Group actions and power maps for groups over non-Archimedean local fields

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Abstract

We consider linear groups and Lie groups over a non-Archimedean local field $\mathbb{F}$ for which the power map $x \mapsto x^k$ has a dense image or it is surjective. We prove that the group of $\mathbb{F}$-points of such algebraic groups is a compact extension of unipotent groups with the order of the compact group being relatively prime to $k$. This in particular shows that the power map is surjective for all $k$ is possible only when the group is unipotent or trivial depending on whether the characteristic of $\mathbb{F}$ is zero or positive. Similar results are proved for Lie groups via the adjoint representation. To a large extent, these results are extended to linear groups over local fields and global fields.

1 Introduction

We will be considering existence of the solution to the equation

$$x^k = g, \quad k > 1 \quad (1)$$

in a group $G$ for every $g \in G$. Our aim is to find structural conditions that are equivalent to equation (1) having a solution in $G$ for every $g \in G$. We first note that equation (1) having a solution in $G$ for every $g \in G$ is equivalent to the corresponding power map $P_k: G \to G$ defined by $P_k(g) = g^k$ being surjective. Thus, we pay attention surjectivity of the power maps. For a (topological) group $G$ and $k \geq 1$, we say that $P_k$ is surjective on $G$ (resp. dense in $G$) if $P_k: G \to G$ defined by $P_k(x) = x^k$ for all $x \in G$ is surjective (resp. has dense image).

Surjectivity of power maps have been studied for algebraic groups over local fields and algebraically closed fields of characteristic zero in [3]-[5] and for semisimple algebraic groups over algebraically closed fields in [20]: see also the references
cited therein [5]. Certain class of solvable (not necessarily algebraic) group is considered in [9].

Density of the power map has also been considered for connected Lie groups in [1], [11], [13] and for real algebraic groups in [12].

We consider groups that have linear representation over non-Archimedean local fields and our approach involves linear dynamics and tidy subgroups: see [8] for linear dynamics and [21] for tidy subgroups.

Let \( \mathbb{F} \) be a field, and let \( G \) be a smooth affine \( \mathbb{F} \)-group. Then the \( \mathbb{F} \)-unipotent radical \( R_{u,\mathbb{F}}(G) \) of \( G \) is defined to be the largest smooth connected normal unipotent \( \mathbb{F} \)-subgroup of \( G \). We say that \( G \) is pseudo-reductive \( \mathbb{F} \)-group if \( R_{u,\mathbb{F}}(G) = \{ e \} \) where \( e \) is the identity element of \( G \). An \( \mathbb{F} \)-split unipotent radical \( R_{us,\mathbb{F}}(G) \) is a maximal smooth connected \( \mathbb{F} \)-split unipotent normal subgroup of \( G \) and \( G \) is said to be quasi-reductive if \( R_{us,\mathbb{F}}(G) = \{ e \} \). Note that \( R_{us,\mathbb{F}}(G) \subset R_{u,\mathbb{F}}(G) \). We equip \( G(\mathbb{F}) \) with topology inherited from \( \mathbb{F} \) when \( \mathbb{F} \) is a local field. If \( \mathbb{F} \) is a non-Archimedean local field, then \( G(\mathbb{F}) \) is totally disconnected locally compact group.

It is well-known that if \( G \) is a smooth connected affine \( \mathbb{F} \)-group then \( G/R_{u,\mathbb{F}}(G) \) is a pseudo-reductive \( \mathbb{F} \)-group and \( G/R_{us,\mathbb{F}}(G) \) is a quasi-reductive \( \mathbb{F} \)-group (see [7] Corollary B.3.5).

We first prove surjectivity and density are equivalent for \( G(\mathbb{F}) \) and characterize the surjectivity/density of the \( k \)-th power map \( P_k \) for \( G(\mathbb{F}) \) in terms of the \( \mathbb{F} \)-points of the quasi-reductive quotient group \( G/R_{us,\mathbb{F}}(G) \). We recall that the canonical map from \( G(\mathbb{F})/R_{us,\mathbb{F}}(G)(\mathbb{F}) \) into \( (G/R_{us,\mathbb{F}}(G))(\mathbb{F}) \) is an isomorphism as \( H^1(\mathbb{F}, R_{us,\mathbb{F}}(G)) = 0 \).

**Theorem 1.1.** Let \( \mathbb{F} \) be a non-Archimedean local field. Let \( G \) be an \( \mathbb{F} \)-group, and \( R_{us,\mathbb{F}}(G) \) be the \( \mathbb{F} \)-split unipotent radical of \( G \). Suppose that the characteristic of \( \mathbb{F} \) does not divide \( k \). Then the following are equivalent:

(a) \( P_k \) is dense in \( G(\mathbb{F}) \);

(b) \( G(\mathbb{F})/R_{us,\mathbb{F}}(G)(\mathbb{F}) \) is compact and \( P_k \) is surjective on \( G(\mathbb{F})/R_{us,\mathbb{F}}(G)(\mathbb{F}) \);

(c) \( P_k \) is surjective in \( G(\mathbb{F}) \).

Suppose the residual characteristic of \( \mathbb{F} \) divides \( k \). Then density of \( P_k \) on \( G(\mathbb{F}) \) implies that \( G(\mathbb{F}) \) is a finite extension of a split unipotent group. In addition if characteristic of \( \mathbb{F} \) is positive, then \( G(\mathbb{F}) \) is finite.

If the field \( \mathbb{F} \) is perfect then for an smooth connected \( \mathbb{F} \)-group \( G \), \( R_{us,\mathbb{F}}(G) = R_{u,\mathbb{F}}(G) = R_u(G) \), the unipotent radical of \( G \), and \( G/R_u(G) \) is a reductive \( \mathbb{F} \)-group. The following Corollary is an immediate consequence of Theorem 1.1 generalizes Theorem 1.2 of [4] - characteristic zero case: refer Section 2.2 for details on order of compact groups.

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Corollary 1.2. Let $F$ be non-Archimedean perfect field and $G$ be an algebraic group over $F$. Suppose that the characteristic of $F$ does not divide $k$. Then $P_k$ is dense in $G(F)$ if and only if $P_k$ is dense in $(G/R_u(G))(F)$ if and only if $(G/R_u(G))(F)$ is compact and $k$ is co-prime to the order of $(G/R_u(G))(F)$.

Next we consider inheritance of surjectivity of $P_k$ for algebraic groups: recall that not all (even closed) subgroups inherit surjectivity or density of $P_k$, e.g., $\mathbb{Z}_p$ (resp. $\mathbb{Z}$) is a closed subgroup of $\mathbb{Q}_p$ (resp. $\mathbb{R}$) and $P_k$ is surjective on $\mathbb{Q}_p$ (resp. $\mathbb{R}$) for all $k \geq 1$ but $P_k$ is not even dense in $\mathbb{Z}_p$ (resp. $\mathbb{Z}$) for any $k$ divisible by $p$ (resp. for any $k > 1$): compare with Corollary 1.3 of [4].

