Abstract. We present some new nonparametric estimators of entropies and we establish almost sure consistency and central limit Theorems for some of the most important entropies in the discrete case. Our theoretical results are validated by simulations.

1. Introduction

1.1. Motivation. Consider an outcome $A$ of a random experiment on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. The information amount or content of the outcome $A$ is (see Carter (2014))

$$I(A) = \log_2 \frac{1}{\mathbb{P}(A)},$$

where $\log_2$ is the logarithm base 2.

Let $X$ be a discrete random variable defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and taking values in the finite countable space $X = \{c_1, c_2, \cdots, c_r\}$ with set of all possible values of $X$.

The probability distribution $p = (p_j)_{j=1, \cdots, r}$ of the events $(X = c_j)$, coupled with the information amount of every event $I(X = c_j)_{j=1, \cdots, r}$, forms a random variable whose expected value is the average amount of information, or entropy (more specifically, Shannon entropy) generated by this distribution.

**Definition 1.** Let $X$ be a discrete random variable defined on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and taking values in the finite countable space $X = \{c_1, c_2, \cdots, c_r\}$ with $r \geq 2$ with probabilities mass function (p.m.f.) $(p_{\cdot, j})_{j=1, \cdots, r}$, that is, $p_j = \mathbb{P}(X = c_j) \quad \forall j \in J = \{1, \cdots, r\}$.

The Shannon entropy of the random variable $X$ is given by

$$E_{Sh}(X) = \sum_{j=1}^{r} p_j \log_2 \frac{1}{p_j} = \mathbb{E}(\log_2(p)).$$

Entropy is usually measured in bits (binary information unit) (if $\log_2$), nats (if $\ln$), or hartley (if $\log_{10}$), depending on the base of the logarithm which is used to define it.

For ease of computations and notation convenience, we use the natural logarithm ($\ln$) since logarithms of varying bases are related by a constant.

In the sequel, we consider the entropy of the discrete random variable $X$ as a function of discrete probabilities $p = (p_j)_{j \in J}$.

1.2. Generalizations of Shannon entropy. Inspired by the study of $\alpha$-deformed algebras and special functions, various generalizations have been investigated.

Most notably, Rényi (1960) proposed a one parameter family of entropies extending Shannon entropy.
(b) The $\alpha$–Rényi entropy of the random variable $X$ is defined by

$$E_{R,\alpha}(p) = \frac{1}{1-\alpha} \ln \left( \sum_{j=1}^{r} p_j^\alpha \right),$$

with $\alpha \in (0,1) \cup (1, +\infty)$, which, in particular, reduces to the Shannon entropy in the limit $\alpha \to 1$.

(c) Also, the $\alpha$–Tsallis entropy of the random variable $X$ defined by (see Tsallis (1988)) :

$$E_{T,\alpha}(p) = \frac{1}{1-\alpha} \left( \sum_{j=1}^{r} p_j^\alpha - 1 \right), \quad \alpha \in (0,1) \cup (1, +\infty)$$

has generated a large burst of research activities.

Let cite a few other examples of entropies.

(d) The $\alpha$–Landsberg-Vedral entropy also called normalized Shannon entropy of the random variable $X$ is defined by (see Landsberg & Vedral (1998)) :

$$E_{L.V,\alpha}(p) = \frac{1}{1-\alpha} \left( 1 - \frac{1}{\sum_{j=1}^{r} p_j^\alpha} \right) = \frac{E_{T,\alpha}(p)}{\sum_{j=1}^{r} p_j^\alpha}, \quad \alpha \in (0,1) \cup (1, +\infty).$$

(e) The $\alpha$–Abe entropy of the random variable $X$ is defined by (see Abe (1997)) :

$$E_{Ab,\alpha}(p) = -\frac{1}{\alpha - \alpha^{-1}} \sum_{j=1}^{r} (p_j^\alpha - p_j^{\alpha^{-1}}), \quad \alpha \in (0,1) \cup (1, +\infty).$$

(f) The $\kappa$-entropy of the random variable $X$ is defined by the following expression (see Kaniadakis (2002)) :

$$E_{\kappa}(p) = \frac{1}{2\kappa} \sum_{j=1}^{r} (p_j^{1-\kappa} - p_j^{1+\kappa}), \quad \kappa \in (0,1).$$

(g) The Varma’s entropy of order $\alpha$ and type $\beta$ of the random variable $X$ is defined by

$$E_{V,\alpha,\beta}(p) = \frac{1}{\beta - \alpha} \ln \left( \sum_{j=1}^{r} p_j^{\alpha + \beta - 1} \right), \quad \text{for} \quad \beta - 1 < \alpha < \beta, \quad \beta \geq 1.$$

Interestingly, the Landsberg-Vedral and $\kappa$ entropies reduce to the Shannon entropy in the limit $\alpha \to 1$ and $\kappa \to 0$ respectively.

From this small sample of entropies, we may give the following remarks :

(a) For most entropies, we may have computation problems. So without loss of generality, suppose

$$p_j > 0, \quad \forall j \in J = \{1, \cdots, r\} \quad \text{(BD)}.$$
and Poczos (2014), Krishnamurthy et al. (2014), Hall (1987), to cite a few.

