Gaussian fluctuation corrections to the BCS mean field gap amplitude at zero temperature

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The leading (Gaussian) fluctuation correction to the weak coupling zero temperature BCS superconducting gap equation is computed. We find that the dominant contribution comes from the high energies and momenta (compared to the gap) and gives a correction smaller by the weak-coupling factor $gN_0$ than the mean-field terms. This correction is small due to cancellation of singular contributions from the amplitude and phase mode at high energies and momenta.

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I. INTRODUCTION

The Bardeen-Cooper-Schrieffer (BCS) theory of superconductivity is important both as an empirically highly accurate and successful model of a nontrivial physical phenomenon and as a paradigm for theoretical study of a wide class of models involving a logarithmically divergent susceptibility. This paper focusses on the latter aspect. From this point of view the essence of BCS theory is the observation that in a generic, weakly coupled fermion system in $d \geq 2$ dimensions, all susceptibilities are non-negative and remain finite as temperature $T \to 0$ except for particle-particle susceptibilities such as

$$\chi_{\mu\nu}(q) = \int d^d r \int_0^\beta d e^{i\mu r - i\mathbf{q} \cdot \mathbf{r}} \left\langle T_\uparrow \left[ \psi_{\downarrow}(\tau, r) \psi_{\uparrow}(\tau, r), \psi_{\downarrow}^+ (0, 0) \psi_{\uparrow}^+ (0, 0) \right] \right\rangle,$$

which diverge logarithmically as $q, \nu$ and temperature $T$ tend to 0. (By the phrase ‘generic’ we mean to rule out for example the nesting and van Hove instabilities occurring in particular models at particular band fillings).

BCS showed that the logarithmic divergence of $\chi$ signalled the appearance, at a transition temperature $T_c$, of a new order parameter $\Delta \sim < \psi_{\downarrow} \psi_{\uparrow} >$. They further argued that $T_c$, the magnitude of $\Delta$ and many other physical consequences of the ordering could be accurately determined by mean field theory. The stunning agreement between predictions of the BCS theory and data on conventional superconductors lends very strong support to this view.

The approach pioneered by BCS has been applied by many workers in many contexts involving logarithmically or more strongly diverging susceptibilities; for example to spin and charge density wave instabilities driven by nesting effects in half filled particle-hole symmetric bands, or, in the two dimensional case, at fillings that give rise to a van-Hove singularity in the density of states.

Some aspects of fluctuations have been understood in detail. Around the time BCS theory was developed it was understood that in $d < 4$ dimensions and for temperatures sufficiently near to $T_c$, nonlinear interactions among very long wavelength order parameter fluctuations would invalidate a mean field treatment of the thermally driven transition, and the (rather modest) temperature window within which conventional superconductors would exhibit non-mean-field behavior was estimated. The effects of order parameter fluctuations with momenta and energies smaller than the inverse correlation length and $\Delta$, respectively, on the near-$T_c$ properties of conventional superconductors in different dimensions has been examined in detail and found, for example, to give a small negative correction to the mean-field value of the order parameter $\Delta$. Such corrections have been studied in detail and found, for example, to give a small negative correction to the mean-field value of the order parameter $\Delta$ in $d = 2$ to have a larger effect on $T_c$. Engelbrecht and co-workers used a functional integral method to investigate the effect of long wavelength fluctuations on the BCS-Bose-Einstein crossover. Varlamov et al. studied the effect of these long wavelength fluctuations on the normal state of layered superconductors in the context of high-temperature superconductivity.

In this paper we investigate corrections to the BCS mean field approximation due to Gaussian fluctuations. We use a standard functional integral formalism to study the fluctuations over a wide range of momenta and energies at $T = 0$, where the order parameter is believed to be well developed, and thus the fluctuations corrections are supposed to be small. Surprisingly, this basic question seems not to have been addressed in the literature. We calculate the corrections and find that they are, indeed, small compared to the mean-field terms. There are three important features: 1. The dominant correction comes from energies and momenta high compared to the mean-field gap. 2. The dominant contribution to the correction comes from processes in which the electrons are scattered nearly parallel to the Fermi surface. 3. The fluctuation corrections from each of the amplitude and the phase mode diverge logarithmically at high energies, but with opposite sign so these divergences cancel leaving a small overall correction.

