The Quantum Noether Condition
in terms of Interacting Fields

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Abstract. - We review our recent work, hep-th/9803030, on the constraints imposed by global or local symmetries on perturbative quantum field theories. The analysis is performed in the Bogoliubov-Shirkov-Epstein-Glaser formulation of perturbative quantum field theory. In this formulation the S-matrix is constructed directly in the asymptotic Fock space with only input causality and Poincaré invariance. We reformulate the symmetry condition proposed in our earlier work in terms of interacting Noether currents.

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1 Introduction

The relation between symmetries and quantum theory is an important and fundamental issue. For instance, symmetry relations among correlation functions (Ward identities) are often used in order to prove that a quantum field theory is unitary and renormalizable. Conversely, the violation of a classical symmetry at the quantum level (anomalies) often indicates that the theory is inconsistent. Furthermore, in recent years symmetries (such as supersymmetry) have been instrumental in uncovering non-perturbative aspects of quantum theories (see, for example, [1]). It is, thus, desirable to understand the interplay between symmetries and quantization in a manner which is free of the technicalities inherent in the conventional Lagrangian approach (regularization/renormalization) and in a way which is model independent as much as possible.

In a recent paper [2] we have presented a general method, the Quantum Noether Method, for constructing perturbative quantum field theories with global symmetries. Gauge theories are within this class of theories, the global symmetry being the BRST symmetry [3]. The method is established in the causal approach to quantum field theory introduced by Bogoliubov and Shirkov [4] and developed by Epstein and Glaser [5, 6]. This explicit construction method rests directly on the axioms of relativistic quantum field theory. The infinities encountered in the conventional approach are avoided by a proper handling of the correlation functions as operator-valued distributions. In particular, the well-known problem of ultraviolet (UV) divergences is reduced to the mathematically well-defined problem of splitting an operator-valued distribution with causal support into a distribution with retarded and a distribution with advanced support or, alternatively [6, 7], to the continuation of time-ordered products to coincident points. Implicitly, every consistent renormalization scheme solves this problem. Thus, the explicit Epstein-Glaser (EG) construction should not be regarded as a special renormalization scheme but as a general framework in which the conditions posed by the fundamental axioms of quantum field theory (QFT) on any renormalization scheme are built in by construction. In this sense our method is independent from the causal framework. Any renormalization scheme can be used to work out the consequences of the general symmetry conditions proposed in [2].

In the EG approach the $S$-matrix is directly constructed in the Fock space of free asymptotic fields in a form of formal power series. The coupling constant is replaced by a tempered test function $g(x)$ (i.e. a smooth function rapidly decreasing at infinity) which switches on the interaction. Instead of evaluating the $S$-matrix by first computing off-shell Greens functions by means of Feynman rules and then applying the LSZ formalism, the $S$-matrix is directly obtained by imposing causality and Poincaré invariance. The method can be regarded as an "inverse" of the cutting rules. One builds $n$-point functions out of $m$-point functions ($m < n$) by suitably “gluing” them together. The precise manner
in which this is done is dictated by causality and Poincaré invariance (see appendix A for details). One shows, that this process uniquely fixes the $S$-matrix up to local terms (which we shall call “local normalization terms”). At tree level these local terms are nothing but the Lagrangian of the conventional approach\cite{2}.

The problem we set out to solve in \cite{4} was to determine how to obtain a quantum theory which, on top of being causal and Poincaré invariant, is also invariant under a global symmetry. For linear symmetries such as global internal symmetries or discrete $C$, $P$, $T$ symmetries the solution is well-known: one implements the symmetry in the asymptotic Fock space by means of an (anti-) unitary transformation. The focus of our investigation in \cite{4} was symmetries that are non-linear in the Lagrangian formulation. The prime examples are BRST symmetry and supersymmetry (in the absence of auxiliary fields). The main puzzle is how a theory formulated in terms of asymptotic fields only knows about the inherent non-linear structure.

The solution to the problem is rather natural. One imposes that the Noether current that generates the asymptotic symmetry is conserved at the quantum level, i.e. inside correlation functions. This condition, the Quantum Noether Condition (QNC), constrains the local normalization terms left unspecified by causality and Poincaré invariance. At tree-level one finds that the asymptotic Noether current renormalizes such that it generates the full non-linear transformation rules. At the quantum level the same condition yields the corresponding Ward identities. The way the methods works is analogous to the classical Noether method \cite{8, 9}, hence its name. In addition, we have shown that the QNC is equivalent to the condition that the $S$-matrix is invariant under the symmetry under question (i.e. the $S$-matrix commutes with the generator of the asymptotic symmetry).

Quantum field theory, however, is usually formulated in terms of interacting fields. In the Lagrangian formulation, the symmetries of the theory are the symmetries of the action (or more generally of the field equations) that survive at the quantum level. These symmetries are generated by interacting Noether currents. It will, thus, be desirable to express the QNC in terms of the latter. As we shall see, this is indeed possible. The QNC in term of the interacting current is given in (3.1). If the symmetry is linear then the condition is that the interacting current is conserved (as expected). If the symmetry, however, is non-linear the interacting current is only conserved in the adiabatic limit ($g \rightarrow \text{const}$).

One important example is Yang-Mills theory. In this case, the corresponding Noether current is the BRST current. Because there are unphysical degrees of freedom present in gauge theories, one needs a subsidiary condition in order to project out the unphysical states. The subsidiary condition should remain invariant under time evolution. This means that it should be expressed in terms of a conserved charge. The appropriate
charge for gauge theories is the BRST charge \([10]\). The subsidiary condition is that physical states should be annihilated by the BRST charge \(Q_{\text{int}}\) (and not be \(Q_{\text{int}}\)-exact).

The considerations in \([10]\), however, (implicitly) assumed the naive adiabatic limit. For pure gauge theories this limit seem not to exist. Then from the Quantum Noether Condition \((3.1)\) follows that the interacting BRST current is not conserved before the adiabatic limit. We stress, however, that the Quantum Noether Condition allows one to work out all consequences of non-linear symmetries for time-ordered operator products before the adiabatic limit is taken. As we shall see, one can even identify the non-linear transformation rules.

We organize this paper as follows: In the next section we shortly review the Quantum Noether Method. In section 3 we express the Quantum Noether Condition in terms of the interacting Noether current. Section 4 contains a discussion of future directions. In the appendix we present the main formulae of the causal framework and our conventions.

## 2 The Quantum Noether Method

In the EG approach one starts with a set of free fields in the asymptotic Fock space. These fields satisfy their (free) field equations and certain commutation relations. To define the theory one still needs to specify \(T_1\), the first term in the \(S\)-matrix. (Actually, as we shall see, even \(T_1\) is not free in our construction method but is also constrained by the Quantum Noether Condition). Given \(T_1\) one can, in a well defined manner, construct iteratively the perturbative \(S\)-matrix. The requirements of causality and Poincaré invariance completely fix the \(S\)-matrix up to local terms. The additional requirement that the theory is invariant under a global and/or local symmetry imposes constraints on these local terms.

