Multifractal Analysis of inhomogeneous Bernoulli products

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Abstract We are interested to the multifractal analysis of inhomogeneous Bernoulli products which are also known as coin tossing measures. We give conditions ensuring the validity of the multifractal formalism for such measures. On another hand, we show that these measures can have a dense set of phase transitions.

Keywords : Hausdorff dimension, multifractal analysis, Gibbs measure, phase transition.

1 Introduction

Let us consider the dyadic tree T (even though all the results in this paper can be easily generalised to any ℓ-adic structure, ℓ ∈ N), let Σ = {0, 1}N be its limit (Cantor) set and denote by (F_n)_{n∈N} the associated filtration with the usual 0 − 1 encoding.

For ε_1, ..., ε_n ∈ {0, 1} we denote by I_{ε_1...ε_n} the cylinder of the nth generation defined by I_{ε_1...ε_n} = \{x = (i_1, ..., i_n, i_{n+1}, ...) ∈ Σ, \; i_1 = ε_1, ..., i_n = ε_n\}. For every x ∈ Σ, I_n(x) stands for the cylinder of F_n containing x.

If (p_n)_n is a sequence of weights, p_i ∈ (0, 1), we are interested in Borel measures μ on σ defined in the following way

$$\mu(I_{ε_1...ε_n}) = \prod_{j=1}^{n} p_j^{1-ε_j}(1-p_j)^{ε_j}. \tag{1}$$

This type of measure will be referred to as an inhomogeneous Bernoulli product. The aim of this paper is to study multifractal properties of such measures.

The particular case where the sequence (p_n) is constant is well-known and provides an example of measure satisfying the multifractal formalism (see e.g [Fal97]). In the general case, Bisbas in [Bis95] gave a sufficient condition on the sequence (p_n) ensuring that μ is a multifractal measure (i.e. the level sets are not empty) . However, the work
of Bisbas does not provide the dimension of the level sets $E_\alpha$ associated to the measure $\mu$.

Let us give a brief description of multifractal formalism. For a probability measure $m$ on $\Sigma$, we define the local dimension (also called Hölder exponent) of $m$ at $x \in \Sigma$ by

$$\alpha(x) = \lim \inf_{n \to +\infty} \alpha_n(x) = \lim \inf_{n \to +\infty} \frac{-\log m(I_n(x))}{n \log 2}.$$ 

The aim of multifractal analysis is to find the Hausdorff dimension, $\dim(E_\alpha)$, of the level set $E_\alpha = \{ x : \alpha(x) = \alpha \}$ for $\alpha > 0$. The function $f(\alpha) = \dim(E_\alpha)$ is called the singularity spectrum (or multifractal spectrum) of $m$ and we say that $m$ is a multifractal measure when $f(\alpha) > 0$ for several $\alpha$'s.

The concepts underlying the multifractal decomposition of a measure go back to an early paper of Mandelbrot [Man74]. In the 80’s multifractal measures were used by physicists to study various models arising from natural phenomena. In fully developed turbulence they were used by Frisch and Parisi [FP83] to investigate the intermittent behaviour in the regions of high vorticity. In dynamical system theory they were used by Benzi et al. [BPPV84] to measure how often a given region of the attractor is visited. In diffusion-limited aggregation (DLA) they were used by Meakin et al. [MCSW86] to describe the probability of a random walk landing to the neighborhood of a given site on the aggregate.

In order to determine the function $f(\alpha)$, Hentschel and Procaccia [HP83] used ideas based on Renyi entropies [Rén70] to introduce the generalized dimensions $D_q$ defined by

$$D_q = \lim_{n \to +\infty} \frac{1}{q - 1} \log \left( \sum_{I \in \mathcal{F}_n} m(I)^q \right),$$ 

(see also [GP83, Gra83]). From a physical and heuristic point of view, Halsey et al. [HJK+86] showed that the singularity spectrum $f(\alpha)$ and the generalized dimensions $D_q$ can be derived from each other. The Legendre transform turned out to be a useful tool linking $f(\alpha)$ and $D_q$. More precisely, it was suggested that

$$f(\alpha) = \dim(E_\alpha) = \tau^*(\alpha) = \inf(\alpha q + \tau(q), \ q \in \mathbb{R}),$$ 

(2)

where

$$\tau(q) = \lim \sup_{n \to +\infty} \tau_n(q) \quad \text{with} \quad \tau_n(q) = \frac{1}{n \log 2} \log \left( \sum_{I \in \mathcal{F}_n} m(I)^q \right).$$ 

(The sum runs over the cylinders $I$ such that $m(I) \neq 0$.) The function $\tau(q)$ is called the $L^q$-spectrum of $m$ and if the limit exists $\tau(q) = (q - 1)D_q$.

Relation (2) is called the multifractal formalism and in many aspects it is analogous to the well-known thermodynamic formalism developed by Bowen [Bow75] and Ruelle [Rue78].
For number of measures, relation (2) can be verified rigorously. In particular, if the sequence \( (p_n) \) is constant or periodic, the measure \( \mu \) given by (1) satisfies the multifractal formalism (e.g. [Fal97]). Moreover some rigorous results have already been obtained for some invariant measures in dynamical systems (e.g. [Col88, Pan94, Ran89]), for some self-similar measures under separation conditions (e.g. [CM92, LN99, Ols95]) and for quasiindependent measures(e.g [BMP92, Heu98, Tes06]).

The minoration of \( \dim(E_\alpha) \) usually follows on the existence of a shift-invariant and ergodic measure \( m_q \) (the so-called Gibbs measure [Mic83]), satisfying
\[
\forall n, \forall I \in F_n, \quad \frac{1}{C} m(I)^{2^{-n\tau(q)}} \leq m_q(I) \leq C m(I)^{2^{-n\tau(q)}},
\]
where the constant \( C > 0 \) is independent of \( n \) and \( I \). If \( \tau \) is differentiable at \( q \), the measure \( m_q \) is supported by \( E_{-\tau'(q)} \) and Brown, Michon and Peyrière established [BMP92, Pey92] that
\[
\dim(E_{-\tau'(q)}) = \tau^*(-\tau'(q)) = -q\tau'(q) + \tau(q).
\]

If the weights \( p_n \) are not all the same, the measure \( \mu \) is in general no shift-invariant and we cannot apply classical tools of ergodic theory, as Shannon-McMillan theorem (e.g. [Bil65]), to get a lower bound of \( \dim(E_\alpha) \).

