CRYSTAL BASES AND TWO–SIDED CELLS OF QUANTUM AFFINE ALGEBRAS

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1. Introduction

Let g be an affine Kac-Moody Lie algebra. Let U = U(g) be its quantum enveloping algebra introduced by Drinfeld and Jimbo, and let U+ be its positive part. The purpose of this paper is twofold. First, we define a basis B of U+ which is an analog of a PBW basis of U for a finite dimensional simple Lie algebra. It has the following properties (see Theorem 3.13):

1. Each element of B is a product of a monomial in ‘real root vectors’ and a Schur function in ‘imaginary root vectors’.
2. The transition matrix between B and Kashiwara–Lusztig’s global crystal basis (or canonical basis) G(B(−∞)) is upper-triangular with 1’s on the diagonal (with respect to a certain explicitly defined ordering) and with above diagonal entries in \( q^{-1} \mathbb{Z}[q^{-1}] \).

When g is symmetric or of type \( A_{2}^{(2)} \), our basis coincides with the one constructed by Beck-Chari-Pressley [6] or Akasaka [1] respectively, where a property weaker than (2) was established.

Second, we study the global crystal basis B(U) of the modified quantum enveloping algebra \( \tilde{U} \) defined by Lusztig [22]. We obtain a Peter-Weyl like decomposition of the crystal B(U) (Theorem 4.18), as well as an explicit description of two-sided cells of B(U) and the limit algebra of \( \tilde{U} \) at \( q = 0 \) (Theorem 6.44). These results had been conjectured by Nakashima [31], Kashiwara [18] and Lusztig [24] respectively. For type \( A^{(1)}_{1} \), the former was proved in [31].

Our results are based on the study of “extremal weight modules” \( V(\lambda) \) for \( \lambda \in P \), introduced by Kashiwara [17]. These \( \tilde{U} \)-modules have global crystal bases \( G(B(\lambda)) \), and for \( \lambda \in \bigcup_{w \in W} wP_{+} \) (the Tits cone) are isomorphic to irreducible highest weight modules [loc. cit.]. Outside of the Tits cone, or equivalently in the affine case for level zero weights, the structure of these modules has been studied by Akasaka-Kashiwara [3] and Kashiwara [15] and Lusztig [24] respectively. In particular, Kashiwara made conjectures on the crystal \( B(\lambda) \) [18, §13]. The present authors independently proved his conjectures for symmetric g [5, 30]. Both proofs used [6]. (The relevance of [6] to his conjectures was already pointed out in [18].) In this paper, we first generalize [6] to the nonsymmetric cases modulo sign and then proceed to prove the conjectures on \( B(\lambda) \) modulo sign. Next we remove the sign ambiguity. Finally, the above mentioned properties of B(U) are established.

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The construction of the basis $B$ is similar to the previous construction [6], although there are two new ideas worth remarking on. First, we use the level zero extremal weight modules $V(\lambda)$ in an essential way to check that after specializing at $q = \infty$ our basis elements equal canonical basis elements (rather than equal up to sign, which is easier and is also proved here). Note that in [6] the sign check depends on a positivity result of Lusztig which is available only for the symmetric case. Second, we obtain the fact that our basis $B$ is an integral basis (i.e., a basis of the Lusztig $\mathbb{Z}[q, q^{-1}]$–form of $U^+$) in an interesting way. This result follows from the upper triangularity (2) above and uses the canonical basis and the extremal weight modules in an essential way. Our proof is quite different from the proofs [1, 6] where the result is obtained by explicitly checking commutation relations between root vectors.

Furthermore, it should be mentioned that the integrality property of $B$ also gives (when specializing $q = 1$) a construction of an easily expressed basis of the $\mathbb{Z}$–form of the universal enveloping algebra of $\mathfrak{g}$. Such a basis has been constructed [14], but these results rely on directly examining all possible commutation relations between elements of the monomials forming the basis. Here we obtain this result as a corollary to the existence of the canonical basis of $U^+$.

Let us make one more remark regarding $B$. Recall that Lusztig used a PBW basis to give an alternative definition of the canonical basis for finite type $\mathfrak{g}$ (an ‘elementary algebraic definition’) [21]. Namely, the canonical basis is characterized as 1) an integral basis, 2) invariant under the bar involution, and 3) upper triangular with respect to the PBW basis. The existence of such a basis is guaranteed by the upper triangular property of the bar involution with respect to the PBW basis. Our definition of elements of $B$ is completely elementary and we can prove the upper triangular property without using the global crystal basis. For symmetric or type $A_{2}^{(2)}$, [1, 6] proved that $B$ is a basis of the integral form of $U^+$. The same argument as in the finite case then gives us the ‘elementary algebraic definition’ of the global crystal basis. (See Theorem 3.45.) However, when $B$ is not a priori known to be an integral basis (i.e., not in the symmetric or $A_{2}^{(2)}$ case), we only show the matrix expressing the bar involution has entries in $\mathbb{Q}(q)$, and we do not give an alternative algebraic definition of $G(B(-\infty))$. We hope that we might avoid the integrality requirement completely in the near future. In any case, our basis $B$ gives a parametrization of the global crystal basis and serves us here to prove the above mentioned conjectures on $B(\tilde{U})$.

