Parameters of cosmological models and recent astronomical observations

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Received July 22, 2014
Revised September 6, 2014
Accepted October 7, 2014
Published October 22, 2014

Abstract. For different gravitational models we consider limitations on their parameters coming from recent observational data for type Ia supernovae, baryon acoustic oscillations, and from 34 data points for the Hubble parameter $H(z)$ depending on redshift. We calculate parameters of 3 models describing accelerated expansion of the universe: the ΛCDM model, the model with generalized Chaplygin gas (GCG) and the multidimensional model of I. Pahwa, D. Choudhury and T.R. Seshadri. In particular, for the ΛCDM model 1σ estimates of parameters are: $H_0 = 70.262 \pm 0.319$ km $c^{-1}$Mpc$^{-1}$, $\Omega_m = 0.276^{+0.009}_{-0.008}$, $\Omega_\Lambda = 0.769 \pm 0.029$, $\Omega_k = -0.045 \pm 0.032$. The GCG model under restriction $\alpha \geq 0$ is reduced to the ΛCDM model. Predictions of the multidimensional model essentially depend on 3 data points for $H(z)$ with $z \geq 2.3$.

Keywords: supernova type Ia - standard candles, dark matter theory, cosmological applications of theories with extra dimensions, dark energy theory

ArXiv ePrint: 1407.5405
1 Introduction

The most important challenge for cosmologists is to explain the accelerated expansion of our universe that was directly measured for the first time from Type Ia supernovae observations [1, 2]. These supernovae were used as standard candles, because one can measure their redshifts $z$ and luminosity distances $D_L$. The observed dependence $D_L(z)$ based on further measurements [3, 4] argues for the accelerated growth of the cosmological scale factor $a(t)$ at late stage of its evolution.

This result was confirmed via observations of cosmic microwave background anisotropy [5], baryon acoustic oscillations (BAO) or large-scale galaxy clustering [4, 6, 7] and other observations [4, 5, 8]. In particular, our attention should be paid to measurements of the Hubble parameter $H(z)$ for different redshifts $z$ [9–20]. The results of these measurements and estimations are represented below in table 6 of appendix.

The values $H(z)$ were calculated with two methods: evaluation of the age difference for galaxies with close redshifts in refs. [9–15] and the method with BAO analysis [16–20]. In the first method the equality

$$a(t) = a_0/(1 + z)$$

and its consequence

$$H(z) = \frac{1}{a(t)} \frac{da}{dt} = -\frac{1}{1 + z} \frac{dz}{dt}$$

are used. Here $a_0 \equiv a(t_0)$ is the current value of the scale factor $a$.

Baryon acoustic oscillations (BAO) are disturbances in the cosmic microwave angular power spectrum and in the correlation function of the galaxy distribution, connected with acoustic waves propagation before the recombination epoch [4, 6]. These waves involved baryons coupled with photons up to the end of the drag era corresponding to $z_d \simeq 1059.3$ [8], when baryons became decoupled and resulted in a peak in the galaxy-galaxy correlation function at the comoving sound horizon scale $r_s(z_d)$ [6, 8].

In table 5 of appendix we represent estimations of two observational manifestations of the BAO effect. These values are taken from refs. [5, 21, 22], they confirm the conclusion...
about accelerated expansion of the universe. In addition, this data with observations of Type Ia supernovae and the Hubble parameter $H(z)$ are stringent restrictions on possible cosmological theories and models.

To explain accelerated expansion of the universe various cosmological models have been suggested, they include different forms of dark matter and dark energy in equations of state and various modifications of Einstein gravity [23–25]. The most popular among cosmological models is the $\Lambda$CDM model with a $\Lambda$ term (dark energy) and cold dark matter (see reviews [23, 25]). This model with 5% fraction of visible baryonic matter nowadays ($\Omega_b = 0.05$), 24% fraction of dark matter ($\Omega_c = 0.24$) and 71% fraction of dark energy ($\Omega_\Lambda = 0.71$) [5] successfully describes observational data for Type Ia supernovae, anisotropy of cosmic microwave background, BAO effects and $H(z)$ estimates [4, 5, 8].

However, there are some problems in the $\Lambda$CDM model connected with vague nature of dark matter and dark energy, with fine tuning of the observed value of $\Lambda$, which is many orders of magnitude smaller than expected vacuum energy density, and with surprising proximity $\Omega_\Lambda$ and $\Omega_m = \Omega_b + \Omega_c$ nowadays, though these parameters depend on time in different ways (the coincidence problem) [23–26].

Therefore a large number of alternative cosmological models have been proposed. They include modified gravity with $f(R)$ Lagrangian [27, 28], theories with scalar fields [29, 30], models with nontrivial equations of state [31–39], with extra dimensions [40–47] and many others [23–26].

Among these gravitational models we concentrate here on the model with generalized Chaplygin gas (GCG) [31–37]. The equation of state in this model

$$p = -B_0/\rho^{\alpha}$$

generalizes the corresponding equation $p = -B/\rho$ for the original Chaplygin gas model [31]. Generalized Chaplygin gas with EoS (1.2) plays the roles of both dark matter and dark energy, it is applied to describing observations of type Ia supernovae, BAO effects, the Hubble parameter $H(z)$ and other observational data in various combinations [33–37].

The equation of state similar to eq. (1.2) is used in the multidimensional gravitational model of I. Pahwa, D. Choudhury and T.R. Seshadri [46] (the PCS model in references below). In this model the $1 + 3 + d$ dimensional spacetime is symmetric and isotropic in two subspaces: in 3 usual spatial dimensions and in $d$ additional dimensions. Matter has zero (dust-like) pressure in usual dimensions and negative pressure $p_e$ in the form (1.2) in extra dimensions:

$$T^\mu_\nu = \text{diag} \left( -\rho, 0, 0, p_e, \ldots, p_e \right), \quad p_e = -B_0\rho^{-\alpha}$$

(1.3)

(in sections 1, 2 we use units with $c = 1$).

In ref. [46] the important case $d = 1$ was omitted. This case was considered in ref. [47], where we analyzed singularities of cosmological solutions in the PCS model [46] and suggested how to modify the equation of state (1.3) for the sake of avoiding the finite-time future singularity (“the end of the world”) which is inevitable in the PCS model. Main advantages of the multidimensional models [46] and [47] are: naturally arising dynamical compactification and successful description of the Type Ia supernovae observations.

In this paper we compare the $\Lambda$CDM model, the model with generalized Chaplygin gas (GCG) [31, 32], and also the models PCS [46] and [47] with $d$ extra dimensions from the point of view of their capacity to describe recent observational data for type Ia supernovae, BAO and $H(z)$. In the next section we briefly summarize the dynamics of the mentioned
models, in section 3 we analyze parameters of the mentioned models resulting in the best description of the observational data from ref. [3] and appendix.

2 Models

For all cosmological models in this paper the Einstein equations

\[ G_{\mu\nu} = 8\pi G T_{\mu\nu} + \Lambda \delta_{\mu\nu}, \]  

(2.1)
determine dynamics of the universe. Here \( T_{\mu\nu} \) and \( G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R \delta_{\mu\nu} \) are the energy momentum tensor and the Einstein tensor, \( \Lambda \) is nonzero only in the ΛCDM model. The energy momentum tensor has the form (1.3) in the multidimensional models [46, 47] and the standard form

\[ T_{\mu\nu} = \text{diag} (-\rho, p, p, p) \]  

(2.2)
in models with 3 + 1 dimensions. In the ΛCDM model baryonic and dark matter may be considered as one component of dust-like matter with density \( \rho = \rho_b + \rho_{dm} \), so we suppose \( p = 0 \) in eq. (2.2). The fraction of relativistic matter (radiation and neutrinos) is close to zero for observable values \( z \leq 2.3 \). In the GCG model [31–37] pressure \( p \) in the form (1.2) plays the role of dark energy, corresponding to the \( \Lambda \) term in the ΛCDM model.

For the Robertson-Walker metric with the curvature sign \( k \)

\[ ds^2 = -dt^2 + a^2(t) \left[ (1 - kr^2)^{-1} dr^2 + r^2 d\Omega \right] \]  

(2.3)
the Einstein equations (2.1) are reduced to the system

\[ 3 \frac{\dot{a}^2 + k}{a^2} = 8\pi G \rho + \Lambda, \]  

(2.4)
\[ \dot{\rho} = -3 \frac{\dot{a}}{a} (\rho + p). \]  

(2.5)

Eq. (2.5) results from the continuity condition \( T^\mu_{\nu;\mu} = 0 \), the dot denotes the time derivative.

