THE EINSTEIN-VLASOV SYSTEM IN MAXIMAL AREAL COORDINATES—LOCAL EXISTENCE AND CONTINUATION

SEBASTIAN GÜNTER AND GERHARD REIN

Department of Mathematics
University of Bayreuth, Germany

Dedicated to the memory of Prof. Robert T. Glassey

Abstract. We consider the spherically symmetric, asymptotically flat Einstein-Vlasov system in maximal areal coordinates. The latter coordinates have been used both in analytical and numerical investigations of the Einstein-Vlasov system [3, 8, 18, 19], but neither a local existence theorem nor a suitable continuation criterion has so far been established for the corresponding nonlinear system of PDEs. We close this gap. Although the analysis follows lines similar to the corresponding result in Schwarzschild coordinates, essential new difficulties arise from the much more complicated form which the field equations take, while at the same time it becomes easier to control the necessary, highest order derivatives of the solution. The latter observation may be useful in subsequent investigations.

1. Introduction and main result. Consider a large ensemble of massive particles which interact only by the gravitational field which they create collectively, for example a galaxy or a globular cluster. We wish to model such a system in the framework of general relativity. We let $M$ denote a smooth four-dimensional spacetime manifold and $g_{\mu\nu}$ a Lorentz metric on $M$ with signature $(- + + +)$. In local coordinates $x^\mu$ the corresponding line element takes the form

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu;$$

Greek indices run from 0 to 3, and we use the Einstein summation convention: indices which appear as a lower and as an upper index in some expression are summed over; our terminology regarding general relativity largely follows [33]. The particle ensemble has a density $f = f(x^\mu, p^\nu) \geq 0$ defined on the cotangent bundle $TM^*$, where $p^\nu = g_{\nu\gamma}p^\gamma$ and $p^\gamma$ denote the canonical momenta corresponding to the local spacetime coordinates $x^\mu$. As usual in this context we restrict ourselves to the situation where all the particles have the same rest mass, which we normalize to 1. This means that $f$ is supported on the mass shell

$$PM^* := \{(x^\mu, p^\nu) \in TM^* \mid g^{\mu\nu}p_{\mu}p_{\nu} = -1, p^0 \text{ future pointing}\},$$

and we assume that on $PM^*$ the component $p^0$ can be expressed by the coordinates $(x^\mu, p_j)$ where Latin indices run from 1 to 3. In what follows we write $x^\mu = (t, x^j)$ and view $t$ as a time-like coordinate. The basic unknowns in the Einstein-Vlasov
system (besides the spacetime $M$) are the Lorentz metric $g_{\mu\nu}$ and the particle density $f$. They obey the Einstein equations

$$G_{\mu\nu} = 8\pi T_{\mu\nu}$$  \hfill (1.1)$$
coupled to the Vlasov equation

$$\partial_t f + \frac{p^i}{p^0} \partial_{x^i} f - \frac{1}{2p^0} \frac{\partial g^{\rho\delta}}{\partial x^i} p_{\nu} p_\delta \partial_{p_\nu} f = 0.$$  \hfill (1.2)$$
Here $G_{\mu\nu}$ is the Einstein tensor induced by the Lorentz metric $g_{\mu\nu}$, and the two equations are coupled by the prescription that the energy momentum tensor $T_{\mu\nu}$ is given as

$$T_{\mu\nu} = |g|^{-\frac{1}{2}} \int p_\mu p_\nu f \frac{dp_1 dp_2 dp_3}{p^0},$$  \hfill (1.3)$$
where $g$ is the determinant of the metric $g_{\mu\nu}$; we use units so that the speed of light and the gravitational constant equal unity. In passing we notice that the geodesic equations

$$\frac{dx^\mu}{d\tau} = p^\mu = g^{\mu\nu} p_\nu, \quad \frac{dp_\mu}{d\tau} = -\frac{1}{2} \frac{\partial g^{\rho\delta}}{\partial x^i} p_{\nu} p_\delta,$$

which describe the worldlines of test particles on the spacetime manifold $(M, g_{\mu\nu})$, correspond to the characteristic equations of the Vlasov equation 1.2 for $f$ supported on the mass-shell $PM^*$, the latter being invariant under the geodesic flow.

We wish to study the Einstein-Vlasov system 1.1, 1.2, 1.3 under the assumptions of spherical symmetry and asymptotic flatness. The former assumption is made to simplify the system while still allowing for interesting dynamics, while the latter means that we model an isolated system such as a galaxy in an otherwise empty universe. These assumptions still allow for a variety of possible coordinate choices with various advantages and disadvantages. Our choice are maximal areal coordinates, where the line element is given by

$$ds^2 = (-\alpha^2 + a^2 \beta^2)dt^2 + 2a^2 \beta dt dr + a^2 dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2).$$  \hfill (1.4)$$
The metric coefficients $a = a(t, r)$, $\beta = \beta(t, r)$, and $\alpha = \alpha(t, r)$ are functions on $I \times [0, \infty]$ where $I \subset \mathbb{R}$ is an interval, and $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$ are the angular coordinates. We impose that $\alpha$ and $a$ are strictly positive. Asymptotic flatness and a regular center imply the boundary conditions

$$a(t, 0) = a(t, \infty) = \alpha(t, \infty) = 1, \quad \beta(t, \infty) = 0.$$  \hfill (1.5)$$
A key question in general relativity is which initial data lead to the formation of a spacetime singularity. Assuming the weak cosmic censorship hypothesis a trapped surface should form before a spacetime singularity does. A surface, say of constant $t$ and $r$, is trapped if along both ingoing and outgoing null geodesics, i.e., light rays, the area of the surface decreases. In a metric of the form 1.4 a surface of constant $t$ and $r$ is trapped iff

$$\frac{1}{a^2} < \frac{\beta^2}{\alpha^2}. $$

Hence in particular, $\beta(t, r) \neq 0$ at a trapped surface, and this corresponds to the fact that the choice $\beta = 0$ turns 1.4 into the Schwarzschild form of the metric, and it is well known that Schwarzschild coordinates cannot cover regions of spacetime which contain a trapped surface. Maximal areal coordinates can, which is a major advantage of these coordinates when studying the formation of spacetime singularities.
The observation that the choice $\beta = 0$ turns 1.4 into the Schwarzschild metric shows that 1.4 allows for an additional gauge condition, and we demand the maximal slicing condition, i.e., each hypersurface of constant $t$ has vanishing mean curvature:

$$\dot{a} = (a\beta)' + 2a\alpha\kappa,$$

(1.6)

where we have introduced the abbreviation

$$\kappa = \frac{\beta}{r\alpha},$$

(1.7)

which is the $\theta\theta$ component of the extrinsic curvature of the hypersurface of constant $t$, and $'$ and $''$ denote partial derivatives with respect to $t$ or $r$ respectively.

The field equations 1.1 now take the following form:

$$a' = 4\pi r \rho a^3 + \frac{3}{2} r\kappa^2 a^3 + \frac{a}{2r} (1 - a^2),$$

(1.8)

$$\kappa' = -3\frac{\kappa}{r} - 4\pi a\beta,$$

(1.9)

$$\alpha'' = \alpha' \left( \frac{a'}{a} - \frac{2}{r} \right) + 6\alpha a^2 \kappa^2 + 4\pi a^2 (S + \rho),$$

(1.10)

$$\dot{\kappa} = \frac{3}{2} \kappa^2 \alpha - \frac{a'}{a} \kappa^2 + \frac{\alpha}{2r^2} \left( 1 - \frac{1}{a^2} \right) + 4\pi (a\beta - a\beta \beta).$$

(1.11)

The source terms $\rho, \beta, S,$ and $p$ are defined below. Strictly speaking, the above equations are not simply equal to the field equations 1.1, but they are equivalent and derived in the framework of the so-called $3 + 1$ formalism which we briefly review in Appendix B.

It remains to couple the above field equations to the Vlasov equation, but if the latter is formulated in the coordinates $(t, r, \theta, \phi)$ and the corresponding $p_j$, then it becomes technically very unpleasant to obtain a characteristic flow which is smooth at the center $r = 0$. This is simply because polar coordinates introduce artificial singularities at the center; they do not constitute proper coordinates in a neighborhood of $r = 0$. To avoid this difficulty we introduce the corresponding Cartesian coordinates:

$$x = (x^1, x^2, x^3) = r \left( \sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta \right) \in \mathbb{R}^3.$$ 

In addition, it is advantageous to introduce non-canonical momentum coordinates (frame components) $v_i$ on $PM^*$, cf. [27, 30]:

$$v_i := p_i + \left( \frac{1}{a} - 1 \right) \frac{x^j p_j}{r} x_i,$$

where $x_i := \delta_{ij} x^j; \delta_{ij}$ is the Kronecker symbol. In $(t, x^1, x^2, x^3, v_1, v_2, v_3)$ coordinates the Vlasov equation takes the form

$$\partial_t f + \left[ \frac{\alpha}{a} \frac{v}{\sqrt{1 + |v|^2}} - \beta \frac{x}{r} \right] \cdot \partial_x f + \left[ \left( -\frac{a'}{a} \sqrt{1 + |v|^2} + \frac{(a\beta)' - \dot{a} \cdot v}{a} \right) \frac{x}{r} + \beta \left( v - \frac{x \cdot v}{r} \right) \right] \cdot \partial_v f = 0,$$

(1.12)
where $x \cdot v := x^i v_i$, $|v|^2 := \delta^{ij} v_i v_j$. The source terms appearing in the field equations become

\begin{align}
\rho(t, x) &= \int \sqrt{1 + |v|^2} f(t, x, v) \, dv, \\
\mathcal{J}(t, x) &= \int \frac{x \cdot v}{r} f(t, x, v) \, dv, \\
S(t, x) &= \int \frac{|v|^2}{\sqrt{1 + |v|^2}} f(t, x, v) \, dv, \\
p(t, x) &= \int \left( \frac{x \cdot v}{r} \right)^2 \frac{f(t, x, v)}{\sqrt{1 + |v|^2}} \, dv.
\end{align}

integrals without a subscript always extend over $\mathbb{R}^3$. The equations 1.6–1.16 supplemented with the boundary conditions 1.5 constitute a closed system which we refer to as the spherically symmetric, asymptotically flat Einstein-Vlasov system in maximal areal coordinates. In order to be consistent, $f$ must be spherically symmetric, i.e.,

$$f(t, x, v) = f(t, A x, A v)$$

for all $A \in \text{SO}(3)$, $x, v \in \mathbb{R}^3$, which implies that the source terms $\rho, \mathcal{J}, S, p$ can be viewed as functions of $(t, x)$ or $(t, r)$ with $r = |x|$. The advantage of using the non-canonical variables $v_j$ is that the source terms are completely determined by $f$ and the metric does not enter there. In particular, on the mass shell $PM^*$,

$$p^0 = \frac{1}{\alpha} \sqrt{1 + |v|^2};$$

in canonical variables the metric would also appear under the square root.

The system as stated above has been used both in analytical and numerical investigations, cf. [3, 8, 18, 19], even though the switch to Cartesian coordinates is not done in all of these papers, and the momentum variable $v$ is sometimes replaced by

$$w = \frac{x \cdot v}{r}, \quad L = |x \times v|^2.$$ 

This has the advantage that $L$, the modulus of angular momentum squared, is conserved along characteristics of the Vlasov equation, and the disadvantage that the latter then contains terms which are singular at $r = 0$. In Appendix B we comment on how the system above is derived, but the main purpose of the present paper is to provide a basis for the investigations mentioned above (and possible future ones) and to establish the following result:

**Theorem 1.1.** Let $\hat{f} \in C^1_c(\mathbb{R}^6)$ be non-negative, spherically symmetric, and such that for the induced source terms

$$\dot{\rho}(r) := \int \sqrt{1 + |v|^2} \hat{f}(x, v) \, dv, \quad \dot{\mathcal{J}}(r) := \int \frac{x \cdot v}{r} \hat{f}(x, v) \, dv,$$

the constraints 1.8, 1.9 have a regular solution on $[0, \infty)$. Then there exists $T > 0$ and a unique regular solution $f$ to the spherically symmetric, asymptotically flat Einstein-Vlasov system in maximal areal coordinates on $[0, T] \times \mathbb{R}^6$ with $f(0) = \hat{f}$. If $T > 0$ is chosen maximal and

$$\sup \{ |v| \mid (t, x, v) \in \text{supp} \, f \} + \sup \{ a(t, r) \mid (t, r) \in [0, T] \times [0, \infty) \} < \infty,$$

then $T = \infty.$
Here $C^1_0$ denotes the space of continuously differentiable functions with compact support, and "regular" essentially means that all relevant derivatives exist, cf. Definitions 2.1 and 3.1. It is clear that some restrictions on the initial data are required in order that the constraint equations 1.8, 1.9 have a solution at least initially, cf. 2.6 and Proposition 2.5. A result analogous to Theorem 1.1 was shown in Schwarzschild coordinates in [28, 27], but there are important differences. In Schwarzschild coordinates it is sufficient to control the $v$-support of $f$ in order to extend the solution; this continuation criterion is well known and very basic for the Vlasov-Poisson system [9, 26] and the Vlasov-Maxwell system [15]. In the present case, the field equations are much more difficult to handle than in Schwarzschild coordinates, where they can be solved explicitly in terms of the source terms, and this more complex structure of the field equations accounts for the more demanding continuation criterion. An analogous result in maximal isotropic coordinates has been shown in [30], based on the earlier local existence result in [11]. The latter avoids symmetry assumptions and uses high order Sobolev spaces and energy-type estimates, but we prefer to prove our result directly for the system stated above, since this yields more understanding of the problem and avoids unnecessary assumptions on the initial data.

Besides being able to cope with trapped surfaces an important technical advantage of maximal areal coordinates is the following. As it stands, the system above seems overdetermined. In an equivalent, reduced system, which is introduced in Section 2, the only derivative of a metric component which appears in the modified Vlasov equation is $\alpha'$, and the radial derivative of the latter can be controlled via 1.10 without control of any derivatives of the source terms. The analogous argument is much more difficult in Schwarzschild coordinates; Eqn. 1.10 simply has a fairly nice elliptic structure.

After introducing the reduced system in Section 2 we analyze the relevant field equations, the modified Vlasov equation, and the equivalence to the full system in subsections of Section 2. In Section 3 we prove Theorem 1.1 and also include a subsection on conservation laws. The derivation of the above system, in particular some details on the 3+1 formalism, and a particularly lengthy Gronwall loop which is needed in the proof of Theorem 1.1 are dealt with in two appendices.

Before we proceed, some further comments to the literature are in order. The asymptotically flat Einstein-Vlasov system has global, geodesically complete solutions for small initial data; this was first shown in spherical symmetry in [28] and more recently without symmetry restriction in [14, 23, 32]. A global result for large, spherically symmetric, "outgoing" data was shown in [3], while gravitational collapse for certain classes of spherically symmetric data was shown in [1, 4, 5, 7]. Based on [12] it has been shown in [13] that for the spherically symmetric, asymptotically flat Einstein-Vlasov system the possible formation of a spacetime singularity is preceded by the formation of a trapped surface. There is also a plethora of stationary and static solutions, spherically symmetric ones, cf. [25] and the references there, and axially symmetric ones, cf. [2, 6]. A challenging problem is the stability or instability of such steady states. On the linearized level, both stability and instability results have been proven in [22, 21, 20]. The stability question has also been analyzed numerically, cf. [8, 18, 19] and the references there, where in some cases maximal areal coordinates were used. It turns out that if an unstable steady state is perturbed "towards collapse", then a black hole forms. We conjecture that...
in a future proof of this interesting fact, coordinates which, like the ones we use in this paper, can cope with trapped surfaces will play an important role.

2. The reduced Einstein-Vlasov system. The system 1.6–1.16 seems to be overdetermined; there are more equations than unknowns. For given \( \rho, j, \) and \( S \) the metric coefficients \( a, \alpha, \) and \( \beta \) are determined by the equations 1.7–1.10. However, restricting ourselves to only these four field equations and leaving the rest of the system unchanged produces a problem. Since no time derivative of \( a \) is included in the equations 1.7–1.10, it becomes very difficult to control \( \dot{a} \), which appears in the Vlasov equation 1.12. To avoid this, we insert Eqn. 1.6 into the Vlasov equation to eliminate \((a\beta)' - \dot{a}\). We then show that we can derive the time evolution equations 1.6 and 1.11 a posteriori. This motivates to define the reduced Einstein-Vlasov system by 1.7–1.10 and 1.13–1.15 coupled to the modified Vlasov equation

\[
\partial_t f + \left[ \frac{\alpha}{a} \frac{v}{\sqrt{1 + |v|^2}} - \beta \frac{x}{r} \right] \cdot \partial_x f + \left[ -\frac{\alpha'}{a} \sqrt{1 + |v|^2} + \alpha \kappa \left( v - 3 \frac{x \cdot v}{r^2} \right) \right] \cdot \partial_v f = 0.
\]

Before we analyze this equation and the reduced field equations, we make precise the regularity properties required for the various quantities appearing in the system.

**Definition 2.1.** Let \( I \subset \mathbb{R} \) be an interval.

(a) \( f : I \times \mathbb{R}^6 \to [0, \infty) \) is regular if \( f \in C^1(I \times \mathbb{R}^6) \), \( f(t) \) is spherically symmetric for \( t \in I \), and \( \text{supp} \, f(t) \) is compact, locally uniformly in \( t \in I \).

(b) \( \rho \) (or \( \rho, S \)) : \( I \times \mathbb{R}^3 \to [0, \infty) \) is regular if \( \rho \in C^1(I \times \mathbb{R}^3) \), \( \rho(t) \) is spherically symmetric for \( t \in I \), and \( \text{supp} \, \rho(t) \) is compact, locally uniformly in \( t \in I \).

(c) \( j : I \times \mathbb{R}^3 \to \mathbb{R} \) is regular if \( j \in C(I \times \mathbb{R}^3) \cap C^1(I \times \mathbb{R}^3 \setminus \{0\}) \), \( j(t) \) is spherically symmetric for \( t \in I \), \( \text{supp} \, j(t) \) is compact, locally uniformly in \( t \in I \), \( j \in C^1(I \times [0, \infty]) \) as a function in \((t, r)\), \( j(t, 0) = 0 \), and \( |j| \leq \rho \).

(d) \( a : I \times [0, \infty] \to [0, \infty] \) is regular if \( a \in C^1(I \times [0, \infty]) \), \( a' \in C^1(I \times [0, \infty]) \), and the boundary conditions \( a(t, 0) = a(t, \infty) = 1 \) and \( a'(t, 0) = 0 \) hold.

