AN EXAMPLE OF AN INFINITE SET OF ASSOCIATED PRIMES OF A
LOCAL COHOMOLOGY MODULE

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0. Introduction

Let \((R, m)\) be a local Noetherian ring, let \(I \subset R\) be any ideal and let \(M\) be a finitely generated \(R\)-module. It has been long conjectured that the local cohomology modules \(H^i_I(M)\) have finitely many associated primes for all \(i\) (see Conjecture 5.1 in [H] and [L]).

If \(R\) is not required to be local these sets of associated primes may be infinite, as shown by Anurag Singh in [S], where he constructed an example of a local cohomology module of a finitely generated module over a finitely generated \(\mathbb{Z}\)-algebra with infinitely many associated primes. This local cohomology module has \(p\)-torsion for all primes \(p \in \mathbb{Z}\).

However, the question of the finiteness of the set of associated primes of local cohomology modules defined over local rings and over \(k\)-algebras (where \(k\) is a field) has remained open until now. In this paper I settle this question by constructing a local cohomology module of a local finitely generated \(k\)-algebra with an infinite set of associated primes, and I do this for any field \(k\).

1. The example

Let \(k\) be any field, let \(R_0 = k[x, y, s, t]\) and let \(S = R_0[u, v]\). Define a grading on \(S\) by declaring \(\text{deg}(x) = \text{deg}(y) = \text{deg}(s) = \text{deg}(t) = 0\) and \(\text{deg}(u) = \text{deg}(v) = 1\). Let \(f = sx^2v^2 - (t+s)xyuv + ty^2u^2\) and let \(R = S/fS\). Notice that \(f\) is homogeneous and hence \(R\) is graded. Let \(S_+\) be the ideal of \(S\) generated by \(u\) and \(v\) and let \(R_+\) be the ideal of \(R\) generated by the images of \(u\) and \(v\).

Consider the local cohomology module \(H^2_{R_+}(R)\): it is homogeneously isomorphic to \(H^2_{S_+}(S/fS)\) and we can use the exact sequence

\[
H^2_{S_+}(S) \xrightarrow{f} H^2_{S_+}(S) \rightarrow H^2_{S_+}(S/fS) \rightarrow 0
\]

of graded \(R\)-modules and homogeneous homomorphisms (induced from the exact sequence

\[
0 \rightarrow S(-2) \xrightarrow{f} S \rightarrow S/fS \rightarrow 0
\]
to study $H^2_{R_k}(R)$. Furthermore, we can realize $H^2_{S_k}(S)$ as the module $R_0[u^-, v^-]$ of inverse polynomials described in [3, 12.4.1]: this graded $S$-module vanishes beyond degree $-2$, and, for each $d \geq 2$, its $(-d)$-th component is a free $R_0$-module of rank $d - 1$ with base $(u^{-\alpha}v^{-\beta})_{\alpha, \beta > 0, \alpha + \beta = -d}$. We will study the graded components of $H^2_{S_k}(S/fS)$ by considering the cokernels of the $R_0$-homomorphisms

$$f_{-d} : R_0[u^-, v^-]_{-d-2} \to R_0[u^-, v^-]_{-d} \quad (d \geq 2)$$

given by multiplication by $f$. In order to represent these $R_0$-homomorphisms between free $R_0$-modules by matrices, we specify an ordering for each of the above-mentioned bases by declaring that

$$u^{\alpha_1}v^{\beta_1} < u^{\alpha_2}v^{\beta_2}$$

(where $\alpha_1, \beta_1, \alpha_2, \beta_2 < 0$ and $\alpha_1 + \beta_1 = \alpha_2 + \beta_2$) precisely when $\alpha_1 > \alpha_2$. If we use this ordering for both the source and target of each $f_d$, we can see that each $f_d$ ($d \geq 2$) is given by multiplication on the left by the tridiagonal $d-1$ by $d+1$ matrix

$$A_{d-1} := \begin{pmatrix}
    sx^2 & -xy(t+s) & ty^2 & 0 & \ldots & 0 \\
    0 & sx^2 & -xy(t+s) & ty^2 & 0 & \ldots & 0 \\
    0 & 0 & sx^2 & -xy(t+s) & ty^2 & \ldots & 0 \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & \ldots & sx^2 & -xy(t+s) & ty^2 & \ldots & 0
\end{pmatrix}.$$

