Abstract

The research in this paper gives a systematic investigation on the asymptotic behaviours of four inverse probability weighting (IPW)-based estimators for conditional average treatment effect, with nonparametrically, semiparametrically, parametrically estimated and true propensity score, respectively. To this end, we first pay a particular attention to semiparametric dimension reduction structure such that we can well study the semiparametric-based estimator that can well alleviate the curse of dimensionality and greatly avoid model misspecification. We also derive some further properties of existing estimator with nonparametrically estimated propensity score. According to their asymptotic variance functions, the studies reveal the general ranking of their asymptotic efficiencies; in which scenarios the asymptotic equivalence can hold; the critical roles of the affiliation of the given covariates in the set of arguments of propensity score, the bandwidth and kernel selections. The results show an essential difference from the IPW-based (unconditional) average treatment effect (ATE). The numerical studies indicate that for high-dimensional paradigms, the semiparametric-based estimator performs well in general whereas nonparametric-based estimator, even sometimes, parametric-based estimator, is more affected by dimensionality. Some numerical studies are carried out to examine their performances. A real data example is analysed for illustration.

*The authors gratefully acknowledge two grants from the University Grants Council of Hong Kong and a NSFC grant (NSFC11671042).
**Corresponding author

Email address: lzhu@hkbnu.edu.hk (Lixing Zhu)
1. Introduction

Treatment effects have been widely analyzed by economists and statisticians in diverse fields. In this paper, we focus on estimating treatment effect under the potential outcomes framework and the unconfoundedness assumption with binary treatment. Let $D = 0, 1$ mean that the individual does not receive or receives treatment and the response $Y$ be the corresponding potential outcome as $Y(0)$ or $Y(1)$. To conveniently identify the quantities measuring treatment effects, the unconfoundedness assumption in \cite{Rosenbaum and Rubin 1983} is generally considered, that is, the assignment to treatment is independent of the potential outcomes given a $k$-dimensional vector $X$ of covariates, i.e.

\begin{equation}
(Y(0), Y(1)) \perp D \mid X.
\end{equation}

Further, we in this paper consider the dimension of $X$ to be fixed throughout this paper, but in some cases it can be high.\footnote{Although the word "high dimension" is usually conjunct with $k$ being divergent with sample size in recent years, when we say $X$ is of high dimension in this paper, it only means $X$ contains many but fixed number of covariates. For ease of explanation, we still use the word "high dimension" whenever no confusion will be caused.} As $Y(0)$ and $Y(1)$ cannot be simultaneously observed for any individual, the observed outcome can be written as $Y = DY(1) + (1 - D)Y(0)$. Since estimating the $i$-th individual treatment effect $(Y_i(1) - Y_i(0))$ is unrealistic, an important trend in the literature turns to estimate the average treatment effect ($ATE$: $\mu = E(Y(1) - Y(0))$). See for instance \cite{Rosenbaum and Rubin 1983} and \cite{Hirano et al. 2003}.

Recently, there is an increasing interest in estimating conditional (or heterogeneous) average treatment effects: $CATE(X) = E(Y(1) - Y(0) \mid X)$, which is designed to reflect how treatment effects vary across different subpopulations. Note that
even thought receiving a treatment may have no effect on outcomes for the overall population, i.e. $ATE = 0$, the treatment can still be effective for a subpopulation defined by specific observable characteristics, i.e. for some $x$ such that $CATE(x) \neq 0$. Thus heterogeneous treatment effects are more informative and can play important roles in personalized medicine or policy intervention. Most of existing estimation methods for the heterogeneous treatment effects are conditional on the full set of variables, $X$, see e.g. Crump et al. (2008), Wager and Athey (2018), where the multivariate variable $X$ are designed to make the unconfoundedness assumption plausible. After 2015, researchers consider to estimate more general conditional/heterogeneous treatment effects, in which the conditioning covariates $Z$ with $Z$ being a subset of covariates, i.e.

$$X = (Z^T, U^T)^T \in \mathbb{R}^l \times \mathbb{R}^m, k = l + m < \infty.$$ 

See e.g. Abrevaya et al. (2015) and Lee et al. (2017). Note that treatment effects conditioning on a subset of $X$, rather than the high dimensional covariates $X$, can provide desirable flexibility and can help making policy decision.

Based on the assumption (1), Abrevaya et al. (2015) used the inverse probability weighting (IPW)-based method, which is popularly used in literature (Robins et al., 1994), to estimate

$$CATE(Z) = E[Y(1) - Y(0) \mid Z]$$

when the propensity score function is estimated parametrically (IPW-P) and nonparametrically (IPW-N). Abrevaya et al. (2015) gave a deep investigation on the asymptotic properties of the estimators. There are two main conclusions in Abrevaya et al. (2015): one is $IPW-N$ can be asymptotically more efficient than $IPW-P$ in the sense that the asymptotic variance function of $IPW-N$ can be uniformly smaller than that of $IPW-P$, the another is the asymptotic variance function of $IPW-P$ is equal to that of $IPW-O$ which is defined as the oracle estimator with the true propensity score. It is noteworthy that the last conclusion is different from that of IPW-type ATE estimators, because the IPW-type ATE estimator based on parametrically estimated propensity score can be more efficient than the one with true propensity score.

As is known, to make the unconfoundedness assumption be plausible, it is often the case that we need to include many covariates in the analysis. Thus we say $X \in \mathbb{R}^k$ is
of high dimension with \( k < \infty \). In this case, on one hand, it is often not easy to choose a parametric specification that can sufficiently capture all the important nonlinear and interaction effects to have IPW-\( P \). On the other hand, any nonparametric estimation of propensity score clearly suffers from the curse of dimensionality and then IPW-\( N \) does not work any more.

Therefore, in this paper, we suggest a semiparametric IPW-based \( CATE(Z) \) estimation procedure to simultaneously alleviate the propensity score misspecification problem and particularly the curse of dimensionality. To this end, we consider a semiparametric dimension reduction structure of the propensity score and the unconfoundedness assumption (1) can have a dimension reduction version. It is worth pointing out that the general nonparametric structure can be regarded as a special case of the dimension reduction structure we consider with an orthonormal projection matrix of full rank. We will call the estimator IPW-\( S \) and give the details about the model setting and the estimation procedure in the next section.

For theoretical development, we will give the asymptotically linear representation and asymptotic normality of IPW-\( S \). We will also give some further properties of existing IPW-\( N \) in Abrevaya et al. (2015). Based on the theoretical studies, we give a systematic comparison on the asymptotic efficiency amongst IPW-\( O \), IPW-\( P \), IPW-\( S \) and IPW-\( N \).

Combining the results of Abrevaya et al. (2015) and the further properties of IPW-\( N \) we derive in this paper, the comparison reveals some very interesting and important phenomena. Specifically, letting \( A \preceq B \) mean that the asymptotic variance of estimator \( A \) is not greater than that of estimator \( B \) and \( A \cong B \) stand for that \( A \) has the same asymptotic variance function as \( B \), we have the following observations in theory.

First, in general IPW-\( N \preceq IPW-S \preceq IPW-P \cong IPW-O \).

Second, the affiliation of \( Z \) to the set of arguments of the propensity score plays an important role in the asymptotic efficiency of IPW-\( S \) and IPW-\( N \). That is, when \( Z \) is a subset of arguments of the propensity score, IPW-\( S \preceq IPW-P \cong IPW-O \) and IPW-\( N \preceq IPW-P \cong IPW-O \), otherwise, IPW-\( N \cong IPW-S \cong IPW-P \cong IPW-O \). Note that this newly found phenomenon provides a deep insight into the performances of IPW-\( S \) and IPW-\( N \), which is also useful in practice.
Third, when the propensity score function is smooth enough, then even in general cases we can also have the asymptotic equivalence by carefully choosing the bandwidths and using high order kernel functions: $IPW-N \cong IPW-S \cong IPW-P \cong IPW-O$. This also gives us a better understanding for the asymptotic performance of different estimators. Of course, this part mainly serves as a theoretical exploration. For practical use, we would have no interest to wilfully choose those kernel function and bandwidths, which are very difficult to implement and make the estimator with worse performance. But it reminds the researchers that a “good” estimator of the propensity score would not be helpful for the performance of the CATE estimator.

Fourth, owing to the dimension reduction structure of $p(X)$, the requirements for bandwidths and the order of kernel function for $IPW-S$ are much milder than those for $IPW-N$. Thus when the dimension is high, even though $IPW-N$ has the superior efficiency in theory, $IPW-S$ is preferable.

The rest of the paper is organized as follows. In Section 2, we first introduce the estimation procedure for $IPW-S$. Also we investigate its asymptotic properties and the theoretical comparisons between the four CATE estimators. Section 3 contains some numerical studies to examine the performance of the CATE estimators. In Section 4, we apply the CATE estimators to analyse a real data set for illustration. Section 5 contains some conclusions and a further discussion. The regularity conditions are listed in Appendix and all the technical proofs are relegated to Supplementary Materials to save space.

2. Semiparametric estimation procedure and asymptotic properties

2.1. Preliminary of estimation

Assume that covariates $X = (Z^T, U^T)^T$ are absolutely continuous, under the unconfoundedness assumption [1], recall that CATE function $\tau(z)$ can be rewritten as

$$
\tau(z) = E \left[ \frac{DY}{p(X)} - \frac{(1 - D)Y}{1 - p(X)} \mid Z = z \right], Z \in R^l.
$$

(2)
If \( p(X) \) is given, we can estimate \( \tau(z) \) immediately via the Nadaraya-Watson kernel method by regarding \( \frac{DY}{p(X)} - \frac{(1-D)Y}{1-p(X)} \) as response:

\[
\hat{\tau}_O(z) = \left( \frac{\sum_{i=1}^{n} \left[ D_i Y_i - \frac{(1-D_i)Y_i}{1-p(X_i)} \right] K_h(Z_i - z) }{\sum_{i=1}^{n} K_h(Z_i - z)} \right) / \sum_{i=1}^{n} K_h(Z_i - z).
\]

Here \( K(\cdot) \) is a multivariate kernel function, \( K_h(u) = \frac{1}{h} K \left( \frac{u}{h} \right) \) and \( l = \text{dim}(Z) \). This \( CATE \) estimator is \( IPW-O \) we mentioned before.

Based on existing results for nonparametric estimation, it is easy to derive the asymptotic distribution of \( IPW-O \) which will be used as the benchmark to make comparisons among all estimators studied in this paper.

**Proposition 1.** Suppose the conditions (C1)-(C4) in Appendix are satisfied, the following statements hold for each point \( z \) in the support of \( Z \):

\[
\sqrt{nh} \left( \hat{\tau}_O(z) - \tau(z) \right) \overset{D}{\rightarrow} N \left( 0, \frac{\| K \|_2^2 \sigma^2_O(z)}{f(z)} \right).
\]

Here \( \sigma^2_O(z) = E \left( \left[ \frac{DY}{p(X)} - \frac{(1-D)Y}{1-p(X)} - \tau(z) \right]^2 \mid Z = z \right) \).

When \( p(X) \) is an unknown function, we then first estimate \( p(X) \) to define a final \( CATE \) estimator \( \hat{\tau}(z) \). We propose the estimator under semiparametric structure below.

### 2.2. Semiparametric estimation for conditional average treatment effect

Assume the propensity score has a semiparametric dimension reduction structure:

\[
p(X) = q(V^\top X),
\]

where both the function \( q(\cdot) \) and the \( r \) projection directions in \( V \) are unknown with \( V \) being a \( k \times r \) orthonormal matrix. It is noteworthy that this structure is general, which covers the structures of some important semiparametric models such as single-index models. From the definition of propensity score, (3) implies the indicator \( D \) depends on \( X \) through the projected variable \( V^\top X \). Thus, we can use the following conditional independence to present the above semiparametric structure:

\[
D \perp X \mid V^\top X.
\]
It follows that \((Y(0), Y(1)) \perp D \mid V^TX\). We call the intersection of all \(V\)'s satisfying the above independence the central subspace, see \cite{Li1991}. Usually \(V^* = V \times C\) can be identified. As this identification issue does not affect the related estimation of \(p(X)\), we then still use \(V\) without confusion. Relevant references are \cite{Li2017} and \cite{Ma2019}. This is a dimension reduction framework, so that the corresponding estimation could be less affected by the curse of dimensionality. For such a dimension reduction structure, we can also consider variable selection as \cite{Ma2019} did. But as this is not a focus of this paper, we then just work on this model and assume the existence of consistent estimation later on.

If we postulate that the information about \(D\) from \(X\) can be completely captured by \(r \) linear combinations \(V^TX\) of \(X\) with \(r \ll k\), the propensity score can be estimated by replaced the original \(X\) with \(V^TX\). That is, we can use lower dimensional kernel function \(\mathcal{H}(u)\) to get a nonparametric estimator \(\hat{q}(\hat{V}^TX)\) of \(q(V^TX) = E(D \mid V^TX)\),

\[
\hat{q}(\hat{V}^TX_i) = \frac{\sum_{j \neq i}^n D_j \mathcal{H}_{h_2}(\hat{V}^TX_j - \hat{V}^TX_i)}{\sum_{j \neq i}^n \mathcal{H}_{h_2}(V^TX_j - V^TX_i)},
\]

where \(h_2\) is the bandwidth, \(\mathcal{H}_{h_2}(u) = h_2^{-r}\mathcal{H}(u/h_2)\) and \(\hat{V}\) is a consistent estimator derived by a sufficient dimension reduction method. There are several methods available in the literature, such as inverse regression methods in \cite{Cook2002} and minimum average variance estimation (MAVE) in \cite{Xia2002, Xia2007}.

Recall that CAT\(E\) can be rewritten as \((2)\). Thus, based on \(\hat{q}(\hat{V}^TX_i)\), the IPW-\(S\) of \(\tau(z)\) is defined as

\[
\hat{\tau}_S(z) = \sum_{i=1}^n \left[ \frac{D_iY_i}{\hat{q}(\hat{V}^TX_i)} - \frac{(1 - D_i)Y_i}{1 - \hat{q}(\hat{V}^TX_i)} \right] K_h(Z_i - z) / \sum_{i=1}^n K_h(Z_i - z).
\]

Since both \(Z\) and \(V^TX\) are low-dimension random vectors, \(\hat{\tau}_S(z)\) can well alleviate the propensity score misspecification problem and the curse of dimensionality simultaneously.

