On the use of complex functions in solving non-stationary problems of heat conductivity in a multilayered medium by the Fourier method

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Abstract. The article shows that the difficulties encountered in solving thermal conductivity problems in a multilayer medium by the Fourier analytic method can be overcome by using the apparatus of complex functions. These are the problems of finding the eigenvalues, normalizing the basis functions and finding the expansion coefficients in a Fourier series. The results can be used to solve other non-stationary problems of transport theory.

1. Introduction
The method of approximating functions using orthogonal decompositions, first proposed by Fourier, is currently developed for a wide range of problems and has achieved great theoretical perfection. However, in the general opinion [1, 2, 3], the problems of transport theory in a multilayer solid medium are difficult both theoretically and computationally, since they are related to the construction of solutions of differential equations with variable coefficients. For example, in multilayer media, these coefficients have a finite number of discontinuities of the first kind. The difficulties are connected not only with the construction of the solution itself, but with its further application for finding eigenvalues and normalization of basic solutions, as well as with the further determination of the expansion coefficients.

We give a short review of some recent research on this issue. In a paper [4], an example was considered when the coefficient of thermal conductivity changes exponentially depending on the coordinate. The resulting solution is expressed in terms of Bessel functions. A paper [5] includes conditions for the electromagnetic exchange of energy between layers, and the material layers, except for the extreme ones, are considered quite thin. In a paper [6], the most general formulation of the heat transfer problem in the fuel piping system is studied at a high theoretical level, including the possibility of solving it in various classes of functions.

In [7 – 12], we gave an application of the matrix method to solving problems of transport theory in a multilayer medium, and for the first time, for solving non-stationary problems, we used [9 – 12] the apparatus of generalized powers of Bers [13].

In this paper, we present a method that uses the functions of a complex variable and allows both to significantly simplify the construction of these solutions by the Fourier method and to facilitate the finding of eigenvalues and normalization of basis functions. Therefore, the solution by the Fourier method of the thermal conduction problem for a multilayer medium is not only of independent interest but can also serve as an approximate solution for any inhomogeneous medium.
2. Statement of the problem and the method of calculation

We give the statement of the problem. A system of flat layers with a total number \( n \) makes up a plate. We direct the \( x \) axis normal to the plane of the plate. The coordinates of the layers are denoted by \( x_1, x_2, \ldots, x_n \) (figure 1), where \( x_1, x_{n+1} \) are the coordinates of the outer boundaries of the plate.

![Layer system](attachment:Layer_system.png)

Figure 1. Layer system.

We will number all layers and the quantities related to them by the number of the left coordinate of the segment \([x_i, x_{i+1}]\) defining the layer. We will put the layer number in the upper index in brackets. For example, \( T^{(k)}(x,t) \) means the temperature field in the \( k \)th layer. We will consider a one-dimensional problem when the heat flux density \( j^{(k)}(x,t) \) in the layer is directed normally to the plane of the layer and is equal to

\[
j^{(k)} = -\lambda^{(k)} \frac{dT^{(k)}}{dx},
\]

where \( \lambda^{(k)} \) is the heat conductivity coefficient of the \( k \)th layer.

The initial system of equations that defines the process in a multilayer plate consists of equations of the form [1]

\[
\frac{\partial}{\partial x} \left( \lambda^{(k)} \frac{\partial T^{(k)}}{\partial x} \right) - \rho^{(k)} c^{(k)} \frac{\partial T^{(k)}}{\partial t} = 0, \quad k = 1, \ldots, n
\]

and conditions for matching the type of ideal contact, consisting in the continuity of temperature and flow at the boundaries of the contact layers

\[
T^{(k)}(x,t) \big|_{x=x_{i+1}} = T^{(k+1)}(x,t) \big|_{x=x_{i+1}},
\]

\[
-\lambda^{(k)} \frac{\partial T^{(k)}(x,t)}{\partial x} \bigg|_{x=x_{i+1}} = -\lambda^{(k+1)} \frac{\partial T^{(k+1)}(x,t)}{\partial x} \bigg|_{x=x_{i+1}}.
\]

at any point \( t, \ k = 1, n \). In equations (3), (4) all quantities are understood as limits to the left and right.

