SYMPLECTIC GROUPS ARE $N$-DETERMINED 2-COMPACT GROUPS

ALEŠ VAVPETIČ AND ANTONIO VIRUEL

Abstract. We show that for $n \geq 3$ the symplectic group $Sp(n)$ is as a 2-compact group determined up to isomorphism by the isomorphism type of its maximal torus normalizer. This allows us to determine the integral homotopy type of $Sp(n)$ among connected finite loop spaces with maximal torus.

1. Introduction

The advent of $p$-compact groups in the celebrated work of Dwyer and Wilkerson [9] is the culmination of a research program that can be traced back to the work of Hopf and Serre on $H$-spaces and loop spaces, and fits within the philosophy of Hilbert’s Fifth Problem: which are the non differential (here homotopy theoretical) properties that characterize compact Lie groups?

A $p$-compact group is a loop space $(X, BX, e)$, i.e. $e : X \simeq \Omega(BX)$ for a pointed space $BX$, such that $H^*(X; \mathbb{F}_p)$ is finite and $BX$ is $p$-complete in the sense of Bousfield and Kan [4]. As expected, examples of $p$-compact groups are given by $p$-completion of compact Lie groups $G$ for which $\pi_0 G$ is a $p$-group, since $G^\wedge$ is homotopy equivalent to $\Omega(BG_p^\wedge)$. In this way a $p$-compact torus $T$ of rank $n$ is the $p$-completion of an ordinary torus, hence $BT$ is the Eilenberg-MacLane space $K((\mathbb{Z}/p)^\oplus_n, 2)$. Further examples are given by the realization of polynomial algebras, i.e. loop spaces $\Omega BX$, where $BX$ is $p$-complete and has polynomial mod-$p$ cohomology ([1], [5], [11], [27], [32], [37]). The importance of $p$-compact groups consists of a dictionary (reviewed in Section 2) that translates much of the rich internal algebraic structure of compact Lie groups to the homotopy theoretical setting of $p$-compact groups, so the challenge is then to give homotopy theoretical proofs of classical algebraic Lie group theory results.

One of those challenges quoted above is the following: $p$-compact groups admit maximal tori, Weyl groups and maximal torus normalizers in a way that extends the classical concepts in Lie group theory [9, 8.13 & 9.5], so can we “reproof” the Lie group theoretical Curtis-Wiederhold-Williams’ theorem [6] in the setting of $p$-compact groups? Recall that Curtis-Wiederhold-Williams’ theorem states that two compact connected Lie groups are isomorphic if and only if their maximal torus normalizers are isomorphic, hence we are led to the following conjecture [7, Conjecture 5.3]
Conjecture 1.1. Let X be a connected p-compact group with maximal torus $T_X$. Then X is determined up to equivalence by the loop space $NT_X$.

We shall say that a p-compact group X is N-determined if X verifies Conjecture 1.1 even if the hypothesis “connected” is dropped, i.e. X is N-determined if every p-compact group $Y$, with the normalizer of a maximal torus isomorphic to that of X, is isomorphic to X.

Given an odd prime $p > 2$, p-compact groups are known to be N-determined [2], what leads to the classification of p-compact groups for $p$ odd. But the situation is quite different at $p = 2$: there exist 2-compact groups which are not N-determined. For example $O(n)_2^\wedge$ and $SO(n+1)_2^\wedge$ are non isomorphic 2-compact groups that have isomorphic maximal torus normalizers. So at $p = 2$ we cannot drop the hypothesis “connected” in Conjecture 1.1.

We say that a 2-compact group X is weakly N-determined if every 2-compact group $Y$, for which there exists a homotopy equivalence $BN_X \simeq BY$ between the maximal torus normalizers of X and Y, inducing an isomorphism $\pi_0X \simeq \pi_0Y$, is isomorphic to X. From the definitions it follows that an N-determined 2-compact group is also weakly N-determined.

It has been shown that the 2-compact groups $O(n)_2^\wedge$, $SO(2n+1)_2^\wedge$ and $Spin(2n+1)_2^\wedge$ [25] are weakly N-determined 2-compact groups (and they are not N-determined), and that $U(n)_2^\wedge$ for $n \neq 2$ [23], $(G_2)_2^\wedge$ [31], $(F_4)_2^\wedge$ [33], and $DI(4)$ [26] are N-determined. $(U(2)_2^\wedge$ is only weakly N-determined, because the normalizer $N$ of a maximal torus of $U(2)_2^\wedge$ is also a 2-compact group but N is not isomorphic to $U(2)_2^\wedge$). In this paper we prove that the symplectic groups $Sp(n)_2^\wedge$ are N-determined 2-compact groups for $n \geq 3$.

Theorem 1.2. Let $n \geq 3$ and let X be a 2-compact group with the maximal torus normalizer $f_N$: $N \longrightarrow X$ isomorphic to that of $Sp(n)_2^\wedge$. Then X and $Sp(n)_2^\wedge$ are isomorphic 2-compact groups.

Proof. First we prove that X is connected in Section 3 (Proposition 3.1). In Section 4 we show that mod-2 cohomology of BX is isomorphic to that of $BSp(n)$ as algebras over the Steenrod algebra, which implies that the Quillen categories associated to X and $Sp(n)$ are isomorphic. In section 5 we describe the 2-stubborn decomposition of the group $Sp(n)$, which allows us to define a map from $BSp(n)_2^\wedge$ to BX that happens to be an equivalence. This is done in Section 6. □

Notice that the hypothesis $n \geq 3$ is necessary as $Sp(1)_2^\wedge = SU(2)_2^\wedge$ and $Sp(2)_2^\wedge = Spin(5)_2^\wedge$ are only weakly N-determined 2-compact groups.

The combination of the results in [2] and Theorem 1.2 shows that if G is a connected compact Lie group, then BG is in the adic genus of $BSp(n)$ if and only if $G = Sp(n)$, which in view of [30] characterize the integral homotopy type of $BSp(n)$ as a loop space. Thus our final result is

Theorem 1.3. Let L be a connected finite loop space with a maximal torus normalizer isomorphic to that of $Sp(n)$. Then BL is homotopy equivalent to $BSp(n)$.
Notation. Here all spaces are assumed to have the homotopy type of a CW-complex. Completion means Bousfield-Kan completion \([4]\). For a given space \(X\), we write \(H^*(X; \mathbb{F}_2)\) for the mod-2 cohomology of \(X\). For a prime \(p\), we write \(X^\wedge_p\) for the Bousfield-Kan \(p\)-completion ((\(\mathbb{Z}_p\))\(\infty\)-completion in the terminology of Bousfield and Kan) of the space \(X\). We assume that the reader is familiar with Lannes’ theory [17].

2. The dictionary

As announced in the introduction, this section is devoted to a brief review of the dictionary translating constructions and arguments from the algebraic theory of groups to the homotopical setting of \(p\)-compact groups. The aim of the minimalist style of this section is to ease the search of concepts by the reader who will find a more detailed exposition in the original [8], or the reviews [7], [21] and [28] if needed.

Along this section \(X\) and \(Y\) will denote \(p\)-compact groups whose classifying spaces are \(BX\) and \(BY\) respectively. By \(T\) we shall denote a \(p\)-compact torus, i.e. \(BT \cong K(\mathbb{Z}_p^n, 2)\) where \(n\) is the rank of \(T\). Finally, we define:

- **Homomorphisms** [8, 3.1]: A homomorphism \(X \xrightarrow{f} Y\) of \(p\)-compact groups is a pointed map \(BX \xrightarrow{Bf} BY\). The homomorphism \(f\) is an isomorphism if \(Bf\) is a homotopy equivalence. It is a monomorphism if the homotopy fiber \(Y/X\) of \(Bf\) is \(\mathbb{F}_p\)-finite or equivalently if \(H^*(BX, \mathbb{F}_p)\) is a finitely generated module over \(H^*(BY, \mathbb{F}_p)\) via \(Bf^*\).

