Relationship between the Bregman divergence and beta-divergence and their Applications

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Abstract

The Bregman divergence have been the subject of several studies. We do not go to do an exhaustive study of its subclasses, but propose a proof that shows that the $\beta$-divergence are subclasses of the Bregman divergences. It is in this order of idea that we will make a proposition of demonstration which shows that the $\beta$-divergence are particular cases of the Bregman divergence. And also we will propose algorithms and their applications to show the consistency of our approach.

This is of interest for numerous applications since these divergences are widely used for instant non-negative matrix factorization (NMF).

Key words: Bregman-divergence, $\beta$-divergence, non negative matrix factorization (NMF).

1 Introduction

Divergence are used in this work to evaluate the disimilarities (similarity) between two objects.

Bregman divergence is know a generalization of some divergences. For example, the Kullback-Leibler (KL) divergence, the Itakura-Saito (IS) divergence etc. The Bregman divergence are several application for example in pattern reconnaissance, classification and clustering etc. In non-negative matrix factorization divergence (NMF) (Lee and Sun 1999), divergence are used as cost function:

For given a data matrix $V$ of dimensions $F \times N$ with nonnegative entries, NMF is the problem of finding a factorization

$$V \approx WH$$

where $W$ and $H$ are non-negative matrices of $F \times K$ and $K \times N$, respectively. $K$ is usually chosen such that $FK + KN \ll FN$, hence reducting the data dimension. The factorization is in general only approximate, so that the terms ”approximate non-negative matrix factorization” or ”nonnegative matrix approximation” also appear in the literature.

NMF provides a lower rank approximation of a matrix. It is a dimensionality reduction method. It approximates a matrix $V$ by a product of two lower rank matrices $W$ and $H$ with non-negative entries minimizing the divergence between $V$ and $WH$. There are many ways to quantify the difference between $V$ and $WH$. NMF has been used for various problems in diverse fields, to cite a few, let us mention the problem of learning parts of face and semantic features of text (Lee and Seung 1999), in numerous areas such as image processing (Daniel D.Lee 1999) polyphonic music transcription (Smaragdis and Brown 2003), text mining (V.Paul and al 2004), DNA gene expression analysis (Brunet and al 2004), spectroscopy (Cyril Gobinet and al 2004), email surveillance (Michael W. Berry and Murray Browne 2005), spectral data analysis (Micheal W.Berry and al 2006), scalable internet distance prediction (Yun Mao and al 2006), object characterization by reflectance spectra analysis (Berry and al 2007), portfolio diversification (Drakakis and al 2007), non-stationary speech denoising (Schmidt M.N and al 2007), biomathematics (Hyunsoo Kim and Hasun Park 2007) clustering of protein interaction (Greene and al,2008), audio processing (Cédric Févotte and al 2009), in signal processing (Liangda,Guy Lebanon and Haesun Park 2012), in hierarchical reinforcement learning and task decomposition in humans (Diuk and al 2013), in lifelong reinforcement learning (Brunskill and al 2014), in MLMDP (Adam C.Earle and al 2017) etc.
The factorization (1) is usually sought after through the minimization problem
\[
\min_{W,H} D_{\phi}(V|WH), W \geq 0, H \geq 0 \tag{2}
\]
where the notation \( A \geq 0 \) expresses non-negative of the entries of matrix \( A \) (and not semi definite positiveness), and where \( D(V|WH) \) is a separable measure of fit such that
\[
D_{\phi}(V|WH) = \sum_{f=1}^{F} \sum_{n=1}^{N} d([V]_{fn}|[WH]_{fn}) \tag{3}
\]
where \( d(x|y) \) is a scalar cost function. What we intend by "cost function" is a positive function of \( y \in \mathbb{R}^+ \) given \( x \in \mathbb{R}^+ \), with a single minimum for \( x = y \). (see Févotte and al 2011).

The generalized divergences which encompass these classical divergence (KL, IS, ..), was given by many authors in the literature:

- Csiszar’s divergence (Andrzej Cichocki and al 2006), which is a generalization of Amari’s \( \alpha \)-divergence (Andrzej Cichocki 2007). Both these divergences encompass the KL divergence and its dual.

- AB-divergence ( generalized Alpha-Beta-divergence ) (Andrzej Cichocki, Sergio Cruces, and Shun-ichi Amari January 2011 ), which encompass the KL divergence, Hellinger distance, Jensen-Shannon divergence, J-divergence, Chi-square divergence, Triangular discrimination and Arithmetic-Geometric divergence.

