Sums and Differences of Correlated Random Sets

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Introduction

Given $A \subset \mathbb{Z}$, let

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$$A - A = \{a_1 - a_2 : a_1, a_2 \in A\}.$$
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Theorem

There exists a positive constant $c$ such that for any $n$ large, the proportion of sets $A \subset \{0, \ldots, n\}$ with $|A + A| > |A - A|$ is greater than $c$. (Martin and O’Bryant 2006)

Such sets are called More Sums Than Differences (MSTD) sets, or sum-dominant sets.
Correlated Random Pairs

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We investigate sums and differences of *pairs* of subsets \((A, B) \subset \{0, \ldots, n\}\). We select such pairs according to the dependent random process:

\[
P(k \in A) = p; \quad P(k \in B | k \in A) = \rho_1; \quad P(k \in B | k \notin A) = \rho_2
\]

Let \(\vec{\rho} = (p, \rho_1, \rho_2)\). We call \((A, B)\) a \(\vec{\rho}\)-correlated pair.
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\[ \rho_1 = \rho_2, \implies (A, B) \text{ independent.} \]
Let $P(\tilde{\rho}, n)$ be the probability that a $\tilde{\rho}$-correlated pair $(A, B) \subset \{0, \ldots, n\}$ is MSTD, that is

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**Note:** $P(1/2, 1, 0, n), P(1/2, 0, 1, n)$, and $P(1/2, 1/2, 1/2, n)$ can be thought of as *proportions* of pairs $(A, A), (A, A^c)$, resp. $(A, B)$ which are MSTD, while other values of $P(\tilde{\rho}, n)$ must be thought of as *probabilities*. 
Main Results on Correlated Pairs

**Theorem**

*For any $\bar{\rho} \in [0, 1]^3$, the limit*

$$\lim_{n \to \infty} P(\bar{\rho}, n) =: P(\bar{\rho})$$

*exists. Moreover, as long as $p \notin \{0, 1\}$ and $(\rho_1, \rho_2) \neq (0, 0), (1, 1)$, then $P(\bar{\rho})$ is strictly positive.*
The function $P(\bar{\rho})$

*Proof idea:*
To show that $P(\bar{\rho})$ exists, we build on the idea of Zhao (2010) and count MSTD pairs by their *minimal fringe profiles.*
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We call a pair a *rich MSTD pair* if the sumset wins over the difference set near the edges, while both the sumset and the difference set contain all the middle elements.
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We show that as $n \to \infty$, a $\vec{\rho}$-pair $(A, B) \subset \{0, \ldots, n\}$ which is an MSTD pair is rich with probability 1.
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Thus, by summing the probabilities that the edges of $(A, B)$ have a given MSTD fringe profile and that $(A, B)$ is rich over all such minimal fringe profiles, we can get the limit $P(\rho)$. 
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If $\rho_1 = 0$, but $\rho_2 p > 0$, we can use the fringe profile $L = R = \{1, 2, 3, 5, 7, 8\}$, $L' = R' = L^c$. (This means that the left and the right edges of $A$ look like

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For any $\rho_1, \rho_2$, the function $P(p, \rho_1, \rho_2)$ is a differentiable function of $p \in [0, 1]$. 

Maximizing $P(\vec{\rho})$

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For each $n$, $P_n(\vec{\rho})$ denotes the proportion of MSTD pair of subsets of $[1, \ldots, n]$. $P_n$ is a polynomial of $p, \rho_1, \rho_2$ based on the sizes of all MSTD pairs and their intersection.
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We fix $n = 9$, do an exhaustive search to find all MSTD pairs and calculate $P_9$. 
Fix \((\rho, \rho_1)\)

