THE RATE OF CONVERGENCE TO THE ASYMPTOTICS FOR THE WAVE EQUATION IN AN EXTERIOR DOMAIN

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Abstract. In this paper we consider the mixed problem for the wave equation exterior to a non-trapping obstacle in odd space dimensions. We derive a rate of the convergence of the solution for the mixed problem to a solution for the Cauchy problem. As a by-product, we are able to find out the radiation field of solutions to the mixed problem in terms of the scattering data.

1. Introduction

This paper is concerned with the global behavior of solutions to the mixed problem for the wave equation in an exterior domain:

\begin{align}
(\partial_t^2 - \Delta)u(t, x) &= 0, & (t, x) &\in (0, T) \times \Omega, \\
u(t, x) &= 0, & (t, x) &\in (0, T) \times \partial\Omega, \\
u(0, x) &= f_0(x), & (\partial_t u)(0, x) &= f_1(x), & x &\in \Omega,
\end{align}

where $\Omega = \mathbb{R}^n \setminus \mathcal{O}$, and $\mathcal{O}$ is a bounded open set in $\mathbb{R}^n$ with smooth boundary. Throughout this paper, we suppose that $n$ is an odd integer with $n \geq 3$. We assume that $\Omega$ is connected and that the initial data $\vec{f} = (f_0, f_1)$ belongs to the associated energy space $\mathcal{H}_D(\Omega)$. Here and in the following, for an open set $Y \subset \mathbb{R}^n$, $\mathcal{H}_D(Y)$ stands for the completion of $(C_0^\infty(Y))^2$ with respect to $\| \vec{f} \|_{\mathcal{H}_D(Y)} = \| \nabla f_0 \|_{L^2(Y)} + \| f_1 \|_{L^2(Y)}$. $U(t)$ denotes the propagator of the mixed problem (1.1) with (1.2) and (1.3); in other words, we define

$$U(t) \vec{f} = (u(t, \cdot), \partial_t u(t, \cdot))$$

for $\vec{f} \in \mathcal{H}_D(\Omega)$, where $u$ is the solution to (1.1)–(1.3).

It is well known that the asymptotic behavior of the solution to the above problem is approximated by a solution to the Cauchy problem.

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More precisely, for a given initial data \( \tilde{f} \in \mathcal{H}_D(\Omega) \) there exists uniquely a scattering data \( \tilde{f}^+ \in \mathcal{H}_D(\mathbb{R}^n) \) such that

\[
(1.4) \quad \| U(t)\tilde{f} - U_0(t)\tilde{f}^+ \|_{\mathcal{H}_D(\Omega)} \to 0 \quad (t \to \infty),
\]

where, for \( \tilde{g} = (g_0, g_1) \in \mathcal{H}_D(\mathbb{R}^n), \) \( U_0(t)\tilde{g} \) is given by

\[
U_0(t)\tilde{g} = (u_0(t, \cdot), \partial_t u_0(t, \cdot))
\]

with \( u_0 \) being the solution to the Cauchy problem

\[
(1.5) \quad (\partial_t^2 - \Delta) u_0(t, x) = 0, \quad (t, x) \in (0, T) \times \mathbb{R}^n,
\]

\[
(1.6) \quad u_0(0, x) = g_0(x), \quad (\partial_t u_0)(0, x) = g_1(x), \quad x \in \mathbb{R}^n.
\]

On the other hand, the local energy of \( U(t)\tilde{f} \) decays to zero as \( t \) tends to infinity. Namely, for any \( R > 0 \) and any \( \tilde{f} \in \mathcal{H}_D(\Omega) \), we have

\[
(1.7) \quad \lim_{t \to \infty} \int_{\{x \in \Omega; |x| < R\}} \left\{ |\partial_t u(t, x)|^2 + |\nabla u(t, x)|^2 \right\} \, dx = 0,
\]

where \( u \) is the solution to \( (1.1)-(1.3) \).

In view of these facts, we see that the main part of the perturbed wave \( U(t)\tilde{f} \) escapes from any ball with a fixed radius as \( t \to \infty \) and that it approaches to some unperturbed wave \( U_0(t)\tilde{f}^+ \) in the sense of the energy. However, to our knowledge, the rate of the convergence in \( (1.4) \) is not found explicitly in the literature. Therefore, it is natural to ask at which rate the perturbed wave tends to an unperturbed wave. In addition, we are interested in the regularity and decay properties of the scattering data \( \tilde{f}^+ \). Namely, we wish to know whether the scattering data becomes smoother and decays faster at the spatial infinity or not, if the initial data is smooth and compactly supported. This consideration might be useful for the application to the nonlinear wave equation in an exterior domain. For instance, we are able to obtain a precise lower bound of the lifespan in our forthcoming paper.

Here we introduce notation in order to state our main result. Let \( m \) be a nonnegative integer and \( Y \) be an open set in \( \mathbb{R}^n \). We set \( \mathcal{H}^m(Y) = H^{m+1}(Y) \times H^m(Y) \) and \( \| \tilde{f} \|_{\mathcal{H}^m(Y)} = \| f_0 \|_{H^{m+1}(Y)} + \| f_1 \|_{H^m(Y)} \) for \( \tilde{f} = (f_0, f_1) \in \mathcal{H}^m(Y) \). Similarly, we put \( \mathcal{W}^{m, \infty}(Y) = W^{m+1, \infty}(Y) \times W^{m, \infty}(Y) \) and \( \| \tilde{f} \|_{\mathcal{W}^{m, \infty}(Y)} = \| f_0 \|_{W^{m+1, \infty}(Y)} + \| f_1 \|_{W^{m, \infty}(Y)} \) for \( \tilde{f} \in \mathcal{W}^{m, \infty}(Y) \). Here \( H^m(Y) \) \( (\text{resp. } W^{m, \infty}(Y)) \) stands for the Sobolev space based on \( L^2(Y) \) \( (\text{resp. } L^\infty(Y)) \). In addition, we denote by \( \mathcal{X}^m(\Omega) \) the set of all \( \tilde{f} = (f_0, f_1) \in \mathcal{H}^m(\Omega) \) satisfying the compatibility condition of the \( m \)-th order for the problem \( (1.1)-(1.3) \), that is \( f_j = 0 \) on \( \partial \Omega \) for
any \( j = 0, \ldots, m \), where we have set
\[
(1.8) \quad f_j(x) \equiv \Delta f_{j-2}(x) \quad \text{for} \quad x \in \overline{\Omega} \quad \text{and} \quad j \geq 2.
\]
Besides, we put \( \mathcal{H}^\infty(Y) = \bigcap_{m=0}^\infty \mathcal{H}^m(Y) \) and \( X^\infty(\Omega) = \bigcap_{m=0}^\infty X^m(\Omega) \).

We will use the notation \( \partial_{\alpha} = \partial_{x_j} \) for \( 1 \leq j \leq n \), and \( \partial_{\alpha}^2 = \partial_{\alpha_1}^2 \cdots \partial_{\alpha_n}^2 \)
for a multi-index \( \alpha = (\alpha_1, \ldots, \alpha_n) \). We set
\[
\Gamma = (\Gamma_0, \Gamma_1, \ldots, \Gamma_N) = (\partial_{\alpha}, \partial_{\alpha_1}, \ldots, \partial_{\alpha_n}, (O_{ij})_{1 \leq i < j \leq n})
\]
with \( N = n(n+1)/2 \), and \( \Gamma^\beta = \Gamma^\beta_0 \Gamma^\beta_1 \cdots \Gamma^\beta_N \) for a multi-index \( \beta = (\beta_0, \beta_1, \ldots, \beta_N) \), where \( O_{ij} \) for \( 1 \leq i, j \leq n \) is given by \( O_{ij} = x_i \partial_j - x_j \partial_i \).

For \( r > 0 \) and \( y \in \mathbb{R}^n \), \( B_r(y) \) stands for an open ball of radius \( r \) centered at \( y \). We write \( B_r \) for \( B_r(0) \). Besides, we set \( \Omega_r = \Omega \cap B_r \).

In what follows, for a constant \( C \), when we write \( C \equiv (p_1, \ldots, p_m) \) with \( p_1, \ldots, p_m \) being some given constants or functions, it means that, with the space dimension \( n \) and the obstacle \( \mathcal{O} \) being fixed, \( C \) is a constant depending only on \( p_1, \ldots, p_m \) (thus \( C \) may depend also on \( n \) and \( \mathcal{O} \) actually).

Then our main result reads as follows.

**Theorem 1.1.** Let the space dimension \( n \) be odd, and \( n \geq 3 \). Assume that \( \mathcal{O} \) is non-trapping, and \( \mathcal{O} \subset B_1 \). Let \( a(>1) \) be a fixed number. Then for any \( \tilde{f} \in X^\infty(\Omega) \) with supp \( \tilde{f} \subset \overline{\Omega}_a \), there exists uniquely \( \tilde{f}_+ \in \mathcal{H}^\infty(\mathbb{R}^n) \) satisfying (1.4). Moreover, there exists a positive constant \( \mu = \mu(a) \) having the following property: For any nonnegative integer \( k \), there exists a positive constant \( C = C(k, a) \) such that
\[
(1.9) \quad \left\| \exp(\mu \langle \cdot \rangle) \left( U(t)\tilde{f} - U_0(t)\tilde{f}_+ \right) \right\|_{\mathcal{H}^k(\Omega)} \leq C \exp(-\mu t) \left\| \tilde{f} \right\|_{\mathcal{H}^k(\Omega)} \quad \text{for} \quad t \geq 0,
\]
\[
(1.10) \quad \left\| \exp(2\mu \langle \cdot \rangle) \tilde{f}_+ \right\|_{W^{k,\infty}(\mathbb{R}^n)} \leq C \left\| \tilde{f} \right\|_{\mathcal{H}^{k+[n/2]+1}(\Omega)},
\]
where \( \langle x \rangle = \sqrt{1+|x|^2} \) for \( x \in \mathbb{R}^n \), and \( [n/2] \) denotes the largest integer not exceeding \( n/2 \).

Theorem 1.1 will be proved in Section 3. Our proof of Theorem 1.1 relies on the exponential decay of the local energy (see Lemma 2.3 below), and this is the reason why \( n(\geq 3) \) is assumed to be odd and the obstacle \( \mathcal{O} \) to be non-trapping. For the notion of the non-trapping obstacle, we refer to Melrose [10] for instance (see also Shibata–Tsuntsu [11, 12]). For example, star-shaped obstacles are known to be non-trapping.

