Mathematical model of flexible dimension-dependent mesh plates

E Yu Krylova¹, I V Papkova², A O Sinichkina², T B Yakovleva² and V A Krysko-yang²
¹Saratov National Research University named after NG Chernyshevsky, Astrakhanskaya st., 83, Saratov, 410012, Russia
²Saratov State Technical University named after Gagarin YA, Polytechnicheskaya st., 73, Saratov, 410054, Russia

Abstract. The theory of a flexible dimensional-dependent plate of the network structure is constructed in this paper. The plate is considered as a Cosserat continuum with constrained particle rotation (pseudocontinuum). The lattice structure is constructed according to the theory of Pshenichnov G.I. The equilibrium equations for the plate element and the boundary conditions are obtained from the Lagrange variation principle on the basis of Kirchhoff's kinematic hypotheses. The geometric nonlinearity is taken into account by the model of Theodor von Karman. By the Bubnov-Galerkin method an analytical solution was found for a square hinge supported by the ends reticulated plate, consisting of two families of ribs under normal load.

1. Introduction
Plates and shells are elements of engineering structures, instruments and mechanisms both at the macro and nano level. An investigation of the static and dynamic behavior of such mechanical objects under the action of external fields of different nature is devoted to the works [1-3].

Reticulated plates and shells due to their lightness and increased strength characteristics are in the area of scientific interests of many Russian and foreign scientists. In work [4.] problems of bending of reticulated shells on the basis of the continuum model are considered. The dynamic stability loss of reticulated plates with different lattice character in a geometricaly nonlinear formulation is investigated in [5]. The paper [6] is devoted to the stability of a carbon fiber plate with different inclination angles of the ribs. The authors [7] study the chaotic dynamics of a reticulated cylindrical shell. The development of micro system technologies and nano technologies leads to the need to build models of the mechanical behavior of reticulated plates, as nano scale objects.

2. Problem formulation
The investigation object is a rectangular plate occupying in space $\mathbb{R}^3$ a region
$$\Omega = \left\{ 0 \leq x \leq b; 0 \leq y \leq s; -\frac{h}{2} \leq z \leq \frac{h}{2} \right\}$$. Cartesian coordinate system, introduced according to Figure1.
Figure 1. Diagram of micro-plate: kinematic parameters, coordinate system and geometry

In view of Kirchhoff’s hypotheses, the components of the displacement vector \( \mathbf{u} \) take the form:

\[
\begin{align*}
    u_x &= u(x, y, t) - z \frac{\partial w}{\partial x}(x, y, t); \\
    u_y &= v(x, y, t) - z \frac{\partial w}{\partial y}(x, y, t); \\
    u_z &= w(x, y, t).
\end{align*}
\]  

Here \( u, v, w \) – the axial displacements of the plate middle surface in the directions \( x, y, z \) respectively.

The non-zero components of the strain tensor \( \mathbf{e} \) in the case of first-approximation hypotheses can be written in the form:

\[
\begin{align*}
    e_{xx} &= e_{xx} - z \kappa_{xx}; \\
    e_{yy} &= e_{yy} - z \kappa_{yy}; \\
    e_{xy} &= e_{xy} - z \kappa_{xy}.
\end{align*}
\]  

Here the components of tangential \( e_{xx}, e_{yy}, e_{xy} \) and flexural \( \kappa_{xx}, \kappa_{yy}, \kappa_{xy} \) deformation of the middle surface of the plate, having the form:

\[
\begin{align*}
    e_{xx} &= \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2; \\
    e_{yy} &= \frac{\partial v}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2; \\
    e_{xy} &= \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y};
\end{align*}
\]  

Classic continual models do not take into account scale effects at the nanoscale level. The paper deals with a nonclassical continuum shell model based on the Cosserat medium with constrained particle rotation (pseudo continuum). It is assumed here that the displacement and rotation fields are not independent [8]. In this case, the symmetric tensor components of the curvature gradient \( \chi \) will be written as follows: \( \chi_{ij} = \frac{1}{2} \left( \theta_{ij} + \theta_{ji} \right) \), where \( \theta_x, \theta_y, \theta_z \) – are the vector components of micro-rotations. \( \theta_i = \frac{1}{2} \left( rot(u) \right)_i, i, j = \{ x, y, z \} \). For the plate material the defining relations are taken in the form [9]:

