Residues formulae for volumes and Ehrhart polynomials of convex polytopes.

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January 2001

Introduction

These notes on volumes and Ehrhart polynomials of convex polytopes are afterwards thoughts on lectures already delivered in Roma, by the second author in December 1999. The subject of these lectures was to explain Jeffrey-Kirwan residue formula for volumes of convex polytopes [J-K] and the residue formula for Ehrhart polynomials of rational polytopes [B-V 1, 2].

The main concept used in these formulae is the study of rational functions with poles on an arrangement of hyperplanes and the total residue of such a function, which can be computed recursively. Now, what about “concrete” polytopes? The residue formula leads in principle to an answer, but to be carried out in practice, without further thinking, requires too many steps. Thus we have been fascinated by calculations of Chan-Robbins in [CR], and the conjectures of Chan-Robbins-Yuen [C-R-Y] on the volume of the a certain polytope (proved by Zeilberger [Z]) and of some of its faces, as well as by other examples given by Pitman-Stanley [Pi-S], Stanley [S]. All these examples are related with the root system $A_n$. Therefore we will study here in more details what can be said on residues formulae for this root system and subsets of this system.

Let us sketch briefly the content of these notes. Short bibliographical comments are also given at the end of each section. We first recall the definition of total residue in Section 1 and Jeffrey-Kirwan residue formula for volumes of general convex polytopes (and give a proof) in Section 2 and Appendix (Section 14). A formula for change of variables in total residues
is obtained in Section 9. We go on considering flow polytopes (see Examples below, and Definition in Section 3), and defining a special class of flow polytopes: the cascade polytopes. In Section 4, we show that the Jeffrey-Kirwan residue formula for the volume of cascade polytopes is an iterated constant term formula. We then give a proof of a divisibility property between the volume of a cascade polytope and the volume of some particular face (this implies Conjecture 4 of [C-R-Y]) and a similar divisibility property for Ehrhart polynomial. We carry out the computation of the residue for the complete flow polytope, following the indications of Zeilberger [Z]. It thus leads to the known formula [Z] for the volume of the complete flow polytope. Our method is algebraic though combinatorists would probably prefer another type of proof. For example, Chan-Robbins-Yuen proved that their polytope $P$ could be explicitly decomposed in a union of $K$ elementary simplices (each of volume equal $\frac{1}{(\dim P)!}$). Later $K$ was computed by Zeilberger. Here we apply directly the residue formula for the volume, and we do not use any simplicial decomposition.

In Section 8, we state a residue formula for Ehrhart polynomials of flow polytopes (the proof is not given here, but is similar to the one given for volumes; it is a reformulation of Khovanskii-Pukhlikov theorem [KP]).

In Section 10, as suggested by R. Stanley, we show that formulae for volumes of a family of flow polytopes can be transmuted very explicitly to formulae for Ehrhart polynomials, due to the formula of change of variables in residues. Sections 4, 5, 6 and 7 can be skipped if the reader is interested only in the nice formula of Section 10. Finally, we state a symmetry property that must satisfy the Kostant partition function.

Let us now state here more precisely the setting and the contents of these notes.

Let $V$ be a $r$-dimensional real vector space and $V^*$ its dual vector space. Let $\Phi = \{\alpha^1, \alpha^2, ..., \alpha^N\}$ be a sequence of non-zero linear forms on $V^*$ all lying in an open half space (we do not assume the $\alpha^k$ to be distinct) and spanning $V^*$. We denote by $\Delta^+$ the set $\{\Phi\}$ (we mean $\Delta^+$ and $\Phi$ are the same sets, but $\Phi$ may have multiplicities), and by $\Delta$ the set $\Delta^+ \cup -\Delta^+$. The closed convex cone $C(\Delta^+)$ generated by $\Delta^+$ is decomposed as a union of closure of big chambers $c$ as in [B-V 1]. For $a \in V^*$, we denote by $P_{\Phi}(a) \subset \mathbb{R}_+^N$ the convex polytope consisting of all solutions $(u_1, u_2, ..., u_N)$, in non negative
real numbers $u_k$, of the equation

$$\sum_{k=1}^{N} u_k \alpha^k = a$$

and by $\text{vol} P_\Phi(a)$ its volume. Any convex polytope can be realized in that way.

Consider the space $R_\Delta$ of rational functions of $x \in V_C$ with poles on the union $H_C$ of the hyperplanes $\alpha^k(x) = 0$. A subset $\sigma$ of $\Delta$ is called a basis of $\Delta$, if the elements $\alpha \in \sigma$ form a basis of $V^*$. For such $\sigma$, set

$$f_\sigma(x) := \frac{1}{\prod_{\alpha \in \sigma} \alpha(x)}.$$ 

In appropriate coordinates $x_1, \ldots, x_r$, the function $f_\sigma$ is simply $\frac{1}{x_1 x_2 \ldots x_r}$ and we denote by $S_\Delta$ the subspace of $R_\Delta$ spanned by such “simple” elements $f_\sigma$. The vector space $S_\Delta$ is contained in the homogeneous component of degree $-r$ of $R_\Delta$ and we have the direct sum decomposition

$$R_\Delta = S_\Delta \oplus \left( \sum_{k=1}^{r} \partial_k R_\Delta \right).$$

We call the projection map

$$T res_\Delta : R_\Delta \to S_\Delta$$

according to this decomposition the total residue map. This projection vanishes outside the homogeneous component of degree $-r$ of $R_\Delta$. The total residue of a function is again a function. It consists of computing a rational function $f(x)$ up to derivatives, in other words, $f(x)dx$ and $(T res_\Delta(f))(x)dx$ are top dimensional holomorphic forms representing the same cohomology class on $V_C - H_C$.

We consider $a \in V^*$. Then

$$J_\Phi(a)(x) = T res_\Delta \left( \frac{e^{\langle a, x \rangle}}{\prod_{k=1}^{N} \alpha^k(x)} \right) = \frac{1}{(N-r)!} T res_\Delta \left( \frac{\langle a, x \rangle^{N-r}}{\prod_{k=1}^{N} \alpha^k(x)} \right)$$

is an element of the vector space $S_\Delta$. 

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We denote by
\[ K_\Phi(a) = \text{Tr} e_s \Delta \left( \frac{e^{(a,x)}}{\prod_{k=1}^N (1 - e^{(a^k,x)})} \right) \]
the “periodic” version of \( J_\Phi \). This is also an element of the vector space \( S_\Delta \).

Now, to each big chamber \( c \) of the subdivision of \( C(\Delta^+) \) is associated a linear form \( f \to \langle c, f \rangle \) on \( S_\Delta \). It takes value 1 or 0 on a normalized multiple of \( f_\sigma \) whether or not \( c \) is contained in \( C(\sigma) \). This is explained in Section 2 and in the Appendix (Section 14). Here are the two fundamental formulae we will use in these notes.

**Formula 1:** for \( a \in \mathfrak{c} \), we have
\[ \text{vol} P_\Phi(a) = \langle c, J_\Phi(a) \rangle. \]

If \( \Phi \) spans a lattice in \( V^* \) and if \( a \) belongs to this lattice, we denote by \( k_\Phi(a) \) the number of all solutions \((u_1, u_2, \ldots, u_N)\), in non negative integral numbers \( u_k \) of the equation
\[ \sum_{k=1}^N u_k \alpha^k = a. \]

Similarly (under simplifying assumptions that we do not state in the introduction), the following formula holds.

**Formula 2:** for \( a \in \mathfrak{c} \), we have
\[ k_\Phi(a) = \langle c, K_\Phi(a) \rangle. \]

In particular, as well known, the functions \( a \mapsto \text{vol} P_\Phi(a) \) and the Ehrhart function \( a \mapsto k_\Phi(a) \) are polynomials on a big chamber \( c \). (The notion of Minkowski sums and of big chambers are intimately related: if \( a_1, \ldots, a_m \) are vectors in the closure of a big chamber \( c \), then the polytope \( P_\Phi(a_1 + \cdots + a_m) \) is isomorphic to the Minkowski sum of the polytopes \( P_\Phi(a_k) \).

This shows that calculation of volumes (or of Ehrhart polynomials) is done by an algebraic version of “integration”: it consists of computing particular rational function \( f(x) \) on \( V_\mathfrak{c} \) modulo derivatives. Furthermore this formula shows clearly that the \( S_\Delta \)-valued polynomial function \( a \to J_\Phi(a) \) determines entirely the locally polynomial function \( a \to \text{vol} P_\Phi(a) \).

Thus the calculation of the volume of a flow polytope \( P_\Phi(a), a \in \mathfrak{c} \), is divided in TWO problems.
A) Compute the linear form $f \rightarrow \langle \langle \mathbf{c}, f \rangle \rangle$.

B) Compute the function

$$
\frac{e^{(a,x)}}{\prod_{k=1}^{N} \alpha^k(x)}
$$

up to derivatives.

We will now study some particular cases, where A) or B) can be solved.

The Chan-Robbins-Yuen polytope $CRY_r$ consists of solutions $(x_{ij}) \geq 0$ where $1 \leq i < j \leq (r + 1)$ of the linear equations:

- For $2 \leq j \leq r$,
  $$
  \sum_{i=1}^{j-1} x_{ij} = \sum_{k=j+1}^{r+1} x_{jk}.
  $$

- $r+1$ \sum_{i=2}^{r+1} x_{1,i} = 1,$
- $r+1$ \sum_{i=1}^{r} x_{i,r+1} = 1,$
The CRY polytope is also called the complete flow polytope. Indeed, consider a graph $G$ with $r + 1$ vertices $1, 2, 3, ..., r, r + 1$ and edges $i \mapsto j$ ($1 \leq i < j \leq r + 1$), for all $i < j$.

We can imagine that the positive quantity $x_{ij}$ is the quantity of liquid at time $t$ in the branch $i \mapsto j$ of this cascade. Thus the linear equations above reflect the constant flow of the cascade (See Figure 1).

We can also consider other graphs with possibly multiple edges. The associated polytope will be called a flow polytope. See the precise definition in Section 3. For example, Pitman-Stanley polytope is the polytope consisting of solutions in non-negative numbers of the inequations

$$y_{1,r+1} \leq a_1,$$

$$y_{1,r+1} + y_{2,r+1} \leq a_1 + a_2,$$
\[ y_{1,r+1} + y_{2,r+1} + y_{3,r+1} \leq a_1 + a_2 + a_3, \]

... where \( a_i \) are non-negative real numbers.

It is associated to a flow graph, with \( r + 1 \) vertices, and edges from \( i \) to \( r + 1 \) and from \( i \) to \( i + 1 \) and last edge \( r \mapsto r + 1 \) of multiplicity 2.

We can imagine that \( a_1, a_2, ..., a_r \) are additional sources of water, so that the quantities of water \( y_{i,r+1} \) in the streams \( i \mapsto r + 1 \) satisfies the inequalities stated above, with a leakage of \( z_{i,i+1} \). (See g 2).

In the rest of these notes, we consider the positive root system

\[ A_r^+ = \{(e^i - e^j), 1 \leq i < j \leq (r + 1)\}. \]

The system \( A_r^+ \) spans the vector space

\[ E_r = \{a = a_1 e^1 + a_2 e^2 + ... + a_{r+1} e^{r+1}, \sum_{i=1}^{r+1} a_i = 0\}. \]

The cone \( C(A_r^+) \subset E_r \) generated by positive roots is the cone \( a_1 \geq 0, a_1 + a_2 \geq 0, ..., a_1 + a_2 + ... + a_r \geq 0, a_1 + a_2 + ... + a_{r+1} = 0 \). We denote by \( c^+ \) the open set of \( C(\Delta^+) \) defined by

\[ c^+ = \{a \in C(A_r^+) \text{ such that } a_i > 0, i = 1, ..., r\}. \]

It is a big chamber of our subdivision, and will be called the nice chamber.

Thus the functions \( a \mapsto \text{vol} P_\Phi(a) \) and the Ehrhart function \( a \mapsto k_\Phi(a) \) on the chamber \( c^+ \) are the mixed volumes and the mixed lattice point enumerators of the fountain polytopes \( P_k = P_\Phi(e^k - e^{r+1}) \).

Let \( \Phi \) be a sequence of \( N \) elements of \( A_r^+ \) spanning \( E_r \). By definition, a flow polytope is a polytope isomorphic to a polytope \( P_\Phi(a) \), where \( a \) is an element of \( C(A_r^+) \) and a cascade polytope the special class of flow polytopes \( P_\Phi(a) \), where \( a \) is constrained to be in the nice chamber of \( C(A_r^+) \). The flow graph associated to \( \Phi \) has \( m_{ij} \) edges from \( i \) to \( j \), if \( (e^i - e^j) \) has multiplicity \( m_{ij} \geq 0 \) in \( \Phi \). In fact, any polytope associated to a smooth toric variety (i.e. Delzant polytopes and limits) can be realized as a flow polytope (Szemes: personal communication).

The set of simple roots

\[ \Pi = \{(e^1 - e^2), (e^2 - e^3), ..., (e^{r-1} - e^r), (e^r - e^{r+1})\} \]
$y_{14} \leq 0$

$y_{14} + y_{24} \leq a_1 + a_2$

$y_{14} + y_{24} + y_{34} \leq a_1 + a_2 + a_3$

Figure 2: Graph for $\pi_3(a)$. 
is a basis of $A_r^+$ and the element

$$f_{II} = \frac{1}{(e^1 - e^2)(e^2 - e^3) \cdots (e^{r-1} - e^r)(e^r - e^{r+1})}$$

is an element of $S_{A_r}$. The dimension of $S_{A_r}$ is $r!$ and a particularly nice basis of $S_{A_r}$ is given by the elements $w \cdot f_{II}$ where $w \in \Sigma_r$, the permutation group of $r$ elements, acts by permutation on the set $\{1, 2, ..., r\}$. By THEOREM 19, the linear form $f \mapsto \langle\langle \epsilon^+, f \rangle\rangle$ is the “simplest” possible: the iterated residue $I_{res_{x=0}}$. This is defined as follows: a function in $R_{A_r}$ is identified with a rational function $f(x_1, ..., x_r)$ (setting $e^{r+1} = 0$) with poles on the hyperplanes $x_i = x_j$, or $x_i = 0$, and we have

$$\langle\langle \epsilon^+, f \rangle\rangle = I_{res_{x=0}} = \text{Res}_{x_1=0} \text{Res}_{x_2=0} \cdots \text{Res}_{x_r=0} f(x_1, x_2, ..., x_r).$$

Thus the volume of a cascade polytope is given by an iterated residue formula. But problem B still remains to be solved.

When $a$ is a root of the system $A_{r+1}$, the calculation of the total residue of the particular rational function

$$J_\Phi(a)(x) = \frac{\langle a, x \rangle^{r(r+1)/2}}{\prod_{\alpha \in A_{r+1}} \langle \alpha, x \rangle}$$

(as well as very similar examples) will be done in Section 6. Let $a = e^0 - e^{r+1}$. Setting as before $e^{r+1} = 0$,

$$J_\Phi(a)(x_0, ..., x_r) = \frac{x_0^{r(r+1)/2}}{x_0 x_1 \cdots x_r \prod_{0 \leq i < j \leq r} (x_i - x_j)}.$$

The “integration” of this function is very familiar in the context of Selberg integral ([Se]). However the usual factor $\prod_{0 \leq i < j \leq r}(x_i - x_j)$ here is in the denominator and “Integration” means that we explicitly compute

$$\frac{1}{x_1 \cdots x_r \prod_{0 \leq i < j \leq r} (x_i - x_j)}$$

modulo partial derivatives $\partial_{x_1}, \partial_{x_2}, ..., \partial_{x_r}$, following Aomoto [A]. In THEOREM 26, we obtain the formula.
\[ T_{res}A^+_r \left( \frac{(e^0 - e^{r+1})^{r+1/2}}{\prod_{0 \leq i < j \leq r+1} (e^i - e^j)} \right) \]

\[ = \prod_{i=1}^{r} \frac{(2i)!}{i!(i+1)!} \sum_{w \in \Sigma_r} c(w)w \cdot \frac{1}{(e^0 - e^1)(e^1 - e^2) \cdots (e^{r-1} - e^r)(e^r - e^{r+1})} \]

where \( \Sigma_r \) acts by permutations on \( \{1, 2, \ldots, r\} \). Similar formulae are also obtained. As pointed out by Zeilberger [Z], these calculations are mere reformulations of Morris Identity [M]. The total residue formula replaces here the iterated constant term. It would be interesting to generalize them to other root systems.

The Chan-Robbins polytope is described as a face of the polytope of doubly stochastic matrices in Section 7, as Chan-Robbins-Yuen originally described it. We apply then THEOREM 27, to prove a formula for the volume of \( CRY_n \), and PROPOSITION 23 for the volume of a particular face of it.

In Section 9, we prove a formula of Change of variables in total residue.

As a corollary, the \( S_\Delta \)-valued polynomial function \( K_\Phi(a) \) for \( \Phi \) a sequence of \( N \) elements of \( A^+_r \) is deduced from the function \( J_\Phi(a) \), in a way which will be explained now. Define

\[ t^\Phi_j = \sum_{k=j+1}^{r+1} m_{jk} - 1, \]

\[ s^\Phi_j = 1 - \sum_{k=1}^{j-1} m_{kj}. \]

We write the vector valued polynomial \( J_\Phi \), which is homogeneous of degree \( (N - r) \),

\[ J_\Phi(a_1e^1 + a_2e^2 + \ldots + a_re^r - (a_1 + a_2 + \ldots + a_r)e^{r+1}) \]

\[ = \sum_{i_1+i_2+\ldots+i_r=N-r} \frac{a_1^{i_1}a_2^{i_2} \cdots a_{r-1}^{i_{r-1}}a_r^{i_r}}{i_1!i_2! \cdots i_{r-1}!i_r!} f_\Phi(i) \]

where \( f_\Phi(i) \) are elements of \( S_\Delta \).

Then we obtain THEOREM 10:

\[ K_\Phi(a_1e^1 + a_2e^2 + \ldots + a_re^r - (a_1 + a_2 + \ldots + a_r)e^{r+1}) \]
Applying FORMULAE (1) and (2) for the volume and Ehrhart polynomials, we see that the Ehrhart polynomial on a big chamber c is immediately deduced from the polynomial function vol $P_\Phi(a)$ (i.e; its highest degree component) by replacing the monomial $\binom{a_k}{i_k}$ by the function $\binom{a_k+t_k^\Phi}{i_k}$ (with same leading term). The function $a \mapsto \binom{a}{i}$ is more adequate than the monomial $a^k/k!$ in the integral context, as it takes integral values on integers.

THEOREM [16] is a generalization of B.V. Lidskii formula for Kostant partition function [Li]. It was also proven by Stanley for general flow polytopes. R. Stanley suggested to look for a proof via residues.

In Section [14], we apply the “nice formula” to a polytope considered by Pitman and Stanley [Pi-S].

Finally, in Section [13], using the action of the full Weyl group $\Sigma_{r+1}$ on $S_{A_r}$, we list $r-1$ symmetries properties of the Kostant polynomial

$$k(A^+, c^+) = (a_1(e_1-e_r)+\cdots+a_{r-2}(e_{r-2}-e_r)+a_{r-1}(e_{r-1}-e_r)+a_r(e_r-e_1))$$

(This is the mixed lattice point enumerator $N(\sum a_k P_k)$ of the Chan-Robbins-Yuen polytopes $P_k = P^+_{A_r}(e_k-e_r)$, $1 \leq k \leq r$.) The function $k(A^+, c^+) (a)$ is a function of $(r-1)$ variables $k^+(a_1, a_2, \ldots, a_{r-1})$. This function satisfies for example, the properties:

- for any values $x_1, \ldots, x_r$, we have

$$k^+(x_1, \ldots, x_{r-2}, -(x_1 + \cdots + x_{r-2} + x_{r-1} + x_r + 2)) =$$

$$-k^+(x_1, x_2, \ldots, x_{r-2}, x_{r-1}) - k^+(x_1, \ldots, x_{r-2}, x_r - 1).$$

- for any values $x_1, \ldots, x_r$, we have

$$k^+(x_1, x_2, \ldots, x_{r-3}, -(x_1 + x_2 + \cdots + x_r + 3), x_{r-1}) =$$

$$-k^+(x_1, x_2, \ldots, x_{r-3}, x_{r-2}, x_r - 1) + k^+(x_1, x_2, \ldots, x_{r-3}, x_{r-1} - 1, x_{r-2} + 1)$$

$$-k^+(x_1, x_2, \ldots, x_{r-3}, x_{r-2} - 2, x_{r-2} + 1) + k^+(x_1, x_2, \ldots, x_{r-3}, x_{r-1} - 1, x_r - 1).$$

We do hope to have shown in these notes that the space $S_\Delta$, well known in the study of hyperplanes arrangements, is providing an efficient tool in the computation of Volumes and Ehrhart polynomials of polytopes. However, we also learnt in writing these notes how difficult and exciting are “concrete polytopes”.

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1 Total residues

Let $V$ be a $r$-dimensional real vector space and $V^*$ its dual vector space. Let $\Delta \subset V^*$ be a finite subset of non-zero linear forms. We assume $\Delta$ symmetric: $\Delta = -\Delta$. Each $\alpha \in \Delta$ determines a linear form on $V$ and a complex hyperplane $\{x \in V_C; \alpha(x) = 0\}$ in $V_C$. Consider the hyperplane arrangement

$$\mathcal{H}_C = \bigcup_{\alpha \in \Delta} \{x \in V_C, \alpha(x) = 0\}.$$  

The ring $R_\Delta$ of rational functions with poles on $\mathcal{H}_C$ is the ring $\Delta^{-1}S(V^*)$ generated by the ring $S(V^*)$ of polynomial functions on $V$, together with inverses of the linear functions $\alpha \in \Delta$.

A subset $\sigma$ of $\Delta$ is called a **basis of $\Delta$**, if the elements $\alpha \in \sigma$ form a basis of $V^*$. We denote by $\mathcal{B}(\Delta)$ the set of bases of $\Delta$. For $\sigma$ a basis of $\Delta$, set

$$f_\sigma(x) := \frac{1}{\prod_{\alpha \in \sigma} \alpha(x)}.$$  

**Definition 1** The subspace $S_\Delta$ of $R_\Delta$ spanned by the elements $f_\sigma$, $\sigma \in \mathcal{B}(\Delta)$, will be called the space of simple elements of $R_\Delta$:

$$S_\Delta = \sum_{\sigma \in \mathcal{B}(\Delta)} \mathbb{R}f_\sigma.$$  

The vector space $S_\Delta$ is contained in the homogeneous component of degree $-r$ of $R_\Delta$. If $\Delta$ does not span $V^*$, then the set $\mathcal{B}(\Delta)$ is empty and $S_\Delta = 0$.

In general, elements $f_\sigma$ are not linearly independent.

**Example 1.**

Let $V$ be a 2-dimensional vector space with basis $e_1, e_2$. Let $\Delta$ be the set

$$\Delta = \{ \pm e^1, \pm e^2, \pm (e^1 - e^2) \}.$$  

Then we have the linear relation

$$\frac{1}{e^1 e^2} = \frac{1}{e^2(e^1 - e^2)} - \frac{1}{e^1(e^1 - e^2)}$$  

between elements $f_{\sigma_1}, f_{\sigma_2}, f_{\sigma_3}$ with $\sigma_1 = \{ e^1, e^2 \}$, $\sigma_2 = \{ e^1, (e^1 - e^2) \}$ and $\sigma_3 = \{ e^2, (e^1 - e^2) \}$ bases of $\Delta$.  

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We let elements $v$ of $V$ act on $R_\Delta$ by differentiation:

$$(\partial(v)f)(x) := \frac{d}{d\epsilon}f(x + \epsilon v) |_{\epsilon=0}.$$ 

Then the following holds ([B-V 2], Proposition 7.)

**Theorem 2**

$$R_\Delta = \partial(V)R_\Delta \oplus S_\Delta.$$ 

As a corollary of this decomposition, we can define the projection map $Tres_\Delta : R_\Delta \to S_\Delta.$

The projection $(Tres_\Delta f)(x)$ of a function $f(x)$ is again a function of $x$ that we called the **total residue** of $f.$ By definition, this function can be expressed as a linear combination of the simple fractions $f_\sigma(x).$ The main property of the map $Tres_\Delta$ is that it vanishes on derivatives. If $f \in R_\Delta = \prod_{k} P_k^{\alpha_k}$ ($P \in S(V^*), \alpha_k \in \Delta$) has a denominator product of linear forms $\alpha_k \in \Delta$ which do not generate $V^*$, then it is easy to see that $f$ is a derivative and the total residue of $f$ is equal to 0.