In the next section, we investigate the asymptotic properties of \(\hat{\tau}_S(z)\) and derive some further properties of existing IPW-\(N\) under certain regularity conditions.
2.3. Asymptotic properties for IPW-S

Denote $|A|$ as the cardinality of set $A$. We first give some notations.

- $\mathcal{W} = (X, D, Y)$ and the observation data $\mathcal{W}_i = (X_i, D_i, Y_i)^n_{i=1}$ are the independent copies of $\mathcal{W}$;
- $m_j(V^TX) = E[Y(j) \mid V^TX], j = 0, 1, \text{ and } K_i = K((Z_i - z)/h);$  
- $\psi(p(V^TX), \mathcal{W}) = DY/(q(V^TX)) - (1 - D)Y/(1 - q(V^TX));$
- $\psi^*(q(V^TX), \mathcal{W}) = [D\{Y - m_1(V^TX)\}]/\{q(V^TX)\} - [(1 - D)\{Y - m_0(V^TX)\}]/\{1 - q(V^TX)\}.$
- For two vectors $A$ and $B$, we use intersection notation $A \cap B$ to write, without confusion, as all components that are contained in both $A$ and $B$. $|A \cap B| = t$ stands for the number of components in the intersection of $A$ and $B$. Particularly, when $t = 0, A \cap B = \emptyset$, and $t = |A|$ implies $A \cap B = A$.

Both $\psi(q(V^TX), \mathcal{W})$ and $\psi^*(q(V^TX), \mathcal{W})$ are the central parts of influence function for IPW-S.

**Theorem 1.** Suppose all the conditions in Appendix are satisfied, the following statements hold for each point $z$ in the support of $Z$:

1. When $|Z \cap V^TX| = t < l$ with $s_2[2 - l/(l - t)] + l > 0$, the asymptotically linear representation is

$$\sqrt{nh^l} \left( \hat{\tau}_S(z) - \tau(z) \right) = \frac{1}{\sqrt{nh^l} f(z)} \sum_{i=1}^n [\psi^*(q(V^TX_i), \mathcal{W}_i) - \tau(z)] K_i + o_p(1)$$

and the asymptotic distribution of $\hat{\tau}_S(z)$ is

$$\sqrt{nh^l} \left( \hat{\tau}_S(z) - \tau(z) \right) \xrightarrow{D} N(0, \Sigma_S(z)).$$

2. When $|Z \cap V^TX| = l$, the asymptotically linear representation is

$$\sqrt{nh^l} \left( \hat{\tau}_S(z) - \tau(z) \right) = \frac{1}{\sqrt{nh^l} f(z)} \sum_{i=1}^n [\psi^*(q(V^TX_i), \mathcal{W}_i) - \tau(z)] K_i + o_p(1)$$

and the asymptotic distribution of $\hat{\tau}_S(z)$ is

$$\sqrt{nh^l} \left( \hat{\tau}_S(z) - \tau(z) \right) \xrightarrow{D} N(0, \Sigma_S^*(z)).$$
Here $s_2$ is the order of $H()$, $\Sigma_S(z) = \| K \|_2^2 \sigma^2_S(z) / f(z)$, $\Sigma^*_S(z) = \| K \|_2^2 \sigma^2_S^*(z) / f(z)$ with $\sigma^2_S(z) = E[(\psi(q(V^T X), W) - \tau(z))^2 | Z = z]$ and $\sigma^2_S^*(z) = E[(\psi^*(q(V^T X), W) - \tau(z))^2 | Z = z]$.

**Remark 1.** These results show a very interesting and somewhat unexpected phenomenon that the asymptotic behaviors of $\hat{\tau}_S(z)$ also depend on whether some of elements of $Z$ belong to $V^T X$. Recall that $| Z \cap V^T X | = t$ means $t$ elements of $Z$ also are $t$ linear combinations of $V^T X$, i.e. we can rewrite $V^T X = (Z_1, \cdots, Z_t, (\hat{V}^T X)^T)^T$ with $V^T = \begin{pmatrix} \hat{I}_{(t) \times k} \\
\tilde{V}^T \end{pmatrix}$ and $\tilde{I}_{(t) \times l} = \begin{pmatrix} I_{l \times t} & 0 \end{pmatrix}$. Here $I_{l \times t}$ is an identity matrix, $\hat{V}^T$ is a $(r-t) \times k$ matrix. The asymptotic behaviors with $t = l$ and any $0 \leq t < l$ are very different. A natural question is whether we can, if possible, choose a dimension reduced vector $V^T X$ such that IPW-S works best. The question is related to IPW-P and IPW-N, we will have some detailed discussion in Subsection 3.3 below.

Next, we present the estimators for $\Sigma_S(z)$ and $\Sigma^*_S(z)$ under $| Z \cap V^T X | < l$ and $| Z \cap V^T X | = l$ respectively as

$$\hat{\Sigma}_S(z) = \frac{\| K \|_2^2 \hat{\sigma}^2_S(z)}{\hat{f}(z)} \quad \text{and} \quad \hat{\Sigma}^*_S(z) = \frac{\| K \|_2^2 \hat{\sigma}^2_S^*(z)}{\hat{f}(z)},$$

(7)

where $\hat{\sigma}_S(z)$ and $\hat{\sigma}^2_S$ are estimators for $\sigma_S(z)$ and $\sigma^2_S$ with

$$\hat{\sigma}_S^2 = \frac{1}{nh^l} \sum_{i=1}^{n} \frac{[(\psi(q_i, W_i) - \hat{\tau}_S(z))^2 \hat{K}_i]}{\hat{f}(z)}$$

and

$$\hat{\sigma}^2_S = \frac{1}{nh^l} \sum_{i=1}^{n} \frac{[(\psi^*(\hat{q}_i, W_i) - \hat{\tau}_S(z))^2 \hat{K}_i]}{\hat{f}(z)},$$

where $\hat{\Sigma}_S(z)$ and $\hat{\Sigma}^*_S(z)$ under $| Z \cap V^T X | < l$ and $| Z \cap V^T X | = l$ respectively as

$$\hat{\sigma}_S^2 = \frac{1}{nh^l} \sum_{i=1}^{n} \frac{[(\psi(q_i, W_i) - \hat{\tau}_S(z))^2 \hat{K}_i]}{\hat{f}(z)}$$

and

$$\hat{\sigma}^2_S = \frac{1}{nh^l} \sum_{i=1}^{n} \frac{[(\psi^*(\hat{q}_i, W_i) - \hat{\tau}_S(z))^2 \hat{K}_i]}{\hat{f}(z)},$$

$\hat{f}(z) = \sum_{i=1}^{n} K_h(Z_i - z) / n$ is a kernel-based estimator of $f(z)$,

$$\psi(q_i, W_i) = \frac{D_i Y_i}{\hat{q}(V^T X_i)} - \frac{(1 - D_i) Y_i}{1 - \hat{q}(V^T X_i)} \quad \text{and} \quad \psi^*(\hat{q}_i, W_i) = \frac{[D_i \{ Y_i - \hat{m}_1(V^T X_i) \}]}{\hat{q}(V^T X_i)}$$

$$- \frac{[(1 - D_i) \{ Y_i - \hat{m}_0(V^T X_i) \}]}{1 - \hat{q}(V^T X_i)} \quad \text{and} \quad \hat{m}_1(V^T X_i) - \hat{m}_0(V^T X_i)$$

with $\hat{m}_3(\hat{V}^T X) = \sum_{t: D_t = 1}^{n} h_3(\hat{V}^T X_t - \hat{V}^T X) Y_t / \sum_{t: D_t = 1}^{n} h_2(\hat{V}^T X_t - \hat{V}^T X)$,

$\hat{m}_0(\hat{V}^T X) = \sum_{t: D_t = 0}^{n} h_3(\hat{V}^T X_t - \hat{V}^T X) Y_t / \sum_{t: D_t = 0}^{n} h_2(\hat{V}^T X_t - \hat{V}^T X)$

being the estimators of $m_1(V^T X)$ and $m_0(V^T X)$. 

9
Further we can state the consistency of the proposed estimators in the following theorem.

**Theorem 2.** Suppose all the conditions in Appendix are satisfied, we have that

\[ \hat{\Sigma}_S(z) = \Sigma_S(z) + o_p(1), \quad \text{and} \quad \hat{\Sigma}_S^*(z) = \Sigma_S^*(z) + o_p(1). \]

By Theorem 2, we can obtain the pointwise consistent estimator for standard error of \( \sqrt{nh}(\hat{\tau}_S(z) - \tau(z)) \), so that we are able to construct a \((1 - \alpha)100\%\) pointwise confidence interval for \( \tau(z) \), i.e.

\[ \hat{\tau}_S(z) \pm (nh)^{-1/2} c_{\alpha/2} \left( \hat{\Sigma}_S(z) \right)^{1/2} \]  

or

\[ \hat{\tau}_S(z) \pm (nh)^{-1/2} c_{\alpha/2} \left( \hat{\Sigma}_S^*(z) \right)^{1/2}, \]  

with \( c_{\alpha/2} \) being the \((1 - \alpha/2)\) quantile of the standard normal distribution. Note that the specification formula of confidence interval depends on whether the condition \(|Z \cap V^\top X| < l\) or \(|Z \cap V^\top X| = l\). One possible way to make choice between (8) and (9) is based on the value of \(|Z \cap \hat{V}^\top X|\).

To be specified, taking MAVE (Xia et al., 2002) as an example dimension reduction method, we proposed an estimation and inference procedure of \( \tau(z) \) based on IPW-S by carrying out the following steps.

**Step 1:** Obtain the estimator of \( V \) by solving the minimizing problem

\[
\min_{V, a, b} \sum_{i,j=1}^n \{ D_i - a_j - b_j^\top V^\top (X_i - X_j) \}^2 \omega_{ij}.
\]

Here \( \omega_{ij} = \mathcal{H}_{h_2} \{ V^\top (X_i - X_j) \} / \sum_{l=1}^n \mathcal{H}_{h_2} \{ V^\top (X_l - X_j) \}, \quad a = (a_1, \ldots, a_n), \quad b = (b_1, \ldots, b_n). \)

Denote the resulting estimator by \( \hat{V} \).

**Step 2:** Given \( \hat{V} \), estimate the propensity score \( E(D \mid \hat{V}^\top X) \) via (5).

**Step 3:** Obtain the semiparametric CATE estimator, \( \hat{\tau}_S(z) \), via (6).
Step 4: Given \( Z = z \), a \((1-\alpha)\) pointwise confidence interval for the true CATE, \( \tau(z) \), can be constructed as follows. If \( |Z \cap \hat{V}^\top X| \approx l \), the confidence interval of \( \tau(z) \) can be constructed in the form of (9), i.e. \( \hat{\tau}_S(z) \pm (nh)^{-1/2} c_{\alpha/2} \left( \hat{\Sigma}_S(z) \right)^{1/2} \), Otherwise, the pointwise confidence interval of \( \tau(z) \) would be constructed in the form of (8), that is, \( \hat{\tau}_S(z) \pm (nh)^{-1/2} c_{\alpha/2} \left( \hat{\Sigma}_S(z) \right)^{1/2} \).

Note that the first step can be implemented using the R package MAVE. Based on this estimation and inference procedure, the empirical analysis in section 5 can be implemented.

2.4. Extension of existing IPW-N

Recall IPW-N proposed by Abrevaya et al. (2015) is

\[
\hat{\tau}_N(z) = \left( \sum_{i=1}^{n} \left[ \frac{D_iY_i}{\hat{p}(X_i)} - \frac{(1-D_i)Y_i}{1-\hat{p}(X_i)} \right] K_h(Z_i - z) \right) / \sum_{i=1}^{n} K_h(Z_i - z), \tag{10}
\]

with \( \hat{p}(X_i) = \sum_{j \neq i} D_jL_{h_1}(X_j - X_i) / \sum_{j \neq i} L_{h_1}(X_j - X_i) \). Here \( L(\cdot) \) is also a multivariate kernel function with \( L_{h_1}(\cdot) = h_1^{-k}L(\cdot/h_1) \), and \( h_1 \) is the corresponding bandwidth.

Note that the asymptotic properties of IPW-S is influenced by the affiliation of \( Z \), we in this paper try to analyse the asymptotic properties of IPW-N in different scenarios similarly as the ones in Theorem 1. Suppose

\[
D \perp X \mid \tilde{X}, \tilde{X} \subseteq X, \tilde{k} = \text{dim}(\tilde{X}) \leq k. \tag{11}
\]

To extend the asymptotic results of IPW-N in Abrevaya et al. (2015), we derive the following theorem that also confirms the influence of the affiliation of \( Z \) to \( \tilde{X} \) in the asymptotic properties of IPW-N. Abrevaya et al. (2015) only considered a special situation in the following Theorem 3: \( |Z \cap \tilde{X}| = l \) and \( \tilde{X} = X \).

