REALIZING SUBEXPONENTIAL ENTROPY GROWTH RATES
BY CUTTING AND STACKING

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Abstract. We show that for any concave positive function $f$ defined on $[0, \infty)$ with $\lim_{x \to \infty} f(x)/x = 0$ there exists a rank one system $(X_f, T_f)$ such that
$$\limsup_{n \to \infty} H(\alpha_{n^{-1}0})/f(n) \geq 1$$
for all nontrivial partitions $\alpha$ of $X_f$ into two sets and that there is one partition $\alpha$ of $X_f$ into two sets for which the limit superior of $H(\alpha_{n^{-1}0})/f(n)$ is equal to one whenever the condition $\lim_{x \to \infty} \ln x/f(x) = 0$ is satisfied. Furthermore, for each system $(X_f, T_f)$ we also identify the minimal entropy growth rate in the limit inferior.

1. Introduction. In order to enable the reader to place the present paper’s subject matter in a broader conceptual context, we will briefly review to begin with some prior results concerning entropy growth rates. To do so, we need to introduce the pertinent standard notation: if $(X, \mathcal{B}, \mu)$ is a probability space, $T : X \to X$ a measure-preserving transformation and $\alpha$ a finite partition of $X$, then the $n$-th refinement of $\alpha$ under $T$ is denoted by $\alpha_{n^{-1}0}$, and the entropy of $\alpha$ is
$$H(\alpha) := \sum_{A \in \alpha} h(\mu(A)),$$
where
$$h(x) := -x \log_2 x$$
for all $x \in [0, 1]$. Given this notational setup, it can be shown that the limit
$$h(\alpha, T) := \lim_{n \to \infty} \frac{H(\alpha_{n^{-1}0})}{n} \tag{1}$$
always exists (see [10], p.240). Furthermore, the supremum
$$h(T) := \sup_{\alpha} h(\alpha, T),$$
taken over all finite partitions $\alpha$, is an isomorphism invariant that can be viewed to be a measure for the maximal exponential rate at which the elements of a partition $\alpha$ become dispersed in the space $X$ under the action of $T$. Put differently, $h(T)$ is a measure for the increase in disorder under the repeated application of $T$ and is said to be the entropy of $T$.

As it turns out, however, within the group $G$ of measure-preserving transformations on a Lebesgue space $X$ the set of positive-entropy transformations is of

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first category with respect to the weak topology on $G$ (see [9], p.103), that is, with respect to the topology that is generated by basis sets of the form

$$\{S \in G \mid \mu(T(B_1) \triangle S(B_1)), \ldots, \mu(T(B_n) \triangle S(B_n)) < \varepsilon\},$$

where $\varepsilon > 0$, $T \in G$, and $B_1, \ldots, B_n \in B$. In other words, in this well defined topological sense, almost every measure-preserving transformation that we may happen to construct or encounter has entropy zero because the set of zero-entropy transformations is residual. Consequently, entropy as an isomorphism invariant is useful only for a very thin set of systems, and the question that therefore naturally arises is whether there are entropy-type invariants that can be used to distinguish zero-entropy systems. The most obvious way to define such an invariant is to replace $n$ in the denominator in the defining equation (1) by a sublinear rate $a_n$. So for a given $s = (a_n)_{n \in \mathbb{N}}$, with

$$\lim_{n \to \infty} \frac{a_n}{n} = 0,$$

we may perhaps attempt to define

$$h_s(T) := \sup_{\alpha} \limsup_{n \to \infty} \frac{H(\alpha_0^{n-1})}{a_n}.$$  

Unfortunately, though, according to Theorem 2.6 in [7], it is the case that $h_s(T) = \infty$ universally for any aperiodic $T$ and any positive increasing sequence $s = (a_n)_{n \in \mathbb{N}}$ that satisfies (2). That is to say, the invariant $h_s(T)$ as a generalization of $h(T)$ is altogether useless. As it turns out, however, the matter is different if we do not aim to find the maximal growth rate by taking the supremum over the partitions $\alpha$ but rather focus our attention on determining lower-bound growth rates that cannot be undercut. The following general result that establishes the existence of such lower-bound rates was proven in [6]:

1.1. **Theorem.** Let $(X,T)$ be an aperiodic measure-preserving system and assume that $g$ is a positive monotone increasing function defined on $[0, \infty)$ which satisfies the condition $\int_1^\infty g(x)/x^2 \, dx < \infty$. If $\alpha$ is a partition of $X$ into two sets such that

$$\lim_{n \to \infty} \max\{\mu(A) \mid A \in \alpha_0^{n-1}\} = 0,$$

then

$$\limsup_{n \to \infty} \frac{H(\alpha_0^{n-1})}{g(\log_2 n)} = \infty.$$  

In the given generality this statement cannot be improved as was shown in [7]. More precisely, for each sufficiently regular function $g : [0, \infty) \to \mathbb{R}$ with $\lim_{n \to \infty} g(x)/x = 0$ and $\int_1^\infty g(x)/x^2 \, dx = \infty$ there exists a weakly mixing system $(X_g, T_g)$ such that $(g(\log_2 n))_{n \in \mathbb{N}}$ is the fastest growing sequence (up to trivial adjustments) for which $\limsup_{n \to \infty} H(\alpha_0^{n-1})/g(\log_2 n) \geq 1/8$ for all nontrivial partitions $\alpha$ of $X_g$ into two sets. Furthermore, in [7] we also constructed weakly mixing rank one systems for which the strongest lower-bound growth rate (in the limit superior) was given by $(\log_2 n)_{n \in \mathbb{N}}$.

The next three theorems show that improvements of Theorem 1.1 are possible if the requirement that $T$ be aperiodic is replaced by the stronger requirement that $T$ be completely ergodic, rank-one, or rank-one mixing, respectively. The relevant proofs can be found in [2], [3], and [7].
1.2. **Theorem.** If \((X,T)\) is completely ergodic (i.e., \(T^n \) is ergodic for all \( n \in \mathbb{Z} \)), then there exists a positive concave function \( g \) defined on \([0, \infty)\) with \( \int_1^\infty g(x)/x^2 \, dx = \infty \) such that
\[
\limsup_{n \to \infty} \frac{H(\alpha_n^{n-1})}{g(\log_2(n))} \geq 1
\]
for all partitions \( \alpha \) of \( X \) into two sets of positive measure.

1.3. **Theorem.** Let \((X,T)\) be a rank-one system and assume that \( g \) is a positive monotone increasing function defined on \([0, \infty)\) which satisfies the condition \( \int_1^\infty g(x)^{1/3}/x \, dx < \infty \). If \( \alpha \) is a partition of \( X \) into two sets such that
\[
\lim_{n \to \infty} \max \{ \mu(A) \mid A \in \alpha_n^{n-1} \} = 0,
\]
then
\[
\limsup_{n \to \infty} \frac{H(\alpha_n^{n-1})}{\log_2(n)} = \infty.
\]

1.4. **Theorem.** If \((X,T)\) is rank-one mixing, then
\[
\limsup_{n \to \infty} \frac{H(\alpha_n^{n-1})}{\log_2(n)} > 0
\]
for all partitions \( \alpha \) of \( X \) into two sets of positive measure.

Furthermore, the nearly universal existence of nontrivial lower-bound growth rates in the limit inferior was established in [2] where we proved the following general theorem:

1.5. **Theorem.** If \((X,T)\) is completely ergodic, then there exists a positive monotone increasing sequence \((a_n)_{n \in \mathbb{N}}\) with \( \lim_{n \to \infty} a_n = \infty \) such that
\[
\liminf_{n \to \infty} \frac{H(\alpha_n^{n-1})}{a_n} \geq 1
\]
for all partitions \( \alpha \) of \( X \) into two sets of positive measure.

Some additional theorems concerning the estimation of minimal entropy growth rates for interval-exchange transformations and the relation of these rates to the average computational complexity of dynamical-system trajectories can be found in [1] and [4], but more important with respect to the topic of the present paper are the topological results established in [5]. Setting
\[
ES(s) := \left\{ T \in G \mid \limsup_{n \to \infty} \frac{H(\alpha_n^{n-1})}{a_n} > 0 \text{ for all finite nontrivial partitions } \alpha \right\}
\]
and
\[
EI(s) := \left\{ T \in G \mid \liminf_{n \to \infty} \frac{H(\alpha_n^{n-1})}{a_n} > 0 \text{ for all finite nontrivial partitions } \alpha \right\},
\]
these latter results show that \( EI(s) \) is of first category in \( G \) whenever \( \limsup s = \limsup_{n \to \infty} a_n = \infty \) and that \( ES(s) \) is residual whenever the growth rate of \( s \) is sublinear.

Given the topological abundance of the transformations that make up the sets \( ES(s) \), the question naturally arises whether for any limsup-rate \( s \) that falls between the lower limit set by Theorem 1.1 and the upper limit given by the positive-entropy rate \((n)_{n \in \mathbb{N}}\) there exists a corresponding transformation that realizes this rate. Since we already know that the range of rates from the lower limit set by Theorem 1.1 to
the rate \((\log_2(n))_{n \in \mathbb{N}}\) can be realized by the systems \((X_g, T_g)\) (as explained above), the purpose of the present paper is to demonstrate that the remaining range from \((\log_2 n)_{n \in \mathbb{N}}\) to \((n)_{n \in \mathbb{N}}\) can be realized as well. Moreover, for any system that realizes a rate in this latter range we will exhibit as well a minimal entropy growth rate in the limit inferior.

**Remark.** One of the problems in determining minimal entropy growth rates—either in the limit inferior or limit superior—is that these minimal rates cannot be computed by considering only generating partitions. Consequently, it is worth mentioning that an alternative counting-type entropy measure for zero-entropy systems that does allow for this reduction to generating partitions was introduced by Katok and Thouvenot in [8].

In order to construct, for a given positive concave function \(f\) that satisfies the conditions \(\lim_{x \to \infty} f(x)/x = 0 = \lim_{x \to \infty} \ln(x)/f(x)\), a measure-preserving transformation \(T_f\) on an interval \(J_f\) that realizes the \(\limsup\)-rate \(s = (f(n))_{n \in \mathbb{N}}\), we will employ a rank-one cutting-and-stacking procedure that produces a sequence of towers \(\tau_i\) whose union equals \(J_f\). That is to say, we will construct a tower \(\tau_i\) from a given tower \(\tau_{i-1}\) by cutting \(\tau_{i-1}\) into a large number of vertical subtowers, placing on top of each of them either one or no spacers according to an essentially random pattern, and then stacking up these resulting subtowers with the spacers added. As we will see, the number of vertical subtowers depends on the asymptotic behavior of \(f\): the faster \(f\) diverges to \(\infty\) the larger this number will be. However, before we can begin to discuss this construction in any more detail, we need to list several simple facts and definitions.