- Bregman divergence (L.M Bregman 1967), (Inderjit S. Dhillon and Survit Sra 2006), which encompass the EUC distance , the KL divergence and the IS divergence.

- \( \beta \)-divergence , introduced by Basu and al (1998) and Eguchi and Kano (2001), which also encompass the EUC distance , KL divergence and the IS divergence.

In this paper, we will improve the proof that the \( \beta \)-divergence is a particular case of Bregman divergence by given several ways to prove the \( \beta \)-divergence is in fact the subclass of Bregman divergence. This result is assumed to be known in a certain community , (Frank Nielsen and Richard Nock 2009) and (Andrzej Cichocki June 2010), (Romain Hennequin and al 2011).

The remain of this paper is organized as following. Section 2 , we introduced definitions and basic notion of the Bregman divergence and \( \beta \)-divergence . Section 3, we showed how the Bregman divergence encompass the \( \beta \)-divergence . Section 4 and 5, we presente our results and some applications in the nonnegative matrix factorization.

2 Definitions and Notations

In this section , we define the concept of divergence, element-wise divergence, and the particular case of Bregman divergence and \( \beta \)-divergence. Divergences are distance-like functions which measure the separation between two elements.

**Definition 2.1.** Let \( S \) be a set. A divergence on \( S \) is a function \( D : S \times S \to \mathbb{R} \) satisfying: \( \forall (p,q) \in S \times S, D(p|q) \geq 0 \) and \( D(p|q) = 0 \) if \( p = q \).

As a distance , a divergence should be non-negative and separable. However, a divergence does not necessarily satisfy the triangle inequality and the symmetry axiom of a distance. In order to avoid the confusion with distances, the notation \( D(p|q) \) is often used instead of the classical distance notation \( D(p,q) \).

2.1 Bregman divergence

**Definition 2.2.** Let \( S \) be a convex subset of a Hilbert space and \( \Phi : S \to \mathbb{R} \) a continuously differentiable strictly convex function. The Bregman divergence (L.M. Bregman 1967) \( D_{\Phi} : S \times S \to \mathbb{R}_+ \) (where \( \mathbb{R}_+ \) is the set of non-negative real numbers) is defined as
\[
D_{\Phi}(x,y) = \Phi(x) - \Phi(y) - <x-y, \nabla \Phi(y)>
\]
where $\nabla \Phi(y)$ stands for the gradient of $\Phi$ evaluated at $y$ and $\langle \ldots \rangle$ is the standard Hermitian dot product.

The value of Bregman divergence $D_\Phi(x||y)$ can be viewed as the difference between the function $\Phi$ and its first order Taylor series at $y$. Thus, adding an affine form to $\Phi$ does not change $D_\Phi$.

### 2.2 Element-wise divergence

#### Definition 2.3.

In this section, $S = \mathbb{R}_+^N$ or $S = (\mathbb{R}_+ \setminus \{0\})^N$. On such sets, one can define element-wise divergence: a divergence on $\mathbb{R}_+^N$ (resp. $(\mathbb{R}_+ \setminus \{0\})^N$) is called element-wise if there exists a divergence $d$ on $\mathbb{R}_+^N$ (resp. $(\mathbb{R}_+ \setminus \{0\})^N$) such that:

$$D(x||y) = \sum_{n=1}^{N} d(x_n|y_n)$$

### 2.3 Element-wise Bregman divergence

Element-wise Bregman divergence are a subclass of Bregman divergence for which $\Phi$ is the sum of $N$ scalar, continuously differentiable and strictly convex element-wise function:

$$\forall x = (x_1, \ldots, x_N) \in S, \Phi(x) = \sum_{n=1}^{N} \phi(x_n)$$

Then $D_\Phi(x||y) = \sum_{i=1}^{N} d_\phi(x_i)$ where $d_\phi(x|y) = \phi(x) - \phi(y) - \phi'(y)(x - y)$ and thus, the divergence is element-wise. For element-wise Bregman divergence, we can equivalently denote the divergence $D_\Phi$ or $D_\phi$.

