On Relation Between Moyal and Kontsevich Quantum Products. Direct Evaluation up to the $\hbar^3$-Order

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Abstract

In his celebrated paper Kontsevich has proved a theorem which manifestly gives a quantum product (deformation quantization formula) and states that changing coordinates leads to gauge equivalent star products. To illuminate his procedure, we make an arbitrary change of coordinates in the Moyal product and obtain the deformation quantization formula up to the third order. In this way, the Poisson bi-vector is shown to depend on $\hbar$ and not to satisfy the Jacobi identity. It is also shown that the values of coefficients in the formula obtained follow from associativity of the star product.

1 Introduction

1. Star product. Let us introduce, following [1] a star product that is an associative $R[[\hbar]]$ – bilinear product on algebra $A[[\hbar]]$ of smooth functions on a finite-dimensional $C^\infty$-manifold:

$$f \star g = fg + \hbar B_1(f,g) + \hbar^2 B_2(f,g) + \ldots \subset A[[\hbar]]$$

where $\hbar$ is a formal variable and $B_i(f,g)$ – bidifferential operators.

Assocativity for the $n^{th}$ order means: $(f \star g) \star h = f \star (g \star h) + O(\hbar^{n+1})$.

2. Gauge group. There is a natural gauge group which acts on star products:

$$\star \rightarrow \star', f'(\hbar) = Df(\hbar), f' \star g' = D(D^{-1}f' \star D^{-1}g')$$

where $D = 1 + \sum_{i \geq 1} \hbar^i D_i$ and $D_i$’s are arbitrary differential operators

$$f \rightarrow f + \hbar D_1 f + \hbar^2 D_2 f + O(\hbar^3)$$

Assocativity of the new star product is obvious since $f' \star g' \star h' = D(f \star g \star h)$.

It follows from above

$$B_1'(f,g) = B_1(f,g) + D_1(fg) - f D_1(g) - g D_1(f)$$

$B_1(f,g)$ can be chosen to be a skew-symmetric bi-vector field (see [1]). Then, we put

$$B_1(f,g) = \alpha^{ab} \partial_a f \partial_b g, \alpha^{ab} = -\alpha^{ba}.$$ The Poisson bi-vector may depend on $\hbar$:

$$\alpha^{ab}(\hbar) = \sum_{i \geq 0} \hbar^i \alpha_i^{ab}$$

The second order term $O(\hbar^2)$ in the associativity equation $f \star (g \star h) = (f \star g) \star h$ implies that $\alpha$ gives a Poisson structure on $X$,

$$\forall f,g,h \quad \{\{f,\{g,h\}\}\} + \{\{g,\{h,f\}\}\} + \{\{h,\{f,g\}\}\} = 0,$$

where $\{f,g\} := \frac{f \hbar g - \hbar g f}{\hbar |_{\hbar = 0}}$.
3. Moyal product. An example of the star product is the Moyal product:

\[ f \star g = fg + \hbar \theta^{ij} \partial_i f \partial_j g + \frac{\hbar^2}{2!} \theta^{ij} \theta^{kl} \partial_i \partial_k f \partial_j \partial_l g + \frac{\hbar^3}{3!} \theta^{ij} \theta^{kl} \theta^{mn} \partial_i \partial_k \partial_m f \partial_j \partial_l \partial_n g + \ldots = \]

\[ = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \sum_{i_1, \ldots, i_n, j_1, \ldots, j_n}^{n} \prod_{k=1}^{n} \theta^{i_k j_k} (\prod_{k=1}^{n} \partial_{i_k})(f) \times (\prod_{k=1}^{n} \partial_{j_k})(g) = \]

\[ = e^{\hbar \theta^{ij} \partial^{(1)}_i \partial^{(2)}_j} f(x^{(1)}) g(x^{(2)}) \bigg|_{x^{(1)}=x^{(2)}=x} \]

where \( \theta^{ij} \) is constant and skew-symmetric.

4. Kontsevich formula. In paper [1] the following theorem was stated

**Theorem.** (1) Let \( \alpha \) be a Poisson bi-vector field in a domain of \( \mathbb{R}^d \). The formula:

\[ f \star g := \sum_{n=0}^{\infty} \hbar^n \sum_{\Gamma \in G_n} \omega_{\Gamma} B_{\Gamma, \alpha}(f, g) \]

defines an associative product. (2) Changing coordinates, one obtains a gauge equivalent star product.

In formula (6) expressions \( B_{\Gamma, \alpha}(f, g) \) are constructed with the help of Kontsevich’s diagrams \( \Gamma \in G_n \), \( G_n \) being a set of the \( n^{\text{th}} \)-order diagrams and \( \omega_{\Gamma} \) are constant coefficients corresponding to diagrams \( \Gamma \in G_n \). The values of these coefficients can be computed by the prescription given in [1].