Using the present time values of the Hubble constant and the critical density

\[ H_0 = \frac{\dot{a}}{a} \bigg|_{t=0} = H \bigg|_{z=0}, \quad \rho_{cr} = \frac{3H_0^2}{8\pi G}, \]  

(2.6)
we introduce dimensionless time \( \tau \), densities \( \tilde{\rho}_i \), pressure \( \tilde{p} \) and logarithm of the scale factor [46, 47]:

\[ \tau = H_0 t, \quad \tilde{\rho} = \frac{\rho}{\rho_{cr}}, \quad \tilde{\rho}_b = \frac{\rho_b}{\rho_{cr}}, \quad \tilde{p} = \frac{p}{\rho_{cr}}, \quad A = \log \frac{a}{a_0}. \]  

(2.7)

We denote derivatives with respect to \( \tau \) as primes and rewrite the system (2.4), (2.5)

\[ A'(\tau) = \sqrt{\tilde{\rho}} + \Omega_A + \Omega_k e^{-2A}, \]  

(2.8)
\[ \tilde{\rho}'(\tau) = -3A'(\tilde{\rho} + \tilde{p}). \]  

(2.9)

Here

\[ \Omega_m = \frac{\rho(t_0)}{\rho_{cr}}, \quad \Omega_A = \frac{\Lambda}{3H_0^2}, \quad \Omega_k = -\frac{k}{a_0^2 H_0^2} \]  

(2.10)
are present time fractions of matter \( (\Omega_m = \Omega_b + \Omega_c) \), dark energy and curvature in the equality
\[
\Omega_m + \Omega_A + \Omega_k = 1, \tag{2.11}
\]
resulting from eq. (2.4) if we fix \( t = t_0 \).

If we know an equation of state \( p = \rho(\rho) \) for any model, we can solve the Cauchy problem for the system (2.8), (2.9) including initial conditions for variables (2.7) at the present epoch \( t = t_0 \) (here and below \( t = t_0 \) corresponds to \( \tau = 1 \))
\[
A|_{\tau=1} = 0, \quad \bar{\rho}|_{\tau=1} = \Omega_m. \tag{2.12}
\]

In the \( \Lambda \)CDM model eq. (2.9) yields \( \bar{\rho} = \Omega_m e^{-3A} = \Omega_m (1 + z)^3 \), so we solve only equation (2.8)
\[
A^2 = \frac{H^2}{H_0^2} = \Omega_m e^{-3A} + \Omega_A + \Omega_k e^{-2A}. \tag{2.13}
\]
with the first initial condition (2.12).

Equation (2.9) may be solved also and in the GCG model, but in this case we are to decompose all matter into two components [34–38]. One of these components is usual dust-like matter including baryonic matter; the other component is generalized Chaplygin gas with density \( \rho_g \equiv \rho_{GCG} \) (and corresponding \( \bar{\rho}_g = \rho_g / \rho_{cr} \)). If the first component is pure baryonic and the latter describes both dark matter and dark energy, equations of state are:
\[
\bar{\rho} = \bar{\rho}_b + \bar{\rho}_g, \quad \bar{\rho}_b = 0, \quad \bar{\rho}_g = -B (\rho_g)^{-\alpha}. \tag{2.14}
\]

If we use the integrals \( \bar{\rho}_b = \Omega_b e^{-3A} \) and \( \bar{\rho}_g = \left[ B + Ce^{-3A(1+\alpha)} \right]^{1/(1+\alpha)} \) of eq. (2.9) for these components, equation (2.8) takes the form [33–38]
\[
A^2 = \frac{H^2}{H_0^2} = \Omega_b e^{-3A} + (1 - \Omega_b - \Omega_k) \left[ B_s + (1 - B_s) e^{-3A(1+\alpha)} \right]^{1/(1+\alpha)} + \Omega_k e^{-2A}. \tag{2.15}
\]

We solve this equation with the initial condition (2.12) \( A|_{\tau=1} = 0 \). The dimensionless constant \( B_s \) [37, 38] (it is denoted \( A_s \) in refs. [34, 35]) is expressed via \( B \) or \( B_0 \):
\[
B_s = B \cdot (1 - \Omega_b - \Omega_k)^{-1-\alpha}, \quad B = B_0 \rho_{cr}^{-1-\alpha}. \tag{2.16}
\]

For the multidimensional model PCS [46] and the model [47] in spacetime with \( 1 + 3 + d \) dimensions the following metric is used [46]:
\[
ds^2 = -dt^2 + a^2(t) \left( \frac{dr^2}{1-kr^2} + r^2 d\Omega \right) + b^2(t) \left( \frac{dR^2}{1-k_2R^2} + R^2 d\Omega_{d-1} \right). \tag{2.17}
\]

Here \( b(t) \) and \( k_2 \) are the scale factor and curvature sign in extra dimensions (along with \( a \) and \( k \) for usual dimensions). For cosmological solutions in refs. [46, 47] the scale factor \( a(t) \) grows while \( b(t) \) diminishes, in other words, some form of dynamical compactification [40–46] takes place, a size of compactified \( b \) is small enough to play no essential role at the TeV scale.

In refs. [46, 47] the authors considered only one component of their matter. Here we generalize these models and introduce the “usual” component with density \( \bar{\rho}_b \) and the “exotic” component with \( \bar{\rho}_e = \rho_e / \rho_{cr} \) and pressure \( \bar{p}_e = p_e / \rho_{cr} \) in extra dimensions similarly to eq. (2.14):
\[
\bar{\rho} = \bar{\rho}_b + \bar{\rho}_e, \quad \bar{p}_e = -B (\bar{\rho}_e)^{-\alpha}. \tag{2.18}
\]
Dynamical equations for the models [46, 47] result from the Einstein equations (2.2) with \( \Lambda = 0 \) and the energy momentum tensor (1.3), (2.18). In our notation (2.7) with \( B = \log (b/b_0) \) (where \( b_0 = b(t_0) \)) these equations for \( k_2 = 0 \) and \( d > 1 \) are [46, 47]

\[
A'' = \frac{1}{d + 2} \left[ d(d - 1) B' \left( \frac{1}{2} B' - A' \right) - 3(d + 1) A'^2 - 3d \bar{\rho}_c + (2d + 1) \Omega_k e^{-2A} \right],
\]

\[
\dot{\rho}_b = -\dot{\rho}_0 (3A' + d B'), \quad \dot{\rho}_c = -3 \dot{\rho}_c A' - d(\dot{\rho}_c + \dot{\rho}_e) B',
\]

\[
B' = (d - 1)^{-1} \left[ -3A' + \sqrt{3[(d + 2) A'^2 + 2(d - 1)(\dot{\rho} + \Omega_k e^{-2A})]} / d \right].
\]

If \( d = 1 \) one should use [47]

\[
B' = (\dot{\rho} + \Omega_k e^{-2A}) / A' - A'
\]

instead of eq. (2.21).

For the system (2.19)–(2.20) the initial conditions include eqs. (2.12) and the additional condition

\[
A'|_{\tau=1} = 1
\]

resulting from definitions of \( A \) (2.7) and \( H_0 \) (2.6):

\[
A'(\tau) = \frac{d}{d\tau} \log \frac{a}{a_0} = \frac{1}{H_0 a} \frac{\dot{a}}{a}.
\]

For the model PCS [46, 47] we have the analog of eq. (2.11)

\[
\Omega_m + \Omega_B + \Omega_k = 1,
\]

resulting from eqs. (2.21) or (2.22) at \( \tau = 1 \). Here \( \Omega_B = -d \left( B' + \frac{d-1}{6} B'^2 \right) \bigg|_{\tau=1} \) is the contribution from \( d \) extra dimensions.

The models \( \Lambda \)CDM, GCG, PCS with suitable values of model parameters have cosmological solutions describing accelerated expansion of the universe [5, 8, 33–37, 46, 47]. We consider restrictions on these parameters coming from recent observational data for type Ia supernovae [3], BAO [5, 21, 22] and from measuring the Hubble parameter \( H(z) \) [9–20], (tables 5, 6).