Note that (1.10) implies that each component of \( \tilde{f}_+ \) belongs to the Schwartz class \( \mathcal{S} \), the class of rapidly decreasing functions.
Now we turn our attention to the asymptotic pointwise behavior of the perturbed wave $U(t)\vec{f}$. To describe the result, we define the Friedlander radiation field $\mathcal{F}_0[\vec{g}]$ by

$$
\mathcal{F}_0[\vec{g}](s, \eta) = \frac{1}{2(2\pi)^{n/2}} \sum_{j=0}^{1} (-\partial_s)^{n/2-j} R[g_j](s, \eta)
$$

for $\vec{g} = (g_0, g_1) \in (S(\mathbb{R}^n))^2$. Here $R[\varphi]$ denotes the Radon transform of $\varphi = \varphi(x)$, that is

$$
R[\varphi](s, \eta) = \int_{y \cdot \eta = s} \varphi(y) dS_y,
$$

where $dS_y$ denotes the area element on the hyperplane $\{y; y \cdot \eta = s\}$. The radiation field $\mathcal{F}_0[\vec{g}]$ is introduced to describe the main part of the unperturbed wave $U(t)\vec{g}$ for $\vec{g} \in (C^{\infty}_0(\mathbb{R}^3))^2$ in Friedlander [2]. Lax–Phillips [9] showed that the main part of the perturbed wave can also be written in terms of the Friedlander radiation field of some function, but the convergence rate seems not to have been obtained. Thus we would like to investigate the convergence rate of the perturbed waves to the pointwise asymptotics described by the Friedlander radiation field. Our result is the following.

**Theorem 1.2.** Let the assumptions of Theorem 1.1 hold. Then for any $\vec{f} \in X^\infty(\Omega)$ with $\text{supp} \vec{f} \subset \overline{\Omega}_a$, there exists $\vec{f}_+ \in (S(\mathbb{R}^n))^2$ satisfying the following property: For any nonnegative integer $k$, there exists a positive constant $C = C(k, a, \vec{f})$ such that, writing $x = r\omega$, for $r \geq t/2 \geq 1$ and $\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in S^{n-1}$ we have

$$
\sum_{|\alpha| \leq k} \left| \Gamma^\alpha \left\{ u(t, x) - r^{-\frac{n+1}{2}} \mathcal{F}_0[\vec{f}_+](r - t, \omega) \right\} \right| \leq C(1 + r + t)^{-\frac{n+1}{2}} \exp\left(-\frac{\mu}{2} r - t\right),
$$

$$
\sum_{|\alpha| \leq k} \left| \Gamma^\alpha \left\{ \partial_t u(t, x) - (-1) r^{-\frac{n+1}{2}} (\partial_s \mathcal{F}_0[\vec{f}_+])(r - t, \omega) \right\} \right|
$$

$$
+ \sum_{|\alpha| \leq k} \sum_{j=1}^{n} \left| \Gamma^\alpha \left\{ \partial_j u(t, x) - \omega_j r^{-\frac{n+1}{2}} (\partial_s \mathcal{F}_0[\vec{f}_+])(r - t, \omega) \right\} \right|
$$

$$
\leq C(1 + r + t)^{-\frac{n+1}{2}} \exp\left(-\frac{\mu}{2} r - t\right),
$$

where $u(t, x)$ is the solution to (1.1)–(1.3) and $\mu = \mu(a)$ is the positive constant from Theorem 1.1.
The proof of Theorem 1.2 will be given in Section 5, after obtaining the detailed convergence rate for the Cauchy problem in Section 4 (see Proposition 4.1).

We underline that the decaying factor \( \exp(-\mu|r-t|/2) \) in the above estimates is quite meaningful even if the initial data is compactly supported, say \( \text{supp} \tilde{f} \subset \Omega_a \), unlike the case of the Cauchy problem.

In fact, the solution \( u(t,x) \) for the mixed problem is identically zero for \( r-t \geq a \) and \( t \geq 0 \), in view of the domain of dependence (see Lemma 2.1 below). On the other hand, it is not expected to vanish for \( r-t \leq -a \) in general, because of the presence of the obstacle. Accordingly, the radiation field \( F^0[\tilde{f}^0](s,\omega) \) for the solution to the mixed problem vanishes for \( s \geq a \) and \( \omega \in S^{n-1} \) due to (1.12), although it is not supposed to be zero for \( s \leq -a \) and \( \omega \in S^{n-1} \) in general. In contrast to this, if there is no obstacle, it is known that the radiation field for compactly supported data vanishes also for \( s \leq -a \) (this property is closely connected to the Huygens principle; see Lemma 2.2 below).

In conclusion, it is essential to extract the factor \( \exp(-\mu|r-t|/2) \), in order to describe the behavior for the mixed problem in the region \( r-t \leq -a \).

2. Preliminaries

Let \( Y \) be an open subset of \( \mathbb{R}^n \), and \( \Omega \) be as in the previous section. For the notational convenience, we put

\[
\mathcal{H}^\infty_a(Y) = \{ \tilde{f} = (f_0, f_1) \in \mathcal{H}^\infty(Y); \text{supp} \tilde{f} \subset Y \cap B_a \},
\]

\[
\mathcal{X}^\infty_a(\Omega) = \{ \tilde{f} = (f_0, f_1) \in \mathcal{X}^\infty(\Omega); \text{supp} \tilde{f} \subset \Omega_a \}
\]

for \( a > 0 \).

The following property is well known.

**Lemma 2.1** (Domain of dependence). Let \( n \) be a positive integer. Let \( \tau, t_0 \in \mathbb{R} \) with \( \tau < t_0 \), and \( x_0 \in \mathbb{R}^n \). We define

\[
\Lambda(t_0, x_0, \tau) = \{(t,x) \in (\tau, t_0) \times \mathbb{R}^n; |x-x_0| < t_0-t\}.
\]

Suppose that \( \psi = \psi(t,x) \) satisfies

\[
(\partial_t^2 - \Delta)\psi(t,x) = 0, \quad (t,x) \in \Lambda(t_0, x_0, \tau).
\]

Then we have

\[
\|\partial_t\psi(t)\|_{L^2(B_{t_0-t}(x_0))} \leq \|\partial_t\psi(\tau)\|_{L^2(B_{t_0-t}(x_0))}, \quad t \in (\tau, t_0),
\]

where \( \partial \psi = (\partial_t \psi, \nabla \psi) \). As a consequence, if we also assume

\[
\psi(\tau,x) = (\partial_t \psi)(\tau,x) = 0, \quad x \in B_{t_0-t}(x_0),
\]

then...
then we have $\psi(t, x) = 0$ for any $(t, x) \in \Lambda(t_0, x_0, \tau)$.

The above assertions are also valid if we replace $\Lambda(t_0, x_0, \tau)$ by

$\Lambda^{\ast}(t_0, x_0, \tau) = \{(t, x) \in (2\tau - t_0, \tau) \times \mathbb{R}^n; |x - x_0| < t + t_0 - 2\tau\}$,

and (2.3) by

$$\|\partial \psi(t)\|_{L^2(B_{t_0 + t_0 - 2\tau}(x_0))} \leq \|\partial \psi(\tau)\|_{L^2(B_{t_0 - \tau}(x_0))}, \quad t \in (2\tau - t_0, \tau).$$

From the lemma above, we see that $\tilde{f} \in X_\infty^\ast(\Omega)$ (resp. $\tilde{g} \in \mathcal{H}_\infty^\ast(\mathbb{R}^n)$) implies $\text{supp}(U(t)\tilde{f}) \subset \overline{\Omega_{|t|+a}}$ (resp. $\text{supp}(U_0(t)\tilde{g}) \subset \overline{\mathbb{B}_{|t|+a}}$).

In odd space dimensions, we have a stronger result.

**Lemma 2.2** (The Huygens principle). Let $n$ be an odd integer with $n \geq 3$. Then $\tilde{g} \in \mathcal{H}_\infty^\ast(\mathbb{R}^n)$ implies

$$\text{supp}(U_0(t)\tilde{g}) \subset \{x \in \mathbb{R}^n; |t| - a \leq |x| \leq |t| + a\}, \quad t \in \mathbb{R}.$$

This result follows immediately from the explicit expression of $U_0(t)\tilde{g}$ (see (2.3) below).

Next we introduce the local energy decay of the perturbed wave at exponential rate (for the proof, see for instance Melrose [10]; see also Shibata–Tsutsumi [11]).

**Lemma 2.3.** Let $n$ be odd and $n \geq 3$. Assume that $\mathcal{O}$ is non-trapping, and $\mathcal{O} \subset B_1$. Suppose that $a, b > 1$, and $k$ is a nonnegative integer. Then there exist two positive constants $C = C(k, a, b)$ and $\sigma = \sigma(a, b)$ such that for any $\tilde{f} \in X_\infty^\ast(\Omega)$ we have

$$\|U(t)\tilde{f}\|_{\mathcal{H}^k(\mathbb{R}^n)} \leq C \exp(-\sigma t) \|\tilde{f}\|_{\mathcal{H}^k(\Omega)} \quad \text{for } t \geq 0.$$

The following lemma, motivated by the arguments in Ikawa [7], tells us that the perturbed wave can be decomposed into the unperturbed wave and the correction term. The former is the main part of the perturbed wave, while the latter takes care of the effect from the boundary and its size can be small compared with the initial energy. This lemma is crucial for proving Theorem 1.1.

**Lemma 2.4.** Let $n, \mathcal{O},$ and $a$ be as in Theorem 1.1. Then, for any $\tilde{f} \in X_\infty^\ast(\Omega)$ and $T(\geq a + 2)$, there exist $\tilde{g}_1 \in \mathcal{H}_T^\infty(\mathbb{R}^n)$ and $\tilde{f}_1 \in X_3^\infty(\Omega)$ satisfying

$$U(t)\tilde{f} = U_0(t - T)\tilde{g}_1 + U(t - T)\tilde{f}_1, \quad t \geq T,$$

$$\|\tilde{g}_1\|_{\mathcal{H}^k(\mathbb{R}^n)} \leq C_0(1 + T)\|\tilde{f}\|_{\mathcal{H}^k(\Omega)},$$

$$\|\tilde{f}_1\|_{\mathcal{H}^k(\Omega)} \leq C_0 \exp(-\sigma T) \|\tilde{f}\|_{\mathcal{H}^k(\Omega)}$$

for any nonnegative integer $k$ with some positive constants $C_0 = C_0(k, a)$ and $\sigma = \sigma(a)$.
Proof. In this proof, various positive constants depending only on k will be indicated by the same $C_k$.

