UNBOUNDED PERIODIC CONSTANT MEAN CURVATURE GRAPHS ON CALIBRABLE CHEEGER SERRIN DOMAINS

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Abstract. We prove a general result characterizing a specific class of Serrin domains as supports of unbounded and periodic constant mean curvature graphs. We apply this result to prove the existence of a family of unbounded periodic constant mean curvature graphs, each supported by a Serrin domain and intersecting its boundary orthogonally, up to a translation. We also show that the underlying Serrin domains are calibrable and Cheeger in a suitable sense, and they solve the 1-Laplacian equation.

Keywords: Overdetermined problems, Cheeger sets, calibrable sets, Serrin domains, mean curvature.

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1. Introduction and main result

In the more recent literature, a domain $\Omega$ in the Euclidean space $\mathbb{R}^N$, $N \geq 2$ is called Serrin domain if it admits a positive solution to the following so called Serrin’s overdetermined problem

$$-\Delta u = 1 \quad \text{in } \Omega \quad \quad (1.1)$$
$$u = 0, \quad \partial_\nu u = -\beta \quad \text{on } \partial \Omega, \quad (1.2)$$

where $\nu$ is the unit outer normal on $\partial \Omega$ and $\beta$ is a positive constant. In [29] Serrin proved that the only $C^2$ bounded domains $\Omega$ where problem (1.1)-(1.2) admits a solution are balls. This parallels the result of Alexandrov [1] that smooth bounded domains with constant boundary mean curvature are balls.

In general, it is a challenging question to relate constant mean curvature (CMC) surfaces with domains where Serrin’s problem (1.1)-(1.2) is solvable. In lower dimension however, Del Pino, Pacard and Wei [10] succeeded to provide examples for domains whose boundary is close to large dilations of a given CMC surface where Serrin’s overdetermined problem is solvable. In [13], A. Farina and E. Valdinoci, and independently H. Berestycki, L. A. Caffarelli, and L. Nirenberg [6] also studied the conditions under which an epigraph domain solving an overdetermined problem must be a half-space. Observe in this case that the boundary of $\Omega$ is then a minimal surface.

Under further assumptions, one might also expect a Serrin domain to support a graph with CMC. For instance if a Serrin domain $\Omega$ is bounded, it is known from our result in [24] that $\Omega$ is uniquely self-Cheeger set. Thanks to E. Guisti [16], there
exists a unique (up to a constant) bounded solution \( w \in C^2(\Omega) \) to the prescribed mean curvature equation

\[
\begin{cases}
-\text{div} \frac{\nabla w}{\sqrt{1 + |\nabla w|^2}} = h(\Omega) & \text{in } \Omega \\
-\frac{\nabla w}{\sqrt{1 + |\nabla w|^2}} \cdot \eta = 1 & \text{on } \partial \Omega,
\end{cases}
\]

where \( \eta \) is the unit outer normal to \( \partial \Omega \) and \( h(\Omega) \) is the Cheeger constant of \( \Omega \). That is \( \Omega \) supports a graph with constant mean curvature equals to \( h(\Omega) \), and this graph intersects perpendicularly the horizontal plane, up to a translation.

In this paper, we examine the analogue of this result on a specific class of unbounded Serrin domains. Namely we aim to prove the existence of constant mean curvature graphs \( w \) satisfying (1.3), with the property that they are supported by unbounded and periodic calibrable Cheeger Serrin domains \( \Omega \). The class of Serrin domains under consideration are domains \( \Omega \subset \mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^m \) that are radial and bounded in the variable \( z \in \mathbb{R}^n \), periodic and symmetric in the variables \( t_1, \ldots, t_m \) in \( \mathbb{R}^m \). A prototype of these domains was already constructed by M. M. Fall, T. Weth and the author in [12, Theorem 1.1].

