HALF-INTEGRAL ERDŐS-PÓSA PROPERTY OF DIRECTED ODD CYCLES

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Abstract. We prove that there exists a function \( f : \mathbb{N} \to \mathbb{R} \) such that every digraph \( G \) contains either \( k \) directed odd cycles where every vertex of \( G \) is contained in at most two of them, or a vertex set \( X \) of size at most \( f(k) \) hitting all directed odd cycles. This extends the half-integral Erdőş-Pósa property of undirected odd cycles, proved by Reed [Mangoes and blueberries. Combinatorica 1999], to digraphs.

1. Introduction

Erdős and Pósa [6] proved that for every (undirected) graph \( G \) and every positive integer \( k \), \( G \) contains either \( k \) pairwise vertex-disjoint cycles, or a vertex set of size \( O(k \log k) \) that hits all cycles of \( G \). This result has been extended to cycles satisfying various constraints: long cycles [28, 2, 8, 21, 4], cycles with modularity constraints [29, 9], cycles intersecting a prescribed vertex set [11, 22, 4, 9], and holes [15]. We refer to a survey of Raymond and Thilikos [23] for more examples. We say that a family \( F \) of graphs satisfies the Erdőş-Pósa property, if there is a function \( f : \mathbb{N} \to \mathbb{R} \) satisfying that every graph contains either \( k \) pairwise vertex-disjoint subgraphs each isomorphic to a graph in \( F \), or a vertex set of size at most \( f(k) \) hitting all subgraphs isomorphic to a graph in \( F \). It is also known that some variants of cycles do not satisfy this property. Reed [24] provided an example, called an Escher wall, which is illustrated in Figure 1 and argued that odd cycles do not satisfy the Erdőş-Pósa property. The Escher wall can be embedded into the projective plane, and every odd cycle must pass the crosscap odd times. Thus, there are no two vertex-disjoint odd cycles, but one can increase the minimum size of a hitting set for odd cycles as much as we want, by taking a larger construction. It is known that the family of odd cycles has the Erdőş-Pósa property on planar graphs [7].

On the positive side, Reed [24] showed that the half-integral relaxation of Erdőş-Pósa property holds for odd cycles. A family \( F \) of graphs satisfies the half-integral Erdőş-Pósa property, if there is a function \( f : \mathbb{N} \to \mathbb{R} \) satisfying that every graph \( G \) contains either \( k \) pairwise subgraphs \( H_1, H_2, \ldots, H_k \) each isomorphic to a graph in \( F \) such that every vertex of \( G \) is contained at most two of \( H_1, H_2, \ldots, H_k \) (so called a half-integral packing), or a vertex set of size at most \( f(k) \) hitting all subgraphs isomorphic to a graph in \( F \). Huynh, Joos, and Wollan [9] extended this half-integral property to \((\Gamma_1, \Gamma_2)\)-non-zero cycles in doubly-group labelled graphs, which include odd cycles, cycles not homologous to zero, odd \( S \)-cycles, and so on. Briefly speaking, each edge is oriented and labelled by elements from two groups \( \Gamma_1 \) and \( \Gamma_2 \), and a \((\Gamma_1, \Gamma_2)\)-non-zero cycle is an undirected cycle where the sum over all edges in the cycle is non-zero in each group. They conjectured that even for \( k > 2 \), the family of \((\Gamma_1, \Gamma_2, \ldots, \Gamma_k)\)-non-zero cycles has the half-integral Erdőş-Property. We remark that the orientation is simply used to define the sum over edges of a cycle, that is, when we consider an edge labelled by a group element \( g \) and contained in a cycle going through...
one fixed direction, we add \( g \) if the orientation of the edge is in the same direction, and otherwise, subtract \( g \). So, the general result on doubly-group labelled graphs does not imply the half-integral Erdős-Pósa properties for natural objects in digraphs.

The Erdős-Pósa property of cycles has been extended to \( H \)-minors or \( H \)-topological minors. As an application of the grid-minor theorem, Robertson and Seymour \[28\] proved that the family of \( H \)-minors has the Erdős-Pósa property if and only if \( H \) is planar. For all planar graphs \( H \) having a cycle, van Batenburg, Huynh, Joret, and Raymond \[30\] established the asymptotically tight bounding function \( O(k \log k) \) for \( H \)-minors. For \( H \)-topological minors, it is more complicated, and Liu, Postle and Wollan \[18\] announced that they characterized \( H \) for which \( H \)-topological minors have the Erdős-Pósa property. Thomas (See \[13\]) conjectured that for every graph \( H \), the family of \( H \)-minors has the half-integral Erdős-Pósa property, and recently, Liu \[17\] proved that both \( H \)-minors and \( H \)-topological minors satisfy the half-integral Erdős-Pósa property.

For digraphs, Reed, Robertson, Seymour, and Thomas \[25\] showed that the family of directed cycles has the Erdős-Pósa property. Masarík et al. \[20\] proved that while the bounding function of \[25\] is highly exponential, the bounding function for the half-integral Erdős-Pósa property of directed cycles is \( O(k^6) \). Kawarabayashi, Kráľ, Krčál, and Kreutzer \[12\] showed that the family of directed \( S \)-cycles (directed cycles meeting a prescribed set \( S \)) does not satisfy the Erdős-Pósa property, but it has the half-integral Erdős-Pósa property (which extends the earlier result by Kakimura and Kawarabayashi \[10\]). For directed odd cycles, it is easier to construct an example to show that they have no Erdős-Pósa property even on planar digraphs. We give a construction
in Figure $2$. It is natural to ask whether the family of directed odd cycles has the half-integral Erdős-Pósa property or not. We give a positive answer to this question.

A family of subgraphs $G_1, G_2, \ldots, G_m$ in a digraph $G$ is a half-integral packing if every vertex of $G$ is contained in at most two of $G_1, G_2, \ldots, G_m$.

**Theorem 1.1.** There is a function $f : \mathbb{N} \to \mathbb{R}$ such that for every digraph $G$ and every positive integer $k$, $G$ contains either a half-integral packing of $k$ directed odd cycles, or a vertex set of size at most $f(k)$ hitting all directed odd cycles.

A direct algorithmic application of Theorem 1.1 is that given a digraph $G$ and a positive integer $k$, one can in time $O(|V(G)|^{f(k)})$ output a half-integral packing of $k$ directed odd cycles, or a vertex set of size at most $f(k)$ hitting all directed odd cycles. We think that it can be used to solve the HALF-INTEGRAL $m$-DIRECTED ODD CYCLE PACKING problem in a similar running time, which asks whether a given digraph $G$ contains a half-integral packing of $m$ directed odd cycles. When Theorem 1.1 outputs a hitting set for directed odd cycles, the problem can be naturally reduced to the HALF-INTEGRAL $m$-DISJOINT PATHS problem on $G - T$ with additional parity constraints, where we want to find a half-integral packing of paths $P_1, \ldots, P_m$ that link given pairs of terminals $(a_1, b_1), \ldots, (a_m, b_m)$ with required parities. Unfortunately, even without parity constraints, it is not known whether HALF-INTEGRAL $m$-DISJOINT PATHS can be solved in polynomial time. We leave open questions related to Theorem 1.1 in this context. These problems are trivial when $m \leq 2$.

**Question 1.** For every fixed positive integer $m \geq 3$, can HALF-INTEGRAL $m$-DIRECTED ODD CYCLE PACKING be solved in polynomial time?

**Question 2.** For every fixed positive integer $m \geq 3$, can HALF-INTEGRAL $m$-DISJOINT PATHS be solved in polynomial time?

A closely related covering problem is the DIRECTED ODD CYCLE TRANSVERSAL problem which asks whether a given digraph has a set of at most $k$ vertices hitting all directed odd cycles. It was an open problem for a long time whether or not this problem is fixed parameter tractable (that is, solvable in time $f(k)\text{poly}(n)$). Recently, Lokshtanov, Ramanujan, Saurabh, and Zehavi \cite{19} proved that it is W[1]-hard, but admits an FPT approximation algorithm with approximation ratio 2.

