Connection between Coulomb and harmonic oscillator potentials in relativistic quantum mechanics

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Abstract
The Levi-Civita transformation is applied in the two-dimensional (2D) Dirac and
Klein–Gordon (KG) equations with equal external scalar and vector potentials. The Coulomb
and harmonic oscillator problems are connected via the Levi-Civita transformation. These
connections lead to an approach to solve the Coulomb problems using the results of the
harmonic oscillator potential in the relativistic systems mentioned above.

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1. Introduction
The hydrogen atom and the harmonic oscillator are usually
given in textbooks as two of several solvable problems in
both classical and quantum physics [1]. The former, whose
potential term takes the Coulomb form, is regarded as a
fundamental prototype to investigate some more complicated
many-electron atoms. And the latter is of great importance
because many complicated potentials can approximate to a
harmonic oscillator in the vicinity of their equilibrium points.

There is a close relationship between the two fundamental
models [2]. This topic has attracted considerable interest,
partly because these simple systems involve concepts such as accidental degeneracy, dynamical symmetry, etc
and also because the relationship between them is of
practical value, for example in the computation of matrix
elements [3], in the derivation of the Coulomb path
integral [4] and in the construction of coherent states [5]. A
convenient method of establishing the connection between
the hydrogen atom and the harmonic oscillator is the
Kustaanheimo–Stiefel transformation [6–8], which arises
from the relation between the Kepler problem and the classical
oscillator in celestial mechanics. It is obtained by generalizing
the Levi-Civita transformation [9, 10], which works only for
the two-dimensional (2D) cases. This method can be naturally
applied to transform the Schrödinger equation of the hydrogen
atom to that of the harmonic oscillator in non-relativistic
quantum mechanics. Only the Levi-Civita transformation
would be taken into account in the remaining sections of the
paper since we shall discuss the problems only in 2D space.

It is well known that non-relativistic quantum mechanics
is an approximate theory of the relativistic one. In relativistic
quantum mechanics, the motion of spin-0 and spin-1/2
particles satisfies the Klein–Gordon (KG) and the Dirac
equations, respectively.
Kustaanheimo–Stiefel transformation and Levi-Civita
transformation are in fact related to the well-known dynamical
symmetries of the two non-relativistic models [11, 12].
For instance, the conserved quantities of the 2D hydrogen
atom generate the Lie group SO(3), whereas those of the
2D harmonic oscillator generate the Lie group SU(2) [13].
The correspondence between the motion equations of the
two models is to a large extent decided by the relation
between the two Lie groups [14]. Recently, dynamical
symmetries in both the KG and the Dirac equations have
been reported. Namely, in the KG and the Dirac systems,
Hamiltonians with equal scalar and vector Coulomb or
harmonic oscillator potentials have the same dynamical
symmetries as their non-relativistic counterparts [15–17].
The discussion above suggests that there should be a coordinate
transformation connecting relativistic systems with SO(3) and
SU(2) dynamical symmetries. In this paper, we will show that
it is nothing but the Levi-Civita transformation.

For the sake of brevity, the spin-0 (or 1/2) particles
in equal scalar and vector Coulomb or harmonic oscillator
potentials are called KG (or Dirac) hydrogen atoms or
harmonic oscillators, respectively. The relativistic units are chosen in this paper as \( c = \hbar = 1 \). Before we go further, in section 2, we first briefly review the Levi-Civita transformation in non-relativistic quantum mechanics [10]. In sections 3 and 4, we show the Levi-Civita transformation in the KG equation and the Dirac equation separately. The conclusion and discussion is given in section 5.

