Extreme and periodic $L_2$ discrepancy of plane point sets

by

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1. Introduction. We study several discrepancy notions of two well-known instances of plane point sets, namely the Hammersley point set and rational lattices. The discrepancies are considered with respect to the $L_2$ norm and a variety of test sets. We define (standard) $L_2$ discrepancy, extreme $L_2$ discrepancy and periodic $L_2$ discrepancy.

Let $\mathcal{P} = \{x_0, x_1, \ldots, x_{N-1}\}$ be an arbitrary $N$-element point set in the unit square $[0,1)^2$. For any measurable subset $B$ of $[0,1]^2$ we define the counting function

$$A(B, \mathcal{P}) := \left| \left\{ n \in \{0,1,\ldots,N-1\} : x_n \in B \right\} \right|,$$

i.e., the number of elements from $\mathcal{P}$ that belong to the set $B$. By the local discrepancy of $\mathcal{P}$ with respect to a given measurable “test set” $B$ one understands the expression

$$A(B, \mathcal{P}) - N\lambda(B),$$

where $\lambda$ denotes the Lebesgue measure of $B$. A global discrepancy measure is then obtained by considering a norm of the local discrepancy with respect to a fixed class of test sets. Here we restrict ourselves to the $L_2$ norm, but we vary the class of test sets.

The (standard) $L_2$ discrepancy uses the class of axis-parallel squares anchored at the origin as test sets. The formal definition is

$$L_{2,N}(\mathcal{P}) := \left( \int_{[0,1]^2} |A([0,t], \mathcal{P}) - N\lambda([0,t])|^2 \, dt \right)^{1/2},$$

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where for $t = (t_1, t_2) \in [0,1]^2$ we set $[0,t) = [0,t_1) \times [0,t_2)$ with area $\lambda((0,t)) = t_1 t_2$.

The extreme $L_2$ discrepancy uses arbitrary axis-parallel rectangles contained in the unit square as test sets. For $x = (x_1,x_2)$ and $y = (y_1,y_2)$ in $[0,1]^2$ and $x \leq y$ let $[x,y) = [x_1,y_1) \times [x_2,y_2)$, where $x \leq y$ means $x_1 \leq y_1$ and $x_2 \leq y_2$. The extreme $L_2$ discrepancy of $P$ is then defined as

$$L_{2,N}^{\text{extr}}(P) := \left( \int_{[0,1]^2} \int_{[0,1]^2, x \leq y} |A([x,y), P) - N \lambda([x,y))|^2 \, dx \, dy \right)^{1/2}.$$  

Note that the only difference between the standard and the extreme $L_2$ discrepancy is the use of anchored and arbitrary rectangles in $[0,1]^2$, respectively. The term “extreme” is used in order to distinguish this notion of $L_2$ discrepancy from the standard $L_2$ discrepancy and refers to the corresponding nomenclature for $L_\infty$ discrepancies (see, e.g., [25, Definitions 2.1 and 2.2]).

The periodic $L_2$ discrepancy uses periodic rectangles as test sets, which are defined as follows: For $x, y \in [0,1]$ set

$$I(x,y) = \begin{cases} [x,y) & \text{if } x \leq y, \\ [0,y) \cup [x,1) & \text{if } x > y, \end{cases}$$

and for $x, y$ as above we set $B(x,y) = I(x_1,y_1) \times I(x_2,y_2)$. We define the periodic $L_2$ discrepancy of $P$ as

$$L_{2,N}^{\text{per}}(P) := \left( \int_{[0,1]^2} \int_{[0,1]^2} |A(B(x,y), P) - N \lambda(B(x,y))|^2 \, dx \, dy \right)^{1/2}.$$  

These discrepancy notions can also be defined for point sets in the $d$-dimensional unit cube $[0,1]^d$ in an obvious way.

The standard $L_2$ discrepancy is a well-known measure for the irregularity of distribution of point sets in the unit square with a close relation to the integration error of quasi-Monte Carlo rules via a Koksma–Hlawka type inequality (see, for example, [9, 26]). In contrast, the extreme and the periodic $L_2$ discrepancies are not so familiar. For this reason we summarize a few facts about these discrepancy notions.

According to [26], extreme $L_2$ discrepancy was first considered by Morokoff and Caflisch [23] since it is more symmetric than standard $L_2$ discrepancy, which prefers the lower left vertex of the unit square. Morokoff and Caflisch could not state a Koksma–Hlawka type inequality for extreme $L_2$ discrepancy, but later it has been shown that this quantity is the worst-case integration error of a certain space of periodic functions with a boundary condition (see [26] and the proof of Theorem 5 in Section 2).
The notion of periodic $L_2$ discrepancy is known from a paper by Lev [22], but as a matter of fact, it is just a geometric interpretation of diaphony according to Zinterhof [31] (see Proposition 3 in Section 2). Its relation to the integration error of quasi-Monte Carlo rules is well known: see, e.g., [16].

The celebrated lower bound of Roth [28] states that there exists a $c > 0$ such that for every $N$-element point set $\mathcal{P}$ in $[0, 1)^2$ the standard $L_2$ discrepancy satisfies $L_{2,N}(\mathcal{P}) \geq c\sqrt{1 + \log N}$. A general lower bound of the same order of magnitude also holds for the periodic $L_2$ discrepancy (see Corollary 2 in Section 2). In the present paper we adapt the proof of Roth to show that also the extreme $L_2$ discrepancy satisfies $L_{2,N}^{\text{extr}}(\mathcal{P}) \geq c\sqrt{1 + \log N}$ (see Theorem 6 in Section 2).

For every $\mathcal{P}$ it is obviously true that

$$L_{2,N}^{\text{per}}(\mathcal{P}) \geq L_{2,N}^{\text{extr}}(\mathcal{P}).$$

This is because when restricting the range of integration in the definition of periodic $L_2$ discrepancy to $x \leq y$, the test sets are exactly those used for extreme discrepancy. In [23] the authors further conjectured that the extreme $L_2$ discrepancy is smaller than standard $L_2$ discrepancy. They could not prove this, but their conjecture was supported by numerical experiments. We will show that this order relation indeed holds true (see Theorem 5 in Section 2).

We mention some further results about extreme and periodic $L_2$ discrepancy: The exact asymptotic behaviour of the average of standard, extreme and periodic $L_2$ discrepancy of random point sets is given in [14] and [17]. See also [12] for an upper bound in the case of extreme $L_2$ discrepancy. Bounds on the periodic $L_2$ discrepancy of certain multi-dimensional point sets (Korobov’s $p$-sets) can be found in [7]. There the dependence of the bounds on the dimension $d$ is of particular interest.

In the present paper we prove exact formulas for the aforementioned $L_2$ discrepancies for Hammersley point sets and for rational lattices. In the next section we present some further information and new results about periodic and extreme $L_2$ discrepancy. There we also prove the already mentioned “Roth-type” lower bound on extreme $L_2$ discrepancy and the order relation between standard and extreme $L_2$ discrepancy that was already conjectured by Morokoff and Caflisch. The exact discrepancy formulas for Hammersley point sets (Theorem 8) and for rational lattices (Theorem 10) will then be presented in Section 3. Their proofs are given in Sections 4–7.

2. More results about periodic and extreme $L_2$ discrepancy.

For a point set $\mathcal{P} = \{x_0, x_1, \ldots, x_{N-1}\}$ and a real vector $\delta \in [0, 1]^d$ the shifted point set $\mathcal{P} + \delta$ is defined as $\mathcal{P} + \delta = \{x_0 + \delta, \ldots, x_{N-1} + \delta\}$, where $\{x_j + \delta\}$ means that the fractional-part function $\{x\} = x - \lfloor x \rfloor$ for
non-negative real numbers \( x \) is applied componentwise to the vector \( x_j + \delta \). We call this kind of shift a *geometric shift*—in contrast to the digital shift as explained in Section 3. The root-mean-square \( L_2 \) discrepancy of a shifted point set \( P \) with respect to all uniformly distributed shift vectors \( \delta \in [0,1]^d \) is

\[
\sqrt{\mathbb{E}_\delta [L_{2,N}(P + \delta)]^2} = \left( \int_{[0,1]^d} L_{2,N}(P + \delta)^2 \, d\delta \right)^{1/2}.
\]

The following relation between periodic \( L_2 \) discrepancy and root-mean-square \( L_2 \) discrepancy of a shifted point set \( P \) holds (see [7, 22] for proofs):

**Proposition 1.** For every \( N \)-element point set \( P \) in \([0,1]^d\) we have

\[
L_{2,N}^{\text{per}}(P) = \sqrt{\mathbb{E}_\delta [L_{2,N}(P + \delta)]^2}.
\]

From this relation we can deduce the following general lower bound on the periodic \( L_2 \) discrepancy of point sets in \([0,1]^d\):

**Corollary 2.**

For every dimension \( d \) there exists \( c_d > 0 \) such that every \( N \)-element point set \( P \) in \([0,1)^d\) satisfies

\[
L_{2,N}^{\text{per}}(P) \geq c_d (1 + \log N)^{(d-1)/2}.
\]

**Proof.** Let \( P \) be an arbitrary \( N \)-element point set in \([0,1)^d\). Then

\[
L_{2,N}^{\text{per}}(P) = \sqrt{\mathbb{E}_\delta [L_{2,N}(P + \delta)]^2} \geq \inf_{\delta \in [0,1]^d} L_{2,N}(P + \delta) \geq c_d (1 + \log N)^{(d-1)/2},
\]

where we have used Roth’s lower bound on standard \( L_2 \) discrepancy.

Another important fact is that periodic \( L_2 \) discrepancy can be expressed in terms of exponential sums.

**Proposition 3 ([16, p. 390]).** For \( P = \{x_0, x_1, \ldots, x_{N-1}\} \) in \([0,1)^d\) we have

\[
L_{2,N}^{\text{per}}(P)^2 = \frac{1}{3^d} \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \frac{1}{r(k)^2} \left| \sum_{h=0}^{N-1} \exp(2\pi i k \cdot x_h) \right|^2,
\]

where \( i = \sqrt{-1} \) and for \( k = (k_1, \ldots, k_d) \in \mathbb{Z}^d \) we set

\[
r(k) = \prod_{j=1}^d r(k_j) \quad \text{and} \quad r(k_j) = \begin{cases} 1 & \text{if } k_j = 0, \\ 2\pi |k_j|/\sqrt{6} & \text{if } k_j \neq 0. \end{cases}
\]

The above formula shows that periodic \( L_2 \) discrepancy is—up to a multiplicative factor—exactly diaphony which is a well-known measure for the irregularity of distribution of point sets and which was introduced by Zinterhof [31] in 1976 (see also [10]).

From this viewpoint we immediately find an order relation between standard and periodic \( L_2 \) discrepancy in the one-dimensional case.
Corollary 4. For every $N$-element point set $P$ in $[0, 1)$ we have

$$L_{2,N}^{\text{per}}(P) \leq \sqrt{2} L_{2,N}(P).$$

We have equality if $N$ is even and $P$ is symmetric, i.e., with every $x_n$ also $1 - x_n$ belongs to $P$.

Proof. In the one-dimensional case the well-known formula of Koksma (see [21, p. 110]) establishes a connection between $L_2$ discrepancy and diaphony. This formula follows easily from an application of Parseval’s identity to local discrepancy. From this we have

$$L_{2,N}(P)^2 \geq \frac{1}{2\pi^2} \sum_{k=1}^{\infty} \left| \frac{1}{k^2} \sum_{h=0}^{N-1} \exp(2\pi i k x_h) \right|^2 = \frac{1}{2} L_{2,N}^{\text{per}}(P)^2,$$

where we have used Proposition 3 in the last step. The result follows by multiplying by 2 and taking the square root. For symmetric $P$ we have equality in (4), because then $\sum_{n=0}^{N-1} (1/2 - x_n)$ equals 0.

We now show that extreme $L_2$ discrepancy is indeed always smaller than standard $L_2$ discrepancy as conjectured in [23]. This is actually implied by the known relationships of extreme and standard $L_2$ discrepancy to worst-case errors of quasi-Monte Carlo rules for numerical integration.

Theorem 5. For every $N$-element point set $P$ in $[0, 1)^d$ we have

$$L_{2,N}^{\text{extr}}(P) \leq L_{2,N}(P).$$

Proof. As already mentioned, we need the relationship between extreme and standard $L_2$ discrepancy, respectively, and worst-case errors of quasi-Monte Carlo rules for numerical integration. The quoted facts can all be found in [26].

