Entanglement measure for any quantum states

Hyuk-jae Lee\textsuperscript{1*}, Sung Dahm Oh\textsuperscript{2†} and Doyeol Ahn\textsuperscript{1,3‡}

\textsuperscript{1}Institute of Quantum Information Processing and Systems, University of Seoul, Seoul, 130-743, Korea
\textsuperscript{2}Department of Physics, Sookmyung Women’s University, Seoul,140-742, Korea
\textsuperscript{3}Department of Electrical and computer Engineering, University of Seoul, Seoul, 130-743, Korea

The entanglement measure for multiqudits is proposed. This measure calculates the partial entanglement distributed by subsystems and the complete entanglement of the total system. This shows that we need to measure the subsystem entanglements to explain the full description for multiqudit entanglement. Furthermore, we extend the entanglement measure to mixed multiqubits and the higher dimension Hilbert spaces.

\textsuperscript{*}e-mail:lhjae@iquips.uos.ac.kr
\textsuperscript{†}e-mail:sdoh@sookmyung.ac.kr
\textsuperscript{‡}e-mail:dahn@uoscc.uos.ac.kr
Entanglement has been recently recognized as the most essential ingredient in the quantum information technology as can be seen by the cases of the superdense coding [1], quantum computation, teleportation [2], clock synchronization [3], [4] and quantum cryptography [5].

In order to clarify the entanglement characteristics, the entanglement degree has to be quantifiable. The existence of entanglement for two-qubit cases can be established by negative partial transpose property of the density matrices [6] [7] and entanglement measures have been quantized by concurrence [8], negativity [9] and entanglement of formation [10]. The mathematical and physical structures of entanglement have not been yet fully understood for multipartite cases even in the two dimensional space. Entanglement in the case of the multipartite systems and the higher dimensional systems has remained unsolved even though there were several investigations to quantify the entanglement and classify types of the entangled states. There have been investigations on the multipartite entanglement measure by the tangle [11] which computes the concurrence between two intentionally divided subsystems in effectively two dimensional Hilbert space. However, tangles cannot properly express the entanglement degree for any $W$-states. There have been also recent proposals of an entanglement measure by the operator norm [12] and the hyperdeterminant [13] of the given quantum states. However, these schemes cannot explain the full entanglement structure of the composite system. Here, we propose a direct measure of the entanglement distributed in the subsystems and the total system for the given entangled states.

The entanglements in multipartite qubits are complex because the quantum states can have various types of entanglement sharing among the subsystems. In order to illuminate this situation, consider a state of the form $|Ψ_5⟩ = |Bell⟩ ⊗ |GHZ⟩$ in a five-qubit system, where $|Bell⟩ = \frac{1}{\sqrt{2}}(|01⟩ − |10⟩)$, between the first and the second qubits and $|GHZ⟩$ the Greenberger-Horne-Zeilinger(GHZ) state among the third, the fourth and the fifth qubits. Next, consider a state of the form $|Φ_5⟩ = |GHZ⟩ ⊗ |Bell⟩$. Then, both quantum states have no complete entanglement for the total system but have different degrees in the specific subsystems. $|Φ_5⟩$ exhibits GHZ entangled in the first, the second and the third qubits but $|Ψ_5⟩$ does not show this property. Until now, there were no any entanglement measures that can distinguish this situation. Then, entanglement measure for multipartite systems has to predict the magnitude of all types of entanglements which exist among the constituents. In this letter, we present a general entanglement measure for multipartite qubits, and introduce examples for pure multiqubit systems. Finally, we explain that our entanglement measure can be extended to multiqubit mixed states and the higher dimensional Hilbert space.

