Bremsstrahlung emission of photons accompanying ternary fission of $^{252}$Cf

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Abstract. We present the first results on the bremsstrahlung emission of photons accompanying ternary spontaneous fission of the $^{252}$Cf nucleus. We also compare our calculations on the basis of quantum model with preliminary experimental data and find a good agreement between theory and experiment for photon energies up to 500 keV, when the $α$-particle emission is in presence of the field of two fission fragments of the daughter nucleus.

1. Introduction

Nuclear fission accompanied by light charged particle emission (often called ternary fission) has been widely studied [1–4]. The main interest about this process is related to the possibility of obtaining new additional information about the fission dynamics. The study of the bremsstrahlung emission of photons accompanying ternary fission allows us other opportunities in understanding of this process. This process has never been studied elsewhere. In this paper we present our model and calculations of the bremsstrahlung probabilities in the ternary fission of the $^{252}$Cf nucleus.

After review in Sect. 2 of our results in determination of bremsstrahlung probabilities during the spontaneous fission of $^{252}$Cf, in Sects. 3 and 4 we present our first results in the theoretical study of bremsstrahlung photons emitted during the ternary fission of the $^{252}$Cf nucleus. In Sect. 5 we summarize the paper.

2. Model of the bremsstrahlung emission accompanying the spontaneous fission

2.1. Shape of the nuclear system undergoing fission

In the fission process it is necessary to describe in a continuous way the sequence of shapes of a fissioning nucleus from its ground state, through its saddle and scission configurations, to the separated fragments at infinity. In a study of ground state shapes one has to be able to describe a sphere, oblate and prolate spheroids, octupole deformations, and positive and
negative hexadecapole deformations. Since many questions studied in the fission process do not sensitively depend upon deviations from axial symmetry, we shall initially apply the model to shapes of nuclear systems that are axially symmetric.

We chose to specify the nuclear shape in terms of smoothly joined portions of three quadratic surfaces of revolution: two spheroids connected by a hyperboloidal neck (see Fig. 3 of Ref. [6]). In terms of the cylindrical coordinate system, we shall use the equation for the drop’s surface written explicitly as [6]

\[
\rho^2 = \begin{cases} 
  a_1^2 - \left(\frac{a_1^2}{c_1^2}\right)(z - l_1)^2, & \text{for } l_1 - c_1 \leq z \leq z_1 \\
  a_2^2 - \left(\frac{a_2^2}{c_2^2}\right)(z - l_2)^2, & \text{for } z_2 \leq z \leq l_2 + c_2 \\
  a_3^2 + \left(\frac{a_3^2}{c_3^2}\right)(z - l_3)^2, & \text{for } z_1 \leq z \leq z_2.
\end{cases}
\]

Here, the quantity \(l_i\) specifies the position of the center of the \(i\)th quadratic surface, \(c_i\) is its symmetry axis, and \(a_i\) is its transverse semiaxis \((i = 1, 2, 3)\). There are nine coordinates in the specification of a nuclear shape. However, following the description reported in the paper of Bolsterli et al. [6], one coordinate is eliminated by the assumption that the volume of the nuclear system remains constant during its evolution, and other two coordinates are eliminated by the requirement that the middle surface joins smoothly with both surfaces of the forming fragments (at points denoted by \(z_1\) and \(z_2\)). This introduces three relations between the original nine degrees of freedom, and reduces the number to six. Elimination of the center of mass coordinate finally reduces the number of shape coordinates to five. In Fig. 1 (a) one can see what shapes we have obtained by such a way at different distances between centers of the \(^{12}\text{C}\) fragment and the daughter nucleus formed in fission of the \(^{252}\text{Cf}\) nucleus.

![Figure 1](image.png)

Figure 1. (a) Shapes of the nuclear surface for emission of the \(^{12}\text{C}\) fragment calculated for different distances \(r\) between centers of the daughter nucleus and the fragment during spontaneous fission process of the \(^{252}\text{Cf}\) nucleus (axes \(z\) and \(\rho\) are defined from eq. (1)). Potential obtained in this paper and spherically symmetric potential proposed by Denisov and Ikezoe in [8] for the \(\alpha\) decay of the \(^{252}\text{Cf}\) nucleus for (b) the Coulomb components and (c) the nuclear components.

