PRIMALITY OF MULTIPLY CONNECTED POLYOMINOES

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Abstract. It is known that the polyomino ideal of simple polyominoes is prime. In this paper, we focus on multiply connected polyominoes, namely polyominoes with holes, and observe that the non-existence of a certain sequence of inner intervals of the polyomino, called zig-zag walk, gives a necessary condition for the primality of the polyomino ideal. Moreover, by computational approach, we prove that for all polyominoes with rank less than or equal to 14 the above condition is also sufficient. Lastly, we present an infinite class of prime polyomino ideals.

1. Introduction

Polyominoes are two-dimensional objects obtained by joining edge by edge squares of the same size. They are studied from the point of view of combinatorics, e.g. in tiling problems of the plane, as well as from the point of view of commutative algebra, associating binomial ideals to polyominoes. The latter have been introduced by Qureshi in [9]. In particular, she introduces a binomial ideal generated by the inner 2-minors of a polyomino, called polyomino ideal. We refer the reader to Section 2 for the notation.

Two pending and of interest questions regarding polyomino ideals are to classify those that are prime and to prove if they are radical ideals. In this work, we focus on the first question, giving a partial answer in terms of their geometric realization. A polyomino is briefly called prime if its polyomino ideal is prime. In [5], [6] and [10], the authors prove that a polyomino is prime if and only if it is balanced, and that the simple polyominoes are prime. A simple polyomino is a polyomino without holes. Whereas, polyominoes having one or more holes are called multiply connected polyominoes, using the terminology adopted in [2], an introductory book on polyominoes.

In general, giving a complete characterization of the primality of multiply connected polyomino ideals seems to be not so easy. A family of prime polyominoes obtained by removing a convex polyomino by a given rectangle has been showed in [8] and [11].

In Section 3, we give a necessary condition for the primality of the polyomino ideal with respect to the geometric representation of the polyomino. This condition is related to a sequence of inner intervals contained in the polyomino, called a zig-zag walk (see Definition 3.2), whose existence determines the non-primality of the polyomino ideal.

It is known that a prime polyomino ideal is a toric ideal. In Section 3, we present a toric ideal associated to a polyomino, generalizing Shikama’s definition in [11]. This toric ideal contains the polyomino ideal (see Proposition 3.1). Moreover, if the polyomino contains a zig-zag walk, the binomial associated to the zig-zag walk belongs to the toric ideal and the above inclusion is strict.
The condition on zig-zag walks gives a good filtering with respect to primality. In fact, as an application, by the implementation of the algorithm described in [7], we compute all the polyominoes with rank less than or equal to 14 that are 123851. By symbolic computation, and using Macaulay2 [3], we obtain the following

**Theorem 1.1.** Let $P$ be a polyomino with $\text{rank}(P) \leq 14$. The following conditions are equivalent:

1. the polyomino ideal $I_P$ is prime;
2. $P$ contains no zig-zag walk.

In the last section, we observe that by removing 5 squares in a particular position from a given rectangle, we obtain a non-prime polyomino that contains a zig-zag walk (see Figure 3(B)). We define a new infinite family of polyominoes, called grid polyominoes, that are obtained by removing inner intervals from a given rectangle in a way that avoids the existence of zig-zag walks. By using a Gröbner basis technique and lattice ideals, we prove that grid polyominoes are prime.

Therefore, the natural conjecture arises:

**Conjecture 1.2.** Let $P$ be a polyomino. The following conditions are equivalent:

1. the polyomino ideal $I_P$ is prime;
2. $P$ contains no zig-zag walk.

## 2. Preliminaries

In this section we recall definitions and notation first introduced by Qureshi in [9]. We refer the reader to [4] for a self-contained presentation on polyomino ideals.

Let $a = (i,j), b = (k,\ell) \in \mathbb{N}^2$, with $i \leq k$ and $j \leq \ell$, the set $[a,b] = \{(r,s) \in \mathbb{N}^2 : i \leq r \leq k \text{ and } j \leq s \leq \ell\}$ is called an interval of $\mathbb{N}^2$. If $i < k$ and $j < \ell$, $[a,b]$ is called a proper interval, and the elements $a, b, c, d$ are called corners of $[a,b]$, where $c = (i,\ell)$ and $d = (k,j)$. In particular, $a, b$ are called diagonal corners and $c, d$ anti-diagonal corners of $[a,b]$. The corner $a$ (resp. $c$) is also called the left lower (resp. upper) corner of $[a,b]$, and $d$ (resp. $b$) is the right lower (resp. upper) corner of $[a,b]$. A proper interval of the form $C = [a, a+(1,1)]$ is called a cell. Its vertices $V(C)$ are $a, a+(1,0), a+(0,1), a+(1,1)$ and its edges $E(C)$ are

$$\{a, a+(1,0)\}, \{a, a+(0,1)\}, \{a+(1,0), a+(1,1)\}, \{a+(0,1), a+(1,1)\}.$$  

Let $P$ be a finite collection of cells of $\mathbb{N}^2$, and let $C$ and $D$ be two cells of $P$. Then $C$ and $D$ are said to be connected if there is a sequence of cells $C = C_1, \ldots, C_m = D$ of $P$ such that $C_i \cap C_{i+1}$ is an edge of $C_i$ for $i = 1, \ldots, m - 1$. In addition, if $C_i \neq C_j$ for all $i \neq j$, then $C_1, \ldots, C_m$ is called a path (connecting $C$ and $D$). A finite collection of cells $P$ is called a polyomino if any two cells of $P$ are connected. We denote by $V(P) = \cup_{C \in P} V(C)$ the vertex set of $P$. The number of cells of $P$ is called the rank of $P$, and we denote it by $\text{rank}(P)$.

A proper interval $[a,b]$ is called an inner interval of $P$ if all cells of $[a,b]$ belong to $P$. We say that a polyomino $P$ is simple if for any two cells $C$ and $D$ of $\mathbb{N}^2$ not belonging to $P$, there exists a path $C = C_1, \ldots, C_m = D$ such that $C_i \notin P$ for any $i = 1, \ldots, m$. If the polyomino is not simple then it is said multiply connected (see [2]).

