Character analogue of the Boole summation formula with applications

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Abstract

In this paper, we present the character analogue of the Boole summation formula. Using this formula, an integral representation is derived for the alternating Dirichlet $L$–function and its derivative is evaluated at $s = 0$. Some applications of the character analogue of the Boole summation formula and the integral representation are given about the alternating Dirichlet $L$–function. Moreover, the reciprocity formulas for two new arithmetic sums, arising from utilizing the summation formulas, and for Hardy–Berndt sum $S_p (b, c : \chi)$ are proved.

Keywords: Boole summation formula, Dirichlet $L$–function, Hardy-Berndt sum, Bernoulli and Euler polynomials.

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1 Introduction

The Euler–MacLaurin summation formula is a well-known formula from classical analysis giving a relation between the finite sum of values of a function and its integral. One of the generalizations of the Euler–MacLaurin summation formula is the character analogue due to Berndt [4] which is presented here in the following form.

**Theorem 1.1 (4, Theorem 4.1)** Let $\chi$ be a primitive character of modulus $k$ with $k > 1$. For $f \in C^{(l+1)} [\alpha, \beta]$, $-\infty < \alpha < \beta < \infty$

$$\sum_{\alpha \leq n \leq \beta} \chi(n) f(n) = \chi(-1) \sum_{j=0}^{l} \frac{(-1)^{j+1}}{(j+1)!} \left( B_{j+1, \chi}(\beta) f^{(j)}(\beta) - B_{j+1, \chi}(\alpha) f^{(j)}(\alpha) \right)$$

$$+ \chi(-1) \frac{(-1)^l}{(l+1)!} \int_{\alpha}^{\beta} B_{l+1, \chi}(u) f^{(l+1)}(u) du,$$

where the dash indicates that if $n = \alpha$ or $n = \beta$, then only $\frac{1}{2} \chi(\alpha) f(\alpha)$ or $\frac{1}{2} \chi(\beta) f(\beta)$ is counted, respectively. Also, $B_{p, \chi} (x)$ denotes the generalized Bernoulli function defined by (3).
The alternating version of the Euler–MacLaurin summation formula is the Boole summation formula ([24, 24.17.1–2]), as pointed out by Nörlund [25], this formula is also due to Euler.

**Theorem 1.2 (Boole summation formula)** For integers $\alpha, \beta$ and $l > 0$

\[
2 \sum_{n=\alpha}^{\beta-1} (-1)^n f(n) = \sum_{j=0}^{l-1} \frac{E_j(0)}{j!} \left( (-1)^{\beta-1} f^{(j)}(\beta) + (-1)^\alpha f^{(j)}(\alpha) \right) \\
+ \frac{1}{(l-1)!} \int_{\alpha}^{\beta} f^{(l)}(x) \overline{E}_{l-1}(-x) \, dx,
\]

where $f^{(l)}(x)$ is absolutely integrable over $[\alpha, \beta]$, and $\overline{E}_p(x)$ is the Euler function defined by (1).

By the authors’ knowledge, the character generalization of the Boole summation formula is not available.

In this paper, we first present the character analogue of the Boole summation formula as

**Theorem 1.3** Let $\chi$ be a primitive character of modulus $k$ with $k > 1$ odd. If $f \in C^{(l+1)}[\alpha, \beta], -\infty < \alpha < \beta < \infty$, then

\[
2 \sum_{\alpha < n < \beta} (-1)^n \chi(n) f(n) = \chi(-1) \sum_{j=0}^{l} \frac{(-1)^j}{j!} \left( \overline{E}_{j, \chi}(\beta) f^{(j)}(\beta) - \overline{E}_{j, \chi}(\alpha) f^{(j)}(\alpha) \right) \\
- \chi(-1) \frac{(-1)^l}{l!} \int_{\alpha}^{\beta} \overline{E}_{l, \chi}(t) f^{(l+1)}(t) \, dt,
\]

where $\overline{E}_{l, \chi}(t)$ is the generalized Euler function defined by (4).

Later, we give applications of this formula on two subjects. The first one is around the alternating Dirichlet $L$–function. For $a \neq -1, -2, -3, \ldots$, let $\ell(s, a, \chi)$ denote the alternating Dirichlet $L$–function

\[
\ell(s, a, \chi) = \sum_{n=1}^{\infty} (-1)^n \frac{\chi(n)}{(n + a)^s}, \quad \text{Re}(s) > 0,
\]

which can be written in terms of Hurwitz zeta function $\zeta(s, a)$ as

\[
\ell(s, a, \chi) = (2k)^{-s} \sum_{j=1}^{2k-1} (-1)^j \chi(j) \zeta \left( s, \frac{a + j}{2k} \right)
\]
for \( \text{Re} (s) > 1 \). Also let
\[
\ell_s (x, a, \chi) = \sum_{1 \leq n \leq x} (-1)^n \chi (n) (n + a)^s, \quad x \geq 0.
\]
Then, the integral representations for \( \ell (s, a, \chi) \) and \( \ell_s (x, a, \chi) \) are derived as in the following.

**Theorem 1.4** Let \( \chi \) be a primitive character of modulus \( k \) with \( k > 1 \) odd. For \( l \geq 0 \) with \( l > \text{Re} (s) \) and for any \( x \geq 0 \), we have the integral representation
\[
2 \ell_s (x, a, \chi) = \chi (-1) \sum_{j=0}^{l} (-1)^j \frac{s_j}{j!} E_{j, \chi} (x) (x + a)^{s-j} + 2 \ell (-s, a, \chi)
\]
\[
- \frac{(s)_{l+1}}{l!} \int_{x}^{\infty} E_{l, \chi} (-t) (t + a)^{s-l-1} dt,
\]
where \( (s)_{j} = s(s-1) \cdots (s-j+1) \) with \( (s)_0 = 1 \).
Moreover, for \( x = 0 \) we have
\[
2 \ell (-s, a, \chi) = \sum_{j=0}^{l} \frac{(s)_j}{j!} E_{j, \chi} (0) a^{s-j} + \frac{(s)_{l+1}}{l!} \int_{0}^{\infty} E_{l, \chi} (-t) (t + a)^{s-l-1} dt.
\]

Besides, some formulas, such as character analogue of the Lerch’s formula for Hurwitz zeta function (see (25)), character analogues of the Stirling’s formula for \( \log \Gamma (a) \) and of the Weierstrass product for \( \Gamma (a) \) (see Propositions 4.3 and 4.6 below, respectively) are deduced via Theorems 1.3 and 1.4.

