GENERIC IMMERSIONS AND TOTALLY REAL EMBEDDINGS

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Dedicated to Professor Takashi Nishimura on the occasion of his 60th birthday

Abstract. We show that, for a closed orientable $n$-manifold, with $n$ not congruent to 3 modulo 4, the existence of a CR-regular embedding into complex $(n-1)$-space ensures the existence of a totally real embedding into complex $n$-space. This implies that a closed orientable $(4k+1)$-manifold with non-vanishing Kervaire semi-characteristic possesses no CR-regular embedding into complex $4k$-space. We also pay special attention to the cases of CR-regular embeddings of spheres and of simply-connected 5-manifolds.

1. Introduction

Let $f: M^n \to \mathbb{C}^{q}$ be a smooth immersion of a $n$-dimensional manifold into $\mathbb{C}^{q}$. Then we see that, for a point $x \in M^n$ and the standard complex structure $J$ on $\mathbb{C}^{q}$,

$$\dim_{\mathbb{C}}(df_x(T_xM^n) \cap J df_x(T_xM^n)) \geq n - q.$$ 

The point $x$ is said to be **CR-regular** if the equality holds (and **CR-singular** otherwise) in the above inequality. A CR-regular point (resp. a CR-singular point) is called a **RC-regualar point** (resp. a **RC-singular point**) in [19]. The immersion $f$ is said to be a **CR-regular immersion** if it has no CR-singular point. A CR-regular immersion $f: M^n \to \mathbb{C}^{q}$ is also called a **generic immersion** (e.g., see Jacobowitz and Landweber [15]), and when $n = q$ it has yet another name, a **totally real immersion**.

We will work in the smooth category; throughout the paper all manifolds and immersions are supposed to be differentiable of class $C^\infty$. Manifolds are further supposed to be connected.

The following is our main theorem.

**Theorem 1.1.** Let $M^n$ be a closed orientable $n$-manifold with $n > 3$. Then,

(a) when $n \equiv 0 \mod 2$, $M^n$ admits a totally real embedding into $\mathbb{C}^{n}$ if and only if it admits a CR-regular immersion into $\mathbb{C}^{n-1}$;

(b) when $n \equiv 1 \mod 4$ and $w_2 w_{n-2} [M^n] = 0$, $M^n$ admits a totally real embedding into $\mathbb{C}^{n}$ if and only if it admits a CR-regular immersion into $\mathbb{C}^{n-1}$;

(c) when $n \equiv 1 \mod 4$ and $w_2 w_{n-2} [M^n] \neq 0$, if $M^n$ admits a totally real embedding into $\mathbb{C}^{n}$ then it does not admit a CR-regular immersion into $\mathbb{C}^{n-1}$.

We will see later that the condition $w_2 w_{n-2} [M^n] = 0$ (vanishing de Rham invariant) above can be replaced with $\tilde{w}_2 \tilde{w}_{n-2} [M^n] = 0$ (see Remark [3,4]). Furthermore, the normal bundle of an embedding of a closed orientable $n$-manifold into a Euclidean space necessarily has trivial Euler class (see [25] Corollary 11.4)). Hence a closed orientable $n$-manifold $M^n$ embeddable in $\mathbb{R}^{2n-2}$ must have trivial $\tilde{w}_{n-2} (M^n)$. Therefore, we have the following.

**Corollary 1.2.** If a closed orientable $n$-manifold, with $n \neq 3 \mod 4$, admits a CR-regular embedding into $\mathbb{C}^{n-1}$, then it admits a totally real embedding into $\mathbb{C}^{n}$.

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Note that the converse of Corollary 1.2 does not hold in general (see Remark 3.8). The proof of Theorem 1.1 will be given in §3. We will discuss CR-regular immersions of simply-connected 5-manifolds into \( \mathbb{C}^4 \) in §4 and CR-regular embeddings of spheres in more general codimensions in §5.

2. BACKGROUND

2.1. Totally real embeddings. The problem of determining when a manifold admits a totally real immersion or embedding has been extensively studied by many authors including Gromov [11, 12, 13], Forstneriˇc [10] and Audin [4].