Before stating the result as theorem, let us define some important quantities:

\[
\psi(p(\tilde{X}), W) = \frac{DY}{p(\tilde{X})} - \frac{(1-D)Y}{1-p(\tilde{X})},
\]

\[
\psi^*(p(\tilde{X}), W) = \frac{D[Y - m_1(\tilde{X})]}{p(X)} - \frac{(1-D)[Y - m_0(\tilde{X})]}{1-p(X)} + m_1(\tilde{X}) - m_0(\tilde{X}).
\]
Theorem 3. Suppose all the conditions in Appendix are satisfied, the following statements hold for each point \( z \) in the support of \( Z \):

1. When \(|Z \cap \tilde{X}| = t < l\) with \(s_1|2 - l/(l-t)| + l > 0\), the asymptotically linear representation is

\[
\sqrt{nh^l} (\hat{\tau}_N(z) - \tau(z)) = \frac{1}{\sqrt{nh^l} f(z)} \sum_{i=1}^{n} [\psi(p(\tilde{X}_i), W_i) - \tau(z)]K_i + o_p(1);
\]

the asymptotic distribution of \( \hat{\tau}_N(z) \) is

\[
\sqrt{nh^l} (\hat{\tau}_N(z) - \tau(z)) \overset{D}{\rightarrow} N(0, \Sigma_N(z)).
\]

2. When \(|Z \cap \tilde{X}| = l\), the asymptotically linear representation is

\[
\sqrt{nh^l} (\hat{\tau}_N(z) - \tau(z)) = \frac{1}{\sqrt{nh^l} f(z)} \sum_{i=1}^{n} [\psi^*(p(\tilde{X}_i), W_i) - \tau(z)]K_i + o_p(1);
\]

the asymptotic distribution of \( \hat{\tau}_N(z) \) is

\[
\sqrt{nh^l} (\hat{\tau}_N(z) - \tau(z)) \overset{D}{\rightarrow} N(0, \Sigma_N^*(z)).
\]

Here \( s_1 \) is the order of \( L(.) \), \( \Sigma_N(z) = \|K \|_2^2 \sigma_N^2(z)/f(z) \), \( \Sigma_N^*(z) = \|K \|_2^2 \sigma_N^2(z)/f(z) \), \( \sigma_N^2(z) = E[\{\psi(p(\tilde{X}_i), W_i) - \tau(z)\}^2 | Z = z] \), \( \sigma_N^2(z) = E[\{\psi^*(p(\tilde{X}_i), W_i) - \tau(z)\}^2 | Z = z] \).

Similarly as IPW-S, we also proposed the estimators for \( \Sigma_N(z) \) and \( \Sigma_N^*(z) \) under \(|Z \cap \tilde{X}| < l\) and \(|Z \cap \tilde{X}| = l\) respectively as

\[
\hat{\Sigma}_N(z) = \frac{\|K\|_2^2 \hat{\sigma}_N^2(z)}{f(z)}, \quad \text{and} \quad \hat{\Sigma}_N^*(z) = \frac{\|K\|_2^2 \hat{\sigma}_N^2(z)}{f(z)},
\]

where \( \hat{\sigma}_N^2 \) and \( \hat{\sigma}_N^2 \) are estimators for \( \sigma_N^2(z) \) and \( \sigma_N^2(z) \) with

\[
\hat{\sigma}_N^2 = \frac{1}{nh^l} \sum_{i=1}^{n} \frac{(\psi(\hat{p}, W_i) - \hat{\tau}_N(z))^2K_i}{f(z)}, \quad \hat{\sigma}_N^2 = \frac{1}{nh^l} \sum_{i=1}^{n} \frac{(\psi^*(\hat{p}, W_i) - \hat{\tau}_N(z))^2K_i}{f(z)},
\]

\[
\psi(\hat{p}, W_i) = \frac{D_i Y_i}{\hat{p}(\tilde{X}_i)} - \frac{(1 - D_i) Y_i}{1 - \hat{p}(\tilde{X}_i)} \quad \text{and} \quad \psi^*(\hat{p}, W_i) = \frac{D_i [Y_i - \hat{m}_1(\tilde{X}_i) - \hat{m}_0(\tilde{X}_i)]}{\hat{p}(\tilde{X}_i)} - \frac{[1 - D_i] Y_i - \hat{m}_0(\tilde{X}_i)]}{1 - \hat{p}(\tilde{X}_i)} + \hat{m}_1(\tilde{X}_i) - \hat{m}_0(\tilde{X}_i).
\]
\[ \hat{m}_1(\tilde{X}) = \sum_{i:D_i=1} \mathcal{L}_{h_1}(\tilde{X}_i - \tilde{X}) Y_i / \sum_{i:D_i=1} \mathcal{L}_{h_1}(\tilde{X}_i - \tilde{X}), \hat{m}_0(\tilde{X}) = \sum_{i:D_i=0} \mathcal{L}_{h_1}(\tilde{X}_i - \tilde{X}) Y_i / \sum_{i:D_i=0} \mathcal{L}_{h_1}(\tilde{X}_i - \tilde{X}) \] being the estimators of \( m_1(\tilde{X}) \) and \( m_0(\tilde{X}) \). Further we can show the consistency of proposed asymptotic variance function estimators via the following theorem.

**Theorem 4.** Suppose all the conditions in Appendix are satisfied, we have that

\[ \hat{\Sigma}_N(z) = \Sigma_N(z) + o_p(1), \text{ and } \hat{\Sigma}^*_N(z) = \Sigma^*_N(z) + o_p(1). \]

**Remark 2.** Based on Theorem 4, we can also get the consistent estimator for standard error of \( \sqrt{nh}^l(\hat{\tau}_N(z) - \tau(z)) \) and construct a pointwise confidence interval of \( \tau(z) \) based on \( \hat{\tau}_N(z) \). However, we first need to estimate the true active arguments of propensity score \( \tilde{X} \), denoting the corresponding estimator as \( \hat{X} \), which can be done by variable selection method, to decide the proper form of confidence interval. To be specified, if \(|Z \cap \hat{X}| \approx l\), the pointwise confidence interval can be constructed as

\[ \hat{\tau}_N(z) \pm (nh)^{-1/2} c_{\alpha/2} \left( \hat{\Sigma}^*_N(z) \right)^{1/2}. \]

Otherwise, we would construct the pointwise confidence interval as

\[ \hat{\tau}_N(z) \pm (nh)^{-1/2} c_{\alpha/2} \left( \hat{\Sigma}_N(z) \right)^{1/2}. \]

**2.5. Some further studies on estimation efficiency**

When \( \tilde{X} = X \), as proved by Abrevaya et al. (2015), IPW-N can be asymptotically more efficient than IPW-P:

\[ \sigma^2_T(z) = \sigma^2_N(z) + E \left[ p(X) \{1 - p(X)\} \left\{ \frac{m_1(X)}{p(X)} + \frac{m_0(X)}{1 - p(X)} \right\}^2 \bigg| Z = z \right], \]

and IPW-P \( \cong \) IPW-O. Here \( m_j(X) = E\{Y(j) \mid X\} \). Thus, with \( p(X) = p(\tilde{X}) = q(V^T X) \), we can give the ranking for the estimation efficiency of the four estimators in the following corollary.

**Corollary 1.** Suppose all the assumptions and conditions in Appendix are satisfied and \( p(X) = p(\tilde{X}) = q(V^T X) \), the following statements hold for each point \( z \) in the support of \( Z \):

\[ 13 \]
Case 1: When $|Z \cap \tilde{X}| = l$ with $\tilde{X} = X$ and $|Z \cap V^\top X| = l$, 

$$IPW \cdot N \leq IPW \cdot S \leq IPW \cdot P \cong IPW \cdot O,$$ with

$$\sigma_P^2(z) = \sigma_S^2(z) + E \left[ q(V^\top X)(1 - q(V^\top X)) \left\{ \frac{m_1(V^\top X)}{q(V^\top X)} + \frac{m_0(V^\top X)}{1 - q(V^\top X)} \right\}^2 \mid Z = z \right],$$

$$\sigma_S^2(z) = \sigma_N^2(z) + E \left[ q(V^\top X)(1 - q(V^\top X)) \left\{ \frac{\Delta m_1}{q(V^\top X)} + \frac{\Delta m_0}{1 - q(V^\top X)} \right\}^2 \mid Z = z \right],$$

where $\Delta m_j = m_j(X) - m_j(V^\top X)$.

Case 2: When $|Z \cap \tilde{X}| = l$ with $\tilde{X} = X$ but $|Z \cap V^\top X| = t$ with $0 \leq t < l$, 

$$IPW \cdot N \leq IPW \cdot S \cong IPW \cdot P \cong IPW \cdot O$$

with $\sigma_S^2(z) = \sigma_P^2(z) = \sigma_O^2(z)$.

Case 3: When $|Z \cap \tilde{X}| = t$ with $\tilde{X} \subseteq X$ and $|Z \cap V^\top X| = t$ with $0 \leq t < l$, 

$$IPW \cdot N \cong IPW \cdot S \cong IPW \cdot P \cong IPW \cdot O$$

with $\sigma_N^2(z) = \sigma_S^2(z) = \sigma_P^2(z) = \sigma_O^2(z)$.

Remark 3. In Case 1, the equality in the first inequality holds when both $m_1(V^\top X)$ and $m_0(V^\top X)$ equal to zero, and the equality in the second inequality holds when $m_j(X) = m_j(V^\top X)$ for all $j = 0, 1$. A sufficient condition to make $m_j(X) = m_j(V^\top X)$ hold is $E(Y_j \mid X) = V^\top X$ meaning that $Y(1)$ and $Y(0)$ share the same central mean subspace.

Remark 4. Here, we discuss another special case: $V^\top X = Z$ in Corollary 1 such that $q(V^\top X) = p(Z)$. It follows that $IPW \cdot S \leq IPW \cdot P$ with $\sigma_P^2(z) = \sigma_S^2(z) + p(z)(1 - p(z)) [m_1(z)/\{p(z)\} + m_0(z)/\{1 - p(z)\}]^2$. Similarly, $IPW \cdot N \leq IPW \cdot P$ if $\tilde{X} = Z$: $\sigma_P^2(z) = \sigma_S^2(z) + p(z)(1 - p(z)) [m_1(z)/\{p(z)\} + m_0(z)/\{1 - p(z)\}]^2$. Thus, if $Z = V^\top X = \tilde{X}$, we have $\sigma_S^2(z) = \sigma_N^2(z) \leq \sigma_P^2(z)$.

Remark 5. Although $IPW \cdot S$ cannot be more efficient than $IPW \cdot N$ in theory, it has an obvious advantage due to its dimension reduction structure. This can be very useful in practice as when $X$ is of high dimension, $IPW \cdot N$ is hard to use as it has to adopt very high order kernel function and delicately chosen bandwidths. The numerical
studies in the next section show that when the dimension of $X$ is only 4, IPW-S can perform better than IPW-N in some cases. Thus, in the numerical studies, when the in high dimension $k = 20$, we do not consider IPW-N.

Another issue is also relevant. Generally speaking, combining the results in Subsections 3.1 and 3.3, when the dimension reduced vector $V^\top X$ cannot fully cover the given covariates $Z$, the IPW-S is less efficient. It seems that we can add $Z$ into the covariates to be $(Z, V^\top X)$ to enhance the estimation efficiency in theory. However, this causes the estimation procedure much more complicated (with higher order kernel and more delicately selected bandwidths) and less accurate due to the dimension increasing as described above. Thus, balancing the theoretical merit and practical usefulness, we still prefer using IPW-S without adding more covariates.

From the above discussion, we can find that the asymptotic efficiency comparison result of IPW-type CATE estimators is different from that of IPW-type ATE estimators, because ATE estimator using nonparametric estimated $p(X) \preceq$ that using parametric estimated $p(X) \preceq$ that using true $p(X)$. Thus it is worthwhile to give a further exploration on the reasons. From our study, we find that it is mainly because of the different convergence rates of the estimated propensity scores under different scenarios. In the following corollary, we show that when the convergence rate of the nonparametrically estimated propensity score can be fast enough, IPW-N and IPW-S can also be asymptotically equivalent to IPW-P, so is IPW-O. This is the case when the propensity score function is smooth sufficiently and the kernel and bandwidths are chosen delicately to meet the mentioned condition in Corollary[2].

**Corollary 2.** Suppose all the conditions in Appendix are satisfied.

1. When $\sqrt{nh^l} \left( h_2^a + \sqrt{\log(n)/nh^2} \right) = o(1)$, it follows that
   \[ IPW-S \cong IPW-P \cong IPW-O. \]

2. When $\sqrt{nh^l} \left( h_1^a + \sqrt{\log(n)/nh^k} \right) = o(1)$, it follows that
   \[ IPW-N \cong IPW-P \cong IPW-O. \]
When \( \sqrt{nh} \left( h_s^2 + \sqrt{\log(n)/nh_s^2} \right) = o(1) \) and \( \sqrt{nh} \left( h_s^1 + \sqrt{\log(n)/nh_s^1} \right) = o(1) \), it follows that

\[
IPW - N \cong IPW - S \cong IPW - P \cong IPW - O.
\]

**Remark 6.** Corollary 2 implies that when the convergence rate of estimated propensity score is fast enough, the corresponding CATE estimator would be asymptotically equivalent to \( IPW - O \), which is based on true propensity score, even though the condition \( |Z \cap V^\top X| \) in Theorem 1 or \( |Z \cap \tilde{X}| \) in Theorem 3 is satisfied. In this sense, we can say that the convergence rate of estimated propensity score is dominant the role of affiliation of \( Z \) in the set of arguments of propensity score in comparing the asymptotic efficiencies among the CATE estimators. It is well known that the convergence rate of non-parametric estimator is possibly close to \( n^{-1/2} \) if the estimated function is very smooth and the higher kernel function is utilized, see Li and Racine (2007). Thus the conditions \( \sqrt{nh} \left( h_s^2 + \sqrt{\log(n)/nh_s^2} \right) = o(1) \) and \( \sqrt{nh} \left( h_s^1 + \sqrt{\log(n)/nh_s^1} \right) = o(1) \) could hold. As the choices for such kernel and bandwidths often make no sense for practical use, this investigation only serves as a theoretical exploration with a reminder that a “good” estimator for the propensity score may not be helpful for constructing a “good” CATE.

3. Simulation study

3.1. Preliminary of the simulation

To evaluate the finite sample performances of \( IPW - S \), we consider the comparisons with \( IPW - P \), \( IPW - N \) and \( IPW - O \). To save space, we only present the simulations in the case \( Z \in R \). To make the comparisons more convincing, we consider two scenarios with two low dimensions of \( X = (Z, U_1, \cdots, U_{k-1}) \) equal to \( k = 2 \) and \( 4 \), and higher dimensions \( k = 20 \). In the latter, \( IPW - N \) is not included as very high order kernel and very delicately selected bandwidths are required and then it is very difficult to implement. Several criteria are used to evaluate the estimation efficiency: Bias; estimated standard deviation \( \text{Est}_SD \); mean square error (MSE). As the asymptotic
distributions are standard normal, we then also report the proportions outside the critical values $\pm 1.645$. Further, to make the efficiency ranking in finite sample setting more visible, we report, as relative efficiency, the $Est_{SD}$ results via dividing each $Est_{SD}$ by $Est_{SD}$ of IPW-O that is used as the benchmark. Thus, when the ratio is smaller than 1, the corresponding estimator is more efficient than IPW-O.