1.6. **Definition.** If \(\beta\) is a finite partition of \(X\) and \(F \in \mathcal{B}\), then the **entropy of \(\beta\)** restricted to \(F\) is

\[
H_F(\beta) := \sum_{B \in \beta} h(\mu(B \cap F)).
\]

1.7. **Lemma.** If \(\beta\) is a finite partition of \(X\) and \(F \in \mathcal{B}\), then

\[
H(\beta) > H_F(\beta) - 2.
\]

**Proof.** Since \(h\) is increasing on \([0, 1/e]\) and decreasing on \([1/e, 1]\) and since there can be at most two sets in \(\beta\) whose measure is greater than \(1/e\), it follows that \(H_F(\beta) \leq H(\beta) + 2h(1/e) < H(\beta) + 2\), as desired. \(\square\)

1.8. **Lemma.** If \(\beta\) is a finite partition of \(X\), \(F \in \mathcal{B}\), and \(\lambda > 0\) such that \(\mu(B \cap F) \leq \lambda \) for all \(B \in \beta\), then

\[
H_F(\beta) \geq \mu(F) \log_2(1/\lambda).
\]

**Proof.**

\[
H_F(\beta) = \sum_{B \in \beta} h(\mu(B \cap F)) \geq -\sum_{B \in \beta} \mu(B \cap F) \log_2(\lambda) = \mu(F) \log_2(1/\lambda).
\]

\(\square\)

1.9. **Definition.** If \(\tau = (M_0, \ldots, M_{m-1})\) and \(\sigma = (N_0, \ldots, N_{n-1})\) are towers of measurable sets with \(\mu(M_k) = \mu(N_l)\) for all \(k \in \{0, \ldots, m-1\}\) and \(l \in \{0, \ldots, n-1\}\) then the **concatenation of \(\tau\) and \(\sigma\)** is

\[
\tau * \sigma := (M_0, \ldots, M_{m-1}, N_0, \ldots, N_{n-1}).
\]
Furthermore, by $|\tau|$ we denote the union of the levels of $\tau$, that is,

$$|\tau| = \bigcup_{k=0}^{n-1} M_k.$$  

1.10. Definition. If $\tau = (I_0, \ldots, I_{n-1})$ is a tower of intervals $I_k = [a_k, b_k)$ and $J \subset \mathbb{R}$ is an interval containing $|\tau|$. Then we define $T_\tau : J \to J$ via the equation

$$T_\tau(x) := \begin{cases}  
    x + a_{k+1} - a_k, & \text{whenever } x \in [a_k, b_k) \text{ and } k \in \{0, \ldots, n-2\} \\
    x + a_0 - a_{n-1}, & \text{whenever } x \in [a_{n-1}, b_{n-1}) \\
    x, & \text{whenever } x \in J \setminus \tau. 
\end{cases}$$

Note: this definition implies that $T_\tau(I_k) = I_{k+1}$ for all $k \in \{0, \ldots, n-2\}$, $T_\tau(I_{n-1}) = I_0$, and $T_\tau|_{J \setminus \tau} = \text{id}$.

1.11. Definition. If $\tau = (M, TM, \ldots, T^{n-1}M)$ is a Rokhlin tower and $F \in B$ such that $T^k M \subset F$ or $T^k M \subset X \setminus F$ for all $k \in \{0, \ldots, n-1\}$, then

$$\frac{\mu(F \cap T^k M)}{\mu(M)} \in \{0, 1\}$$

for all $k \in \{0, \ldots, n-1\}$ and the 01-name of $F$ in $\tau$ is

$$s(F, \tau) := \left( \frac{\mu(F \cap M)}{\mu(M)}, \frac{\mu(F \cap TM)}{\mu(M)}, \ldots, \frac{\mu(F \cap T^{n-1}M)}{\mu(M)} \right) \in (0,1)^n.$$

1.12. Definition. If $E \in B$, $x \in X$, and $n \in \mathbb{N}$, then

$$\alpha(E) := \{E, X \setminus E\}$$

and

$$s_n^F(x) := (\chi_E(x), \chi_E(Tx), \ldots, \chi_E(T^{n-1}x)) \in \{0, 1\}^n.$$  

1.13. Lemma. Let $\tau = (M, TM, \ldots, T^{n-1}M)$ be a Rokhlin tower and $\varepsilon > 0$. If $E, F \in B$, such that $\mu(E \Delta F) < \varepsilon \mu(|\tau|)$, then for

$$D := \{x \in M \mid \|s_n^F(x) - s_n^E(x)\|_1 < n\sqrt{\varepsilon}\},$$

it is the case that

$$\mu(D) > (1 - \sqrt{\varepsilon}) \mu(M).$$

Proof. Since

$$\varepsilon \mu(|\tau|) > \mu(E \Delta F) \geq \mu((E \Delta F) \cap |\tau|) = \int_M \|s_n^E(x) - s_n^F(x)\|_1 \, d\mu$$

$$\geq \int_{M \setminus D} \|s_n^E(x) - s_n^F(x)\|_1 \, d\mu \geq n\sqrt{\varepsilon} \mu(M \setminus D)$$

$$= \sqrt{\varepsilon} \mu(|\tau|) \left(1 - \frac{\mu(D)}{\mu(M)}\right),$$

we may infer that $\mu(D) > (1 - \sqrt{\varepsilon}) \mu(M)$, as desired. \qed
2. **Construction of the transformations.** To begin with, we assume that \( f : \mathbb{R} \to \mathbb{R} \) is a concave monotone increasing function that is strictly positive on \([0, \infty)\) and satisfies the following growth-rate conditions:

\[
\lim_{x \to \infty} \frac{f(x)}{x} = 0 \tag{3}
\]

and

\[
\lim_{x \to \infty} \frac{\ln x}{f(x)} = 0. \tag{4}
\]

**Remark.** For the proof of the fact that the sequence \((f(n))_{n \in \mathbb{N}}\) represents a minimal entropy growth rate of the system \((J_f, T_f)\) assumption (4) can be replaced with the weaker condition \(\lim_{x \to \infty} f(x) = \infty\). The stronger condition (4) is needed only afterwards, when we show that the rate given by \(f\) is optimal.

In order to control the placement of the spacers, in the cutting-and-stacking procedure described in the Introduction, we enumerate, for a given \(n \in \mathbb{N}\), the elements of \(
\{0, 1\}^n \)

in the natural way by considering them to be binary expansions of natural numbers:

\[
s(0) := (0, 0, \ldots, 0, 0)
\]

\[
s(1) := (0, 0, \ldots, 0, 1)
\]

\[
s(2) := (0, 0, \ldots, 1, 0)
\]

\[
\vdots
\]

\[
s(2^n - 1) := (1, 1, \ldots, 1, 1).
\]

The concatenation of these 01-names—with \(s(0)\) at the beginning and the end—we denote by \(u_n\), that is,

\[
u_n = (u_n(0), \ldots, u_n(n2^n + n - 1)) := (s(0), s(1), \ldots, s(2^n - 1), s(0)).
\]

**2.1. Lemma.** For all \(n \in \mathbb{N}\) and all \(r \in \{0, 1\}^n\) it is the case that

\[
\# \{k \in \{0, \ldots, n2^n - 1\} \mid (u_n(k), \ldots, u_n(k + n - 1)) = r\} = n.
\]

**Proof.** Using elementary (but slightly tedious) combinatorial arguments, it is easy to see that

\[
\{(u_n(k + jn), \ldots, u_n(k + jn + n - 1)) \mid j \in \{0, \ldots, 2^n - 1\}\} = \{0, 1\}^n
\]

for all \(k \in \{0, \ldots, n - 1\}\), and this immediately implies the statement of the lemma.

}\)

Now let \(l_0 := 1, x_0 := 1, \) and \(\tau_0 := ([0, 1))\) (so \(\tau_0\) is a tower consisting of only one level), and assume that \(l_{i-1}, x_{i-1}, \) and \(\tau_{i-1}\) have been defined in such a way that

\[
\tau_{i-1} = (I_{i-1}(0), \ldots, I_{i-1}(l_{i-1} - 1))
\]

is a tower of half open intervals \(I_{i-1}(k)\) whose union is \(|\tau_{i-1}| = [0, x_{i-1})\). Using assumption (3), it follows that

\[
n_i := \min\{n \in \mathbb{N} \mid f(n(l_{i-1} + 1))/n < 1\} \tag{5}
\]

is well defined. Next we cut \(\tau_{i-1}\) into \(n_i2^{n_i} + n_i - 1\) vertical subtowers

\[
s_i(0), \ldots, s_i(n_i2^{n_i} + n_i - 2)
\]
of equal width $\mu(I_{i-1}(0))/(n_i2^{n_i}+n_i-1)$ and place a spacer on top of those $\sigma_i(k)$ for which $u_{n_i}(k) = 1$. The resulting towers—with the spacers added—we denote by $\eta_i(0), \ldots, \eta_i(n(2^{n_i}+n_i-2))$ and define

$$\tau_i := \eta_i(0) * \cdots * \eta_i(n(2^{n_i}+n_i-2))$$

(see Definition 1.9).

Moreover, the bottom intervals of the towers $\eta_i(k)$ we denote by $K_i(k)$ and by $l_i$ we denote the length of $\tau_i$.

Illustration of the construction:

For convenience we arrange the spacers in such a way that

$$|\tau_i| = [0, x_i)$$

for some $x_i > x_{i-1}$. Then

$$x_i - x_{i-1} \leq \frac{x_{i-1}}{l_{i-1}} \leq \frac{x_{i-1}}{2^{i-1}},$$

and therefore,

$$\lim_{i \to \infty} x_i \leq \lim_{i \to \infty} \prod_{k=0}^{i-1} \left(1 + \frac{1}{2^{k-1}}\right) < \infty.$$

Thus we may define

$$J_f := \left[0, \lim_{i \to \infty} x_i\right]$$

and

$$\mu_f := \frac{m}{\mu(J_f)},$$

where $m$ denotes Lebesgue measure on $\mathbb{R}$. If we denote by $B_f$ the $\sigma$-algebra of all Lebesgue measurable subsets of $J_f$, then $(J_f, B_f, \mu_f)$ is a probability space and the preceding tower construction induces a measure-preserving rank one transformation $T_f : J_f \to J_f$ in the obvious way via the defining equation

$$T_f(x) := \lim_{i \to \infty} T_{\tau_i}(x)$$

(see Definition 1.10).
3. Identifying similar 01-names. Our purpose in this section is to prove the central technical result of this paper (Lemma 3.9) concerning the similarity of certain 01-names. To begin with, we need to establish several simple facts that are mostly analytic in character.

