Fast estimation of multivariate stochastic volatility

Kostas Triantafyllopoulos∗ Giovanni Montana†

December 12, 2008

Abstract

In this paper we develop a Bayesian procedure for estimating multivariate stochastic volatility (MSV) using state space models. A multiplicative model based on inverted Wishart and multivariate singular beta distributions is proposed for the evolution of the volatility, and a flexible sequential volatility updating is employed. Being computationally fast, the resulting estimation procedure is particularly suitable for on-line forecasting. Three performance measures are discussed in the context of model selection: the log-likelihood criterion, the mean of standardized one-step forecast errors, and sequential Bayes factors. Finally, the proposed methods are applied to a data set comprising eight exchange rates vis-à-vis the US dollar.

Some key words: multivariate time series, stochastic volatility, GARCH, state space models, Bayesian forecasting, Kalman filter, Wishart distribution.

1 Introduction

Over the last two decades, considerable effort has been devoted to the development of time-varying volatility models and related computational algorithms. It is widely recognized that volatility modeling has important implications for the analysis of returns on stocks and exchange rates. More recently, attention has moved to examining the implications of volatility for other financial applications such as derivatives pricing, optimal portfolio selection, and risk management (for instance, to enable efficient forecasting of Value-at-Risk). Although several univariate volatility models have been developed and are routinely used, the time-changing feature of the volatility is better described by multivariate models that explicitly account for cross-correlations among asset returns. A multivariate framework is desirable because assets

∗Department of Probability and Statistics, Hicks Building, University of Sheffield, Sheffield S3 7RH, UK, email: k.triantafyllopoulos@sheffield.ac.uk
†Department of Mathematics, Statistics Section, Imperial College London, London SW7 2AZ, UK, email: g.montana@imperial.ac.uk
can be formally linked together and can be influenced by common unobserved factors; as a consequence of this, we often observe related movements between markets, or sectors, or exchange rates.

The many efforts to model multivariate volatility fall into two main classes of models: multivariate generalized auto-regressive heteroscedastic (M-GARCH) models and multivariate stochastic volatility (MSV) models. The review paper by Bauwens et al. [2006] well describes the capabilities and limitations of M-GARCH models. In brief, the large number of parameters, which are typically specified by maximum likelihood estimation, and the fact that the unobserved volatility is not modelled as a stochastic process, somehow limit the applicability of these models. On the other hand, MSV models are more flexible, because the volatility is assumed to change stochastically according to a latent process. However, most stochastic volatility models, as reviewed for instance in Yu and Meyer [2006], Liesenfeld and Richard [2006], Asai et al. [2006], and Maasoumi and McAleer [2006], need essentially to resort to stochastic simulation schemes such as Markov chain Monte Carlo methods (MCMC), which may be heavily computationally intensive. Although much progress has been made on the front of simulation-based procedures, and more efficient algorithms are now available, the iterative nature of such procedures hampers the applicability of multivariate stochastic volatility estimation in real-time applications where, for instance, prompt user interventions may be required [Salvador and Gargallo, 2004]. For such reasons, it would be desirable to rely on analytic solutions that translate into fast and flexible algorithms, while still enjoying some of the advantages offered by MSV models.

Computational solutions that trade off the complexity of the model for speed are valuable, and have been explored in the literature. A simplification that facilitates the development of inferential procedures is to assume that the volatility follows a random walk (RW) evolution. This assumption has been often adopted in the relevant literature, for instance in the works of Quintana and West [1987], Putnam and Quintana [1994], Quintana and Putnam [1996], West and Harrison [1997], Uhlig [1997], Liu [2000], Soyer and Tanyeri [2006], Carvalho and West [2007], and references therein. For instance, Harvey et al. [1994] suggest an approximate inferential method for a MSV model based on the extended Kalman filter using crude mean and variance approximations; although the evolution of the volatility matrix is defined as an autoregressive (AR) process, the authors suggest that a RW evolution works equally well.

In this work we elaborate on some of the results that have already been proposed in the literature mentioned above. Using the convolution of the Wishart and singular multivariate beta distributions, which was first proved in Uhlig [1994], we construct a RW model for the evolution of the volatility. In the works of Aguilar and West [2000], Liu [2000], Soyer and Tanyeri [2006], and Carvalho and West [2007], all adopting the RW assumption, the multivariate
volatility estimators resemble their counterpart univariate estimators based on gamma and beta distributions [West and Harrison, 1997, Triantafyllopoulos, 2007]. However, we have noticed that these estimators are incorrectly derived, in that they give rise to a shrinkage volatility evolution, which is not a realistic choice. In particular, we demonstrate how the multivariate beta density has been overlooked in the above references to the point that the updating equation for the degrees of freedom has been wrongly computed. The resulting volatility estimator proposed in this paper is a weighted average of the square logarithmic returns. Thus, with proper choice of the weights, the modeller obtains volatility estimators that guarantee mean reversion over time and are appropriate to analyze volatility.

