EXISTENCE, RENORMALIZATION, AND REGULARITY PROPERTIES OF HIGHER ORDER DERIVATIVES OF SELF-INTERSECTION LOCAL TIME OF FRACTIONAL BROWNIAN MOTION

KAUSTAV DAS1 AND GREG MARKOWSKY†

Abstract. In a recent paper by Yu (arXiv:2008.05633, 2020), higher order derivatives of self-intersection local time of fractional Brownian motion were defined, and existence over certain regions of the Hurst parameter $H$ was proved. Utilizing the Wiener chaos expansion, we provide new proofs of Yu’s results, and show how a Varadhan-type renormalization can be used to extend the range of convergence for the even derivatives.

Keywords: Self-intersection local time; derivatives of self-intersection local time; fractional Brownian motion; Wiener chaos.

1. Introduction

Let $(B^H_t)$ be a fractional Brownian motion (fBm) with Hurst parameter $H \in (0, 1)$. Namely, $(B^H_t)$ is the unique centred Gaussian process with covariance function

$$R_H(t, s) = \frac{1}{2}(s^{2H} + t^{2H} - |s-t|^{2H}), \quad s, t \geq 0.$$ 

The self-intersection time of fBm is defined as

$$\alpha_t := \int_0^t \int_0^s \delta(B^H_s - B^H_r - y)drds,$$

where $\delta$ is the Dirac delta function. Intuitively, $\alpha_t(0)$ measures the amount of time that the process $(B^H_t)$ spends revisiting prior values on the interval $[0, t]$.

The object of focus in this paper is the $k$-th derivative of self-intersection time (DSLT) of fBm, which was introduced in the interesting recent paper [45]. To be precise, we define

$$\alpha_{t, \varepsilon}(y) := \int_0^t \int_0^s f_{\varepsilon}(B^H_s - B^H_r - y)drds,$$

where $f_{\varepsilon}(x) := \frac{1}{\sqrt{2\pi\varepsilon}} e^{-x^2/(2\varepsilon)}$ is the centred Gaussian density with variance $\varepsilon$. Since $f_{\varepsilon} \to \delta$ as $\varepsilon \downarrow 0$ weakly, then one can think of $\alpha_{t, \varepsilon}$ as an approximation of $\alpha_t$ for small $\varepsilon$. Consider

$$\alpha_{t, \varepsilon}^{(k)}(y) := (-1)^k \int_0^t \int_0^s f_{\varepsilon}^{(k)}(B^H_s - B^H_r - y)drds,$$

where $f_{\varepsilon}^{(k)}(x) := \frac{d^k}{dx^k} f_{\varepsilon}(x)$. The $k$-th derivative of $\alpha_t$ is then naturally defined as a limit in some sense of $\alpha_{t, \varepsilon}^{(k)}$ as $\varepsilon \downarrow 0$. Yu proved a number of facts related to $\alpha_t^{(k)}$ in [45], and in fact defined the process in arbitrarily high dimensions with mixed partial derivatives allowed; however we will be content to remain in one dimension, let $y = 0$, and address the following result.

†School of Mathematics, Monash University, Victoria, 3800 Australia.
E-mail addresses: kaustav.das@monash.edu, greg.markowsky@monash.edu.
Theorem 1.1 ([45]). \( \alpha_t^{(k)}(0) \) exists in \( L^2(\Omega) \) whenever

\[
H < \begin{cases} 
\frac{2}{k+1}, & \text{if } k \text{ is odd}, \\
\frac{1}{k+1}, & \text{if } k \text{ is even}.
\end{cases}
\]

We will apply the Wiener chaos expansion, which was not used for this purpose by Yu, in order to give a new proof of this theorem, and in fact will be able to prove the following extension.

**Theorem 1.2.** If \( k > 0 \) is even and \( \frac{1}{k+1} < H < \frac{2}{k+1} \), then \( \alpha_t^{(k)}(0) - \mathbb{E}[\alpha_t^{(k)}(0)] \) converges in \( L^2(\Omega) \) as \( \varepsilon \downarrow 0 \).

This kind of result is commonly referred to as a *Varadhan-type renormalization*, due to its origin in the important result from [39], which addressed the self-intersection local time of Brownian motion in two dimensions. We note also the similarity of Theorem 1.2 with the main results in [15], which addressed the self-intersection local time of fractional Brownian motion in arbitrarily many dimensions, and similarly showed convergence of the process, renormalized by subtracting the mean, for certain ranges of \( H \). We remark that such a renormalization to increase the range of convergence in \( H \) for the odd derivatives is not possible: since \( \delta^{(k)} \) is odd in this case, the expectation will be 0. In the process of proving these results, we will also deduce the Wiener chaos decompositions of \( \alpha_t^{(k)} \) and its renormalization (see the statements of Theorems 3.1 to 3.3 below).

Our results fit naturally with a number of other results in the field, and for that reason in Section 2 we provide a brief survey of the existing knowledge on this topic. In Section 3 we give precise statements of our main results, and in Section 4 we prove them. The final sections, Appendices A and B, contain a discussion on the Malliavin calculus content that we employ within our methodology, as well as the proofs of a number of technical lemmas respectively.

**Remark 1.1 (Notation).** We will make use of the following notation extensively.

| (i) | \( H := L^2([0, T]) \) is the Hilbert space of square integrable functions on \([0, T]\) with respect to Lebesgue measure. |
| (ii) | \( L^2(\Omega, \mathcal{G}, \mathbb{P}) \equiv L^2(\Omega) \) denotes the Hilbert space of square integrable random variables on \((\Omega, \mathcal{G}, \mathbb{P})\), where \( \mathcal{G} \) is the \( \sigma \)-field generated by the isonormal Gaussian process \( W = \{W(\phi) := \int_0^T \phi_t dB_t : \phi \in H\} \). |
| (iii) | \( D_t := \{(r, s) \in \mathbb{R}^2 : 0 \leq r \leq s \leq t\} \) is a simplex in \( \mathbb{R}^2 \). |
| (iv) | \( N = \{1, 2, \ldots\} \) and \( N_0 := N \cup \{0\} \). |
| (v) | \( P = \{(n', k') \in N_0^2 : n' + k' = 0 \pmod{2}\} \) is the set of tuples of non-negative integers whose sum is even. |
| (vi) | We will utilize the following standard notation utilized in the area of DSLT: \( \lambda := \text{Var}(B_s^H - B_r^H) = |s - r|^{2H} \), \( \rho := \text{Var}(B_s^H - B_r^H) = |s - r'|^{2H} \), \( \mu := \text{Cov}(B_s^H - B_r^H, B_{s'}^H - B_{r'}^H) = \frac{1}{2} (|s - r'|^{2H} + |r - s'|^{2H} - |s - s'|^{2H} - |r - r'|^{2H}) \). |
| (vii) | For a function \( f : [a, b]^n \to \mathbb{R} \), \( I_n(f) \) denotes the multiple Wiener integral, \( I_n(f) = \int_{[a, b]^n} f(v_1, \ldots, v_n) dB_{v_1} \ldots dB_{v_n} \). |
2. Literature review

The topic of self-intersection local times of stochastic processes has received a great deal of attention in recent decades. Originally studied for Brownian motion due to connections with theoretical physics (see [8, 25, 26, 39]), it has been generalized to a wide variety of processes, and has become a major focus of research for a number of theoretical probabilists. This paper lies in the intersection of several different research threads, and we will take the time to discuss these before stating our results.

The process which is our focus is fractional Brownian motion. The self-intersection local time of this process was first studied by Rosen in [35], and this has led to a large literature on the subject, including [4, 5, 6, 9, 10, 14, 15, 16, 21, 22, 31, 33, 34, 40, 46]. A key difficulty with this process, in comparison to ordinary Brownian motion, is that in general the increments of the process are not independent, and this brings considerable difficulties into the calculations. The key method for overcoming this difficulty is to use the property of local nondeterminism (see [2] and Lemma B.1 below).

An important tool in the analysis of local times and intersection local times is the Wiener chaos expansion. This was first discovered in [32], and the chaos expansion has since become a standard method in the field (see [1, 3, 7, 14, 17, 18, 23, 29, 30], to name a few).

The derivative of self-intersection local time of Brownian motion was also introduced by Rosen, in [36], and has blossomed into a research topic in its own right ([11, 12, 13, 19, 20, 23, 24, 27, 28, 29, 37, 38, 41, 42, 43, 44, 45, 46]). The original motivation discussed by Rosen was the Tanaka-style formula

\[ \frac{1}{2} \alpha''_v(x) + \frac{1}{2} \text{sgn}(x) t = \int_0^t L_s B_{s-x} dB_s - \int_0^t \text{sgn}(B_t - B_u - x) du. \]

Rosen only stated this formally, but later it was proved rigorously in [27]. The formal argument for this identity comes from applying Ito’s Lemma to the non-differentiable function \( \text{sgn} \) and then integrating: since the derivatives of \( \text{sgn} \) are the Dirac delta function and its derivatives, we end up with DSLT in the equation. Similar formal calculations applied to \( \delta \) itself will require higher order derivatives of self-intersection local time. For instance, the same argument applied directly to \( \delta \) yields

\[ \frac{1}{2} \alpha''_v(x) + \frac{1}{2} \delta(x) t = \int_0^t \int_0^t \delta'(B_s - B_u - x) du dB_s - \int_0^t \delta(B_t - B_u - x) du \]

We note that the final term is simply the local time of the Brownian motion \( \hat{B}_u := B_t - B_{t-u} \) for \( 0 \leq u \leq t \) at time \( t \). However, each of the other terms is problematic: it is not clear that the stochastic integral exists, the term containing \( \delta(x) \) without an integral is worrisome, and Yu’s results show that \( \alpha''_v(x) \) does not exist for Brownian motion. It is therefore unclear whether any meaning can be assigned to this identity; however, it is possible that some method of renormalization can be devised which clarifies it.