We put $T_0 = T - 2 (\geq a)$. If we set $\tilde{\phi} = U(T_0)\tilde{f}$, then $\tilde{\phi} \in X^\infty(\Omega)$ and

$$
\|\tilde{\phi}\|_{H^k(\Omega)} \leq C_k (1 + T_0) \|\tilde{f}\|_{H^k(\Omega)}
$$

for any nonnegative integer $k$. Indeed, (2.8) follows from the fact that we have

$$
\|U(t)\tilde{f}\|_{H^k(\Omega)} \leq C_k (1 + |t|) \|\tilde{f}\|_{H^k(\Omega)}, \quad t \in \mathbb{R}
$$

for any $\tilde{f} \in X^\infty(\Omega)$. This estimate is a simple consequence of the energy estimate and an elementary inequality

$$
\|v(t)\|_{L^2(\Omega)} \leq \|v(t_0)\|_{L^2(\Omega)} + \int_{t_0}^t \|\partial_t v(\tau)\|_{L^2(\Omega)} d\tau, \quad t \geq t_0,
$$

which is valid for any smooth function $v$. Besides, in view of the domain of dependence (see Lemma 2.1), we have $\text{supp} \tilde{\phi} \subset \Omega_{T_0 + a}$, since $\text{supp} \tilde{f} \subset \Omega_0$.

Next we extend $\tilde{\phi}$ to $\tilde{\psi} \in H^\infty_{T_0 + a}(\mathbb{R}^n)$ in such a way that $\tilde{\psi} = \tilde{\phi}$ in $\Omega$ and

$$
\|\tilde{\psi}\|_{H^k(\mathbb{R}^n)} \leq C_k (1 + T_0) \|\tilde{f}\|_{H^k(\Omega)}.
$$

To do this, we set $\tilde{\phi}_0 = \chi \tilde{\phi}$ and $\tilde{\phi}_\infty = (1 - \chi) \tilde{\phi}$, where $\chi$ is a smooth function on $\mathbb{R}^n$ satisfying $\chi(x) = 1$ for $|x| \leq 5$ and $\chi(x) = 0$ for $|x| \geq 6$. Then $\tilde{\phi}_0$ can be regarded as a function on a bounded domain $\Omega_0$, and we see from the Stein extension theorem that $\tilde{\phi}_0$ can be extended to $\tilde{\psi}_0 \in H^\infty(\mathbb{R}^n)$ such that $\tilde{\psi}_0 = \tilde{\phi}_0$ in $\Omega$ and

$$
\|\tilde{\psi}_0\|_{H^k(\mathbb{R}^n)} \leq C_k \|\tilde{\phi}_0\|_{H^k(\Omega)}
$$

for any nonnegative integer $k$ (refer to [13]). Recalling (2.8) and setting $\tilde{\psi} := \tilde{\psi}_0 + \tilde{\phi}_\infty$, we see that $\tilde{\psi}$ has the desired properties.

Next we let $v$ be the solution of

$$
(\partial_t^2 - \Delta) v(t, x) = 0, \quad (t, x) \in (T_0, \infty) \times \mathbb{R}^n,
$$

$$(v(T_0, x), (\partial_t v)(T_0, x)) = \psi(x), \quad x \in \mathbb{R}^n,
$$

and we define $w = u - v$ in $[T_0, \infty) \times \Omega$, so that

$$
(\partial_t^2 - \Delta) w(t, x) = 0, \quad (t, x) \in (T_0, \infty) \times \Omega,
$$

$$
w(T_0, x) = (\partial_t w)(T_0, x) = 0, \quad x \in \Omega,
$$

where $u, \psi \in L^2(\Omega)$.

We put $T_0 = T - 2 (\geq a)$. If we set $\tilde{\phi} = U(T_0)\tilde{f}$, then $\tilde{\phi} \in X^\infty(\Omega)$ and

$$
\|\tilde{\phi}\|_{H^k(\Omega)} \leq C_k (1 + T_0) \|\tilde{f}\|_{H^k(\Omega)}
$$

for any nonnegative integer $k$. Indeed, (2.8) follows from the fact that we have

$$
\|U(t)\tilde{f}\|_{H^k(\Omega)} \leq C_k (1 + |t|) \|\tilde{f}\|_{H^k(\Omega)}, \quad t \in \mathbb{R}
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for any $\tilde{f} \in X^\infty(\Omega)$. This estimate is a simple consequence of the energy estimate and an elementary inequality

$$
\|v(t)\|_{L^2(\Omega)} \leq \|v(t_0)\|_{L^2(\Omega)} + \int_{t_0}^t \|\partial_t v(\tau)\|_{L^2(\Omega)} d\tau, \quad t \geq t_0,
$$

which is valid for any smooth function $v$. Besides, in view of the domain of dependence (see Lemma 2.1), we have $\text{supp} \tilde{\phi} \subset \Omega_{T_0 + a}$, since $\text{supp} \tilde{f} \subset \Omega_0$.

Next we extend $\tilde{\phi}$ to $\tilde{\psi} \in H^\infty_{T_0 + a}(\mathbb{R}^n)$ in such a way that $\tilde{\psi} = \tilde{\phi}$ in $\Omega$ and

$$
\|\tilde{\psi}\|_{H^k(\mathbb{R}^n)} \leq C_k (1 + T_0) \|\tilde{f}\|_{H^k(\Omega)}.
$$

To do this, we set $\tilde{\phi}_0 = \chi \tilde{\phi}$ and $\tilde{\phi}_\infty = (1 - \chi) \tilde{\phi}$, where $\chi$ is a smooth function on $\mathbb{R}^n$ satisfying $\chi(x) = 1$ for $|x| \leq 5$ and $\chi(x) = 0$ for $|x| \geq 6$. Then $\tilde{\phi}_0$ can be regarded as a function on a bounded domain $\Omega_0$, and we see from the Stein extension theorem that $\tilde{\phi}_0$ can be extended to $\tilde{\psi}_0 \in H^\infty(\mathbb{R}^n)$ such that $\tilde{\psi}_0 = \tilde{\phi}_0$ in $\Omega$ and

$$
\|\tilde{\psi}_0\|_{H^k(\mathbb{R}^n)} \leq C_k \|\tilde{\phi}_0\|_{H^k(\Omega)}
$$

for any nonnegative integer $k$ (refer to [13]). Recalling (2.8) and setting $\tilde{\psi} := \tilde{\psi}_0 + \tilde{\phi}_\infty$, we see that $\tilde{\psi}$ has the desired properties.

Next we let $v$ be the solution of

$$
(\partial_t^2 - \Delta) v(t, x) = 0, \quad (t, x) \in (T_0, \infty) \times \mathbb{R}^n,
$$

$$(v(T_0, x), (\partial_t v)(T_0, x)) = \psi(x), \quad x \in \mathbb{R}^n,
$$

and we define $w = u - v$ in $[T_0, \infty) \times \Omega$, so that

$$
(\partial_t^2 - \Delta) w(t, x) = 0, \quad (t, x) \in (T_0, \infty) \times \Omega,
$$

$$
w(T_0, x) = (\partial_t w)(T_0, x) = 0, \quad x \in \Omega,
$$

where $u, \psi \in L^2(\Omega)$.
where \( u(t, \cdot) \) denotes the first component of \( U(t)\tilde{f} \). Furthermore, we define
\[
\tilde{g}_1(x) := (v(T, x), (\partial_t v)(T, x)) = U_0(2)\tilde{\psi}(x),
\]
\[
\tilde{f}_1(x) := (w(T, x), (\partial_t w)(T, x)) = U(T)\tilde{f}(x) - U_0(2)\tilde{\psi}(x)
\]
(recall \( T = T_0 + 2 \)). Then we easily get (2.6) from (2.11), because we have
\[
\|U_0(t)\tilde{\psi}\|_{\mathcal{H}^k(\mathbb{R}^n)} \leq C_k (1 + |t|)\|\tilde{\psi}\|_{\mathcal{H}^k(\mathbb{R}^n)}, \quad t \in \mathbb{R}
\]
for \( \tilde{\psi} \in \mathcal{H}^\infty(\mathbb{R}^n) \). This estimate is shown similarly to (2.9). Taking the domain of dependence into account, we have \( \tilde{g}_1 \in \mathcal{H}^\infty_{T+a}(\mathbb{R}^n) \).

Next we consider \( \tilde{f}_1 \). Note that (2.15) and (2.16) imply
\[
w(t, x) = 0 \quad \text{for} \ |x| \geq t - T_0 + 1, \ t \geq T_0
\]
in view of the domain of dependence, because we have \( \mathcal{O} \subset B_1 \). Hence \( \text{supp } \tilde{f}_1 \subset \Omega_3 \), so that
\[
\|\tilde{f}_1\|_{\mathcal{H}^k(\Omega)} \leq \|U(T)\tilde{f}\|_{\mathcal{H}^k(\Omega_3)} + \|U_0(2)\tilde{\psi}\|_{\mathcal{H}^k(\Omega_3)}
\]
\[
\leq C \exp(-\sigma T) \|\tilde{f}\|_{\mathcal{H}^k(\Omega)} + C_k\|\tilde{\psi}\|_{\mathcal{H}^k(\mathbb{R}^n)},
\]
thanks to (2.4), (2.3) and (2.10), where \( C = C(k, a) \) and \( \sigma = \sigma(a) \) are positive constants. Since \( \tilde{\psi} = \tilde{\psi}_0 \) in \( B_5 \), (2.12) yields
\[
\|\tilde{\psi}\|_{\mathcal{H}^k(\mathbb{R}^n)} \leq \|\tilde{\psi}_0\|_{\mathcal{H}^k(\mathbb{R}^n)} \leq C_k\|\tilde{\phi}_0\|_{\mathcal{H}^k(\Omega)} \leq C_k\|\tilde{\phi}\|_{\mathcal{H}^k(\Omega_6)}.
\]
Recalling \( \tilde{\phi} = U(T_0)\tilde{f} \) and using (2.4) again, we obtain (2.7).

