Fundamental properties of solar-like oscillating stars from frequencies of minimum $\Delta \nu$: II. Model computations for different chemical compositions and mass

M. Yıldız*, Z. Çelik Orhan and C. Kayhan

Department of Astronomy and Space Sciences, Science Faculty, Ege University, 35100, Bornova, Izmir, Turkey.

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ABSTRACT

The large separations between the oscillation frequencies of solar-like stars are measures of stellar mean density. The separations have been thought to be mostly constant in the observed range of frequencies. However, detailed investigation shows that they are not constant, and their variations are not random but have very strong diagnostic potential for our understanding of stellar structure and evolution. In this regard, frequencies of the minimum large separation are very useful tools. From these frequencies, in addition to the large separation and frequency of maximum amplitude, Yıldız et al. recently have developed new methods to find almost all the fundamental stellar properties. In the present study, we aim to find metallicity and helium abundances from the frequencies, and generalize the relations given by Yıldız et al. for a wider stellar mass range and arbitrary metallicity ($Z$) and helium abundance ($Y$). We show that the effect of metallicity is significant for most of the fundamental parameters. For stellar mass, for example, the expression must be multiplied by $(Z/Z_\odot)^{0.12}$. For arbitrary helium abundance, $M \propto (Y/Y_\odot)^{0.25}$. Methods for determination of $Z$ and $Y$ from pure asteroseismic quantities are based on amplitudes (differences between maximum and minimum values of $\Delta \nu$) in the oscillatory component in the spacing of oscillation frequencies. Additionally, we demonstrate that the difference between the first maximum and the second minimum is very sensitive to $Z$. It also depends on $\nu_{\text{min}1}/\nu_{\text{max}}$ and small separation between the frequencies. Such a dependence leads us to develop a method to find $Z$ (and $Y$) from oscillation frequencies. The maximum difference between the estimated and model $Z$ values is about 14 per cent. It is 10 per cent for $Y$.

Key words: stars: evolution – stars: interior – stars: late-type – stars: oscillations.

1 INTRODUCTION

Determination of fundamental stellar properties is required in many sub-fields of astrophysics. In this regard, the promise of asteroseismology is very important, in particular for solar-like oscillating stars. Most of the oscillations, at least for main-sequence (MS) stars, are acoustic pressure waves and their frequencies depend on sound speed throughout stellar interior. Sound speed, however, depends on the first adiabatic exponent $\Gamma_1$, which is very low in the He ionization zone. The He ionization zone of the solar-like oscillating stars is not too shallow, and its upper and lower borders coincide with the nodes of certain modes. It has so significant effect on the oscillation frequencies that one can infer basic properties of such stars from these frequencies. The present study develops new methods for this purpose.

Solar-like oscillating stars have such regular oscillation frequencies ($\nu$) that the frequency of a mode linearly depends on its order $n$ and degree $l$. This dependence is known as the asymptotic relation. According to this idea, the frequencies of modes with adjacent orders and the same $l$ are evenly spaced by the so-called large separation, $\Delta \nu = \nu_{nl} - \nu_{n-1,l}$. In reality, however, there are some deviations from this simple relation, and these variations lead us to discover parameters related to stellar structure and evolution. For this purpose, Yıldız et al. (2014; hereafter Paper I) introduce two new reference frequencies $\nu_{\text{min}1}$ and $\nu_{\text{min}2}$, which are the frequencies at which the large separations between the oscillation frequencies are minimum. In Paper I, new expressions for fundamental stellar properties, such as stellar mass ($M$), radius ($R$), surface gravity ($g$), luminosity ($L$), effective temperature ($T_{\text{eff}}$) and age ($t$), are derived from the interior models of 0.8–1.3 $M_\odot$ with solar composition. The present study aims to generalize these expressions for the wider mass range (1.0–1.6 $M_\odot$) than this range and to test the effects of the metallicity ($Z$) and helium abundance ($Y$) on these relations. We also try to determine $Y$ and $Z$ from the oscillation frequencies, at least for interior models.
The frequencies of MS models are used in deriving expressions for age and other quantities. For sub-giants and giants, these relations must be tested.