The rest of this paper is organized as follows: in Section II we present the formalism to be used. In Section...
III, we evaluate the polarization kernel and its derivatives, and use it in Section IV to calculate the fluctuation correction. Section V is a discussion and conclusion.

II. GENERAL FORMULAE

We consider for definiteness a model of fermions in $d \geq 2$ spatial dimensions with energy dispersion $\varepsilon_p = p^2 / 2m - \mu$ (as will become evident below, our results may be trivially generalized to more realistic dispersions provided nesting and van Hove singularities are absent). We take the fermions to interact via a short ranged instantaneous attractive interaction parametrized by a coefficient $g > 0$. We follow Shankar and consider only states within a cutoff $\Lambda << p_F$ of the fermi surface, expand the density of states $N(\varepsilon) = \int \frac{dp}{(2\pi)^d} \delta(\varepsilon - \varepsilon_p)$ around the Fermi energy as

$$N(\varepsilon) = N_0 + N_1 \varepsilon / v_F \Lambda + ..., \quad (2)$$

where $N_0$ and $N_1$ are of the same order of magnitude. The Hamiltonian thus becomes

$$H = N_0 \sum_\sigma \int d\varepsilon \varepsilon_p c_{p\sigma}^+ c_{p\sigma} - g \int (dp_1 ... dp_4) c_{p_1 \uparrow}^+ c_{p_2 \downarrow}^+ c_{p_3 \downarrow} c_{p_4 \uparrow}$$

where $(dp) = \frac{dp}{(2\pi)^d}$ and the prime on the integral indicates that all momenta are restricted to the shell $-\Lambda < |p| - p_F < \Lambda$ and that there is a momentum conserving delta function.

To analyse the model we write it as a functional integral, decouple the interaction via a Hubbard-Stratonovich transformation and perform the integral over the fermionic fields obtaining for the partition function

$$Z = \int D\Delta^*(\tau, r) D\Delta(\tau, r) e^S \quad (4)$$

with action $S$ given by

$$S = \text{Tr} \ln( - \partial_\tau + H(\Delta(\tau, r))) - \int d\tau dr |\Delta(\tau, r)|^2 / g \quad (5)$$

and

$$H(\Delta(\tau, r)) = N_0 \sum_\sigma \int d\varepsilon \varepsilon_p c_{p\sigma}^+ c_{p\sigma} + \int d\tau dr \left( \Delta(\tau, r) \psi_\uparrow^+ (r, \tau) \psi_\downarrow^+ (r, \tau) + H.c \right) \quad (6)$$

The BCS mean field theory corresponds to a saddle-point approximation to Eq (11) at the saddle point we have

$$\Delta(\tau, r) = \Delta^*(\tau, r) = \Delta_0, \quad (7)$$

and the saddle-point approximation to the action is

$$S_0(\Delta_0) - S_0(\Delta_0 = 0) = \beta V \left( N_0 \Delta_0^2 \log \frac{2v_F \Lambda}{\Delta_0} - \frac{\Delta_0^2}{g} + ... \right). \quad (8)$$

where the ellipsis denotes terms of the order of $\Delta_0^2$ with coefficient of order unity, and $v_F \equiv d\varepsilon_p / dp$ at $p_F$ is the Fermi velocity. Our calculations are valid in the weak-coupling regime $gN_0 << 1$, and these omitted terms are small by at least one power of $gN_0$ relative to terms which we retain. Finally, the saddle point value of $\Delta_0$ is fixed by extremizing $S$ with respect to $\Delta_0$ yielding the familiar BCS gap equation

$$\frac{1}{g} = N_0 \left( \frac{\log \frac{v_F \Lambda}{\Delta_0}}{\Delta_0} + ... \right) \quad (9)$$

where again the ellipsis indicates terms of order unity. Thus, as is well known, within the BCS approximation the $T = 0$ gap value $\Delta_0$ is determined only to logarithmic accuracy, i.e. $\log(\Lambda / \Delta_0)$ is known up to terms of relative order unity.

