On-Line Permutation Routing
in Partitioned Optical Passive Star Networks

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Abstract—This paper establishes the state of the art in both deterministic and randomized online permutation routing in the POPS network. Indeed, we show that any permutation can be routed online on a POPS(d,g) network either with \(O(\frac{d}{g})\) deterministic slots, or, with high probability, with \(5c[d/g] + o(d/g) + O(\log g)\) randomized slots, where constant \(c = \exp(1+e^{-1}) \approx 3.927\). When \(d = \Theta(g)\), that we claim to be the “interesting” case, the randomized algorithm is exponentially faster than any other algorithm in the literature, both deterministic and randomized ones. This is true in practice as well. Indeed, experiments show that it outperforms its rivals even starting from as small a network as a POPS(2,2), and the gap grows exponentially with the size of the network. We can also show that, under proper hypothesis, no deterministic algorithm can asymptotically match its performance.

Index Terms—Optical interconnections, partitioned optical passive star network, permutation routing.

I. INTRODUCTION

The ever-growing demand of fast interconnections in multiprocessor systems has fostered a large interest in optical technology. All-optical communication benefits from a number of good characteristics such as no opto-electronic conversion, high noise immunity, and low latency. Optical technology can provide an enormous amount of bandwidth and, most probably, will have an important role in the future of distributed and parallel computing systems.

The Partitioned Optical Passive Stars (POPS) network [1], [2], [3], [4] is a SIMD parallel architecture that uses a fast optical network composed of multiple Optical Passive Star (OPS) couplers. A \(d \times d\) OPS coupler is an all-optical passive device which is capable of receiving an optical signal from one of its \(d\) sources and broadcasting it to all of its \(d\) destinations. The number of processors of the network is denoted by \(n\), and each processor has a distinct index in \(\{0, \ldots, n-1\}\). The \(n\) processors are partitioned into \(g\) groups of \(d\) processors, \(n = dg\), in such a way that processor \(i\) belongs to group \(\text{group}(i) = [i/d]\) (see Figure 1). For each pair of groups \(a, b \in \{0, \ldots, g-1\}\), a coupler \(c(b,a)\) is introduced which has all the \(d\) processors of group \(a\) as sources and all the \(d\) processors of group \(b\) as destinations. During a computational step (also referred to as a slot), each processor \(i\) receives a single message from one of the \(g\) couplers \(c(\text{group}(i), a), a \in \{0, \ldots, g-1\}\), performs some local computations, and sends a single message to a subset of the \(g\) couplers \(c(b, \text{group}(i)), b \in \{0, \ldots, g-1\}\). The couplers are broadcast devices, so this message can be received by more than one processor in the destination groups. In agreement with the literature, in the case when multiple messages are sent to the same coupler, we assume that no message is delivered. This architecture is denoted by POPS(d,g).

One of the advantages of a POPS(d,g) network is that its diameter is one. A packet can be sent from processor \(i\) to processor \(j\), \(i \neq j\), in one slot by using coupler \(c(\text{group}(j), \text{group}(i))\). However, its bandwidth varies according to \(g\). In a POPS(n,1) network, only one packet can be sent through the single coupler per slot. On the other extreme, a POPS(1,n) network is a highly expensive, fully interconnected optical network using \(n^2\) OPS couplers. A one-to-all communication pattern can also be performed in only one slot in the following way: Processor \(i\) (the speaker) sends the packet to all the couplers \(c(a, \text{group}(i)), a \in \{0, \ldots, g-1\}\), during the same slot all the processors \(j, j \in \{0, \ldots, n-1\}\), can receive the packet through coupler \(c(\text{group}(j), \text{group}(i))\).

The POPS network has been shown to support a number of non trivial algorithms. Several common communication patterns are realized in [3]. Simulation algorithms for the ring, the mesh, and the hypercube interconnection networks can be found in [5] and [6]. Some reliability issues are analyzed in [7]. Algorithms for data sum, prefix sums, consecutive sum, adjacent sum, and several data movement operations are also described in [6] and [8]. Later, both the algorithms for hypercube simulation and prefix sums have been improved in [9]. An algorithm for matrix multiplication is provided in [10]. Moreover, [11] shows that POPS networks can be modeled by directed and complete stack graphs with loops, and uses this formalization to obtain optimal embeddings of rings and de Bruijn graphs into POPS networks.

In [8], Datta and Soundaralakshmi claim that in most practical POPS(d,g) networks it is likely that \(d > g\). We believe that they are only partly right. While it is true that systems with \(d \ll g\) are too expensive, it is also true that systems with \(d \gg g\) give too low parallelism to be worth building. We illustrate our point with an example. Consider the problem of summing \(16n\) data values on a POPS(d,g) network, \(d = g = \sqrt{n}\). This network has \(n\) processors. Therefore, the algorithm can work as follows: we input 16 data values per processor, let each processor sum its 16 data values, and finally we use the algorithm in [8] to get the overall sum. This algorithm requires 16 steps to input the data values and compute the local sums, plus \(2 \log \sqrt{n} = \log n\) slots for computing the final result. A total of \(16 + \log n\) slots. With the idea of upgrading our system, we
buy additional 15n processors and build a 16n processor POPS(d', g') network with d' = 16d = 16√n and g' = g = √n. Now, we can use just one step to input the data values, one per processor, and then use the same algorithm in [8] to get the overall sum. Unfortunately, this algorithm still requires 16 + log n slots, even though we are solving a problem of the same size using a system with 16 times more processors!

The problem is not on the data sum algorithm in [8]. Essentially the same thing happens with the prefix sums algorithm in [8], the simulations in [6], and all the other algorithms in the literature for the POPS network we know of, including the ones presented in this paper. The point is that a POPS(d, g) network can exchange g² messages at most in a slot. This is an unavoidable bottleneck for networks where d is much larger than g, resulting in the poor parallelism of these systems. Also, experience says that the case d = g is the most interesting from a “mathematical” point of view. In the past literature, the case d > g and symmetrically the case d < g are always dealt with by reducing them to the case d = g, that usually contains the “core” of the problem in its purest form. This work is not an exception to this empirical yet general rule. So, it is probably more reasonable to assume that practical POPS networks will have d = Θ(g), that is d/g, and similarly g/d, bounded by a constant.

In any case, finding good algorithms for the case d ≠ g, both d < g and d > g, is of absolute importance, since it is not clear what is the optimal tradeoff between d, g, and the cost of the network yet. Furthermore, an optimal tradeoff may not exist in general, since it probably depends on the specific problem being solved. By the way, such algorithms are often non trivial, as, for example, in [8]. Therefore, we partly accept the claim in [8] that the number of groups cannot substantially exceed the number of processors per group. So, throughout the whole paper, we will discuss our asymptotical results assuming that g grows and that d = Ω(g). Nonetheless, we will keep in mind that the “important” case is likely to be d = Θ(g).

Here, we consider the permutation routing problem: Each of the n processors of the POPS network has a packet that is to be sent to its destination of the packet it stores. This is a fundamental problem in parallel computing and interconnection networks, and the literature on this topic is vast. As an excellent starting point, the reader can see [12]. On the POPS network, this problem has been studied in two different versions: the offline and the online permutation routing problem. In the former, the permutation to be routed is globally known in the network. Therefore, every processor can pre-compute the routing of its packet taking advantage of this information. This version of the problem has been implicitly studied, for particular permutations, in all the simulation algorithms we reviewed above. Later, most of these results have been unified by proving that any permutation can optimally be routed off-line in one slot, when d = 1, and 2[d/g] slots, when d > 1 [13].

In the online version, every processor knows only the destination of the packet it stores. This problem has been attacked in [8]. The solution iteratively makes use of a sub-routing that sorts g² items in POPS(g, g) subnetworks of the larger POPS(d, g) network. The sub-routing is built by hypercube simulation starting from either Cypher and Plaxton’s O(log n log log n) sorting algorithm for the n-processor hypercube or from Leighton’s implementation [12] on the n-processor hypercube of Batcher’s odd-even merge sort algorithm [14]. In the first case, Datta and Sounderalakshmi get the asymptotically fastest algorithm for routing in the POPS network, running in O(16log n log log n) slots. In the second, they get an algorithm that turns out to be the fastest in practice, running in \( \frac{4}{9} g^2 \log g + \frac{4}{9} + 3 \log g + 7 \) slots. Recently, and independently of this work, Rajasekaran and Davila have presented a randomized algorithm for online permutation routing that runs in O(\( \frac{4}{9} \log g + \log n \)) slots [15].