Theorem 1.3. Let $F$ be a non-Archimedean local field and $G$ be an algebraic group defined over $F$. Then we have the following:

(1) If $P_k$ is surjective on $G(F)$ and $H$ is an algebraic subgroup of $G$ defined over $F$, then $P_k$ is surjective on $H(F)$.

(2) If $H$ is a closed (not necessarily algebraic) normal subgroup in $G(F)$ and $P_k$ is dense in $H$ as well as in $G(F)/H$, then $P_k$ is surjective on $G(F)$.

In particular, for any algebraic normal subgroup $H$ of $G$ defined over $F$, $P_k$ is surjective on $G(F)$ if and only if $P_k$ is surjective on both $H(F)$ and $G(F)/H(F)$.

We have the following corollary regarding infinitely divisible algebraic groups over non-Archimedean local fields.

Corollary 1.4. Let $G$ be an algebraic group over a non-Archimedean local field. Suppose $P_k$ is surjective on $G(F)$ for all $k \in \mathbb{N}$. Then $G(F)$ is unipotent. In addition if characteristic of $F$ is positive, then $G(F) = \{e\}$.

The above results are proved using canonical realization of algebraic groups as subgroups of matrix groups. We note that apart from algebraic groups, there are some other groups that have interesting and enough linear representations. Therefore, we consider linear representation of groups and prove the results for general groups in terms of their linear representations.

Recall that a linear representation of a group $G$ over a local field $F$ is a continuous homomorphism $\rho: G \to GL(n, F)$.

Apart from algebraic groups, Lie groups over local fields is an interesting class that admit a linear (not necessarily injective) representation, namely the adjoint representation: recall that Ad is the adjoint representation of $G$ defined on the Lie algebra of $G$ (see [2] and [16] for more information on Lie groups).

Various classes of $p$-adic Lie groups were introduced using the adjoint representation and interesting results were obtained (ref. [15] and [14]). Motivated by
these studies we now introduce the following: let $\rho: G \to \text{GL}(n, \mathbb{F})$ be a linear representation of $G$.

We say that $G$ is called $\rho$-type if eigenvalues of $\rho(g)$ are of absolute value 1 for all $g \in G$ and $G$ is called $\rho$-unipotent (resp., $\rho$-compact) if $\rho(G)$ is contained in an unipotent (resp., compact) subgroup of $\text{GL}(n, \mathbb{F})$.

We now give the results for Lie groups.

**Theorem 1.5.** Let $G$ be a Lie group over a non-Archimedean local field $\mathbb{F}$ and $P_k$ is dense in $G$ for $k > 1$. Then we have the following:

1. $G$ is type $R$.
2. $\text{Ad}(G)$ is contained in a compact extension of an unipotent normal subgroup.
3. If $G$ is compactly generated, then $G$ is $\text{Ad}$-compact.
4. If the residue characteristic divides $k$ and the characteristic of $\mathbb{F}$ is zero, then $\text{Ad}(G)$ is a finite extension of an unipotent group.
5. If the characteristic $p > 0$ divides $k$, $\text{Ad}(G)$ is finite.
6. If $P_k$ is dense in $G$ for all $k \geq 1$, then $\text{Ad}(G)$ is a $\mathbb{F}$-split unipotent group, in particular, $G$ is $\text{Ad}$-unipotent. In addition if the characteristic of $\mathbb{F}$ is positive, then $\text{Ad}$ is trivial.

2 Preliminaries

2.1 Semi-direct product

Let $H$ and $N$ be locally compact groups. We say that $H$ acts on $N$ by automorphisms if there is a homomorphism $\phi: H \to \text{Aut}(N)$ such that the map $(g, x) \mapsto \phi(g)(x)$ is continuous. In this case we define the semi-direct product $H \ltimes N$ of $H$ and $N$ as the product space $H \times N$ with binary operation:

$$(g, x)(h, y) = (gh, xg(y))$$

for all $g, h \in G$ and $x, y \in X$.

Then $H \ltimes N$ is a locally compact group. Identifying $g \in H$ with $(g, e) \in H \ltimes N$ and $x \in N$ with $(e, x) \in H \ltimes N$, we may view $H$ and $N$ as closed subgroups of $H \ltimes N$. This in particular implies that $N$ is a normal subgroup of $H \ltimes N$. Semi-direct product is a useful technique, particularly helps us to prove Lemma 3.1.

**Example 2.1.** 1. Any closed subgroup of $\text{GL}(n, \mathbb{F})$ acts on $\mathbb{F}^n$ by linear transformations for a local field $\mathbb{F}$. 
2. Generally, any closed subgroup of $\text{Aut}(G)$ acts canonically on $G$ by automorphisms.

3. Given two closed subgroups $H$ and $N$ of a locally compact group $G$ such that $N$ is normalized by $H$. Then taking $\phi(g)$ to be inner automorphism restricted to $N$, defines an action of $H$ on $N$ by automorphisms. In this case, if $HN = G$, then $G$ is called semidirect product of $H$ and $N$.

### 2.2 Profinite groups

A topological group $G$ is said to be profinite if $G$ is a inverse limit of finite groups. It is easy to see that any profinite group is a totally disconnected compact group and the converse is also true (see Corollary 1.2.4 of [24]).

A Steinitz number or supernatural number is a formal infinite product $\prod p^{n(p)}$, over all primes $p$, where $n(p)$ is a non-negative integer or infinity. One may define the product and l.c.m of super-natural numbers in the natural way: see 2.1 of [24].

For a finite set $X$, Ord($X$) denotes the order of $X$ which is the number of elements in $X$.

For a pro-finite group $G$, the order of $G$ (possibly a supernatural number) denoted by Ord($G$) is defined by

$$\text{Ord}(G) := \text{l.c.m}\{\text{Ord}(G/U) : \text{for any open subgroup } U \subset G\}.$$  

Since $G$ has arbitrarily small compact open normal subgroups, we may replace open subgroups in the above definition of order, by open normal subgroups.

We recall the following result from [17] and include a simple proof.

**Proposition 2.2.** Let $G$ be a profinite group. Then $P_k : G \to G$ is surjective if and only if $k$ is coprime to Ord($G$), that is for any compact open normal subgroup $U$ of $G$, $k$ is coprime to the order of the finite group $G/U$.

**Proof.** It is sufficient to prove the if direction. Since $G$ is a profinite group, $G$ has a basis ($U_i$) of compact open normal subgroups at $e$. If $k$ is co-prime to the order of $G/U_i$ for all $i$, then $P_k$ is surjective $G/U_i$ for all $i$, hence for $g \in G$ there exist $x_i \in G$ and $u_i \in U_i$ such that $g = x_i^k u_i$. Since ($U_i$) is a basis at $e$, ($gU_i$) is a basis at $g$, hence any neighborhood of $g$ contains $x_i^k$ for some $i$. Thus, $P_k$ is dense in $G$. Since $G$ is compact, $P_k$ is surjective on $G$.  

