Towards a Canonical Formalism of Field Theory on Discrete Spacetime

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It is shown that the difficulties in formulating the quantum field theory on discrete spacetime appear already in classical dynamics of one degree of freedom on discrete time. The difference equation of motion which maintains a conserved quantity like energy has a very restricted form that is not probably derived by the least action principle. On the other hand, the classical dynamics is possible to be canonically formulated and quantized, if the equation is derived from an action. The difficulties come mainly from this incompatibility of the conserved quantity and the action principle. We formulate a quantum field theory canonically on discrete spacetime in the case where the field equation is derived from an action, though there may be no exactly conserved quantity. It may, however, be expected that a conserved quantity exists for a low "energy" region.

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I. INTRODUCTION

Since the classical dynamics was established by Newton, space and time have been considered to be continuous. In the theory the existence of space and time is an ad hoc assumption and any physical object is supposed to be in this spacetime. The location of the object is described by continuous coordinates taken in this spacetime which satisfy a differential equation. The tradition to describe physical phenomena in the continuous spacetime has been inherited up to the contemporary physics and succeeded very much in almost all area. However, when it was applied to the field theory, it was confronted with the difficulty of infinity. In the quantum electrodynamics the difficulty is avoided by the renormalization theory. The theory is so beautiful and attractive that many people considered the renormalizability as a fundamental principle of the field theory. However, there are still some physicists who are not totally satisfied with this theory. Dirac, for example, mentioned in a lecture [1] in 1968 as follows:

"But still one is using working rules and not regular mathematics. Most theoretical physicists nowadays appear to be satisfied with this situation, but I am not. I believe that theoretical physics has gone on the wrong track with such developments and one should not be complacent about it." "Worrying over this point (=difficulty of infinity) may lead to an important advance."

Originally the spacetime in physics is an ad hoc assumption and its continuity has never been confirmed by experiments. Furthermore, the existence of spacetime is assumed independently of the existence of matter. However, the spacetime is impossible to exist without matter. How do we measure a distance in space without matter? The same question holds for time. The space is recognized by disposition of matter and the time is recognized through the change of this disposition.

When we accept the above argument, we easily understand that spacetime and matter are inseparable and can not exist independently. In fact the existence of matter means at the same time the existence of spacetime. The particle physics of present-days seems to
endeavor at finding the fundamental or minimal constituents of matter. If there exists the minimum constituent of the elementary particles, there should exist the minimum element of spacetime. This means that we have no means to recognize a distance much smaller than the length of this constituent. Considering in this way we may easily accept the existence of minimum length in spacetime. This length is probably in the same order as the size of minimum constituents. This is the reason, why we introduced the discrete spacetime.

The experimental researches to examine the continuity of spacetime are rare. Hawking and Ellis [2] wrote as follows:

"So far this continuity has been established for distance down to about $10^{-15}$ cm by experiments on pion scattering (Foley et al. (1967)). It may be difficult to extend this to much smaller lengths as to do so would require a particle of such high energy that several other particles might be created and confuse the experiment. Thus it may be that a manifold model for space-time is inappropriate for distances less than $10^{-15}$ cm and that we should use theories in which space-time has some other structure on this scale."

The above suggestion is along the same line as the lecture given by Heisenberg at Cavendish Laboratory, Cambridge in 1949 [3].

"Any future theory of the elementary particles must contain, besides the fundamental constants $\hbar$ and $c$, a third fundamental constant of the dimensions of a length or a mass. This follows simply from the fact that, on account of purely dimensional reasons, one cannot derive the mass of an elementary particle from the constants $\hbar$ and $c$. The actual mass of the main elementary particles suggests that this new constant may be considered as a length $l$ of the order of magnitude $l \sim 10^{-13}$ cm. If the future theory contains such a constant in whatever form, it is natural to assume that the usual correspondence between the classical wave description and its quantum-theoretical analogue only holds for very much greater than $l$, but fails in the region of smaller distances."

We have introduced the discrete spacetime and tried to formulate a canonical quantum field theory on this spacetime in the previous papers [4]. There are also some approaches in the same direction by other physicists [5]. We have seen from the form of propagators
that there occurs no divergence difficulty at all. In the case of fields with interaction we have not yet succeeded to formulate the quantum theory and to find a conserved quantity like energy that plays a role of time developing generator. Therefore, the purpose of this paper is to find the conserved quantity and the time developing operator and to quantize the field canonically. Here we should emphasize our standpoint that the field theory on discrete spacetime is never an approximation of a continuous theory but a true theory. Conversely we consider that the continuous theory is rather an approximation of the discrete theory. At this point the theory on discrete spacetime is basically different from the lattice gauge theory which is considered as an approximation of the continuous theory. While the lattice spacing in the lattice gauge theory is expected to be set to zero at the end, the fundamental length in the discrete spacetime is never set to zero but remains finite.

In the theory on continuous spacetime Lagrangian is invariant under infinitesimal translations and from Noether’s theorem the corresponding conserved quantities are easily obtained. The energy and momenta thus obtained are the translation generators of time and space respectively. In contrast to this, Lagrangian on discrete spacetime is invariant under the translation of a finite distance. In this case Noether’s theorem is not applicable and we have no procedure to get conserved quantities. Moreover, we don’t even know whether or not a conserved quantity does exist. In this sense Lagrangian might have no important meaning. To get a conserved quantity and to find an operator of translation are different tasks. The operator of time translation is indispensable for canonical formulation. At the same time conserved energy is necessary for physical interpretation of the theory. There is of course no verification that the generator and the energy are identical to each other.

In the next section we show that the above mentioned is not only the problem of field theory, but also the problem of classical dynamics on discrete time. There exists no conserved quantity, if time is discretized simply in an equation of motion of classical dynamics. We begin with an one-dimensional Newtonian equation and the time derivatives are replaced by time differences in naïve way. The Hamiltonian defined following to the Lagrangian formalism of continuous time is not conserved. The equation is interpreted as a periodical
map on the phase space of coordinate and momentum. If there exists a conserved quantity, the map remains on a certain closed curve. However, in our case the map behaves chaotically, if the coordinate or momentum becomes large. In order to get a conserved quantity the time should be carefully discretized. Furthermore, for the sake of physical interpretability the area in the phase space should be conserved. This gives another restriction on the way of discretization.

Next we consider the case where the equation of motion is derived from an action or Lagrangian. The time translation operator is possible to define using Poisson bracket. However, it is still unknown whether the conserved energy or the time translation generator exists and is derived from the operator. Only in special cases the conserved quantity does exist and it is identical to the generator. The advantage of this method is the easiness of quantization, because it is enough to replace the Poisson brackets by commutators.

In the section III we show an attempt at formulating a quantum field theory with interaction on discrete spacetime. In the free field case it is easily seen that the time translation generator is identical to the conserved energy operator. In the case with special interaction the conserved energy is obtained, but it is not the time translation generator in general. Conversely we can write down the time translation operator, but we obtain neither generator nor energy in a compact form. If we start with the translation operator, we can quantize a scalar field canonically and have a method to calculate S-matrix perturbatively. In this sense we have the quantum field theory on discrete spacetime, though we have no conserved quantity.

In the section IV we show that our canonical formalism is identical with path integral method on discrete spacetime. The relation between quantities on Euclidean and Minkowski spacetimes is also discussed. Contrary to continuum theories, they are not simply connected to each other by analytic continuation.