3.1. Theorem. If \( \alpha \) is a nontrivial partition of \( J_f \) into two sets, then

\[
\limsup_{n \to \infty} \frac{H(\alpha^{n-1})}{f(n)} \geq 1.
\]

3.2. Lemma. For every \( K \in \mathbb{N} \), there is an \( L \in \mathbb{N} \), such that for all \( n \geq L \) we have

\[
\forall m \in \mathbb{N}, \quad \frac{f(mn)}{m} < 1 \Rightarrow m \geq K.
\]

Proof. Since \( f \) is concave on \( \mathbb{R} \), it follows that \( f \) is continuous, and therefore, the assumed strict positivity of \( f \) on \( (0, \infty) \) in conjunction with (3) implies that the equation \( f(x)/x = 1/n \) has a solution \( z_n \geq 1 \) for all sufficiently large values \( n \in \mathbb{N} \). If it were not the case that \( \lim_{n \to \infty} z_n = \infty \), then the sequence \( (z_n) \) would contain a bounded subsequence which in turn would contain a convergent subsequence \( (z_{n_k})_{k \in \mathbb{N}} \). Denoting by \( z \) the limit of this latter subsequence, it would be the case that \( z \geq 1 > 0 \), and the continuity of \( f \) would thus allow us to infer that \( f(z)/z = \lim_{k \to \infty} f(z_{n_k})/z_{n_k} = \lim_{k \to \infty} 1/n_k = 0 \) in contradiction to the assumed strict positivity of \( f \) on \( (0, \infty) \). Having thus shown that \( \lim_{n \to \infty} z_n = \infty \), we may apply (4) (or the weaker assumption mentioned above) to conclude that

\[
\infty = \lim_{n \to \infty} f(z_n) = \lim_{n \to \infty} \frac{z_n}{n}.
\]

Consequently, for a given \( K \in \mathbb{N} \) we can find an \( L \in \mathbb{N} \) such that \( z_n/n > K \) for all \( n \geq L \). Since \( f \) is concave, we further find that

\[
f(x) \geq \frac{x}{z_n} f(z_n) + \left(1 - \frac{x}{z_n}\right) f(0) \geq \frac{x}{z_n} f(z_n) = \frac{x}{n}
\]

for all \( x \in [0, z_n] \) and therefore, in particular, for all \( x \in [0, nK] \) whenever \( n \geq L \) (because if \( n \geq L \) then \( nK < z_n \)). So for all \( n \geq L \) it is the case that

\[
\forall x > 0, x < nK \Rightarrow \frac{f(x)}{x} \geq \frac{1}{n}
\]

or, equivalently,

\[
\forall x > 0, \frac{f(x)}{x} < \frac{1}{n} \Rightarrow x \geq nK.
\]

Setting \( x := mn \), the statement of the lemma follows as desired. \( \Box \)

3.3. Corollary. For the values \( n_i \) defined in (5), it is the case that

\[
\lim_{i \to \infty} n_i = \infty.
\]

Proof. Let \( K \in \mathbb{N} \) and let \( L \in \mathbb{N} \) be given as in Lemma 3.2. Since \( \lim_{i \to \infty} l_i = \infty \), we have \( l_{i-1} \geq L \) for all sufficiently large \( i \), and therefore, Lemma 3.2 in conjunction with the definition of \( n_i \) in (5) implies that \( n_i \geq K \) for all sufficiently large \( i \). \( \Box \)

3.4. Lemma. Let \( \varepsilon > 0 \) and \( 0 \leq x_1, \ldots, x_n \leq 1 \).

a) If \( \sum_{k=1}^{n} x_k > (1 - \varepsilon)n \), then \( \# \{ k \in \{1, \ldots, n \} \mid x_k \leq 1 - \sqrt[3]{\varepsilon} \} < \sqrt[3]{\varepsilon}n \).

b) If \( \sum_{k=1}^{n} x_k < \varepsilon n \), then \( \# \{ k \in \{1, \ldots, n \} \mid x_k \geq \sqrt[3]{\varepsilon} \} < \sqrt[3]{\varepsilon}n \).
Proof. a) Setting \( \lambda := \# \{ k \in \{1, \ldots, n\} | x_k \leq 1 - \sqrt{\varepsilon} \} \), it follows that
\[
\sum_{k=1}^{n} x_k \leq \lambda (1 - \sqrt{\varepsilon}) + (n - \lambda),
\]
and therefore, \( \lambda < \sqrt{\varepsilon} n \) whenever \( (1 - \varepsilon)n < \sum_{k=1}^{n} x_k \).

b) Here we set \( \lambda := \# \{ k \in \{1, \ldots, n\} | x_k \geq \sqrt{\varepsilon} \} \) and observe that
\[
\sum_{k=1}^{n} x_k \geq \lambda \sqrt{\varepsilon}.
\]
This yields \( \lambda < \sqrt{\varepsilon} n \) whenever \( \varepsilon n > \sum_{k=1}^{n} x_k \). \( \square \)

3.5. Lemma. Let \( \varepsilon > 0 \), \( n \in \mathbb{N} \), and \( \Omega \subset \{0, \ldots, n - 1\} \) such that \( \# \Omega/n < \varepsilon \). If \( m \in \mathbb{N} \) and \( m \leq n/2 \), then the set
\[
\Psi := \left\{ k \in \{0, \ldots, n - m\} \left| \# \left( \frac{k, \ldots, k + m - 1}{m} \cap \Omega \right) < \sqrt{\varepsilon} \right\}
\]
satisfies the inequality
\[
\frac{\# \Psi}{n - m + 1} > 1 - 2\sqrt{\varepsilon}.
\]
Proof. Since each \( i \in \{0, \ldots, n - 1\} \) can be an element of at most \( m \) of the sets \( \vartheta_k := \{k, \ldots, k + m - 1\} \) for \( k \in \{0, \ldots, n - m\} \), it follows that
\[
\sum_{k=0}^{n-m} \frac{\#(\vartheta_k \cap \Omega)}{m} = \sum_{i=0}^{n-1} \sum_{k=0}^{n-m} \frac{\#(\{i\} \cap \vartheta_k \cap \Omega)}{m} \leq \sum_{i=0}^{n-1} \#(\{i\} \cap \Omega) = \# \Omega < n\varepsilon,
\]
and using Lemma 3.4b, we may thus infer that
\[
n\sqrt{\varepsilon} > \# \left\{ k \in \{0, \ldots, n - m\} \left| \frac{\#(\vartheta_k \cap \Omega)}{m} \geq \sqrt{\varepsilon} \right\} = n - m + 1 - \# \Psi.
\]
Hence
\[
\frac{\# \Psi}{n - m + 1} > 1 - \frac{n\sqrt{\varepsilon}}{n - m + 1} > 1 - \frac{n\sqrt{\varepsilon}}{n - n/2} = 1 - 2\sqrt{\varepsilon},
\]
as desired. \( \square \)

3.6. Lemma. Let \( \varepsilon > 0 \), \( n \in \mathbb{N} \) and \( \Delta \subset \{0, \ldots, n - 1\} \) such that \( \# \Delta/n < \varepsilon \) and \( 1/n < \varepsilon \) (in case that \( \Delta = \emptyset \)). If \( m \in \mathbb{N} \) such that \( m \leq 1/\sqrt{\varepsilon} \), then the set
\[
\Upsilon := \{ k \in \{0, \ldots, n - m\} \mid \{k, \ldots, k + m - 1\} \cap \Delta = \emptyset \}
\]
satisfies the inequality
\[
\frac{\# \Upsilon}{n} > 1 - 2\sqrt{\varepsilon}.
\]
Proof. Arguing as in the proof of Lemma 3.5 that each \( i \in \{0, \ldots, n - 1\} \) can be an element of at most \( m \) of the sets \( \vartheta_k \), it follows that
\[
n - m + 1 - \# \Upsilon = \#(\{0, \ldots, n - m\} \setminus \Upsilon) \leq m \# \Delta < mn\varepsilon,
\]
and therefore,
\[
\frac{\# \Upsilon}{n} > 1 - \frac{m}{n} - m\varepsilon > 1 - \frac{\varepsilon}{\sqrt{\varepsilon}} - \frac{\varepsilon}{\sqrt{\varepsilon}} = 1 - 2\sqrt{\varepsilon}.
\]
\( \square \)
3.7. Definition. Let \( k, n \in \mathbb{N} \) and \( k \leq n \). We call \( r, s \in \{0, 1 \}^n \) \( k \)-similar, if there are indices \( 0 \leq i_1 < i_2 < \cdots < i_k \leq n - 1 \) and \( 0 \leq j_1 < j_2 < \cdots < j_k \leq n - 1 \) such that

\[
s_{i_1} = r_{j_1}, s_{i_2} = r_{j_2}, \ldots, s_{i_k} = r_{j_k}.
\]

Let \( E \in \mathcal{B}f \) and \( \varepsilon > 0 \) such that \( 0 < \mu_f(E) < 1 \) and \( \varepsilon < \min \{ \mu_f(E), 1 - \mu_f(E), 1/(13e) \} \).

\[
(6)
\]

Since \((Jf, T_f)\) is a rank one system, it is in particular ergodic, and, using the ergodic theorem, we can therefore find a \( K_E \in \mathbb{N} \) and a set \( X_E \subset Jf \) such that

\[
\mu_f(X_E) > 1 - \frac{\varepsilon}{16}
\]

and

\[
\left| \frac{1}{n} \sum_{k=0}^{n-1} \chi_E(T_f^k x) - \mu_f(E) \right| < \frac{\varepsilon}{4}
\]

for all \( x \in X_E \) and \( n \geq K_E \). Moreover, we can choose \( K_E \in \mathbb{N} \) so large that for all \( i \geq K_E \) we have

\[
\mu_f(|\tau_{i-1}|) > 1 - \frac{\varepsilon}{16}
\]

Combining (6), (7), and (9), we find that

\[
\mu_f(|\tau_{i-1}| \triangle X_E) \leq 1 - \mu_f(X_E) + 1 - \mu_f(|\tau_{i-1}|) < \frac{\varepsilon}{8} < \frac{\varepsilon}{4} \mu_f(|\tau_{i-1}|).
\]