A finite collection $H$ of cells not in $P$ is called a hole of $P$, if any two cells in $H$ are connected through a path of cells in $H$, and $H$ is maximal with respect to the inclusion. Note that a hole $H$ of a polyomino $P$ is itself a simple polyomino.
Following [6], an interval $[a, b]$ with $a = (i, j)$ and $b = (k, ℓ)$ is called a *horizontal edge interval* of $P$ if $j = ℓ$ and the sets $\{(r, j), (r + 1, j)\}$ for $r = i, \ldots, k - 1$ are edges of cells of $P$. If a horizontal edge interval of $P$ is not strictly contained in any other horizontal edge interval of $P$, then we call it *maximal horizontal edge interval*. Similarly one defines vertical edge intervals and maximal vertical edge intervals of $P$.

Let $a = (a_1, a_2)$ and $b = (b_1, b_2) \in V(P)$, we define on the vertices of $P$ the following total orders:

1. $a <^1 b$ if $a_1 < b_1$ or $a_1 = b_1$ and $a_2 < b_2$;
2. $a <^2 b$ if $b_1 < a_1$ or $a_1 = b_1$ and $a_2 < b_2$.

Let $P$ be a polyomino, and let $K$ be a field. We denote by $S$ the polynomial ring over $K$ with variables $x_v$, where $v \in V(P)$. The binomial $x_a x_b - x_c x_d \in S$ is called an *inner 2-minor* of $P$ if $[a, b]$ is an inner interval of $P$, where $c, d$ are the anti-diagonal corners of $[a, b]$. We denote by $M$ the set of all inner 2-minors of $P$. The ideal $I_P \subset S$ generated by $M$ is called the *polynomino ideal* of $P$. We also set $K[\!\!\![P]\!\!] = S/I_P$.

### 3. The toric ring of polyominos and zig-zag walks

Let $P$ be a polyomino. Let $S = K[x_v \mid v \in V(P)]$ and $I_P \subset S$ the polynomino ideal associated to $P$. Let $H$ be a hole of $P$. We call *lower left corner* $e$ of $H$ the minimum, with respect to $<^1$, of the vertices of $H$.

Let $H_1, \ldots, H_r$ be holes of $P$. For $k = 1, \ldots, r$, we denote by $e_k = (i_k, j_k)$ the lower left corner of $H_k$. For $k \in K = \{1, \ldots, r\}$, we define the following subset of $V(P)$

$$F_k = \{(i, j) \in V(P) \mid i \leq i_k \text{ and } j \leq j_k\}.$$

Let $\{V_i\}_{i \in I}$ be the set of all the maximal vertical edge intervals of $P$, and $\{H_j\}_{j \in J}$ be the set of all the maximal horizontal edge intervals of $P$. Let $\{v_i\}_{i \in I}, \{h_j\}_{j \in J}$, and $\{w_k\}_{k \in K}$ be three sets of variables associated to $\{V_i\}_{i \in I}, \{H_j\}_{j \in J}$, and $\{F_k\}_{k \in K}$, respectively. We consider the map

$$\alpha : V(P) \rightarrow K[\!\!\![h_i, v_j, w_k] \mid i \in I, j \in J, k \in K]\!\!\!]$$

$$a \mapsto \prod_{a \in H_i \cap V_j} h_i v_j \prod_{a \in F_k} w_k$$

The *toric ring* $T_P$ associated to $P$ is defined as $T_P = K[\alpha(a) \mid a \in V(P)] \subset K[\!\!\![h_i, v_j, w_k] \mid i \in I, j \in J, k \in K]\!\!\!]$. The homomorphism

$$\varphi : S \rightarrow T_P$$

$$x_a \mapsto \alpha(a)$$

is surjective and the *toric ideal* $J_P$ is the kernel of $\varphi$.

**Lemma 3.1.** Let $P$ be a polyomino and $(J_P)_2$ the homogeneous part of degree 2 of $J_P$. Then $I_P = (J_P)_2$.

**Proof.** First of all we show that $I_P \subset (J_P)_2$. Let $f \in M$, with $f = x_a x_b - x_c x_d$. Since $[a, b]$ is an inner interval of $P$, the corners $a$ and $d$ (resp. $b$ and $c$) lie on the same horizontal edge interval $H_i$ (resp. $H_j$). In the same way, it holds that $a$ and
$c$ (resp. $b$ and $d$) lie on the same vertical edge interval $V_k$ (resp. $V_m$). Therefore,
\[
\varphi(x_a x_b) = h_1 h_2 v_{i+1} v_m \prod_{k=1,...,r} w_k^{p_k}
\]
and
\[
\varphi(x_c x_d) = h_1 h_2 v_{j+1} v_m \prod_{k=1,...,r} w_k^{n_k}
\]
for some $p_k, n_k \in \{0, 1, 2\}$. We have to show that for any $k \in \{1, \ldots, r\}$ $p_k = n_k$.

If $\mathcal{P}$ has no holes, then $n_k = p_k = 0$ and $\varphi(x_a x_b) = \varphi(x_c x_d)$, that is $f \in \mathcal{J}_p$.

Suppose that $\mathcal{H}_1, \ldots, \mathcal{H}_r$ are holes of $\mathcal{P}$ and consider $\mathcal{H}_k$ for $k = 1, \ldots, r$. Observe that the left lower corner $e_k$ of $\mathcal{H}_k$ satisfies one of the following

(I) $e_k < a$;
(II) $a \leq e_k \leq d$;
(III) $d < e_k$,

where $<$ stands for $<^1$. Case (I). $w_k$ does not divide neither $\varphi(x_a x_b)$ nor $\varphi(x_c x_d)$ (see Figure (I)). Case (II). $w_k$ divides either both $\varphi(x_a)$ and $\varphi(x_c)$ (see Figure (II)) or it does not divide neither $\varphi(x_a x_b)$ nor $\varphi(x_c x_d)$. Case (III). $w_k$ divides either $\varphi(x_a)$ and $\varphi(x_d)$ (see Figure (III)) or all $\varphi(x_a), \varphi(x_b), \varphi(x_c)$ and $\varphi(x_d)$ (see Figure (IIIA)) or $w_k$ does not divide neither $\varphi(x_a x_b)$ nor $\varphi(x_c x_d)$. Therefore $n_k = p_k$, and it holds for any $k = 1, \ldots, r$. It follows $\varphi(x_a x_b) = \varphi(x_c x_d)$, and $f \in \ker \varphi = \mathcal{J}_p$. Since all generators of $\mathcal{I}_p$ belong to $\mathcal{J}_p$, the inclusion $\mathcal{I}_p \subseteq (\mathcal{J}_p)_2$ is proved.

