On the Problem of Multiple M2 Branes
A.Morozov

ITEP, Moscow, Russia

ABSTRACT

A simplified version of 3d BL theory is considered, which allows any number $N$ of M2 branes in $d = 11$. The underlying 3-algebra structure is provided by degenerate $U(N)$ Nambu bracket $[X,Y,Z] = \text{tr}(X) \cdot [Y,Z] + \text{tr}(Y) \cdot [Z,X] + \text{tr}(Z) \cdot [X,Y]$, the corresponding $f^{abcd}$ is not totally antisymmetric and the $\mathcal{N} = 8$ supersymmetry of the action remains to be checked. All the fields, including auxiliary non-propagating gauge fields, are in adjoint representation of $SU(N)$ and the only remnant of 3-algebra structure is an octuple of gauge singlets, acquiring vacuum expectation value in transition to $D2$ branes in $d = 10$.

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1 Introduction

In [1] John Schwarz explicitly formulated a problem to find the world-volume 3d non-Abelian action with $OSp(8|4)$ symmetry, including the $\mathcal{N} = 8$ supersymmetry and conformal invariance, which could be used in description of $N$ parallel fundamental M2 branes embedded into 11d space-time. It was explained in [1] that 3d gauge fields in this theory should be non-dynamical and governed by a Chern-Simons action. After important work [2], the problem was finally resolved by Jonathan Bagger and Neil Lambert (BL) in a series of papers
[3, 5, 6] and was further elaborated on in [4, 7, 8, 9, 10, 11]. The BL construction is based on non-trivial 3-algebra structure and involves a new kind of gauge fields, which do not belong to adjoint representation of naive gauge group $U(N)$. Moreover, additional antisymmetry requirement imposed in [5] on the 3-algebra structure constants $f^{abcd}$, leaves only one non-trivial example $f^{abcd} \sim \epsilon^{abcd}$, restricting the choice of the "gauge group" to $U(2)$ (or, perhaps, $SO(4)$). A possible way out was actually suggested in [7] and [8], where $f^{abcd}$ were linked to the structure constants $f^{abc}$ of the ordinary gauge group $G$, say, $G = U(N)$: $f^{abcd} = f^{abc}$. As already noted at the very end of [7], if one relaxes the unnecessary antisymmetry constraint, this identification essentially implies the use of the standard quantum Nambu 3-bracket for ordinary matrices [12]

$$[X, Y, Z] = \text{tr}(X) \cdot [Y, Z] + \text{tr}(Y) \cdot [Z, X] + \text{tr}(Z) \cdot [X, Y]$$

in the role of BL 3-algebra structure.

In what follows we explicitly describe this simplified version of BL construction for arbitrary gauge group $G$, including $G = U(N)$ relevant for the stack of $N$ $M2$ branes. The non-trivial BL gauge fields reduce to the pair of ordinary adjoint auxiliary fields, one gauge and one not, very much in the spirit of [8]. We begin in s.2 and s.3 from reminding respectively the $N = 1$ and $N = 8$ SUSY non-gauged 3d actions from [1] with adjoint octuplet $(\phi^I_j, \psi^A_{ij})$, $I = 1 \ldots 8$, $A = 1 \ldots 8$, $i, j = 1 \ldots N$. Then in s.4 its simplest gauged generalization is considered, with two auxiliary vector fields: gauge $A^i_{\mu}$ and additional adjoint $B^a_{\mu}$. A non-linear potential $a$ a la BL is introduced in s.5, involving additional octuplet of gauge singlets $(\varphi^I, \chi^A)$. SUSY invariance of this action is addressed in s.6, where some representative but non-exhaustive examples are presented. In s.7 we briefly remind the main points of original BL construction for $M2$ in $d = 11$ and the way [8] it reduces to $D2$ in $d = 10$. Finally in s.8 we demonstrate that the substitution of (1) converts this BL action into the simple formula (21) from our s.5, which is the main suggestion of the present paper.

2 Non-gauged $N = 1$ SUSY action in 3d

Consider a real (Grassmann-valued) 3d spinor $\psi = \left( \begin{array}{c} \psi_+ \\ \psi_- \end{array} \right)$. It exists if space-time signature is $(-++)$ and three real-valued gamma-matrices are $\gamma_0 = i \sigma_2 = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right)$, $\gamma_1 = \sigma_1 = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)$ and $\gamma_2 = \sigma_3 = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right)$.

Then the Dirac Lagrangian

$$\mathcal{L}_{\text{Dir}} = i \bar{\psi} \gamma^\mu \partial_\mu \psi = -\psi_+ \gamma^0 \sigma_2 \left( -i \sigma_2 \partial_0 + \sigma_1 \partial_1 + \sigma_3 \partial_2 \right) \psi = i \psi^\dagger \left( \partial_0 + \sigma_3 \partial_1 - \sigma_1 \partial_2 \right) \psi$$

$$= \left( \psi_+, \psi_- \right) \left( \begin{array}{cc} i(\partial_0 + \partial_1) & -i \partial_2 \\ -i \partial_2 & i(\partial_0 - \partial_1) \end{array} \right) \left( \begin{array}{c} \psi_+ \\ \psi_- \end{array} \right)$$

$$= i \psi_+ (\partial_0 + \partial_1) \psi_+ + i \psi_- (\partial_0 - \partial_1) \psi_- - i \psi_+ \partial_2 \psi_- - i \psi_- \partial_2 \psi_+$$

(2)

is non-vanishing, because of the anticommuting nature of $\psi$-fields. It is related to bosonic Lagrangian

$$\mathcal{L}_{\text{bos}} = -\partial^\mu \phi \partial_\mu \phi = (\partial_0 \phi)^2 - (\partial_1 \phi)^2 - (\partial_2 \phi)^2$$

(3)

by an elementary supersymmetry transformation

$$\delta \phi = i \bar{\psi} \varepsilon = \left( \psi_+, \psi_- \right) i \sigma_2 \left( \begin{array}{c} \varepsilon_+ \\ \varepsilon_- \end{array} \right) = \psi_+ \varepsilon_- - \psi_- \varepsilon_+ = \varepsilon_+ \psi_- - \varepsilon_- \psi_+ = i \bar{\psi} \varepsilon;$$

$$\delta \psi = \left( \begin{array}{c} \delta \psi_+ \\ \delta \psi_- \end{array} \right) = -\partial_\mu \phi \varepsilon = \left( i \sigma_2 \partial_0 \phi - \sigma_1 \partial_1 \phi - \sigma_3 \partial_2 \phi \right) \left( \begin{array}{c} \varepsilon_+ \\ \varepsilon_- \end{array} \right)$$