First, as for totally real immersions, the following theorem due to Gromov [11] and Lees [20] is fundamental. (see also [9]).

**Theorem 2.1** (Gromov [11] and Lees [20]). An \( n \)-manifold \( M^n \) admits a totally real immersion into \( \mathbb{C}^n \) if and only if the complexified tangent bundle \( \mathbb{C}T M^n \) is trivial.

This is called the \( h \)-principle for totally real immersions. For totally real embeddings, Gromov [13] and Forstneriˇc [10] have proved the following \( h \)-principle.

**Theorem 2.2** (Gromov [13] and Forstneriˇc [10]). Let \( M^n \) be a closed orientable \( n \)-manifold with \( n \geq 3 \). Then, \( M^n \) admits a totally real embedding into \( \mathbb{C}^n \) if and only if it admits a totally real immersion into \( \mathbb{C}^n \) which is regularly homotopic to an embedding.

Using these \( h \)-principles, Audin [4] has given a necessary and sufficient condition for the existence of totally real embeddings. In order to state her results, we need the notion of semi-characteristic.

**Definition 2.3** (Kervaire [17] and Lusztig–Milnor–Peterson [21]). Let \( F \) be either the field \( \mathbb{R} \) or \( \mathbb{Z}/2\mathbb{Z} \). For a closed orientable \((2r+1)\)-manifold \( M \), the semi-characteristic of \( M \) with respect to a coefficient field \( F \) is defined to be

\[
\hat{\chi}_F(M) = \sum_{i=0}^r \text{rank } H_i(M; F) \in \mathbb{Z}/2\mathbb{Z}
\]

taken as an integer modulo 2. We call \( \hat{\chi}_R(M) \) the real semi-characteristic and \( \hat{\chi}_{\mathbb{Z}/2\mathbb{Z}}(M) \) the Kervaire semi-characteristic, respectively.

**Theorem 2.4** (Audin [4]). A closed orientable \( n \)-manifold \( M^n \) admits a totally real embedding into \( \mathbb{C}^n \) if and only if

(a) when \( n \equiv 0 \mod 2 \), \( \mathbb{C}T M^n \) is trivial and the Euler characteristic \( \chi(M^n) \) vanishes;
(b) when \( n \equiv 1 \mod 4 \), \( \mathbb{C}T M^n \) is trivial and the Kervaire semi-characteristic \( \hat{\chi}_{\mathbb{Z}/2\mathbb{Z}}(M^n) \) vanishes.

Note that Audin [4] has given some partial results also in the case where \( n \equiv 3 \mod 8 \).

2.2. CR-regular immersions and embeddings. CR-regular immersions and embeddings also have been studied by many authors from various viewpoints (see e.g., Cartan [6], Tanaka [29, 40], Wells [35, 36], Lai [19], Jacobowitz and Landweber [15], Ho, Jacobowitz and Landweber [14], Slapar [27], Torres [33, 34]). Our previous paper [16] sheds some new light on a relation between CR-singularity and differential topology.

In the recent paper [15] by Jacobowitz and Landweber the following generalisation of Gromov–Lees Theorem (Theorem 2.1) has been shown.

**Theorem 2.5** (Jacobowitz and Landweber [15]). Let \( n < 2q \). Then, an \( n \)-manifold \( M^n \) admits a CR-regular immersion into \( \mathbb{C}^q \) if and only if the complexified tangent bundle \( \mathbb{C}T M^n \) splits as

\[
\mathbb{C}T M^n = A \oplus B,
\]

where \( A \) is the trivial complex vector bundle of rank \( q \), and \( B \) is a complex vector bundle of rank \( n - q \) with \( B \cap \overline{B} = \{0\} \).
The "only if" part of Theorem 2.5 is straightforward: if \( M^n \) admits a CR-regular immersion into \( \mathbb{C}^q \) then we can find inside \( TM^n \) a suitable complex bundle of rank \( n - q \), whose fibre at \( x \in M^n \) is mapped onto \( df_x(T_xM^n) \cap Jd(df_x(T_xM^n)) \) via \( df_x \) (see the argument in the proof of Proposition 2.6). The converse is the essence of Theorem 2.5, which clarifies the condition for the existence of CR-regular immersions. The next goal is to write down the condition for the existence of CR-regular immersions. In particular, the case of CR-regular embeddings of \( n \)-manifolds \( M^n \) into \( \mathbb{C}^{n+1} \), \( n \geq 3 \), seems interesting and has been extensively studied in recent years.