#### Definition 2.4.

Let $\beta \in \mathbb{R}$. The $\beta$-divergence on $\mathbb{R}_+ \setminus \{0\}$ was introduced by (Basu and al 1998) and (Eguchi and Kano 2001) and can be defined as:

$$d_\beta(x|y) = \begin{cases} 
  x^{\beta+1} - x^\beta y^{\beta-1} & \beta \in \mathbb{R} \setminus \{1\} \\
  x(\log x - \log y) + (y - x) & \beta = 1 \\
  \frac{x}{y} - \log(\frac{x}{y}) - 1 & \beta = 0
\end{cases}$$ (4)

#### Lemma 1. (see Schoenberg)

A metric $d^2(x|y)$ is $\lambda$-homogeneous if and only if they satisfy: for $\forall c \in \mathbb{R}_+$

$$d(cx|cy) = c^\lambda d(x|y)$$

We give some proprieties of the $\beta$-divergence and for more detailed exposition see Cichocki and Amari (2010).

#### Properties 1.

- One should notice that the previous definition of $\beta$-divergence is continuous with respect to $\beta$ in the sense that:

$$\forall \beta_0 \in \mathbb{R}_+ \setminus \{0\}, \lim_{\beta \to \beta_0} d_\beta(x,y) = d_{\beta_0}(x,y)$$

particularly for $\beta_0 = 0$ and $\beta_0 = 1$

- The first and second derivative of $d_\beta(x|y)$ w.r.t $y$ are continuous in $\beta$, and write:

$$d'_\beta(x|y) = y^{\beta-2}(y - x)$$

$$d''_\beta(x|y) = y^{\beta-3}[(\beta - 1)(y - 1)y - (\beta - 2)x]$$

. The derived shows that $d_\beta(x|y)$ as a function of $y$, has a single minimum in $x = y$. The second derivative shows that the $\beta$-divergence is convex w.r.t $y$ for $\beta \in [1, 2]$
It is easy to check that the equality $d_{\beta}(x|y) = \lambda^2 d_{\beta}(x|y)$

*(see Févotte and al. 2009)*

Form this divergence on $\mathbb{R}_+ \setminus \{0\}$, on can define an element-wise $\beta$-divergence on $(\mathbb{R}_+ \setminus \{0\})^N$:

$$D_{\beta}(x||y) = \sum_{n=1}^{N} d_{\beta}(x|y)$$

### 3 $\beta$-Divergence as a Bregman divergence

In this section, we show that the Bregman divergence encompasses the $\beta$-divergence.

The $\beta$-divergence belongs to the family of Bregman divergences. For $\beta \in \mathbb{R} \setminus \{1, 0\}$, a suitable Bregman generating function is $\phi(x) = y^\beta/(\beta(\beta - 1))$, as noted by (Févotte and Cemgil 2009). This function however, cannot generate the IS-KL divergence by continuous when $\beta$ tends to 0 or 1. The latter divergence may nonetheless be generated "separately", using the function $\phi_{y} = -\log y$ and $\phi_{y}(y) = y \log y$, respectively. Cichocki and Amari (2010) give a general Bregman generating function of the $\beta$-divergence, defined for all $\beta \in \mathbb{R}$, in the form of $\phi_{\beta}(y) = (y^\beta - \beta y + 1)/(\beta(\beta - 1))$, $\phi_{\beta=0}(y) = y - \log y - 1$ and $\phi_{\beta=1}(y) = y \log y - y + 1$.

We will present the both $\phi_{\beta}$ to show that the $\beta$-divergence is a subclass of Bregman divergence.

**Proposition 3.1.** For $\beta \in \mathbb{R}$, let $\phi_{\beta} : \mathbb{R}_+ \setminus \{0\} \rightarrow \mathbb{R}$ be the function defined as:

$$\forall y \in \mathbb{R}_+ \setminus \{0\}, \phi_{\beta}(y) = \begin{cases} \frac{y^\beta}{\beta(\beta - 1)} & \beta \in \mathbb{R} \setminus \{1, 0\} \\ y \log y & \beta = 1 \\ -\log y & \beta = 0 \end{cases}$$

If $\phi_{\beta}$ is strictly convex ( $\phi''(y) = y^{\beta-2} \geq 0$), then $D_{\phi}(x||y) = d_{\beta}(x|y)$ ie we can define the Bregman divergence $D_{\phi_{\beta}}$ associated to $\phi_{\beta}$.  

