ON SOME OF BROUWER’S AXIOMS

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1. Introduction

1.1. Bishop’s disagreement with Brouwer. E. Bishop, who founded constructive analysis, has an ambivalent attitude towards L.E.J. Brouwer, who, a generation earlier, began intuitionistic mathematics. On the one hand, Bishop recognizes that Brouwer was the first to raise his voice against the disturbing fact that many mathematical theorems lack constructive content. He also judges that Brouwer made a good beginning with the necessary reconstruction of parts of mathematics. He agrees with him that disjunction and the existential quantifier should be interpreted constructively and that, as a consequence, the principle of the excluded third \( X \lor \neg X \) should be rejected.

On the other hand, he thinks Brouwer went the wrong way by introducing ‘semi-mystical elements’ into mathematics in order to prove the theorem that every (effectively total) function from \([0,1]\) to \(\mathbb{R}\) is uniformly continuous.

This theorem may be split up into two statements:
1. Every function from \([0,1]\) to \(\mathbb{R}\) is pointwise continuous.
2. Every pointwise continuous function from \([0,1]\) to \(\mathbb{R}\) is uniformly continuous.

The first statement is a consequence of Brouwer’s Continuity Principle and the second one follows from his Fan Theorem.

Bishop rejects Brouwer’s argument for the first conclusion. He says that a set like ‘the set of all functions from \([0,1]\) to \(\mathbb{R}\)’ seems to have little practical interest, meaning probably, that one does not need general statements about such functions like the statement 1.

In addition, he decides not to use the notion of a pointwise continuous function, see [1, page 66]. He defines a function from \(\mathbb{R}\) to \(\mathbb{R}\) to be continuous if and only if the function is uniformly continuous on every closed interval \([a,b]\), see [1, Chapter 2, Definition 4.5].

The statement 2 then becomes a tautology.

Brouwer’s Fan Theorem also implies that, given a closed interval \([a,b]\) and a sequence \(f_0,f_1,\ldots\) of functions from \([a,b]\) to \(\mathbb{R}\) that converges pointwise to a a function \(f\) from \([a,b]\) to \(\mathbb{R}\) will converge uniformly to \(f\). Bishop avoids the notion of pointwise convergence, see [1, Chapter 2, Definition 4.7].

Brouwer derived his Fan Theorem from a much stronger statement: the Bar Theorem. Bishop does not discuss this stronger statement.

1.2. Going back to Brouwer’s basic assumptions. Brouwer’s arguments for the statements 1 and 2 necessarily are of a philosophical rather than a mathematical nature. One should not put them aside as non-mathematical and, therefore, not worth a mathematician’s attention.

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1This paper has been written as a contribution for the Handbook of Constructive Mathematics, soon to appear with Springer Verlag. Unfortunately, it was completed so late that the editors were unable to consider it for inclusion in this handbook.

2This strategy may work for a locally compact space like \(\mathbb{R}\), but it does not help if one starts thinking on, for instance, continuous functions from Baire space \(\mathcal{N}\) to \(\omega\).
Besides, these arguments might have consequences that go further than the statements 1 and 2 in Subsection 1.1 and are possibly important for the development of constructive mathematics.

Brouwer is trying to redefine the game of mathematics and make it a better game than it has been up to now. The starting points of the game have to be agreed upon, and are a topic of ongoing discussion. They should be called axioms, although Brouwer avoids this expression. Brouwer, somewhat misleadingly, presents his axioms as being beyond doubt and speaking for themselves, and he does not distinguish the arguments supporting them from his more mathematical arguments.

1.3. The three varieties. Constructive mathematics is often described as having three varieties: BISH, i.e. Bishop style constructive mathematics, INT, i.e. intuitionistic mathematics and RUSS, i.e. Russian style constructive mathematics, where the notion of a computable function sets the scene, see [4]. This division is problematic.

It is difficult to make sense of the slogan:

In essence, BISH is simply mathematics with intuitionistic logic explained and defended in the preface to [3].

The slogan suggests that, for a classical mathematician who decides to join the constructive enterprise, mathematical objects remain the same in spite of the fact that he is changing the language he uses for describing them. This suggestion is wrong. It sounds as if the objects are and remain there, like animals in a zoo, while we, the visitors, start babbling about them in a foreign tongue.

More fundamentally, it does not seem to make sense to say:

‘We may prove statement X classically as well as intuitionistically’

as the statement X does not mean the same intuitionistically as it does classically. Classical and intuitionistic mathematicians do not speak the same language.

Of course, if we formalize mathematics we may prove combinatorial facts of the form:

Formula $\phi$ is provable in the classical as well as in the intuitionistic system.

But the meaning of the formula would change with the user of the formalisms. It probably would be better to say

BISH is part of INT. Only, an intuitionistic result is reckoned to belong to BISH if it can be proven without making use of either the Continuity Principle, the Fan Theorem or the Bar Theorem.

The third variety, RUSS, arises from BISH by adding the assumption that every real is given by an algorithm in the Church-Turing sense. The study of computable functions, however, is part of intuitionistic mathematics, in fact a part of intuitionistic number theory, and not an alternative for intuitionistic mathematics, see [34]. Unfortunately, the theory of computable functions, up to now, is mostly done from a classical point of view.

So, we propose the following picture:

RUSS $\subseteq$ BISH $\subseteq$ INT

where each of the three RUSS, BISH and INT is considered as a body of proven intuitionistically meaningful results.

CLASS, the collection of results obtained classically, does not occur in the picture. The constructive mathematician has no immediate understanding of results in CLASS. The inclusion BISH $\subseteq$ CLASS makes no sense.

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3: The picture is slightly inaccurate as the intuitionistic mathematician does not want to use Markov’s Principle, unlike some members of the Russian school.
1.4. The need for axioms. The constructive rebuilding of mathematics forces one to rethink radically the meaning of mathematical statements. One should keep in mind that the meaning of a mathematical statement is ultimately given by its proof. The proof should be seen as an explication or unfolding of the meaning of the statement.

In the proof we may refer to constructions we did earlier but, sometimes, we come to invoke ‘axioms’. An axiom might be seen as a stipulation on the meaning of some of the expressions we are using in our language. As such, axioms come very close to ‘definitions’.

Axioms arise from situations we want to consider as canonical, as setting an example.

For instance, once we have seen Euclid’s proof that there exist infinitely many primes, we see how we want to prove such a thing, and we define: ‘\(X \subseteq \omega\) is infinite if one is able to indicate an algorithm providing, given any finite list \((n_0, n_1, \ldots, n_{k-1})\) of natural numbers, an element of \(X\) not occurring in the list.’

Or, observing our own use of the disjunction, we may decide to lay down: ‘a proof of \(X \lor Y\) should consist either in a proof of \(X\) or in a proof of \(Y\).’

Setting up our common mathematical discourse, we have to discuss carefully the question which principles deserve the status of a canonical starting point for our arguments, i.e., the status of an axiom.

An axiom is not a truth solid as a rock that is beyond doubt and discussion. On the contrary, it is a proposal that invites and shapes discussion. It might be compared to an hypothesis or a thought experiment. Like Gödel suggested in the context of axiomatic set theory, an axiom may prove its value when we decide to follow its lead and try to see what we find when using it.

It is an illusion however that one can do without axioms, and it is wrong to condemn arguments defending axioms as philosophical and unmathematical and therefore not relevant.

Brouwer’s Continuity Principle, the Fan Theorem and the Bar Theorem, to be discussed later in this paper, may be seen as agreements on the meaning of certain statements of the form \(\forall x \exists y [xRy]\).

1.5. The contents of the paper. In this paper, we introduce three main axioms of intuitionistic mathematics: the Continuity Principle, the Fan Theorem and the Thesis on Bars in \(\mathcal{N}\). We briefly discuss their plausibility and then show some of their applications in intuitionistic mathematics.

Apart from this introductory Section, the paper contains eight Sections.

In Section 2 we explain Brouwer’s Continuity Principle. We show its famous consequence: the pointwise continuity of real functions. We also introduce axioms of countable choice.

In Section 3 we give a more sophisticated application of Brouwer’s Continuity Principle: the proof of the Borel Hierarchy Theorem, see [23, Chapter 9] and [31, Section 7]. The Continuity Principle is also important at other points in the development of intuitionistic descriptive set theory. It is crucial for proving the fine structure of the hierarchy, see [33], and a strong formulation of the Principle leads to the collapse of the projective hierarchy, see [23, Chapter 14] and [36, Section 7].

In Section 4 we explain the Fan Theorem. We give its first and most famous application: functions with domain \([0, 1]\) that are pointwise continuous are also uniformly continuous.

In Section 5 we sketch the intuitionistic development of the theory of measure and integration, as begun by Brouwer and developed further by some students of Heyting, see [15, Chapter VI]. Bishop, see [1, Chapter 6] decided not to follow Brouwer’s lead in this field, probably out of fear of the Fan Theorem.
In Section 6, we explain Brouwer’s Thesis on bars in $\mathcal{N}$. Brouwer introduced it for proving the Fan Theorem but it has stronger consequences than that. As an example, we prove the equivalence of two definitions of the class of decidable and well-founded subsets of the set of the rationals.

In Section 7, we explain the Almost-Fan Theorem. Like the Fan Theorem itself, the Almost-Fan Theorem follows from the Bar Theorem. For a classical spectator, the Almost-Fan Theorem is difficult to distinguish from the Fan Theorem itself. The Almost-Fan Theorem implies the Fan Theorem but it is a stronger statement and we will see it has other important consequences too.

In Section 8, we explain our notations.

2. Axioms of Continuity and Choice

2.1. The Continuity Principle.

**Axiom 1** (Brouwer’s Continuity Principle). For every relation $R \subseteq \omega^\omega \times \omega$, if $\forall \alpha \exists n[\alpha R n]$, then $\forall \alpha \exists n \forall \beta [\alpha n \sqsubseteq \beta \rightarrow \beta R n]$.

The Principle came on the scene in 1918. In \[6, p. 13\], Brouwer explains that there can not exist an injective function from the set $\mathcal{N}$ of all infinite sequences of natural numbers to the set $\omega$ of the natural numbers. He says: if we should have such a function, say $f$, and an element $\alpha$ of $\mathcal{N}$ is given, then $f(\alpha)$, the value of $f$ at $\alpha$, would have to be decided upon at a point of time at which only finitely many values of $\alpha$, say, $\alpha(0), \alpha(1), \ldots, \alpha(m-1)$, have become known. So all infinite sequences $\beta$ that take the same values as $\alpha$ on the arguments $0, 1, \ldots, m-1$ would be allotted the same value by $f$ as $\alpha$, and $f$ would be non-injective.

The principle Brouwer is using here, for the first time, is what we now call his Continuity Principle. In our formulation of the principle, the starting point seems to be a little bit more general: $\forall \alpha \exists n[\alpha R n]$, but we interpret this as: we have a method to assign to any given $\alpha$ a suitable $n$, i.e. there exists a function $f$ from $\mathcal{N}$ to $\omega$ such that $\forall \alpha[\alpha R f(\alpha)]$.

Brouwer is clearly imagining that the sequence $\alpha = \alpha(0), \alpha(1), \ldots$ is given to us only step by step. We learn its values one by one and have no information on the development of the sequence as a whole.

The first and foremost example of an infinite sequence is the sequence $\alpha(0) = 0, \alpha(1) = 1, \alpha(2) = 2, \ldots$ of the natural numbers themselves. Even this infinite sequence is growing step by step and never fully realized. Its construction is a job that we started to carry out and always will work on without ever finishing it. As one sometimes says, it is a project rather than an object.

Another infinite sequence that grows step by step is the sequence $\alpha(0) = 1, \alpha(1) = 4, \alpha(2) = 1, \ldots$ of the decimals of $\pi$.

These two examples do not give us a complete picture. Although we construct the natural numbers one by one and also calculate the decimals of $\pi$ one by one, we in both cases have a key for finding all values that precludes surprises. Brouwer calls such algorithmic infinite sequences lawlike sequences.

We want to make room for other infinite sequences too. We admit the possibility that the values of $\alpha$ are disclosed to us, or chosen by us, one by one, and that we do not know any finite algorithm that determines the values of $\alpha$.

Every individual member of the set $\mathcal{N}$ of the infinite sequences of natural numbers may be imagined to be always under construction and never completed.
Brouwer thus saw that ‘sets’ like $\mathcal{N}$ deserve careful treatment.

Cantor’s idea that a set is the result of collecting certain already existing objects, to be called its elements, into a new whole, is wrong. In this picture, the elements of the set are ‘earlier there’ than the set itself. The intuitionistic mathematician proposes to view a set like $\mathcal{N}$ as a realm of possibilities. A set is like a musical instrument, on which many tunes will be played in the future.

One should keep in mind that it does not make sense, intuitionistically, that something is the case without our knowing so. The meaning of a statement
\[ \forall \alpha \exists n[\alpha Rn] \]
must be that I see that I am able to effectively find a suitable $n$ to any possible $\alpha$. $\alpha$ may be given to me, or created by myself making free choices, value by value without, at any point of time, any further information on the whole of its course, and, what is very important, even if $\alpha$ is not given in this way but, somehow, at one stroke, $\alpha$ might have been given to me in this way.

In this way, we defend the claim that, if $\forall \alpha \exists n[\alpha Rn]$, then, given any $\alpha$, one must be able to come up with a suitable $n$ for $\alpha$ knowing only finitely many values of $\alpha$.

Note that an infinite sequence that I am creating freely, value by value, may:

\[ \exists x \forall n[\alpha_{n+1}] = (\varphi(\alpha))(n) \]

Note that an infinite sequence that I am creating freely, value by value, may turn out, in the end, to be a ‘simple’ one that admits of a finite description. The decimal expansion of $\pi$, for instance, may be the result of an infinite sequence of free choices.

### 2.2. A first application.

#### Theorem 1.
For every relation $R \subseteq \mathcal{R} \times \omega$, if $\forall x \exists n[x Rn]$, then $\forall x \exists m \forall y[|x - y| < \frac{1}{2^m} \rightarrow y Rn]$.

**Proof.** Assume $\forall x \in \mathcal{R} \exists n[x Rn]$. We prove the promised conclusion for the case $x = 0$, i.e. $\exists m \forall y[|y| < \frac{1}{2^m} \rightarrow y Rn]$. The general case is proven similarly.

We first define a function $\varphi$ from $\mathcal{N}$ to $\mathcal{N}$.

Let $q_0 := 0, q_1, q_2, \ldots$ be an enumeration of the rationals.

Let $\alpha$ be given.

We have to define the infinite sequence $\varphi(\alpha) = (\varphi(\alpha))(0), (\varphi(\alpha))(1), \ldots$ and we do so as follows.

We define $\varphi(\alpha)(0) = 0$.

For each $n$, if $|q_{\alpha(n+1)} - q_{\varphi(\alpha)(n)}| \leq \frac{1}{2^n}$, we define $\varphi(\alpha)(n + 1) = \alpha(n + 1)$, and, if not, we define $\varphi(\alpha)(n + 1) = (\varphi(\alpha))(n)$.

Note that, for each $\alpha$, the sequence $q_{\varphi(\alpha)(0)}, q_{\varphi(\alpha)(1)}, \ldots$ converges.

Also note that, if, for all $n$, $|q_{\alpha(n+1)} - q_{\alpha(n)}| \leq \frac{1}{2^n}$, then $\varphi(\alpha) = \alpha$.

We define, for each $\alpha$, $x_\alpha := \lim_{n \rightarrow \infty} q_{\varphi(\alpha)(n)}$.

Note: $\forall \alpha \exists n[x_\alpha Rn]$.

Apply the Continuity Principle and find $m, n$ such that $\forall \alpha[\exists m \supset \alpha \rightarrow x_\alpha Rn]$.

Note that, for each $x$ in $[-\frac{1}{2^n}, \frac{1}{2^n}]$ there exists $\alpha$ such that $\exists m \supset \alpha$ and $x = x_\alpha$.

Conclude: $\forall x \in [-\frac{1}{2^m}, \frac{1}{2^m}]$ there exists $\alpha$ such that $\exists m \supset \alpha$ and $x = x_\alpha$.

**Corollary 2.** Every function from $\mathcal{R}$ to $\mathcal{R}$ is pointwise continuous.

**Proof.** Let a function $f$ from $\mathcal{R}$ to $\mathcal{R}$ be given.

Let $q_0, q_1, \ldots$ be an enumeration of the rationals. Let $p$ be given.

Note $\forall x \in \mathcal{R} \exists n[|f(x) - q_n| < \frac{1}{2^{m+1}}]$.

Let $x$ be given. Applying Theorem 1 find $m, n$ such that $\forall y \in \mathcal{R}[|x - y| < \frac{1}{2^m} \rightarrow |f(x) - q_n| < \frac{1}{2^{m+1}}]$.

Conclude: $\forall y \in \mathcal{R}[|f(x) - f(y)| < \frac{1}{2^m} \rightarrow |f(x) - f(y)| < \frac{1}{2^{m+1}}]$.

We thus see: $\forall \exists n \exists y \in \mathcal{R}[|f(x) - f(y)| < \frac{1}{2^m} \rightarrow |f(x) - f(y)| < \frac{1}{2^{m+1}}]$.

i.e. $f$ is continuous at $x$. \qed
We thus see that Corollary 2 follows from the Continuity Principle alone. Brouwer seems to have thought the Fan Theorem is needed for this result, see [21, 24] and [27].

2.3. Spreads. The ‘set’ \( \omega^\omega \) is an example of a kind of sets that are called spreads. A spread is given by a spread-law \( \beta \). The elements of the spread will be certain infinite sequences of natural numbers that, in general, are unfinished and are created step-by-step. The spread-law is there to regulate the process of defining elements of the spread. It informs me, whenever I have completed a finite initial part

\[
\alpha(0), \alpha(1), \ldots, \alpha(n-1)
\]

of an element \( \alpha \) of the spread, which numbers I may choose for the next value: \( \alpha(n) \).

The spread-law \( \beta \) itself is an element of \( \omega^\omega \). For every \( s \), the informal meaning of \( '\beta(s) = 0' \) is: ‘the finite sequence coded by \( s \) is admitted by \( \beta \).’ \( \beta \) has to satisfy the following condition:

\[
\forall s[\beta(s) = 0 \leftrightarrow \exists n[\beta(s \ast \langle n \rangle) = 0]]
\]

The ‘set’ consisting of all \( \alpha \) such that \( \forall n[\beta(\alpha n) = 0] \) will be called \( \mathcal{F}_\beta \). \( \mathcal{F}_\beta \) is the spread determined by the spread-law \( \beta \).

The condition just imposed on the spread-law \( \beta \) guarantees that, when I am creating an element \( \alpha \) of \( \mathcal{F}_\beta \), then, at every stage \( n \), having chosen \( \alpha(0), \alpha(1), \ldots, \alpha(n-1) \), I am able to decide, for each \( m \), if I may define \( \alpha(n):= m \), and: there will be at least one such \( m \), i.e. I never will get ‘stuck’.

**Theorem 3** (Brouwer’s Continuity Principle, extended to spreads).

Let \( \beta \) be a spread-law. For every relation \( R \subseteq \mathcal{F}_\beta \times \omega \), if \( \forall \alpha \in \mathcal{F}_\beta \exists n[\alpha Rn] \), then \( \forall \alpha \in \mathcal{F}_\beta \exists m \exists n[\forall \gamma \in \mathcal{F}_\beta[\alpha m \subseteq \gamma \rightarrow \gamma Rn]] \).

**Proof.** One might defend this theorem like Axiom 4 itself, as a more general formulation of the same principle. One may also derive the Theorem from the Axiom, as follows.

Let a spread-law \( \beta \) be given.

We define \( \rho : \omega^\omega \rightarrow \omega^\omega \) such that, for each \( \alpha \), for each \( n \), if \( \beta(\alpha(n + 1)) = 0 \), then \( (\rho(\alpha))(n) = \alpha(n) \), and, if not, then \( (\rho(\alpha))(n) = mk[\beta(\rho(\alpha \ast k))] = 0 \).

Then \( \forall \alpha[\rho(\alpha) \in \mathcal{F}_\beta] \) and \( \forall \alpha \in \mathcal{F}_\beta[\rho(\alpha) = \alpha] \).

The function \( \rho \) is called a retraction of \( \omega^\omega \) onto \( \mathcal{F}_\beta \).

Now assume \( \forall \alpha \in \mathcal{F}_\beta \exists n[\alpha Rn] \).

Then \( \forall \alpha \exists n[(\rho(\alpha)) Rn] \).

By Axiom 4, for any given \( \alpha \) in \( \mathcal{F}_\beta \), one may find \( m, n \) such that \( \forall \gamma[\alpha m \subseteq \gamma \rightarrow (\rho(\gamma)) Rn] \) and, therefore, \( \forall \gamma \in \mathcal{F}_\beta[\alpha m \subseteq \gamma \rightarrow \gamma Rn] \).

Conclude: \( \forall \alpha \in \mathcal{F}_\beta \exists m \exists n[\gamma \in \mathcal{F}_\beta[\alpha m \subseteq \gamma \rightarrow \gamma Rn]] \).

\( \square \)

2.4. Axioms of Countable Choice.

**Axiom 2** (First Axiom of Countable Choice). For every relation \( R \subseteq \omega \times \omega \), if \( \forall m \exists n[\alpha Rn] \), then \( \exists n \forall m \exists n[\alpha Rn] \).

This axiom seems to be a good proposal as we decided to allow the possibility that an infinite sequence \( \alpha \) is created step by step.

\( ^4 \)As was observed by Serge Bozon, there is a mistake in the proof of [27] Theorem 2.6. Here is a correction. After the first two sentences of the proof, continue as follows: ‘Find \( p \) such that \( |\alpha - \alpha(n-1)| + 1/p^p < 1/2^p \). Then, for every canonical real number \( \beta \), if \( |\alpha - \beta| < 1/4 \), there exists a canonical real \( \gamma \) such that \( \beta = \gamma \) and, for all \( i < n \), \( \alpha(i) = \gamma(i) \), and, therefore, \( f(\beta) = f(\gamma) \) and \( |f(\alpha - f(\beta)| < 1/4^p \).

6
Assume $\forall m \exists n [(m, n)]$. We first find $n$, such that $0Rn$ and define $\alpha(0) = n$, we then find $n$ such that $1Rn$ and define $\alpha(1) = n$, and so on.