Technically speaking, showing (half-integral) Erdős-Pósa property for digraphs is much harder than showing that for undirected graphs. This is because we need many more technical developments in (mostly) structure graph theory. For example, Erdős-Pósa’s result was proved in 1960s, but it took 30 years to extend this result to digraphs by Reed et al. \cite{25}. Let us also remark that the undirected version of Theorem 1.1 was shown by Reed \cite{24} more than 20 years ago. So it makes sense to sketch our proof of Theorem 1.1 based on technical developments on (directed) structure graph theory. For a digraph $G$, we denote by $\nu_2(G)$ the maximum size of a half-integral packing of directed odd cycles in $G$, and denote by $\tau(G)$ the minimum size of a hitting set for directed odd cycles in $G$. For each positive integer $k$, we define $t_k$ as the minimum integer such that for every digraph $G$ with $\nu_2(G) < k$, we have $\tau(G) \leq t_k$. Clearly, $t_1 = 0$.

We show in Lemma 4.1 that if $t_{k-1}$ exists and a digraph $G$ with $\nu_2(G) < k$ has a hitting set $T$ of directed odd cycles with $|T| = \tau(G)$, then $T$ is $2t_{k-1}$-well-linked, that is, for any $r$-subsets $A$ and $B$ of $T$ with $r \geq 2t_{k-1}$, there is a linkage from $A$ to $B$ of order $r$ in $G - (T \setminus (A \cup B))$. This is an adaption of \cite{25} (2.1]). Starting from this well-linked set, we can find a large cylindrical wall $W$ as a subgraph, along the directed grid theorem by Kawarabayashi and Kreutzer \cite{14}, so that there is no small separator between $W$ and $T$. See Figure 3 for an illustration of a cylindrical wall. We take $k$ disjoint subwalls of $W$ in a natural way, and we may assume that one of them, say $W_1$, has no directed odd cycles. Thus, $W_1$ is bipartite. Let $N$ be a large set of vertices of $W_1$ such that they have out-degree 2 or in-degree 2, and they are in the same part of the bipartition of $W_1$.

One of the main contributions in the paper is the half-integral min-max property of directed odd $X$-walks. More precisely, we show in Lemma 3.1 that given a digraph $G$ and a vertex set $X$, $G$
contains either $\ell$ directed odd $X$-walks where every vertex is used at most twice, or a vertex set of size less than $\ell$ hitting all directed odd $X$-walks. This can be used to find a large half-integral packing of directed odd cycles, a large half-integral packing of directed odd $X$-paths, or a small vertex set hitting all directed odd $X$-walks. We remark that for undirected graphs, odd $X$-paths have the Erdős-Pósa property [5], but for directed graphs, directed odd $X$-paths do not have the Erdős-Pósa property, see [3] Section 7.

We apply the lemma for odd $X$-walks to the set $X = N$ of the wall $W_1$. In case when there is a small hitting set $Y$ of directed odd $N$-walks, there is a strongly connected component $H$ of $G - Y$ containing most part of the well-linked set $T$, and it implies that most part of the cylindrical wall $W$ is also contained in $H$. On the other hand, if $H$ has a directed odd cycle, then one can find a directed odd $N$-walk, which leads to a contradiction. So, $Y$ together with $T \setminus V(H)$ gives a hitting set for all directed odd cycles, which is small. In the case when there are many odd $N$-walks, we show in Section [5] that we can use the bipartite cylindrical wall to find a half-integral packing of $k$ directed odd cycles. This completes the proof.

We remark that our approach can be adapted to simplify the proof of the half-integral Erdős-Pósa property of (undirected) odd cycles, given by Reed [24]. We can use the tangle truncation version of the grid-minor theorem [27] to obtain a similar result in Section [4] and then use the odd $A$-path theorem [5] in a similar way.

2. Preliminaries

Let $G$ be a digraph. We denote by $V(G)$ and $E(G)$ the vertex set and the edge set of $G$, respectively. If $(v,w)$ is an edge, then $v$ is its tail and $w$ is its head. For a directed path $P$, we denote by $end_i(P)$ the first vertex of $P$ and $end_k(P)$ the last vertex of $P$. For two vertex sets $A$ and $B$ in $G$, a directed path is an $(A,B)$-path if it starts at $A$ and ends at $B$, and all its internal vertices are not in $A \cup B$. We say that $G$ is strongly connected if for every pair of two vertices $v$ and $w$ in $G$, there are a directed path from $v$ to $w$ in $G$, and a directed path from $w$ to $v$ in $G$. A strongly connected component of $G$ is a maximal subgraph of $G$ that is strongly connected.

For a vertex set $A$ of $G$, we denote by $G - A$ the graph obtained from $G$ by removing all the vertices in $A$, and denote by $G[A]$ the subgraph of $G$ induced by $A$. For two digraphs $G$ and $H$, let $G \cup H := (V(G) \cup V(H), E(G) \cup E(H))$ and $G \cap H := (V(G) \cap V(H), E(G) \cap E(H))$.

For a set of vertex-disjoint directed paths $P_1, P_2, \ldots, P_m$ and a directed path or a directed cycle $Q$ such that $P_i \cap Q$ is a directed path for each $i \in \{1, 2, \ldots, m\}$, we say that $P_1, P_2, \ldots, P_m$ appear in this order on $Q$ if for all $1 \leq j < m$, $P_j \cap Q$ appears strictly before $P_{j+1} \cap Q$ when traversing $Q$ from $P_1 \cap Q$ to $P_m \cap Q$. If $P_i$ appears before $P_j$, then we denote by $Q[P_i : P_j]$ the minimal subpath of $Q$ containing all vertices of $(Q \cap P_i) \cup (Q \cap P_j)$ and the subpath of $Q$ from $P_i$ to $P_j$.

Let $t$ be a positive integer. A family of subgraphs $G_1, G_2, \ldots, G_m$ in a digraph $G$ is a (1/t)-integral packing if every vertex of $G$ is contained in at most $t$ of $G_1, G_2, \ldots, G_m$. When $t = 2$, we say that it is a half-integral packing.

2.1. Cylindrical walls. A cylindrical wall of order $k$, for some $k \geq 1$, is a digraph consisting of $k$ pairwise vertex-disjoint directed cycles $C_1, \ldots, C_k$, called columns, and a set of $2k$ pairwise vertex-disjoint paths $P_1, \ldots, P_{2k}$, called rows, such that

- for each $i \in \{1, 2, \ldots, k\}$ and $j \in \{1, 2, \ldots, 2k\}$, $C_i \cap P_j$ is a directed path with at least one edge,
- both endpoints of $P_i$ are in $V(C_1) \cup V(C_k)$,
- the paths $P_1, \ldots, P_{2k}$ appear in this order on each $C_i$ and
- for odd $i$ the cycles $C_i, \ldots, C_k$ occur on all $P_i$ in this order and for even $i$ they occur in reverse order $C_k, \ldots, C_1$. 
Figure 3. The cylindrical wall of order 4. The cycle $C_4$ is depicted using thick edges.

See Figure 3 for an illustration of a cylindrical wall of order 4. A vertex of a cylindrical wall with in-degree 2 or out-degree 2 is called a nail. A directed path of a cylindrical wall between nails whose all internal vertices are not nails is called a certifying path.

We will especially use cylindrical walls that do not contain directed odd cycles. Because of the following fact, the underlying undirected graph of such a wall is bipartite. So, we can naturally consider its bipartition as the bipartition of the underlying undirected graph.

**Proposition 2.1** (Folklore). Let $D$ be a strongly connected digraph that does not contain a directed odd cycle. Then, the underlying undirected graph of $D$ is a bipartite graph.

We say that a cylindrical wall is bipartite if its underlying undirected graph is bipartite.

### 2.2. Linkage.

For vertex sets $A$ and $B$ in $G$, a set of pairwise vertex-disjoint directed $(A, B)$-paths $\{P_1, P_2, \ldots, P_m\}$ is called a **linkage** from $A$ to $B$, and its order is defined to be $m$. For a positive integer $t$ and for vertex sets $A$ and $B$ in $G$, a set of directed $(A, B)$-paths $\{P_1, P_2, \ldots, P_m\}$ in $G$ is a $(1/t)$-integral linkage of order $m$ from $A$ to $B$ if every vertex of $G$ is contained in at most $t$ of $P_1, P_2, \ldots, P_m$. When $t = 2$, we say that it is half-integral.