(e) \( \alpha : I \times [0, \infty] \to [0, \infty] \) is regular if \( \alpha \in C^1(I \times [0, \infty]) \), \( \alpha' \in C^1(I \times [0, \infty]) \), and the boundary conditions \( \alpha(t, 0) = \alpha(t, \infty) = 1 \) and \( \alpha'(t, 0) = 0 \) hold.

(f) \( \kappa : I \times [0, \infty] \to [0, \infty] \) is regular if \( \kappa \in C^1(I \times [0, \infty]) \), \( \kappa' \in C^1(I \times [0, \infty]) \), and the boundary conditions \( \kappa(t, 0) = \kappa(t, \infty) = \kappa'(t, 0) = 0 \) hold.

For intervals \( I \) and \( J \) we set

\[
C^{1, 0}(I \times J) := \{ g \in C(I \times J) \mid \partial_t g \in C(I \times J) \text{ exists} \}.
\]

**Remark 1.** If \( f \) is regular, then so are \( \rho, j, S, \) and \( p, \) cf. [27]. We will see that the imposed regularity properties fit together and lead to the existence of solutions of the field equations and the characteristic system. We do not need to specify the regularity of \( \beta \), since by 1.7, \( \beta = \alpha \kappa r \).

The aim of this section is to show the equivalence of the full and the reduced system and to derive existence results for the reduced field equations and the modified Vlasov equation for given regular source terms or metric coefficients respectively. We first study the reduced field equations 1.8–1.10 and establish a local existence result for them.

2.1. Investigation of the reduced field equations. We aim to solve the reduced field equations 1.8–1.10, where we of course keep in mind the definition 1.7. We assume that \( \rho, j, \) and \( S \) are regular in the sense of Definition 2.1. The Hamiltonian
constraint 1.8 and the momentum constraint 1.9 decouple from the slicing condition 1.10 for \( \alpha \). To motivate our strategy let us assume that we already have a sufficiently regular solution of 1.8 and 1.9 on \( I \times [0, \infty] \) where \( I \) is an interval with \( 0 \in I \), satisfying the boundary conditions. As in [3, 8] we define on \( I \times [0, \infty] \) the auxiliary quantity
\[
\mu := \frac{r}{2} \left( 1 - \frac{1}{a^2} \right).
\] (2.3)
Differentiating this with respect to \( r \) and using 1.8 gives
\[
\mu' = r \frac{a'}{a^3} + \frac{1}{2} \left( 1 - \frac{1}{a^2} \right) = 4\pi r^2 \rho + \frac{3}{2} r^2 \kappa^2
\]
which upon integration implies that
\[
\mu(t, r) = \int_0^r \left( 4\pi \rho(t, s) + \frac{3}{2} \kappa^2(t, s) \right) s^2 ds.
\] (2.4)
The momentum constraint 1.9 can be rewritten as
\[
(r^2 \kappa)' = -4\pi r^3 a j
\]
so that upon integration,
\[
\kappa(t, r) = -\frac{4\pi}{r^3} \int_0^r a(t, s) j(t, s) s^3 ds
\] (2.5)
which still contains \( a \). Note that 2.5 satisfies the required boundary conditions for \( \kappa \) if \( j \) has compact support in the spatial variable. We can now insert 2.5 into 2.4, solve 2.3 for \( a \), and treat the result as a fixed-point problem for \( a \):
\[
a(r) = \left( 1 - \frac{8\pi}{r} \int_0^r \rho(s) s^2 ds - \frac{3}{2} \int_0^r \frac{16\pi^2}{s^4} \left( \int_0^s a(\sigma) j(\sigma) s^3 d\sigma \right)^2 d\sigma \right)^{-\frac{1}{2}};
\]
we suppress the dependence on \( t \) which merely acts as a parameter. Once \( a \) and \( \kappa \) exist, the equation for \( \alpha \) can be solved by a fixed-point method as well. Since
\[
\frac{1}{a^2} = 1 - \frac{2\mu}{r}
\]
needs to be positive, necessarily
\[
\frac{2\mu}{r} < 1
\] (2.6)
on \( I \times [0, \infty] \); the coordinate system breaks down at any point \((t, r)\) where \( 2\mu(t, r) = r \). A sufficient condition for a solution to at least exist for some fixed time \( t \) on \([0, \infty]\) is given in Proposition 2.5.

For the proof of the continuation criterion in Section 3 it will be crucial that we are not bound to any conditions on \( \rho \) and \( j \) that would be necessary to solve the field equations. Instead, we need to be able to continue a solution locally in \( t \) once a solution is given at \( t = 0 \). This local solution must be continuously differentiable in \( t \) so that \( a, \alpha, \) and \( \kappa \) are regular in the sense of Definition 2.1. We achieve all this in the next result.

**Proposition 2.2.** For some \( T_0 > 0 \), let \( \rho \), \( j \), and \( S \) be regular on \([0, T_0] \times [0, \infty]\), and let \( a, \kappa \) be regular solutions to the constraints 1.8 and 1.9 on \([0, \infty]\) for \( t = 0 \), i.e., with the source terms \( \dot{\rho} = \rho(0, \cdot) \) and \( \dot{j} = j(0, \cdot) \). Then there exist \( 0 < T \leq T_0 \) and \( a, \alpha, \kappa \) which are regular on \([0, T] \times [0, \infty]\) and solve the field equations 1.8, 1.9, and 1.10. If \( T \) is chosen maximal, then \( T = T_0 \) or
\[
\lim_{t \to T} \|a(t)\| = \infty.
\]
Furthermore, \( \alpha(t, \cdot) \) is increasing, \( 0 < \alpha \leq 1 \), and on \( [0, T] \times [0, \infty] \) the following identities hold:

\[
\mu(t, r) = \frac{r}{2} \left( 1 - \frac{1}{a^2(t, r)} \right) = \int_0^r \left( 4\pi \rho(t, s) + \frac{3}{2} \kappa^2(t, s) \right) s^2 \, ds, \tag{2.7}
\]

\[
\kappa(t, r) = -\frac{4\pi}{r^3} \int_0^r a(t, s)j(t, s) s^3 \, ds, \tag{2.8}
\]

\[
r^2 \alpha'(t, r) = \int_0^r \left( 4\pi a\alpha + 6a\alpha \kappa \right) s^2 \, ds, \tag{2.9}
\]

\[
\left( \frac{r^2 \alpha'}{a^2} - \alpha \mu \right)' = 4\pi r^2 (\alpha S - \rho \alpha') + \frac{3}{2} r^2 \kappa^2 (3\alpha - \rho \alpha'). \tag{2.10}
\]

The identity 2.10 was also used in [3]. It will be essential in Subsection 2.3. Since the proof of this result is lengthy and technical, we split it up into three parts. First, we show that a regular solution of the constraints exists locally in \( t \) and \( r \). In the proof, one problem will be to guarantee that the time derivatives exist and are continuous.

**Lemma 2.3.** Let \( 0 < \delta < T_0 \) be arbitrary. Under the assumptions of Proposition 2.2 there exist \( \eta > 0 \) and unique functions \( a \) and \( \kappa \) which solve the constraint equations 1.8, 1.9 and are regular on \( [0, \delta] \times [0, \eta] \), i.e., \( a \) and \( \kappa \) satisfy the regularity conditions given in Definition 2.1 on \( [0, \delta] \times [0, \eta] \).

**Proof.** We fix \( 0 < \delta < T_0 \) and let \( \eta > 0 \) be small enough such that for \( (t, r) \in [0, \delta] \times [0, \eta] \),

\[
8\pi \int_0^r \rho(t, s)s^2 \, ds + \frac{12}{r} \int_0^r \frac{16\pi^2}{s^4} \left( \int_0^s |j(t, \sigma)| \sigma^3 \, d\sigma \right)^2 \, ds \leq \frac{3}{4}, \tag{2.11}
\]

and

\[
\frac{4\pi}{r} \int_0^r \rho s^2 \, ds + \frac{6}{r} \int_0^r \frac{16\pi^2}{s^4} \left( \int_0^s |j| \sigma^3 \, d\sigma \right) \left( \int_0^s (2|j| + |j|) \sigma^3 \, d\sigma \right) \, ds \leq \frac{1}{8}; \tag{2.12}
\]

note that \( \rho, \dot{\rho}, j, \) and \( j \), which stands for the time derivative of \( j \), are bounded on \( [0, \delta] \times [0, \eta] \) so that the left hand sides are proportional to \( r^2 \) for small \( r \). We consider the complete metric space

\[
\mathcal{X} := \{ a \in C^{1,0}([0, \delta] \times [0, \eta]) \mid 1 \leq a \leq 2, |\dot{a}| \leq 1 \}
\]
equipped with the norm

\[
\|a\|_\mathcal{X} := \|a\| + \|\dot{a}\|,
\]

where \( \| \cdot \| \) denotes the sup norm, and we recall 2.2. We claim that

\[
\mathcal{F}(a)(t, r) := \left( 1 - \frac{8\pi}{r} \int_0^r \rho s^2 \, ds - \frac{3}{r} \int_0^r \frac{16\pi^2}{s^4} \left( \int_0^s a \sigma^3 \, d\sigma \right)^2 \, ds \right)^{-\frac{1}{2}}
\]
defines a contraction \( \mathcal{F} : \mathcal{X} \to \mathcal{X} \), provided that \( \eta > 0 \) is sufficiently small.

First, we show that \( \mathcal{F}(a) \in \mathcal{X} \). The estimate 2.11 and \( a \leq 2 \) imply that \( |\mathcal{F}(a)(t, r)| \leq 2 \). Since \( \rho, j, \) and \( a \) are continuously differentiable with respect to \( t \in [0, \delta] \), so is \( \mathcal{F}(a) \) with

\[
\dot{\mathcal{F}}(a) = \mathcal{F}(a)^3 \left( \frac{4\pi}{r} \int_0^r \dot{\rho} s^2 \, ds + \frac{3}{r} \int_0^r \frac{16\pi^2}{s^4} \left( \int_0^s a \sigma^3 \, d\sigma \right) \left( \int_0^s (aj + \dot{a}j) \sigma^3 \, d\sigma \right) \right)
\]

(2.13)
for \(r > 0\). The fact that \(F(a)(t,0) = 1\) for \(t \in [0,\delta]\) implies that
\[
F(a)(t,0) = 0 = \lim_{r \to 0} F(a)(t,r).
\]
With 2.12 and the bounds on \(a\) and \(\dot{a}\) from the definition of \(X\), 2.13 implies that
\[
|\dot{F}(a)| \leq 1.
\]
In order to prove that \(F\) is a contraction on \(X\) we define for \(a_1, a_2 \in X\),
\[
\mu_i(t,r) := \frac{r}{2} \left( 1 - \frac{1}{F(a_i)(t,r)^2} \right) = 4\pi \int_0^r \rho_s^2 s^2 ds + \frac{3}{2} \int_0^r \frac{16\pi^2}{s^4} \left( \int_0^s a_i \sigma^3 d\sigma \right)^2 ds,
\]
and \(i = 1, 2\). Then
\[
|\mu_1(t,r) - \mu_2(t,r)| \leq C \int_0^r \left( \int_0^s |a_1 + a_2||\dot{a}| d\sigma \right) \left( \int_0^s |a_1 - a_2||\dot{a}| d\sigma \right) s^2 ds \leq Cr^4 ||a_1 - a_2||
\]
for \((t,r) \in [0,\delta] \times [0,\eta]\); \(C > 0\) denotes a constant which depends only on \(\eta\). Since
\[
F(a_i) \leq 2, \quad \frac{\mu_i(t,r)}{r} \in [0,\frac{1}{2}] \quad \text{for} \quad i = 1, 2, \quad \text{and with 2.14 it follows that}
\]
\[
|F(a_1)(t,r) - F(a_2)(t,r)| \leq \sqrt{2} \left| \frac{\mu_1(t,r)}{r} - \frac{\mu_2(t,r)}{r} \right| \leq C \eta^4 ||a_1 - a_2||. \quad (2.15)
\]
Due to the definition of \(||\cdot||_X\) we need to control the time derivatives as well. Using
\(F(a_i) \leq 2\) and 2.15,
\[
|F(a_1)(t,r) - \dot{F}(a_2)(t,r)| \leq Cr^3 |F(a_1)^3(t,r) - F(a_2)^3(t,r)|
\]
\[
+ C \int_0^r s \left( \int_0^s |a_1 - a_2||\dot{a}| d\sigma \right) \left( \int_0^s |\dot{a}_1| + a_1 ||\dot{a}| d\sigma \right) ds
\]
\[
+ C \int_0^r s \left( \int_0^s a_2 ||\dot{a}| d\sigma \right) \left( \int_0^s (|\dot{a}_1 - \dot{a}_2| + |a_1 - a_2||\dot{a}|) d\sigma \right) ds
\]
\[
\leq C \eta^4 ||a_1 - a_2||_X;
\]
the constant \(C > 0\) is independent of \(\eta, a_1,\) and \(a_2\). By these inequalities,
\[
||F(a_1) - F(a_2)||_X \leq C \eta^4 ||a_1 - a_2||_X,
\]
and \(F\) is a contraction on \(X\), if \(\eta > 0\) is chosen such that \(C \eta^4 < 1\).

By Banach’s fix point theorem there exists a unique solution \(a \in X\) that satisfies
\(F(a) = a\) on \([0,\delta] \times [0,\eta]\). This equation immediately implies that
\[
\mu(t,r) = \frac{r}{2} \left( 1 - \frac{1}{a^2(t,r)} \right) = 4\pi \int_0^r \rho(t,s)^2 s^2 ds + \frac{3}{2} \int_0^r \frac{16\pi^2}{s^4} \left( \int_0^r a(t,\sigma) |\dot{a}(t,\sigma)\sigma^3 d\sigma \right)^2 ds.
\]
Thus \(\mu \sim r^3\) for small \(r\) and \(\mu \in C^1([0,\delta] \times [0,\eta])\). We conclude that \(a \in C^1([0,\delta] \times [0,\eta])\). We define
\[
\kappa(t,r) := \frac{4\pi}{r^3} \int_0^r a(t,s)|\dot{a}(t,s)s^3 ds, \quad r \in [0,\delta],
\]
which implies that
\[
a' = 4\pi r \rho a^3 + \frac{3}{2} r \kappa^2 a^3 + \frac{a}{2r} (1 - a^2)
\]
and \(\kappa \in C^1([0,\delta] \times [0,\eta])\); the uniqueness of \(a\) implies the uniqueness of \(\kappa\). The differential equations yield the continuous differentiability of \(a'\) and \(\kappa'\) on \([0,\delta] \times [0,\eta]\).
For $a$ and $\kappa$ to be regular, we need to show that $\dot{a}'$, $a''$, and $\dot{\kappa}'$ are continuous in $r = 0$. In order to show the regularity of $a$ it is convenient to consider $\mu$ instead of $a$. Since
\[
\mu' = 4\pi r^2 \rho + \frac{3}{2} r^2 \kappa^2,
\]
it follows that
\[
\dot{\mu}' = 4\pi r^2 \dot{\rho} + 3 r^2 \kappa \dot{\kappa},
\]
exist and are continuous in $r = 0$. In particular, the quantities
\[
\frac{\mu''}{r}, \frac{\mu'}{r^2}, \frac{\mu}{r^3}, \frac{\dot{\mu}'}{r^2}, \frac{\dot{\mu}}{r^3}, \frac{\dot{\mu}}{r^2}
\]
are continuous in $r = 0$. Hence the continuity of $\dot{a}'$ and $a''$ follows via
\[
a' = \left(1 - \frac{2\mu}{r}\right)^{-\frac{3}{2}} \left(\frac{\mu'}{r} - \frac{\mu}{r^2}\right).
\]

The time derivative of $\kappa$ becomes
\[
\dot{\kappa}(t, r) = -\frac{4\pi}{r^3} \int_0^r (\dot{a}(t,s)j(t,s) + a(t,s)\dot{j}(t,s)) s^3 ds
\]
which together with $\dot{a}(t, 0) = 0 = j(t, 0)$ implies that
\[
\lim_{r \to 0} \frac{\dot{\kappa}(t, r)}{r} = 0.
\]
This shows the continuity of
\[
\kappa' = -\frac{3}{r} \dot{\kappa} - 4\pi (\dot{a}j + a\dot{j}).
\]

In the next lemma we show that local solutions can be continued until $a$ blows up.

\begin{lemma}
Under the assumptions of Proposition 2.2 there exist $0 < T \leq T_0$ and unique, regular solutions $a$ and $\kappa$ to the constraints 1.8 and 1.9, defined on $[0, T] \times [0, \infty]$. If $T$ is chosen maximal then $T = T_0$ or $\lim_{t \to T} \|a(t)\| = \infty$.
\end{lemma}

\begin{proof}
We choose $0 < \delta < T_0$ which we will fix later. By Lemma 2.3, regular solutions $a$, $\kappa$ to 1.8 and 1.9 exist on $[0, \delta] \times [0, \eta]$, where $\eta > 0$. We consider 1.8 and 1.9 as a system of ODEs on $[\eta, \infty]$. By the Picard-Lindelöf theorem $a$ and $\kappa$ can be uniquely extended to maximal solutions $a(t, \cdot)$, $\kappa(t, \cdot)$ on $[0, R_t]$ where $R_t \in [\eta, \infty]$ may depend on $t \in [0, \delta]$. Because of uniqueness, $a(0, \cdot) = \dot{a}$ and $\kappa(0, \cdot) = \dot{\kappa}$. The required regularity in $t$ follows directly from the continuous differentiable dependence on the parameter $t$, cf. Lemma 2.3. By the assumptions in Proposition 2.2, $R_0 = \infty$. We will show that $R_t = \infty$ on $[0, \delta]$ for some $\delta > 0$ which is small enough. To achieve this, we define for $(t, r, a, \kappa) \in [0, T_0] \times [\eta, \infty] \times [1, \infty] \times \mathbb{R}$ the right hand side of the differential equations as
\[
g(t, r, a, \kappa) := \begin{pmatrix}
4\pi \rho(t, r)a^3 + \frac{3}{2} \kappa a^3 + \frac{a}{2r}(1 - a^2) \\
-\frac{3}{r} \kappa - 4\pi a j(t, r)
\end{pmatrix}.
\]
We write \( x(t, r) := (a(t, r), \kappa(t, r)) \) and choose \( R > \eta \) such that for \( t \in [0, \delta] \),

\[
\text{supp} \, \rho(t), \text{supp} \, j(t) \subset [0, 4^{-1/3} R].
\]

This is possible because of the locally uniformly compact support of \( \rho(t) \) and \( j(t) \).