(b) The power sum of order \( \alpha \in (0, 1) \cup (1, +\infty) \) of the distribution \( p \) over \( \{c_j, j \in J\} \) is

\[
S_\alpha(p) = \sum_{j \in J} p_j^\alpha,
\]

and, is related to Reyni, Tsallis, Landsberg-Vedral, Abel, \( \kappa \), and Varma entropies via

\[
E_{R,\alpha}(p) = \frac{1}{1 - \alpha} \ln (S_\alpha(p)), \quad E_{T,\alpha}(p) = \frac{1}{1 - \alpha} (S_\alpha(p) - 1),
\]

\[
E_{L.V,\alpha}(p) = \frac{1}{1 - \alpha} \left(1 - \frac{1}{S_\alpha(p)}\right), \quad E_{Ab,\alpha}(p) = -\frac{1}{\alpha - \alpha^{-1}} (S_\alpha(p) + S_{\alpha^{-1}}(p)),
\]

\[
E_{\kappa}(p) = \frac{1}{2\kappa} (S_{1-\kappa}(p) - S_{1+\kappa}(p)), \quad \text{and} \quad E_{V,\alpha,\beta}(p) = \frac{1}{\beta - \alpha} \ln (S_{\alpha+\beta-1}(p)).
\]

Hence establishing asymptotic limits of estimators of these ones is equivalent to establishing asymptotic limits of \( S_\alpha(\hat{p}_n) \).

1.3. Bibliography and applications. Although we are focusing on the aforementioned entropies in this paper, it is worth mentioning that there exist quite a few number of them.

Let us cite for example the ones named after: Fuzzy Entropy (see Luca & Termini (1972), Bhandari & Pal (1993), Kosko (1986), Pal & Bezdek (1994), Yager (2000)), Havrda-Charvát entropy (see Havrda & Charvát (1967)), Generalized Entropy also called \( f \)-divergence (see Liese & Vajda (2006), Balestrino et al. (2009)), Frank-Daffertshofer entropy (see Frank & Daffertshofer (2000)), Kapur measure (see Kapur (1986)), Hartley entropy, min entropy and max entropy (see Dodis el al. (2008)), collision entropy etc.

Recently, there have been made several successful attempts in order to categorize the various entropy classes and their properties: Hanel & Thurner (2011), Hanel et al. (2014) classified the entropies according to their asymptotic scaling. Tempesta (2011) studied the Generalized entropies also called \( f \)-divergence (see Liese & Vajda (2006), Balestrino et al. (2009)), Frank-Daffertshofer entropy (see Frank & Daffertshofer (2000)), Kapur measure (see Kapur (1986)), Hartley entropy, min entropy and max entropy (see Dodis el al. (2008)), collision entropy etc.

Before coming back to our entropies estimation of interest, we want to highlight some important applications of them.

Indeed, entropy has proven to be useful in applications. Let us cite some of them:

(a) The entropy concept was born initially in thermodynamics by Clausius (1870) to measure the ratio of transferred heat through a reversible process in an isolated system and to measure of uncertainty about the system that remains after observing its macroscopic properties (pressure, temperature or volume). Since then, entropy has been of great theoretical and applied interest.

(b) In finance, Philippatos & Wilson (1972) were the first two authors who applied the concept of entropy to portfolio selection. It has been used as a risk measure for stock, for portfolio returns, for portfolio diversifications (see Ormos & Zibriczky (2014)), it has been applied as measure of investment risk in the discrete case (see Nawrocki & Harding (1986)), as well as a measure of dependence in return time series (see Maasoumi & Racine (2002)).

(c) While a significant number of other entropies have since been introduced, Rényi entropy is especially important because it is a well known one parameter generalization of Shannon entropy. It is often used as a bound on Shannon entropy (see Mokkadem (1989), Nemenman (2004), Harvey (2008), and
it replaces Shannon entropy as a measure of randomness (see Csiszár (1995), Massey (1994), Arikan (1996), etc.). It generalizes also Hartley, collision, and min-entropy. It has successfully been used in a
number of different fields, such as statistical physics, quantum mechanics, communication theory and
data processing (see Jizba & Arimitsu (2004), Csiszár (1995)), in the context of channel coding (see
Arimoto (1977)), secure communication (see Cachin (1997)), and Ayashi (2011)), multifractal analysis
(see Jizba & Arimitsu (2004)).

In the context of fractal dimension estimation, Rényi entropy forms the basis of the concept of gen-
eralized dimensions. It intervene as well in ecology and statistics as index of diversity.

Rényi entropy is also of interest in its own right, with diverse applications in unsupervised learning (see
Xu (1998), Jenssen et al. (2003), source adaptation (see Mansour et al. (2012), image registration (see
Ma et al. (2000), see Neemuchwala et al. (2006), and password guessability (see Arikan (1996), Pfister
and W. Sullivan (2004), Hanawal and R. Sundaresan (2011) among others. In particular, the Rényi
entropy of order 2 measures the quality of random number generators (see Knut (1973)), Oorschot and
M. J. Wiener (1999), determines the number of unbiased bits that can be extracted from a physical
source of randomness see Impagliazzo and Zuckerman (1989), Bennett et al. (1995), helps test graph
expansion Goldreich and Ron (2000) and closeness of distributions Batu et al. (2013), and characterizes
the number of reads needed to reconstruct a DNA sequence Motahari et al. (2013).

The Rényi entropy is important in ecology and statistics as index of diversity. It is also important in
quantum information, where it can be used as a measure of entanglement.

e) Varma’s entropy plays a vital role as a measure of complexity and uncertainty in different areas
such as physics, electronics and engineering to describe many chaotic systems

f) In the context of multi-dimensional harmonic oscillator systems, the Sharma–Mittal entropy has
previously been studied (see Uzengi et al. (2008)).

1.4. Previous work. The estimation of entropies have become growingly important for their wide
applications in the fields of neural science and information theory, etc.

For example Shannon entropy estimation has several applications, including measuring genetic diversity
(see Shenkin et al. (1991), quantifying neural activity (see Paninski (2003)), see Nemenman (2004),
network anomaly detection Lall et al. (2006), and others.