(4)

with constant infinitesimal spinor $\varepsilon$, so that

$$\delta \mathcal{L}_{\text{bos}} + \delta \mathcal{L}_{\text{Dir}} = \partial_\mu \left( -i \bar{\psi} \partial_\mu \phi \varepsilon \right)$$

(5)

and the action

$$\int \left( \mathcal{L}_{\text{bos}} + \mathcal{L}_{\text{Dir}} \right) d^3x$$

(6)

remains invariant.
3 Non-gauged $\mathcal{N} = 8$ SUSY action in 3d

The number of fields can be easily increased: just take $M$ copies of $\phi$ and $\psi$. What is non-trivial, for the special value of $M = 8$ the number of supersymmetries can also be increased to $M$. This can be done by using the special triality relation between three different 8-dimensional representations of $SO(8)$ (or, what is essentially the same, triality relation in octonionic algebra). For reasons explained in [9] we separate 3d and 8d gamma-matrices and also denote the scalar fields by $\phi^I$ rather than $X^I$.

**Fields:** Scalars $\phi^I$ and 3d Majorana spinors $\psi_A$, with $I$ and $A$ labeling components of $V_{\bar{s}}$ and $S_{\bar{s}}^-$ – the vector and spinor representations of $SO(8)$. 4d spinor indices are suppressed.

$\mathcal{N} = 8$ SUSY transformation:

$$\delta \phi^I = i \varepsilon^A \Gamma^I_{AA} \psi^A = i \bar{\psi}^A \Gamma^I_{AA} \varepsilon^A,$$

$$\delta \psi_A = - \partial \phi^I \Gamma^I_{AA} \varepsilon^A$$

(7)

Here $\varepsilon = \{\varepsilon^A\}$ belongs to the second spinor representation $S_{\bar{s}}^-$ of $SO(8)$, the $8 \times 8$ real-valued structure constants $\Gamma^I_{AA}$ define the triality relation between the three 8-dimensional representations of $SO(8)$. They are $8 \times 8$ off-diagonal blocks in $16 \times 16$ gamma-matrices in 8d:

$$\gamma^I = \begin{pmatrix} 0 & \Gamma^I_{AA} & 0 \\ \Gamma^I_{AA} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$\gamma^I \gamma^J + \gamma^J \gamma^I = 2 \delta^I_J$$

where $\Gamma$ denotes transposed matrix. In particular basis $\Gamma^I_{\alpha\beta} = \delta_{\alpha\beta}$ while for $i, j, k = 1 \ldots 7$ we have $\Gamma^I_{ij} = -\Gamma^I_{ji} = \delta_{ij}$ and $\Gamma^I_{jk} = c_{ijk}$ where $c_{ijk} = -c_{ikj} = c_{jki}$ are octonionic structure constants, non-vanishing for the following triples:

$$c_{124} = c_{137} = c_{156} = c_{235} = c_{267} = c_{346} = c_{457} = 1$$

(8)

Explicitly in this basis

$$\Gamma^I = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Gamma^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}$$

$$\Gamma^3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Gamma^4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Gamma^5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Gamma^6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Gamma^7 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Gamma^8 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Indices $I, A$ and $\bar{A}$ are raised and lowered with the help of invariant metrics $g_{IJ}, g_{AB}$ and $g_{\bar{A} \bar{B}}$.

**Invariant action:** Lagrangian

$$\mathcal{L}_{\text{free}} = - \partial^\mu \phi^I \partial_\mu \phi^I + i \bar{\psi}^A \partial \psi^A$$

(9)

changes under (7) by total derivative:

$$-2 \partial \phi^I \partial_\mu (i \bar{\psi}^A \Gamma^I_{AA} \varepsilon^A) \rightarrow + 2 i \partial^2 \phi^I (\bar{\psi}^A \Gamma^I_{AA} \varepsilon^A),$$

$$2 i \bar{\psi}^A \partial (\bar{\psi}^A \Gamma^I_{AA} \varepsilon^A) = -2 i \partial^2 \phi^I (\bar{\psi}^A \Gamma^I_{AA} \varepsilon^A)$$

(10)

so that the action

$$\int \mathcal{L}_{\text{free}} d^3 x$$

(11)

remains invariant.
4 Gauged $\mathcal{N}=8$ SUSY action in 3$d$

If $\phi$ and $\psi$ are promoted to $N \times N$ matrices, or to elements of adjoint representation of any other group $G$, $\phi^I_a$, $\psi^A_a$, $a = 1 \ldots \dim(G)$, the action acquires a global $G$-symmetry, which can further be gauged by introduction of the gauge field $A^I_a$:

$$\mathcal{L}_{\text{kin}} = -(D^\mu \phi^I)_a (D_\mu \phi^I)_a + i \bar{\psi}^A (\overline{D} \psi^A)_a$$ (12)

with

$$D^I_\mu = \eta^{ab} \partial_\mu + f^{abc} A^c_\mu$$ (13)

where $f^{abc}$ are the structure constants of $G$, satisfying Jacobi identity. Indices $a$ are raised and lowered with the help of the Killing metric $\eta_{ab}$. However, the action

$$\int \mathcal{L}_{\text{kin}} d^3 x$$ (14)

is not invariant under the $\mathcal{N}=8$ SUSY transformations

$$\delta \phi^I_a = i \varepsilon^A I^I A^I_\mu \phi^A_\mu = i \bar{\psi}^A I^I A^I_\mu \gamma^A$$

$$\delta \psi^A_a = -(\overline{D} \phi^I)_a I^I A^I_\mu \gamma^A$$ (15)

because $(\overline{D} - D^2)_{ab} = \frac{1}{2} f_{abc} F^{bc}_\mu \sigma_{\mu \nu} \neq 0$, where $F^{bc}_\mu = \partial_\mu A^c_\nu - \partial_\nu A^c_\mu + f^{abc} A^b_\mu A^c_\nu$ and $\sigma_{\mu \nu} = \frac{1}{2} [\gamma_\mu, \gamma_\nu] = \epsilon_{\lambda \mu \nu} \gamma^\lambda$. The variation of the action is