The work does not use the dimensionless quantities that are usually accepted in solving problems of the theory of heat conduction, since in this case the fulfillment of the matching conditions is difficult.

The boundary conditions for temperature \( T \) at the outer boundaries of the layer system are accepted equal to zero, as is most often assumed when using the Fourier method,

\[
T^{(1)}(x_1, t) = 0, \quad T^{(n)}(x_{n+1}, t) = 0.
\]

The problem is to find the temperature of the system at all points in time and at all points of the plate, if the initial temperature distribution over the plate at some initial point in time \( t = 0 \)

\[
T(x, 0) = f(x).
\]

The initial temperature distribution can be set for individual layers, that is, for segments
\[ T^{(k)}(x,0) = f^{(k)}(x), \ x \in [x_k, x_{k+1}]. \]  
\[ (7) \]

This generally means the discontinuity of the initial temperature at the contact points of the layers.

It is shown that the solution of the problem under condition (7) exists and is unique \[9\].

The solution of problem (2) – (6) by the Fourier method suggests the possibility of constructing a sequence of particular solutions

\[ T_{j}^{(k)}(x,0) = u_{j}^{(k)}(x) \exp(-\mu_{j}^{2}t), \ k = 1, n, \ j = 1, \ldots, \infty. \]
\[ (8) \]

Here \( k \) is the layer number, and \( j \) numbers the terms of the Fourier series.

Unlike the usual form of the time factor, when the coefficient of thermal diffusivity is included in the indicator, which depends on the parameters of the medium, this should not be done for a multilayer plate, since this would introduce time factors into the matching conditions and would greatly complicate the task.

The solution of the problem is reduced to finding a sequence of functions of the form (8), where \( u_{j}^{(k)} \) must satisfy all the required conditions, and is represented by the Fourier series for individual segments

\[ T^{(k)}(x,t) = \sum_{j=1}^{\infty} u_{j}^{(k)} \exp(-\mu_{j}^{2}t), \ k = 1, n, \ x \in [x_k, x_{k+1}). \]

From equations (2) and (8) it follows that the functions \( u_{j}^{(k)} \), that determine the basis solution are subordinate to the system

\[ \frac{d}{dx} \left( \lambda^{(k)} \frac{\partial u_{j}^{(k)}(x)}{\partial x} \right) = -m^{(k)} \mu_{j}^{2} u_{j}^{(k)}(x), \ k = 1, n, \ x \in [x_k, x_{k+1}], \ j = 1, 2, \ldots, \infty, \]
\[ (9) \]

where is taken

\[ m^{(k)} = \rho^{(k)} c^{(k)}. \]
\[ (10) \]

Here \( \rho^{(k)} \), \( c^{(k)} \) are the density and heat capacity of the substance of the \( k \)th layer accordingly.

Then we introduce the notation

\[ s^{(k)} = \sqrt{\frac{\rho^{(k)} c^{(k)}}{\lambda^{(k)}}} = \sqrt[m^{(k)}]{\lambda^{(k)}}, \]
\[ (11) \]

The function \( u_{j}^{(k)} \) satisfies the equation

\[ \frac{d^{2} u_{j}^{(k)}}{dx^{2}} = -\left(s^{(k)}\right)^{2} \mu_{j}^{2} u_{j}^{(k)}, \ k = 1, n, \ j = 1, 2, \ldots, \infty. \]

The agreement conditions for \( u^{(k)} \) considering equations (3), (4), (8) should be written as

\[ u_{j}^{(k)}(x) \bigg|_{x_{j+1}} = u_{j}^{(k+1)}(x) \bigg|_{x_{j+1}}, \]
\[ (12) \]

\[ \lambda^{(k)} \frac{du_{j}^{(k)}}{dx} \bigg|_{x_{j+1}} = \lambda^{(k+1)} \frac{du_{j}^{(k+1)}}{dx} \bigg|_{x_{j+1}}. \]
\[ (13) \]

Therefore, they are independent of time, since \( \mu_{j} \) is independent of the layer number \( k \).

