On the Least counterexample to Robin hypothesis

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Abstract

Let $G(n) = \frac{\sigma(n)}{n \log \log n}$. Robin made hypothesis that $G(n) < e^\gamma$ for all integer $n > 5040$. If Robin hypothesis fails, there will be a least counterexample. This article collects the requirements the least counterexample should satisfy.

Introduction and notations

Robin made a hypothesis [Robin 1984] that the Robin’s inequality

$$\sigma(n) < e^\gamma n \log \log n,$$

holds for all integers $n > 5040$. Here $\sigma(n) = \sum_{d|n} d$ is the divisor sum function, $\gamma$ is the Euler-Mascheroni constant, $\log$ is the nature logarithm.

For calculation convenience, we define

$$\rho(n) := \frac{\sigma(n)}{n}, \quad G(n) := \frac{\rho(n)}{\log \log n}.$$  (1)
Then Robin inequality can also be written as

\[ \rho(n) < e^{\gamma \log \log n}, \quad n > 5040. \]  

(RI)

and as

\[ G(n) < e^{\gamma}, \quad n > 5040. \]  

(RI)

If Robin hypothesis fails, there will be a least counterexample. This article collects the conditions the least counterexample should satisfy. Theorems A - F are known properties of the least counterexample.

Let \( n > 5040 \) be an integer. Write the factorization of \( n \) as

\[ n = \prod_{i=1}^{r} p_i^{a_i}, \]  

(2)

where \( p_i \) is the \( i \)-th prime, \( p_r \) is the largest prime divisor of \( n \). Theorems 1 and 2 prove that for the least counterexample \( n \), we have

\[ p_r < \log n < p_r \left( 1 + \frac{0.005589}{\log p_r} \right). \]  

(3)

Theorems 3 and 4 show that for the least counterexample \( n \), we have

\[ \left\lfloor \frac{\log p_r}{\log p_i} \right\rfloor \leq a_i \leq \left\lfloor \frac{\log(a_i \log n)}{\log p_i} \right\rfloor. \]  

(4)

**Theorem A.** Let \( n \) be the least counterexample to Robin hypothesis. Then \( n \) is superabundant.

*Proof.* See [AF 2009] Theorem 3.

**Theorem B.** The following properties of superabundant numbers are satisfied by the least counterexample to Robin hypothesis. Notations as in (2).

1) \( a_i \geq a_j \) if \( i < j \).
2) \( \left| a_j - \left[ a_j \frac{\log p_j}{\log p_i} \right] \right| \leq 1 \) if \( i < j \).
3) \( a_r = 1 \) except for \( n = 4 \) and \( n = 36 \) in which cases we have \( a_r = 2 \).
4) \( p_i^{a_i} < 2^{a_i+2} \) for \( 2 \leq i \leq r \).
5) \( a_i \geq \left\lfloor \frac{\log p_r}{\log p_i} \right\rfloor \) for \( 2 \leq i \leq r \).
6) Define $\epsilon(x) := \frac{1}{\log x} \left( 1 + \frac{3}{2 \log x} \right)$ and write $\phi(n)$ for Euler totient function. Then

$$\frac{\sigma(n)}{n} > (1 - \epsilon(p_r)) \frac{n}{\phi(n)}.$$

Proof. See [Broughan 2017] Section 6.2.

Theorem C. Let $n$, $5040 < n \leq 10^{(10^{13.099})}$, be an integer. Then $n$ satisfies (RI). Hence the least counterexample to (RI) must be greater than $10^{(10^{13.099})}$.

Proof. See [MP 2018] Theorem 5.

Theorem D. Let $n$ be the least counterexample to (RI). Define

$$M(k) := e^{-\gamma} f(N_k) - \log N_k,$$

where $N_k$ is the product of first $k$ primes and $f(n) = \prod_{p \mid n} \frac{p}{p-1}$. Then

1) $r > 969672728$,
2) $\# \{i \leq r; a_i \neq 1\} < \frac{r}{14}$,
3) $e^{-1/\log p_r} < \frac{p_r}{\log n} < 1$,
4) for all $1 < i \leq r$,

$$p_i^{a_i} < \min \left( 2^{a_i + 2}, p_i e^{M(r)} \right).$$

Proof. See [Vojak 2020] Theorem 1.6.