The notion of Cheeger sets we are dealing with in this work generalizes the classical one in [22] and involves the definition of relative perimeter. We let \( S \) and \( \Omega \) be open subsets of \( \mathbb{R}^N \). For a given Borel set \( A \) in \( \mathbb{R}^N \), the relative perimeter of \( A \) in \( S \) is given by

\[
P(A, S) := \sup \left\{ \int_A \text{div}(\xi) \, dx : \xi \in C^\infty_c(S; \mathbb{R}^N), \ |\xi| \leq 1 \right\}.
\]

The perimeter, \( P(A) \), of a Borel set \( A \subset \mathbb{R}^N \) is the relative perimeter \( P(A, \mathbb{R}^N) \) of \( A \) in \( \mathbb{R}^N \). If a set \( A \) is of finite perimeter, denoting by \( \partial^* A \) the reduced boundary of \( A \), then by De Giorgi’s structure Theorem [13, Theorem 2.2] (see also [17, Remark on p.161] or [3]), \( P(A, \Omega) = H^{N-1}(\partial^* A \cap \Omega) \). In addition, we have the Gauss-Green formula

\[
\int_A \text{div}(\xi) \, dx = \int_{\partial^* A} \langle \xi, \nu_A \rangle \, dH^{N-1} \quad \text{for all} \quad \xi \in C^1_c(\mathbb{R}^N; \mathbb{R}^N).
\]

(1.4)

Here and in the following, \( H^{N-1} \) denotes the \((N-1)\)-dimensional Hausdorff measure. The Cheeger constant of an open set \( \Omega \) relative to \( S \) is by definition

\[
h(\Omega, S) := \inf_{A \subseteq \Omega \cap S} \frac{P(A, S)}{|A|},
\]

(1.5)

where the infimum is taken over Borel subsets \( A \subseteq \Omega \cap S \) with finite perimeter. If this constant is attained by some Borel subset \( A \subset \Omega \cap S \) with finite perimeter, then \( A \) will be called a Cheeger set of \( \Omega \) relative to \( S \). If \( A = \Omega \cap S \) attains the constant \( h(\Omega, S) \) in (1.5), we say that \( \Omega \) is self-Cheeger relative to \( S \). Moreover, if any Cheeger set in \( \Omega \cap S \) is equal to \( \Omega \cap S \) up to a set with zero Lebesgue measure, we say that \( \Omega \) is uniquely self-Cheeger relative to \( S \).
Related to the notion Cheeger sets is the concept calibrable sets defined in [2], where convex bounded calibrable sets were characterized. In the same fashion, we introduce the notion of relative calibrable sets. We say that a set $E$ is calibrable relative to $S$, if there exists a vector field $\xi \in L^\infty(E, \mathbb{R}^N)$, $|\xi|_\infty \leq 1$ such that
\[
-\text{div} \xi = \lambda_E 1_E \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^N), \quad \xi \cdot \nu = -1 \quad \text{on} \quad \partial^*E \cap S, \quad \xi \cdot \nu = 0 \quad \text{on} \quad E \cap \partial S,
\]
where $1_E$ denotes the characteristic function of $E$. Observe that if $E$ is calibrable with respect to $S$, then $\lambda_E = \frac{P(E, S)}{|E|}$. With these notations and definitions, we can now state our first result.

**Theorem 1.1.** Let $\Omega \subset \mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^m$ be a radial and bounded domain in the variable $z \in \mathbb{R}^n$, $2\lambda$-periodic and symmetric in the directions $t_1, \ldots, t_m$ in $\mathbb{R}^m$. Assume that the overdetermined problem (1.1)-(1.2) admits a solution $u \in C^{2,\alpha}(\Omega)$.

Then there a function $w \in C^\infty(\Omega)$ which is radial in $z \in \mathbb{R}^n$, even and $2\lambda$-periodic in the variables $t_1, \ldots, t_m$ and such that
\[
\begin{cases}
-\text{div} \frac{\nabla w}{\sqrt{1 + |\nabla w|^2}} = \frac{1}{\beta} & \text{in} \quad \Omega \\
- \frac{\nabla w}{\sqrt{1 + |\nabla w|^2}} \cdot \eta = 1 & \text{on} \quad \partial \Omega
\end{cases}
\]  

(1.7)

Furthermore, the function $w$ is unique (up to additive constant) in the class of even and $2\lambda$-periodic functions in the variables $t_1, \ldots, t_m$.