Most texts on entropy estimation deal with Shannon entropy estimation and use the plug-in method.

Xing (2013) showed that, if \{p_j, j \geq 1\} is non uniform distribution satisfying \(\mathbb{E}(\log P_X)^2 < \infty\),
and if there exists an integer valued function \(J(n)\) such that, \(J(n) \to +\infty, \quad J(n) = o(\sqrt{n})\) and
\(\sqrt{n} \sum_{j \geq J(n)} p_j \log p_j \to 0\), as \(n \to \infty\), then
\[
\sqrt{n}(\mathcal{E}_{Sh}(\hat{p}_n) - \mathcal{E}_{Sh}(p)) \sim N(0, \sigma_{Sh}^2(p)) \quad \text{as} \quad n \to +\infty
\]
where \(\sigma_{Sh}^2(p) = Var(-\log P_X) > 0\).

Zhang (2012) proposed a non parametric estimator of Shannon’s entropy on a countable alphabet
\[
\mathcal{E}^{(Z)}(\hat{p}_n) = \sum_{\ell=1}^{n-1} \ell \left\{ \frac{n^{\ell+1}[n-(\ell+1)]!}{n!} \sum_j \hat{p}_{nj}^{\ell-1} \prod_{i=0}^{\ell-1} \left( 1 - \hat{p}_{ni} - \frac{i}{n} \right) \right\}
\]
and established that
\[
\mathbb{E} \left( \mathcal{E}^{(Z)}(\hat{p}_n) \right) - \mathcal{E}(p) = O \left( \frac{(1-p_0)^n}{n} \right)
\]
where \( p_0 = \min_{j \in J} \{ p_j \} \).

Later on Miller (1955), Basharin (1959), and Harris (1975) established that

\[
\mathbb{E} (\mathcal{E}_{Sh}(\hat{p}_n) - \mathcal{E}_{Sh}(p)) = -\frac{r-1}{2n} + \frac{1}{12n^2} \left( 1 - \sum_{j=1}^r \frac{1}{p_k} \right) + O(n^{-3})
\]

\[
\text{Var} (\mathcal{E}_{Sh}(\hat{p}_n)) = \frac{1}{n} \left( \sum_{j=1}^r p_j \ln p_j \right)^2 - \left( \mathcal{E}_{Sh}(p) \right)^2 + \frac{r-1}{2n^2} + O(n^{-3})
\]

Antos & Kontoyiannis (2001) proved that

\[
\mathbb{E} (\mathcal{E}(\hat{p}_n) - \mathcal{E}(p)) \sim n^{-(\lambda - 1)/\lambda} \quad \text{and} \quad \text{Var} (\mathcal{E}(\hat{p}_n)) \leq O \left( \frac{(\log n)^2}{n} \right)
\]

provided that the probability distribution \((p_j)_{j \in J}\) satisfies \( p_j = C \lambda^j \), where \( \lambda > 1 \).

Under distributions \( p_j = C j^{-\lambda} \), a necessary condition for

\[
\sqrt{n} (\mathcal{E}(\hat{p}_n) - \mathcal{E}(p))
\]

to hold asymptotic normality is \( \lambda \geq 2 \).

Acharya (2016) focused on the number of samples needed to estimate the \( \alpha \)-Reyni entropy.

However, to our knowledge, no results regarding the almost sure consistency and the asymptotic normality of the most of entropies, are known.

1.5. Main contribution.

Most texts on entropy estimation deal with Shannon entropy estimation whereas we deal with estimation of the most common entropies including Shannon, Tsallis, Reyni, Landsberg-Vedral, Abel entropies, etc by deriving their almost sure convergence and central limit Theorems.

Our method consist in getting first general laws for an arbitrary summation of the form

\[
J(p) = \sum_{j \in J} \phi(p_j),
\]

where \( \phi : (0, 1) \to \mathbb{R} \) is a twice continuously differentiable function.

The results on the summation \( J(p) \), which is also known under the name of \( \phi \)-entropy summation, will lead to results of entropies already mentioned above.
1.6. Overview of the paper.

The rest of the paper is organized as follows. In Section 2, we define estimators \( p_n^c_j \) of the p.m.f \( p_j \) and construct the plug-in estimators of the \( \phi \)–entropy summation \( J(p) = \sum_{j \in J} \phi(p_j) \), where \( \phi \) is a twice continuous differentiable function, from an i.i.d. sample of size \( n \) and according to \( p \). We end this section by giving our full results for the summation \( J(p) \).

In Section 3, we will particularize the results for specific entropies we already described. Section 4 provides the proofs and in Section 5 we present some simulations confirming our results. Finally, in Section 6, we conclude.

2. \( \phi \)–Entropy summation

2.1. Notations and main results.

Let \( X \) be a random variable defined on the probability space \((\Omega, \mathcal{A}, P)\) and taking values \( X = \{c_1, c_2, \cdots, c_r\} \) with p.m.f \( p = (p_j)_{1 \leq j \leq r} \) i.e.,

\[ p_j = P(X = c_j), \quad \forall j \in J = \{1, 2, \cdots, r\}. \]

In general, the full probability distribution \( p = (p_j)_{1 \leq j \leq r} \) is not known and, in particular, in many situations only sets from which to infer entropies are available.

For example, it could be of interest to determine the entropies of a given DNA sequence. In such a case, one could estimate the probability of each element \( c_i \) to occur, \( p_i \).