2.2. Interaction potential between daughter nucleus and emitted fragments

Once the nuclear shape has been specified, the next step is to generate an interaction potential between the daughter nucleus and fragment with respect to the shape and distance between such nucleus and fragment. If such a procedure is useful in the general theory of fission, it has to be capable of handling very deformed shapes and the evolution from one parent nucleus to two final fragments (the daughter nucleus and the related fission fragment). The interaction potential between the daughter nucleus and the fragment could be written as

\[
V_{\text{total}}(r) = V_N(r) + V_C(r).
\]
We define the nuclear component \( V_N(r) \) and the Coulomb component \( V_C(r) \) as
\[
V_N(r) = E_{N,\text{nucleus}}(r) - E_{N,\text{fragment}}, \quad V_C(r) = E_{C,\text{nucleus}}(r) - E_{C,\text{fragment}},
\]
where \( E_{N,\text{nucleus}}(r) \) and \( E_{C,\text{nucleus}}(r) \) are nuclear and Coulomb energies for the total nuclear system, \( E_{N,\text{fragment}}(r) \) and \( E_{C,\text{fragment}}(r) \) are nuclear and Coulomb energies for that fragment only, defined in the folding form (see also Ref. [6]):
\[
E_{N,\text{nucleus}}(r) = -\lambda_N \int_V \frac{dr^3}{1 + \exp(|r - r'|/a)}, \quad E_{N,\text{fragment}} = -\lambda_N \int_{V, r \neq r'} \frac{dr^3}{1 + \exp(|r'|/a)},
\]
\[
E_{C,\text{nucleus}}(r) = \lambda_C \int_{V, r \neq r'} \frac{dr^3}{|r - r'|}, \quad E_{C,\text{fragment}} = \lambda_C \int_{V, r \neq 0} \frac{dr^3}{|r|}.
\]
The integration is over the volume \( V \) of the shape defined relatively given distance \( r \) between the daughter nucleus and the fragment emitted. Introduced parameters \( \lambda_C = Z_1 Z_2 e^2 / V_p \) and \( \lambda_N = M_p / V_p \) (where \( V_p \) and \( M_p \) are volume and mass of the parent nucleus) give us practically very precise coincidence between the potential defined by (2)–(4) and the potential obtained by procedure of parametrization proposed by Denisov and Ikezoe in Ref. [8] for \( \alpha \)-decay (see Fig. 1 (b, c) for \( 252 \text{Cf} \)).

The potential total depends on the difference between centers of the daughter nucleus and the emitted fragment, and we use the spherically symmetric approximation for further calculation of the wave functions.

The \( Q \) value concerning each emitted fragment in fission is calculated by standard procedure as
\[
Q = M_p - M_d - M_f,
\]
where \( M_p, M_d \) and \( M_f \) are the masses of the parent, daughter and specific fragment nuclei, respectively (see Ref. [7]).

2.3. Photon emission probability accompanying the spontaneous fission

We define the photon emission probability formed by a separate fragment during fission in terms of the transition matrix elements for the composite quantum system (the daughter nucleus and the fragment) from its state before the photon emission (initial state \( i \)) into its state after the photon emission (finale state \( f \)), as [9]
\[
\frac{dP(w, \theta)}{dE_\gamma} = N_0 w |p(w, \theta)|^2, \quad k_{i,f} = \sqrt{2mE_{i,f}}, \quad w = E_i - E_f,
\]
where
\[
p(w, \theta) = \frac{1}{3} \sum_{l=0}^{\infty} i^l (-1)^l (2l + 1) P_l(\cos \theta) \sum_{\mu=-1,1} h_\mu J_{m_f}(l, w), \quad h_\pm = \frac{1}{\sqrt{2}} (1 \pm i).
\]
Here, the vector \( k \) represent the photon impulse in the direction of its propagation, the vector \( r \) is the radius vector marking the position of center of the emitted fragment relative to the center of the daughter nucleus, \( \theta \) is the angle between the direction \( n_1 = r/|r| \) of the fragment motion and the propagation direction \( n_2 = k/|k| \) of the photon emitted, where \( k = |k| \) and \( r = |r| \). \( E_{i,f} \) and \( k_{i,f} \) are total energy and wave vector of the system in the initial state \( i \) or in the final state \( f \). \( \psi_i(r) \) and \( \psi_f(r) \) are the wave functions of the system in the initial and final state \( i \) and \( f \), \( w = k = |k| \) is the photon frequency (energy) (see Refs. [10–13] for a detailed description.
of all parameters, calculation of wave functions, boundary conditions etc.). The radial integral
\[ J_{m_f}(l, w) = \int_0^{\infty} r^2 R_i(r, E_i) \frac{\partial R_f(r, E_f)}{\partial r} j_l(kr) \, dr, \] (8)
where \( R_i(r) \) and \( R_f(r) \) are the radial components of the total wave functions \( \psi_i(r) \) and \( \psi_f(r) \),
\( j_l(kr) \) is the spherical Bessel function of order \( l \). \( N_0 \) is a coefficient calculated by (see Ref. [14])
\[ N_0 = \frac{Z_{\text{eff}}^2 e^2}{(2\pi)^4 m}, \] (9)
where \( Z_{\text{eff}} \) and \( m \) are the effective charge and reduced mass of the daughter-fragment system, respectively.