Recall that the worst-case error $e(I, Q, H(K_d))$ of the quasi-Monte Carlo rule

$$Q(f) = \frac{1}{N} \sum_{k=0}^{N-1} f(x_k)$$

for the integration problem

$$I(f) = \int_{[0,1]^d} f(x) \, dx$$

of functions $f : [0, 1]^d \to \mathbb{R}$ in a reproducing kernel Hilbert space $H(K_d)$ with kernel $K_d : [0, 1]^d \times [0, 1]^d \to \mathbb{R}$ is given as

$$e(I, Q, H(K_d)) = \sup_{\|f\|_{H(K_d)} \leq 1} |I(f) - Q(f)|.$$
A closed formula involving the kernel and the Riesz representer \( h_d \in H(K_d) \) of the integration functional \( I \) is

\[
e(I, Q, H(K_d))^2 = \| h_d \|_{H(K_d)}^2 - \frac{2}{N} \sum_{k=0}^{N-1} h_d(x_k) + \frac{1}{N^2} \sum_{k, \ell=0}^{N-1} K(x_k, x_\ell)
\]

(see [26, (9.31)]).

We now introduce the relevant reproducing kernel Hilbert spaces. They are Hilbert space tensor products of Sobolev spaces of univariate functions. Let \( W^{1,2}([0,1]) \) be the Sobolev space of absolutely continuous functions \( f : [0,1] \to \mathbb{R} \) with weak first derivative \( f' \) in \( L^2([0,1]) \). Let \( H \) be the subspace of all functions \( f \in W^{1,2}([0,1]) \) satisfying the boundary condition \( f(1) = 0 \) equipped with the norm \( \| f \|_H = \| f' \|_{L^2} \). Let \( H^{\text{extr}} \) be the subspace of all \( f \in W^{1,2}([0,1]) \) satisfying \( f(0) = f(1) = 0 \) equipped with the norm \( \| f \|_{H^{\text{extr}}} = \| f' \|_{L^2} \). Obviously, \( H^{\text{extr}} \) consists of the 1-periodic functions in \( H \). Both \( H \) and \( H^{\text{extr}} \) are reproducing kernel Hilbert spaces. The kernels are \( K(x,y) = \min \{1-x, 1-y\} \) for \( H \) and \( K^{\text{extr}}(x,y) = \min \{x, y\} - xy \) for \( H^{\text{extr}} \). Denote the \( d \)-fold Hilbert space tensor products of these spaces by \( H_d \) and \( H^{\text{extr}}_d \), respectively. Their kernels \( K_d \) and \( K_d^{\text{extr}} \) are the \( d \)-fold tensor products of the corresponding univariate kernels.

Now, using the above formula for the worst-case error of the integration problem and comparing to the formulas for standard and extreme \( L^2 \) discrepancy in Proposition [13] in Section 4 below shows that

\[
Ne(I, Q, H_d) = L_{2,N}(\mathcal{P}) \quad \text{and} \quad Ne(I, Q, H_d^{\text{extr}}) = L_{2,N}^{\text{extr}}(\mathcal{P}),
\]

where \( \mathcal{P} = \{x_0, x_1, \ldots, x_{N-1}\} \) is the point set used by the quasi-Monte Carlo rule \( Q \). A complete derivation of the first equality is given in [26, Section 9.5.1]; for the second identity we refer to [26, Section 9.5.5].

But since \( H_d^{\text{extr}} \) is a subspace of \( H_d \) (with the induced scalar product and norm), the inequality \( e(I, Q, H_d^{\text{extr}}) \leq e(I, Q, H_d) \) is obvious from the definition of the worst-case error.

Next, we show how to adapt the proof of Roth’s lower bound to extreme \( L^2 \) discrepancy.

**Theorem 6.** For every dimension \( d \) there exists \( c_d > 0 \) such that every \( N \)-element point set \( \mathcal{P} \) in \([0,1]^d\) satisfies

\[
L_{2,N}^{\text{extr}}(\mathcal{P}) \geq c_d(1 + \log N)^{(d-1)/2}.
\]

**Proof.** We assume some familiarity with the proof of Roth in the language of Haar functions, as can be found, e.g., in [2] or [6]. We only prove the case \( d = 2 \); the extension to general \( d \) is done as for Roth’s lower bound.

A dyadic interval in \([0,1]\) is of the form \( I = [2^{-m}n, 2^{-m}(n+1)) \) with non-negative integers \( m, n \) satisfying \( 0 \leq n < 2^m \). The Haar function supported
on $I$ is the function $h_I : [0, 1] \to \mathbb{R}$ which is +1 on the left half of $I$, −1 on the right half, and 0 outside of $I$. The Haar functions form an orthogonal system in $L_2([0, 1])$.

Haar functions in $[0, 1]^2$ are tensor products of univariate Haar functions. A dyadic rectangle in $[0, 1]^2$ is a product $R = I \times J$ of two dyadic intervals $I$ and $J$. The Haar function supported on $R$ is the function $h_R : [0, 1]^2 \to \mathbb{R}$ given as $h_R(x, y) = h_I(x)h_J(y)$. The Haar functions form an orthogonal system in $L_2([0, 1]^2)$.

Roth’s method for proving an order optimal lower bound for standard $L_2$ discrepancy uses the orthogonal expansion of the discrepancy function into a series of Haar functions. To adapt the proof to extreme $L_2$ discrepancy, we first fix a natural number $m$ satisfying $2^{m-3} \leq 2N \leq 2^{m-2}$ and consider all dyadic rectangles $R = I \times J$ of area $2^{-m}$. They come in $m + 1$ different shapes according to the side lengths of $R$, i.e., the lengths of $I$ and $J$. There are $2^m$ dyadic rectangles of the same shape tiling the unit square. There are $m − 1$ shapes where both side lengths are at most $1/2$, and one quarter, that is, $2^{m-2}$, of the dyadic rectangles $R$ of that shape satisfy $R \subseteq [1/2, 1)^2$. Since $2N \leq 2^{m-2}$, at least half of those rectangles also satisfy $P \cap R = \emptyset$.

Now Bessel’s inequality implies

$$\int_{[0,1]^2} D(y)^2 \, dy \geq \sum_R \frac{\langle D, h_R \rangle^2}{\|h_R\|^2_{L_2}},$$

where the sum is taken over all dyadic rectangles $R$. Using just the dyadic rectangles with area $2^{-m}$ and satisfying $R \subseteq [1/2, 1)^2$ as well as $P \cap R = \emptyset$, of which there are at least $(m − 1)2^{m-3}$, we obtain

$$\int_{[0,1]^2} D(y)^2 \, dy \geq (m − 1)2^{m-3}2^{-8}N^22^{-4m} = 2^{-11}(m − 1)2^{-2m}N^2.$$

Now using $2^{-m}N \geq 2^{-4}$ and $m − 1 \geq 2 + \log_2 N$ we arrive at

$$\int_{[0,1]^2} D(y)^2 \, dy \geq 2^{-19}(2 + \log_2 N).$$
Since this holds for any fixed $x \in [0, 1/2)^2$, we can finally integrate over all those $x$ to obtain

$$L_{2,N}^{\text{extr}}(P)^2 \geq 2^{-21}(2 + \log_2 N).$$

Hence the desired result follows.

In dimension one we have the following surprising relationship between periodic and extreme $L_2$ discrepancy. Whether a corresponding relation also holds in higher dimensions is an open question (see also the brief discussion at the end of Section 3).

**Theorem 7.** For every $N$-element point set $P$ in $[0, 1)$ we have

$$L_{2,N}^{\text{per}}(P)^2 = 2L_{2,N}^{\text{extr}}(P)^2.$$

**Proof.** Let $P = \{x_0, x_1, \ldots, x_{N-1}\}$. We may assume that the points are ordered, i.e., $x_0 \leq x_1 \leq \cdots \leq x_{N-1}$. An easy computation (see also [20, Eq. (1.3)]) shows that

$$L_{2,N}^{\text{extr}}(P)^2 = \frac{1}{12} + \frac{1}{2} \sum_{n,m=0}^{N-1} \left( x_n - x_m - \frac{n-m}{N} \right)^2.$$

From this formula and since $\sum_{n,m=0}^{N-1} (n-m)^2 = N^2(N^2 - 1)/6$ we obtain

$$L_{2,N}^{\text{extr}}(P)^2 = \frac{1}{2} \left( \frac{N^2}{6} + \sum_{n,m=0}^{N-1} (x_n - x_m)^2 - 2 \sum_{n,m=0}^{N-1} (x_n - x_m)(n-m) \right).$$

We have

$$\sum_{n,m=0}^{N-1} (x_n - x_m)(n-m) = \sum_{n,m=0}^{N-1} (nx_n - mx_n - nx_m + mx_m)$$

and hence

$$L_{2,N}^{\text{extr}}(P)^2 = \frac{1}{2} \left( \frac{N^2}{6} + \sum_{n,m=0}^{N-1} (x_n - x_m)^2 - 4 \sum_{n=0}^{N-1} nx_n + 2(N-1) \sum_{n=0}^{N-1} x_n \right).$$

For periodic $L_2$ discrepancy in dimension one we know (see, e.g., Proposition 13 below or [16, pp. 389–390]) that

$$L_{2,N}^{\text{per}}(P)^2 = \sum_{n,m=0}^{N-1} B_2(|x_n - x_m|),$$

where $B_2(x) = x^2 - x + 1/6$ is the second Bernoulli polynomial. Inserting
the formula for $B_2$ we obtain

$$L_{2,N}^{\text{per}}(\mathcal{P})^2 = \frac{N^2}{6} + \sum_{n,m=0}^{N-1} (x_n - x_m)^2 - \sum_{n,m=0}^{N-1} |x_n - x_m|.$$ 

We have further

$$\sum_{n,m=0}^{N-1} |x_n - x_m| = \sum_{n=0}^{N-1} \sum_{m=0}^{n} (x_n - x_m) + \sum_{n=0}^{N-1} \sum_{m=n+1}^{N-1} (x_m - x_n)$$

$$= \sum_{n=0}^{N-1} x_n(n+1) - \sum_{n=0}^{N-1} \sum_{m=0}^{n} x_m + \sum_{n=0}^{N-1} \sum_{m=n+1}^{N-1} x_m - \sum_{n=0}^{N-1} x_n(N-1-n)$$

$$= 2 \sum_{n=0}^{N-1} x_n(n+1) - N \sum_{n=0}^{N-1} x_n - N \sum_{m=0}^{N-1} x_m \sum_{m=0}^{1} + N \sum_{n=0}^{1} x_n \sum_{n=0}^{N-m} = N-m$$

$$= 4 \sum_{n=0}^{N-1} nx_n - 2(N-1) \sum_{n=0}^{N-1} x_n.$$ 

Hence

$$L_{2,N}^{\text{per}}(\mathcal{P})^2 = \frac{N^2}{6} + \sum_{n,m=0}^{N-1} (x_n - x_m)^2 - 4 \sum_{n=0}^{N-1} nx_n + 2(N-1) \sum_{n=0}^{N-1} x_n.$$ 

A comparison of (5) and (6) shows the result.

Note that Theorem 7 in combination with Corollary 4 gives another proof of Theorem 5 for the one-dimensional case.

**Summary.** In this section we have presented a number of inequalities and relations between three types of $L_2$ discrepancy. We briefly summarize these relations here: For every $N$-element point set $\mathcal{P}$ in $[0, 1)^d$ we have

$$L_{2,N}^{\text{extr}}(\mathcal{P}) \leq L_{2,N}^{\text{per}}(\mathcal{P}) \quad \text{and} \quad L_{2,N}^{\text{extr}}(\mathcal{P}) \leq L_{2,N}(\mathcal{P}).$$

Furthermore, there exists $c_d > 0$ such that for every $N$-element point set $\mathcal{P}$ in $[0, 1)^d$ we have

$$c_d(1 + \log N)^{(d-1)/2} \leq L_{2,N}^{\text{extr}}(\mathcal{P}).$$

In the one-dimensional case we even know that

$$L_{2,N}^{\text{per}}(\mathcal{P}) = \sqrt{2} L_{2,N}^{\text{extr}}(\mathcal{P}) \quad \text{and} \quad L_{2,N}^{\text{per}}(\mathcal{P}) \leq \sqrt{2} L_{2,N}(\mathcal{P}).$$

**3. Exact discrepancy formulas.** In this section we present exact formulas for the $L_2$ discrepancies of Hammersley point sets and of rational lattices, two well established constructions of point sets in discrepancy theory.
**Hammersley point set.** We calculate the extreme and the periodic $L_2$ discrepancy of the 2-dimensional Hammersley point set in base 2, which for $m \in \mathbb{N}$ is given as the set of $N = 2^m$ points

$$H_m = \left\{ \left( \frac{t_m}{2} + \cdots + \frac{t_1}{2^m}, \frac{t_1}{2} + \cdots + \frac{t_m}{2^m} \right) : t_1, \ldots, t_m \in \{0, 1\} \right\}.$$  

The Hammersley point set is the prototype of low-discrepancy point sets whose construction is based on digit representations. Its elements $(x_k, y_k)$ for $k = 0, 1, \ldots, 2^m - 1$ can also be written in the form

$$x_k = k / 2^m \quad \text{and} \quad y_k = \varphi_2(k),$$

where $\varphi_2(k)$ is the van der Corput digit reversal function $\varphi_2(k) = \frac{\kappa_0}{2} + \frac{\kappa_1}{2^2} + \cdots + \frac{\kappa_r}{2^{r+1}}$ whenever $k$ has dyadic expansion $k = \kappa_0 + \kappa_1 2 + \cdots + \kappa_r 2^r$ with $\kappa_i \in \{0, 1\}$. Note that the Hammersley point set is symmetric with respect to the main diagonal in $\mathbb{R}^2$. Another viewpoint on Hammersley point sets as a special instance of digital nets will be used in Section 6.

