Scattering in the Presence of a Reflecting and Transmitting Impurity

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Abstract

We investigate factorized scattering from a reflecting and transmitting impurity. Bulk scattering is non trivial, provided that the bulk scattering matrix depends separately on the spectral parameters of the colliding particles, and not only on their difference. We show that a specific extension of a boundary algebra encodes the underlying scattering theory. The total scattering operator is constructed in this framework and shown to be unitary.
1 Introduction

Integrable quantum field theories with boundaries [1, 2, 3] have been subject of intense study during the past decade. The great interest in such theories stems from the large number of potential applications in different physical areas, including open strings, branes, boundary conformal field theory, dissipative quantum phenomena and impurity problems. The investigations have been mainly focussed on purely reflecting boundaries. In realistic impurity problems [4, 5] however, one must often deal with defects, which both reflect and transmit. In spite of some progress, the results [1, 4, 8] on this subject derived by the inverse scattering method, are not very encouraging. The present status of the problem can be summarized as follows [8]: simultaneous reflection and transmission are possible only if the two-body bulk scattering matrix is constant (momentum and energy independent). This condition severely restricts the class of admissible systems. If the scattering matrix is diagonal for instance, one is left [3] with free bosons, free fermions and Federbush type models [7], all of them representing a limited physical interest. In the present paper we explore the possibility to weaken the assumptions adopted in [1, 2, 3] in order to avoid the above mentioned no-go theorem, preserving at the same time the basic physical features of integrability, reflection and transmission. Our analysis is constructive and uses a certain extension of the boundary algebra [10], previously applied [11, 12, 13] in the case of reflecting boundaries. The framework covers a large class of integrable systems with reflecting and transmitting impurities and allows to construct explicitly the total scattering operator and to prove unitarity and asymptotic completeness.

The paper is organized as follows. The next section is devoted to the consistency relations, following from three-body factorized scattering in presence of impurity. In Sect. 3 we describe the general solution of these relations and give some examples. An algebraic framework for deriving the scattering amplitudes is developed in Sect. 4. Here we construct also the asymptotic states. In Sect. 5 we compute some amplitudes explicitly, we define the total scattering operator and discuss unitarity. The last section contains our conclusions.
2 Kinematics and consistency relations

Following the basic ideas of the quantum inverse scattering method in 1+1 dimensions, we parameterize the asymptotic particles by their energy $E$, momentum $p$ and an “isotopic” index $i = 1, ..., N$, the latter describing the internal degrees of freedom. Usually $E$ and $p$ are not independent and obey some dispersion relation. It is conveniently implemented by parameterizing both $E$ and $p$ in terms of one parameter $\chi \in \mathbb{R}$, i.e.

$$E = E(\chi), \quad p = p(\chi).$$

(2.1)

The conventional relativistic dispersion relation reads

$$E(\chi) = m \cosh(\chi), \quad p(\chi) = m \sinh(\chi),$$

(2.2)

where $m$ is the mass and $\chi$ the rapidity. A non relativistic example is

$$E(\chi) = \frac{m\chi^2}{2} + U, \quad p(\chi) = m\chi,$$

(2.3)

$\chi$ being the velocity and $U$ some constant. Notice that a Lorentz boost in (2.2) and a Galilean transformation in (2.3) are both realized by a translation $\chi \mapsto \chi + \alpha$.

In what follows we adopt a generic dispersion relation (2.1) and parameterize each asymptotic particle by $\chi$ and its isotopic type $i$, referring to $\chi$ as spectral parameter. We start by considering an impurity localized at $x = 0$ and without internal degrees of freedom. In this case the fundamental building blocks of factorized scattering are:

(i) the two-body bulk scattering matrix $S_{ij}^{j_1j_2}(\chi_1, \chi_2)$;

(ii) the reflection matrix $R_i^j(\chi)$, describing the reflection of a particle from the impurity;

(iii) the transmission matrix $T_i^j(\chi)$, describing the transmission of a particle by the impurity.

It is worth stressing that $S$ is allowed to depend on $\chi_1$ and $\chi_2$ separately [14]. This feature represents an essential difference with respect to the framework of [3, 4, 8], where $S$ is assumed to depend on $\chi_1 - \chi_2$ only. Notice that this last condition imposes on $S$ some symmetries; with the dispersion relation
for instance, $S_{i_1 j_1}^{i_2 j_2} (\chi_1 - \chi_2)$ turns out to be Lorentz invariant. The same conclusion holds for Galilean invariance in the non-relativistic case with dispersion relation \((2.3)\). However, one can expect in general that the effect of the impurity is not localized only at $x = 0$, but propagates also in the bulk, breaking down the Lorentz or Galilean invariance of $S$. For this reason we find the framework of \([1, 7, 8]\) too restrictive and a bit artificial at this point. In fact, we will show below that allowing $S$ to depend on $\chi_1$ and $\chi_2$ separately, leads to a natural generalization of the inverse scattering method, which avoids the no-go theorem of \([6, 7, 8]\) and describes a large set of integrable systems, not covered there.