### 3 Observational data and model parameters

Recent observational data on Type Ia supernovae in the Union2.1 compilation [3] include redshifts \( z = z_i \) and distance moduli \( \mu_i \) with errors \( \sigma_i \) for \( N_S = 580 \) supernovae. The distance modulus \( \mu_i = \mu(D_L) = 5 \log \left( \frac{D_L}{10 \text{pc}} \right) \) is logarithm of the luminosity distance [8, 23]:

\[
D_L(z) = \frac{c(1 + z)}{H_0 \sqrt{|\Omega_k|}} \sin_k \left( H_0 \sqrt{|\Omega_k|} \int_0^z \frac{d\tilde{z}}{H(\tilde{z})} \right), \quad \sin_k(x) = \begin{cases} \sin x, & \Omega_k > 0, \\ x, & \Omega_k = 0, \\ \sin x, & \Omega_k < 0. \end{cases}
\]

In particular, for the flat universe \( (k = \Omega_k = 0) \) the expression (3.1) is

\[
D_L = c (1 + z) \int_0^z \frac{d\tilde{z}}{H(\tilde{z})} = \frac{ca_0^2}{H_0 a(\tau)} \int_\tau^1 \frac{d\tilde{\tau}}{a(\tilde{\tau})},
\]
To describe the Type Ia supernovae data [3] we fix values of model parameters \( p_1, p_2, \ldots \) for the chosen model ΛCDM, GCG or PCS and calculate dependence of the scale factor \( a(\tau) \) on dimensionless time \( \tau \). Further, we calculate numerically the integral expression (3.1) and the distance modulus \( \mu(\tau) \). For each value of redshift \( z_i \) in the table 3 we find the corresponding \( \tau_i = \tau(z_i) \) with using linear approximation in eq. (1.1) and the theoretical value \( \mu_{th} = \mu(\tau_i, p_1, p_2, \ldots) \) from the dependence \( \mu(\tau) \) (3.1).

We search a good fit between theoretical predictions \( \mu_{th} \) and the observed data \( \mu_i \) as the minimum of

\[
\chi^2_{S}(p_1, p_2, \ldots) = \sum_{i=1}^{N_S} \left[ \frac{\mu_i - \mu_{th}(z_i, p_1, p_2, \ldots)}{\sigma_i^2} \right]^2
\]

or the maximum of the corresponding likelihood function \( L_S(p_1, p_2, \ldots) = \exp(-\chi^2_S/2) \) in the space of model parameters \( p_1, p_2, \ldots \).

The Type Ia supernovae data [3] and the best fits for the mentioned models ΛCDM, GCG and PCS are shown in figure 1b in \( z, D_L \) plane. Details of the optimization procedure are described below.

Model predictions for the Hubble parameter \( H(z) = \dot{a}/a = H_0A'(\tau) \) we compare with observational data [9–20], from table 6 (figure 1c) and use the \( \chi^2 \) function similar to (3.2):

\[
\chi^2_H(p_1, p_2, \ldots) = \sum_{i=1}^{N_H} \frac{[H_i - H_{th}(z_i, p_1, p_2, \ldots)]^2}{\sigma_{H,i}^2}.
\]

Here \( N_H = 34 \), theoretical values \( H_{th}(z_i, \ldots) = H_0A'(\tau(z_i)) \) are obtained from the calculated dependence \( A(\tau) \) and the equality (1.1) \( z = e^{-A} - 1 \).

The observational data for BAO [5, 21, 22] (table 5) includes two measured values [6]

\[
d_z(z) = \frac{r_s(z_d)}{D_V(z)}
\]

and

\[
A(z) = \frac{H_0\sqrt{\Omega_m}}{cz}D_V(z).
\]

They are connected with the distance [5, 6, 8]

\[
D_V(z) = \left[ \frac{czD_L^2(z)}{(1 + z)^2H(z)} \right]^{1/3},
\]

expressed here via the luminosity distance (3.1).

The BAO observations [5, 21, 22] in table 5 are not independent. So the \( \chi^2 \) function for the values (3.4) and (3.5)

\[
\chi^2_B(p_1, p_2, \ldots) = (\Delta d)^T C_d^{-1} \Delta d + (\Delta A)^T C_A^{-1} \Delta A.
\]

includes the columns \( \Delta d = [d_{z,th}(z_i, p_1, \ldots) - d_z(z_i)], \Delta A = [A_{th}(z_i, p_1, p_2, \ldots) - A(z_i)], \) \( i = 1, \ldots, N_B \) and the covariance matrices \( C_d^{-1} \) and \( C_A^{-1} \) [5, 21] described in appendix.

The best fits to the observational data for Type Ia supernovae [3], \( H(z) \) and BAO data from tables 5, 6 are presented in figure 1 for the models ΛCDM, GCG and PCS (with \( d = 1 \).
Predictions of different models in figure 1 are rather close, in particular, the curves for the models ΛCDM and GCG practically coincide. The Hubble parameter $H(z)$ in figure 1c is measured in km $c^{-1}$Mpc$^{-1}$, the distances $D_L(z)$ and $D_V(z)$ in figure 1b, d are in Gpc.

The data points for $D_V(z) = r_s(z_d)/d_z(z)$ in figure 1d are calculated from $d_z(z_i)$ in table 5. Here the error boxes include the data spread between the recent estimations of the comoving sound horizon size:

$$r_s(z_d) = 147.49 \pm 0.59 \, \text{Mpc} \ [8], \quad r_s(z_d) = 153.3 \pm 2.0 \, \text{Mpc} \ [17, 21].$$

### 3.1 ΛCDM model

In the ΛCDM model we use three free parameters $H_0$, $\Omega_m$, and $\Omega_\Lambda$ in eq. (2.13) for describing the considered observational data at $z \leq 2.3$. For the Hubble constant $H_0$ different approaches result in different estimations. In particular, observations of Cepheid variables in the project Hubble Space Telescope (HST) give the recent estimate $H_0 = 73.8 \pm 2.4 \, \text{km} \, c^{-1} \text{Mpc}^{-1} \ [48]$. On the other hand, the satellite projects Planck Collaboration (Planck) [8] and Wilkinson Microwave Anisotropy Probe (WMAP) [5] for observations of cosmic microwave background anisotropy result in the following values (in km $c^{-1}$Mpc$^{-1}$):

$$H_0 = 67.3 \pm 1.2 \quad (\text{Planck} \ [8]),$$
$$H_0 = 69.7 \pm 2.4 \quad (\text{WMAP} \ [5]),$$
$$H_0 = 73.8 \pm 2.4 \quad (\text{HST} \ [48]).$$

(3.10)
The nine-year results from WMAP [5] include also the estimate $H_0 = 69.33 \pm 0.88$ km c$^{-1}$ Mpc$^{-1}$ with added recent BAO and $H_0$ observations.

For the $\Lambda$CDM model many authors [5, 8, 49–55] calculated the best fits for parameters $H_0$, $\Omega_m$ and $\Omega_\Lambda$ for describing the Type Ia supernovae, $H(z)$ and BAO data in various combinations. In refs. [52–55] some other cosmological models were compared with the $\Lambda$CDM model. In particular, the authors [52] compared 8 models with two information criteria including minimal $\chi^2$ and the number of model parameters. Optimal values of these parameters were pointed out in ref. [52] with the exception of $H_0$, though $H_0$ is the important parameter for all 8 models.

In refs. [53–55] the $\Lambda$CDM, XCDM and $\phi$CDM models were applied to describe the supernovae, $H(z)$ and BAO data. For all mentioned models the authors [53–55] fixed two values of the Hubble constant $H_0 = 68 \pm 2.8$ [56] and $H_0 = 73.8 \pm 2.4$ km c$^{-1}$Mpc$^{-1}$ [48] and searched optimal values of other model parameters. But they did not estimated the best choice of $H_0$ among these two values and in the segment between them.

In this paper we pay the special attention to dependence of $\chi^2$ minima on $H_0$. This dependence is very important if we compare different cosmological models.

The results of calculations [5, 8, 51–55], as usual, are presented as level lines for the functions $\chi^2(p_1, p_2)$ or $\mathcal{L}_S(p_1, p_2) = \exp(-\chi^2/2)$ of two parameters at 1$\sigma$ (68.27$\%$), 2$\sigma$ (95.45$\%$) and 3$\sigma$ (99.73$\%$) confidence levels. In particular, if a value $H_0$ is fixed, these two parameters for the $\Lambda$CDM model may be $\Omega_m$ and $\Omega_\Lambda$.