- **Centralizers** [8, 3.4]: For a homomorphism \(Y \xrightarrow{f} X\) of \(p\)-compact groups, the centralizer \(C_X(f(Y))\) is defined by the equation \(BC_X(f(Y)) := \text{Map}(BY, BX)_{Bf}\).

- **Maximal tori** [8, Definition 8.9]: A monomorphism \(T \hookrightarrow X\) of a \(p\)-compact torus into a \(p\)-compact group \(X\) is a maximal torus if \(C_X(T)\) is a \(p\)-compact toral group and if \(C_X(T)/T\) is homotopically discrete. Every \(p\)-compact group admits maximal tori [8, Theorem 8.13].

- **Weyl group** [8, Definition 9.2]: Let \(BT_X \xrightarrow{Bf_T} BX\) be a maximal torus of a \(p\)-compact group \(X\). Assume that \(Bf_T\) is already a fibration and treat \(W_X\) as the space of self-maps of \(BT_X\) over \(BX\). Composition gives \(W_X\) the structure of an associative topological monoid. It is shown [8, Proposition 9.5] that \(W_X\) is homotopically discrete and therefore \(W_X := \pi_0 W_X\) is a (finite) group. Moreover, if \(X\) is connected, the action of \(W_X\) on \(BT_X\) induces a faithful representation

\[
W_X \to \text{GL}(H^*_\mathbb{Q}_p BT_X) \cong \text{GL}_n(\mathbb{Q}_p^\wedge)
\]

whose image is generated by pseudoreflections, i.e. \(W_X\) is a pseudo reflection group [8, Theorem 9.7].

- **Maximal torus normalizers** [8, Definition 9.8]: Let \(BT_X \xrightarrow{Bf_T} BX\) be a maximal torus of a \(p\)-compact group \(X\). The normalizer of \(T_X\) denoted by \(NT_X\), or simply by \(N_X\) or \(N\), is the loop space such that \(BNT_X\) is the Borel construction associated to the action of \(W_X\) on \(BT_X\).
All these concepts generalize the classical algebraic definitions. In particular, if \( G \) is a compact Lie group such that \( \pi_0G \) is \( p \)-group, \( i: T \longrightarrow G \) a maximal torus of \( G \), \( W \) the Weyl group of \( G \), and \( N \) is the normalizer of the maximal torus \( T \), then the \( p \)-completion \( i_p^\wedge: T_p^\wedge \longrightarrow G_p^\wedge \) is a maximal torus of the \( p \)-compact group \( G_p^\wedge \). The Weyl group \( W \) is naturally isomorphic to the \( p \)-Weyl group \( W_{G_p} \). The classifying space \( BN \) of the normalizer \( N \) sits in the fibration \( BT \longrightarrow BN \longrightarrow BW \), and a normalizer of the maximal torus \( T_p^\wedge \) of \( G_p^\wedge \) is isomorphic to the fiberwise \( p \)-completion \( BN_\pi^\wedge \) by [23] Proposition 1.8, or [34] Lemma 6.1.

3. Connectedness

In this section we proceed with the first step in the proof of Theorem 1.2 by proving the following proposition.

**Proposition 3.1.** Let \( X \) be a 2-compact group with the normalizer of a maximal torus isomorphic to that of \( Sp(n)_2^\wedge \), where \( n \geq 3 \), then \( X \) is connected.

The proof of the result above requires calculating the Weyl group of some centralizer in the connected component of \( X \). This is done by means of the technics developed by Dwyer and Wilkerson in [8] that we recall now.

An extended \( p \)-discrete torus \( P \) is an extension of a \( p \)-discrete torus \( (\mathbb{Z}/p^\infty)^n \) by a finite group. A discrete approximation for an extended \( p \)-compact torus \( P \) is a homomorphism \( f: \check{P} \longrightarrow P \), where \( \check{P} \) is an extended \( p \)-discrete torus and \( f \) induces an isomorphism \( \check{P}/\check{P}_0 \longrightarrow \pi_0 P \) and an isomorphism \( H^*B\check{P}_0 \longrightarrow H^*BP_0 \). Every extended \( p \)-compact torus has a discrete approximation [8] Proposition 3.13.

**Definition 3.2.** [8] Definition 7.3] If \( s \in W \) is a pseudoreflection of order \( \text{ord}(s) \), then

1. the fixed point set \( F(s) \) of \( s \) is the fixed point set of the action of \( x \) on \( \check{T} \) by conjugation, where \( x \in \check{N}(T) \) is an element which projects on \( s \) by natural projection \( \check{N}(T) \longrightarrow W \),
2. the singular hyperplane \( H(s) \) of \( s \) is the maximal divisible subgroup of \( F(s) \) (so \( H(s) \cong (\mathbb{Z}/p^\infty)^{-1} \)),
3. the singular coset \( K(s) \) of \( s \) is the subset of \( \check{T} \) given by elements of the form \( x^{\text{ord}(s)} \), as \( x \) runs through elements of \( \check{N}(T) \), which project to \( s \) in \( W \), and
4. the singular set \( \sigma(s) \) of \( s \) is the union \( \sigma(s) = H(s) \cup K(s) \).

Notice that there are inclusions \( H(s) \subset \sigma(s) \subset F(s) \) [8] Remark 7.7[.

Let \( A \subset \check{T} \) be a subgroup. Let \( W_X(A) \) denote the Weyl group of \( C_X(A) \), and \( W_X(A)_1 \) the Weyl group of the unit component \( C_X(A)_0 \) of \( C_X(A) \). There are inclusions \( W_X(A)_1 \subset W_X(A) \subset W \), where the last follows from [8] §4. The next theorem tells how to calculate \( W_X(A) \) and \( W_X(A)_1 \).

**Theorem 3.3.** [8] Theorem 7.6] Let \( X \) be a connected \( p \)-compact group with maximal torus \( T \) and Weyl group \( W \). Suppose that \( A \subset \check{T} \) is a subgroup. Then
(1) $W_X(A)$ is the subgroup of $W$ consisting of the elements which, under the conjugation action of $W$ on $\tilde{T}$, pointwise fix the subgroup $A$, and
(2) $W_X(A)_1$ is the subgroup of $W_X(A)$ generated by those elements $s \in W_X(A)$ such that $s \in W$ is a reflection and $A \subset \sigma(s)$.

Now we have all the ingredients needed for the proof of Proposition 3.1.

Proof of Proposition 3.1 Let $X_0$ be the unit component of $X$, and let $W_{X_0}$ be the Weyl group of $X_0$. Then $W_{X_0}$ is a normal subgroup of $W_X$ of index a power of 2 by Proposition 3.8]. The minimal normal subgroup of $W_X$ of 2 power index, usually denoted by $O^2(W_X)$ in the literature, equals $(\mathbb{Z}/2)^{n-1} \rtimes A_n$, i.e. the sequence
\[(\mathbb{Z}/2)^{n-1} \rtimes A_n = O^2(W_X) \longrightarrow (\mathbb{Z}/2)^n \rtimes \Sigma_n \longrightarrow (\mathbb{Z}/2)^2,\]
where $A_n$ is the alternating group, is exact. The group $(\mathbb{Z}/2)^2$ has 5 subgroups: the trivial subgroup 1, the first and the second factor $Z_1$ and $Z_2$, the diagonal $D$, and the whole group $(\mathbb{Z}/2)^2$. Hence, there are 5 normal subgroups of $W_X$ of index a power of 2:
1. $\pi^{-1}(1) = (\mathbb{Z}/2)^{n-1} \rtimes A_n$,
2. $\pi^{-1}(Z_1) = (\mathbb{Z}/2)^n \rtimes A_n$,
3. $\pi^{-1}(Z_2) = (\mathbb{Z}/2)^{n-1} \rtimes \Sigma_n$,
4. $\pi^{-1}(D)$, and
5. $\pi^{-1}((\mathbb{Z}/2)^2) = W_X$.