To study fluctuation corrections we write:

$$\Delta(\tau, r) = \Delta_0 + \eta(\tau, r). \quad (10)$$

We write the fluctuation $\eta$ in terms of its real and imaginary parts

$$\eta(\tau, r) = \eta_1(\tau, r) + i \eta_2(\tau, r), \quad (11)$$

which, for the gauge in which $\Delta_0$ is real, correspond to the amplitude and phase fluctuation respectively.

Substitution of Eq (11) into Eq (1) and expansion in powers of $\delta$ yields $Z = \int D\eta^*(\tau, r) D\eta(\tau, r) e^{S_0 + i \delta S}$ with

$$\delta S = - \int \frac{dv}{2\pi} \int \frac{dq}{2\pi} \int \int \frac{d\omega}{2\pi} \int \frac{dk}{2\pi} \left( \frac{\delta_{ab}}{g} + \Pi_{ab}(i\nu, q) \right) \quad (12)$$

$$\eta_a^*(i\nu, q) \eta_b(i\nu, q) + ..., \quad (13)$$

where the ellipsis denotes higher powers of $\eta, a, b = 1, 2$ and

$$\Pi_{ab}(i\nu, q) = - \frac{(-1)^{a+b}}{2} \int \frac{d\omega}{2\pi} \int \frac{dk}{2\pi} \left[ \text{Tr} G(i\omega_+, k_+) \tau_a G(i\omega_-, k_-) \tau_b \right]$$

Here $\omega_{\pm} = \omega \pm \frac{q}{2}$ and $k_{\pm} = k \pm \frac{q}{2}$ while $G$ is the superconducting Green function in Nambu matrix notation and $\tau_a, a = 1, 2$ are the Pauli matrices in particle-hole space. Note that in the instantaneous interaction model it suffices to place a cutoff on the momentum integral; no frequency cutoff is required.

We see from Eq (13) that the propagator corresponding to $\eta$ is of order $g$ (except at long wavelengths and low energy) so an evaluation of $Z$ via an expansion in powers of $\eta$ yields a series expansion in powers of $gN_0$ for the free energy and other physical quantities. For example,
restricting the expansion to Gaussian order leads to a correction to the free energy given by

\[ F_1 = -TS_1 \]

\[ = VT \sum_{\nu}^A \left( \frac{dq}{2\pi} \right)^d \text{Tr} \ln(1 + g\Pi(i\nu, q)) \]

\[ = VT \sum_{\nu}^A \left( \frac{dq}{2\pi} \right)^d \frac{1}{2} \ln((1 + g\Pi_0)^2 - g^2(\Pi_1^2 + \Pi_2^2 + \Pi_3^2)). \]

Here

\[ \Pi \equiv \Pi_0 I + \Pi_1 \tau_1 + \Pi_2 \tau_2 + \Pi_3 \tau_3 \]

is the polarization matrix written in terms of the Pauli matrices. The correction to the saddle point equation arising from the Gaussian fluctuation term is

\[ \frac{dS_1}{d\Delta_0} = -\beta V \int \frac{d\nu}{2\pi} \int \frac{d\omega}{2\pi} \left( \frac{dq}{2\pi} \right)^d \frac{(1 + g\Pi_0)g\frac{d\Pi_0}{d\nu} - g^2(\Pi_1\frac{d\Pi_1}{d\nu} + \Pi_2\frac{d\Pi_2}{d\nu} + \Pi_3\frac{d\Pi_3}{d\nu})}{(1 + g\Pi_0)^2 - g^2(\Pi_1^2 + \Pi_2^2 + \Pi_3^2)}. \]

In the next section, we shall estimate the relative contributions of Eq 17 and Eq 9 to the gap equation.

III. CALCULATION OF $\Pi$

This section computes the polarizabilities appearing in Eq 17. To simplify notation we work with the action density (i.e. divide by $\beta V$) and omit the subscript 0 of $\Delta_0$.