The theory on discrete spacetime may seem a kind of cutoff theory and hence the relativistic non-invariance comes to question. At the end of this introduction we cite again the words of Dirac.
"The relativistic invariance of the theory is then destroyed. This is a pity, but it is a lesser evil than a departure from logic would be."

II. SYSTEM WITH ONE DEGREE OF FREEDOM ON DISCRETE TIME

Though our final aim is to construct a canonical formalism of field theories on discrete space-time, almost all difficulties exist already in simple systems with few degrees of freedom. In this section, we will examine the systems with only one degree of freedom on discrete time and see what problems arise.

First, we will see that it is difficult to define Hamiltonian by naïve discretization of the time in the usual canonical formalism. When we want to discretize the time, there might be many possible ways, but the simplest one is probably to replace time derivatives by time differences. If the minimum unit of time is \( \tau \), the difference operators are defined by

\[
\Delta^R x(t) \equiv \frac{1}{\tau} [x(t + \tau) - x(t)], \\
\Delta^L x(t) \equiv \frac{1}{\tau} [x(t) - x(t - \tau)].
\]

Of course there are many operators other than \( \Delta^R \) and \( \Delta^L \) that tend to the differential operator \( d/dt \) as \( \tau \to 0 \). However, we restrict ourselves to consider only the above two for simplicity.

Now we assume the Lagrangian:

\[
L(t) = \frac{1}{2}[\Delta^R x(t)]^2 - V[x(t)],
\]

where \( V[x(t)] \) is potential and a function of \( x(t) \). Clearly the action:

\[
S = \sum_t L(t)
\]

is invariant under the finite translation:

\[
t \longrightarrow t + \tau.
\]
The first problem is whether there exists a conserved quantity corresponding to this invariance.

The equation of motion is obtained so that the action Eq. (2.3) takes the least value. That is

\[
\Delta^R \Delta^L x(t) = -\frac{\partial V[x(t)]}{\partial x(t)} \tag{2.5a}
\]

or

\[
\frac{1}{\tau^2} \{x(t + \tau) - 2x(t) + x(t - \tau)\} = -\frac{\partial V[x(t)]}{\partial x(t)}. \tag{2.5b}
\]

The momentum conjugate to \( x(t) \) is

\[
p(t) = \frac{\partial L(t)}{\partial \Delta^R x(t)} = \Delta^R x(t) + \frac{\tau}{2} \frac{\partial V[x(t)]}{\partial x(t)}. \tag{2.6}
\]

The second term comes from the fact that \( x(t) \) and \( \Delta^R x(t) \) are not independent [1]:

\[
x(t) = \frac{1}{2} \{x(t + \tau) + x(t)\} - \frac{1}{2} \{x(t + \tau) - x(t)\}. \tag{2.7}
\]

Hence we have

\[
\frac{\partial x(t)}{\partial \Delta^R x(t)} = -\frac{\tau}{2}. \tag{2.8}
\]

According to the usual Lagrangian formalism the conserved quantity is Hamiltonian defined by

\[
H = p(t) \Delta^R x(t) - L(t)
\]

\[
= \frac{1}{2} [\Delta^R x(t)]^2 + V[x(t)] + \frac{\tau}{2} \frac{\partial V[x(t)]}{\partial x(t)} \Delta^R x(t). \tag{2.9}
\]

Let us see what happens if the potential is given by \( V[x(t)] = \Lambda x^n(t)/n \). The equation of motion is

\[
\Delta^R \Delta^L x(t) = -\Lambda x^{n-1}(t),
\]

and Hamiltonian defined by Eq. (2.9) is given by
\[ H = \frac{1}{2} \left[ \Delta^R x(t) \right]^2 + \frac{1}{n} \Lambda x^n(t) + \frac{1}{2} \Lambda x^{n-1}(t) \{ x(t + \tau) - x(t) \} \]. \quad (2.10) 

It is readily verified that the above Hamiltonian is conserved only when \( n = 1 \) (uniform gravitation) or \( n = 2 \) (harmonic oscillator). Thus we conclude that the Hamiltonian Eq. (2.9) obtained by the naïve discretization is not conserved in general:

\[ \Delta^L H \neq 0. \quad (2.11) \]

Furthermore, the above Hamiltonian cannot be regarded as a time-developing generator.

An interesting question would be whether there exists a conserved quantity corresponding to energy in the case of nonlinear forces, and if it is the case, how it is related to the time-developing generator. We have two different approaches to study these problems. The first one is described in Subsec. [II A], where we show that the equation of motion can be discretized so that the system has an energy-like conserved quantity, though it is not generally the time-developing generator. In the second approach given in Subsec. [II B], we can define a time-developing operator and construct a canonical formalism. Although the construction is rather formal, this method is useful if the time scale of the motion is much larger than the minimum unit of time \( \tau \). Unfortunately the two approaches are not related to each other except for only special cases.

A. Conserved quantity

We assume the equation of motion:

\[ \Delta^R \Delta^L x(t) = f(t). \quad (2.12) \]

When the equation is derived by the principle of least action, it is the same as Eq. (2.5a). We can easily verify that the following quantity:

\[ \tilde{H} = \frac{1}{2} \left[ \Delta^R x(t) \right]^2 - \frac{1}{2} \sum_{t' = t_0}^{t} \{ x(t' + \tau) - x(t' - \tau) \} f(t') \quad (2.13) \]

is conserved:
\[ \Delta^L \tilde{H} = 0. \] (2.14)

It must be noticed here that \( \tilde{H} \) includes the summation of time-dependent quantities over all times from a certain past \( t_0 \) to the present \( t \).

When the force is given by \( f(t) = -\Lambda x^{n-1}(t) \) as before, \( \tilde{H} \) takes the form:

\[
\tilde{H} = \frac{1}{2} [\Delta^R x(t)]^2 + \frac{1}{2} \Lambda \sum_{t'=t_0}^{t} \{x(t' + \tau) - x(t' - \tau)\} x^{n-1}(t').
\]

If \( n = 1 \) or \( n = 2 \), the terms in the summation cancel each other and \( \tilde{H} \) is reduced to \( H \) obtained by the naïve discretization. We, however, see that the terms in summation (2.13) do not cancel each other in general. Because the conserved quantity should be expressed by \( x(t) \) and \( x(t + \tau) \) or equivalently \( x(t) \) and \( p(t) \):

\[
\tilde{H} = \tilde{H}[x(t), x(t + \tau)] \quad \text{or} \quad \tilde{H}[x(t), p(t)],
\]

\( \tilde{H} \) cannot generally be called a conserved quantity.

So far, we discretized only the term of time derivative in the equation of motion on continuous time. However, if we consider our guiding principle that the discretized equation should return to the original differential equation of motion as \( \tau \to 0 \), we may not disregard the discretization of the interaction term. That is, if the force term \( f(t) \) satisfies

\[
\lim_{\tau \to 0} f(t) = -\frac{\partial V[x(t)]}{\partial x(t)},
\]

(2.16)

the difference equation (2.12) tends to the original differential equation:

\[
\frac{d^2 x}{dt^2} = -\frac{\partial V}{\partial x}.
\]

(2.17)

Therefore, we may discretize the interaction term so that the obtained equation has a conserved quantity.