3. Main results

First, we define the following object, which will be utilized in the subsequent results:

\[
\begin{align*}
g_{\ell,\epsilon}(n, k) &\equiv g_{\ell,\epsilon}(n; k; v_1, \ldots, v_n) \\
&= \left( \frac{(-1)^{(3k+n)/2} \Gamma(n+k-1)!}{n! \sqrt{2\pi}} \right) \left( \int_0^t \int_0^s \prod_{j=1}^n K_{r,s}(v_j) (s-r)^{2H + \epsilon} [(k+n+1)/2] \, dr \, ds \right) \mathbf{1}_P(n, k),
\end{align*}
\]

where \( K_{r,s} \) is given in eq. (A.1). The results pertaining to \( \alpha_{\ell,\epsilon}^{(k)} \) and \( \alpha_{\ell}^{(k)} \) will depend on the parity of \( k \).
Lemma 3.1. Let $k, k' \in \mathbb{N}$. Then $\alpha_{t, \varepsilon}^{(2k-1)}(0)$ and $\alpha_{t, \varepsilon}^{(2k')}(0)$ admit the Wiener chaos expansions

$$\alpha_{t, \varepsilon}^{(2k-1)}(0) = \sum_{m=1}^{\infty} I_{2m-1}(g_{t, \varepsilon}(2m-1, 2\hat{k} - 1)),$$

and

$$\alpha_{t, \varepsilon}^{(2k')}(0) = \sum_{l=0}^{\infty} I_{2l}(g_{t, \varepsilon}(2l, 2k')).$$

Theorem 3.1. Let $k = 2\hat{k} - 1$, where $\hat{k} \in \mathbb{N}$. Suppose $H \in (0, \frac{2}{4k-1})$. Then $\alpha_{t}^{(2k-1)}(0)$ exists in $L^2(\Omega)$. Moreover, $\alpha_{t}^{(2k-1)}(0)$ exhibits the Wiener chaos expansion

$$\alpha_{t}^{(2k-1)}(0) = \sum_{m=1}^{\infty} I_{2m-1}(g_{t, 0}(2m-1, 2\hat{k} - 1)).$$

Theorem 3.2. Let $k = 2k'$, where $k' \in \mathbb{N}$. Suppose $H \in (0, \frac{1}{4k'+1})$. Then $\alpha_{t}^{(2k')}(0)$ exists in $L^2(\Omega)$. Moreover, $\alpha_{t}^{(2k')}(0)$ exhibits the Wiener chaos expansion

$$\alpha_{t}^{(2k')}(0) = \sum_{l=0}^{\infty} I_{2l}(g_{t, 0}(2l, 2k')).$$

We reiterate that the existence parts of Theorems 3.1 and 3.2 were proved in [45] by different methods, although the formulas for the Wiener chaos expansions are new. The following result is the Varadhan-type of renormalization for the even derivatives.

Theorem 3.3. Let $k = 2k'$, where $k' \in \mathbb{N}$. Suppose $H \in (0, \frac{2}{4k'+1})$. Then $\alpha_{t}^{(2k')}(0) - E[\alpha_{t}^{(2k')}(0)]$ exists in $L^2(\Omega)$. Moreover, $\alpha_{t}^{(2k')}(0) - E[\alpha_{t}^{(2k')}(0)]$ exhibits the Wiener chaos expansion

$$\alpha_{t}^{(2k')}(0) - E[\alpha_{t}^{(2k')}(0)] = \sum_{l=1}^{\infty} I_{2l}(g_{t, 0}(2l, 2k')).$$

4. PROOFS

Proof of Lemma 3.1. The main tools are Theorem A.2 (Stroock’s formula), as well as the Fourier identity

$$f_{\varepsilon}(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ipx} e^{-xp^2/2} dp$$

which is a simple consequence of the characteristic function of a $\mathcal{N}(0, \varepsilon)$ random variable. This leads to the representation

$$f^{(k)}_{\varepsilon}(x) = \frac{i^k}{2\pi} \int_{\mathbb{R}} p^k e^{ipx} e^{-xp^2/2} dp.$$

Since

$$\alpha_{t, \varepsilon}^{(k)}(0) = (-1)^k \int_{0}^{t} \int_{0}^{s} f^{(k)}_{\varepsilon}(B^H_s - B^H_r) dr ds,$$

then obtaining the Wiener chaos expansion of $f^{(k)}_{\varepsilon}(B^H_s - B^H_r)$ will in turn result in the Wiener chaos expansion of $\alpha_{t, \varepsilon}^{(k)}(0)$. By Theorem A.2 (Stroock’s formula),

$$f^{(k)}_{\varepsilon}(B^H_s - B^H_r) = \sum_{n=0}^{\infty} I_{n}(h_{\varepsilon}(n, k))$$

where

$$h_{\varepsilon}(n, k) \equiv h_{\varepsilon}(n, k; s, r; v_1, \ldots, v_n) = \frac{1}{n!} E[D_{v_1, \ldots, v_n} f^{(k)}_{\varepsilon}(B^H_s - B^H_r)].$$
From Appendix A.2, we have that
\[ D^n f_{\varepsilon}^{(k)}(B_{s_i}^H - B_{s_j}^H) = f_{\varepsilon}^{(k+n)}(B_{s_i}^H - B_{s_j}^H)(K_{r,s})^{\otimes n}, \]
where \( K_{r,s} \) is defined in eq. (A.1). Thus
\[ \mathbb{E}[D^n_{v_1, \ldots, v_n} f_{\varepsilon}^{(k)}(B_{s_i}^H - B_{s_j}^H)] = \mathbb{E}[f_{\varepsilon}^{(k+n)}(B_{s_i}^H - B_{s_j}^H)] \prod_{j=1}^{n} K_{r,s}(v_j). \]
Using the Fourier identity, we have that
\[ \mathbb{E}[f_{\varepsilon}^{(k+n)}(B_{s_i}^H - B_{s_j}^H)] = \frac{i^{k+n}}{2\pi} \int_{\mathbb{R}} p^{k+n} e^{ip(B_{s_i}^H - B_{s_j}^H)} e^{-\varepsilon p^2/2} dp \]
\[ = \frac{i^{k+n}}{2\pi} \int_{\mathbb{R}} p^{k+n} e^{-\varepsilon (s-r)^2 + \varepsilon / 2} dp. \]
Changing variable \( p \leftarrow p((s-r)^2 + \varepsilon)^{1/2} \), then
\[ \mathbb{E}[f_{\varepsilon}^{(k+n)}(B_{s_i}^H - B_{s_j}^H)] = \frac{i^{k+n}}{2\pi ((s-r)^2 + \varepsilon)^{(k+n+1)/2}} \int_{\mathbb{R}} p^{k+n} e^{-\varepsilon p^2/2} dp. \]
The proceeding integral is 0 if \( k + n \) is odd. Recall \( P \) is the set of tuples of non-negative integers whose sum is even. Then
\[ \mathbb{E}[f_{\varepsilon}^{(k+n)}(B_{s_i}^H - B_{s_j}^H)] = \frac{i^{k+n} \sqrt{2\pi}}{2\pi ((s-r)^2 + \varepsilon)^{(k+n+1)/2}} \prod_{j=1}^{n} K_{r,s}(v_j) \]
\[ = \frac{(-1)^{(k+n)/2}(n + k - 1)!}{2\pi ((s-r)^2 + \varepsilon)^{(k+n+1)/2}} \prod_{j=1}^{n} K_{r,s}(v_j). \]
Thus
\[ h_{\varepsilon}(n, k) = \left( \frac{1}{n!} \prod_{j=1}^{n} K_{r,s}(v_j) \right) \frac{(-1)^{(k+n)/2}(n + k - 1)!}{\sqrt{2\pi ((s-r)^2 + \varepsilon)^{(k+n+1)/2}}} \prod_{j=1}^{n} K_{r,s}(v_j). \]
If \( k \) is odd then \( k + n \) is even if and only if \( n \) is odd. If \( k \) is even then \( k + n \) is even if and only if \( n \) is even. So we have two cases, depending on the parity of \( k \). Let \( k, k' \in \mathbb{N} \). Hence,
\[ f_{\varepsilon}^{(2k-1)}(B_{s_i}^H - B_{s_j}^H) = \sum_{n=1}^{\infty} I_{2m-1}(h_{\varepsilon}(2m - 1, 2k - 1)), \]
and
\[ f_{\varepsilon}^{(2k')}(B_{s_i}^H - B_{s_j}^H) = \sum_{l=0}^{\infty} J_l(h_{\varepsilon}(2l, 2k')). \]
Lastly, take
\[ g_{l,\varepsilon}(n, k) \equiv g_{l,\varepsilon}(n, k; v_1, \ldots, v_n) \]
\[ = (-1)^k \int_{0}^{t} \int_{0}^{s} h_{\varepsilon}(n, k) drds \]
\[ = \left( (-1)^{(3k+n)/2}(n + k - 1)! \right) \left( \int_{0}^{t} \int_{0}^{s} \prod_{j=1}^{n} K_{r,s}(v_j) \frac{drds}{\sqrt{2\pi ((s-r)^2 + \varepsilon)^{(k+n+1)/2}}} \right) \prod_{j=1}^{n} K_{r,s}(v_j). \]
Then the Wiener chaos expansions of \( \alpha_{l,\varepsilon}^{(2k-1)}(0) \) and \( \alpha_{l,\varepsilon}^{(2k')}(0) \) follow. \( \square \)