**Proof 1.** For $\beta \in \mathbb{R} \setminus \{0,1\}$:

$$d_{\phi_{\beta}}(x|y) = \frac{x^\beta}{\beta(\beta - 1)} - \frac{x^{\beta - 1}}{\beta - 1} + \frac{y^\beta}{\beta(\beta - 1)} + \frac{y^{\beta - 1}}{\beta - 1} - (\frac{y^{\beta - 1}}{\beta - 1} - \frac{1}{\beta - 1})(x - y)$$

$$= d_{\beta}(x|y)$$

It is easy to check that the equality $d_{\phi_{\beta}}(x|y) = d_{\phi_{\beta}}(x|y)$ also holds for $\beta \in \{0,1\}$:

$$d_{\phi_{0}}(x|y) = -\log x + x - (\log y + y) = -\frac{1}{y} + 1) - (x - y)$$

$$= -\log x + \log y + (x - y) + \frac{x}{y} - 1 - (x - y)$$

$$= -\log(x/y) + \frac{x}{y} - 1$$

$$= d_{0}(x|y)$$

$$d_{\phi_{1}}(x|y) = (x \log x + 1 - (y \log y + y + 1) - \log y(x - y)$$

$$= x(\log x - \log y) + (y - x)$$

$$= d_{1}(x|y)$$

□
Proposition 3.2. For $\beta \in \mathbb{R}$, let $\phi_\beta : \mathbb{R}_+ \setminus \{0\} \to \mathbb{R}$ be the function defined as:

$$
\forall y \in \mathbb{R}_+ \setminus \{0\}, \phi_\beta(y) = \begin{cases} 
\frac{y^\beta - 2y + \beta - 1}{\beta(\beta - 1)} & \beta \in \mathbb{R} \setminus \{1, 0\} \\
y \log y - y + 1 & \beta = 1 \\
-\log y + y - 1 & \beta = 0
\end{cases}
$$

(8)

If $\phi_\beta$ is strictly convex ($\phi''(y) = y^{\beta - 2} \geq 0$), then $D_\phi(x||y) = d_\beta(x||y)$ ie we can define the Bregman divergence $D_{\phi_\beta}$ associated to $\phi_\beta$

**Proof 2.** For $\beta \in \mathbb{R} \setminus \{0, 1\}$ $\phi_\beta(y) = \frac{y^\beta}{\beta(\beta - 1)}$

$$
d_\phi(x||y) = \phi_\beta(x) - \phi_\beta(y) - \phi'_\beta(y)(x - y)
$$

$$
d_\phi(x||y) = \frac{x^\beta}{\beta(\beta - 1)} - \frac{y^\beta}{\beta(\beta - 1)} - \frac{\beta y^{\beta - 1}}{\beta(\beta - 1)}(x - y)
$$

$$
d_\phi(x||y) = \frac{1}{\beta(\beta - 1)}(x^\beta - y^\beta - \beta y^{\beta - 1}x + \beta y^\beta)
$$

For $\beta = 1$

$$
d_\phi(x||y) = x \log x - y \log y - (\log y + 1)(x - y)
$$

$$
d_\phi(x||y) = x \log x - y \log y - x \log y - x + y \log y + y
$$

$$
= x \log x - x \log y - x + y
$$

$$
x (\log x - \log y) - x + y
$$

$$
d_\phi(x||y) = x \log \left(\frac{x}{y}\right) - x + y
$$

$$
d_\phi(x||y) = d_\beta(x||y)
$$

For $\beta = 0$

$$
d_\phi(x||y) = -\log x + \log y + \frac{1}{y}(x - y)
$$

$$
d_\phi(x||y) = \log y - \log x = \frac{x}{y} - 1
$$

$$
d_\phi(x||y) = \log \left(\frac{y}{x}\right) + \left(\frac{x}{y}\right) - 1 = -\log \left(\frac{x}{y}\right) + \left(\frac{x}{y}\right) - 1
$$

$$
d_\phi(x||y) = \frac{x}{y} - \log \left(\frac{x}{y}\right) - 1
$$

$$
d_\phi(x||y) = d_\beta(x||y)
$$

□

To give a full demonstration of this theorem we need the following proprieties:

**Lemma 2.** (Romain Hennequin 2011)

- This function $\phi_\beta$ in this case is continuous with respect to $\beta$ in the sense that:

$$
\forall \beta_0 \in \mathbb{R}, \forall y \in \mathbb{R}_+ \setminus \{0\}, \lim_{\beta \to \beta_0} \phi_\beta(y) = \phi_{\beta_0}(y)
$$