Because $W_{X_0}$ is the Weyl group of a connected 2-compact group, $W_{X_0}$ is a pseudoreflection group. According to the Clark-Ewing list [5], only the cases (3) and (5) are pseudoreflection groups (note that $n \geq 3$). We complete the proof showing that the case $W_{X_0} = (\mathbb{Z}/2)^{n-1} \rtimes \Sigma_n$ is not possible.

Suppose that $X$ is non connected, and let $X_0$ be the unit component. By the arguments above $W_{X_0}$ is $(\mathbb{Z}/2)^{n-1} \rtimes \Sigma_n$. Let $V$ be the subgroup of the maximal torus $T$ of $X$ (and also of $X_0$) generated by elements $(-1,-1,1,\ldots), (1,1,-1,-1,1,\ldots)$, and so on. Then $V$ is an elementary abelian 2-group of rank $m = \lfloor \frac{n}{2} \rfloor$. Let us write $n = 2m + r$ where $r$ is 0 or 1, and let $C$ denote the centralizer $C_{X_0}(V)$. By Theorem 3.3(1), we get
\[W_C = \{ s \in W_0 \mid s|_V = id_V \} = (\mathbb{Z}/2)^{n-1} \rtimes (\mathbb{Z}/2)^{m},\]
where the subgroup $(\mathbb{Z}/2)^m \subset \Sigma_n$ is generated by the transpositions $\tau_{2i-1,2i}$ for $i = 1,\ldots m$. Let $C_0$ be the unit component of $C$. By Theorem 3.3(2), the Weyl group of $C_0$ is
\[W_{C_0} = \langle s \in W_C \mid s \text{ is a reflection and } V \subset \sigma(s) \rangle.\]

An element $s \in W_C = (\mathbb{Z}/2)^{n-1} \rtimes (\mathbb{Z}/2)^{m}$ is a reflection if and only if $s$ equals $((1,\ldots,1), \tau_{2i-1,2i})$ or $((1,\ldots,1), (-1,-1,1,\ldots,1), \tau_{2i-1,2i})$ for some $i$, where both entries “$-1$” are in the $(2i-1)^{th}$ and $(2i)^{th}$ positions. We analyze both cases:
- If $s = ((1,\ldots,1), \tau_{2i-1,2i})$, then $F(s) = \{ (x_1,\ldots,x_n) \in (\mathbb{Z}/2^\infty)^n \mid x_{2i-1} = x_{2i} \}$ and $H(s) = F(s)$. Therefore $\sigma(s) = F(s)$.
If \( s = ((1, \ldots, 1, -1, -1, 1, \ldots, 1), \tau_{2i-1,2i}) \), then \( F(s) = \{ (x_1, \ldots, x_n) \in (\mathbb{Z}/2^\infty)^n \mid x_{2i-1} = x_{2i}^{-1}, i = 1, \ldots, m \} \). Hence also in this case \( H(s) = F(s) = \sigma(s) \).

Since \((-1)^{-1} = -1 \in \mathbb{Z}/2^\infty\), the group \( V \) is a subgroup of \( \sigma(s) \) in both cases, and by Theorem 3.3, we get

\[
W_{C_0} = \langle s \in W_C \mid s \text{ is a reflection} \rangle = ((\mathbb{Z}/2)^2)^m = (\mathbb{Z}/2)^{2m}.
\]

Hence the normalizer of a maximal torus of \( C_0 \) has the form \( M^m \), where \( M \) is the subgroup of the normalizer of the maximal torus of \( Sp(2)^\wedge_2 \) corresponding to the subgroup \( \langle ((1, 1), \tau_{1,2}), ((-1, -1), \tau_{1,2}) \rangle < (\mathbb{Z}/2)^2 \times \mathbb{Z}/2 = W_{Sp(2)^\wedge_2} \). By [12] Theorem 6.1 and [10] Theorem 0.5B (5), the group \( 2 \)-compact group \( C_0 \) splits into a product \( C_0 \cong X_1 \times \cdots \times X_m \), where each \( X_i \) is isomorphic to \( (SU(2)^2/E_i)^\wedge_2 \) for some subgroup \( E_i < Z(SU(2)^2) = (\mathbb{Z}/2)^2 \), and \( M \) is isomorphic to the maximal torus normalizer of \( X_i \). Among the 5 possibilities for each \( X_i \):

1. \( SU(2) \times SU(2) \wedge_2 = Spin(4)^\wedge_2 \),
2. \( SU(2)/(\mathbb{Z}/2) \times SU(2)^\wedge_2 \cong (SO(3) \times SU(2))^\wedge_2 \),
3. \( SU(2) \times SU(2)/(\mathbb{Z}/2)^2 \wedge_2 \cong (SU(2) \times SO(3))^\wedge_2 \),
4. \( SU(2) \times \mathbb{Z}/2 SU(2)^\wedge_2 \cong SO(4)^\wedge_2 \), and
5. \( (SU(2) \times SU(2))/(\mathbb{Z}/2)^2 \wedge_2 \cong (SO(3) \times SO(3))^\wedge_2 \),

only \( SO(4) \) produces a pseudoreflection group which is equivalent to that given by \( M \). But while the maximal torus normalizer of \( SO(4) \) is an split extension \( T : (\mathbb{Z}/2 \times \mathbb{Z}/2) \), \( M \) is not. Therefore there is no 2-compact group \( X_i \) whose maximal torus normalizer is \( M \), what contradicts our initial assumption of \( X \) being non connected.

\[ \square \]

4. **MOD-2 COHOMOLOGY OF THE 2-COMPACT GROUP \( X \)**

In this section we calculate the mod-2 cohomology of a 2-compact group \( X \) whose maximal torus normalizer is isomorphic to that of \( Sp(n)^\wedge_2 \). This is done under the induction hypothesis that \( Sp(m)^\wedge_2 \) is \( N \)-determined for \( 2 < m < n \). Notice that we already know that \( Sp(1) \) and \( Sp(2) \) are weakly \( N \)-determined.

First we need some information about the centralizers of elementary abelian subgroups in \( Sp(n) \). It is well known that these centralizers are isomorphic to products \( Sp(n_1) \times \cdots \times Sp(n_k) \), where \( n_1 + \cdots + n_k = n \). The next lemma shows that those centralizers are \( N \)-determined if each factor is so.

**Lemma 4.1.** Let \( Sp(m)^\wedge_2 \) be an \( N \)-determined 2-compact group for \( 2 < m \leq n \). Then the product \( Sp(n_1)^\wedge_2 \times \cdots \times Sp(n_k)^\wedge_2 \) is weakly \( N \)-determined if all \( n_i \leq n \) and is \( N \)-determined if \( 2 < n_i \leq n \).

