Super-A-polynomials of Twist Knots

joint work with Ramadevi and Zodinmawia
to appear soon

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Polynomial knot invariants

- **Alexander polynomials** $\Delta(K; q)$
  \[ \Delta \begin{tikzpicture}[baseline=0.2cm] \draw (-0.15,0) arc (180:360:0.15); \draw (0.15,0) arc (0:180:0.15); \end{tikzpicture} - \Delta \begin{tikzpicture}[baseline=0.2cm] \draw (-0.15,0) arc (180:0:0.15); \draw (0.15,0) arc (0:180:0.15); \end{tikzpicture} = \left(q^{1/2} - q^{-1/2}\right) \Delta \begin{tikzpicture}[baseline=0.2cm] \draw (-0.2,0.2) arc (180:360:0.2); \draw (0.2,0.2) arc (0:180:0.2); \end{tikzpicture} \]

  For trefoil, $\Delta \begin{tikzpicture}[baseline=0.2cm] \draw (-0.15,0) arc (180:360:0.15); \draw (0.15,0) arc (0:180:0.15); \draw (0,0) circle (0.15); \end{tikzpicture} = q - 1 + q^{-1}$

- **Jones polynomials** $J(K; q)$
  \[ q^{1/2} J \begin{tikzpicture}[baseline=0.2cm] \draw (-0.15,0) arc (180:360:0.15); \draw (0.15,0) arc (0:180:0.15); \end{tikzpicture} - q^{-1/2} J \begin{tikzpicture}[baseline=0.2cm] \draw (-0.15,0) arc (180:0:0.15); \draw (0.15,0) arc (0:180:0.15); \end{tikzpicture} = \left(q^{1/2} - q^{-1/2}\right) J \begin{tikzpicture}[baseline=0.2cm] \draw (-0.2,0.2) arc (180:360:0.2); \draw (0.2,0.2) arc (0:180:0.2); \end{tikzpicture} \]

  For trefoil, $J \begin{tikzpicture}[baseline=0.2cm] \draw (-0.15,0) arc (180:360:0.15); \draw (0.15,0) arc (0:180:0.15); \draw (0,0) circle (0.15); \end{tikzpicture} = q + q^3 - q^4$

- **HOMFLY polynomials** $P(K; a, q)$
  \[ a^{1/2} P \begin{tikzpicture}[baseline=0.2cm] \draw (-0.15,0) arc (180:360:0.15); \draw (0.15,0) arc (0:180:0.15); \end{tikzpicture} - a^{-1/2} \Delta \begin{tikzpicture}[baseline=0.2cm] \draw (-0.15,0) arc (180:0:0.15); \draw (0.15,0) arc (0:180:0.15); \end{tikzpicture} = \left(q^{1/2} - q^{-1/2}\right) P \begin{tikzpicture}[baseline=0.2cm] \draw (-0.2,0.2) arc (180:360:0.2); \draw (0.2,0.2) arc (0:180:0.2); \end{tikzpicture} \]

  For trefoil, $P \begin{tikzpicture}[baseline=0.2cm] \draw (-0.15,0) arc (180:360:0.15); \draw (0.15,0) arc (0:180:0.15); \draw (0,0) circle (0.15); \end{tikzpicture} = aq^{-1} + aq - a^2$
Categoifications

- the Poincaré polynomial of colored Khovanov homology $\mathcal{H}_{i,j}^{sI_2,R}$ [Khovanov ’00]

$$Kh_R(K; q, t) = \sum_{i,j} t^i q^j \dim \mathcal{H}_{i,j}^{sI_2,R}(K),$$

- $q$-graded Euler characteristic gives colored Jones polynomial:

$$J_R(K; q) = Kh(q, t = -1) = \sum_{i,j} (-1)^j q^i \dim \mathcal{H}_{i,j}^{sI_2,R}(K).$$