3.2. Experiment 1 (low-dimensional setting)

In the low-dimensional setting, we consider the covariates $X = (Z, U_1, \cdots, U_{k-1})$ are given by the following procedure. When $k = 2$, $X = (Z, U_1)$ with $Z = \varepsilon_1$ and $U_1 = (1 + 2Z)^2(-1 + Z)^2 + \varepsilon_2$. When $k = 4$, $X = (Z, U_1, U_2, U_3)$ are given by $Z = \varepsilon_1$, $U_1 = (1 + 2Z) + \varepsilon_1$, $U_2 = (1 + 2Z) + \varepsilon_2$, $U_3 = (-1 + Z)^2 + \varepsilon_3$. $\varepsilon_i \sim \text{unif}[-0.5, 0.5]$ for $i = 1, 2, 3$, and they are mutually independent.

To easily compare the theoretical results under parametric, nonparametric and semiparametric structure, we consider four models:

- **Model 1** ($k=2, r=1$ with $|Z \cap X| = 1$ but $|Z \cap V^T X| = 0$):
  
  $Y(1) = \beta_1^T X + \gamma_1 Z U_1 + \nu$, $Y(0) = 0$ and $p_1(X) = \Lambda(1/\sqrt{2}(Z + U_1))$.

- **Model 2** ($k=2, r=1$ with $|Z \cap \bar{X}| = |Z \cap U_1| = 0$ and $|Z \cap V^T X| = 0$):
  
  $Y(1) = \beta_1^T X + \gamma_1 Z U_1 + \nu$, $Y(0) = 0$ and $p_2(X) = \Lambda(U_1)$.

- **Model 3** ($k=4, r=1$ with $|Z \cap X| = 1$ but $|Z \cap V^T X| = 0$):
  
  $Y(1) = \beta_2^T X + \gamma_2 Z U_1 U_2 U_3 + \nu$, $Y(0) = 0$, and $p_3(X) = \Lambda(0.5(Z + U_1 + U_2 + U_3))$.

- **Model 4** ($k=4, r=2$ with $|Z \cap X| = 1$ and $|Z \cap V^T X| = 1$):
  
  $Y(1) = \beta_2^T X + \gamma_2 Z U_1 U_2 U_3 + \nu$, $Y(0) = 0$ and $p_4(X) = \Lambda\left(\frac{\sqrt{3}(1 + Z)}{\sqrt{3} + U_1 + U_2 + U_3}\right)$.

Here $\nu \sim N(0, 0.25^2)$, $\Lambda(\cdot)$ is the c.d.f. of the logistic distribution. Given that the matrix $V$ satisfies $E(D|X) \perp X|V^T X$, we consider these four types of propensity score models to satisfy the conditions in different scenarios. Under Model 1 and
Obviously, when a nonlinear model is taken into account, IPW-N and  can be chosen via replacing  \( \tilde{V} \) in Model 3, the dimension of  \( V^T = (1, \cdots, 1)_{1 \times k} \) is \( \text{dim}(V) = r = 1 \), \( |Z \cap X| = 1 \), but \( |Z \cap V^T X| = 0 \). Thus, we aim to examine whether  IPW-N \( \preceq \) IPW-S \( \preceq \) IPW-P \( \preceq \) IPW-O. In order to examine the theoretical results in Case 3 of Corollary [1] in the finite sample scenario, we consider  \( p_2(X) \) in Model 2. In this setting,  \( D \perp X | \tilde{X} = U_1 \), and for  \( V^T = (0, 1) \),  \( p_2(X) \perp X | V^T X \). Obviously, \( |Z \cap \tilde{X}| = |Z \cap U_1| = 0 \) and \( |Z \cap V^T X| = 0 \) in Model 2, thus it can be used to examine whether  IPW-N \( \preceq \) IPW-S \( \preceq \) IPW-P \( \preceq \) IPW-O. \( p_4(X) \) in Model 4 is also set to verify the results in Corollary [1]. This propensity score function has  \( Z \) itself as an individual argument. Namely,  \( p_4(X) \perp X | V^T X \) with  \( V^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix} \) and \( \text{dim}(V) = r = 2 \), and \( |Z \cap X| = 1 \) and \( |Z \cap V^T X| = 1 \). We will examine whether  IPW-S can be more efficient than  IPW-P and IPW-O. As for the parameters  \( \beta_i, \gamma_i, i = 1, 2 \), we consider following scenarios:

- **Scenario I:** \( \beta_1^T = (0, 0), \gamma_1 = 1, \beta_2^T = (1/10, 1/\sqrt{2}, -1/\sqrt{2}, -1/10), \gamma_2 = 0; \)

- **Scenario II:** \( \beta_1^T = (1/2, -1/5), \gamma_1 = 0, \beta_2^T = (0, 0, 0, 0), \gamma_2 = 1. \)

Obviously, when  \( r_i = 0 \), the linear model is being considered, while  \( r_i = 1 \), the nonlinear model is taken into account,  \( i = 1, 2 \).

Next, we determine the order of kernels  \( L(\cdot), H(\cdot) \) and  \( K(\cdot) \) to guarantee the regularity condition 5 in Appendix. As there is no data-driven or optimal selection method available for  IPW-N and IPW-S, we use the rule of thumb to select them as suggested by Abrevaya et al. (2015) for fair comparisons. The principle of selection is based on proper rates of convergence in the form of  \( h = a \cdot n^{-\eta} \) for  \( a > 0 \) and  \( \eta > 0 \). Since IPW-S can be regarded as a low-dimensional type IPW-N, the bandwidths can be chosen via replacing  \( \tilde{k} \) by  \( r \) as follows:

\[
h_2 = a_2 n^{-1/(2r+\delta_r+\delta_2)}, \quad h = an^{-1/(t+4+2\delta_r+\delta_1)},
\]

where  \( a, a_2, \delta_2, \delta_1 \) are positive. Note that  \( \delta_2 \) and  \( \delta_1 \) can be as small as desired, thus we let them to be zero in the simulations for simplicity. Further, the order of  \( H \) is  \( s_2 = r + \delta_r \) with  \( \delta_r = 0 \) for even  \( r \),  \( \delta_r = 1 \) for odd  \( r \), and  \( s = s_2 + 2 = r + \delta_r + 2 \). Due to the semiparametric nature, we construct two consistent estimators  \( \hat{V} \) via MAVE.
proposed by (Xia et al., 2002; Xia, 2007), which can be implemented in the R package MAVE.

Further, to fairly examine the performances, the parameters \( s, h \) for \( K(u) \) are the same for all these four \( CATE \) estimators. Since \( r \leq \tilde{k} \), the choices of \( s \) and \( h \) for IPW-N can be used for all the \( CATE \) estimators. Taking all into account, the corresponding bandwidths are summarized in Table 1.

Table 1: The order of bandwidths in the simulations.

| \( \tilde{k} \) | \( h \) | \( h_1 \) | \( h_2 \) |
|---------------|----------|----------|----------|
| \( \tilde{k}=1 \) | \( a \cdot n^{-1/9} \) | \( a_1 \cdot n^{-1/3} \) | \( a_2 \cdot n^{-1/(2r+\delta_r)} \) |
| \( \tilde{k}=2 \) | \( a \cdot n^{-1/9} \) | \( a_1 \cdot n^{-1/4} \) | \( a_2 \cdot n^{-1/(2r+\delta_r)} \) |
| \( \tilde{k}=4 \) | \( a \cdot n^{-1/13} \) | \( a_1 \cdot n^{-1/8} \) | \( a_2 \cdot n^{-1/(2r+\delta_r)} \) |

As for the tuning parameters \( a, a_1, a_2 \), we consider the following two groups of values: Group 1: \( \{ a = 0.55, a_1 = 1.05, a_2 = 0.75 \} \), Group 2: \( \{ a = 0.55, a_1 = 1.05, a_2 = 0.69 \} \).

Specifically, we estimate \( \tau(Z) \) at \( Z \in \{-0.4, -0.2, 0, 0.2, 0.4\} \). The sample sizes are \( n = 500 \) and \( 1000 \). The replication time is 500. We choose the Gaussian kernel and higher order kernels derived from it throughout this section. Further, we should point out that the estimated propensity score is trimmed to lie in the interval \([0.005, 0.995]\) as Abrevaya et al. (2015) did. We give the observations from the simulation results reported in Tables 2-9. To save space, we only report the results about the relative efficiency with \( \{ a = 0.55, a_1 = 1.05, a_2 = 0.75 \} \) under Scenario I. See Figure 1. We have the following observations.

Observation 1. As expected, larger sample size leads to smaller bias and standard deviation in most cases. When \( k = 4 \) and the sample size goes from \( n = 500 \) up to \( n = 1000 \), the bias and variance reduction are more significant and the empirical values of \( P_{1.645} \) and \( P_{-1.645} \) are closer to the nominal level 0.05. That implies that the normal approximation works well.

Observation 2. When the dimension \( k \) increases to 4, \( Bias \), \( Est_{SD} \) and \( MSE \) also increase. Further, the dimension does have impact on the performance of IPW-
N. In Tables S.1 under Model 1 with \( k = 2 \), \( IPW-N \) is uniformly more efficient than all the others. But under the models with \( k = 4 \), especially with \( k = 4 \) and \( r = 2 \), the superiority of \( IPW-N \) becomes less significant. We can see that \( IPW-S \) can even be more efficient than \( IPW-N \) sometimes, mainly due to its dimension reduction structure.

**Observation 3.** Taking into account all the simulation results in Tables 2-9, the estimated standard deviation (\( Est\_SD \)) of all CATE estimations increase as \( z \) is close to the boundary of the support of \( Z \). This phenomenon should be mainly because of the nonparametric estimation of \( CATE(z) \) function with respect to \( Z \). Note that \( IPW-O \) also involves nonparametric estimation for the conditional expectation over \( Z \), thus the boundary effect also takes place for it. Further, empirically, the \( Est\_SD \) of \( IPW-O \) often increases, in the numerical studies we conduct, relatively more quickly than \( IPW-P \) or \( IPW-S \) in the cases we will discuss in Observation 4 below when \( z \) is close to the boundary. Figure 1 about the relative efficiency compared with \( IPW-O \) shows this though for different models, at the boundary, \( IPW-O \) has different relative efficiency with \( IPW-P \). Combining the information that ATE with estimated unknowns, the final estimators could be more efficient in general, less relative efficiency of \( IPW-O \) in finite sample scenarios may be understandable although in the case that the asymptotic efficiency should be equivalent. On the other hand, when we look at the original values of \( IPW-O \), the differences with \( IPW-P \) is not significant.

**Observation 4.** We also check the effect caused by the inclusiveness of the given covariates \( Z \) in the set of the arguments of the propensity score for \( IPW-S \) and \( IPW-N \). Under Model 1 or Model 3 with \( |Z \cap \tilde{X}| = 1 \) but \( |Z \cap V^\top X| = 0 \), Figure 1 shows that the \( Est\_SD \) of \( IPW-N \) is uniformly smaller than those of the other CATE estimators. While, \( IPW-S \) and \( IPW-P \) have similar performance. This coincides with the theory. In contrast, under Model 2 where \( |Z \cap \tilde{X}| = |Z \cap U_1| = 0 \), \( IPW-N \) loses its superiority of efficiency to share similar performance to \( IPW-S \) and \( IPW-P \). Under Model 4 where \( |Z \cap X| = 1 \) and \( |Z \cap V^\top X| = 1 \), \( IPW-S \) outperforms \( IPW-O \) and \( IPW-P \), and can even be comparable with \( IPW-N \) sometimes. These results also coincide with the theory in Corollary 1.
3.3. Experiment 2 (high dimensional setting).

Consider models with much higher dimensional $X$: $k = \text{dim}(X) = 20$. As $IPW-N$ obviously suffers from the curse of dimensionality and thus does not work at all, we then only focus on $IPW-O$, $IPW-P$ and $IPW-S$. To better examine the corresponding finite sample performances, we consider the model settings which are similar to Models 3 and 4 with uniformed $Z$, but with more zero coefficients for ease of comparison.

Given $X = (Z, U_1, \cdots, U_{k-1})$, $X$ is generated by $Z \sim \text{unif}(-0.5, 0.5)$, $U_1 = (1 + 2Z) + e_1$, $U_2 = (1 + 2Z) + e_2$, $U_3 = (1 + Z)^2 + e_3$, and independent $e_j \sim \text{unif}(-0.5, 0.5)$, for $j = 1, 2, 3$. The other variables $U'_j$s are generated as: when $3 < j \leq 9$, $U_j = |1 + 1/(11 - j)Z| - |1 + 1/j\epsilon|$; when $9 < j \leq 19$, $U_j = |1 + 1/(21 - j)Z| - |1 + 1/j\epsilon|$; and $U_j = |1 + 1/(31 - j)Z| - |1 + 1/j\epsilon|$ for
\[ j > 19, \epsilon \sim \text{unif}(-0.5, 0.5). \] We consider the following models in high dimensional setting.

- **Model 5** (\( r=1 \) with uniformed \(|Z \cap V^\top X| = 0\)):
  \[ Y(1) = \beta_3^\top X + \gamma_3 ZU_1U_2U_3 + \nu, \quad Y(0) = 0 \text{ and } p_5(X) = \Lambda(1 + V_3^\top X). \]

- **Model 6** (\( r=2 \) with uniformed \(|Z \cap V^\top X| = 1\)):
  \[ Y(1) = \beta_3^\top X + \gamma_3 ZU_1U_2U_3 + \nu, \quad Y(0) = 0, \quad \text{and } p_6(X) = \Lambda(g(\tilde{V}_3^\top X)). \]

As for the propensity score, we set
\[ V_3^\top = \left( -1, \cdots, -1, 0, \cdots, 0, 4, \cdots, 4, 5, \cdots, 5, 10, \cdots, 10 \right) / \sqrt{20}, \tilde{\alpha} = \left( 0, \cdots, 0, -1, \cdots, -1, 0, \cdots, 0, 1, \cdots, 1 \right) / \sqrt{19}, \]
and \( g(\tilde{V}_3^\top X) = (1 + \tilde{\alpha}^\top X)/(1 + Z) \) with \( \text{dim}(\tilde{V}_3^\top X) = r = 2, \) while \(|Z \cap V_3^\top X| = 1\).