When $\mathcal{P}$ is a linkage or a $(1/t)$-integral linkage, we consider $\mathcal{P}$ as the subgraph which is the union of all paths in $\mathcal{P}$. Thus, we denote by $V(\mathcal{P})$ and $E(\mathcal{P})$ as the set of all vertices and edges used by paths in $\mathcal{P}$, respectively.

A **separation** of a digraph $G$ is a pair $(A, B)$ of vertex sets of $G$ such that $A \cup B = V(G)$ and there are no edges from $A \setminus B$ to $B \setminus A$. The **order** of the separation $(A, B)$ is $|A \cap B|$. When $(A, B)$ is a separation, we call $A \cap B$ a **separator**. We use the Menger’s theorem.

**Theorem 2.2** (Menger’s theorem). Let $A$ and $B$ be vertex sets in a digraph $G$, and $k$ be a positive integer. Then $G$ contains either a linkage of order $k$ from $A$ to $B$, or a separation $(X, Y)$ of order less than $k$ such that $A \subseteq X$ and $B \subseteq Y$.

We will use the following observation.

**Lemma 2.3.** Let $t, m$ be positive integers, and $A, B$ be vertex sets in a digraph $G$. If there is a $(1/t)$-integral linkage $\mathcal{P}_1$ of order $mt$ from $A$ to $B$, then there is a linkage $\mathcal{P}_2$ of order $m$ from $A$ to $B$ such that $\mathcal{P}_2$ is a subgraph of $\mathcal{P}_1$. 
Lemma 3.1. Let \( \ell \) be a positive integer. Every digraph \( D \) with a vertex subset \( X \) contains either

1. a set of \( \ell \) directed odd \( X \)-walks such that every vertex of \( D \) is used at most two of them (so if a vertex is already used twice in some odd \( X \)-walk, then it does not appear in other odd \( X \)-walk), or

2. a vertex set \( Y \) of order at most \( \ell - 1 \) such that \( D - Y \) has no directed odd \( X \)-walk.

Proof. We obtain a bipartite digraph \( G = (A \cup A', B \cup B') \) from \( D \) as follows. Each vertex \( v \) in \( V(D) \setminus X \) is split in four vertices \( v_A, v_A', v_B, v_B' \) such that both \( v_A \) and \( v_A' \) have all incident edges of \( v \) with tail \( v \) and both \( v_B \) and \( v_B' \) have all incident edges of \( v \) with head \( v \), and moreover if there is an edge from \( v \) to \( u \), \( v_A \) (\( v_A' \), resp.) is adjacent only to \( u_B \) (\( u_B' \), resp.). For each \( v \in V(D) \setminus X \), add two edges \( v_e = (v_B, v_A') \) and \( v'_e = (v_B', v_A) \). Moreover, each vertex \( v \) in \( X \) is split in two vertices \( v_A, v_B \) such that \( v_A \) has all incident edges of \( v \) with tail \( v \) and \( v_B \) has all incident edges of \( v \) with head \( v \). Let \( A := \{v_A : v \in V(D)\}, B := \{v_B : v \in V(D)\}, A' := \{v_A' : v \in V(D) \setminus X\}, B' := \{v_B' : v \in V(D) \setminus X\}, \) and let \( X_A := \{v_A : v \in X\}, X_B := \{v_B : v \in X\} \). See Figure 4 for an illustration. This graph \( G \) has the following properties:

1. Each edge in \( E(D) \) corresponds to at most two edges in \( G \). More precisely, each edge with no endpoint in \( X \) corresponds to two edges in \( G \) (one in the induced bipartite graph \( (A, B) \) and the other in the induced bipartite graph \( (A', B') \), and each edge with an endpoint in

**Figure 4.** Examples of \( D \) and \( G \) in Lemma 3.1. The directed odd \( X \)-walk \( abde \) in \( D \) corresponds to the directed odd \( X \)-path \( a_A b_B b_A d_B' d_A e_B \) that has length 5.

Proof. We may assume that \( G = \mathcal{P}_1 \). We construct a graph \( G' \) by making \( t \) copies of every vertex. Let \( A' \) and \( B' \) be the vertex sets in \( G' \) obtained from \( A \) and \( B \), respectively. From \( \mathcal{P}_1 \), we can observe that there is a linkage of order \( m \) from \( A' \) to \( B' \) in \( G' \). It implies that there is no separator \((X, Y)\) of order less than \( m \) where \( A \subseteq X \) and \( B \subseteq Y \). Thus, by Menger’s theorem, there is a linkage \( \mathcal{P}_2 \) of order \( m \) from \( A \) to \( B \) such that \( \mathcal{P}_2 \) is a subgraph of \( \mathcal{P}_1 \).

A set \( T \) of vertices in a digraph \( G \) is an \( r \)-well-linked set if for all subsets \( A \) and \( B \) of \( T \) with \( |A| = |B| \geq r \), there is a linkage of order \( |A| \) from \( A \) to \( B \) in \( G - (T \setminus (A \cup B)) \) and there is a linkage of order \( |A| \) from \( B \) to \( A \) in \( G - (T \setminus (A \cup B)) \). A 1-well-linked set is called a well-linked set.

3. A lemma on directed odd \( X \)-walks

Given a digraph \( D \) and a vertex set \( X \subseteq V(D) \), a directed \( X \)-walk is a directed walk with both endpoints in \( X \) and all internal vertices in \( V(D) \setminus X \). Note that the two endpoints of a directed \( X \)-walk may be the same vertex. A directed \( X \)-walk is closed if its endpoints are the same. It is odd if its length is odd and even otherwise. A directed \( X \)-walk is a directed \( X \)-path if it is a path.
X corresponds to a single edge in G.) Moreover, for each \( v \in V(D) \setminus X \), we also add two "matching edges" \( v_e, v'_e \) to \( G \).

(2) For any vertex \( v \in V(D) \setminus X \), both \( v_A \) and \( v_A' \) (\( v_B \) and \( v_B' \), resp.) have only one edge \( v_e, v'_e \) with head \( v_A, v_A' \), respectively (\( v_B, v_B' \), resp.).

We claim that

(*) every directed path \( P \) from \( X_A \) to \( X_B \) in \( G \) corresponds to a directed odd \( X \)-walk in \( D \).

By construction of the bipartite digraph, every such a path \( P \) is of length 1 (mod 4) and alternates an edge in \( E(D) \) and the matching edges \( v_e, v'_e \) (for some \( v \in V(D) \setminus X \)). Thus \( P \) contains the even number of the matching edges. They correspond to a single vertex in \( V(D) \setminus X \). Thus \( P \) is of odd length and yields a directed odd \( X \)-walk.

Next, we claim that

(**) if there are \( \ell \) vertex-disjoint paths from \( X_A \) to \( X_B \), then the corresponding paths in \( D \) form a set of \( \ell \) directed odd \( X \)-walks such that every vertex of \( D \) is used at most two of them.

Each vertex in \( X \) appears twice in \( G \), so it is used in at most two of them. Each vertex in \( V(D) \setminus X \) appears four times in \( G \), but each path from \( X_A \) to \( X_B \) alternates an edge in \( E(D) \) and the matching edges \( v_e, v'_e \) (for some \( v \in V(D) \setminus X \)), so if one such a path contains \( v_B \) (\( v_B' \), resp.), then it must contain \( v'_e \) as well (\( v_A \), resp.). Hence no vertex of \( V(D) \setminus X \) is in at least three of the paths.

By (*) and (**), if there are \( \ell \) vertex-disjoint paths from \( X_A \) to \( X_B \) in \( G \), then we get the first conclusion. As an alternate, by Menger’s theorem, there is a vertex set \( Y' \) of order at most \( \ell - 1 \) in \( G \) such that there is no path from \( X_A \) to \( X_B \). We now take the corresponding vertex set \( Y' \) of order at most \( \ell - 1 \) in \( D \), and this corresponds to the second conclusion.

Lemma 3.1 leads the following consequence. We will use the following lemma on the set of nails of a cylindrical wall.