**Example 2:** Let us do a computation. Let $V$ be with basis $e_0, e_1, e_2$ and let

$$\Delta = \{ \pm e^1, \pm e^2, \pm (e^0 - e^1), \pm (e^0 - e^2), \pm (e^1 - e^2) \}.$$ 

We write $x \in V$ as $x = x_0 e_0 + x_1 e_1 + x_2 e_2.$ The reason, for writing $x_i$ instead of $x^i$ is in order not to misinterpret an upper index with a power. Consider the following function $W_3$ of $R_\Delta$:

$$W_3(x_0, x_1, x_2) = \frac{x_0^2}{(x_0 - x_1)(x_0 - x_2)(x_1 - x_2)x_1 x_2}.$$ 

Then $W_3$ is homogeneous of degree $-3.$ To compute the total residue of $W_3(x_0, x_1, x_2),$ we write $x_0$ as a linear combination of linear forms in the denominator of $W_3,$ in order to reduce the degree of the denominator. For example, writing $x_0^2 = ((x_0 - x_1) + x_1)((x_0 - x_2) + x_2),$ we obtain

$$W_3(x_0, x_1, x_2) = \frac{1}{(x_1 - x_2)x_1 x_2} + \frac{1}{(x_0 - x_2)(x_1 - x_2)x_1} + \frac{1}{(x_0 - x_1)(x_1 - x_2)x_2} + \frac{1}{(x_0 - x_1)(x_0 - x_2)(x_1 - x_2)}.$$ 

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The first and last fractions have denominators with linearly dependent forms, so that their total residue is zero and we obtain:

\[ T_{\text{res}}(W_3(x)) = \frac{1}{(x_0 - x_1)(x_1 - x_2)x_2} + \frac{1}{(x_0 - x_2)(x_1 - x_2)x_1}. \]

(More precisely, in the direct sum decomposition,

\[ R_{\Delta} = S_{\Delta} \oplus \partial_{x_0} R_{\Delta} \oplus \partial_{x_1} R_{\Delta} \oplus \partial_{x_2} R_{\Delta}, \]

we have,

\[ W_3(x_0, x_1, x_2) - T_{\text{res}}(W_3(x)) = U_3(x_0, x_1, x_2), \]

with

\[ U_3(x_0, x_1, x_2) = -\partial_{x_1} \frac{x_0 - 2x_2}{x_2(x_0 - x_2)(x_1 - x_2)} - \partial_{x_2} \frac{x_0 - 2x_1}{x_1(x_0 - x_1)(x_1 - x_2)}. \]

If \( V \) is one dimensional, and \( \Delta = \{ \pm e^1 \} \), then \( R_{\Delta} \) is the ring of Laurent series

\[ L = \{ f(x) = \sum_{k \geq -q} a_k x^k \}. \]

The total residue of a function \( f(x) \in L \) is the function \( \frac{a_{-1}}{x} \). The usual residue denoted \( \text{Res}_{x=0} f \) is the constant \( a_{-1} \). The constant term \( a_0 \) of the Laurent series \( f(x) \) is denoted by \( \text{CT}_{x=0} f \). We have \( \text{CT}_{x=0} x f(x) = \text{Res}_{x=0} f(x) \).

We will also use the following obvious properties.

**Lemma 3**

- Assume \( \Gamma \subset \Delta \) is a subset of \( \Delta \). Then
  \[ R_{\Gamma} \subset R_{\Delta}, \]
  \[ S_{\Gamma} \subset S_{\Delta}. \]

  Furthermore, if \( f \in R_{\Gamma} \), then \( T_{\text{res}}(\Delta)(f) \) belongs to \( S_{\Gamma} \) and
  \[ T_{\text{res}}(\Gamma)(f) = T_{\text{res}}(\Delta)(f). \]

- Assume that \( V = V_1 \oplus V_2 \) and \( \Delta = \Delta_1 \cup \Delta_2 \), with \( \Delta_i \subset V_i \), then
  \[ R_{\Delta} = R_{\Delta_1} \otimes R_{\Delta_2}, \]
  \[ S_{\Delta} = S_{\Delta_1} \otimes S_{\Delta_2}, \]
  \[ T_{\text{res}}(\Delta) = T_{\text{res}}(\Delta_1) \otimes T_{\text{res}}(\Delta_2). \]
Consider the vector space $\hat{S}(V^*)$ of formal power series on $V$. Define $R_\Delta = \Delta^{-1}\hat{S}(V^*)$. As the total residue vanishes outside the homogeneous component of degree $-r$ of $R_\Delta$, the map $Tres_\Delta$ extends as a map

$$Tres_\Delta : R_\Delta \to S_\Delta.$$  

Consider the open set $U_\Delta = V_C - H_C$, complement of the union of hyperplanes $\{\alpha = 0\}$. Choose a basis $e^i$ of $V^*$. This gives coordinates $x^i$ on $V_C$. Let $dx = dx^1 \wedge dx^2 \wedge \cdots \wedge dx^r$. Let $f \in R_\Delta$. Consider the $r$-holomorphic differential form $f(x)dx$, defined on the open set $U_\Delta$. Then $(f - Tres_\Delta(f))dx$ is an exact form on this open set: by definition of the total residue, the function $f - Tres_\Delta(f)$ is in the span of $\partial_k R_\Delta$, so $(f - Tres_\Delta(f))dx$ is the differential of some $(r-1)$-holomorphic differential form $\sum_{k=1}^r f_k dx^1 \wedge dx^2 \wedge \cdots \wedge dx^r$.

**Bibliographical remarks.**

The space $S_\Delta$ is isomorphic to the highest degree component of the Orlik-Solomon algebra $H^*(U_\Delta, \mathbb{C})$, described in [O-T] by generators and relations. The definition of the total residue is given in [B-V 2]. It formalizes notions introduced in Jeffrey-Kirwan [J-K]. Proposition 2 is proven in [B-V 2].

## 2 Jeffrey-Kirwan residue formula for volumes of convex polytopes

Now the vector space $V$ is equipped with a fixed Lebesgue measure $dx$. We denote by $da$ the dual measure on $V^*$. If $\sigma$ is a basis of $\Delta$, we denote by vol($\sigma$) the volume of the parallelepiped $\oplus_{\alpha \in \sigma}[0, 1] \alpha$, for our Lebesgue measure $da$.

We consider $\mathbb{R}^N$ with basis $(w^1, \ldots, w^N)$ and corresponding Lebesgue measure $dw$. Let $p$ be a surjective linear map from $\mathbb{R}^N$ to the vector space $V^*$. Then the vector space $\text{Ker}(p) = p^{-1}(0)$ is of dimension $d := N - r$. It is equipped with a Lebesgue measure $dm$ quotient of $dw$ by $da$. For $a \in V^*$, $p^{-1}(a)$ is an affine space parallel to $\text{Ker}(p)$, thus also equipped with the Lebesgue measure $dm$. Volumes of subsets of $p^{-1}(a)$ are computed for this measure.

We set $p(w^k) := \alpha^k$ for $1 \leq k \leq N$, and

$$\Phi = (\alpha^1, \ldots, \alpha^N).$$
We have thus, for \( u_i \in \mathbb{R} \),

\[
p(u_1w^1 + u_2w^2 + \ldots + u_Nw^N) = u_1\alpha^1 + u_2\alpha^2 + \cdots + u_N\alpha^N.
\]

The \( N \) elements \( \alpha^k \) of the sequence \( \Phi \) need not to be distinct. We denote by \( C_N \) the closed convex cone in \( \mathbb{R}^N \) generated by \( w^1, \ldots, w^N \), and we set \( C(\Phi) := p(C_N) \), the cone generated by \( (\alpha^1, \ldots, \alpha^N) \).

We assume that \( p^{-1}(0) \cap C_N = \{0\} \). Then 0 is not in the convex hull of the vectors \( \alpha^k \) and \( C(\Phi) \) is an acute cone.

**Definition 4** Let \( a \in V^* \). We define

\[
P_{\Phi}(a) := p^{-1}(a) \cap C_N.
\]

We immediately see that the set \( P_{\Phi}(a) \) is the convex polytope consisting of all solutions \((u_1, u_2, \ldots, u_N)\), in non negative real numbers \( u_k \), of the equation

\[
\sum_{k=1}^{N} u_k\alpha^k = a.
\]

In particular, the polytope \( P_{\Phi}(a) \) is empty if \( a \) is not in the cone \( C(\Phi) \).

**Remark:** Any convex polytope can be realized canonically as a polytope \( P_{\Phi}(a) \), see for example [B-V 1, Section 4.1].

The following lemma is obvious.

**Lemma 5** For any invertible transformation \( w \) of \( V^* \), then

\[
P_{w\Phi}(w \cdot a) = P_{\Phi}(a).
\]

For \( a \) in the interior of \( C(\Phi) \), the dimension of \( P_{\Phi}(a) \) is \( N - r \). The function \( a \mapsto \text{vol} P_{\Phi}(a) \) is a continuous function on \( C(\Phi) \), homogeneous of degree \( N - r \). This function is locally polynomial. Let us recall the description of regions where we are sure that this function is given by a polynomial formula.

We consider the set \( \{\Phi\} \subset V^* \) consisting of elements of \( \Phi \): we mean \( \Phi \) and \( \{\Phi\} \) are the same sets, but \( \Phi \) may have multiplicities. Let \( \Delta^+ \) be a finite subset of \( V^* \) containing the set \( \Phi \) and such that the cone \( C(\Delta^+) \) is acute. Usually, we take \( \Delta^+ = \{\Phi\} \). We define

\[
\Delta = \Delta^+ \cup -\Delta^+.
\]
For any subset $\nu^+$ of $\Delta^+$, we denote by $C(\nu^+)$ the closed cone generated by $\nu^+$. We denote by $C(\Delta^+)^{sing}$ the union of the cones $C(\nu^+)$ where $\nu^+$ is any subset of $\Delta^+$ of cardinal strictly less than $r = \dim(V)$. By definition, the set $C(\Delta^+)^{reg}$ of $\Delta^+$-regular elements is the complement of $C(\Delta^+)^{sing}$. A connected component of $C(\Delta^+)^{reg}$ is called a big chamber. If $c$ is a big chamber, and $\sigma^+$ a basis of $\Delta^+$, then either $c \subset C(\sigma^+)$, or $c \cap C(\sigma^+) = \emptyset$, as the boundary of $C(\sigma^+)$ does not intersect $c$. Thus the closure of the big chamber $c$ is the intersection of the simplicial cones $C(\sigma^+)$, $\sigma^+$ a basis of $\Delta^+$. Hence each small chamber is contained in a big chamber.

In the appendix, we describe big and small chambers $c$ when $\Delta^+$ is the positive root system of $A_n$, for $n = 2, 3$. See Figure 4 in Section 15.

Remark. When $a$ varies in a big chamber $c$, the combinatorial nature of $P_\Phi(a)$ remains the same, and the family of polytopes $P_\Phi(a)$ have parallel facets. The notion of Minkowski sums and of big chambers are intimately related: if $a_1, \ldots, a_m$ are vectors in the closure of a big chamber $c$, then the polytope $P_\Phi(a_1 + \cdots + a_m)$ is isomorphic to the Minkowski sum of the polytopes $P_\Phi(a_k)$. This follows for example of ([B-V 1 Section 3.1]).

The function $a \mapsto \text{vol} P_\Phi(a)$ is given by a polynomial formula, when $a$ varies in a big chamber $c$. Let us recall this formula.

The Jeffrey-Kirwan residue ([J-K]) associate to a big chamber $c$ of $C(\Delta^+)$ a linear form $f \mapsto \langle \langle c, f \rangle \rangle$ on the vector space $S_\Delta$ of simple fractions. To determine the linear map $f \mapsto \langle \langle c, f \rangle \rangle$, it is enough to determine it on the generating set $f_\sigma$, with $\sigma$ a basis of $\Delta$. If $\sigma = \sigma^+ \cup \sigma^-$ with $\sigma^+ \subset \Delta^+$ and $\sigma^- \subset \Delta^-$, then

$$f_\sigma = (-1)^{|\sigma^-|} f_{(\sigma^+ \cup (-\sigma^-))}.$$ 

Thus any function $f$ in $S_\Delta$ can be written as a linear combination of functions $f_\sigma$, with $\sigma$ a basis of $\Delta$ consisting of positive elements.

The linear form $\langle \langle c, f \rangle \rangle$ has the following properties:

A) If $\sigma$ is a basis of $\Delta^+$ (it is important to require that $\sigma$ consists of positive elements), then

a) If $c \subset C(\sigma)$,

$$\langle \langle c, f_\sigma \rangle \rangle = \frac{1}{\text{vol}(\sigma)},$$

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b) If $c \cap C(\sigma) = \emptyset$,

$$\langle\langle c, f_\sigma \rangle\rangle = 0.$$ 

B) If $\sigma = \sigma^+ \cup \sigma^-$ with $\sigma^+ \subset \Delta^+$ and $\sigma^- \subset \Delta^-$, then

$$\langle\langle c, f_\sigma \rangle\rangle = (-1)^{\sigma^-} \langle\langle c, f_{(\sigma^+ \cup \{-\sigma^-\})} \rangle\rangle.$$ 

Remark. We can interpret the linear form $f \mapsto \langle\langle c, f \rangle\rangle$ in terms of Laplace transform (see Section 13). Assume $\sigma$ is a basis of $\Delta$ contained in $\Delta^+$. Then we have:

$$\text{vol}(\sigma) f_\sigma(x) = \int_{C(\sigma)} e^{-(x,a)} da$$

whenever $x$ is in the dual cone of $C(\sigma)$. In particular, a function $f$ in $S_\Delta$, being a linear combination of such functions $f_\sigma$, coincide on the dual cone to $C(\Delta^+)$ with the Laplace transform of a function $\hat{f}$ locally constant on big chambers of $C(\Delta^+)$, and the linear form $\langle\langle c, f \rangle\rangle$ consists in evaluating the function $\hat{f}$ at a point of $c$.

Let $a \in V^*$. Let $f \in R_\Delta$. Consider

$$e^a f = \sum_{k=0}^{\infty} \frac{a^k}{k!} f.$$ 

This is an element of $\hat{R}_\Delta$. If $e_k$ is a basis of $V$, and we write $a \in V^* = \sum_{i=1}^r a_i e^i$ and $x \in V$ as $x = \sum_{k=1}^r x_k e_k$ (here, and in many other local calculations, we write $x_i$ instead of $x^i$, as we do not want to misinterpret an upper index as a power), then $e^a f$ is the analytic function on $V_C - \mathcal{H}_C$, given by $x \mapsto e^{a_1 x_1 + a_2 x_2 + \cdots + a_r x_r} f(x)$. Its total residue is defined by

$$\text{Tres}_\Delta(e^a f) = \sum_{k=0}^{\infty} \text{Tres}_\Delta(\frac{a^k}{k!} f).$$

There are only a finite number of non-zero terms in this sum. More precisely, if $f$ is homogeneous of degree $-q$, then

$$\text{Tres}_\Delta(e^a f) = \frac{1}{(q-r)!} \text{Tres}_\Delta(a^{q-r} f).$$
Definition 6 Define

\[ J_\Phi(a) := \text{Res}_\Delta \left( \frac{e^a}{\prod_{\alpha \in \Phi} \alpha} \right). \]

This is a polynomial function of \( a \in V^* \) with value in the vector space \( S_\Delta \).

More precisely, we choose a basis \( e_1, \ldots, e_r \) of \( V^* \), and write \( a = \sum_{i=1}^r a_i e_i \), \( x = \sum_{i=1}^r x_i e_i \). Let \( i \) be a sequence \( (i_1, i_2, \ldots, i_r) \) of non-negative integers. We write \( |i| := i_1 + i_2 + \cdots + i_r \). If \( |i| = N - r \), then the function

\[ \frac{x_1^{i_1} x_2^{i_2} \cdots x_r^{i_r}}{\prod_{\alpha \in \Phi} \alpha(x)} \]

is of homogeneous degree \(-r\).

Definition 7 Let \( f_\Phi(i) \) be the following element of \( S_\Delta \):

\[ f_\Phi(i)(x) = \text{Res}_\Delta \left( \frac{x_1^{i_1} x_2^{i_2} \cdots x_r^{i_r}}{\prod_{\alpha \in \Phi} \alpha(x)} \right). \]

Lemma 8 We have

\[ J_\Phi(a) = \sum_{|i| = N - r} \frac{a_1^{i_1} a_2^{i_2} \cdots a_r^{i_r}}{i_1! i_2! \cdots i_r!} f_\Phi(i). \]

Proof. We have

\[ J_\Phi(a)(x) = \text{Res}_\Delta \left( \frac{e^{a_1 x_1 + \cdots + a_r x_r}}{\prod_{\alpha \in \Phi} \alpha(x)} \right) \]

\[ = \sum_{i_1, i_2, \ldots, i_r} \frac{a_1^{i_1} a_2^{i_2} \cdots a_r^{i_r}}{i_1! i_2! \cdots i_r!} \text{Res}_\Delta \left( \frac{x_1^{i_1} x_2^{i_2} \cdots x_r^{i_r}}{\prod_{\alpha \in \Phi} \alpha(x)} \right). \]

The total residue is zero outside the homogeneous component of degree \(-r\). This gives the formula of the lemma.

QED

Thus the function \( a \mapsto J_\Phi(a) \) is an homogeneous polynomial in the variable \( a \) of degree \( N - r \), with value in the finite dimensional vector space \( S_\Delta \).
Theorem 9 (Jeffrey-Kirwan) Let $c$ be a big chamber of $C(\Delta^+)$. Then, if $a \in \overline{c}$, we have
\[ \text{vol } P_{\Phi}(a) = \langle \langle c, J_{\Phi}(a) \rangle \rangle. \]

We give the proof of this theorem in the appendix (Section 14).

Definition 10 Let $c$ be a big chamber. We denote by $v(\Phi, c)(a)$ the polynomial function on $V^*$ such that
\[ v(\Phi, c)(a) = \text{vol } P_{\Phi}(a) \]
when $a$ varies in $\overline{c}$.

More explicitly, when $a$ varies in the closure of a big chamber $c$, we have the polynomial formula for the volume:

\[ v(\Phi, c)(a) = \sum_{|i|=N-r} \langle \langle c, f_{\Phi}(i) \rangle \rangle a_1^{i_1} a_2^{i_2} \ldots a_{r-1}^{i_{r-1}} a_r^{i_r} i_1! i_2! \ldots i_{r-1}! i_r!. \]

(1)

Remark. The function $a \rightarrow J_{\Phi}(a) \in S_\Delta$ is a vector-valued polynomial function on $V^*$ of degree $N - r$. It is thus determined by $\dim S_\Delta$ scalar homogeneous polynomial functions of degree $(N - r)$ in $r$ variables. As a result of THEOREM 9, the collection of polynomial functions $v(\Phi, c)$ is entirely determined by the function $J_{\Phi}(a)$. It is very difficult to determine the number of big chambers $c$. It is usually much larger than $\dim S_\Delta$. For example, consider $A_n^+$ the positive root system of $A_n$. For $n = 1, 2, 3$, the number of big chambers is 1, 2, 7; for $n = 4, 5, 6$, it is known that their number is greater or equal to than 48, 820, 51133 ($[S]$). The dimension of $S_\Delta$ is $n!$, so respectively 1, 2, 6, 24, 120, 720. So, there are many linear relations between the different linear functions $f \rightarrow \langle \langle c, f \rangle \rangle$ on $S_\Delta$, thus many linear relations between the polynomials $v(\Phi, c)$.

The relative volume of $P_{\Phi}(a)$ compare the volume of $P_{\Phi}(a)$ to the volume of the standard simplex, so is defined by
\[ \text{vol}_{\text{rel}} P_{\Phi}(a) = (N - r)! \text{vol } P_{\Phi}(a). \]

We have:
\[ \text{vol}_{\text{rel}} P_{\Phi}(a) = \langle \langle c, T_{\text{res}} \Delta \left( \prod_{\alpha \in \Phi} a^{\alpha} \right) \rangle \rangle. \]
Bibliographical remarks.
Results on this section are mainly due to Jeffrey-Kirwan [J-K]. The stratification on big chambers is introduced in [B-V 1]. The fact that the volume is polynomial on big chambers follows from the proof given in the Appendix (Section ??). More details on the relations between Jeffrey-Kirwan formulae and Laplace transforms are given in [B-V 2].

3 Flow polytopes

Consider a \((r + 1)\) dimensional vector space, with basis \(e^1, e^2, ..., e^{r+1}\), and let \(A^{+}_r\) (the positive root system of \(A_r\)) be

\[
A^{+}_r = \{(e^i - e^j), 1 \leq i < j \leq (r+1)\}.
\]

Let \(E_r\) be the vector space spanned by the elements \((e^i - e^j), a = a_1e^1 + \cdots + a_re^r + a_{r+1}e^{r+1}\) with \(a_1 + a_2 + \cdots + a_r + a_{r+1} = 0\).

The vector space \(E_r\) is of dimension \(r\) and is provided with the lattice generated by \(A^{+}_r\), so has a canonical measure \(da\). We have \(da = da_1...da_r\). If \(\sigma\) is any basis of \(A_r\), then \(\text{vol}(\sigma) = 1\). The cone \(C(A^{+}_r)\) generated by positive roots is the cone \(a_1 \geq 0, a_1 + a_2 \geq 0, ..., a_1 + a_2 + \cdots + a_r \geq 0\).

**Definition 11** If \(a = (a_1, a_2, ..., a_r) \in \mathbb{R}^r\), we denote by \(a \in E_r\) the element

\[
a = a_1e^1 + \cdots + a_re^r - (a_1 + \cdots + a_r)e^{r+1}.
\]

Let \(\Phi\) be a sequence of \(N\) elements of \(A^{+}_r\) spanning \(E_r\).

**Definition 12** We define \(m_{ij}(i < j)\) to be the multiplicity of the root \((e^i - e^j)\) in \(\Phi\). Thus \(m_{ij} \geq 0\).

Associate to \(\Phi\) a graph with \(r + 1\) vertices \(1, 2, ..., r + 1\), and \(m_{ij}\) edges from \(i \rightarrow j\) if \((e^i - e^j)\) is in \(\Phi\) (thus \(i < j\)). This graph is called a flow graph. The graph associated to \(A^{+}_r\) is the complete flow graph (with 1 edge \(i \leftrightarrow j\) for any \(i < j\)).

**Definition 13** A flow polytope is a polytope isomorphic to a polytope \(P_{\Phi}(a)\) where \(\Phi\) is a sequence of elements of \(A^{+}_r\) and \(a = (a_1, a_2, ..., a_r) \in \mathbb{R}^r\).
• The polytope $P_{\Phi}(e^1 - e^{r+1})$ will be called the fountain polytope of shape $\Phi$.

• A flow polytope $P_{\Phi}(a)$ with $a_1 \geq 0, a_2 \geq 0, \ldots, a_r \geq 0$ will be called a cascade polytope.

The flow polytope $P_{\Phi}(a)$ is of dimension $N-r$ when $a_1 > 0$. The polytope $P_{A^+_r}(e^1 - e^{r+1})$ associated to the complete flow graph is the Chan-Robbins-Yuen polytope. It will be described in Section 7.

Remark 1. If $a_1, \ldots, a_r$ are greater or equal to 0, the polytope $P_{\Phi}(a)$ is the Minkowski sum of the polytopes $a_k P_{\Phi}(e^k - e^{r+1})$. If $k > 1$, the vector $e^k - e^{r+1}$ is on the boundary of $C(\Delta^+)$ and the polytope $P_{\Phi}(e^k - e^{r+1})$ is of smaller dimension than $N-r$. It is the fountain polytope of shape $\Phi'$ where $\Phi'$ is the system where we have deleted all $(e^i - e^j)$ with $i < k$.

Remark 2. Let $d_a$ be the dimension of $P_{\Phi}(a)$. If $s$ is a vertex of $P_{\Phi}(a)$ and $s + \mathbb{R}^+ v_1, s + \mathbb{R}^+ v_2, \ldots, s + \mathbb{R}^+ v_{d_a}$ are $d_a$ edges passing through $s$, directed by integral vectors $v_i$ with minimal length, then the volume of the parallelepiped spanned by $v_k, 1 \leq k \leq d_a$ is equal to 1. Such polytope is associated to an ample line bundle on a smooth toric variety: a limit of Delzant polytopes. Reciprocally, any polytope associated to a smooth toric variety can be realized as a flow polytope $P_{\Phi}(a)$ when $\Phi$ is a sequence of elements of the positive root system $A^+_r$ (Szenes: personal communication).

In the rest of this article, we study volumes and Ehrhart polynomials of flow polytopes. Our method is based on the study of the vector space $R_{A_r}$ and of the vector space $S_{A_r}$.

Consider the system $A_{r-1} = \{ (e^i - e^j), 1 \leq i < j \leq r \}$.

**Proposition 14** The map from

$$\sum_{i=1}^{r} S_{A_{r-1}} \otimes \frac{1}{(e^i - e^{r+1})} \rightarrow S_{A_r}$$

given by

$$\sum_{i=1}^{r} \frac{f_i}{(e^i - e^{r+1})} \rightarrow \sum_{i=1}^{r} \frac{f_i}{(e^i - e^{r+1})}$$

is a bijection.
Proof. If $f$ is in $S_{A_{r-1}}$, then for any $1 \leq i \leq r$, the element $f\frac{1}{(e^i - e^{r+1})}$ is in $S_{A_r}$, and it is easy to see that the map above is injective.

To prove that it is surjective, observe first that for any set of elements $A \subset \{1, ..., r\}$:

$$\prod_{i \in A} \frac{1}{(e^{r+1} - e^i)} = \sum_{i \in A} \prod_{j \in A : i \neq j} \frac{1}{(e^i - e^j)} \frac{1}{(e^{r+1} - e^i)}.$$ 

If $\nu$ is a subset of $A_{r-1}$, we denote by $f_\nu = \frac{1}{\prod_{i \in \nu} e^i}$. Then

$$\prod_{i \in A} \frac{1}{(e^{r+1} - e^i)} f_\nu = \sum_{i \in A} \prod_{j \in A : i \neq j} \frac{1}{(e^i - e^j)} f_\nu \frac{1}{(e^{r+1} - e^i)}.$$

If $\nu \cup \{(e^i - e^{r+1}), i \in A\}$ is a basis of $A_r$, then for every $i \in A$,

$$\nu \cup \{(e^j - e^{r+1}), j \in A, j \neq i\} \cup \{(e^i - e^{r+1})\}$$

is a basis of $A_r$. The lemma follows.

QED

It follows from the lemma above that the dimension of $S_{A_r}$ is $r!$. We denote by $\Sigma_r$ the set of permutations of $\{1, 2, ..., r\}$. As seen from PROPOSITION 14 above, a particularly nice basis of $S_{A_r}$ is given by the elements

$$f_w = w \cdot \frac{1}{(e^1 - e^2)(e^2 - e^3) \cdots (e^{r-1} - e^r)(e^r - e^{r+1})}$$

where $w \in \Sigma_r$ acts by permutation on the set $\{1, 2, ..., r\}$.

