A Quadratic Deformation of the Heisenberg-Weyl
and Quantum Oscillator Enveloping Algebras.

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ABSTRACT

A new 2-parameter quadratic deformation of the quantum oscillator algebra
and its 1-parameter deformed Heisenberg subalgebra are considered. An infinite
dimensional Fock module representation is presented which at roots of unity con-
tains null vectors and so is reducible to a finite dimensional representation. The
cyclic, nilpotent and unitary representations are discussed. Witten’s deformation
of $sl_2$ and some deformed infinite dimensional algebras are constructed from the
$1d$ Heisenberg algebra generators. The deformation of the centreless Virasoro alge-
bra at roots of unity is mentioned. Finally the $SL_q(2)$ symmetry of the deformed
Heisenberg algebra is explicitly constructed.
1. Introduction

In the recent intense study of quantum groups and quantum (enveloping) algebras\(^1,2,3,4\) the \(q\)-analogues of the simple Lie algebra \(sl_2\) and of the non-semisimple quantum oscillator Lie algebra \(h_4\) have played an important rôle. In this paper I consider a new deformation of the quantum oscillator algebra \(h_4\), which is nearer to the spirit of the Woronowicz program in quantum groups than the deformations of \(h_4\) previously considered.

The deformations of \(sl_2\), \(U_q(sl_2)\), have been studied in detail: both the transcendental deformations (\([X_+, X_-] = [N]_q, [N, X_\pm] = \pm X_\pm\)) of Jimbo\(^1\) and Drinfeld\(^2\), and the quadratic deformations of Sklyanin\(^5\), Woronowicz\(^6\), Witten\(^7\) and Fairlie\(^8\) (see section 3). When the deforming parameter is not a root of unit \(y\), these deformations have a representation theory essentially equivalent to that of the Lie algebra \(sl_2\)\(^9,10\). The most interesting aspect of quantum algebras is when the deformation parameter is a root of unity (see later), then their properties are quite different from those of the corresponding Lie algebras.

The Heisenberg algebra \(h_3\) and the quantum oscillator algebra \(h_4\) are undoubtedly two of the most important non-semisimple Lie algebras in modern quantum physics. Deformations of the universal enveloping algebra of the quantum oscillator algebra have also been investigated\(^11-16\). Macfarlane and Biedenharn\(^11\) were the first to discuss the non-linear \(q\)-deformation of the harmonic oscillator algebra in the context of quantum groups, and from two independent \(q\)-oscillators they realised a Jimbo-Drinfeld-deformation of \(sl_2\). I call it the transcendental deformation of \(h_4\); it is generated by \(\{N, a_+, a_-\}\), which satisfy the relations:

\[
[N, a_\pm] = \pm a_\pm \quad a_-a_+ - qa_+a_- = q^{-N}.
\] (1.1)

Chan et al.\(^15\) studied the transcendentally deformed \(su(2)\) algebra (again using the Jordan-Schwinger construction) both classically as a \(q\)-deformed Poisson bracket algebra and in the quantum case as a deformed Lie algebra, emphasising that deformation (\(q\)) and quantisation (\(\hbar\)) are different concepts. Yan\(^14\) presented the Hopf
algebra structure of a different transcendental deformation of the quantum oscillator algebra (constructing its coproduct, coinverse, counit, and so on). Celeghini et al.\textsuperscript{16} have produced another simple deformation of the quantum oscillator algebra as a quantum group with the deformation in the Heisenberg subalgebra \([a_-, a_+] = [e]_q, e\) being central). Also Gelfand and Fairlie\textsuperscript{17} have studied \(q\)-symmetrised polynomial algebras and their central extension using a \(q\)-Heisenberg algebra.

In this paper, a 2-parameter quadratic deformation of the quantum oscillator algebra \(h_4\) (and consequently also a 1-parameter deformation of its Heisenberg subalgebra \(h_3\)) is studied. In the next section the notion of quadratic deformations of enveloping algebras is reviewed. Then in section 3, the deformed quantum oscillator algebra \(U_{q,r}(h_4)\) and deformed Heisenberg subalgebra \(U_r(h_3)\) are presented. Section 4 deals with their representation theory, with emphasis on the possibility of finite dimensional representations. In particular I discuss the cyclic and ‘nilpotent’ algebras and representations. A unitary representation is also presented. Section 5 contains the construction of some algebras from \(U_q(h_3)\) generators, including a 1-parameter deformation of \(h_4\), a quadratically deformed \(su(1,1)\) algebra and some deformed infinite dimensional algebras, including the \(q\)-Witt algebra. Before concluding, I present an \(SL_{s^2}(2)\) symmetry of \(U_s(h_3)\).

2. Quadratic Deformations

The \(q\)-analogues of the simple Lie algebras are all known, though they are still being studied actively. Much less is known about non-semisimple Lie algebras and about their \(q\)-analogues (transcendental or quadratic).

It seems that the transcendental quantum algebras are often surprisingly easy to work with and tend to enjoy a pleasing Hopf algebra structure. On the other hand from the viewpoint of non-commutative geometry quadratically deformed quantum (Lie) algebras are more natural\textsuperscript{18,19,20}. In classical differential geometry it is the commutative \(k\)-algebra of smooth \(k\)-valued functions \(C^\infty(M, k)\) on a smooth
manifold $M$ that are of central importance. $C^\infty(M, k)$ contains in particular the functions, whose restriction to local open sets in $M$, gives $M$ a local coordinate structure. In non-commutative differential geometry, non-commutative algebras take the rôle $C^\infty(M, k)$ had classically.

