To the theory of $q$-ary Steiner and other-type trades

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Abstract

We introduce the concept of a clique bitrade, which generalizes several known types of bitrades, including latin bitrades, Steiner $T(k-1, k, v)$ bitrades, extended 1-perfect bitrades. For a distance-regular graph, we show a one-to-one correspondence between the clique bitrades that meet the weight-distribution lower bound on the cardinality and the bipartite isometric subgraphs that are distance-regular with certain parameters. As an application of the results, we find the minimal cardinality of $q$-ary Steiner $T_q(k-1, k, v)$ bitrades and show a connection of such bitrades with dual polar subgraphs of the Grassmann graph $J_q(v, k)$.

Keywords: trades, bitrades, Steiner system, $q$-ary designs

1. Introduction

In this paper, we prove some statements on a rather general class of combinatorial bitrades, mainly concentrating on bitrades of minimal possible cardinality, and obtain a partial result concerning minimal $q$-ary Steiner bitrades. Bitrades (trades) are used in combinatorics (including combinatorial design theory and combinatorial coding theory) to study possible differences between two combinatorial objects from the same class and to obtain new objects from a given one. In particular, small bitrades are used to construct large classes of objects with the same parameters, see e.g. [22], [2], [1], [24], [19], [11]; often minimal trades are utilized to get a lower bound on the number of objects. Trades related to “complete” objects are also known under term “switching components” [17]; but in general, bitrades (and trades) are defined independently, using the local relation that defines the “complete” objects. Trades can exist even if complete objects with the corresponding parameters do not exist. This gives additional possibilities to study trades and, as a result, to develop the theory of “complete” objects of given type.

In Section 2, we define the main notations and concepts (Subsection 2.1), including the concept of a clique bitrade, and prove four general theorems. Theorem 1 (Subsection 2.2) shows that the existence of a clique bitrade in a regular graph is equivalent to the existence of an eigenfunction with certain restrictions and to the existence of a bipartite regular subgraph of certain degree. In Subsection 2.3, we remind the concepts of Delsarte cliques and pairs and establish some useful properties of the Delsarte cliques.

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As a corollary, we prove an intersecting characterization of the eigenfunctions with the minimal eigenvalue, related to these concepts (Theorem 2). In Subsection 2.4 we remind the weight-distribution lower bound on the number of nonzeros of an eigenfunction. Theorem 3 in Subsection 2.5 shows for distance-regular graphs that the existence of a clique bitrade whose cardinality meets the weight-distribution lower bound is equivalent to the existence of a bipartite regular isometric subgraph of certain degree. Theorem 4 states that the isometric subgraph mentioned above is distance-regular.

In Section 3 we illustrate the theory by known examples from design theory, coding theory, and the theory of latin hypercubes.

In Section 4, based on the results of Section 2 and known facts about the dual polar graphs, we find \( q \)-ary Steiner \( T_q(d − 1, d, n) \) bitrades of minimal cardinality.

2. General theory

2.1. Basic definitions

Given a connected graph \( \Gamma \), by the distance \( d_{\Gamma}(x,y) \) between two vertices \( x \) and \( y \), we will mean the length of a shortest path from \( x \) to \( y \). For a graph \( \Gamma = (V, E) \) and a vertex \( x \in V \) or a set of vertices \( x \subseteq V \), \( \Gamma_i(x) \) will denote the \( i \)th neighborhood of \( x \), that is, the set of vertices at distance \( i \) from \( x \). The diameter \( D(\Gamma) \) of \( \Gamma \) is the maximal distance between two vertices of \( \Gamma \).

An eigenfunction of a graph \( \Gamma = (V, E) \) is a function \( f : V \to \mathbb{R} \) that is not constantly zero and satisfies

\[
\sum_{y \in \Gamma_1(x)} f(y) = \theta f(x)
\]

for all \( x \) from \( V \) and some constant \( \theta \), which is called an eigenvalue of \( \Gamma \).