Note that the axiom is not so plausible if we should require that an infinite sequence $\alpha$ is given by means of an algorithm.

The classical mathematician would suggest to define $\alpha$: for each $m$, $\alpha(m)$ should be the least $n$ such that $mRn$. This suggestion does not work in a constructive context, except for the case that one may decide, for all $m, n$, if $mRn$ or not $mRn$.

**Axiom 3** (Second Axiom of Countable Choice). For every relation $R \subseteq \omega \times \omega$, if $\forall n \exists \alpha(mRn)$, then $\exists \alpha \forall m [(m \alpha m)]$.

This axiom also seems to be a good proposal as, in general, we think it possible to start a project for an infinite sequence, do some work on it, then start a second project for an infinite sequence, and do some work on it, then return to the first project and do some further work on it, and so on.

Assume $\forall n \exists \alpha(mRn)$. Start a project for an infinite sequence suitable to 0, calling it $\alpha^0$ and find $\alpha^0(0)$. Start a project for an infinite sequence suitable to 1, calling it $\alpha^1$, and find $\alpha^1(0)$ and $\alpha^0(1)$. Start a project for an infinite sequence suitable to 2, calling it $\alpha^2$ and calculate $\alpha^2(0)$, $\alpha^1(1)$ and $\alpha^0(2)$. And so on.

2.5. Sharpened versions of the Continuity Principle.

**Axiom 4** (First Axiom of Continuous Choice). For every relation $R \subseteq \omega^\omega \times \omega$, if $\forall n \exists \alpha(mRn)$, then there exists $\varphi : \omega^\omega \to \omega$ such that $\forall \alpha [\alpha R \varphi (\alpha)]$.

Assume $\forall \alpha \exists n [\alpha R n]$. We find the promised $\varphi$ recursively, as follows. Given any $s$, we first ask if $\exists \forall \exists s [\varphi (t) \neq 0]$. If so, we define $\varphi (s) = 0$. If not, we imagine the finite sequence (coded by) $s$ as the beginning of an infinite sequence $\alpha$ that is created step by step, and we ask ourselves: ‘does $s$ contain sufficient information for finding $n$ such that $\alpha R n$?’ If so, we choose such $n$ and define $\varphi (s) = n + 1$, and, if not, we define $\varphi (s) = 0$.

**Axiom 5** (Second Axiom of Continuous Choice). For every relation $R \subseteq \omega^\omega \times \omega^\omega$, if $\forall n \exists \beta [\alpha R \beta]$, then there exists $\varphi : \omega^\omega \to \omega^\omega$ such that $\forall \alpha [\alpha R \varphi (\alpha)]$.

Assume $\forall \alpha \exists \beta [\alpha R \beta]$. We find the promised $\varphi$ recursively, as follows. Given any $s$, we first ask if $\forall \exists \psi [\varphi (t) \neq 0]$. We imagine the finite sequence (coded by) $s$ as the beginning of an infinite sequence $\alpha$ that is created step by step. Clearly, we convinced ourselves that $s$ contains sufficient sufficient information for finding the first $p$ values of a sequence $\beta$ satifying $\alpha R \beta$. We now ask ourselves: ‘does $s$ contain sufficient information for finding the next value of the sequence $\beta$ such that $\alpha R \beta$, i.e. for finding $n$ such that $n = \beta (p)$?’ If so, we choose such $n$ and define $\varphi^p (s) = n + 1$, and, if not, we define $\varphi^p (s) = 0$.

3. The Borel Hierarchy Theorem

Although the early descriptive set theorist had their doubts about some of Cantor’s assumptions, they never questioned the use of classical logic. The symmetry of classical logic is heavily used in the classical proof of the Borel Hierarchy Theorem. It is not so easy to formulate and prove a satisfying similar result in a constructive context. Brouwer’s Continuity Principle comes to the rescue, and the resulting theorem may be considered a significant and surprising application of the principle.

3.1. Introducing stumps. Generalized inductive definitions like the following one are acceptable intuitionistically.

**Definition 1. Stp.** A collection of subsets of $\omega$, called stumps, is defined as follows.

(i) $\emptyset \in \text{Stp}$, and
(ii) for every infinite sequence $S_0, S_1, \ldots$ of elements of $\text{Stp}$, the set 
$S := \{\langle \rangle \} \cup \bigcup_{n \in \omega} \langle n \rangle * S_n$ is again an element of $\text{Stp}$, and

(iii) nothing more: every element of $\text{Stp}$ is obtained by starting from $\emptyset$ and applying the operation mentioned in (ii) repeatedly.

For every non-empty stump $S$, for every $n$, $S \upharpoonright \langle n \rangle = \{s \mid \langle n \rangle * s \in S\}$ is called the $n$-th immediate substump of $S$.

**Definition 2.** For every stump $S$ we define $S' := \{s \mid s \notin S \land \forall t \exists s \to t \in S\}$. The set $S'$ is called the border of the stump $S$.

The border $S'$ of $S$ consists of those (code numbers of) finite numbers that are just outside $S$.

We shall call $\text{Stp}$ a class or a set, although it is a totality of a different kind than $\omega^\omega$ or $\omega$. Again, we have a general idea how its members are created but only a very few of them have been realized until now.

Stumps take the role fulfilled by countable ordinals in classical analysis.

As we accept Definition 4, we also feel entitled to use the following axiom.

**Axiom 6 (Induction on $\text{Stp}$).** Let $P \subseteq \text{Stp}$ be given. If

(i) $\emptyset \in P$ and,

(ii) for every nonempty stump $S$, if, for all $n$, $S \upharpoonright \langle n \rangle \in P$, then $S \in P$,

then $\text{Stp} = P$.

The following definition introduces a subclass of the class of stumps, useful for the treatment of Borel sets.

**Definition 3.** $\text{Hrs}$, a collection of subsets of $\omega$, called hereditarily repetitive nonzero stumps, is defined as follows.

(i) $\{\langle \rangle \} \in \text{Hrs}$, and

(ii) for every infinite sequence $S_0, S_1, \ldots$ of elements of $\text{Hrs}$, the set $S := \{\langle \rangle \} \cup \bigcup_{m,n} (\langle m, n \rangle) * S_n$ is again an element of $\text{Hrs}$, and applying the operation mentioned in (ii) repeatedly.

(iii) nothing more: every element of $\text{Hrs}$ is obtained by starting from $\{\langle \rangle \}$ and applying the operation mentioned in (ii) repeatedly.

$1^* := \{\langle \rangle \}$ is called the basic element of $\text{Hrs}$. Note that, for every $S$ in $\text{Hrs}$, $S = 1^*$ if and only if $0 \notin S$, so one may decide if $S = 1^*$ or not.

For every $S$ in $\text{Hrs}$, for every $s$, $s$ is called an endpoint of $S$ if and only if $s \in S$ and $s * 0 \notin S$. Note that $\langle \rangle$ is an endpoint of $1^*$ and that, for every $S \neq 1^*$ in $\text{Hrs}$, for every $s$, $s$ is and endpoint of $S$ if and only if there exist $n, t$ such that $s = \langle n \rangle * t$ and $t$ is an endpoint of $S \upharpoonright \langle n \rangle$.

Note that, for every $S$ in $\text{Hrs}$, for each $s$, $s$ is an endpoint of $S$ if and only if, for each $n$, $s * \langle n \rangle$ is an element of the border $S'$ of $S$.

Note: $(1^*)' = \{\langle n \rangle \mid n \in \omega\}$.

3.2. (Positively) Borel sets. The class of the (positively) Borel subsets of $\omega^\omega$ is the least class of subsets of $\omega^\omega$ containing the open subsets of $\omega^\omega$ and the closed subsets of $\omega^\omega$ that is closed under the operations of countable union and countable intersection. We define this class using hereditarily repetitive nonzero stumps as indices.

The constructive mathematician avoids negation\footnote{A negative statement $\neg P$ reports the failure of obtaining a construction validating $P$.} and the operation of taking the complement $X \setminus Y$ of a given set $Y \subseteq X$ as much as possible. It is not true, intuitionistically, that the complement $\omega^\omega \setminus Y$ of a of a given positively Borel set $Y \subseteq \omega^\omega$ is again positively Borel.
Definition 4 (Borel sets and Borel classes). For every $S$ in $\text{Hrs}$, for every $\beta$, we define subsets $G^{S}_{\beta}$ and $F^{S}_{\beta}$ of $\omega^{\omega}$, by induction, as follows.

(i) $G^{S}_{\beta} = \{ \alpha \mid \exists n[\beta(\tau n) \neq 0] \}$ and $F^{S}_{\beta} = \{ \alpha \mid \forall n[\beta(\tau n) = 0] \}$.

(ii) For every $S \neq 1^{\ast}$, $G^{S}_{\beta} = \bigcup_{n} F^{S(\pi n)}_{\beta}$ and $F^{S}_{\beta} = \bigcap_{n} G^{S(\pi n)}_{\beta}$.

For every $S$ in $\text{Hrs}$, we define classes $\Sigma^{S}_{\beta}$ and $\Pi^{S}_{0}$ of subsets of $\omega^{\omega}$ as follows.

For every $X \subseteq \omega^{\omega}$, $X$ is $\Sigma^{0}_{\beta}$ if and only if $\exists \beta[X = G^{S}_{\beta}]$ and $X$ is $\Pi^{0}_{\beta}$ if and only if $\exists \beta[X = F^{S}_{\beta}]$.

$X \subseteq \omega^{\omega}$ is open if and only if $X$ is $\Sigma^{0}_{\beta}$, and closed if and only if $X$ is $\Pi^{0}_{\beta}$.

In each Borel class, we single out a special element that will turn out to be an element of the class of maximal complexity.

Definition 5 (The leading sets of the hierarchy). For every $S$ in $\text{Hrs}$ we define subsets $E_{S}$ and $A_{S}$ of $\omega^{\omega}$ as follows.

(i) $E_{1} := G^{1}_{\omega} := \{ \alpha \mid \exists n[\alpha(n) \neq 0] \}$ and $A_{1} := F^{1}_{\omega} := \{ \alpha \mid \forall n[\alpha(n) = 0] \}$.

(ii) For each hereditarily repetitive nonzero stamp $S$, if $S \neq 1^{\ast}$, then $E_{S} := G^{S}_{1} := \{ \alpha \mid \exists n[\alpha(n) \in A_{S} \langle n \rangle] \}$ and $A_{S} := F^{S}_{1} := \{ \alpha \mid \forall n[\alpha(n) \in E_{S} \langle n \rangle] \}$.

Remark 1. One may prove, by induction on $\text{Hrs}$:

for each $S$ in $\text{Hrs}$, for all $\alpha, \beta$, if $\alpha \in A_{S}$ and $\beta \in E_{S}$, then $\alpha \neq \beta$.

3.3. Games. It is very useful to think of the leading sets of the Borel hierarchy in a game theoretic way, as follows.

Definition 6 (Introducing games and strategies). Let $a$ hereditarily repetitive nonzero stamp $S$ be given, and let also $\alpha$ in $\omega^{\omega}$ be given. We introduce $G_{S}(\alpha)$, the game in $S$ for $\alpha$.

A play in $G_{S}(\alpha)$ goes as follows. Players I, II start constructing an infinite sequence $\gamma$ in $\omega^{\omega}$.

I chooses $\gamma(0)$, II chooses $\gamma(1)$, I chooses $\gamma(2)$, …

The play ends as soon as a position $\tau n = (\gamma(0), \gamma(1), \ldots, \gamma(n-1))$ in the border $S'$ of $S$ is reached. If $n$ is even, then Player I wins the play if and only if $\alpha(\tau n) = 0$ and, if $n$ is odd, then Player I wins the play if and only if $\alpha(\tau n) \neq 0$. Player II wins the play if and only if Player I does not win the play.

For all $\sigma, \tau$, for all $s$, we define: $s \in I \sigma$, ‘$s$ is played by Player I according to the strategy $\sigma'$, if and only if $\forall n[2n < length(s) \rightarrow s(2n) = \sigma(\tau 2n)]$, and $s \in I \tau$, ‘$s$ is played by Player II according to the strategy $\tau'$, if and only if $\forall n[2n + 1 < length(s) \rightarrow s(2n + 1) = \tau(2n + 1)]$.

For each $S$ in $\text{Hrs}$, for each $\alpha$, for all $\sigma, \tau$, we define:

$I_{S}(\sigma, \alpha), \sigma$ is a winning strategy for Player I in $G_{S}(\alpha)$, and if only if $\forall \forall s \in S'[s \in I \sigma \rightarrow ((s \in \omega^{2n} \rightarrow \alpha(s) = 0) \land (s \in \omega^{2n+1} \rightarrow \alpha(s) \neq 0))]$, and: $I_{S}(\tau, \alpha), \tau$ is a winning strategy for Player II in $G_{S}(\alpha)$, and if only if $\forall \forall s \in S'[s \in I \tau \rightarrow ((s \in \omega^{2n} \rightarrow \alpha(s) \neq 0) \land (s \in \omega^{2n+1} \rightarrow \alpha(s) = 0))]$.

Theorem 4. For every $S$ in $\text{Hrs}$,

(i) for every $\alpha$, for every $\sigma$, if $I_{S}(\sigma, \alpha)$, then $\alpha \in E_{S}$ and

(ii) for every $\alpha$, for every $\tau$, if $I_{S}(\tau, \alpha)$, then $\alpha \in A_{S}$.

(iii) for every $\alpha$, if $\alpha \in E_{S}$, then, for some $\sigma$, $I_{S}(\sigma, \alpha)$, and

(iv) for every $\alpha$, if $\alpha \in A_{S}$, then, for some $\tau$, $I_{S}(\tau, \alpha)$. 

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Proof. (i) and (ii). The proof is by induction on \( \text{Hrs} \).

The case \( S = 1^* \) is easy and left to the reader.

Now let \( S \) be given such that \( S \neq 1^* \) and, for each \( n \), for each \( \alpha \), if \( \exists \nu[\mathcal{W}^I_{S(n)}(\sigma, \alpha)] \), then \( \alpha \in \mathcal{E}_{S(n)} \), and, if \( \exists \tau[\mathcal{W}^II_{S(n)}(\tau, \alpha)] \), then \( \alpha \in A_{S(n)} \).

Let \( \sigma, \alpha \) be given such that \( \mathcal{W}^I_{S(n)}(\sigma, \alpha) \). Define \( n_0 := \sigma(\langle \rangle) \).

Define \( \tau := \sigma(\langle n_0 \rangle) \) and note: \( \mathcal{W}^II_{S(n_0)}(\tau, \alpha \upharpoonright \langle n_0 \rangle) \).

Conclude: \( \alpha \uparrow \langle n_0 \rangle \in A_{S(n_0)} \) and \( \alpha \in \mathcal{E}_S \).

Let \( \tau, \alpha \) be given such that \( \mathcal{W}^II_{S(n)}(\tau, \alpha) \).

Then, for each \( n \), \( W_{S(n)}(\tau \upharpoonright \langle n \rangle, \alpha \upharpoonright \langle n \rangle) \).

Conclude: for each \( n \), \( \alpha \uparrow \langle n \rangle \in \mathcal{E}_{S(n)} \) and: \( \alpha \in A_S \).

(iii) and (iv). The proof is by induction on \( \text{Hrs} \).

The case \( S = 1^* \) is easy and left to the reader.

Now let \( S \) be given such that \( S \neq 1^* \) and, for each \( n \), for each \( \alpha \), if \( \alpha \in \mathcal{E}_{S(n)} \), then, for some \( \sigma, \alpha \in \mathcal{W}^I_{S(n)}(\sigma, \alpha) \), and,

if \( \alpha \in A_{S(n)} \), then for some \( \tau, \alpha \in \mathcal{W}^II_{S(n)}(\tau, \alpha) \),

Let \( \alpha \) be given such that \( \alpha \in \mathcal{E}_S \). Find \( n \) such that \( \alpha \uparrow \langle n \rangle \in A_{S(n)} \).

Find \( \tau \) such that \( \mathcal{W}^II_{S(n)}(\tau, \alpha \upharpoonright \langle n \rangle) \).

Define \( \sigma \) such that \( \sigma(\langle \rangle) = n \) and \( \sigma \uparrow \langle n \rangle = \tau \) and note: \( \mathcal{W}^I_{S(n)}(\sigma, \alpha) \).

Let \( \alpha \) be given such that \( \alpha \in A_S \).

Then \( \forall n[\alpha \uparrow \langle n \rangle \in \mathcal{E}_{S(n)}] \) and \( \forall n \exists \sigma[\mathcal{W}^I_{S(n)}(\sigma, \alpha \upharpoonright \langle n \rangle)] \).

Using the Second Axiom of Countable Choice, Axiom \( \spadesuit \) find \( \tau \) such that \( \forall n[\mathcal{W}^I_{S(n)}(\tau, \alpha \upharpoonright \langle n \rangle)] \) and note: \( \mathcal{W}^II_{S(n)}(\tau, \alpha) \).

\( \square \)

**Definition 7.** For every \( S \) in \( \text{Hrs} \), for every \( \tau \), for every \( \alpha \), we let \( \tau \triangleright S^I \alpha \), \( \langle \alpha \rangle \)-as-corrected by the strategy \( \tau \) for Player II in \( G_2(\alpha) \), be the element of \( \omega^\omega \) satisfying: for all \( s \) in the border \( S' \) of \( S \),

if \( \text{length}(s) \) is even, then \( \tau \triangleright S^I \alpha(s) = \max(1, \alpha(s)) \), and,

if \( \text{length}(s) \) is odd, then \( \tau \triangleright S^I \alpha(s) = 0 \), and,

for all \( s \), if \( s \) is not in the border \( S' \) of \( S \), then \( \tau \triangleright S^I \alpha(s) = \alpha(s) \).

**Remark 2.** Note: for every \( S \) in \( \text{Hrs} \), for every \( \alpha \), \( \alpha \in A_S \) if and only if, for some \( \tau, \alpha = \tau \triangleright S^I \alpha \).

The following Lemma is crucial. With this tool we will prove the Hierarchy Theorem.

**Lemma 5** (A consequence of Brouwer’s Continuity Principle).

For every \( S \) in \( \text{Hrs} \), for every relation \( R \subseteq \omega^\omega \times \omega \), if \( \forall \alpha \in A_S \exists n[\alpha R n] \), then \( \forall \alpha \in A_S \exists \beta \exists n[\beta(0) = \alpha(0) \land (\forall n < p[\beta \upharpoonright \langle n \rangle] = \alpha \upharpoonright \langle n \rangle) \rightarrow (\beta R n)] \).

**Proof.** Assume: \( \forall \alpha \in A_S \exists n[\alpha R n] \). Conclude: \( \forall \alpha \forall \exists \exists n[\tau \triangleright S^I \alpha(n)] \).

Let \( \alpha \in A_S \) be given. Using Theorem \( \clubsuit \) (iv), find \( \tau \) such that \( \alpha = \tau \triangleright S^I \alpha \).

Using Brouwer’s Continuity Principle, Axiom \( \spadesuit \), find \( m, q \) such that \( \forall \beta[\beta(0) = \alpha(0) \land (\forall n < p[\beta \upharpoonright \langle n \rangle] = \alpha \upharpoonright \langle n \rangle)] \rightarrow (\beta R n) \).

Find \( p \) such that, for all \( s \), if \( s \leq m \), then \( s(0) < p \).

Note that, for all \( \beta \) in \( A_S \), if \( \beta(0) = \alpha(0) \) and \( \forall n < p[\beta \upharpoonright \langle n \rangle] = \alpha \upharpoonright \langle n \rangle \), then \( \exists m \underset{\beta}{\triangleright} \beta \) and, for some \( \sigma, \exists m \underset{\sigma}{\triangleright} \beta \) and \( \beta \triangleright S^I \beta \), and, therefore, \( \beta R \).

3.4. (Wadge-)reducibility. The following notion of reducibility plays a key rôle. In classical descriptive set theory this notion is called Wadge-reducibility. Its analog in computability theory is many-one-reducibility.

**Definition 8** (Reducibility). For all \( X, Y \subseteq \omega^\omega \), for all \( \varphi : \omega^\omega \rightarrow \omega^\omega \) we define: \( \varphi \) reduces \( X \) to \( Y \) if and only if, for every \( \alpha, \varphi(\alpha) \in Y \).
For all $\mathcal{X}, \mathcal{Y} \subseteq \omega^\omega$, we define: $\mathcal{X} \preceq \mathcal{Y}$, $\mathcal{X}$ reduces to $\mathcal{Y}$ if and only if there exists $\varphi : \omega^\omega \to \omega^\omega$ reducing $\mathcal{X}$ to $\mathcal{Y}$.

**Theorem 6** ($\mathcal{E}_S, \mathcal{A}_S$ are complete elements of $\Sigma^0_S$, $\Pi^0_S$, respectively).

For each $S$ in $\mathcal{Hrs}$, for every $\mathcal{X} \subseteq \omega^\omega$, $\mathcal{X} \preceq \omega^\omega$ is $\Sigma^0_S$ if and only if $\mathcal{X} \preceq \mathcal{E}_S$, and $\mathcal{X} \subseteq \omega^\omega$ is $\Pi^0_S$ if and only if $\mathcal{X} \preceq \mathcal{A}_S$.

**Proof.** The proof that, for each $S$ in $\mathcal{Hrs}$, $\mathcal{E}_S$ is $\Sigma^0_S$ and $\mathcal{A}_S$ is $\Pi^0_S$ is left to the reader. Also the proof that, for each $S$ in $\mathcal{Hrs}$, for all $\mathcal{X}, \mathcal{Y} \subseteq \omega^\omega$, if $\mathcal{X} \preceq \mathcal{Y}$, then, if $\mathcal{Y}$ is $\Sigma^0_S$, then $\mathcal{X}$ is $\Sigma^0_S$ and, if $\mathcal{Y}$ is $\Pi^0_S$, then $\mathcal{X}$ is $\Pi^0_S$, is left to the reader.

We now prove that, for each $S$ in $\mathcal{Hrs}$, for each $\beta$, there exists $\varphi : \omega^\omega \to \omega^\omega$ reducing both $G_{\beta}^S$ to $\mathcal{E}_S$ and $F_{\beta}^S$ to $\mathcal{A}_S$ and we do so by induction on $\mathcal{Hrs}$.