According to Lemma 1.13, we can therefore find a \( D_i \subset I_{i-1}(0) \) (recall that \( I_{i-1}(0) \) is the bottom-level interval of \( \tau_{i-1} \)) with

\[
\frac{\mu_f(D_i)}{\mu_f(I_{i-1}(0))} > 1 - \frac{\varepsilon^4}{2}
\]

and

\[
\frac{1}{l_{i-1}} \left\| s(|\tau_{i-1}|, \tau_{i-1}) - s_{\tau_{i-1}}^{X_E}(x) \right\|_1 < \frac{\varepsilon^4}{2}
\]

for all \( x \in D_i \) where \( s(|\tau_{i-1}|, \tau_{i-1}) = (1, 1, \ldots, 1) = s_{\tau_{i-1}}^{X_E}(x) \) for all \( x \in I_{i-1}(0) \) (by Definitions 1.11 and 1.12). Now let \( \delta > 0 \) such that

\[
\delta < \frac{\varepsilon^4}{2}
\]

and

\[
K_E^2 < \frac{1}{12\delta}.
\]

Setting

\[
F_i := \bigcup \left\{ I_{i-1}(k) \middle| k \in \{0, \ldots, l_{i-1} - 1\} \land \frac{\mu_f(I_{i-1}(k) \cap E)}{\mu_f(I_{i-1}(0))} \geq \frac{1}{2} \right\},
\]

we may apply the Lebesgue density theorem, to infer that there is an \( L_E \in \mathbb{N} \) such that for all \( i \geq L_E \) it is the case that

\[
\mu_f(F_i \triangle E) < \delta^2 \mu_f(|\tau_{i-1}|).
\]
Furthermore, according to Corollary 3.3, we can choose $L_E$ so large that for all $i \geq L_E$ we have

\[
\frac{1}{n_i} < \min\{12\delta, \varepsilon\}, \quad (15)
\]

\[
l_{i-1} > \frac{1}{4\delta}, \quad (16)
\]

\[
\frac{n_i 2^{n_i}}{n_i 2^{n_i} + n_i - 1} > 1 - 5\varepsilon, \quad (17)
\]

and

\[
\frac{(n_i - 1)l_{i-1}}{n_i(l_{i-1} + 1)} - \frac{2}{n_i} > 1 - \varepsilon. \quad (18)
\]

Using Lemma 1.13 in conjunction with (14), we may conclude that for any $i \geq L_E$ there exists a set $C_i \subset I_{i-1}(0)$ such that

\[
\frac{\mu_f(C_i)}{\mu_f(I_{i-1}(0))} > 1 - \delta \quad (19)
\]

and

\[
\frac{1}{l_{i-1}} \left\| \delta(F_i, \tau_{i-1}) - s_{l_{i-1}}(x) \right\|_1 < \delta. \quad (20)
\]

for all $x \in C_i$. To proceed we set $N_E := \max\{K_E, L_E\}$, and for all $i \geq N_E$ we define $B_i := C_i \cap D_i$. Then, according to (10), (12), and (19), we have

\[
\frac{\mu_f(B_i)}{\mu_f(I_{i-1}(0))} > 1 - \varepsilon^4. \quad (21)
\]

Now let

\[
\Omega_i := \left\{ k \in \{0, \ldots, n_i 2^{n_i} + n_i - 2\} \mid \frac{\mu_f(B_i \cap K_i(k))}{\mu_f(K_i(0))} \leq 1 - \varepsilon^2 \right\}.
\]

Since

\[
\frac{1}{n_i 2^{n_i} + n_i - 1} \sum_{k=0}^{n_i 2^{n_i} + n_i - 2} \frac{\mu_f(B_i \cap K_i(k))}{\mu_f(K_i(0))} = \frac{\mu_f(B_i)}{\mu_f(I_{i-1}(0))} > 1 - \varepsilon^4,
\]

we may apply Lemma 3.4a to infer that

\[
\#\Omega_i / n_i 2^{n_i} + n_i - 1 < \varepsilon^2. \quad (21)
\]

Now we define

\[
\Psi_i := \left\{ k \in \{0, \ldots, n_i 2^{n_i} - 1\} \mid \frac{1}{n_i} \#(\{k, \ldots, k + n_i - 1\} \cap \Omega_i) < \varepsilon \right\}
\]

and

\[
\Theta_i := \{ k \in \Psi_i \mid k + 1 \in \Psi_i \}.
\]

Using (21) and Lemma 3.5 (with $m := n_i \leq n_i 2^{n_i}/2$) yields

\[
\#\Psi_i / n_i 2^{n_i} > 1 - 2\varepsilon, \quad (22)
\]

and therefore, (15) implies that

\[
\#\Theta_i / n_i 2^{n_i} \geq \frac{\#\Psi_i - (n_i 2^{n_i} - \#\Psi_i) - 1}{n_i 2^{n_i}} > 1 - 4\varepsilon - \frac{1}{n_i 2^{n_i}} > 1 - 5\varepsilon. \quad (23)
\]

Since

\[
\#(\{k, \ldots, k + n_i - 1\} \cap \Omega_i) > (1 - \varepsilon)n_i > 0
\]
for all \( k \in \Psi_i \), it follows that for any \( k \in \Psi_i \) there are integers \( m_k \in \mathbb{N} \) and \( j_k(1), \ldots, j_k(m_k) \in \mathbb{N} \) such that

\[
k \leq j_k(1) < \cdots < j_k(m_k) \leq k + n_i - 1
\]

and

\[
\{ j_k(1), \ldots, j_k(m_k) \} = \{ k, \ldots, k + n_i - 1 \} \setminus \Omega_i.
\]

To proceed we artificially define, for any \( f \in \Psi_i \), the tower

\[
\rho_i(k) := (K_i(j_k(1)), \ldots, K_i(j_k(m_k)))
\]

and assign to it the transformation \( T_{\rho_i(k)} \) as specified in Definition 1.10. Setting further

\[
E_i(k) := B_i \cap |\rho_i(k)|
\]

for all \( k \in \Psi_i \), the definitions of \( \Omega_i \) and \( \rho_i(k) \) imply that

\[
\mu_f(|\rho_i(k)| \triangle E_i(k)) = \mu_f(|\rho_i(k)| \setminus E_i(k)) = \sum_{l=1}^{m_k} \mu_f(K_i(j_k(l)) \setminus B_i)
\]

\[
< \sum_{l=1}^{m_k} \varepsilon^2 \mu_f(K_i(0)) = \varepsilon^2 \mu_f(|\rho_i(k)|)
\]

for all \( k \in \Psi_i \). Introducing the definition

\[
t_i^k(x) := (\chi_{E_i(k)}(x), \chi_{E_i(k)}(T_{\rho_i(k)}x), \ldots, \chi_{E_i(k)}(T_{\rho_i(k)}^{m_k-1}x))
\]

for all \( k \in \Psi_i \) and \( x \in K_i(j_k(1)) \), we may use (24) and Lemma 1.13 to infer that there exists a set \( B_i(k) \subset K_i(j_k(1)) \) such that

\[
\frac{\mu_f(B_i(k))}{\mu_f(K_i(0))} > 1 - \varepsilon
\]

and

\[
\frac{1}{m_k} ||s(|\rho_i(k)|, \rho_i(k)) - t_i^k(x)||_1 < \varepsilon
\]

for all \( x \in B_i(k) \) and all \( k \in \Psi_i \). Now let \( A_i(k) \) be the projection of \( B_i(k) \) onto \( K_i(k) \), that is,

\[
A_i(k) := T_f^{-((j_k(1)-k)_{l_i-1} + \sum_{m=0}^{j_k(m_k)-1} u_{\Omega_i}(k+m))}B_i(k),
\]

for all \( k \in \Psi_i \) and let

\[
L_i := \bigcup_{k \in \Psi_i} \bigcup_{m=0}^{l_i-1+u_{\Omega_i}(k)-1} T_f^m(A_i(k))
\]

and

\[
M_i := \bigcup_{k \in \Theta_i} \bigcup_{m=0}^{l_i-1+u_{\Omega_i}(k)-1} T_f^m(A_i(k)).
\]

Given these definitions, we notice that

\[
(A_i(k), T_f(A_i(k)), \ldots, T_f^{l_i-1+u_{\Omega_i}(k)-1}(A_i(k)))
\]
is the vertical subtower of $\eta_i(k)$ generated by $A_i(k)$. Moreover,

$$\mu_f(M_i) = \sum_{k \in \Theta_i} (l_{i-1} + u_{n_i}(k)) \mu_f(A_i(k))$$

$$> \# \Theta_i l_{i-1} (1 - \varepsilon) \mu_f(K_i(0))$$

(by (25) and the definition of $A_i(k)$)

$$= \# \Theta_i (1 - \varepsilon) \mu_f(|\sigma_i(0)|)$$

$$> (1 - 5\varepsilon)(1 - \varepsilon) \mu_f(|\sigma_i(0)|) n_i 2^{n_i}$$

(by (23))

$$> (1 - 5\varepsilon)^2 n_i 2^{n_i} \mu_f(\tau_{i-1})$$

and therefore,

$$\mu_f(M_i) > (1 - 5\varepsilon)^4$$

(by (9) and (17)).

(27)

3.8. Definition. For any $x \in L_i$ we denote by $k_x$ the unique integer in $\Psi_i$ for which there is a unique $m \in \{0, \ldots, l_{i-1} + u_{n_i}(k_x) - 1\}$ with $x \in T_f^m(A_i(k_x))$. Using this integer $k_x$, we define

$$r_i(x) := (u_{n_i}(k_x), \ldots, u_{n_i}(k_x + n_i - 1)).$$

3.9. Lemma. Let $i \geq N_E$ and $x, y \in M_i$ such that

$$s_{n_i(l_i-1+1)}^E(x) = s_{n_i(l_i-1+1)}^E(y).$$

Then $r_i(x)$ and $r_i(y)$ are $[(1 - 11\varepsilon) n_i]$-similar.

Proof. Our first goal is to show that the assumption

$$s_{n_i(l_i-1+n_i)}^E(x) = s_{n_i(l_i-1+n_i)}^E(y)$$

(28)

implies that $r_i(x)$ and $r_i(y)$ are $[(1 - 11\varepsilon) n_i]$-similar whenever $x \in A_i(k_x)$ and $y \in A_i(k_y)$. To do so, we will use $r_i(x)$ and $r_i(y)$ to construct a finite sequence of pairs $(a_0, b_0), \ldots, (a_{l_i-1}, b_{l_i-1}) \in \{0, \ldots, n_i-3\}^2$. Setting $(a_0, b_0) := (0, 0)$, we assume that $(a_{j-1}, b_{j-1}) \in \{0, \ldots, n_i-5\}^2$ has been defined in a such a way that the sums

$$S_a(j-1) := \sum_{m=0}^{a_{j-1}-1} u_{n_i}(k_x + m)$$

and

$$S_b(j-1) := \sum_{m=0}^{b_{j-1}-1} u_{n_i}(k_y + m)$$

satisfy one of the following two pairs of inequalities (which are both trivially satisfied for $j-1 = 0$):

$$b_{j-1}l_{i-1} + S_b(j-1) \leq a_{j-1}l_{i-1} + S_a(j-1) \leq b_{j-1}l_{i-1} + \frac{2}{3} l_{i-1} + S_b(j-1)$$

(29)

or

$$a_{j-1}l_{i-1} + S_a(j-1) \leq b_{j-1}l_{i-1} + S_b(j-1) \leq a_{j-1}l_{i-1} + \frac{2}{3} l_{i-1} + S_a(j-1).$$

(30)

(Note: For the definition of $S_a(j-1)$ and $S_b(j-1)$ we agree that $\sum_{m=0}^{l_i-1} = 0$.)

