Black hole with a Dirac field in 3+1 dimensions

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We study a complex Dirac field in the quiral representation minimally coupled to gravity in 3 + 1 dimensions in the context of Einstein-Cartan theory. Generically the matter content gravitates in two different ways: On the one hand, the energy-momentum induces spacetime curvature; on the other hand, the presence of spin acts as a source for the spacetime torsion, which does not propagate. In this setup we consider the most general static spherically symmetric solution and we find an analytic black hole solution that supports a nontrivial spinor configuration. The spinor field affects the geometry by inducing spacetime torsion, though, remarkably, it does not alter the black hole metric, which retains its Schwarzschild form. We find solutions both in asymptotically flat and asymptotically (Anti) de Sitter spaces. Additionally, we consider how the solution gets deformed when the so-called Holst term is included in the gravity action. We discuss possible observational effects due to the coupling of fermions in the nonvanishing torsion background. Finally we include a Maxwell field into the problem and find that it forces the spinor field to be zero.

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I. INTRODUCTION

The Einstein Cartan Sciama Kibble (EC) theory is an extension of Einstein theory of General Relativity (GR) where spacetime is promoted from a pseudo-Riemannian manifold to a Riemann-Cartan one, which has a connection that is not the symmetric Christoffel symbol but a more general one, with an antisymmetric part defined by the torsion tensor \( T^{\alpha}_{\mu\nu} = \Gamma^\alpha_{\nu\mu} - \Gamma^\alpha_{\mu\nu} \). This theory was first formulated by Elie Cartan in 1922 [1-3] and hardly studied since 1961-62, when Sciama and Kibble recovered it from a localization of the Lorentz group [4,5], now called Poincaré Gauge Theory (PGT), for a detailed study on PGT and spacetime torsion see [6-14]. The fact that the connection is not symmetric has several consequences, including:

- There are now two inequivalent geodesics, one which extremizes the length and another whose tangent vector is parallel transported along the path. Both geodesics coincide provided torsion is completely antisymmetric \( T^{\alpha}_{\mu\nu} = T^{\alpha}_{[\mu\nu]} \), otherwise they deviate.
- Bianchi identities are modified and, as a consequence, Einstein tensor is neither symmetric nor covariantly constant.

The EC theory is described by the same action as GR, with the difference that the connection is not necessarily symmetric and is regarded as an independent dynamical field in the action. Its variation gives, besides Einstein’s equations, a new equation, called Cartan’s equation

\[
T^\alpha_{\mu\nu} + 2T^\lambda_{\lambda[\mu,\nu]} = 8\pi G \sigma^\alpha_{\mu\nu},
\]

where the spin density \( \sigma^\alpha_{\mu\nu} \) is the source of torsion. This equation predicts that torsion does not propagate, in the sense that it must vanish in vacuum, or wherever spin density is zero, even if there is a source nearby. In consequence EC theory completely agrees with all the observational tests of GR, which all of them occur in vacuum. On the other hand, because of the equivalence between EC and GR in vacuum, the only hope of finding a deviation between both theories is in the presence of spinning matter; moreover, even if there is a difference it is going to be very small unless the spin density is huge,
so EC theory should have a relevant effect in very extreme conditions, such as a very dense star on in the early universe \[15\]\[19\].

The treatment of a real astrophysical object like a star or of a cosmological scenario in the presence of torsion could be done by including matter with stress and spin described by a Weyssenhoff perfect spinning fluid \[7\]. The stress tensor of this fluid is \(\tau_{\mu\nu} = (\rho + p)u_{\mu}u_{\nu} - pg_{\mu\nu}\) and the spin density \(\sigma_{\mu\nu} = u^aS_{\mu\nu}\), where \(\rho\) is the energy density, \(p\) the pressure, \(u_{\mu}\) the velocity vector and \(S_{\mu\nu}\) the spin density of the fluid, with the condition \(S_{\mu\nu}u^\nu = 0\). Einstein’s and Cartan’s equations can then be written completely in terms of the metric and the affine connection. Unfortunately there seems to be no satisfactory Lagrangian from which these equations can be obtained, so they must be postulated as a phenomenological fact.

A more formal treatment of spinning matter described by an action principle requires the use of spinorial fields\[2\] but in this case it is necessary to introduce a co-tetrad (vielbein) and spin connection, otherwise it is impossible to define a covariant derivative and use the minimal coupling procedure.

In this work we will explore EC theory in one of the most extreme objects in nature, the black hole. In section II the first order formalism together with the elements of the spinorial representation for the Lorentz group are presented. In section III the field equations are obtained from an action principle in 3+1 dimensions. Section IV focuses on stationary and spherically symmetric solutions discussing the results. In section V there is a discussion about possible observable effects of the solution. Section VI extends the problem including a Maxwell field, while section VII sums up the conclusions.