We also define

$$\overline{A}_{d-1} := \begin{pmatrix}
    s & -(t+s) & t & 0 & \ldots & 0 \\
    0 & s & -(t+s) & t & 0 & \ldots & 0 \\
    0 & 0 & s & -(t+s) & t & \ldots & 0 \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & \ldots & s & -(t+s) & t & \ldots & 0
\end{pmatrix}$$

obtained by substituting $x = y = 1$ in $A_{d-1}$.

Let also $\tau_i = (-1)^i(t^i + st^{i-1} + \cdots + s^{i-1}t + s^i)$.

1.1. Lemma.

(i) Let $B_i$ be the submatrix of $\overline{A}_i$ obtained by deleting its first and last columns. Then $\det B_i = \tau_i$ for all $i \geq 1$.

(ii) Let $S$ be an infinite set of positive integers. Suppose that either $k$ has characteristic zero or that $k$ has prime characteristic $p$ and $S$ contains infinitely many integers of the form $p^m - 2$. The $(k[s, t]-)$irreducible factors of $\{\tau_i\}_{i \in S}$ form an infinite set.
Proof. We prove the first statement by induction on \( i \). Since

\[
\det B_1 = \det (-t - s) = -t - s \quad \text{and} \quad \det B_2 = \det \begin{pmatrix} -t - s & t \\ s & -t - s \end{pmatrix} = t^2 + st + s^2,
\]

the lemma holds for \( i = 1 \) and \( i = 2 \). Assume now that \( i \geq 3 \). Expanding the determinant of \( B_i \) by its first row and applying the induction hypothesis we obtain

\[
\det B_i = (-t - s) \det B_{i-1} - st \det B_{i-2}
\]

\[
= (-1)^{i-1}(-t - s)(t^{i-1} + \cdots + s^{i-2}t + s^{i-1}) - (-1)^{i-2}st(t^{i-2} + \cdots + s^{i-3}t + s^{i-2})
\]

\[
= (-1)^i \left[ (t^i + \cdots + s^{i-2}t^2 + s^{i-1}t) + (st^{i-1} + \cdots + s^{i-1}t + s^i) - (st^{i-1} + \cdots + s^{i-2}t^2 + s^{i-1}t) \right]
\]

\[
= (-1)^i(t^i + st^{i-1} + \cdots + s^{i-1}t + s^i).
\]

We now prove the second statement. Define \( \sigma_i = t^i + t^{i-1} + \cdots + t + 1 \) and notice that it is enough to show that the set of irreducible factors of \( \{ \sigma_i \}_{i \in S} \) is infinite. Let \( \mathcal{I} \) be the set of irreducible factors of \( \{ \sigma_i \}_{i \in S} \). If \( k \) has characteristic zero consider \( \mathbb{Q}[\mathcal{I}] \supseteq \mathbb{Q} \), the splitting field of this set of irreducible factors. If \( \mathcal{I} \) is finite, \( \mathbb{Q}[\mathcal{I}] \supseteq \mathbb{Q} \) is finite extension which contains all \( m \)th roots of unity for all \( i \in S \), which is impossible.

Assume now that \( k \) has prime characteristic \( p \). Let \( \mathbb{F} \) be the algebraic closure of the prime field of \( k \). For any positive integer \( m \)

\[
\frac{d}{dt}(t^{p^{m-1}} - 1) = -1
\]

so \( \sigma_{p^m-2} = (p^{m-1} - 1)/(t - 1) \) has \( p^m - 2 \) distinct roots in \( \mathbb{F} \) and, therefore, the roots of \( \{ \sigma_s \}_{s \in S} \) form an infinite set.