In order to show that \( \tilde{f}_1 \in X^\infty_3(\Omega) \), it suffices to prove
\[
w(t, x) = 0 \quad \text{for} \ (t, x) \in [T_0 + 2, \infty) \times \partial \Omega.
\]
Indeed, we already know \( \tilde{f}_1 \in \mathcal{H}^\infty(\Omega) \); as for the compatibility condition, writing \( \tilde{f}_1 = (f_{1,0}, f_{1,1}) \) and \( f_{1,j} = \Delta f_{1,j-2} \) for \( j \geq 2 \), we find \( f_{1,j}(x) = (\partial_{ij} w)(T, x) \) for \( j \geq 0 \), and (2.21) immediately leads to \( f_{1,j} = 0 \) on \( \partial \Omega \) for \( j \geq 0 \). Since \( w = u - v \) and \( \partial \Omega \subset B_1 \), (2.21) is a consequence of (1.2) and
\[
v(t, x) = 0 \quad \text{for} \ t \geq |x| + T_0 + 1.
\]
To prove (2.22), we define a function \( z \) on \( [0, \infty) \times \mathbb{R}^n \) by
\[
z(t, x) = \begin{cases} 
  u(t, x) & \text{for} \ (t, x) \in [0, T_0] \times \Omega, \\
  0 & \text{for} \ (t, x) \in [0, T_0] \times \overline{\Omega}, \\
  v(t, x) & \text{for} \ (t, x) \in (T_0, \infty) \times \mathbb{R}^n.
\end{cases}
\]
For $\varepsilon > 0$, let $\xi_\varepsilon$ be a smooth function on $[0, \infty) \times \mathbb{R}^n$ such that

$$
\xi_\varepsilon(t, x) = \begin{cases} 
1 & \text{for } |x| \geq 1 \text{ or } t \geq T_0 + \varepsilon, \\
0 & \text{for } (t, x) \in [0, T_0] \times \mathcal{O},
\end{cases}
$$

and $\xi_\varepsilon(t, x) = \xi_\varepsilon(0, x)$ for $(t, x) \in [0, T_0] \times \mathbb{R}^n$. Then we have

$$
supp \Box(\xi_\varepsilon z) \subset [0, T_0 + \varepsilon] \times \overline{B_1},$$

$$
supp (\xi_\varepsilon z)(0, \cdot) \cup supp \partial_t (\xi_\varepsilon z)(0, \cdot) \subset \overline{B_{T_0}},$$

since $a \leq T_0$. From the Duhamel principle, we have

$$
(\xi_\varepsilon z)(t, \cdot) = U_0(t) ((\xi_\varepsilon z)(0), \partial_t (\xi_\varepsilon z)(0)) + \int_0^t U_0(t - \tau) (0, \Box(\xi_\varepsilon z)(\tau)) d\tau.
$$

Thus by the Huygens principle (Lemma 2.2) we see that $v(t, x) = (\xi_\varepsilon z)(t, x) = 0$ for $t \geq |x| + T_0 + \varepsilon + 1$, which implies (2.22) because $\varepsilon$ is arbitrary.

Finally, we prove (2.5). We see from (2.17) and (2.18) that (2.5) holds at $t = T$. Besides, for $(t, x) \in [T, \infty) \times \partial \Omega$ we have

$$
(U_0(t - T)\bar{g}_1)(x) + (U(t - T)\bar{f}_1)(x) = (U_0(t - T)\bar{g}_1)(x)
$$

$$
= (v(t, x), (\partial_t v)(t, x)) = (0, 0)
$$

by (2.22). It is apparent that we have $\Box(U_0(t - T)\bar{g}_1 + U(t - T)\bar{f}_1) = 0$ for $t \geq T$. Hence we find (2.5) by the uniqueness of the solution for the mixed problem. This completes the proof. $\square$

3. Proof of Theorem 1.1

In this section we prove Theorem 1.1. The uniqueness is deduced from the following assertion: For given $\bar{f} \in X^{\infty}_a(\Omega)$, if $\bar{f}_+ \in \mathcal{H}^0(\mathbb{R}^n)$ satisfies

$$
\lim_{t \to \infty} \|U(t)\bar{f} - U_0(t)\bar{f}_+\|_{\mathcal{H}_D(\Omega)} = 0,
$$

then $\bar{f}_+$ is determined uniquely. To verify this assertion, suppose that $\bar{g}_+ \in \mathcal{H}^0(\mathbb{R}^n)$ also satisfies $\lim_{t \to \infty} \|U(t)\bar{f} - U_0(t)\bar{g}_+\|_{\mathcal{H}_D(\Omega)} = 0$, so that

$$
\lim_{t \to \infty} \|U_0(t)(\bar{f}_+ - \bar{g}_+\|_{\mathcal{H}_D(\Omega)} = 0.
$$

We also have

$$
\lim_{t \to \infty} \|U_0(t)(\bar{f}_+ - \bar{g}_+)\|_{\mathcal{H}_D(B_1)} = 0.
$$
In fact, for any \( \varepsilon > 0 \), there exists \( \tilde{h} \in (C_0^\infty(\mathbb{R}^n))^2 \) such that \( \|(f_+ - g_+) - \tilde{h}\|_{\mathcal{H}_D(\mathbb{R}^n)} < \varepsilon \). Let \( \text{supp} \; \tilde{h} \subset \overline{B_M} \). Since the Huygens principle implies \( U_0(t)\tilde{h} = 0 \) for \( |x| \leq 1 \) and \( t \geq M + 1 \), we obtain
\[
\|U_0(t)(f_+ - g_+)\|_{\mathcal{H}_D(B_1)} = \|U_0(t)(f_+ - g_+ - \tilde{h})\|_{\mathcal{H}_D(B_1)} \\
\leq \|f_+ - g_+ - \tilde{h}\|_{\mathcal{H}_D(\mathbb{R}^n)} < \varepsilon
\]
for \( t \geq M + 1 \), which leads to (3.3). Here we have used the unitarity of \( U_0(t) \) on \( \mathcal{H}_D(\mathbb{R}^n) \). From (3.2) and (3.3), we see that
\[
\|f_+ - g_+\|_{\mathcal{H}_D(\mathbb{R}^n)} = \|U_0(t)(f_+ - g_+)\|_{\mathcal{H}_D(\mathbb{R}^n)} \rightarrow 0 \quad (t \rightarrow \infty),
\]
which implies \( f_+ = g_+ \) in \( \mathcal{H}_D(\mathbb{R}^n) \). Since the H"{o}lder inequality and the Sobolev imbedding theorem imply that, for any \( R > 0 \), there exists a positive constant \( C_R \) such that we have
\[
\|v\|_{L^2(B_R)} \leq C_R \|v\|_{L^{2n/(n-2)}(\mathbb{R}^n)} \leq C_R \|\nabla v\|_{L^2(\mathbb{R}^n)}
\]
for any \( v \in \dot{H}^1(\mathbb{R}^n) \), we conclude that \( f_+ = g_+ \in \mathcal{H}^0(\mathbb{R}^n) \).

Next we consider the existence part. We set \( a_* = \max\{a, 3\} \), and we fix a nonnegative integer \( k \). We put \( \mu = \sigma(a_*)/4 \) and \( C_1 = C_0(k, a_*) \), where \( \sigma \) and \( C_0 \) are from Lemma 2.4. We choose \( T(\geq a_* + 2) \) to be so large that \( C_1 \exp(-\mu T) \leq 1 \). Then we see from Lemma 2.4 that for \( f \in X_0^\infty(\Omega) \), there exist \( g_1 \in \mathcal{H}_D^{\infty}(\mathbb{R}^n) \) and \( \tilde{f}_1 \in X_3^\infty(\Omega) \) satisfying (2.5),
\[
\|g_1\|_{\mathcal{H}^k(\mathbb{R}^n)} \leq C_1(1 + T)\|\tilde{f}_1\|_{\mathcal{H}^k(\Omega)},
\]
and
\[
\|\tilde{f}_1\|_{\mathcal{H}^k(\Omega)} \leq C_1 \exp(-4\mu T) \|\tilde{f}_1\|_{\mathcal{H}^k(\Omega)} \leq \exp(-3\mu T) \|\tilde{f}_1\|_{\mathcal{H}^k(\Omega)}.
\]
We apply Lemma 2.4 to \( \tilde{f}_1 \) again to find \( g_2 \in \mathcal{H}_D^{\infty} (\mathbb{R}^n) \) and \( \tilde{f}_2 \in X_3^\infty(\Omega) \) for which we have
\[
U(t - T)\tilde{f}_1 = U_0(t - 2T)g_2 + U(t - 2T)\tilde{f}_2 \quad \text{for} \; t \geq 2T,
\]
\[
\|g_2\|_{\mathcal{H}^k(\mathbb{R}^n)} \leq C_1(1 + T)\|\tilde{f}_1\|_{\mathcal{H}^k(\Omega)} \leq C_1(1 + T) \exp(-3\mu T) \|\tilde{f}_1\|_{\mathcal{H}^k(\Omega)},
\]
and
\[
\|\tilde{f}_2\|_{\mathcal{H}^k(\Omega)} \leq \exp(-3\mu T) \|\tilde{f}_1\|_{\mathcal{H}^k(\Omega)} \leq \exp(-6\mu T) \|\tilde{f}_1\|_{\mathcal{H}^k(\Omega)}.
\]
Repeating the same procedure, we can construct sequences \( \{g_j\}_{j=1}^\infty \subset \mathcal{H}_D^{\infty}(\mathbb{R}^n) \) and \( \{\tilde{f}_j\}_{j=1}^\infty \subset X_3^\infty(\Omega) \) in such a way that
\[
U(t - (j - 1)T)\tilde{f}_{j-1} = U_0(t - jT)g_j + U(t - jT)\tilde{f}_j, \quad t \geq jT,
\]
\[
\|g_j\|_{\mathcal{H}^k(\mathbb{R}^n)} \leq C_1(1 + T) \exp(-3\mu(j - 1)T) \|\tilde{f}_j\|_{\mathcal{H}^k(\Omega)},
\]
and

\[(3.6) \quad \|\tilde{f}_j\|_{\mathcal{H}^k(\Omega)} \leq \exp(-3\mu j T) \|\tilde{f}\|_{\mathcal{H}^k(\Omega)}\]

for \( j \geq 1 \), where we have put \( \tilde{f}_0 = \tilde{f} \).

Now, we define \( \tilde{f}_+ = \sum_{j=1}^{\infty} U_0(-jT)\tilde{g}_j \), which belongs to \( \mathcal{H}^\infty(\mathbb{R}^n) \).