**Remark 1.2.** We do not know that $w \in C(\Omega)$ and that $w|_{\partial \Omega} \equiv \text{Const}$. This would implies that the graph of $w$ is a half of Delaunay hypersurface [11].

Notice that the existence of solution to (1.3) as stated in [16] is strongly related to the fact that $\Omega$ is uniquely self-Cheeger set. It is known from our result in [24] that any bounded Serrin domain is uniquely self-Cheeger set. For Serrin domains consider in Theorem 1.1 we show that they are uniquely self-Cheeger and calibrable relatively to a tubular domain in $\mathbb{R}^N$. In addition, they are support of a solution to the 1-Laplacian equation.

**Theorem 1.3.** Let $\Omega \subset \mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^m$ be a radial and bounded domain in the variable $z \in \mathbb{R}^n$, $2\lambda$-periodic and symmetric in the directions $t_1, \ldots, t_m$ in $\mathbb{R}^m$. Assume that the overdetermined problem (1.1)-(1.2) admits a solution $u \in C^{2,\alpha}(\Omega)$ and let $a, b \in \lambda \mathbb{Z}^m$ with $a_i < b_i$ for $i = 1, \ldots, m$.

Then $\Omega$ is uniquely self-Cheeger relative to the set
\[
S^b_a := \mathbb{R}^n \times (a_1, b_1) \times \cdots \times (a_m, b_m)
\]
with corresponding relative Cheeger constant $h(\Omega, S^b_a) = \frac{1}{\beta}$.
Moreover,

$$-\text{div} \left( \frac{\nabla 1_\Omega}{|\nabla 1_\Omega|} \right) = \frac{1}{\beta} 1_\Omega$$

and $\Omega$ is calibrable relative to $S^b_a$.

**Remark 1.4.** As a remark, the first statement in Theorem 1.3 is the result of [12, Corollary 1.2], where the Cheeger constant was defined taking the infimum over subsets $A \subset \Omega \cap S$ with Lipschitz boundary. The same conclusion in [12, Corollary 1.2] holds even for current definition of the Cheeger constant in (1.5). Indeed working with (1.5), the main challenge in the proof of the first part of Theorem 1.3 is to show the uniqueness of $\Omega$ as a self-Cheeger relative $S^b_a$. But scanning [12, Section 5], this follows from the last part of the proof Theorem 1.2 in [24] after replacing $\Omega$ by $\Omega \cap S^b_a$.

**Corollary 1.5.** There exists a family of unbounded constant mean curvature graphs $(\Gamma_s)_{s \in (-\varepsilon_0, \varepsilon_0)}$, each supported by a calibrable Cheeger Serrin domain $E_s$ and intersecting its boundary orthogonally (up to a translation).

For every $s \in (-\varepsilon_0, \varepsilon_0)$, the graph $\Gamma_s$ is $2\pi$-periodic in some variables and radial in the others, and has mean curvature equal to $\frac{n}{\lambda_s}$, with $\lambda_s > 0$.

The existence of Serrin domains stated in Corollary 1.5 relies on our recent result in [12, Theorem 1.1], where we constructed a family $(E_s)_{s \in (-\varepsilon_0, \varepsilon_0)}$ of unbounded periodic Serrin domains of the form

$$E_s := \{(z, t) \in \mathbb{R}^n \times \mathbb{R}^m : |z| < \phi_s(t)\} \subset \mathbb{R}^N,$$

where $N = n + m$ and $\phi_s : \mathbb{R}^m \to (0, \infty)$ is an even and $2\pi \mathbb{Z}^m$-periodic function. We note that, each of $E_s$ solves the overdetermined (1.1)-(1.2) with $\beta = \frac{\lambda_s}{n}$.

From Theorem 1.3 Serrin domains $E_s$ are calibrable and Cheeger. The proof of Corollary 1.5 is then complete once we prove the more general result in Theorem 1.1, namely the existence of a solution to (1.7).