A set \( C \) of vertices of a regular graph \( \Gamma \) of degree \( k \) is said to be completely regular with covering radius \( \rho \) if \( \Gamma_{\rho}(C) \neq \emptyset = \Gamma_{\rho+1}(C) \) and there is a sequence \((b_0, \ldots, b_{\rho-1}; c_1, \ldots, c_{\rho})\), which is named the intersection array, such that \( \Gamma_{i+1}(C) \cap \Gamma_1(y) = b_i \) and \( \Gamma_{i-1}(C) \cap \Gamma_1(y) = c_i \) hold for every \( i \in \{0, \ldots, \rho\} \), and every \( y \in \Gamma_i(C) \), where \( b_\rho = c_0 = 0 \). The numbers \( b_0, \ldots, b_\rho, c_0, \ldots, c_\rho \), and \( a_0, \ldots, a_\rho \), where \( a_i = k - b_i - c_i \), are referred to as the intersection numbers, and the tridiagonal matrix \((a_{i,j})_{i,j=0}^{\rho}\), where \( a_{i,i} = a_i \), \( a_{i,i+1} = b_i \), \( a_{i,i-1} = c_i \), is called the intersection matrix of \( C \). By the eigenvalues of a completely regular code, we will mean the eigenvalues of its intersection matrix. Given a completely regular code \( C \) of covering radius \( \rho \) and one of its eigenvalues \( \theta \), by \( \delta_C^\theta \) we denote the function on the vertex set that equals \( \nu_i \) on \( \Gamma_i(C) \), where \( (1 = \nu_0, \nu_1, \ldots, \nu_\rho) \) is an eigenvector of the intersection matrix corresponding to the eigenvalue \( \theta \) (it is easy to see for a tridiagonal matrix with nonzero lower- and upper-diagonal elements that an eigenvector is uniquely determined by its first element and the eigenvalue, which means that for each eigenvalue there is a unique eigenvector starting with 1; in particular, there are \( \rho + 1 \) different eigenvalues). It is straightforward that \( \delta_C^\theta \) is an eigenfunction of the graph with the same eigenvalue \( \theta \), which proves the known fact that an eigenvalue of a completely regular set is necessarily an eigenvalue of the graph.
A connected graph $\Gamma$ is called distance-regular if every one-vertex set is completely regular with the same intersection array (independent on the choice of the vertex), which is called the intersection array of $\Gamma$.

Let $\Gamma$ be a connected regular graph of degree $k$. Assume that $S$ is a set of $(s+1)$-cliques in $\Gamma$ such that every edge of $\Gamma$ is included in exactly $m$ cliques from $S$; in this case, we will say that the pair $(\Gamma, S)$ is a $(k, s, m)$ pair. A couple $(T_0, T_1)$ of mutually disjoint nonempty sets is called an $S$-bitrade, or a clique bitrade, if every clique from $S$ either intersects with each of $T_0$ and $T_1$ in exactly one vertex or does not intersect with both of them (in particular, this means that each of $T_0$, $T_1$ is an independent set in $\Gamma$). A set of vertices $T_0$ is called an $S$-trade if there is another set $T_1$ (known as a mate of $T_0$) such that the pair $(T_0, T_1)$ is an $S$-bitrade.

### 2.2. A bitrade criterion

We start with a criterion, which can be used as alternative definition of a clique bitrade.

**Theorem 1.** Let $\Gamma$ be a regular graph of degree $k$. Let $(\Gamma, S)$, where $S$ is a set of cliques in $\Gamma$, be a $(k, s, m)$ pair. Let $T = (T_0, T_1)$ be a pair of disjoint nonempty independent sets of vertices of $\Gamma$. The following assertions are equivalent.

(a) $T$ is an $S$-bitrade.

(b) The function

$$f^T(x) = \begin{cases} (-1)^i & \text{if } \bar{x} \in T_i, \ i \in \{0, 1\} \\ 0 & \text{otherwise} \end{cases}$$

(2)

is an eigenfunction of $\Gamma$ with eigenvalue $\theta = -k/s$.

(c) The subgraph $\Gamma^T$ of $\Gamma$ generated by the vertex set $T_0 \cup T_1$ is regular with degree $-\theta = k/s$ (as $T_0$ and $T_1$ are independent sets, this subgraph is bipartite).

**Proof.** All proofs are based on counting arguments, mainly utilizing the definition of a $(k, s, m)$ pair.

(a)$\Rightarrow$(b): Assume $(T_0, T_1)$ is an $S$-bitrade.

At first, we show that (1) holds for every vertex $x \not\in T_0 \cup T_1$. Indeed, double-counting the number of triples $(u, t_0, t_1)$ such that $t_0, t_1, x \in S$ and $t_1 \in \Gamma_1(x) \cap T_1$ gives $m|\Gamma_1(x) \cap T_0| = m|\Gamma_1(x) \cap T_1|$. Thus, (1) holds with both sides being equal to zero.