Conjecture 1: For any fixed \((\rho, \rho_1)\) with \(\rho_1\) not too big \((\rho_1 \leq 0.4)\) then \(P\) as a function of \(\rho_2\) is strictly increasing in \([0, 1]\) and reaches its maximum at \(\rho_2 = 1\).
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Conjecture 2: For any fixed \((\rho, \rho_2)\) with \(\rho_2\) not too small (\(\rho_2 \geq 0.5\)) then \(P\) as a function of \(\rho_1\) is strictly decreasing in \([0, 1]\) and reaches its maximum at \(\rho_1 = 0\).
**Conjecture 2:** For any fixed \((p, \rho_2)\) with \(\rho_2\) not too small \((\rho_2 \geq 0.5)\) then \(P\) as a function of \(\rho_1\) is strictly decreasing in \([0, 1]\) and reaches its maximum at \(\rho_1 = 0\).
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**Conjecture 3:** For any fixed \((\rho_1, \rho_2)\), \(P\) as a function of \(p\) in \((0, 1)\) has a maximum at \(1/2\).
A and A complement

From Conjectures 1 and 2, it makes sense that the maximum of $P$ is at $\rho_1 = 0, \rho_2 = 1$ or when it is the case of $A$ and $A^c$. 
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**Conjecture 4:** The maximum of the function $P(p, \rho_1, \rho_2)$ in $[0, 1]^3$ occurs at $P(1/2, 0, 1) \approx 0.03$. 
Some notation

- Big $O$: $f(n) = O(g(n))$ if $\exists c, n_0 > 0$ s.t $f(n) > cg(n)$ for all $n > n_0$. 
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- **Little o**: \( f(n) = o(g(n)) \) if \( \lim_{n \to \infty} \frac{g(n)}{f(n)} = \infty \).

- \( X \sim f(N) \) if for any \( \epsilon_1, \epsilon_2 > 0 \) there exists \( N_{\epsilon_1, \epsilon_2} > 0 \) such that for all \( N > N_{\epsilon_1, \epsilon_2} \)

\[
P \left( X \notin [(1 - \epsilon_1)f(N), (1 + \epsilon_1)f(N)] \right) < \epsilon_2
\]
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Hegarty-Miller investigated this for $(\rho_1, \rho_2) = (1, 0)$ and $p = p(n) = o(1), n^{-1} = o(p(n))$. The first condition indicates $p$ decays with $n$ while the second one guarantees the expected size of $A$ grow with $n$. 
Theorem (Hegarty-Miller)

Given a function \( p : \mathbb{N} \rightarrow (0, 1) \) such that \( p(N) = o(1) \) and \( N^{-1} = o(p(N)) \). As \( N \rightarrow \infty \), the probability \( A \) as a subset of \([1, \ldots, N]\) is MSTD tends to 0. Let \( S = |A + A|, D = |A - A| \) and \( S^C, D^C \) denote their complements.
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Given a function \( p : \mathbb{N} \to (0, 1) \) such that \( p(N) = O(1) \) and \( N^{-1} = o(p(N)) \). As \( N \to \infty \), the probability \( A \) as a subset of \([1, \ldots, N]\) is \( \text{MSTD} \) tends to 0. Let \( S = |A + A|, D = |A - A| \) and \( S^C, D^C \) denote their complements.

(i) \( p = o(N^{-1/2}) \): Then \( D \sim 2S \sim (N.p)^2 \)

(ii) \( p = c.N^{-1/2} \) for \( c \in (0, \infty) \). Let \( g(x) = 2(e^{-x} - (1 - x))/x \):

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S \sim g \left( \frac{c^2}{2} \right) N \quad \text{and} \quad D \sim g(c^2)N
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(iii) \( N^{-1/2} = o(p) \) : \( S^C \sim 2.D^C \sim \frac{4}{p^2} \)
Our result

We prove a similar result:

Let \( \hat{p} = p^2(2\rho_1 - \rho_1^2) + 2p(1 - p)\rho_2 \) be depend on \( N \). Then

(i) \( \hat{p} = o(N^{-1}) \): Then \( \mathcal{D} \sim 2S \sim N^2\hat{p} \)

(ii) \( \hat{p} = cN^{-1} \) for \( c \in (0, \infty) \). Let \( g(x) = 2(e^{-x} - (1 - x))/x \):