The simplest example is a (finite) $n$-dimensional vector space $V$ over a field $k$ (of characteristic zero). Let $\{x_i \mid i = 1, \ldots, n\}$ be a basis of $V$ and let $V^*$ be the vector space dual to $V$. A (dual) basis in $V^*$ forms a set of linear coordinates on $V$. Smooth functions on $V$ can be represented as polynomials in these linear coordinates with coefficients in $k$. The set of all such polynomials has a natural structure of a commutative algebra (the function algebra on $V$) and is isomorphic to the symmetric algebra $S(V^*)$ over $V^*$. I will briefly recall the construction of $S(V)$ from the tensor algebra $T(V)$ of $V$, before turning to some more interesting examples. In the basis given for $V$, $T(V)$ is isomorphic to the free unital algebra of formal (non-commuting) polynomials in the generating basis (denoted $k[x_1, \ldots, x_n]$). I will be using this isomorphism in what follows. The (associative) tensor product in $T(V)$ will be denoted by ‘$\otimes$’. In this section I am going to be interested in 2-sided ideals $I$ of $T(V)$ and particularly in the quotient algebras $T(V)/I$. (Remember that for an algebra $A$ with a linear subspace $R$, $I := A \cdot R \cdot A \subseteq A$ is in general a proper 2-sided ideal in $A$, and so $A/I$ is a quotient algebra.) As a first example, consider the symmetrising ideal $I_{sym} \subset T(V)$:

$$I_{sym} := \langle v \otimes w - w \otimes v \mid v, w \in V \rangle := T(V) \otimes \{v \otimes w - w \otimes v \mid v, w \in V\} \otimes T(V) \quad (2.1)$$

which is 2-sided by construction. I will often use the notation $\langle \cdot \rangle$ to denote the 2-sided ideal (without unity) in the appropriate tensor algebra generated by the elements in the angled brackets. The quotient algebra $T(V)/I_{sym}$ defines the symmetric algebra on $V$:

$$S(V) := T(V)/I_{sym} \equiv T(V)/\langle v \otimes w - w \otimes v \mid v, w \in V \rangle \quad (2.2)$$

Similarly $A(V) := T(V)/\langle a \otimes b + b \otimes a \mid a, b \in V \rangle$ defines a realisation of the
antisymmetrised exterior (grassmann) algebra on \(V\), which is \(2^n\) dimensional. From here on I will take \(k\) to be the complex numbers \(\mathbb{C}\).

Recall that a \((n\text{ dimensional})\) quantum vector space \(^{19,20}\) has the relations \(y_iy_j = qy_jy_i\) (or \(q^{-\frac{1}{2}}y_iy_j - q^{\frac{1}{2}}y_jy_i = 0\)) for \(i < j\) \((i, j = 1, \ldots, n \geq 2)\) between the elements of a basis \(\{y_a\mid a = 1, \ldots, n\}\) of its non-commutative coordinate-functions. So it is natural to think of

\[S_q(V) := T(V)/\langle y_i \otimes y_j - qy_j \otimes y_i \mid i < j \rangle\]  

(2.3)

or equivalently \(T(V)/\langle q^{-\frac{1}{2}}y_i \otimes y_j - q^{\frac{1}{2}}y_j \otimes y_i \rangle\) as the non-commutative algebra of functions of a quantum vector space. Although the 2-sided ideal is written in a coordinate dependent way, the defining relations are in fact \(GL_q(V)\)-covariant. The space \(S_q(V)\) has an equivalent interpretation as the algebra of \(q\)-symmetrised polynomials\(^{17}\). In the same way \(Q_\hbar := \mathbb{C}[x, p]/\langle xp - px - \hbar i \rangle\) can heuristically be thought of as the general ‘quantum phase space of a particle in one dimension’ (with ‘non-commutative phase space coordinates’ \(x\) and \(p\)) and \(Q_{q,\hbar} := \mathbb{C}[x, p]/\langle xp - qpx - \hbar i \rangle\) as a ‘deformed quantum phase space’. Here again the \(\mathbb{C}[\cdot]\) symbol means the associative, unital \(\mathbb{C}\)-algebra of formal (non-commuting) polynomials generated freely by the elements inside its brackets. As \(\hbar \to 0\) they reduce to the symmetric algebra \(S(V(2))\) and the quantum vector space \(S_q(V(2))\) respectively \((V(2)\) is a 2-dimensional vector space). Note that this deformation \(Q_{q,\hbar}\) of the Heisenberg algebra \(h_3\) is slightly different from the \(U_q(h_3)\) that I consider in detail later. (1-parameter deformations of bosonic and fermionic quantum mechanical phase space and their symmetries have been studied by Zumino\(^{21}\) in the \(R\)-matrix formalism.) Note the difference between quantisation deformation (\(\hbar\)) and non-commutative deformations \((q)\)\(^{15}\).

Universal enveloping algebras play a crucial rôle in Lie algebra and Lie group theory, specially in their representation theory\(^{22}\). The exponential map from the Lie algebra \(g\) to the Lie group \(G\) gives an (algebraic) embedding \(G \hookrightarrow U(g)\). The universal enveloping algebra \(U(g)\) is the topological dual of the algebra of
continuous (representative) functions on $G$ ($\text{fun}(G)$). The universal enveloping algebra $U(g)$ of a Lie algebra $g$ (with Lie bracket $[\cdot \cdot]$) can be constructed from the tensor algebra of its underlying vector space:

$$U(g) := T(g)/\langle V \otimes W - W \otimes V - [VW] \mid V, W \in g \rangle$$

so that the ideal $\langle \cdot \rangle$ gives $U(g)$ equivalence relations of the form $V \cdot W - W \cdot V = [VW]$. The definition of this (2-sided) ideal can also be written as $\langle (id_2 - P - F)(V \otimes W) \mid V, W \in g \rangle$, with $id_2(a \otimes b) := a \otimes b$ (the identity operator, $id_2 : g \otimes g \mapsto g \otimes g$), $P(a \otimes b) := b \otimes a$ (the ‘flip operator’, $P : g \otimes g \mapsto g \otimes g$) and $F(V \otimes W) := [VW]$ (the structure tensor, $F : g \otimes g \mapsto g$).

In a basis $\{Y^i \mid i = 1, \ldots, n := \dim(g)\}$ of $g$, $T(g)$ can be identified with $\mathbb{C}[Y^1, \ldots, Y^n]$ and the enveloping algebra is constructed as $U(g) := T(g)/\langle Y^i \otimes Y^j - Y^j \otimes Y^i - f^{ij}_k Y^k \mid i, j = 1, \ldots, n \rangle$. Using these structure constants of the undeformed Lie algebra $g$ in this basis ($[Y^i Y^j] = f^{ij}_k Y^k$), I can then construct a quadratic $q$-analogue of its enveloping algebra—an quantum enveloping algebra—with (non-zero) deformation parameters $q(i,j) \in \mathbb{C}^*$:

$$U_q(g) := T(g)/\langle q(i,j) X^i \otimes X^j - q(j,i) X^j \otimes X^i - f^{ij}_k X^k \mid i, j = 1, \ldots, n \rangle. \quad (2.5)$$

Here the $\{X^a \mid a = 1, \ldots, \dim(g)\}$ are the generators of $g$ considered as a vector space and summation over the index $k$ (only) is understood. Symmetry requires that $q(j,i) = (q(i,j))^{-1}$.