II. THE INGREDIENTS

In this section we present the elements necessary in order to construct an action principle for gravity coupled to spinning matter in 3+1 dimension. For this purpose we use the first order formalism of gravity\[2\] and a spinorial representation of the Lorentz group to describe matter.

A. First order formalism: the gravity part

The EC theory of gravity is best described using an orthonormal frame (vielbein) \(e^a = e^a_\mu dx^\mu\), which represents a local inertial observer, and the spin (or Lorentz) connection \(\omega^{ab} = \omega^{ab}_\mu dx^\mu\). They are both 1-forms and transform as a vector and a gauge field under local Lorentz (\(SO(1,3)\)) transformations, respectively \[20\]. The vielbein is related to spacetime metric via \(g_{\mu\nu} = \eta_{ab}e^a_\mu e^b_\nu\), while spin connection defines a covariant derivative for a field \(\phi\) that is in a representation of the Lorentz group \(D\phi = d\phi + \frac{1}{2}\omega^{ab}\Sigma_{ab}\phi\), where \(\Sigma_{ab}\) are the generators of the Lorentz algebra in the field representation, in the sense that \(\phi\) transforms like \(d\phi = \frac{1}{2}\lambda^{ab}(x)\Sigma_{ab}\phi\) under infinitesimal local Lorentz transformations \(\lambda^{ab} = -\lambda^{ba}\) \[6\].

The only Lorentz tensors that can be constructed up with first derivatives of this fields are the Lorentz Curvature and Torsion 2-forms

\[
R^{ab} = d\omega^{ab} + \omega^a_c \wedge \omega^{cb} \\
T^a = de^a + \omega^a_b \wedge e^b.
\]

They can be regarded as Field-Strengths of the \(\omega\) and \(e\) fields respectively. Bianchi identities ensure that there are not Lorentz tensors involving higher order derivatives of \(e\) and \(\omega\) made up of exterior products and exterior derivatives only:

\[
DT^a = R^a_b \wedge e^b \\
DR^{ab} = 0.
\]

It is possible to split spin connection \(\omega^{ab} = \tilde{\omega}^{ab} + \kappa^{ab}\) into a torsion-free connection \(\tilde{\omega}^{ab}\), defined by \(de^a + \tilde{\omega}^a_b \wedge e^b = 0\) and contorsion \(\kappa^{ab}\), that is related to torsion by \(\kappa^a_b \wedge e^b = T^a\). The torsion 2-form components \(T^a = \frac{1}{2}T^{ab}e^b \wedge e^c\) can be decomposed, without loss of generality, into its three irreducible

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1 Massless vector fields are also spinning matter, however coupling them with torsion is inconsistent with gauge invariance, so the usual assumption is that photons should neither produce nor feel torsion. See section V for a more extended discussion on the subject.

2 For a modern aproach in terms of differential forms see \[21\].
the generators of the Lorentz algebra are $\Gamma_{ab}$, the Minkowski metric $\eta_{a}$, whose components are complex numbers, and a chiral representation of the Clifford algebra $\sigma$. For the matter part we consider a spinorial representation of the Lorentz group in our case will be shown to be identically zero, the most general action that can be constructed for the gravity part (vielbein and spin connection) in the so-called Einstein-Cartan-Holst action $3+1$ dimensions, which is invariant under local Lorentz transformations and gives first order equations is $Riemann-Cartan$ manifold field. The most general real matter action which gives first order equations is the Dirac action on a spinor field and its conjugate, such as $\bar{\psi}$, and its conjugate are with the property $[D, \Gamma_a] = 0$. Finally, it is useful to introduce a short hand notation for vector and axial currents $J_a = i\bar{\psi}\Gamma_a\psi$, $J^5_a = i\bar{\psi}\Gamma^5\Gamma_a\psi$, with $\Gamma^5 = i\Gamma_0\Gamma_1\Gamma_2\Gamma_3 = \text{diag}(1, -1)$. 

III. EINSTEIN-CARTAN-HOLST-DIRAC ACTION

The most general action that can be constructed for the gravity part (vielbein and spin connection) in $3+1$ dimensions, which is invariant under local Lorentz transformations and gives first order equations is the so-called Einstein-Cartan-Holst action $I_G[e, \omega] = \frac{1}{4\kappa} \int \epsilon_{abcd} (R^a_{\ b c} \pm \frac{1}{2\ell^2} e^a e^b) e^c e^d + \frac{2}{\gamma} R_{ab} e^a e^b$. (2)

Here $\kappa = 8\pi G$ is the gravitational constant, $c = 1$, $\ell$ is the $(A)dS$ radius and $\gamma$ is the Immirzi parameter. Wedge products $\wedge$ of differential forms are implicitly assumed.