- The Poincaré polynomial of the Khovanov-Rozansky homology [Khovanov-Rozansky ’04]

$$Kh_{RR}(K; q, t) = \sum_{i,j} t^i q^j \dim \mathcal{H}_{i,j}^{sI_2,R}(K).$$

is related to the colored HOMFLY polynomial via

$$Kh_{RR}(K; q, t = -1) = P_R(K; a = q^N, q)$$
Colored superpolynomials

- the Poincaré polynomial of triply-graded homology $\mathcal{H}_{i,j,k}^{s_1^2, R}$ [Dunfield-Gukov-Rasmussen ‘05]

$$\mathcal{P}_R(K; a, q, t) = \sum_{i,j,k} a^i q^j t^k \dim \mathcal{H}_{i,j,k}^R(K).$$

- The $(a, q)$-graded Euler characteristic of the triply-graded homology theory is equivalent to the colored HOMFLY polynomial

$$P_R(K; a, q) = \sum_{i,j,k} (-1)^k a^i q^j \dim \mathcal{H}_{i,j,k}^R(K).$$
The correspondence between the twist number and the knots in Rolfsen’s table

Colored Jones polynomials and $A$-polynomials are known
Quantum $A$-polynomials were already computed for $p = -14 \cdots 15$. 
Colored Jones polynomials of twist knots

The double-sum expressions [Habiro '03][Masbaum '03]

\[
J_n(K_p; q) = \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} q^k (q^{1-n}; q)_k (q^{n+1}; q)_k \\
\times (-1)^\ell q^{\ell(\ell+1)p+\ell(\ell-1)/2} (1 - q^{2\ell+1})^{(q; q)_{k+\ell+1}(q; q)_{k-\ell}}.
\]

The multi-sum expressions

\[
J_n(K_{p>0}; q) = \sum_{s_p \geq \cdots \geq s_1 \geq 0} q^{s_p} (q^{1-n}; q)^{s_p} (q^{1+n}; q)^{s_p} \prod_{i=1}^{p-1} q^{s_i(s_i+1)} \left[ \begin{array}{c} s_i+1 \\ s_i \end{array} \right]_q
\]

\[
J_n(K_{p<0}; q) = \sum_{s_{|p|} \geq \cdots \geq s_1 \geq 0} (-1)^{s_{|p|}} q^{-\frac{s_{|p|}(s_{|p|}+1)}{2}} (q^{1-n}; q)^{s_{|p|}} (q^{1+n}; q)^{s_{|p|}} \\
\times \prod_{i=1}^{|p|-1} q^{-s_i(s_i+1)} \left[ \begin{array}{c} s_i+1 \\ s_i \end{array} \right]_q
\]
Colored superpolynomials of trefoil and figure-8 are known
[Fuji Gukov Sulkowski ’12][Itoyama, Mironov, Morozov2 ’12]

\[ P_n(3_1; a, q, t) = (-t)^{-n+1} \sum_{k=0}^{\infty} q^k \frac{(-atq^{-1}; q)_k}{(q; q)_k} (q^{1-n}; q)_k (-at^3 q^{n-1}; q)_k, \]

\[ P_n(4_1; a, q, t) = \sum_{k=0}^{\infty} (-1)^k a^{-k} t^{-2k} q^{-k(k-3)/2} \frac{(-atq^{-1}; q)_k}{(q; q)_k} (q^{1-n}; q)_k (-at^3 q^{n-1}; q)_k. \]

Colored superpolynomials \( P_n(a, q, t) \) for \( 5_2 \) and \( 6_1 \) are also known up to \( n = 3 \) [Gukov Stosic ’11]

One can do educated guess on colored superpolynomials of twist knots
Colored superpolynomials of twist knots

- The double-sum expressions

\[ P_n(K_p; a, q, t) = \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} q^k \left(\frac{-atq^{-1}; q}{(q; q)_k}\right) (q^{1-n}; q)_k (-at^3 q^{n-1}; q)_k \]
\[ \times (-1)^\ell \ t^{2\ell} q^{(p+1/2)\ell(\ell-1)} \frac{1 - at^2 q^{2\ell-1}}{(at^2 q^{\ell-1}; q)_{k+1}} \left[ \begin{array}{c} k \\ \ell \end{array} \right]_q. \]

