Abstract. Generalizations of the AGT correspondence between 4D \( \mathcal{N} = 2 \) SU(2) supersymmetric gauge theory on \( \mathbb{C}^2 \) with \( \Omega \)-deformation and 2D Liouville conformal field theory include a correspondence between 4D \( \mathcal{N} = 2 \) SU(\( N \)) supersymmetric gauge theories, \( N = 2, 3, \ldots \), on \( \mathbb{C}^2 / \mathbb{Z}_n \), \( n = 2, 3, \ldots \), with \( \Omega \)-deformation and 2D conformal field theories with \( \mathcal{W}_{para}^{\text{para}} \) (\( n \)-th parafermion \( \mathcal{W} \)) symmetry and \( \widehat{\mathfrak{sl}}(n)_N \) symmetry. In this work, we trivialize the factor with \( \mathcal{W}_{para}^{\text{para}} \) symmetry in the 4D SU(\( N \)) instanton partition functions on \( \mathbb{C}^2 / \mathbb{Z}_n \) (by using specific choices of parameters and imposing specific conditions on the \( N \)-tuples of Young diagrams that label the states), and extract the 2D \( \widehat{\mathfrak{sl}}(n)_N \) WZW conformal blocks, \( n = 2, 3, \ldots \), \( N = 1, 2, \ldots \).

1. Introduction

1.1. Algebras on the equivariant cohomology of instanton moduli spaces. In [1], Alday, Gaiotto and Tachikawa conjectured a profound correspondence between SU(2) instanton partition functions in \( \mathcal{N} = 2 \) supersymmetric gauge theories on \( \mathbb{C}^2 \), with \( \Omega \)-deformation [2], and Virasoro conformal blocks on the sphere and on the torus (see [3] for a proof). Their conjecture was further generalized to correspondences between SU(\( N \)) instanton partition functions on \( \mathbb{C}^2 \) and \( \mathcal{W}_N \) conformal blocks [7, 8]. SU(2) instanton partition functions on \( \mathbb{C}^2 / \mathbb{Z}_2 \) and \( \mathcal{N} = 1 \) super-Virasoro conformal blocks [9, 10, 11, 12, 13, 14], SU(2) instanton partition functions on \( \mathbb{C}^2 / \mathbb{Z}_4 \) and conformal blocks of \( S_3 \) parafermion algebra [15, 16], etc.

In [17], by considering \( N \) M5-branes compactified on \( \mathbb{C}^2 / \mathbb{Z}_n \) with \( \Omega \)-deformation, Nishioka and Tachikawa, following a proposal in [9], suggested that \( \mathcal{N} = 2 \) SU(\( N \)) supersymmetric gauge theories on \( \mathbb{C}^2 / \mathbb{Z}_n \) are in correspondence with 2D CFTs with \( n \)-th parafermion \( \mathcal{W}_N \) symmetry, which we refer to as \( \mathcal{W}_{para}^{\text{para}} \), and affine \( \widehat{\mathfrak{sl}}(n)_N \) symmetry.

In [18], it was proposed that the AGT correspondence for U(\( N \)) supersymmetric gauge theory on \( \mathbb{C}^2 / \mathbb{Z}_n \) can be understood in terms of a 2D CFT based on the algebra

\[
\mathcal{A}(N; n; p) = \mathcal{H} \oplus \widehat{\mathfrak{sl}}(n)_N \oplus \frac{\widehat{\mathfrak{sl}}(N)_n \oplus \widehat{\mathfrak{sl}}(N)_{p-N}}{\mathfrak{sl}(N)_{n+p-N}},
\]

which acts on the equivariant cohomology of the moduli space of U(\( N \)) instantons on \( \mathbb{C}^2 / \mathbb{Z}_n \), \( n = 2, 3, \ldots \). Here, the first factor \( \mathcal{H} \cong \mathfrak{u}(1) \) is the affine Heisenberg algebra, the second factor is

---

1 In the context of geometric representation theory, the AGT correspondence for pure SU(\( N \)) supersymmetric gauge theory on \( \mathbb{C}^2 \) was proved in [4, 5] (see [6] for a generalization to all simply-laced gauge groups).
the affine \(\mathfrak{sl}(n)\) level-\(N\) algebra, and the third (coset) factor is the \(\mathcal{W}_{N,n}^{\text{para}}\) algebra, whose parameter \(p\), which controls the central charge \(c\), is related to the \(\Omega\)-deformation parameters \(\epsilon_1, \epsilon_2\) by

\[
\frac{\epsilon_1}{\epsilon_2} = -1 - \frac{n}{p}
\]

The coset factor gives a Virasoro algebra when \((N, n) = (2, 1)\), a \(\mathcal{W}_N\) algebra when \((N, n) = (N, 1)\), an \(\mathcal{N} = 1\) super-\(\mathcal{W}_N\) algebra when \((N, n) = (N, N)\), and an \(S_3\) parafermion algebra when \((N, n) = (2, 4)\).

1.2. Burge conditions. Let \(p \geq N\) be a positive integer. For \(n = 1\), the \(\hat{\mathfrak{sl}}(n)\) factor in the algebra \(A(N, n; p)\) is trivialised, while the coset (third) factor describes the \(\mathcal{W}_N\) \((p, p+1)\)-minimal model. In \([19, 20]\) for \((N, n) = (2, 1)\) and further in \([21]\) for \((N, 1), N = 3, 4, \ldots\), it was shown that to obtain minimal model conformal blocks from the \(SU(N)\) instanton partition functions on \(\mathbb{C}^2\) with \(\Omega\)-deformation \((1.2)\), we need to remove the non-physical poles, corresponding to \(\mathcal{W}_N\) minimal model null states, from the instanton partition functions. These non-physical poles emerge when the Coulomb and mass parameters of the gauge theory take special values labeled by integers \(r_I, s_I, I = 1, \ldots, N - 1\) with \(N - 1 \leq \sum_{I=1}^{N-1} r_I \leq p - 1, N - 1 \leq \sum_{I=1}^{N-1} s_I \leq p\). The conditions that exclude the non-physical poles were shown to be \((N-)\)Burge conditions (see \([22, 23]\) for \(N = 2\) and \([24, 25, 26]\) for general \(N\))

\[
Y_{I,i} \geq Y_{I+1,i+r_I-1} - s_I + 1, \quad I = 1, \ldots, N,
\]

for \(N\) tuple of Young diagrams \(Y_1, \ldots, Y_N\) which defines the instanton partition functions, where \(Y_{N+1} = Y_1\), and \(r_N = p - \sum_{I=1}^{N-1} r_I, s_N = p + 1 - \sum_{I=1}^{N-1} s_I\). For \(n \geq 2\), the coset factor in the algebra \(A(N, n; p)\) is considered to describe a \(\mathcal{W}_{N,n}^{\text{para}}\) \((p, p+n)\)-minimal model. In this paper, we show that the same \((N-)\)Burge conditions above also remove the non-physical poles from the \(SU(N)\) instanton partition functions on \(\mathbb{C}^2/\mathbb{Z}_n\) with \(\Omega\)-deformation \((1.2)\).

1.3. Trivialization of the coset factor. For \(p = N\), the coset factor in the algebra \(A(N, n; p)\) is trivialized (the partition function reduces to 1),

\[
A(N, n; N) = \mathcal{H} \oplus \hat{\mathfrak{sl}}(n)_N,
\]

and the \(SU(N)\) instanton partition functions on \(\mathbb{C}^2/\mathbb{Z}_n\) provide the \(\hat{\mathfrak{sl}}(n)_N\) WZW conformal blocks. Since all parameters are now integral (or at least non-generic), the affine factor will include non-physical poles due to null states. To remove these, we need to impose the appropriate Burge conditions \((1.3)\) on the gauge theory side. In the present work, we show that the integrable \(\hat{\mathfrak{sl}}(n)_N\) WZW conformal blocks can be extracted from the instanton partition functions by an appropriate choice of the parameters and imposing the appropriate Burge conditions.

\(^2\) In general \(p \in \mathbb{C}\).
1.4. Plan of the paper. In Section 2 we briefly recall the characters and the instanton partition functions in $\mathcal{N} = 2 U(N)$ supersymmetric gauge theories on $\mathbb{C}^2/\mathbb{Z}_n$ with $\Omega$-deformation. The relevant AGT-corresponding 2D CFTs are reviewed in Section 3. In Section 4 we derive the Burge conditions (Proposition 4.1) from the requirement that the $SU(N)$ instanton partition functions on $\mathbb{C}^2/\mathbb{Z}_n$, with $\Omega$-deformation (1.2), labeled by a positive integer $p$, do not have non-physical poles of the type described in Section 1.2. In subsequent sections, we only consider the Burge conditions that correspond taking $p = N$, which we need to trivialize the coset factor. In Section 5, by imposing the Burge conditions, we introduce what we refer to as reduced characters, and show, using a result in [27], that these coincide with the integrable $\hat{\mathfrak{sl}}(n)_N$ WZW characters (Proposition 5.9). In Section 6, we introduce what we refer to as reduced instanton partition functions, by imposing the appropriate Burge conditions, and find that specific integrable $\hat{\mathfrak{sl}}(n)_N$ WZW conformal blocks are obtained from them (Conjectures 6.5, 6.6 and 6.7). Our proposal, for computing WZW conformal blocks from models based on the $\mathcal{A}(N,n;N)$ algebra, is tested in Section 7 for $(N, n) = (2, 2), (2, 3)$ and $(3, 2)$. Finally, in Section 8 we make some remarks. In Appendix A we review some AGT correspondences to confirm our conventions, and in Appendix B we recall a class of integrable $\hat{\mathfrak{sl}}(n)_N$ WZW 4-point conformal blocks computed in [28], which we compare in Section 7 with our results.

2. $U(N)$ Instanton Counting on $\mathbb{C}^2/\mathbb{Z}_n$

We review how the moduli space of $U(N)$ instantons on $\mathbb{C}^2/\mathbb{Z}_n$ with $\Omega$-deformation is characterized by coloured Young diagrams, and define the characters and the instanton partition functions in terms of coloured Young diagrams.

2.1. Characterization of the instanton moduli space by coloured Young diagrams. Consider the $U(N)$ instantons on $\mathbb{C}^2/\mathbb{Z}_n$, where $\mathbb{Z}_n$ acts on $(z_1, z_2) \in \mathbb{C}^2$ by

$$Z_n : \begin{pmatrix} z_1, z_2 \end{pmatrix} \rightarrow \begin{pmatrix} e^{2\pi i/n} z_1, e^{-2\pi i/n} z_2 \end{pmatrix}, \tag{2.1}$$

and introduce the $\Omega$-deformation parameters $(\epsilon_1, \epsilon_2)$ [2, 29], by

$$U(1)^2 : \begin{pmatrix} z_1, z_2 \end{pmatrix} \rightarrow \begin{pmatrix} e^{\epsilon_1} z_1, e^{\epsilon_2} z_2 \end{pmatrix}. \tag{2.2}$$

Using localization, the $U(N)$ instantons on $\mathbb{C}^2/\mathbb{Z}_n$ are described by the fixed points, on the instanton moduli space, of the $U(1)^2 \times U(1)^N$ torus generated by $e^{\epsilon_1}$, $e^{\epsilon_2}$ and $e^{\sigma_I}$, where $a_I$, $I = 1, \ldots, N$, are the Coulomb parameters in the $U(N)$ gauge theory. The Coulomb parameters have charges $\sigma_I \in \{0, 1, \ldots, n - 1\}$ under the $\mathbb{Z}_n$ action

$$Z_n : a_I \rightarrow e^{2\pi i \sigma_I} a_I \tag{2.3}$$

Let $Y^\sigma$ be a coloured Young diagram, with a $\mathbb{Z}_n$ charge $\sigma \in \{0, 1, \ldots, n - 1\}$, in other words, $Y^\sigma$ is composed of boxes such that the box at position $(i, j) \in Y^\sigma$ is assigned the colour $\sigma - i + j \pmod{n}$. The fixed points of $U(N)$ $k$-instanton moduli space on $\mathbb{C}^2/\mathbb{Z}_n$ are labeled by $N$ tuples of coloured

---

3 In this work, by ‘reduced’ we mean ‘Burge reduced’ so as to remove the null states.
Young diagrams $Y^\sigma = (Y_1^\sigma, \ldots, Y_N^\sigma)$ with $k = \sum_{i=1}^N |Y_i^\sigma|$ total number of boxes \cite{30,31}, where $Y_i^\sigma$ are charged by (2.3). Let $N_\sigma$ and $k_\sigma$ be the number of Young diagrams with charge $\sigma$ and the total number of boxes with colour $\sigma$, respectively. Then,

\begin{equation}
\sum_{\sigma=0}^{n-1} N_\sigma = N, \quad \sum_{\sigma=0}^{n-1} k_\sigma = k
\end{equation}

Figure 1 shows an example of a coloured Young diagram. The $U(N)$ instantons on $\mathbb{C}^2/\mathbb{Z}_n$ are characterized by the first Chern class of the gauge bundle

\begin{equation}
c_1 = \sum_{\sigma=0}^{n-1} c_\sigma c_1(T_\sigma),
\end{equation}

where

\begin{equation}
c_\sigma = N_\sigma + \delta k_{\sigma-1} - 2\delta k_{\sigma} + \delta k_{\sigma+1} = N_\sigma - \sum_{i=1}^{n-1} A_{\sigma,i} \delta k_i, \quad \sigma = 1, \ldots, n-1, \quad \delta k_\sigma := k_\sigma - k_0,
\end{equation}

with $k_n = k_0$. Here, $c_1(T_\sigma)$ is the first Chern class of the vector bundle $T_\sigma$ on the ALE space with holonomy $e^{2\pi i \sigma/\alpha}$, and $A$ is the Cartan matrix of type $A_{n-1}$. Note that $c_1(T_0) = 0$, and the instanton moduli space is labelled by the $n-1$ integers $c = (c_1, \ldots, c_{n-1})$. In the following, we refer to the conditions (2.6) as Chern conditions. The Chern conditions can be inverted for $\delta k_\sigma$ as,

\begin{equation}
\delta k_\sigma = \sum_{i=1}^{n-1} \left( A^{-1} \right)_{\sigma,i} \left( N_i - c_i \right), \quad \left( A^{-1} \right)_{\sigma,i} = \min(\sigma, i) - \frac{\sigma i}{n}
\end{equation}

2.2. Characters. Let $P(\sigma; \delta k)$ be the sets of $N$ tuples of coloured Young diagrams $Y^\sigma$ which have charges $\sigma = (\sigma_1, \ldots, \sigma_N)$ and $\delta k = (\delta k_1, \ldots, \delta k_{n-1})$ with $c = (c_1, \ldots, c_{n-1})$ in (2.6). We introduce a generating function, which is referred to as the character, that counts the number of torus fixed points of the $U(N)$ instanton moduli space on $\mathbb{C}^2/\mathbb{Z}_n$, as

\begin{equation}
\chi(\sigma; \delta k)(q) = \sum_{Y^\sigma \in P(\sigma; \delta k)} q^{\sum_{i=1}^N |Y_i^\sigma|}
\end{equation}
Example 2.1 \((n = 1)\). For \(n = 1, \delta k = \emptyset\), the character (2.8) is

\[
\chi_{(0,0)}(q) = \chi_N(q)^N := \frac{1}{\left(\frac{q}{q}\right)_\infty^N}
\]

\[
= 1 + Nq + \frac{N(N+3)}{2}q^2 + \frac{N(N+1)(N+8)}{6}q^3
\]

\[
+ \frac{N(N+1)(N+3)(N+14)}{24}q^4 + \frac{N(N+3)(N+6)(N^2+21N+8)}{120}q^5 + \cdots ,
\]

where

\[
\left(a; q\right)_\infty = \prod_{n=0}^{\infty} \left(1 - a q^n\right)
\]

Example 2.2 \((N = 1, \text{see [32, 33]}\). For \(N = 1\), the character (2.8) with a charge \(\sigma \in \{0, 1, \ldots, n-1\}\) and \(\delta k = (\delta k_1, \ldots, \delta k_{n-1})\) is

\[
\chi_{(\sigma; \delta k)}(q) = \frac{1}{\left(\frac{q}{q}\right)_\infty} q^{\sum_{i=1}^{n-1} \left(\delta k_i^2 + \frac{\delta k_i}{n} - \delta k_i \delta k_{i+1} - \delta_{\sigma,i} \delta k_i\right)}
\]

For example, for \((N, n) = (1, 2)\), the characters are

\[
\chi_{(0,-\ell)}(q) = \frac{1}{\left(\frac{q}{q}\right)_\infty^2} q^{\ell \left(\frac{1}{2} - \frac{1}{2}\right)} , \quad \chi_{(1,-\ell)}(q) = \frac{1}{\left(\frac{q}{q}\right)_\infty^2} q^{\ell \left(\frac{1}{2} + \frac{1}{2}\right)}
\]

Example 2.3 \((N = 2, n = 2, \text{see [14]}\). For \((N, n) = (2, 2)\), the characters (2.8) are

\[
\chi_{(0,0;-\ell)}(q) + \chi_{(1,1;-\ell)}(q) = \frac{\chi_{NS}(q)^2}{\left(\frac{q}{q}\right)_\infty^\ell} q^{\ell (-1)} ,
\]

\[
\chi_{(0,1;-\ell)}(q) = \chi_{(1,0;-\ell)}(q) = \frac{\chi_{R}(q)^2}{\left(\frac{q}{q}\right)_\infty^{\ell^2}}
\]

where \(\chi_{NS}(q)\) and \(\chi_{R}(q)\) are, respectively, the NS sector and Ramond sector characters in \(\mathcal{N} = 1\) super-Virasoro algebra

\[
\chi_{NS}(q) = \left(\frac{-q^{\frac{1}{2}}; q}{q; q}\right)_\infty = 1 + q^\frac{1}{2} + q + 2q^2 + 3q^3 + 4q^4 + 5q^5 + 7q^6 + 10q^7 + 13q^8 + \cdots ,
\]

\[
\chi_{R}(q) = \left(\frac{-q; q}{q; q}\right)_\infty = 1 + 2q + 4q^2 + 8q^3 + 14q^4 + 24q^5 + 40q^6 + 64q^7 + 100q^8 + \cdots
\]

2.3. Instanton partition functions. To define instanton partition function, we introduce a fundamental building block, which is associated with \(U(N) \times U(N)\) gauge symmetry, with coloured
Young diagrams $Y^\sigma = (Y^\sigma_1, \ldots, Y^\sigma_N)$ and $W^\sigma' = (W^\sigma'_1, \ldots, W^\sigma'_N)$ by

$$Z_{\text{bif}}(a, Y^\sigma; a', W^\sigma') = \prod_{I,J=1}^{N} \prod_{\square \in Y^\sigma_{I,J}}^{*} E \left( -a_I + a'_J, Y^\sigma_{I,J}(\square), W^\sigma'_{I,J}(\square) \right)$$

(2.15)

$$\times \prod_{\square \in W^\sigma'_{I,J}} \left( \epsilon_1 + \epsilon_2 - E \left( a_I - a'_J, W^\sigma'_{I,J}(\square), Y^\sigma_{I,J}(\square) \right) \right),$$

where

$$E(P, Y(\square), W(\square)) = P - \epsilon_1 L_W(\square) + \epsilon_2 \left( A_Y(\square) + 1 \right)$$

(2.16)

Here the arm length $A_Y(\square)$ and the leg length $L_Y(\square)$ are defined by

$$A_Y(\square) = Y_I - j, \quad L_Y(\square) = Y_J^T - i, \quad \text{for} \quad \square = (i,j) \in Y,$$

where $Y_i$ (resp. $Y^T_j$) is the length of the $i$-row in $Y$ (resp. the $j$-row in the transposed Young diagram $Y^T$ of $Y$, i.e. the $j$-column in $Y$). The product $\prod_{\square \in Y}$ in (2.15) means to take the $Z_n$ invariant factors in the product, modulo $2\pi i$, under the shift of parameters following (2.1) and (2.3),

$$\epsilon_1 \to \epsilon_1 + \frac{2\pi i}{n}, \quad \epsilon_2 \to \epsilon_2 - \frac{2\pi i}{n}, \quad a_I \to a_I + \frac{2\pi i}{n}, \quad a'_J \to a'_J + \frac{2\pi i}{n}$$