For the matter part, the Lorentz invariant terms that can be constructed are bilinear forms of the spinor field and its conjugate, such as $\bar{\psi}^* \Gamma \psi$ and $D\bar{\psi}^* \Gamma \psi$, where $\Gamma = \Gamma_a e^a$ is a one-form and $^*$ is the hodge dual. \footnote{Hodge dual is defined as the linear map from $p$-forms to $(d-p)$-forms in $d$ dimensions, such that $^** p = (-1)^{p(d-p)} p$, for instance $^* \Gamma = \frac{1}{d!} \epsilon_{\ b c d} \Gamma_a e^b e^c e^d$.} Other possible invariants are the mass term $\bar{\psi}^* \Gamma \psi e^4$ and the axial mass term $\bar{\psi}^* \Gamma^5 \psi e^4$, where $e^4 = e^0 e^1 e^2 e^3 = |e| d^4 x$ is the invariant volume form, but in this work we just consider a massless Dirac field. The most general real matter action which gives first order equations is the Dirac action on a Riemann-Cartan manifold $I_M[e, \omega, \psi] = - \int \frac{\alpha}{2} \bar{\psi}^* \Gamma D \psi + \alpha^* \frac{\gamma}{2} D\bar{\psi}^* \Gamma \psi$, (3)
with \( \alpha \) an arbitrary constant complex number. By a rescaling of the spinor field one can normalize the real part of \( \alpha \) to 1, but the imaginary part then remains as a free parameter here called \( n \), so from now on we take \( \alpha = 1 + in \). Note that if \( n \neq 0 \) the Dirac Lagrangian acquires a term proportional to torsion \( 3n/4V_d J^a \) plus a boundary term, which is absent in the limit of torsion-free spacetime, so \( \boxed{3} \) recovers the standard Dirac action in Minkowski spacetime.

In this work we take the simplest situation, when the Dirac field is minimally coupled to gravity

\[
I[e, \omega, \psi] = I_G[e, \omega] + I_M[e, \omega, \psi]. \tag{4}
\]

### A. Equations of motion

The minimal action principle applied to \( \boxed{4} \) gives us the following field equations

\[
\begin{align*}
\delta e & : -\frac{1}{2} \epsilon_{abcd} \left( R^{ab} \pm \frac{1}{\ell^2} e^a e^b \right) e^c + \frac{1}{\gamma} R_{da} e^a = \kappa \tau_d \\
\delta \omega & : \epsilon_{abcd} T^c e^d + \frac{2}{\gamma} J_{[ab]} e_{[b]} = \kappa \sigma_{ab} \\
\delta \psi & : \Gamma^a \vartheta_a \psi - \frac{3i}{4} (A_a \Gamma^a - n V_a \Gamma^a) \psi = 0,
\end{align*}
\]

where \( \tau_d \) and \( \sigma_{ab} \) are the stress and spin 3-forms defined by \( \delta I_M = \int -\delta e^d \wedge \tau_d - \frac{1}{2} \delta \omega^{ab} \wedge \sigma_{ab} + \cdots \) \( \boxed{21} \), and \( \vartheta = d + \bar{\omega} \) denotes the covariant derivative for the torsion-free connection. In particular

\[
\begin{align*}
\tau_d & = \frac{\alpha}{2} \vartheta^* (\Gamma e_d) \vartheta \psi - \frac{\alpha}{2} \vartheta^* (\Gamma e_d) \psi \\
\sigma_{ab} & = \frac{1}{2} \epsilon_{abcd} J^{ad} \vartheta^* \vartheta^* + n J_{[a} e_{[b]}.
\end{align*}
\]

### B. Effective theory

Cartan’s equation \( \boxed{6} \) can be solved algebraically for torsion in terms of the vector and chiral currents, \( J, J^5 \). In fact, as shown in appendix A, replacing \( \boxed{1} \) in \( \boxed{6} \) one obtains

\[
\begin{align*}
A_a & = \frac{k \gamma^2}{2(1 + \gamma^2)} \left( J^5_a - \frac{n}{\gamma} J_a \right) \\
V_a & = \frac{k \gamma^2}{2(1 + \gamma^2)} \left( n J_a + \frac{1}{\gamma} J^5_a \right),
\end{align*}
\]

while the mixed part vanishes \( M_{abc} = 0 \).