This paper is organized as follows. In Section 2, we make general considerations about the reference frequencies. Section 3 is devoted to generalizing the relations obtained in Paper I for wider mass range and arbitrary metallicity and helium abundance. We present the sensitivity of the adiabatic oscillation frequencies to the metallicity and helium abundance, and new methods for determination of $Y$ and $Z$ from oscillation frequencies in Section 4. Finally, in Section 5, we draw our conclusions.

2 GENERAL CONSIDERATIONS

The asteroseismic parameters that can be extracted from oscillation frequencies and related to the stellar parameters are $\Delta \nu$, the frequency of the maximum amplitude ($\nu_{\text{max}}$), and small separation between the oscillation frequencies ($\delta \nu_{\text{min}} = \nu_{\text{min1}} - \nu_{\text{min2}}$). Brown et al. (1991) give $\nu_{\text{max}}$ as (see also Kjeldsen & Bedding 1995)

$$\nu_{\text{max}} \propto \frac{M}{R^2} T_{\text{eff}}^{-1/2}.$$  \hspace{1cm} (1)

In addition to these parameters, we introduce $\nu_{\text{min1}}$ and $\nu_{\text{min2}}$. Their ratio to $\nu_{\text{max}}$ gives us the stellar mass:

$$\nu_{\text{max}} = 1.188 \nu_{\text{min1}} \frac{M_{\odot}}{M} = 1.623 \nu_{\text{min2}} \frac{M_{\odot}}{M}.$$  \hspace{1cm} (2)

where the numeric values 1.188 and 1.623 come from the ratio $\nu_{\text{max}}/\nu_{\text{min1}}$ and $\nu_{\text{max}}/\nu_{\text{min2}}$, respectively. If we insert equation (2) in equation (1), expression for $\nu_{\text{min1}}$ and $\nu_{\text{min2}}$ in terms of fundamental stellar parameters is derived as

$$\nu_{\text{min1}} \propto \nu_{\text{min2}} \propto \frac{M^2}{R^2} T_{\text{eff}}^{-1/2}.$$  \hspace{1cm} (3)

Equation (3) is valid at least for the models with solar values and mass ranging from 0.8 $M_{\odot}$ to 1.3 $M_{\odot}$ presented in Paper I.

Stellar parameters change with chemical composition. Therefore, one can expect that $\nu_{\text{max}}$ is also a sensitive function of $Z$ and $Y$, and thus equation (1) will need to be modified to take these effects into account. For the relations derived in Paper I, it is also important to understand how the other asteroseismic quantities are influenced by change in $Z$ and $Y$ (see Section 3.2).

Inference of the solar helium surface abundance ($Y_s$) from high degree ($l > 40$) oscillation frequencies of the Sun is the subject of many studies. Basu & Antia (1995), for example, give $Y_s$ as 0.25 ± 0.01. Unfortunately, high degree oscillations are not observable for other stars. Houdek & Gough (2007) confirm that the amplitude of the second difference of the frequencies depends on the helium abundance. Recently, Verma et al. (2014) find helium abundances from Keplar data for 16 Cyg A and B. In Paper I, we have used the frequencies at which $\Delta \nu$ is minimum for determination of stellar parameters. In the present study, however, we use the difference between the maximum and minimum values of $\Delta \nu$ for determination of helium abundance (see Section 4.1). A similar approach may also lead us to develop a new method for inference of metallicity from oscillation frequencies.

3 GENERAL RELATIONS FOR STELLAR PARAMETERS FROM ASTEROSEISMIC QUANTITIES

In this study, as in Paper I, the stellar interior models with solar chemical composition are constructed by using the ANKI code (Ezer & Cameron 1965; Yildiz 2011). The solar chemical composition is taken as $X = 0.7024$, $Y = 0.2804$ and $Z = 0.0172$. The adiabatic oscillation frequencies of these models are computed by ADIPLS (Christensen-Dalsgaard 2008). In this section, we test if the relations derived in Paper I are also valid for $1.3 M_{\odot} < M \leq 1.6 M_{\odot}$ and arbitrary $Z$ and $Y$.