$$\delta \int \mathcal{L}_{\text{kin}} d^3 x = i \epsilon^{\lambda \mu \nu} \left( \int \bar{\psi}^A F^{bc}_\mu \Gamma^A_{\lambda \mu \nu} \gamma^A d^3 x \right) f^{abc} \Gamma^I_{\lambda \mu \nu}$$ (16)

and can be easily compensated by adding a Chern-Simons-like term

$$S_{\text{CS}} = \epsilon^{\lambda \mu \nu} \int B^a_\mu F^{a \mu \nu} d^3 x$$ (17)

with additional auxiliary pseudovector field $B^a_\mu$ in adjoint representation of $G$, which varies under $\mathcal{N}=8$ SUSY transformation:

$$\delta B^c_\mu = -i \left( \bar{\psi}^A \gamma^c I^I A^I_\mu \right) \phi^I f^{abc} \Gamma^I_{\lambda \mu \nu}$$ (18)

while

$$\delta A^c_\mu = 0$$ (19)

Note that there is no $B^3$ term in Chern-Simons action. Therefore one can say that this auxiliary field $B$ works as Lagrange multiplier, nullifying the gauge curvature $F^{\mu \nu}_c$ on-shell, what makes it the theory essentially linear in flat-connection background. Still this theory

$$\int \mathcal{L}_{\text{kin}} d^3 x + S_{\text{CS}},$$ (20)

though rather trivial, possesses all the desired properties: $\mathcal{N}=8$ supersymmetry, conformal invariance (at least classical) and it is also $P$-invariant, provided $B^a_\mu$ is a pseudovector.

5 Non linear $U(N)$ gauged $\mathcal{N}=8$ SUSY action in 3$d$

The theory can be made more interesting by introducing non-linear potential and accompanying $\psi^2$ terms. This can be done for any group $G$, but we write it in terms of the most interesting $G = U(N)$, to avoid repetition of formulas from the previous section. Adjoint representation of $SU(N)$ can be described as anti-Hermitian traceless $N \times N$ matrices, so that the single index $a = 1, \ldots, N^2 - 1$ turns into a pair of indices $a = (ij)$, $i, j = 1 \ldots N$, the structure constants $f^{abc}$ are induced by matrix commutators and Killing metric – by matrix trace. Still, formulation in terms of $f^{abc}$ in the previous section is also useful not only for more complicated Lie algebras, it also allows to neglect details, associated with the complex-valuedness of anti-Hermitian matrices.
One can also reduce $SU(N)$ to $SO(N)$, represented by real antisymmetric $N \times N$ matrices, to fully avoid this kind of problems.

Non-linearization turns to be related to the new $U(1)$ octuplet $(\varphi^I, \chi_A)$, it can be associated with the unit matrix in $U(N)$, but in this section we treat these fields simply as gauge singlets, without indices $i\bar{j}$ at all.

**Fields:**
- scalars $\phi^I_{ij}$, $\varphi^I$ with $I = 1, \ldots, 8$, $i, \bar{j} = 1, \ldots, N$
- spinors $\psi_{ij}^A$, $\chi_A$ with $A = 1, \ldots, 8$, $i, \bar{j} = 1, \ldots, N$
- vectors $A^\mu_{ij}$, $B^\mu_{ij}$ with $\mu = 0, 1, 2, 3$, $i, \bar{j} = 1, \ldots, N$

**Lagrangian:**

$$
-\text{tr}(D_\mu \phi^I)^2 + i \text{tr} \tilde{\psi}^A \slashed{D} \psi^A + \epsilon^{\lambda\mu\nu} \text{tr} F_{\mu\nu} B_\lambda - \\
-(\partial_\mu \varphi^I)^2 + i \bar{\chi}^I \partial \chi^I + 2i \varphi^I \text{tr} \left( \varphi^J \left[ \bar{\psi}^A, \psi^B \right] \right) \Gamma^I_{AB} + i \text{tr} \left( \left[ \varphi^I, \bar{\varphi}^J \right] \psi^A \right) \chi B \Gamma^J_{AB} + \sum_{k \neq I, J} (\varphi^K)^2 \text{tr} \left( \left[ \varphi^I, \bar{\varphi}^J \right] \right)^2 
$$

(21)

Sums are taken over repeated indices of all kinds. The long derivatives here are

$$
(D_\mu \phi^I)^{ij} = \partial_\mu \phi^I_{ij} + [A^\mu, \phi^I]_{ij} + B^\mu_{ij} \varphi^I,
$$

$$
(D_\mu \psi^A)^{ij} = \partial_\mu \psi^A_{ij} + [A^\mu, \psi^A]_{ij} + 2B^\mu_{ij} \chi^A.
$$

(22)

$N = 8$ SUSY transformations:

$$
\delta \varphi^I = i \tilde{\epsilon}^A \Gamma^I_{AA} \chi^A,
\delta \chi_A = -\tilde{\partial} \varphi^I \Gamma^I_{AA} \epsilon^A,
\delta \phi^I_{ij} = i \tilde{\epsilon}^A \Gamma^I_{AA} \psi^A_{ij},
(\delta \psi^A)_{ij} = -(\tilde{\partial} \phi^I)_{ij} \Gamma^I_{AA} \epsilon^A - [\varphi^I, \phi^J]_{ij} \varphi^K \Gamma^I_{KK} \epsilon^A,
\delta A^\lambda_{ij} = \left( \varphi^I \tilde{\psi}^A_{ij} + \phi^I \epsilon_{ij} \chi^A \right) \gamma^I \Gamma^I_{AA} \epsilon^A,
\delta B^\lambda_{ij} = [\tilde{\psi}^A, \phi^I]_{ij} \gamma^I \Gamma^I_{AA} \epsilon^A.
$$

(23)

The "linear" action (20) – the first line in (21) – and the SUSY transformations (15), (18), (19) from the previous section are reproduced if we put all $\varphi^I = \chi^A = 0$, what is a consistent reduction of the theory.

