From Entropic Dynamics to Quantum Theory*

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Abstract
Non-relativistic quantum theory is derived from information codified into an appropriate statistical model. The basic assumption is that there is an irreducible uncertainty in the location of particles: positions constitute a configuration space and the corresponding probability distributions constitute a statistical manifold. The dynamics follows from a principle of inference, the method of Maximum Entropy. The concept of time is introduced as a convenient way to keep track of change. A welcome feature is that the entropic dynamics notion of time incorporates a natural distinction between past and future. The statistical manifold is assumed to be a dynamical entity: its curved and evolving geometry determines the evolution of the particles which, in their turn, react back and determine the evolution of the geometry. Imposing that the dynamics conserve energy leads to the Schroedinger equation and to a natural explanation of its linearity, its unitarity, and of the role of complex numbers. The phase of the wave function is explained as a feature of purely statistical origin. There is a quantum analogue to the gravitational equivalence principle.

1 Introduction
Our subject has been very succinctly stated by Jaynes: “Our present QM formalism is a peculiar mixture describing in part realities in Nature, in part incomplete human information about Nature—all scrambled up by Heisenberg and Bohr into an omelette that nobody has seen how to unscramble.” He also understood where to start looking: “We suggest that the proper tool for incorporating human information into science is simply probability theory—not the currently taught ‘random variable’ kind, but the original ‘logical inference’ kind of James Bernoulli and Laplace” which he explains “is often called Bayesian inference” and is “supplemented by the notion of information entropy”. Bohr, Heisenberg and other founders of quantum theory might have agreed. They were keenly aware of the epistemological and pragmatic elements in quantum

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mechanics (see e.g., [2]) but they wrote at a time when the language and tools of quantitative epistemology had not yet been sufficiently developed.

Our goal is to derive quantum theory as an example of entropic inference. A central feature is the privileged role we assign to position over and above all other observables. Position is, strictly, the only observable. This is one important difference from other approaches that also emphasize notions of information (see e.g., [3]-[20]). The theory has formal similarities with Nelson’s stochastic mechanics [21]-[26], but there are important conceptual differences. Stochastic mechanics operates at the ontological level; its goal is a realistic interpretation of quantum theory as arising from a deeper, possibly non-local, but essentially classical “reality”. In contrast, the entropic dynamics advocated here operates almost completely at the epistemological level.

The statistical manifold associated to the configuration space is introduced in section 2. The basic dynamical question — the probability of a small step — is answered in section 3 using the method of maximum entropy [27]. (Earlier versions of related ideas were discussed in [28]-[31].) The introduction of time as a device to keep track of the accumulation of small changes is discussed in section 4 and the Schroedinger equation is derived in section 5. An important new element is that the geometry of the statistical manifold is not a fixed background but a dynamical entity. This leads to a certain similarity with the theory of general relativity including a new quantum equivalence principle (section 6). Our conclusions are summarized in section 7.

2 The statistical model

We assume the usual configuration space $\mathcal{X}$ for a single particle, namely, a flat three-dimensional space. We further assume — and this is the crucial new element — that there is a very small but ultimately irreducible uncertainty in the location of the particle. The particle cannot be localized; at best we can specify its expected position $x^a = \langle y^a \rangle$ and give a probability distribution $p(y|x)$. Thus, when we say that the particle is at $x$ what we mean is that the true but unknown position $y$ is somewhere in the vicinity of $x$ with probability $p(y|x)$.

To each point $x \in \mathcal{X}$ there corresponds a probability distribution $p(y|x)$ and the set of these distributions is also a three-dimensional manifold — a statistical manifold which we call $\mathcal{M}$. The same label $x$ is used to denote points in $\mathcal{X}$ and the corresponding points in $\mathcal{M}$. Points in $\mathcal{M}$ are not structureless dots but probability distributions.

The origin of the irreducible uncertainty is left unspecified. Indeed, the arguments below turn out to be remarkably independent of the particular choice of the functional form of $p(y|x)$. Nevertheless, for the sake of clarity it is convenient to choose a specific statistical model. We can reasonably assume that the distributions $p(y|x)$ arise as the result of unknown microscopic influences, in which case, general arguments such as the central limit theorem lead us to expect that for a very wide variety of microscopic conditions the plausible dis-
tributions are Gaussians. We further assume that the Gaussians are spherically symmetric with a small non-uniform variance expressed as a small constant $\sigma^2$ modulated by a positive scalar field $\Phi(x)$,

$$p(y|x) = \frac{\Phi^{3/2}(x)}{(2\pi\sigma^2)^{3/2}} \exp \left[ -\frac{1}{2} \Phi(x) \gamma_{ab} (y^a - x^a)(y^b - x^b) \right],$$

(1)

where

$$\gamma_{ab} = \frac{\delta_{ab}}{\sigma^2}$$

(2)

is the metric in $X$.