As usual $U_q(g) \to U(g)$ in the limit as all the deforming parameters $q(i,j) \to 1$. The reader may wonder how this relates to the theory of non-commutative (covariant) differential geometry developed by Woronowicz and others \cite{18, 19, 20}; so I will just comment that a general quadratic deformation of the above type (2.5) will not be bicovariant and may not even be left covariant, though a number of important examples are both ‘quadratic’ and covariant \cite{6, 23}. Quadratic quantum algebras have the advantage that their representations are rather easier to study.
(A typical left covariant quantum (Lie) algebra has relations \((id \otimes id - R - C)(Z^i \otimes Z^j) = 0\) on its generators \(Z^a\) (i.e. \(Z^i Z^j - R^{ij}_{\; \; kl} Z^k Z^l = C^i_{\; \; k} Z^k\)), with \(R : g \otimes g \to g \otimes g\) and \(C : g \otimes g \to g\) satisfying the braid Yang-Baxter equation \((R_{12} R_{23} R_{12} = R_{23} R_{12} R_{23})\) and the \(R\)-Jacobi identity respectively. The \(C\) and \(R\) tensors must satisfy some additional relations to have bicovariance \(^1\). The undeformed case then corresponds to \(R = P\) and \(C = F\).) Often it is desirable to endow the quadratic quantum algebra \(U_q(g)\) with a non-trivial (non-cocommutative) Hopf algebra structure, but in general this is not possible for arbitrary choices of \(q_{(i,j)}\). When it is possible, the particular \(U_q(g)\) is called a “quantum group”. (From here on, the formal associative ‘\(\otimes\)’ product symbol of \(T(g)\) will either be implicit or denoted by ‘·’, and I will not distinguish the product symbols of \(T(g)\) and \(U_q(g)\).)

For the quantum vector space (2.3) the coordinate function algebra \(S(V)\) of a vector space \(V\) was deformed to a non-commutative algebra \(S_q(V)\), making the geometry ‘non-commutative’. So in the same way the non-commutative differential analogues of the Lie algebra of left-invariant vector fields on a Lie group lose the anti-commutativity of Lie brackets, as happens for example in \(U_q(g)\). So relations of the form “\(qXY - q^{-1}YX = A\)” defining the the quantum universal enveloping algebra are more appropriate geometrically than the much used transcendental deformations mentioned above (\([X, Y] = [A]_q\)) which have (rightly) received an enormous amount of attention recently in algebraic contexts. The “\(qXY - q^{-1}YX = A\)” form of deformation is more symmetric than the “\(XY - qYX = A\)” form that is sometimes considered. However unlike in the latter and transcendental deformations, \(q = 0\) is not allowed.
3. The quadratically deformed quantum oscillator algebra

The original (1-parameter) quadratic deformation of the $sl_2$ (complexified $su(2)$) was invented by Woronowicz\(^6\). A similar deformation was found by Witten to occur in the context of vertex models\(^7\) and generalised by Fairlie\(^8,10\) to a 2-parameter deformation of the universal enveloping algebra, denoted $U_{q,r}(sl_2)$. It is generated freely by $\{W_+, W_-, W_0\}$ respecting the relations over $\mathbb{C}$:

$$
[W_0, W_+]_q = W_+
$$
$$
[W_-, W_0]_q = W_-
$$
and $$
[W_+, W_-]_r = W_0,
$$
where the notation $[X, Y]_s := sX \cdot Y - s^{-1}Y \cdot X$ is introduced; $q, r, s \in \mathbb{C}^*$ (the set of non-zero complex numbers). For example the first relation reads $q W_0 \cdot W_+ - q^{-1}W_+ \cdot W_0 = W_+$.

So $U_{q,r}(sl_2) := T(sl_2)/I_{sl_2,qr}$, where $I_{sl_2,qr}$ is the 2-sided ideal in $T(sl_2)$ generated by elements corresponding to the relations in (3.1), i.e. $I_{sl_2,qr} \equiv \langle qW_0W_+ - q^{-1}W_+W_0 - W_+, \ qW_-W_0 - q^{-1}W_0W_- - W_-, \ r^{-1}W_+W_- - rW_-W_+ - W_0 \rangle$. It seems that this 2-parameter deformation is only a Hopf algebra for certain values of $q$ and $r$: when $q := r^2$ or $7 \ r := q^2$. Woronowicz’s deformation for real $q$ corresponds to a left-covariant differential calculus on the quantum group $SU_q(2)$, i.e. the $q$-analogue of the Lie algebra of left-invariant vector fields on $SU(2)$\(^6\).

The 1d quantum oscillator algebra is a non-semisimple Lie algebra $h_4$ which has 4 generators ($n, a_+$, $a_-$ and $e$ which is central). The Lie brackets are:

$$
[na_+] = a_+
$$
$$
[a_-n] = a_-
$$
$$
[a_-a_+] = e
$$
and $$
[xe] = 0, \ \forall x \in h(4).
$$

I will now consider a deformation (similar to the one above) of the 1d quantum oscillator algebra: a 2-parameter deformation of the $h_4$ universal enveloping algebra,
which I denote $U_{q,r}(h_4)$, generated by $\{A_+, A_-, N, E\}$ over $\mathbb{C}$ with the relations:

$$
\begin{align*}
[N, A_+]_q &= A_+ \\
[A_-, N]_q &= A_- \\
[A_-, A_+]_r &= E
\end{align*}
$$

(3.3)

and $[x, E] = 0 \ (\forall x \in U_{q,r}(h_4))$.

In other words $U_{q,r}(h_4) := T(h_4)/I_{h_4,qr}$, where corresponding to (3.3):

$$
I_{h_4,qr} := \langle [N, A_+]_q - A_+, [A_-, N]_q - A_-, [A_-, A_+]_r - E, [x, E] | x \in T(h_4) \rangle;
$$

see (2.1) for explanation of the $\langle \cdot \rangle$ notation. In order to discuss the 1d deformed Heisenberg subalgebra, I also define:

$$
I_{h_3,r} := \langle [A_-, A_+]_r - E, [x, E] | x \in T(h_3) \rangle
\equiv T(h_3) \cdot \mathbb{C}[[A_-, A_+]_r - E, [x, E] | x \in T(h_3)]_0 \cdot T(h_3)
$$

(3.4)

an ideal (without unity) in $T(h_3)$ (the 0 subscript denoting that ‘1’ is not a generator), so I can then construct the quotient algebra $U_r(h_3) := T(h_3)/I_{h_3,r}$. Clearly $T(h_3) \subset T(h_4)$ and $I_{h_3,r} \subset I_{h_4,qr} \subset T(h_4)$, so $U_r(h_3) \subset U_{q,r}(h_4)$.