Up to the second order, this formula can be written as follows

\[ f \star g = fg + \hbar \alpha^{ab} \partial_a f \partial_b g + \frac{1}{2} \hbar^2 \alpha^{ab} \alpha^{cd} \partial_a \partial_c f \partial_b \partial_d g + \]

\[ + \frac{1}{3} \hbar^2 \alpha_{as} \partial_s \alpha^{bc} (\partial_a \partial_b f \partial_c g + \partial_a \partial_b g \partial_c f) \right) + O(\hbar^3) \]

5. Purpose of the paper. We are going to make a direct check of Kontsevich’s statement that changing coordinates one obtains a gauge equivalent star product, up to the \( \hbar^3 \)-order starting from the constant bi-vector (in this case, the star product is Moyal’s one). For the symplectic case, this fact was stated in [2].

We make a change of variables in the Moyal product, which is nothing but Kontsevich’s star product with the constant Poisson bi-vector (let us denote it \( \vartheta^{ij} \)), and try to represent it with a help of gauge transformations in the form of (6) in new coordinates \( \tilde{x} \) using only the Poisson bi-vector field \( \alpha^{ab} (\hbar) \) \( \alpha_0^{ab} = \vartheta^{ij} \frac{\partial x^a}{\partial x^i} \frac{\partial x^b}{\partial x^j} \). The result should be compared with Kontsevich’s formula for non-constant (non-Moyal) \( \vartheta^{ij} \). In this way, we obtain formula (8). In the third order, we see that bi-vector terms can not be rewritten in new coordinates and thus are interpreted as \( \alpha_2^{ab} \) in formula (4): \( \alpha^{ab} (\hbar) = \sum_{i \geq 0} \hbar^i \alpha_2^{ab} \). The definition of \( \alpha_2 \) is not unique as we may add to the formula (8) a bi-vector term (for example, \( \partial_s \alpha^{ab} \partial_p \alpha^{cd} \partial_t \partial_k \alpha^{ab} \partial_n f \partial_0 g \) and thus subtract it from \( \alpha_2 \). However, this kind of terms always correspond to loop diagrams. If one considers the case when there are no loops, this consequently fixes the value of \( \alpha \). In this case, the answer we are going to get is given by the following formula.
\[ f \ast g = fg + \hbar \alpha^{ab} \partial_a f \partial_b g + \]
\[ + \hbar^2 \left[ \frac{1}{2} \alpha^{ab} \alpha^{cd} \partial_a \partial_c f \partial_b \partial_d g + \frac{1}{3} \alpha^{as} \alpha^{bc} (\partial_a \partial_b f \partial_c g + \partial_a \partial_b g \partial_c f) \right] + \]
\[ + \hbar^3 \left[ \frac{1}{6} \alpha^{ab} \alpha^{cd} \alpha^{ho} \partial_a \partial_c \partial_h f \partial_b \partial_d \partial_o g + \frac{1}{3} \alpha^{ap} \partial_p \alpha^{as} \partial_a \partial_b \partial_c f \partial_d g \right] + \]
\[ + \frac{2}{3} \alpha^{ap} \partial_p \alpha^{as} \partial_a \partial_b \partial_c f \partial_d g + \frac{1}{3} \alpha^{as} \partial_a \partial_b \partial_c f \partial_d g + \]
\[ + \frac{1}{6} \alpha^{as} \partial_a \partial_b \partial_c f \partial_d g + \frac{1}{6} \alpha^{as} \partial_a \partial_b \partial_c f \partial_d g + \]
\[ + \frac{1}{3} \alpha^{as} \partial_a \partial_b \partial_c f \partial_d g + \frac{1}{3} \alpha^{as} \partial_a \partial_b \partial_c f \partial_d g + \]
\[ + O(\hbar^4) \]
where
\[ \alpha^{ab} = \partial^{ij} \partial^a \partial^b - \frac{1}{18} \alpha^{ij} \alpha^{kl} \partial^a \partial^b - \frac{1}{4} \partial^{ij} \partial^a \partial^b \partial^c \partial^d \partial^e \partial^f \partial^g \partial^h \partial^i \partial^j \partial^k \partial^l \partial^m \partial^n - \]
\[ - \frac{1}{4} \partial^i \partial^j \partial^k \partial^l \partial^m \partial^n - \frac{1}{4} \partial^i \partial^j \partial^k \partial^l \partial^m \partial^n - \frac{1}{4} \partial^i \partial^j \partial^k \partial^l \partial^m \partial^n - \frac{1}{4} \partial^i \partial^j \partial^k \partial^l \partial^m \partial^n - \frac{1}{4} \partial^i \partial^j \partial^k \partial^l \partial^m \partial^n - \frac{1}{4} \partial^i \partial^j \partial^k \partial^l \partial^m \partial^n - \]
and \( S^{abc} \) is determined in (15). The differential operator in the gauge transformation (2) necessary for obtaining (8),(9) is the following
\[ D = 1 + \hbar^2 \left[ - \frac{1}{18} \partial^{ij} \partial^a \partial^b \partial^c \partial^d \partial^e \partial^f \partial^g \partial^h \partial^i \partial^j \partial^k \partial^l \partial^m \partial^n - \right] + O(\hbar^4) \]
We show that \( \alpha(\hbar) \) defined by (9) do not satisfy the Jacobi identity: \( \alpha^{as} \partial_a \partial^bc + \alpha^{as} \partial_a \partial^bc + \alpha^{as} \partial_a \partial^bc = 0 \). Still \( \alpha_0 \) does satisfy the Jacobi identity and we will use this fact to compute coefficients in (8) from the requirement of associativity. These coefficients are in agreement with [3]. It is discussed in section 5.