2.4. Calculation of bremsstrahlung spectra for the spontaneous fission of \(^{252}\text{Cf}\)
We applied the above-described method to calculate the spectrum of photons emitted during
the spontaneous fission of the \(^{252}\text{Cf}\) nucleus [9]. At first, we had to check our model and
calculations on a simple problem: the estimation of the bremsstrahlung radiation formed by only
one fragment with arbitrary mass and charge numbers \( A \) and \( Z \), respectively which participates
in the complex process of fission.

![Figure 2.](image-url)

**Figure 2.** (a) Calculation of the bremsstrahlung photon emission probabilities caused by heavy
fragments included in mass region \( A = 95 - 115 \) during the spontaneous fission of \(^{252}\text{Cf}\) [9]; (b) Total
bremsstrahlung photon probability in the spontaneous fission of \(^{252}\text{Cf}\) [9]: calculation of the spectrum
(solid line) obtained by averaging the spectra for contributions of all fragments; in comparison with the
sets of experimental data given by squares [16], triangles [17], diamonds [18], stars [19], and circles [20].

Calculations for heavy fragments are the most difficult. Fragments with large masses
participate in fission with larger \( Q \)-values. In order to achieve stable spectra, it is necessary
to take a larger number of oscillations in the integrant function into account for the integration
of the matrix element. Moreover, the high energies of emitted photons reinforce such difficulty.
After explicit integration of the matrix element in far asymptotic region, we have improved
our code that allows us to achieve such stability for different heavy fragments. Results of such
calculations for some fragments within the 95-115 mass region are presented in Fig. 2 (a).

The total spectrum of photons which are emitted during the spontaneous fission of \(^{252}\text{Cf}\) was obtained by averaging the spectra for contributions of all fragments, and normalizing the
resulting spectrum to the experimental data at lower energies. Such obtained spectrum is
presented in Fig. 2 (b) by solid line. Our calculations were obtained in the framework of a fully quantum approach and agree well with the experimental data in literature concerning bremsstrahlung photon emission in the spontaneous fission of $^{252}$Cf in the $E_{\gamma}$ energy range up to about 20 MeV. In the 20-38 MeV range, however, the calculations agree only with Eremin et al.’s experimental data [20]. So, this model could be a convenient basis of developing the study of photon emission in the ternary fission of the same $^{252}$Cf nucleus.

3. Model for the bremsstrahlung emission accompanying ternary fission

In order to generalize the above-described model for the emission probability in the ternary fission it is necessary to solve the following steps:

(i) to define the geometry of the surface of nuclear system undergoing ternary fission;
(ii) to calculate the interaction potential between the $\alpha$-particle and the other two fragments of fissioning nucleus;
(iii) to calculate the matrix element of the $\alpha$-emission by taking the motion of the two fragments into account;
(iv) to reach the condition that the final spectra have to be stable for light and heavy fragments of the fissioning nucleus, for different geometries of fragments in the fissioning process.

3.1. Geometry of the fissioning system and interaction potential

In the first step we define the geometry of the nuclear system which decays into three fragments. In Fig. 3 (a) are reported some parameters which we use.

![Diagram of the fissioning system](image)

**Figure 3.** Geometry of the fissioning system: (a) separation into three fragments; (b) transition to the effective charge $Z_{\text{eff}}$ at far distances.

In order to study the ternary fission process, as emitted fragment we shall choose $\alpha$-particle. By using the surface of the nuclear system reported in Fig.3 (a), we calculate the interaction potential between the $\alpha$-particle and daughter nucleus which is composed of two fragments. Here, we use standard folding procedure. The new potential $V_{\text{total}}$ depends on the geometry of the relative positions of the two fragments and the $\alpha$-particle. Here, as the main parameter, we consider the relative distance $R_{12}$ between the centers of masses of two fragments. So, we write the total potential as

$$V_{\text{total}}(r, R_{12}) = V_N(r, R_{12}) + V_C(r, R_{12}).$$  

(10)
where the nuclear component $V_N(r)$ and the Coulomb component $V_C(r)$ are

$$V_N(r,R_{12}) = V_{N,nucleus}(r,R_{12}) - E_{N,\text{fragment}},$$

$$V_C(r,R_{12}) = V_{C,nucleus}(r,R_{12}) - E_{C,\text{fragment}}. \tag{11}$$

Here, $V_{N,nucleus}(r,R_{12})$ and $V_{C,nucleus}(r,R_{12})$ are the nuclear and Coulomb potential components of the total daughter nuclear system, while $E_{N,\text{fragment}}(r)$ and $E_{C,\text{fragment}}(r)$ are the nuclear and Coulomb potential components for the separated fragments, defined by the folding form:

$$V_{N,nucleus}(r,R_{12}) = -\lambda_N \int_{V(R_{12})} \frac{dr^3}{1 + \exp(|r-r'|/a) - E_{N,\text{fragment}}} = -\lambda_N \int_{V(R_{12})} \frac{dr^3}{1 + \exp(|r-r'|/a) - E_{N,\text{fragment}}} \tag{12}$$

$$V_C(r,R_{12}) = \lambda_C \int_{V(R_{12}), r \neq r'} \frac{dr^3}{r' - r} - E_{C,\text{fragment}} = \lambda_C \int_{V(R_{12}), r \neq r'} \frac{dr^3}{r} - E_{C,\text{fragment}}$$