Remark: $U_{q,r}(h_4)$ is a 2-parameter generalisation of the transcendental oscillator algebra, since $U_{1,q^{-1}}(h_4)$ is equivalent to (1.1) under the identifications $E \equiv 1$ and $A_\pm \equiv a_{\pm q^{1/2}}N^{1/2}$. It is easy to check that the quantum algebra relations (3.3) satisfy the ‘braid-associativity consistency conditions’: that is, it does not matter which way a cubic monomial $X_1X_2X_3$ in the generators of the quantum algebra is re-ordered: the result is the same answer either way. For example in $U_{q,r}(h_4)$ the ‘braidings’

$NA_+A_- = r^2NA_-A_+ - rEN = q^2r^2A_-NA_+ - qr^2A_-A_+ - rEN = r^2A_-A_+N - rEN$ and $NA_+A_- = q^{-2}A_+NA_- + q^{-1}A_+A_- = A_+A_-N = r^2A_-A_+N - rEN$

give the same expression.
4. Finite Representations

The following identities can easily be proved by induction for all positive integer \( m \):

\[
[N, A_+^m]_{q^m} \equiv [m]_q A_+^m
\]
\[
[A_-^m, N]_{q^m} \equiv [m]_q A_-^m
\]
\[
[A_-, A_+^m]_{r^m} \equiv [m]_r E \cdot A_+^{m-1},
\]

where I define the (usual) ‘\( q \)-number’ to be \([m]_q := q^m (m)_q\) and \((m)_s := \sum_{i=1}^{m} s^{1-2i}, \forall m \in \mathbb{Z}^+\). Any deformation parameter is always assumed to be non-zero complex, unless reality is specified. Also \((0)_q := 0\), so \([0]_q \equiv 0\). \((m)_1 \equiv m \equiv [m]_1\) and \([1]_s \equiv 1 \equiv s(1)_s\).

As I am dealing with a deformation of the quantum oscillator algebra, I want to start by considering a Fock-module representation of \( U_{q,r}(h_4) \). A linear representation of an algebra \( A \) on a vector space \( V \) is defined to be a homomorphic linear action \( \cdot \) of the algebra on \( V \) (i.e. a homomorphism \( A \to \text{End}(V) \)). For the vector space of my representation I define \( V_{q,r}(j,c) := \bigoplus_{n=0}^{\infty} \mathbb{C} u_n \), such that \( \{u_n \mid n \in \mathbb{N}\} \) is a basis of \( V_{q,r}(j,c) \) (\( \mathbb{N} \) is the set of natural numbers) and \( u_0 \) is a vacuum vector (lowest vector):

\[
N \cdot u_0 = j u_0
\]
\[
A_- \cdot u_0 = 0
\]
\[
E \cdot u_0 = c u_0.
\]

\( j \) and \( c \neq 0 \) are numbers. It is natural to associate \( c \) with \( \hbar \) (Planck’s constant). The following consistent left-action of \( U_{q,r}(h_4) \) then makes \( V_{q,r}(j,c) \) a left \( U_{q,r}(h_4) \)-module (4.1):

\[
A_+ \cdot u_k = u_{k+1}
\]
\[
A_- \cdot u_k = c(k)_r u_{k-1}
\]
\[
N \cdot u_k = (q^{-2k} j + (k)_q) u_k
\]
\[
E \cdot u_k = c u_k \quad k \in \mathbb{N}
\]

So \( u_k = A_+^k \cdot u_0 \). The representations at different \( j \) and different \( c \) are generically
inequivalent. Remark: the defining relations (3.3) of $U_{q,r}(h_4)$ are invariant under the automorphism $A_+ \mapsto -A_+$, $A_- \mapsto A_-$, $N \mapsto N$ and $E \mapsto -E$, so another representation exists which is inequivalent to (4.3): $A_+ \cdot u_k := -u_{k+1}$, $A_- \cdot u_k := c(k)ru_{k-1}$ and $E \cdot u_k := -cu_k$ ($N$ unchanged). Matrix representations of $U_{q,r}(h_4)$ can also be constructed. From here on I will concentrate on the $j = 0$ representations, the natural vacuum representations, defining $V^{(c)}_{q,r} := V^{(j=0,c)}_{q,r}$.

Since $U_r(h_3) \subset U_{q,r}(h_4)$, the restriction of the above representation of $U_{q,r}(h_4)$ to the subalgebra $U_r(h_3)$ induces a representation of it on the same space; I denote the $U_r(h_3)$-submodule by $V^{(c)}_r \equiv U_r(h_3) \cdot u_0$.

For a simple Lie algebra $g$ and a generic value of $q$, the centre of $U_q(g)$ is just generated by the $q$-analogues of the (universal) Casimirs. However for the deformation parameter at a root of unity (denoted $q_p$), this is no longer true and the representation theory changes drastically. Basically this is because the centre of $U_{q_p}(g)$ is larger than that of $U_q(g)$. This is also the case for $q$-analogues of non-semisimple Lie algebras: the centres are also enlarged at roots of unity.

Specifically: when $r$ is a non-trivial $2n$-th root of unity $r_n (n > 1)$, i.e. $(r_n)^{2n} = 1$ and $r_n \neq 1$, $A_+^n$ and $A_-^n$ are additional generators of the centre of $U_{r_n}(h_3)$, as can be seen from (4.1) using $[n]_{r_n} \equiv 0$. If additionally $q$ is a non-trivial $2n$-th root of unity $q_n$ then $A_+^n$ and $A_-^n$ lie in the centre of $U_{q_n,r_n}(h_4)$. Remark: when $q = r$ there exists a 2-Casimir of $U_q(h_4)$, $K := A_+ \cdot A_- - E \cdot N$, which coincides in form with the Casimir of $h_4$.