In high dimensional setting, we only consider the nonlinear model where the parameters are set as \( \beta_3^\top = (0, \ldots, 0) \) and \( r_3 = 1. \)

The sample size is taken to be \( n = 500. \) Estimate \( \tau(Z) \) at \( Z \in \{-0.4, -0.2, 0, 0.2, 0.4\} \) with 500 simulation realizations. As for the bandwidth choice we adopt the same rule in (13) of Experiment 1 to have \( h = an^{-1/(l+4+2r+2\delta_r)} \) and \( h_2 = a_2n^{-1/(2r+\delta_r)}. \) Consider two groups of \( \{a, a_2\}, Group~1: \{a = 0.55, a_2 = 0.75\} \) and \( Group~2: \{a = 0.55, a_2 = 0.69\}. \) For the kernel function in the estimated propensity score, we also use the Gaussian kernel and higher order kernels derived from it since the distribution of \( X \) is bounded. All the original simulation results are reported in Table S.5 in the Supplement and the relative efficiency results are plotted in Figure 2.

From the simulation results, we also have the following findings.

1. The high dimensionality of \( X \) has relatively weak influence on \( IPW-S. \) All the values of \( Bias, Est_{\text{SD}} \) and \( MSE \) are rather stable and the values of \( P_{\pm 1.645} \) are closer to the nominal value 0.05 as the dimension of \( X \) goes from 4 to 20, especially in the case of \(|Z \cap V^\top X| = 1\). This is very informative because it implies that \( IPW-S \) can greatly avoid the curse of dimensionality due to its dimension reduction structure.

2. \( IPW-S \) not only shows its superiority in dealing with the curse of dimensionality,
but also inherits the efficiency superiority of IPW-N in low-dimensional cases. Under such high dimensional scenarios, the values of EstSD and MSE of IPW-S are smaller than those of the parametric competitors in some cases even \( |Z \cap V^\top X| = 0 \). When in Model 6, IPW-S is uniformly more efficient than IPW-P. This is consistent with the theoretical results in Corollary\(^\text{1}\) as in the model \(|Z \subseteq V^\top X| = 1\), IPW-S is asymptotically more efficient than IPW-P.

![Graph showing ARE comparison between IPW-S and IPW-P](image)

**Figure 2**: The asymptotic relative efficiency (ARE) about EstSD against that of IPW-O under high dimensional setting.

### 4. Data Analysis

In this section, we consider a dataset collected by Ichino et al. (2008), which can be obtained from the internet.\(^2\) We apply the proposed method to estimate the CATE

---

\(^2\) The data is publicly available at [http://qed.econ.queensu.ca/iae/2008-v23.3/ichino-maalli-nannicini/](http://qed.econ.queensu.ca/iae/2008-v23.3/ichino-maalli-nannicini/)
function to investigate the treatment effect of temporary work assignment (TWA) on permanent employment over worker’s age.

First introduce some details and setting about the dataset. Restricting the sample to Tuscany and aged 17-39, the resulting sample size is \( n = 901, 281 \) of which were on a TWA during the first semester of 2001. That is, the binary treatment variable \( D = 0, 1 \) means that the individual was not on or was on a TWA during the first six mouths of 2001. The outcome \( Y \) here is a dummy variable: \( Y = 1 \) if the subject is permanently employed at the end of 2002, and \( Y = 0 \) otherwise. Choose \( X_1 \) as the worker’s age and a set of 25 covariates as \( X \) adopted by Ichino et al. (2008) to guarantee the unconfoundedness assumption. The set of covariates is about demographic characteristics, family background, educational achievements and work experience (See Table 1 in Ichino et al. (2008)). This dataset was first analyzed by Ichino et al. (2008), who estimated the parameter \( ATT = E(Y_1 - Y_0 \mid D) \) and showed that TWA can increase the probability of getting a permanent employment. Ichino et al. (2008) pointed out that the TWA effect is heterogeneous for the individuals older than 30 and younger than 30.

In order to catch more specific heterogeneity of the TWA effect across individuals’ age, we estimate the \( CATE \) function \( \tau(Z) \) in the interval between ages 20 and 35. As the number of covariates is large (=30), we then use a semiparametric single-index model to estimate the propensity score such that the dimensionality problem and model misspecification problem can be greatly alleviated. Given that \( D \perp X \mid \alpha^T X \), we can get the \( IPW-S \) and pointwise confidence band of \( \tau(Z) \) by carrying out the estimation procedure proposed in subsection 2.3. As for nonparametric estimation part, we use the Gaussian kernel and choose the bandwidths to be \( h = 0.85 \times \hat{\sigma}_1 n^{-1/9} = 2.22 \), and \( h_2 = 1.15 \times \hat{\sigma}_d n^{-1/3} = 0.04 \ll h \), where \( \hat{\sigma}_1 = \sqrt{\text{var}(x_1)} \) and \( \hat{\sigma}_d = \sqrt{\text{var}(\alpha^T X)} \). We also estimate \( IPW-P \) as a benchmark to analyse the TWA effect over worker’s age.

Figure 3 presents the results of \( IPW-S \) and \( IPW-P \) as a function of worker’s age in the range of 20 to 35 years old, which can be regarded as an extension of Ichino et al. (2008) in a certain sense. Furthermore, the 95% pointwise confidence band of \( IPW-S \) and \( IPW-P \) have been also reported in Figure 3. There are several points we want
to highlight: 1). *both IPW-S and IPW-P suggest that, from age 20 to 35, a TWA assignment uniformly increases the probability of finding a stable job with the range roughly between 0.05 and 0.35. It means that if a worker with a TWA experience would more likely to get a permanent job. This finding is in accordance with, but extends the conclusion of Ichino et al. (2008). 2). The trend of $CATE(x_1)$ varies with worker’s age and has two peaks. From Figure 3 we can also find that there are two peaks at around age 24 and age 32, while the trough appears at around age 29. That implies the TWA experience has different effect for the workers older than 29 and under 29, which was also similarly discussed by Ichino et al. (2008). However, comparing the details in the curves of $IPW-S$ and $IPW-P$, the effect of TWA on finding a stable job for the subpopulation aged under 29 is greater than the ones older 29 in the $IPW-S$ curve, while things are opposite in the $IPW-P$ curve. It seems that the $IPW-S$ curve provides a more reasonable explanation on the effect of TWA: younger individuals receiving TWA could have better chance to get a stable job than older individuals who need to receive TWA.

Figure 3: The curves of conditional average treatment effects (CATE) over worker’s age with the 95% pointwise confidence band.
5. Conclusion

In this paper, we propose an estimation (IPW-S) of conditional average treatment effect with semiparametric propensity score and investigate its asymptotic properties which can be used to construct pointwise confidence intervals. We give a relatively complete picture about the asymptotic efficiency of different estimators with nonparametric, parametric and true propensity score when model is correctly specified. Further, when the dimension of covariates is high, by the numerical studies, we demonstrate the advantages of IPW-S in alleviating the curse of dimensionality and inheriting the theoretical superiority of IPW-N in estimation efficiency. But a challenging topic is how to develop a good uniform confidence band of the whole function $\tau(z)$ although the Bonferroni confidence band could be applied. Further, a research topic is about the situation that not all of the covariates are important for propensity score. Thus, by incorporating variable selection, we can simultaneously identify important confounders and guarantee the unconfoundedness assumption. The dimension reduction and variable selection have been investigated by, say, Ma et al. (2019) for the model under sparsity structure. This topic is also related to variable selection and thus we will try to have a computationally inexpensive algorithm for this purpose and study its asymptotic behaviours. Another topic is about the model misspecification even when the semiparametric model is used. We will study the relevant asymptotic behaviours in the near future.

Acknowledgement

The authors’ research was supported by grants from NSFC grants (NSFC11671042, NSFC11601227) and the University Grants Council of Hong Kong.

Supplementary material

The supplementary file covers the detailed proofs to Theorems and Corollaries.
Appendix: Technical conditions

The following regularity conditions are required to get the theoretical results.

(C1) (Strong ignorability)

(i) Unconfoundedness: \((Y(0), Y(1)) \perp D \mid X\).

(ii) Common support: For some very small \(c > 0, c < p(X) < 1 - c\).

(C2) (on distribution):

(i) The set \(\chi\) that is the support of the \(k\)-dimensional covariate vector \(X\) is a Cartesian product of compact intervals.

(ii) The density function of \(Z, f_1(Z)\), and the density function of \(X\), are bounded away from zero and infinity and \(s \geq r\) times continuously differentiable.

(C3) (Conditional moments and smoothness)

(i) \(\sup_{x \in \chi} E[Y(j)^2 \mid X = x] < \infty\) for \(j = 0, 1\);

(ii) the functions \(m_j(V^T X) = E[Y(j) \mid V^T X], j = 0, 1\) are \(s \geq r\) times continuously differentiable.

(C4) (on kernel function)

(i) \(L(u)\) is a kernel of order \(s_1\), is symmetric around zero, has finite support \([-1, 1]^k\), and is continuously differentiable.

(ii) \(H(u)\) is a kernel of order \(s_2\), is symmetric around zero, has finite support \([-1, 1]^r\), and is continuously differentiable.

(iii) \(K(u)\) is of order \(s\), is symmetric around zero, and is \(s\) times continuously differentiable.

(C5) (on bandwidths)

(i) \(h \to 0, nh^l \to \infty, nh^{2s+l} \to 0\).

(ii) \(h_1, h_2 \to 0, \log(n)/(nh_2^{r+s_2}) \to 0, \) and \(\log(n)/(nh_1^{k+s_1}) \to 0\).

(iii) \(h_i^{2s_i}h^{-2s_i-l} \to 0, nh_i^{-2s_i} \to 0, i = 1, 2\).
(C6) (on dimension reduction structure) the dimension of \( V, r \), is given and \( \hat{V} - V = O_p(n^{-1/2}) \).

Recall the definition of high order kernel in the literature. We say a function \( g: \mathbb{R}^r \rightarrow \mathbb{R} \) is a kernel of order \( s \) if it integrates to one over \( \mathbb{R}^s \), and \( \int u^{p_1} \cdots u^{p_r} g(u) du = 0 \) for all nonnegative integers \( p_1, \cdots, p_r \) such that \( 1 \leq \sum_i p_i < s \), and it is nonzero when \( \sum_i p_i = s \).

References

P. R. Rosenbaum, D. B. Rubin, The central role of the propensity score in observational studies for causal effects, Biometrika 70 (1983) 41–55.

K. Hirano, G. W. Imbens, G. Ridder, Efficient estimation of average treatment effects using the estimated propensity score, Econometrica 71 (2003) 1161–1189.

R. K. Crump, V. J. Hotz, G. W. Imbens, O. A. Mitnik, Nonparametric tests for treatment effect heterogeneity, Rev. Econom. Statist. 90 (2008) 389–405.

S. Wager, S. Athey, Estimation and inference of heterogeneous treatment effects using random forests, J. Amer. Statist. Assoc. 113 (2018) 1228–1242.

J. Abrevaya, Y.-C. Hsu, R. P. Lieli, Estimating conditional average treatment effects, J. Bus. Econom. Statist. 33 (2015) 485–505.

S. Lee, R. Okui, Y.-J. Whang, Doubly robust uniform confidence band for the conditional average treatment effect function, J. Appl. Econometries 32 (2017) 1207–1225.

J. M. Robins, A. Rotnitzky, L. P. Zhao, Estimation of regression coefficients when some regressors are not always observed, J. Amer. Statist. Assoc. 89 (1994) 846–866.

K.-C. Li, Sliced inverse regression for dimension reduction, J. Amer. Statist. Assoc. 86 (1991) 316–327.
W. Luo, Y. Zhu, D. Ghosh, On estimating regression-based causal effects using sufficient dimension reduction, Biometrika 104 (2017) 51–65.

S. Ma, L. Zhu, Z. Zhang, C.-L. Tsai, R. J. Carroll, A robust and efficient approach to causal inference based on sparse sufficient dimension reduction, Ann. Statist. 47 (2019) 1505–1535.

R. D. Cook, B. Li, Dimension reduction for conditional mean in regression, Ann. Statist. 30 (2002) 455–474.

Y. Xia, H. Tong, W. K. Li, L.-X. Zhu, An adaptive estimation of dimension reduction space, J. R. Stat. Soc. Ser. B Stat. Methodol. 64 (2002) 363–410.