2 Diagram representation

There is a natural way to represent separate terms in Kontsevich’s formula by diagrams.

For the \( n \)-th-order term one needs \( n \) vertices, each vertex containing \( \alpha \), and two more vertices containing functions \( f \) and \( g \). Vertices can be connected by arrows. Each vertex is an origin of two ordered arrows. If an arrow ends in some vertex, it describes a partial derivative acting on this vertex.

The following diagrams correspond to formula (8) (here we do not draw the arrows as all of them point to the right):
Note that there are no loop diagrams in this picture. In particular, there is no structure of the type $\partial_c \alpha^{da} \partial_d \alpha^{bc}$.

3 Calculations up to the second order terms

Let us consider the terms up to the second order ($\alpha = \alpha_0$) in the Moyal formula (5) and make a change of variables in it. We are going to obtain formula (7) with the help of gauge transformations. The change of variables gives the following expression for the Moyal product

$$f \ast g = fg + \hbar \alpha^{ab} \partial_a f \partial_b g +$$

$$+ \frac{1}{2} \hbar^2 \partial^i \partial^j \partial^k \partial^l \left( \frac{\partial^2 z^a}{\partial x^i \partial x^k \partial x^j \partial x^l} \partial_a \partial_a f \partial_b g + \frac{\partial^2 z^b}{\partial x^i \partial x^k \partial x^j \partial x^l} \partial_a \partial_b f \partial_c g + \frac{\partial^2 z^c}{\partial x^i \partial x^k \partial x^j \partial x^l} \partial_a \partial_c f \partial_b g + \frac{\partial^2 z^d}{\partial x^i \partial x^k \partial x^j \partial x^l} \partial_a \partial_d f \partial_b g + O(\hbar^3) \right)$$

(10)

The last term is a symmetric bi-vector and can be canceled by the following gauge transformation

$$D'_2 = -\frac{1}{4} \alpha^{ij} \alpha^{kl} \frac{\partial^2 z^a}{\partial x^i \partial x^k} \frac{\partial^2 z^b}{\partial x^j \partial x^l} \partial_a \partial_b$$

(11)

Now we pay attention only to the terms $\partial^2 f \partial g, \partial^2 g \partial f$ which are

$$\frac{1}{2} P_1 = \frac{1}{2} \frac{\partial^2 z^c}{\partial x^i \partial x^j} \frac{\partial^2 z^b}{\partial x^k \partial x^l} \partial_a \partial_b f \partial_c g + \partial_a \partial_b \partial_c \partial_f$$

(12)

If one puts $\alpha^{ab} = \delta^{ij} \delta^{kl} \frac{\partial^2 z^c}{\partial x^i \partial x^j} \frac{\partial^2 z^b}{\partial x^k \partial x^l}$ in $K_0 \alpha^{ab} \partial_a \partial_c f \partial_b g + \partial_a \partial_b \partial_c \partial_f$, where $K_0$ is constant, this gives

$$K_0 \left( -\frac{\partial^2 z^c}{\partial x^i \partial x^j} \frac{\partial^2 z^b}{\partial x^k \partial x^l} \partial_a \partial_c f \partial_b g + \partial_a \partial_b \partial_c \partial_f \right) = K_0 P_1 - K_0 P_2$$

(13)

where $P_2 = \frac{\partial^2 z^c}{\partial x^i \partial x^j} \frac{\partial^2 z^b}{\partial x^k \partial x^l} \partial_a \partial_b f \partial_c g + \partial_a \partial_b \partial_c \partial_f$.