Our contribution is both theoretical and practical. We show that any permutation can be routed on a POPS(d, g) network either with O(\( \frac{4}{9} \log g \)) deterministic slots, or, with high probability, with 5c(d/g) + o(d/g) + O(\log log g) randomized slots, where constant c = exp(1 + e⁻¹) ≈ 3.927. The deterministic algorithm is based on a direct simulation of the AKS network, and it is the first that requires only O(\( \frac{4}{9} \log g \)) slots. When d = Ω(g), that we claim to be the “interesting” case, the randomized algorithm is exponentially faster than any other algorithm in the literature, both deterministic and randomized ones. This is true in practice as well. Indeed, our experiments show that it outperforms its rivals even starting from as small a network as a POPS(2, 2), and the gap grows exponentially with the size of the network. We can also show that, under proper hypothesis, no deterministic algorithm can asymptotically match its performance.

This paper also presents a strong separation theorem between determinism and randomization. We build a meaningful and natural problem inspired on permutation routing in the POPS network such that there exists a O(\log log g) slots randomized solution, and such that no deterministic solution can do better than O(\log g) slots, that is exponentially slower. To the best of our knowledge, this is the first strong separation result from log g to log log g, and, quite interestingly, it does not make use of the notion of oblivious routing, that we show to be essentially out of target in the context of routing in the POPS network.

II. A Deterministic Algorithm

Let \( N_m := \{0, 1, \ldots, m - 1\} \) denote the set of the first m natural numbers. In the on-line permutation routing problem we are given n packets, one per processor. Packet \( p_i, i \in N_m \), originates at processor \( i \), the source processor, and has processor \( \pi(i) \) as destination, where \( \pi \) is a permutation of \( N_m \).

The problem is to route all the packets to destination with as few slots as possible. Crucially, permutation \( \pi \) is not known in advance—at the beginning of the computation, each processor knows only the destination of the packet it stores.

A. The Upper Bound

So far, the best deterministic algorithm for online permutation routing on the POPS(d, g) network is presented in [8]. The algorithm runs in O(\( \frac{4}{9} \log^2 g \)) slots. The computational bottleneck is a O(\log^2 g) sorting sub-routing that sorts g² data value [d/g] times, each on one of the \( \lfloor d/g \rfloor \) POPS(g, g) sub-networks into which the larger POPS(d, g) network is partitioned. The idea in [8] is to make each POPS(g, g) network simulate Leighton’s O(\log^2 n) sorting algorithm for the n-processor hypercube [12], that is, in turn, an implementation of Batcher’s odd-even merge sort. This is carried out by using a general result due to Sahni [6], showing that every move of a normal algorithm for the hypercube (where only one dimension is used for communication at each step) can be simulated with 2\( \lfloor d/g \rfloor \) slots on a POPS network of the same size. Since Leighton’s algorithm is normal, and since the sub-routing is always used on POPS(g, g) sub-networks, we get a constant factor slow-down.

The algorithm in [8] is fairly good in practice, since hidden constants are small. However, we are interested in the best asymptotical result. So, as suggested in [8], we can replace the Leighton implementation of Batcher’s odd-even merge sort with Cypher and Plaxton’s routing algorithm for the hypercube, that is asymptotically faster (though slower for networks of practical size), since it runs in O(\log n log log n) time [16]. This yields a O(\( \frac{4}{9} \log g \log \log g \)) slots algorithm for permutation routing on the POPS network, that is a good improvement. Nonetheless, here we do even better. Our
simple key idea is to simulate a fast sorting network directly on the POPS, instead of going through hypercube simulation. By giving an improved $O(\log g)$ upper bound for sorting on the POPS network, we also get an asymptotically faster algorithm for online permutation routing.

A comparator $[i : j]$, $i, j \in \mathbb{N}_n$ sorts the $i$-th and $j$-th element of a data sequence into non-decreasing order. A comparator stage is a composition of comparators $[i_1 : j_1] \circ \cdots \circ [i_k : j_k]$ such that all $i_i$ and $j_j$ are distinct, and a sorting network is a sequence of comparator stages such that any input sequence of $n$ data elements is sorted into non-decreasing order. An introduction to sorting networks can be found in [17]. Crucially, we can show that a POPS$(d, g)$ network can efficiently simulate any comparator stage.

**Theorem 2.1 ([13]):** A POPS$(d, g)$ network can route off-line any permutation among the $n = dg$ processors using one slot when $d = 1$ and $2[d/g]$ slots when $d > 1$.

**Lemma 2.2:** A POPS$(d, g)$ network, $n = dg$, can simulate a comparator stage in one slot, when $d = 1$, and in $2[d/g]$ slots, when $d > 1$.

**Proof:** Let $[i_1 : j_1] \circ \cdots \circ [i_k : j_k]$ be a comparator stage. We define a function $\pi$ such that $\pi(i_i) = j_i$ and $\pi(j_j) = i_i$ for all $r$. Since all $i_i$ are distinct, and so are all $j_j$, $\pi$ can arbitrarily be extended in such a way to be a permutation. By Theorem 2.1 $\pi$ can be routed in one slot when $d = 1$, and in $2[d/g]$ slots when $d > 1$. During this routing, for every $r$, processor $i_i$ sends its data value to processor $j_j$ and vice-versa. Then, processor $i_i$ discards the maximum of the two data values, while processor $j_j$ discards the minimum.

In [18], the AKS sorting network is presented. This network is able to sort any data sequence with only $O(\log n)$ comparator stages, which is optimal. By simulating the AKS network on a POPS network using Lemma 2.2, we easily get the following theorem.

**Theorem 2.3:** A POPS$(g, g)$ network can sort $g^2$ data values in $O(\log g)$ slots.

The above result is the key to improve on the best deterministic algorithm for online permutation routing in the literature.

**Corollary 2.4:** A POPS$(d, g)$ network can route on-line any permutation in $O(\sqrt{\frac{d}{g}} \log g)$ slots.

**Proof:** To get the claim, it is enough to plug the sorting algorithm of Theorem 2.3 into Stage 1 of the deterministic routing algorithm proposed in [8].

This algorithm is not very practical. Indeed, it is based on the AKS network that, in spite of being optimal, is not efficient when $n$ is small due to very large hidden constants. However, the result is important from a theoretical point of view because of two facts: it establishes that, in principle, $O(\sqrt{\frac{d}{g}} \log g)$ slots are enough to solve deterministically the online permutation routing problem; and, when $d = O(g)$ and under proper hypothesis, it matches one of the lower bounds for deterministic algorithms in the next section.

**B. A Few Lower Bounds**

Borodin et al. [19] study the extent to which both complex hardware and randomization can speed up routing in interconnection networks. One of the questions they address is how oblivious routing algorithms (in which the possible paths followed by a packet depend only on its own source and destination) compare with adaptive routing algorithms. Since oblivious routing can usually be implemented by using limited hardware resources on each node, it is important to understand whether it is worth using the more complex hardware required by adaptive routing. Here, we address similar questions. In the following, our discussion will be limited to the case $d = \Theta(g)$.

Unfortunately, the concept of oblivious routing does not seem to be useful for POPS networks. Indeed, by adapting the ideas first used in [20], we can prove that any oblivious deterministic routing algorithm needs $\Omega(\sqrt{g})$ slots to deliver correctly every permutation. Moreover, by customizing and slightly adapting the approach developed in [19] (that makes use of Yao’s minimax principle [21]), it is also possible to show that any oblivious randomized routing algorithm must use $\Omega(\log g / \log \log g)$ slots on the average.

**Theorem 2.5:** For any POPS$(d, g)$ network, $d = \Theta(g)$, and any oblivious deterministic routing algorithm, there is a permutation for which the routing time is $\Omega(\sqrt{g})$ slots.

**Proof:** We essentially customize the proof in [20] to POPS networks, but also some minor modifications are in order to allow for passive devices and a few different assumptions.

We assume $d = g$, the extension to $d = \Theta(g)$ or wider involving no further ideas, only more technical fuss. Consider the bipartite digraph $D = (V, A)$ having the set $P$ of processors and the set $C$ of couplers as color classes and having as arcs in $A$ those pairs $(p, c)$ such that processor $p$ can send to coupler $c$ plus those pairs $(c, p)$ such that processor $p$ can listen from coupler $c$. We have $|P| = n = dg = g^2$ processors and $|C| = g^2 = n$ couplers, $|V| = |P| + |C| = 2n$; all nodes have in-degree and out-degree both equal to $g$.