In fact, (2.19) and (3.5) lead to

\[(3.7) \quad \|U_0(-jT)\tilde{g}_j\|_{\mathcal{H}^k(\mathbb{R}^n)} \leq C(1 + jT) \exp(-3\mu j - 1) \|\tilde{f}\|_{\mathcal{H}^k(\Omega)} \leq C \exp(-2\mu(j - 1)T) \|\tilde{f}\|_{\mathcal{H}^k(\Omega)},\]

where \( C \) is a constant depending on \( k \) and \( T \), but is independent of \( j \). Here we have used \( (1 + T + y) \exp(-\mu y) \leq \mu^{-1} \exp(\mu(1 + T) - 1) \) for \( y \in \mathbb{R} \). Therefore we have

\[(3.8) \quad \|\tilde{f}_+\|_{\mathcal{H}^k(\mathbb{R}^n)} \leq \sum_{j=1}^{\infty} C \exp(-2\mu T)\|\tilde{f}\|_{\mathcal{H}^k(\Omega)} \leq C\|\tilde{f}\|_{\mathcal{H}^k(\Omega)} .\]

Next we prove (1.10). For \( \tilde{h} = (h_0, h_1) \), we write

\[(3.9) \quad |\tilde{h}(x)|_k = \sum_{|\alpha| \leq k+1} |\partial^\alpha_x h_0(x)| + \sum_{|\alpha| \leq k} |\partial^\alpha_x h_1(x)|\]

in what follows. Since \( \text{supp} \tilde{g}_j \subset B_{T+a_*} \), the Huygens principle implies

\[(3.10) \quad \text{supp} (U_0(t - jT)\tilde{g}_j) \subset \{ x \in \mathbb{R}^n; \ |x| - |jT - t| \leq T + a_* \} \]

for any natural number \( j \) and \( t \in \mathbb{R} \). Hence it follows from the Sobolev imbedding theorem and (3.7) that

\[(3.11) \quad |(U_0(-jT)\tilde{g}_j)(x)|_k \leq C \|U_0(-jT)\tilde{g}_j\|_{\mathcal{H}^{k+[n/2]+1}(\mathbb{R}^n)} \leq C \exp(-2\mu(j - 1)T) \|\tilde{f}\|_{\mathcal{H}^{k+[n/2]+1}(\Omega)} \leq C \exp(-2\mu|x|) \|\tilde{f}\|_{\mathcal{H}^{k+[n/2]+1}(\Omega)} \]

for \( x \in \text{supp} (U_0(-jT)\tilde{g}_j) \), where \( C \) is a constant depending on \( k, a \) and \( T \), but is independent of \( j \) and \( x \). Noting that, for each fixed \( x \in \Omega \), the number of \( j \) for which we have \( x \in \text{supp} (U_0(-jT)\tilde{g}_j) \) is at most \( \lfloor 2(T + a_*)/T \rfloor + 1 \) (cf. (3.10)), we obtain (1.10) from (3.11).

Next we prove (1.9). For \( t \geq T \), we find a positive integer \( J \) such that \( t \in [JT, (J + 1)T) \). By (3.4) with \( j = 1, \ldots, J \) we have

\[ U(t)\tilde{f} = \sum_{j=1}^{J} U_0(t - jT)\tilde{g}_j + U(t - JT)\tilde{f}_J. \]
Since $U_0(t)\tilde{f}_+ = \sum_{j=1}^{\infty} U_0(t-jT)\tilde{g}_j$, we get

\begin{eqnarray}
(3.12) & \|e^{\mu(\cdot)}U(t)\tilde{f} - U_0(t)\tilde{f}_+\|_{\mathcal{H}_k(\Omega)} \\
& \leq \sum_{j=1}^{\infty} \|e^{\mu(\cdot)}U_0(t-jT)\tilde{g}_j\|_{\mathcal{H}_k(\Omega)} + \|e^{\mu(\cdot)}U(t-JT)\tilde{f}_J\|_{\mathcal{H}_k(\Omega)}.
\end{eqnarray}

Note that (3.12) is also valid for $0 \leq t < T$, by regarding $J = 0$ and $\tilde{f}_0 = \tilde{f}$. So we assume $J \geq 0$ and $t \in [JT, (J+1)T)$ in the following. Since $|t-JT| \leq T$ for $t \in [JT, (J+1)T)$, we get

$$\sum_{|\alpha| \leq k+1} |\partial_\alpha^\alpha \exp(\mu(x))| \leq C_k \exp(\mu(2T + a_*))$$

for $x \in \text{supp}(U(t-JT)\tilde{f}_J)$ with some positive constant $C_k$ depending only on $k$ and $\mu(= \sigma(a_*)/4)$. Thus the second term on the right-hand side of (3.12) is estimated by

$$C(1 + |t-JT|)\|\tilde{f}_J\|_{\mathcal{H}_k(\Omega)} \leq C(1 + T) \exp(-3\muJT) \|\tilde{f}\|_{\mathcal{H}_k(\Omega)}.$$  

Here we have used (2.3) and (3.6), and the constant $C = C(k, a, T)$ is independent of $J$. From (3.10), we get

$$\sum_{|\alpha| \leq k+1} |\partial_\alpha^\alpha \exp(\mu(x))| \leq C_k \exp(\mu(1 + T + a_* + JT - t))$$

$$\leq C_k \exp(\mu(1 + T + a_* + JT))$$

for $x \in \text{supp}(U_0(t-jT)\tilde{g}_j)$ with $j \geq J + 1$, where $C_k$ is a positive constant depending only on $k$ and $\mu(= \sigma(a_*)/4)$. Hence, it follows from (2.19) and (3.5) that

$$\|e^{\mu(\cdot)}U_0(t-jT)\tilde{g}_j\|_{\mathcal{H}_k(\Omega)} \leq C e^{\mu(jT)}(1 + |t-jT|)\|\tilde{g}_j\|_{\mathcal{H}_k(\mathbb{R}^n)}$$

$$\leq C(1 + jT)e^{\mu(jT-3\mu(j-1)T)}\|\tilde{f}\|_{\mathcal{H}_k(\mathbb{R}^n)}$$

$$\leq C e^{-\mu(j-1)T}\|\tilde{f}\|_{\mathcal{H}_k(\Omega)}$$

for $j \geq J + 1$ and $t \in [JT, (J+1)T)$, where $C$ is a constant independent of $j$ and $J$. Thus the first term on the right-hand side of (3.12) is evaluated by $C \exp(-\mu JT) \|\tilde{f}\|_{\mathcal{H}_k(\Omega)}$, where $C$ is a constant independent of $J$. Therefore, (1.9) holds for $t \in [JT, (J+1)T)$ with $J \geq 0$, and hence for all $t \geq 0$.

Finally, we remark that by the uniqueness result, $\tilde{f}_+$ being constructed in the above is independent of $k$, although the construction itself depends on $k$ through the choice of $T$. In fact, let $\tilde{f}_+^{(1)}$ and $\tilde{f}_+^{(2)}$ denote $\tilde{f}_+$ constructed in the above with the choice of $k = k_1$ and $k = k_2$, respectively, where $k_1$ and $k_2$ are nonnegative integers. Then, from
(1.9), (3.1) is valid for $\vec{f}_+ = \vec{f}_+^{(1)}$ and $\vec{f}_+ = \vec{f}_+^{(2)}$. Hence the uniqueness of $\vec{f}_+$ satisfying (3.1) implies $\vec{f}_+^{(1)} = \vec{f}_+^{(2)}$.

This completes the proof of Theorem 1.1. □

4. The Friedlander Radiation Field for Rapidly Decreasing Data

Our aim in this section is to discuss the Friedlander radiation field for the Cauchy problem with rapidly decreasing data. The case of compactly supported data is well known (see Friedlander [2, 3, 4]; see also Hörmander [6] and John [8]). The case of rapidly decreasing data was also treated in [6] through the conformal compactification of the Minkowski space. But the decay away from the light cone was neglected there. Hence we would like to obtain a more detailed estimate, restricting our attention to the odd space dimensional case.

As is known, the behavior of the solution away from the cone is closely related to the decay property of the data. Because the scattering data $\vec{f}_+$ obtained in Theorem 1.1 satisfies the stronger decay property than general functions in $\mathcal{S}(\mathbb{R}^n)$, we introduce the following class of the data. Throughout this section, $\chi = \chi(s)$ is some given non-decreasing function of $s \geq 0$, satisfying $\chi(s) \geq 1$ for all $s \geq 0$. For $\varphi \in C^\infty(\mathbb{R}^n)$, $m \geq 0$ and a nonnegative integer $k$, we define

$$
\|\varphi\|_{\chi,k,m} = \left( \sup_{x \in \mathbb{R}^n} \sum_{|\alpha| \leq k} (1 + |x|^2)^m \chi^2(|x|) |\partial_x^\alpha \varphi(x)|^2 \right)^{1/2},
$$

and let $\mathcal{S}_\chi(\mathbb{R}^n)$ be the set of all $\varphi \in C^\infty(\mathbb{R}^n)$ satisfying $\|\varphi\|_{\chi,m,k} < \infty$ for any nonnegative integers $m$ and $k$. Apparently we have $\mathcal{S}_\chi(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$, where $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz class, the set of rapidly decreasing functions. Note that $\mathcal{S}_\chi(\mathbb{R}^n) = \mathcal{S}(\mathbb{R}^n)$ if $\chi$ is identically equal to 1. Our main result in this section is the following.

**Proposition 4.1.** Let $n$ be an odd integer with $n \geq 3$, and let $\nu \geq 0$. For any $\vec{f} \in (\mathcal{S}_\chi(\mathbb{R}^n))^2$ and any multi-index $\alpha$, there exists a positive
constant $C = C(\alpha, \nu, \vec{f})$ such that we have
\[
\begin{align*}
(4.1) \quad & \left| \Gamma^\alpha \left\{ u(t, x) - r^{-\frac{n-1}{2}} F_0[\vec{f}](r - t, \omega) \right\} \right| \\
& \quad \leq C(1 + t + r)^{-\frac{n+1}{2}} (1 + |r - t|)^{-\nu} \chi^{-1}(|r - t|), \\
(4.2) \quad & \left| \Gamma^\alpha \left\{ \partial_t u(t, x) - (-1)r^{-\frac{n-1}{2}} (\partial_s F_0[\vec{f}])(r - t, \omega) \right\} \right| \\
& \quad + \sum_{j=1}^n \left| \Gamma^\alpha \left\{ \partial_j u(t, x) - \omega_j r^{-\frac{n-1}{2}} (\partial_s F_0[\vec{f}])(r - t, \omega) \right\} \right| \\
& \quad \leq C(1 + t + r)^{-\frac{n+1}{2}} (1 + |r - t|)^{-\nu} \chi^{-1}(|r - t|)
\end{align*}
\]
for $r \geq t/2 \geq 1$ with $r = |t|$ and $\omega = (\omega_1, \ldots, \omega_n) = r^{-1}x$, where $u(t, \cdot)$ is the first component of $U_0(t)\vec{f}$, and the radiation field $F_0[\vec{f}](s, \eta)$ is given by (4.4).

We will give a proof of this proposition, taking a fundamental approach based on the explicit representation of $U_0(t)\vec{f}$, instead of using the conformal compactification.