We are going to find a gauge transformation that makes (12) equal to expression (13). Consider

$$D_2 = K_1 S^{abc} \partial_a \partial_b \partial_c$$

(14)

$$S^{abc} = \delta^{ij} \delta^{kl} \left( \frac{\partial^2 z^a}{\partial x^i \partial x^j} \frac{\partial^2 z^b}{\partial x^k \partial x^l} + \frac{\partial^2 z^a}{\partial x^i \partial x^j} \frac{\partial^2 z^b}{\partial x^k \partial x^l} + \frac{\partial^2 z^b}{\partial x^i \partial x^j} \frac{\partial^2 z^a}{\partial x^k \partial x^l} \right)$$

(15)
Note that $S^{abc}$ is symmetric with respect to $a, b, c$. Transformation (14-15) adds to (12) the following terms

$$D_2(fg) - fD_2(g) - gD_2(f) = 3S^{abc}(\partial_a \partial_b f \partial_c g + \partial_a \partial_b g \partial_c f) = 3K_1(P_1 + 2P_2)$$  \(16\)

The equality (12) + (16) = (13) looks like

$$\frac{1}{2} P_1 + 3K_1(P_1 + 2P_2) = K_0(P_1 - P_2)$$

$$K_0 = \frac{1}{3}, K_1 = -\frac{1}{18}$$

This means we should substitute in formula (3) the gauge transformations $D'_2$ (11) and $D_2$ (14-15), where $K_1 = -\frac{1}{18}$. Note that one can add to the star product in formula (3) the symmetric bi-vector

$$\lambda h^2 \partial_d \alpha^{ac} \partial_e \alpha^{bd}$$  \(17\)

(where $\lambda$ is a constant) which corresponds to loop diagram.

It is easy to get corrections to the expression for $B_3(f, g)$ from formula (1) under gauge transformations $D_2$ made in the second order:

$$B'_3(f, g) = B_3(f, g) + D_2B_1(f, g) - B_1(D_2f, g) - B_1(f, D_2g)$$  \(18\)

Thus, it is possible to change higher order terms by changing $\lambda$. We will put $\lambda = 0$.

### 4 Calculations up to the third order terms

Let us write down the terms of the third order from the Moyal product in the new coordinates:

$$\frac{1}{6} g^{ij} g^{kl} g^{mn} \left( \frac{\partial^3 f}{\partial x^i \partial x^j \partial x^m} \frac{\partial z^c}{\partial x^l} \frac{\partial z^d}{\partial x^n} + \frac{\partial^2 f}{\partial x^i \partial x^j} \frac{\partial^2 z^a}{\partial x^m \partial x^k} \frac{\partial z^c}{\partial x^n} + \frac{\partial^2 z^a}{\partial x^i \partial x^m} \frac{\partial^2 z^c}{\partial x^n \partial x^k} \frac{\partial z^d}{\partial x^n} + \frac{\partial^2 z^a}{\partial x^i \partial x^j} \frac{\partial^2 z^c}{\partial x^n \partial x^k} \frac{\partial z^d}{\partial x^n} \right)$$

$$+ \frac{\partial^2 z^a}{\partial x^i \partial x^j} \frac{\partial^2 z^c}{\partial x^n \partial x^k} \frac{\partial z^d}{\partial x^n} + \frac{\partial^2 z^a}{\partial x^i \partial x^j} \frac{\partial^2 z^c}{\partial x^n \partial x^k} \frac{\partial z^d}{\partial x^n} + \frac{\partial^2 z^a}{\partial x^i \partial x^j} \frac{\partial^2 z^c}{\partial x^n \partial x^k} \frac{\partial z^d}{\partial x^n}$$

Due to (18) we are to add to the above expression

$$D_2B_1(f, g) - B_1(D_2f, g) - B_1(f, D_2g) + D'_2B_1(f, g) - B_1(D'_2f, g) - B_1(f, D'_2g)$$  \(20\)

where $D_2$ and $D'_2$ are given by (14-15) and (11).

Again we are going to represent the sum of the Moyal product and gauge terms in diagrams in the new coordinates.

An important remark is that there is no necessity to make gauge transformations in the third order, since the corrections to the third order terms $B_3(f, g)$ coming from the gauge transformation of the second order (20) are enough to represent this expression in the form of Kontsevich’s diagrams.