Let \( \epsilon > 0 \) be arbitrary. We can choose \( \delta > 0 \) small enough such that for \( t \in [0, \delta] \),

\[
x = (a, \kappa) \in [1, 2||\hat{a}||] \times \mathbb{R}, \quad \text{and } r \in [\eta, R],
\]

\[
|g(t, r, x) - g(0, r, x)| \leq 4\pi \rho a^3 |\rho(t, r) - \rho(0, r)| + 4\pi a |j(t, r) - j(0, r)| < \epsilon, \quad (2.16)
\]

\[
|x(t, \eta) - x(0, \eta)| < \epsilon. \quad (2.17)
\]

These inequalities follow from the continuity of \( \rho \) and \( j \) and the fact that \( x(\cdot, \eta) \) is continuous due to Lemma 2.3. Since \( g \) is continuously differentiable there exists for every \( K > 0 \) a constant \( L > 0 \) such that for all \( x_1, x_2 \in [1, 2||\hat{a}||] \times [-K, K] \), and \( r \in [\eta, R] \),

\[
|g(0, r, x_1) - g(0, r, x_2)| \leq L|x_1 - x_2|, \quad (2.18)
\]

In order to show that \( R_t = \infty \) on \([0, \delta]\) we define

\[
r^*_t := \sup \{ r \in [\eta, R_t] : a(t, r) \leq 2||\hat{a}|| \};
\]

the set is non-empty for \( \epsilon \) small enough. Using 2.8 we can choose \( K > 0 \) such that for \( t \in [0, \delta] \) and \( r \in [\eta, \min \{ r^*_t, R_t \}], \)

\[
|\kappa(t, r)| \leq 8\pi R ||\hat{a}|| \|j(t)\| \leq K.
\]

Hence for every \( t \in [0, \delta] \) and \( r \in [\eta, \min \{ r^*_t, R_t \}], \)

\[
|x(t, r) - x(0, r)| \leq |x(t, \eta) - x(0, \eta)| + \int_\eta^r |g(t, s, x(t, s)) - g(0, s, x(t, s))| \, ds
\]

\[
+ \int_\eta^r |g(0, s, x(t, s)) - g(0, s, x(0, s))| \, ds.
\]

We insert 2.16, 2.17, and 2.18 to find that

\[
|x(t, r) - x(0, r)| \leq \epsilon(1 + R - \eta) + L \int_\eta^r |x(t, s) - x(0, s)| \, ds.
\]

Applying Gronwall’s inequality and choosing \( \epsilon < \frac{1}{2(1 + R - \eta)} e^{-L(R - \eta)} \), yields

\[
|x(t, r) - x(0, r)| \leq \frac{1}{2}.
\]

In particular, \( a(t, r) \leq \frac{1}{2} + ||\hat{a}|| < 2||\hat{a}|| \) since \( \hat{a} \geq 1 \). This implies \( R_t > r^*_t > R \). We claim that \( \kappa \) and \( a \) are decreasing for \( r > R \) which proves that \( R_t = \infty \) for \( t \in [0, \delta] \), since \( a(t, \cdot) \) and \( \kappa(t, \cdot) \) remain bounded and must therefore exist on \([0, \infty[\). In order to show this claim, we let \( R_t > r > 4^{-1/3} R =: \hat{R} \). From the choice of \( R \) we get \( \rho = j = 0 \) on \([0, \delta] \times [\hat{R}, \infty[ \). Therefore, the formulas 2.7 and 2.8 for \( a \) and \( \kappa \) yield

\[
\kappa(t, r) = \kappa(t, \hat{R}) \frac{\hat{R}^3}{r^3}
\]

and as a result,

\[
\frac{2\mu(t, r)}{r} = \frac{8\pi}{r} \int_0^{\hat{R}} \rho(t, s)s^2 ds + \frac{3}{r} \int_0^{\hat{R}} \kappa^2(t, s)s^2 ds + \frac{\kappa^2(t, \hat{R})\hat{R}^6}{r} \left( \frac{1}{\hat{R}^3} - \frac{1}{r^3} \right).
\]

Hence \( 2\mu/r \) and \( a \) are decreasing for \( r > R = 4^{1/3}\hat{R} \). Thus \( a \) and \( \kappa \) remain bounded which implies \( R_t = \infty \) for \( t \in [0, \delta] \).
The boundary conditions at 0 follow from the implicit formulas and the differential equations. We can estimate \( \kappa(t, r) \leq C(t)(1 + r^3)^{-1} \) for fixed \( t \) and deduce \( \kappa(t, \infty) = 0 \) and \( a(t, \infty) = 1 \) from the compact support of \( \rho \) and \( \lambda 
olimits \).

If \( T \) is chosen maximal and \( T < T_0 \), we obtain \( R_t = \infty \) for \( t \in [0, T] \). Assume that \( R_T = \infty \) holds; notice that we can apply Lemma 2.3 with \( \delta = T \) so that \( R_T \) is defined. Then we can carry out the above process again for \( t = T \) instead of \( t = 0 \), which extends the solution in contradiction to \( T \) being maximal. Hence \( R_T < \infty \) which implies

\[
\lim_{r \to R_T} |a(t, r)| = \infty
\]

from the boundary behavior of maximal solutions. Since \( a \in C([0, T] \times [0, R_T]) \),

\[
\lim_{t \to T} \|a(t)\| = \infty.
\]

We are now ready to prove Proposition 2.2.

**Proof of Proposition 2.2.** Let \( a \) and \( \kappa \) be the regular solution of 1.8 and 1.9 on \( [0, T] \times [0, \infty) \) obtained in Lemma 2.4. We must show the existence and uniqueness of a regular solution to 1.10, which we can write as

\[
\left( \frac{\alpha'}{a} \right)' = 6\kappa^2 a + 4\pi(S + \rho) a' r^2 \alpha.
\]

We recall that \( a \geq 1 \) and abbreviate

\[
q := 6\kappa^2 a + 4\pi(S + \rho) a,
\]

which now is a known quantity. A problem arises for the differential equation for \( \alpha \) from the fact that the only boundary condition for \( \alpha \) is given at \( r = \infty \). We solve the equation for some prescribed value at \( r = 0 \), using a fixed point method. Then we show that this solution, called \( \tilde{\alpha} \), has a positive, finite limit for \( r \to \infty \), and due to the linearity of the differential equation we obtain the desired solution \( \alpha \) as a suitable multiple of \( \tilde{\alpha} \). We consider the initial value problem

\[
\left( \frac{\tilde{\alpha}'}{\tilde{a}} \right)' = qr^2 \tilde{\alpha}, \quad \tilde{\alpha}(0) = 1.
\]

After integrating twice, this yields the integral equation

\[
\tilde{\alpha}(t, r) = 1 + \int_0^r \frac{a(t, s)}{s^2} \int_0^s q(t, \sigma)\tilde{\alpha}(t, \sigma)\sigma^2 d\sigma ds.
\]

The same norm as used in the proof of Lemma 2.3 turns

\[
\mathcal{X} := \{ \tilde{\alpha} \in C^{1,0}([0, \delta] \times [0, \eta]) \mid \tilde{\alpha} \geq 0 \}
\]

into a complete metric space for \( 0 < \delta < T, \eta > 0 \). The map

\[
\mathcal{F}(\tilde{\alpha})(t, r) := 1 + \int_0^r \frac{a(t, s)}{s^2} \int_0^s q(t, \sigma)\tilde{\alpha}(t, \sigma)\sigma^2 d\sigma ds
\]

is a contraction on \( \mathcal{X} \). To see this we first note that \( \mathcal{F}(\tilde{\alpha}) \in C^{1,0}([0, \delta] \times [0, \eta]) \) and \( \mathcal{F}(\tilde{\alpha}) \geq 0 \). For \( \tilde{\alpha}_1, \tilde{\alpha}_2 \in \mathcal{X} \) and \( (t, r) \in [0, \delta] \times [0, \eta] \),

\[
|\mathcal{F}(\tilde{\alpha}_1)(t, r) - \mathcal{F}(\tilde{\alpha}_2)(t, r)| \leq C\eta^2 \|\tilde{\alpha}_1 - \tilde{\alpha}_2\|
\]

and

\[
|\tilde{\mathcal{F}}(\tilde{\alpha}_1)(t, r) - \tilde{\mathcal{F}}(\tilde{\alpha}_2)(t, r)| \leq C\eta^2 \left( \|\tilde{\alpha}_1 - \tilde{\alpha}_2\| + \|\tilde{\alpha}_1 - \tilde{\alpha}_2\| \right).
\]
For $\eta > 0$ small enough Banach’s fix-point theorem yields a unique solution $\tilde{\alpha} \in \mathcal{X}$ with $\mathcal{F}(\tilde{\alpha}) = \tilde{\alpha}$. We immediately get $\tilde{\alpha} \in C^1([0, \delta] \times [0, \eta])$, since the identity

$$\tilde{\alpha}'(t, r) = \frac{a(t, r)}{r^2} \int_0^r q(t, s)\tilde{\alpha}(t, s)s^2ds, \quad r > 0$$

(2.22) implies that $\lim_{r \to 0} \tilde{\alpha}'(t, r) = 0 = \tilde{\alpha}'(t, 0)$. Furthermore, 2.22 implies that $\tilde{\alpha}' \in C^1([0, \delta] \times [0, \eta])$. To show this, we only have to prove the continuity of $\tilde{\alpha}''$ and $\tilde{\alpha}'$ in $r = 0$. For $r > 0$,

$$\frac{\tilde{\alpha}'(t, r)}{r} = \frac{a(t, r)}{r^3} \int_0^r q(t, s)\tilde{\alpha}(t, s)s^2ds \to \frac{1}{3}q(t, 0)\tilde{\alpha}(t, 0) = \frac{q(t, 0)}{6} \text{ as } r \to 0$$

because of 2.22. Thus

$$\tilde{\alpha}'' = \left(\frac{\tilde{\alpha}'}{a} - \frac{2}{r}\right)\tilde{\alpha}' + \left(6a^2 \kappa^2 + 4\pi a^2(S + \rho)\right)\tilde{\alpha}$$

is continuous on $[0, \delta] \times [0, \eta]$. Differentiating 2.22 with respect to $t$ yields

$$\tilde{\alpha}' = \frac{\dot{\alpha}}{r^2} \int_0^r q\dot{\alpha}s^2ds + \frac{\dot{\alpha}}{r^2} \int_0^r (\dot{q}\dot{\alpha} + q\dot{\alpha})s^2ds,$$

which converges to zero for $r \to 0$.

To show that $\tilde{\alpha}(t, \cdot)$ can be continued to $[0, \infty]$, we fix $t \in [0, \delta]$. The positivity of $q$, the formula 2.22, and $\tilde{\alpha}(t, 0) = 1 > 0$ imply that $\tilde{\alpha}' \geq 0$, i.e., $\tilde{\alpha}(t, \cdot)$ is increasing. In addition,

$$q(t, r) \leq C \frac{1}{1 + r^6}, \quad r \in [0, \infty], \quad t \in [0, \delta]$$

for some $C > 0$, which follows if we use the locally uniform compact support of $S + \rho$ and the estimate $|\kappa(t, r)| \leq C(1 + r^3)^{-1}$ which follows from Eqn. 2.8. This inequality for $q$ together with 2.22 and the monotonicity of $\tilde{\alpha}$ gives

$$\tilde{\alpha}'(t, r) \leq C \frac{a(t, r)}{r^2} \left(\int_0^r \frac{s^2}{1 + s^6}ds\right) \tilde{\alpha}(t, r) \leq C \frac{a(t, r)}{r^2} \tilde{\alpha}(t, r).$$

Gronwall’s lemma and the boundedness of $a$ imply that

$$\tilde{\alpha}(t, r) \leq \tilde{\alpha}(t, 0) \exp\left(C \int_0^r \frac{a(t, s)}{s^2}ds\right) \leq C.$$

This upper bound for $\tilde{\alpha}(t, \cdot)$ implies that 2.20 has a unique solution on $[0, \infty]$ for every $t \in [0, \delta]$. Since $\tilde{\alpha}(t, \cdot)$ is increasing and bounded on $[0, \infty]$, the limit

$$\tilde{\alpha}_\infty(t) := \lim_{r \to \infty} \tilde{\alpha}(t, r) = 1 + \int_0^\infty \frac{a(t, s)}{s^2} \int_0^s q(t, \sigma)\tilde{\alpha}(t, \sigma)s^2d\sigma ds$$

(2.23) exists. Since the differential equation 2.19 is linear in $\alpha$,

$$\alpha(t, r) := \frac{\tilde{\alpha}(t, r)}{\tilde{\alpha}_\infty(t)}$$

solves 2.19 and for fixed $t \in [0, \delta]$ converges to 1 as $r \to \infty$. We need to check that $\alpha$ is regular in $t$. The functions $\tilde{\alpha}$ and $\tilde{\alpha}'$ are continuously differentiable on $[0, \delta] \times [0, \infty]$ by the fixed-point method above and the continuously differentiable dependence on parameters for ODEs. To show that $\tilde{\alpha}_\infty$ is differentiable with respect to $t \in [0, \delta]$, we let $0 < \delta < T$ be arbitrary. There exists $C > 0$ such that for $(t, r) \in [0, \delta] \times [0, \infty[$,

$$\alpha(t, r), \tilde{\alpha}(t, r), |\tilde{\alpha}(t, r)|, |\tilde{\alpha}(t, r)|, (1 + r^3)|\kappa(t, r)|, (1 + r^3)|\dot{\kappa}(t, r)| \leq C.$$
These estimates and the locally uniformly compact support of $\rho$, $j$, and $S$ imply that
\[ |q(t, r)|, \ |\dot{q}(t, r)| \leq \frac{C}{1 + r^2}, \quad (t, r) \in [0, \delta] \times [0, \infty[. \]
Hence we can differentiate under the integral in 2.23, and $\alpha$ has the required regularity and is the solution we wanted to derive. Since $\alpha(t, \cdot)$ is increasing, it follows that $0 < \alpha \leq 1$.

To deduce the assertion 2.10 we use 2.7 and 2.9 to find that
\[
\left( r^2\frac{\alpha'}{a^2} - \alpha\mu \right)' = 4\pi r^2 \alpha(S + \rho) + 6\pi r^2 \alpha \kappa^2 - \frac{a'}{a^3} r^2 \alpha' - \alpha r^2 \left( 4\pi \rho + \frac{3}{2}\kappa^2 \right) - \alpha' \mu = 4\pi r^2 \alpha S + \frac{9}{2} r^2 \alpha \kappa^2 - \alpha' r \left( \frac{a'}{a^3} r + \frac{1}{2} \left( 1 - \frac{1}{a^2} \right) \right). \]

The Hamiltonian constraint 1.8 yields
\[ \frac{a'}{a^3} r + \frac{1}{2} \left( 1 - \frac{1}{a^2} \right) = \mu' = \frac{3}{2} r^2 \kappa^2 + 4\pi r^2 \rho, \]
and this leads to 2.10.

In view of the assumption both in Proposition 2.2 and in Theorem 1.1 it is of interest to know which data $\dot{\rho}$ and $\dot{j}$ allow us to obtain a solution of the constraints at $t = 0$ that is defined on $[0, \infty[$. A sufficient condition is given in the next result.

**Proposition 2.5.** Let $\rho \in C^1_c(\mathbb{R}^3)$, $\rho \geq 0$, $j \in C^1(\mathbb{R} \cap C^1(\mathbb{R}^3 \setminus \{0\})$ be spherically symmetric and $j \in C^1([0, \infty])$ as a function in $r$, and assume that for some $c_0 > 1$ these data satisfy the following condition for all $r \in [0, \infty[$:
\[
4\pi \int_0^r \rho(s) s^2 ds + \frac{3}{2} c_0^2 \int_0^r \frac{16\pi^2}{8^4} \left( \int_0^s |j(\sigma)| \sigma^3 d\sigma \right)^2 ds < \frac{r}{2} \left( 1 - \frac{1}{c_0^2} \right). \quad (2.24)
\]
Then there exist unique functions $a \in C^2([0, \infty])$ and $\kappa \in C^1([0, \infty])$ which satisfy 1.8 and 1.9 and the boundary conditions
\[ a(0) = a(\infty) = 1, \quad \kappa(0) = \kappa(\infty) = a'(0) = \kappa'(0) = 0. \]

**Proof.** For some $\eta > 0$ a local solution $a \in C^2([0, \eta])$, $\kappa \in C^1([0, \eta])$ is obtained as in the proof of Lemma 2.3. We have to show that under the condition 2.24 it can be extended to $[0, \infty[. \] Viewing the constraints as a system of differential equations on $[\eta, \infty[$, it is sufficient to bound $a$ on the maximal interval of existence. Let $[0, R]$ be some arbitrary interval such that $a(r) < c_0$ for $0 < r < R$. Using 2.8 we can estimate
\[
\int_0^R \kappa^2(s) s^2 ds \leq c_0^2 \int_0^R \frac{16\pi^2}{8^4} \left( \int_0^s |j(\sigma)| \sigma^3 d\sigma \right)^2 ds. \]
After inserting this into the implicit formula for $a$ and using condition 2.24 we find that
\[ a^2(R) \leq \left[ 1 - \frac{8\pi}{R} \int_0^R \rho(s) s^2 ds - \frac{3}{R} c_0^2 \int_0^R \frac{16\pi^2}{8^4} \left( \int_0^s |j(\sigma)| \sigma^3 d\sigma \right)^2 ds \right]^{-1} < c_0^2 \]
and hence $a(R) < c_0$. Since $[0, R]$ is arbitrary, $a(r) \leq c_0$ on $[0, \infty[$ which, in particular, implies that $a$ and $\kappa$ exist on $[0, \infty[$. The boundary conditions follow from the compact support of $\rho$ and $j$, the implicit formulas, and the differential equations.\[\square\]
Remark 2. For \( j = 0 \) condition 2.24 reduces to

\[
4\pi \int_0^r \rho(s) s^2 ds < \frac{r}{2}
\]

which is known to be necessary in the case of Schwarzschild coordinates, cf. [28]; notice that \( j = 0 \) implies that \( \kappa = 0 \), and the metric 1.4 reduces to Schwarzschild form. This shows that 2.24 is not an unnatural condition; there is no reason to expect it to be necessary.

Having obtained sufficient control of the field equations we now examine the modified Vlasov equation.