To construct a theory with global and/or local symmetry we introduce the coupling \(g_{\mu j_0}^\mu\) in the theory, where \(j_0^\mu\) is the Noether current that generates the asymptotic (linear) symmetry transformations, and we impose the condition that “the Noether current is conserved at the quantum level”

\[
\partial_{\mu} J_\mu^n(x_1, \ldots, x_n; \hbar) = 0, \tag{2.1}
\]

where we introduce the notation (we use the abbreviation \(\partial/\partial x_i^\mu = \partial_i^\mu\))

\[
\partial_{\mu} J_\mu^n(x_1, \ldots, x_n; \hbar) = \sum_{l=1}^{n} \partial_i^l J_\mu^{n/l}, \tag{2.2}
\]

and

\[
J_\mu^{n/l} = T[T_1(x_1) \cdots j_0^\mu(x_l) \cdots T_1(x_n)]. \tag{2.3}
\]

(for \(n = 1, J_\mu^1(x_1) = j_0^\mu(x_1)\)). In other words we consider an \(n\)-point function with one insertion of the current \(j_0^\mu\) at the point \(x_l\). Notice that since the left hand side of \((2.1)\) is a formal Laurent series in \(\hbar\), this condition is actually a set of conditions.
One may apply the inductive EG construction to work out the consequences of (2.1). This may be done by first working out \(T[j_0T_1...T_1]\) and then constructing (2.2). However, there is an alternative route [2]. One relaxes the field equations of the fields \(\phi^A\). Then the inductive hypothesis takes the form: for \(m < n\),

\[
\sum_{i=1}^{m} \partial^i_{\mu} J^{\mu}_{m/l} = \sum_{A} R^{A;m}(h)K_{AB}\phi^B \delta(x_1, \ldots, x_m),
\]

(2.4)

where

\[
K_{AB}\phi^B = \partial^\mu \frac{\partial L_0}{\partial (\partial^\mu \phi^A)} - \frac{\partial L_0}{\partial \phi^A},
\]

(2.5)

are the free field equations (\(L_0\) is the free Lagrangian that yields (2.5); the present formulation assumes that such a Lagrangian exists). The coefficients \(R^{A;m}(h)\) are defined by (2.4) and are formal series in \(\bar{h}\).

Clearly, if we impose the field equation we go back to (2.1). The converse is also true. Once one relaxes the field equations in the inductive step, (2.1) implies (2.4) as was shown in [2]. The advantage of the off-shell formulation is that it makes manifest the non-linear structure: the coefficients \(R^{A;m}(h)\) are just the order \(m\) part of the non-linear transformation rules. In addition, the calculation of local on-shell terms arising from tree-level graphs simplifies:

We now discuss the condition (2.1) at tree-level. For the analysis at loop level we refer to [3]. At tree-level we only need the \(h^0\) part of (2.4). Let us define

\[
s_{(m-1)}\phi^A = \frac{1}{m!}R^{A;m}(h^0).
\]

(2.6)

Depending on the theory under consideration the quantities \(R^{A;m}(h^0)\) may be zero after some value of \(m\). Without loss of generality we assume that they are zero for \(m > k + 1\), for some integer \(k\) (which may be infinity; the same applies for \(k'\) below.). One shows that

\[
s\phi^A = \sum_{m=0}^{k} g^m s_m \phi^A
\]

(2.7)

are symmetry transformation rules that leave the Lagrangian,

\[
\mathcal{L} = \sum_{m=0}^{k'} g^m \mathcal{L}_m,
\]

(2.8)

invariant (up to total derivatives), where \(k'\) is also an integer (generically not equal to \(k\)). The Lagrangian \(\mathcal{L}\) will be determined from the tree-level normalization conditions as follows,

\[
\mathcal{L}_m = \frac{h N_m}{i m!}, \quad \text{for} \quad m > 1,
\]

(2.9)

where \(N_m\) denotes the local normalization ambiguity of \(T_m[T_1(x_1)...T_1(x_m)]\) in tree graphs defined with respect to the naturally split solution (i.e. the Feynman propagator is used
in tree-graphs). For \( m = 1 \), \( \mathcal{L}_1 = (h/i)T_1 \). The factor \( m! \) reflects the fact that \( T_m[...] \) appears in (A.2) with a combinatorial factors \( m! \) while the factor \( h/i \) is there to cancel the overall factor \( i/h \) that multiplies the action in the tree-level \( S \)-matrix. Notice that we regard (2.9) as definition of \( L_m \). Let us further define \( j^\mu_h \) as the local normalization ambiguity of \( T_n[j_0T_1...T_1] \).\footnote{We use the following abbreviations for the delta function distributions \( \delta^{(m)} = \delta(x_1, \ldots, x_n) = \delta(x_1 - x_2) \cdots \delta(x_{n-1} - x_n). \)}

\[
T_n[j^\mu_h(x_1)T_1(x_2) \cdots T_1(x_n)] = T_{c,n}[j^\mu_h(x_1)T_1(x_2) \cdots T_1(x_n)] + j^\mu_{n-1}\delta^{(n)} \tag{2.10}
\]

where \( T_{n,c} \) denotes the naturally splitted solution. We shall see that the normalization terms \( j_n \) complete the asymptotic current \( j_0 \) to the Noether current that generates the non-linear symmetry transformations (2.4).

We wish to calculate the tree-level terms at \( n \)th order. The causal distribution \( \sum_{l=1}^{n} \partial^{(l)}_\mu \mathcal{D}_n^{\mu/l} \) at the \( n \)th order consists of a sum of terms each of these being a tensor product of \( T_m[T_1...T_1 \partial \cdot j_0T_1...T_1] \) \( (m < n) \) with \( T \)-products that involve only \( T_1 \) vertices according to the general formulae (A.8,A.9,A.15). By the off-shell induction hypothesis, we have for all \( m < n \)

\[
\sum_{l=1}^{m} \partial^{(l)}_\mu \mathcal{J}_n^{\mu/l} = \sum_{A} (m!s_{m-1}\phi^A)K_{AB}\phi^B \delta^{(m)}. \tag{2.11}
\]

As explained in detail in [2], at order \( n \) one obtains all local on-shell terms by performing the so-called “relevant contractions”, namely the contractions between the \( \phi^B \) in the right hand side of (2.11) and \( \phi \) in local terms. In this manner we get the following general formula for the local term \( A_{c,n} \) arising through tree-level contractions at level \( n \),

\[
A_{c,n}(\text{tree}) = \sum_{\pi \in \Pi^n} \sum_{m=1}^{n-1} \partial^{(m)}_\mu \mathcal{J}_n^{\mu}(x_{\pi(1)}, \ldots, x_{\pi(m)}) N_{n-m}\delta(x_{\pi(k+1)}, \ldots, x_{\pi(n)}) \tag{2.12}
\]

where it is understood that in the right hand side only “relevant contractions” are made. The factors \( N_{n-m} \) are tree-level normalization terms of the \( T \)-products that contain \( n-m \) \( T_1 \) vertices.

In [2] we have provided a detailed analysis of (2.12) for any \( n \) (under the assumption that the Quantum Noether Method is not obstructed). In the next section, we will need these results in order to show that condition (3.1) is equivalent to condition (2.1). We therefore list them here without proofs.