The following lemma is easy to see and known and will be used often.

**Lemma 2.3.** Let $G$ be a profinite group and $H$ be a closed subgroup of $G$. If $P_k$ is surjective on $G$ then $P_k$ is surjective on $H$. Conversely, if $H$ is normal in $G$ and $P_k$ is surjective on $H$ as well as on $G/H$, then $P_k$ is surjective on $G$.  

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Proof. As $G$ is profinite, $H$ is also profinite. Note that, Lagrange’s Theorem holds for profinite group (see Proposition 2.1.2 of [24]), so $\text{Ord}(H)$ divides $\text{Ord}(G)$. Now, $P_k : G \to G$ is surjective if and only if $(k, \text{Ord}(G)) = 1$. This implies that $(k, \text{Ord}(H)) = 1$, and hence $P_k : H \to H$ is surjective.

Conversely, if $H$ is normal in $G$ and $P_k$ is surjective on both $H$ and $G/H$, then by Proposition 2.2 we get that $k$ is co-prime to both $\text{Ord}(H)$ and $\text{Ord}(G/H)$. By Lagrange’s Theorem on profinite groups, we get that $k$ is co-prime to $\text{Ord}(G)$ (ref. Proposition 2.1.2 of [24]). Another application of Proposition 2.2 proves that $P_k$ is surjective on $G$.

Next lemma relates order of a profinite group and its open subgroups.

**Lemma 2.4.** Let $K$ be a profinite group and $L$ be an open subgroup of $K$. Then $\text{Ord}(K) = \text{l.c.m}\{|K/U| : U \text{ is open normal in } L\} = [K : L]\text{Ord}(L)$.

**Proof.** Note that $L$ is a finite index subgroup in $K$. For any open normal subgroup $U$ of $L$, $|K/U| = |L/U|[K : L]$ (see Proposition 2.1.2 of [24]). This proves the second equality.

Let $V$ be any open normal subgroup of $K$. Then $U = V \cap L$ is an open normal subgroup of $L$. Also, $K/V \cong (K/U)/(V/U)$ and hence $|K/V||V/U| = |K/U|$, that is $K/V$ divides $K/U$. This proves the first equality. □

### 2.3 Algebraic groups

Let $\mathbb{F}$ be a local field and $G$ be defined over $\mathbb{F}$. The $\mathbb{F}$-unipotent (resp., $\mathbb{F}$-split unipotent) radical of $G$ denoted by $R_{u,\mathbb{F}}(G)$ (resp., $R_{us,\mathbb{F}}(G)$) is defined to be the maximal connected unipotent normal subgroup of $G$ that is defined over $\mathbb{F}$ (resp., $\mathbb{F}$-split). These subgroups are contained in the usual unipotent radical of $G$. Every connected linear algebraic $\mathbb{F}$-group $G$ has $R_{us,\mathbb{F}}(G)$. We say that $G$ is pseudo-reductive over $\mathbb{F}$ if $R_{u,\mathbb{F}}(G) = 1$ and $G$ is quasi-reductive over $\mathbb{F}$ if $R_{us,\mathbb{F}}(G) = 1$. Since $R_{us,\mathbb{F}}(G) \subset R_{u,\mathbb{F}}(G)$, every pseudo-reductive $\mathbb{F}$-group is also quasi-reductive. If $\mathbb{F}$ is perfect, then any connected unipotent group defined over $\mathbb{F}$ and is $\mathbb{F}$-split, hence quasi-reductive $\mathbb{F}$-group and pseudo-reductive are equivalent and in fact reductive, that is unipotent radical is trivial. When $\mathbb{F}$ is not perfect, there do exist quasi-reductive $\mathbb{F}$-groups that are not pseudo-reductive (ref. [19]).

A connected linear algebraic group $G$ over a field of characteristic 0 admits a Levi decomposition, that is, it can be written as the semidirect product of its unipotent radical and a reductive subgroup known as Levi factor. In the case the field $\mathbb{F}$ is of positive characteristic, Levi factors need not exist, even if $\mathbb{F}$ is algebraically closed (cf. [7, A.6]) the unipotent radical need not be defined over $\mathbb{F}$.

However, we can get following two short exact sequences.

$$1 \to R_{u,\mathbb{F}}(G) \to G \to G/R_{u,\mathbb{F}}(G) \to 1.$$
and
\[ 1 \to R_{us,F}(G) \to G \to G/R_{us,F}(G) \to 1. \]

Let \( N = R_{us,F}(G) \) and \( M = R_{u,F}(G) \). Let \( P = G/R_{u,F}(G) \) and \( Q = G/R_{us,F}(G) \).

The map \( G(F) \to (G/N)(F) \) is a surjective submersion of \( F \)-analytic manifolds. In particular, this map induces an isomorphism between \( G(F)/R_{us,F}(G)(F) \) and \( Q(F) \) as \( H^1(F, R_{us,F}(G)) = 0. \)

### 2.4 \( F \)-nilpotent groups

By an \( F \)-nilpotent group, we mean a nilpotent group \( N \) such that if \( N = N_0 \supset N_1 \supset \cdots \supset N_m = \{e\} \) is a central series of \( N \), then each \( N_j/N_{j+1} \) is a finite-dimensional \( F \)-vector space. Let \( N \) be an \( F \)-nilpotent group and \( N_j \) be a central series. Let \( G \) be a group acting on \( N \) as a group of automorphisms of \( N \). The \( G \)-action on \( N \) is said to be \( F \)-linear if the induced action of \( G \) on \( N_j/N_{j+1} \) is \( F \)-linear for all \( j \).

**Fact 1 (see [9]):** Let \( G \) be a group and \( N \) a normal subgroup of \( G \). Suppose that \( N \) is \( F \)-nilpotent with respect to a field \( F \), and that the conjugation action of \( G \) on \( N \) is \( F \)-linear. Let \( N = N_0 \supset N_1 \supset \cdots \supset N_r = \{e\} \) be the central series of \( N \). Let \( A = G/N \), \( x \in G \) and \( a = xN \in A \). Let \( k \in \mathbb{N} \) be co-prime to the characteristic of \( F \). Let \( B = \{b \in A \mid b^k = a\} \) and let \( B^* \) be the subset consisting of all \( b \) in \( B \) such that for any \( j \), any element of \( N_j/N_{j+1} \) which is fixed under the action of \( a \) is also fixed under the action of \( b \). Then for any \( b \in B^* \), \( u \in N \), there exists \( y \in G \) such that \( yN = b \) and \( y^k = xu \). But when \( G/N \) is a profinite group, we have the following:

**Lemma 2.5.** Let \( F \) be a non-archimedean local field and \( G \) be a locally compact group containing a closed normal subgroup \( N \) such that \( N \) is \( F \)-nilpotent and \( G/N \) is a profinite group. Assume that the conjugation action of \( G \) on \( N \) is \( F \)-linear. Suppose \( k \) is co-prime to the characteristic of \( F \) and \( P_k \) is surjective on \( G/N \). Then \( P_k \) is surjective on \( G \).