We define as usual the character $\epsilon(w) = \pm 1$ of $\Sigma_r$ with value $-1$ on symmetries.

Bibliographical remarks. Polytopes associated to subsets of $A^+_r$ are related to graphs and called flow polytopes by Stanley [S]. The representation of the Weyl group in the space $S_\Delta$, for $\Delta$ any root system, has been studied extensively by G. Lehrer [L]. In particular, for $A_r$, the space $S_\Delta$ carries the regular representation of the subgroup $\Sigma_r$ of the Weyl group $\Sigma_{r+1}$. We employ the corresponding basis $f_w (w \in \Sigma_r)$ in these notes.
4 Chambers and Iterated residues for $A_r$.

Let 
$$ V = \{ \sum_{i=1}^{r+1} x_i e_i, x_{r+1} = 0 \}.$$  
(as before, we write $x_i$ instead of $x^i$). The vector space $V$ has basis $e_1, ..., e_r$. An element $a = \sum_{i=1}^{r+1} a_i e^i$ of $E_r$ gives the linear form $\sum_{i=1}^r a_i x_i$ on $V$. This identifies $E_r$ with $V^*$. In this identification 
$$ A_r^+ = \{ x_i, 1 \leq i \leq r, (x_i - x_j), 1 \leq i < j \leq r \}. $$

A function in $R_{A_r}$ is thus a rational function $f(x_1, x_2, ..., x_r)$ on $V_C$, with poles on the hyperplanes $x_i = x_j$ or $x_i = 0$. The basis $f_w$ considered in the preceding section is given by the elements 
$$ f_w(x_1, ..., x_r) = w \cdot \frac{1}{(x_1 - x_2)(x_2 - x_3) \cdots (x_{r-1} - x_r)x_r} $$

where $w \in \Sigma_r$ acts by permutation on the set $\{1, 2, ..., r\}$.

The following linear form $f \mapsto Ires_{x=0} f$ defined by 
$$ Ires_{x=0} f = Res_{x_1=0} Res_{x_2=0} \cdots Res_{x_r=0} f(x_1, x_2, ..., x_r) $$

is a linear form on $R_{A_r}$ which vanishes on the vector space of derivatives $\sum_{i=1}^r \partial_i R_{A_r}$. It will be called the iterated residue. It provides a linear form on $S_{A_r}$. If we compute it on the basis $f_w$ of $S_{A_r}$ indexed by the symmetric group, we have $Ires_{x=0}(f_w) = \delta^1_w$. The iterated residue depends on the order on variables. Permuting the variables by the group $\Sigma_r$, we obtain $r!$ linear forms on $S_{A_r}$, dual to the basis $f_w$. Precisely set 
$$ Ires^\sigma_{x=0} f = Res_{x_{\sigma(1)}=0} Res_{x_{\sigma(2)}=0} \cdots Res_{x_{\sigma(r)}=0} f(x_1, x_2, ..., x_r) = $$ 
$$ Res_{x_1=0} Res_{x_2=0} \cdots Res_{x_r=0} f(x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, ..., x_{\sigma^{-1}(r)}), $$

it is not difficult to check that $\sigma \cdot Ires_{x=0} f_w = Ires_{x=0} \sigma^{-1} f_w = Ires^\sigma_{x=0} f_w = \delta^\sigma_w$ while $Ires^1_{x=0} = Ires_{x=0}$.

Iterated residues are particularly easy to calculate, thus it is important to express the linear forms associated to big chambers in terms of iterated residues.

Consider the set $H^*$ of hyperplanes in $E_r$ spanned by $(r - 1)$ linearly independent vectors of $A_r$. 

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Lemma 15 An element $H$ of $\mathcal{H}^*$ is the kernel of a linear form $\sum_{i \in A} a_i$ where $A$ is a subset of $1, 2, \ldots, r$.

Proof. Let $\nu$ be a subset of $A_r^+$ spanning a $(r - 1)$-dimensional vector space. If $\nu$ is contained in the set $\pm (e^i - e^j), 1 \leq i < j \leq r$, the hyperplane spanned by $\nu$ is the hyperplane $\sum_{i=1}^{r} a_i = 0$. If not, then $\nu$ contains a vector $(e^i - e^{r+1})$, and we conclude by induction.

QED

Definition 16 We denote by $c^+$ the open set

$$c^+ = \{a \in C(A_r^+) \text{ such that } a_i > 0, i = 1, \ldots, r\}.$$

Lemma 17 The set $c^+$ is a small and a big chamber for $A_r^+$. It will be called the nice chamber.

Proof. The set $c^+$ is a small chamber since it doesn’t meet any hyperplane. But $c^+$ is the simplicial cone generated by the elements $(e^i - e^{r+1}), 1 \leq i \leq r$, so that $c^+$ is also a big chamber.

QED

Let $w \in \Sigma_r$ and $n(w)$ be the number of elements such that $w(i) > w(i+1)$. We denote by $C^+_w \subset C(A_r^+)$ the simplicial cone generated by the vectors

$$\epsilon(1)(e^{w(1)} - e^{w(2)}), \epsilon(2)(e^{w(2)} - e^{w(3)}), \ldots, \epsilon(r-1)(e^{w(r-1)} - e^{w(r)}), (e^{w(r)} - e^{r+1}),$$

where $\epsilon(i)$ is 1 or -1 depending whether $w(i) < w(i+1)$ or not. When $w = 1$, then $C_1 = C(A_r^+)$. The dual basis for the vectors generating $C^+_w$ is given by

$$\epsilon(1)e_{w(1)}, \epsilon(2)(e_{w(1)} + e_{w(2)}), \ldots, \epsilon(r-1)(e_{w(1)} + \ldots + e_{w(r-1)}), (e_{w(1)} + \ldots + e_{w(r)}).$$

If we write $a = \sum_{j=1}^{r+1} a_j e^j$ in $E_r$, then the cone $C^+_w$ is given by the following system of inequalities $\sum_{j=1}^{i} a_w(j) \geq 0$, for all $i$ such that $w(i) < w(i+1)$, but $\sum_{j=1}^{i} a_w(j) \leq 0$ if $w(i) > w(i+1)$.

Lemma 18 Let $c$ be a big chamber. Consider the set of elements $w \in \Sigma_r$ such that $c \subset C^+_w$. Then, for $f \in S_{A_r}$,

$$\langle\langle c, f \rangle\rangle = \sum_{w \in \Sigma_r, c \subset C^+_w} (-1)^{n(w)} \text{Res}_{x=0}^w f.$$
Proof. Let \( f \in S_{A_r} \) and write \( f = \sum_{w \in \Sigma_r} a^f_w f_w \). Using the dual base, we compute \( a^f_w = Ires_{x=0}^s(f) \) and we find that the linear form \( f \mapsto \langle \langle c, f \rangle \rangle \) is the sum over all elements \( w \in \Sigma_r \) such that \( c \subset C^+_w \) of the linear form \((-1)^{n(w)} Ires^w_{x=0} = (-1)^{n(w)} Ires^w_{x=0} \).

QED

Theorem 19 The linear form \( f \mapsto \langle \langle c^+, f \rangle \rangle \) coincide with the iterated residue \( Ires_{x=0} \).

Proof. As we just observed, the linear form \( f \mapsto \langle \langle c^+, f \rangle \rangle \) is the sum over all elements \( w \) such that \( c^+ \subset C^+_w \) of the linear form \((-1)^{n(w)} Ires^w_{x=0} \). So, to prove the lemma, it is enough to prove that if \( c^+ \subset C^+_w \), then \( w = 1 \). Assume that \( w \neq 1 \). We let \( k \) be the smallest integer such that \( w(k) > w(k+1) \). In particular if \( a \in C^+_w \) then

\[
\begin{align*}
a_w(1) & \geq 0, a_w(1) + a_w(2) \geq 0, \ldots, a_w(1) + a_w(2) + \ldots + a_w(k-1) \geq 0, \\
a_w(1) + a_w(2) + \ldots + a_w(k) \leq 0
\end{align*}
\]

which forces \( a_w(k) \leq 0 \), and therefore, if \( w \neq 1 \), then \( c^+ \) is not contained in \( C^+_w \).

QED

Bibliographical remarks.

Bases of \( S^*_\Delta \) given by iterated residues are constructed for any system \( \Delta \) by Szenes [Sz]. The linear forms \( f \mapsto \langle \langle c, f \rangle \rangle \) are called Jeffrey-Kirwan residues. It is usually difficult to express them in functions of iterated residues.

5 Volumes of flow polytopes

Let \( \Phi \) be a sequence of \( N \) vectors in \( A_r^+ \) spanning \( E_r \). The set of regular elements for the system \( A_r^+ \) is clearly contained in the set of regular elements for the smaller system \( \{ \Phi \} \). Recall the map \( a = (a_1, a_2, \ldots, a_r) \to a \) given in DEFINITION 11. We will sometimes identify implicitly \( E_r \) with \( \mathbb{R}^r \) using this map. Thus if \( c \) is a big chamber for \( A_r^+ \), the function \( a \mapsto \text{vol} P_\Phi(a) \) is polynomial on \( c \). We denote it by \( v(\Phi, c) \). We have (see Section 2, Formula 1):

\[
v(\Phi, c)(a) = \langle \langle c, J_\Phi(a) \rangle \rangle.
\]
The function $v(\Phi, c^+)$, attached to the nice chamber $c^+$ is particularly important, due to the following lemma.

**Lemma 20** Let $c$ be a big chamber. Let $\Sigma_r(c)$ be the set of elements $w \in \Sigma_r$ such that $c \subset C^+_w$. Then

$$v(\Phi, c)(a) = \sum_{w \in \Sigma_r(c)} (-1)^{\mu(w)} v(\Phi, c^+)(w^{-1}a).$$

**Proof.** Due to THEOREM 19, the linear form $f \mapsto \langle\langle c^+, f \rangle\rangle$ is the iterated residue $I_{res_{x=0}}$. Thus $v(\phi, c^+)(a) = I_{res_{x=0}} J_\Phi(a)$. By LEMMA 18, the form $\langle\langle c, f \rangle\rangle$ is a signed sum of iterated residues $I_{res_{x=0}}$ over the elements $w$ in $\Sigma_r(c)$ and we obtain the lemma.

QED

We write

$$v(\Phi, c)(a_1, \ldots, a_r) = \sum_{i_1+i_2+\ldots+i_r=N-r} v(\Phi, c, i) \frac{a_1^{i_1} a_2^{i_2} \ldots a_{r-1}^{i_{r-1}} a_r^{i_r}}{i_1! i_2! \ldots i_{r-1}! i_r!},$$

where

$$v(\Phi, c, i) = \langle\langle c, f_\Phi(i) \rangle\rangle.$$

**Lemma 21** The coefficient $v(\Phi, c^+, i)$ is the non negative integer defined by:

$$v(\Phi, c^+, i) = I_{res_{x=0}} \frac{x_1^{i_1} \ldots x_r^{i_r}}{\prod_{1 \leq i < j \leq r} (x_i - x_j)^{m_{ij}} \prod_{1 \leq i \leq r} x_i^{m_{i,r+1}}}. $$

**Proof.** It is immediate to check that

$$v(\Phi, c^+, i) = I_{res_{x=0}} \frac{x_1^{i_1} \ldots x_r^{i_r}}{\prod_{1 \leq i < j \leq r} (x_i - x_j)^{m_{ij}} \prod_{1 \leq i \leq r} x_i^{m_{i,r+1}}}$$

is a non negative integer. We will give a combinatorial interpretation of this integer in LEMMA 13 in Section 8.

QED

**Proposition 22** The polynomial $v(\Phi, c^+)(a_1, a_2, \ldots, a_r)$ is homogeneous of degree $|\Phi| - r$. It is divisible by $a_1^{\sum_{k=2}^{r+1} m_{1k}-1}$ and is of degree less or equal to $(m_{r,r+1} - 1)$ in $a_r$. 

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Proof. We have

\[ J_\Phi(a)(x) = \frac{e^{a_1x_1 + \ldots + a_{r-1}x_{r-1} + a_r x_r}}{\prod_{1 \leq i < j \leq r} (x_i - x_j)^{m_{ij}} \prod_{1 \leq i \leq r-1} x_i^{m_{i,r+1}} x_r^{m_{r+1}}} \]

\[ = \frac{e^{a_1x_1 + \ldots + a_{r-1}x_{r-1} + a_r x_r}}{x_1^{m_{1,r+1}} \prod_{1 < j \leq r} (x_1 - x_j)^{m_{1j}} \prod_{2 \leq i < j \leq r} (x_i - x_j)^{m_{ij}} \prod_{2 \leq i \leq r} x_i^{m_{i,r+1}}}. \]

The function \( v(\Phi, c^+)(a_1, a_2, \ldots, a_r) \) is the iterated residue of \( J_\Phi(a)(x) \). If we start by taking the residue in \( x_r = 0 \), (use the first expression), we have to develop the term \( e^{a_r x_r} \) up to order \( m_{r,r+1} - 1 \), so we obtain the second property.

On the other hand, \( x_1 \) is considered as generic up to the last step. Consider the function

\[ F(x_1, a_2, \ldots, a_r) = e^{a_2 x_2 + \ldots + a_{r-1} x_{r-1} + a_r x_r} \]

\[ Ires_{x_2=0, \ldots, x_r=0} \frac{e^{a_1 x_1}}{x_1^{N_1}} F(x_1, a_2, \ldots, a_r), \]

with \( N_1 = \sum_{k=2}^{r+1} m_{1k} \).

Thus, expanding the exponential, we obtain

\[ v(\Phi, c^+)(a_1, a_2, \ldots, a_r) = \sum_{k \geq 0} C_k(a_2, \ldots, a_r) \frac{a_1^{N_1+k-1}}{(N_1+k-1)!}. \]

This establishes the first property.

QED

**Proposition 23 (Schmidt-Bincer)**

Let \( c^+ \) be the nice chamber. Let \( v_\tau^+(a_1, a_2, \ldots, a_r) \). The function \( v_\tau^+(a_1, a_2, \ldots, a_r) \) is independent of \( a_r \). It is of homogeneous degree \( r(r-1)/2 \), of degree less than 1 in the variable \( a_{r-1} \) and is divisible by \( a_1^{-1}(a_1 + a_2 + \ldots + a_{r-2} + 3a_{r-1}) \).

More precisely, we have:

\[ 3v_\tau^+(a_1, a_2, \ldots, a_r) = (a_1 + a_2 + \ldots + a_{r-2} + 3a_{r-1}) v(A_1^+ \text{ minus } (e^{r-1} - e^r), c^+)(a_1, a_2, \ldots, a_{r-2}, 0, 0). \]
Proof. Let
\[
J(a, x) = \frac{e^{a_1 x_1 + \cdots + a_r x_r}}{\prod_{1 \leq i < j \leq r-1} (x_i - x_j) \prod_{1 \leq i \leq r-1} (x_i - x_r) \prod_{1 \leq i \leq r} x_i}.
\]
We have
\[
v_r^+(a_1, a_2, \ldots, a_r) = I_{\text{res}}_{x=0} J(a, x).
\]
We first take the residue in \(x_r = 0\). We obtain
\[
\text{Res}_{x_r=0} J(a, x) = \frac{e^{a_1 x_1 + \cdots + a_{r-1} x_{r-1}}}{\prod_{1 \leq i \leq r-1} x_i^2 \prod_{1 \leq i < j \leq r-1} (x_i - x_j)}
\]
\[
= \frac{e^{a_1 x_1 + \cdots + a_{r-1} x_{r-1}}}{\prod_{1 \leq i \leq r-2} x_i^3 \prod_{1 \leq i < j \leq r-2} (x_i - x_j) \prod_{1 \leq i \leq r-2} (1 - x_{r-1}/x_i)x_{r-1}^2}.
\]
This shows already that \(v_r^+(a_1, a_2, \ldots, a_r)\) is independent of \(a_r\). We proceed to take the residue in \(x_{r-1} = 0\). There is a double pole in \(x_{r-1}\), so that the dependence in \(a_{r-1}\) is of degree at most 1. More precisely,

\[
\text{Res}_{x_{r-1}=0} \left[ \frac{e^{a_1 x_1 + \cdots + a_{r-1} x_{r-1}}}{\prod_{1 \leq i \leq r-2} x_i^3 \prod_{1 \leq i < j \leq r-2} (x_i - x_j) \prod_{1 \leq i \leq r-2} (1 - x_{r-1}/x_i)x_{r-1}^2} \right]
\]
\[
= \frac{e^{a_1 x_1 + \cdots + a_{r-2} x_{r-2}}}{\prod_{1 \leq i \leq r-2} x_i^3 \prod_{1 \leq i < j \leq r-2} (x_i - x_j)} (a_{r-1} + \sum_{i=1}^{r-2} \frac{1}{x_i})
\]
\[
= (a_{r-1} + \frac{1}{3} (a_1 + a_2 + \cdots + a_{r-2})) \frac{e^{a_1 x_1 + \cdots + a_{r-2} x_{r-2}}}{\prod_{1 \leq i \leq r-2} x_i^3 \prod_{1 \leq i < j \leq r-2} (x_i - x_j)}
\]
\[- \frac{1}{3} (\partial_1 + \partial_2 + \cdots + \partial_{r-2}) \frac{e^{a_1 x_1 + \cdots + a_{r-2} x_{r-2}}}{\prod_{1 \leq i \leq r-2} x_i^3 \prod_{1 \leq i < j \leq r-2} (x_i - x_j)},
\]
as \((\partial_1 + \partial_2 + \cdots + \partial_{r-2})\) annihilates functions \(x_i - x_j\). Residues vanishes on derivatives, so that we obtain

\[
3 v_r^+(a_1, a_2, \ldots, a_r) = (3a_{r-1} + a_1 + a_2 + a_3 + \cdots + a_{r-2}) \times
\]
\[ \text{Res}_{x_1=0} \cdots \text{Res}_{x_{r-2}=0} e^{a_1 x_1 + \cdots + a_{r-2} x_{r-2}} \prod_{1 \leq i \leq r-2} x_i^3 \prod_{1 \leq i < j \leq r-2} (x_i - x_j) \].

On the other hand, the residue computation of

\[ v(A_r^+ \text{ minus } (e^{r-1} - e^r), c^+)(a_1, a_2, \ldots, a_{r-2}, 0, 0) \]

gives

\[ v(A_r^+ \text{ minus } (e^{r-1} - e^r), c^+)(a_1, a_2, \ldots, a_{r-2}, 0, 0) = \text{Res}_{x_1=0} \cdots \text{Res}_{x_{r-2}=0} e^{a_1 x_1 + \cdots + a_{r-2} x_{r-2}} \prod_{1 \leq i \leq r-2} x_i^3 \prod_{1 \leq i < j \leq r-2} (x_i - x_j) \]
as the step \( x_r = 0 \) as well as the step \( x_{r-1} = 0 \) involves only simple poles, and we obtain the divisibility property announced.

QED

Example.

The dimension of the polytope \( P_{A_2^+}(a) \) is 1, of \( P_{A_3^+}(a) \) is 3 and of \( P_{A_4^+}(a) \) is 6. Thus the corresponding polynomials \( v(A_2^+, c^+), v(A_3, c^+), v(A_4, c^+) \) are homogeneous of degrees 1, 3, 6.

We have:

\[ v(A_2^+, c^+)(a_1, a_2) = a_1, \]

\[ v(A_3^+, c^+)(a_1, a_2, a_3) = \frac{1}{3!} a_1^3 + \frac{1}{2} a_1^2 a_2 = \frac{1}{6} a_1^2 (a_1 + 3a_2), \]

\[ v(A_4^+, c^+)(a_1, a_2, a_3, a_4) = \frac{1}{120} a_1^2 (a_1 + a_2 + 3a_3) (a_1^2 + 5a_1 a_2 + 10a_2^2). \]

More generally, we have the following proposition, with same proof.

**Proposition 24** Let \( \Phi \) be a sequence of \( N \) vectors in \( A_r^+ \), generating \( E_r \).

Assume \( m_{r+1} = 1 \) and \( m_{r-1,r+1} + m_{r-1} = 2 \). Furthermore, assume that

\[ \frac{m_{j,r+1} + m_{j,r} + m_{j,r-1}}{m_{j,r-1}} = c \]

is independent of \( j \) for \( 1 \leq j \leq (r-2) \), then

\[ v(\Phi, c^+)(a_1, \ldots, a_{r-2}, a_r) = v(\Phi, c^+)(a_1, \ldots, a_{r-1}, 0) \]

\[ = \left( \frac{a_1 + \cdots + a_{r-2}}{c} + a_{r-1} \right) v(\Phi \text{ minus } (e^{r-1} - e^r), c^+)(a_1, a_2, \ldots, a_{r-2}, 0, 0). \]
Consider the permutation $w_0 : [1, 2, ..., r, r+1] \mapsto [r+1, r, ..., 2, 1]$. Then $-w_0$ preserves $A_r^+$ and the vector $e^1 - e^{r+1}$. It transforms $\Phi = A^+_r$ minus $(e^{r-1} - e^r)$ in $A^+_r$ minus $(e^2 - e^3)$. Thus we obtain from LEMMA 3.

**Corollary 25** We have

$$3v(A^+_r, c^+)(e^1 - e^{r+1}) = v(A^+_r \text{ minus } (e^{r-1} - e^r), c^+)(e^1 - e^{r+1})$$

$$= v(A^+_r \text{ minus } (e^2 - e^3), c^+)(e^1 - e^{r+1}).$$

**Bibliographical remarks.**

Proposition 23 is due to [S-B].

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### 6 Calculation of some total residues for the system $A_{r+1}$

Consider a $(r+2)$ dimensional real vector space, with basis $e^0, e^1, e^2, ..., e^r, e^{r+1}$. In this Section, we consider $A_{r+1}$ as the collection of elements $(e^i - e^j)$ with $0 \leq i \leq r+1, 0 \leq j \leq r+1,$ and $i \neq j$.

Let

$$\Pi = \{(e^0 - e^1), (e^1 - e^2), ..., (e^r - e^{r+1})\}$$

be the set of simple roots. Let

$$f_\Pi = \frac{1}{(e^0 - e^1)(e^1 - e^2)\cdots(e^r - e^{r+1})}.$$  

This is an element of $S_{A_{r+1}}$ (in particular is homogeneous of degree $-(r + 1)$).

As explained in the introduction, we are particularly interested in the $\Sigma_r$ anti-invariant function of homogeneous degree $-(r + 1)$ given by

$$W_{r+1} = \frac{(e^0 - e^{r+1})^{r(r+1)/2}}{\prod_{0 \leq i < j \leq r+1}(e^i - e^j)}.$$  

As seen by the Example 2 of Section [4], this function is not in the space $S_\Delta$. However, its projection on $S_\Delta$ is particularly nice.

We prove in this section the following Theorem.
Theorem 26. We have

\[ T_{res}A^+_r \left( \frac{(e^0 - e^{r+1})^{r(r+1)/2}}{\prod_{0 \leq i < j \leq r+1} (e^i - e^j)} \right) \]

\[ = \prod_{i=1}^{r-1} \frac{(2i)!}{i!(i+1)!} \sum_{w \in \Sigma_r} \epsilon(w) w \cdot \frac{1}{(e^0 - e^1)(e^1 - e^2) \cdots (e^{r-1} - e^r)(e^r - e^{r+1})}. \]

In fact, we will prove more general identities, which are reformulation of Morris identity.

Consider the group \( \Sigma_r \) of permutations of \( \{1, \ldots, r\} \). Let \( 0 \leq \ell \leq r \) and denote by \( P_{\ell,r} \) the \( \Sigma_r \)-invariant polynomial

\[ P_{\ell,r} = \sum_{w \in \Sigma_r} w \cdot [(e^1 - e^{r+1})(e^2 - e^{r+1}) \cdots (e^\ell - e^{r+1})]. \]

In particular

\[ P_{0,r} = r! \quad P_{r,r} = r! \prod_{j=1}^{r-1} (e^j - e^{r+1}). \]

When \( r \) is fixed, we will write \( P_\ell \) for \( P_{\ell,r} \), when \( \ell > 0 \).

We consider the rational function given by

\[ \phi_{r+1}(\ell, k_1, k_2, k_3) \]

\[ = \frac{P_\ell}{(\prod_{j=1}^{r} (e^j - e^{r+1}))^{k_1} (\prod_{j=1}^{r} (e^0 - e^j))^{k_2} (\prod_{1 \leq i < j \leq r} (e^i - e^j))^{k_3}}. \]

In particular

\[ \phi_{r+1}(0, k_1, k_2, k_3) \]

\[ = \frac{1}{(\prod_{j=1}^{r} (e^j - e^{r+1}))^{k_1} (\prod_{j=1}^{r} (e^0 - e^j))^{k_2} (\prod_{1 \leq i < j \leq r} (e^i - e^j))^{k_3}}. \]

Here, \( k_1, k_2 \) and \( k_3 \) are non negative integers, so that \( \phi_{r+1}(\ell, k_1, k_2, k_3) \) is an element of \( R_{A^+_r} \) of homogeneity degree \( \ell - (k_1 + k_2)r - k_3 \frac{r(r-1)}{2} \). If \( k_3 \) is odd, this function is anti-invariant under the group \( \Sigma_r \) of permutations of \( \{1, \ldots, r\} \). If \( k_3 \) is even, this function is invariant.
Remark that if \( k_1 \geq 1 \),

\[
\phi_{r+1}(r, k_1, k_2, k_3) = \phi_{r+1}(0, k_1 - 1, k_2, k_3).
\]

Let \( k_1, k_2, k_3 \geq 0, 0 \leq \ell \leq r \). Let \( D = (k_1 + k_2)r + k_3 \frac{r(r-1)}{2} - \ell \).