It is convenient to define at this stage the matrices

$$
R^\pm (\chi) \equiv \theta (\pm \chi) R(\chi), \quad T^\pm (\chi) \equiv \theta (\pm \chi) T(\chi),
$$

which have a simple physical interpretation: $R^\pm$ describe the reflection of a particle propagating $R_\pm$, whereas $T^\pm$ correspond to the transmission of a particle from $R_-$ to $R_+$ and vice versa. If one interprets the impurity as an infinitely heavy body, the data $\{S, R^\pm, T^\pm\}$ capture all two-body interactions. They are represented graphically in Fig. 1.

![Figure 1: The two-body processes.](image)

Time is flowing in these diagrams along the vertical direction and single lines represent particles. The double line corresponds to the world line of
the impurity, which is vertical because the impurity does not move from the point \( x = 0 \).

The construction of the possible three-body processes in terms of \( S, R^\pm \) and \( T^\pm \) leads to a series of relations \([6, 7, 8]\) among them. The consistency condition, stemming from the scattering of three particles among themselves, is the well known quantum Yang-Baxter equation (in its braid form)

\[
S_{12}(\chi_1, \chi_2)S_{23}(\chi_1, \chi_3)S_{12}(\chi_2, \chi_3) = S_{23}(\chi_2, \chi_3)S_{12}(\chi_1, \chi_3)S_{23}(\chi_1, \chi_2), \tag{2.5}
\]

where standard tensor notation has been adopted. Eq. (2.5) has a familiar graphic representation, which we omit for conciseness.

The consistency conditions implied by the scattering of two particles among each other and with the impurity are conveniently organized in the following three groups:

(a) pure reflection:

\[
S_{12}(\chi_1, \chi_2)R^+_2(\chi_1)S_{12}(\chi_2, -\chi_1)R^+_2(\chi_2) = \\
R^+_2(\chi_2)S_{12}(\chi_1, -\chi_2)R^+_2(\chi_1)S_{12}(-\chi_2, -\chi_1), \tag{2.6}
\]

\[
S_{12}(\chi_1, \chi_2)R^-_1(\chi_2)S_{12}(-\chi_2, \chi_1)R^-_1(\chi_1) = \\
R^-_1(\chi_1)S_{12}(-\chi_1, \chi_2)R^-_1(\chi_2)S_{12}(-\chi_2, -\chi_1). \tag{2.7}
\]

Eqs. (2.6) and (2.7) concern the reflection on \( \mathbb{R}_+ \) and \( \mathbb{R}_- \) respectively. Using the rules in Fig. 1 and moving back in time, one gets the graphic representation of (2.6) shown in Fig. 2.

![Figure 2: Pure reflection.](image)

The picture associated to (2.7) is obtained from Fig. 2 by reflection with respect to the impurity world line.

(b) pure transmission:

\[
T^+_1(\chi_1)S_{12}(\chi_1, \chi_2)T^-_1(\chi_2) = T^-_2(\chi_2)S_{12}(\chi_1, \chi_2)T^+_2(\chi_1), \tag{2.8}
\]
As before, the picture corresponding to eq. (2.10) is obtained from Fig. 3(b). Eqs. (2.8) and (2.9) are represented in Fig. 3(a) and Fig. 3(b) respectively.

This completes our discussion of the three-body processes.

(c) mixed relations:

\[ R_1^+(\chi_1)T_2^-(\chi_2) = T_2^-(\chi_2)S_{12}(\chi_1, \chi_2)R_2^+(\chi_1)S_{12}(\chi_2, -\chi_1), \]  
(2.11)

\[ T_1^+(\chi_1)R_2^-(\chi_2) = T_1^+(\chi_1)S_{12}(\chi_1, \chi_2)R_1^-(\chi_2)S_{12}(-\chi_2, \chi_1), \]  
(2.12)

\[ R_1^+(\chi_1)T_2^+(\chi_2) = S_{12}(\chi_1, \chi_2)R_2^+(\chi_1)S_{12}(\chi_2, -\chi_1)T_2^+(\chi_2), \]  
(2.13)