**Proof.** Let \( Y \) be a 2-compact group with maximal torus normalizer \( N_Y \) isomorphic to that of \( Sp(n_1)^\wedge_2 \times \cdots \times Sp(n_k)^\wedge_2 \). If at least one \( n_i \) is 1 or 2, assume that \( Y \) is connected. Since \( N_Y \) is a product \( N_1 \times \cdots \times N_k \), where \( N_i \) is the normalizer of a maximal torus of \( Sp(n_i)^\wedge_2 \), the space \( Y \) is by [12] Theorem 6.1 isomorphic to a product \( Y_1 \times \cdots \times Y_k \), where \( N_i \) is the normalizer of a maximal torus of \( Y_i \). Hence \( Y_i \) is isomorphic to \( Sp(n_i)^\wedge_2 \), and therefore \( Y \) is isomorphic to \( Sp(n_1)^\wedge_2 \times \cdots \times Sp(n_k)^\wedge_2 \). So \( Sp(n_1)^\wedge_2 \times \cdots \times Sp(n_k)^\wedge_2 \) is weakly \( N \)-determined if at least one \( n_i \) is 1 or 2, otherwise \( Y \) is \( N \)-determined. \( \square \)
As $X$ and $Sp(n)\wedge_2$ “share” the same maximal torus normalizer $N$, they both “share” the same maximal torus $T$. Let $E_T < T$ be the maximal toral elementary abelian 2-group in both $X$ and $Sp(n)\wedge_2$. Call $f_{E_T}$ the monomorphism $E_T \longrightarrow X$. Next lemma shows that $E_T$ is in fact the maximal elementary abelian subgroup of $X$ (up to conjugation).

**Lemma 4.2.** Let $g: E \longrightarrow X$ be an elementary abelian subgroup of $X$. Then $g$ factors through $f_{E_T}$.

**Proof.** If $g: E \longrightarrow X$ is central, then by [22] Lemma 4.1 or [8] Theorem 1.2] $g$ factors though $f_{E_T}$ (recall that $X$ is connected by Proposition 3.1).

Now assume that $g: E \longrightarrow X$ is not central, thus there exists a subgroup $V < E$ of rank 1 which is noncentral. By [18] Proof of Theorem 1.3] there exists $\tilde{g}: E \longrightarrow N$ such that $Bg \simeq f_N \tilde{g}$, the centralizer $C_N(\tilde{g})$ is the maximal torus normalizer of $C_X(g)$, and $\tilde{g}|_V$ factors through $f_{E_T}$. Because $V$ is a toral subgroup, the centralizer $C_N(V)$ is the maximal torus normalizer of both $C_{Sp(n)\wedge_2}(V)$ and $C_X(V)$ by [18] Theorem 1.3]. So the calculation of $W_X(V)$ and $W_X(V)_1$ by means of Theorem 3.3 equals the calculation of $W_{Sp(n)\wedge_2}(V)$ and $W_{Sp(n)\wedge_2}(V)_1$ what implies that $C_X(V)$ is connected and since by induction, the centralizer $C_{Sp(n)\wedge_2}(V) = BS_{Sp(m)\wedge_2} \times BS_{Sp(n-m)\wedge_2}$, $m > 0$, is weakly $N$-determined (Lemma 4.3, then $C_X(V)$ is isomorphic to $C_{Sp(n)\wedge_2}(V)$.

The map $g: E \longrightarrow X$ has a lift to a map $g': E \longrightarrow C_X(E_T) \cong Sp(m)\wedge_2 \times Sp(n-m)\wedge_2$. Up to conjugacy every elementary abelian subgroup of $Sp(m) \times Sp(n-m)$ is toral. Hence $g$ is toral, i.e. factors through $f_{E_T}$. □

We can calculate the centralizer of $E_T$ in $X$:

**Lemma 4.3.** The centralizer $C_X(E_T)$ is isomorphic to the 2-compact group $(Sp(1)^n)\wedge_2$.

**Proof.** As $E_T$ is toral, the centralizer $C_N(E_T)$ is the maximal torus normalizer of both $C_{Sp(n)\wedge_2}(E_T)$ and $C_X(E_T)$ [19] Proposition 3.4(3)]. So the calculation of $W_X(E_T)$ and $W_X(E_T)_1$ by means of Theorem 3.3 equals the calculation of $W_{Sp(n)\wedge_2}(E_T)$ and $W_{Sp(n)\wedge_2}(E_T)_1$ what implies that $C_X(E_T)$ is connected. Since $C_{Sp(n)\wedge_2}(E_T) = (C_{Sp(n)\wedge_2}(E_T))\wedge_2 = (Sp(1)^n)\wedge_2$ is weakly $N$-determined, the centralizer $C_X(E_T)$ is isomorphic to $C_{Sp(n)\wedge_2}(E_T)$ by Lemma 4.3. Finally $C_{Sp(n)\wedge_2}(E_T) \cong (Sp(1)^n)\wedge_2$.

The action of $\Sigma_n < W_{Sp(n)} = W_X$ on $BE_T$ induces an action of $\Sigma_n$ on $BC_X(E_T) = \text{Map}(BE_T, BX)_{Bf_{E_T}} \cong (Sp(1)^n)\wedge_n$ that permutes the copies $Sp(1)^n$. Define $BY = BC_X(E_T) \times \Sigma_n E\Sigma_n$ and consider the diagram

\[
\begin{array}{ccc}
(BSp(1)^n)\wedge_2 & \longrightarrow & BY \\
(BT)\wedge_2 & \longrightarrow & \text{Map}(BT, BX)_{Bf_T} \times \Sigma_n E\Sigma_n \\
(BT)\wedge_2 & \longrightarrow & \text{Map}(BT, BX)_{Bf_T} \times W_{Sp(n)} E\Sigma_n \\
& & \text{BW}_{Sp(n)}
\end{array}
\]
where all rows are fibrations. The space \( \text{Map}(BT, BX)_{Bf_T} \times_{\text{Map}(BT, BX)_{Bf_T}} \) is the normalizer of the maximal torus \( T \), so it is isomorphic to \( B(T \rtimes W_{Sp(n)}) \). Therefore the space \( \text{Map}(BT, BX)_{Bf_T} \times_{\Sigma_n} E\Sigma_n \) is isomorphic to \( B(T \rtimes \Sigma_n) \). Which means that the middle row has a section, and hence also the top row has a section. It follows that \( BY \) is homotopic to \( B((Sp(1)^2)^n \rtimes \Sigma_n) \).

**Proposition 4.4.** The cohomology \( H^*BX \) is detected by elementary abelian 2-subgroups.

**Proof.** The cohomology \( H^*BSp(1)^n \) is detected by elementary abelian 2-subgroups, hence by \cite{13}, \( H^*BY \) is detected by elementary abelian subgroups. The normalizer \( Bf_N \) factors through the map \( Bf_Y \). Because \( Bf_N^* \) is a monomorphism, \( Bf_Y^* \) is a monomorphism. Hence \( H^*BX \) is detected by elementary abelian 2-subgroups. \( \square \)

We can now identify the algebra \( H^*BX \):

**Proposition 4.5.** The cohomology \( H^*BX \) is isomorphic to \( H^*BSp(n) \) as an algebra over the mod-2 Steenrod algebra.

**Proof.** By Proposition \cite{13}, the cohomology \( H^*BX \) is detected by elementary abelian 2-subgroups, and by Lemma \ref{lem} every elementary abelian subgroup of \( X \) factors through \( E_T \). Therefore \( H^*BX \) injects into \( H^*BE_T \) and therefore into \( H^*BC_X(E_T) \). If we take trivial action of \( \Sigma_n \) on \( X \), the inclusion \( \Sigma_{C_X(E_T)} \rightarrow X \) is a \( \Sigma_n \)-equivariant map. Hence the cohomology \( H^*BX \) is a subalgebra of \( (H^*BSp(1)^n)^{\Sigma_n} = H^*BSp(n) \).