2.2. The modified Vlasov equation 2.1. In the present section we deduce some properties of the modified Vlasov equation 2.1 and its characteristic system under the assumption that the quantities \( a, \alpha, \kappa \), and \( \beta \) are known and regular in the sense of Definition 2.1. The characteristic system is given as

\[
\dot{x} = \frac{\alpha}{a} \frac{v}{\sqrt{1 + |v|^2}} - \frac{\beta x}{r},
\]

\[
\dot{v} = -\frac{\alpha'}{a} \frac{\sqrt{1 + |v|^2} x}{r} + \alpha \kappa \left( v - 3 \frac{x \cdot v}{r} \right),
\]

where \((x, v) \in \mathbb{R}^6, |x| = r, \) and \( t \in I \) for some interval \( I \subset \mathbb{R} \). In order to control the spatial support of the solution \( f \) to the Vlasov equation, we apply a general result from [24] for which we give a proof adapted to our setting. It becomes essential in many different places in what follows.

Lemma 2.6. Let \( I \subset \mathbb{R} \) be an open interval, let \( \rho \) and \( j \) be regular on \( I \times \mathbb{R}^3 \), and let \( a \) and \( \kappa \) be a regular solution of the constraints 1.8 and 1.9 on \( I \times [0, \infty] \). Then for every \( t \in I, r \in [0, \infty) \) the following estimates hold:

\[
\left| \frac{1}{a(t, r)} + r \kappa(t, r) \right| \leq 1, \quad \left| \frac{1}{a(t, r)} - r \kappa(t, r) \right| \leq 1. \tag{2.25}
\]

Proof. Since the proof can be carried out with any fixed \( t \in I \), we drop the dependence on \( t \) and define

\[
\phi_1(r) := \frac{1}{a(r)} - r \kappa(r), \quad \phi_2(r) := \frac{1}{a(r)} + r \kappa(r).
\]

The constraints imply that

\[
\phi'_1 = -\frac{a'}{a^2} - \kappa - r \kappa' = -4\pi ra(\rho - j) - \frac{3}{2} r a \kappa^2 - \frac{1 - a^2}{2ra} + 2\kappa, \tag{2.26}
\]

\[
\phi'_2 = -\frac{a'}{a^2} + \kappa + r \kappa' = -4\pi ra(\rho + j) - \frac{3}{2} r a \kappa^2 - \frac{1 - a^2}{2ra} - 2\kappa.
\]

The right hand side of these equations can be expressed by \( \phi_1 \) and \( \phi_2 \) via the identities

\[
2\phi_1^2 - \phi_1 \phi_2 = 2 \left( \frac{1}{a^2} - 2 \frac{r \kappa}{a} + r^2 \kappa^2 \right) + r^2 \kappa^2 - \frac{1}{a^2} = \frac{r}{a} \left( \frac{1}{ar} + 3r \kappa^2 - 4 \kappa \right),
\]

\[
2\phi_2^2 - \phi_1 \phi_2 = 2 \left( \frac{1}{a^2} + 2 \frac{r \kappa}{a} + r^2 \kappa^2 \right) + r^2 \kappa^2 - \frac{1}{a^2} = \frac{r}{a} \left( \frac{1}{ar} + 3r \kappa^2 + 4 \kappa \right).
\]
Then the following assertions hold:

Inserting this into \(2.26\) yields ordinary differential equations for \(\phi_1\) and \(\phi_2\):

\[
\phi'_1 = -4\pi ra(\rho - \rho_{1\text{a}}) - \frac{a}{2r} \left( \phi_2^2 - 1 + \phi_1(\phi_1 - \phi_2) \right),
\]

\[
\phi'_2 = -4\pi ra(\rho + \rho_{1\text{a}}) - \frac{a}{2r} \left( \phi_2^2 - 1 + \phi_2(\phi_2 - \phi_1) \right).
\]

The boundary conditions of \(a\) and \(r\kappa\) imply that \(\phi_i(0) = 1 = \phi_i(\infty)\). We claim that \(-1 \leq \phi_1, \phi_2 \leq 1\). Assume that there exists \(r \geq 0\) such that \(\phi_1(r) > 1\). Because of the boundary conditions at zero and infinity, \(\phi_1\) then takes on its maximum at some point \(r_{\text{max}} > 0\). If \(\phi_1(r_{\text{max}}) \geq \phi_2(r_{\text{max}})\), then the weak energy condition \(|j| \leq \rho\) and the differential equations above imply that \(\phi'_1(r_{\text{max}}) < 0\), which contradicts the fact that \(\phi_1\) has an interior maximum at \(r_{\text{max}}\). Hence \(\phi_1(r_{\text{max}}) < \phi_2(r_{\text{max}})\), and \(\phi_2\) also takes on its maximum at some point \(\tilde{r}_{\text{max}} > 0\). From the differential equations this implies \(\phi_2'(\tilde{r}_{\text{max}}) < 0\) which leads to the same contradiction as above. Therefore, \(\phi_1 \leq 1\). The remaining bounds are obtained by similar arguments. \(\square\)

**Remark 3.** If \(\alpha\) is a regular solution of the slicing condition 1.10 with \(\beta = \alpha \kappa r\), then

\[
|r\kappa| = \frac{|\beta|}{\alpha} \leq 1, \quad \frac{1}{a} \leq 1, \quad 0 \leq |\beta| \leq \alpha
\]

follow by adding and subtracting the estimates from the lemma above.

The next result is quite standard and can, for example, be found in [27] for the case of Schwarzschild coordinates.

**Proposition 2.7.** Let \(a\), \(\alpha\), and \(\kappa\) be regular solutions to the constraints 1.8, 1.9, and the slicing condition 1.10 on \(I \times [0, \infty)\) for some interval \(I \subset \mathbb{R}\) with \(0 \in I\). Let \(\beta = \alpha \kappa r\) and define \(F(\tau, z) := (F_1, F_2)(\tau, x, v)\) for \(\tau \in I\), \(z = (x, v) \in \mathbb{R}^6\) by

\[
F_1(\tau, x, v) := \frac{\alpha}{\alpha^{\prime} \sqrt{1 + |v|^2}} v - \beta \frac{x}{r},
\]

\[
F_2(\tau, x, v) := \begin{cases} \frac{\alpha^{\prime}}{\alpha} 1 + |v|^2 \frac{x}{r} + \alpha \kappa \left( v - 3 \frac{v \cdot x}{r} \frac{x}{r} \right), & x \neq 0, \\ 0, & x = 0. \end{cases}
\]

Then the following assertions hold:

(a) \(F \in C^{0,1}(I \times \mathbb{R}^6)\).

(b) For every \(t \in I\) and \(z = (x, v) \in \mathbb{R}^6\) the characteristic system

\[
\dot{z} = F(\tau, z)
\]

has a unique solution

\[
I \ni \tau \mapsto Z(\tau, t, z) = (X, V)(\tau, t, x, v)
\]

satisfying \(Z(t, t, z) = z\). In addition, \(Z \in C^1(I^2 \times \mathbb{R}^6)\), \(Z(\tau, t, \cdot, \cdot, \cdot)\) is a \(C^1\)-diffeomorphism of \(\mathbb{R}^6\) with inverse \(Z(t, \tau, \cdot, \cdot, \cdot)\), and

\[
(X, V)(\tau, t, Ax, Av) = (AX, AV)(\tau, t, x, v)
\]

for \(A \in \text{SO}(3)\), \((x, v) \in \mathbb{R}^6\), and \(\tau, t \in I\).

(c) For \(\hat{f} \in C^1_+ (\mathbb{R}^6)\) non-negative and spherically symmetric

\[
f(t, z) := \hat{f} \left( Z(0, t, z) \right), \quad t \in I, \quad z \in \mathbb{R}^6
\]

defines the unique regular solution of the modified Vlasov equation 2.1 with \(f(0) = \hat{f}\). Its support

\[
\text{supp } f(t) = Z(t, 0, \hat{f}) = \left\{ Z(t, 0, z) \mid z \in \text{supp } \hat{f} \right\}
\]
is compact for every \( t \in I \).

**Proof.** The crucial point in part (a) is the regularity at \( r = 0 \) which follows from the boundary conditions there; cf. [27]. From the assumptions on \( a, \alpha, \) and \( \kappa \) as well as Lemma 2.6 and Proposition 2.2 it follows that

\[
|\dot{X}| = \left| \frac{\alpha}{a} \frac{V}{\sqrt{1 + |V|^2}} - \beta \frac{X}{|X|} \right| \leq \alpha \left( \frac{1}{a} + \frac{\beta}{a} \right) \leq \alpha \leq 1.
\]

For arbitrary compact subintervals \( J \subset I \),

\[
|F_2(\tau, x, v)| \leq \frac{|\alpha'|}{a} (1 + |v|) + 4|\alpha \kappa v| \leq C(1 + |v|), \; \tau \in J,
\]

where \( C > 0 \) denotes a constant depending on \( J \) and the given metric coefficients. These two estimates imply that the characteristics exist on \( I \), and the rest of the assertions is standard. \( \square \)

In [29, 27] control of certain derivatives of the characteristics was needed and required quite non-trivial additional arguments. These are not needed in the present situation, the main reason being that the only derivative of a metric component contained in the modified Vlasov equation is \( \alpha' \), but due to the slicing condition 1.10, \( \alpha' \) is differentiable and controlling \( \alpha'' \) does not involve derivatives of the source terms.

We now study the connection between the full and the reduced Einstein-Vlasov system.

### 2.3. Equivalence of the reduced and the full system

In this subsection we prove that any regular solution of the reduced system 1.7–1.10, 1.13–1.15, and 2.1 is a solution to the full system 1.6–1.16; the reverse assertion is obvious. Once 1.6 holds, the Vlasov equations 1.12 and 2.1 become equal.

**Proposition 2.8.** Let \((f, a, \alpha, \kappa)\) be a regular solution of the reduced system 1.7–1.10, 1.13–1.15, and 2.1 satisfying the boundary conditions 1.5. Then the equations 1.6 and 1.11 hold, with \( p \) given by 1.16.

**Proof.** We start by investigating \( \dot{a} \), since in the process of deriving 1.6 we also obtain an equation for \( \dot{\kappa} \).

**Step 1: Computation of \( \dot{\rho}, \dot{j}, \) and \( \dot{a} \).** The key to deducing Eqn. 1.6 is calculating the time derivatives of \( \rho \) and \( j \). We insert the modified Vlasov equation 2.1 and integrate by parts:

\[
\dot{\rho} = \text{div} \left[ \int \left( \beta \sqrt{1 + |v|^2} \frac{x}{r} - \frac{\alpha}{a} v \right) f \, dv \right] - \left( \frac{\beta' + 2\beta}{r} \right) \rho - \left( \frac{\alpha'}{a} + \frac{\alpha a'}{a^2} \right) j + \alpha \kappa (S - 3p), \tag{2.27}
\]

\[
\dot{j} = \int \frac{x \cdot v}{r} \left( \frac{\beta}{r} \frac{x}{\sqrt{1 + |v|^2}} - \frac{\alpha}{a} \frac{v}{\sqrt{1 + |v|^2}} \right) : \partial_x f \, dv - \frac{\alpha'}{a} (p + \rho) - 2\alpha \kappa j. \tag{2.28}
\]

By Proposition 2.2,

\[
\frac{1}{a^2(t, r)} = 1 - \frac{8\pi}{r} \int_0^r \rho(t, s) s^2 ds - \frac{3}{r} \int_0^r \kappa^2(t, s) s^2 ds
\]
which, after taking the derivative with respect to $t$, gives

$$\frac{\dot{a}}{a^3} = \frac{4\pi}{r} \int_0^r \rho(t,s)s^2 ds + \frac{3}{r} \int_0^r \kappa(t,s)\kappa(t,s)s^2 ds. \quad (2.29)$$

We calculate the terms on the right hand side separately using 2.27 and 2.28.

**Step 2: The integral over $\dot{\rho}$**. The constraint equations 1.8 and 1.9 imply that

$$\beta' = \alpha \kappa + \alpha' \kappa r + \alpha \kappa r = -\frac{2\beta}{r} + \frac{\alpha'}{\alpha} \beta - 4\pi a\alpha r\gamma,$$

$$\alpha \frac{d'}{a^2} = \frac{3}{2} \alpha \kappa^2 r + 4\pi a\alpha r + \alpha \left(\frac{1}{a} - a\right).$$

Using 2.27, Gauß’s theorem, spherical symmetry, and the above relations we find that

$$4\pi \int_0^r \rho(t,s)s^2 ds = 4\pi r^2 \left(\beta \rho - \frac{\alpha'}{a} j\right)$$

$$+ 4\pi \int_0^r \left(\left(-\frac{\alpha'}{a} - \frac{3}{2} a \alpha \kappa^2 + \frac{\alpha}{2s} \left(a - \frac{1}{a}\right)\right) j - \frac{\alpha' \beta}{\alpha} \rho + \alpha \left(S - 3p\right)\right) s^2 ds. \quad (2.30)$$

**Step 3: The integral over $\dot{\kappa}$**. Eqn. 2.8 implies that

$$\dot{\kappa} = -\frac{4\pi}{r^3} \int_0^r (\dot{a} j + a j) s^3 ds. \quad (2.31)$$

Using 2.28, Gauß’s theorem, and spherical symmetry implies that

$$4\pi \int_0^r a \ j \ s^3 ds = 4\pi r^3 (\alpha \beta \gamma - \alpha p)$$

$$+ 4\pi \int_0^r (\alpha S - 3\alpha j - \alpha' \rho s) s^2 ds - 4\pi \int_0^r (\alpha S - 3\alpha j) s^2 ds.$$

Next we integrate equation 2.10 observing the boundary conditions $\alpha'(t,0) = \mu(t,0)$ = 0, and divide by $r^3$ to find that

$$\frac{\alpha'}{r a^2} - \frac{\alpha}{2r^2} \left(1 - \frac{1}{a^2}\right) = \frac{1}{r^3} \int_0^r \left(4\pi (\alpha S - \alpha' \rho s) + \frac{9}{2} \kappa^2 s^2 \right) s^2 ds - \frac{1}{r^3} \int_0^r \kappa^2 a' s^3 ds$$

$$= \frac{4\pi}{r^3} \int_0^r (\alpha S - \alpha' \rho s - 3a \beta j) s^2 ds - \frac{3}{2} \kappa^2 a,$$

where we integrated by parts to get rid of the term involving $\frac{3}{2} \kappa^2 \alpha' s^3$ and used the momentum constraint 1.9 to deal with the term arising from $\kappa'$. If we combine 2.31, 2.32, and 2.33 it follows that

$$\dot{\kappa} = 4\pi (\alpha p - a \beta j) - \frac{\alpha'}{r a^2} + \frac{\alpha}{2r^2} \left(1 - \frac{1}{a^2}\right) - \frac{3}{2} \kappa^2 a + \frac{4\pi}{r^3} \int_0^r (\alpha S - \alpha' \rho s - 3a \beta j) s^2 ds.$$

We observe that, apart from the integral which vanishes once 1.6 holds, this is precisely field equation 1.11.

By reinserting $\kappa'$ via 1.9 into 2.34,

$$3\kappa^2 = 12\pi r^2 a \rho - \frac{3\alpha'}{a^2} + \frac{3\alpha}{2} \left(1 - \frac{1}{a^2}\right) + \frac{9}{2} r^2 \kappa^2 a + 3r^2 \beta \kappa' + 3r^2 I_1(r)$$
where we abbreviate
\[ I_1(r) := \frac{4\pi}{r^3} \int_0^r ((a\beta)' + 2a\alpha\kappa - \dot{a}) s^3 ds. \]

We use 2.10 again to obtain the following lengthy expression:
\[
3 \int_0^r \kappa s^2 ds = \int_0^r \left( 12\pi \alpha \kappa s^2 - \frac{3\alpha' \kappa}{a^2} s + \frac{3\alpha \kappa}{2} \left( 1 - \frac{1}{a^2} \right) \right)
+ 4\pi s^2 \kappa (\alpha' \rho s - \alpha S) + \frac{3}{2} s^3 \kappa^3 \alpha'^2 \right) ds + 3 \int_0^r \kappa I_1(s) s^2 ds
+ 3 \int_0^r \beta \kappa s^2 ds + \int_0^r \kappa \left( \frac{s^2}{a^2} \alpha' - \alpha \mu \right)' ds \tag{2.35}
\]

We integrate the last term by parts and use the field equation 1.9 to find that
\[
\int_0^r \kappa \left( \frac{s^2}{a^2} \alpha' - \alpha \mu \right)' ds = \kappa \left( \frac{s^2}{a^2} \alpha' - \alpha \mu \right)
+ \int_0^r \left( \frac{3\alpha' \kappa}{a^2} s - \frac{3\alpha \kappa}{2} \left( 1 - \frac{1}{a^2} \right) + 4\pi s^2 \frac{\alpha'}{a} j - 4\pi a \alpha \mu j \right) ds
\]
which eliminates several terms in 2.35 above and yields
\[
3 \int_0^r \kappa s^2 ds = \kappa \left( \frac{s^2}{a^2} \alpha' - \alpha \mu \right) + 4\pi \int_0^r \left( \frac{\alpha'}{a} j + \kappa \alpha' \rho s - \alpha \kappa S + 3\alpha \kappa p \right) s^2 ds
- 4\pi \int_0^r a \alpha \mu j ds + \int_0^r \left( \frac{3}{2} \kappa^3 \alpha' s + 3\beta \kappa \kappa \right) s^2 ds + 3 \int_0^r \kappa I_1(s) s^2 ds. \tag{2.36}
\]

**Step 4: Calculation of \( \dot{a} \).** The constraints 1.8 and 1.9 give rise to the equation
\[
\frac{r}{a^3} (\dot{a} - (a\beta)' - 2a\alpha\kappa) = \frac{r\dot{a}}{a^3} + 4\pi r^2 \left( \frac{\alpha}{a} j - \beta \rho \right) - \frac{3}{2} \kappa^3 \alpha' + \alpha \kappa \mu - \frac{3\kappa}{a^2} \alpha'.
\]
We insert this as well as 2.30 and 2.36 into 2.29, use 1.9, and integrate by parts to finally arrive at
\[
\frac{r}{a^3} (\dot{a} - (a\beta)' - 2a\alpha\kappa) = 12\pi \int_0^r \kappa(s) \left( \int_0^s \frac{j(s') \sigma^3}{s} ((a\beta)' + 2a\alpha\kappa - \dot{a}) ds' \right) ds.
\]

Gronwall’s lemma implies that the left hand side of this identity vanishes on \( I \times [0, \infty] \), and thus equation 1.6 is satisfied. Considering 2.34, we also get 1.11 on \( I \times [0, \infty] \), and the proof is complete. \( \square \)

3. **Local existence and continuation of solutions; conservation laws.** In this section we prove our local existence and continuation result, Theorem 1.1. As we showed in Subsection 2.3 it is sufficient to do this for the reduced system 1.7–1.10, 1.13–1.15, 2.1. Once Theorem 1.1 is established we also consider some conservation laws for the system, which however are not used in the proof of that theorem. Before we proceed, we make precise the relevant solution concept.