Let us start with the pure states for multipartite qubit system as
\[
\Psi(1, 2, 3, \ldots, n) = \sum_{ij\ldots k=0}^{1} a_{ij\ldots k}|i\rangle_{1} \otimes |j\rangle_{2} \otimes \cdots \otimes |k\rangle_{n},
\]  
where \(n\) denotes the number of qubits and \(\sum_{ij\ldots k=0}^{1}|a_{ij\ldots k}|^2 = 1\). Our question is how to distinguish separated states from entangled states given in eq. (1). The three-qubit state, \(|\Psi(1, 2, 3)\rangle\), has two types of entanglement different from two qubits system. The first is the entanglement between two particles and the second is among three particles. Furthermore, entanglement between two particles has three possibilities, depending on which qubit is separated. The increase of the qubit numbers in the system produces the increase of the possibilities in the entanglement types. Then, we have to differentiate all these situations if we suggest the entanglement measure.

For this purpose, we consider the quantum correlations for the given multipartite qubit system \(|\Psi\rangle\) as the following:

\[
M_{ijkl\ldots}(\alpha, \beta, \gamma, \ldots; |\Psi\rangle) = \langle (\sigma_i(\alpha) - \lambda_i(\alpha)) \otimes (\sigma_j(\beta) - \lambda_j(\beta)) \otimes (\sigma_k(\gamma) - \lambda_k(\gamma)) \otimes \cdots \rangle,
\]  
where \(\sigma_i(\alpha)\) denotes the \(i\)th-component Pauli’s matrix of \(\alpha\)-th qubit and \(\lambda_i(\alpha) = \langle I(1) \otimes I(2) \otimes I(3) \cdots \sigma_i(\alpha) \otimes I((\alpha + 1) \otimes \cdots)\rangle\). Here, \(I(\alpha)\) is the identity operator in \(\alpha\)-th qubit. We will show that \(M\) is zero for the completely separable state later. Define the tensor form as

\[
M'_{ijkl\ldots}(\alpha, \beta, \gamma, \ldots; |\Psi\rangle) = M_{ijkl\ldots}(\alpha, \beta, \gamma, \ldots; |\Psi\rangle)
- \sum (\text{all the possible partitions of indices of } M_{ijkl\ldots}(\alpha, \beta, \gamma, \ldots; |\Psi\rangle)).
\]  
The sum of the second term in the right side of eq. (3) appears in the case of the system which is composed by more than three qubits. In the four-qubit case,

\[
M'_{ijkl}(1, 2, 3, 4; |\Psi\rangle) = M_{ijkl}(1, 2, 3, 4; |\Psi\rangle) - M_{ij}(1, 2; |\Psi\rangle) M_{kl}(3, 4; |\Psi\rangle)
- M_{ik}(1, 3; |\Psi\rangle) M_{jl}(2, 4; |\Psi\rangle) - M_{il}(1, 4; |\Psi\rangle) M_{jk}(2, 3; |\Psi\rangle).
\]

Define the entanglement measure from \(M'\):

\[
B^{(m)}(\alpha, \beta, \cdots \gamma; |\Psi\rangle) = \frac{1}{N} \sum_{ijkl\ldots} M'_{ijkl\ldots}(\alpha, \beta, \cdots \gamma; |\Psi\rangle) M'_{ijkl\ldots}(\alpha, \beta, \cdots \gamma; |\Psi\rangle),
\]  
where \(N\) is a normalization constant which depends on the number of qubits, \(m\). \(B^{(m)}(\alpha, \beta, \cdots \gamma; |\Psi\rangle)\) calculates the entanglement magnitude among \(m\) qubits labelled by \(\alpha, \beta, \cdots \gamma\). For example, \(B^{(2)}(\alpha, \beta; |\Psi\rangle)\) describes the entanglement magnitude between \(\alpha\) and \(\beta\) qubits and \(B^{(3)}(\alpha, \beta, \gamma; |\Psi\rangle)\) the entanglement degree among \(\alpha\), \(\beta\) and \(\gamma\) qubits and so on. In two-qubit systems, there only exists \(B^{(2)}(1, 2; |\Psi\rangle)\) which is the same measure as Schlinz and Mahler’s entanglement measure [14]. Measures of eq. (5) satisfy the following properties:
\[ B^{(m)} = 0 \] for completely separable states.

\[ B^{(m)} \geq 0. \]

\[ B^{(m)} \] is invariant under any local unitary transformations.