The proof of Theorem 1.1 is inspired by [15] and [16]. We first apply [12, Lemma 5.1] to the overdetermined problem (1.1)-(1.2) and show the conditions (1.4) and (1.4) of [15, Theorem 2.1] are fulfill by the minimization problem associated with (1.3). The result of [15, Theorem 2.1] then yields the existence of a function $v_\varepsilon \in BV(F_\varepsilon) \cap C^2(F_\varepsilon)$ solution to (3.2). By compactness, the sequence $(v_\varepsilon)_{n \in \mathbb{N}}$ converges to a generalized solution $v : F \to [0, +\infty]$, which we prove to be a classical solution by showing it is bounded. The function $v$ is even in the variables $t_1, \ldots, t_m$ and this allows to derive the second equation of (3.8). To finish the proof, we make the shift $\tilde{v}(z, t) := v(z, t_1 - 2\lambda, \ldots, t_m - 2\lambda)$, and finally deduce in (3.11) the existence of a function $w \in C^\infty(\Omega)$, radial in the $z$ variable and $2\lambda$-periodic in the variables $t_1, \ldots, t_m$, which in addition satisfies (1.7).
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2. Proof of Theorem 1.3

From Remark 1.4, we only need to prove the last statement of Theorem 1.3.

We start by explaining the meaning of (1.9) (see also [27]).

Let $\Omega$ be an open set of $\mathbb{R}^N$ and $u \in L^1(\Omega)$, the total variation of $u$ in $\Omega$ is defined by

$$|\nabla u|(\Omega) = \sup \left\{ \int_{\Omega} u \text{div}(\xi) \, dx : \xi \in C_0^\infty(\Omega, \mathbb{R}^N), |\xi| \leq 1 \right\}.$$  

The space of functions with bounded variation in an open $\Omega$ is denoted by $BV(\Omega)$. For a given Borel set $A$, it is plain that $P(A, S) = |\nabla 1_A|(S)$.

Let $f \in L^1_{loc}(\Omega)$. A function $v \in BV_{loc}(\Omega) \cap L^\infty_{loc}(\Omega)$ solves the equation

$$-\text{div} \frac{\nabla v}{|\nabla v|} = f \quad \text{in } \Omega$$

if there exists a sub-unit vector field $\xi \in W^{1,1}_{loc}(\Omega, \mathbb{R}^N)$ such that

$$(\xi, \nabla v) = |\nabla v| \quad \text{as Radon measures in } \Omega \quad (2.1)$$

and

$$-\text{div} \xi = f \quad \text{in the sense of distribution in } \Omega. \quad (2.2)$$

We recall the Radon measure $(\xi, \nabla v)$ in (2.1) is defined (see Anzelotti [4]) thanks to Riesz representation theorem via distributional sense as

$$(\xi, \nabla v)(\varphi) = -\int_{\Omega} v \varphi \text{div} \xi \, dx - \int_{\Omega} v \xi \cdot \nabla \varphi \, dx \quad \text{for every } \varphi \in C_0^\infty(\Omega). \quad (2.3)$$

We remark, that if $\Omega$ is a Lipschitz open set and $v = 1_\Omega$, we have

$$(\xi, \nabla v) = -\xi \cdot \nu_\Omega \mathcal{H}^{N-1}|_{\partial \Omega}. \quad (2.4)$$

by integration by parts. In addition, it is well known from the divergence theorem that

$$|\nabla 1_\Omega| = \mathcal{H}^{N-1}|_{\partial \Omega}, \quad (2.5)$$

where $\mathcal{H}^{N-1}$ is the $(N-1)$-dimensional Hausdorff measure.

Since $u = 0$ on $\partial \Omega$, we have $\nabla u = -|\nabla u| \eta$ on $\partial \Omega$, where $\eta$ denotes the outer unit normal vector field of $\partial \Omega$. We put

$$\xi = \frac{1}{\beta} \nabla u \quad \text{and } v = 1_\Omega.$$

It is plain that the solution $u$ of problem (1.1)-(1.2) satisfies the gradient estimate

$$|\nabla u| < \beta \quad \text{in } \Omega. \quad (2.6)$$
This follows from [12, Lemma 5.1], see also [24, Section 2] for the corresponding estimate in compact Riemannian manifolds.