Proof of Theorem 3.1. The idea is to utilize Lemma A.2 as well as the Wiener chaos expansion of \( \alpha_{l,\varepsilon}^{(k)}(0) \), which is given in Lemma 3.1. Since all homogeneous chaoses are orthogonal to each other, the \( L^2(\Omega) \) norm of the Wiener chaos expansion is the infinite sum of the \( L^2(\Omega) \) norms of each individual chaos term. So it suffices to study the object
\[ \mathbb{E}\left[ I_n(g_{l,\varepsilon}(n, k)) \right]^2 \]
for when \((k, n) \in P\). The objective then is to show that the Wiener chaos expansion is bounded uniformly with respect to \(\varepsilon\) in \(L^2(\Omega)\), and then appeal to Lemma A.2. Since the function 
\(g_{t, \varepsilon}(n, k) \equiv g_{t, \varepsilon}(n; k; v_1, \ldots, v_n)\) is symmetric in \(\vec{v} := (v_1, \ldots, v_n)\), then a well known identity 
pertaining to multiple Wiener integrals yields
\[
\mathbb{E}[I_n(g_{t, \varepsilon}(n, k))^2] = n!\|g_{t, \varepsilon}(n, k)\|^2_{H^0}.
\]
Fix \((k, n) \in P\). Recall the simplex \(D_t = \{(r, s) \in \mathbb{R}^2 : 0 \leq r \leq s \leq t\}\). Then
\[
\|g_{t, \varepsilon}(n, k)\|^2_{H^0} = \frac{[(n + k - 1)!]^2}{(n!)^2 2\pi} \int_{D_t^2} \frac{1}{[(s-r)^{2H} + \varepsilon][\gamma(k+1/2) + \varepsilon][(s'-r')^{2H} + \varepsilon][\gamma(k+1/2) + \varepsilon]} \cdot \int_{[0,T]^n} \left( \prod_{j=1}^{n} K_{r, s}(v_j)K_{r', s'}(v_j) \right) d\vec{v} ds' dr' ds'.
\]
Since \((k, n) \in P\), then \(n + k - 1\) is odd, and hence \((n + k - 1)! = \frac{(n+k-1)!}{2^{(\frac{n+k-1}{2})}}\). Recall
\[
\lambda = \text{Var}(B_s^H - B_r^H) = |s - r|^{2H}
\]
\[
\rho = \text{Var}(B_s^H - B_r^H) = |s' - r'|^{2H}
\]
\[
\mu = \text{Cov}(B_s^H - B_r^H, B_s^H - B_r^H) = \frac{1}{2} |(s - r)^{2H} + |r - s'|^{2H} - |s - s'|^{2H} - |r - r'|^{2H}|.
\]
Furthermore, from Appendix A.2, \(\int_0^T K_{r, s}(u)K_{r', s'}(u) du = \langle K_{r, s}, K_{r', s'} \rangle_{H} = \mu\), and thus
\[
\int_{[0,T]^n} \left( \prod_{j=1}^{n} K_{r, s}(v_j)K_{r', s'}(v_j) \right) d\vec{v} = \mu^n.
\]
Now take \(\varepsilon = 0\) to maximize the integrand over \(D_t^2\). This yields,
\[
n!\|g_{t, 0}(n, k)\|^2_{H^0} = \frac{[(n + k - 1)!]^2}{n!(2\pi)^{2n+2}} \int_{D_t^2} \frac{\mu^n}{\lambda^{k+\frac{1}{2}\gamma^2} \rho^{\frac{1}{2}\gamma^2}} dr ds ds' dr'.
\]
Let \(\gamma := \mu/\sqrt{\lambda\rho}\). Then we get
\[
\sum_{n=0}^{\infty} \mathbb{E}[I_n(g_{t, 0}(n, k))]^2 = \frac{1}{2^{k-1}} \int_{D_t^2} \left\{ \sum_{n=0}^{\infty} \frac{[(n + k - 1)!]^2 \gamma^n}{n!2^n \left(2^{n+2}\right)} \right\} \frac{1}{\lambda^{k+\frac{1}{2}\gamma^2} \rho^{\frac{1}{2}\gamma^2}} dr ds ds' dr'.
\]
Now take \(k = 2\hat{k} - 1\), for \(\hat{k} \in \mathbb{N}\). Then the \(L^2(\Omega)\) norm of \(\alpha_{t, 0}^{(2\hat{k}-1)}(0)\) is
\[
\sum_{m=1}^{\infty} \mathbb{E}[J_{m-1}(g_{t, 0}(2m - 1, 2\hat{k} - 1))]^2
\]
\[
= \frac{1}{2^{2k-2}} \int_{D_t^2} \left\{ \sum_{m=1}^{\infty} \frac{[(2m + 2\hat{k} - 3)!2^{m+2m-1} \gamma^{2m-1}}{(2m - 1)!2^{m-1} \left(2^{m+2}\right)} \right\} \frac{1}{\lambda^{k} \rho^{k}} dr ds ds' dr'.
\]
After a tedious rearrangement, we can write the proceeding infinite sum in terms of the Hypergeometric function, specifically
\[
\sum_{m=1}^{\infty} \frac{[(2m + 2\hat{k} - 3)!2^{m-1}}{(2m - 1)!2^{m-1} \left(2^{m+2}\right)} \right\} = \left( \frac{2\hat{k} - 1}{2} \right) \left( \frac{2\hat{k} - 1}{2} \right)^2 \gamma_{F_{1,2}} \left( 2\hat{k} - 1, \frac{1}{2} \hat{k} + \frac{1}{2}; 3/2; \gamma^2 \right),
\]
where \(F_{1,2}\) is the Hypergeometric function
\[
F_{1,2}(a, b; c; z) := \sum_{d=0}^{\infty} \frac{a^d b^d z^d}{e^d \cdot d!}
\]
and $a^d := \prod_{j=1}^{d} (a + j - 1)$ is the rising factorial. Then, using the Euler identity, $F_{1,2}(a, b; c; z) = (1 - z)^{-a-b}F_{1,2}(c - a, c - b; c; z)$, as well as $(-a)^d = 0$ for $d > a > 0$ yields
\[
F_{1,2} \left( \hat{k} + \frac{1}{2}, \hat{k} + \frac{1}{2}; 3/2; \gamma^2 \right) = \frac{1}{(1 - \gamma^2)(4k - 1)/2} \sum_{d=0}^{k-1} C_d(\hat{k}) \gamma^{2d}
\]
where
\[
C_d(\hat{k}) := \frac{[(1 - \hat{k})^d_{1/2}]}{(3/2)^d d!}.
\]

Thus, in order to show existence of $\alpha^{(2k-1)}_t(0)$ in $L^2(\Omega)$, it suffices to determine for which values of $\hat{k} \in \mathbb{N}$ and $H \in (0, 1)$ the integral
\[
\int_{D_t^2} \frac{1}{(1 - \gamma^2)(4k - 1)/2} \lambda^{k^2} \sum_{d=0}^{k-1} \gamma^{2d+1} d\gamma d'^{k+1} d's'^{\gamma^2}
\]
is finite. From Lemma B.2, this integral is finite for $H \in (0, \frac{2}{4k-1})$ for all $\hat{k} \in \mathbb{N}$. The Wiener chaos expansion of $\alpha^{(2k-1)}_t(0)$ then follows from Lemma 3.1 by letting $\varepsilon = 0$. \hfill \Box

**Proof of Theorem 3.3.** The first part of the proof is identical to that of the proof of Theorem 3.1. However, for the second part, instead take $k = 2k'$, for $k' \in \mathbb{N}$. Notice also that $E[\alpha^{(2k')}_t(0)] = I_0(g_{t,0}(0, 2k'))$. Leading on from eq. (4.1), the $L^2(\Omega)$ norm of $\alpha^{(2k')}_t(0) - E[\alpha^{(2k')}_t(0)]$ is
\[
\sum_{l=1}^{\infty} \mathbb{E}[I_2(g_{t,0}(2l, 2k'))]^2 = \frac{1}{2^{2k'-1} \pi} \int_{D_t^2} \sum_{l=1}^{\infty} \left[ \frac{(2l + 2k' - 1)!}{(l + k' - 1)!} \right]^2 \gamma^{2l} d\gamma d'^{k'+1} d's'^{\gamma^2}
\]
It can be seen that
\[
\sum_{l=1}^{\infty} \left[ \frac{(2l + 2k' - 1)!}{(l + k' - 1)!} \right]^2 \gamma^{2l} = \frac{(2k' - 1)!}{(k' - 1)!} 2^{-2l} \Gamma(1/2) \Gamma(3/2 - l) \Gamma(1/2 - l) \Gamma(3/2)
\]
\[
= \frac{(2k' - 1)!}{(k' - 1)!} \frac{1}{(1 - \gamma^2)(4k'+1)/2} \sum_{d=1}^{k'} D_d(k') \gamma^{2d},
\]
where
\[
D_d(k') := \frac{[(k')^d_{1/2}]}{(1/2)^d d!}.
\]
Thus, in order to show existence of $\alpha^{(2k')}_t(0) - E[\alpha^{(2k')}_t(0)]$ in $L^2(\Omega)$, it suffices to determine for which values of $k' \in \mathbb{N}$ and $H \in (0, 1)$ the integral
\[
\int_{D_t^2} \frac{1}{(1 - \gamma^2)(4k'+1)/2} \lambda^{k'+1/2} \rho^{k'+1/2} \sum_{d=1}^{k'} \gamma^{2d} d\gamma d'^{k'+1} d's'^{\gamma^2}
\]
is finite. From Lemma B.3, this integral is finite for $H \in (0, \frac{2}{4k'+1})$ for all $k' \in \mathbb{N}$. The Wiener chaos expansion of $\alpha^{(2k')}_t(0) - E[\alpha^{(2k')}_t(0)]$ then follows from Lemma 3.1 by subtracting the 0-th order term and letting $\varepsilon = 0$. \hfill \Box

**Remark 4.1.** We have omitted the proof of Theorem 3.2, since it is essentially identical to the proof of Theorem 3.3.
References

[1] Albeverio, S., Hu, Y., and Zhou, X. (1997). A remark on non-smoothness of the self-intersection local time of planar Brownian motion. *Statistics & Probability Letters*, 32(1):57–65.

