Tornheim-like series, harmonic numbers and zeta values

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Abstract
Explicit evaluations of the Tornheim-like double series in the form
\[ \sum_{n,m=1}^{\infty} \frac{H_{n+m+s}}{nm(n+m+s)}, \quad s \in \mathbb{N} \cup \{0\} \]
and their extensions are given. Furthermore, series of the type
\[ \sum_{m=1}^{\infty} \frac{2H_{2m+1} - H_m}{2m(2m+1)} \]
and some other Tornheim-like multiple series are evaluated in terms of the zeta values.

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1 Introduction

Riemann zeta function is defined by

\[ \zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}, \]

where \( s = \sigma + it \) and \( \sigma > 1 \). For even positive integers, one has the well-known relationship between zeta values and Bernoulli numbers:

\[ \zeta(2n) = (-1)^{n+1} \frac{(2\pi)^{2n}}{2(2n)!} B_{2n}. \]  

(1)

Here \( B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_4 = -\frac{1}{30}, B_6 = \frac{1}{42}, \ldots \) and \( B_{2n+1} = 0 \) for \( n \geq 1 \) (this result first published by Euler in 1740).

For odd positive integers, no such simple expression as (1) is known. Roger Apéry [1] proved the irrationality of \( \zeta(3) \) and after that \( \zeta(3) \) was named as Apéry’s constant. Rivoal [13] has shown that infinitely many of the numbers \( \zeta(2n+1) \) must be irrational. Besides Zudilin [18] has shown that at least one of the numbers \( \zeta(5), \zeta(7), \zeta(9) \) and \( \zeta(11) \) is irrational.

The \( n \)-th harmonic number \( H_n \) is the \( n \)-th partial sum of the harmonic series:

\[ H_n := \sum_{k=1}^{n} \frac{1}{k}. \]

For a positive integer \( n \) and an integer \( m \) the \( n \)-th generalized harmonic number of order \( m \) is defined by

\[ H_n^{(m)} := \sum_{k=1}^{n} \frac{1}{k^m}, \]

which is the \( n \)-th partial sum of the Riemann zeta function \( \zeta(m) \).

Tornheim double series [15] (or so called Witten’s zeta function [17]) is defined by

\[ S(a, b, c) := \sum_{m,n=1}^{\infty} \frac{1}{m^a n^b (m+n)^c}. \]  

(2)

This series has attracted considerable attention in recent years and been proved to be powerful tool to find numerous interesting relations between various zeta values ([2, 3, 4, 5, 6, 9, 10, 11, 16]). Boyadzhiev [7, 8] described a simple method to evaluate multiple series of the form (2) in terms of zeta values.

It is well known that there exist deep connections between Tornheim type series, harmonic numbers and zeta values. As a simple and nice example the following equation can be given (see [4, 5, 9, 10, 12]):

\[ \sum_{n,m=1}^{\infty} \frac{1}{nm (n+m)} = \sum_{m=1}^{\infty} \frac{H_m}{m^2} = 2\zeta(3). \]  

(3)
Kuba [11] considered the following general sum:

\[
V = \sum_{j,k=1}^{\infty} \frac{H_{j+k}^{(n)}}{j^r k^s (j+k)^t}.
\]

This sum includes the Tornheim’s double series [2] as special case, Kuba [11] proved that whenever \( w = r + s + t + u \) is even, for \( r, s, t, w \in \mathbb{N} \), the series \( V \) can be explicitly evaluated in terms of zeta functions.