Y. Xia, A constructive approach to the estimation of dimension reduction directions, Ann. Statist. 35 (2007) 2654–2690.

Q. Li, J. S. Racine, Nonparametric econometrics, Princeton University Press, Princeton, NJ, 2007. Theory and practice.

A. Ichino, F. Mealli, T. Nannicini, From temporary help jobs to permanent employment: what can we learn from matching estimators and their sensitivity?, J. Appl. Econometrics 23 (2008) 305–327.
Table 2: The simulation results under Model 1 with $|Z \cap X| = 1$ but $|Z \cap V^T X| = 0$, $\beta_1 = (0, 0)$ and $\gamma_1 = 1$

| n  | Z   | -0.4 | -0.2 | 0    | 0.2  | 0.4  |
|-----|-----|------|------|------|------|------|
| 500 | Bias | -0.0262 | 0.0607 | 0.0553 | -0.0278 | -0.1072 |
|     | $E_{est}D$ | 0.0269 | 0.0199 | 0.0190 | 0.0215 | 0.0271 |
|     | MSE  | 0.0014 | 0.0041 | 0.0034 | 0.0012 | 0.0122 |
|     | $P_{-1,4}D$ | 0.0540 | 0.0440 | 0.0580 | 0.0560 | 0.0560 |
|     | $P_{0,5}D$ | 0.0480 | 0.0460 | 0.0400 | 0.0360 | 0.0540 |
| 1000 | Bias | -0.0254 | 0.0551 | 0.0499 | -0.0186 | -0.0888 |
|      | $E_{est}D$ | 0.0230 | 0.0145 | 0.0134 | 0.0154 | 0.0192 |
|      | MSE  | 0.0010 | 0.0032 | 0.0026 | 0.0007 | 0.0093 |
|      | $P_{-1,4}D$ | 0.0480 | 0.0540 | 0.0500 | 0.0640 | 0.0440 |
|      | $P_{0,5}D$ | 0.0540 | 0.0560 | 0.0420 | 0.0540 | 0.0580 |
| 500 | Bias | -0.0263 | 0.0608 | 0.0566 | -0.0233 | -0.0991 |
|     | $E_{est}D$ | 0.0254 | 0.0195 | 0.0190 | 0.0201 | 0.0236 |
|     | MSE  | 0.0013 | 0.0041 | 0.0036 | 0.0009 | 0.0104 |
|     | $P_{-1,4}D$ | 0.0480 | 0.0540 | 0.0500 | 0.0640 | 0.0440 |
|     | $P_{0,5}D$ | 0.0540 | 0.0560 | 0.0420 | 0.0540 | 0.0580 |
| 1000 | Bias | -0.0254 | 0.0551 | 0.0499 | -0.0186 | -0.0888 |
|      | $E_{est}D$ | 0.0230 | 0.0145 | 0.0134 | 0.0154 | 0.0192 |
|      | MSE  | 0.0010 | 0.0032 | 0.0027 | 0.0005 | 0.0082 |
|      | $P_{-1,4}D$ | 0.0480 | 0.0540 | 0.0500 | 0.0640 | 0.0440 |
|      | $P_{0,5}D$ | 0.0540 | 0.0560 | 0.0420 | 0.0540 | 0.0580 |

$IPW-O$ $IPW-P$ $IPW-N$ $IPW-S$
Table 3: The simulation results under Model 1 with $|Z \cap U| = 0$ and $|Z \cap V^T X| = 0$, $\beta_1^T = (1/2, -1/5)$ and $\gamma_1 = 0$

| $k=2$, $r=1$ under Group 1: $\{a_i = 0.55, a_2 = 0.75\}$ |
|---|---|---|---|---|---|---|---|
| $n$ | $Z$ | -0.4 | -0.2 | 0 | 0.2 | 0.4 |
| 500 | IPW - O | 0.0027 | -0.0006 | 0.0174 | 0.0147 | 0.0504 |
| | IPW - P | 0.0021 | -0.0010 | 0.0173 | 0.0148 | 0.0503 |
| | MSE | 0.0009 | 0.0001 | 0.0006 | 0.0006 | 0.0030 |
| | 0.0006 | 0.0000 | 0.0005 | 0.0005 | 0.0030 |
| | $P_{-1,645}$ | 0.0540 | 0.0500 | 0.0420 | 0.0500 | 0.0420 |
| | 0.0460 | 0.0520 | 0.0420 | 0.0540 | 0.0440 |
| | $P_{0,645}$ | 0.0440 | 0.0440 | 0.0440 | 0.0480 | 0.0560 |
| | 0.0480 | 0.0480 | 0.0480 | 0.0560 | 0.0560 |
| 1000 | IPW - O | 0.0297 | 0.0215 | 0.0185 | 0.0187 | 0.0211 |
| | IPW - P | 0.0191 | 0.0141 | 0.0123 | 0.0127 | 0.0147 |
| | MSE | 0.0005 | 0.0006 | 0.0006 | 0.0006 | 0.0303 |
| | 0.0002 | 0.0002 | 0.0002 | 0.0002 | 0.0002 |
| | $P_{-1,645}$ | 0.0500 | 0.0460 | 0.0400 | 0.0600 | 0.0440 |
| | 0.0440 | 0.0480 | 0.0440 | 0.0560 | 0.0460 |
| | $P_{0,645}$ | 0.0460 | 0.0340 | 0.0460 | 0.0460 | 0.0600 |
| | 0.0440 | 0.0440 | 0.0500 | 0.0500 | 0.0560 |

| $k=2$, $r=1$ under Group 2: $\{a_i = 0.55, a_2 = 0.69\}$ |
|---|---|---|---|---|---|---|---|
| $n$ | $Z$ | -0.4 | -0.2 | 0 | 0.2 | 0.4 |
| 500 | IPW - O | 0.0029 | -0.0005 | 0.0155 | 0.0153 | 0.0448 |
| | IPW - P | 0.0029 | -0.0005 | 0.0156 | 0.0153 | 0.0448 |
| | MSE | 0.0225 | 0.0162 | 0.0135 | 0.0131 | 0.0146 |
| | 0.0192 | 0.0140 | 0.0125 | 0.0127 | 0.0146 |
| | $E_{S,t}\hat{S}$ | 0.0005 | 0.0003 | 0.0004 | 0.0004 | 0.0022 |
| | 0.0004 | 0.0002 | 0.0004 | 0.0004 | 0.0022 |
| | $P_{-1,645}$ | 0.0500 | 0.0460 | 0.0400 | 0.0600 | 0.0440 |
| | 0.0440 | 0.0480 | 0.0440 | 0.0560 | 0.0460 |
| | $P_{0,645}$ | 0.0460 | 0.0340 | 0.0460 | 0.0460 | 0.0600 |
| | 0.0440 | 0.0440 | 0.0500 | 0.0500 | 0.0560 |
| 1000 | IPW - O | 0.0297 | 0.0215 | 0.0185 | 0.0187 | 0.0211 |
| | IPW - P | 0.0191 | 0.0141 | 0.0123 | 0.0127 | 0.0147 |
| | MSE | 0.0005 | 0.0006 | 0.0006 | 0.0006 | 0.0303 |
| | 0.0002 | 0.0002 | 0.0002 | 0.0002 | 0.0002 |
| | $E_{S,t}\hat{S}$ | 0.0500 | 0.0460 | 0.0400 | 0.0600 | 0.0440 |
| | 0.0420 | 0.0480 | 0.0460 | 0.0560 | 0.0460 |
| | $P_{0,645}$ | 0.0500 | 0.0460 | 0.0400 | 0.0600 | 0.0440 |
| | 0.0420 | 0.0480 | 0.0460 | 0.0560 | 0.0460 |
| | $P_{0,645}$ | 0.0460 | 0.0340 | 0.0460 | 0.0460 | 0.0600 |
| | 0.0440 | 0.0440 | 0.0500 | 0.0500 | 0.0560 |

| 500 | IPW - O | 0.0297 | 0.0215 | 0.0185 | 0.0187 | 0.0211 |
| | IPW - P | 0.0191 | 0.0141 | 0.0123 | 0.0127 | 0.0147 |
| | MSE | 0.0005 | 0.0006 | 0.0006 | 0.0006 | 0.0303 |
| | 0.0002 | 0.0002 | 0.0002 | 0.0002 | 0.0002 |
| | $E_{S,t}\hat{S}$ | 0.0500 | 0.0460 | 0.0400 | 0.0600 | 0.0440 |
| | 0.0420 | 0.0480 | 0.0460 | 0.0560 | 0.0460 |
| | $P_{0,645}$ | 0.0500 | 0.0460 | 0.0400 | 0.0600 | 0.0440 |
| | 0.0420 | 0.0480 | 0.0460 | 0.0560 | 0.0460 |

| 1000 | IPW - O | 0.0297 | 0.0215 | 0.0185 | 0.0187 | 0.0211 |
| | IPW - P | 0.0191 | 0.0141 | 0.0123 | 0.0127 | 0.0147 |
| | MSE | 0.0005 | 0.0006 | 0.0006 | 0.0006 | 0.0303 |
| | 0.0002 | 0.0002 | 0.0002 | 0.0002 | 0.0002 |
| | $E_{S,t}\hat{S}$ | 0.0500 | 0.0460 | 0.0400 | 0.0600 | 0.0440 |
| | 0.0420 | 0.0480 | 0.0460 | 0.0560 | 0.0460 |
| | $P_{0,645}$ | 0.0500 | 0.0460 | 0.0400 | 0.0600 | 0.0440 |
| | 0.0420 | 0.0480 | 0.0460 | 0.0560 | 0.0460 |

31
Table 4: The simulation results under Model 2 with $|Z \cap U| = 0$ and $|Z \cap V^T X| = 0$, $\beta_1^T = (0, 0)$ and $\gamma_1 = 1$

| n   | $\gamma$ | $\beta_1$ | $\beta_2$ | $\gamma_1$ |
|-----|----------|-----------|-----------|------------|
| 500 | 0.55     | 1         | 0.02      | 0          |
|     | Z        | IPW-O     | IPW-P     | IPW-S      |
|     |          | Bias      | MSE       | MSE        |
|     |          | -0.0269   | 0.0060    | 0.0550     |
|     | $E_{IPW_D}$ | 0.0255   | 0.0192    | 0.0186     |
|     | MSE       | 0.0014    | 0.0040    | 0.0034     |
|     | $P_{1.465}$ | 0.0520   | 0.0480    | 0.0520     |
|     | $P_{1.465}$ | 0.0520   | 0.0480    | 0.0520     |
| Z   | -0.4     | -0.2      | 0.2       | 0.4       |
| MSE | 0.0269   | 0.0060    | 0.0551    | 0.0278     |
| MSE | 0.0252   | 0.0190    | 0.0186    | 0.0199     |
| MSE | 0.0014   | 0.0040    | 0.0034    | 0.0121     |
| MSE | 0.0520   | 0.0480    | 0.0540    | 0.0440     |
| MSE | 0.0520   | 0.0480    | 0.0540    | 0.0440     |
| MSE | 0.0440   | 0.0520    | 0.0480    | 0.0420     |
| MSE | 0.0440   | 0.0520    | 0.0480    | 0.0420     |
| MSE | 0.0269   | 0.0060    | 0.0551    | 0.0278     |
| MSE | 0.0252   | 0.0190    | 0.0186    | 0.0199     |
| MSE | 0.0014   | 0.0040    | 0.0034    | 0.0121     |
| MSE | 0.0520   | 0.0480    | 0.0540    | 0.0440     |
| MSE | 0.0520   | 0.0480    | 0.0540    | 0.0440     |
| MSE | 0.0440   | 0.0520    | 0.0480    | 0.0420     |
| MSE | 0.0269   | 0.0060    | 0.0551    | 0.0278     |
| MSE | 0.0252   | 0.0190    | 0.0186    | 0.0199     |
| MSE | 0.0014   | 0.0040    | 0.0034    | 0.0121     |
| MSE | 0.0520   | 0.0480    | 0.0540    | 0.0440     |
| MSE | 0.0520   | 0.0480    | 0.0540    | 0.0440     |
| MSE | 0.0440   | 0.0520    | 0.0480    | 0.0420     |

| 1000 | 0.55     | 1         | 0.02      | 0          |
|      | Z        | IPW-O     | IPW-P     | IPW-S      |
|      |          | Bias      | MSE       | MSE        |
|      |          | -0.0260   | 0.0543    | 0.0489     |
|      | $E_{IPW_D}$ | 0.0186   | 0.0313    | 0.0127     |
|      | MSE       | 0.0010    | 0.0031    | 0.0026     |
|      | $P_{1.465}$ | 0.0560   | 0.0580    | 0.0540     |
|      | $P_{1.465}$ | 0.0560   | 0.0580    | 0.0540     |
| Z   | -0.4     | -0.2      | 0.2       | 0.4       |
| MSE | 0.0260   | 0.0543    | 0.0489    | 0.0216     |
| MSE | 0.0187   | 0.0313    | 0.0127    | 0.0011     |
| MSE | 0.0011   | 0.0031    | 0.0026    | 0.0006     |
| MSE | 0.0520   | 0.0480    | 0.0420    | 0.0040     |
| MSE | 0.0520   | 0.0480    | 0.0420    | 0.0040     |
| MSE | 0.0011   | 0.0031    | 0.0026    | 0.0006     |
| MSE | 0.0520   | 0.0480    | 0.0420    | 0.0040     |
| MSE | 0.0520   | 0.0480    | 0.0420    | 0.0040     |
| MSE | 0.0011   | 0.0031    | 0.0026    | 0.0006     |
| MSE | 0.0520   | 0.0480    | 0.0420    | 0.0040     |