**Lemma 3.2.** Let \( k \) be a positive integer. Every digraph \( D \) with a vertex subset \( X \) contains either

(1) a half-integral packing of \( k \) directed odd cycles,

(2) a half-integral packing of \( k \) directed odd \( X \)-paths such that endpoints of them are distinct, or

(3) a vertex set \( Y \) of order at most \( 4k - 1 \) such that \( D - Y \) has no directed odd \( X \)-walk.

**Proof.** We apply Lemma 3.1 to \( D \) and \( X \) with \( \ell = 4k \). If the third conclusion happens, then we are done. Suppose there are \( 4k \) directed odd \( X \)-walks such that every vertex of \( D \) is used at most twice. If there are \( k \) directed odd \( X \)-walks such that each of them contains a directed odd cycle, then the first conclusion holds. So we may assume that there are at least \( 3k \) odd \( X \)-walks without directed odd cycles.

It is straightforward to check that every directed closed odd walk contains a directed odd cycle. Let \( W \) be a directed odd \( X \)-walk without directed odd cycles. We know that the two endpoints of \( W \) are distinct. Let \( W' \) be a subgraph of \( W \) that is a directed odd \( X \)-walk having the same endpoints as \( W \) such that \( W' \) is shortest. If \( W' \) contains a vertex that appears in two times, then \( W' \) contains a proper subgraph that is a closed walk. If this walk is odd, then it contains a directed odd cycle, a contradiction. On the other hand, if this closed walk is even, then we can remove this part to obtain a shorter odd walk. It contradicts the minimality of \( W' \). Thus, \( W' \) has no repeated vertex. Thus, \( W' \) is a directed odd \( X \)-path.

As there are at least \( 3k \) directed odd \( X \)-walks without odd cycles, by the above observation, we have at least \( 3k \) odd \( X \)-paths such that each vertex of \( D \) is used in at most two of them. By greedily choosing one \( X \)-path and removing two possible \( X \)-paths sharing an endpoint with it, we take at least \( k \) of them that have pairwise distinct endpoints, that is, each vertex in \( X \) is used in at most one of them. \( \square \)
4. From a well-linked set to a cylindrical wall

When Reed, Robertson, Seymour, and Thomas [25] proved that the family of directed cycles has the Erdős-Pósa property, they showed that in a minimal counterexample, the minimum-sized hitting set is well-linked. The lemma in fact can be applied to any type of connected families, and furthermore, applied to half-integrality. For completeness, we start with giving a proof for it.

We recall that \( t_k \) is the minimum integer such that for every graph \( G \) with \( \nu_2(G) < k \), we have \( \tau(G) \leq t_k \).

**Lemma 4.1.** Let \( k \geq 2 \) be an integer such that \( t_{k-1} \) exists. Let \( G \) be a digraph with \( \nu_2(G) < k \) and let \( T \subseteq V(G) \) with \( |T| = \tau(G) \) meeting all directed odd cycles in \( G \). Then \( T \) is \( (2t_{k-1}) \)-well-linked.

**Proof.** Let \( A, B \subseteq T \) be disjoint sets with \( |A| = |B| = r \) with \( r \geq 2t_{k-1} \). We claim that there is a linkage in \( G \) from \( A \) to \( B \) of order \( r \) containing no vertex in \( T \setminus (A \cup B) \).

Suppose that there is no such a linkage. Let \( Z = T \setminus (A \cup B) \). By Menger’s theorem applied to \( G - Z \), there is a separation \( (X, Y) \) of \( G \) with \( A \subseteq X, B \subseteq Y \) such that \( Z \subseteq X \cap Y \) and \( |(X \cap Y) \setminus Z| < r \). Let \( W := (X \cap Y) \setminus Z \).

Let \( T_A := (T \setminus A) \cup W \) and \( T_B := (T \setminus B) \cup W \). Since \( |W| = |(X \cap Y) \setminus Z| < r \), we obtain that \( |T_A| < |T| = \tau(G) \) and \( |T_B| < |T| = \tau(G) \). Thus, none of \( T_A \) and \( T_B \) is a hitting set for directed odd cycles. It means that there are a directed odd cycle \( C_A \) in \( G - T_A \), and a directed odd cycle \( C_B \) in \( G - T_B \). Since \( T \) is a hitting set for directed odd cycles, \( C_A \) must contain a vertex of \( A \) and \( C_B \) must contain a vertex of \( B \). So, \( G - Y \) contains \( C_A \) and \( G - X \) contains \( C_B \) while \( V(G - Y) \cap V(G - X) = \emptyset \).

By the definition of \( t_{k-1} \), \( G - Y \) has a hitting set \( M_Y \) of size at most \( t_{k-1} \), and \( G - X \) has a hitting set \( M_X \) of size at most \( t_{k-1} \). Then \( M_X \cup M_Y \cup (X \cap Y) \) is a hitting set for directed odd cycles in \( G \) of size at most \( 2t_{k-1} + (r - 1) + |Z| \leq 2t_{k-1} + (|T| - r) - 1 \). So, \( \tau(G) \leq 2t_{k-1} + \tau(G) - 1 - r \), and we have that \( r < 2t_{k-1} \), which contradicts the choice of \( r \).

In the proof of the result of Reed, Robertson, Seymour, and Thomas [25], they constructed a grid-like acyclic structure, called a \((p, q)\)-fence, starting from a well-linked set using a lemma analogous to Lemma 4.1 and showed that together with many disjoint paths from one terminal set (bottom) to the other terminal set (top), one can find many vertex-disjoint cycles. Motivated from their work, Kawarabayashi and Kreutzer [14] developed the directed grid theorem for digraphs. What they essentially argued is that if a digraph has sufficiently large directed tree-width, then it contains many disjoint paths from one terminal set to the other.

A cylindrical wall is similar to the cylindrical wall, but the difference is that for every \( P_i \) and \( C_j \), they intersect on a vertex. We will not define the butterfly-minor operation. It is well known that every graph having a large cylindrical grid as a butterfly-minor also has a large cylindrical wall as a subgraph.

For our purpose, starting from a large \((2t_{k-1})\)-well-linked set given from Lemma 4.1, we want to find a large cylindrical wall as a subgraph, and furthermore, find such a wall so that there is no separation of small order between the wall and the well-linked set. Slightly modifying the proof in the directed grid theorem by Kawarabayashi and Kreutzer [14], we can state as follows.

**Theorem 4.2 (Kawarabayashi and Kreutzer [14]).** For positive integers \( r \) and \( w \) with \( w \geq r \), there exists an integer \( f_{\text{wall}}(r, w) \) satisfying the following. Let \( G \) be a digraph and \( T \) be an \( r \)-well-linked set of size \( f_{\text{wall}}(r, w) \). Then \( G \) has a cylindrical wall \( W \) of order \( w \) as a subgraph such that for every subset \( A \) of \( T \) of size at least \( |T|/2 \) and every \( t \)-subset \( B \) of \( V(W) \) consisting of elements in distinct columns with \( r \leq t \leq w \), there are a linkage from \( A \) to \( B \) of order \( t \) in \( G \), and a linkage from \( B \) to \( A \) of order \( t \) in \( G \).

We explain how this modification works. We introduce some necessary concepts that appear in the proof of the directed grid theorem [14].
A bramble in a digraph $G$ is a set $\mathcal{B}$ of strongly connected subgraphs of $G$ such that for all $B_1, B_2 \in \mathcal{B}$, $V(B_1) \cap V(B_2) \neq \emptyset$ or there are edges $e_1$ and $e_2$ such that $e_1$ links from $B_1$ to $B_2$ and $e_2$ links from $B_2$ to $B_1$. A cover of a bramble is a vertex set $X$ of $G$ such that $V(B) \cap X = \emptyset$ for all $B \in \mathcal{B}$. The order of a bramble is the minimum size of a cover of $\mathcal{B}$. The bramble number of $G$, denoted by $bn(G)$, is the maximum order of a bramble in $G$.

For a positive integer $k$, a set $S$ of vertices in a digraph $G$ is $k$-linked if for every set $X \subseteq V(G)$ with $|X| < k$, there is a unique strongly connected component of $G - X$ that contains more than half of the vertices in $S$. The linkedness of $G$, denoted by $\text{link}(G)$, is the maximum integer $k$ for which $G$ contains a $k$-linked set.