But \( H^*(BX; \mathbb{Q}) = H^*(BT; \mathbb{Q})^{W_X} = \mathbb{Q}[x_4, \ldots, x_{4n}] \), hence the Bockstein spectral sequence associated to \( H^*BX \subset H^*Sp(n) = \mathbb{F}_2[x_4, \ldots, x_{4n}] \) converges to \( \mathbb{F}_2[x_4, \ldots, x_{4n}] \), and therefore \( H^*BX \cong H^*BSp(n) \). \( \square \)

Recall that the Quillen category \( Q_p(G) \) of a group \( G \) at a prime \( p \) is the category with objects \( (V, \alpha) \), where \( V \) is a nontrivial elementary abelian \( p \)-group and \( \alpha: V \rightarrow G \) is a monomorphism, and \( \text{Mor}_{Q_p(G)}((V, \alpha), (V', \alpha')) \) is the set of group morphisms \( f: V \rightarrow V' \) such that \( \alpha = \alpha' \circ f \). By Lannes’ theory (\cite{17}) and Dwyer-Zabrodsky theorem (\cite{14}, \cite{29}), the set of monomorphisms \( \alpha: V \rightarrow G \) is in bijections with the set of morphisms \( B\alpha^*: H^*BG \rightarrow H^*BV \) of unstable algebras over the Steenrod algebra \( \mathcal{A}_p \) such that \( H^*BV \) is a finitely generated module over \( B\alpha^*(H^*BG) \). Hence, there is an equivalent description of the Quillen category which can be use also for \( p \)-compact groups (\cite{13} §2). If \( X \) is a \( p \)-compact group, then \( Q_p(X) \) is the category with objects \( (V, \alpha) \), where \( V \) is a nontrivial elementary abelian \( p \)-group and \( \alpha: H^*BX \rightarrow H^*BV \) is a monomorphism of unstable algebras over the Steenrod algebra \( \mathcal{A}_p \) such that \( H^*BV \) is a finitely generated module over \( \alpha^*(H^*BX) \), and \( \text{Mor}_{Q_p(G)}((V, \alpha), (V', \alpha')) \) is the set of group morphisms \( f: V \rightarrow V' \) such that \( \alpha = Bf^*\alpha' \). If \( X \) is a compact Lie group than the definitions agree (\cite{13} Proposition 2.2).

By the definition of the Quillen category and propositions \ref{prop} we get the following proposition.

**Proposition 4.6.** The categories \( Q_2(Sp(n)) \) and \( Q_2(X) \) are isomorphic.
5. 2-STUBBORN DECOMPOSITION OF $Sp(n)$

A 2-stubborn subgroup of a Lie group $G$ is a 2-toral group $P$ such that $N_G(P)/P$ is a finite group which has no nontrivial normal 2-subgroup. Let $\mathcal{R}_2(Sp(n))$ be the 2-stubborn category of $Sp(n)$, which is the full subcategory of the orbit category of $Sp(n)$ with objects $Sp(n)/P$, where $P \subseteq Sp(n)$ is a 2-stubborn subgroup. The natural map

$$\hocolim_{Sp(n)/P \in \mathcal{R}_2(Sp(n))} ES_p(n)/P \longrightarrow BSp(n)$$

induces an isomorphism of homology with $\mathbb{Z}_2$-coefficients [16] Theorem 4. Hence, defining a map $f: Sp(n)/2 \longrightarrow X$ is equivalent to defining a family of compatible maps $\{f_P: ES_p(n)/P \simeq BP \longrightarrow X \mid Sp(n)/P \in \text{ob}(\mathcal{R}_2(Sp(n)))\}$.

We first recall the 2-stubborn subgroups of $Sp(n)$ calculated in [31]. Let the permutations $\sigma_0, \ldots, \sigma_{k-1}$ in $\Sigma_{2^k}$ be defined by

$$\sigma_r(s) = \begin{cases} s + 2^r; & s \equiv 1, \ldots, 2^r \mod 2^{r+1} \\ s - 2^r; & s \equiv 2^r + 1, \ldots, 2^{r+1} \mod 2^{r+1} \end{cases}$$

Let $A_0, \ldots, A_{k-1} \in Sp(2^k)$ be diagonal matrices with

$$(A_r)_{ss} = (-1)^{\left\lfloor \frac{s-1}{2^r} \right\rfloor},$$

where $[-]$ denotes greatest integer, and $B_0, \ldots, B_{k-1}$ be the permutation matrices for the $\sigma_0, \ldots, \sigma_{k-1}$.

**Definition 5.1.** For every $k \geq 0$, the subgroups $E_{2^k} \subset \Sigma_{2^k}$ and $\Gamma_{2^k}, \overline{\Gamma}_{2^k} \subset Sp(2^k)$ are defined by

$$E_{2^k} = \langle \sigma_0, \ldots, \sigma_{k-1} \rangle \cong (\mathbb{Z}/2)^k,$$

$$\Gamma_{2^k} = \langle uI, A_r, B_r \mid u \in Q(8), 0 \leq r < k \rangle,$$

$$\overline{\Gamma}_{2^k} = \langle uI, A_r, B_r \mid u \in S^1(j), 0 \leq r < k \rangle,$$

where $Q(8) = \{ \pm 1, \pm i, \pm j, \pm k \}$ is the quaternion group and $S^1(j) = \{ a + bi, aj + bk \mid a^2 + b^2 = 1 \}$ is the normalizer of the maximal torus in $Sp(1) = S^3$.

**Remark 5.2.** Let $P$ be $\Gamma_{2^k}$ or $\overline{\Gamma}_{2^k}$ and $P_D$ subgroup of all diagonal matrices in $P$. Then $P_D$ is $Q(8) \times E_{2^k}$ or $S^1(j) \times E_{2^k}$ and the extension $P_D \longrightarrow P \longrightarrow (\mathbb{Z}/2)^k$ splits.

**Theorem 5.3.** [31] Theorem 3]

1. A 2-stubborn group $P < Sp(n)$ is irreducable if it is conjugate to either

$$P = \Gamma_{2^k} \lhd E_{2^r_1} \lhd \cdots \lhd E_{2^r_s}$$

or

$$P = \overline{\Gamma}_{2^k} \lhd E_{2^r_1} \lhd \cdots \lhd E_{2^r_s}$$

where $n = 2^{k+r_1+\cdots+r_s}$.

2. A group $P < Sp(n)$ is a 2-stubborn group if it is conjugate to $P_1 \times \cdots \times P_s$, where $P_i$ is an irreducible 2-stubborn subgroup of $Sp(n_i)$ and $n = n_1 + \cdots + n_s$. 


Let $\tilde{R}_2(Sp(n))$ be the full subcategory of $R_2(Sp(n))$ with objects $Sp(n)/P$, where $P$ is one of the representative 2-stubborn groups from the previous theorem. The category $\tilde{R}_2(Sp(n))$ is equivalent to $R_2(Sp(n))$, so the natural map

$$\text{hocolim}_{Sp(n)/P \in \tilde{R}_2(Sp(n))} ESp(n)/P \longrightarrow BSp(n)$$

is also a homotopy equivalence up to 2-completion.

**Proposition 5.4.** Let $Sp(n)/P \in \tilde{R}_2(Sp(n))$ and define $P_D = P \cap Sp(1)^n$ and $P_T = P \cap T_{Sp(n)}$. Then

1. $C_{Sp(n)}(P_T) = T_{Sp(n)}$ and $C_{Sp(n)}(P_D) = (\mathbb{Z}/2)^n$,
2. for any extension $\alpha: P \longrightarrow Sp(n)$ of $i: P_T \longrightarrow Sp(n)$, we have $C_{Sp(n)}(\alpha(P)) = Z(P)$ and
3. the canonical map

$$\pi_0(\text{Map}(BP, BSp(n)^\wedge)_{Ba|BP_T = Bi_{P_T}}) \longrightarrow \text{Hom}(H^*BSp(n), H^*BP)$$

is an injection.