- The multi-sum expressions

\[ P_n(K_p > 0; a, q, t) = (-t)^{-n+1} \sum_{s_p \geq \cdots \geq s_1 \geq 0} q^{sp} \left(\frac{-atq^{-1}; q}{(q; q)_{sp}}\right) (q^{1-n}; q)_{sp} (-at^3 q^{n-1}; q)_{sp} \]
\[ \times \prod_{i=1}^{p-1} (at^2)^{s_i} q^{s_i(s_i-1)} \left[ \begin{array}{c} s_i+1 \\ s_i \end{array} \right]_q. \]

\[ P_n(K_p < 0; a, q, t) \]
\[ = \sum_{s_{|p|} \geq \cdots \geq s_1 \geq 0} (-1)^k a^{-s_{|p|}} t^{-2s_{|p|}} q^{-s_{|p|}(s_{|p|}-3)/2} \left(\frac{-atq^{-1}; q}{(q; q)_{s_{|p|}}}\right) (q^{1-n}; q)_{s_{|p|}} (-at^3 q^{n-1}; q)_{s_{|p|}} \]
\[ \times \prod_{i=1}^{p-1} (at^2)^{-s_i} q^{-s_i(s_i+1-1)} \left[ \begin{array}{c} s_i+1 \\ s_i \end{array} \right] q. \]
Colored Jones polynomials of twist knots

- The double-sum expressions [Habiro '03][Masbaum '03]

\[ J_n(K_p; q) = \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} q^k (q^{1-n}; q)_k (q^{n+1}; q)_k \]

\[ \times (\ell) (q^{\ell+1})^{p+\ell-1}/2(1 - q^{2\ell+1}) \left( \frac{q; q)_k}{(q; q)_{k+\ell+1}(q; q)_{k-\ell}} \right). \]

- The multi-sum expressions

\[ J_n(K_{p>0}; q) = \sum_{s_p \geq \cdots \geq s_1 \geq 0} q^{s_p} (q^{1-n}; q)_{s_p} (q^{1+n}; q)_{s_p} \prod_{i=1}^{p-1} q^{s_i(s_i+1)} \left[ \begin{array}{c} s_i+1 \\ s_i \end{array} \right]_{q} \]

\[ J_n(K_{p<0}; q) = \sum_{s_{|p|} \geq \cdots \geq s_1 \geq 0} (-1)^{s_{|p|}} q^{-s_{|p|}(s_{|p|}+1)/2} (q^{1-n}; q)_{s_{|p|}} (q^{1+n}; q)_{s_{|p|}} \]

\[ \times \prod_{i=1}^{\left| p \right|-1} q^{-s_i(s_i+1)} \left[ \begin{array}{c} s_i+1 \\ s_i \end{array} \right]_{q} \]
Colored superpolynomials of twist knots

- The double-sum expressions

\[ \mathcal{P}_n(K_p; a, q, t) = \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} q^k \frac{(-atq^{-1}; q)_k}{(q; q)_k} (q^{1-n}; q)_k (-at^3 q^{n-1}; q)_k \]

\[ \times (-1)^{\ell} a^{\ell} t^{2\ell} q^{(p+1/2)\ell(\ell-1)} \frac{1 - at^2 q^{2\ell-1}}{(at^2 q^{\ell-1}; q)_{k+1}} \left[ \begin{array}{c} k \\ \ell \end{array} \right]_q. \]