It remains to prove (1) for $x \in T_0$ (the case $x \in T_1$ is similar). Double-counting the number of triples $(u, t, t_1)$ such that $t, t_1, x \in S$, $t \in \Gamma_1(x) \setminus T_1$ and $t_1 \in \Gamma_1(x) \cap T_1$ gives $m|\Gamma_1(x) \setminus T_1| = (s - 1)m|\Gamma_1(x) \cap T_1|$. This implies $|\Gamma_1(x) \cap T_1| = |\Gamma_1(x)|/s = k/s$.

Since $T_0$ is independent and thus $|\Gamma_1(x) \cap T_0| = 0$, we find that (1) turns to $\theta = \theta$.

(b)$\Rightarrow$(c) is trivial.

(c)$\Rightarrow$(a): Let $\Gamma^T$ be regular of degree $k/s$. Let us consider some $x$ from $T_0$. There are $km/s$ cliques from $S$ containing $x$ (as well as any other fixed vertex). On the other hand, every $y \in \Gamma_1(x) \cap T_1$ is in $m$ of them. Since, by the hypothesis, there are $k/s$ such $y$, every clique from $S$ containing $x$ contains some $y \in \Gamma_1(x) \cap T_1$. By the definition of an $S$-bitrade, the claim follows.
2.3. Delsarte cliques, Delsarte pairs, and eigenfunctions

We can say more if $\Gamma$ is distance-regular. A clique in a distance-regular graph is called a \textit{Delsarte clique} if it has exactly $1 - k/\theta$ elements, where $\theta$ is the minimal eigenvalue of the graph. A $(k, s, m)$ pair $(\Gamma, S)$ is known as a \textit{Delsarte pair} if the graph $\Gamma$ is distance-regular and $S$ consists of Delsarte cliques \cite{Delsarte}.

\textbf{Proposition 1}. If, under notation and hypothesis of Theorem 1 (a)--(c) hold and, additionally, the graph $\Gamma$ is distance-regular, then $\theta$ is the minimal eigenvalue of $\Gamma$, $s + 1$ is the maximal order of a clique in $\Gamma$, and $(\Gamma, S)$ is a Delsarte pair.

\textbf{Proof}. As proved in [12], see also [5, Proposition 4.4.6], any clique of order $M$ satisfies $M \leq 1 - k/\theta_{\text{min}}$, where $\theta_{\text{min}}$ is the minimal eigenvalue of $\Gamma$. From (b) we have $s + 1 = 1 - k/\theta$. Thus, $\theta = \theta_{\text{min}}$ and $s + 1$ is the maximal order of a clique in $\Gamma$. The pair $(\Gamma, S)$ is a Delsarte pair by the definition. ▲

We will use the following properties of the Delsarte cliques:

\textbf{Proposition 2}. Let $\Gamma$ be a distance-regular graph of degree $k$ and diameter $D$, and let $\theta$ be the minimal eigenvalue of $\Gamma$. Then

(i) every Delsarte clique is a completely-regular set of covering radius $D - 1$, and $\theta$ is not one of its eigenvalues;

(ii) the sum of any eigenfunction with eigenvalue $\theta$ over a Delsarte clique is zero;

(iii) there are positive numbers $s_0^+, \ldots, s_{D-1}^+, s_0^-, \ldots, s_D^-$ such that for every vertex $x$ and every Delsarte clique $C$ at distance $i$ from $x$, there hold $|\Gamma_i(x) \cap C| = s_i^+$ and $|\Gamma_{i+1}(x) \cap C| = s_{i+1}^-$. \hfill ▲

\textbf{Proof}. It is shown in [5, Proposition 4.4.6, Remark] that a Delsarte clique is a completely-regular set and its covering radius $\rho$ is less than $D$; hence, $\rho = D - 1$. The second claim of (i) can also be retrieved from the proof of [5, Proposition 4.4.6] and some algebraic background, but we will give another proof. Let $\theta_0, \ldots, \theta_{D-1}, \theta'$ be the eigenvalues of $\Gamma$, where $\theta'$ is not an eigenvalue of the given Delsarte clique $C$. Then the characteristic function $\chi_C$ of $C$ is a linear combination of $\delta_{C_0}^\theta, \ldots, \delta_{C_{\rho-1}}^\theta$ (as well as the vector $(1, 0, \ldots, 0)$) is a linear combination of the eigenfunctions of the intersection matrix of $C$). Thus, $\chi_C$ is orthogonal to any eigenfunction with eigenvalue $\theta'$; in other words, the sum of any eigenfunction with eigenvalue $\theta'$ over $C$ is zero. Let us consider the eigenfunction $\delta_{\{x\}}^\theta$ for some $x$ from $C$. We have $\delta_{\{x\}}^\theta(x) = 1$ and $\delta_{\{x\}}^\theta(y) = \theta'/k$ for all $y$ from $C \setminus \{x\}$. Since, by the definition of a Delsarte clique, $|C \setminus \{x\}| = -k/\theta$, we have $1 + (k/\theta')(\theta'/k) = 0$. So, $\theta' = \theta$, and (i) and (ii) hold.