Let us replace \( \boxed{10, 11} \) into the Einstein’s equations \( \boxed{5} \), and lift all terms depending on currents \( J \) and \( J^5 \) to the right-hand side obtaining effective Einstein’s equations

\[
\tilde{G}_d = -\frac{1}{2} \epsilon_{abcd} \left( \tilde{R}^{ab} \pm \frac{1}{\ell^2} e^a e^b \right) e^c = \kappa \tau_d^{eff}
\]

for the usual Riemannian curvature \( \tilde{R}^{ab} = d \tilde{\omega}^{ab} + \bar{\omega}^a \omega^b \) of \( \text{GR} \) and with \( \tau_d^{eff} \) an effective stress 3-form given by

\[
\tau_d^{eff} = \frac{1}{2} \vartheta^* (\Gamma e_d) \vartheta \psi - \frac{1}{2} \vartheta^* (\Gamma e_d) \psi + \frac{1}{2} \vartheta^* J^5 e_d
\]

\[
+ \frac{3k \gamma^2}{16(1 + \gamma^2)} \left( J^5 \right)^2 - \frac{n^2}{\gamma} J^2 - \frac{2n}{\gamma} J^5 \cdot J \right] e_d,
\]

where \( J^5 = J^5_a e^a \). We can see that the effect of torsion in this case is to produce an effective potential that is a contact term between vector and chiral currents\( \boxed{4} \). Parity breaking of the Holst term reflects in

\[\text{(In fact what happens is that spinors produce torsion and then they are affected by it, but this interaction is local because torsion does not propagate, which means that spinors at one spacetime point are not affected by the torsion produced by spinors at a different point.)}\]
the product $J^5 \cdot J$. Spin-Spin interaction was first found by Kerlick [22]. Some possible effects of these contact terms are proposed to be detected in LHC experiments [23][26], and have been also studied in the context of Loop Quantum Gravity, considering specially the role of the Holst term [27][29], and in the cosmological scenario [30].

### IV. BLACK HOLE SOLUTIONS

Now we focus on spherically symmetric solutions of (7,12). Solutions to the Einstein-Cartan-Dirac equations have been reported in 2+1 dimensions [31][32], but to the best of my knowledge there are not known solutions in 3+1. No hair theorem [33][34] may discourage searching for nontrivial black hole solutions to this theory, nevertheless some of the assumptions of this theorem are relaxed in this work.

The metric
\[ ds^2 = -f(r)dt^2 + h(r)dr^2 + r^2(d\theta^2 + \sin \theta d\phi^2) \]

admits an $SO(3)$ group of isometries (i.e., it is invariant under rotations $\mathcal{L}_\xi g_{\mu\nu} = 0$, with $\xi_i$ the generators of rotations: $[\xi_i, \xi_j] = e_{ijk} \xi_k$). The vielbein can be chosen as $e^0 = f^{1/2}dt$, $e^1 = h^{1/2}dr$, $e^2 = r d\theta$, and $e^3 = r \sin \theta d\phi$, compatible with this metric.

A similar assumption could be made for the spinors, in order to respect spherical symmetry: $\mathcal{L}_\xi \psi = 0$, however this is an unnecessarily strong condition. Instead it seems preferable to impose the physically more realistic and lighter condition that the observable currents ($J^a$) be spherically symmetric. This is guaranteed if all the tangential components of the currents vanish, $J_\theta = J_{\phi} = 0$. This condition could seem suspicious, but it is necessary in view of (11) and considering that torsion tensor (i.e., $\Lambda$ and $V$ vectors) must also respect the spherical symmetry. The form that the spinors must take to fulfill these conditions in the chiral representation is
\[ \psi_L = \phi_L \begin{pmatrix} 1 \\ s_L \end{pmatrix}, \quad \psi_R = \phi_R \begin{pmatrix} 1 \\ s_R \end{pmatrix}, \quad (14) \]

where $\phi_{L,R}$ are complex numbers, and $s_{L,R}^2 = 1$. As shown in appendix B, consistency in Eq. (11) implies that $|\phi_L| = |\phi_R| \equiv \Phi(r)$, with $\Phi(r)$ a real number, and $s_L = -s_R = \pm 1$. Writing $\phi_L = \Phi(r)e^{i\lambda(x)}$ and $\phi_R = \Phi(r)e^{i\rho(x)}$ we obtain that $\partial_a(\lambda(x) + \rho(x)) = 0$ for each coordinate $x^a$. Under these conditions, the vector current is necessarily null, $J_a = 0$. The chiral current vanishes $J_a^5 = 0$ and the effective stress 3-form also vanishes
\[ \epsilon_{abc} J^a = 0. \]

So, Einstein’s equations (12) imply that the metric is that of Schwarzschild-(A)dS,
\[ f(r) = \frac{1}{h(r)} = 1 - \frac{2GM}{r} \pm \frac{r^2}{l^2}. \]

On the other hand, the Dirac equation (7) gives
\[ \Phi(r) = \frac{\Phi_0}{f^{1/4}r}, \]

with $\Phi_0$ and integration constant. The solutions for the phases $\lambda$ and $\rho$ are irrelevant because the only nonvanishing observable quantity ($J_a$) does not depend on them. The result is a nonvanishing spinor field that has no backreaction on the metric, but it does generate torsion, in fact the vector $V_a = \kappa \gamma^2/[2(1 + \gamma^2)]J_a$ and the pseudovector $A_a = -\kappa \gamma^2/[2(1 + \gamma^2)]J_a$ are proportional to the current vector $J_a = 4\Phi_0^2/(f^{1/2}r^2)(1, \pm 1, 0, 0)$. The torsion 2-form can then be written as
\[ T^a = \frac{\kappa \gamma^2}{2(1 + \gamma^2)} \left( \frac{1}{2} \epsilon^{ab} J - \frac{1}{\gamma} (e^a, J) \right), \]

where $J = J_\theta e^\phi$. Note that torsion in this solution is proportional to the parameter $n$, which goes to show that this parameter has a nontrivial contribution to the action only in a Cartan spacetime. Also observe that torsion can be tuned with $\Phi_0$, which could be regarded as a hair of the black hole.