We first consider the case that $S = \{(\phi)\} = 1^*$ is the basic element of $\mathcal{Hrs}$. Let $\beta$ be given. Define $\varphi : \omega^\omega \to \omega^\omega$ such that, for each $\alpha$, for each $n$, $(\varphi(\alpha))(\langle n \rangle) = \beta(\langle n \rangle)$, and note that $\varphi$ reduces $G_{\beta}^S$ to $\mathcal{E}_S$ and $F_{\beta}^S$ to $\mathcal{A}_S$.

Now let $S$ be a non-basic element of $\mathcal{Hrs}$. Let $\beta$ be given. Using the induction hypothesis and the Second Axiom of Countable Choice (a) find $\varphi$ such that, for each $n$, $\varphi^n : \omega^\omega \to \omega^\omega$ reduces $G_{\beta}^{S_{\langle n \rangle}}$ to $\mathcal{E}_{S_{\langle n \rangle}}$ and $F_{\beta}^{S_{\langle n \rangle}}$ to $\mathcal{A}_{S_{\langle n \rangle}}$.

Define $\psi : \omega^\omega \to \omega^\omega$ such that, for each $\alpha$, for each $n$, $(\psi(\alpha))(\langle n \rangle) = \varphi^n(\alpha \langle n \rangle)$ and note that $\psi$ reduces $G_{\beta}^S$ to $\mathcal{E}_S$ and $F_{\beta}^S$ to $\mathcal{A}_S$.

**Definition 9.** For each $S$ in $\mathcal{Hrs}$, we define canonical elements $\varepsilon^*_S$, $\alpha^*_S$, of the sets $\mathcal{E}_S$, $\mathcal{A}_S$, respectively, as follows.

- $\forall s \in S \neq \emptyset$ $\varepsilon^*_S(s) = 0$
- $\forall \alpha \in S \neq \emptyset$ $\alpha^*_S = \alpha$
- $\forall s, \alpha \in S \neq \emptyset$ $\varepsilon^*_S(s) = \alpha^*_S(s) = 0$
- $\forall s \in S \neq \emptyset$ $\varepsilon^*_S(s) = 0$
- $\forall s \in S \neq \emptyset$ $\alpha^*_S(s) = 0$
- $\forall s, \alpha \in S \neq \emptyset$ $\varepsilon^*_S(s) = \alpha^*_S(s) = 0$

**Remark 3.** Note that, for every $S$ in $\mathcal{Hrs}$, for every $\sigma$, $\mathcal{W}^S_\omega(\sigma, \varepsilon^*_S)$, and, for every $\tau$, $\mathcal{W}^S_\omega(\tau, \alpha^*_S)$.

3.5. **The Hierarchy Theorem.** We would like to prove the statement that, for each $S$ in $\mathcal{Hrs}$, $\Pi^0_S$ is not a subclass of $\Sigma^0_S$ and, conversely, $\Sigma^0_S$ is not a subclass of $\Pi^0_S$. We are going to prove a stronger and more positive statement. We shall make use of the following technical notion.

**Definition 10** (Freedom in spreads). Let $\beta$ be a spread-law and let $\mathcal{F} = F_{\beta}$ be the corresponding spread.

Let $t$ be given.

- We define: $t$ is free in $\mathcal{F}$ if and only if, for all $s$ in $\omega^\omega$, if $\beta(s) = 0$, then, for all $m$, $\beta(s \ast \langle m \rangle) = 0$.
- We define: $t$ is completely free in $\mathcal{F}$ if and only if, for all $u$ such that $t \sqsubseteq u$, $u$ is free in $\mathcal{F}$.

**Remark 4.** If $t$ is free in $\mathcal{F} = F_{\beta}$, then, when building, step-by-step, an element $\alpha$ of $\mathcal{F}$, and having defined $\alpha(0), \ldots, \alpha(t - 1)$, we may choose any number $m$ as a value of $\alpha(t)$. The spread-law $\beta$ does not impose any restriction on our freedom of choice at $t$.

We shall prove that, for each $S$ in $\mathcal{Hrs}$, $\Pi^0_S$ is not a subclass of $\Sigma^0_S$ by showing that every $\varphi : \omega^\omega \to \omega^\omega$ mapping $\mathcal{A}_S$ into $\mathcal{E}_S$ positively fails to reduce $\mathcal{A}_S$ to $\mathcal{E}_S$, and, similarly, we prove that, for each $S$ in $\mathcal{Hrs}$, $\Sigma^0_S$ is not a subclass of $\Pi^0_S$ by showing that every $\varphi : \omega^\omega \to \omega^\omega$ mapping $\mathcal{E}_S$ into $\mathcal{A}_S$ positively fails to reduce $\mathcal{E}_S$ to $\mathcal{A}_S$, i.e.:

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6The use of this axiom at this place may be avoided.
Theorem 7 (Borel Hierarchy Theorem).

(i) For every $S$ in $\text{Hrs}$, for every $\varphi : \omega^\omega \rightarrow \omega^\omega$, if $\forall \alpha [\alpha \in A_S \rightarrow \varphi|\alpha \in E_S]$, then $\exists \alpha [\alpha \in E_S \land \varphi|\alpha \in E_S]$.

(ii) For every $S$ in $\text{Hrs}$, for every $\varphi : \omega^\omega \rightarrow \omega^\omega$, if $\forall \alpha [\alpha \in E_S \rightarrow \varphi|\alpha \in A_S]$, then $\exists \alpha [\alpha \in A_S \land \varphi|\alpha \in A_S]$.

Proof. (i) Let $S$ in $\text{Hrs}$ be given.

Let $\varphi : \omega^\omega \rightarrow \omega^\omega$ be given such that $\forall \alpha [\alpha \in A_S \rightarrow \varphi|\alpha \in E_S]$.

We are going to construct $\alpha, \sigma, \tau$ such that $W^2_0(\alpha, \sigma)$ and $W^2_0(\tau, \varphi|\alpha)$, and, therefore, both $\alpha$ and $\varphi|\alpha$ are in $E_S$. We do so by building an infinite sequence $F_0, F_1, F_2, \ldots$ of spreads such that

a. $F_0 = \omega^\omega$ and for each $t$, $F_{i+1} \subseteq F_i$, and, for each $s$, if $\text{length}(s) < \text{length}(t)$, then $s$ is admitted by the spread-law defining $F_i$ if and only if $s$ is admitted by the spread-law defining $F_{i+1}$.

b. for all $i, t, u$,

- if $t, u \in \omega^2$ and $t \in S$ and $t \in I$ and $u \in I$ and $t_{II} = u_{II}$, then $b_1$. $t$ is completely free in $F_i$, and
- $b_2$. $u \in S$ and $S \nmid t = S \nmid u$, and
- $b_3$. $\alpha \in F_i[\alpha \mid t \in A_{S|t} \rightarrow (\varphi|\alpha) \mid u \in E_{S|u}]$, and,
- $b_4$. if $t$ is an endpoint of $S$, then $u$ is an endpoint of $S$ and $\alpha \in F_{i+1}[\alpha \mid t \in A_{S|t} \rightarrow (\varphi|\alpha) \mid u \in E_{S|u}]$, and,
- $b_5$. if $t \in I$ then $u$ is an endpoint of $S$ and $\alpha \in F_{i+1}[\alpha \mid t \in A_{S|t} \rightarrow (\varphi|\alpha) \mid u \in E_{S|u}]$, and
- $b_6$. if $t \in I$ and $u$ is an endpoint of $S$ and $\alpha \in F_{i+1}[\alpha \mid t \in A_{S|t} \rightarrow (\varphi|\alpha) \mid u \in E_{S|u}]$, and

Note that, as $F_0 = \omega^\omega$, conditions $b_1, b_2$ and $b_3$ are satisfied for $i = 0$.

Now let $t$ be given. We want to define $F_{i+1}$ and distinguish two cases.

Case (i) $t \notin S$, or $\exists i[t \in \omega^{2i+1}]$, or $t \notin I$. We define $F_{i+1} = F_i$.

Case (ii). $t \in S$, and $\exists i[t \in \omega^{2i}]$, and $t \in I$.

We now determine $u$ such that $\text{length}(u) = \text{length}(t)$ and $u \in I$ and $t_{II} = u_{II}$.

We then define $S \nmid t = S \nmid u$ and $t$ is completely free in $F_i$ and $\forall \alpha \in F_i[\alpha \mid t \in A_{S|t} \rightarrow (\varphi|\alpha) \mid u \in E_{S|u}]$.

We again distinguish two cases.

Case (ii)(I). $S \nmid t = S \nmid u = 1^*$, i.e. $t, u$ are endpoints of $S$.

Note: $\forall \alpha \in F_i[\alpha \mid t \in A_{I \cdot} \rightarrow (\varphi|\alpha) \mid u \in E_{I \cdot}]$.

Find $\eta$ in $F_i$ such that $\eta \mid t \nmid \alpha_{S|t} = \alpha_1 \in A_{I \cdot}$, i.e. $\forall n[\eta(t \nmid \langle n \rangle) = 0]$.

Note: $(\varphi|\eta) \mid u \in E_{I \cdot}$, and find $q$ such that $(\varphi|\eta)(u \nmid \langle q \rangle) \neq 0$.

Define $\tau(u) := q$.

Find $p$ such that $\varphi|\eta(u \nmid \langle \tau(u) \rangle + 1) \subseteq \varphi(p)$.

Find $r$ such that $t \nmid r \geq p$. Note that $t \nmid r$ is free in $F_i$.

Define $\sigma(t) := r$.

Find $c$ such that $\tau(r) \cap c$ and $c(t \nmid (\sigma(t))) = 1$ and $c$ is admitted by $F_i$.

Define $F_{i+1} = F_i \cap c$.

Note: for every $\alpha$ in $F_{i+1}$, $\alpha(t \nmid (\sigma(t))) = 1$ and $(\varphi|\alpha)(u \nmid (\tau(u))) = 1$.

Also note: for all $v \perp t$, if $v$ is completely free in $F_i$, then $v$ is completely free in $F_{i+1}$.

Case (ii)(II). $S \nmid t = S \nmid u \neq 1^*$, i.e. $t, u$ are no endpoints of $S$.

We have to define $F_{i+1}$ but also $\sigma(t)$ and $\tau(u)$.

Find $\eta$ in $F_i$ such that $\eta \mid t = \alpha_{S|t} \in A_{S|t}$.

We define $\psi : \omega^\omega \rightarrow \omega^\omega$ such that, for each $\beta$,

1. $\forall u [u \nmid \langle t \nmid u \rangle] \rightarrow (\psi|\beta)(u) = \eta(u)$, and
2. $(\varphi|\beta) \mid t = \beta$.
Note that, for all $\beta$, if $\beta \in A_{S|t}$, then $\langle \psi|\beta \rangle \upharpoonright t \in A_{S|t}$ and $\langle \varphi|\psi|\beta \rangle \upharpoonright u \in E_{S|u}$.  

In particular, this is true for $\beta = \alpha_{S|t}$.  

Note: $\varphi|\langle \alpha_{S|t} \rangle = \eta$.  

Using Lemma 5, find $p, q$ such that, for each $\beta$, if $\beta \in A_{S|t}$ and $\beta(0) = \alpha_{S|t}(0)$ and $\forall n < p/\beta \upharpoonright \langle n \rangle = \alpha_{S|t}(\langle n \rangle)$, then $\langle \varphi|\psi|\beta \rangle \upharpoonright u \ast \langle q \rangle \in A_{S|u \ast \langle q \rangle}$.  

Define $\tau(u) = q$.  

Find $r > p$ such that $S \upharpoonright t \ast \langle r \rangle = S \upharpoonright u \ast \langle q \rangle = S \upharpoonright u \ast \langle \tau(u) \rangle$ and define $\sigma(t) = r$.  

Define $F = \{ \alpha \in F \mid \forall u(t \subseteq u \land u \uplus t \ast \langle \gamma(t) \rangle \rightarrow \alpha(u) = \eta(u) \}$.  

Note that $t \ast \langle \sigma(t) \rangle$ is completely free in $F$.  

Also note that, for each $v$, if $v \uplus t$ and $v$ is completely free in $F_t$, then $v$ is completely free in $F$.  

Note: $\forall \alpha \in F = \{ \alpha \in F \mid \forall n(\alpha(t \ast (\sigma(t), n), n) = 0) \}$.  

We distinguish two subcases of Case (ii)/(II).  

Case (ii)/(II)a. $t \ast \langle \sigma(t) \rangle$ and $u \ast \langle \tau(u) \rangle$ are endpoints of $S$.  

Define $F_{t+1} = \{ \alpha \in F \mid \forall n(\alpha(t \ast (\sigma(t), n), n) = 0) \}$.  

We claim: $\forall \alpha \in F_{t+1} \forall n(\langle \varphi|\alpha|u \ast \langle \tau(u), n \rangle \rangle = 0)$.  

We prove this claim as follows.  

Assume we find $\alpha$ in $F_{t+1}$ and $n$ such that $\langle \varphi|\alpha|u \ast \langle \tau(u), n \rangle \rangle \neq 0$.  

Find $m$ such that $\varphi|\langle \alpha|u \ast \langle \tau(u), n \rangle \rangle = 0$.  

This concludes the definition of the sequence $F_0, F_1, \ldots$.  

Using the fact that condition a is satisfied, find $\alpha$ such that for each $t, \alpha \in F_t$.  

Conclude from the fact that conditions b4 and b5 are satisfied:  

for all $t$ in the border $S'$ of $S$ such that $t \upharpoonright t \in \tau$,

- if length$(t)$ is even, then $\alpha(t) = 0$ and, if length$(t)$ is odd, then $\alpha(t) = 1$, and also:
  - for all $u$ in the border $S'$ of $S$ such that $u \upharpoonright t \in \tau$,
  - if length$(u)$ is even, then $\langle \varphi|\alpha|u \rangle = 0$ and, if length$(u)$ is odd, then $\langle \varphi|\alpha|u \rangle = 1$.  

Conclude $W^t_S(\sigma, \alpha)$ and $W^t_S(\tau, \varphi|\alpha)$ and: both $\alpha$ and $\varphi|\alpha$ are in $E_S$.  

(ii) Let $S$ in $Hrs$ and $\varphi : \omega^\omega \rightarrow \omega^\omega$ be given such that $\forall \alpha \in E_S \rightarrow \varphi|\alpha \in A_S$.  

Find $T$ in $Hrs$ such that $\forall n[T \upharpoonright \langle n \rangle = S]$.  

Define $\psi : \omega^\omega \rightarrow \omega^\omega$ such that $\forall \alpha \forall n(\langle \varphi|\psi|\alpha \rangle \upharpoonright \langle n \rangle = \varphi|\langle n \rangle \rangle)$.  

Note: $\forall \alpha \in \mathbb{A}_T \rightarrow \forall \alpha(\langle \varphi|\psi|\alpha \rangle \upharpoonright \langle n \rangle \in A_S)$.  

Find $\rho : \omega^\omega \rightarrow \omega^\omega$ such that $\forall \beta[\langle \varphi|\psi|\beta \rangle \upharpoonright \langle n \rangle \in A_S \leftrightarrow \rho|\beta \in A_S]$.  

Define $\zeta : \omega^\omega \rightarrow \omega^\omega$ such that, for all $\alpha$, for all $n$, $\langle \zeta|\alpha \rangle \upharpoonright \langle n \rangle = \rho|\langle \psi|\alpha \rangle \rangle$.  

Note: for every $\alpha$, if $\alpha \in \mathbb{A}_T$, then $\zeta|\alpha \in E_T$.  

Using (i), find $\alpha$ in $E_T$ such that $\zeta|\alpha \in E_T$.  

Find $n$ such that $\alpha \upharpoonright \langle n \rangle \in A_{T|\langle n \rangle = A_S}$.  

---

7$T$ might be called the successor of $S$.

8Proving the existence of $\rho$ is equivalent to proving that the class $\Pi^S_2$ is closed under the operation of countable intersection. We leave it to the reader to find this proof.
Find \( m \) such that \( (\zeta | \alpha) \downarrow \langle m \rangle \in A_{T(\zeta)} = A_S \).

Note: \( (\zeta | \alpha) \downarrow \langle m \rangle = \rho(\psi | \alpha) \in A_S \) and conclude:

\[
(\psi | \alpha) \downarrow \langle n \rangle = \varphi(\alpha \downarrow \langle n \rangle) \in A_S.
\]

Defining \( \beta := \alpha \downarrow \langle n \rangle \), we thus see: both \( \beta \) and \( \varphi | \beta \) are in \( A_S \).

\[ \square \]

4. The Fan Theorem

4.1. Finitary spreads.

**Definition 11.** A spread-law \( \beta \) will be called finitary if it satisfies the following condition:

\[
\forall s[\beta(s) = 0 \rightarrow \exists n \forall \alpha[\beta(s * \langle n \rangle) = 0 \rightarrow n \leq m]].
\]

If the spread-law \( \beta \) is finitary, the corresponding spread \( F_\beta \) will be called a finitary spread or a fan.

When I am creating an element \( \alpha \) of a fan \( F_\beta \), then, at each stage \( n \), having completed

\[
\alpha(0), \alpha(1), \ldots, \alpha(n - 1)
\]

I only have finitely many choices for the next value, \( \alpha(n) \).

An important example of a fan is Cantor space \( 2^\omega := C := \{ \alpha | \forall n[\alpha(n) \leq 1] \} \).

**Definition 12.** For all \( X \subseteq \omega^\omega \), for all \( B \subseteq \omega \), we define: \( B \) is a bar in \( X \), \( Bar_X(B) \), if and only if \( \forall \alpha \in X \exists n[\alpha(n) \in B] \).

**Theorem 8** (Fan Theorem). Let \( \beta \) be a finitary spread-law.

If \( B \subseteq \omega \) is a bar in \( F_\beta \), some finite \( B' \subseteq B \) is bar in \( F_\beta \).

**Proof.** Assume \( \beta \) is a finitary spread-law and let \( B \subseteq \omega \) be a bar in \( F_\beta \).

How may I have convinced myself that \( B \) is indeed a bar in \( F_\beta \)?

(Under what circumstances shall we say that this conclusion is justified? Some agreement here is, intuitionistically, the only way to make sense of the statement).

Let us define, for each \( s \) such that \( \beta(s) = 0 \),

B bars \( s \) in \( F_\beta \) if and only if \( \forall \alpha \in F_\beta[s \sqsubseteq \alpha \rightarrow \exists n[\alpha(n) \in B]] \).

Now observe the following:

(i) For each \( s \), if \( \beta(s) = 0 \) and \( s \in B \), then \( B \) bars \( s \) in \( F_\beta \).

(ii) For each \( s \), if \( \beta(s) = 0 \) and, for every \( n \) such that \( \beta(s * \langle n \rangle) = 0 \), \( B \) bars \( s * \langle n \rangle \) in \( F_\beta \), then \( B \) bars \( s \) in \( F_\beta \).

(iii) For each \( s \), if \( \beta(s) = \beta(s * \langle n \rangle) = 0 \) and \( B \) bars \( s * \langle n \rangle \) in \( F_\beta \), then \( B \) bars \( s * \langle n \rangle \) in \( F_\beta \).

Note that one may prove a statement of the form ‘\( B \) bars \( s \) in \( F_\beta \)’ by starting from observations of the form (i) and using observations of the form (ii) and (iii) as reasoning steps.

Let us now agree to consider the statement ‘\( B \) bars \( s \) in \( F_\beta \)’ as established or true if and only if we are able to provide such a canonical proof.

This agreement marks an important point in the development of our intuitionistic mathematics. We are introducing an axiomatic assumption.

If we do so, we may argue as follows.

Assume \( \text{Bar}_{F_\beta}(B) \), i.e. \( B \) bars \( \langle \rangle \) in \( F_\beta \).

Find a canonical proof of this statement.

Now replace in this canonical proof every statement ‘\( B \) bars \( s \) in \( F_\beta \)’ by ‘\( B \) finitely bars \( s \) in \( F_\beta \)’ where the latter means:

some finite \( B' \subseteq B \) bars \( s \) in \( F_\beta \).
Under this replacement our canonical proof changes into another valid proof. In order to see this, we have to use the fact that a finite union of finite sets of integers is itself a finite set of integers.

The conclusion of the new proof will be: ‘some finite \( B' \subseteq B \) bars ⟨ ⟩ in \( \mathcal{F}_B \)’ and that is what we wanted to establish. □

4.2. The Uniform-Continuity Theorem. The following result is the first application of the Fan Theorem.

**Theorem 9.** Every pointwise continuous function from \([0, 1]\) to \( \mathcal{R} \) is uniformly continuous on \([0, 1]\).

**Proof.** Let \( f \) be a pointwise continuous function from \([0, 1]\) to \( \mathcal{R} \).

We first define \( \rho \) such that, for each \( s \) in \( 2^{<\omega} \), \( \rho(s) = (\rho'(s), \rho''(s)) \) is a pair of rationals. We define \( \rho \) by induction on the length of the argument.

We define \( \rho(\epsilon) = (0, 1) \) and, for each \( s \) in \( 2^{<\omega} \),
\[
\rho(s * (0)) = (\rho'(s), \frac{1}{3}\rho'(s) + \frac{2}{3}\rho''(s)) \quad \text{ and } \quad \rho(s * (1)) = (\frac{2}{3}\rho'(s) + \frac{1}{3}\rho''(s), \rho''(s)).
\]

We intend to prove, for each \( m \),
\[
\exists n \forall x \in [0, 1] \forall y \in [0, 1] [[x − y] < \frac{1}{3^n} \rightarrow |f(x) − f(y)| < \frac{1}{3^n}].
\]

Let \( m \) be given.

We define, for all rationals \( p, q \) such that \( 0 \leq p < q \leq 1 \),
\[
[p, q] \text{ is fine if and only if } \exists n \forall x \in [p, q] \forall y \in [p, q] [[x − y] < \frac{1}{3^n} \rightarrow |f(x) − f(y)| < \frac{1}{3^n}].
\]

We want to prove: \([0, 1]\) is fine.

Let \( B \) be the set of all \( s \) in \( 2^{<\omega} \) such that \( [\rho'(s), \rho''(s)] \) is fine.

We first prove: \( B \) is a bar in Cantor space \( 2^{<\omega} \).

Let \( \alpha \) in \( 2^{<\omega} \) be given. Find a real \( x \) such that, for all \( n, \rho'(\overline{m}n) \leq x \leq \rho''(\overline{m}n) \). As \( f \) is continuous at \( x \), find \( l \) such that \( \forall y \in [0, 1] [[x − y] < \frac{1}{3^n} \rightarrow |f(x) − f(y)| < \frac{1}{3^n}] \).