This paper is organized as follows. Section 2.1 defines the model and the Bayesian estimation procedure is given in Section 2.2. Section 2.3 is concerned with model assessment and selection, and three performance measures are derived, namely the log-likelihood criterion, the mean of the standardized one-step forecast errors, and sequential Bayes factors. Section 3 applies our methods to a data set comprising eight foreign exchange rates vis-à-vis the US dollar. A proof of Section 2.3 can be found in the appendix.

2 Stochastic volatility

2.1 The model

Consider a p-variate vector of log-returns \( \{y_t\}_{t=1,\ldots,N} \), where \( t \) is the time index, for some positive integer \( N \). The zero-drift conditional volatility model assumes

\[
y_t = \Sigma_t^{1/2} \epsilon_t, \quad \epsilon_t \sim N_p(0, I_p), \quad t = 1, \ldots, N,
\]

where \( \Sigma_t \) is the conditional volatility matrix of \( y_t \), \( \epsilon_t \) is a p-variate innovation vector following a p-variate Gaussian distribution with zero mean vector and identity covariance matrix; finally, \( \Sigma_t^{1/2} \) denotes the square root of \( \Sigma_t \), using the Choleski decomposition or the spectral decomposition [Gupta and Nagar, 2000].

At time \( t \), let \( y^t = \{y_1, \ldots, y_t\} \) denote the information set, comprising data up to time \( t = 1, \ldots, N \). In order to estimate \( \Sigma_t \), we need to define an evolution law for \( \Sigma_t \). A sensible law postulates that

\[
\mathbb{E}(\Sigma_t^{-1}|y^t) = \mathbb{E}(\Sigma_t^{-1}|y^t),
\]

namely the expectation from time \( t \) to \( t+1 \) remains unchanged, and

\[
\text{Var}(\text{vecp}(\Sigma_t^{-1}|y^t)) \geq \text{Var}(\text{vecp}(\Sigma_t^{-1}|y^t)),
\]

where \( \text{vecp}(\Sigma_t^{-1}) \) denotes the column stacking operator of the covariance matrix \( \Sigma_t^{-1} \). These assumptions define a random-walk type evolution law for \( \Sigma_t^{-1} \), i.e. \( \Sigma_{t+1}^{-1} = \Sigma_t^{-1} + \Gamma_t \), where
\( \Gamma_t \) has zero mean. Such an evolution is possible under the multiplicative law of covariance matrices of Uhlig [1994], that is

\[
\Sigma_{t+1}^{-1} = kU(\Sigma_t^{-1})'B_{t+1}U(\Sigma_t^{-1}), \quad t = 0, 1, \ldots, N - 1,
\]

where \( U(\Sigma_t^{-1}) \) denotes the upper triangular matrix of the Choleski decomposition of \( \Sigma_t^{-1} \), so that \( \Sigma_t^{-1} = U(\Sigma_t^{-1})'U(\Sigma_t^{-1}) \). Here \( B_{t+1} \) follows, independently of \( \Sigma_t^{-1} \), the singular multivariate beta distribution (whose density is given in equation (A-1) of the appendix). Initially, we assume the inverted Wishart prior

\[
\Sigma_0 \sim IW_p(n + 2p, S_0), \quad n = \frac{1}{1 - \delta},
\]

with density function

\[
p(\Sigma_0) = \frac{|S_0|^{(n+p-1)/2} \text{etr}(-S_0 \Sigma_0^{-1})}{2^{p(n+p-1)/2} \Gamma_p((n + p - 1)/2)|\Sigma_0|^{(n+2p)/2}},
\]

where \( 0 < \delta < 1 \) is a discount factor, \( |S_0| \) is the determinant of \( S_0 \), \( \text{etr}(\cdot) \) stands for the exponent of a trace of a matrix, and \( \Gamma_p(\cdot) \) denotes the multivariate gamma function. It is also assumed that the innovation sequence \( \{\epsilon_t\} \) is uncorrelated and that \( \{\epsilon_t\} \) is uncorrelated with \( \Sigma_0 \), i.e. \( \mathbb{E}(\epsilon_t\epsilon_s') = 0 \) (for any \( t \neq s \)) and \( \mathbb{E}(\epsilon_t\text{vecp}(\Sigma_0)') = 0 \) (for all \( t \)). From the above inverted Wishart prior it turns out that \( \Sigma_0^{-1} \) follows the Wishart distribution with \( n + p - 1 \) degrees of freedom and scale matrix \( S_0^{-1} \), i.e. \( \Sigma_0^{-1} \sim W_p(n + p - 1, S_0^{-1}) \).

