The Four Intersection-and-Difference Model for Line-Line Topological Relations

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Abstract  The description of line-line topological relations is still an unsolved issue although much effort has been done. The problem is involved in many practical applications such as spatial query, spatial analysis and cartographic generalization. To develop a sound and effective approach to describe line-line relations, it is first necessary to define the topology of an individual line, i.e., local topology. The concept of connective degree is used for the identification of topological differences in the geometric structure of a line. The general topological definition of a line is given, i.e., endpoints set and interior point set. This definition can be applied to the embedded spaces of different dimensions, whether co-dimension is equal to or larger than zero. On this basis, a generic model called the 4 intersection-and-difference is set up for the description of basic line-line topological relations, upon which a conceptual neighborhood graph is built with consideration of topological distance. It is concluded that the proposed model can represent the property of topological changes, and basic relations between line segments in $IR^1$ and $IR^2$.

Keywords  topological relations; line object; topological distance; conceptual neighborhood; co-dimension

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Introduction

The representation of spatial relations, which essentially reflect the spatial configuration between spatial objects, is one of the key issues in GIS. To a large degree, knowledge on spatial relations is used for describing some kinds of geometric constraints to spatial objects and is classified into three: topological, metric, and order. These relations have been found useful for spatial query, analysis and reasoning. In this study, we will concentrate on topological relations. It has played important roles in many aspects, such as reducing burden of geometric computation, speeding up spatial queries, and improving spatial analysis.

As early as the 1980s, the importance of spatial relations theory has already been recognized[1]. In the last decade, many researchers from the computing science and GIS fields have paid much attention to the formal description and determination of spatial relations, especially of topological relations and their applications in various domains such as GIS, spatial database, CAD/CAM systems, image databases, spatial analysis, computer vision, artificial intelligence, linguistics, cognitive science, psychology, and robot-

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ics. Considerable papers on this topic have been also published. The approaches used in these works can be classified into two categories: decomposition-based and whole-based[2]. In the former, a spatial object is represented in terms of the set of its components (e.g. boundary, interior) and spatial relations are described and determined by the combinatorial relations of those components. In the latter, a spatial object is considered a whole and spatial relations are differentiated by the interaction between these objects themselves instead of their components. To date, most models used for the identification of topological relations are built upon point-set topology[3,4]. One of the most commonly used is the 4-intersection model presented by Egenhofer and Franzosa[4], which represents topological relations by calculating the four intersections of the interior and the boundary of two regions without holes, i.e., regions with connected boundaries. According to the model, eight unique and mutually exclusive relations can be identified. To reduce much confusion for the identification of topological relations between two lines, and between lines and regions, Egenhofer and Herring expanded the 4-intersection model to a 9-intersection model[5], in which the exterior of a spatial object is also included. The 9-intersection model can distinguish 33 line-line relations and 19 line-area relations. However, such an extension has a fundamental deficiency in theory. Thus, the 9-intersection model was later modified by Chen and his collaborators, with the exterior of a spatial object replaced by its Voronoi region[6]. In addition, these vector-based models have been also implemented in raster mode through vector representation for raster cells[7].

To describe the topological equivalence of two spatial configurations, Egenhofer and Franzosa developed a set of invariants for describing the topological relations between spatial regions in detail[8]. Clementini and Di Felice developed a set of invariants for line-line topological relations[9]. Li et al. proposed a spatial algebraic model, in which multiple operators (e.g. union, intersection, difference), spatial objects and their Voronoi regions are utilized for detailed identification of topological relations[2].

In spite of these efforts and progress in this area, it is undeniable that there are still many imperfections associated with existing work. An effective and general approach for formal description and determination of topological relations between spatial objects is needed. In this paper, much attention is paid to line-line topological relations. A general method is at first proposed to formally define the local topology of an individual line in the next section.

1 Topology of an individual line

In terms of codimension, two cases are discussed. One case is that codimension is equal to zero; the other is that codimension is larger than zero. A unified method is also presented for topology of an individual line.