We are going to prove the other inclusion, namely $(\mathcal{J}_p)_2 \subseteq \mathcal{I}_p$. Let $f \in \mathcal{J}_p$, $f = x_a x_b - x_c x_d$. We start observing that if $a = b$ or $a \in \{c, d\}$ we obtain that $f$ is null. Hence we assume without loss of generality $a < b$ and $c < d$. Since $\varphi(x_a x_b) = \varphi(x_c x_d)$, by (1) and (2) the vertices $a$ and $d$ (resp. $b$ and $c$) lie on the same horizontal edge interval of $\mathcal{P}$, and $a$ and $c$ (resp. $b$ and $d$) lie on the same vertical edge interval of $\mathcal{P}$, and all the vertices of these edge intervals belong to $\mathcal{P}$. Therefore, the vertices $a, b, c,$ and $d$ are the corners of the interval $[a, b]$. By contradiction, we assume that $[a, b]$ is not an inner interval of $\mathcal{P}$, namely exists a set $\mathcal{C}$ of cells of $[a, b]$ that do not belong to $\mathcal{P}$.

Since $[a, d], [a, c], [b, c]$ and $[b, d]$ are edge intervals in $\mathcal{P}$, $\mathcal{C}$ is a set of holes of $\mathcal{P}$ which are properly contained in $[a, b]$. Let $\mathcal{H}_1$ be a hole of $\mathcal{P}$ in $[a, b]$ with lower left corner $e = (i, j)$. Let $\mathcal{F}_1 = \{ (m, n) \in V(\mathcal{P}) \mid m \leq i \text{ and } n \leq j \}$, then $a$ is the unique vertex in $\{a, b, c, d\}$ such that $a \in \mathcal{F}_1$, namely $w_1 | \varphi(x_a x_b)$ but $w_1 \nmid \varphi(x_c x_d)$, and $f \notin \mathcal{J}_p$. The assertion follows. \qed
Describing completely the elements of $J_{\mathcal{P}} \setminus I_{\mathcal{P}}$ is not an easy task. However, if the polyomino has a particular collection of inner intervals, then we have some partial information on the elements of $J_{\mathcal{P}} \setminus I_{\mathcal{P}}$.

**Definition 3.2.** Let $\mathcal{P}$ be a polyomino. A sequence of distinct inner intervals $\mathcal{W}: I_1, \ldots, I_\ell$ of $\mathcal{P}$ such that $v_i$, $z_i$ are diagonal (resp. anti-diagonal) corners and $v_i$, $v_{i+1}$ the anti-diagonal (resp. diagonal) corners of $I_i$, for $i = 1, \ldots, \ell$, is a zig-zag walk of $\mathcal{P}$, if

1. \( |I_1 \cap I_{\ell}| = \{v_1 = v_{\ell+1}\} \) and \( I_i \cap I_{i+1} = \{v_{i+1}\} \), for $i = 1, \ldots, \ell - 1$,
2. $v_i$ and $v_{i+1}$ are on a same edge interval of $\mathcal{P}$, for $i = 1, \ldots, \ell$,
3. for any $i, j \in \{1, \ldots, \ell\}$, with $i \neq j$, does not exist an inner interval $J$ of $\mathcal{P}$ such that $z_i, z_j \in J$.

**Remark 3.3.** Let $\mathcal{W}: I_1, \ldots, I_\ell$ be a zig-zag walk of $\mathcal{P}$. Then

1. if $v_i$ is a diagonal vertex of $I_i$ then $v_{i+1}$ is an anti-diagonal vertex of $I_{i+1}$;
2. $\ell$ is even.

**Proof.** (1) Assume that $v_k$, with $k \in \{1, \ldots, \ell - 1\}$ is a diagonal corner of $I_k$. From condition (Z2), $v_{k+1}$ lies on the same edge interval of $v_k$, say $E$, and is an anti-diagonal corner of $I_k$. The line containing $E$ divides $\mathbb{N}^2$ in two semi-planes. From condition (Z1), we have $I_k \cap I_{k+1} = \{v_{k+1}\}$, hence $I_k$ and $I_{k+1}$ do not lie on the same semi-plane. Therefore, $v_{k+1}$ is anti-diagonal corner of $I_{k+1}$, as well. Observe that the latter justifies the name “zig-zag”.

(2) Assume that the starting point $v_1$ is a diagonal corner of $I_1$. From (1) it follows that the vertex $v_k$ is a diagonal corner of $I_k$ if and only if $k$ is odd (resp. anti-diagonal corner if and only if $k$ is even). Since $v_{\ell+1} = v_1$, $\ell + 1$ is odd. \( \square \)

**Remark 3.4.** Let $\mathcal{P}$ be a polyomino and $I_{\mathcal{P}} \subset S$ the polyomino ideal associated to $\mathcal{P}$. If $f \in I_{\mathcal{P}}$, then

\[ f = \sum f_{I_i} h_j = \sum x_{a_i} x_{b_j} h_j - \sum x_{c_i} x_{d_j} h_j \]

where $f_{I_i} = x_{a_i} x_{b_j} - x_{c_i} x_{d_j} \in \mathcal{M}$, hence for every $m$, monomial of $f$, there are two variables in $m$ that are (anti-)diagonal corners of an inner interval of $\mathcal{P}$.

**Proposition 3.5.** Let $\mathcal{P}$ be a polyomino and $I_{\mathcal{P}}$ the polyomino ideal associated to $\mathcal{P}$. If there exists a zig-zag walk $\mathcal{W}: I_1, \ldots, I_\ell$ in $\mathcal{P}$ then

\[ x_{v_1}, \ldots, x_{v_\ell} \text{ and } f_\mathcal{W} = \prod_{k=1, \ldots, \ell} x_{z_k} - \prod_{j=1, \ldots, \ell} x_{u_j} \]

are zero divisors of $K[\mathcal{P}]$ with $x_{v_i} f_{I_i} \in I_{\mathcal{P}}$ for $i = 1, \ldots, \ell$.