Theorem E. Let $n = \prod_{i=1}^{r} p_i^{a_i}$ be the least counterexample to (RI). Then $a_1 > 19$, $a_2 > 12$, $a_3 > 7$, $a_4 > 6$ and $a_5 > 5$.

Proof. See [Hertlein 2016] Theorems 1 and 2.

Theorem F. If $n > 5040$ is a sum of two squares, then $n$ satisfies (RI).

Proof. See [BHMN 2008] Theorem 2.

Theorem 1. Let $n$ be the least counterexample to (RI). Then $\log n > p_r$. 


Proof. Write $p := p_r$. By Theorem A, we know $n$ is superabundant, so the exponent of $p$ in $n$ is 1. Assume, to the contrary, $\log n \leq p$, we have

$$\frac{G(n)}{G(n/p)} = \frac{\rho(n) \log \log (n/p)}{\rho(n/p) \log \log n}$$

$$= \left(1 + \frac{1}{p}\right) \frac{\log (\log n - \log p)}{\log \log n}$$

$$= \left(1 + \frac{1}{p}\right) \frac{\log \log n + \log \left(1 - \frac{\log p}{\log n}\right)}{\log \log n}$$

$$< \left(1 + \frac{1}{p}\right) \left(1 - \frac{\log p}{\log n \log \log n}\right)$$

$$= 1 + \frac{\log n \log \log n - p \log p - \log p}{p \log n \log \log n} < 1.$$ \hfill (1.1)

That is, $G(n) < G(n/p)$, which means $n/p$ is also a counterexample of (RI). This contradicts to the minimality of $n$. \hfill \Box

**Theorem 2.** let $N > 10^{10^{13}}$ be an integer. If

$$p_r \leq (\log N) \left(1 - \frac{0.005587}{\log \log N}\right),$$ \hfill (2.1)

or Conversely,

$$\log N > p_r \left(1 + \frac{0.005589}{\log p_r}\right).$$ \hfill (2.2)

Then $N$ satisfies (RI).

Hence the least counterexample $n$ of (RI) satisfies $\log n \leq p_r \left(1 + \frac{0.005589}{\log p_r}\right)$.

**Proof.** The proof is almost identical to Theorem 9 of [Wu 2019]. \hfill \Box

**Definition 1.** Let $n$ be the least counterexample of (RI). Define

$$x_k = (k \log n)^{1/k}, \quad k = 1, 2, \ldots, \text{until } x_k < 2$$ \hfill (D1.1)

Then define a function

$$U_n(p_i) := \left\lfloor \frac{\log (k \log n)}{\log p_i} \right\rfloor, \quad \text{when } x_{k+1} < p_i \leq x_k.$$ \hfill (D1.2)

Since $p_r < \log n$, $U_n(p_i)$ is well defined for all primes $p_i \leq p_r$.

**Lemma 1.** For $x_{k+1} < p_i \leq x_k$, we have $U_n(p_i) = k$. 

Proof. Since $x_{k+1} < p_i \leq x_k$, we have $\log x_{k+1} < \log p_i \leq \log x_k$. By (D1.1)
\[
\frac{\log((k+1) \log n)}{k+1} < \log p_i \leq \frac{\log(k \log n)}{k},
\]
(L1.1)
The left inequality means
\[
k + 1 > \frac{\log((k+1) \log n)}{\log p_i} \geq \left\lfloor \frac{\log(k \log n)}{\log p_i} \right\rfloor = U_n(p_i).\]
(L1.2)
The right inequality means
\[
k \leq \frac{\log(k \log n)}{\log p_i} \implies k \leq \left\lfloor \frac{\log(k \log n)}{\log p_i} \right\rfloor = U_n(p_i).\]
(L1.3)
So, we must have $k = U_n(p_i)$.

**Theorem 3.** Let $n = \prod_{i=1}^{r} p_i^{a_i} > 10^{10^3}$ be an integer. Assume $p_s > U_n(p_s)$ for some index $s$. Then $G(n) < G(n/p_s)$.
This means that if $n$ is the least counterexample of (RI), then $a_i \leq U_n(p_i)$ for all $i$, $1 \leq i \leq r$.