Table continued...
Table 5: The simulation results under Model 2 with $|Z \cap U| = 0$ and $|Z \cap V^T X| = 0$, $\beta_1^T = (1/2, -1/5)$ and $\gamma_1 = 0$

| n  | Z   | -0.4 | -0.2 | 0   | 0.2 | 0.4 | -0.4 | -0.2 | 0   | 0.2 | 0.4 |
|-----|-----|------|------|-----|-----|-----|------|------|-----|-----|-----|
|     |     |      |      |     |     |     |      |      |     |     |     |
|     |     |      |      |     |     |     |      |      |     |     |     |
|     |     |      |      |     |     |     |      |      |     |     |     |
|     | Bias| 0.0015 | -0.0014 | 0.0172 | 0.0148 | -0.0504 | 0.0171 | 0.0147 | -0.0504 |
|     | MSE | 0.0007 | 0.0004 | 0.0001 | 0.0005 | 0.0003 | 0.0005 | 0.0006 | 0.0005 |
|     | $P_{-1.645}$ | 0.0580 | 0.0460 | 0.0520 | 0.0540 | 0.0460 | 0.0580 | 0.0460 | 0.0580 |
|     | $P_{1.645}$ | 0.0480 | 0.0460 | 0.0540 | 0.0440 | 0.0480 | 0.0580 | 0.0460 | 0.0580 |
| 500 |     |      |      |     |     |     |      |      |     |     |     |
|     | Bias| 0.0010 | -0.0022 | 0.0161 | 0.0137 | -0.0513 | 0.0154 | 0.0133 | -0.0515 |
|     | MSE | 0.0238 | 0.0184 | 0.0170 | 0.0180 | 0.0210 | 0.0181 | 0.0181 | 0.0209 |
|     | $E_{S_{q,D}}$ | 0.0006 | 0.0003 | 0.0005 | 0.0005 | 0.0031 | 0.0006 | 0.0003 | 0.0005 |
|     | $P_{-1.645}$ | 0.0580 | 0.0460 | 0.0500 | 0.0540 | 0.0440 | 0.0590 | 0.0460 | 0.0590 |
|     | $P_{1.645}$ | 0.0500 | 0.0500 | 0.0480 | 0.0460 | 0.0500 | 0.0500 | 0.0460 | 0.0500 |
| 1000| Z   | -0.4 | -0.2 | 0   | 0.2 | 0.4 | -0.4 | -0.2 | 0   | 0.2 | 0.4 |
|     |     |      |      |     |     |     |      |      |     |     |     |
|     |     |      |      |     |     |     |      |      |     |     |     |
|     |     |      |      |     |     |     |      |      |     |     |     |
|     | Bias| 0.0025 | -0.0010 | 0.0153 | 0.0152 | -0.0449 | 0.0153 | 0.0153 | -0.0448 |
|     | MSE | 0.0201 | 0.0142 | 0.0124 | 0.0126 | 0.0144 | 0.0128 | 0.0128 | 0.0145 |
|     | $E_{S_{q,D}}$ | 0.0004 | 0.0002 | 0.0004 | 0.0004 | 0.0023 | 0.0002 | 0.0002 | 0.0023 |
|     | $P_{-1.645}$ | 0.0420 | 0.0500 | 0.0460 | 0.0520 | 0.0500 | 0.0460 | 0.0520 | 0.0500 |
|     | $P_{1.645}$ | 0.0560 | 0.0440 | 0.0360 | 0.0500 | 0.0560 | 0.0440 | 0.0360 | 0.0560 |
| 500 | Z   | -0.4 | -0.2 | 0   | 0.2 | 0.4 | -0.4 | -0.2 | 0   | 0.2 | 0.4 |
|     |     |      |      |     |     |     |      |      |     |     |     |
|     |     |      |      |     |     |     |      |      |     |     |     |
|     |     |      |      |     |     |     |      |      |     |     |     |
|     | Bias| 0.0010 | -0.0015 | 0.0148 | 0.0147 | -0.0453 | 0.0142 | 0.0144 | -0.0454 |
|     | MSE | 0.0180 | 0.0131 | 0.0120 | 0.0125 | 0.0144 | 0.0132 | 0.0132 | 0.0146 |
|     | $E_{S_{q,D}}$ | 0.0003 | 0.0002 | 0.0004 | 0.0004 | 0.0023 | 0.0002 | 0.0002 | 0.0023 |
|     | $P_{-1.645}$ | 0.0440 | 0.0520 | 0.0460 | 0.0500 | 0.0440 | 0.0460 | 0.0500 | 0.0440 |
|     | $P_{1.645}$ | 0.0460 | 0.0480 | 0.0440 | 0.0540 | 0.0480 | 0.0460 | 0.0540 | 0.0480 |
| 1000| Z   | -0.4 | -0.2 | 0   | 0.2 | 0.4 | -0.4 | -0.2 | 0   | 0.2 | 0.4 |
|     |     |      |      |     |     |     |      |      |     |     |     |
|     |     |      |      |     |     |     |      |      |     |     |     |
|     |     |      |      |     |     |     |      |      |     |     |     |
|     | Bias| 0.0025 | -0.0010 | 0.0153 | 0.0152 | -0.0449 | 0.0153 | 0.0153 | -0.0448 |
|     | MSE | 0.0201 | 0.0142 | 0.0124 | 0.0126 | 0.0144 | 0.0128 | 0.0128 | 0.0145 |
|     | $E_{S_{q,D}}$ | 0.0004 | 0.0002 | 0.0004 | 0.0004 | 0.0023 | 0.0002 | 0.0002 | 0.0023 |
|     | $P_{-1.645}$ | 0.0420 | 0.0500 | 0.0460 | 0.0520 | 0.0500 | 0.0460 | 0.0520 | 0.0500 |
|     | $P_{1.645}$ | 0.0560 | 0.0440 | 0.0360 | 0.0500 | 0.0560 | 0.0440 | 0.0360 | 0.0560 |
Table 6: The simulation results under Model 3 with $|Z \cap X| = 1$ but $|Z \cap Y^TX| = 0$, $\beta_2^T = (1/10, 1/\sqrt{2}, -1/\sqrt{3}, -1/10)$ and $\gamma_2 = 0$

| $k=4, r=1$ under Group 1: $(a = 0.05, a_1 = 1.05, a_2 = 0.75)$ | $k=4, r=1$ under Group 2: $(a = 0.05, a_1 = 1.05, a_2 = 0.75)$ |
|---|---|
| $n$ | $Z$ | $0$ | $0.2$ | $0.4$ | $0$ | $0.2$ | $0.4$ |
| $IPW-O$ | $IPW-P$ | $IPW-N$ | $IPW-S$ |
| Bias | $0.0271$ | $0.0610$ | $0.0556$ | $-0.0280$ | $-0.1086$ | $-0.0271$ | $0.0611$ | $0.0557$ | $-0.0279$ | $-0.1083$ |
| $SE_{4.4D}$ | $0.0225$ | $0.0179$ | $0.0176$ | $0.0232$ | $0.0334$ | $0.0225$ | $0.0179$ | $0.0177$ | $0.0231$ | $0.0329$ |
| MSE | $0.0012$ | $0.0040$ | $0.0034$ | $0.0013$ | $0.0129$ | $0.0012$ | $0.0040$ | $0.0034$ | $0.0013$ | $0.0128$ |
| $P_{-1.645}$ | $0.0360$ | $0.0380$ | $0.0540$ | $0.0420$ | $0.0400$ | $0.0340$ | $0.0420$ | $0.0520$ | $0.0480$ | $0.0460$ |
| $P_{-1.645}$ | $0.0480$ | $0.0420$ | $0.0460$ | $0.0500$ | $0.0440$ | $0.0460$ | $0.0440$ | $0.0520$ | $0.0500$ | $0.0420$ |
| $Z$ | $-0.4$ | $-0.2$ | $0$ | $0.2$ | $0.4$ | $-0.4$ | $-0.2$ | $0$ | $0.2$ | $0.4$ |
| $IPW-O$ | $IPW-P$ |
| Bias | $0.0260$ | $0.0570$ | $0.0512$ | $-0.0231$ | $-0.0993$ | $0.0261$ | $0.0569$ | $0.0512$ | $-0.0230$ | $-0.0993$ |
| $SE_{4.4D}$ | $0.0151$ | $0.0120$ | $0.0119$ | $0.0159$ | $0.0236$ | $0.0150$ | $0.0119$ | $0.0121$ | $0.0155$ | $0.0233$ |
| MSE | $0.0009$ | $0.0034$ | $0.0028$ | $0.0008$ | $0.0104$ | $0.0009$ | $0.0034$ | $0.0028$ | $0.0008$ | $0.0104$ |
| $P_{-1.645}$ | $0.0520$ | $0.0500$ | $0.0420$ | $0.0540$ | $0.0400$ | $0.0560$ | $0.0480$ | $0.0440$ | $0.0580$ | $0.0440$ |
| $P_{-1.645}$ | $0.0360$ | $0.0440$ | $0.0560$ | $0.0380$ | $0.0620$ | $0.0400$ | $0.0480$ | $0.0520$ | $0.0600$ | $0.0580$ |

| $k=4, r=1$ under Group 2: $(a = 0.05, a_1 = 1.05, a_2 = 0.09)$ |
| $n$ | $Z$ | $0$ | $0.2$ | $0.4$ | $0$ | $0.2$ | $0.4$ |
| $IPW-O$ | $IPW-P$ |
| Bias | $0.0270$ | $0.0563$ | $0.0517$ | $-0.0201$ | $-0.0940$ | $0.0260$ | $0.0570$ | $0.0521$ | $-0.0199$ | $-0.0935$ |
| $SE_{4.4D}$ | $0.0148$ | $0.0118$ | $0.0120$ | $0.0155$ | $0.0231$ | $0.0151$ | $0.0119$ | $0.0123$ | $0.0164$ | $0.0250$ |
| MSE | $0.0009$ | $0.0033$ | $0.0028$ | $0.0006$ | $0.0094$ | $0.0009$ | $0.0034$ | $0.0029$ | $0.0007$ | $0.0094$ |
| $P_{-1.645}$ | $0.0560$ | $0.0480$ | $0.0500$ | $0.0500$ | $0.0500$ | $0.0560$ | $0.0480$ | $0.0440$ | $0.0580$ | $0.0440$ |
| $P_{-1.645}$ | $0.0380$ | $0.0480$ | $0.0560$ | $0.0460$ | $0.0480$ | $0.0400$ | $0.0480$ | $0.0520$ | $0.0600$ | $0.0580$ |

| $k=4, r=1$ under Group 2: $(a = 0.05, a_1 = 1.05, a_2 = 0.09)$ |
| $n$ | $Z$ | $0$ | $0.2$ | $0.4$ | $0$ | $0.2$ | $0.4$ |
| $IPW-O$ | $IPW-P$ |
| Bias | $0.0280$ | $0.0604$ | $0.0563$ | $-0.0247$ | $-0.1026$ | $0.0270$ | $0.0614$ | $0.0577$ | $-0.0213$ | $-0.0967$ |
| $SE_{4.4D}$ | $0.0224$ | $0.0178$ | $0.0176$ | $0.0231$ | $0.0330$ | $0.0222$ | $0.0177$ | $0.0179$ | $0.0243$ | $0.0358$ |
| MSE | $0.0013$ | $0.0040$ | $0.0035$ | $0.0011$ | $0.0116$ | $0.0012$ | $0.0040$ | $0.0037$ | $0.0010$ | $0.0106$ |
| $P_{-1.645}$ | $0.0380$ | $0.0480$ | $0.0480$ | $0.0460$ | $0.0420$ | $0.0400$ | $0.0540$ | $0.0580$ | $0.0420$ | $0.0400$ |
| $P_{-1.645}$ | $0.0500$ | $0.0440$ | $0.0360$ | $0.0500$ | $0.0440$ | $0.0500$ | $0.0420$ | $0.0520$ | $0.0500$ | $0.0420$ |

34
Table 7: The simulation results under Model 3 with \(|Z \cap X| = 1\) but \(|Z \cap X^\top| = 0\), \(\beta_2^\top = (1/10, 1/\sqrt{3}, -1/\sqrt{3}, -1/10)\) and \(\gamma_2 = 0\).

| n   | Z     | -0.4 | -0.2 | 0    | 0.2 | 0.4 |
|------|-------|------|------|------|-----|-----|
|      | Bias  | 0.0333 | 0.0427 | -0.0219 | -0.0228 |      |
|      | MSE   | 0.0201 | 0.0097 | 0.0006 | 0.0013 |      |
|      | \(P_{1.645}\) | 0.0440 | 0.0520 | 0.0560 | 0.0500 |      |
|      | \(P_{1.645}\) | 0.0480 | 0.0440 | 0.0400 | 0.0580 | 0.0480 |

|      | -0.4 | -0.2 | 0 | 0.2 | 0.4 |
|------|------|------|---|-----|-----|
| MSE  | 0.0202 | 0.0097 | 0.0005 | 0.0006 | 0.0013 |
| \(P_{1.645}\) | 0.0480 | 0.0520 | 0.0560 | 0.0500 | 0.0480 |
| \(P_{1.645}\) | 0.0480 | 0.0440 | 0.0400 | 0.0580 | 0.0480 |

| 500  | Bias  | 0.0338 | 0.0401 | -0.0026 | -0.0072 | -0.0232 |
|      | MSE   | 0.0293 | 0.0245 | 0.0211 | 0.0228 | 0.0275 |
|      | \(P_{1.645}\) | 0.0540 | 0.0560 | 0.0520 | 0.0520 | 0.0602 |
|      | \(P_{1.645}\) | 0.0480 | 0.0440 | 0.0400 | 0.0580 | 0.0480 |

|      | -0.4 | -0.2 | 0 | 0.2 | 0.4 |
| MSE  | 0.0292 | 0.0245 | 0.0211 | 0.0228 | 0.0273 |
| \(P_{1.645}\) | 0.0540 | 0.0560 | 0.0520 | 0.0520 | 0.0602 |
| \(P_{1.645}\) | 0.0480 | 0.0440 | 0.0400 | 0.0580 | 0.0480 |

| 1000 | Bias  | 0.0316 | 0.0018 | -0.0039 | -0.0070 | -0.0221 |
|      | MSE   | 0.0225 | 0.0182 | 0.0157 | 0.0156 | 0.0190 |
|      | \(P_{1.645}\) | 0.0580 | 0.0520 | 0.0580 | 0.0580 | 0.0800 |
|      | \(P_{1.645}\) | 0.0480 | 0.0560 | 0.0440 | 0.0440 | 0.0560 |

|      | -0.4 | -0.2 | 0 | 0.2 | 0.4 |
| MSE  | 0.0224 | 0.0182 | 0.0157 | 0.0156 | 0.0190 |
| \(P_{1.645}\) | 0.0580 | 0.0520 | 0.0580 | 0.0580 | 0.0800 |
| \(P_{1.645}\) | 0.0480 | 0.0560 | 0.0440 | 0.0440 | 0.0560 |

| 500  | Bias  | 0.0338 | 0.0041 | -0.0026 | -0.0072 | -0.0232 |
|      | MSE   | 0.0293 | 0.0245 | 0.0221 | 0.0228 | 0.0275 |
|      | \(P_{1.645}\) | 0.0540 | 0.0560 | 0.0520 | 0.0520 | 0.0602 |
|      | \(P_{1.645}\) | 0.0480 | 0.0440 | 0.0400 | 0.0580 | 0.0480 |

|      | -0.4 | -0.2 | 0 | 0.2 | 0.4 |
| MSE  | 0.0292 | 0.0245 | 0.0221 | 0.0228 | 0.0273 |
| \(P_{1.645}\) | 0.0540 | 0.0560 | 0.0520 | 0.0520 | 0.0602 |
| \(P_{1.645}\) | 0.0480 | 0.0440 | 0.0400 | 0.0580 | 0.0480 |