First we suppose the following force:

\[
f(t) = \frac{V(t + \tau) - V(t - \tau)}{x(t + \tau) - x(t - \tau)}
\]

(2.18)
where we abbreviated $V[x(t)]$ to $V(t)$ for simplicity. The force satisfies obviously the above condition (2.16). However, the equation of motion:

$$
\Delta^R \Delta^L x(t) = -\frac{V(t + \tau) - V(t - \tau)}{x(t + \tau) - x(t - \tau)},
$$

(2.19)
does not seem to be derived from Lagrangian by the principle of least action as usual. Now we substitute Eq. (2.18) in Eq. (2.13) and we have

$$
\tilde{H} = \frac{1}{2} [\Delta^R x(t)]^2 + \frac{1}{2} \{V(t + \tau) + V(t)\} - \frac{1}{2} \{V(t_0) + V(t_0 - \tau)\}.
$$

(2.20)

Since $\tilde{H}$ includes only $x(t)$ and $x(t + \tau)$, or satisfies the condition Eq. (2.13), we may call it the conserved quantity.

There exist other possibilities. If the potential is

$$
V(t) = \frac{\Lambda}{2m} x^{2m}(t), \quad (m : \text{integer})
$$

(2.21)

we assume

$$
f(t) = -\frac{\Lambda}{m} \cdot \frac{x^m(t + \tau) - x^m(t - \tau)}{x(t + \tau) - x(t - \tau)} x^m(t),
$$

(2.22)

and then we have

$$
\tilde{H} = \frac{1}{2} [\Delta^R x(t)]^2 + \frac{\Lambda}{2m} x^m(t + \tau) x^m(t) - \frac{\Lambda}{2m} x^m(t_0) x^m(t_0 - \tau).
$$

(2.23)

In the case where the potential is

$$
V(t) = \frac{\Lambda}{2m + 1} x^{2m+1}(t),
$$

(2.24)

substituting

$$
f(t) = -\frac{\Lambda}{2m + 1} \left\{ \frac{x^m(t + \tau) - x^m(t - \tau)}{x(t + \tau) - x(t - \tau)} x^{m+1}(t) \\
- \frac{x^{m+1}(t + \tau) - x^{m+1}(t - \tau)}{x(t + \tau) - x(t - \tau)} x^m(t) \right\}
$$

(2.25)

we have
\[
\hat{H} = \frac{1}{2} [\Delta^R x(t)]^2 + \frac{\Lambda}{2(2m + 1)} x^m(t + \tau)x^m(t)\{x(t + \tau) + x(t)\} \\
- \frac{\Lambda}{2(2m + 1)} x^m(t_0)x^m(t_0 - \tau)\{x(t_0) + x(t_0 - \tau)\}. \tag{2.26}
\]

Thus we found that for an arbitrary potential the equation of motion can be discretized so that the system still has an energy-like conserved quantity. As we will see in section III, the similar discretization method can be applied to field theories.

### B. Time developing operator

The approach mentioned in Subsec. [IIA](#) does not generally give the canonical conjugate momentum which, together with the coordinate \(x\), defines an area preserving dynamics. Furthermore, it is hard to establish relation between the conserved quantity obtained in this way and the time developing generator of the system. The problems will be more serious, when we try to quantize the system canonically. We speculate that the difficulties arise from the fact that the newly discretized equation of motion can not generally be derived from the least action principle. Keeping this in mind, we next attempt to define the time developing operator in the case where the equation of motion is derived from the action (2.3). Once the time developing generator is defined, it is clearly a conserved quantity.

We begin with the equation of motion Eq. (2.5a), which is derived from the na"ively discretized action Eq. (2.3). Defining the canonical momentum \(p(t)\) conjugate to \(x(t)\) as

\[
p(t) = \Delta^R x(t), \tag{2.27}
\]

we look for the conserved quantity in a power series of \(\tau\):

\[
H_\tau = \frac{1}{2} p^2(t) + V[x(t)] + \sum_{n=1}^{\infty} \tau^n H_n[x(t), p(t)]. \tag{2.28}
\]

Of course \(H_\tau\) should tend to the Hamiltonian of continuous time for \(\tau \to 0\).

Now we consider the time developing operator of the form:

\[
\Omega(t + \tau) = U^{-1}\Omega(t), \tag{2.29}
\]
\[ U^{-1} \equiv e^{-\tau \{ \frac{1}{2} p^2(t), \ P, B \}} e^{-\tau \{ V[x(t)], \ P, B \}}, \quad (2.30) \]

where
\[ e^{-\tau \{ P, B, \Omega \}} = \sum_{n=0}^{\infty} \frac{(-\tau)^n}{n!} \{ P, \{ P, \{ \ldots \{ P, \Omega \} \ldots \} \} \} \quad (2.31) \]

and \( \{ P, Q \}_{P,B} \) means Poisson bracket:
\[ \{ P, Q \}_{P,B} = \frac{\partial P}{\partial x} \frac{\partial Q}{\partial p} - \frac{\partial Q}{\partial x} \frac{\partial P}{\partial p}. \quad (2.32) \]

It should be noted that we defined \( U^{-1} \) rather than \( U \) as in Eq. (2.29) so that \( U \) will be the usual time developing operator after quantization. We see easily
\[ x(t + \tau) = U^{-1} x(t) = x(t) + \tau p(t), \quad (2.33a) \]
\[ p(t + \tau) = U^{-1} p(t) = p(t) - \tau \frac{\partial V[x(t + \tau)]}{\partial x(t + \tau)}. \quad (2.33b) \]

It is clear that the above equations reproduce the original equation of motion Eq. (2.5a).

The question is whether the operator \( U \) can be expressed in the form:
\[ U^{-1} = e^{-\tau \{ H_\tau, \ P, B \}}. \quad (2.34) \]

If it is possible, \( H_\tau \) is a conserved quantity, since
\[ H_\tau(x(t + \tau), p(t + \tau)) = U^{-1} H_\tau(x(t), p(t)) = H_\tau(x(t), p(t)). \quad (2.35) \]

Using Hausdorf’s formula:
\[ e^{\tau X} e^{\tau Y} = e^{\tau (X+Y)+\{\tau^2/2\}X,Y} + \ldots, \quad (2.36) \]

we formally obtain
\[ H_\tau = \frac{1}{2} p^2(t) + V[x(t)] + \frac{\tau}{2} \frac{\partial}{\partial t} V'[x(t)] + \frac{\tau^2}{12} \left[ \frac{\partial^2}{\partial t^2} V''[x(t)] + \frac{1}{2} \frac{\partial^2}{\partial t^2} V'[x(t)] \right] + O(\tau^3). \quad (2.37) \]

In fact this quantity is of the form of \( \tau \)-expansion in Eq. (2.28). The same approach was also studied by Yoshida [7] in the context of numerical integration method for continuous equations of motion.
In the case of the uniform gravitational potential, the power series in $\tau$ ends at the second order as follows:

$$H_\tau = \frac{1}{2} p^2(t) + gx(t) + \frac{\tau}{2} gp(t) + \frac{\tau^2}{12} g^2$$

$$= \frac{1}{2} [\Delta^R x(t)]^2 + \frac{1}{2} g \{x(t + \tau) + x(t)\} + \frac{\tau^2}{12} g^2.$$

This agrees with $H$ and $\tilde{H}$ obtained previously, except for a constant term.

When the potential is harmonic, this series in $H_\tau$ again takes the closed form:

$$H_\tau = K(\tau) \{p^2(t) + \omega^2 x^2(t) + \tau \omega^2 p(t)x(t)\}$$

$$= K(\tau) \{[\Delta^R x(t)]^2 + \omega^2 x(t)x(t + \tau)\},$$

with

$$K(\tau) = \frac{1}{\omega \tau \sqrt{4 - (\omega \tau)^2}} \cos^{-1}(1 - \frac{(\omega \tau)^2}{2}).$$

This also agrees with the conserved quantity obtained before, except for a factor. In passing we see

$$\lim_{\tau \to 0} K(\tau) = \frac{1}{2},$$

thus $H_\tau$ recovers the harmonic hamiltonian in the continuous time as $\tau$ goes to 0.