The solution determined segment wise for the \( k \)th layer has the form
$$f^{(k)}(x,t) = \sum_{j=1}^{\infty} F_j u^{(k)}_j(x) \exp(-\mu_j^2 t), \quad x_k < x < x_{k+1}. $$

Here $F_j$ is the Fourier series expansion coefficient common to all layers, determined from the orthogonality condition.

The systems of functions $u^{(k)}$ are orthogonal and normalized in accordance with the scalar product

$$\left( u_j, u_j \right) = \sum_{k=1}^{\infty} \left( u_j^{(k)}, u_j^{(k)} \right).$$

The constant $\mu_j$ is independent of $k$ and is an eigenvalue of an operator of the form

$$a_2 \frac{d}{dx} \left( a_1 \frac{du_j}{dx} \right) = -\mu_j^2 u_j,$$

where $a_2, a_1$ has gaps of the first kind. Such an approach includes matching conditions in an operator, as is customary in graph theory [15].

Based on the form of equation (14), as the scalar product of two solutions $u_1(x), u_2(x)$, it is taken

$$\left( u_1, u_2 \right) = \sum_{k=1}^{\infty} \int_{x_k}^{x_{k+1}} u_1^{(k)} u_2^{(k)} m^{(k)} \, dx.$$

The method of constructing the solution $u_j^{(k)}$, proposed below, differs from the generally accepted one. Usually looking for solutions for the $k$th layer in the form

$$u_j^{(k)}(x) = C_1^{(k)} \cos \mu_j s^{(k)} x + C_2^{(k)} \sin \mu_j s^{(k)} x.$$

Next, based on the matching conditions (12), (13), the corresponding system of linear algebraic equations is established to find $C_1^{(k)}, C_2^{(k)}$.

Below, we use solution (16) in a complex form

$$u_j^{(k)}(x) = \frac{1}{2} \left( C_1^{(k)} \exp(i \mu_j s^{(k)} x) + \overline{C_1^{(k)}} \exp(-i \mu_j s^{(k)} x) \right), \quad k = 1, n,$$

for

$$C_1^{(k)} = C_1^{(k)} - i C_2^{(k)}, \quad \overline{C_1^{(k)}} = C_1^{(k)} + i C_2^{(k)}.$$

In the paper, it is proposed to construct a solution sequentially from one layer to another (similar to how it is done in the matrix solution method [5–7]) for which a recurrence formula is obtained that relates the solutions of the $k$th and $(k + 1)$th layers. In a complex form, the matching relations (12), (13) for $u^{(k)}$, $u^{(k+1)}$ take the form

$$C_1^{(k)} \exp(i \mu_j s^{(k)} x_{k+1}) + \overline{C_1^{(k)}} \exp(-i \mu_j s^{(k)} x_{k+1}) = C_1^{(k+1)} \exp(i \mu_j s^{(k+1)} x_{k+1}) + \overline{C_1^{(k+1)}} \exp(-i \mu_j s^{(k+1)} x_{k+1}),$$

$$\lambda^{(k+1)} s^{(k)} \left[ C_1^{(k)} \exp(i \mu_j s^{(k)} x_{k+1}) - \overline{C_1^{(k)}} \exp(-i \mu_j s^{(k)} x_{k+1}) \right] = \lambda^{(k+1)} s^{(k)} \left[ C_1^{(k+1)} \exp(i \mu_j s^{(k+1)} x_{k+1}) - \overline{C_1^{(k+1)}} \exp(-i \mu_j s^{(k+1)} x_{k+1}) \right].$$
Relations (19), (20) give a system of algebraic equations for finding $C^{(k+1)}$, $\overline{C}^{(k+1)}$ through $C^{(k)}$, $\overline{C}^{(k)}$. The solution to this system can be written as

