Abstract. The Chernoff bound is an important inequality relation in probability theory. The original version of the Chernoff bound is to give an exponential decreasing bound on the tail distribution of sums of independent random variables. Recent years, several works have been done by extending the original version of the Chernoff bound to high-dimensional random objects, e.g., random matrices, or/and to consider the relaxation that there is no requirement of independent assumptions among random objects. In this work, we generalize the matrix expander Chernoff bound studied by Garg et al. [8] to tensor expander Chernoff bounds. Our main tool is to develop new tensor norm inequalities based on log-majorization techniques. These new tensor norm inequalities are used to bound the expectation of Ky Fan norm of the random tensor exponential function, then tensor expander Chernoff bounds can be established. Compared with the matrix expander Chernoff bound, the tensor expander Chernoff bounds proved at this work contributes following aspects: (1) the random objects dimensions are increased from matrices (two-dimensional data array) to tensors (multidimensional data array); (2) this bound generalizes the identity map of the random objects summation to any polynomial function of the random objects summation; (3) Ky Fan norm, instead only the maximum or the minimum eigenvalues, for the function of the random objects summation is considered; (4) we remove the restriction about the summation of all mapped random objects is zero, which is required in the matrix expander Chernoff bound derivation.

Key words. Random Tensors, Tail Bound, Ky Fan Norm, Log-Majorization, Graph.

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1. Introduction. In probability theory, the Chernoff bound provides the exponential decreasing bound on tail distribution of sums of independent random variables. This bound has various applications in science and engineering. For example, the Chernoff bound is utilized in mathematical learning theory to prove that a learning algorithm is probably approximately correct [15]. On the engineering side, the Chernoff bound is also used to obtain tight bounds for packet routing problems which reduce wireless communication congestion while routing packets in sparse networks [14].

It is a tighter bound than the known first- or second-moment-based tail bounds such as Markov’s inequality or Chebyshev’s inequality, which only yield power-law bounds on tail distribution. Nevertheless, neither Markov’s inequality nor Chebyshev’s inequality requires that the variates are independent, which is necessary by the Chernoff bound [5]. Given \( n \) independent and identically distributed (i.i.d.) random variables \( X_1, X_2, \cdots, X_n \) taking values in \( \{0,1\} \) with \( \mathbb{E}[X_i] = q \) and \( \epsilon > 0 \), Chernoff bound for the version of \( n \) i.i.d. random variables is

\[
\text{Pr}\left( \frac{1}{n} \sum_{i=1}^{n} X_i \geq q + \epsilon \right) \leq \exp(-nD(q + \epsilon \| q)),
\]

where \( D(x \| y) \overset{\text{def}}{=} x \ln \frac{x}{y} + (1 - x) \ln \frac{1 - x}{1 - y} \) is the Kullback–Leibler divergence between Bernoulli distributed random variables with parameters \( x \) and \( y \) respectively.

There are various directions to generalize the Chernoff bound from Eq. (1.1), one major direction is to increase the dimension of random objects from random variables to random matrices. The works of Rudelson [21], Ahlswede-Winter [1] and Tropp [23] demonstrated that a similar concentration bound is also valid for matrix-valued random variables. If \( \mathbf{X}_1, \mathbf{X}_2, \cdots, \mathbf{X}_n \) are independent \( m \times m \) Hermitian complex random...
matrices with $\|X_i\| \leq 1$ for $1 \leq i \leq n$, where $\|\cdot\|$ is the spectral norm, we have following Chernoff bound for the version of $n$ i.i.d. random matrices:

$$\Pr \left( \left\| \frac{1}{n} \sum_{i=1}^{n} X_i - E[X] \right\| \geq \vartheta \right) \leq m \exp(-\Omega n\vartheta^2),$$

(1.2)

where $\Omega$ is a constant related to the matrix norm. This is also called “Matrix Chernoff Bound” and is applied to many fields, e.g., spectral graph theory, numerical linear algebra, machine learning and information theory [24]. Recently, Shih Yu generalized matrix bounds to various tensors bounds, e.g., Chernoff, Bennett, and Bernstein inequalities associated with tensors in [3].

Another direction to extend from the basic Chernoff bound is to consider non-independent assumptions for random variables. By Gillman [9] and its refinement works [6, 20], they changed the independence assumption to Markov dependence and we summarize their works as follows. We are given $\mathcal{G}$ as a regular $\lambda$-expander graph with vertex set $\mathcal{V}$, and $g : \mathcal{V} \to \mathbb{C}$ as a bounded function. Suppose $v_1, v_2, \ldots, v_\kappa$ is a stationary random walk of length $\kappa$ on $\mathcal{G}$, it is shown that:

$$\Pr \left( \left\| \frac{1}{\kappa} \sum_{j=1}^{\kappa} g(v_j) - \mathbb{E}[g] \right\| \geq \vartheta \right) \leq 2 \exp(-\Omega(1 - \lambda)\kappa\vartheta^2).$$

(1.3)

The value of $\lambda$ is also the second largest eigenvalue of the transition matrix of the underlying graph $\mathcal{G}$. The bound given in Eq. (1.3) is named as “Expander Chernoff Bound”. It is natural to generalize Eq. (1.3) to “Matrix Expander Chernoff Bound”. Wigderson and Xiao in [25] began first attempt to obtain partial results of “Matrix Expander Chernoff Bound” and the complete solution is given later by Garg et al. [8]. Their results can be summarized by the following theorem.

**Theorem 1.1.** Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a regular graph whose transition matrix has second largest eigenvalue $\lambda$ and let $g : \mathcal{V} \to \mathbb{C}^{m \times m}$ be a function satisfy following:

1. For each $v \in \mathcal{V}$, $g(v)$ is a Hermitian matrix with $\|g(v)\| \leq 1$;
2. $\sum_{v \in \mathcal{V}} g(v) = 0$.

Then, for a stationary random walk $v_1, \ldots, v_\kappa$ with $\epsilon \in (0, 1)$, we have

$$\Pr \left( \lambda_{g, \max} \left( \frac{1}{\kappa} \sum_{j=1}^{\kappa} g(v_j) \right) \geq \epsilon \right) \leq m \exp(-\Omega(1 - \lambda)\kappa\epsilon^2),$$

$$\Pr \left( \lambda_{g, \min} \left( \frac{1}{\kappa} \sum_{j=1}^{\kappa} g(v_j) \right) \leq -\epsilon \right) \leq m \exp(-\Omega(1 - \lambda)\kappa\epsilon^2),$$

(1.4)

where $\lambda_{g, \max}, (\lambda_{g, \min})$ is the largest (smallest) eigenvalue of the summation of $\kappa$ matrices obtained by the mapping $g$.

In this work, we generalize matrix expander Chernoff bound to tensor expander Chernoff bounds by allowing more general norm for tensors, Ky Fan norm and general convex function, instead identity function, of the tensors summand. We first extend the setting of Hermitian matrices to Hermitian tensors in [12] by utilizing majorization techniques to prove the following theorem 1.2, which will play a crucial role in our proof of tensor expander Chernoff bounds.
THEOREM 1.2. Let $C_i \in \mathbb{C}^d \times \cdots \times I_N \times I_N \times \cdots \times I_N$ be positive Hermitian tensors for $1 \leq i \leq n$ with Hermitian rank $r$, $\|\cdot\|_{(k)}$ be a Ky Fan $k$-norm with corresponding gauge function $\rho$. For any continuous function $f : (0, \infty) \to [0, \infty)$ such that $x \to \log f(e^x)$ is convex on $\mathbb{R}$, we have

$$\left\| f \left( \exp \left( \sum_{i=1}^{n} \log C_i \right) \right) \right\|_{(k)} \leq \exp \left( \int_{-\infty}^{\infty} \log \left\| f \left( \prod_{i=1}^{n} C_i^{\lambda_i} \right) \right\|_{(k)} \beta_0(t) dt, \right.$$

where $\lambda$ is $\sqrt{-1}$ and $\beta_0(t) = \frac{\pi}{2(\cosh(\lambda t))^{1+1/\lambda}}$. For any continuous function $g(0, \infty) \to (0, \infty)$ such that $x \to g(e^x)$ is convex on $\mathbb{R}$, we have

$$\left\| g \left( \exp \left( \sum_{i=1}^{n} \log C_i \right) \right) \right\|_{(k)} \leq \int_{-\infty}^{\infty} \left\| g \left( \prod_{i=1}^{n} C_i^{\lambda_i} \right) \right\|_{(k)} \beta_0(t) dt.$$ 

The main contribution of this work is the following tensor expander Chernoff bounds.

THEOREM 1.3. Let $G = (\mathcal{V}, \mathcal{E})$ be a regular undirected graph whose transition matrix has second eigenvalue $\lambda$, and let $g : \mathcal{V} \to \mathbb{C}^d \times \cdots \times I_M \times I_M \times \cdots \times I_M$ be a function. We assume following:

1. For each $v \in \mathcal{V}$, $g(v)$ is a Hermitian tensor;
2. $\|g(v)\| \leq r$;
3. A nonnegative coefficients polynomial raised by the power $s \geq 1$ as $f : x \to (a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n)^s$ satisfying $f \left( \exp \left( t \sum_{j=1}^{n} g(v_j) \right) \right) \geq \exp \left( rf \left( \sum_{j=1}^{n} g(v_j) \right) \right)$ almost surely;
4. For $\tau \in [\infty, \infty)$, we have constants $C$ and $\sigma$ such that $\beta_0(\tau) \leq \frac{C \exp(-\tau^2)}{\sigma \sqrt{2\pi}}$.

Then, we have

$$\Pr \left( \left| \int_{\mathcal{V}} g(v_j) \right| \geq \vartheta \right) \leq \min \left\{ \frac{(n+1)^{(s-1)} e^{-\vartheta t} \left( a_0 k + C \left( k + \sqrt{\frac{1}{k}} \right) \right)}{\left( \sum_{i=1}^{n} a_i \exp(8\kappa \lambda + 2(\kappa + 8\lambda)lsr + 2(\sigma(\kappa + 8\lambda)lsr)^2 t^2) \right)}, \right.$$

where $\prod_{k=1}^{M} I_k$ is a positive integer obtained from $\prod_{k=1}^{M} I_k$. 

By comparing matrix expander Chernoff bound, the tensor expander Chernoff bounds derived at this paper makes following relaxation: (1) the random objects dimensions are increased from matrices (2D data array) to tensors (multidimensional data array); (2) this bound generalizes the identity map to the power of polynomial functions as shown by the third assumption of the function $f$ at Theorem 1.3; (3) Ky Fan norm of $f \left( \sum_{j=1}^{n} g(v_j) \right)$ is considered instead only the maximum or the minimum eigenvalues of $f \left( \sum_{j=1}^{n} g(v_j) \right)$ being evaluated; (4) there are no restriction for $\sum_{v \in \mathcal{V}} g(v)$, but this
summation is required to be a zero matrix in the matrix expander Chernoff bound derivation.

The paper is organized as follows. Preliminaries of tensors and basic majorization notations are given in Section 2. In Section 3, multivariate tensor norm inequalities are established and these inequalities will become our main ingredients to prove tensor expander Chernoff bounds. Main theorem 1.3 is discussed and proved in Section 4. Finally, the conclusion and potential future works are given in Section 5.

2. Fundamentals of Tensors and Majorization. The purpose of this section is to provide fundamental facts about tensors and introduce notions about majorization.

2.1. Tensors Preliminaries. Throughout this work, scalars are represented by lower-case letters (e.g., $d, e, f, \ldots$), vectors by boldfaced lower-case letters (e.g., $\mathbf{d}, \mathbf{e}, \mathbf{f}, \ldots$), matrices by boldfaced capitalized letters (e.g., $\mathbf{D}, \mathbf{E}, \mathbf{F}, \ldots$), and tensors by calligraphic letters (e.g., $\mathcal{D}, \mathcal{E}, \mathcal{F}, \ldots$), respectively. Tensors are multiarrays of values which are higher-dimensional generalizations from vectors and matrices. Given a positive integer $N$, let $[N] = \{1, 2, \ldots, N\}$. An order-$N$ tensor (or $N$th order tensor) denoted by $X \equiv (x_{i_1, i_2, \ldots, i_N})$, where $1 \leq i_j = 1, 2, \ldots, I_j$ for $j \in [N]$, is a multidimensional array containing $\prod_{n=1}^{N} I_n$ entries. Let $\mathbb{C}^{I_1 \times \cdots \times I_N}$ and $\mathbb{R}^{I_1 \times \cdots \times I_N}$ be the sets of the order-$N$ $I_1 \times \cdots \times I_N$ tensors over the complex field $\mathbb{C}$ and the real field $\mathbb{R}$, respectively. For example, $X \in \mathbb{C}^{I_1 \times \cdots \times I_N}$ is an order-$N$ multiarray, where the first, second, ..., and $N$th dimensions have $I_1, I_2, \ldots, I_N$ entries, respectively. Thus, each entry of $X$ can be represented by $x_{i_1, \ldots, i_N}$. For example, when $N = 3$, $X \in \mathbb{C}^{I_1 \times I_2 \times I_3}$ is a third-order tensor containing entries $x_{i_1, i_2, i_3}$'s.