We have the following exact result on the extreme and the periodic $L_2$ discrepancy of the Hammersley point set. For comparison only, we also include the formula for the standard $L_2$ discrepancy.

**Theorem 8.** We have

$$L_{2,2^m}^2(H_m)^2 = \frac{m^2}{64} + \frac{29m}{192} + \frac{3}{8} - \frac{m}{2^{m+4}} + \frac{1}{2^{m+2}} - \frac{1}{9 \cdot 2^{2m+3}},$$

$$L_{2,2^m}^{\text{extr}}(H_m)^2 = \frac{m}{64} + \frac{1}{72} - \frac{1}{9 \cdot 4^{m+2}},$$

$$L_{2,2^m}^{\text{per}}(H_m)^2 = \frac{m}{16} + \frac{1}{9} + \frac{1}{9 \cdot 4^{m+1}}.$$

The result for the standard $L_2$ discrepancy is well known. A proof can be found, for example, in [13, 27]. The results for the extreme and periodic $L_2$ discrepancy are new. The proofs—along with a new proof for the standard $L_2$ discrepancy—will be presented in Section 4.

An immediate consequence of Theorem 8 is that—in contrast to the standard $L_2$ discrepancy—the extreme and periodic $L_2$ discrepancy of the Hammersley point set are each of the optimal order $\sqrt{\log N}$. The $L_2$ discrepancy of the Hammersley point set is only of order $\log N$, which is not the optimal order according to the aforementioned lower bound of Roth [28]. Several modifications such as digital shifts or symmetrization are necessary to overcome this defect of the Hammersley point set (see e.g. [11, 13, 15, 19]), which for the other two notions of $L_2$ discrepancy are not necessary. Considering the fact the periodic $L_2$ discrepancy can be understood as a root-mean-square $L_2$ discrepancy of shifted point sets (see Proposition 1 in Section 2) and with inequality (1) in mind, this result does not come unexpected.
Theorem 8 further demonstrates that standard and extreme $L_2$ discrepancy are not equivalent in general. This is in contrast to the $L_{\infty}$ extreme/star discrepancies $D_N(P)$ and $D_N^*(P)$, which are defined as

$$D_N(P) = \sup_{x, y \in [0,1]^2, x \leq y} |A([x, y], P) - N\lambda([x, y])|,$$

$$D_N^*(P) = \sup_{t \in [0,1]^2} |A([0, t], P) - N\lambda([0, t])|,$$

for two-dimensional point sets. For these notions we have the almost trivial inequalities $D_N^*(P) \leq D_N(P) \leq 4D_N^*(P)$.

Another obvious implication of Theorem 8 in conjunction with Proposition 1 is that there exists a geometric shift $\delta \in [0,1]^2$ such that the point set $H_m + \delta$ achieves the optimal order of $L_2$ discrepancy. In fact, Roth [29] used geometric shifts (but only in one coordinate) to prove for the first time the existence of point sets in $[0,1)^d$ with optimal $L_2$ discrepancy rate $(\log N)^{(d-1)/2}$. He could show that the average of the $L_2$ discrepancy of higher dimensional versions of the Hammersley point set over all possible shifts achieves this bound; hence it was a probabilistic existence result. In dimension 2, Roth’s result has later been derandomized by Bilyk [1] who could find an explicit geometric shift $\delta = (\delta, 0) \in [0,1]^2$ such that $H_m + \delta$ has the optimal order of $L_2$ discrepancy.

Since periodic $L_2$ discrepancy equals the root-mean-square discrepancy with respect to geometric shifts, we would like to compare the result on $L_{2,2m}^{\text{per}}(H_m)$ with the root-mean-square $L_2$ discrepancy of the Hammersley point set with respect to digital shifts, which are often studied in this context.

This kind of shifts are based on digitwise addition modulo 2. In more detail, for $x, y \in [0,1)$ with dyadic expansions

$$x = \sum_{i=1}^{\infty} \frac{\xi_i}{2^i} \quad \text{and} \quad y = \sum_{i=1}^{\infty} \frac{\eta_i}{2^i}$$

with digits $\xi_i, \eta_i \in \{0,1\}$ for all $i \geq 1$ we define

$$x \oplus y := \sum_{i=1}^{\infty} \frac{\xi_i + \eta_i \ (\text{mod} \ 2)}{2^i}.$$

For vectors $x, y \in [0,1)^d$ the digitwise addition $x \oplus y$ is defined component-wise.

For a point set $P = \{x_0, x_1, \ldots, x_{N-1}\}$ and a real vector $\delta \in [0,1]^d$ we define the digitally shifted point set $P \oplus \delta$ as

$$P \oplus \delta = \{x_0 \oplus \delta, x_1 \oplus \delta, \ldots, x_{N-1} \oplus \delta\}.$$
The root-mean-square $L_2$ discrepancy of a digitally shifted point set $P$ with respect to all uniformly distributed (digital) shift vectors $\delta \in [0, 1]^d$ is

\[
\sqrt{\mathbb{E}_\delta[L_2^d(P \oplus \delta)]} = \left( \int_{[0,1]^d} L_2^d(P \oplus \delta)^2 \, d\delta \right)^{1/2}.
\]

This is the digital equivalent to the root-mean-square $L_2$ discrepancy of a geometrically shifted point set $P$ given in \[2\] and therefore to the periodic $L_2$ discrepancy.

We compute $E_\delta[L_2^d(\mathcal{H}_m \oplus \delta)^2]$ and obtain the following result:

**Theorem 9.** For the $2^m$-element Hammersley point set $\mathcal{H}_m$ we have

\[
E_\delta[L_2^d(\mathcal{H}_m \oplus \delta)^2] = \frac{m}{24} + \frac{5}{36}.
\]

The proof of Theorem 9 will be presented in Section 6. Note that the root-mean-square $L_2$ discrepancy for digitally shifted Hammersley points is about a factor $\sqrt{2}/3$ lower than for geometrically shifted Hammersley points.

**Rational lattices.** We will also calculate the extreme and the periodic $L_2$ discrepancy of rational lattices. First we define them. Let $\alpha \in \mathbb{R}$ be an irrational number. Then for $N \in \mathbb{N}$ we define the point set

\[
\mathcal{A}_N(\alpha) := \left\{ \left( \frac{k}{N}, \{k\alpha\} \right) : k = 0, 1, \ldots, N-1 \right\},
\]

where $\{k\alpha\}$ denotes the fractional part of the real $k\alpha$. Let $\alpha = [a_0; a_1, a_2, \ldots]$ be the continued fraction expansion of $\alpha$ and $p_n/q_n$ for $n \in \mathbb{N}$ be the $n$th convergent of $\alpha$, i.e. $p_n/q_n = [a_0; a_1, \ldots, a_n]$. Further we consider the sets

\[
\mathcal{L}_n(\alpha) := \left\{ \left( \frac{k}{q_n}, \left\{ \frac{k\alpha}{q_n} \right\} \right) : k = 0, 1, \ldots, q_n - 1 \right\},
\]

which are an approximation of the set $\mathcal{A}_N(\alpha)$. We call a point set $\mathcal{L}_n(\alpha)$ a rational lattice. A special instance of a rational lattice is the Fibonacci lattice $\mathcal{F}_n$, which is obtained for $\alpha = \frac{1}{2}(\sqrt{5} + 1)$, the golden ratio. Then $\alpha = [1; 1, 1, \ldots]$, $(p_n, q_n) = (F_{n-1}, F_n)$ and

\[
\mathcal{F}_n := \left\{ \left( \frac{k}{F_n}, \left\{ \frac{kF_{n-1}}{F_n} \right\} \right) : k = 0, 1, \ldots, F_n - 1 \right\},
\]

where the Fibonacci numbers are defined recursively via $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.

We have the following formula for the $L_2$ discrepancies of rational lattices.
THEOREM 10. Let $\alpha$ be as above. Then

$$L_{2,q_n}(\mathcal{L}_n(\alpha))^2 = \frac{1}{16q_n^2} \sum_{r=1}^{q_n-1} \left( 1 + 2 \cos^2 \left( \frac{\pi rp_n}{q_n} \right) \right) \sin^2 \left( \frac{\pi rp_n}{q_n} \right) + \left( D(p_n, q_n) + \frac{3}{4} \right)^2$$

$$= \frac{1}{18} - \frac{1}{144q_n^2},$$

$$L_{2,q_n}^{\text{extr}}(\mathcal{L}_n(\alpha))^2 = \frac{1}{16q_n^2} \sum_{r=1}^{q_n-1} \frac{1}{\sin^2 \left( \frac{\pi r}{q_n} \right) \sin^2 \left( \frac{\pi rp_n}{q_n} \right)} + \frac{1}{72} - \frac{1}{144q_n^2},$$

$$L_{2,q_n}^{\text{per}}(\mathcal{L}_n(\alpha))^2 = \frac{1}{4q_n^2} \sum_{r=1}^{q_n-1} \frac{1}{\sin^2 \left( \frac{\pi r}{q_n} \right) \sin^2 \left( \frac{\pi rp_n}{q_n} \right)} + \frac{1}{9} + \frac{1}{36q_n^2},$$

where in the first formula $D(p, q)$ is the inhomogeneous Dedekind sum

$$D(p, q) = \sum_{k=1}^{q-1} \rho \left( \frac{k}{q} \right) \rho \left( \frac{kp}{q} \right) \text{ where } \rho(x) = \frac{1}{2} - \{x\}.$$
connected to such an $\alpha$ can be shifted geometrically in a way such that the resulting point set achieves the optimal order of $L_2$ discrepancy. From the same paper it is known that the unshifted lattice $L_n(\alpha)$ has the optimal order of $L_2$ discrepancy if and only if
\[
\sum_{k=0}^{n} (-1)^k a_k \leq c \sqrt{n}
\]
for a constant $c > 0$.

**Remark 11.** It follows from Theorem 10 and (8) that
\[
\liminf_{N \to \infty} \inf_{\#P=N} \frac{L_{2,N}^{\text{extr}}(P)}{\sqrt{\log N}} \leq \eta := \frac{1}{\sqrt{60\sqrt{5}\log(\sqrt{5}+1)}} = 0.124455 \ldots,
\]
and
\[
\liminf_{N \to \infty} \inf_{\#P=N} \frac{L_{2,N}^{\text{per}}(P)}{\sqrt{\log N}} \leq 2\eta = 0.248910 \ldots.
\]
Note that the corresponding constants that can be derived from the results on the Hammersley point set in Theorem 8 are larger. For the standard $L_2$ discrepancy we have
\[
\liminf_{N \to \infty} \inf_{\#P=N} \frac{L_{2,N}(P)}{\sqrt{\log N}} \leq \sqrt{2} \eta = 0.176006 \ldots,
\]
where this constant is attained by symmetrized Fibonacci lattices (see [3]).

**Brief discussion of possible relationships between $L_2$ discrepancies.** We point out the following peculiarity, which follows from Theorems 8 and 10:

**Remark 12.** If $P$ is either the Hammersley point set $\mathcal{H}_m$ or a rational lattice $L_n(\alpha)$, then
\[
L_{2,N}^{\text{per}}(P)^2 = 4L_{2,N}^{\text{extr}}(P)^2 + \frac{1}{18} + \frac{1}{18N^2},
\]
where $N = 2^m$ or $N = q_n$, respectively.

From Remark 12 and other observations (e.g. the one-element point set $P = \{(0,0)\}$ satisfies (9) because, as is easily checked, $L_{2,N}^{\text{per}}(P)^2 = 5/36$ and $L_{2,N}^{\text{extr}}(P)^2 = 1/144$) one might conjecture that (9) holds for arbitrary $N$-element point sets in the unit square.