Let \( X_1, \cdots, X_n \) be \( n \) i.i.d. random variables according to \( p \). For a given \( j \in J \), define the easiest and most objective estimator of \( p_j \), based on the i.i.d sample \( X_1, \cdots, X_n \), by

\[
\hat{p}_n^c_j = \frac{1}{n} \sum_{i=1}^{n} 1_{c_j}(X_i)
\]

where \( 1_{c_j}(X_i) = \begin{cases} 1 & \text{if } X_i = c_j \\ 0 & \text{otherwise} \end{cases} \) for any \( j \in J \).

For a given \( j \in J \), this empirical estimator \( \hat{p}_n^j \) of \( p_j \) is strongly consistent and asymptotically normal. Precisely, when \( n \) tends to infinity,

\[
\hat{p}_n^j - p_j \overset{a.s.}{\to} 0
\]

\[
\sqrt{n}(\hat{p}_n^j - p_j) \overset{D}{\to} Z_{p_j}
\]

where \( Z_{p_j} \sim N(0, p_j(1 - p_j)) \)

We denote by \( \overset{a.s.}{\to} \) the almost sure convergence and \( \overset{D}{\to} \) the convergence in distribution. The notation \( \overset{d}{\sim} \) denote the equality in distribution.

These asymptotic properties derive from the law of large numbers and central limit theorem.

The entropy of \( p \) can be approximated by simply replacing the probabilities \( p_j \) by \( \hat{p}_n^j \) in the entropy summation. For example, the Shannon entropy \( \mathcal{E}_{Sh}(p) \) can be estimated by its counter part plug-in

\[
\mathcal{E}_{Sh}(\hat{p}_n) = - \sum_{j=1}^{r} \hat{p}_n^j \ln(\hat{p}_n^j)
\]
2.2. $\phi$–entropy summation.

**Definition 2.** Let $\phi : (0, 1) \to \mathbb{R}$ a twice continuously differentiable function. The $\phi$–entropy summation of the probability distribution $p = (p_j)_{j \in J}$ is given by

\[
J(p) = \sum_{j \in J} \phi(p_j).
\]

The results on the summation $J(p)$ will lead to those on the particular cases of the Shannon, Rényi, Tsallis, Landsberg-Vedral, Abe, Varma and $\kappa$–entropies.

Based on (2.1), we will use the following $\phi$–entropy summation.

\[
J(\hat{p}_n) = \sum_{j \in J} \phi(\hat{p}_n^j).
\]

2.3. **Statement of the main result.** It concerns the almost sure efficiency and the asymptotic normality of the summation $\phi$–entropy $J(\hat{p}_n)$.

Denote

\[
A_J(p) = \sum_{j \in J} |\phi'(p_j)|
\]

and

\[
\sigma^2(p) = \sum_{j \in J} p_j(1 - p_j)(\phi'(p_j))^2 - 2 \sum_{(i,j) \in J^2, i \neq j} (p_ip_j)^{3/2} \phi'(p_i)\phi'(p_j).
\]

**Theorem 1.** Let $p = (p_j)_{j \in J}$ a probability distribution and $\hat{p}_n = (\hat{p}_n^j)_{j \in J}$ be generated by i.i.d. sample $X_1, \ldots, X_n$ copies of a random variable $X$ according to $p$ and (1.6) be satisfied. Then the following asymptotic results hold

\[
\limsup_{n \to +\infty} \frac{|J(\hat{p}_n) - J(p)|}{a_n} \leq A_J(p), \quad a.s.,
\]

\[
\sqrt{n}(J(\hat{p}_n) - J(p)) \overset{D}{\to} N(0, \sigma^2_J(p)), \quad \text{as } n \to +\infty.
\]

3. **Entropies asymptotic limit law**

(A-) Asymptotic behavior of $S_\alpha(\hat{p}_n)$.

For $\alpha \in (0, 1) \cup (1, +\infty)$, denote

\[
A_S(\alpha) = \sum_{j \in J} \left| \sum_{j \in J} \alpha^{\alpha - 1} p_j^\alpha - \alpha^{\alpha - 1} \right|
\]

and

\[
\sigma^2_S(\alpha) = \alpha^2 \left( \sum_{j \in J} (1 - p_j)p_j^{2\alpha - 1} - 2 \sum_{(i,j) \in J^2, i \neq j} (p_ip_j)^\alpha\right).
\]

**Corollary 1.** Under the same assumptions as in Theorem 1 and for $\alpha \in (0, 1) \cup (1, +\infty)$, the following hold

\[
\limsup_{n \to +\infty} \frac{|S_\alpha(\hat{p}_n) - S_\alpha(p)|}{a_n} \leq A_S(\alpha), \quad a.s.
\]

\[
\sqrt{n}(S_\alpha(\hat{p}_n) - S_\alpha(p)) \overset{D}{\to} N(0, \sigma^2_S(\alpha)), \quad \text{as } n \to +\infty.
\]
(B)- Asymptotic behavior of Shannon entropy estimator.

Let
\[ A_{S_h}(\mathbf{p}) = \sum_{j \in J} |1 + \ln(p_j)| \]
and
\[ \sigma^2_{S_h}(\mathbf{p}) = \sum_{j \in J} p_j (1 - p_j)(1 + \ln(p_j))^2 - 2 \sum_{(i,j) \in J^2, i \neq j} (p_i p_j)^{3/2}(1 + \ln(p_i))(1 + \ln(p_j)). \]

**Corollary 2.** Under the same assumptions as in Theorem 1, the following hold
\[
\limsup_{n \to +\infty} \frac{|\mathcal{E}_{S_h}(\hat{\mathbf{p}}_n) - \mathcal{E}_{S_h}(\mathbf{p})|}{a_n} \leq A_{S_h}(\mathbf{p}), \quad \text{a.s.}
\]
\[
\sqrt{n} (\mathcal{E}_{S_h}(\hat{\mathbf{p}}_n) - \mathcal{E}_{S_h}(\mathbf{p})) \overset{D}{\to} N(0, \sigma^2_{S_h}(\mathbf{p})), \quad \text{as } n \to +\infty.
\]

(C)- Asymptotic behavior of the Renyi entropy estimator.