Without loss of generality, one can partition the dimensions of a tensor into two groups, say $M$ and $N$ dimensions, separately. Thus, for two order-$(M+N)$ tensors: $X \equiv (x_{i_1, \ldots, i_M, j_1, \ldots, j_N}) \in \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}$ and $Y \equiv (y_{i_1, \ldots, i_M, j_1, \ldots, j_N}) \in \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}$, according to [16], the tensor addition $X + Y \in \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}$ is given by

$$
(X + Y)_{i_1, \ldots, i_M, j_1 \times \cdots \times j_N} \equiv x_{i_1, \ldots, i_M, j_1, \ldots, j_N} + y_{i_1, \ldots, i_M, j_1, \ldots, j_N}.
$$

On the other hand, for tensors $X \equiv (x_{i_1, \ldots, i_M, j_1, \ldots, j_N}) \in \mathbb{C}^{J_1 \times \cdots \times J_N \times K_1 \times \cdots \times K_L}$ and $Y \equiv (y_{j_1, \ldots, j_N, k_1, \ldots, k_L}) \in \mathbb{C}^{J_1 \times \cdots \times J_N \times K_1 \times \cdots \times K_L}$, according to [16], the Einstein product (or simply referred to as tensor product in this work) $X \odot Y \in \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N \times K_1 \times \cdots \times K_L}$ is given by

$$
(X \odot Y)_{i_1, \ldots, i_M, j_1 \times \cdots \times j_N, k_1 \times \cdots \times k_L} \equiv \sum_{j_1, \ldots, j_N} x_{i_1, \ldots, i_M, j_1, \ldots, j_N} y_{j_1, \ldots, j_N, k_1, \ldots, k_L}.
$$

Note that we will often abbreviate a tensor product $X \odot Y$ to “$XY$” for notational simplicity in the rest of the paper. This tensor product will be reduced to the standard matrix multiplication as $L = M = N = 1$. Other simplified situations can also be extended as tensor–vector product ($M > 1$, $N = 1$, and $L = 0$) and tensor–matrix product ($M > 1$ and $N = L = 1$). In analogy to matrix analysis, we define some basic tensors and elementary tensor operations as follows.

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1 We have another work [4] apply same majorization techniques to build bounds for tail bounds for the summation of random tensors under independent assumptions.
**Definition 2.1.** A tensor whose entries are all zero is called a zero tensor, denoted by \( \mathcal{O} \).

**Definition 2.2.** An identity tensor \( I \in \mathbb{C}^{I_1 \times \cdots \times I_N \times J_1 \times \cdots \times J_N} \) is defined by

\[
(I)_{i_1 \times \cdots \times i_N \times j_1 \times \cdots \times j_N} \overset{\text{def}}{=} \prod_{k=1}^{N} \delta_{i_k,j_k},
\]

where \( \delta_{i_k,j_k} \overset{\text{def}}{=} 1 \) if \( i_k = j_k \); otherwise \( \delta_{i_k,j_k} \overset{\text{def}}{=} 0 \).

In order to define Hermitian tensor, the conjugate transpose operation (or Hermitian adjoint) of a tensor is specified as follows.

**Definition 2.3.** Given a tensor \( \mathcal{X} \overset{\text{def}}{=} (x_{i_1,\ldots,i_M,j_1,\ldots,j_N}) \in \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N} \), its conjugate transpose, denoted by \( \mathcal{X}^H \), is defined by

\[
(\mathcal{X}^H)_{j_1 \times \cdots \times j_N,i_1,\ldots,i_M} \overset{\text{def}}{=} \overline{x_{i_1,\ldots,i_M,j_1,\ldots,j_N}},
\]

where the overline notation indicates the complex conjugate of the complex number \( x_{i_1,\ldots,i_M,j_1,\ldots,j_N} \). If a tensor \( \mathcal{X} \) satisfies \( \mathcal{X}^H = \mathcal{X} \), then \( \mathcal{X} \) is a Hermitian tensor.

**Definition 2.4.** Given a tensor \( \mathcal{U} \overset{\text{def}}{=} (u_{i_1,\ldots,i_M,i_1,\ldots,i_M}) \in \mathbb{C}^{I_1 \times \cdots \times I_M \times l_1 \times \cdots \times l_M} \), if

\[
\mathcal{U}^H \ast_M \mathcal{U} = \mathcal{U} \ast_M \mathcal{U}^H = I \in \mathbb{C}^{I_1 \times \cdots \times I_M \times l_1 \times \cdots \times l_M},
\]

then \( \mathcal{U} \) is a unitary tensor. In this work, the symbol \( \mathcal{U} \) is reserved for a unitary tensor.

Following definition is provided to define the inverse of a given tensor.

**Definition 2.5.** Given a square tensor \( \mathcal{X} \overset{\text{def}}{=} (x_{i_1,\ldots,i_M,j_1,\ldots,j_M}) \in \mathbb{C}^{I_1 \times \cdots \times I_M \times l_1 \times \cdots \times l_M} \), if there exists \( \mathcal{X} \in \mathbb{C}^{I_1 \times \cdots \times I_M \times l_1 \times \cdots \times l_M} \) such that

\[
\mathcal{X} \ast_M \mathcal{X} = \mathcal{X} \ast_M \mathcal{X} = I,
\]

then \( \mathcal{X} \) is the inverse of \( \mathcal{X} \). We usually write \( \mathcal{X} \overset{\text{def}}{=} \mathcal{X}^{-1} \) thereby.

We also list other crucial tensor operations here. The trace of a square tensor is equivalent to the summation of all diagonal entries such that

\[
\text{Tr}(\mathcal{X}) \overset{\text{def}}{=} \sum_{1 \leq i \leq j \leq |M|} x_{i_1,\ldots,i_M,i_1,\ldots,i_M}.
\]

The inner product of two tensors \( \mathcal{X}, \mathcal{Y} \in \mathbb{C}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N} \) is given by

\[
\langle \mathcal{X}, \mathcal{Y} \rangle \overset{\text{def}}{=} \text{Tr} (\mathcal{X}^H \ast_M \mathcal{Y}).
\]

According to Eq. (2.8), the Frobenius norm of a tensor \( \mathcal{X} \) is defined by

\[
\| \mathcal{X} \| \overset{\text{def}}{=} \sqrt{\langle \mathcal{X}, \mathcal{X} \rangle}.
\]

From Theorem 5.2 in [19], every Hermitian tensor \( \mathcal{H} \in \mathbb{C}^{I_1 \times \cdots \times I_M \times l_1 \times \cdots \times l_N} \) has following decomposition

\[
\mathcal{H} = \sum_{i=1}^{r} \lambda_i \mathcal{U}_i \otimes \mathcal{U}_i^H,
\]

with \( \langle \mathcal{U}_i, \mathcal{U}_i \rangle = 1 \) and \( \langle \mathcal{U}_i, \mathcal{U}_j \rangle = 0 \) for \( i \neq j \).
where $\lambda_i \in \mathbb{R}$ and $\otimes$ denotes for Kronecker product. The values $\lambda_i$ are named as Hermitian eigenvalues, and the minimum integer of $r$ to decompose a Hermitian tensor as in Eq. (2.10) is called Hermitian tensor rank. A positive Hermitian tensor is a Hermitian tensor with all Hermitian eigenvalues are positive. A nonnegative Hermitian tensor is a Hermitian tensor with all Hermitian eigenvalues are nonnegative. The Hermitian determinant, denoted as $\det_H(\mathcal{A})$, is defined as the product of $\lambda_i$ of the tensor $\mathcal{A}$.

2.2. Unitarily Invariant Tensor Norms. Let us represent the Hermitian eigenvalues of a Hermitian tensor $H \in \mathbb{C}^{l_1 \times \cdots \times l_N \times l_1 \times \cdots \times l_N}$ in decreasing order by the vector $\vec{\lambda}(H) = (\lambda_1(H), \ldots, \lambda_r(H))$, where $r$ is the Hermitian rank of the tensor $H$. We use $\mathbb{R}_{\geq 0}(\mathbb{R}_{\geq 0})$ to represent a set of nonnegative (positive) real numbers. Let $\|\cdot\|_\rho$ be a unitarily invariant tensor norm, i.e.,

$$
\|H\|_\rho = \|\mathcal{U} \ast N \mathcal{H}\|_\rho = \|\mathcal{H}\|_\rho,
$$

where $\mathcal{U}$ is any unitary tensor. Let $\rho : \mathbb{R}_{\geq 0}^r \to \mathbb{R}_{\geq 0}$ be the corresponding gauge function that satisfies Hölder’s inequality so that

$$
\|H\|_\rho = \|\mathcal{U}\|_\rho = \rho(\vec{\lambda}(\mathcal{H})),
$$

where $|\mathcal{H}| \overset{\text{def}}{=} \sqrt{H^\dagger \ast N H}$. The bijective correspondence between symmetric gauge functions on $\mathbb{R}_{\geq 0}^r$ and unitarily invariant norms is due to von Neumann [7].

Several popular norms can be treated as special cases of unitarily invariant tensor norm. The first one is Ky Fan like $k$-norm [7] for tensors. For $k \in \{1, 2, \ldots, r\}$, the Ky Fan $k$-norm [7] for tensors $\mathcal{H} \in \mathbb{C}^{l_1 \times \cdots \times l_N \times l_1 \times \cdots \times l_N}$, denoted as $\|\mathcal{H}\|_{(k)}$, is defined as:

$$
\|\mathcal{H}\|_{(k)} \overset{\text{def}}{=} \sum_{i=1}^k \lambda_i(\mathcal{H}).
$$

If $k = 1$, the Ky Fan $k$-norm for tensors is the tensor operator norm (or spectral norm), denoted as $\|\mathcal{H}\|$. The second one is Schatten $p$-norm for tensors, denoted as $\|\mathcal{H}\|_p$, is defined as:

$$
\|\mathcal{H}\|_p \overset{\text{def}}{=} (\text{Tr}|\mathcal{H}|^p)^{\frac{1}{p}},
$$

where $p \geq 1$. If $p = 1$, it is the trace norm. The third one is $k$-trace norm, denoted as $\text{Tr}_k[\mathcal{H}]$, defined by [13]. It is

$$
\text{Tr}_k[\mathcal{H}] \overset{\text{def}}{=} \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq r} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}
$$

where $1 \leq k \leq r$. If $k = 1$, $\text{Tr}_1[\mathcal{H}]$ is reduced as trace norm. In our later derivation for multivariate tensor norm inequalities in Section 3 and tensor expander bounds in Section 4, we will focus on Ky Fan $k$-norm.

Following inequality is the extension of Hölder inequality to gauge function $\rho$ which will be used by later majorization proof arguments.

**Lemma 2.6.** For $n$ nonnegative real vectors with the dimension $r$, i.e., $\mathbf{b}_i = (b_{i_1}, \cdots, b_{i_r}) \in \mathbb{R}_{\geq 0}^r$, and $\alpha > 0$ with $\sum_{i=1}^n \alpha_i = 1$, we have

$$
\rho \left( \prod_{i=1}^n b_{i_1}^{\alpha_1} b_{i_2}^{\alpha_2} \cdots b_{i_r}^{\alpha_r} \right) \leq \prod_{i=1}^n \rho^{\alpha_i}(\mathbf{b}_i)
$$

where $\lambda_i \in \mathbb{R}$ and $\otimes$ denotes for Kronecker product. The values $\lambda_i$ are named as Hermitian eigenvalues, and the minimum integer of $r$ to decompose a Hermitian tensor as in Eq. (2.10) is called Hermitian tensor rank. A positive Hermitian tensor is a Hermitian tensor with all Hermitian eigenvalues are positive. A nonnegative Hermitian tensor is a Hermitian tensor with all Hermitian eigenvalues are nonnegative. The Hermitian determinant, denoted as $\det_H(\mathcal{A})$, is defined as the product of $\lambda_i$ of the tensor $\mathcal{A}$.
Proof: This proof is based on mathematical induction. The base case for \( n = 2 \) has been shown by Theorem IV.1.6 from [2].

We assume that Eq. (2.15) is true for \( n = m \), where \( m > 2 \). Let \( \odot \) be the component-wise product (Hadamard product) between two vectors. Then, we have

\[
(2.16) \quad \rho \left( \prod_{i=1}^{m+1} b_i^{a_i} \right) = \rho \left( \odot_{i=1}^{m+1} b_i^{a_i} \right),
\]

where \( \odot_{i=1}^{m+1} b_i^{a_i} \) is defined as \( \left( \prod_{i=1}^{m+1} b_i^{a_i} \right) \). Under such notations, Eq. (2.16) can be bounded as

\[
(2.17) \quad \rho \left( \odot_{i=1}^{m+1} b_i^{a_i} \right) \leq \rho^{m+1} \left( \prod_{i=1}^{m+1} b_i^{a_i} \right).
\]

By mathematical induction, this lemma is proved. \( \square \)

2.3. Antisymmetric Tensor Product. Let \( \mathcal{H} \) be a space of Hermitian tensors with Hermitian rank \( r \). Two tensors \( C, B \in \mathcal{H} \) is said \( C \succeq B \) if \( C - B \) is a nonnegative Hermitian tensor. For any \( k \in \{1, 2, \ldots, r\} \), let \( \mathcal{H}^{\otimes k} \) be the \( k \)-th tensor power of the tensor space \( \mathcal{H} \) and let \( \mathcal{H}^{\wedge k} \) be the antisymmetric subspace of \( \mathcal{H}^{\otimes k} \). The \( k \)-th antisymmetric tensor power, \( \wedge^k : \mathcal{H} \to \mathcal{H}^{\wedge k} \), maps any Hermitian tensor \( C \) to the restriction of \( C^{\otimes k} \in \mathcal{H}^{\otimes k} \) to the antisymmetric subspace \( \mathcal{H}^{\wedge k} \). Following lemma summarizes several useful properties of such antisymmetric tensor products.

Lemma 2.7. Let \( C, B, D \in \mathcal{H}^{I_1 \times \cdots \times I_N} \) be Hermitian tensors from \( \mathcal{H} \) with Hermitian rank \( r \). For any \( k \in \{1, 2, \ldots, r\} \), we have

1. \( (C^{\wedge k})^{H} = (C^{H})^{\wedge k} \).
2. \( (C^{\wedge k})^{*} (B^{\wedge k}) = (C^{*} B)^{\wedge k} \).
3. \( \lim_{i \to \infty} ||C_i - C|| \to 0 \), then \( \lim_{i \to \infty} ||C^{\wedge k}_i - C^{\wedge k}|| \to 0 \).
4. If \( C \succeq 0 \) (zero tensor), then \( C^{\wedge k} \succeq 0 \) and \( (C^p)^{\wedge k} = (C^{\wedge k})^p \) for all \( p \in \mathbb{R}_{\geq 0} \).
5. \( |C^{\wedge k}| = |C^{\wedge k}| \).
6. If \( D \succeq 0 \) and \( D \) is invertible, \( (D^z)^{\wedge k} = (D^{\wedge k})^z \) for all \( z \in \mathbb{C} \).
7. \( ||C^{\wedge k}|| = \prod_{i=1}^{k} \lambda_i(|C|) \).