However, let us consider the regular grid
\[
\Gamma_{m,d} = \left\{ 0, \frac{1}{m}, \ldots, \frac{m-1}{m} \right\}^d
\]
consisting of $N = m^d$ points in $[0,1]^d$, where $m \in \mathbb{N}$. For this point set the $L_2$ discrepancies are easily computed using the formulas of Koksma [18] and Warnock [30] (see the forthcoming Proposition 13). As a result one obtains
\[
L_{2,m^d}^{\text{per}}(\Gamma_{m,d})^2 = \left( \frac{m^2}{3} + \frac{1}{6} \right)^d - \left( \frac{m^2}{3} \right)^d,
\]
\[ L_{2,m}^{\text{extr}}(\Gamma_m, d)^2 = \frac{m^{2d} - (m^2 - 1)^d}{12^d}. \]

For \( d = 1 \) we have
\[ L_{2,m}^{\text{per}}(\Gamma_m, 1)^2 = \frac{1}{6} \quad \text{and} \quad L_{2,m}^{\text{extr}}(\Gamma_m, 1)^2 = \frac{1}{12}, \]
which nicely confirms the relation from Theorem 7.

For \( d = 2 \) we have
\[ L_{2,m^2}^{\text{extr}}(\Gamma_m, 2)^2 = \frac{2m^2 - 1}{144} \quad \text{and} \quad L_{2,m^2}^{\text{per}}(\Gamma_m, 2)^2 = \frac{m^2}{9} + \frac{1}{36}. \]

If \( m = 1 \), then \( \Gamma_{1,2} = \{(0, 0)\} \) and (9) is still satisfied. But if \( m > 1 \), then the relation (9) does not hold anymore for \( \Gamma_{m,2} \). Not even the implied multiplier 4 obtains, because
\[ \lim_{m \to \infty} \frac{L_{2,m^2}^{\text{per}}(\Gamma_m, 2)^2}{L_{2,m^2}^{\text{extr}}(\Gamma_m, 2)^2} = 8. \]

These observations raise some interesting questions about relationships between periodic and extreme \( L_2 \) discrepancy. In particular: Which plane point sets satisfy (9)? Are the periodic and extreme \( L_2 \) discrepancies in arbitrary dimension \( d \) equivalent (like for \( d = 1 \) according to Theorem 7)?

4. The proof of Theorem 8. We use the following well-known formulas for the standard, extreme and periodic \( L_2 \) discrepancy of point sets. Although we only need the two-dimensional versions in our proofs, we state the results for arbitrary \( d \).

**Proposition 13.** Let \( P = \{x_0, x_1, \ldots, x_{N-1}\} \) be a point set in \([0, 1)^d\), where we write \( x_k = (x_{k,1}, \ldots, x_{k,d}) \) for \( k \in \{0, 1, \ldots, N-1\} \). Then

\[
L_{2,N}(P)^2 = \frac{N^2}{3^d} - \frac{N}{2^{d-1}} \sum_{k=0}^{N-1} \prod_{i=1}^{d} (1 - x_{k,i}^2) \\
+ \sum_{k,l=0}^{N-1} \prod_{i=1}^{d} \min(1 - x_{k,i}, 1 - x_{l,i}),
\]

\[
L_{2,N}^{\text{extr}}(P)^2 = \frac{N^2}{12^d} - \frac{N}{2^{d-1}} \sum_{k=0}^{N-1} \prod_{i=1}^{d} x_{k,i}(1 - x_{k,i}) \\
+ \sum_{k,l=0}^{N-1} \prod_{i=1}^{d} (\min(x_{k,i}, x_{l,i}) - x_{k,i}x_{l,i}),
\]

\[
L_{2,N}^{\text{per}}(P)^2 = -\frac{N^2}{3^d} + \sum_{k,l=0}^{N-1} \prod_{i=1}^{d} \left( \frac{1}{2} - |x_{k,i} - x_{l,i}| + (x_{k,i} - x_{l,i})^2 \right).
\]
Proof. The first formula is well known and easily proved by direct integration (see [18, 30]). Sometimes this formula is attributed to Warnock [30], which is not entirely correct, since it was already proved by Koksma [18] in 1942 for $d = 1$ using the same method as later applied by Warnock [30] for arbitrary dimension (see also [24]). Also the second formula follows by simple direct integration and can be found in [30] and [23, 26], respectively. The last formula can be found in [16, 26], where it was derived in the context of the worst-case error in a certain reproducing kernel Hilbert space. This formula can also be derived more directly from Proposition 1 and equation (10). To this end, we observe that for $x, y \in [0, 1]$ we have
\[
\int_{0}^{1} \{x + \delta\} \, d\delta = \frac{1}{2}, \quad \int_{0}^{1} \{x + \delta\}^2 \, d\delta = \frac{1}{3},
\]
and
\[
\int_{0}^{1} \max\{\{x + \delta\}, \{y + \delta\}\} \, d\delta = \frac{1}{2} + |y - x| - (y - x)^2.
\]
This is an easy calculation. We just show the third formula. Assume without loss of generality that $0 \leq x \leq y \leq 1$. Then
\[
\int_{0}^{1} \max\{\{x + \delta\}, \{y + \delta\}\} \, d\delta = \int_{0}^{1-y} \{y + \delta\} \, d\delta + \int_{1-y}^{1-x} \{x + \delta\} \, d\delta + \int_{1-x}^{1} \{y + \delta\} \, d\delta
\]
\[
= \int_{y}^{1} u \, du + \int_{1-(y-x)}^{1} u \, du + \int_{1+(y-x)}^{1+y} (u - 1) \, du.
\]
Now the result follows from evaluating the elementary integrals. Formula (12) follows as well.

Remark 14. Using formulas (10)–(12) and the fact that
\[
\min\{x, y\} = \frac{1}{2}(x + y - |x - y|) \quad \text{for } x, y \in \mathbb{R},
\]
we find that for a two-dimensional point set
\[
\mathcal{P} = \{(x_k, y_k) : k = 0, 1, \ldots, N - 1\}
\]
we have
\[
L_{2,N}(\mathcal{P})^2 = \frac{N^2}{9} - \frac{N}{2} \sum_{k=0}^{N-1} (1 - x_k^2)(1 - y_k^2)
\]
\[
+ \frac{1}{4} \sum_{k,l=0}^{N-1} (2 - x_k - x_l - |x_k - x_l|)(2 - y_k - y_l - |y_k - y_l|),
\]
\[ L_{2,N}^{\text{extr}}(\mathcal{P})^2 = \frac{N^2}{144} - \frac{N}{2} \sum_{k=0}^{N-1} x_k(1 - x_k)y_k(1 - y_k) \]

\[ + \frac{1}{4} \sum_{k,l=0}^{N-1} (x_k + x_l - 2x_kx_l - |x_k - x_l|)(y_k + y_l - 2y_ky_l - |y_k - y_l|), \]

\[ L_{2,N}^{\text{per}}(\mathcal{P})^2 = -\frac{N^2}{9} \]

\[ + \sum_{k,l=0}^{N-1} \left( \frac{1}{2} - |x_k - x_l| + (x_k - x_l)^2 \right) \left( \frac{1}{2} - |y_k - y_l| + (y_k - y_l)^2 \right). \]

The following lemma giving the exact values of various sums involving the components of the Hammersley point set is crucial.

**Lemma 15.** Let \( \mathcal{H}_m = \{(x_k, y_k) : k = 0, 1, \ldots, 2^m - 1\} \) be the Hammersley point set. Then

\[ S_1 := \sum_{k=0}^{2^m-1} x_k = \sum_{k=0}^{2^m-1} y_k = \frac{2^m - 1}{2}, \]

\[ S_2 := \sum_{k=0}^{2^m-1} x_k^2 = \sum_{k=0}^{2^m-1} y_k^2 = \frac{(2^m - 1)(2^{m+1} - 1)}{6 \cdot 2^m}, \]

\[ S_3 := \sum_{k=0}^{2^m-1} x_ky_k = 2^{m-2} + \frac{m}{8} - \frac{1}{2} + \frac{1}{2^{m+2}}, \]

\[ S_4 := \sum_{k=0}^{2^m-1} x_ky_k^2 = \sum_{k=0}^{2^m-1} x_k^2y_k = \frac{(2^m - 1)(4^{m+1} + 3 \cdot 2^m(m - 2) + 2)}{3 \cdot 2^{2m+3}}, \]

\[ S_5 := \sum_{k=0}^{2^m-1} x_k^2y_k^2 = 8(2^{2m+1} - 3 \cdot 2^m + 1)^2 + 9m2^m(4^{m+1} + 2^m(m - 9) + 4) \]

\[ 9 \cdot 2^{3m+5} \]

\[ S_6 := \sum_{k,l=0}^{2^m-1} |x_k - x_l| = \sum_{k,l=0}^{2^m-1} |y_k - y_l| = \frac{4^m - 1}{3}, \]

\[ S_7 := \sum_{k,l=0}^{2^m-1} x_k|y_k - y_l| = \sum_{k,l=0}^{2^m-1} y_k|x_k - x_l| = \frac{(2^m - 1)^2(2^m + 1)}{6 \cdot 2^m}, \]

\[ S_8 := \sum_{k,l=0}^{2^m-1} x_k^2|y_k - y_l| = \sum_{k,l=0}^{2^m-1} y_k^2|x_k - x_l| \]

\[ = \frac{16(2^m - 1)^2(2^{2m+1} + 2^m - 1) + 9m(m - 1)4^m}{9 \cdot 2^{2m+5}}. \]
We defer the technical proofs of these formulas to the next section. We are ready to prove the discrepancy formulas for the Hammersley point set:

Proof of Theorem 8. We expand the formulas for $L_{2,2^m}(H_m)^2$, $L_{2,2^m}^{\text{extr}}(H_m)^2$ and $L_{2,2^m}^{\text{per}}(H_m)^2$ as given in Remark 14 and express them in terms of the sums which appear in Lemma 15. We obtain

$$L_{2,N}(H_m)^2 = \frac{11 \cdot 4^m}{18} - \frac{2^m}{2} (S_5 - 2S_2) + \frac{1}{4} (-2^{m+3}S_1 + 2^{m+1}S_3 + 2S_1^2 - 4S_6 + 4S_7 + S_{10}),$$

$$L_{2,N}^{\text{extr}}(H_m)^2 = \frac{4^m}{144} - \frac{2^m}{2} (S_3 - 2S_4 + S_5) + \frac{1}{4} (2^{m+1}S_3 + 2S_1^2 - 8S_1S_3 + 4S_3^2 - 4S_7 + 4S_9 + S_{10}),$$

$$L_{2,N}^{\text{per}}(H_m)^2 = \frac{5 \cdot 4^m}{36} - 4S_8 + 4S_9 - S_6 + 2^{m+1}S_2 - 2S_1^2 + 2^{m+1}S_5 - 8S_1S_4 + 4S_3^2 + 2S_2^2 + S_{10}.$$

The remaining trivial task is to insert the expressions for $S_i$, $1 \leq i \leq 10$, given in Lemma 15.

5. The proof of Lemma 15

Calculation of $S_1$, $S_2$ and $S_6$. We have

$$S_1 = \sum_{k=0}^{2^m-1} \frac{k}{2^m}, \quad S_2 = \sum_{k=0}^{2^m-1} \left( \frac{k}{2^m} \right)^2,$$

$$S_6 = \frac{2}{2^m} \sum_{k=1}^{2^m-1} \sum_{l=0}^{k-1} (k - l),$$

which yields the results for these sums.

Calculation of $S_3$, $S_4$ and $S_5$. Since the proofs of the formulas for these sums are very similar, we only sketch the proof for the most complicated sum $S_5$. We have
Extreme and periodic $L_2$ discrepancy

\[
S_5 = \sum_{t_1, \ldots, t_m = 0}^{1} \left( \sum_{j_1=1}^{m} \frac{t_{j_1}}{2^{m+1-j_1}} \right)^2 \left( \sum_{j_2=1}^{m} \frac{t_{j_2}}{2^{j_2}} \right)^2
\]

\[
= \sum_{a,b,c,d=1}^{m} \frac{2^{m-4}}{2^{m+2-a-b+c+d}} + \sum_{a,b,c=1}^{m} \frac{2^{m-3}}{2^{m+2-a-b+2c}} + \sum_{a,b=1}^{m} \frac{2^{m-2}}{2^{m+2-a+b+2c}} + \sum_{a=1}^{m} \frac{2^{m-1}}{2^{m+2-a-c}}
\]

where “p.d.” stands for “pairwise different”. For the first sum on the right hand side we obtain

\[
\sum_{a,b,c,d=1}^{m} \frac{2^{m-4}}{2^{m+2-a-b+c+d}} = \frac{1}{2^{m+6}} \left( \sum_{a,b,c,d=0}^{m} 2^{a+b-c-d} - \sum_{a,b,c,d=0}^{m} 2^{2a-c-d} \right)
\]

\[
- \sum_{a,b,c=1}^{m} 2^{a+b-2c} - 4 \sum_{a,b,d=1}^{m} 2^{b-d} - \sum_{a,b,c,d=1}^{m} 2^{2a-2c}
\]

\[
- 2 \sum_{a,b=1}^{m} 1 - 4 \sum_{a,c,d=1}^{m} 2^{2a-d} - \sum_{a,b,c,d=1}^{m} 1
\]

The calculation of these sums is straightforward. The remaining summands in the expression for $S_5$ can be computed analogously. This leads to the final result.

