NOD joint mixes. As we have seen from Example 2, not all $n$-completely mixable marginal distributions support NOD joint mixes.
Using the fact that the distribution of a joint mix can be written as a mixture of DU distributions on \( n \) points in \( \mathbb{R}^n \), we have the following result.

**Proposition 6.** The distribution of any exchangeable joint mix with center \( \mu \) can be written as a mixture of distributions of exchangeable NOD joint mixes with center \( \mu \).

**Proof.** Let \( G \) be the joint distribution of an exchangeable joint mix. Let us write

\[
G(A) = \int_{\mathbb{R}^n} \delta_a(A)dG(a), \quad A \in \mathcal{B}(\mathbb{R}^n).
\]

By exchangeability, we have \( G(A^\pi) = G(A) \) for \( \pi \in \Pi_n \) and \( A \in \mathcal{B}(\mathbb{R}^n) \), where \( A^\pi \) is \( \pi \) applied to elements of \( A \). Therefore,

\[
G(A) = \int_{\mathbb{R}^n} \delta_{a^\pi}(A)dG(a).
\]

Taking an average of the above formula over \( \Pi_n \), we have

\[
G(A) = \int_{\mathbb{R}^n} U_a(A)dG(a).
\]

Using Proposition 5, we know that each \( U_a \) is both NLOD and NUOD. Moreover, the center of the joint mix distributed as \( U_a \) is \( \mu \) since \( G \) is supported on \( \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 + \cdots + x_n = \mu\} \).

\[\Box\]

Recall that by de Finetti’s Theorem, any infinite exchangeable sequence can be written as a mixture of iid sequences. Proposition 6 may be vaguely seen as a finite version of de Finetti’s theorem for joint mixes.

**Remark 3.** Although a mixture of NOD joint mixes with a common center is again a joint mix, it is not necessarily NOD. To see this, let \( G_p, p \in (0, 1) \), be the joint distribution considered in Example 2. Then \( G_p = (1 - 3p) \delta_{(0,0,0)} + 3p U_{(-1,-1,2)} \), where \( \delta_{(0,0,0)} \) and \( U_{(-1,-1,2)} \) are the distributions of exchangeable NOD joint mixes with center 0. However, \( G_p \) is neither NLOD nor NUOD for \( p = 1/5 \) as seen in Example 2.

### 5.2 Elliptical distributions

An \( n \)-dimensional *elliptical distribution* is a family of multivariate distributions defined through the characteristic function

\[
\phi_X(t) = \mathbb{E}\left[\exp\left(\text{i} t^\top X\right)\right] = \exp\left(\text{i} t^\top \mu\right) \psi(t^\top \Sigma t), \quad t \in \mathbb{R}^n,
\]

\[\tag{2}\]
for some location parameter $\mu \in \mathbb{R}^n$, $n \times n$ positive semi-definite symmetric matrix $\Sigma \in \mathbb{R}^{n \times n}$ and the so-called characteristic generator $\psi : \mathbb{R}_+ \to \mathbb{R}$, where $\mathbb{R}_+ = \{ x \in \mathbb{R} : x \geq 0 \}$. See Section 6 of McNeil et al. (2015) for more properties. We denote an elliptical distribution by $E_n(\mu, \Sigma, \psi)$ and refer to $\mu$ as the location vector and $\Sigma$ the dispersion matrix. As presented in Proposition 6.27 of McNeil et al. (2015), a random vector $X \sim E_n(\mu, \Sigma, \psi)$ with $\text{rank}(\Sigma) = k$ admits the stochastic representation $X = \mu + RAS$, where $S$ is the uniform distribution on the unit sphere on $\mathbb{R}^k$, the radial random variable $R \geq 0$ is independent of $S$, and $A \in \mathbb{R}^{n \times k}$ is such that $AA^\top = \Sigma$. With this representation, we have that $\mathbb{E}[X] = \mu$ and $\text{Cov}(X) = \mathbb{E}[R^2] \Sigma/k$ provided $\mathbb{E}[R^2] < \infty$.

Below we present a simple lemma on elliptical joint mixes which will be useful for later discussions.

**Lemma 1.** An $n$-dimensional elliptically distributed random vector $X \sim E_n(\mu, \Sigma, \psi)$ is a joint mix if and only if $1_n^\top \Sigma 1_n = 0$ or $\psi = 1$ on $\mathbb{R}_+$.

**Proof.** One of the key properties of elliptical distributions is that they are closed under linear transformations, which is clear from (2). Hence, for $X \sim E_n(\mu, \Sigma, \psi)$, the random variable $\sum_{i=1}^n X_i$ follows $E_n(1_n^\top \mu, 1_n^\top \Sigma 1_n, \psi)$, which is degenerate if and only if $1_n^\top \Sigma 1_n = 0$. Hence, $X$ is a joint mix if and only if $1_n^\top \Sigma 1_n = 0$ or $\psi = 1$ on $\mathbb{R}_+$. \qed

Negative orthant dependence of such an elliptical joint mix is the main topic of this section. We begin with the following characterization result on supermodular order among elliptical distributions.

**Lemma 2** (Theorem 1 of Ansari and Rüschendorf (2021)). Let $X \sim E_n(\mu, \Sigma, \psi)$ and $Y \sim E_n(\mu', \Sigma', \psi')$ be two elliptical distributions. Then $X \leq_{\text{sm}} Y$ if and only if $\mu = \mu'$, $\psi = \psi'$, $\sigma_{ii} = \sigma'_{ii}$ for all $i \in [n]$ and $\sigma_{ij} \leq \sigma'_{ij}$ for all $i \neq j$, where $\Sigma = (\sigma_{ij})$ and $\Sigma' = (\sigma'_{ij})$.

Among elliptical distributions, multivariate Gaussian distribution is the only one such that the independence of $X$ is equivalent to $\sigma_{i,j} = 0$ for all $i \neq j$. Hence, for Gaussian random vectors, NOD is equivalent to non-positive bivariate correlations. This observation leads to the following result.

**Proposition 7.** A Gaussian random vector $X \sim \mathcal{N}_n(\mu, \Sigma)$ is an NOD $n$-joint mix if and only if $1_n^\top \Sigma 1_n = 0$ and $\sigma_{ij} \leq 0$ for all $i \neq j$, where $\sigma_{ij}$ is the $(i,j)$-entry of $\Sigma$.

**Proof.** By Lemma 1, $1_n^\top \Sigma 1_n = 0$ is equivalent to $X$ being a joint mix. It remains to verify that NOD is equivalent to non-positivity of $\sigma_{ij}$ for all $i \neq j$. The “only if” statement comes from the
fact that NOD implies non-positive bivariate correlation. To see the “if” statement, we assume
\( \sigma_{ij} \leq 0 \) for all \( i \neq j \) and use Lemma 2, which yields that \( X \preceq_{\text{sm}} X^\perp \) where \( X^\perp \) is an independent Gaussian random vector with the same marginal distributions as \( X \). Since supermodular order is stronger than concordance order, we know that \( X \) is NOD.

A natural choice of \( \Sigma \) that satisfies \( 1_n^\top \Sigma 1_n = 0 \) is the equicorrelated matrix \( P_n^* \) which has diagonal entries being 1 and off-diagonal entries being \( -1/(n - 1) \). This matrix \( P_n^* \) is the only choice of \( \Sigma \) with diagonal entries being 1 such that \( X \sim N_n(\mu, \Sigma) \) is exchangeable. In what follows, we write \( P_n^\perp \) as the identity matrix, which is the correlation matrix of an independent random vector.

Although Lemma 2 implies that \( E_n(\mu, P_n^*, \psi) \preceq_{\text{sm}} E_n(\mu, P_n^\perp, \psi) \) for general elliptical distributions, \( E_n(\mu, P_n^*, \psi) \) is not necessarily NOD in general since \( E_n(0_n, P_n^\perp, \psi) \) does not have independent components. In fact, an elliptical distribution \( E_n(0_n, P_n^\perp, \psi) \) is neither POD nor NOD unless it is Gaussian.

**Lemma 3.** The elliptical distribution \( E_n(\mu, \Sigma, \psi) \) where \( \Sigma \) is diagonal is neither POD nor NOD unless it is Gaussian.

**Proof.** Assume that \( X \sim E_n(\mu, \Sigma, \psi) \) is NOD. Since NOD is location invariant, it suffices to show the case when \( \mu = 0_n \). When \( X \) is NOD, then so is \( (X_1, X_2) \), that is,

\[
P(X_1 \leq x_1, X_2 \leq x_2) \leq P(X_1 \leq x_1)P(X_2 \leq x_2) \quad \text{for all } (x_1, x_2) \in \mathbb{R}^2.
\]

Since \( (X_1, X_2) \) and \( (-X_1, X_2) \) are identically distributed, we have

\[
P(X_1 \geq x_1, X_2 \leq x_2) \leq P(X_1 \geq x_1)P(X_2 \leq x_2) \quad \text{for all } (x_1, x_2) \in \mathbb{R}^2,
\]

and similarly, by symmetry,

\[
P(X_1 \geq x_1, X_2 \geq x_2) \leq P(X_1 \geq x_1)P(X_2 \geq x_2) \quad \text{for all } (x_1, x_2) \in \mathbb{R}^2,
\]

\[
P(X_1 \leq x_1, X_2 \geq x_2) \leq P(X_1 \leq x_1)P(X_2 \geq x_2) \quad \text{for all } (x_1, x_2) \in \mathbb{R}^2.
\]

Adding the above four inequalities together, we get \( 1 \leq 1 \). Hence, each of them is an equality. However, \( (X_1, X_2) \) follows a bivariate elliptical distribution with generator \( \psi \), and thus \( X_1 \) and \( X_2 \) are not independent unless it is Gaussian; see Theorem 4.11 of Fang et al. (1990). Therefore, \( X \) cannot be NOD unless it is Gaussian. The case of POD follows by the same argument. \( \square \)
A natural question is whether there exists a non-Gaussian elliptical distribution that represents an NOD joint mix for $n \geq 3$. The following theorem states that Gaussian distribution is characterized as the only elliptical distribution which admits an NOD $n$-joint mix for all $n$ large enough.