- For all $\beta \in \mathbb{R}, \phi_\beta$ is smooth on $\mathbb{R}_+ \setminus \{0\}$ and is second derivative is:

$$
\phi''_\beta(y) = y^{\beta - 2}
$$

(9)

Thus $\phi_\beta$ is strictly convex and one can define the Bregman divergence $D_{\phi_\beta}$ associated to $\phi_\beta$:

$$
D_{\phi_\beta}(x||y) = \sum_{n=1}^{N} \phi_\beta(x_n) - \phi_\beta(y_n) - \phi'_\beta(y_n)(x_n - y_n)
$$

Straightforward calculations show that for all

$$
\beta \in \mathbb{R}, D_{\phi_\beta} = D_{\beta}
$$

is a $\beta$-divergence. Thus the Bregman divergence encompasses $\beta$-divergence.
4 Main Results

We proposed the function to prove that the $\beta$-divergence is a subclass of Bregman divergence, for the prove we used the definition \ref{2.2}.

Theorem 4.1. For $\beta \in \mathbb{R}$, let $\phi_\beta : \mathbb{R}_+ \setminus \{0\} \rightarrow \mathbb{R}$ be the function defined as

\[
\phi_\beta(y) = \begin{cases} 
  y^{1-\beta} + (\beta - 1)y + \beta & \beta \in (-\infty, 0) \cup (1, +\infty) \\
  (1 - \beta)y - y^{1-\beta} - \beta & \beta \in (0, 1) \\
  y - \log y - 1 & \beta = 1 \\
  y\log y - y + 1 & \beta = 0 
\end{cases} \tag{10}
\]

If $\phi_\beta$ is strictly convex, then $D_\phi(x||y) = d_\beta(x|y)$ ie we can define the Bregman divergence $D_{\phi_\beta}$ associated to $\phi_\beta$.

Proof 3. For all $\beta \in \mathbb{R}, \phi_\beta$ is smooth on $\mathbb{R}_+ \setminus \{0\}$ and is second derivative is :

\[
\phi''(y) = \begin{cases} 
  \frac{\beta(\beta-1)}{y^{\beta+1}} & \beta \in (-\infty, 0) \cup (1, +\infty) \\
  \frac{\beta(1-\beta)}{y^{\beta+1}} & \beta \in (0, 1) 
\end{cases} \tag{11}
\]

Thus $\phi_\beta$ is strictly convex and one can define the Bregman divergence $D_{\phi_\beta}$ associated to $\phi_\beta$:

\[
D_{\phi_\beta}(x||y) = \sum_{n=1}^{N} \phi_\beta(x_n) - \phi_\beta(y_n) - \phi'_\beta(y_n)(x_n - y_n)
\]

Straightforward calculations show that for all $\beta \in \mathbb{R}$, $D_{\phi_\beta} = D_\beta$

is a $\beta$-divergence. Thus the Bregman divergence encompasses $\beta$-divergence.

For $\beta = 0$

\[
d_{\phi_0}(x|y) = x\log x - x + 1 - (y\log y - y + 1) - (\log y)(x - y)
\]

\[
d_{\phi_0}(x|y) = x\log \left( \frac{x}{y} \right) - (x - y)
\]

\[
d_{\phi_0}(x|y) = d_{\beta=0}(x|y)
\]

For $\beta = 1$

\[
d_{\phi_1}(x|y) = (x - \log x - 1) - (y - \log y - 1) - (1 - \frac{1}{eta})(x - y)
\]

\[
d_{\phi_1}(x|y) = -\log \left( \frac{x}{y} + \frac{1}{eta} - 1 \right)
\]

\[
d_{\phi_1}(x|y) = d_{\beta=1}(x|y)
\]

For $\beta \in (-\infty, 0) \cup (1, +\infty)$

\[
d_{\phi_\beta}(x|y) = (x^{1-\beta} + (\beta - 1)x + \beta) - (y^{1-\beta} + (\beta - 1) - ((1 - \beta)y^{-\beta} + (\beta - 1))(x - y)
\]

\[
d_{\phi_\beta}(x|y) = \frac{1}{\beta(1-\beta)}(\beta(1 - \beta)x^{1-\beta} + \beta^2y^{1-\beta} - \beta y^{-\beta}x)
\]

\[
d_{\phi_\beta}(x|y) = d_\beta(x|y)
\]