While the configuration space $X$ is flat, the statistical manifold $M$ turns out, in general, to be curved. $M$ inherits its unique geometry from the distributions $p(y|x)$. The distance, $d\ell^2 = g_{ab} dx^a dx^b$, between $x$ and $x + dx$, or better, between $p(y|x)$ and $p(y|x + dx)$, is given by the information metric [27][32][33],

$$g_{ab} = \int dy p(y|x) \frac{\partial \log p(y|x)}{\partial x^a} \frac{\partial \log p(y|x)}{\partial x^b}.$$  

(3)

Substituting (1) and (2) into (3) gives

$$g_{ab}(x) = \Phi(x) \frac{\delta_{ab}}{\sigma^2} + \frac{3}{2\Phi} \partial_a \Phi \partial_b \Phi.$$  

(4)

We will be interested in situations where the intrinsic uncertainties are very small. More precisely, when the change in $\Phi(x)$ over the support of $p(y|x)$ is negligible, $|\partial \Phi/\Phi|^2 \ll \Phi/\sigma^2$, the metric simplifies considerably,

$$g_{ab}(x) \approx \frac{\Phi(x)}{\sigma^2} \delta_{ab} = \Phi(x) \gamma_{ab},$$  

(5)

and we see that $\Phi(x)$ plays the role of a conformal factor.

For future reference, the entropy of $p(y|x)$ relative to the flat measure of $X$ is

$$S(x) = -\int dy p(y|x) \log \frac{p(y|x)}{\gamma^{1/2}} = \frac{3}{2} \left[ 1 - \log \frac{\Phi(x)}{2\pi} \right].$$  

(6)

(The volume element in $X$ is given by $dv = \gamma^{1/2} d^3 x$ where $\gamma = \det \gamma_{ab}$.)

The generalization to $N$ particles is straightforward. The 3$N$-dimensional configuration space $X_N$ remains flat but it is no longer isotropic. For example for $N = 2$ particles its metric is

$$\gamma_{AB} = \begin{bmatrix} \delta_{a_1 b_1}/\sigma_1^2 & 0 \\ 0 & \delta_{a_2 b_2}/\sigma_2^2 \end{bmatrix}.$$  

(7)

The position uncertainty is given by a Gaussian distribution $p(y|x)$ in 3$N$ dimensions,

$$p(y|x) \propto \exp \left[ -\frac{1}{2} \Phi(x) \gamma_{AB} (y^A - x^A)(y^B - x^B) \right],$$  

(8)

where the index $A$ takes the values $1 \ldots 3N$. The statistical manifold $M_N$ has metric $g_{AB}(x) \approx \Phi(x) \gamma_{AB}$.
3 Law without law: entropic dynamics

The basic dynamical information is that changes from one state to another are possible and do, in fact, happen. We do not explain why they happen but, given the information that changes occur, we want to venture a guess about what changes to expect. What gives this program some hope of success is the assumption that large changes result from the accumulation of many small changes. Therefore, our job separates into two parts, first we consider a small change, and then we figure out how small changes add up.

Consider a single particle that moves from an initial position \( x \) to an unknown final position \( x' \). (The generalization to more particles is immediate.) What can we say about \( x' \) when all we know is that it is near \( x \)? Since \( x \) and \( x' \) represent probability distributions we use the method of maximum entropy (ME) \(^{27}\). As in all ME problems success hinges on appropriate choices of entropy, prior distribution, and constraints. Since neither the new \( x' \) nor the new microstate \( y' \) are known, the relevant universe of discourse is \( \mathcal{X} \times \mathcal{X} \) and what we want to find is the joint distribution \( P(x', y'|x) \). The appropriate (relative) entropy is

\[
S[P, \pi] = - \int dx' dy' P(x', y'|x) \log \frac{P(x', y'|x)}{\pi(x', y')} . \tag{9}
\]