Every oblivious algorithm defines a directed $a, b$-path, denoted with $(a, b)$, for every pair $(a, b) \in P^2$. Namely, the directed path of $D$ followed by a packet with destination in $b$ and origin in $a$. The characteristic vector $\chi_{a, b}$ of a path $(a, b)$ is defined by regarding the path has the set of its nodes including $b$ but not $a$. The congestion of a family of $\Pi$ of directed paths is defined as $c(\Pi) := \max_{p \in V} \sum_{(a, b) \in \Pi} \chi_{a, b}(v)$. It is clear that the congestion of $\Pi$ gives a lower bound on the number of steps required to move a packet along each path in $\Pi$ since no processor in $P$ and no coupler in $C$ can receive more than one different packet within a single slot. To prove the theorem we do the following: with reference to the path family $\{(a, b) \mid (a, b) \in P^2 \}$ determined by the oblivious algorithm under consideration, we show how to construct a permutation $\pi : P \rightarrow P$ such that $c(\{(a, \pi(a)) \mid a \in P\}) \geq \sqrt{\frac{g}{2}}$. This will imply the stated lower bound regardless of the queueing discipline, however omniscient, employed by the algorithm. For every $b \in P$, let $S_b := \{v \in V \mid \sum_{a \in P \setminus \{b\}} \chi_{a, b}(v) \geq \sqrt{g/2}\}$. Clearly, every path $(a, b)$, $a \notin S_b$, must have a last node not in $S_b$. Moreover, since $b \in S_b$, the next node on the path $(a, b)$ must be in $S_b$. Let $X_b$ be the set of the last nodes when $a$ ranges in $P \setminus S_b$. By definition, $S_b$ in $X_b$ can be the last node outside $S_b$ for more than $\sqrt{g/2}$ such paths, hence $|P \setminus S_b| \leq |X_b| / \sqrt{g/2}$, which implies $|S_b| \geq g / \sqrt{g}$ in case $|X_b| < g \sqrt{g}$. Moreover, $|X_b| \geq |S_b|$ since the in-degree of the network is bounded by $g$. This implies $|S_b| \geq \sqrt{g}$ in the complementary case that $|X_b| \geq g \sqrt{g}$. In conclusion, $|S_b| \geq \sqrt{g}$ holds for every $b \in P$. Therefore, by averaging argument, there must exist a $v \in V$ which belongs to at least $\frac{|P| \cdot \sqrt{g}}{2}$ of these sets $S_b$, $b \in P$. Let $B = \{b \in P \mid v \in S_b\}$. Let $b_1, b_2, \ldots, b_{\frac{|P| \cdot \sqrt{g}}{2}}$ be distinct processors in $B$ and run the following greedy algorithm where for all processors $p$ in $P$ the value $\pi(p)$ is initially undefined.

For $i := 1$ to $\sqrt{g/2}$, let $a$ be any processor in $S_b$ such that $\pi(a)$ is undefined and define $\pi(a) := b_i$.

Notice that such an $a$ can be found at each step $i \leq \sqrt{g/2}$ since at step $i$ at most $i$ values of $\pi$ have been defined, while $S_b \geq \sqrt{g}$. Moreover, $\pi$ can be clearly extended to a full permutation, while already $c(\{(a, \pi(a)) \mid (a, \pi(a)) \text{ is defined} \}) \geq |\{(a, \pi(a)) \text{ is defined} \}| / \sqrt{g/2}$ since node $v$ belongs to each path $(a, \pi(a))$ by construction.

**Theorem 2.6:** For any POPS$(d, g)$ network, $d = \Theta(g)$, and any oblivious deterministic routing algorithm, the expected routing time for a random permutation (with each permutation chosen with uniform probability) is $\Omega(\log g / \log \log g)$.

**Proof:** The proofs to be customized and adapted here come
This bound applies to both the $O(\log^2 n)$ algorithm in [8] and to our deterministic algorithm in the previous section. Therefore, within the class of rigid algorithms, our proposed routing scheme is optimal.

Now, we prove a strong separation theorem. Under restricted hypotheses, we can show that randomization can give an exponential speed-up over determinism. Here, we address a class of routing algorithms we call two-hops algorithms. A two-hops algorithm has the following properties:

1. Every processor has two buffers, an A-buffer and a B-buffer;
2. At the beginning, the packets are stored in the A-buffer of each processor;
3. At each odd slot $2r+1$, $r = 0, 1, \ldots$, every processor $i$ with a packet in the A-buffer sends the packet to group $c_{\text{out}}(i, 2r+1)$ (two-hops algorithms can only use unicast), listens to incoming packets from group $c_{\text{in}}(i, 2r+1)$, and store the incoming packet (if any) into the B-buffer;
4. At each even slot $2t$, $t = 1, \ldots$, every processor $i$ sends the packet in the B-buffer to destination, reset the B-buffer, and listens to incoming packets from coupler $c_{\text{in}}(i, 2t)$.

Also, we will make the following assumptions:

5. When multiple packets use the same coupler (multiple packets from a group sent to the same group), no packet is delivered.
6. When a packet arrives to any processor in the destination group, it is considered to be successfully routed, and disappears from the network (from the original A-buffer as well);

The last hypothesis simplifies the job of routing all the packets to destination—we don’t have to take care of acks when packets reach their destination. However, since we are proving a lower bound, we don’t lose generality. Now, our goal is to show that for every deterministic choice of functions $c_{\text{in}}$ and $c_{\text{out}}$, there exists an input permutation such that the routing is completed in $\Omega(\log^2 n)$ slots. On the other hand, our randomized algorithm shows that there exists a deterministic $c_{\text{in}}$ and a randomized $c_{\text{out}}$ such that all the packets are routed to destination in $O(\log\log n)$ slots with high probability.

Consider a deterministic two-hops algorithm. Assume that the algorithm stops after $T < \frac{1}{2}\min\{\log d, \log g\}$ slots, $T$ even. We will say that processor $i$ shoots on group $a$ in the first $T$ slots if there exists an odd $t < T$ such that $c_{\text{out}}(i, t) = a$.

**Lemma 2.9:** There exists a group $a_0$ such that at most $dT$ processors shoot on $a_0$ in the first $T$ slots.

**Proof:** By counting.

**Corollary 2.10:** There are at least $n - dT = dg - dT > dg/2$ processors $i$ such that processor $i$ does not shoot on $a_0$ in the first $T$ slots.

Let $P(a_0)$ be the set of processors $i$ such that processor $i$ does not shoot on $a_0$ in the first $T$ slots. By Corollary 2.10, $|P(a_0)| > dg/2$. A subset $A \subseteq P(a_0)$ is $\sqrt{g}$-robust if for every $i \in A$ and for every $t < T$ there are at least $\sqrt{g}$ processors $j$ in $A$ such that $c_{\text{out}}(i, t) = c_{\text{out}}(j, t)$.

**Lemma 2.11:** There exists a $\sqrt{g}$-robust subset $P'(a_0) \subset P(a_0)$ such that $|P'(a_0)| \geq \frac{dg}{Tg} > \sqrt{g}$.

**Proof:** If $P(a_0)$ is not $\sqrt{g}$-robust, then there must be a processor $i \in P(a_0)$ and a $t < T$ such that $c(i, t) = c(j, t)$ for less than $\sqrt{g}$ processors $j \in P(a_0)$. This means that all the processors $j$ such that $c(i, t) = c(j, t)$ (including $i$) must be removed from $P(a_0)$ to get a $\sqrt{g}$-robust subset. So, let $P_1(a_0)$ be obtained from $P(a_0)$ by removing all these processors and mark the pair $(i, c(i, t))$. Start now from $P_1(a_0)$ in place of $P(a_0)$ and keep iterating. Notice that no pair can be marked twice in the process. The number of pairs is at most $Tg$, and each time we mark a pair we drop at most $\sqrt{g}$ processors.
Theorem 2.12: Any deterministic and two-hops algorithm for online permutation routing on the POPS\((d,g)\) network, \(d = \Theta(g)\), must use \(\Omega(\log n)\) slots.

Proof: We will show that for every processor \(i\) in \(P(a_0)\) there exists an input permutation such that \(p_i\) will not reach destination. The idea of the proof is as follows: we can build an input permutation such that \(p_i\) has to perform two hops to get to destination, and that has a conflict at every even slot. Take a packet \(p_i\) such that \(i \in P'(a_0)\) and mark the packet. Now, for \(t := T - 1\) downto 1, \(t\) odd, do the following:

for every marked packet \(p_j\),
1) take an unmarked packet \(p_h\) such that \(c(h,t) = c(j,t)\);
2) mark packet \(p_h\).

Then, set the destination of all marked packets to processors in group \(g\) and mark the packet. Now, for every group is responsibility for listening to coupler \(c(a,\pi(i))\) for the message possibly coming from group \(\Delta(i)\). This way, every conflict-less communication successfully completes and no packet is lost. Indeed, during slots 1 and 2, in every group \(a, a \in \mathbb{N}_g\), the processor with index \(b\) within the group, \(b \in \mathbb{N}_g\), receives the packet that is possibly coming from group \(b\). In slot 5, every processor \(\pi(i)\) that still has to receive packet \(p_i\) hopefully receives its packet from group \(\Delta(\pi(i))\), the temporary destination group of packet \(p_i\). Slots 3 and 4 behave differently. Indeed, each ack sent during slot 3 is received by the same processor that sent the packet in slot 2. Similarly, each ack sent during slot 4 is received by the same processor that sent the packet in slot 1.