On the other hand, Xu and Li [16] used the Tornheim type series computations for evaluation of non-linear Euler sums. Among other results they obtained

\[
\sum_{m=1}^{\infty} \frac{H_{m+k}}{m(m+k)} = \frac{H_k^2 + H_k^{(2)}}{k}, k \in \mathbb{N} = \{1, 2, 3, \ldots\}.
\]

From (3) and (4) it is easy to see that the value of series

\[
a(k) = \sum_{m=1}^{\infty} \frac{H_{m+k}}{m(m+k)}, k \in \mathbb{N} \cup \{0\}
\]

is irrational for \( k = 0 \) and rational for every \( k \in \mathbb{N} \). Hence the following questions arise naturally: for the integer \( s \in \mathbb{N} \cup \{0\} \), the values of the double series

\[
\sum_{n,m=1}^{\infty} \frac{H_{n+m+s}}{nm(n+m+s)},
\]

and more generally, multiple series

\[
A_n(s) = \sum_{k_1=1}^{\infty} \cdots \sum_{k_{n-1}=1}^{\infty} \frac{H_{k_1+\cdots+k_{n-1}+s}}{k_1 \cdots k_{n-1} (k_1+\cdots+k_{n-1}+s)}
\]

are rational or irrational? In this work we answer these questions. Moreover, we give explicit evaluation formulas for some Tornheim-like series via zeta values.

### 2 Formulations and proofs of the main results

**Theorem 1** Consider the double series

\[
A(s) = \sum_{n,m=1}^{\infty} \frac{H_{n+m+s}}{nm(n+m+s)}, \quad s \in \mathbb{N} \cup \{0\}.
\]

For any \( s \in \mathbb{N} \) the value of \( A(s) \) is rational but \( A(0) \) is irrational. More precisely,

\[
A(s) = \begin{cases} 
6 \zeta(4) & \text{if } s = 0 \\
6 \sum_{j=0}^{s-1} (-1)^j (s-1)_{(j+1)}^{-1} & \text{if } s \geq 1
\end{cases}
\]
Proof. By telescoping series formula we have

\[ 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n + m + s} = \sum_{k=1}^{\infty} \left( \frac{1}{k} - \frac{1}{k + n + m + s} \right) = (n + m + s) \sum_{k=1}^{\infty} \frac{1}{k(k + n + m + s)}. \]

It then follows that

\[ A(s) = \sum_{m=1}^{\infty} \frac{1}{nm(n + m + s)} \left( 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n + m + s} \right) \]

\[ = \sum_{n,m,k=1}^{\infty} \frac{1}{nmk(n + m + k + s)} \]

\[ = \sum_{n,m,k=1}^{\infty} \left( \int_0^1 x^{n-1} dx \right) \left( \int_0^1 y^{m-1} dy \right) \left( \int_0^1 z^{k-1} dz \right) \left( \int_0^1 t^{n+m+k+s-1} dt \right) \]

\[ = \int_0^1 t^{s+2} \left[ \int_0^1 \left( \sum_{n=1}^{\infty} (xt)^{n-1} \right) dx \right] \left( \int_0^1 \left( \sum_{m=1}^{\infty} (yt)^{m-1} \right) dy \right) \left( \int_0^1 \left( \sum_{k=1}^{\infty} (zt)^{k-1} \right) dz \right) dt \]

\[ = \int_0^1 t^{s+2} \left[ \int_0^1 \frac{1}{1-xt} dx \right] \left( \int_0^1 \frac{1}{1-yt} dy \right) \left( \int_0^1 \frac{1}{1-zt} dz \right) dt. \]

Since

\[ \int_0^1 \frac{1}{1-ut} du = -\frac{1}{t} \ln(1-t), \]

we have

\[ A(s) = -\int_0^1 t^{s-1} \ln^3(1-t) dt = -\int_0^1 (1-t)^{s-1} \ln^3 t dt. \] 

Setting \( s = 0 \), it follows that

\[ A(0) = -\int_0^1 \frac{1}{1-t} \ln^3 t dt = -\sum_{j=0}^{\infty} \int_0^1 t^j \ln^3 t dt \]

\[ = -\sum_{j=0}^{\infty} \left( -\frac{6}{(j+1)^4} \right) = 6 \zeta(4) = \frac{\pi^4}{16}. \]