In the case of nonlinear potential we cannot have any explicit form of $H_\tau$. However, it does not mean non-convergence of the series. As an example we study numerically the nonlinear iterative map Eqs. (2.33a,b) with the potential $V(x) = \Lambda x^4/4$. In Fig. 1 we plot points generated by the map in terms of scaled variables $\tilde{x} = \tau \sqrt{\Lambda} x$ and $\tilde{p} = \tau^2 \sqrt{\Lambda} p$. For initial values, we fix $\tilde{p}(0)$ to be 0 and take several different $\tilde{x}(0)$'s. We see that for relatively small initial values of $\tilde{x}(0)$ the map draws a smooth curve, KAM-curve, whereas the map shows chaotic behaviour for larger initial values of $\tilde{x}(0)$. The smooth curve should be defined by a function of $x(t)$ and $p(t)$. This means there exists a conserved quantity, which is presumably $H_\tau$. In conclusion we may consider that the expansion Eq. (2.37) converges for small $\tilde{x}(t)$ and $\tilde{p}(t)$.
As we have seen in Subsec. II A, we can discretize the equation of motion so that the system has a conserved quantity, but in this case it is not easy to define the canonical variables and construct the time developing operator. On the other hand, in the second approach (Subsec. II B), one can construct the time developing operator in terms of canonical variables and obtain the formally conserved Hamiltonian \( H_\tau \). Although we cannot say much about the convergence of \( H_\tau \), we speculate from the example that the series converge, if the minimum unit of time \( \tau \) is much less than the time scale of the motion considered. Assuming the convergence, we proceed to quantize the dynamics in the second approach. The quantization is straightforward: the canonical variables \( x \) and \( p \) are replaced by the operators \( \hat{x} \) and \( \hat{p} \) satisfying the usual canonical commutation relation \([\hat{x}, \hat{p}] = i\). Using the quantized time-developing operator

\[
\hat{U} = e^{-i\tau V(\hat{x})}e^{-i\tau \hat{p}^2/2} = e^{-i\tau H_\tau(\hat{x}, \hat{p})},
\]

we easily obtain

\[
\hat{x}(t + \tau) = \hat{U}^{-1}\hat{x}(t)\hat{U} = \hat{x}(t) + \tau \hat{p}(t), \tag{2.39a}
\]

\[
\hat{p}(t + \tau) = \hat{U}^{-1}\hat{p}(t)\hat{U} = \hat{p}(t) - \tau \frac{\partial V[\hat{x}(t + \tau)]}{\partial \hat{x}(t + \tau)}. \tag{2.39b}
\]

which have the same form as the classical equations of motion. In the Schrödinger representation, the wave function \( \psi_S(t) \) develops in time according to the following integrated form of the Schrödinger equation:

\[
\psi_S(t + \tau) = \hat{U}\psi_S(t). \tag{2.40}
\]

Since the equation of motion is derived from the action in this case, we could quantize the system by the use of the path integral method. In the last section [V], we will discuss how our canonical quantization is related to the path integral.
III. FIELD THEORY

In this section we consider the formulation of field theory on discrete spacetime. The field theory is not much different from the classical dynamics except for that it is a multi-freedom system, because every point of discrete spacetime has a dynamical freedom. Hence we refer to the foregoing section and consider a scalar field.

We begin with the following difference equation, which is obtained by replacing Klein-Gordon operator with a naive difference operator:

$$\sum_{\mu, \nu=0}^{3} \eta^{\mu\nu} \Delta^R_{\mu} \Delta^L_{\nu} \phi (x) = G (\phi (x)),$$  \hspace{1cm} (3.1)

where

$$\Delta^R_{\mu} \phi (x) = \frac{1}{\sigma (\mu)} \left[ \phi (x + \hat{\sigma} (\mu)) - \phi (x) \right],$$  \hspace{1cm} (3.2a)

$$\Delta^L_{\mu} \phi (x) = \frac{1}{\sigma (\mu)} \left[ \phi (x) - \phi (x - \hat{\sigma} (\mu)) \right].$$  \hspace{1cm} (3.2b)

$\hat{\sigma} (\mu)$ is a vector directing to $\mu$-axis with length $\sigma (\mu)$. We assume for simplicity

$$\sigma (0) = \tau, \quad \sigma (i) = \sigma \quad (i = 1, 2, 3).$$  \hspace{1cm} (3.3)

The spacetime points are then expressed by

$$x = \sum_{\mu=0}^{3} \hat{\sigma} (\mu) n_{\mu} \quad (n_{\mu} : \text{integers}).$$  \hspace{1cm} (3.4)

We are considering Minkowski spacetime and the metric tensor is $\eta^{\mu\nu} = (1, -1, -1, -1)$. Equation (3.1) is then written as

$$\frac{1}{\tau^2} \left[ \phi \left( x^0 + \tau, x \right) - 2\phi \left( x^0, x \right) + \phi \left( x^0 - \tau, x \right) \right] + \frac{1}{\sigma^2} \sum_{i=1}^{3} \left[ \phi \left( x^0, x + \hat{\sigma} (i) \right) - 2\phi \left( x^0, x \right) + \phi \left( x^0, x - \hat{\sigma} (i) \right) \right] + G (\phi (x)).$$  \hspace{1cm} (3.5)

$G (\phi)$ represents the mass term and interaction terms. In the continuous spacetime limit or in the continuous approximation $G (\phi)$ agrees with what is derived from the interaction Lagrangian or potential $V (\phi)$:

$$\lim_{\sigma, \tau \to 0} G (\phi) = -\frac{\partial V}{\partial \phi}.$$  \hspace{1cm} (3.6)
A. Free field and canonical quantization

In the field theory on continuous spacetime there are two ways of quantization, that is, path integral method and canonical quantization. If one wants to quantize a field by path integral, it is necessary to have the Lagrangian or action. Equation (3.1) or (3.5) is not always derived from Lagrangian, because $G(\phi)$ is not necessarily derived from a potential $V(\phi)$:

$$G(\phi) \neq -\frac{\partial V}{\partial \phi}.$$  

The case where $G(\phi) = -\frac{\partial V}{\partial \phi}$ will be considered in Subsec. III C.