In figure 2 we use this scheme for 3 fixed values $H_0$ (3.10) indicated on the panels (including the optimal value $H_0 = 70.262$ km c$^{-1}$Mpc$^{-1}$) and draw level lines of the functions (3.2), (3.3), (3.7) and (3.8) $\chi^2(\Omega_m, \Omega_\Lambda)$ in the $\Omega_m, \Omega_\Lambda$ plane and for $\chi^2(\Omega_m, H_0)$ with fixed $\Omega_\Lambda = 0.769$ in the bottom-right panel. The points of minima are marked in figure 2 as hexagrams for $\chi^2_\Sigma$, pentagrams for $\chi^2_H$, diamonds for $\chi^2_B$ and circles for $\chi^2_\Sigma$. Minimal values of the functions $\chi^2$ (3.2), (3.3), (3.7) and (3.8) at these points are tabulated in table 1 so we can compare efficiency of this description for different $H_0$. For the same purpose we point out the corresponding values $\chi^2$ for some level lines in figure 2 and present the dependence of minima $\min \chi^2 (H_0)$ on $H_0$ and on $\Omega_m$ in the left bottom panels of figure 2. Here we denote $\min \chi^2 (H_0) = \min \chi^2 (\Omega_m, \Omega_\Lambda) = \min \chi^2 (H_0, \Omega_m, \Omega_\Lambda)$ and graphs of the fractions $\chi^2_\Sigma$, $\chi^2_H$, $\chi^2_B$, $\chi^2_\Sigma$ in $\min \chi^2 (H_0)$ are also shown.

In the bottom panels we present how parameters of a minimum point of $\chi^2_\Sigma$ depend on $H_0$ and on $\Omega_m$. In particular, for the dependence on $H_0$ the coordinates $\Omega_m(H_0)$ and $\Omega_\Lambda(H_0)$ of this point are calculated, the value $\Omega_k$ is determined from eq. (2.11). For the dependence on $\Omega_m$ we also present the graph $h(\Omega_m)$, where $h = H_0/100$.

We see in figure 2 and in table 1 that the dependence of $\min \chi^2 (H_0)$ is appreciable and significant. This function has the distinct minimum and achieves its minimal value 585.35 at $H_0 \approx 70.26$. The optimal values of the $\Lambda$CDM model parameters $\Omega_m \approx 0.276$, $\Omega_\Lambda \approx 0.769$, corresponding to this minimum are presented in table 2, these values are taken for the $\Lambda$CDM curves in figure 1.

The mentioned sharp dependence of $\min \chi^2_\Sigma$ on $H_0$ is connected with two factors: (1) the similar dependence of the main contribution $\chi^2_\Sigma(H_0)$ shown in the same panel; (2) the large shift of the minimum point for $\chi^2_\Sigma$ in the $\Omega_m, \Omega_\Lambda$ plane corresponding to $H_0$ growth. For $H_0 = 68$ and 73.8 km c$^{-1}$Mpc$^{-1}$ this minimum point is far from the similar points of $\chi^2_H$ and $\chi^2_B$. Only for $H_0$ close to 70 km c$^{-1}$Mpc$^{-1}$ all these three minimum points are near each other (the top-right panel in figure 2).
Figure 2. The ΛCDM model. For the values $H_0$ (3.10) and the optimal value $H_0 = 70.26$ km $c^{-1}$Mpc$^{-1}$ level lines are drawn at 1σ, 2σ and 3σ (thick solid) for $\chi^2_S(\Omega_m, \Omega_\Lambda)$ (black), for $\chi^2_H(\Omega_m, \Omega_\Lambda)$ (green) and $\chi^2_B(\Omega_m, \Omega_\Lambda)$ (red in the top row), the sum (3.8) $\chi^2_S(\Omega_m, \Omega_\Lambda)$ (the middle row), $\chi^2_S(\Omega_m, H_0)$ for $\Omega_\Lambda = 0.758$ (the bottom-right panel); dependence of min $\chi^2_S$, its fractions $\chi^2_H$ and $\chi^2_B$ and parameters of a minimum point on $H_0$ and on $\Omega_m$.

| $H_0$ | $\text{min } \chi^2_S$ | $\text{min } \chi^2_H$ | $\text{min } \chi^2_B$ | $\text{min } \chi^2_S$ | $\Omega_m$ | $\Omega_\Lambda$ | $\Omega_k$ |
|-------|-----------------|-----------------|-----------------|-----------------|---------|---------|---------|
| 67.3  | 599.37          | 18.492          | 5.548           | 673.64          | 0.285   | 0.568   | 0.147   |
| 69.7  | 562.73          | 17.993          | 3.517           | 588.53          | 0.278   | 0.734   | −0.012  |
| 73.8  | 639.90          | 19.466          | 5.322           | 707.84          | 0.269   | 0.961   | −0.230  |

Table 1. The ΛCDM model. For given $H_0$ (3.10) the calculated minima of $\chi^2_S$, $\chi^2_H$, $\chi^2_B$ and $\chi^2_S$ with $\Omega_m$, $\Omega_\Lambda$, $\Omega_k$ correspond to min $\chi^2_S$. 
Only the value $H_0 = 69.7 \text{ km c}^{-1}\text{Mpc}^{-1}$ in table 1 is close to the optimal value in table 2. We may conclude that the values of the Hubble constant $H_0 = 68$ and $73.8 \text{ km c}^{-1}\text{Mpc}^{-1}$ taken in refs. [53–55], unfortunately, lie to the left and to the right from the optimal value $H_0 \simeq 70 \text{ km c}^{-1}\text{Mpc}^{-1}$. We see the significant difference between the large values $\chi^2_{\Sigma} = 673.64$ or $707.84$ for the too small and too large values of $H_0$ in table 1 and the optimal value $\chi^2_{\Sigma} = 585.35$ for $H_0 = 70.262$ in table 2.

In the middle row panels of figure 2 with $\chi^2_{\Sigma}$ the flatness line $\Omega_m + \Omega_A = 1$ (or $\Omega_k = 0$) is shown as the black dashed straight line. This line shows that only for $H_0$ close to the optimal value from table 2 the following recent observational limitations on the $\Lambda$CDM model parameters (2.10) from surveys [5, 8]

\begin{equation}
\begin{aligned}
\Omega_m &= 0.279 \pm 0.025, \\
\Omega_k &= -0.0027^{-0.0037}_{+0.0039}, \\
\Omega_{\Lambda} &= 0.721 \pm 0.025, \\
\Omega_{\Lambda} &= 0.686 \pm 0.025, \\
\Omega_k &= -0.0005^{-0.0066}_{+0.0065},
\end{aligned}
\end{equation}

are satisfied on $1\sigma$ or $2\sigma$ level. For $H_0 = 67.3$ and $73.8 \text{ km c}^{-1}\text{Mpc}^{-1}$ the optimal values of parameters $\Omega_m$, $\Omega_{\Lambda}$, $\Omega_k$ in table 1 are far from restrictions (3.11) for $\Omega_k$ even on $3\sigma$ level.

Graphs of the optimal values $\Omega_m$, $\Omega_{\Lambda}$ and $\Omega_k$ depending on $H_0$ are presented in the second bottom panel. We see that the value $\Omega_m$ weakly depends on $H_0$, but $\Omega_{\Lambda}$ and $\Omega_k$ satisfy conditions (3.11) only for $H_0 \simeq 70 \text{ km c}^{-1}\text{Mpc}^{-1}$.

The dependence of $\chi^2_{\Sigma}$ on $\Omega_m$ is rather sharp because of the correspondent dependence of its fraction $\chi^2_B$. This fact for $\chi^2_B$ is connected with the contribution from the value $A(z)$ (3.5) measurements, because $A(z)$ is proportional to $\sqrt{\Omega_m}$ and $\chi^2_B$ is very sensitive to $\Omega_m$ values. Note that the fractions $\chi^2_{3}\Lambda$ and $\chi^2_{3}\Lambda$ (in $\chi^2_{\Sigma}$) weakly depend on $\Omega_m$.

Dependencies of $\chi^2_{\Sigma}$ on $H_0$, $\Omega_m$ and also $\Omega_{\Lambda}$, $\Omega_k$ let us calculate estimates of acceptable values for these model parameters. They are presented below in table 3.

Coordinates $h = H_0/100$ and $\Omega_A$ of the minimum point for $\chi^2_{\Sigma}$ depend on $\Omega_m$ in a such manner that only for $\Omega_m \simeq 0.27$ values $\Omega_{\Lambda}$ and $\Omega_k$ satisfy conditions (3.11). Note that the optimal value of $h$ is close to 0.7 for all $\Omega_m$ in the limits $0 < \Omega_m < 1$.