(2.18)

Thus, the factors in the first and second products of (2.15) are constrained, respectively, by

$$-\sigma_I + \sigma'_J - L_{W^\sigma'_{I,J}}(\square) - A_{Y^\sigma}(\square) - 1 \equiv 0 \pmod{n},$$

$$\sigma_I - \sigma'_J - L_{Y^\sigma_I}(\square) - A_{W^\sigma'_{I,J}}(\square) - 1 \equiv 0 \pmod{n}$$

(2.19)

**Definition 2.4.** Using the building block (2.15), the $U(N)$ instanton partition function on $\mathbb{C}^2/\mathbb{Z}_n$ with $N$ fundamental and $N$ anti-fundamental hypermultiplets, which is defined by an equivariant integration over the moduli space of instantons $2$ (see also $29, 34, 35$), is $16$ (see also $31, 14$),

$$Z_{(\sigma, \delta k)}^{b, b'}(a, m, m'; q) = \sum_{Y^\sigma \in \mathcal{P}(\sigma, \delta k)} \frac{Z_{\text{bif}}(m, \theta^b; a, Y^\sigma) Z_{\text{bif}}(a, Y^\sigma; -m', \theta^{b'})}{Z_{\text{vec}}(a, Y^\sigma) q^{\frac{1}{n} \sum_{I=1}^{N} |V^\sigma_I|}},$$

(2.20)

where $m = (m_1, \ldots, m_N)$ and $m' = (m'_1, \ldots, m'_N)$ are the mass parameters, associated with $U(N)^2$ flavor symmetry, of $N$ fundamental and $N$ anti-fundamental hypermultiplets, respectively. The denominator, which is the contribution from the $U(N)$ vector multiplet with Coulomb parameters $a = (a_1, \ldots, a_N)$, is

$$Z_{\text{vec}}(a, Y^\sigma) = Z_{\text{bif}}(a, Y^\sigma; a, Y^\sigma)$$

(2.21)

The instanton partition function (2.20) depends on not only the integers $c = (c_1, \ldots, c_{n-1})$, in the first Chern class, but also the boundary charges $b = (b_1, \ldots, b_N)$ and $b' = (b'_1, \ldots, b'_N)$, which take values in $\{0, 1, \ldots, n-1\}$, assigned to the empty Young diagrams.

\begin{footnotesize}
4 By shifting $a'_J \to a'_J - \mu$, it is possible to introduce the mass parameter $\mu$ of bifundamental hypermultiplet.
\end{footnotesize}
3. 2D CFT for $U(N)$ instantons on $\mathbb{C}^2/\mathbb{Z}_n$

We recall various versions of the AGT correspondence, focusing on the algebra acting on the equivariant cohomology of the moduli space of $U(N)$ instantons on $\mathbb{C}^2/\mathbb{Z}_n$, and on explicit parameter relations.

3.1. Algebra on the moduli space of instantons and 2D CFT. In [9, 18], it has been proposed that the algebra

$$A(N, n; p) = \mathcal{H} \oplus \hat{sl}(n)_N \oplus \frac{\hat{sl}(N)_{n} \oplus \hat{sl}(N)_{p-N}}{\hat{sl}(N)_{p'-N}}, \quad p' = p + n,$$

(3.1)

naturally acts on the equivariant cohomology of the moduli space of $U(N)$ instantons on $\mathbb{C}^2/\mathbb{Z}_n$ with $\Omega$-deformation, where $\mathcal{H} \cong u(1)$ is the Heisenberg algebra. \(^5\) The parameter $p$ is identified with the $\Omega$-deformation parameters $\epsilon_1$, $\epsilon_2$ by the relation

$$\frac{\epsilon_1}{\epsilon_2} = -\frac{p'}{p} = -1 - \frac{n}{p},$$

(3.2)

This proposal implies that there exists a combined system of 2D CFTs, one with $\mathcal{H} \oplus \hat{sl}(n)_N$ symmetry and the other with $\mathcal{W}_{\text{para}}^{\text{para}}$ symmetry, corresponding to 4D $\mathcal{N} = 2$ $U(N)$ supersymmetric gauge theory on $\mathbb{C}^2/\mathbb{Z}_n$ with $\Omega$-deformation [17]. The central charges of these CFTs are

$$c\left(\mathcal{H} \oplus \hat{sl}(n)_N\right) = 1 + \frac{N (n^2 - 1)}{n + N},$$

$$c\left(\mathcal{W}_{\text{para}}^{\text{para}}\right) = \frac{n (N^2 - 1)}{n + N} + \frac{N (N^2 - 1) (\epsilon_1 + \epsilon_2)^2}{\epsilon_1 \epsilon_2} - \frac{n N (N^2 - 1)}{p (p + n)}.$$  

(3.3)

In (3.1), the first and second factors are realized by the $\hat{sl}(n)_N$ WZW model with an additional $u(1)$ symmetry, and the third (coset) factor is realized by a $\mathcal{W}_{\text{para}}^{\text{para}} (p, p + n)$-minimal model \(^6\), where $p$ is taken to be a positive integer with $p \geq N$.

3.2. Instanton partition functions as 4-point conformal blocks. Let $(\ast, \ast)$ be the standard inner product in the $A_{N-1}$ root space, and take the orthogonal basis $e_I, I = 1, \ldots, N$ with $(e_I, e_J) = \delta_{I,J}$. Then the $A_{N-1}$ simple roots and the $A_{N-1}$ fundamental weights are, respectively,

$$\alpha_I = e_I - e_{I+1} \quad \text{and} \quad \Lambda_I = \sum_{J=1}^{I} e_J - \frac{I}{N} e_0, \quad e_0 := \sum_{I=1}^{N} e_I,$$

(3.4)

for $I = 1, \ldots, N - 1$, and satisfy the defining conditions $(\alpha_I, \Lambda_J) = \delta_{I,J}$ and $(e_0, \Lambda_I) = 0.$

\(^5\) The first works on this subject, in the absence of an $\Omega$-deformation, are by Nakajima [36, 37].

\(^6\) While the $\mathcal{W}_{\text{para}}^{\text{para}} (p, p + n)$-minimal models are in general not well-understood except in special cases, see [38] and references therein, in this work, we only need to assume that they exist.
3.2.1. Parameter relations. We now provide the relations between the parameters of the instanton partition function (2.20) for \( N \geq 2 \) and those of the conformal blocks of the 4-point function on \( \mathbb{P}^1 \) of primary fields \( \psi_{\mu_r} \) with momenta \( \mu_r, r = 1, 2, 3, 4 \) (see Remark 3.2).

\[
\langle \psi_{\mu_1}(\infty) \psi_{\mu_2}(1) \psi_{\mu_3}(q) \psi_{\mu_4}(0) \rangle_\mathcal{W}_{N,n}^{para}
\]

in the \( \mathcal{W}_{N,n}^{para} \) CFT described by the coset factor in (3.1). We propose that the mass parameters \( m \) and \( m' \) in (2.20) are related to the external momenta \( \mu_r \) of the four primary fields by

\[
2 \mu_1 = \left( \epsilon_1 + \epsilon_2 \right) + \sum_{l=1}^{N-1} \left( m_l - m_{l+1} \right) \Lambda_l, \quad 2 \mu_2 = \sum_{l=1}^{N} m_l \Lambda_l, \\
2 \mu_4 = \left( \epsilon_1 + \epsilon_2 \right) - \sum_{l=1}^{N-1} \left( m'_l - m'_{l+1} \right) \Lambda_l, \quad 2 \mu_3 = \sum_{l=1}^{N} m'_l \Lambda_l,
\]

where

\[
\rho = \sum_{l=1}^{N-1} \Lambda_l = \frac{1}{2} \sum_{l=1}^{N} (N - 2I + 1)
\]

is the Weyl vector. We consider this as a generalisation of the \( n = 1 \) case in [1, 7, 8, 39, 40] to positive integer \( n \). By writing \( \mu_2 = \mu_2 \Lambda_{N-1}, \mu_3 = \mu_3 \Lambda_1, \) and \( \mu_r = \sum_{I=1}^{N-1} m_{\mu,I} \Lambda_I \) for \( r = 1, 4 \), the relations (3.6) are equivalent to

\[
2 \mu_{1,I} = \left( \epsilon_1 + \epsilon_2 \right) + \left( m_I - m_{I+1} \right), \quad 2 \mu_2 = \sum_{l=1}^{N} m_l, \\
2 \mu_{4,I} = \left( \epsilon_1 + \epsilon_2 \right) - \left( m'_I - m'_{I+1} \right), \quad 2 \mu_3 = \sum_{l=1}^{N} m'_l,
\]

\[
\iff \left\{ \begin{array}{l}
I = \frac{N + 1}{2} \quad \epsilon_1 + \epsilon_2 + \frac{2}{N} \left( \sum_{j=1}^{I-1} \mu_{1,J} + \sum_{J=1}^{N-1} \left( N - J \right) \mu_{1,J} + \mu_2 \right), \\
I = \frac{N + 1}{2} \quad \epsilon_1 + \epsilon_2 + \frac{2}{N} \left( \sum_{j=1}^{I-1} \mu_{4,J} - \sum_{J=1}^{N-1} \left( N - J \right) \mu_{4,J} + \mu_3 \right)
\end{array} \right.
\]

Note that the momenta \( \mu_2 \) and \( \mu_3 \) of two of the primary fields are taken to be proportional to \( \Lambda_1 \) or \( \Lambda_{N-1} \), i.e. \( \mathcal{W} \)-null, which ensures the matching of the number of free parameters \( \{ m_I, m'_I \} \{ I = 1, \ldots, N \} \) and \( \{ \mu_{1,I}, \mu_{2,J}, \mu_{3,J}, \mu_{4,J} \} \{ I = 1, \ldots, N-1 \} \). The Coulomb parameters \( a \) in (2.20) are related to the internal momenta \( \mu^v = \sum_{I=1}^{N-1} \mu^v_I \Lambda_I \) by

\[
2 \mu^v = \left( \epsilon_1 + \epsilon_2 \right) + \sum_{I=1}^{N} a_I \epsilon_I, \quad \epsilon_I := \epsilon_I - \frac{1}{N} e_0,
\]

\[
\iff a_I - \frac{1}{N} \sum_{I=1}^{N} a_I = \left( 2 \mu^v - \left( \epsilon_1 + \epsilon_2 \right) \rho, \epsilon_I \right)
\]

Remark 3.1 (\( U(1) \) factor). The \( U(N) \) instanton partition function (2.20) contains a \( U(1) \) factor coming from the Heisenberg algebra \( \mathcal{H} \) in the algebra \( \mathcal{A}(N,n;p) \). To obtain it, we need to impose
the traceless condition

\[ \sum_{I=1}^{N} a_I = 0 \]  

Then, following [11, 17, 8, 11, 16], we find an overall \( U(1) \) factor for general \( N \) and \( n \) in the instanton partition function (2.20),

\[ Z_H(m, m'; q) := \left( 1 - q \right) \left( \sum_{I=1}^{N} a_I \right) \left( \sum_{I=1}^{N} b_I \right) \]

In Appendix A, we confirm the above parameter relations and the \( U(1) \) factor by checking some AGT correspondences.

By analogy with known minimal model CFTs, we propose that, in the \( W_{N,p} \) (\( p, p + n \))-minimal models, the momenta should take the degenerate values

\[ r_N = p - \sum_{I=1}^{N-1} r_I, \quad s_N = p' - \sum_{I=1}^{N-1} s_I \]

\[ N - 1 \leq \sum_{I=1}^{N-1} r_I \leq p - 1, \quad N - 1 \leq \sum_{I=1}^{N-1} s_I \leq p' - 1 = p + n - 1, \]

and, for later convenience, we define

\[ r_N = p - \sum_{I=1}^{N-1} r_I, \quad s_N = p' - \sum_{I=1}^{N-1} s_I \]

Remark 3.2 (Free field realization). We check our normalization conventions by focusing on the well-understood \( n = 1 \) CFT with \( W_N \) symmetry. In this case, one can introduce the energy-momentum tensor by

\[ T(z) = \frac{1}{2} \sum_{I=1}^{N} : \partial \phi_I(z)^2 : + Q \left( \partial^2 \phi(z) \right), \quad Q = \frac{\epsilon_1 + \epsilon_2}{g_s}, \quad g_s^2 = -\epsilon_1 \epsilon_2, \]

where \( : \cdot : \) is the normal ordered product,

\[ \phi(z) = \sum_{I=1}^{N} \phi_I(z) \varepsilon_I, \quad \sum_{I=1}^{N} \phi_I(z) = 0, \quad \varepsilon_I = e_I - \frac{1}{N} e_0, \]

are \( N \) free chiral bosons with

\[ \partial \phi_I(z) \phi_J(w) = \frac{\varepsilon_I \varepsilon_J}{z - w} + \partial \phi_I(z) \phi_J(w), \quad \left( \varepsilon_I, \varepsilon_J \right) = \delta_{I,J} - \frac{1}{N}, \]

and \( g_s \) is introduced as a mass parameter just for a convention. Then, the Virasoro central charge

\[ c = (N - 1) - N \left( N^2 - 1 \right) Q^2 = (N - 1) + N \left( N^2 - 1 \right) \frac{\left( \epsilon_1 + \epsilon_2 \right)^2}{\epsilon_1 \epsilon_2}, \]
which is the one in (3.3) for \( n = 1 \), is obtained. One can also introduce the primary field with momenta \( \mu \) by

\[
\psi_\mu(z) = e^{\left( 2 \frac{\mu}{g_s} \phi(z) \right)}, \quad \mu = \sum_{I=1}^{N-1} \mu_I \Lambda_I,
\]

which has the conformal dimension

\[
\Delta_\mu = 2 \left( \frac{\mu}{g_s} \right) = -\frac{2}{\epsilon_1 \epsilon_2} \left( \mu, \mu - \left( \epsilon_1 + \epsilon_2 \right) \rho \right),
\]

under the action of the energy-momentum tensor (3.15). For example, when \( N = 2 \) with the \( \Omega \)-background \( \frac{\mu}{\epsilon_2} = -\frac{\rho}{p} \) (Virasoro \((p,p')\)-minimal model case), the conformal dimension of the primary field with degenerate momentum \( 2 \mu^{r,s} = -(r-1)\epsilon_1 - (s-1)\epsilon_2 \) is

\[
\Delta_{\mu^{r,s}} = \frac{\mu^{r,s} \left( \epsilon_1 + \epsilon_2 - \mu^{r,s} \right)}{\epsilon_1 \epsilon_2} = \left( \frac{rp' - sp}{4pp'} \right)^2 - \left( \frac{p' - p}{4pp'} \right)^2.
\]

We assume that there exist similar free field realizations for general \( n \), and the central charge and the conformal dimension of the primary field \( \psi_\mu(z) \) are \( c(W_{N,n}^{para}) \) in (3.3) and \( \Delta_\mu/n \) in (3.20), respectively.

4. Burge conditions from \( SU(N) \) instanton partition functions on \( \mathbb{C}^2/\mathbb{Z}_n \)

We deduce the Burge conditions in Proposition 4.1 by looking at the non-physical poles of the \( SU(N) \) instanton partition function (2.20), with \( \sum_{I=1}^N a_I = 0 \), on \( \mathbb{C}^2/\mathbb{Z}_n \) with the rational \( \Omega \)-deformation (3.2).

For the rational \( \Omega \)-background (3.2), i.e. \( p \epsilon_1 + p' \epsilon_2 = 0 \), we see that the instanton partition function (2.20) with \( \sum_{I=1}^N a_I = 0 \) has poles at the values

\[
a_I = a_I^{r,s} := -\sum_{J=1}^{N-1} \left( \Lambda_J, e_I \right) \left( r_J \epsilon_1 + s_J \epsilon_2 \right) = -\sum_{J=1}^{N-1} \left( r_J \epsilon_1 + s_J \epsilon_2 \right) + \frac{1}{N} \sum_{J=1}^{N-1} J \left( r_J \epsilon_1 + s_J \epsilon_2 \right)
\]

of the Coulomb parameters (3.9) corresponding to the degenerate momenta (3.12) (see (4.5)). These poles correspond to the propagation of null-states and need to be removed. Taking a shift of the central \( U(1) \) factor in the \( U(N) \) gauge symmetry, from (2.1) and (2.3), into account, the integral charges \( \sigma_I \) assigned to \( a_I \) are related to \( r \) and \( s \) by

\[
\sigma_I - \sigma_{I+1} \equiv -r_I + s_I \pmod{n}, \quad I = 1, \ldots, N - 1
\]

We refer the conditions (4.2) as charge conditions.

**Proposition 4.1.** If the following conditions for an \( N \)-tuple of coloured Young diagrams \( Y^\sigma \), which are referred to as the Burge conditions,

\[
Y^\sigma_{I,i} \geq Y^\sigma_{I+1,i+r_I-1} - s_I + 1, \quad I = 1, \ldots, N,
\]


are satisfied, the instanton partition function (2.20) at \( a_I = a_I^{r,s} \), in the background \( p \epsilon_1 + p' \epsilon_2 = 0 \), \( p' = p + n \), does not have poles, where \( Y_N^{\sigma_{N+1}} = Y_1^{\sigma_1} \). The Burge conditions (4.3) are equivalent to
\[
Y^T_{I,j} \geq Y^T_{I+1,j} + r_I - 1, \quad I = 1, \ldots, N
\]

**Proof.** We follow the proof of [21] for \( n = 1 \) (see also [19] [20] for \( N = 2, n = 1 \)). For notational simplicity, we abbreviate the charges \( \sigma_I \) assigned to Young diagrams as \( Y_I = Y_I^{\sigma_I} \). At \( a_I = a_I^{r,s} \) with (3.2), the instanton partition function (2.20) has poles if and only if the denominator vanishes, i.e. there exists \( \Box \in Y_I \) such that
\[
E_{I,J}^{r,s}(\Box) + \delta = 0, \quad \delta = 0 \text{ or } n(= p' - p),
\]
where
\[
E_{I,J}^{r,s}(\Box) = \frac{p}{\epsilon_2} E \left( a_I^{r,s} - a_I^{r,s}, Y_I(\Box), Y_J(\Box) \right)
\]
\[
= \sum_{K=1}^{N-1} \left( \Lambda_K, e_J - e_I \right) \left( r_K p' - s_K p \right) + p' L_{Y_J}(\Box) + p \left( A_{Y_I}(\Box) + 1 \right)
\]
Because \( E_{I,I}^{r,s}(\Box) \neq 0 \) for \( \Box \in Y_I \), to find \( \Box \in Y_I \) which satisfies (4.5), we only need to consider the case (i) \( I > J \) and case (ii) \( I < J \).