Using the explicit mean-field form for $G$, namely

\[ G(i\omega, k) = -\frac{i\omega + \epsilon \tau_3 + \Delta \tau_1}{\omega^2 + \epsilon^2(k) + \Delta^2} \]

we find (here $\epsilon_{\pm} = \epsilon_{k,\pm \eta/2}$)

\[ \Pi_0 = -\int \frac{d\omega}{2\pi} \int \frac{dk}{2\pi} \left( \frac{d\omega}{2\pi} \right)^d \frac{i\omega + \epsilon \tau_3 + \Delta \tau_1}{[\omega^2 + \epsilon^2 + \Delta^2][\omega^2 + \epsilon^2 + \Delta^2]} \]

\[ \Pi_2 = -\int \frac{d\omega}{2\pi} \int \frac{dk}{2\pi} \left( \frac{d\omega}{2\pi} \right)^d \frac{i\omega + \epsilon \tau_3 - i\omega - \epsilon \tau_3}{[\omega^2 + \epsilon^2 + \Delta^2][\omega^2 + \epsilon^2 + \Delta^2]} \]

\[ \Pi_3 = \int \frac{d\omega}{2\pi} \int \frac{dk}{2\pi} \left( \frac{d\omega}{2\pi} \right)^d \frac{\Delta^2}{[\omega^2 + \epsilon^2 + \Delta^2][\omega^2 + \epsilon^2 + \Delta^2]} \]

and $\Pi_1 = 0$.

First, we analyze $\Pi_3$. We linearize the fermion dispersion around the Fermi surface

\[ \epsilon_+ = \epsilon + v_F \frac{q}{2} \mu \]

where $v_F$ is the Fermi velocity, and $\mu$ is the cosine of the angle between $k$ and $q$. For a spherical Fermi surface in $d$ dimensions, $v_F$ is a constant, and

\[ \frac{(dk)}{(2\pi)}^d = \int_{-v_F \Lambda}^{v_F \Lambda} N(\epsilon) d\epsilon \frac{1}{\kappa_d} \frac{d\mu(1 - \mu^2)^{d-3}}{\kappa_d}; \]

with

\[ \kappa_d = \frac{\sqrt{\pi} \Gamma \left( \frac{d-1}{2} \right)}{2 \Gamma \left( \frac{d}{2} \right)}. \]

Then

\[ \Pi_3(i\nu, q) = N_0 \int_0^{v_F \Lambda} \frac{d\mu(1 - \mu^2)^{d-3}}{\kappa_d} I(\nu, v_F q \mu, \Delta; \Lambda) \]

with

\[ I(\nu, v_F q \mu, \Delta; \Lambda) = \int \frac{d\omega}{2\pi} \int_{-v_F \Lambda}^{v_F \Lambda} \frac{\Delta^2}{[\omega_+^2 + \epsilon_+^2 + \Delta^2][\omega_-^2 + \epsilon_-^2 + \Delta^2]} \]

and the combination $\Pi_{\nu,v}$ appearing in $I(\nu, v_F q \mu, \Delta; \Lambda)$.

If $\nu << v_F \Lambda$ and $q << \Lambda$ then $I$ depends on $\nu, \mu$ only via the combination

\[ r = \sqrt{\nu^2 + (v_F q \mu)^2} \]

so we may evaluate the $\epsilon$ and $\omega$ integrals at $q = 0$ and $\nu = 2\Delta r$ obtaining

\[ I(r; \Lambda) \equiv \int \frac{d\epsilon}{v_F \Lambda} \int_{-v_F \Lambda}^{v_F \Lambda} \frac{d\omega}{2\pi} \]

\[ = \frac{1}{2r \sqrt{1 + r^2}} \ln \left( r + \sqrt{1 + r^2} \right) + ... \]

where the last approximation applies for $r \Delta << v_F \Lambda$.

Next, to analyze $\Pi_0$, we separate $\Pi_0(0,0)$ from the formula by adding and subtracting $\frac{1}{2} \left[ \omega_+^2 + \epsilon_+^2 + \Delta^2 \right] + \frac{1}{2} \left[ \omega_-^2 + \epsilon_-^2 + \Delta^2 \right]$ to get

\[ \Pi_0(i\nu, q) = \Pi_0(0,0) + N_0 \int_0^{v_F \Lambda} \frac{d\mu(1 - \mu^2)^{d-3}}{\kappa_d} (2r^2 + 1) I(r, \Lambda) \]
For the static uniform polarization, we obtain for $\Delta \ll v_F\Lambda$

$$\Pi_0(0, 0) = -N_0 \int_0^1 \frac{d\mu(1 - \mu^2)^{\frac{d-3}{2}}}{\kappa_d}$$  \hspace{1cm} (31)$$

$$\int \frac{d\omega}{2\pi} \int \frac{dk}{\omega^2 + \epsilon^2 + \Delta^2}$$

$$=-N_0 \left( \ln \left( \frac{v_F\Lambda}{\Delta} \right) + \ldots \right)$$  \hspace{1cm} (32)

where again the ellipsis denotes terms of order unity.