The second is around the Hardy–Berndt sums. Let us mention that utilizing the summation formulas, alternative proofs of the reciprocity formulas of certain Dedekind sums and their analogues have been offered in [9, 10, 14, 15, 18]. Here, we reveal that new arithmetic sums obeying reciprocity law may be defined by the summation formulas aforementioned above. We describe two of such sums as
\[
S_p^{(1)} (b, c : \chi) = \sum_{n=1}^{ck} (-1)^n \mathcal{E}_{p, \chi} \left( \frac{bn}{c} \right),
\]
\[
S_p^{(2)} (b, c : \chi) = \sum_{n=1}^{ck} (-1)^n \chi (n) \mathcal{E}_{p} \left( \frac{bn}{c} \right)
\]
and prove the following reciprocity formula.

**Theorem 1.5** Let \( b \) and \( c \) be positive integers with \( (b + c) \) odd and \( \chi (-1) (-1)^p = 1 \). Then, the reciprocity formula holds:
\[
c^p S_p^{(1)} (b, c : \chi) + b^p S_p^{(2)} (c, b : \chi) = 2 \sum_{j=0}^{p} \binom{p}{j} c^j b^{p-j} \mathcal{E}_{j, \chi} (0) \mathcal{E}_{p-j} (0).
\]
In fact, these sums are generalizations of one of the Hardy–Berndt sums \[ S(b, c) = \sum_{n=1}^{c-1} (-1)^{n+1+[bn/c]}. \]

For various generalizations and properties of Hardy–Berndt sums, the reader may consult to [5, 6, 8, 9, 13, 16, 17, 21, 22, 23, 26, 27, 28, 29, 30] and [29, 30] for association between \( S(b, c) \) and Dirichlet \( L \)-function. One of the generalizations of \( S(b, c) \) has been given by [9]

\[ S_p(b, c : \chi) = \sum_{n=1}^{ck} \chi(n) B_p \left( \frac{b+ck}{2c} \right) \]

and the corresponding reciprocity formula is proved via transformation formulas. Here, utilizing Theorem 1.3, we give a new proof for the following reciprocity formula by refining the conditions.

**Theorem 1.6** Let \( p > 1 \) be odd and let \( b \) and \( c \) be positive integers with \( (b+c) \) odd. Then, the reciprocity formula holds:

\[
\begin{align*}
\bar{\chi}(-2) b^c S_p(b, c : \chi) + \chi(-2) c^b S_p(c, b : \bar{\chi}) &= \frac{p}{2p+1} \sum_{j=1}^{p} (-1)^j \left( \frac{p-1}{j-1} \right) c^j b^{p+1-j} E_{j-1, \chi}(0) E_{p-j, \bar{\chi}}(0). \\
\end{align*}
\]

The remainder of this paper is organized as follows: Section 2 is the preliminary section where we give definitions and known results needed. In Section 3, we prove Theorem 1.3 and Theorem 1.4. Some applications of the integral representation and the character analogue of the Boole summation formula are given in Section 4. The final section is devoted to prove the reciprocity formulas for the Hardy–Berndt sums mentioned above via summation formulas.

## 2 Preliminaries

The Bernoulli and Euler polynomials \( B_n(x) \) and \( E_n(x) \) are defined by means of the generating functions [2]

\[
\begin{align*}
\frac{te^t}{e^t - 1} &= \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad (|t| < 2\pi), \\
\frac{2e^{xt}}{e^t + 1} &= \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \quad (|t| < \pi),
\end{align*}
\]

respectively. In particular, the rational numbers \( B_n = B_n(0) \) and integers \( E_n = 2^n E_n(1/2) \) are called classical Bernoulli numbers and Euler numbers, respectively.

For \( 0 \leq x < 1 \) and \( m \in \mathbb{Z} \), the Bernoulli functions \( \overline{B}_n(x) \) are defined by

\[
\overline{B}_n(x + m) = B_n(x) \text{ when } n \neq 1 \text{ and } x \neq 0, \text{ and } \overline{B}_1(m) = \overline{B}_1(0) = 0
\]
and the Euler functions $E_n(x)$ are defined by [11]
\[
E_n(x + m) = (-1)^m E_n(x) \quad \text{and} \quad E_n(x) = E_n(x).
\]

The Bernoulli functions satisfy the Raabe or multiplication formula
\[
r^{n-1} \sum_{j=0}^{r-1} B_n \left( x + \frac{j}{r} \right) = B_n(rx)
\]
and also following property is valid for even $r$
\[
r^{n-1} \sum_{j=0}^{r-1} (-1)^j B_n \left( \frac{x + j}{r} \right) = -\frac{n}{2} E_{n-1}(x).
\]

$B_{m,\chi}(x)$ denotes the generalized Bernoulli function, with period $k$, defined by Berndt [4]. We will often use the following property that can confer as a definition
\[
B_{m,\chi}(x) = k^{m-1} \sum_{j=0}^{k-1} \chi(j) B_m \left( \frac{j+x}{k} \right), \quad m \geq 1.
\]

For the convenience with the definition of $B_{m,\chi}(x)$, let the character Euler function $E_{m,\chi}(x)$ be defined by
\[
E_{m,\chi}(x) = k^m \sum_{j=0}^{k-1} (-1)^j \chi(j) E_m \left( \frac{j+x}{k} \right), \quad m \geq 0
\]
for odd $k$, the modulus of $\chi$.

List some properties that we need in the sequel
\[
\frac{d}{dx} E_m(x) = mE_{m-1}(x), \quad m > 1,
\]
\[
\frac{d}{dx} E_{m,\chi}(x) = mE_{m-1,\chi}(x), \quad m \geq 1,
\]
\[
E_{m,\chi}(-x) = (-1)^{m-1} \chi(-1) E_{m,\chi}(x),
\]
\[
E_{m,\chi}(x + nk) = (-1)^n E_{m,\chi}(x).
\]

In the sequel, unless otherwise stated, we assume that $\chi$ is a primitive character of modulus $k$ with $k > 1$ odd.