\[ T_1^-(\chi_1)R_2^-(\chi_2) = S_{12}(\chi_1, \chi_2)R_1^-(\chi_2)S_{12}(-\chi_2, \chi_1)T_1^-\]  
(2.14)

\[ R_1^+(\chi_1)T_2^-(\chi_2)S_{12}(-\chi_1, \chi_2) = T_2^-(\chi_2)S_{12}(\chi_1, \chi_2)R_2^+(\chi_1), \]  
(2.15)

\[ T_1^+(\chi_1)R_2^-(\chi_2)S_{12}(\chi_1, -\chi_2) = T_1^+(\chi_1)S_{12}(\chi_1, \chi_2)R_1^-(\chi_2), \]  
(2.16)

\[ R_2^+(\chi_1)S_{12}(\chi_2, -\chi_1)T_2^+(\chi_2) = S_{12}(\chi_2, \chi_1)R_1^+(\chi_1)T_2^+(\chi_2), \]  
(2.17)

\[ R_1^-\]  
(2.18)

Eqs. (2.11) and (2.15) are shown in Fig. 4(a) and 4(b) respectively, whereas eqs. (2.13) and (2.17) are drawn in Fig. 4(c) and 4(d). The pictures related to the remaining four mixed equations are obtained from Fig. 4 by reflection. This completes our discussion of the three-body processes.

Let us focus finally on the requirements of unitarity and Hermitian analyticity. For \( S \) one has the familiar \[1, 2, 3\] conditions

\[ S_{12}(\chi_1, \chi_2)S_{12}(\chi_2, \chi_1) = 1, \]  
(2.19)

\[ [S_{12}]^\dagger(\chi_1, \chi_2) = S_{12}(\chi_2, \chi_1), \]  
(2.20)
where the dagger stands for Hermitian conjugation. Concerning $R$ and $T$, the conditions of unitarity and Hermitian analyticity read

$$T(\chi)T(\chi) + R(\chi)R(-\chi) = 1, \quad (2.21)$$
$$T(\chi)R(\chi) + R(\chi)T(-\chi) = 0, \quad (2.22)$$

and

$$[T]^\dagger(\chi) = T(\chi), \quad [R]^\dagger(\chi) = R(-\chi), \quad (2.23)$$

respectively. We will come back to eqs. (2.21-2.23) in Sect. 5, observing for the moment that (2.19) implies the equivalence of the two sets of equations (2.11-2.14) and (2.15-2.18). Therefore one is left with the study eqs. (2.5-2.14) and (2.19-2.23), which is the main subject of the next section.

3 Solutions of the consistency relations

Let us assume in what follows that $S$ obeys (2.5, 2.19, 2.20). Our aim below is to find the general solution of eqs. (2.6-2.18) when the conditions of unitarity and Hermitian analyticity (2.21-2.23) are satisfied and the matrix $T$ is invertible. We shall present also some families of non trivial solutions, which are of physical interest.

Because of (2.23), $T(\chi)T(\chi)$ and $R(\chi)R(-\chi)$ are non negative Hermitian matrices. From (2.21) it follows that they are diagonalizable simultaneously and that the corresponding eigenvalues satisfy

$$\lambda_i(\chi) + \mu_i(\chi) = 1, \quad \lambda_i(\chi) \geq 0, \quad \mu_i(\chi) \geq 0, \quad i = 1, ..., N. \quad (3.1)$$
Now, using eq. (2.21), one can express $T$ as a function of $R$:

$$T(\chi) = t(\chi) \sqrt{1 - R(\chi)[R(\chi)]^\dagger} = t(\chi) \sum_{n=0}^{\infty} \alpha_n \left(R(\chi)R(-\chi)\right)^n, \quad (3.2)$$

where $t(\chi) \in \{-1, +1\}$ is some unknown function and the real numbers $\alpha_n$ are defined through the expansion $\sqrt{1 - x} = \sum_{n=0}^{\infty} \alpha_n x^n$. Notice, that the conditions (3.1) ensure that this series is convergent. Demanding $T$ to satisfy eq. (2.22) shows that $t(\chi)$ must be an odd function. If $t(\chi)$ is in addition continuous for $\chi \neq 0$, one concludes that $t(\chi) = \pm \epsilon(\chi)$, where $\epsilon$ is the sign function.