We say that an elliptical distribution $E_n(\mu, \Sigma, \psi)$ is non-degenerate if its all components are non-degenerate. Equivalently, the diagonal entries of $\Sigma$ are positive. For a characteristic generator $\psi$, denote by $E(\psi)$ the class of all non-degenerate random vectors following an elliptical distribution with characteristic generator $\psi$.

**Theorem 1.** For a characteristic generator $\psi$, the following are equivalent.

(i) The class $E(\psi)$ contains an NOD $n$-joint mix for all $n \geq 2$;

(ii) the class $E(\psi)$ contains an NOD $n$-random vector for all $n \geq 2$;

(iii) the class $E(\psi)$ contains an NOD $n$-random vector for all $n \geq m$ for some fixed $m \in \mathbb{N}$;

(iv) the class $E(\psi)$ is Gaussian.

**Proof.** The implications (i)$\Rightarrow$(ii) and (ii)$\Rightarrow$(iii) are trivial. The implication (iv)$\Rightarrow$(i) follows from Proposition 7. It remains to show (iii)$\Rightarrow$(iv).

To this end, let $\psi$ be a characteristic generator different from that of the Gaussian distribution. For $n \geq m$, let $X \sim E_n(\mu, \Sigma, \psi)$ be an NOD joint mix where $\Sigma$ has positive diagonal entries. We start by observing from Lemma 2 that if $\sigma_{ij} > 0$ for $i \neq j$, then the bivariate projection $(X_i, X_j)$ of $X$ satisfies $(X_i, X_j) \succeq_{sm} (X'_i, X'_j)$ where $(X'_i, X'_j) \sim E_n(\mu, \Sigma'_{ij}, \psi)$ with

$$
\Sigma'_{ij} = \begin{pmatrix}
\sigma_{ii} & 0 \\
0 & \sigma_{jj}
\end{pmatrix}.
$$

Using Lemma 3, we know that $(X'_i, X'_j)$ is not NOD, that is, there exists $(x_i, x_j) \in \mathbb{R}^2$ such that

$$
\mathbb{P}(X'_i \leq x_i, X'_j \leq x_j) \geq \mathbb{P}(X'_i \leq x_i)\mathbb{P}(X'_j \leq x_j);
$$

(3)

note that it suffices to consider the inequality needed for NLOD (not NUOD) by symmetry of the elliptical distribution and location invariance of NOD. Since supermodular order is stronger than concordance order, we know that

$$
\mathbb{P}(X_i \leq x_i, X_j \leq x_j) \geq \mathbb{P}(X'_i \leq x_i, X'_j \leq x_j).
$$

(4)
The two inequalities (3) and (4) imply that
\[
P(X_i \leq x_i, X_j \leq x_j) \geq P(X_i \leq x_i)P(X_j \leq x_j),
\]
that is, \((X_i, X_j)\) is not NOD. This leads to a contradiction.

Next, we assume \(\sigma_{ij} \leq 0\) for all \(i \neq j\). Since \(a^\top \Sigma a \geq 0\) for all \(a \in \mathbb{R}^n\) and \(\Sigma\) has positive diagonal entries, we can take \(a = (1/\sqrt{\sigma_{11}}, \ldots, 1/\sqrt{\sigma_{nn}})\), and this yields
\[
\sum_{i,j=1}^{n} \frac{\sigma_{ij}}{\sqrt{\sigma_{ii}\sigma_{jj}}} = n + \sum_{i \neq j} \frac{\sigma_{ij}}{\sqrt{\sigma_{ii}\sigma_{jj}}} \geq 0.
\]
Hence, there exist \(i, j\) with \(i \neq j\) such that
\[
\rho_{ij} := \frac{\sigma_{ij}}{\sqrt{\sigma_{ii}\sigma_{jj}}} \geq -\frac{1}{n-1}.
\]
Since NOD is location-scale invariant, the NOD of \((X_i, X_j)\) implies that \(E_2(0, P_{ij}, \psi)\) is NOD, where
\[
P_{ij} = \begin{pmatrix} 1 & \rho_{ij} \\ \rho_{ij} & 1 \end{pmatrix}.
\]
Taking a limit as \(n \to \infty\), and using (v) of Proposition 2 in the form of Remark 2, we conclude that \(E_2(0, P_{ij}^\perp, \psi)\) is also NOD, which contradicts Lemma 3.

**Remark 4.** For a characteristic generator \(\psi\), each statement in Theorem 1 implies that \(\mathcal{E}(\psi)\) contains at least one \(n\)-random vector for each \(n \in \mathbb{N}\). Note that \(\psi\) generates an \(n\)-dimensional elliptical distribution for each \(n \in \mathbb{N}\) if and only if the corresponding elliptical distribution is a scale mixture of Gaussian distribution (this class includes e.g., \(t\) distributions); see Fang et al. (1990, Section 2.6).

### 6 Which marginal distributions support an NOD joint mix?

In this section, we address a different question from Section 5: instead of asking which joint mixes are NLOD (or NUOD, NOD), we ask which marginal distributions allow us to construct an NLOD joint mix. Thus, the dependence structure is free to choose in this question, instead of being specified as in Section 5.
6.1 A necessary condition

Next, we present a necessary condition for a tuple of distributions to support any NOD joint mixes.

**Proposition 8.** If the tuple of distributions \((F_1, \ldots, F_n)\) with finite variance vector \((\sigma_1^2, \ldots, \sigma_n^2)\) supports an NOD joint mix, then

\[
2 \max_{i \in [n]} \sigma_i^2 \leq \sum_{i \in [n]} \sigma_i^2. \tag{5}
\]

**Proof.** Without loss of generality, assume \(\sigma_n^2\) is the maximum of \(\{\sigma_1^2, \ldots, \sigma_n^2\}\). Note that NOD implies that the bivariate correlations are non-positive. If \((X_1, \ldots, X_n)\) is an NOD joint mix where \(X_i \sim F_i, i \in [n]\), then

\[
\sigma_n^2 = \text{Var}(X_n) = \text{Var}(X_1 + \cdots + X_{n-1}) \leq \sum_{i=1}^{n-1} \text{Var}(X_i) = \sum_{i=1}^{n-1} \sigma_i^2,
\]

which yields (5) by adding \(\sigma_n^2\) to both sides. \(\square\)

For a given tuple of distributions \((F_1, \ldots, F_n)\) with finite variance vector \((\sigma_1^2, \ldots, \sigma_n^2)\), the condition (5) is not necessary for the existence of an NOD random vector, since an independent random vector supported by \((F_1, \ldots, F_n)\) always exists and it is NOD. More interestingly, the condition (5) is not necessary for the existence of a joint mix either. Indeed, as shown by Wang and Wang (2016, Corollary 2.2), a popular necessary condition for a joint mix supported by \((F_1, \ldots, F_n)\) to exist is

\[
2 \max_{i \in [n]} \sigma_i \leq \sum_{i \in [n]} \sigma_i. \tag{6}
\]

We note that (5) is strictly stronger than (6), because for any \(j \in [n]\), (5) gives

\[
\sigma_j^2 \leq \sum_{i \in [n] \setminus \{j\}} \sigma_i^2 \implies \sigma_j \leq \left( \sum_{i \in [n] \setminus \{j\}} \sigma_i^2 \right)^{1/2} \leq \sum_{i \in [n] \setminus \{j\}} \sigma_i,
\]

which implies (6). It is clear that (5) and (6) are not equivalent; e.g., \((\sigma_1, \sigma_2, \sigma_3) = (2, 2, 3)\) satisfies (6) but not (5).

By Proposition 2.4 of Wang et al. (2013), if the marginal distributions \(F_1, \ldots, F_n\) are Gaussian, then the condition (6) is necessary and sufficient for a joint mix supported by \((F_1, \ldots, F_n)\) to exist. Hence, the condition (5), which is strictly stronger than (6), is not necessary for a joint mix to exist. On the other hand, (5) is generally not sufficient for an NOD joint mix to exist.
either, since it is well known that a joint mix may not exist even if the marginal distributions are identical. Nevertheless, it turns out that (5) is necessary and sufficient for an NOD joint mix to exist for Gaussian marginals, which is the main result of Section 6.2.

### 6.2 Gaussian marginal distributions

In this section, we give a full characterization of the existence of an NOD joint mix for Gaussian marginals. Following immediately from Proposition 7 by choosing any covariance matrix \( \Sigma \) satisfying \( \mathbf{1}_n^\top \Sigma \mathbf{1}_n = 0 \), the univariate Gaussian distribution supports an exchangeable NOD \( n \)-joint mix for all \( n \geq 2 \). Clearly, not all tuples of Gaussian marginal distributions support an NOD joint mix, as (5) may not hold, even if a joint mix exists with (6) holding. In the next result, we establish that the necessary condition (5) in Proposition 8 is also sufficient for Gaussian marginals.

**Theorem 2.** A tuple of Gaussian distributions with variance vector \((\sigma_1^2, \ldots, \sigma_n^2)\) supports an NOD joint mix if and only if (5) holds, that is, \( 2 \max_{i \in [n]} \sigma_i^2 \leq \sum_{i \in [n]} \sigma_i^2 \). Moreover, such an NOD joint mix can be chosen as multivariate Gaussian.