Finally, we analyze $\Pi_2$. We see that $\Pi_2(i\nu, q) \to 0$ as $\nu \to 0$, so

$$\Pi_2 = -i\nu J(i\nu, q).$$  \hspace{1cm} (33)

To the accuracy with which we discussed $\Pi_3$ and $\Pi_0$, we may say that $J$ is a function of $q$, and we evaluate it at $q = 0, \nu = 2\Delta r$. Thus,

$$J = \frac{1}{v_F\Lambda} \int_0^1 \frac{d\mu(1 - \mu^2)^{\frac{d-3}{2}}}{\kappa_d}$$

$$\int \frac{d\epsilon}{v_F\Lambda} \ln \frac{\epsilon^2 + \Delta^2}{(\epsilon^2 + \Delta^2 + \Delta^2 r^2)}$$

We separate out the logarithmically divergent term and perform the integral over $\epsilon$ obtaining

$$J = \frac{N_1}{4v_F\Lambda} \int_0^1 \frac{d\mu(1 - \mu^2)^{\frac{d-3}{2}}}{\kappa_d}$$

$$\left[ \ln \frac{v_F\Lambda}{\Delta} - \frac{r^2 + 1}{r} \ln(r + \sqrt{r^2 + 1}) \right]$$

(34)

The derivatives $d\Pi/d\Delta^2$ needed for Eq (17) may be computed by straightforward differentiation (recall that $r = \sqrt{\nu^2 + (v_F q)^2}/2\Delta$). We summarize the results:

$$\Pi_0 = -N_0 \ln \left( \frac{v_F\Lambda}{\Delta} \right)$$

$$+ N_0 \int_0^1 \frac{d\mu(1 - \mu^2)^{\frac{d-3}{2}}}{\kappa_d} \left( \frac{2\nu^2 + 1}{2r^2 + 1} \right)$$

$$\Pi_2 = N_1 \frac{i\nu}{2v_F\Lambda} \left[ -\ln \left( \frac{v_F\Lambda}{\Delta} \right) \right] (38)$$

$$+ \int_0^1 \frac{d\mu(1 - \mu^2)^{\frac{d-3}{2}}}{\kappa_d} \frac{\sqrt{r^2 + 1} \ln(r + \sqrt{r^2 + 1})}{r}$$

$$\Pi_3 = N_0 \int_0^1 \frac{d\mu(1 - \mu^2)^{\frac{d-3}{2}}}{\kappa_d} \frac{\ln(r + \sqrt{r^2 + 1})}{2r\sqrt{r^2 + 1}}$$

$$\left[ -\frac{\ln(\sqrt{r^2 + 1} + r)}{r}\left(\frac{r^2 + 1}{r^2 + 1}\right)^{3/2} - 1 \right]$$

IV. EVALUATION OF FLUCTUATION CORRECTION.

This section uses the results of the previous section to evaluate Eq (17). There are two regimes in Eq (17): small $r$ ($\nu^2, (v_F q)^2 \ll \Delta^2$) and large $r$. We consider them in turn.