1.2. Theorem. For every \( d \geq 2 \) the \( R_0 \)-module \( H^2_{R_1}(R)_{-d} \) has \( \tau_{d-1} \)-torsion. Hence \( H^2_{R_1}(R) \) has infinitely many associated primes.

Proof. For the purpose of this proof we introduce a bigrading in \( R_0 \) by declaring \( \deg(x) = (1,0) \), \( \deg(y) = (1,1) \) and \( \deg(t) = \deg(s) = (0,0) \).

We also introduce a bigrading on the free \( R_0 \)-modules \( R^0_0 \) by declaring \( \deg(x^\alpha y^\beta s^at^b e_j) = (\alpha + \beta, \beta + j) \) for all non-negative integers \( \alpha, \beta, a, b \) and all \( 1 \leq j \leq n \). Notice that \( R^0_0 \) is a bigraded \( R_0 \)-module when \( R_0 \) is equipped with the bigrading mentioned above.

Consider the \( R_0 \)-module \( \text{Coker} \, A_{d-1} \); the columns of \( A_{d-1} \) are bihomogeneous of bidegrees

\[
(2,1), (2,2), \ldots, (2,d+1).
\]

We can now consider \( \text{Coker} \, A_{d-1} \) as a \( k[s,t] \) module generated by the natural images of \( x^\alpha y^\beta e_j \) for all non-negative integers \( \alpha, \beta \) and all \( 1 \leq j \leq d - 1 \). The \( k[s,t] \)-module of relations among
these generators is generated by \( k[x, y] \)-linear combinations of the columns of \( A_{d-1} \), and since these columns are bigraded, the \( k[s, t] \)-module of relations will be bihomogeneous and we can write

\[
\text{Coker } A_{d-1} = \bigoplus_{0 \leq D, 1 \leq j} (\text{Coker } A_{d-1})_{(D, j)}.
\]

Consider the \( k[s, t] \)-module \( (\text{Coker } A_{d-1})_{(d, d)} \), the bihomogeneous component of \( \text{Coker } A_{d-1} \) of bidegree \((d, d)\). It is generated by the images of

\[
xy^{d-1}e_1, x^2y^{d-2}e_2, \ldots, x^{d-2}y^2e_{d-2}, x^{d-1}ye_{d-1}
\]

and the relations among these generators are given by \( k[s, t] \)-linear combinations of

\[
y^{d-2}c_2, xy^{d-3}c_3, \ldots, x^{d-3}y^c_{d-1}, x^{d-2}c_d
\]

where \( c_1, \ldots, c_{d+1} \) are the columns of \( A_{d-1} \). So we have

\[
(\text{Coker } A_{d-1})_{(d, d)} = \text{Coker } B_{d-1}
\]

where \( B_{d-1} \) is viewed as a \( k[s, t] \)-homomorphism \( k[s, t]^{d-1} \to k[s, t]^{d-1} \).

Using Lemma 1.1(i) we deduce that for all \( d \geq 2 \) the direct summand \( (\text{Coker } A_{d-1})_{(d, d)} \) of \( \text{Coker } A_{d-1} \) has \( \tau_{d-1} \)-torsion, and so does \( \text{Coker } A_{d-1} \) itself.

Lemma 1.1(ii) applied with \( S = \mathbb{N} \) now shows that there exist infinitely many irreducible homogeneous polynomials \( \{p_i \in k[s, t] : i \geq 1\} \) each one of them contained in some associated prime of the \( R_0 \)-module \( \oplus_{d \geq 2} \text{Coker } A_{d-1} \). Clearly, if \( i \neq j \) then any prime ideal \( P \subset R_0 \) which contains both \( p_i \) and \( p_j \) must contain both \( s \) and \( t \).