Both of the minima do not appear in all the models within the mass range we deal with. For example, the second minimum is not seen in the oscillation frequencies of the models with $M < 1.0 M_{\odot}$ (see table 1 of Paper I) while the first minima disappears in some models within the upper mass range of 1.3-1.6 $M_{\odot}$. Therefore, our present analysis is based on the frequency of the second minimum of the models with mass range 1.0-1.6 $M_{\odot}$.

3.1 Expressions for stellar parameters for the mass range 1.0-1.6 $M_{\odot}$

In Paper I, the expressions for stellar mass in terms of $\nu_{\text{max}}$ and one of $\nu_{\text{min1}}$ or $\nu_{\text{min2}}$ are derived from the interior models for the mass range 0.8-1.3 $M_{\odot}$. For the Sun, $\nu_{\text{max}}/\nu_{\text{min1}} = 3050$ $\mu$Hz is higher than both $\nu_{\text{min1}} = 2555.18$ $\mu$Hz and $\nu_{\text{min2}} = 1879.52$ $\mu$Hz. For 1.2 $M_{\odot}$ models with solar composition $\nu_{\text{max}}$ is equal to $\nu_{\text{min1}}$. For the models with 1.2 $M_{\odot} < M < 1.45 M_{\odot}$, $\nu_{\text{min2}} < \nu_{\text{min1}} < \nu_{\text{min1}}$. And, $\nu_{\text{min2}} > \nu_{\text{max}}$ for the models with $M > 1.45 M_{\odot}$ (see Fig. 2). Oscillation amplitudes of these models with frequencies around $\nu_{\text{min2}}$ are greater than that of around $\nu_{\text{min1}}$. $M_2$ is the mass computed from $\nu_{\text{min2}}$ (see equation 10 of Paper I). $M_2/M_\odot = 1.623 \nu_{\text{min2}}/\nu_{\text{max}}$. In Fig. 1, $M_2$ with $X_c = 0.17$, 0.35, 0.53 and 0.7 is plotted with respect to model mass. The model mass and mass found from the ratio of frequencies are in very good agreement for $M < 1.3 M_{\odot}$. For higher mass models, however, a deviation occurs. Here, we obtain a relation between mass and the frequency ratio $\nu_{\text{min2}}/\nu_{\text{max}}$ for the range $M = 1.3-1.6 M_{\odot}$ of

$$\frac{M}{M_\odot} = (0.462 \nu_{\text{min2}}/\nu_{\text{max}} - 0.356)^{0.74} + 1.254.$$  \hspace{1cm} (4)

We note that the mass still is found from the same frequency ratio $\nu_{\text{min2}}/\nu_{\text{max}}$. The reason for the deviation of mass from the expression given in equation 10 of Paper I is due to change in properties of $\nu_{\text{min2}}$ for relatively higher masses. In Fig. 2, $\nu_{\text{min2}}, \nu_{\text{min1}}$ and $\nu_{\text{max}}$ of models are plotted with respect to model mass. $\nu_{\text{max}}$ gradually decreases from 2600 to 1000 $\mu$Hz as model mass increases. $\nu_{\text{min2}}$ and $\nu_{\text{min1}}$, however, decrease in the low-mass range (1.0-1.4 $M_{\odot}$) and increase in the high-mass range (1.4-1.6 $M_{\odot}$). There is a minimum for $\nu_{\text{min2}}$ of about 1.4 $M_{\odot}$. Therefore, the ratio $\nu_{\text{min2}}/\nu_{\text{max}}$ does not directly give mass. If $M > 1.3 M_{\odot}$, the method for computation of stellar mass can be still based on this ratio. First, we can compute stellar mass from equation 10 of Paper I. If the mass is greater than 1.3 $M_{\odot}$, then we use equation (4). The maximum difference between the mass computed from equation (4) and model mass for $M > 1.3 M_{\odot}$ is less than 0.025 $M_{\odot}$, as in Paper I. More realistic error analysis than this can be made in terms of uncertainties in observed frequencies and will be given in our third paper of this series.