Note: \( \rho''(\overline{m}(2l)) − \rho'(\overline{m}(2l)) = (\frac{2}{3})^{2l} < (\frac{1}{3})^l \). Conclude: \( \overline{\alpha}(2l) \in B \).

We thus see that \( B \) is a bar in \( 2^{<\omega} \).

One easily verifies: for all \( s \) in \( 2^{<\omega} \),
\[
s \in B \text{ if and only if both } s * (0) \in B \text{ and } s * (1) \in B.
\]

Now find a canonical proof of: ‘\( B \) bars ⟨ ⟩ in \( 2^{<\omega} \)’ and replace, in this proof, every statement: ‘\( B \) bars \( s \) in \( 2^{<\omega} \)’ by ‘\( s \in B \)’.

The result will be a valid proof, and the conclusion of the proof is: ‘⟨ ⟩ \in B', i.e. ‘\([0, 1] \text{ is fine} \).’ □

5. Measure and Integration

Brouwer worked on the theory of measure and integration, following the lead of H. Lebesgue, see [13] Chapter VI. Bishop chose for an approach inspired by P.J. Daniell see [11] Chapter 6. We here return to Brouwer’s approach.

5.1. A note on real numbers. A real number is an infinite sequence \( x = x(0), x(1), \ldots \) of pairs \( x(n) = (x'(n), x''(n)) \) of rationals such that

1. \( x \) is shrinking, i.e. for all \( n, x'(n) \leq x'(n + 1) \leq x''(n + 1) \leq x''(n) \), and
2. \( x \) is dwindling, i.e. for every \( m \), there exists \( n \) such that \( x''(n) − x'(n) < \frac{1}{2^m} \).

\( \mathcal{R} \) denotes the set of the real numbers. For all \( x, y \) in \( \mathcal{R} \), one defines

1. \( x <_\mathcal{R} y \) if and only if, for some \( n, x''(n) < y'(n) \), and
2. \( x \leq_\mathcal{R} y \) if and only if, for all \( n, x''(n) \leq y''(n) \), and
3. \( x \equiv_\mathcal{R} y (x \text{ really-coincides with } y, x \text{ is (really) equal to } y) \),
   if and only if \( x \leq_\mathcal{R} y \text{ and } y \leq_\mathcal{R} x \).

If confusion seems unlikely, we omit the subscript ‘\( \mathcal{R} \)’.

One may prove *Cantor’s Intersection Theorem:*
Given an infinite sequence \((x_0, y_0), (x_1, y_1), \ldots\) of pairs of reals that is shrinking, i.e., for all \(n\), \(x_n \leq x_{n+1} \leq y_{n+1} \leq y_n\), and dwindling, i.e., for all \(m\) there exists \(n\) such that \(y_n - x_n < \frac{1}{m}\), then there exists a real \(z\) such that, for all \(n\), \(x_n \leq z \leq y_n\), and, for each real \(t\), if, for all \(n\), \(x_n \leq t \leq y_n\), then \(t = z\).

\([0, 1] := \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}\).

We will treat rationals and also pairs of rationals as if they were natural numbers.

This approach may be made precise by suitable coding functions, see [33, Section 8].

5.2. Measurable open sets.

**Definition 13.** Let \((q_0, r_0), (q_1, r_1), \ldots\) be an enumeration of all pairs \((q, r)\) of rationals such that \(q \leq r\).

For all \(n\), for all \(a \in \omega^n\), we define \(\mathcal{H}_a := \{x \in \mathbb{R} \mid \exists j < n[q_{a(j)} < x < r_{a(j)}]\}\).

For all \(\alpha \in \omega^\omega\), we define \(\mathcal{H}_\alpha := \{x \in \mathbb{R} \mid \exists j[q_{\alpha(j)} < x < r_{\alpha(j)}]\} = \bigcup_n \mathcal{H}_{\alpha^n}\).

\(\mathcal{X} \subseteq [0, 1]\) is called open if and only if there exists \(\alpha\) such that \(\mathcal{X} = \mathcal{H}_\alpha \cap [0, 1]\).

**Definition 14.** For each \(n\), for each \(b \in \omega^n\), \(b\) is neatly increasing if and only if \(\forall j < n[q_{b(j)} < r_{b(j)}] \text{ and } \forall j < n-1[r_{b(j)} \leq q_{b(j+1)}]\).

**Remark 5.** For each \(a\), there exists exactly one \(b\) such that \(b\) is neatly increasing and \(\mathcal{H}_a = \mathcal{H}_b\).

The proof of this fact is left to the reader.

**Definition 15.** For each \(a\), we define \(\mu(a) := \sum_{j<\text{length}(a)} r_{b(j)} - q_{a(j)}\), where \(b\) is neatly increasing and satisfies \(\mathcal{H}_a = \mathcal{H}_b\).

\(\alpha\) is called measurable if and only if \(\mu(\alpha) := \lim_{n \to \infty} \mu(\alpha^n)\) exists.

**Definition 16.** For all rationals \(q, r, s, t\) such that \(q \leq r\) and \(s \leq t\) we define a pair of rationals called \((q, r) \cap (s, t)\) as follows.

If \(r < s\) or \(t < q\), we define \((q, r) \cap (s, t) = (0, 0)\), and, if \(s \leq r\) and \(q \leq t\) we define: \((q, r) \cap (s, t) = (\max(q, s), \min(r, t))\).

**Definition 17.** Let \(\alpha\) be measurable and let rationals \(q, r\) be given such that \(q < r\).

Find \(\beta\) such that, for each \(j\), \((q_{\beta(j)}, r_{\beta(j)}) = (q_{\alpha(j)}, r_{\alpha(j)}) \cap (q, r)\). Note that \(\beta\) is measurable. We define: \(\mu \upharpoonright [q, r](\alpha) := \mu(\beta)\).

For each \(n\), we define: \(\alpha\) covers \((q, r)\) for more than \(1 - \frac{1}{n}\) if and only if \(\mu \upharpoonright [q, r](\alpha) > (1 - \frac{1}{n})(r - q)\).

We also define: \(\alpha\) never covers \((q, r)\) if and only if \(\mu \upharpoonright [q, r](\alpha) < r - q\).

**Lemma 10.** Let \(\alpha\) be measurable and let rationals \(p, q, r\) be given such that \(q < p < r\).

(i) For each \(n\), either \(\alpha\) covers \((q, r)\) for more than \(1 - \frac{1}{n}\), or \(\alpha\) never covers \((q, r)\).

(ii) \(\mu \upharpoonright [q, r](\alpha) = \mu \upharpoonright [q, p](\alpha) + \mu \upharpoonright [p, r](\alpha)\).

(iii) For all \(n\), if \(\alpha\) covers both \((p, q)\) and \((q, r)\) for more than \(1 - \frac{1}{n}\), then \(\alpha\) covers \((q, r)\) for more than \(1 - \frac{1}{n}\).

(iv) If \(\alpha\) never covers \((q, r)\), then either \(\alpha\) never covers \((q, p)\) or \(\alpha\) never covers \((p, r)\).

**Proof.** The proof of these statements is left to the reader. \(\square\)

**Lemma 11.** Let \(\alpha, \beta\) be given such that both \(\alpha, \beta\) are measurable.

(i) If \(\mu(\alpha) < \mu(\beta)\) and \(\mathcal{H}_\alpha \subseteq \mathcal{H}_\beta\), then one may find an element of \(\mathcal{H}_\beta \setminus \mathcal{H}_\alpha\).

(ii) If \(\mathcal{H}_\beta \subseteq \mathcal{H}_\alpha\), then \(\mu(\beta) \leq \mu(\alpha)\), and, if \(\mathcal{H}_\beta = \mathcal{H}_\alpha\), then \(\mu(\beta) = \mu(\alpha)\).
Proof. (i) Find \( n > 0 \) such that \( \mu(\overline{M}_n) > \mu(\alpha) \). Find \( b \) such that \( b \) is neatly increasing and \( H_b = \overline{H}_n \). Find \( j < \text{length}(b) \) such that \( \alpha \) never covers \((q_{b(j)}, r_{b(j)})\).

Using Lemma 10, define a real \( x \) such that \( x(0) = (q_{b(j)}, r_{b(j)}) \) and, for each \( n, x'(n) < x(n+1) < x''(n+1) < x''(n) \) and \( \alpha \) never covers \( x(n) \).

Note: \( x \in H_\beta \setminus H_\alpha \).

(ii) This easily follows from (i).

\( \square \)

Note that, in the proof of Lemma 11, we did not use the Fan Theorem.

5.3. On the complement of a measurable open set.

Definition 18. We define a function \( B \) associating to every \( a \) in \( 2^{<\omega} \) a pair of rationals \( B(a) = (B'(a), B''(a)) \).

\( B(\langle \rangle) = (0,1) \) and for all \( a \) in \( 2^{<\omega} \), we consider \( M(a) := \frac{B'(a)+B''(a)}{2} \) and then define \( B(a \ast (0)) = (B'(a), M(a)) \) and \( B(a \ast (1)) = (M(a), B''(a)) \).

We also define a function \( \varphi_{bin} \) from Cantor space \( 2^\omega \) to \( [0,1] \), by the following.

For each \( \gamma \) in \( 2^\omega \), for each \( n, (\varphi_{bin}(\gamma))(n) = B(\gamma _n) \).

Note that, for all \( \gamma \) in \( 2^\omega \), \( \varphi_{bin}|\gamma \) := \( \sum_{n=0}^{\infty} \gamma(n) \cdot 2^{-n-1} \).

Note that, constructively, \( \varphi_{bin} \) is not a surjective mapping of Cantor space \( C \) onto \( [0,1] \). The set \( \varphi_{bin}[2^\omega] \) consists of all \( x \) in \( [0,1] \) that admit of a binary expansion, i.e. for all \( m \), for all \( i < 2^m \), one may decide: \( x \leq \frac{i}{2^m} \) or \( \frac{i}{2^m} < x \).

The following Lemma shows that, within the complement of a ‘small’ measurable subset of \([0,1]\), one may construct ‘large’, ‘compact’ sets.

Lemma 12. Let \( n > 0 \) and \( \alpha \) be given such that \( \alpha \) is measurable and \( \mu(\alpha) < \frac{1}{\sqrt{m-n}} \).

There exists a fan-law \( \beta \) in \( 2^\omega \) such that

1. for all \( a \), if \( \beta(a) = 0 \), then \( a \in 2^{<\omega} \), and \( \alpha \) never covers \( B(a) \),
2. for all \( a \) in \( 2^{<\omega} \), for all \( i < 2 \), if \( \beta(a) = 0 \) and \( \beta(a \ast (i)) = 1 \), then \( \alpha \) covers \( B(a \ast (i)) \) for more than \( \frac{1}{2} \), and
3. there exists a measurable \( \alpha^+ \) such that \( \mu(\alpha^+) < \frac{1}{\sqrt{m-n}} \), and, for all \( x \) in \([0,1]\), if \( x \notin H_\alpha^+ \), then \( x \notin H_\alpha \) and \( \exists \gamma \in F_\beta[\varphi_{bin}|\gamma = x] \).

Proof. Let \( n > 0 \) and \( \alpha \) be given such that \( \alpha \) is measurable and \( \mu(\alpha) < \frac{1}{\sqrt{m-n}} \).

We define the promised fan-law \( \beta \) as follows.

For each \( a \), if \( a \notin 2^{<\omega} \), then \( \beta(a) = 1 \).

For a in \( 2^{<\omega} \), \( \beta(a) \) is defined by induction on \( \text{length}(a) \). We will take care that, for each \( a \) in \( 2^{<\omega} \), if \( \beta(a) = 0 \), then \( \alpha \) never covers \( B(a) \).

Define \( \beta(\langle \rangle) = 0 \) and note: \( \alpha \) never covers \( B(\langle \rangle) \).

Now assume: \( a \in 2^{<\omega} \), and \( \beta(a) \) has been defined.

If \( \beta(a) = 1 \), define \( \beta(a \ast (0)) = \beta(a \ast (1)) = 1 \).

If \( \beta(a) = 0 \), we may assume: \( \alpha \) never covers \( B(a) \).

Find \( i < 2 \) such that \( \alpha \) never covers \( B(a \ast (i)) \) and define \( \beta(a \ast (i)) = 0 \).

Then consider \( a \ast (1-i) \) and note:

either \( \alpha \) never covers \( B(a \ast (1-i)) \) or \( \alpha \) covers \( B(a \ast (1-i)) \) for more than \( \frac{1}{2} \).

Define \( \beta(a \ast (1-i)) \) such that, if \( \beta(a \ast (1-i)) = 0 \), then \( \alpha \) never covers \( B(a \ast (1-i)) \), and, if \( \beta(a \ast (1-i)) = 1 \), then \( \alpha \) covers \( B(a \ast (1-i)) \) for more than \( \frac{1}{2} \).

Note that we have some freedom in carrying out this step as the conditions ‘\( a \) never covers \( B(a \ast (1-i)) \)’ and ‘\( \alpha \) covers \( B(a \ast (1-i)) \) for more than \( \frac{1}{2} \)’ do not exclude each other.

Define \( C := \{ a \ast (i) \mid a \in 2^{<\omega}, i < 2 \mid \beta(a) = 0 \land \beta(a \ast (i)) = 1 \} \).

Note that, for all \( a, b \) in \( C \), if \( a \neq b \) then \( B(a) \cap B(b) = (0,0) \).

Define \( \alpha^+ \) such that, for each \( n, \) if there exists \( a \) in \( C \) such that \((q_{a}, r_{a}) = (B'(a), B''(a))\), then \( \alpha^+(n) = n \) and, if not, then \( q_{\alpha^+(n)} = r_{\alpha^+(n)} = 0 \).
Note: $\mathcal{H}_\alpha = \bigcup_{a \in C} (B'(a), B''(a))$.

We now prove that $\alpha^+$ is measurable.

Let $k$ be given. Find $p$ such that $\mu(\alpha) - \mu(\gamma_p) < \frac{1}{2^k}$. Find $m$ such that $\mu(\alpha^+ m) > \mu(\gamma_p) - \frac{1}{2^k}$. Note that, for all $n > m$, $\mu(\alpha^+ n) - \mu(\alpha^+ m) < 2 \cdot \frac{1}{2^k} + 2 \cdot \frac{1}{2^k} = \frac{1}{2^k}$. We thus see that $\mu(\alpha^+) = \lim_{n \to \infty} \mu(\alpha^+ n)$ exists, i.e. $\alpha^+$ is measurable.

Note: $\mu(\alpha) < \frac{1}{2^{k+1}}$ and for each $a$ in $C$, $\alpha$ covers $B(a)$ for more than $\frac{1}{2}$. Conclude: $\mu(\alpha^+) \leq 2\mu(\alpha) < \frac{1}{2^k}$.

Finally, note that, for each $\gamma$ in $\mathcal{F}_\beta$, for each $n$, $\alpha$ does not cover $(\varphi_{bin}|\gamma)(n) = B(\gamma_n)$, and, therefore: $\varphi_{bin}|\gamma \notin \mathcal{H}_\alpha$.

Now define $\alpha^+$ such that, for each $a$,

$$\alpha^+(2a) = \alpha^+(a), \text{ and } \alpha(2a + 1) = (q_0 - \frac{1}{2^k + 1}, q_0 + \frac{1}{2^k + 1}).$$

Note that $\alpha^+$ is measurable and $\mu(\alpha^+) \leq \mu(\alpha^+) + \frac{1}{2^k} < \frac{1}{2^k}$.

Assume $x \in [0, 1]$ and $x \notin \mathcal{H}_{\alpha^+}$. Then $\forall q \in \mathbb{Q} [q \# x]$. We thus may find $\gamma$ in $2^\omega$ such that $\varphi_{bin}|\gamma = x$. Note: $x \notin \mathcal{H}_{\alpha^+}$ and thus, for each $n$, $\gamma_n \notin C$. Conclude: $\forall n(\beta(\gamma_n) = 0)$ and: $\gamma \in \mathcal{F}_\beta$.

We thus see: $\forall x \in [0, 1] [x \notin \mathcal{H}_{\alpha^+} \rightarrow \exists \gamma \in \mathcal{F}_\beta [\varphi_{bin}|\gamma = x]]$.

$$\Box$$

5.4. Almost-full subsets of $[0, 1]$ and almost-full functions from $[0, 1]$ to $[-1, 1]$.

**Definition 19.** $X \subseteq [0, 1]$ will be called almost-full if and only if, for each $n$, there exists a measurable $\alpha$ such that $\mu(\alpha) < \frac{1}{2^n}$ and $\forall x \in [0, 1] [x \notin \mathcal{H}_\alpha \rightarrow x \in X]$.

A partial function $f$ from $[0, 1]$ to $[-1, 1]$ will be called almost-full if and only if its domain $\text{Dom}(f) = \{x \in [0, 1] \mid f(x) \text{ is defined}\}$ is almost-full.

**Definition 20.** Let $f$ be an almost-full function from $[0, 1]$ to $[-1, 1]$.

$f$ is called measurable if and only if, for each $n$, one may find $m$ and rationals $u_0, u_1, \ldots, u_{2^m - 1}$ such that for each $i < 2^m$, $-1 \leq u_i \leq 1$, and a measurable $\alpha$ such that $\mu(\alpha) < \frac{1}{2^m}$ and, for each $x$ in $\text{Dom}(f)$, for each $i < 2^m$, if $x \notin \mathcal{H}_\alpha$ and $\frac{1}{2^n} < x < \frac{i + 1}{2^n}$, then $|f(x) - u_i| < \frac{1}{2^n}$.

The number $\sum_{i < 2^m} u_i \cdot \frac{1}{2^m}$ will be called an estimate of the integral of $f$ of accuracy $\frac{1}{2^n}$.

**Theorem 13.** Let $f$ be an almost-full and measurable function from $[0, 1]$ to $[-1, 1]$.

(i) For all rationals $q, r$, if $q, r$ are estimates of the integral of $f$ of accuracy $\frac{1}{2^n}$, $\frac{1}{2^n}$, respectively, then $|q - r| \leq \frac{1}{2^n} + \frac{1}{2^n}$.

(ii) There exists a real $x$ such that, for all $n$, for all estimates $q$ of the integral of $f$ of accuracy $\frac{1}{2^n}$, $|x - q| \leq \frac{1}{2^n}$.

**Proof.** The proof of this Theorem is left to the reader. $\Box$

The number intended in Theorem 13(ii) is unique up to the relation of real coincidence and will be called: $\int_0^1 f$, the integral of $f$ on $[0, 1]$.

For the next result, see [21, page 7] and [15, Section 6.2.2, Theorem 1].

**Theorem 14 (van Rootselaar).** Every almost-full function from $[0, 1]$ to $[-1, 1]$ is measurable.

**Proof.** Let $f$ be an almost-full function from $[0, 1]$ to $[-1, 1]$.

Let $n$ be given.

Find a measurable $\alpha$ such that $\mu(\alpha) < \frac{1}{2^m}$ and $\forall x \in [0, 1] [x \notin \mathcal{H}_\alpha \rightarrow x \in \text{Dom}(f)]$.

Using Lemma 12 find a fan-law $\beta$ in $2^\omega$ such that
Lemma 16. Let $f$ be a measurable function from $[0, 1]$ to $[-1, 1]$. Let rationals $q, r$ be given such that $-1 \leq q < r \leq 1$. Let $n$ be given.

One may find rationals $s, t$ such that $q \leq s < t \leq r$ and a measurable $\alpha$ such that $\mu(\alpha) < \frac{1}{2^n}$ and $\forall x \in Dom(f) \rightarrow (f(x) < s \lor t < f(x))$.

Proof. Let $f$ be a measurable function from $[0, 1]$ to $[-1, 1]$. Let rationals $q, r$ be given such that $-1 \leq q < r \leq 1$. Let $n$ be given.

Find $l$ such that $\frac{1}{2^n} < \frac{r - q}{2^n}$. Note: $r - q \leq 2$ and $l > n$.

Using Definition $[20]$ find $m$ and rationals $u_0, u_1, \ldots, u_{2^m - 1}$ such that, for each $i \leq 2^m$, $-1 \leq u_i \leq 1$, and a measurable $\alpha$ such that $\mu(\alpha) < \frac{1}{2^{2n}}$ and, for each $x$ in $Dom(f)$, for each $i < 2^m$, if $x \notin H_\alpha$ and $\frac{1}{2^m} < u_i < \frac{r + q}{2^n}$, then $|f(x) - u_i| < \frac{1}{2^n}$.

Define $j_0 := \mu[j \mid q < q + \frac{r + q}{2^n}]$.

Note: $2^{n+1}, \frac{1}{2^n} < r - q$, so $2^{n+1} \leq j_0$.

For each $j < j_0$, consider the set $A_j := \{i \leq 2^m \mid q + \frac{i}{2^m} \leq u_i < q + \frac{r + q}{2^n}\}$.
Theorem 17. Let each $u,v$ such that, for all $u \in A_j$, $\text{Card}(A_j) \leq \text{Card}(A_h)$.

Note: Card$(A_j) \leq \frac{2^m}{2^n} = 2^{m-n-1}$.

Define $s := q + \frac{1}{2^n} + \frac{1}{2^n}$ and $t := q + \frac{1}{2^n} - \frac{1}{2^n}$ and note: $q < s < t < r$.

Note: for each $x$ in $\text{Dom}(f) \cap (\frac{1}{2^n}, \frac{1}{2^n})$, if $x \notin H_\alpha$, then $f(x) < q + \frac{1}{2^n} + \frac{1}{2^n} = s$, or $q + \frac{1}{2^n} \leq u_t$ and,

for all $x$ in $\text{Dom}(f) \cap (\frac{1}{2^n}, \frac{1}{2^n})$, if $x \notin H_\alpha$, then $t = q + \frac{1}{2^n} - \frac{1}{2^n} < f(x)$.

One may define $\alpha^+$ such that $\alpha^+$ is measurable and $H_\alpha \subseteq H_{\alpha^+}$ and, for each $i \in A_j$, $(\frac{1}{2^n} + \frac{1}{2^n}) \subseteq H_{\alpha^+}$ and, for each $i \leq 2^n$, $\frac{1}{2^n} \in H_{\alpha^+}$ and $\mu(\alpha^+) \leq \mu(\alpha) + \sum_{i \in A_j} \frac{1}{2^n} \leq \frac{1}{2^n} < \frac{1}{2^n}$.