Let us introduce the other following level sets defined by
\[
E_\alpha = \{ x ; \alpha(x) \leq \alpha \}, \quad \overline{F}_\alpha = \left\{ x ; \limsup_{n \to \infty} \alpha_n(x) \geq \alpha \right\},
\]
and
\[
F_\alpha = \left\{ x ; \limsup_{n \to \infty} \alpha_n(x) = \alpha \right\}.
\]

We can now state our main results. In section 2, we prove the following.

**Theorem 1.1** Let \( \mu \) be an inhomogeneous Bernoulli product on \( \Sigma \) and \( q \in \mathbb{R} \). We have
\[
\liminf_{n \to \infty} -q\tau_{\mu,n}(q) + \tau_{\mu,n}(q) \leq \dim \left( E_{-\tau'(q^-)} \cap \overline{F}_{-\tau'(q^+)} \right) \leq \inf \{ \tau^*(-\tau'(q^+)), \tau^*(-\tau'(q^-)) \}.
\]

The proof of the lower bound relies on the construction of a special inhomogeneous Bernoulli product which has the dimension of the level set studied.

In section 3 we are interested to the case where the sequence \( \tau_{\mu,n}(q) \) converges. In this situation, we prove that the multifractal formalism holds for \( \alpha = -\tau_{\mu}'(q) \) -if it exists. More precisely, we have
**Theorem 1.2** Suppose that the sequence \((\tau_{\mu,n}(q))\) converges at a point \(q > 0\). If \(\tau_\mu'(q)\) exists and if \(\alpha = -\tau_\mu'(q)\), we have

\[
\dim (E_\alpha \cap F_\alpha) = \tau_\mu^*(\alpha) = \alpha q + \tau_\mu(q).
\] (3)

Theorem 1.2 lead us to study the differentiability of the \(L^q\)-spectrum \(\tau_\mu(q)\). A point \(q\) will be called a phase transition if \(\tau_\mu'(q)\) does not exist. In section 4, we are interested to the existence of phase transitions. More precisely, we prove the following.

**Theorem 1.3** There exist inhomogeneous Bernoulli products \(\mu\) presenting a dense set of phase transitions.

## 2 Proof of theorem 1.1

We begin by a preliminary result.

**Lemma 2.1** If \(\mu\) is an inhomogeneous Bernoulli product, then \((\tau_{\mu,n}''(q))\) are locally uniformly bounded on \((0, +\infty)\).

**Proof** We denote by \(\beta(p_i)\) the Bernoulli homogeneous measure of parameter \(p_i\) and by \(\tau(p_i, q)\) it’s \(\tau\) function, \(\tau(p_i, q) = \log(p_i^q + (1-p_i)^q)\). Using the fact that \(\mu\) is the product of \(\beta(p_i)\) we easily obtain

\[
\tau_{\mu,n}(q) = \frac{1}{n} \sum_{i=0}^{n} \tau(p_i, q) \quad q > 0.
\]

It is therefore sufficient to show that, for any \(q_0 > 0\), there exists a constant \(C = C(q_0)\) such that for all \(p \in (0, 1)\) and all \(q > q_0\), \(\frac{\partial^2 \tau(p, q)}{\partial q^2} \leq C\). The proof is straightforward:

\[
\frac{\partial^2 \tau(p, q)}{\partial q^2} = \frac{(p^q(\log p)^2 + (1-p)^q(\log(1-p))^2)}{(p^q + (1-p)^q)} - \frac{(p^q \log p + (1-p)^q \log(1-p))^2}{(p^q + (1-p)^q)^2}
\]

\[
= \frac{p^q(1-p)^q ((\log p)^2 + (\log(1-p))^2 - 2 \log p \log(1-p))}{(p^q + (1-p)^q)^2}
\]

\[
= \frac{p^q(1-p)^q (\log \frac{p}{1-p})^2}{(p^q + (1-p)^q)^2} \leq [4p(1-p)]^q(\log p)^2 \leq [4p(1-p)]^q_0(\log p)^2,
\]

which is uniformly bounded on \(p \in (0, 1)\) and the proof is complete. •
Lemma 2.1 allows us to give estimates for the lower and the upper Hausdorff dimension of the measure \( \mu \). They are respectively defined by

\[
\dim_\ast(\mu) = \inf \{ \dim(E), \ \mu(E) > 0 \}; \quad \dim^*(\mu) = \inf \{ \dim(E), \ \mu(E) = 1 \}.
\]

We say that \( \mu \) is exact if \( \dim_\ast(\mu) = \dim(\mu) \) and we note \( \dim(\mu) \) the common value. In the same way, we can define the lower and the upper packing dimension of the measure \( \mu \). It is well known that there exist some relations between these quantities and the derivatives of the function \( \tau_\mu(q) \) at \( q = 1 \). More precisely, it is proved in [Fan93, Heu98] that

\[
-\tau'_\mu(1+) \leq \dim_\ast(\mu) \leq h_\ast(\mu) \leq \dim^*(\mu) \leq h^*(\mu) \leq -\tau'_\mu(1-),
\]

where \( h_\ast(\mu) \) and \( h^*(\mu) \) stand for the lower and the upper entropy of the measure \( \mu \), defined as

\[
h_\ast(\mu) = \liminf -\frac{1}{n \log 2} \sum_{I \in \mathcal{F}_n} \mu(I) \log \mu(I) = \liminf -\tau'_{\mu_n}(1)
\]

and

\[
h^*(\mu) = \limsup -\frac{1}{n \log 2} \sum_{I \in \mathcal{F}_n} \mu(I) \log \mu(I) = \limsup -\tau'_{\mu_n}(1).
\]

By Lemma 2.1, we deduce (see [BH02, Heu98]) the following remark.