2. Non-relativistic quantum mechanics

We start with the time-independent Schrödinger equation for a 2D hydrogen atom

\[
H\Psi = E\Psi, \quad H = \frac{p^2}{2\mu} - \frac{\kappa}{r},
\]

(1)

where \( \mu \) is the reduced mass of the hydrogen atom, \( \kappa = e^2 \), \( p^2 = -\sum_{i=1}^{2}(\partial^2/\partial x_i^2) \), the \( x_i \) being Cartesian coordinates, and \( r = (x_1^2 + x_2^2)^{1/2} \). We now begin to transform the problem into a 2D harmonic oscillator via the Levi-Civita transformation. Introducing the variables \( u_1 \) and \( u_2 \), the transformation can be written as

\[
x_1 = u_1^2 - u_2^2, \quad x_2 = 2u_1u_2.
\]

(2)

Under this transformation we have \( r = u^2 = u_1^2 + u_2^2 \). The Schrödinger equation (1) becomes

\[
\left[ -\frac{1}{8\mu u^2} \sum_{i=1}^{2} \frac{\partial^2}{\partial u_i^2} + \frac{\kappa}{r} \right] \Psi = E\Psi.
\]

(3)

After multiplying on both sides of equation (3) by \( u^2 \), we obtain

\[
\left[ -\frac{1}{8\mu} \sum_{i=1}^{2} \frac{\partial^2}{\partial u_i^2} - Eu^2 \right] \Psi = \kappa \Psi.
\]

(4)

This equation turns into the form of the Schrödinger equation of the 2D harmonic oscillator after assuming that \( E < 0 \) (for bound motions), and making the definitions

\[
m = 4\mu, \quad \omega = (-E/2\mu)^{1/2}, \quad \epsilon = \kappa.
\]

(5)

Then, we obtain

\[
\left( -\frac{1}{2m} \sum_{i=1}^{2} \frac{\partial^2}{\partial u_i^2} + \frac{\epsilon}{2} \right) \Psi = \epsilon \Psi
\]

(6)

or \( \mathcal{H} \Psi = \epsilon \Psi \), with

\[
\mathcal{H} = -\frac{1}{2m} \sum_{i=1}^{2} \frac{\partial^2}{\partial u_i^2} + \frac{1}{2} \epsilon \omega^2 u^2.
\]

(7)

Here, \( \mathcal{H} \) and \( \epsilon \) are consistent with the Hamiltonian and the energy eigenvalue of a 2D harmonic oscillator, respectively. Considering the energy spectrum of the 2D harmonic oscillator in non-relativistic quantum mechanics, one obtains

\[
\kappa = \epsilon = (n_1 + n_2 + 1)\omega, \quad n_1, n_2 = 0, 1, 2, \ldots.
\]

(8)

From equation (5), the energy levels of the 2D hydrogen atom can be derived immediately:

\[
E = E_n = \frac{\kappa}{2\alpha^2 n^2}, \quad n = 1, 2, 3, \ldots.
\]

(9)

where \( \alpha = 1/(\mu \kappa) \) is the Bohr radius and \( n = \frac{1}{2}(n_1 + n_2 + 1) \). It is obvious that the value of \( n_1 + n_2 \) must be an odd number to guarantee that \( n \) is a positive integer. Hence, the expression in equation (9) is the familiar Bohr formula for the energy eigenvalues of the hydrogen atom. The correspondence of the energy level is not complete when the energy level of the harmonic oscillator only partly corresponds with that of the hydrogen atom. The cause of this problem will be illustrated in the following section.

3. KG equation

The KG equation with scalar potential \( V_S \) and vector potential \( V_V \) is given by

\[
\left\{ p^2 + [M + V_S]^2 - \left[ \frac{\partial}{\partial t} - V_V \right]^2 \right\} \Psi = 0.
\]

(10)

where \( M \) is the mass and \( p^2 = -\sum_{i=1}^{2}(\partial^2/\partial x_i^2) \) is the momentum. For the time-independent potentials \( V_S = V_V = V(r)/2 \), the KG equation (10) becomes

\[
\left[ -\sum_{i=1}^{2} \frac{\partial^2}{\partial x_i^2} + (M + E) V(r) - (E^2 - M^2) \right] \Psi = 0.
\]

(11)

where \( E \) is the relativistic energy. If the potential term \( V(r) \) takes the Coulomb form, i.e. \( V(r) = -\kappa/r \), we obtain the eigenvalue of the KG hydrogen atom:

\[
\left[ -\sum_{i=1}^{2} \frac{\partial^2}{\partial x_i^2} - (M + E)\kappa - 4(E^2 - M^2)u^2 \right] \Psi = 0.
\]