Then the function \((e^0 - e^{r+1})D^{-(r+1)}\phi_{r+1}(\ell, k_1, k_2, k_3)\) is of homogeneity degree equal to \(-(r+1)\).

We have in particular

\[
W_{r+1} = \frac{1}{r!}(e^0 - e^{r+1})^{r(r+1)/2} \phi_{r+1}(0, 1, 1, 1).
\]

**Theorem 27** Let \( k_1, k_2, k_3 \geq 0, 0 \leq \ell \leq r \). Let \( D = (k_1 + k_2)r + k_3 \frac{r(r-1)}{2} - \ell \).

Then the function \((e^0 - e^{r+1})D^{-(r+1)}\phi_{r+1}(\ell, k_1, k_2, k_3)\) is of homogeneity degree equal to \(-(r+1)\), and we have

- If \( k_3 \) is odd,
  
  \[
  \text{Tres}_{A_{r+1}^+} ((e^0 - e^{r+1})D^{-(r+1)}\phi_{r+1}(\ell, k_1, k_2, k_3))
  = C_{r+1}(\ell, k_1, k_2, k_3)[\sum_{w \in \Sigma_r} e(w)w.f_{\Pi}].
  \]

- If \( k_3 \) is even,
  
  \[
  \text{Tres}_{A_{r+1}^+} ((e^0 - e^{r+1})D^{-(r+1)}\phi_{r+1}(\ell, k_1, k_2, k_3))
  = C_{r+1}(\ell, k_1, k_2, k_3)[\sum_{w \in \Sigma_r} w.f_{\Pi}].
  \]

The constants \( C_{r+1}(\ell, k_1, k_2, k_3) \) are determined uniquely by the relations:

- for \( 1 \leq \ell \leq r \),

\[
(k_1 + k_2 - 2 + \frac{k_3}{2}(2r - \ell - 1))C_{r+1}(\ell, k_1, k_2, k_3)
= (k_1 - 1 + \frac{k_3}{2}(r - \ell))C_{r+1}(\ell - 1, k_1, k_2, k_3),
\]

\[
(k_1 - 2 + \frac{k_3}{2}(2r - \ell))C_{r+1}(\ell, k_1, k_2, k_3)
= (k_1 - 1 + \frac{k_3}{2}(r - \ell))C_{r+1}(\ell - 1, k_1, k_2, k_3),
\]

\[
(k_2 - 2 + \frac{k_3}{2}(2r - \ell))C_{r+1}(\ell, k_1, k_2, k_3)
= (k_2 - 1 + \frac{k_3}{2}(r - \ell))C_{r+1}(\ell - 1, k_1, k_2, k_3),
\]

\[
(k_3 - 2 + \frac{k_3}{2}(2r - \ell))C_{r+1}(\ell, k_1, k_2, k_3)
= (k_3 - 1 + \frac{k_3}{2}(r - \ell))C_{r+1}(\ell - 1, k_1, k_2, k_3).
\]
• $C_{r+1}(r, k_1, k_2, k_3) = C_{r+1}(0, k_1 - 1, k_2, k_3)$.

• $C_{r+1}(r - 1, 1, k_2, k_3) = C_r(0, k_3, k_2, k_3)$.

• $C_{r+1}(0, k_1, k_2, k_3) = C_{r+1}(0, k_2, 0, k_3)$. 

• $C_{r+1}(0, 1, 1, 0) = r!$.

• If $k_1$ or $k_2 = 0$,
  
  $C_{r+1}(0, k_1, k_2, k_3) = 0$.

Remark 1. The function $(e^0 - e^{r+1})^{D-(r+1)}\phi_{r+1}(\ell, k_1, k_2, k_3)$ is invariant or anti-invariant under the group $\Sigma_r$ depending on the parity of $k_3$, then its total residue must be an element of $S_{A_{r+1}}$ which is invariant or anti-invariant by $\Sigma_r$. There are $(r + 1)$ linearly independent such functions. Let us consider the basis $w \cdot f_{\Pi}$ of $S_{A_{r+1}}$ with $w$ a permutation of $\{0, 1, 2, ..., r\}$. For homogeneity reasons, it is easy to see that the iterated residue $\text{Ires}_{x=0}^\sigma$ of the function $x^D_{0}^{-(r+1)} \phi_{r+1}(\ell, k_1, k_2, k_3)(x_0, x_1, ..., x_r, 0)$ is equal to 0, if the permutation $\sigma$ of $\{0, 1, ..., r\}$ does not leave 0 fixed. Thus the total residue of $(e^0 - e^{r+1})^{D-(r+1)}\phi_{r+1}(\ell, k_1, k_2, k_3)$ belongs to the vector space spanned by elements $f_w$ with $w \in \Sigma_r$, and the total residue of the function

$$(e^0 - e^{r+1})^{D-(r+1)}\phi_{r+1}(\ell, k_1, k_2, k_3))$$

is proportional to either $\left[\sum_{w \in \Sigma_r} \epsilon(w)w \cdot f_{\Pi}\right]$ or $\left[\sum_{w \in \Sigma_r} w \cdot f_{\Pi}\right]$. The calculation of the constant of proportionality is thus equivalent to the Morris iterated constant term identity. However, we will give here a direct proof.

The recurrence formula above determines entirely the constants

$C_{r+1}(\ell, k_1, k_2, k_3)$.

Indeed, assume first $k_3 > 0$. Then, in the first relation, when $k_3 > 0$ and $r > 1$, all constants $(k_1 - 1) + \frac{k_1}{2}(r - \ell)$ are strictly positive, so if $k_1 > 1$, we can increase $\ell$ to $\ell = r$, then using the second relation, we decrease $k_1$ to $k_1 - 1$. Now, if $k_1 - 1 > 1$, then using one we can again increase $\ell = 0$. 

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to $\ell = r$, and using the second decrease to $k_1 - 2$. In conclusion, we may determine using alternatively one and two, the constant $C_{r+1}(\ell, k_1, k_2, k_3)$ from $C_{r+1}(\ell, 1, k_2, k_3)$. If $k_1 = 1$, $k_3 > 0$ and $r > 1$, we can increase $\ell$ up to $(r - 1)$, using the first relation. Then using the third, we decrease $r + 1$ to $r$. In conclusion we determine $C_{r+1}(\ell, k_1, k_2, k_3)$, if $k_3 > 0$ from the value of $C_2(\ell, k_1, k_2, k_3)$. But if $r = 1$, there is no factor corresponding to $k_3$, so that $C_2(\ell, k_1, k_2, k_3) = C_2(\ell, k_1, k_2, 0)$. Now, check that constants $C_{r+1}(\ell, k_1, k_2, k_3)$ are uniquely determined by the recurrence relations above. The first one reads, for $1 \leq \ell \leq r$,

$$(k_1 + k_2 - 2)C_{r+1}(\ell, k_1, k_2, 0) = (k_1 - 1)C_{r+1}(\ell - 1, k_1, k_2, 0).$$

In the same way, using alternatively one and two, we compute $C_{r+1}(0, k_1, k_2, k_3)$ from the value of $C_{r+1}(\ell, 1, k_2, 0)$. Relation above shows that $C_{r+1}(\ell, 1, k_2, 0) = 0$, if $\ell$ is not equal to 0. It remains to compute $C_{r+1}(0, 1, k, 0)$. By the symmetry relation, we can also assume $k = 1$. We are finally reduced to $C_{r+1}(0, 1, 1, 0).

**Corollary 28** Assume $r > 1$.

- If $k_3 > 0$, or if $k_1 + k_2 > 2$, then for $1 \leq \ell \leq r$,

$$C_{r+1}(\ell, k_1, k_2, k_3) = \prod_{j=1}^{\ell} \frac{k_1 - 1 + (r - j)\frac{k_2}{2}}{k_1 + k_2 - 2 + (2r - j - 1)\frac{k_3}{2}}C_{r+1}(0, k_1, k_2, k_3).$$

- If $k_1 + k_2 \geq 2$,

$$C_{r+1}(0, k_1, k_2, k_3) = r!\prod_{j=0}^{r-1} \frac{\Gamma(1 + \frac{k_3}{2})\Gamma(k_1 + k_2 - 1 + (r + j - 1)\frac{k_3}{2})}{\Gamma(1 + (j + 1)\frac{k_3}{2})\Gamma(k_1 + j\frac{k_3}{2})\Gamma(k_2 + j\frac{k_3}{2})}.$$ 

**Corollary 29** We have

- $$C_{r+1}(\ell, 1, 1, 1) = \prod_{j=1}^{\ell} \frac{(r - j)}{(2r - j - 1)}C_{r+1}(0, 1, 1, 1).$$
where \( C_i = \frac{(2i)!}{i!(i+1)!} \) is the \( i \)-th Catalan number.

- \( C_{r+1}(r, k, 1, 1) = \frac{r!}{(r-1)!} \prod_{i=k-1}^{r+k-3} \frac{1}{2i+1} \binom{r+k+i-1}{2i} \).

- \( C_{r+1}(\ell, k, 1, 1) = \prod_{j=1}^{\ell} \frac{2(k-1)+(r-j)}{2(k-1)+(2r-j-1)} C_{r+1}(0, k, 1, 1) \).

The second corollary is of course a consequence of the first, using several times the duplication formula for the Gamma function, but it is somewhat easier to use directly the recurrence formulas in \( k_1, k_2, k_3 \), with \( k_3 = 1 \) as the value of \( k_3 \) remains constant and equal to 1, through the recurrence.

\( \text{THEOREM 26} \) is then a corollary of \( \text{THEOREM 27} \) and \( \text{COROLLARY 29} \).

Remark that\[ \frac{1}{r!} C_{r+1}(0, 1, 1, 1) = \frac{1}{(r-1)!} C_r(0, 2, 1, 1) = \prod_{i=1}^{r-1} C_i = \prod_{i=1}^{r-2} \frac{1}{2i+1} \binom{r+i}{2i}. \]

Let us first verify the corollaries, assuming \( \text{THEOREM 27} \).

To verify the first corollary, we verify the recurrence relations, the first being obvious, we check the second:

\[ C_{r+1}(r, k_1, k_2, k_3) = C_{r+1}(0, k_1-1, k_2, k_3). \]

We write

\[ C_{r+1}(r, k_1, k_2, k_3) = \]

\[ = r! \prod_{j=1}^{r} \frac{k_1-1+(r-j)\frac{k_3}{2}}{k_1+k_2-2+(2r-j-1)\frac{k_3}{2}} \]

\[ \times \prod_{j=0}^{r-1} \frac{\Gamma(1+\frac{k_3}{2})\Gamma(k_1+k_2-1+(r+j-1)\frac{k_3}{2})}{\Gamma(1+(j+1)\frac{k_3}{2})\Gamma(k_1+j\frac{k_3}{2})\Gamma(k_2+j\frac{k_3}{2})}. \]
In the first product, we change $j$ in $(r - j)$, in the second we use $\Gamma(z + 1) = z\Gamma(z)$ and we obtain:

\[
C_{r+1}(r-1, 1, k_2, k_3)
= r! \prod_{j=0}^{r-1} \frac{k_1 - 1 + j\frac{k_3}{2}}{k_1 + k_2 - 2 + (r + j - 1)\frac{k_3}{2}} \frac{\Gamma(1 + \frac{k_3}{2})(k_1 + k_2 - 2 + (r + j - 1)\frac{k_3}{2})}{\Gamma(1 + (j + 1)\frac{k_3}{2})(k_1 - 1 + j\frac{k_3}{2})} \\
\times \prod_{j=0}^{r-1} \frac{\Gamma(k_1 - 1 + k_2 - 1 + (r + j - 1)\frac{k_3}{2})}{\Gamma(k_1 - 1 + j\frac{k_3}{2})\Gamma(k_2 + j\frac{k_3}{2})} \\
= C_{r+1}(0, k_1 - 1, k_2, k_3).
\]

We verify the third condition.

We write

\[
C_{r+1}(r-1, 1, k_2, k_3) \\
= r! \prod_{j=0}^{r-1} \frac{(r - j)\frac{k_3}{2}}{k_2 - 1 + (2r - j - 1)\frac{k_3}{2}} \prod_{j=0}^{r-1} \frac{\Gamma(1 + \frac{k_3}{2})\Gamma(k_2 + (r + j - 1)\frac{k_3}{2})}{\Gamma(1 + (j + 1)\frac{k_3}{2})\Gamma(1 + j\frac{k_3}{2})\Gamma(k_2 + j\frac{k_3}{2})} \\
\times \Gamma(k_2 + (r - 1)\frac{k_3}{2}) \prod_{j=1}^{r-1} \frac{\Gamma(k_2 + (r + j - 1)\frac{k_3}{2})}{\Gamma(1 + j\frac{k_3}{2})} \\
\]

In the first product, we change $j$ in $(r - j)$, in the last we use $\Gamma(z + 1) = z\Gamma(z)$, and we obtain after simplification that $C_{r+1}(r - 1, 1, k_2, k_3)$ is equal to

\[
r! \prod_{j=0}^{r-1} \frac{\Gamma(1 + \frac{k_3}{2})}{\Gamma(1 + (j + 1)\frac{k_3}{2})\Gamma(k_2 + j\frac{k_3}{2})} \Gamma(k_2 + (r - 1)\frac{k_3}{2}) \\
\times \prod_{j=1}^{r-1} \frac{\Gamma(k_2 - 1 + (r + j - 1)\frac{k_3}{2})}{\Gamma(j\frac{k_3}{2})} \\
= r! \prod_{j=0}^{r-2} \frac{\Gamma(1 + \frac{k_3}{2})}{\Gamma(1 + (j + 1)\frac{k_3}{2})\Gamma(k_2 + j\frac{k_3}{2})} \Gamma(k_2 - 1 + (r + j - 1)\frac{k_3}{2}) \\
\prod_{j=1}^{r-1} \frac{\Gamma(k_2 - 1 + (r + j - 1)\frac{k_3}{2})}{\Gamma(j\frac{k_3}{2})}.
\]
\[= (r-1)! \prod_{j=0}^{r-2} \frac{\Gamma(1 + \frac{k_j}{2})}{\Gamma(1 + (j + 1)\frac{k_j}{2})} \frac{\Gamma(\frac{k_j}{2})}{\Gamma(\frac{k_j}{2})} \frac{r-1}{j=1} \frac{\Gamma(k_j - 1 + (r + j - 1)\frac{k_j}{2})}{\Gamma(k_j + (j+1)\frac{k_j}{2})}\]

while

\[C_r(0, k_3, k_2, k_3) = (r-1)! \prod_{j=0}^{r-2} \frac{\Gamma(1 + \frac{k_j}{2})\Gamma(k_j + k_2 - 1 + (r - 1 + j - 1)\frac{k_j}{2})}{\Gamma(1 + (j + 1)\frac{k_j}{2})\Gamma(k_j + j\frac{k_j}{2})\Gamma(k_j + (j+1)\frac{k_j}{2})}\]

\[= (r-1)! \prod_{j=0}^{r-2} \frac{\Gamma(1 + \frac{k_j}{2})}{\Gamma(1 + (j + 1)\frac{k_j}{2})} \frac{\Gamma(k_j - 1 + (r + j)\frac{k_j}{2})}{\Gamma((j+2)\frac{k_j}{2})}\]

It remains to verify

\[\frac{\Gamma(\frac{k_j}{2})}{\Gamma(\frac{k_j}{2})} \frac{r-1}{j=1} \frac{\Gamma(k_j - 1 + (r + j - 1)\frac{k_j}{2})}{\Gamma(k_j + (j+1)\frac{k_j}{2})} = \prod_{j=0}^{r-2} \frac{\Gamma(k_j - 1 + (r + j)\frac{k_j}{2})}{\Gamma((j+2)\frac{k_j}{2})}\]

which is true.

The remaining properties are obvious.

We now prove THEOREM 27 by induction on \(r\).

**Proof.** If \(k_1 = 0\), the remaining roots \((e^i - e^j)\) occurring in the denominator of \(\phi_{r+1}(\ell, 0, k_2, k_3)\) are contained in the hyperplane \(\sum_{i=0}^{r-1} e^i = 0\). So the total residue of \(\phi_{r+1}(\ell, 0, k_2, k_3)\) is 0. The same argument shows that \(\phi_{r+1}(\ell, k_1, 0, k_3)\) is 0.

We thus may assume that \(k_1, k_2 > 0\). We first show that the function \((e^0 - e^{r+1})\phi_{r+1}(\ell, 1, k_1, k_2, k_3)\) is proportional to the function \(\phi_{r+1}(\ell, k_1, k_2, k_3)\) modulo \(\sum_{i=1}^{r} \partial_i R_1^{r+1}\). Thus the functions \((e^0 - e^{r+1})^{D-(r+1)}\phi_{r+1}(\ell-1, k_1, k_2, k_3)\) and \((e^0 - e^{r+1})^{D-r}\phi_{r+1}(\ell, k_1, k_2, k_3)\) will be proportional too, modulo the vector space \(\sum_{i=1}^{r} \partial_i R_1^{r+1}\), their total residues will be proportional and we will get the first recursive relations for the constant \(C_{r+1}\).

In the following, we use \(x_i\) instead of \(e^i\), etc... as it is a more familiar notation for computing derivatives.

We write \(U = (\prod_{j=1}^{r} (x_j - x_{r+1}))^{-k_1}(\prod_{j=1}^{r} (x_0 - x_j))^{-k_2}(\prod_{1 \leq i < j \leq r} (x_i - x_j))^{-k_3}\) so that \(P_t U = \phi_{r+1}(\ell, k_1, k_2, k_3)\).
We compute

$$\partial_1[(x_0 - x_1)(x_1 - x_{r+1})(x_2 - x_{r+1}) \cdots (x_\ell - x_{r+1})U]$$

This is equal to

$$-(1 - k_2)(x_1 - x_{r+1})(x_2 - x_{r+1}) \cdots (x_\ell - x_{r+1})U$$

$$+ (1 - k_1)(x_0 - x_1)(x_2 - x_{r+1}) \cdots (x_\ell - x_{r+1})U$$

$$- k_3(x_0 - x_1)(x_1 - x_{r+1})(x_2 - x_{r+1}) \cdots (x_\ell - x_{r+1}) \sum_{j=2}^{r} \frac{1}{x_1 - x_j} U.$$

Using \((x_0 - x_1) = (x_0 - x_{r+1}) + (x_{r+1} - x_1)\), this is also equal to

$$= (k_1 + k_2 - 2)(x_1 - x_{r+1})(x_2 - x_{r+1}) \cdots (x_\ell - x_{r+1})U$$

$$+ (1 - k_1)(x_0 - x_{r+1})(x_2 - x_{r+1}) \cdots (x_\ell - x_{r+1})U$$

$$- k_3(x_0 - x_1)(x_1 - x_{r+1})(x_2 - x_{r+1}) \cdots (x_\ell - x_{r+1}) \sum_{j=2}^{r} \frac{1}{x_1 - x_j} U.$$

Assume first that \(k_3\) is odd, so that \(U\) is anti-invariant by the group \(\Sigma_r\). Let us antisymmetrize over permutations. We obtain

$$\sum_{w \in \Sigma_r} \epsilon(w) w \cdot (\partial_1((x_0 - x_1)(x_1 - x_{r+1})(x_2 - x_{r+1}) \cdots (x_\ell - x_{r+1})U)) =$$

$$(k_1 + k_2 - 2)P_U + (1 - k_1)(x_0 - x_{r+1})P_{\ell-1}U$$

$$- k_3 \sum_{w \in \Sigma_r} w \cdot ((x_0 - x_1)(x_1 - x_{r+1})(x_2 - x_{r+1}) \cdots (x_\ell - x_{r+1}) \sum_{j=2}^{r} \frac{1}{x_1 - x_j} U).$$

To compute \(\sum_{w \in \Sigma_r} w \cdot ((x_0 - x_1)(x_1 - x_{r+1})(x_2 - x_{r+1}) \cdots (x_\ell - x_{r+1}) \frac{1}{x_1 - x_j})\), we first sum over the transposition \((j, 1)\).

If \(2 \leq j \leq \ell\), we use

$$\frac{(x_0 - x_1)(x_1 - x_{r+1})(x_j - x_{r+1})}{x_1 - x_j} + \frac{(x_0 - x_j)(x_j - x_{r+1})(x_1 - x_{r+1})}{x_j - x_1}$$

$$= -(x_1 - x_{r+1})(x_j - x_{r+1}).$$
If \( j > \ell \), we use
\[
\frac{(x_1 - x_{r+1})(x_0 - x_1)}{x_1 - x_j} + \frac{(x_j - x_{r+1})(x_0 - x_j)}{x_j - x_1} = (x_0 - x_1) + (x_{r+1} - x_j) = (x_0 - x_{r+1}) + (x_{r+1} - x_1) + (x_{r+1} - x_j).
\]
We obtain that
\[
2 \sum_{w \in \Sigma_r} w \cdot ((x_0 - x_1)(x_1 - x_{r+1})(x_2 - x_{r+1})...(x_\ell - x_{r+1}) \sum_{j=2}^{r} \frac{1}{x_1 - x_j})
\]
is equal to
\[
(-(\ell - 1) - 2(r - \ell))P_\ell + (x_0 - x_{r+1})(r - \ell)P_{\ell - 1}.
\]
Thus finally, we obtain
\[
\sum_{w \in \Sigma_r} e(w)w \cdot (\partial_1((x_0 - x_1)(x_1 - x_{r+1})(x_2 - x_{r+1})...(x_\ell - x_{r+1})\phi_{r+1}(0, k_1, k_2, k_3)) =
\]
\[
(k_1 + k_2 - 2 + \frac{k_3}{2}(2r - \ell - 1))\phi_{r+1}(\ell, k_1, k_2, k_3)
\]
\[
+(x_0 - x_{r+1})(-k_1 + 1 - \frac{k_3}{2}(r - \ell))\phi_{r+1}(\ell - 1, k_1, k_2, k_3).
\]
If \( k_3 \) is even, we also obtain
\[
\sum_{w \in \Sigma_r} w \cdot (\partial_1((x_0 - x_1)(x_1 - x_{r+1})(x_2 - x_{r+1})...(x_\ell - x_{r+1})\phi_{r+1}(0, k_1, k_2, k_3)) =
\]
\[
(k_1 + k_2 - 2 + \frac{k_3}{2}(2r - \ell - 1))\phi_{r+1}(\ell, k_1, k_2, k_3)
\]
\[
+(x_0 - x_{r+1})(-k_1 + 1 - \frac{k_3}{2}(r - \ell))\phi_{r+1}(\ell - 1, k_1, k_2, k_3).
\]
Thus we see that
\[
(x_0 - x_{r+1})\phi_{r+1}(\ell - 1, k_1, k_2, k_3)
\]
is proportional to \( \phi_{r+1}(\ell, k_1, k_2, k_3) \), modulo derivatives with respect to \( x_1, x_2, ..., x_r \). In particular the total residue of the function \( (x_0 - x_{r+1})^{D-(r+1)}\phi_{r+1}(\ell, k_1, k_2, k_3) \) is proportional to...
the total residue of the function $(x_0 - x_{r+1})^{D-(r+1)+1}\phi_{r+1}(\ell - 1, k_1, k_2, k_3)$. This proves the first property.

We proceed to the proof of the second property. We return to the notation $x_i = e^i$.

To avoid confusion in the following argument we will write explicitly the dependence by the parameters of $D$, that is we will write, whenever necessary, $D = D_{A_r+1}(\ell, k_1, k_2, k_3) = (k_1 + k_2)r + k_3 \frac{r(r-1)}{2} - \ell$.

We now compute the total residue of

$$(e^0 - e^{r+1})^{D-(r+1)}\phi_{r+1}(r - 1, 1, k_2, k_3)$$

with $D = D_{A_r+1}(r - 1, 1, k_2, k_3) = (1 + k_2)r + k_3 \frac{r(r-1)}{2} - (r - 1)$.

We have:

$$r!(e^0 - e^{r+1})^{D-(r+1)}\phi_{r+1}(r - 1, 1, k_2, k_3) = (e^0 - e^{r+1})^{D-(r+1)}P_{r-1}(e^1 - e^{r+1})(e^2 - e^{r+1})...(e^{r-1} - e^{r+1})\phi_{r+1}(0, 0, k_2, k_3).$$

with

$$P_{r-1} = \frac{\sum_{w \in \Sigma_r} w \cdot [ (e^1 - e^{r+1})(e^2 - e^{r+1})...(e^{r-1} - e^{r+1}) ]}{(e^1 - e^{r+1})(e^2 - e^{r+1})...(e^{r-1} - e^{r+1})} = (r - 1)! \sum_{j=1}^r \frac{1}{(e^j - e^{r+1})}.$$

Consider the subgroup $Cyc_r$ generated by the circular permutation of $1, ..., r$. Then

$$\sum_{j=1}^r \frac{1}{(e^j - e^{r+1})} = \sum_{w \in Cyc_r} w \cdot \frac{1}{(e^r - e^{r+1})}.$$

Assume $k_3$ odd. Thus

$$\frac{r!}{(r-1)!} \phi_{r+1}(r - 1, 1, k_2, k_3) = \left[ \sum_{w \in Cyc_r} w \cdot \frac{1}{(e^r - e^{r+1})} \right] \phi_{r+1}(0, 0, k_2, k_3)$$
\[ \sum_{w \in \text{Cyc}_r} \epsilon(w)w \cdot \left[ \frac{1}{(e^r - e^{r+1})k_2} \phi_{r+1}(0, 0, k_2, k_3) \right] \]

as \( \phi_{r+1}(0, 0, k_2, k_3) \) is anti-invariant under \( \Sigma_r \).