Now we assume

$$G(\phi) = -m^2 \phi,$$  

that corresponds to a free field with mass $m$. The field equation is

$$\sum_{\mu,\nu=0}^{3} \eta^{\mu\nu} \Delta^R_\mu \Delta^L_\nu \phi(x) + m^2 \phi(x) = 0.$$  

This equation is identical with the following canonical equation:

$$\phi(x^0 + \tau, \mathbf{x}) = \phi(x^0, \mathbf{x}) + \tau \pi(x^0, \mathbf{x}),$$  

$$\pi(x^0 + \tau, \mathbf{x}) = \pi(x^0, \mathbf{x}) - \tau \left[ m^2 \phi(x^0 + \tau, \mathbf{x}) - \sum_{i=1}^{3} \Delta^R_i \Delta^L_i \phi(x^0 + \tau, \mathbf{x}) \right].$$

If we regard $\pi(x^0, \mathbf{x})$ as the conjugate to $\phi(x^0, \mathbf{x})$, Poisson bracket is

$$\{ \phi(x^0, \mathbf{x}), \pi(x^0, \mathbf{x'}) \}_{P.B.} = \delta_{x,x'}.$$  

Using Equations (3.9a) and (3.9b), we easily verify

$$\{ \phi(x^0 + \tau, \mathbf{x}), \pi(x^0 + \tau, \mathbf{x'}) \}_{P.B.} = \delta_{x,x'}.$$  

$$= \sum_{y} \left[ \frac{\partial \phi(x^0 + \tau, \mathbf{x})}{\partial \phi(x^0, \mathbf{y})} \frac{\partial \pi(x^0 + \tau, \mathbf{x'})}{\partial \pi(x^0, \mathbf{y})} - \frac{\partial \phi(x^0 + \tau, \mathbf{x})}{\partial \pi(x^0, \mathbf{y})} \frac{\partial \pi(x^0 + \tau, \mathbf{x'})}{\partial \phi(x^0, \mathbf{y})} \right] = \delta_{x,x'}.$$  

16
This means that Poisson bracket is conserved. The equation (3.8) is easily solved to give
\[
\phi(x) = \left(\frac{\sigma}{2\pi}\right)^\frac{3}{2} \int_R d^3k \sqrt{\frac{\tau}{2 \sin \tau \omega}} \left[ a(k) e^{-i(\omega x^0 - k \cdot x)} + \text{c.c.} \right],
\]
where \( \omega \) is defined by
\[
\frac{1}{\tau^2 \sin^2 \frac{\tau \omega}{2}} = \frac{1}{\sigma^2} \sum_{i=1}^3 \sin^2 \frac{\sigma k_i}{2} + \frac{m^2}{4},
\]
and \( a(k) \) is an arbitrary function of \( k \). The domain of integration \( R \) is
\[
-\frac{\pi}{\sigma} \leq k_i \leq \frac{\pi}{\sigma} \quad (i = 1, 2, 3).
\]
Using Eq. (3.9a), \( \pi(x) \) is also expressed by \( a(k) \) and \( a^*(k) \). From Eq. (3.10) we have Poisson bracket of \( a(k) \) and \( a^*(k) \):
\[
\{ a(k), a^*(k') \}_{\text{P.B.}} = -i\delta(k - k').
\]
Now we look for the time developing generator \( H_0 \), which satisfies
\[
\phi(x^0 + \tau, x) = e^{-\tau H_0} \phi(x^0, x).
\]
The notation of the above equation is the same as Eq. (2.32). This means
\[
e^{-\tau H_0} \{ \text{P.B.} a(k) \} = e^{-i\omega \tau} a(k),
\]
\[
e^{-\tau H_0} \{ \text{P.B.} a^*(k) \} = e^{i\omega \tau} a^*(k).
\]
Therefore, we have
\[
H_0 = \frac{1}{2} \int_R d^3k \omega \left[ a(k) a^*(k) + a^*(k) a(k) \right],
\]
or using \( \phi(x) \) and \( \pi(x) \),
\[
H_0 = \left(\frac{\sigma}{2\pi}\right)^3 \sum_{x,x'} \int_R d^3k \cos k \cdot (x - x')
\times \left[ \frac{\omega \tau}{4 \tan \frac{\omega \tau}{2}} \pi(x^0, x) \pi(x^0, x') + \frac{\sin \omega \tau}{2 \omega \tau} \omega^2 \phi(x^0, x) \phi(x^0, x') - \frac{\omega \sin \omega \tau}{4} \{ \phi(x^0, x) \pi(x^0, x') + \pi(x^0, x') \phi(x^0, x) \} \right].
\]
This $H^0$ is rather complicated, but the time developing operator $U_0$ is expressed in a more simplified form:

$$U_0^{-1} = e^{-\tau(H_0, \cdot)_{\text{P.B.}}}$$

$$= \exp \left\{ -\tau \sum_x \frac{1}{2} \pi^2 \left( x^0, x \right) \right\}_{\text{P.B.}}$$

$$\times \exp \left\{ -\tau \sum_x \left[ \frac{1}{2} \sum_{i=1}^3 \left( \Delta_i^R \phi \left( x^0, x \right) \right)^2 + \frac{1}{2} m^2 \phi^2 \left( x^0, x \right) \right] \right\}_{\text{P.B.}}. \quad (3.19)$$

As we have seen above, the conserved energy and the time developing generator are identical in the case of free field. Therefore, the field is easily quantized by replacing Poisson bracket with commutation relations, e.g.

$$\{ A, B \}_{\text{P.B.}} \rightarrow \frac{1}{i} [ A, B ] = \frac{1}{i} (AB - BA). \quad (3.20)$$

The time developing operator is

$$U_0 (\phi, \pi) = e^{-i\tau H_0}$$

$$= \exp \left[ -i \tau \sum_x \left\{ \frac{1}{2} \sum_{i=1}^3 \left( \Delta_i^R \phi \left( x^0, x \right) \right)^2 + \frac{1}{2} m^2 \phi^2 \right\} \right] \exp \left[ -i \tau \sum_x \frac{1}{2} \pi^2 \right], \quad (3.21)$$

with

$$\phi \left( x^0 + \tau, x \right) = U_0^{-1} (\phi, \pi) \phi \left( x^0, x \right) U_0 (\phi, \pi). \quad (3.22)$$

Feynman propagator of the fields \[4\] is

$$\langle 0 \mid T (\phi (x) \phi (x')) \mid 0 \rangle = \left( \frac{\sigma}{2\pi} \right)^3 \int_R d^3k \left( \frac{\tau}{2 \sin \omega \tau} \right)$$

$$\times \left[ \theta \left( x^0 - x'^0 \right) \exp \left\{ -i\omega \left( x^0 - x'^0 \right) + i \mathbf{k} \cdot \left( \mathbf{x} - \mathbf{x}' \right) \right\} \right.$$

$$\left. + \theta \left( x'^0 - x^0 \right) \exp \left\{ i\omega \left( x'^0 - x^0 \right) - i \mathbf{k} \cdot \left( \mathbf{x} - \mathbf{x}' \right) \right\} \right]$$

$$= i \left( \prod_{\mu=0}^3 \int_{\sigma(\mu)}^{\sigma(\mu)} \frac{\sigma(\mu)}{2\pi} dk_\mu \right) \exp \left\{ -ik_\nu \left( x^\nu - x'^\nu \right) \right\}$$

$$\times \left[ \sum_{\mu, \nu=0}^3 \eta^{\mu\nu} \frac{4}{\sigma(\mu) \sigma(\nu)} \sin \frac{\sigma(\mu) k_\mu}{2} \sin \frac{\sigma(\nu) k_\nu}{2} - m^2 + i\epsilon \right]^{-1}. \quad (3.23)$$
B. Interacting field with a conserved quantity

In this subsection we consider a field with interaction, where we regard the existence of a conserved quantity as the most important requirement. Under the condition the form of interaction $G(\phi)$ in Eq. (3.1) or (3.5) is very much restricted. We show two cases in the following.

a) We assume Eq. (3.1) or (3.5) and

$$G(\phi) = -\frac{g}{2} \left\{ \phi(x^0 + \tau, \mathbf{x}) + \phi(x^0 - \tau, \mathbf{x}) \right\} \phi^2(x) - m^2 \phi(x),$$

(3.24)

that tends to

$$\lim_{\sigma, \tau \to 0} G(\phi) = -\frac{\partial V}{\partial \phi} \quad \text{with} \quad V = \frac{g}{4} \phi^4 + \frac{m^2}{2} \phi^2,$$

