Super-Wavelets Versus Poly-Bergman Spaces

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Abstract. We investigate a vector-valued version of the classical continuous wavelet transform. Special attention is given to the case when the analyzing vector consists of the first elements of the basis of admissible functions, namely the functions whose Fourier transform is a Laguerre function. In this case, the resulting spaces are, up to a multiplier isomorphism, poly-Bergman spaces. To demonstrate this fact, we introduce a new map and call it the polyanalytic Bergman transform. Our method of proof uses Vasilevski’s restriction principle for Bergman-type spaces. The construction is based on the idea of multiplexing of signals.

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1. Introduction

In this paper, inspired by signal-theoretical ideas, we find a relation between two seemingly unrelated theories: wavelets and polyanalytic Bergman spaces. The cornerstone of this relation is a unitary map called the polyanalytic Bergman transform and its construction is based on the idea of multiplexing of signals.

1.1. The Heuristics of the Multiplexing Problem

The problem of multiplexing —encoding several signals as a single one with the purpose of sharing a communication channel—is a classical one in the theory of signals [5].

To gain some intuition concerning the multiplexing problem, consider two functions $f_1$ and $f_2$ in $H^2(\mathbb{C}^+)$, the classical Hardy space of the upper
half plane. If we transmit the signal
\[ f = f_1 + f_2 \]
throughout a channel, then we may be able to recover the complete information concerning \( f \), but it will be impossible to separate \( f_1 \) from \( f_2 \) without more information about their structure. Thus, before combining the two signals into a single one, we need two operators mapping each of the signals to new signals with some additional structure known \textit{a priori}, so that they can be later separated.

1.2. Multiplexing with True Polyanalytic Bergman Spaces

Our ideas are motivated by an operator theoretical approach to the multiplexing problem based on \textit{Vasilevski orthogonal decomposition} of polyanalytic Bergman spaces. To describe the basic ideas we need some definitions. Let \( \mathbb{C}^+ = \{ z \in \mathbb{C} : \text{Im} \, z \geq 0 \} \) stand for the upper half plane. The \textit{polyanalytic Bergman space}, \( A^n(\mathbb{C}^+) \) (poly-Bergman space, for short), consists of polyanalytic complex valued functions (see [6] for a text about this functions) defined on the upper half plane, square integrable with respect to area measure. More precisely, \( f \in A^n(\mathbb{C}^+) \) if
\[ \left( \frac{d}{dz} \right)^n f(z) = 0 \text{ and } \int_{\mathbb{C}^+} |f(z)|^2 \, d\mu^+(z) < \infty. \]

The space \( A^1(\mathbb{C}^+) = A(\mathbb{C}^+) \) is the usual (analytic) Bergman space in the upper half plane. Vasilevski [14] has introduced and characterized (by a restriction principle which will be used by us later on) an orthogonal sequence of orthogonal spaces called \textit{true polyanalytic Bergman spaces}, setting
\[ A^{n-1}(\mathbb{C}^+) = A^n(\mathbb{C}^+) \ominus A^{n-1}(\mathbb{C}^+), \]
so that
\[ A^n(\mathbb{C}^+) = A^0(\mathbb{C}^+) \oplus \cdots \oplus A^{n-1}(\mathbb{C}^+). \tag{1.1} \]

Now, if we are given \( n \) signals, \( f_0, \ldots, f_{n-1} \), all of them with “positive frequencies and finite energy” (the physical description of the Hardy space by its Fourier transform), we would like to “process” each \( f_k \in H^2(\mathbb{C}^+) \) by “sending” it to a space \( A^k(\mathbb{C}^+) \). Once the signals are processed in this way, they can be combined into a single one and subsequently separated by orthogonal projection. Thus, if we find a sequence of unitary operators between \( H^2(\mathbb{C}^+) \) and each of the spaces \( A^k(\mathbb{C}^+) \), we have a solution for our multiplexing problem. We define such a sequence in the next paragraph.

1.3. An Orthogonal Sequence of Unitary Operators

As the first element in our sequence of unitary operators we consider the (analytic) Bergman transform \( \text{Ber} : H^2(\mathbb{C}^+) \to A^0(\mathbb{C}^+) \) defined as
\[ \text{Ber} \, f(z) = \int_0^\infty t^{\frac{n}{2}} (\mathcal{F}f)(t) e^{itz} \, dt. \]
Write \( z = x + is \in \mathbb{C}^+ \). To process the remaining signals we introduce the 
**true polyanalytic Bergman transform**

\[
\text{Ber}^n f(z) = \frac{1}{(2i)^n n!} \left( \frac{d}{dz} \right)^n [s^n \text{Ber} f(z)], \quad n > 1,
\]

with \( \text{Ber}^0 = \text{Ber} \). As we will see in this paper, the transforms \( \text{Ber}^n : H^2(\mathbb{C}^+) \to \mathbb{A}^n(\mathbb{C}^+) \) are indeed unitary operators. Now, let \( H^2(\mathbb{C}^+, \mathbb{C}^n) \) stand for the space of vector-valued functions \( f = (f_0, \ldots, f_{n-1}) \) with each component in \( H^2(\mathbb{C}^+) \), together with the inner product

\[
\langle f, g \rangle_{H^2(\mathbb{C}^+, \mathbb{C}^n)} = \sum_{k=0}^{n-1} \langle f_k, g_k \rangle_{H^2(\mathbb{C}^+)}.
\]

Then, starting with a vector \( (f_0, \ldots, f_{n-1}) \in H^2(\mathbb{C}^+, \mathbb{C}^n) \) we construct a 
**single “multiplexed function”**,

\[
\text{Ber}^n f = \text{Ber}^0 f_0 + \ldots + \text{Ber}^{n-1} f_{n-1} \in \mathbb{A}^n(\mathbb{C}^+).
\]

The transform \( \text{Ber}^n : H^2(\mathbb{C}^+, \mathbb{C}^n) \to \mathbb{A}^n(\mathbb{C}^+) \) is also a unitary operator.