The expressions for other stellar parameters given in Paper I must also be tested for the mass range 1.3-1.6 $M_{\odot}$. For radius, we obtain

$$\frac{R}{R_\odot} = 1.024 \left( \frac{\nu_{\text{min2}}}{\nu_{\text{min1}}/2} \right)^{0.09} \left( \frac{\Delta \nu_{\text{min2}}}{\Delta \nu_{\text{min1}}/2} \right)^{0.83},$$  \hspace{1cm} (5)
where $\langle \Delta \nu \rangle$ is the mean of $\Delta \nu$. In Fig. 3, radii computed by using equation 17 of Paper I and equation (5) are plotted with respect to model radii. The radii computed from equation (5) are in very good agreement with model radii. The difference between these two radii is mostly less than 2 per cent. For the models with $M > 1.3 \, M_{\odot}$, a slight difference appears between model radius and radius from equation 17 of Paper I.

We want to check if expressions for $L$ and $t$ for the mass range $0.8-1.3 \, M_{\odot}$, given in equations 20 and 22 of Paper I, respectively, are also valid for the range $1.3-1.6 \, M_{\odot}$. We have obtained an expression for luminosity for the entire range $1.0-1.6 \, M_{\odot}$, but that expression has very high departure from the model values for the lower part of the range, about 20 per cent. Therefore, we derive a separate expression for the range $1.3-1.6 \, M_{\odot}$:

$$
\frac{L}{L_{\odot}} = 2.016 \left( \frac{\nu_{\text{min}2}}{\nu_{\text{min}1 \odot}} \right)^{1.3} \frac{\langle \Delta \nu \odot \rangle}{\langle \Delta \nu \rangle} \frac{\nu_{\max \odot}}{\nu_{\max}} - 0.456. \quad (6)
$$

The agreement between the estimated and model luminosities is shown in Fig. 4. For the estimated luminosity, equation 20 of Paper I is used if $M \leq 1.3 \, M_{\odot}$ and equation (6) is employed if $M > 1.3 \, M_{\odot}$.

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**Figure 1.** $M_2/M_\odot = 1.623 \nu_{\text{min}2}/\nu_{\max}$ with respect to model mass for $X_e = 0.17$, 0.35, 0.53 and 0.7.

**Figure 2.** $\nu_{\max}$ (circles), $\nu_{\text{min}1}$ (filled circles) and $\nu_{\text{min}2}$ (boxes) with respect to model mass, for $X_e = 0.17$.

**Figure 3.** Asteroseismic $R$ with respect to model radius. The open circles show the radii computed from equation 17 of Paper I and the filled circles are for the radii from equation (5). The radii from these equations are very close but equation (5) is in better agreement with the radii of models with $M > 1.3 \, M_{\odot}$.

**Figure 4.** The asteroseismic luminosity with respect to model luminosity. Equation 20 of Paper I and equation (6) are used for the models with $M \leq 1.30 \, M_{\odot}$ (circles) and $M > 1.30 \, M_{\odot}$ (filled circles), respectively.
Metallicity strongly influences structure and evolution of stars, and consequently asteroseismic properties. Luminosity among the non-asteroseismic quantities is the most influenced parameter by $Z$ and $Y$. It decreases as metallicity increases, while radius slightly diminishes, in the range we consider. This implies that effective temperature also decreases. Then, we confirm from equation (1) that there is a direct relation between $\nu_{\text{max}}$ and $Z$. The effect of metallicity on the relation for stellar mass can be given as

$$
\frac{M}{M_{\odot}} = \frac{\nu_{\text{min2}}}{\nu_{\text{min2,0}}} \frac{\nu_{\text{max,0}}}{\nu_{\text{max}}} \left( \frac{Z}{Z_{\odot}} \right)^{0.12}.
$$