1.1 Local topology with codimension zero

For any object \(O\), its set-theoretic boundary (\(\partial O\)) in topology is defined to be intersection of the closure of \(O\) (\(\overline{O}\)) and the closure of the complement of \(O\) (\(\overline{O}^c\)), which can be represented as follows:

\[
\partial O = \overline{O} \cap \overline{O}^c
\]  

(1)

Thus the interior of object \(O (O^0)\) is obtained by:

\[
O^0 = O - \partial O
\]  

(2)

It is clear that \(\partial O\) and \(O^0\) form a separation of the object space. According to Eq.(1), the boundary of a line in a one-dimensional vector space (\(IR^1\) for short) is its two endpoints, while its interior is the line itself except for the two endpoints. By means of the concept of neighborhood, one can find that each of the two endpoints of a line in \(IR^1\) does not have its neighborhood, i.e., there is no such open interval of the form \((p_i - \varepsilon, p_i + \varepsilon)\) contained in a line[10]. Conversely, all other points in a line except its two endpoints have their corresponding neighborhoods, and the union of their neighborhoods forms the interior of a line. To make a clear distinction between endpoints and points in the interior from the view of topology, especially in the case when a line is embedded into a higher dimensional space (i.e., codimension larger than zero), we introduce the concept of connective degree. The connective degree is a topological invariant.

For any point \(p_i\) of a line in \(IR^1\), an intuitive method for determining its connective degree (de-
noted by \( D_n(p_i) \) is to compute the number of separated parts generated by the deletion of point \( p_i \) from a line. In this way, we can obtain that the connective degree of its endpoints (equivalent to the boundary) in a line equals 1 and that the connected degree of the other points (equivalent to the interior) in the line equals 2, as shown in Fig.1. One can see from this figure that the local topology (i.e., boundary, interior) of a line in \( \mathbb{R}^1 \) can be distinguished effectively in terms of the connective degree, and the result obtained is completely consistent with that obtained based on point set topology. We represent the local topology of an individual line in the following forms as:

\[
\partial A = \{ p_i \mid D_n(p_i) = 1, p_i \in A \} \quad (3)
\]

\[
A^0 = \{ p_i \mid D_n(p_i) = 2, p_i \in A \} \quad (4)
\]

where \( \partial A \) and \( A^0 \) denote the boundary and the interior of a line \( A \) in \( \mathbb{R}^1 \), respectively.

![Fig.1 Local topology of an individual line in \( \mathbb{R}^1 \)](image)

**1.2 Local topology with codimension larger than zero**

In the case of the codimension larger than zero, we can obtain from Eq.(1) that the point-set topological boundary of a line object is the line itself and its interior an empty set, as has been also pointed by Li et al\(^{[11]}\). In this case, only fewer topological relations can be identified based on the 4-intersection model than those with codimension zero; in other words, many distinguishable topological relations for line objects in \( \mathbb{R}^1 \) will be possibly confused when the 4-intersection model is applied to the case of \( \mathbb{R}^2 \). But Li et al. did not provide a further solution of how to identify local topological differences for a single line, and to define formally its local topology\(^{[11]}\). In the following, we still utilize the connective degree invariant to analyze and identify topological differences of the geometry of an individual line.

For a simply connected line in \( \mathbb{R}^2 \), it can be obtained that the connective degree of its two endpoints is both one, while that of all other points in the line is two. Here, the method used is similar to the previous section. Another method is to calculate the number of intersections between the neighborhood of point \( p_i \) (denoted by a small circle) and the line (see Fig.2(a)). Compared to the corresponding topological components for a line in \( \mathbb{R}^1 \), there are no changes in the connective degree. Likewise, the above-mentioned methods can be extended to a line embedded in \( \mathbb{R}^3 \), even in a higher dimensional space. For example, in \( \mathbb{R}^3 \) we can draw a small sphere around the point of interest and calculate the number of intersections between the sphere and the line to obtain the connective degree of this point (see Fig.2(b)). As a result, Eqs.(3) and (4) are a general expression of the formal definition for the topology of a line. In the following discussion, the terms endpoints and interior refer to those defined by Eqs.(3) and (4), respectively.