In order to completely specify this model, a value for the parameter \( k \) has to be specified. In Section 2.2 it is shown that in order to guarantee the expectation invariance property (2) of the RW model, it is necessary to specify \( k \) as

\[
k = \frac{\delta(1 - p) + p}{\delta(2 - p) + p - 1}.
\]

### 2.2 Estimation

Suppose that at time \( t \), the posterior distribution of \( \Sigma_t \) is

\[
\Sigma_t|y^t \sim IW_p(n + 2p, S_t),
\]

where \( n = 1/(1 - \delta) \) and \( S_t \) is known. For the singular multivariate beta density of \( B_{t+1} \), we write \( B_{t+1} \sim B_p(m/2, 1/2) \), where \( m = \delta(1 - \delta)^{-1} + p - 1 \). The “singularity” of the distribution derives from \( 1 < p - 1 \), for any \( p > 1 \) and so the matrix \( I_p - B_{t+1} \) is singular (for more details the reader is referred to Uhlig [1994] and Díaz-García and Gutiérrez [1997]). The choice of \( m \) is conveniently made so that two of the assumptions of the beta density are satisfied, that is \( m > p - 1 \) and \( (1 - \delta)n \) has to be an integer (see also the last paragraph of Section 2.2).
Since $\Sigma_t^{-1}|y^t \sim W_p(n + p - 1, S_t^{-1})$, from the evolution (4) and from Uhlig [1994], it follows that $k^{-1}\Sigma_{t+1}^{-1}|y^t \sim W_p(n + p - 1, S_t^{-1})$ or $\Sigma_{t+1}^{-1}|y^t \sim W_p(n + p - 1, kS_t^{-1})$ and so the prior distribution of $\Sigma_{t+1}$ is

$$\Sigma_{t+1}|y^t \sim IW_p(\delta n + 2p, k^{-1}S_t).$$  \(7\)

From (9) we have $\mathbb{E}(\Sigma_t^{-1}|y^t) = (n + p - 1)S_t^{-1}$ and from (11) we have $\mathbb{E}(\Sigma_{t+1}^{-1}|y^t) = (\delta n + p - 1)kS_t^{-1}$, and so by equalizing these two expectations we obtain

$$k = \frac{n + p - 1}{\delta n + p - 1} = \frac{\delta(1 - p) + p}{\delta(2 - p) + p - 1},$$

as in (5). Using properties of the Wishart distribution, and adopting $k$ as proposed above, one can verify that $\text{Var}(\text{vec}(\Sigma_{t+1}^{-1})|y^t) \geq \text{Var}(\text{vec}(\Sigma_t^{-1})|y^t)$, thus the RW type evolution (3) is verified.

Proceeding now with the posterior distribution at time $t + 1$, we apply Bayes theorem by noting that the likelihood function from the single observation $y_{t+1}$ is $p(y_{t+1}|\Sigma_{t+1})$, which from (4) is the $p$-variate Gaussian distribution $N_p(0, \Sigma_{t+1})$. Thus

$$p(\Sigma_{t+1}|y^{t+1}) = \frac{p(y_{t+1}|\Sigma_{t+1}, y^t)p(\Sigma_{t+1}|y^t)}{p(y_{t+1}|y^t)}$$

$$\propto \frac{\exp(-y_{t+1}'\Sigma_{t+1}^{-1}y_{t+1}/2)[k^{-1}S_t^{-(\delta n + p - 1)/2} \exp(-k^{-1}S_t\Sigma_{t+1}^{-1}/2)]}{[\Sigma_{t+1}]^{1/2}[\Sigma_{t+1}]^{((\delta n + 2p)/2)}$$

$$= [\Sigma_{t+1}]^{-(\delta n + 1 + 2p)/2} \exp(-(y_{t+1}'y_{t+1}/2) + k^{-1}S_t\Sigma_{t+1}^{-1}/2),$$

which is proportional to

$$\Sigma_{t+1}|y^{t+1} \sim IW_p(n + 2p, S_{t+1}),$$  \(8\)

where $S_{t+1} = k^{-1}S_t + y_{t+1}'y_{t+1}$, since $\delta n + 1 = n$.

Equations (6), (7) and (8), together with the prior (4) constitute a full algorithm, for $t = 1, \ldots, N - 1$. We remark that, for $p = 1$ and $k = 1/\delta$, the above results reduce to the usual algorithm for univariate stochastic volatility estimation, as reported in West and Harrison [1997] and Triantafyllopoulos [2007].

For $p \geq 1$, we see that, since $\delta < 1$, we have $\delta(1 - p) + p > \delta(2 - p) + p - 1$ and so $0 < k^{-1} < 1$. Thus by expanding $S_t$ as

$$S_t = k^{-t}S_0 + \sum_{j=1}^{t} k^{j-t}y_j'y_j', \quad t = 1, \ldots, N,$$

we can approximate $S_t$ by

$$S_t \approx \sum_{j=1}^{t} k^{j-t}y_j'y_j'$$  \(9\)
and exclude the influence of the prior \(S_0\), which anyway is deflated as \(t\) increases. We note that \(S_t\) is just a weighted average of the log-returns \(\{y_jy_j'\}_{j=1,...,t}\) with weights \(k^{-1}\). From this it follows that even if \(\{\Sigma^{-1}_t\}\) follows a random walk, the estimator \(S_t\) is still capable of exploiting mean reversion of the log-returns (as it is a weighted average of the squares of log-returns) and thus it is a suitable estimator for the volatility. The posterior mean of \(\Sigma_t\) and the prior mean at \(\Sigma_{t+1}\) can be derived easily from the inverted Wishart densities, i.e.

\[
E(\Sigma_t|y^t) = \frac{S_t}{n-2} = \frac{(1-\delta)S_t}{2\delta-1} \quad \text{and} \quad E(\Sigma_{t+1}|y^t) = \frac{k^{-1}S_t}{\delta n-2} = \frac{(1-\delta)S_t}{k(3\delta-2)},
\]

the posterior mean being defined for \(\delta > 1/2\) and the prior mean being defined for \(\delta > 2/3\).