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Footnotes:
- Torsion-free condition for the connection, for instance.
- There is a generalization of the concept of Lie derivative for objects defined on a fiber bundle [35].
V. OBSERVABLE EFFECTS

In our solution the spacetime metric, in the presence of a complex Dirac field, does not depend on $\Phi_0$, and therefore it is the same as it would be in its absence. The spacetime geometry, however, is not the same, because the connection is different due to torsion. In order to observe the difference and to test the theory it would be necessary to measure the torsion of spacetime.

The question about measuring torsion is still open (in particular because it has never been measured). Let me cite a brief list of established facts

- **Massive point particles** are expected to be unaffected by torsion because they couple only to the metric in the action $I_{pp} = -m \int ds = -m \int \sqrt{-g^{00}} \dot{x}^2 d\tau$, where $\dot{x} = dx/d\tau$ denotes derivative with respect to proper time. This particles should follow geodesics of stationary length.
- **Minimally coupled scalar fields** do not couple to torsion either because are spinless fields, so the covariant derivative is $\nabla \phi = \partial \phi$ and therefore they do not couple to the connection. However, when the scalar field is nonminimally coupled to curvature may interact with torsion [35].
- **Gauge fields** are supposed not to couple minimally to the connection because that would break gauge invariance of the field strength $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu + T^{\alpha\mu\nu}_\alpha A_\alpha$. Actually, just because $A = A_\mu dx^\mu$ is a gauge connection, the $\mu$ index is not a Lorentz frame index but a coordinate one (A is one-form), so the exterior derivative $dA$ is invariant under Lorentz and coordinate transformations and gauge invariance is automatically protected. Nevertheless, some arrangements allow to minimally couple a Maxwell field to torsion and also preserve gauge invariance by restricting torsion to be pure trace, proportional to the gradient of a scalar field [37,39]. On the other hand, as in the scalar field case, it is always possible to couple a Maxwell field nonminimally with curvature and torsion [40,41].
- **Spinning tops**, like Gravity Probe B, are one of the most discussed possible candidates to detect torsion. For a long time it has been believed that orbital angular momentum is not sensitive to torsion [12,13]. Nevertheless Mao et al. [41] argue that it is possible that Gravity Probe B will give some information about torsion, depending on how matter is coupled to gravity, and that the ultimate answer should be given by experiments. March et al. [45] follow the argument and propose to use the Moon and Mercury prescesion as a test of torsion. Finally Hehl et al. [46] show that the conclusions of Mao et al. do not yield any information on torsion, and at the moment most experts seems to agree, however the subject is on the table and looks like an open question still.

- **Dirac particles**: Beyond the previous discussion, the consensus seems to be that if there is torsion in spacetime, it may be measured by means of test particles with intrinsic spin possessing a canonical energy-momentum tensor with an antisymmetric part [47]. The Dirac field is a good example of this case.

Let us consider the problem of a test Dirac field propagating in a background geometry with torsion, described by the action

$$ I_{Dirac}[\Psi] = - \int \frac{\alpha}{2} \bar{\Psi} \Gamma D \Psi + \frac{\alpha^*}{2} D \bar{\Psi} \Gamma \Psi + m \bar{\Psi} \Psi e^4, $$

where $e^4 = |e|d^4x$ is the invariant volume form and $\alpha = 1 + i n$ again. Recall that $\Psi$ is an external test field, unrelated to the solution (14). From the Dirac equation

$$ \Gamma^a \bar{D}_a \Psi - \frac{3i}{4} (A_\alpha \Gamma^5 - n V_\alpha \Gamma^a) \Psi - m \Psi = 0, $$

the role of the vectors $A$ and $V$ in the dynamics of the test field is manifest: they represent some kind of axial and vector Maxwell fields minimally coupled to the test fermion. So it is expected that they would produce a force on an electron beam passing through, and by means of this force it would be possible to detect the difference between the current solution and the usual (vacuum) Schwarzschild-(A)dS spacetime. It is possible to compute this force by looking to the conservation laws. Local Lorentz and diffeomorism invariance guarantee the identities [47]

$$ D \tau_d = \frac{1}{2} \xi_a \Gamma^{ad} \sigma_{ab} + \xi_d T^a \tau_a, \quad (15) $$