In the following we assume that $T$ is invertible, which is equivalent to $\lambda_i(\chi) \neq 0$ for all $i = 1, \ldots, N$. Physically, this means that there is no isotopic channel with pure reflection only. If such channels exist, one can separate the corresponding isotopic degrees of freedom and treat them as a system with pure reflection. For invertible $T$, one gets from eqs. (2.11) and (2.12) that the matrices $R^\pm$ obey $S_{12}(\chi_1, \chi_2)R^\pm_2(\chi_1) = R^\mp_1(\chi_1)S_{12}(\chi_1, \chi_2)$, which can be compactly rewritten as

$$S_{12}(\chi_1, \chi_2)R_2(\chi_1) = R_1(\chi_1)S_{12}(\chi_1, \chi_2). \quad (3.3)$$

It is a simple matter to prove that (3.3) solves all the mixed relations (2.11-2.18), as well as the pure reflection ones (2.6-2.7). Moreover, from the expression (3.2) and the eq. (3.3), one easily deduces that

$$S_{12}(\chi_1, \chi_2)T_2(\chi_1) = T_1(\chi_1)S_{12}(\chi_1, \chi_2). \quad (3.4)$$

Finally, a direct inspection shows that any matrix $T$ satisfying eq. (3.4) is a solution to the pure transmission eqs. (2.8-2.10).

Let us stress that, when the unitarity conditions (2.19-2.22) and the invertibility of $T$ are assumed, the whole set of equations (2.6-2.18) is equivalent to the two simple linear equations (3.3) and (3.4). Summarizing, we proved that the Hermitian matrix

$$T(\chi) = \pm \epsilon(\chi) \sqrt{1 - R(\chi)[R(\chi)]^\dagger} \quad (3.5)$$

with $R$ obeying eq. (3.3), is the general solution to the eqs. (2.6-2.18).

Collecting the results of this section, we have reduced the original problem to the solution of eqs. (2.21-2.23) and (3.3-3.4). It is instructive at this point
to produce some explicit examples. We start with the \(gl(N)\)-invariant \(S\)-matrix

\[
S_{12}(\chi_1, \chi_2) = \frac{1}{s(\chi_1) - s(\chi_2) + ig} \left\{ [s(\chi_1) - s(\chi_2)] P_{12} + ig \mathbb{I} \otimes \mathbb{I} \right\}, \tag{3.6}
\]

where \(P_{12}\) is the standard flip operator, \(\mathbb{I}\) is the \(N \times N\) identity matrix, \(g \in \mathbb{R}\) and \(s(\chi)\) is any real valued even function. For \(R\) and \(T\) one easily derives

\[
R(\chi) = [\cos p(\chi)] \exp [iq(\chi)] \mathbb{I}, \quad T(\chi) = [\sin p(\chi)] \mathbb{I}, \tag{3.7}
\]

\(p(\chi)\) and \(q(\chi)\) being real valued odd functions. In this example both reflection and transmission preserve the isotopic type. Moreover, all isotopic types have the same reflection and transmission coefficient.

Slightly more complicated is the Toda type \(S\)-matrix

\[
S_{ij12}(\chi_1, \chi_2) = \exp \left[ is_{ij12}(\chi_1, \chi_2) \right] \delta_{ij}^1 \delta_{ij}^2, \tag{3.8}
\]

where \(s_{ij12}(\chi_1, \chi_2)\) are real valued functions obeying

\[
s_{ii12}(\chi_1, \chi_2) = -s_{i2i1}(\chi_2, \chi_1), \quad s_{i1i2}(\chi_1, \chi_2) = s_{i1i2}(\chi_1, -\chi_2). \tag{3.9}
\]

If we assume furthermore that for any couple of indices \((i, j)\) with \(i \neq j\), there exists an index \(k\) such that \(s_{ik}(\chi_1, \chi_2)\) is different from \(s_{jk}(\chi_1, \chi_2)\), one finds

\[
R_{ij}^i(\chi) = [\cos p_i(\chi)] \exp [iq_i(\chi)] \delta_{ij}^i, \quad T_{ij}^i(\chi) = [\sin p_i(\chi)] \delta_{ij}^i, \tag{3.10}
\]

with all \(p_i(\chi)\) and \(q_i(\chi)\) real valued odd functions. Also here the impurity interaction preserves the isotopic type, but the individual reflection and transmission coefficients may be different.

On the other hand, whenever a couple of indices \((i_0, j_0)\) with \(i_0 \neq j_0\) exists, such that \(s_{i_0k}(\chi_1, \chi_2) = s_{j_0k}(\chi_1, \chi_2)\) for any \(k\), one has in general that the corresponding off-diagonal matrix elements \(R_{i_0}^{j_0}(\chi)\) and \(R_{j_0}^{i_0}(\chi)\), as well as \(T_{i_0}^{j_0}(\chi)\) and \(T_{j_0}^{i_0}(\chi)\), do not vanish. They describe impurity interactions, which does not preserve the isotopic type.