**Remark 2.2** If \( \mu \) is an inhomogeneous Bernoulli product then

\[
\dim \mu = \liminf_{n \to \infty} -\tau'_\mu_n(1) = -\tau'_\mu(1^+) = h_\ast(\mu).
\]

and

\[
\text{Dim } \mu = \limsup_{n \to \infty} -\tau'_\mu_n(1) = -\tau'_\mu(1^-) = h^*(\mu).
\]

Fix \( q \in \mathbb{R} \). To prove theorem 1.1, we construct an auxiliary measure \( \nu \) supported by the set \( \mathcal{E}_{-\tau'(q^-)} \cap \mathcal{F}_{-\tau'(q^+)} \). More precisely, we consider a sequence of measures \( \nu_n \) satisfying

\[
\nu_n(I) = \frac{\mu(I)^q}{\sum_{I \in \mathcal{F}_n} \mu(I)^q} = \mu(\mathcal{I}|I)^q|I|^{\tau_\mu_n(q)},
\]

if \( I \in \mathcal{F}_n \). The following lemma implies that the sequence \( (\nu_n) \) converges in the weak* sense to a probability measure \( \nu \) which is also an inhomogeneous Bernoulli product.

**Lemma 2.3** Let \( n \in \mathbb{N} \) and \( I \in \mathcal{F}_n \). If \( \mu \) is an inhomogeneous Bernoulli product, we have \( \nu_n(I) = \nu_{n+1}(I) \).
Proof Take $n > 0$ and $I \in \mathcal{F}_n$. We can compute

$$
\nu_{n+1}(I) = \frac{\sum_{J \in \mathcal{F}_1} \mu(IJ)^q}{\sum_{I \in \mathcal{F}_n} \sum_{J \in \mathcal{F}_1} \mu(IJ)^q} = \frac{\mu(I)^q(p_n^q + (1 - p_{n+1})^q)}{\sum_{I \in \mathcal{F}_n} (p_n^q + (1 - p_{n+1})^q) \mu(I)^q}
$$

and therefore $\nu_{n+1}(I) = \nu_n(I)$ for all $I \in \mathcal{F}_n$.

By remark 2.2, we then deduce that the Hausdorff and the packing dimension of $\nu$ are given by an entropy formula. In other terms, we have

$$
dim \nu = \liminf_{n \to \infty} -\tau'_{\nu,n}(1) = h^*(\nu)
$$

and

$$
\text{Dim} \nu = \limsup_{n \to \infty} -\tau'_{\nu,n}(1) = h^*(\nu).
$$

Now we can prove Theorem 1.1.

Proof of Theorem 1.1 The upper bound is a well known fact of multifractal formalism (see for instance [BMP92]). In fact we have

1. If $\alpha \leq -\tau'(0^+)$ then $\dim E_\alpha \leq \text{Dim} E_\alpha \leq \tau^*(\alpha)$.
2. If $\alpha \geq -\tau'(0^-)$ then $\dim F_\alpha \leq \text{Dim} F_\alpha \leq \tau^*(\alpha)$.
3. $-\tau'(0^+) \leq \alpha \leq -\tau'(0^-)$ then $\tau^*(\alpha) = \tau(0)$ and the upper bound is trivial.

Lemma 2.3 and a straightforward computation imply $\tau_{\nu,n}(s) = \tau_{\mu,n}(qs) - s\tau_{\mu,n}(q)$. Using once again the (inhomogeneous) Bernoulli property of $\mu$ and remark 2.2 we deduce that

$$
-\tau'_\nu(1^+) = \liminf -\tau'_\nu,n(1) = \liminf \left(-q\tau'_{\nu,n}(q) + \tau_{\mu,n}(q)\right).
$$

The following lemma then implies the lower bound.

Lemma 2.4 We have $\nu(\Sigma \setminus E_{-\tau'(q^-)} \cap F_{-\tau'(q^+)}) = 1$.

Proof For $\eta > 0$ we put $\beta = -\tau'_\mu(q^-) + \eta$ and we prove that $\nu(\Sigma \setminus E_\beta) = 0$; it can be shown in a similar way that $\nu(\Sigma \setminus F_\gamma) = 0$ for $\gamma < -\tau'_\mu(q^+)$. The lemma then easily follows.

It suffices to show that $\Sigma \setminus E_\beta = \left\{ x \in \Sigma ; \liminf_{k \to \infty} \alpha_n(x) > \beta \right\}$ is of 0 $\nu$-measure.

Consider the collection $\mathcal{R}_n(\beta)$ of cylinders $I \in \mathcal{F}_n$ satisfying $\frac{\log \mu(I)}{\log |I|} > \beta$. It is clear that $\Sigma \setminus E_\beta = \limsup_{n \to \infty} \mathcal{R}_n(\beta)$.
Let \((\tau_{\mu,n})_{k \in \mathbb{N}}\) be the subsequence of \((\tau_{\mu,n})_{n \in \mathbb{N}}\) such that \(\lim_{k \to \infty} \tau_{\mu,n_k}(q) = \tau_{\mu}(q)\). Using the convergence of \(\tau_{\mu,n_k}(q)\) we can choose (and fix) \(t < 0\) such that for \(k\) big enough

\[
\tau_{\mu}(q+t) - \tau_{\mu,n_k}(q) < -\left(\beta - \frac{\eta}{2}\right) t = \left(\tau'_{\mu}(q^-) - \frac{\eta}{2}\right) t
\]

We get \(\mu(I)^{-t} |I|^{\beta t} \leq 1\) and hence

\[
\sum_{I \in R_{\nu_k}(\beta)} \nu(I) = \sum_{I \in R_{\nu_k}(\beta)} \mu(I)^q |I|^\tau_{\mu,n_k}(q) = \sum_{I \in R_{\nu_k}(\beta)} \mu(I)^{q+t} |I|^\tau_{\mu,n_k}(q-\beta t) \mu(I)^{-t} |I|^{\beta t} \\
\leq \sum_{I \in R_{\nu_k}(\beta)} \mu(I)^{q+t} |I|^\tau_{\mu,n_k}(q-\beta t) \mu(I)^{-t} |I|^{\beta t} \\
\leq \sum_{I \in F_{\nu_k}} \mu(I)^{q+t} |I|^\tau_{\mu,n_k}(q+t) = 1,
\]

where for the last inequality we used the fact that \(\tau_{\mu}(q+t) = \limsup \tau_{\mu,n}(q+t)\). It easily follows that \(\limsup_{k \to \infty} \sum_{I \in R_{\nu_k}(\beta)} \nu(I) = 0\) and the lemma is proved.