(3.25)

in the continuous limit. In this case the field equation (3.1) or (3.5) is rewritten canonically and shown to have a conserved quantity corresponding to the energy. It is easily understood if we notice that the Equation is a simple extension of the classical dynamics with $x^4$-potential shown in section II. The equation is written in a canonical form:

$$\phi \left( x^0 + \tau, \mathbf{x} \right) = \phi \left( x^0, \mathbf{x} \right) + \tau \pi \left( x^0, \mathbf{x} \right),$$

(3.26a)

$$\pi \left( x^0 + \tau, \mathbf{x} \right) = \pi \left( x^0, \mathbf{x} \right) + \tau \sum_{x'} F \left( \phi \left( x^0 + \tau, \mathbf{x} \right) \right)_{x', x} \phi \left( x^0 + \tau, \mathbf{x}' \right),$$

(3.26b)

where

$$F \left( \phi \left( x^0 + \tau, \mathbf{x} \right) \right)_{x, x'} = \left\{ \frac{1}{2} g \phi^2 \left( x^0, \mathbf{x} \right) + \frac{1}{\tau^2} \right\}^{-1} \left( \frac{2}{\tau^2} \delta_{x, x'} - \Omega_{x, x'} \right) - \frac{2}{\tau^2} \delta_{x, x'},$$

(3.27)

and

$$\Omega_{x, x'} = m^2 \delta_{x, x'} - \frac{1}{\sigma^2} \sum_{i=1}^{3} \left( \delta_{x+\delta(i), x'} - 2 \delta_{x, x'} + \delta_{x-\delta(i), x'} \right).$$

(3.28)

From the equation we see that Poisson bracket is independent of time $x^0$:

$$\{ \phi \left( x^0, \mathbf{x} \right), \pi \left( x^0, \mathbf{x}' \right) \}_{P.B.} = \delta_{x, x'},$$

(3.29)
This corresponds to that the equal-time commutation relation in quantum theory is independent of time.

The difference equations \((3.26a)\) and \((3.26b)\) conserve the following quantity:

\[
\mathcal{H} = \sum_x \frac{1}{2} \pi^2 (x^0, x) + \frac{T}{2} \sum_{x,x'} \phi \left( x^0, x \right) \Omega_{x,x'} \pi (x^0, x') \\
+ \frac{1}{2} \sum_{x,x'} \phi \left( x^0, x \right) \Omega_{x,x'} \phi \left( x^0, x' \right) + \frac{g}{4} \sum_x \left\{ \phi \left( x^0, x \right) + \tau \pi \left( x^0, x \right) \right\}^2 \phi^2 \left( x^0, x \right)
\]

\[
= \sum_x \frac{1}{2} \left[ \pi^2 (x^0, x) + \sum_{i=1}^3 \Delta_i^R \phi \left( x^0, x \right) \left\{ \Delta_i^R \phi \left( x^0, x \right) + \tau \Delta_i^R \pi \left( x^0, x \right) \right\} \right] \\
+ \frac{m^2}{2} \phi \left( x^0, x \right) \left\{ \phi \left( x^0, x \right) + \tau \pi \left( x^0, x \right) \right\}^2 \phi^2 \left( x^0, x \right) .
\]

As we have seen above, the field with the interaction \((3.24)\) is formulated canonically and the conserved quantity is obtained. However, we cannot find the time developing operator.

b) Next we assume

\[
G (\phi) = - \frac{V \left( x^0 + \tau, x \right) - V \left( x^0 - \tau, x \right)}{\phi \left( x^0 + \tau, x \right) - \phi \left( x^0 - \tau, x \right)},
\]

that satisfied Eq. \((3.6)\), where we used the abbreviation: \(V \left( x^0, x \right) = V \left( \phi \left( x^0, x \right) \right)\). In this case we cannot find the momentum conjugate to \(\phi \left( x^0, x \right)\), so as to rewrite the equation in a canonical form. However, we can define energy-momentum tensor \(T_{\mu \nu}\), that satisfies the continuity condition:

\[
\sum_{\mu, \rho = 0}^3 \eta^{\mu \rho} \Delta_\mu^R T_{\nu \rho} = 0,
\]

\[
T_{\rho \mu} = I_{\rho \mu} - \eta_{\rho \mu} K (\rho),
\]

\[
I_{\rho \mu} = \frac{1}{2} \left[ \Delta_\rho^R \phi \left( x \right) \Delta_\mu^L \phi \left( x \right) + \sigma \left( \mu \right) \Delta_\rho^R \phi \left( x \right) \Delta_\mu^L \phi \left( x \right) \right],
\]

\[
K (\rho) = \frac{1}{2} \sum_{\mu, \nu = 0}^3 \eta^{\mu \nu} \Delta_\mu^R \phi \left( x \right) \Delta_\nu^L \phi \left( x \right) + \frac{1}{2} \sigma \left( \rho \right) \Delta_\rho^L \phi \left( x \right) \mathcal{G} \left( \phi \left( x \right) \right) \\
+ \frac{1}{2} \sum_{y^\rho = -\infty}^{x^\rho - \sigma (\rho)} \left( \Delta_\rho^R \phi \left( y \right) + \Delta_\rho^L \phi \left( y \right) \right) \mathcal{G} \left( \phi \left( y \right) \right) \bigg|_{y^\sigma = x^\sigma (\rho \neq \sigma)} ,
\]

20
Therefore, the conserved quantities are

\[ P_0 = \sum_x T_{00} \]
\[ = \frac{1}{2} \sum_x \left\{ (\Delta_0^R \phi(x))^2 + \sum_{i=1}^{3} \Delta_i^R \phi(x^0 + \tau, x) \Delta_i^R \phi(x^0, x) \right\} + V(x^0, x) + V(x^0 + \tau, x) \right\}, \tag{3.36} \]

\[ P_k = \sum_x T_{k0} \]
\[ = -\frac{1}{2\sigma \tau} \sum_x \left\{ \phi(x^0, x + \hat{\sigma}(k)) - \phi(x^0, x - \hat{\sigma}(k)) \right\} \phi(x^0 - \tau, x). \tag{3.37} \]

These quantities correspond to energy and momentum in the continuous limit respectively.

We have found the conserved quantities in the case of arbitrary interaction \( V(\phi) \). However, we could not find the canonical formalism and the time developing operator.

C. Quantum field theory with interaction

In this section we give up for a moment to obtain conserved quantities, and try to formulate the quantum field theory with interaction on discrete spacetime. For this purpose it is necessary to have the time developing operator. We assume the existence of Lagrangian, i.e.,

\[ G(\phi) = -\frac{\partial V}{\partial \phi}. \tag{3.38} \]

In a very simple case

\[ G(\phi) = -g\phi^3 - m^2\phi \tag{3.39a} \]

or

\[ V(\phi) = g\frac{\phi^4}{4} + m^2\frac{\phi^2}{2}, \tag{3.39b} \]

it is supposed from the case \( V(x) = \Lambda x^4/4 \) in Sec. II that in high "energy" region the system has no conserved quantity and behaves chaotically.
Now we assume the time developing operator $U$ as follows:

\[
U(\phi, \pi) = \exp \left\{ -\tau \sum_x \frac{1}{2} \pi^2 (x^0, x), P.B. \right\} \times \exp \left\{ -\tau \sum_x \left[ 1/2 \sum_{i=1}^3 \left( \Delta_i^R \phi (x^0, x) \right)^2 + V (x^0, x) \right], P.B. \right\}, (3.40)
\]

where

\[
\pi (x) = \Delta_0^R \phi (x), \quad (3.41)
\]

and Poisson bracket is the same that of free field:

\[
\{ \phi (x^0, x), \pi (x^0, x') \}_{P.B.} = \delta_{x,x'}. \quad (3.42)
\]

Using the operator we see that the field equation (3.1) or (3.5) is equivalent to the following canonical equations:

\[
\phi (x^0 + \tau, x) = U \phi (x^0, x) = \phi (x^0, x) + \tau \pi (x^0, x), \quad (3.43a)
\]

\[
\pi (x^0 + \tau, x) = U \pi (x^0, x) = \pi (x^0, x) + \tau \left[ \Delta_1^R \phi (x^0 + \tau, x) + \sum_{i=1}^3 \Delta_i^L \Delta_i^R \phi (x^0 + \tau, x) + G (x^0 + \tau, x) \right]. \quad (3.43b)
\]

Needless to say, Eq. (3.43a) is the same as Eq. (3.41).