On the other hand, if \( s \geq 1 \), then utilizing the formulas

\[ (1-t)^{s-1} = \sum_{j=0}^{s-1} (-1)^j \binom{s-1}{j} t^j \]

and

\[ \int_0^1 t^j \ln^3 t dt = -\frac{3!}{(j+1)^4}, \]

it follows that

\[ \int_0^1 t^{s+2} \left[ \int_0^1 \frac{1}{1-xt} dx \right] \left( \int_0^1 \frac{1}{1-yt} dy \right) \left( \int_0^1 \frac{1}{1-zt} dz \right) dt. \]

...
can computed explicitly as

\[ A(s) = -\int_0^1 (1 - t)^{s-1} \ln^3 t dt = 3! \sum_{j=0}^{s-1} (-1)^j \binom{s-1}{j} \frac{1}{(j+1)^4}. \]

This proves the stated result. ■

In the same way as in Theorem 1, by making use of the formulas,

\[ (1 - t)^{s-1} = \sum_{j=0}^{s-1} (-1)^j \binom{s-1}{j} t^j \] and \[ \int_0^1 t^j \ln^k t dt = (-1)^k \frac{k!}{(j+1)^{k+1}}, \]

one can prove the following more general result.

**Theorem 2** Denote

\[ A_n(s) = \sum_{k_1=1}^{\infty} \cdots \sum_{k_n=1}^{\infty} \frac{H_{k_1+\cdots+k_{n-1}+s}}{k_1 \cdots k_{n-1} (k_1 + \cdots + k_{n-1} + s)}, \quad s \in \mathbb{N} \cup \{0\}. \]

Then

\[ A_n(s) = \begin{cases} n! \zeta(n+1) & \text{if } s = 0, \\ n! \sum_{j=0}^{s-1} (-1)^j \binom{s-1}{j} \frac{1}{(j+1)^{n+2}} & \text{if } s \geq 1. \end{cases} \quad (6) \]

Two special cases of the theorem are as follows:

\[ A_2(s) = \sum_{k=1}^{\infty} \frac{H_{k+s}}{k (k+s)} = \begin{cases} 2\zeta(3) & \text{if } s = 0, \\ 2! \sum_{j=0}^{s-1} (-1)^j \binom{s-1}{j} \frac{1}{(j+1)^3} & \text{if } s \geq 1, \end{cases} \]

and

\[ A_4(s) = \sum_{k,m=1}^{\infty} \frac{H_{k+m+n+s}}{k m n (k+m+n+s)} = \begin{cases} 4! \zeta(5) & \text{if } s = 0, \\ 4! \sum_{j=0}^{s-1} (-1)^j \binom{s-1}{j} \frac{1}{(j+1)^5} & \text{if } s \geq 1. \end{cases} \]

**Remark 3** It can be easily seen from (6) that the expression \( A_n(s) \) is a rational number for all \( s \geq 1 \), but \( A_2(0) = 2\zeta(3) \) is irrational. If \( n \geq 4 \) and even, it is not known whether the numbers \( A_n(0) = n! \zeta(n+1) \) are irrational or not. On the other hand, for any odd \( n \in \mathbb{N} \) we have \( A_n(0) = n! \zeta(n+1) = r_n \pi^{n+1} \) (see (11)) is also irrational because of \( r_n \) is rational and \( \pi^{n+1} \) is irrational. Notice that the irrationality of \( \pi^n \) is a consequence of the transcendentality of \( \pi \). Indeed, if \( \pi^n \) is rational, say \( \pi^n = \frac{p}{q} \) where \( p \) and \( q \) are integers, then \( \pi \) is a solution of the equation \( qx^n - p = 0 \) and therefore \( \pi \) must be an algebraic number, which is false. More generally, if \( \alpha \) is a transcendental number and \( r = \frac{p}{q} \) is a rational number, then \( \alpha^r \) becomes irrational number.
Corollary 4 For any $k \in \mathbb{N}$ we have

$$
\sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} \frac{1}{(j+1)^2} = \frac{H_k^2 + H_k^{(2)}}{2k}.
$$

Proof. By applying Theorem 2 in the case of $n = 2$ and considering (4) we arrive at the stated result. ■

The next theorem gives a new relationship between harmonic numbers and $\zeta(2)$.