When \( n = 3 \), each immersion or embedding of 3-manifolds in \( \mathbb{C}^2 \) is CR-regular. When \( n = 4 \), Slapar [27] has shown that a closed orientable 4-manifold \( M^4 \) admits a CR-regular immersion in \( \mathbb{C}^3 \) if and only if the first Pontryagin class \( p_1(M^4) \) and the Euler characteristic \( \chi(M^4) \) vanish, and admits a CR-regular embedding if and only if in addition the second Stiefel–Whitney class \( w_2(M^4) \) vanishes.

In the 5-dimensional case, Jacobowitz and Landweber [15, p. 163] posed a problem: "one could ask if every \( M^5 \) which admits an embedding into \( \mathbb{C}^7 \) and satisfies the necessary condition for generic immersions given by Theorem 1.2 has a generic embedding into \( \mathbb{C}^7 \) or if the Kervaire semi-characteristic is still restricted." In the present paper we give an answer to this question. Our main theorem, in fact, claims that vanishing Kervaire semi-characteristic is a necessary condition for the existence of "a generic embedding" into \( \mathbb{C}^{4k} \) for a closed orientable \((4k + 1)\)-manifold (see Corollary 3.5).

For the 5-dimensional sphere \( S^5 \), Elgindi has already proved in [31] the non-existence of a CR-regular embedding of \( S^5 \) into \( \mathbb{C}^4 \). We show that the proof found in [31] can be applied to prove the following.

**Proposition 2.6.** If a closed orientable \( n \)-manifold \( M^n \) admits a CR-regular immersion into \( \mathbb{C}^{n+1} \) with trivial normal bundle, then \( M^n \) is parallelisable.

**Proof.** Suppose that there exists a CR-regular immersion of \( M^n \) into \( \mathbb{C}^{n+1} \) with normal bundle \( N \). Then, in each \( T_xM^n \), we can take such a complex line that is mapped onto \( df_x(T_xM^n) \cap Jd(df_x(T_xM^n)) \) through \( df_x \). Thus we have a complex line bundle \( L \) over \( M^n \) and a decomposition

\[
TM^n \cong L_R \oplus M,
\]

where \( L_R \) is the underlying real bundle of \( L \) and \( M \) is the complementary bundle. Therefore we have

\[
L_R \oplus M \oplus N \cong f^*T\mathbb{R}^{2n-2} = f^*T\mathbb{C}^{n+1}.
\]

Since \( M \) and \( N \), viewed in \( T\mathbb{C}^{n+1} \), should be mapped isomorphically onto each other via \( J \), we see that \( M \cong N \), more precisely

\[
M \oplus N \cong M \oplus \mathbb{C}.
\]

Hence we have the bundle isomorphism

\[
L \oplus (M \oplus \mathbb{C}) \cong f^*T\mathbb{C}^{n+1}.
\]

It follows from \( w_1(M) = w_1(M) = 0 \) that \( c_1(M \oplus \mathbb{C}) = 0 \) (see [25, Problem 15-D]). Therefore the complex line bundle \( L \) has trivial first Chern class \( c_1(L) \) and hence is trivial.

Now since \( N \) is assumed to be trivial, \( M \) is trivial. Consequently, \( TM^n \cong L_R \oplus M \) is trivial, that is, \( M^n \) is parallelisable. \( \square \)

**Remark 2.7.** Torres [33] has studied CR-regular immersions and embeddings of orientable 6-manifolds into \( \mathbb{C}^7 \). Another interesting aspect proved in [34], based on Slapar [26] and Di Scala–Kasuya–Zuddas [7], is that an almost-complex \( 2n \)-manifold \( M^{2n} \) admits a pseudo-holomorphic embedding into \( \mathbb{R}^{2n+2} \) endowed with a suitable almost-complex structure if and only if it admits a CR-regular embedding into \( \mathbb{C}^{n+1} \).
3. THE PROOF OF THEOREM 1.1 AND DISCUSSION

We first prove the following observation.