**Proof.** For any vertex $v_i$ in $v_1, \ldots, v_\ell$, after relabelling, we may assume $j = 1$ and, without loss of generality, that $v_1$ is a diagonal corner of $I_1$. Let $f_{I_i} \in \mathcal{M}$ be associated to the inner interval $I_i$.

We define the following polynomial

\[ \hat{f} = \omega_1 f_{I_1} + \cdots + (-1)^{i+1} \omega_i f_{I_i} + \cdots + (-1)^{\ell+1} \omega_\ell f_{I_\ell}, \]

where, for $i = 1, \ldots, \ell$,

\[ \omega_i = \prod_{j < i} x_{u_j} \prod_{k > i} x_{z_k}. \]
Let \( i = 1, \ldots, \ell - 1 \). Suppose that \( v_i \) is a diagonal corner of \( I_i \), hence \( v_{i+1} \) is an anti-diagonal corner of \( I_{i+1} \). It holds

\[
\omega_i f_{I_i} - \omega_{i+1} f_{I_{i+1}}
\]

is

\[
\omega_i (x_{v_i} x_{z_i} - x_{v_{i+1}} x_{u_{i+1}}) - \omega_{i+1} (x_{v_{i+2}} x_{u_{i+1}} - x_{v_{i+1}} x_{z_{i+1}}),
\]

where

\[
\omega_i x_{u_i} = \omega_{i+1} x_{z_{i+1}} \text{ for all } i.
\]

That is (3) becomes

\[
(\omega_i x_{z_i}) x_{v_i} - (\omega_{i+1} x_{u_{i+1}}) x_{v_{i+2}}.
\]

Due to the alternation of the signs in \( \tilde{f} \), Remark 3.3, Equation (4) and since \( v_1 = u_{l+1} \), it follows that

\[
\tilde{f} = \left( \prod_{k=1, \ldots, \ell} x_{z_k} \right) x_{v_1} - \left( \prod_{j=1, \ldots, \ell} x_{u_j} \right) x_{v_{l+1}} = x_{v_1} f_W.
\]

Since \( \tilde{f} \) is sum of polynomials in \( I_P \), then \( \tilde{f} \in I_P \). Observe that, by hypothesis, for \( i \neq j \), \( z_i, z_j \) do not belong to the same inner interval of \( P \), and the same fact holds for \( u_i \) and \( u_j \), with \( i \neq j \). Due to this fact and by Remark 3.4, \( f_W \notin I_P \). Therefore, \( x_{v_1} \) and \( f_W \) are zero divisors of \( K[P] \).

**Corollary 3.6.** Let \( P \) be a polyomino and \( I_P \) the polyomino ideal associated to \( P \). If there exists a zig-zag walk in \( P \), then \( I_P \) is not prime.

**Remark 3.7.** Let \( W : I_1, \ldots, I_\ell \) be a zig-zag walk of \( P \) and let \( f_W = \prod_{k=1, \ldots, \ell} x_{z_k} - \prod_{j=1, \ldots, \ell} x_{u_j} \) be its associated binomial. The ideal \( J_P \) contains the binomials associated to zig-zag walks. Indeed, by Proposition 3.5, it arises that

\[
x_{v_1} f_W \in I_P \subseteq J_P
\]

and, due to primality of \( J_P \), it follows \( f_W \in J_P \).

We give some examples to better understand the structure of \( J_P \).

**Example 3.8.** We consider the polyomino \( P \subset [(1, 1), (8, 4)] \) in Figure 2. By using

![Figure 2](image_url)

Macaulay2 [3], we computed the ideal \( J_P \) associated to \( P \). \( J_P \) has 50 generators, 46 having degree 2, corresponding to the inner 2-minors of \( P \), and 4 having degree 4 that do not belong to \( I_P \). The latter are:

\[
\begin{align*}
f_1 &= x_{(1,3)} x_{(3,1)} x_{(7,4)} x_{(8,2)} - x_{(1,2)} x_{(3,4)} x_{(7,1)} x_{(8,3)}, \\
f_2 &= x_{(1,3)} x_{(2,1)} x_{(7,4)} x_{(8,2)} - x_{(1,2)} x_{(2,4)} x_{(7,1)} x_{(8,3)}, \\
f_3 &= x_{(1,3)} x_{(3,1)} x_{(6,4)} x_{(8,2)} - x_{(1,2)} x_{(3,4)} x_{(6,1)} x_{(8,3)},
\end{align*}
\]
\[ f_4 = x_{(1,3)}x_{(2,1)}x_{(6,4)}x_{(8,2)} - x_{(1,2)}x_{(2,4)}x_{(6,1)}x_{(8,3)}. \]

The four binomials above correspond to the four zig-zag walks drawn in Figure 3. In this case, the generators of \( J_P \) in \( J_P \setminus I_P \) are all related to zig-zag walks.

Figure 3. The zig-zag walks related to \( f_1, \ldots, f_4 \).

However, we computed \( J_P \) for the polyomino \( P \subset [(1,1), (8,6)] \) in Figure 4, and we found that there are generators of degree 6 that are not related to zig-zag walks, for example

\[ g = x_{(1,4)}x_{(3,1)}x_{(4,6)}x_{(5,1)}x_{(6,6)}x_{(8,3)} - x_{(1,3)}x_{(3,6)}x_{(4,1)}x_{(5,6)}x_{(6,1)}x_{(8,4)}. \]

In Figure 5(A), we highlight the intervals related to \( g \). On the other hand, there are two zig-zag walks that arises from \( g \), as in Figure 5(B).

(A) \( g \) is not related to a zig-zag walk...  
(B) ...but there are two zig-zag walks.

Figure 5
Verifying that the non-existence of zig-zag walks is a sufficient condition for the primality of $I_P$, for any multiply connected polyomino $P$ of rank $\leq 14$, is not an easy task. In fact, the cardinality of the set of polyominoes grows exponentially with respect to the rank. In Table 1, we show the numbers of distinct free multiply connected polyominoes, the ones there are not a translation, rotation, reflection or glide reflection of another polyomino, of rank $\leq 14$, obtained by the implementation in [7] (see also [2, Chapter 6]).

| Rank | Free multiply connected polyominoes |
|------|-------------------------------------|
| 7    | 1                                  |
| 8    | 6                                  |
| 9    | 37                                 |
| 10   | 195                                |
| 11   | 979                                |
| 12   | 4663                               |
| 13   | 21474                              |
| 14   | 96496                              |

Table 1. Numbers of distinct free multiply connected polyominoes.

