APPROXIMATION OF GENERAL 3-VARIABLE JENSEN \( \rho \)-FUNCTIONAL INEQUALITIES IN COMPLEX BANACH SPACES

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Abstract. In this paper, we introduce and investigate general 3-variable Jensen \( \rho \)-functional equation, and prove the Hyers-Ulam stability of the Jensen functional equations associated with the general 3-variable Jensen \( \rho \)-functional inequalities in complex Banach spaces.

1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [26] concerning the stability of group homomorphisms. The essence of the problem is, under what condition does there exists a homomorphism near an approximate homomorphism? The study of stability for functional equation arises from the Ulam’s problem. In 1941, Hyers [11] gave the first affirmative answer to the question of Ulam for Banach spaces. His method was called the direct method. Later, Hyers’ theorem was generalized by Aoki [1] for additive mappings and by Rassias [23] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [10] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th.M. Rassias’ approach. The stability problems for several functional equations or inequalities have been extensively investigated by a number of authors (see [2]–[9], [12]–[19], [22], [24]–[27]).

The function equations

\[
\begin{align*}
 f(x + y + z) + f(x + y - z) - 2f(x) - 2f(y) &= 0 \tag{1.1} \\
 f(x + y + z) - f(x - y - z) - 2f(y) - 2f(z) &= 0. \tag{1.2}
\end{align*}
\]

are called 3-variable Jensen. In [18, 20, 25], Lu et al. investigated the 3-variable functional inequalities and proved their stability.

In this paper, we consider the following functional equations

\[
\begin{align*}
 f(x + y + \alpha z) + f(x + y - \alpha z) - 2f(x) - 2f(y) &= 0, \tag{1.3} \\
 f(x + \beta y + \alpha z) - f(x - \alpha z) - \beta f(y) - 2f(\alpha z) &= 0. \tag{1.4}
\end{align*}
\]
where \( \beta \) and \( \alpha \) are nonzero real numbers. And discuss the Hyers-Ulam stability of general 3-variable Jensen \( \rho \)-functional equations associated with functional inequalities in complex Banach spaces.

Throughout this paper, assume that \( X \) is a complex normed vector space with norm \( \| \cdot \| \) and that \( Y \) is a complex Banach space.

2. Hyers-Ulam stability of (1.3)

In this section, we prove the Hyers-Ulam stability of the 3-variable function inequality

\[
\| f(x + y + \alpha z) + f(x + y - \alpha z) - 2f(x) - 2f(y) \|
\leq \| \rho_1 (f(x + y + \alpha z) - f(x + y) - f(\alpha z)) \|
\leq \| \rho_2 (f(x + y - \alpha z) + f(-x) + f(\alpha z - y)) \|
\]

in complex Banach spaces, where \( \rho_1 \) and \( \rho_2 \) are fixed complex numbers with \( |\rho_1| + 3|\rho_2| < 2 \).

**Lemma 2.1.** Let \( f : X \to Y \) be a mapping. If it satisfies (2.1) for all \( x, y, z \in X \), then \( f \) is additive.

**Proof.** Letting \( x = y = z = 0 \) in (2.1), we get

\[
2\| f(0) \| \leq (|\rho_1| + 3|\rho_2|)\| (f(0) \|
\]

and thus \( f(0) = 0 \), \( |\rho_1| + 3|\rho_2| < 2 \).

Letting \( x = y = 0 \) in (2.1), we get

\[
\| f(\alpha z) + f(-\alpha z) \| \leq \| \rho_2 (f(-\alpha z) + f(\alpha z)) \|
\]

and so \( f(-x) = -f(x) \) for all \( x \in X \).