Instead one can ascribe an average value to the gauge singlet $\varphi$, say

$$
\left\langle \varphi^{I=8} \right\rangle = g_{YM}
$$

(24)

then the $B$ field acquires a "mass term" from $\text{tr} \left( (D\phi)^2 \right)^2 \rightarrow g_{YM}^2 \text{tr} (B_\mu)^2$ and integrating $B$ out one obtains a kinetic term $\frac{1}{g_{YM}^2} \text{tr} F_{\mu\nu} F^{\mu\nu}$ for the gauge field. This is the $M2 \rightarrow D2$ transition, described in [8] for the general BL theory. Of course, eq.(24) is inconsistent with equations of motion and is not an allowed VEV in the theory (21). One can cure this by adding non-trivial potential to the field $\varphi$, what necessarily breaks a part of supersymmetry as well as conformal invariance – as required in transition to $D2$ branes.

6 On SUSY invariance of (21)

Detailed check of invariance of (21) is somewhat sophisticated and remains beyond the scope of the present paper, we give in this section only some illustrative examples. The real problem is of course the lack of superfield or any other explicitly supersymmetric formulation. Non-linear terms in BL action are not expressed
even through an $N = 1$ superpotential, i.e. do not look like $\left(\frac{\partial W}{\partial \phi^I}\right)^2 + \bar{\psi}^I \frac{\partial^2 W}{\partial \phi^I \partial \phi^J} \psi^J$ for some $W(\phi)$, though they look surprisingly close to this for a theory with highly extended supersymmetry – what can serve as additional inspiration for the search of invariant formulations. We remind the "natural" logic of invariance check for BL action in the next s.7, while here we directly collect the terms of some different structures in the variation of (21). Of primary importance is variation of the potential, since it has the open dependence on the matrix $h_{ab}$ and thus can cause problems for non-antisymmetric $f^{abcd}$.

### 6.1 $\phi^4 \chi$ terms

These terms come from the variation of $\varphi$ in the potential and from non-linear variation of $\psi$ in the last fermionic item of (21):

$$
\sum_{K \neq I,J} 2i(\varepsilon \Gamma^K \chi) \varphi^K \text{tr}(\left[\phi^I, \phi^J\right]^2) + i \text{tr}(\left[\phi^I, \phi^J\right] \left[\phi^K, \phi^M\right]) (\varepsilon \Gamma^{KLM} \Gamma^{IJ} \chi) \varphi^K = 
$$

$$
i \sum_{K \neq I,J} \varphi^K \text{tr}\left\{\left[\phi^I, \phi^J\right] \left(2\left[\phi^I, \phi^J\right] (\varepsilon \Gamma^K \chi) + \left[\phi^K, \phi^M\right] (\varepsilon \Gamma^{IJK} \Gamma^{LM} \chi)\right)\right\}
$$

(25)

How can these very different structures cancel each other? The reason is that antisymmetry in $KLM$ and in $IJ$ allows one to arbitrarily change the order of $\Gamma$-matrices in $\Gamma^{KLM}$ and $\Gamma^{IJ}$ and then Jacobi identity for the trace over $\phi$-fields can be applied to ensure the cancelation.

Let $K = 8$. Then there are three different kinds of terms in remaining sum over $IJLM$.

- The pair $LM$ coincides with $IJ$, say, $I, J = 1, 2$ and either $L, M = 1, 2$ or $L, M = 2, 1$. These terms contribute:

$$
2[\phi^1, \phi^2] \Gamma^8 + [\phi^1, \phi^2] \Gamma^{812} \Gamma^{12} + [\phi^2, \phi^1] \Gamma^{812} \Gamma^{21} = 0
$$

because $\Gamma^{812} \Gamma^{12} = (-\Gamma^8 \Gamma^2 \Gamma^1)(\Gamma^1 \Gamma^2) = -\Gamma^8$, while $\Gamma^{812} \Gamma^{21} = +\Gamma^8$.

- Both indices $L$ and $M$ is different from $I$ and $J$, say $I, J = 1, 2$ and $L, M = 3, 4$. The corresponding contribution is obtained by reordering commutators under the sign of trace:

$$
\text{tr}\left\{\phi^I \left(\left[\phi^1, [\phi^1, \phi^3]\right] \Gamma^{812} \Gamma^{13} + \left[\phi^2, [\phi^1, \phi^3]\right] \Gamma^{813} \Gamma^{42} + \left[\phi^1, [\phi^2, \phi^3]\right] \Gamma^{823} \Gamma^{41}\right)\right\} = 0
$$

due to Jacobi identity for the commutators, which can be applied because all $\Gamma$-matrix structures are the same: proportional to $\Gamma^{814} \Gamma^{123}$. Note that one of the indices $L, M$ (but not $I, J$) could be equal to $K = 8$.

- One of the indices $LM$ coincides with one of $IJ$, another – not, say, $I, J = 1, 2$ and $L, M = 1, 3$ or $I, J = 1, 3$ and $L, M = 2, 1$. Such terms contribute

$$
\text{tr}\left(\left[\phi^1, \phi^3\right]\left[\phi^1, \phi^3\right]\right) \Gamma^{812} \Gamma^{13} + \text{tr}\left(\left[\phi^1, \phi^3\right]\left[\phi^2, \phi^2\right]\right) \Gamma^{813} \Gamma^{12} = -\text{tr}\left(\left[\phi^1, \phi^2\right]\left[\phi^1, \phi^3\right]\right) \Gamma^{81} \left(\Gamma^3 \Gamma^2 + \Gamma^2 \Gamma^3\right) = 0
$$

because of anti-commutativity of $\Gamma$-matrices. Once again, one of the indices $L, M$ could be $K = 8$.

It is important here that the terms with $K = I$ or $J$ are excluded from the potential, because such terms would contribute to the first item in (25), but not to the second one (where $L$ and $M$ are different from $K$) thus no cancelation occurs and supersymmetry would be broken.