- The multi-sum expressions

\[ \mathcal{P}_n(K_{p>0}; a, q, t) = (-t)^{-n+1} \sum_{s_p \geq \cdots \geq s_1 \geq 0} q^{s_p} \frac{(-atq^{-1}; q)_{s_p}}{(q; q)_{s_p}} (q^{1-n}; q)_{s_p} (-at^3 q^{n-1}; q)_{s_p} \]

\[ \times \prod_{i=1}^{p-1} (at^2)^{s_i} q^{s_i(s_i-1)} \left[ \begin{array}{c} s_i+1 \\ s_i \end{array} \right]_q. \]

\[ \mathcal{P}_n(K_{p<0}; a, q, t) = \sum_{s_{|p|} \geq \cdots \geq s_1 \geq 0} (-1)^k a^{-s_{|p|}} t^{-2s_{|p|}} q^{-s_{|p|}(s_{|p|}-3)/2} \frac{(-atq^{-1}; q)_{s_{|p|}}}{(q; q)_{s_{|p|}}} (q^{1-n}; q)_{s_{|p|}} (-at^3 q^{n-1}; q)_{s_{|p|}} \]

\[ \times \prod_{i=1}^{\lfloor p \rfloor-1} (at^2)^{-s_i} q^{-s_i(s_i+1-1)} \left[ \begin{array}{c} s_i+1 \\ s_i \end{array} \right]_q. \]
Checks

- For $a = q^2$ and $t = -1$, the above formulae reduce to the colored Jones polynomials.
- For $t = -1$, they reduce to the colored HOMFLY polynomials. We checked they agree with the colored HOMFLY polynomials computed by $SU(N)$ Chern-Simons theory up to 10 crossings.
- The colored HOMFLY polynomials can be reformulated into the Ooguri-Vafa polynomials. We checked that the Ooguri-Vafa polynomials have integer coefficients up to specific factors.
- We checked that the special polynomials which are the limits $q \to 1$ of the colored HOMFLY polynomials have the property,

$$\lim_{q \to 1} P_n(K_p; a, q) = \left[ \lim_{q \to 1} P_2(K_p; a, q) \right]^{n-1}.$$
Cancelling differentials and Rasmussen s-invariants

- The action of the differential $d_1$ on the colored superpolynomials

\[
P_{n+1}(K_p>0; a, q, t) = a^n q^{-n} + (1 + a^{-1} qt^{-1}) Q_{n+1}^{s_1}(K_p>0; a, q, t),
\]
\[
P_{n+1}(K_p<0; a, q, t) = 1 + (1 + a^{-1} qt^{-1}) Q_{n+1}^{s_1}(K_p<0; a, q, t),
\]

- The action of the differential $d_{-n}$ on the colored superpolynomials

\[
P_{n+1}(K_p>0; a, q, t) = a^n q^n t^{2n} + (1 + a^{-1} q^{-n} t^{-3}) Q_{n+1}(K_p>0; a, q, t),
\]
\[
P_{n+1}(K_p<0; a, q, t) = 1 + (1 + a^{-1} q^{-n} t^{-3}) Q_{n+1}(K_p<0; a, q, t).
\]

The exponents of the remaining monomials are consistent with the Rassmusen’s s-invariant of the twist knots, $s(K_p>0) = 1$ and $s(K_p<0) = 0$ [Rasmussen '04]

\[
\deg \left( \mathcal{H}_{*,*,*}^n(K), d_1 \right) = \left( n s(K), -n s(K), 0 \right),
\]
\[
\deg \left( \mathcal{H}_{*,*,*}^n(K), d_{-n} \right) = \left( n s(K), n^2 s(K), 2n s(K) \right).
\]
Volume conjecture and $A$-polynomials

- The volume conjecture relates “quantum invariants” of knots to “classical” 3d topology [Kashaev '99][Murakami '00]

$$\lim_{n \to \infty} \frac{2\pi}{n} \log \left| J_n(K; q = e^{\frac{2\pi i}{n}}) \right| = \text{Vol}(S^3 \setminus K).$$

- The relation b/w volume conjecture and $A$-polynomial [Gukov '03]

$$\log y = -x \frac{d}{dx} \lim_{n,k \to \infty} e^{i\pi n/k = x} \frac{1}{k} \log J_n(K; q = e^{\frac{2\pi i}{k}}),$$

gives the zero locus of the $A$-polynomial $A(K; x, y)$ of the knot $K$.