3.2. Calculations of the Coulomb component at far distances

In principle, by using the above-described procedure, one can calculate the wave functions and bremsstrahlung probabilities. However, in order to realize such a procedure we have to solve some particular problems: the time of calculations of spectra on the basis of the folding procedure is very long, and it is very difficult to obtain convergent calculations and stable spectra because we have to take far distances into account where it is needed to calculate the wave function starting from the corresponding folding potential.

For such reasons, in order to calculate the Coulomb component at far distances we change two fragments (located at distances $R_{1\alpha}$ and $R_{2\alpha}$ from the $\alpha$-particle) with a new effective charge located at the distance $R_{\text{eff}}$ from the $\alpha$-particle with the aim to have the same influence on the $\alpha$-particle as the two original fragments (see Fig. 3 (b)). Therefore, in eqs. (10)–(12) we use $R_{\text{eff}}$ instead of the variable $r$ for a fixed angular geometry. We determine this $Z_{\text{eff}}$ charge from the following condition:

$$Z_{\text{eff}}(R_{\text{eff}}) = R_{\text{eff}} \left\{ \frac{Z_1}{R_{1\alpha}} + \frac{Z_2}{R_{2\alpha}} \right\}, \tag{13}$$

where $R_{\text{eff}} = |R_{\text{eff}}|$, $R_{1\alpha} = |R_{1\alpha}|$, $R_{2\alpha} = |R_{2\alpha}|$. It turns out that such an approach allows us to achieve stability in calculations of the bremsstrahlung spectra which are convergent. Another important result of this procedure is that the spectra are sensitive to the chosen geometry of the fissioning nucleus.

3.3. Results

We shall start from considerations of calculations obtained for the $\alpha$-decay of $^{252}$Cf. In Fig. 4 (a) we present our results for the bremsstrahlung emission of photons (full line) caused by the $\alpha$-particle of $E_\alpha$=6.2 MeV when the daughter nucleus is not separated into two fragments. As one can see this spectrum has the similar order as in our previous results for the spectra of the $\alpha$-decay of $^{214}$Po (red dashed line) and $^{220}$Ra (blue dashed line) nuclei. In Fig. 4 (b) we present new calculation for the emission of photons caused by the $\alpha$-particle emitted from the same $^{254}$Cf nucleus but when the daughter nucleus is separated into two fragments (dash-dotted line) but in this case is $E_\alpha=16$ MeV [21]. In this last figure we also report for a comparison the result (full line) of the previous Fig. 4 a of the photon spectrum for the $\alpha$-decay at $E_\alpha=6.2$ MeV. As one can see the relevant difference with the last spectrum is partially explained by the higher energy $E_\alpha=16$ MeV of the emitted $\alpha$-particle in presence of the two fragments.
In Fig. 5 we present the calculated spectra obtained by the above-described approach in comparison with the preliminary experimental data [22] of photon emission in presence of the ternary fission. These data were obtained analyzing the event of the photon emission in coincidence with the α-particle, emitted from the $^{252}$Cf nucleus, having a kinetic energy $E_\alpha > 6.2$ MeV corresponding to motion of α-particles in presence of the field of the two fission fragments of daughter nucleus. By comparing the experimental data with the calculated spectra, we can conclude that: (1) the general tendency of the calculated spectra is similar to the one of the experimental data; (2) there is some deviation between calculation and data of emitted photons in the 200–400 keV energy region of $E_\gamma$. On such a basis we can rise the following question: what mechanism could improve the description of the experimental data?

4. Improvement of the model
In order to resolve the above-mentioned question, let us collect such aspects of the model which influence in an appreciable mode the photon emission spectra.

(i) In calculation of the photon emission probability caused by the α-particle in presence of the binary nucleus, we consider:

(a) different separations of the same binary nucleus into two fragments;
(b) the geometry of the fissioning nucleus;
(c) the neck of the binary nucleus at formation of the α-particle;
(d) the dynamics of the relative motion between α-particle and fragments of the binary nucleus;
(e) the angles between emitted photons and fragments.