Letting \( z = 0 \) in (2.1), we have

\[
\| 2f(x + y) - 2f(x) - 2f(y) \| \leq \| \rho_2 (f(x + y) - f(x) - f(y)) \|
\]

and so \( f(x + y) = f(x) + f(y) \) for all \( x, y \in X \). Hence \( f : X \to Y \) is additive. \( \square \)

**Corollary 2.2.** Let \( f : X \to Y \) be a mapping satisfying

\[
\| f(x + y + \alpha z) + f(x + y - \alpha z) - 2f(x) - 2f(y) \|
\leq \| \rho_1 (f(x + y + \alpha z) - f(x + y) - f(\alpha z)) \|
\leq \| \rho_2 (f(x + y - \alpha z) + f(-x) + f(\alpha z - y)) \|
\]

for all \( x, y, z \in X \). Then \( f : X \to Y \) is additive.

We prove the Hyers-Ulam stability of the additive functional inequality (2.1) in complex Banach spaces.
Theorem 2.3. Let $f : X \to Y$ be a mapping. If there is a function $\varphi : X^3 \to [0, \infty)$ with $\varphi(0,0,0) = 0$ such that
\[
\|f(x + y + \alpha z) + f(x + y - \alpha z) - 2f(x) - 2f(y)\|
\leq \rho_1(f(x + y + \alpha z) - f(x + y) - f(\alpha z))
+ \rho_2(f(x + y - \alpha z) + f(-x) + f(\alpha z - y)) + \varphi(x, y, z)
\]
and
\[
\lim_{j \to \infty} \frac{1}{2^j} \varphi(2^j x, 2^j y, 2^j z) = 0
\]
for all $x, y, z \in X$, then there exists a unique additive mapping $A : X \to Y$ such that
\[
\|f(x) - A(x)\| \leq \tilde{\varphi}(x)
\]
for all $x \in X$.

Proof. Letting $x = y = z = 0$ in (2.3), we get
\[
2\|f(0)\| \leq (|\rho_1| + 3|\rho_2|)\|f(0)\|.
\]  
(2.7)
So $f(0) = 0$. Letting $x = y = 0$ in (2.1), we get
\[
\|f(\alpha z) + f(-\alpha z)\| \leq \rho_2(f(-\alpha z) + f(\alpha z)) + \varphi(0, 0, z)
\]
and so
\[
\|f(z) + f(-z)\| \leq \frac{\varphi(0, 0, \frac{z}{\alpha})}{1 - |\rho_2|}
\]
for all $z \in X$.

Letting $y = x$ and $z = 0$ in (2.3), we get
\[
\|2f(2x) - 4f(x)\| \leq |\rho_2|\|f(2x) - 2f(x)\| + 2|\rho_2|\|f(x) + f(-x)\| + \varphi(x, x, 0)
\]
(2.8)
and so
\[
\|f(2x) - 2f(x)\| \leq \frac{1}{2 - |\rho_2|} \left( \varphi(x, x, 0) + \frac{2|\rho_2|}{1 - |\rho_2|} \varphi(0, 0, \frac{x}{\alpha}) \right)
\]
for all $x \in X$. Thus
\[
\left\| \frac{f(x) - f(2x)}{2} \right\| \leq \frac{1}{2(2 - |\rho_2|)} \left( \varphi(x, x, 0) + \frac{2|\rho_2|}{1 - |\rho_2|} \varphi(0, 0, \frac{x}{\alpha}) \right)
\]
for all \( x \in X \). Hence one may have the following formula for positive integers \( m, l \) with \( m > l \),

\[
\left\| \frac{1}{2^l} f \left( 2^l x \right) - \frac{1}{2^m} f \left( 2^m x \right) \right\| \\
\leq \sum_{i=l}^{m-1} \frac{1}{2^{i+1}} \frac{1}{(2 - |\rho_2|)} \left( \phi(2^i x, 2^i x, 0) + \frac{2|\rho_2|}{1 - |\rho_2|} \phi \left( 0, 0, \frac{2^i x}{\alpha} \right) \right)
\]

(2.9)

for all \( x \in X \).

It follows from (2.5) that the sequence \( \left\{ \frac{f(2^k x)}{2^k} \right\} \) is a Cauchy sequence for all \( x \in X \).