### 3 Proofs of Theorems 1.3 and 1.4

#### 3.1 Proof of Theorem 1.3

Firstly, we write
\[
\sum_{\alpha<n<\beta} (-1)^n \chi(n) f(n) = \sum_{\alpha<n<\beta} \chi(2n) f(2n) - \sum_{\alpha<2n+1<\beta} \chi(2n+1) f(2n+1)
\]
\[
2 \chi(2) \sum_{\alpha/2<n<\beta/2} \chi(n) f(2n) - \sum_{\alpha<n<\beta} \chi(n) f(n).
\]

Applying Theorem 1.1 on the right-hand side, one has
\[
\sum_{\alpha<n<\beta} (-1)^n \chi(n) f(n)
= \chi(-1) \sum_{j=0}^{l} \frac{(-1)^{j+1}}{(j+1)!} \left( \left( 2^{j+1} \chi(2) \overline{B}_{j+1,\pi} \left( \frac{\beta}{2} \right) - \overline{B}_{j+1,\pi}(\beta) \right) f^{(j)}(\beta)
- \left( 2^{j+1} \chi(2) \overline{B}_{j+1,\pi} \left( \frac{\alpha}{2} \right) - \overline{B}_{j+1,\pi}(\alpha) \right) f^{(j)}(\alpha) \right)
+ \chi(-1) \frac{(-1)^l}{(l+1)!} \int_{\alpha}^{\beta} \left( 2^{l+1} \chi(2) \overline{B}_{l+1,\pi}(u) \right) f^{(l+1)}(u) du.
\] (9)

On the other hand, taking \( x \to x/2 \) in [9, Eq. (3.13)] gives
\[
\overline{B}_{m,\chi}(\frac{x}{2}) + \overline{B}_{m,\chi}(\frac{x+k}{2}) = 2^{1-m} \chi(2) \overline{B}_{m,\chi}(x).
\] (10)

By using (3) and (2) for \( r = 2 \) we can write
\[
\overline{B}_{m,\chi}(\frac{x}{2}) - \overline{B}_{m,\chi}(\frac{x+k}{2}) = -\frac{m}{2^m} k^{m-1} \sum_{v=0}^{k-1} \chi(v) \overline{E}_{m-1}(\frac{2v+x}{k}).
\] (11)

Employing basic manipulations, (11) becomes
\[
k^{m-1} \left\{ \sum_{v=0}^{k-1} \chi(2v) \overline{E}_{m-1}(\frac{2v+x}{k}) + \sum_{v=\frac{k+1}{2}}^{k-1} \chi(2v) \overline{E}_{m-1}(\frac{2v+x}{k}) \right\}
= k^{m-1} \left\{ \sum_{v=0}^{k-1} \chi(2v) \overline{E}_{m-1}(\frac{2v+x}{k}) - \sum_{v=0}^{k-3} \chi(2v+1) \overline{E}_{m-1}(\frac{2v+1+x}{k}) \right\}
= k^{m-1} \sum_{v=0}^{k-1} (-1)^v \chi(v) \overline{E}_{m-1}(\frac{v+x}{k})
= \overline{E}_{m-1,\chi}(x),
\] (12)

where we have used (1) and (4). Thus, combining (10) and (12) leads to
\[
2^m \chi(2) \overline{B}_{m,\chi}(\frac{x}{2}) - \overline{B}_{m,\chi}(x) = -\frac{m}{2} \overline{E}_{m-1,\chi}(x).
\] (13)

Substituting (13) in (9) completes the proof.
3.2 Proof of Theorem 1.4

The method used here have already been employed by Kanemitsu et al. \cite{19} to the Euler–MacLaurin summation formula to obtain integral representations for Hurwitz zeta function and its partial sum.

For $\alpha = 0$ and $\beta = x$, let $f(t) = (t + a)^s$ in Theorem 1.3. Then, from (7),

$$2\ell_s(x, a, \chi) = 2 \sum_{0 \leq n \leq x} (-1)^n \chi(n)(n + a)^s$$

$$= \chi(-1) \sum_{j=0}^{l} (-1)^j \frac{(s)_j}{j!} \left( \sum_{j=0}^{l} E_j(x + a)^{s-j} - \sum_{j=0}^{l} E_j(0) a^{s-j} \right)$$

$$+ \frac{(s)_{l+1}}{l!} \int_{0}^{x} \sum_{j=0}^{l} E_j(-t)(t + a)^{s-l-1} dt.$$

Since

$$|E_l(x)| \leq \frac{l!}{(\pi/k)^{l+1}} \xi(l + 1), \ l \geq 1,$$

the integral

$$\int_{0}^{x} \sum_{j=0}^{l} E_j(-t)(t + a)^{s-l-1} dt$$

is absolutely convergent for $\text{Re}(s) < l$. So, we may write

$$2\ell_s(x, a, \chi)$$

$$= \chi(-1) \sum_{j=0}^{l} (-1)^j \frac{(s)_j}{j!} \left( \sum_{j=0}^{l} E_j(x + a)^{s-j} - \sum_{j=0}^{l} E_j(0) a^{s-j} \right)$$

$$+ \frac{(s)_{l+1}}{l!} \int_{0}^{x} \sum_{j=0}^{l} E_j(-t)(t + a)^{s-l-1} dt - \frac{(s)_{l+1}}{l!} \int_{x}^{\infty} \sum_{j=0}^{l} E_j(-t)(t + a)^{s-l-1} dt. \quad (14)$$

Now, for $\text{Re}(s) < 0$, letting $x$ tends to $\infty$ in (14), we arrive at

$$2\ell(-s, a, \chi) = -\chi(-1) \sum_{j=0}^{l} (-1)^j \frac{(s)_j}{j!} E_j(0) a^{s-j}$$

$$+ \frac{(s)_{l+1}}{l!} \int_{0}^{\infty} \sum_{j=0}^{l} E_j(-t)(t + a)^{s-l-1} dt, \quad (15)$$

where the integral converges absolutely for $\text{Re}(s) < l$ and represents an analytic function. Substituting (15) in (14) gives

$$2\ell_s(x, a, \chi) = \chi(-1) \sum_{j=0}^{l} (-1)^j \frac{(s)_j}{j!} E_j(x + a)^{s-j} + 2\ell(-s, a, \chi)$$

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\[- \frac{(s)_{l+1}}{l!} \int_{x}^{\infty} E_{l, \overline{\chi}}(-t) (t + a)^{s-l-1} dt, \quad (16)\]

for \(\text{Re}(s) < l\) and \(x \geq 0\).

Writing \(x = 0\) in (16) yields

\[2\ell (-s, a, \chi) = \sum_{j=0}^{l} \frac{(s)_j}{j!} E_{j, \overline{\chi}}(0) a^{s-j} + \frac{(s)_{l+1}}{l!} \int_{0}^{\infty} \frac{E_{l, \overline{\chi}}(-t)}{(t + a)^{l+1-s}} dt, \quad (17)\]

which is valid for \(\text{Re}(s) < l\).

4 Some consequences

This section is concerned with some formulas about the alternating Dirichlet \(L\)–function and counterparts of the Examples 6–10 of [4].