\[
\begin{align*}
    \sigma_{xx} &= \frac{E}{1 - \nu^2} \left[ e_{xx} + \nu e_{yy} \right]; \\
    \sigma_{yy} &= \frac{E}{1 + \nu} e_{yy}, \\
    \left( m_{xx}, m_{yy}, m_{xy} \right) &= \frac{E l^2}{1 - \nu^2} \left( \chi_{xx} + \chi_{yy} + \chi_{xy} \right),
\end{align*}
\]  

where \( \sigma \) – Cauchy tensor, \( m \) – symmetric moment tensor of higher order, \( E \) – the Young's modulus, \( \nu \) – the Poisson's ratio. The parameter \( l \) is an additional independent material length parameter associated with the symmetric rotational gradient tensor. [9].

The equilibrium equations for the plate and the boundary conditions are obtained from Lagrange variation principle, according to which from all possible displacements the true impart the total potential energy of the stationary value

\[
\delta \Pi = 0. 
\]  

Here \( \Pi = U - W \) – the Lagrangian (the total potential energy of the elastic system). Taking into account the moment theory [9] the potential energy \( U \) in an elastic body for infinite small deformations is written in the form:

\[
U = \frac{1}{2} \left( \int_{\Omega} \left( \sigma_{xx} e_{xx} + 2 \sigma_{xy} e_{xy} + \sigma_{yy} e_{yy} + m_{xx} \chi_{xx} + m_{yy} \chi_{yy} + 2 m_{xy} \chi_{xy} + 2 m_{xx} \chi_{xx} + 2 m_{yy} \chi_{yy} \right) dv \right),
\]
external forces \( W = \int_0^h \int_0^a q(x,y) \, dw \, dx \, dy \) is the external normal load. Using the notation
\[
\begin{align*}
(N_{xx}, N_{yy}, T) &= \int_0^h \left( \sigma_{xx}, \sigma_{yy}, \sigma_{xy} \right) \, dz, \\
(M_{xx}, M_{yy}, H) &= \int_0^h \left( \sigma_{xx}, \sigma_{yy}, \sigma_{xy} \right) \, dz,
\end{align*}
\]
\[
(Y_{xx}, Y_{yy}, Y_{xy}, Y_{xy}) = \int_0^h \left( m_{xx}, m_{yy}, m_{xy}, m_{yx} \right) \, dz,
\]
where \( N_{xx}, N_{yy}, T, M_{xx}, M_{yy}, H \) the forces and moments of the plate, the forces caused by the moment stresses, we obtain the resolving equations:
\[
\begin{align*}
\frac{\partial N_{xx}}{\partial x} + \frac{\partial T}{\partial y} - \frac{1}{2} \frac{\partial^2 Y_{xy}}{\partial x^2} - \frac{1}{2} \frac{\partial^2 Y_{xx}}{\partial x^2} &= 0, \\
\frac{\partial N_{yy}}{\partial y} + \frac{\partial T}{\partial x} + \frac{1}{2} \frac{\partial^2 Y_{xy}}{\partial y^2} + \frac{1}{2} \frac{\partial^2 Y_{yy}}{\partial y^2} &= 0, \\
\frac{\partial N_{xx}}{\partial x} \frac{\partial N_{yy}}{\partial y} + N_{xx} \frac{\partial^2 W}{\partial x^2} + N_{yy} \frac{\partial^2 W}{\partial y^2} + N_{xy} \frac{\partial^2 W}{\partial x \partial y} + \frac{\partial^2 M_{xx}}{\partial x^2} + \frac{\partial^2 M_{yy}}{\partial y^2} + \frac{1}{2} \frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} &= 0,
\end{align*}
\]
where
\[
\begin{align*}
\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + 4 \frac{\partial^2 W}{\partial x \partial y} - \frac{\partial^2 Y_{xx}}{\partial x^2} - \frac{\partial^2 Y_{yy}}{\partial y^2} + \frac{\partial^2 Y_{xy}}{\partial x \partial y} + \frac{\partial^2 Y_{yx}}{\partial x \partial y} &= -2q.
\end{align*}
\]
Assume that the plate consists of \( n \) families of edges, \( a_j \) – the distance between the edges of the \( j \)th family, \( \delta_j \) – the width of the \( j \)th family, \( \varphi_j \) – the angle between the axis \( x \) and the axis of the \( j \)th family edges (Fig. 2).