### 3.2 GCG model

Let us apply the model with generalized Chaplygin gas (GCG) [31–37] to describing the same observational data for Type Ia supernovae, $H(z)$ and BAO. We use here eq. (2.15) with the initial condition $A |_{\tau = 1} = 0$, so we have 5 independent free parameters in this model: $H_0$, $\Omega_b$, $\Omega_k$, $\alpha$ and $B_s$. However we really used only 4 free parameters, because the fraction $\Omega_b$ may

| Model          | $\chi^2_{\Sigma}$ | $H_0$ | $\Omega_m$ | $\Omega_{\Lambda}$ | $\Omega_k$ |
|----------------|-------------------|-------|------------|-------------------|------------|
| $\Lambda$CDM  | 585.35            | 70.262| 0.276      | $\Omega_{\Lambda} = 0.769$, $\Omega_k = -0.045$ |
| GCG           | 584.54            | 70.093| 0.277      | $\Omega_{\Lambda} = -0.019$, $\alpha = -0.066$, $B_s = 0.759$ |
| PCS, $d = 1$  | 588.41            | 69.52 | 0.286      | $\Omega_{\Lambda} = -0.040$, $\alpha = -0.256$, $B = 2.067$ |
| PCS, $d = 2$  | 591.10            | 69.49 | 0.288      | $\Omega_{\Lambda} = -0.017$, $\alpha = -0.372$, $B = 1.599$ |
| PCS, $d = 3$  | 592.18            | 69.34 | 0.288      | $\Omega_{\Lambda} = -0.027$, $\alpha = -0.431$, $B = 1.461$ |
| PCS, $d = 6$  | 592.65            | 69.29 | 0.290      | $\Omega_{\Lambda} = -0.029$, $\alpha = -0.493$, $B = 1.302$ |

Table 2. Optimal values of model parameters ($\Omega_b = 0.047$, for the GCG model $\Omega_m = \Omega_{\Lambda}^{\text{opt}}$ (3.13)).
include not only baryonic but also a part of cold dark matter. Our calculations yield that the minimum over remaining 4 parameters \(\min_{\mathcal{A}, \Omega_k, \alpha, B_s} \chi^2_\Sigma\) practically does not depend on \(\Omega_b\) in the range \(0 \leq \Omega_b \leq 0.25\) (see figure 3). So in our analysis presented in figure 3 (except for 3 bottom-right panels) we fixed the value

\[ \Omega_b = 0.047, \]

that is the simple average of the WMAP \(\Omega_b = 0.0464\) [5] and Planck \(\Omega_b = 0.0485\) [8] estimations.

In the GCG model \(\Omega_A = 0\) and \(\Omega_m = 1 - \Omega_k\) in accordance with eq. (2.11) and the formal definition (2.10). However we should use the effective value \(\Omega_m^{\text{eff}}\) in this model, in particular, in expression (3.5). In refs. [34–38] the following effective value is used

\[ \Omega_m^{\text{eff}} = \Omega_b + (1 - \Omega_b - \Omega_k)(1 - B_s)^{1/(1 + \alpha)}. \]  

(3.12)

This value results from correspondence between the models \(\Lambda\)CDM with eq. (2.13) and GCG with eq. (2.15) in the early universe at \(z \gg 1\).

But in our investigation the majority of observational data is connected with redshifts \(0 < z < 1\), so in eq. (3.5) we are to consider the present time limit of the value \(\Omega_m^{\text{eff}} = \lim_{z \to 0} \Omega_m^{\text{eff}}\). If we compare limits of the right hand sides of eqs. (2.13) and (2.15) at \(z \to 0\) or \(\mathcal{A} \to 0\), we obtain another effective value

\[ \Omega_m^{\text{eff}} = \Omega_b + (1 - \Omega_b - \Omega_k)(1 - B_s). \]  

(3.13)

Values \(\chi^2_B\) calculated with expressions (3.12) and (3.13) are different if \(\alpha \neq 0\). This difference looks like rather small if we compare minima of the sum (3.8) \(\min_{\Omega_k, \alpha, B_s} \chi^2_\Sigma\) depending on \(H_0\). In figure 3 this dependence with eq. (3.13) for \(\Omega_m^{\text{eff}}\) is the blue solid line and for the case with eq. (3.12) it is the violet dash-and-dot line. We see that the lines closely converge in the vicinity of the minimum point \(H_0 \approx 70\) km\(^{-1}\)Mpc\(^{-1}\). The dependence \(\min_{\Omega_k}(H_0)\) in both cases (3.12) and (3.13) has the sharp minimum and resembles the case of the \(\Lambda\)CDM model in figure 2. The value \(\min \chi^2_\Sigma \approx 584.54\) of this minimum, its parameters in table 2, graph of the contribution \(\chi^2_S\) and dependence on \(H_0\) for parameters \(\alpha, \Omega_k, B_s\) of the minimum point in the bottom-left panel in figure 3 are presented for the case with eq. (3.13).

One should note that all mentioned dependencies are different for the case (3.12), in particular, the absolute minimum of \(\chi^2_\Sigma\) is 584.31. This difference is illustrated in the central panels in figure 3 with level lines of \(\chi^2_\Sigma(\alpha, B_s)\) for \(H_0 = 73.8\) and 70.093 km\(^{-1}\)Mpc\(^{-1}\) (with the specified values \(\Omega_k\), optimal for these \(H_0\)). These level lines are blue for the expression (3.13) and they are thin violet for eq. (3.12). Positions of the optimal points are close only if \(H_0\) is close to its optimal value in table 2.

We suppose that the estimation of \(\chi^2_B\) with the expression (3.13) is more adequate to the considered values \(z\). So in table 2 and in other panels of figure 3 we use only eq. (3.13). Notations in figure 3 correspond to figure 2.

The similar dependence of \(\min \chi^2_\Sigma\) on \(H_0\) for the \(\Lambda\)CDM and GCG models results in unsuccessful description of the data with \(H_0 = 67.3\) and 73.8 km\(^{-1}\)Mpc\(^{-1}\) with the corresponding optimal values \(\Omega_k = 0.247\) and \(-0.295\). Figure 3 illustrates large distances between minimum points of \(\chi^2_S\), \(\chi^2_B\) and \(\chi^2_\Sigma\) in these cases. The mentioned distances are small for the optimal values from table 2 \(H_0 = 70.093\) km\(^{-1}\)Mpc\(^{-1}\) and \(\Omega_k = -0.19\). For these optimal
values we present level lines of $\chi^2_\Sigma$ in $\alpha, B_s$; $\alpha, H_0$; $\Omega_k, H_0$ and $\Omega_b, H_0$ planes. In these panels other model parameters are fixed and specified.

When we test dependence of the minimum $\min \chi^2_\Sigma$ on $H_0, \Omega_k, \alpha$ and $\Omega_b$ in figure 3, we minimize this value over all other parameters (except for the above mentioned $\Omega_b$). In particular, $\min \chi^2_\Sigma(\Omega_k) = \min_{H_0, \alpha, B_s} \chi^2_\Sigma$, this function has the distinct minimum near $\Omega_k \approx 0$ and resembles the dependence $\min \chi^2_\Sigma(H_0)$. The optimal value of $H_0$ or $h = H_0/100$ is practically constant and close to $h \approx 0.7$ if we vary $\Omega_k, \alpha$ or $\Omega_b$. As mentioned above the dependence of $\min \chi^2_\Sigma$ on $\Omega_k$ is very weak, so we fixed in our previous analysis $\Omega_k = 0.047$.

For the graph $\min \chi^2_\Sigma(\alpha) = \min_{H_0, \Omega_k, B_s} \chi^2_\Sigma$ the correspondent minimum is achieved if $\alpha$ is negative: $\alpha = -0.066$ (see table 2). In the GCG model this parameter is connected with the square of adiabatic sound speed [33, 36, 37]

$$c_s^2 = \frac{\delta p}{\delta \rho} = -\alpha \frac{p}{\rho}. \quad (3.14)$$

If we accept the restriction $\alpha \geq 0$ (equivalent to $c_s^2 \geq 0$) in our investigation with the mentioned observational data, we obtain the optimal value $\alpha = 0$ and the GCG model will
be reduced to the ΛCDM model with $\Omega_\Lambda = B = B_s(1 - \Omega_b - \Omega_k)$. The dependence of $\min \chi^2_2$ and other parameters on $\alpha$ in figure 3 show that for $\alpha = 0$ we have $\min \chi^2_2 \approx 585.35$ and the optimal values of $H_0$, $\Omega_k$, $\Omega_\Lambda = B$ corresponding to the ΛCDM model in table 2.