Proof: Facts 1 and 2 are the restrictions of the associated relations (\( C^{H}\otimes k = (C^{\otimes k})^{H} \) and (\( C^{*} B^{\otimes k} = (C^{*} B)^{\otimes k} \)) to \( \mathcal{H}^{\otimes k} \). The fact 3 is true since, if \( \lim_{i \to \infty} ||C_i - C|| \to 0 \), we have \( \lim_{i \to \infty} ||C_i^{\otimes k} - C^{\otimes k}|| \to 0 \) and the associated restrictions of \( C_i^{\otimes k}, C^{\otimes k} \) to the antisymmetric subspace \( \mathcal{H}^{\wedge k} \).

For the fact 4, if \( C \succeq 0 \), then we have \( C^{\wedge k} = ((C^{1/2})^{\wedge k})^{H} \succeq O \), \((C^{1/2})^{\wedge k} \succeq O \) from facts 1 and 2. If \( p \) is rational, we have \( (C^p)^{\wedge k} = (C^{\wedge k})^p \) from the fact 2, and the equality \( (C^p)^{\wedge k} = (C^{\wedge k})^p \) is also true for any \( p > 0 \) if we apply the fact 3 to approximate any irrational numbers by rational numbers.
Because we have
\begin{equation}
|C|^{\wedge k} = \left(\sqrt{C^H C}\right)^{\wedge k} = \sqrt{(C^{\wedge k})^H C^{\wedge k}} = |C^{\wedge k}|,
\end{equation}
from facts [1], [2] and [4], so the fact [5] is valid.

For the fact [6], if \( z < 0 \), the fact [6] is true for all \( z \in \mathbb{R} \) by applying the fact [4] to \( D^{-1} \). Since we can apply the definition \( D^z \overset{\text{def}}{=} \exp(z \ln D) \) to have
\begin{equation}
C^p = D^z \Leftrightarrow C = \exp\left(\frac{z}{p} \ln D\right),
\end{equation}
where \( C \geq \mathcal{O} \). The general case of any \( z \in \mathbb{C} \) is also true by applying the fact [4] to \( C = \exp(\frac{z}{p} \ln D) \).

For the fact [7] proof, it is enough to prove the case that \( C \geq \mathcal{O} \) due to the fact [5]. Then, there exists a set of orthogonal tensors \( \{U_1, \cdots, U_r\} \) such that \( C \ast_N U_i = \lambda_i U_i \) for \( 1 \leq i \leq r \). We then have
\begin{equation}
C^{\wedge k} (U_1 \wedge \cdots \wedge U_k) = C \ast_N U_1 \wedge \cdots \wedge C \ast_N U_k = \left(\prod_{i=1}^k \lambda_i(|C|)\right) U_1 \wedge \cdots \wedge U_k.
\end{equation}
Hence, \( \|C^{\wedge k}\| = \prod_{i=1}^k \lambda_i(|C|) \). \hfill \Box

\subsection*{2.4. Majorization}
In this subsection, we will discuss majorization and several lemmas about majorization which will be used at later proofs.

Let \( \mathbf{x} = [x_1, \cdots, x_r] \in \mathbb{R}^r \), \( \mathbf{y} = [y_1, \cdots, y_r] \in \mathbb{R}^r \) be two vectors with following orders among entries \( x_1 \geq \cdots \geq x_r \) and \( y_1 \geq \cdots \geq y_r \), weak majorization between vectors \( \mathbf{x}, \mathbf{y}, \) represented by \( \mathbf{x} \prec_w \mathbf{y} \), requires following relation for vectors \( \mathbf{x}, \mathbf{y} \):
\begin{equation}
\sum_{i=1}^k x_i \leq \sum_{i=1}^k y_i,
\end{equation}
where \( k \in \{1, 2, \cdots, r\} \). Majorization between vectors \( \mathbf{x}, \mathbf{y} \), indicated by \( \mathbf{x} \prec \mathbf{y} \), requires following relation for vectors \( \mathbf{x}, \mathbf{y} \):
\begin{align*}
\sum_{i=1}^k x_i & \leq \sum_{i=1}^k y_i, \quad \text{for } 1 \leq k < r; \\
\sum_{i=1}^r x_i & = \sum_{i=1}^r y_i, \quad \text{for } k = r.
\end{align*}

For \( \mathbf{x}, \mathbf{y} \in \mathbb{R}_{\geq 0}^r \) such that \( x_1 \geq \cdots \geq x_r \) and \( y_1 \geq \cdots \geq y_r \), weak log majorization between vectors \( \mathbf{x}, \mathbf{y}, \) represented by \( \mathbf{x} \prec_w \log \mathbf{y} \), requires following relation for vectors \( \mathbf{x}, \mathbf{y} \):
\begin{equation}
\prod_{i=1}^k x_i \leq \prod_{i=1}^k y_i,
\end{equation}
where \( k \in \{1, 2, \cdots, r\} \), and log majorization between vectors \( \mathbf{x}, \mathbf{y} \), represented by \( \mathbf{x} \prec_{\log} \mathbf{y} \), requires equality for \( k = r \) in Eq. (2.23). If \( f \) is a single variable function, \( f(\mathbf{x}) \) represents a vector of \( [f(x_1), \cdots, f(x_r)] \). From Lemma 1 in [12], we have
3. Multivariate Tensor Norm Inequalities. In this section, we will develop several theorems about majorization in Section 3.1, and log majorization with integral average in Section 3.2. These majorization related theorems will provide us tools in deriving bounds for Ky Fan $k$-norms of multivariate tensors in Section 3.3.

3.1. Majorization with Integral Average. Let $\Omega$ be a $\sigma$-compact metric space and $\nu$ a probability measure on the Borel $\sigma$-field of $\Omega$. Let $C, D_r \in \mathbb{C}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N}$ be Hermitian tensors with Hermitian rank $r$. We further assume that tensors $C, D_r$ are uniformly bounded in their norm for $\tau \in \Omega$. Let $\tau \in \Omega \to D_r$ be a continuous function such that $\sup \{\|D_r\| : \tau \in \Omega\} < \infty$. For notational convenience, we define the following relation:

\[
\left(\int_{\Omega}^{x_1} \lambda_1(D_r)d\nu(\tau), \ldots, \int_{\Omega}^{x_r} \lambda_r(D_r)d\nu(\tau)\right) \overset{\text{def}}{=} \int_{\Omega}^{\lambda}(D_r)d\nu(\tau).
\]

If $f$ is a single variable function, the notation $f(C)$ represents a tensor function with respect to the tensor $C$. We want to prove the following theorem about weak majorization of the integral average.

**Theorem 3.1.** Let $\Omega, \nu, C, D_r$ be defined as the beginning part of Section 3.1, and $f : \mathbb{R} \to [0, \infty)$ be a non-decreasing convex function, we have following two equivalent statements:

\[
\begin{align*}
\tilde{\lambda}(C) &\prec_w \int_{\Omega}^{\tilde{\lambda}(D_r)d\nu(\tau)} \iff \|f(C)\|_{(k)} \leq \int_{\Omega}^{\|f(D_r)\|_{(k)}d\nu(\tau)},
\end{align*}
\]

where $\|\cdot\|_{(k)}$ is the Ky Fan $k$-norm defined by Eq. (2.12).

**Proof:** We assume that the left statement of Eq. (3.2) is true and the function $f$ is a non-decreasing convex function. From Lemma 2.8, we have

\[
\tilde{\lambda}(f(C)) = f(\tilde{\lambda}(C)) \prec_w f\left(\int_{\Omega}^{\tilde{\lambda}(D_r)d\nu(\tau)}\right).
\]

From the convexity of $f$, we also have

\[
f\left(\int_{\Omega}^{\tilde{\lambda}(D_r)d\nu(\tau)}\right) \leq \int_{\Omega}^{f(\tilde{\lambda}(D_r))d\nu(\tau)} = \int_{\Omega}^{\tilde{\lambda}(f(D_r))d\nu(\tau)}.
\]
Then, we obtain \( \tilde{\lambda}(f(C)) \sim_w f_{\Omega} \tilde{\lambda}(f(D_\tau))d\nu^r(\tau) \). By applying Lemma 4.4.2 in [11] to both sides of \( \tilde{\lambda}(f(C)) \sim_w f_{\Omega} \tilde{\lambda}(f(D_\tau))d\nu^r(\tau) \) with gauge function \( \rho \) of Ky Fan \( k \)-norm, we obtain

\[
\|f(C)\|_{(k)} \leq \rho \left( \int_{\Omega} \tilde{\lambda}(f(D_\tau))d\nu^r(\tau) \right)
\]

(3.5)

\[
\leq \int_{\Omega} \rho(\tilde{\lambda}(f(D_\tau)))d\nu(\tau) = \int_{\Omega} \|f(D_\tau)\|_{(k)} d\nu(\tau).
\]

Therefore, the right statement of Eq. (3.2) is true from the left statement.

On the other hand, if the right statement of Eq. (3.2) is true, we select a function \( f \overset{\text{def}}{=} \max\{x+c,0\} \), where \( c \) is a positive real constant satisfying \( C+cI \geq 0, D_\tau+cI \geq 0 \) for all \( \tau \in \Omega \), and tensors \( C+cI, D_\tau+cI \) having Hermitian rank \( r \). If the Ky Fan norm \( k \)-norm at the right statement of Eq. (3.2) is applied, we have

\[
\sum_{i=1}^{k} \lambda_i(C) + c \leq \sum_{i=1}^{k} \int_{\Omega} (\lambda_i(D_\tau) + c)d\nu(\tau).
\]

(3.6)

Hence, \( \sum_{i=1}^{k} \lambda_i(C) \leq \sum_{i=1}^{k} \int_{\Omega} \lambda_i(D_\tau)d\nu(\tau) \), this is the left statement of Eq. (3.2).

Next theorem will provide a stronger version of Theorem 3.1 by enhancing weak majorization to majorization.

**Theorem 3.2.** Let \( \Omega, \nu, C, D_\tau \) be defined as the beginning part of Section 3.1, and \( f : \mathbb{R} \to [0, \infty) \) be a convex function, we have following two equivalent statements:

\[
\tilde{\lambda}(C) \sim \int_{\Omega} \tilde{\lambda}(D_\tau)d\nu^r(\tau) \iff \|f(C)\|_{(k)} \leq \int_{\Omega} \|f(D_\tau)\|_{(k)} d\nu(\tau).
\]

(3.7)

**Proof:** We assume that the left statement of Eq. (3.7) is true and the function \( f \) is a convex function. Again, from Lemma 2.8, we have

\[
\tilde{\lambda}(f(C)) = f(\tilde{\lambda}(C)) \sim_w \left( \int_{\Omega} \tilde{\lambda}(D_\tau)d\nu^r(\tau) \right) \leq \int_{\Omega} f(\tilde{\lambda}(D_\tau))d\nu^r(\tau),
\]

(3.8)

then,

\[
\|f(C)\|_{(k)} \leq \rho \left( \int_{\Omega} f(\tilde{\lambda}(D_\tau))d\nu^r(\tau) \right)
\]

(3.9)

\[
\leq \int_{\Omega} \rho \left( f(\tilde{\lambda}(D_\tau)) \right) d\nu(\tau) = \int_{\Omega} \|f(D_\tau)\|_{(k)} d\nu(\tau),
\]

where \( \rho \) is the gauge function of Ky Fan \( k \)-norm. This proves the right statement of Eq. (3.7).

Now, we assume that the right statement of Eq. (3.7) is true. From Theorem 3.1, we already have \( \tilde{\lambda}(C) \sim_w \int_{\Omega} \tilde{\lambda}(D_\tau)d\nu^r(\tau) \). It is enough to prove \( \sum_{i=1}^{r} \lambda_i(C) \geq \int_{\Omega} \sum_{i=1}^{r} \lambda_i(D_\tau)d\nu(\tau) \). We define a function \( f \overset{\text{def}}{=} \max\{c-x,0\} \), where \( c \) is a positive real constant satisfying \( C \leq cI, D_\tau \leq cI \) for all \( \tau \in \Omega \) and tensors \( cI - C, cI - D_\tau \) having
Hermitian rank $r$. If the trace norm is applied, i.e., the sum of the absolute value of all eigenvalues of a Hermitian tensor, then the right statement of Eq. (3.7) becomes

\begin{equation}
\sum_{i=1}^{r} \lambda_i (cI - \mathcal{C}) \leq \int_{\Omega} \sum_{i=1}^{r} \lambda_i (cI - \mathcal{D}_\tau) d\nu(\tau).
\end{equation}

The desired inequality $\sum_{i=1}^{r} \lambda_i(\mathcal{C}) \geq \int_{\Omega} \sum_{i=1}^{r} \lambda_i(\mathcal{D}_\tau) d\nu(\tau)$ is established. \hfill \Box

3.2. Log-Majorization with Integral Average. The purpose of this section is to consider log-majorization issues for Ky Fan $k$-norm of Hermitian tensors. In this section, let $\mathcal{C}, \mathcal{D}_\tau \in \mathbb{C}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N}$ be nonnegative Hermitian tensors with Hermitian rank $r$, i.e., all Hermitian eigenvalues are positive, and keep other notations with the same definitions as at the beginning of the Section 3.1. For notational convenience, we define the following relation for logarithm vector:

\begin{equation}
\int_{\Omega} \log \lambda_1(\mathcal{D}_\tau) d\nu(\tau), \cdots, \int_{\Omega} \log \lambda_r(\mathcal{D}_\tau) d\nu(\tau) \quad \text{def} \quad \int_{\Omega'} \log \bar{\lambda}(\mathcal{D}_\tau) d\nu'(\tau).
\end{equation}

Following theorem is used to build the relationship between weak log-majorization of eigenvalues and Ky Fan $k$-norm.