Now consider an example of computation for the terms of the type $(\partial^2 f \partial^2 g)$. Such terms in the Moyal product (19) are

$$\frac{1}{6} g^{ij} g^{kl} g^{mn} \left( \frac{\partial^2 z^a}{\partial x^i \partial x^j} \frac{\partial^2 z^b}{\partial x^k \partial x^m} + \frac{\partial^2 z^a}{\partial x^i \partial x^j} \frac{\partial^2 z^c}{\partial x^k \partial x^n} + \frac{\partial^2 z^a}{\partial x^i \partial x^j} \frac{\partial^2 z^c}{\partial x^k \partial x^m} \frac{\partial z^d}{\partial x^n} + \frac{\partial^2 z^a}{\partial x^i \partial x^j} \frac{\partial^2 z^c}{\partial x^k \partial x^m} \frac{\partial z^d}{\partial x^n} \right) \frac{\partial^2 z^d}{\partial x^i \partial x^j} \frac{\partial^2 z^d}{\partial x^k \partial x^m} \frac{\partial z^d}{\partial x^n}$$

$$= g^{ij} g^{kl} g^{mn} \left( \frac{1}{2} \frac{\partial^2 z^a}{\partial x^i \partial x^j} \frac{\partial^2 z^b}{\partial x^k \partial x^m} + \frac{\partial^2 z^a}{\partial x^i \partial x^j} \frac{\partial^2 z^c}{\partial x^k \partial x^n} \frac{\partial z^d}{\partial x^n} + \frac{\partial^2 z^a}{\partial x^i \partial x^j} \frac{\partial^2 z^a}{\partial x^k \partial x^m} \frac{\partial z^d}{\partial x^n} \frac{\partial z^d}{\partial x^n} \right) \frac{\partial^2 z^d}{\partial x^i \partial x^j} \frac{\partial^2 z^d}{\partial x^k \partial x^m} \frac{\partial z^d}{\partial x^n} \frac{\partial z^d}{\partial x^n}$$

$$+ \frac{\partial^2 z^a}{\partial x^i \partial x^j} \frac{\partial^2 z^b}{\partial x^k \partial x^m} + \frac{\partial^2 z^a}{\partial x^i \partial x^j} \frac{\partial^2 z^c}{\partial x^k \partial x^n} + \frac{\partial^2 z^a}{\partial x^i \partial x^j} \frac{\partial^2 z^c}{\partial x^k \partial x^m} \frac{\partial z^d}{\partial x^n} + \frac{\partial^2 z^a}{\partial x^i \partial x^j} \frac{\partial^2 z^c}{\partial x^k \partial x^m} \frac{\partial z^d}{\partial x^n} \frac{\partial z^d}{\partial x^n}$$

$$\frac{\partial^2 z^d}{\partial x^i \partial x^j} \frac{\partial^2 z^d}{\partial x^k \partial x^m} \frac{\partial z^d}{\partial x^n} \frac{\partial z^d}{\partial x^n}$$

$$= g^{ij} g^{kl} g^{mn} \left( \frac{1}{2} \frac{\partial^2 z^a}{\partial x^i \partial x^j} \frac{\partial^2 z^b}{\partial x^k \partial x^m} + \frac{\partial^2 z^a}{\partial x^i \partial x^j} \frac{\partial^2 z^c}{\partial x^k \partial x^n} \frac{\partial z^d}{\partial x^n} + \frac{\partial^2 z^a}{\partial x^i \partial x^j} \frac{\partial^2 z^a}{\partial x^k \partial x^m} \frac{\partial z^d}{\partial x^n} \frac{\partial z^d}{\partial x^n} \right) \frac{\partial^2 z^d}{\partial x^i \partial x^j} \frac{\partial^2 z^d}{\partial x^k \partial x^m} \frac{\partial z^d}{\partial x^n} \frac{\partial z^d}{\partial x^n}$$

$$+ \frac{\partial^2 z^a}{\partial x^i \partial x^j} \frac{\partial^2 z^b}{\partial x^k \partial x^m} + \frac{\partial^2 z^a}{\partial x^i \partial x^j} \frac{\partial^2 z^c}{\partial x^k \partial x^n} + \frac{\partial^2 z^a}{\partial x^i \partial x^j} \frac{\partial^2 z^c}{\partial x^k \partial x^m} \frac{\partial z^d}{\partial x^n} + \frac{\partial^2 z^a}{\partial x^i \partial x^j} \frac{\partial^2 z^c}{\partial x^k \partial x^m} \frac{\partial z^d}{\partial x^n} \frac{\partial z^d}{\partial x^n}$$