**Proof.** The necessity follows from Proposition 8, and below we show sufficiency. Suppose that (5) holds. Without loss of generality, we can assume \( \sigma_n \geq \sigma_{n-1} \geq \cdots \geq \sigma_1 \). It suffices to consider \( n \geq 3 \) and \( \sigma_{n-1} > 0 \), and otherwise the problem is trivial. Moreover, the location parameters of the Gaussian distributions are not relevant, and they are assumed to be 0.

Let \( \lambda \) be a constant such that

\[
\lambda^2 \sum_{i=1}^{n-1} \sigma_i^2 + (1 - \lambda^2) \sigma_{n-1}^2 = \sigma_n^2.
\]

By (5), we have \( \sum_{i=1}^{n-1} \sigma_i^2 \geq \sigma_n^2 \geq \sigma_{n-1}^2 \), and this ensures that we can take \( \lambda \in [0, 1] \).

Let \( Y = (Y_1, \ldots, Y_{n-1}) \sim N_{n-1}(\mathbf{0}_{n-1}, P_{n-1}^\perp) \) and

\[
Z^{(m)} = (Z^{(m)}_1, \ldots, Z^{(m)}_n) \sim N_{n-m+1}(\mathbf{0}_{n-m+1}, P^*_m), \quad m = 1, \ldots, n-1,
\]

such that \( Y, Z^{(1)}, \ldots, Z^{(n-1)} \) are independent. Note that \( Z^{(n-1)} \) is 2-dimensional, and each \( Z^{(m)} \) is a joint mix.

For notational simplicity, let the function \( d \) be given by \( d(a, b) = (a^2 - b^2)^{1/2} \) for \( a \geq b \geq 0 \). Note that \( a^2 = d(a, b)^2 + b^2 \). Moreover, for \( k = 1, \ldots, n \), let

\[
\alpha_k = d(\sigma_k, \sigma_{k-1}) = (\sigma_k^2 - \sigma_{k-1}^2)^{1/2},
\]
with $\sigma_0 = 0$, and thus $\alpha_1 = \sigma_1$. For $k = 1, \ldots, n - 1$, let

$$X_k = \lambda \sigma_k Y_k + d(1, \lambda) \sum_{j=1}^{k} \alpha_j Z^{(j)}_k.$$ 

Moreover, let

$$X_n = -\lambda Y^* + d(1, \lambda) \sum_{j=1}^{n-1} \alpha_j Z^{(j)}_n,$$

where $Y^* = \sum_{k=1}^{n-1} \sigma_k Y_k$.

For $k = 1, \ldots, n - 1$, using independence among $Z^{(1)}_k, \ldots, Z^{(k)}_k$, we get

$$\text{Var} \left( \sum_{j=1}^{k} \alpha_j Z^{(j)}_k \right) = \sum_{i=1}^{k} \alpha^2_j = \sigma^2_1 + d(\sigma_2, \sigma_1)^2 + \cdots + d(\sigma_k, \sigma_{k-1})^2 = \sigma^2_k.$$

Hence, $\sum_{j=1}^{k} \alpha_j Z^{(j)}_k \sim N(0, \sigma^2_k)$, and again using independence of $Y_k$ and $\sum_{j=1}^{k} \alpha_j Z^{(j)}_k$, we get $X_k \sim N(0, \sigma^2_k)$. By (7), we have

$$\text{Var}(X_n) = \text{Var} \left( \lambda \sum_{k=1}^{n-1} \sigma_k Y_k \right) + \text{Var} \left( d(1, \lambda) \sum_{j=1}^{n-1} \alpha_j Z^{(j)}_n \right) = \lambda^2 \sum_{i=1}^{n-1} \sigma^2_i + (1 - \lambda^2) \sigma^2_{n-1} = \sigma^2_n.$$

Hence, $X_n \sim N(0, \sigma^2_n)$.

Next, we show that $(X_1, \ldots, X_n)$ is a joint mix. We can directly compute

$$\sum_{k=1}^{n} X_k = \sum_{i=k}^{n-1} \lambda \sigma_k Y_k + d(1, \lambda) \sum_{k=1}^{n-1} \sum_{j=1}^{k} \alpha_j Z^{(j)}_k - \lambda \sum_{k=1}^{n-1} \sigma_k Y_k + d(1, \lambda) \sum_{j=1}^{n-1} \alpha_j Z^{(j)}_n$$

$$= d(1, \lambda) \sum_{j=1}^{n-1} \sum_{k=j}^{n} \alpha_j Z^{(j)}_k = 0,$$

where the last equality follows from the fact that $Z^{(j)}_k$ is a joint mix for each $j = 1, \ldots, n - 1$.

We check that $(X_1, \ldots, X_n)$ is NOD. This follows from the fact that $(X_1, \ldots, X_n)$ is the weighted sum of several independent NOD random vectors $(\sigma_1 Y_1, \ldots, \sigma_{n-1} Y_{n-1}, -Y^*)$ and $(0_{m-1}, Z^{(m)})$ for $m = 1, \ldots, n - 1$. Alternatively, one can check that all non-zero terms in $\text{Cov}(X_i, X_j)$ are negative for $i \neq j$.

Finally, the statement that the joint mix can be chosen as multivariate Gaussian holds by construction of $(X_1, \ldots, X_n)$ as the sum of Gaussian vectors.

**Example 6.** In case $n = 3$, for any marginals with variance vector $(\sigma^2_1, \sigma^2_2, \sigma^2_3)$, the covariance
of a joint mix $X$ is uniquely given by

$$\Sigma = \begin{pmatrix}
\sigma_1^2 & \frac{1}{2}(\sigma_3^2 - \sigma_1^2 - \sigma_2^2) & \frac{1}{2}(\sigma_2^2 - \sigma_1^2 - \sigma_3^2) \\
\frac{1}{2}(\sigma_3^2 - \sigma_1^2 - \sigma_2^2) & \sigma_2^2 & \frac{1}{2}(\sigma_1^2 - \sigma_2^2 - \sigma_3^2) \\
\frac{1}{2}(\sigma_2^2 - \sigma_1^2 - \sigma_3^2) & \frac{1}{2}(\sigma_1^2 - \sigma_2^2 - \sigma_3^2) & \sigma_3^2
\end{pmatrix};$$

the uniqueness was observed by Xiao and Yao (2020, Corollary 6). If $X$ is Gaussian, it is clear that $X$ is NOD if and only if (5) holds. In case $n \geq 4$, for Gaussian marginals we can obtain an explicit covariance matrix of an NOD joint mix from the proof of Theorem 2.

### 6.3 Uniform distributions

We consider next the case where the marginal distributions are uniform and identical. For the standard uniform distribution, various explicit constructions of joint mixes are known in the literature. The following examples show that some of them are not NOD.

**Example 7.** We provide two 3-joint mixes of uniform distributions introduced in the literature but are not NOD.

1. Rüschendorf and Uckelmann (2002) constructed a 3-joint mix $X = (X_1, X_2, X_3)$ of uniform distributions on $[-1, 1]$ via

$$X = (U, -2U - 1, U + 1)\mathbf{1}_{\{-1 \leq U \leq 0\}} + (U, 1 - 2U, U - 1)\mathbf{1}_{\{0 \leq U \leq 1\}},$$

where $U \sim \text{U}[-1, 1]$. For $x = (-1/2, 1, 1/2)$, we have

$$\mathbb{P}(X \leq x) = \mathbb{P}\left(U \leq -\frac{1}{2}, -2U + 1 \leq 1, U + 1 \leq \frac{1}{2}\right) = \mathbb{P}(U \leq -\frac{1}{2}) = \frac{1}{4},$$

whereas

$$\mathbb{P}(X_1 \leq x_1) \mathbb{P}(X_2 \leq x_2) \mathbb{P}(X_3 \leq x_3) = \frac{1}{4} \times \frac{3}{4} \times \frac{3}{4} < \frac{1}{4} = \mathbb{P}(X \leq x).$$

2. Nelsen and Úbeda-Flores (2012) introduced a random vector $X = (X_1, X_2, X_3)$ whose distribution function has a probability mass uniformly distributed on the edges of the triangle in $[0, 1]^3$ which vertices $(0, 1/2, 1), (1/2, 1, 0)$ and $(1, 0, 1/2)$. A simple calculation shows that

$$\mathbb{P}(X \leq x) = \frac{1}{3} \sum_{i=1}^{3} \max\{0, \min(2x_i, 2x_{[i+1;3]} - 1) - (1 - x_{[i+2;3]})\},$$

where $[i+2;3]$ denotes the index $i+2$ modulo 3.
where \([i; 3]\) denotes the remainder of the division of \(i\) by 3. For \(= (1, 1/4, 3/4)\), we have

\[
P(X \leq x) = \frac{1}{3} \times 0 + \frac{1}{3} \times \frac{1}{2} + \frac{1}{3} \times \frac{1}{4} = \frac{1}{4},
\]

whereas

\[
P(X_1 \leq x_1)P(X_2 \leq x_2)P(X_3 \leq x_3) = 1 \times \frac{1}{4} \times \frac{3}{4} \times \frac{1}{4} = \frac{1}{4} = P(X \leq x).
\]

Despite the counterexamples shown in Example 7, uniform distributions support NOD \(n\)-joint mixes constructed by the so-called iterative Latin hypercube sampling (McKay et al., 1979; Craiu and Meng, 2005), and thus we have the following result. The proof is put in Appendix A.

**Proposition 9.** For \(-\infty < a < b < \infty\), the uniform distribution on an interval \([a, b]\) supports an exchangeable NOD \(n\)-joint mix for all \(n \geq 2\).