To proceed, we will define $a_j$ and $b_j$ in dependence on which of the two pairs of
inequalities—(29) or (30)—is satisfied. If (29) is satisfied, then we set \( a_j := a_{j-1} + 1 \) and \( b_j := b_{j-1} + 1 \) in case that

\[
(a_{j-1} + 1)l_{i-1} + S_a(j - 1) + u_n(k_x + a_{j-1}) \leq (b_{j-1} + 1)l_{i-1} + \frac{2}{3}l_{i-1} + S_b(j - 1) + u_n(k_y + b_{j-1}),
\]

(31)

and otherwise, if this inequality is violated, we set \( a_j := a_{j-1} + 1 \) and \( b_j := b_{j-1} + 2 \). Similarly, if (30) is satisfied, then we set \( a_j := a_{j-1} + 1 \) and \( b_j := b_{j-1} + 1 \) in case that

\[
(b_{j-1} + 1)l_{i-1} + S_b(j - 1) + u_n(k_y + b_{j-1}) \leq (a_{j-1} + 1)l_{i-1} + \frac{2}{3}l_{i-1} + S_a(j - 1) + u_n(k + x + a_{j-1}),
\]

(32)

and otherwise, if this inequality is violated, we set \( a_j := a_{j-1} + 2 \) and \( b_j := b_{j-1} + 1 \). For clarity we wish to point out that there is no ambiguity in the case where (29) and (30) are valid simultaneously. For in this case we evidently have

\[
 b_{j-1}l_{i-1} + S_b(j - 1) = a_{j-1}l_{i-1} + S_a(j - 1),
\]

and therefore, \( a_j := a_{j-1} + 1 \) and \( b_j := b_{j-1} + 1 \) because both (31) and (32) are satisfied. Furthermore, in order to explain why the pair \((a_j, b_j)\), defined in the manner just described, satisfies either

\[
b_jl_{i-1} + S_b(j) \leq a_jl_{i-1} + S_a(j) \leq b_jl_{i-1} + \frac{2}{3}l_{i-1} + S_b(j)
\]

(33)

or

\[
a_jl_{i-1} + S_a(j) \leq b_jl_{i-1} + S_b(j) \leq a_jl_{i-1} + \frac{2}{3}l_{i-1} + S_a(j),
\]

(34)

we will assume w.l.o.g. that (29) is satisfied for \((a_{j-1}, b_{j-1})\). Given this assumption, we consider first the case that (31) is satisfied. In that case we have \( a_j = a_{j-1} + 1 \) and \( b_j = b_{j-1} + 1 \) and (31) is therefore equivalent to the second inequality in (33). Thus (33) is satisfied whenever

\[
b_jl_{i-1} + S_b(j) \leq a_jl_{i-1} + S_a(j).
\]

Consequently, we only need to consider the case in which this latter inequality is violated, that is, the case in which

\[
(b_{j-1} + 1)l_{i-1} + S_b(j - 1) + u_n(k_y + b_{j-1}) > (a_{j-1} + 1)l_{i-1} + S_a(j - 1) + u_n(k_x + a_{j-1}).
\]

Given that we assumed (29) to be satisfied, this strict inequality can be valid only if \((b_{j-1} + 1)l_{i-1} + S_b(j - 1) = (a_{j-1} + 1)l_{i-1} + S_a(j - 1), u_n(k_y + b_{j-1}) = 1, \) and \( u_n(k_x + a_{j-1}) = 0 \). Since these three conditions together imply that (34) is satisfied, we only need to examine the remaining case where (31) is violated and \((a_j, b_j) = (a_{j-1} + 1, b_{j-1} + 2)\). Here we find that

\[
a_jl_{i-1} + \frac{2}{3}l_{i-1} + S_a(j) = (a_{j-1} + 1)l_{i-1} + \frac{2}{3}l_{i-1} + S_a(j - 1) + u_n(k_x + a_{j-1})
\]

\[
> (b_{j-1} + 1)l_{i-1} + \frac{4}{3}l_{i-1} + S_b(j - 1) + u_n(k_y + b_{j-1})
\]

(because (31) is violated)

\[
= bj_{i-1} + \frac{1}{3}l_{i-1} + S_b(j) - u_n(k_y + b_{j-1} + 1)
\]

\[
> bj_{i-1} + S_b(j)
\]
and

\[ a_j l_{i-1} + S_d(j) = (a_j - 1) l_{i-1} + S_d(j - 1) + u_n_i(k_x + a_j - 1) \]
\[ \leq (b_j - 1) l_{i-1} + S_b(j - 1) + u_n_i(k_x + a_j - 1) \quad \text{(by (29))} \]

\[ < b_j l_{i-1} + S_b(j), \]

and therefore, (34) is satisfied. Having thus established the validity of (33) or (34), we define

\[ \partial_x := \{ k_x, \ldots, k_x + n_i - 1 \}, \]
\[ \partial_y := \{ k_y, \ldots, k_y + n_i - 1 \}, \]
\[ R_x(k) := (k - k_x) l_{i-1} + \sum_{m=0}^{k - 2^{k_x} - 1} u_n_i(k_x + m) \]

for all \( k \in \partial_x, \)
\[ R_y(k) := (k - k_y) l_{i-1} + \sum_{m=0}^{k - 2^{k_y} - 1} u_n_i(k_y + m) \]

for all \( k \in \partial_y, \)
\[ \Phi_x := \left\{ k \in \partial_x \setminus \{ k_x + n_i - 1 \} \mid T_j^{R_x(k)} x \in B_i \land T_j^{R_x(k+1)} x \in B_i \right\}, \]

and
\[ \Phi_y := \left\{ k \in \partial_y \setminus \{ k_y + n_i - 1 \} \mid T_j^{R_y(k)} y \in B_i \land T_j^{R_y(k+1)} y \in B_i \right\}. \]

Using the notation that we introduced in the context of defining the towers \( \rho_i(k), \) we may infer that
\[ \#\Phi_x \geq n_i - 2 \# \left\{ k \in \partial_x \mid T_j^{R_x(k)} x \notin B_i \right\} - 1 \]
\[ = n_i - 2 \# \left\{ k \in \partial_x \cap \Omega_i \mid T_j^{R_x(k)} x \notin B_i \right\} \]
\[ - 2 \# \left\{ k \in \partial_x \setminus \Omega_i \mid T_j^{R_x(k)} x \notin B_i \right\} - 1 \]
\[ \geq n_i - 2 \# (\partial_x \cap \Omega_i) - 2 \# \left\{ l \in \{1, \ldots, m_{k_x}\} \mid T_j^{R(\rho_i(l))} x \notin B_i \right\} - 1 \]
\[ = n_i - 2 \# (\partial_x \cap \Omega_i) \]
\[ - 2 \# \left\{ l \in \{0, \ldots, m_{k_x} - 1\} \mid T_{\rho_i(k_x)}^l \left( T_j^{R(\rho_i(1))} x \right) \notin E_i(k_x) \right\} - 1 \]
\[ = n_i - 2 \# (\partial_x \cap \Omega_i) - 2 \left\| s(\rho_i(k_x), \rho_i(k_x)) - t_i k_x \left( T_j^{R(\rho_i(1))} x \right) \right\| - 1 \]
\[ > n_i - 2 \# (\partial_x \cap \Omega_i) - 2 \varepsilon m_{k_x} - 1 \]

(by (26), because \( x \in A_i(k_x) \) implies \( T_j^{R(\rho_i(1))} x \in B_i(k_x) \))
\[ > n_i - 2 \varepsilon n_i - 2 \varepsilon m_{k_x} - 1 \quad \text{(because \( k_x \in \Psi_i \)),} \]

and therefore,
\[ \#\Phi_x > n_i - 5 \varepsilon n_i \quad \text{(by (15)).} \quad (35) \]

Similarly, we find that
\[ \#\Phi_y > n_i - 5 \varepsilon n_i. \quad (36) \]
To proceed, we denote by $\Phi$ the set of all pairs $(a_j, b_j)$ for which the following conditions are satisfied: $k_x + a_j \in \Phi_x$, $k_y + b_j \in \Phi_y$,

$$(a_j + 2)l_{i-1} + \sum_{m=0}^{a_j+1} u_{n_i}(k_x + m) \leq (n_i - 1)l_{i-1}, \quad (37)$$

and

$$(b_j + 2)l_{i-1} + \sum_{m=0}^{b_j+1} u_{n_i}(k_y + m) \leq (n_i - 1)l_{i-1}. \quad (38)$$

It is easy to see that the number of pairs $(a_j, b_j)$ satisfying (37) and (38) becomes minimal if either $u_{n_i}(k_x + m) = 1$ for all $m \in \{0, \ldots, n_i - 1\}$ or $u_{n_i}(k_y + m) = 1$ for all $m \in \{0, \ldots, n_i - 1\}$. This shows that the number of pairs $(a_j, b_j)$ satisfying (37) and (38) is greater than or equal to $c$, where $c$ is given by the equation

$$(c + 2)l_{i-1} + (c + 2) = (n_i - 1)l_{i-1},$$

that is,

$$c = \frac{(n_i - 1)l_{i-1} - 2}{1 + l_{i-1}}.$$  

Using (35), (36), and (18), we may thus conclude that

$$\frac{\#\Phi}{n_i} > \frac{1}{n_i} \left(\frac{(n_i - 1)l_{i-1} - 2}{1 + l_{i-1}}\right) - 10\varepsilon > 1 - 11\varepsilon. \quad (39)$$

Next we claim that

$$u_{n_i}(k_x + a_j) = u_{n_i}(k_y + b_j) \quad (40)$$

for all $(a_j, b_j) \in \Phi$. To give a proof by contradiction, we assume that there is a pair $(a_j, b_j) \in \Phi$ for which

$$u_{n_i}(k_x + a_j) \neq u_{n_i}(k_y + b_j).$$

or equivalently

$$(u_{n_i}(k_x + a_j) = 1 \land u_{n_i}(k_y + b_j) = 0) \lor (u_{n_i}(k_x + a_j) = 0 \land u_{n_i}(k_y + b_j) = 1).$$