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S \sim g \left( \frac{c^2}{2} \right) N \quad \text{and} \quad \mathcal{D} \sim g(c^2)N
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(iii) \( N^{-1} = o(\hat{p}) \): \( \mathbb{E}(S^C) = \mathbb{E}(2\mathcal{D}^C) = 4/\hat{p} \)
In our result, if we let $\rho_1 = 1, \rho_2 = 0$ then $\hat{p} = p^2$, consistent with the result in Hegarty-Miller.
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If $\rho_1 = \rho_2 = p$ then the critical phase happens when $p^2 = \Theta(1/N)$ or $p = \Theta(N^{-1/2})$. 
Notes in Our result

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If $\rho_1 = \rho_2 = p$ then the critical phase happens when $p^2 = \Theta(1/N)$ or $p = \Theta(N^{-1/2})$.

The interesting case is $A$ and $A^C$: $\hat{p} = 2p(1 - p) = \Theta(1/N)$. If we let $p = o(1)$, $p = \Theta(1/N)$ which implies the expected number of elements of $A$ is $pN = \text{constant}$. 
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Examples of minimal size MSTD pair:

\[ A = \{1, 2, 5, 7\}, \quad B = \{1, 3, 6, 7\} \]

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Proof of Minimal MSTD pair

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Lemma

If $A, B$ is a MSTD pair then there exist $a_1 < a_2 < a_3 \in A$ and $b_1 > b_2 > b_3 \in B$ such that $a_1 + b_1 = a_2 + b_2 = a_3 + b_3$. 
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Idea of the proof: Consider all sums and differences $a \pm b$ where $a \in A, b \in B$. Each collapsed sum implies one collapsed difference.
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*Corollary:* There is no MSTD pair of size \((1, k)\) and \((2, k)\) for \(k > 0\).
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*Corollary:* There is no MSTD pair of size \((1, k)\) and \((2, k)\) for \(k > 0\).

We use some tedious checking to eliminate the case \((3, 3)\) and \((3, 4)\).
Summary of Results

We prove for each $\rho = (p, \rho_1, \rho_2)$ the limiting probability $P(\rho)$ of picking an MSTD $\rho$-correlated pair exists and positive (except in some extreme cases).
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- We prove for each \( \vec{\rho} = (p, \rho_1, \rho_2) \) the limiting probability \( P(\vec{\rho}) \) of picking an MSTD \( \vec{\rho} \)-correlated pair exists and positive (except in some extreme cases).

- The function \( P(\vec{\rho}) \) is continuous and differentiable.
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We show that $P(\vec{\rho})$ approaches zero and characterize the phase transition when we let $\vec{\rho}$ decay with $n$. 
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- The function \( P(\vec{\bar{\rho}}) \) is continuous and differentiable.

- We show that \( P(\vec{\bar{\rho}}) \) approaches zero and characterize the phase transition when we let \( \vec{\bar{\rho}} \) decay with \( n \).

- We find the minimal size of an MSTD pair \((A, B)\).
Future Research

- Prove Conjecture 4: $\sup P(\tilde{\rho}) = P(1/2, 0, 1)$. 
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- Prove Conjecture 4: \( \sup P(\hat{\rho}) = P(1/2, 0, 1) \).

- Find an efficient way to calculate values of \( P \) and investigate more analytic properties of \( P \).

- Prove the strong concentration of \( S^C \) and \( D^C \) in the case of slow decay (i.e. when \( N^{-1/2} = o(\hat{p}) \)).
Future Research

- Prove Conjecture 4: \( \sup P(\hat{\rho}) = P(1/2, 0, 1) \).

- Find an efficient way to calculate values of \( P \) and investigate more analytic properties of \( P \).

- Prove the strong concentration of \( S^C \) and \( D^C \) in the case of slow decay (i.e. when \( N^{-1/2} = o(\hat{\rho}) \)).

- Prove the uniqueness of the MSTD pairs of size \((4, 4)\) and \((3, 5)\), up to translation/dilation.
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