[2] Berman, S. (1973). Local nondeterminism and local times of Gaussian processes. *Indiana University Mathematics Journal*, 23(1):69–94.

[3] Bornales, J., Oliveira, M., and Streit, L. (2016). Chaos decomposition and gap renormalization of Brownian self-intersection local times. *Reports on Mathematical Physics*, 77(2):141–152.

[4] Chen, C. and Yan, L. (2011). Remarks on the intersection local time of fractional Brownian motions. *Statistics & Probability Letters*, 81(8):1003–1012.

[5] Chen, X., Li, W., Rosiński, J., and Shao, Q. (2011). Large deviations for local times and intersection local times of fractional Brownian motions and Riemann–Liouville processes. *The Annals of Probability*, 39(2):729–778.

[6] Chen, Z., Sang, L., and Hao, X. (2018). Renormalized self-intersection local time of bifractional Brownian motion. *Journal of Inequalities and Applications*, 2018(1):326.

[7] de Faria, M., Drumond, C., and Streit, L. (2000). The renormalization of self-intersection local times I. The chaos expansion. *Infinite Dimensional Analysis, Quantum Probability, and Related Topics*, 3(2).

[8] Edwards, S. (1965). The statistical mechanics of polymers with excluded volume. *Proceedings of the Physical Society*, 85(4):613.

[9] Evans, S., Pitman, J., and Tang, W. (2017). The spans in Brownian motion. *Annales de l’Institut Henri Poincaré, Probabilités et Statistiques*, 53(3):1108–1135.

[10] Grothaus, M., Oliveira, M., da Silva, J., and Streit, L. (2011). Self-avoiding fractional Brownian motion-the Edwards model. *Journal of Statistical Physics*, 145(6):1513–1523.

[11] Guo, J., Hu, Y., and Xiao, Y. (2019). Higher-order derivative of intersection local time for two independent fractional Brownian motions. *Journal of Theoretical Probability*, 32(3):1190–1201.

[12] Guo, J. and Xiao, Y. (2018). Higher-order derivative local time for fractional Ornstein-Uhlenbeck processes. arXiv preprint arXiv:1810.12772.

[13] Hong, M. and Xu, F. (2020). Derivatives of local times for some Gaussian fields. *Journal of Mathematical Analysis and Applications*, 484(2):123716.

[14] Hu, Y. (2001). Self-intersection local time of fractional Brownian motions-via chaos expansion. *Journal of Mathematics of Kyoto University*, 41(2):233–250.

[15] Hu, Y. and Nualart, D. (2005). Renormalized self-intersection local time for fractional Brownian motion. *The Annals of Probability*, 33(3):948–983.

[16] Hu, Y., Nualart, D., and Song, J. (2008). Integral representation of renormalized self-intersection local times. *Journal of Functional Analysis*, 255(9):2507–2532.

[17] Hu, Y. and Øksendal, B. (2002). Chaos expansion of local time of fractional Brownian motions. *Stochastic Analysis and Applications*.

[18] Imkeller, P., Perez-Abreu, V., and Vives, J. (1995). Chaos expansions of double intersection local time of Brownian motion in $rd$ and renormalization. *Stochastic Processes and Their Applications*, 36(1):1–34.

[19] Jaramillo, A., Nourdin, I., and Peccati, G. (2019). Approximation of fractional local times: Zero energy and derivatives. arXiv preprint arXiv:1903.08683.

[20] Jaramillo, A. and Nualart, D. (2017). Asymptotic properties of the derivative of self-intersection local time of fractional Brownian motion. *Stochastic Processes and their Applications*, 127(2):669–700.

[21] Jaramillo, A. and Nualart, D. (2019). Functional limit theorem for the self-intersection local time of the fractional Brownian motion. *Annales de l’Institut Henri Poincaré, Probabilités et Statistiques*, 55(1):480–527.

[22] Jiang, Y. and Wang, Y. (2007). On the collision local time of fractional Brownian motions. *Chinese Annals of Mathematics, Series B*, 28(3):311–320.
[23] Jung, P. and Markowsky, G. (2014). On the Tanaka formula for the derivative of self-intersection local time of fractional Brownian motion. *Stochastic Processes and their Applications*, 124(11):3846–3868.

[24] Jung, P. and Markowsky, G. (2015). Hölder continuity and occupation-time formulas for fBm self-intersection local time and its derivative. *Journal of Theoretical Probability*, 28(1):299–312.

[25] Le Gall, J.-F. (1986). Propriétés d’intersection des marches aléatoires. *Communications in Mathematical Physics*, 104(3):471–507.

[26] Le Gall, J.-F. (1988). Fluctuation results for the Wiener sausage. *The Annals of Probability*, pages 991–1018.

[27] Markowsky, G. (2008a). Proof of a Tanaka-like formula stated by J. Rosen in Séminaire XXXVIII. In *Séminaire de Probabilités XLI*, pages 199–202. Springer.

[28] Markowsky, G. (2008b). Renormalization and convergence in law for the derivative of intersection local time in $r^2$. *Stochastic Processes and their Applications*, 118(9):1552–1585.

[29] Markowsky, G. (2012). The derivative of the intersection local time of Brownian motion through Wiener chaos. In *Séminaire de Probabilités XLIV*, pages 41–148. Springer.

[30] Mataramvura, S., Øksendal, B., and Proske, F. (2004). The Donsker delta function of a Lévy process with application to chaos expansion of local time. In *Annales de l’IHP Probabilités et statistiques*, volume 40, pages 553–567.

[31] Nualart, D. and Ortiz-Latorre, S. (2007). Intersection local time for two independent fractional Brownian motions. *Journal of Theoretical Probability*, 20(4):759–767.

[32] Nualart, D. and Vives, J. (1992). Chaos expansions and local times. *Publicacions Matemàtiques*, pages 827–836.

[33] Oliveira, M., Da Silva, J., and Streit, L. (2011). Intersection local times of independent fractional Brownian motions as generalized white noise functionals. *Acta Applicandae Mathematicae*, 113(1):17–39.

[34] Rezgui, A. (2007). The renormalization of self intersection local times of fractional Brownian motion. In *Int. Math. Forum*, volume 2, pages 2161–2178.

[35] Rosen, J. (1987). The intersection local time of fractional Brownian motion in the plane. *Journal of multivariate analysis*, 23(1):37–46.

[36] Rosen, J. (2005). Derivatives of self-intersection local times. In *Séminaire de Probabilités XXXVIII*, pages 263–281. Springer.

[37] Shi, Q. (2020). Fractional smoothness of derivative of self-intersection local times with respect to bi-fractional Brownian motion. *Systems & Control Letters*, 138:104627.

[38] Shi, Q. and Yu, X. (2017). Fractional smoothness of derivative of self-intersection local times. *Statistics & Probability Letters*, 121:18–28.

[39] Varadhan, S. (1969). Appendix to *euclidean quantum field theory* by K. Symanzik. *Local Quantum Theory*. Academic Press, Reading, MA, 1:219–226.

[40] Wu, D. and Xiao, Y. (2010). Regularity of intersection local times of fractional Brownian motions. *Journal of Theoretical Probability*, 23(4):972–1001.

[41] Yan, L. (2014). Derivative for the intersection local time of fractional Brownian motions. *arXiv preprint arXiv:1403.4102*.

[42] Yan, L. and Yu, X. (2015). Derivative for self-intersection local time of multidimensional fractional Brownian motion. *Stochastics An International Journal of Probability and Stochastic Processes*, 87(6):966–999.

[43] Yan, L., Yu, X., and Chen, R. (2017). Derivative of intersection local time of independent symmetric stable motions. *Statistics & Probability Letters*, 121:18–28.

[44] Yu, Q. (2020a). Asymptotic properties for $q$-th chaotic component of derivative of self-intersection local time of fractional Brownian motion. *Journal of Mathematical Analysis and Applications*, 492(2):124477.

[45] Yu, Q. (2020b). Higher order derivative of self-intersection local time for fractional Brownian motion. *arXiv preprint arXiv:2008.05633*.

[46] Yu, X. (2019). Smoothness of self-intersection local time of multidimensional fractional Brownian motion. *Communications in Statistics-Theory and Methods*, 48(17):4278–4293.
A.1. Multiple Wiener integrals and Wiener chaos.

Lemma A.1. Let \((B_t)\) be an ordinary Brownian motion. The multiple Wiener integral of a function \(f : [a, b]^n \to \mathbb{R}\), denoted by \(I_n(f)\), can be written as an \(n\)-fold iterated Itô integral with respect to \((B_t)\). Specifically,

\[
I_n(f) := \int_{[a,b]^n} f(v_1, \ldots, v_n) dB_{v_1} \cdots dB_{v_n}
\]

\[= n! \int_a^b \left( \int_a^{v_1} \cdots \left( \int_a^{v_{n-1}} \hat{f}(v_1, v_2, \ldots, v_n) dB_{v_n} \right) \cdots dB_{v_2} \right) dB_{v_1},
\]

where \(\hat{f}(v_1, \ldots, v_n) := \frac{1}{n!} \sum_{\sigma \in S_n} f(v_{\sigma(1)}, \ldots, v_{\sigma(n)})\) is the symmetric version of \(f\).

Theorem A.1 (Wiener chaos expansion). Let \(\mathcal{F}\) be the \(\sigma\)-field generated by an isonormal Gaussian process \(W\) with the underlying real separable Hilbert space \(\mathcal{H} = L^2([a, b])\). Let \(\xi \in L^2(\Omega, \mathcal{F}, \mathbb{P})\). Then there exists functions \(f_n : [a, b]^n \to \mathbb{R}\), such that \(\xi\) admits the representation

\[\xi = \sum_{n=0}^{\infty} I_n(f_n),\]

which is called the Wiener chaos expansion of \(\xi\).