The same equation also holds for $\nu_{\text{min1}}$. Equation (8) is more precise than equation 23 in Paper I, in which the power of $(Z/Z_{\odot})$ is found to be roughly 0.1. These equations are derived for the first time. The $Z$ dependence in equation (8) is not large. For a metallic rich star, say $Z = 2Z_{\odot}$, the effect is about 9 per cent. However, for stellar mass such an effect is very significant.

Radii computed by using equation (5) are plotted with respect to model radii in Fig. 6. The slope slightly changes with $Z$; it decreases as $Z$ increases. Equation (5) for radius for solar metallicity is generalized for arbitrary $Z$ as

$$
\frac{R}{R_{\odot}} = a(Z) \left( \frac{\nu_{\text{min2}}}{\nu_{\text{min2,0}}} \right)^{0.13} \left( \frac{\Delta \nu_{\odot}}{\Delta \nu} \right)^{0.87} + b(Z),
$$

where

$$
a(Z) = 0.210 \left( \frac{Z}{Z_{\odot}} \right) + 0.783
$$

and

$$
b(Z) = -0.019 \left( \frac{Z}{Z_{\odot}} \right)^{3.32} + 0.011.
$$

Radii computed from equation (9) are plotted with respect to model radii in Fig. 7. The agreement is very good for the full range of $Z = 0.0172-0.0322$.

One of the very important relations obtained in Paper I is the one between $T_{\text{eff}}$ and $\Delta \nu_{\text{X1}}$, defined as $(\nu_{\text{max}} - \nu_{\text{min1}})/\Delta \nu$ (see Fig. 6 in Paper I). We consider if this relation varies with $Z$. In

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5.png}
\caption{Age inferred from asteroseismic quantities is the most influenced parameter by $Z$. The difference between the estimated and model luminosities is mostly less than 8 per cent. We do not derive a separate fitting curve for $T_{\text{eff}}$ since $T_{\text{eff}}$ can easily be obtained from $L$ and $R$. For alternative expressions, see equations (12) and (13).}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure6.png}
\caption{$R$ computed from equation (5) is plotted with respect to $R_{\text{mod}}$ for different metallicities. The upper solid line is the fitted line for $Z = 0.0172$ and the lower dotted line is for $Z = 0.0322$. The other lines between them are for the intermediate values of $Z$.}
\end{figure}
model \( T_{\text{eff}} \) is mostly less than 100 K. Similarly, we derive an alternative expression for \( T_{\text{eff}} \) in terms of \( \Delta n_{x_2} \), defined as \((\nu_{\text{max}} - \nu_{\text{min}_2})/\Delta \nu\):

\[
\frac{T_{\text{eff}}(Z, \Delta n_{x_2})}{T_{\text{eff}}(\odot)} = 1.275 - 0.061 \left( \frac{Z}{Z_{\odot}} \right) - 6.06 \times 10^{-4} (\Delta n_{x_2} + 6)^{2.2}.
\]  

(13)

These two expressions (equations 12 and 13) for \( T_{\text{eff}} \) are very important for determination of stellar parameters from oscillation frequencies. They can be used to check how precise is \( T_{\text{eff}} \) found by conventional methods. If observational \( T_{\text{eff}} \) is precisely determined, then one can obtain metallicity from these relations. The difference between two curves in Fig. 8 for a given \( \Delta n_{x_1} \) is about 250 K. The metallicity difference for the models represented by these curves is 0.0150. This implies that we can find \( Z \) with an uncertainty of \( \Delta Z = 0.015 \). If uncertainty in \( T_{\text{eff}} \) is 250 K. However, if \( \Delta T_{\text{eff}} = 100 \) K, for example, \( \Delta Z = 0.006 \).