(ii) One can assume that the two fragments of the binary nucleus are in motion during the fission in order to give an additional contribution to the emission probability of the bremsstrahlung.
Figure 5. Bremsstrahlung photon emission probabilities in the ternary fission of the $^{252}$Cf nucleus: calculated spectra accompanying of $\alpha$-emission in presence of different kind of fragmentation of the daughter nucleus as presented in Fig 4 (b), in comparison with the preliminary experimental data [22] (full squares).

photons.

The last point seems to have the largest influence on the spectra. For this reason, we shall continue our analysis on this item.

4.1. Contribution of the binary nucleus

Let us consider the operator of cluster emission from the total nuclear system ($\alpha$-particle and daughter nucleus) where both $\alpha$-particle and residual nucleus are composed by nucleons (we indicate nucleons of $\alpha$-particle by the index $i$, and nucleons of the residual nucleus by the index $j$):

$$\hat{H}_\gamma = -e \sqrt{\frac{2\pi\hbar}{w}} \sum_{n=1,2} e^{(n),*} \left\{ \sum_{i=1}^{4} \frac{Z_i}{m_i} e^{-ikr_i} \hat{p}_i + \sum_{j=1}^{A} \frac{Z_j}{m_j} e^{-ikr_j} \hat{p}_j \right\}. \quad (14)$$

We pass to new variables, where we take into account centers of masses of the $\alpha$-particle and of the two fragments in the daughter nucleus. We define coordinates of centers of masses of the $\alpha$-particle by $r_\alpha$, of the daughter nucleus by $R_A$, and of the total nuclear system by $R$, as

$$r_\alpha = \frac{1}{m_\alpha} \sum_{i=1}^{4} m_i r_{\alpha i}, \quad R_A = \frac{1}{M} \sum_{j=1}^{A} m_j r_{A j}, \quad R = \frac{M R_A + m_\alpha r_\alpha}{M + m_\alpha}. \quad (15)$$

Introducing the new relative coordinates $\rho_\alpha$, $\rho_A$, and $r$:

$$r_i = r_\alpha + \rho_\alpha, \quad r_j = R_A + \rho_A, \quad r = r_\alpha - R_A \quad (16)$$

we obtain:

$$r_i = R + \frac{M}{M + m_\alpha} r + \rho_\alpha, \quad r_j = R - \frac{m_\alpha}{M + m_\alpha} r + \rho_A. \quad (17)$$
As a result, we obtain the matrix element in the following form (see Appendix A):

$$
\langle f | \hat{H}_\gamma | i \rangle = - e \sqrt{\frac{2 \pi \hbar}{w}} \sum_{n=1,2} \epsilon^{(n)*} \delta(K_f - k) \cdot \left\{ \langle f_A, f_\alpha | \hat{Z}_{\text{eff}}(r) \hat{e}^{-ikr} \hat{p} | i_A, i_\alpha \rangle + \langle f_\alpha | e^{ikr} \frac{m_\alpha}{M + m_\alpha} \langle f_A Z_A(k) \hat{p}_{A_j} | i_A \rangle | i_\alpha \rangle \right\},
$$

(18)

where the indexes $i$ and $f$ denote the initial state (the state before photon emission) and the final state (after photon emission), $| s_A \rangle$ is wave function, describing the internal states of the daughter nucleus, $s_\alpha$ is wave function describing relative motion (with possible tunnelling) of the $\alpha$-particle concerning the daughter nucleus. Here, $\hat{Z}_{\text{eff}}(r)$, $Z_\alpha(k)$ and $Z_A(k)$ are, respectively, the effective charge of the system composed by the $\alpha$-particle and daughter nucleus, the charge form-factor of the $\alpha$-particle and the charge form-factor of the daughter (binary) nucleus, as defined in Appendix A.

Calculation of various contributions are presented in Fig. 6 (a). In order to analyze how the various contributions influence the spectra, we made calculation for the same fragmentation of the binary nucleus (we chose one fragment with $A_1 = 120$ and $Z_1 = 45$) and normalized the spectra to the experimental data. Such calculations are presented in Fig. 6 (b). This last figure shows that the component of emission caused by fragments of the binary nucleus well describes the experimental data at energies higher than 300 keV, while at lower energies the emission is mainly caused by the motion of the $\alpha$-particle in the field of the daughter nucleus.