Definition A.1 (Malliavin derivative). Let \(W\) be an isonormal Gaussian process with respect to an underlying real separable Hilbert space \(\mathcal{H}\). Let \(f \in C^\infty_b(\mathbb{R}^n)\), that is, \(f\) is a bounded smooth function on \(\mathbb{R}^n\). Define the random variable \(F = f(W(h_1), \ldots, W(h_n))\), where \(h_i \in \mathcal{H}\) for each \(i = 1, \ldots, n\). Then the Malliavin derivative of \(F\), denoted by \(DF\), is given by

\[DF = \sum_{i=1}^{n} \partial_i f(W(h_1), \ldots, W(h_n)) h_i.\]

The \(k\)-fold iterated Malliavin derivative of \(F\), denoted by \(D^k F\), is given by

\[D^k F = \sum_{i_1, i_2, \ldots, i_k}^{n} \partial_{i_1, i_2, \ldots, i_k} f(W(h_1), \ldots, W(h_n)) \bigotimes_{j=1}^{k} h_{i_j}.
\]

In particular, for \(n = 1\), then \(DF = f'(W(h)) h\) and \(D^k F = f^{(k)}(W(h)) h^{\otimes k}\).

Remark A.1. Consider the Malliavin derivative \(DF\) when \(n = 1\). When the underlying Hilbert space is \(\mathcal{H} = L^2([a, b])\), we will write

\[D_v F = f'(W(h)) h(v)\]

and

\[D_{v_1, v_2, \ldots, v_k} F = f^{(k)}(W(h)) \prod_{j=1}^{k} h(v_j).
\]

Theorem A.2 (Stroock’s formula). Let \(\xi\) possess a Wiener chaos expansion as in Theorem A.1. Then Stroock’s formula states that the functions \(f_n : [a, b]^n \to \mathbb{R}\) have the explicit representation

\[f_n = \frac{1}{n!} \mathbb{E}[D^n \xi].\]

\(^{1}\)Here, \(\partial_i f(x_1, \ldots, x_i, \ldots, x_n) \equiv \frac{\partial}{\partial x_i} f(x_1, \ldots, x_i, \ldots, x_n)\).
Lemma A.2. Let $F_\varepsilon$ be a collection of $L^2(\Omega)$ random variables with Wiener chaos expansions $F_\varepsilon = \sum_{n=0}^{\infty} I_n(f_n^\varepsilon)$. Let $H := L^2([0, T])$. Suppose that for each $n$, $f_n^\varepsilon$ converges in $H^{\otimes n}$ to $f_n$ as $\varepsilon \downarrow 0$, and that
\[
\sum_{n=0}^{\infty} \sup_{\varepsilon} \mathbb{E}[I_n(f_n^\varepsilon)]^2 = \sum_{n=0}^{\infty} \sup_{\varepsilon} \{n! \|f_n^\varepsilon\|_{H^{\otimes n}}\} < \infty
\]
where $f_n^\varepsilon(v_1, \ldots, v_n) := \frac{1}{n!} \sum_{\sigma \in S_n} f_n^\varepsilon(v_{\sigma 1}, \ldots, v_{\sigma n})$ is the symmetric version of $f_n^\varepsilon$. Then $F_\varepsilon$ converges in $L^2(\Omega)$ to $F = \sum_{n=0}^{\infty} I_n(f_n)$ as $\varepsilon \downarrow 0$.

A.2. Malliavin calculus relating to fBm. Let $(B_t)$ be an ordinary Brownian motion. As utilized previously in this paper, let $H := L^2([0, T])$. Then the collection $W = \{W(\phi) := \int_0^T \phi dB_t : \phi \in H\}$ is an isonormal Gaussian process with respect to the underlying Hilbert space $H$. Let $\mathcal{E}$ denote space of the indicator functions on $[0, T]$. For $u \leq t$, let
\[
K_{0,t}(u) := c_H u^{1/2} - H \int_u^t (r - u)^{H - 2} r^{1/2} dr,
\]
where
\[
c_H := \left[ \frac{H(2H - 1)}{\beta(2 - 2H, H - 1/2)} \right]^{1/2},
\]
and $\beta(\cdot, \cdot)$ is the Beta function. Define the operator $M^H : \mathcal{E} \to H$ such that
\[
M^H 1_{[s,t]}(u) := K_{s,t}(u) := K_{0,t}(u) 1_{[0,t]}(u) - K_{0,s}(u) 1_{[0,s]}(u). \tag{A.1}
\]
It can be shown that
\[
\langle K_{0,t}(\cdot) 1_{[0,t]}(\cdot), K_{0,s}(\cdot) 1_{[0,s]}(\cdot) \rangle_H = R^H(t, s).
\]
Define $\mathcal{H}$ to be the Hilbert space given as the completion of $\mathcal{E}$ with respect to the inner product
\[
\langle 1_{[0,s]}, 1_{[0,t]} \rangle_{\mathcal{H}} := \langle K_{0,t}(\cdot) 1_{[0,t]}(\cdot), K_{0,s}(\cdot) 1_{[0,s]}(\cdot) \rangle_H = R^H(t, s).
\]
The operator $M^H$ is clearly a linear isometry between $\mathcal{E}$ and $H$, and can be extended to map $\mathcal{H}$ to $H$.

Remark A.2. It is clear that
\[
\int_0^T K_{r,s}(u)K_{r',s'}(u) du = \langle K_{r,s}, K_{r',s'} \rangle_H
\]
\[
= \langle K_{0,s}(\cdot) 1_{[0,s]}(\cdot) - K_{0,r}(\cdot) 1_{[0,r]}(\cdot), K_{0,s'}(\cdot) 1_{[0,s']}(\cdot) - K_{0,r'}(\cdot) 1_{[0,r']}(\cdot) \rangle_H
\]
\[
= R^H(s, s') + R^H(r, r') - R^H(s, r') - R^H(r, s')
\]
\[
= \text{Cov}(B^H_s - B^H_r, B^H_s - B^H_{r'})
\]
\[
= \mu.
\]
It is well known that the process $(B^H_t)$ defined as
\[
B^H_t := \int_0^t K_{0,t}(u) dB_u \tag{A.2}
\]
is an fBm with Hurst parameter $H$. Let $f \in C_b^\infty(\mathbb{R})$. Consider the collection of random variables $W^H = \{W^H(\varphi) := \int_0^T M^H \varphi(u) dB_u : \varphi \in \mathcal{H}\}$. Then $W^H$ is an isonormal Gaussian process with respect to the underlying Hilbert space $\mathcal{H}$. Define the random variable $F = f(W^H(\varphi))$.

However, since for any $\varphi \in \mathcal{H}$, we have $M^H \varphi \in H$, then this implies $W^H(\varphi) = W(M^H \varphi) \in W$.

So the Malliavin derivative of $F$ with respect to the isonormal Gaussian process $W$ is
\[
DF = f'(W^H(\varphi)) M^H \varphi,
\]
and the $k$-fold Malliavin derivative is
\[
D^k F = f^{(k)}(W^H(\varphi))(M^H \varphi)^{\otimes k}.
\]
In particular, when $\varphi = 1_{[s,t]}$, then $DF = f'(B^H_t - B^H_s) K_{s,t}$, and $D^k F = f^{(k)}(B^H_t - B^H_s)(K_{s,t})^{\otimes k}$. 

Remark A.3. Since $W^H$ is an isonormal Gaussian process with respect to the underlying Hilbert space $\mathcal{H}$, then it is of course possible to apply the Malliavin derivative to $W^H$-measurable random variables with respect to $W^H$ itself. So the Malliavin derivative of $F = f(W^H(\varphi))$ with respect to $W^H$, denoted by $D^H F$, is $D^H F = f'(W^H(\varphi))\varphi$. However, for our purposes, it will be simpler to compute the Malliavin derivative with respect to $W$ instead, since the Wiener chaos machinery utilizes multiple Wiener integrals with respect to ordinary Brownian motion.

Appendix B. Finiteness of key integrals

In this appendix, we prove the finiteness of key integrals encountered in Theorem 3.1 and Theorem 3.3 with restrictions imposed on the Hurst parameter $H$ and order of derivative $k$. Before we proceed, we first present some necessary preliminary information. First, recall the region $D_t = \{(r,s) : 0 \leq r < s \leq t\}$. It is then true that the region $D^2_t$ can be partitioned into three regions, given by

$$D^2_{1,t} := \{(r,s,r',s') \in D^2_t : r < r' < s < s'\},$$
$$D^2_{2,t} := \{(r,s,r',s') \in D^2_t : r < r' < s' < s\},$$
$$D^2_{3,t} := \{(r,s,r',s') \in D^2_t : r < s < r' < s'\}. \tag{B.1}$$

Recall the notation

$$\lambda = \text{Var}(B^H_s - B^H_r) = |s - r|^{2H},$$
$$\rho = \text{Var}(B^H_s - B^H_{r'}) = |s' - r'|^{2H},$$
$$\mu = \text{Cov}(B^H_s - B^H_r, B^H_{s'} - B^H_{r'}) = \frac{1}{2}(|s - r'|^{2H} + |r - s'|^{2H} - |s - s'|^{2H} - |r - r'|^{2H}).$$

The following lemma from Hu [14] (see also Jung and Markowsky [23]) will be essential.

Lemma B.1.

(i) Let $(r,s,r',s') \in D^2_{1,t}$ (that is, $r < r' < s < s'$). Define $a = r' - r$, $b = s - r'$ and $c = s' - s$. Then

$$\lambda \rho - \mu^2 \geq K((a+b)^{2H}c^{2H} + a^{2H}(b+c)^{2H}).$$

(ii) Let $(r,s,r',s') \in D^2_{2,t}$ (that is, $r < r' < s' < s$). Define $a = r' - r$, $b = s' - r'$ and $c = s - s'$. Then

$$\lambda \rho - \mu^2 \geq K\beta^{2H}(a^{2H} + c^{2H}).$$

(iii) Let $(r,s,r',s') \in D^2_{3,t}$ (that is, $r < s < r' < s'$). Define $a = s - r$, $b = r' - s$ and $c = s' - r'$. Then

$$\lambda \rho - \mu^2 \geq K(a^{2H}c^{2H}).$$

Remark B.1 (Standard Inequalities). We will make use of the following standard inequalities from analysis. Let $K, K_1$ and $K_2$ be a positive constants that may change from line to line.