$$ D \sigma_{ab} = c_a \bar{\tau}_b - c_b \bar{\tau}_a, \quad (16) $$
which are equivalent to geodesic equations, here $\eta_d$ denotes the contraction operation\footnote{The contraction operation is defined as a linear map from $p$-forms to $(p - 1)$-forms which satisfies the Leibniz rule, i.e., $\iota_d(p \wedge q) = \iota_d p \wedge q + (-1)^p p \wedge \iota_d q$ and such that $\iota_d e^a = \delta^a_d$.} with the inverse vielbein field $E_d = E^a_d \partial_a$, and

$$
\tau_d = \frac{\alpha}{2} \Psi^*(\Gamma e_d)\mathcal{D}\Psi - \frac{\alpha^*}{2} \mathcal{D}\Psi^*(\Gamma e_d)\Psi + m \Psi \Psi^* e_d
$$

are the stress and spin 3-forms respectively, with $j_a = i \bar{\Psi} \Gamma_a \Psi$ and $j^5_a = i \bar{\Psi} \Gamma^5 \Gamma_a \Psi$ the corresponding vector and axial currents of the test field. Let us shift everything depending on torsion to the right hand side of (15) and identify it as an effective force due to torsion, $\mathcal{D} \tilde{T} - \frac{4}{3} \iota_d \tilde{R}^{ab} \sigma_{ab} = F_d e^a$, where $\tilde{T} = \frac{1}{2} \Psi^*(\Gamma e_d)\mathcal{D}\Psi - \frac{1}{2} \mathcal{D}\Psi^*(\Gamma e_d)\Psi - \frac{5}{2} \mathcal{D}^* (j e_d) + m \Psi \Psi^* e_d$ is a torsion-free stress and $j = j_a e^a$. Using (16) one finds $F_d = \frac{1}{2} (-A^5 \mathcal{D}_a j^5_a + n V^a \mathcal{D}_a j_a)$. Then, replacing (1011), the effect of the bulk Dirac field on the test Dirac field via the torsional interaction can be summarized as

$$
F_d = \frac{3 \kappa \hbar^2}{8(1 + \gamma^2)} J^a \mathcal{D}_a \left( \frac{1}{\gamma} j^5_a + n j_a \right),
$$

where $J_a = 4 \Psi^2 / (f^{1/2} r^2)(1, \pm 1, 0, 0)$. Observe that $F_d$ is of the order of Newton’s constant $\kappa = 8\pi G$, so the effect should be hard to detect unless the source density ($\Phi^2$) is large enough, as anticipated at the beginning, or if the test beam passes near an event horizon ($f \to 0$).

**VI. CHARGED CASE**

Let us include a Maxwell field $A = A_a dx^a$ into our problem just by extending our covariant derivative to $\mathcal{D}\psi = d\psi + ieA\psi + \frac{1}{4} \omega^{ab} \Gamma_{ab} \psi$ and considering the Maxwell action

$$
I_M[A, \psi] = -\frac{1}{2} \int \mathcal{F} \wedge^* \mathcal{F},
$$

with $\mathcal{F} = dA$. Note that Maxwell field does not couple to spin connection, so Cartan’s equations, which come from variations respect to $\omega$, are not going to change, and the effective theory is going to have the same form as (12) with the same effective stress tensor as (13) plus the usual Maxwell stress. Additionally we have Maxwell’s equations

$$
d^*\mathcal{F} = -e^* J. \tag{17}
$$

Under our assumptions of spherical symmetry the vector field can be gauge fixed as $A = \phi(r) dt$. Conclusions of appendix B remain the same, but (17) impose additionally that $J_1 = 0$, it is $\Phi^2 = 0$. In conclusion the inclusion of a Maxwell field into the problem results in killing off the Dirac field and in consequence the torsion. The solution is the usual Reissner-Nordstrom spacetime.

**VII. CONCLUSIONS**

The presence of spinning matter, in the form of a Dirac field, minimally coupled to gravity in a stationary and spherically symmetric configuration has the effect of inducing a nontrivial torsion filling the spacetime only if we admit that the constant $\alpha$ in the Dirac action has a nonvanishing imaginary part $n$, however it has no effect on the metric. In summary, for the matter part we have a nonvanishing spinor that has a vanishing stress tensor and a vanishing axial current but a nonzero null vector current that is singular at the black hole horizon(s) and at the origin, while for the gravity part we have Schwarzschild-(A)dS metric together with a nonzero torsion that is proportional to the vector current of the Dirac field.

The Immirzi parameter, i.e., the Holst term, changes very little this conclusions, it just has the effect to modulate the components of torsion. In the limit $\gamma \to \infty$ (Einstein-Cartan theory alone) the
completely antisymmetric part of torsion vanishes, but the vector part still remains. For $\gamma \to 0$ torsion goes to zero like $\gamma$ for the completely antisymmetric part and like $\gamma^2$ for the vector part.