**Theorem 3.3.** Let $\mathcal{C}, \mathcal{D}_\tau$ be nonnegative Hermitian tensors, $f : (0, \infty) \to [0, \infty)$ be a continous function such that the mapping $x \to \log f(e^x)$ is convex on $\mathbb{R}$, and $g : (0, \infty) \to [0, \infty)$ be a continous function such that the mapping $x \to g(e^x)$ is convex on $\mathbb{R}$, then we have following three equivalent statements:

\begin{equation}
\bar{\lambda}(\mathcal{C}) \prec_w \exp \int_{\Omega'} \log \bar{\lambda}(\mathcal{D}_\tau) d\nu'(\tau);
\end{equation}

\begin{equation}
\|f(\mathcal{C})\|_{(k)} \leq \exp \int_{\Omega} \|f(\mathcal{D}_\tau)\|_{(k)} d\nu(\tau);
\end{equation}

\begin{equation}
\|g(\mathcal{C})\|_{(k)} \leq \int_{\Omega} \|g(\mathcal{D}_\tau)\|_{(k)} d\nu(\tau).
\end{equation}

**Proof:** The roadmap of this proof is to prove equivalent statements between Eq. (3.12) and Eq. (3.13) first, followed by equivalent statements between Eq. (3.12) and Eq. (3.14).

**Eq. (3.12) \implies Eq. (3.13)**

There are two cases to be discussed in this part of proof: $\mathcal{C}, \mathcal{D}_\tau$ are positive Hermitian tensors, and $\mathcal{C}, \mathcal{D}_\tau$ are nonnegative Hermitian tensors. We consider the case that $\mathcal{C}, \mathcal{D}_\tau$ are positive Hermitian tensors first.

Since $\mathcal{D}_\tau$ are positive, we can find $\varepsilon > 0$ such that $\mathcal{D}_\tau \geq \varepsilon I$ for all $\tau \in \Omega$. From Eq. (3.12), the convexity of $\log f(e^x)$ and Lemma 2.8, we have

\begin{equation}
\bar{\lambda}(f(\mathcal{C})) = f \left( \exp \left( \log \bar{\lambda}(\mathcal{C}) \right) \right) \prec_w f \left( \exp \int_{\Omega'} \log \bar{\lambda}(\mathcal{D}_\tau) d\nu'(\tau) \right)
\leq \exp \left( \int_{\Omega'} \log f \left( \bar{\lambda}(\mathcal{D}_\tau) \right) d\nu'(\tau) \right).
\end{equation}

Then, from Eq. (2.11), we obtain

\begin{equation}
\|f(\mathcal{C})\|_{\rho} \leq \rho \left( \exp \left( \int_{\Omega'} \log f \left( \bar{\lambda}(\mathcal{D}_\tau) \right) d\nu'(\tau) \right) \right).
\end{equation}
From the function \( f \) properties, we can assume that \( f(x) > 0 \) for any \( x > 0 \). Then, we have bounded and continuous maps on \( \Omega \): \( \tau \rightarrow \log f(\lambda_i(\mathcal{D}_\tau)) \) for \( i \in \{1, 2, \ldots, r\} \), and \( \tau \rightarrow \log \|f(\mathcal{D}_\tau)\|_{(k)} \). Because we have \( \nu(\Omega) = 1 \) and \( \sigma \)-compactness of \( \Omega \), we have \( \tau^{(n)}_k \in \Omega \) and \( \alpha_k^{(n)} \) for \( k \in \{1, 2, \ldots, n\} \) and \( n \in \mathbb{N} \) with \( \sum_{k=1}^{n} \alpha_k^{(n)} = 1 \) such that

\[
(3.17) \int_{\Omega} \log f(\lambda_i(\mathcal{D}_\tau)) d\nu(\tau) = \lim_{n \to \infty} \sum_{k=1}^{n} \alpha_k^{(n)} \log f(\lambda_i(\mathcal{D}_{\tau^{(n)}_k})), \text{ for } i \in \{1, 2, \ldots, r\};
\]

and

\[
(3.18) \int_{\Omega} \log \|f(\mathcal{D}_\tau)\|_{(k)} d\nu(\tau) = \lim_{n \to \infty} \sum_{k=1}^{n} \alpha_k^{(n)} \log \|f(\mathcal{D}_{\tau^{(n)}_k})\|_{(k)}.
\]

By taking the exponential at both sides of Eq. (3.17) and apply the gauge function \( \rho \), we have

\[
(3.19) \rho \left( \exp \int_{\Omega^r} \log f(\lambda_i(\mathcal{D}_\tau)) d\nu^\tau(\tau) \right) = \lim_{n \to \infty} \rho \left( \prod_{k=1}^{n} f \left( \lambda_i(\mathcal{D}_{\tau^{(n)}_k}) \right)^{\alpha_k^{(n)}} \right).
\]

Similarly, by taking the exponential at both sides of Eq. (3.18), we have

\[
(3.20) \exp \left( \int_{\Omega} \log \|f(\mathcal{D}_\tau)\|_{(k)} d\nu(\tau) \right) = \lim_{n \to \infty} \prod_{k=1}^{n} \|f(\mathcal{D}_{\tau^{(n)}_k})\|_{(k)}^{\alpha_k^{(n)}}.
\]

From Lemma 2.6, we have

\[
\rho \left( \prod_{k=1}^{n} f \left( \lambda_i(\mathcal{D}_{\tau^{(n)}_k}) \right)^{\alpha_k^{(n)}} \right) \leq \prod_{k=1}^{n} \rho^{\alpha_k^{(n)}} \left( f \left( \lambda_i(\mathcal{D}_{\tau^{(n)}_k}) \right) \right)
\]

\[= \prod_{k=1}^{n} \rho^{\alpha_k^{(n)}} \left( \lambda_i(\mathcal{D}_{\tau^{(n)}_k}) \right)
\]

\[= \prod_{k=1}^{n} \|f(\mathcal{D}_{\tau^{(n)}_k})\|_{(k)}^{\alpha_k^{(n)}}.
\]

(3.21)

From Eqs. (3.19), (3.20) and (3.21), we have

\[
(3.22) \rho \left( \exp \int_{\Omega^r} \log f(\lambda_i(\mathcal{D}_\tau)) d\nu^\tau(\tau) \right) \leq \exp \int_{\Omega} \log \|f(\mathcal{D}_\tau)\|_{(k)} d\nu(\tau).
\]

Then, Eq. (3.13) is proved from Eqs. (3.16) and (3.22).

Next, we consider that \( \mathcal{C}, \mathcal{D}_\tau \) are nonnegative Hermitian tensors. For any \( \delta > 0 \), we have following log-majorization relation:

\[
(3.23) \prod_{i=1}^{k} \left( \lambda_i(\mathcal{C}) + \epsilon_\delta \right) \leq \prod_{i=1}^{k} \exp \int_{\Omega} \log (\lambda_i(\mathcal{D}_\tau) + \delta) d\nu(\tau),
\]
where $\delta > \epsilon > 0$ and $k \in \{1, 2, \ldots, r\}$. Then, we can apply the previous case result about positive Hermitian tensors to positive Hermitian tensors $C + \epsilon I$ and $D_\tau + \delta I$, and get

$$
\|f(C) + \epsilon I\|_{(k)} \leq \exp \int_\Omega \log \|f(D_\tau) + \delta I\|_{(k)} d\nu(\tau)
$$

As $\delta \to 0$, Eq. (3.24) will give us Eq. (3.13) for nonnegative Hermitian tensors by the monotone convergence theorem.

**Eq. (3.12) $\iff$ Eq. (3.13)**

We consider positive Hermitian tensors at first phase by assuming that $D_\tau$ are positive Hermitian for all $\tau \in \Omega$. We may also assume that the tensor $C$ is a positive Hermitian tensor. Since if this is a nonnegative Hermitian tensor, i.e., some $\lambda_i = 0$, we always have following inequality valid:

$$
\prod_{i=1}^k \lambda_i(C) \leq \prod_{i=1}^k \exp \int_\Omega \log \lambda_i(D_\tau) d\nu(\tau)
$$

If we apply $f(x) = x^p$ for $p > 0$ and $\|\cdot\|_{(k)}$ as Ky Fan $k$-norm in Eq. (3.13), we have

$$
\frac{1}{p} \log \left( \frac{1}{k} \sum_{i=1}^k \lambda_i^p(C) \right) \leq \int_\Omega \frac{1}{p} \log \left( \frac{1}{k} \sum_{i=1}^k \lambda_i^p(D_\tau) \right) d\nu(\tau).
$$

If we add $\log \frac{1}{p}$ and multiply $\frac{1}{p}$ at both sides of Eq. (3.26), we have

$$
\frac{1}{p} \log \left( \frac{1}{k} \sum_{i=1}^k \lambda_i^p(C) \right) + \frac{1}{p} \log \lambda_i(C) \to \frac{1}{k} \sum_{i=1}^k \log \lambda_i(C),
$$

and

$$
\frac{1}{p} \log \left( \frac{1}{k} \sum_{i=1}^k \lambda_i^p(D_\tau) \right) + \frac{1}{p} \log \lambda_i(D_\tau) \to \frac{1}{k} \sum_{i=1}^k \log \lambda_i(D_\tau),
$$

where $\tau \in \Omega$. Applying Eqs. (3.28) and (3.29) into Eq. (3.27) and taking $p \to 0$, we have

$$
\sum_{i=1}^k \lambda_i(C) \leq \int_\Omega \sum_{i=1}^k \log \lambda_i(D_\tau) d\nu(\tau).
$$

Therefore, Eq. (3.12) is true for positive Hermitian tensors.

For nonnegative Hermitian tensors $D_\tau$, since Eq. (3.13) is valid for $D_\tau + \delta I$ for any $\delta > 0$, we can apply the previous case result about positive Hermitian tensors to $D_\tau + \delta I$ and obtain

$$
\prod_{i=1}^k \lambda_i(C) \leq \prod_{i=1}^k \exp \int_\Omega \log (\lambda_i(D_\tau) + \delta) d\nu(\tau),
$$
where \( k \in \{1, 2, \cdots, r\} \). Eq. (3.12) is still true for nonnegative Hermitian tensors as \( \delta \to 0 \).

**Eq. (3.12) \implies Eq. (3.14)**

If \( C, D_\tau \) are positive Hermitian tensors, and \( D_\tau \geq \delta I \) for all \( \tau \in \Omega \). From Eq. (3.12), we have

\[
\vec{\lambda}(\log C) = \log \vec{\lambda}(C) \prec_k \int \log \vec{\lambda}(D_\tau) d\nu(\tau) = \int \vec{\lambda}(D_\tau) d\nu(\tau).
\]

If we apply Theorem 3.1 to \( \log C, \log D_\tau \) with function \( f(x) = g(e^x) \), where \( g \) is used in Eq. (3.14), Eq. (3.14) is implied.

If \( C, D_\tau \) are nonnegative Hermitian tensors and any \( \delta > 0 \), we can find \( \epsilon_\delta \in (0, \delta) \) to satisfy following:

\[
\prod_{i=1}^{k} (\lambda_i(C) + \epsilon_\delta) \leq \prod_{i=1}^{k} \exp \int \log (\lambda_i(D_\tau) + \delta) d\nu(\tau).
\]

Then, from positive Hermitian tensor case, we have

\[
\|g(C + \epsilon_\delta I)\|_{(k)} \leq \int \|g(D_\tau + \delta I)\|_{(k)} d\nu(\tau).
\]

Eq. (3.14) is obtained by taking \( \delta \to 0 \) in Eq. (3.34).

**Eq. (3.12) \implies Eq. (3.14)**

For \( k \in \{1, 2, \cdots, r\} \), if we apply \( g(x) = \log(\delta + x) \), where \( \delta > 0 \), and Ky Fan \( k \)-norm in Eq. (3.14), we have

\[
\sum_{i=1}^{k} \log (\delta + \lambda_i(C)) \leq \sum_{i=1}^{k} \int \log (\delta + \lambda_i(D_\tau)) d\nu(\tau).
\]

Then, we have following as \( \delta \to 0 \):

\[
\sum_{i=1}^{k} \log \lambda_i(C) \leq \sum_{i=1}^{k} \int \log \lambda_i(D_\tau) d\nu(\tau).
\]

Therefore, Eq. (3.12) can be derived from Eq. (3.14). \( \square \)

Next theorem will extend Theorem 3.3 to non-weak version.

**Theorem 3.4.** Let \( C, D_\tau \) be nonnegative Hermitian tensors with \( \int \|D_\tau^p\|_p d\nu(\tau) < \infty \) for any \( p > 0 \), \( f : (0, \infty) \to [0, \infty) \) be a continuous function such that the mapping \( x \to \log f(e^x) \) is convex on \( \mathbb{R} \), and \( g : (0, \infty) \to [0, \infty) \) be a continuous function such that the mapping \( x \to g(x) \) is convex on \( \mathbb{R} \), then we have following three equivalent statements:

\[
\vec{\lambda}(C) \prec \log \exp \int \log \vec{\lambda}(D_\tau) d\nu(\tau);
\]

\[
\|f(C)\|_{(k)} \leq \exp \int \|f(D_\tau)\|_{(k)} d\nu(\tau);
\]

\[
\|g(C)\|_{(k)} \leq \int \|g(D_\tau)\|_{(k)} d\nu(\tau).
\]
Proof:

The proof plan is similar to the proof in Theorem 3.3.

Eq. (3.37) \implies Eq. (3.38)

First, we assume that $\mathcal{C}, \mathcal{D}_r$ are positive Hermitian tensors with $\mathcal{D}_r \geq \delta I$ for all $\tau \in \Omega$. The corresponding part of the proof in Theorem 3.3 about positive Hermitian tensors $\mathcal{C}, \mathcal{D}_r$ can be applied here.

For case that $\mathcal{C}, \mathcal{D}_r$ are nonnegative Hermitian tensors, we have

\begin{equation}
\prod_{i=1}^{k} \lambda_i(C) \leq \prod_{i=1}^{k} \exp \int_{\Omega} \log (\lambda_i(D_r) + \delta_n) d\nu(\tau),
\end{equation}

where $\delta_n > 0$ and $\delta_n \to 0$. Because $\int_{\Omega} \log (\tilde{\lambda}(D_r) + \delta_n) d\nu'(\tau) \to \int_{\Omega} \log \tilde{\lambda}(D_r) d\nu(\tau)$ as $n \to \infty$, from Lemma 2.9, we can find $\mathbf{a}^{(n)}$ with $n \geq n_0$ such that $a_1^{(n)} \geq \cdots \geq a_r^{(n)} > 0$, $\mathbf{a}^{(n)} \rightarrow \tilde{\lambda}(C)$ and $\mathbf{a}^{(n)} \rightarrow \tilde{\lambda}(C)$ for all $\tau \in \Omega$.