Thanks to the **orthogonal decomposition** (1.1) of \( \mathbb{A}^n(\mathbb{C}^+) \), it is now possible 
to transmit the multiplexed signal \( F = \text{Ber}^n f \) using a single channel and to 
recover at the receiver each of the “processed signals” \( \text{Ber}^k f_k \) by **orthogonal projection** over \( \mathbb{A}^k(\mathbb{C}^+) \). 
The initial signals \( f_0, \ldots, f_{n-1} \) are obtained by inverting the transforms \( \text{Ber}^k \). With two signals \( f_0, f_1 \), this can be outlined in the following scheme:

\[
\begin{array}{cccc}
\text{f}_0 \rightarrow \text{Ber}f_0 & \quad \rightarrow \quad & \text{Ber}f_0 \rightarrow \text{f}_0 \\
\text{Ber}f_0 + \text{Ber}^1f_1 = \text{BF} & \quad \rightarrow \quad & \text{Ber}^1f_1 \rightarrow \text{f}_1 \\
\text{f}_1 \rightarrow \text{Ber}^1f_1 & \quad \rightarrow \quad & \text{P}^1 \quad \rightarrow \quad & \text{P}^0
\end{array}
\]

The explicit formula for the orthogonal projection \( P_k : \mathbb{A}^n(\mathbb{C}^+) \to \mathbb{A}^k(\mathbb{C}^+) \) is known from the reproducing kernel \( K^k(z,w) \) of the spaces \( \mathbb{A}^k(\mathbb{C}^+) \). 
Thus, given \( F \in \mathbb{A}^n(\mathbb{C}^+) \), its true polyanalytic component \( F_k \in \mathbb{A}^k(\mathbb{C}^+) \) can be recovered by the orthogonal projection of \( F \) over the space \( \mathbb{A}^k(\mathbb{C}^+) \):

\[
F_k(z) = (P_k F)(z) = \langle F(w), K^k(z,w) \rangle_{\mathbb{A}^n(\mathbb{C}^+)}. 
\]

An explicit formula for the orthogonal projections \( P_k : \mathbb{A}^n(\mathbb{C}^+) \to \mathbb{A}^k(\mathbb{C}^+) \) has already been derived in [14]. We will obtain an alternative formula, using an interpretation of the polyanalytic transforms as special wavelet transforms. Define the functions \( \psi_n \) on the Fourier transform side as

\[
\mathcal{F}\psi_n(t) = t^{\frac{1}{2}l_n^0}(2t),
\]

where \( l_n^0 \) is a Laguerre function (see Sect. 2 for definition). The true polyanalytic Bergman transform is related to the wavelet transform

\[
W_g f(x,s) = \langle f, s^{-\frac{1}{2}} g(s^{-1}(\cdot - x)) \rangle_{H^2(\mathbb{C}^+)}
\]

by the formula

\[
\text{Ber}^n f(z) = s^{-1}W_{\psi_n} f(x,s).
\]
1.4. Forerunners
The construction in this paper parallels the connection between Gabor analysis with Hermite functions and polyanalytic Fock spaces. See the papers [2–4, 9]. It has also been motivated by the study of spaces of wavelet transform using restriction principles [11, 12]. In the short note [1] we have constructed discrete versions (frames) of our wavelet transforms. Another well-known approach to the vector-valued wavelet transform is [10].

1.5. Organization of the Paper
We have a background section where we review some facts concerning wavelets, Laguerre functions and the fundamental facts about analytic and polyanalytic Bergman spaces as well as the connection between Bergman spaces and wavelets provided by the analytic Bergman transform. In the third section, we define a vector-valued version of the continuous wavelet transform and some of its elementary properties. In the fourth section we introduce the true polyanalytic and the polyanalytic transforms and describe their relation with wavelets and vector-valued wavelets. In Sect. 5, the structure of the polyanalytic Bergman spaces is investigated using these new tools. We obtain a sequence of rational functions which is orthogonal in the upper half plane, defined by its Rodrigues-type formula. This sequence of rational functions is a basis of the polyanalytic Bergman space. Then we obtain an explicit formula for the reproducing kernel of the polyanalytic Bergman spaces, in the form of a differential operator also in the form of a Rodrigues-type formula.