For some solar-like oscillating stars, only one minimum appears in the \( \Delta \nu - \nu \) diagram. In order to use asteroseismic relations for determination of fundamental stellar parameters, we must find out whether the seen minimum is \( \text{min}_1 \) or \( \text{min}_2 \) (see Fig. 12). If \( T_{\text{eff}} \) and \( Z \) of a given star are known, then we can overcome this problem again by using equations (12) and (13). However, role of the convective parameter \( \alpha \) must also be tested.

The mass of the convective zone (\( M_{\text{CZ}} \)) significantly depends on metallicity. The generalized form of \( M_{\text{CZ}}/M_{\odot} = 0.066 \Delta n_{x_1} \) (see eq. 6 of Paper I) is

\[
\frac{M_{\text{CZ}}}{M_{\odot}} = (0.225Z + 0.0027) \Delta n_{x_1} + 0.627Z - 0.011
\]  

(14)

for arbitrary \( Z \). \( M_{\text{CZ}} \) is directly proportional to \( Z \), as expected.

### 3.2.2 Effects of helium abundance

All of the models used in analysis of the metallicity effect are constructed with the helium abundance \( Y = 0.2804 \). In order to test influence of helium abundance on various asteroseismic relations, we also obtain models with \( Y = 0.2404, 0.2604, 0.2804, 0.3004 \) and 0.3204. \( T_{\text{eff}} - \Delta n_{x_1} \) and \( T_{\text{eff}} - \Delta n_{x_2} \) relations do not depend on \( Y \). For stellar mass, we confirm that there is a moderate \( Y \) dependence:

\[
\frac{M}{M_{\odot}} = \frac{\nu_{\text{min}_2}}{\nu_{\text{min}_2}^\odot} \frac{\nu_{\text{max}_2}}{\nu_{\text{max}_2}^\odot} \left( \frac{Y}{Y_{\odot}} \right)^{0.25}.
\]  

(15)

Mass computed from equation (15) is plotted with respect to model mass in Fig. 9. Although the data are scattered, the models are populated around the \( M_Y = M_{\text{mod}} \) line. This implies that scaling relation also depends on helium abundance. Thus, in order to find stellar mass and radius from asteroseismic quantities we also need the helium abundance. Therefore, we either have to assume that the helium abundance does not vary much from star to star or find a new method for determination of the helium abundance (see Section 4.1).

If we combine equations (8) and (15), then the seismic mass is proportional to \((Z/Z_{\odot})^{0.12}(Y/Y_{\odot})^{0.25} \). Note especially that the power of \( Y \) is twice the power of \( Z \). The effect of metallicity on stellar mass is in general more important than that of \( Y \), because range of \( Z/Z_{\odot} \) is much greater than \( Y/Y_{\odot} \).

Equation (5) for radius is not sensitive to \( Y \) and therefore it can also be used for models with any helium abundance different from \( Y_{\odot} \).
4 DETERMINATION OF CHEMICAL COMPOSITION FROM OSCILLATION FREQUENCIES

The sound speed in a stellar interior is a function of the first adiabatic exponent $\Gamma_1$, pressure ($P$) and density ($\rho$): $c = \sqrt{\Gamma_1 P/\rho}$. The local values of $\Gamma_1$ in the He ionization zones, however, depend on the He (or H) abundance and influence the spacing of oscillation frequencies. The amplitude of the oscillatory component in the spacing is determined by the He abundance (Houdek & Gough 2011). Similar correlation can be sought for metallicity. In the present section, we plot $\Delta \nu - \langle \Delta \nu \rangle$ with respect to $n$ for interior models with different chemical composition and try to find relations between amplitudes and chemical abundances.