**Case (i) \( I > J \).** In this case, the zero-condition (4.5) is \( E_{I+\ell,J}^{r,s}(\Box) + \delta = 0 \) for \( \Box \in Y_{I+\ell} \), where \( 1 \leq I \leq N - 1 \) and \( 1 \leq \ell \leq N - I \). By \( \sum_{K=1}^{N-1} (\Lambda_K, e_I - e_{I+\ell}) = \sum_{K=1}^{N-1} \sum_{J=1}^{\ell} \delta_{K,I+J-1} \), this zero-condition is written as
\[
\sum_{J=1}^{\ell} \left( r_{I+J-1} p' - s_{I+J-1} p \right) + p' L_{Y_J}(\Box) + p \left( A_{Y_{I+\ell}}(\Box) + 1 \right) + \delta = 0, \quad \Box \in Y_{I+\ell}
\]
Let \( d = \gcd(p, p') \), \( p = dp_d \) and \( p' = dp'_d \), then the zero-condition (4.7) is equivalent to
\[
L_{Y_I}(\Box) = - \sum_{J=1}^{\ell} r_{I+J-1} - \gamma p_d - \delta_{\delta,n},
\]
\[
A_{Y_{I+\ell}}(\Box) = \sum_{J=1}^{\ell} s_{I+J-1} + \gamma p'_d - 1 + \delta_{\delta,n}, \quad \Box \in Y_{I+\ell},
\]
where \( \gamma \) is an indeterminate integer. For \( \Box = (i,j) \in Y_{I+\ell} \), using \( L_{Y_I}(\Box) = Y^T_{I,j} - i \), the zero-conditions (4.8) imply that an obvious condition for any Young diagrams,
\[
Y^T_{I+\ell,j + A_{Y_{I+\ell}}(\Box)} \geq i,
\]
yields
\[
Y^T_{I+\ell,j + \sum_{J=1}^{\ell} s_{I+J-1} + \gamma p'_d - 1 + \delta_{\delta,n}} \geq Y^T_{I,j} + \sum_{J=1}^{\ell} r_{I+J-1} + \gamma p_d + \delta_{\delta,n}
\]
For the above zero-conditions, \( \Box \in Y_{I+\ell} \) needs to be restricted by the \( \mathbb{Z}_n \) condition like (2.19)
\[
\sigma_I - \sigma_{I+\ell} - L_{Y_I}(\Box) - A_{Y_{I+\ell}}(\Box) - 1 \equiv 0 \pmod{n}
\]
By the charge conditions (4.2) and the zero-conditions (4.8), the \( \mathbb{Z}_n \) condition (3.11) yields
\[
0 \equiv \gamma (p_d - p'_d) \equiv -\frac{n}{d} \gamma \pmod{n} \iff \gamma = d \gamma_d,
\]
where the indeterminate integer \( \gamma_d \) should be \( \gamma_d \geq 0 \) by \( A_{Y_{I+t,\ell}}(\square) \geq 0 \) and (3.13). As a result, the zero-condition (4.10) yields
\[
Y^T_{I+\ell,j} + \sum_{j=1}^\ell s_{I+j-1} + \gamma_d p' - 1 + \delta_{\gamma,n} \geq Y^T_{I,j} + \sum_{j=1}^\ell r_{I+j-1} + \gamma_d p + \delta_{\gamma,n}
\]
Therefore, if conditions
\[
Y^T_{I,j} \geq Y^T_{I+\ell,j} + \sum_{j=1}^\ell s_{I+j-1} + \gamma_d p' - 1 + \delta_{\gamma,n} - \sum_{j=1}^\ell r_{I+j-1} + \gamma_d p + 1 - \delta_{\gamma,n}, \quad \gamma_d \geq 0
\]
are satisfied, there does not exist \( \square \in Y_{I+\ell} \) such that \( E_{r_{I+\ell},I}^s(\square) + \delta = 0 \). These non-zero conditions follow from the ones for \( \gamma_d = 0 \) and \( \delta = 0 \):
\[
Y^T_{I,j} \geq Y^T_{I+\ell,j} + \sum_{j=1}^\ell s_{I+j-1} - \sum_{j=1}^\ell r_{I+j-1} + 1
\]
All these non-zero conditions (4.15) for \( 1 \leq \ell \leq N - I \) are obtained from the ones for \( \ell = 1 \), i.e.
we arrive at the strongest non-zero conditions among them as
\[
Y^T_{I,j} \geq Y^T_{I+1,j} + s_{I-1} - r_I + 1, \quad I = 1, \ldots, N - 1,
\]
which are the transposed Burge conditions (4.4) for \( I = 1, \ldots, N - 1 \).

Case (ii) \( I < J \). In this case, the zero-condition (4.5) is \( E_{I,I+\ell}^{r,s}(\square) + \delta = 0 \) for \( \square \in Y_I \), where
\( 1 \leq I \leq N - 1 \) and \( 1 \leq \ell \leq N - I \). We repeat the proof of case (i). As (4.7) and (4.8), the zero-condition \( E_{I,I+\ell}^{r,s}(\square) + \delta = 0 \) is
\[
- \sum_{j=1}^\ell \left( r_{I+j-1} p' - s_{I+j-1} p \right) + p' L_{Y_{I+t,\ell}}(\square) + p \left( A_{Y_I}(\square) + 1 \right) + \delta = 0, \quad \square \in Y_I,
\]
which is equivalent to
\[
L_{Y_{I+t,\ell}}(\square) = \sum_{j=1}^\ell r_{I+j-1} - \gamma_d p - \delta_{\gamma,n},
\]
\[
A_{Y_I}(\square) = - \sum_{j=1}^\ell s_{I+j-1} + \gamma_d p' - 1 + \delta_{\gamma,n}, \quad \square \in Y_I,
\]
where we have used (4.12) obtained from the \( \mathbb{Z}_n \) condition. From \( A_{Y_I}(\square) \geq 0 \) and (3.13), the indeterminate integer \( \gamma_d \) should be \( \gamma_d \geq 1 \). As in (4.10), these conditions yield a zero-condition
\[
Y^T_{I,j} + \sum_{j=1}^\ell s_{I+j-1} + \gamma_d p' - 1 + \delta_{\gamma,n} \geq Y^T_{I+\ell,j} - \sum_{j=1}^\ell r_{I+j-1} + \gamma_d p + \delta_{\gamma,n}
\]
Therefore, if conditions

\[(4.20) \quad Y^T_{I+\ell,j} \geq Y^T_{I,j} - \sum_{J=1}^{\ell} r_{I+J-1} - \gamma_d p + 1 - \delta_{d,n}, \quad \gamma_d \geq 1\]

are satisfied, there does not exist \(\varnothing \in Y_I\) such that \(E^{r,s}_{I,\ell}(\varnothing) + \delta = 0\). Among the non-zero conditions \((4.20)\), the strongest ones are \(\gamma_d = 1\) and \(\delta = 0\):

\[(4.21) \quad Y^T_{I+\ell,j} \geq Y^T_{I,j} - \sum_{J=1}^{\ell} r_{I+J-1} - p + 1\]

In particular, for \(\ell = N - I\), one obtains

\[(4.22) \quad Y^T_{N,j} \geq Y^T_{1,j+sN-1} - r_N + 1,\]

which is the transposed Burge condition for \(I = N\) in \((4.4)\). Combining this condition with the non-zero conditions derived in the case (i), one obtains the transposed Burge conditions \((4.4)\). It is straightforward to see that the conditions \((4.4)\) are stronger than the non-zero conditions \((4.21)\).

Following Section 4.10 of \([19]\), one finds that the transposed Burge conditions \((4.4)\) are equivalent to the Burge conditions \((4.3)\). This completes the proof of Proposition 4.1.

5. \(t\)-refined reduced characters and \(\widehat{\mathfrak{sl}}(n)_N\) WZW characters

In this and in subsequent sections, we concentrate on the case of \(p = N\) in the algebra \(\mathcal{A}(N, n; p)\). This choice of parameters trivializes the coset factor, and we obtain \(\mathcal{A}(N, n; N) = \mathcal{H} \oplus \widehat{\mathfrak{sl}}(n)_1\).

In this case, imposing the Burge conditions in Proposition 4.1 on the characters and instanton partition functions, the \(\widehat{\mathfrak{sl}}(n)_N\) WZW characters and conformal blocks emerge. In this section we discuss the characters\(^7\) and the instanton partition functions are discussed in Section 6.

5.1. \(U(1)\) \(t\)-refined character. Consider the characters for \(U(1)\) instantons on \(\mathbb{C}^2/\mathbb{Z}_n\). In this case, the coset factor in the algebra \((3.1)\) is absent for generic \(p\) (generic \(\Omega\)-background), i.e. \(\mathcal{A}(1, n; p) = \mathcal{H} \oplus \widehat{\mathfrak{sl}}(n)_1\), an algebra whose highest-weight representations have no null-states, so there are no Burge conditions to impose. Subtracting the Heisenberg factor \(\mathcal{H}\), whose character is \(\chi_\mathcal{H}(q) = (q; q)^{-1}_\infty\) in \((2.9)\), from the algebra \(\mathcal{A}(1, n; p) = \mathcal{H} \oplus \widehat{\mathfrak{sl}}(n)_1\), we introduce a \(t\)-refined character, with fugacities \(t = (t_1, \ldots, t_{n-1})\), labeled by a charge \(\sigma \in \{0, 1, \ldots, n - 1\}\).

\(^7\) When \(p = N\), the central charge \(c(W^\text{para})_{N,N,n} = 0\) in \((3.3)\).

\(^8\) Without the Burge conditions, the characters correspond to the partition functions of \(\mathcal{N} = 4\) topologically twisted \(U(N)\) supersymmetric gauge theories \([36]\). A description of the characters/WZW characters in terms of ‘orbifold partitions’, and a realization in terms of intersecting D4 and D6-branes can be found in \([22]\).
Definition 5.1 \((N = 1 t\text{-refined character})\). The \(N = 1 t\text{-refined character}\) is defined by

\[
\hat{\chi}_N(q, t) = \left( \frac{q}{\ell} \right) \sum_{\delta k \in \mathbb{Z}^{n-1}} \chi_{(\sigma; \delta k)}(q) \prod_{i=1}^{n-1} t_i^{\frac{1}{n} \ell_i},
\]

where \(N = [N_0, \ldots, N_{n-1}]\) with \(N_i = \delta_{i,0}\) in \eqref{eq:11}. Here \(\chi_{(\sigma; \delta k)}(q)\) is the character for \(N = 1\) defined in \eqref{eq:23}, and \(c_i = \delta_{\sigma,i} - \sum_{j=1}^{n-1} A_{i,j} \delta_{k,j}\), by the Chern conditions \eqref{eq:26}. In the second equality, \eqref{eq:11} was used.

The affine version of the notation of Section 5.2 for the affine \(A_{n-1}\) root space. Using the orthogonal basis \(\{e_1, \ldots, e_n\}\) and \(e_0 = \sum_{i=1}^n e_i\), the affine \(A_{n-1}\) simple roots and the affine \(A_{n-1}\) fundamental weights are, respectively,

\[
\alpha_0 = e_n - e_1 + \delta, \quad \alpha_i = e_i - e_{i+1}, \quad \text{and } \Lambda_i = \sum_{j=1}^i e_j - \frac{i}{n} e_0 + \Lambda_0,
\]

for \(i = 1, \ldots, n - 1\), and satisfy \((\alpha_i, \Lambda_j) = \delta_{i,j}\). Here the null root \(\delta\) and the basic fundamental weight \(\Lambda_0\), with inner products \((\delta, \Lambda_0) = 1\) and \((\delta, \delta) = (\Lambda_0, \Lambda_0) = (\delta, e_i) = (\Lambda_0, e_i) = 0\), are introduced.

**Definition 5.2.** The \(\hat{\mathfrak{sl}}(n)_N\) WZW character of level-N dominant integral highest-weight

\[
\hat{\chi}_{\Lambda}^{(n)}(q, t') = \text{Tr}_{\Lambda}^{\hat{\mathfrak{sl}}(n)} q^{L_0 - h_{\Lambda}} \prod_{i=1}^{n-1} t_i^{\frac{1}{n} \ell_i}(H, \alpha_i), \quad t'_i := q^{-\frac{i(n-i)}{2}} t_i,
\]

where the Virasoro generator \(L_0\) and the \(\hat{\mathfrak{sl}}(n)\) current \(H = \sum_{i=0}^{n-1} H_i \Lambda_i\), \((H, \alpha_i) = H_i\), act on the highest-weight module \(|\Lambda\rangle\) as

\[
L_0 |\Lambda\rangle = h_{\Lambda} |\Lambda\rangle := \frac{(\Lambda, \Lambda + 2\rho)}{2(n + N)} |\Lambda\rangle, \quad H_i |\Lambda\rangle = 0, \quad \rho = \sum_{i=1}^{n-1} \Lambda_i.
\]

Using

\[
\frac{1}{2} \left( \sum_{i=1}^{n-1} \ell_i \alpha_i, \sum_{i=1}^{n-1} \ell_i \alpha_i \right) = \sum_{i=1}^{n-1} \ell_i^2 - \sum_{i=1}^{n-2} \ell_i \ell_{i+1}, \quad \left( \sum_{i=1}^{n-1} \ell_i \alpha_i, \Lambda_j \right) = \ell_j,
\]

\[
\left( \Lambda_i, \Lambda_j \right) = \left( A^{-1} \right)_{ij} = \min(i,j) - \frac{i j}{n}, \quad \left( \Lambda_i, \rho \right) = \frac{n}{2} \left( \Lambda_i, \Lambda_i \right) = \frac{i}{n} \left( n - i \right),
\]

An explicit formula for the \(\hat{\mathfrak{sl}}(n)_N\) WZW character is given by the Weyl-Kac character formula in Proposition 5.11.
we arrive at the following proposition.

**Proposition 5.3** ($\hat{\mathfrak{g}}(n)_1$ WZW character). The $N = 1$ t-refined character \(^{(5.1)}\) can be rewritten as

\[
\hat{\chi}_{N_{\sigma}}(q, t) = \frac{1}{(q; q)_{\infty}} \sum_{\mu \in \mathfrak{g}^{n-1}_{i=1}Z\alpha_i + \Lambda_{\sigma}} q^{\frac{1}{2}(\mu, \mu)} t^{\chi(\mu, \rho)} \prod_{i=1}^{n-1} t_i^{\frac{1}{2}(\mu, \alpha_i)},
\]

and agrees with the $\hat{\mathfrak{g}}(n)_1$ WZW character \(^{(5.2)}\) of highest-weight $\Lambda_{\sigma}$ as

\[
\hat{\chi}_{N_{\sigma}}(q, t) = q^{\frac{1}{2}\sigma(n-\sigma)} \hat{\chi}_{\Lambda_{\sigma}}(q, t'),
\]

where \(^{(5.3)}\) now becomes,

\[
L_0 |\Lambda_{\sigma}\rangle = h_{\Lambda_{\sigma}} |\Lambda_{\sigma}\rangle = \left(\frac{\Lambda_{\sigma}, \Lambda_{\sigma}}{2}\right) |\Lambda_{\sigma}\rangle = \frac{\sigma(n-\sigma)}{2n} |\Lambda_{\sigma}\rangle, \quad H_i |\Lambda_{\sigma}\rangle = \delta_{\sigma,i} |\Lambda_{\sigma}\rangle
\]

**Example 5.4** ($N = 1, n = 2$). For $n = 2$, Proposition \(^{5.3}\) says

\[
\hat{\chi}_{(1,0)}(q, t) = \left(\frac{q; q}{q^{\frac{1}{2}}t^{\frac{1}{2}}H}\right)_{\infty} \sum_{\ell \in \mathbb{Z}} \chi(0; -\ell) \left(t^{\ell+\frac{1}{2}}H - t^{\ell-\frac{1}{2}}H\right) = 1 + (1 + f_1(t')) q + (2 + f_1(t')) q^2 + (3 + 2f_1(t')) q^3 + \cdots,
\]

\[
\hat{\chi}_{(0,1)}(q, t) = \left(\frac{q; q}{q^{\frac{1}{2}}t^{\frac{1}{2}}H}\right)_{\infty} \sum_{\ell \in \mathbb{Z}} \chi(1; -\ell) \left(t^{\ell+\frac{1}{2}}H - t^{\ell-\frac{1}{2}}H\right) = f_1(t') + f_2(t') q + (2f_1(t') + f_3(t')) q^2 + \cdots,
\]

where $t' = q^{-\frac{1}{2}} t$ and $f_j(t') = t'^j + t'^{-j}$.

**Remark 5.5** (Principal specialization). In the formula \(^{(5.6)}\), the q-shift factor

\[
\left(\mu, \rho\right) = \sum_{i=1}^{n-1} \left(\mu, \Lambda_i\right) = \sum_{i, j=1}^{n-1} \left(\Lambda_i, \Lambda_j\right) \left(\mu, \alpha_j\right) = \sum_{j=1}^{n-1} \left(\rho, \Lambda_j\right) \left(\mu, \alpha_j\right)
\]

is understood as the principal grading in the $\hat{\mathfrak{g}}(n)_N$ WZW character \(^{(5.4)}\):

\[
\prod_{i=1}^{n-1} t_i^{\frac{1}{2}(H, \alpha_i)} = q^{-\frac{n}{2}(n-1)\ell} \prod_{i=1}^{n-1} t_i^{\frac{1}{2}(H, \alpha_i)}
\]

By taking $t'_i = q^{-\frac{(n-1)\ell}{2}}$ (i.e. $t_i = 1$) in \(^{(5.4)}\), we define the principal $\hat{\mathfrak{g}}(n)_N$ WZW character by

\[
\text{Pr}_{\Lambda} \hat{\chi}_{\Lambda}^{(n)}(q) := \text{Tr}_{\Lambda} \hat{\mathfrak{g}}(n)_N q^{L_0 - h_{\Lambda} + \sum_{i=1}^{n-1} \frac{i(n-1)}{2n} (d_i - H_i)},
\]

where a normalization factor in \(^{(5.7)}\) (and \(^{(5.21)}\) for general $N$) is introduced, and an explicit formula is given in Corollary \(^{5.12}\).
Example 5.6 (Principal $\widehat{\mathfrak{sl}}(n)_1$ WZW character). Proposition 5.3 and Corollary 5.12 give
\begin{equation}
\hat{\chi}_{N_s}(q, 1) = \frac{1}{(q; q)_\infty} \sum_{(\ell_1, \ldots, \ell_{n-1}) \in \mathbb{Z}^{n-1}} q^{\sum_{i=1}^{n-1} \left( \ell_i - \frac{i-1}{n-1} \right)} = \frac{(q; q)_{\infty}}{(q^\frac{n}{n-1}; q^\frac{n}{n-1})_{\infty}} = \Pr \chi_{\widehat{\mathfrak{sl}}(n)_1}^{\hat{\widehat{r}}}(q)
\end{equation}

5.2. $SU(N)$ $t$-refined reduced characters. We consider the cases of $N \geq 2$. Let $\mathcal{P}^s(\sigma; \delta k)$ be the subset of $\mathcal{P}(\sigma; \delta k)$,
\begin{equation}
\mathcal{P}^s(\sigma; \delta k) \subset \mathcal{P}(\sigma; \delta k),
\end{equation}
whose elements satisfy the Burge conditions (4.3) with $p = N$, where by (3.13), $r_I = 1$, $I = 1, \ldots, N - 1$, are fixed. The specialized Burge conditions
\begin{equation}
Y_{I,i}^{\sigma_I} \geq Y_{I+1,i}^{\sigma_{I+1}} - s_I + 1, \quad I = 1, \ldots, N,
\end{equation}
are parametrized by the positive integers $s = (s_1, \ldots, s_{N-1})$ and $s_N = N + n - \sum_{I=1}^{N-1} s_I$ with
\begin{equation}
N - 1 \leq \sum_{I=1}^{N-1} s_I \leq N + n - 1
\end{equation}
We now introduce a reduced version of the characters $\chi_{(\sigma, \delta k)}(q)$ in (2.8) by imposing the Burge conditions above as
\begin{equation}
\chi_{(\sigma, \delta k)}^{(s)}(q) = \sum_{Y^\sigma \in \mathcal{P}^s(\sigma; \delta k)} q^{\sum_{I=1}^{N-1} \left| Y_{I,i}^{\sigma_I} \right|_n},
\end{equation}
where $\sigma$ and $s$ should satisfy the charge conditions (4.2),
\begin{equation}
\sigma_I - \sigma_{I+1} \equiv s_I - 1 \pmod{n}, \quad I = 1, \ldots, N - 1
\end{equation}
Remark 5.7 (Fixing the parameters $s$). The charge conditions (5.17) fix $s$ modulo $n$. By (5.15), this ambiguity arises only when $\sigma_1 = \sigma_2 = \ldots = \sigma_N$, and one element of $s$, say $s_K$, can be $s_K = n + 1$ whereas $s_{I \neq K} = 1$, $I = 1, \ldots, N$. In this case, by cyclically shifting the labels $I$ in the Young diagrams as $I \rightarrow I + N - K$, modulo $n$, the Burge conditions (5.14) become the same one in the case of $s_I = 1$, $I = 1, \ldots, N - 1$, $s_N = n + 1$. By relabeling the labels $I$, the charges $\sigma$ are ordered as $\sigma_1 \geq \ldots \geq \sigma_N$, and by shifting them cyclically when $\sigma_1 = \sigma_2 = \ldots = \sigma_N$, we can fix $s$ as
\begin{equation}
s_I = s_I^* = \sigma_I - \sigma_{I+1} + 1,
\end{equation}
where, by $\sigma_I \in \{0, 1, \ldots, n - 1\}$, $s = s^*$ satisfies condition (5.15).