**Remark 5.** Let \(U, V \sim U[0, 1]\) be two independent uniform distributions. Then \(X = (U, 1 - U, V, 1 - V)\) is an NOD joint mix of \(U[0, 1]\) by (iii) of Proposition 2. As constructed in the proof of Proposition 3, its randomly permuted version \(X^\pi\) is an exchangeable NOD joint mix of \(U[0, 1]\). Since \(X^\pi\) differs from the one constructed in the proof of Proposition 9, it turns out that an exchangeable NOD joint mix is not unique.

**Remark 6.** A joint mix of \(U[-1, 1]\) is known as a random balanced sample (Gerow and Holbook, 1996). Although Bubenik and Holbrook (2007) studied densities of some random balanced samples on simplices for certain dimensions, we are not aware of negative orthant dependence of such interesting joint mixes.

### 6.4 Symmetric distributions

In this section we consider symmetric distributions with finite mean. A univariate distribution \(F\) with finite mean is called symmetric if \(X - E[X] \overset{d}{=} E[X] - X\) for \(X \sim F\). Although the class of symmetric distributions is identical to the class of one-dimensional elliptical distributions, multivariate elliptical distributions may not yield joint mixes as we have seen in Section 5.2.

For even \(n\), an exchangeable NOD joint mix can be constructed by concatenation of countermonotonic structures, which yields the following proposition.

**Proposition 10.** Any symmetric distribution supports an exchangeable NOD \(n\)-joint mix for all even \(n \geq 2\).
Proof. Let \( k = n/2 \). Without loss of generality, assume that the distribution \( F \) is symmetric with respect to 0. Let \( X_1, \ldots, X_k \) be independent and identically distributed to \( F \), and define \( Y_i = -X_i \) for \( i = 1, \ldots, k \). By symmetry of \( F \), the random vector \( \mathbf{X} = (X_1, \ldots, X_k, Y_1, \ldots, Y_k) \) is a joint mix of \( F \). Since \( (X_i, Y_i) \) is NOD for \( i = 1, \ldots, k \), so is \( \mathbf{X} \) by (iii) of Proposition 2. Finally, using Proposition 3, we obtain a desired exchangeable NOD joint mix. \( \square \)

In contrast to multivariate \( t \) distribution which is not NOD as seen in Example 3, the exchangeable NOD joint mix constructed in the proof of Proposition 10 does not have upper or lower tail dependence.

7 A multi-marginal optimal transport problem

In the previous sections, we have studied joint mixes that are NOD or not. An important remaining question is whether NOD joint mixes have some additional attractive properties in applications that are not shared by other joint mixes. The objective of this section is to address this issue, although we admit that a full picture is far from clear.

7.1 Repulsive harmonic cost and variance minimization under uncertainty

We discuss the connection of NOD joint mixes to a class of optimization problems. In multi-marginal optimal transport theory (Santambrogio, 2015), the general objective is the Monge-Kantorovich problem

\[
\text{to minimize } \mathbb{E}[c(X_1, \ldots, X_n)] \quad \text{subject to } X_i \sim F_i, \ i \in [n],
\]

where \( c : \mathbb{R}^n \to \mathbb{R} \) is a cost function, and \( F_1, \ldots, F_n \) are specified marginal distributions. In the context of this paper, the distributions \( F_1, \ldots, F_n \) are on \( \mathbb{R} \), although in optimal transport applications the underlying spaces are often more complicated (such as \( \mathbb{R}^d \)). We consider the repulsive harmonic cost function

\[
c(x_1, \ldots, x_n) = -\sum_{i,j=1}^{n} (x_i - x_j)^2, \quad (x_1, \ldots, x_n) \in \mathbb{R}^n.
\]

This cost function originates from the so-called weak interaction regime in Quantum Mechanics; see e.g., Di Marino et al. (2017). Any joint mix minimizes the expected repulsive harmonic cost.

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To see this, we can rewrite

\[
\mathbb{E}[c(X_1, \ldots, X_n)] = -2n \sum_{i=1}^{n} \mathbb{E}[X_i^2] + 2\mathbb{E} \left( \left( \sum_{i=1}^{n} X_i \right)^2 \right) \\
= -2n \sum_{i=1}^{n} \mathbb{E}[X_i^2] + 2 \left( \mathbb{E} \left[ \sum_{i=1}^{n} X_i \right] \right)^2 + 2\text{Var} \left( \sum_{i=1}^{n} X_i \right). 
\]

(8)

Since the first and second terms on the right-hand side of (8) do not depend on the dependence structure of \((X_1, \ldots, X_n)\), minimizing \(\mathbb{E}[c(X_1, \ldots, X_n)]\) is equivalent to minimizing \(\text{Var}(\sum_{i=1}^{n} X_i)\), which is clearly minimized if \((X_1, \ldots, X_n)\) is a joint mix. By (8), the optimal transport problem is equivalent to variance minimization with given marginals, that is,

\[
\text{to minimize} \ Var \left( \sum_{i=1}^{n} X_i \right) \text{ subject to } X_i \sim F_i, \ i \in [n], 
\]

(9)

which is a classic problem in Monte Carlo simulation (Craiu and Meng, 2001, 2005) and risk management (Rüschendorf, 2013). Below, we will focus on (9). In all optimization problems we discuss in this section, the constraint is always \(X_i \sim F_i\) for each \(i \in [n]\) with \(F_1, \ldots, F_n\) given.

Clearly, a joint mix supported by \((F_1, \ldots, F_n)\), when it exists, is not necessarily unique, and each of the joint mixes minimizes the objective in (8) or (9). Among these optimizers, we wonder whether an NOD joint mix plays a special role.

Although all joint mixes minimize (8) and (9), their distributions can be quite different. For a concrete example, suppose that the marginal distributions are standard Gaussian, and let \(n\) be even. With these marginals, \(X^E \sim N_n(0_n, P_n^*)\) is an NOD joint mix, and \(X^A = ((-1)^iZ)_i\in[n]\) where \(Z \sim N_1(0, 1)\) is another joint mix which is not NOD. Here, “E” stands for “exchangeable” and “A” stands for “alternating”. These two joint mixes have the same value for (8) and (9). Intuitively, \(X^A\) may be seen as undesirable in some situations, because some subgroups of its components are comonotonic. Inspired by this, we consider the cost of a subset \(K \subseteq [n]\) of risks given by, using the simpler formulation (8),

\[
\text{Var} \left( \sum_{i \in K} X_i \right) = \sum_{i,j \in K} \sigma_{ij}. 
\]

If \(K\) is known to the decision maker, then we are back to (8) with \((X_i)_i \in [n]\) replaced by \((X_i)_i \in K\).

In different applications, choosing \(K \subsetneq [n]\) may represent the absence of some risks in a risk aggregation pool, missing particles in a quantum system, unspecified number of simulation
size in a sampling program, or uncertainty on the participation of some agents in a risk sharing game. In each context above, a decision maker may not know $K$, and hence she is interested in a worst-case value of the cost, that is,

$$\max_{K \subseteq [n]} \text{Var} \left( \sum_{i \in K} X_i \right), \quad (10)$$

where the variance is set to 0 if $K$ is empty. A related problem is to minimize, for a fixed $k \in [n]$,

$$\max_{K \subseteq [n], |K|=k} \text{Var} \left( \sum_{i \in K} X_i \right)$$

(11)

where $|K|$ is the cardinality of $K$; this represents the situation where one knows how large the subgroup is, but not precisely how the subgroup is constructed. We will study the two problems of minimizing (10) and (11) in what follows.

### 7.2 Homogeneous marginals

To make our discussions more concrete, we first focus on Gaussian distributions. Two properties of the Gaussian distribution are particularly useful: all positive semi-definite matrices are possible covariance matrices of Gaussian random vectors, and NOD is equivalent to bivariate non-positive covariances by Proposition 7. By (9), the location parameter $\mu$ of a Gaussian random vector is not important, and we will safely choose $\mu = 0_n$. Using Proposition 7, the Gaussian joint mixes are given by $N_n(0_n, \Sigma)$ with $1_n^\top \Sigma 1_n = 0$, and it is NOD if (and only if) $\sigma_{ij} \leq 0$ for all $i \neq j$, where $\sigma_{ij}$ is the $(i,j)$-entry of $\Sigma$.

In Theorem 3 below, we will see that the exchangeable NOD joint mix $X^E$ minimizes (11) for each $k \in [n]$, and this minimizer is unique among Gaussian vectors with standard Gaussian marginals for each $k \in [n] \setminus \{1, n-1, n\}$. As a consequence, $X^E$ is also the unique minimizer to (10) among all Gaussian vectors (this holds for $n \geq 3$).

**Remark 7.** We briefly comment on the three cases of $k$ excluded from the statement regarding the unique minimizer of (11), and it will be clear that uniqueness cannot be expected in these cases. Recall that the marginal distributions of $X$ are assumed identical.

1. If $k = 1$, then $\text{Var} \left( \sum_{i \in K} X_i \right) = \text{Var}(X_1)$ which does not depend on the dependence structure of $X$, and hence any coupling minimizes (11).

2. If $k = n$, then $K = [n]$ and thus any joint mix minimizes (11).
3. If \( k = n - 1 \), then \( \text{Var} \left( \sum_{i \in K} X_i \right) = \text{Var}(c - X_1) = \text{Var}(X_1) \) for any joint mix \( X \) with center \( c \). Hence, any joint mix has the same value for (11).