First we state some basic properties of the Radon transform. We recall that the Radon transform $R[\varphi](s, \eta)$ for $\varphi \in \mathcal{S}(\mathbb{R}^n)$ is defined by
\[
R[\varphi](s, \eta) = \int_{\Pi(s, \eta)} \varphi(y) dS_y, \quad (s, \eta) \in \mathbb{R} \times S^{n-1},
\]
where $\Pi(s, \eta) = \{ y \in \mathbb{R}^n; y \cdot \eta = s \}$, and $dS_y$ denotes the area element on $\Pi(s, \eta)$. For $\eta \in S^{n-1}$ and a smooth function $\varphi = \varphi(y)$ on $\mathbb{R}^n$, $D_\eta \varphi$ denotes the directional derivative of $\varphi$ in the direction $\eta$; in other words, we define $(D_\eta \varphi)(y) = \eta \cdot \nabla_y \varphi(y)$. We write
\[
o_{ij} = \eta_i \partial_{\eta_j} - \eta_j \partial_{\eta_i}, \quad 1 \leq i, j \leq n.
\]
We put $o = (o_1, \ldots, o_{n(n-1)/2}) = (o_{ij})_{1 \leq i < j \leq n}$, where $o_{ij}$'s are regarded to be arranged in dictionary order. We write $o^\alpha = o_1^{\alpha_1} \cdots o_d^{\alpha_d}$ with a multi-index $\alpha$, where $d = n(n-1)/2$. $O^\alpha$ is similarly defined using $O_{ij}$ instead of $o_{ij}$, where $(O_{ij} \varphi)(y) = y_i (\partial_j \varphi)(y) - y_j (\partial_i \varphi)(y)$ as before.

It is easy to check
\[
(4.3) \quad \partial_s R[\varphi](s, \eta) = R[D_\eta \varphi](s, \eta) \left( = \int_{\Pi(s, \eta)} (D_\eta \varphi)(y) dS_y \right),
\]
\[
(4.4) \quad o_{ij} R[\varphi](s, \eta) = R[O_{ij} \varphi](s, \eta), \quad 1 \leq i < j \leq n
\]
for $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Because integrals over $\Pi(s, \eta)$ of directional derivatives of $\varphi$ in directions proportional to $\Pi(s, \eta)$ vanish, we get
\[
(4.5) \quad R[\partial_i \varphi](s, \eta) = R[\eta_i D_\eta \psi](s, \eta) = \eta_i \partial_s R[\varphi](s, \eta)
\]
for $1 \leq i \leq n$.

We observe that if $\varphi \in \mathcal{S}_\chi(\mathbb{R}^n)$, then we have

$$\| D_s^\beta O^\alpha \mathcal{R}[\varphi](s, \eta) \| \leq C_{j,\alpha} \| \varphi \|_{\chi,j+|\alpha|,\mu+n+|\alpha|}(1 + s^2)^{-\beta/2} \chi^{|\alpha|}(|s|)$$

for any $(s, \eta) \in \mathbb{R} \times S^{n-1}$, any $\mu \geq 0$, any nonnegative integer $j$, and any multi-index $\alpha$. Here $C_{j,\alpha}$ denotes a positive constant depending only on $j$ and $\alpha$. In fact, writing $\rho = |y - (y, \eta)|$, we have $|y|^2 = s^2 + \rho^2$ for $y \in \Pi(s, \eta)$. Hence we get

$$\| (D^\beta \varphi)(y) \| \leq C_{j,\alpha}(1 + s^2 + \rho^2)^{-\beta/2} (1 + \rho)^{-n} \chi^{|\alpha|}(|s|)$$

for $y \in \Pi(s, \eta)$. In view of (4.3) and (4.4), we find (4.6).

We also notice that if $\varphi \in \mathcal{S}_\chi(\mathbb{R}^n)$, then we have

$$\| \partial_s^\beta O^\alpha \mathcal{R}[\varphi](s, \eta) \|$$

$$\| \varphi \|_{\chi,k+|\alpha|+|\beta|,\nu+n+|\alpha|+|\beta|} \chi(|s|)$$

for $|s| \geq t/2 \geq 1, \nu \geq 0$, any nonnegative integer $k$, and any multi-indices $\alpha, \beta, \nu$, where $C = C(k, \nu, \alpha, \beta)$ is a positive constant. Since for any $\psi \in C^\infty(S^{n-1})$ we have $O^\beta \{ \psi(|x|^{-1} x) \} = (\alpha^\beta \psi)(|x|^{-1} x)$, it suffices to show (4.8) for $\beta = 0$, thanks to (4.4). By (4.3) we have

$$\partial_s \{ (\partial_s^\beta \mathcal{R}[\varphi])(|x| - t, |x|^{-1} x) \}$$

$$= \left( \eta_i \partial_{s}^{k+1} \mathcal{R}[\varphi](s, \eta) - |x|^{-1} \sum_{j=1}^{n} \eta_j a_{ij} \partial_{s}^{k} \mathcal{R}[\varphi](s, \eta) \right)_{(s, \eta) = (|x| - t, |x|^{-1} x)}$$

$$= (\partial_s^k \mathcal{R}[(\partial_s^\beta \varphi)](|x| - t, |x|^{-1} x) - \sum_{j=1}^{n} \frac{x_j}{|x|^2} (a_{ij} \partial_s^k \mathcal{R}[\varphi])(|x| - t, |x|^{-1} x)$$

for $1 \leq i \leq n$ and any nonnegative integer $k$. Therefore, (4.6) implies

$$\| \partial_s \{ (\partial_s^k \mathcal{R}[\varphi])(|x| - t, |x|^{-1} x) \} - (\partial_s^k \mathcal{R}[(\partial_s^\beta \varphi)](|x| - t, |x|^{-1} x) \|$$

$$\leq C \frac{\| \varphi \|_{\chi,k+1,\nu+n+1}}{(1 + t + |x|) (1 + |x|^{-1} t) \chi(|x|^{-1} t)}$$

for $|x| \geq t/2 \geq 1, 1 \leq i \leq n$, and $\nu \geq 0$, where $C$ is a positive constant depending on $k$ and $\nu$. Hence (4.8) holds for $|\alpha| = 1$. Similarly we obtain it for general $\alpha$.  

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We now turn our attention to the explicit representation of \( u(t, \cdot) \), the first component of \( U_0(t) \mathbf{f} \). Let \( n(\geq 3) \) be an odd integer. It is known that when \( \mathbf{f} = (0, \varphi) \), \( u(t, x) \) is expressed by the following integral:

\[
E[\varphi](t, x) = \frac{\sqrt{\pi}}{2\Gamma(n/2)} \left( \frac{1}{2t} \frac{\partial}{\partial t} \right)^{\frac{n-1}{2}} (t^{n-2} Q[\varphi](t, x)),
\]

where \( \Gamma(s) \) is the Gamma function and we put

\[
Q[\varphi](t, x) = \frac{1}{A_n} \int_{\theta \in S^{n-1}} \varphi(x + t\theta) dS'_\theta
\]

for \( \varphi \in \mathcal{S}(\mathbb{R}^n) \) and \((t, x) \in (0, \infty) \times \mathbb{R}^n \). Here \( A_n \) is the total measure of \( S^{n-1} \), that is \( A_n = 2\pi^{n/2}/\Gamma(n/2) \), and \( dS'_\theta \) is the area element on \( S^{n-1} \) (see, e.g., Courant and Hilbert [1, Chapter VI, Section 12]). Therefore, in general, \( u(t, x) \) can be written as

\[
u(t, x) = \partial_t E[f_0](t, x) + E[f_1](t, x).
\]

We also have

\[
\partial_t^k \partial_x^\alpha O^\beta u(t, x) = \partial_t^{k+1} E[\partial_x^\alpha O^\beta f_0](t, x) + \partial_t^k E[\partial_x^\alpha O^\beta f_1](t, x)
\]

for any nonnegative integer \( k \) and any multi-indices \( \alpha, \beta \) (note that \( \varphi \in \mathcal{S}_\chi(\mathbb{R}^n) \) implies \( \partial_x^\alpha O^\beta \varphi \in \mathcal{S}_\chi(\mathbb{R}^n) \) for any multi-indices \( \alpha \) and \( \beta \)). Hence, once we establish that there exist a large integer \( N \) and a positive constant \( C = C(k) \) such that

\[
\left| \partial_t^k E[\varphi](t, x) - \frac{1}{2(2\pi r)^{n/2}} \left( -\partial_s \right)^{\frac{n-1}{2} + k} \mathcal{R}[\varphi] \right| (r - t, \omega)
\]

\[
\leq C \|\varphi\|_{\chi, \frac{n-1}{2} + k, N}(1 + t + r)^{-\frac{n-1}{2}}(1 + |r - t|)^{-\nu} \chi^{-1}([r - t])
\]

holds for \( \varphi \in \mathcal{S}_\chi(\mathbb{R}^n) \), \( \nu \geq 0 \), \( r(\geq |x|) \geq t/2 \geq 1 \), and \( \omega = r^{-1}x \), we can conclude that Proposition 4.1 is valid, in view of (1.11), (4.5), (4.6), (4.8), and (4.11).

In order to prove (4.12), we observe that

\[
\partial_t^k E[\varphi](t, x) = \frac{\sqrt{\pi}}{2^{\frac{n-1}{2}} \Gamma(n/2)} \sum_{\ell=0}^{\frac{n-1}{2} + k} a_\ell t^{1-k+\ell} \partial_t^\ell Q[\varphi](t, x)
\]

for \( k \geq 0 \), where \( a_\ell \) are suitable constants with \( a_\ell = 1 \) for \( \ell = \frac{n-3}{2} + k \). Let \( \varphi \in \mathcal{S}_\chi(\mathbb{R}^n) \), \( \nu \geq 0 \), \( r(\geq |x|) \geq t/2 \geq 1 \), and \( \omega = r^{-1}x \) in the
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following. Then we see that (4.12) follows from the estimate

\[
\left| \partial^\ell_t Q[\varphi](t, x) - \frac{1}{A_n t^{n-1}} \left( (-\partial_s) t^\ell R[\varphi] \right) (r - t, \omega) \right|
\leq C \|\varphi\| \chi, \frac{n+1}{2} k, \nu (1 + t + r)^{-n} (1 + \mu r) \chi^{-1} (|r - t|).
\]

In fact, when \( 2t \geq r \geq t/2 \geq 1 \), since we have

\[
\left| t^{-\frac{n+1}{2}} - r^{-\frac{n+1}{2}} \right| \leq C (1 + t + r)^{-\frac{n+1}{2}} (1 + |r - t|),
\]

from (4.6) we see that it suffices to prove (4.12) with \((2\pi r)^{-\frac{n+1}{2}}\) in its left-hand side being replaced by \((2\pi t)^{-\frac{n+1}{2}}\). This replaced estimate can be easily proved by (4.13), (4.14) and (4.6). On the other hand, when \( r > 2t \), from (4.6) we get

\[
\left| (-\partial_s) t^\ell R[\varphi](r - t, \omega) \right| \leq C \|\varphi\| \chi, \ell, \mu (1 + t + r)^{-\mu} \chi^{-1} (|r - t|)
\]

for any \( \mu \geq 0 \). Hence, using (4.13) and (4.14), we find that both \(|\partial^\ell_t E[\varphi](t, x)|\) and \(|(-\partial_s) t^\ell R[\varphi](r - t, \omega)|\) are bounded from above by the right-hand side of (4.12). Now we have seen that our task is to prove (4.14).