Similarly to (5.1), we define a $t$-refined reduced character, with fugacities $t = (t_1, \ldots, t_{n-1})$, labeled by the non-negative integers $N = [N_0, \ldots, N_{n-1}]$, $\sum_{\sigma=0}^{n-1} N_\sigma = N$ in (2.4).
Definition 5.8 \((SU(N) t\text{-}refined reduced character). The \(SU(N) t\text{-}refined reduced character, which is reduced by imposing the specialized Burge conditions \((5.13)\), is defined by

\[
(5.19) \quad \hat{\chi}^\text{red}_{\mathcal{N}}(q,t) = \left( q; q \right)_\infty \times \sum_{\delta k \in \mathbb{Z}^{n-1}} \chi^{(s_0^*)}_{\left( \sigma, \delta k \right)}(q) \bigg|_{\sigma_1 \geq \ldots \geq \sigma_N} \prod_{i=1}^{n-1} t_i^{\frac{s_i}{2}},
\]

Here, for a choice of \(\mathcal{N}\) the ordered charges \(\sigma_1 \geq \ldots \geq \sigma_N\) are uniquely fixed, and by \((5.13)\) \(s_1 = s_1^*\) are also fixed. The set \(c_i\) is related to the set \(\delta k\) by the Chern conditions \((2.6)\).

As a natural generalization of Proposition 5.3 for \(N \geq 2\), we find that Theorem 1.2 in \([27]\), which was first conjectured in the context of solvable lattice models in statistical mechanics \([44]\), is translated into the following proposition.

Proposition 5.9 (Combinatorial \(\widehat{\mathfrak{sl}}(n)_N\) WZW character formula \([27]\)). The \(t\text{-}refined reduced character \((5.19)\) agrees with the \(\widehat{\mathfrak{sl}}(n)_N\) WZW character \((5.4)\) of level-\(N\) dominant integral highest-weight

\[
(5.20) \quad \mathcal{N} = \sum_{i=0}^{n-1} N_i \Lambda_i = [N_0, N_1, \ldots, N_{n-1}], \quad \sum_{i=0}^{n-1} N_i = N,
\]
as

\[
(5.21) \quad \hat{\chi}^\text{red}_{\mathcal{N}}(q,t) = q^{\sum_{i=1}^{n-1} \frac{1}{2} \sum_{j=0}^{N_i} \Lambda_{ij}^* \delta k_{ij}} \times \chi^{\widehat{\mathfrak{sl}}(n)_N}(q,t'),
\]

where \(t'_i = q^{-\frac{1}{2}} t_i\).

Remark 5.10. Propositions 5.3 and 5.9 say that the integers \(c_i\) in the first Chern class on the gauge side are identified with the eigenvalues of \(\widehat{\mathfrak{sl}}(n)_N\) currents \(H_i\) on a module \([\mathcal{N}; \delta k]\) specified by \(\mathcal{N}\) and \(\delta k\),

\[
(5.22) \quad H_i = \left( H, \alpha_i \right)_{\text{eigenvalue}} \quad c_i = N_i - \sum_{j=0}^{n-1} \hat{A}_{ij} \delta k_j, \quad i = 0, 1, \ldots, n - 1,
\]

where \(\hat{A}\) is the affine Cartan matrix of type \(A_{n-1}\).

In Section 7 we give examples of Proposition 5.9 for \((N,n) = (2,2), (2,3)\) and \((3,2)\) by comparing with the \(\widehat{\mathfrak{sl}}(n)_N\) WZW characters computed using the following proposition.

Proposition 5.11 (Weyl-Kac character formula \([45]\), see also Appendix B.2 in \([46]\)). The \(\widehat{\mathfrak{sl}}(n)_N\) WZW character of level-\(N\) dominant integral highest-weight \(\Lambda = \sum_{i=0}^{n-1} d_i \Lambda_i, \sum_{i=0}^{n-1} d_i = N\) is

\[
(5.23) \quad \chi^{\widehat{\mathfrak{sl}}(n)_N}(q,t') = \frac{\mathcal{N}_\Lambda(q,x)}{\left( q; q \right)_\infty \prod_{1 \leq i < j \leq n} \left( x_i / x_j ; q \right)_\infty \left( x_j / x_i ; q \right)_\infty^{n-1} \prod_{i=1}^{n-1} t_i^{\frac{1}{2} d_i}},
\]

where

\[
(5.24) \quad \mathcal{N}_\Lambda(q,x) = \sum_{(k_1, \ldots, k_n) \in \mathbb{Z}^n} \det_{1 \leq i < j \leq n} \left( x_i^{n+n} k_i - \mu_i + \mu_j - j \right) q^{\frac{1}{2} (n+n) k_i^2 + (\mu_i - j) k_i}
\]
where $A$ is the Cartan matrix of type $A_{n-1}$.

By the principal specialization in Remark 5.5, the following corollary of Propositions 5.9 and 5.11 is proved \[46\].

**Corollary 5.12** (Equations (5) and (12) in \[46\]). The $t$-refined reduced character \[5.19\] at $t_i = 1$ (see Remark 5.5),

\[
\hat{\chi}_{\mathbf{N}}^{\text{red}}(q, 1) = \left( q; q \right)_{\infty} \times \sum_{\delta k \in \mathbb{Z}^{n-1}} \chi_{(\delta, \delta k)}^{(\ast)}(q) \bigg|_{\sigma_1 \geq \cdots \geq \sigma_N},
\]

agrees, by \[5.20\] (i.e. $d_i = N_i$), with the principal $\mathfrak{sl}(n)_\mathbb{N}$ WZW character \[5.11\]

\[
\text{Pr} \chi_{\mathbf{N}}^{\mathfrak{sl}(n)_\mathbb{N}}(q) = \left( q; q \right)_{\infty} \prod_{1 \leq i < j \leq n+N} \left( q^{\frac{1}{\omega_i}; q^{\frac{1}{\omega_j}}} \right)_{\infty} \prod_{1 \leq i < j \leq n+N} \frac{1}{q^{\frac{1}{\omega_i} + \frac{1}{\omega_j}}},
\]

where $\omega_1 \omega_2 \cdots \omega_{n+N}$ is a binary word of length $n+N$, associated with $\mathbf{N}$, and defined by

\[
\omega_{\ell} = \begin{cases} 
0 & \text{if } \ell \in \{ j + \sum_{i=0}^{j-1} N_i, \ j = 1, \ldots, n \}, \\
1 & \text{if } \ell \notin \{ j + \sum_{i=0}^{j-1} N_i, \ j = 1, \ldots, n \}.
\end{cases}
\]

### 6. Reduced instanton partition functions and $\mathfrak{sl}(n)_\mathbb{N}$ WZW conformal blocks

We discuss how the integrable $\mathfrak{sl}(n)_\mathbb{N}$ WZW conformal blocks are extracted from the $SU(N)$ instanton partition functions on $\mathbb{C}^2/\mathbb{Z}_n$ with $\sum_{i=1}^N a_i = 0$.

#### 6.1. $U(1)$ instanton partition function

In the $U(1)$ case, as was mentioned in Section 5.1 for generic $p$ (generic $\Omega$-background) one obtains the algebra $\mathcal{A}(1, n; p) = \mathcal{H} \oplus \mathfrak{sl}(n)_1$ acting on the equivariant cohomology of the moduli space of $U(1)$ instantons on $\mathbb{C}^2/\mathbb{Z}_n$. Let us consider the instanton partition function \[2.20\] for $N = 1$ with vanishing Coulomb parameter $a = 0$ and first Chern class $c = N_\sigma = [N_0, \ldots, N_{n-1}]$, $N_i = \delta_{i,\sigma}$. Following Remark 5.10, the corresponding module in $\mathfrak{sl}(n)_1$ is the highest-weight module with $\Lambda = \Lambda_\sigma$. We define

\[
Z^b_{\mathbf{N}_\sigma}(m, m'; q) = Z^b_{\sigma, 0}(0, m, m'; q),
\]

and make the following conjecture.
Conjecture 6.1. The $U(1)$ instanton partition function \([6.1]\) on $\mathbb{C}^2/\mathbb{Z}_n$ with $b' = b$ and $N_i = \delta_i, 0$ is

\[ Z_{b,b'}^b(m,m';q) = \left(1 - q\right)^{-2h_b}, \]

where $h_b = h_{N_b} = \frac{b(n-b)}{2n}$ is the conformal dimension of the highest-weight module $|N_b\rangle$ in the $\hat{sl}(n)$ WZW model. The first factor is the $U(1)$ factor $Z_b(m,m';q)$ in (3.11) for $N = 1$, and the second factor is the 2-point function of $\hat{sl}(n)_1$ WZW primary fields with integrable representation $\Lambda = \Lambda_b$.

6.2. $SU(N)$ reduced instanton partition functions. For $N \geq 2$, in the same way that we defined the reduced character \([5.16]\), we now introduce a reduced version of the instanton partition function \([2.20]\) by imposing the specialized Burge conditions \([5.14]\) with $p = N$ and $r_I = 1$, $I = 1, \ldots, N-1$,

\[ Z_{(s),b,b'}^{(s),b}(a, m, m'; q) = \sum_{Y^\sigma \in P^s(\sigma, \delta k)} \frac{Z_{bif}(m, \theta^b; a, Y^\sigma) Z_{bif}(a, Y^\sigma; -m', \theta^{b'})}{Z_{vec}(a, Y^\sigma)} q^{\frac{1}{p} \sum_{I=1}^N |y_{ij}^\sigma|}, \]

where $\sum_{I=1}^N a_I = 0$ is imposed. The Coulomb parameters $a = (a_1, \ldots, a_N)$, and the mass parameters $m = (m_1, \ldots, m_N)$, $m' = (m'_1, \ldots, m'_N)$, are related to the internal momenta $\mu^v$, and the external momenta $\mu_{r=1,2,3,4}$, of a 4-point conformal block in a $\mathcal{W}_{N,n}^{para}$ CFT, by the relations \([3.9]\) and \([3.8]\), respectively. The gauge theory in the rational $\Omega$-background \([3.2]\) for $p = N$,

\[ \frac{\epsilon_1}{\epsilon_2} = -1 - \frac{n}{N}, \]

is expected to describe a minimal model CFT whose momenta take values in the degenerate momenta \([3.12]\) for $r_I = 1$,

\[ 2\mu^v = \sum_{I=1}^{N-1} \left( s_I - 1 \right) \epsilon_2 \Lambda_I, \]

\[ a_I = a_I^s := -\sum_{J=1}^{N-1} \left( \Lambda_J, \epsilon_1 \right) \left( s_J - 1 - \frac{n}{N} \right) \epsilon_2 \]

\[ = -\sum_{J=1}^{N-1} \left( s_J - 1 - \frac{n}{N} \right) \epsilon_2 + \frac{1}{N} \sum_{J=1}^{N-1} J \left( s_J - 1 - \frac{n}{N} \right) \epsilon_2, \]
parametrized by positive integers \( s = (s_1, \ldots, s_{N-1}) \), with \( \sum_{I=1}^{N-1} s_I \leq N + n - 1 \), and

\[
2 \mu_1 = - \sum_{I=1}^{N-1} \left( s_{1,I} - 1 \right) \epsilon_2 \Lambda_I, \quad 2 \mu_2 = - \left( s_{1,N} - 1 \right) \epsilon_2 \Lambda_{N-1},
\]

\[
2 \mu_4 = - \sum_{I=1}^{N-1} \left( s_{2,I} - 1 \right) \epsilon_2 \Lambda_I, \quad 2 \mu_3 = - \left( s_{2,N} - 1 \right) \epsilon_2 \Lambda_1,
\]

\[
(6.6)
\]

\[
m_I = m_{I}^{s_1} := - \left( I - \frac{N + 1}{2} \right) \frac{n}{N} \epsilon_2
\]

\[
+ \frac{1}{N} \left( \sum_{J=1}^{I-1} J \left( s_{1,J} - 1 \right) - \sum_{J=I}^{N-1} \left( N - J \right) \left( s_{1,J} - 1 \right) - \left( s_{1,N} - 1 \right) \right) \epsilon_2,
\]

\[
m'_I = m_{I}^{s_2} := \left( I - \frac{N + 1}{2} \right) \frac{n}{N} \epsilon_2
\]

\[
+ \frac{1}{N} \left( - \sum_{J=1}^{I-1} J \left( s_{2,J} - 1 \right) + \sum_{J=I}^{N-1} \left( N - J \right) \left( s_{2,J} - 1 \right) - \left( s_{2,N} - 1 \right) \right) \epsilon_2,
\]

parametrized by positive integers \( s_1 = (s_1, \ldots, s_{1,N}) \), and \( s_2 = (s_2, \ldots, s_{2,N}) \), with

\[
(6.7) \quad \sum_{I=1}^{N-1} s_{1,I} \leq N + n - 1, \quad s_{1,N} \leq n + 1, \quad \sum_{I=1}^{N-1} s_{2,I} \leq N + n - 1, \quad s_{2,N} \leq n + 1
\]

Here from (3.6), \( \mu_2 \propto \Lambda_{N-1} \) and \( \mu_3 \propto \Lambda_1 \), and the corresponding degenerate momenta are parametrized as

\[
(6.8)
\]

\[
s_c := (s_{c,1}, s_{c,2}, \ldots, s_{c,N-1}) = (1, 1, \ldots, 1, s_{1,N}), \quad 1 \leq s_{1,N} \leq n + 1,
\]

\[
s'_c := (s'_{c,1}, s'_{c,2}, \ldots, s'_{c,N-1}) = (s_{2,N}, 1, 1, \ldots, 1), \quad 1 \leq s_{2,N} \leq n + 1
\]

**Remark 6.2** (Fixing \( s_1, s_2 \)). From Remark 5.7, the (Coulomb) parameters \( s_I \) are determined as \( s_I = s^*_I = \sigma_I - \sigma_{I+1} + 1 \), from the ordered charges \( \sigma_1 \geq \ldots \geq \sigma_N \). Similarly, we fix the (mass) parameters \( s_1 \) and \( s_2 \). Taking a shift by the central \( U(1) \) factor in the \( U(N) \) flavor symmetry, into account, from (3.8) one obtains the boundary charge conditions

\[
(6.9) \quad s_{1,I} - 1 \equiv b_I - b_{I+1} \pmod{n}, \quad s_{2,I} - 1 \equiv b'_I - b'_{I+1} \pmod{n}, \quad I = 1, \ldots, N - 1,
\]

where the parameters \( s_{1,N} \) and \( s_{2,N} \) in (6.8) are not constrained. By relabeling the labels \( I \) of the boundary charges, we order them as \( b_1 \geq \ldots \geq b_N, b'_1 \geq \ldots \geq b'_N \), and then determine the parameters \( s_{1,I} \) and \( s_{2,I} \), \( I = 1, \ldots, N - 1 \), as

\[
(6.10) \quad s_{1,I} = s^*_{1,I} := b_I - b_{I+1} + 1, \quad s_{2,I} = s^*_{2,I} := b'_I - b'_{I+1} + 1, \quad I = 1, \ldots, N - 1
\]

The remaining parameters \( s_{1,N} \) and \( s_{2,N} \) are fixed in Remark 6.4.

Taking the \( U(1) \) factor (3.11) into account, as in the case of Definition 5.8 of the t-refined reduced character, we define a reduced instanton partition function labeled by the non-negative integers \( \mathcal{N} = [N_0, \ldots, N_{n-1}] \), \( \sum_{\sigma=0}^{n-1} N_{\sigma} = N \) in (2.4), the set of integers \( \delta k \), and the boundary charges \( b, b' \).
Definition 6.3 (SU(N) reduced instanton partition function). The SU(N) reduced instanton partition function is defined by

\[
\mathcal{Z}_{\{N;\delta k\}}^{b,b'}(q) = Z_H(m^{s_1}, m^{s_2}; q)^{-1} \times \mathcal{Z}_{\{\sigma; \delta k\}}^{a,a'}(a^{s_1}, m^{s_1}, m^{s_2}; q) \bigg|_{\sigma_1 \geq \ldots \geq \sigma_N},
\]

where, for a choice of N, the ordered charges \(\sigma_1 \geq \ldots \geq \sigma_N\) are uniquely fixed, and \(s_l = s_l^* = \sigma_l - \sigma_{l+1} + 1\) in (6.18), are determined. Then, the Coulomb parameters \(a^{s_1} = (a_1^{s_1}, \ldots, a_N^{s_1})\) are determined by (6.5). Similarly, for fixed ordered boundary charges \(b_1 \geq \ldots \geq b_N\) and \(b'_1 \geq \ldots \geq b'_N\), \(s_{1,l} = s_{1,l}^*\) and \(s_{2,l} = s_{2,l}^*,\) \(I = 1, \ldots, N\) in \((6.10)\) and Remark \(6.4\) are determined. The mass parameters \(m^{s_1} = (m_1^{s_1}, \ldots, m_N^{s_1})\) and \(m^{s_2} = (m_1^{s_2}, \ldots, m_N^{s_2})\) are determined by (6.6). \(^{10}\)

By Proposition \(5.9\) the set \(N\), determined from the integral charges \(\sigma\), indicates level-N dominant integral highest-weight in the \(\hat{sl}(n)_N\) WZW model. We propose that, in the \(\hat{sl}(n)_N\) WZW 4-point conformal blocks, the integrable representations of two of the four external primary fields are also determined by the \(\mathbb{Z}_n\) boundary charges \(b\) and \(b'\) as

\[
b = (b_1, \ldots, b_N) \rightarrow B = \sum_{i=0}^{n-1} B_i \Lambda_i = [B_0, B_1, \ldots, B_{n-1}], \quad \sum_{i=0}^{n-1} B_i = N, \tag{6.12}\n\]

\[
b' = (b'_1, \ldots, b'_N) \rightarrow B' = \sum_{i=0}^{n-1} B'_i \Lambda_i = [B'_0, B'_1, \ldots, B'_{n-1}], \quad \sum_{i=0}^{n-1} B'_i = N, \tag{6.12}\n\]

where \(B_i\) (resp. \(B'_i\)) is the cardinality of subset in \(b\) (resp. \(b'\)) with charge \(i\). Here, for a highest-weight representation \(\Lambda = \sum_{i=0}^{n-1} d_i \Lambda_i\), \(\sum_{i=0}^{n-1} d_i = N\), we can identify \(\Lambda\) with a Young diagram \(\mu\) of \(i\)-row length

\[
\mu_i = \sum_{j=i}^{n-1} d_j, \quad i = 1, \ldots, n-1 \tag{6.13}\n\]

We represent the reduced instanton partition function (6.11), graphically, as

\[
\mathcal{Z}_{\{N;\delta k\}}^{b,b'}(q) =
\]

\[
B_c \quad B_c' \quad B \quad N \quad B' \quad B'
\]

We also represent (6.14) schematically by \(B - B_c - (N) - B_c' - B'\). The representations \(B_c\) and \(B'_c\) of the remaining two of the four external primary fields need to be taken so that they respect the fusion rules, which apply from right to left in (6.14), of the \(\hat{sl}(n)_N\) WZW model when \(N, b\) and \(b'\) are fixed (see e.g. Chapter 16 of \(\text{[17]}\)). Then, the choice of the labels \(\delta k\) on the left hand side of (6.14), which indicate the states of internal channel following Remark \(5.10\) is also restricted by the fusion rules of \(B'\) and \(B'_c\).