Note: $\alpha^+$ is measurable and $\mu(\alpha^+) < \frac{1}{2^n}$, and, for all $x$ in $\text{Dom}(f)$, if $x \notin H_{\alpha^+}$, then either $f(x) < s$ or $t < f(x)$.

\[\square\]

Theorem 17. Let $f$ be a measurable function from $[0,1]$ to $[-1,1]$. Let rationals $u,v$ be given such that $-1 \leq u < v \leq 1$.

There exists $y$ in $(u,v)$ such that, for almost all $x$ in $[0,1]$, $x \in \text{Dom}(f)$ and $f(x) < y$ or $y < f(x)$.

Proof. Applying Lemma 18 we find an infinite sequence $(u_0,v_0),(u_1,v_1),\ldots$ of pairs of rationals, and an infinite sequence $\alpha_0,\alpha_1,\ldots$ of measurable elements of $\omega^\omega$ such that

1. $(u_0,v_0) = (u,v)$,
2. for each $n$, $u_n < v_{n+1} < v_n$, and $v_{n+1} - u_{n+1} < \frac{1}{n}$,
3. for each $n$, $\mu(\alpha_n) < \frac{1}{n}$ and $\forall x \in \text{Dom}(f) \setminus H_{\alpha_n} [f(x) < u_n \lor v_n < f(x)]$.

Find $y$ such that, for each $n$, $u_n \leq y \leq v_n$ and note that $y$ satisfies the requirements.

\[\square\]

Corollary 18. Let $f$ be a measurable function from $[0,1]$ to $[-1,1]$.

The set $\{y \in [-1,1] \mid \{x \in [0,1] \mid f(x) < y\} \text{ is measurable}\}$ is dense in $[-1,1]$.

Proof. Use Theorem 17 and Corollary 18.

\[\square\]

Lemma 19. Let $h$ be a partial function from $[-1,1]$ to $[0,1]$ such that $\text{Dom}(h)$ is dense in $[-1,1]$ and $h$ is non-decreasing, i.e.

$\forall x \in \text{Dom}(h) \forall y \in \text{Dom}(h) | x < y \rightarrow h(x) \leq h(y)$.

Let $x,y$ in $\text{Dom}(h)$ be given such that $x < y$ and $h(x) < h(y)$.

There exists $z$ in $[x,y]$ such that, for all $u$ in $\text{Dom}(h)$, if $u \neq z$, then $\exists t \in \text{Dom}(h) | \exists w \in \text{Dom}(h) | t < u < w \land h(w) - h(t) < \frac{1}{2}(h(y) - h(x))$.

Proof. Define $\varepsilon := h(y) - h(x)$.

A point $u$ in $[x,y]$ will be called neat if and only if

$\exists t \in \text{Dom}(h) \exists w \in \text{Dom}(h) | t < u < w \land h(w) - h(t) < \frac{1}{2} \varepsilon$.

We define an infinite sequence $(x_0,y_0),(x_1,y_1),\ldots$ of pairs of elements of $\text{Dom}(h)$, such that, for all $n$, $x \leq x_n \leq x_{n+1} \leq y_{n+1} \leq y_n \leq y$ and $y_n - x_n \leq (\frac{1}{2})^n (y - x)$ and every $u$ in $[x,x_n) \cup (y_n,y]$ is neat.

We first define: $(x_0,y_0) := (x,y)$.

Now let $n$ be given such that $(x_n,y_n)$ has been defined already such that $x_n < y_n$.

Determine $a$ in $\text{Dom}(h)$ such that $\frac{1}{2}x_n + \frac{1}{2}y_n < a < \frac{1}{2}x_n + \frac{1}{2}y_n$.

Note: $a = x_n < \frac{1}{2}(y_n - x_n)$ and also: $y_n - a < \frac{1}{2}(y_n - x_n)$.

Note: $h(x) \leq h(a) \leq h(y)$. 

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Either $\frac{1}{4}\varepsilon < h(a) - h(x) \text{ or } h(a) - h(x) < \frac{1}{4}\varepsilon$, and also: either $\frac{1}{4}\varepsilon < h(y) - h(a)$ or $h(y) - h(a) < \frac{1}{4}\varepsilon$, but not both $\frac{1}{4}\varepsilon < h(a) - h(x)$ and $\frac{1}{4}\varepsilon < h(y) - h(a)$.

Note that, if $h(a) - h(x) < \frac{1}{4}\varepsilon$, then every $u$ in $[x, a]$ will be neat, and, if $h(y) - h(a) < \frac{1}{4}\varepsilon$, then every $u$ in $[a, y]$ will be neat.

We define $(x_{n+1}, y_{n+1})$ in such a way that either $(x_{n+1}, y_{n+1}) = (x_n, a)$ and $h(y) - h(a) < \frac{1}{2}\varepsilon$, or $(x_{n+1}, y_{n+1}) = (a, y_n)$ and $h(a) - h(x) < \frac{1}{2}\varepsilon$.

Then every $u$ in $[x, x_n+1] \cup [y_n+1, y]$ will be neat.

Clearly, the infinite sequence $(x_0, y_0), (x_1, y_1), \ldots$ satisfies our requirements.

Using Cantor’s Intersection Theorem, find $z$ such that, for all $n$, $x_n \leq z \leq y_n$ and note: for all $u$ in $[0, 1]$, if $u \neq z$, then, for some $n$, $u < x_n$ or $y_n < u$ and: $u$ is neat. □

**Definition 22.** Let $h$ be a partial function from $[-1, 1]$ to $[0, 1]$ such that $\text{Dom}(h)$ is dense in $[-1, 1]$ and $h$ is non-decreasing.

Let $u$ in $[-1, 1]$ be given. $u$ is a point of continuity for $h$ if and only if $\forall n \exists x \in \text{Dom}(h) \exists w \in \text{Dom}(h)[t < u < w \land h(w) - h(t) < \frac{1}{n}]$.

**Remark 6.** If $u$ is a point of continuity for $h$, there exists $y$ in $[0, 1]$ such that $\forall n \exists m \in \text{Dom}(h)[u - \frac{1}{m} < t < u + \frac{1}{m} \rightarrow |h(t) - y| < \frac{1}{n}]$. $y$ is unique up to the relation of real coincidence and will be called the value of $h$ at $u$. One may extend the partial function $h$ to the partial function $g \subseteq [-1, 1] \times [0, 1]$ consisting of all pairs $(u, y)$ such that either $(u, y) \in h$ or $u$ is a point of continuity for $h$ and $y$ is the value of $h$ at $u$.

**Theorem 20.** Let $h$ be a partial function from $[-1, 1]$ to $[0, 1]$ such that $\text{Dom}(h)$ is dense in $[-1, 1]$ and $h$ is non-decreasing. All but countably many elements of $[-1, 1]$ are points of continuity for $h$.

**Proof.** Find an infinite sequence $x_0, x_1, \ldots$ of elements of $\text{Dom}(h)$ such that $\forall p \in \mathbb{Q} \forall q \in \mathbb{Q}[0 \leq p < q \leq 1 \rightarrow \exists n[p \leq x_n \leq q]]$.

Define, for all $m, n$ such that $x_m < x_n$ and $h(x_m) < h(x_n)$, for all $z$ in $[0, 1]$, $z$ resolves $(x_m, x_n)$ if and only if $x_m \leq z \leq x_n$ and, for all $u$ in $[x_m, x_n]$, if $u \neq z$, then $\exists \in \text{Dom}(h) \exists w \in \text{Dom}(h)[t < u < w \land h(w) - h(t) < \frac{1}{n} (h(x_n) - h(x_m))]$.

Using Lemma 19 define an infinite sequence $z_0, z_1, \ldots$ of elements of $[0, 1]$ such that $z_0 = 0$ and $z_1 = 1$ and, for each $n$,

if $(x_{n(0)})'((n(2))) < (x_{n(1)})'((n(2)))$ and $(h(x_{n(0)}))''((n(3))) < (h(x_{n(1)}))''((n(3)))$,

then $z_{n+2}$ resolves $(x_{n(0)}, x_{n(1)})$.

Let $u$ be given such that, for each $n$, $u \neq z_n$. We prove that $u$ is a point of continuity for $h$, by showing, inductively:

For each $n$, there exist $t, w$ in $\text{Dom}(h)$ such that $t < u < w$ and $h(w) - h(t) < (\frac{1}{4})^n \cdot 3$.

The case $n = 0$ obviously holds: find $t, w$ in $\text{Dom}(h)$ such that $t < u < w$ and note: $h(w) - h(t) \leq |h(w)| + |h(t)| \leq 2 < 3$.

Now let $n, t, w$ be given such that $t, w$ in $\text{Dom}(h)$ and $t < u < w$ and $h(w) - h(t) < (\frac{1}{4})^n \cdot 3$. Find $m, p$ such that $t < x_m < u < x_p < w$ and note: $h(x_p) - h(x_m) < (\frac{1}{4})^n \cdot 3$. Now distinguish two cases.

Case (1). $h(x_p) - h(x_m) < (\frac{1}{4})^{n+1} \cdot 3$ and we are done, or

Case (2). $0 < h(x_p) - h(x_m)$.

In the latter case, find $q, r$ such that $x_n''(q) < x_n'(q)$ and $(h(x_n))''(r) < (h(x_n))'(r)$.

Then find $s$ such that $s(0) = m$, $s(1) = p$, $s(2) = q$ and $s(3) = r$.

As $s_{n+2}$ resolves $(x_{n+2}, x_{n+2})$ and $u \neq s_{n+2}$, find $t, w$ in $\text{Dom}(h)$ such that $t < u < w$ and $h(w) - h(t) < (\frac{1}{4})^{n+1} \cdot 3$.
We thus see that \( u \) is indeed a point of continuity for \( h \). 

\[ \square \]

**Remark 7.** Using Remark [7] and Theorem [9], observe that a non-decreasing partial function \( h \) from \([-1, 1]\) to \([0, 1]\) such that \( \text{Dom}(h) \) is dense in \([-1, 1]\) may be extended to a non-decreasing partial function \( g \) from \([-1, 1]\) to \([0, 1]\) such that \( \text{Dom}(g) \) is co-enumerable, i.e. there exists an infinite sequence \( z_0, z_1, \ldots \) of elements of \([-1, 1]\) such that, for every \( u \) in \([0, 1]\), if \( \forall n[u \neq z_n] \), then \( u \in \text{Dom}(g) \).

**Theorem 21.** Let \( f \) be a measurable function from \([0, 1]\) to \([-1, 1]\). The domain of the partial function \( D(f) := \{(y, m) \in [-1, 1] \times [0, 1] \mid m = \mu(\{x \in [0, 1] \mid f(x) < y\})\} \) is a co-enumerable subset of \([-1, 1]\).

**Proof.** Note that \( D(f) \) is non-decreasing and that, by Corollary [18] \( \text{Dom}(D(f)) \) is a dense subset of \([-1, 1]\). Now use Remark [7].

\[ \square \]

6. The Bar Theorem

Brouwer, when first proving the Fan Theorem, obtained the Fan Theorem as a Corollary of a more general result, see [7] and [8].

**Theorem 22** (Bar Theorem).

If \( B \subseteq \omega \) be a bar in \( \omega^\omega \), there exists a stump \( S \) such that \( S \cap B \) is a bar in \( \omega^\omega \).

**Proof.** Let us define, for each \( B \subseteq \omega \), for each \( s \),

\[ B \text{ bars } s \text{ if and only if } \text{Bar}_{\omega^\omega}(B), \text{ i.e. } \forall \alpha[s \subseteq \alpha \rightarrow \exists n[\text{Dom}(B) \cap n \in B]] . \]

Observe the following:

(i) For all \( s \), if \( s \in B \), then \( B \) bars \( s \).

(ii) If, for all \( n \), \( B \) bars \( s \upuparrows \langle n \rangle \), then \( B \) bars \( s \) in \( \omega^\omega \).

(iii) For all \( s \), if \( B \) bars \( s \), then, for all \( n \), \( B \) bars \( s \upuparrows \langle n \rangle \).

Now let \( B \subseteq \omega \) be given such that \( \text{Bar}_{\omega^\omega}(B) \), i.e. \( B \) bars \( \langle \rangle \).

Under what circumstances should we say that we are entitled to affirm this statement?

Note that one may prove a statement of the form ‘\( B \) bars \( s \)’ by starting from observations of the form (i) and using observations of the form (ii) and (iii) as reasoning steps.

Let us now agree to consider the statement ‘\( B \) bars \( s \)’ as established or true if and only if we are able to provide such a canonical proof.

Note that such a canonical proof is no longer a finite ‘tree’, like in the case of the Fan Theorem, but an infinitary one. The structure of a canonical proof is comparable to the structure of a stump.

The above agreement marks an important point in the development of our intuitionistic mathematics. We are introducing an axiomatic assumption.

After shaking hands, we argue as follows.

Take a canonical proof of ‘\( B \) bars \( \langle \rangle \)’.

In this canonical proof, replace every statement: ‘\( B \) bars \( s \)’ by the statement: ‘there exists a stump \( S \) such that \( (s \upuparrows S) \cap B \) bars \( s \)’.

We now verify that the new ‘proof’ is a valid proof.

(i) If \( s \in B \) we can take \( S = \{\langle \rangle \} \).

(ii) Let \( s \) be given such that, for each \( n \), there exists a stump \( S \) such that \( (s \upuparrows \langle n \rangle \upuparrows S) \cap B \) bars \( s \upuparrows \langle n \rangle \). Using the Second Axiom of Countable Choice [4] we build, step by step, a stump \( S \) such that, for each \( n \), \( (s \upuparrows \langle n \rangle \upuparrows (S \upuparrows n)) \cap B \) bars \( s \upuparrows \langle n \rangle \). Note that \( (s \upuparrows S) \cap B \) bars \( s \).

\[ \text{For Stumps, as decidable subsets of } \omega, \text{ may be identified with their characteristic functions.} \]
(iii) Let $s, n$ be given. Let $S$ be a stump such that $(s * (n) * S)) \cap B$ bars $s$. Now distinguish two cases.

Case (1). $S \not\in \langle n \rangle$. Conclude: $\exists t \subseteq s[t \in B]$. Define $T := \{s\}$ and note:

$(s * (n) * T) \cap B$ bars $s * (n)$.

Case (2). $\langle \rangle \in S \not\in \langle n \rangle$. Then $(s * (n) * S \not\in \langle n \rangle) \cap B$ bars $s * (n)$.

We thus see that our new ‘proof’ is a valid proof indeed.

We may affirm its conclusion: ‘there exists a stump $S$ such that $S \cap B$ bars $\langle \rangle$, i.e. $\text{Bar}_{\omega} (S \cap B)$.’ □

6.1. An application.

**Definition 23.** For all $A, B \subseteq \mathbb{Q}$, we define:

$A < B$ if and only if $\forall q \in A \forall r \in B[q < r]$.

We define a collection $WO$ of subsets of $\mathbb{Q}$ by the following inductive definition.

The elements of $WO$ are called the inductively well-ordered subsets of $\mathbb{Q}$.

(i) $\emptyset \in WO$, and, for each $q$ in $\mathbb{Q}$, $\{q\} \in WO$.

These are the basic elements of $WO$.

(ii) For every sequence $A_0, A_1, \ldots$ of elements of $WO$ such that, for each $n$, $A_n < A_{n+1}$, also $\bigcup_n A_n \in WO$.

This is the construction step of the set $WO$.

(iii) Every element of $WO$ is obtained from basic elements of $WO$ by applying the construction step repeatedly.

We let $q_0, q_1, \ldots$ be some canonical enumeration without repetitions of $\mathbb{Q}$.

$A \subseteq \mathbb{Q}$ is a decidable subset of $\mathbb{Q}$ if and only if $\exists \alpha \forall n[q_n \in A \leftrightarrow \alpha(n) \neq 0]$.

Let $A \subseteq \mathbb{Q}$ and $\alpha$ be given. $\alpha$ is called an enumeration of $A$ if and only if

$\forall n[q_n \in A \leftrightarrow \exists m(\alpha(n) = m + 1)]$.

$A \subseteq \mathbb{Q}$ is enumerable if and only if there exists an enumeration of $A$.

$A \subseteq \mathbb{Q}$ is well-founded if and only if $\forall \gamma[\forall n[q_{\gamma(n)} \in A] \rightarrow \exists m[q_m(n) \leq q_{\gamma(n+1)}]]$.

**Lemma 23.** Let $\alpha$ be an enumeration of $A \subseteq \mathbb{Q}$.

$A$ is well-founded if and only if $\forall \gamma[\forall n[q_{\gamma(n)} \in A] \rightarrow \exists m[q_m(n) \leq q_{\alpha(n+1)}]]$.

**Proof.** Let $\alpha$ be an enumeration of $A \subseteq \mathbb{Q}$.

First assume $A$ is well-founded. Let $\gamma$ be given. Distinguish two cases.

Case (a). $\alpha \circ \gamma(0) = 0$. Then $\forall i \leq 2[\alpha \circ \gamma(i) > 0] \rightarrow q_{\alpha \circ \gamma(i)} \leq q_{\alpha \circ \gamma(1)}$.

Cases (b). $\alpha \circ \gamma(0) > 0$. Now define $\beta$ such that, for each $n$, if $\alpha(n) > 0$, then $\beta(n) = \alpha(n)$, and, if $\alpha(n) = 0$, then $\beta(n) = \alpha \circ \gamma(0)$. Note: $\forall n[\beta(n) > 0 \land q_{\beta(n)} < q_{\gamma(n+1)}]$.

Find $n$ such that $q_{\beta(n)} \leq q_{\alpha(n)+1}$. Note that, if $\forall i \leq n + 1[\alpha(i) > 0]$, then $\alpha(n) = \beta(n)$ and $\alpha(n+1) = \beta(n+1)$ and $q_{\alpha(n)-1} \leq q_{\alpha(n)+1}$.

Now assume $\forall \gamma[\forall n[q_{\gamma(n)} \in A] \rightarrow \exists m[q_m(n) \leq q_{\alpha(n+1)}]]$.

Let $\gamma$ be given such that $\forall n[q_{\gamma(n)} \in A]$. Find $\delta$ such that, for each $n$, $\delta(n) > 0$ and $q_{\alpha(n)+1} = q_{\gamma(n)}$. Find $n$ such that $q_{\alpha(n)+1} \leq q_{\alpha(n)+\delta(n)+1}$ and conclude:

$q_{\gamma(n)} \leq q_{\alpha(n)+1}$.

Let $\gamma$ be given such that $\forall n[q_{\gamma(n)} \in A]$. Find $\delta$ such that, for each $n$, $\delta(n) > 0$ and $q_{\alpha(n)+1} = q_{\gamma(n)}$. Find $n$ such that $q_{\alpha(n)+1} \leq q_{\alpha(n)+\delta(n)+1}$ and conclude:

$q_{\gamma(n)} \leq q_{\alpha(n)+1}$.

Conclusion: $\forall \gamma[\forall n[q_{\gamma(n)} \in A] \rightarrow \exists m[q_m(n) \leq q_{\gamma(n+1)}]]$, i.e. $A$ is well-founded. □

**Lemma 24.**

(i) Every $A$ in $WO$ is enumerable and well-founded.

(ii) For all $A$ in $WO$, for all $a, b$ in $A$, the set $\{c \in A \mid a < c \leq b\}$ belongs to $WO$.

(iii) For every enumerable subset $A$ of $\mathbb{Q}$, if, for all $a$ in $A$, $A_{\leq a} := \{b \in A \mid b \leq a\}$ belongs to $WO$, then $A \in WO$. 23
Proof. The proof of (i) and (ii) is by straightforward induction on \(WO\) and left to the reader.

(iii) Let \(\alpha\) be an enumeration of \(A\). Define \(\beta\) such that, for each \(n\), \(\alpha(n) > 0\) and \(\forall i < n[\beta(i) > 0 \rightarrow q_{\beta(i)-1} < q_{\alpha(n)-1}]\), then \(\beta(n) = \alpha(n)\), and, if \(\alpha(n) = 0\), then \(\beta(n) = 0\).

Define an infinite sequence \(A_0,A_1,A_2,\ldots\) of subsets of \(A\), such that, for each \(n\), if \(\beta(n) > 0\), then \(A_n := \{r \in A \mid r \leq q_{\beta(n)-1} \land \forall i < n[\beta(i) > 0 \rightarrow r > q_{\beta(i)-1}\}\}\), and, if \(\beta(n) = 0\), then \(A_n = \emptyset\).

Using (iii), note: for all \(n\), \(A_n \in WO\).

Then note: for all \(n\), \(A_n < A_{n+1}\) and \(A := \bigcup_n A_n\) and conclude: \(A \in WO\). \(\square\)

The next result may be compared to results in [16] §5.

**Theorem 25.** \(\Leftrightarrow\) Let \(\alpha\) be an enumeration of \(A \subseteq \mathbb{Q}\).

If \(\forall \gamma \exists n[\gamma i \leq n + 1[\alpha \circ \gamma(i) > 0] \rightarrow q_{\alpha \circ \gamma(n)} \leq q_{\alpha \circ \gamma(n+1)}]\), then \(A \in WO\).

**Proof.** For every \(\alpha\), define \(B_{\alpha} := \bigcup_n \{s \in \omega_n+1 \mid \exists \gamma \leq n[\alpha \circ s(i) = 0]\} \cup \bigcup_n \{s \in \omega_n+2 \mid \alpha \circ s(n) > 0 \land \alpha \circ s(n+1) > 0 \land q_{\alpha \circ s(n)-1} \leq q_{\alpha \circ s(n+1)-1}\}\

Note: if \(\forall \gamma \exists n[\gamma i \leq n + 1[\alpha \circ \gamma(i) > 0] \rightarrow q_{\alpha \circ \gamma(n)} \leq q_{\alpha \circ \gamma(n+1)}]\), then \(B_{\alpha}\) is a bar in \(\omega^\omega\), and, by Theorem 22, there exists a stump \(S\) such that \(S \cap B_{\alpha}\) is a bar in \(\omega^\omega\).

We will say say that a stump \(S\) has the property \((*)\) if and only if \((*)\) for every \(\alpha\), if \(S \cap B_{\alpha}\) is a bar in \(\omega^\omega\), then \(\{q_{\alpha(n)} \mid n \in \omega\} \in WO\).