The \( n = 1 \) case is trivial. One just gets that \( R^{A;1}(\hbar^0) = s_0\phi^A \). For \( 2 \leq n \leq k + 1 \), the condition (2.4) at tree-level yields the following constraint on the local normalization terms of the \( T_m, m < n \),

\[
s_0\mathcal{L}_{n-1} + s_1\mathcal{L}_{n-2} + \cdots + s_{n-2}\mathcal{L}_1 = \partial^{(m)}_\mu \mathcal{L}_n^{\mu} + s_{n-1}\phi^A K_{AB}\phi^B \tag{2.13}
\]
and, furthermore, determines $j_{n-1}^\mu$,

$$j_{n-1}^\mu = -n! L_{n-1}^\mu + (n-1)! \sum_{l=0}^{n-2} (l+1) \frac{\partial L_{n-1-l}}{\partial (\partial_\mu \phi^A)} s_l \phi^A. \quad (2.14)$$

For $n > k + 1$ we obtain,

$$s_0 L_{n-1} + s_1 L_{n-2} + \cdots + s_k L_{n-1-k} = \partial_\mu L_{n-1}, \quad (2.15)$$

and

$$j_{n-1}^\mu = -n! L_{n-1}^\mu + (n-1)! \sum_{l=0}^{k} \frac{\partial L_{n-l}}{\partial (\partial_\mu \phi^A)} s_{l-1} \phi^A. \quad (2.16)$$

Depending on the theory under consideration the $L_n$’s will be zero for $n > k'$, for some integer $k'$. Given the integers $k$ and $k'$, there is also an integer $k''$ (determined from the other two) such that $L_n^\mu = 0$, for $n > k''$.

Summing up the necessary and sufficient conditions (2.13), (2.15) for the Quantum Noether method to hold at tree level we obtain,

$$s \sum_{l=1}^{k'} g_l L_l = \sum_{l=1}^{k''} \partial_\mu L_l^\mu + (\sum_{l=1}^{k} g_l s_l \phi^A) K_{AB} \phi^B \quad (2.17)$$

Using $s_0 L_0 = \partial_\mu k_0^\mu$ and for $l \leq k$

$$s_l \phi^A K_{AB} \phi^B = \partial_\mu \left( \frac{\partial L_0}{\partial (\partial_\mu \phi^A)} s_l \phi^A \right) - s_l L_0 \quad (2.18)$$

we obtain,

$$s L = \partial_\mu \left( \sum_{l=0}^{k''} g_l^l k_l^\mu \right) \quad (2.19)$$

where, for $1 < l \leq k$,

$$k_l^\mu = L_l^\mu + \frac{\partial L_0}{\partial (\partial_\mu \phi^A)} s_l \phi^A \quad (2.20)$$

and for $l > k$, $k_l^\mu = L_l^\mu$. We therefore find that $L$ is invariant under the symmetry transformation,

$$s \phi^A = \sum_{l=0}^{k} g_l^l s_l \phi^A. \quad (2.21)$$

According to Noether’s theorem there is an associated Noether current. One may check that the current normalization terms $j_m^\mu$ (2.14), (2.16) are in one-to-one correspondence with the terms in the Noether current. Therefore the current $j_0$ indeed renormalizes to the full non-linear current.
3 Conservation of the Interacting Noether Current

The Quantum Noether Condition (2.1) can be reformulated in terms of interacting fields. Let \( j_{\mu,\text{int}}^0 \) and \( \tilde{j}_{\mu,\text{int}}^1 \) be the interacting currents corresponding to free field operators \( j_0^\mu \) and \( j_1^\mu \), respectively, perturbatively constructed according to (A.29). \( \tilde{j}_{\mu}^1 \) is equal to \(-L_{\mu}^1\) (defined in (2.13)) as will see below. Then the general Ward identity

\[
\partial_\mu j_{\mu,\text{int}}^0 = \partial_\mu g \tilde{j}_{\mu,\text{int}}^1
\]

is equivalent to condition (2.1). According to condition (3.1) the interacting Noether current \( j_{\mu,\text{int}}^0 \) is conserved only if it generates a linear symmetry, i.e. \( \tilde{j}_{\mu}^1 \) vanishes, or otherwise in the adiabatic limit \( g(x) \to 1 \), provided this limit exists. In the following we shall show that the condition (3.1) yields the same conditions on the time-ordered products \( T_n[T_1...T_1] \) as the Quantum Noether condition (2.1). In this sense the two general symmetry conditions are considered equivalent.

Because Poincaré invariance and causality already fix the time-ordered products \( T_n[T_1...T_1] \) up to the local normalization ambiguity \( N_n \), we only have to show that these local normalization terms \( N_n \) are constrained in the same way by both conditions, (3.1) and (2.1).

First, we translate the condition (3.1) to a condition on time-ordered products using the formulae given in the appendix:

The perturbation series for the interacting field operator \( j_{\mu,\text{int}}^0 \) of a free field operator \( j^\mu \) is given by the advanced distributions of the corresponding expansion of the S-matrix (see (A.29)):

\[
j_{\mu,\text{int}}^0(g,x) = j^\mu(x) + \sum_{n=1}^{\infty} \frac{1}{n!} \int d^4x_1 \ldots d^4x_n Ad_{n+1} [T_1(x_1) \ldots T_1(x_n); j^\mu(x)] g(x_1) \ldots g(x_n),
\]

(3.2)

where \( Ad_{n+1} \) denotes the advanced operator-valued distribution with \( n \) vertices \( T_1 \) and one vertex \( j^\mu(x) \) at the \( (n+1) \)th position. This distribution is only symmetric in the first \( n \) variables \( x_1, \ldots, x_n \). The support properties are defined with respect to the unsymmetrized variable \( x \).

With the help of (3.2), we rewrite the left hand side of equation (3.1)

\[
\partial_\mu j_{\mu,\text{int}}^0(x) = \partial_\mu j_0^\mu(x) + \sum_{n=1}^{\infty} \frac{1}{n!} \int d^4x_1 \ldots d^4x_n \partial_\mu Ad_{n+1} [T_1(x_1) \ldots T_1(x_n); j_0^\mu(x)] g(x_1) \ldots g(x_n)
\]

(3.3)

and the right hand side of (3.1)

\[
\tilde{j}_{\mu,\text{int}}^1(x) \partial_\mu g(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \int d^4x_1 \ldots d^4x_n d^4x_{n+1} Ad_{n+1} [T_1(x_1) \ldots T_1(x_n); \tilde{j}_{\mu}^1(x)] \delta(x - x_{n+1}) g(x_1) \ldots g(x_n) \partial_{\mu}^{x_{n+1}} g(x_{n+1})
\]

(3.4)
After partial integration, symmetrization of the integrand in the variable \((x_1, \ldots, x_{n+1})\) and shifting the summation index, the right hand side of (3.1) can be further rewritten as

\[
\tilde{J}^\mu_{1,\text{int}}(x)\partial_\mu g(x) = -\sum_{n=1}^{\infty} \frac{1}{n!} \int d^4x_1 \ldots d^4x_n \sum_{j=1}^{n} \left\{ \text{Ad}_n \left[ T_1(x_1) \ldots T(x_j) \ldots T(x_n); \tilde{J}^\mu_1(x) \right] \partial_\mu^s \delta(x_j - x) \right\} g(x_1) \ldots g(x_n)
\]

where the hat indicates that this coupling has to be omitted. Equation (3.1) reads then

\[
\partial_\mu j^\mu_0 = 0, \quad (n = 0)
\]

\[
\partial_\mu^s \text{Ad}_{n+1} \left[ T_1(x_1) \ldots T(x_n); j^\mu_0(x) \right] + \sum_{j=1}^{n} \text{Ad}_n \left[ T_1(x_1) \ldots T(x_j) \ldots T(x_n); \tilde{J}^\mu_1(x) \right] \partial_\mu^s \delta(x_j - x) = 0, \quad (n > 0)
\]

where the local normalization terms of the Ad-distributions with respect to a specified splitting solution will be given below.