2. Background
2.1. The Wavelet Transform
For standard references about time-frequency analysis and wavelet theory see [7, 8]. We restrict ourselves to functions \( f \in H^2(\mathbb{C}^+) \), the Hardy space in the upper half plane, which constituted by analytic functions \( f \) such that

\[
\|f\|_{H^2(\mathbb{C}^+)}^2 = \sup_{0 < s < \infty} \int_{-\infty}^{\infty} |f(x + is)|^2 \, dx < \infty.
\]

Let the Fourier transform be \((\mathcal{F}f)(\xi) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-i\xi t} f(t) \, dt\). The functions in the space \( H^2(\mathbb{C}^+) \) are said to be of “positive frequency” since, a well known Paley-Wiener theorem says that \( \mathcal{F}(H^2(\mathbb{C}^+)) = L^2(0, \infty) \). For this reason it is common to study \( H^2(\mathbb{C}^+) \) on the “frequency side”, where many calculation become easier. Now consider \( \mathbb{C}^+ = \{ z \in \mathbb{C} : \text{Re}(z) > 0 \} \). For every \( x \in \mathbb{R} \) and \( s \in \mathbb{R}^+ \), let \( z = x + is \in \mathbb{C}^+ \) and define

\[
\pi_z g(t) = s^{-\frac{1}{2}} g(s^{-1}(t-x)).
\]

A function \( g \neq 0 \) such that

\[
\int_{0}^{\infty} \frac{|g(\xi)|^2}{\xi} \, d\xi = C_g.
\]
is called \textit{admissible}. The constant $C_g$ is the \textit{admissibility constant}. Then the \textit{continuous wavelet transform} of a function $f$ with respect to a wavelet $g$ is defined, for every $x \in \mathbb{R}, s > 0$ as

$$W_g f(x, s) = \langle f, \pi_z g \rangle_{H^2(C^+)}.$$  \hfill (2.1)

Let $d\mu^+(z)$ stand for the standard normalized area in $\mathbb{C}^+$. The orthogonal relations for the wavelet transform

$$\int_{\mathbb{C}^+} W_{g_1}(x, s) W_{g_2}(x, s) s^{-2} d\mu^+(z) = \langle \mathcal{F} g_1, \mathcal{F} g_2 \rangle_{L^2(\mathbb{R}^+, t^{-1})} \langle f_1, f_2 \rangle_{H^2(C^+)}.$$  \hfill (2.2)

are valid for all $f_1, f_2 \in H^2(C^+)$ and $g_1, g_2 \in H^2(C^+)$ and admissible. As a result, the continuous wavelet transform provides an isometric inclusion

$$W_g : H^2(C^+) \rightarrow L^2(C^+, s^{-2} d\mu^+(z)),$$

which is an isometry for $C_g = 1$.

\textbf{2.2. The Laguerre functions}

The Laguerre polynomials will play a central role in our discussion. For $\alpha \geq 0$, one way to define them is by the power series

$$L_n^\alpha(x) = \frac{e^{x}x^{-\alpha}}{n!} \frac{d^n}{dx^n} \left[ e^{-x} x^{\alpha+n} \right] = \sum_{k=0}^{n} (-1)^k \binom{n+\alpha}{n-k} \frac{x^k}{k!}.$$  \hfill (2.3)

They satisfy the orthogonality relation

$$\int_0^{+\infty} L_n^\alpha(x) L_m^\alpha(x) x^\alpha e^{-t} dt = \frac{\Gamma(n+\alpha+1)}{n!} \delta_{n,m}.$$  \hfill (2.4)

Therefore, to obtain orthogonal functions in $L^2(\mathbb{R}^+)$ we define the Laguerre functions as

$$l_n^\alpha(x) = 1_{[0, \infty)}(x)e^{-x/2}x^{\alpha/2}L_n^\alpha(x).$$

For $\alpha \geq 0$, the Laguerre functions constitute an orthogonal basis for the space $L^2(0, \infty)$.

\textbf{2.3. Analytic and Polyanalytic Bergman Spaces}

With the Wirtinger differential operator notation,

$$\frac{d}{dz} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial s} \right), \quad \frac{d}{dz} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial s} \right),$$

a complex valued function $f$, is said to be analytic in a domain, if, for every $z$ in such domain, it satisfies

$$\frac{d}{dz} f(z) = 0.$$  

More generally, $f$ is said to be polyanalytic of order $n$ if

$$\left( \frac{d}{dz} \right)^n f(z) = 0.$$
This is equivalent to saying that $F(z)$ is a polynomial of order $n - 1$ in $z$ with analytic functions $\{\varphi_k(z)\}_{k=0}^{n-1}$ as coefficients:

$$F(z) = \sum_{k=0}^{n-1} \beta^k \varphi_k(z)$$  \hspace{1cm} (2.5)

Then $A(\mathbb{C}^+)$ stands for the Bergman space in the upper half plane, constituted by the analytic functions in $\mathbb{C}^+$ such that

$$\int_{\mathbb{C}^+} |f(z)|^2 \, d\mu^+(z) < \infty.$$  \hspace{1cm} (2.6)

The space constituted by the polyanalytic functions of order $n$, equipped with the same norm as the Bergman space is called the polyanalytic Bergman space, $A^n(\mathbb{C}^+)$ [14]. With this notation, $A^1(\mathbb{C}^+) = A(\mathbb{C}^+)$. Consider also the true polyanalytic Bergman space, $A^n(\mathbb{C}^+)$, defined by

$$A^{n-1}(\mathbb{C}^+) = A^n(\mathbb{C}^+) \oplus A^{n-1}(\mathbb{C}^+).$$

Then, the following decomposition holds:

$$A^n(\mathbb{C}^+) = A^0(\mathbb{C}^+) \oplus \ldots \oplus A^{n-1}(\mathbb{C}^+).$$  \hspace{1cm} (2.7)