## 4 Boundary algebra with transmission

The goal of this section is to provide an algebraic framework for integrable models with impurity, analogous to the Zamolodchikov-Faddeev (ZF) algebra
\( A \) operating in the case without impurities. The main idea is to adapt the concept of boundary algebra \( B \) to the case in which the boundary is both reflecting and transmitting. It might be useful to recall in this respect that in the case with pure reflection all scattering processes take place only in \( \mathbb{R}_+ \) (or \( \mathbb{R}_- \)). The whole line \( \mathbb{R} \) is involved instead, if nontrivial transmission is present. Moreover, the impurity at \( x = 0 \) breaks down translation invariance. In order to take into account these two facts, we equip the generators of our algebra with a double index \( \alpha = (\xi, i) \). As before, the index \( i = 1, \ldots, N \) denotes the isotopic type. The index \( \xi = \pm \) indicates the half line \( \mathbb{R}_\pm \) where the particle is created or annihilated. With this notation, we consider an associative algebra \( C \) with identity element \( 1 \), generated by \( \{a_\alpha(\chi), a^{*\alpha}(\chi)\} \), which are subject to the constraints:

\[
\begin{align*}
  a_{\alpha_1}(\chi_1) a_{\alpha_2}(\chi_2) - S_{\alpha_2\alpha_1}(\chi_2, \chi_1) a_{\beta_2}(\chi_2) a_{\beta_1}(\chi_1) &= 0, \\
  a^{*\alpha_1}(\chi_1) a^{*\alpha_2}(\chi_2) - a^{*\beta_2}(\chi_2) a^{*\beta_1}(\chi_1) S_{\alpha_2\beta_2\beta_1}(\chi_2, \chi_1) &= 0, \\
  a_{\alpha_1}(\chi_1) a^{*\alpha_2}(\chi_2) - a^{*\beta_2}(\chi_2) S_{\alpha_1\beta_2\beta_1}(\chi_1, \chi_2) a_{\beta_1}(\chi_1) &= \\
  \delta(\chi_1 - \chi_2) \left[ \delta^{\alpha_2}_{\alpha_1} + T^{\alpha_2}_{\alpha_1}(\chi_1) \right] 1 + \delta(\chi_1 + \chi_2) R^{\alpha_2}_{\alpha_1}(\chi_1) 1.
\end{align*}
\]

Here \( \{S, R, T\} \) are related to the starting data \( \{S, R, T\} \) in the following way:

\[
\begin{align*}
  S^{(m,j_1)(n,j_2)}_{(\xi_1,i_1)(\xi_2,i_2)}(\chi_1, \chi_2) &\equiv \delta^{m}_{\xi_1} \delta^{n}_{\xi_2} S^{j_1j_2}_{i_1i_2}(\chi_1, \chi_2), \\
  R^{(n,j)}_{(\xi,i)}(\chi) &\equiv \delta^{n}_{\xi} R^{j}_{i}(\chi), \\
  T^{(n,j)}_{(\xi,i)}(\chi) &\equiv \epsilon^{n}_{\xi} T^{j}_{i}(\chi),
\end{align*}
\]

where

\[
\epsilon = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

This algebra differs from the one proposed in [3, 4, 8]. Comparing \( C \) to the ZF algebra \( A \), the presence of the new terms \( R \) and \( T \) in (4.3) must be emphasized. Only the \( R \)-term appears in the boundary algebra \( B \), but in general as a new generator not proportional to the identity \( 1 \). This difference of \( B \) with respect to \( C \) can be traced back to condition (3.3), which is stronger than (2.6,2.7). Let us remark also that the \( R \)-term in (4.3) breaks down both Lorentz (Galilean) and translation invariance.

It is easy to verify that each triplet \( \{S, R, T\} \), obeying eqs. (2.3,2.19,2.23) and (3.3,3.4), determines a triplet \( \{S, R, T\} \), which satisfies the same equations. Using this fact and following closely the formalism developed in [10, 14], one can construct the Fock representation \( \mathcal{F} \) of \( C \). Referring for the
details to [13], let us collect here those basic features of $\mathcal{F}$, needed in the next section for the definition of the scattering operator. After smearing, \(\{a_\alpha(\chi), a^{*\alpha}(\chi)\}\) are represented by densely defined operators, acting in a Hilbert space $\mathcal{H}$ with scalar product $(\cdot, \cdot)$. There exist a cyclic vacuum state $\Omega \in \mathcal{H}$, which is annihilated by $\{a_\alpha(\chi)\}$ and satisfies $(\Omega, \Omega) = 1$.