### 6.2 $\phi^2 \phi^3 \psi$ terms

The story about these terms is very similar: they come from the variation of $\phi$ in the potential and from non-linear variation of $\psi$ in the first bi-fermionic item in the second line of (21):

$$
\sum_{K \neq I,J} 4i(\varphi^K)^2 \text{tr}\left\{\left[\phi^I, \phi^J\right] \left[\phi^J, \bar{\psi}^A\right] \Gamma^{IJ}_{AA} \varepsilon^A + 4i\varphi^I \text{tr}\left(\left[\phi^J, \bar{\psi}^A\right] \left[\phi^K, \phi^L\right]\right) \varphi^L \Gamma^{IJ}_{AB} \Gamma^{KLM} \varepsilon^B\right\}
$$

(26)

The second term can be rewritten as

$$
\varphi^I \varphi^M \text{tr}\left(\bar{\psi}^A \left[\phi^J, [\phi^K, \phi^L]\right]\right) \Gamma^{IJ}_{AB} \Gamma^{KLM} \varepsilon^B
$$

Now we take into account Jacobi identity for commutators and $I \leftrightarrow M$ symmetry.

- Let first $I \neq M$, say, $I, M = 1, 2$. If all $J, K, L \neq 1, 2$ then $\Gamma^{1J} \Gamma^{K2L} + \Gamma^{2J} \Gamma^{K1L} = 0$. If at least two of the three indices $J, K, L$ coincide with $I$ or $M$, one of the two $\Gamma$-factors vanishes. The only interesting case is when one of the three coincides with $I$ or $M$, say, $J, K, L = 1, 3, 4$ Then we get:
\[
\varphi^1 \varphi^2 \left\{ \left[ \varphi^1, [\varphi^3, \varphi^4] \right] (\Gamma^{11} \Gamma^{342} + \Gamma^{21} \Gamma^{341}) + \left[ \varphi^3, [\varphi^1, \varphi^4] \right] (\Gamma^{13} \Gamma^{142} + \Gamma^{23} \Gamma^{141}) + \left[ \varphi^4, [\varphi^1, \varphi^3] \right] (\Gamma^{14} \Gamma^{132} + \Gamma^{24} \Gamma^{131}) \right\} = \\
= \varphi^1 \varphi^2 \left\{ \left[ \varphi^1, [\varphi^3, \varphi^4] \right] - [\varphi^3, [\varphi^1, \varphi^4]] + [\varphi^4, [\varphi^1, \varphi^3]] \right\} \Gamma^{234} = 0
\]

because of Jacobi identity.

\[\bullet\] Let now \( I = M \). Then in the second term in (26) all \( J, K, L \neq I \), and it is equal to

\[
\sum_{I \neq J, K, L} (\varphi^I)^2 \text{tr} \left( \tilde{\psi}^A \left[ \phi^J, \left[ \phi^K, \phi^L \right] \right] \right) \Gamma^J \Gamma^{KL} A_A \varepsilon \hat{A}
\]

(27)

If all the three indices \( J, K, L \) are different, this sum vanishes due to Jacobi identity; if all the three coincide than \( \Gamma^{KL} \) is identical zero. The only interesting cases are \( J = K \) and \( J = L \), when (27) exactly cancels the first term in (26). Note that again it is important that terms with \( K = I, J \) are excluded from the potential in (21).

### 6.3 Terms with \( DB \)

Such terms appear from variation of kinetic part of the action in the first line of (21) and are compensated by the variation of \( A \)-field in the Chern-Simons action: both are proportional to the long derivative

\[
\epsilon_{\lambda \mu \nu} (D^\mu B^\nu)_{ij} = \epsilon_{\lambda \mu \nu} \left( \partial^\mu B^\nu_{ij} + \left[ A^\mu, B^\nu \right]_{ij} \right)
\]

Note once again that there are no \( B^3 \) terms in (21), there is nothing to cancel their variation and they would violate supersymmetry. At the same time the \( \varphi^2 B^2 \) term is present in the first line of (21): as we discussed, it plays the role in transition to \( D2 \) branes, but its variation cancels against that of the \( \psi B \chi \) and does not produce \( B^2 \) terms. Only \( DB \) is present in the variation.

In a little more detail, the first line in (21) can be rewritten as

\[
-\text{tr} (D_\mu \phi^I)^2 - 2 \varphi^I \text{tr} (B^\mu D_\mu \phi^I) - (\varphi^I)^2 \text{tr} B^2_\mu + i \text{tr} \tilde{\psi}^A \hat{D} \psi^A + 2i \text{tr} (\tilde{\psi}^A \hat{B}) \chi^A
\]

(28)

where \( D_\mu \phi = \partial_\mu \phi + [A_\mu, \phi] \) is an ordinary adjoint long derivative with the gauge field \( A_\mu \). The \( B^2 \) terms in the SUSY variation are

\[
-2i \varphi^I (\bar{\epsilon} \Gamma^I \chi) \text{tr} B^2_\mu + 2i \text{tr} (\bar{\epsilon} \Gamma^I \hat{B}) \chi = 0
\]

because \( \text{tr} \hat{B}^2 = \text{tr} B^2 \). Note that coefficient 2 in front of the last terms in (28) – and thus in the second long derivative in (22), – as well as anti-Hermiticity (antisymmetry) of \( B \) are important for this cancelation.

Similarly the variation of other terms in (28) provide terms with \( F = \partial A + [A, A] \) and \( DB = \partial B + [A, B] \) canceled respectively by the variations of \( B \) and \( A \) in the Chern-Simons term: \( \delta \int \text{tr} FB = \int \text{tr} (F \delta B - DB \delta A) \).

### 6.4 Other terms

The terms which are cubic in fermion fields, \( \varphi \chi \psi^2 \) and \( \varphi \psi^3 \), come from the variation of gauge fields in the first line of (21) and from variation of scalars in bi-fermion terms in the second line. Their cancelation requires adjustment of the relative coefficient between the first and the second lines of (21) and depends on the Fierz identities for gamma-matrices. The latter can be handled by making use of explicit \( \Gamma \)-matrix representation from s.3. Most numerous are terms of the type \( \varphi \phi^2 \psi \), they come from many places in (21) and we do not analyze them in this paper. Only when such analysis is completed one can take the action (21) really serious.

### 7 BL construction

In the remaining part of this paper we comment on the place of (21) in the general BL theory. As already mentioned in the introduction, (21) is associated with the special version (1) of the 3-bracket – the only one known for generic group \( U(N) \). It does not satisfy the antisymmetry requirement for \( f^{abcd} \) and original \( SO(4) = SU(2) \times SU(2) \) example of [5] with \( f^{abcd} = \epsilon^{abcd} \) is not the same as (21) for \( N = 2 \). The \( SO(4) \) example is associated with the deformation of (1) by addition of \( I \cdot \text{tr}(A[B, C]) \), with the unit matrix \( I \in U(N) \), which exists – at least in such simple form – only for \( N = 2 \).
7.1 Action and its invariance

This section is a very brief summary of BL construction, based on arbitrary (axiomatically defined) 3-product. The story begins at the end of s.3. Note that we use the same notations \(a, b, c, d\) and "tr" as in ss.4,5, but here they correspond to a somewhat different group \(\tilde{G} \neq G\): the simplified BL action (21) with the gauge group \(G = SU(N)\) is associated with \(\tilde{G} = U(N)\).