**Proof.** Let \( x \in G \) and \( a = xN \). Since \( P_k \) is surjective on the compact group \( G/N \), Lemma 2.3 implies that \( P_k \) is surjective on \( \langle a \rangle \). Let \( N = N_0 \supset N_1 \supset \cdots \supset N_r = e \) be the central series of \( N \). We observe that, for each \( j \), any element of \( N_j/N_{j+1} \) fixed by \( a \) is also fixed by the group \( \langle a \rangle \). In view of Fact 1, we conclude that \( x \in P_k(G) \), that is \( P_k \) is surjective on \( G \). \( \square \)

### 2.5 Scale function

Let \( G \) be a locally compact totally disconnected group. Then \( G \) has arbitrarily small compact open subgroups \( U \) of \( G \). Let \( \text{Cos}(G) \) be the set of all compact open subgroup of \( G \).
Let Aut($G$) be the collection of all (continuous) automorphisms of $G$. Then the scale function $s : \text{Aut}(G) \rightarrow \mathbb{N}$ is defined as follows:

$$s(\alpha) := \min\{[\alpha(U) : U \cap \alpha(U)]|U \in \text{Cos}(G)\}$$

for any $\alpha \in \text{Aut}(G)$ and the compact open subgroup for which the minimum is attained is called tidy subgroup of $\alpha$ (see [22]). The scale function was introduced by G. Willis [21] and it has proved to be useful. A property of scale function that we often uses is the following: $s(\alpha) = 1 = s(\alpha^{-1})$ if and only if $G$ contains a $\alpha$-invariant compact open subgroup (ref. Proposition 4.3 of [21]).

For each $x \in G$, let $\alpha_x : G \rightarrow G$ be the inner-automorphism defined by $x$, that is $\alpha_x(y) = xyx^{-1}$ for all $y \in G$. Now define $s(x) = s(\alpha_x)$ and the tidy subgroup of $\alpha_x$ is defined to be the tidy subgroup of $x$.

It is known that $s(\alpha^n) = s(\alpha)^n$: see [22].

### 3 Representations and Lie groups

We now prove a dynamic consequence of density of the power map using semidirect product technique and scale function: recall that $P_k(x) = x^k$ is the $k$-th power map for $k \geq 1$.

**Lemma 3.1.** Let $G$ be a locally compact totally disconnected group and $K$ be a compact normal subgroup of $G$. Suppose $G$ acts on a totally disconnected locally compact group $X$ by automorphisms. Then we have the following:

1. If $g \in \overline{P_k(G)K}$ for infinitely many $k$, then the $g$ action on $X$ fixes a compact open subgroup $K_g$ of $X$.

2. If $P_k$ is dense in $G/K$ for some $k > 1$, then the $g$ action on $X$ fixes a compact open subgroup $K_g$ of $X$.

**Proof.** Let $Y = G \ltimes X$ be the semidirect product of $G$ and $X$ for the given action of $G$ on $X$. Then $Y$ is a totally disconnected locally compact group containing $G$ as a closed subgroup and $X$ as a closed normal subgroup. Let $s$ be the scale function on $Y$. Then $s$ is continuous on $Y$ (ref. Corollary 4 of [21]).

Let $V$ be a compact open subgroup of $G$ containing $K$ and $W$ be a compact open subgroup of $X$ fixed by $V$. Then $V \ltimes W$ is a compact open subgroup of $Y$ invariant under conjugation by elements of $K$.

Let $x \in G$. Then by Lemma 4.2 of [18], the group generated by $x$ and $K$ has a common tidy subgroup in $Y$. This implies by Corollary 2.7 of [18] that $s(ab) \leq s(a)s(b)$ for all $a$ and $b$ in the group generated by $x$ and $K$ (see also
Proposition 7.2 of [10]). Since \( s(h) = 1 \) for all \( h \in K \), we have \( s(xh) \leq s(x) \leq s(xh)s(h^{-1}) = s(xh) \). Therefore, \( s(xh) = s(x) \) for all \( h \in K \).

For \( g \in G \), let \( V_g = \{ x \in G \mid s(x) = s(g) \} \). Then since \( s \) is continuous on \( Y \), \( V_g \) is an open neighborhood of \( g \) in \( G \) such that \( V_gK = V_g \).

Suppose \( g \in \overline{P_k(G)K} \) for infinitely many \( k \). Then there are infinitely many \( k \) such that \( x_k^k \in V_g \) for some \( x_k \in G \). This implies that \( s(g) = s(x_k)^k \) for infinitely many \( k \). Thus, \( s(g) \in \mathbb{N} \) has infinitely many roots, hence \( s(g) = 1 \). Similarly \( s(g^{-1}) = 1 \). Now the first part follows from Proposition 4.3 of [21].

Suppose \( P_k \) is dense in \( G/K \) for some \( k > 1 \). Let \( g \in G \). Then there is a \( x_1 \in G \) such that \( x_1^k \in V_g \), hence \( s(x_1)^k = s(g) \). Now by considering \( V_{x_1} \), there is a \( x_2 \in G \) such that \( x_2^k \in V_{x_1} \). This implies that \( s(x_2)^k = s(x_1)^k = s(g) \). Inductively, we get a sequence \( (x_n) \) in \( G \) such that \( s(x_n)^k = s(g) \). Thus, \( s(g) \in \mathbb{N} \) has infinitely many roots, hence \( s(g) = 1 \). Similarly \( s(g^{-1}) = 1 \). Now the second part follows from Proposition 4.3 of [21].

In case, the dynamics in the above Lemma 3.1 is linear, then we can proceed further.

**Lemma 3.2.** Let \( G \) be a locally compact totally disconnected group and \( V \) be a finite-dimensional vector space over a non-Archimedean field \( \mathbb{F} \). Suppose \( \rho: G \to \text{GL}(V) \) is the map defining an action of \( G \) on \( V \) and \( s(x) = 1 \) for all \( x \in G \) where \( s \) is the scale function on \( V \). Then \( G \) is \( \rho \)-type \( R \) and there exists a compact group \( K \subseteq \text{GL}(V) \) and a \( \mathbb{F} \)-split unipotent algebraic group \( U \subseteq \text{GL}(V) \) normalized by \( K \) such that \( K \cap U \) is trivial, \( \rho(G) \subseteq KU \) and \( \rho(G)U \) is dense in \( KU \).

**Proof.** Let \( L = \rho(G) \). Then \( s(g) = 1 \) for all \( g \in L \), hence each \( g \in L \) fixes a compact open subgroup of \( V \). Therefore, it follows that all eigenvalues of \( g \in L \subseteq \text{GL}(V) \) are absolute value 1. Thus \( G \) is \( \rho \)-type \( R \).