**Definition 3.1.** Let \( I \subset \mathbb{R} \) be an interval. A regular function \( f : I \times \mathbb{R}^6 \to [0, \infty] \) is a regular solution of the spherically symmetric, asymptotically flat Einstein-Vlasov system in maximal areal coordinates 1.6–1.16 if the source terms defined in 1.13–1.16
are regular, the field equations 1.6–1.11 have regular solutions \((a, \alpha, \kappa)\) on \(I \times [0, \infty]\) obeying the boundary conditions 1.5, and the Vlasov equation 1.12 holds on \(I \times \mathbb{R}^6\).

The concept of a regular solution to the reduced system 1.7–1.10, 1.13–1.15, 2.1 is defined analogously.

Due to 1.7, \(\beta = r \alpha \kappa \) inherits its regularity from \(\alpha \) and \(\kappa\), cf. Remark 1.

By the results in the previous section we can solve the field equations for given source terms, and given the metric we can solve the Vlasov equation, the solution of which yields new source terms. The resulting iteration scheme can be shown to converge to the desired solution. This procedure is quite standard in kinetic theory and has been applied to, for example, the Vlasov-Poisson system [26], the Vlasov-Maxwell system [15], and the Einstein-Vlasov system in Schwarzschild coordinates [27]. As opposed to the latter investigation, the structure of the modified Vlasov equation 2.1 ensures that we do not need to control derivatives of the metric coefficients that are not directly controlled by the source terms.

Throughout the rest of this section we assume that \(\hat{f}\) be as assumed in Theorem 1.1; without loss of generality, \(\hat{f} \neq 0\).

### 3.1. The iterative scheme.

We start the iteration by defining

\[
\begin{align*}
ap_0 &= a_0 := 1, \quad \kappa_0 := 0, \quad \rho_0 := \gamma_0 := S_0 := p_0 := 0, \quad f_0 := \hat{f}, \\
T_0 &= \infty, \quad R_0 := \|\hat{f}\|_{\text{supp} \hat{f}}, \quad V_0 := \|v\|_{(x,v) \in \text{supp} \hat{f}}.
\end{align*}
\]

We can assume \(R_0, V_0 \geq 1\) by making them larger if necessary. If \(a_{n-1}, \alpha_{n-1}\), and \(\kappa_{n-1}\) are already defined and regular on \([0, T_{n-1}] \times [0, \infty]\) with \(T_{n-1} > 0\), we define \(F_{n-1}(\tau, x, v)\) for \((\tau, x, v) \in [0, T_{n-1}] \times \mathbb{R}^6\) exactly as \(F\) in Proposition 2.7, with \(\alpha, \alpha\), and \(\kappa\) replaced by \(a_{n-1}, \alpha_{n-1}\), and \(\kappa_{n-1}\) For every \(t \in [0, T_{n-1}]\) and \(z = (x, v) \in \mathbb{R}^6\) we let

\[
Z_n(\cdot, t, z) = (X_n, V_n)(\cdot, t, x, v)
\]

be the unique solution to the initial value problem

\[
\dot{z} = F_{n-1}(\tau, z), \quad Z_n(t, t, z) = z.
\]

Then the solution to the corresponding Vlasov equation is given by

\[
f_n(t, z) := \hat{f}(Z_n(0, t, z)), \quad t \in [0, T_{n-1}], \quad z \in \mathbb{R}^6.
\]

The phase space density \(f_n\) induces new source terms \(\rho_n, j_n, S_n, p_n\) as in 1.13–1.16. To complete the iteration step we use Proposition 2.2 and define \(a_n, \alpha_n\), and \(\kappa_n\) as the regular solutions to the field equations 1.8–1.10 on \([0, T_n] \times [0, \infty]\), with source terms given by \(\rho_n, j_n\), and \(S_n\). The solutions can be defined as long as \(a_n\) remains bounded, i.e., for \(t\) smaller than

\[
T_n := \sup \left\{ t \in [0, T_{n-1}[ \mid 4\pi \int_0^r \rho_n(\tau, s)s^2ds + \frac{3}{2} \int_0^r \kappa_n^2(\tau, s)s^2ds < \frac{r}{2}, \quad r \geq 0, \quad \tau \in [0, t] \right\}.
\]

**Lemma 3.2.** The iterative scheme is well defined, i.e., \(T_n > 0\), and the functions \(f_n, \rho_n, j_n, S_n, p_n, a_n, \alpha_n\), and \(\kappa_n\) exist and are regular on \([0, T_n]\). For every \(n \in \mathbb{N}\) the estimates

\[
\left| \frac{1}{a_n} \pm r\kappa_n \right| = \left| \frac{1}{a_n} \pm \frac{\beta_n}{\alpha_n} \right| \leq 1, \quad 0 < \alpha_n \leq 1
\]

and the formulas of Proposition 2.2 hold on \([0, T_n] \times [0, \infty]\).
3.2. Uniform bounds. We need to make sure that $T_n$ does not converge to zero and $\|a_n(t)\|$ does not blow up for $n$ tending to infinity. To ensure this, we control the momentum support of $f_n$ and the supremum of $a_n$ uniformly in $n$ via

$$P_n(t) := \sup \{ |v| \mid (x,v) \in \text{supp} f_n(\tau), \ 0 \leq \tau \leq t \}$$

$$= \sup \left\{ |V_n(\tau,0,x,v)| \mid (x,v) \in \text{supp} \tilde{f}, \ 0 \leq \tau \leq t \right\}, \quad (3.1)$$

$$Q_n(t) := \sup \left\{ a_n(\tau,r) \mid 0 \leq \tau \leq t, \ r \geq 0 \right\}. \quad (3.2)$$

First, we bound various quantities by $P_n(t)$ and $Q_n(t)$.

**Lemma 3.3.** There exists a constant $C = C(\|\tilde{f}\|) > 0$ such that for $n \in \mathbb{N}$ and $t \in [0, T_n]$,\n
(a) $\|\rho_n(t)\|, \|j_n(t)\|, \|S_n(t)\|, \|p_n(t)\| \leq C (1 + P_n(t))^4$

(b) $\|\kappa_n(t)\| \leq C Q_n^{1/2}(t) (1 + P_n(t))^2$, $\left\| \frac{\kappa_n(t)}{r} \right\| \leq C Q_n(t) (1 + P_n(t))^4$

(c) $\left\| \frac{\alpha_n' (t)}{a_n(t)} \right\| \leq C (R_0 + t) Q_n^2(t) (1 + P_n(t))^4$

(d) $\left\| r \frac{\beta_n' (t)}{a_n^2(t)} \right\| \leq C (R_0 + t)^2 Q_n(t) (1 + P_n(t))^4$

(e) $\left\| \beta_n (t) + 2 \frac{\beta_n}{r}(t) \right\| \leq C (R_0 + t) Q_n^3(t) (1 + P_n(t))^4$

**Proof.** Due to Proposition 2.7, $Z_n(\tau, t, \cdot)$ is, for every $\tau, t \in [0, T_n]$ and $n \in \mathbb{N}$, a $C^1$-diffeomorphism of $\mathbb{R}^6$. Hence $\|\tilde{f}\| = \|f_n(t)\|$, and part (a) follows from the definition of $P_n(t)$. By Lemma 3.2,

$$|X_{n+1}| \leq |\alpha_n| \left( \frac{1}{a_n} + \frac{|\beta_n|}{a_n} \right) \leq 1,$$

which implies $f_n(t, x, v) = 0$ for $r > R_0 + t, t \in [0, T_n], n \in \mathbb{N}$. Hence the source terms vanish for $r > R_0 + t$ as well. To estimate $\kappa$, we let $R > 0$ and use part (a) to obtain

$$|\kappa_n(t, r)| \leq C R |a_n(t)| |j_n(t)| \leq C R Q_n(t) (1 + P_n(t))^4, \quad r < R,$$

$$|\kappa_n(t, r)| \leq \frac{1}{r} \leq \frac{1}{R}, \quad r \geq R,$$

where the second line follows from the estimates in Lemma 3.2. Choosing

$$R = R(t) = Q_n^{-1/2}(t) (1 + P_n(t))^{-2}$$

we arrive at the first estimate in part (b). The second estimate in part (b) is a direct consequence of estimates which have already been carried out. For part (c) we consider the implicit formula for $\alpha_n'$ which follows from 2.9. Thus

$$0 < \frac{\alpha_n'(t,r)}{a_n(t,r)} \leq C (R_0 + t) Q_n^2(t) (1 + P_n(t))^4$$

via (a), (b), and $r \leq R_0 + t$. We prove the assertion in part (d) by using the constraint 1.8, i.e.,

$$r \frac{\alpha_n'}{a_n^2} = 4\pi r^2 \rho_n a_n + \frac{3}{2} r^2 \kappa_n^2 a_n + \frac{1 - a_n^2}{2 a_n}.$$
This implies that
\[ r \frac{d'_{n}}{a_{n}^2} \leq C(R_0 + t)^2 Q_n(t) (1 + P_n(t))^4, \]
where we used the previous estimates, and the facts that \( \kappa_n^2 r^2 \leq 1 \)—cf. Lemma 3.2—
and \( |1 - a_n^2| \leq a_n^2 \) since \( a_n \geq 1 \). Expressing the constraint 1.9 via \( \beta_n = r \alpha_n \kappa_n \)
gives
\[ \beta_n' + \frac{2 \beta_n}{r} \leq 4 \pi a_n \alpha_n |j_n r| + |\alpha_n' \kappa_n r| \leq C(R_0 + t) Q_n^3(t) (1 + P_n(t))^4, \]
after inserting the estimates for \( \alpha_n', j_n, \) and \( |\kappa_n r| \leq 1. \]

The essential step is to bound \( T_n \) away from zero and to bound \( P_n \) and \( Q_n \)
uniformly in \( n \) on a suitable time interval.

**Lemma 3.4.** There exists \( T > 0 \) and continuous functions \( z_1, z_2 : [0, T] \rightarrow [0, \infty] \)
such that for every \( t \in [0, T] \) and \( n \in \mathbb{N} \),
\[ T_n \geq T, \quad P_n(t) \leq z_1(t), \quad Q_n(t) \leq z_2(t). \]

**Proof.** Let \( n \in \mathbb{N} \) be arbitrary. Lemma 3.3 implies that for every \( 0 \leq \tau \leq t < T_{n+1} \)
and \( z \in \mathbb{R}^6 \),
\[ |\tilde{V}_{n+1}(\tau, 0, z)| \leq C(R_0 + t) Q_n^2(t) (1 + P_n(t))^4 (1 + P_{n+1}(t)); \]
as above \( C = C(\tilde{f}) > 0 \) denotes a constant which is independent of \( t \) and \( n \) and
may from now on change from line to line. By the above inequality,
\[ P_{n+1}(t) \leq V_0 + C \int_0^t (R_0 + \tau) Q_n^2(\tau) (1 + P_n(\tau))^4 (1 + P_{n+1}(\tau)) \, d\tau. \]  
(3.3)
The more difficult issue is to estimate \( \dot{a}_{n+1} \); we cannot use the field equation 1.6,
so we cannot directly use the reduced system. We consider the
implicit formula for \( a \) in 2.7 together with the formulas for \( \dot{\rho} \) and \( k \) in 2.30 and 2.36
applied to the iterates, and then try to generate a Gronwall loop. This results in the estimate
\[ Q_{n+1}(t) \leq Q_0 + C \int_0^t (R_0 + \tau)^7 Q_n^5(\tau) Q_{n+1}^7(\tau) (1 + P_n(\tau))^4 (1 + P_{n+1}(\tau))^{12} \cdot \exp \left( C(R_0 + \tau)^5 Q_{n+1}^4(\tau) (1 + P_{n+1}(\tau))^8 \right) \, d\tau \]  
(3.4)
which holds for \( t \in [0, T_{n+1}] \), where we set
\[ Q_0 := Q_n(0) = \sup_{r > 0} \left( 1 - \frac{8 \pi}{r} \int_0^r \rho_n(0,s)s^2 ds - 3 \frac{r}{\kappa_n^2(s)} \int_0^r \kappa_n^2(s) s^2 ds \right)^{-\frac{1}{2}}. \]
In Lemma A.1 we show how this is done; note that \( Q_0 \) is actually independent of \( n \)
since \( \rho_n(0, \cdot) = \dot{\rho}, \quad j_n(0, \cdot) = j, \) and the corresponding solutions \( a_n(0, \cdot), \kappa_n(0, \cdot) \) are
unique, cf. Proposition 2.2.
To prove the assertions of the lemma, let \((z_1, z_2): [0, T] \rightarrow \mathbb{R}^2\) be the maximal solution to the following system of integral equations:

\[
    z_1(t) = V_0 + C \int_0^t (R_0 + \tau) z_2^2(\tau) (1 + z_1(\tau))^5 \, d\tau,
\]

\[
    z_2(t) = Q_0 + C \int_0^t (R_0 + \tau)^7 z_2^{10}(\tau) (1 + z_1(\tau))^{16}
    \cdot \exp \left( C(R_0 + \tau)^5 z_2^4(\tau) (1 + z_1(\tau))^8 \right) \, d\tau.
\]

By induction,

\[
P_{n+1}(t) \leq z_1(t), \quad Q_{n+1}(t) \leq z_2(t), \quad t \in [0, T] \cap [0, T_{n+1}].
\]

Since \(\lim_{t \to T_{n+1}} |Q_{n+1}(t)| = \infty\), this bound implies that \(T_{n+1} \geq T\). This completes the proof. \(\square\)

### 3.3. Uniform bounds for certain derivatives and uniform convergence.

We need to establish uniform bounds on certain derivatives. This will lead to the uniform convergence of these derivatives and \(f_n\).

**Lemma 3.5.** There exists an increasing, continuous function \(C: [0, T] \rightarrow [0, \infty]\) such that for all \(t \in [0, T]\), \(n \in \mathbb{N}\), and \(U > 0\),

\[
    \|a'_n(t)\|, \|\alpha'_n(t)\|, \|\beta'_n(t)\|, \|\alpha''_n(t)\|, \|\alpha'''_n(t)\| \leq C(t),
\]

\[
    \sup \{ \|\partial_z F_n(t, x, v)\| \mid x \in \mathbb{R}^3, |v| \leq U \} \leq C(t) (1 + U).
\]

**Proof.** By Lemma 3.3 and Lemma 3.4,

\[
    \|a'_n(t)\| \leq C(R_0 + t) z_2^2(t) (1 + z_1(t))^4 \leq C(t), \quad t \in [0, T].
\]

The bound on \(\|a'_n(t)\|\) is obtained similarly. By 1.9 and Lemma 3.3,

\[
    |\alpha'_n(t, r)| \leq 3 \left\| \frac{\kappa_n(t)}{r} \right\| + 4\pi \|J_n(t)\| \|a_n(t)\| \leq C(t).
\]

A bound for \(\beta'_n\) follows from \(\beta_n = r\alpha_n \kappa_n\). To control \(\alpha''_n\) we use 2.9, applied to our case with index \(n\), and take its derivative with respect to \(r\). By the previous bounds this gives the estimate for \(\|\alpha''_n(t)\|\). Examining \(\partial_z F_n(t, x, v)\) proves the remaining assertion. \(\square\)

We now estimate the difference of the source or metric terms corresponding to two consecutive iterates in terms of the difference in \(f_n\).

**Lemma 3.6.** Let \(0 < \delta < T\). There exists \(C = C(\delta, \hat{f}) > 0\) such that for all \(t \in [0, \delta]\) and \(n \in \mathbb{N}\),

\[
    \|\rho_{n+1}(t) - \rho_n(t)\|, \|S_{n+1}(t) - S_n(t)\|, \|J_{n+1}(t) - J_n(t)\|, \|p_{n+1}(t) - p_n(t)\|,
    \|a_{n+1}(t) - a_n(t)\|, \|\alpha_{n+1}(t) - \alpha_n(t)\|, \|\kappa_{n+1}(t) - \kappa_n(t)\|, \|\beta_{n+1}(t) - \beta_n(t)\|,
    \left\| \frac{a_{n+1}(t)}{r} - \frac{a_n(t)}{r} \right\|, \left\| \frac{\kappa_{n+1}(t)}{r} - \frac{\kappa_n(t)}{r} \right\| \leq C\|f_{n+1}(t) - f_n(t)\|.
\]

**Proof.** We fix some \(0 < \delta < T\). In what follows, every assertion holds for \(t \in [0, \delta]\), \(r \in [0, \infty]\), and \(n \in \mathbb{N}\) unless stated otherwise. By Lemma 3.3 and Lemma 3.4,

\[
    \|\rho_{n+1}(t) - \rho_n(t)\|, \|S_{n+1}(t) - S_n(t)\|, \|J_{n+1}(t) - J_n(t)\|, \|p_{n+1}(t) - p_n(t)\| \leq C\|f_{n+1}(t) - f_n(t)\|; \quad (3.5)
\]
here and in what follows, \( C > 0 \) denotes a constant that may only depend on \( f \) and \( \delta \). Every quantity from Lemma 3.3 and Lemma 3.5 remains bounded uniformly on \([0, \delta]\) and in \( n \). In particular, 
\[
|\kappa_n(t, r)| \leq \frac{C}{1 + r^3}.
\]
Let us examine the assertions for \( a_{n+1} - a_n \) and \( \kappa_{n+1} - \kappa_n \). We define 
\[
\mu_n(t, r) := \frac{r}{2} \left(1 - \frac{1}{a_n^2(t, r)}\right) = 4\pi \int_0^r \rho_n(t, s)s^2 ds + \frac{3}{2} \int_0^r \kappa_n^2(t, s)s^2 ds
\]
and proceed similarly to the first step of the proof of Lemma 2.3. We use the mean value theorem, the formulas for \( \mu_n, \kappa_n \) as well as \( \rho_n = j_n = 0 \) for \( r > R_0 + t \), and 3.5 to find that 
\[
|a_{n+1}(t, r) - a_n(t, r)| \leq C |\mu_{n+1}(t, r) - \mu_n(t, r)|
\]
\[
\leq \|f_{n+1}(t) - f_n(t)\| + C \int_0^r \frac{s}{1 + s^3} \sup_{\sigma \in [0, s]} |a_{n+1}(t, \sigma) - a_n(t, \sigma)| ds.
\]
Thus 
\[
\sup_{s \in [0, r]} |a_{n+1}(t, s) - a_n(t, s)| \leq C \|f_{n+1}(t) - f_n(t)\|
\]
\[
+ C \int_0^r \frac{s}{1 + s^3} \sup_{\sigma \in [0, s]} |a_{n+1}(t, \sigma) - a_n(t, \sigma)| ds
\]
so that by Gronwall’s lemma, 
\[
\|a_{n+1}(t) - a_n(t)\| \leq C \|f_{n+1}(t) - f_n(t)\|.
\]

The identity 
\[
\kappa_{n+1}(t, r) - \kappa_n(t, r) = \frac{4\pi}{r^3} \int_0^r (a_n(t, s)j_n(t, s) - a_{n+1}(t, s)j_{n+1}(t, s)) s^3 ds
\]
readily implies the claim for \( \kappa_{n+1} - \kappa_n \). The differences 
\[
\frac{a_{n+1}}{r} - \frac{a_n}{r}, \quad \frac{\kappa_{n+1}}{r} - \frac{\kappa_n}{r}
\]
can be estimated analogously, since there still is some power of \( s \) left in the corresponding integrals, which can be used to eliminate \( r^{-1} \).