The relevant information is introduced through the prior \( \pi(x', y') \) and the constraints that specify the family of acceptable posteriors \( P(x', y'|x) \). Consider the prior, \( \pi(x', y') = \pi(x')\pi(y'|x') \). Before the relation between the variables \( x' \) and \( y' \) is known the state of extreme ignorance is represented by a product, \( \pi(x') = \pi(y'|x') \) — knowledge of \( x' \) tells us nothing about \( y' \) and vice versa — and the probabilities \( \pi(y')d^3y' \) and \( \pi(x')d^3x' \) are uniform, that is, proportional to the respective volume elements \( \gamma^{1/2}d^3y' \) and \( \gamma^{1/2}d^3x' \). Proportionality constants are not essential here; we set \( \pi(y') = \gamma^{1/2} \) and \( \pi(x') = \gamma^{1/2} \) and the prior is

\[
\pi(x', y') = \gamma . \tag{10}
\]

Next consider the possible posteriors \( P(x', y'|x) = P(x'|x)P(y'|x', x) \). Besides normalization we impose two constraints. First we have the known relation between \( x' \) and \( y' \): the particular functional form of \( P(y'|x'), x \) is given by eq.\(^{11}\). Therefore,

\[
P(x', y'|x) = P(x'|x)p(y'|x') . \tag{11}
\]

The second constraint concerns the factor \( P(x'|x) \): we know that \( x' \) is only a short step away from \( x \). Let \( x' = x + \Delta x \). We require that the expectation

\[
\langle \Delta t^2 \rangle = \langle \gamma_{ab} \Delta x^a \Delta x^b \rangle = \lambda^2(x) \tag{12}
\]

be some small but for now unspecified numerical value \( \lambda^2(x) \) which might perhaps depend on \( x \). (Provided the steps are sufficiently short their actual length is not particularly critical; they do not even have to be all of the same length.)
Once prior and constraints have been specified the ME method takes over. Substituting (10) and (11) into (9) gives

\[ S[P, \pi] = - \int dx' P(x'|x) \log \frac{P(x'|x)}{\gamma^{1/2}} + \int dx' P(x'|x) S(x'), \tag{13} \]

where \( S(x) \) is given in eq.(6). Varying \( P(x'|x) \) to maximize \( S[P, \pi] \) subject to (12) and normalization gives

\[ P(x'|x) = \frac{1}{\zeta(x, \alpha)} e^{S(x') - \frac{1}{2} \alpha(x) \Delta \ell^2}, \tag{14} \]

where \( \zeta(x, \alpha) \) is the normalization constant, and the Lagrange multiplier \( \alpha(x) \) is determined from \( \partial \log \zeta / \partial \alpha = -\lambda^2 / 2 \). We see that large values of \( \alpha(x) \) clearly lead to short steps.

\( P(x'|x) \) gives the probability of a step from \( x \) to \( x' \). It is the basic building block out of which all dynamics is constructed. It implements what Wheeler foresaw as “law without law”: “that every law of physics, pushed to the extreme, will be found statistical and approximate, not mathematically perfect and precise.”

The most probable displacement \( \Delta \bar{x}^a \) is that which maximizes the scalar probability density. For large \( \alpha(x) \) we expect \( \Delta x^a \) to be small. Then

\[ 0 = \frac{\partial}{\partial x^a} \left[ S(x') - \frac{1}{2} \alpha(x) \gamma_{bc} \Delta x^b \Delta x^c \right]_{\Delta x = \Delta \bar{x}} = \partial_a S(x) - \alpha(x) \gamma_{ab} \Delta \bar{x}^b \tag{15} \]

so that the maximum occurs at

\[ \Delta \bar{x}^a = \frac{1}{\alpha(x)} \gamma^{ab} \partial_b S(x), \tag{16} \]

which shows that the particle tends to drift up the entropy gradient. Expanding the exponent of (14) about its maximum gives

\[ P(x'|x) \approx \frac{1}{Z(x)} \exp \left[ -\frac{\alpha(x)}{2\sigma^2} \delta_{ab} (\Delta x^a - \Delta \bar{x}^a) (\Delta x^b - \Delta \bar{x}^b) \right], \tag{17} \]

with a new normalization \( Z(x) \). The displacement \( \Delta x^a \) can be written as its expectation plus a “fluctuation”,

\[ \Delta x^a = \Delta \bar{x}^a + \Delta w^a, \tag{18} \]

where

\[ \langle \Delta x^a \rangle = \Delta \bar{x}^a = \frac{\sigma^2}{\alpha(x)} \delta^{ab} \partial_b S(x), \tag{19} \]

\[ \langle \Delta w^a \rangle = 0 \quad \text{and} \quad \langle \Delta w^a \Delta w^b \rangle = \frac{\sigma^2}{\alpha(x)} \delta^{ab}. \tag{20} \]