7.1.1 Kinetic terms

Let \(\phi\) and \(\psi\) belong to some \textit{real} representation \(R\) of some gauge group \(\tilde{G}\) with connection \(\tilde{A}\). Then short derivatives in (9) are substituted by the long ones, and the \(\tilde{G}\)-invariant scalar product is involved (denoted by "tr"):

\[
\mathcal{L}_{\text{kin}} = -\text{tr}_R \tilde{D}_\mu \phi^I \tilde{D}_\mu \phi^I + i\text{tr}_R \bar{\psi}^A \tilde{D} \psi^A
\]  

(29)

Under the same transformation (7) we get:

\[
\delta S_{\text{kin}} = \int \delta \mathcal{L}_{\text{kin}} d^3x = i \int \text{tr}_R \bar{\psi}^A \tilde{F}_{\mu\nu} \phi^I \Gamma^I_{\tilde{A}\tilde{A}} \epsilon^\tilde{A},
\]  

(30)

(of course, \(\epsilon^\tilde{A}\), is also an element of \(R\)).

7.1.2 Chern-Simons self-interaction of the gauge field

In order to compensate for this change one can add to \(S_{\text{kin}}\) a Chern-Simons action for \(\tilde{A}\),

\[
S_{\text{CS}} = \text{Tr} \int \left( \tilde{A}_d \tilde{A} + \frac{2}{3} \tilde{A}^3 \right) = \epsilon^{\lambda\mu\nu} \text{Tr} \int \left( \tilde{A}_\lambda \partial_\mu \tilde{A}_\nu + \frac{2}{3} \tilde{A}_\lambda \tilde{A}_\mu \tilde{A}_\nu \right) d^3x
\]  

(31)

which varies as

\[
\delta S_{\text{CS}} = \epsilon^{\lambda\mu\nu} \text{Tr} \int \tilde{F}_{\lambda} \delta \tilde{A}_\lambda = \epsilon^{\lambda\mu\nu} \int \tilde{F}^{ab} \delta \tilde{A}^{ab}
\]  

(32)

and can compensate the change in (30) if (7) is complemented by

\[
\delta \tilde{A}_\lambda = -i\phi^I \otimes \bar{\psi}^A \gamma_\lambda \Gamma^I_{\tilde{A}A} \epsilon^\tilde{A} \quad \text{or} \quad \delta \tilde{A}^\lambda = -i\bar{\psi}_a \gamma_\lambda \Gamma^I_{\tilde{A}A} \epsilon^\tilde{A}\phi^I_b
\]  

(33)

Note that "Tr" is different from "tr": while the latter one is for representation \(R\) where \(\phi, \psi\) and \(\epsilon\) belong, the former one is over representation \(R \otimes R\), where connection \(\tilde{A}\) is taking its values.

7.1.3 Twisted Chern-Simons

The same result can be achieved for any action, which changes by

\[
\delta \tilde{S} = \epsilon^{\lambda\mu\nu} \text{Tr} \int \tilde{F}_{\mu\nu} \delta \tilde{A}_\lambda
\]  

(34)

provided

\[
\delta \tilde{A}_\lambda = -i\phi^I \otimes \bar{\psi}^A \gamma_\lambda \Gamma^I_{\tilde{A}A} \epsilon^\tilde{A} \quad \text{or} \quad \delta \tilde{A}^\lambda = -i\bar{\psi}_a \gamma_\lambda \Gamma^I_{\tilde{A}A} \epsilon^\tilde{A}\phi^I_b
\]  

(35)

An example is provided by "twisted Chern-Simons" action

\[
\tilde{S}_{\text{CS}} = \text{Tr} \int \left( \tilde{A}_d \tilde{A} + \frac{2}{3} \tilde{A}^2 \right) = \epsilon^{\lambda\mu\nu} \text{Tr} \int \left( \tilde{A}_\lambda \partial_\mu \tilde{A}_\nu + \frac{2}{3} \tilde{A}_\lambda \tilde{A}_\mu \tilde{A}_\nu \right) d^3x
\]  

(36)

with

\[
\tilde{A}^{ab} = f^{abcd} \tilde{A}_{cd}
\]  

(37)
7.1.4 Interaction terms

Non-trivial transformation (33) of the $\hat{A}$-field contributes new terms to (30):

$$\text{Tr} \int \delta \hat{A}_\mu \left( 2\phi^I \otimes D^\mu \phi^I - i\bar{\psi}^A \otimes \gamma^\mu \psi^A \right) d^3x$$

(38)

These can be compensated by simultaneous addition of interaction terms to the action non-linear terms to the SUSY transformation (7). As shown in [5] this can be done if (35) is used instead of (33), and all additions are expressed in terms of the "structure constant" $f^{abcd}$ from (37), which can be used to define a 3-product

$$R \otimes R \otimes R \rightarrow R : [X, Y, Z]^a = f^{abcd} X_b Y_c Z_d$$

(39)

To raise and lower indices one also needs a metric $h_{ab}$, which, however, does not show up in the supersymmetry transformations. In these terms

$$\mathcal{L}_{\text{int}} = \frac{1}{6} \text{tr}_R[\phi^I, \phi^J, \phi^K]^2 + \frac{1}{2} \text{tr}_R \bar{\psi}^A \Gamma^{IJ}_{AB} [\phi^I, \phi^J, \psi^B]$$

(40)

and

$$\delta \psi_A = \dot{\phi}^I \Gamma^I_{AA} \bar{\epsilon}^A + \frac{1}{6} [\phi^I, \phi^J, \phi^K] \Gamma^{JJK} \bar{\epsilon}^A$$

(41)

The full action [5]

$$\int \mathcal{L}_{\text{kin}} d^3x + \hat{S}_{CS} + \int \mathcal{L}_{\text{int}} d^3x$$

(42)

is invariant, provided the 3-product satisfies the Jacobi-like "fundamental identity" [13, 4, 5, 6]

$$[A, B, [C, D, E]] = [[A, B, C], D, E] + [C, [A, B, D], E] + [C, D, [A, B, E]]$$

(43)

and $f^{abcd}$ has certain symmetry properties. It is usually required to be totally antisymmetric, though this requirement can probably be relaxed [7].