**Proposition 3.1.** A closed orientable \( n \)-manifold \( M^n \) with \( n \geq 3 \) admits a CR-regular immersion into \( \mathbb{C}^{n-1} \) if and only if the complexified tangent bundle \( \mathbb{C}T M^n \) is trivial and \( M^n \) has an oriented 2-plane field with trivial Euler class.

**Proof of Proposition 3.1.** Suppose that there exists a CR-regular immersion of \( M^n \) into \( \mathbb{C}^{n-1} \) with normal bundle \( N \). Then, by exactly the same argument as in the proof of Proposition 2.6, for a trivial complex line bundle \( \mathcal{L} \) over \( M^n \) we have the decomposition

\[
TM^n \cong \mathcal{L}_\mathbb{R} \oplus M,
\]

(1)

where \( \mathcal{L}_\mathbb{R} \) is the underlying real bundle of \( \mathcal{L} \) and \( M \) is isomorphic to \( N \).

From the bundle isomorphism (1), we see that the complexified tangent bundle is described as

\[
\mathbb{C}T M^n \cong \mathcal{L} \oplus (M \otimes \mathbb{C})
\]

(see [25, Lemma 15.4, p. 176]), which is trivial since \( \mathcal{L} \) is trivial.

Conversely, suppose that the complexified tangent bundle \( \mathbb{C}T M^n \) is trivial and \( M^n \) has an oriented 2-plane field with trivial Euler class. Then the tangent bundle \( TM \) splits as

\[
TM = P \oplus Q,
\]

where \( P \) is a trivial 2-plane bundle and \( Q \) is its complementary bundle. Then, the complexified tangent bundle splits as

\[
\mathbb{C}T M^n = (P \otimes \mathbb{C}) \oplus (Q \otimes \mathbb{C})
\]

where \( \epsilon \) is the trivial complex line bundle. Thus we see that the complex vector bundle \( \epsilon \oplus (Q \otimes \mathbb{C}) \) of rank \( n-1 \) is stably trivial, and actually trivial by the dimensional reason [15, Lemma 1.2]. Therefore, by Theorem 2.5 (putting \( A = \epsilon \oplus (Q \otimes \mathbb{C}) \)), \( M^n \) admits a CR-regular immersion into \( \mathbb{C}^{n-1} \). \( \square \)

The problem of determining when an orientable manifold has an oriented 2-plane field has been solved due to Atiyah–Dupont [3] and a series of papers by Thomas (see [32]). We recall here the following special case where the Euler class of the 2-plane field determined by the 2-plane field is trivial.

**Theorem 3.2** (Atiyah–Dupont [3] and Thomas [32]). A closed orientable smooth \( n \)-manifold \( M^n \) has an oriented 2-plane field with trivial Euler class if and only if either of the following holds:

- (a) \( n \equiv 0 \mod 4 \) and the signature \( \sigma(M^n) \) of \( M^n \) is congruent to zero modulo 4;
- (b) \( n \equiv 1 \mod 4 \) and \( \chi_{\mathbb{R}}(M^n) = 0 \);
- (c) \( n \equiv 2 \mod 4 \) and \( \chi(M^n) = 0 \);
- (d) \( n \equiv 3 \mod 4 \).

By combining Proposition 3.1 with Theorem 3.2, we obtain Corollary 3.3. below. Compare it with Theorem 2.4 (Audin). Note that if \( \mathbb{C}T M^n \) is trivial then all the Pontryagin classes of \( M^n \) vanish and therefore we have \( \sigma(M^n) = 0 \).

**Corollary 3.3.** A closed orientable \( n \)-manifold \( M^n \) admits a CR-regular immersion into \( \mathbb{C}^{n-1} \) if and only if

- (a) when \( n \equiv 0 \mod 2 \), \( \mathbb{C}T M^n \) is trivial and \( \chi(M^n) = 0 \);
- (b) when \( n \equiv 1 \mod 4 \), \( \mathbb{C}T M^n \) is trivial and \( \chi_{\mathbb{R}}(M^n) = 0 \);
(c) when \( n \equiv 3 \mod 4 \), \( CTM^n \) is trivial.