(i) Small $r$: in this limit both $\Pi_0 d\Pi_0/d\Delta^2$ in the numerator and $\Pi_2^2$ in the denominator of (17) are suppressed by the small factor $(\nu/v_F\Lambda)^2$ relative to the other terms, so we shall neglect the contribution from $\Pi_0$. In $d = 2$ with the parabolic dispersion, $N_1 = 0$ identically, so this term vanishes altogether. Then the leading behavior of both $1 + g\Pi_0$ and $g\Pi_3$ is $gN_0/2$; the leading behavior of $d\Pi_0/d\Delta^2$ is $N_0/2\Delta^2$, and the leading behavior of $d\Pi_3/d\Delta^2$ is reduced compared to $N_0/2\Delta^2$ by $r^2$. Thus, the numerator behaves as $(gN_0)^2/4\Delta^2$, and we factor the denominator as

$$1 + g\Pi_0 + g\Pi_3 \simeq (gN_0)^2 1 \times r^2$$

identifying the two factors with amplitude and phase. That way, we find

$$d\Pi_0/d\Delta^2_{\text{small}} \simeq -\frac{\Delta^{d-1}}{v_F} \int_0^{\pi} \frac{dv}{\pi} \int_0^{1} \frac{S_{d-1} u^{d-1} du}{\pi^{d-1} r^2}$$

$$\simeq -\frac{S_{d-1} \Delta^{d-1}}{d\pi^{d-1} v_F} = -N_0 \frac{2^d}{\pi^d} \left( \frac{\Delta}{v_F\Lambda} \right)^{d-1},$$

with $u = \frac{v_F q}{2\Delta}$ and $v = \frac{\nu}{2\Delta}$, since, for a spherical Fermi
\[ N_0 = \frac{S_{d-1} \Lambda^{d-1}}{(2\pi)^d v_F^d}. \]

(ii) Large \( r \): The leading behavior of the individual terms in the numerator and denominator of \ref{eq:17} is

\[ (1 + g\Pi_0) g \frac{d\Pi_0}{d\Delta^2} = \left( \frac{gN_0}{4\Delta^2} \right)^2 \left( \int_0^1 \frac{d\mu(1-\mu^2)^{\frac{d-3}{2}}}{\kappa_d} \ln r \right)^2. \]

\[ g^2 \Pi_2 \frac{d\Pi_2}{d\Delta^2} = \left( \frac{gN_1}{4\Delta^2} \right)^2 \left( \int_0^1 \frac{d\mu(1-\mu^2)^{\frac{d-3}{2}}}{\kappa_d} \ln r \right)^2. \]

\[ g^2 \Pi_3 \frac{d\Pi_3}{d\Delta^2} = \left( \frac{gN_0}{4\Delta^2} \right)^2 \left( \int_0^1 \frac{d\mu(1-\mu^2)^{\frac{d-3}{2}}}{\kappa_d} \ln r \right)^2. \]

We see that the leading contribution to both the numerator and denominator comes from \( \Pi_0 \), so

\[ \frac{dS_1}{d\Delta^2} \simeq -\int \frac{d\nu}{2\pi} \int \frac{d\nu}{2\pi} \frac{gN_0}{1 + g\Pi_0} \] \[ \simeq -\frac{\Delta^{d-1} S_{d-1}}{2v_F^d \pi^{d+1}} \int \frac{d\mu}{2\pi} \int \frac{d\nu}{2\pi} \] \[ \frac{d\Pi_0}{d\Delta^2} \left( 1 + g\Pi \right)^{-1}. \]

In the first region, we neglect the \( u \) dependence of \( r \), so the \( \mu \) (angular) integration becomes trivial, and we get

\[ -\frac{\Delta^{d-1} S_{d-1}}{2v_F^d \pi^{d+1}} \int \frac{d\nu}{2\pi} \int \frac{d\nu}{2\pi} \frac{\min(v, v_F \Lambda)}{\nu^d \ln v} \] \[ \approx -\frac{\Delta^{d-1} S_{d-1}}{2v_F^d \pi^{d+1}} \frac{1}{d-1} \ln \frac{v_F \Lambda}{v} = -\frac{N_0}{\pi(d-1) \ln \frac{v_F \Lambda}{v}}. \]