7.1.5 Summary of transformations

$$\delta \left( (\hat{D}\phi^I)^2 + \bar{\psi}^A (\hat{D}\psi^A)_b + \hat{S}_{CS} \right)$$

$$\delta \hat{A}^{ab}_\mu \left( \phi^I (\hat{D}_\mu \phi^I)_b + \bar{\psi}^A \gamma^\mu \psi^A \right) + \bar{\psi}^a \frac{\partial H^{ab}}{\partial a} (\hat{D}\phi^I)_b \bar{\epsilon}^A$$

(44)

$$\delta \left( \bar{\psi}^a \psi^b \Gamma_{IJ}^{IJ} T_{ab} (\phi) \right)$$

$$+ \bar{\psi}^a \psi^b \Gamma_{IJ}^{IJ} T_{ab} \frac{\partial T_{IJ}^{IJ}}{\partial a} \bar{\psi}^c \Gamma^K_{CC} \bar{\epsilon}^C + 2 \bar{\psi}^a \Gamma_{AB}^{IJ} T_{ij} \Gamma^K_{BB} (\hat{D}\phi^K)_b \bar{\epsilon}^B + 2 \bar{\psi}^a \Gamma_{AB}^{IJ} T_{ij} H^b_{BB} \bar{\epsilon}^B$$

(45)

$$\delta \left( V(\phi) \right)$$

$$+ \frac{\partial V}{\partial a} \bar{\psi}^a \Gamma^I_{AA} \bar{\epsilon}^A$$

(46)

There are three terms in the first line, three in the second and one in the forth. Enumerating them, from 1 to 7, we have the following cancelations to take place:

$$2 + 4 \quad \delta \hat{A}^{ab}_\mu \bar{\psi}^a \gamma^\mu \psi^A + \bar{\psi}^a \psi^b \Gamma^{IJ} \Lambda_{\mu AB} \bar{\psi}^c \Gamma^K_{CC} \bar{\epsilon}^C = 0$$

$$6 + 7 \quad 2 \bar{\psi}^a \Gamma^{IJ} \Lambda_{\mu AB} T_{ij} H^b_{BB} \bar{\epsilon}^B + \frac{\partial V}{\partial a} \bar{\psi}^a \Gamma^I_{AA} \bar{\epsilon}^A = 0$$

$$1 + 3 + 5 \quad \delta \hat{A}^{ab}_\mu \phi^I (\hat{D}_\mu \phi^I)_b + \bar{\psi}^a \frac{\partial H^{ab}}{\partial a} (\hat{D}\phi^I)_b \bar{\epsilon}^A + 2 \bar{\psi}^a \Gamma_{AB}^{IJ} T_{ij} \Gamma^K_{BB} (\hat{D}\phi^K)_b \bar{\epsilon}^B = 0$$

Direct check of these cancelations involves application of Fierz and other $\gamma$-matrix identities and is rather tedious.
7.2 From $M2$ to $D2$

A very important and instructive transition from BL action for $M2$ branes in $d = 11$ with 8 transverse fields $\phi^I$ to $D2$ branes in $d = 10$ with only 7 transverse fields $\phi^I$ an the action

$$
\int d^3 x \left( \sum_{i=1}^{7} (D_\mu \phi^i)_a (D^\mu \phi^i)_a + \frac{1}{4 g_{YM}^2} \text{tr} F_{a\mu}^\mu F_{\mu\nu}^a + \text{fermions} \right)
$$

was considered in [8]. Note that both $\phi^a$ and $A^a_\mu$ are now in the same adjoint representation of the gauge group.

As explained in [8], transition from $M2$ action to $D2$ one in (44) takes place when one of the scalar fields, associated with the 8-th transverse direction, acquires vacuum expectation value:

$$
< \phi^I_{a=8} > = g_{YM}
$$

and $a$ in (44) takes one less value than $a$ in BL action (42), so that $\dim(G) = \dim(G) + 1$. We label this extra value of $a$ in $G$ by 0 (note that it is not the additional 0-generator of [5!]). Then for $a \neq 0$

$$(\tilde{D}_\mu \phi^I)_a = \partial_\mu \phi^I_a - \tilde{A}^{ab}_\mu \phi^I_b = \partial_\mu \phi^I_a - f_{abc} A^c_\mu \phi^I_b = \partial_\mu \phi^I_a - f_{abc} A^c_\mu \phi^I_b - f_{a0cd} A^c_\mu \phi^0 - B^a_\mu \phi^I_0$$

where

$$
 f^{abc} = f^{abc0}
$$

are the ordinary structure constants of $G$.

The terms with $I \neq 8$ in (44) provide

$$
\sum_{i=1}^{7} \left( \partial_\mu \phi^I_a + f_{abc} A^c_\mu \phi^I_b \right)^2 + O(B\phi^2) = \sum_{i=1}^{7} \left( (D_\mu \phi^i)_a \right)^2 + O(B\phi^2)
$$

(47)

to the action, while those with $I = 8$ contribute

$$
g_{YM}^2 B^a_\mu B^a_\mu
$$

(48)

In combination with

$$
S_{CS} = \int (BF + B^3)
$$

(49)

it provides the kinetic term $\frac{1}{4 g_{YM}^2} \int \text{tr} F^2 d^3 x$ for Yang-Mills field.

Note that the number of $B$-fields

$$
B^a_\mu = f_{a0cd} A^c_\mu
$$

(50)

is smaller than that of $A^c_\mu$, so that in the quantum case the measure $[DA]$ in functional integral would differ from $[DB]$ by non-trivial Jacobian factor.