Now we prove our main theorem.

Proof of Theorem 1.1

Theorem 1.1 (a) is just a combination of Theorem 2.4 (a) and Corollary 3.3 (a).

Suppose that \( n \equiv 1 \mod 4 \). Then, according to Lusztig–Milnor–Peterson’s formula (21) (see also [2] Remark on p. 16]) and Remark 3.4 below, we have

\[
\hat{\chi}_R(M^n) - \hat{\chi}_{Z/2Z}(M^n) = w_2w_{n-2}[M^n].
\]

Therefore, if \( w_2w_{n-2}[M^n] = 0 \) then \( \hat{\chi}_R(M^n) \) and \( \hat{\chi}_{Z/2Z}(M^n) \) have the same parity, so that Theorem 2.4 (b) and Corollary 3.3 (b) prove the Theorem 1.1 (b). On the contrary, if \( w_2w_{n-2}[M^n] \neq 0 \) then either one of \( \hat{\chi}_R(M^n) \) and \( \hat{\chi}_{Z/2Z}(M^n) \) must be non-zero and the other zero; Theorem 2.4 (b) and Corollary 3.3 (b) then imply Theorem 1.1 (c). \( \square \)

Remark 3.4. In view of (1) in the proof of Proposition 3.1 we see that for a closed orientable \( n \)-manifold \( M^n \) admitting a CR-regular immersion into \( \mathbb{C}^{n-1} \) and each \( i \),

\[
w_i(M^n) = w_i(L_\varepsilon \oplus \mathcal{M}) = w_i(\mathcal{M}) = w_i(N) = \hat{w}_i(M^n).
\]

Hence the square of each Stiefel–Whitney class of \( M^n \) vanishes. This is also seen from the triviality of \( CTM^n \cong TM \oplus TM \) (see [4]). Therefore, the condition on the Stiefel–Whitney number in Theorem 1.1 can be replaced with that on the normal Stiefel–Whitney number.

As mentioned in §4 we have \( \hat{w}_{n-2}(M^n) = 0 \) for a closed orientable \( n \)-manifold \( M^n \) embeddable in \( \mathbb{R}^{2n-2} \), and hence Corollary 1.2 follows in view of Remark 3.4. As a consequence of Theorem 1.1 and Corollary 3.3 we have the following.

Corollary 3.5. If a closed orientable \( n \)-manifold, with \( n \equiv 1 \mod 4 \) (resp. \( n \equiv 0 \mod 2 \)), admits a CR-regular embedding into \( \mathbb{C}^{n-1} \), then it has trivial Kervaire semi-characteristic (resp. trivial Euler characteristic).

Note that for \( n \equiv 0 \mod 2 \) we will have the more general Corollary 3.2.

Remark 3.6. Let \( M^n \) be a closed orientable \( n \)-manifold \( M^n \) which admits a CR-regular immersion into \( \mathbb{C}^{n-1} \). Then, we see from (1) that span \( M^n \geq 2 \), where span \( M^n \) stands for the maximum number of everywhere linearly independent vector fields on \( M^n \).

Remark 3.7. Let \( M^n \) be a closed orientable \( n \)-manifold and \( n > 4 \). When \( n \) is odd \( M^n \) can be immersed in \( \mathbb{R}^{2n-2} \), when \( n \) is even \( M^n \) can be immersed in \( \mathbb{R}^{2n-2} \) if and only if \( \hat{w}_2\hat{w}_{n-2}[M^n] = 0 \) (22 Theorem 7.1.1).

If \( n \equiv 3 \mod 4 \) then \( \hat{w}_{n-2}(M^n) \) always vanishes ([17], see [24, Theorem III]) and hence \( \hat{w}_2\hat{w}_{n-2}[M^n] = 0 \). This is not the case where \( n \equiv 1 \mod 4 \); e. g., the 5-dimensional Dold manifold \( P(1, 2) \) can be immersed in \( \mathbb{R}^8 \) and has non-trivial \( \hat{w}_2\hat{w}_3 \). Note that \( P(1, 2) \) does not admit a CR-regular immersion into \( \mathbb{C}^4 \) since its span equals 1 (see Remark 3.6).