**Calculation of $S_7$, $S_8$ and $S_9$**. These sums can be treated similarly. Therefore we will only show how to evaluate the probably most complicated sum $S_9$. We write it in the following way:

\[
S_9 = \sum_{t_1^{(k)}, \ldots, t_m^{(k)}, t_1^{(l)}, \ldots, t_m^{(l)} = 0}^{1} \left( \sum_{j_1=1}^{m} \frac{t_{j_1}^{(k)}}{2^{m+1-j_1}} \right) \left( \sum_{j_2=1}^{m} \frac{t_{j_2}^{(l)}}{2^{m+1-j_2}} \right) \left| \sum_{j_3=1}^{m} \frac{t_{j_3}^{(k)} - t_{j_3}^{(l)}}{2^{j_3}} \right|
\]
\[
\sum_{r=0}^{m-1} \sum_{t_r^{(k)}, t_r^{(l)} : t_r^{(k)}=t_r^{(l)} \forall i=1,\ldots,r, t_{r+1}^{(k)} \neq t_{r+1}^{(l)}} \left( \sum_{j=1}^{m} \frac{t_j^{(k)}}{2^{m+1-j}} \right) \left( \sum_{j=2}^{m} \frac{t_j^{(l)}}{2^{m+1-j}} \right) \sum_{j_3=r+1}^{m} \frac{t_j^{(k)}-t_j^{(l)}}{2^{j_3}}.
\]

We define
\[
P_0(t_{r+1}^{(k)}) := \sum_{j_1=1}^{m} \frac{t_{j_1}^{(k)}}{2^{m+1-j_1}} + \frac{t_{r+1}^{(k)}}{2^{m-r}}, \quad T := \sum_{j_3=r+2}^{m} \frac{t_j^{(k)}-t_j^{(l)}}{2^{j_3}},
\]
\[
P_1(t_{r+1}^{(l)}) := \sum_{j_1=1}^{r} \frac{t_{j_1}^{(k)}}{2^{m+1-j_1}} + \frac{t_{r+1}^{(l)}}{2^{m-r}} + \sum_{j_1=r+2}^{m} \frac{t_{j_1}^{(l)}}{2^{m+1-j_1}}
\]
to write (after summation over the indices \(t_{r+1}^{(k)}\) and \(t_{r+1}^{(l)}\) with \(t_{r+1}^{(k)} \neq t_{r+1}^{(l)}\))
\[
S_9 = \sum_{r=0}^{m-1} \sum_{t_r^{(k)}, t_r^{(l)} : t_r^{(k)}=t_r^{(l)} \forall i=1,\ldots,r, t_{r+2}^{(k)}, t_m^{(k)}, t_{r+2}^{(l)}, t_m^{(l)} = 0} \left( \frac{P_0(1)P_1(0) + P_0(0)P_1(1)}{2^{r+1}} \right) + T \left( P_0(1)P_1(0) - P_0(0)P_1(1) \right).
\]

Since
\[
P_0(1)P_1(0) - P_0(0)P_1(1) = -\frac{1}{2^{m-r}} \sum_{j=r+2}^{m} \frac{t_j^{(k)}-t_j^{(l)}}{2^{m+1-j}},
\]
we obtain
\[
\sum_{t_1^{(k)}, \ldots, t_r^{(k)}, t_{r+2}^{(k)}, \ldots, t_m^{(k)}, t_r^{(l)}, \ldots, t_m^{(l)} = 0} T \left( P_0(1)P_1(0) - P_0(0)P_1(1) \right)
\]
\[
= -\frac{1}{2^{m-r}} \sum_{j_1,j_3=r+2}^{m+1} \frac{1}{2^{j_3}} \sum_{t_1^{(k)}, \ldots, t_r^{(k)}, t_{r+2}^{(k)}, \ldots, t_m^{(k)}, t_1^{(l)}, \ldots, t_r^{(l)}, t_{r+2}^{(l)}, \ldots, t_m^{(l)} = 0} \left( t_{j_1}^{(k)} - t_{j_1}^{(l)} \right) \left( t_{j_3}^{(k)} - t_{j_3}^{(l)} \right)
\]
\[
= 0 \text{ for } j_1 \neq j_3
\]
\[
= -\frac{1}{2^{m-r}} \sum_{j=r+2}^{m+1} \frac{1}{2^{m+1-j}} 2^{m-r-3} = -\frac{m-r-1}{16}.
\]
Observe that

$$P_0(1)P_1(0) + P_0(0)P_1(1) = 2 \left( \sum_{j_1=1}^{m} \frac{t_{j_1}^{(k)}}{2^{m+1-j_1}} \right) \left( \sum_{j_2=1}^{m} \frac{t_{j_2}^{(l)}}{2^{m+1-j_2}} \right) + \frac{1}{2^{m-r}} \sum_{j_1=1}^{m} \frac{t_{j_1}^{(k)}}{2^{m+1-j_1}} + \frac{1}{2^{m-r}} \sum_{j_2=1}^{m} \frac{t_{j_2}^{(l)}}{2^{m+1-j_2}}$$

$$=: A + B + C.$$  

It is straightforward to prove that

$$B = \sum_{k=1}^{1 \ldots, t_r \ldots, t_r = 0} 2^{j_1+j_2} = \frac{1}{16} \sum_{j=1}^{m} 2^j.$$  

Further we have

$$A = \frac{2}{4^{m+1}} \sum_{j_1, j_2=1}^{m} 2^{j_1+j_2} = \frac{1}{2^{r+1}} \sum_{j_2=r+2}^{m} 2^{j_2} \left( \sum_{j_1=1}^{m} 2^{j_1} - 2^{r+1} \right),$$

while in the first sum it is necessary to distinguish between the cases $j_1 = j_2$ and $j_1 \neq j_2$. We obtain for this sum the result

$$\frac{2}{4^{m+1}} 2^{2m-r-4} \sum_{j_1, j_2=1}^{m} 2^{j_1+j_2} = \frac{1}{2^{r+1}} \sum_{j_2=r+2}^{m} 2^{j_2} \left( \sum_{j_1=1}^{m} 2^{j_1} - 2^{r+1} \right),$$

We put everything together to find the claimed result for $S_9$. 

$$\sum_{j_1, j_2=1}^{m} 2^{j_1+j_2} = \frac{r}{4^{m+1}} 2^{2m-r-1} \left( 2^{m-3} \sum_{j_1=r+2}^{m} \sum_{j_2=1}^{r} 2^{j_1+j_2} \right) + \frac{r}{4^{m+1}} 2^{2m-r-3} \sum_{j_1, j_2=1}^{m} 2^{j_1+j_2}.$$
Calculation of $S_{10}$. We have

$$S_{10} = \left| \sum_{j_1=1}^{m} \frac{t_{j_1}^{(k)} - t_{j_1}^{(l)}}{2j_1} \right| \sum_{j_2=1}^{m} \frac{t_{j_2}^{(k)} - t_{j_2}^{(l)}}{2m+1-j_2}$$

$$= \sum_{r=0}^{m-1} \sum_{s=0}^{m-r-2} 2^{r+s} t_{r+1}^{(k)} \cdots t_{m-s}^{(k)} t_{r+1}^{(l)} \cdots t_{m-s}^{(l)} = 0$$

$$= \sum_{r=0}^{m-1} \sum_{s=0}^{m-r-1} 2^{r+s} \left| \sum_{j_1=r+2}^{m-s-1} \frac{t_{j_1}^{(k)} - t_{j_1}^{(l)}}{2j_1} \right| \sum_{j_2=r+2}^{m-s-1} \frac{t_{j_2}^{(k)} - t_{j_2}^{(l)}}{2m+1-j_2}$$

$$= \sum_{r=0}^{m-1} \sum_{s=0}^{m-r-1} 2^{r+s} t_{r+1}^{(k)} \cdots t_{m-s}^{(k)} t_{r+1}^{(l)} \cdots t_{m-s}^{(l)} = 0$$

$$= \sum_{r=0}^{m-1} \sum_{s=0}^{m-r-1} 2^{r+s} \left| \sum_{j_1=r+2}^{m-s-1} \frac{t_{j_1}^{(k)} - t_{j_1}^{(l)}}{2j_1} \right| \sum_{j_2=r+2}^{m-s-1} \frac{t_{j_2}^{(k)} - t_{j_2}^{(l)}}{2m+1-j_2}$$

We write $S_{10} = P_1 + P_2$, where $P_1$ is the part of the last expression where $s = m - r - 1$ and $P_2$ is the part where $s \leq m - r - 2$. For $P_1$ we have

$$P_1 = \sum_{r=0}^{m-1} 2^{m-1} \sum_{t_{r+1}^{(k)}=0}^{1} \sum_{t_{r+1}^{(l)}=1}^{1} t_{r+1}^{(k)} - t_{r+1}^{(l)} \left| \frac{t_{r+1}^{(k)} - t_{r+1}^{(l)}}{2r+1} \right| \left| \frac{t_{r+1}^{(k)} - t_{r+1}^{(l)}}{2m-r} \right|$$

$$= \sum_{r=0}^{m-1} \frac{1}{2} = \frac{m}{2}.$$ 

For the evaluation of $P_2$ we abbreviate

$$T_1 := \sum_{j_1=r+2}^{m-s-1} \frac{t_{j_1}^{(k)} - t_{j_1}^{(l)}}{2j_1} \quad \text{and} \quad T_2 := \sum_{j_2=r+2}^{m-s-1} \frac{t_{j_2}^{(k)} - t_{j_2}^{(l)}}{2m+1-j_2}$$

(which are empty sums for $s = m - r - 2$). Then we sum the expression over $t_{r+1}^{(k)}$, $t_{r+1}^{(l)}$, $t_{m-s}^{(k)}$ and $t_{m-s}^{(l)}$, where the first two and the last two must be different. We get

$$P_2 = \sum_{r=0}^{m-1} \sum_{s=0}^{m-r-2} 2^{r+s} \left| \sum_{t_{r+1}^{(k)}=0}^{1} \sum_{t_{r+1}^{(l)}=0}^{1} t_{r+1}^{(k)} - t_{r+1}^{(l)} \left| \frac{t_{r+1}^{(k)} - t_{r+1}^{(l)}}{2r+1} \right| + T_1 + \frac{t_{m-s}^{(k)} - t_{m-s}^{(l)}}{2m-s} \right| \left| \frac{t_{m-s}^{(k)} - t_{m-s}^{(l)}}{2s+1} \right| + T_2 + \frac{t_{r+1}^{(k)} - t_{r+1}^{(l)}}{2r+1} \right|$$
Extreme and periodic $L_2$ discrepancy

\[
= \sum_{r=0}^{m-1} \sum_{s=0}^{m-r-2} 2^{r+s} \times \sum_{t_r^{(k)} \in \mathbb{Z}_2} \sum_{t_s^{(l)} \in \mathbb{Z}_2} \left\{ \left( \frac{1}{2^{r+1}} + T_1 + \frac{1}{2^{m-s}} \right) \left( \frac{1}{2^{s+1}} + T_2 + \frac{1}{2^{m-r}} \right) \right. \\
+ \left( \frac{1}{2^{r+1}} + T_1 - \frac{1}{2^{m-s}} \right) \left( \frac{1}{2^{s+1}} - T_2 - \frac{1}{2^{m-r}} \right) \\
+ \left. \left( \frac{1}{2^{r+1}} - T_1 - \frac{1}{2^{m-s}} \right) \left( \frac{1}{2^{s+1}} + T_2 - \frac{1}{2^{m-r}} \right) \right\}.
\]

The expression in curled brackets simplifies very nicely and we get

\[
P_2 = 4 \sum_{r=0}^{m-1} \sum_{s=0}^{m-r-2} 2^{r+s} \sum_{t_r^{(k)} \in \mathbb{Z}_2} \sum_{t_s^{(l)} \in \mathbb{Z}_2} \frac{1}{2^{r+s+2}} + \frac{1}{2^{2m-r-s}} = 4^{m-1} \sum_{r=0}^{m-1} \sum_{s=0}^{m-r-2} 2^{-r-s} \left( \frac{1}{2^{r+s+2}} + \frac{1}{2^{2m-r-s}} \right)
\]

\[
= \frac{8(4^m - 1) + 9m^2 - 33m}{72}.
\]

The formula for $S_{10}$ follows.