(iii) Let $v^\theta = (\nu_0, \nu_1, \ldots, \nu_D)$ be the eigenvector of the intersection matrix of $\Gamma$ corresponding to $\theta$ and starting with 1. Let us consider a vertex $x$ at distance $i$ from the given Delsarte clique $C$. Since the sum of the eigenfunction $\delta_{\{x\}}^\theta$ over $C$ is zero, we have

$$|\Gamma_i(x) \cap C|\nu_i + |\Gamma_{i+1}(x) \cap C|\nu_{i+1} = 0.$$  \hfill (3)

If the first summand of the left part is nonzero, then the second one is nonzero too; it follows by induction that all $\nu_i$, $i = 0, 1, \ldots, D$, are nonzero. Then, (3) implies that $|\Gamma_i(x) \cap C|/|\Gamma_{i+1}(x) \cap C| = -\nu_{i+1}/\nu_i$. Claim (iii) follows. ▲
As a corollary, for the distance-regular graphs, we can formulate a more strong analog of the equivalence of (a) and (b) in Theorem 1.

**Theorem 2.** Let \((\Gamma, S)\) be a Delsarte pair and \(\theta\) be the minimal eigenvalue of \(\Gamma\). Then

(i) A function \(f\) over the vertex set of \(\Gamma\) is an eigenfunction with the eigenvalue \(\theta\) if and only if for every clique \(C\) from \(S\) it holds \(\sum_{x\in C} f(x)\).

(ii) A proper subset \(B\) of the vertex set of \(\Gamma\) is a completely regular code of radius 1 with eigenvalue \(\theta\) if and only if it has a constant number of elements in any clique from \(B\).

**Proof.** In (i), the “if” statement follows from direct checking of (1), while “only if” comes from Proposition 2(ii). In (ii), again, “if” is straightforward, while “only if” follows from claim (i) if we consider the eigenfunction \(\delta^\theta_B\). ▲

### 2.4. The weight-distribution bound

The rest of Section 2 is devoted to the bitrades that are minimal in the sense that their cardinality meets a special lower bound. In this subsection, we define the concept of the weight distribution and the weight-distribution bound.

By the weight distribution of a function \(f : V \to \mathbb{R}\) with respect to a vertex \(x\) of a graph \(\Gamma = (V, E)\) we will mean the sequence \(W(x) = (W^i(f))_{i=0}^{D(\Gamma)}\), where \(W^i = \sum_{y\in \Gamma_i(x)} f(y)\). The following fact is well known and easy to derive from definitions, by induction on \(i\).

**Lemma 1.** The weight distribution \(W(x)\) of an eigenfunction \(f\) of a distance-regular graph \(\Gamma\) is calculated as \((f(x)W^i_{A,\theta})_{i=0}^{D(\Gamma)}\) where the coefficients \(W^i_{A,\theta}\) are derived from the intersection array \(A = (b_0, \ldots, c_{D(\Gamma)})\) of \(\Gamma\) and the eigenvalue \(\theta\) that corresponds to \(f\):

\[
W^0_{A,\theta} = 1, \quad W^1_{A,\theta} = \theta, \quad W^i_{A,\theta} = ((\theta - a_{i-1})W^{i-1}_{A,\theta} - b_{i-2}W^{i-2}_{A,\theta}) / c_i, \quad i \geq 2.
\]

(4)

To read more about how to calculate the weight distribution of eigenfunctions and generalizations of eigenfunctions, see [14]. Using Lemma 1 it is easy to derive the following lower bound on the support of an eigenfunction. The bound is also known; a partial case of this argument was used in [9] to find the minimal cardinality of a switching component of binary 1-perfect codes (which can also be treated in terms of eigenfunctions).