**Theorem 3.9.** Let $P$ be a polyomino with rank($P$) $\leq 14$. The following conditions are equivalent:

1. the polyomino ideal $I_P$ is prime;
2. $P$ contains no zig-zag walk.

**Proof.** (1) $\Rightarrow$ (2) It is an immediate consequence of Corollary 3.6

(2) $\Rightarrow$ (1) By Corollary 3.6 simple polyominoes have no zig-zag walk, since they are prime. Therefore, we have to consider only multiply connected polyominoes. We prove that if $P$ is a non prime multiply connected polyomino with rank($P$) $\leq 14$, then $P$ has a zig-zag walk. To this aim, we implemented a computer program that performs the following 3 steps:

1. Compute the set of all multiply connected polyominoes with rank $\leq 14$, namely $P$.
2. Compute the set of polyominoes $NP \subset P$ whose polyomino ideal is not prime. We used a routine developed in Macaulay2 [3].
3. Verify that all polyominoes in NP have at least one zig-zag walk.

We refer to [7] for a complete description of the algorithm that we used.

4. Grid Polyominoes

From a view point of finding a new class of prime polyomino ideals, due to Corollary 3.6 it is reasonable to consider multiply connected polyominoes without zig-zag walks. In this section, we consider polyominoes obtained by subtracting some inner intervals by a given interval of $\mathbb{N}^2$, similarly as done in [8] and [11]. However, if the cells are removed without a specific pattern, one can easily obtain a zig-zag walk in this case, too (see Figure 6(B)). Hence we define an infinite family of polyominoes with no zig-zag walks by their intrinsic shape: the grid polyominoes. We prove that their polyomino ideal is prime by using Gröbner basis technique and lattice ideals. To this aim, we define the following monomial orders.

The total orders $<_1$ and $<_2$ on the vertices of $P$ induce in a natural way the following monomial orders on $S = \mathbb{K}[x_v | v \in V(P)]$, respectively:

1. $x_a <_1 x_b$ if $a <_1 b$;
2. $x_a <_2 x_b$ if $a <_2 b$. 


In [9], the author provides a necessary and sufficient condition for the set $\mathcal{M}$ of inner 2-minors to be a reduced Gröbner basis of $I_P$, where $\mathcal{P}$ is a collection of cells of $\mathbb{N}^2$. In the following, we state the result when $\mathcal{P}$ is a polyomino.

**Proposition 4.1.** Let $\mathcal{P}$ be a polyomino. $\mathcal{M}$ forms a reduced Gröbner basis of $I_P$ with respect to $<_{1_{\text{lex}}}$ if and only if for any two intervals $[a, b]$ and $[b, e]$ of $\mathcal{P}$, at least one between $[a, f]$ and $[a, g]$ is an inner interval of $\mathcal{P}$, where $f$ and $g$ are the anti-diagonal corners of $[b, e]$. Similarly, $\mathcal{M}$ forms a reduced Gröbner basis of $I_P$ with respect to $<_{2_{\text{lex}}}$ if and only if for any two inner intervals $[a, b]$ and $[e, f]$ of $\mathcal{P}$, with $d$ anti-diagonal corner of both the inner intervals, either $a, e$ or $b, f$ are anti-diagonal corners of an inner interval of $\mathcal{P}$.

Let $V(\mathcal{P}) = \{v_1, \ldots, v_n\}$. Given a monomial order $<$ such that

\[ x_{v_1} < x_{v_2} < \cdots < x_{v_n}, \]

we define $<_v$, with $v = v_k \in V(\mathcal{P})$, as the following monomial order:

\[ x_{v_k} < x_{v_{k+1}} < \cdots < x_{v_n} < x_{v_1} < x_{v_2} < \cdots < x_{v_{k-1}}. \]

From now on, we will respectively denote $<_{1_{\text{lex}}}$ and $<_{2_{\text{lex}}}$ by $<_v$.

**Definition 4.2.** Let $\mathcal{P} \subseteq I := [(1, 1), (m, n)]$ be a polyomino such that

\[ \mathcal{P} = I \setminus \{H_{ij} : i \in \{1, \ldots, r\}, j \in \{1, \ldots, s\}\}, \]

where $H_{ij} = [a_{ij}, b_{ij}]$, with $a_{ij} = ((a_{ij})_1, (a_{ij})_2)$, $b_{ij} = ((b_{ij})_1, (b_{ij})_2)$, $1 < (a_{ij})_1 < (b_{ij})_1 < m$, $1 < (a_{ij})_2 < (b_{ij})_2 < n$, and

1. for any $1 \leq i \leq r$ and $1 \leq k \leq s$, \((a_{ik})_1 = (a_{jk})_1\) and \((b_{ik})_1 = (b_{jk})_1\);
2. for any $1 \leq j \leq s$ and $1 \leq k \leq r$, \((a_{kj})_2 = (a_{kj})_2\) and \((b_{kj})_2 = (b_{kj})_2\);
3. for any $1 \leq i \leq r - 1$ and $1 \leq j \leq s - 1$, \((a_{i+1j})_1 = (b_{ij})_1 + 1\) and \((a_{ij+1})_2 = (b_{ij})_2 + 1\).

We call $\mathcal{P}$ a grid polyomino.

![A grid polyomino.](image)

![A non-grid polyomino, with a zig-zag walk.](image)

**Figure 6**

**Definition 4.3.** Let $\mathcal{P}$ be a polyomino and let $v \in V(\mathcal{P})$. We say that $v$ satisfies the condition (II) if it fulfills at least one of the following:

1. there exist two inner intervals $I = [a, b]$ and $K = [b, c]$ of $\mathcal{P}$, with $c$ upper left corner of $I$, $d$ lower right corner of $I$, $v$ upper left corner of $K$, and $g$ lower right corner of $K$, such that $J = [c, v]$ is inner interval of $\mathcal{P}$, whereas $L = [d, g]$ is not (see Figure 4 Case (I)).
There exist two inner intervals $J = [a, b]$ and $L = [c, f]$ of $P$, with $d$ lower right corner of $J$ and upper left corner of $L$, such that the interval $K = [d, v]$ having $a$ and $c$ as anti-diagonal corners is inner interval of $P$, whereas the interval $I$ having $a$ and $e$ as anti-diagonal corners is not (see Figure 7 Case (II)).