6. The proof of Theorem 9. In this proof we consider the Hammersley point set as a digital net with generating matrices

\[
C_1 = \begin{pmatrix}
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0 \\
0 & 0 & \cdots & 0 & 1
\end{pmatrix}
\quad \text{and} \quad
C_2 = \begin{pmatrix}
0 & 0 & \cdots & 0 & 1 \\
0 & 0 & \cdots & 1 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 1 & \cdots & 0 & 0 \\
1 & 0 & \cdots & 0 & 0
\end{pmatrix}.
\]

Let $k \in \{0, 1, \ldots, 2^m - 1\}$ with dyadic expansion $k = \kappa_0 + \kappa_1 2 + \cdots + \kappa_{m-1} 2^{m-1}$ and corresponding digit vector $\vec{k} = (\kappa_0, \kappa_1, \ldots, \kappa_{m-1})^\top$ over $\mathbb{Z}_2$. Then the $k$th element $(x_k, y_k)$ of the Hammersley point set is given by $x_k = \xi_{k,1} \frac{1}{2} + \xi_{k,2} \frac{1}{2^2} + \cdots + \xi_{k,m} \frac{1}{2^m}$ and $y_k = \eta_{k,1} \frac{1}{2} + \eta_{k,2} \frac{1}{2^2} + \cdots + \eta_{k,m} \frac{1}{2^m}$, where

\[
(\xi_{k,1}, \xi_{k,2}, \ldots, \xi_{k,m})^\top = C_1 \vec{k} \quad \text{and} \quad (\eta_{k,1}, \eta_{k,2}, \ldots, \eta_{k,m})^\top = C_2 \vec{k}.
\]
Proof of Theorem 9. In [8] the analogous quantity, but for digital shifts of depth $m$, was computed. The present case can be interpreted as digital shifts of depth $m = \infty$. Let $(x_k, y_k)$ for $k = 0, 1, \ldots, 2^m - 1$ denote the elements of the Hammersley point set. A slight modification of the proof in [8] shows that

$$\mathbb{E}_\delta[L_{2,N}(H_m \oplus \delta)^2] = -\frac{1}{4} \sum_{k=1}^{\infty} \tau(k) \sum_{n,h=0}^{2^m-1} \text{wal}_k(x_n \oplus x_h) - \frac{1}{4} \sum_{l=1}^{\infty} \tau(l) \sum_{n,h=0}^{2^m-1} \text{wal}_l(y_n \oplus y_h)$$

$$+ \frac{1}{4} \sum_{k,l=0}^{\infty} \tau(k) \tau(l) \sum_{n,h=0}^{2^m-1} \text{wal}_k(x_n \oplus x_h) \text{wal}_l(y_n \oplus y_h),$$

where $\text{wal}_k$ denotes the $k$th dyadic Walsh function given by

$$\text{wal}_k(x) = (-1)^{\kappa_0 \xi_1 + \kappa_1 \xi_2 + \cdots + \kappa_{r-1} \xi_r}$$

when $k \in \mathbb{N}_0$ and $x \in [0, 1)$ have dyadic expansions $k = \kappa_0 + \kappa_1 2 + \cdots + \kappa_{r-1} 2^{r-1}$ and $x = \xi_1 + \xi_2 2 + \cdots$, respectively. Further $\tau(0) = \frac{1}{3}$ and $\tau(k) = -\frac{1}{6 \cdot 4^{r\tau(k)}}$ for $k > 0$, where $r(k)$ denotes the unique integer $r$ such that $2^r \leq k < 2^{r+1}$.

We have

$$\sum_{n,h=0}^{2^m-1} \text{wal}_k(x_n \oplus x_h) = \left| \sum_{n=0}^{2^m-1} \text{wal}_k(x_n) \right|^2 = \begin{cases} 4^m & \text{if } C_1^\top \vec{k} = \vec{0}, \\ 0 & \text{otherwise,}\end{cases}$$

where we have used a well-known relation between digital nets and Walsh functions (see, for example, [9 Lemma 4.75] or [8 Lemma 2]). Although this relation is only stated for $0 \leq k \leq 2^m - 1$, it also holds for $k \geq 2^m$ with dyadic expansion $k = \sum_{i=0}^{s} \kappa_i 2^i$, where $s \geq m$, if we set $\vec{k} = (\kappa_0, \ldots, \kappa_{m-1})^\top$. Since $C_1$ is regular, the condition $C_1^\top \vec{k} = \vec{0}$ is equivalent to $k = 2^m k'$ with $k' \in \mathbb{N}$. Therefore we obtain

$$\sum_{k=1}^{\infty} \tau(k) \sum_{n,h=0}^{2^m-1} \text{wal}_k(x_n \oplus x_h) = 4^m \sum_{k'=1}^{\infty} \tau(2^m k') = \sum_{u=0}^{\infty} \left(-\frac{1}{6 \cdot 4^u}\right) 2^u = -\frac{1}{3}.$$ 

Likewise,

$$\sum_{l=1}^{\infty} \tau(l) \sum_{n,h=0}^{2^m-1} \text{wal}_l(y_n \oplus y_h) = -\frac{1}{3}.$$ 

(1) Set $m = \infty$ in [8] Lemma 3 and take care of the resulting consequences.
Furthermore,
\[
\sum_{n,h=0}^{2^m-1} \text{wal}_k(x_n \oplus x_h)\text{wal}_l(y_n \oplus y_h) = \left| \sum_{n=0}^{2^m-1} \text{wal}_k(x_n)\text{wal}_l(y_n) \right|^2
\]
\[
= \begin{cases} 4^m & \text{if } C_1^\top \vec{k} + C_2^\top \vec{l} = \vec{0}, \\ 0 & \text{otherwise,} \end{cases}
\]
where we have used \cite[Lemma 4.75]{9} (or \cite[Lemma 2]{8}) again. Hence
\[
\mathbb{E}_\delta[L_{2,N}(\mathcal{P} \oplus \delta)^2] = \frac{1}{6} + 4^{m-1} \sum_{k,l=0}^{\infty} \tau(k)\tau(l).
\]
We have
\[
\sum_{k,l=0}^{\infty} \tau(k)\tau(l) = \sum_{k=1}^{\infty} \tau(k)\tau(0) + \sum_{l=1}^{\infty} \tau(0)\tau(l)
\]
\[
+ \sum_{k,l=1}^{\infty} \tau(k)\tau(l)
\]
\[
= - \frac{2}{9} \cdot 4^m + \sum_{k,l=1}^{\infty} \tau(k)\tau(l).
\]
Hence
\[
\mathbb{E}_\delta[L_{2,N}(\mathcal{P} \oplus \delta)^2] = \frac{1}{9} + 4^{m-1} \sum_{k,l=1}^{\infty} \tau(k)\tau(l).
\]
We have
\[
\Sigma := \sum_{k,l=1}^{\infty} \tau(k)\tau(l) = \frac{1}{36} \sum_{u,v=0}^{\infty} \frac{1}{4^{u+v}} \sum_{k=2^u}^{2^{u+1}-1} \sum_{l=2^v}^{2^{v+1}-1} 1.
\]
Denote by \(e_1, \ldots, e_m\) the row vectors of \(C_1\) and by \(d_1, \ldots, d_m\) the row vectors of \(C_2\). Set \(e_i = d_i = \vec{0}\) for \(i \geq m + 1\). The condition \(C_1^\top \vec{k} + C_2^\top \vec{l} = \vec{0}\) can be rewritten as
\[
e_1 \kappa_0 + \cdots + e_u \kappa_{u-1} + e_{u+1} + d_1 \lambda_0 + \cdots + d_v \lambda_{v-1} + d_{v+1} = \vec{0},
\]
where \(k = \kappa_0 + \kappa_1 + \cdots + \kappa_{u-1}2^{u-1} + 2^u\) and \(l = \lambda_0 + \lambda_1 + \cdots + \lambda_{v-1}2^{v-1} + 2^v\).
Since \( e_1, \ldots, e_{u+1}, d_1, \ldots, d_{v+1} \) are linearly independent as long as \( u + 1 + v + 1 \leq m \) we must have \( u + v \geq m - 1 \). Hence

\[
\Sigma = \frac{1}{36} \sum_{u,v=0}^{\infty} \frac{1}{4^{u+v}} \sum_{\kappa_{u-1}, \ldots, \kappa_0=0}^{1} \sum_{\lambda_{v-1}, \ldots, \lambda_0=0}^{1} e_1 \kappa_0 + \cdots + e_{u+1} \kappa_u + d_1 \lambda_0 + \cdots + d_{v+1} \lambda_v = \vec{0}
\]

Now we split the range of summation over \( u \) and \( v \). Then

\[
\Sigma = \frac{1}{36} \sum_{u,v=0}^{u+v \geq m-1} \frac{1}{4^{u+v}} \sum_{\kappa_{u-1}, \ldots, \kappa_0=0}^{1} \sum_{\lambda_{v-1}, \ldots, \lambda_0=0}^{1} e_1 \kappa_0 + \cdots + e_{u+1} \kappa_u + d_1 \lambda_0 + \cdots + d_{v+1} \lambda_v = \vec{0}
\]

\[
+ \frac{1}{36} \sum_{u=0}^{m-1} \sum_{v=0}^{\infty} \frac{1}{4^{u+v}} \sum_{\kappa_{u-1}, \ldots, \kappa_0=0}^{1} \sum_{\lambda_{v-1}, \ldots, \lambda_0=0}^{1} e_1 \kappa_0 + \cdots + e_{u+1} \kappa_u + d_1 \lambda_0 + \cdots + d_{v+1} \lambda_v = \vec{0}
\]

\[
+ \frac{1}{36} \sum_{u=0}^{m-1} \sum_{v=m}^{\infty} \frac{1}{4^{u+v}} \sum_{\kappa_{u-1}, \ldots, \kappa_0=0}^{1} \sum_{\lambda_{v-1}, \ldots, \lambda_0=0}^{1} e_1 \kappa_0 + \cdots + e_{u+1} \kappa_u + d_1 \lambda_0 + \cdots + d_{v+1} \lambda_v = \vec{0}
\]

\[
+ \frac{1}{36} \sum_{u,v=m}^{\infty} \frac{1}{4^{u+v}} \sum_{\kappa_{u-1}, \ldots, \kappa_0=0}^{1} \sum_{\lambda_{v-1}, \ldots, \lambda_0=0}^{1} e_1 \kappa_0 + \cdots + e_{u+1} \kappa_u + d_1 \lambda_0 + \cdots + d_{v+1} \lambda_v = \vec{0}
\]

We consider the first sum where \( u, v \in \{0, 1, \ldots, m-1\} \) and \( \tau := u + v \geq m - 1 \). Then we have

\[
e_1 \kappa_0 + \cdots + e_{u+1} \kappa_u + d_1 \lambda_0 + \cdots + d_{v+1} \lambda_v = \vec{0}
\]

iff

\[
\begin{pmatrix}
\kappa_0 \\
\vdots \\
\kappa_{m-\tau+u-2} \\
\kappa_{m-\tau+u-1} \\
\vdots \\
\kappa_u = 1 \\
0 \\
\vdots \\
0
\end{pmatrix}
+ \begin{pmatrix}
0 \\
\vdots \\
0 \\
\lambda_{\tau-u} = 1 \\
\vdots \\
\lambda_{m-u-1} \\
\lambda_{m-u-2} \\
\vdots \\
\lambda_0
\end{pmatrix}
= \vec{0},
\]

i.e., iff \( \tau = m - 1 \) and
\( \kappa_0 = \cdots = \kappa_{u-1} = 0 \) and  
\( \kappa_u = \lambda_v = 1 \) and  
\( \lambda_0 = \cdots = \lambda_{v-1} = 0, \)

or \( \tau \in \{m, \ldots, 2m-2\} \) and  
\( \kappa_0 = \cdots = \kappa_{m-\tau+u-2} = 0, \kappa_{m-\tau+u-2} = 1 \) and  
\( \lambda_0 = \cdots = \lambda_{m-u-2} = 0, \lambda_{m-u-1} = 1 \) and  
\( \kappa_i = \lambda_{m-1-i} \) for \( i = m-\tau+u, \ldots, u-1. \)