- $A$-polynomial $A(K; x, y)$ is a character variety of $SL(2, \mathbb{C})$-representation of the fundamental group of the knot complement.

$$\mathcal{M}_{\text{flat}}(SL(2, \mathbb{C}), T^2) \supset \mathcal{M}_{\text{flat}}(SL(2, \mathbb{C}), S^3 \setminus K) = \{(x, y) \in \mathbb{C}^\times \times \mathbb{C}^\times | A(K; x, y) = 0\}.$$
AJ conjecture

- Quantum version of the volume conjecture → AJ conjecture \cite{Gukov '03, Garoufalidis '03}
  \[ \hat{A}(K; \hat{x}, \hat{y}, q) J_n(K; q) = 0 \]
  where action of \( \hat{x} \) and \( \hat{y} \) on the set of colored Jones polynomials as
  \[ \hat{x} J_n(K; q^n) = q^n J_n(K; q^n) , \quad \hat{y} J_n(K; q) = J_{n+1}(K; q) . \]

- Find difference equations of colored Jones polynomials
  \[ a_k J_{n+k}(K; q) + \ldots + a_1 J_{n+1}(K; q) + a_0 J_n(K; q) = 0 \]
  where \( a_k = a_k(K; \hat{x}, q) \) and
  \[ \hat{A}(K; \hat{x}, \hat{y}; q) = \sum a_i(K; \hat{x}, q) \hat{y}^i \]

- Taking the classical limit \( q = e^{\frac{\hbar}{i}} \to 1 \), quantum (non-commutative) \( A \)-polynomials reduces to ordinary \( A \)-polynomials
  \[ \hat{A}(K; \hat{x}, \hat{y}; q) \to A(K; x, y) \quad \text{as} \quad q \to 1 \]
**Super-A-polynomials**

- Refinement of quantum and classical A-polynomials [Fuji Gukov Sulkowski '12]

| Quantum operator | provides recursion for | classical limit |
|------------------|------------------------|-----------------|
| \( \hat{A}^{\text{super}}(\hat{x}, \hat{y}; a, q, t) \) | colored superpolynomial | \( A^{\text{super}}(x, y; a, t) \) |
| \( \hat{A}^{\text{ref}}(\hat{x}, \hat{y}; q, t) \) | colored Khovanov-Rozansky homology | \( A^{\text{ref}}(x, y, t) \) |
| \( \hat{A}^{\text{Q-def}}(\hat{x}, \hat{y}; a, q) \) | colored HOMFLY | \( A^{\text{Q-def}}(x, y; a) \) |
| \( \hat{A}(\hat{x}, \hat{y}; q) \) | colored Jones | \( A(x, y) \) |

- \( a = q^2 \)
- \( t = -1 \)
- \( q = 1 \)
Classical super-A-polynomials

- Refinement of volume conjecture [Fuji Gukov Sulkowski '12]

\[
\log y = -x \frac{d}{dx} \lim_{n,k \to \infty, e^{i\pi n/k} = x} \frac{1}{k} \log \mathcal{P}_n(K; a, q = e^{\frac{2\pi i}{k}}, t),
\]

gives the zero locus of the super-A-polynomial \( A(K; x, y; a, q, t) \) of the knot \( K \).