In the following I want to concentrate on the case when $r$ is a primitive $2p$-th root of unity $r_p (p > 1)$

$$r = r_p \quad r_p := e^{i\pi/p} \quad (4.4)$$

Then considering the representation of $U_{q,r_p}(h_4)$ on $V^{(c)}_{q,r_p}$ (and at the same time the representation of $U_{r_p}(h_3)$ on $V^{(c)}_{r_p}$), it is seen that $u_p$ has become a null vector (or singular vector) in the representation:

$$A_- \cdot u_p = c(p)r_{p}u_{p-1} \equiv 0. \quad (4.5)$$
since \((p)_{r_p} \equiv 0\). This can also be seen as \(A_- \cdot u_p = A_- \cdot A_+^p \cdot u_0 = A_+^p \cdot A_- \cdot u_0 \equiv 0\) (4.2). In fact \(\{u_{kp} \mid k \in \mathbb{Z}^+\}\) are all null vectors. Consequently the infinite dimensional representation on \(V^{(c)}_{q,r_p} (V^{(c)}_{r_p})\) is reducible to a finite dimensional one. Note however that these representations are not completely reducible. If I define \(B_+ := \{A_+^n \mid n = 1, \ldots, \infty\}\), a (trivial) subalgebra of \(U_{q,r_p}(h_4) (U_{r_p}(h_3))\), the set \(B_+ \cdot u_p\) generates a subspace of \(V^{(c)}_{q,r_p} (V^{(c)}_{r_p})\). Then the space \(T_q^{(p)} := V^{(c)}_{q,r_p} / \text{Span}(B_+ \cdot u_p)\) \((T(p) := V^{(c)}_{r_p} / \text{Span}(B_+ \cdot u_p))\) carries a quotient module representation of \(U_{q,r_p}(h_4) (U_{r_p}(h_3))\), which is (finite) \(p\)-dimensional and irreducible. In other words the quotient representation is constructed by identifying \(u_p \in V^{(c)}_{q,r_p} (V^{(c)}_{r_p})\) with 0, \(u_p = 0\), as is normally done with null vectors. So \(T_q^{(p)} = \sum_{n=0}^{p-1} C u_n = T^{(p)}\). Remark: in this case the matrix representations mentioned earlier can also be reduced to a finite \(p \times p\) dimensional matrix representation.

I will briefly discuss the reducibility of these quotient representations at a general (non-primitive) \(2p\)-th root of unity. Then \(r\) can take the values \(r = (r_p)^n = e^{n\pi i/p} (n = 1, \ldots, p - 1)\). When \(p\) is a prime number, the quotient representations are still irreducible for all \(n = 1, \ldots, p - 1\). If \(p\) is not prime and \(m := \gcd(p, n) = 1\), then the quotient representations are also irreducible. In the final case of \(p\) not prime and \(m = \gcd(p, n) > 1\) the quotient representation is reducible to a \(\frac{p}{m}\) dimensional representation of \(U_{q,r_p/m}(h_4) (U_{r_p/m}(h_3))\), i.e. \(u_{p/m}, u_{2p/m}, \ldots, u_{(m-1)p/m}\) are null vectors in \(T_q^{(p)}\) \((T^{(p)}))\). Here the \(\gcd(\cdot, \cdot)\) function takes the value of the greatest common (integer) divisor of its arguments.

I will just mention that there is also a representation of \(U_{q,r}(h_4) (U_r(h_3))\) on \(\bigoplus_{n \in \mathbb{N}} C u_n\), that treats \(A_+\) and \(A_-\) symmetrically:

\[
A_+ \cdot u_k = ((k + 1)r)^\frac{n}{2} u_{k+1} \quad A_- \cdot u_k = c ((k)r)^\frac{n}{2} u_{k-1},
\]

\((N\) and \(E\) acting as in (4.3)). For \(r\) not at a (non-trivial) root of unity, it is equivalent to (4.3). But when \(r = r_p\) for example, \(A_- \cdot u_{kp} \equiv 0\) and \(A_+ \cdot u_{kp-1} \equiv 0\) (for all \(k \in \mathbb{Z}^+\)), and this representation is therefore completely reducible to a \(p\)-dimensional one.
Since at \((q, r) = (q_p, q_p)\), \(A_+^p\) and \(A_-^p\) are central in \(U_{q_p}(h_4) := U_{q_p, q_p}(h_4)\) \((U_{q_p}(h_3))\), as I mentioned earlier, it is natural to consider realisations with them as complex numbers: say \(\lambda := A_+^p\) and \(\mu := A_-^p\). Since I am concentrating on lowest vector representations, I will just deal with \(\lambda\), taking \(\mu = 0\). \(A_+^p\) and \(A_-^p\) generate a subalgebra of the centre of \(U_{q_p}(h_4)\) \((U_{q_p}(h_3))\), which is an ideal. I define:

\[
U_{q_p}(h_4)^{(\lambda)} := U_{q_p}(h_4) / \langle A_+^p - \lambda, A_-^p \rangle
\]

(with a similar definition of \(U_{q_p}(h_3)^{(\lambda)}\)). There are two particular cases I want to discuss:

(i) In the case \(\lambda \neq 0\), I call the algebra \(U_{q_p}(h_4)^{(\lambda \neq 0)}\) \((U_{q_p}(h_3)^{(\lambda \neq 0)})\) cyclic and its representations correspond to the subset of (lowest vector) representations of \(U_{q_p}(h_4)\) \((U_{q_p}(h_3))\) which are called cyclic\(^{25}\). For example a representation on the \(p\)-dimensional vector space \(T_q^{(p)}\) \((T^{(p)})\), similar to above (4.3) except that \(A_+\) acts as \(A_+ \cdot u_k := u_{(k+1)}\) \((k = 0, 1, \ldots, p - 2)\) and \(A_+ \cdot u_{p-1} := \lambda u_0\), is cyclic. In particular if \(\lambda = 1\) the action of the subalgebra \(\{A_+^k \mid k = 1, \ldots, p\}\) on the cyclic representation space carries a representation of the cyclic group \(\mathbb{Z}_p\). The cyclic representation space is obtained from \(V_{q_p, r_p}^{(c)} \ (V_{r_p}^{(c)})\) with the identification \(u_k \equiv u_{k \mod p} \ (\forall k \in \mathbb{N})\). **Remark:** I should mention that with \(\mu = A_-^p \neq 0\) the fully cyclic representations can be constructed. They have neither highest nor lowest vectors.