The first property can be shown easily with simple calculations since
\[ \langle (\sigma_i(\alpha) - \lambda_i(\alpha)) \otimes (\sigma_j(\beta) - \lambda_j(\beta)) \otimes (\sigma_k(\gamma) - \lambda_k(\gamma)) \otimes \cdots \rangle = (\langle \sigma_i(\alpha) \rangle - \lambda_i(\alpha))(\langle \sigma_j(\beta) \rangle - \lambda_j(\beta))(\langle \sigma_k(\gamma) \rangle - \lambda_k(\gamma)) \cdots, \]
in completely separate states. The second property is true since
\[ M = 0 \] in completely separable states and \( B \) is defined by the square of real numbers. The third property is shown easily by using
\[ U^\dagger \sigma_i U = T_{ij} \sigma_j \] and
\[ \sum_i T_{ij} T_{ik} = \delta_{jk} \] where \( U \) is an unitary matrix and \( T \) is a
\[ 3 \times 3 \] orthogonal matrix [14].

Let us explain entanglement degrees for two-, three- and four-qubit cases through direct calculations. For pure two-qubit, an arbitrary pure entangled state can be written by
\[ |\Psi_2\rangle = a|00\rangle + b|11\rangle \]
where \( a \) and \( b \) are the nonnegative real number coefficients with normalization \( |a|^2 + |b|^2 = 1 \) appearing in the Schmidt’s decomposition. The entanglement of this state can be calculated simply by using eq. (5). Then, \( B^{(2)}(1, 2; |\Psi_2\rangle) = 4a^2b^2 = C^2 \) where \( C \) is the concurrence.

This explains that \( B^{(2)}(1, 2; |\Psi_2\rangle) \) is a monotonically increasing function of \( C \) in the region of \( 0 \leq C \leq 1 \). Then, one can see that our measure is appropriate in bipartite qubits.

In the three-qubit cases, we consider the two complete entanglements such as \( |GHZ_3\rangle \) and \( |W_3\rangle \) states and a partially entangled state as \( |\Psi_3\rangle = |Bell\rangle \otimes |0\rangle \), that the first and the second qubits are entangled and the third qubit is separated. We are summarizing the obtained results for all the possible three-qubit states in the following table:

| State  | \( B^{(2)}(\alpha, \beta; |\Psi\rangle) \) | \( B^{(3)}(1, 2, 3; |\Psi\rangle) \) |
|--------|---------------------------------|---------------------------------|
| \( |GHZ_3\rangle \) | \( \frac{1}{2} \) | 1 |
| \( |W_3\rangle \) | \( \frac{28}{243} \) | \( \frac{280}{243} \) |
| \( |\Psi_3\rangle \) | 1 or 0 | 0 |

Here, 1 or 0 in \( B^{(2)} \) of \( |\Psi_3\rangle \), represents \( B^{(2)}(1, 2; |\Psi_3\rangle) = 1 \), and \( B^{(2)}(1, 3; |\Psi_3\rangle) = B^{(2)}(2, 3; |\Psi_3\rangle) = 0 \), respectively. This explains the situation of entanglement for individual subsystems in \( |\Psi_3\rangle \) well. The entanglement of \( |W_3\rangle \) is stronger than \( |GHZ_3\rangle \) between two particles but the entanglement of \( |W_3\rangle \) is weaker than \( |GHZ_3\rangle \) for the complete entanglement. Note that the bipartite entanglement does not disappear in GHZ state. It has been generally accepted that the partial entanglement degree of \( (N - 1) \) qubits of \( |GHZ_N\rangle \) is zero with an artificial reduction of a qubit. We here point out that it is inappropriate to view the partial entanglement, tracing out on qubit.