We now prove that every stump has the property \((*)\), by induction on the set \(Stp\) of stumps.

Let a stump \(S\) be given such that every immediate substump of \(S\) has the property \((*)\).

Let \(\alpha\) be given such that \(S \cap B_{\alpha}\) is a bar in \(\omega^\omega\).

We want to prove: \(A \in WO\).

According to Lemma 24 it suffices to prove:

for all \(q \in \mathbb{Q}\), if \(q \in A\), then \(A_{\leq q} = \{r \in A \mid r \leq q\} \in WO\).

Let \(q \in A\) be given. Find \(n\) such that \(\alpha(n) > 0\) and \(q_{\alpha(n)-1} = q\).

Define \(\beta\) such that, for all \(m\), if \(\alpha(m) > 0\) and \(q_{\alpha(m)-1} = q\), then \(\beta(m) = \alpha(m)\), and, if and only if \(\beta(m) = 0\). Note that \(\beta\) enumerates \(\{q \in A \mid q < q_n\}\).

Let \(\delta\) be given. Find \(p\) such that \(\langle n \rangle \ast \delta p \in S \cap B_{\alpha}\). Note: \(\alpha(n) > 0\) and: \(p > 0\).

Now distinguish two cases.

**Case (a):** \(\alpha \circ \delta(p - 1) = 0\). Then also \(\beta \circ \delta(p - 1) = 0\) and \(\delta p \in B_{\beta}\).

**Case (b):** \(\alpha \circ \delta(p - 1) > 0\). We distinguish two subcases.

**Case (bi):** \(p = 1\) and \(q_{\alpha_0(p)-1} \leq q_{\alpha_0(p)-1}\). Then \(\beta \circ \delta(1) = 0\) and \(\delta 1 = \delta p \in B_{\beta}\).

**Case (bii):** \(p > 1\) and \(\alpha \circ \delta(p - 2) > 0\) and \(\alpha \circ \delta(p - 1) > 0\) and \(q_{\alpha \circ \delta(p-2)-1} \leq q_{\alpha \circ \delta(p-1)-1}\). Now either \(\beta \circ \delta(p - 2) = \alpha \circ \delta(p - 2)\) and \(\beta \circ \delta(p - 1) = \alpha \circ \delta(p - 1)\) and \(\delta p \in B_{\beta}\), or \(\exists i < p[\beta \circ \delta(i) = 0]\) and again: \(\delta p \in B_{\beta}\).

Conclude: \(\forall \delta \exists p[\langle n \rangle \ast \delta p \in S \land \delta p \in B_{\beta}]\).

We thus see: \(\langle S \mid \langle n \rangle \rangle \cap B_{\beta}\) is a bar in \(\omega^\omega\).

As \(S \mid \langle n \rangle\) has the property \((*)\), conclude: \(A_{\leq q_n} \in WO\).

But then also \(A_{\leq q_n} = A_{\leq q_n} \cup \{q_n\} \in WO\).

We thus see: for all \(n\), if \(q_n \in A\), then \(A_{\leq q_n} \in WO\), and conclude: \(A = \bigcup_n A_n \in WO\).

Using Theorem 22 we conclude: for every decidable subset \(A\) of \(\mathbb{Q}\), if \(\forall \gamma \exists n[q_{\gamma(n)} \leq q_{\gamma(n+1)} \lor \gamma(n) \notin A]\), then \(A \in WO\). \(\square\)

\[\text{In [22] Section 4, Theorem 6] one finds an intuitionistic version of a more difficult but related result: F. Hausdorff's Theorem on scattered subsets of } \mathbb{Q}. \text{ The proof is wrong however and the result is doubtful.}\]
6.2. Bar Induction.

**Theorem 26.** (Principle of Bar Induction) Let $B, C \subseteq \omega$ be given such that $\text{Bar}_{\omega^n}(B)$ and $B \subseteq C$ and $\forall s [s \in C \leftrightarrow \forall n [s * \langle n \rangle \in C]$. Then $\langle \rangle \in C$.

**Proof.** Assume $\text{Bar}_{\omega^n}(B)$, i.e. $B$ bars $\langle \rangle$.

Find a canonical proof of: ‘$B$ bars $\langle \rangle$’.

Assume also: $B \subseteq C$ and $\forall s [s \in C \leftrightarrow \forall n [s * \langle n \rangle \in C]$.

In the canonical proof, replace every statement: ‘$B$ bars $s$’ by the statement ‘$s \in C$’.

Note that the result is another valid proof, with conclusion: ‘$\langle \rangle \in C$’. □

We now may give a second proof of Theorem 26.

**Proof.** Let $\alpha$ be given such that

$$\forall \gamma \forall n [\forall i \leq n + 1 [\alpha \circ \gamma(i) > 0] \rightarrow q_{\alpha \circ \gamma(n)} \leq q_{\alpha \circ \gamma(n+1)}].$$

Define $B = B_{\alpha} := \bigcup_n \{s \in \omega^{n+1} \mid \exists i \leq n [\alpha \circ s(i) = 0]\} \cup \bigcup_n \{s \in \omega^{n+2} \mid \alpha \circ s(n) > 0 \wedge (\alpha \circ s(n+1) > 0 \wedge q_{\alpha \circ s(n)} \leq q_{\alpha \circ s(n+1)} - 1\}.

Let $C$ be the set of all $s$ such that either $\exists t \subseteq s[t \in B]$ or $s = \langle \rangle$ and $A \in WO$, or $n := \text{length}(s) > 0$ and $\alpha \circ s(n - 1) > 0$ and $A_{<q_{\alpha \circ s(n-1)-1}} \in WO$.

Note: $\text{Bar}_{\omega^n}(B)$ and $B \subseteq C$.

Also note: for all $s$, if $s \in C$, then $\forall n [s * \langle n \rangle \in C]$.

Finally, let $s$ be given such that $\forall n [s * \langle n \rangle \in C]$. We distinguish two cases.

**Case (1).** $s = \langle \rangle$. For all $n$, we may consider $\langle n \rangle$, and conclude: if $q_n \in A$, then $A_{<q_n} \in WO$.

By Lemma 24 $A \in WO$ and $\langle \rangle \in C$.

**Case (2).** $p := \text{length}(s) > 0$.

For all $n$, such that $q_n < q_{n(p-1)}$, we may consider $s(n)$ and conclude: $A_{<q_n} \in WO$.

By Lemma 24 $A_{<q_{n(p-1)}} \in WO$ and $s \in C$.

Using Theorem 26 we conclude: $\langle \rangle \in C$ and $A \in WO$. □

7. The Almost-Fan Theorem

This Section has seven Subsections. In Subsection 7.1.1 we introduce the notion of an almost-finite subset of $\omega$. In Subsection 7.2 we introduce almost-finitary spreads and we show that, like the Fan Theorem, the Almost-Fan Theorem follows from the Bar Theorem. We show that the Almost-Fan Theorem implies the Fan theorem. In Subsection 7.3 we formulate the Principle of Open Induction on $[0, 1]$ and show that it follows from the Almost-Fan Theorem. In Subsection 7.4 we see that the Principle of Open Induction on $[0, 1]$ implies a version of Dedekind’s Theorem. In Subsection 7.5 we prove that the Almost-Fan Theorem also implies an intuitionistic version of the Infinite Ramsey Theorem. In Subsection 7.6 we use this Ramsey Theorem together with Dedekind’s Theorem in order to prove an intuitionistic version of the Bolzano-Weierstrass Theorem. In Subsection 7.7 we show that the Infinite Ramsey Theorem implies the Paris-Harrington-Ramsey Theorem.

The results of this Section may be seen as intuitionistic comments on results in [22] Chapter III.

7.1. Almost-finite subsets of $\omega$. One may formulate many notions of finiteness, even for decidable subsets of $\omega$, see [25], [26] and [29] Section 3. We need three of them.

**Definition 24.** $B \subseteq \omega$ is a decidable subset of $\omega$ if and only if

$$\exists n \forall n [n \in B \leftrightarrow \alpha(n) \neq 0].$$

A decidable subset $B$ of $\omega$ is
Theorem 27.  
(i) For every decidable \( B \subseteq \omega \), if \( B \) is finite, then \( B \) is bounded-in-number, but not conversely.

(ii) For every decidable \( B \subseteq \omega \), if \( B \) is bounded-in-number, then \( B \) is almost-finite, but not conversely.

Proof.  
(i) Let \( B \subseteq \omega \) be decidable and finite. Find \( n \) be such that \( \forall m > n [m \notin B] \).

Clearly, for all \( s \) in \( \omega \), \( s(n + 1) > n \) and \( s(n + 1) \notin B \). We thus see that \( B \) is bounded-in-number.

As to the converse, we give a counterexample in Brouwer’s style.

Let \( d : \omega \rightarrow \{0, 1, \ldots, 9\} \) be the decimal expansion of \( \pi \).

Define \( B := \{k_{99}\} := \{n \mid n = \mu k^i \cdot 9 \iff \exists i \leq n (d(k + i) = 9)\} \).

\( B \) is a decidable subset of \( \omega \) and \( B \) has at most one member, but we are unable to find \( n \) such that \( \forall m > n [m \notin B] \).

(ii) Let \( B \subseteq \omega \) be decidable and bounded-in-number.

Find \( k \) such that \( \forall s \in [\omega]^k+1 \exists i \leq k [s(i) \notin B] \).

Conclude: \( \forall \zeta \in [\omega]^2 \exists i \leq k [\zeta(i) \notin B] \) and: \( B \) is almost-finite.

As to the converse, we give a counterexample in Brouwer’s style.

Define \( B := \{n \mid k_{99} \leq n \leq 2 \cdot k_{99}\} := \{n \mid \mu k^i \cdot 9 \equiv \exists i \leq n (d(k + i) = 9) \leq n \leq 2 \cdot \mu k^i \cdot 9 \equiv \exists i \leq n (d(k + i) = 9)\} \).

We now prove that \( B \) is almost-finite.

Let \( \zeta \in [\omega]^\omega \) be given.

Either \( \zeta(0) < k_{99} \) and \( \zeta(0) \notin B \), or \( \zeta(0) \geq k_{99} \) and \( \zeta(2 \cdot k_{99} + 1) \notin B \).

We are unable, however, to find \( n \) such that \( B \) has at most \( n \) members. \( \square \)

We now prove that the union of two almost-finite subsets of \( \omega \) is almost-finite, and that a decidable subset of \( \omega \) that is the union of almost-finitely many almost-finite subsets of \( \omega \), is itself almost-finite.

Lemma 28.  
(i) For all decidable subsets \( B, C \) of \( \omega \), if \( B, C \) are almost-finite, then \( B \cup C \) is almost-finite.

(ii) For every decidable subset \( B \) of \( \omega \), if there exists an infinite sequence \( B_0, B_1, B_2, \ldots \) of decidable and almost-finite subsets of \( \omega \) such that \( B = \bigcup_n B_n \) and \( \forall \zeta \in [\omega]^\omega \exists i [B_{\zeta(i)} = \emptyset] \), then \( B \) itself is almost-finite.

Proof.  
(i) Let \( \zeta \in [\omega]^\omega \) be given. Find \( \eta \in [\omega]^\omega \) such that, for each \( n \), \( \eta(n) = \mu \exists i \leq n (p > \eta(i)) \) and \( \zeta(p) \notin B \). Note: \( \forall p [\zeta \circ \eta(p) \notin B] \).

Find \( p \) such that \( \zeta \circ \eta(p) \notin C \). Define \( m := \eta(p) \) and note: \( \zeta(m) \notin B \cup C \).

We thus see: \( \forall \zeta \in [\omega]^\omega \exists m [\zeta(m) \notin B \cup C] \).

(ii) Let \( \zeta \in [\omega]^\omega \) be given. We want to prove: \( QED := \exists p [\zeta(p) \notin \bigcup_n B_n] \).

Find \( \alpha \) such that, for all \( p \), if \( \zeta(p) \in B = \bigcup_n B_n \), then \( \alpha(p) = \mu \exists [\zeta(p) \in B_n] \).

We claim: \( \forall \exists q > p \exists q > p \circ QED \). We prove this claim as follows.

Let \( p \) be given.

Using (i), observe that \( \bigcup_{i \leq p} B_i \) is almost-finite.

Find \( q > p \) such that \( \zeta(q) \notin \bigcup_{i \leq p} B_i \).

If \( \zeta(q) \in \bigcup_n B_n, \) then \( \alpha(q) > p \).

If \( \zeta(q) \notin \bigcup_n B_n, \) then \( QED \).

Now find \( \eta \in [\omega]^\omega \) such that \( \eta(0) = 0 \) and \( \forall p [\alpha \circ \eta(p + 1) > \alpha \circ \eta(p) \circ QED] \).

Find \( p \) such that \( B_{\alpha \circ \eta(p)} = \emptyset \) and conclude: either \( QED \), or \( \zeta \circ \eta(p) \notin \bigcup_n B_n \).

and again: \( QED \). \( \square \)

\(^{11}\)QED: ‘quod est demonstrandum’, ‘what we (still) have to prove’ rather than ‘quod erat demonstrandum’, what we did have to prove. \( \)
7.2. Almost-finitary spreads.

**Definition 25.** A spread-law \( \beta \) will be called almost-finitary if it satisfies the following condition:

\[ \forall s [ \beta(s) = 0 \rightarrow \forall \zeta \in [\omega]^s \exists n [\beta(s \ast \langle n \rangle)) \neq 0]]. \]

If the spread-law \( \beta \) is almost-finitary, the corresponding spread \( F_\beta \) will be called an almost-finitary spread or an almost-fan.

If \( \beta \) is an almost-finitary spread-law and \( s \) is admitted by \( \beta \), there are only almost-finitely many immediate successors \( s \ast \langle n \rangle \) of \( s \) that are admitted by \( \beta \).

**Definition 26.** \( P \subseteq \omega \) is almost-full if and only if \( \forall \zeta \in [\omega]^s \exists n [\zeta(n) \in P] \).

We need the following Lemma.

**Lemma 29.**

(i) For all \( P, Q \subseteq \omega \), if \( P, Q \) are almost-full, then \( P \cap Q \) is almost-full.

(ii) For all \( n > 0 \), for all \( P_0, P_1, \ldots, P_{n-1} \subseteq \omega \), if, for all \( i < n \), \( P_i \) is almost-full, then \( \bigcap_{i<n} P_i \) is almost-full.

**Proof.** (i) Assume \( P, Q \subseteq \omega \) are almost-full. Let \( \zeta \in [\omega]^s \) be given. Using the First Axiom of Countable Choice, find \( \eta \in [\omega]^\omega \) such that, for each \( n \), \( \eta(n) \in P \).

Find \( p \) such that \( \zeta \ast \eta(p) \in Q \). Define \( m := \eta(p) \) and note: \( \zeta(m) \in P \cap Q \).

We thus see: \( \forall \zeta \in [\omega]^s \exists m [\zeta(m) \notin P \cap Q] \), i.e. \( P \cap Q \) is almost-full.

(ii) Use (i) and induction. \(\square\)

The following theorem appears in [27] and [35, Section 7.2].

**Theorem 30 (Almost-Fan Theorem).** Let \( \beta \) be an almost-finitary spread-law and let \( B \subseteq \omega \) be a bar in \( F_\beta \).

(i) There exists \( B' \subseteq B \) such that \( B' \) is a bar in \( F_\beta \) and \( B' \) is a decidable and almost-finite subset of \( \omega \).

(ii) \( \forall \zeta \in [\omega]^s \exists n [\beta(\zeta(n)) \neq 0 \lor \exists t \subseteq \zeta(n) \forall t \in B]] \), and, therefore: \( \forall \zeta \in [\omega]^s \forall n [\beta(\zeta(n)) = 0] \rightarrow \exists t \subseteq \zeta(n) \forall t \in B] \).

**Proof.** Let \( \beta \) be an almost-finitary-spread-law and let \( B \subseteq \omega \) be a bar in \( F_\beta \).

Define \( B^+ := B \cup \{ s \mid \beta(s) \neq 0 \} \).

We claim that \( B^+ \) is a bar in \( \omega^\omega \).

In order to see this, we let \( p : \omega^\omega \rightarrow \omega^\omega \) be a retraction\(^1\) of \( \omega^\omega \) onto \( F_\beta \), i.e.

\( \forall \alpha \in F_\beta \) and even \( \forall \alpha \exists m [\langle p(\alpha) \rangle \neq \overline{m} \rightarrow \beta(\overline{m}) \neq 0] \), and \( \forall \alpha \in F_\beta [\langle p(\alpha) \rangle = \alpha] \).

Given \( \alpha \), find \( n \) such that \( \langle p(\alpha) \rangle n \in B \) and distinguish two cases. Either \( \langle p(\alpha) \rangle n = \overline{m} \) and \( \overline{m} \in B^+ \), or \( \langle p(\alpha) \rangle n \neq \overline{m} \) and \( \beta(\overline{m}) \neq 0 \) and, again, \( \overline{m} \in B^+ \).

We thus see that, indeed, \( B^+ \) is a bar in \( \omega^\omega \).

(i) Let \( C \) be the set of all \( s \) such that either \( \beta(s) \neq 0 \) or \( \beta(s) = 0 \) and there exists \( B' \subseteq B \) such that \( \text{Bar}_{F_\beta}(B') \) and \( B' \) is a decidable and almost-finite subset of \( \omega \).

Note the following:

(i)1. \( B^+ \subseteq C \).

(i)2. For all \( s \), if \( s \in C \), then for all \( n, s \ast \langle n \rangle \in C \).

(i)3. Let \( s \) be given such that \( \forall n [s \ast \langle n \rangle \in C] \).

Find, using axioms\(^3\) an infinite sequence \( B_0, B_1, \ldots \) of subsets of \( B \) such that, for each \( n \), if \( \beta(s \ast \langle n \rangle) \neq 0 \), then \( B_n = 0 \) and, if \( \beta(s \ast \langle n \rangle) = 0 \), then \( B_n \) is a decidable and almost-finite subset of \( \omega \) and \( B_n \) is a bar in \( F_\beta \cap s \ast \langle n \rangle \).

Define \( E := \bigcup_{n} B_n \) and note: \( E \) is a decidable subset of \( \omega \) and \( \forall \zeta \in [\omega]^s \exists n [B_{\zeta}(n) = 0] \) and \( E \) is a bar.

\(^1\)See the proof of Theorem [3]

\(^3\)
in $F_\beta \cap s$ and $E \subseteq B$.

According to Lemma 28, $E$ is almost-finite. One may conclude: $s \in C$.

Using Theorem 26 conclude: $\langle \rangle \in C$, i.e. there exists $B' \subseteq B$ such that $B'$ is bar in $F_\beta$ and $B'$ is a decidable and almost-finite subset of $\omega$.

(ii) Let $D$ be the set of all $s$ such that either: $\beta(s) \neq 0$ or:

\[ \beta(s) = 0 \text{ and } \forall \zeta \in [\omega]^\omega \exists n[\neg(s \subseteq \zeta(n)) \lor \beta(\zeta(n)) \neq 0 \lor \exists t \subseteq \zeta(n)[t \in B]]. \]

Note the following:

(iii)1. $B^+ \subseteq D$.

(ii)2. For all $s$, if $s \in D$, then for all $i$, $s * \langle i \rangle \in D$.

(ii)3. For all $s$, if $\forall i[s * \langle i \rangle \in D]$, then $s \in D$.

We prove (ii)3 as follows.

Let $s$ be given such that $\forall i[s * \langle i \rangle \in D]$.

We want to prove: $s \in D$ and may assume: $\beta(s) = 0$.

Let $\zeta$ in $[\omega]^\omega$ be given.

Define $QED := \exists n[\neg(s \subseteq \zeta(n)) \lor \beta(\zeta(n)) \neq 0 \lor \exists t \subseteq \zeta(n)[t \in B]]$.

Define $\zeta^*$ in $[\omega]^\omega$ such that, for each $n$,

if $\forall i \leq n + 1 [s \subseteq \zeta(i) \land \beta(\zeta(i)) = 0]$, then $\zeta^*(n) = \zeta(n + 1)$ and,

if not, then $\zeta^*(n)$ is the least $k$ such that $\forall i < n + 1 [s \subseteq \zeta(i)]$ and $s \subseteq k$ and $\beta(k) = 0$.

Note that, for each $n$, $s \subseteq \zeta^*(n)$ and $\beta(\zeta^*(n)) = 0$.

Define $QED^* := \exists n \forall t \subseteq \zeta^*(n)[t \in B]$.

Note that, by assumption, for each $i$,

the set $P_i := \{ u \mid \neg(s \subseteq \zeta(i) \cup u) \lor \beta(u) \neq 0 \lor \exists t \subseteq u[t \in B] \}$ is almost-full.

Using Lemma 28, we conclude that, for each $n$, the set $\bigcap_{i \leq n} P_i$ is almost-full.

Using the First Axiom of Countable Choice, we determine $\eta$ in $[\omega]^\omega$ such that, for all $n$, $\zeta^* \circ \eta(n) \in \bigcap_{i \leq n} P_i$.

Now find $k := \text{length}(s)$ and define $\gamma$ such that, for each $n$, $\gamma(n) = (\zeta^* \circ \eta(n))(k)$.

Note that, for each $n$, if $\gamma(n) \leq n$, then $s \star (\gamma(n)) \subseteq \zeta^* \circ \eta(n)$, and,

as $\zeta^* \circ \eta(n) \in P_{\gamma(n)}$, one may conclude $\exists t \subseteq \zeta^* \circ \eta(n)[t \in B]$ and $QED^*$.

Define $\zeta^*$ such that, for each $m$,

if $\forall n \leq m [\gamma(n) > n]$, then $\gamma^*(m) = \gamma(m)$, and, if not, then $\gamma^*(m) = \eta$.

Note that, for each $m$, if $\gamma^*(m) \neq \gamma(m)$, then $\exists n \leq m [\gamma(n) \leq n]$ and $QED^*$.

Moreover, $\forall p \exists q [\gamma^*(q) > p]$.

Find $\theta$ in $[\omega]^\omega$ such that $\theta(0) = 0$ and $\forall n [\gamma^* \circ \theta(n + 1) > \gamma \circ \theta(n)]$.

Using the fact that $\beta$ is an almost-fan-law and find $n$ such that $\beta(s \star \gamma^* \circ \theta(n)) = 0$.