We see that as \( \alpha \to \infty \) the steps get smaller, \( \Delta \bar{x}^a \to 0 \) as \( \alpha^{-1} \), but the fluctuations become dominant because \( \Delta w^a \to 0 \) only as \( \alpha^{-1/2} \). This implies that as \( \alpha \to \infty \) the trajectory is continuous, but not differentiable—just as in Brownian motion.
4 Time

To keep track of the accumulation of small changes we need to introduce a notion of time. Our task here is to develop a model that includes (a) something one might identify as an “instant”, (b) a sense in which these instants might be “ordered”, and (c) a useful concept of “duration” measuring the interval between instants. This set of concepts constitutes what we will call “time” in entropic dynamics. A welcome bonus is that there is an intrinsic directionality from past to future instants; an arrow of time is generated automatically and need not be externally imposed.

The foundation to any notion of time is dynamics. Given an initial position we have some idea, given by \( P(x'|x) \), of what the next position might be. For all steps after the first, however, we are uncertain about both the initial \( x \) and the final step \( x' \), which means we must deal with the joint probability \( P(x'|x)P(x) \). Using the product rule \( P(x',x) = P(x'|x)P(x) \) and integrating over \( x \), we get

\[
P(x') = \int dx P(x'|x)P(x).
\]

If \( P(x) \) happens to be the probability of different values of \( x \) at a given instant of time \( t \), then it is tempting to interpret \( P(x') \) as the probability of values of \( x' \) at a later instant of time \( t' = t + \Delta t \). Accordingly, we write \( P(x) = \rho(x,t) \) and \( P(x') = \rho(x',t') \) so that

\[
\rho(x',t') = \int dx P(x'|x)\rho(x,t).
\]

Nothing in the laws of probability forces this interpretation on us—it is an independent assumption about what constitutes time in the model. We use eq. (22) to define what we mean by an instant: if \( \rho(x,t) \) refers to an “initial” instant, then we use \( \rho(x',t') \) to define what we mean by the “next” instant. Thus, eq. (22) allows time to be constructed, step by step, as a succession of instants.

Specifying the interval of time \( \Delta t \) between successive instants amounts to tuning the steps, or equivalently \( \alpha(x) \), appropriately. To model a “Newtonian” time that flows “equably” everywhere, that is, at the same rate at all places and times we define \( \Delta t \) as being independent of \( x \), and such that every \( \Delta t \) is as long as the previous one. Inspection of the actual dynamics as given in eqs. (17-20) shows that this is achieved if we choose \( \alpha(x) \) so that

\[
\alpha(x) = \frac{\tau}{\Delta t} = \text{constant}.
\]

where \( \tau \) is a constant introduced so that \( \Delta t \) has units of time.

Thus, it is the equable flow of time that leads us to impose uniformity on the step sizes \( \lambda^a(x) \) and the corresponding multipliers \( \alpha(x) \). This completes the implementation of Newtonian time in entropic dynamics. In the end, however, the only justification for any definition of duration is that it simplifies the description of motion, and indeed, eqs. (18-20) are simplified to

\[
\Delta x^a = b^a(x)\Delta t + \Delta w^a.
\]
where the drift velocity $b^a(x)$ and the fluctuation $\Delta w^a$ are

$$b^a(x) = \frac{\sigma^2}{\tau} \delta^{ab} \partial_b S(x), \quad (25)$$

$$\langle \Delta w^a \rangle = 0 \text{ and } \langle \Delta w^a \Delta w^b \rangle = \frac{\sigma^2}{\tau} \Delta t \delta^{ab}. \quad (26)$$