$$C^{(k+1)} = \frac{1}{2} \left[ p^{(k+1)} C^{(k)} \exp(i\mu \alpha^{(k)} x_{k+1}) + q^{(k+1)} \overline{C}^{(k)} \exp(-i\mu \beta^{(k)} x_{k+1}) \right],$$

where

$$p^{(k+1)} = 1 + \frac{s^{(k)} \lambda^{(k)}}{s^{(k+1)} \lambda^{(k+1)}}, \quad q^{(k+1)} = 1 - \frac{s^{(k)} \lambda^{(k)}}{s^{(k+1)} \lambda^{(k+1)}},$$

(22)

$$a^{(k+1)} = s^{(k)} - s^{(k+1)}, \quad b^{(k+1)} = s^{(k)} + s^{(k+1)}.$$  

(23)

We note right away that the eigenvalue $\mu_j$ is not included in $p^{(k+1)}$, $q^{(k+1)}$, but is present only in the exponent, which greatly simplifies the computational process.

The eigenvalues of the operator are determined when solving $u^{(k)}$, $k = 1, n$ from the boundary conditions. To satisfy the first boundary condition for $x = x_1 = 0$, it suffices to put

$$C^{(1)} + \overline{C}^{(1)} = 0$$

(24)

or $C^{(1)} = -i\overline{C}^{(1)}$.

The second condition for $x = x_{n1}$ is written

$$C^{(n)} \exp(i\mu \alpha^{(n)} x_{n1}) + \overline{C}^{(n)} \exp(-i\mu \beta^{(n)} x_{n1}) = 0.$$  

(25)

This condition determines the eigenvalues $\mu$. For $x = x_{n1}$ it has the form

$$C^{(n)} \cos \mu \alpha^{(n)} x_{n1} + \overline{C}^{(n)} \sin \mu \alpha^{(n)} x_{n1} = 0.$$  

Here $C^{(n)}$, $C^{(n)}$, or $C^{(n)}$, $\overline{C}^{(n)}$, found by recurrence relations (21).

With a large number of layers, the proposed method is simpler than the known ones. Although the number of terms is equal to $2^{n-1}$, as in the matrix method [9 – 12], however, the terms are expressed by only one trigonometric function, and not by the product of trigonometric functions.

We give examples of using the result of equation (21). The results for the case of a single layer are known [14]. According to the boundary conditions adopted in the Fourier method, we define $C^{(1)} = 0$, $C^{(1)} \neq 0$. For the case of two layers, we have

$$u^{(2)}_{j} = C^{(2)}_{j} \sin \mu \alpha^{(2)} x, \quad x_1 \leq x \leq x_2,$$

(26)

$$u^{(2)}_{j} = \frac{1}{2} \overline{C}^{(2)}_{j} \left( p^{(2)} \sin \mu \alpha^{(2)} x - s^{(2)} x + s^{(2)} x + q^{(2)} \sin \mu \beta^{(2)} x \right),$$

(27)

Then, in the case of two layers, according to equation (24), $\mu_j$ is determined from the equation

$$p^{(2)} \sin \mu \alpha^{(2)} x - s^{(2)} x + s^{(2)} x + q^{(2)} \sin \mu \beta^{(2)} x = 0.$$  

(28)

In the case of three layers, the expressions (26), (27) for the first and second layers remain unchanged, and for the third layer we find
The eigenvalues in the case of three layers, according to equation (24), are determined from the equation

\( u_j^{(3)}(x) = \frac{1}{4} C_2^{(1)} \left[ p^{(1)} p^{(2)} \sin \mu_j (s^{(1)} x_2 - s^{(2)} x_2 + s^{(3)} x_3) + p^{(3)} q^{(3)} \sin \mu_j (s^{(1)} x_2 + s^{(2)} x_2 - s^{(3)} x_3) + p^{(2)} q^{(3)} \sin \mu_j (s^{(1)} x_2 - s^{(2)} x_2 + s^{(3)} x_3 - s^{(3)} x) \right] \),

(29)

\( x_2 \leq x \leq x_3 \).