Since \( Y \) is complete, the sequence \( \left\{ \frac{f(2^k x)}{2^k} \right\} \) converges. So one may define the mapping \( A : X \to Y \) by

\[
A(x) := \lim_{k \to \infty} \left\{ \frac{f(2^k x)}{2^k} \right\}, \quad \forall x \in X.
\]

Taking \( l = 0 \) and letting \( m \) tend to \( \infty \) in (2.9), we get (2.6).

It follows from (2.3) that

\[
\| A(x + y + \alpha z) + A(x + y - \alpha z) - 2A(x) - 2A(y) \| \\
= \lim_{n \to \infty} \frac{1}{2^n} \left\| f \left[ 2^n (x + y + \alpha z) \right] + f \left[ 2^n (x + y - \alpha z) \right] - 2 f \left( 2^n x \right) - 2 f \left( 2^n y \right) \right\| \\
\leq \lim_{n \to \infty} \frac{1}{2^n} \left\| \rho_1 ( f \left[ 2^n (x + y + \alpha z) \right] - f \left( 2^n x + 2^n y \right) - f \left( 2^n \alpha z \right) ) \right\| \\
+ \lim_{n \to \infty} \frac{1}{2^n} \left\| \rho_2 ( f \left[ 2^n (x + y - \alpha z) \right] + f \left( -2^n x \right) + f \left( -2^n y + 2^n \alpha z \right) ) \right\| \\
+ \lim_{n \to \infty} \frac{1}{2^n} \phi \left( 2^n x, 2^n y, 2^n \alpha z \right) \\
= \| \rho_1 (A(x + y + \alpha z) - A(x + y) - A(\alpha z)) \| \\
+ \| \rho_2 (A(x + y - \alpha z) + A(-x) + A(-y + \alpha z)) \| \\
\]

(2.10)

for all \( x, y, z \in X \). One can see that \( A \) satisfies the inequality (2.1) and so it is additive by Lemma 2.1.

Now, we show that the uniqueness of \( A \). Let \( T : X \to Y \) be another additive mapping satisfying (2.3). Then one has

\[
\| A(x) - T(x) \| = \left\| \frac{1}{2^k} A \left( 2^k x \right) - \frac{1}{2^k} T \left( 2^k x \right) \right\| \\
\leq \frac{1}{2^k} \left( \| A \left( 2^k x \right) - f \left( 2^k x \right) \| \\
+ \| T \left( 2^k x \right) - f \left( 2^k x \right) \| \right) \\
\leq 2 \frac{1}{2^k} \tilde{\phi}(2^k x) = \sum_{i=k}^{\infty} \frac{1}{2^{i+1}} \frac{1}{(2 - |\rho_2|)} \left( \phi(2^i x, 2^i x, 0) + \frac{2|\rho_2|}{1 - |\rho_2|} \phi \left( 0, 0, \frac{2^i x}{\alpha} \right) \right),
\]

(2.11)
which tends to zero as $k \to \infty$ for all $x \in X$. So we can conclude that $A(x) = T(x)$ for all $x \in X$. 

\textbf{Corollary 2.4.} Let $r < 1$ and $\theta$ be nonnegative real numbers and $f : X \to Y$ be a mapping such that

$$\|f(x + y + \alpha z) + f(x + y - \alpha z) - 2f(x) - 2f(y)\|$$

$$= \|\rho_1(f(x + y + \alpha z) - f(x + y) - f(\alpha z))\|$$

$$+ \|\rho_2(f(x + y - \alpha z) + f(-x) + f(\alpha z - y))\| + \theta(\|x\|^{r} + \|y\|^{r} + \|z\|^{r})$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{2\theta}{(2 - 2^{r})} \cdot \frac{1}{(1 - |\rho_2|)(2 - |\rho_2|)} \|x\|^{r}$$

for all $x \in X$.