For \( \alpha_{n+1} - \alpha_n \) we need to choose a different approach. We recall the strategy employed in the proof of Proposition 2.2, where we solved the initial value problem 
\[
\left( \frac{\ddot{\alpha}r^2}{a} \right)' = \left( 6\kappa^2 a + 4\pi (S + \rho)a \right) r^2 \alpha, \quad \ddot{\alpha}(0) = 1 \quad \text{(3.6)}
\]
and set \( \alpha = \ddot{\alpha}/\ddot{\alpha}_\infty \) to solve the actual equation. We employ the same strategy here and denote by \( \ddot{\alpha}_n(t) \) the solution to 3.6 where \( a, \kappa, S, \) and \( \rho \) are replaced by the corresponding iterates with subscript \( n \). As shown in the proof of Proposition 2.2, \( \ddot{\alpha}_{n, \infty}(t) := \lim_{r \to \infty} \ddot{\alpha}_n(t, r) \geq 1 \) exists. We will prove 
\[
\|\ddot{\alpha}_{n+1}(t) - \ddot{\alpha}_n(t)\| \leq C \|f_{n+1}(t) - f_n(t)\| \quad \text{(3.7)}
\]
and Lemma 3.6 it follows that
\[ |\alpha_{n+1}(t,r) - \alpha_n(t,r)| = \left| \frac{\hat{\alpha}_{n+1}(t,r) - \hat{\alpha}_n(t,r)}{\hat{\alpha}_{n+1,\infty}(t) - \hat{\alpha}_{n,\infty}(t)} \right| \leq |\hat{\alpha}_{n+1} - \hat{\alpha}_n| + |\hat{\alpha}_{n+1,\infty} - \hat{\alpha}_{n,\infty}| \leq C \|f_{n+1}(t) - f_n(t)\|, \]
and so complete the proof. In order to prove 3.7, we integrate the initial value problem twice in \( r \), use the boundedness of various quantities, and apply the triangle inequality several times to get
\[ |\hat{\alpha}_{n+1}(t,r) - \hat{\alpha}_n(t,r)| \leq C \int_0^r \frac{1}{s^2} \int_0^s \left( |S_{n+1}(\sigma) - S_n(\sigma)| + |\rho_{n+1}(\sigma) - \rho_n(\sigma)| \right) \sigma^2 d\sigma ds + \int_0^r \frac{1}{s^2} \int_0^s \left( |a_{n+1}(s) - a_n(s)| + |a_{n+1}(\sigma) - a_n(\sigma)| \right) \sigma^2 \kappa_n^2(\sigma) d\sigma ds + \int_0^r \frac{1}{s^2} \int_0^s |\kappa_{n+1}(\sigma) + \kappa_n(\sigma)| |\kappa_{n+1}(\sigma) - \kappa_n(\sigma)| \sigma^2 d\sigma ds. \]
The first integral can be controlled by 3.5 and \( \rho_{n+1}(t,r) = S_{n+1}(t,r) = 0 \) for \( r > R_0 + t \). By
\[ |\kappa_n(t,\sigma)| \leq \frac{C}{1 + \sigma^3} \]
and the estimates for \( \kappa_{n+1} - \kappa_n \) and \( a_{n+1} - a_n \),
\[ \sup_{\sigma \in [0,r]} |\hat{\alpha}_{n+1}(t,\sigma) - \hat{\alpha}_n(t,\sigma)| \leq C \|f_{n+1}(t) - f_n(t)\| + C \int_0^r \sup_{\sigma \in [0,s]} |\hat{\alpha}_{n+1}(t,\sigma) - \hat{\alpha}_n(t,\sigma)| ds. \]
Now Gronwall’s lemma proves 3.7, the difference \( \beta_{n+1} - \beta_n \) can be dealt with via \( \beta_n = r\alpha_n\kappa_n \), and the proof is complete.

For the uniform convergence of \( f_n \) we need to control \( F_{n+1} - F_n \). In order to achieve this, we establish further estimates for certain differences of derivatives.

**Lemma 3.7.** Let \( 0 < \delta < T \). There exists \( C = C(\delta, \hat{f}) > 0 \) such that for \( t \in [0, \delta] \) and \( n \in \mathbb{N} \),
\[ ||a_n'(t) - a_n'(t)||, ||\kappa_n'(t) - \kappa_n'(t)||, ||\alpha_n'(t) - \alpha_n'(t)||, ||\alpha_n''(t) - \alpha_n''(t)|| \leq C \|f_{n+1}(t) - f_n(t)\|. \]

**Proof.** We fix some \( 0 < \delta < T \); every statement in this proof holds for \( t \in [0, \delta] \), \( r \in [0, \infty] \), and \( n \in \mathbb{N} \) unless stated otherwise. From
\[ a_n' = 4\pi r \rho_n a_n^3 + \frac{3}{2} n^2 a_n^3 + \frac{a_n}{2r} \left( 1 - a_n^2 \right) \]
and Lemma 3.6 it follows that
\[ |a_{n+1}'(t) - a_n'(t)| \leq C \|f_{n+1}(t) - f_n(t)\|. \]
The momentum constraint 1.9 allows for the estimate
\[ |\kappa_{n+1}'(t,r) - \kappa_n'(t,r)| \leq 3 \left( \frac{\kappa_{n+1}}{r} - \frac{\kappa_n}{r} \right) + 4\pi |a_{n+1}j_{n+1} - a_n j_n| \leq C \|f_{n+1}(t) - f_n(t)\|. \]
Eqn. 2.9 for $\alpha'_n$ implies the claim for the difference $\alpha'_{n+1} - \alpha'_n$ after using the asymptotic behavior

$$|\kappa_n(t, r)| \leq \frac{C}{1 + r^3}$$

to keep the integral in 2.9 bounded. In addition,

$$\left| \frac{\alpha'_{n+1}(t, r)}{r} - \frac{\alpha'_n(t, r)}{r} \right| \leq \|f_{n+1}(t) - f_n(t)\|$$

which similarly to above gives

$$|\alpha''_{n+1}(t, r) - \alpha''_n(t, r)| \leq C\|f_{n+1}(t) - f_n(t)\|$$

because $\alpha_n$ solves the differential equation

$$\alpha''_n = \alpha'_n \left( \frac{a'_n}{a_n} - \frac{2}{r} \right) + (6\kappa^2_n + 4\pi(S_n + \rho_n)) \frac{a^2_n}{a_n}$$

the same kind of arguments were already carried out earlier, and the proof is complete.

3.4. **Proof of the main result.** After the above preparations we can complete the proof of our main result, Theorem 1.1, proceeding in four steps: in Step 1 we prove the convergence of the iterates, in Step 2 we show that the corresponding limit is a local solution of the reduced, and hence by Proposition 2.8, of the full system, uniqueness is shown in Step 3, and the continuation criterion in Step 4.

**Step 1: Convergence of the iterates.** We prove that the sequence of iterates $(f_n)$ and all the quantities the differences of which were estimated in Lemma 3.6 and Lemma 3.7 converge uniformly on $[0, \delta]$. Furthermore, for fixed $t \in [0, \delta]$, $z = (x, v)$ with $|v| \leq z_1(\delta)$, and $x \in \mathbb{R}^3$ the functions $Z_n(\cdot, t, z)$, $\tilde{Z}_n(\cdot, t, z)$ converge uniformly on $[0, t]$, where $z_1(\delta)$ is given by Lemma 3.4. The right hand side $F_n$ of the characteristic system converges uniformly on $[0, \delta] \times \mathbb{R}^3 \times B_R(0)$ for every fixed $R > 0$.

To see this, we first examine the relevant characteristics further and combine previous results to gain essential estimates for $F_n$, $Z_n$, and $\tilde{Z}_n$. We first bound $Z_n$ on a set that is independent of $n$ and that contains all supports of $f_n(t)$. For $0 \leq \tau \leq t \leq \delta$, $x \in \mathbb{R}^3$, $|v| \leq z_1(\delta)$, and $z = (x, v)$,

$$|X_n(\tau, t, z)| \leq 1, \quad |V_n(\tau, t, z)| \leq z_1(\delta) + C \int_\tau^t (1 + |V_n(s, t, z)|) \, ds,$$

and by Gronwall’s lemma,

$$|Z_n(\tau, t, z)| \leq C_\delta;$$

the constant $C_\delta$ only depends on $\delta$ and $\dot{f}$. By Lemma 3.5,

$$\sup \{|\partial_z F_n(t, z)| \mid |z| \leq C_\delta, t \in [0, \delta]\} \leq C. \quad (3.8)$$

The already proven estimates imply that

$$|F_{n+1}(t, x, v) - F_n(t, x, v)| \leq C\|f_{n+1}(t) - f_n(t)\|, \quad t \in [0, \delta], \quad x \in \mathbb{R}^3, \quad |v| \leq C_\delta. \quad (3.9)$$
Hence we can apply the mean value theorem together with 3.8 and 3.9 for $x \in \mathbb{R}^3$ and $|v| \leq z_1(\delta)$ to find that

$$\left| \dot{Z}_{n+1}(\tau, t, z) - \dot{Z}_n(\tau, t, z) \right|$$

$$\leq \left| F_n(\tau, Z_{n+1}(\tau, t, z)) - F_n(\tau, Z_n(\tau, t, z)) \right|$$

$$+ \left| F_n(\tau, Z_n(\tau, t, z)) - F_{n-1}(\tau, Z_n(\tau, t, z)) \right|$$

$$\leq C \left| Z_{n+1}(\tau, t, z) - Z_n(\tau, t, z) \right| + C \|f_n(\tau) - f_{n-1}(\tau)\|, \quad 0 \leq \tau \leq t \leq \delta; \quad (3.10)$$

note that for $|v| \leq z_1(\delta)$ we showed $|Z_n(\tau)| \leq C_\delta$ which is the set where we bounded $\partial_z F_n$ and estimated $F_n - F_{n-1}$ above. After integrating this from $\tau$ to $t$,

$$|Z_{n+1}(\tau, t, z) - Z_n(\tau, t, z)| \leq C \int_\tau^t \|f_n(s) - f_{n-1}(s)\| \, ds$$

$$+ C \int_\tau^t |Z_{n+1}(s, t, z) - Z_n(s, t, z)| \, ds$$

for $0 \leq \tau \leq t \leq \delta$, $x \in \mathbb{R}^3$, and $|v| \leq z_1(\delta)$. By Gronwall’s lemma,

$$|Z_{n+1}(\tau, t, z) - Z_n(\tau, t, z)| \leq C \int_\tau^t \|f_n(s) - f_{n-1}(s)\| \, ds, \quad 0 \leq \tau \leq t \leq \delta. \quad (3.11)$$

This leads to the estimate

$$\|f_{n+1}(t) - f_n(t)\|$$

$$\leq \|\partial_z \dot{f}\| \sup \{|Z_{n+1}(0, t, z) - Z_n(0, t, z)| \mid z \in \text{supp } f_{n+1}(t) \cup \text{supp } f_n(t)\}$$

$$\leq C \sup \{|Z_{n+1}(0, t, z) - Z_n(0, t, z)| \mid x \in \mathbb{R}^3, |v| \leq z_1(\delta)\}$$

$$\leq C \int_0^t \|f_n(\tau) - f_{n-1}(\tau)\| \, d\tau.$$

A straightforward inductive argument implies that

$$\|f_{n+1}(t) - f_n(t)\| \leq 2 \|\dot{f}\| \frac{C n t^n}{n!}, \quad t \in [0, \delta], \quad n \in \mathbb{N},$$

and hence

$$\|f_{n+k}(t) - f_n(t)\| \leq 2 \|\dot{f}\| \sum_{m=k}^{\infty} \frac{(C \delta)^m}{m!}, \quad t \in [0, \delta], \quad n, k \in \mathbb{N}.$$

This proves that the iterates $f_n$ converge uniformly on $[0, \delta]$. From 3.10 and 3.11 we deduce the uniform convergence of $Z_n(\cdot, t, z)$ and $\dot{Z}_n(\cdot, t, z)$ on $[0, t]$ for fixed $t \in [0, \delta]$, $z \in \mathbb{R}^3 \times B_{z_1}(\delta)$. $F_n$ converges uniformly on $[0, \delta] \times \mathbb{R}^3 \times B_R$ via a similar argument as in 3.9 where we replace $C_\delta$ with an arbitrary $R > 0$.

**Step 2: Local existence.** By the previous step, Lemma 3.6, and Lemma 3.7 the quantities

$$f_n, a_n, \frac{a_n}{r}, a_n', \alpha_n, \alpha_n', \alpha_n'', \kappa_n, \frac{\kappa_n}{r}, \kappa_n', \rho_n, j_n, S_n, p_n, Z_n, \dot{Z}_n, F_n$$

converge uniformly on $[0, \delta]$. We define the corresponding limits by dropping the subscripts. The uniform convergence implies that

$$a, \kappa \in C^{0,1}([0, \delta] \times [0, \infty[), \quad a \in C^{0,2}([0, \delta] \times [0, \infty[),$$
where $C^{0,1}$ and $C^{0,2}$ are defined similarly to 2.2. By sending $n$ to infinity in the differential equations these functions solve the constraints 1.8, 1.9, and the slicing condition 1.10 with source terms $\rho$, $S$, and $j$. They also satisfy the required boundary conditions 1.5. This together with Proposition 2.7 (a) shows that

$$F \in C^{0,1}([0, \delta] \times \mathbb{R}^6).$$

(3.12)

Let us fix $t \in [0, \delta]$, $x \in \mathbb{R}^3$, and $|v| \leq z_1(\delta)$. By Lemma 3.7,

$$Z(\cdot, t, z) \in C^1([0, t]).$$

Passing to the limits in the characteristic system yields

$$\dot{Z}(\tau, t, z) = F(\tau, Z(\tau, t, z)),$$

where we use the uniform convergence of $Z_n(\cdot, t, z)$ as well as $F_n$ and the uniform boundedness of $\partial_z F_n$. Thus, $Z(\cdot, t, z)$ is a solution to the characteristic system of the modified Vlasov equation 2.1 with $Z(t, t, z) = z$. By Proposition 2.7 (b), (c) this solution is unique, by 3.12

$$Z \in C^1([0, \delta]^2 \times \mathbb{R}^3 \times B_{z_1(\delta)}),$$

and hence

$$f \in C^1([0, \delta] \times \mathbb{R}^3 \times B_{z_1(\delta)}).$$

The $v$ supports of $f_n(t)$ were uniformly bounded in $t$ and $n$ by $z_1(\delta)$ which leads to

$$f(t, z) = 0, \ |v| > z_1(\delta).$$

Thus

$$f \in C^1([0, \delta] \times \mathbb{R}^6)$$

after making $z_1(\delta)$ bigger if needed, such that the support of $f(t, x, \cdot)$ is compactly contained in $B_{z_1(\delta)}$. The regularity of $f$ implies the regularity of $\rho$, $j$, $S$, and $p$ by Remark 1. By Proposition 2.2 the solutions to the field equations $a$, $\alpha$, and $\kappa$ are regular on $[0, \delta] \times [0, \infty[$. This proves that $f$ is a regular solution to the reduced Einstein-Vlasov system.

Step 3: Uniqueness. Let $f$ and $g$ be two regular solutions with $f(0) = g(0)$. The series of inequalities employed to estimate $\|f_{n+1}(t) - f_n(t)\|$ in Step 1 can analogously be applied to $\|f(t) - g(t)\|$ to find that

$$\|f(t) - g(t)\| \leq C \int_0^t \|f(\tau) - g(\tau)\| d\tau,$$

and by Gronwall’s lemma $f(t) = g(t)$ for every $t$ where both $f$ and $g$ are defined.

Step 4: Continuation criterion. Let $f: [0, T] \times \mathbb{R}^6 \to \mathbb{R}$ be the maximal solution, and let us assume that $T < \infty$ and

$$P^* := \sup \{|v| \mid (t, x, v) \in \text{supp} \ f\} < \infty,$$

$$a^* := \sup \{a(t, r) \mid (t, r) \in [0, T] \times [0, \infty[\} < \infty.$$
steps above. Via a similar procedure as in Lemma 3.4 the solution $\tilde{f}$ must exist on the common existence interval of the solutions of

$$
\begin{align*}
& z_1(t) = V_{t_0} + C \int_{t_0}^{t} (R_{t_0} + \tau) z_2^2(\tau) (1 + z_1(\tau))^5 \, d\tau, \\
& z_2(t) = Q_{t_0} + C \int_{t_0}^{t} (R_{t_0} + \tau)^7 z_2^{10}(\tau) (1 + z_1(\tau))^{16} \\
& \quad \cdot \exp \left( C (R_{t_0} + \tau)^5 z_2^4(\tau) (1 + z_1(\tau))^8 \right) \, d\tau,
\end{align*}
$$

(3.13)

where we set

$$
\begin{align*}
R_{t_0} &:= \sup \{|x| \mid (x, v) \in \text{supp } f(t_0)\}, \\
V_{t_0} &:= \sup \{|v| \mid (x, v) \in \text{supp } f(t_0)\}, \\
Q_{t_0} &:= \sup \{a(t_0, r) \mid r \in [0, \infty[\}.
\end{align*}
$$

We can establish the bounds

$$
R_{t_0} \leq R_0 + T, \quad V_{t_0} \leq P^*, \quad Q_{t_0} \leq a^*
$$

independently of $t_0$. Hence there exists $\epsilon > 0$ independent of $t_0$ such that the maximal solution to the system 3.13 exists at least on the interval $[t_0, t_0 + \epsilon]$. This contradicts the maximality of $T$, since $t_0 \in [0, T]$ can be chosen arbitrarily close to $T$. Thus $T = \infty$, and the proof of Theorem 1.1 is complete.

In the following result we provide some alternative continuation criteria.