Further, we can differentiate other types of line objects in \( \mathbb{R}^2 \), including loops, forked lines, self-intersecting lines, and line groups. Fig.3 illustrates some types of lines with different local topologies,
where the topological invariants as connectedness and separation number are utilized and the numeric labels in vertices of each line denote the connective degree of the vertices.

![Fig.3 Different types of lines differentiated by connective degree](image)

2 Line-line topological relations in $IR^1$

2.1 An intersection-and-difference model

As shown in Fig.4, there exist only eight possible topological relations between two lines in $IR^1$ with names of the characterized topological relations: disjoint, meet, overlap, covers, contain, coveredby, containedby (or called inside) and equal. In practice, a simpler method is to enumerate those relations with corresponding spatial configurations. In theory, however, there is a question of how to describe and determine those relations. In current literature, the 4-intersection model is commonly used for identifying line-line relations in $IR^1$. However, it should be pointed out that this model is not very general. For this purpose, a new topological model is proposed for the line-line topological relations in $IR^1$, i.e.,

$$ID(A,B) = \begin{bmatrix} A^0 \cap B^0 & A-B \\ B-A & A \cap B \end{bmatrix}$$

where $A^0$ and $B^0$ denote the interior of $A$ and $B$, respectively; the values of elements in Eq.(5) are represented by the separation number of corresponding intersections or difference sets. Therefore, Eq.(5) is also termed as the intersection-and-difference model based on separation number ($ID$ model for brevity). By using the $ID$ model, the eight line-line relations can be identified similar to those obtained by the 4-intersection model. Moreover, compared to the 4-intersection model, this model has three advantages: ① it avoids operations between topological components of different dimensions (i.e. $\partial A \cap B^0$ and $A^0 \cap \partial B$); ② it can be extended to the case of $IR^2$; ③ it reflects the order of transformation among topological relations. In the following, we will discuss the third aspect in detail.

![Fig.4 Eight kinds of topological relations for lines in $IR^1$](image)

2.2 Conceptual neighborhood of line-line topological relations

As pointed out by Egenhofer and Al-Taha\cite{12}, a topological relation has the property of order. In other words, the change or transformation of topological relations is of a certain order under changes in geometry of one or two of the involved objects. Theoretically, if a topological relation changes, it should be transformed into its most similar relation. Based on this idea, we introduce the concept of topological distance as a measure of the similarity between topological relations, which can be defined as:

$$d_T(ID^1, ID^2) = \sum_{i=1}^{3} \sum_{j=1}^{3} |\chi^1_{ij} - \chi^2_{ij}|$$

where $ID^1$ and $ID^2$ denote the intersection-and-difference matrix of two relations; $\chi^1_{ij}$ and $\chi^2_{ij}$ (1 ≤ i, j ≤ 2) are respectively the separation number of corresponding elements in $ID^1$ and $ID^2$; and $d_T(ID^1, ID^2)$ is the topological distance between $ID^1$ and $ID^2$. Therefore, the topological distance between two relations can be computed and obtained according to Eq. (6). Taking disjoint and overlap as an example, and letting $ID^1$ denote a disjoint relation and $ID^2$ an overlap relation, the topological distance between meet and overlap can be calculated as follows:

$$ID^1 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \text{ and } ID^2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \text{ thus having } d_T(ID^1, ID^2) = |0-1| + |1-1| + |1-1| + |0-1| = 2$$

Similarly, all of the topological distances can be obtained, as listed in Table 1.
Table 1  Topological distances of eight line-line relations based on the ID model

| d(−,−)     | Disjoint | Meet | Overlap | Covers | Contains | Covered by | Inside | Equal |
|------------|----------|------|---------|--------|----------|------------|--------|-------|
| Disjoint   | 0        | 1    | 2       | 3      | 4        | 3          | 4      | 4     |
| Meet       | 1        | 0    | 1       | 2      | 3        | 2          | 3      | 3     |
| Overlap    | 2        | 1    | 0       | 1      | 2        | 1          | 2      | 2     |
| Covers     | 3        | 2    | 1       | 0      | 1        | 2          | 3      | 1     |
| Contains   | 2        | 3    | 2       | 1      | 0        | 3          | 4      | 2     |
| Covered by | 3        | 2    | 1       | 2      | 3        | 0          | 1      | 1     |
| Inside     | 4        | 3    | 2       | 3      | 4        | 1          | 0      | 2     |
| Equal      | 4        | 3    | 2       | 1      | 2        | 1          | 2      | 0     |