In related work, a number of authors such as [Quintana and West 1987, West and Harrison 1997, Chapter 16, Aguilar and West 2000, Liu 2000, Soyer and Tanyeri 2006, and Carvalho and West 2007] have suggested to use \(k=1/\delta\). Although it is easily verified that this is a correct choice when \(p=1\), setting \(k=1/\delta\) when \(p>1\) results in a shrinkage-type evolution for \(\Sigma^{-1}_t\). This can be seen by first noting that, with \(k=1/\delta\), we have

\[
E(\Sigma^{-1}_{t+1}|y^t) - E(\Sigma^{-1}_t|y^t) = (p-1)(\delta^{-1} - 1)S^{-1}_t
\]

and therefore the expectation is not preserved from time \(t\) to \(t+1\), as we have \(E(\Sigma^{-1}_{t+1}|y^t) > E(\Sigma^{-1}_t|y^t)\).

In particular, when \(p\) is large, even if \(\delta \approx 1\), the above model postulates that the estimate of \(\Sigma^{-1}_{t+1}\) is larger than that of \(\Sigma^{-1}_t\). In other words \(\{\Sigma^{-1}_t\}\) follows an AR model \(\Sigma^{-1}_t = \alpha \Sigma^{-1}_{t-1} + \Gamma_t\), where \(\alpha > 1\); such a setting is clearly inappropriate. With the RW type evolution of \(\Sigma^{-1}_t\), claimed in all the above references, assuming that the limit of \(S_t\) exists, it follows from \(10\) that \(0 = (p-1)(\delta^{-1} - 1) \lim_{t \to \infty} S_t\). This, for \(p > 1\), implies that \(\delta = 1\) or \(\lim_{t \to \infty} S^{-1}_t = 0\), two meaningless results. Our suggestion is that \(\delta\) should be replaced by \(k^{-1}\), as in \([5]\), a choice that now preserves the expectations.

Furthermore, for \(p > 1\), the updating equation of the degrees of freedom of the Wishart distribution suggested in the above references, namely

\[
n_t + 2p = \delta n_{t-1} + 1 + 2p = n_0 \delta^t + (1-\delta^t)/(1-\delta) + 2p,
\]

does not seem to be correct. The reason for this lies in the multivariate singular beta distribution, \(B_p(m_1/2, m_2/2)\) which is only defined for \(m_2\) being a positive integer [Uhlig 1994]. Setting \(m_2 = (1-\delta)n_t\), as in [West and Harrison 1997] and [Soyer and Tanyeri 2006], results in \(m_2\) not being a positive integer. In our algorithm, we resolve this issue by setting \(n_t = n = 1/(1-\delta)\) so that \(m_2 = (1-\delta)n = 1\). For more details on the multivariate singular beta distribution the reader is referred to [Uhlig 1994, Díaz-García and Gutiérrez 1997, and Srivastava 2003]; the density function of this distribution is given in equation \([A-1]\) of the appendix.
2.3 Performance measures

2.3.1 The likelihood function

One method of model judgement and model comparison is via the likelihood function. In this section, first we derive the likelihood of our model in closed form. Adopting approximation (9), the only parameters that need to be selected in order to fully specify the model is the scalar $\delta$, since $k$ is specified in (5). Using the following result of Theorem 1, one possibility is to choose the value of $\delta$ that maximizes the log-likelihood function (under the restriction $2/3 < \delta < 1$).

**Theorem 1.** In model (1)-(3) the log-likelihood function of $\Sigma_1, \ldots, \Sigma_N$, based on data $y_1, \ldots, y_N$ is

$$c - \frac{1}{2} \sum_{t=1}^{N} y_i' \Sigma_t^{-1} y_t + \frac{2\delta - 1}{2(1 - \delta)} \sum_{t=1}^{N} \log |\Sigma_t^{-1}| - \frac{p}{2} \sum_{t=1}^{N} \log |L_t| - \frac{3\delta - 2}{2(1 - \delta)} \sum_{t=1}^{N} \log |\Sigma_t|,$$

for

$$c = -\frac{Np}{2} \log \pi - \frac{N}{2} \log 2\pi - \frac{Np(2\delta - 1)}{2(1 - \delta)} \log k + N \log \frac{\Gamma_p\{2^{-1}(1 - \delta)^{-1}(\delta(1 - p) + p)\}}{\Gamma_p\{2^{-1}(1 - \delta)^{-1}(\delta(2 - p) + p - 1)\}},$$

where $\delta > 2/3$, $k$ is as in (5) and $L_t$ is the diagonal matrix with diagonal elements the positive eigenvalues of $I_p - k^{-1}\{U(\Sigma_{t-1}^{-1})'\}^{-1}\Sigma_t^{-1}\{U(\Sigma_{t-1}^{-1})\}^{-1}$, with $\Sigma_t^{-1} = U(\Sigma_t^{-1})'U(\Sigma_t^{-1})$.