Equation (25) gives the mean velocity to the future or future drift,

$$b^a(x) = \lim_{\Delta t \to 0^+} \frac{\langle x^a(t + \Delta t) \rangle_{x(t)} - x^a(t)}{\Delta t} = \lim_{\Delta t \to 0^+} \frac{1}{\Delta t} \int dx' P(x'|x) \Delta x^a, \quad (27)$$

where $x = x(t)$, $x' = x(t + \Delta t)$, and $\Delta x^a = x'^a - x^a$. The expectation in (27) is conditional on the earlier position $x = x(t)$. One can also define a mean velocity from the past or past drift,

$$b^a_\ast(x) = \lim_{\Delta t \to 0^+} \frac{x^a(t) - \langle x^a(t - \Delta t) \rangle_{x(t)}}{\Delta t} \quad (28)$$

where the expectation is conditional on the later position $x = x(t)$. Shifting the time by $\Delta t$, $b^a_\ast$ can be equivalently written as

$$b^a_\ast(x') = \lim_{\Delta t \to 0^+} \frac{x^a(t + \Delta t) - \langle x^a(t) \rangle_{x(t + \Delta t)}}{\Delta t} = \lim_{\Delta t \to 0^+} \frac{1}{\Delta t} \int dx P(x'|x) \Delta x^a, \quad (29)$$

with the same definition of $\Delta x^a$ as in eq.(27).

The two mean velocities, to the future $b^a$, and from the past $b^a_\ast$, need not coincide. The connection between them is well known [21][23],

$$b^a_\ast(x, t) = b^a(x) - \frac{\sigma^2}{\tau} \partial^a \log \rho(x, t), \quad (30)$$

where $\partial^a = \delta^{ab} \partial_b$ and $\rho(x, t) = P(x)$. What might not be widely appreciated is that eq.(30) is a straightforward consequence of Bayes’ theorem,

$$P(x|x') = \frac{P(x)}{P(x')} P(x'|x). \quad (31)$$

(For a related idea see [36].) The proof of eq.(30) is lengthy but straightforward; the crucial step is to Taylor expand $P(x')$ about $x$ in (31) to get

$$P(x|x') = \left[ 1 - (\partial_b \log \rho) \Delta x^b + \ldots \right] P(x'|x), \quad (32)$$

which accounts for the $\partial \log \rho$ term in eq.(30).

The fact that $b^a \neq b^a_\ast$ is very significant because it signals an asymmetry in time. The arrow of time, constitutes a puzzle that has plagued physics ever since Boltzmann. The standard formulation of the problem is that the laws of nature are symmetric under time reversal but everything else in nature indicates a clear asymmetry. How can that be?
The notion that the laws of physics might be rules for the manipulation of information and not necessarily laws of nature offers a new perspective on this old puzzle. Note that entropic dynamics does not assume any underlying laws of nature — whether they be symmetric or not — and it makes no attempt to explain the asymmetry between past and future. The asymmetry is the inevitable consequence of entropic inference. From the point of view of entropic dynamics the challenge does not consist in explaining the arrow of time; on the contrary, it is the reversibility of the laws of physics that demands an explanation. Indeed, time itself always remains intrinsically directional even when the derived laws of physics turn out to be fully reversible.

5 The Schroedinger equation

Time has been introduced as a useful device to keep track of the accumulation of small changes. The technique to do this is well known from diffusion theory. The equation of evolution for the distribution \( \rho(x, t) \), derived from eq. (22) together with (24)–(26), is the Fokker-Planck equation, [23][37]

\[
\frac{\partial}{\partial t} \rho = -\frac{\partial}{\partial a} \left( b^a \rho \right) + \frac{\sigma^2}{2\tau} \nabla^2 \rho \, ,
\]

(33)

where \( \delta^{ab} \partial_a \partial_b = \nabla^2 \). Using eq. (30) it can be expressed in terms of \( b^a_* \),

\[
\frac{\partial}{\partial t} \rho = -\frac{\partial}{\partial a} \left( b^a_* \rho \right) - \frac{\sigma^2}{2\tau} \nabla^2 \rho \, .
\]

(34)

Adding eqs. (33) and (34) gives a continuity equation

\[
\frac{\partial}{\partial t} \rho = -\frac{\partial}{\partial a} \left( v^a \rho \right) \quad \text{where} \quad v^a \overset{\text{def}}{=} \frac{1}{2} \left( b^a + b^a_* \right) \, ,
\]

(35)

where \( v^a \) is interpreted as the velocity of the probability flow or the current velocity. On the other hand, eq. (30) gives yet another “velocity”

\[
u^a \overset{\text{def}}{=} \frac{1}{2} \left( b^a_* - b^a \right) = -\frac{\sigma^2}{2\tau} \partial^a \log \rho \, .
\]

(36)