\textbf{Theorem 2.5.} Let $f : X \to Y$ be a mapping with $\varphi(0, 0, 0) = 0$. If there is a function $\varphi : X^3 \to [0, \infty)$ satisfying (2.13) such that

$$\lim_{j \to \infty} 2^{j} \varphi \left(\frac{x}{2^{j}}, \frac{y}{2^{j}}, \frac{z}{2^{j}}\right) = 0$$

for all $x, y, z \in X$, then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \tilde{\varphi} \left(\frac{x}{2}\right) := \sum_{i=0}^{\infty} \frac{1}{2^{i}} \cdot \frac{1}{2 - |\rho_2|} \left(\varphi \left(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}, 0\right) + \frac{2|\rho_2|}{1 - |\rho_2|} \varphi(0, 0, \frac{x}{2^{i+1}})\right)$$

for all $x \in X$.

\textit{Proof.} Similar to the proof of Theorem 2.3, we can get

$$\left\|f(x) - 2f \left(\frac{x}{2}\right)\right\| \leq \frac{1}{2 - |\rho_2|} \left(\varphi \left(\frac{x}{2}, \frac{x}{2}, 0\right) + \frac{2|\rho_2|}{1 - |\rho_2|} \varphi(0, 0, \frac{x}{2\alpha})\right)$$

for all $x \in X$.

Next, we can prove that the sequence $\{2^{n}f \left(\frac{x}{2^{n}}\right)\}$ is a Cauchy sequence for all $x \in X$, and define a mapping $A : X \to Y$ by

$$A(x) := \lim_{n \to \infty} 2^{n}f \left(\frac{x}{2^{n}}\right)$$

for all $x \in X$.

The rest proof is similar to the corresponding part of the proof of Theorem 2.3. \hfill \Box

\textbf{Corollary 2.6.} Let $r > 1$ and $\theta$ be nonnegative real numbers and $f : X \to Y$ be a mapping such that

$$\|f(x + y + \alpha z) + f(x + y - \alpha z) - 2f(x) - 2f(y)\|$$

$$\leq \|\rho_1(f(x + y + \alpha z) - f(x + y) - f(\alpha z))\|$$

$$+ \|\rho_2(f(x + y - \alpha z) + f(-x) + f(\alpha z - y))\| + \theta(\|x\|^{r} + \|y\|^{r} + \|z\|^{r})$$

(2.14)
for all \(x, y, z \in X\). Then there exists a unique additive mapping \(A : X \rightarrow Y\) such that
\[
\|f(x) - A(x)\| \leq \frac{2^{1+r} \theta}{2^r - 1} \frac{1}{(1 - |\rho_2|)(2 - |\rho_2|)}\|x\|^r
\]  
for all \(x \in X\).

3. Hyers-Ulam stability of (1.4)

In this section, we prove that the Hyers-Ulam stability of the 3-variable functional inequality
\[
\|f(x + \beta y + \alpha z) - f(x - \alpha z) - \beta f(y) - 2f(\alpha z)\|
\leq \|\rho_1(f(x + \alpha z) - f(x) - f(\alpha z))\|
+ \|\rho_2(f(x + \beta y - \alpha z) - f(x) - \beta f(y) + f(\alpha z))\|
\] in complex Banach space, where \(\rho_1\) and \(\rho_2\) are fixed complex numbers with \(|\rho_2| < 1\) and \(|\beta + 2| \geq |\rho_1| + |\rho_2(1 - \beta)|\).

Lemma 3.1. Let \(f : X \rightarrow Y\) be a mapping. If it satisfies (3.1) for all \(x, y, z \in X\), then \(f\) is additive.

Proof. Letting \(x = y = z = 0\) in (3.1) for all \(x, y, z \in X\), we get
\[
\|(\beta + 2)f(0)\| \leq (|\rho_1| + |\rho_2||\beta - 1|)\|f(0)\|. 
\] Thus \(f(0) = 0\).

Letting \(x = y = 0\) in (3.1), we get
\[
(1 - |\rho_2|)\|f(\alpha z) + f(-\alpha z)\| \leq 0
\] and so \(f(-x) = -f(x)\) for all \(x \in X\).