Theorem 5 Let

$$H_m = \sum_{k=1}^{m} \frac{1}{k} \text{ and } O_m = \sum_{k=1}^{m} \frac{1}{2k-1}.$$  

Then the formulas

$$\sum_{m=1}^{\infty} \frac{2H_{2m+1} - H_m}{2m(2m+1)} = 2(2 - \ln 2) - \zeta(2) \tag{7}$$

and

$$\sum_{m=1}^{\infty} \frac{O_m}{2m(2m+1)} = \frac{1}{4} \zeta(2) \tag{8}$$

are valid.

Proof. Denote

$$A = \sum_{m,n=0}^{\infty} \frac{1}{(2m+1)(2n+1)(2m+2n+1)}$$

and

$$B = \sum_{m,n=1}^{\infty} \frac{1}{(2m+1)(2n+1)(2m+2n+1)}.$$  

From the equation

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8},$$

we have

$$A = \frac{\pi^2}{4} - 1 + B. \tag{9}$$
Further, by the telescoping series formula, we find

\[
B = \sum_{m=1}^{\infty} \frac{\frac{1}{2m + 1} \sum_{n=1}^{\infty} \left( \frac{1}{2n + 1} - \frac{1}{2n + 1 + 2m} \right)}{2m (2m + 1)}
= \sum_{m=1}^{\infty} \frac{O_m - \frac{2m}{2m + 1}}{2m (2m + 1)}
= \sum_{m=1}^{\infty} \frac{O_m}{2m (2m + 1)} - \sum_{m=1}^{\infty} \frac{1}{(2m + 1)^2}
= \sum_{m=1}^{\infty} \frac{O_m}{2m (2m + 1)} - \frac{\pi^2}{8} + 1
\]

Hence we obtain that

\[
B = \sum_{m=1}^{\infty} \frac{O_m}{2m (2m + 1)} + 1 - \frac{3}{4} \zeta(2). \quad (10)
\]

On the other hand, it is clear that the expression \( B \) can also be written as

\[
B = \sum_{m=1}^{\infty} \frac{H_{2m+1} - 1 - \frac{1}{2} H_m}{(2m + 1) 2m} \quad (11)
\]

Now let us evaluate \( A \).

\[
A = \sum_{m,n=0}^{\infty} \left( \int_0^1 x^{2m} dx \right) \left( \int_0^1 y^{2n} dy \right) \left( \int_0^1 t^{2m+2n} dt \right)
= \int_0^1 \left( \int_0^1 \sum_{m=0}^{\infty} (xt)^{2m} dx \int_0^1 \sum_{n=0}^{\infty} (yt)^{2n} dy \right) dt
= \int_0^1 \left( \int_0^1 \frac{1}{1 - (xt)^2} dx \int_0^1 \frac{1}{1 - (yt)^2} dy \right) dt
= \frac{1}{4} \int_0^1 \frac{1}{t^2} \ln^2 \left( \frac{1 + t}{1 - t} \right) dt.
\]

The substitution \( \frac{1+t}{1-t} = u \) immediately leads to the following equality:

\[
A = \frac{1}{2} \int_1^\infty \frac{1}{(1-u)^2} \ln^2 u du.
\]