Its interpretation follows from \( v^a = b^a + u^a \). The drift \( b^a \) represents the tendency of the probability \( \rho \) to flow up the entropy gradient while \( u^a \) represents its tendency to slide down the density gradient. The situation is analogous to Brownian motion where the drift velocity is the response to the gradient of an external potential, while \( u^a \) is a response to the concentration gradient — the so-called osmotic force. Accordingly, \( u^a \) is called the osmotic velocity. Its contribution to the probability flow is the actual diffusion current,

\[
\rho u^a = -\frac{\sigma^2}{2\tau} \partial^a \rho \, ,
\]

(37)

which can be recognized as Fick’s law, with a diffusion coefficient given by \( \sigma^2/2\tau \).
Since both the drift $b^a$ and the osmotic velocity $u^a$ are gradients, it follows that $v^a = b^a + u^a$ is a gradient too,

$$v^a = \frac{\sigma^2}{\tau} \partial^a \phi \quad \text{where} \quad \phi(x, t) = S(x) - \log \rho^{1/2}(x, t) . \quad (38)$$

Eqs.(33-38) provide the complete answer to the problem as originally posed: what is the expected evolution of a particle afflicted by the intrinsic uncertainties described by $p(y|x)$ in eq.(1)? Unfortunately, we can see that the dynamics described by eqs.(33-38) is not quantum mechanics; it is just diffusion. Indeed, in order to construct a wave function, $\Psi = \rho^{1/2} e^{i \phi}$, in addition to the density $\rho$ we need a second degree of freedom, a phase $\phi$.

Note that the function $\phi(x, t)$ in eq.(38) does not (yet) qualify as an independent degree of freedom because the entropy $S(x)$ — or equivalently the conformal factor $\Phi(x)$ — is an externally prescribed field. As long as the statistical manifold is a fixed static manifold there is no logical room for additional degrees of freedom. To promote $\phi(x, t)$ to an independent degree of freedom we are forced to allow the manifold $\mathcal{M}$ itself to participate in the dynamics.

To specify the dynamics of the manifold we follow Nelson and assume that it is “conservative” [22]. Requiring that some “energy” be conserved may seem natural in that it clearly represents physically relevant information but we feel that it is an assumption that demands a deeper justification. Normally energy is whatever happens to be dynamically conserved as a result of invariance under translations in time. But our dynamics has not yet been fully defined; what, then, is “energy” and why should it be conserved in the first place? This is a question we leave for the future. At this early stage, for the purpose of deriving a non-relativistic model, we just propose an intuitively reasonable conserved energy and proceed.

The energy is chosen to be a local functional that includes a term representing a potential energy and includes terms in the velocities that are invariant under time reversal and under rotations. Under time reversal, $t \rightarrow -t$, we have $b^a \rightarrow -b^*_a$, $v^a \rightarrow -v^a$, $u^a \rightarrow u^a$. For low velocities this means we need only include quadratic terms in the velocities, $v^2$ and $u^2$ [26]. The proposed energy functional is

$$E[\rho, v] = \int d^3 x \rho(x, t) \left( A \gamma_{ab} v^a v^b + B \gamma_{ab} u^a u^b + V(x) \right) , \quad (39)$$

where $A$ and $B$ are constants. In order that $E$ have units of energy $A/\sigma^2$ and $B/\sigma^2$ must have units of mass. Then

$$E[\rho, v] = \int d^3 x \rho(x, t) \left( \frac{1}{2} m v^2 + \frac{1}{2} \mu u^2 + V(x) \right) , \quad (40)$$

where $m = 2A/\sigma^2$ and $\mu = 2B/\sigma^2$ will be referred to as the “mass” and the “osmotic mass” respectively. It is further convenient to combine the constants $\tau$ and $A$ into yet a new constant $\eta$, which relates the units of mass or energy
with those of time,
\[ \eta = \frac{2A}{\tau} \text{ so that } \frac{\sigma^2}{\tau} = \frac{\eta}{m}. \] (41)

Then the current and osmotic velocities, eqs. (38) and (36), are
\[ mv_a = \eta \partial_a \phi \quad \text{and} \quad mu_a = \eta \partial_a \log \rho^{1/2}, \] (42)
while (40) becomes
\[ E = \int dx \rho \left( \frac{\eta^2}{2m} (\partial_a \phi)^2 + \frac{\mu \eta^2}{8m^2} (\partial_a \log \rho)^2 + V \right). \] (43)