Conclude: $\gamma \circ \theta(n) \neq \gamma \circ \theta(n)$ and: $QED^*$.

Find $n, t$ such that $t \subseteq \zeta^*(n)$ and $t \in B$. There are two cases.

Case (ii)3a. $\zeta^*(n) = \zeta(n + 1)$. Then $QED$.

Case (ii)3b. $\zeta^*(n) \neq \zeta(n + 1)$.

Then $\exists i \leq n + 1 [\neg(s \subseteq \zeta(i)) \lor \beta(\zeta(i)) \neq 0]$ and again: $QED$.

We thus see: for all $s$, if $\forall i[s * \langle i \rangle \in D]$, then $s \in D$.

This concludes our proof of (ii)3.

Using Theorem 26 conclude: $\langle \rangle \in D$, i.e.

$\forall \zeta \in [\omega]^\omega [\exists n [\beta(\zeta(n)) \neq 0 \lor \exists t \subseteq \zeta(n)[t \in B]]].$ 

□

The Fan Theorem may be derived from the Almost-Fan Theorem, as follows.

**Corollary 31.** Let $\beta$ be a finitary spread-law.

If $B \subseteq \omega$ is a a decidable subset of $\omega$ and a bar in $F_\beta$, some finite $B' \subseteq B$ is bar in $F_\beta$.

**Proof.** Let $\beta$ be a finitary spread-law and let $B \subseteq \omega$ be a decidable subset of $\omega$ and a bar in $F_\beta$. 

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Applying the Almost-Fan Theorem, find $B' \subseteq B$ such that $B'$ is a bar in $F_3$ and $B'$ is a decidable and almost-finite subset of $\omega$.

Note that $F_3$ is a fan. Therefore, for all $n$, for all $s$ in $\omega^n$, one may decide: either $\forall n \in F_3 \exists i < n[s(i) \subseteq \alpha]$, or $\exists \alpha \in F_3 \forall i < n[s(i) \subseteq \alpha]$

Now define $\eta$ such that, for all $n$,

if $\exists \alpha \in F_3 \forall i < n[\alpha \not\subseteq s(i)]$, then $\eta(n) = \mu \in B' \forall i < n[i \not\in \eta(i)]$, and,

if $\forall n \in F_3 \exists i < n[\eta(i) \not\subseteq \alpha]$, then $\eta(n) = \mu[\eta(p) \not\subseteq 0 \land \forall i < n[\eta(i) < p]]$.

Note $\eta \in [\omega]_{\omega}^*$. Define $k := \mu[\eta(n) \not\subseteq B']$ and conclude: $\{\eta(0), \eta(1), \ldots, \eta(k-1)\}$ is a finite subset of $B$ and a bar in $F_3$.

7.3. Open Induction in $[0,1]$.

**Definition 27.** For every $n > 0$, for every finite sequence $((a_0,b_0),(a_1,b_1),\ldots,(a_{n-1},b_{n-1}))$ of pairs of rationals, for every rational $c \geq 0$, we define the relation

\[
( (a_0,b_0),(a_1,b_1),\ldots,(a_{n-1},b_{n-1}) ) \text{ covers } [0,c]
\]
as follows, by induction.

(i) $n = 1$ and either: $a_0 < 0 < b_0$ and $c = b_0$, or: $c = 0$.

(ii) $n > 1$ and there exists $d$ such that $((a_0,b_0),(a_1,b_1),\ldots,a_{n-2},b_{n-2})$ covers $[0,d)$ and either: $a_{n-1} < d < b_{n-1}$ and $c = b_{n-1}$ or: $c = d$.

If $A := \{(a_0,b_0),(a_1,b_1),\ldots,(a_{n-1},b_{n-1})\}$ is a finite set of pairs of rationals, then, for every rational $c \geq 0$, $A$ covers $[0,c)$ if and only if, for some permutation $\pi$ of $\{0,1,\ldots,n-1\}$, the finite sequence $((a_{\pi(0)},b_{\pi(0)}),(a_{\pi(1)},b_{\pi(1)}),\ldots,(a_{\pi(n-1)},b_{\pi(n-1)})$ and $\alpha \in F_3$.

Note that one may decide, for all $n > 0$, for every finite set $A$ of pairs of rationals, for every rational $c \geq 0$, if $A$ covers $[0,c)$ or not.

**Definition 28.** $X \subseteq [0,1]$ is called progressive in $[0,1]$ if and only if, for every $x$ in $[0,1]$, if $[0,x) \subseteq X$, then $x \in X$.

The following principle was used by É. Borel in the proof of what is now called the Heine-Borel Theorem, see [2] and [35] Section 4.3. Its first proof in an intuitionistic context is due to Th. Coquand, see [11] and [35] Theorem 4.1.

**Theorem 32** (Principle of Open Induction in $[0,1]$).

If $\mathcal{H} \subseteq \mathcal{R}$ is open and progressive in $[0,1]$, then $[0,1] \subseteq \mathcal{H}$.

*Proof.** We first construct $\rho$. For each $s$, $\rho(s) = (\rho'(s),\rho''(s))$ will be a pair of rationals such that $0 \leq \rho'(s) \leq \rho''(s) \leq 1$.

Define $\rho(\langle \rangle) := (0,1)$ and, for each $s$ in $2^{<\omega}$, $\rho(s * 0)) := (\rho'(s),\frac{1}{2}\rho'(s) + \frac{1}{2}\rho''(s))$ and,

for each $n > 0$, $\rho(s * n)) := (\frac{1}{2}\rho'(s) + \frac{1}{4}\rho''(s),\rho''(s))$.

Let $\mathcal{H} \subseteq \mathcal{R}$ be given such that $\mathcal{H}$ is open and progressive in $[0,1]$. Using Definition [13] find $a$ such that $\forall x \in \mathcal{R}[x \in \mathcal{H} \leftrightarrow \exists n[q_{a(n)} < x < r_{a(n)}]]$.

Now define $\beta$ such that $\beta(\langle \rangle) = 0$ and, for all $s$,

(1) if $\beta(s) = 0$, then $\beta(s * 0)) = 0$ and,

(2) for all $n$, $\beta(s * (n + 1)) = 0$ and only if $n$ is the least $p$ such that $\{(q_{a(i)},r_{a(i)}) | i < p\}$ covers $[0,\rho'(s * n)]$.

One may prove: for each $s$, if $\beta(s) = 0$, then $[0,\rho'(s)) \subseteq \mathcal{H}$. The proof is by induction on $\text{length}(s)$.

Note that $\beta$ is a spread-law, and that, for each $s$, if $\beta(s) = 0$, there are at most two numbers $n$ such that $\beta(s * (n)) = 0$, so $\beta$ is an almost-finitary spread-law.

We let $B$ be the set of all $s$ such that $\beta(s) = 0$ and, for some $n < \text{length}(s)$, $q_{a(n)} < \rho'(s) < \rho''(s) < r_{a(n)}$.
The following argument shows that $B$ is a bar in $\mathcal{F}_\beta$.

Let $\gamma$ in $\mathcal{F}_\beta$ be given.

Note that, for each $n$, $[0, \rho'(\gamma n)] \subseteq \mathcal{H}$.

Let $x$ be the real number such that, for each $n$, $x(n) = \rho(\gamma n)$.

Note: $[0, x) \subseteq \mathcal{H}$, and, therefore, $x \in \mathcal{H}$.

Find $n$ such that $q_{\alpha(n)} < x < r_{\alpha(n)}$.

Find $m > n$ such that $q_{\alpha(n)} \rho'(\gamma m) \rho'(\gamma m) < r_{\alpha(n)}$ and note: $\gamma m \in B$.

We thus see: $\text{Bar}_{\mathcal{F}_\beta}(B)$.

We now apply Theorem 32 and find $B' \subseteq B$ such that $B'$ is a bar in $\mathcal{F}_\beta$ and a decidable and almost-finite subset of $\omega$.

Define $\zeta$ in $\omega^\omega$ as follows, by induction.

Let $\zeta(0)$ the least element $s$ of $B'$ such that $\rho'(s) < 0 < \rho''(s)$.

Note: $\{\rho'(\zeta(0))\}$ covers $[0, \rho''(\zeta(0))]$, so $[0, \rho''(\zeta(0))] \subseteq \mathcal{H}$.

Now let $n$ be given such that we defined $\zeta(0), \zeta(1), \ldots, \zeta(n)$ and

\[ \{\rho'(\zeta(0)), \rho'(\zeta(1)), \ldots, \rho'(\zeta(n))\} \text{ covers } [0, \rho''(\zeta(n))], \text{ so } [0, \rho''(\zeta(n))] \subseteq \mathcal{H}. \]

Consider $t := \rho''(\zeta(n))$ and note: $[0, t) \subseteq \mathcal{H}$ and: $t \in \mathcal{H}$.

If $t < q_1$, let $\zeta(n + 1)$ be the least $s$ in $B'$ such that $\rho'(s) < t < \rho''(s)$ and

\[ \forall i < \text{length}([s]) \mathcal{H} \not\subseteq B. \]

Note: $\forall i < n + 1(\zeta(j) \neq \zeta(n + 1))$.

If $t \geq q_1$, let $\zeta(n + 1)$ be the least $s$ such that $s \notin B'$ and $\forall i \leq n|s \neq \zeta(i)$.

Note: $\forall i \forall j(i < j \rightarrow (\zeta(i) \neq \zeta(j))$. For all $\gamma$, if $\forall n[q_{\tau(n)} < q_{\tau(n+1)}]$, and $\forall \zeta \in [\omega^\omega]^{\geq 13}q_{\gamma \circ \zeta (n)} + \frac{1}{2^n} < q_{\gamma \circ \zeta (n+1)}$, then $\exists n[1 < q_{\gamma \circ \zeta (n)}]$. 

\[ \text{Note that the condition: } \forall \zeta \in [\omega^\omega]^{\geq 13}q_{\gamma \circ \zeta (n)} + \frac{1}{2^n} < q_{\gamma \circ \zeta (n+1)} \text{ says: the sequence } q_{\tau(0)}, q_{\tau(1)}, \ldots \text{ positively fails to be convergent.} \]

Also note that, once one sees how to obtain the conclusion ‘$\exists n[1 < q_{\gamma \circ \zeta (n)}]$’, one will also see how to obtain the conclusion: ‘$\forall M \in \mathcal{Q} \exists n[M < r_{\tau(n)}]$’, i.e. the sequence $q_{\tau(0)}, q_{\tau(1)}, \ldots$ grows beyond all bounds.

**Proof.** Define $\mathcal{H} := \{x \in \mathbb{R} \mid \exists n[x < q_{\gamma_{\tau(n)}}]\}$.

Note that $\mathcal{H}$ is an open subset of $\mathcal{R}$.

We now prove that $\mathcal{H}$ is progressive in $[0, 1]$.

Let $x$ in $[0, 1]$ be given such that $[0, x) \subseteq \mathcal{H}$.

Note that $\mathcal{H}$ is a progressive subset of $\mathcal{R}$.

In particular, for each $n$, $x - \frac{1}{2^n} \in \mathcal{H}$.

Find $\zeta$ in $[\omega^\omega]$ such that, for each $n$, $x - \frac{1}{2^n} < q_{\gamma \circ \zeta (n)}$, i.e. $x < q_{\gamma \circ \zeta (n)} + \frac{1}{2^n}$.

Find $n$ such that $q_{\gamma \circ \zeta (n)} + \frac{1}{2^n} < q_{\gamma \circ \zeta (n+1)}$ and conclude: $x < q_{\gamma \circ \zeta (n+1)}$, and: $x \in \mathcal{H}$.

Conclusion: $\forall x \in [0, 1][[0, x) \subseteq \mathcal{H} \rightarrow x \in \mathcal{H}$, i.e. $\mathcal{H}$ is progressive in $[0, 1]$.

Using Theorem 32 conclude: $1 \in \mathcal{H}$, i.e. $\exists n[1 < q_{\tau(n)}]$. \[ \square \]
7.5. **Ramsey’s Theorem.** The usual formulation of (the two-dimensional case of) the Infinite Ramsey Theorem is the following:

Given \( R \subseteq [\omega]^2 \), there exists an infinite subset \( Z \) of \( \omega \) such that either \([Z]^2 \subseteq R \) or \([Z]^2 \subseteq [\omega]^2 \setminus R \).

Given any \( X \subseteq \omega \), \([X]^2 \) denotes the collection of the 2-element-subsets of \( X \).

We prefer to define \([\omega]^\omega \) as the collection of strictly increasing sequences of elements of \( X \) of length 2.

We also use the set \([\omega]^\omega \) of the infinite strictly increasing sequences of natural numbers rather than the collection of the infinite subsets of \( \omega \).

We reformulate Ramsey’s theorem as follows:

Given \( R \subseteq [\omega]^2 \), there exists \( \zeta \) in \([\omega]^\omega \) such that either \( \forall s \in [\omega]^2 [\zeta \circ s \in R] \) or \( \forall s \in [\omega]^2 [\zeta \circ s \not\in R] \).

As finite sequences of natural numbers are coded by natural numbers, we may further simplify this to:

Given \( R \subseteq [\omega]^2 \), there exists \( \zeta \) in \([\omega]^\omega \) such that either \( \forall s \in [\omega]^2 [\zeta \circ s \in R] \) or \( \forall s \in [\omega]^2 [\zeta \circ s \not\in \omega \setminus R] \).

If we consider this statement from a constructive point of view, we soon discover, thinking of the constructive interpretation of ‘or’, that it can not be true.

A counterexample in Brouwer’s style is given by the set

\( R := \{ s \in [\omega]^2 \mid s(0) < k_{99} \land s(1) < k_{99} \} \).

(Note that, for every \( \zeta \) in \([\omega]^\omega \), if \( \forall s \in [\omega]^2 [\zeta \circ s \in R] \), then \( \forall n [n < k_{99}] \), and, if \( \forall s \in [\omega]^2 [\zeta \circ s \not\in R] \), then \( \exists n \leq \zeta(0)[n = k_{99}] \).

One might hope however, that the following holds:

Given \( R \subseteq [\omega], \) a contradiction follows from:

\( \neg \exists \zeta \in [\omega]^\omega \forall s \in [\omega]^2 [\zeta \circ s \in R] \) and \( \neg \exists \zeta \in [\omega]^\omega \forall s \in [\omega]^2 [\zeta \circ s \in \omega \setminus R] \).

One might be even more hopeful about:

Given \( R \subseteq [\omega], \) a contradiction follows from:

\( \forall \zeta \in [\omega]^\omega \exists s \in [\omega]^2 [\zeta \circ s \in \omega \setminus R] \) and \( \forall \zeta \in [\omega]^\omega \exists s \in [\omega]^2 [\zeta \circ s \in \omega \setminus R] \).

Indeed, this may be proven intuitionistically. But one may do better and show:

Given \( R, T \subseteq [\omega], \)

if \( \forall \zeta \in [\omega]^\omega \exists s \in [\omega]^2 [\zeta \circ s \in \omega \setminus R] \) and \( \forall \zeta \in [\omega]^\omega \exists s \in [\omega]^2 [\zeta \circ s \in T] \), then \( \forall \zeta \in [\omega]^\omega \exists s \in [\omega]^2 [\zeta \circ s \in \omega \setminus R \cap T] \).

We will prove Ramsey’s Theorem and its extension to higher dimensions in the above form. The proof uses the Almost-Fan Theorem. This theorem enables one to use a version of the ‘Erdős-Rado compactness argument’.

**Definition 29.** For each \( \alpha \), \( D_\alpha := \{ n \mid \alpha(n) \neq 0 \} \).

\( D_\alpha \) is called the **subset of \( \omega \) decided by \( \alpha \)**.

For each \( k > 0 \), for each \( \alpha \), \( D_\alpha \) is **k-almost-full**\(^{14}\) if and only if

\( \forall \zeta \in [\omega]^\omega \exists s \in [\omega]^k [\zeta \circ s \in D_\alpha] \).

For all \( n, k \) such that \( k \leq n \), for all \( s \) in \([\omega]^n \), for all \( \alpha \),

1. \( s \) is \((\alpha, k)\)-monochromatic if and only if
   \( \forall t \in [n]^k \exists u \in [n]^k [\alpha(s \circ t) = \alpha(s \circ u)] \), and
2. \( s \) is \((\alpha, k + 1)\)-pre-monochromatic if and only if
   \( \forall t \in [n]^k \exists q \in [n]^{k + 1} [t \circ q \in \alpha(t \circ q)] \).

Note that, if \( D_\alpha \) is \( k \)-almost-full, then \( \neg \exists \zeta \in [\omega]^\omega \forall s \in [\omega]^k [\zeta \circ s \in \omega \setminus D_\alpha] \).

---

\(^{14}\)See also **Definition 29**.
Theorem 34 (Ramsey’s Theorem, the infinite case).

For all $k > 0$, for all $\alpha, \beta$,
if $D_\alpha$, $D_\beta$ are $k$-almost-full, then $D_\alpha \cap D_\beta$ is $k$-almost-full.

Proof. We use induction, and start with the case $k = 1$.
Let $\alpha, \beta$ be given such that $D_\alpha, D_\beta$ are 1-almost-full.
Let $\zeta$ in $[\omega]^\omega$ be given. Find $\eta$ in $[\omega]^\omega$ such that $\forall n \in \omega[[\zeta \circ \eta(n)] \in D_\alpha]$. Find $p$ such that $(\zeta \circ \eta(p)) \in D_\beta$. Define $q := \eta(p)$ and note: $(\zeta(q)) \in D_\alpha \cap D_\beta$.
We thus see: $\forall \zeta \in [\omega]^\omega \exists q[[\zeta(q)] \in D_\alpha \cap D_\beta$, i.e. $D_\alpha \cap D_\beta$ is 1-almost-full.

Now assume $k > 0$ is given such that for all $\alpha, \beta$,
if $D_\alpha$, $D_\beta$ are $k$-almost-full, then $D_\alpha \cap D_\beta$ is $k$-almost-full.
Let $\alpha, \beta$ be given such that $D_\alpha, D_\beta$ are $k$-almost-full.
We want to prove: $D_\alpha \cap D_\beta$ is $k+1$-almost-full.

We use induction, and start with the case $k = 1$.
Let $\zeta$ in $[\omega]^\omega$ be given.
We want to prove: QED := $\exists s \in [\omega]^{k+1}[\zeta \circ s \in D_\alpha \cap D_\beta]$.

We define: $\alpha^\uparrow = \alpha \circ \zeta$ and $\beta^\uparrow = \beta \circ \zeta$.

We define $\delta$, as follows, by induction. $\delta(\zeta, s) = 0$, and, for all $s$, for all $n$,
$\delta(s \circ (n)) = 0$ if and only if $\delta(s) = 0$ and $s$ is the largest $t < n$ such that $\delta(t) = 0$ and $t \circ (n)$ is both $(k+1, \alpha^\uparrow)$-pre-monochromatic and $(k+1, \beta^\uparrow)$-pre-monochromatic.

The set $D_\delta$ has the property that, for all $s, n$, if $s \circ (n) \in D_\delta$, then $s \in D_\delta$, and for this reason, is called a tree. $D_\delta$ should be called the $(k+1, \alpha^\uparrow, \beta^\uparrow)$-Erdős-Rado-tree.

Let $\zeta$ in $[\omega]^\omega$ be given.

We want to prove: QED := $\exists s \in [\omega]^{k+1}[\zeta \circ s \in D_\alpha \cap D_\beta]$.
We define: $\alpha^\uparrow = \alpha \circ \zeta$ and $\beta^\uparrow = \beta \circ \zeta$.

We define $\delta$, as follows, by induction. $\delta(\zeta, s) = 0$, and, for all $s$, for all $n$,
$\delta(s \circ (n)) = 0$ if and only if $\delta(s) = 0$ and $s$ is the largest $t < n$ such that $\delta(t) = 0$ and $t \circ (n)$ is both $(k+1, \alpha^\uparrow)$-pre-monochromatic and $(k+1, \beta^\uparrow)$-pre-monochromatic.

The set $D_\delta$ has the property that, for all $s, n$, if $s \circ (n) \in D_\delta$, then $s \in D_\delta$, and for this reason, is called a tree. $D_\delta$ should be called the $(k+1, \alpha^\uparrow, \beta^\uparrow)$-Erdős-Rado-tree.

Note: $D_\delta \subseteq [\omega]^{<\omega}$.

Note: for each $n$, there exists exactly one $s$ such that $s \circ (n) \in D_\delta$.

Note: for each $n$, for each $s$ in $D_\delta \subseteq [\omega]^{<\omega}$, there are at most $4^2$ numbers $i$ such that $s \circ (i) \in D_\delta$. This is because, for each $n$, the set $[n]^{k}$ has $4$ elements, and for each $t$ in $[n]^{k}$, for each $i$, $(s \circ t) \circ (i)$ belongs to one of the four sets $D_{\alpha^\uparrow} \cap D_{\beta^\uparrow}$, $D_{\alpha^\uparrow} \setminus D_{\beta^\uparrow}$, $D_{\beta^\uparrow} \setminus D_{\alpha^\uparrow}$ and $\omega \setminus (D_{\alpha^\uparrow} \cup D_{\beta^\uparrow})$.

Define $\varepsilon$ such that $\forall s \varepsilon(s) = 0 \iff \exists n \exists t \in D_\delta \land s = t \setminus U[n]]$.

Note that $\varepsilon$ is an almost-finitary spread-law and that the set $F_\varepsilon$ is an almost-
finitary spread.

Define $B := \bigcup_s \{s \in [\omega]^\omega \mid \varepsilon(s) = 0 \land (\exists t \in [n]^{k+1}[s \circ t \in D_{\alpha^\uparrow} \cap D_{\beta^\uparrow}] \lor s \notin D_\delta)\}$. We now prove that $B$ is a bar in the almost-finitary spread $F_\varepsilon$.

Assume: $\gamma \subseteq F_\varepsilon$.

Define $\gamma^+ \subseteq [\omega]^{k+1}$ that, for each $n$, if $\gamma(n+1) \in D_\delta$ then $\gamma^+(n+1) = \gamma(n+1)$, and, if not, then $\gamma^+(n+1) \subseteq [\omega]^{n+1} \setminus D_\delta$. Note: $\gamma^+ \in [\omega]^\omega$ and $\forall n[\gamma^+ \neq \gamma \setminus n \setminus D_\delta]$.