**Theorem 3.** Subject to standard Gaussian marginals, \( X^E \sim N_n(0_n, P^*_n) \) is a minimizer to both (10) and (11) for each \( k \in [n] \). If \( n \geq 3 \), this minimizer is unique among Gaussian vectors in both cases of (10) and (11) with \( k \in [n] \setminus \{1, n - 1, n\} \).

**Proof.** We start with the problem (11). Take any \( X \) with standard Gaussian marginals, and let \( \Sigma \) be the covariance matrix of \( X \). Here, we do not need to assume that \( X \) is multivariate Gaussian; only the marginal distributions and the covariance matrix are needed. We assume \( n \geq 2 \) since the statement is trivial if \( n = 1 \).

First, note from \( 1^T_n \Sigma 1_n \geq 0 \) that
\[
\sum_{i,j \in [n], i \neq j} \sigma_{ij} \geq -n. \tag{12}
\]

Fix \( k \in [n] \). Let \( K_\ell, \ell = 1, \ldots, c_k \), be all subsets of \( [n] \) with cardinality \( k \), where \( c_k = \binom{n}{k} \). We can compute
\[
\frac{1}{c_k} \sum_{\ell=1}^{c_k} \text{Var} \left( \sum_{i \in K_\ell} X_i \right) = \frac{1}{c_k} \sum_{\ell=1}^{c_k} \left( \sum_{i,j \in K_\ell} \sigma_{ij} \right) = k + \frac{1}{c_k} \sum_{\ell=1}^{c_k} \left( \sum_{i,j \in K_\ell, i \neq j} \sigma_{ij} \right). \tag{13}
\]

Note that by symmetry, all terms \( \sigma_{ij} \) for \( i \neq j \) appear the same number of times in the last summation in (13), and the total number of appearance is \( k(k - 1)c_k \). Therefore, using (12), we have
\[
\frac{1}{c_k} \sum_{\ell=1}^{c_k} \left( \sum_{i,j \in K_\ell, i \neq j} \sigma_{ij} \right) = k(k - 1) \frac{(n-1)}{n(n-1)} \left( \sum_{i,j \in [n], i \neq j} \sigma_{ij} \right) \geq - k(k - 1) \frac{n-1}{n-1}. \tag{14}
\]

Since a maximum is no smaller than an average, (13) and (14) yield
\[
\max_{K \subseteq [n], |K| = k} \text{Var} \left( \sum_{i \in K} X_i \right) \geq \frac{1}{c_k} \sum_{\ell=1}^{c_k} \text{Var} \left( \sum_{i \in K_\ell} X_i \right) \geq k - \frac{k(k - 1)}{n-1} = \frac{k(n-k)}{n-1}. \tag{15}
\]

Recall that \( X^E = (X^E_1, \ldots, X^E_n) \sim N_n(0_n, P^*_n) \). For \( i, j \in [n] \), let \( \sigma^*_{ij} \) be \((i, j)\)-entry of \( P^*_n \). We have, for any \( K \subseteq [n] \) with \( |K| = k \),
\[
\text{Var} \left( \sum_{i \in K} X^E_i \right) = k + \sum_{i,j \in K, i \neq j} \frac{-1}{n-1} = k - \frac{k(k - 1)}{n-1} = \frac{k(n-k)}{n-1}. \tag{16}
\]
Putting (15) and (16) together, we obtain

\[
\min_{X_i \sim \mathcal{N}_1(0,1), \ i \in [n]} \left\{ \max_{K \subseteq [n], |K| = k} \text{Var} \left( \sum_{i \in K} X_i \right) \right\} = \frac{k(n-k)}{n-1}.
\]  

(17)

Therefore, \(X^E\) minimizes (11) for each \(k \in [n]\), and hence it also minimizes (10).

We next show that the minimizer to (11) among Gaussian vectors is uniquely given by \(X^E = (X^E_1, \ldots, X^E_n)\) for \(1 < k < n-1\). Note that the optimal value for (10) is given by, using (17),

\[
\min_{X_i \sim \mathcal{N}_1(0,1), \ i \in [n]} \left\{ \max_{K \subseteq [n]} \text{Var} \left( \sum_{i \in K} X_i \right) \right\} = \max_{K \subseteq [n]} \text{Var} \left( \sum_{i \in K} X^E_i \right) = \frac{k^*(n-k^*)}{n-1},
\]

(18)

where \(k^* = \lfloor n/2 \rfloor\).

First, we consider the case \(n = 3\). In this case, \([n] \setminus \{1, n-1, n\}\) is empty, and we only need to show that \(X^E\) is the unique minimizer to (10). Suppose that the Gaussian vector \(X = (X_1, \ldots, X_n) \sim \mathcal{N}_n(0_n, \Sigma)\) is a minimizer to (10). By (18), optimal value for (10) is 1. Hence,

\[
\text{Var} \left( \sum_{i \in K} X_i \right) \leq 1 \quad \text{for each } K \text{ with } |K| = 2,
\]

and this implies

\[
\sigma_{ij} \leq -1/2, \quad \text{for } i \neq j.
\]

(19)

Since \(\sum_{i,j \in [n]} \sigma_{ij} \geq 0\), we have \(3 + 2\sigma_{12} + 2\sigma_{13} + 2\sigma_{23} \geq 0\), implying \(\sigma_{12} + \sigma_{13} + \sigma_{23} \geq -3/2\). Together with (19), we get \(\sigma_{12} = \sigma_{13} = \sigma_{23} = -1/2\), and hence \(\Sigma = P^*_n\).

Next, we consider the case \(n \geq 4\). Fix \(k \in [n] \setminus \{1, n-1, n\}\). Suppose that the Gaussian vector \(X = (X_1, \ldots, X_n) \sim \mathcal{N}_n(0_n, \Sigma)\) is a minimizer to (11). Our goal is to show \(\Sigma = P^*_n\). Since \(X\) is a minimizer, we have

\[
\text{Var} \left( \sum_{i \in K} X_i \right) = \frac{k(n-k)}{n-1} \quad \text{for each } K \text{ with } |K| = k,
\]

(20)

where the “\(\leq\)’’ sign is implied by (17), and the “\(\geq\)’’ sign is implied by the second inequality in (15). This further implies that (12) is an equality. Hence, \(1_n^\top \Sigma 1_n = 0\).

Note that each of the row sums and the column sums of \(P^*_n\) is 0, i.e., \(\sum_{j \in [n]} \sigma^*_tj = 0\) for
each $\ell \in [n]$. We first show that the same holds for $\Sigma$. Using $\mathbf{1}_n^T \Sigma \mathbf{1}_n = 1_n^T \Sigma 1_n = 0 = 1_n^T \Sigma 1_n$, we have

$$
\sum_{i,j \in [n]} \sigma_{ij} = \sum_{i,j \in [n]} \sigma_{ij}^*.
$$

(21)

Fix $\ell \in [n]$. Applying the same argument for (13) and (14) to $[n] \setminus \{\ell\}$, we have, by letting $d_k = \binom{n-1}{k}$,

$$
\frac{1}{d_k} \sum_{K \subseteq [n] \setminus \{\ell\}, |K| = k} \text{Var} \left( \sum_{i \in K} X_i \right) = k + \frac{k(k-1)}{(n-1)(n-2)} \left( \sum_{i,j \in [n] \setminus \{\ell\}, i \neq j} \sigma_{ij} \right)
$$

and

$$
\frac{1}{d_k} \sum_{K \subseteq [n] \setminus \{\ell\}, |K| = k} \text{Var} \left( \sum_{i \in K} X_i^E \right) = k + \frac{k(k-1)}{(n-1)(n-2)} \left( \sum_{i,j \in [n] \setminus \{\ell\}, i \neq j} \sigma_{ij}^* \right).
$$

Using (16) and (20), this implies

$$
\sum_{i,j \in [n] \setminus \{\ell\}, i \neq j} \sigma_{ij} = \sum_{i,j \in [n] \setminus \{\ell\}, i \neq j} \sigma_{ij}^*.
$$

Hence, using $\sigma_{ii} = \sigma_{ii}^* = 1$ for $i \in [n]$, we have

$$
\sum_{i,j \in [n] \setminus \{\ell\}} \sigma_{ij} = \sum_{i,j \in [n] \setminus \{\ell\}} \sigma_{ij}^*.
$$

(22)

Together with (21), this implies

$$
2 \sum_{j \in [n]} \sigma_{\ell j} = \sum_{i,j \in [n]} \sigma_{ij} - \sum_{i,j \in [n] \setminus \{\ell\}} \sigma_{ij} + \sigma_{\ell \ell} = \sum_{i,j \in [n]} \sigma_{ij}^* - \sum_{i,j \in [n] \setminus \{\ell\}} \sigma_{ij}^* + \sigma_{\ell \ell}^* = 2 \sum_{j \in [n]} \sigma_{\ell j}^*.
$$

(23)

Therefore, we obtain $\sum_{j \in [n]} \sigma_{\ell j} = \sum_{j \in [n]} \sigma_{\ell j}^* = 0$ for each $\ell \in [n]$.

We fix an index $\ell \in [n]$, and remove it from $[n]$. Note that $\sum_{i,j \in [n] \setminus \{\ell\}} \sigma_{ij} = \sum_{i,j \in [n] \setminus \{\ell\}} \sigma_{ij}^*$ as shown in (22). Following the same arguments for (21), (22) and (23) by replacing $[n]$ with $[n] \setminus \{\ell\}$, we obtain

$$
\sum_{j \in [n] \setminus \{\ell\}} \sigma_{ij} = \sum_{j \in [n] \setminus \{\ell\}} \sigma_{ij}^*.
$$
for each \( t \neq \ell \). This implies \( \sigma_{t\ell} = \sigma^*_{t\ell} \) for all \( t \neq \ell \) because

\[
\sigma_{t\ell} + \sum_{j \in [n] \setminus \{\ell\}} \sigma_{tj} = \sum_{j \in [n]} \sigma_{tj} = \sum_{j \in [n]} \sigma^*_{tj} = \sum_{j \in [n] \setminus \{\ell\}} \sigma^*_{tj}.
\]

Since \( \ell \) is arbitrary, we conclude that \( \Sigma = P^*_n \).