In the following we discuss the equivalent condition of the time-ordered distributions instead of the advanced ones in order to compare the unsymmetrized condition (3.1) with the symmetrized Quantum Noether Condition (2.1). We get instead of (3.6)

\[
\partial_\mu T_{n+1} \left[ T_1(x_1) \ldots T_1(x_n); j^\mu_0(x) \right] = -\sum_{j=1}^{n} T_n \left[ T_1(x_1) \ldots T(x_j) \ldots T_1(x_n); \tilde{J}^\mu_1(x) \right] \partial_\mu^s \delta(x_j - x)
\]

These distributions get smeared out by \(g(x_1) \ldots g(x_n)\tilde{g}(x)\), where the test-function \(\tilde{g}\) differs from \(g\). One easily verifies the left hand side of (3.7) is just the Quantum Noether Condition (2.1) but without the symmetrization; the missing symmetrization produces the extra terms on the right hand side of (3.7) as we shall see.

We shall use the same off-shell procedure in order to fix the local on-shell obstruction terms (which is explained in detail in [3], section 4.2). The starting point \((n = 0)\) of both conditions is the same

\[
\partial_\mu j^\mu_0(x) = s_0 \phi^A \mathcal{K}_{AB} \phi^B
\]

We have now for \(n = 1\),

\[
\partial_\mu (T_{2,c}[T_1(x_1)j^\mu_0(x)] + j^\mu_1\delta(x_1 - x)) = -\tilde{J}_1^\mu(x)\partial_\mu^s \delta(x_1 - x)
\]

Working out the left hand side (and using \(T_1 = \frac{i}{\hbar} \mathcal{L}_1\)) we obtain,

\[
\partial_\mu (j^\mu_1\delta(x_1 - x)) + s_0 \mathcal{L}_1 \delta(x_1 - x) - \partial_\mu \left( \frac{\partial \mathcal{L}_1}{\partial (\partial_\mu \phi^A)} \right) s_0 \phi^A \delta(x_1 - x) = \tilde{J}_1^\mu(x) \partial_\mu^s \delta(x_1 - x)
\]
This condition fixes the local renormalization of $j_0^\mu$ at order $g$, denoted by $j^\mu_1$ (defined with respect to the natural splitting solution $T_{2,c}$) and also $\tilde{j}^\mu_1$ in condition (3.11). The latter term, proportional to the derivative of the $\delta$-distribution, is left over in our new unsymmetrized condition. Note that in the symmetrized case, we reduced these kind of terms to ones proportional to the $\delta$-distribution with the help of distributional identities.

The condition (3.10) can be fulfilled for some local operators $j^\mu_1$ and $\tilde{j}^\mu_1$ if and only if $s_0L_1$ is a divergence up to field equation terms,

$$s_0L_1 = \partial_\mu L_1^\mu + s_1 \phi^A \mathcal{K}_{AB} \phi^B. \quad (3.11)$$

In the absence of real obstructions this equation has solutions and we get

$$j^\mu_1 = -L_1^\mu + \frac{\partial L_1}{\partial(\partial_\mu \phi^A)} s_0 \phi^A \quad (3.12)$$
as local renormalization of $j_0^\mu_{int}$ at order $g^1$ and

$$\tilde{j}^\mu_1 = -L_1^\mu. \quad (3.13)$$

Equation (3.12) should be compared with the analogous formulae (2.14) for $n = 2$. We finally have

$$\partial_\mu T_2 \left[ T_1(x_1)j_0^\mu(x) \right] + \tilde{j}_1^\mu(x) \partial_\mu \delta(x_1 - x) = s_1 \phi^A \mathcal{K}_{AB} \phi^B \delta(x_1 - x). \quad (3.14)$$

The off-shell term on the right hand side of (3.14) is responsible for local obstruction terms at the next order, $n = 2$. We get (taking special care of derivative terms and advantage of our off-shell procedure):

$$\partial_\mu T_{3,c} \left[ T_1(x_1)T_1(x_2)j_0^\mu(x) \right] + \left( T_{2,c} \left[ T_1(x_1)j_1^\mu(x) \right] \right) \partial_\mu \delta(x_2 - x) + [x_1 \leftrightarrow x_2] \quad (3.15)$$

$$= \frac{1}{2} \left[ 2s_1 T_1 \delta^{(3)} - \left( 2\partial_\mu + \partial_\nu + \partial_\mu \right) \left( \frac{\partial T_1}{\partial(\partial_\mu \phi^A)} s_1 \phi^A \delta^{(3)} \right) + s_0 N_2 \delta^{(3)} - \partial_\mu \left( \frac{\partial N_2}{\partial(\partial_\mu \phi^A)} s_0 \phi^A \delta^{(3)} \right) \right]$$

where $N_2$ denotes the tree-normalization term of $T_2[T_1T_1]$ which is uniquely defined with respect to the natural splitting solution $T_{2,c}[T_1T_1]$. Now we include also the normalization ambiguity of the other distributions involved:

$$T_{3} \left[ T_1(x_1)T_1(x_2)j_0^\mu(x) \right] = T_{3,c} \left[ T_1(x_1)T_1(x_2)j_0^\mu(x) \right] + j_2^\mu(x) \delta(x_1, x_2, x) \quad (3.16)$$

$$T_{2} \left[ T_1(x_1)j_1^\mu(x) \right] = T_{2,c} \left[ T_1(x_1)j_1^\mu(x) \right] + \tilde{j}_2^\mu(x) \delta(x_1 - x) \quad (3.17)$$

According to (2.13) the Quantum Noether Condition (2.1) at order $n = 3$ is fulfilled if and only if

$$s_1 L_1 + s_0 L_2 = \partial_\mu L_2^\mu + s_2 \phi^A \mathcal{K}_{AB} \phi^B \quad (3.17)$$

$^4$Notice that $n$ in the present section should be compared with $n + 1$ in section 2.
where the definition $L_n = (\hbar/i)(N_n/n!)$ is used. Now the same is true for condition (3.16). Only if (3.17) holds one can absorb the local terms on the right hand side of (3.16) in the normalization terms $j_2^\mu(x)$ and $\tilde{j}_2^\mu(x)$ given in (3.18). The reasoning is again slightly different from the one in the symmetrized case. The distributions are only symmetric in the variables $x_i$, but $x$ is a distinguished variable. This means that the two local operator-valued distributions

\[
\hat{A}_0\delta(x_1, x_2, x); \sum_{i=1}^2 \partial_{x_i} \left( \hat{A}_1\delta(x_1, x_2, x) \right),
\]

(3.18)

where $\hat{A}_0(x)$ and $\hat{A}_1(x)$ are local operators, are independent (on the test functions $\check{g}(x_1, x_2) := g(x_1)g(x_2)\check{g}(x)$ with $g \neq \check{g}$).