The treatment of the asymptotic behavior of the Renyi-\(\alpha\) entropies estimator and of the \(\alpha, \beta\)-Varma entropy estimator is obtained by the application of the delta method.

For \(\alpha \in (0, 1) \cup (1, +\infty)\), denote
\[ A_{R,\alpha}(\mathbf{p}) = \frac{\alpha}{|\alpha - 1|} S_{\alpha}(\mathbf{p}) \sum_{j \in J} p_j^{\alpha-1} \]
and
\[ \sigma^2_{R,\alpha}(\mathbf{p}) = \left( \frac{\alpha}{|\alpha - 1|} S_{\alpha}(\mathbf{p}) \right)^2 \left( \sum_{j \in J} (1 - p_j)p_j^{2\alpha-1} - 2 \sum_{(i,j) \in J^2, i \neq j} (p_i p_j)^{\alpha+1/2} \right). \]

**Corollary 3.** Under the same assumptions as in Theorem 1 and for any \(\alpha \in (0, 1) \cup (1, +\infty)\), the following hold
\[
\limsup_{n \to +\infty} \frac{|\mathcal{E}_{R,\alpha}(\hat{\mathbf{p}}_n) - \mathcal{E}_{R,\alpha}(\mathbf{p})|}{a_n} \leq A_{R,\alpha}(\mathbf{p}), \quad \text{a.s.}
\]
\[
\sqrt{n} (\mathcal{E}_{R,\alpha}(\hat{\mathbf{p}}_n) - \mathcal{E}_{R,\alpha}(\mathbf{p})) \overset{D}{\to} N(0, \sigma^2_{R,\alpha}(\mathbf{p})), \quad \text{as } n \to +\infty.
\]

(D)- Asymptotic behavior of the Tsallis entropy estimator.

For \(\alpha \in (0, 1) \cup (1, +\infty)\), denote
\[ A_{T,\alpha}(\mathbf{p}) = \frac{\alpha}{|\alpha - 1|} \sum_{j \in J} p_j^{\alpha-1} \]
and
\[ \sigma^2_{T,\alpha}(\mathbf{p}) = \left( \frac{\alpha}{|\alpha - 1|} \right)^2 \left( \sum_{j \in J} (1 - p_j)p_j^{2\alpha-1} - 2 \sum_{(i,j) \in J^2, i \neq j} (p_i p_j)^{\alpha+1/2} \right). \]

**Corollary 4.** Under the same assumptions as in Theorem 1 and for \(\alpha \in (0, 1) \cup (1, +\infty)\), the following hold
\[
\limsup_{n \to +\infty} \frac{|\mathcal{E}_{T,\alpha}(\hat{\mathbf{p}}_n) - \mathcal{E}_{T,\alpha}(\mathbf{p})|}{a_n} \leq A_{T,\alpha}(\mathbf{p}), \quad \text{a.s.}
\]
\[
\sqrt{n} (\mathcal{E}_{T,\alpha}(\hat{\mathbf{p}}_n) - \mathcal{E}_{T,\alpha}(\mathbf{p})) \overset{D}{\to} N(0, \sigma^2_{T,\alpha}(\mathbf{p})), \quad \text{as } n \to +\infty.
\]
(E-) Asymptotic behavior of the Landsberg-Vedral entropy estimator.

For \( \alpha \in (0, 1) \cup (1, +\infty) \), denote
\[
A_{L.V,\alpha}(p) = \frac{\alpha}{|\alpha - 1|} \sum_{j \in J} p_j^{\alpha - 1}
\]
and \( \sigma^2_{L.V,\alpha}(p) = \left( \frac{\alpha}{|\alpha - 1|} \sum_{j \in J} (1 - p_j) p_j^{2\alpha - 1} - 2 \sum_{(i,j) \in J^2, i \neq j} (p_i p_j)^{\alpha + 1/2} \right)^2 \).

**Corollary 5.** Under the same assumptions as in Theorem 1 and for \( \alpha \in (0, 1) \cup (1, +\infty) \), the following hold
\[
\limsup_{n \to +\infty} \frac{|E_{L.V,\alpha}(\hat{p}_n) - E_{L.V,\alpha}(p)|}{a_n} \leq A_{L.V,\alpha}(p), \text{ a.s.}
\]
\[
\sqrt{n} (E_{L.V,\alpha}(\hat{p}_n) - E_{L.V,\alpha}(p)) \text{ } \overset{D}{\to} \text{ } N(0, \sigma^2_{L.V,\alpha}(p)) \text{ as } n \to +\infty.
\]

(F-) Asymptotic behavior of \( \alpha \)-Abel entropy estimator.