4.2. Delay in the emission of the binary nucleus

Following the previous assumption, the total matrix element of emission can be written in two parts:

$$
\langle f | \hat{H}_\gamma | i \rangle = \langle f | \hat{H}_\gamma | i \rangle_\alpha + \langle f | \hat{H}_\gamma | i \rangle_{\text{binary}}.
$$

(19)
Here, the second term represents the emission of photons caused by the relative motion of two charged fragments of the daughter nucleus; it can be written as

\[
\langle f | \hat{H}_\gamma | i \rangle_{\text{binary}} = -e \sqrt{\frac{2\pi \hbar}{w}} \sum_{n=1,2} e^{(n),}\langle f | e^{i \mathbf{k} \cdot \mathbf{r}} \frac{m_\alpha}{M + m_\alpha} | i \rangle \cdot \langle f | Z_{\text{eff},A}(\rho) \mathbf{p}_\rho | i \rangle\frac{1}{i} = -e \sqrt{\frac{2\pi \hbar}{w}} \sum_{n=1,2} e^{(n),}\langle f | e^{i \mathbf{k} \cdot \mathbf{r}} \frac{m_\alpha}{M + m_\alpha} | i \rangle \cdot \langle f | Z_{\text{eff},A}(\rho) \mathbf{p}_\rho | i \rangle = \langle f | e^{i \mathbf{k} \cdot \mathbf{r}} \frac{m_\alpha}{M + m_\alpha} | i \rangle = 0 \tag{20}
\]

In this expression calculation of the first matrix element \(\langle f | \exp[i \mathbf{k} \cdot \mathbf{r}] \frac{m_\alpha}{M + m_\alpha} | i \rangle\) represents a relevant difficulty. In order to estimate it, in first approximation we use the following assumption:

\[
\langle f | e^{i \mathbf{k} \cdot \mathbf{r}} \frac{m_\alpha}{M + m_\alpha} | i \rangle \simeq N_i^2 = \frac{\mu}{k_i}, \tag{21}
\]

where \(N_i\) is the normalization factor of the wave function for the state before emission of photon (see Ref.s [10–13, 15] for details). In particular, applying such approximation, we obtained the spectra for the contributions of the binary nucleus presented in Fig. 6 (a) by lines which beyond the experimental data.

In order to find the next correction to the approximation (21), we should take into account dependence of this matrix element both on the wave vector \(\mathbf{k}\) of the emitted photon, and on the angle between vectors \(\mathbf{k}\) (i.e. direction of the emission of photon) and \(\mathbf{r}\) (i.e. direction of motion of \(\alpha\)-particle). On such a basis, we write the following (second) approximation as

\[
\langle f | e^{i \mathbf{k} \cdot \mathbf{r}} \frac{m_\alpha}{M + m_\alpha} | i \rangle \simeq \frac{\mu}{k_i} \cdot F[k, \cos(\mathbf{k}, \mathbf{r})]. \tag{22}
\]

where \(F[k, \cos(\mathbf{k}, \mathbf{r})]\) is a new unknown function. We find this function by such a way that it should include better description of the experimental data at low energies by the emission of the \(\alpha\)-binary nucleus and at high energies by the contribution of the fragment emission from the binary nucleus. In particular, such situation can be present in the following dynamical scheme of the ternary fission process:

(i) In the starting stage of ternary fission, \(\alpha\)-particle leaves a region of the nuclear system. It moves and emits photons at low energies. The fragments of binary nucleus are moved very slowly, and emission of photons can be neglected.

(ii) After, in the next stage of ternary fission, two fragments of the binary nucleus begin to move outside with acceleration. They give emission of photons at higher energies, which becomes prevailing.

Taking into account such a scheme, we assume for the function \(F[k, \cos(\mathbf{k}, \mathbf{r})]\) the form \(F = 1 - f_0/(1 + \exp((w - w_0)/w_1))\). Then, we obtain the unknown parameters by the fitting procedure where its inclusion into calculations should describe effectively experimental data in the whole energy region of the emitted photons (we have \(f_0 = 1 + \exp(-w_0/w_1)\), \(w_0 = w_1 \cdot \ln(\exp(w_2/w_1) - 2)\), \(w_1 = 0.057\), \(w_2 = 0.24\), and \(w\) is the energy of the emitted photons).

Results of calculation of spectra on basis of such assumptions are presented in Fig. 7 (a). As one can see, the new spectrum, (presented by solid green line) constructed on the basis of the second approximation (22) and the function \(F[k, \cos(\mathbf{k}, \mathbf{r})]\), is in agreement with the experimental data [20] for the spontaneous fission of the \(^{252}\text{Cf}\) nucleus.
According to this description, at higher energies (above 300 keV) the emission of photons is mainly determined by the contribution of the binary nucleus. So, at lower energies we have mainly the emission caused by the $\alpha$-particle in the field of the binary nucleus. By other words, in the energy range smaller than 300 keV, the analysis of the dependence of the spectrum on the length of the neck is useful. The results of calculations are presented in Fig. 7 (b). From this result we deduce that the optimal value of the neck is about of 11 fm, when the calculated curve is closest to the experimental data, in the assumed case of separation of the binary nucleus into two fragments with $A_1 = 120$, $Z_1 = 45$, and $A_2 = 128$, $Z_2 = 51$.