**Remark 5.5.** By $\text{Map}(BP, BSp(n)^\wedge)_{Ba|BP_T = Bi_{P_T}}$ we denote the components of the mapping space $\text{Map}(BP, BSp(n)^\wedge)$ given by maps $Ba: BP \longrightarrow Sp(n)^\wedge$, such that $Ba|BP_T \simeq Bi_{P_T}$.

**Proof.** Part (1) is obvious for $P = \Gamma_{2^k}$ and $P = \Gamma_{2^k}$. If $P = Q \wr E_{2^q}$, where $Q$ is irreducible 2-stubborn subgroup of $Sp(2^{k-r})$, then $C_{Sp(2^k)}(P_T) = C_{Sp(2^{k-r})(Q_T)^{2^r}}$ which is, by induction, $(T_{Sp(2^{k-r})})^{2^r} = T_{Sp(2^k)}$. If $P = P_1 \times \ldots \times P_s$ is a product of irreducible 2-stubborn groups, then $C_{Sp(n)}(P_T) = C_{Sp(n)}((P_1)_T) \times \ldots \times C_{Sp(n)}((P_s)_T) = T_{Sp(n)} \times \ldots \times T_{Sp(n)}$. Analogously we prove that $C_{Sp(n)}(P_D) = (\mathbb{Z}/2)^n$.

Every map $g \in \text{Map}(BP, BSp(n)^\wedge)$ is homotopic to $Ba: BP \longrightarrow BSp(n)^\wedge$, where $\alpha$ is the 2-completion of a homomorphism of groups $P \longrightarrow Sp(n)$ ([14], [29]).

Let $P$ be an irreducible 2-stubborn subgroup of $Sp(2^k)$ and let $\alpha: P \longrightarrow Sp(2^k)$ be a homomorphism such that $\alpha|P_T = i_{P_T}$. The extensions $Ba|BP_D: BP_D \longrightarrow BSp(2^k)^\wedge$ of $Bi_{P_T}$ are classified by obstruction groups $H^m(P_D/P_T; \pi_m(\text{Map}(BP_T, BSp(2^k)^\wedge)_{Bi_{P_T}}))$. By [16], $\text{Map}(BP_T, BSp(2^k)^\wedge)_{Bi_{P_T}}$ is homotopy equivalent to $BC_{Sp(2^k)}(P_T)^\wedge$, which is isomorphic to $(BS^1)^{2^k}$, by part (1). Then

$$H^m(P_D/P_T; \pi_m(\text{Map}(BP_T, BSp(2^k)^\wedge)_{Bi_{P_T}})) = H^m(P_D/P_T; \pi_m(BS^1)^{2^k})$$

and the only possible nontrivial group is for $m = 1$. And

$$H^1(P_D/P_T; \pi_m(\text{Map}(BP_T, BSp(2^k)^\wedge)_{Bi_{P_T}})) = H^1(\mathbb{Z}/2; (\mathbb{Z}_2)^{2^k})$$

where the group $\mathbb{Z}/2$ acts on $(\mathbb{Z}_2)^{2^k}$ by reflection on each component; this action comes from the action of the Weyl group $\mathbb{Z}/2$ of the group $Sp(1)$ on the maximal torus $S^1$. By Shapiro’s lemma [31, III, Proposition 6.2], the group $H^1(\mathbb{Z}/2; (\mathbb{Z}_2)^{2^k})$ is trivial, so all obstruction groups vanish. Hence if $Ba|BP_T = Bi_{P_T}$ then $Ba|BP_D = Bi_{P_D}$. 
First we will prove part (2) and (3) for the case $P$ is $\Gamma_{2k}$ or $\overline{\Gamma}_{2k}$. Let $\alpha : P \longrightarrow Sp(2^k)$ be a homomorphism such that $\alpha|_{P_r} = i_{P_r}$. Then by above paragraph $B\alpha|_{BP_D}$ is homotopical to $Bi_{P_r}$, the homomorphisms $\alpha|_{BP_D} = i_{BP_D}$ be two homomorphisms such that $B\alpha = B\beta$ and $\alpha|_{BP_D} = \beta|_{BP_D}$. We proved that $\alpha|_{BP_D} = i_{BP_D} = \beta|_{BP_D}$. Let $\tilde{\alpha}, \tilde{\beta} : Q^{2r} \longrightarrow Sp(2^k)$ be the restrictions of $\alpha$ and $\beta$. Because $Z(Q^{2r}) = Z(Q)^{2r} = (\mathbb{Z}/2)^{2r}$, the homomorphisms $\tilde{\alpha}$ and $\tilde{\beta}$ factor trough homomorphisms $\tilde{\alpha}, \tilde{\beta} : Q^{2r} \longrightarrow Sp(2k) = 1$.

The map $B\tilde{\alpha}$ is homotopic to the map $BQ^{2r} \cong \text{Map}(BZ(Q^{2r}), BQ^{2r})|_{Bi} \cong H^*(B\tilde{\alpha} ; \mathbb{F}_2) = H^*(\text{Map}(BZ(Q^{2r}), B\tilde{\alpha}); \mathbb{F}_2)$. Analogously $H^*(B\tilde{\beta} ; \mathbb{F}_2) = H^*(\text{Map}(BZ(Q^{2r}), B\tilde{\beta}); \mathbb{F}_2)$.

By Lannes’ theory [17],

$$H^*(\text{Map}(BZ(Q^{2r}), B\tilde{\alpha}); \mathbb{F}_2) = \mathcal{T}_{B\tilde{\alpha}}(Q^{2r}) = \mathcal{T}_{B\tilde{\beta}}(Q^{2r}) = H^*(\text{Map}(BZ(Q^{2r}), B\tilde{\beta}); \mathbb{F}_2),$$

so $B\tilde{\alpha} = B\tilde{\beta}$. The homomorphisms $\tilde{\alpha}$ and $\tilde{\beta}$ are matrices of dimension $2^r \times 2^r$, where entries are $\tilde{\alpha}_{i,j}, \tilde{\beta}_{i,j} : Q_i \longrightarrow Sp(2^{k-r})$. The indexes $i$ and $j$ indicate the components in the products. By induction, $B\tilde{\alpha}_{i,i}$ and $B\tilde{\beta}_{i,i}$ are homotopical and therefore $\tilde{\alpha}_{i,i}$ and $\tilde{\beta}_{i,i}$ are conjugate [17 Théorème 3.4.5]. We can assume that $\tilde{\alpha}_{i,i} = \tilde{\beta}_{i,i}$. Because $Q_i$ and $Q_j$ commutes for $i \neq j$, the homomorphisms $\tilde{\alpha}_{i,j}$ and $\tilde{\beta}_{i,j}$ factor trough homomorphisms $\tilde{\alpha}_{i,j}, \tilde{\beta}_{i,j} : Q_i \longrightarrow C_{Sp(2^{k-r})}(\tilde{\alpha}_{j,j}(Q))$. By induction, the centralizer $C_{Sp(2^{k-r})}(\tilde{\alpha}_{j,j}(Q))$ equals $(\mathbb{Z}/2)_j$. Because $\tilde{\alpha}|_{P_D} = \tilde{\beta}|_{P_D}$, the homomorphism $H^*(\text{Map}(BZ(Q^{2r}), B\tilde{\alpha}); \mathbb{F}_2) = H^*(\text{Map}(BZ(Q^{2r}), B\tilde{\beta}); \mathbb{F}_2)$.
The only possible nontrivial obstruction group is for \( \tilde{\gamma}_{i,j} \) to \( (Q/Q_D) \). Therefore all obstruction groups vanish, so \( \tilde{\gamma}_{i,j} \) induces trivial map on mod-2 cohomology. Because \( Q/Q_D \) is an iterated wreath product of elementary abelian groups, the map \( \tilde{\gamma}_{i,j} \) is constant \([23, \text{Lemma 6.10}]\). Hence \( \tilde{\alpha}_{i,j} = \tilde{\beta}_{i,j} \) and so \( \tilde{\alpha} = \tilde{\beta} \) and the centralizer \( C_{Sp(2^k)}(\tilde{\alpha}) \) is given by the fixed-point set \( C_{Sp(2^k)}(\tilde{\alpha}) = (C_{Sp(2^k)}(\tilde{\alpha}))_2 = ((C_{Sp(2^k)}(Q))_2)_{2} = (\mathbb{Z}/2)_2 = \mathbb{Z}/2 \). This proves part (2).