### 3.3 PCS model

The multidimensional gravitational model of I. Pahwa, D. Choudhury and T.R. Seshadri [46] has the set of model parameters $H_0$, $\Omega_b$, $\Omega_m$, $\Omega_k$, $\alpha$, $B$ similar to the GCG model, but also it has the additional integer-valued parameter $d$ (the number of extra dimensions). Our calculations of optimal parameters in table 2 demonstrate that the value $d = 1$ is the most preferable for describing the observational data for supernovae, BAO and $H(z)$. So it is the case $d = 1$ that we present in detail in figure 4 (analysis for $d \geq 2$ is shown below in figure 5).

In figure 4 for the PCS model we draw the same graphs and level lines in correspondent panels as in figure 3 for the GCG model. Here we use the similarity of model parameters for these models, but, naturally, in figure 4 the value $B$ is shown instead of $B_s$. Colors of correspondent lines also coincide.

The minimum $\min \chi^2_2$ (over all other parameters) increases when the baryon fraction $\Omega_b$ grows in figure 4. This dependence is more distinct than in the GCG case (figure 3), but it is
also rather weak for small $\Omega_0$. So for the multidimensional model PCS we also fix $\Omega_b = 0.047$ and really use only 5 remaining parameters $H_0$, $\Omega_m$, $\Omega_k$, $\alpha$, $B$. The value $\Omega_b = 0.047$ is fixed in all panels of figures 4 and 5 (except for 3 bottom-right panels in figure 4).

The dependence of $\min \chi^2_{\Sigma} = \min_{\Omega_m, \Omega_k, \alpha, B} \chi^2_{\Sigma}$ on $H_0$ has the distinct minimum at $H_0 \approx 69.52$ for $d = 1$ (the solid blue line in figure 4). The black dashed line corresponds to the fraction $\chi^2_{\Sigma}$ in $\chi^2_{\Sigma}$.

If we compare the minimal value of $\chi^2_{\Sigma}$ for the PCS model with the results of the $\Lambda$CDM and GCG models in table 2, we see that even for the best case $d = 1$ the result of the PCS model $\min \chi^2_{\Sigma} \approx 588.41$ is larger than for other two models. This disadvantage of the PCS model is connected with description of the $H(z)$ recent data with high $z$ ($z > 2$ in table 6). When we excluded 3 data points [14, 19, 20] for $H(z)$ with $z \geq 2.3$, we obtained absolutely other results presented below in table 4.

In figure 4 all level lines and graphs correspond to the whole $H(z)$ data with $N_H = 34$ points. But only one except is done for the dependence of $\min \chi^2_{\Sigma}$ on $H_0$: here $N_H = 31$, this graph is shown as the red dash-and-dot line. The minimum value for this line $\min \chi^2_{\Sigma} \approx 582.68$ is in table 4.

Level lines of functions $\chi^2$ are shown in figure 4 in the same panels as for the GCG model in figure 3, in particular, for the values (3.10) $H_0 = 67.3, 73.8$ and the optimal value 69.52 km c$^{-1}$Mpc$^{-1}$. If $H_0$ is too large, the domain of acceptable level of $\chi^2_{\Sigma}$ becomes very narrow. One should note that for all level lines we change only two parameters, all remaining model parameters are fixed (they are from table 2 or optimal for a given $H_0$).

In 6 top-left panels with the $\alpha, B$ plane we draw thin purple lines bounding the domain of regular solutions (below these lines). The upper domain (for larger $B$) consists of singular solutions, they have singularities in the past with infinite value of density $\rho$ corresponding to nonzero value of the scale factor $a$ [47]. These solutions are nonphysical and should be excluded. It is interesting that the optimal solutions in figure 4 and in tables 2 and 4 are near this border, but they are regular and describe the standard Big Bang $\rho \to \infty \iff a \to 0$ with dynamical compactification of extra dimensions.
The case with high number of extra dimensions $d \geq 2$ is presented in figure 3, where we see the similar behavior of the model with different $d$. For any $d$ the dependence of $\min \chi^2_{\Sigma} = \min_{\Omega_m, \Omega_k, \alpha, B} \chi^2_{\Sigma}$ on $H_0$ has the distinct minimum, but the more $d$ the higher value of this minimum (see table 2). So it is the case $d = 1$ that results in the best fit for the considered data in tables 2 and 4.

4 Conclusion

We investigated how the $\Lambda$CDM, GCG and PCS models describe the observational data for type Ia supernovae, BAO and $H(z)$ [3], tables 5, 6. These observations distinctly restrict acceptable values for the Hubble constant $H_0$ and other parameters of the mentioned models. We used our calculations for dependance $\min \chi^2_{\Sigma}(p)$, where the absolute minimum (over other parameters) of the value (3.8) $\chi^2_{\Sigma}$ depend on a fixed parameter $p$. On the base of these calculations (presented partially in figures 2, 3, 4) we obtained the following 1σ estimates for parameters of the $\Lambda$CDM, GCG and PCS ($d = 1$) models:

Our estimates for the $\Lambda$CDM model are in agreement with the WMAP observational restrictions (3.11) on $\Omega_m$, $\Omega_A$, $\Omega_k$ [5], but they are in tension with the Planck data [8]. This fact is connected with too low value $H_0 = 67.3$ km c$^{-1}$Mpc$^{-1}$ (3.10) in the Planck survey [8].

For the GCG model min $\chi^2_{\Sigma}$ is slightly better and our limitations on $H_0$ and $\Omega_k$ in table 3 are rather close to the $\Lambda$CDM case. However, if we require $\alpha \geq 0$ in accordance with eq. (3.14) and refs. [36, 37], the GCG model with the optimal value $\alpha = 0$ will be reduced to the $\Lambda$CDM model with its optimal parameters in tables 2, 3 and the same min $\chi^2_{\Sigma}$.

Values $\chi^2_B$ and $\chi^2_{\Sigma}$ for the GCG model essentially depend on the expression for $\Omega_m$ (3.12) or (3.13). But the optimal parameters in table 2 for these expressions are rather close.

We mentioned above that the multidimensional model PCS is less effective in description of the considered observational data, and that the main problem of this model is connected with the $H(z)$ recent data with high $z$ ($z > 2$). We excluded 3 $H(z)$ data points [14, 19, 20] with $z = 2.3, 2.34, 2.36$ and for remaining $N_H = 31$ points of $H(z)$ and the same SN and BAO data from [3], table 5. we calculated min $\chi^2_{\Sigma}$ and optimal values of model parameters presented here in table 4.
We see that the model PCS [46] describes the reduced set of data with \( z < 2 \) better than other models. The best fit is for \( d = 1 \), the optimal value of \( H_0 \) close to 70 km c\(^{-1}\)Mpc\(^{-1}\).

This example demonstrates that predictions of any cosmological model essentially depend on data selection. Moreover, there is the important problem of model dependence (in addition to mutual dependence) of observational data, in particular, data in tables 5, 6.

Leaving the last problem beyond this paper, we can conclude that the considered observations of type Ia supernovae [3], BAO (table 5) and the Hubble parameter \( H(z) \) (table 6) confirm effectiveness of the ΛCDM model, but they do not deny other models. The important argument in favor of the ΛCDM model is its small number \( N_p \) of model parameters (degrees of freedom). This number is part of information criteria of model selection statistics, in particular, the Akaike information criterion is [52] \( AIC = \min \chi^2_2 + 2N_p \). This criterion supports the leading position of the ΛCDM model.

### Table 5

| \( z \) | \( d(z) \) | \( \sigma_d \) | \( A(z) \) | \( \sigma_A \) | Refs |
|-----|-----|-----|-----|-----|-----|
| 0.106 | 0.336 | 0.015 | 0.526 | 0.028 | [5] |
| 0.20 | 0.1905 | 0.0061 | 0.488 | 0.016 | [5] |
| 0.35 | 0.1097 | 0.0036 | 0.484 | 0.016 | [5] |
| 0.44 | 0.0916 | 0.0071 | 0.474 | 0.034 | [21] |
| 0.57 | 0.07315 | 0.0012 | 0.436 | 0.017 | [5, 22] |
| 0.60 | 0.0726 | 0.0034 | 0.442 | 0.020 | [21] |
| 0.73 | 0.0592 | 0.0032 | 0.424 | 0.021 | [21] |

Table 5. Values of \( d(z) = r_s(z_d)/D_V(z) \) (3.4) and \( A(z) \) (3.5) with corresponding errors [5, 21, 22].