The proof of Theorem 1.1 is now completed.

Let \(f\) and \(g\) be the functions defined by \(f(t) = \dim E_t\) and \(g(t) = \dim F_t\). Obviously, \(f\) is increasing and \(g\) is decreasing. Recall that \(t\) is a non-stationary point of a monotone function \(h\) if \(h(s) \neq h(t)\) for all \(s \neq t\).

Since \(E_\alpha = E_\alpha \setminus \bigcup_{\beta < \alpha} E_\beta\), we deduce from theorem 1.1 the following.

**Remark 2.5** If \(\alpha = -\tau'(q^-)\) for \(q > 0\) is a non-stationary point of \(f\) there a sequence of \(q_m \leq q\) such that \(\alpha_m = -\tau'(q^-_m)\) are non-stationary points of \(f\) converging to \(\alpha\) and

\[
\liminf_{n \to \infty} -q_m \tau'_{\mu,n}(q_m) + \tau_{\mu,n}(q_m) \leq \dim E_{\alpha_m} = \dim E_{\alpha_m} \leq \tau^*(\alpha_m).
\]

If \(\alpha = -\tau'(q^+)\) for \(q < 0\) is a non-stationary point of \(g\) there a sequence of \(q_m \geq q\) such that \(\alpha_m = -\tau'(q^+_m)\) are non-stationary points of \(g\) converging to \(\alpha\) and

\[
\liminf_{n \to \infty} -q_m \tau'_{\mu,n}(q_m) + \tau_{\mu,n}(q_m) \leq \dim F_{\alpha_m} = \dim F_{\alpha_m} \leq \tau^*(\alpha_m).
\]

We conjecture that under the same conditions on \(\alpha\) we should also have \(\dim E_\alpha = \dim E_\alpha\) (\(\dim F_\alpha = \dim F_\alpha\) respectively).
3 Some conditions ensuring the validity of multifractal formalism

In this section we prove Theorem 1.2. We will use the following result.

**Proposition 3.1** For \( q > 0 \) consider \((\tau_{\mu,n})\) the subsequence of \((\tau_{\mu,n})\) such that

\[
\lim_{k \to \infty} \tau_{\mu,n_k}(q) = \limsup_{n \to \infty} \tau_{\mu,n}(q).
\]

Then if \( q \) is a differentiability point of \( \tau_\mu \) we have

\[
\lim_{k \to \infty} \tau'_{\mu,n_k} = \tau'_\mu(q).
\]

**Proof** The proposition is a immediate consequence of the following lemmas.

**Lemma 3.2** Under the assumptions of proposition 3.1

\[
\tau'_\mu(q^+) \geq \limsup_{k \to \infty} \tau'_{\mu,n_k}(q),
\]

where \( \tau'_\mu(q^+) \) stands for the right hand derivative of \( \tau_\mu \) at \( q \).

On the other hand, we get

**Lemma 3.3** Under the assumptions of proposition 3.1

\[
\tau'_\mu(q^-) \leq \liminf_{k \to \infty} \tau'_{\mu,n_k}(q),
\]

where \( \tau'_\mu(q^-) \) stands for the left hand derivative of \( \tau_\mu \) at \( q \).

**Proof of lemma 3.2** Take \( \epsilon > 0 \) and \( \tilde{q} > q \) satisfying

\[
\left| \frac{\tau_\mu(\tilde{q}) - \tau_\mu(q)}{\tilde{q} - q} - \tau'_\mu(q^+) \right| < \epsilon/8
\]

\[
|\tilde{q} - q| \sup_{n \in \mathbb{N}} \|\tau''_{\mu,n}\|_\infty < \epsilon/8.
\]

and consider \((\tilde{n}_k)\) such that \( \lim_{k \to \infty} \tau_{\mu,\tilde{n}_k}(\tilde{q}) = \limsup_{n \to \infty} \tau_{\mu,n}(\tilde{q}) \).
We can chose \( k \) big enough to have
\[
\frac{\left| \tau\mu,n_k(q) - \tau\mu(q) \right|}{\left| q - q \right|} < \frac{\epsilon}{8} \\
\frac{\left| \tau\mu,\tilde{n}_k(q) - \tau\mu(q) \right|}{\left| q - q \right|} < \frac{\epsilon}{8} \\
\tau\mu,n_k(q) \leq \tau\mu,\tilde{n}_k(q) + (q - q)\epsilon/8.
\]
We then obtain
\[
\tau'\mu(q) \geq \frac{\tau\mu(q) - \tau\mu(q)}{q - q} - \epsilon/8 \geq \frac{\tau\mu,\tilde{n}_k(q) - \tau\mu,n_k(q)}{q - q} - \epsilon/4
\]
\[
\geq \frac{\tau\mu,n_k(q) - \tau\mu,n_k(q)}{q - q} - 3\epsilon/8 \geq \tau'\mu,n_k(q) - |q - q| \sup_{n \in \mathbb{N}} \left| \tau''\mu,n_k(q) \right| - \epsilon/8
\]
\[
\geq \tau'\mu,n_k(q) - \epsilon.
\]
and the proof is completed. 

Lemma 3.3 is proven in a similar manner and together with lemma 3.2 provide the proposition’s proof.