**Corollary 1 (the weight-distribution (w.d.) bound).** An eigenfunction \(f\) of a distance-regular graph has at least \(\sum_{i=0}^{D(\Gamma)} |W^i_{A,\theta}|\) nonzeros, in notation of Lemma 1.

**Proof.** Considering the weight distribution with respect to a vertex \(x\) with the maximal value of \(|f(x)|\), we see that the number of nonzeros in \(\Gamma_i(x)\) is at least \(|W^i_{A,\theta}|\). ▲

We will say that an eigenfunction of a distance-regular graph (or the corresponding clique bitrade, if any) meets the w.d. bound if it has exactly \(\sum_{i=0}^{D(\Gamma)} |W^i_{A,\theta}|\) nonzeros.

### 2.5. Trades that meet the w.d. bound

**Theorem 3.** Let \(\Gamma\) be a distance-regular graph of degree \(k\). Under notation and hypothesis of Theorem 1 the following assertions are equivalent.
(a') $T$ is an $S$-bitrade meeting the w.d. bound.
(b') The function $f^T$ is an eigenfunction of $\Gamma$ meeting the w.d. bound with eigenvalue $-k/s$.
(c') The subgraph $\Gamma^T$ is a regular isometric subgraph with degree $k/s$.

**Proof.** (a')$\Leftrightarrow$(b') is straightforward from (a)$\Leftrightarrow$(b) of Theorem 1 and the definition of the concept “to meet the w.d. bound.”

(c')$\Rightarrow$(b'). Assume (c') holds. Consider some $x$ from $T_0$. By the isometry property,

$$\Gamma_i(x) \cap (T_0 \cup T_1) = \Gamma^T_i(x) \cap (T_0 \cup T_1) = \Gamma^T_i(x) \cap T_{i \mod 2}$$

(the last equality holds because $\Gamma^T$ is bipartite with parts $T_0$ and $T_1$). It follows that $f^T$ is either non-negative or non-positive on $\Gamma_i(x)$ and $|W^i(f^T)| = |\Gamma_i(x) \cap (T_0 \cup T_1)|$. Thus, $|T_0 \cup T_1| = \sum_{i=0}^d |W^i(f^T)|$, and $f^T$ meets the w.d. bound.

(a',b')$\Rightarrow$(c'). Assume (b') holds. Then for every $x \in T_0 \cup T_1$ and for every $i$, the function $f^T$ is either non-negative, or non-positive on $\Gamma_i(x)$. That is, $\Gamma_i(x)$ does not intersect with either $T_0$ or $T_1$. Let us prove by induction on $d_{\Gamma}(x,y)$ that

$$d_{\Gamma^T}(x,y) = d_{\Gamma}(x,y)$$

for every $x$, $y \in T_0 \cup T_1$. For $d_{\Gamma}(x,y) = 0$, this is trivial. Let $x$, $y \in T_0$ and $d_{\Gamma}(x,y) = i$ (the case when $x$, or $y$, or both belong to $T_1$ is similar). There is a vertex $v$ in $\Gamma_{i-1}(x) \cap \Gamma_1(y)$. A clique from $S$ that contains both $v$ and $x$ has a vertex $z$ from $T_1$. All the vertices of this clique lie in $\Gamma_{i-1}(x) \cup \Gamma_1(y)$. But $z$ cannot belong to $\Gamma_i(x)$ as $\Gamma_i(x)$ already contains a vertex from $T_0$. Hence, $z \in \Gamma_{i-1}(x)$. By the induction hypothesis, $d_{\Gamma^T}(x,z) = i - 1$. Therefore, $d_{\Gamma^T}(x,y) = (i - 1) + 1 = d_{\Gamma}(x,y)$, which proves the statement. ▲

**Theorem 4.** Assume that, under the notation and the hypothesis of Theorem 3, (a')–(c') hold. Then the graph $\Gamma^T$ is distance-regular.