Therefore

\[
\frac{1}{36} \sum_{\substack{u,v=0 \\
u+v \geq m-1}}^{m-1} \frac{1}{4u+v} \sum_{\kappa_{u-1}, \ldots, \kappa_0=0}^{1} \sum_{\lambda_{v-1}, \ldots, \lambda_0=0}^{1} 1 \\
\epsilon_1 \kappa_0 + \cdots + \epsilon_1 \kappa_u \lambda_0 + \cdots + d_v \lambda_v = 0 \\
= \frac{1}{36} \left[ \frac{1}{4m-1} \sum_{\substack{u,v=0 \\
u+v \geq m-1}}^{m-1} 1 + \sum_{\tau=m}^{2m-2} \frac{2m-2-2\tau-m}{2^\tau} \sum_{\substack{u,v=0 \\
u+v = \tau}}^{m-1} 1 \right].
\]

For \( m-1 \leq \tau \leq 2m-2 \) we have

\[
\sum_{\substack{u,v=0 \\
u+v = \tau}}^{m-1} 1 = 2m - \tau - 1.
\]

Hence

\[
\frac{1}{36} \sum_{\substack{u,v=0 \\
u+v \geq m-1}}^{m-1} \frac{1}{4u+v} \sum_{\kappa_{u-1}, \ldots, \kappa_0=0}^{1} \sum_{\lambda_{v-1}, \ldots, \lambda_0=0}^{1} 1 \\
\epsilon_1 \kappa_0 + \cdots + \epsilon_1 \kappa_u \lambda_0 + \cdots + d_v \lambda_v = 0 \\
= \frac{1}{36} \left[ \frac{m}{4m-1} + \frac{1}{2m} \sum_{\tau=m}^{2m-2} \frac{2m-2-2\tau-m}{2^\tau} \right].
\]

Now we use

\[
\sum_{\tau=m}^{2m-2} \frac{2m-2-2\tau-m}{2^\tau} = \frac{2m}{2^m} + \frac{4(1-2^m)}{4^m}
\]

and hence

\[
\frac{1}{36} \sum_{\substack{u,v=0 \\
u+v \geq m-1}}^{m-1} \frac{1}{4u+v} \sum_{\kappa_{u-1}, \ldots, \kappa_0=0}^{1} \sum_{\lambda_{v-1}, \ldots, \lambda_0=0}^{1} 1 \\
\epsilon_1 \kappa_0 + \cdots + \epsilon_1 \kappa_u \lambda_0 + \cdots + d_v \lambda_v = 0 \\
= \frac{1}{36} \left[ \frac{m}{4m-1} + \frac{2m}{4^m} + \frac{4(1-2^m)}{8m} \right] = \frac{m}{6 \cdot 4^m} + \frac{1}{9 \cdot 8^m} - \frac{1}{9 \cdot 4^m}.
\]
Next we consider the second sum where \( u \in \{m, m+1, \ldots\} \) and \( v \in \{0, 1, \ldots, m-1\} \). Then we have

\[
e_1 \kappa_0 + \cdots + e_{u+1} \kappa_u + d_1 \lambda_0 + \cdots + d_{v+1} \lambda_v = \vec{0}
\]

iff

\[
\begin{pmatrix}
\kappa_0 \\
\vdots \\
\kappa_{m-v-2} \\
\kappa_{m-v-1} \\
\kappa_{m-v} \\
\vdots \\
\kappa_{m-1}
\end{pmatrix}
+ 
\begin{pmatrix}
0 \\
\vdots \\
0 \\
\lambda_v = 1 \\
\lambda_{v-1} \\
\vdots \\
\lambda_0
\end{pmatrix}
= \vec{0},
\]

i.e., iff

- \( \kappa_0 = \cdots = \kappa_{m-v-2} = 0 \), \( \kappa_{m-v-1} = 1 \), and
- \( \kappa_{m-v} = \lambda_{v-1}, \ldots, \kappa_{m-1} = \lambda_0 \).

The digits \( \kappa_m, \ldots, \kappa_{u-1} \) are arbitrary. Hence

\[
\sum_{\kappa_{u-1}, \ldots, \kappa_0=0}^{1} \sum_{\lambda_{v-1, \ldots, \lambda_0=0}}^{1} 1 = 2^{u-m}2^v = 2^{u+v-m}.
\]

This yields for the second sum

\[
\frac{1}{36} \sum_{u=m}^{\infty} \sum_{v=0}^{m-1} \frac{1}{4^u v} e_1 \kappa_0 + \cdots + e_{u+1} \kappa_u + d_1 \lambda_0 + \cdots + d_{v+1} \lambda_v = \vec{0}
\]

\[
= \frac{1}{36 \cdot 2^m} \sum_{u=m}^{\infty} \frac{1}{2^u v} \sum_{v=0}^{m-1} \frac{1}{9 \cdot 4^v} = \frac{1}{9 \cdot 4^m} - \frac{1}{9 \cdot 8^m}.
\]

In the same way we can calculate the third sum:

\[
\frac{1}{36} \sum_{u=0}^{m-1} \sum_{v=m}^{\infty} \frac{1}{4^u v} e_1 \kappa_0 + \cdots + e_{u+1} \kappa_u + d_1 \lambda_0 + \cdots + d_{v+1} \lambda_v = \vec{0}
\]

\[
= \frac{1}{9 \cdot 4^m} - \frac{1}{9 \cdot 8^m}.
\]

It remains to evaluate the last sum where \( u, v \in \{m, m+1, \ldots\} \). Then we have

\[
e_1 \kappa_0 + \cdots + e_{u+1} \kappa_u + d_1 \lambda_0 + \cdots + d_{v+1} \lambda_v = \vec{0}
\]
iff

\[
\begin{pmatrix}
\kappa_0 \\
\vdots \\
\kappa_{m-1}
\end{pmatrix} + \begin{pmatrix}
\lambda_{m-1} \\
\vdots \\
\lambda_0
\end{pmatrix} = \vec{0},
\]

i.e., iff \(\kappa_i = \lambda_{m-i-1}\) for \(i = 0, \ldots, m - 1\). The digits \(\kappa_m, \ldots, \kappa_{u-1}\) and \(\lambda_m, \ldots, \lambda_{v-1}\) are arbitrary. Hence

\[
\begin{align*}
\sum_{e_1 \kappa_0 + \cdots + e_{u-1} \kappa_{u-1} = \lambda_0 + \cdots + d_{v+1} \lambda_{v+1}} & 1 \\
\sum_{\kappa_{u-1}, \ldots, \kappa_0 = 0}^1 \sum_{\lambda_{u-1}, \ldots, \lambda_0 = 0}^1 & 1 \\
1 & = 2^m 2^{u-m} 2^{v-m} = 2^{u+v-m}.
\end{align*}
\]

This yields for the last sum

\[
\frac{1}{36} \sum_{u,v=m}^{\infty} \frac{1}{4^{u+v}} \sum_{\kappa_{u-1}, \ldots, \kappa_0 = 0}^1 \sum_{\lambda_{u-1}, \ldots, \lambda_0 = 0}^1 1 = \frac{1}{36} \sum_{u,v=m}^{\infty} \frac{1}{4^{u+v}} 2^{u+v-m}
\]

\[
= \frac{1}{36 \cdot 2^m} \left( \sum_{u=m}^{\infty} \frac{1}{2^u} \right)^2 = \frac{1}{9 \cdot 8^m}.
\]

Putting all four sums together we obtain

\[
\Sigma = \frac{m}{6 \cdot 4^m} + \frac{1}{9 \cdot 8^m} - \frac{1}{9 \cdot 4^m} + \frac{1}{9 \cdot 4^m} - \frac{1}{9 \cdot 8^m} + \frac{1}{9 \cdot 8^m}
\]

\[
= \frac{m}{6 \cdot 4^m} + \frac{1}{9 \cdot 4^m}.
\]

Finally, this yields

\[
\mathbb{E}_\delta [L_{2,N}(P \oplus \delta)^2] = \frac{1}{9} + 4^{m-1} \Sigma = \frac{m}{24} + \frac{5}{36}.
\]

Remark 16. If we restrict to the average over all digital \(m\)-bit shifts \(\delta = \frac{\delta^{(1)}}{2} + \frac{\delta^{(2)}}{2^2} + \cdots + \frac{\delta^{(m)}}{2^m}\) per coordinate, then it follows easily from [19, Theorem 1] that

\[
\mathbb{E}_{\delta_m} [L_{2,N}(P \oplus \delta_m)^2] = \frac{m}{24} + \frac{3}{8} + \frac{1}{4 \cdot 2^m} - \frac{1}{72 \cdot 4^m}.
\]

Remark 17. It can be shown that Theorem 9 holds not only for the Hammersley point set, but for all \((0, m, 2)\)-nets over \(\mathbb{F}_2\). The proof is similar, but a bit more involved than for \(\mathcal{H}_m\).

7. The proof of Theorem 10. We need the following lemma, which has essentially been proven in [3, 4]. Since this result is crucial for the com-
putation of the periodic and extreme $L_2$ discrepancy of rational lattices, we repeat the short proof. Let $Z^* := Z \setminus \{0\}$.

**Lemma 18.** With the notation explained in the lines before Theorem 10 we have

$$\sum_{k_1, k_2 \in Z^*} \frac{1}{k_1^2 k_2^2} = \pi^4 q_n^{-1} \sum_{r=1}^{q_n-1} \frac{1}{\sin^2 \left( \frac{\pi r}{q_n} \right) \sin^2 \left( \frac{\pi r p_n}{q_n} \right)}.$$  

**Proof.** We make use of the formula

$$\sum_{k \in Z} \frac{1}{(k + x)^2} = \frac{\pi^2}{\sin^2(\pi x)} \quad \text{for } x \in \mathbb{R} \setminus \mathbb{Z}.$$  

For $k_1, k_2 \in Z^*$ with $k_1, k_2 \not\equiv 0 \pmod{q_n}$ and $k_1 + k_2 p_n \equiv 0 \pmod{q_n}$ we write $k_1 + k_2 p_n = l q_n$ with $l \in \mathbb{Z}$, and $k_2 = m q_n + r$ for $m \in \mathbb{Z}$ and $r \in \{1, \ldots, q_n - 1\}$. Then

$$\sum_{k_1, k_2 \in Z^*} \frac{1}{k_1^2 k_2^2} = \sum_{k_2 \in Z} \frac{1}{k_2^2} \sum_{l \in \mathbb{Z}} \frac{1}{(l q_n - k_2 p_n)^2}$$

$$= \frac{1}{q_n^2} \sum_{k_2 \not\equiv 0 \pmod{q_n}} \frac{1}{k_2^2} \sum_{l \in \mathbb{Z}} \frac{1}{(l - k_2 p_n q_n)^2}$$

$$= \frac{1}{q_n^2} \sum_{r=1}^{q_n-1} \sum_{m \in \mathbb{Z}} \frac{1}{(m + r q_n)^2} \frac{\pi^2}{\sin^2 \left( \frac{\pi r p_n}{q_n} \right)}$$

$$= \frac{\pi^4 q_n^{-1}}{q_n^4} \sum_{r=1}^{q_n-1} \frac{1}{\sin^2 \left( \frac{\pi r}{q_n} \right) \sin^2 \left( \frac{\pi r p_n}{q_n} \right)}. \quad \square$$

**Proof of Theorem 10.** First we prove the result on the periodic $L_2$ discrepancy of $L_n(\alpha)$. To this end we use the representation of the periodic $L_2$ discrepancy in terms of exponential sums as given in Proposition 3. Writing $L_n(\alpha) = \{x_0, \ldots, x_{q_n-1}\}$, where $x_h = (h q_n, \{h p_n q_n\})$ for $h = 0, 1, \ldots, q_n - 1$,

$$L_{2,q_n}^{\text{per}}(L_n(\alpha))^2 = \frac{1}{9} \sum_{k \in Z^2 \setminus \{0\}} \frac{1}{r(k)^2} \left| \sum_{h=0}^{q_n-1} \exp(2\pi i k \cdot x_h) \right|^2,$$

where the $r(k)$ are defined in [3]. Note that the following arguments are similar to those used in the proof of [4, Theorem 3]. In order to study the sum (13) we need to distinguish different instances for the vector $k$. 