From Table 1, it can be found that topological distance based upon the ID model satisfies the following three properties:

1) \(d_T(ID^1, ID^2) = d_T(ID^2, ID^1)\);

2) \(0 \leq d_T(ID^1, ID^2) \leq 4\), if and only if \(ID^1 = ID^2\), \(d_T(ID^1, ID^2) = 0\) is found;

3) \(d_T(ID^1, ID^2) + d_T(ID^2, ID^3) \geq d_T(ID^1, ID^3)\).

In (1) - (3), \(ID^1\), \(ID^2\) and \(ID^3\) represent three topological relations, respectively. Compared to the definition of distance in metric space, it is easily observed that the topological distance has closely similar properties to Manhattan distance. In essence, the topological distance can be understood as a measure of topological relations. The properties (1), (2) and (3) can also be called symmetric property, non-minus property, and triangular inequality, respectively.

On the other hand, it has been also shown in Table 1 that the minimum distance between two different relations is 1, i.e.,

\[d_T(ID^1, ID^2) \geq 1 \quad (ID^1 \neq ID^2)\]  (7)

Further, we defined two such topological relations satisfying \(d_T(ID^1, ID^2) = 1\) as neighborhood relations. For example, disjoint and meet and meet and overlap are two pairs of neighborhood relations. If all pairs of neighborhood relations are connected, a conceptual neighborhood graph will be formed as shown in Fig.5. It can be easily found that the conceptual neighborhood graph is consistent with the process of continuous topological changes.

3 Line-line topological relations in \(IR^2\)

For two line segments in \(IR^2\), there are eleven kinds of topological relations according to the 4 intersection-and-difference model, consistent with results obtained by enumeration. Fig.6 shows three additional cases besides the eight line-line relations in \(IR^1\).

For two lines in \(IR^2\), the possible topological relations are infinite in theory, because two lines may have any number of intersections and various types of intersection. This can be embodied by the ranges of values of elements in the 4 intersection-and-difference model. In \(IR^1\), they are:
\[0 \leq \chi(A^0 \cap B^0) \leq \chi(A \cap B) \leq 1\]
\[0 \leq \chi(A - B), \chi(B - A) \leq 2\]  

(8)

In \(IR^2\), the ranges of values are changed as
\[0 \leq \chi(A^0 \cap B^0) \leq \chi(A \cap B) \leq +\infty\]
\[0 \leq \chi(A - B), \chi(B - A) \leq +\infty\]  

(9)

Accordingly, the 4 intersection-and-difference model is able to describe an infinite number of topological relations of two lines in \(IR^2\). However, in practice, the easiest spatial query asks whether two lines intersect or not, how many times they intersect, whether they have a common piece, or even into how many parts one line is separated by another. All these queries indicate that the spatial configuration consisting of two lines should satisfy one or more invariants. Based on this purpose, a hierarchical representation of line-line topological relations in \(IR^2\) is very necessary. This is also our future work.

4 Conclusions

The formalization of local topology for an individual object is a prerequisite for description of topological relations between spatial objects. Since topological properties for spatial objects of different dimensions may vary as the dimension of the embedded space, this paper proposes to utilize connective degree as an invariant for discrimination and description of the local topology of an individual line, and a unified definition is given. This definition is suitable for the cases whether codimension is equal to or larger than zero.

A general model, the 4 intersection and difference, is developed for a full description of line-line topological relations in \(IR^1\). This model can effectively represent the partial order property of such relations, i.e., the order of topological relations transformation. In addition, this model can be extended to the case of line-line topological relations in \(IR^2\).

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