$$\frac{\partial^2 z^d}{\partial x^i \partial x^j} \frac{\partial^2 z^d}{\partial x^k \partial x^m} \frac{\partial z^d}{\partial x^n} \frac{\partial z^d}{\partial x^n}$$
The gauge terms are
\[ D_2' : -\frac{1}{2} \partial^{ij} \partial^{kl} \frac{\partial^2 z^a}{\partial x^i \partial x^k} \frac{\partial^2 z^b}{\partial x^j \partial x^l} \frac{\partial^2 z^c}{\partial x^m \partial x^n} \partial_x f \partial_y g \] (22)
(Note that this gauge transformation cancels the first term in the right side of (21).)
\[ D_2 : \frac{1}{3} S^{scd} \partial_x \alpha^{ab} \partial_y f \partial_y g \] (23)
where \( S^{scd} \) is defined by (15). One may check that (23) is equal to
\[ -\frac{1}{3} S^{scd} \partial_x \alpha^{ab} \partial_x f \partial_y g = -\frac{1}{3} Y_1 - \frac{1}{3} Y_2 - \frac{1}{3} Y_3 \] (24)
where \( Y_1, Y_2 \) and \( Y_3 \) are
\[ Y_1 = \partial^{ij} \partial^{kl} \partial_x z^a \left( \frac{\partial^2 z^c}{\partial x^i \partial x^k} \frac{\partial^2 z^b}{\partial x^j \partial x^l} \right) \] (25)
\[ Y_2 = \partial^{ij} \partial^{kl} \partial_x z^a \left( \frac{\partial^2 z^c}{\partial x^i \partial x^k} \frac{\partial^2 z^b}{\partial x^j \partial x^l} \right) \] (26)
\[ Y_3 = \partial^{ij} \partial^{kl} \partial_x z^a \left( \frac{\partial^2 z^c}{\partial x^i \partial x^k} \frac{\partial^2 z^b}{\partial x^j \partial x^l} \right) \] (27)
We will try to find an answer in the form
\[ [A \alpha^{ap} \partial_x \alpha^{as} \partial_x \alpha^{bc}] \partial_x f \partial_y g \] (28)
In old coordinates it looks like
\[ [A \alpha^{ap} \partial_x \alpha^{as} \partial_x \alpha^{bc}] \partial_x f \partial_y g = A(Y_2 - Y_1) + B(Y_3 - Y_2) \] (29)
where \( A \) and \( B \) are constants. Note also that the last term in the right part of (21) is equal to \( Y_3 \) since
\[ \partial^{ij} \partial^{kl} \partial_x z^a \left( \frac{\partial^2 z^c}{\partial x^i \partial x^k} \frac{\partial^2 z^b}{\partial x^j \partial x^l} \right) \partial_x f \partial_y g = 0 \]
due to the skew symmetry with respect to the interchange \( (a \leftrightarrow c \text{ and } d \leftrightarrow b) \). Now we can find \( A \) and \( B \)
\[ Y_3 - \frac{1}{3} Y_1 - \frac{1}{3} Y_2 - \frac{1}{3} Y_3 = A(Y_3 - Y_2) + B(Y_2 - Y_1) \]
\[ A = \frac{2}{3}, \quad B = \frac{1}{3} \]
Thus, we have found the \( (\partial^2 f \partial^2 g) \)-terms in the deformation quantization formula (8):
\[ h^2 \left[ \frac{2}{3} \alpha^{ap} \partial_x \alpha^{as} \partial_x \alpha^{bc} + \frac{1}{3} \alpha^{ap} \partial_x \alpha^{as} \partial_x \alpha^{bc} \right] \partial_x f \partial_y g \]
All other terms can be found in the same manner. The only exception happens with the \( \partial f \partial g \)-terms. For these terms, one has
\[ \text{Moyal term} : \frac{1}{6} \partial^{ij} \partial^{kl} \frac{\partial^2 z^a}{\partial x^i \partial x^k} \frac{\partial^2 z^b}{\partial x^j \partial x^l} \] (30)
\[ \text{Gauge } D_2 : -\frac{1}{18} S^{apq} \partial_x \partial_y \partial_x \alpha^{ab} \partial_x f \partial_y g \] (31)
\[ \text{Gauge } D_2' : -\frac{1}{4} \partial^{ij} \partial^{kl} \frac{\partial^2 z^a}{\partial x^i \partial x^k} \frac{\partial^2 z^b}{\partial x^j \partial x^l} \partial_x \alpha^{ab} \partial_x f \partial_y g \] (32)
These terms cannot be canceled by gauge transformations as they are skew symmetric, and none of them can be represented in the form of Kontsevich’s diagrams. Therefore, we are going to treat them as \( \alpha^{ab}_2 \partial_x f \partial_y g \). The definition of \( \alpha^{ab}_2 \) is not unique for the following reason. Formula (32) is contained in \( \partial_x \alpha^{as} \partial_x \alpha^{ap} \partial_x \partial_x \alpha^{ab} f \partial_y g \), however, this expression contains other terms. Formula (31) can not be realized in the form of Kontsevich’s diagrams.
at all because of the third derivative. The Moyal term (30) contains $\partial^3 z^a \partial^3 z^b$ and thus is contained in the only diagram: $\alpha^{pt} \partial_t \alpha^{ao} \partial_a \alpha^{bo}$ but it again contains many other terms.