(12)

Under the Levi-Civita transformation in equation (2), the KG equation (12) becomes

\[
\left[ -\sum_{i=1}^{2} \frac{\partial^2}{\partial u_i^2} - 4(M + E)\kappa - 4(E^2 - M^2)u^2 \right] \Psi = 0.
\]

(13)

Setting

\[
M + E = m + \epsilon \Rightarrow \kappa = \frac{1}{2}(\epsilon - m), \quad M + E = m + \epsilon, \quad \kappa = \frac{1}{2}(\epsilon - m), \quad \epsilon = \frac{1}{2}m\omega^2,
\]

(14)

the KG equation (13) becomes

\[
\left[ -\sum_{i=1}^{2} \frac{\partial^2}{\partial u_i^2} + \frac{1}{2} m\omega^2 u^2 (m + \epsilon) - (\epsilon^2 - m^2) \right] \Psi = 0.
\]

(15)

It is consistent with the eigenvalue of the 2D KG harmonic oscillator, with mass \( m \), frequency \( \omega \) and energy level \( \epsilon \).

The Levi-Civita transformation also reveals the relationship between the energy eigenvalues of the 2D KG hydrogen atom and the 2D KG harmonic oscillator. The energy spectrum of the 2D harmonic oscillator is the real root of the cubic equation [17]:

\[
(\epsilon - m)^2(\epsilon + m)^2 = 2m^{2}\epsilon^2(\epsilon + m)(n + 1)^2, \quad n = 2j = 0, 1, 2, \ldots.
\]

(16)
Substituting equation (14) into cubic equation (16), one obtains
\[ E = \frac{\pm s^2 - k^2}{s^2 + k^2} M, \quad s = 2j + 1, 2, 3, \ldots \] (17)

It takes the same form as the energy level of the KG hydrogen atom but differs in the values of \( n \) [17]. In the energy level of the KG hydrogen atom, the value of \( n \) is a positive odd number, i.e. \( n = 2j + 1 = 1, 3, 5, \ldots \). Hence, the correspondence of the energy levels of the two systems is not complete. This problem results from the value of \( j \) in the expression of energy spectra. From [17], \( j(j + 1) \) is the eigenvalue of the Casimir operator. We introduce the orbit angular momenta \( \ell_h \) and \( \ell_o \) for the 2D KG hydrogen atom and the harmonic oscillator, respectively,
\[
\ell_h = \frac{1}{i} \frac{\partial}{\partial x_2} - \frac{1}{i} \frac{\partial}{\partial x_1}, \quad \ell_o = \frac{1}{i} \frac{\partial}{\partial u_2} - \frac{1}{i} \frac{\partial}{\partial u_1}, \] (18)

which are the conserved quantities of the two systems and whose eigenvalue is an integer. The relationship between these two conserved quantities can be established via the Levi-Civita transformation. Combining the definition of the angular momentum with the Levi-Civita transformation (2), we obtain the relation
\[
\ell_h = \ell_o \frac{1}{2}. \] (19)

\( \ell_h \) and \( \ell_o / 2 \) are just normalized generators of the two systems. The relation of the angular orbit momentum between the two systems is the same as the relations of the other normalized generators of the KG hydrogen atom and the KG harmonic oscillator. Hence the value of \( j \) in the KG hydrogen atom is a non-negative integer whereas the value of \( j \) in the KG harmonic oscillator is a non-negative integer and a half-odd-integer, which leads to an incomplete correspondence of the energy spectrum of the KG hydrogen atom and the KG harmonic oscillator. The illustration for the controversy of the correspondence of the energy level of the two KG systems can also be applied to the inconsistency in sections 2 and 4.