Similar to the undirected case, it is well known that the directed tree-width, bramble number, linkedness, and the maximum size of a well-linked set are all related to each other; we refer to [11, Section 9], [16, Section 6], and [26] for discussions. Among the relations, we use the following.

**Lemma 4.3** (See Lemma 6.4.23 of [16]). Let $k, r$ be positive integers with $k \geq r$. Every $r$-well-linked set of order $4k + 1$ is a $k$-linked set.

Also, we use the following property of brambles.

**Lemma 4.4** (Lemma 4.3 of [14]). Let $G$ be a digraph and $\mathcal{B}$ be a bramble of $G$. Then there is a path $P$ intersecting every set in $\mathcal{B}$.

In [14, Lemma 4.4], they showed that if $G$ contains a bramble $\mathcal{B}$ of order $k(k + 2)$, then it contains a path $P$ intersecting every element of $\mathcal{B}$ and a set $A \subseteq V(P)$ of order $k$ such that $A$ is well-linked. From these $A$ and $P$, they constructed a large cylindrical grid as a butterfly-minor.

There are two points to consider when we modify the proof. First, we are starting with a specific bramble $\mathcal{B}$ rather than a bramble. We want to find a closely related wall, and so, we need to find a bramble related to $\mathcal{B}$. This is not difficult; we can adapt the known proof that gives a relation between a bramble number and the maximum size of a well-linked set (along the linkedness of $G$). More importantly, we have to impose that when we apply [14, Lemma 4.4] to the obtained bramble and have a pair of $A$ and $P$, we need an additional property that $T$ and $A$ are well connected. This additional property will be used in the next section.

**Lemma 4.5.** For all positive integers $r$ and $p$, there exists an integer $f_{\text{path}}(r,p)$ satisfying the following. If $G$ contains an $r$-well-linked set $T$ of size at least $f_{\text{path}}(r,p)$, then it contains a path $P$ and $A \subseteq V(P)$ such that

- $A$ is a well-linked set of size $p$, and
- for all subsets $X$ of $A$ and subsets $Y$ of $T$ with $r \leq t \leq \left\lfloor \frac{p+1}{2} \right\rfloor$, $|X| = t$ and $|Y| \geq |T|/2$, there is a linkage of order $t$ from $X$ to $Y$ in $G$ and a linkage of order $t$ from $Y$ to $X$ in $G$.

**Proof.** We set

- $f_1(r,p) = p(p + 2)$,
- $f_{\text{path}}(r,p) = 4f_1(r,p) + 1$.

Let $T$ be an $r$-well-linked set of size $m \geq f_{\text{path}}(r,p)$. By Lemma 4.3, $T$ is an $f_1(r,p)$-linked set.

We construct a bramble $\mathcal{B}$ of order $f_1(r,p)$ as follows. By definition of a $k$-linked set, for every vertex subset $X$ of $G$ with $|X| < f_1(r,p)$, $G - X$ has a unique strongly connected component $C_X$ containing more than half of the vertices of $T$. We define $\mathcal{B} := \{C_X : X \subseteq V(G), |X| < f_1(r,p)\}$. As every pair of two sets in $\mathcal{B}$ intersects on $T$, $\mathcal{B}$ is a bramble. The order of $\mathcal{B}$ is at least $f_1(r,p)$, because for every set $Y$ of size less than $f_1(r,p)$, $Y$ does not hit $C_Y \in \mathcal{B}$. On the other hand, since $T$ is a cover of size $f_1(r,p)$, the order of $\mathcal{B}$ is exactly $f_1(r,p)$.

By Lemma 4.4, there is a path $P$ intersecting every element of $\mathcal{B}$. To find a proper set $A$ in $P$, we follow the proof of [14, Lemma 4.4].

We construct sequences of subpaths $P_1, \ldots, P_{2p}$ of $P$ and brambles $\mathcal{B}_1, \ldots, \mathcal{B}_{2p} \subseteq \mathcal{B}$. Let $P_1$ be the minimal initial subpath of $P$ such that $\mathcal{B}_1 = \{B \in \mathcal{B} : V(B) \cap V(P_1) \neq \emptyset\}$ is a bramble of order
$\lceil \frac{p+1}{2} \rceil$. Let $v$ be the last vertex of $P_i$ and $s$ be the successor of $v$ in $P$. Let $P_{i+1}$ be the minimal subpath of $P$ starting at $s$ such that
\[
B_{i+1} = \left\{ B \in \mathcal{B} : V(B) \cap \left( \bigcup_{1 \leq j \leq i} V(P_j) \right) = \emptyset \text{ and } V(B) \cap V(P_{i+1}) \neq \emptyset \right\}
\]
has order $\lceil \frac{p+1}{2} \rceil$. As $\mathcal{B}$ has order $p(p+2)$, we can construct the sequences $P_1, \ldots, P_{2p}$ and $B_1, \ldots, B_{2p} \subseteq \mathcal{B}$. For each $i \in \{1, 2, \ldots, p\}$, let $a_i$ be the first vertex of $P_{2i}$, and let $A = \{a_1, \ldots, a_p\}$.

In \cite[Proposition 5.1.]{14}, they showed that $A$ is well-linked. So, we concentrate on showing the third property. Let $X \subseteq A$ and $Y \subseteq T$ with $r \leq t \leq p$, $|X| = t$ and $|Y| \geq \lceil T \rceil/2$. Let $X = \{a_{i_1}, a_{i_2}, \ldots, a_{i_t}\}$.

Suppose for contradiction that there is no linkage of order $t$ from $X$ to $Y$ in $G$. Then, by Menger’s theorem, there is a separation $(C, D)$ of order less than $t$ with $X \subseteq C$ and $Y \subseteq D$.

As $|C \cap D| < t \leq \lceil \frac{p+1}{2} \rceil$ and each $B_i$ has order exactly $\lceil \frac{p+1}{2} \rceil$, there is an index $\ell$ and $B \in B_{2\ell}$ such that $S \cap (V(P_{2\ell}) \cup V(B)) = \emptyset$. Then $V(B) \subseteq C \setminus D$, $B$ does not intersect $Y \subseteq D \setminus C$. It contradicts the fact that every set of $B$ contains more than half of the vertices in $T$.

We conclude that there is a linkage of order $t$ from $X$ to $Y$, and in the same way, we can show that there is a linkage of order $t$ from $Y$ to $X$.

In the rest of the proof of the directed grid theorem \cite[14], they kept having a large portion of the set $A$ in the intermediate structures. Thus, we have a proof of Theorem 4.2 following the original proof.

5. Half-integral Erdős-Pósa property of directed odd cycles

We prove Theorem 4.2. We may assume that a given graph $G$ has a large cylindrical wall, say $W$, by Lemma 4.1, and Theorem 4.2. By considering one of $k$ vertex-disjoint subgrids of $W$, we may assume that $W$ has a large bipartite subgrid $W'$. We will apply Lemma 3.2 to the set of nails of $W'$ on the same part. The following proposition tells that in the case when we have a small hitting set from Lemma 3.2 we can hit all directed odd cycles by a small set.

**Proposition 5.1.** Let $f_{wall}$ be the function defined in Theorem 4.2. Let $r, t, w$ be positive integers with $t \geq r$ and $w \geq t+1$, and let $G$ be a digraph. Let $T$ be a hitting set of directed odd cycles that is also an $r$-well-linked set of size $f_{wall}(r, w)$ in $G$, and $W$ be a bipartite cylindrical wall of order $w$ in $G$ satisfying that for every subset $A$ of $T$ of size at least $|T|/2$ and every $q$-subset $B$ of $V(W)$ consisting of vertices in distinct columns with $r \leq q \leq w$, there are a linking from $A$ to $B$ of order $q$ in $G$, and a linking from $B$ to $A$ of order $q$ in $G$. Let $N$ be a set of nails in the same bipartition of $W$ having at least half of the nails of $W$.

If $X$ is a vertex set of size less than $t$ hitting all directed odd $N$-walks, then $G$ has a vertex set of size at most $3t$ hitting all directed odd cycles.