Finally, note that \( k^* \) in (18) satisfies \( 1 < k^* < n - 1 \) for \( n \geq 4 \). The above arguments have justified that the optimizer to (11) with \( k = k^* \) is unique among Gaussian vectors. Therefore, the minimizer to (10) is also unique by using (18).

**Remark 8.** As we have seen in Remark 7, if \( n = 3 \), then any joint mix minimizes (11) for each \( k \in [n] \). The uniqueness statement in Theorem 3 implies that the covariance structure of a joint mix is unique in this setting; see also Example 6.

Although Theorem 3 is specific to standard Gaussian marginals, the general observation that negative dependence yields more stable costs in the presence of subgroup uncertainty applies to other distributions. Since the proof technique of Theorem 3 applies to any \( n \)-completely mixable distribution, we immediately obtain the following proposition.

**Proposition 11.** Suppose that the distribution \( F \) is \( n \)-completely mixable with a finite variance. Subject to the marginal distribution \( F \), an exchangeable \( n \)-joint mix with correlation matrix \( P^*_n \) is a minimizer to (11) for each \( k \in [n] \) and hence also to (10). For \( n \geq 3 \), the correlation matrix of the minimizer is unique in both cases of (10) and (11) with \( k \in [n] \setminus \{1, n - 1, n\} \).

**Proof.** Since \( F \) supports an \( n \)-joint mix, it also supports an exchangeable \( n \)-joint mix \( X \), as we have seen in the proof of Proposition 3. Since \( X \) is exchangeable, it is equicorrelated, and hence its correlation matrix is \( P^*_n \). The fact that \( X \) minimizes (10) and (11) and the uniqueness of the correlation of the minimizer follow from the same proof as in Theorem 3.

The following corollary follows by combining Propositions 3 and 11.

**Corollary 1.** In the setting of Propositions 11, if \( F \) supports an NOD \( n \)-joint mix, then an exchangeable NOD \( n \)-joint mix is a minimizer to (11) for each \( k \in [n] \) and (10).

To interpret the results of Theorem 3, Proposition 11 and Corollary 1, it is intuitively clear that for the minimization of (10) or (11), having a negative covariance among each bivariate projection \((X_i, X_j)\) of \( X \) is desirable and reduces the worst-case cost. This statement holds true regardless of the distribution type of \( X \). It just happens with the Gaussian class that negative
covariance is equivalent to bivariate NOD (Proposition 7). For other distributions we only have the one-sided implication that NOD implies negative covariance.

**Remark 9.** Let \( c_K(x_1, \ldots, x_n) = -\sum_{i,j \in K} (x_i - x_j)^2, (x_1, \ldots, x_n) \in \mathbb{R}^n, \) for \( K \subseteq [n]. \) For \( k \in [n], \) minimizing

\[
\max_{K \subseteq [n], |K| = k} \mathbb{E}[c_K(X)]
\]

under the marginal constraint is equivalent to minimizing (11) by (8). Therefore, the statements in Theorem 3 and Proposition 11 remain true when (11) is replaced by (24). On the other hand, without fixing \( |K| = k, \) the worst-case value \( \max_{K \subseteq [n]} \mathbb{E}[c_K(X)] \) is always 0 by choosing \( K \) with \( |K| = 1, \) and hence this problem is trivial.

### 7.3 Heterogeneous Gaussian marginals

In Theorem 3 and Proposition 11, we assumed that the marginal distributions are identical. This assumption is not dispensable, as the situation for heterogeneous marginals is drastically different. In this section, we obtain a result in the simple case \( n = 3 \) and provide several examples to discuss some subtle issues and open questions.

**Proposition 12.** Let \( n = 3. \) For any tuple of marginal distributions with finite variance vector, any joint mix, if it exists, minimizes (10). If an NOD joint mix exists, then no random vector with any positive bivariate covariance can minimize (10).

**Proof.** For any \((Y_1, \ldots, Y_n)\) with variance vector \((\sigma_1^2, \ldots, \sigma_n^2)\),

\[
\max_{K \subseteq [n]} \Var \left( \sum_{i \in K} Y_i \right) \geq \max_{i \in [n]} \Var(Y_i) = \max_{i \in [n]} \sigma_i^2.
\]

In case \( n = 3, \) a joint mix \( X \) with variance vector \((\sigma_1^2, \ldots, \sigma_3^2)\) satisfies

\[
\max_{K \subseteq [3], |K| = 1} \Var \left( \sum_{i \in K} X_i \right) = \max_{K \subseteq [3], |K| = 2} \Var \left( \sum_{i \in K} X_i \right) = \max_{i \in [3]} \Var(X_i) = \max_{i \in [3]} \sigma_i^2,
\]

and \( \Var(X_1 + X_2 + X_3) = 0. \) Hence, the joint mix minimizes (10).

To show that no positive covariance is allowed, suppose that \((X_1, X_2, X_3)\) is a minimizer to (10) and \( \Cov(X_i, X_j) > 0 \) for some \( i \neq j. \) We have

\[
\max_{K \subseteq [n]} \Var \left( \sum_{i \in K} X_i \right) \geq \Var(X_i + X_j) > \sigma_i^2 + \sigma_j^2 \geq \max (\sigma_1^2, \sigma_2^2, \sigma_3^2),
\]
where the last inequality follows from the necessary condition (5) of the existence of an NOD joint mix. Since we have seen that the optimal value of (10) is \( \max_{i \in [3]} \sigma_i^2 \), (25) implies that \((X_1, X_2, X_3)\) does not minimize (10).

Remark 10. Proposition 12 states that, if a Gaussian triplet supports an NOD joint mix, then it minimizes (10), and all Gaussian minimizers must be NOD. It is not clear whether this observation can be extended to \( n \geq 4 \).

Unlike the situation in Theorem 3 and Proposition 11, uniqueness of the covariance matrix does not hold in the setting of Proposition 12, as illustrated in the following example.

Example 8. Consider Gaussian marginal distributions with variance vector \((\sigma_1^2, \sigma_2^2, \sigma_3^2) = (2, 1, 1)\). In this case, (5) holds, and an NOD joint mix exists by Theorem 2. Both the covariance matrices \( \Sigma \) and \( \Sigma' \) defined by

\[
\Sigma = \begin{pmatrix}
2 & -1 & -1 \\
-1 & 1 & 0 \\
-1 & 0 & 1
\end{pmatrix}
\quad \text{and} \quad
\Sigma' = \begin{pmatrix}
2 & -1/2 & -1 \\
-1/2 & 1 & 0 \\
-1 & 0 & 1
\end{pmatrix}
\]

minimize (10) subject to the marginal distributions. We can see that \( \Sigma \) corresponds to an NOD joint mix, whereas \( \Sigma' \) corresponds to an NOD Gaussian random vector, but not a joint mix.

The next example illustrates that, although a joint mix generally minimizes (10) in case \( n = 3 \), NOD may be more relevant than joint mixes for minimizing (11) with some \( k \neq n \) when the two dependence requirements cannot be simultaneously achieved.

Example 9. Consider Gaussian marginal distributions with variance vector \((\sigma_1^2, \sigma_2^2, \sigma_3^2) = (4, 1, 1)\). In this case, (5) does not hold, and no NOD joint mix exists. Both the covariance matrices \( \Sigma \) and \( \Sigma' \) defined by

\[
\Sigma = \begin{pmatrix}
4 & -2 & -2 \\
-2 & 1 & 1 \\
-2 & 1 & 1
\end{pmatrix}
\quad \text{and} \quad
\Sigma' = \begin{pmatrix}
4 & -1 & -1 \\
-1 & 1 & 0 \\
-1 & 0 & 1
\end{pmatrix}
\]

minimize (10) subject to the marginal distributions. The covariance matrix \( \Sigma \) corresponds to a joint mix, but not NOD. The covariance matrix \( \Sigma' \) corresponds to an NOD Gaussian random vector, but not a joint mix. Thus, the problem (10) admits an NOD minimizing distribution \( N_3(0_3, \Sigma') \). Moreover, for (11) with \( k = 2 \), the NOD distribution \( N_3(0_3, \Sigma') \) has a maximum of 3 which is strictly better than the joint mix distribution \( N_3(0_3, \Sigma) \) with a maximum of 4.
Example 9 suggests, informally, that there is a trade-off between a joint mix and NOD when both cannot be attained simultaneously, with a joint mix minimizing (11) for $k = n$, and an NOD random vector improving (11) from the case of a joint mix for some $1 < k < n$. In fact, (11) is not always minimized by NOD random vectors as seen in the following example.

**Example 10.** Consider Gaussian marginal distributions with variance vector $(\sigma_1^2, \sigma_2^2, \sigma_3^2) = (\sigma^2, 1, 1)$, where $\sigma > 3$. For any $(X_1, X_2, X_3)$ with the given margins, we have

$$\text{Var}(X_2 + X_3) \leq 4 < (\sigma - 1)^2 \leq \min(\text{Var}(X_1 + X_2), \text{Var}(X_1 + X_3)),$$

and hence

$$\max_{K \subseteq [3], |K| = 2} \text{Var} \left( \sum_{i \in K} X_i \right) = \sigma^2 + 1 + 2\sigma \max(\rho_{12}, \rho_{13}),$$

(26)

where $\rho_{ij}$, $i, j \in [3]$ for is the correlation coefficient of $(X_i, X_j)$. The minimum of (26) is attained if and only if $\rho_{12} = \rho_{13} = -1$. In this case, $\rho_{23} = 1$ is an only possible correlation, and thus the minimizer to (26) cannot be NOD. Note that (26) is minimized by a Gaussian random vector $(\sigma Z, -Z, -Z)$ where $Z \sim \mathcal{N}(0, 1)$.