Remark that:

\[ \phi_{r+1}(0, 0, k_2, k_3) = r \frac{1}{(e^0 - e^r)k_2} \phi_r(0, k_3, k_2, k_3) \]

so that

\[ (e^0 - e^{r+1})^{D-(r+1)} \frac{1}{(e^r - e^{r+1})} \phi_{r+1}(0, 0, k_2, k_3) \]

\[ = r(e^0 - e^{r+1})^{D-(r+1)} \frac{1}{(e^r - e^{r+1})} \frac{1}{(e^0 - e^r)k_2} \phi_r(0, k_3, k_2, k_3). \]

It follows that we have

\[ (e^0 - e^{r+1})^{D-(r+1)} \phi_{r+1}(r - 1, 1, k_2, k_3) \]

\[ = \sum_{w \in \text{Cyc}_r} \epsilon(w)w \cdot \left[ (e^0 - e^{r+1})^{D-(r+1)} \frac{1}{(e^r - e^{r+1})} \frac{1}{(e^0 - e^r)k_2} \phi_r(0, k_3, k_2, k_3) \right]. \]

We now use LEMMA 3 to compute the total residue of the last term in the equality.

We write the vector space \( E_{r+1} \) as \( E_r \oplus \mathbb{R}(e^r - e^{r+1}) \), and we consider

\[ \Delta' = A_r \cup \{(e^r - e^{r+1})\} \]

Using the decomposition \( (e^0 - e^{r+1}) = (e^r - e^{r+1}) + (e^0 - e^r) \), we write

\[ (e^0 - e^{r+1})^{D-(r+1)} = \sum_{i \geq 0, j \geq 0, i+j=D-(r+1)} c_{ij} (e^r - e^{r+1})^i (e^0 - e^r)^j. \]

Thus

\[ (e^0 - e^{r+1})^{D-(r+1)} \frac{1}{(e^r - e^{r+1})} \frac{1}{(e^0 - e^r)k_2} \phi_r(0, k_3, k_2, k_3) \]

\[ = \sum_{i \geq 0, j \geq 0, i+j=D-(r+1)} c_{ij} \frac{(e^r - e^{r+1})^i}{(e^r - e^{r+1})} \frac{(e^0 - e^r)^j}{(e^0 - e^r)k_2} \phi_r(0, k_3, k_2, k_3) \]

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belongs to the vector space $R^{(e^r - e^{r+1})} \otimes R_{A_r}$ and we can easily compute the total residue using LEMMA 3 of Section 1, as well as the obvious calculation for a one dimensional space. Precisely

$$Tres_{A_{r+1}}[(e^0 - e^{r+1})^{D-(r+1)} \frac{1}{(e^r - e^{r+1})} \frac{1}{(e^0 - e^r)k_2} \phi_r(0, k_3, k_2, k_3)]$$

$$= \sum_{i \geq 0, j \geq 0, i+j = D-(r+1)} c_{ij} Tres_{(e^r - e^{r+1})^i} \frac{(e^r - e^{r+1})^i}{(e^r - e^{r+1})^j} Tres_{A_r}[(e^0 - e^r)^{D-(r+1)-k_2} \phi_r(0, k_3, k_2, k_3)].$$

Only the term $i = 0$ gives a non zero residue, so we obtain

$$Tres_{A_{r+1}}[(e^0 - e^{r+1})^{D-(r+1)} \frac{1}{(e^r - e^{r+1})} \frac{1}{(e^0 - e^r)k_2} \phi_r(0, k_3, k_2, k_3)]$$

$$= \frac{1}{(e^r - e^{r+1})} Tres_{A_r}[(e^0 - e^r)^{D-(r+1)-k_2} \phi_r(0, k_3, k_2, k_3)].$$

Now

$$D - (r+1) - k_2 = D_{A_{r+1}}(r - 1, 1, k_2, k_3) - (r+1) - k_2 = D_{A_r}(0, k_3, k_2, k_3) - r.$$ 

So we obtain

$$Tres_{A_{r+1}}[(e^0 - e^{r+1})^{D-(r+1)} \frac{1}{(e^r - e^{r+1})} \frac{1}{(e^0 - e^r)k_2} \phi_r(0, k_3, k_2, k_3)]$$

$$= \frac{1}{(e^r - e^{r+1})} Tres_{A_r}[(e^0 - e^r)^{D_{A_r}(0, k_3, k_2, k_3) - r} \phi_r(0, k_3, k_2, k_3)].$$

We apply induction hypothesis on $r$. We have

$$Tres_{A_r}[(e^0 - e^r)^{D_{A_r}(0, k_3, k_2, k_3) - r} \phi_r(0, k_3, k_2, k_3)]$$

$$= C_r(0, k_3, k_2, k_3) \sum_{w' \in \Sigma_{r-1}} \epsilon(w')w' \cdot \left[ \frac{1}{(e^0 - e^1)(e^1 - e^2)\ldots(e^{r-1} - e^r)} \right].$$

By FORMULA 3, as the total residue commutes with the action of $W$, we obtain:
\[Tres_{A_{r+1}}(e^0 - e^{r+1})D^{-(r+1)}\phi_{r+1}(r - 1, 1, k_2, k_3)\]

\[= C_r(0, k_3, k_2, k_3) \sum_{w \in Cyc_r} \epsilon(w)w \cdot [\frac{1}{(e^r - e^{r+1})}]
\times \sum_{w' \in \Sigma_{r-1}} \epsilon(w')w' \cdot \left[\frac{1}{(e^0 - e^1) \cdots (e^{r-1} - e^r)}\right].\]

But

\[\sum_{w \in Cyc_r} \epsilon(w)w \cdot [\frac{1}{(e^r - e^{r+1})}] \sum_{w' \in \Sigma_{r-1}} \epsilon(w')w' \cdot \left[\frac{1}{(e^0 - e^1) \cdots (e^{r-1} - e^r)}\right]\]

\[= \sum_{ww' \in Cyc_r, \Sigma_{r-1}=\Sigma_r} \epsilon(ww')ww' \cdot \left[\frac{1}{(e^0 - e^{r+1})(e^0 - e^1) \cdots (e^{r-1} - e^r)}\right] = \sum_{w \in \Sigma_r} \epsilon(w)w.f_{\Pi}.\]

Thus we obtain the second relation. The case \(k_3\) even is completely analogous, so the proof of the first and second relation is complete. The symmetry property in \(k_1, k_2\) is obvious.

Let us check

\[C_{r+1}(0, 1, 1, 0) = r!\]

More precisely, we have the following exact formula (without projection on \(S_\Delta\)).

**Lemma 30**

\[\frac{(e^0 - e^{r+1})^{(r-1)}}{\prod_{i=1}^{r}(e^0 - e^i)(e^i - e^r)} = \sum_{w \in \Sigma_r} w \cdot f_{\Pi}.\]

Indeed, by reduction to the same denominator, the right hand side can be written as

\[P \frac{\prod_{i=1}^{r}(e^0 - e^i)(e^i - e^r)}{\prod_{1 \leq i < j \leq r}(e^i - e^j)}\]

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\( \Sigma_r \), so is divisible by \( \prod_{1 \leq i < j \leq r} (e^i - e^j) \). From degree consideration, we obtain the desired equality, and \( C_{r+1}(0, 1, 1, 0) = r! \).

QED

Bibliographical remarks.
For the proofs of Theorem 26 and 27, we followed the indications of Zeilberger [Z]. Indeed, as explained in the remark following Theorem 27, these formulae are closely related to calculations of Selberg integral [Se], and to Morris identity [M]. We follow here closely Aomoto’s [A0] proof of the Selberg formula.

7 Chan-Robbins-Yuen polytope

Let \( B_n \) be the polytope of \( n \times n \) doubly stochastic matrices; that is the set of \( n \times n \) matrices with non-negative entries, and such that the sum of entries in any row or column is equal to 1. The vector space of \( n \times n \) matrices is equipped with the lattice of matrices with integral coefficients. The intersection of this lattice with the affine space spanned by \( B_n \) is an affine lattice, thus determines an unity of volume. The volume of \( B_n \) is defined with respect to this unity. It has been computed only up to \( n = 8 \) ([CY]). In [C-R-Y], some conjectures on the volume of some faces of \( B_n \) were stated. We show that they follow from Jeffrey-Kirwan formula for volumes, together with the explicit calculation of the preceding paragraph.

Definition 31 Define the vector space \( Y_n \) as the vector space of \( (n \times n) \) matrices, with \( x_{ij} = 0 \), if \( j \geq (i + 2) \). The Chan-Robbins-Yuen polytope is the polytope \( CRY_n = B_n \cap Y_n \).

Consider the vector space of \( (n + 1) \times (n + 1) \) lower triangular matrices such that first and last coefficients in the diagonal are equal to 0. Clearly this vector space is isomorphic to \( Y_n \), by adding to a matrix \( X \in Y_n \) a row above identically equal to 0, and a last column identically equal to 0.

Example \( n = 5 \).

The vector space \( Y_5 \) consists of \( 5 \times 5 \) matrices \( X = (x_{ij}) \) such that
The augmented matrix \( \text{aug}(X) \) is the \( 6 \times 6 \) lower triangular matrix

\[
\begin{pmatrix}
x_{11} & x_{12} & 0 & 0 & 0 & 0 \\
x_{21} & x_{22} & x_{23} & 0 & 0 & 0 \\
x_{31} & x_{32} & x_{33} & x_{34} & 0 & 0 \\
x_{41} & x_{42} & x_{43} & x_{44} & x_{45} & 0 \\
x_{51} & x_{52} & x_{53} & x_{54} & x_{55} & 0 \\
x_{61} & x_{62} & x_{63} & x_{64} & x_{65} & 0
\end{pmatrix}
\]

It is clear that the transformation \( X \mapsto \text{aug}(X) \) transforms the polytope \( CRY_n \) to the polytope \( C'_n \) of \( (n + 1) \times (n + 1) \) lower triangular matrices with non-negative entries, first row and last column identically equal to 0, and other rows and columns summing to 1.

**Example** \( n = 5 \). The polytope \( C'_5 \) consists of matrices

\[
A = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
a_{21} & a_{22} & 0 & 0 & 0 & 0 \\
a_{31} & a_{32} & a_{33} & 0 & 0 & 0 \\
a_{41} & a_{42} & a_{43} & a_{44} & 0 & 0 \\
a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & 0 \\
a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & 0
\end{pmatrix}
\]

such that \( a_{ij} \geq 0 \), and such that

\[
a_{21} + a_{31} + a_{41} + a_{51} + a_{61} = 1,
\]

\[
a_{21} + a_{22} = 1 = a_{22} + a_{32} + a_{42} + a_{52} + a_{62},
\]

\[
\ldots
\]

\[
a_{51} + a_{52} + a_{53} + a_{54} + a_{55} = 1 = a_{55} + a_{65},
\]

\[
a_{61} + a_{62} + a_{63} + a_{64} + a_{65} = 1.
\]
Let \( n \) be the vector space of strictly lower triangular \((n+1) \times (n+1)\)-matrices, with basis \( E_{ij}; (j < i) \) with unique non zero entry at the \( i \)-th row and \( j \) column. Consider a \((n+1)\) dimensional vector space with basis \( e_i, 1 \leq i \leq (n+1) \) and dual basis \( e^i \). Let \( A_n = \{(e^k - e^\ell); k \neq \ell \} \) with positive root system \( \{(e^k - e^\ell); 1 \leq k < \ell \leq (n+1)\} \). Let \( p \) be the map sending the basis \( E_{ji} \) of the vector space \( n \) to \( (e^j - e^i) \), for \( 1 \leq j < i \leq (n+1) \). The span of \( A_n \) is the \( n \)-dimensional vector space \( E_n \) consisting of elements \( a = \sum_{i=1}^{n+1} a_i e^i \), with \( \sum_{i=1}^{n+1} a_i = 0 \)

**Lemma 32** The polytope \( CRY_n \) is isomorphic to \( P_{A_n^+}(e^1 - e^{n+1}) \).

We give the proof for \( n = 5 \), as it is most transparent on an example. By definition the polytope \( P_{A_5^+}(e^1 - e^6) \) consists of all \((6 \times 6)\) lower triangular matrices

\[
B = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
b_{21} & 0 & 0 & 0 & 0 & 0 \\
b_{31} & b_{32} & 0 & 0 & 0 & 0 \\
b_{41} & b_{42} & b_{43} & 0 & 0 & 0 \\
b_{51} & b_{52} & b_{53} & b_{54} & 0 & 0 \\
b_{61} & b_{62} & b_{63} & b_{64} & b_{65} & 0
\end{pmatrix}
\]

with non negative coefficients \( b_{ij} \) and such that

\[
\sum b_{ij}(e^j - e^i) = e^1 - e^6.
\]

This gives the 6 equations:

\[
\begin{align*}
b_{21} + b_{31} + b_{41} + b_{51} + b_{61} &= 1, \\
-b_{21} + b_{32} + b_{42} + b_{52} + b_{62} &= 0, \\
-b_{31} - b_{32} + b_{43} + b_{53} + b_{63} &= 0, \\
&\ldots\ldots
\end{align*}
\]

\[
-b_{61} - b_{62} - b_{63} - b_{64} - b_{65} = -1.
\]

Define

\[
\begin{align*}
b'_{22} &= 1 - b_{21}, & b'_{33} &= 1 - (b_{31} + b_{32}) \\
b'_{44} &= 1 - (b_{41} + b_{42} + b_{43}) \\
b'_{55} &= 1 - (b_{51} + b_{52} + b_{53} + b_{54}).
\end{align*}
\]

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These coefficients are positive: indeed the total sum of coefficients of the entries of the matrix $B$ in the rectangle $R_{ij}$ consisting on elements in a row of index greater than $i$ and on a column of index less than $j$ is identically equal to 1, so that any partial sum is less then 1. For example, adding first 3 inequalities, we obtain that

$$b_{41} + b_{42} + b_{43} + b_{51} + b_{52} + b_{53} + b_{61} + b_{62} + b_{63}$$

is equal to 1, so that $b_{41} + b_{42} + b_{43}$ is less than 1.

Then the matrix

$$B' = \begin{pmatrix}
  b_{21} & b'_{22} & 0 & 0 & 0 \\
  b_{31} & b_{32} & b'_{33} & 0 & 0 \\
  b_{41} & b_{42} & b_{43} & b'_{44} & 0 \\
  b_{51} & b_{52} & b_{53} & b_{54} & b'_{55} \\
  b_{61} & b_{62} & b_{63} & b_{64} & b_{65}
\end{pmatrix}$$

is in the polytope $CRY_5$.

We now show that the conjectures of Chan-Robbins-Yuen on their polytope $CRY_n$ and related polytopes are consequences of THEOREM 26 and PROPOSITION 23.

**Theorem 33 (Zeilberger)** The relative volume of $CRY_n$ is equal to

$$\prod_{i=1}^{n-2} \frac{1}{i+1} \binom{2i}{i}.$$  

**Proof.** The polytope $CRY_n$ is isomorphic to the polytope $P_{A_n^+}(e^1 - e^{n+1})$. The linear isomorphism described above using the map $aug$ preserves the volume, as it preserves the corresponding lattices. We thus compute the relative volume of $P_{A_n^+}(e^1 - e^{n+1})$. Consider

$$J_n = \frac{(e^1 - e^{n+1})^{(n-1)/2}}{\prod_{1 \leq i < j \leq (n+1)} (e^i - e^j)}.$$  

It is an element of $R_{A_n}$ of homogeneous degree $-n$. We choose the big chamber $c^+$ in $E_n$ consisting of elements $a = \sum_{i=1}^{n+1} a_i e^i$, with $a_i > 0$, for
\( i = 1, 2, \ldots, n \). The element \( a = e^1 - e^{n+1} \) is in the closure of this big chamber. Jeffrey-Kirwan formula is:

\[
\text{vol}_{\text{rel}} P_{A_n^+}(e^1 - e^{n+1}) = \langle \langle \epsilon^+, \text{Tres}_{A_n}(J_n) \rangle \rangle.
\]

The function \( J_n \) up to renumbering has been introduced in Section 5. This is the function \( W_n \). Recall the formula for its total residue given in THEOREM 24.

Let

\[
f_{\Pi} = \frac{1}{(e^1 - e^2)(e^2 - e^3) \cdots (e^n - e^{n+1})}
\]

be the particular element of \( S_\Delta \) obtained by taking the inverse of the product of simple roots. Then

\[
\text{Tres}_{A_n}(J_n) = \prod_{i=1}^{n-2} \frac{1}{i + 1} \binom{2i}{i} \left[ \sum_{w \in \Sigma[2, \ldots, n]} \epsilon(w) w . f_{\Pi} \right].
\]

The linear form \( \langle \langle \epsilon^+, f \rangle \rangle \) has been computed in Section 4. We write

\[
j_n(x) = \frac{x_1^{n(n-1)/2}}{x_1 \ldots x_n \prod_{1 \leq i < j \leq n} (x_i - x_j)}.
\]

It is the function \( J_n \) restricted at elements \( \sum_{i=1}^n x_i e_i \). Then

\[
\langle \langle \epsilon^+, J_n \rangle \rangle = \text{Ires}_{x=0} j_n(x).
\]

As iterated residues form a dual basis to the basis \( w . f_{\Pi} \) of \( S_\Delta \) (with \( w \in \Sigma[1, 2, \ldots, n] \)), we obtain \( \langle \langle \epsilon^+, J_n \rangle \rangle = \prod_{i=1}^{n-2} \frac{1}{i + 1} \binom{2i}{i} \).

QED

Consider the polytope \( CRY_n((2, 2)) \), consisting on elements of \( CRY_n \) with the entry \( (2, 2) \) equal to 0. It is a face of codimension 1 of \( CRY_n \), isomorphic to the polytope \( P_{A_n^+ \text{ minus } (e^2 - e^3)}(e^1 - e^{n+1}) \). Taking in account the fact that the dimension of \( CRY_n \) is \( \binom{n}{2} \), we obtain from PROPOSITION 23 the following corollary.

\textbf{Lemma 34 (Conjecture 4 of [C-R-Y])}

We have

\[
\binom{n}{2} \text{vol}_{\text{rel}} CRY_n((2, 2)) = 3 \text{vol}_{\text{rel}} CRY_n.
\]
Bibliographical remarks.

The Chan-Robbins-Yuen polytope is described in [C-R-Y]. Its volume is computed by Zeilberger [Z], using a formula of Postnikov-Stanley [S] or of Chan-Robbins-Yuen [C-R-Y] for the volume of $CRY_n$, as a particular value of Kostant partition function (see Lemma 44 in next Section 8). Our proof follows directly from the Jeffrey-Kirwan residue formulae for the volume, but we use also the crucial identity of Theorem 27. The conjectured relations between volumes of flow polytopes and the volume of a particular face is deduced from Proposition 23. However, the volume of the entire boundary of the polytope $CRY_n$ is unknown.

8 Ehrhart polynomial of a flow polytope.

We consider the positive root system of $A_r$ realized in $E_r \subset \mathbb{R}^{r+1}$. We will use sometimes implicitly the map

$$a = (a_1, a_2, ..., a_r) \rightarrow a = a_1 e^1 + ... + a_r e^r - (a_1 + \cdots + a_r)e^{r+1}$$

to identify $E_r$ and $\mathbb{R}^r$.

Let $M = r(r+1)/2$. Let $a \in \mathbb{Z}^{r+1} \cap C(A^+_r)$. Then $a_1 \geq 0$, $a_1 + a_2 \geq 0$, ..., $a_1 + a_2 + \cdots + a_r \geq 0$, $a_1 + a_2 + \cdots + a_{r+1} = 0$. Let $k_{A^+_r}(a)$ be the number of solutions $(u_1, u_2, ..., u_M)$, in non negative integers $u_m$, of the equation

$$\sum_{m=1}^{M} u_m \alpha^m = a.$$ 

Here $\alpha^m$ runs though the $r(r+1)/2$ positive roots $(e^i - e^j)$ of $A_r$. The function $k_{A^+_r}(a)$ is the number of integral points in the polytope $P_{A^+_r}(a)$ and is called the Kostant partition function (for $A_r$).

Consider the permutation $w_0 : [1, 2, ..., r, r+1] \mapsto [r+1, r, ..., 2, 1]$. Then $-w_0$ preserves $A^+_r$. A solution $(u_1, u_2, ..., u_M)$, in non negative integers $u_m$, of the equation

$$\sum_{m=1}^{M} u_m \alpha^m = a$$

gives a solution of the equation

$$\sum_{m=1}^{M} u_m(-w_0 \cdot \alpha^m) = -w_0 \cdot a.$$
So for $a_1 + a_2 + \ldots + a_{r+1} = 0$, we have

$$k_{A_r^+}^{-}(a_1 e^1 + a_2 e^2 + \cdots + a_r e^r + a_{r+1} e^{r+1})$$

$$= k_{A_r^+}^{-}(-a_{r+1} e^1 - a_r e^2 - \cdots - a_2 e^r - a_1 e^{r+1}).$$

Almost by definition, the function $k_{A_r^+}$ is given by an iterated constant term.

**Lemma 35** Let $a = a_1 e^1 + a_2 e^2 + \ldots + a_r e^r + a_{r+1} e^{r+1}$, with $a_1 + a_2 + \ldots + a_r + a_{r+1} = 0$. The value $k_{A_r^+}(a)$ is given by the iterated constant term formula:

$$k_{A_r^+}(a) = CT_{z_1=0} CT_{z_2=0} \cdots CT_{z_{r+1}=0} \left( \frac{z_1^{a_1} z_2^{a_2} \cdots z_{r+1}^{a_{r+1}}}{\prod_{1 \leq i < j \leq r+1} (1 - z_j/z_i)} \right).$$

Here $\frac{1}{(1-z_{r+1}/z_i)}$ is expanded as a power series in $z_{r+1}$. The constant term at $z_{r+1} = 0$ of $\prod_{1 \leq i \leq r+1} (1 - z_{r+1}/z_i)$ is in the ring $R[z_1^{-1}, z_2^{-1}, \ldots, z_r^{-1}]$ and we reiterate.

Thus, some explicit values for Kostant partition function can be obtained from the total residue formula of Section 6, as iterated constant term of a function $f$ coincide with the iterated residue of this function divided by $z_1 z_2 \ldots z_r$.

For example

**Lemma 36** For $r \geq 2$,

$$k_{A_r^+}((1 + d), (2 + d), \ldots, (r + d)) = \prod_{i=d+1}^{r+d-1} \frac{1}{2i+1} \binom{r + d + i + 1}{2i},$$

$$k_{A_r^+}(1, 2, \ldots, r) = \prod_{i=1}^{r} C_i.$$

Let $\Phi = \{\alpha^1, \alpha^2, \ldots, \alpha^N\}$ be a sequence of elements of $A_r^+$ generating $E_r$.

Consider the surjective linear map $p : \mathbb{R}^N \to E_r$ such that $p(w') = \alpha^i$.

Then, for $a \in \mathbb{Z}^r$, the polytope $P_{\Phi}(a) \subset \mathbb{R}^N \cap p^{-1}(a)$ has vertices with integral coordinates.
Definition 37. For \( a \in \mathbb{Z}^r \), define

\[
k_{\Phi}(a) = |P_{\Phi}(a) \cap \mathbb{Z}^N|.
\]

Thus \( k_{\Phi}(a) \) is the number of solutions \((u_1, u_2, ..., u_N)\), in non negative integers \( u_m \), of the equation

\[
\sum_{m=1}^{N} u_m \alpha_m = a.
\]

Define \( k'_{\Phi}(a) \) as the number of solutions \((u_1, u_2, ..., u_N)\), in strictly positive integers \( u_m \), of the equation

\[
\sum_{m=1}^{N} u_m \alpha_m = a.
\]

Remark. Let \( a = \sum_{i=1}^{r+1} a_i e_i \). If \( k'_{\Phi}(a) \) is non zero, then necessarily \( a_1 \geq \sum_{j=2}^{r+1} m_{1j} \).

We consider the system \(-w_0(\Phi)\). Then clearly \( k_{\Phi}(-w_0(a)) = k_{-w_0(\Phi)}(a)\).

Lemma 38. Let \( a = a_1 e_1 + a_2 e_2 + ... + a_r e_r + a_{r+1} e_{r+1} \), with \( a_1 + a_2 + ... + a_r + a_{r+1} = 0 \). The value \( k_{\Phi}(a) \) is given by the iterated constant term formula:

\[
k_{\Phi}(a) = \text{CT}_{z_1=0} \text{CT}_{z_2=0} \cdots \text{CT}_{z_{r+1}=0} \left( \frac{z_1^{a_1} z_2^{a_2} \cdots z_{r+1}^{a_{r+1}}}{\prod_{1 \leq i < j \leq r+1} (1 - z_j/z_i)^{m_{ij}}} \right).
\]

We also consider the number \( k'(\Phi, c)(a) \) of solutions in strictly positive integers \( u_m \) of the equation

\[
\sum_{m=1}^{N} u_m \alpha_m = a.
\]

Let \( c \) be a big chamber for the system \( \Delta^+ = \{ \Phi \} \). It follows from Ehrhart results on polytopes with integral vertices that the function \( k_{\Phi}(a) \) is given by a polynomial formula when \( a \) varies in \( c \cap \mathbb{Z}^r \). This polynomial is called the Ehrhart polynomial of the family of polytopes \( P_{\Phi}(a) \), on the big chamber \( c \).

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For $\alpha \in \Phi$, the rational function on $E_r^*$

$$\frac{\alpha}{1 - e^{-\alpha}} = 1 + \frac{1}{2}\alpha + \frac{1}{12}\alpha^2 + ...$$

is analytic at the origin, and can be expanded as a Taylor series. Thus

$$\frac{1}{1 - e^{-\alpha}} = \frac{1}{\alpha}(1 - e^{-\alpha})$$

as an element of $\hat{R}_\Delta$, and for any $a \in E_r$, the function $\prod_{i=1}^{N} e^{a_i} (1 - e^{-\alpha_i})$ is an element of $\hat{R}_\Delta$.