We can now prove Theorem 1.2

**Proof of Theorem 1.2** Let \( \nu \) be the Gibbs-measure defined in lemma 2.3. Since
\[
\tau\nu,n(s) = \tau\mu,n(qs) - s\tau\mu,n(q)
\]
we get
\[
\tau'\nu,n(1) = q\tau'\mu,n(q) - \tau\mu,n(q).
\]
Using the convergence of \( \tau\mu,n(q) \) we deduce from Proposition 3.1 that
\[
\lim_{n \to \infty} \tau'\nu,n(1) = \lim_{n \to \infty} \left( q\tau'\mu,n(q) - \tau\mu,n(q) \right) = q\tau'\mu(q) - \tau\mu(q).
\]
Lemma 2.3 then implies that \( \tau'\nu(1) \) exists and
\[
\dim \nu = \text{Dim} \nu = -\tau'\nu(1) = -q\tau'\mu(q) + \tau\mu(q).
\]
On the other hand, for \( I \in F_n \), we have
\[
\frac{\log \nu(I)}{\log |I|} = q\frac{\log \mu(I)}{\log |I|} + \tau\mu,n(q)
\]
Since
\[
\lim_{n \to \infty} \frac{\log \nu(I_n(x))}{\log |I_n(x)|} = \dim \nu = \text{Dim } \nu; \nu\text{-a.s.}
\]
we obtain that
\[
\lim_{n \to \infty} \frac{\log \mu(I_n(x))}{\log |I_n(x)|} = -\tau'(q), \nu\text{-a.s.}
\]
We conclude that
\[
\dim (E_{\alpha} \cap F_{\alpha}) \geq \dim = \tau^*_{\mu}(\alpha).
\]
The opposite inequality being always valid, the proof is done.

4 Phase transitions

**Theorem 4.1** Let \( \tau \) be a convex combination of functions \( \tau(p_i, .) \) where \( 0 < p_i \leq 1/2 \), \( i = 1, ..., n \). For any \( 1 < q_1 < q_2 < \infty \) there exists another convex combination \( \bar{\tau} \) of functions \( \tau(p'_j, .) \) such that
- \( \bar{\tau}(q_i) = \tau(q_i) \) and \( \bar{\tau}'(q_i) \neq \tau'(q_i), i = 1, 2 \),
- for \( q \in (q_1, q_2) \), \( \bar{\tau}(q) > \tau(q) \),
- else, for \( q \notin [q_1, q_2] \), \( \bar{\tau}(q) < \tau(q) \).

5 Proof of Theorem 4.1

In this section whenever we use the notation \( p_i \) for a weight in \( (0, 1) \) we will also note \( \tau_i = \tau(p_i, .) \).

**Lemma 5.1** Take \( \tau = \lambda \tau(p_1, .) + (1 - \lambda) \tau(p_2, .) \) with \( 0 < p_1 < p_2 < 1/2 \) and \( \lambda \in (0, 1) \). For \( p_0 \in (0, 1/2) \) one of the following occurs :
1. either \( \tau(q) \neq \tau(p_0, q) \) for all \( q > 1 \),
2. or, there exists \( q_0 > 1 \) such that \( \tau(q) > \tau(p_0, q) \) for \( q < q_0 \) and \( \tau(q) < \tau(p_0, q) \) for \( q > q_0 \). The point \( q_0 \) is, then, the unique point of equality between these functions.

To prove this lemma we need the following subsidiary result.

**Lemma 5.2** Let \( p_1 < p_2 < p_3 \) take values in \( (0, 1/2) \) and \( \tau_1, \tau_2, \tau_3 \) be the functions \( \tau(p_1, .), \tau(p_2, .), \tau(p_3, .) \) respectively. Then \( \frac{\tau_1 - \tau_2}{\tau_2 - \tau_3} \) is decreasing on \( (1, +\infty) \).

Although the proof only uses elementary calculus, it is a little bit “tricky” and cannot be omitted.
Proof of Lemma 5.2 Taking into account the trivial equality
\[ \tau(p', q) - \tau(p'', q) = \int_{p''}^{p'} \frac{\partial \tau}{\partial p}(p, q) dp \]
we only need to show that if \( p' < p'' \) then \( \frac{\partial \tau}{\partial p}(p', q) : \frac{\partial \tau}{\partial p}(p'', q) \) is decreasing on \( q \in (1, \infty) \). We get
\[
\frac{\partial \tau}{\partial p}(p', q) : \frac{\partial \tau}{\partial p}(p'', q) = \frac{1 - (-1 + 1/p')^{q-1}}{1 + (-1 + 1/p')^q} : \frac{1 - (-1 + 1/p'')^{q-1}}{1 + (-1 + 1/p'')^q} = p - s_1^{q-1} : p - s_2^{q-1},
\]
where \( s_1 = -1 + 1/p' > 1 \) and \( s_2 = -1 + 1/p'' > 1 \).

If we set \( f(s, q) = \ln \frac{1 - s^{q-1}}{1 + s^q} \), with \( s, q > 1 \), it is sufficient to prove that \( \frac{\partial f}{\partial s} f(s, q) \) is decreasing in \( q \). We calculate
\[
\frac{\partial f}{\partial s} f(s, q) = \frac{(q - 1)s^{q-2} - qs^{q-1}}{s^q - 1} - \frac{qs^q}{s + 1}.
\]
We multiply by \( s \) and need to show that \( \frac{(q - 1)s^{q-1}}{s^q - 1} - \frac{qs^q}{s + 1} \) is decreasing which is equivalent to \( q - 1 + \frac{q-1}{s^q - 1} - q + \frac{q}{s + 1} \) being decreasing.

Put \( Q = q - 1 \); it remains to show that \( \frac{Q}{s^q - 1} + \frac{Q}{s + 1} \) decreases in \( Q > 0 \) and since the last term is decreasing it suffices to show that \( \frac{Q}{s^q - 1} + \frac{Q}{s^q + 1} \) is doing the same. By taking derivatives we need to show that
\[
(s^Q - 1)(s^{Q+1} + 1) - s^Q \ln s^Q(s^{Q+1} + 1) - s^{Q+1} \ln s^Q(s^Q - 1)
\]
is negative for \( Q > 0 \), which is trivial since \( s^Q \ln s^Q > s^Q - 1 \).