(ii) The second case is \(\lambda = 0\). Then I call the algebra \(U_{q_p}(h_4)^{(0)}\) \((U_{q_p}(h_3)^{(0)})\) ‘nilpotent’. The irreducible lowest vector representations of the nilpotent algebras are also finite dimensional. The finite \(p\)-dimensional representations of \(U_{q_p}(h_4)\) \((U_{q_p}(h_3))\) I mentioned above is an example of what I call a nilpotent representation. It would appear that \(U_{q_p}(h_3)\) plays a significant rôle in parafermionic quantum mechanics. A nilpotent algebra very similar to \(U_{q_p}(h_3)^{(0)}\) has recently been discussed in the context of paragrassmann algebras\(^{27}\).

In the case \(q = 1\) \(U_{1, r_p}(h_4)\) has the finite representation on \(T_1^{(p)}\) described above. This is possible since \(A_+^p\) can still act nilpotently, even though it is not in the
centre of the algebra. Then the \( N \)-eigenvalues become integer, and it is meaningful to call \( N \) the number operator: \( N \cdot u_k = (k)_1 u_k \equiv k u_k \) \((k = 0, 1, \ldots, p - 1)\). The representation of \( U_{1,r_p}(h_4) \) on \( T_{q=1}^{(p)} \) corresponds to a \( p \)-paragrassmann \((p\)-parafermionic) oscillator:

\[
[N, A_+] = A_+ \quad [A_-, N] = A_- \quad [A_-, A_+]_{r_p} = E \quad (4.6)
\]

Of course as \( p \to \infty \), then \( r_p \to 1 \) and the usual infinite dimensional bosonic Fock space representation of \( U(h_4) \) is recovered. The nilpotent algebra \( U_{r_p}(h_4)^{(0)} \) is a \( p \)-paragrassmann algebra, with \( N \) interpreted as a \( q \)-number operator.

To conclude this section, I will discuss a (complex) unitary representation. Note that I now take \( q \) to be real and \( r \) to be a positive real number. As in the case of the transcendental oscillator (1.1) there exists an anti-automorphism \( \omega \) of \( U_{q,r}(h_4) \) \((U_{r}(h_3))\) that maps \( U_{q,r}(h_4) \to U_{q,r}(h_4) \) \((U_{r}(h_3) \to U_{r}(h_3))\):

\[
A_+ \mapsto A_- \quad A_- \mapsto A_+ \quad N \mapsto N \quad E \mapsto E \quad \alpha \mapsto \alpha^* \quad \forall \alpha \in \mathbb{C} \quad (4.7)
\]

preserving the defining relations (3.3); \( \omega(x \cdot y) = \omega(y) \cdot \omega(x) \) \((\forall x, y \in U_{q,r}(h_4)) \) \((U_{r}(h_3)))\). I define the following positive definite sesquilinear scalar product \( \langle \cdot, \cdot \rangle \) on the complex vector space \( V_{q,r}^{(j,c)} \) \((V_{r}^{(c)})\):

\[
(u_k, u_l) := \delta_{k,l} \prod_{m=1}^{k} (m)_r \equiv \delta_{k,l}(k)_r! \quad k, l \in \mathbb{N} \quad (4.8)
\]

which being contravariant with respect to \( \omega \) \((i.e. \ (x \cdot u_k, u_l) = (u_k, \omega(x) \cdot u_l)\) and \((u_k, y \cdot u_l) = (\omega(y) \cdot u_k, u_l) \forall x, y \in U_{q,r}(h_4) \) \((U_{r}(h_3))\) and \( k, l \in \mathbb{N} \)), affords a unitary representation of \( U_{q,r}(h_4) \) \((U_{r}(h_3))\). I define \((k)_r! := (k)_r(k-1)_r \ldots (1)_r \) \((k \in \mathbb{Z}^+)\) and \((0)_r! := 1 \). The basis can be normalised as \( u_k' := \frac{1}{((k)_r!)^{1/2}} u_k \) \((k \in \mathbb{N})\), so that \( (u_k', u_l') = \delta_{k,l} \). There are two reasons why this unitary representation unfortunately cannot be extended to the case at roots of unity: when \( q \) and \( r \) are not real but
treated as complex numbers (i) the map \( \omega \) is no longer an anti-automorphism and (ii) the scalar product is no longer positive definitive: \( (u_k, u_k) = (k)_r \) is not positive real.

5. Construction of other algebras

In this section I will construct some well known finite and infinite dimensional algebras from the generators of \( U_s(h_3) \), which I denote in this section by \( \{a_+, a_-, e\} \) (not to be confused with (3.2)). I will also briefly present the contraction of \( U_{1,r}(sl_2) \) to \( U_r(h_3) \). First I repeat the definition of \( U_s(h_3) \) for completeness.

The defining relations which generate \( U_s(h_3) \) are

\[
\begin{align*}
[a_-, a_+]_s &= e, \\
[e, a_\pm] &= 0,
\end{align*}
\] (5.1)

where \( U_s(h_3) := T(h_3)/I_{h_3,s} \), where the ideal \( I_{h_3,s} \) defined earlier (3.4), corresponds to the relations of (5.1). In this section it will be useful to identify \( e \) with a scalar (i.e. a multiple of unity) in \( U_s^0 h_3 = \mathbb{C} \), (as is normally done for central terms). So I choose ‘\( e = 1 \)’ and work with the realisation \( U_s(h_3)/\{e - 1\} \). Actually all the constructions can be made from \( U_s(h_3) \), but then factors of ‘\( e \)’ appear regularly on the right hand sides.