It follows from (4.10) that

\[
\partial^\ell_t Q[\varphi](t, x) = \frac{1}{A_n} \int_{S_{n-1}} (D_y^\ell \varphi)(x + t\theta)dS'_\theta
\]

\[
= \frac{1}{A_n} \sum_{|\alpha| = \ell} \int_{S_{n-1}} c_{\ell, \alpha}(\theta)(\partial^\alpha \varphi)(x + t\theta)dS'_\theta
\]

\[
= \frac{1}{A_n t^{n-1}} \sum_{|\alpha| = \ell} \int_{S(t, x)} c_{\ell, \alpha} (t^{-1}(y - x)) \partial^\alpha_y \varphi(y)dS^*_y
\]

with some polynomial \( c_{\ell, \alpha} \), where \( S(t, x) = \{ y \in \mathbb{R}^n; |y - x| = t \} \) and \( dS^*_y \) stands for the area element on \( S(t, x) \) (recall that \( D_\theta \varphi \) is the directional derivative of \( \varphi \) in the direction \( \theta \)). On the other hand, (4.3) implies that

\[
(-\partial_s) t^\ell R[\varphi](r - t, \omega) = \sum_{|\alpha| = \ell} c_{\ell, \alpha}(-\omega) R[\partial^\alpha_y \varphi](r - t, \omega).
\]
Thus our task of proving \((4.14)\) is reduced to the estimate
\[
\left| \int_{S(t,x)} c_{\ell,\alpha} (t^{-1}(y - x)) \partial_y^\alpha \varphi(y) dS_y^{*} 
\right.
\]
\[
- c_{\ell,\alpha}(-\omega) R[\partial_y^\alpha \varphi](r - t, \omega)
\]
\[
\leq C\|\varphi\|_{\chi, \frac{n-1}{2} + \kappa, \chi}(1 + t + r)^{-1}(1 + |r - t|)^{-\nu} \chi^{-1}(|r - t|).
\]

In order to proceed further, we decompose the integral over \(S(t, x)\) as follows. Let \(\varepsilon\) be a small and positive constant. For \(r > 0, t > 0\) and \(\omega \in S^{n-1}\), we set
\[
\Lambda^1_0(t, r, \omega) = \{ y \in S(t, r\omega); |y| > (t + r)^{\varepsilon}\},
\]
\[
\Lambda^2_0(t, r, \omega) = \{ y \in S(t, r\omega); |y| \leq (t + r)^{\varepsilon}\}.
\]

When \(|r - t| > (t + r)^{\varepsilon}\), we have \(S(t, r\omega) = \Lambda^1_0(t, r, \omega)\). Therefore, using \((4.6)\) and Lemma 4.2 below to estimate \(\int_{S(t,x)} c_{\ell,\alpha} (t^{-1}(y - x)) \partial_y^\alpha \varphi(y) dS_y^{*}\) and \(c_{\ell,\alpha}(-\omega) R[\partial_y^\alpha \varphi](r - t, \omega)\), respectively, we obtain \((4.17)\). On the other hand, when \(|r - t| \leq (t + r)^{\varepsilon}\), \((4.17)\) is a consequence of Lemmas 4.2 and 4.3 below.

**Lemma 4.2.** Let \(c\) be a bounded function on \(S^{n-1}\), and \(\varphi \in \mathcal{S}_\chi(\mathbb{R}^n)\). Let \(\varepsilon > 0\) and \(\kappa > 0\). Suppose that \(N_1\) is a positive integer satisfying \(N_1\varepsilon \geq \kappa + n - 1\). Then there exists a positive constant \(C = C(\varepsilon, \kappa, N_1)\) such that we have
\[
\left| \int_{\Lambda^1_0(t, r, \omega)} c (t^{-1}(y - r\omega)) \varphi(y) dS_y^{*} \right|
\]
\[
\leq C(1 + t + r)^{-\kappa} \chi^{-1}(|r - t|)\|c\|_{L^\infty(S^{n-1})}\|\varphi\|_{\chi, 0, N_1}
\]
for any \((t, r, \omega) \in [0, \infty) \times [0, \infty) \times S^{n-1}\).

**Proof.** Observing that the total measure of \(S(t, r\omega)\) is bounded by \(A_n t^{n-1}\), and that we have
\[
|\varphi(y)| \leq (1 + |t + r|^{2\varepsilon})^{-N_1/2} \chi^{-1}(|r - t|)\|\varphi\|_{\chi, 0, N_1}
\]
\[
\leq C(1 + t + r)^{-\kappa - n/2} \chi^{-1}(|r - t|)\|\varphi\|_{\chi, 0, N_1}
\]
for any \(y \in \Lambda^1_0(t, r, \omega)\) with some positive constant \(C\), because \(|y| \geq |r - t|\) for any \(y \in S(t, r\omega)\). Thus we obtain the desired result. \(\square\)

**Lemma 4.3.** Let \(c \in C^1(\overline{B_1})\), and \(\varphi \in \mathcal{S}_\chi(\mathbb{R}^n)\). Let \(0 < \varepsilon \leq 1/4\), and \(\nu \geq 0\). Suppose that \(N_2\) is a positive integer satisfying \(N_2 \geq n + 2 + \nu + (1/\varepsilon)\). Then there exists a positive constant \(C = C(\varepsilon, \nu, N_2)\)
such that we have

\[
\begin{align*}
(4.18) \quad & \left| \int_{\Lambda_2^*(t,r,\omega)} c \left( t^{-1}(y - r\omega) \right) \varphi(y) dS^*_y - c(-\omega)R[\varphi](r - t, \omega) \right| \\
& \leq C(1 + t + r)^{-1} (1 + |r - t|)^{-\nu} \chi^{-1}(|r - t|) \\
& \times \|c\|_{C^1(B_1)} \|\varphi\|_{\chi,1,N_2}
\end{align*}
\]

for any \((t, r, \omega) \in [0, \infty) \times [0, \infty) \times S^{n-1}\) with \(r \geq t/2 \geq 1\) and \(|r - t| \leq (t + r)^\varepsilon\), where

\[
\|\psi\|_{C^1(B_1)} = \sup_{y \in B_1} \left( |\psi(y)|^2 + |\nabla_y \psi(y)|^2 \right)^{1/2}
\]

for \(\psi \in C^1(B_1)\).

**Proof.** Since the right-hand side of (4.18) is invariant under the orthogonal transforms, we may assume \(\omega = e_n\) without loss of generality, where \(e_n = (0, \ldots, 0, 1)\).

Suppose \(r \geq t/2 \geq 1\), \(|r - t| \leq (t + r)^\varepsilon\) and \(0 < \varepsilon \leq 1/4\), in the following. Then, since \(t + r \geq 3\), we get \((t + r)^\varepsilon \leq 3^{-3/4} < 1/2\), which implies \((t + r)^\varepsilon < (t + r)/2\). If \(r > 3t\), then we get \(r - t > (t + r)/2 > (t + r)^\varepsilon\), which contradicts the assumption. Hence we obtain

\[
(4.19) \quad 1 \leq \frac{t}{2} \leq r \leq 3t,
\]

and we find that \(t, r\) and \(1 + t + r\) are equivalent to each other.

First we prove that

\[
(4.20) \quad \left| \int_{\Lambda_2^*(t,r,e_n)} c \left( t^{-1}(y - re_n) \right) \varphi(y) dS^*_y \\
- c(-e_n) \int_{\Lambda_2^*(t,r,e_n)} \varphi(y) dS^*_y \right| \\
\leq 2^{\frac{n-2}{2}} A_{n-1} \|c\|_{C^1(B_1)} \|\varphi\|_{\chi,1,N_2} t^{-1} (1 + |r - t|^2)^{-\frac{\varepsilon}{2}} \chi^{-1}(|r - t|).
\]

We put

\[
\lambda_0(t, r) = \frac{(t + r)^{2\varepsilon} - (r - t)^2}{2rt}.
\]

Note that \(0 \leq \lambda_0(t, r) \leq 1\). Writing \(t\lambda = y_n - (r - t)\), we find that \(\Lambda_2^*(t, r, e_n)\) is equal to

\[
\left\{ y = \left( t\sqrt{\lambda(2 - \lambda)} \zeta, r - t + t\lambda \right); \zeta \in S^{n-2}, 0 \leq \lambda \leq \lambda_0(t, r) \right\}.
\]

For the coordinate system \((\zeta, \lambda)\) in the above, we have

\[
(4.21) \quad dS_y^* = t^{n-1} \lambda^{\frac{n-1}{2}} (2 - \lambda)^{\frac{n-1}{2}} d\lambda dS^*_\zeta,
\]
where $dS_y^*$ denotes the area element on $S^{n-2}$. We also note that
\begin{equation}
(4.22) \quad |y|^2 = (r - t)^2 + 2rt\lambda.
\end{equation}

We put $\eta(y, r) = |y - re_n|^{-1}(y - re_n) \in S^{n-1}$. Then we get
\begin{equation}
|\eta(y, r) - (-e_n)| = \sqrt{2 + 2\eta(y, r)} = \sqrt{2}\lambda
\end{equation}
for any $y \in \Lambda^2_n(t, r, e_n)$. Hence, by the mean value theorem, we get
\begin{equation}
\left| \int_{\Lambda^2_n(t, r, e_n)} \{c(\eta(y, r)) - c(-e_n)\} \varphi(y)dS_y^* \right| \leq \sqrt{2}\|c\|_{C^1(B_1)} J(t, r),
\end{equation}
where we put
\begin{equation}
J(t, r) = \int_{\Lambda^2_n(t, r, e_n)} \lambda^{\frac{3}{2}}|\varphi(y)|dS_y^*
\end{equation}
with $t\lambda = y_n - (r - t)$. Recalling (4.21) and (4.22), we obtain
\begin{equation}
(4.23) \quad \chi(|r - t|)J(t, r)
\end{equation}
\begin{align}
&\leq A_{n-1} t^{n-1} \|\varphi\|_{\chi_{0, N_2}} \int_0^{t_0(t, r)} \lambda^{\frac{n-2}{2}}(2 - \lambda)^{\frac{n-3}{2}} \frac{1}{(1 + (r - t)^2 + 2rt\lambda)^\frac{n-2}{2}} d\lambda \\
&\leq 2^{\frac{3}{2}} A_{n-1} \left(\frac{t}{r}\right)^{\frac{n}{2}} \|\varphi\|_{\chi_{0, N_2}} \\
&\quad \times \int_0^{t_0(t, r)} (2r\lambda)^{\frac{n-2}{2}} \frac{1}{(1 + |r - t|^2)^\frac{n-2}{2}} (1 + 2rt\lambda)^\frac{n-2}{2} d\lambda \\
&\leq 2^{\frac{n-1}{2}} A_{n-1} r (1 + |r - t|^2)^{-\frac{n}{2}} \|\varphi\|_{\chi_{0, N_2}} \int_0^\infty \frac{1}{(1 + 2rt\lambda)^2} d\lambda \\
&\leq 2^{\frac{n-3}{2}} A_{n-1} t^{-1} (1 + |r - t|^2)^{-\frac{n}{2}} \|\varphi\|_{\chi_{0, N_2}}.
\end{align}

This estimate yields (4.20) immediately.