Proof of lemma 5.1. Let us first remark that \( \tau \) and \( \tau(p_0, .) \) can coincide at one point only if \( p_0 \in (p_1, p_2) \). Moreover, \( \tau(q) = \tau(p_0, q) \) implies
\[
\frac{\tau(p_1, q) - \tau(p_0, q)}{\tau(p_0, q) - \tau(p_2, q)} = \frac{\lambda}{1 - \lambda}.
\]
By lemma 5.2 this can only occur once. The lemma 5.1 easily follows on the decreasing property of the ratio.
The following two lemmas prove Theorem 4.1 in the particular case $n = 2$.

**Lemma 5.3** Take $\lambda_1, \lambda_2 \in (0, 1)$ such that $\lambda_1 + \lambda_2 = 1$, $1 < p_1 < p_2 < 1/2$ and set $\tau = \lambda_1 \tau_1 + \lambda_2 \tau_2$. Fix $1 < q_1 < q_2 < +\infty$ and consider $p_1 < q_4 < p_2 < p_5 < 1/2$ such that $\tau(q_4, q) = \tau(q)$. Then there is a unique convex combination $\tilde{\tau}$ of $\tau_1, \tau_4$ and $\tau_5$ such that

$$\tilde{\tau}(q_1) = \tau(q_1) \text{ and } \tilde{\tau}(q_2) = \tau(q_2).$$

Furthermore, for $i = 1, 2$, we have $\tau'(q_i) \neq \tilde{\tau}'(q_i)$ and $\tau(q) \neq \tilde{\tau}(q)$ if $q \neq q_i$.

**Proof** It suffices to show that the linear system

$$\begin{cases}
\lambda_3 \tau_1(q_1) + \lambda_4 \tau_4(q_1) + \lambda_5 \tau_5(q_1) &= \tau(q_1) \\
\lambda_3 \tau_1(q_2) + \lambda_4 \tau_4(q_2) + \lambda_5 \tau_5(q_2) &= \tau(q_2) \\
\lambda_3 + \lambda_4 + \lambda_5 &= 1
\end{cases} \quad (S)$$

has a unique positive solution $(\lambda_3, \lambda_4, \lambda_5)$. The existence of a unique solution is easy to verify. Let us show that this solution is positive.

Since $\tau(q_1) = \tau_4(q_1)$, we have

$$\lambda_3 \tau_1(q_1) + \lambda_5 \tau_5(q_1) = (1 - \lambda_4)(\lambda_1 \tau_1(q_1) + \lambda_2 \tau_2(q_1))$$

which is equivalent to

$$\frac{\lambda_3}{\lambda_3 + \lambda_5} \tau_1(q_1) + \frac{\lambda_5}{\lambda_3 + \lambda_5} \tau_5(q_1) = \lambda_1 \tau_1(q_1) + \lambda_2 \tau_2(q_1). \quad (4)$$

This implies that $\lambda_3 \lambda_5 > 0$. Moreover, since $\tau_5 < \tau_2$, we also have $\frac{\lambda_5}{\lambda_3 + \lambda_5} > \lambda_1$.

Let us show that $\lambda_3$ and $\lambda_5$ are positive. Otherwise, by the above remark, we have $\lambda_3 < 0$, $\lambda_5 < 0$ and $\lambda_4 > 0$. By the system (S) we have

$$\tau_4(q) = \frac{\lambda_1 - \lambda_3}{\lambda_4} \tau_1(q) + \frac{\lambda_2}{\lambda_4} \tau_2(q) + \frac{\lambda_5}{\lambda_4} \tau_5(q)$$

at the points $q = q_1$ and $q = q_2$. We then obtain that

$$\frac{\lambda_1 - \lambda_3}{\lambda_4} \tau_1(q) - \frac{\lambda_4}{\lambda_4} \tau_4(q) = \frac{\lambda_2}{\lambda_4} \tau_2(q) - \frac{\lambda_5}{\lambda_4} \tau_5(q)$$

for $q = q_1$ and $q = q_2$. Since $p_1 < p_4 < p_2$, by Lemma 5.2 the function $\frac{\tau_1 - \tau_4}{\tau_4 - \tau_2}$ is decreasing. On the other hand, since $p_4 < p_2 < p_5$, Lemma 5.2 implies that the function $\frac{\tau_4 - \tau_5}{\tau_4 - \tau_2} = 1 + \frac{\tau_2 - \tau_4}{\tau_4 - \tau_2}$ is increasing. Thus, these functions cannot coincide at two points so we conclude that $\lambda_3$ and $\lambda_5$ are positive.
Let us now prove that $\lambda_4 > 0$. By (4) we have
\[
\frac{\lambda_3}{\lambda_3 + \lambda_5} \tau_1(q_1) + \frac{\lambda_5}{\lambda_3 + \lambda_5} \tau_5(q_1) = \lambda_1 \tau_1(q_1) + \lambda_2 \tau_2(q_1)
\]
which gives that
\[
\lambda_2 \tau_2(q_1) = \left( \frac{\lambda_3}{\lambda_3 + \lambda_5} - \lambda_1 \right) \tau_1(q_1) + \frac{\lambda_5}{\lambda_3 + \lambda_5} \tau_5(q_1).
\]
Using Lemma 5.2 for $q > q_1$ we get
\[
\lambda_2 \tau_2(q) > \left( \frac{\lambda_3}{\lambda_3 + \lambda_5} - \lambda_1 \right) \tau_1(q) + \frac{\lambda_5}{\lambda_3 + \lambda_5} \tau_5(q)
\]
and
\[
\lambda_1 \tau_1(q) + \lambda_2 \tau_2(q) > \frac{\lambda_3}{\lambda_3 + \lambda_5} \tau_1(q) + \frac{\lambda_5}{\lambda_3 + \lambda_5} \tau_5(q).
\]
In particular, for $q = q_2$ we find that
\[
\lambda_3 \tau_1(q_2) + \lambda_5 \tau_5(q_2) + \lambda_4 \tau(q_2) < \tau(q_2) = \lambda_3 \tau_1(q_2) + \lambda_4 \tau_4(q_2) + \lambda_5 \tau_5(q_2)
\]
and we deduce that
\[
\lambda_4 \tau(q_2) < \lambda_4 \tau_4(q_2).
\]
It follows from Lemma 5.2 $\lambda_4 > 0$.