Selecting $\mathcal{C}^{(n)}$ with $\tilde{\lambda}(\mathcal{C}^{(n)}) = \mathbf{a}^{(n)}$ and applying positive Hermitian tensors case to $\mathcal{C}^{(n)}$ and $\mathcal{D}_r + \delta_n I$, we obtain

\begin{equation}
\left\| f(\mathcal{C}^{(n)}) \right\|_{(k)} \leq \exp \int_{\Omega} \log \left\| f(\mathcal{D}_r + \delta_n I) \right\|_{(k)} d\nu(\tau)
\end{equation}

where $n \geq n_0$.

There are two situations for the function $f$ near 0: $f(0^+) < \infty$ and $f(0^+) = \infty$. For the case with $f(0^+) < \infty$, we have

\begin{equation}
\left\| f(\mathcal{C}^{(n)}) \right\|_{(k)} = \rho(f(\mathbf{a}^{(n)})) \rightarrow \rho(f(\tilde{\lambda}(C))) = \left\| f(\mathcal{C}) \right\|_{(k)}
\end{equation}

and

\begin{equation}
\left\| f(\mathcal{D}_r + \delta_n I) \right\|_{(k)} \rightarrow \left\| f(\mathcal{D}_r) \right\|_{(k)}
\end{equation}

where $\tau \in \Omega$ and $n \to \infty$. From Fatou–Lebesgue theorem, we then have

\begin{equation}
\limsup_{n \to \infty} \int_{\Omega} \log \left\| f(\mathcal{D}_r + \delta_n I) \right\|_{(k)} d\nu(\tau) \leq \int_{\Omega} \log \left\| f(\mathcal{D}_r) \right\|_{(k)} d\nu(\tau).
\end{equation}

By taking $n \to \infty$ in Eq. (3.41) and using Eqs. (3.42), (3.43), (3.44), we have Eq. (3.38) for case that $f(0^+) < \infty$.

For the case with $f(0^+) = \infty$, we assume that $\int_{\Omega} \log \left\| f(\mathcal{D}_r) \right\|_{(k)} d\nu(\tau) < \infty$ (since the inequality in Eq. (3.38) is always true for $\int_{\Omega} \log \left\| f(\mathcal{D}_r) \right\|_{(k)} d\nu(\tau) = \infty$). Since $f$ is decreasing on $(0, \epsilon)$ for some $\epsilon > 0$. We claim that the following relation is valid: there are two constants $a, b > 0$ such that

\begin{equation}
a \leq \left\| f(\mathcal{D}_r + \delta_n I) \right\|_{(k)} \leq \left\| f(\mathcal{D}_r) \right\|_{(k)} + b,
\end{equation}

for all $\tau \in \Omega$ and $n \geq n_0$. If Eq. (3.45) is valid and $\int_{\Omega} \log \left\| f(\mathcal{D}_r) \right\|_{(k)} d\nu(\tau) < \infty$, from Lebesgue’s dominated convergence theorem, we also have Eq. (3.38) for case that $f(0^+) = \infty$ by taking $n \to \infty$ in Eq. (3.41).

Now, we want to prove the claim shown by Eq. (3.45). By the uniform boundedness of tensors $\mathcal{D}_r$, there is a constant $\chi > 0$ such that

\begin{equation}
0 < \mathcal{D}_r + \delta_n I \leq \chi I,
\end{equation}

where $\mathcal{D}_r$ is decreasing on $(0, \epsilon)$.
where $\tau \in \Omega$ and $n \geq n_0$. From Eq. (2.10), we have

$$f(D_\tau + \delta_n I) = \sum_{\nu \text{ s.t. } \lambda_\nu(D_\tau) + \delta_n < \epsilon} f(\lambda_\nu(D_\tau) + \delta_n) U_\nu \otimes U_\nu^H +$$

$$+ \sum_{\nu \text{ s.t. } \lambda_\nu(D_\tau) + \delta_n \geq \epsilon} f(\lambda_\nu(D_\tau) + \delta_n) U_\nu \otimes U_\nu^H$$

$$\leq f(\lambda_\nu(D_\tau)) U_\nu \otimes U_\nu^H +$$

$$\sum_{\nu \text{ s.t. } \lambda_\nu(D_\tau) + \delta_n \geq \epsilon} f(\lambda_\nu(D_\tau) + \delta_n) U_\nu \otimes U_\nu^H$$

(3.47)

$$\leq f(D_\tau) + \sum_{\nu \text{ s.t. } \lambda_\nu(D_\tau) + \delta_n \geq \epsilon} f(\lambda_\nu(D_\tau) + \delta_n) U_\nu \otimes U_\nu^H.$$ 

Therefore, the claim in Eq. (3.45) follows by the triangle inequality for $\|\cdot\|_H$ and $f(\lambda_\nu(D_\tau) + \delta_n) < \infty$ for $\lambda_\nu(D_\tau) + \delta_n \geq \epsilon$.

**Eq. (3.37) $\iff$ Eq. (3.38)**



The weak majorization relation

$$\prod_{i=1}^{k} \lambda_i(C) \leq \prod_{i=1}^{k} \exp \int_{\Omega} \log \lambda_i(D_\tau) d\nu(\tau),$$

(3.48)

is valid for $k < r$ from Eq. (3.12) $\implies$ Eq. (3.13) in Theorem 3.3. We wish to prove that Eq. (3.48) becomes equal for $k = r$. It is equivalent to prove that

$$\log \det_H(C) \geq \int_{\Omega} \log \det_H(D_\tau) d\nu(\tau),$$

(3.49)

where $\det_H(\cdot)$ is the Hermitian determinant. We can assume that $\int_{\Omega} \log \det_H(D_\tau) d\nu(\tau) \geq -\infty$ since Eq. (3.49) is true for $\int_{\Omega} \log \det_H(D_\tau) d\nu(\tau) = -\infty$. Then, $D_\tau$ are positive Hermitian tensors.

If we scale tensors $C, D_\tau$ as $aC, aD_\tau$ by some $a > 0$, we can assume $D_\tau \leq I$ and $\lambda_i(D_\tau) \leq 1$ for all $\tau \in \Omega$ and $i \in \{1, 2, \ldots, r\}$. Then for any $p > 0$, we have

$$\frac{1}{p} \frac{\|D_\tau^{-p}\|_1}{r} \leq \lambda_i^{-p}(D_\tau) \leq (\det_H(D_\tau))^{-p},$$

(3.50)

where $\|\cdot\|_1$ represents the tensor trace norm, and

$$\frac{1}{p} \log \left( \frac{\|D_\tau^{-p}\|_1}{r} \right) \leq -\log \det_H(D_\tau).$$

(3.51)

If we use tensor trace norm as unitarily invariant tensor norm and $f(x) = x^{-p}$ for any $p > 0$ in Eq. (3.38), we obtain

$$\log \|C^{-p}\|_1 \leq \int_{\Omega} \log \|D_\tau^{-p}\|_1 d\nu(\tau).$$

(3.52)

By adding $\frac{1}{p}$ and multiplying $\frac{1}{p}$ for both sides of Eq. (3.52), we have

$$\frac{1}{p} \log \left( \frac{\|C^{-p}\|_1}{r} \right) \leq \int_{\Omega} \frac{1}{p} \log \left( \frac{\|D_\tau^{-p}\|_1}{r} \right) d\nu(\tau).$$

(3.53)
Similar to Eqs. (3.28) and (3.29), we have following two relations as $p \to 0$:

\begin{equation}
\frac{1}{p} \log \left( \frac{\|C^{-p}\|_1}{r} \right) \to -\log \det_H(C),
\end{equation}

and

\begin{equation}
\frac{1}{p} \log \left( \frac{\|D_{\tau}^{-p}\|_1}{r} \right) \to -\log \det_H(D_\tau).
\end{equation}

From Eq. (3.51) and Lebesgue’s dominated convergence theorem, we have

\begin{equation}
\lim_{p \to 0} \int_\Omega \frac{1}{p} \log \left( \frac{\|D_{\tau}^{-p}\|_1}{r} \right) d\nu(\tau) = \frac{-1}{r} \int_\Omega \log \det_H(D_\tau) d\nu(\tau)
\end{equation}

Finally, we have Eq. (3.49) from Eqs. (3.53) and (3.56).

**Eq. (3.37) \implies Eq. (3.39)**

First, we assume that $C, D_\tau$ are positive Hermitian tensors and $D_\tau \geq \delta I$ for $\tau \in \Omega$. From Eq. (3.37), we can apply Theorem 3.2 to $C, \log D_\tau$ and $f(x) = g(e^r)$ to obtain Eq. (3.39).

For $C, D_\tau$ are nonnegative Hermitian tensors, we can choose $a^{(n)}$ and corresponding $C^{(n)}$ for $n \geq n_0$ given $\delta_n \to 0$ with $\delta_n > 0$ as the proof in Eq. (3.37) \implies Eq. (3.38). Since tensors $C^{(n)}, D_\tau + \delta_n I$ are positive Hermitian tensors, we then have

\begin{equation}
\left\| g(C^{(n)}) \right\|_{(k)} \leq \int_\Omega \left\| g(D_\tau + \delta_n I) \right\|_{(k)} d\nu(\tau).
\end{equation}

If $g(0^+) < \infty$, Eq. (3.39) is obtained from Eq. (5.7) by taking $n \to \infty$. On the other hand, if $g(0^+) = \infty$, we can apply the argument similar to the portion about $f(0^+) = \infty$ in the proof for Eq. (3.37) \implies Eq. (3.38) to get $a, b > 0$ such that

\begin{equation}
a \leq \left\| g(D_\tau + \delta_n I) \right\|_{\rho} \leq \left\| g(D_\tau) \right\|_{\rho} + b,
\end{equation}

for all $\tau \in \Omega$ and $n \geq n_0$. Since the case that $\int_\Omega \left\| g(D_\tau) \right\|_{(k)} d\nu(\tau) = \infty$ will have Eq. (3.39), we only consider the case that $\int_\Omega \left\| g(D_\tau) \right\|_{(k)} d\nu(\tau) < \infty$. Then, we have Eq. (3.39) from Eqs. (5.7), (5.8) and Lebesgue’s dominated convergence theorem.

**Eq. (3.37) \iff Eq. (3.39)**

The weak majorization relation

\begin{equation}
\sum_{i=1}^k \log \lambda_i(C) \leq \sum_{i=1}^k \int_\Omega \log \lambda_i(D_\tau) d\nu(\tau)
\end{equation}

is true from the implication from Eq. (3.14) to Eq. (3.12) in Theorem 3.3. We have to show that this relation becomes identity for $k = r$. If we apply $\left\| \right\|_{\rho} = \left\| \right\|_1$ and $g(x) = x^{-p}$ for any $p > 0$ in Eq. (3.39), we have

\begin{equation}
\frac{1}{p} \log \left( \frac{\|C^{-p}\|_1}{r} \right) \leq \frac{1}{p} \log \int_\Omega \frac{\|D_{\tau}^{-p}\|_1}{r} d\nu(\tau).
\end{equation}

Then, we will get

\begin{equation}
-\frac{\log \det_H(C)}{r} = \lim_{p \to 0} \frac{1}{p} \log \left( \frac{\|C^{-p}\|_1}{r} \right)
\end{equation}

\begin{equation}
\leq \lim_{p \to 0} \frac{1}{p} \log \left( \int_\Omega \frac{\|D_{\tau}^{-p}\|_1}{r} d\nu(\tau) \right) = -\int_\Omega \log \det_H(D_\tau) d\nu(\tau),
\end{equation}
which will proves the identity for Eq. (3.59) when \( k = r \). The equality in \( =_1 \) will be proved by the following Lemma 3.5.

**Lemma 3.5.** Let \( \mathcal{D}_\tau \) be nonnegative Hermitian tensors with \( \int_\Omega \|\mathcal{D}_\tau^{-p}\|_1 d\nu(\tau) < \infty \) for any \( p > 0 \), then we have

(3.62) \[ \lim_{p \to 0} \left( \frac{1}{p} \log \int_\Omega \|\mathcal{D}_\tau^{-p}\|_1^{\frac{1}{r}} d\nu(\tau) \right) = -\frac{1}{r} \int_\Omega \log \det_H(\mathcal{D}_\tau) d\nu(\tau) \]

**Proof:** Because \( \int_\Omega \|\mathcal{D}_\tau^{-p}\|_p d\nu(\tau) < \infty \), we have that \( \mathcal{D}_\tau \) are positive Hermitian tensors for \( \tau \) almost everywhere in \( \Omega \). Then, we have

\[ \lim_{p \to 0} \left( \frac{1}{p} \log \int_\Omega \|\mathcal{D}_\tau^{-p}\|_1^{\frac{1}{r}} d\nu(\tau) \right) = \frac{1}{r} \int_\Omega \sum_{i=1}^r \log \lambda_i(\mathcal{D}_\tau) d\nu(\tau) \]

(3.63)

where \( =_1 \) is from L’Hopital’s rule, and \( =_2 \) is obtained from \( \det_H \) definition.

### 3.3. Multivariate Tensor Norm Inequalities

In this section, we will apply derived majorization inequalities for tensors to multivariate tensor norm inequalities which will be used to derive tensor expander bounds. We will begin to present a Lie-Trotter product formula for tensors.