Clearly, during slots 1 and 2, multiple conflicts on the couplers should be expected, and many of the communications may not complete. For example, two packets in the same group can choose the same random intermediate group during slot 1, or two packets willing to go to the same temporary destination group are currently in the same random intermediate group during slot 2. On the contrary, slots 3, 4, and 5 do not generate any conflict, as shown in the following proposition.

Proposition 3.2: At all steps, slots 3, 4, and 5 of the routing algorithm do not generate any conflict.

Proof: Consider packet \(p_i\) stored at processor \(i\) in group \(a\). Assume that, during an arbitrary step, its random intermediate group is \(r(i)\), chosen uniformly at random. In the case when packet \(p_i\) survives slot 1 and arrives to its random intermediate group \(r(i)\), we know that coupler \(c(r(i),a)\) has been used to send packet \(p_i\) only, otherwise a conflict would have stopped the packet. Moreover, since there is only one processor in group \(r(i)\) that is responsible for receiving packet \(p_i\), namely processor \(r(i)+a\), there will be only one ack message corresponding to packet \(p_i\) to be sent in slot 4, and this ack message is the only one that uses the symmetric coupler \(c(a,r(i))\) during slot 4. In conclusion, slot 4 is conflict-free. A similar argument shows that slot 3 is conflict-free as well.

Consider now slot 5. Assume that, after step 4, packet \(p_j\) has arrived at the same temporary destination group as packet \(p_i\). This means that \(\Delta(\pi(i)) = \Delta(\pi(j))\). That is, \(\pi(i) \equiv \pi(j) \mod g\). In this case, it is not possible that \(\pi(i) \equiv \pi(j) \mod g\); otherwise we would have \(\pi(i) = \pi(j)\), in contrast with the fact that \(\pi\) is a permutation. Therefore, packets \(p_i\) and \(p_j\) go to different groups from their temporary destination group. In other words, step 5 is conflict-free as well.

By Proposition 3.2, if packet \(p_i\) survives the first two slots of a step, then, in the very same step, it will be routed to its destination, and an ack will be successfully returned to source processor \(i\). When the ack arrives, the source processor can delete the packet, since it knows it will be safely stored by the destination processor. Conversely, if no ack arrives, the packet is not deleted, and the processor tries again to deliver it in the next step, choosing again a possibly different random temporary group.
By the above discussion, we can safely concentrate on slots 1 and 2. A useful way to visualize the conflicts in slots 1 and 2 of an arbitrary step is shown in Figure 3(b). At any given step of the routing algorithm, let \( \pi \) be the restriction of the input permutation to those packets that have not been successfully routed yet (during previous steps). We build the graph of conflicts, a bipartite multi-graph \( G_\pi \) on node classes \( S := N_g \) and \( D := N_g \). For every group \( a \) and for each packet \( p_i \) in group \( a \) and yet to be routed, we introduce an edge with one endpoint in \( a \in S \) and the other endpoint in the temporary destination group \( \Delta(\pi(i)) \in D \). During slot 1 of the step, every edge (packet yet to be routed) randomly and uniformly chooses a color in \( N_g \) (the random intermediate group). Clearly, a same packet can choose different colors in different steps of the routing algorithm. Now we can exactly characterize the conflicts in the first two slots of the routing algorithm during step \( s \). Packet \( p_i \) in group \( a \) (represented by an edge from \( a \in S \) to \( \Delta(\pi(i)) \in D \)) has a conflict during slot 1 if and only if there is another edge incident to \( a \in S \) with the same random color. Moreover, if we remove all edges relative to packets that have a conflict in slot 1 (see Figure 3(b)), every remaining packet \( p_i \) has a conflict during slot 2 if and only if there is another remaining edge incident to \( \Delta(\pi(i)) \in D \) with the same random color. Figure 3(c) shows which packets of Figure 3(a) survive both slots and are hence delivered to destination by Proposition 3.2.

Our first result shows that, in case the packets are “sparse” in the network, then all the packets can be delivered in a constant number of slots with high probability.

**Lemma 3.3:** If the maximum degree of the conflict graph is \( g^\alpha \) for some constant \( \alpha < 1 \), then the routing algorithm delivers all the packets to destination in a constant number of slots with high probability.

**Proof:** Since the maximum degree of the conflict graph is \( g^\alpha \), in every group of the POPS network there are at most \( g^\alpha \) packets left to be routed, and every group of the POPS network is the temporary destination group of at most \( g^\alpha \) packets. Let \( \beta = 1 - \alpha \). We show that the probability that all packets get routed to destination within \( 3/\beta \) steps is at least \( 1 - \beta^g/g \), where \( \beta^g/g = \frac{2^g}{g^2} \) is a constant depending only on (the constant) \( \beta \). Consider a generic packet \( p_i \) in group \( a \). The probability that packet \( p_i \) has a conflict in one step is at most equal to the probability that either one of the packets in group \( a \) or one of the packets with temporary destination group \( \Delta(\pi(i)) \) chooses the same random intermediate group as packet \( p_i \). Since at most \( g^\alpha - 1 \) other packets are in group \( a \), and similarly at most \( g^\alpha - 1 \) have temporary destination group \( \Delta(\pi(i)) \), this probability cannot be larger than \( 2g^\alpha/g = 2g^{-\beta} \). Therefore, the probability that the packet is not routed in each of the \( 3/\beta \) steps is at most

\[
\left( \frac{2}{g^\beta} \right)^{3/\beta} = \frac{2^{3/\beta}}{g^2} = \frac{c_\beta}{g^2}.
\]

By the union bound, the probability that any of the \( g^1 + \alpha < g^2 \) packets in the network has not been routed in \( 3/\beta \) steps is at most \( c_\beta/g \).

As a matter of fact, the hard part of the job is to reduce the initial number of \( g \) packets in each group in such a way to get a “sparse” set of remaining packets. We can prove that this is done quickly by our randomized algorithm by providing sharp bounds on the number \( X \) of packets that are successfully delivered in a step. We define \( X \) as a sum of indicator random variables \( Z_i \), where \( Z_i \) is equal to 1 if the \( i \)-th packet is delivered in this step, and 0 otherwise. It is important to realize that these random variables are not independent: the event that one packet has a conflict influences the probability that another packet has a conflict as well. As a consequence, we cannot use the well-known Chernoff bound to get sharp estimates of the value of \( X \) since there does not seem to be any way to describe the process as a sum of independent random variable. So, we need a more sophisticated mathematical tool. Specifically, we will see that slots 1 and 2 of one step of the routing algorithm can be modeled by a set of martingales. Martingale theory is useful to get sharp bounds when the process is described in terms of not necessarily independent random variables.

For an introduction to martingales, the reader is referred to [22]. Also [23], [24], [25], and [26] give a description of martingale theory. Here, we give a brief review of the main definitions and theorems we will be using in the following.

**Definition 3.4 ([22]):** Given the \( \sigma \)-field \( (\Omega,F) \) with \( F = 2^\Omega \), a filter is a nested sequence \( F_0 \subseteq F_1 \subseteq \cdots \subseteq F_m \) of subsets of \( 2^\Omega \) such that

1) \( F_0 = \{\emptyset, \Omega\} \);
2) \( F_m = 2^\Omega \);
3) for \( 0 \leq h \leq m \), \( (\Omega,F_h) \) is a \( \sigma \)-field.

**Definition 3.5 ([22]):** Let \( (\Omega,F,\mathbb{P}) \) be a probability space with a filter \( F_0,\ldots,F_m \). Suppose that \( Y_0,\ldots,Y_m \) are random variables such that for all \( h \geq 0 \), \( Z_h = F_h \)-measurable. The sequence \( Z_0,\ldots,Z_m \) is a martingale provided that, for all \( h \geq 0 \),

\[
\mathbb{E}[Z_{h+1} | F_h] = Z_h.
\]

The next tail bound for martingales is similar to the Chernoff bound for the sum of Poisson trials.
Moreover, let random variable $X_s$ be the number of packets with temporary destination group that survive slot 1. Random variable $X_s$ may depend on $h$. In the following analysis of the network, the conflict graph at step $s$ is denoted by $G_s$. Theorem 3.6 (Azuma's Inequality [22]): Let $Z_0, \ldots, Z_m$ be a martingale such that for each $h$,

$$
|Z_h - Z_{h-1}| \leq c_h,
$$

where $c_h$ may depend on $h$. Then, for all $t \geq 0$ and any $\lambda > 0$,

$$
\Pr[|Z_t - Z_0| \geq \lambda] \leq 2e^{-\lambda^2 / \sum c_h^2}.
$$

Theorem 3.7: A POPS($g, g$) network can route any permutation in $O(\log \log g)$ slots with high probability.