![Figure 7. Condition (II)](image)

**Proposition 4.4.** Let $P$ be a grid polyomino. For all $v \in V(P)$, $M$ forms a reduced Gröbner basis of $I_P$ with respect to either $<^1_v$ or $<^2_v$.

**Proof.** Let $P$ be a grid polyomino. We observe that $M$ forms a Gröbner basis of $I_P$ with respect to $<^1_v$ or $<^2_v$ since $P$ satisfies the conditions of Proposition 4.1.

Let $f, g \in M$, where $f = x_ax_b - x_cx_d$ is associated to the inner interval $[a, b]$ of $P$ and $g = x_px_q - x_rx_s$ is associated to the inner interval $[p, q]$ of $P$. Let $v \in V(P)$. We have to show that for each pair of inner 2-minors, $f$ and $g$, the corresponding $S$-polynomial reduces to 0 with respect to a fixed monomial order $<_v \in \{<^1_v, <^2_v\}$. In the following, we denote by $S$ the $S$-polynomial between $f$ and $g$, by $\text{in}(h)$ the leading monomial of a polynomial $h$ with respect to $<_v$, and by $f_{m,n}$ the inner 2-minor associated to the inner interval $[m, n]$ of $P$.

We leave to the reader the trivial cases $\{a, b, c, d\} \cap \{p, q, r, s\} = \emptyset$, and $|\{a, b, c, d\} \cap \{p, q, r, s\}| = 2$ where $S$ reduces to 0 since the polyomino ideal is generated by all inner 2-minors of $P$.

Note that if for all vertices $w \in \{a, b, c, d, p, q, r, s\}$ and a monomial order $<_v \in \{<^1_v, <^2_v\}$, it holds $x_w <_v x_v$ or $x_v <_v x_w$, then $S$ reduces to 0 with respect to $<_v$, since it reduces by 0 with respect to $<_v$. Therefore, we consider the cases $u <^1 v \leq u$ for $u, w \in \{a, b, c, d, p, q, r, s\}$ with $|\{a, b, c, d\} \cap \{p, q, r, s\}| = 1$.

If one of the inner intervals, namely $[a, b]$, is contained in the second one, namely $[p, q]$, $S$ reduces to 0 since the polyomino ideal is generated by all inner 2-minors of $P$. Without loss of generality, let $a \leq p$. The possible situations are:

- $a = p$, $b, d \in \{p, q, r, s\}$, $c \in \{p, r\}$.

If $v$ does not satisfy the condition (II), we fix the monomial order $<_v$. Otherwise, we fix $<_v^2$. Assume $v$ does not satisfy (II). In the following cases, denote by $<_v$ the total order $<^1_v$ on the vertices of $P$.

Let $a = p$, that is $f = x_ax_b - x_cx_d$ and $g = x_ax_q - x_rx_s$, and assume $a < r < c < d < b < s < q$ as in Figure 8.
We start observing that if $r < v \leq q$, then $\gcd(\text{in}(f), \text{in}(g)) = 1$. If $v \in \{a, r\}$, then $S = x_r x_s x_b - x_c x_d x_q$ and $\text{in}(S) = x_q x_c x_d$. Therefore,

$$S = -x_c(x_q x_d - x_s x_b) + x_s(x_b x_r - x_c x_e),$$

that is $S$ reduces to 0 with respect to $\mathcal{M}$.

Let $b = p$, then $a < c < d < b < r < s < q$, as in Figure 8.

If either $c < v \leq b$ or $r < v \leq q$, then $\gcd(\text{in}(f), \text{in}(g)) = 1$. In the other cases, namely $a \leq v \leq c$ and $b < v \leq r$, we have $S = x_a x_r x_s - x_q x_c x_d$. If $v = a$, then $\text{in}(S) = x_q x_c x_d$. By hypothesis, there exist the inner interval $[c, q]$ or $[d, q]$, with $\text{in}(f_{c,q}) = x_c x_q$ and $\text{in}(f_{d,q}) = x_d x_q$, and then

$$S = -x_d(x_c x_q - x_s x_e) + x_e(x_a x_r - x_d x_e)$$

or

$$S = -x_e(x_d x_q - x_r x_t) + x_t(x_a x_s - x_c x_t),$$

that is $S$ reduces to 0 with respect to the inner 2-minors $f_{c,q}$ and $f_{a,r}$ or $f_{d,q}$ and $f_{a,s}$. If $a < v \leq c$, then $\text{in}(S) = x_a x_r x_s$. By hypothesis, there exists the inner interval $[a, r]$ or $[a, s]$, with $\text{in}(f_{a,r}) = x_a x_r$ and $\text{in}(f_{a,s}) = x_a x_s$. Similarly, one shows that $S$ reduces to 0. If $b < v \leq r$, since $v$ does not satisfy the condition (II), $[d, s]$ is an inner interval of $\mathcal{P}$, whereas $[c, v]$ is not. Therefore, $[d, q]$ is an inner interval of $\mathcal{P}$. Since $\text{in}(S) = x_q x_c x_d$, and

$$S = x_c(x_d x_q - x_r x_t) - x_r(-x_c x_t + x_a x_s)$$

it follows that $S$ reduces to 0.

Note that when $b < v \leq r$, if there exists the inner interval $[c, v]$ but $[d, s]$ does not, then $v = r$, since $\mathcal{P}$ is a grid polyomino. Therefore, $v$ satisfies condition (II) and $S$ does not reduce to 0 with respect to $\mathcal{M}$ and $\leq^1_v$. In fact, $\text{in}(S) = x_q x_c x_d$, but the
monomials $x_cr, x_cq,$ and $x_dxq$ are not leading monomials of any inner 2-minors of $\mathcal{P}$. This situation justifies the hypothesis $v$ not satisfying the condition (II), and in particular the case (I) in Definition 4.2.