\(^{10}\) The ordering of the charges \(\sigma, b\) and \(b'\) is a matter of convention, and it is possible to reorder them and fix \(s, s_1\) and \(s_2\) from the charge conditions \((5.17)\) and \((6.9)\).
Remark 6.4 (Fixing the remaining parameters $s_{1,N}, s_{2,N}$). In Remark 6.2 the parameters $s_{1,I}$ and $s_{2,I}$, $I = 1, \ldots, N-1$, were fixed using the boundary charge conditions. In the following, we fix the remaining parameters $s_{1,N}, s_{2,N}$ using the fusion rules. Let $b_\epsilon = (b_{c,1}, \ldots, b_{c,N})$ and $b'_\epsilon = (b'_{c,1}, \ldots, b'_{c,N})$ be $\mathbb{Z}_n$ boundary charges associated with $B_\epsilon$ and $B'_\epsilon$, respectively. We propose that they satisfy the same type of boundary charge conditions with $[6.3]$ as $s_{c,I} = b_{c,I+1} - b_{c,I}$ (mod $n$) and $s'_{c,I} = b'_{c,I} - b'_{c,I+1}$ (mod $n$) for the (mass) parameters $s_{c,I}$ and $s'_{c,I}$ in $[6.8]$. As a result, these boundary charges are

\[
\begin{align*}
    b_\epsilon &\equiv (b_{c,1}, \ldots, b_{c,N} + s_{1,N}^* - 1) \pmod{n}, \\
    b'_\epsilon &\equiv (b'_{c,1}, \ldots, b'_{c,N} - 1, b'_{c,1}, \ldots, b'_{c,N}) \pmod{n},
\end{align*}
\]

where $b_{c,1}, b'_{c,1} \in \{0, 1, \ldots, n-1\}$, and $s_{1,N} = s_{1,N}^*$, $s_{2,N} = s_{2,N}^*$ should be determined by the fusion rules. For definiteness, we restrict $s_{1,N}^*, s_{2,N}^* \in \{1, \ldots, n\}$, and if $N = 2$ we take $b_{c,1} + s_{1,2}^* \leq n$, $b'_{c,1} + s_{2,2}^* \leq n$ so that the boundary charges are $b_\epsilon = (b_{c,1} + s_{1,2}^* - 1)$ and $b'_\epsilon = (b'_{c,1} + s_{2,2}^* - 1)$.

6.2.1. Conjectures. We propose the following conjectures on the relation between the $SU(N)$ reduced instanton partition functions $[6.11]$ on $\mathbb{C}^2/\mathbb{Z}_n$ and the $\mathfrak{s}\mathfrak{l}(n)_N$ WZW conformal blocks.\(^1\)

Conjecture 6.5 ($\emptyset - \emptyset - (\emptyset) - \emptyset - \emptyset$). The $\mathfrak{s}(n)_N$ WZW 2-point conformal block of the type

\[
\langle \emptyset(1) \emptyset(q) \rangle_{\mathfrak{s}(n)_N}
\]

agrees with the following reduced instanton partition function

\[
\begin{align*}
B_\epsilon &= \emptyset & B'_\epsilon &= \emptyset \\
B &= \emptyset & N &= \emptyset & B' &= \emptyset
\end{align*}
\]

\[
\mathcal{Z}_{[N,0,\ldots,0]}^{0,0}(q) = \mathcal{Z}_{[N,0,\ldots,0]}(q) = \left(1 - q\right)^{-2h_0} = 1
\]

Here $s = (1, \ldots, 1)$, $s_1 = (1, \ldots, 1)$ and $s_2 = (1, \ldots, 1)$ are fixed by $[6.18]$, $[6.10]$ and $[6.15]$, and $h_0 = 0$ is the conformal dimension for the representation $\emptyset = [N,0,\ldots,0]$.

Conjecture 6.6 ($\emptyset - [N-1,0,\ldots,0,1] - (\square) - \square - \emptyset$). The $\mathfrak{s}(n)_N$ WZW 2-point conformal block of the type

\[
\langle \bigcirc(1) \square(q) \rangle_{\mathfrak{s}(n)_N}
\]

agrees with the following reduced instanton partition function

\[
\begin{align*}
B_\epsilon &= \bigcirc(n-1) & B'_\epsilon &= \square \\
B &= \emptyset & N &= \emptyset & B' &= \emptyset
\end{align*}
\]

\[
\mathcal{Z}_{[N-1,1,0,\ldots,0]}^{0,0}(q) = \mathcal{Z}_{[N-1,1,0,\ldots,0]}(q) = \left(1 - q\right)^{-2h_\square}
\]

\(^1\) In $[6.11]$, we have subtracted the $U(1)$ factor $Z_H(m^{s_1}, m^{s_2}; q)$. If this factor is included, $\mathfrak{s}(n)_N$ WZW conformal blocks should be obtained.
Here \( s = (2, 1, \ldots, 1), \ s_1 = (1, \ldots, 1, n) \) and \( s_2 = (1, \ldots, 1, 2) \) are fixed by (5.18), (6.10) and (6.15), and \( h_\square = \frac{n^2 - 1}{2(n + N)} \) is the conformal dimension for the representation \( \square = [N - 1, 1, 0 \ldots, 0] \).

**Conjecture 6.7** \( (\square \square - (\emptyset \text{ or } [N - 2, 1, 0 \ldots, 0, 1]) - \square - [N - 1, 0 \ldots, 0, 1]) \). The \( \widehat{\mathfrak{sl}}(n)_N \) WZW 4-point conformal blocks of the type

\[
(\square(\infty) \square(1) \square(q) \square(0))_{\widehat{\mathfrak{sl}}(n)_N},
\]

which are \([B.3] \) in Appendix \([B] \) agree with, up to certain overall factors, the following reduced instanton partition functions \([12] \).

\[
(6.18) \quad \tilde{Z}^{(1,0,\ldots,0),(n-1,0,\ldots,0)}_{[N',0,\ldots,0;\delta_k]}(q) = b_{(1,0,\ldots,0)} \sigma_{(0,\ldots,0)} b'_{(n-1,0,\ldots,0)} \quad \begin{array}{c}
B = \square \quad B' = \square \quad B = \square \quad N = \emptyset \quad B' = \begin{vertiarray}{c}
\vdots
\end{vertiarray}\end{array}
\]

\[
= \begin{cases}
1 - q^{\frac{2h}{n+1}} \prod_{n+1}^{2F_1} \left(- \frac{n-1}{n+1}; \frac{n}{n+1}; q \right), & \text{for } \delta k = 0, \\
\frac{1}{n} q^{\frac{1}{n} - \frac{h}{2}} \left(1 - q^{\frac{2h}{n+1}} \prod_{n+1}^{2F_1} \left(- \frac{n-1}{n+1}; \frac{n}{n+1}; q \right), & \text{for } \delta k = (-1, \ldots, -1),
\end{cases}
\]

and

\[
(6.19) \quad \tilde{Z}^{(1,0,\ldots,0),(n-1,0,\ldots,0)}_{[N-2,1,0,\ldots,0;1;\delta_k]}(q) = b_{(1,0,\ldots,0)} \sigma_{(n-1,1,0,\ldots,0)} b'_{(n-1,0,\ldots,0)} \quad \begin{array}{c}
B = \square \quad B' = \square \quad B = \square \quad N = \begin{vertiarray}{c}
\vdots
\end{vertiarray}\end{array}
\]

\[
= \begin{cases}
1 - q^{\frac{2h}{n+1}} \prod_{n+1}^{2F_1} \left(- \frac{1}{n+1}; \frac{n-1}{n+1}; \frac{n}{n+1}; q \right), & \text{for } \delta k = 0, \\
\frac{1}{n} q^{\frac{1}{n} - \frac{h}{2}} \left(1 - q^{2h} \prod_{n+1}^{2F_1} \left(- \frac{1}{n+1}; \frac{n-1}{n+1}; \frac{n}{n+1}; q \right), & \text{for } \delta k = (1, \ldots, 1)
\end{cases}
\]

Here, by (5.18), (6.10), and (6.15), for (6.18) the parameters \( s = (1, \ldots, 1), \ s_1 = (2, 1, \ldots, 1, 2) \) and \( s_2 = (n, 1, \ldots, 1, 2) \) are fixed, and for (6.19) the parameters \( s = (n - 1, 2, 1, \ldots, 1), \ s_1 = (2, 1, \ldots, 1, 2) \) and \( s_2 = (n, 1, \ldots, 1, 2) \) are fixed. The integral charges \( \delta k \) are taken so that the corresponding modules on the CFT side, following Remark 5.10, are in the fundamental chamber under the action of affine Weyl group of \( \widehat{\mathfrak{sl}}(n) \), and the second ones in (6.18) and (6.19) respect the fusion rules by

\[
(6.20) \quad \begin{array}{c}
N = [N, 0, \ldots, 0] = \emptyset \quad \delta k = (-1, \ldots, -1) \quad [N - 2, 1, 0, \ldots, 0, 1],
\end{array}
\]

\[
(6.20) \quad \begin{array}{c}
N = [N - 2, 1, 0, \ldots, 0, 1] \quad \delta k = (1, \ldots, 1) \quad [N, 0, \ldots, 0] = \emptyset,
\end{array}
\]

where, when \( n = 2 \), \( [N - 2, 1, 0, \ldots, 0, 1] \) means \( [N - 2, 2] = \square \) and then \( \sigma = (1, 1, 0, \ldots, 0) \).

\( [B.5] \) and (6.18) correspond to, respectively, the 4-point WZW conformal blocks \( \tilde{Z}(0)_{i=1,2}(q) \) and \( \tilde{Z}(1)_{i=1,2}(q) \) in [B.5].
7. Examples of SU(N) reduced instanton counting on \( \mathbb{C}^2/\mathbb{Z}_n \)

We illustrate the statement of Proposition 5.7 and check Conjectures 6.5, 6.6 and 6.7 for \((N, n) = (2, 2), (2, 3)\) and \((3, 2)\). In particular we demonstrate how one can extract their \(\widehat{\mathfrak{sl}}(n)\) WZW conformal blocks from the reduced instanton partition functions.\(^{13}\)

7.1. \((N, n) = (2, 2)\) and \(\widehat{\mathfrak{sl}}(2)_2\) WZW model. For \((N, n) = (2, 2)\), there are three highest-weight representations

\[
\emptyset = [2, 0], \quad \blacksquare = [1, 1], \quad \boxdot = [0, 2],
\]

with conformal dimensions

\[
h_{k\Lambda_1} = \frac{k(k+2)}{16}; \quad h_{\emptyset} = 0, \quad h_{\blacksquare} = \frac{3}{16}, \quad h_{\boxdot} = \frac{1}{2}.
\]

7.1.1. \(t\)-refined reduced characters. The \(t\)-refined reduced characters (7.19) for \((N, n) = (2, 2)\) are obtained as

\[
\hat{\chi}_{(2,0)}^{\text{red}}(q,t) = \left( q; q \right) \sum_{\ell \in \mathbb{Z}} \chi_{(0,0; -\ell)}^{(1)}(q) t^\ell = \chi_{\text{even}}(q) f_0(q,t') + \chi_{\text{odd}}(q) f_1(q,t'),
\]

\[
\hat{\chi}_{(0,2)}^{\text{red}}(q,t) = \left( q; q \right) \sum_{\ell \in \mathbb{Z}} \chi_{(1,1; -\ell)}^{(1)}(q) t^{\ell+1} = \chi_{\text{even}}(q) f_1(q,t') + \chi_{\text{odd}}(q) f_0(q,t'),
\]

\[
\hat{\chi}_{(1,1)}^{\text{red}}(q,t) = \left( q; q \right) \sum_{\ell \in \mathbb{Z}} \chi_{(1,0; -\ell)}^{(2)}(q) t^{\ell+\frac{1}{2}} = \chi_R(q) g(q,t'),
\]

where \(t' = q^{-\frac{1}{2}} t\),

\[
\chi_{\text{even}}(q) + \chi_{\text{odd}}(q) = \chi_{\text{NS}}(q) = \left( \frac{-q^{\frac{1}{2}}; q}{q; q} \right)_{\infty}, \quad \chi_R(q) = \left( \frac{-q; q}{q; q} \right)_{\infty},
\]

(7.4)

\[
\chi_{\text{even}}(q) = 1 + q + 3q^2 + 5q^3 + 10q^4 + 16q^5 + 28q^6 + 43q^7 + 70q^8 + 105q^9 + 161q^{10} + \cdots,
\]

\[
\chi_{\text{odd}}(q) = q^{\frac{1}{2}} + 2q^\frac{3}{2} + 4q^2 + 7q^2 + 13q^2 + 21q^3 + 35q^3 + 55q^4 + 86q^4 + 130q^5 + \cdots,
\]

and

\[
f_\sigma(q,t') = \sum_{j \in \mathbb{Z}+\frac{1}{2}} q^{\frac{1}{2}j^2} t'^j, \quad \sigma = 0, 1, \quad g(q,t') = \sum_{j \in \mathbb{Z}+\frac{1}{2}} q^{\frac{1}{2}j^2+\frac{1}{2}} t'^j\]

The characters (7.3) agree with the \(\widehat{\mathfrak{sl}}(2)_2\) WZW characters computed by (5.23),

\[
\hat{\chi}_{(2,0)}^{\text{red}}(q,t) = \hat{\chi}_{(2,0)}^{\widehat{\mathfrak{sl}}(2)}(q,t'), \quad \hat{\chi}_{(0,2)}^{\text{red}}(q,t) = \hat{\chi}^{\frac{1}{2}} \chi_{(0,2)}^{\widehat{\mathfrak{sl}}(2)}(q,t'), \quad \hat{\chi}_{(1,1)}^{\text{red}}(q,t) = q^{\frac{1}{2}} \hat{\chi}^{\frac{1}{2}} \chi_{(1,1)}^{\widehat{\mathfrak{sl}}(2)}(q,t'),
\]

and Proposition 5.9 is confirmed. Using the Jacobi triple product identity

\[
\sum_{\ell \in \mathbb{Z}} x^\ell y^{2\ell-1} = \left( -x; y \right)_{\infty} \left( -\frac{y}{x}; y \right)_{\infty} \left( y; y \right)_{\infty},
\]

\(^{13}\) The computations in this section heavily rely on Mathematica. We have also checked Conjectures 6.5, 6.6 and 6.7 for \((N, n) = (2, 4)\) up to \(O(q^5)\).
we can also confirm Corollary 5.12 for the principal characters of $\hat{\mathfrak{l}}(2)_2$,

\[
\hat{\chi}^{\text{red}}_{[2,0]}(q, 1) = \hat{\chi}^{\text{red}}_{[0,2]}(q, 1) = \Pr_{\hat{\chi}^{\text{red}}_{[2,0]}(q)} = \left\langle -q^{\frac{3}{2}}; q^\frac{3}{2} \right\rangle_\infty \left\langle -q; q \right\rangle_\infty,
\]

\[
\hat{\chi}^{\text{red}}_{[1,1]}(q, 1) = \Pr_{\hat{\chi}^{\text{red}}_{[1,1]}(q)} = \left\langle -q^{\frac{3}{2}}; q^\frac{1}{2} \right\rangle_\infty \left\langle -q^{\frac{1}{2}}; q^{\frac{1}{2}} \right\rangle_\infty.
\]

**Remark 7.1.** Using the above notation, the characters in Example 2.3 can be written as

\[
\left\langle q; q \right\rangle_\infty \sum_{\ell \in \mathbb{Z}} \chi_{(0,0; -\ell)}(q) t^\ell = (\chi_{\text{even}}(q)^2 + \chi_{\text{odd}}(q)^2) f_0(q, t') + 2 \chi_{\text{even}}(q) \chi_{\text{odd}}(q) f_1(q, t'),
\]

\[
\left\langle q; q \right\rangle_\infty \sum_{\ell \in \mathbb{Z}} \chi_{(1,1; -\ell)}(q) t^{\ell+1} = (\chi_{\text{even}}(q)^2 + \chi_{\text{odd}}(q)^2) f_1(q, t') + 2 \chi_{\text{even}}(q) \chi_{\text{odd}}(q) f_0(q, t'),
\]

\[
\left\langle q; q \right\rangle_\infty \sum_{\ell \in \mathbb{Z}} \chi_{(1,0; -\ell)}(q) t^{\ell+\frac{3}{2}} = \chi_R(q)^2 g(q, t').
\]

By comparing these characters with the reduced characters in (7.3), we see that the Burge conditions indeed trivialize the characters in $\mathcal{N} = 1$ super-Virasoro algebra as $\chi_{\text{even}}(q) = 1$, $\chi_{\text{odd}}(q) = 0$, (i.e. $\chi_{\text{NS}}(q) = 1$ and $\chi_R(q) = 1$). More precisely, we see that $2 \chi_{\text{even}}(q) \chi_{\text{odd}}(q)$ means $\chi_{\text{even}}(q) \chi_{\text{odd}}(q) + \chi_{\text{odd}}(q) \chi_{\text{even}}(q)$ and the first factor in each term describes the NS sector characters.

### 7.1.2. Reduced instanton partition functions

For $N = 2$ with general $n$, the reduced instanton partition functions (6.11) are determined by the parameters $s, s_1 = (s_{1,1}, s_{1,2})$ and $s_2 = (s_{2,1}, s_{2,2})$, which take values in $\{1, \ldots, n\}$, fixed in (5.18), (6.10):

\[
s = \sigma_1 - \sigma_2 + 1, \quad s_{1,1} = b_1 - b_2 + 1, \quad s_{2,1} = b'_1 - b'_2 + 1,
\]

and (6.15) from the ordered charges $\sigma_1 \geq \sigma_2$, $b_1 \geq b_2$ and $b'_1 \geq b'_2$. The Coulomb parameters are then determined from the parameter $s$ by (6.5):

\[
a_1 = -\frac{1}{2} \left( s - 1 - \frac{n}{2} \right) \epsilon_2, \quad a_2 = \frac{1}{2} \left( s - 1 - \frac{n}{2} \right) \epsilon_2,
\]

and the mass parameters $m = (m_1, m_2)$ and $m' = (m'_1, m'_2)$ are determined from the parameters $s_1$ and $s_2$, respectively, by (6.6).

Let us consider the case of $(N, n) = (2, 2)$ with the rational $\Omega$-background $\epsilon_1/\epsilon_2 = -2$ in (6.4)\textsuperscript{14}

**Example 7.2** $(\emptyset - \emptyset - (\emptyset) - \emptyset - (\emptyset))$. Consider the reduced instanton partition function $\hat{Z}_{[2,0; \ell]}^{(0,0),(0,0)}(q)$ and take $\ell = 0$ in the fundamental chamber, which respects the fusion rules, as in Conjecture 6.5

Here the parameters $s = 1$, $s_1 = (1, 1)$ and $s_2 = (1, 1)$ are fixed. Then, the reduced instanton partition function is obtained as

\[
\hat{Z}_{[2,0;0]}^{(0,0),(0,0)}(q) = \left( 1 - q \right)^{-2h_0} = 1, \quad h_0 = 0,
\]

and Conjecture 6.5 is confirmed.

\textsuperscript{14} Examples 7.2, 7.3 and 7.4 are confirmed up to $O(q^6)$. 

Example 7.3 ($\emptyset - \Box - (\Box) - \Box - \emptyset$). Consider the reduced instanton partition function $\hat{Z}^{(0,0),(0,0)}_{[1,1,\ell]}(q)$ and take $\ell = 0$ in the fundamental chamber as in Conjecture 6.6. Here the parameters $s = 2$, $s_1 = (1,2)$ and $s_2 = (1,2)$ are fixed. Then we see that the reduced instanton partition function is

$$\hat{Z}^{(0,0),(0,0)}_{[1,1,0]}(q) = \left(1 - q \right)^{-2h_{\Box}} = 1 + \frac{3q}{8} + \frac{33q^2}{128} + \frac{209q^3}{1024} + \frac{5643q^4}{32768} + \frac{39501q^5}{262144} + \cdots ,$$

where $h_{\Box} = 3/16$, and Conjecture 6.6 is confirmed.