Define

$$K_\Phi(a) = Tres_\Delta \left( \frac{e^a}{\prod_{i \in \Phi} (1 - e^{-\alpha})} \right).$$

As $\frac{1}{\prod_{i=1}^{N} (1 - e^{-\alpha_i})} = \frac{1}{\prod_{i=1}^{N} \alpha_i} + \sum_{k>N} f_k$, we have

$$Tres_\Delta \left( \frac{e^a}{\prod_{i=1}^{N} (1 - e^{-\alpha_i})} \right)$$

$$= \frac{1}{(N - r)!} Tres_\Delta \left( \frac{a^{N-r}}{\prod_{i=1}^{N} \alpha_i} \right) + \sum_{q<(N-r)} \frac{1}{q!} Tres_\Delta (a^q f_{-q}).$$

Thus the function

$$a \mapsto Tres_\Delta \left( \frac{e^a}{\prod_{i=1}^{N} (1 - e^{-\alpha_i})} \right)$$

is a polynomial function of degree $(N - r)$ with value in $S_\Delta$. Recall DEFINITION 5 of the function $J_\Phi$. We see that the homogeneous component of degree $(N - r)$ of $K_\Phi(a)$ is

$$J_\Phi(a) = Tres_\Delta \left( \frac{e^a}{\prod_{i=1}^{N} \alpha_i} \right)$$

$$= \sum_{|i|=N-r} f_\Phi(i) \frac{a_i^1 a_i^2 \cdots a_i^{r-1} a_i^r}{i_1! i_2! \cdots i_{r-1}! i_r!}.$$
Section 3.5) from the locally polynomial function \( \text{vol} P_\Phi(a) \) via a Todd operator. Using Jeffrey-Kirwan residue formula for the volume, we obtain (see [B-V 1, Section 3.5]) the following residue formula for the Ehrhart polynomials \( k_\Phi(a) \) and \( k'_\Phi(a) \).

**Theorem 39 (Multidimensional residue theorem)**

*Let \( a \in C(\Delta^+) \). Let \( c \subset C(\Delta^+) \) be a big chamber such that \( a \in \mathcal{C} \), then*

\[
k_\Phi(a) = \langle \langle c, K_\Phi(a) \rangle \rangle.
\]

*If \( a \in c \), then*

\[
k'_\Phi(a) = (-1)^{|\Phi|} \langle \langle c, K_\Phi(-a) \rangle \rangle.
\]

For example, in case \( \Delta = A_1 \) and \( \Phi \) with multiplicity \( m \), the preceding theorem asserts that for \( a \geq 0 \)

\[
k_\Phi(ae^1 - ae^2) = CT_{z=0} \frac{z^{-a}}{(1-z)^m} = Res_{x=0} \frac{e^{ax}}{(1-e^{-x})^m}.
\]

The first expression is the residue of the 1-form \( \frac{z^{-a}}{(1-z)^m} \frac{dz}{z} \) at \( z = 0 \). The second is the residue at \( z = 1 \) of this 1-form (we choose \( z = e^{-x} \) as local coordinate, near \( x = 0 \)). As \( a + m > 0 \), the rational function \( \frac{z^{-a}}{(1-z)^m} \frac{dz}{z} \) has no poles at \( z = \infty \) and we obtain the result from the one dimensional residue theorem.

**Definition 40** We denote by \( k(\Phi, c)(a) \) the polynomial such that \( k(\Phi, c)(a) = k_\Phi(a) \) when \( a \in \mathcal{C} \)

Thus

\[
k(\Phi, c)(a) = \langle \langle c, K_\Phi(a) \rangle \rangle.
\]

This is a polynomial of degree \( N - r \) and the homogeneous component of degree \( N - r \) of \( k(\Phi, c) \) is the polynomial

\[
v(\Phi, c)(a) = \langle \langle c, J_\Phi(a) \rangle \rangle.
\]

As in LEMMA 20, the knowledge of \( k(\Phi, c^+)(a) \) determines (in principle) the other functions \( k(\Phi, c) \).
Lemma 41 Let $c$ be a big chamber. Let $\Sigma_r(c)$ be the set of elements $w \in \Sigma_r$ such that $c \subset C^+_w$. Then

$$k(\Phi, c)(a) = \sum_{w \in \Sigma_r(c)} (-1)^{n(w)} k(\Phi, c^+)(w^{-1}a).$$

We write

$$v(\Phi, c)(a_1, ..., a_r) = \sum_{i_1 + i_2 + ... + i_r = N-r} v(\Phi, c, i) \frac{a_1^{i_1} a_2^{i_2} ... a_r^{i_r}}{i_1! i_2! ... i_r!}.$$  

Definition 42 Define

$$t^\Phi_j = \sum_{k=j+1}^{r+1} m_{jk} - 1,$$

$$s^\Phi_j = 1 - \sum_{k=1}^{j-1} m_{kj}.$$

We consider the nice chamber $c^+$. Then we can express the coefficients of the function $v(\Phi, c^+)(a)$ as values at very particular points depending on $i$ of the function $k_{\Phi'}$, where $\Phi'$ is the system where we have deleted all roots $e^j - e^{r+1}$.

The system $\Phi'$ is a sequence of vectors in the positive root system of $A_{r-1}$, which spans the vector space $E_{r-1} = \{ a = \sum_{i=1}^{r} a_i e^i, \text{with } \sum_{i=1}^{r} a_i = 0 \}$. Remark that if $i_1 + i_2 + ... + i_r = N-r$, then the vector $(i_1 - t_1)e^1 + (i_2 - t_2)e^2 + ... + (i_r - t_r)e^r$ is in $E_{r-1}$.

Proposition 43 (Postnikov-Stanley)

For the big chamber $c^+$ of $C(A_+^{r})$ we have

$$v(\Phi, c^+, i) = k_{\Phi'}((i_1 - t_1)e^1 + (i_2 - t_2)e^2 + ... + (i_r - t_r)e^r).$$

Proof. We use the iterated residue formula for $f \mapsto \langle \langle c^+, f \rangle \rangle$ and the iterated constant term formula (Lemma 38) for $k_{\Phi'}$. Indeed

$$v(\Phi, c^+, i)$$

$$= Res_{x_1=0} Res_{x_2=0} ... Res_{x_r=0} \left( \frac{x_1^{i_1} x_2^{i_2} ... x_r^{i_r}}{x_1^{m_1,r+1} ... x_r^{m_r,r+1} \prod_{1 \leq i < j \leq r} (x_i - x_j)^{m_{ij}}} \right)$$

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\[
\begin{align*}
&= CT_{x_1=0}CT_{x_2=0} \cdots CT_{x_r=0} \left( \frac{x_1^{i_1+1-m_{1,r+1}} x_2^{i_2+1-m_{2,r+1}} \cdots x_r^{i_r+1-m_{r,r+1}}}{\prod_{1 \leq i < j \leq r} (x_i - x_j)^{m_{ij}}} \right) \\
&= CT_{x_1=0}CT_{x_2=0} \cdots CT_{x_r=0} \left( \frac{x_1^{i_1-t_1} x_2^{i_2-t_2} \cdots x_r^{i_r-t_r}}{\prod_{1 \leq i < j \leq (1 - x_j/x_i)^{m_{ij}}}^{m_{ij}}} \right) \\
&= k_{\Phi'}((i_1 - t_1)e^1 + (i_2 - t_2)e^2 + \cdots + (i_r - t_r)e^r).
\end{align*}
\]

QED

Consider now the system \( \Phi = A^+_r \). We have \( t_i^\Phi = (r - i) \). The system \( \Phi' \) is \( A^+_{r-1} \). We have thus

\[
v(A^+_r, c^+, i) = k_{A^+_{r-1}}((i_1 - (r - 1))e^1 + (i_2 + (r - 2))e^2 + \cdots + i_{r-1}e^{r-1} + i_re^r)
\]

In the right hand side, if \( i_r > 0 \), the element \((i_1 - (r - 1))e^1 + \cdots + i_{r-1}e^{r-1} + i_re^r\) cannot be in the cone generated by \((e^i - e^j)\) with \(1 \leq i < j \leq r\), as seen by looking at the component on \( e^r \) which would be negative. Furthermore, if \( i_r = 0 \), we see that the coefficients of roots \( e^i - e^r \) in a solution of the equation

\[
\sum_{\alpha^m \in A^+_{r-1}} x_m \alpha^m = (i_1 - (r - 1))e^1 + (i_2 + (r - 2))e^2 + \cdots + i_{r-1}e^{r-1}
\]

(with \( x_m \geq 0 \)) are necessarily 0. So, consider the system \( A^+_{r-2} \) realized as \((e^i - e^j), 1 \leq i < j \leq (r - 1)\). Then we see that

\[
v(A^+_r, c^+; (i_1, i_2, \ldots, i_{r-1}, i_r)) = 0
\]

if \( i_r > 0 \) and

\[
v(A^+_r, c^+; (i_1, i_2, \ldots, i_{r-1}, 0))
\]

\[
= k_{A^+_{r-2}}((i_1 - (r - 1))e^1 + (i_2 - (r - 2))e^2 + \cdots + (i_{r-1} - 1)e^{r-1}).
\]

In particular, we obtain for monomials, the relation between the volume of the CRY polytope \( P_{A_r}(e^1 - e^{r+1}) \) and Kostant Partition function. Indeed, the only \( i \) to consider is \( i_1 = r(r - 1)/2, i_2 = 0, \ldots, i_r = 0 \). The point \((i_1 - (r - 1))e^1 - (r - 2)e^2 + \cdots + (r - 2)e^{r-2} - e^{r-1}\) is flipped to the point \( e^1 + 2e^2 + \cdots + (r - 2)e^{r-2} + (r - 1) - i_1)e^{r-1}\) under the transformation \(-w_0\) of the system \( A^+_{r-2} \). We then obtain:
Corollary 44 (Postnikov-Stanley)

\[ \text{vol } P_{A_r^+}(e^1 - e^{r+1}) = k_{A_{r-2}^+}(1, 2, 3, 4, \ldots, (r - 2)). \]

Similarly, for any sequence \( \Phi \), the relative volume of the flow polytope \( P_{\Phi}(e^1 - e^{r+1}) \) is an integer, given by the Kostant partition function \( k_{\Phi} \) at a particular point. Combinatorists are happy of this result, only if they can explain this by giving an explicit simplicial decomposition of the corresponding flow polytope.

Bibliographical remarks.

Lemma 36, as well as many other results on Kostant Partition function is in [Ki]. Theorem 39 follows from [K-P] and [J-K], at least for generic values. A generalization to any rational polytope is given in [B-V 1]. Proposition 43 and corollary 44 are due to Postnikov- Stanley [P-S].

9 Change of variables for the total residue

We return to the notations of Section 2.

Let \( F : V_C \to V_C \) be an analytic map, such that \( F(0) = 0 \) and preserving the open set \( U_\Delta = V_C - \mathcal{H}_C \) of \( V_C \). If \( f \in \hat{R}_\Delta \), the function \( (F^* f)(x) = f(F(x)) \) is again in \( \hat{R}_\Delta \). Let \( \text{Jac}(F) \) be the Jacobian of the map \( F \). We assume \( \text{Jac}(F)(0) \neq 0 \). We write \( F(x) = L(F)(x) + r(x) \) where \( L(F) \) is a linear invertible map, and \( r(x) \) vanishes at 0 at order 2. Thus \( L(F) \) permutes the hyperplanes \( \{ \alpha = 0 \} \). If \( f \) is in \( S_\Delta \), the function \( L(F)^*(f) \) is again in \( S_\Delta \).

Theorem 45 For any \( f \) in \( \hat{R}_\Delta \), we have the equality in \( S_\Delta \):

\[ \text{Jac}(F)(0)L(F)^*(\text{Tres}_\Delta(f)) = \text{Tres}_\Delta(\text{Jac}(F)(F^* f)). \]

Proof. Let \( f \in \hat{R}_\Delta \). Then \( (f - \text{Tres}_\Delta(f))dx \) is the differential of some \((r-1)\)-form \( \sum_{k=1}^{r} f_k dx_1 \wedge dx_2 \wedge \cdots \wedge dx_r \), with \( f_k \in \hat{R}_\Delta \). The vector space \( \sum_{k=1}^{r} f_k dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n \) with \( f_k \in \hat{R}_\Delta \) is stable by the action of \( F^* \) on differential forms. As \( F^* \) commutes with \( d \), \( (F^* f)F^* dx - F^*(\text{Tres}_\Delta(f))F^* dx \) is the differential of some \((r-1)\)-form \( \sum_{k=1}^{r} \Phi_k dx_1 \wedge dx_2 \wedge \cdots \wedge dx_r \), with \( \Phi_k \in \hat{R}_\Delta \). We have \((F^* f)F^* dx = \text{Jac}(F)(F^* f)dx \).
We now analyze the differential form \((F^*f)(F^*dx) = (F^*f)\text{Jac}(F)dx\) with \(f \in S_\Delta\). The function \(f\) is the inverse of the product of \(r\) linear forms \(\alpha^k\). As \(F\) preserves \(U_\Delta\), we have \(\alpha^k(F(x)) = \beta^k(x)g^k(x)\) where \(\beta^k(x)\) is another linear form in the system \(\Delta\) and \(g^k(x)\) is holomorphic at 0. Thus \((F^*f)(x) = \frac{1}{\prod_k \beta^k(x)} \prod_k (g^k(x))^{-1}\) is again in \(\hat{R}_\Delta\).

Furthermore, we see that \((F^*f) - (L(F)^*f)\) is an element of \(R_\Delta\) of degree \(> -r\). Thus \((F^*f)\text{Jac}(F) - (L(F)^*f)J(F)(0)\) is an element of \(\hat{R}_\Delta\) of degree \(> -n\). It follows that \((F^*f)\text{Jac}(F)dx - (L(F)^*f)J(F)(0)dx\) is the differential of some \((r-1)\)-form \(\sum_{k=1}^r h_k(x)dx_1 \wedge dx_2 \wedge dx_k \wedge \cdots \wedge dx_r,\) with \(h_k \in \hat{R}_\Delta\).

Adding these two informations, we see that for any \(f \in \hat{R}_\Delta\),

\[\text{Jac}(F)(F^*f)dx - J(F)(0)(L(F)^*\text{Tr}_\Delta(f))dx\]

is the differential of some \(r-1\)-form \(\sum_{k=1}^r m_k(x)dx_1 \wedge dx_2 \wedge dx_k \wedge \cdots \wedge dx_r,\) with \(m_k \in \hat{R}_\Delta\). This implies the formula of the lemma.

QED

10 A nice formula for Kostant restricted partition function

Let \(\Phi\) be a sequence of \(N\) vectors in \(A^\Delta_+ \subset E_r\) generating \(E_r\). In the same spirit that Lidskii formula for Kostant partition function ([L]), we will give a closed formula for \(K_\Phi(a)\) in function of \(\text{vol} P_\Phi(a)\). In fact, we will express the \(S_\Delta\)-valued polynomial \(K_\Phi(a)\) in function of the \(S_\Delta\)-valued polynomial \(J_\Phi(a)\) of Section 12.

Let

\[
\binom{\binom{u}{k}}{k} = \frac{u(u+1)(u+2) \cdots (u+(k-1))}{k!}.
\]

**Theorem 46** Let

\[
J_\Phi(a) = \sum_{i_1+i_2+\cdots+i_r = N-r} f_\Phi(i) \frac{a_{i_1}^{i_1} a_{i_2}^{i_2} \cdots a_{i_{r-1}}^{i_{r-1}} a_{i_r}^{i_r}}{i_1! i_2! \cdots i_{r-1}! i_r!}.
\]

Then

\[
K_\Phi(a) = \sum_{|i| = N-r} f_\Phi(i) \binom{a_1 + t_1^\Phi}{i_1} \binom{a_2 + t_2^\Phi}{i_2} \cdots \binom{a_{r-1} + t_{r-1}^\Phi}{i_{r-1}} \binom{a_r + t_r^\Phi}{i_r}.
\]
We have as well

\[ K_\Phi(a) = \sum_{|\mathbf{i}| = N-r} f_\Phi(\mathbf{i}) \left( \begin{pmatrix} a_1 + s_{\Phi_1} \\ i_1 \end{pmatrix} \right) \cdots \left( \begin{pmatrix} a_{r-1} + s_{\Phi_{r-1}} \\ i_{r-1} \end{pmatrix} \right) \left( \begin{pmatrix} a_r + s_{\Phi_r} \\ i_r \end{pmatrix} \right). \]

**Proof.** We realize \( A_r \) in \( \mathbb{R}^r \) as the system \((e^i - e^j), 1 \leq i < j \leq r, e^i, 1 \leq i \leq r\).

Then

\[ K_\Phi(a) = Tres_\Delta \left[ e^{a_1 x_1} e^{a_2 x_2} \cdots e^{a_r x_r} \right. \]

\[ \left. \prod_{i=1}^r (1 - e^{-x_i})^{m_{i,r+1}} \prod_{1 \leq i < j \leq r} (1 - e^{-(x_i - x_j)})^{m_{ij}} \right]. \]

We use LEMMA 45 for changing variables in residues. Let \( x = \sum_{i=1}^r x_i e_i \) in \( V \). Define \( F(x) = \sum_{i=1}^r (e^{x_i} - 1) e_i \). This change of variables preserves the vector space \( R_\Delta \). Indeed \((e^{x_i} - e^{x_j})\) is divisible by \((x_i - x_j)\). The differential of \( F \) at the origin is the identity. We have \( \text{Jac}(F) = e^{x_1} e^{x_2} \cdots e^{x_r} \).

We write \( t_k, s_k \) instead of \( t^{\Phi_k}, s^{\Phi_k} \). Let \( a_1, a_2, \ldots, a_r \) be integers and let

\[ f(a_1, a_2, \ldots, a_r)(x) = \frac{(1 + x_1)^{a_1+t_1} (1 + x_2)^{a_2+t_2} \cdots (1 + x_r)^{a_r+t_r}}{x_1^{m_{1,r+1}} x_2^{m_{2,r+2}} \cdots x_r^{m_{r,r+1}} \prod_{1 \leq i < j \leq r} (x_i - x_j)^{m_{ij}}} \]

Then the function \((F^* f) \text{Jac}(F))\) is equal to

\[ e^{a_1 x_1} e^{a_2 x_2} \cdots e^{a_r x_r} \]

\[ \frac{1}{(1 - e^{-x_1})^{m_{1,r+1}} (1 - e^{-x_2})^{m_{2,r+2}} \cdots (1 - e^{-x_r})^{m_{r,r+1}}} \]

\[ \times \prod_{1 \leq i < j \leq r} (1 - e^{-(x_i - x_j)})^{m_{ij}}. \]

Thus, we obtain from LEMMA 45

\[ Tres_\Delta \left( \frac{e^{(a,x)}}{\prod_{\alpha \in \Phi} (1 - e^{-(a,x)})} \right) = Tres_\Delta \left( \frac{(1 + x_1)^{a_1+t_1} \cdots (1 + x_r)^{a_r+t_r}}{\prod_{\alpha \in \Phi} \alpha(x)} \right). \]

To compute the total residue of the last expression, as the denominator \( \prod_{\alpha \in \Phi} \alpha(x) \) is homogeneous of degree \( N \), we have to seek for the term in the numerator which is homogeneous of degree \( N - r \), thus we seek the coefficient
of each term of the form $x_1^{i_1} \cdots x_r^{i_r}$ with $i_1 + i_2 + \cdots + i_n = N - r$. We thus obtain the first part of Proposition 46.

The second is proved the same way, using

$$F(x) = \sum_{i=1}^r (1 - e^{-x_i}) c_i$$

and the function

$$\frac{(1 + x_1)^{-a_1} \cdots (1 + x_r)^{-a_r}}{x_1^{m_{1,r+1}} x_2^{m_{2,r+1}} \cdots x_r^{m_{r,r+1}} \prod_{1 \leq i < j \leq r} (x_i - x_j)^{m_{ij}}}$$

QED

Thus we obtain the following formula for $k_\Phi(a)$.

**Theorem 47** Let $\mathfrak{c}$ be a big chamber and $a \in \mathfrak{c}$. Then

$$k_\Phi(a) = \sum_{|i| = N - r} f_\mathfrak{c}(i) \left( \binom{a_1 + s_1}{i_1} \binom{a_2 + s_2}{i_2} \cdots \binom{a_{r-1} + s_{r-1}}{i_{r-1}} \binom{a_r + s_r}{i_r} \right).$$

We have as well

$$k_\Phi(a) = \sum_{|i| = N - r} f_\mathfrak{c}(i) \left( \binom{a_1 + s_1}{i_1} \binom{a_2 + s_2}{i_2} \cdots \binom{a_{r-1} + s_{r-1}}{i_{r-1}} \binom{a_r + s_r}{i_r} \right).$$

Comparing with formula [1] for the volume, we see that the function $k_\Phi(a)$ is immediately deduced from the polynomial function $\text{vol} P_\Phi(a)$ (i.e.; its highest degree component) by replacing the monomial $\frac{a_k^i}{i!}$ by the function $\binom{a_k + t_k}{i_k}$ (with same leading term).

**Corollary 48** Let $q_\Phi = \left( \sum_{k=2}^{r+1} m_{1k} \right) - 1$. The polynomial $k(\Phi, c^+)$ is divisible by $(a_1 + 1)(a_1 + 2) \cdots (a_1 + q_\Phi)$.

**Proof.** Indeed, on the big chamber $c^+$, we have seen in PROPOSITION 22 that that multiindices $i$ such that $f_\Phi(i)$ is non zero are such that $i_1 \geq q_\Phi$. We have $s_1 = 1$, thus the corollary follows.

QED

**Remark** This follows also from the remark following definition 37, and Theorem 39.

If $\Phi = A_r^+$, we then obtain the relation for Kostant partition $k_\mathfrak{c}$ on a big chamber $\mathfrak{c}$. Define

$$f_\mathfrak{c}(i) = \langle \langle \mathfrak{c}, Tres_\Delta \left( \frac{x_1^{i_1} x_2^{i_2} \cdots x_r^{i_r}}{x_1 x_2 \cdots x_r \prod_{1 \leq i < j \leq r} (x_i - x_j)} \right) \rangle \rangle.$$

Then we have

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Proposition 49 (Lidskii)

We have:

\[ k(A^+_r, c)(a) = \sum_{|i|=\binom{r}{2}} f_\ell(i) \binom{a_1 + r - 1}{i_1} \binom{a_2 + r - 2}{i_2} \cdots \binom{a_{r-1} + 1}{i_{r-1}} \binom{a_r}{i_r}. \]

We have as well:

\[ k(A^+_r, c)(a) = \sum_{|i|=\binom{r}{2}} f_\ell(i) \left( \binom{a_1 + 1}{i_1} \right) \left( \binom{a_2}{i_2} \right) \cdots \left( \binom{a_{r-1} + 3 - r}{i_{r-1}} \right) \left( \binom{a_r + 2 - r}{i_r} \right). \]

Bibliographical remarks.

Proposition 49 is due to B.V. Lidskii [L]. Theorem 47 is due to Stanley, who suggested to look for a proof via residues, as given here.

11 Volumes and Ehrhart polynomials of the Stanley-Pitman polytope

Section 10 reduces the computation of the Ehrhart polynomial to the computation of mixed volumes.

Here is an example. Let

\[ \Pi_r(a) = \{ y \in \mathbb{R}^r; y_1 \geq 0, y_1 + \cdots + y_i \leq a_1 + a_2 + \cdots + a_i \} \]

with \( a_i \geq 0 \).

Let \( \Phi \) be the following sequence of \( 2r \) elements \( \{ \beta^i, \gamma^j \}, 1 \leq i \leq r, 1 \leq j \leq r \) of \( A^+_r \)

\[ \Phi = \{ \beta^1 = (e^1 - e^{r+1}), \ldots, \beta^r = (e^r - e^{r+1}) \} \]

\[ \cup \{ \gamma^1 = (e^1 - e^2), \gamma^2 = (e^2 - e^3), \cdots \gamma^{r-1} = (e^{r-1} - e^r), \gamma^r = (e^r - e^{r+1}) \} \]
The multiplicity of \((e^r - e^{r+1})\) in \(\Phi\) is 2. We write \(t_i = t_i^\Phi\). Thus \(t_1 = t_2 = \ldots = t_{r-1} = t_r = 1\), while \(s_1 = 1, s_2 = s_3 = \ldots = s_r = 0\).

Let \(n_\Phi = \mathbb{R}^{2r}\) with basis \(v_{\beta k}, w_{\gamma k}\). Let \(C^+_{2r}\) be the standard cone
\[
\bigoplus_{k=1}^r y_{i,k,r+1} v_{\beta k} \bigoplus_{k=1}^r z_{k,k+1} w_{\gamma k}
\]
with \(y_{i,r+1} \geq 0, z_{k,k+1} \geq 0\).

Let \(p : \mathbb{R}^{2r} \to E_r\) be the map sending \(v_{\beta}\) to \(\beta\) and \(w_{\gamma}\) to \(\gamma\). Then for \(a = a_1 e^1 + \cdots + a_r e^r\), with \(a_i \geq 0\) the polytope \(\Pi_r(a)\) is isomorphic to \(P_\Phi(a) = p^{-1}(a) \cap C^+_{2r}\). Indeed the point \(\sum_{k=1}^r y_{k,r+1} v_{\beta k} + \sum_{k=1}^r z_{k,k+1} w_{\gamma k}\) with \(y_{i,r+1} \geq 0, z_{i,i+1} \geq 0\) is in \(P_\Phi(a)\) if and only if
\[
\sum_{k=1}^r y_{k,r+1}(e^k - e^{r+1}) + \sum_{i=1}^r z_{i,i+1}(e^i - e^{i+1}) = a_1 e^1 + \cdots + a_r e^r - (a_1 + a_2 + \ldots + a_r) e^{r+1}.
\]
This gives
\[
y_{1,r+1} + z_{1,2} = a_1,
\]
\[
y_{2,r+1} + z_{2,3} = a_2 + z_{1,2}
\]
\[
y_{3,r+1} + z_{3,4} = a_3 + z_{1,2} + z_{2,3}
\]
\[
\ldots
\]
so that
\[
y_{1,r+1} \leq a_1,
\]
\[
y_{1,r+1} + y_{2,r+1} \leq a_1 + a_2,
\]
\[
y_{1,r+1} + y_{2,r+1} + y_{3,r+1} \leq a_1 + a_2 + a_3
\]
\[
\ldots
\]
and the point \((y_{1,r+1}, y_{2,r+1}, \ldots, y_{r,r+1})\) \(\in \Pi_r(a)\).