4.1 Determination of helium abundance from oscillation frequencies

Determination of helium abundance from the second difference of oscillation frequencies is extensively discussed in several papers (see, e.g. Miglio et al. 2010; Mazumdar et al. 2014). Due to the effect of the He n ionization zone on the sound speed, helium (or hydrogen) abundance influences the spacing between the oscillation frequencies. We plot $\Delta \nu - \langle \Delta \nu \rangle$ with respect to $n$ for the interior models with the same input parameters but helium abundance. In general, two maxima and two minima are seen in such a graph. We already call the minima with the higher frequency (or order) as min1 and the lower one as min2. In a similar manner, we define max1 and max2 ($A_Y$) is a function of helium abundance. This is shown in Fig. 10, in which $\Delta \nu - \langle \Delta \nu \rangle$ is plotted with respect to order $n$. The frequencies are from three 1.0 $M_\odot$ models with $Z = 0.0172$ and $X_e = 0.17$, but different helium abundances. Their helium abundances are $Y = 0.2804$, 0.3004 and 0.3204. The highest amplitude occurs for the model with $Y = 0.3204$. $A_Y$ gradually increases as $Y$ increases. This dependence has a diagnostic potential for determination of helium abundance. However, $A_Y$ is not only a function of $Y$ but also $\Delta \nu$, $M$ (or $\nu_{\text{min}}/\nu_{\text{max}}$) and $\langle \delta \nu_2 \rangle$. Using the models with $Y = 0.2404, 0.2604, 0.2804, 0.3004$ and 0.3204, and $M = 1-1.3$ $M_\odot$, we derive an expression for $Y$ as

$$Y = 10.74 \frac{A_Y}{\Delta \nu} \left( \frac{M_\odot}{M} \right)^{1.5} + 0.0034 \langle \delta \nu_2 \rangle + 0.071. \quad (16)$$

$Y$ is not only a function of $Y$ but also $\Delta \nu$, $M$ (or $\nu_{\text{min}}/\nu_{\text{max}}$) and $\langle \delta \nu_2 \rangle$. Using the models with $Y = 0.2404, 0.2604, 0.2804, 0.3004$ and 0.3204, and $M = 1-1.3$ $M_\odot$, we derive an expression for $Y$ as

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$$Y = 10.74 \frac{A_Y}{\Delta \nu} \left( \frac{M_\odot}{M} \right)^{1.5} + 0.0034 \langle \delta \nu_2 \rangle + 0.071. \quad (16)$$
The estimated \( Y \) by using equation (16) is plotted with respect to model \( Y \) in Fig. 11. There is a good agreement between estimated and model \( Y \) values. The maximum difference between the estimated and model helium abundances is about 10 per cent. Equation (16) can be used to estimate helium abundance of \textit{Kepler} and \textit{CoRoT} target stars.

Since helium abundance in the He \( n \) ionization zone moderately changes due to microscopic diffusion, equation (16) does not directly give us initial helium abundance if age is not very small. The value we find is the present value of \( Y \), and it can be used as a constraint during calibration of interior models. However, there is significant difference between \( A_Y \) of the Sun and solar model. While \( A_Y \) of the Sun is about 3 \( \mu \)Hz, it is about 2 \( \mu \)Hz for the solar models. The so-called near-surface effects may influence \( A_Y \). If the oscillation frequencies are corrected by using the method given by Kjeldsen, Bedding & Christensen-Dalsgaard (2008), we obtain \( A_Y = 2.3 \mu \)Hz. Although the near-surface effects decrease the discrepancy between observed and model values, the remaining part is still significant. Consideration of \textit{Kepler} and \textit{CoRoT} stars is required if this problem can be solved by a simple method based on calibration approach.