Remark 3.8. The Wu manifold \( X_{3^n} = SU(3)/SO(3) \) ([7]) is a closed simply-connected 5-manifold with \( w_2w_3[X_{3^n}] \neq 0 \) ([4], p. 80). It appears in Barden’s list [5] as one of the “building blocks” of simply-connected 5-manifolds (see [4]), and has been proved to admit no embedding into \( \mathbb{R}^8 \) ([5, Theorem 2.5]). On the other hand, Audin ([4, Proposition 0.8]) has shown that \( X_{3^n} \) has trivial Kervaire semi-characteristic and admits a totally real embedding into \( \mathbb{C}^5 \). This implies that the converse of Corollary 1.2 does not hold in general. We may ask the following question: if a closed orientable \( n \)-manifold, with \( n \equiv 3 \mod 4 \), admits a totally real embedding into \( \mathbb{C}^n \) and an embedding into \( \mathbb{R}^{2n-2} \), then does it admit a CR-regular embedding into \( \mathbb{C}^{n-1} \)?
4. Generic immersions of simply-connected 5-manifolds into $\mathbb{C}^4$

Let $M^5$ be a closed simply-connected 5-manifold. Then, by definition,

$$\hat{\chi}_R(M^5) = \text{rank } H_2(M^5; \mathbb{Z}) + 1 \pmod{2}.$$ 

Therefore, Corollary 3.3 together with Slapar–Torres’s result [28, Theorem C] claiming that every closed simply-connected 5-manifold has trivial complexified tangent bundle, implies the following.

**Theorem 4.1.** A closed simply-connected 5-manifold $M^5$ admits a CR-regular immersion into $\mathbb{C}^4$ if and only if the rank of $H_2(M^5; \mathbb{Z})$ is odd.

Barden [5] has proved that every closed simply-connected 5-manifold $M^5$ can be expressed as a connected sum of the manifolds listed in Table 1 more precisely, $M^5$ is diffeomorphic to a connected sum $M_{k_1} \# \cdots \# M_{k_l} \# X_m$, where $-1 \leq m \leq \infty$, $l \geq 0$, $k_i > 1$, and $k_i$ divides $k_{i+1}$ or $k_{i+1} = \infty$. Since each manifold in Barden’s list is irreducible, with respect to connected sum, except $X_1 \equiv X_m \# X_1$, we may refer to the above decomposition just as the connected sum decomposition of $M^5$. Then, the following is an easy interpretation of Theorem 4.1

**Corollary 4.2.** A closed simply-connected 5-manifold $M^5$ admits a CR-regular immersion into $\mathbb{C}^4$ if and only if the sum of the numbers of copies of $S^2 \times S^3$ and $S^2 \times S^3$, appearing in the connected sum decomposition of $M^5$, is odd.

It seems interesting to compare Corollary 4.2 with [28] Theorem C and Remark 2.

5. Generic immersions of spheres

The basic ideas of the proofs of Propositions 2.6 and 3.1 are applicable for CR-regular immersions of spheres in more general codimensions.

We start with a very easy observation.

**Proposition 5.1.** Suppose that a closed orientable $n$-manifold $M^n$ admits a CR-regular immersion $f : M^n \to \mathbb{C}^q$ with normal bundle $N$. If the normal bundle $N$ possesses $r$ linearly independent sections, then span $M^n \geq r$, that is, $M^n$ admits $r$ everywhere linearly independent vector fields.

**Proof.** A similar argument can be found in the proof of Propositions 2.6 In each $T_x M^n$, we can find a complex $(n-q)$-space, isomorphic to $df_x(T_x M^n) \cap J df_x(T_x M^n)$ through $df_x$. Thus we have a $(n-q)$-dimensional complex bundle $\mathcal{L}$ over $M^n$ and the decomposition

$$TM^n \cong \mathcal{L} \oplus M,$$

where $\mathcal{L}_R$ is the underlying real bundle of $\mathcal{L}$ and $M$ is the complementary bundle. Furthermore we have

$$\mathcal{L}_R \oplus M \oplus N \cong f^* T\mathbb{R}^{2d} = f^* T\mathbb{C}^q,$$
where \( M \) and \( N \) are isomorphic through \( J \). Namely a section of \( N \) determines a section of \( M \), that is, a vector field on \( M^n \).