4.1 Around the alternating Dirichlet \(L\)–function

It is clear from (17) that for \(l = p\) and \(s = p - 1\) with \(0 < a < 1\),

\[2\ell (1-p, a, \chi) = \sum_{j=0}^{p-1} \binom{p-1}{j} E_{j, \overline{\chi}}(0) a^{p-1-j} = E_{p-1, \overline{\chi}}(a), \quad p \geq 1.\]

Also, for \(\text{Re}(s) > 0 = l\)

\[2\ell (s, a, \chi) = -\chi(-1) E_{0, \overline{\chi}}(0) a^{-s} + \chi(-1) s \int_{0}^{\infty} \frac{E_{0, \overline{\chi}}(t)}{(t + a)^{1+s}} dt\]

and for \(\text{Re}(s) > -1, (l = 1)\)

\[2\ell (s, a, \chi) = E_{0, \overline{\chi}}(0) a^{-s} - s E_{1, \overline{\chi}}(0) a^{-s-1} + s(s+1) \chi(-1) \int_{0}^{\infty} \frac{E_{1, \overline{\chi}}(t)}{(t + a)^{2+s}} dt. \quad (18)\]

Differentiating both sides of (18) with respect to \(s\) at \(s = 0\) gives

\[2 \frac{d}{ds} \ell (s, a, \chi) |_{s=0} = 2\ell' (0, a, \chi)\]

\[= -E_{0, \overline{\chi}}(0) \log a - \frac{1}{a} E_{1, \overline{\chi}}(0) + \chi(-1) \int_{0}^{\infty} \frac{E_{1, \overline{\chi}}(x)}{(x + a)^2} dx. \quad (19)\]
Similar results for \( \ell (s, \chi) = \ell (s, 0, \chi) \) can be achieved by applying Theorem 1.3 to \( f(x) = x^{-s} \), \( \Re (s) > 0 \), where \( \alpha = 1 \) and \( \beta = 2kN, \; N \in \mathbb{N} \). Following the arguments in the proof of (17) and then letting \( N \to \infty \) give rise to

\[
2 \ell (s, \chi) + 2 = -\chi (-1) \sum_{j=1}^{l} \frac{s (s+1) \ldots (s+j-1)}{j!} \bar{E}_{j, \chi} (1)
+ \chi (-1) \frac{s (s+1) \ldots (s+l)}{l!} \int_{1}^{\infty} \frac{\bar{E}_{l, \chi} (x)}{x^{s+l+1}} dx,
\]

where the integral is analytic for \( \Re (s) > -l \). In particular, for \( l = 1 \),

\[
2 \ell (s, \chi) = -2 - \chi (-1) \bar{E}_{0, \chi} (1) - \chi (-1) s \bar{E}_{1, \chi} (1) + \chi (-1) s (s+1) \int_{1}^{\infty} \frac{\bar{E}_{1, \chi} (x)}{x^{s+2}} dx.
\]

Differentiating both sides of the equality above with respect to \( s \) at \( s = 0 \) gives

\[
2 \ell' (0, \chi) = -\chi (-1) \bar{E}_{1, \chi} (1) + \chi (-1) \int_{1}^{\infty} \frac{\bar{E}_{1, \chi} (x)}{x^2} dx.
\] (20)

Observe that the integrals in (19) and (20) can be emerged from Theorem 1.3 by taking the logarithm function. So, we may establish a connection between generalized Euler functions and some identities for logarithmic means.

**Proposition 4.1** As \( t \) tends to \( +\infty \), we have the following asymptotic expansion,

\[
2 \sum_{1 \leq n < t} (-1)^n \chi (n) \log (t/n) \sim 2 \ell' (0, \chi) + \ell (0, \chi) \log t + \chi (-1) \sum_{j=1}^{\infty} \frac{\bar{E}_{j, \chi} (t)}{jt^j}.
\]

**Proof.** Let \( f(x) = \log (t/x) \), \( \alpha = 1 \) and \( \beta = t \) and \( l = 1 \) in Theorem 1.3. Then

\[
2 \chi (-1) \sum_{1 \leq n < t} (-1)^n \chi (n) \log (t/n)
= 2 \chi (-1) \sum_{1 \leq n < t} (-1)^n \chi (n) \log (t/n) + 2 \log t
= -\bar{E}_{1, \chi} (1) - \bar{E}_{0, \chi} (1) \log t + \frac{\bar{E}_{1, \chi} (t)}{t} + \int_{1}^{t} \frac{\bar{E}_{1, \chi} (x)}{x^2} dx
= 2 \chi (-1) \ell' (0, \chi) - \bar{E}_{0, \chi} (1) \log t + \frac{\bar{E}_{1, \chi} (t)}{t} - \int_{t}^{\infty} \frac{\bar{E}_{1, \chi} (x)}{x^2} dx,
\] (21)

where we have used (20). Using that \( \bar{E}_{0, \chi} (1) = \bar{E}_{0, \chi} (0) - 2 \chi (-1) \) and \( \ell (0, \chi) = \bar{E}_{0, \chi} (0) \), and integrating by parts repeatedly with the use of (21), one arrives at the asymptotic formula. \( \blacksquare \)
We now apply Theorem 1.3 to the function \( f(x) = \log (x + a), \) \(-\pi < \arg a < \pi,\) where \( \alpha = 0, \beta = 2kN, N \in \mathbb{N}, \) and \( l = 1 \) to obtain
\[
2 \sum_{n=1}^{2kN} (-1)^n \chi(n) \log(n + a) = -E_{0,\chi}(0) \log(2kN + a) - \frac{E_{1,\chi}(0)}{2kN + a} \\
+ E_{0,\chi}(0) \log a + \chi(-1) \frac{E_{1,\chi}(0)}{a} - \chi(-1) \int_{0}^{2Nk} \frac{E_{1,\chi}(x)}{(x + a)^2} dx. \tag{22}
\]

Gathering (21) and (22) for \( t = 2kN, N \in \mathbb{N}, \) then letting \( N \to \infty \) and using (20), we find that
\[
-2 \sum_{n=1}^{\infty} (-1)^n \chi(n) \log(n) - \log(n + a) = 2 \ell'(0, \chi) + \chi(-1) \frac{E_{1,\chi}(0)}{a} + E_{0,\chi}(0) \log a - \chi(-1) \int_{0}^{\infty} \frac{E_{1,\chi}(x)}{(x + a)^2} dx. \tag{23}
\]

Note that the sum in (23) is reminiscent of the definition of character analogue of the gamma function defined by Berndt [4, Definition 4]. This motivates us to make the following definition.