We have a similar demonstration for $\beta \in (0, 1), d_{\phi_\beta}(x|y) = d_\beta(x|y)\square$

Proposition 4.2. We obtain the $\beta$ divergence defined by:

\[
d_\beta(x|y) = \begin{cases} 
  \frac{\beta(\beta-1)x^{1-\beta} + \beta^2y^{1-\beta} - \beta y^{-\beta}x}{\beta(1-\beta)} & \beta \in \mathbb{R} \setminus \{0, 1\} \\
  \frac{x}{y} - \log \left( \frac{x}{y} \right) - 1 & \beta = 1 \\
  x\log \left( \frac{x}{y} \right) - y - x & \beta = 0 
\end{cases} \tag{12}
\]
Using the lemma[?] 1 we obtained the IS divergence ($\beta = 1$) is scale-invariant i.e., $d_{IS}(\lambda x|\lambda y) = d_{IS}(x|y)$, and it is the only one in the family of $\beta$-divergence.

We show that for $\beta \in \mathbb{R}$, $D_{\phi_{\beta}} = D_{\beta}$ is a $\beta$-divergence. Thus the Bregman divergence encompasses $\beta$-divergence.

The specific choice of $\phi_{\beta}$ depends on the application. For example, the Euclidian distance ($\beta = 2$) has been used successfully in text clustering (A. Banerjee December 2005), KL divergence is well suited for many problems in signal processing (A. Cichocki and A. H.Phan 2009) while IS divergence has been shown to perform well in music recommendation (C. Févotte March 2009). For other choices of $\phi_{\beta}$ and areas where these divergences are utilized (C. Févotte USA 2009).

5 Applications

5.1 In Non-negative Matrix Factorization

In this section, we presented our multiplicative update rules Scalar Block Coordinate Descent (SBCD) Algorithm and their applications.

The multiplication update rules

The multiplication update rule of $H$ (resp. $W$) : for minimizing a Bregman divergence cost function given by Inderjit S.Dhillon and Suvrit Sra December (2006) is as

$$H \leftarrow H \frac{W^T (\phi (WH)V)}{W^T (\phi (WH)WH)}$$  \hspace{1cm} (13)

resp. $W$

$$W \leftarrow W \frac{(\phi (WH)V)H^T}{(\phi (WH)WH)H^T}$$  \hspace{1cm} (14)

The product ".", the fraction bar and $\phi$ are element wise operation on the corresponding matrices. We can directly derive the results (already well known by Andrzej Cichocki, Rafa Zdunek and Sun-Ichi Amari March 2006). This illustrates the interest of deriving general properties about the Bregman divergence instead of the $\beta$-divergence.

Algorithms

The above expression is a generalization of the algorithm of Bregman divergence. When we replacing our $\phi_{\beta}$ function in the algorithm we obtain the follows expression.

$$H \leftarrow H \frac{W^T [(WH)^{-\beta-1}V]}{W^T [(WH)^{-\beta}]}$$  \hspace{1cm} (15)

Resp. $W$

$$W \leftarrow W \frac{[(WH)^{-\beta-2}V]H^T}{[(WH)^{-\beta}]H^T}$$  \hspace{1cm} (16)

Generally, in NMF given a matrix $V = [v_{ij}] \in \mathbb{R}^{F \times N}$, and an integer $K \leq \min (F,N)$, and we are to find $W = [w_{1}, w_{2}, ..., w_{K}] \in \mathbb{R}^{F \times K}$ and $H = [h_{1}, h_{2}, ..., h_{K}] \in \mathbb{R}^{N \times K}$, such that

$$V \approx V' = WH$$  \hspace{1cm} (17)