7.3 3-algebra and associated gauge fields

The key point of BL construction is the very interesting 3-algebra structure, supposedly generalizing the Lie-algebra structure underlying the ordinary gauge symmetry. BL construction associates with 3-algebras a new kind of gauge fields $A^a_\mu$. If the $\phi^I$-fields are interpreted as $X^I$, describing 8 transverse directions to the $M2$ brane world volume in embedding $d = 11$ space time, then index $a$ is naturally treated as $a = (ij)$ where $i$ and $j$ label the copies of $M2$ brane and associated string stretched between them. Despite this interpretation is natural for $D2$ and not obligatory to $M2$ branes, the reduction procedure [8], briefly described in the previous s.7.2, strongly suggests that it also holds for $M2$. Then $A^{ab}_\mu$ is actually carrying two pairs of indices $A^{ab}_\mu = A^{ij,kl}_\mu = A^{ij}_k$ and looks like associated with a pair of strings, i.e. with a (fundamental) 2-brane stretching between the two strings which stretch between the two pairs of $M2$ branes. This is of course more than natural for the $M$-theory and this is what makes the BL construction so attractive. Unfortunately, this intriguing structure is fully suppressed in the simplified action (21), where gauge fields are the ordinary adjoints, associated with pairs of branes rather than pairs of strings between them.
The big problem of BL construction is that no fully-non-trivial examples of 3-algebras are known. Spectacular original SO(4) example of [5] appeared to be difficult to generalize, despite certain effort in [4]-[11]. Immediate idea about octonionic generalizations was shown in [9] to be not so straightforward. Even the very inspiring reformulations of [4, 5, 7] do not provide immediate new examples. One could hope that such examples are obliged to exist, because N branes with arbitrary N certainly exist. Unfortunately, eq.(21) demonstrates that these configurations can probably be described by a very primitive BL action, which is associated with degenerate 3-bracket (1), as we shall see in the next section.

Before proceeding to this last subject of the present paper, it deserves emphasizing that the total-antisymmetry requirement imposed in [5] over-constrains the possible 3-algebra structures. The 3-bracket (1) corresponds to \( f^{abcd} \) which is not totally antisymmetric, in particular, \( f^{abcd} = 0 \) while \( f^{a0bd} = f^{abc} \neq 0 \). Only for \( N = 2 \) it can be promoted to the totally antisymmetric \( f^{abcd} = \epsilon^{abcd} \) without violating the "fundamental identity" (43). Fortunately, as mentioned in [7], the antisymmetry condition depends on the metric \( h_{ab} \), which does not appear in supersymmetry transformations and does not affect the closure of the algebra.

A way from Lie algebra structure constants to generic 3-algebras lies through solving a linear equation for a linear operator \( t_{AB} \) [4, 7, 8], \( t_{AB}(P) = [A, B, P] \):

\[
t_{AB}([C, D]) - t_{CD}([A, B]) = [C, t_{AB}(D)] - [D, t_{AB}(C)]
\]

which should afterwards be constrained by a non-linear equation

\[
t_{AB}(t_{CD}(E)) - t_{CD}(t_{AB}(E)) = t_{AB(C),D}(E) + t_{C,t_{AB}(D)}(E)
\]

8 The simplified action (21) in the general BL framework

The \( U(N) \) Lie algebra decomposes as

\[
U(N) = U(1) + SO(N) + S(N)
\]

(anti-Hermitian matrix is a sum of imaginary unit, real antisymmetric and imaginary symmetric) and this is a symmetric decomposition: \( S(N) \) is not a subalgebra of \( U(N) \), only a representation of \( SO(N) \), but with the special property

\[
[S(N), S(N)] \subset SO(N)
\]

- this is important for the algebraic discussion of [4]. It is also important that in \( U(N) \) one can multiply matrices, not only commute (there are \( d^{abc} \) constants in addition to \( f^{abc} \)). However, this possibility is actually not used in construction of Nambu bracket (1) [12, 7]

\[
[A, B, C] = \text{tr}(A) \cdot [B, C] + \text{tr}(B) \cdot [C, A] + \text{tr}(C) \cdot [A, B]
\]

As already mentioned in the previous sections, there is no term with \( I \) (with non-vanishing trace) at the r.h.s. - this makes the bracket a kind of degenerate and \( f^{abcd} \) very asymmetric. In the case of \( U(2) \) one can add \( I \cdot \text{tr}([A, B]C) \), so that \( [A, B, C] = \epsilon^{abcd} A_a B_b C_c \). It is easy to see that the linear operator

\[
t_{AB}(*) = \text{tr}(A) \cdot [B, *] - \text{tr}(B) \cdot [A, *] + [A, B] \cdot \text{tr}(*)
\]

which defines (55), satisfies equations (51) and (52).

It remains to rewrite the action (42) in terms of this 3-product (taking care of the lack of total antisymmetry). We denote the extra 0-components (traces) of the fields \( \phi^i_A \) and \( \psi^A_{ij} \) through \( \varphi^i \) and \( \chi^A \), while the BL field \( A_{ab} = A_{ijkl} = A_{ik\ell}^j \) is split into \( A_{ik\ell}^j = \frac{1}{2} A_i^\ell \) and \( A_{ikj} = \frac{1}{4} \) (this is not a one-to-one change, as already mentioned in the end of s.7.2). With these notations we get, for example, from (37):

\[
A_{ik\ell}^j \phi_p^i = A_{ik\ell}^j \phi^i_p - A_{ikl}^j \phi^i_\ell - A^{lk}_i \phi_j^i + A^{lk}_i \phi^i_j + i \left( A_{ik\ell}^j - A_{ikj} \right) = [A, \phi^i_p] + B_i^\ell \phi^i_j
\]

Expressions (22) for long derivatives are immediate corollaries of this relation. In the same way one can deduce from (42) all other terms of the action (21).
9 Conclusion

To summarize, eq.(21) explicitly describes a presumably-$OSp(8|4)$-invariant BL world-volume action for an arbitrary number $N$ of coincident $M2$ branes. Despite being based on a very primitive 3-algebra, associated with the $U(N)$ Nambu bracket (1) [12, 7] and thus lacking the mysterious non-adjoint gauge fields $A_{\mu}^{ab}$, this simplified action is sufficient to describe transition $a la$ [8] to generic $D2$ branes with arbitrary gauge group. Thus it can provide a far simpler and down-to-Earth solution to the problem posed in [1] than the generic BL construction, deeply hiding the underlying 3-algebra structure. As explained in s.7.3 above, this is rather a drawback than an advantage of eq.(21).

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