So if and only if (3.17) is true the condition (3.1) can be fulfilled at order $n = 2$ and the local normalization terms of the interacting currents, $j_0^{\mu,\text{int}}$ and $\tilde{j}_1^{\mu,\text{int}}$, get fixed to

\[
j_2^\mu = 2! \left( -L_2^\mu + \frac{\partial L_2}{\partial(\partial_\mu \phi^A)} s_0 \phi^A + \frac{\partial L_1}{\partial(\partial_\mu \phi^A)} s_1 \phi^A \right)
\]

\[
\tilde{j}_2^\mu = -2! L_2^\mu + \frac{\partial L_1}{\partial(\partial_\mu \phi^A)} s_1 \phi^A
\]

(3.19)

Note the different symmetry factors in $j_2^\mu$ compared with the symmetrized case (2.14). With these normalizations we get

\[
\partial_{x_3} T_3 \left[ T_1(x_1)T_1(x_2)j_0^{\mu}(x) \right] + \left( T_2 \left[ T_1(x_1)\tilde{j}_1(x) \right] \partial_{x_2} \delta(x_2 - x) + [x_1 \leftrightarrow x_1] \right)
\]

\[
= 2! s_2 \phi^A K_{AB} \phi^B \delta(x_1, x_2, x)
\]

(3.20)

This corresponds to (3.1) at order $n = 2$:

\[
\partial_{x_3} j_0^{\mu}(x) \bigg|_{g^2} = \tilde{j}_1^{\mu}(x) \bigg|_{g^1} \partial_{x_3} g(x) + 2! s_2 \phi^A K_{AB} \phi^B(x).
\]

(3.21)

From these first two steps of the inductive construction, one already realizes that in general the additional terms proportional to $\partial_\mu g$ in (3.1) correspond to terms proportional to $\partial_\mu \delta^n$ which are now independent. In the former condition (2.1) we got rid of these terms by symmetrization and moding out the general formula $\sum_{i=1}^n \partial^i \delta^n = 0$. This formula is a direct consequence of translation invariance. Regardless this slight technical difference both conditions, (2.1) and (3.1), pose the same consistency conditions on the physical normalization ambiguity.

---

5 One could also choose as a basis $\hat{A}_0^i \delta(x_1, x_2, x); \partial^i \left( \hat{A}_1^i \delta(x_1, x_2, x) \right)$.

6 In the symmetrized case, where one smears out with totally symmetric test functions $g(x_1, x_2, x_3) := g(x_1)g(x_2)g(x_3)$, one has $\sum_{i=1}^2 \partial_{x_i} \left( \hat{A}_1^i \delta(x_1, x_2, x) \right) = (2/3) \partial \hat{A}_1^i \delta(x_1, x_2, x)$.
For $0 < n \leq k$, (where $k$ is the minimal integer such that $\forall m > k, s_m = 0$), condition (3.7) yields
\[
\partial^x_\mu (j_n \delta^{(n+1)}) + n! \left( \sum_{l=0}^{n-1} s_l \mathcal{L}_{n-l} \right) \delta^{(n+1)} - \\
- \sum_{l=0}^{n-1} \left( n! \partial^x_\mu + l \ (n - 1)! \sum_{i=1}^{n} (\partial^x_i) \right) \left( \frac{\partial \mathcal{L}_{n-l}}{\partial (\partial^x_\mu \phi^A)} s_l \phi^A \delta^{(n+1)} \right) = \tilde{j}^\mu_n (x) \partial^x_\mu \delta^{(n+1)}
\] (3.22)
where $j^\mu_n$ and $\tilde{j}^\mu_n$ are defined by analogous to (3.16) formulae. The sufficient and necessary condition for this equation to have a solution is
\[
s_0 \mathcal{L}_n + \cdots + s_{n-1} \mathcal{L}_1 = \partial_\mu \mathcal{L}^\mu_n + s_n \phi^A \mathcal{K}_{AB} \phi^B.
\] (3.23)
This agrees with (2.13) (we remind the reader that $n$ in present section corresponds to $n + 1$ in section 2). Then the current normalization terms are given by
\[
j^\mu_n = n! \left( -\mathcal{L}^\mu_n + \sum_{l=0}^{n-1} \frac{\partial \mathcal{L}_{n-l}}{\partial (\partial^x_\mu \phi^A)} s_l \phi^A \right)
\] (3.24)
\[
\tilde{j}^\mu_n = -n! \mathcal{L}^\mu_n + (n - 1)! \sum_{l=0}^{n-1} l \frac{\partial \mathcal{L}_{n-l}}{\partial (\partial^x_\mu \phi^A)} s_l \phi^A
\] (3.25)
and we have
\[
\partial^x_\mu j^\mu_{0, int} (x) \bigg|_{g^n} = \tilde{j}^\mu_{1, int} \bigg|_{g^{n-1}} \partial_\mu g (x) + n! s_n \phi^A \mathcal{K}_{AB} \phi^B (x)
\] (3.26)
For $n > k$, equation (3.7) yields
\[
\partial^x_\mu (j_n \delta^{(n+1)}) + n! \left( \sum_{l=0}^{k} s_l \mathcal{L}_{n-l} \right) \delta^{(n+1)} - \\
- \sum_{l=0}^{k} \left( n! \partial^x_\mu + l \ (n - 1)! \sum_{i=1}^{n} (\partial^x_i) \right) \left( \frac{\partial \mathcal{L}_{n-l}}{\partial (\partial^x_\mu \phi^A)} s_l \phi^A \delta^{(n+1)} \right) = \tilde{j}^\mu_n (x) \partial^x_\mu \delta^{(n+1)}
\] (3.27)
This equation now implies
\[
s_0 \mathcal{L}_n + \cdots + s_k \mathcal{L}_{n-k} = \partial_\mu \mathcal{L}^\mu_n.
\] (3.28)
We further obtain for the current normalization terms,
\[
j^\mu_n = n! \left( -\mathcal{L}^\mu_n + \sum_{l=0}^{k} \frac{\partial \mathcal{L}_{n-l}}{\partial (\partial^x_\mu \phi^A)} s_l \phi^A \right)
\] (3.29)
\[
\tilde{j}^\mu_n = -n! \mathcal{L}^\mu_n + (n - 1)! \sum_{l=0}^{k} l \frac{\partial \mathcal{L}_{n-l}}{\partial (\partial^x_\mu \phi^A)} s_l \phi^A
\] (3.30)
Therefore,
\[
\partial^x_\mu j^\mu_{0, int} (x) \bigg|_{g^n} = \tilde{j}^\mu_{1, int} \bigg|_{g^{n-1}} \partial_\mu g (x)
\] (3.31)
without using the free field equations.