**Proof.** Let $H_1, H_2, \ldots, H_m$ be the set of all strongly connected components of $G - X$, and assume that it is ordered in a topological ordering, that is, for distinct integers $i, j \in \{1, 2, \ldots, m\}$, there is an edge from $H_i$ to $H_j$ only if $j > i$.

As $T$ is an $r$-well-linked set of size $g \geq 4t+1$, by Lemma 4.3, $T$ is $t$-linked. Since $X$ has size less than $t$, by the definition of a $t$-linked set, $G - X$ has a unique strongly connected component, say $H_x$, having more than half of the vertices in $T$. Because $T$ is $r$-well-linked and $t \geq r$, $H_1 \cup \cdots \cup H_{x-1}$ contains less than $t$ vertices of $T$, and similarly, $H_{x+1} \cup \cdots \cup H_m$ contains less than $t$ vertices of $T$.

As $w \geq t+1$, there are at least $2w - (t-1) \geq 2w + 2$ rows $P$ of $W$ containing no vertices of $T$. If $B$ is a set of $t$ nails in $P$ that are contained in distinct columns of $W$, then by the assumption, there are a linking of order $t$ from $B$ to $T \cap V(H_x)$, and a linking of order $t$ from $T \cap V(H_x)$ to $B$ in $G$. It implies that $P$ has to be contained in $H_x$, and thus $H_x$ fully contains $w + 2$ rows of $W$. 

Since \( N \) contains at least half of the nails of \( W \) and \( H_x \) contains \( w + 2 \) rows of \( P \), \( H_x \) contains at least two nails in \( N \), say \( v \) and \( z \).

Assume that \( H_x \) contains a directed odd cycle \( H \). Since \( H_x \) is strongly connected, there is a directed path \( P_v \) from \( v \) to \( H \) in \( H_x \), and there is a directed path \( P_z \) from \( H \) to \( z \) in \( H_x \). In \( H \cup P_v \cup P_z \), there are two directed odd walks from \( v \) to \( z \), namely, one is obtained by using the shortest path from the endpoint of \( P_v \) to the endpoint of \( P_z \) in \( H \), and the other one is obtained by traversing \( H \) one more. As \( H \) is an odd cycle, the two walks have different parities. So \( G \) contains a directed odd walk between two nails of \( W \) that is contained in \( H_x \), a contradiction. Thus, \( H_x \) has no directed odd cycle.

For other strongly connected components \( H_y \neq H_x \), if it contains a directed odd cycle, then \( T \cap V(H_y) \) must intersect this odd cycle. Therefore, \((T \cap (V(G) \setminus V(H_x))) \cap X \) hits all directed odd cycles, and it has size at most \( 3 \ell \).

By Proposition 5.1, we may assume that Lemma 3.2 outputs a large half-integral packing of odd paths whose endpoints are distinct nails of \( W' \). The rest part devotes to find a half-integral packing of \( k \) directed odd cycles from it.

Let \( W \) be a bipartite cylindrical wall in a digraph \( G \). A walk \( P \) of \( G \) with endpoints \( v \) and \( w \) in \( W \) is parity-breaking if the parity of the path from \( v \) to \( w \) in \( P \) is different from the parity of a path from \( v \) to \( w \) in \( W \). If the parities are the same, then we say that \( P \) is parity-preserving.

**Proposition 5.2.** There is a function \( g : \mathbb{N} \rightarrow \mathbb{R} \) satisfying the following. Let \( k \) be a positive integer, and let \( W \) be a bipartite cylindrical wall of order at least \( (2k + 1)(6g(k) + 1) \) in a digraph \( G \). Let \( N \) be a set of nails of \( W \) such that the vertices in \( M \) are contained in the same part of the bipartition of \( W \). Let \( U \) be a half-integral packing of \( 12g(k) - 1 \) odd \( N \)-paths in \( G \) such that the endpoints of paths in \( U \) are disjoint. Then \( G \) contains a half-integral packing of \( k \) directed odd cycles.

The following lemma will be crucial in the proof of Proposition 5.2. We use a duplication argument as in Lemma 3.1.

**Lemma 5.3.** Let \( k \) and \( m \) be positive integers. Let \( G \) be a digraph, \( W \) be a bipartite cylindrical wall in \( G \), and \( A, B, C, D \) are subsets of \( V(W) \) of size \( m \). Let \( Q \) be a linkage of order \( m \) from \( A \) to \( B \) in \( W \), and let \( R \) be a linkage of order \( m \) from \( C \) to \( D \) in \( W \). Let \( U \) be a half-integral packing of \( m \) parity-breaking paths from \( B \) to \( C \) in \( G \). If \( m \geq 8k \), then either there is a half-integral packing of \( k \) directed odd cycles, or there is a half-integral packing of \( k \) parity-breaking paths from \( A \) to \( D \) in \( Q \cup R \cup U \).

**Proof.** Note that \( Q \cup R \) is a subgraph of \( W \). We construct a graph \( F_1 \) starting from the vertex set \( V(Q) \cup V(R) \) and the empty edge set such that

- for every edge \((u, v)\) in \( E(Q) \cup E(R) \), we add a new vertex \( x_{uv} \) and two edges \((u, x_{uv})\) and \((x_{uv}, v)\),
- for every \( W \)-subpath \( P \) of some path in \( U \) that is from a vertex \( u \) to a vertex \( v \), if \( P \) is parity-breaking, then we add an edge \((u, v)\), and if otherwise, we add a vertex \( z_{uv} \) and two edges \((u, z_{uv})\) and \((z_{uv}, v)\),
- for every \( q \in A \), we add two new vertices \( q^1 \) and \( q^2 \) and add edges \((q^1, q^2)\) and \((q^2, q)\), and
- for every \( r \in B \), we add two new vertices \( r^1 \) and \( r^2 \) and add edges \((r, r^2)\) and \((r^2, r^1)\).

We assign \( A_1 := \{q^1 : q \in A\} \) and \( D_1 := \{r^1 : r \in D\} \). It is not difficult to observe that a walk between two vertices of \( W \) in \( Q \cup R \cup U \) is parity-breaking if and only if the corresponding walk in \( F_1 \) is odd. Note that two paths in \( U \) may share a vertex on a path in \( Q \cup R \). Thus, there is a set of \( m \) odd walks from \( A_1 \) to \( D_1 \) in \( F_1 \) such that every vertex of \( G \) is used at most 4 times. Observe that for each vertex \( w \in A_1 \cup D_1 \), there is exactly one walk containing \( w \) in the \( m \) odd walks.

Now, we construct a bipartite digraph \( F_2 = (X \cup X', Y \cup Y') \) from \( F_1 \) as follows. Each vertex \( v \) of \( F_1 - (A_1 \cup D_1) \) is split in four vertices \( v_X, v_{X'}, v_Y, v_{Y'} \) such that both \( v_X \) and \( v_{X'} \) have all incident edges.
edges of \( v \) with tail \( v \) and both \( v_Y \) and \( v_Y' \) have all incident edges of \( v \) with head \( v \), and moreover if there is an edge from \( v \) to \( u \), \( v_X \) (\( v_Y', \) resp.) is adjacent only to \( u_Y \) (\( u_Y', \) resp.). For each vertex \( v \) in \( A_1 \), let \( v_X := v \), and for each vertex \( v \) in \( D_1 \), let \( v_Y := v \). For each \( v \in V(F_1) \setminus (A_1 \cup D_1) \), add an edge \( v_e \) with tail \( v_Y \) and head \( v_X \) and add an edge \( v'_e \) with tail \( v_Y' \) and head \( v_X \). Let \( A_2 \) be the vertex set in \( F_2 \) obtained from \( A_1 \), and \( B_2 \) be the vertex set in \( F_2 \) obtained from \( B_1 \).