Let $b = r$. We have to distinguish two different situations: $d < p$ (see Figure 10(A)) or $p < d$ (see Figure 10(B)).

**Figure 10. Case $b = r$.**

Assume $d < p$. If $a < v \leq b$, then $\gcd(\text{in}(f), \text{in}(g)) = 1$. In the other cases, namely $b < v \leq q$, $S = x_axp, x_q - x_cxd$. When $b < v \leq s$, $\text{in}(S) = x_axp, x_q$, whereas, when $s < v \leq q$, that is $v = q$, $\text{in}(S) = x_cxd$. We consider the inner intervals $[e, q]$ and $[a, p]$. In both cases, $\text{in}(f_e, q) = x_cxs$ and $\text{in}(a, p) = x_axp$, and we have

$$S = xd(-sx + x_qx) + xq(x_axp - x_cxd),$$

that is $S$ reduces to 0.

Assume $p < d$. If $a < v \leq b$, then $\gcd(\text{in}(f), \text{in}(g)) = 1$. Otherwise, $S = x_axp, x_q - x_cxd$. Let $b < v \leq q$. First of all, note that since $\mathcal{P}$ is a grid polyomino, and since $v$ does not satisfy (II), by hypothesis, then the interval with anti-diagonal corners $a$ and $p$ is not an inner interval of $\mathcal{P}$ and $v \neq q$. Therefore, let $b < v < q$, and, in particular, $b < v \leq e$. In this case, $\text{in}(S) = x_cxd$. Since $\text{in}(f_p, c) = x_axe$, we have

$$S = x_c(xp, x_e - xdx) + x_p(xq - x_cxe),$$

that $S$ reduces to 0.

Note that $v = q$ satisfies condition (II) and $S$ does not reduces to 0, since neither $x_cxd$, nor $x_cxe$, nor $x_dx$ are leading monomials of any inner 2-minor of $\mathcal{P}$. This situation justifies the hypothesis $v$ not satisfying condition (II), and in particular the case (II) in Definition 4.2.

Let $d = q$, with $a < c < p < r < s < d < b$ (see Figure 11). If either $a < v \leq c$ or $r < v \leq d$, then $\gcd(\text{in}(f), \text{in}(g)) = 1$. Otherwise, $S = x_axb, x_p - x_cxe$. If $c < v \leq p$, then $\text{in}(S) = x_cxe$. Since $\text{in}(f_a, e) = x_cxe$, we have

$$S = x(sx_cxe - x_cxe) + x_a(xp, x_p - x_cxe),$$

that is $S$ reduces to 0. If $p < v \leq r$, then $\text{in}(S) = x_axb, x_p$. Since $\text{in}(f_p, b) = x_ixb$, we have

$$S = x_a(xp, x_p - x_cxe) - x_a(xq - x_cxe),$$

that is $S$ reduces to 0. Let $d < v \leq b$, that is $v = b$. Since $b$ satisfies the condition (II), we do not consider this case.
Let \( c = r \), with \( a < p < c < d < b < s < q \) (see Figure 12). If either \( a \leq v \leq c \) or \( b < v \leq q \), then \( \text{gcd}(\text{in}(f), \text{in}(g)) = 1 \). Otherwise, namely \( c < v \leq b \), we have \( S = x_ax_bx_c - x_dx_px_q \) and \( \text{in}(S) = x_dx_px_q \).

Due to \( v \) does not satisfy the condition (II), \( v \neq b \), that is \( c < v \leq e \). Since \( a < p < c \), then \( \text{in}(f_{a,e}) = x_px_d \), we have

\[
S = x_q(x_ax_c - x_px_d) - x_a(x_cx_q - x_px_a).
\]

We leave to the reader to check, in a similar way, that if \( b \in \{q, s\} \), \( d \in \{p, r, s\} \), and \( c = p \), then all the \( S \)-polynomials reduce to 0, and no one of the corners in these cases satisfy the condition (II).

Note that the orders \(<_{\text{ex}}^1\) and \(<_{\text{ex}}^2\) are symmetric, i.e. if \([a, b]\) is an inner interval of \( P \) with anti-diagonal corners \( c \) and \( d \), then \( \text{in}_{<_{\text{ex}}^1}(f_{a,b}) = x_ax_b \) and \( \text{in}_{<_{\text{ex}}^2}(f_{a,b}) = x_cx_d \). This property reflects naturally on the orders \(<_{v}^1\) and \(<_{v}^2\). Hence, it is possible to verify that all \( S \)-polynomials of inner 2-minors of \( P \) reduce to 0 with respect to \( \mathcal{M} \) and \(<_{v}^2\), except for these four cases:

A) when \( b = q \) and \( v = r \) (see Figure 13(A)), analogous to the case \( c = r \) and \( v = b \) treated above;

B) when \( b = r \) and \( v = c \) (see Figure 13(B)) analogous to the case \( b = r \) and \( v = q \) treated above;

C) when \( c = p \) and \( v = r \) (see Figure 13(C)) analogous to the case \( d = q \) and \( v = b \) treated above;

D) when \( d = r \) and \( v = b \), and the interval of \( P \) with anti-diagonal corners \( v, q \) is an inner interval of \( P \), but the one with anti-diagonal corners \( a \) and \( p \) is not (see Figure 13(D)), analogous to the case \( b = p \) and \( v = r \) treated above.

Since \( P \) is a grid polyomino, it does not happen simultaneously that \( v \) satisfies the condition (II) and one of the above situations A)–D). This implies that if \( v \) satisfies
the condition (II) and we fix the monomial order $<_v$, all $S$-polynomials reduce to 0 and $\mathcal{M}$ is a reduced Gröbner basis of $I_P$.

\[\square\]

We recall that given a lattice $\Lambda \subseteq \mathbb{Z}^m \times \mathbb{Z}^n$, is defined a binomial ideal $I_\Lambda$ called the lattice ideal of $\Lambda$ such that

\[x^a - x^b \in I_\Lambda \iff a - b \in \Lambda.\]

We say that a lattice $\Lambda$ is saturated if for any $a \in \mathbb{Z}^m \times \mathbb{Z}^n$, $c \in \mathbb{Z}$ such that $ca \in \Lambda$, we have $a \in \Lambda$. It is known that $\Lambda$ is saturated if and only if $I_\Lambda$ is prime. Let $P \subseteq [(1, 1), (m, n)]$ be a polyomino. Let

\[B = \{e_{ij} : i \in \{1, \ldots, n\}, j \in \{1, \ldots, n\}\}\]

be the canonical basis of $\mathbb{Z}^{m \times n}$ and let $C = \{C_1, \ldots, C_r\}$ be the set of cells of $P$. Let $\alpha : C \rightarrow \mathbb{Z}^{m \times n}$ such that $\alpha(C_k) = c_k = e_{ij} + e_{i+1j+1} - e_{i+1j} - e_{ij+1}$ where $(i, j)$ is the lower left corner of the cell $C$.