**Definition 4.2** Let \( \chi \) be a real primitive character. Define
\[
\Gamma^*(a, \chi) = \prod_{n=1}^{\infty} \left( \frac{n}{n + a} \right)^{(-1)^n \chi(n)}.
\]

In the light of this definition, (23) becomes
\[
2 \log \Gamma^*(a, \chi) = -E_{0,\chi}(0) \log a - 2 \ell'(0, \chi) - \chi(-1) \frac{E_{1,\chi}(0)}{a} + \chi(-1) \int_{0}^{\infty} \frac{E_{1,\chi}(x)}{(x + a)^2} dx, \tag{24}
\]

which shows that \( \Gamma^*(a, \chi) \) is well defined and analytic for \(-\pi < \arg a < \pi.\)

Combining (19) and (24), we infer the Lerch’s formula for \( \ell(s, a, \chi) \) as
\[
\ell'(0, a, \chi) = \log \Gamma^*(a, \chi) + \ell'(0, \chi), \tag{25}
\]
which is the character analogue of the familiar formula
\[
\zeta'(0, z) = \log \Gamma(z) + \zeta'(0),
\]
where \( \Gamma(z) \) and \( \zeta(z) \) are the Euler gamma and Riemann zeta functions, respectively.

Furthermore, in (24), integrating by parts repeatedly in view of (6), we arrive at the following asymptotic formula, the counterpart of [4, Proposition 5.3].
Proposition 4.3 (Stirling’s formula for $\log \Gamma^{*}(a, \chi)$) For $-\pi < \arg a < \pi$, as $a$ tends to $\infty$,

$$\log \Gamma^{*}(a, \chi) \sim -\frac{1}{2} \ell(0, \chi) \log a - \ell'(0, \chi) - \frac{\chi(-1)}{2} \sum_{j=1}^{\infty} \frac{E_{2n\chi}(0)}{ja^{j}},$$

where the principal branch of the logarithm is taken.

Next we write the integral in (19) as in the form

$$\int_{0}^{\infty} \frac{E_{1,\chi}(x)}{(x+a)^2} dx = \sum_{n=0}^{\infty} \frac{2(n+1)k}{2n+2} \int_{0}^{\infty} \frac{E_{1,\chi}(x)}{(x+a)^2} dx$$

$$= \frac{1}{(2k)^2} \int_{0}^{2k} E_{1,\chi}(t) \sum_{n=0}^{\infty} \left( n + \frac{t+a}{2k} \right)^{-2} dt$$

$$= \frac{1}{(2k)^2} \int_{0}^{2k} E_{1,\chi}(t) \zeta\left( 2, \frac{t+a}{2k} \right) dt.$$

So, Eq. (19) becomes

$$2\ell'(0, a, \chi) = -E_{0,\chi}(0) \log a - \frac{1}{a} \int E_{1,\chi}(0) + \frac{\chi(-1)}{(2k)^2} \int_{0}^{2k} E_{1,\chi}(x) \zeta\left( 2, \frac{x+a}{2k} \right) dx. \quad (26)$$

Since

$$\frac{d^2}{dz^2} \log \Gamma(z) = \frac{d}{dz} \psi(z) = \zeta(2, z), \quad (27)$$

where $\psi(z)$ is the digamma function, the integral in (26) may be arisen from Theorem 1.3 by setting $f(x) = \log \Gamma((x+a)/2k), \alpha = 0, \beta = 2k$ and $l = 1$. Under the circumstances,

$$2 \sum_{n=0}^{2k-1} (-1)^n \chi(n) \log \Gamma\left( \frac{n+a}{2k} \right)$$

$$= -E_{0,\chi}(0) \log \frac{a}{2k} - \frac{1}{a} E_{1,\chi}(0) + \frac{\chi(-1)}{(2k)^2} \int_{0}^{2k} E_{1,\chi}(x) \zeta\left( 2, \frac{x+a}{2k} \right) dx, \quad (28)$$

where we have used that $\Gamma(z+1) = z\Gamma(z)$ and $\psi(z+1) - \psi(z) = 1/z$. Assembling (26) and (28), we have

$$2\ell'(0, a, \chi) = -E_{0,\chi}(0) \log (2k) + 2 \sum_{n=1}^{2k-1} (-1)^n \chi(n) \log \Gamma\left( \frac{n+a}{2k} \right). \quad (29)$$

The following proposition shows that $\Gamma^{*}(a, \chi)$ is a quotient of ordinary gamma functions.
Proposition 4.4 We have
\[ \Gamma^* (a, \chi) = \prod_{n=1}^{2k-1} \left( \frac{\Gamma \left( \frac{n+a}{2k} \right)}{\Gamma (n/2k)} \right)^{(-1)^n \chi(n)}. \]

Proof. From
\[ \ell (s, 2k, \chi) = \ell (s, \chi) - \sum_{n=1}^{2k-1} (-1)^n \chi(n) n^{-s}, \]
it is seen that
\[ \ell' (0, 2k, \chi) = \ell' (0, \chi) - \sum_{n=1}^{2k-1} (-1)^n \chi(n) \log n. \]  
(30)

Setting \( a = 2k \) in (29) and then comparing with (30) give
\[ \ell' (0, \chi) = -\frac{1}{2} \ell (0, \chi) \log (2k) + \sum_{n=1}^{2k-1} (-1)^n \chi(n) \log \left( \frac{n}{2k} \right). \]
Substituting this in (25) and then combining with (29) lead to
\[ \log \Gamma^* (a, \chi) = \sum_{n=1}^{2k-1} (-1)^n \chi(n) \left( \log \left( \frac{n+a}{2k} \right) - \log \left( \frac{n}{2k} \right) \right) \]
(31)
\[ = \sum_{n=1}^{2k-1} (-1)^n \chi(n) \log \left( \frac{\Gamma \left( \frac{n+a}{2k} \right)}{\Gamma (n/2k)} \right), \]
which is the desired result.

Let us continue by differentiating both sides of (31) with respect to \( a \). Then, we have
\[ \frac{d}{da} \log \Gamma^* (a, \chi) = -\frac{1}{2} \sum_{n=1}^{2k-1} (-1)^n \chi(n) \psi \left( \frac{n+a}{2k} \right), \]
(32)
by (27). For the convenience with (27), the right-hand side of (32) can be denoted by \( \psi^* (a, \chi) \), i.e.,
\[ \psi^* (a, \chi) = \frac{1}{2k} \sum_{n=1}^{2k-1} (-1)^n \chi(n) \psi \left( \frac{n+a}{2k} \right). \]
On the other hand, in the light of (25), differentiating both sides of (19) with respect to \( a \), and then comparing with (18) for \( s = 1 \), we see that
\[ \ell (1, a, \chi) = -\psi^* (a, \chi) = \frac{1}{2k} \sum_{n=1}^{2k-1} (-1)^n \chi(n) \psi \left( \frac{n+a}{2k} \right). \]
(33)
In general, for \( m \geq 0 \) we have
\[ \frac{d^m}{da^m} \psi^* (a, \chi) = (-1)^{m+1} m! \ell (m+1, a, \chi), \]
(34)
which implies the following identity, viewed as the Taylor expansion of \( \ell (s, a, \chi) \) in the second variable \( a \).
Proposition 4.5  For \( |z| < 1 \) we have
\[
\sum_{m=2}^{\infty} \ell(m, a, \chi) z^{m-1} = \psi^*(a, \chi) - \psi^*(a - z, \chi)
\]  
(35)

Proof. The statement follows from the Taylor expansion of \( \psi^*(z, \chi) \) at \( z = a \).  