### 4.2 Determination of metallicity from oscillation frequencies

The effect of metallicity on the amplitude is much stronger if we take the amplitude between max1 and min2. This fact is sketched in Fig. 12 in which \( \Delta \nu - \langle \Delta \nu \rangle \) is plotted with respect to order \( n \) for 1.25 \( M_{\odot} \) models with \( X_c = 0.35 \) and different \( Z \) values. The first maximum (in the right part of Fig. 12) and the second minimum (in the left part) are significantly sensitive to metallicity. The difference between max1 and min2 (\( A_Z \)) varies very rapidly as \( Z \) changes. It is slightly less than 4 \( \mu \)Hz for \( Z = 0.0172 \) and about 2 \( \mu \)Hz for \( Z = 0.0222 \). It is very small for \( Z = 0.0322 \) (about 0.5 \( \mu \)Hz) and becomes negative for some higher values of \( Z \). The amplitude \( A_Z \) is not only function of \( Z \) but also \( M \) (or \( \nu_{\text{min1}}/\nu_{\text{max}} \)) and \( \langle \delta \nu_{\text{02}} \rangle \):

\[
A_Z = 3.606 \frac{M}{M_{\odot}} \frac{Z}{Z_{\odot}} + 1.345 \frac{M}{M_{\odot}} - 0.141 \langle \delta \nu_{\text{02}} \rangle - 2.105. \quad (17)
\]

From equation (17), we take \( Z/Z_{\odot} \) to the left-hand side and find an expression for \( Z \) in terms of asteroseismic quantities as

\[
Z = \frac{Z_{\odot}}{A_Z - 1.345(M/M_{\odot}) + 0.141 \langle \delta \nu_{\text{02}} \rangle + 2.105}. \quad (18)
\]

In Fig. 13, estimated metallicity given in equation (18) is plotted with respect to model metallicity. The agreement between these two metallicities is very good. The maximum difference between the two metallicities is about 14 per cent.

For 1.0 \( M_{\odot} \) models with \( Z > 0.0322 \), the second minimum disappears. This implies that, at least for MS stars, if the second minimum of a star is not observed then either its mass is less than 0.9 \( M_{\odot} \) (see table 1 of Paper I) or its metallicity is higher than 0.0322.

### 5 CONCLUSION

In Paper I, we have found new reference frequencies from the oscillation frequencies and derived new relations between asteroseismic quantities and all the fundamental stellar parameters. These relations are based on interior models for the mass range 0.8-1.3 \( M_{\odot} \) with the solar composition. In this study, the mass range is extended to 1.6 \( M_{\odot} \) and obtain new relations for arbitrary \( Z \) and \( Y \).

For the mass range \( M > 1.3 \ M_{\odot} \), the expressions given in Paper I are not valid any more and therefore we derive new relations for \( M, R, L, T_{\text{eff}}, M_{\text{CZ}} \) and \( t \).

Metallicity also affects stellar structure and evolution significantly, and consequently affects oscillation frequencies. The derived relations are in general different for different \( Z \). We develop new relations valid for arbitrary \( Z \). The relation between \( T_{\text{eff}}, Z \) and \( Z_{\odot} \) is plotted with respect to order \( n \) for 1.25 \( M_{\odot} \) models with \( X_c = 0.35 \) and different metallicities (\( Z = 0.0172, 0.0222 \) and 0.0322) is plotted with respect to \( n \). The amplitude is defined by the vertical arrows. It strongly depends on \( Z \). It decreases as metallicity increases. The amplitude is very small for the model with \( Z = 0.0322 \).
and $\Delta n_{x1}$ (equation 12) is in particular very useful and can be employed to determine of any of these quantities. We also obtain a similar relation for $\Delta n_{x2}$ (equation 13).

The relations between asteroseismic and non-asteroseismic quantities are in general less sensitive to helium abundance in comparison with metallicity. However, the relation for mass is significantly changed by $Y$. We find that estimated mass is inversely proportional to $Y^{0.25}$ (equation 15).

We also develop new methods for determination of $Y$ and $Z$ from oscillation frequencies. We plot $\Delta \nu - \langle \Delta \nu \rangle$ with respect to $n$ for interior models with the same input parameters but $Y$ or $Z$. These methods are based on the amplitudes in such diagrams. Usefulness of these methods will be clear when they are applied to the Kepler and CoRoT targets. The difference between model and estimated $Z$ (equation 18) values is about 14 per cent at most. This difference is about 10 per cent for $Y$.

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