The advantage of the method is that if a program is compiled or calculations are made for \( n \) layers, then all previous calculations remain valid except for eigenvalues. But here it is easy to follow their change. Calculations can be carried out starting from any layer of the system, for example, from the \( n \).

In some cases, the matrix form of these recurrence relations is useful. We introduce matrices and column vectors

\[
S_j^{(k)} = \begin{pmatrix}
\exp(i \mu s^{(k)} x_{k+1}) & 0 \\
0 & \exp(-i \mu s^{(k)} x_{k+1})
\end{pmatrix},
\]

\[
S_j^{(k+1)} = \begin{pmatrix}
\exp(-i \mu s^{(k+1)} x_{k+1}) & 0 \\
0 & \exp(i \mu s^{(k+1)} x_{k+1})
\end{pmatrix},
\]

\[
T^{(k+1)} = \begin{pmatrix}
p^{(k+1)} & q^{(k+1)} \\
q^{(k+1)} & p^{(k+1)}
\end{pmatrix},
V^{(k+1)} = \begin{pmatrix}
C^{(k)} \\
\overline{C}^{(k)}
\end{pmatrix},
V^{(k+1)} = \begin{pmatrix}
C^{(k+1)} \\
\overline{C}^{(k+1)}
\end{pmatrix}.
\]

Here \( S_j^{(k)} \) obtained as the limit on the left, and \( S_j^{(k+1)} \) as the limit on the right at the point \( x_{k+1} \).

The main relations between the coefficients are presented in a simpler form

\[
V^{(k+1)} = S_j^{(k)} T^{(k+1)} S_j^{(k+1)} V^{(k+1)}.
\]

(31)

From the matrix form, the coefficients \( C^{(k)}, \overline{C}^{(k)} \) are linear functions of \( C^{(1)}, \overline{C}^{(1)} \).

The next problem is to find the norm of the basis element, which, according to equation (5), reduces to finding the sum of integrals of the form

\[
N_j^2 = \sum_{k=1}^{n} \int_{x_k}^{x_{k+1}} (u_j^{(k)})^2 m^{(k)} \, dx.
\]

(32)

As \( k \) increases, the expression for \( u_j^{(k)} \), as already noted, becomes more complicated, especially since it is squared. We show that the norm is expressed as the sum of terms of the form \( C^{(k)} \overline{C}^{(k)} \)

\[
N_j^2 = \frac{1}{2} \sum_{k=1}^{n} m^{(k)} (x_{k+1} - x_k) C^{(k)} \overline{C}^{(k)}.
\]

(33)

Denote the integral over the \( k \) layer in the sum (32) by \( I^{(k)} \)

\[
I^{(k)} = \int_{x_k}^{x_{k+1}} (u_j^{(k)})^2 m^{(k)} \, dx.
\]
Considering for a pair of layers $k$ and $k+1$ sum $I^{(k)} + I^{(k+1)}$, after integration and algebraic operations, we make sure that the terms related to the point $x_{k+1}$, separating the layers $k$ and disappear due to the matching conditions and only terms that belong to the extreme points remain. This is true for any pair. $I^{(k)}$, $I^{(k+1)}$ if points $x_k$, $x_{k+1}$ are different from the points $x_1$, $x_{n+1}$, in which the boundary conditions are specified. The latter are equal to zero by virtue of equations (24) and (25).