We compute the volume
\[
\text{vol } \Pi_r(a) = \sum_{i_1+i_2+\cdots+i_r=r} f_\Phi(i)\frac{a_{i_1} a_{i_2}}{i_1! i_2!} \cdots \frac{a_{i_{r-1}} a_{i_r}}{i_{r-1}! i_r!}
\]
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As \( a \in \mathfrak{c}^+ \), we have, by LEMMA 43,
\[
f_\Phi(i) = k_{\Phi'}((i_1 - 1)e^1 + (i_2 - 1)e^2 + \ldots + (i_r - 1)e^r).
\]

The system \( \Phi' \) is the set of simple roots \( (e^1 - e^2), \ldots, (e^{r-1} - e^r) \). They are linearly independent and generate a simplicial cone \( C(\Phi') \). Thus the function \( k_{\Phi'} \) is identically 1 on the cone \( C(\Phi') \cap \mathbb{Z}^r \). Thus we obtain \( f(i_1, i_2, i_3, \ldots, i_r) = 0 \) or 1. It is 1 if and only \( (i_1 - 1)e^1 + (i_2 - 1)e^2 + \ldots + (i_r - 1)e^r \) is in the cone \( C(A_{r-1}^+) \). We thus need
\[
(i_1, i_2, i_3, \ldots, i_r) \in K_r
\]
where
\[
K_r = \{(i_1, i_2, i_3, \ldots, i_r), i_1 \geq 1, i_1 + i_2 \geq 2, \ldots, i_1 + i_2 + \cdots + i_r = r\}.
\]

We obtain the formula for the volume of \( \Pi_r(a) \) and its Ehrhart polynomial, given in [Pi-S].

**Proposition 50 (Pitman-Stanley)**
\[
\text{vol } \Pi_r(a) = \sum_{i \in K_r} a_{i_1}^{i_{i_2}} \cdots a_{i_{r-1}}^{i_r},
\]
\[
k_{\Phi}(a) = \sum_{i \in K_r} \binom{a_1 + 1}{i_1} \binom{a_2}{i_2} \cdots \binom{a_{r-1}}{i_{r-1}} \binom{a_r}{i_r}.
\]

The same beautiful occurs for any family of polytopes associated to a smooth toric variety, giving rather nice formulae deduced immediately from the mixed volumes.

**Bibliographical remarks.** Results of this section are due to [Pi-S].

**12 Divisibility property of the Kostant partition function**

Here we list some properties of the polynomial \( k(A_{r}^+, \mathfrak{c}^+) \). They are similar to the properties of its highest degree term \( v(A_{r}^+, \mathfrak{c}^+) \) established in Section 5. For example, the following Lemma implies of course LEMMA 23.
Proposition 51 (Schmidt-Bincer)

The function $k(A_r^+, c^+)(a_1, a_2, ..., a_r)$ is independent of $a_r$. It is of degree less or equal to 1 in the variable $a_{r-1}$ and is divisible by $(a_1 + a_2 + a_3 + ... + a_{r-2} + 3a_{r-1} + 3)$.

More precisely, we have:

$$3k(A_r^+, c^+)(a_1, a_2, ..., a_{r-1}, a_r) = 3k(A_r^+, c^+)(a_1, a_2, ..., a_{r-1}, 0)$$

$$= (a_1 + a_2 + ... + a_{r-2} + 3a_{r-1} + 3)k(A_r^+, c^+)(a_1, a_2, ..., a_{r-1}, 0)$$

$$= (a_1 + a_2 + ... + a_{r-2} + 3a_{r-1} + 3)k(A_r^+ \text{ minus } (e^{r-1} - e^r), c^+)(a_1, a_2, ..., a_{r-2}, 0, 0).$$

Proof. The proof is almost identical to the proof of PROPOSITION 23. Let

$$K(a, x) = e^{a_1 x_1 + ... + a_{r-1} x_{r-1} + a_r x_r} \prod_{1 \leq i < j < r-1}(1 - e^{-(x_i - x_j)}) \prod_{1 \leq i \leq r-1}(1 - e^{-(x_i - x_r)}) \prod_{1 \leq i \leq r}(1 - e^{-x_i}).$$

We have:

$$k(A_r^+, c^+)(a_1, a_2, ..., a_r) = Ires_{x=0}K(a, x).$$

We write $k(A_r^+, c^+) = k_r^+$. We first take the residue of $K(a, x)$ in $x_r = 0$. We obtain

$$Res_{x_r=0}K(a, x) = \frac{e^{a_1 x_1 + ... + a_{r-1} x_{r-1}}}{\prod_{1 \leq i < j < r-1}(1 - e^{-(x_i - x_j)}) \prod_{1 \leq i \leq r-1}(1 - e^{-x_i})^2}.$$

This shows already that $k_r^+(a_1, a_2, ..., a_r)$ is independent of $a_r$. We proceed now to take the residue in $x_{r-1} = 0$. There is a double pole in $x_{r-1}$, so that the dependence in $a_{r-1}$ is of degree at most 1. More precisely, a simple calculation shows that

$$Res_{x_{r-1}=0} = \frac{e^{a_1 x_1 + ... + a_{r-1} x_{r-1}}}{\prod_{1 \leq i < j < r-2}(1 - e^{-(x_i - x_j)}) \prod_{1 \leq i \leq r-2}(1 - e^{-x_i})^2}$$

$$= \frac{e^{a_1 x_1 + ... + a_{r-2} x_{r-2}}}{\prod_{1 \leq i < j < r-2}(1 - e^{-(x_i - x_j)}) \prod_{1 \leq i \leq r-2}(1 - e^{-x_i})^3}(a_{r-1} + 1 + \sum_{i=1}^{r-2} \frac{e^{-x_i}}{1 - e^{-x_i}}).$$
\[
= (a_{r-1} + 1 + \frac{1}{3}(a_1 + a_2 + \ldots + a_{r-2})) e^{a_1 x_1 + \ldots + a_{r-2} x_{r-2}} \prod_{1 \leq i < j \leq r-2} (1 - e^{- (x_i - x_j)}) \prod_{1 \leq i \leq r-2} (1 - e^{-x_i})^3
\]

\[
- \frac{1}{3}(\partial_1 + \partial_2 + \ldots + \partial_{r-2}) e^{a_1 x_1 + \ldots + a_{r-2} x_{r-2}} \prod_{1 \leq i < j \leq r-2} (1 - e^{- (x_i - x_j)}) \prod_{1 \leq i \leq r-2} (1 - e^{-x_i})^3.
\]

As residue vanishes on derivatives, we obtain

\[
k^+_r(a_1, a_2, \ldots, a_{r-1}, a_r)
= (a_{r-1} + 1 + \frac{1}{3}(a_1 + a_2 + \ldots + a_{r-2})) \times
\]

\[
Res_{x_1=0} \ldots Res_{x_{r-2}=0} e^{a_1 x_1 + \ldots + a_{r-2} x_{r-2}} \prod_{1 \leq i < j \leq r-2} (1 - e^{- (x_i - x_j)}) \prod_{1 \leq i \leq r-2} (1 - e^{-x_i})^3.
\]

On the other hand, the residue computation of

\[
k(A^+_r \text{ minus } (e^{r-1} - e^r), c^+)(a_1, a_2, ..., a_{r-2}, 0, 0)
\]

gives

\[
k(A^+_r \text{ minus } (e^{r-1} - e^r), c^+)(a_1, a_2, ..., a_{r-2}, 0, 0)
\]

\[
= Res_{x_1=0} \ldots Res_{x_{r-2}=0} e^{a_1 x_1 + \ldots + a_{r-2} x_{r-2}} \prod_{1 \leq i < j \leq r-2} (1 - e^{- (x_i - x_j)}) \prod_{1 \leq i \leq r-2} (1 - e^{-x_i})^3
\]

as the step $x_r = 0$ as well as the step $x_{r-1} = 0$ involves only simple poles, and we obtain the divisibility property announced.

QED

More generally, we have the following lemma, with same proof.

**Lemma 52** Let $\Phi$ a sequence of vectors in $A^+_r$ generating $E_r$. Assume $m_{r,r+1} = 1$ and $m_{r-1,r+1} + m_{r-1,r} = 2$. Furthermore, assume that

\[
m_{j,r+1} + m_{j,r} + m_{j,r-1} = c
\]

is independent of $j$ for $1 \leq j \leq (r - 2)$, then

\[
k(\Phi, c^+)(a_1, \ldots, a_{r-1}, a_r) = k(\Phi, c^+)(a_1, \ldots, a_{r-1}, 0)
= (\frac{a_1 + \cdots + a_{r-2}}{c} + a_{r-1} + 1)v(\Phi \text{ minus } (e^{r-1} - e^r), c^+)(a_1, a_2, \ldots, a_{r-2}, 0, 0).
\]
13 A “not so obvious” symmetry property of the volume and Ehrhart polynomial

The full Weyl group $\Sigma_{r+1}$ of the root system $A_r$ acts on $S_{A_r}$ as it permutes the elements $(e^i - e^j), 1 \leq i < j \leq (r + 1)$. We show that the existence of this action implies some constraints on the coefficients of the $S_{\Delta}$-valued polynomial function $J_{A_r^+}(a)$ and $K_{A_r^+}(a)$. In particular, the Kostant polynomial $k(A_r^+, c^+)(a) = \text{Ires}_{x=0}(K_{A_r^+}(a))$ satisfies some symmetries.

Lemma 53 For any $w \in \Sigma_{r+1}$, we have

$$J_{A_r^+}(w.a) = \epsilon(w)w \cdot J_{A_r^+}(a).$$

Let $q$ be a linear form on $S_{\Delta}$. Then the map $a \mapsto \langle q, J_{A_r^+}(a) \rangle$ is a polynomial of degree $|A_r-1| = r(r-1)/2$ on $E_r$. This way, we obtain a map from the $r!$-dimensional space $S_{\Delta}$ to the space of polynomials of degree $|A_r-1| = r(r-1)/2$ on $E_r$, commuting with the representation of $\Sigma_{r+1}$.

A subset of $\{1, 2, ..., N\}$ will be called an interval, if it is of the form $\{a, a+1, a+2, ..., a+k\}$, for $a$ an integer between 1 and $N$ (and $k$ an integer between 0 and $N - a$). Let us represent a permutation in $\Sigma_N$ as a list of $N$ elements. For example, if we write $w = [Id_{(N-3)}, N, N-2, N-1]$, the first $N - 3$ indices are fixed by the permutation $w$, $N-2$ is sent to $N$, $N-1$ to $N-2$ and $N$ to $N-1$.

Definition 54 The subset $B_N$ is the subset of elements $w$ of $\Sigma_N$ such that for any $1 \leq k \leq N$ the subset $w^{-1}\{1, 2, 3, ..., k\}$ is an interval.

Lemma 55 The subset $B_N$ is of cardinal $2^{N-1}$.

Proof. We prove it by induction on $N$. If $v \in B_{N+1}$, the set $v^{-1}\{1, 2, 3, ..., N\}$ is an interval. So it is either equal to $\{1, 2, 3, ..., N\}$ or to $\{2, 3, ..., N, N+1\}$. Thus a transformation $w$ of $B_N$ gives rise to two transformations of $B_{N+1}$ namely $[w, N+1]$ and $[N+1, w]$.

QED
Definition 56 For $1 \leq i \leq r$, we define $W^{i,r}$ as the subset of $\Sigma_r$ consisting of elements $w$ such that:

- $w(k) = k$, $1 \leq k \leq i - 1$.
- For $0 \leq s \leq (r - i)$, the set $w^{-1}\{i, r, r - 1, r - 2, r - 3, ..., r - s\}$ is an interval (of $\{i, ..., r\}$).

Clearly the subset $W^{i,r}$ is isomorphic to $B_{r-(i-1)}$ by relabelling indices. Thus the set $W^{i,r}$ is of cardinality $2^{r-i}$. A list of $r$ elements representing an element of $W^{i,r}$ is constructed as follows: we start writing $i$ in the middle of a horizontal line, then we write $r$ either at the immediate left of $i$, or at the immediate right. After, we write $(r - 1)$ either at the immediate left of the unordered set $\{i, r\}$, or at the immediate right, and we go on until we have written all elements of the set $\{i, r, ..., i + 1\} = \{i, i + 1, ..., r\}$ on this line. At this step, we finally write the ordered set $\{1, 2, ..., i - 1\} = \{i, i + 1, ..., r\}$ at the immediate left of the unordered set $\{i, i + 1, ..., r\}$.

Example. Let us list elements of $W^{i,r} \subset \Sigma_r$ for $i = r, r - 1, r - 2$.

We have

$$W^{r,r} = \{[Id_r]\}$$
$$W^{(r-1),r} = \{[Id_r], [Id_{r-2}, r, r - 1]\}.$$  

$$W^{(r-2),r} = \{[Id_{r-3}, r-2, r, r-1], [Id_{r-3}, r, r-2, r-1], [Id_{r-3}, r-2, r-1, r-2, r], [Id_{r-3}, r-1, r, r-2]\}.$$  

Let $w \in \Sigma_r$. Consider the element $\hat{r}(w)$ of $\Sigma_{r-1}$ such that the list representing $\hat{r}(w)$ is the list $w$ where the element $r$ is omitted. In other words, if $u$ is such that $w(u) = r$, then $\hat{r}(w)(k) = w(k)$ if $k < u$, and $\hat{r}(w)(k) = w(k+1)$ if $k \geq u$. We have the lemma.

Lemma 57 The element $w \in W^{i,r}$ if and only if:

- $w^{-1}\{i, r\}$ is an interval.
- $\hat{r}(w) \in W^{i,r-1}$.

We denote by $T(i, r+1) \in \Sigma_{r+1}$ the transposition of $i$ and $(r+1)$. It acts on $S_\Delta$, thus on $S^*_\Delta$.  

67
Proposition 58 Let $1 \leq i \leq r$. Then the following holds:
\[ T(i, r + 1) \cdot I_{\text{res}_{x=0}} = \sum_{w \in W^{i,r}} (-1)^{r+1-w^{-1}(i)} w \cdot I_{\text{res}_{x=0}}. \]

Proof. If $g \in \Sigma_{r+1}$, we have $g \cdot I_{\text{res}_{x=0}} = \sum_{w \in \Sigma_r} c^g_w I_{\text{res}_{x=0}}$, as elements $I_{\text{res}_{x=0}} = w \cdot I_{\text{res}_{x=0}}$ form a basis of linear forms on $S_\Delta$. Let, for $w \in \Sigma_r$,
\[ f_w = \frac{1}{\prod_{1 \leq p \leq r-1} (e^{w(p)} - e^{w(p+1)})(e^{w(r)} - e^{r+1})}. \]

The coefficient $c^g_w$ is equal to $\langle g \cdot I_{\text{res}_{x=0}}, f_w \rangle = I_{\text{res}_{x=0}}(g^{-1} f_w)$. We need to prove

- If $w$ is not in $W^{i,r}$, then
  \[ \langle I_{\text{res}_{x=0}}, T(i, r + 1)f_w \rangle = 0. \]

- If $w \in W^{i,r}$, then
  \[ \langle I_{\text{res}_{x=0}}, T(i, r + 1)f_w \rangle = (-1)^{r+1-j} \]
  with $j = w^{-1}(i)$.

Let $\phi = \prod_{\alpha \in \sigma} \frac{1}{\alpha}$ be in $S_{\Delta_r}$. The set $\sigma$ consists on $r$ linearly independent elements of $E_r$. The partial residue $Res_{x=0}\phi$ is non zero if and only one of the elements $\alpha$ in the denominator of $\phi$ is proportional to $(e^r - e^{r+1})$. Indeed, in computing $Res_{x=0}\phi$, we replace $e^i$ by $x_i$, when $i \leq r$, and $e^{r+1}$ by 0. Thus the only factor creating a pole on $x_r = 0$ is $(e^r - e^{r+1})$. It follows that $Res_{x=0}(T(i, r + 1)f_w)$ is zero unless there exists $p, 1 \leq p \leq r$, such that $(e^{w(p)} - e^{w(p+1)})$ is proportional to the linear form $T(i, r + 1) \cdot (e^r - e^{r+1})$.

If $i = r$, then $T(r, r + 1) \cdot (e^r - e^{r+1}) = -(e^r - e^{r+1})$ and we see that necessarily $p = r$ and $w(r) = r$. Then
\[ I_{\text{res}_{x=0}} T(r, r + 1) \cdot f_w \]
\[ = Res_{x=1} \ldots Res_{x_{r-1}} = \frac{-1}{\prod_{k=1}^{r-2} (x_{w(k)} - x_{w(k+1)}) x_{w(r-1)}} = -\delta^1_w \]
since $w(r) = r$ and that the basis $f_w$, with $w \in \Sigma_{r-1}$ is dual to the basis of iterated residues on $S_{\Delta_{r-1}}$, as seen in Section 4. On the other hand $W^{r,r} = [Id_{(r)}]$, and thus the case $i = r$ is completed.
Let \( i < r \). We proceed by induction on \( r \). Let \( w \in \Sigma_r \) and \( j \) such that \( w(j) = i \). Then, with \( g = T(i, r + 1) \),

\[
g \cdot f_w = \frac{1}{(e^{w(1)} - e^{w(2)})(e^{w(2)} - e^{w(3)}) \cdots (e^{w(j-2)} - e^{w(j-1)})} \times
\]

\[
\frac{1}{(e^{w(j-1)} - e^{r+1})(e^{r+1} - e^{w(j+1)}) \cdots (e^{w(j+1)} - e^{w(j+2)}) \cdots (e^{w(r)} - e^1)}
\]

and

\[
I_{res_{x=0}} g \cdot f_w = -I_{res_{x=0}} \frac{1}{(x_{w(1)} - x_{w(2)})(x_{w(2)} - x_{w(3)}) \cdots (x_{w(j-2)} - x_{w(j-1)})}
\]

\[
\times \frac{1}{x_{w(j-1)}x_{w(j+1)}(x_{w(j+1)} - x_{w(j+2)}) \cdots (x_{w(r)} - x_i)}.
\]

The function \( g \cdot f_w \) has only two simple poles in \( x_{w(j-1)} \) and in \( x_{w(j+1)} \), thus \( Res_{x=0} g \cdot f_w \neq 0 \) iff \( w(j - 1) = r \), or \( w(j + 1) = r \). Precisely

- if \( w(j - 1) = r \), then \( Res_{x=0} g \cdot f_w \)
  \[
  = \frac{1}{(x_{w(1)} - x_{w(2)}) \cdots (x_{w(j-2)} - x_{w(j-1)})x_{w(j-1)}(x_{w(j+1)} - x_{w(j+2)}) \cdots (x_{w(r)} - x_i)}
  \]

- and if \( w(j + 1) = r \), then \( Res_{x=0} g \cdot f_w \)
  \[
  = \frac{1}{(x_{w(1)} - x_{w(2)}) \cdots (x_{w(j-2)} - x_{w(j-1)})x_{w(j-1)}(x_{w(j+1)} - x_{w(j+2)}) \cdots (x_{w(r)} - x_i)}
  \]

In particular we see that if \( w^{-1}\{i, r\} \) is not an interval, then \( Res_{x=0} g \cdot f_w \) is zero, so a fortiori \( I_{res_{x=0}} g \cdot f_w \) is equal to 0.

If \( w^{-1}\{i, r\} \) is an interval, then we check on the preceding formula that

- if \( w(j - 1) = r \), then
  \[
  I_{res_{x=0}} T(i, r + 1) \cdot f_w = Res_{x_1=0} \cdots Res_{x_{r-1}=0} T(i, r) \cdot f_{\hat{\tau}(w)}.
  \]

- if \( w(j + 1) = r \), then
  \[
  I_{res_{x=0}} T(i, r + 1) \cdot f_w = -Res_{x_1=0} \cdots Res_{x_{r-1}=0} T(i, r) \cdot f_{\hat{\tau}(w)}.
  \]

We conclude by induction, using Lemma [51].

QED

We now consider the volume polynomial \( v(A_r^+, c^+)(a) = I_{res_{x=0}} (J_{A^+}(a)) \).

As the function \( J_{A^+}(a) \) is anti-invariant under the full group \( \Sigma_{r+1} \), we obtain from PROPOSITION 58.
Let \( a = \sum_{i=1}^{r+1} a_i e^i \) with \( \sum_{k=1}^{r+1} a_k = 0 \). Then we have

\[
v(A^+_r, c^+)(T(i, r + 1) \cdot a) = \sum_{w \in W_{i,r}} \epsilon(w)(-1)^{-w^{-1}(i)}v(A^+_r, c^+)(w^{-1} \cdot a).
\]

**Example:** We know that the function \( v(A^+_r, c^+)(a) \) is a function of \((r - 1)\) variables \( v^+(a_1, a_2, ..., a_{r-1}) \). Then, for \( i = r, r - 1, r - 2 \), we get from Proposition 59, and the description given of the corresponding sets \( W_{i,r} \), that the function \( v^+(x) \) satisfies the identities:

- \( i = r \): \( v^+(x) = v^+(x) \).
- \( i = r - 1 \): for any values \( x_1, ..., x_r \), we have
  \[
  v^+(x_1, ..., x_{r-2}, -(x_1 + ... + x_{r-2} + x_{r-1} + x_r)) =
  -v^+(x_1, x_2, ..., x_{r-2}, x_{r-1}) - v^+(x_1, ..., x_{r-2}, x_r).
  \]
- \( i = r - 2 \): for any values \( x_1, ..., x_r \), we have
  \[
  v^+(x_1, x_2, ..., x_{r-3}, -(x_1 + x_2 + ... + x_r), x_{r-1}) =
  -v^+(x_1, x_2, ..., x_{r-3}, x_{r-2}, x_r) + v^+(x_1, x_2, ..., x_{r-3}, x_{r-1}, x_{r-2})
  -v^+(x_1, x_2, ..., x_{r-3}, x_r, x_{r-2}) + v^+(x_1, x_2, ..., x_{r-3}, x_{r-1}, x_r).
  \]

**Remark:** A function \( v(x_1, ..., x_{r-1}) \) of the form \( w(x_1, ..., x_{r-2})(x_1 + x_2 + ... + x_{r-2} + 3x_{r-1}) \) satisfies (2). This is in agreement with the divisibility of \( v(A^+_r, c^+)(a) \) by the linear factor \((a_1 + a_2 + ... + a_{r-2} + 3a_{r-1})\) proved in Section 5.

We now give a stronger result on the symmetry for the Kostant partition polynomial \( k(A^+_r, c^+) \) attached to the nice chamber \( c^+ \).

Let \( \rho = \frac{1}{2} \sum_{\alpha \in A^+_r} \alpha \). We have

\[
\rho = \frac{1}{2}(re^1 + (r - 2)e^2 + \cdots - (r - 2)e^r - re^{r+1}).
\]

As usual, we have a shifted symmetry property for \( K_{A^+_r} \).
Lemma 60 For any $w \in \Sigma_{r+1}$, the function

$$K_{A_r^+}(a) = \text{Res}_{\Delta} \left( \frac{e^a}{\prod_{\alpha \in \Phi}(1 - e^{-\alpha})} \right)$$

satisfies the relation:

$$w \cdot K_{A_r^+}(a) = \epsilon(w)K_{A_r^+}(w \cdot a + w \cdot \rho - \rho).$$

Thus PROPOSITION 13 implies the following symmetry relation for $k(A_r^+, c^+)(a) = \text{Res}_{x=0} K_{A_r^+}(a)$.

Proposition 61 Let $a = \sum_{k=1}^{r+1} a_k e^k$ with $\sum_{k=1}^{r+1} a_k = 0$. Then we have for, every $1 \leq i \leq r$,

$$k(A_r^+, c^+)(T(i, r + 1) \cdot a - (r - i + 1)(e^i - e^{r+1})) = \sum_{w \in W_{i,r}} \epsilon(w)(-1)^{r-w-1}(i)k(A_r^+, c^+)(w^{-1}a + w^{-1} \cdot \rho - \rho).$$

Example: We know that the function $k(A_r^+, c^+)(a)$ is a function of $(r-1)$ variables $k^+(a_1, a_2, ..., a_{r-1})$. Then, for $i = r$, $r - 1$, $r - 2$, we get from Proposition 61, that the function $k^+(x)$ satisfies the identities:

- $i = r$: $k^+(x) = k^+(x)$.
- $i = r - 1$: for any values $x_1, ..., x_r$, we have

$$k^+(x_1, ..., x_{r-2}, -(x_1 + ... + x_{r-2} + x_{r-1} + x_r + 2)) = -k^+(x_1, x_2, ..., x_{r-2}, x_{r-1}) - k^+(x_1, ..., x_{r-2}, x_r - 1).$$

- $i = r - 2$: for any values $x_1, ..., x_r$, we have

$$k^+(x_1, x_2, ..., x_{r-3}, -(x_1 + x_2 + ... + x_r + 3), x_{r-1}) = -k^+(x_1, x_2, ..., x_{r-3}, x_{r-2}, x_r - 1) + k^+(x_1, x_2, ..., x_{r-3}, x_{r-1} - 1, x_{r-2} + 1) - k^+(x_1, x_2, ..., x_{r-3}, x_{r-2} + 1) + k^+(x_1, x_2, ..., x_{r-3}, x_{r-1} - 1, x_{r-1}).$$

Remark: A function $k(x_1, ..., x_{r-1})$ of the form $w(x_1, ..., x_{r-2})(x_1 + x_2 + ... + x_{r-2} + 3x_{r-1} + 3)$ satisfies (2). This is in agreement with the divisibility of $k(A_r^+, c^+)(a)$ by the linear factor $(a_1 + a_2 + ... + a_{r-2} + 3a_{r-1} + 3)$ proved in Section 13.