Thus we may assume w.l.o.g. that

$$u_{n_i}(k_x + a_j) = 1 \land u_{n_i}(k_y + b_j) = 0. \quad (41)$$

Since $(a_j, b_j)$ satisfies either (33) or (34), we will assume w.l.o.g. that (33) is satisfied, because the alternative case, where (34) is valid, is completely analogous. Setting

$$\alpha := S_x(j) + a_j l_{i-1} = R_x(k_x + a_j),$$

$$\beta := S_y(j) + b_j l_{i-1} = R_y(k_y + b_j),$$

and

$$\gamma := \beta + l_{i-1} - \alpha,$$

we apply (33) to infer that

$$\beta \leq \alpha \leq \beta + 2l_{i-1}/3 \quad (42)$$

and, by implication, that

$$\frac{1}{3}l_{i-1} \leq \gamma \leq l_{i-1}. \quad (43)$$

Since $(a_j, b_j) \in \Phi$, we have $k_x + a_j \in \Phi_x$ and $k_y + b_j \in \Phi_y$, and therefore, (41) implies that
Combining (49) and (50) yields

\( T_f^\alpha x = T_f^{R_x(k^x+a_j^x)} x \in B_i \land T_f^{\alpha+l_i-1+1} x = T_f^{R_x(k^x+a_j^x+1)} \in B_i \)  \hfill (44)

and

\( T_f^\beta y = T_f^{R_y(k^y+b_j^y)} y \in B_i \land T_f^{\beta+l_i-1} y = T_f^{R_y(k^y+b_j^y+1)} y \in B_i \).  \hfill (45)

Since \( B_i = C_i \cap D_i \), we may apply (20) to conclude that

\[ \frac{1}{t_i-1} \| s(F_i, \tau_{i-1}) - s_{i-1}^E(T_f^\alpha x) \|_1 < \delta \]

and

\[ \frac{1}{t_i-1} \| s(F_i, \tau_{i-1}) - s_{i-1}^E(T_f^{\alpha+l_i-1+1} x) \|_1 < \delta. \]

Hence

\[ \left\| s_{i-1}^E(T_f^\alpha x) - s_{i-1}^E(T_f^{\alpha+l_i-1+1} x) \right\|_1 < 2t_i-1\delta \]  \hfill (46)

because the second inequality in (43) implies that \( s_{i-1}^E(T_f^\alpha x) \) and \( s_{i-1}^E(T_f^{\alpha+l_i-1+1} x) \) are subnames of \( s_{i-1}^E(T_f^\alpha x) \) and \( s_{i-1}^E(T_f^{\alpha+l_i-1+1} x) \), respectively. Since \( \beta \leq \alpha \) (by (42)), we also find that \( s_{i-1}^E(T_f^\beta y) \) and \( s_{i-1}^E(T_f^{\beta+l_i-1} y) \) are subnames of \( s_{i-1}^E(T_f^\beta y) \) and \( s_{i-1}^E(T_f^{\beta+l_i-1} y) \), respectively. Thus, using (45), we may deduce in a completely analogous fashion that

\[ \left\| s_{i-1}^E(T_f^\beta y) - s_{i-1}^E(T_f^{\beta+l_i-1} y) \right\|_1 < 2t_i-1\delta. \]  \hfill (47)

Setting

\[ \Delta := \left\{ k \in \{0, \ldots, \gamma - 1\} \mid \chi_E(T_f^{\alpha+k} x) \neq \chi_E(T_f^{\alpha+k+l_i-1+1} x) \right\} \]

\[ \cup \left\{ k \in \{0, \ldots, \gamma - 1\} \mid \chi_E(T_f^{\alpha+k} y) \neq \chi_E(T_f^{\alpha+k+l_i-1} y) \right\}, \]

we may combine (46) and (47) with (43) to infer that

\[ \frac{\#\Delta}{\gamma} < \frac{4t_i-1\delta}{\gamma} \leq 12\delta. \]  \hfill (48)

Similarly, for the set

\[ \Gamma := \left\{ k \in \{0, \ldots, \gamma - 1\} \mid T_f^{\alpha+k} x \notin X_E \lor T_f^{\alpha+k} y \notin X_E \right\} \]

\[ \cup \left\{ k \in \{0, \ldots, \gamma - 1\} \mid T_f^{\alpha+k+l_i-1+1} x \notin X_E \lor T_f^{\alpha+k+l_i-1} y \notin X_E \right\}, \]

we combine (44) and (45) with (11) to infer that \( \#\Gamma < 2\varepsilon^4t_i-1 \) and that

\[ \frac{\#\Gamma}{\gamma} < 6\varepsilon^4 < \frac{1}{2} \quad \text{(by (43) and (6))}. \]  \hfill (49)

Moreover, according to (13), (16) and (43), we have

\[ K_E^2 < \frac{1}{12\delta} < \frac{1}{3}t_i-1 \leq \gamma. \]

Consequently, we may apply Lemma 3.6 in conjunction with (48), (12), and (6) to the set

\[ \Upsilon := \left\{ k \in \{0, \ldots, \gamma - K_E\} \mid \{k, \ldots, k + K_E - 1\} \cap \Delta = \emptyset \right\} \]

to conclude that

\[ \frac{\#\Upsilon}{\gamma} > 1 - 2\sqrt{12\delta} > 1 - 2\sqrt{6}\varepsilon^2 > \frac{1}{2}. \]  \hfill (50)

Combining (49) and (50) yields

\[ \Upsilon \setminus \Gamma \neq \emptyset. \]
So let \( k_0 \in \mathfrak{T} \setminus \Gamma \). Then
\[
T_f^{\alpha + k_0} x, T_f^{\alpha + k_0 + l_{i-1} + 1} x, T_f^{\alpha + k_0} y, T_f^{\alpha + k_0 + l_{i-1} + 1} y \in X_E
\]
and
\[
\{k_0, \ldots, k_0 + K_E - 1\} \cap \Delta = \emptyset.
\]
Using the definition of \( \Delta \), it follows that
\[
\chi_E \left( T_f^{\alpha + k} x \right) = \chi_E \left( T_f^{\alpha + k + l_{i-1} + 1} x \right)
\]
and
\[
\chi_E \left( T_f^{\alpha + k} y \right) = \chi_E \left( T_f^{\alpha + k + l_{i-1} + 1} y \right)
\]
for all \( k \in \{k_0, \ldots, k_0 + K_E - 1\} \). Consequently, since
\[
\alpha + k + l_{i-1} + 1 \leq \alpha + k_0 + K_E + l_{i-1} \leq \alpha + \gamma + l_{i-1} = \beta + 2l_{i-1}
\]
\[
= S_n(j) + b_j l_{i-1} + 2l_{i-1} \leq b_j (1 + l_{i-1}) + 2l_{i-1}
\]
\[
\leq (n_i - 3)(l_{i-1} + 1) + 2l_{i-1} < (n_i - 1)l_{i-1} + n_i - 1
\]
for all \( k \in \{k_0, \ldots, k_0 + K_E - 2\} \), and therefore,
\[
\sum_{k=k_0}^{K_E} s^E_{K_E} \left( T_f^{\alpha + k} x \right) = (1, 1, \ldots, 1)
\]
or
\[
\sum_{k=k_0}^{K_E} s^E_{K_E} \left( T_f^{\alpha + k} y \right) = (0, 0, \ldots, 0).
\]
Hence
\[
\left| \frac{1}{K_E} \sum_{k=k_0}^{k_0 + K_E - 1} \chi_E \left( T_f^{\alpha + k} x \right) - \mu_E \left( T_f \right) \right| \geq \min \left\{ 1 - \mu_f(E), \mu_f(E) \right\}.
\]
But according to (51), (8) and (6) we also have
\[
\left| \frac{1}{K_E} \sum_{k=k_0}^{k_0 + K_E - 1} \chi_E \left( T_f^{\alpha + k} x \right) - \mu_E \left( T_f \right) \right| \leq \min \left\{ 1 - \mu_f(E), \mu_f(E) \right\}.
\]
Since this is a contradiction, the proof of (40) is complete. Using (39) we have
\[\# \Phi > \left[ (1 - 11\varepsilon)n_i \right] \] and therefore we have shown that assumption (28) does indeed imply that \( r_i(x) \) and \( r_i(y) \) are \( (1 - 11\varepsilon)n_i \)-similar. In order to complete the proof of the lemma, we proceed to assume that \( x, y \in M_i \) and
\[
s^E_{n_i, \ell_{i-1} + 1}(x) = s^E_{n_i, \ell_{i-1} + 1}(y),
\]
Since the definitions of \( M_i \) and \( \Theta_i \) imply that \( k_x + 1, k_y + 1 \in \Psi_i \), we can find an \( l_x \in \{0, \ldots, l_{i-1} + u_{n_i}(k_x) - 1\} \) and an \( l_y \in \{0, \ldots, l_{i-1} + u_{n_i}(k_y) - 1\} \) such that for \( x_0 := T_f^{l_x} x \) and \( y_0 := T_f^{l_y} y \) we have \( x_0 \in A_i(k_{x_0}) \) and \( y_0 \in A_i(k_{y_0}) \). For clarification we notice here that \( k_{x_0} = k_x \) if \( x \in A_i(k_x) \) and otherwise \( k_{x_0} = k_x + 1 \) (and analogously for \( k_y \) and \( k_{y_0} \)). Using (52), it follows that
\[
s^E_{(n_i - 1)l_{i-1} + n_i}(x_0) = s^E_{(n_i - 1)l_{i-1} + n_i}(y_0),
\]
and therefore, \( r_i(x_0) \) and \( r_i(y_0) \) are \([(1-11\varepsilon)n_i]-\)similar, and, by implication, \( r_i(x) \) and \( r_i(y) \) are \([(1-11\varepsilon)n_i]-1\) \(-\)similar. To complete the proof, we apply (15) to infer that
\[
[(1-11\varepsilon)n_i]-1 = [(1-11\varepsilon)n_i-1] \geq [(1-11\varepsilon)n_i-\varepsilon n_i] = [(1-12\varepsilon)n_i].
\]

\[\Box\]

4. Minimal entropy growth rates. In this concluding section we will show, by means of Lemma 3.9, how the sequence \( \langle f(n) \rangle_{n \in \mathbb{N}} \) defines a minimal entropy growth rate for the system \( \langle J_f, T_f \rangle \) in the limit superior as well as—in somewhat altered form—in the limit inferior.