Now using (2.4), (2.5) and (2.6), we find that (2.1) and (2.2) are satisfied with 
\[ f = \frac{1}{\beta} 1_{\Omega} \]
and this yields (1.9).

We now end the proof by showing that the the set \( \Omega \) is calibrable relative to \( S_a^b \).

We observe that we have just constructed the vector field \( \xi = \frac{1}{\beta} \nabla u \) which by clearly satisfies \( |\xi| \leq 1 \) by (2.6).

From the first part of the proof, we also have
\[ -\text{div} \xi = \frac{1}{\beta} 1_{\Omega} = h(\Omega, S_a^b) 1_{\Omega} \] (2.7)
\[ \xi = -\eta \quad \text{on} \quad \partial \Omega \cap S_a^b, \]
where for \( a, b \in \lambda \mathbb{Z}^m \) with \( a_i < b_i \) \( (i = 1, \ldots, m) \), the set \( S_a^b \) is defined as in (1.8).

We emphasize that the boundary \( \partial S_a^b \) can be decomposed into a disjoint union \( \partial S = K \cup S^1 \cup \cdots \cup S^m \), where
\[ S^i := \mathbb{R}^n \times \{ t \in \mathbb{R}^m : t_i \in \{ a_i, b_i \}, t_j \in (a_j, b_j) \text{ for } j \neq i \} \quad \text{for } i = 1, \ldots, m, \]
and \( K \) has zero \((N - 1)\)-dimensional Hausdorff measure. Since by hypothesis \( \Omega \) is \( 2\lambda \)-periodic and symmetric in \( t_1, \ldots, t_m \), the solution \( u \) of problem (1.1)-(1.2) is \( 2\lambda \)-periodic and even in \( t_1, \ldots, t_m \) which leads to
\[ \frac{\partial u}{\partial t_i} \equiv 0 \quad \text{on } S^i \quad \text{for } i = 1, \ldots, m. \]

Observe that the outer unit normal to \( S^i \) coincides with \((0, e_i)\) or \((0, -e_i)\), where \( 0 \in \mathbb{R}^n \) and \( e_i \) denotes the \( i \)-th coordinate vector in \( \mathbb{R}^m \) and we have
\[ \xi \cdot e_i = \frac{1}{\beta} \frac{\partial u}{\partial t_i} \equiv 0 \quad \text{on } S^i \quad \text{for } i = 1, \ldots, m. \] (2.8)

Hence, following [2] and our definition (1.6), we can say from (2.7) and (2.8) that the set \( \Omega \) is calibrable relative to \( S_a^b \).

3. Proof of Theorem 1.1

We prove that under the hypotheses on the Serrin domain \( \Omega \) in Theorem 1.1 there a function \( w \in C^\infty(\Omega) \) which is even and \( 2\lambda \)-periodic in the variables \( t_1, \ldots, t_m \), radial in \( z \in \mathbb{R}^n \), and such that (1.7) holds.