The asymptotic states are prepared in $\mathcal{H}$ as follows. In-states are created from the vacuum by $\{a^{*(-i)}(\varphi) : \varphi > 0\}$ and $\{a^{*(+i)}(\varphi) : \varphi < 0\}$. The out-states are generated instead by $\{a^{*(-j)}(\chi) : \chi < 0\}$ and $\{a^{*(+j)}(\chi) : \chi > 0\}$. This choice corresponds to incoming particles traveling towards the impurity and outgoing particles moving in the opposite directions. Without loss of generality [1, 2], one can also order the creation operators using the values of the spectral parameter (rapidity). We thus define

$$|\varphi_1, \alpha_1; \ldots; \varphi_m, \alpha_m\rangle_{\text{in}} = a^{*\alpha_1}(\varphi_1) \cdots a^{*\alpha_m}(\varphi_m)\Omega, \quad (4.7)$$

with

$$\varphi_1 < \ldots < \varphi_m, \quad \alpha_k = (\xi_k, i_k), \quad \xi_k = -\epsilon(\varphi_k), \quad (4.8)$$

$\epsilon$ being the sign function. Analogously

$$\langle \chi_1, \beta_1; \ldots; \chi_n, \beta_n| = a^{*\beta_1}(\chi_1) \cdots a^{*\beta_n}(\chi_n)\Omega, \quad (4.9)$$

where

$$\chi_1 > \ldots > \chi_n, \quad \beta_l = (\eta_l, j_l), \quad \eta_l = \epsilon(\chi_l). \quad (4.10)$$

The asymptotic spaces $\mathcal{F}_{\text{in}}$ and $\mathcal{F}_{\text{out}}$ are generated by finite linear combinations of vectors of the type (4.7) and (4.9) respectively. Each of these spaces is dense in $\mathcal{H}$, ensuring asymptotic completeness.

Finally, one can express via

$$\langle \chi_1, \beta_1; \ldots; \chi_n, \beta_n|\varphi_1, \alpha_1; \ldots; \varphi_m, \alpha_m\rangle_{\text{in}} = (4.11)$$

$$\langle a^{*\beta_1}(\chi_1) \cdots a^{*\beta_n}(\chi_n)\Omega, a^{*\alpha_1}(\varphi_1) \cdots a^{*\alpha_m}(\varphi_m)\Omega \rangle$$

a generic scattering amplitude in terms of the correlation functions, which can be computed in turn by means of the exchange relation (4.3). The Fock structure implies that this amplitude vanishes unless $m = n$, which corresponds physically to the absence of particle production due to integrability.
5 The total scattering operator

In order to become more familiar with the scattering theory, encoded in the representation $F$ of $\mathcal{C}$, we derive now some transition amplitudes in explicit form. The simplest ones are the one-particle amplitudes. All of them can be deduced from the correlation function

$$\langle \alpha^* \beta \chi \rangle \left( \alpha^* \alpha \varphi \Omega \right) = 
\left[ \delta^\alpha_\beta + T^\alpha_\beta (\chi) \right] \delta (\chi - \varphi) + R^\alpha_\beta (\chi) \delta (\chi + \varphi).$$

Taking into account eqs. (4.8-4.10), one has the following four possibilities:

$$\text{out} \langle \chi, (\eta, j) | \varphi, (\xi, i) \rangle \text{in} = \begin{cases} 
T^{\pm i}_j (\chi) \delta (\chi - \varphi), & \xi = +, \eta = +, \\
R^{-i}_j (\chi) \delta (\chi + \varphi), & \xi = +, \eta = -, \\
R^{+i}_j (\chi) \delta (\chi + \varphi), & \xi = -, \eta = +, \\
T^{-i}_j (\chi) \delta (\chi - \varphi), & \xi = -, \eta = -.
\end{cases} \quad (5.2)$$

These amplitudes have transparent physical interpretation and describe the particle-impurity interaction. Notice that eqs. (2.21-2.23) ensure one-particle unitarity.