**Proof.** Consider a vertices $x$ and $y$ of $\Gamma^T$ at distance $i$ from each other. Without loss of generality we assume $y \in T_0$. We know that the cardinality of $\Gamma_{i+1}(x) \cap \Gamma_1(y)$ is $b_i$, the corresponding intersection number of $\Gamma$. Every vertex from this set is in $m$ cliques of $S$ containing $y$. On the other hand, every such clique contains exactly $s^-_{i+1}$ vertices of $\Gamma_{i+1}(x) \cap \Gamma_1(y)$. So, the number of cliques containing $y$ and intersecting with $\Gamma_{i+1}(x)$ is $mb_i/s^-_{i+1}$. Each such clique contains one element from $T_1$, and this element lies in $\Gamma_{i+1}(x)$, because the considered bitrade meets the w.d. bound. On the other hand, every such element belongs to $m$ cliques containing $y$. So, $\Gamma_{i+1}(x) \cap \Gamma_1(y) \cap T_1 = (mb_i/s^-_{i+1})/m = b_i/s^-_{i+1}$. By the isometry property, we have $\Gamma^T_{i+1}(x) \cap \Gamma^T_1(y) = b_i/s^-_{i+1}$. Similarly, $\Gamma^T_{i-1}(x) \cap \Gamma^T_1(y) = c_i/s^+_{i-1}$, and the graph $\Gamma^T$ is distance-regular by the definition. ▲

**Corollary 2.** For every distance-regular graph $\Gamma$ admitting a Delsarte pair, there is a sequence $A' = (b'_0, \ldots, b'_d_{\Gamma}-1; c'_1, \ldots, c'_d_{\Gamma})$ such that the existence of a clique bitrade in $\Gamma$ is equivalent to the existence of an isometric distance-regular subgraph with intersection array $A$.

In the following sections, we will consider examples of such subgraph.
3. Known examples

By a clique design, we will mean a set of vertices that has exactly one vertex in common with each maximal clique. The difference pair $(D_1 \setminus D_2, D_2 \setminus D_1)$ of two different clique designs is always a clique bitrade, while the existence of a clique trade does not imply the existence of a clique design in the same graph. In this section, we will consider classes of distance-transitive graphs for which the theory of clique designs and clique trades, in different notations, is more-or-less developed.

Example 1. We start with a very simple example, when the graph is an $n$-dimensional octahedron, a regular graph with $2^n$ vertices of degree $2n - 2$. There are $2^n$ maximal cliques of cardinality $n$; a clique design consists of two non-adjacent vertices; a minimal bitrade corresponds to a square subgraph. A less trivial problem is to characterize all $(n, m)$ systems of cliques (for different $m$). One can find that such systems are in one-to-one correspondence with the Boolean functions with $4^m$ ones whose correlation immunity \[20\] is at least 2.

Example 2. The vertex set of the Hamming graph $H(n, q)$ is the set $\{0, \ldots, q - 1\}^n$ of words of length $n$ over the alphabet $\{0, \ldots, q - 1\}$. The graph $H(n, 2)$ is also known as the $n$-cube, or the hypercube of dimension $n$. Two words are adjacent whenever they differ in exactly one position. The clique designs in Hamming graphs are known as the latin hypercubes (in coding theory, these objects are known as the distance-2 MDS codes), and the clique bitrades, as the latin bitrades \[18\]. The most studied case, which corresponds to the latin squares, is $n = 3$, see e.g. \[6\]. The graph corresponding to a minimal bitrade is $H(n, 2)$ \[18\].

Example 3. The vertices of the Johnson graph $J(n, w)$ are the binary words of length $n$ and weight (the number of ones) $w$. Two words are adjacent whenever they differ in exactly two positions. The graphs $J(n, w)$ and $J(n, n - w)$ are isomorphic, and below we assume $2w \leq n$. The clique designs in Johnson graphs are known as the Steiner $S(w - 1, w, n)$ systems (see e.g. \[8\]), and the clique bitrades, as the Steiner $T(w - 1, w, n)$ bitrades, alternatively, Steiner $T(w - 1, w, n)$ trades, see e.g. \[10\]. The subgraph corresponding to a minimal bitrade is $H(w, 2)$; an example of the vertex set of such subgraph is $\{(x, \bar{x}, 0, \ldots, 0) \mid x, \bar{x} \in \{0, 1\}^{w}, \bar{x} \text{ is opposite to } x\}$. The minimal bitrade cardinality was found in \[13\]. In the case $w = 3$, the minimal trade is known as the Pasch configuration, or the quadrilateral.