Since the localisation of \( (\text{Coker } A_{d-1})_{(d, d)} \) at \( s \) does not vanish, there exist \( P_i, P_j \in \text{Ass}_{R_0} \text{Coker } A_{d-1} \) which do not contain \( s \) and such that \( p_i \subset P_i, p_j \subset P_j \), and the previous paragraph shows that \( P_j \neq P_j \).

The second statement now follows from the fact that \( H^2_{R_+}(R) \) is \( R_0 \)-isomorphic to \( \oplus_{d \geq 2} \text{Coker } A_{d-1} \).

\[\square\]

1.3. Corollary. Let \( T \) be the localisation of \( R \) at the irrelevant maximal ideal \( m = (s, t, x, y, u, v) \).

Then \( H^2_{(u, v)T}(R) \) has infinitely many associated primes.

Proof. Since \( \tau_i \in m \) for all \( i \geq 1 \), \( H^2_{(u, v)T}(R) \cong (H^2_{(u, v)R}(R))_m \) has \( \tau_i \)-torsion for all \( i \geq 1 \).

\[\square\]

2. A connection with associated primes of Frobenius powers

In this section we apply a technique similar to the one used in section 1 to give a proof of a slightly more general statement of Theorem 12 in \[9\]. The new proof is simpler, open to generalisations and
it gives a connection between associated primes of Frobenius powers of ideals and of local cohomology modules, at least on a purely formal level.

Let \( k \) be any field, let \( S = k[x, y, s, t] \), let \( F = xy(x - y)(sx - ty) = sx^3y - (t + s)x^2y^2 + txy^3 \) and let \( R = S/F S \).

2.1. Theorem. Let \( S \) be an infinite set positive integers and suppose that either \( k \) has characteristic zero or that \( k \) has characteristic \( p \) and that \( S \) contains infinitely many powers of \( p \). The set

\[
\bigcup_{n \in S} \text{Ass}_R \left( \frac{R}{(x^n, y^n)} \right)
\]

is infinite.

Proof. We introduce a grading in \( S \) by setting \( \deg(x) = \deg(y) = 1 \) and \( \deg(s) = \deg(t) = 0 \). Since \( F \) is homogeneous, \( R \) is also graded.

Fix some \( n > 0 \) and consider the graded \( R \)-module \( T = R/(x^n, y^n) \). For each \( d > 4 \) consider \( T_d \), the degree \( d \) homogeneous component of \( T \), as a \( k[s, t] \)-module. If \( d < n \), \( T_d \) is generated by the images of \( y^d, xy^{d-1}, \ldots, x^{d-1}y, x^d \) and the relations among these generators are obtained from \( y^{d-4}F, xy^{d-5}F, \ldots, x^{d-5}yF, x^{d-4}F \). Using these generators and relations, in the given order, we write \( T_d = \text{Coker} M_d \) where

\[
M_d = \begin{pmatrix}
0 & 0 & \ldots & 0 \\
t & -t - s & t \\
- t - s & t & s & \ldots \\
s & - t - s & \ldots & t & -t - s \\
0 & 0 & \ldots & 0
\end{pmatrix}
\]

When \( d = n \), \( T_d \) is isomorphic to the cokernel of the submatrix of \( M_d \) obtained by deleting the first and last rows which correspond to the generators \( y^n, x^n \) of \( T_n \).

When \( d = n + 1 \), \( T_d \) is isomorphic to the cokernel of the submatrix of \( M_d \) obtained by deleting the first two rows and and last two rows which correspond to the generators \( y^{n+1}, xy^n, x^n y, x^{n+1} \) of \( T_{n+1} \), and the resulting submatrix is \( B_{n-2} \) defined in Lemma 1.1, the result now follows from that lemma. \( \square \)
This technique for finding associated primes of non-finitely generated graded modules and of sequences of graded modules has been applied in BKS and KS to yield further new and surprising properties of top local cohomology modules.

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