We will find the recurrence relation for the squared modulus of numbers $C^{(k)}$. Using equation (20), we can easily find the connection

$$C^{(k+1)}C^{(k+1)} = \frac{1}{4} C^{(k)} C^{(k)} \left[ \left( \rho^{(k+1)} \right)^2 + \left( \eta^{(k+1)} \right)^2 + 2 \rho^{(k+1)} \eta^{(k+1)} \cos 2 \mu \eta^{(k+1)} x_{k+1} \right],$$

or

$$C^{(k+1)}C^{(k+1)} = C^{(k)} C^{(k)} \left( \cos^2 \mu \eta^{(k+1)} x_{k+1} + \eta_k^2 \sin^2 \mu \eta^{(k+1)} x_{k+1} \right),$$

where

$$\eta_k = \frac{\eta^{(k+1)}}{\eta^{(k+1)}}.$$

If $\eta < 1$, then $C^{(k+1)}C^{(k)} < C^{(k)}C^{(k)}$, if $\eta > 1$, then $C^{(k+1)}C^{(k+1)} > C^{(k)}C^{(k)}$. The obtained recurrence equation (34) makes it quite simple to find all terms $I^{(k)}$.

Thus, the solution to the problem

$$T^{(k)}(x,t) = \sum_{j=1}^{n} F_j \frac{u_j^{(k)}(x)}{N_j} \exp(-\mu_j^2 t),$$

where

$$F_j = \sum_{k=1}^{n} \int_{x_k}^{x_{k+1}} T^{(k)}(x,0) \frac{u_j^{(k)}(x)m^{(k)}}{N_j} \, dx, k = 1, n.$$

3. Results of calculations and their discussion

According to the proposed method, in the Maple computing environment, calculations were performed for a symmetric three-layer medium with parameters: $x_1 = 0$ m, $x_2 = 0.1$ m, $x_3 = 0.2$ m, $x_4 = 0.3$ m, that is, the thickness of each layer 0.1 m, $\lambda^{(1)} = \lambda^{(2)} = 0.7$ W/(m K) , $c^{(3)} = c^{(3)} = 800$ J/(kg \cdot K), $\rho^{(3)} = \rho^{(3)} = 1800$ kg/m$^3$, $\lambda^{(2)} = 58$ W/(m K), $c^{(2)} = 462$ J/(kg \cdot K), $\rho^{(2)} = 7860$ kg/m$^3$. The parameters of the layers correspond to the “brick-steel-brick” system of materials. Using the recurrence relations (21) – (23), we found, $C^{(n)}$, $\eta^{(n)}$, and then the corresponding eigenfunctions of the temperature field are constructed. In the calculations, it is observed that the periodicity of $u^{(n)}(x_{n+1}, \mu)$ function is possible. Figure 2 shows the graphs of the first three eigenfunctions. The results completely coincide with the values found earlier by the matrix method [5] up to a factor in the eigenfunctions. The graphs have a symmetrical shape, which corresponds to the specified symmetric parameter values.

Figure 3 shows the temperature distribution for this system at a given initial temperature. $T(x,0) = -1000x(x - 0.3)$, constructed using the 20 terms of the Fourier series. Figure 4 shows the temperature distribution constructed by 40 terms of the Fourier series. The need to take a large number of terms in a series is due to the multi-layered environment.
Figure 2. The first three eigenfunctions of the temperature field for a symmetric three-layer planar medium, the coefficients of which are found using recurrence relations.

Figure 3. The temperature distribution as a function of time (a) and at time $t = 0$ (b) in a symmetric three-layer plane medium, constructed from 20 terms of the Fourier series.
Figure 4. The temperature distribution as a function of time (a) and at time $t = 0$ (b) in a symmetric three-layer plane medium, constructed from 40 terms of the Fourier series.

4. The conclusion
The proposed method can also be used for setting other boundary conditions, for example, the presence of one

$$\frac{dT^{(1)}}{dx} \bigg|_{x=x_{i}} = 0, \quad T^{(n)}(x_{i+1}) = 0$$  \hspace{1cm} (37)

or two

$$\frac{dT^{(1)}}{dx} \bigg|_{x=x_{i}} = \frac{dT^{(n)}}{dx} \bigg|_{x=x_{i+1}} = 0$$  \hspace{1cm} (38)

adiabatically isolated walls.

The method proposed in the work allows us to construct an approximation scheme for calculating the unsteady transport process from given point measurements of the parameters of the medium.

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