| Knot | \( A^{\text{super}}(K; x, y; a, t) \) |
|------|----------------------------------|
| 5_2  | \((1 + at^3 x)^3 y^4 \) |
|      | \(-a(1 + at^3 x)^2 (2 - x + tx - 2t^2 x + 3t^2 x^2 + at^2 x^2 + 4at^3 x^2 - 2at^3 x^3 + 2at^4 x^3 + 2at^5 x^3 - at^5 x^4 + 2a^2 t^5 x^4 + 2a^2 t^6 x^4 - a^2 t^7 x^5 + a^2 t^8 x^6) y^3 \) |
|      | \(-a^2 (x - 1) (1 + at^3 x) (1 + tx - 2t^2 x + 2t^2 x^2 - 2t^3 x^2 + 4at^3 x^2 + t^4 x^2 - 3t^4 x^3 + at^4 x^3 - 2at^5 x^3 + 4at^5 x^4 - 4at^6 x^4 + 6a^2 t^6 x^4 - 4at^7 x^4 + 3at^7 x^5 - a^2 t^7 x^5 + 2a^2 t^8 x^5 + 2a^2 t^8 x^6 - 2a^2 t^9 x^6 + 4a^3 t^9 x^6 + a^2 t^{10} x^6 - a^3 t^{10} x^7 + 2a^3 t^{11} x^7 + a^4 t^{12} x^8) y^2 \) |
|      | \(+a^3 t^3 x^2 (x - 1)^2 (1 + tx - t^2 x - t^3 x^2 + 2at^3 x^2 + 2at^4 x^2 + 2at^4 x^3 - 2at^5 x^3 - 2at^6 x^3 + 3at^6 x^4 + a^2 t^6 x^4 + 4a^2 t^7 x^4 + a^2 t^7 x^5 - a^2 t^8 x^5 + 2a^2 t^9 x^5 + 2a^3 t^{10} x^6) y \) |
|      | \(-a^5 t^{11} x^7 (x - 1)^3 \) |
Quantum super-$A$-polynomials

- Refinement of AJ conjecture
  \[ \hat{A}^{\text{super}}(K; \hat{x}, \hat{y}; a, q, t) P_n(K; a, q, t) = 0 \]
  where the operators $\hat{x}$ and $\hat{y}$ acts on the set of colored superpolynomials as
  \[ \hat{x}P_n(K; a, q, t) = q^n P_n(K; a, q, t) , \quad \hat{y}P_n(K; q) = P_{n+1}(K; a, q, t) . \]

- Find difference equations of colored superpolynomials
  \[ a_k P_{n+k}(K; q) + \ldots + a_1 P_{n+1}(K; q) + a_0 P_n(K; q) = 0 \]
  where $a_k = a_k(K; \hat{x}; a, q, t)$ and
  \[ \hat{A}^{\text{super}}(K; \hat{x}, \hat{y}; a, q, t) = \sum a_i(K; \hat{x}; a, q, t) \hat{y}^i \]

- Taking the classical limit $q = e^\hbar \to 1$, quantum (non-commutative) super-$A$-polynomials reduces to classical super-$A$-polynomials
  \[ \hat{A}^{\text{super}}(K; \hat{x}, \hat{y}; a, q, t) \to A^{\text{super}}(K; x, y; a, q, t) \quad \text{as} \quad q \to 1 \]
Quantum super-\(A\)-polynomials