On the other hand, the next example shows that, if $n = 3$, there always exists an NOD minimizer to (10) under Gaussian marginal constraint.

**Example 11.** Let $(X_1, X_2, X_3)$ follow a multivariate Gaussian distribution with the prescribed marginal distributions and an equicorrelation matrix $P_3^*$, i.e., all pair-wise correlation coefficients are $-1/2$. The variances $\sigma_1^2$, $\sigma_2^2$, and $\sigma_3^2$ are assumed to satisfy $\sigma_1 \leq \sigma_2 \leq \sigma_3$ without the loss of generality. We can easily verify that each of $\text{Var}(X_1 + X_2 + X_3)$ and $\text{Var}(X_i + X_j)$, $i, j \in [3]$, is smaller than or equal to $\sigma_3^2$. Hence, $(X_1, X_2, X_3)$ attains the lower bound

$$\max_{K \subseteq [3]} \text{Var} \left( \sum_{i \in K} X_i \right) = \sigma_3^2 = \max_{i \in [3]} \sigma_i^2,$$

and thus it minimizes (10).

**Remark 11.** For $n \geq 4$, it is not clear whether there always exists an NOD minimizer to (10) under a general heterogeneous Gaussian marginal constraint.

8 Conclusion

We revealed that joint mixability is not always consistent with the classic notion of negative dependence called negative orthant dependence (NOD). Some tuples of distributions do not even
support any NOD joint mix, and joint mixes with lower or upper tail dependence cannot be NOD. On the other hand, joint mixability can be regarded as a different concept of negative dependence since every joint mix is never positively orthant dependent. We also showed that some natural classes of joint mixes, particularly those with discrete uniform distributions and Gaussian distributions, are NOD. Moreover, we derived a new characterization of the Gaussian class as the only elliptical class which supports NOD joint mixes of arbitrary dimension. We also characterized all tuples of Gaussian distributions which support NOD joint mixes. Finally, we showed that an exchangeable NOD Gaussian joint mix solves a multi-marginal optimal transport problem under uncertainty on the participation of agents. Uniqueness of this optimizer was shown and the results were extended to other marginal distributions.

The following questions were partially addressed in this paper, but they remain largely open when formulated with greater generality.

1. Under what conditions (stronger than exchangeability) do we know a joint mix has to be NOD?

2. Under what conditions do we know a tuple of distributions supports an NOD joint mix?

3. Does there exist an elliptical distribution, other than Gaussian, which supports NOD joint mixes for a fixed $n \geq 3$?

4. Do NOD joint mixes play an important role in optimization problems other than the ones considered in Section 7? It is also unclear how the results in Section 7 can be extended to general marginal distributions with dimension higher than 3; see the unsolved questions in Remarks 10 and 11.

These questions yield new challenges to dependence theory and require future research.

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A Omitted proofs

Proof of Proposition 2. (i): Let $a_1, \ldots, a_n \in \mathbb{R}$ and $b > 0$. If $X$ is NLOD, then

$$
P(a_1 + bX_1 \leq x_1, \ldots, a_n + bX_n \leq x_n) = \mathbb{P}
\left( X_1 \leq \frac{x_1 - a_1}{b}, \ldots, X_n \leq \frac{x_n - a_n}{b} \right) \\
\leq \prod_{i=1}^{n} \mathbb{P}(X_i \leq \frac{x_i - a_i}{b}) \\
= \prod_{i=1}^{n} \mathbb{P}(a_i + bX_i \leq x_i)
$$

for every $\mathbf{x}$, and thus $(a_1 + bX_1, \ldots, a_n + bX_n)$ is NLOD. Similarly, if $X$ is NUOD, so is $(a_1 + bX_1, \ldots, a_n + bX_n)$. Finally, if $\sum_{i=1}^{n} X_i = c$ a.s. for some $c \in \mathbb{R}$, then

$$
\sum_{i=1}^{n} (a_i + bX_i) = \left( \sum_{i=1}^{n} a_i \right) + bc \quad \text{a.s.,}
$$

and thus we have the statement (i).

(ii): Let $\pi \in \Pi_n$. If $X$ is NLOD, then

$$
P(X^\pi \leq \mathbf{x}) = \mathbb{P}(X_1 \leq x_{\pi^{-1}(1)}, \ldots, X_n \leq x_{\pi^{-1}(n)}) = \prod_{i=1}^{n} \mathbb{P}(X_i \leq x_{\pi^{-1}(i)}) = \prod_{i=1}^{n} \mathbb{P}(X_{\pi(i)} \leq x_i),
$$

for every $\mathbf{x}$, and thus $X^\pi$ is NLOD. Similarly, if $X$ is NUOD, so is $X^\pi$. Finally, if $\sum_{i=1}^{n} X_i = c$ a.s. for some $c \in \mathbb{R}$, then

$$
\sum_{i=1}^{n} X_{\pi(i)} = \sum_{i=1}^{n} X_i = c \quad \text{a.s.,}
$$

which leads to the statement (ii).
(iii) Suppose that an \( n_1 \)-dimensional random vector \( \mathbf{X} \) and an \( n_2 \)-dimensional random vector \( \mathbf{Y} \) are NLOD and independent with each other. Then we have
\[
\mathbb{P}(\mathbf{X} \leq \mathbf{x}, \mathbf{Y} \leq \mathbf{y}) = \mathbb{P}(\mathbf{X} \leq \mathbf{x})\mathbb{P}(\mathbf{Y} \leq \mathbf{y}) \leq \prod_{i=1}^{n} \mathbb{P}(X_i \leq x_i) \prod_{i=1}^{n} \mathbb{P}(Y_i \leq y_i)
\]
fors every \( \mathbf{x} \in \mathbb{R}^{n_1} \) and \( \mathbf{y} \in \mathbb{R}^{n_2} \). Therefore, the joint random vector \((\mathbf{X}, \mathbf{Y})\) is NLOD. Similarly, if the two independent random vectors \( \mathbf{X} \) and \( \mathbf{Y} \) are NUOD, then so is \((\mathbf{X}, \mathbf{Y})\). Finally, if \( \mathbf{1}_{n_1}^\top \mathbf{X} = c_1 \) a.s. and \( \mathbf{1}_{n_2}^\top \mathbf{Y} = c_2 \) a.s. for some \( c_1, c_2 \in \mathbb{R} \), then \( \mathbf{1}_{n_1+n_2}^\top (\mathbf{X}, \mathbf{Y}) = \mathbf{1}_{n_1}^\top \mathbf{X} + \mathbf{1}_{n_2}^\top \mathbf{Y} = c_1 + c_2 \) a.s., which yields (iii).

(iv) Suppose that \( \mathbf{X} \) and \( \mathbf{Y} \) are NLOD and independent of each other. For \( \mathbf{X}^\perp \) and \( \mathbf{Y}^\perp \) independent of each other and of \((\mathbf{X}, \mathbf{Y})\), we have
\[
\mathbb{P}(\mathbf{X} + \mathbf{Y} \leq \mathbf{t}) = \int_{\mathbb{R}^n} \mathbb{P}(\mathbf{X} \leq \mathbf{t} - \mathbf{y}) P_Y(\mathrm{d}\mathbf{y})
\]
\[
\leq \int_{\mathbb{R}^n} \mathbb{P}(\mathbf{X}^\perp \leq \mathbf{t} - \mathbf{y}) P_Y(\mathrm{d}\mathbf{y})
\]
\[
= \mathbb{P}(\mathbf{X}^\perp + \mathbf{Y} \leq \mathbf{t})
\]
\[
= \int_{\mathbb{R}^n} \mathbb{P}(\mathbf{Y} \leq \mathbf{t} - \mathbf{x}) P_{\mathbf{X}^\perp}(\mathrm{d}\mathbf{x})
\]
\[
\leq \int_{\mathbb{R}^n} \mathbb{P}(\mathbf{Y}^\perp \leq \mathbf{t} - \mathbf{x}) P_{\mathbf{X}^\perp}(\mathrm{d}\mathbf{x}) = \mathbb{P}(\mathbf{X}^\perp + \mathbf{Y}^\perp \leq \mathbf{t}),
\]
for all \( \mathbf{t} \in \mathbb{R}^n \), where \( P_Y \) and \( P_{\mathbf{X}^\perp} \) are the probability measures corresponding to \( \mathbf{Y} \) and \( \mathbf{X}^\perp \), respectively. Therefore, \( \mathbf{X} + \mathbf{Y} \) is NLOD since \( \mathbf{X}^\perp + \mathbf{Y}^\perp \) has the same marginals with \( \mathbf{X} + \mathbf{Y} \) with independent components. The case of NUOD can be shown similarly. Finally, if \( \sum_{i=1}^{n} X_i = c_1 \) a.s. and \( \sum_{i=1}^{n} Y_i = c_2 \) a.s., then \( \sum_{i=1}^{n} (X_i + Y_i) = c_1 + c_2 \) a.s., and thus \( \mathbf{X} + \mathbf{Y} \) is a joint mix. Therefore, we have the statement (iv).