Letting \(x = 0\) in (3.1), we have
\[
\|f(\beta y + \alpha z) - f(\alpha z) - \beta f(y)\| \leq \|\rho_2(f(\beta y - \alpha z) - \beta f(y) + f(\alpha z))\|
\] for all \(y, z \in X\).

Letting \(z = -z\) in (3.3), we get
\[
\|f(\beta y - \alpha z) + f(\alpha z) - \beta f(y)\| \leq |\rho_2|\|f(\beta y + \alpha x) - \beta f(y) - f(\alpha z)\|
\] for all \(y, z \in X\). Thus
\[
\|f(\beta y - \alpha z) - \beta f(y) + f(\alpha z)\| \leq 0
\] and so
\[
\|f(y + z) - f(y) - f(z)\| = 0
\] for all \(y, z \in X\). Hence \(f : X \rightarrow Y\) is additive. \(\square\)
Corollary 3.2. Let \( f : X \rightarrow Y \) be a mapping satisfying
\[
\|f(x + \beta y + \alpha z) - f(x - \alpha z) - \beta f(y) - 2f(\alpha z)\| \\
= \|\rho_1(f(x + \alpha z) - f(x) - f(\alpha z))\| \\
+ \|\rho_2(f(x + \beta y - \alpha z) - f(x) - \beta f(y) + f(\alpha z))\| \\
\tag{3.6}
\]
for all \( x, y, z \in X \). Then \( f : X \rightarrow Y \) is additive.

We prove the Hyers-Ulam stability of the functional inequality (3.1) in complex Banach spaces.

Theorem 3.3. Let \( f : X \rightarrow Y \) be a mapping. Assume that there is a function \( \phi : X^3 \rightarrow [0, \infty) \) with \( \phi(0, 0, 0) = 0 \) such that
\[
\|f(x + \beta y + \alpha z) - f(x - \alpha z) - \beta f(y) - 2f(\alpha z)\| \\
\leq \|\rho_1(f(x + \alpha z) - f(x) - f(\alpha z))\| \\
+ \|\rho_2(f(x + \beta y - \alpha z) - f(x) - \beta f(y) + f(\alpha z))\| + \phi(x, y, z) \\
\tag{3.7}
\]
and
\[
\lim_{j \to \infty} \frac{1}{|1 + \beta|^j} \phi((1 + \beta)^j x, (1 + \beta)^j y, (1 + \beta)^j z) = 0 \tag{3.8}
\]
for all \( x, y, z \in X \). Then there exists a unique additive mapping \( A : X \rightarrow Y \) such that
\[
\|f(x) - A(x)\| \leq \tilde{\phi}(x, x, 0) \tag{3.9}
\]
for all \( x \in X \), where
\[
\tilde{\phi}(x, y, z) := \frac{1}{|1 + \beta|(1 - |\rho_1|)} \sum_{j=0}^{\infty} \frac{1}{|1 + \beta|^j} \phi((1 + \beta)^j x, (1 + \beta)^j y, (1 + \beta)^j z) < \infty \tag{3.10}
\]
for all \( x, y, z \in X \).

Proof. Letting \( x = y = z = 0 \) in (3.7), we get
\[
\| (\beta + 2)f(0) \| \leq (|\rho_1| + |(1 - \beta)\rho_2|) \|f(0)\|. \tag{3.11}
\]
So \( f(0) = 0 \).

Letting \( z = 0 \) and \( y = x \) in (3.7), we get
\[
\|f((1 + \beta)x) - (1 + \beta)f(x)\| \leq |\rho_2|\|f((1 + \beta)x) - (1 + \beta)f(x)\| + \phi(x, x, 0) \tag{3.12}
\]
for all \( x \in X \).