Proof: Let $G_\pi = (S, D, E)$ be the conflict graph at step $s$ of the routing algorithm, where $\pi$ is the input permutation restricted to those packets that still have to be routed at the beginning of step $s$. Let $d_a$ be the maximum degree of $G_\pi$. So, at step $s$ there are at most $d_a$ packets left to be routed in every group, and at most $d_a$ packets are willing to go to the same temporary destination group. Clearly, $d_1 \leq d$. We will show that after $O(\log \log g)$ steps the conflict graph has maximum degree at most $g^{5/6}$. This is enough to prove this theorem by Lemma 3.3.

Assume to be at step $s$. If $d_a \leq g^{5/6}$, then we are done. So, we can assume that $d_a > g^{5/6}$. Let $S_a, a \in S$, be the set of indices of the packets of group $a$ that still have to be delivered at the beginning of step $s$. Similarly, let $D_a, b \in D$, be the set of indices of the packets in the whole network that still have to be delivered and that have group $b$ as temporary destination group. Clearly, $|S_a|$ and $|D_b|$ are the degrees of nodes $a \in S$ and $b \in D$ in the conflict graph at step $s$. Therefore, $|S_a| \leq d_a$ and $|D_b| \leq d_a$ for every $a \in S$ and $b \in D$. For every packet $p_i$ still to be routed, we define the following indicator random variable,

$$
Z_a^i = \begin{cases} 1 & \text{if packet } p_i \text{ survives slot 1 in step } s, \\ 0 & \text{otherwise}. \end{cases}
$$

Random variable $X_a^1 = \sum_{i \in S_a} Z_a^i$ tells the number of packets from group $a$ that survive slot 1; random variable $Y_a^1 = \sum_{j \in D_a} Z_a^i$ tells the number of packets with temporary destination group $b$ that survive slot 1. Moreover, let random variable $C_b$ be equal to the color chosen by packet $p_i$ in step $s$.

Clearly, we have nothing to show about the nodes in $G_a$ that have degree smaller than or equal to $g^{5/6}$. So, we define sets $S^+ \subseteq S$ and $D^+ \subseteq D$, which collect the nodes with degree larger that $g^{5/6}$, and focus on the nodes in these sets. Consider an arbitrary node $a \in S^+$. The expectation of $Z_a^i$, $i \in S_a$, can be bounded as follows:

$$
E[Z_a^i] = \Pr[\forall h \in S_a \setminus \{i\}, C_h \neq C_i] = \prod_{h \in S_a \setminus \{i\}} \Pr(C_h \neq C_i) = (1 - \frac{1}{g})^{|S_a| - 1} \geq e^{-|S_a|/g}.
$$

So, the expected number of packets in group $a$ that survive slot 1 can be bounded accordingly,

$$
E[X_a^1] = E\left[\sum_{i \in S_a} Z_a^i\right] = E\left[\sum_{i \in S_a} E[Z_a^i]\right] \geq |S_a| e^{-|S_a|/g}. \tag{2}
$$

In order to show that random variable $X_a^1$ is not far from its expectation with high probability, we now define random variables $W_h = E[X_a^1|F_h]$, $h = 0, \ldots, |S_a|$, where $F_h$ is the $\sigma$-field generated by the random color chosen by the first $h$ packets in $S_a$. Filter $F_h$, $h = 0, \ldots, |S_a|$, is such that $W_0, \ldots, W_{|S_a|}$ is a martingale and that $|W_h - W_{h-1}| \leq 2$, since fixing the random color chosen by the $h$-th packet in $S_a$ can only affect the expected value of the sum $X_a^1$ at most by two. By the Azuma’s inequality, for every $\delta > 0$,

$$
\Pr\left[\|X_a^1 - E[X_a^1]\| \geq \delta E[X_a^1]\right] = \Pr\left[|W_{|S_a|} - W_0| \geq \delta E[X_a^1]\right] \leq 2 e^{-\frac{\delta^2 E[X_a^1]^2}{2|S_a|}} \leq 2 e^{-\frac{\delta^2 |S_a|^2}{2g}}. \tag{3}
$$

To prove a similar result for $Y_b^1$, $b \in D^+$, we must recast the above general martingale arguments into a more structured approach. This is because $Y_b^1$ may depend on the random colors chosen by all the packets in the network, and not only on those chosen by the packets in $D_b$.

Consider an arbitrary node $b \in D^+$. In the following analysis of the expectation and concentration of $Y_b^1$, we can clearly pretend that the random colors are first chosen for the packets outside $D_b$ and later for the packets in $D_b$. This will not invalidate our conclusions about the whole of the $Y_b^1$s, $b \in D^+$, since these will be derived from the solid claims about any single $Y_b^1$ by the union bound. For every $a \in S_a$, we define set $C_{a, b}$ as $\mathbb{B} \setminus C_{a, b}$, where $C_{a, b}$ is the set of colors that are chosen in step $s$ by a packet in group $a$ that has temporary
destination group different from $b$,
\[
\overline{C}_{a,b} = |N_{a,b}| \cup \left\{ C_i \right\}.
\]

The average size of $\overline{C}_{a,b}$ is
\[
E[|P_{a,b}|] = g \left( 1 - \frac{1}{g} \right)^{|S_a|/|D_b|}.
\]

Being just a classical ball and bins problem [22], we know that random variable $\overline{C}_{a,b}$ is not far from its expectation with probability
\[
\Pr[|\overline{C}_{a,b}| < (1 - \delta)E[|\overline{C}_{a,b}|]] \leq e^{-\frac{\delta^2E[|\overline{C}_{a,b}|]}{2g}} \leq e^{-\frac{\delta^2}{8g}},
\]
for every $\delta > 0$. By the union bound over the $g$ nodes in $S$, for every $\delta > 0$, we know that for every node $a \in S$
\[
|\overline{C}_{a,b}| \leq (1 - \delta)g \left( 1 - \frac{1}{g} \right)^{|S_a|/|D_b|},
\]
with probability
\[
1 - ge^{-\frac{\delta^2}{8g}}.
\]

Under the hypothesis that Equation (4) holds for every $a \in S$, we can bound the expectation of $Z_j^1$, $j \in D_b$, as follows:
\[
E[Z_j^1] = \Pr \left( \forall h \in D_b \cap S_a, \{C_i \} \land (C_j \notin P_{a,h}) \right),
\]
where $a_j$ is the group of packet $p_j$. So,
\[
E[Z_j^1] \geq \left( 1 - \frac{1}{g} \right)^{|D_b|/|S_a|} (1 - \delta) \left( 1 - \frac{1}{g} \right)^{|S_a|/|D_b|} = (1 - \delta) \left( 1 - \frac{1}{g} \right)^{|S_a|/|D_b|} / g.
\]

The expectation of $Y_b^1$ can be bounded accordingly,
\[
E[Y_b^1] = \sum_{j \in D_b} E[Z_j^1] \geq (1 - \delta)|D_b|e^{-|D_b|/g}.
\]

In order to show that random variable $Y_b^1$ is not far from its expectation with high probability, we now define random variables $W_k = E[Y_b^1|F_k]$, $k = 0, \ldots, |D_b|$, where $F_k$ is the $\sigma$-field generated by the random color chosen by the $k$th packet in $D_b$. Filter $F_k$, $k = 0, \ldots, |D_b|$, is such that $W_0, \ldots, W_{|D_b|}$ is a martingale and that $|W_k - W_{k-1}| \leq 2$, since fixing the random color chosen by the $k$th packet in $D_b$ can only affect the expected value of the sum $Y_b^1$ at most by two. By the Azuma’s inequality, for every $\delta > 0$
\[
\Pr \left[ \left| Y_b^1 - E[Y_b^1] \right| \geq \delta E[Y_b^1] \right] \leq 2e^{-\frac{\delta^2E[Y_b^1]}{2|D_b|}} \leq 2e^{-\frac{\delta^2}{8g}},
\]

Let $G_{s'} = (S, D; E')$ be the conflict graph at step $s$, where $\pi'$ is the input permutation restricted to those packets that survive slot 1 in step $s$. Hence, $E' \subseteq E$. Our goal is to bound the number of packets that survive slot 2 as well, and are thus delivered to destination during this step. Let $Z_j^2$ be equal to one if packet $p_j$ survives both slots 1 and 2, and zero otherwise. Also, let $S_{a,b}^2$, $a \in S$, be the set of indices of the packets of group $a$ that have survived slot 1. Similarly, let $D_b^2$, $b \in D$, be the set of indices of the packets in the whole network that have survived slot 1 and have group $b$ as temporary destination group. Clearly, for every $a \in S$, $|S_{a,b}^2|$ is equal to $X_a^1$ and is the degree of node $a$ in $G_{s'}$; while for every $b \in D$, $|D_b^2|$ is equal to $Y_b^1$ and is the degree of node $b$ in $G_{s'}$. Random variables
\[
X_a^2 = \sum_{i \in S_{a,b}^2} Z_i^2,
\]
$a \in S$, tell the number of packets in group $a$ that are delivered during step $s$; similarly, random variables
\[
Y_b^2 = \sum_{j \in D_b^2} Z_j^2
\]
b $D$, tell the number of packets willing to go to temporary destination group $b$ that are delivered during step $s$.