| Knot | \(\hat{A}^{\text{super}}(K; \hat{x}, \hat{y}; a, q, t)\) |
|------|--------------------------------------------------|
| 52   | \(q^6 t^4 \left(1 + at^3 \hat{x}\right) \left(1 + aqt^3 \hat{x}\right) \left(1 + aq^2 t^3 \hat{x}\right) \left(1 + at^3 \hat{x}^2\right) \left(q + at^3 \hat{x}^2\right) \left(1 + aqt^3 \hat{x}^2\right) \hat{y}^4 - aq^5 t^4 (1 + at^3 \hat{x})(1 + aqt^3 \hat{x})(1 + at^3 \hat{x}^2)(q + at^3 \hat{x}^2)(1 + aq^4 t^3 \hat{x}^2)(1 + q - q^3 \hat{x} - q^2 t^2 \hat{x} - q^3 t^2 \hat{x} + q^4 t^2 \hat{x}^2 + aq^4 t^2 \hat{x}^2 + q^5 t^2 \hat{x}^2 + q^6 t^2 \hat{x}^2 + aq^3 t^2 \hat{x}^2 + aq^5 t^2 \hat{x}^2 + aq^6 t^2 \hat{x}^2 - aq^4 t^3 \hat{x}^3 - aq^8 t^3 \hat{x}^3 + aq^4 t^4 \hat{x}^3 + aq^5 t^4 \hat{x}^3 + aq^6 t^4 \hat{x}^3 + aq^8 t^4 \hat{x}^3 + aq^9 t^4 \hat{x}^3 + aq^{10} t^4 \hat{x}^3) \hat{y}^3 - a^2 q^5 t^4 (-1 + q^2 \hat{x})(1 + at^3 \hat{x})(q + at^3 \hat{x}^2)(1 + aq^2 t^3 \hat{x}^2)(1 + aq^5 t^3 \hat{x}^2)(1 + q^2 t^2 \hat{x} - q^2 t^2 \hat{x} - q^3 t^2 \hat{x} + q^4 t^2 \hat{x}^2 + at^3 \hat{x}^2 + aq^3 t^3 \hat{x}^2 + aq^3 t^3 \hat{x}^2 - q^4 t^3 \hat{x}^2 + aq^4 t^3 \hat{x}^2 + q^3 t^4 \hat{x}^2 + aq^4 t^4 \hat{x}^3 - q^4 t^4 \hat{x}^3 - aq^4 t^5 \hat{x}^3 - q^6 t^4 \hat{x}^3 + aq^4 t^5 \hat{x}^3 - aq^5 t^5 \hat{x}^3 + aq^6 t^5 \hat{x}^3 - aq^{10} t^6 \hat{x}^3) \hat{y}^2 + a^3 q^7 t^7 \hat{x}^2 (-1 + q \hat{x})(-1 + q^2 \hat{x})(1 + at^3 \hat{x}^2)(1 + aq^4 t^3 \hat{x}^2)(q + q^2 t^2 \hat{x} - q^2 t^2 \hat{x} - aq^4 t^2 \hat{x}^2 - aq^4 t^2 \hat{x}^2 + aq^2 t^4 \hat{x}^2 + aq^4 t^4 \hat{x}^2 + aq^4 t^4 \hat{x}^2 - aq^5 t^5 \hat{x}^3 - aq^5 t^5 \hat{x}^3 - aq^5 t^5 \hat{x}^3 - aq^5 t^5 \hat{x}^3 - aq^5 t^5 \hat{x}^3 - aq^6 t^6 \hat{x}^4 + aq^6 t^6 \hat{x}^4 + aq^7 t^6 \hat{x}^5 + aq^7 t^7 \hat{x}^5 + aq^8 t^7 \hat{x}^5 + aq^{10} t^8 \hat{x}^5 + a^2 q^8 t^{15} (-1 + \hat{x}) \hat{x}^7 (-1 + q \hat{x})(-1 + q^2 \hat{x})(1 + aq^3 t^3 \hat{x}^2)(1 + aq^4 t^3 \hat{x}^2)(1 + aq^5 t^3 \hat{x}^2) |
$Q$-deformed $A$-polynomial is equivalent to augmentation polynomial of knot contact homology [Ng ’10][Aganagic Vafa ’12]

$$A^{\text{super}}\left(K_p^{>0}; x = -\mu, y = \frac{1 + \mu}{1 + U\mu} \lambda; a = U, t = -1\right)$$

$$= \frac{(-1)^{p-1}(1 + \mu)^{(2p-1)}}{1 + U\mu} \text{Aug}(K_p^{>0}; \mu, \lambda; U, V = 1),$$

$$A^{\text{super}}\left(K_p^{<0}; x = -\mu, y = \frac{1 + \mu}{1 + U\mu} \lambda; a = U, t = -1\right)$$

$$= \frac{(-1)^p(1 + \mu)^{-2p}}{1 + U\mu} \text{Aug}(K_p^{<0}; \mu, \lambda; U, V = 1),$$
Prospects and Future Directions

- Anti-symmetric representation. Mirror symmetry
- Quantum $6j$-symbolos for $U_q(\mathfrak{sl}_N)$ and their refinement
- Colored superpolynomials of other non-torus knots
- $SU(N)$ analogue of WRT invariants and one-parameter deformations of Mock modular forms.
- Refinement of colored Kaufmann polynomials
Thank you