The last assertion follows directly from the indipendancy of the vector families
\[
\left\{ \begin{pmatrix} \tau_1(q_1) \\ \tau_4(q_1) \\ \tau_5(q_1) \end{pmatrix}, \begin{pmatrix} \tau_1(q_2) \\ \tau_4(q_2) \\ \tau_5(q_2) \end{pmatrix}, \begin{pmatrix} \tau_1'(q_i) \\ \tau_4'(q_i) \\ \tau_5'(q_i) \end{pmatrix} \right\}
\]
and
\[
\left\{ \begin{pmatrix} \tau_1(q_1) \\ \tau_4(q_1) \\ \tau_5(q_1) \end{pmatrix}, \begin{pmatrix} \tau_1(q_2) \\ \tau_4(q_2) \\ \tau_5(q_2) \end{pmatrix}, \begin{pmatrix} \tau_1(q) \\ \tau_4(q) \\ \tau_5(q) \end{pmatrix} \right\},
\]
which can be easily established.

**Lemma 5.4** The functions $\tau$ and $\tilde{\tau}$ defined in lemma 5.3 verify $\tilde{\tau}(q) > \tau(q)$ if and only if $q \in (q_1, q_2)$. 

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Proof Let us first remark that for \( \lambda_3, \lambda_4 \) and \( \lambda_5 \) defined by the linear system (S) we have
\[
\frac{\lambda_3}{1-\lambda_4} \tau_1(q_1) + \frac{\lambda_5}{1-\lambda_4} \tau_5(q_1) = \tau(q_1) = \tau_4(q_1).
\]
Put \( \rho = \frac{\lambda_3}{\lambda_3+\lambda_5} \tau_1 + \frac{\lambda_5}{\lambda_3+\lambda_5} \tau_5 \) and consider the function \( \Lambda : [0, 1] \to C^\infty ([1, \infty), \mathbb{R}) \) that assigns \( \mu \in [0, 1] \) to \( \Lambda(\mu) = \mu \tau_4 + (1-\mu) \rho \).
Let us also take \( q_1 = q_2 \) so that
\[
\Lambda(\lambda_4)'(q_1) = \tau'(q_1)
\]
(the parameter \( \lambda_4 \) depends on \( q_2 \)). It is sufficient to show that \( \Lambda(\lambda_4) \leq \tau \) : to obtain that \( \Lambda(\lambda_4) \leq \tau \) outside \([q_1, q_2]\), for \( q_1 < q_2 \), one can use a simple continuity argument on the graph of \( \Lambda(\lambda_4) \), seen as a function of \( q_2 \).

By (5) we obtain \( \frac{(\tau-\rho)'(q_1)}{(\tau_4-\rho)'(q_1)} = 1 > \lambda_4 \). The function \( \frac{\tau-\rho}{\tau_4-\rho} \) being increasing in a neighborhood of \( q_1 \) (as we will show below) we get that for \( q > q_1 \) \( (\tau-\rho)(q) > \lambda_4(\tau_4-\rho)(q) \) which implies \( \Lambda(\lambda_4)(q) < \tau(q) \) for \( q \neq q_1 \).

To finish the proof we need to show that \( \frac{\tau-\rho}{\tau_4-\rho} \) is increasing. Put \( s_1 = \frac{\lambda_3}{\lambda_3+\lambda_5} - \lambda_1 \), \( k_5 = \frac{\lambda_5}{\lambda_3+\lambda_5} = 1 - k_1 \) and \( k_1 = \frac{\lambda_3}{\lambda_3+\lambda_5} \), all positive. We can write :
\[
\frac{\rho - \tau}{\rho - \tau_4} = \frac{s_1(\tau_1 - \tau_2) + k_5(\tau_5 - \tau_2)}{k_1(\tau_1 - \tau_4) + k_5(\tau_5 - \tau_4)} =
\]
\[
= \frac{1}{1 - \lambda_1} \frac{s_1 - \frac{\tau_2 - \tau_5}{\tau_1 - \tau_5}}{k_1 - \frac{\tau_4 - \tau_5}{\tau_1 - \tau_5}} = \frac{k_1}{s_1(1 - \lambda_1)} \frac{1 - f}{1 - g}
\]
where \( f, g \) are both positive increasing functions by lemma 5.2. Moreover \( f/g \) is increasing which implies \( f'g - g'f > 0 \) and \( g'(q_1) = f'(q_1) \) hence \( (f-g)'(g+f) + g' - f' < 0 \). The result follows.

The proof of theorem 4.3 in the case \( n > 2 \) is now easy to derive : suppose \( \tau = \sum_{k=1}^n \lambda_k \tau(p_k,.) \) and let \( \tau(p_1,.) \) and \( \tau(p_2,.) \) be the first two functions of the convex combination. Be the previous two lemmas there exist a convex combination \( \tilde{\tau} \) of three \( \tau(p_i,.) \) functions such that
1. \( \frac{1}{\lambda_1+\lambda_2} (\lambda_1 \tau_1(q_i) + \lambda_2 \tau_2(q_i)) = \tilde{\tau}(q_i) \), for \( i = 1, 2 \)
2. \( \frac{1}{\lambda_1+\lambda_2} (\lambda_1 \tau_1'(q_1) + \lambda_2 \tau_2(q_1)) < \tilde{\tau}'(q_1) \), \( \frac{1}{\lambda_1+\lambda_2} (\lambda_1 \tau_1'(q_2) + \lambda_2 \tau_2(q_2)) > \tilde{\tau}'(q_2) \)
3. \( \frac{1}{\lambda_1+\lambda_2} (\lambda_1 \tau_1 + \lambda_2 \tau_2) < \tilde{\tau} \) on \((q_1, q_2)\) and \( \frac{1}{\lambda_1+\lambda_2} (\lambda_1 \tau_1 + \lambda_2 \tau_2) > \tilde{\tau} \) on \((1, \infty) \setminus [q_1, q_2]\).
The function \( \tilde{\tau} = (\lambda_1 + \lambda_2) \tilde{\tau} + \sum_{k=3}^n \lambda_k \tau(p_k,.) \) satisfies then the conclusion of theorem 4.1.