**Lemma 3.6.** Let \( m \in \mathbb{N} \) and \( (\mathcal{L}_k)_{k=1}^m \) be a finite sequence of bounded tensors with dimensions \( \mathcal{L}_k \in \mathbb{C}^{I_1 \times \cdots \times I_M \times I_1 \times \cdots \times I_M} \), then we have

(3.64) \[ \lim_{n \to \infty} \left( \prod_{k=1}^m \exp \left( \frac{\mathcal{L}_k}{n} \right) \right)^n = \exp \left( \sum_{k=1}^m \mathcal{L}_k \right) \]

**Proof:**

We will prove the case for \( m = 2 \), and the general value of \( m \) can be obtained by mathematical induction. Let \( \mathcal{L}_1, \mathcal{L}_2 \) be bounded tensors act on some Hilbert space. Define \( \mathcal{C} \overset{\text{def}}{=} \exp((\mathcal{L}_1 + \mathcal{L}_2)/n) \), and \( \mathcal{D} \overset{\text{def}}{=} \exp(\mathcal{L}_1/n) *_M \exp(\mathcal{L}_2/n) \). Note we have following estimates for the norm of tensors \( \mathcal{C}, \mathcal{D} \):

(3.65) \[ \|\mathcal{C}\|, \|\mathcal{D}\| \leq \exp \left( \frac{\|\mathcal{L}_1\| + \|\mathcal{L}_2\|}{n} \right) = \left[ \exp \left( \|\mathcal{L}_1\| + \|\mathcal{L}_2\| \right) \right]^{1/n} . \]

From the Cauchy-Product formula, the tensor \( \mathcal{D} \) can be expressed as:

\[ \mathcal{D} = \exp(\mathcal{L}_1/n) *_M \exp(\mathcal{L}_2/n) = \sum_{i=0}^\infty \frac{(\mathcal{L}_1/n)^i}{i!} *_M \sum_{j=0}^\infty \frac{(\mathcal{L}_2/n)^j}{j!} \]

(3.66)

\[ = \sum_{i=0}^\infty \frac{\mathcal{L}_1^i}{i!} *_M \frac{\mathcal{L}_2^{i-1}}{(i-i)!} , \]
then we can bound the norm of $\mathcal{C} - \mathcal{D}$ as

$$\|\mathcal{C} - \mathcal{D}\| = \left\| \sum_{i=0}^{\infty} \frac{\left(\mathcal{L}_1 + \mathcal{L}_2\right)^i}{i!} - \sum_{i=0}^{\infty} \frac{\mathcal{L}_1^i}{i!} \cdot \mathcal{L}_2 \right\|$$

$$= \left\| \sum_{i=2}^{\infty} \frac{k^{-2} \left(\mathcal{L}_1 + \mathcal{L}_2\right)^i}{i!} - \sum_{i=0}^{\infty} \frac{\mathcal{L}_1^i}{i!} \cdot \mathcal{L}_2 \right\|$$

$$\leq \frac{1}{k^2} \left[ \exp(\|\mathcal{L}_1\| + \|\mathcal{L}_2\|) + \sum_{i=2}^{\infty} n^{-i} \left(\|\mathcal{L}_1\| + \|\mathcal{L}_2\|\right)^i \right]$$

$$= \frac{1}{n^2} \left[ \exp(\|\mathcal{L}_1\| + \|\mathcal{L}_2\|) + \sum_{i=2}^{\infty} n^{-i} \left(\|\mathcal{L}_1\| + \|\mathcal{L}_2\|\right)^i \right]$$

$$\leq \frac{2 \exp(\|\mathcal{L}_1\| + \|\mathcal{L}_2\|)}{n^2}.$$  \hspace{1cm} (3.67)

For the difference between the higher power of $\mathcal{C}$ and $\mathcal{D}$, we can bound them as

$$\|\mathcal{C}^n - \mathcal{D}^n\| = \left\| \sum_{i=0}^{n-1} \mathcal{C}^m (\mathcal{C} - \mathcal{D}) \mathcal{C}^{n-i-1} \right\|$$

$$\leq 1 \exp(\|\mathcal{L}_1\| + \|\mathcal{L}_2\|) \cdot n \cdot \|\mathcal{L}_1 - \mathcal{L}_2\|,$$  \hspace{1cm} (3.68)

where the inequality $\leq 1$ uses the following fact

$$\|\mathcal{C}\|^n \|\mathcal{D}\|^{n-1} \leq \exp(\|\mathcal{L}_1\| + \|\mathcal{L}_2\|) \leq \exp(\|\mathcal{L}_1\| + \|\mathcal{L}_2\|).$$  \hspace{1cm} (3.69)

based on Eq. (3.65). By combining with Eq. (3.67), we have the following bound

$$\|\mathcal{C}^n - \mathcal{D}^n\| \leq \frac{2 \exp(2 \|\mathcal{L}_1\| + 2 \|\mathcal{L}_2\|)}{n}. \hspace{1cm} (3.70)$$

Then this lemma is proved when $n$ goes to infinity. \hspace{1cm} \square

**Theorem 3.7.** Let $\mathcal{C}_i \in \mathbb{C}^{I_i \times \cdots \times I_n \times I_i \times \cdots \times I_N}$ be positive Hermitian tensors for $1 \leq i \leq n$ with Hermitian rank $r_i$. \(\|\mathcal{C}_i\|_{(k)}\) be Ky Fan $k$-norm. For any continuous function $f : (0, \infty) \rightarrow (0, \infty)$ such that $x \rightarrow \log f(e^x)$ is convex on $\mathbb{R}$, we have

$$\left\| f \left( \exp \left( \sum_{i=1}^{n} \log \mathcal{C}_i \right) \right) \right\|_{(k)} \leq \exp \int_{-\infty}^{\infty} \log \left\| f \left( \prod_{i=1}^{n} \mathcal{C}_i^{1+it} \right) \right\|_{(k)} \beta_0(t) \, dt,$$  \hspace{1cm} (3.71)

where $\beta_0(t) = \frac{n}{2(\cosh(\pi t) + 1)}$.

For any continuous function $g(0, \infty) \rightarrow (0, \infty)$ such that $x \rightarrow g(e^x)$ is convex on $\mathbb{R}$, we have

$$\left\| g \left( \exp \left( \sum_{i=1}^{n} \log \mathcal{C}_i \right) \right) \right\|_{(k)} \leq \int_{-\infty}^{\infty} \left\| g \left( \prod_{i=1}^{n} \mathcal{C}_i^{1+it} \right) \right\|_{(k)} \beta_0(t) \, dt.$$  \hspace{1cm} (3.72)

**Proof:** From Hirschman interpolation theorem [22] and $\theta \in [0, 1]$ and setting $\sqrt{-1}$ as $i$, we have

$$\log |h(\theta)| \leq \int_{-\infty}^{\infty} \log |h(t)|^{1-\theta} \beta_{1-\theta}(t) \, dt + \int_{-\infty}^{\infty} \log |h(1+it)|^\theta \beta_\theta(t) \, dt,$$  \hspace{1cm} (3.73)
where \( h(z) \) be uniformly bounded on \( S \) \( \overset{\text{def}}{=} \{ z \in \mathbb{C} : 0 \leq \Re(z) \leq 1 \} \) and holomorphic on \( S \). The term \( \beta_\theta(t) \) is defined as:

\[
\beta_\theta(t) \overset{\text{def}}{=} \frac{\sin(\pi \theta)}{2\theta(\cos(\pi t) + \cos(\pi \theta))}.
\]

Let \( H(z) \) be a uniformly bounded holomorphic function with values in \( \mathbb{C}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N} \). Fix some \( \theta \in [0, 1] \) and let \( U, V \in \mathbb{C}^{I_1 \times \cdots \times I_N \times I_1} \) be normalized tensors such that \( \langle U, \mathcal{H}(\theta)^* V \rangle = \|H(\theta)\| \). If we define \( h(z) \) as \( h(z) \overset{\text{def}}{=} \langle U, \mathcal{H}(z)^* V \rangle \), we have following bound: \( |h(z)| \leq \|H(z)\| \) for all \( z \in S \). From Hirschman interpolation theorem, we then have following interpolation theorem for tensor-valued function:

\[
\log \|H(\theta)\| \leq \int_{-\infty}^{\infty} \log \|H(it)\|^{1-\theta} \beta_{1-\theta}(t) dt + \int_{-\infty}^{\infty} \log \|H(1+it)\|^\theta \beta_\theta(t) dt.
\]

Let \( H(z) = \prod_{i=1}^n C_i^z \). Then the first term in the R.H.S. of Eq. (3.75) is zero since \( H(it) \) is a product of unitary tensors. Then we have

\[
\log \left\| \prod_{i=1}^n C_i^t \right\|^\frac{1}{2} \leq \int_{-\infty}^{\infty} \log \left\| \prod_{i=1}^n C_i^{1+it} \right\| \beta_\theta(t) dt.
\]

From Lemma 2.7, we have following relations:

\[
\left\| \prod_{i=1}^n \left( \wedge^k C_i \right)^t \right\|^\frac{1}{k} = \wedge^k \left\| \prod_{i=1}^n C_i^t \right\|^\frac{1}{k},
\]

and

\[
\left\| \prod_{i=1}^n \left( \wedge^k C_i \right)^{1+it} \right\| = \wedge^k \left\| \prod_{i=1}^n C_i^{1+it} \right\|.
\]

If Eq. (3.76) is applied to \( \wedge^k C_i \) for \( 1 \leq k \leq r \), we have following log-majorization relation from Eqs. (3.77) and (3.78):

\[
\log \wedge \left( \left\| \prod_{i=1}^n C_i^t \right\|^\frac{1}{k} \right) \times \int_{-\infty}^{\infty} \log \wedge \left\| \prod_{i=1}^n C_i^{1+it} \right\| \beta_\theta(t) dt.
\]

Moreover, we have the equality condition in Eq. (3.79) for \( k = r \) due to following identities:

\[
\det_H \left\| \prod_{i=1}^n C_i^t \right\|^\frac{1}{k} = \det_H \left\| \prod_{i=1}^n C_i^{1+it} \right\| = \prod_{i=1}^n \det_H C_i.
\]

At this stage, we are ready to apply Theorem 3.4 for the log-majorization provided by Eq. (3.79) to get following facts:

\[
\left\| f \left( \left\| \prod_{i=1}^n C_i^t \right\| \right) \right\|_\theta \leq \exp \int_{-\infty}^{\infty} \log \left\| f \left( \left\| \prod_{i=1}^n C_i^{1+it} \right\| \right) \right\|_\theta \beta_\theta(t) dt.
\]
and
\[
(3.82) \quad \left\| g \left( \prod_{i=1}^{n} c_{i}^{\beta} \right) \right\|_{(k)} \leq \int_{-\infty}^{\infty} \left\| g \left( \prod_{i=1}^{n} c_{i}^{1+t} \right) \right\|_{(k)} \beta_{g}(t) dt.
\]

From Lie product formula for tensors given by Lemma 3.6, we have
\[
(3.83) \quad \prod_{i=1}^{n} c_{i}^{\beta} \rightarrow \exp \left( \sum_{i=1}^{n} \log C_{i} \right).
\]

By setting \( \theta \rightarrow 0 \) in Eqs. (3.81), (3.82) and using Lie product formula given by Eq. (3.83), we will get Eqs. (3.71) and (3.72).

4. Tensor Expander Chernoff Bounds Derivation by Majorization. In this section, we will begin with the derivation for the expectation bound of Ky Fan \( k \)-norm for the product of positive Hermitian tensors in Section 4.1. This bound will play a key role in the next Section 4.2 by establishing tensor expander Chernoff bounds.

4.1. Expectation Estimation for Product of Tensors. The main purpose of this section is to bound the expectation of Ky Fan \( k \)-norm for the product of positive Hermitian tensors. We extend scalar valued expander Chernoff bound proof in [10] and matrix valued expander Chernoff bound proof in [8] to context of tensors and remove the restriction that the summation of all mapped tensors should be zero tensor, i.e., \( \sum_{v \in \mathcal{V}} g(v) = 0 \).

Let \( A \) be the normalized adjacency matrix of the underlying graph \( \mathcal{G} \) and let \( \hat{A} = A \otimes I_{(1M)^2} \), where the identity tensor \( I_{(1M)^2} \) has dimensions as \( I_{1} \times \cdots \times I_{M} \times I_{1} \times \cdots \times I_{M} \). We use \( \mathcal{F} \in \mathbb{C}^{(n \times I_{1} \times \cdots \times I_{M}) \times (n \times I_{1} \times \cdots \times I_{M})} \) to represent block diagonal tensor valued matrix where the \( v \)-th diagonal block is the tensor
\[
(4.1) \quad T_{v} = \exp \left( \frac{t g(v)(a + ib)}{2} \right) \otimes \exp \left( \frac{t g(v)(a - ib)}{2} \right).
\]

The tensor \( F \) can also be expressed as
\[
(4.2) \quad F = \begin{bmatrix}
T_{v_{1}} & O & \cdots & O \\
O & T_{v_{2}} & \cdots & O \\
\vdots & \vdots & \ddots & \vdots \\
O & O & \cdots & T_{v_{n}}
\end{bmatrix}.
\]

Then the tensor \( \left( F \ast_{M+1} \hat{A} \right)^{\kappa} \) is a block tensor valued matrix whose \((u, v)\)-block is a tensor with dimensions as \( I_{1} \times \cdots \times I_{M} \times I_{1} \times \cdots \times I_{M} \) expressed as :
\[
(4.3) \quad \sum_{u_{1}, \ldots, u_{n-1} \in \mathcal{U}} A_{u_{1}, v_{1}} \prod_{j=1}^{n-2} A_{v_{j}, u_{j+1}} A_{v_{n-1}, v} \left( T_{v_{1}} \ast_{2M} T_{v_{2}} \ast_{2M} \cdots \ast_{2M} T_{v_{n}} \right)
\]

Let \( u_{0} \in \mathbb{C}^{n \times I_{1} \times \cdots \times I_{M}} \) be the tensor obtained by \( \frac{1}{\sqrt{n}} \otimes \operatorname{col}(I_{1M}) \), where \( 1 \) is the all ones vector with size \( n \) and \( \operatorname{col}(I_{1M}) \in \mathbb{C}^{I_{1M} \times 1} \times 1 \) is the column tensor of the
identity tensor $I_{M} ∈ \mathbb{C}^{I_{1}×⋯×I_{M}}$. By applying the following relation:

\begin{align}
\langle \text{col}(I_{M}^{k}), C \otimes B *_{M} \text{col}(I_{M}^{k}) \rangle = \text{Tr} (C * M B^{T})
\end{align}

where $C, B ∈ \mathbb{C}^{I_{1}×⋯×I_{M}×I_{1}×⋯×I_{M}}$, we will have following expectation of $κ$ steps transition of Hermitian tensors from the vertex $v_{1}$ to the vertex $v_{κ}$,

\begin{align}
E \left[ \text{Tr} \left( \prod_{i=1}^{κ} \exp \left( \frac{tg(v_{i})(a + ib)}{2} \right) *_{M} \prod_{i=κ}^{1} \exp \left( \frac{tg(v_{i})(a - ib)}{2} \right) \right) \right] = \\
(4.5) = E \left[ \langle \text{col}(I_{M}^{κ}), \prod_{i=1}^{κ} \mathcal{T}_{vi} *_{M} \text{col}(I_{M}^{κ}) \rangle \right] = \langle u_{0}, (F *_{M+1} \tilde{A})^{κ} *_{M+1} u_{0} \rangle.
\end{align}

If we define $(F *_{M+1} \tilde{A})^{κ} *_{M+1} u_{0}$ as $u_{κ}$, the goal of this section is to estimate $\langle u_{0}, u_{κ} \rangle$.