The proof of this result can be found in the appendix. A common modelling strategy in Bayesian inference is to plug the posterior mean of $\Sigma_t$ in to the likelihood function and then to compare models by comparing their likelihood functions (e.g. see Leonard and Hsu [1999]). This approach has common roots to estimation methods using the profile likelihood Lütkepohl [2005], Leonard and Hsu [1999], and clearly it has the advantage of combining Bayes estimation with likelihood-based inference. In addition to that, this approach can be very useful for choosing nuisance parameters, such as the discount factor $\delta$. The maximization of the log-likelihood function with respect to $\delta$ may be slow because this is a non-linear function in $\delta$. A possibility would be to evaluate the log-likelihood function only on a few admissible values for $\delta$ ($2/3 < \delta < 1$). Values of $\delta$ lower than 0.7 can result in very volatile, not smooth, and thus unstable posterior estimates of $\Sigma_t$; values of $\delta$ larger than 0.95 can result in very smooth estimates of $\Sigma_t$, not able to capture the clusters and the spikes of the volatility. In this paper (see the illustration of Section 3), we recommend exploring values of $\delta$ in the range $0.7, 0.75, 0.8, 0.85, 0.9, 0.95$. West and Harrison [1997] and Triantafyllopoulos and Nason [2007] have some discussion on the performance of the posterior estimates at the boundary values of discount factors $\delta > 0.95$.
2.3.2 One-step forecast error

Other than the log-likelihood function, the mean of square standardized one-step forecast error vector (MSSE) provides another performance measure. From (1) the one-step forecast distribution of \( y_{t+1}|y^t \) is a \( \hat{p} \)-variate Student \( t \) density with \( \delta/(1-\delta) \) degrees of freedom, mean vector 0 and scale matrix \( k^{-1}S_t \), written \( y_{t+1}|y^t \sim t_p(\delta/(1-\delta),0,k^{-1}S_t) \) [Gupta and Nagar, 2000]. It then follows that, for \( \delta > 2/3 \),

\[
\text{Var}(y_{t+1}|y^t) = \frac{k^{-1}S_t}{\delta/(1-\delta) - 2} = \frac{(1-\delta)S_t}{(3\delta - 2)k},
\]

which also can be derived from Section 2.2 using conditional expectations, i.e.

\[
\text{Var}(y_{t+1}|y^t) = \mathbb{E}(\text{Var}(y_{t+1}|\Sigma_{t+1}, y^t)|y^t) = \mathbb{E}(\Sigma_{t+1}|y^t) = \frac{(1-\delta)S_t}{(3\delta - 2)k},
\]

since from model (1), it is \( \text{Var}(\mathbb{E}(y_{t+1}|\Sigma_{t+1}, y^t)|y^t) = 0 \). Having obtained an expression for the variance, we can now write the standardized one-step forecast error vector \( u_{t+1} \) as

\[
u_{t+1} = \sqrt{kS_t^{-1/2}}y_{t+1} \quad \text{with} \quad u_{t+1}|y^t \sim t_p\left(\frac{\delta}{1-\delta},0,I_p\right)
\]

so that the vector

\[
u_{t+1}^* = \left\{ \frac{(1-\delta)S_t}{(3\delta - 2)k} \right\}^{-1/2}y_{t+1}
\]

has \( \mathbb{E}(u_{t+1}^*|y^t) = 0 \) and \( \mathbb{E}(u_{t+1}^*(u_{t+1}^*)'|y^t) = I_p \). Then the MSSE vector is given by

\[
\text{MSSE} = \frac{1}{N} \sum_{t=1}^{N} \left\{ (u_{1t}^*)^2, \ldots, (u_{pt}^*)^2 \right\}',
\]

where \( u_t^* = (u_{1t}^*, \ldots, u_{pt}^*)' \). Models that fit well the data are expected to yield \( \text{MSSE} \approx (1, \ldots, 1)' \).