Next we impose that the energy \( E[\rho, \phi] \) be conserved. After some manipulations involving integration by parts and the continuity equation,
\[ \dot{\rho} = -\partial_a (\rho v^a) = -\frac{\eta}{m} \partial_a (\rho \partial_a \phi) = -\frac{\eta}{m} \left( \partial^a \rho \partial_a \phi + \rho \nabla^2 \phi \right), \] (44)
the time derivative \( \dot{E} \) of (43) is
\[ \dot{E} = \int dx \dot{\rho} \left[ \eta \dot{\phi} + \frac{\eta^2}{2m} (\partial_a \phi)^2 + V - \frac{\mu \eta^2}{2m^2} \nabla^2 \rho^{1/2} \right]. \] (45)

Requiring that \( \dot{E} = 0 \) for arbitrary choices of \( \dot{\rho} \) [which follows from arbitrary choices of \( \rho \) and \( \phi \) in eq. (44)] we get
\[ \eta \dot{\phi} + \frac{\eta^2}{2m} (\partial_a \phi)^2 + V - \frac{\mu \eta^2}{2m^2} \nabla^2 \rho^{1/2} = 0. \] (46)

Equations (44) and (46) are the coupled dynamical equations we seek. The evolution of \( \rho(x, t) \) in eq. (44) is determined by \( \phi(x, t) \); the evolution of \( \phi(x, t) \) in eq. (46), is determined by \( \rho(x, t) \). The evolving geometry of the manifold enters through \( \phi(x, t) \).

Next we show that, with one very interesting twist, the dynamical equations turn out to be equivalent to the Schroedinger equation. We can always combine the functions \( \rho \) and \( \phi \) into a complex function \( \Psi = \rho^{1/2} \exp(i\phi) \). Then eqs. (44) and (46) can be rewritten as
\[ i\eta \dot{\Psi} = -\frac{\eta^2}{2m} \nabla^2 \Psi + V \Psi + \frac{\eta^2}{2m} \left( 1 - \frac{\mu}{m} \right) \nabla^2 (\Psi \Psi^*)^{1/2} \Psi. \] (47)

This reproduces the Schroedinger equation,
\[ i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \Psi + V \Psi, \] (48)
provided the osmotic mass is identified with the mass, \( \mu = m \), and \( \eta \) is identified with Planck’s constant, \( \eta = \hbar \). Setting \( S_J = \eta \phi \) in eq. (46) and letting \( \eta \to 0 \)
leads to the Hamilton-Jacobi equation and thus the classical limit \( \eta = \hbar \to 0 \) allows one to identify \( m \) with the inertial mass of the particle.

But why should \( \mu = m \)? This question is so central that we devote the next section to it. But before that we note that the non-linearity in eq. (47) is undesirable both for experimental and theoretical reasons. From the experimental side non-linear terms have been ruled out to an extreme degree through precision experiments on the Lamb shift [24] and even more so in hyperfine transitions [38]. From the theory side there is a consistency argument that links the linearity of the Hilbert space with the linearity of time evolution; retaining one and not the other leads to inconsistently assigned amplitudes [9]. And, further, it has been argued that the non-linear terms can lead to superluminal communication [39]. Therefore it is extremely probable that the identity of inertial and osmotic mass is exact.

Among the many mysteries of quantum theory there is one — the central role played by complex numbers — that turns out to be related to these issues. The dynamical equations (44) and (46) contain no complex numbers but they can always be written in terms \( \Psi \) and \( \Psi^* \) instead of \( \rho \) and \( \phi \). There is no mystery there. The statement that complex numbers play a fundamental role in quantum theory is the non-trivial assertion that the equation of evolution contains only \( \Psi \) and not both \( \Psi \) and \( \Psi^* \). In the entropic approach both the linear time evolution and the special role of complex numbers are linked through the equality \( \mu = m \).

6 A new equivalence principle
The generalization to many particles is easy. The conserved energy (for \( N = 2 \)) [see eq. (7)] is

\[
E = \int d^3x \rho(x,t) \left( A\gamma_{AB}v^A v^B + B\gamma_{AB} u^A u^B + V(x) \right)
\]

\[
= \int d^6x \rho(x,t) \left( \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 + \frac{1}{2} \mu_1 u_1^2 + \frac{1}{2} \mu_2 u_2^2 + V(x) \right),
\]