Now it follows from Theorem 1 of [8] that there exists a flag \( \{0\} = V_0 \subset V_1 \subset \cdots \subset V_{m-1} \subset V_m = V \) such that \( L \) on \( V_i/V_{i-1} \) has only bounded orbits for any \( i \geq 1 \). Let \( U = \{ \alpha \in \text{GL}(V) \mid \alpha(v + V_{i-1}) = v + V_{i-1} \text{ for all } v \in V_i \text{ and } i \geq 1 \} \) and \( K \) be the direct product of closure of the image of \( L \) in \( \text{GL}(V_i/V_{i-1}) \). Then \( U \) is a split unipotent algebraic group and \( K \) is a compact group that normalizes \( U \) such that \( K \cap U \) is trivial and \( L \subseteq KU \) - note that \( KU \) is the semidirect product \( K \ltimes U \) of \( K \). This implies that \( KU/U \simeq K \), hence replacing \( K \simeq KU/U \) by the closure of \( LU/U \subseteq KU/U \), we may assume that \( LU \) is dense in \( KU \).

We next obtain results in terms of the representation of the groups.

**Proposition 3.3.** Let \( \mathbb{F} \) be a non-Archimedean local field and \( H \) be a group with a linear representation \( \rho: H \to \text{GL}(d, \mathbb{F}) \). Suppose that \( P_k \) is dense in \( H \) for some \( k > 1 \). Then we have the following:

1. \( H \) is \( \rho \)-type \( R \).
(2) There exists a compact group $K \subset GL(d, \mathbb{F})$ and a split unipotent algebraic group $U \subset GL(d, \mathbb{F})$ normalized by $K$ such that $K \cap U$ is trivial, $\rho(H) \subset KU$ and $\rho(H)U$ is dense in $KU$. Moreover, $P_k$ is surjective on $KU/U \simeq K$.

(3) If $k$ is co-prime to the characteristic of $\mathbb{F}$, then $P_k$ is surjective on $KU$.

(4) If the residual characteristic $p$ of $\mathbb{F}$ divides $k$, then $K$ is finite, that is $\rho(H)$ is contained in a finite extension of a split unipotent algebraic group $U$ and $P_k$ is dense in $\rho(H) \cap U$.

(5) If the residual characteristic $p$ of $\mathbb{F}$ divides $k$ and the characteristic of $\mathbb{F}$ is zero, then $\rho(H)$ is a finite extension of a split unipotent algebraic group.

(6) If the characteristic $p$ of $\mathbb{F}$ divides $k$, then $\rho(H)$ is finite.

(7) If $G$ is an $\mathbb{F}$-group and $\rho(H) = G(\mathbb{F})$, then $G$ has no $\mathbb{F}$-split torus.

Remark 3.4. The above results (5) and (6) generalize Corollary 1.7 of [5] to any linear group (not necessarily algebraic) over any non-Archimedean local field.

Proof of Proposition 3.3: Let $L = \rho(H)$. Then $P_k$ is dense in $L$. By Lemma 3.1, each $g \in L$ fixes a compact open subgroup of $\mathbb{F}^n$. Therefore, it follows that all eigenvalues of $g \in L \subset GL(d, \mathbb{F})$ are absolute value 1. This proves (1).

It follows from Lemma 3.1 that the scale function is trivial on $L$. Therefore by Lemma 3.2, there exists a compact group $K \subset GL(d, \mathbb{F})$ and a split unipotent algebraic group $U \subset GL(d, \mathbb{F})$ normalized by $K$ such that $K \cap U$ is trivial, $\rho(H) \subset KU$ and $\rho(H)U$ is dense in $KU$. Since $P_k$ is dense in $L$, $P_k$ is dense in $LU/U$. Since $LU$ is dense in $KU$ and $KU/U$ is compact, we get that $P_k$ is surjective on $KU/U \simeq K$. This proves (2).

Suppose $k$ is co-prime to the characteristic of $\mathbb{F}$. Then since $P_k$ is surjective on the compact group, $KU/U$, Lemma 2.5 implies that $P_k$ is surjective on $KU$. This proves (3).

Suppose the residual characteristic $p$ of $\mathbb{F}$ divides $k$. Since $K$ is a compact linear group over $\mathbb{F}$, we get that $K$ contains an open subgroup $K_0$ such that $K_0$ is pro-$p$ group. By Lemma 2.3, $P_k$ is surjective on $K_0$. Since $p$ divides $k$, $K_0$ is trivial. This implies that $K$ is a finite group and hence $U$ is an open subgroup of $KU$. Therefore, $L \cap U$ is open in $L$. Since $P_k$ is dense in $L$, $P_k(L) \cap U$ is dense in $L \cap U$. Let $g \in P_k(L) \cap U$. Then there exist $x \in L$ such that $x^k = g \in L \cap U$. Since $L \subset KU$, there exist $a \in K$ and $v \in U$ such that $x = av$, hence $x^k = (av)^k = a^kv_k$ where $v_k = \prod_{j=1}^k a^{-k+j}va^{k-j} \in U$. This implies that $a^k \in K \cap U$. Since $K \cap U$ is trivial, $a^k = e$, hence $k$ divides the order of $K$ or $a = e$. Since $P_k$ is surjective on $K$, by Proposition 2.2, we get that $a = e$. Thus, $x = v \in L \cap U$. Therefore, $g \in P_k(L \cap U)$. Thus, $P_k$ is dense in $L \cap U$. This proves (4).
Suppose the characteristic of $\mathbb{F}$ is zero and the residual characteristic $p$ of $\mathbb{F}$ divides $k$. Since $P_k$ is dense in $L$, $P_k$ is dense in $\overline{L}$. Therefore, replacing $L$ by $\overline{L}$, we may assume that $L$ is closed. Since $L/L \cap U \simeq LU/U$ is finite, it is sufficient to claim that $L \cap U$ is a unipotent group. By (4), $P_k$ is dense in $L \cap U$, we may assume that $L$ is a closed subgroup of $U$ and $U$ is the smallest unipotent group containing $L$. If $U$ is abelian, then $U$ is the vector space spanned by $L$. Let $V$ the maximal vector space contained in $L$. Then $L/V$ is compact. Since $P_k$ is dense in $L$, $P_k$ is surjective on $L/V$ but $p$ divides $k$, hence $L/V$ is finite. Since $L/V$ is a subgroup of the unipotent group $U/V$ which has no elements of finite order, we get that $L = V$, hence $L = U$. If $U$ is a general unipotent group, let $Z$ be the center of $U$. Then since $P_k$ is dense in $L$, $P_k$ dense in $LZ/Z \subset U/Z$, hence by induction $\overline{LZ} = U$. This proves that $[U,U] \subset L$. Using the commutative case for $U/[U,U]$, we may conclude that $L = U$. This proves (5).

Suppose characteristic of $\mathbb{F}$ divides $k$. Since $U$ is a split unipotent group and $p$ divides $k$, we get that $P_k(U) = \{e\}$. Since $P_k$ is dense in $L \cap U$, $L \cap U$ is trivial. Since $L$ is contained in a finite extension of $U$, $L$ is finite. This proves (6).