**Corollary 3.8.** Let $f$ be a regular solution of the spherically symmetric, asymptotically flat Einstein-Vlasov system in maximal areal coordinates on $[0, T]$ where $T > 0$ is chosen maximal. If

- (I) $\sup \{|v| \mid (t, x, v) \in \text{supp } f\} < \infty$  \\
- or  \\
- (II) $\sup \{\|\rho(t)\| \mid t \in [0, T]\} < \infty$,

and in addition,

- (i) $\sup \{a(t, r) \mid (t, r) \in [0, T] \times [0, \infty[\} < \infty$  \\
- or  \\
- (ii) $\sup \left\{ \left| \frac{a'(t, r)}{a(t, r)} + \frac{\alpha'(t, r)}{\alpha(t, r)} \right| \mid (t, r) \in [0, T] \times [0, \infty[ \right\} < \infty$,

then $T = \infty$.

**Proof.** We show that conditions (I) and (i) hold, when any other combination does, beginning with the combination of (I) and (ii). In this case,

$$
P(t) := \sup \{|v| \mid (x, v) \in \text{supp } f(\tau), \, 0 \leq \tau \leq t\} \leq C, \, t \in [0, T],
$$

which yields $\|\rho(t)\| \leq C$. Because of (ii) and the boundary condition $a(t, 0) = 1$,  

$$
\ln \left( \frac{a(t, r)\alpha(t, r)}{a(t, 0)} \right) = \int_{0}^{r} \left( \frac{a'}{a} + \frac{\alpha'}{\alpha} \right) \, ds \leq Cr,
$$

and since $\alpha(t, 0) \leq 1$,  

$$
0 \leq a(t, r)\alpha(t, r) \leq e^{Cr}, \quad t \in [0, T], \quad r \in [0, \infty[.
$$
By the evolution equation 1.6 and the momentum constraint 1.9,
\[ \dot{a} = a'\beta + a'\beta + 2a\kappa = a\beta \left( \frac{a'}{a} + \frac{\alpha'}{\alpha} \right) - 4\pi ra^2 j. \]
Assume that \( T < \infty \). Since \( j(t, r) = 0 \) for \( r > R_0 + T \), and \( |\beta| \leq 1 \),
\[ |\dot{a}(t, r)| \leq C \left( 1 + (R_0 + T)e^{C(R_0 + T)} \right) a(t, r), \quad t \in [0, T], \quad r \in [0, \infty[ , \]
which implies (i). By Theorem 1.1, this contradicts the assumption \( T < \infty \) above.
Next, we consider (II) and (i). The bound on \( \|\rho(t)\| \) yields bounds on \( \|j(t)\| \) and \( \|S(t)\| \). Together with Eqn. 2.9, the boundedness of \( a \), and the estimate \( |\kappa(t, r)| \leq C(1 + r^3)^{-1} \) this implies that
\[ |a'(t, r)| \leq Cr \left( 4\pi (\|\rho(t)\| + \|S(t)\|) + 6 \int_0^r \frac{s^2}{1 + s^6} ds \right) \leq Cr. \]
Now assume that \( T < \infty \), and let \((X, V) ; 0, x, v)\) be a characteristic starting in \( \text{supp } f \) Then \(|X| \leq R_0 + T \) and the inequalities above together with \(|\alpha| < 1 \) imply that for \( 0 \leq t < T \),
\[ |\dot{V}(t)| \leq C \left( 1 + (R_0 + T)^2 \right) (1 + |V(t)|). \]
By Gronwall’s lemma, \( |V(t, 0, x, v)| \leq C \) for \( t \in [0, T[ \) and \((x, v) \in \text{supp } f \). This gives \( P(t) \leq C \), and therefore (I) and (i) hold which again contradicts \( T < \infty \).
To finish the proof, we need to show that (II) and (ii) imply (I) and (i). From (II) we get \( \|j(t)\| \leq C \) which was essentially everything we needed to deduce \( |a(t, r)| \leq C \) from (ii). Hence, (II) and (i) hold. This case has already been covered.  

3.5. Conservation laws. We close this paper by deriving some basic conservation laws.

**Proposition 3.9.** Let \( f \) be a regular solution of the spherically symmetric, asymptotically flat Einstein-Vlasov system in maximal areal coordinates on \( I \times \mathbb{R}^3 \). Then the ADM mass
\[ M_{ADM} := \int_0^\infty \left( 4\pi \rho(t, r) + \frac{3}{2} \kappa^2(t, r) \right) r^2 dr \]  
(3.14)
and the number of particles
\[ N := \int \int a(t, x) f(t, x, v) \, dx \, dv \]  
(3.15)
are conserved on \( I \).

**Proof.** Conservation of the ADM mass is basically a consequence of the computations in the proof of Proposition 2.8. As in that proof,

\[ 4\pi \int_0^\infty \dot{\rho}(t, r)r^2 dr \]
\[ = 4\pi \int_0^\infty \left( \frac{\alpha'}{a} - \frac{3}{2} r a^2 \alpha + \alpha \frac{a - 1}{a} \right) \rho - \frac{\alpha' \beta}{\alpha} \rho + a \kappa (S - 3p) \]  
r\[ 3 \int_0^\infty \kappa \dot{r}^2 dr = 4\pi \int_0^\infty \left( \frac{\alpha'}{a} r + k a \rho r - a \kappa S + 3 a \kappa p \right) r^2 dr - 4\pi \int_0^\infty a a \mu j dr \]
\[ + \int_0^\infty \left( \frac{3}{2} \kappa^3 \alpha' r + 3 \beta \kappa \right) r^2 dr; \]
in the equations corresponding to 2.30 and 2.36 the boundary terms arising from the integration by parts now vanish, because we integrate over $[0, \infty]$ and $r^2 \kappa$ converges to zero for $r$ tending to infinity since $\kappa \sim r^{-3}$ for large $r$. Using $\beta = \alpha \kappa r$, and integrating by parts to get rid of $\alpha'$ we arrive at
\[
d\frac{d}{dt} M_{ADM} = \frac{3}{2} \int_0^{\infty} \left( -4 \pi \alpha \kappa - \frac{3 \kappa}{r} - \kappa' \right) \alpha \kappa^2 r^3 dr,
\]
which vanishes due to field equation 1.9.

For the second conservation law, we differentiate the integrand of the spatial integral with respect to $t$ and integrate by parts after inserting the Vlasov equation. This gives
\[
\partial_t \left( a(t, x) \int f(t, x, v) dv \right) = -\text{div}_x \left( \int \left( \alpha \sqrt{1 + |v|^2} - a \beta x \right) f dv \right).
\]
After integrating with respect to $x$ over $\mathbb{R}^3$ and applying Gauß’s theorem 3.15 follows.

The ADM mass is a general concept which comes up through the foliation of spacetime into spacelike hypersurfaces and is one way of defining mass or energy in general relativity, cf. [16, 33]. We observe that
\[
M_{ADM} = \lim_{r \to \infty} \mu(t, r).
\]
The factor $a(t, x)$ in 3.15 appears due to the fact that we use non-canonical momentum coordinates. In canonical variables $p_i$ the factor $a$ is replaced by 1.

**Appendix A. The Gronwall loop for $a_n$.** The construction of the Gronwall loop for the bound of $a_n$ in the iterative scheme will now be discussed. Some arguments are very similar to the proof of Proposition 2.8.

**Lemma A.1.** In the context of the iterative scheme in Section 3.1 the quantity $Q_n$ defined in 3.2 satisfies the integral inequality
\[
Q_{n+1}(t) \leq Q_0 + C \int_0^t (R_0 + \tau)^7 Q_n^3(\tau) Q_{n+1}^7(\tau) (1 + P_n(\tau))^4 (1 + P_{n+1}(\tau))^{12} \cdot \exp \left( C(R_0 + \tau)^5 Q_{n+1}(\tau) (1 + P_{n+1}(\tau))^8 \right) d\tau.
\]
**Proof.** We consider the implicit formula for $a$ in 2.7 applied to $a_{n+1}$. By taking the derivative with respect to $t$,
\[
\dot{a}_{n+1}(t, r) = a_{n+1}^3(t, r) \left( \frac{4 \pi}{r} \int_0^r \dot{\rho}_{n+1}(t, s)s^2 ds + \frac{3}{r} \int_0^r \kappa_{n+1}(t, s) \kappa_{n+1}(t, s)s^2 ds \right);
\]
throughout this proof, $0 \leq \tau \leq t < T_{n+1}$ and $r \in [0, \infty]$. The term containing $\dot{\rho}_{n+1}$ can be calculated similarly to the proof of Proposition 2.8 in 2.27, but we have to be careful with the indices $n$ and $n+1$ arising from the iterative scheme. As a result,\[
\dot{\rho}_{n+1} = \text{div}_x \left( \int \left( \beta_n \sqrt{1 + |v|^2} \frac{r}{\alpha_n a_n} \right) f_{n+1} dv \right) - \left( \beta'_n + \frac{2 \beta_n}{r} \right) \rho_{n+1} - \left( \frac{\alpha'_n}{a_n} + \frac{\alpha_n a'_n}{a_n^2} \right) j_{n+1} - \alpha_n \kappa_n (3 p_{n+1} - S_{n+1}).
\]
Gauß’s theorem and the estimates at hand imply that
\[ \frac{4\pi}{r} \int_0^r \rho_{n+1}(t, s)s^2 ds \leq C(R_0 + t)^3 Q_n^3(t) (1 + P_n(t))^4 (1 + P_{n+1}(t))^4, \] (A.2)
cf. Lemma 3.3. The more difficult part is to estimate \( \dot{\kappa}_{n+1} \) and its corresponding integral in A.1. Similarly to \( \rho_{n+1} \), we can recycle the previous results from the proof of Proposition 2.8 in 2.32. The time derivative of \( j_{n+1} \) becomes
\[ j_{n+1} = \int x \cdot \nu \left( \beta_n \frac{x}{r} - \frac{\alpha_n}{a_n} \frac{v}{\sqrt{1 + |v|^2}} \right) \partial_x f_{n+1} dv - \frac{\alpha'_n}{a_n} (p_{n+1} + \rho_{n+1}) - 2\alpha_n \kappa_n j_{n+1}. \] (A.3)
After defining
\[ A_n(t, r) := \sup \{ |\dot{a}_n(t, s)| : s \in [0, r] \}, \quad n \in \mathbb{N}, \quad (t, r) \in [0, T_n] \times [0, \infty[, \]
we get
\[ |\kappa_{n+1}(t, r)| \leq C (1 + P_{n+1}(t))^4 (R_0 + t) A_{n+1}(t, r) + \frac{4\pi}{r^3} \int_0^r \left( a_{n+1} j_{n+1} \right) s^3 ds. \] (A.4)
The latter integral can be treated by inserting (A.3), using spherical symmetry, and integrating by parts to get rid of the derivative \( \partial_x f \). This yields
\[ \int_0^r a_{n+1} \dot{a}_{n+1} s^3 ds \]
\[ = r^3 \left( a_{n+1} \beta_n j_{n+1} - \frac{a_{n+1}}{a_n} \alpha_n p_{n+1} \right) \]
\[ + \int_0^r \left( \frac{a_{n+1}}{a_n} \alpha_n s_{n+1} - 5a_{n+1} \beta_n j_{n+1} \right) s^2 ds \]
\[ + \int_0^r \left( \frac{a_{n+1}}{a_n} \right) \left( a_{n+1} \beta_n j_{n+1} - \frac{a_{n+1}}{a_n} \alpha_n (p_{n+1} + \rho_{n+1}) \right)s^3 ds; \]
a similar computation appeared in the proof of Proposition 2.8 in 2.32. We can now use the results of Lemma 3.3 to estimate
\[ \frac{4\pi}{r^3} \int_0^r a_{n+1} \dot{a}_{n+1} s^3 ds \leq C(R_0 + t)^3 Q_n^3(t) Q_{n+1}^3(t) (1 + P_n(t))^4 (1 + P_{n+1}(t))^8. \]
We insert this into (A.4) to obtain
\[ |\kappa_{n+1}(t, r)| \leq C(R_0 + t)^3 Q_n^3(t) Q_{n+1}^3(t) (1 + P_n(t))^4 (1 + P_{n+1}(t))^8 \]
\[ + C(R_0 + t) (1 + P_{n+1}(t))^4 A_{n+1}(t, r). \]
The implicit formula 2.8 implies that
\[ |\kappa_{n+1}(t, r)| \leq C \frac{(R_0 + t)^4}{1 + r^3} Q_{n+1}(t) (1 + P_{n+1}(t))^4, \]
and hence
\[
\frac{1}{r} \left| \int_0^r \kappa_{n+1}(t, s) \kappa_{n+1}(t, s) s^2 ds \right|
\leq \int_0^r |\kappa_{n+1}(t, s)| s ds
\leq C(R_0 + t)^7 Q_n^3(t) Q_{n+1}^7(t) (1 + P_n(t))^4 (1 + P_{n+1}(t))^{12}
\]
\[+ C(R_0 + t)^5 Q_{n+1}(t) (1 + P_{n+1}(t))^8 \int_0^r \frac{s}{1 + s^3} A_{n+1}(t, s) ds. \tag{A.5} \]

We put A.2 and A.5 into A.1 and arrive at the inequality
\[
A_{n+1}(t, r) \leq C(R_0 + t)^7 Q_n^3(t) Q_{n+1}^7(t) (1 + P_n(t))^4 (1 + P_{n+1}(t))^{12}
\]
\[+ C(R_0 + t)^5 Q_{n+1}(t) (1 + P_{n+1}(t))^8 \int_0^r \frac{s}{1 + s^3} A_{n+1}(t, s) ds \]
which by Gronwall’s lemma implies that
\[
A_{n+1}(t, r) \leq C(R_0 + t)^7 Q_n^3(t) Q_{n+1}^7(t) (1 + P_n(t))^4 (1 + P_{n+1}(t))^{12}
\]
\[\cdot \exp \left(C(R_0 + t)^5 Q_{n+1}(t) (1 + P_{n+1}(t))^8\right). \]

By the definition of \(A_{n+1}\) this completes the proof. \(\square\)

**Appendix B. Derivation of the relevant equations; the 3+1 formalism.**

In this appendix we give some details of the derivation of the spherically symmetric, asymptotically flat Einstein-Vlasov system in maximal areal coordinates. We will try to keep all these computations short and refer the reader to \([17]\) for the details. We begin by expressing the metric 1.4 in the corresponding Cartesian coordinates. The usual tensor transformation law implies:

**Lemma B.1.** In the Cartesian coordinates
\[
x^0 := t, \quad x := (x^1, x^2, x^3) := r(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \in \mathbb{R}^3 \tag{B.1}
\]
the Lorentz metric defined in 1.4 takes the form
\[
g_{00} = -\alpha^2 + a^2 \beta^2, \quad g_{ii} = a^2 \beta x^i r, \quad g_{ij} = \delta_{ij} + (a^2 - 1) \frac{x^i x^j}{r^2},
\]
its inverse metric is
\[
g^{00} = -\frac{1}{\alpha^2}, \quad g^{0i} = \frac{\beta}{\alpha^2} \frac{x^i}{r}, \quad g^{ij} = \delta^{ij} + \left(\frac{1}{\alpha^2} - \frac{\beta^2}{\alpha^2} - 1\right) \frac{x^i x^j}{r^2}.
\]
and its determinant is
\[
g = -a^2 \alpha^2.
\]
We recall that \(x_i = \delta_{ij} x^j\) and that Latin indices run from 1 to 3.

Next we comment on the change to non-canonical variables
\[
v_i := p_i + \left(\frac{1}{a} - 1\right) \frac{x^i}{r} x_i, \quad p_i = v_i + (a - 1) \frac{x^i}{r} x_i,
\]
the second equation is the corresponding inverse transformation, and by definition,
\[
x \cdot p := \delta^{ij} x_i p_j = x^i p_i, \quad |p|^2 := \delta^{ij} p_i p_j, \quad x \cdot v := \delta^{ij} x_i v_j = x^i v_i, \quad |v|^2 := \delta^{ij} v_i v_j.
\]
It is only a lengthy calculation to show that the Vlasov equation 1.2 takes the form 1.12. For this and other computations we note that under suitable regularity assumptions, cf. Definition 2.1, and for fixed $t$ the transformation
\[ \Phi : \mathbb{R}^6 \to \mathbb{R}^6, \ (x,p) \mapsto (x, \frac{x \cdot p}{r}) = (x,v) \]
is a $C^1$-diffeomorphism with inverse
\[ \Phi^{-1} : \mathbb{R}^6 \to \mathbb{R}^6, \ (x,v) \mapsto (x, \frac{x + (a-1)x \cdot v}{r}) = (x,p). \]
In addition,
\[ x \cdot p = a x \cdot v, \quad |p|^2 = |v|^2 + (a^2 - 1) \left( \frac{x \cdot v}{r} \right)^2, \quad p^0 = \frac{1}{\alpha} \sqrt{1+|v|^2}, \quad \det \left( \frac{\partial p}{\partial v} \right) = a. \] (B.2)

Next we consider the relation of the energy-momentum tensor to the source terms introduced in 1.13–1.16. Change of variables and exploiting spherical symmetry implies:

**Lemma B.2.** The energy-momentum tensor becomes
\[ T_{00} = \alpha^2 \rho + a^2 \beta^2 p - 2a\alpha\beta j, \]
\[ T_{0i} = (a^2 \beta p - a\alpha j) \frac{x_i}{r}, \]
\[ T_{ij} = a^2 p \frac{x_i x_j}{r^2} + \frac{1}{2} (S-p) \left( \delta_{ij} - \frac{x_i x_j}{r^2} \right), \]
and its trace is given by
\[ T = g^{\mu\nu} T_{\mu\nu} = S - \rho. \] (B.3)

We now come to the derivation of the field equations 1.8–1.11; recall that 1.6 is the gauge condition and 1.7 is only a definition. An important feature of the Einstein field equations is that in a suitable formulation they can be split into constraint equations, which hold on each hypersurface $\Sigma_t$ of constant $t$ separately, and evolution equations, which propagate the geometry from one $\Sigma_t$ to the next. One way to obtain such a formulation is the 3+1 formalism; for general background of this concept we refer to [10, 16, 31, 34]. A unit normal vector field on $\Sigma_t$ is given by\[ (n^\mu) = (\alpha, 0, 0, 0) \] with covariant form
\[ (n^\mu) = \left( \frac{1}{\alpha}, -\beta x^1 / \alpha r, -\beta x^2 / \alpha r, -\beta x^3 / \alpha r \right). \]
This is used to project the metric $g_{\mu\nu}$ onto $\Sigma_t$,
\[ \gamma_{\mu\nu} := g_{\mu\nu} + n_\mu n_\nu, \]
which amounts to the components
\[ \gamma_{00} = a^2 \beta^2, \quad \gamma_{0i} = a^2 \beta \frac{x_i}{r}, \quad \gamma_{ij} = \delta_{ij} + (a^2 - 1) \frac{x_i x_j}{r^2}, \]
\[ \gamma^{00} = 0, \quad \gamma^{0i} = 0, \quad \gamma^{ij} = \delta^{ij} + \left( \frac{1}{a^2} - 1 \right) \frac{x^i x^j}{r^2}. \] (B.4)

We also need the mixed components
\[ \gamma^0_0 = 0, \quad \gamma^i_0 = \beta \frac{x^i}{r}, \quad \gamma^\mu_\mu = \delta^\mu_\mu. \] (B.5)
The metric $\gamma_{ij}$ has signature $(+++)$ as long as $a \geq 1$. Thus, $\Sigma_t$ is a spacelike hypersurface, endowed with the Riemannian metric $\gamma_{ij}$ which is the projection of the Lorentz metric to this hypersurface.