• The case $k = (k, 0), k \neq 0$. Then we have
\[
\sum_{k=(k,0), k \neq 0}^{\infty} \frac{1}{r(k)^2} \left| \sum_{h=0}^{q_n-1} \exp \left( \frac{2\pi i k}{q_n} \frac{h}{q_n} \right) \right|^2 + \sum_{k=-(k,0), k \neq 0}^{\infty} \frac{1}{r(k)^2} \left| \sum_{h=0}^{q_n-1} \exp \left( -\frac{2\pi i k}{q_n} \frac{h}{q_n} \right) \right|^2
\]
\[
= 2 \sum_{k=1}^{\infty} \frac{q_n^2}{r(k)^2} = 2 \frac{6}{4\pi^2} \sum_{l=1}^{\infty} \frac{q_n^2}{(lq_n)^2} = \frac{1}{2},
\]
where we have used the well-known identity
\[
\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}
\]
and the fact that
\[
\sum_{h=0}^{q_n-1} \exp \left( \pm 2\pi i k \frac{h}{q_n} \right) = \begin{cases} q_n & \text{if } k \equiv 0 \pmod{q_n}, \\ 0 & \text{otherwise.} \end{cases}
\]

• The case $k = (0, k), k \neq 0$. This case can be treated analogously to the previous one and yields the same result. One has to use $\gcd(p_n, q_n) = 1$, which is a well-known fact from the theory of continued fractions. Therefore
\[
\sum_{h=0}^{q_n-1} \exp \left( \pm 2\pi i k \frac{hp_n}{q_n} \right) = \begin{cases} q_n & \text{if } k \equiv 0 \pmod{q_n}, \\ 0 & \text{otherwise.} \end{cases}
\]

• The case $k = (k_1, k_2)$, where $k_1, k_2 \neq 0$ and $k_1 \equiv 0 \pmod{q_n}$, but $k_2 \neq 0 \pmod{q_n}$. In this case we find
\[
\sum_{k=(k_1,k_2)\in\mathbb{Z}^2\setminus\{0\}, k_1 \equiv 0 \pmod{q_n}, k_2 \neq 0 \pmod{q_n}} \frac{1}{r(k)^2} \left| \sum_{h=0}^{q_n-1} \exp \left( 2\pi i k_2 \frac{hp_n}{q_n} \right) \right|^2 = 0.
\]

• The case $k = (k_1, k_2)$, where $k_1, k_2 \neq 0$ and $k_2 \equiv 0 \pmod{q_n}$, but $k_1 \neq 0 \pmod{q_n}$, can be treated analogously to the previous one and yields the same result.

• The case $k = (k_1, k_2)$, where $k_1, k_2 \neq 0$ and $k_1 \equiv 0 \pmod{q_n}$ as well as $k_2 \equiv 0 \pmod{q_n}$. In this case we find
\[
\sum_{k=(k_1,k_2)\in\mathbb{Z}^2\setminus\{0\}, k_1 \equiv 0 \pmod{q_n}, k_2 \equiv 0 \pmod{q_n}} \frac{q_n^2}{r(k)^2} = q_n^2 \left( \frac{6}{4\pi^2} \right)^2 \sum_{l_1,l_2\in\mathbb{Z}^*} \frac{1}{(q_n l_1)^2 (q_n l_2)^2}
\]
\[
= \frac{1}{q_n^2} \left( \frac{6}{4\pi^2} \right)^2 \left( \frac{\pi^2}{6} \right)^2 = \frac{1}{4q_n^2}.
\]
- The case $k = (k_1, k_2)$, where $k_1, k_2 \neq 0$ and $k_1 \not\equiv 0 \pmod{q_n}$ as well as $k_2 \not\equiv 0 \pmod{q_n}$. In this case we have to evaluate the sum

$$q_n^2 \sum_{k_1, k_2 \in \mathbb{Z}^*} \frac{1}{r(k)^2},$$

which equals

$$q_n^2 \left( \frac{6}{4\pi^2} \right)^2 \sum_{k_1, k_2 \not\equiv 0 \pmod{q_n}} \frac{1}{k_1^2 k_2^2} = \frac{9q_n^2}{4q_n^2} \sum_{r=1}^{q_n-1} \frac{1}{\sin^2 \left( \frac{\pi r}{q_n} \right) \sin^2 \left( \frac{\pi r p_n}{q_n} \right)}$$

by Lemma 18.

The result on $L_{2,q_n}^\text{ber} (\mathcal{L}_n(\alpha))^2$ follows.

Finally, it remains to prove the result for the extreme $L_2$ discrepancy of $\mathcal{L}_n(\alpha)$. Recall from Remark 14 that for $\mathcal{P} = \{(x_h, y_h) : h = 0, 1, \ldots, N-1\}$,

$$(14) \quad L_{2,N}^{\text{extr}}(\mathcal{P})^2 = \frac{N^2}{144} - \frac{N}{2} \sum_{h=0}^{N-1} f(x_h)f(y_h) + \frac{1}{4} \sum_{h,l=0}^{N-1} g(x_h, x_l)g(y_h, y_l),$$

where $f(x) := x(1-x)$ and $g(x,y) := x + y - 2xy - |x-y|$. We compute the Fourier series of these two functions. For $k, k_1, k_2 \in \mathbb{Z}$ let

$$\hat{f}(k) = \int_0^1 f(x) \exp(-2\pi i k x) \, dx,$$

$$\hat{g}(k_1, k_2) = \int_0^1 \int_0^1 g(x,y) \exp(-2\pi i (k_1 x + k_2 y)) \, dx \, dy.$$

It is not difficult to find that $\hat{f}(0) = \frac{1}{6}$ and $\hat{f}(k) = -\frac{1}{2\pi^2 k^2}$ for $k \in \mathbb{Z}^*$. Therefore

$$f(x) = \frac{1}{6} - \sum_{k \in \mathbb{Z}^*} \frac{\exp(-2\pi i k x)}{2\pi^2 k^2} = \sum_{k \in \mathbb{Z}^*} \frac{1 - \exp(-2\pi i k x)}{2\pi^2 k^2}.$$

For the function $g$ we find

$$\hat{g}(k_1, k_2) = \begin{cases} 
\frac{1}{6} & \text{if } k_1 = k_2 = 0, \\
-\frac{1}{2\pi^2 k_1^2} & \text{if } k_1 \in \mathbb{Z}^* \text{ and } k_2 = 0, \\
-\frac{1}{2\pi^2 k_2^2} & \text{if } k_1 = 0 \text{ and } k_2 \in \mathbb{Z}^*, \\
\frac{1}{2\pi^2 k_1^2} & \text{if } k_1 \in \mathbb{Z}^* \text{ and } k_2 = -k_1, \\
0 & \text{otherwise.}
\end{cases}$$
Extreme and periodic $L_2$ discrepancy

Therefore

$$g(x, y) = \frac{1}{6} - \sum_{k_1 \in \mathbb{Z}^*} \frac{\exp(-2\pi i k_1 x)}{2\pi^2 k_1^2} - \sum_{k_2 \in \mathbb{Z}^*} \frac{\exp(-2\pi i k_2 y)}{2\pi^2 k_2^2}$$

$$+ \sum_{k_1 \in \mathbb{Z}^*} \frac{\exp(-2\pi i k_1 x) \exp(2\pi i k_1 y)}{2\pi^2 k_1^2}$$

$$= \sum_{k \in \mathbb{Z}^*} \frac{1}{2\pi^2 k^2} - \sum_{k \in \mathbb{Z}^*} \frac{\exp(-2\pi i k x)}{2\pi^2 k^2} - \sum_{k \in \mathbb{Z}^*} \frac{\exp(2\pi i k y)}{2\pi^2 k^2}$$

$$+ \sum_{k \in \mathbb{Z}^*} \frac{\exp(-2\pi i k x) \exp(2\pi i k y)}{2\pi^2 k^2}$$

$$= \sum_{k \in \mathbb{Z}^*} \frac{(1 - \exp(-2\pi i k x))(1 - \exp(2\pi i k y))}{2\pi^2 k^2}.$$ 

We insert the Fourier expansions of $f$ and $g$ into (14) to obtain, after some simplifications,

$$L_{2, N}^\text{extr}(P)^2 = \frac{N^2}{144}$$

$$- \frac{N}{2} \sum_{k_1, k_2 \in \mathbb{Z}^*} \frac{1}{4\pi^4 k_1^2 k_2^2} \sum_{h=0}^{N-1} (1 - \exp(-2\pi i k_1 x_h))(1 - \exp(-2\pi i k_2 y_h))$$

$$+ \frac{1}{4} \sum_{k_1, k_2 \in \mathbb{Z}^*} \frac{1}{4\pi^4 k_1^2 k_2^2} \sum_{h=0}^{N-1} (1 - \exp(-2\pi i k_1 x_h))(1 - \exp(-2\pi i k_2 y_h))^2.$$ 

In order to find the exact formula for $L_{2, q_n}^\text{extr}(\mathcal{L}_n(\alpha))$, we need to investigate the expression

$$\Sigma_{k_1, k_2} := \sum_{h=0}^{q_n-1} \left(1 - \exp\left(-2\pi i k_1 \frac{h}{q_n}\right)\right)\left(1 - \exp\left(-2\pi i k_2 \frac{hp_n}{q_n}\right)\right)$$

for non-zero integers $k_1$ and $k_2$. We observe that $\Sigma_{k_1, k_2}$ can have the following values:

$$\Sigma_{k_1, k_2} = \begin{cases} q_n & \text{if } k_1, k_2 \not\equiv 0 \pmod{q_n} \text{ and } k_1 + k_2 p_n \not\equiv 0 \pmod{q_n}, \\ 2q_n & \text{if } k_1, k_2 \not\equiv 0 \pmod{q_n} \text{ and } k_1 + k_2 p_n \equiv 0 \pmod{q_n}, \\ 0 & \text{otherwise.} \end{cases}$$

This leads to
\[ L^\text{extr}_2(L_n(\alpha))^2 = \frac{q_n^2}{144} \]

\[ - \frac{q_n}{2} \left( \sum_{\substack{k_1, k_2 \in \mathbb{Z}^* \setminus \{0\} \mid k_1 + k_2 p_n \equiv 0 \mod q_n}} \frac{q_n}{4\pi^2 k_1^2 k_2^2} + \sum_{\substack{k_1, k_2 \in \mathbb{Z}^* \setminus \{0\} \mid k_1 + k_2 p_n = 0 \mod q_n}} \frac{2q_n}{4\pi^2 k_1^2 k_2^2} \right) \]

\[ + \frac{1}{4} \left( \sum_{\substack{k_1, k_2 \in \mathbb{Z}^* \setminus \{0\} \mid k_1 + k_2 p_n \equiv 0 \mod q_n}} \frac{q_n^2}{4\pi^4 k_1^2 k_2^2} + \sum_{\substack{k_1, k_2 \in \mathbb{Z}^* \setminus \{0\} \mid k_1 + k_2 p_n = 0 \mod q_n}} \frac{4q_n^2}{4\pi^4 k_1^2 k_2^2} \right) \]

\[ = \frac{q_n^2}{144} - \frac{q_n^4}{16\pi^4} \sum_{\substack{k_1, k_2 \in \mathbb{Z}^* \setminus \{0\} \mid k_1 + k_2 p_n \equiv 0 \mod q_n}} \frac{1}{k_1^2 k_2^2} \]

We have

\[ (15) \sum_{\substack{k_1, k_2 \in \mathbb{Z}^* \setminus \{0\} \mid k_1 + k_2 p_n \equiv 0 \mod q_n}} \frac{1}{k_1^2 k_2^2} = \sum_{\substack{k_1, k_2 \in \mathbb{Z}^* \setminus \{0\} \mid k_1 + k_2 p_n \equiv 0 \mod q_n}} \frac{1}{k_1^2 k_2^2} - \sum_{\substack{k_1, k_2 \in \mathbb{Z}^* \setminus \{0\} \mid k_1 + k_2 p_n \equiv 0 \mod q_n}} \frac{1}{k_1 k_2} \]

For the first sum on the right hand side we find

\[ \sum_{\substack{k_1, k_2 \in \mathbb{Z}^* \setminus \{0\} \mid k_1 + k_2 p_n \equiv 0 \mod q_n}} \frac{1}{k_1^2 k_2^2} = \left( \sum_{\substack{k \in \mathbb{Z}^* \setminus \{0\} \mid k \equiv 0 \mod q_n}} \frac{1}{k^2} \right)^2 = \left( \sum_{\substack{k \in \mathbb{Z}^* \setminus \{0\} \mid kq_n \equiv 0 \mod q_n}} \frac{1}{k^2} - \sum_{\substack{k \in \mathbb{Z}^* \setminus \{0\} \mid k \equiv 0 \mod q_n}} \frac{1}{kq_n} \right)^2 \]

\[ = \frac{\pi^4}{9} \left( 1 - \frac{1}{q_n^2} \right)^2 \]

The value of the second sum in (15) is known by Lemma 18. Now the result follows.

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