Suppose $H$ is the group of $\mathbb{F}$-points of an algebraic group $G$ defined over $\mathbb{F}$. Then the set of eigenvalues of elements of any non-trivial split torus in $H$ is $\mathbb{F}^*$. Thus, (1) implies that $G$ has no $\mathbb{F}$-split torus.

It is known that $P_k$ is not dense in any finitely generated infinite abelian groups: any such group is isomorphic to $F \times \mathbb{Z}^d$ for $d \geq 1$ for some finite group $F$, hence have $\mathbb{Z}$ as a quotient. We extend this to compactly generated groups and its linear representations over non-Archimedean local fields.

**Corollary 3.5.** Let $\mathbb{F}$ be a non-Archimedean local field and $H$ be a group with a linear representation $\rho: H \to GL(d, \mathbb{F})$. If $P_k$ is dense in $H$ for some $k > 1$ and $H$ is compactly generated then $H$ is $\rho$-compact.

**Proof.** Let $L = \rho(H)$. Then $P_k$ is dense in $L$. By (2) of Proposition 3.3, there is a unipotent group $U$ and a compact linear group $K$ normalizing $U$ such that $L \subset KU$. Let $C$ be a compact generating subset of $L$. Then $C \subset KM$ where $M$ is a compact subgroup of $U$. Since $K$ normalizes $U$, any compact subset of $U$ is a contained in a compact $K$-invariant subset of $U$. Thus, we may assume that $M$ is a $K$-invariant compact subgroup. This implies that $KM$ is a compact subgroup. Since $C \subset KM$ and $L$ is generated by $C$, we get that $\overline{L}$ is compact. \qed

The next result shows that unipotent groups are the only infinite divisible linear groups.

**Corollary 3.6.** Let $\mathbb{F}$ be a non-Archimedean local field and $H$ be a group with a linear representation $\rho: H \to GL(d, \mathbb{F})$. Suppose $P_k$ is dense in $H$ for all $k$. Then $\rho(H)$ is a split unipotent algebraic group. In addition if $\mathbb{F}$ has positive characteristic, $\rho$ is trivial.
**Proof.** If the characteristic of $\mathbb{F}$ is zero, then by (5) of Proposition 3.3, $\overline{\rho(H)}$ is a finite extension of an unipotent algebraic group $U$. This implies that $\overline{\rho(H)/U} = (\rho(H)U)/U$. Let $k = |\overline{\rho(H)/U}|$. Then since $P_k$ is dense in $H$, $P_k$ is surjective on $\rho(H)U/U = \overline{\rho(H)/U}$ which has order $k$, hence $\rho(H)/U$ is trivial. Thus, $\overline{\rho(H)} = U$.

If $\mathbb{F}$ has positive characteristic, then by (6) of Proposition 3.3, $\rho(H)$ is a finite group, hence $\rho(H) = \rho(H)/\rho(U)$. This implies that $U$ is finite, hence trivial. □

For subgroups of linear groups over global fields: compare with Section 6 of [4].

**Corollary 3.7.** Let $\mathbb{E}$ be a global field and $H$ be a subgroup of $\text{GL}(d, \mathbb{E})$. Assume that $P_k$ is surjective on $H$ for some $k > 1$.

1. If the characteristic of $\mathbb{E}$ is 0, then $H$ contains an unipotent normal subgroup of finite index.

2. If the characteristic of $\mathbb{E}$ is $p > 0$, then $H$ is locally finite, that is any finitely generated subgroup of $H$ is finite.

3. If the characteristic $p$ of $\mathbb{E}$ divides $k$, then $H$ is finite.

4. If $P_k$ is surjective on $H$ for all $k \geq 1$, then either $H$ is a unipotent group or $H$ is trivial depending on characteristic of $\mathbb{E}$ is 0 or positive.

**Proof.** Suppose the characteristic of $\mathbb{E}$ is 0. Let $E_p$ be the $p$-adic completion for $p$ dividing $k$. Then by (5) of Proposition 3.3, $H$ is contains an unipotent normal subgroup of finite index.

Suppose the characteristic of $\mathbb{E}$ is $p > 0$. Then any completion $E_v$ of $\mathbb{E}$ is non-Archimedean of characteristic $p > 0$. By (2) of Proposition 3.3, $H$ is contained in a compact extension of a split unipotent group in $\text{GL}(d, E_v)$. If $N$ is a finitely generated subgroup of $H$, then $N$ is contained in a compact subgroup of $\text{GL}(d, E_v)$. This implies that $N$ is finite.

Suppose the characteristic $p$ of $\mathbb{E}$ divides $k$. Then by (6) of Proposition 3.3, in any completion of $\mathbb{E}$, $H$ is finite. Thus, $H$ is finite.

Last part follows from 1, 3 and Proposition 2.2. □

**Proof of Theorem 1.5:** Let $d$ be the dimension of $G$. Then $\text{Ad}(G)$ is a subgroup of $\text{GL}(d, \mathbb{F})$. Thus, Proposition 3.3 and its Corollaries 3.5, 3.6.

### 4 Proof of Theorems 1.1, 1.3 and Corollary 1.4

We first deal the following:
Lemma 4.1. Let \( \mathbb{F} \) be a non-Archimedean local field and \( H \) be a closed subgroup of \( \text{GL}(d, \mathbb{F}) \). Suppose the smallest algebraic group over \( \mathbb{F} \) containing \( H \) has no \( \mathbb{F} \)-split unipotent normal subgroup. Then \( P_k \) is dense in \( H \) if and only if \( H \) is compact and \( (k, \text{Ord}(H)) = 1 \) (if and only if \( P_k \) is surjective on \( H \)).

**Proof.** Let \( s \) be the scale function on \( \mathbb{F}^d \). Applying Lemma 3.1 to the canonical action of \( H \) on \( \mathbb{F}^d \), we get that \( s(x) = 1 \) for all \( x \in H \). Now by Lemma 3.2, there exists a compact group \( K \subset \text{GL}(d, \mathbb{F}) \) and a split unipotent algebraic group \( U \subset \text{GL}(d, \mathbb{F}) \) normalized by \( K \) such that \( K \cap U \) is trivial, \( H \subset KU \) and \( HU \) is dense in \( KU \). Let \( G \) be the normalizer of \( U \). Then \( G \) is defined over \( \mathbb{F} \) and contains \( H \). Thus, the smallest algebraic group, say \( G_1 \), defined over \( \mathbb{F} \) containing \( H \) normalizes \( U \). Since \( G_1 \) and \( U \) are algebraic groups, \( G_1 \mathbb{F} U \) is closed. This implies that \( KU \subset G_1 \mathbb{F} U \). Since \( G_1 \) has no \( \mathbb{F} \)-split unipotent normal subgroup, \( G_1 \mathbb{F} \cap U \) is trivial, hence the map \( f : G_1 \mathbb{F} \rightarrow G_1 \mathbb{F} U / U \) given by \( f(g) = gU \) is an isomorphism of topological groups. In particular, \( f(H) \) is a closed subgroup. Since \( HU \subset KU \), \( f(H) \) is a closed subgroup of \( KU/U \). Therefore \( f(H) \) is compact.