Example 4. The vertices of the halved $n$-cube are the even-weight binary words of length $n$ (i.e., a part of the bipartite $n$-cube). Two words are adjacent whenever they differ in exactly two positions. A maximal clique is the set of binary $n$-words adjacent in $H(n, 2)$ to a fixed odd-weight word. The clique designs in halved $n$-cubes are the extended 1-perfect codes. Such codes exist if and only if $n$ is a power of two, see e.g. \[15\]. The minimal cardinality $2^{n/2}$ of a bitrade was found in \[9\] (the authors considered a special type of 1-perfect trades, but the argument works for the general case; the 1-perfect trades in $H(n - 1, 2)$ are in one-to-one correspondence with the extended 1-perfect trades in the halved $n$-cube). An example of a minimal bitrade is $\{(x, x) \mid x \in \{0, 1\}^{n/2}\}$; bitrades
exist if and only if \( n \) is even (for odd \( n \), the value \(-\frac{1}{n-1}\) is not an eigenvalue of the halved \( n \)-cube). The graph corresponding to a minimal bitrade is \( H(n/2, 2) \).

**Example 5.** If for every vertex \( x \) of a distance-regular graph \( \Gamma \), there is exactly one vertex \( y \) at distance \( D(\Gamma) \) from \( x \), then identifying all such pairs \( x, y \) results in a distance-regular graph of diameter \( \lceil D(\Gamma)/2 \rceil \), known as the folded \( \Gamma \). It is not difficult to see that the bipartite isometric subgraph corresponding to a minimal clique bitrade will be also folded under this operation. Examples are the folded \( J(2d, d) \) and the folded halved \( H(2d, 2) \), where the corresponding subgraph is the folded \( H(d, 2) \).

The next example shows that analogs of the clique trades can be considered even if the graph has no cliques of required cardinality.

**Example 6.** The Shrikhande graph can be defined on the 16 quaternary pairs from \( \mathbb{Z}_4^2 \), where two pairs are adjacent if and only if their element-wise difference is one of \((0, 1), (0, 3), (1, 0), (3, 0), (1, 1), (3, 3)\). The Doob graph \( D(m, n) \) is the Cartesian product of \( m > 0 \) copies of the Shrikhande graph and \( n \) copies of the complete graph on 4 vertices. This graph is completely regular with the same intersection array as the Hamming graph \( H(2m + n, 4) \). It follows that it has the same minimal eigenvalue \( \theta = -2m - n \), and the w.d. bound on the number of nonzeros of an eigenfunction is the same too, i.e., \( 2^{2m+n} \), for \( \theta \). However, the Doob graph does not admit a Delsarte pair; moreover, Delsarte cliques, which have cardinality 4, does not occur in \( D(m, 0) \). So, we cannot apply the definition of a clique design. Nevertheless, we can apply an alternative definition using Theorem 1(b): let us say that a pair of two disjoint independent vertex sets is a pseudo-clique bitrade if the difference of their characteristic functions is an eigenfunction with minimal eigenvalue. An example of a minimal bitrade is \( \{ (0, 0), (0, 1), (0, 2), (0, 3) \}^m \{ 0, 1 \}^n \); it is not difficult to find that the subgraph generated by any minimal bitrade is \( H(2n + m, 2) \). In a same manner, a pseudo-clique design can be defined as an independent completely regular set with minimal eigenvalue and covering radius 1. Such sets are the maximal independent sets in the Doob graph; we leave constructing an example as an exercise.

From the last example, we see that defining bitrades in terms of eigenfunction is a more general approach than in terms of Delsarte cliques. In a similar manner, bitrades with other eigenvalues can be defined. For example, the bitrades with eigenvalue \(-1\) (1-perfect bitrades) are studied in the theory of 1-perfect codes, see e.g. [23].

### 4. Minimal \( q \)-ary Steiner bitrades

Let \( F_q^n \) be an \( n \)-dimensional vector space over the Galois field \( F_q \) of prime-power order \( q \). The Grassmann graph \( J_q(n, d) \) is defined as follows. The vertices are the \( d \)-dimensional subspaces of \( F_q^n \). Two vertices are adjacent whenever they intersect in a \((d - 1)\)-dimensional subspace. The Grassmann graph is a distance-transitive graph of degree \( q \left[ \begin{array}{c} d \\ 1 \end{array} \right] \left[ \begin{array}{c} n-d \\ 1 \end{array} \right] q^d \), where \( \left[ \begin{array}{c} a \\ b \end{array} \right] = \prod_{i=0}^{b-1} \frac{q^{a+i} - 1}{q^{b+i} - 1} \) see e.g. [5, Theorem 9.3.3].