Following [15, Section 1], we consider the set \( A_1 := (\Omega \cup \{ (z, t) \in \mathbb{R}^n \times \mathbb{R}^m : \text{dist}(z, \partial \Omega) < 1 \}) \cap S \), where \( S := \mathbb{R}^n \times (-2\lambda, 2\lambda)^m \).
For \( \varepsilon > 0 \), we also define the Lipschitz domains
\[
\Omega_\varepsilon := \{(z, t) \in \Omega : \text{dist}(z, \partial \Omega) > \varepsilon\}, \quad F_\varepsilon := \Omega_\varepsilon \cap S, \quad F := \Omega \cap S.
\]
We further fix notations for boundaries:
\[
\partial_1 F_\varepsilon = \partial F_\varepsilon \cap A_1 \quad \text{and} \quad \partial_2 F_\varepsilon = \overline{F_\varepsilon} \cap \partial A_1.
\]
Let \( B \) be a Borel set in \( F_\varepsilon \). We use (1.4) (2.8) and integrate (1.1)-(1.2) over \( B \) to get
\[
|B| \leq \sup_{F_\varepsilon} |\nabla u| |\partial^* B \cap S|.
\]
We put
\[
c_\varepsilon := 1 - \frac{1}{\beta} \sup_{F_\varepsilon} |\nabla u|.
\]
For every \( \varepsilon > 0 \), \( F_\varepsilon \subset \Omega \) and by (2.6), \( c_\varepsilon > 0 \) and we have for any Borel set \( B \subset F_\varepsilon \),
\[
\frac{1}{\beta} |B| \leq (1 - c_\varepsilon) |\nabla 1_B|(S). \quad (3.1)
\]
Thanks to [15], Theorem 2.1, we can find a function \( v_\varepsilon \in BV(F_\varepsilon) \cap C^2(F_\varepsilon) \) solution to
\[
\begin{cases}
- \text{div} \frac{\nabla v_\varepsilon}{\sqrt{1 + |\nabla v_\varepsilon|^2}} = \frac{1}{\beta} & \text{in } F_\varepsilon \\
v_\varepsilon = 0 & \text{on } \partial_1 F_\varepsilon \\
\frac{\nabla v_\varepsilon}{\sqrt{1 + |\nabla v_\varepsilon|^2}} \cdot e_i = 0, \quad i = 1, \ldots, m. & \text{on } \partial_2 F_\varepsilon.
\end{cases} \quad (3.2)
\]
The second condition is understood in the sense of trace and the third condition in the following sense
\[
\lim_{\varepsilon \to 0} \int_{\partial_2 F_\varepsilon} \frac{\nabla v_\varepsilon}{\sqrt{1 + |\nabla v_\varepsilon|^2}} \cdot e_i \, d\sigma = 0; \quad i = 1, \ldots, m. \quad (3.3)
\]
Hence the solution is unique up to a additive constant.

Let \( \tilde{v}_\varepsilon(z, t) := v_\varepsilon(z, -t) \) then the function \( \tilde{v}_\varepsilon \) satisfies similar equation as \( v_\varepsilon \) and thus \( v_\varepsilon \) is even in the variables \( t_1, \ldots t_m \) and we have \( \partial_{t_i} v_\varepsilon(z, 0) = 0, \quad \text{for } i = 1, \ldots, m. \)

By a similar argument, we can obtain \( v_\varepsilon(z_1, \ldots, -z_i, \ldots, z_n, t) = v_\varepsilon(z, t) \) and thus
\[
v_\varepsilon(z, t) = \tilde{v}_\varepsilon(|z|, t).
\]
We recall that \( v_\varepsilon \) minimizes the functional
\[
\mathcal{G}(w) = \int_{A_1} \sqrt{1 + |\nabla w|^2} - \frac{1}{\beta} \int_{A_1} w \, dx \quad (3.4)
\]
in the class
\[
\mathcal{C} := \{w \in BV(A_1) : w = 0 \text{ in } A_1 \setminus F_\varepsilon\}, \quad (3.5)
\]
where

\[
\int_E \sqrt{1 + |\nabla w|^2} = \sup \left\{ \int_E [u \text{div}(\zeta) + \zeta N_{N+1}] \, dx : \zeta \in C^\infty_c(E; \mathbb{R}^{N+1}), |\zeta| \leq 1 \right\}.
\]

Since \( \mathcal{G}(|w|) \leq \mathcal{G}(w) \), we can assume that \( v_\varepsilon \geq 0 \) in \( F_\varepsilon \).

By compactness (see [23]) the sequence \( (v_\varepsilon)_{\varepsilon \in \mathbb{N}} \) converges to a generalized solution \( v : F \to [0, +\infty] \). We now show that \( v \) is bounded.

We define \( P := \{ v = +\infty \} \). By simple arguments, using also the co-area formula (see also [26]), one has that \( P \) minimizes the functional

\[
E \mapsto P(E, S) - \frac{1}{\beta} |E \cap S|.
\]

In addition, by Theorem 1.3 \( P(\Omega, S) = \frac{1}{\beta} |\Omega \cap S| \). Hence using similar argument as in [16], Lemma 1.2, we deduce that the set \( P = \emptyset \). We then conclude that \( v \) is a classical solution and moreover since it is even in the variables \( t_1, \ldots, t_m \) (as pointwise limit of even functions), we get \( \partial_i v(z, 0) = 0, \quad i = 1, \ldots, m \).