Since \( f \) is an isomorphisms, \( H \) is compact. \( \square \)

**Proof of Theorem 1.1:** Let \( N = R_{us,\mathbb{F}}(G) \) and \( Q = G/R_{us,\mathbb{F}}(G) \).

\((a) \Rightarrow (b) : \) Recall that \( G(\mathbb{F})/R_{us,\mathbb{F}}(G)(\mathbb{F}) \) is isomorphic to \( Q(\mathbb{F}) \) as \( H^1(\mathbb{F}, R_{us,\mathbb{F}}(G)) = 0 \). Since \( P_k \) is dense in \( G(\mathbb{F}) \), \( P_k \) is dense in \( G(\mathbb{F})/R_{us,\mathbb{F}}(G)(\mathbb{F}) \) and hence by Lemma 4.1, we get that \( G(\mathbb{F})/R_{us,\mathbb{F}}(G)(\mathbb{F}) \) is compact. Compactness of \( G(\mathbb{F})/R_{us,\mathbb{F}}(G)(\mathbb{F}) \) implies that \( P_k \) is surjective on \( G(\mathbb{F})/R_{us,\mathbb{F}}(G)(\mathbb{F}) \).

We next observe that \((b) \Rightarrow (c) \) follows from Lemma 2.5 and that \((c) \Rightarrow (a) \) is trivial. \( \square \)

**Proof of Theorem 1.3:** Suppose that \( P_k : G(\mathbb{F}) \rightarrow G(\mathbb{F}) \) is surjective. Let \( N = R_{us,\mathbb{F}}(G) \), and \( N_H = R_{us,\mathbb{F}}(H) \).

Assume that the characteristic of \( \mathbb{F} \) does not divide \( k \). By Theorem 1.1, we have \( G(\mathbb{F})/N(\mathbb{F}) \) is compact and \( P_k \) is surjective on \( G(\mathbb{F})/N(\mathbb{F}) \). Let \( \phi : G(\mathbb{F}) \rightarrow G(\mathbb{F})/N(\mathbb{F}) \) be the canonical quotient. Then \( \phi(H(\mathbb{F})) \) is a closed subgroup of \( G(\mathbb{F})/N(\mathbb{F}) \). By Lemma 2.3, we have \( P_k \) is surjective on \( \phi(H(\mathbb{F})) \). Since \( \ker \phi \cap H(\mathbb{F}) = N(\mathbb{F}) \cap H(\mathbb{F}) \subset N_H(\mathbb{F}) \), we get that \( H(\mathbb{F})/N_H(\mathbb{F}) \) is a quotient of \( \phi(H(\mathbb{F})) \), hence \( H(\mathbb{F})/N_H(\mathbb{F}) \) is compact and \( P_k \) is surjective on \( H(\mathbb{F})/N_H(\mathbb{F}) \). Now the result follows from Theorem 1.1.

Suppose the characteristic of \( \mathbb{F} \) divides \( k \). Then \( G(\mathbb{F}) \) is finite, hence the result follows from Lemma 2.3.

Now let \( H \) be a closed normal subgroup of \( G(\mathbb{F}) \) and \( P_k \) is dense in \( H \) as well as in \( G(\mathbb{F})/H \).

If characteristic of \( \mathbb{F} \) divides \( k \), then by (6) of Proposition 3.3, \( H \) is finite and \( (\text{Ord}(H), k) = 1 \). Since \( H \) is finite, \( G(\mathbb{F})/H \) is also linear group, hence (6) of Proposition 3.3 implies that \( G(\mathbb{F})/H \) is also finite and \( k \) is co-prime to the order.
of \( G(\mathbb{F})/H \). Thus, \( G(\mathbb{F}) \) is finite and \(|G(\mathbb{F})| = |H||G(\mathbb{F})/H|\), hence \( k \) is co-prime to the order of \( G(\mathbb{F}) \). Thus, \( P_k \) is surjective on \( G(\mathbb{F}) \).

We may now assume that the characteristic of \( \mathbb{F} \) does not divide \( k \).

Let \( Q = G/R_{us,\mathbb{F}}(G) \) and \( M = G(\mathbb{F})/R_{us,\mathbb{F}}(G)(\mathbb{F}) \). Then \( M \) is isomorphic to \( Q(\mathbb{F}) \). So we may assume that \( M = Q(\mathbb{F}) \). Let \( M_1 \) be the closure of \( HR_{us,\mathbb{F}}(G)(\mathbb{F})/R_{us,\mathbb{F}}(G)(\mathbb{F}) \). Then \( M_1 \) is a closed normal subgroup of \( M \). Since \( P_k \) is dense in \( H \), \( P_k \) is dense in \( M_1 \). Since \( Q \) has no \( \mathbb{F} \)-split unipotent normal subgroup and \( M_1 \) is a closed subgroup of \( M = Q(\mathbb{F}) \), Lemma 4.1 implies that \( M_1 \) is compact.

Let \( d \geq 1 \) be such that \( M(= Q(\mathbb{F})) \) is a closed subgroup of \( GL(d, \mathbb{F}) \) and \( s \) be the scale function on \( \mathbb{F}^d \). Since \( P_k \) is dense in \( G(\mathbb{F})/H \), \( P_k \) is dense on \( M/M_1 \). Therefore by Lemma 3.1, \( s \) is trivial on \( M \). By Lemma 3.2 implies that there is a compact subgroup \( K \subset GL(d, \mathbb{F}) \) and a split unipotent algebraic group \( U \subset GL(d, \mathbb{F}) \) normalized by \( K \) such that \( K \cap U \) is trivial, \( M \subset KU \) and \( MU \) is dense in \( KU \). Since \( Q \) is an algebraic group and \( M = Q(\mathbb{F}) \), \( MU \) is closed, hence \( MU = KU \) and since \( Q \) has no \( \mathbb{F} \)-split unipotent normal subgroup, \( M \cap U \) is trivial. Therefore, \( M \simeq MU/U = KU/U \simeq K \). Thus, \( M \) is compact. Therefore, \( P_k \) is surjective on the compact groups \( M_1 \) and \( M/M_1 \), hence by Lemma 2.3 we get that \( P_k \) is surjective on the compact group \( M = G(\mathbb{F})/R_{us,\mathbb{F}}(G)(\mathbb{F}) \). Now applying Lemma 2.5, we get that \( P_k \) is surjective on \( G(\mathbb{F}) \).

\[ \square \]

**Proof of Corollary 1.4** Since \( G(\mathbb{F}) \) is a closed subgroup of \( GL(d, \mathbb{F}) \) for some \( d \geq 1 \), Corollary 3.6 implies that \( G(\mathbb{F}) \) is split unipotent.