| 1000 | Bias  | 0.0316 | 0.0018 | -0.0039 | -0.0070 | -0.0221 |
|      | MSE   | 0.0225 | 0.0182 | 0.0157 | 0.0156 | 0.0190 |
|      | \(P_{1.645}\) | 0.0580 | 0.0520 | 0.0580 | 0.0580 | 0.0800 |
|      | \(P_{1.645}\) | 0.0480 | 0.0560 | 0.0440 | 0.0440 | 0.0560 |

|      | -0.4 | -0.2 | 0 | 0.2 | 0.4 |
| MSE  | 0.0224 | 0.0182 | 0.0157 | 0.0156 | 0.0190 |
| \(P_{1.645}\) | 0.0580 | 0.0520 | 0.0580 | 0.0580 | 0.0800 |
| \(P_{1.645}\) | 0.0480 | 0.0560 | 0.0440 | 0.0440 | 0.0560 |

| 500  | Bias  | 0.0338 | 0.0041 | -0.0026 | -0.0072 | -0.0232 |
|      | MSE   | 0.0293 | 0.0245 | 0.0221 | 0.0228 | 0.0275 |
|      | \(P_{1.645}\) | 0.0540 | 0.0560 | 0.0520 | 0.0520 | 0.0602 |
|      | \(P_{1.645}\) | 0.0480 | 0.0440 | 0.0400 | 0.0580 | 0.0480 |

|      | -0.4 | -0.2 | 0 | 0.2 | 0.4 |
| MSE  | 0.0292 | 0.0245 | 0.0221 | 0.0228 | 0.0273 |
| \(P_{1.645}\) | 0.0540 | 0.0560 | 0.0520 | 0.0520 | 0.0602 |
| \(P_{1.645}\) | 0.0480 | 0.0440 | 0.0400 | 0.0580 | 0.0480 |

| 1000 | Bias  | 0.0316 | 0.0018 | -0.0039 | -0.0070 | -0.0221 |
|      | MSE   | 0.0225 | 0.0182 | 0.0157 | 0.0156 | 0.0190 |
|      | \(P_{1.645}\) | 0.0580 | 0.0520 | 0.0580 | 0.0580 | 0.0800 |
|      | \(P_{1.645}\) | 0.0480 | 0.0560 | 0.0440 | 0.0440 | 0.0560 |

|      | -0.4 | -0.2 | 0 | 0.2 | 0.4 |
| MSE  | 0.0224 | 0.0182 | 0.0157 | 0.0156 | 0.0190 |
| \(P_{1.645}\) | 0.0580 | 0.0520 | 0.0580 | 0.0580 | 0.0800 |
| \(P_{1.645}\) | 0.0480 | 0.0560 | 0.0440 | 0.0440 | 0.0560 |

35
Table 8: The simulation results under Model 4 with \( k=4, r=2 \), \( Z \cap X = 1 \), \( Z \cap V^\top X = 1 \), \( \beta_2^* = (0, 0, 0, 0) \) and \( \psi_2 = 1 \)

| n  | \( Z \) | \( -0.4 \) | \( -0.2 \) | 0  | 0.2 | 0.4 | \( -0.4 \) | \( -0.2 \) | 0  | 0.2 | 0.4 |
|-----|--------|--------|--------|----|----|----|--------|--------|----|----|----|----|
| 500 | \( -0.258 \) | 0.0569 | 0.0512 | -0.225-0.0979 | \( -0.247 \) | 0.0631 | 0.0575 | -0.254-0.1046 | \( -0.258 \) | 0.0569 | 0.0511 | -0.229-0.0985 |
|     | \( E_{IPW-M} \) | 0.0169 | 0.0133 | 0.0144 | 0.0205 | 0.0319 | \( 0.0167 \) | 0.0133 | 0.0141 | 0.0186 | 0.0294 |
|     | \( MSE \) | 0.0909 | 0.0043 | 0.0034 | 0.0038 | 0.0040 | \( 0.0099 \) | 0.0043 | 0.0036 | 0.0015 | 0.0134 |
|     | \( P_{1.645} \) | 0.0380 | 0.0540 | 0.0520 | 0.0480 | 0.0420 | \( 0.0460 \) | 0.0540 | 0.0540 | 0.0520 | 0.0540 |
|     | \( \hat{P}_{1.645} \) | 0.0440 | 0.0520 | 0.0520 | 0.0460 | 0.0440 | \( 0.0460 \) | 0.0540 | 0.0500 | 0.0480 | 0.0480 |

| 1000 | \( -0.258 \) | 0.0569 | 0.0512 | -0.225-0.0979 | \( -0.248 \) | 0.0578 | 0.0521 | -0.205-0.0943 | \( -0.258 \) | 0.0569 | 0.0511 | -0.229-0.0985 |
|     | \( E_{IPW-M} \) | 0.0166 | 0.0133 | 0.0139 | 0.0179 | 0.0270 | \( 0.0164 \) | 0.0132 | 0.0140 | 0.0184 | 0.0293 |
|     | \( MSE \) | 0.0909 | 0.0043 | 0.0034 | 0.0038 | 0.0040 | \( 0.0099 \) | 0.0043 | 0.0036 | 0.0015 | 0.0134 |
|     | \( P_{1.645} \) | 0.0380 | 0.0540 | 0.0520 | 0.0480 | 0.0420 | \( 0.0460 \) | 0.0540 | 0.0540 | 0.0520 | 0.0540 |
|     | \( \hat{P}_{1.645} \) | 0.0440 | 0.0520 | 0.0520 | 0.0460 | 0.0440 | \( 0.0460 \) | 0.0540 | 0.0500 | 0.0480 | 0.0480 |

| 500 | \( -0.258 \) | 0.0569 | 0.0512 | -0.225-0.0979 | \( -0.247 \) | 0.0631 | 0.0575 | -0.254-0.1046 | \( -0.258 \) | 0.0569 | 0.0511 | -0.229-0.0985 |
|     | \( E_{IPW-M} \) | 0.0169 | 0.0133 | 0.0144 | 0.0205 | 0.0319 | \( 0.0167 \) | 0.0133 | 0.0141 | 0.0186 | 0.0294 |
|     | \( MSE \) | 0.0909 | 0.0043 | 0.0034 | 0.0038 | 0.0040 | \( 0.0099 \) | 0.0043 | 0.0036 | 0.0015 | 0.0134 |
|     | \( P_{1.645} \) | 0.0380 | 0.0540 | 0.0520 | 0.0480 | 0.0420 | \( 0.0460 \) | 0.0540 | 0.0540 | 0.0520 | 0.0540 |
|     | \( \hat{P}_{1.645} \) | 0.0440 | 0.0520 | 0.0520 | 0.0460 | 0.0440 | \( 0.0460 \) | 0.0540 | 0.0500 | 0.0480 | 0.0480 |

| 1000 | \( -0.258 \) | 0.0569 | 0.0512 | -0.225-0.0979 | \( -0.249 \) | 0.0577 | 0.0521 | -0.205-0.0943 | \( -0.258 \) | 0.0569 | 0.0511 | -0.229-0.0985 |
|     | \( E_{IPW-M} \) | 0.0166 | 0.0133 | 0.0139 | 0.0179 | 0.0270 | \( 0.0165 \) | 0.0132 | 0.0140 | 0.0184 | 0.0292 |
|     | \( MSE \) | 0.0909 | 0.0043 | 0.0034 | 0.0038 | 0.0040 | \( 0.0099 \) | 0.0043 | 0.0036 | 0.0015 | 0.0106 |
|     | \( P_{1.645} \) | 0.0380 | 0.0540 | 0.0520 | 0.0480 | 0.0420 | \( 0.0540 \) | 0.0500 | 0.0420 | 0.0440 | 0.0580 |
|     | \( \hat{P}_{1.645} \) | 0.0460 | 0.0540 | 0.0540 | 0.0460 | 0.0470 | \( 0.0460 \) | 0.0540 | 0.0540 | 0.0520 | 0.0440 |

36
Table 9: The simulation results under Model 4 with $k=4$, $r=2$, $|Z \cap X| = 1$ and $|Z \cap V^TX| = 1$, $\beta^2_1 = (1/10, 1/\sqrt{2}, -1/\sqrt{2}, -1/10)$ and $\gamma_2 = 0$

| $n$ | $Z$ | $\beta$ | $IPW-O$ | $IPW-P$ | $IPW-N$ | $IPW-S$ |
|-----|-----|--------|--------|--------|--------|--------|
|     |     | .4     | -.2    | 0      | .2     | .4     |
| Bias | 0.0339 | 0.0050 | -0.0009 | -0.0057 | -0.0224 | 0.0543 | 0.0053 | -0.0008 | -0.0055 | -0.0222 |
| MSE  | 0.0025 | 0.0099 | 0.0008  | 0.0009  | 0.0017  | 0.0024 | 0.0009 | 0.0008  | 0.0009  | 0.0017  |
| $P_{-1,645}$ | 0.0520 | 0.0480 | 0.0560  | 0.0460  | 0.0480  | 0.0520 | 0.0460 | 0.0580  | 0.0380  | 0.0560  |
| $P_{1,645}$  | 0.0440 | 0.0520 | 0.0420  | 0.0520  | 0.0440  | 0.0380 | 0.0440 | 0.0400  | 0.0540  | 0.0440  |

| $n$ | $Z$ | $\beta$ | $IPW-O$ | $IPW-P$ | $IPW-N$ | $IPW-S$ |
|-----|-----|--------|--------|--------|--------|--------|
|     |     | .4     | -.2    | 0      | .2     | .4     |
| Bias | 0.0330 | 0.0040 | -0.0019 | -0.0064 | -0.0228 | 0.0372 | 0.0064 | -0.0010 | -0.0060 | -0.0224 |
| $E_{4,645}$D | 0.0264 | 0.0215 | 0.0253  | 0.0269  | 0.0320  | 0.0254 | 0.0279 | 0.0258  | 0.0277  | 0.0332  |
| MSE  | 0.0017 | 0.0005 | 0.0004  | 0.0004  | 0.0011  | 0.0016 | 0.0004 | 0.0004  | 0.0004  | 0.0010  |
| $P_{-1,645}$ | 0.0560 | 0.0480 | 0.0520  | 0.0500  | 0.0600  | 0.0520 | 0.0460 | 0.0500  | 0.0460  | 0.0680  |
| $P_{1,645}$  | 0.0540 | 0.0500 | 0.0500  | 0.0540  | 0.0520  | 0.0460 | 0.0480 | 0.0500  | 0.0520  | 0.0520  |

$\alpha = 0.55, a_1 = 1.05, a_2 = 0.09$
Table 10: The simulation results under high dimensional setting

| Group | Z     | IPW-O | IPW-P | IPW-S |
|-------|-------|-------|-------|-------|
|       |       | -0.4  | -0.2  | 0     | 0.2   | 0.4   | -0.4 | -0.2 | 0    | 0.2   | 0.4   |       |
| Bias  | 0.0275| 0.0598| 0.0554| -0.0271| -0.1064| -0.0273| 0.0601| 0.0556| -0.0270| -0.1062|
|       | MSE   | 0.0014| 0.0040| 0.0035 | 0.0016 | 0.0136| 0.0014| 0.0040| 0.0035 | 0.0014| 0.0131| 0.0012| 0.0041| 0.0033| 0.0019| 0.0164|
| P_{1.645} | 0.0420| 0.0520| 0.0420| 0.0380| 0.0420| 0.0580| 0.0500| 0.0420| 0.0540| 0.0540| 0.0540| 0.0520| 0.0560| 0.0540| 0.0560| 0.0540| 0.0560|
| Group 2 |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |
| Bias  | -0.0274| 0.0598| 0.0554| -0.0271| -0.1064| -0.0273| 0.0601| 0.0556| -0.0270| -0.1062|
|       | MSE   | 0.0014| 0.0040| 0.0035 | 0.0016 | 0.0136| 0.0014| 0.0040| 0.0035 | 0.0014| 0.0131| 0.0012| 0.0041| 0.0033| 0.0019| 0.0164|
| P_{1.645} | 0.0420| 0.0520| 0.0420| 0.0380| 0.0420| 0.0580| 0.0500| 0.0420| 0.0540| 0.0540| 0.0540| 0.0520| 0.0560| 0.0540| 0.0560| 0.0540| 0.0560|

dim(X)=20 with \( |Z \cap V| = 0; r = 1 \)

| Group | Z     | IPW-O | IPW-P | IPW-S |
|-------|-------|-------|-------|-------|
|       |       | -0.4  | -0.2  | 0     | 0.2   | 0.4   | -0.4 | -0.2 | 0    | 0.2   | 0.4   |       |
| Bias  | 0.0274| 0.0599| 0.0553| -0.0272| -0.1065| -0.0270| 0.0603| 0.0559| -0.0261| -0.1046|
|       | MSE   | 0.0014| 0.0040| 0.0035 | 0.0016 | 0.0136| 0.0014| 0.0040| 0.0035 | 0.0014| 0.0131| 0.0012| 0.0041| 0.0033| 0.0019| 0.0164|
| P_{1.645} | 0.0540| 0.0520| 0.0360| 0.0440| 0.0480| 0.0440| 0.0480| 0.0380| 0.0480| 0.0500| 0.0480| 0.0500| 0.0480| 0.0500| 0.0480| 0.0500| 0.0480| 0.0500| 0.0480| 0.0500| 0.0480| 0.0500| 0.0480| 0.0500|
| Group 2 |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |
| Bias  | -0.0274| 0.0599| 0.0553| -0.0272| -0.1065| -0.0270| 0.0603| 0.0559| -0.0261| -0.1046|
|       | MSE   | 0.0014| 0.0040| 0.0035 | 0.0016 | 0.0136| 0.0014| 0.0040| 0.0035 | 0.0014| 0.0131| 0.0012| 0.0041| 0.0033| 0.0019| 0.0164|
| P_{1.645} | 0.0540| 0.0520| 0.0360| 0.0440| 0.0480| 0.0440| 0.0480| 0.0380| 0.0480| 0.0500| 0.0480| 0.0500| 0.0480| 0.0500| 0.0480| 0.0500| 0.0480| 0.0500| 0.0480| 0.0500| 0.0480| 0.0500| 0.0480| 0.0500|

dim(X)=20 with \( |Z \cap V| = 1; r = 2 \)