By $\gamma_{ij}$, $\gamma_{i}^{j}$, and $\Gamma_{ij}^{k}$ we denote the three-dimensional Ricci tensor, Ricci scalar, and Christoffel symbols generated by the metric $\gamma_{ij}$ on $\Sigma_t$. Together with the three-dimensional Riemann tensor these quantities determine the intrinsic geometry of the hypersurface and are defined in the same way as in four dimensions with the zeroth index removed. The way $\Sigma_t$ is embedded into the four-dimensional space is fixed by the extrinsic curvature tensor $K_{ij}$. Since $\Sigma_t$ is a member of a foliation, it can be defined as

$$K_{ij} := -\frac{1}{2a} \left( \partial_t \gamma_{ij} - \partial_i \gamma_{0j} - \partial_j \gamma_{0i} + 2 \Gamma^k_{ij} \gamma_{0k} \right),$$

(B.6)

cf. [16]. We further introduce the trace of the extrinsic curvature tensor $K = \gamma^{ij} K_{ij}$ which is also called the mean curvature of the hypersurface $\Sigma_t$; note that the Latin indices of three-dimensional objects can now be lowered, raised, and contracted by the 3-metric $\gamma_{ij}$. The covariant derivative associated with $\gamma_{ij}$ can be obtained by projecting every appearing index in the covariant derivative associated with $g_{\mu\nu}$ onto $\Sigma_t$, cf. [16]. For example, the covariant derivative of a 2-tensor $X^\nu_{\mu}$ becomes

$$D_i X^\nu_{\mu} = \gamma^\epsilon_i \gamma^j_{\delta} \gamma^\sigma_{\mu} \nabla_\epsilon X^\delta_{\sigma}.$$  

It is easily checked that the covariant derivative is compatible with the 3-metric, i.e., $D_i \gamma_{jk} = 0$. This leads to the following identities which will be used below:

$$D_i D_j \alpha = \partial_i \partial_j \alpha - 3 \Gamma^k_{ij} \partial_k \alpha,$$

$$D_j K^i_{j} = \partial_x K^i_{j} + 3 \Gamma^j_{jk} K^k_{i} - 3 \Gamma^k_{ij} K^j_{k},$$

$$D_i K = \partial_i K.$$  

(B.7)

If $R_{\mu\nu}$ denotes the Ricci tensor and $R$ its trace, the Ricci scalar, then the Einstein equations 1.1 read

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu}.$$  

The trace of this tensor equation gives $R = -8\pi T$, where $T$ is the trace of the energy-momentum tensor $T_{\alpha\beta}$. Hence

$$R_{\mu\nu} + 4\pi T g_{\mu\nu} = 8\pi T_{\mu\nu}$$

is equivalent to the Einstein equations 1.1. We project these equations onto the hypersurface and along the normal vector as follows:

(1) Full projection onto $\Sigma_t$:

$$(R_{\delta\epsilon} + 4\pi T g_{\delta\epsilon}) \gamma^\delta_{\mu} \gamma^\epsilon_{\nu} = 8\pi T_{\delta\epsilon} \gamma^\delta_{\mu} \gamma^\epsilon_{\nu}.$$  

(B.8)

(2) Mixed projection once onto $\Sigma_t$ and once perpendicular to $\Sigma_t$:

$$(R_{\mu\delta} + 4\pi T g_{\mu\delta}) n^\mu \gamma^\delta_{\nu} = 8\pi T_{\mu\delta} n^\mu \gamma^\delta_{\nu}.$$  

(B.9)

(3) Full projection perpendicular to $\Sigma_t$:

$$(R_{\mu\nu} + 4\pi T g_{\mu\nu}) n^\mu n^\nu = 8\pi T_{\mu\nu} n^\mu n^\nu.$$  

(B.10)
We introduce some notation for the projections of the energy momentum tensor:

\[ S_{\mu\nu} := T_{\delta\epsilon} \gamma_{\mu}^{\delta} \gamma_{\nu}^{\epsilon}, \quad E := T_{\mu\nu} n^{\mu} n^{\nu}, \quad J_{\mu} := -T_{\nu\delta} n^{\nu} \gamma_{\mu}^{\delta}. \]

As a consequence of B.5, \( S_{ij} = T_{ij} \), which together with Lemma B.2 implies that

\[ S = S_{ij} \gamma^{ij} = S_{\mu\nu} g^{\mu\nu}. \]

Hence \( S \), which is defined as the integral 1.15 in terms of the density \( f \), is the trace of the tensor \( S_{\mu\nu} \). As in [16], we interpret \( E \) as a matter-energy density, \( J_i \) as a matter-momentum density, and \( S_{ij} \) as a matter-stress tensor as measured by an Eulerian observer, i.e., an observer who moves forward in time and orthogonal to the hypersurfaces \( \Sigma_t \). Some differential geometric relations and equations provide the reduction of the various projections to known quantities. The contracted Gauß relation [16, Eqn. (3.74)] and the Ricci equation [16, Eqn. (4.41)] yield

\[ (R_{\delta\epsilon} + 4\pi T g_{\delta\epsilon}) \gamma^{\delta}_{i} \gamma^{\epsilon}_{j} = -\frac{1}{\alpha} (\partial_{t} - L_{\beta}) K_{ij} - \frac{1}{\alpha} D_{j} D_{j} \alpha + 3 R_{ij} + K K_{ij} - 2 K_{ik} K^{k}_{j} + 4\pi \gamma^{ij}. \]

We restrict ourselves to the spatial indices \( i \) and \( j \) since, geometrically speaking, the tensor field is tangent to \( \Sigma_t \). Here

\[ L_{\beta} K_{ij} := \beta^{k}_{i} \partial_{x} K^{k}_{j} + \partial_{x}^{i} \left( \beta^{k}_{j} \right) K^{k}_{j} + \partial_{x}^{j} \left( \beta^{k}_{i} \right) K^{k}_{i} \]  

is the Lie derivative along the shift vector field \( \beta^{i}_{j} \). From B.8 and \( T = S - \rho \) we get

\[ (\partial_{t} - L_{\beta}) K_{ij} = -D_{i} D_{j} \alpha + \alpha \left( 3 R_{ij} + K K_{ij} - 2 K_{ik} K^{k}_{j} + 4\pi ((S - \rho) \gamma_{ij} - 2 S_{ij}) \right), \]  

an evolution equation for the geometry of spacetime. By the mixed projection B.9 and the contracted Codazzi equation [16, Eqn. (3.82)],

\[ D_{j} K^{i}_{j} - D_{i} K = 8\pi J_{i} \]

which is the so-called momentum constraint. Again, we restrict ourselves to the spatial components. Lastly, equation B.10 gives

\[ 3 R + K^2 - K_{ij} K^{ij} = 16\pi E \]

after applying the scalar Gauß equation [16, Eqn. (3.75)]. This is the so-called the Hamiltonian constraint. Both B.13 and B.14 can be directly compared to the constraints arising in electrodynamics. They only involve quantities which reside on the hypersurface \( \Sigma_t \) and must be satisfied separately on each hypersurface.

We call B.12–B.14 the 3+1 Einstein equations. We have to include the slicing condition

\[ K = 0 \]

into the set of field equations, and B.12–B.15 are the 3+1 field equations in maximal areal coordinates. In order to reduce them to a system of differential equations for the metric coefficients we recall the field term \( \kappa \) defined in 1.7. From Lemma B.2
we obtain the projections of the energy-momentum tensor:
\[ S_{ij} = T_{ij} = a^2 \frac{p_{ix_j}}{r^2} + \frac{1}{2} (S - p) \left( \delta_{ij} - \frac{x_i x_j}{r^2} \right), \]
\[ J_i = -T_{ri} n^r = a \frac{x_i}{r}, \]
\[ E = T_{\mu \nu} n^\mu n^\nu = \rho. \]

Next, we compute that
\[ 3 R_{ij} = \frac{2 a' x_i x_j}{r a} + \frac{1}{r} \left( \frac{a'}{a^3} + \frac{1}{r} \left( 1 - \frac{1}{a^2} \right) \right) \left( \delta_{ij} - \frac{x_i x_j}{r^2} \right), \]
\[ 3 R = 4 \frac{a'}{r a^3} + \frac{2}{r^2} \left( 1 - \frac{1}{a^2} \right), \]
\[ 3 \Gamma^i_{jk} = \frac{a}{a} \left( (a \beta)' - \dot{a} \right) \frac{x_i x_j}{r^2} + \frac{\beta}{a r} \left( \delta_{ij} - \frac{x_i x_j}{r^2} \right), \]
\[ K_{ij} = \frac{1}{a \alpha} \left( (a \beta)' - \dot{a} \right) + \frac{2 \beta}{a r}, \]
\[ K = \frac{1}{a \alpha} \left( (a \beta)' - \dot{a} \right) + \frac{2 \beta}{a r}. \]

We see that \( K = 0 \) holds if and only if 1.6 does. With \( K = 0 \), the covariant form of the extrinsic curvature tensor
\[ K^{ij} = \gamma^{ik} \gamma^{jl} K_{kl} = \frac{1}{a^3 \alpha} \left( (a \beta)' - \dot{a} \right) \frac{x^i x^j}{r^2} + \frac{\beta}{a r} \left( \delta^{ij} - \frac{x^i x^j}{r^2} \right), \]
and \( 3 R \) from B.16, the Hamiltonian constraint B.14 becomes
\[ \frac{4 a'}{r a^3} + \frac{2}{r^2} \left( 1 - \frac{1}{a^2} \right) - \left( (a \beta)' - \dot{a} \right)^2 - \frac{2 \beta^2}{a^2 r^2} = 16 \pi \rho, \]
which by 1.7 and 1.6 can be rewritten in the form 1.8. For the calculation of the momentum constraint we need the covariant derivative of
\[ K^i_j = \gamma^{ik} K_{ki} = \frac{1}{a \alpha} \left( (a \beta)' - \dot{a} \right) \frac{x^i x^j}{r^2} + \frac{\beta}{a r} \left( \delta^i_j - \frac{x^i x^j}{r^2} \right). \]

Using B.7, B.16, and 1.6 this amounts to
\[ D_j K^j_i = -2 \kappa \frac{x_i}{r} - 6 \kappa \frac{x_i}{r^2}. \]

From the momentum constraint B.13 we thus obtain Eqn. 1.9. It remains to derive 1.10 and 1.11, and we have yet to use B.12, which obviously is the most complicated equation of the 3+1 equations. It turns out that the elliptic equation 1.10 for \( \alpha \) and the evolution equation 1.11 for \( \kappa \) constitute a set of equations which is equivalent to B.12, if the equations 1.6–1.9 are already satisfied.

We now have derived the full set of field equations 1.6–1.11 from the 3+1 Einstein equations in maximal areal coordinates. It turns out that 1.6–1.11 are, in fact, equivalent to Einstein’s equations 1.1 when imposing the gauge condition \( K = 0 \). Loosely speaking, no information gets lost by the projections: it is possible to reconstruct \( T_{\mu \nu} \) and \( G_{\mu \nu} \) from the projected quantities. A detailed argument for this is given in [17, Prop. 2.5].
REFERENCES

[1] H. Andréasson, Black hole formation from a complete regular past for collisionless matter, *Ann. Henri Poincaré*, 13 (2012), 1511–1536.

[2] H. Andréasson, M. Kunze and G. Rein, Existence of axially symmetric static solutions of the Einstein-Vlasov system, *Comm. Math. Phys.*, 308 (2011), 23–47.

[3] H. Andréasson, M. Kunze and G. Rein, Global existence for the spherically symmetric Einstein-Vlasov system with outgoing matter, *Commun. Partial Differential Equations*, 33 (2008), 656–668.

[4] H. Andréasson, M. Kunze and G. Rein, Gravitational collapse and the formation of black holes for the spherically symmetric Einstein-Vlasov system, *Quart. Appl. Math.*, 68 (2010), 17–42.

[5] H. Andréasson, M. Kunze and G. Rein, The formation of black holes in spherically symmetric gravitational collapse, *Math. Ann.*, 350 (2011), 683–705.

[6] H. Andréasson, M. Kunze and G. Rein, Rotating, stationary, axially symmetric spacetimes with collisionless matter, *Comm. Math. Phys.*, 329 (2014), 787–808.

[7] H. Andréasson and G. Rein, A numerical investigation of the stability of steady states and critical phenomena for the spherically symmetric Einstein–Vlasov system, *Classical Quantum Gravity*, 23 (2006), 3659–3677.

[8] H. Andréasson and G. Rein, Formation of trapped surfaces for the spherically symmetric Einstein-Vlasov system, *J. Hyperbolic Differ. Equ.*, 7 (2010), 707–731.

[9] J. Batt, Global symmetric solutions of the initial value problem of stellar dynamics, *J. Differential Equations*, 25 (1977), 342–364.

[10] T. Baumgarte and S. Shapiro, *Numerical Relativity: Solving Einstein’s Equations on the Computer*, Cambridge University Press, 2010.

[11] Y. Choquet-Bruhat, Problème de Cauchy pour le système intégral différentiel d’Einstein-Liouville, *Ann. Inst. Fourier (Grenoble)*, 21 (1971), 181–201.

[12] M. Dafermos, Spherically symmetric spacetimes with a trapped surface, *Classical Quantum Gravity*, 22 (2005), 2221–2232.

[13] M. Dafermos and A. D. Rendall, An extension principle for the Einstein-Vlasov system in spherical symmetry, *Ann. Henri Poincaré*, 6 (2005), 1137–1155.

[14] D. Fajman, J. Joudioux and J. Smulevici, The stability of the Minkowski space for the Einstein-Vlasov system, *Anal. PDE*, 14 (2021), 425–531.

[15] R. T. Glassy and W. A. Strauss, Singularity formation in a collisionless plasma could occur only at high velocities, *Arch. Rational Mech. Anal.*, 92 (1986), 59–90.

[16] É. Gourgoulhon, *3 + 1 Formalism in General Relativity. Bases of Numerical Relativity*, Lecture Notes in Physics, 846, Springer, Heidelberg, 2012.

[17] S. Günther, *The Einstein-Vlasov System in Maximal Areal Coordinates*, Masters thesis, University of Bayreuth, 2019.

[18] S. Günther, J. Körner, T. Lebeda, B. Pötzl, G. Rein, C. Straub and J. Weber, A numerical stability analysis for the Einstein-Vlasov system, *Classical Quantum Gravity*, 38 (2021), 27pp.

[19] S. Günther, C. Straub and G. Rein, Collisionless equilibria in general relativity: Stable configurations beyond the first binding energy maximum, *Astrophysical J.*, 918 (2021), 48pp.

[20] M. Hadžić, Z. Lin and G. Rein, Stability and instability of self-gravitating relativistic matter distributions, *Arch. Ration. Mech. Anal.*, 241 (2021), 1–89.

[21] M. Hadžić and G. Rein, On the small redshift limit of steady states of the spherically symmetric Einstein-Vlasov system and their stability, *Math. Proc. Cambridge Philos. Soc.*, 159 (2015), 529–546.

[22] M. Hadžić and G. Rein, Stability for the spherically symmetric Einstein-Vlasov system—A coercivity estimate, *Math. Proc. Cambridge Philos. Soc.*, 155 (2013), 529–556.

[23] H. Lindblad and M. Taylor, Global stability of Minkowski space for the Einstein-Vlasov system in the harmonic gauge, *Arch. Ration. Mech. Anal.*, 235 (2020), 517–633.

[24] E. Malec and N. Ó Murchadha, Optical scalars and singularity avoidance in spherical spacetimes, *Phys. Rev. D*, 50 (1994), R6033–R6036.

[25] T. Ramming and G. Rein, Spherically symmetric equilibria for self-gravitating kinetic or fluid models in the non-relativistic and relativistic case—A simple proof for finite extension, *SIAM J. Math. Anal.*, 45 (2013), 900–914.
[26] G. Rein, Collisionless kinetic equations from astrophysics—The Vlasov-Poisson system, in Handbook of Differential Equations: Evolutionary Equations, Handb. Differ. Equ., 3, Elsevier/North-Holland, Amsterdam, 2007, 383–476.

[27] G. Rein, The Vlasov-Einstein System with Surface Symmetry, Habilitation thesis, Ludwig-Maximilians-Universität in Munich, 1995.

[28] G. Rein and A. D. Rendall, Global existence of solutions of the spherically symmetric Vlasov-Einstein system with small initial data, Comm. Math. Phys., 150 (1992), 561–583.

[29] G. Rein and A. D. Rendall, Erratum: “Global existence of solutions of the spherically symmetric Vlasov-Einstein system with small initial data”, Comm. Math. Phys., 176 (1996), 475–478.

[30] A. D. Rendall, An introduction to the Vlasov-Einstein system, in Mathematics of Gravitation, Part I, Banach Center Publ., 41, Part I, Polish Acad. Sci. Inst. Math., Warsaw, 1997, 35—68.

[31] A. D. Rendall, Partial Differential Equations in General Relativity, Oxford Graduate Texts in Mathematics, 16, Oxford University Press, Oxford, 2008.

[32] M. Taylor, The global nonlinear stability of Minkowski space for the massless Einstein-Vlasov system, Ann. PDE, 3 (2017), 177pp.

[33] R. M. Wald, General Relativity, University of Chicago Press, Chicago, IL, 1984.

[34] J. York, Kinematics and dynamics of general relativity, in Sources of Gravitational Radiation, Cambridge University Press, 1979, 83–126.

Received for publication May 2021; early access November 2021.

E-mail address: sebastian.guenther@uni-bayreuth.de
E-mail address: gerhard.rein@uni-bayreuth.de