Example 7.4 ($\Box - \Box - (\emptyset) - \Box - \Box$ and $\Box - \Box - (\emptyset) - \Box - \Box$). As Conjecture 6.7, consider, first, the reduced instanton partition function $\hat{Z}^{(1,0),(1,0)}_{[2,0,\ell]}(q)$, where the parameters $s = 1$, $s_1 = (2,2)$ and $s_2 = (2,2)$ are fixed. Then we find that the reduced instanton partition functions for $\ell = 0, -1$ in the fundamental chamber are

$$\hat{Z}^{(1,0),(1,0)}_{[2,0,0]}(q) = \left(1 - q \right)^{2h_{\Box}} \frac{3}{2} F_1 \left( -\frac{1}{4}, \frac{1}{4}; \frac{1}{2}; q \right)$$

$$= 1 + \frac{q}{4} + \frac{11q^2}{64} + \frac{35q^3}{256} + \frac{949q^4}{8192} + \frac{3333q^5}{32768} + \frac{47909q^6}{524288} + \cdots ,$$

(7.14)

$$\hat{Z}^{(1,0),(1,0)}_{[2,0,-1]}(q) = \frac{q^{1/2}}{2} \left(1 - q \right)^{2h_{\Box}} \frac{3}{2} F_1 \left( \frac{3}{4}, \frac{3}{4}; \frac{3}{2}; q \right)$$

$$= \frac{q^{1/2}}{2} + \frac{q^2}{4} + \frac{23q^3}{128} + \frac{37q^4}{256} + \frac{2013q^5}{16384} + \frac{3537q^6}{32768} + \cdots ,$$

where $h_{\Box} = 3/16$, and the second one respects the fusion rules by (6.20). Consider, next, the reduced instanton partition function $\hat{Z}^{(1,0),(1,0)}_{[0,2,\ell]}(q)$, where the parameters $s = 1$, $s_1 = (2,2)$ and $s_2 = (2,2)$ are fixed. Then we obtain the reduced instanton partition functions for $\ell = 0, 1$ in the fundamental chamber as

$$\hat{Z}^{(1,0),(1,0)}_{[0,2,0]}(q) = \left(1 - q \right)^{2h_{\Box}} \frac{3}{2} F_1 \left( -\frac{1}{4}, \frac{1}{4}; \frac{1}{2}; q \right) = \hat{Z}^{(1,0),(1,0)}_{[2,0,0]}(q),$$

(7.15)

$$\hat{Z}^{(1,0),(1,0)}_{[0,2,1]}(q) = \frac{q^{1/2}}{2} \left(1 - q \right)^{2h_{\Box}} \frac{3}{2} F_1 \left( \frac{3}{4}, \frac{3}{4}; \frac{3}{2}; q \right) = \hat{Z}^{(1,0),(1,0)}_{[2,0,-1]}(q),$$

where the second one respects the fusion rules by (6.20). The above results (7.14) and (7.15) support Conjecture 6.7. By

$$2F_1 \left( -\frac{1}{4}, \frac{1}{4}; \frac{1}{2}; q \right) = \left( 1 + \sqrt{1 - q} \right) ^{1/2}, \quad \frac{q^{1/2}}{2} 2F_1 \left( \frac{3}{4}, \frac{3}{4}; \frac{3}{2}; q \right) = \left( 1 - \sqrt{1 - q} \right) ^{1/2},$$

(7.16)

they are also consistent with the results in [13] by Belavin and Mukhametzhanov.\footnote{More precisely, in [13], the generic $\Omega$-background, without the Burge conditions, was discussed. Then the first one of (7.14) and the second one of (7.15), with $\ell = 0$, were obtained as prefactors combined with the $N = 1$ super-Virasoro Ramond conformal blocks $H_{\pm}(q)$, $F_{\pm}(q)$, $\bar{H}_{\pm}(q)$ and $\bar{F}_{\pm}(q)$. What we found is that, when we impose the specific Burge conditions, the conformal blocks are trivialized as $H_{\pm}(q) = F_{\pm}(q) = 1$ and $\bar{H}_{\pm}(q) = \bar{F}_{\pm}(q) = 0$, and only the prefactors are obtained.}
7.2. \((N, n) = (2, 3)\) and \(\widehat{sl}(3)_2\) WZW model. For \((N, n) = (2, 3)\), there are six highest-weight representations

\[(7.17) \quad \emptyset = [2, 0, 0], \quad \mathfrak{a} = [1, 1, 0], \quad \mathfrak{m} = [0, 2, 0], \quad \mathfrak{b} = [1, 0, 1], \quad \mathfrak{f} = [0, 1, 1], \quad \mathfrak{m} = [0, 0, 2],\]

with conformal dimensions

\[(7.18) \quad h_{k_1\Lambda_1+k_2\Lambda_2} = \frac{k_1^2 + k_2^2 + k_1k_2 + 3k_1 + 3k_2}{15}; \quad h_\emptyset = 0, \quad h_{\mathfrak{a}} = h_{\mathfrak{b}} = \frac{4}{15}, \quad h_{\mathfrak{m}} = h_{\mathfrak{f}} = \frac{2}{3}, \quad h_{\mathfrak{m}} = \frac{3}{5}.\]

7.2.1. \(t\)-refined reduced characters. The \(t\)-refined reduced characters \((7.19)\) for \((N, n) = (2, 3)\) are obtained as

\[(7.19) \quad \check{\chi}_{[2,0,0]}^\text{red}(q, t_1, t_2) = \left( q; q \right) \sum_{(\ell_1, \ell_2) \in \mathbb{Z}^2} \chi_{(0,0; -\ell_1, -\ell_2)}(q) t_1^{2\ell_1 - \ell_2} t_2^{\frac{1}{3}\left(-\ell_1 + 2\ell_2\right)}
\]

\[= \chi_A(q) f_{00}(q, t_1', t_2') + \chi_B(q) g_{00}(q, t_1', t_2'),\]

\[\check{\chi}_{[0,2,0]}^\text{red}(q, t_1, t_2) = \left( q; q \right) \sum_{(\ell_1, \ell_2) \in \mathbb{Z}^2} \chi_{(1,1; -\ell_1, -\ell_2)}(q) t_1^{2\ell_1 - \ell_2} t_2^{\frac{1}{3}\left(-\ell_1 + 2\ell_2\right)}
\]

\[= \chi_A(q) f_{10}(q, t_1', t_2') + \chi_B(q) g_{10}(q, t_1', t_2'),\]

\[\check{\chi}_{[0,0,2]}^\text{red}(q, t_1, t_2) = \left( q; q \right) \sum_{(\ell_1, \ell_2) \in \mathbb{Z}^2} \chi_{(2,2; -\ell_1, -\ell_2)}(q) t_1^{2\ell_1 - \ell_2} t_2^{\frac{1}{3}\left(2 - \ell_1 + 2\ell_2\right)}
\]

\[= \chi_A(q) f_{01}(q, t_1', t_2') + \chi_B(q) g_{01}(q, t_1', t_2'),\]

\[\check{\chi}_{[1,1,0]}^\text{red}(q, t_1, t_2) = \left( q; q \right) \sum_{(\ell_1, \ell_2) \in \mathbb{Z}^2} \chi_{(1,0; -\ell_1, -\ell_2)}(q) t_1^{1+2\ell_1 - \ell_2} t_2^{\frac{1}{3}\left(-\ell_1 + 2\ell_2\right)}
\]

\[= \chi_C(q) g_{01}(q, t_1', t_2') + \chi_D(q) f_{01}(q, t_1', t_2'),\]

\[\check{\chi}_{[0,1,1]}^\text{red}(q, t_1, t_2) = \left( q; q \right) \sum_{(\ell_1, \ell_2) \in \mathbb{Z}^2} \chi_{(2,1; -\ell_1, -\ell_2)}(q) t_1^{1+2\ell_1 - \ell_2} t_2^{\frac{1}{3}\left(1-\ell_1 + 2\ell_2\right)}
\]

\[= \chi_C(q) g_{00}(q, t_1', t_2') + \chi_D(q) f_{00}(q, t_1', t_2'),\]

\[\check{\chi}_{[1,0,1]}^\text{red}(q, t_1, t_2) = \left( q; q \right) \sum_{(\ell_1, \ell_2) \in \mathbb{Z}^2} \chi_{(2,0; -\ell_1, -\ell_2)}(q) t_1^{2\ell_1 - \ell_2} t_2^{\frac{1}{3}\left(1-\ell_1 + 2\ell_2\right)}
\]

\[= \chi_C(q) g_{10}(q, t_1', t_2') + \chi_D(q) f_{10}(q, t_1', t_2'),\]

where \(t_1' = q^{-1} t_1, \quad t_2' = q^{-1} t_2,\)

\[(7.20) \quad \chi_A(q) = 1 + 2q + 8q^2 + 20q^3 + 52q^4 + 116q^5 + 256q^6 + 522q^7 + \cdots,\]

\[\chi_B(q) = q^{\frac{1}{3}} + 4q^{\frac{2}{3}} + 12q^{\frac{5}{3}} + 32q^2 + 77q^{\frac{8}{3}} + 172q^{\frac{11}{3}} + 365q^3 + 740q^{\frac{14}{3}} + \cdots,\]

\[\chi_C(q) = 1 + 4q + 13q^2 + 36q^3 + 89q^4 + 204q^5 + 441q^6 + 908q^7 + \cdots,\]

\[\chi_D(q) = 2q^{\frac{2}{3}} + 7q^{\frac{5}{3}} + 22q^{\frac{8}{3}} + 56q^2 + 136q^{\frac{11}{3}} + 300q^3 + 636q^{\frac{14}{3}} + 1280q^4 + \cdots,\]
and \( f_{00}, f_{10}, f_{01}, g_{00}, g_{10}, g_{01} \) are

\[
\begin{align*}
\text{Example 7.5} & \quad (\theta - \theta - (\theta - \theta)) = (0, 0) \\
\text{Example 7.7.} & \quad \text{Reduced instanton partition functions. For } (N, n) = (2, 3), \text{ the rational } \Omega\text{-background } (6.4) \text{ yields } \epsilon_1/\epsilon_2 = -5/2. \text{ The parameters } s, s_1 = (s_{1,1}, s_{1,2}) \text{ and } s_2 = (s_{2,1}, s_{2,2}), \text{ which determine the reduced instanton partition functions, are fixed as in } (7.10).\]

\[
(7.24) \quad \mathcal{Z}_{[2,0,0,0]}^{(0,0),(0,0)}(q) = \left( 1 - q \right)^{-2h_0} = 1, \quad h_0 = 0,
\]

and Conjecture 6.5 is confirmed. 

\[16\] Examples 7.5, 7.4 and 7.7 are confirmed up to \( O(q^0) \).
Example 7.6 \((\emptyset - \sbullet - (\sbullet) - \sbullet - \emptyset)\). Consider the reduced instanton partition function \(\widehat{Z}_{[1,1,0,\ell_1,\ell_2]}^{(0,0),(0,0)}(q)\) and take \((\ell_1,\ell_2) = (0,0)\) in the fundamental chamber as in Conjecture 6.6. Here the parameters \(s = 2, s_1 = (1,3)\) and \(s_2 = (1,2)\) are fixed. Then the reduced instanton partition function is

\[
(7.25) \quad \widehat{Z}_{[1,1,0,0,0]}^{(0,0),(0,0)}(q) = \left(1 - q\right)^{-2h_{\sbullet}} = 1 + \frac{8q}{15} + \frac{92q^2}{225} + \frac{3496q^3}{10125} + \frac{4632q^4}{151875} + \frac{3149896q^5}{11390625} + \cdots ,
\]

where \(h_{\sbullet} = 4/15\), and Conjecture 6.6 is confirmed.

Example 7.7 \((\sbullet - \sbullet - \emptyset) - \sbullet - \emptyset\) and \(\sbullet - \sbullet - (\sbullet) - \sbullet - \emptyset\). As Conjecture 6.7 consider, first, the reduced instanton partition function \(\widehat{Z}_{[2,0,0,\ell_1,\ell_2]}^{(1,0),(2,0)}(q)\), where the parameters \(s = 1, s_1 = (2,2)\) and \(s_2 = (3,2)\) are fixed. Then we find that the reduced instanton partition functions for \((\ell_1,\ell_2) = (0,0)\) and \((-1,-1)\) in the fundamental chamber are

\[
\widehat{Z}_{[2,0,0,0,0]}^{(1,0),(2,0)}(q) = \left(1 - q\right)^{2h_{\sbullet}^{-4}} \frac{1}{2} F_1 \left(-\frac{1}{5}, \frac{2}{5}; q\right)
= 1 + \frac{q}{6} + \frac{34q^2}{315} + \frac{67q^3}{810} + \frac{49309q^4}{72295} + \frac{254267q^5}{4337550} + \cdots ,
\]

(7.26)

\[
\widehat{Z}_{[2,0,0,-1,-1]}^{(1,0),(2,0)}(q) = \frac{q^{\sbullet}}{2} \left(1 - q\right)^{2h_{\sbullet}^{-4}} \frac{1}{2} F_1 \left(-\frac{1}{5}, \frac{4}{5}; q\right)
= \frac{q^{\sbullet}}{2} + \frac{4q^2}{21} + \frac{79q^3}{630} + \frac{4619q^4}{48195} + \frac{16237q^5}{206550} + \cdots ,
\]

where \(h_{\sbullet} = 4/15\), and the second one respects the fusion rules by (6.20). Consider, next, the reduced instanton partition function \(\widehat{Z}_{[0,1,0,\ell_1,\ell_2]}^{(1,0),(2,0)}(q)\), where the parameters \(s = 2, s_1 = (2,2)\) and \(s_2 = (3,2)\) are fixed. Then we see that the reduced instanton partition functions for \((\ell_1,\ell_2) = (0,0)\) and \((1,1)\) in the fundamental chamber are

\[
\widehat{Z}_{[0,1,1,0,0]}^{(1,0),(2,0)}(q) = \left(1 - q\right)^{2h_{\sbullet}^{-4}} \frac{1}{2} F_1 \left(-\frac{1}{5}, \frac{2}{5}; q\right)
= 1 + \frac{2q}{15} + \frac{13q^2}{150} + \frac{8792q^3}{131625} + \frac{218507q^4}{394875} + \frac{54190157q^5}{1135265625} + \cdots ,
\]

(7.27)

\[
\widehat{Z}_{[0,1,1,1,1]}^{(1,0),(2,0)}(q) = \frac{q^{\sbullet}}{3} \left(1 - q\right)^{2h_{\sbullet}^{-4}} \frac{1}{2} F_1 \left(-\frac{1}{5}, \frac{4}{5}; q\right)
= \frac{q^{\sbullet}}{3} + \frac{7q^2}{45} + \frac{1867q^3}{17550} + \frac{32582q^4}{394875} + \frac{18575621q^5}{272463750} + \cdots ,
\]

where the second one respects the fusion rules by (6.20). The above results (7.26) and (7.27) support Conjecture 6.7.

7.3. \((N, n) = (3, 2)\) and \(\widehat{sl}(2)_3\) WZW model. For \((N, n) = (3, 2)\), there are four highest-weight representations

\[
(7.28) \quad \emptyset = [3, 0], \quad \sbullet = [2, 1], \quad \sbullet = [1, 2], \quad \sbullet = [0, 3],
\]

with conformal dimensions

\[
(7.29) \quad h_{k\Lambda_1} = \frac{k (k + 2)}{20}, \quad h_\emptyset = 0, \quad h_{\sbullet} = \frac{3}{20}, \quad h_{\sbullet} = \frac{3}{5}, \quad h_{\sbullet} = \frac{3}{4}.
\]
7.3.1. t-refined reduced characters. The t-refined reduced characters (5.19) for \((N, n) = (3, 2)\) are obtained as

\[
\hat{\chi}_{[3,0]}^{\text{red}}(q, t) = \left( q; q \right) \sum_{\ell \in \mathbb{Z}} \chi_{(0,0,0; -\ell)}^{(1,1)}(q) t^{\ell} = \chi_{A}(q) f_{0}(q, t') + \chi_{B}(q) g_{0}(q, t'),
\]
\[
\hat{\chi}_{[0,3]}^{\text{red}}(q, t) = \left( q; q \right) \sum_{\ell \in \mathbb{Z}} \chi_{(1,1,0; -\ell)}^{(1,1)}(q) t^{\ell+\frac{1}{2}} = \chi_{A}(q) f_{1}(q, t') + \chi_{B}(q) g_{1}(q, t'),
\]
\[
\hat{\chi}_{[2,1]}^{\text{red}}(q, t) = \left( q; q \right) \sum_{\ell \in \mathbb{Z}} \chi_{(1,0,0; -\ell)}^{(2,1)}(q) t^{\ell+\frac{1}{2}} = \chi_{C}(q) g_{1}(q, t') + \chi_{D}(q) f_{1}(q, t'),
\]
\[
\hat{\chi}_{[1,2]}^{\text{red}}(q, t) = \left( q; q \right) \sum_{\ell \in \mathbb{Z}} \chi_{(1,1,0; -\ell)}^{(1,2)}(q) t^{\ell+1} = \chi_{C}(q) g_{0}(q, t') + \chi_{D}(q) f_{0}(q, t'),
\]

where \(t' = q^{-\frac{1}{2}} t\),

\[
\chi_{A}(q) = 1 + q + 3q^{2} + 6q^{3} + 12q^{4} + 21q^{5} + 39q^{6} + 64q^{7} + 108q^{8} + \cdots,
\]
\[
\chi_{B}(q) = q^{\frac{1}{2}} + 2q^{\frac{3}{2}} + 5q^{2} + 9q^{\frac{5}{2}} + 18q^{3} + 31q^{\frac{7}{2}} + 55q^{4} + 90q^{\frac{9}{2}} + 149q^{5} + \cdots,
\]
\[
\chi_{C}(q) = 1 + 2q + 5q^{2} + 10q^{3} + 20q^{4} + 36q^{5} + 64q^{6} + 108q^{7} + 180q^{8} + \cdots,
\]
\[
\chi_{D}(q) = q^{\frac{1}{2}} + 3q^{\frac{3}{2}} + 6q^{2} + 13q^{\frac{5}{2}} + 24q^{3} + 44q^{\frac{7}{2}} + 76q^{4} + 129q^{\frac{9}{2}} + 210q^{5} + \cdots,
\]

and

\[
f_{\sigma}(q, t') = \sum_{j \in \mathbb{Z} + \frac{1}{2} \sigma} q^{j} t^{j}, \quad g_{\sigma}(q, t') = \sum_{j \in \mathbb{Z} + \frac{1}{2} \sigma} q^{j} t^{j}, \quad \sigma = 0, 1
\]

The characters (7.30) agree with the \(\widehat{sl}(2)_{3}\) WZW characters computed by (5.23),

\[
\hat{\chi}_{[3,0]}^{\text{red}}(q, t) = \chi_{[3,0]}^{\widehat{sl}(2)_{3}}(q, t'), \quad \hat{\chi}_{[0,3]}^{\text{red}}(q, t) = q^{\frac{1}{2}} \chi_{[0,3]}^{\widehat{sl}(2)_{3}}(q, t'),
\]
\[
\hat{\chi}_{[2,1]}^{\text{red}}(q, t) = q^{\frac{1}{2}} \chi_{[2,1]}^{\widehat{sl}(2)_{3}}(q, t'), \quad \hat{\chi}_{[1,2]}^{\text{red}}(q, t) = q^{\frac{1}{2}} \chi_{[1,2]}^{\widehat{sl}(2)_{3}}(q, t'),
\]

and Proposition 5.12 is confirmed. By taking \(t = 1\), the principal characters of \(\widehat{sl}(2)_{3}\) are obtained as in Corollary 5.12

\[
\hat{\chi}_{[3,0]}^{\text{red}}(q, 1) = \hat{\chi}_{[0,3]}^{\text{red}}(q, 1) = \Pr \chi_{[3,0]}^{\widehat{sl}(2)_{3}}(q) = \left( -q^{\frac{3}{2}}; q^{2} \right)_{\infty} \left( q^{2}; q^{2} \right)_{\infty},
\]
\[
\hat{\chi}_{[2,1]}^{\text{red}}(q, 1) = \hat{\chi}_{[1,2]}^{\text{red}}(q, 1) = \Pr \chi_{[2,1]}^{\widehat{sl}(2)_{3}}(q) = \left( -q^{\frac{3}{2}}; q^{2} \right)_{\infty} \left( q^{2}; q^{\frac{3}{2}} \right)_{\infty},
\]

Note that, these principal characters coincide with the principal characters of \(\widehat{sl}(3)_{2}\) in (7.23).