(i) $xy \leq \frac{x^p}{p} + \frac{y^q}{q} \leq K(x^p + y^q)$, where $p, q > 0$ and $q = \frac{p - 1}{p}$ (Young’s inequality).

(ii) $x^\alpha y^\beta \leq \alpha x + \beta y \leq K(x + y)$, where $\alpha, \beta > 0$ and $\alpha + \beta = 1$ (Young’s inequality variation).

(iii) $K_1(x + y)^\xi \leq (x^\xi + y^\xi) \leq K_2(x + y)^\xi$ for $a, b \geq 0$ and $\xi \in \mathbb{R}$.

(iv) $(x + y)^\xi \leq (x^\xi + y^\xi)$, where $\xi \in (0, 1)$.
Lemma B.2. Consider eq. (4.2), the key integral from the proof of Theorem 3.1. Namely,

$$\int_{D^2} \frac{1}{(1 - \gamma^2)^{(4k-1)/2} \lambda^k \rho^k} \sum_{d=0}^{k-1} \gamma^{2d+1} \text{d}r \text{d}s \text{d}r' \text{d}s'$$

(B.2)

where we recall $\gamma = \mu/(\sqrt{\lambda \rho})$ and $\hat{k} \in \mathbb{N}$. Then this integral is finite if $H \in (0, \frac{2}{4k-1})$, where $\hat{k} \in \mathbb{N}$.

Proof. First, notice that we can take out the sum from the integral, so that the finiteness condition of eq. (B.2) can be replaced with finiteness of

$$\int_{D^2} \frac{\gamma^{2d+1}}{(1 - \gamma^2)^{(4k-1)/2} \lambda^k \rho^k} \text{d}r \text{d}s \text{d}r' \text{d}s'$$

for each $d = 0, 1, \ldots, \hat{k} - 1$. Now since $|\gamma| \leq 1$, we have that $|\gamma|^{d+1} \leq |\gamma|^d$, so it suffices to determine the convergence of the above integral for $d = 0$. Next, notice that

$$\frac{\gamma}{(1 - \gamma^2)^{(4k-1)/2} \lambda^k \rho^k} = \frac{\mu \lambda^{\hat{k}-1} \rho^{\hat{k}-1}}{(\lambda \rho - \mu^2)^{(4k-1)/2}}$$

Moreover, utilizing that the region $D^2$ can be partitioned into $D^2_{1,t}, D^2_{2,t}$ and $D^3_{3,t}$ given in eq. (B.1), then finiteness of the integral eq. (B.2) is guaranteed if and only if

$$\int_{D^2_{i,t}} \frac{\mu \lambda^{\hat{k}-1} \rho^{\hat{k}-1}}{(\lambda \rho - \mu^2)^{(4k-1)/2}} \text{d}r \text{d}s \text{d}r' \text{d}s'$$

is finite for each $i = 1, 2, 3$. Define $\bar{H}(\hat{k}) := \frac{2}{4k-1}$, which is the proposed upper bound on $H$. In what follows, we will make use of these facts:

- We will denote $K$ to be a strictly positive constant whose value may change line by line.
- $\bar{H}(\hat{k})$ is strictly decreasing and its maximum is $\bar{H}(1) = 2/3$.
- We will often need to divide the cases into $H \geq 1/2$ and $H < 1/2$. Notice that $\bar{H}(\hat{k}) \geq 1/2$ if and only if $\hat{k} = 1$.

We now proceed case by case.

Case 1: We start by considering the integration over $D^2_{1,t}$. Following the notation in Lemma B.1 item (i), we have that $a + b = s - r$ and $b + c = s' - r'$. This gives that $\lambda = (a + b)^{2H}$ and $\rho = (b + c)^{2H}$. Now utilizing the lemma, we have

$$\int_{D^2_{1,t}} \frac{\mu \lambda^{\hat{k}-1} \rho^{\hat{k}-1}}{(\lambda \rho - \mu^2)^{(4k-1)/2}} \text{d}r \text{d}s \text{d}r' \text{d}s' \leq K \int_{[0,t]^3} \frac{|\mu|(a + b)^{2H} \bar{H}^{(\hat{k})}(a + c)^{2H} \bar{H}^{(\hat{k})}}{(a + b)^{2H} + a^{2H} c^{2H} \bar{H}^{(4k-1)/2}} \text{d}a \text{d}b \text{d}c$$

$$\leq K \int_{[0,t]^3} \frac{|\mu|}{(a + b)^{2H} c^{2H}} \text{d}a \text{d}b \text{d}c \quad \text{(B.3)}$$

$$= K \int_{[0,t]^3} \frac{|\mu|}{(a + b)^{2H} c^{2H}} \text{d}a \text{d}b \text{d}c$$

(B.4)

where we used Young’s inequality, $x^{1/2} y^{1/2} \leq \frac{x}{2} + \frac{y}{2}$ in eq. (B.3). Now

$$2\mu = ((a + b) + b)^{2H} + b^{2H} - a^{2H} - c^{2H}$$

which gives the trivial bound

$$|\mu| \leq K (a^{2H} + b^{2H} + c^{2H})$$


Thus we need to show that
\[
K \int_{[0,t]^3} \frac{a^{2H} + b^{2H} + c^{2H}}{(a + b)^{\frac{3H}{2}}(b + c)^{\frac{3H}{2}} a^{\frac{H}{2}(4k-1)} c^{\frac{H}{2}(4k-1)}} \, dadbdc
\]
is finite. So we look at each of the three terms involving \(a^{2H}, b^{2H}\) and \(c^{2H}\) in the numerator separately. First start with the \(b^{2H}\) term. Then,
\[
K \int_{[0,t]^3} \frac{b^{2H}}{(a + b)^{\frac{3H}{2}}(b + c)^{\frac{3H}{2}} a^{\frac{H}{2}(4k-1)} c^{\frac{H}{2}(4k-1)}} \, dadbdc
\]
\[
\leq K \int_{[0,t]^3} \frac{1}{b^{H} a^{\frac{H}{2}(4k-1)} c^{\frac{H}{2}(4k-1)}} \, dadbdc
\]
which is clearly finite for \(H < \bar{H}(\hat{k})\). For the \(a^{2H}\) term we have
\[
K \int_{[0,t]^3} \frac{a^{2H}}{(a + b)^{\frac{3H}{2}}(b + c)^{\frac{3H}{2}} a^{\frac{H}{2}(4k-5)} c^{\frac{H}{2}(4k-1)}} \, dadbdc
\]
\[
= K \int_{[0,t]^3} \frac{1}{a^{(2H)(4k-5)} c^{\frac{H}{2}(4k-1)}} \, dadbdc
\]
Clearly the \(c\) term exponent is less than 1 for \(H < \bar{H}(\hat{k})\). Looking at the \(b\) term exponent, we have that \(H < \bar{H}(\hat{k}) < \bar{H}(1) = 2/3\), and so the \(b\) term exponent is less than 1. Observing the \(a\) term exponent, we have
\[
\frac{H}{2} (4k - 2) < \frac{\bar{H}(\hat{k})}{2} (4k - 2) = \frac{4k - 2}{4k - 1} < 1
\]
for any \(\hat{k} \geq 1\). So this integral is finite. For the \(c^{2H}\) term, notice that there is symmetry between \(a\) and \(c\) in the integrand of eq. (B.3). Thus, this case is identical to the case of \(a^{2H}\). Hence, we have completed Case 1.

Case 2: Next we look at the integration over \(D_{2^H}^3\). Following the notation in Lemma B.1 item (ii), we have that \(a + b + c = s - r\) and \(b = s' - r'\). This gives that \(\lambda = (a + b + c)^{2H}\) and \(\rho = b^{2H}\). Now utilizing the lemma, we have
\[
\int_{D_{2^H}^3} \frac{|\mu| \lambda^{k-1} \rho^{k-1}}{(\lambda \rho - \mu^2)^{4(4k-1)/2}} \, drdsdr'ds' \leq K \int_{[0,t]^3} \frac{|\mu|(a + b + c)^{2H(k-1)} b^{2H(k-1)}}{(b^{2H}(a^{2H} + c^{2H}))^{4(4k-1)/2}} \, dadbdc
\]
\[
= K \int_{[0,t]^3} \frac{|\mu|(a + b + c)^{2H(k-1)}}{b^{H(2k+1)}(a^{2H} + c^{2H})^{4(4k-1)/2}} \, dadbdc
\]
\[
\leq K \int_{[0,t]^3} \frac{|\mu|(a + b + c)^{2H(k-1)}}{b^{H(2k+1)}(a + c)^{H(4k-1)}} \, dadbdc \quad (B.5)
\]
where we have used \(K_1(x^p + y^p) \leq (x + y)^p \leq K_2(x^p + y^p)\) in eq. (B.5). Now we have
\[
2\mu = (a + b)^{2H} - a^{2H} + (b + c)^{2H} - c^{2H},
\]
which by Hu [14], can be rewritten as
\[
2\mu = 2Hb \int_0^1 ((a + bu)^{2H-1} + (c + bu)^{2H-1}) \, du. \quad (B.6)
\]
By this integral representation, it is clear that $\mu$ is non-negative. So we bracket the following way:

$$2\mu = ((a + b)^{2H} - a^{2H}) + ((b + c)^{2H} - c^{2H}) .$$

Now for $H < 1/2$, we have $(a + b)^{2H} \leq (a^{2H} + b^{2H})$, and similar for the other term, since the exponents are between 0 and 1. This gives $\mu \leq K b^{2H}$. Now for $H \geq 1/2$, following the integral representation eq. (B.14), notice that the exponents within the integrand are non-negative. Thus we can bound the integral by a constant. Succinctly

$$\mu \leq \begin{cases} K b^{2H}, & H < 1/2, \\ K b, & H \geq 1/2. \end{cases}$$