We can now prove theorem 1.3:
There exists an inhomogeneous Bernoulli product $\mu$ such that the spectrum $\tau$ of $\mu$ is not derivable on a dense subset of $[1, \infty)$.

The strategy of the demonstration of this theorem is the following: we first find inhomogeneous Bernoulli products that are not derivable at a finite number of predefined points and we construct the measure $\mu$ using Cantor’s diagonal argument.

**Lemma 5.5** For any $p_1, \ldots, p_n$ and any convex combination $\tau$ of $\tau(p_1, \ldots, \tau(p_n, \ldots)$ there exist an inhomogeneous Bernoulli measure $\mu$ whose multifractal spectrum equals $\tau$.

The proof of this lemma is not difficult and left to the reader. Let us now prove theorem 1.3.

**Proof of Theorem 1.3.** Fix $(q_n)_n$ a sequence of real numbers, dense in $[1, \infty)$ and nested in the sense that $q_{2n+1} < q_{2n+2}$ and $\{q_1, \ldots, q_{2n}\} \cap [q_{2n+1}, q_{2n+2}] = \emptyset$ for all $n \geq 0$. Let $p_1, p_2 \in (0, 1)$ and $\tau_1 = \frac{1}{2} \tau(p_1, \ldots) + \frac{1}{2} \tau(p_2, \ldots)$. By the previous lemma we can construct a Bernoulli product $\mu_1$ of spectrum $\tau_1$. Theorem 4.1 implies then the existence of a convex combination $\tau_2$ of $\tau(p_i, \ldots)$’s functions, such that

1. $\tau_1(q_1) = \tau_2(q_1)$, for $i = 1, 2$, $\tau'_1(q_1) < \tau'_2(q_1)$, $\tau'_1(q_2) > \tau'_2(q_2)$
2. $\tau_2 > \tau_1$ on $(q_1, q_2)$ and $\tau_2 < \tau_1$ on $(1, \infty) \setminus [q_1, q_2]$.

We can therefore define a measure $\mu_2$ of spectrum $\tau_2$; Using $\mu_1$ and $\mu_2$ we can construct a measure $\nu_2$ of spectrum $\max\{\tau_1, \tau_2\}$: Let $\mu_i$ me the inhomogeneous Bernoulli measure of spectrum $\tau_i, i = 1, 2$. Take $(\ell_k)_k$ a sequence of integers such that $\sum_{k=1}^{\infty} \ell_k \to \infty$. On dyadique intervals of length between $2^{-\ell_{2k}}$ and $2^{-\ell_{2k+1}}$ apply the weight distribution of $\mu_1$ and on dyadique intervals of length between $2^{-\ell_{2k+1}}$ and $2^{-\ell_{2k+2}}$ apply the weight distribution of $\mu_2$, where $k \in \mathbb{N}$. It is easy to verify that the resulting inhomogeneous measure $\nu_2$ has spectrum $\max\{\tau_1, \tau_2\}$. Remark that this spectrum equals $\tau_2$ on $[q_1, q_2]$ and $\tau_1$ elsewhere on $[1, \infty)$.

We proceed by induction. Suppose the measures $\nu_1 = \mu_1, \mu_2, \nu_2, \ldots, \mu_n, \nu_n$ defined and denote $\tau_i$ the spectrum of the measure $\mu_i, i \in \{1, \ldots, n\}$. We assume that that on every interval $[q_{2i+1}, q_{2i+2}]$, where $i \leq n$, the spectrum of $\nu_n$ equals $\max\{\tau_1, \ldots, \tau_n\}$ and is realized by $\tau_i$. Let us construct $\mu_{n+1}$ and $\nu_{n+1}$. Consider $\tau_j$ the function that equals the max$\{\tau_1, \ldots, \tau_n\}$ on $[q_{2(n+1)+1}, q_{2(n+1)+2}]$. By theorem 4.1 we can find a function $\tau_{n+1}$ satisfying:

1. $\tau_{n+1}(q_{2(n+1)+i}) = \tau_j(q_{2(n+1)+i})$, for $i = 1, 2$, $\tau'_{n+1}(q_{2(n+1)+1}) > \tau'_j(q_{2(n+1)+1})$, $\tau_{n+1}(q_{2(n+1)+2}) < \tau'_j(q_{2(n+1)+2})$
2. $\tau_{n+1} > \tau_j$ on $(q_{2(n+1)+1}, q_{2(n+1)+2})$ and $\tau_{n+1} < \tau_j$ on $(1, \infty) \setminus [q_{2(n+1)+1}, q_{2(n+1)+2}]$.

Let $\nu_{n+1}$ be the inhomogeneous Bernoulli measure of spectrum $\tau_{n+1}$. To define the measure $\nu_{n+1}$ we use the previous procedure convenably adapted: Take $(\ell_k)_k$ a sequence of
integers such that \( \sum_{k=1}^{\infty} \ell_i \rightarrow \infty \). On dyadic intervals of length between \( 2^{-\ell(n+1)k+i+1} \) and \( 2^{-\ell(n+1)k+i+1} \) apply the weight distribution of \( \mu_i \), where \( i = 1, \ldots, n+1, k \in \mathbb{N} \). It is easy to verify that the resulting inhomogeneous measure \( \nu_{n+1} \) has spectrum \( \max\{\tau_1, \ldots, \tau_{n+1}\} \) on \( (1, \infty) \). Remark that this spectrum equals \( \tau_{n+1} \) on \( [q_2(n+1)+1, q_2(n+1)+2] \) and \( \max\{\tau_1, \ldots, \tau_n\} \) elsewhere on \( [1, \infty) \).

To end the proof we use Cantor’s diagonal argument: take \((\ell_k)_k\) comme ci-dessus et define the measure \( \nu \) by applying the weight distribution of \( \nu_k \) on dyadic intervals of length between \( 2^{-\ell_k} \) and \( 2^{-\ell+1} \). The spectrum of the measure \( \nu \) equals then \( \tau = \sup_{n \in \mathbb{N}} \tau_n \). By construction the function \( \tau \) is not derivable at the points \((q_k)_k\) and the proof of theorem 1.3 is complete.

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