If the time developing operator is written in the following form:

\[
U^{-1} = e^{-\tau (H, P.B.). (3.44)}
\]

$H$ is conserved and interpreted as the energy or time developing generator. As is mentioned above, this quantity $H$ is not supposed to exist always.

The quantization of the field is straightforward. The canonical commutation relation is

\[
[\phi (x^0, x), \pi (x^0, x')] = i \delta_{x,x'}, \quad (3.45)
\]

and the time developing operator is
\[ U(\phi, \pi) = \exp \left[ -i\tau \sum_{x} \left\{ \frac{1}{2} \sum_{i=1}^{3} (\Delta^R_i \phi \left(x^0, x \right))^2 + V \left(x^0, x \right) \right\} \right] \]
\[ \times \exp \left[ -i\tau \sum_{x} \frac{1}{2} \pi \left(x^0, x \right)^2 \right], \tag{3.46} \]

\[ \phi \left(x^0 + \tau, x \right) = U(\phi, \pi)^{-1} \phi \left(x^0, x \right) U(\phi, \pi), \tag{3.47a} \]

\[ \pi \left(x^0 + \tau, x \right) = U(\phi, \pi)^{-1} \pi \left(x^0, x \right) U(\phi, \pi). \tag{3.47b} \]

These equations are the same as Heisenberg equations (3.43a) and (3.43b).

After we have obtained the time developing operator, the next problem is how to calculate the scattering matrix. Following the method used in the case of continuous spacetime, we express the field in the interaction representation. A physical state in Schrödinger representation at time \(x^0\) is translated to the state at the next time \(x^0 + \tau\) by the time developing operator in Schrödinger representation:

\[ |\psi_{S}, x^0 + \tau) = U(\phi_{S}, \pi_{S}) |\psi_{S}, x^0). \tag{3.48} \]

\(U(\phi_{S}, \pi_{S})\) is given by Eq. (3.47), in which \(\phi\) and \(\pi\) of Heisenberg operator are replaced by \(\phi_{S}\) and \(\pi_{S}\) of Schrödinger operator respectively. We simply assume that \(\phi\) and \(\pi\) coincide with \(\phi_{S}\) and \(\pi_{S}\) at \(x^0 = 0\).

Now the time developing operator in Schrödinger representation is rewritten as follows:

\[ U^{-1}(\phi_{S}, \pi_{S}) = \exp \left[ i\tau \sum_{x} \frac{1}{2} \pi_{S}^2 \left(x \right) \right] \exp \left[ i\tau \sum_{x} \left\{ \frac{1}{2} \sum_{i=1}^{3} (\Delta^R_i \phi_{S} \left(x \right))^2 + V \left(\phi_{S} \left(x \right) \right) \right\} \right] \]
\[ = \exp \left[ i\tau \sum_{x} \frac{1}{2} \pi_{S}^2 \left(x \right) \right] \exp \left[ i\tau \sum_{x} \left\{ \frac{1}{2} \sum_{i=1}^{3} (\Delta^R_i \phi_{S} \left(x \right))^2 + \frac{1}{2} m^2 \phi_{S}^2 \left(x \right) \right\} \right] \]
\[ \times \exp \left[ i\tau \sum_{x} V_{I} \left(\phi_{S} \right) \right] \]
\[ = \exp \left( i\tau H_{0} \right) \exp \left[ i\tau \sum_{x} V_{I} \left(\phi_{S} \right) \right] \]
\[ = U_0^{-1}(\phi_{S}, \pi_{S}) \exp \left[ i\tau \sum_{x} V_{I} \left(\phi_{S} \right) \right], \tag{3.49} \]
where
\[ V_I (\phi_S) = V (\phi_S) - \frac{1}{2} m^2 \phi^2_S. \] (3.50)

and \( U_0 (\phi, \pi) \) is defined in Eq. (3.21). The field operators in interaction representation are obtained from those in Schrödinger representation using the time developing operator of free field in Schrödinger representation:

\[
\begin{align*}
\phi_I \left(x^0, x\right) &= U_0^{x^0/\tau} (\phi_S, \pi_S) \phi_S (x) U_0^{x^0/\tau} (\phi_S, \pi_S), \\
\pi_I \left(x^0, x\right) &= U_0^{x^0/\tau} (\phi_S, \pi_S) \pi_S (x) U_0^{x^0/\tau} (\phi_S, \pi_S).
\end{align*}
\] (3.51a, b)

Therefore, we see that the time developing operator for the field operator in interaction representation is the same as that for free field in Schrödinger representation:

\[
\begin{align*}
\phi_I \left(x^0 + \tau, x\right) &= U_0^{x^0/\tau} (\phi_I, \pi_I) \phi_I \left(x^0, x\right) U_0 (\phi_I, \pi_I), \\
\pi_I \left(x^0 + \tau, x\right) &= U_0^{x^0/\tau} (\phi_I, \pi_I) \pi_I \left(x^0, x\right) U_0 (\phi_I, \pi_I),
\end{align*}
\] (3.51a, b)

where
\[
U_0 (\phi_I, \pi_I) \equiv U_0 \left(\phi_I \left(x^0, x\right), \pi_I \left(x^0, x\right)\right)
= U_0^{x^0/\tau} (\phi_S (x), \pi_S (x)) U_0 (\phi_S (x), \pi_S (x)) U_0^{x^0/\tau} (\phi_S (x), \pi_S (x)).
\]

The physical state of interaction representation is expressed as follows:

\[
| \psi_I, x^0 \rangle = U_0^{x^0/\tau} (\phi_S, \pi_S) | \psi_S, x^0 \rangle
= U_0^{x^0/\tau} (\phi_S, \pi_S) U_0^{x^0/\tau} (\phi_S, \pi_S) | \psi_S, 0 \rangle
= U_0^{x^0/\tau} (\phi_S, \pi_S) \left[ \exp \left\{ -i \tau \sum_x V_I (\phi_S (x)) \right\} U_0 (\phi_S, \pi_S) \right]^{x^0/\tau} | \psi_S, 0 \rangle
= \prod_{l=0}^{(x^0/\tau)^{-1}} \exp \left\{ -i \tau \sum_x V_I \left(\phi_I \left(x^0 - l\tau, x\right)\right) \right\} | \psi_S, 0 \rangle.
\] (3.52)

The physical state of interaction representation is thus developed in the following form:
\[ | \psi_I, x^0 + \tau \rangle = \exp \left\{ -i \tau \sum_x V_I (\phi_I (x^0 + \tau, x)) \right\} | \psi_I, x^0 \rangle. \] (3.53)