The extensions \( B\alpha: BP \longrightarrow BSp(2^k)_2 \) of \( B\tilde{\alpha} \) are classified by the obstruction groups \( H^m(P/Q^{2^k}; \pi_m(\text{Map}(BQ^{2^k}, BSp(2^k)_2)) \). By \([16]\), the mapping space \( \text{Map}(BQ^{2^k}, BSp(2^k)_2) \) is homotopy equivalent to \( BC_{Sp(2^k)}(Q^2)_2 = (\mathbb{Z}/2)_2 \). Hence the obstruction groups are

\[
H^m(P/Q^{2^k}; \pi_m(\text{Map}(BQ^{2^k}, BSp(2^k)_2))) = H^m(P/Q^{2^k}; \pi_m(\mathbb{B}\mathbb{Z}/2))
\]

The only possible nontrivial obstruction group is for \( m = 1 \). The group \( P/Q^{2^k} = E_{2^k} \) acts by permutation on \( BC_{Sp(2^k)}(Q^2) = (\mathbb{B}\mathbb{Z}/2)_{2^k} \), hence \( \text{Ind}_{E_{2^k}}^{E_{2^k}}(\mathbb{Z}/2)^{2^k} = (\mathbb{Z}/2)^{2^k} \), by Shapiro’s lemma \([3, \text{III. Proposition 6.2}]\). Therefore

\[
H^1(K \ltimes E_{2^k}; (\mathbb{Z}/2)^{2^k}) = H^1(K; (\mathbb{Z}/2)^{2^k})
\]

Therefore all obstruction groups vanish, so \( B\alpha \) is homotopic to \( Bi_{P} \).

Finally let \( P = P_1 \times \ldots \times P_s \), where \( P_i \) is an irreducible 2-stubborn subgroup of \( Sp(n_i) \). Let \( B\alpha: P \longrightarrow Sp(n) \) be two homomorphisms such that \( B\alpha = B\beta \) and \( \alpha|_{P_i} = \beta|_{P_i} \). Both homomorphisms factor through \( \alpha, \beta: P \longrightarrow C_{Sp(2^k)}(\mathbb{Z}(P)) = C_{Sp(2^k)}((\mathbb{Z}/2)^s) = Sp(n_1) \times \ldots \times Sp(n_s) \). In the same way as in the case \( P \) is irreducible 2-stubborn group, we can show that \( B\alpha = B\beta \). Maps \( \alpha \) and \( \beta \) are matrices of dimension \( s \times s \) with entries maps \( \tilde{\alpha}_{i,j}; \tilde{\beta}_{i,j}: P_i \longrightarrow Sp(n_i) \). Analogously as before we can show that \( \tilde{\alpha}_{i,j} = \tilde{\beta}_{i,j} \), so \( B\alpha \cong B\beta \). The equation \( \mathbb{Z}(P) = \mathbb{Z}(P_1) \times \ldots \times \mathbb{Z}(P_s) \) finishes the proof. \(\square\)

6. THE MAP FROM \( Sp(n)_2 \) TO \( X \)

For every object \( Sp(n)/P \) in \( \tilde{\mathcal{R}}(Sp(n)) \) we define a map \( f_P: P \longrightarrow X \) as the composition of the two inclusions \( i_P: P \longrightarrow N \) and \( f_N: N \longrightarrow X \). We will prove that for every morphism \( c_\varphi: Sp(n)/P \longrightarrow Sp(n)/Q \) in \( \tilde{\mathcal{R}}(Sp(n)) \), the diagram

\[
\begin{array}{ccc}
BP & \xrightarrow{Bi_P} & BN \\
\downarrow{Bc_\varphi} & & \downarrow{Bf_N} \\
BQ & \xrightarrow{Bi_Q} & BN \\
& & \downarrow{Bf_N}
\end{array}
\]

(1)

commutes up to homotopy.
Let us define $\alpha = f_N \circ i_P$ and $\beta = f_N \circ i_Q \circ c_g$. Then $B\alpha^* = B\alpha^*$. The group $P_T = P \cap Sp(n)$ is 2-toral. The restrictions $\alpha|_{P_T}$ and $\beta|_{P_T}$ are conjugate in $Sp(n)$, and hence by [23 Proposition 4.1], they are also conjugate in the normalizer $N$ of the maximal torus. So $B\alpha|_{P_T} \simeq B\beta|_{P_T}$. By the next proposition, $B\alpha \simeq B\beta$.

Let $K \longrightarrow G \longrightarrow H$ be an exact sequence of groups. Then $H$ acts freely on $BK = EG/K \simeq BK$, and $BK/H$ equals $BG$. For any space $BX$ with trivial action of the group $H$, we have

\[(2) \quad \text{Map}(BG, BX) = \text{Map}(\overline{BK}/H, BX) = \text{Map}_H(\overline{BK}, BX) \simeq \text{Map}_H(EH \times \overline{BK}, BX) = \text{Map}_H(EH, \text{Map}(\overline{BK}, BX)) = \text{Map}(\overline{BK}, BX)^{hH}.
\]

**Proposition 6.1.** For every $Sp(n)/P \in \text{ob}(\overline{R}(Sp(n)))$, the canonical map

$$\pi_0(\text{Map}(BP, BX)|_{B\alpha|_{BP_T} = Bf_P}) \longrightarrow \text{Hom}(H^*BX, H^*BP)$$

is an injection.

*Proof.* Consider the diagram

$$\text{Map}(\overline{BP_T}, BY)_{B_P} \xrightarrow{	ext{Map}(\overline{BP_T}, BSp(n)_2^h)_{B_{P_T}}} \text{Map}(\overline{BP_T}, BX)_{B_{P_T}}.$$ 

By [29], the mapping space $\text{Map}(\overline{BP_T}, BSp(n)_2^h)_{B_{P_T}}$ is homotopy equivalent to $BC_{Sp(n)}(BP_T)_2^h$ and by proposition 5.4 this space is homotopy equivalent to $(BT_{Sp(n)}^h)_2^h$. Analogously $\text{Map}(\overline{BP_T}, BY)_{B_{P_T}}$ is homotopy equivalent to $(BT_{Sp(n)}^h)_2^h$. The mapping space $\text{Map}(\overline{BP_T}, BX)_{B_{P_T}}$ is the classifying space of a 2-compact group ([9]). Its Weyl group is $\text{Iso}(Bf_N \circ Bi_{P_T}) = \{w \in W_N \mid w \circ Bf_N \circ Bi_{P_T} \simeq Bf_N \circ Bi_{P_T}\}$ ([34 Proposition 4.3]). By the construction of the map $f_N$, the group $\text{Iso}(Bf_N \circ Bi_{P_T})$ equals $\text{Iso}(Bi_N \circ Bi_{P_T})$. Because $\text{Iso}(Bi_N \circ Bi_{P_T})$ is the Weyl group of the mapping space $\text{Map}(BP_T, BSp(n)_2^h)_{B_{P_T}} \simeq (BT_{Sp(n)}^h)_2^h$, the group $\text{Iso}(Bf_N \circ Bi_{P_T})$ is trivial, hence $\text{Map}(\overline{BP_T}, BX)_{B_{P_T}} \simeq (BT_{Sp(n)}^h)_2^h$. Therefore both maps in the diagram (6) are homotopy equivalences.