7.3.2. Reduced instanton partition functions. For \(N = 3\) with general \(n\), using the relations (5.18) and (6.10) with (6.15), the reduced instanton partition functions (6.11) are determined by the parameters \(s = (s_1, s_2), s_1 = (s_{1,1}, s_{1,2}, s_{1,3})\) and \(s_2 = (s_{2,1}, s_{2,2}, s_{2,3})\) in \(\{1, \ldots, n\}\) as

\[
s_{1} = \sigma_{1} - \sigma_{2} + 1, \quad s_{2} = \sigma_{2} - \sigma_{3} + 1,
\]
\[
s_{1,I} = b_{I} - b_{I+1} + 1, \quad s_{2,I} = b'_{I} - b'_{I+1} + 1, \quad I = 1, 2,
\]
when the ordered charges $\sigma_1 \geq \sigma_2 \geq \sigma_3$, $b_1 \geq b_2 \geq b_3$, $b'_1 \geq b'_2 \geq b'_3$ are fixed. The Coulomb parameters are then determined from $s$ by (6.5):

\[
a_1 = \frac{1}{3} \sum_{l=1,2} \left( I - 3 \right) \left( s_l - 1 - \frac{n}{3} \right) \epsilon_2, \\
a_2 = \frac{1}{3} \sum_{l=1,2} \left( 3 - 2I \right) \left( s_l - 1 - \frac{n}{3} \right) \epsilon_2, \\
a_3 = \frac{1}{3} \sum_{l=1,2} I \left( s_l - 1 - \frac{n}{3} \right) \epsilon_2,
\]

(7.36)

and the mass parameters $m = (m_1, \ldots, m_N)$ and $m' = (m'_1, \ldots, m'_N)$ are determined from the parameters $s_1$ and $s_2$, respectively, by (6.6).

We now consider the case of $(N, n) = (3, 2)$ with the rational $\Omega$-background $\epsilon_1/\epsilon_2 = -5/3$ in (6.4)\textsuperscript{17}

Example 7.8 ($\emptyset - \emptyset - (\emptyset) - \emptyset - \emptyset$). Consider the reduced instanton partition function $\hat{Z}_{[\emptyset, \emptyset]}^{(0,0),(0,0),0}(q)$ and take $\ell = 0$ in the fundamental chamber, which respects the fusion rules, as in Conjecture 6.5. Here the parameters $s = (1, 1)$, $s_1 = (1, 1, 1)$ and $s_2 = (1, 1, 1)$ are fixed. Then we see that the reduced instanton partition function is

\[
\hat{Z}_{[\emptyset, \emptyset]}^{(0,0),(0,0),0}(q) = \left( 1 - q \right)^{-2h_\emptyset} = 1, \quad h_\emptyset = 0,
\]

and Conjecture 6.5 is confirmed.

Example 7.9 ($\emptyset - \emptyset - (\emptyset) - \emptyset - \emptyset$). Consider the reduced instanton partition function $\hat{Z}_{[\emptyset, \emptyset]}^{(0,0),(0,0),0}(q)$ and take $\ell = 0$ in the fundamental chamber as in Conjecture 6.6, where the parameters $s = (2, 1)$, $s_1 = (1, 1, 2)$ and $s_2 = (1, 1, 2)$ are fixed. Then the reduced instanton partition function is obtained as

\[
\hat{Z}_{[\emptyset, \emptyset]}^{(0,0),(0,0),0}(q) = \left( 1 - q \right)^{-2h_\emptyset} = 1 + \frac{3q}{10} + \frac{39q^2}{200} + \frac{299q^3}{2000} + \frac{9867q^4}{80000} + \frac{42481q^5}{4000000} + \cdots,
\]

where $h_\emptyset = 3/20$, and Conjecture 6.6 is confirmed.

Example 7.10 ($\emptyset - \emptyset - (\emptyset) - \emptyset - \emptyset$ and $\emptyset - \emptyset - (\emptyset) - \emptyset - \emptyset$). As Conjecture 6.7 consider, first, the reduced instanton partition function $\hat{Z}_{[\emptyset, \emptyset]}^{(1,0),(1,0),0}(q)$, where the parameters $s = (1, 1)$, $s_1 = (2, 1, 2)$ and $s_2 = (2, 1, 2)$ are fixed. Then, we find that the reduced instanton partition functions for $\ell = 0, -1$ in the fundamental chamber are

\[
\hat{Z}_{[\emptyset, \emptyset]}^{(1,0),(1,0),0}(q) = \left( 1 - q \right)^{2h_\emptyset} \frac{1}{5} 2F_1 \left( -\frac{1}{5}, \frac{2}{5}; \frac{3}{5}; q \right)
= 1 + \frac{q}{6} + \frac{13q^2}{120} + \frac{87q^3}{1040} + \frac{8669q^4}{124800} + \frac{344797q^5}{5740800} + \cdots,
\]

(7.39)

\[
\hat{Z}_{[\emptyset, \emptyset]}^{(1,0),(1,0),0}(q) = \frac{q^{1/3}}{3} \left( 1 - q \right)^{2h_\emptyset} \frac{1}{5} 2F_1 \left( -\frac{4}{5}, \frac{8}{5}; \frac{3}{5}; q \right)
= \frac{q^{1/3}}{3} + \frac{q^{2/3}}{6} + \frac{61q^{5/3}}{520} + \frac{289q^{7/3}}{3120} + \frac{222529q^{9/3}}{2870400} + \frac{25723q^{11/3}}{382720} + \cdots,
\]

\textsuperscript{17} Examples 7.8, 7.9, and 7.10 are confirmed up to $O(q^{13})$.\textsuperscript{18}
where \( h_{\Omega} = 3/20 \), and the second one respects the fusion rules by (6.20). Consider, next, the reduced instanton partition function \( \hat{Z}_{[1:2;\ell]}^{(1,0,0),(1,0,0)}(q) \), where the parameters \( s = (1, 2) \), \( s_1 = (2, 1, 2) \) and \( s_2 = (2, 1, 2) \) are fixed. Then we find that the reduced instanton partition functions for \( \ell = 0, 1 \) in the fundamental chamber are

\[
\hat{Z}_{[1:2;\ell]}^{(1,0,0),(1,0,0)}(q) = \left(1 - q\right)^{2h_{\Omega} - \frac{3}{5}} \frac{1}{2} \, _2F_1\left(-\frac{1}{5}, \frac{1}{5}; \frac{2}{5}; q\right)
\]

(7.40)

\[
= 1 + \frac{q^\frac{1}{2}}{5} + \frac{183q^2}{1400} + \frac{353q^3}{3500} + \frac{796073q^4}{9520000} + \frac{17182143q^5}{238000000} + \cdots,
\]

\[
\hat{Z}_{[1:2;1]}^{(1,0,0),(1,0,0)}(q) = \frac{q^\frac{1}{2}}{2} \left(1 - q\right)^{2h_{\Omega} - \frac{3}{5}} \frac{1}{2} \, _2F_1\left(-\frac{1}{5}, \frac{4}{5}; \frac{7}{5}; q\right)
\]

\[
= \frac{q^\frac{1}{2}}{2} + \frac{29q^3}{140} + \frac{393q^5}{2800} + \frac{51949q^7}{476000} + \frac{1725293q^9}{19040000} + \frac{7443271q^{11}}{952000000} + \cdots,
\]

where the second one respects the fusion rules by (6.20). The above results (7.39) and (7.40) support Conjecture 6.7.

8. Summary of results and remarks

8.1. Summary of Results. The point of this paper is to compute conformal blocks in integral-level WZW models. Starting from the \( SU(N) \) instanton partition functions on \( \mathbb{C}^2/\mathbb{Z}_n \), with rational \( \Omega \)-deformation, based on the algebra \( A(N, n; p) \) in (1.1), we proposed (in Conjectures 6.5, 6.6 and 6.7) a way to compute integral-level, integrable \( \hat{sl}(n)_N \) WZW conformal blocks, with rational central charges, where one has to deal with the issue of null states. By considering a rational \( \Omega \)-background \( \frac{\ell_1}{\ell_2} = -\frac{1}{N} \) in (6.4) and imposing appropriate Burge conditions in (5.14) to eliminate the null states, we trivialized the coset factor in the algebra \( A(N, n; N) \) as in (1.4), and were left with an integral-level WZW model. Further, we showed, in Remark 5.10, that the first Chern class (2.5) of the gauge bundle, which labels the instanton partition functions on the gauge side, can be interpreted as the eigenvalues of \( \hat{sl}(n)_N \) currents on the CFT side.

8.2. The work of Alday and Tachikawa. In [48], Alday and Tachikawa, using results from [49, 50, 51, 52], as well as AGT, found that \( SU(2) \) instanton partition functions on \( (z_1, z_2) \in \mathbb{C}^2 \) with generic \( \Omega \)-deformation, and in the presence of a full surface operator at \( z_2 = 0 \), agree with \( \hat{sl}(2) \) conformal blocks that are modified by a \( K \)-operator insertion, at generic-level \( k = -2 - \frac{\ell_2}{\ell_1} \). A generalization to the relation between \( SU(N) \) instanton partition functions in the presence of a full surface operator and modified \( \hat{sl}(N) \) conformal blocks at generic-level

\[
k = -N - \frac{\ell_2}{\ell_1},
\]

was proposed in [53].

In analogy with the moduli space of \( U(N) \) instantons on \( \mathbb{C}^2/\mathbb{Z}_n \) without surface operators described in Section 2 to describe the moduli space of \( U(N) \) instantons on \( \mathbb{C}^2 \) in the presence of a full surface operator, one can use the moduli space of \( U(N) \) instantons on \( \mathbb{C} \times (\mathbb{C}/\mathbb{Z}_N) \) [51, 54, 55]. Unlike
the $\widehat{\mathfrak{sl}}(n)_N$ conformal blocks discussed in our work, these conformal blocks are at generic-level, and modified by the $K$-operator insertion.

8.3. The work of Belavin and Mukhametzhanov. In [11], Belavin and Mukhametzhanov obtained integrable WZW conformal blocks for $(N, n) = (2, 2)$, (see footnote [15]). They found that, starting from the $SU(2)$ instanton partition functions on $\mathbb{C}^2/\mathbb{Z}_2$ with generic $\Omega$-deformation, the $\widehat{sl}(2)_2$ WZW conformal blocks in Examples 7.2, 7.3 and 7.4 are obtained as prefactors of $\mathcal{N} = 1$ super-Virasoro conformal blocks with generic central charge. In our work, with suitable rational choices of the parameters and by imposing Burge conditions, we trivialized the super-Virasoro conformal blocks (and their higher $(N, n)$ analogues), and computed conformal blocks for rational central charges, for more values of $(N, n)$. We conjecture that our approach works, for rational central charges, for all $(N, n), N, n \in \mathbb{Z}_{\geq 1}$.

Acknowledgements

We would like to thank Piotr Su/suppress lkowski for useful comments on the manuscript. OF wishes to thank Vladimir Belavin, Jean-Emile Bourgine and Raoul Santachiara for discussions on the subject of this work and related topics. We thank the Australian Research Council for support of this work.

Appendix A. Some AGT Correspondences

Following [17, 8, 11], we summarize some explicit AGT correspondences to identify our conventions in Section 3.2 and to confirm the $U(1)$ factor $Z_{\mathcal{H}}(m, m'; q)$ in (3.11).

A.1. $(N, n) = (2, 1)$ and Virasoro conformal blocks. For $(N, n) = (2, 1)$, the $SU(2)$ instanton partition function (2.24) with $a_1 = -a_2 = a$ is computed as

(A.1)

\[
Z_{(0,0)}^{0,0}(a, m, m'; q) = 1 + q \left\{ \frac{(a - m_1) (a - m_2) (a + m'_1 - \epsilon_1 - \epsilon_2) (a + m'_2 - \epsilon_1 - \epsilon_2)}{2 a \epsilon_1 \epsilon_2 (-2 a + \epsilon_1 + \epsilon_2)} - \frac{(a + m_1) (a + m_2) (a - m'_1 + \epsilon_1 + \epsilon_2) (a - m'_2 + \epsilon_1 + \epsilon_2)}{2 a \epsilon_1 \epsilon_2 (2 a + \epsilon_1 + \epsilon_2)} \right\} + O \left( q^2 \right)
\]

In [17] (see [56, 57] for non-conformal/Whittaker limits) it was found that, by subtracting the $U(1)$ factor (3.11) for $(N, n) = (2, 1)$, the normalized instanton partition function

(A.2)

\[
\tilde{Z}_{(0,0)}^{0,0}(a, m, m'; q) := \left( 1 - q \right) \left\{ \frac{1}{\epsilon_1 \epsilon_2} \frac{(\Sigma_{j=1}^2 m_j)}{\epsilon_1 \epsilon_2} \right\} Z_{(0,0)}^{0,0}(a, m, m'; q)
\]

gives the $c = 1 + 6 \frac{(\epsilon_1 + \epsilon_2)^2}{\epsilon_1 \epsilon_2}$ Virasoro conformal blocks of 4-point function (3.3) on $\mathbb{P}^1$ by the parameter identifications (3.8) and (3.9):

\[
\mu'' = \frac{\epsilon_1 + \epsilon_2}{2} + a, \quad \mu_1 = \frac{\epsilon_1 + \epsilon_2}{2} + \frac{m_1 - m_2}{2}, \quad \mu_2 = \frac{m_1 + m_2}{2},
\]

\[
\mu_3 = \frac{m'_1 + m'_2}{2}, \quad \mu_4 = \frac{\epsilon_1 + \epsilon_2}{2} - \frac{m'_1 - m'_2}{2}
\]

Here, note that, in the $n = 1$ cases, the WZW factor $\hat{\mathfrak{sl}}(n)_N$ in the algebra $[3.1]$ is absent. For example, the Virasoro conformal block at level 1,

$$\frac{\left(\Delta_{\mu^v} - \Delta_{\mu_1} + \Delta_{\mu_2}\right)}{2\Delta_{\mu^v}} \frac{\left(\Delta_{\mu^v} + \Delta_{\mu_3} - \Delta_{\mu_4}\right)}{2\Delta_{\mu^v}}, \quad \Delta_\mu = \frac{\mu}{\epsilon_1 \epsilon_2},$$

agrees with the coefficient of $q$ in $[A.2]$.

A.2. $(N, n) = (3, 1)$ and $\mathcal{W}_3$ conformal blocks. For $(N, n) = (3, 1)$, the $SU(3)$ instanton partition function (2.21) with $a_3 = -a_1 - a_2$ is computed as

$$\tilde{Z}^0_{(0,0)}(a, m, m'; q) = 1 + q \left( (a_1 - m_1)(a_1 - m_2)(a_1 - m_3)(-a_1 - m_1' + \epsilon_1 + \epsilon_2)(-a_1 - m_2' + \epsilon_1 + \epsilon_2)(-a_1 - m_3' + \epsilon_1 + \epsilon_2) + \frac{2}{\epsilon_1 \epsilon_2} (a_1 - a_2)(-a_1 - a_2 + \epsilon_1 + \epsilon_2) + \frac{2}{\epsilon_1 \epsilon_2} (a_1 - a_2)(a_1 - a_2 + \epsilon_1 + \epsilon_2) + \frac{2}{\epsilon_1 \epsilon_2} (a_1 + a_2 + m_1)(a_1 + a_2 + m_2)(a_1 + a_2 + m_3)(a_1 + a_2 - m_1' + \epsilon_1 + \epsilon_2)(a_1 + a_2 - m_2' + \epsilon_1 + \epsilon_2)(a_1 + a_2 - m_3' + \epsilon_1 + \epsilon_2) \right) + O(q^2)
$$

By subtracting the $U(1)$ factor $[3.11]$ for $(N, n) = (3, 1)$, one finds that the normalized instanton partition function

$$\tilde{Z}^0_{(0,0)}(a, m, m'; q) := \left(1 - q\right)^{-\left(\sum_{l=1}^{\infty} m_l \frac{l+2}{l+1}\left(1+\frac{\epsilon_1}{\epsilon_2}+\frac{\epsilon_2}{\epsilon_1}\right)\right)} \tilde{Z}^0_{(0,0)}(a, m, m'; q)
$$

gives the $\mathcal{W}_3$ conformal blocks of 4-point function $[3.5]$ on $\mathbb{P}^1$, with $c = 2 + 24\frac{(\epsilon_1 + \epsilon_2)^2}{\epsilon_1 \epsilon_2}$, by the parameter identifications $[3.8]$ and $[3.9]$ [7, 8] (see [58, 59, 60] for non-conformal/Whittaker limits):

$$\begin{align*}
\mu_{1,1} &= \frac{\epsilon_1 + \epsilon_2}{2} + \frac{m_1 - m_2}{2}, & \mu_{1,1}' &= \frac{\epsilon_1 + \epsilon_2}{2} + \frac{m_1 - m_2}{2}, \\
\mu_{1,2} &= \frac{\epsilon_1 + \epsilon_2}{2} + \frac{m_2 - m_3}{2}, & \mu_{2,2} &= \frac{\epsilon_1 + \epsilon_2}{2} + \frac{m_2 - m_3}{2}, & \mu_2 &= \frac{m_1 + m_2 + m_3}{2}, \\
\mu_{4,1} &= \frac{\epsilon_1 + \epsilon_2}{2} - \frac{m_1' - m_2'}{2}, & \mu_{4,2} &= \frac{\epsilon_1 + \epsilon_2}{2} - \frac{m_1' - m_2'}{2}, & \mu_3 &= \frac{m_1' + m_2' + m_3'}{2}.
\end{align*}
$$

For example, the $\mathcal{W}_3$ conformal block at level 1,

$$\begin{align*}
\frac{\left(\Delta_{\mu^v} - \Delta_{\mu_1} + \Delta_{0,\mu_2}\right)}{2\Delta_{\mu^v}} & \left(\Delta_{\mu^v} - \Delta_{\mu_3} + \Delta_{\mu_4}\right) \\
+ & \left\{ -\frac{w_{\mu^v}}{2} - w_{\mu_1} + \frac{w_{0,\mu_2}}{2} + \frac{3}{2} \left(\Delta_{\mu^v} - \Delta_{\mu_1}\right) w_{0,\mu_2} - \frac{3}{2} \left(\Delta_{0,\mu_2} - \Delta_{\mu_1}\right) w_{\mu^v} \right\} \\
\times & \left\{ -\frac{w_{\mu^v}}{2} - w_{\mu_4} + \frac{w_{\mu_3,0}}{2} + \frac{3}{2} \left(\Delta_{\mu^v} - \Delta_{\mu_1}\right) w_{\mu_3,0} - \frac{3}{2} \left(\Delta_{\mu_3,0} - \Delta_{\mu_4}\right) w_{\mu^v} \right\} \\
\times & \left(\Delta_{\mu^v} - \frac{3}{4} \epsilon_1 \epsilon_2 \Delta_{\mu^v} - \frac{3}{4} \epsilon_1 \epsilon_2 \frac{w_{\mu^v}}{w_{\mu_1}} \right) - \frac{9}{2} \Delta_{\mu^v}^{-1},
\end{align*}
$$

where $w_{\mu^v}$ is the conformal weight of the $\Delta_{\mu^v}$ conformal block at level 1.
agrees with the coefficient of \( q \) in (A.6), where
\[
\Delta_\mu = \Delta_{\mu_1, \mu_2} = -\frac{2 \left( 2 \mu_1^2 + 2 \mu_1 \mu_2 + 2 \mu_2^2 - 3 (\epsilon_1 + \epsilon_2) (\mu_1 + \mu_2) \right)}{3 \epsilon_1 \epsilon_2},
\]
(A.9)
\[
w_\mu = w_{\mu_1, \mu_2} = \frac{1}{\epsilon_1 \epsilon_2} \left( \frac{2}{3} (2 \mu_1 + \mu_2) - (\epsilon_1 + \epsilon_2) \right) \left( \frac{2}{3} (\mu_1 + 2 \mu_2) - (\epsilon_1 + \epsilon_2) \right) \left( \frac{2}{3} (\mu_1 - \mu_2) \right)
\times \left( \frac{-6}{4 \epsilon_1 \epsilon_2 + 15 (\epsilon_1 + \epsilon_2)^2} \right)^\frac{1}{2}.
\]

A.3. \((N, n) = (2, 2)\) and \( \mathcal{N} = 1 \) super-Virasoro conformal blocks. For \((N, n) = (2, 2)\), the \( SU(2) \) instanton partition functions (2.21) with \( a_1 = -a_2 = a \) are computed as e.g.,
\[
Z_{0,0}^{(0,0),(0,0)}(a, m, m'; q)
= 1 + q \left( \frac{(a - m_1)(a - m_2)(a + m'_1 - \epsilon_1 - \epsilon_2)(a + m'_2 - \epsilon_1 - \epsilon_2)}{4 a \epsilon_2 (\epsilon_1 - \epsilon_2)(2a + \epsilon_1 + \epsilon_2)} + \frac{(a - m_1)(a - m_2)(a + m'_1 - \epsilon_1 - \epsilon_2)(a + m'_2 - \epsilon_1 - \epsilon_2)}{4 a \epsilon_2 (\epsilon_1 - \epsilon_2)(-2a + \epsilon_1 + \epsilon_2)} \right) + O \left( q^2 \right),
\]
(A.10)
\[
Z_{(1,1)}^{(0,0),(0,0)}(a, m, m'; q) = q^{2 \Delta} \left( \frac{1}{2 a (-2a + \epsilon_1 + \epsilon_2)} - \frac{1}{2 a (2a + \epsilon_1 + \epsilon_2)} \right) + O \left( q^2 \right),
\]
for the vanishing first Chern class \( \epsilon_1 = 0 \) in (2.5).