4. Dirac equation

The Hamiltonian of the Dirac hydrogen atom with equal scalar and vector Coulomb potentials is given by
\[
H = \alpha_1 p_1 + \alpha_2 p_2 + \beta M + (1 + \beta) \frac{V(r)}{2},
\]
(20)

where
\[
\alpha_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

are the Pauli matrices, \( (p_1, p_2) \) are the 2D linear momenta, \( (x_1, x_2) \) are the spatial coordinates with magnitude \( r \) and \( M \) is the mass. The Dirac equation can be written as
\[
H \Psi = E \Psi, \quad H = \begin{pmatrix} M - \frac{\kappa}{r} p_{1x} - i p_{2x} \\ p_{1x} + i p_{2x} \end{pmatrix}. \] (21)

We obtain the relation of linear momentum via the Levi-Civita transformation as
\[
p_{1u} - i p_{2u} = 2(u_1 + i u_2)(p_{1x} - i p_{2x}), \]
\[
p_{1u} + i p_{2u} = 2(p_{1x} + i p_{2x})(u_1 - i u_2), \] (22)

where \( p_{1x} \) is the 2D linear momentum of the hydrogen atom and \( p_{2u} \) is the 2D linear momentum of the harmonic oscillator. To give an obvious result, we introduce the following transformation into the Dirac equation (21):
\[
C(H - E)C^\dagger D\Psi = 0,
\]
(23)

where
\[
C = \begin{pmatrix} 2r & 0 \\ 0 & 1 \end{pmatrix}, \quad C^\dagger = \begin{pmatrix} 2r^\dagger & 0 \\ 0 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} \frac{\tau}{2u^2} & 0 \\ 0 & 1 \end{pmatrix},
\]

\( \tau = u_1 + i u_2 \), \( \tau^\dagger = u_1 - i u_2 \) and \( \Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} \)
is the eigenfunction of the Hamiltonian for a relativistic hydrogen atom. Then, the Dirac equation (21) becomes
\[
\left( \frac{4Mu^2 - 4\kappa^2 - 4Em^2}{4Mu^2 - 2Em^2} \right) \left( \begin{array}{c} u_1 + i u_2 \\ \frac{u_1 + i u_2}{2u^2} \end{array} \right) = 0.
\]
(24)

From equation (14), the Dirac equation (21) becomes
\[
(H - \epsilon)\Phi = 0, \quad H = \begin{pmatrix} m + \frac{1}{2}\omega^2 u^2 & p_{1u} - i p_{2u} \\ p_{1u} + i p_{2u} & -m \end{pmatrix}, \] (25)

where
\[
\Phi = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = \begin{pmatrix} \frac{u_1 + i u_2}{2u^2} \Psi_1 \\ \Psi_2 \end{pmatrix}.
\]

Equation (25) is consistent with the Dirac equation for the Dirac harmonic oscillator [16], where \( H \) can be seen as the Hamiltonian of the Dirac harmonic oscillator and the wave function \( \Phi \) can be considered as the eigenfunction of Hamiltonian \( H \).

Following the same procedure as in section 3, we also obtain the energy levels of the 2D Dirac hydrogen atom from the result of the 2D Dirac harmonic oscillator. The energy spectrum of the Dirac harmonic oscillator given in [16] can be expressed as the real roots of the following equation:
\[
\frac{(\epsilon - m)^2(\epsilon + m)^2}{2m\omega^2} - (\epsilon + m)(n + 1)^2 = 0,
\]
(26)

where \( n = 2j = 0, 1, 2, \ldots \). Substituting definitions (5) into the energy spectrum expression of the 2D Dirac harmonic oscillator, we obtain the following expression:
\[
E = \frac{\pm n^2 - k^2}{n^2 + \kappa^2} M, \quad n = 2j + 1 = 1, 3, 5, \ldots,
\]
(27)

which is consistent with the energy eigenvalue of the 2D Dirac hydrogen atom [16], where \( E \) and \( M \) can be considered as the relativistic energy and mass of the 2D hydrogen atom. Only a part of the energy level of the 2D Dirac harmonic oscillator corresponds to that of the 2D hydrogen atom. The
main reason is the same as that given in section 3. The conserved angular momenta of the 2D Dirac hydrogen atom and harmonic oscillator are given by [16]