Thus we have following expression for the Poisson bi-vector

$$\alpha^{ab} = \theta^{ij} \frac{\partial z^a}{\partial x^i} \frac{\partial z^b}{\partial x^j} + \hbar^2 \left[ \frac{1}{3!} \theta^{ij} \theta^{kl} \theta^{mn} \frac{\partial^3 z^a}{\partial x^i \partial x^k \partial x^m} \frac{\partial^3 z^b}{\partial x^j \partial x^l \partial x^n} \right] - \frac{1}{18} \sum_{p=1}^{3} \partial_p \partial_z \alpha^{ab} - \frac{1}{4} \theta^{ij} \theta^{kl} \frac{\partial^2 z^a}{\partial x^i \partial x^k} \frac{\partial^2 z^b}{\partial x^j \partial x^l} \partial_l \alpha^{ab} \right] + O(\hbar^3) \tag{33}$$

So we found $\alpha = \alpha_0 + \hbar^2 \alpha_2$. It is obvious that $\alpha^{ab} = \theta^{ij} \frac{\partial z^a}{\partial x^i} \frac{\partial z^b}{\partial x^j}$ satisfy the Jacobi identity

$$\alpha_{0}^{as} \partial_s \alpha_{0}^{bc} + \alpha_{0}^{cs} \partial_s \alpha_{0}^{ab} + \alpha_{0}^{bs} \partial_s \alpha_{0}^{ca} = 0 \tag{34}$$

The Poisson bi-vector $\alpha(h)$ does not satisfy the Jacobi identity in the $\hbar^2$-order. This statement may be proved in the following way. One may consider a certain type of terms in the right hand side of the Jacobi identity and show that there is no way to drop them off.

In the $\hbar^2$-order, the Jacobi identity looks like

$$\alpha_{0}^{as} \partial_s \alpha_{0}^{bc} + \alpha_{0}^{as} \partial_s \alpha_{0}^{bc} + \alpha_{0}^{cs} \partial_s \alpha_{0}^{ab} + \alpha_{0}^{cs} \partial_s \alpha_{0}^{ab} + \alpha_{0}^{bs} \partial_s \alpha_{0}^{ca} + \alpha_{0}^{bs} \partial_s \alpha_{0}^{ca} = 0 \tag{35}$$

To see it is not true for $\alpha$ from (33) one should have a look on the terms $\partial z \partial^2 z \partial^3 z$. The coordinate $z$ can have different types of indices: external $a, b, c$ and internal $o, t, p$. Thus, there is a number of the above expressions with different types of indices and each of them should satisfy the Jacobi identity in order to have it for the whole $\alpha$. This kind of terms appears from the two first terms in (33). The indices have the following structures: $\partial z^a \partial^2 z^b \partial^3 z^c$ from the first term, $\partial z^a \partial^4 z^b \partial^3 z^c$ and $\partial z^a \partial^4 z^b \partial^3 z^c$ from the second one. Each of the above expressions makes the Jacobi identity non-valid. Still, we should remember that $\alpha_2$ can be defined in a different way if one adds $\partial f \partial g$ terms to the star product (8). The only such a term which gives the same type of terms in the Jacobi identity is $\partial_p \alpha^{ao} \partial_a \alpha^{tp} \partial_t \alpha^{ab}$. It can cancel $\partial z^a \partial^4 z^b \partial^3 z^c$ and add $\partial z^a \partial^4 z^o \partial^3 z^b$, $\partial z^a \partial^4 z^b \partial^3 z^c$ and $\partial z^a \partial^4 z^o \partial^3 z^c$. In this case, the expression $\partial z^a \partial^4 z^o \partial^3 z^c$ violates the Jacobi identity. Note that we did not consider the "tadpole" diagrams (which contain $\partial \alpha^{as}$), since there are no such terms in Kontsevich's formula but they save the Jacobi identity neither.

### 5 Values of coefficients from associativity

In this section, we are going to obtain the coefficients in formula (8) from the condition of associativity and the Jacobi identity for $\alpha_0$. Associativity for the $n^{th}$-order means $f \ast (g \ast h) = f \ast (g \ast h) + O(\hbar^{n+1})$. For the second and the third orders, we have (star product is defined by (1)):

$$B_2(f, g, h) + B_1(B_1(f, g, h) + B_2(f, g)h = B_2(f, gh) + B_1(f, B_1(g, h)) + fB_2(g, h) \tag{34}$$

$$B_3(f, g, h) + B_2(B_1(f, g), h) + B_1(B_2(f, g), h) + B_3(f, g)h =$$

$$= B_3(f, gh) + B_2(f, B_1(g, h)) + B_1(f, B_2(g, h)) + fB_3(g, h) \tag{35}$$

In the second order condition (34), one may put $\alpha = \alpha_0$. In the third order (35), $B_3(f, g)$ contains the term depending on $\alpha_2$, but this is a bi-vector $\alpha_2^{ab} \partial_a f \partial_b g$ and, thus, cancels from (35) due to the Leibnitz rule. It means there is no possibility to obtain some knowledge about $\alpha_2$ from associativity in the $\hbar^3$-order. So we may use the Jacobi identity for $\alpha$ in equations (34),(35).