Expressions for classical stresses and stresses of higher order in the \( j \)-th family of edges in the Pshenichnov theory [10]

\[
\begin{align*}
\sigma^j &= \frac{E}{1-v^2} \left( e_{xx} \cos^2 \varphi_j + e_{yy} \sin^2 \varphi_j + e_{xy} \cos \varphi_j \sin \varphi_j \right), \\
\sigma^j &= \frac{E}{2(1+\nu)} \left( e_{xx} \cos \varphi_j + e_{yy} \sin \varphi_j \right), \\
m^j &= \frac{E \ell^2}{1+\nu} \left( \chi_{xx} \cos^2 \varphi_j + \chi_{yy} \sin^2 \varphi_j + \chi_{xy} \cos \varphi_j \sin \varphi_j \right), \\
m^j &= \frac{E \ell^2}{1+\nu} \left( \chi_{xx} \cos \varphi_j + \chi_{yy} \sin \varphi_j \right).
\end{align*}
\]

Here \( e_{xx}, e_{yy}, e_{xy}, e_{yx}, e_{zz} \) – the solid plates deformations written in (2), \( \chi_{xx}, \chi_{yy}, \chi_{xy}, \chi_{yx}, \chi_{zz} \) – are the components of the symmetric tensor of the gradient of the curvature of the solid plate \( \chi \). Non-zero stresses and higher voltages for the reticulated plate consisting of \( n \) families of rods will have the form (8):
\[\sigma_{xx} = \sum_{j=1}^{n} \sigma_j \delta_j \cos^2 \varphi_j, \quad \sigma_{yy} = \sum_{j=1}^{n} \sigma_j \delta_j \sin^2 \varphi_j, \quad \sigma_{xy} = \sum_{j=1}^{n} \sigma_j \delta_j \cos \varphi_j \sin \varphi_j,\]
\[m_{sx} = \sum_{j=1}^{n} m_j \delta_j \cos \varphi_j, \quad m_{sy} = \sum_{j=1}^{n} m_j \delta_j \sin \varphi_j, \quad m_{xy} = \sum_{j=1}^{n} m_j \delta_j \cos \varphi_j \sin \varphi_j.\]  
(9)

Taking into account relations (8) and (9), it is easy to write down the expressions for the forces and moments of the reticulated plate, substituting them into the system (7) we obtain the equilibrium equations for a reticulated rectangular plate.

Consider a grid with two families of rods \(\varphi_1 = 0, \ \varphi_2 = 90^\circ, \ a_1 = a_2 = a, \ \delta_1 = \delta_2 = \delta.\)

We introduce the force function in the middle plane \(F\) by the formulas:
\[N_{xx} = h \frac{\partial^2 F}{\partial x^2}, \quad N_{yy} = h \frac{\partial^2 F}{\partial y^2}, \quad T = -h \frac{\partial^2 F}{\partial x \partial y},\]
so that the first two equations (7) are automatically satisfied. Then, using the dimensionless parameters 
\[x = \tilde{b} \bar{x}, \ y = s \tilde{y}, \ w = h \bar{w}, \ \delta = b \tilde{\delta}, \ l = \tilde{l}, \ a = b \tilde{a}, \ F = Eh^2 \bar{F},\]
\[t = \frac{b^2}{h} \sqrt{\frac{\rho}{E}}, \ \epsilon = \frac{\rho}{E} \bar{F}; \ q = \frac{Eh^2}{s^2} \bar{q}.\]

The third equation of the system (7) for the reticulated plate is written as:
\[\frac{\partial^4 F}{\partial x^4} + \frac{\partial^4 F}{\partial x^2 \partial y^2} + \frac{\partial^4 F}{\partial x^2 \partial y^2} - \frac{1}{12(1-\nu^2)} \left( \frac{s^2}{b^2} \frac{\partial^4 w}{\partial x^4} + \frac{b^2}{s^2} \frac{\partial^4 w}{\partial y^4} \right) - \frac{2l^2}{(1+\nu) \frac{h^2}{b^2 \partial^2}} = -2 \frac{a^2}{\delta} q.\]  
(10)