In the second region, we find, with logarithmic precision,

\[ \int_0^1 \frac{d\mu(1-\mu^2)^{\frac{d-3}{2}}}{\kappa_d} \ln \frac{2}{\nu^2 + (u \mu)^2} = \ln u, \]

and

\[ \int_0^1 \frac{d\mu(1-\mu^2)^{\frac{d-3}{2}}}{\kappa_d} \frac{1}{\nu^2 + (u \mu)^2} \approx \frac{1}{\kappa_d \nu^2} \]

so

\[ \frac{\Delta^{d-1} S_{d-1}}{2v_F^d \pi^{d+1}} \int \frac{d\nu}{2\pi} \int \frac{d\nu}{2\pi} \frac{u^{d-1}}{\kappa_d \nu \ln u} \]

\[ \approx -\frac{\Delta^{d-1} S_{d-1}}{2v_F^d \pi^{d+1}} \frac{1}{d-1} \left( \frac{v_F \Lambda}{2\kappa_d} \right)^{d-1} = -\frac{N_0}{\pi(d-1) \kappa_d}, \]

so it is the dominant fluctuation contribution to the fluctuation correction of the gap equation. We see that it is by factor \( gN_0 \) or \( 1/\ln(v_F \Lambda/\Delta) \) smaller than the mean-field terms, and that it is negative, so it decreases the value of the gap. This contribution will change only the prefactor in the solution of the gap equation. Other effects, not considered in the present calculation, will also make corrections of the same order. Comparing formula \ref{eq:51} to formula \ref{eq:57}, we see that the contribution from processes that scatter electrons along the Fermi surface dominate other contributions by factor \( \ln(v_F \Lambda/\Delta) \).

It is interesting to note that the result \ref{eq:57} is small because of cancellation of large terms. Indeed, returning to \ref{eq:54}, we can write the fluctuation correction to the gap equation \ref{eq:57} as

\[ \frac{dS_1}{d\Delta^2} = -\beta V \int \frac{d\nu}{2\pi} \int \frac{d\nu}{2\pi} \Tr \left[ \frac{g\Pi}{d\Delta^2} (1 + g\Pi)^{-1} \right]. \]

At high frequencies and momenta, the leading behavior of the two eigenvalues of the matrix

\[ g \frac{d\Pi}{d\Delta^2} (1 + g\Pi)^{-1}, \]

that is, of the amplitude and phase mode, is \( \pm f \) where

\[ f = \int \frac{d\nu}{2\pi} \int \frac{d\nu}{2\pi} \frac{gN_0}{1 + g\Pi_0}. \]
\[ f \simeq \frac{\Delta^{d-1}}{v_F^{d-1}} \int_1^{\infty} \frac{d\nu}{\pi} \int_1^{\infty} \frac{S_{d-1} v^{d-1} du}{\pi^d} \]  
\[ \quad \simeq \frac{1}{v_F^{d-1}} \frac{\Delta^{d-1} S_{d-1}}{\kappa_d} \int_1^{\infty} v^{d-1} du \int_1^{\infty} \frac{du (1-\mu^2) d^{d-1} \ln r}{\kappa_d} \]  
\[ \quad \simeq \frac{1}{\pi (d-1) \kappa_d} \ln \frac{v_F^2 \Lambda}{\Delta} \]  

\[ f \simeq \frac{1}{v_F^{d-1} \pi^{d-1}} \frac{\Delta^{d-1} S_{d-1}}{\kappa_d} \int_1^{\infty} v^{d-1} du \int_1^{\infty} \frac{du (1-\mu^2) d^{d-1} \ln r}{\kappa_d} \]  
\[ \quad \simeq \frac{N_0}{\pi (d-1) \kappa_d} \ln \frac{v_F^2 \Lambda}{\Delta} , \]  

(61)

Thus each of the amplitude and phase mode gives a correction apparently large enough to call the BCS approximation into question, but the two terms cancel in the trace.

V. SUMMARY AND CONCLUSIONS

We have used a functional integral formulation to study fluctuation corrections to the weak coupling BCS mean field expression for the superconducting gap amplitude. We find that in both two and three dimensions and in the weak coupling limit, the correction is smaller by the weak-coupling factor \( gN_0 \) or \( 1/(\ln(v_F \Lambda/\Delta)) \) than the mean-field terms. This correction comes from energies and momenta large compared to the mean-field gap, and, more specifically, from the region in the phase space where the electrons are scattered nearly parallel to the Fermi surface. We note that individually, both the contribution from the amplitude mode and the contribution from the phase mode diverge logarithmically in this region, but the divergencies have opposite sign, and thus cancel when we take the trace.

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