Bibliographical remarks.

As explained at the beginning, the relations given here are transcription of the "hidden" action of $\Sigma_{r+1}$ on $S(A_r)$.
14 Appendix 1: Jeffrey-Kirwan residue formula for the volume.

We return to the notations of Section 3. Let \( \Phi = \{ \alpha^1, ..., \alpha^N \} \) be a sequence of elements of \( \Delta^+ \). Let \( p : \mathbb{R}^N \rightarrow V^* \) be the map such that

\[
p(u_1w^1 + u_2w^2 + ... + u_Nw^N) = u_1\alpha^1 + u_2\alpha^2 + \cdots + u_N\alpha^N.
\]

Consider the family of polytopes \( P_\Phi(a) \) where \( a \) varies in \( C(\Delta^+) \subset V^* \). Let \( x \in V \), such that \( \langle \alpha^k, x \rangle > 0 \) for all \( \alpha^k \in \Phi \). We have

\[
\prod_{k=1}^N \frac{1}{\langle \alpha^k, x \rangle} = \int_{\mathbb{R}^N_+} e^{-\sum_{k=1}^N u_k \langle \alpha^k, x \rangle} du_1 \cdots du_N = \int_{\mathbb{R}^N} e^{-(p(w),x)} dw.
\]

Let \( s : V^* \rightarrow \mathbb{R}^N \) be a section from \( V^* \) to \( E \) such that \( \mathbb{R}^N = s(V^*) \oplus \text{Ker}(p) \).

We write \( w = s(a) + m \), with \( p(w) = a \) and \( p(m) = 0 \). Our choice of measure is such that \( dw = dadm \). By Fubini, we obtain

\[
\int_{\mathbb{R}^N} e^{-(p(w),x)} dw = \int_{C(\Delta^+)} e^{-\langle x, a \rangle} \left( \int_{P_\Phi(a)} dm \right) da.
\]

Thus

\[
\prod_{k=1}^N \frac{1}{\langle \alpha^k, x \rangle} = \int_{C(\Delta^+)} e^{-\langle x, a \rangle} \text{vol} P_\Phi(a) da,
\]

i.e. the Laplace transform of the function \( \text{vol} P_\Phi(a) \) is \( \prod_{k=1}^N \frac{1}{\alpha^k(x)} \). We need to inverse this formula to find \( \text{vol} P_\Phi(a) \).

We denote by \( G_\Delta \) the subspace of \( R_\Delta \) spanned by the functions of the form

\[
\prod_{\alpha \in \sigma} \alpha^{n_\alpha}
\]

where \( \sigma \) is a basis of \( \Delta \) and the \( n_\alpha \) are positive integers.

**Lemma 62** Let \( \kappa \) be a sequence of elements of \( \Delta^+ \) such that \( \kappa \) generates \( V^* \). Then the function

\[
\prod_{\alpha \in \kappa} \alpha
\]

belongs to the vector space \( G_\Delta \).
Proof. We argue on the cardinal on the underlying set \( \{ \kappa \} \) to \( \kappa \). If the cardinal of \( \{ \kappa \} \) is minimum, then \( \{ \kappa \} \) is a basis of \( \Delta \). If not, there is a linear relation \( \beta = \sum_{j} c_j \alpha^j \) between elements \( \beta, \alpha^j \) belonging to \( \kappa \). Then

\[
\frac{1}{\beta \prod_j \alpha^j} = \sum_{j} \frac{c_j}{\beta^2 \prod_{i \neq j} \alpha^i}.
\]

We conclude by induction.

QED

Let \( \sigma = \{ \alpha^1, \alpha^2, \ldots, \alpha^r \} \) be a basis of \( \Delta \) consisting of elements of \( \Delta^+ \). Let

\[
f_\sigma = \frac{1}{\alpha^1 \cdots \alpha^r}
\]

and let \([C(\sigma)]\) be the characteristic function of the cone \( C(\sigma) \).

Consider the function

\[
F(x) = \frac{1}{\langle \alpha^1, x \rangle_{k_1+1} \cdots \langle \alpha^r, x \rangle_{k_r+1}}
\]

where \( k_j \) are non negative integers. From the one dimensional formula:

\[
\frac{1}{x^{k+1}} = \int_{\mathbb{R}^+} e^{-ax} \frac{a^k}{k!} da
\]

we see that

\[
F(x) = \int_{V^*} e^{-\langle a, x \rangle} v(a) [C(\sigma)](a) da
\]

where \( v \) is a polynomial.

Let \( c \) be a big chamber of \( \Delta^+ \). The verification of the inversion formula:

\[
[C(\sigma)](a) v(a) = \langle \langle c, \text{Res}_\Delta(e^a F) \rangle \rangle (2)
\]

for \( a \in c \) is straightforward. Indeed, we write \( a = \sum_{k=1}^r a_k \alpha^k \), then \( da = \text{vol}(\sigma) da_1 da_2 \ldots da_r \) and \( v(a) = \frac{1}{\text{vol}(\sigma)} \frac{a_{k_1}^{k_1}}{k_1!} \cdots \frac{a_{k_r}^{k_r}}{k_r!} \). On the other hand the function \( F(x) e^{\langle a, x \rangle} \) is equal to

\[
\frac{e^{a_1 \langle \alpha^1, x \rangle}}{\langle \alpha^1, x \rangle_{k_1+1}} \cdots \frac{e^{a_r \langle \alpha^r, x \rangle}}{\langle \alpha^r, x \rangle_{k_r+1}}
\]
and its total residue is the function

$$
\frac{a_1^{k_1}}{k_1!} \cdots \frac{a_r^{k_r}}{k_r!} \frac{1}{\langle \alpha^1, x \rangle} \cdots \frac{1}{\langle \alpha^r, x \rangle}.
$$

So

$$Tres_{\Delta}(e^a F) = \frac{a_1^{k_1}}{k_1!} \cdots \frac{a_r^{k_r}}{k_r!} f_{\sigma}.$$

Let $c$ be a big chamber. If $c \subset C(\sigma)$, then the left hand side of the equation above is $v(a)$, and the right hand side is

$$\langle\langle c, f_{\sigma} \rangle\rangle \frac{a_1^{k_1}}{k_1!} \cdots \frac{a_r^{k_r}}{k_r!} = \frac{1}{\text{vol}(\sigma)} \frac{a_1^{k_1}}{k_1!} \cdots \frac{a_r^{k_r}}{k_r!} = v(a).$$

If $c$ is not contained in $C(\sigma)$, then $c \cap C(\sigma) = \emptyset$ and both sides are equal to 0.

Thus the inversion formula for any function of $G_{\Delta}$ is established and we obtain THEOREM 9.
15 Appendix 2. Chambers and basis for $A_2$ and $A_3$

We briefly recall the setting. Following Section 4 we realize $V^*$ as $E_r$ with basis \( \{e^1, e^2, \ldots, e^r\} \), so that we may write:

\[
A_r = \{ \pm e^1, \pm e^2, \ldots, \pm e^r, \pm (e^i - e^j), 1 \leq i < j \leq r \}.
\]

Then \( |A_r^+| = N = r(r + 1)/2 \) and the set of positive roots is

\[
A_r^+ = \{ e^1, e^2, \ldots, e^r, (e^i - e^j), 1 \leq i < j \leq r \}.
\]

We consider the group \( \Sigma_r \) of permutations of \( \{1, 2, \ldots, r\} \). We denote a permutation in \( \Sigma_r \) as a list of \( r \) elements. For example \([231]\) represents the permutation in 3 elements \( \{1, 2, 3\} \) sending 1 to 2, 2 to 3 and 3 to 1. The group \( \Sigma_r \) acts naturally on \( V^* \).

The knowledge of the \( S_\Delta \)-valued polynomial function

\[
J_{A_r^+}(a)(x) = Tres_{A_r^+}\left[ \frac{e^{(a,x)}}{\prod_{\alpha \in A_r^+} \langle \alpha, x \rangle} \right] = \frac{1}{(r(r-1)/2)!} Tres_{A_r^+}\left[ \frac{\langle a, x \rangle^{r(r-1)/2}}{\prod_{\alpha \in A_r^+} \langle \alpha, x \rangle} \right]
\]

determines the various polynomial functions associated to big chambers.

The function \( J_{A_r^+} \) is anti-invariant under the group \( \Sigma_r \):

\[
J_{A_r^+}(w \cdot a) = \epsilon(w) J_{A_r^+}(a)
\]

thus we need only to determine the polynomial \( res_{x=0} J_{A_r^+}(a) \). It is an homogeneous polynomial of degree \( |A_{r-1}| \). (In fact, \( J_{A_r^+}(a) \) is also anti-invariant under \( \Sigma_{r+1} \) leading to some constraints on coefficients of \( J_{A_r^+}(a) \).)

Recall that \( C(\sigma) \) denote the simplicial cone determined by a basis \( \sigma \) of \( A_r^+ \). If we write \( a = \sum_{i=1}^r a_i e^i \) for \( a \in V^* \), the cone generated by \( A_r^+ \) is

\[
C(A_r^+) = \{ a \in V^* \text{such that } a_1 \geq 0, a_1 + a_2 \geq 0, \ldots, a_1 + \ldots + a_r \geq 0 \}
\]

The small chambers, denoted by \( a \), are defined as the open connected component of \( V^* - \mathcal{H}^* \) in \( C(A_r^+) \). To any \( h \in C(A_r^+) \) we associate the intersection of all the simplicial cones \( C(\sigma) \) which contain \( h \). The interior of the maximal cones of this subdivision of \( C(A_r^+) \) into polyhedral cones are called big chambers. The relevance of the big chambers lies in the fact that
the polynomial volume is the same on all the small chambers that make up a big chamber. Recall that the nice chamber \( c^+ \) is given by \( a_i > 0 \).

The nice chamber \( c^+ \) is the cone \( \sum_{i=1}^{r} a_i e^i \) with \( a_i > 0 \), \( 1 \leq i \leq r \). Using the permutation \( w_0 \in \Sigma_{r+1} \) reversing order on \( \{1,2,...,(r+1)\} \), there is the “opposite” nice chamber \( c^- \), which is the cone spanned by the roots \( \{e^1, (e^1 - e^2), (e^1 - e^3),...,(e^1 - e^r)\} \).

If \( w \in \Sigma_r \), we denote by \( C^+_w \subset C(A_r^+) \) the simplicial cone generated by the vectors

\[
\epsilon(1)(e^{w(1)} - e^{w(2)})\, ..., \epsilon(r-1)(e^{w(r-1)} - e^{w(r)})\, , (e^{w(r)} - e^{r+1}) ,
\]
where \( \epsilon(i) \) is 1 or -1 depending whether \( w(i) < w(i+1) \) or not.

The space \( S_{A_r^+} \) is of dimension \( r! \). As basis of \( S_{A_r^+} \), we choose elements \( f_w \) indexed by \( w \in \Sigma_r \), with

\[
f_w = w \cdot \frac{1}{(e^1 - e^2)(e^2 - e^3)...(e^{r-1} - e^r)e^r} .
\]

To a big chamber \( c \) is associated a linear form \( f \rightarrow \langle \langle c, f \rangle \rangle \) on \( S_{A_r} \).

By definition, \( \langle \langle c, f_w \rangle \rangle \) is equal to 0 if \( c \) is not contained in \( C^+_w \). Otherwise, \( \langle \langle c, f_w \rangle \rangle = (-1)^{n(w)} \), where \( n(w) \) be the number of elements such that \( w(i) > w(i+1) \). Thus to compute \( \langle \langle c, f \rangle \rangle = (-1)^{n(w)} \text{Res}_{x=0}^w f \), we need to determine elements \( w \) such that \( c \subset C^+_w \) and \( n(w) \).

We compute the big chambers forms \( c \), and the corresponding form \( \langle \langle c, f \rangle \rangle \), in the \( A_2 \) and \( A_3 \) case. Only the form associated to \( c^+ \) has a simple expression in the basis \( f_w \). (Of course this basis is somewhat arbitrary).

For \( a \in \mathbb{C} \), \( c \) a big chamber we recall the transmutation formula for the volume and the Kostant partition function as given in Sections \([\text{II}]\):

\[
\text{vol } P_{A_r^+}(a) = v(A_r^+, c) = \sum_{|i|=\binom{r}{2}} f_i(i) \frac{a_1^{i_1}}{i_1!} \frac{a_2^{i_2}}{i_2!} \cdots \frac{a_{r-1}^{i_{r-1}}}{i_{r-1}!} \frac{a_r^{i_r}}{i_r!}
\]

and

\[
k(A_r^+, c)(a) = \sum_{|i|=\binom{r}{2}} f_i(i) \frac{(a_1 + r - 1)}{i_1} \frac{(a_2 + r - 2)}{i_2} \cdots \frac{(a_{r-1} + 1)}{i_{r-1}} \frac{a_r}{i_r} .
\]

where \( \langle \langle c, f_{A_r^+}(i) \rangle \rangle = f_i(i) \)

For \( A_2 \) we may draw the following picture:
The equations defined by the hyperplanes are:

\[ a_1 = 0, a_2 = 0, a_1 + a_2 = 0. \]

In dimension 2, small and big chambers coincide.

There are 3 bases in \( A_2^+ \), \( \sigma_1 = \{e^1, e^2\} \), \( \sigma_2 = \{e^1, (e^1 - e^2)\} \), and \( \sigma_3 = \{e^2, (e^1 - e^2)\} \) bases of \( \Delta \). The space \( S_\Delta \) is spanned by \( f_{\sigma_2}, f_{\sigma_3} \). We have the linear relation \( f_{\sigma_1} = f_{\sigma_3} - f_{\sigma_2} \).

Therefore, if \( a = a_1 e^1 + a_2 e^2 \), then

\[ J_{A_2^+}(a) = a_1 f_{[12]} - a_2 f_{[21]}, \]

\[ v( A_2^+, c^+) = a_1, \]

\[ k( A_2^+, c^+) = a_1 + 1. \]

\[ \text{vol} \, P_{A_2^+}(a) = a_1, \text{ if } a \in c_1 \]
and
\[ \text{vol } P_{A^+}(a) = a_1 + a_2, \text{ if } a \in c_2. \]

As for the Ehrhart polynomials, we have
\[ k(A^+_1, c_1)(a) = a_1 + 1 \text{ and } k(A^+_2, c_2)(a) = a_1 + a_2 + 1. \]

When \( r = 3 \) we may draw the following picture on a plane with equation \( e_1 = \text{constant} \).

We see that there are 8 small chambers \( a_i \) and 7 big chambers \( c_k \).

The big chambers are
- \( c_1 = a_1 = C(e_1, e_2, e_3) \).
- \( c_2 = a_2 = C(e_1, e_1 - e_2, e_1 - e_3) \).
- \( c_3 = a_3 = C(e_1, e_2 - e_3, e_2) \cap C(e_1, e_2, e_1 - e_3) \).
- \( c_4 = a_4 = C(e_1 - e_3, e_2 - e_3, e_1) \cap C(e_1, e_2, e_1 - e_3) \).
- \( c_5 = a_5 = C(e_1, e_2 - e_3, e_2) \cap C(e_2, e_1 - e_3, e_2 - e_3) \).
- \( c_6 = a_6 = C(e_2, e_1 - e_3, e_2 - e_3) \cap C(e_1, e_2 - e_3, e_1 - e_3) \).
- \( c_7 = a_7 \cup a_8 = C(e_1 - e_2, e_1, e_3). \)

For all \( w \in \Sigma_3 \), we have \( n(w) = 1 \), except if \( w = [123] \) where \( n(w) = 0 \) and \( w = [321] \) where \( n(w) = 2 \). We have
\[
\begin{align*}
C^+_{[123]} &= c_1 \cup c_2 \cup c_3 \cup c_4 \cup c_5 \cup c_6 \cup c_7 \\
C^+_{[213]} &= c_2 \cup c_7 \\
C^+_{[132]} &= c_5 \cup c_6 \\
C^+_{[231]} &= c_4 \cup c_6 \\
C^+_{[312]} &= c_2 \cup c_3 \cup c_4 \\
C^+_{[321]} &= c_2 \cup c_4 \cup c_6.
\end{align*}
\]

Thus, we have rather complicated formula for the linear forms \( \langle \langle c, f \rangle \rangle \), except on the “nice” chamber \( c^+ = c_1 \).

\[
\begin{align*}
\langle \langle c_1, f \rangle \rangle &= \text{Ires}_{x=0}^{[123]} f, \\
\langle \langle c_2, f \rangle \rangle &= \text{Ires}_{x=0}^{[123]} f - \text{Ires}_{x=0}^{[213]} f - \text{Ires}_{x=0}^{[312]} f + \text{Ires}_{x=0}^{[321]} f, \\
\langle \langle c_3, f \rangle \rangle &= \text{Ires}_{x=0}^{[123]} f - \text{Ires}_{x=0}^{[312]} f, \\
\langle \langle c_4, f \rangle \rangle &= \text{Ires}_{x=0}^{[123]} f - \text{Ires}_{x=0}^{[231]} f - \text{Ires}_{x=0}^{[312]} f + \text{Ires}_{x=0}^{[321]} f, \\
\langle \langle c_5, f \rangle \rangle &= \text{Ires}_{x=0}^{[123]} f - \text{Ires}_{x=0}^{[132]} f, \\
\langle \langle c_6, f \rangle \rangle &= \text{Ires}_{x=0}^{[123]} f - \text{Ires}_{x=0}^{[132]} f - \text{Ires}_{x=0}^{[231]} f + \text{Ires}_{x=0}^{[321]} f, \\
\langle \langle c_7, f \rangle \rangle &= \text{Ires}_{x=0}^{[123]} f - \text{Ires}_{x=0}^{[213]} f.
\end{align*}
\]

We easily compute the coefficients \( f_i \) of the function
\[
J_{A^+_3}(a_1 e^1 + a_2 e^2 + a_3 e^3) = \frac{1}{6} \text{Ires}_{A^+_3} \left( \frac{(a_1 x_1 + a_2 x_2 + a_3 x_3)^3}{x_1 x_2 x_3 (x_1 - x_2) (x_1 - x_3) (x_2 - x_3)} \right).
\]
Figure 4: Chambers for $A_3$ with $a = a_1 e_1 + a_2 e_2 + a_3 e_3$
Let \( v_+(a) = \frac{1}{6}a_1^3 + \frac{1}{2}a_1^2a_2 \). We obtain:

\[
J_{A^+}(a_1e^1 + a_2e^2 + a_3e^3) = \sum_{w \in \Sigma_v} \epsilon(w)v_+(w^{-1} \cdot a)f_w.
\]

Thus, we see that

\[
v(A^+_3, c^+)(a) = \frac{1}{6}a_1^2(a_1 + 3a_2).
\]

\[
k(A^+_3, c^+)(a) = \frac{1}{6}(a_1 + 1)(a_1 + 2)(a_1 + 3a_2 + 3).
\]

Thus here is the list of formulas for the volume and partition functions. We see that the polynomials \( v(A^+_3, c_i) \) are all different so that the big chambers are indeed the minimal domains where the function \( \text{vol} \) is expressed by a polynomial formula.

\[
v(A_3^+, c_1)(a) = v_1(a_1 + 3a_2).
\]

\[
k(A_3^+, c_1)(a) = k_1(a_1 + 1)(a_1 + 2)(a_1 + 3a_2 + 3).
\]

\[
v(A_3^+, c_2)(a) = \frac{1}{6}(a_1 + a_2 + a_3 + 1)(a_1 + a_2 + a_3 + 2)(a_1 + a_2 - 2a_3 + 3).
\]

\[
k(A_3^+, c_2)(a) = \frac{1}{6}(a_1 + a_2 + a_3 + 1)(a_1 + a_2 + a_3 + 2)(a_1 + a_2 - 2a_3 + 3).
\]

\[
v(A_3^+, c_3)(a) = v(A_3^+, c_3)(a) + a_1^2 + \frac{1}{2}a_1a_2 + \frac{1}{2}a_1a_3 - \frac{1}{2}a_2^2 + \frac{1}{2}a_2^2 + \frac{1}{2}a_2^2 + \frac{1}{2}a_3 + 1.
\]

\[
k(A_3^+, c_3)(a) = v(A_3^+, c_3)(a) + a_1^2 + \frac{1}{2}a_1a_2 + \frac{1}{2}a_1a_3 - \frac{1}{2}a_2^2 + \frac{1}{2}a_2^2 + \frac{1}{2}a_3 + 1.
\]

\[
v(A_3^+, c_4)(a) = v(A_3^+, c_4)(a) + a_1^2 + \frac{1}{2}a_1a_2 + \frac{1}{2}a_1a_3 - \frac{1}{2}a_2^2 + \frac{1}{2}a_2^2 + \frac{1}{2}a_3 + 1.
\]

\[
k(A_3^+, c_4)(a) = v(A_3^+, c_4)(a) + a_1^2 + \frac{1}{2}a_1a_2 + \frac{1}{2}a_1a_3 - \frac{1}{2}a_2^2 + \frac{1}{2}a_2^2 + \frac{1}{2}a_3 + 1.
\]

Volume functions vanish on the boundary of the cone \( C(\Delta^+) \), thus all functions \( v(A_3^+, c_k) \) for \( k \neq 3, 4 \) must have a linear factor. The order of this factor is computed as \( L - 1 \), where \( L \) is the number of roots not on the face bounding the chamber (see for example jump formula in [B-V 2]), thus is 2 for \( c_1, c_2, c_3, c_6 \) and 3 for \( c_7 \).

It is also reassuring to check that functions \( k(A_3^+, c_k) \) define a continuous function on \( C(\Delta^+) \) (polynomials \( k(A_3^+, c_{k_1}) \) and \( k(A_3^+, c_{k_2}) \) agree on \( c_{k_1} \cap \ldots \)
$c_k$. For example, if $a_2 = a_3 = 0$, the five polynomials $k(A_3^+, c_k)(x e_1)$ $k = 1, 2, 3, 4, 7$ restricts to $\frac{1}{6}(x + 1)(x + 2)(x + 3)$.

**Bibliographical remarks.** Tables for $k(A_3^+, c)$ are given for example in [S-B].
REFERENCES

- [Ao] K. Aomoto: Jacobi polynomials associated with Selberg integrals. SIAM Math analysis (1987), p 545-549
- [B-S] A. Bincer and J. Schmidt: The Kostant partition function for simple Lie algebras. J. Math. Phys. 25, (1984), p. 2367-2373.
- [B-V 1] M. Brion and M. Vergne: Residue formulae, vector partition functions and Lattice points in rational polytopes. J. A. M. S vol 10, (1997), pp 797-833.
- [B-V 2] M. Brion and M. Vergne: Arrangement of hyperplanes I Rational functions and Jeffrey-Kirwan residue. Ann. scient. Éc. Norm. Sup., vol 32, (1999), p. 715-741.
- [C-R] C. S. Chan and D. P. Robbins: On the volume of the polytope of doubly stochastic matrices. Electronic preprint (1998). math.CO/9806076. Experimental Mathematics.
- [C-R-Y] C. S. Chan, D. P. Robbins, D. S. Yuen: On the volume of a certain polytope. Electronic preprint (1998). math.CO/9810154.
- [E] E. Ehrhart. Sur un problème de géométrie diophantienne linéaire II, J. Reine Angew. Math. 226, (1967); p. 1-29.
- [J-K] L. C. Jeffrey and F. C. Kirwan: Localization for non abelian group actions. Topology 34, (1995), p 291-327.
- [K-P] G. Khovanskii, A. V. Pukhlikov: A Riemann-Roch theorem for integrals and sums of quasipolynomials over virtual polytopes. St. Petersburg Math. J., vol 4, (1993), p. 789-812.
- [Ki1] A. N. Kirillov: Ubiquity of Kostka polynomials. Electronic preprint (1999). arXiv:math.QA/9912094.
- [Ki2] A. N. Kirillov: Kostant partition functions for the root system of type A. Preprint 1999.
- [Le] G. I. Lehrer: On the Poincaré series associated with Coxeter group actions on complement of hyperplanes. J. London Math. Society 35(2), (1987), pp. 275-294.
• [Li] B.V. Lidskii: Kostant Function of the root system $A_n$. Funktsional’nyi Analiz i Ego Prilozhenia, vol 18, (1983), p. 76-77

• [M] W. Morris: Constant term identities for finite and affine root systems. (1982). PHD thesis, University of Wisconsin, Madison, Wisconsin.

• [O-T] P. Orlik and H. Terao: Arrangements of Hyperplanes; Grundlehren der Mathematischen Wissenschaften. vol 300, Springer-Verlag, Berlin 1992.

• [Pi- S] J. Pitman and R. Stanley: A polytope related to empirical distribution, plane trees, parking functions, and the associahedron. Electronic preprint (1999). math.CO/9908029

• [Po-St] R.P. Stanley: Acyclic flow polytopes and Kostant’s partition function. Conference. Transparencies (www-math.mit.edu/ rstan/trans.html).

• [Se] A. Selberg: Bemerkninger om et multipelt integral, Norsk Mat. Tidsskr., vol 26, (1994) pp. 71-78

• [Sz] A. Szenes: Iterated residues and multiple Bernoulli polynomials. hep-th/9707114. International Mathematical Research Notices.

• [Z] D. Zeilberger: A conjecture of Chan, Robbins, and Yuen. Electronic preprint (1998). math.CO/9811108

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