Recall: $k > 0$. For each $t$ in $[\omega]^k$, we let $t^\uparrow$ be the element of $[\omega]^{k+1}$ satisfying $t \subseteq t^\uparrow$ and $t^\uparrow(k) = t(k - 1) + 1$.

Define $\alpha^\ast$ and $\beta^\ast$ such that, for each $t$ in $[\omega]^k$,
$\alpha^\ast(t) = \alpha^\uparrow(\gamma^\circ t^\uparrow)$ and $\beta^\ast(t) = \beta^\uparrow(\gamma^\circ t^\uparrow)$.

We now prove: $\forall s \in [\omega]^\omega \exists t \in [n]^{k+1}[\gamma^\circ t \in D_{\alpha^\ast} \lor \exists n[\gamma^\circ t \notin D_\delta]]$.

Assume: $\eta \subseteq [\omega]^\omega$. Find $s$ in $[\omega]^{k+1}$ such that $\gamma^\circ \eta \circ s \subseteq D_{\alpha^\ast}$.
Define: $n := (\eta \circ s)(k) + 1$.
Define $t := \pi k$ and $i := s(k)$.

Note: $\gamma^\circ \eta \circ t \subseteq D_{\alpha^\ast}$, i.e. $\gamma^\circ (\eta \circ t \circ (i)) \subseteq D_{\alpha^\ast}$. Conclude: either: $\gamma^\circ (\eta \circ t \circ (i)) \subseteq D_{\alpha^\ast}$, and therefore: $\eta \circ t \subseteq D_{\alpha^\ast}$, or: $\eta \circ t \subseteq D_{\alpha^\ast}$, and therefore: $\gamma^\circ t \subseteq D_{\alpha^\ast}$, and therefore: $\gamma^\circ t \subseteq D_{\alpha^\ast}$.

One may prove by a similar argument:
$\forall \eta \in [\omega]^\omega \exists t \in [\omega]^{k+1}[\eta \circ t \subseteq D_{\alpha^\ast} \lor \exists n[\delta(\gamma \circ t) = 0]]$.

\footnote{QED: ‘quod est demonstrandum’, ‘what we have to prove’ rather than ‘quod erat demonstrandum’, what we did have to prove.}
Define $\alpha^{**}, \beta^{**}$ such that $\forall s[\alpha^{**(s)} \neq 0 \leftrightarrow (\alpha^*(s) \neq 0 \lor \exists n \leq s[\delta(\gamma n) = 0])]$
and $\forall s[\beta^{**(s)} \neq 0 \leftrightarrow (\beta^*(s) \neq 0 \lor \exists n \leq s[\delta(\gamma n) = 0])]$.

Conclude: $\forall \eta \in [\omega]^\omega \exists t \in D_{\alpha^{**}} \land \forall \eta \in [\omega]^\omega \exists t \in [\omega]^k[\eta \circ t \in D_{\beta^{**}}]$.

Using the induction hypothesis, we conclude:
$\forall \eta \in [\omega]^\omega \exists t \in [\omega]^k[\eta \circ t \in D_{\alpha^{**}} \cap D_{\beta^{**}}]$.

Find $t$ in $[\omega]^k$ such that $t \in D_{\alpha^{**}} \cap D_{\beta^{**}}$.

Either: $t \in D_{\alpha^{**}} \cap D_{\beta}$, or: $\exists n[\delta(\gamma n) = 0]$, that is, either: $\gamma^* \circ (t \ast j) = (\gamma^* \circ t) \ast (\gamma^* \ast j) \in D_{\alpha^{**}} \cap D_{\beta}$, where $j = t(k - 1) + 1$, or:
$\exists n[\delta(\gamma n) = 0]$. In both cases, we find $n$ such that $\gamma n \in B$.

Either: $\gamma n = \gamma n$ or: $\gamma n \notin \gamma n$. In both cases: $\gamma n \in B$.
We thus see: $\forall \gamma \in F, \forall \gamma n \in B$, i.e. $\text{Bar}_F(B)$.

We now use Theorem 30(ii).
Find $\eta$ in $[\omega]^\omega$ such that $D_k = \{ \eta(n) \mid n \in \omega \}$.
Then find $n, m$ such that $\eta(n)m \in B$.

Conclude: $\exists \eta \in [\omega]^k \ast \exists \eta \in [\omega]^k[\eta \circ t \in D_{\alpha^{**}} \cap D_{\beta^{**}}]$, i.e. $\exists t \in [\omega]^k[\eta \circ t \in D_{\alpha^{**}} \cap D_{\beta^{**}}]$.

We thus see: $\forall \eta \in [\omega]^\omega \exists t \in [\omega]^k[\eta \circ t \in D_{\alpha^{**}} \cap D_{\beta^{**}}]$, i.e. $D_{\alpha^{**}} \cap D_{\beta^{**}}$ is $(k + 1)$-almost-full. \(\square\)

7.6. The Bolzano-Weierstrass Theorem. The Bolzano-Weierstrass Theorem:

'An infinite sequence of reals bounded both from above and from below must have a convergent subsequence'.

is a strengthening of Dedekind’s Theorem, see Subsection 7.3. As Dedekind’s Theorem, in its usual formulation, already fails to be true constructively, the case of the usual formulation of the Bolzano-Weierstrass Theorem is also hopeless.

**Theorem 35** (Bolzano-Weierstrass-Theorem). For all $\gamma$,
if $\forall n [\omega] \exists n[\frac{1}{\gamma n} < |q_{\gamma \circ \eta(n)} - q_{\gamma \circ \eta(n + 1)}|]$, then $\exists n[1 < |q_{\gamma(n)}|]$.

Note that the condition: $\forall \eta \in [\omega]^\omega \exists n[\frac{1}{\gamma n} < |q_{\gamma \circ \eta(n)} - q_{\gamma \circ \eta(n + 1)}|]$ says: every subsequence of the sequence $q_{\gamma(0)}, q_{\gamma(1)}, \ldots$ positively fails to be convergent.

Also note that, once one sees how to obtain the conclusion $\exists n[1 < |q_{\gamma(n)}|]$,
one will also see how to obtain the conclusion: $\forall M \in \mathbb{Q} \exists n[|M| < |q_{\gamma(n)}|]$, i.e. the sequence $q_{\gamma(0)}, q_{\gamma(1)}, \ldots$ grows beyond all bounds.

Our proof uses both Dedekind’s Theorem and Ramsey’s Theorem.

**Proof.** Let $\gamma$ be given such that $\forall \eta \in [\omega]^\omega \exists n[\frac{1}{\gamma n} < |q_{\gamma \circ \eta(n)} - q_{\gamma \circ \eta(n + 1)}|]$.

We first prove:

(\ast) $\forall \eta \in [\omega]^\omega \exists n[q_{\gamma \circ \eta(n)} < q_{\gamma \circ \eta(n + 1)} \lor 1 < q_{\gamma(n)}]$

Let $\eta \in [\omega]^\omega$ be given. Define $\delta = \gamma \circ \epsilon$.

Define $\delta^*$ such that $\delta^*(0) = \delta(0)$ and, for each $i$, if $\forall i \leq n[q_{\delta(i)} \leq q_{\delta(i + 1)}]$, then $\delta^*(n + 1) = \delta(n + 1)$ and, if not, then $q_{\delta^*(n + 1)} = q_{\delta(n + 1)} + 1$.

Note: $\forall n[q_{\delta^*(n)} \leq q_{\delta^*(n + 1)}]$.

We now prove: $\forall \eta \in [\omega]^\omega \exists n[q_{\delta^*(\eta(n))} + \frac{1}{\gamma n} < q_{\delta^*(\eta(n + 1))}]$.

Let $\eta \in [\omega]^\omega$ be given. Find $n$ such that $\frac{1}{\gamma n} < |q_{\delta^*(\eta(n))} - q_{\delta^*(\eta(n + 1))}|$. Either $q_{\delta^*(\eta(n))} = q_{\delta^*(\eta(n + 1))}$ and $q_{\delta^*(\eta(n + 1))} = q_{\delta^*(\eta(n + 1))}$ and $q_{\delta^*(\eta(n + 1))} = q_{\delta^*(\eta(n))}$, or $q_{\delta^*(\eta(n))} + \frac{1}{\gamma n} < q_{\delta^*(\eta(n + 1))}$, or $\exists m < n[\delta^*(m + 1) < q_{\delta^*(m)}]$. In the latter case, for all sufficiently large $i$, $q_{\delta^*(\eta(n))} + 1 \leq q_{\delta^*(\eta(n + 1))}$.

Using Theorem 33, we find $n$ such that $1 < q_{\delta^*(n)}$.

If $q_{\delta^*(n)} = q_{\delta(n)}$, we conclude: $\exists n[1 < q_{\delta(n)}]$, and if not, we conclude: $\exists n[q_{\delta(n + 1)} < q_{\delta(n)}]$, i.e. $\exists n[q_{\delta^*(n + 1)} < q_{\delta^*(n)}]$.

This concludes our proof of (\ast).

One may also prove:  

33
\[
(*) \forall \zeta \in [\omega]^\omega \exists n[q_{\gamma \smallfrown \zeta(n)} < q_{\gamma \smallfrown \zeta(n+1)} \land q_\gamma(n) < -1].
\]

(Find \(\gamma^*\) such that \(\forall n[q_{\gamma^*(n)} = -q_\gamma(n)]\) and use \((*)\), but now for \(\gamma^*\) rather than for \(\gamma\) itself.)

Using both \((*)\) and \((***)\) and Theorem 34 conclude:
\[
\exists n[q_\gamma(n) < -1 \lor 1 < q_\gamma(n)], \text{ i.e. } \exists n[1 < |q_\gamma(n)|].
\]

\[\text{□}\]

7.7. The Paris-Harrington-Ramsey Theorem. F.P. Ramsey proved the Infinite Ramsey Theorem, in \([20]\), in order to make his reader gain experience before attacking the Finite Ramsey Theorem. Later, it turned out that one may prove the Finite Ramsey Theorem from the Infinite Ramsey Theorem by a so-called ‘compactness argument’, see \([10]\). Paris and Harrington then saw that one may prove also certain strengthenings of the Finite Ramsey Theorem from the Infinite Ramsey Theorem, see \([13]\) Sections 1.5 and 6.3. One such statement turned out to be expressible in the language of first-order arithmetic but unprovable from Peano’s axioms. We want to show that the ‘compactness argument’ works also intuitionistically, thanks to the intuitionistic version of the Infinite Ramsey Theorem proven in Subsection 7.6 and the Fan Theorem.

We need some terminology in order to introduce the Finite Ramsey Theorems.

**Definition 30.** For all \(r > 0\), \(r^{<\omega} := \{c \mid \forall i < \text{length}(s)[c(i) < r]\}\).

For all positive integers \(m, k\), \([m]^k := \{s \in [\omega]^k \mid \forall i[k(i) < m]\}\).

One may consider elements of \([m]^k\) as \(k\)-element subsets of \(m = \{0, 1, \ldots, m-1\}\).

For all positive integers \(c, m, k, r\),
\[
c : [m]^k \to r \text{ if and only if } c \in r^{<\omega} \text{ and } \forall s \in [m]^k[s < \text{length}(c)].
\]

One may consider \(c : [m]^k \to r\) as an \(r\)-colouring of the \(k\)-element subsets of \(m\).

For all positive integers \(c, k, m, r, t, n\), if \(c : [m]^k \to r\) and \(n \leq m\) and \(t \in [m]^n\),
\[
\text{then } t \text{ is } c, k\text{-monochromatic if and only if } \exists j < r \forall u \in [n]^k[c(t \circ u) = j].
\]

For all positive integers \(k, r, n, M, M \to (n)^k_r\) if and only if,
for every \(c : [M]^k \to r\), there exists \(t \in [M]^n\) such that \(t\) is \(c, k\)-monochromatic.

If \(M \to (n)^k_r\), then, for every \(r\)-colouring \(c\) of the \(k\)-element subsets of \(M = \{0, 1, \ldots, M-1\}\) there exists an \(n\)-element subset \(A = \{t(0), t(1), \ldots, t(n-1)\}\) of \(M\) such that all \(k\)-element subsets of \(A\) obtain, from \(c\), one and the same colour.

\(t \in \bigcup_n [\omega]^n\) is relatively large if and only if \(n > 0\) and \(\text{length}(t) \geq t(0)\).

The expression relatively large is used in \([19]\). A finite subset \(A\) of \(\omega\) is relatively large if the number of elements of \(A\) is at least as big as the smallest member of \(A\).

For all positive integers \(k, r, n, M, M \to (n)^k_r\) if and only if,
for every \(c : [M]^k \to r\), there exist \(t \in [M]^{<\omega}\) such that \(\text{length}(t) \geq n\) and \(t\) is relatively large and \(c, k\)-monochromatic.

We want to call a collection \(A\) of finite subsets of \(\omega\) omnipresent if and only if every infinite subset of \(\omega\) has a subset in \(A\). With our terminology, the definition takes the following form.

**Definition 31.** \(A \subseteq [\omega]^{<\omega}\) is called omnipresent if and only if
\[
\forall \zeta \in [\omega]^{<\omega} \exists s \in [\omega]^{<\omega}[\zeta \circ s \in A];
\]

For every positive integer \(r\), \(r^{<\omega} := \{\chi \mid \forall i[\chi(i) < r]\}\).

Note that \(r^{<\omega}\) is a fan.

One may consider an element \(\chi\) of \(r^{<\omega}\) as an \(r\)-colouring of \(\omega\).

**Theorem 36.** Let \(A \subseteq [\omega]^{<\omega}\) be a decidable subset of \(\omega\) and omnipresent.

(i) For all positive integers \(k, r\), for all \(\chi\) in \(r^{<\omega}\), the set \(B = B(A, k, r, \chi) := \{s \in A \mid s \text{ is } (\chi, k)\text{-monochromatic}\}\) is omnipresent.
Corollary 37.

(i) The Finite Ramsey Theorem: \( \forall k \forall r \forall n \exists M[M \to \langle n \rangle^k_r] \).

(ii) The Paris-Harrington-Ramsey Theorem: \( \forall k \forall r \forall n \exists M[M \to^* \langle n \rangle^k_r] \).

Proof. (i) Define \( A := [\omega]^n \). Note that \( A \) is omnipresent. Let positive integers \( k, r \) be given. Using Theorem 36 conclude that the set \( C = \{ s \in [\omega]^{n+1} | \forall \chi \in r^\omega \exists t \in [\omega]^{n+1} | s \circ t \in A \land \bigwedge \chi \text{ is } (\chi, k)\text{-monochromatic} \} \) is a bar in \([\omega]^{n+1}\). Consider \( Id_\omega \), the element of \([\omega]^{n+1}\) such that \( \forall n[Id_\omega(n) = n] \). Find \( M \) such that \( \overline{Id_\omega}M \in C \) and note: \( M \to \langle n \rangle^k_r \).

(ii) Start with \( A := \bigcup_n \{ s \in [\omega]^{n+1} | s(0) \leq n \} \). Note that \( A \) is omnipresent and repeat the argument given for (i). \( \square \)

8. Notation and conventions

8.1. \( \omega \) denotes the set of the natural numbers 0, 1, 2, ...

We use \( a, b, \ldots, m, n, \ldots, p, q, r, s, \ldots \) as variables over \( \omega \).

We assume a bijective function \( J : \omega \times \omega \to \omega \) has been defined with inverse functions \( K, L : \omega \to \omega \) such that \( \forall n[J(K(n), L(n)) = n] \).

We assume a function \( < : \bigcup_k \omega^k \to \omega \) has been defined that is a bijection.

If \( \langle m_0, m_1, \ldots, m_{k-1} \rangle = s \) then \( \text{length}(s) = k \), and for each \( i < k, s(i) = m_i \).

\( \omega^k := \{ s | \text{length}(s) = k \} \).

\( \omega^\omega := \bigcup_k \omega^k \).
For all $k, l$, for all $s$ in $\omega^k$, for all $t$ in $\omega^l$, $s \ast t$ is the element $u$ of $\omega^{k+l}$ satisfying 
$\forall i < k[u(i) = s(i)]$ and $\forall j < l[u(k + j) = t(j)]$.

For every $s$ in $\omega$, for every $A \subseteq \omega$, $s \ast A := \{s \ast t \mid t \in A\}$.

For all $s, t$ such that $\forall n < length(t)[t(n) < length(s)]$, $s \ast t$ is the number $u$ satisfying $length(u) = length(t)$ and $\forall n < length(u)[u(n) = s(t(n))].$

$\langle \rangle$ denotes the empty sequence, that is, the unique $s$ such that

$length(s) = 0.$

$s \subseteq t \iff \exists u[t = s \ast u].$

$s \sqsubset t \iff (s \sqsubseteq t \land s \neq t).$

$s \perp t \iff \neg(s \sqsubseteq t \lor t \sqsubseteq s).$

For all $s$, for all $n \leq length(s)$, $\overline{n}$ is the unique $u$ in $\omega^n$ such that $u \sqsubseteq s.$

For all $s, n$, $s^n$ is the largest $u$ such that $\forall m < length(u)[\langle n \rangle \ast m < length(s) \land u(m) = s(\langle n \rangle \ast m)].$

$2^{<\omega} := \{s \mid \forall n < length(s)[s(n) < 2]\}.$

$Bin_n := \{s \in 2^{<\omega} \mid length(s) = n\}.$

$[\omega]^k := \{s \in \omega^k \mid \forall n < k[s(n) < n + 1]\}.$

$[\omega]^{<\omega} := \bigcup_k [\omega]^k.$

$\mu_0[P(n)] = k \iff \{P(k) \land \forall n < k[P(n)]\}.$

8.2. $\omega^\omega = \mathcal{N}$ is the set of all functions from $\omega$ to $\omega$.

We use $\alpha, \beta, \ldots, \zeta, \eta, \ldots, \varphi, \psi, \ldots$ as variables over $\omega^\omega$.

$Id$ is the element of $\omega^\omega$ satisfying $\forall n[Id(n) = n].$

$C = 2^\omega$ is the set of all $\alpha$ in $\omega^\omega$ such that $\forall n[\alpha(n) < 2].$

For each $n$, $\overline{n}$ is the element $\beta$ of $\omega^\omega$ such that $\forall m[\beta(m) = n].$

$\alpha \# \beta \iff \alpha \perp \beta \iff \exists n[\alpha(n) \neq \beta(n)].$

$\overline{\alpha} := \langle \alpha(0), \alpha(1), \ldots, \alpha(k - 1) \rangle.$

$s \sqsubset \alpha \iff \overline{s}(length(s)) = s.$

For all $X \subseteq \omega^\omega$, for all $s$, $X \cap s := \{\alpha \in X \mid s \sqsubset \alpha\}.$

$s \perp \alpha \iff \alpha \perp s \iff \neg(s \sqsubset \alpha).$

$(\alpha \ast s)(t) := \alpha(s \ast t).$

$(\alpha \ast \langle n \rangle)(m) := \alpha^n(m) := \alpha(n \ast m).$

For every $X \subseteq \omega$, for every $s$, $X \ast s := \{t \mid s \ast t \in X\}.$

$[\omega]^\omega$ is the set of all $\alpha$ in $\omega^\omega$ such that $\forall n[\alpha(n) < \alpha(n + 1)].$

8.3. $\mathcal{F} \subseteq \omega^\omega$ is a spread if and only if

$\exists \beta[\forall s[\beta(s) = 0 \iff \exists n[\beta(s \ast \langle n \rangle) = 0]] \land \forall \alpha[\alpha \in \mathcal{F} \iff \forall n[\beta(\overline{n}) = 0]].$

Let $\mathcal{F} \subseteq \omega^\omega$ be a spread.

$\varphi : \mathcal{F} \rightarrow \omega \iff \forall \alpha \in \mathcal{F}[\exists n[\varphi(\overline{n}) \neq 0]].$

If $\varphi : \mathcal{F} \rightarrow \omega$ then, for each $\alpha$ in $\mathcal{F}$, $\varphi(\alpha)$ is the number $q$ such that $\exists n[\varphi(\overline{n}) = q + 1 \land \forall m < n[\varphi(\overline{m}) = 0]].$

$\varphi : \mathcal{F} \rightarrow [\omega]^\omega \iff \forall n[\varphi^n : \mathcal{F} \rightarrow \omega].$

If $\varphi : \mathcal{F} \rightarrow [\omega]^\omega$, then, for each $\alpha$ in $\mathcal{F}$, $\varphi(\alpha)$ is the element $\beta$ of $\omega^\omega$ such that $\forall n[\beta(n) = \varphi^n(\beta)].$

If $\varphi : \mathcal{F} \rightarrow [\omega]^\omega$, then, for each $s$ such that $\exists \delta \in \mathcal{F}[a \sqsubset \delta]$,

$\varphi[s]$ is the greatest number $t \leq s$ such that

$\forall i < length(t)\exists n < length(s)[\varphi(\overline{n}) = t(i) + 1 \land \forall j < n[\varphi(\overline{j}) = 0]].$
Note: if \( \varphi : \mathcal{F} \rightarrow \omega^\omega \), then, for all \( \alpha \) in \( \mathcal{F} \), for all \( \beta \),
\[
\varphi|_{\alpha} = \beta \iff \forall n \exists m (|m| \leq |\varphi|_{\alpha})
\]
For all \( \varphi : \omega^\omega \rightarrow \omega^\omega \), for all \( \mathcal{X} \subseteq \omega^\omega \), \( \varphi|_{\mathcal{X}} := \{ \varphi|_{\alpha} \mid \alpha \in \mathcal{X} \} \).
For each \( V \subseteq \omega^\omega \), \( \varphi : \mathcal{F} \leftrightarrow V \) if and only if \( \varphi : \mathcal{F} \rightarrow \omega^\omega \) and
\[
\forall \alpha \in \mathcal{F} \wedge \exists \beta \in \mathcal{F} (\alpha \neq \beta \wedge \varphi|_{\alpha} \neq \varphi|_{\beta}) \text{ and } \forall \alpha \in \mathcal{F} (\varphi|_{\alpha} \in V).
\]
For each \( \mathcal{X} \subseteq \omega^\omega \), \( \mathcal{X}^- := \{ \alpha \mid \alpha \notin \mathcal{X} \} \).
\( \mathcal{R} \), the set of the real numbers, may be defined as a subset of \( \omega^\omega \).
For \( x, y \) in \( \mathcal{R} \), \( x \neq y \iff \exists n (|x - y| > n) \).

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