Now, in \( F_2 \), there is a set of \( m \) walks from \( A_2 \) to \( D_2 \) in \( F_2 \) such that every vertex is used at most 4 times, because each walk from \( A_2 \) to \( D_2 \) in \( F_2 \) corresponds to an odd walk from \( A_1 \) to \( D_1 \) in \( F_1 \). So, there is a 1/4-integral packing of \( m \) paths from \( A_2 \) to \( D_2 \) in \( F_2 \). Since \( m \geq 8k \), by Lemma 2.3 there is a linkage of order \( 2k \) from \( A_2 \) to \( D_2 \) in \( F_2 \). It implies that there is a set \( L \) of \( 2k \) odd walks from \( A_1 \) to \( D_1 \) in \( F_1 \) such that every vertex of \( F_1 \) is used at most twice.

Note that a directed odd cycle in \( F_1 \) corresponds to a parity-breaking cycle in \( G \). A parity-breaking cycle should be a directed odd cycle, because its parity is different from a possible cycle in the bipartite digraph \( W \).

Each odd walk from \( A_1 \) to \( D_1 \) contains either a directed odd cycle, or a directed odd path from \( A_1 \) to \( D_1 \). So, either there is a half-integral packing of \( k \) directed odd cycles in \( F_1 \), or there is a half-integral packing of \( k \) directed odd paths from \( A_1 \) to \( D_1 \) in \( F_1 \). In the former case, we find a half-integral packing of \( k \) odd cycles in \( G \). In the latter case, there is a half-integral packing of \( k \) parity-breaking paths from \( A \) to \( D \) in \( G \), as required. \( \Box \)

**Proof of Proposition 5.2.** Let \( w \) be the order of \( W \). We set

- \( g_2(k) = 8k \),
- \( g_1(k) = 4g_2(k) \),
- \( g(k) = 256g_1(k)^4 \).

Since every path in \( U \) is an odd path between two nails in the same part of the bipartition of \( W \), every path in \( U \) is parity-breaking. We start with finding subpaths of some paths in \( U \) so that they are still parity-breaking and do not intersect many certifying paths of \( W \).

We claim that for all \( t \leq g(k) \), there is a half-integral packing of paths \( U_1, U_2, \ldots, U_t \) in \( G \) satisfying that for each \( i \in \{1, 2, \ldots, t\} \),

- \( U_i \) is a parity-breaking path whose endpoints are in \( W \),
- \( U_1 \cup \cdots \cup U_i \) intersects at most \( 6t \) certifying paths of \( W \), and
- the certifying paths containing the endpoints of \( U_i \) are not used by paths in \( U_1, \ldots, U_{i-1} \).

We prove the statement by induction on \( t \). Assume that such a set of paths \( U_1, \ldots, U_{t-1} \) has been constructed for some \( t \leq g(k) \). By the second assumption, \( U_1 \cup \cdots \cup U_{t-1} \) intersects a set \( A \) of at most \( 6(t - 1) \leq 6(g(k) - 1) \) certifying paths of \( W \). Let \( B := \bigcup_{Q \in A} V(Q) \). Note that \( B \) contains at most \( 12(g(k) - 1) \) nails. Since \( |U| = 12(g(k) - 1) + 1 \) and the endpoints of paths in \( U \) are disjoint, there is a path \( U \in U \) such that the endpoints of \( U \) are not contained in \( B \).

Let \( U = u_{1}u_{2}\cdots u_{m} \). We color \( u_{i} \) red if \( u_{i} \in V(W) \setminus B \), and color blue for other vertices. By the choice of \( U \), the endpoints of \( U \) are colored by red. A subpath \( Q \) of \( U \) is a blue-interval if its endpoints are colored by red, and all its internal vertices are colored by blue. Clearly \( U \) is decomposed into edge-disjoint blue-intervals.

Since \( U \) is parity-breaking, at least one of blue-intervals of \( U \) is parity-breaking. Let \( U' \) be a parity-breaking blue-interval of \( U \). Note that every vertex of \( W \) is contained in at most 4 certifying paths. Since all the internal vertices of \( U' \) are contained in \( B \), \( U_1 \cup U_2 \cup \cdots \cup U_{t-1} \cup U' \) uses at most \( 6(t - 1) + 6 \leq 6t \) certifying paths of \( W \), and the certifying paths containing the endpoints of \( U_i \) are not used by paths in \( U_1, \ldots, U_{t-1} \). Thus, the claim holds.

So, by the above claim, there is a half-integral packing of parity-breaking paths \( U_1, U_2, \ldots, U_{g(k)} \) that intersect at most \( 6g(k) \) certifying paths.

Note that the order of \( W \) is at least \( 2k(6g(k)+1) = (2k-1)(6g(k)+1)+6g(k)+1 \). So, there is a set of consecutive columns, say \( C_{x+1}, C_{x+2}, \ldots, C_{x+2k} \), containing no vertices of \( B \). Also, \( W \) has at
then we can take a half of the tuples so that $c_i$.

Note that if we have a subset $d_i$ such that $i \in I$ is distinct. Furthermore, we take a subset $I$ containing no vertices of $B$ such that $P_{y+1}$ is a row from $C_1$ to $C_w$. We define $W^*$ as the union of $P_{y+1}, P_{y+2}, \ldots, P_{y+4k}$. See Figure 5 for an illustration of $W^*$. Observe that $V(W^*) \cap B = \emptyset$.

We assign a tuple $L(v)$ for each nail $v$. Let $i \in \{1, 2, \ldots, w\}$ and $j \in \{1, 2, \ldots, 2w\}$. If we traverse $C_i$ from $P_1$ to $P_{2w}$, then $C_i$ contains two nails in $P_j$, and for the first vertex $v$, we assign $L(v) := (i, j, 1)$ and for the second vertex $v$, we assign $L(v) := (i, j, 2)$.

For each $i \in \{1, 2, \ldots, g(k)\}$, we define that

- if $\text{end}_i(U_i)$ is a nail, then $(a_i, b_i, c_i) := L(\text{end}_i(U_i))$ and $A_i := G[\{\text{end}_i(U_i)\}]$, and otherwise, $(a_i, b_i, c_i) = L(v)$, where $v$ is the first vertex $v$ of the certifying path containing $\text{end}_i(U_i)$ and $A_i$ is the subpath from $v$ to $\text{end}_i(U_i)$ in the certifying path,

- if $\text{end}_b(U_i)$ is a nail, then $(a_i, b_i, c_i) := L(\text{end}_b(U_i))$ and $B_i := G[\{\text{end}_b(U_i)\}]$, and otherwise, $(a_i, b_i, c_i) = L(v)$, where $v$ is the first vertex $v$ of the certifying path containing $\text{end}_b(U_i)$ and $B_i$ is the subpath from $\text{end}_b(U_i)$ to $v$ in the certifying path.

By the pigeon-hole principle, there is a subset $I \subseteq \{1, 2, \ldots, g(k)\}$ of size $\frac{g(k)1^4}{4} = g_1(k)$ such that

- either (all integers in $A_i : i \in I$ are distinct), or (all integers in $A_i : i \in I$ are the same, all integers in $B_i : i \in I$ are distinct, and all integers in $c_i : i \in I$ are the same), and

- either (all integers in $d_i : i \in I$ are distinct), or (all integers in $d_i : i \in I$ are the same, all integers in $e_i : i \in I$ are distinct, and all integers in $f_i : i \in I$ are the same).

Note that if we have a subset $J$ of $\{1, 2, \ldots, g(k)\}$ so that all integers in $A_i : i \in J$ are the same, then we can take a half of the tuples so that $c_i$ are all the same. The same argument holds for $d_i$ and $e_i$ as well. Thus, we can achieve the bound $\frac{g(k)1^4}{4}$ (we can take half of the sets at the end together). Furthermore, we take a subset $I_1 \subseteq I$ of size $g_1(k)/4 = g_2(k)$ such that

- if $a_{i_1} = a_{i_2}$ for some $i_1, i_2 \in I_1$, then $|b_{i_1} - b_{i_2}| \geq 2 \mod w$,
• if \( d_{i_1} = d_{i_2} \) for some \( i_1, i_2 \in I_1 \), then \( |e_{i_1} - e_{i_2}| \geq 2 \) (mod \( w \)).

We can greedily choose elements of \( I_1 \) from \( I \).