5. Conclusions

We have elaborated a new model describing the emission of the bremsstrahlung photons accompanying the ternary fission of heavy nuclei. For the chosen $^{252}$Cf nucleus, the general tendency of our calculated spectra of bremsstrahlung photon emission is in good agreement with the preliminary experimental data [22] at lower and higher photon energies in the energy range up to 500 keV (see Fig. 5). However, there is some disagreement between theory and experiment in region 200–400 keV of the emitted photons.

In order to improve the description of experimental data, also in this intermediate energy range we suggest a new way for the dynamics of the ternary fission process:

(i) when the $\alpha$-particle leaves the region of the ternary nuclear system, the binary nucleus is separated very slowly and gives an unimportant contribution to the total spectrum of the emitted photons;

(ii) at this stage, $\alpha$-particle moves fast and emits photons of low energies;

(iii) then, the binary nucleus begins to separate quickly, and fragments move with acceleration giving appreciable photon emission at high energies.

This model and mechanism allows us to achieve better results.
By comparing our spectra with the preliminary experimental data we find that: (1) the \( \alpha \)-particle is emitted when the length of the neck of the binary nucleus is 11 fm; (2) the probable separation of the binary nucleus is into two similar fragments.

Appendix A.

Let us write the operator of photon emission of the nuclear system decaying in the laboratory system by \( \alpha \)-particle and daughter nucleus, where \( \alpha \)-particle and daughter nucleus are composed of nucleons (we describe nucleons of \( \alpha \)-particle by index \( i \), and nucleons of the nucleus by index \( j \)):

\[
\hat{H}_\gamma = - \sum_{i=1}^{4} \frac{e Z_i}{m_i c} A(r_i, t) \hat{p}_i - \sum_{j=1}^{A} \frac{e Z_j}{m_j c} A(r_j, t) \hat{p}_j.
\]  

(A.1)

By using the following form for the potential of the electromagnetic field:

\[
A(r, t) = \sqrt{\frac{2 \pi \hbar c^2}{w}} \sum_{n=1,2} e^{(n)*} e^{-ikr},
\]  

(A.2)

we rewrite the operator of emission (A.1) as the used expression (14).

By using the relative coordinates \( \rho_{\alpha i} \), \( \rho_{A j} \) and \( r \) defined in relation (16) we find the corresponding momenta:

\[
\begin{align*}
\mathbf{p}_i &= -i \hbar \frac{d}{dr_i} = \mathbf{p}_\alpha + \mathbf{p}_{\alpha i}, \\
\mathbf{p}_\alpha &= -i \hbar \frac{d}{dr_\alpha}, \\
\mathbf{p}_{\alpha i} &= -i \hbar \frac{d}{dr_{\alpha i}}, \\
\mathbf{p}_j &= -i \hbar \frac{d}{dr_j} = \mathbf{p}_A + \mathbf{p}_{A j}, \\
\mathbf{p}_A &= -i \hbar \frac{d}{d\mathbf{R}_A}, \\
\mathbf{p}_{A j} &= -i \hbar \frac{d}{d\rho_{A j}}, \\
\mathbf{p} &= \mathbf{p}_\alpha - \mathbf{p}_A.
\end{align*}
\]  

(A.3)

Taking into account the relative coordinate (16), we obtain

\[
\mathbf{R}_A = \mathbf{R} - \frac{m_\alpha}{M + m_\alpha} \mathbf{r}, \\
\mathbf{r}_\alpha = \mathbf{R} + \frac{M}{M + m_\alpha} \mathbf{r}
\]  

(A.4)

and we can rewrite the first and second expressions in (16) as

\[
\begin{align*}
\mathbf{r}_i &= \mathbf{R} + \frac{M}{M + m_\alpha} \mathbf{r} + \rho_{\alpha i}, \\
\mathbf{r}_j &= \mathbf{R} - \frac{m_\alpha}{M + m_\alpha} \mathbf{r} + \rho_{A j}.
\end{align*}
\]  

(A.5)