*Proof.* By definition of $U_n$, we have $x_{k+1} < p_s \leq x_k$, for some $k$. $a_s > U_n(p_s)$ means that
\[
\frac{\log((k+1) \log n)}{k+1} < \log p_s \leq \frac{\log(k \log n)}{k},
\]
(3.1)
and $a_s > U_n(p_s) = k$. Hence $a_s \geq k + 1$, and
\[
\log p_s > \frac{\log((k+1) \log n)}{k+1} \geq \frac{\log(a_s \log n)}{a_s}.
\]
(3.2)
We have $p_s^{a_s} > a_s \log n$, and hence
\[
p_s > (a_s \log n)^{1/a_s}.
\]
(3.3)
Write $n_1 = n/p_s$. It is easy to verify that
\[
\frac{G(n)}{G(n_1)} = \frac{\rho(n) \log \log n_1}{\rho(n_1) \log \log n} < \left(1 + \frac{1}{p_s^{a_s}}\right) \left(1 - \frac{\log p_s}{\log n \log \log n}\right)
\leq \left(1 + \frac{1}{a_s \log n}\right) \left(1 - \frac{\log(a_s \log n)}{a_s \log n \log \log n}\right)
\]
\[
\frac{1}{a_s \log n} - \log a_s - \frac{\log(a_s \log n)}{(a_s \log n)^2 \log \log n}
\]

That is, \( G(n) < G(n/p_s) \).

**Definition 2.** Define

\[
L(p_i) = L_{p_r}(p_i) := \left\lfloor \frac{\log p_r}{\log p_i} \right\rfloor \text{ for } i \leq r. \quad (D2.1)
\]

**Theorem 4.** Let \( n > 10^{10^{13}} \) be an integer. If \( a_s < L(p_s) \) for some index \( s < r \), then \( G(n) < G(np_s/p_r) \).

Hence the least counterexample of (RI) must have \( a_i \geq L(p_i) \) for all \( i \), \( 1 \leq i \leq r \).

**Proof.** As \( n \) being superabundant, we know \( a_r = 1 = L(p_r) \). Define \( n_1 = \frac{p_1}{p_r} n \).

Then \( n_1 < n \). \( a_s < L(p_s) = \left\lfloor \frac{\log p_r}{\log p_s} \right\rfloor \) means \( a_s + 1 \leq \left\lfloor \frac{\log p_r}{\log p_s} \right\rfloor \leq \frac{\log p_r}{\log p_s} \). Hence

\[
p_{s+1}^{a_s+1} \leq p_r \quad \text{and} \quad \log p_s \leq \frac{1}{a_s + 1} \log p_r. \quad (4.1)
\]

It is easy to deduce

\[
\frac{G(n)}{G(n_1)} = \frac{\rho(n) \log \log n_{1}}{\rho(n_1) \log \log n} \leq \left( 1 + \frac{1}{p_r} \right) \left( \frac{p_s(p_s^{a_s} + \cdots + 1)}{p_{s+1}^{a_s+1} + \cdots + 1} \right) \left( \frac{\log n - \log p_r + \log p_s}{\log \log n} \right)
\]

\[
< \left( 1 + \frac{1}{p_r} \right) \left( 1 - \frac{1}{p_{s+1}^{a_s+1} + \cdots + 1} \right) \left( 1 - \frac{\log p_r - \frac{1}{a_{s+1}} \log p_r}{\log n \log \log n} \right)
\]

\[
= \left( 1 + \frac{1}{p_r} \right) \left( 1 - \frac{1}{p_{s+1}^{a_s+1} + \cdots + 1} \right) \left( 1 - \left( \frac{a_s}{a_s + 1} \right) \frac{\log p_r}{\log n \log \log n} \right) \quad (4.2)
\]

By Theorem 2,

\[
\log n \leq p_r \left( 1 + \frac{0.005589}{\log p_r} \right).
\]
Noting \( n > 10^{10^{13}} \), we have
\[
\log n \leq c p_r, \quad c := 1 + \frac{0.005589}{\log(2.3 \times 10^{13})} = 1.000235.
\] (4.3)

Since \( \log(c p_r) < c \log p_r \), (4.2) can be simplified to
\[
\frac{G(n)}{G(n_1)} < \left(1 + \frac{1}{p_r}\right) \left(1 - \frac{1}{p_s^{a_s + 1} + \cdots + 1}\right) \left(1 - \frac{a_s}{a_s + 1}\right) \frac{\log p_r}{(c p_r) \log(c p_r)} < 1.
\] (4.4)

Now we split the proof into two cases.