It is easy to check that \( \{a_+, a_-, M\} \) with \( M := a_+ \cdot a_- \), satisfy the generator relations isomorphic to those of \( U_s(h_4) := U_{s,s}(h_4) \):

\[
\begin{align*}
[M, a_+]_s &= a_+ \\
[a_-, M]_s &= a_- \\
[a_-, a_+]_s &= 1(\equiv e),
\end{align*}
\] (5.2)

so \( U_s(h_4) \) is a subalgebra of \( U_s(h_3) \). \textbf{Remark}: it is equally good to define \( M \) as
\( \frac{1}{2} (a_+ \cdot a_- + a_- \cdot a_+ + \alpha e), \alpha \) a complex number. Next defining:

\[
\begin{align*}
B_+ &:= a_+ \cdot a_+ & B_- &:= a_- \cdot a_- \\
B_0 &:= \frac{1}{2} (s^2 a_- \cdot a_+ + s^{-2} a_+ \cdot a_-)
\end{align*}
\]  
(5.3)

I obtain the following realisation of a deformation of \( su(1,1) \), provided \( s \notin \{ e^{\frac{\pi}{4}}, e^{\frac{3\pi}{4}}, e^{\frac{5\pi}{4}} \} \):

\[
[ B_0, B_+ ]_{s^2} = \frac{1}{2} [2]_s^2 [2]_s B_+ & & [ B_-, B_0 ]_{s^2} = \frac{1}{2} [2]_s^2 [2]_s B_- \\
[ B_-, B_+ ]_{s^4} = 2 [2]_s B_0
\]
(5.4)

and so \( U_{s^2} su(1,1) \subset U_s(h_3) \). If I define:

\[
W_0 := 2 ( [2]_s^2 [2]_s)^{-1} B_0 & & W_\pm := \pm \left( ( [2]_s^2 )^\frac{1}{2} [2]_s \right)^{-1} B_\pm
\]
(5.5)

then it is easy to check that \( \{ W_0, W_+, W_- \} \) satisfy the defining relations (3.1) of Witten’s deformation of \( su(2) \), \( U_{s^2} (sl_2) \equiv U_{q=s^2,r=s^4} (sl_2) \), contained in \( U_s(h_3) \) as a subalgebra. Since \( sl_2 \simeq sp_2 \), I am also free to call this a deformation of the Lie algebra \( sp_2 \).

Next I mention that elements \( \{ a_-, a_+^k \mid k \in \mathbb{N} \} \) of \( U_s(h_3) \) generate a subalgebra with \( [a_-, a_+^k]_{s^k} = [k]_s a_+^{k-1} \). This subalgebra has a natural interpretation as the algebra of polynomials in one variable with a \( q \)-derivation and then \( U_s(h_3) \) is the algebra of polynomials and \( q \)-differential operators. When \( s = s_p \) (4.4), then repeating the methods of section 4, the cyclic and nilpotent forms of this algebras can be studied.

To my knowledge, contractions of quantum groups were first performed in ref. 28, where the transcendental deformation of \( sl_2 \) was contracted to the Heisenberg and Euclidean algebras. Here I show that this is also possible with the \((q=1)\) quadratic deformation of \( sl_2 \), \( U_{1,r} (sl_2) \) (3.1). I scale the generators: \( W_0 \rightarrow X_0 := \)
\[ \xi W_0 \text{ and } W_\pm \to X_\pm := \xi \hat{\xi} W_\pm \ (\xi \geq 0). \] Then in the limit \( \xi \to 0 \), I find that the new ‘contracted’ algebra has \( X_0 \) central:

\[
[X_\pm, X_0] = 0 \\
[X_+, X_-] = X_0. 
\] (5.6)

These are the defining relations of \( U_r(h_3) \). So I have contracted \( U_{1,r}(sl_2) \) to \( U_r(h_3) \).

I can also scale the generators like this: \( W_\pm \to \xi^{-1} W_\pm \) and \( W_0 \to W_0 \), and in the limit as \( \xi \to 0 \) the contracted algebra becomes a quadratic deformation of the Euclidean algebra: \( [W_0, W_+]_s = W_+, [W_-, W_0]_s = W_- \) and \( [W_+, W_-] = 0 \).

Recently Kassel has found a \( q \)-analogue of the Virasoro algebra 2-cocycle\(^{29}\), i.e. a general central extension for the \( q \)-Virasoro algebra (he also briefly discusses the \( q \)-Heisenberg relation). Here I construct a \( q \)-deformation of the Witt algebra \( Witt \) (the (centreless Virasoro) Lie algebra of vector fields on the circle). It is simpler than that of ref. 12 which involves the \( N \) of the transcendental \( q \)-oscillator algebra (1.1). My deformation is similar to the centreless one in ref. 29. Incidentally

The set \( \{ L_{-1} := a_-, L_0 := a_+ \cdot a_-, L_1 := (a_+)^2 \cdot a_- \} \) generates a deformation of the usual \( su(1,1) \) subalgebra of \( Witt \):

\[
[L_1, L_0]_{s^{-1}} = L_0 \quad [L_1, L_{-1}]_{s^{-2}} = [2]_s L_0 \quad [L_0, L_{-1}]_{s^{-1}} = L_{-1}.
\]

The set \( \{ L_m := -(a_+)^{1+m} \cdot a_- \mid m = -1, 0, 1, 2, \ldots \} \) generates a deformation of the ‘constraint subalgebra’ of the full \( s \)-Virasoro algebra, which I construct next. I formally extend \( U_s(h_3) \) to an algebra \( (U_s(h_3)') \) additionally generated by \( a_{+1} \), with the extra relations:

\[
a_{+1} \cdot a_+ = 1 = a_+ \cdot a_{+1} \\
[a_{+1}, a_-]_s = e \cdot a_{+1}^1 \cdot a_{+1}^1 \\
[a_{+1}, e] = 0. 
\] (5.7)

then \( L_m := -(a_+)^{1+m} \cdot a_- \ (m \in \mathbb{Z}) \) generates a \( q \)-deformation of \( Witt \) realised in
$U_s(h_3')$:

$$[L_m, L_n]_{s^{n-m}} = [m - n]_s L_{m+n}.$$  \hspace{1cm} (5.8)

Here $(k)_q$ and $[k]_q$ are extended to non-positive integers: $(-k)_q = -q^{-2k}(k)_q$ and $[-k]_q = -[k]_q$, $\forall k \in \mathbb{Z}^+$. \textit{Remark:} it is also possible to construct a centreless $q$-deformed $w_\infty$ algebra$^{12}$ ($W_n^{(k+1)} := a_{+n+k} \cdot a_{-k}$, $k \in \mathbb{Z}^+$), as was done in ref. 17 using a $q$-Heisenberg algebra.

Considering this construction of $s_p$-Witt instead with the cyclic algebra of $U_s(h_3)^{(\lambda=1)}$, then the $s_p$-Witt algebra has the relations: $L_{m+kp}^{cycl} \equiv L_m^{cycl}$ ($k \in \mathbb{Z}$ and $m \in \mathbb{N}$), and $L_{-m} := L_{p-m}$ ($m \in \mathbb{N}$). So this deformation of $s_p$-Witt realised in $U_s(h_3)$ has $p$ generators $\{L_0^{cycl}, L_1^{cycl}, \ldots, L_{p-1}^{cycl}\}$. It might be interesting to study this algebra in more detail. \textit{Note added:} after finishing this work I became aware that cyclic representations of a different cyclic $q$-Witt algebra have been considered previously$^{30}$.