For \( \alpha \in (0, 1) \cup (1, +\infty) \), denote
\[
A_{Ab,\alpha}(p) = \frac{1}{|\alpha^2 - 1|} \sum_{j \in J} |\alpha^2 p_j^{\alpha - 1} - p_j^{(1/\alpha) - 1}| \]
and \( \sigma^2_{Ab,\alpha}(p) = \left( \frac{1}{|\alpha^2 - 1|} \sum_{j \in J} (1 - p_j) \left( \alpha^2 p_j^{\alpha - 1/2} - p_j^{(1/\alpha) - 1/2} \right)^2 \right.
\]
\[
- 2 \sum_{(i,j) \in J^2, i \neq j} \left[ \alpha^2 p_i^{\alpha + 1/2} - p_i^{(1/\alpha) + 1/2} \right] \left[ \alpha^2 p_j^{\alpha + 1/2} - p_j^{(1/\alpha) + 1/2} \right].
\]

**Corollary 6.** Under the same assumptions as in Theorem 1 and for any \( \alpha \in (0, 1) \cup (1, +\infty) \), the following hold
\[
\limsup_{n \to +\infty} \frac{|E_{Ab,\alpha}(\hat{p}_n) - E_{Ab,\alpha}(p)|}{a_n} \leq A_{Ab,\alpha}(p)
\]
\[
\sqrt{n} (E_{Ab,\alpha}(\hat{p}_n) - E_{Ab,\alpha}(p)) \text{ } \overset{D}{\to} \text{ } N(0, \sigma^2_{Ab,\alpha}(p)), \text{ as } n \to +\infty.
\]

(G-) Asymptotic behavior of \( \kappa \)-entropy

For \( \kappa \in (0, 1) \), denote
\[
A_{\kappa}(p) = \frac{1}{2\kappa} \sum_{j \in J} |(1 - \kappa)p_j^{-\kappa} - (1 + \kappa)p_j^\kappa|
\]
and \( \sigma^2_{\kappa}(p) = \frac{1}{4\kappa^2} \left( \sum_{j \in J} (1 - p_j) \left( (1 - \kappa)p_j^{-\kappa + 1/2} - (1 + \kappa)p_j^{\kappa + 1/2} \right)^2 \right.
\]
\[
- 2 \sum_{(i,j) \in J^2, i \neq j} \left[ (1 - \kappa)p_i^{-\kappa + 3/2} - (1 + \kappa)p_i^{\kappa + 3/2} \right] \left[ (1 - \kappa)p_j^{-\kappa + 3/2} - (1 + \kappa)p_j^{\kappa + 3/2} \right].
\]

**Corollary 7.** Under the same assumptions as in Theorem 1 and for any \( \kappa \in (0, 1) \), the following hold
\[
\limsup_{n \to +\infty} \frac{|E_{\kappa}(\hat{p}_n) - E_{\kappa}(p)|}{a_n} \leq A_{\kappa}(p), \text{ a.s}
\]
\[
\sqrt{n}(E_{\kappa}(\hat{p}_n) - E_{\kappa}(p)) \text{ } \overset{D}{\to} \text{ } N(0, \sigma^2_{\kappa}(p)), \text{ as } n \to +\infty.
\]
(H -) Asymptotic behavior of Varma’s entropy of order α and type β.

For \( \beta - 1 < \alpha < \beta, \quad \beta \geq 1 \) denote

\[
A_{V, \alpha, \beta}(p) = \frac{\alpha + \beta - 1}{S_{\alpha + \beta - 1}} \sum_{j \in J} p_j^{\alpha + \beta - 2}
\]

and \( \sigma_{V, \alpha, \beta}^2(p) = \left( \frac{\alpha + \beta - 1}{(\beta - \alpha)S_{\alpha + \beta - 1}(p)} \right)^2 \left( \sum_{j \in J} (1 - p_j)p_j^{\alpha + 2\beta - 3} - 2 \sum_{(i,j) \in J^2, i \neq j} (p_ip_j)^{\alpha + 1/2} \right). \]

**Corollary 8.** Under the same assumptions as in Theorem 1 and for \( \beta - 1 < \alpha < \beta, \quad \beta \geq 1 \), the following hold

\[
\limsup_{n \to +\infty} \frac{|\mathcal{E}_{V, \alpha, \beta}(\hat{p}_n) - \mathcal{E}_{V, \alpha, \beta}|}{a_n} \leq A_{V, \alpha, \beta}(p), \quad a.s.
\]

\[
\sqrt{n}(\mathcal{E}_{V, \alpha, \beta}(\hat{p}_n) - \mathcal{E}_{V, \alpha, \beta}(p)) \xrightarrow{D} \mathcal{N}(0, \sigma_{V, \alpha, \beta}^2(p)), \quad \text{as } n \to +\infty.
\]

4. The proofs

Before we state our main results we introduce the following notations. For a fixed \( j \in J \), denote

\[
\Delta_{p_n}^{e_j} = \hat{p}_n - p_j, \quad \delta_n(p_j) = \sqrt{n/p_j} \Delta_{p_n}^{e_j},
\]

and \( a_n = \sup_{j \in J} |\Delta_{p_n}^{e_j}|. \)

We recall that, since for a fixed \( j \in D, \; n \hat{p}_n^{e_j} \) has a binomial distribution with parameters \( n \) and success probability \( p_j \), we have

\[
\mathbb{E}[\hat{p}_n^{e_j}] = p_j \quad \text{and} \quad \mathbb{V}[(\hat{p}_n^{e_j})] = \frac{p_j(1 - p_j)}{n}.
\]

And finally, by the asymptotic Gaussian limit of the multinomial law (see for example Lo et al. (2016), Chapter 1, Section 4), we have

\[
\left( \delta_n(p_j), \ j \in J \right) \overset{D}{\rightarrow} Z(p) \sim \mathcal{N}(0, \Sigma_p), \quad \text{as } n \to +\infty,
\]

where \( Z(p) = (Z_{p_j}, j \in J)^t \) is a centered Gaussian random vector of dimension \( \#(J) \) having the following elements:

\[
(\Sigma_p)_{(i,j)} = (1 - p_j)\delta_{ij} - \sqrt{p_ip_j(1 - \delta_{ij})}, \quad (i, j) \in J^2,
\]

where \( \delta_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases} \).