**Corollary 5.2.** If a closed orientable \( n \)-manifold admits a CR-regular embedding into some complex space \( \mathbb{C}^q \) then it has trivial Euler characteristic.

**Proof.** Since the normal bundle of an embedding of a closed orientable \( n \)-manifold \( M^n \) into a Euclidean space has trivial Euler class (see [25, Corollary 11.4]), it has a nowhere-zero section. By Proposition 5.1, \( M^n \) has a nowhere-zero vector field and hence vanishing Euler characteristic by the Poincaré–Hopf theorem.

Now we recall Adams’ fundamental result [11] on vector fields on spheres.

Given a positive integer \( N \), find integers \( a \) and \( b \) such that \( N = (2a + 1)2^b \). Then the Radon–Hurwitz number is defined as

\[
\rho(N) = \begin{cases} 
2b + 1 & b \equiv 0 \mod 4, \\
2b & b \equiv 1, 2 \mod 4, \\
2b + 2 & b \equiv 3 \mod 4.
\end{cases}
\]

**Theorem 5.3 (Adams [11]).** The maximum number of linearly independent vector fields on \( S^{N-1} \) is equal to \( \rho(N) - 1 \).

The following generalises Elgindi’s theorem [8, Theorem 2].

**Theorem 5.4.** There does not exist a CR-regular embedding of \( S^n \) into \( \mathbb{C}^q \) in each case of the following:

(a) \( n \equiv 0 \mod 2 \);

(b) \( 4q \geq 3n + 2 \) and \( n \neq 1, 3, 7 \);

(c) \( 2q = n + 3 > 6 \) and \( n \equiv 3 \mod 4 \).

**Proof.** (a) This is a direct consequence of Corollary 5.2.

(b) For \( 4q \geq 3n + 2 \), any embedding of \( S^n \) into \( \mathbb{C}^q \) has trivial normal bundle (Kervaire [18, Theorem 8.2]). In view of Proposition 5.1 and Theorem 5.3, it suffices to show that \( 2q - n \geq \rho(n + 1) \).

Let \( n + 1 = (2a + 1)2^b \) for non-negative integers \( a \) and \( b \). As a result of (a), we may assume that \( n \) is odd, that is, \( b \geq 1 \). Thus, the condition \( 4q \geq 3n + 2 \) implies that \( 2q - n \geq \frac{n + 3}{2} \).

If \( a \geq 1 \) we have

\[ 2q - n \geq \frac{n + 3}{2} = \frac{2a + 1}{2}2^b + 1 \geq 2^b + 2 \geq 2b + 2 \geq \rho(n + 1). \]

If \( a = 0 \) then we have \( b \geq 4 \) by \( n \neq 1, 3, 7 \). Then if \( b = 4 \), we have \( n + 1 = 16 \) and hence

\[ 2q - n \geq \frac{n + 3}{2} = 2^{b-1} + 1 = 9 = \rho(16). \]

If \( a = 0 \) and \( b \geq 5 \), we see that

\[ 2q - n \geq \frac{n + 3}{2} = 2^{b-1} + 1 \geq 2b + 2 \geq \rho(n + 1). \]

This completes the proof.

(c) An embedding of \( S^n \) into \( \mathbb{R}^{n+3} \) has trivial normal bundle (Massey [23, Corollary, p. 960]). Therefore, by Proposition 5.1, the existence of a CR-regular embedding of \( S^{4k+3} \) into \( \mathbb{C}^{2k+3} \), \( k \geq 1 \), would require \( S^{4k+3} \) to have three linearly independent vector fields and hence vanishing real semi-characteristic by Atiyah’s theorem (see [32, p. 108]). This, however, is impossible.

**Remark 5.5.** It would be interesting to know whether there exists a CR-regular embedding of \( S^n \) into \( \mathbb{C}^q \) for \((n, q)\) outside the range of Theorem 5.3.
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