Integration by parts gives

\[
A = \int_1^{\infty} \frac{1}{u (u-1)} \ln u \, du = \int_1^{\infty} \frac{1}{1 - \frac{1}{u}} \frac{\ln u}{u^2} \, du
= \sum_{k=0}^{\infty} \int_1^{\infty} u^{-k-2} \ln u \, du = \sum_{k=0}^{\infty} \frac{1}{(k+1)^2} = \zeta(2) = \frac{\pi^2}{6}. \quad (12)
\]
We use (9), (11) and (12) to conclude that

\[
\sum_{m=1}^{\infty} \frac{H_{2m+1} + \frac{1}{2}H_m}{2m (2m + 1)} = 1 - \frac{\pi^2}{4} + \frac{\pi^2}{6} = 1 - \frac{\pi^2}{12}
\]

from which we obtain

\[
\sum_{m=1}^{\infty} \frac{H_{2m+1} + \frac{1}{2}H_m}{2m (2m + 1)} = 1 - \frac{\pi^2}{12} + \sum_{m=1}^{\infty} \frac{1}{2m (2m + 1)}
\]

From the formulas

\[
\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \cdots = 1
\]

and

\[
\frac{1}{1.2} + \frac{1}{3.4} + \frac{1}{5.6} + \cdots = \ln 2,
\]

it follows that

\[
\sum_{m=1}^{\infty} \frac{1}{2m (2m + 1)} = 1 - \ln 2
\]

and therefore

\[
\sum_{m=1}^{\infty} \frac{H_{2m+1} + \frac{1}{2}H_m}{2m (2m + 1)} = 2 \left(1 - \frac{\pi^2}{12} + 1 - \ln 2\right)
\]

\[
= 2 (2 - \ln 2) - \zeta(2).
\]

Similarly, from (9), (10) and (12) we have

\[
\frac{\pi^2}{6} = \frac{\pi^2}{4} - 1 + \sum_{m=1}^{\infty} \frac{O_m}{2m (2m + 1)} = 3 \zeta(2) + 1
\]

and as a result

\[
\sum_{m=1}^{\infty} \frac{O_m}{2m (2m + 1)} = \frac{1}{4} \zeta(2).
\]

In the following theorem we give some interesting relationships between the Tornheim-like series and the zeta values \(\zeta(2)\) and \(\zeta(3)\).

**Theorem 6** We have the following series evaluations:

(a) \[
\sum_{m,n=0}^{\infty} \frac{1}{(m + \frac{1}{2}) (n + \frac{1}{2}) (m + n + \frac{1}{2}) (m + n + 1)} = 16 \zeta(2) - 14 \zeta(3).
\]

(b) \[
\sum_{m,n=0}^{\infty} \frac{1}{(m + \frac{1}{2}) (n + \frac{1}{2}) (m + n + 1) (m + n + \frac{3}{2})} = 14 \zeta(3) - 8 \zeta(2).
\]

(c) \[
\sum_{m,n=0}^{\infty} \frac{1}{(m + \frac{1}{2}) (n + \frac{1}{2}) (m + n + \frac{3}{2}) (m + n + 1) (m + n + \frac{5}{2})} = 24 \zeta(2) - 28 \zeta(3).
\]
Proof. (a) We have
\begin{align*}
\sum_{m,n=0}^{\infty} \frac{1}{(2m+1)(2n+1)(2m+2n+2)} &= \sum_{m,n=0}^{\infty} \left( \int_0^1 x^{2m} \, dx \right) \left( \int_0^1 y^{2n} \, dy \right) \left( \int_0^1 t^{2m+2n+1} \, dt \right) \\
&= \int_0^1 t \left( \int_0^1 \sum_{m=0}^{\infty} (xt)^{2m} \, dx \int_0^1 \sum_{n=0}^{\infty} (yt)^{2n} \, dy \right) t \, dt \\
&= \frac{1}{4} \int_0^1 \frac{1}{t} \ln^2 \left( \frac{1+t}{1-t} \right) \, dt.
\end{align*}