The scattering matrix element between an initial state and a final state is written as
\[
S_{fi} \equiv \langle \psi_I^{\text{final}}, x^0 = +\infty | \psi_I^{\text{initial}}, x^0 = -\infty \rangle
= \langle \psi_I^{\text{final}}, x^0 = -\infty | \prod_{t=-\infty}^{\infty} e^{-i \tau \sum_x V_I (\phi_I (t\tau, x))} | \psi_I^{\text{initial}}, x^0 = -\infty \rangle. \] (3.54)

If \( V_I \) contains a small parameter, the scattering matrix is expanded in a power series of \( V_I \):
\[
S_{fi} = \langle f | i \rangle - i \tau \langle f | \sum_x V_I (x) | i \rangle
- \frac{\tau^2}{2} \langle f | \sum_{x,y} \left[ \theta (x^0 - y^0) V_I (x) V_I (y) + \theta (y^0 - x^0) V_I (y) V_I (x) \right] | i \rangle
+ \cdots. \] (3.55)

where we wrote \( | i \rangle \) and \( | f \rangle \) for \( | \psi_I^{\text{initial}}, x^0 = -\infty \rangle \) and \( | \psi_I^{\text{final}}, x^0 = -\infty \rangle \) for simplicity and used the abbreviation:

\[ V_I (x) = V (\phi_I (x^0, x)). \]

The summation \( \sum_x \) means the sum over \( x^0 \) and \( x \).

**IV. DISCUSSION**

We formulated quantum mechanics on discrete time in the framework of canonical formalism in Sec. II. We will briefly comment here on the relation between the canonical method and path integral quantization on discrete time.

The transition matrix element on usual continuous time is given by the following formula in path integral formulation,
\[
\langle x_f, t_f | x_i, t_i \rangle = \int_{(x_f, t_f) \sim (x_i, t_i)} \mathcal{D}x \ e^{iS}, \] (4.1)

where \( S \) is an action. The quantization on discrete time by path integral is obtained only by replacing the continuous action in Eq. (4.1) by the discrete action (2.3). The path integral
quantization is equivalent to the canonical one in continuum theories, but we have to check the equivalence for discrete time. This can be easily seen as follows. The transition matrix element is written like

\[
\langle x_f, t_f | x_i, t_i \rangle = \langle x_f | \prod e^{-i\tau V(\hat{x})} e^{-i\tau \hat{p}^2/2} | x_i \rangle
\]  

in our canonical formalism. We insert complete sets \( 1 = \int dx |x\rangle\langle x| \) and \( 1 = \int dp |p\rangle\langle p| \) between exponentials and write them as an integral over c-numbers. After the integrations over \( p \)'s using \( \langle x|p\rangle = e^{ixp}/\sqrt{2\pi} \), we have (4.1) with the discrete action (2.3).

In a continuous time theory, an evolution operator \( U_{\text{cont}}(t) \) is given by

\[
U_{\text{cont}}(t) = \exp(-itH_{\text{cont}}) = \left[ \exp(-i\Delta t H_{\text{cont}}) \right]^{t/\Delta t},
\]

where \( H_{\text{cont}} = \hat{p}^2/2 + V(\hat{x}) \). We have to use the approximation:

\[
\exp(-i\Delta t H_{\text{cont}}) \cong \exp(-i\Delta t V(\hat{x})) \exp(-i\Delta t \hat{p}^2/2),
\]

for short time distance \( \Delta t \) in order to show the equivalence. The calculation for the proof is completely same as above [8]. However, the right hand side of Eq. (4.4) is an exact time evolution operator in our framework:

\[
\exp(-i\tau H_{\tau}) = \exp(-i\tau V(\hat{x})) \exp(-i\tau \hat{p}^2/2),
\]

though \( H_{\tau} \) becomes very complicated (see Eq. (2.37)). If we consider \( \exp(-i\tau H_{\text{cont}}) \) with discrete variables as an time evolution operator, the equation of motion becomes very complicated.

Generally, it is not evident what kind of quantity corresponds to usual momentum in a discrete time theory. We see that the usual discrete approximation holds exactly under the definition of momentum (2.27) for our simple equation of motion. This definition is natural in this sense. In other words, this discretization scheme has a natural continuum limit and is appropriate on discrete time.

In Sec. [11] we formulated quantum field theory on discrete Minkowski spacetime. Lattice theories are usually formulated on Euclidean spacetime because of difficulties of indefinite
metric on Minkowski spacetime. The relation between continuum theories on Euclidean and
Minkowski spacetime has been investigated by many authors [9]-[11]. Sufficient conditions
for field operators of two spacetimes to be connected with each other by simple analytic con-
tinuation is given by Osterwalder and Schrader [9]. In their proof, Minkowski and Euclidean
invariances are essential. We have no such invariances on discrete spacetime and this simple
relation does not hold. As an example, the Euclidean propagator of free scalar field is

\[
\langle 0 \mid T(\phi(x)\phi(x')) \mid 0 \rangle = -\left(3 \prod_{\mu=0}^{3} \int_{\frac{\pi}{2\pi}}^{\frac{\pi}{2\pi}} \frac{\sigma(\mu)}{2\pi} dk_{\mu}\right) \exp\left\{i \sum_{\mu,\nu=0}^{3} \delta^{\mu\nu} k_{\mu} (x_{\nu} - x'_{\nu})\right\} \\
\times \left[3 \sum_{\mu,\nu=0}^{3} \delta^{\mu\nu} \frac{4}{\sigma(\mu)\sigma(\nu)} \sin \frac{\sigma(\mu)k_{\mu}}{2} \sin \frac{\sigma(\nu)k_{\nu}}{2} + m^2 \right]^{-1},
\]  

(4.6)

and the Minkowski propagator is

\[
\langle 0 \mid T(\phi(x)\phi(x')) \mid 0 \rangle = i \left(3 \prod_{\mu=0}^{3} \int_{\frac{\pi}{2\pi}}^{\frac{\pi}{2\pi}} \frac{\sigma(\mu)}{2\pi} dk_{\mu}\right) \exp\left\{-ik_{\nu} (x'_{\nu} - x_{\nu})\right\} \\
\times \left[3 \sum_{\mu,\nu=0}^{3} \eta^{\mu\nu} \frac{4}{\sigma(\mu)\sigma(\nu)} \sin \frac{\sigma(\mu)k_{\mu}}{2} \sin \frac{\sigma(\nu)k_{\nu}}{2} - m^2 + i\epsilon \right]^{-1}.
\]  

(4.7)

They are not connected by analytic continuation. We always have to be careful about
the above when we extract Minkowski information from quantities calculated on Euclidean
spacetime.
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FIGURES

FIG. 1. Behavior of the iterative map of Eq. (2.33a,b) with the potential $V(x) = \Lambda x^4/4$. Variables are scaled so that $\tilde{x} = \tau \sqrt{\Lambda} x$ and $\tilde{p} = \tau^2 \sqrt{\Lambda} p$. Initial values are $\tilde{p}(0) = 0$ and $\tilde{x}(0) = 0.17, 0.47, 0.62, 0.67, 0.76, 0.775, 0.80$. For small initial values of $\tilde{x}(0)$, the map draws a smooth curve, which indicates the existence of a conserved quantity $H_\tau$. The chaotic behavior for larger initial values of $\tilde{x}(0)$ suggests the divergence of the series of $H_\tau$. 