4.1. Lemma. Let \( k, n \in \mathbb{N}, k \leq n \) and \( s \in \{0,1\}^n \), then
\[
\# \{r \in \{0,1\}^n \mid r \text{ is } k\text{-similar to } s \} \leq \left( \frac{n}{k} \right)^2 2^{n-k}.
\]

Proof. For any \( s \in \{0,1\}^n \) there are at most \( \binom{n}{k} \) different subnames \( (s_{i_1}, \ldots, s_{i_k}) \) of \( s \), because this is the number of possible choices for the indices \( i_1, \ldots, i_k \in \{0, \ldots, n-1\} \). Furthermore, for each such choice there are \( \binom{n-i}{k-i} \) possible ways to choose the indices \( j_1, \ldots, j_k \in \{0, \ldots, n-1\} \) that specify a subname \( (r_{j_1}, \ldots, r_{j_k}) \). Having chosen \( j_1, \ldots, j_k \) there are \( 2^{n-k} \) possibilities to fill in the remaining positions in \( r \). This completes the proof. \[\Box\]

4.2. Lemma. For all \( i \in \mathbb{N} \) that are sufficiently large it is the case that
\[
H\left( r \left( \alpha(E)^{n_i(l_i-1)+1} - 1 \right) > (1-13\varepsilon)^{-1} (1-13\varepsilon)^{-1} \sum \log_2 \varepsilon - 1 - 2,
\]
where
\[
c(\varepsilon) := \left( (1-13\varepsilon)^{-1-13\varepsilon} (13\varepsilon)^{13\varepsilon} \right)^2 2^{13\varepsilon}.
\]

Proof. Let \( i \geq N_E \) and \( x \in M_i \). Setting
\[
Q := \{ r \in \{0,1\}^{n_i} \mid r \text{ is } [(1-12\varepsilon)n_i]-\text{similar to } r_i(x) \}
\]
and
\[
P(r) := \{ k \in \{0, \ldots, n_i 2^{n_i} - 1 \} \mid (u_{n_i}(k), \ldots, u_{n_i}(k+n_i-1)) = r \}.
\]
for all \( r \in \{0,1\}^{n_i} \), we find that
\[
\mu_f \left( \left\{ y \in M_i \mid s_{n_i(l_i-1)+1}(y) = s_{n_i(l_i-1)+1}(x) \right\} \right)
\leq \mu_f \left( \left\{ y \in M_i \mid r_i(y) \text{ is } [(1-12\varepsilon)]-\text{similar to } r_i(x) \right\} \right)
\text{ (by Lemma 3.9)}
= \sum_{r \in Q} \mu_f \left( \left\{ y \in M_i \mid r_i(y) = r \right\} \right)
\leq \sum_{r \in Q} (l_i-1+1) \mu_f \left( \left\{ y \in \bigcup_{k=0}^{n_i} K_i(k) \mid r_i(y) = r \right\} \right)
= \sum_{r \in Q} (l_i-1+1) \mu_f \left( K_i(0) \right) \#P(r)
= \sum_{r \in Q} (l_i-1+1) \mu_f \left( K_i(0) \right) n_i \quad \text{ (by Lemma 2.1)}
\]
According to Stirling’s estimate, we can find an \( N \in \mathbb{N} \) such that for all \( n \geq N \) we have
\[
\frac{1}{2} \sqrt{2\pi n} \frac{n^n}{e^n} \leq n! \leq \frac{3}{2} \sqrt{2\pi n} \frac{n^n}{e^n}.
\]
Let \( i \in \mathbb{N} \) such that
\[
n_i - [(1 - 12\varepsilon)n_i] \geq N \land [(1 - 12\varepsilon)n_i] \geq N \land (1 - 12\varepsilon)n_i - 1 > 1.
\]
Then a few simple calculations using Stirling’s estimate show that
\[
\left( \frac{n_i}{[(1 - 12\varepsilon)n_i]} \right) \leq \frac{1}{\sqrt{e}} \left( (1 - 12\varepsilon - 1/n_i)^{(1-12\varepsilon-1/n_i)(12\varepsilon-1/n_i)} \right)^{n_i}.
\]
Since \( x^{-x} \) is increasing for \( x \leq 1/e \) and decreasing for \( x \geq 1/e \), we can use (6) and (15) to conclude that
\[
\left( \frac{n_i}{[(1 - 12\varepsilon)n_i]} \right) \leq \frac{1}{\sqrt{e}} \left( (1 - 13\varepsilon)^{-1(1-13\varepsilon)} \right)^{n_i}.
\]
Combining this estimate with (53), yields
\[
\mu_f \left( \left\{ y \in J_f \mid s_{n_i(l_i-1+1)}^E \left( M_i \cap y \right) \right\} \right) < \frac{2}{\varepsilon} \alpha(\varepsilon)^{-n_i},
\]
for all sufficiently large \( i \). Since \( x \) was an arbitrary element of \( M_i \), we have
\[
\mu_f \left( \left\{ y \in J_f \mid s_{n_i(l_i-1+1)}^E \left( M_i \cap y \right) \right\} \right) < \frac{2}{\varepsilon} \alpha(\varepsilon)^{-n_i},
\]
for all \( s \in \{0, 1\}^{n_i(l_i-1+1)} \). This implies that for all sufficiently large \( i \) we have
\[
H \left( \alpha(E)^{n_i(l_i-1+1)-1} \right) > H_{M_i} \left( \alpha(E)^{n_i(l_i-1+1)-1} \right) - 2 \quad \text{(by Lemma 1.7)}
\]
\[
\geq \mu_f(M_i) \log_2 \frac{\varepsilon \alpha(\varepsilon)^{n_i}}{2} - 2 \quad \text{(by Lemma 1.8)}
\]
\[
> (1 - 5\varepsilon)^4 (n_i \log_2 \varepsilon + \log_2 \varepsilon - 1) - 2 \quad \text{(by (27))},
\]
as desired.

\[\square\]

**4.3. Theorem.** If \( \alpha \) is a nontrivial partition of \( J_f \) into two sets, then
\[
\limsup_{n \to \infty} \frac{H(\alpha_0^{n-1})}{f(n)} \geq 1.
\]
Proof. According to Lemma 4.2 we have
\[
\limsup_{n \to \infty} \frac{H \left( \alpha(E)^{n-1}_0 \right)}{f(n)} \geq \limsup_{i \to \infty} \frac{H \left( \alpha(E)^{n_i(l_i-1+1)-1}_0 \right)}{f(n_i(l_i-1+1))} \\
\geq \limsup_{i \to \infty} \frac{(1 - 5\varepsilon)^4(n_i \log_2 c(\varepsilon) + \log_2 \varepsilon - 1) - 2}{f(n_i(l_i-1+1))} \\
= (1 - 5\varepsilon)^4 \log_2 c(\varepsilon) \limsup_{i \to \infty} \frac{n_i}{f(n_i(l_i-1+1))} \\
\geq (1 - 5\varepsilon)^4 \log_2 c(\varepsilon) \quad \text{(by (5)).}
\]
Since \( \lim_{\varepsilon \to 0} (1 - 5\varepsilon)^4 \log_2 c(\varepsilon) = 1 \), we may conclude, as desired, that
\[
\limsup_{n \to \infty} \frac{H \left( \alpha(E)^{n-1}_0 \right)}{f(n)} \geq 1
\]
for all \( E \in B_J \) with \( 0 < \mu_J(E) < 1 \). \( \square \)

Since the length-sequence \((l_i)_{i \in \mathbb{N}}\) is evidently strictly increasing, it follows that the sequence \((n_i)_{i \in \mathbb{N}}\) is increasing as well (by (5)), and therefore, the piecewise constant function
\[
g : [0, \infty) \to \mathbb{R} \\
x \mapsto \begin{cases} 
  f(0) & \text{if } x \in [0, n_1(l_0 + 1)) = [0, 2n_1], \\
  f(n_i(l_{i-1} + 1)) & \text{if } x \in [n_i(l_{i-1} + 1), n_{i+1}(l_i + 1)) \text{ and } i \in \mathbb{N}.
\end{cases}
\]
is well defined.

4.4. Theorem. If \( \alpha \) is a nontrivial partition of \( J_f \) into two sets, then
\[
\liminf_{n \to \infty} \frac{H \left( \alpha^0 \right)}{g(n)} \geq 1.
\]

Proof. Let \( \alpha(E) \) be a nontrivial partition of \( J_f \). Since \((H(\alpha(E)^{n-1}_0))_{n \in \mathbb{N}}\) is increasing, we may apply Lemma 4.2 to infer that for all sufficiently large values \( i \) and all \( n \in [n_i(l_{i-1} + 1), n_{i+1}(l_i + 1)) \) it is the case that
\[
\frac{H \left( \alpha(E)^{n-1}_0 \right)}{g(n)} = \frac{H \left( \alpha(E)^{n_i(l_{i-1}+1)-1}_0 \right)}{f(n_i(l_{i-1} + 1))} \geq \frac{H \left( \alpha(E)^{n_i(l_{i-1}+1)-1}_0 \right)}{f(n_i(l_{i-1} + 1))} \\
\geq (1 - 5\varepsilon)^4 \log_2 c(\varepsilon) \liminf_{i \to \infty} \frac{n_i}{f(n_i(l_{i-1} + 1))} - 2.
\]
Consequently, as in the proof of Theorem 4.3, we find that
\[
\liminf_{n \to \infty} \frac{H \left( \alpha(E)^{n-1}_0 \right)}{g(n)} \geq (1 - 5\varepsilon)^4 \log_2 c(\varepsilon) \liminf_{i \to \infty} \frac{n_i}{f(n_i(l_{i-1} + 1))} \\
\geq (1 - 5\varepsilon)^4 \log_2 c(\varepsilon)
\]
for all \( \varepsilon > 0 \), and therefore,
\[
\liminf_{n \to \infty} \frac{H \left( \alpha(E)^{n-1}_0 \right)}{g(n)} \geq 1,
\]
as desired. \( \square \)
To conclude our discussion, we will now go on to show that assumption (4) implies that the rates \((f(n))_{n \in \mathbb{N}}\) and \((g(n))_{n \in \mathbb{N}}\) are optimal in the sense that there is a partition of \(J_f\) into two sets by which these rates are realized.