2.3.3 Bayes factors

A third approach for model diagnostics is based on sequential Bayes factors [West and Harrison, 1997, Salvador and Gargallo, 2004, Triantafyllopoulos, 2006]. Suppose we have two competing models, \( M_1 \) and \( M_2 \), parameterized in terms of \( \delta_1 \) and \( \delta_2 \), respectively. First, a Bayes factor is obtained as the logarithm of the ratio between the density of \( u_t \equiv u_t(\delta_1) \) (under \( M_1 \)) and the density of \( u_t \equiv u_t(\delta_2) \) (under \( M_2 \)). Specifically, at each time \( t \) we have

\[
H_t = \log \frac{p(u_t(\delta_1)|y^{t-1}, M_1)}{p(u_t(\delta_2)|y^{t-1}, M_2)}, \quad t = 1, \ldots, N,
\]

and, from the Student \( t \) density (11), this becomes

\[
H_t = \frac{\Gamma\left((n_1+p)/2\right)\Gamma\left(n_2/2\right)}{\Gamma\left((n_2+p)/2\right)\Gamma\left(n_1/2\right)} \left\{ \frac{1 + u_t(\delta_2)'u_t(\delta_2)}{1 + u_t(\delta_1)'u_t(\delta_1)} \right\}^{1/2},
\]
where $\Gamma(.)$ denotes the gamma function and $n_i = \delta_i/(1 - \delta_i)$, for $i = 1, 2$.

A value of $H_t > 0$ then suggests that model $\mathcal{M}_1$ has to be preferred over $\mathcal{M}_2$, in the sense that $\mathcal{M}_1$ is associated with a superior forecast distribution. Alternative, negative values for $H_t$ suggest that $\mathcal{M}_2$ is the preferred model. In situation where $H_t = 0$, both models are deemed equivalent. One point of interest is what decision can we make when $H_t$ fluctuates around zero. In such a case one may select a threshold value in order to decide which model to choose, as in West and Harrison [1997].

3 An illustration using foreign exchange rates

In this section we present an analysis of eight exchange rates vis-à-vis the US dollar. The exchange rates are the Australian dollar (AUS), British pounds (GBP), Canadian dollar (CAD), German Deutschmark (GDM), Dutch guilder (DUG), French frank (FRF), Japanese yen (JPY) and Swiss franc (SWF), all expressed as number of units of the foreign currency per US dollar. The sample period runs from 2 January 1980 until 31 December 1997, and corresponds to 4774 observations, sampled at daily frequencies. This data set was originally
Figure 2: Sequential Bayes factor $H_t$ of the standardized one-step forecast errors of model $\mathcal{M}_1 (\delta = 0.7)$ vs model $\mathcal{M}_2 (\delta = 0.95)$. 

obtained from the New York Federal Reserve, and then discussed in Franses and van Dijk [2000]. Figure 1 illustrates the daily observations on the level of all eight exchange rates.

We have applied the stochastic volatility model of Section 2.2 to the logarithmic returns, which have been collected in a vector $y_t = (y_{1t}, \ldots, y_{8t})'$. Following the empirical studies of exchange rates, as in Quintana and West [1987], Putnam and Quintana [1994], and Quintana and Putnam [1996], we adopt the random walk for the evolution of the volatility and thus we specify $k$ as in (5). In order to choose a suitable value for the parameter $\delta$, we have used the performance measures described in Section 2.3. Following suggestions in that section, we have only considered a few selected values of $\delta$ in the range $0.7 \leq \delta \leq 0.95$. The results from this analysis are summarized in Table 1 which provides the mean of the MSSE (MMSSE), the log-likelihood function (evaluated at the posterior mean of the volatility), and the mean of the Bayes factors of the standardized one-step forecast errors. For the computation of the Bayes factors here, each $\mathcal{M}_1$ is based on the current value of $\delta$, and is compared against a baseline model $\mathcal{M}_2$ that uses $\delta = 0.95$.

From Table 1 it can be observed that for small values of $\delta$, the MMSSE also attains small values, indicating poor performance, when compared to an ideal MMSSE value of one. This result seems to suggest that the forecast covariance matrix of $y_t$ has been over-estimated. As $\delta$ gets close to one, the MMSSE also gets close to one, which underlines an improvement in
Table 1: Mean (over the eight exchange rates) of the mean square one-step forecast standardized errors (MMSSE), log-likelihood function (LogL) evaluated at the posterior mean of the volatility, and mean of the log Bayes factor $H_t$ ($t = 1, \ldots, 4774$).

| $\delta$ | MMSSE | LogL   | $H$   |
|----------|-------|--------|-------|
| 0.70     | 0.072 | -12857.59 | -6.269 |
| 0.75     | 0.194 | -12395.30 | -5.681 |
| 0.80     | 0.337 | -11721.93 | -4.950 |
| 0.85     | 0.506 | -10644.32 | -3.982 |
| 0.90     | 0.701 | -8627.03  | -2.564 |
| 0.95     | 0.912 | -3458.23  | 0      |

the estimation of the forecast covariance matrix of $y_t$. The log-likelihood function attains its largest value at $\delta = 0.95$. For each $\delta < 0.95$, the Bayes factor mean $H$ is negative and this indicates a preference in favour of model $M_2$ (for $\delta = 0.95$). In particular we note that the model performance deteriorates as $\delta$ decreases, a fact that is captured by all three diagnostic measures considered here. As a result of this, we conclude that $\delta = 0.95$ produces the best model.