If \( p > 0 \) is the characteristic of \( \mathbb{F} \), then \( P_p \) is surjective on \( R_{us,\mathbb{F}}(G) \) implies that \( G(\mathbb{F}) = R_{us,\mathbb{F}}(G) \) is trivial.

\[ \square \]

## 5 Lattices

We now consider the situation when the group \( G \) has a finite co-volume subgroup or a co-compact subgroup on which \( P_k \) is dense.

**Proposition 5.1.** Let \( G \) be a totally disconnected locally compact group acting on a totally disconnected locally compact group \( X \) by automorphisms. Suppose \( G \) has a finite co-volume or cocompact subgroup \( H \) and \( P_k \) is dense in \( H \). Then every element of \( G \) fixes a compact open subgroup of \( X \).

**Proof.** Since \( P_k \) is dense in \( H \), by Lemma 3.1 applied to the conjugate action of \( H \) of \( G \), we get that every element of \( H \) normalizes a compact open subgroup of \( G \). This implies that the modular function of \( G \) is trivial on \( H \) and \( H \) is unimodular. If \( G/H \) is compact, we get that \( G \) is unimodular. Thus, \( G/H \) has an invariant measure. Since \( G/H \) is compact, \( G/H \) has finite volume. Thus, we may assume that \( H \) has finite co-volume.

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Since every element of $H$ normalizes a compact open subgroup of $G$, by Theorem 2.5 of \cite{23}, every element of $G$ also normalizes a compact open subgroup of $G$.

Let $g \in G$ and $V$ be a compact open subgroup of $G$ normalized by $g$. Then there exists a $n \geq 1$ such that $g^nV \cap H \neq \emptyset$. Let $h \in H$ be such that $h \in g^nV$.

Let $s$ be the scale function on $X$. Then by Lemma 4.2 of \cite{18}, the group generated by $g$ and $V$ has a common tidy subgroup in $X$. This implies by Corollary 2.7 of \cite{18} that $s(ab) \leq s(a)s(b)$ for all $a$ and $b$ in the group generated by $g$ and $V$ (see also Proposition 7.2 of \cite{10}). Since $s(a) = 1$ for all $a \in V$, we have

$$s(g^n a) \leq s(g^n) \leq s(g^n) s(a^{-1}) = s(g^n a).$$

Therefore, $s(g^n a) = s(g^n)$ for all $a \in V$. Since $h \in g^nV$, $s(h) = s(g^n)$. It now follows from Lemma 3.1 that $1 = s(h) = s(g^n) = s(g)^n$, hence $s(g) = 1$.

It can easily be seen that we can’t expect $P_k$ to be dense in $G$ if $P_k$ is dense in a co-compact or a finite co-volume subgroup $H$. For example, take $H$ to be the trivial subgroup of a compact group which forces us to ask is this the only obstruction. In case $G$ is a group of $\mathbb{F}$-points of an algebraic group defined over $\mathbb{F}$, we have the following affirmative answer.

**Proposition 5.2.** Let $G$ be an algebraic group defined over a non-Archimedean local field $\mathbb{F}$ and $H$ be a closed subgroup of $G(\mathbb{F})$ with finite co-volume or co-compact. Suppose $P_k$ is dense in $H$. Then we have the following:

1. $G(\mathbb{F})$ is a compact extension of $R_{us,\mathbb{F}}(G)(\mathbb{F})$.

2. If the residual characteristic of $\mathbb{F}$ does not divide $k$, then $G(\mathbb{F})$ contains an open subgroup $G_0$ of finite index such that $P_k$ is surjective on $G_0$ and $G_0$ contains $H$.

3. If the residual characteristic of $\mathbb{F}$ divides $k$, then the characteristic of $\mathbb{F}$ is zero implies that $H$ is a finite extension of $R_{us,\mathbb{F}}(G)(\mathbb{F})$ and the characteristic of $\mathbb{F}$ is positive implies that $G(\mathbb{F})$ is compact.

**Proof.** Let $M = G(\mathbb{F})/R_{us,\mathbb{F}}(G)(\mathbb{F})$ and $Q = G/R_{us,\mathbb{F}}(G)$. Then $M \simeq Q(\mathbb{F})$, hence we may assume that $M = Q(\mathbb{F})$. Then $M$ is a closed subgroup of $GL(d, \mathbb{F})$ for some $d \geq 1$. Let $s$ be the scale function on $\mathbb{F}^d$. Then by Proposition 5.1 $s$ is trivial on $M$. Now applying Lemma 3.2 to $M$, we get that there is a compact group $K \subset GL(d, \mathbb{F})$ and a split unipotent group $U \subset GL(d, \mathbb{F})$ normalized by $K$ such
that $M \subset KU$ and $MU$ is dense in $KU$. Since both $Q$ and $U$ are algebraic, $MU$ is closed, hence $MU = KU$.

Let $f : KU \to KU/U$ be the canonical quotient map. Since $Q$ has no $F$-split unipotent normal subgroup, $M \cap U$ is trivial. Therefore $f$ restricted to $M$ is an isomorphism onto $KU/U \simeq K$. Thus, $M$ is compact.

Suppose the residual characteristic $p$ of $F$ does not divide $k$. Let $M_0$ be an open normal pro-$p$ subgroup of $M$. Then by Proposition 2.2, we get that $P_k$ is surjective on $M_0$. Let $M_1$ be the closure of $(HR_{us,F}(G)(F)/R_{us,F}(G)(F))M_0$. Then $M_1$ is an open subgroup of finite index in $M$ and by Lemma 2.3, $P_k$ is surjective on $M_1$. Let $G_0$ be the subgroup of $G(F)$ containing $R_{us,F}(G)(F)$ such that $G_0/R_{us,F}(G)(F) = M_1$. Then $G_0$ is an open subgroup of finite index in $G(F)$. Since the residual characteristic $p$ of $F$ does not divide $k$ and $G_0/R_{us,F}(G)(F)$ is compact on which $P_k$ is surjective, Lemma 2.5 implies that $P_k$ is surjective on $G_0$.

If the residual characteristic $p$ of $F$ divides $k$ and the characteristic of $F$ is zero, then by (5) of Proposition 3.3 we get that $H$ is a finite extension of a split unipotent algebraic group $V$. Since $H$ is a finite co-volume or co-compact subgroup of $G(F)$, $V$ is also finite co-volume or co-compact subgroup of $G(F)$. Since $G(F)/R_{us,F}(G)(F)$ is compact, $V \subset R_{us,F}(G)(F)$. In particular, $V$ is a finite co-volume subgroup of $R_{us,F}(G)(F)$. Now, it can be easily shown that $V = R_{us,F}(G)(F)$. Thus, $H$ is a finite extension of $R_{us,F}(G)(F)$.

If the residual characteristic $p$ of $F$ divides $k$ and the characteristic of $F$ is positive, then by (6) of Proposition 3.3, $H$ is finite. This implies that $G(F)$ itself has finite volume, hence $G(F)$ is compact.

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