The particle-particle interaction shows up in the two-particle amplitudes, which can be deduced from the correlator

$$\langle \alpha^{* \beta_1} (\chi_1) | \alpha^{* \beta_2} (\chi_2) \rangle \left( \alpha^{* \alpha_1} (\varphi_1) | \alpha^{* \alpha_2} (\varphi_2) \Omega \right) = 
\left[ \delta^\mu_\beta_2 + T^\mu_\beta_2 (\chi_2) \right] S^{\alpha_1 \mu}_\beta_1 (\chi_1, \chi_2) \left[ \delta^\nu_\alpha_2 + T^\nu_\alpha_2 (\chi_1) \right] \delta (\chi_1 - \varphi_2) \delta (\chi_2 - \varphi_1) + 
R^\mu_\beta_2 (\chi_2) S^{\alpha_1 \mu}_\beta_1 (\chi_1, \chi_2) \delta (\chi_1 - \varphi_2) \delta (\chi_2 + \varphi_1) + 
\left[ \delta^\mu_\beta_2 + T^\mu_\beta_2 (\chi_2) \right] S^{\alpha_1 \mu}_\beta_1 (\chi_1, \chi_2) \left[ \delta^\nu_\alpha_2 + T^\nu_\alpha_2 (\chi_1) \right] \delta (\chi_1 + \varphi_2) \delta (\chi_2 - \varphi_1) + 
R^\mu_\beta_2 (\chi_2) S^{\alpha_1 \mu}_\beta_1 (\chi_1, \chi_2) \delta (\chi_1 + \varphi_2) \delta (\chi_2 + \varphi_1) + 
\left[ \delta^\mu_\beta_2 + T^\mu_\beta_2 (\chi_2) \right] S^{\alpha_1 \mu}_\beta_1 (\chi_1, \chi_2) \delta (\chi_1 - \varphi_1) \delta (\chi_2 - \varphi_2) + 
R^\mu_\beta_2 (\chi_2) S^{\alpha_1 \mu}_\beta_1 (\chi_1, \chi_2) \delta (\chi_1 - \varphi_1) \delta (\chi_2 + \varphi_2) + 
\left[ \delta^\mu_\beta_2 + T^\mu_\beta_2 (\chi_2) \right] S^{\alpha_1 \mu}_\beta_1 (\chi_1, \chi_2) \delta (\chi_1 + \varphi_1) \delta (\chi_2 - \varphi_2) + 
R^\mu_\beta_2 (\chi_2) S^{\alpha_1 \mu}_\beta_1 (\chi_1, \chi_2) \delta (\chi_1 + \varphi_1) \delta (\chi_2 + \varphi_2). \quad (5.3)$$

Using the definition (4.7-4.10) of asymptotic states, one has various kinematic domains, depending on the sign of $\varphi_i$ and $\chi_j$. There are four cases with $\epsilon (\varphi_1) = \epsilon (\varphi_2)$ and $\epsilon (\chi_1) = \epsilon (\chi_2)$, the corresponding amplitudes being:

$$\text{out} \langle \chi_1, (\pm, j_1); \chi_2, (\pm, j_2) | \varphi_1, (\pm, i_1); \varphi_2, (\pm, i_2) \rangle \text{in} = 
T^{\pm k}_{j_x} (\chi_2) S^{\alpha_1 \mu}_{j_x k} (\chi_1, \chi_2) T^{\pm i}_l (\chi_1) \delta (\chi_1 - \varphi_2) \delta (\chi_2 - \varphi_1), \quad (5.4)$$
There exist other four cases in which $\epsilon(\varphi_1) \neq \epsilon(\varphi_2)$ and $\epsilon(\chi_1) = \epsilon(\chi_2)$ or $\epsilon(\varphi_1) = \epsilon(\varphi_2)$ and $\epsilon(\chi_1) \neq \epsilon(\chi_2)$. From eq. (5.3) one gets the amplitudes:

\[
\begin{align*}
\text{out} \langle \chi_1, (\pm, j_1); \chi_2, (\pm, j_2)|\varphi_1, (\mp, i_1); \varphi_2, (\mp, i_2) \rangle \text{in} &= \\
R^{\pm}_{j_1}(\chi_1) R^{\pm}_{j_2}(\chi_2) \delta(\chi_1 + \varphi_1) \delta(\chi_2 + \varphi_2),
\end{align*}
\]

\[\text{(5.5)}\]