2.4. The Bergman Transform

We can relate the wavelet transform to Bergman spaces of analytic functions, by choosing the window $\psi^\alpha$ such that

$$\mathcal{F}\psi^\alpha(t) = \frac{1}{c_\alpha} 1_{[0,\infty]} t^\alpha e^{-t},$$  \hspace{1cm} (2.8)

where $1_X$ stands for the characteristic function of the set $X$ and

$$c_\alpha^2 = \int_0^\infty t^{2\alpha-1} e^{-2t} \, dt = 2^{2\alpha-1} \Gamma(2\alpha),$$

where $\Gamma$ is the Gamma function. The constant $c_\alpha$ leads to $C_{\psi,\alpha} = 1$ and the corresponding wavelet transform is isometric. The Bergman transform of order $\alpha$ is the isometric inclusion map $Ber_\alpha : H(\mathbb{C}^+) \to A_\alpha(\mathbb{C}^+)$ such that

$$Ber_\alpha f(z) = s^{-\frac{\alpha}{2}-1} W_{\frac{\alpha+1}{2}} f(-x, s) = \frac{1}{c_{\alpha+1}^{\alpha+1}} \int_0^\infty t^{\frac{\alpha+1}{2}} (\mathcal{F}f)(t) e^{itz} \, dt.$$  \hspace{1cm} (2.9)

Since $c_\frac{1}{2} = 1$, the following special case

$$Ber f(z) = Ber_0 f(z) = \int_0^\infty \frac{1}{2} (\mathcal{F}f)(t) e^{itz} \, dt,$$  \hspace{1cm} (2.10)

is an isometric isomorphism

$$Ber : H^2(\mathbb{C}^+) \to A(\mathbb{C}^+).$$

The surjectivity follows from the fact that the Laguerre functions $l_n^1$ are mapped to an orthogonal basis of $A(\mathbb{C}^+)$. We give some details about this in
the last section, where we will need the explicit formulas of such an orthogonal basis.

3. A Continuous Vector Valued Wavelet Transform

The Hilbert space $\mathcal{H} = H^2(\mathbb{C}^+, \mathbb{C}^n)$ consists of vector-valued functions $f = (f_0, \ldots, f_{n-1})$ with the inner product

$$\langle f, g \rangle_{\mathcal{H}} = \sum_{k=0}^{n-1} \langle f_k, g_k \rangle_{H^2(\mathbb{C}^+)}.$$  \hfill (3.1)

The formal considerations of this section are also valid for $L^2(\mathbb{R}, \mathbb{C}^n)$.

**Definition 1.** Let $g = (g_0, \ldots, g_{n-1})$ be a vector of functions in $\mathcal{H}$ such that

$$\langle \mathcal{F}g_i, \mathcal{F}g_j \rangle_{L^2(\mathbb{R}, t^{-1}dt)} = \delta_{i,j} \hfill (3.2)$$

The continuous vector valued wavelet transform of a vector function $f = (f_1, \ldots, f_{n-1})$ with respect to the vectorial window $g$ is defined, for every $x \in \mathbb{R}, s \in \mathbb{R}^+$, as

$$W_g f(x, \omega) = \langle f, D_s T_x g \rangle_{\mathcal{H}}. \hfill (3.3)$$

We can also write

$$W_g f(x, s) = \sum_{k=0}^{n-1} W_{g_k} f_k(x, s).$$

This defines a map

$$W_g f : \mathcal{H} \to L^2(\mathbb{C}^+, s^{-2}d\mu^+).$$

The orthogonality condition imposed on the vector $g$ allows the continuous vector valued wavelet to retain most of the properties of the scalar Wavelet transform. In particular, we have vector valued versions of the isometric property and orthogonality relations.

**Proposition 1.** Let $g$ satisfy (3.2). Then, for $f_1, f_2, \in \mathcal{H}$,

$$\langle W_g f_1, W_g f_2 \rangle_{L^2(\mathbb{C}^+, s^{-2}d\mu^+)} = \langle f_1, f_2 \rangle_{\mathcal{H}}. \hfill (3.4)$$

In particular, $W_g f$ is an isometry between Hilbert spaces, that is

$$\|W_g f\|_{L^2(\mathbb{C}^+, s^{-2}d\mu^+)} = \|f\|_{\mathcal{H}}. \hfill (3.5)$$

**Proof.** From (2.2) and (3.2),

$$\langle W_{g_k} f_k, W_{g_j} f_j \rangle_{L^2(\mathbb{C}^+, s^{-2}d\mu^+)} = \langle f_k, f_j \rangle_{L^2(\mathbb{R})} \times \delta_{k,j}. \hfill (3.6)$$
Then,

\[
\langle W_g f_1, W_g f_2 \rangle_{L^2(\mathbb{C}^+, s^{-2}d\mu^+)} = \sum_{k=0}^{n-1} \langle W_{g_k} f_{1,k}, W_{g_j} f_{2,k} \rangle_{L^2(\mathbb{C}^+, s^{-2}d\mu^+)}
\]

\[
= \sum_{k,j=0}^{n-1} \langle f_{1,k}, f_{2,j} \rangle_{L^2(\mathbb{C}^+, s^{-2}d\mu^+)} \times \delta_{k,j}
\]

\[
= \sum_{k=0}^{n-1} \langle f_{1,k}, f_{2,k} \rangle_{L^2(\mathbb{C}^+, s^{-2}d\mu^+)}
\]

\[
= \langle f_1, f_2 \rangle_H.
\]

Now let \( W_g \) stand for the subspace of \( L^2(\mathbb{C}^+, s^{-2}d\mu^+) \) constituted by the image of \( \mathcal{H} \) under the vector valued wavelet transform \( W_g f \):

\[
W_g = \{ W_g f : f \in \mathcal{H} \}.
\]