The trick is to separate the space of $u$ as the subspace spanned by the $(I_{M}^{κ})^{2}$ tensors $1 \otimes e_{i}$ denoted by $u^{∥}$, where $1 \leq i \leq (I_{M}^{κ})^{2}$ and $e_{i} ∈ \mathbb{C}^{I_{1}×⋯×I_{M}×1}$ is the column tensor of size $(I_{M}^{κ})^{2}$ with 1 in position $i$ and 0 elsewhere, and its orthogonal complement space, denoted by $u^{⊥}$. Following lemma is required to bound how the tensor norm is changed in terms of aforementioned subspace and its orthogonal space after acting by the tensor $F *_{2M+1} \tilde{A}$.

**Lemma 4.1.** Given paramters $λ ∈ (0, 1), a ≥ 0, r > 0$, and $t > 0$. Let $\mathfrak{G} = (\mathfrak{V}, \mathfrak{E})$ be a regular $λ$-expander graph on the vertices set $\mathfrak{V}$ and $\|g(v_{i})\| ≤ r$ for all $v_{i} ∈ \mathfrak{V}$. Each vertex $v$ will be assigned a tensor $\tilde{T}_{v}$, where $\tilde{T}_{v} \triangleq \frac{g(v)(a + ib)}{2} \otimes I_{M}^{κ} \otimes I_{1}^{κ} × I_{2}^{κ} × I_{3}^{κ}$. Let $F ∈ \mathbb{C}^{(n×I_{1}^{κ}×⋯×I_{M}^{κ})×(n×I_{1}^{κ}×⋯×I_{M}^{κ})}$ to represent block diagonal tensor valued matrix where the $v$-th diagonal block is the tensor $\exp(\tilde{T}_{v}) = \mathcal{T}_{v}$. For any tensor $u ∈ \mathbb{C}^{n×I_{1}^{κ}×⋯×I_{M}^{κ}}$, we have

1. $\| (F *_{M+1} \tilde{A} *_{M+1} u^{∥}) \| ≤ γ_{1} \| u^{∥} \|$, where $γ_{1} = \text{exp}(tr\sqrt{a^{2} + b^{2}})$;
2. $\| (F *_{M+1} \tilde{A} *_{M+1} u^{⊥}) \| ≤ γ_{2} \| u^{⊥} \|$, where $γ_{2} = \lambda(\text{exp}(tr\sqrt{a^{2} + b^{2}}) - 1)$;
3. $\| (F *_{M+1} \tilde{A} *_{M+1} u^{∥}) \| ≤ γ_{3} \| u^{∥} \|$, where $γ_{3} = \text{exp}(tr\sqrt{a^{2} + b^{2}}) - 1$;
4. $\| (F *_{M+1} \tilde{A} *_{M+1} u^{⊥}) \| ≤ γ_{4} \| u^{⊥} \|$, where $γ_{4} = \lambda\text{exp}(tr\sqrt{a^{2} + b^{2}})$.

**Proof:**

For Part 1, let $1 ∈ \mathbb{C}^{n}$ be all ones vector, and let $u^{∥} = 1 ⊗ v$ for some $v ∈ \mathbb{C}^{(I_{M}^{κ})^{2}}$. Then, we have

\begin{align}
(4.6) \quad (F *_{M+1} \tilde{A} *_{M} u^{∥}) = (F *_{M} u^{∥}) = 1 ⊗ \left( \frac{1}{n} \sum_{v ∈ \mathfrak{V}} \text{exp}(t\tilde{T}_{v}) *_{M} v \right)
\end{align}
and we can bound \( \left( \frac{1}{n} \sum_{v \in \mathcal{V}} \exp(t \tilde{T}_v) \right) \) further as

\[
\left\| \frac{1}{n} \sum_{v \in \mathcal{V}} \exp(t \tilde{T}_v) \right\| = \left\| \frac{1}{n} \sum_{v \in \mathcal{V}} \sum_{i=0}^{\infty} \frac{t^i \tilde{T}_v^i}{i!} \right\| \\
= \left\| \mathcal{I} + \frac{1}{n} \sum_{v \in \mathcal{V}} \sum_{i=1}^{\infty} \frac{t^i \tilde{T}_v^i}{i!} \right\| \\
\leq 1 + \frac{1}{n} \sum_{v \in \mathcal{V}} \sum_{i=1}^{\infty} \frac{t^i \tilde{T}_v^i}{i!} \\
\leq 1 + \sum_{i=1}^{\infty} \frac{(tr \sqrt{a^2 + b^2})^i}{i!} = \exp(tr \sqrt{a^2 + b^2}),
\]

(4.7)

where the last inequality is due to the fact that \( \left\| \frac{g(v)(a+b)}{2} \otimes \mathcal{I}_v^M + \mathcal{I}_v^M \otimes \frac{g(v)(a-b)}{2} \right\| \leq 2tr \times \sqrt{a^2 + b^2}. \)

Then Part 1. of this lemma is established due to

\[
\left\| (\mathcal{F} \star_{M+1} \tilde{A} \star_{M+1} \mathbf{u})^\perp \right\| = \sqrt{n} \left\| \frac{1}{n} \sum_{v \in \mathcal{V}} \exp(t \tilde{T}_v) \right\| \\
\leq \sqrt{n} \| \mathbf{v} \| \exp(tr \sqrt{a^2 + b^2}) = \exp(tr \sqrt{a^2 + b^2}) \left\| \mathbf{u} \right\|.
\]

(4.8)

For Part 2, since \((\tilde{A} \star_{M+1} \mathbf{u})^\perp = 0\), we have

\[
\left\| (\mathcal{F} \star_{M+1} \tilde{A} \star_{M+1} \mathbf{u})^\perp \right\| = \left\| (\mathcal{F} - \mathcal{I}) \star_{M+1} \tilde{A} \star_{M+1} \mathbf{u}^\perp \right\| \\
\leq \left\| (\mathcal{F} - \mathcal{I}) \star_{M+1} \tilde{A} \star_{M+1} \mathbf{u}^\perp \right\| \\
\leq \max_{v \in \mathcal{V}} \left\| \exp(t \tilde{T}_v) - \mathcal{I} \right\| \cdot \left\| \tilde{A} \star_{M+1} \mathbf{u}^\perp \right\| \\
\leq \max_{v \in \mathcal{V}} \sum_{i=1}^{\infty} \frac{t^i \tilde{T}_v^i}{i!} \cdot \left\| \tilde{A} \star_{M+1} \mathbf{u}^\perp \right\| \leq (\exp(tr \sqrt{a^2 + b^2}) - 1)\lambda \left\| \mathbf{u} \right\|,
\]

(4.9)

where the last inequality uses that the underlying graph \( \mathcal{G} \) is a \( \lambda \)-expander graph, i.e., \( \| \mathbf{A} \mathbf{x} \| \leq \lambda \cdot \| \mathbf{x} \|. \) Therefore, Part 2 is also valid.

For Part 3, because \((\mathbf{u})^\perp = 0\), we have \((\mathcal{F} \star_{M+1} \tilde{A} \star_{M+1} \mathbf{u})^\perp = (\mathcal{F} \star_{M+1} \mathbf{u})^\perp = ((\mathcal{F} \star_{M+1} \mathbf{u})^\perp)^\perp. \) Then, we can upper bound as

\[
\left\| (\mathcal{F} - \mathcal{I}) \star_{M+1} \mathbf{u}^\perp \right\| \leq \left\| (\mathcal{F} - \mathcal{I}) \star_{M+1} \mathbf{u} \right\| \\
\leq \left\| (\mathcal{F} - \mathcal{I}) \right\| \cdot \left\| \mathbf{u} \right\| \\
= \max_{v \in \mathcal{V}} \left\| \exp(t \tilde{T}_v) - \mathcal{I} \right\| \cdot \left\| \mathbf{u} \right\| \\
\leq \max_{v \in \mathcal{V}} \sum_{i=1}^{\infty} \frac{t^i \tilde{T}_v^i}{i!} \cdot \left\| \mathbf{u} \right\| \leq (\exp(tr \sqrt{a^2 + b^2}) - 1) \left\| \mathbf{u} \right\|,
\]

(4.10)
hence, Part 3 is also proved.

Finally, for Part 4, we have

\[
\left\| \left( F^*M+1 \hat{A}^*M+1 u^i \right)^\dagger \right\| \leq \left\| F^*M+1 \hat{A}^*M+1 u^i \right\|
\]

\[
\leq \|F\| \cdot \left\| \hat{A}^*M+1 u^i \right\| \leq \exp(tr\sqrt{a^2+b^2})\lambda \| u^i \|,
\]

where we use \( \|F\| \leq \exp(tr\sqrt{a^2+b^2}) \) (shown at previous part) and the underlying graph \( \mathcal{G} \) is a \( \lambda \)-expander graph.

In the following, we will apply Lemma 4.1 to bound the following term provided by Eq. (4.5)

\[
(F \ast M+1 \hat{A})^\astM+1 u_0
\]

This bound is formulated by the following Lemma 4.2.

**Lemma 4.2.** Let \( \mathcal{G} \) be a regular \( \lambda \)-expander graph on the vertex set \( \mathcal{V} \), \( g : \mathcal{V} \to \mathbb{C}^l \times \cdots \times \mathbb{C}^l \times \cdots \times \mathbb{C}^l \), and let \( v_1, \ldots, v_n \) be a stationary random walk on \( \mathcal{G} \). If \( tr\sqrt{a^2+b^2} < 1 \) and \( \lambda(2 \exp(tr\sqrt{a^2+b^2})-1) \leq 1 \), we have:

\[
\mathbb{E} \left[ \text{Tr} \left( \prod_{i=1}^{\kappa} \exp \left( \frac{tg(v_i)(a+ib)}{2} \right) \prod_{i=\kappa}^1 \exp \left( \frac{tg(v_i)(a-ib)}{2} \right) \right) \right] \leq \exp \left[ \kappa \left( 2tr\sqrt{a^2+b^2} + \frac{8}{1-\lambda} + \frac{16tr\sqrt{a^2+b^2}}{1-\lambda} \right) \right].
\]

**Proof:** There are two phases for this proof. The first phase is to bound the evolution of tensor norms \( \|u^i\| \) and \( \|u^i\| \), respectively. The second phase is to bound \( \gamma_i \) for \( 1 \leq i \leq 4 \) in Lemm 4.1. We begin with the derivation for the bound \( \|u^i\| \), where \( u_i \) is the output tensor after acting by the tensor \( F \ast M+1 \hat{A} \) for \( i \) times. It is

\[
\|u^i\| = \left\| (F \ast M+1 \hat{A} \ast M+1 u_{i-1})^\dagger \right\|
\]

\[
\leq \left\| (F \ast M+1 \hat{A} \ast M+1 u_{i-1}) \right\| + \left\| (F \ast M+1 \hat{A} \ast M+1 u_{i-1})^\dagger \right\|
\]

\[
\leq \gamma_3 \| u_{i-1} \| + \gamma_4 \| u_{i-1} \|
\]

\[
\leq 2 (\gamma_3 + \gamma_3 \gamma_4 + \gamma_3 \gamma_4^2 + \cdots) \max_{j<i} \| u_j \| \leq \frac{\gamma_3}{1-\gamma_4} \max_{j<i} \| u_j \|.
\]

where \( \leq_1 \) is obtained from Lemma 4.1, \( \leq_2 \) is obtained by applying the inequality at \( \leq_1 \) repeatedly. The next task is to bound \( \|u^i\| \), we have

\[
\|u^i\| = \left\| (F \ast M+1 \hat{A} \ast M+1 u_{i-1}) \right\|
\]

\[
\leq \left\| (F \ast M+1 \hat{A} \ast M+1 u_{i-1}) \right\| + \left\| (F \ast M+1 \hat{A} \ast M+1 u_{i-1}) \right\|
\]

\[
\leq \gamma_1 \| u_{i-1} \| + \gamma_2 \| u_{i-1} \|
\]

\[
\leq 2 \left( \gamma_1 + \frac{\gamma_2 \gamma_3}{1-\gamma_4} \right) \max_{j<i} \| u_j \|,
\]
where \( \leq 1 \) is obtained from Lemma 4.1, \( \leq 2 \) is obtained from Eq. (4.14). From Eqs (4.5), (4.14) and (4.15), we have

\[
\mathbb{E} \left[ \text{Tr} \left( \prod_{i=1}^{\kappa} \exp \left( \frac{t g(v_i)(a + \sqrt{M})}{2} \right) \right) \right.
\]

\[
= \langle u_0, u_\kappa \rangle = \langle u_0, u_\kappa \rangle \leq \|z_0\| \cdot \|z_\kappa\| = \sqrt{\|I_1^M \cdot z_\kappa\|}
\]

(4.16)

\[
\leq \sqrt{I_1^M} \left( \gamma_1 + \frac{\gamma_2 \gamma_3}{1 - \gamma_4} \right)^{\kappa} \cdot \|z_0\| \leq I_1^M \left( \gamma_1 + \frac{\gamma_2 \gamma_3}{1 - \gamma_4} \right)^{\kappa}.
\]

The second phase of this proof requires us to bound following four terms: \( \gamma_i \) for \( 1 \leq i \leq 4 \). Since \( \text{tr} \sqrt{a^2 + b^2} < 1 \), we can bound \( \gamma_1 \) as following:

(4.17)

\[
\gamma_1 = \exp(\text{tr} \sqrt{a^2 + b^2}) \leq 1 + 2\text{tr} \sqrt{a^2 + b^2}.
\]

(4.18)

\[
\gamma_2 = \lambda(\exp(\text{tr} \sqrt{a^2 + b^2}) - 1) \leq 2\lambda\text{tr} \sqrt{a^2 + b^2};
\]

(4.19)

\[
\gamma_3 = \exp(\text{tr} \sqrt{a^2 + b^2}) - 1 \leq 2\text{tr} \sqrt{a^2 + b^2};
\]

and the condition \( \lambda(2 \exp(\text{tr} \sqrt{a^2 + b^2}) - 1) \leq 1 \), we have

(4.20)

\[
1 - \gamma_4 = 1 - \lambda \exp(\text{tr} \sqrt{a^2 + b^2}) \geq \frac{1 - \lambda}{2}.
\]

By applying Eqs. (4.17), (4.18), (4.19) and (4.20) to the upper bound in Eq. (4.16), we also have

\[
I_1^M \left( \gamma_1 + \frac{\gamma_2 \gamma_3}{1 - \gamma_4} \right)^{\kappa} \leq I_1^M \left[ 1 + 2(\text{tr} \sqrt{a^2 + b^2}) + \frac{8\lambda^2 r^2(a^2 + b^2)}{1 - \lambda} \right]^{\kappa}
\]

\[
\leq I_1^M \left[ \left( 1 + 2\text{tr} \sqrt{a^2 + b^2} \right) \left( 1 + \frac{8}{1 - \lambda} \right) \right]^{\kappa}
\]

(4.21)

\[
\leq I_1^M \exp \left[ \kappa \left( 2\text{tr} \sqrt{a^2 + b^2} + \frac{8}{1 - \lambda} + \frac{16\text{tr} \sqrt{a^2 + b^2}}{1 - \lambda} \right) \right]
\]

This lemma is proved. \( \square \)

### 4.2. Tensor Expander Chernoff Bounds.

We begin with a lemma about a Ky Fan \( k \)-norm inequality for the sum of tensors.