### Table 6

| \( z \) | \( H(z) \) | \( \sigma_H \) | Refs. | \( z \) | \( H(z) \) | \( \sigma_H \) | Refs |
|-----|-----|-----|-----|-----|-----|-----|-----|
| 0.070 | 0.69 | 19.6 | [13] | 0.57 | 92.9 | 7.855 | [17] |
| 0.090 | 0.69 | 12 | [9] | 0.593 | 104 | 13 | [11] |
| 0.120 | 0.686 | 26.2 | [13] | 0.600 | 87.9 | 6.1 | [12] |
| 0.170 | 0.83 | 8 | [9] | 0.680 | 92 | 8; | [11] |
| 0.179 | 0.75 | 4 | [11] | 0.730 | 97.3 | 7.0 | [12] |
| 0.199 | 0.75 | 5 | [11] | 0.781 | 105 | 12 | [11] |
| 0.200 | 0.729 | 29.6 | [13] | 0.875 | 125 | 17 | [11] |
| 0.240 | 0.7969 | 2.65 | [16] | 0.880 | 90 | 40 | [10] |
| 0.270 | 0.77 | 14 | [9] | 0.900 | 117 | 23 | [9] |
| 0.280 | 0.888 | 36.6 | [13] | 1.037 | 154 | 20 | [11] |
| 0.300 | 0.817 | 6.22 | [18] | 1.300 | 168 | 17 | [9] |
| 0.350 | 0.827 | 8.4 | [15] | 1.430 | 177 | 18 | [9] |
| 0.352 | 0.83 | 14 | [11] | 1.530 | 140 | 14 | [9] |
| 0.400 | 0.95 | 17 | [9] | 1.750 | 202 | 40 | [9] |
| 0.430 | 0.8645 | 3.68 | [16] | 2.300 | 224 | 8 | [14] |
| 0.440 | 0.826 | 7.8 | [12] | 2.340 | 222 | 7 | [19] |
| 0.480 | 0.97 | 62 | [10] | 2.360 | 226 | 8 | [20] |

Table 6. Values of the Hubble parameter \( H(z) \) with errors \( \sigma_H \) from refs. [9–20].
A Measurements of $d_z(z)$, $A(z)$ and $H(z)$

Measurements of $d_z(z)$ and $A(z)$ in ref. [21] are not independent, they are described with the following elements of covariance matrices $C^{-1}_d = ||c_{d_{ij}}||$ and $C^{-1}_A = ||c_{A_{ij}}||$ in eq. (3.7) [5, 21]:

- $c_{d_{44}} = 24532.1$, $c_{d_{46}} = -25137.7$, $c_{d_{47}} = 12099.1$,
- $c_{d_{66}} = 134598.4$, $c_{d_{67}} = -64783.9$, $c_{d_{77}} = 128837.6$,
- $c_{A_{44}} = 1040.3$, $c_{A_{46}} = -807.5$, $c_{A_{47}} = 336.8$,
- $c_{A_{66}} = 3720.3$, $c_{A_{67}} = -1551.9$, $c_{A_{77}} = 2914.9$.

These matrices are symmetric ones, their remaining elements are $c_{ii} = 1/\sigma_i^2$, $c_{ij} = 0$, $i \neq j$.

Acknowledgments

G.S. would like to acknowledge the support of the Ministry of education and science of Russia (grant No. 1.476.2011).

References

[1] Supernova Search Team collaboration, A.G. Riess et al., Observational evidence from supernovae for an accelerating universe and a cosmological constant, Astron. J. 116 (1998) 1009 [astro-ph/9805201] [SPIRE].

[2] Supernova Cosmology Project collaboration, S. Perlmutter et al., Measurements of Omega and Lambda from 42 high redshift supernovae, Astrophys. J. 517 (1999) 565 [astro-ph/9812133] [SPIRE].

[3] N. Suzuki, D. Rubin, C. Lidman, G. Aldering, R. Amanullah et al., The Hubble Space Telescope Cluster Supernova Survey: V. Improving the Dark Energy Constraints Above z > 1 and Building an Early-Type-Hosted Supernova Sample, Astrophys. J. 746 (2012) 85 [arXiv:1105.3470] [SPIRE].

[4] D.H. Weinberg, M.J. Mortonson, D.J. Eisenstein, C. Hirata, A.G. Riess et al., Observational Probes of Cosmic Acceleration, Phys. Rept. 530 (2013) 87 [arXiv:1201.2434] [SPIRE].

[5] WMAP collaboration, G. Hinshaw et al., Nine-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Cosmological Parameter Results, Astrophys. J. Suppl. 208 (2013) 19 [arXiv:1212.5226] [SPIRE].

[6] SDSS collaboration, D.J. Eisenstein et al., Detection of the baryon acoustic peak in the large-scale correlation function of SDSS luminous red galaxies, Astrophys. J. 633 (2005) 560 [astro-ph/0501171] [SPIRE].

[7] B.A. Reid, W.J. Percival, D.J. Eisenstein, L. Verde, D.N. Spergel et al., Cosmological Constraints from the Clustering of the Sloan Digital Sky Survey DR7 Luminous Red Galaxies, Mon. Not. Roy. Astron. Soc. 404 (2010) 60 [arXiv:0907.1659] [SPIRE].

[8] PLANCK collaboration, P.A.R. Ade et al., Planck 2013 results. XVI. Cosmological parameters, Astron. Astrophys. (2014) [arXiv:1303.5076] [SPIRE].

[9] J. Simon, L. Verde and R. Jimenez, Constraints on the redshift dependence of the dark energy potential, Phys. Rev. D 71 (2005) 123001 [astro-ph/0412269] [SPIRE].

[10] D. Stern, R. Jimenez, L. Verde, M. Kamionkowski and S.A. Stanford, Cosmic Chronometers: Constraining the Equation of State of Dark Energy. I: $H(z)$ Measurements, JCAP 02 (2010) 008 [arXiv:0907.3149] [SPIRE].
The WiggleZ Dark Energy

C. Blake, S. Brough, M. Colless, C. Contreras, W. Couch et al., Four new observational constraints on the expansion rate of the Universe up to $z \sim 1.1$ from the spectroscopic evolution of cosmic chronometers, *JCAP* 08 (2012) 006 [arXiv:1201.3609] [SPIRE].

C. Blake, M. Colless, C. Contreras, W. Couch et al., The WiggleZ Dark Energy Survey: Joint measurements of the expansion and growth history at $z < 1$, *Mon. Not. Roy. Astron. Soc.* 425 (2012) 405 [arXiv:1204.3674] [SPIRE].

C. Zhang, H. Zhang, S. Yuan, T.-J. Zhang and Y.-C. Sun, Four new observational $H(z)$ data from luminous red galaxies in the Sloan Digital Sky Survey data release seven, *Res. Astron. Astrophys.* 14 (2014) 1221 [arXiv:1207.4541] [SPIRE].

N.G. Busca, T. Delubac, J. Rich, S. Bailey, A. Font-Ribera et al., Baryon Acoustic Oscillations in the Ly-$\alpha$ forest of BOSS quasars, *Astron. Astrophys.* 552 (2013) A96 [arXiv:1211.2616] [SPIRE].

E.J. Copeland, M. Sami and S. Tsujikawa, Dynamics of dark energy

M. Kunz, The phenomenological approach to modeling the dark energy, *Comptes Rendus Physique* 13 (2012) 539 [arXiv:1204.5482] [SPIRE].

E. Gaztanaga, A. Cabre and L. Hui, Clustering of Luminous Red Galaxies IV: Baryon Acoustic Peak in the Line-of-Sight Direction and a Direct Measurement of $H(z)$, *Mon. Not. Roy. Astron. Soc.* 399 (2009) 1663 [arXiv:0807.3551] [SPIRE].

L. Anderson, E. Aubourg, S. Bailey, F. Beutler, A.S. Bolton et al., The clustering of galaxies in the SDSS-III Baryon Oscillation Spectroscopic Survey: Measuring $D_A$ and $H$ at $z=0.57$ from the Baryon Acoustic Peak in the Data Release 9 Spectroscopic Galaxy Sample, *Mon. Not. Roy. Astron. Soc.* 439 (2014) 83 [arXiv:1303.4486] [SPIRE].

A. Oka, S. Saito, T. Nishimichi, A. Taruya and K. Yamamoto, Simultaneous constraints on the growth of structure and cosmic expansion from the multipole power spectra of the SDSS DR7 LRG sample, *Mon. Not. Roy. Astron. Soc.* 439 (2014) 2515 [arXiv:1310.2820] [SPIRE].