Since

\[
W_g f = \sum_{k=0}^{n-1} W_{g_k} f_k
\]

and

\[
\langle W_{g_k} f_k, W_{g_j} f_j \rangle_{L^2(\mathbb{C}^+, s^{-2}d\mu^+)} = \delta_{k,j},
\]

we know that every \( F \in W_g \) can be written in a unique way in the form

\[
F = F_0 + \ldots + F_{n-1}.
\]

As a result,

\[
W_g = W_{g_0} \oplus \ldots \oplus W_{g_{n-1}},
\]

where

\[
W_{g_j} = \{ W_{g_j} f : f \in L^2(\mathbb{R}) \}.
\]

**Proposition 2.** The space \( W_g \) is a Hilbert space with reproducing kernel given by

\[
k(z, w) = \langle T_\eta D_u g, T_x D_s g \rangle_{\mathcal{H}} = \sum_{j=0}^{n-1} k_j (z, w),
\]

where \( k_j (z, w) \) is the reproducing kernel of \( W_{g_j} \).

**Proof.** Let \( F \in W_g \). There exists \( f \in \mathcal{H} \) such that \( F = W_g f \). By definition, \( k(z, \cdot) = W_g (T_x D_s g) \). Thus, using (3.4),

\[
\langle F, k(z, \cdot) \rangle_{L^2(\mathbb{C}^+, s^{-2}d\mu^+)} = \langle W_g f, W_g (T_x D_s g) \rangle_{L^2(\mathbb{C}^+, s^{-2}d\mu^+)}
\]

\[
= \langle f, T_x D_s g \rangle_{\mathcal{H}}
\]

\[
= F(z).
\]

The second inequality follows from the well known fact that the reproducing kernel of the space \( W_{g_j} \) is given by \( \langle T_\eta D_u g, T_x D_s g \rangle_{H^2(\mathbb{C}^+)} \).
4. The Polyanalytic Bergman Transform

In this section we will study a special case of the continuous vector valued wavelet transform, when the vector is defined in terms of Laguerre functions. First we treat the scalar case, which originates a unitary map onto the true polyanalytic Bergman space. We show that the required unitary mappings can be related to special wavelet transforms and, via a connection to the previous section, we define the vector valued polyanalytic transform onto the polyanalytic space (which we call the polyanalytic Bergman transform).

4.1. The True Polyanalytic Bergman Transform

First we will study the transform that allows, in the multiplexing context explained in the introduction, to send each signal \( f_k \in H^2(C^+) \) to a space \( \mathcal{A}^k(C^+) \).

**Definition 2.** The true polyanalytic Bergman transform of order \( n \) is the transform mapping every \( f \in H^2(C^+) \) to

\[
Ber^n f(z) = \frac{1}{(2i)^n n!} \left( \frac{d}{dz} \right)^n \left[ s^n F(z) \right],
\]

where \( F = Ber^2 f \) and \( f \in H^2(C^+) \).

**Remark 1.** In an effort to make the notations clear, we have choose the \( n \) as superscript in \( Ber^n f \) to avoid confusion with the weighted Bergman transform \( Ber_{\alpha} \) (2.9), where the weight is indicated by an underscript.

The purpose of this section is to prove that the transform \( Ber^n : H^2(C^+) \rightarrow \mathcal{A}^n(C^+) \) is unitary. This is done using a connection to wavelets. First observe that, since

\[
L^0_n(t) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{t^k}{k!},
\]

we have

\[
l_0^n(t) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{k!} 1_{[0,\infty]}(x) t^{k} e^{-t/2} = t^{-\frac{1}{2}} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \mathcal{F}\psi^{k+\frac{1}{2}}(\frac{t}{2}).
\]

Due to this observation, it is reasonable to expect that the functions \( \psi_n \), defined by

\[
\mathcal{F}\psi_n(t) = \sqrt{2t^2} l_0^n(2t),
\]

normalized such that \( C_{\psi_n} = 1 \), will play a distinguished role in our analysis. Indeed, an essential step in the proof of the unitary property is to write (4.1) in terms of a wavelet transform with analyzing wavelet \( \psi_n \). Observe that

\[
\psi_0(t) = \psi^{\frac{1}{2}}(t)
\]

**Proposition 3.** The true polyanalytic Bergman transform of order \( n \) can be written as:
1. A polyanalytic function of order $n + 1$:

$$Ber^n f(z) = \frac{1}{(2i)^n n!} \sum_{k=0}^{n} (2i)^k \binom{n}{k} \frac{1}{k!} s^k F^{(k)}(z), \quad (4.4)$$

where $F(z) = Ber_0 f(z)$

2. In terms of analytic Bergman transforms of different orders:

$$Ber^n f(z) = \sum_{k=0}^{n} (-2)^k \binom{n}{k} s^k Ber_{2k} f(z). \quad (4.5)$$

3. In terms of a wavelet transform:

$$Ber^n f(z) = s^{-1} W_{\psi_n} f(x, s). \quad (4.6)$$

Proof. The first identity follows from a standard application of Leibnitz formula. Then, since

$$\frac{d}{dz} s^k = \frac{i}{2} k s^{k-1},$$

we have

$$\left( \frac{d}{dz} \right)^{n+1} Ber^n f(z) = 0,$$

and $Ber^n f$ is polyanalytic of order $n + 1$. To prove (4.5), observe that differentiation of (2.10) under the integral sign gives

$$F^{(k)}(z) = \left( \frac{d}{dz} \right)^k Ber_0 f(z) = i^k Ber_{2k} f(z). \quad (4.7)$$

Applying this to (4.4) gives (4.5).