4.1. **Proof of Theorem 1.** For a fixed \( j \in J \), we have

\[
\phi(\hat{p}_n^{e_j}) = \phi(p_j + \Delta_{p_n}^{e_j}) = \phi(p_j) + \Delta_{p_n}^{e_j} \phi'(p_j + \theta_1(j)\Delta_{p_n}^{e_j})
\]

by the mean value Theorem applied to the function \( \phi \) and where \( \theta_1(j) \in (0, 1) \).

Apply again the mean value Theorem to the derivative of the function \( \phi' \)

\[
\phi'(p_j + \theta_1(j)\Delta_{p_n}^{e_j}) = \phi'(p_j) + \theta_1(j)\Delta_{p_n}^{e_j} \phi''(p_j + \theta_2(j)\Delta_{p_n}^{e_j})
\]

where \( \theta_2(j) \in (0, 1) \). We can write (4.3) as

\[
\phi(\hat{p}_n^{e_j}) = \phi(p_j) + \Delta_{p_n}^{e_j} \phi'(p_j) + \theta_1(j)\Delta_{p_n}^{e_j} \phi''(p_j + \theta_2(j)\Delta_{p_n}^{e_j})
\]
Now we have, by summation over \( j \in J \)

\[
J(\hat{p}_n) - J(p) = \sum_{j \in J} \Delta^{\epsilon_j}_{p_n} \phi'(p_j) \\
+ \sum_{j \in J} \theta_1(j)(\Delta^{\epsilon_j}_{p_n})^2 \phi''(p_j + \theta_2(j)\Delta^{\epsilon_j}_{p_n})
\]

Hence

\[
|J(\hat{p}_n) - J(p)| \leq a_n \sum_{j \in J} |\phi'(p_j)| + a_n^2 \sum_{j \in J} |\phi''(p_j + \theta_2(j)\Delta^{\epsilon_j}_{p_n})|,
\]

Therefore

\[
\limsup_{n \to +\infty} \frac{|J(\hat{p}_n) - J(p)|}{a_n} \leq A_J(p), \text{ a.s.,}
\]

since \( a_n \to 0 \) as \( n \to +\infty \) and

\[
\sum_{j \in J} |\phi''(p_j + \theta_2(j)\Delta^{\epsilon_j}_{p_n})| \to \sum_{j \in J} |\phi''(p_j)| < \infty \quad \text{as} \quad n \to +\infty.
\]

This prove (2.5).

Let prove (2.6). By going back to (4.4), we get

\[
\sqrt{n}(J(\hat{p}_n) - J(p)) = \sum_{j \in J} \sqrt{p_j} \delta_n(p_j) \phi'(p_j) + \sqrt{n} R_n,
\]

where

\[
R_n = \sum_{j \in J} \theta_1(j)(\Delta^{\epsilon_j}_{p_n})^2 \phi''(p_j + \theta_2(j)\Delta^{\epsilon_j}_{p_n}).
\]

Using Formula (4.1) above, we get

\[
\sum_{j \in J} \sqrt{p_j} \delta_n(p_j) \phi'(p_j) \overset{D}{\to} \sum_{j \in J} \phi'(p_j) \sqrt{p_j} Z_{p_j}, \quad \text{as} \quad n \to +\infty,
\]

which follows a centered normal law of variance \( \sigma^2_j(p) \) since

\[
\text{Var} \left( \sum_{j \in J} \phi'(p_j) \sqrt{p_j} Z_{p_j} \right) = \sum_{j \in J} \text{Var} \left( \phi'(p_j) \sqrt{p_j} Z_{p_j} \right) + 2 \sum_{j \in J} \text{Cov} \left( \phi'(p_j) \sqrt{p_j} Z_{p_j}, \phi'(p_j) \sqrt{p_j} Z_{p_j} \right)
\]

\[
= \sum_{j \in J} p_j(1-p_j)(\phi'(p_j))^2 - 2 \sum_{(i,j) \in J^2, i \neq j} p_ip_j \sqrt{p_i p_j} \phi'(p_i) \phi'(p_j).
\]

The proof will be complete if we show that \( \sqrt{n} R_n \) converges to zero in probability.

We have

\[
|\sqrt{n} R_n| \leq \sqrt{n} a_n^2 \sum_{j \in J} \phi''(p_j + \theta_2(j)\Delta^{\epsilon_j}_{p_n}).
\]

By the Bienaymé-Tchebychev inequality, we have, for any fixed \( \epsilon > 0 \) and for any \( j \in J \)

\[
\mathbb{P}(\sqrt{n}(\hat{\epsilon}_n - p_j)^2 \geq \epsilon) = \mathbb{P} \left( |\hat{\epsilon}_n - p_j| \geq \frac{\sqrt{\epsilon}}{n^{1/2}} \right) \leq \frac{p_j(1-p_j)}{\epsilon n^{1/2}}.
\]

Hence \( \sqrt{n} a_n^2 = o_P(1) \), which proves (2.6).

All this ends the proof of Theorem 1.
4.2. Proofs of Corollaries.

A-) The Proofs of Corollaries 1 and 2 are direct adaptations of Theorem 1 with respectively \( \phi(s) = s^\alpha \) and \( \phi(s) = -s \ln s \).

B-) Proof of Corollary 3. For \( \alpha \in (0, 1) \cup (1, +\infty) \), \( \alpha \)-Reyni entropy is expressed through the power sum \( S_\alpha(p) = \sum_{j \in J} \phi(p_j) \) with \( \phi(s) = s^\alpha \). We have

\[
E_{R,\alpha}(\mathcal{P}_n) - E_{R,\alpha}(p) = \frac{1}{\alpha - 1} (\ln S_\alpha(\mathcal{P}_n) - \ln S_\alpha(p)),
\]

by using a Taylor expansion of \( \ln(1 + y) \) it follows that, almost surely,

\[
\ln S_\alpha(\mathcal{P}_n) - \ln S_\alpha(p) = \ln \left( 1 + \frac{S_\alpha(\mathcal{P}_n) - S_\alpha(p)}{S_\alpha(p)} \right) = \frac{S_\alpha(\mathcal{P}_n) - S_\alpha(p)}{S_\alpha(p)} + O_{a.s}(\alpha_n^2).
\]

Finally this, combined with (3.1) of Corollary 1, proves (3.5).