In exactly the same way as in section 2, we deduce that the sum of all tree-level local normalization terms constitute a Lagrangian which is invariant (up to a total derivative) under the symmetry transformation $s\phi^A = \sum s_i \phi^A$. Inserting now the local normalization terms (3.24) and (3.29) into (3.2) we obtain,

$$j_{\mu,\text{int}}^\mu = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi^A)} s\phi^A - k^\mu$$

(3.32)

where we have used the definitions (2.8), (2.7), and (2.20). The combinatorial factor $n!$ in (3.24) and (3.29) exactly cancels the same factor in (3.2). We, therefore, see that the interacting free current exactly becomes the full non-linear current.

We have, thus, found that going from condition (2.1) to condition (3.1) just corresponds to a different technical treatment of the $\partial_\mu \delta^{(n)}$ terms which has no influence on the fact that both conditions pose the same conditions on the normalization ambiguity of the physical $T_n$ distributions, namely the consistency conditions of the classical Noether method. Our analysis of the condition (2.1) at the loop level is also independent of this slight technical rearrangement of the derivative terms. Thus, the issue of stability can be analyzed in exactly the same way as before (see section 4.3 of [2]). One shows (under the assumption that the Wess-Zumino consistency condition has only trivial solutions) that condition (3.1) at loop level also implies that the normalization ambiguity at the loop level, $N_n(h)$, is constrained in the same way as the tree-level normalizations, $N_n(h^0)$. Once the stability has been established the equivalence of (2.1) and (3.1) at loop level follows.

Summing up, we have shown that conditions (2.1)-(3.1) yield all consequences of non-linear symmetries for time-ordered products before the adiabatic limit. So at that level currents seem to be sufficient. As mentioned in the introduction, however, if one wants to identify the physical Hilbert space, one may need to use the Noether charge $Q_{\text{int}} = \int d^3x j_{\text{int}}^\mu(x)$. As our Quantum Noether Condition (3.1) shows, only in the adiabatic limit (provided the latter exists) the interacting Noether current is conserved. Moreover, there is an additional technical obstacle. In the construction of the BRST charge a volume divergence occurs. In [11] a resolution was proposed for the case of QED. It was also described there how the analysis of Kugo-Ojima may hold locally. One may expect more technical problems in the construction of the BRST charge in the case non-abelian gauge theories where the free non-interacting Noether current includes two quantum fields. However, at least for the implementation of the symmetry transformations in correlation functions, such an explicit construction of the BRST charge is not necessary, as we have shown. Symmetries are implemented with the help of Noether currents only.
4 Discussion

We have presented a general method for constructing perturbative quantum field theories with global and/or local symmetries. The analysis was performed in the Bogoliubov-Shirkov-Epstein-Glaser approach. In this framework the perturbative $S$-matrix is directly constructed in the asymptotic Fock space with only input causality and Poincaré invariance. The construction directly yields a finite perturbative expansion without the need of intermediate regularization. The invariance of the theory under a given symmetry is imposed by requiring that the asymptotic Noether current is conserved at the quantum level.

The novel feature of the present discussion with respect to the usual approach is that our results are manifestly scheme independent. In addition, in the conventional approach one implicitly assumes the naive adiabatic limit. Our construction is done before the adiabatic limit is taken. The difference between the two approaches is mostly seen when the symmetry condition is expressed in terms of the interacting Noether current. If the interacting current generates non-linear symmetries, it is not conserved before the adiabatic limit is taken. An important example is pure gauge theory. In this case, the global symmetry is BRST symmetry. The interacting BRST current is not conserved before the adiabatic limit. Nevertheless, one may still construct correlation functions that satisfy the expected Ward identities.

In the present contribution and in \cite{2} we analyzed the symmetry conditions assuming that there are no true tree-level or loop-level obstructions. The algebra of the symmetry transformation imposes integrability conditions on the possible form of these obstructions \cite{12}. Therefore, to analyze the question of anomalies in the present context one would have to understand how to implement the algebra of symmetry transformations in this framework. This is expected to be encoded in multi-current correlation functions. We will report on this issue in a future publication \cite{13}.

The Quantum Noether Condition (2.1) or (3.1) leads to specific constraints (equations (2.13), (2.15)) that the local normalization terms should satisfy. We have seen that these conditions are equivalent to the condition that one has an invariant action. So, one may infer the most general solution of equations (2.13), (2.15) from the most general solution of the problem of finding an action invariant under certain symmetry transformation rules.

For the particular case of gauge theories the global symmetry used in the construction is BRST symmetry. In EG one always works with a gauged fixed theory since one needs to have propagators for all fields. Therefore, the symmetry transformation rules are the gauged fixed ones. Physics, however, should not depend on the particular gauge fixing chosen. The precise connection between the results of the gauge invariant cohomology
(which may be derived with the help of the antifield formalism\cite{13,14}) and the present gauged-fixed formulation will be presented elsewhere \cite{14}.

The symmetry condition we proposed involves the (Lorentz invariant) condition of conservation of the Noether current. There are cases, however, where one has a charge that generates the symmetry but not a Noether current (for this to happen the theory should not possess a Lagrangian). A more fundamental formulation that will also cover these cases may be to demand that the charge that generates the symmetry is conserved at the quantum level (i.e. inside correlation functions). A precise formulation of this condition may require a Hamiltonian reformulation of the EG approach. Such a reformulation may be interesting on its own right.

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**A Appendix**

In this appendix we give the basic conventions and formulae of the causal framework, in particular the definition of the interacting field. A self-contained introduction to the EG construction may be found in section 3 of \cite{2}. For further technical details we refer the reader to the literature \cite{5,6,17,18}.

We describe the construction for the case of a massive scalar field. The very starting point is the Fock space $\mathcal{F}$ of the massive scalar field (based on a representation space $H_{s}^{m}$ of the Poincaré group) with the defining equations

\begin{equation}
(\Box + m^2)\varphi = 0 \quad (a), \quad [\varphi(x), \varphi(y)] = i\hbar D_m(x-y) \quad (b),
\end{equation}

where $D_m(x-y) = \frac{i}{(2\pi)^4} \int dk^4 \delta(k^2 - m^2) \text{sgn}(k^0) \exp(-ikx)$ is the Pauli-Jordan distribution. In contrast to the Lagrangian approach, the $S$-matrix is directly constructed in this Fock space in the form of a formal power series

\begin{equation}
S(g) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int dx_1^4 \cdots dx_n^4 \ T_n(x_1, \cdots, x_n; \hbar) \ g(x_1) \cdots g(x_n). \quad (A.2)
\end{equation}

The coupling constant $g$ is replaced by a tempered test function $g(x) \in \mathcal{S}$ (i.e. a smooth function rapidly decreasing at infinity) which switches on the interaction.