A tentative observational test is proposed via the detection of a force predicted on a test electron beam, this force should be very small unless it is measured in a very huge spin density region or very near the event horizon.

The presence of a Maxwell field completely eliminates the hair of the black hole, forcing the spinor field to be zero.

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**APPENDIX A**

Taking the hodge dual at both sides of (6) we obtain the component version of Cartan’s equation

$$T_{abc} + 2\eta_{[a}T_{c]d} + \frac{1}{\gamma} \epsilon^{lm}_{[a}T_{c]lm} = \kappa \left( \frac{1}{2} \epsilon_{abcd}J^{5d} - \eta_{[a}J_{c]} \right). \quad (A1)$$

Reminding the decomposition $T_{abc} = \epsilon_{abcd}A^d + \eta_{[a[b}V_{c]} + M_{abc}$, and if contracting (A1) with $\epsilon^{abcd}$ one obtains $\text{(A2)}$ and if with $\eta^{ab}$ obtains $\text{(A3)}$

$$A_a + \frac{1}{\gamma} V_a = \frac{\kappa}{2} J^5_a \quad (A2)$$

$$V_a - \frac{1}{\gamma} A_a = \frac{\kappa n}{2} J_a. \quad (A3)$$

Solution to these equations is given by $\text{(10,11)}$. Now replacing this into (A1) one obtains an equation for the mixed part

$$M_{abc} + \frac{1}{\gamma} \epsilon^{lm}_{[a}M_{c]lm} = 0,$$

which when iterated gives $(1 + \gamma^{-2})M_{abc} = 0$, and the mixed part vanishes.

**APPENDIX B**

From the vielbein choice $e^0 = f^{1/2}dt$, $e^1 = h^{1/2}dr$, $e^2 = r d\theta$, and $e^3 = r \sin \theta d\phi$, one can compute the torsion-free spin connection: $\tilde{\omega}^{01} = f'/2(fh^{1/2})e^0$, $\tilde{\omega}^{02} = \tilde{\omega}^{03} = 0$, $\tilde{\omega}^{12} = -1/(rh^{1/2})e^2$, $\tilde{\omega}^{13} = -1/(rh^{1/2})e^3$, and $\tilde{\omega}^{23} = -\cot \theta / re^3$, and the Riemann curvature $\tilde{R}^{ab} = d\omega^{ab} + \omega^{ac}\omega^{cb}$ gives $\tilde{R}^{01} = -1/[2(fh)^{1/2}][f'/((fh)^{1/2})e^0 e^1$, $\tilde{R}^{02} = -f'/2(fh)e^1 e^2$, $\tilde{R}^{03} = h'(2rh^2)e^1 e^3$, and $\tilde{R}^{23} = r^{-2}(1 - 1/h)e^2 e^3$, where $I = 2, 3$ and $'$ denotes derivative respect to $r$ $[48]$. With this in hand we compute the
left hand side of (12), $G_d = -\frac{1}{2} \epsilon_{abcd}(\tilde{R}^{ab} + \ell^{-2} e^c e^d) e^e$

\[
G_0 = \left[ \frac{h'}{r h} + \frac{1}{r^2} \left( 1 - \frac{1}{h} \right) \right] e^e e^3
\]

\[
G_1 = -\left[ -\frac{f'}{r f h} + \frac{1}{r^2} \left( 1 - \frac{1}{h} \right) \right] e^e e^3
\]

\[
G_2 = \left[ -\frac{1}{2 (f h)^{1/2}} \left( \frac{f'}{(f h)^{1/2}} \right)^2 + \frac{r' h}{r f h^2} - \frac{3}{r^2} \right] e^0 e^1 e^3
\]

\[
G_3 = -\left[ -\frac{1}{2 (f h)^{1/2}} \left( \frac{f'}{(f h)^{1/2}} \right)^2 + \frac{r' h}{r f h^2} - \frac{3}{r^2} \right] e^0 e^2 e^3.
\]

On the other hand, we compute the effective stress 3-form $\tau_{eff}^d$ given by (13) obtaining

\[
\tau_{00}^{eff} = -\left( \frac{i}{2 h^{1/2}} / \right)^{2} \psi \Gamma_{01} \partial_\tau \psi + \frac{i}{2 h^{1/2}} \psi \Gamma_{02} \partial_\theta \psi + \frac{i}{2 r \sin \theta} \psi \Gamma_{03} \partial_\varphi \psi + U_{ef}
\]

\[
\tau_{01}^{eff} = -\left( \frac{i}{2 h^{1/2}} / \right)^{2} \psi \Gamma_{01} \partial_\theta \psi + \frac{i}{2 r \sin \theta} \psi \Gamma_{02} \partial_\varphi \psi + \frac{i}{2 r \sin \theta} \psi \Gamma_{03} \partial_\varphi \psi + U_{ef}
\]