Now recall by going back to (4.4), we can write

\[
\sqrt{n}(S_\alpha(\mathcal{P}_n) - S_\alpha(p)) = \sqrt{n} \sum_{j \in J} \Delta_{\mathcal{P}_n}^{\alpha} \phi'(p_j) + o_P(1)
\]

here \( \phi'(p_j) = \alpha p_j^{\alpha - 1} \).

Hence dividing each member by \( \sqrt{n} S_\alpha(p) \), we get

\[
\frac{S_\alpha(\mathcal{P}_n)}{S_\alpha(p)} = 1 + \sum_{j \in J} \frac{\Delta_{\mathcal{P}_n}^{\alpha} \phi'(p_j)}{S_\alpha(p)} + o_P(1).
\]

Now by Taylor expansion of \( \ln(1 + y) \), it follows that, almost surely,

\[
\ln S_\alpha(\mathcal{P}_n) - \ln S_\alpha(p) = \ln \left( 1 + \frac{\sum_{j \in J} \Delta_{\mathcal{P}_n}^{\alpha} \phi'(p_j)}{S_\alpha(p)} \right) = \sum_{j \in J} \frac{\Delta_{\mathcal{P}_n}^{\alpha} \phi'(p_j)}{S_\alpha(p)} + O_P \left( \frac{1}{n} \right)
\]

therefore

\[
\sqrt{n} (E_{R,\alpha}(\mathcal{P}_n) - E_{R,\alpha}(p)) = \frac{1}{\alpha - 1} \sum_{j \in J} \sqrt{n} \frac{\Delta_{\mathcal{P}_n}^{\alpha} \phi'(p_j)}{S_\alpha(p)} + o_P(1)
\]

using (4.5) and where

\[
\sigma_{R,\alpha}^2(p) = \left( \frac{\alpha}{(\alpha - 1)S_\alpha(p)} \right)^2 \left( \sum_{j \in J} (1 - p_j)p_j^{2\alpha - 2} - 2 \sum_{(i,j) \in J^2, i \neq j} (p_i p_j)^{\alpha - 1/2} \right).
\]

This proves (3.6) and ends the proof of the Corollary 3.

C-) Proof of Corollary 4. Since \( \alpha \)-Tsallis entropy is related to the power sum \( S_\alpha(p) \), the proof follows directly from Corollary 1.

D-) Proof of Corollary 5. Since Landsberg-Vedral and Tsallis \( \alpha \)-entropies are related by
\[
\mathcal{E}_{L,V,\alpha}(\mathbf{p}) = \frac{\mathcal{E}_{T,\alpha}(\mathbf{p})}{S_{\alpha}(\mathbf{p})},
\]

the proof of this Corollary results directly from the Corollary 4.

**E-)** The proof of Corollary 8 is similar to the one of Corollary 3 with the power sum
\[S_{\alpha+\beta-1}(\mathbf{p}) = \sum_{j \in J} \phi(p_j)\]
with \(\phi(s) = s^{\alpha+\beta-1}\).

**F-)** Corollaries 6 and 7 are, as for Corollaries 1 and 2, adaptations of Theorem 1 with this time
\[\phi(s) = \frac{1}{s^{\alpha-\alpha}}(s^{\alpha} - s^{\alpha-1})\]
and \(\phi(s) = \frac{1}{2\kappa}(s^{1+\kappa} - s^{1-\kappa}), \quad \kappa \in (0, 1)\), respectively.

5. Simulation

To assess the performance of our estimators, we present a simulation study.

Let \(X\) a random variable defined on a measurable space \((\Omega, \mathcal{A}, \mathbb{P})\) and with range \(\mathcal{X} = \{1, 2, 3\}\) with their respective probabilities mass
\[p_1 = 0.4, \quad p_2 = 0.25, \quad p_3 = 0.35.
\]
We plot the entropies estimators and construct histograms and Q-Q plots to see whether data are normally distributed.

In each figure, the left panel represents the plot of entropy estimator, built from sample sizes of \(n = 100, 200, \ldots, 30000\), and the true entropy (represented by horizontal black line). The middle panel shows the histogram of the data and the red line represents the plot of the theoretical normal distribution calculated from the same mean and the same standard deviation of the data. The right panel concerns the Q-Q plot of the data which display the observed values against normally distributed data (represented by the red line).

As we can see from **FIGURES 1 2, 3 and 4**, our entropies estimators are asymptotically normally distributed.

![Figure 1](image)

**Figure 1.** Plots of Shannon and Renyi entropies estimators when samples sizes increase, histograms and normal Q-Q plots versus \(\mathcal{N}(0, 1)\).
Figure 2. Plots of Tsallis and Landsberg-Vedral entropies estimators when samples sizes increase, histograms and normal Q-Q plots versus $N(0,1)$.

Figure 3. Plots of Abel and Kappa entropies estimators when samples sizes increase, histograms and normal Q-Q plots versus $N(0,1)$.
6. Conclusion

We have derived a new nonparametric estimator for entropies in the discrete case and on finite sets. We adopted the plug-in method and we derived almost sure rates of convergence and central limit Theorems for some of the most important entropies in the discrete case. We also demonstrated their efficiency using a simulation study.

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