Now (4.6). Combining (4.2) with (4.3) and inverting the Fourier transform gives

$$\psi_n = \sum_{k=0}^{n} (-2)^k \binom{n}{k} \psi^{k+\frac{1}{2}}(t).$$

Then,

$$W_{\psi_n} f(x, s) = \sum_{k=0}^{n} (-2)^k \binom{n}{k} W_{\psi^{k+\frac{1}{2}}} f(-x, s)$$

$$= \sum_{k=0}^{n} (-2)^k \binom{n}{k} s^{k+1} Ber_{2k} f(z)$$

$$= s Ber^n f(z).$$

and (4.6) follows. □

Now we can prove the main result. The proof consists of writing the wavelet transform (4.6) as a composition of several unitary operators and is
suggested by the techniques used in [12,14]. We need to introduce two auxiliary operators. For convenience write $L^2(C^+) = L^2(R) \otimes L^2(R^+)$ and define the unitary operators $U_{1,2} : L^2(R) \otimes L^2(R^+) \to L^2(R) \otimes L^2(R^+)$:

\[
U_1(F)(x,s) = (\mathcal{F}^{-1} \otimes I)(F)(x,s) \\
U_2(F)(x,s) = \frac{1}{\sqrt{2|x|}} F(x,\frac{s}{2|x|}).
\]

The following theorem is Vasilevski’s restriction principle for the true polyanalytic Bergman space [14].

**Theorem A.** [14, Corollary 4.2] Let $L_n$ stand for the space generated by $1_{[0,\infty]}l^0_n$. The operator $U = U_2U_1$,

\[
U : \mathcal{A}^n(C^+) \to L^2(R^+) \otimes L_n
\]

such that, given $f \in \mathcal{A}^n(C^+)$,

\[
(Uf)(x,s) = 1_{[0,\infty]}(x)f(x)l^0_n(s),
\]

is unitary.

Now consider the unitary operator

\[
R_n : L^2(R^+) \to L^2(R^+) \otimes L_n
\]

defined by

\[
(R_nf)(x,s) = 1_{[0,\infty]}(x)f(x)l^0_n(s).
\]

In the next theorem we prove the unitarity of $Ber^n : H^2(C^+) \to \mathcal{A}^n(C^+)$ by combining Theorem A and Proposition 3.

**Theorem 4.** The transform $Ber^n : H^2(C^+) \to \mathcal{A}^n(C^+)$ is unitary.

**Proof.** The idea of the proof is to decompose $Ber^n$ according to the following commutative diagram:

\[
\begin{array}{ccc}
H^2(C^+) & \xrightarrow{Ber^n} & \mathcal{A}^n(C^+) \\
\mathcal{F} & \downarrow & \downarrow U \\
L^2(R^+) & \xrightarrow{R_n} & L^2(R^+) \otimes L_n
\end{array}
\]

Clearly, the composition

\[
U^{-1}R_n : L^2(R^+) \to \mathcal{A}^k(C^+),
\]

is unitary. In view of the above diagram we should have $U^{-1}R_n = Ber^n \mathcal{F}^{-1}$. Let us see that this is indeed the case. From the definition of $U_2$ it is easy to see that

\[
U_2^{-1}(F)(x,s) = \sqrt{2|x|} F(x,2|x|s)
\]

and

\[
(U_2^{-1}R_nf)(x,s) = 1_{[0,\infty]}(x)\sqrt{2xf(x)}l^0_n(2xs).
\]
Applying $U_1^{-1}$ to $R_n$ gives
\[(U_1^{-1}R_nf)(x,s) = s^{-1} \int_0^\infty f(t)s^{\frac{1}{2}}(2ts)^{\frac{1}{2}}l_0(t)e^{ixt}dt\]
\[= s^{-1} \int_\mathbb{R} (F^{-1}f)(t)s^{-\frac{1}{2}}\psi_n(s^{-1}(t-x))dt\]
\[= s^{-1}W_{\psi_n}(F^{-1}f)(x,s)\]
\[= \text{Ber}^n(F^{-1}f)(z),\]
by using the identity (4.6). Thus, $U_1^{-1}R_n = \text{Ber}^nF^{-1}$ and we are done: since $U_1^{-1}R_n : L^2(\mathbb{R}^+) \to A^n(\mathbb{C}^+)$ is unitary, the result follows from the Paley-Wiener isomorphism $F : H^2(\mathbb{C}^+) \to L^2(\mathbb{R}^+)$.

4.2. The Polyanalytic Bergman Transform

**Definition 3.** The polyanalytic Bergman transform of order $n$ is defined, for $f \in \mathcal{H}$ as
\[\text{Ber}^n f = \sum_{k=0}^{n-1} \text{Ber}^k f_k.\]

**Theorem 5.** The polyanalytic Bergman transform of order $n$ is a unitary operator
\[\text{Ber}^n : \mathcal{H} \to A^n(\mathbb{C}^+)\]