By (4.20), we find that, in order to show (4.18), it suffices to prove
\begin{equation}
(4.24) \quad \left| \int_{\Lambda^2_n(t, r, e_n)} \varphi(y)dS_y^* - R[\varphi](r - t, e_n) \right| \leq C t^{-1} (1 + |r - t|)^{-\nu} \chi^{-1}(|r - t|) \|\varphi\|_{\chi_{1, N_2}}
\end{equation}
with some positive constant $C$. We observe that
\begin{equation}
\left| \int_{\Lambda^2_n(t, r, e_n)} \lambda\varphi(y)dS_y^* \right| \leq C (rt)^{-1} (1 + |r - t|)^{-\nu} \chi^{-1}(|r - t|) \|\varphi\|_{\chi_{0, N_2}},
\end{equation}
which can be shown similarly to (4.23). Therefore, (4.24) follows from
\begin{equation}
(4.25) \quad |I_1(r, t) - R[\varphi](r - t, e_n)| \leq C t^{-1} (1 + |r - t|)^{-\nu} \chi^{-1}(|r - t|) \|\varphi\|_{\chi_{1, N_2}},
\end{equation}
where
where we put

\[ I_1(r, t) = \int_{\Lambda(t, e_n)} (1 - \lambda) \varphi(y) dS_y. \]

Introducing a new coordinate \( \rho = t \sqrt{\lambda(2 - \lambda)} \), we get

\[ I_1(t, r) = \int_{S^{n-2}} \int_0^{\rho_0(t, r)} \varphi(\rho \zeta, r - \sqrt{t^2 - \rho^2}) \rho^{n-2} d\rho \] \( dS'_{\zeta} \),

where \( \rho_0(t, r) = t \sqrt{\lambda_0(t, r)(2 - \lambda_0(t, r))} \). While, we have

\[ R[\varphi](r - t, e_n) = I_2(t, r) + I_3(t, r), \]

where we put

\[ I_2(t, r) = \int_{S^{n-2}} \int_0^{\rho_0(t, r)} \varphi(\rho \zeta, r - t) \rho^{n-2} d\rho \] \( dS'_{\zeta} \),

\[ I_3(t, r) = \int_{S^{n-2}} \int_{\rho_0(t, r)}^{\infty} \varphi(\rho \zeta, r - t) \rho^{n-2} d\rho \] \( dS'_{\zeta} \).

Since \( t - \sqrt{t^2 - \rho^2} = \rho^2 \left( t + \sqrt{t^2 - \rho^2} \right)^{-1} \), we get

\[
\left| \varphi(\rho \zeta, r - \sqrt{t^2 - \rho^2}) - \varphi(\rho \zeta, r - t) \right| \\
\leq \left( t - \sqrt{t^2 - \rho^2} \right) \int_0^1 \left| \partial_\tau \varphi \right| (\rho \zeta, r - t + \tau \left( t - \sqrt{t^2 - \rho^2} \right)) d\tau \\
\leq \frac{\rho^2 \| \varphi \|_{H^1, N_2}}{t(1 + |r - t|^2)^{\frac{\nu}{2}}(1 + \rho^2)^{\frac{\nu}{2}} \chi(|r - t|)}
\]

for \( 0 \leq \rho \leq \rho_0(t, r) \), which yields

\[ |I_1(t, r) - I_2(t, r)| \]

\[
\leq \frac{A_{n-1} \| \varphi \|_{H^1, N_2}}{t(1 + |r - t|^2)^{\frac{\nu}{2}}\chi(|r - t|)} \int_0^\infty \frac{\rho}{(1 + \rho^2)^{\frac{\nu}{2}}} d\rho \\
\leq CA_{n-1} \| \varphi \|_{H^1, N_2} t^{-1} (1 + |r - t|)^{-\nu} \chi^{-1}(|r - t|),
\]

where \( C \) is a positive constant depending only on \( \nu \).

Finally, we evaluate \( I_3(t, r) \). Notice that

\[ (\rho_0(t, r))^2 + (r - t)^2 \geq \frac{3}{32} (t + r)^2. \]

In fact, it is trivial when \( (t + r)^{2\varepsilon}/2 \leq (r - t)^2 \leq (t + r)^{2\varepsilon} \). On the other hand, when \( (r - t)^2 \leq (t + r)^{2\varepsilon}/2 \), recalling that we have \( (t + r)^{\varepsilon} <
(t + r)/2, we get
\[
(p_0(t, r))^2 = \frac{(t + r)^2 - (t + r)^{2\varepsilon}}{4r^2} \{(t + r)^{2\varepsilon} - (r - t)^2\} \geq \frac{3}{32}(t + r)^{2\varepsilon},
\]
which shows (4.28). We thus find
\[
\psi
\]
for any \(x\) apparent, and the first half is nothing but the latter half with \(r_1.1, \) there exists \(f\) respectively.

Let the assumptions in Theorem 1.2 be fulfilled. Then, by Theorem 1.1, there exists \(f_+ \in H^\infty(\mathbb{R}^n)\) satisfying (1.9) and (1.10). We write \(u(t, \cdot)\) and \(u_+(t, \cdot)\) for the first components of \(U(t)\) and \(U_0(t)\) respectively.

First we claim that we have
\[
C^{-1}e^{A(x)} \sum_{|\alpha| \leq k} |\partial^\alpha_x \psi(x)| \leq \sum_{|\alpha| \leq k} |\partial^\alpha_x \psi(x)| \leq Ce^{A(x)} \sum_{|\alpha| \leq k} |\partial^\alpha_x \psi(x)|
\]
for any \(x \in \mathbb{R}^n\) and any \(\psi \in C^\infty(\mathbb{R}^n)\). In fact, the latter half is almost apparent, and the first half is nothing but the latter half with \(A\) and \(e^{A(x)}\psi(x)\) being replaced by \(-A\) and \(e^{A(x)}\psi(x)\), respectively.

Let the assumptions in Theorem 1.2 be fulfilled. Then, by Theorem 1.1, there exists \(f_+ \in H^\infty(\mathbb{R}^n)\) satisfying (1.9) and (1.10). We write \(u(t, \cdot)\) and \(u_+(t, \cdot)\) for the first components of \(U(t)\) and \(U_0(t)\) respectively.

First we claim that we have
\[
|\Gamma^\alpha u(t, \cdot) - \Gamma^\alpha u_+(t, \cdot)| \leq C \exp(-mt)\|f\|_{H^{[n/2]+1}(\Omega)}, \quad t \geq 0
\]
for any multi-index \(\alpha\). Let \(\Gamma^\alpha = \partial^\beta_x \partial^\gamma O^\nu\) with a nonnegative integer \(j,\) and multi-indices \(\beta, \gamma,\) and let \(\alpha'\) be a multi-index satisfying \(|\alpha'| \leq [n/2] + 1.\) If \(j\) is even, then we get
\[
|\partial^\gamma_x \Gamma^\alpha (u - u_+(t, x))| = |\Delta^{j/2} \partial^\alpha_x \partial^\beta O^\nu (u - u_+) (t, x)|
\]
\[
\leq C(1 + |x|) \sum_{|\beta'| \leq |\alpha| + |\alpha'|} |\partial^\beta_x (u - u_+) (t, x)|
\]
\[
\leq Ce^{\mu(x)/4} \sum_{|\beta'| \leq |\alpha| + |\alpha'|} |\partial^\beta_x (u - u_+) (t, x)|
\]

5. Proof of Theorem 1.2

To begin with, we note that, for any \(A \in \mathbb{R}\) and any nonnegative integer \(k,\) there exists a positive constant \(C\) such that we have

\[
C^{-1}e^{A(x)} \sum_{|\alpha| \leq k} |\partial^\alpha_x \psi(x)| \leq \sum_{|\alpha| \leq k} |\partial^\alpha_x \psi(x)| \leq Ce^{A(x)} \sum_{|\alpha| \leq k} |\partial^\alpha_x \psi(x)|
\]
with some positive constant $C = C(\alpha, \alpha', \mu)$. Similarly, if $j$ is odd, we get
\[
|\partial^\gamma_x \Gamma^\alpha (u - u_+)(t, x)| \leq C e^{\mu (x)/4} \sum_{|\beta'| \leq |\alpha| + |\alpha'|-1} |\partial^\beta_x \partial_t (u - u_+)(t, x)|.
\]
Hence the left-hand side of (5.1) is bounded by
\[
C \| \exp(\mu \langle \cdot \rangle)(U(t) \vec{f} - U_0(t) \vec{f}_+)|_{H^{n/2} + |\alpha| (\Omega)}
\]
and (1.9) implies (5.1).

By (5.1) and the Sobolev imbedding theorem, we get
\[
|\Gamma^\alpha (u - u_+)(t, x)|
\leq C_k e^{-3\mu (x)/4} \| e^{3\mu \langle \cdot \rangle/4} \Gamma^\alpha (u - u_+)(t, \cdot) \|_{H^{n/2} + 1 (\Omega)}
\leq C_k e^{-3\mu (t + |x|)/4} \| \vec{f} \|_{H^{k+|n/2|} (\Omega)}
\leq C_k (1 + t + |x|)^{-(n+1)/2} e^{-\mu (t + |x|)/2} \| \vec{f} \|_{H^{k+|n/2|} (\Omega)}
\]
for $|x| \geq t/2 \geq 1$ and $|\alpha| \leq k$, where $C_k$ is a positive constant. Hence we find that our task is to show (1.12) and (1.13) with $u$ being replaced by $u_+$.

From (1.10) we see that $\vec{f}_+ \in (S_\chi (\mathbb{R}^n))^2$ with $\chi(s) = \exp(\mu s/2)$. Therefore Proposition 4.1 with $\nu = 0$ immediately implies (1.12) and (1.13) with $u$ being replaced by $u_+$. This completes the proof. \hfill \square

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