To (10) we have to add the equation of continuity of deformations, written in a dimensionless form:
\[\frac{\partial^4 F}{\partial x^4} - \frac{2vb^2}{s^2} \frac{\partial^4 F}{\partial x^2 \partial y^2} + \frac{b^4}{s^4} \frac{\partial^4 F}{\partial y^4} - \frac{a b^2}{s^2} \frac{\partial^2 w}{\partial x \partial y} - \frac{a b^2}{s^2} \frac{\partial^2 w}{\partial x \partial y} - \frac{b^2}{s^2} \frac{\partial^2 w}{\partial y^4} - \frac{a b^4}{s^2} \frac{\partial^2 w}{\partial y^4} = \frac{b^2}{s^2} \frac{\partial^2 w}{\partial y^4} - \frac{a b^4}{s^2} \frac{\partial^2 w}{\partial y^4}.\]  
(11)

To the equations system (10-11) we add the boundary conditions of hinged fastening with the presence of flexible edges on the ends [11]:
\[w = 0; \ \frac{\partial^2 w}{\partial x^2} = 0; \ \frac{\partial^2 F}{\partial x^2} = 0 \text{ at } x = 0; 1 \ \frac{\partial^2 w}{\partial y^2} = 0; \ \frac{\partial^2 F}{\partial y^2} = 0 \ \text{ at } y = 0; 1.\]  
(12)

Consider a square plate with linear dimensions \(s = b = 1, \ h = \frac{b}{40} = 0.025.\) Geometric parameters of the grid are chosen as follows \(a = 2\delta.\) To solve the task, we apply the Bubnov-Galerkin method in the first approximation; for this we represent the deflection and force functions in the form:
\[w(x,y) = A \sin(\pi x) \sin(\pi y), \ F(x,y) = B \sin(\pi x) \sin(\pi y).\]

Following the procedure of the method, we find the unknown coefficients of the expansion from the equations:
\[\frac{64}{9(1-\nu)} \left( \frac{4}{3} A^3 - 10 A^2 \right) + \frac{1}{12(1-\nu^2)} \left( \frac{40 l^2}{(1+\nu)} \right) = 4q,\]
\[B = \frac{A}{\pi^2 (1-\nu)} \left( -\frac{4}{3} A^2 + 10 A \right).\]  
(13)
Figure 3 shows the dependence of the coefficient $A$ in the representation of the deflection function on the value of the normal load $q$ depending on the value of the parameter $l$, appearing in the micropolar theory. With increasing this parameter, the plate's resistance to external normal load increases.

![Figure 3. Dependence of the coefficient A on the normal load.](image)

3. Conclusion

Based on the micropolar theory, taking into account Kirchhoff's kinematic hypotheses (hypothesis of the first nailing), in the paper constructed a theory of geometrically non-linear plates of a reticulated structure. It is shown that with an increase in the dimension-dependent parameter, the load-carrying capacity of the plate to the external normal load increases.

The work was supported by the Russian Foundation for Basic Research № 18-01-00351а, № 18-38-00878 мол_а, № 16–08–01108а, № 16–01–00721a

4. References

[1] Awrejcewicz J, Krysko V A, Papkova I V, Krylova E Yu and Krysko A V 2014 *Nonlinear Studies* 21(2) pp 293-307
[2] Krylova E Y, Yakovleva T V and Bazhenov B G 2016 *PNRPU Mechanics Bulletin* 1 pp 82-92 DOI: 10.15593/perm.mech/2016.1.06
[3] Krylova E Y, Papkova I V, Erofeev N P, Zakharov V M and Krysko V A 2016 *Journal of Applied Mechanics and Technical Physics* 57 pp 714-719
[4] Belikov G I 2014 *Bulletin of the Volgograd State Architectural and Construction University. Series: Construction and architecture* 37 pp 121-128
[5] Trushin S I, Zhuravleva T A and Sysoeva E V 2016 Dynamic loss of stability of nonlinear deformable mesh plates made of composite material with various lattice configurations. *Scientific review* pp 44-51
[6] Azikov N S and Pavlov E A 2016 *Aviation industry* 3 pp 46-50
[7] Wua Q L, Zhang W and Dowell E H 2018 *International Journal of Non-Linear Mechanics* 102 pp 25-40
[8] Erofeev V I 1999 Wave processes in solids with a microstructure (Moscow: Moscow University)
[9] Yang F, Chong A C M, Lam D C C and Tong P 2002 *International Journal of Solids and Structures* 39 pp 2731–2743
[10] Pshenichnov G I 1982 *The theory of thin elastic net shells and plates* (Moscow: Nauka) 352 pp
[11] Kornishin M S 1964 *Nonlinear problems of the theory of plates and shallow shells and methods for their solution* (Moscow: Nauka) 1964