For four-qubit, let us treat four four-qubit quantum states such as \( |GHZ_4\rangle \), \( |W_4\rangle \), \( |\phi_6\rangle = \frac{1}{\sqrt{6}}(|0011\rangle + |0101\rangle + \ldots) \) where \( |\phi_6\rangle \) is a partially entangled state.
We also calculate $B^{(m)}$ for a partially entangled states, $|GHZ_3 \otimes |0\rangle$ and $|Bell \otimes |Bell\rangle$.

| State          | $B^{(2)}(\alpha, \beta; |\Psi\rangle)$ | $B^{(3)}(\alpha, \beta, \gamma; |\Psi\rangle)$ | $B^{(4)}(1, 2, 3, 4; |\Psi\rangle)$ |
|----------------|-------------------------------------|-------------------------------------|-------------------------------------|
| $|GHZ_3\rangle$ | $\frac{1}{3}$                       | 0                                   | $1$                                 |
| $|W_4\rangle$   | $\frac{1}{3}$                       | $\frac{1}{4}$                       | $\frac{1}{4}$                       |
| $|\phi_6\rangle$| $\frac{1}{3}$                       | 0                                   | $\frac{1}{3}$                       |
| $|\phi_4\rangle$| $\frac{1}{3}$ or 0                   | 0                                   | $\frac{1}{3}$                       |
| $|GHZ_3\rangle \otimes |0\rangle$ | $\frac{1}{3}$                       | 1 or 0                              | 0                                   |
| $|Bell\rangle \otimes |Bell\rangle$ | 1 or 0                              | 0                                   | 0                                   |

$\frac{1}{3}$ or 0 in $B^{(2)}$ of $|\phi_4\rangle$ means that $B^{(2)}(1, 2; |\phi_4\rangle) = B^{(2)}(3, 4; |\phi_4\rangle) = \frac{1}{3}$ and $B^{(2)}$ for any other combinations of two qubits are zero, and 1 or 0 in $B^{(2)}$ of $|Bell\rangle \otimes |Bell\rangle$ does $B^{(2)}(1, 2; |Bell\rangle \otimes |Bell\rangle) = B^{(2)}(3, 4; |Bell\rangle \otimes |Bell\rangle) = 1$ and the others are zero. $B^{(3)}(1, 2, 3; |GHZ_3 \otimes |0\rangle) = 1$ and otherwise 0. Our measure gives the ordering for entanglement degrees in multiqubit systems, which depends on $m$.

So far we have focused on the entanglement measure of pure multiqubits. Now we intend to consider whether our measure extends to higher dimensional Hilbert spaces. The quantum correlation for eq. (1) is well defined in higher dimensional Hilbert space if the Pauli’s matrices are substituted by the identity operator and $N^2 - 1$ generators of $SU(N)$. Our definition of $M$ applies to the density operator in the mixed state as $\langle \bullet \rangle = tr(\rho \bullet)$. In the two-qubit mixed state, we get $B^{(2)}$ for Werner’s state;

$$B^{(2)}(1, 2; \rho_W) = \frac{1}{9}(4F - 1)^2$$

where $F$ is the fidelity for the singlet state. However, we know that the state in the region $F \leq \frac{1}{2}$, is separable. $B^{(2)}(1, 2; \rho_W)$ must be zero if $F \leq \frac{1}{2}$. $B$ is not ready to measure the entanglement for mixed states directly. The measure for the mixed state can be defined by convex roof as

$$B^{(m)}(\alpha, \beta, \cdots \gamma; \rho) = \min\left\{ \sum \rho_i B^{(m)}(\alpha, \beta, \cdots \gamma; |\Psi_i\rangle) | \sum \rho_i |\Psi_i\rangle \langle \Psi_i| = \rho \right\}.$$  

(7)

We here present the general measure of entanglement degree for any quantum states utilizing the expectation values of Pauli matrices based on the quantum correlations. Our measure gives full description on the entanglement structure to the given composite quantum systems. For instance, $B^{(4)}$ has the same magnitude in both $|GHZ_3 \otimes |0\rangle$ and $|Bell\rangle \otimes |Bell\rangle$, but $B^{(2)}$ and $B^{(3)}$ present the difference in the entanglement measure of subsystems. Entanglement ordering for comparing state are different depending on $m$. We believe that this feature leads us to find mathematical and physical avenues to construct the classification of multiqudit entanglements.
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