A. Font-Ribera, D. Kirkby, N. Busca, J. Miralda-Escude, N.P. Ross et al., Quasar-Lyman $\alpha$ forest cross-correlation from BOSS DR11: Baryon Acoustic Oscillations

BOSS collaboration, T. Delubac et al., Baryon Acoustic Oscillations in the Ly$\alpha$ forest of BOSS DR11 quasars, arXiv:1404.1801 [SPIRE].

C. Blake, E. Kazin, F. Beutler, T. Davis, D. Parkinson et al., The WiggleZ Dark Energy Survey: mapping the distance-redshift relation with baryon acoustic oscillations, *Mon. Not. Roy. Astron. Soc.* 418 (2011) 1707 [arXiv:1108.2635] [SPIRE].

C.-H. Chuang, F. Prada, A.J. Cuesta, D.J. Eisenstein, E. Kazin et al., The clustering of galaxies in the SDSS-III Baryon Oscillation Spectroscopic Survey: single-probe measurements and the strong power of normalized growth rate on constraining dark energy, *Mon. Not. Roy. Astron. Soc.* 433 (2013) 3559 [arXiv:1303.4486] [SPIRE].

T. Clifton, P.G. Ferreira, A. Padilla and C. Skordis, Modified Gravity and Cosmology, *Phys. Rept.* 513 (2012) 1 [arXiv:1106.2476] [SPIRE].

K. Bamba, S. Capozziello, S. Nojiiri and S.D. Odintsov, Dark energy cosmology: the equivalent description via different theoretical models and cosmography tests, *Astrophys. Space Sci.* 342 (2012) 155 [arXiv:1205.3421] [SPIRE].

E.J. Copeland, M. Sami and S. Tsujikawa, Dynamics of dark energy, *Int. J. Mod. Phys.* D 15 (2006) 1753 [hep-th/0603057] [SPIRE].

M. Kunz, The phenomenological approach to modeling the dark energy, *Comptes Rendus Physique* 13 (2012) 539 [arXiv:1204.5482] [SPIRE].
[27] T.P. Sotiriou and V. Faraoni, f(R) Theories Of Gravity, Rev. Mod. Phys. 82 (2010) 451 [arXiv:0805.1726] [SPIRE].

[28] S. Nojiri and S.D. Odintsov, Unified cosmic history in modified gravity: from F(R) theory to Lorentz non-invariant models, Phys. Rept. 505 (2011) 59 [arXiv:1011.0544] [SPIRE].

[29] R.R. Caldwell, R. Dave and P.J. Steinhardt, Cosmological imprint of an energy component with general equation of state, Phys. Rev. Lett. 80 (1998) 1582 [astro-ph/9708069] [SPIRE].

[30] J. Khoury and A. Weltman, Chameleon cosmology, Phys. Rev. D 69 (2004) 044026 [astro-ph/0309411] [SPIRE].

[31] A.Y. Kamenshchik, U. Moschella and V. Pasquier, An Alternative to quintessence, Phys. Lett. B 511 (2001) 265 [gr-qc/0103004] [SPIRE].

[32] M.C. Bento, O. Bertolami and A.A. Sen, Generalized Chaplygin gas, accelerated expansion and dark energy matter unification, Phys. Rev. D 66 (2002) 043507 [gr-qc/0202064] [SPIRE].

[33] M. Makler, S. Quinet de Oliveira and I. Waga, Constraints on the generalized Chaplygin gas from supernovae observations, Phys. Lett. B 555 (2003) 1 [astro-ph/0209486] [SPIRE].

[34] J. Lu, Y. Gui and L.-X. Xu, Observational constraint on generalized Chaplygin gas model, Eur. Phys. J. C 63 (2009) 349 [arXiv:1004.3365] [SPIRE].

[35] L. Xu, J. Lu and Y. Wang, Revisiting Generalized Chaplygin Gas as a Unified Dark Matter and Dark Energy Model, Eur. Phys. J. C 72 (2012) 1883 [arXiv:1204.4798] [SPIRE].

[36] J. Lu, L. Xu, Y. Wu and M. Liu, Combined constraints on modified Chaplygin gas model from cosmological observed data: Markov Chain Monte Carlo approach, Gen. Rel. Grav. 43 (2011) 819 [arXiv:1105.1870] [SPIRE].

[37] B.C. Paul and P. Thakur, Observational constraints on modified Chaplygin gas from cosmic growth, JCAP 11 (2013) 052 [arXiv:1306.4808] [SPIRE].

[38] N. Mohammadi, Dynamical compactification, standard cosmology and the accelerating universe, Phys. Rev. D 65 (2002) 104018 [hep-th/0202119] [SPIRE].

[39] F. Darabi, Accelerating universe and dynamical compactification of extra dimensions, Class. Quant. Grav. 20 (2003) 3385 [gr-qc/0301075] [SPIRE].

[40] T. Bringmann, M. Eriksson and M. Gustafsson, Cosmological evolution of homogeneous universal extra dimensions, Phys. Rev. D 68 (2003) 063516 [astro-ph/0303497] [SPIRE].

[41] D. Panigrahi, Y.Z. Zhang and S. Chatterjee, Accelerating universe as window for extra dimensions, Int. J. Mod. Phys. A 21 (2006) 6491 [gr-qc/0604079] [SPIRE].

[42] C.A. Middleton and E. Stanley, Anisotropic evolution of 5D Friedmann-Robertson-Walker spacetime, Phys. Rev. D 84 (2011) 085013 [arXiv:1107.1828] [SPIRE].

[43] H. Farajollahi and H. Amiri, A 5D noncompact Kaluza-Klein cosmology in the presence of Null perfect fluid, Int. J. Mod. Phys. D 19 (2010) 1823 [arXiv:1005.3140] [SPIRE].

[44] I. Pahwa, D. Choudhury and T.R. Seshadri, Late-time acceleration in Higher Dimensional Cosmology, JCAP 09 (2011) 015 [arXiv:1104.1925] [SPIRE].

[45] O.A. Grigorieva and G.S. Sharov, Multidimensional gravitational model with anisotropic pressure, Int. J. Mod. Phys. D 22 (2013) 1350075 [arXiv:1211.4992] [SPIRE].
[48] A.G. Riess, L. Macri, S. Casertano, H. Lampeitl, H.C. Ferguson et al., A 3% Solution: Determination of the Hubble Constant with the Hubble Space Telescope and Wide Field Camera 3, Astrophys. J. 730 (2011) 119 [Erratum ibid. 732 (2011) 129] [arXiv:1103.2976] [inSPIRE].

[49] Supernova Search Team collaboration, J.L. Tonry et al., Cosmological results from high-z supernovae, Astrophys. J. 594 (2003) 1 [astro-ph/0305008] [inSPIRE].

[50] Supernova Cosmology Project collaboration, R.A. Knop et al., New constraints on Ω_M, Ω_Λ and w from an independent set of eleven high-redshift supernovae observed with HST, Astrophys. J. 598 (2003) 102 [astro-ph/0309368] [inSPIRE].

[51] Supernova Cosmology Project collaboration, M. Kowalski et al., Improved Cosmological Constraints from New, Old and Combined Supernova Datasets, Astrophys. J. 686 (2008) 749 [arXiv:0804.4142] [inSPIRE].

[52] K. Shi, Y. Huang and T. Lu, A comprehensive comparison of cosmological models from latest observational data, Mon. Not. Roy. Astron. Soc. 426 (2012) 2452 [arXiv:1207.5875] [inSPIRE].

[53] O. Farooq, D. Mania and B. Ratra, Hubble parameter measurement constraints on dark energy, Astrophys. J. 764 (2013) 138 [arXiv:1211.4253] [inSPIRE].

[54] O. Farooq and B. Ratra, Hubble parameter measurement constraints on the cosmological deceleration-acceleration transition redshift, Astrophys. J. 766 (2013) L7 [arXiv:1301.5243] [inSPIRE].

[55] M.O. Farooq, Observational constraints on dark energy cosmological model parameters, Ph.D. Thesis, Kansas State University, [arXiv:1309.3710] [inSPIRE].

[56] I. Gott, J. Richard, M.S. Vogeley, S. Podariu and B. Ratra, Median statistics, H(0) and the accelerating universe, Astrophys. J. 549 (2001) 1 [astro-ph/0006103] [inSPIRE].