First consider $H < 1/2$. Leading on from eq. (B.5), we have

$$K \int_{[0,\xi]^3} \frac{\mu(a + b + c)^{2H(k-1)}}{b^{H(2k+1)}(a + c)^{H(4k-1)}} \, \mathrm{d}abcd$$

$$\leq K \int_{[0,\xi]^3} \frac{b^{2H}(a + b + c)^{2H(k-1)}}{b^{H(2k+1)}(a + c)^{H(4k-1)}} \, \mathrm{d}abcd$$

$$\leq K \int_{[0,\xi]^3} \frac{(a + b + c)^{2H(k-1)}}{b^{H(2k-1)}a^{H(4k-1)}c^{H(4k-1)}} \, \mathrm{d}abcd$$

$$\leq K \int_{[0,\xi]^3} \frac{1}{b^{H(2k-1)}a^{H(4k-1)}c^{H(4k-1)}} \, \mathrm{d}abcd.$$ 

The $a$ term and $c$ term exponents are obviously less than 1 for $H < H(\hat{k})$. For the $b$ term exponent, we have $H(2\hat{k} - 1) < H(\hat{k})(2\hat{k} - 1) < 1$ for all $\hat{k} \geq 1$. So this integral is finite. Now for $H \geq 1/2$, notice that $H(\hat{k}) \geq 1/2$ if and only if $\hat{k} = 1$. So it suffices to look at the case of $H \in \{1/2, 2/3\}$ and $\hat{k} = 1$. Following from eq. (B.4), we have

$$K \int_{[0,\xi]^3} \frac{1}{(x+y)^2} \, \mathrm{d}x \, \mathrm{d}y < \infty \text{ if and only if } \xi < 2.$$ 

This completes Case 2.

**Case 3:** Finally, we look at the integration over $D_{3,1}$. Following the notation in Lemma B.1 item (iii), we have that $a = s - r$ and $c = s' - r'$. This gives that $\lambda = a^{2H}$ and $\rho = c^{2H}$. Now utilizing the lemma, we have

$$\int_{D_{3,1}^2} \frac{1}{(\lambda \rho - \mu^2)(4k-1)/2} \, \mathrm{d}r \, \mathrm{d}s \, \mathrm{d}r' \, \mathrm{d}s' \leq K \int_{[0,\xi]^3} \frac{1}{(a^{2H}c^{2H})(4k-1)/2} \, \mathrm{d}abcd$$

$$\quad = K \int_{[0,\xi]^3} \frac{1}{(ac)^{H(4k-1)}} \, \mathrm{d}abcd$$

$$\quad = K \int_{[0,\xi]^3} \frac{1}{(ac)^{H(2k+1)}} \, \mathrm{d}abcd. \quad \text{(B.7)}$$

Now we have that

$$2\mu = (a + b + c)^{2H} + b^{2H} - (a + b)^{2H} - (b + c)^{2H} .$$
By Hu [14], this is equivalent to

\[ 2\mu = 2H(2H - 1)ac \int_0^1 \int_0^1 (b + au + cv)^{2H-2} dudv. \]  \hspace{2cm} (B.8)

Using Young’s inequality, we have \((b + (au + cv)) \geq K b^\alpha (au + cv)^\beta \geq K b^\alpha (aucv)^{\beta/2}, \) where \(\alpha + \beta = 1,\) and \(\alpha, \beta > 0.\) Notice that the exponent \(2H - 2\) is negative for any \(H.\) Thus we have

\[\begin{align*}
|\mu| & \leq K ac b^{\alpha(2H-2)} \int_0^1 \int_0^1 ((au)^{\beta/2}(cv)^{\beta/2})^{2H-2} dudv \\
& \leq K b^{2\alpha(H-1)} a^{\beta(H-1)+1} c^{\beta(H-1)+1}.
\end{align*}\]

Leading on from eq. (B.7), we have

\[ K \int_{[0,t]^3} \frac{|\mu|}{(ac)^{H(2k+1)}} dadbdc \leq K \int_{[0,t]^3} \frac{b^{2\alpha(H-1)} a^{\beta(H-1)+1} c^{\beta(H-1)+1}}{(ac)^{H(2k+1)}} dadbdc \]

= \( K \int_{[0,t]^3} \frac{1}{b^{2\alpha(1-H)}(ac)^{H(2k+1-\beta)+\beta-1}} dadbdc. \]  \hspace{2cm} (B.9)

Choose \(\alpha = \beta = 1/2.\) Then this is equal to

\[ K \int_{[0,t]^3} \frac{1}{b^{(1-H)}(ac)^{H(2k+1-\beta)+\beta-1}} dadbdc.\]

Clearly the \(b\) term exponent is always less than 1 for any \(H.\) Studying the \(ac\) term exponent, we have

\[ H\left(2\hat{k} + \frac{1}{2}\right) - \frac{1}{2} < H(\hat{k})\left(2\hat{k} + \frac{1}{2}\right) - \frac{1}{2} = \frac{4\hat{k} + 1}{4\hat{k} - 1} - \frac{1}{2}, \]

which is less than 1 for \(\hat{k} > 1.\) For \(\hat{k} = 1,\) the \(ac\) term exponent is less than 1 for \(H \in [0, 3/5].\) So now we must deal with the case of \(\hat{k} = 1\) and \(H \in [3/5, 2/3]\) separately. Instead, consider \(\hat{k} = 1\) and \(H \in [1/2, 2/3],\) as this range of \(H\) will be more telling as to why the subsequent methodology will not work for \(H < 1/2.\) Leading on from eq. (B.9), we must find \(\alpha, \beta > 0\) with \(\alpha + \beta = 1\) such that

\[ K \int_{[0,t]^3} \frac{1}{b^{2\alpha(1-H)}(ac)^{H(3-\beta)+\beta-1}} dadbdc \]

is finite. Focusing on the \(ac\) term exponent, we must demand that \(H(3-\beta)+\beta-1 < 1\) or equivalently, \(\beta < (2 - 3H)/(1 - H).\) Define

\[ \beta(H) := \frac{2 - 3H}{1 - H}. \]

Notice that \(\beta(H) \in (0, 1]\) for \(H \in [1/2, 2/3].\)

\[\square\]

**Lemma B.3.** Consider eq. (4.3), the key integral from the proof of Theorem 3.3. Namely,

\[\int_{D^2} \frac{1}{(1 - \gamma^2)(4k' + 1)^{3/2} \lambda^{k' + 1/2} \rho^{k' + 1/2}} \sum_{d=1}^{k'} \gamma^{2d} dr ds dr'ds' \]  \hspace{2cm} (B.10)

where we recall \(\gamma = \mu/(\sqrt{\lambda \rho})\) and \(k' \in \mathbb{N}.\) Then this integral is finite if \(H \in (0, \frac{2}{4k'+1})\) where \(k' \in \mathbb{N}.\)
Proof. Similar to the argument made in Lemma B.2, we need only look at the finiteness of the above integral for the case $d = 1$. Now notice that

$$\frac{\gamma^2}{(1 - \gamma^2)(4k'+1)/2 \lambda_k' + 1/2 \rho_k' + 1/2} = \frac{\mu^2 \lambda_k' - 1 \rho_k' - 1}{(\lambda \rho - \mu^2)(4k'+1)/2}.$$ 

Thus, finiteness of eq. (B.10) is guaranteed if and only if

$$\int_{D^2} \frac{\mu^2 \lambda_k' - 1 \rho_k' - 1}{(\lambda \rho - \mu^2)(4k'+1)/2} \, \text{d}r \, \text{d}s' \, \text{d}s'$$

is finite for each $i = 1, 2, 3$. Define $\tilde{H}(k') := \frac{2}{4k'+1}$, which is the proposed upper bound on $H$. The following steps are very similar to the steps in the proof of Lemma B.2, and so we will skip some details. In what follows, we will make use of these facts:

- We will denote $K$ to be a strictly positive constant whose value may change line by line.
- $\tilde{H}(k')$ is strictly decreasing and its maximum is $\tilde{H}(1) = 2/5$. This means that, unlike the odd case, we do not need to consider the case of $H \geq 1/2$.

We now proceed case by case.