The central objects are the $n$-point operator-valued distributions $T_n \in \mathcal{S}'$, where $\mathcal{S}'$ denotes the space of functionals on $\mathcal{S}$. They should be viewed as mathematically
well-defined (renormalized) time-ordered products,

$$T_n(x_1, \ldots, x_n; \hbar) = T[T_1(x_1) \cdots T_1(x_n)], \quad (A.3)$$

of a given specific coupling, say $T_1 = \frac{i}{\hbar} : \Phi^4 :$, (c), which is the third defining equation in order to specify the theory in this formalism. Notice that the expansion in (A.2) is not a loop expansion. Each $T_n$ in (A.2) can receive tree-graph and loop-contributions. One can distinguish the various contributions from the power of $\hbar$ that multiplies them.

Epstein and Glaser present an explicit inductive construction of the most general perturbation series in the sense of (A.2) which is compatible with the fundamental axioms of relativistic quantum field theory, causality and Poincaré invariance.

The main guiding principle is the property of causal factorization which can be stated as follows:

- Let $g_1$ and $g_2$ be two tempered test functions. Then causal factorization means that

$$S(g_1 + g_2) = S(g_2)S(g_1) \quad if \quad \text{supp} g_1 \preceq \text{supp} g_2 \quad (A.4)$$

the latter notion means that the support of $g_1$ and the support of $g_2$, two closed subsets of $\mathbb{R}^4$, can be separated by a space like surface.

It is well-known that the heuristic solution for (A.4), namely

$$T_n(x_1, \ldots, x_n; \hbar) = \sum_{\pi} T_1(x_{\pi(1)}) \cdots T_1(x_{\pi(n)}) \Theta(x_{\pi(1)}^0 - x_{\pi(2)}^0) \cdots \Theta(x_{\pi(n-1)}^0 - x_{\pi(n)}^0), \quad (A.5)$$

is, in general, affected by ultra-violet divergences ($\pi$ runs over all permutations of $1, \ldots, n$). The reason for this is that the product of the discontinuous $\Theta$-step function with Wick monomials like $T_1$ which are operator-valued distributions is ill-defined. One can handle this problem by using the usual regularization and renormalization procedures and finally end up with the renormalized time-ordered products of the couplings $T_1$.

Epstein and Glaser suggest another path which leads directly to well-defined $T$-products without any intermediate modification of the theory using the fundamental property of causality (A.4) as a guide. They translate the condition (A.4) into an induction hypothesis, $H_m, m < n$, for the $T_m$-distribution which reads

$$H_m : \left\{ \begin{array}{ll}
T_m(X \cup Y) = T_{m_1}(X) T_{m-m_1}(Y) & \text{if } X \supseteq Y, \quad X, Y \neq \emptyset, \quad 0 < m_1 < m \\
[T_{m_1}(X), T_{m_2}(Y)] = 0 & \text{if } X \sim Y \ (\iff X \supseteq Y \land X \subseteq Y) \ \forall m_1, m_2 \leq m
\end{array} \right. \quad (A.6)$$

Here we use the short-hand notation $T_m(x_1, \ldots, x_m; \hbar) = T(X); \ |X| = m$.

Besides other properties they also include the Wick formula for the $T_m$ distributions into the induction hypothesis. This is most easily done by including the so-called Wick submonomials of the specific coupling $T_1 = (i/\hbar) : \Phi^4 :$ as additional couplings in the
construction \( T^j_1 := (i/\hbar)(4!/(4 - j)!): \Phi^{4-j} : , 0 < j < 4 \). Then the Wick formula for the \( T_n \) products can be written as

\[
T_m[T_1^{j_1}(x_1) \cdots T_1^{j_m}(x_m)] = \sum_{s_1, \ldots, s_m} \langle 0 \mid T[T_1^{j_1+s_1}(x_1) \cdots T_1^{j_m+s_m}(x_m)] \mid 0 \rangle \prod_{i=1}^m \left[ \frac{\Phi^{s_i}(x_1)}{s_i!} \right] : \tag{A.7}
\]

That such a quantity is a well-defined operator-valued distribution in Fock space is assured by distribution theory (see Theorem O in [3], 2. p. 229). Note also that the coefficients in the Wick expansion are now represented as vacuum expectation values of operators.

Now let us assume that \( T_m \) distributions with all required properties are successfully constructed for all \( m < n \). Epstein and Glaser introduce then the retarded and the advanced \( n \)-point distributions (from now on we suppress the \( h \) factor in our notation):

\[
R_n(x_1, \ldots, x_n) = T_n(x_1, \ldots, x_n) + R'_n, \quad R'_n = \sum_{P_2} T_{n-1}(Y, x_n)T_{n_1}(X) \tag{A.8}
\]

\[
A_n(x_1, \ldots, x_n) = T_n(x_1, \ldots, x_n) + A'_n, \quad A'_n = \sum_{P_2} T_{n_1}(X)T_{n-1}(Y, x_n). \tag{A.9}
\]

The sum runs over all partitions \( P_2 : \{x_1, \ldots, x_{n-1}\} = X \cup Y, \ X \neq \emptyset \) into disjoint subsets with \( |X| = n_1 \geq 1, |Y| \leq n - 2 \). The \( \tilde{T} \) are the operator-valued distributions of the inverse S-matrix:

\[
S(g)^{-1} = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int d^4 x_1 \ldots d^4 x_n \tilde{T}_n(x_1, \ldots, x_n) g(x_1) \ldots g(x_n) \tag{A.10}
\]

The distributions \( \tilde{T} \) can be computed by formal inversion of \( S(g) \):

\[
S(g)^{-1} = (1 + T)^{-1} = 1 + \sum_{n=1}^{\infty} (-T)^r \tag{A.11}
\]

\[
\tilde{T}_n(X) = \sum_{r=1}^n (-)^r \sum_{P_r} T_{n_1}(X_1) \cdots T_{n_r}(X_r), \tag{A.12}
\]

where the second sum runs over all partitions \( P_r \) of \( X \) into \( r \) disjoint subsets \( X = X_1 \cup \ldots \cup X_r \), \( X_j \neq \emptyset \), \( |X_j| = n_j \).

We stress the fact that all products of distributions are well-defined because the arguments are disjoint sets of points so that the products are tensor products of distributions. We also remark that both sums, \( R'_n \) and \( A'_n \), in contrast to \( T_n \), contain \( T_j \)'s with \( j \leq n - 1 \) only and are therefore known quantities in the inductive step from \( n - 1 \) to \( n \). Note that the last argument \( x_n \) is marked as the reference point for the support of \( R_n \) and \( A_n \). The following crucial support property is a consequence of the causality conditions (A.6):

\[
\text{supp} R_m(x_1, \ldots, x_m) \subseteq \Gamma^+_m(x_m), \quad m < n \tag{A.13}
\]
where $\Gamma_{m-1}^+$ is the $(m-1)$-dimensional closed forward cone,

$$
\Gamma_{m-1}^+(x_m) = \{(x_1, \ldots, x_{m-1}) \mid (x_j - x_m)^2 \geq 0, x_j^0 \geq x_m^0, \forall j\}.
$$  \tag{A.14}

In the difference

$$
D_n(x_1, \ldots, x_n) \overset{\text{def}}{=} R_n' - A_n'
$$  \tag{A.15}

the unknown $n$-point distribution $T_n$ cancels. Hence this quantity is also known in the inductive step. With the help of the causality conditions (A.6) again, one shows that $D_n$ has causal support

$$
\text{supp}D_n \subseteq \Gamma_{n-1}^+(x_n) \cup \Gamma_{n-1}^-(x_n)
$$  \tag{A.16}

Thus, this crucial support property is preserved in the inductive step from $n - 1$ to $n$.