Consider an arbitrary node $b \in D^+$. The expected value of $Y_b^2$ depends on permutation $\pi'$. Since we are computing a lower bound to $Y_b^2$, the worst case is when all packets in $D_b^2$ originate at different groups. Indeed, if two packets in $D_b^2$ belong to the same $S_{a,b}^2$, we already know that they have chosen two different colors during step $s$, and the expectation of $Y_b^2$ is larger. A formal proof of this intuitive claim can be given, though it’s omitted for the sake of brevity. Assuming that random variable $Y_b^1$ is not far from expectation as in Equation (4) we can bound the expectation of $Y_b^2$:
\[
E[Y_b^2] = |D_b^2| \left( 1 - \frac{1}{g} \right)^{|D_b^2|/|D_b|} \geq (1 - \delta)^2 |D_b^2|e^{-|D_b^2|/g} \geq (1 - \delta)^2 |D_b^2|e^{-2\delta g / g}.
\]

Just as before, also $Y_b^2$ is not far from its expectation. Martingale theory can be used again to show that
\[
\Pr \left| Y_b^2 - E[Y_b^2] \right| \geq \delta E[Y_b^2] \leq 2e^{-\frac{\delta^2E[Y_b^2]}{8|D_b^2|}} \leq 2e^{-\frac{\delta^2}{8g}},
\]

Similarly, by using the same technique that has been used to bound random variable $Y_b^1$, for every node $a \in S^+$ we can show that
\[
E[X_a^2] \geq (1 - \delta)|S_{a,b}^2| \left( 1 - \frac{1}{g} \right)^{|S_{a,b}^2|/|D_b^2|} \geq (1 - \delta)^2 |S_{a,b}^2|e^{-|S_{a,b}^2|/g} \geq (1 - \delta)^2 |S_{a,b}^2|e^{-2\delta g / g},
\]

and that $X_a^2$ is not far from its expectation
\[
\Pr \left| X_a^2 - E[X_a^2] \right| \geq \delta E[X_a^2] \leq 2e^{-\frac{\delta^2E[X_a^2]}{2|S_{a,b}^2|}} \leq 2e^{-\frac{\delta^2}{8g}},
\]

By Equations (9) and the union bound, the number of packets successfully delivered in step $s$ can be bounded as follows: For every $\delta > 0$,
\[
X_a^2 \geq (1 - \delta)^3 |S_{a,b}^2|e^{-2\delta g / g},
\]
\[
Y_b^2 \geq (1 - \delta)^3 |D_b^2|e^{-2\delta g / g},
\]
for every $a \in S^+$ and $b \in D^+$, with probability at least
\[
1 - 9ge^{-\frac{\delta^2}{8g}}.
\]

Now, we divide our analysis into two phases. Phase 1 is composed of a constant number of steps and, with high probability, reduces the maximum degree of the conflict graph from $d_1$ to $g_2$ or less, where $0 \leq x < 1$ is any fixed constant. Phase 2 follows and reduces the maximum degree of the conflict graph to $g_5/2$ or less in $O(\log \log n)$ steps with high probability.

Let us start from Phase 1. For every step $s$ during Phase 1, $g_2 \leq d_b \leq g$. We show that a constant number of steps is enough to make
\(d_i\) fall below \(gx\) with high probability. For all \(a \in S^+\), let us refer to a step such that
\[
X_a^2 \geq \frac{|S_a|e^{-2}}{2}
\]
as a **lucky** step for group \(a\). By Equation 12 and 14 where we fix \(\delta\) such that \((1 - \delta)^3 = 1/2\), step \(s\) is lucky for every group \(a \in S^+\) with probability at least
\[
1 - 9ge^{-a|S_a|} \geq 1 - 9ge^{-ag^{3/6}},
\]
where \(\alpha\) is a positive constant. Therefore, the number of packets that remain after step \(s\) in group \(a \in S^+\) is
\[
|S_a| - X_a^2 \leq |S_a| - \frac{|S_a|e^{-2}}{2} \leq \frac{e^{-2}}{2}(1 - e^{-2}/2)
\]
with high probability. Note the same bound can be shown for sets \(|D_a|, b \in D^+\), with exactly the same analysis (where an analogous notion of lucky step refers to a step such that the degree of group \(b \in D\) reduces by \(|D_b| e^{-2/2}\) at least). Therefore, after
\[
y := \left\lfloor \frac{\log x}{\log(1 - e^{-2}/2)} \right\rfloor
\]
lucky steps for all the groups the maximum degree of the conflict graph reduces to \(gx\) or less. By the union bound, this happens within the very first \(y\) steps with probability at least
\[
1 - 9yge^{-ag^{3/6}},
\]
That is, Phase 1 completes in a constant number of steps with high probability.

We are now at a generic step \(s\) in Phase 2. Our goal is to reduce the degree of the graph of conflicts to \(g^{5/6}\). Let \(\lambda_s = d_s/gx\). We can assume that \(g^{-1/6} \leq \lambda_s < x\), and when \(\lambda_s\) falls below \(g^{-1/6}\) we are done. This time, let’s refer to a step during which at least \((1 - \lambda/s)|S_a|e^{-2}\) packets in group \(a \in S^+\) are delivered as a lucky step for group \(a\).

By Equation 12 and 14 where we take \(\delta_i = \lambda_i/3\) (in such a way that \((1 - \delta)^3 \geq (1 - \lambda))\), step \(s\) is lucky for every group \(a \in S^+\) with probability at least
\[
1 - 9yge^{-\beta g^{3/6}}
\]
where \(\beta\) is a positive constant, since \(|S_a|X_a^2 \geq g^{5/6}(g^{-1/6})^2 = g^{1/2}\). So, the number of packets that remain in group \(a \in S^+\) after step \(s\) is
\[
|S_a| - X_a^2 \leq |S_a| - (1 - \lambda_s)|S_a|e^{-2}\lambda_s \leq \frac{e^{-2}}{2}(1 - (1 - \lambda_s)e^{-2\lambda_s})
\]
with high probability. A similar result can be shown for any group \(b \in D\) such that \(|D_b| > g^{5/6}\) with exactly the same analysis. By the union bound, at the end of step \(s\) the degree of the conflict graph is at most
\[
d_s \geq 1 - (1 - \lambda_s)e^{-2\lambda_s}
\]
with high probability. Now, assuming a sequence of lucky steps, we can set up the following recursion,
\[
\lambda_{r+1} \leq \lambda_r [1 - (1 - \lambda_s)e^{-2\lambda_s}] \leq \lambda_r [1 - (1 - \lambda_s)(1 - 2\lambda_s)] = \\
= \lambda_r [1 - 1 + 3\lambda_s - 2\lambda_s^2] \leq 3\lambda_r^2.
\]
Therefore,
\[
\lambda_s \leq 3\lambda_s^2 \leq \frac{3}{3} \lambda_s^2 \leq \cdots \leq 3^{2r-1} \lambda_s^{2r-1}.
\]
That is,
\[
\log_3 \lambda_s \leq \log_3 \left(3^{2r-1} \lambda_s^{2r-1} \right) = 2r^{3-1} \left(1 + \log_3 \lambda_{r+1}\right).
\]
Since our first goal is to have \(\lambda_s \leq g^{-1/6}\), we should find \(\tilde{s}\) such that
\[
\log_3 \lambda_s \leq -\frac{\log_3 g}{6}.
\]
We can get this by taking \(\tilde{s}\) such that
\[
2^{\tilde{s}-1} \left(1 + \log_3 \lambda_{r+1}\right) \leq -\frac{\log_3 g}{6}.
\]
If we choose the arbitrary constant \(x\) of Phase 1 to be strictly smaller than \(1/3\), we obtain that \(1 + \log_3 \lambda_{r+1}\) is negative, and the above equation comes down to \(\tilde{s} = O(\log \log g)\). Therefore, by the union bound over the \(\tilde{s} - y - 1\) steps of Phase 2, the whole Phase 2 is made of lucky steps for all the groups in \(S^+\) and \(D^+\) with probability at least
\[
1 - 9(\tilde{s} - y - 1)ge^{-\alpha g^{3/6}} = 1 - O \left(ge^{-\alpha g^{3/6}} \log \log g\right).
\]
We have shown that, after \(\tilde{s} = O(\log \log n)\) steps, the maximum degree of the conflict graph \(G_\pi\) is at most \(g^{5/6}\) with high probability. This is enough to get the claim of our theorem by combining Phase 1 and Phase 2, and then using Lemma 28.