**Case 1:** We start by considering the integration over $D^2_{1,t}$. Following the notation in Lemma B.1 item (i), we have that $a + b = s - r$ and $b + c = s' - r'$. This gives that $\lambda = (a + b)^2 H$ and $\rho = (b + c)^2 H$. Now utilizing the lemma, we have

$$\int_{D^2_{1,t}} \frac{\mu^2 \lambda_k' - 1 \rho_k' - 1}{(\lambda \rho - \mu^2)(4k'+1)/2} \, \text{d}r \, \text{d}s' \, \text{d}s' \leq K \int_{[0,t]^3} \frac{\mu^2 (a + b)^{2H(4k'+1)} (b + c)^{2H(4k'+1)}}{(a + b)^{2H} c^{2H} + (b + c)^{2H} (a + c)^{2H} (4k'+1)/2} \, \text{d}a \, \text{d}b \, \text{d}c$$

$$\leq K \int_{[0,t]^3} \frac{\mu^2 (a + b)^{2H(4k'+1)} (b + c)^{2H(4k'+1)}}{(a + b)^{2H} c^{2H} + (b + c)^{2H} (a + c)^{2H} (4k'+1)/2} \, \text{d}a \, \text{d}b \, \text{d}c$$

(B.11)

$$= K \int_{[0,t]^3} \frac{\mu^2 a b c}{a^{2H} (b + c)^{2H} (4k'+1)/2} \, \text{d}a \, \text{d}b \, \text{d}c$$

(B.12)

where we used Young’s inequality, $x^{1/2}y^{1/2} \leq \frac{x}{2} + \frac{y}{2}$ in eq. (B.11). Now

$$2\mu = ((a + b + c)^2 H + b^2 H - a^2 H - c^2 H)$$

which gives the trivial bound

$$\mu^2 \leq K (a^{4H} + b^{4H} + c^{4H}).$$

Thus we need to show that

$$K \int_{[0,t]^3} \frac{a^{4H} + b^{4H} + c^{4H}}{(a + b)^{2H} (b + c)^{2H} (4k'+1)/2} \, \text{d}a \, \text{d}b \, \text{d}c$$

is finite. So we look at each of the three terms in the numerator involving $a^{4H}$, $b^{4H}$ and $c^{4H}$ separately. First start with the $b^{4H}$ term. Then,

$$K \int_{[0,t]^3} \frac{b^{4H}}{(a + b)^{2H} (b + c)^{2H} (4k'+1)/2} \, \text{d}a \, \text{d}b \, \text{d}c$$

$$\leq K \int_{[0,t]^3} \frac{b^{4H}}{(b + c)^{2H} (4k'+1)/2} \, \text{d}a \, \text{d}b \, \text{d}c$$

$$= K \int_{[0,t]^3} \frac{1}{b^{H} (4k'+1)/2} \, \text{d}a \, \text{d}b \, \text{d}c$$
which is clearly finite for $H < \tilde{H}(k')$. For the $a^{4H}$ term we have

$$K \int_{[0,\ell]^3} \frac{a^{4H}}{(a+b)^{2H} (b+c) \rho^H c_{2(k')}^{(4k'+1)} c_{2(k')}} \, da db dc$$

$$= K \int_{[0,\ell]^3} \frac{1}{(a+b)^{2H} (b+c) \rho^H c_{2(k')}^{(4k'+1)}} \, da db dc$$

$$\leq K \int_{[0,\ell]^3} \frac{1}{b^{2H} (b+c)^{2H} (4k'+1)} \, db dc$$

$$= K \int_{[0,\ell]^3} \frac{1}{b^{2H} (4k'+2) c_{2(k'+1)}} \, db dc.$$

Clearly the $c$ term exponent is less than 1 for $H < \tilde{H}(k')$. Looking at the $b$ term exponent, we have that $H < \tilde{H}(k') < \tilde{H}(1) = 2/5$, and so the $b$ term exponent is less than 1. Observing the $a$ term exponent, we have

$$\frac{H}{2} (4k' - 2) < \frac{\tilde{H}(k')}{2} (4k' - 2) = \frac{4k' - 2}{4k' + 1} < 1$$

for any $k' \geq 1$. So this integral is finite. For the $c^{4H}$ term, notice that there is symmetry between $a$ and $c$ in the integrand of eq. (B.11). Thus, this case is identical to the case of $a^{4H}$. Hence, we have completed Case 1.

**Case 2:** Next we look at the integration over $D^2_{2,k'}$. Following the notation in Lemma B.1 item (ii), we have that $a + b + c = s - r$ and $b = s - r'$. This gives that $\lambda = (a+b+c)^{2H}$ and $\rho = b^{2H}$. Now utilizing the lemma, we have

$$\int_{D^2_{2,k'}} \mu^2 \lambda^{k'-1} \rho^{k'-1} \, dr ds dr' ds' \leq K \int_{[0,\ell]^3} \frac{\mu^2 (a + b + c)^{2H} b^{2H(k'-1)}}{(b^{2H} (a^{2H} + c^{2H}))^{(4k'+1)/2}} \, da db dc$$

$$= K \int_{[0,\ell]^3} \frac{\mu^2 (a + b + c)^{2H(k'-1)}}{b^{H(2k'+3)} (a^{2H} + c^{2H})^{(4k'+1)/2}} \, da db dc$$

$$\leq K \int_{[0,\ell]^3} \frac{\mu^2 (a + b + c)^{2H(k'-1)}}{b^{H(2k'+3)} (a + c)^{H(4k'+1)}} \, da db dc$$

(B.13)

where we have used $K_1(x^p + y^p) \leq (x+y)^p \leq K_2(x^p + y^p)$ in eq. (B.13). Now we have

$$2\mu = (a+b)^{2H} - a^{2H} + (b+c)^{2H} - c^{2H},$$

which by Hu [14], can be rewritten as

$$2\mu = 2H b \int_0^1 ((a + bu)^{2H-1} + (c + bu)^{2H-1}) \, du.$$

(B.14)

By this integral representation, it is clear that $\mu$ is non-negative. So we bracket in the following way.

$$2\mu = ((a + b)^{2H} - a^{2H}) + ((b + c)^{2H} - c^{2H}).$$

Now for $H < 1/2$, we have $(a + b)^{2H} \leq (a^{2H} + b^{2H})$, and similar for the other term, since the exponents are between 0 and 1. This gives $\mu^2 \leq Kb^{4H}$. Recall that $H < \tilde{H}(k') < \tilde{H}(1) = 2/5 < 1/2$, so this bound on $\mu^2$ is always valid. Leading on from
eq. (B.13), we have

\[ K \int_{[0,t]^{3}} \frac{\mu^2(a + b + c)^{2H(k'-1)}}{b^H(2k'+3)(a + c)^H(4k'+1)} \, ddadbdc \leq K \int_{[0,t]^{3}} \frac{b^H(a + b + c)^{2H(k'-1)}}{b^H(2k'+3)(a + c)^H(4k'+1)} \, ddadbdc \]

\[ \leq K \int_{[0,t]^{3}} \frac{(a + b + c)^{2H(k'-1)}}{b^H(2k'-1)a^{H(4k'+1)}c^{H(4k'+1)}} \, ddadbdc \]

\[ \leq K \int_{[0,t]^{3}} \frac{1}{b^H(2k'-1)a^{H(4k'+1)}c^{H(4k'+1)}} \, ddadbdc. \]

The \( a \) term and \( c \) term exponents are obviously less than 1 for \( H < \tilde{H}(k') \). For the \( b \) term exponent, we have \( H(2k' - 1) < \tilde{H}(k')(2k' - 1) < 1 \) for all \( k' \geq 1 \). So this integral is finite. This completes Case 2.

**Case 3:** Finally, we look at the integration over \( D_{3,t}^{2} \). Following the notation in Lemma B.1 item (iii), we have that \( a = s - r \) and \( c = s' - r' \). This gives that \( \lambda = a^{2H} \) and \( \rho = c^{2H} \).

Now utilizing the lemma, we have

\[ \int_{D_{3,t}^{2}} \frac{\mu^2 \lambda^{k'-1} \rho^{k'-1}}{(\lambda \rho - \mu^2)(4k'+1)/2} \, drdsdr' \, ds' \leq K \int_{[0,t]^{3}} \frac{\mu^2 a^{2H(k'-1)}b^{2H(k'-1)}}{(a^{2H}c^{2H})(4k'+1)/2} \, ddadbdc \]

\[ = K \int_{[0,t]^{3}} \frac{\mu^2 a^{2H(k'-1)}b^{2H(k'-1)}}{(ac)^H(4k'+1)} \, ddadbdc \]

\[ = K \int_{[0,t]^{3}} \frac{\mu^2}{(ac)^H(2k'+3)} \, ddadbdc. \] (B.15)

Now we have that

\[ 2\mu = (a + b + c)^{2H} + b^{2H} - (a + b)^{2H} - (b + c)^{2H}. \]

By Hu [14], this is equivalent to

\[ 2\mu = 2H(2H - 1)ac \int_{0}^{1} \int_{0}^{1} (b + au + cv)^{2H-2} \, dudv. \] (B.16)

Using Young’s inequality, we have \( (b + (au + cv)) \geq Kb^{\alpha}(au + cv)^{\beta} \geq Kb^{\alpha}(aucv)^{\beta/2} \), where \( \alpha + \beta = 1 \), and \( \alpha, \beta > 0 \). Notice that the exponent \( 2H - 2 \) is negative for any \( H \). Thus we have

\[ |\mu| \leq Kacb^{\alpha(2H-2)} \int_{0}^{1} \int_{0}^{1} ((au)^{\beta/2}(cv)^{\beta/2})^{2H-2} \, dudv \]

\[ \leq Kb^{2\alpha(H-1)}a^{\beta(H-1)+1}c^{\beta(H-1)+1} \]

which yields

\[ \mu^2 \leq Kb^{4\alpha(H-1)}a^{2\beta(H-1)+2}c^{2\beta(H-1)+2}. \]

Leading on from eq. (B.15), we have

\[ K \int_{[0,t]^{3}} \frac{\mu^2}{(ac)^H(2k'+3)} \, ddadbdc \]

\[ \leq K \int_{[0,t]^{3}} \frac{b^{4\alpha(H-1)}a^{2\beta(H-1)+2}c^{2\beta(H-1)+2}}{(ac)^H(2k'+3)} \, ddadbdc \]

\[ = K \int_{[0,t]^{3}} \frac{1}{b^{4\alpha(1-H)}(ac)^H(2k'+3-2\beta)+2\beta-2} \, ddadbdc. \]

Choose \( \alpha = 1/4 \) and \( \beta = 3/4 \). Then this is equal to

\[ K \int_{[0,t]^{3}} \frac{1}{b^{(1-H)}(ac)^H(2k'+\frac{1}{2}-\frac{3}{2})} \, ddadbdc. \]
Clearly the $b$ term exponent is always less than 1 for any $H$. Studying the $ac$ term exponent, we have

$$H\left(2k' + \frac{3}{2}\right) - \frac{1}{2} < \bar{H}(k') \left(2k' + \frac{3}{2}\right) - \frac{1}{2}$$

$$= \frac{4k' + 3}{4k' + 1} - \frac{1}{2}$$

which is less than 1 for $k' \geq 1$. This completes Case 3.