Case 1) \( a_s = 1 \). We have in this case
\[
1 - \left(\frac{a_s}{a_s + 1}\right) \frac{1}{c^2 p_r} < 1 - \frac{1}{2c^2 p_r} < 1 - \frac{0.49}{p_r},
\] (4.5)
\[
1 - \frac{1}{p_s^2 + p_s + 1} \leq 1 - \frac{4}{7p_s^{a_s + 1}} < 1 - \frac{0.57}{p_r}.
\] (4.6)

Substitute (4.5) and (4.6) into (4.4), we get
\[
\frac{G(n)}{G(n_1)} < \left(1 - \frac{0.49}{p_r}\right) \left(1 - \frac{0.57}{p_r}\right) \left(1 + \frac{1}{p_r}\right) < 1.
\] (4.7)

Hence \( G(n) < G(n_1) \).

Case 2) \( a_s > 1 \). We have
\[
1 - \left(\frac{a_s}{a_s + 1}\right) \frac{1}{c^2 p_r} < 1 - \frac{2}{3c^2 p_r} < 1 - \frac{0.66}{p_r},
\] (4.8)
\[
1 - \frac{1}{p_s^{a_s + 1} + \cdots + 1} \leq 1 - \frac{1}{2p_s^{a_s + 1}} < 1 - \frac{0.50}{p_r}.
\] (4.9)

Substitute (4.8) and (4.9) into (4.4), we get
\[
\frac{G(n)}{G(n_1)} < \left(1 - \frac{0.66}{p_r}\right) \left(1 - \frac{0.50}{p_r}\right) \left(1 + \frac{1}{p_r}\right) < 1.
\] (4.10)

Hence \( G(n) < G(n_1) \).

\[ \square \]

**Theorem 5.** Let \( n = \prod_{i=1}^{\ell} p_i^{a_i} > 10^{10^{13}} \) be the least counterexample of (RI). Let \( s \) be the largest index such that \( a_s \neq 1 \). Then \( p_s < 1.414342 \sqrt{p_r} \).
Proof. Since we are searching for the largest index, we may assume \( a_s = 2 \) and set \( k = 2 \) in the definition of \( U_n(p_s) \). By Theorem 3 we have

\[
2 = a_s \leq U_n(p_s) = \left\lfloor \frac{\log(2 \log n)}{\log p_s} \right\rfloor \leq \frac{\log(2 \log n)}{\log p_s}. \tag{5.1}
\]

By Theorem 2,

\[
p_s^2 \leq 2 \log n \leq 2 p_r \left(1 + \frac{0.005589}{\log p_r}\right). \tag{5.2}
\]

\[\square\]

**Theorem 6.** Let \( n = \prod_{i=1}^r p_i^{a_i} > 10^{(10^{13})} \) be the least counterexample of \((RI)\). Let \( s \) be the largest index such that \( a_s \neq 1 \). Then \( p_s > 0.999999 \sqrt{p_r} \).

Proof. Let integer \( t \) be the index such that \( p_t \) is the prime just below \( \sqrt{p_r} \). Then by theorem 9, we have \( a_t \geq L(p_t) = \left\lfloor \frac{\log p_r}{\log p_t} \right\rfloor = 2 \). By Corollary 5.5 of [Dusart 2018], for all \( x \geq 468991632 \), there exists a prime \( p \) such that

\[
x < p \leq x \left(1 + \frac{1/5000}{(\log x)^2}\right). \tag{6.1}
\]

Hence \( p_s \geq p_t > \left(1 - \frac{1/5000}{\log \sqrt{p_r}}\right) \sqrt{p_r} > 0.999999 \sqrt{p_r}. \tag*{\square}

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