Now by integrating over \( F_\varepsilon \) we have

\[
- \int_{\partial_1 F_\varepsilon} \frac{\nabla v}{\sqrt{1 + |\nabla v|^2}} \cdot \eta^\varepsilon \, d\sigma_\varepsilon = \frac{1}{\beta} \frac{|F_\varepsilon|}{|\partial_1 F_\varepsilon|} \left| \partial_1 F_\varepsilon \right|
\]

so that

\[
\lim_{\varepsilon \to 0} \int_{\partial_1 F_\varepsilon} \left\{ - \frac{\nabla v}{\sqrt{1 + |\nabla v|^2}} \cdot \eta^\varepsilon - 1 \right\} \, d\sigma_\varepsilon = \lim_{\varepsilon \to 0} \left[ \frac{F_\varepsilon}{\beta |\partial_1 F_\varepsilon|} - 1 \right] = 0.
\]

We then conclude that \( v \) solves uniquely (up to an additive constant) the equation

\[
\begin{cases}
- \text{div} \frac{\nabla v}{\sqrt{1 + |\nabla v|^2}} = \frac{1}{\beta} & \text{in } F \\
- \frac{\nabla v}{\sqrt{1 + |\nabla v|^2}} : \eta = 1 & \text{on } \partial_1 F \\
\frac{\nabla v}{\sqrt{1 + |\nabla v|^2}} : e_i = 0 & \text{on } \partial_2 F, \quad i = 1, \ldots, m.
\end{cases}
\]

We now make the shift \( \tilde{v}(z, t) := v(z, t_1 - 2\lambda, \ldots, t_m - 2\lambda) \) for \((z, t) \in W \cap F\), where \( W := \mathbb{R}^n \times (0, 2\lambda)^m \).
By a direct computation, we have that both $\tilde{v}$ and $v$ satisfy the equation

$$
\begin{aligned}
-\text{div} \frac{\nabla \omega}{\sqrt{1 + |\nabla \omega|^2}} &= \frac{1}{\beta} \quad \text{in } F \cap W \\
\frac{\nabla \omega}{\sqrt{1 + |\nabla \omega|^2}} \cdot \eta &= 1 \quad \text{on } \partial_1(F \cap W) \\
\frac{\nabla \omega}{\sqrt{1 + |\nabla \omega|^2}} \cdot e_i &= 0, \quad i = 1, \ldots, m \quad \text{on } \partial_2(F \cap W).
\end{aligned}
$$

(3.9)

By uniqueness, we deduce that $\tilde{v}(z, t) = v(z, t_1 - 2\lambda, \ldots, t_m - 2\lambda) = v(z, t) + c$, for some constant $c \in \mathbb{R}$. Since $v(z, \lambda, \ldots, \lambda) = v(z, -\lambda, \ldots, -\lambda)$, we infer that $c = 0$ and thus

$$
v(z, t_1 - 2\lambda, \ldots, t_m - 2\lambda) = v(z, t) \quad \text{for every } (z, t) \in W \cap F.
$$

(3.10)

This implies that $v \in C^\infty(F \cup \partial S)$. For $k \in \mathbb{Z}$, define the sets

$$
S_k := \mathbb{R}^n \times (2k\lambda, 2(k+1)\lambda)^m.
$$

If $(z, t) \in \Omega \cap S_k$, then $(z, t_1 - 2k\lambda, \ldots, t_m - 2k\lambda) \in W \cap F$ and from (3.10) we can define

$$
w(z, t) := v(z, t_1 - 2k\lambda, \ldots, t_m - 2k\lambda) \quad \text{for } (z, t) \in \Omega \cap S_k.
$$

(3.11)

It is then clear from (3.11) and (3.10) that the function $w$ is $2\lambda$-periodic in the variables $t_1, \ldots, t_m$ and satisfies (1.7). In addition $w \in C^\infty(\Omega)$ and is unique, up to a constant, in the class of functions which are even and $2\lambda$-periodic in the variables $t_1, \ldots, t_m$. \qed

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