It is known from [1] that an ideal generated by any set of adjacent 2-minors of a $m \times n$ matrix is a lattice ideal and that its corresponding lattice is saturated. Hence, the lattice $\Lambda = \langle \{c_k\}_{k=1}^r \rangle$ is a saturated lattice, and $I_\Lambda$ is a prime ideal.

In addition, it is known from [9] that for a collection $P$ of cells of $\mathbb{N}^2$, $I_P$ is prime if and only if $I_\Lambda = I_P$. Moreover,

**Lemma 4.5.** Let $P$ be a collection of cells of $\mathbb{N}^2$, let $S$ be the polynomial ring associated to $P$. Then, there exists a monomial $u \in S$ such that

\[I_\Lambda = (I_P : u).\]
Proof. \(\supseteq\). It holds for any monomial \(u \in S\), since \(I_P \subseteq I_A\) and \(I_A\) is a prime ideal. \(\subseteq\). Let \(f_E = x^{E^+} - x^{E^-}\) be a generator of \(I_A\), with

\[
E = E^+ - E^- = \sum_{k=1}^{r} \lambda_k c_k = \sum_{k=1}^{r} ((\lambda_k c_k)^+ - (\lambda_k c_k)^-) \in \Lambda,
\]

where \(v^+\) denotes the vector obtained from \(v \in \mathbb{Z}^{m \times n}\) by replacing all negative components of \(v\) by zero, and \(v^- = -(v - v^+)\).

Let \(\lambda = \sum_{k=1}^{r} (\lambda_k c_k)^+ - E^+ = \sum_{k=1}^{r} (\lambda_k c_k)^- - E^-\). We have that all the components of \(\lambda\) are non-negative, as for any \(k \in \{1, \ldots, r\}\) one has \((c_k^+)^{i_j} \geq (c_k)^{i_j}\), for all \(1 \leq i \leq m\) and \(1 \leq j \leq n\). This implies that the monomial \(x^\lambda \in S\) is such that

\[
x^\lambda(x^{E^+} - x^{E^-}) = \prod_{k=1}^{r} x^{(\lambda_k c_k)^+} - \prod_{k=1}^{r} x^{(\lambda_k c_k)^-} = \sum_{k=1}^{r} \mu_k (x^{c_k^+} - x^{c_k^-}) \in I_P.
\]

If we set \(u\) as the least common multiple of the elements \(x^\lambda\) induced by all the generators \(f_E\) of \(I_A\), the assertion follows. \(\square\)

An immediate consequence is the following

**Corollary 4.6.** Let \(\mathcal{P}\) be a polyomino. Then \(I_{\mathcal{P}} \subseteq J_{\mathcal{P}}\).

**Proof.** Since \(J_{\mathcal{P}}\) is a prime ideal and \(I_{\mathcal{P}} \subseteq J_{\mathcal{P}}\), then for any monomial \(u \in S\), we have

\[
(I_{\mathcal{P}} : u) \subseteq J_{\mathcal{P}}.
\]

From Lemma 4.5, the assertion follows. \(\square\)

We do not know anything about the inclusion \(J_{\mathcal{P}} \subseteq I_{\mathcal{P}}\), that leads to the following

**Question 4.7.** Let \(\mathcal{P}\) be a polyomino. \(I_{\mathcal{P}}\) is prime if and only if \(I_{\mathcal{P}} = J_{\mathcal{P}}\)?

We now prove the main theorem of this section.

**Theorem 4.8.** Let \(\mathcal{P}\) be a grid polyomino. Then \(I_{\mathcal{P}} = I_A\).

**Proof.** By Proposition [4.4] for all \(v \in V(\mathcal{P})\), there exists a monomial order \(<_v\) such that \(x_v\) is the smallest variable with respect to \(<_v\) and \(\mathcal{M}\) forms a reduced Gröbner basis of \(I_{\mathcal{P}}\) with respect to \(<_v\). Fix \(v \in V(\mathcal{P})\). By [12] Lemma 12.1], the reduced Gröbner basis of \((I_{\mathcal{P}} : x_v)\) with respect to \(<_v\) is given by

\[
\{f \in \mathcal{M} | x_v \text{ does not divide } f\} \cup \{f/x_v | f \in \mathcal{M} \text{ and } x_v \text{ divides } f\}.
\]

Since no \(f \in \mathcal{M}\) can be divided by \(x_v\), the reduced Gröbner basis of \((I_{\mathcal{P}} : x_v)\) with respect to \(<_v\) is \(\mathcal{M}\). Therefore \((I_{\mathcal{P}} : x_v) = I_{\mathcal{P}}\), for all \(x_v \in V(\mathcal{P})\). It follows that \((I_{\mathcal{P}} : u) = I_{\mathcal{P}}\) for any monomial \(u \in S\). By Lemma 4.5 we have that there exists a monomial \(u \in S\) such that \(I_A = (I_{\mathcal{P}} : u)\). Then

\[
I_A = (I_{\mathcal{P}} : u) = I_{\mathcal{P}}.
\]

\(\square\)

**Corollary 4.9.** Let \(\mathcal{P}\) be a grid polyomino. Then \(I_{\mathcal{P}}\) is prime.

From the main results of this work, that are Corollary 3.6, Theorem 3.9 and Corollary 4.9, it arises naturally the following:
Conjecture 4.10. Let $\mathcal{P}$ be a polyomino. The following conditions are equivalent:

1. the polyomino ideal $I_{\mathcal{P}}$ is prime;
2. $\mathcal{P}$ contains no zig-zag walk.

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