We remark that all transmissions occurring during slots 3 and 4 are just asking requiring only “empty” messages providing only headers but without payload. When packets are very long, it may be more efficient to divide the 5 slots into 2 “short” slots and only 3 “long” slots, hence profiting from the homogeneity of the operations within a same slot in our routing algorithm.

Note an important property of our algorithm: processor \(i\) requires enough memory to store at most three packets: one is the original packet \(p_i\), the second is the packet whose destination is processor \(i\), and the third is a copy of another packet as received from group \(\Delta(i)\). However, if we can assume that packet \(p_i\) exits the network the slot after \(p_i\) got to its destination \(\pi(i)\), then the requirement on the internal capacity of processors drops to only 2 packets. Similarly, if we can assume that the input packets are stored on an external feeding line, then the internal storage requirement drops to 1.

### B. The General Case

Let start from the case when \(d \geq g\). A natural approach to solve the problem is to perform two stages: Stage 1 routes the packets until the degree of the conflict graph is at most \(g\); then Stage 2 uses the randomized algorithm described in the previous section to route the remaining packets in \(O(\log \log g)\) slots. Since at most \(g\) packets can be moved without conflicts from each group in each slot, \((d - g)/g\) is a simple lower bound to the number of slots used in the first of the two above mentioned stages. In the following, we will show that we are only a constant factor far from the lower bound, and that we can precisely indicate this factor.

Consider a group \(a \in \mathbb{N}_g\). From this group, there are \(d \geq g\) packets willing to go to destination. If we let every packet choose a random destination group and try to reach that group, when \(d\) is large (it is enough that \(d = \Omega(g \log g)\)) every coupler will have a conflict with high probability and no packet is delivered. Clearly, this is not what we like to happen. So, the idea for the first stage of the algorithm is a small modification of the randomized algorithm: Before participating to the step, every processor with a packet tosses a coin that says ‘yes’ with probability \(p\). Only those processors that get a ‘yes’ are allowed to participate and send their packet.

In the first step, it is best to choose \(p\) equal to \(g/d\), in such a way that \(g\) packets are sent on expectation. This value maximizes the expected number of conflict-less communications, and thus the number of packets that survive slot 1 and slot 2. Later on, \(p\) has to be iteratively reduced using a fixed law according to the expected reduction of the number of packets left in each group. When at most
g packets are left in each group with high probability, then we can set p to one, and so proceed with the same algorithm we propose for the case when d = g.

To understand what is the most efficient law, it is important to understand what is the expected number of packets that are delivered in each step of the algorithm. Informally speaking, our hope is that exactly g packets from each group participate to step by step of the first phase of the algorithm. Under this assumption, we know that approximately ge⁻¹ packets of each group will survive the first slot. At the beginning of the second slot, these packets are somewhat randomly scattered in the network (not uniformly at random, unfortunately, as we know from the previous section). If everything goes just like in the first slot, and this is far from being obvious since the destination is not random now and the packets are not distributed uniformly at random, we can hope that \( \exp\{- (1 + e⁻¹)\} \) packets from each group survive the second slot as well, and are thus safely delivered. If this is the case, \( \exp\{1 + e⁻¹\} \) steps are enough to reduce the number of packets from d to g on expectation. The following theorem shows that, eventually, what happens is exactly what we can set p to one for.

Theorem 3.8: Let c = \( \exp\{1 + e⁻¹\} \) ≤ 3.927. A POPS(d, g) network can route any permutation in 5c[d/g] + o(d/g) + \( O(\log \log g) \) slots with high probability.

Proof: The idea of the algorithm is to use \( \lceil (c + e(\varepsilon))(d/g) - 1 \rceil \) steps, where \( \varepsilon(g) = o(1) \), to reduce the maximum degree of the conflict graph to at most g with high probability. Since every step consists of 5 slots, we then get the claim by Theorem 3.7.

Every step \( s, s = 1, \ldots, \lceil (c + e(\varepsilon))(d/g) - 1 \rceil \), is similar to the standard step of the randomized routing algorithm, with the difference that, before choosing its random color during slot 1, every packet independently tosses a coin and participates to the step with probability \( \frac{g}{d - \frac{g}{c + \varepsilon}(g)} \).

Our claim is that, at the beginning of step \( s, s = 1, \ldots, \lceil (c + e(\varepsilon))(d/g) - 1 \rceil + 1 \), the degree of the conflict graph is at most \( d_s := d - \frac{g}{c + \varepsilon}(g) \) with high probability. As a consequence, when \( s = \lceil (c + e(\varepsilon))(d/g) - 1 \rceil + 1 \), we get \( d_s \leq g \) as desired. The claim is certainly true when \( s = 1 \). Assume it is true at the beginning of step \( s \leq \lceil (c + e(\varepsilon))(d/g) - 1 \rceil \). We show that it is true at the beginning of step \( s + 1 \) as well.

Let \( S_a, a \in S \), be the set of indices of the packets in group a that still have to be delivered at the beginning of step s. Similarly, let \( D_b, b \in D \), be the set of indices of the packets in the whole network that still have to be delivered at the beginning of step s and that have group b as temporary destination group. By hypothesis, \( |S_a| \leq d_s \) and \( |D_b| \leq d_s \) for all \( a \in S \) and \( b \in D \). Our first goal is to prove that at the beginning of step \( s + 1 \) the degree of the conflict graph is at most \( d_{s+1} \) with high probability.

For every packet \( P_i \) yet to be routed, let random variable \( P_i \) be equal to 1 if packet \( P_i \) participates to step s, and 0 otherwise. Random variable \( P_a = \sum_{i \in S_a} P_i \) counts the number of packets in group a that participate to step s. The expectation of \( P_a \) can be computed as follows:

\[
\mathbb{E}[P_a] = \sum_{i \in S_a} \mathbb{E}[P_i] = \frac{|S_a|}{d_s} g.
\]

And, clearly, \( \mathbb{E}[P_a] \leq g \). Since random variables \( P_i \) are independent, the Chernoff bound [22], [25] (note that in [22] this bound appears in a different yet stronger form) is enough to claim that for every \( \delta > 0 \)

\[
\Pr \left[ P_a < (1 - \delta) \frac{|S_a|}{d_s} g \right] \leq e^{-\delta^2 \frac{g |S_a|}{d_s}} \leq e^{-\frac{\delta^2 \varepsilon(\varepsilon)}{d_s}} \leq e^{-\frac{\delta^2 g}{4}}.
\]

Let \( S'_a, a \in S \), be the set of indices of the packets in group a that participate to step s. Random variable \( P_a \) is thus equal to \( |S'_a| \). Therefore, for every \( \delta > 0 \)

\[
(1 - \delta) \frac{|S_a|}{d_s} g \leq |S'_a| \leq (1 + \delta) g
\]

with probability at least \( 1 - 2e^{-\delta^2 g/4} \). Since a similar result holds for every \( a \in S \) and \( b \in D \), we also know that for every \( \delta > 0 \)

\[
(1 - \delta) \frac{|S_a|}{d_s} g \leq |S'_a| \leq (1 + \delta) g,
\]

hold for every \( a \in S \) and \( b \in D \), with probability at least

\[
1 - 4ge^{-\delta^2 g/4},
\]

by the union bound over the 2g nodes of the conflict graph.

Clearly, we have nothing to show about the nodes in the conflict graph that have degree smaller than or equal to \( d_{s+1} \). So, we define sets \( S^+ \subseteq S \) and \( D^+ \subseteq D \), which collect the nodes with degree larger that \( d_{s+1} \), and focus on the nodes in these sets. Consider an arbitrary group \( a \in S^+ \), and assume that the bound in Equations 18 and 19 hold for every \( a \in S \) and \( b \in D \). Now, we can perform the same analysis as in the proof of Theorem 3.7. Similarly to Equation 10 we know that

\[
\mathbb{E}[X^2_a] \geq (1 - \delta) |S'_a| \left( 1 - \frac{1}{g} \right)^{|S'_a|} \geq (1 - \delta) |S'_a| e^{-|S'_a|/g},
\]

with high probability. In the next equation, we will use the following two facts: \( xe^{-x} \leq ye^{-y} \) whenever \( x \leq y \leq g \), and \( xe^{-x} \) has maximum when \( x = g \). Clearly, \( |S'_a| \leq g \) (there are only g couplers from group a). So, we get

\[
\mathbb{E}[X^2_a] \geq (1 - \delta) |S'_a| e^{-|S'_a|/g} \geq (1 - \delta) |S'_a| e^{-|S'_a|/g} \geq (1 - \delta) \frac{|S_a|}{d_s} e^{-\delta^2 g/4}.
\]

with high probability. By setting \( \delta = g^{-1/3} \) in the above equation, with high probability we get

\[
X^2_a \geq \frac{|S_a|}{d_s} \frac{g}{c + \varepsilon(g)}
\]

where \( c = e^{1 + e^{-1}} \) and \( \varepsilon(g) = o(1) \). Since \( X^2_a \) is the number of packets in group a that are delivered to destination during slot s, the degree of group a in the conflict graph at the beginning of step \( s + 1 \) is

\[
|S_a| - X^2_a \leq |S_a| - \frac{|S_a|}{d_s} \frac{g}{c + \varepsilon(g)} \leq d_{s+1} - \frac{g}{c + \varepsilon(g)} = d_{s+1}.
\]

The same result can be shown for every \( a \in S^+ \) and \( b \in D^+ \). By the union bound over the \( \lceil (c + e(\varepsilon))(d/g) - 1 \rceil \) steps required, and over the 2g nodes in the conflict graph, and by Equation 20 and a corresponding version of Equation 14 the degree of the conflict graph is reduced below g with probability at least

\[
1 - \left( 9ge^{-\delta^2(1-\delta)^3} + e^{1/3} + 4ge^{-\delta^2 g/4} \right).
\]

Note that this is \( 1 - o(1) \) as \( g \) grows.