(49)

where we introduced inertial and osmotic masses, \( m_n = 2A/\sigma_n^2 \) and \( \mu_n = 2B/\sigma_n^2 \). Note that the ratio of osmotic to inertial mass turns out to be a universal constant, the same for all particles: \( \mu_n/m_n = B/A \). But why should \( \mu_n = m_n \) exactly? To see this let us go back to eq. (43). We can always change units and rescale \( \eta \) and \( \tau \) by some constant \( \kappa \) into \( \eta = \kappa \eta' \), \( \tau = \tau'/\kappa \). If we also rescale \( \phi \) into \( \phi = \phi'/\kappa \), eqs. (44) and (43) become

\[
\dot{\rho} = -\frac{\eta'}{m} \left( \partial_t \rho \partial_t \phi' + \rho \nabla^2 \phi' \right),
\]

(50)

\[
E = \int dx \rho \left( \frac{\eta'^2}{2m} (\partial_t \phi')^2 + \frac{\mu \kappa^2 \eta'^2}{8m^2} (\partial_t \log \rho)^2 + V \right).
\]

(51)
Then we can introduce a different wave function $\Psi'$ as $\Psi' = \rho^{1/2} \exp(i\phi')$ which satisfies
\[
\begin{align*}
\imath \eta' \dot{\Psi}' &= \frac{\eta'^2}{2m} \nabla^2 \Psi' + V \Psi' + \frac{\eta'^2}{2m} \left(1 - \frac{\mu \kappa^2}{m}\right) \frac{\nabla^2 (\Psi' \Psi'^*)^{1/2}}{(\Psi' \Psi'^*)^{1/2}} \Psi'.
\end{align*}
\]

Since the mere rescaling by $\kappa$ can have no physical implications the different regraduated theories are all equivalent and it is only natural to use the simplest one: choose $\kappa = (A/B)^{1/2}$ so that $\mu \kappa^2 = m$ and rescale the old $\mu$ to a new osmotic mass $\mu' = \mu \kappa^2 = m$. We conclude that whatever the value of the original coefficient $\mu$ it is always possible to regraduate $\eta, \phi$ and $\mu$ to an equivalent but more convenient description where the Schroedinger equation is linear and complex numbers attain a special significance. It is the rescaled value $\eta'$ of the linear theory that gets numerically identified with Planck’s constant $\hbar$. From this perspective the linear superposition principle and the complex Hilbert spaces are important because they are convenient, but not because they are fundamental — a theme that was also explored in [9].

These considerations remind us of Einstein’s original argument for the equivalence principle: We accept the complete physical equivalence of a gravitational field with the corresponding acceleration of the reference frame because this offers a natural explanation of the equality of inertial and gravitational masses and opens the door to an explanation of gravity in purely geometrical terms. Similarly, in the quantum case we accept the complete equivalence of quantum and statistical fluctuations because this offers a natural explanation of the Schroedinger equation — its linearity, its unitarity, the role of complex numbers, the equality of inertial and osmotic masses — and opens the door to explaining quantum theory as an example of entropic inference.

7 Conclusions

On epistemology vs. ontology: Quantum theory has been derived as an example of entropic dynamics. The discussion is explicitly epistemological — it is concerned with how we handle information and update probabilities. Of course, once we know $x$ we can immediately make inferences about the unobserved “true” position $y$. But this is precisely the point: the relation between the “laws of physics” (the Schroedinger equation) and “actual reality” is less direct than it is commonly assumed.

On interpretation: Ever since Born the magnitude of the wave function $|\Psi|^2 = \rho$ has received a statistical interpretation. Within the entropic dynamics approach the phase of the wave function is also recognized as a feature of purely statistical origin. We can make this explicit using eq.(38) to write the wave function as
\[
\Psi = \rho^{1/2} \exp[iS]
\]

where $\rho$ is a probability density and $S$ is an entropy — it is the entropy associated to each point on the statistical manifold $\mathcal{M}$. 12
On dynamical laws: The principles of entropic inference form the backbone of this approach to dynamics. Energy conservation was introduced as an important constraint in the present non-relativistic theory. One can safely expect that in a fully relativistic theory it will not survive in its current form. The peculiar features of quantum mechanics such as non-locality and entanglement arise naturally by virtue of the theory being formulated in the $3N$-dimensional configuration space.

On time: Time was introduced to keep track of the accumulation of small changes and its particular form was chosen to simplify the description of evolution. We have proposed a scheme that models temporal order, its duration, and most interestingly, its directionality.

Equivalence principle: The derivation of the Schroedinger equation from entropic inference led to a surprising similarity with general relativity. The statistical manifold $\mathcal{M}$ is not a fixed background but actively participates in the dynamics. The potential for uncovering deeper relations between quantum theory and gravitation theory looks promising.

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