Thus
\[
\left\| \frac{f(x) - f((1 + \beta)x)}{1 + \beta} \right\| \leq \frac{1}{1 - |\rho_2|} \frac{1}{|1 + \beta|} \phi(x, x, 0)
\]
for all \( x \in X \).
Hence one may have the following formula for positive integers \(m, l\) with \(m > l\),

\[
\left\| \frac{1}{|1 + \beta|^l} f \left( (1 + \beta)^l x \right) - \frac{1}{|1 + \beta|^m} f \left( (1 + \beta)^m x \right) \right\| \\
\leq \frac{1}{|1 + \beta|(1 - |\rho_1|)} \sum_{i=l}^{m-1} \frac{1}{|1 + \beta|^i} \varphi \left( |1 + \beta|^i x, |1 + \beta|^i x, 0 \right),
\]

(3.13)

for all \(x \in X\).

It follows from (3.10) that the sequence \(\left\{ \frac{f((1 + \beta)^k x)}{(1 + \beta)^k} \right\}\) is a Cauchy sequence for all \(x \in X\). Since \(Y\) is complete, the sequence \(\left\{ \frac{f((1 + \beta)^k x)}{(1 + \beta)^k} \right\}\) converges. So one may define the mapping \(A : X \to Y\) by

\[
A(x) := \lim_{k \to \infty} \left\{ \frac{f((1 + \beta)^k x)}{(1 + \beta)^k} \right\}, \quad \forall x \in X.
\]

Taking \(m = 0\) and letting \(l\) tend to \(\infty\) in (3.13), we get (3.9).

It follows from (3.7) that

\[
\|A(x + \beta y + \alpha z) - A(x - \alpha z) - \beta A(y) - 2A(\alpha z)\|
\]

\[
= \lim_{n \to \infty} \frac{1}{|1 + \beta|^n} \| f \left[ (1 + \beta)^n (x + \beta y + \alpha z) \right] + f \left[ (1 + \beta)^n (x - \alpha z) \right] - \beta f \left( (1 + \beta)^n y \right) - 2f \left( (1 + \beta)^n \alpha z \right) \|
\]

\[
\leq \lim_{n \to \infty} \frac{1}{|1 + \beta|^n} \| \rho_1 \left( f \left[ (1 + \beta)^n (x + \alpha z) \right] - f \left[ (1 + \beta)^n (x - \alpha z) \right] \right) - f \left( (1 + \beta)^n \alpha z \right) \|
\]

\[
+ \lim_{n \to \infty} \frac{1}{|1 + \beta|^n} \| \rho_2 \left( f \left[ (1 + \beta)^n (x + \beta y - \alpha z) \right] - f \left( (1 + \beta)^n x \right) \right) - \beta f \left( (1 + \beta)^n y \right) + f \left( (1 + \beta)^n \alpha z \right) \|
\]

\[
+ \lim_{n \to \infty} \frac{1}{|1 + \beta|^n} \varphi \left( (1 + \beta)^n x, (1 + \beta)^n y, (1 + \beta)^n z \right)
\]

\[
= \| \rho_1 (A(x + \alpha z) - A(x) - A(\alpha z)) \|
\]

\[
+ \| \rho_2 (A(x + \beta y - \alpha z) - A(x) - \beta A(y) + A(\alpha z)) \|
\]

(3.14)

for all \(x, y, z \in X\). One can see that \(A\) satisfies the inequality (3.1) and so it is additive by Lemma 3.1.

Now, we show that the uniqueness of \(A\). Let \(T : X \to Y\) be another additive mapping satisfying (3.7). Then one has
\[\|A(x) - T(x)\| = \left\| \frac{1}{(1 + \beta)^k} A((1 + \beta)^k x) - \frac{1}{(1 + \beta)^k} T((1 + \beta)^k x) \right\| \]
\[\leq \frac{1}{1 + \beta} \left( \|A((1 + \beta)^k x) - f((1 + \beta)^k x)\| \right. \]
\[+ \left. \|T((1 + \beta)^k x) - f((1 + \beta)^k x)\| \right) \]
\[\leq 2 \frac{1}{1 + \beta|k|} \tilde{\varphi}(x, x, 0), \]

which tends to zero as \(k \to \infty\) for all \(x \in X\). So we can conclude that \(A(x) = T(x)\) for all \(x \in X\). \(\Box\)