\[
\tau_{02}^{eff} = -\left( \frac{i}{2 h^{1/2}} / \right)^{2} \psi \Gamma_{01} \partial_\varphi \psi + \frac{i}{2 r \sin \theta} \psi \Gamma_{02} \partial_\varphi \psi + \frac{i}{2 r \sin \theta} \psi \Gamma_{03} \partial_\varphi \psi + U_{ef}
\]

\[
\tau_{03}^{eff} = -\left( \frac{i}{2 h^{1/2}} / \right)^{2} \psi \Gamma_{01} \partial_\varphi \psi + \frac{i}{2 r \sin \theta} \psi \Gamma_{02} \partial_\varphi \psi + \frac{i}{2 r \sin \theta} \psi \Gamma_{03} \partial_\varphi \psi + U_{ef}
\]

where $\psi \Gamma \partial_\mu \psi = \psi \Gamma \partial_\mu \psi - \partial_\mu \psi \psi$ and $U_{ef} = -3 \kappa \gamma^2 / (16(1 + \gamma^2)) [J^3 - r^2 J^2 - \frac{4\pi}{3} J^3]$. So comparing with $G_d$ we see that off diagonal components of $\tau_{eff}^d$ must vanish. In particular we have

\[
\psi \Gamma_{01} \partial_\tau \psi = \psi \Gamma_{02} \partial_\theta \psi = \psi \Gamma_{03} \partial_\varphi \psi = 0
\]

\[
\psi \Gamma \partial_\varphi \psi - \cos \theta \psi \Gamma \partial_\varphi \psi = 0
\]

\[
\psi \Gamma \partial_\varphi \psi - \cos \theta \psi \Gamma \partial_\varphi \psi = 0
\]

By writing $\phi_L = |\phi_L| e^{i \lambda(x)}$ and $\phi_R = |\phi_R| e^{i \rho(x)}$ we can see that these equations have nontrivial solutions if $|\phi_L| = |\phi_R|$, $s_L = -s_R$, and $\partial_\mu (\lambda(x) + \rho(x)) = 0$. Putting it back to $\tau_{eff}^d$ we observe that it vanishes identically.

**APPENDIX C**

All over this work, spinor fields representing matter are considered to have complex and non-Grassmannian components. Maybe one would be tempted to believe the opposite, because spinors are fermions ($s = \frac{1}{2}$) so
they should anticommute and hence have Grassmannian components. Nevertheless the fermionic nature
of some particles is a quantum property, while what we are studying here is the classical regime of
the theory (field equations derived from a minimal action principle), so at this level Pauli exclusion principle
should not be relevant.

In this appendix we explain why the use of Grassmann spinors does not come to help, but brings some
inconsistencies into the problem. Let us consider that the spinor components are Grassmann numbers
\( \psi_\alpha \psi_\beta = -\psi_\beta \psi_\alpha \), here \( \alpha \) and \( \beta \) are spinorial indices, and proceed under this assumption. The only
difference with the treatment here presented is to change the definition of Dirac conjugate by \( \bar{\psi} = \psi^\dagger \Gamma_0 \)
in order that the Lagrangian \( \mathcal{L} \) stays hermitian. All expressions until \( (13) \) remain the same, except that
\( \psi \) has now complex Grassmann components. The solution to the equations (in the spherically symmetric
case) will naturally change. In fact, the overall factor \( \Phi \) in the current solution would be meaningless,
because all physical observables, such as currents \( (J, J^5) \) and energy \( (\tau) \) which are quadratic in \( \psi \)
would identically vanish.

Nevertheless, the problem is rather ill defined, because, although even products of Grassmann numbers
are commutative, they are not actually real numbers, they are vectors, so in \( (14) \) we have equations
in which the left-hand side is a real number while the right-hand side is not. How can this equality be
regarded? It seems like we are comparing objects of intrinsic different nature. Of course it is possible to
treat such an equation order by order in Grassmann powers:
\[
G^{(0)}_{\mu\nu} = \bar{T}^{(0)}_{\mu\nu}, \quad G^{(2)}_{\mu\nu} = \bar{T}^{(2)}_{\mu\nu} \quad \text{and on.}
\]
But what actually happens is that the left-hand side is of order zero while the right-hand side is of order
two, and under this point of view this theory would give Einstein’s equations in vacuum and the huge
constraint that the stress tensor of \( \psi \) vanishes. In fact the problem comes from the very beginning, when
we constructed an action that is not a real number but a vector, so the sentence “minimal action” is
meaningless. In summary we see that the classical treatment of a quantum object like a fermion leads to
categorical inconsistencies, equivalent to the incongruence of having a quantum stress tensor at the right
hand side of the Einstein’s equations \( G_{\mu\nu} = \bar{T}_{\mu\nu} \).

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