We consider the subtraction of the \( U(1) \) factor (3.11) for \((N, n) = (2, 2)\) from the instanton partition functions
\[
\mathcal{Z}_{(\sigma, \ell)}^{b, b'}(a, m, m'; q) := \left( 1 - q \right) - \frac{\left( \Sigma_{i=1}^2 m_i \right)}{2 \epsilon_1 \epsilon_2} \mathcal{Z}_{(\sigma, \ell)}^{b, b'}(a, m, m'; q)
\]
(A.11)
\[
\text{In} \ [11, 13] \text{ (see also} [12] \text{), it was shown that the normalized instanton partition functions (A.11) give the} \mathcal{N} = 1 \text{ super-Virasoro conformal blocks of 4-point function (3.5) on} \mathbb{P}^1, \text{ with} \ c = \frac{3}{2} + 3 \left( \epsilon_1 + \epsilon_2 \right)^2, \text{by the parameter identifications (A.3) (see} [9, 10, 13] \text{ for non-conformal/Whittaker limits). For example, the instanton partition functions (A.10) correspond to the conformal blocks of four NS primary fields, and actually the conformal block at level 1,}
\]
(A.12)
\[
\frac{\left( \Delta_{\mu'} - \Delta_{\mu_1} + \Delta_{\mu_2} \right)}{2 \Delta_{\mu'}} \frac{\left( \Delta_{\mu'} + \Delta_{\mu_3} - \Delta_{\mu_4} \right)}{2 \Delta_{\mu'}}, \quad \Delta_{\mu} = \frac{\mu (\epsilon_1 + \epsilon_2 - \mu)}{2 \epsilon_1 \epsilon_2},
\]
agrees with the coefficient of \( q \) in \( \mathcal{Z}_{(0,0),(0,0)}^{(0,0)}(a, m, m'; q) \), and the conformal blocks
\[
\text{at level} \ \frac{1}{2} : \quad \frac{1}{2 \Delta_{\mu'}},
\]
(A.13)
\[
\text{at level} \ \frac{3}{2} : \quad \left( 1 + 2 \Delta_{\mu'} - 2 \Delta_{\mu_1} + 2 \Delta_{\mu_2} \right) \left( 1 + 2 \Delta_{\mu'} + 2 \Delta_{\mu_3} - 2 \Delta_{\mu_4} \right)^2
\times \left( \frac{6 \left( \Delta_{\mu_2} - \Delta_{\mu_1} \right) \left( \Delta_{\mu_3} - \Delta_{\mu_4} \right)}{c - \left( 9 - 2c \right) \Delta_{\mu'} + 6 \Delta_{\mu'}^2} \right) \left( 1 + 2 \Delta_{\mu'} \right),
\]
agree with the coefficients of \( q^{\frac{1}{2}} \) and \( q^{\frac{3}{2}} \) in \( 2\epsilon_1 \epsilon_2 \mathcal{Z}_{(1,1)}^{(0,0),(0,0)}(a, m, m'; q) \).
Appendix B. Integrable $\widehat{\mathfrak{sl}}(n)_N$ WZW 4-point conformal blocks for fundamental representations

The integrable $\widehat{\mathfrak{sl}}(n)_N$ WZW conformal blocks of 4-point function on $\mathbb{P}^1$ of primary fields with (anti-)fundamental representations $\Box, \bar{\Box}, \Box^\ast$, and $\bar{\Box}^\ast$, schematically denoted by $\mathcal{F}(0)$, were obtained in [28] (see also [47]), as solutions to the Knizhnik-Zamolodchikov equation, as

\begin{equation}
\mathcal{F}_1^{(0)}(z) = z^{-2h_\Box} \left( 1 - z \right) F_{1,2} \left( \frac{1}{n+N}, \frac{1}{n+N}; \frac{N}{n+N}; z \right),
\end{equation}

\begin{equation}
\mathcal{F}_2^{(0)}(z) = \frac{1}{N} z^{1-2h_\Box} \left( 1 - z \right) F_{1,2} \left( \frac{n-1}{n+N}, \frac{n+1}{n+N}; \frac{n}{n+N}; z \right),
\end{equation}

\begin{equation}
\mathcal{F}_1^{(1)}(z) = z^{h_\theta - 2h_\Box} \left( 1 - z \right) F_{1,2} \left( \frac{n-1}{n+N}, \frac{n+1}{n+N}; \frac{n}{n+N}; z \right),
\end{equation}

\begin{equation}
\mathcal{F}_2^{(1)}(z) = -n z^{h_\theta - 2h_\Box} \left( 1 - z \right) F_{1,2} \left( \frac{n-1}{n+N}, \frac{n+1}{n+N}; \frac{n}{n+N}; z \right),
\end{equation}

where $h_\Box = \frac{n^2 - 1}{2n(n+N)}$ is the conformal dimension of the four primary fields, and $h_\theta = \frac{n}{n+N}$ is the conformal dimension of the adjoint field with weight $\theta = [N-1, 1, 0, \ldots, 0, 1]$. These four solutions correspond to two choices of the representations of states in the internal channel which follow from the fusion of $\Box$ and $\bar{\Box}$, and $\mathcal{F}_1^{(0)}(z), \mathcal{F}_2^{(0)}(z)$ (resp. $\mathcal{F}_1^{(1)}(z), \mathcal{F}_2^{(1)}(z)$) corresponds to the identity (resp. adjoint) field conformal block of “s-channel”. Under a hypergeometric transformation

\begin{equation}
z \rightarrow q := \frac{z}{z-1},
\end{equation}

the Gauss hypergeometric function transforms as

\begin{equation}
\mathcal{F}_1^{(0)}(q) := z^{2h_\Box} \mathcal{F}_1^{(0)}(z) = \left(1 - q\right)^{h_\Box - \frac{n+1}{n+N}} \mathcal{F}_1 \left( \frac{N-1}{n+N}, \frac{N}{n+N}; q \right),
\end{equation}

\begin{equation}
\mathcal{F}_2^{(0)}(q) := z^{2h_\Box} \mathcal{F}_2^{(0)}(z) = -\frac{q}{N} \left(1 - q\right)^{h_\Box - \frac{n+1}{n+N}} \mathcal{F}_1 \left( \frac{N-1}{n+N}, \frac{n}{n+N}; q \right),
\end{equation}

\begin{equation}
\mathcal{F}_1^{(1)}(q) := \frac{z^{h_\theta}}{n} \mathcal{F}_1^{(1)}(z) = \left(-\frac{q}{n}\right)^{h_\theta} \left(1 - q\right)^{h_\theta - \frac{n+1}{n+N}} \mathcal{F}_1 \left( \frac{n-1}{n+N}, \frac{n}{n+N}; q \right),
\end{equation}

\begin{equation}
\mathcal{F}_2^{(1)}(q) := \frac{z^{h_\theta}}{n} \mathcal{F}_2^{(1)}(z) = -\left(-\frac{q}{n}\right)^{h_\theta} \left(1 - q\right)^{h_\theta - \frac{n+1}{n+N}} \mathcal{F}_1 \left( \frac{n-1}{n+N}, \frac{n}{n+N}; q \right),
\end{equation}

References

[1] L F Alday, D Gaiotto and Y Tachikawa, Liouville Correlation Functions from Four-dimensional Gauge Theories, Letters in Mathematical Physics 91, 167-197 (2010) [arXiv:0906.3219 [hep-th]].
[2] N A Nekrasov, "Seiberg-Witten prepotential from instanton counting," Advances in Theoretical and Mathematical Physics 7, no. 5, 831-864 (2003) [hep-th/0206161].

[3] V A Alba, V A Fateev, A V Litvinov and G M Tarnopolskiy, "On combinatorial expansion of the conformal blocks arising from AGT conjecture," Letters in Mathematical Physics 98, 33 (2011) [arXiv:1012.1312 [hep-th]].

[4] O Schiﬀmann and E Vasserot, "Cherednik algebras, W-algebras and the equivariant cohomology of the moduli space of instantons on $A^2$," Publications mathématiques de l'IHÉS 118, 213 (2013) [arXiv:1202.2750 [math.QA]].

[5] D Maulik and A Okounkov, "Quantum Groups and Quantum Cohomology," [arXiv:1211.3287 [math.AG]].

[6] A Braverman, M Finkelberg and H Nakajima, "Instanton moduli spaces and $W$-algebras," Astérisque 385 (2016) [arXiv:1406.2383 [math.QA]].

[7] N Wyllard, "$(N-1)$ conformal Toda field theory correlation functions from conformal $N = 2$ SU$(N)$ quiver gauge theories," Journal of High Energy Physics 0911, 002 (2009) [arXiv:0907.2189 [hep-th]].

[8] A Mironov and A Morozov, "On AGT relation in the case of $U(3)$," Nuclear Physics B 825, 1 (2010) [arXiv:0908.2569 [hep-th]].

[9] V Belavin and B Feigin, "Super Liouville conformal blocks from $N = 2$ SU$(2)$ quiver gauge theories," Journal of High Energy Physics 1107, 079 (2011) [arXiv:1105.3800 [hep-th]].

[10] G Bonelli, K Maruyoshi and A Tanzini, "Instantons on ALE spaces and Super Liouville Conformal Field Theories," Journal of High Energy Physics 1108, 056 (2011) [arXiv:1106.2505 [hep-th]].

[11] A Belavin, V Belavin and M Bershtein, "Instantons and 2d Superconformal field theory," Journal of High Energy Physics 1109, 117 (2011) [arXiv:1106.2001 [hep-th]].

[12] G Bonelli, K Maruyoshi and A Tanzini, "Gauge Theories on ALE Space and Super Liouville Correlation Functions," Letters in Mathematical Physics 101, 103 (2012) [arXiv:1107.4609 [hep-th]].

[13] Y Ito, "Ramond sector of super Liouville theory from instantons on an ALE space," Nuclear Physics B 861, 387 (2012) [arXiv:1110.2176 [hep-th]].

[14] A Belavin and B Mukhametzhanov, "$N = 1$ superconformal blocks with Ramond fields from AGT correspondence," Journal of High Energy Physics 1301, 178 (2013) [arXiv:1210.7454 [hep-th]].

[15] N Wyllard, "Coset conformal blocks and $N = 2$ gauge theories," [arXiv:1109.4264 [hep-th]].

[16] M N Allimov and G M Tarnopolsky, "Parafermionic Liouville field theory and instantons on ALE spaces," Journal of High Energy Physics 1202, 036 (2012) [arXiv:1110.5628 [hep-th]].

[17] T Nishioka and Y Tachikawa, "Central charges of para-Liouville and Toda theories from M5-branes," Physical Review D 84, 046009 (2011) [arXiv:1106.1172 [hep-th]].

[18] A A Belavin, M A Bershtein, B I Feigin, A V Litvinov and G M Tarnopolskiy, "Instanton moduli spaces and bases in coset conformal field theory," Communications in Mathematical Physics 319, 269 (2013) [arXiv:1111.2803 [hep-th]].

[19] M Bershtein and O Foda, "AGT, Burge pairs and minimal models," Journal of High Energy Physics 1406, 177 (2014) [arXiv:1404.7075 [hep-th]].

[20] K B Alkalaev and V A Belavin, "Conformal blocks of $W_N$ minimal models and AGT correspondence," Journal of High Energy Physics 1407, 024 (2014) [arXiv:1404.7093 [hep-th]].

[21] V Belavin, O Foda and R Santachiara, "AGT, $N$-Burge partitions and $W_N$ minimal models," Journal of High Energy Physics 1510, 073 (2015) [arXiv:1507.03540 [hep-th]].

[22] W H Burge, "Restricted partition pairs," Journal of Combinatorial Theory, Series A 63, Issue 2, 210-222 (1993).

[23] O Foda, K S M Lee and T A Welsh, "A Burge tree of Virasoro type polynomial identities," International Journal of Modern Physics A 13, 4967 (1998) [q-alg/9710025].

[24] I M Gessel and C Krattenthaler, "Cylindric Partitions," Transactions of the American Mathematical Society 349, no. 2, 429-479 (1997).

[25] B Feigin, E Feigin, M Jimbo, T Miwa and E Mukhin, "Quantum continuous $gl_{\infty}$: Semi-infinite construction of representations," Kyoto Journal of Mathematics 51, no. 2, 337-364 (2011) [arXiv:1002.3100 [math.QA]].

[26] B Feigin, E Feigin, M Jimbo, T Miwa and E Mukhin, "Quantum continuous $gl_{\infty}$: Tensor products of Fock modules and $W_n$ characters," Kyoto Journal of Mathematics 51, no. 2, 365-392 (2011) [arXiv:1002.3113 [math.QA]].
[27] E Date, M Jimbo, A Kuniba, T Miwa, M Okado, Paths, Maya Diagrams and representations of $\mathfrak{sl}(r, \mathbb{C})$, Integrable Systems in Quantum Field Theory and Statistical Mechanics, 149–191, Mathematical Society of Japan, Tokyo, Japan, 1989.

[28] V G Knizhnik and A B Zamolodchikov, Current Algebra and Wess-Zumino Model in Two-Dimensions, Nuclear Physics B 247, 83 (1984).

[29] N Nekrasov and A Okounkov, Seiberg-Witten theory and random partitions, Progress in Mathematics 244, 525 (2006) [hep-th/0306238].

[30] P B Kronheimer and H Nakajima, Yang-Mills instantons on ALE gravitational instantons, Mathematische Annalen 288, 263-307 (1990).

[31] F Fucito, J F Morales and R Poghossian, Multi instanton calculus on ALE spaces, Nuclear Physics B 703, 518 (2004) [hep-th/0406243].

[32] S Fujii and S Minabe, A Combinatorial study on quiver varieties, SIGMA 13, 052 (2017) [math/0510455 [math.AG]].

[33] M N Alfimov, A A Belavin and G M Tarnopolsky, Coset conformal field theory and instanton counting on $\mathbb{C}^2/\mathbb{Z}_p$, Journal of High Energy Physics 1308, 134 (2013) [arXiv:1306.3938 [hep-th]].

[34] H Nakajima, Instantons on ALE spaces, quiver varieties, and Kac-Moody algebras, Duke Mathematical Journal 76, no. 2, 365 (1994).

[35] H Nakajima, Quiver varieties and Kac-Moody algebras, Duke Mathematical Journal 91, no. 3, 515 (1998).

[36] H Nakajima, Lectures on instanton counting, math/0311058 [math-ag].

[37] S Kanno, Y Matsuo, S Shiba and Y Tachikawa, $\mathcal{N}=2$ gauge theories and degenerate fields of Toda theory, Physical Review D 81, 046004 (2010) [arXiv:0911.4787 [hep-th]].

[38] R Dijkgraaf and P Sulkowski, Instantons on ALE spaces and orbifold partitions, Journal of High Energy Physics 0803, 013 (2008) [arXiv:0712.1427 [hep-th]].

[39] L F Alday and Y Tachikawa, Affine SL(2) conformal blocks from 4d gauge theories, Letters in Mathematical Physics 94, 87 (2010) [arXiv:1005.4469 [hep-th]].

[40] A Braverman, Instanton counting via affine Lie algebras I: Equivariant J-functions of (affine) flag manifolds and Whittaker vectors, CRM Proceedings and Lecture Notes 38, 113-132 (2004) [math/0401409 [math-ago]].

[41] A Braverman and P Etingof, Instanton counting via affine Lie algebras II: From Whittaker vectors to the Seiberg-Witten prepotential, Studies in Lie Theory, 61-78 (2006) [math/0409441 [math-ag]].
[53] C Kozcaz, S Pasquetti, F Passerini and N Wyllard, *Affine $sl(N)$ conformal blocks from $\mathcal{N} = 2$ $SU(N)$ gauge theories*, Journal of High Energy Physics **1101**, 045 (2011) [arXiv:1008.1412 [hep-th]].

[54] M Finkelberg and L Rybnikov, *Quantization of Drinfeld Zastava in type A*, Journal of the European Mathematical Society **16**, no. 2, 235-271 (2014) [arXiv:1009.0676 [math.AG]].

[55] H Kanno and Y Tachikawa, *Instanton counting with a surface operator and the chain-saw quiver*, Journal of High Energy Physics **1106**, 119 (2011) [arXiv:1105.0357 [hep-th]].

[56] D Gaiotto, *asymptotically free $\mathcal{N} = 2$ theories and irregular conformal blocks*, Journal of Physics: Conference Series, Volume **462**, conference 1, 012014 (2013) [arXiv:0908.0307 [hep-th]].

[57] A Marshakov, A Mironov and A Morozov, *On non-conformal limit of the AGT relations*, Physics Letters B **682**, 125 (2009) arXiv:0909.2052 [hep-th].

[58] M Taki, *On AGT Conjecture for Pure Super Yang-Mills and $W$-algebra*, Journal of High Energy Physics **1105**, 038 (2011) arXiv:0912.4789 [hep-th].

[59] C A Keller, N Mekareeya, J Song and Y Tachikawa, *The ABCDEFG of Instantons and $W$-algebras*, Journal of High Energy Physics **1203**, 045 (2012) arXiv:1111.5624 [hep-th].

[60] H Kanno and M Taki, *Generalized Whittaker states for instanton counting with fundamental hypermultiplets*, Journal of High Energy Physics **1205**, 052 (2012) arXiv:1203.1427 [hep-th].

School of Mathematics and Statistics, University of Melbourne, Royal Parade, Parkville, Victoria 3010, Australia

E-mail address: omar.foda@unimelb.edu.au, n.macleod@student.unimelb.edu.au, masahidemanabe@gmail.com, trevviewelsh@yahoo.co.uk