Given this fact, the following inductive construction of the $n$-point distribution $T_n$ becomes possible: Starting off with the known $T_m(x_1, \ldots, x_n)$, $m \leq n - 1$, one computes $A_n', R_n'$ and $D_n = R_n' - A_n'$. With regard to the supports, one can decompose $D_n$ in the following way:

$$
D_n(x_1, \ldots, x_n) = R_n(x_1, \ldots, x_n) - A_n(x_1, \ldots, x_n)
$$  \tag{A.17}

$$
\text{supp}R_n \subseteq \Gamma_{n-1}^+(x_n), \quad \text{supp}A_n \subseteq \Gamma_{n-1}^-(x_n)
$$  \tag{A.18}

Having obtained these quantities we define $T_n'$ as

$$
T_n' = R_n - R_n' = A_n - A_n'
$$  \tag{A.19}

Symmetrizing over the marked variable $x_n$, we finally obtain the desired $T_n$,

$$
T_n(x_1, \ldots x_n) = \sum_{\pi} \frac{1}{n!} T_n'(x_{\pi(1)}, \ldots x_{\pi(n)})
$$  \tag{A.20}

One can verify that the $T_n$ satisfy the conditions (A.6) and all other further properties of the induction hypothesis [5].

Summing up, with the help of the corresponding causal factorization property of the $T_m$-distribution one is able to reduce the problem of constructing well-defined time-ordered products to the following splitting problem of distributions:

Given an operator-valued tempered distribution $D_n \in \mathcal{S}'(\mathbb{R}^{4n})$ with causal support,

$$
\text{supp}D_n \subseteq \Gamma_{n-1}^+(x_n) \cup \Gamma_{n-1}^-(x_n).
$$  \tag{A.21}

one has to find a pair $(R, A)$ of tempered distributions on $\mathbb{R}^{4n}$ with the following characteristics:

- $R, A \in \mathcal{S}'(\mathbb{R}^{4n})$ \hspace{1cm} (A) \tag{A.22}
- $\text{supp}R \subset \Gamma_{n}^+(x_n), \hspace{1cm} \text{supp}A \subset \Gamma_{n}^-(x_n)$ \hspace{1cm} (B) \tag{A.23}
- $R - A = D$ \hspace{1cm} (C) \tag{A.24}
A general solution of this problem was given by the mathematician Malgrange some time ago [19]. As mentioned already, every renormalization scheme solves this problem implicitly. The advantage of the Epstein-Glaser formulation is that it separates the purely technical details (which are essential for explicit calculations) from the simple physical structure of the theory.

The singular behavior of the distributions \( d_n \) for \( x \to 0 \) is crucial for the splitting problem because \( \Gamma^+_{n-1}(0) \cap \Gamma^-_{n-1}(0) = \{0\} \). One therefore has to classify the singularities of distributions in this region. This can be characterized in terms of the singular order \( \omega \) of the distribution under consideration which turns out to be identical with the usual power-counting degree. For details on the theory of distribution splitting we refer to the literature [5, 17].

One has to ask whether the splitting solution of a given numerical distribution \( d \) with singular order \( \omega(d) \) is unique. Let \( r_1 \in S' \) and \( r_2 \in S' \) be two splitting solutions of the given distribution \( d \in S' \). By construction \( r_1 \) and \( r_2 \) have their support in \( \Gamma^+ \) and agree with \( d \) on \( \Gamma^+ \setminus \{0\} \), from which follows that \( (r_1 - r_2) \) is a tempered distribution with point support and with singular order \( \omega \leq \omega(d) \):

\[
\text{supp}(r_1 - r_2) \subset \{0\}, \quad \omega(r_1 - r_2) = \omega(d), \quad (r_1 - r_2) \in S'
\]

(A.25)

According to a well-known theorem in the theory of distributions, we have

\[
r_1 - r_2 = \sum_{|a|=0}^{\omega_0} C_a \partial^a \delta(x).
\]

(A.26)

In the case \( \omega(d) < 0 \) which means that \( d_n \) is regular at the zero point, the splitting solution is thus unique. In the case \( \omega(d) \geq 0 \) the splitting solution is only determined up to a local distribution with a fixed maximal singular degree \( \omega_0 = \omega(d) \). The demands of causality (A.4) and translational invariance leave the constants \( C_a \) in (A.26) undetermined. They have to be fixed by additional normalization conditions.

One shows that, besides this normalization ambiguity, the \( T_n \) distributions are already fixed at all orders by the fundamental axioms of QFT and the defining equations of the specific theory under consideration which includes the definition of the specific coupling \( T_1 \).

Having constructed the most general \( S \)-matrix one can construct interacting field operators (compatible with causality and Poincaré invariance) (second reference in [5], section 8) as follows:

One starts with an extended first order \( S \)-matrix

\[
S(g, g_1, g_2, \ldots) = \int d^4x \{ T_1(x)g(x) + \frac{i}{\hbar} (\Phi_1(x)g_1(x) + \Phi_2(x)g_2(x) + \ldots) \}
\]

(A.27)

where \( \Phi_i \) represent certain Wick monomials like \( \varphi \) or : \( \varphi^3 : \). Following Bogoliubov and Shirkov [4], Epstein and Glaser defined the corresponding interacting fields \( \Phi^\text{int}_i \) as
functional derivatives of the extended $S$-matrix:

$$
\Phi^\text{int}_i(g, x) = \frac{\hbar}{i} S^{-1}(g, g_1, \ldots) \frac{\delta S(g, g_1, \ldots)}{\delta g_i(x)} \bigg|_{g_i=0} \tag{A.28}
$$

One shows that the perturbation series for the interacting fields is given by the advanced distributions of the corresponding expansion of the $S$-matrix, namely

$$
\Phi^\text{int}_i(g, x) = \Phi_i(x) + \sum_{n=1}^{\infty} \frac{1}{n!} \int d^4x_1 \ldots d^4x_n A_{n+1/n+1}(x_1, \ldots, x_n; x), \tag{A.29}
$$

where $A_{n+1/n+1}$ denotes the advanced distributions with $n$ original vertices $T_1$ and one vertex $\Phi_i$ at the $(n + 1)$th position; symbolically we may write:

$$
A_{n+1/n+1}(x_1, \ldots, x_n; x) = Ad_{n+1} [T_1(x_1) \ldots T_1(x_n); \Phi_i(x)] \tag{A.30}
$$

One shows that the perturbative defined object $\Phi^\text{int}_i$ fulfills the properties like locality and field equations in the sense of formal power series. The definition can be regarded as a direct construction of renormalized composite operators. Epstein and Glaser showed that the adiabatic limit $g \to 1$ exists only in the weak sense of expectation values in massive theories. The limit possesses all the expected properties of a Green’s function such as causality, Lorentz covariance and the spectral condition.

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