Figure 2 shows the log-Bayes factor sequence $\{H_t\}$, from which the superiority of model $M_2$ is clear. We observe that, out of $N = 4774$ data points, $\{H_t\}$ is positive at only 37 points (i.e. only 0.77% of the time). Using sequential Bayes factors, the modeler has the extra advantage of choosing the discount factor at each time $t$ according to the sign of $H_t$. This is particularly advantageous in an on-line setting, and when decisions have to be made in real time.

Figure 3 shows the posterior volatilities, i.e. the estimates of $\sigma_{ii,t}$ ($i = 1, \ldots, 8$), for a subset of the data points ($t = 4001, \ldots, 4774$). Most of the volatilities are small, except for the JPY/USD; even for small volatilities, this figure indicates clearly the highly volatile periods for each exchange rate. Figure 4 shows the posterior correlations of GBP/USD versus all the other rates. This figure confirms that the correlations are time-varying. By inspecting Figure 4 we observe that GBP/USD is most correlated with DUG/USD, FRF/USD, GDM/USD, and SWF/USD.

Finally we note that, for this relatively large data set, based on 4774 time points in 8 dimensions, the estimation algorithm (implemented in the R language on a Windows platform) took less than a minute (55 seconds) to complete, on a PC with Intel(R) Celeron(R)M Processor 1.60GHz and 504MB RAM, including the evaluation of the log-likelihood function.
Figure 3: Estimate of the posterior volatility for the FX data, using the model with $\delta = 0.95$. and the Bayes factors.

### 4 Conclusions

In this paper we have described a Bayesian modeling approach for multivariate stochastic volatility. The proposed estimation methodology is delivered in closed form, is easily implementable and efficient, as the model relies on only one parameter.

The models proposed in this paper are closely related to the above mentioned articles as well as to the models of [Uhlig 1997] and [Philipov and Glickman 2006]. Notably, we have shown that similar volatility estimators proposed in the literature are based on a shrinkage-type volatility evolution, which is not a realistic choice. Instead, the estimator described here guarantees a random walk type evolution.

The procedure proposed in this paper attempts to combine the simplicity of non-iterative algorithms with the sophistication of stochastic volatility models. In our view, algorithms such as the one suggested here are particularly attractive because they can model high dimensional data with low computational cost, which is crucial for certain real-time applications in modern
computational finance, such as algorithmic trading. Future research efforts will be directed
towards other financial applications with special focus on optimal portfolio allocation.

Appendix

Proof of Theorem 1. First we derive the density of $\Sigma_t|\Sigma_{t-1}$, for $t = 1, \ldots, N$. From (3), it is $B_t \sim B_p(m/2, 1/2)$, for $m = \delta(1 - \delta)^{-1} + p - 1$, with density

$$p(B_t) = \pi^{-p/2} \Gamma_p((m + 1)/2) |K_t|^{-p/2} |B_t|^{(m-p-1)/2},$$  \hspace{1cm} (A-1)

where $I_p - B_t = H_1 K_t H_1'$, $K_t$ is the diagonal matrix with diagonal elements the positive
eigenvalues of $I_p - B_t$, and $H_1$ is a matrix with orthogonal columns, i.e. $H_1 H_1' = I_p$. For more details on this distribution see [Uhlig 1994].

Now from evolution (3) we have the transformation from $B_t$ to $\Sigma_t = k^{-1}(U(\Sigma_{t-1})^{-1}B_t^{-1} \times (U(\Sigma_{t-1})^{-1})^{-1}$. From [Díaz-García and Gutiérrez 1996] the Jacobian of this transformation is

$$(dB_t) = |K_t|^{p/2} |L_t|^{-p/2} k^{-p/2} |\Sigma_{t-1}|^{1/2} (d\Sigma_t).$$
where $L_t$ is the diagonal matrix with diagonal elements the positive eigenvalues of $I_p - k^{-1}(U(\Sigma_{t-1}^{-1})')^{-1} \Sigma_t^{-1}(U(\Sigma_{t-1}^{-1}))^{-1}$. From the above transformation it is

$$|U(\Sigma_{t-1}^{-1})| = |U(\Sigma_{t-1}^{-1})'U(\Sigma_{t-1}^{-1})|^{1/2} = |\Sigma_{t-1}|^{1/2} \quad \text{and} \quad |B_t| = k^{-p}|\Sigma_{t-1}|^2|\Sigma_t|^{-1}$$

and thus from (A-1)

$$p(\Sigma_t|\Sigma_{t-1}) = \pi^{-p/2} \frac{\Gamma_p((m+1)/2)}{\Gamma_p(m/2)} k^{-p/2}|\Sigma_{t-1}|^{1/2} \times |K_t|^{-p/2}|B_t|^{(m-p-1)/2}|K_t|^{p/2}|L_t|^{-p/2} = \pi^{-p/2} k^{-p(m-p)/2} \frac{\Gamma_p((m+1)/2)}{\Gamma_p(m/2)} |L_t|^{-p/2} \times |\Sigma_{t-1}|^{(m-p)/2}|\Sigma_t|^{-(m-p-1)/2}. \quad (A-2)$$

For the likelihood function $L(\Sigma; y)$, where $\Sigma = (\Sigma_1, \ldots, \Sigma_N)$ and $y = (y_1, \ldots, y_N)$, write

$$L(\Sigma; y) = \prod_{t=1}^N p(y_t|\Sigma_t)p(\Sigma_t|\Sigma_{t-1}).$$

From equation (1) we have $y_t|\Sigma_t \sim N_p(0, \Sigma_t)$, while the density of $\Sigma_t|\Sigma_{t-1}$ is given by (A-2).