The character analogue of the Weierstrass product representation of \( \Gamma(s) \) can be derived from Definition 4.2 and also from Proposition 4.5.

Proposition 4.6  We have for all \( s \)
\[
\Gamma^*(s, \chi) = e^{-s\ell(1, \chi)} \prod_{n=1}^{\infty} \left( (1 + s/n)^{-1} e^{s/n} \right)^{(-1)^n \chi(n)},
\]  
(36)

where the product converges uniformly on any compact set \( S \) which avoids the points \( s = -n \), where \( n \) is a positive integer and \( (-1)^n \chi(n) = 1 \).

Proof. The proof from Definition 4.2 is exactly like the proof of Berndt [4, Proposition 5.4], so we omit it.

For the proof via Proposition 4.5, integrating (35) from 0 to \( s \), we see that
\[
\sum_{m=2}^{\infty} \ell(m, a, \chi) \frac{s^m}{m} = \log \Gamma^*(a - s, \chi) - \log \Gamma^*(a, \chi) + s\psi^*(a, \chi).
\]

Taking \( s \to -s \) and \( a = 0 \), we have
\[
\sum_{m=2}^{\infty} \ell(m, 0, \chi) \frac{(-s)^m}{m} = \log \Gamma^*(s, \chi) - \log \Gamma^*(0, \chi) - s\psi^*(0, \chi)
\]  
(37)

The left-hand side of (37) is
\[
\sum_{n=1}^{\infty} (-1)^n \chi(n) \sum_{m=2}^{\infty} \frac{1}{m} \left( -\frac{s}{n} \right)^m = \sum_{n=1}^{\infty} (-1)^n \chi(n) \left[ \frac{s}{n} - \log \left( 1 + \frac{s}{n} \right) \right],
\]  
(38)

where we have used that
\[
\sum_{m=2}^{\infty} \frac{r^m}{m} = -r - \log (1 - r), \text{ for } |r| < 1.
\]

Combining (37) and (38) gives (36).  

Note that another consequence of (33) with \( \psi(1 - x) - \psi(x) = \pi \cot \pi x \) is
\[
2\ell(m + 1, \chi) = -\frac{(-\pi/2k)^{m+1}}{m!} \sum_{n=1}^{2k-1} (-1)^n \chi(n) \cot(n) \left( \frac{\pi n}{2k} \right), \quad m \geq 0,
\]  
(39)
when $\chi (-1) (-1)^{m+1} = 1$. Indeed, it is easy to see that for $0 \leq a < 1$,
\[
\ell (1, a, \chi) - \chi (-1) \ell (1, -a, \chi) = -\frac{1}{2k} \sum_{n=1}^{2k-1} (-1)^n \chi (n) \left\{ \psi \left( \frac{n+a}{2k} \right) - \psi \left( 1 - \frac{n+a}{2k} \right) \right\} = -\frac{\pi}{2k} \sum_{n=1}^{2k-1} (-1)^n \chi (n) \cot \left( \frac{n+a}{2k} \right).
\]

(40)

Now (39) follows from (34) and (40) for $\chi (-1) (-1)^{m+1} = 1$ and $a = 0$. In particular,
\[
2\ell (1, \chi) = \frac{\pi}{2k} \sum_{n=0}^{2k-1} (-1)^n \chi (n) \cot \left( \frac{n}{2k} \right), \quad \text{for odd } \chi,
\]
\[
2\ell (2, \chi) = \left( \frac{\pi}{2k} \right)^2 \sum_{n=1}^{2k-1} (-1)^n \frac{\chi (n)}{\sin^2 \left( \frac{\pi n}{2k} \right)}, \quad \text{for even } \chi,
\]
\[
2\ell (3, \chi) = \left( \frac{\pi}{2k} \right)^2 \sum_{n=1}^{2k-1} (-1)^n \chi (n) \frac{\cos \left( \frac{\pi n}{2k} \right)}{\sin^3 \left( \frac{\pi n}{2k} \right)}, \quad \text{for odd } \chi,
\]
\[
2\ell (4, \chi) = \frac{1}{3} \left( \frac{\pi}{2k} \right)^3 \sum_{n=1}^{2k-1} (-1)^n \chi (n) \left( \frac{2}{\sin^4 \left( \frac{\pi n}{2k} \right)} + \frac{\cos \left( \frac{\pi n}{2k} \right)}{\sin^4 \left( \frac{\pi n}{2k} \right)} \right), \quad \text{for even } \chi,
\]

which are analogues of Eqs. (5.9)–(5.12) of Alkan [1]. Such sums and many ones can be found in [3, 7, 20].

4.2 Counterparts of the Examples 6–10 of [4]

In this part, we constitute $f (x)$ in Theorem 1.3 in order to give some formulas, the counterparts of the Examples 6–10 of [4].

- Let $f (x) = e^{xt}$, $\alpha = 0$ and $\beta = k$. Then
\[
2 \sum_{n=0}^{k} (-1)^n \chi (n) e^{nt} = \chi (-1) \sum_{j=0}^{l} (-1)^{j+1} \sum_{j=0}^{k} \frac{\psi \left( \frac{n}{k} \right) - \psi \left( 1 - \frac{n+k}{k} \right)}{\psi \left( \frac{n}{k} \right) + \psi \left( \frac{n+k}{k} \right)} (e^{kt} + 1) - R_l,
\]

where
\[
|R_l| \leq \frac{|t^{l+1}|}{l!} \int_{0}^{k} \left| E_l \chi (x) e^{xt} \right| dx
\]
\[
\leq 4e^{kt} \frac{|t^{l+1}|}{(\pi/k)^{l+1} \zeta (l+1)} |t| \to 0 \text{ as } l \to \infty \text{ for } |t| < \pi/k.
\]

(41)