Let us put arbitrary coefficients in formula (8)
\[ f \ast g = fg + \hbar \alpha^{ab} \partial_a f \partial_b g + \hbar^2 [A_1 \alpha^{ab} \alpha^{cd} \partial_a \partial_c f \partial_b \partial_d g + A_2 \alpha^{as} \alpha^{bs} \partial_a \partial_s f \partial_b \partial_s g + A_3 \alpha^{as} \alpha^{bs} \partial_a \partial_s \bar{\partial}_b \partial_s \bar{\partial}_f \partial_b g] + \hbar^3 [C_1 \alpha^{ab} \alpha^{cd} \alpha^{ho} \partial_a \partial_c \partial_h f \partial_b \partial_d \partial_o \partial_s g + C_2 \alpha^{ip} \alpha^{as} \partial_c \partial_s \partial_o \partial_c \partial_p \partial_b \partial_c f \partial_b \partial_o \partial_g + C_3 \alpha^{ip} \partial_p \alpha^{as} \partial_c \partial_s \partial_o \partial_c \partial_p \partial_b \partial_c \partial_g \partial_b f + + [C_4 \alpha^{dp} \partial_p \alpha^{as} \partial_s \bar{\partial}_c \partial_g \partial_d g + C_5 \alpha^{dp} \partial_p \alpha^{as} \partial_s \partial_o \partial_c \partial_p \partial_b \partial_c f \partial_b \partial_o \partial_g + C_6 \alpha^{as} \alpha^{cd} \partial_a \partial_c \partial_d \partial_p \partial_d \partial_a \partial_b \partial_d \partial_g \partial_c f + + C_7 \alpha^{as} \alpha^{cd} \partial_a \partial_c \partial_d \partial_p \partial_d \partial_a \partial_b \partial_d \partial_g \partial_c f + 0(\hbar^4)] \]

Equation (34) simply gives \( A_1 = \frac{1}{2} \) and
\[(1 - A_2)\alpha^{as} \partial_s \alpha^{bc} + (1 - A_3)\alpha^{as} \partial_s \alpha^{ab} + (A_2 + A_3)\alpha^{bs} \partial_s \alpha^{ca} \partial_a f \partial_b \partial_c \partial_\partial g_\partial = 0\]
As the Jacobi identity is known to be satisfied by \( \alpha_0 \), one obtains
\[1 - A_2 = 1 - A_3 = A_2 + A_3\]
and thus \( A_2 = A_3 = \frac{1}{2} \).

Similarly from equation (35) one may obtain the following results
\[C_1 = \frac{1}{6}, C_2 = -C_3 = \frac{1}{3}, C_4 - C_5 = \frac{1}{3}, C_6 = -C_7 = \frac{1}{6}, C_8 = -C_9 = \frac{1}{3}\]
These results are in agreement with formula (8). At the same time, they are in agreement with [3]. The only difference is that the authors of [3] write \( C_4 = -C_5 = \frac{1}{6} \) and, in formula (8), \( C_4 = \frac{1}{2}, C_5 = \frac{1}{3} \). The both cases satisfy the obtained condition \( C_4 - C_5 = \frac{1}{3} \). However, in our case the two coefficients \( C_4 \) and \( C_5 \) are not fixed since the two diagrams corresponding to the terms \( \partial \partial f \partial \partial g \) are dependent through the Jacobi identity for \( \alpha_0 \). Since the entire \( \alpha \) does not satisfy the Jacobi identity, these coefficients in formula (8) can not be changed. We expect that our coefficients will better suit next orders calculations.

6 Conclusion

In the present paper, we checked, up to the third order in \( \hbar \) the statement made in [1] that changing coordinates in Kontsevich’s star product leads to gauge equivalent star products. The manifest calculations were presented starting from the star product with the constant Poisson bi-vector (giving the Moyal product). In this way, we obtained formula (8) for the deformation quantization which is agreement with [3]. We also obtained the same result from the requirement of associativity of the star product and the Jacobi identity for \( \alpha_0 \) (which is not valid for \( \alpha(\hbar) \)).

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