| Updating rules for particularly case of Bregman divergence |
|----------------------------------------------------------|
| Name           | Function $\phi(y)$ | $\phi^\beta(y)$ | $D_{\phi}(v|v')$ | updating rule |
|----------------|---------------------|-----------------|-----------------|---------------|
| Itakura-Saito  divergence | $y - \log(y) + 1$ | $\frac{1}{y^\beta} - \log(y) - 1$ | $\sum_{i=1}^{M} v_{ij}^\beta w_{ik}/v_{ij}$ | $h_{jk} = \frac{\sum_{i=1}^{M} v_{ij}^\beta w_{ik}/v_{ij}^\beta}{\sum_{i=1}^{M} w_{ik} w_{ik}/v_{ij}}$ |
| KL-divergence  | $y \log(y) - y + 1$ | $\frac{1}{y^\beta} - \log(y) - v - v'$ | $\sum_{i=1}^{M} v_{ij}^\beta w_{ik}/v_{ij}$ | $h_{ij} = \frac{\sum_{i=1}^{M} v_{ij}^\beta w_{ik}/v_{ij}^\beta}{\sum_{i=1}^{M} w_{ik} w_{ik}/v_{ij}}$ |
| Beta divergence | $y^{1-\beta} + (\beta - 1)y + \beta$ | $\frac{\beta(1-\beta)}{y^{1-\beta}} + \beta(1-\beta)v^{1-\beta} + \beta v^{1-\beta} - \beta y^{1-\beta} v$ | $\sum_{i=1}^{M} v_{ij}^\beta w_{ik}/v_{ij}$ | $h_{ij} = \frac{\sum_{i=1}^{M} v_{ij}^\beta w_{ik}/v_{ij}^\beta}{\sum_{i=1}^{M} w_{ik} w_{ik}/v_{ij}}$ |
The Scalar Block Coordinate (SBCD)

The solution for scalar block $h_{jk}$ is:

$$h_{jk} = \frac{\sum_{i=1}^{M} \nabla^2 \phi(v_{ij}^{(k)}) v_{ij}^{(k)} w_{ik}}{\sum_{i=1}^{M} \nabla^2 \phi(v_{ij}^{(k)}) w_{ik} w_{ik}}$$

(18)

Similarly, we have the updating rule for $w_{ik}$:

$$w_{ik} = \frac{\sum_{j=1}^{N} \nabla^2 \phi(v_{ij}^{(k)}) v_{ij}^{(k)} h_{jk}}{\sum_{j=1}^{N} \nabla^2 \phi(v_{ij}^{(k)}) h_{jk} h_{jk}}$$

(19)

The summary of the algorithm, which we refer to as SBCD (Scalar Block Coordinate Descent) is shown in the table. Note that the algorithm follows the block coordinate descent framework where each element in $W$ and $H$ is considered as a scalar block that we update in each step. The algorithm above is expressed in a general form for all Bregman divergences. Replacing $\phi(x)$ with the corresponding expression provides the specific algorithm for each specific Bregman divergence. Some particularly case updating rules are enumerated in the Table above.

**Fast algorithm for NMF:**

The two expressions show an interesting relationship between SBCD and two other NMF algorithms, Multiplicative Updating Descent methods. The fast Bregman divergence NMF given by Taylor Expansion and Coordinate Descent was presented by Liangdali, Guy Lebanon and Haesum Park (August 2012). Inspired to these works, we presented another relationship between Bregman divergence and $\beta$-divergence.

**Algorithm Scalar Block Coordinate Descent (SBCD)**

1. Given $V \in \mathbb{R}^{F \times N}$
2. $V' = WH$
3. $E = V - V'$
   repeat
4. $B = \phi''(V')$
5. for $k = 1, 2, 3, ..., K$ do
6. $V^{(k)} = E + w_{k} h_{k}$
7. for $j = 1, 2, 3, ..., N$ do
8. $h_{jk}^{T} = \left[ \frac{\sum_{i=1}^{F} b_{ij} v_{ij}^{(k)} w_{ik}}{\sum_{i=1}^{F} b_{ij} w_{ik} w_{ik}} \right]_{+}$
9. end for
10. for $i = 1, 2, 3, ..., F$ do
11. $w_{ik} = \left[ \frac{\sum_{j=1}^{N} b_{ij} h_{ij}^{(k)} h_{jk}}{\sum_{j=1}^{N} b_{ij} h_{jk} h_{jk}} \right]_{+}$
12. end for
13. $E = V^{(k)} - w_{k} h_{k}$
14. end for
15. $V' = WH$
   until stopping criterion is reached

$\overline{V} \in \mathbb{R}^{F \times N}$, a reduced dimension $K$ and function $\phi$ for Bregman divergence values for $W$ and $H$. where we denoted $[x]_{+} = \max\{x, 0\}$. 