Taking homotopy fixed points we obtain the following diagram

$$\text{Map}(\overline{BP_T}, BY)^{h(P/P_T)}_{B_{P_T}} \xrightarrow{\text{Map}(\overline{BP_T}, BSp(n)_2^h)^{h(P/P_T)}_{B_{P_T}}} \text{Map}(\overline{BP_T}, BX)^{h(P/P_T)}_{B_{P_T}},$$

where both maps are mod-2 equivalences, since an equivariant mod-2 equivalence between 1-connected spaces induces a mod-2 equivalence between the homotopy fixed-point sets.
By proposition 5.4, the components of $\text{Map}(\tilde{BP}_T, BSp(n)_{2}^{h(P/P_T)})_{Bi_T}$ are distinguished by mod 2 cohomology. Any map in $\text{Map}(\tilde{BP}_T, BX)^{h(P/P_T)}_{Bf_T}$ has a lift to $BN$ and therefore to $BY$. The obstruction group which classifies the extensions is

$$H^2(P/P_T;\pi_2\text{Map}(\tilde{BP}_T, BSp(n)_{2}^{h(P/P_T)})_{Bi_T}),$$

so the components of $\text{Map}(\tilde{BP}_T, BX)^{h(P/P_T)}_{Bf_T} \simeq \text{Map}(BP, BX)_{Bf}^{h(P/P_T)}$ are also distinguished by mod-2 cohomology. □

Diagram (1) establishes a map from the 1-skeleton of the homotopy colimit $\{BP\}_{\tilde{R}_2(Sp(n))}$ to $BX$. The obstruction groups for extending a map defined on the 1-skeleton of the homotopy colimit to a map on the total homotopy colimit are

$$\lim_{\leftarrow}^{i+1} \pi_i \text{Map}(BP, BX)_{Bf},$$

for $i \geq 2$, where $\lim^i$ is the $i$-th derived functor of the inverse limit limit functor ([4] and [36]).

Let $Ab$ be the category of Abelian groups and let

$$\Pi_j^X, \Pi_j^{Sp(n)}: \tilde{R}_2(Sp(n)) \longrightarrow Ab$$

be functors defined by

$$\Pi_j^X(Sp(n)/P) = \pi_j \text{Map}(BP, BX)_{Bf},$$

$$\Pi_j^{Sp(n)}(Sp(n)/P) = \pi_j \text{Map}(BP, BSp(n)_{2}^{h(P/P_T)})_{Bi_P}.$$ Note that $\text{Map}(BP, BSp(n)_{2}^{h(P/P_T)})_{Bi_P}$ is isomorphic to $BZ(P)_{2}^{h(P/P_T)}$ ([16], Theorem 3.2) therefore $\Pi_1(Sp(n))(Sp(n)/P)$ is well defined. By the next proposition, also $\Pi_1(X)(Sp(n)/P)$ is well defined.

**Proposition 6.2.** There exists a natural transformation

$$T: \Pi_j^{Sp(n)} \rightarrow \Pi_j^X$$

which is an equivalence.

**Proof.** For every 2-stubborn group $P$ we have homotopy equivalences

$$(3) \quad \text{Map}(BP, BSp(n)_{2}^{h(P/P_T)})_{Bi_P} \simeq \text{Map}(BP, BY)_{Bi_P} \simeq \text{Map}(BP, BX)_{Bf}$$

which depend on the chosen lift $Bi_P: BP \longrightarrow BY$ of the map $Bi_P: BP \longrightarrow BSp(n)_{2}^{h(P/P_T)}$. Because $\text{Rep}(P, Sp(n)) \longrightarrow [BP, BSp(n)]$ is a bijection ([14], [29]), two lifts differ by a conjugation $Bg$. Since $Bf \simeq Bf \circ Bg$, the equivalence (3) induces well defined isomorphisms

$$\Pi_j^{Sp(n)}(Sp(n)/P) \rightarrow \Pi_j^X(Sp(n)/P)$$

which commute with maps induced by morphisms in $\tilde{R}_2(Sp(n))$. □

**Proposition 6.3.** For all $i, j \geq 1$,

$$\lim_{\leftarrow}^{i} \pi_j \text{Map}(BP, BX)_{Bf} = 0.$$
SYMPLECTIC GROUPS ARE N-DETERMINED 2-COMPACT GROUPS

Proof. By the previous lemma,
\[
\lim_{\overline{\mathbb{R}_2(Sp(n))}} \pi_j \text{Map}(BP, BX)_{BP} = \lim_{\overline{\mathbb{R}_2(Sp(n))}} \pi_j \text{Map}(BP, BSp(n) \wedge)_{BP} \\
\]
and the right side is 0 [16, Theorem 4.8]. □

Because all obstructions vanish, there exists a map
\[
f : \text{hocolim}_{\overline{\mathbb{R}_2(Sp(n))}} BP \longrightarrow BX.
\]

By the construction of the map we have a commutative diagram

\[
\begin{array}{ccc}
BY & \longrightarrow & BX \\
\downarrow & & \\
\text{hocolim}_{\overline{\mathbb{R}_2(Sp(n))}} ESp(n)/P & \longrightarrow & BX
\end{array}
\]

where the diagonal maps induce monomorphisms on the cohomology and therefore \( f^* \) is a monomorphism. Since \( H^* BSp(n) \cong H^* BX \), \( f^* \) is an isomorphism and therefore \( f \) is a homotopy equivalence.

7. \( Sp(n) \) AS A LOOP SPACE

The normalizer conjecture can be stated also for finite loop spaces with maximal torus normalizers as a weak version of Wilkerson' conjecture (see [35]).

Theorem 7.1. Let \( L \) be a connected finite loop space with a maximal torus normalizer isomorphic to that of \( Sp(n) \). Then \( BL \) is homotopy equivalent to \( BSp(n) \).

Proof. To prove \( BL \simeq BSp(n) \) is equivalent to showing that \( BL \) and \( BSp(n) \) lie in the same adic genus [30]. The loop spaces \( BL \) and \( BSp(n) \) have the same rational genus. Since \( BL \) is finite and connected, \( L_p^\wedge \) is a \( p \)-compact group. The maximal torus normalizer of \( L_p^\wedge \) is just the fibrewise \( p \)-completion of \( N \) by the fibration \( BT \longrightarrow BN \longrightarrow BW_L \). Hence \( L_p^\wedge \) and \( Sp(n)_p^\wedge \) have isomorphic normalizers of the maximal torus. By [2], \( BSp(n)_p^\wedge \) is \( N \)-determined if \( p \) is an odd prime and by the main theorem of this paper \( BSp(n)_2^\wedge \) is (weakly) \( N \)-determined. So \( BL_p^\wedge \) and \( BSp(n)_p^\wedge \) are homotopy equivalent. □

References

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Fakulteta za matematiko in fiziko, Univerza v Ljubljana, Jadranska 19, SI-1111 Ljubljana, Slovenia
E-mail address: ales.vavpetic@FMF.Uni-Lj.Si

Dpto de Álgebra, Geometría y Topología, Universidad de Málaga, Apdo correos 59, 29080 Málaga, Spain
E-mail address: viruel@agt.cie.uma.es