Here the substitution \( \frac{1+t}{1-t} = u \) leads to the following equality:
\begin{align*}
\frac{1}{4} \int_0^1 \frac{1}{t} \ln^2 \left( \frac{1+t}{1-t} \right) \, dt &= \frac{1}{2} \int_1^\infty \frac{1}{u^2 - 1} \ln^2 u \, du = \frac{1}{2} \int_1^\infty \frac{1}{u^2 (1 - \frac{1}{u^2})} \ln^2 u \, du \\
&= \frac{1}{2} \sum_{k=0}^{\infty} \int_1^\infty u^{-2k-2} \ln u \, du.
\end{align*}

After integration by parts we get
\begin{align*}
\frac{1}{2} \sum_{k=0}^{\infty} \int_1^\infty u^{-2k-2} \ln u \, du &= \sum_{k=0}^{\infty} \frac{1}{(2k+1)^3} = \frac{7}{8} \zeta(3) .
\end{align*}

Hence
\begin{align*}
\sum_{m,n=0}^{\infty} \frac{1}{(2m+1)(2n+1)(2m+2n+2)} &= \frac{7}{8} \zeta(3) . \tag{13}
\end{align*}

On the other hand, according to the formula \([12]\),
\begin{align*}
A &= \sum_{m,n=0}^{\infty} \frac{1}{(2m+1)(2n+1)(2m+2n+1)} = \zeta(2) . \tag{14}
\end{align*}

Now, from \([13]\) and \([14]\) it follows that
\begin{align*}
\zeta(2) - \frac{7}{8} \zeta(3) &= \sum_{m,n=0}^{\infty} \frac{1}{(2m+1)(2n+1)(2m+2n+1)} \\
&\quad - \sum_{m,n=0}^{\infty} \frac{1}{(2m+1)(2n+1)(2m+2n+2)} \\
&= \sum_{m,n=0}^{\infty} \frac{1}{(2m+1)(2n+1)(2m+2n+1)(2m+2n+2)}
\end{align*}

and this proves \((a)\).
(b) By the same method in the proof of (a), we have

\[
\sum_{m,n=0}^{\infty} \frac{1}{(2m+1)(2n+1)(2m+2n+3)} = \sum_{m,n=0}^{\infty} \left( \int_0^1 x^{2m}dx \right) \left( \int_0^1 y^{2n}dy \right) \left( \int_0^1 t^{2m+2n+2}dt \right)
\]

\[
= \int_0^1 t^2 \left( \int_0^1 \frac{1}{1-(xt)^2} dx \right) \left( \int_0^1 \frac{1}{1-(yt)^2} dy \right) dt
\]

\[
= \frac{1}{4} \int_0^1 \ln^2 \left( \frac{1+t}{1-t} \right) dt
\]

\[
= \frac{1}{2} \int_1^\infty \frac{1}{(u+1)^2} \ln^2 u du = -\frac{1}{2} \int_1^\infty \ln^2 u \frac{1}{u+1} du
\]

\[
= \int_1^\infty \frac{1}{u(u+1)} \ln u du = \sum_{k=2}^{\infty} (-1)^k \int_1^\infty u^{-k} \ln u du
\]

\[
= \sum_{k=2}^{\infty} (-1)^k \frac{1}{(k-1)^2} = \frac{\pi^2}{12}
\]

Thus

\[
\sum_{m,n=0}^{\infty} \frac{1}{(2m+1)(2n+1)(2m+2n+3)} = \frac{\pi^2}{12} = \frac{1}{2} \zeta(2).
\] (15)

Now, from (13) and (15) we have

\[
\frac{7}{8} \zeta(3) - \frac{1}{2} \zeta(2) = \sum_{m,n=0}^{\infty} \frac{1}{(2m+1)(2n+1)(2m+2n+2)(2m+2n+3)}
\]

and this proves (b).

Finally, formula (c) can be obtained by substracting the formula (b) from the formula (a).

Remark 7 With the method used in this theorem, series of similar types containing different combinations in the denominator, can be evaluated.

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