**Proof.** To see that it is onto, let $F \in A^n(\mathbb{C}^+)$. Then, using (2.7), write
\[F = F_0 + \ldots + F_{n-1},\]
with $F_k \in A^k(\mathbb{C}^+), k = 0, \ldots, n-1$. Since $\text{Ber}^k$ is onto, for every $k = 0, \ldots, n-1$ there exists $f_k \in H^2(\mathbb{C}^+)$ such that $F_k = \text{Ber}^k f_k$. To prove the isometry, we first relate the polyanalytic Bergman transform to the vector valued wavelet transform with the vectorial window $\psi_n = (\psi_0, \ldots, \psi_{n-1})$, using the identity $\text{Ber}^n f(z) = s^{-1}W_{\psi_n}(f)(x,s) :$
\[W_{\psi_n}(f)(x,s) = \sum_{k=0}^{n-1} W_{\psi_k}(f_k)(x,s) = \sum_{k=0}^{n-1} s\text{Ber}^k f_k = s\text{Ber}^n f.\]
Now, combining this with (3.5),
\[\|\text{Ber}^n f\|_{A^n(\mathbb{C}^+)} = \|W_{\psi_n}(f)\|_{L^2(\mathbb{C}^+, s^{-2}d\mu^+)} = \|f\|_\mathcal{H}.\]

5. The Structure of Polyanalytic Bergman Spaces

The purpose of this section is to apply the connection to wavelet transforms to study polyanalytic Bergman spaces. We will obtain an orthogonal basis for $A^n(\mathbb{C}^+)$ and compute an explicit formula for the reproducing kernel. The reproducing kernel, $K^n(z,w)$, of the true polyanalytic Bergman space $A^n(\mathbb{C}^+)$ is also very important, since once we have a function $F \in A^n(\mathbb{C}^+)$,
we can recover its true polyanalytic component $F_k \in \mathcal{A}^k(\mathbb{C}^+)$ by the orthogonal projection over the space $\mathcal{A}^k(\mathbb{C}^+)$, which is given by the formula

$$F_k(z) = \langle F(w), K^k(z, w) \rangle_{\mathcal{A}^k(\mathbb{C}^+)}.$$

Our formulas are differential operators reminiscent of the Rodrigues formula, a well known structure formula in the theory of classic orthogonal polynomials. An example of a Rodrigues formula is the first formula in the identity (2.3) defining Laguerre polynomials.

5.1. An Orthogonal Basis

Consider the functions $\Psi_n^\alpha$, for every $n \geq 0$ and $\alpha > -1$:

$$\Psi_n^\alpha(z) = (2i)^{\alpha+3} \frac{\Gamma(\alpha + 2 + n)}{n!} \left( \frac{z - i}{z + i} \right)^n \left( \frac{1}{z + i} \right)^{\alpha+2}.$$

It is well known [13] that these functions constitute a basis of $A_\alpha(\mathbb{C}^+)$, the space of analytic functions with the norm

$$\|f\|^2_{A_\alpha(\mathbb{C}^+)} = \int_{\mathbb{C}^+} |f(z)| s^\alpha d\mu^+(z).$$

A calculation using the special function formula

$$\int_0^\infty x^\alpha L_n^\alpha(x)e^{-xs} dx = \frac{\Gamma(\alpha + n + 1)}{n!} s^{-\alpha-n-1}(s-1)^n$$

gives

$$\text{Ber}_\alpha l_n^{\alpha-1} = \Psi_n^\alpha.$$

Now write

$$\Psi_n(z) = \Psi_n^0(z) = (2i)^3 \frac{\Gamma(2 + n)}{n!} \left( \frac{z - i}{z + i} \right)^n \left( \frac{1}{z + i} \right)^2$$

to denote a basis of $A(\mathbb{C}^+)$, so that

$$\text{Ber}_1 l_n^1 = \Psi_n.$$

**Definition 4.** Define a set of functions by

$$e_{n,m}(z) = \frac{1}{(2i)^n n!} \left( \frac{d}{dz} \right)^n [s^n \Psi_m(z)].$$

**Proposition 6.** The set \{e_{k,m}\}_{k \geq 0, 0 \leq m < n} is an orthonormal basis of $\mathcal{A}^n(\mathbb{C}^+)$. 

**Proof.** Let $z = x + is$. Since

$$e_{n,k}(z) = \text{Ber}^n(\mathcal{F}^{-1}l_k^1) = s^{-1}W_{\psi_n}(\mathcal{F}^{-1}l_k^1)(x, s),$$

the orthogonality follows from (2.2):

$$\langle e_{n,k}, e_{l,j} \rangle_{L^2(\mathbb{C}^+, dz)} = \langle W_{\psi_n}(\mathcal{F}^{-1}l_k^1), W_{\psi_l}(\mathcal{F}^{-1}l_j^1) \rangle_{L^2(\mathbb{C}^+, s^{-2}dz)}$$

$$\quad = \langle n^0, l_j^0 \rangle_{L^2(\mathbb{R}^+)} \langle \mathcal{F}^{-1}l_k^1, \mathcal{F}^{-1}l_j^1 \rangle_{H^2(\mathbb{C}^+)}$$

$$\quad = \delta_{n,l} \delta_{k,j}.$$
The unitarity of $Ber^n$ shows that, for every $m$, \( \{e_{k,m}\}_{k \geq 0} \) spans $A^m(\mathbb{C}^+)$, since $\{l_n^1\}_{n \geq 0}$ spans $L^2(\mathbb{R}^+)$. From the decomposition (2.7), every element in $A^n(\mathbb{C}^+)$ can be written as a linear combination of elements of $\{A^m(\mathbb{C}^+)\}_{m<n}$. Therefore $\{e_{k,m}\}_{k \geq 0, 0 \leq m < n}$ spans $A^n(\mathbb{C}^+)$. □