Substituting these expressions in relation (14), we find:

\[
\hat{H}_\gamma = - e \sqrt{\frac{2 \pi \hbar}{w}} \sum_{n=1,2} e^{(n)*} e^{-ik\left[\mathbf{R} - \frac{m_\alpha}{M + m_\alpha} \mathbf{r}\right]} \times
\]

\[
\times \left\{ e^{-ikr} \sum_{i=1}^{4} \frac{Z_i}{m_i} e^{-i k \rho_{\alpha i}} + \sum_{j=1}^{A} \frac{Z_j}{m_j} e^{-i k \rho_{A j}} \right\} \mathbf{p}_+ +
\]

\[
+ \left[ e^{-ikr} \frac{M}{M + m_\alpha} \sum_{i=1}^{4} \frac{Z_i}{m_i} e^{-i k \rho_{\alpha i}} - \frac{m_\alpha}{M + m_\alpha} \sum_{j=1}^{A} \frac{Z_j}{m_j} e^{-i k \rho_{A j}} \right] \mathbf{p}_+ +
\]

\[
+ e^{-ikr} \sum_{i=1}^{4} \frac{Z_i}{m_i} e^{-i k \rho_{\alpha i}} \mathbf{p}_{\alpha i} + \sum_{j=1}^{A} \frac{Z_j}{m_j} e^{-i k \rho_{A j}} \mathbf{p}_{A j} \right\}.
\]  

(A.6)
Now we present the wave function of the total system in the following expression (see Ref. [23]):

$$|s\rangle = e^{-iK_i R} s_\alpha |s_A\rangle,$$

(A.7)

where $s = i$ or $f$ are the indexes denoting the initial state (before emission of photon) and the final state (after emission of photon), $\hbar K_i$ is the full momentum of the total system, $|s_A\rangle$ is the wave function describing internal states of the daughter nucleus, and $s_\alpha$ is the wave function describing the relative motion (with possible tunneling) of the $\alpha$-particle concerning the daughter nucleus. Assuming $K_i = 0$, we calculate the matrix element:

$$\langle f | \hat{H}_\gamma | i \rangle = -e^{\frac{2\pi \hbar}{\omega}} \sum_{n=1,2} e(n) \cdot \left\{ \langle f_\alpha, f_A | e^{i(K_f - k)R} e^{ikr} \frac{m_\alpha}{M + m_\alpha} \times 
\left[ e^{-ikr} \sum_{i=1}^{\frac{A}{2}} Z_i e^{-ikp_{\alpha_i}} + \sum_{j=1}^{A} \frac{Z_j}{m_j} e^{-ikp_{\alpha_j}} \right] \mathcal{P} | i_\alpha, i_A \rangle + 
\langle f_\alpha, f_A | e^{i(K_f - k)R} e^{ikr} \frac{m_\alpha}{M + m_\alpha} \left[ e^{-ikr} \sum_{i=1}^{\frac{A}{2}} Z_i e^{-ikp_{\alpha_i}} \right] \mathcal{P} | i_\alpha, i_A \rangle + 
\langle f_\alpha, f_A | e^{i(K_f - k)R} e^{ikr} \frac{m_\alpha}{M + m_\alpha} \sum_{j=1}^{A} \frac{Z_j}{m_j} e^{-ikp_{\alpha_j}} \mathcal{P}_{\alpha_i} | i_\alpha, i_A \rangle + 
\right\}.$$

(A.8)

In our calculation we shall not use the first term of expression (A.8), since we study process of fission in the center-of-mass system, and the third term because we assume that $\alpha$-particle is not deformed in the studied process of fission. Now, introducing the effective charge of the system ($\alpha$-particle and daughter nucleus) as

$$Z_{\text{eff}}(r) = e^{ikr} \left\{ e^{-ikr} \frac{M Z_\alpha}{M + m_\alpha} - e^{-ikr} \frac{m_\alpha Z_{\alpha_i}}{M + m_\alpha} \right\},$$

(A.9)

the charged form-factor of $\alpha$-particle as

$$Z_{\alpha}(k) = \left\langle \beta_f | \sum_{i=1}^{\frac{A}{2}} \frac{Z_i}{m_i} e^{-ikp_{\alpha_i}} | \beta_i \right\rangle,$$

(A.10)

and the charged form-factor of the daughter (binary) nucleus as

$$Z_{\alpha}(k) = \left\langle \beta_f | \sum_{j=1}^{A} \frac{Z_j}{m_j} e^{-ikp_{\alpha_j}} | \beta_i \right\rangle.$$
and by using (A.9) we obtain

$$\langle f | \hat{H}_\gamma | i \rangle = - e \sqrt{\frac{2\pi}{w}} \sum_{n=1,2} e^{(n)\ast} \delta(K_f - k) \cdot \left\{ \langle f_A, f_\alpha | Z_{\text{eff}}(r) e^{-i kr} p | i_A, i_\alpha \rangle + \right.$$  

$$+ \langle f_\alpha | e^{i kr} \frac{m_\alpha}{M + m_\alpha} \langle f_A | Z_A(k) p_{A_j} | i_A \rangle | i_\alpha \rangle \right\}.$$  

(A.12)

In particular, in the first approximation (so called dipole) we obtain:

$$Z_{\text{eff}}^{(\text{dip})} = \frac{M Z_\alpha - m_\alpha Z_A}{M + m_\alpha}.$$  

(A.13)

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