(v) We first argue that the weak limit of joint mixes is still a joint mix. If \( \mathbf{X} \) is not a joint mix, then there exists a constant \( a \in \mathbb{R} \) such that the two open sets \( A := \{ \mathbf{x} \in \mathbb{R}^n : \mathbf{1}_n^\top \mathbf{x} < a \} \) and \( B := \{ \mathbf{x} \in \mathbb{R}^n : \mathbf{1}_n^\top \mathbf{x} > a \} \) satisfy \( \mathbb{P}(\mathbf{X} \in A) > 0 \) and \( \mathbb{P}(\mathbf{X} \in B) > 0 \). Using the definition of weak convergence, this gives
\[
\liminf_{m \to \infty} \mathbb{P}(\mathbf{X}^{(m)} \in A) \geq \mathbb{P}(\mathbf{X} \in A) > 0 \quad \text{and} \quad \liminf_{m \to \infty} \mathbb{P}(\mathbf{X}^{(m)} \in B) \geq \mathbb{P}(\mathbf{X} \in B) > 0
\]
This is clearly a contradiction, since either \( \mathbb{P}(\mathbf{X}^{(m)} \in A) = 0 \) or \( \mathbb{P}(\mathbf{X}^{(m)} \in B) = 0 \) for all \( m \in \mathbb{N} \). Hence, \( \mathbf{X} \) is a joint mix.

Suppose that \( \mathbf{X}^{(m)} \) is NLOD for every \( m \in \mathbb{N} \). Since \( \mathbf{X}^{(m)} \) converges weakly to \( \mathbf{X} \), we have
that $X_i^{(m)}$ converges weakly to $X_i$ for every $i \in [n]$ by continuous mapping theorem. These marginal convergences imply that $X_i^{(m)}$ converges weakly to $X_i$ by Part (ii) in Theorem 2.8 of Billingsley (1999). Together with $X_i^{(m)} \leq_{c.l.} X_i$ for every $m$, we have that $X_i \leq_{c.l.} X_i$ by (P6) in Theorem 3.8.7 of Muller and Stoyan (2002). Therefore, $X_i$ is NLOD and the case of NUOD is shown analogously.

Proof of Proposition 5. Without loss of generality, assume the order $a_1 \leq \ldots \leq a_n$. Take $X = (X_1, \ldots, X_n) \sim U_a$. We have seen that $X$ is an $n$-dimensional exchangeable joint mix with marginal distributions $F_a$. We will show

$$\mathbb{P}(X \leq x) = \prod_{i=1}^n \mathbb{P}(X_i \leq x_i)$$

(27)

for every $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Without loss of generality, let us order $x_1 \leq \ldots \leq x_n$.

Let $x = (x_1, \ldots, x_n)$ and $x^j = (x_1^j, \ldots, x_n^j)$ be one of the permutation of $x$, $j = 1, \ldots, n!$. As $X$ is exchangeable, $\mathbb{P}(X \leq x) = \mathbb{P}(X \leq x^j)$ for each permutation $x^j$. Clearly, it suffices to consider $x \in \{a_1, \ldots, a_n\}^n$, which is the largest possible support of $X$. Moreover, if $x_i < a_i$, then $x_k \leq x_i < a_i$ for $k = 1, \ldots, i$, which means there are at least $i$ components of $x$ less than $a_i$. This violates the fact that $X$ is in $\Pi(a) = \{a^\pi : \pi \in \Pi_n\}$ with probability one. Thus, we have $\mathbb{P}(X \leq x) = 0$ if there exists $i$ such that $x_i < a_i$. Hence, it suffices to check (27) for $x = (x_1, \ldots, x_n)$ with conditions

(i) $x_1 \leq \ldots \leq x_n$,

(ii) $x_i \geq a_i$ for each $i = 1, \ldots, n$, and

(iii) $x \in \{a_1, \ldots, a_n\}^n$.

Now, for $x$ satisfying condition (i)-(iii), let $m$ be the number of distinct values of components of $x$. We divide $\{1, \ldots, n\}$ into $m$ sets of indices $S_1, \ldots, S_m$, each containing indices of $x_1, \ldots, x_n$ with the same value. Let $k_j = \#S_j$, $\bar{k}_j = \sum_{i=1}^j k_i$, $j = 1, \ldots, m$, and $z_h = \max\{k : a_k \leq x_i, \ i \in S_h\}$. Condition (i) implies $1 \leq z_1 < \cdots < z_m \leq n$. For $h = 1, \ldots, m$, condition (ii) implies $a_{z_h} \geq a_i$ for all $i \in S_h$, and hence we have $z_h \geq \bar{k}_h$, $z_h - \bar{k}_{h-1} \geq k_h$, and $z_m = n$. Therefore,

$$\mathbb{P}(X \leq x) = \mathbb{P}(X_i \leq a_{z_h} \text{ for all } i \in S_h \text{ and } h = 1, \ldots, m)
= \frac{1}{n!} \cdot \frac{(z_1)!}{(z_1 - k_1)!} \cdot \frac{(z_2 - \bar{k}_1)!}{(z_2 - k_2)!} \cdots \frac{(z_m - \bar{k}_{m-1})!}{(z_m - k_m)!}
= \frac{z_1 \cdots (z_1 - \bar{k}_1 + 1)}{n \cdots (n - k_1 + 1)} \cdot \frac{(z_2 - \bar{k}_1) \cdots (z_2 - k_2 + 1)}{(n - k_1) \cdots (n - k_2 + 1)} \cdots \times 1^{k_m}.\]
As \((b - c) / (a - c) \leq b/a\) for \(0 < c < b < a\), we have
\[
\mathbb{P}(X \leq x) \leq \left( \frac{z_1}{n} \right)^{k_1} \times \left( \frac{z_2}{n} \right)^{k_2} \times \ldots \times 1^{k_m} = (F(a_{z_1}))^{k_1} (F(a_{z_2}))^{k_2} \ldots (F(a_{z_m}))^{k_m} = \prod_{i=1}^{n} \mathbb{P}(X_i \leq x_i).
\]

Therefore, for \(x\) satisfying (i)-(iii), the equation (27) holds. That is, \(U_a\) is NLOD. The conclusion of NUOD is symmetric to NLOD.

\[\square\]

**Proof of Proposition 9.** We can assume, without loss of generality, that \(a = 0\) and \(b = 1\). Let \(U_1^{(0)}, \ldots, U_n^{(0)} \sim U[0, 1]\) be iid random variables, and define

\[U^{(0)} = (U_1^{(0)}, \ldots, U_n^{(0)}) \quad \text{and} \quad U^{(m+1)} = \frac{\pi^{(m)} + U^{(m)}}{n}, \quad m = 0, 1, \ldots,
\]

where \(\pi^{(1)}, \pi^{(2)}, \ldots\) are iid random vectors of uniform permutations of \(\{0, \ldots, n-1\}\) independent of \(U^{(0)}\). For every \(m \in \mathbb{N}\), it holds that

1. \(U^{(m)}\) has \(U[0, 1]\) margins,

2. \(\text{Cor}(U_i^{(m)}, U_j^{(m)}) = -\frac{1}{n-1} \left(1 - \frac{1}{n^m}\right)\) for all \(i, j \in [n], i \neq j\), and

3. \(U^{(m)}\) is NSD.

The statements (1) and (2) follow from Theorem 4 of Craiu and Meng (2005). We prove (3) by induction. It is straightforward to check (3) for \(m = 0\). Suppose that \(U^{(m)}\) is NSD for \(m \geq 1\). By Theorem 3.2 of Hu (2000), the random vector \(\pi^{(m)}\) is NSD. Since \(U^{(m)}\) and \(\pi^{(m)}\) are independent, the sum \(\pi^{(m)} + U^{(m)} = nU^{(m+1)}\) is also NSD by P7 of Hu (2000). Finally, since \(g : x \in \mathbb{R} \mapsto x/n\) is an increasing function, we have that \(U^{(m+1)} = (g(nU_1^{(m+1)}), \ldots, g(nU_n^{(m+1)}))\) is NSD by P3 of Hu (2000).

Define

\[\bar{U}^{(m)} = \frac{U^{(0)}}{n^m} + \sum_{l=1}^{m} \frac{\pi^{(l)}}{n^l}.
\]

As documented in Section 5.1 of Craiu and Meng (2006), \(U^{(m)}\) and \(\bar{U}^{(m)}\) have identical joint distributions for every \(m \in \mathbb{N}\), and the sequence \(\{\bar{U}^{(m)}, m = 1, 2, \ldots\}\) converges point-wise to the well-defined limit \(\bar{U}^{(\infty)} := \sum_{l=1}^{\infty} \pi^{(l)}/n^l\). Therefore, \(U^{(m)}\) converges to \(\bar{U}^{(\infty)}\) in distribution. Since \(U^{(m)}\) is NSD for every \(m \in \mathbb{N}\), so is \(U^{(\infty)}\) by Theorem 3.9.12 of Muller and Stoyan (2002).
Therefore, $\tilde{U}^{(\infty)}$ is NSD with standard uniform margins. Moreover, $\tilde{U}^{(\infty)}$ is an $n$-joint mix since

$$\text{Var} \left( \sum_{i=1}^{n} \tilde{U}_i^{(\infty)} \right) = \lim_{m \to \infty} \text{Var} \left( \sum_{i=1}^{n} \tilde{U}_i^{(m)} \right) = \lim_{m \to \infty} \text{Var} \left( \sum_{i=1}^{n} U_i^{(m)} \right)$$

$$= \frac{n}{12} + \frac{n(n-1)}{12} \left( -\frac{1}{n-1} \right) = 0.$$ 

Since NSD implies NOD, we conclude that $\tilde{U}^{(\infty)}$ is the desired NOD $n$-joint mix of $U[0, 1]$. \qed