**Lemma 4.3.** Let \( C_i \in \mathbb{C}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N} \) with Hermitian rank \( r \), then we have

(4.22)

\[
\left\| \sum_{i=1}^{m} C_i \right\|_{(k)}^{s} \leq m^{s-1} \sum_{i=1}^{m} \left\| C_i \right\|_{(k)}^{s}
\]

where \( s \geq 1 \) and \( k \in \{1, 2, \cdots, r\} \).

**Proof:** Since we have

(4.23)

\[
\left\| \sum_{i=1}^{m} C_i \right\|_{(k)}^{s} = \sum_{j=1}^{k} \lambda_j \left( \left\| \sum_{i=1}^{m} C_i \right\|_{(k)}^{s} \right) = \sum_{j=1}^{k} \lambda_j^s \left( \left\| \sum_{i=1}^{m} C_i \right\|_{(k)}^{s} \right) = \sum_{j=1}^{k} \sigma_j^s \left( \sum_{i=1}^{m} C_i \right).
\]
where we have orders for eigenvalues as \( \lambda_1 \geq \lambda_2 \geq \cdots \), and singular values as \( \sigma_1 \geq \sigma_2 \geq \cdots \).

From Theorem G.1.d. in [17] and Theorem 5.2 in [19], we have Ky Fan singular value majorization inequalities:

\[
(4.24) \quad \sum_{j=1}^{k} \sigma_j (\sum_{i=1}^{m} C_i) \leq \sum_{j=1}^{k} \left( \sum_{i=1}^{m} \sigma_j(C_i) \right),
\]

where \( k \in \{1, 2, \cdots, s\} \). Then, we have

\[
\sum_{j=1}^{k} \sigma_j^s (\sum_{i=1}^{m} C_i) \leq \sum_{j=1}^{k} \left( \sum_{i=1}^{m} \sigma_j^s(C_i) \right) \leq m^{s-1} \sum_{j=1}^{k} \left( \sum_{i=1}^{m} \sigma_j^s(C_i) \right) = m^{s-1} \sum_{j=1}^{k} \left( \sum_{i=1}^{m} \sigma_j^s(|C_i|^s) \right) = m^{s-1} \sum_{i=1}^{m} \|C_i|^s\|_{(k)}
\]

We are ready to present our main theorem about the tensor expander bound for Ky Fan \( k \)-norm.

**Theorem 4.4.** Let \( \mathcal{G} = (\mathcal{V}, \mathcal{E}) \) be a regular undirected graph whose transition matrix has second eigenvalue \( \lambda \), and let \( g : \mathcal{V} \rightarrow \mathbb{C}^{I_1 \times \cdots \times I_M \times I_1 \times \cdots \times I_M} \) be a function. We assume following:

1. For each \( v \in \mathcal{V} \), \( g(v) \) is a Hermitian tensor;
2. \( \|g(v)\| \leq r \);
3. A nonnegative coefficients polynomial raised by the power \( s \geq 1 \) as \( f : x \rightarrow (a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n)^s \) satisfying \( f \left( \exp \left( t \sum_{j=1}^{\kappa} g(v_j) \right) \right) \geq \exp \left( tf \left( \sum_{j=1}^{\kappa} g(v_j) \right) \right) \) almost surely;
4. For \( \tau \in [\infty, \infty] \), we have constants \( C \) and \( \sigma \) such that \( \beta_0(\tau) \leq \frac{C \exp(-\frac{\tau^2}{2\pi})}{\sigma \sqrt{2\pi}} \).

Then, we have

\[
\Pr \left( \left\| f \left( \sum_{j=1}^{\kappa} g(v_j) \right) \right\|_{(k)} \geq \vartheta \right) \leq \min_{t > 0} \left[ (n + 1)^{(s-1)} e^{-\vartheta t} \left( a_0^k + C \left( k + \sqrt{\frac{I^2 - k}{k}} \right) \right) \right],
\]

\[
(4.26) \quad \sum_{t=1}^{n} a_t \exp(8\kappa \lambda + 2(\kappa + 8\lambda)lsrt + 2(\sigma(\kappa + 8\lambda)lsr)^2 t_2^2) \right),
\]

where \( \lambda = 1 - \lambda \).
Proof: Let $t > 0$ be a parameter to be chosen later, then we have

$$\Pr \left( \left\| f \left( \sum_{j=1}^{\kappa} g(v_j) \right) \right\|_{(k)} \geq \vartheta \right) = \Pr \left( \exp \left( \left\| t f \left( \sum_{j=1}^{\kappa} g(v_j) \right) \right\|_{(k)} \right) \geq \exp (\vartheta t) \right)$$

$$= \Pr \left( \exp \left( t f \left( \sum_{j=1}^{\kappa} g(v_j) \right) \right) \right) \geq \exp (\vartheta t)$$

$$\leq \exp (-\vartheta t) \mathbb{E} \left( \left\| \exp \left( t f \left( \sum_{j=1}^{\kappa} g(v_j) \right) \right) \right\|_{(k)} \right)$$

$$\leq \exp (-\vartheta t) \mathbb{E} \left( f \left( \exp \left( \sum_{j=1}^{\kappa} g(v_j) \right) \right) \right)$$

(4.27)

where equality $=1$ comes from spectral mapping theorem, inequality $\leq 2$ is obtained from Markov inequality, and the last inequality $\leq 3$ is based on our function $f$ assumption (third assumption).

From Eq. (3.72) in Theorem 3.7, we can further bound the expectation term in Eq. (4.27) as

$$\mathbb{E} \left( \left\| f \left( \exp \left( t \sum_{j=1}^{\kappa} g(v_j) \right) \right) \right\|_{(k)} \right)$$

$$\leq \mathbb{E} \left( \left\| f \left( \prod_{j=1}^{\kappa} \exp (tg(v_j)(1 \pm \iota \tau)) \right) \right\|_{(k)} \beta_0(\tau) d\tau \right)$$

$$= \mathbb{E} \left( \left\| \sum_{l=0}^{n} a_l \prod_{j=1}^{\kappa} \exp (tg(v_j)(1 \pm \iota \tau)) \right\|_{(k)} \beta_0(\tau) d\tau \right)$$

$$\leq (n + 1)^{(s-1)} \mathbb{E} \left( \sum_{l=0}^{n} a_l \left\| \prod_{j=1}^{\kappa} \exp (tg(v_j)(1 \pm \iota \tau)) \right\|_{(k)} \beta_0(\tau) d\tau \right)$$

$$= (n + 1)^{(s-1)} \cdot$$

(4.28)

where equality $=1$ comes from the function $f$ definition, inequality $\leq 2$ is based on
Lemma 4.3. Each summand for \( l \geq 1 \) in Eq. (4.28) can further be bounded as

\[
\mathbb{E} \left( \left\| \left( \prod_{j=1}^{\kappa} \exp \left( t g(v_j)(1 + \iota) \right) \right)^l \right\|_{(k)} \right) \leq 1 \frac{k}{4^l} \text{Tr} \left( \left( \prod_{j=1}^{\kappa} \exp \left( t g(v_j)(1 + \iota) \right) \right)^l \right) + \\
\left\lfloor \frac{\Pi^{M-k} - k}{k} \text{Tr} \left( \left( \prod_{j=1}^{\kappa} \exp \left( t g(v_j)(1 + \iota) \right) \right)^{2l} \right) \right\rfloor^{1/2} \\
\leq 2 k \exp \left[ \kappa \left( 2lstr \sqrt{1 + \tau^2} + \frac{8}{1 - \lambda} + \frac{16ltsr \sqrt{1 + \tau^2}}{1 - \lambda} \right) \right] + \\
\left( \frac{\Pi^{M} - k}{k} \exp \left[ \kappa \left( 4lstr \sqrt{1 + \tau^2} + \frac{8}{1 - \lambda} + \frac{32ltsr \sqrt{1 + \tau^2}}{1 - \lambda} \right) \right] \right)^{1/2} \\
(4.29) \leq (k + \sqrt{\frac{\Pi^{M} - k}{k} \tau^2}) \exp \left[ \kappa \left( 2lstr(1 + \tau) + \frac{8}{1 - \lambda} + \frac{16ltsr(1 + \tau)}{1 - \lambda} \right) \right]
\]

where \( \leq 1 \) comes from Theorem 5 in [18], and \( \leq 2 \) comes from Lemma 4.2, and the last inequality \( \leq 3 \) is obtained by bounding \( \sqrt{1 + \tau^2} \) as \( 1 + \tau \).

From Eqs. (4.27), (4.28), and (4.29), we have

\[
\Pr \left( \left\| f \left( \sum_{j=1}^{\kappa} g(v_j) \right) \right\|_{(k)} \geq \vartheta \right) \leq \min_{t > 0} \left( n + 1 \right)^{s-1} e^{-\vartheta t} \left( a_0 k + \frac{\Pi^{M} - k}{k} \right) \cdot \\
\sum_{l=1}^{\infty} a_l \int_{0}^{\infty} \exp \left[ \kappa \left( 2lstr(1 + \tau) + \frac{8}{1 - \lambda} + \frac{16ltsr(1 + \tau)}{1 - \lambda} \right) \right] \beta_0(\tau) d\tau \\
\leq 1 \min_{t > 0} \left( n + 1 \right)^{s-1} e^{-\vartheta t} \left( a_0 k + \frac{\Pi^{M} - k}{k} \right) \cdot \\
\sum_{l=1}^{\infty} a_l \int_{0}^{\infty} \exp \left[ \kappa \left( 2lstr(1 + \tau) + \frac{8}{1 - \lambda} + \frac{16ltsr(1 + \tau)}{1 - \lambda} \right) \right] \frac{C \exp \left( \frac{-\tau^2}{2\sigma^2} \right)}{\sigma \sqrt{2 \pi}} d\tau \\
= \min_{t > 0} \left( n + 1 \right)^{s-1} e^{-\vartheta t} \left( a_0 k + C \left( k + \sqrt{\frac{\Pi^{M} - k}{k} \tau^2} \right) \right) \\
(4.30) \sum_{l=1}^{\infty} a_l \exp \left( 8k\bar{\lambda} + 2(\kappa + 8\bar{\lambda})ltsr + 2(\sigma(\kappa + 8\bar{\lambda})ltsr)^2 \right) \right),
\]

where inequality \( \leq 1 \) is obtained by the distribution bound for \( \beta_0(\tau) \) via another distribution function \( \frac{C \exp \left( \frac{-\tau^2}{2\sigma^2} \right)}{\sigma \sqrt{2 \pi}} \), and the last equality comes from Gaussian integral with respect to the variable \( \tau \) by setting \( 1 - \lambda \) as \( \bar{\lambda} \).
Following corollary is about a tensor expander bound with identity function $f$.

**Corollary 4.5.** If we consider the special case of Theorem 4.4 by assuming that the function $f : x \to x$ is an identity map, then we have

$$
\Pr \left( \left\| \sum_{j=1}^{\kappa} g(v_j) \right\|_{(k)} \geq \vartheta \right) \leq C \left( k + \sqrt{\frac{4M}{k} - \frac{1}{k}} \right) \cdot \exp \left( -\frac{\vartheta^2}{2 \sigma^2 r^2} + \frac{\vartheta}{2 \sigma^2 r^2} - \frac{1}{2 \sigma^2} + 8 \kappa \lambda \right).
$$

(4.31)

**Proof:** From Theorem 4.4, since the exponent is a quadratic function of $t$, the minimum of this quadratic function is achieved by selecting $t$ as

$$
t = \frac{\vartheta - 2(\kappa + 8\lambda)r}{4\sigma^2 r^2(\kappa + 8\lambda)^2},
$$

then, we have the desired bound after some algebra by applying Eq. (4.32) in Eq. (4.26) and setting $l = s = 1$, all $a_i = 0$ for $1 \leq i \leq n$ except $a_1 = 1$. □

5. **Conclusions.** In this work, we first build tensor norm inequalities based on the concept of log-majorization, and apply these new tensor norm inequalities to derive the tensor expander Chernoff bounds which generalize the matrix expander Chernoff bound by adopting more general norm for tensors, Ky Fan norm, and general convex function, instead of identity function, of random tensors summation.

There are several future directions that can be explored based on the current work. The first is to consider other types of tensor expander Chernoff bounds under other non-independent assumptions among random tensors. The other direction is to characterize random behaviors of other tensor related quantities besides norms or eigenvalues, for example, what is the Hermitian tensor rank behavior for the summation of random tensors.

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