The required formula of the log-likelihood function is obtained by taking the logarithm of $L(\Sigma; y)$. \hfill \Box

References

O. Aguilar and M. West. Bayesian dynamic factor models and portfolio allocation. *Journal of Business and Economic Statistics*, 18:338–357, 2000.

M. Asai, M. McAleer, and J. Yu. Multivariate stochastic volatility: A review. *Econometric Reviews*, 25:145–175, 2006.

L. Bauwens, S. Laurent, and J.V.K. Rombouts. Multivariate GARCH models: A survey. *Journal of Applied Econometrics*, 21:79–109, 2006.

C.M. Carvalho and M. West. Dynamic matrix-variate graphical models. *Bayesian Analysis*, 2:69–98, 2007.

J.A. Díaz-García and J.R. Gutiérrez. Proof of the conjectures of H. Uhlig on the singular multivariate beta and the jacobian of a certain matrix transformation. *Annals of Statistics*, 25:2018–2023, 1997.

P.H. Franses and D. van Dijk. *Nonlinear Time Series Models in Empirical Finance*. Cambridge University Press Cambridge, 2000.
A.K. Gupta and D.K. Nagar. *Matrix Variate Distributions*. Chapman and Hall New York, 2000.

A.C. Harvey, E. Ruiz, and N. Shephard. Multivariate stochastic variance models. *Review of Economic Studies*, 61:247–264, 1994.

T. Leonard and J.S.J. Hsu. *Bayesian Methods*. Cambridge University Press Cambridge, 1999.

R. Liesenfeld and J.F. Richard. Classical and Bayesian analysis of univariate and multivariate stochastic volatility models. *Econometric Reviews*, 25:335–360, 2006.

J. Liu. *Bayesian Time Series: Analysis Methods Using Simulation-Based Computation*. PhD thesis, ISDS, Duke University, 2000.

H. Lütkepohl. *New Introduction to Multiple Time Series Analysis*. Springer New York, second edition, 2005.

E. Maasoumi and M. McAleer. Multivariate stochastic volatility: An overview. *Econometric Reviews*, 25:139–144, 2006.

A. Philipov and M.E. Glickman. Multivariate stochastic volatility via Wishart processes. *Journal of Business and Economic Statistics*, 24:313–328, 2006.

B.H Putnam and J.M. Quintana. New Bayesian statistical approaches to estimating and evaluating models of exchange rates determination. American Statistical Association - Section on Bayesian Statistical Science, pages 232–237, 1994.

J.M Quintana and B.H. Putnam. Debating currency markets efficiency using dynamic multiple-factor models. American Statistical Association - Section on Bayesian Statistical Science, pages 55–60, 1996.

J.M. Quintana and M. West. An analysis of international exchange rates using multivariate dLms. *Statistician*, 36:275–281, 1987.

M. Salvador and P. Gargallo. Automatic monitoring and intervention in multivariate dynamic linear models. *Computational Statistics and Data Analysis*, 47:401–431, 2004.

R. Soyer and K. Tanyeri. Bayesian portfolio selection with multi-variate random variance models. *European Journal of Operational Research*, 171:977–990, 2006.

M.S. Srivastava. Singular Wishart and multivariate beta distributions. *Annals of Statistics*, 31:1537–1560, 2003.
K. Triantafyllopoulos. Feedback quality adjustment with Bayesian state space models. *Applied Stochastic Models in Business and Industry*, 23:145–156, 2007.

K. Triantafyllopoulos. Multivariate control charts based on Bayesian state space models. *Quality and Reliability Engineering International*, 22:693–707, 2006.

K. Triantafyllopoulos and G.P. Nason. A Bayesian analysis of moving average processes with time-varying parameters. *Computational Statistics and Data Analysis*, 52:1025–1046, 2007.

H. Uhlig. On singular Wishart and singular multivariate beta distributions. *Annals of Statistics*, 22:395–405, 1994.

H. Uhlig. Bayesian vector autoregressions with stochastic volatility. *Econometrica*, 65:59–73, 1997.

M. West and P.J. Harrison. *Bayesian Forecasting and Dynamic Models*. Springer-Verlag New York, 2nd edition, 1997.

J. Yu and R. Meyer. Multivariate stochastic volatility models: Bayesian estimation and model comparison. *Econometric Reviews*, 25:361–384, 2006.