To get a feeling of the performance of our randomized algorithm, we can set \( \varepsilon(g) \approx 0.073 \) in the proof of the above theorem, in such a
way that \( c + \varepsilon(g) = 4 \). The result is claimed in the following corollary.

**Corollary 3.9:** A POPS\((d, g)\) network can route any permutation in \( \frac{2d}{g} + O(\log \log g) \) slots with high probability.

## IV. Experiments

Our results in Theorems 3.7 and 3.8 are asymptotic. In principle, it could thus be possible that the randomized algorithm does not perform well in practice. This is not the case. Experiments show that it outperforms the algorithm in [8] even on networks as small as a POPS\((2, 2)\), and prove to be exponentially faster when \( d \) and \( g \) grow.

The algorithm in [8] is claimed to run in \( \frac{8d}{g} \log^2 g + \frac{2d}{g} \log g + 3 \log g + 7 \) slots. However, the authors make a small mistake when saying that Leighton’s implementation of the odd-even merge sort algorithm is composed of \( \log^2 n \) steps. The actual complexity is only \( \log \left( 1 + \frac{1}{g-d} \right) \approx 2 \log^2 g \) steps. So, the running time of the routing algorithm in [8] is \( \frac{8d}{g} \log^2 g + \frac{2d}{g} \log g + \frac{2d}{g} \log g + 3 \log g + 7 \) slots, that is smaller, and this is what we will use in the following.

To perform the experiments, we built a simulator for the POPS network. It is written in C++ and simulates the network at a message level. That is, for every message in the real network, there is a message in the simulator. Processors (implemented as instances of a class Processor) locally take decisions about the next step to perform, and couplers (implemented as instances of a class Coupler) locally propagate messages or stop them in case of conflicts.

Then, we implemented our randomized algorithm in the simulator, slot by slot. We have been conservative, no theoretical result is taken for granted and the randomized algorithm is just simulated by message. Not surprisingly, slots 3, 4, and 5 prove to be conflict-less, supporting what is proven in Proposition 3.2. So, whenever a copy survives slots 1 and 2 it reaches its final destination, and the associated ack successfully gets to the source processor. Moreover, three buffers in every processor \( i \) (one for packet \( p_1 \), one for packet \( p_{g-1} \), and the third for floating copies of other packets) are enough.

In Figure 4, it is shown the average over a large number of experiments in the case when \( d = g \). The number of processors \( n = dg \) goes from 4 to 16,777,216. The permutation in input is chosen uniformly at random from the class of all possible permutations. It is clear, from the results shown in the figure, that our algorithm is much faster than the algorithm in [8] even in practice. Actually, our algorithm outperforms its competitor for all network sizes hence putting aside any possible concern about the hidden constants. The performance of our algorithm is so good that it is actually hard to appreciate it from Figure 4. Hence, Table I shows the exact numerical results.

| \( n \) | \( d = g \) | \( d = 4g \) | \( d = 16g \) |
|-------|-------|-------|-------|
| \( A \) | \( A \) | \( A \) | \( A \) |
| \( 4 \) | 14.75 | 177 | - |
| \( 16 \) | 20.90 | 71.40 | 113 |
| \( 64 \) | 27.35 | 82.80 | 177 |
| \( 256 \) | 30.10 | 87.15 | 268 |
| \( 1,024 \) | 32.50 | 92.60 | 391 |
| \( 4,096 \) | 34.50 | 94.00 | 546 |
| \( 16,384 \) | 35.55 | 94.95 | 733 |
| \( 65,536 \) | 35.55 | 95.15 | 925 |
| \( 262,144 \) | 35.55 | 95.35 | 1,203 |
| \( 1,048,576 \) | 35.45 | 95.65 | 1,486 |
| \( 4,194,304 \) | 35.70 | 96.25 | 1,801 |
| \( 16,777,216 \) | 40.05 | 97.05 | 2,148 |

**TABLE I**

**NUMBER OF SLOTS TO ROUTE A RANDOMLY CHosen PERMUTATION BY OUR RANDOMIZED ALGORITHM (A) AND BY THE ALGORITHM IN [8] (B).**

## V. Conclusion

In this paper, we introduced the fastest algorithms for both deterministic and randomized on-line permutation routing. Indeed, we have shown that any permutation can be routed on a POPS\((d, g)\) network either with \( O(\frac{2d}{g} \log g) \) deterministic slots, or, with high probability, with \( 5c[d/g] + o(d/g) + O(\log \log g) \) randomized slots, where \( c = \exp(1 + e^{-1}) \approx 3.927 \). The randomized algorithm shows that the POPS network is one of the fastest permutation networks ever. This can be of practical relevance, since fast switching is one of the key technologies to deliver the ever-growing amount of bandwidth needed by modern network applications.

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Fig. 4. Performance of our randomized routing algorithm against the routing algorithm proposed in [8]. Case when \( d = g \). The number of processors goes from 4 to 16,777,216 (note that axis \( x \) is in logscale).

Fig. 5. Performance of our randomized routing algorithm against the routing algorithm proposed in [8]. Case when \( d = 4g \). The number of processors goes from 16 to 16,777,216 (note that axis \( x \) is in logscale).
Fig. 6. Performance of our randomized routing algorithm against the routing algorithm proposed in [8]. Case when $d = 16g$. The number of processors goes from 64 to 16,777,216 (note that axis $x$ is in log scale).

TABLE II

| $n$  | $d = g$ | $\mu$ | $\sigma$ | max   | $d = 4g$ | $\mu$ | $\sigma$ | max   | $d = 16g$ | $\mu$ | $\sigma$ | max   |
|------|---------|-------|----------|------|---------|-------|----------|------|---------|-------|----------|------|
| 4    | 3.15    | 1.94  | 12       | -    | 5.05    | 4.13  | 23       | -    | 10.00    | 5.16  | 42       | 82   |
| 16   | 4.43    | 1.03  | 8        | 14.33| 4.22    | 35    | 60.24    | 5.16 | 16.77    | 4.52  | 84       | 84   |
| 64   | 5.39    | 0.79  | 7        | 16.13| 2.31    | 22    | 56.88    | 4.52 | 8.12     | -     | -        | -    |
| 256  | 6.10    | 0.57  | 8        | 18.06| 1.54    | 23    | 62.58    | 3.86 | 8.12     | -     | -        | -    |
| 1,024| 6.50    | 0.53  | 8        | 18.45| 0.86    | 20    | 66.26    | 5.16 | 9.4       | -     | -        | -    |
| 4,096| 6.82    | 0.46  | 8        | 18.81| 0.64    | 21    | 68.21    | 3.94 | 8.6       | -     | -        | -    |
| 16,384| 7.04   | 0.20  | 8        | 18.95| 0.46    | 20    | 67.65    | 1.76 | 7.3       | -     | -        | -    |
| 65,536| 7.16   | 0.37  | 8        | 19.06| 0.34    | 20    | 67.12    | 0.89 | 7.1       | -     | -        | -    |
| 262,144| 7.30  | 0.46  | 8        | 19.09| 0.29    | 20    | 66.88    | 0.59 | 6.9       | -     | -        | -    |
| 1,048,576| 7.59 | 0.49  | 8        | 19.15| 0.36    | 20    | 66.70    | 0.50 | 6.8       | -     | -        | -    |
| 4,194,304| 7.92 | 0.29  | 8        | 19.21| 0.41    | 20    | 66.59    | 0.49 | 6.7       | -     | -        | -    |
| 16,777,216| 8.00 | 0.00  | 8        | 19.41| 0.49    | 20    | 66.79    | 0.41 | 6.7       | -     | -        | -    |

TABLE II

Number of iterations (mean, standard deviation, and worst case over one hundred runs) to route a randomly chosen permutation by our randomized algorithm.