6. The Symmetry of $U_s(h_3)$

Consider the matrix:

$$T := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with the following relations between its non-commuting elements ($q \in \mathbb{C}^*$, a non-zero complex number):

$$ac = qca \quad ad - qcb = 1$$

$$bd = qdb \quad da - q^{-1}bc = 1$$

(6.1)

which are in fact a deformation of the defining relations of the symplectic matrix group $Sp(2n)$ in the case $n=1$ $^{31}$. I call this quantum algebra $Sp'_q(2) := \mathbb{C}[a, b, c, d]/\langle ac - qca, \ bd - qdb, \ ad - qcb - 1, \ da - q^{-1}bc - 1 \rangle$. It contains the
well-studied quantum group $SL_q(2)$ as a subalgebra, since $SL_q(2)$’s defining relations include (6.1). Its additional relations are:

$$bc = cb \quad ab = qba \quad cd = qdc.$$  \hspace{1cm} (6.2)

It is easy to show that $Sp^{'s}_2(2)$ is a symmetry of the ‘deformed quantum phase space’ $U_s(h_3)$: the form of the defining relation $[A_-, A_+]_s = E$ is left-covariant with respect to the left co-action $(U_s(h_3) \to Sp^{'s}_2(2) \otimes U_s(h_3))$:

$$\begin{pmatrix} A_- \\ A_+ \end{pmatrix} \mapsto \begin{pmatrix} A'_- \\ A'_+ \end{pmatrix} := T \otimes \begin{pmatrix} A_- \\ A_+ \end{pmatrix} \quad E \mapsto E' := 1 \otimes E$$ \hspace{1cm} (6.3)

i.e. $[A'_-, A'_+]_s \equiv [c \otimes A_+ + d \otimes A_-, a \otimes A_+ + b \otimes A_-]_s \equiv E'$ follows using (6.1). More precisely under the left co-action: $U_s(h_3) \equiv C[A_+, A_-] E / ([A_-, A_+]_s - E, [E, x] | \forall x) \to C[A'_+, A'_-] E' / ([A'_-, A'_+]_s - E', [E', x] | \forall x) \simeq U_s(h_3)$.

On the other hand it is equally good to consider the right co-action $(U_s(h_3) \to U_s(h_3) \otimes Sp''_{s2}(2))$:

$$(A_+ \quad A_-) \mapsto (A''_+ \quad A''_-) := (A_+ \quad A_-) \hat{\otimes} T \quad E \mapsto E \otimes 1$$ \hspace{1cm} (6.4)

and this results in the relations ($q = s^2$):

$$ab = qba \quad ad - qbc = 1$$

$$cd = qdc \quad da - q^{-1} cb = 1$$ \hspace{1cm} (6.5)

which is another deformation of the $Sp(2)$ defining relations. So requiring both left and right covariance (bicovariance) of the $U_s(h_3)$ symmetry, means combining the relations (6.1) and (6.5), forcing the extra relation $bc = cb$. Then it is seen that the bicovariant symmetry of $U_s(h_3)$ is the quantum group $SL_{s^2}(2)$. From the definitions of the left $SL_{s^2}$-co-action $\Delta_L$ (6.3) and the right co-action $\Delta_R$ (6.4), it is almost obvious that they are compatible with the $SL_{s^2}$-coproduct ($\Delta(T) := T \hat{\otimes} T$),
i.e. that $(\Delta_L \otimes id) \circ \Delta_L = (id \otimes \Delta) \circ \Delta_L$ and $(id \otimes \Delta_R) \circ \Delta_R = (\Delta \otimes id) \circ \Delta_R$.

and also that they co-commute: $(id \otimes \Delta_R) \circ \Delta_L = (\Delta_L \otimes id) \circ \Delta_R$. I will not consider the full linear quantum symmetry of $U_{q,r}(h_4)$ here, though I comment that this $SL(2)_{s2}$ symmetry can be extended to the realisation of $U_s(h_4)$ in $U_s(h_3)$ constructed earlier (5.2).

Finally I want to make a remark about $SL_q(2)$ when $q$ is a non-trivial $p$-th root of unity $q(p)$. Then it turns out that $b^p$ and $c^p$ lie in the centre of the quantum group algebra, whereas $a^p$ and $d^p$ only commute with polynomials in $b$ and $c$. This is significantly similar to the situation discussed in section 4, where $A_+^p$ and $A_-^p$ fell in the centre of $U_q(h_4)$, but $N^p$ did not, and corresponds in fact to the centrality of $X_+^p$ and $X_-^p$ in the transcendental $U_{q(p)}(sl_2)$.

7. Conclusions

It may be that only for $sl_2$ do we have the equivalent Hopf algebras of Drinfeld-Jimbo, Woronowicz and Witten. The relationship between transcendental and quadratic quantum algebras still requires more study. I have tried to construct a coproduct for $U_{q,r}(h_4)$, but this does not seem to be possible. A more complicated quadratic deformation of $h_3$ and $h_4$, such as a general ‘bicovariant q-Lie algebra’, may give a nice Hopf algebra structure. Though I would comment that $U_{q,r}(h_4)$ ($U_r(h_3)$) does not really need to be a Hopf algebra, since it is only really the symmetries of the system that are expected to be quantum groups.

There are still some issues which I have not discussed in this paper. One of these is the $q$-quantum mechanics$^{32}$ of $U_{q,r}(h_4)$ ($U_r(h_3)$) when $q$ and $r$ are complex, in particular when they are roots of unity. If it were possible to find a unitary representation in this case, then the Hilbert space of $U_{q,r}(h_4)$ ($U_r(h_3)$) would be finite dimensional and it would be interesting to study the cyclic and nilpotent representations. At present unitary vector space representations of the transcendental (1.1) and quadratic (3.3) oscillators are only known for positive real deformation
parameters. It may be possible to consider representations on quantum spaces, where a generalisation of unitarity may exist.

Work is now in progress on $q$-deformed affine algebras, moving from this quantum mechanical algebraic framework into the quantum field theory arena.

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