**Corollary 7.** The set $\{e_{k,m}\}_{0 \leq m < n}$ is an orthonormal basis of $A^k(\mathbb{C}^+)$.  

**Proof.** This follows immediately from the decomposition (2.7). □

### 5.2. The Reproducing Kernel

Let $w = u + i\eta \in \mathbb{C}^+$. In our computations of the reproducing kernels we will need the following relation

$$Ber(\pi_w f)(z) = \eta^{-\frac{1}{2}}(\pi_w Ber f)(z), \quad (5.1)$$

which follows from

$$Ber(\pi_w f)(z) = \int_{0}^{\infty} t^\frac{1}{2} e^{itz} (\mathcal{F}(\pi_w f))(t) dt$$

$$= \eta^{-\frac{1}{2}} \int_{0}^{\infty} t^\frac{1}{2} e^{i(z+\eta)t} (\mathcal{F} f)(\eta t) dt$$

$$= \eta^{-1} \int_{0}^{\infty} t^\frac{1}{2} e^{i(z+\eta)t} (\mathcal{F} f)(t) dt.$$  

**Theorem 8.** The reproducing kernel of $\mathcal{W}_n$ is given by

$$k^n(z, w) = \frac{1}{n!(2i)^n s\eta^{-\frac{1}{2}}} \left( \frac{d}{dz} \right)^n [s^n \pi_w \Omega_n(z)],$$

where

$$\Omega_n(z) = 4(n + z - i) \left( \frac{1}{z + i} \right)^3 \left( \frac{z - i}{z + i} \right)^{n-1}.$$

**Proof.** The reproducing kernel of $\mathcal{W}_n$ is

$$k^n(z, w) = \langle \pi_w \psi_n, \pi_z \psi_n \rangle_{H^2(\mathbb{C}^+)}$$

$$= s Ber^n(\pi_w \psi_n)(z)$$

$$= \frac{s}{n!(2i)^n} \left( \frac{d}{dz} \right)^n [s^n Ber(\pi_w \psi_n)(z)].$$

Now, (5.1) gives

$$k^n(z, w) = \frac{1}{(2i)^n n!} s\eta^{-\frac{1}{2}} \left( \frac{d}{dz} \right)^n [s^n \pi_w (Ber \psi_n)(z)].$$
We just need to compute
\[
Ber(\psi_n) = \int_0^\infty tl_n^0(2t)e^{itz}dt = \frac{1}{i} \frac{d}{dz} \int_0^\infty l_n^0(2t)e^{itz}dt
\]
\[
= 4 \frac{d}{dz} \left[ \frac{(z-i)^n}{z+i} \right] = \Omega_n(z),
\]
and the formula is proved. □

The next lemma can be used to transfer properties from the spaces \(W_{\psi_n}\) to the spaces \(A^n(\mathbb{C}^+)\).

**Lemma 9.** The operator \(E\) defined by the correspondence
\[
E : f \rightarrow s^{-1}f(x,s),
\]
is unitary
\[
E : W_{\psi_n} \rightarrow A^n(\mathbb{C}^+)
\]
and
\[
E : W_{\psi_n} \rightarrow A^n(\mathbb{C}^+).
\]

**Proof.** Clearly, \(E\) is isometric. Since \(l_m^1\) is a basis of \(L^2(\mathbb{R}^+)\), then \(W_{\psi_n}(\mathcal{F}^{-1}l_m^1)\) is a basis of \(W_{\psi_n}\). Then
\[
E(W_{\psi_n}(\mathcal{F}^{-1}l_m^1)) = s^{-1}W_{\psi_n}(\mathcal{F}^{-1}l_m^1)(x,s) = Ber^n l_m^1(z) = e_{n,m}(z).
\]
Thus, \(E(W_{\psi_n})\) is dense in \(A^n(\mathbb{C}^+)\). The second assertion follows immediately from (3.9). □

**Theorem 10.** The reproducing kernels of the spaces \(A^n(\mathbb{C}^+), K^n(z,w)\), are given by
\[
K^n(z,w) = \frac{1}{n!(2i)^n} \left( \frac{d}{dz} \right)^n \left[ s^n \pi_w \Omega_n(z) \right],
\]
and the reproducing kernels of the spaces, \(A^n(\mathbb{C}^+), K^n(z,w)\), are given by
\[
K^n(z,w) = \sum_{k=0}^{n-1} \frac{1}{k!(2i)^k} \left( \frac{d}{dz} \right)^k \left[ s^k \pi_w \Omega_k(z) \right].
\]

**Proof.** Let \(f \in A^n(\mathbb{C}^+)\). Then, by the above Lemma, \(sf(z) \in W_{\psi_n}\). Therefore,
\[
sf(z) = \langle k^n(z,w), \eta f(w) \rangle_{W_{\psi_n}},
\]
and
\[
f(z) = \left\langle \frac{\eta}{s} k^n(z,w), f(w) \right\rangle_{A^n(\mathbb{C}^+)}.\]

We conclude that \(K^n(z,w) = \frac{\eta}{s} k^n(z,w)\). The second assertion follows immediately from (3.9). □
Remark 2. An alternative form for the reproducing kernels of $A^n(\mathbb{C}^+)$ has been obtained in [14]:

$$K^n(z, w) = \frac{1}{\pi(z - \bar{w})^2} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} \left( \frac{z - \bar{z}}{z - w} \right)^j \left( \frac{w - \bar{w}}{z - w} \right)^k,$$

$$k_{n,k,j} = (-1)^{j+k+1} \binom{n!}{j!k!}^2 \frac{(j+k+1)!}{(n-j)!(n-k)!}.$$

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