Now, we construct a linkage \( \{ S_i : i \in I_1 \} \) from \( W^* \) to \( \{ \text{end}_i(U_i) : i \in I_1 \} \), and a linkage \( \{ T_i : i \in I_1 \} \) from \( \{ \text{end}_b(U_i) : i \in I_1 \} \) to \( W^* \), and apply Lemma 5.3. If all integers in \( \{ a_i : i \in I_1 \} \) are distinct, then \( S_i \) is the union of \( A_i \) and \( C_{a_i}[P_{y+2k} : P_b] \) (recall that it is a minimal subpath of \( C_{a_i} \) containing \( P_{y+2k} \cap C_{a_i} \), \( P_b \cap C_{a_i} \), and the subpath of \( C_{a_i} \) from \( P_{y+2k} \cap C_{a_i} \) to \( P_b \cap C_{a_i} \)). Otherwise, we know that all integers in \( \{ b_i : i \in I_1 \} \) are distinct. When \( \alpha \) is a path from \( C_w \) to \( C_w \) and \( a_i > z + k \), \( T_i \) is the union of \( B_i \) and \( P_{y}[C_{z + k} : B_i] \). If \( \alpha \) is a path from \( C_w \) to \( C_w \) and \( a_i < z \), then \( \alpha \) is the union of \( B_i \) and \( \alpha_{z-1}[C_{z+1} : C_{a_i}] \), and \( C_{a_i}[P_{b_{i-1}} : P_b] \). We define similarly when \( \alpha \) is a path from \( C_w \) to \( C_w \) by using one of \( P_{b_i} \) and \( P_{b_{i-1}} \). By the choice of \( I_1 \), we can observe that all paths in \( \{ S_i : i \in I_1 \} \) are pairwise vertex-disjoint.

We define paths \( T_i \) in a similar way. If all integers in \( \{ d_i : i \in I_1 \} \) are distinct, then \( T_i \) is the union of \( D_i \) and \( C_{a_i}[P_{d_i} : P_{y+1}] \). Otherwise, we know that all integers in \( \{ e_i : i \in I_1 \} \) are distinct. When \( \beta \) is a path from left to right and \( d_i > z + k \), \( T_i \) is the union of \( E_i \) and \( C_{d_i}[P_{e_i} : P_{y+1}] \), and \( P_{e_i}[C_{d_i} : C_{z + k}] \). If \( \beta \) is a path from left to right and \( a_i < z \), then \( \beta \) is the union of \( E_i \) and \( P_{e_i}[C_{d_i} : C_{z + k}] \). We define similarly when \( \beta \) is a path from right to left by using one of \( P_{e_i} \) and \( P_{e_{i-1}} \). Again, by the choice of \( I_1 \), we can observe that all paths in \( \{ T_i : i \in I_1 \} \) are pairwise vertex-disjoint.

We apply Lemma 5.3 for linkages \( \{ S_i : i \in I_1 \}, \{ T_i : i \in I_1 \}, \) and a half-integral packing of parity-breaking paths \( \{ U_i : i \in I_1 \} \). Since \( g_2(k) = 8k \), by Lemma 5.3 either there is a half-integral packing of \( k \) directed odd cycles, or there is a half-integral packing of parity-breaking paths \( L = \{ L_1, L_2, \ldots, L_k \} \) from \( W^* \cap \{ S_i : i \in I_1 \} \) to \( W^* \cap \{ T_i : i \in I_1 \} \). It is not difficult to see that for each \( i \in \{ 1, 2, \ldots, k \} \), the endpoints of \( L_i \) in \( W^* \) can be linked using \( C_{z + i}, C_{z + i+1}, P_{y+2i-1}, P_{y+2i}, P_{y+2k+2i-1}, P_{y+2k+2i} \) and rows and columns containing the endpoints. If an endpoint of \( L_i \) lies on \( C_{z+i} \), then we use \( C_{z+i} \) and if it is on \( C_{z+4k} \), we use \( C_{z+k+i} \). Similarly, if the endpoint lies on \( P_{y+1} \), then we use \( P_{y+2i-1} \cup P_{y+2i} \), and if it lies on \( P_{y+4k} \), then we use \( P_{y+2k+2i-1} \cup P_{y+2k+2i} \). Let \( Q_i \) be such a path. By the construction, we observe that \( \{ Q_1, Q_2, \ldots, Q_k \} \) is also a half-integral linkage. Since each \( L_i \cup Q_i \) contains a directed odd cycle, we have a half-integral packing of \( k \) directed odd cycles.

We now prove Theorem 1.1. We recall that \( \nu_2(G) \) is the maximum size of a half-integral packing of directed odd cycles in \( G \), and \( \tau(G) \) is the minimum size of a hitting set for directed odd cycles in \( G \). Also, \( t_k \) is the minimum integer such that for every graph \( G \) with \( \nu_2(G) < k \), we have \( \tau(G) \leq t_k \).

Proof of Theorem 7.1 We prove by induction on \( k \). We know \( t_1 = 0 \). So, we may assume that \( k > 1 \).

Let \( f_{wall} \) and \( g \) be the functions defined in Theorem 4.2 and Proposition 5.2 respectively. Let \( r = 2t_{k-1} \). We set

\[
\begin{align*}
    f_\beta(k) &= \max(k, r, 12(g(k) - 1) + 1) \\
    f_2(k) &= \max(f_3(k) + 1, (2k + 1)(6g(k) + 1)) \\
    f_1(k) &= k f_2(k) \\
    f(k) &= \max(4f_3(k), f_{wall}(r, f_1(k)))
\end{align*}
\]

Suppose that \( \nu_2(G) < k \) and \( \tau(G) > f(k) \) for some digraph \( G \). By Lemma 4.1, \( G \) contains an \( r \)-well-linked set \( T \) of size \( \tau(G) > f(k) = f_{wall}(r, f_1(k)) \). By Theorem 4.2 \( G \) has a cylindrical wall \( W \) of order \( f_1(k) \) as a subgraph such that

\((*)\) for every subset \( A \) of \( T \) of size at least \( |T|/2 \) and every \( t \)-subset \( B \) of \( V(W) \) consisting of elements in distinct columns with \( r \leq t \leq f_1(k) \), there are a linkage from \( A \) to \( B \) of order \( t \) in \( G \), and a linkage from \( B \) to \( A \) of order \( t \) in \( G \).

We consider \( k \) vertex-disjoint subwalls of \( W \) of order \( f_1(k)/k = f_2(k) \) where each subwall consists of subpaths of all rows, and \( k \) consecutive columns. If each of the subwalls contain a directed
odd cycle, then we have \( k \) vertex-disjoint directed odd cycles, contradicting the assumption that \( \nu_2(G) < k \). Thus, one of the subwalls does not contain directed odd cycles. Let \( W' \) be the bipartite cylindrical subwall of \( W \) of order \( f_3(k) \). It is clear that \( W' \) also satisfies the property (*).

Let \( N \) be the set of all nails of \( W' \) such that they are in the same bipartition of \( W' \), and it has at least half of the all nails of \( W' \). Now, we apply Lemma 3.2 to \( N \) in \( G \) with \( f_3(k) \). As \( \nu_2(G) < k \leq f_3(k) \), we have the second or third outcome.

Note that since \( f_3(k) \geq r \), we can consider \( T \) as a \( f_3(k) \)-well-linked set. Assume that Lemma 3.2 outputs a vertex set \( Y \) of order at most \( 4f_3(k) - 1 \) such that \( G - Y \) has no directed odd \( N \)-walks. Since \(|T| = \tau(G) \geq 4f_3(k)\) and \( f_2(k) \geq f_3(k) + 1 \), by Proposition 5.1 \( G \) has a hitting set of size at most \( 3f_3(k) \). It contradicts the assumption that \( \tau(G) > f(k) \geq 3f_3(k) \).

Thus, we may assume that Lemma 3.2 outputs the second outcome, that is, a half-integral packing \( L \) of \( f_3(k) \) directed odd \( N \)-paths such that the endpoints of them are distinct. Now, observe that \( W' \) has order \( f_3(k) \leq (2k + 1)(6g(k) + 1) \) and \(|L| = f_3(k) \geq 12(g(k) - 1) + 1 \). So, by Proposition 5.2 \( \nu_2(G) \geq k \), a contradiction.

We conclude that \( \tau(G) \leq f(k) \).  

\[ \square \]

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