All vertices that include a fixed \((d - 1)\)-dimensional subspace form a clique of order \( M = \left[ \begin{array}{c} n-d+1 \\ 1 \end{array} \right] \) in \( J_q(n, d) \); if \( n \geq 2d \) then this clique is maximal. We form an \((M, 1)\) system \( S \) from all cliques that correspond to a \((d - 1)\)-dimensional subspace. A set of
vertices that intersect with every cliques from $S$ in exactly one vertex is known as a $q$-ary Steiner $S_q(d-1,d,n)$ system. Constructing $q$-ary Steiner $S_q(d-1,d,n)$ systems with $d \geq 3$ is not easy; at the moment, only the existence of $S_2(2,3,13)$ is known in this field \cite{1}. An $S$-bitrade is called a Steiner $T_q(d-1,d,n)$ bitrade.

A quadratic form $Q : F_q^n \to F_q$ is said to be nondegenerate if its kernel $\{x \mid Q(y+x) = Q(y) \forall y \in F_q^n\}$ is zero. A subspace $V$ of $F_q^n$ is called totally isotropic whenever the form vanishes completely on $V$, i.e., $Q(V) = \{0\}$. The maximal dimension of a totally isotropic subspace is known as the Witt index of $Q$. If $n = 2d$ then the maximal Witt index of quadratic form is equal to $d$. There exists a unique (up to isomorphism) quadratic form with the Witt index $d$. One of its representations is $Q_0(v_1, \ldots, v_d, u_1, \ldots u_d) = v_1u_1 + \ldots + v_du_d$.

The dual polar graph $D_d(q)$ has as vertices the maximal isotropic subspaces, with respect to $Q_0$; two vertices $\alpha$ and $\beta$ are adjacent whenever $\dim(\alpha \cap \beta) = d-1$ (there are also other types of dual polar graphs, which are not considered here).

**Theorem 5.** The minimal cardinality of a Steiner $T_q(d-1,d,n \geq 2d)$ bitrade is

$$\prod_{i=1}^{d} (q^{d-i} + 1) = \sum_{i=0}^{d} q^{\binom{i}{2}} \binom{d}{i}_q,$$

which is also the minimal number of nonzeros of an eigenfunction with the minimal eigenvalue in $J_q(n,d)$, $n \geq 2d$.

**Proof.** $J_q(2d,d)$ is an isometric subgraph of $J_q(n,d)$; $D_d(q)$ is an isometric subgraph of $J_q(2d,d)$ \cite{5} p.276]. $D_d(q)$ is a bipartite distance-regular graph of degree $(q^d-1)/(q-1)$ (the biparticity and the degree are easily retrieved from the intersection array \cite{5} Theorem 9.4.3]) and order $\prod_{i=1}^{d} (q^{d-i} + 1)$ \cite{5} p.274, Lemma 9.4.1]. A proof of the identity \cite{5} can be found in \cite{21} Equation (1.87)]. Since $(q^d-1)/(q-1) = k(M-1)$ with $k = q \left[ \frac{d}{1} \right]_q \left[ \frac{n-d}{1} \right]_q$ and $M = \left[ \frac{n-d+1}{1} \right]_q$ the result follows from Theorem 3. ▲

**Remark 1.** It can be found that the $i$th summand $S_i = q^{\binom{i}{2}} \left[ \frac{d}{i} \right]_q$ of the right part of \cite{4} coincides with the number $|D_d(q)_{i}(x)|$ of vertices at distance $i$ from a fixed vertex $x$ in $D_d(q)$. One straightforward way to check this is checking the relation $b'_{i-1}S_{i-1} = c'_iS_i$ where $b'_i = q^i \left[ \frac{d-i}{1} \right]_q$ and $c'_i = \left[ \frac{i}{1} \right]_q$ are coefficients from the intersection array of $D_d(q)$, which can be found in \cite{5} Theorem 9.4.3] (this relation correspond to double-counting the edges between $D_d(q)_{i-1}(x)$ and $D_d(q)_{i}(x)$).

**Remark 2.** The minimal $T_q(2,3,n)$ trades can be considered as $q$-ary analogs of the Pasch configuration (quadrilateral). Another representation of the $T_q(2,3,n)$ trades constructed in the current paper was announced in \cite{16}.

**Problem 1.** The following question is natural: is a minimal Steiner $T_q(d-1,d,n)$ bitrade unique, up to isomorphism of the Grassmann graph? As noted in \cite{5} Remark 9.4.6], in general, the dual polar graph $D_d(q)$ is not unique, as a distance regular graph with given intersection array. The question is if there are nonisomorphic isometric embeddings of
such graphs into the Grassmann graph. Note that the minimal trades from the examples of Section 3 are known to be unique.

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