Thus, we have the generating function for the number $E_{j, \chi} (0)$ as
\[
\sum_{n=0}^{k-1} (-1)^n \chi (n) \frac{2e^{nt}}{e^{kt} + 1} = \sum_{j=0}^{\infty} E_{j, \chi} (0) \frac{t^j}{j!}.
\]
• Let \( f(x) = \cos(xt), \alpha = 0 \) and \( \beta = k \). It is obvious from (7) that \( \overline{E}_{j,\chi}(0) = 0 \) if \( \chi \) and \( j \) have the same parity. If \( \chi \) is odd, then
\[
2 \sum_{n=0}^{k-1} (-1)^n \chi(n) \cos(nt) = \chi(-1) \sum_{j=0}^{l} (-1)^{2j+1} E_{2j,\chi}(0) \cos(kt) + t^{2j}(-1)^j - R_l
\]
where, as in (41), \( R_l \) tends to 0 as \( l \to \infty \) for \( |t| < \pi/k \). So, we have
\[
\frac{2 \sum_{n=1}^{k-1} (-1)^n \chi(n) \cos(nt)}{\cos(kt) + 1} = \sum_{j=0}^{\infty} (-1)^j E_{2j,\chi}(0) \frac{t^{2j+1}}{(2j)!}, \text{ for } |t| < \frac{\pi}{k}.
\]
If \( \chi \) is even, then similarly
\[
\frac{2 \sum_{n=1}^{k-1} (-1)^n \chi(n) \cos(nt)}{\sin(kt)} = \sum_{j=0}^{\infty} (-1)^j E_{2j+1,\chi}(0) \frac{t^{2j+1}}{(2j+1)!}, \text{ for } |t| < \frac{\pi}{k}.
\]
• Let \( f(x) = \sin(xt), \alpha = 0 \) and \( \beta = k \). If \( \chi \) is odd, then for \( |t| < \pi/k \)
\[
\frac{2 \sum_{n=1}^{k-1} (-1)^n \chi(n) \sin(nt)}{\sin(kt)} = \sum_{j=0}^{\infty} (-1)^j E_{2j,\chi}(0) \frac{t^{2j}}{(2j)!}
\]
and if \( \chi \) is even
\[
\frac{2 \sum_{n=1}^{k-1} (-1)^n \chi(n) \sin(nt)}{\cos(kt) + 1} = \sum_{j=0}^{\infty} (-1)^j E_{2j+1,\chi}(0) \frac{t^{2j+1}}{(2j+1)!}.
\]
• By the similar way, it can be seen that if \( \chi \) is odd, then for \( |t| < \pi/k \)
\[
\frac{2 \sum_{n=0}^{k} (-1)^n \chi(n) \cosh(nt)}{\cosh(kt) + 1} = \frac{2 \sum_{n=1}^{k-1} (-1)^n \chi(n) \sinh(nt)}{\sinh(kt)}
\]
and if \( \chi \) is even
\[
\frac{2 \sum_{n=0}^{k} (-1)^n \chi(n) \cosh(nt)}{\sinh(kt)} = \frac{2 \sum_{n=0}^{k} (-1)^n \chi(n) \sinh(nt)}{\cosh(kt) + 1}
\]

5 Proofs of reciprocity theorems

Proof of Theorem 1.5. Let \( f(x) = \overline{E}_{p,\chi}(xb/c), \alpha = 0 \) and \( \beta = ck \) in Theorem 1.2. By virtue of (6), for \( 1 \leq l \leq p \), one has
\[
2 \sum_{n=0}^{ck} (-1)^n \overline{E}_{p,\chi}(\frac{b}{c}n) = \sum_{j=0}^{l-1} E_j(0) \frac{b}{c} \frac{j^p}{(p-j)!} \left((-1)^{ck-1} \overline{E}_{p-j,\chi}(bk) + \overline{E}_{p-j,\chi}(0)\right)
\]
\[ \frac{p!}{(l-1)!(p-l)!} \left( \frac{b}{c} \right)^l \int_0^c E_{p-l,\chi} \left( \frac{b}{c} x \right) E_{l-1} (-x) \, dx. \]

For odd \( b + c \), with the use of (8), one can write
\[ 2 \sum_{n=0}^{ck} (-1)^n \xi_p \left( n \frac{b}{c} \right) = 2 \sum_{j=0}^{l-1} \left( \frac{b}{c} \right)^j E_j (0) E_{p-j,\chi} (0) \]
\[ + l \left( \frac{p}{l} \right) \left( \frac{b}{c} \right)^l (-1)^l \int_0^c E_{p-l,\chi} (bx) E_{l-1} (cx) \, dx. \]

Now, let \( f(x) = E_p (xc/b) \), \( \alpha = 0 \) and \( \beta = bk \) in Theorem 1.3. Using (5),
\[ 2 \chi (-1) \sum_{n=0}^{bk} (-1)^n \chi (n) E_p \left( \frac{c}{b} \right) \]
\[ = \sum_{j=0}^l \frac{(-1)^j}{j!} \left( \frac{c}{b} \right)^j \frac{p!}{(p-j)!} \left( E_j \xi (bk) E_{p-j} (ck) - E_j \xi (0) E_{p-j} (0) \right) \]
\[ - \frac{(-1)^l}{l!} \frac{p!}{(p-l-1)!} \left( \frac{c}{b} \right)^{l+1} \int_0^{bk} E_{l,\xi} (x) E_{p-l-1} \left( \frac{c}{b} x \right) \, dx \]
\[ = \sum_{j=0}^l (-1)^j \left( \frac{p}{j} \right) \left( c \right)^j \xi \xi (0) E_{p-j} (0) ((-1)^{b+c} - 1) \]
\[ - (-1)^l \left( \frac{p-1}{l} \right) \left( c \right)^{l+1} b \int_0^k E_{l,\xi} (bx) E_{p-l-1} (cx) \, dx, \]

for \( 0 \leq l \leq p - 2 \). Then, for odd \( b + c \), we have
\[ 2 \chi (-1) \sum_{n=0}^{bk} (-1)^n \chi (n) E_p \left( \frac{c}{b} \right) = 2 \sum_{j=0}^l (-1)^{j+1} \left( \frac{c}{b} \right)^j \xi \xi (0) E_{p-j} (0) \]
\[ - (-1)^l \left( \frac{p-1}{l} \right) \left( c \right)^{l+1} b \int_0^k E_{l,\xi} (bx) E_{p-l-1} (cx) \, dx. \]