**4.5. Lemma.** If \(x_1, \ldots, x_n > 0\) and \(\sum_{k=1}^{n} x_k \leq 1\), then the concavity of the entropy function \(h\) implies that
\[
\sum_{k=0}^{n} h(x_k) \leq n h\left(\frac{1}{n} \sum_{k=0}^{n} x_k\right) < \left(\sum_{k=0}^{n} x_k\right) \log_2 n + 1.
\]

**4.6. Theorem.** For \(\alpha := \alpha([0,1])\) we have
\[
\limsup_{n \to \infty} \frac{H(\alpha^{n-1})}{f(n)} = 1.
\]

**Proof.** Since \([0,1)\) is the only level of the tower \(\tau_0\), it is clear that for any \(i \in \mathbb{N}\) all levels of \(\tau_i\) are either contained in \([0,1)\) or in \(J_f \setminus [0,1)\). This allows us to determine the entropy of the partitions \(\alpha^{n-1}\) by analyzing the 01-names \(s([0,1), \tau_i)\). Let \(n \in \mathbb{N}\) such that
\[
l_{i-1} < n \leq l_i,
\]
and let \(k_i \in \mathbb{N}\) such that
\[
\frac{l_i(l_i + 1)}{l_{i+k_i}} < 1.
\]

Denoting by \(A\) the set \(|\tau_i|\) minus the last \(n\) levels of \(\tau_i\), that is,
\[
A := \bigcup_{j=0}^{l_{i-1} - 1} T_f^j(K_i(0))
\]
and setting
\[
B := J_f \setminus A,
\]
we may invoke the concavity of the entropy function \(h\) to infer that
\[
H(\alpha^{n-1}) \leq H_A(\alpha^{n-1}) + H_B(\alpha^{n-1}).
\]

In order to find an estimate for \(H_A(\alpha^{n-1})\), we need to determine the number of subnames of length \(n\) of \(s([0,1), \tau_i)\) that start on any of the levels of \(\tau_i\) that the set \(A\) consists of. Denoting this number by \(N_i(n)\), the construction of \(\tau_i\) from \(\tau_{i-1}\) readily implies that
\[
N_i(n) \leq \min\{ (l_{i-1} + 1)2^{n/l_{i-1}+1}, l_i - n\}. \quad (57)
\]

**Case 1.** \(n \leq (l_{i-1} + 1)(n_i - 1)\). In this case we apply (57) to infer that
\[
\frac{H_A(\alpha^{n-1})}{f(n)} \leq \frac{\mu_f(A) \log_2 N_i(n) + 1}{f(n)} \quad \text{(by Lemma 4.5)}
\]
\[
\leq \frac{\log_2(l_{i-1} + 1) + n/l_{i-1} + 2}{f(n)}
\]
\[
\leq \frac{\log_2(l_{i-1} + 1)}{f(l_{i-1} + 1)} + \frac{n}{f(l_{i-1} + 1)} + \frac{2}{f(f(n))} \quad \text{(by (54), because \(f\) is monotone increasing).}
\]
Since the concavity of $f$ implies that $f(x)/x$ is monotone decreasing, we may use (5) to conclude that
\[
\frac{n}{l_{i-1}f(n)} \leq \frac{(l_{i-1} + 1)(n_i - 1)}{l_{i-1}f((l_{i-1} + 1)(n_i - 1))} \leq 1 + \frac{1}{l_{i-1}}. \tag{59}
\]
Combining (58) and (59) yields
\[
\frac{H_A(\alpha_0^{n-1})}{f(n)} \leq \frac{\log_2(l_{i-1} + 1)}{f(l_{i-1} + 1)} + 1 + \frac{1}{l_{i-1}} + \frac{2}{f(n)}. \tag{60}
\]

**Case 2.** $n > (l_{i-1} + 1)(n_i - 1)$. Using the second estimate provided by (57), we find that
\[
\frac{H_A(\alpha_0^{n-1})}{f(n)} \leq \frac{\mu_f(A) \log_2 N_i(n) + 1}{f(n)} \quad \text{(by Lemma 4.5)}
\]
\[
\leq \frac{\log_2(l_{i-1} + 1)}{f(n)} + \frac{\log_2((l_{i-1} + 1)(n_i 2^{n_i} + n_i - 1))}{f((l_{i-1} + 1)(n_i - 1))} + \frac{1}{f(n)}
\]
\[
\leq \frac{\log_2(l_{i-1} + 1)}{f(l_{i-1} + 1)} + \frac{\log_2(n_i 2^{n_i})}{f(n)} + \frac{1}{f(n)} \tag{61}
\]
\[
\leq \frac{\log_2(l_{i-1} + 1)}{f(l_{i-1} + 1)} + \frac{\log_2(n_i + 1)}{f(n_i - 1)} + \frac{\log_2(n_i + 1)}{f((l_{i-1} + 1)(n_i - 1))} + \frac{2}{f(n)}
\]
\[
\leq \frac{\log_2(l_{i-1} + 1)}{f(l_{i-1} + 1)} + \frac{\log_2(n_i + 1)}{f(n_i - 1)} + 1 + \frac{2}{f(n)} \quad \text{(by (5))}.
\]
Combining this latter estimate with (60) yields
\[
\frac{H_A(\alpha_0^{n-1})}{f(n)} \leq \frac{\log_2(l_{i-1} + 1)}{f(l_{i-1} + 1)} + \frac{\log_2(n_i + 1)}{f(n_i - 1)} + 1 + \frac{1}{l_{i-1}} + \frac{2}{f(n)}. \tag{62}
\]
To find an estimate for $H_B(\alpha_0^{n-1})$, which is the second term in (56), we observe that
\[B = C \cup D\]
for
\[C := \bigcup_{j=l_i-n}^{l_{i-1}} T_j^f(K_i(0))\]
and
\[D := J_f \setminus (A \cup C) = J_f \setminus |\tau_i|\].

In order to find an upper estimate for the number of 01-names of length $n$ for each of the levels that $C$ consists of, we observe that for every $x \in T_j^{f_{i-n}}(K_i(0))$ the 01-name $s_n^{[0,1]}(x)$ is equal to the last $n$ digits of $s([0,1), \tau_i)$. So all $x \in T_j^{f_{i-n}}(K_i(0))$ have the same 01-name. If $x$ is in $T_j^{f_{i-n+2}}(K_i(0))$—the next higher level—then $s_n^{[0,1]}(x)$ is equal to the last $n - 1$ digits of $s([0,1), \tau_i)$ followed by either the first digit in $s([0,1), \tau_i)$ or by a zero, depending on whether $T_j^n(x) \in K_i(0)$ or $T_j^n(x) \in J_f \setminus |\tau_i|$. So there are at most two 01-names for the elements in the level $T_j^{f_{i-n+1}}(K_i(0))$. Similarly, for $x \in T_j^{f_{i-n+2}}(K_i(0))$, the 01-name $s_n^{[0,1]}(x)$ is equal to the first $n - 2$ digits in $s([0,1), \tau_i)$ followed either by the first two digits in $s([0,1), \tau_i)$ or by a zero and the first digit in $s([0,1), \tau_i)$ or by two zeros. Thus the maximal number of
01-names for this third level is three. Consequently, the total number of 01-names for the elements of $C$ is less than or equal to
\[
\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.
\]

To determine the 01-names of elements in $D$, we notice that every $x \in D$ is contained in $T^{-j}_f(K_i(0))$ for some $j \in \{1, \ldots, n\}$ and that, by implication, the number of 01-names of length $n$ for the elements of $D$ is $n$. Thus the total number of 01-names of length $n$ for the elements in $B$ is less than or equal to
\[
\frac{n(n+1)}{2} + n \leq n(n+1).
\]

Hence
\[
H_B(\alpha_0^{n-1}) < \mu_f(B) \log_2(n(n+1)) + 1 \quad \text{(by Lemma 4.5)}
\]
\[
\leq 2\log_2(n+1) + 1,
\]
and therefore, (56) and (62) together imply that
\[
\frac{H(\alpha_0^{n-1})}{n} \leq \frac{\log_2(l_i-1+1)}{f(l_i-1+1)} + \frac{\log_2(n_i+1)}{f(n_i-1)} + \frac{1}{l_i-1} + \frac{2\log_2(n+1)}{f(n)} + \frac{3}{f(n)}.
\]

Consequently, Theorem 4.3 in conjunction with (4) implies that
\[
1 \leq \limsup_{n \to \infty} \frac{H(\alpha_0^{n-1})}{f(n)} \leq 1,
\]
as desired. 

**4.7. Theorem.** For $\alpha := \alpha([0,1))$ we have
\[
\liminf_{n \to \infty} \frac{H(\alpha_0^{n-1})}{g(n)} = 1.
\]

**Proof.** Using the definition of $g$ in conjunction with Theorem 4.4, estimate (61), and (5), it follows that
\[
1 \leq \liminf_{n \to \infty} \frac{H(\alpha_0^{n-1})}{f(n)} \leq \liminf_{i \to \infty} \frac{H(\alpha_0^{n_i(l_i-1)+1}-1)}{f(n_i(l_i-1)+1)}
\]
\[
\leq \liminf_{i \to \infty} \left( \frac{\log_2(l_i-1+1)}{f_l(l_i-1+1)} + \frac{\log_2(n_i+1)}{f(n_i-1)} + 1 + \frac{2}{f(n_i(l_i-1)+1)} \right) = 1,
\]
as desired. 

**Remark.** The fact that the systems $(J_f, T_f)$ are weakly mixing but not strongly mixing follows in principle from the same argument that is used to show that Chacon's transformation is weakly mixing but not strongly mixing (see [7]). We will therefore omit the proof.

**REFERENCES**

[1] F. Blume, *An entropy estimate for infinite interval exchange transformations*, Mathematische Zeitschrift, 272 (2012), 17–29.
[2] F. Blume, *Minimal rates of entropy convergence for completely ergodic systems*, Israel Journal of Mathematics, 108 (1998), 1–12.
[3] F. Blume, *Minimal rates of entropy convergence for rank one systems*, Discrete and Continuous Dynamical Systems, 6 (2000), 773–796.
[4] F. Blume, On the relation between entropy and the average complexity of trajectories in dynamical systems, Computational Complexity, 9 (2000), 146–155.
[5] F. Blume, On the relation between entropy convergence rates and Baire category, Mathematische Zeitschrift, 271 (2012), 723–750.
[6] F. Blume, Possible rates of entropy convergence, Ergodic Theory and Dynamical Systems, 17 (1997), 45–70.
[7] F. Blume, The Rate of Entropy Convergence, Doctoral Dissertation, University of North Carolina at Chapel Hill, 1995.
[8] A. Katok and J.-P. Thouvenot, Slow entropy type invariants and smooth realization of commuting measure-preserving transformations, Annales de l’Institut Henri Poincare (B) Probability and Statistics, 33 (1997), 323–338.
[9] W. Parry, Entropy and Generators in Ergodic Theory, Benjamin, New York, 1969.
[10] K. E. Petersen, Ergodic Theory, Cambridge University Press, New York, 1983.

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