\[
L_o = \begin{pmatrix} l_o & 0 \\ 0 & U_{px} U_o U_{px} \end{pmatrix}, \quad L_h = \begin{pmatrix} l_o & 0 \\ 0 & U_{pu} U_o U_{pu} \end{pmatrix},
\]

where \( U_{px} = (p_{1x} - ip_{2x})/\sqrt{p_{1x}^2 + p_{2x}^2} \), \( U_{pu} = (p_{1u} - ip_{2u})/\sqrt{p_{1u}^2 + p_{2u}^2} \), and \( l_o \) and \( l_h \) are defined in equation (18). Their relation under the Levi-Civita transformation can be easily derived as

\[
\left( \begin{array}{cc} 1 & 0 \\ 0 & U_u \end{array} \right) L_h \left( \begin{array}{cc} 1 & 0 \\ 0 & U_u^d \end{array} \right) = \frac{L_o}{2},
\]

where \( U_u = (u_1 - iu_2)/u \) and \( U_u^d = (u_1 + iu_2)/u \). This explains the discrepancy of the quantum numbers in the 2D Dirac hydrogen atom and the 2D Dirac harmonic oscillator.

5. Conclusion and discussion

The Levi-Civita transformation connects the 2D Coulomb problem and the 2D harmonic oscillator in non-relativistic quantum mechanics. It is in close touch with the dynamical symmetries of these two models. Both the spin-0 and spin-1/2 particles in equal scalar and vector Coulomb or harmonic oscillator potentials have SO(3) or SU(2) dynamical symmetries. In this paper, we have shown that the Levi-Civita transformation can be applied to transform the 2D KG (or Dirac) hydrogen atom to the KG (or Dirac) harmonic oscillator. Taking the relation between the conserved angular momenta into account, the Levi-Civita transformation leads to an approach to solve the KG (or Dirac) hydrogen atom by using the results of the KG (or Dirac) harmonic oscillator. In addition, the connection between the 3D hydrogen atom and the 4D isotropic harmonic oscillator has also been a subject of considerable interest in the last three decades [8, 18, 19] and the Kustaanheimo–Stiefel transformation between them in non-relativistic quantum mechanics has been well known. Hence, we could go further to generalize it to the relativistic case by a similar procedure.

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References

[1] Greiner W and Müller B 1994 Quantum Mechanics (An Introduction) (Berlin: Springer)
[2] Augustine C C 1980 Phys. Rev. A 22 333
[3] Seligmann T H, Moshinsky M and Wolf K B 1976 J. Math Phys. 13 901
[4] Roger H and Akira I 1982 Phys. Rev. Lett. 48 231
[5] Christopher G C 1986 Phys. Rev. A 33 2207
[6] Kustaanheimo P and Stiefel E 1965 J. Reine. Angew. Math. 218 204
[7] Ikeda M and Miyachi Y 1970 Math. Japan 15 127
[8] Kibler M and Negadi T 1983 J. Phys. A: Math. Gen. 16 4265
[9] Levi-Civita T 1973 Opere Mat. 6 111
[10] Ter-Antoyan V 2003 arXiv:quant-ph/0003106
[11] Greiner W and Müller B 1994 Quantum Mechanics (Symmetries) (Berlin: Springer)
[12] Gross D J 1996 Proc. Natl Acad. Sci. USA 93 14256
[13] Wybourne B G 1974 Classical Groups for Physicists (Canada: Wiley)
[14] Biedenharn L C and Louck J D 1981 Angular Momentum in Quantum Physics Theory and Application (Wesley: Addison)
[15] Ginocchio J N 2005 Phys. Rev. Lett. 95 252501
[16] Zhang F-L, Song C and Chen J-L 2009 Ann. Phys. 324 173
[17] Zhang F-L and Chen J-L 2009 J. Math. Phys. 50 032301
[18] De Lange O L and Rabb R E 1991 Operator Methods in Quantum Mechanics (New York: Oxford University Press)
[19] Chen A C and Kibler M 1985 Phys. Rev. A 31 3960