Taking \( \chi \to \overline{\chi} \) and \( l = 2 \) in (42) leads to
\[ S_p^{(1)} (b, c : \overline{\chi}) = 2 \sum_{n=0}^{ck} (-1)^n E_{p,\overline{\chi}} \left( \frac{b}{c} \right) \]
\[ = 2 E_0 (0) E_{p,\overline{\chi}} (0) + \frac{2bp}{c} E_1 (0) E_{p-1,\overline{\chi}} (0) \]
\[ = 2 E_0 (0) E_{p,\overline{\chi}} (0) + \frac{2bp}{c} E_1 (0) E_{p-1,\overline{\chi}} (0) \]
+ p(p - 1) \left( \frac{b}{c} \right)^2 \int_0^k \overline{E}_{p-2, \chi} (bx) \overline{E}_1 (cx) \, dx. \tag{44}

Taking \( l = p - 2 \) in (43) yields

\[
S_p^{(2)}(c, b : \chi) = 2 \sum_{n=1}^{bk} (-1)^n \chi(n) \overline{E}_p \left( \frac{n}{b} \right) \\
= 2\chi(1) \sum_{j=0}^{p-2} (-1)^{j+1} \binom{p}{j} \left( \frac{c}{b} \right)^j \overline{E}_{j, \chi}(0) \overline{E}_{p-j}(0) \\
- (-1)^p \chi(1) p(p - 1) \left( \frac{c}{b} \right)^{p-1} \int_0^k \overline{E}_{p-2, \chi} (bx) \overline{E}_1 (cx) \, dx. \tag{45}
\]

Combining (44) and (45), one obtains that

\[
c^p S_p^{(1)}(b, c : \chi) + b^p S_p^{(2)}(c, b : \chi) = 2 \sum_{j=0}^{p} \binom{p}{j} c^j b^{p-j} \overline{E}_{j, \chi}(0) \overline{E}_{p-j}(0),
\]

for odd \((b + c)\) and \((-1)^p \chi(1) = 1\). \( \blacksquare \)

**Proof of Theorem 1.6.** The definition of

\[
S_p(b, c : \chi) = \sum_{n=1}^{ck} \chi(n) \overline{B}_{p, \chi} \left( \frac{b + ck}{2c} n \right)
\]

in this form is not convenient to prove reciprocity formula by aid of Euler–MacLaurin or Boole summation formula. So, \( S_p(b, c : \chi) \) should be modified to apply summation formulas. For this, using (13) in the definition of \( S_p(b, c : \chi) \), and then \([12, \text{Lemma 5.5}]\), we see that

\[
S_p(b, c : \chi) = 2^{-p} \chi(2) \sum_{n=1}^{ck} \chi(n) \overline{B}_{p, \chi} \left( \frac{bn}{c} \right) - p \frac{\chi(2)}{2^{p+1}} \sum_{n=1}^{ck} \chi(n) \overline{E}_{p-1, \chi} \left( \frac{bn}{c} + kn \right) \\
= \frac{\chi(2c)}{2^p c^{p-1}} (k^p - 1) \overline{B}_p(0) - p \frac{\chi(2)}{2^{p+1}} \sum_{n=1}^{ck} (-1)^n \chi(n) \overline{E}_{p-1, \chi} \left( \frac{bn}{c} \right). \tag{46}
\]

Now let \( f(x) = \overline{E}_{p-1, \chi}(xb/c) \), \( \alpha = 0 \) and \( \beta = ck \) in Theorem 1.3. Then, in the light of (6), we can write

\[
\sum_{n=0}^{ck} (-1)^n \chi(n) \overline{E}_{p-1, \chi} \left( \frac{n}{c} \right) \\
= \frac{\chi(-1)}{2} \sum_{j=0}^{l} (-1)^j \binom{p-1}{j} \left( \frac{b}{c} \right)^j \left\{ \left( (-1)^{(b+c)} - 1 \right) \overline{E}_{j, \chi}(0) \overline{E}_{p-1-j, \chi}(0) \right\}
\]

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\[-\frac{\chi(-1)}{2} (-1)^l (p-1) \left(\frac{b}{l}\right)^{l+1} \int_0^k \overline{E}_{l,\chi}(x) \overline{E}_{p-2-l,\chi} \left(\frac{b}{c} x\right) dx.\]  

(47)

Following precisely the method in the proof of Theorem 1.5 and using that $B_p(0) = 0$ for odd $p$ yield

\[\overline{\chi} (-2) bc^p S_p (b, c : \chi) + \chi(-2) cb^p S_p (c, b : \overline{\chi})\]

\[= \frac{p}{2p+1} \sum_{j=1}^p (-1)^j \left(\frac{p-1}{j-1}\right) c^j b^{p+1-j} \overline{E}_{j-1,\chi}(0) \overline{E}_{p-j,\chi}(0).\]

\[\blacksquare\]

**Remark 5.1** Taking into consideration (13), this formula coincides with [9, Corollary 4.3] wherein there is the condition $b$ or $c \equiv 0 \pmod{k}$.

We conclude the study with some results for the integral involving character Euler functions in consequence of (47) and (46). We first note that the sum on the left-hand side of (47) is zero when $p$ and $(b+c)$ have opposite parity. Therefore, if $p > 1$ is odd and $(b+c)$ is even, then

\[
\int_0^k \overline{E}_{l,\chi}(x) \overline{E}_{p-2-l,\chi} \left(\frac{b}{c} x\right) dx = 0
\]

and if $p$ is even and $(b+c)$ is odd, then

\[
\int_0^k \overline{E}_{l,\chi}(cx) \overline{E}_{p-2-l,\chi} (bx) dx \\
= \frac{2}{c (p-1)} \left(\frac{b}{l}\right)^{p-2} \sum_{j=0}^l (-1)^j \left(\frac{p-1}{j}\right) \left(\frac{b}{c}\right)^j \overline{E}_{j,\chi}(0) \overline{E}_{p-1-j,\chi}(0).
\]

Let $p$ and $(b+c)$ be even. Gathering $S_p(b, c : \chi) = c^{1-p} \chi(2c) \overline{\chi} (-b) (k^p - 1) B_p(0)$ ([9, Proposition 5.7]) and (46), one arrives

\[
\sum_{n=1}^{ck} (-1)^n \chi(n) \overline{E}_{p-1,\chi} \left(\frac{bn}{c}\right) = \frac{1}{p} 2 \left(1 - 2^p\right) c^{1-p} \chi(c) \overline{\chi} (-b) (k^p - 1) B_p.
\]

Thus, from the fact that $2 \left(2^p - 1\right) B_p = -p E_{p-1}(0)$, we have

\[
\int_0^k \overline{E}_{l,\chi}(cx) \overline{E}_{p-2-l,\chi} (bx) dx = 2 (-1)^{l+1} \frac{\chi(c) \overline{\chi}(b) (k^p - 1) E_{p-1}(0)}{c^{lp-l-1} b^{l+1} \left(\frac{p-2}{l}\right) p - 1}.
\]
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