Finally, the case $\epsilon(\varphi_1) \neq \epsilon(\varphi_2)$ and $\epsilon(\chi_1) \neq \epsilon(\chi_2)$ exhausts all possibilities. The corresponding amplitude is

\[
\begin{align*}
\text{out} \langle \chi_1, (+, j_1); \chi_2, (+, j_2)|\varphi_1, (-, i_1); \varphi_2, (+, i_2) \rangle \text{in} &= \\
R^{+}_{j_2}(\chi_2) S^{ij}_{ik}(\chi_1, -\chi_2) T^{+ij}_{ik}(\chi_1) \delta(\chi_1 - \varphi_2) \delta(\chi_2 + \varphi_1)
+ R^{+}_{j_1}(\chi_1) T^{+ij}_{j_2}(\chi_2) \delta(\chi_1 + \varphi_1) \delta(\chi_2 - \varphi_2),
\end{align*}
\]

\[\text{(5.6)}\]

\[
\begin{align*}
\text{out} \langle \chi_1, (-, j_1); \chi_2, (-, j_2)|\varphi_1, (-, i_1); \varphi_2, (+, i_2) \rangle \text{in} &= \\
T^{-}_{j_2}(\chi_2) S^{ij}_{ik}(\chi_1, \chi_2) R^{-ij}_{i}(\chi_1) \delta(\chi_1 + \varphi_2) \delta(\chi_2 - \varphi_1)
+ T^{-}_{j_1}(\chi_1) R^{-ij}_{j_2}(\chi_2) \delta(\chi_1 - \varphi_1) \delta(\chi_2 + \varphi_2),
\end{align*}
\]

\[\text{(5.7)}\]

\[
\begin{align*}
\text{out} \langle \chi_1, (+, j_1); \chi_2, (-, j_2)|\varphi_1, (+, i_1); \varphi_2, (-, i_2) \rangle \text{in} &= \\
R^{-}_{j_2}(\chi_2) S^{ij}_{ik}(\chi_1, -\chi_2) T^{-ij}_{ik}(\chi_1) \delta(\chi_1 - \varphi_2) \delta(\chi_2 + \varphi_1)
+ T^{+}_{j_1}(\chi_1) R^{-ij}_{j_2}(\chi_2) \delta(\chi_1 - \varphi_1) \delta(\chi_2 + \varphi_2),
\end{align*}
\]

\[\text{(5.8)}\]

\[
\begin{align*}
\text{out} \langle \chi_1, (+, j_1); \chi_2, (-, j_2)|\varphi_1, (-, i_1); \varphi_2, (-, i_2) \rangle \text{in} &= \\
T^{-}_{j_2}(\chi_2) S^{ij}_{ik}(\chi_1, \chi_2) R^{+ij}_{i}(\chi_1) \delta(\chi_1 + \varphi_2) \delta(\chi_2 - \varphi_1)
+ R^{+}_{j_1}(\chi_1) T^{-ij}_{j_2}(\chi_2) \delta(\chi_1 + \varphi_1) \delta(\chi_2 - \varphi_2).
\end{align*}
\]

\[\text{(5.9)}\]

Keeping in mind that $R$ and $T$ satisfy eqs. (5.3, 5.4), one can recover from eqs. (5.4, 5.10) the analytic expressions of all two-particle processes, some of which represented by the diagrams in Figs. 2-4.
Any \(n\)-particle scattering amplitude can be reconstructed with the above algorithm. These amplitudes (4.12) define the total scattering operator \(S : F^{\text{out}} \to F^{\text{in}}\), which acts according to
\[
S : a^{\beta_1}(\chi_1) \cdots a^{\beta_n}(\chi_n)\Omega \mapsto a^{\tilde{\beta}_n}(\chi_n) \cdots a^{\tilde{\beta}_1}(\chi_1)\Omega,
\]
(5.11)
where \(\chi_1 > \cdots > \chi_n\) and \(\beta_l = (\eta_l, j_l), \tilde{\beta}_l = (-\eta_l, j_l)\). One can check that
\[
(S\Psi^{\text{out}} \cdot S\Phi^{\text{out}}) = (\Psi^{\text{out}} \cdot \Phi^{\text{out}})
\]
(5.12)
holds on \(F^{\text{out}}\), which together with asymptotic completeness and invertibility of \(S\), implies its unitarity.

6 Outlook and conclusions

Factorized scattering theory in the presence of a reflecting and transmitting impurity has been investigated. We have shown that relaxing the condition on the bulk scattering matrix to depend only on the difference of the spectral parameters of the colliding particles, allows for non trivial solutions of the three-body consistency relations. We established the general one in the case of invertible transmission factor. Our philosophy in constructing the scattering amplitudes is not to postulate the existence of a boundary state with certain reflection and transmission properties. We rather prefer to deal with an algebra, which admits a Fock representation, whose cyclic (vacuum) state plays the role of boundary state. This approach has already shown some advantages in the derivation of off-shell correlation functions \([1, 2]\) and the study of symmetries \([3]\). We demonstrate above that it works also in the case of impurities and provides a direct and relatively simple construction of the total scattering operator. A further generalization of the algebraic structure, applied in this paper, is presently under investigation \([4]\).

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