**Corollary 3.4.** Let \(r > 1\) and \(\theta\) be nonnegative real numbers and \(f : X \to Y\) be a mapping such that
\[\|f(x + \beta y + \alpha z) - f(x - \alpha z) - \beta f(y) - 2f(\alpha z)\| \]
\[\leq \|\rho_1(f(x + \alpha z) - f(x) - f(\alpha z))\| \]
\[+ \|\rho_2(f(x + \beta y - \alpha z) - f(x) - \beta f(y) + f(\alpha z))\| + \theta(||x||^r + ||y||^r + ||z||^r) \quad (3.15)\]

for all \(x, y, z \in X\) with \(|1 + \beta| > 1\). Then there exists a unique additive mapping \(A : X \to Y\) such that
\[\|f(x) - A(x)\| \leq \frac{2\theta}{|1 + \beta| - |1 + \beta|^r} \frac{1}{1 - |\rho_2|} ||x||^r \quad (3.16)\]

for all \(x \in X\).

**Theorem 3.5.** Let \(f : X \to Y\) be a mapping with \(f(0) = 0\). If there is a function \(\varphi : X^3 \to [0, \infty)\) satisfying (3.7) such that
\[\tilde{\varphi}(x, y, z) := \sum_{j=1}^{\infty} |1 + \beta|^j \varphi \left( \frac{x}{(1 + \beta)^j}, \frac{y}{(1 + \beta)^j}, \frac{z}{(1 + \beta)^j} \right) < \infty \quad (3.17)\]

for all \(x, y, z \in X\), then there exists a unique additive mapping \(A : X \to Y\) such that
\[\|f(x) - A(x)\| \leq \frac{1}{1 - |\rho_2|} \tilde{\varphi} \left( \frac{x}{1 + \beta}, \frac{x}{1 + \beta}, 0 \right) \quad (3.18)\]

for all \(x \in X\).

**Proof.** Similar to the proof of Theorem 3.3 we can get
\[\left\| f(x) - (1 + \beta) f \left( \frac{x}{1 + \beta} \right) \right\| \leq \frac{1}{1 - |\rho_2|} \varphi \left( \frac{x}{1 + \beta}, \frac{x}{1 + \beta}, 0 \right) \]

for all \(x \in X\).
Next, we can prove that the sequence \( \{(1 + \beta)^nf\left(\frac{x}{(1+\beta)^n}\right)\} \) is a Cauchy sequence for all \( x \in X \) and define a mapping \( A : X \to Y \) by
\[
A(x) := \lim_{n \to \infty} (1 + \beta)^nf\left(\frac{x}{(1+\beta)^n}\right)
\]
for all \( x \in X \). The rest of the proof is similar to the corresponding part of the proof of Theorem 3.3. \( \square \)

**Corollary 3.6.** Let \( r > 1 \) and \( \theta \) be nonnegative real numbers and \( f : X \to Y \) be a mapping such that
\[
\|f(x + \beta y + \alpha z) - f(x - \alpha z) - \beta f(y) - 2f(\alpha z)\|
\leq \|\rho_1(f(x + \alpha z) - f(x) - f(\alpha z))\|
+ \|\rho_2(f(x + \beta y - \alpha z) - f(x) - \beta f(y) + f(\alpha z))\|
+ \theta(\|x\|^r + \|y\|^r + \|z\|^r)
\]
for all \( x, y, z \in X \) and \(|1 + \beta| < 1\). Then there exists a unique additive mapping \( A : X \to Y \) such that
\[
\|f(x) - A(x)\| \leq \frac{2\theta}{|1 + \beta|^r - |1 + \beta| |1 - |\rho_2||} \|x\|^r
\]
for all \( x \in X \).

**Competing interests**

The author declares that he has no competing interests.

**Authors’ contributions**

The author conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

**Funding**

This work was supported by National Natural Science Foundation of China (No. 11761074), the Projection of the Department of Science and Technology of JiLin Province and the Education Department of Jilin Province (No. 20170101052JC) and the scientific research project of Guangzhou College of Technology and Business in 2020(No. KA202032).
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