8
Recognition

The $\beta$-divergence takes as special cases the divergence of Kullback-Leibler, the divergence of Itakura saito and euclidian ($\beta = 1; \beta = 0; \beta = 2$). The $\beta$-divergence offers a continuum of noise models interpolating these particular cases. The parameter $\beta$ thus offers a degree of freedom specific to the modeling of the data and its values can be fixed arbitrarily or learned on a learning set for a given context and application. $\beta$-divergence for $y=1$ Considered as a function of $x$ to $y$ fixed, the -divergence is convex for $1 \leq \beta \leq 2$ (figure 1), the parameter $\beta$ thus offers a degree of freedom specific to the modeling of the data and its value can be fixed arbitrarily or learned on a learning set for a context and a given application. For illustration, decomposition of face data using $\beta$-NMF has been presented by Cédric Févotte and Jérôme Idier (March 2011). Performance with various Bregman divergence also was experiments for four Bregman divergence see Liangda Li, Guy Lebanon and Haesun Park (2012).

Data spectroales

The NMF allowing to separate a non-negative signal there composantes different they too non-denials, the method is adapted well to the spectral measures. The $\beta$-divergence has often been considered in audio, for the decomposition of the spectogram into elementary components, where the value of $\beta$ can be set to optimize the results of transcription or separation of sources on learning data. For our application, the decomposition by the NMF of an acquired signal $V$ in $V \approx WH$ can be physically interpreted

\[
V = \begin{pmatrix}
v_{11} & \ldots & v_{1,N} \\
\vdots & \ddots & \vdots \\
v_{N,1} & \ldots & v_{N,N}
\end{pmatrix}
= \begin{pmatrix}
w_{11} & w_{1,2} \\
\vdots & \ddots & \vdots \\
w_{N,1} & w_{N,2}
\end{pmatrix}
\begin{pmatrix}
h_{11} & \ldots & h_{1,N} \\
h_{21} & \ldots & h_{2,N}
\end{pmatrix}
\]

When the signal is supposed trained(formd) by two different sources(springs) of fluorescence, $H$ contains itself the forms of spectres in wavelength and $W$ contains the weight of these spectres for every spatial position, as schematized on the following representation:

\[
\begin{pmatrix}
v_{11} & \ldots & v_{1,N} \\
\vdots & \ddots & \vdots \\
v_{N,1} & \ldots & v_{N,N}
\end{pmatrix}
= \begin{pmatrix}
w_{11} & w_{1,2} \\
\vdots & \ddots & \vdots \\
w_{N,1} & w_{N,2}
\end{pmatrix}
\begin{pmatrix}
h_{11} & \ldots & h_{1,N} \\
h_{21} & \ldots & h_{2,N}
\end{pmatrix}
\]

Every line of the matrix $V$ is a linear combination (overall) of spectres $H_1$ and $H_2$:row $e_i = w_{i1} \times h_{1i} + w_{i2} \times h_{2i}$; $i \in (1, N)$
So, the NMF applied to our spectral data turns a matrix $H$ which contains information on the present spectres of fluorescence of the fluorophore and the autofluorescence in the image of departure $V$ and a matrix $W$ which defines the level-headedness of these spectres of fluorescence in each of the lines of $V$. As shows him(it) the example of Anne-Sophie Montcuquet and al in the application of the Non-negative Matrix Factorization in the elimination of the autofluorescence of biological tissues (2011) is explored to ummix overlapping spectra and thus isolate the specific fluorescence signals from the autofluorescence signal.

**Data from sonar**

The data set contains patterns obtained by rebonding sonar signals on a metal cylinder at different angles and under various conditions. And the file also contains patterns obtained from rocks in similar situations. The transmitted sonar signal is a frequency modulated chirp, increasing in frequency. The data set contains signals obtained from different angles. Each pattern is a set of 60 numbers between 0.0 and 1.0. each number represents the energy in a particular frequency band integrated over a period of time. Thus, to adapt it to nonnegative matrix factorization the label associated with each record contains the number "2" if the object is a rock and "3" if the object is a mine. We will present a comparaison of various results of algorithms using the data sonar. Here the NMF computation is executed with the allowed error parth using the argument .options. The trajector of the objective value is computed during the fit. The trajectroy can be plot with the method plot (figure 2).

![NMF Residuals](image)

**6 Conclusion**

Divergence or distance are of key importance in number of theoretical and application statistical inference and data processing problems, such as estimation, detection, classification, recognition ... In this paper, we have presented a prove that the general class of Bregman divergence encompasses the $\beta$-divergence. This results permits to easily apply theorems about the Bregman divergence to the $\beta$-divergence. We have illustrated that the $\beta$-divergence is widely used in methods such as NMF which has application in numerous areas and we have given algorithms and also some examples.

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