We review the organization of this paper in detail. In §2 we introduce notation and preliminary results from [18, 30] and [5]. Next, in §3 we construct the basis $B$ for the integral form $\mathcal{A}U^+$ with the properties described above (up to sign). In §4 we consider the crystal structure of $\tilde{U}$ in more detail. We verify the conjectures of [18, §13] (up to sign) which describe the the crystal structure $B(\lambda)$ of $V(\lambda)$. $B(\lambda)$ decomposes into a product of $B_W(\lambda) \times \text{Irr} G_{\lambda}$, where for $\lambda = \sum \lambda_i \varpi_i$, $B_W(\lambda)$ denotes the crystal of $W(\lambda) = \bigotimes W(\varpi_i)^{\otimes \lambda_i}$ and $\text{Irr} G_{\lambda}$ denotes irreducible representations of $G_{\lambda} = \prod_i GL_{\lambda_i}(\mathbb{C})$. This decomposition is then used to give a $\mathfrak{g} \times \mathfrak{g}$ bicrystal decomposition of $B(\tilde{U}) \cong \bigsqcup_{\lambda \in P} B_0(\lambda) \times B(\lambda)/W$, where $B_0(\lambda)$ denotes the connected component of $B(\lambda)$ containing the extremal weight vector $v_\lambda$ and $W$ is the affine Weyl group. In §5 we pause to remove the sign ambiguity in §3 and §4.
In §6 we study the global basis of the level zero modified quantum affine algebra \( \tilde{U} = \bigoplus_{\lambda \in \mathcal{P}} U_{\lambda} \). To each \( \lambda \in P_{\mathfrak{cl}}^0 \) we associate a two sided ideal which is the intersection of the annihilators of all \( V(\lambda') \) for \( \lambda' \) outside the cone \( \lambda + P_{\mathfrak{cl}}^0 \) modulo this same ideal further intersected with the annihilator of \( V(\lambda) \). We show that these ideals have crystal bases \( \tilde{\mathcal{B}}[\lambda] \) which have globalizations which partition the global basis of \( \tilde{U} \). We use this partition to describe the cell structure of \( \tilde{\mathcal{B}}(\tilde{U}) \) and to verify the conjectures which appear in [24].

Lusztig’s conjectures on two-sided cells were based on his conjectures [20] on cells of an affine Hecke algebra, which as far as the authors know, are still open. In [24] Lusztig made a deep connection between two–sided cells of the affine Hecke algebra and the geometry of Springer fibers. Our proof is based on extremal weight modules and is purely algebraic. However geometry is in the background, since extremal weight modules are isomorphic [31] to universal standard modules, which are defined as \( K \)-homology groups of certain quiver varieties introduced by the second author [29]. For example, values of the \( a \)-function introduced in §6 are equal to the dimensions of the quiver varieties, where the corresponding result for the affine Hecke algebra was proved in [20]. It is also worthwhile mentioning that the appearance of \( G_\lambda \) is quite natural from quiver varieties.

While the authors were preparing this paper, K. McGerty posted an article [27] to the q-algebra archive where he proves Lusztig’s conjecture for type \( A_n^{(1)} \). His proof is completely different from our proof.

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2. Preliminaries

2.1. Affine Kac–Moody Lie algebras. We fix a realization \( \mathfrak{g} = \mathfrak{g}(X_N^{(r)}) \). Here \( X_N^{(r)} \) is a diagram from Table Aff \( r \) of [12] Section 4.8], except in the case of \( X_N^{(r)} = A_n^{(2)} \) \( (n \geq 1) \), where we reverse the numbering of the simple roots. Let its Cartan subalgebra be \( \mathfrak{h} \). We denote by \( I \) the index set of simple roots. The numbering gives us an identification \( I = \{0, 1, \ldots, n\} \). Let \( \{\alpha_i\}_{i \in I} \subset \mathfrak{h} \) (resp. \( \{h_i\}_{i \in \mathfrak{h}} \subset \mathfrak{h}^* \)) denote the set of simple roots (resp. simple coroots), where \( \langle h_i, \alpha_j \rangle = a_{ij} \), where \( a_{ij} \) is the Cartan matrix of \( \mathfrak{g} \). Fix \( d \) so that \( \langle d, \alpha_j \rangle = \delta_{ij} \). Denote by \( P^* = \bigoplus_{i \in I} \mathbb{Z}h_i \oplus \mathbb{Z}d \) the dual weight lattice and by \( P = \text{Hom}_\mathbb{Z}(P^*, \mathbb{Z}) \) the weight lattice. Let \( Q = \sum_{i \in I} \mathbb{Z}a_i \subset P \) denote the root lattice, \( \Delta \) the root system and \( \Delta^w = \Delta \setminus \mathbb{Z}\delta \) the set of real roots. Fix the fundamental weights \( \Lambda_i \in P \) defined by \( \langle h_i, \Lambda_i \rangle = \delta_{ij}, \langle d, \Lambda_i \rangle = 0 \). Denote by \( Q_+ \) the semigroup generated by positive roots \( \sum_{i \in I} \mathbb{Z}_{\geq 0}a_i \); \( P_+ = \{\lambda \in P \mid \langle h_i, \lambda \rangle \geq 0 \text{ for all } i \in I\} \) the semigroup of integral dominant weights. Let \( \Delta^\pm = \Delta \cap (\pm Q_+) \) be the set of positive and negative roots respectively.

The center of \( \mathfrak{g} \) is 1–dimensional and is spanned by \( c = \sum_{i \in I} a_i^\vee h_i \), where \( a_i^\vee \) are the labels of the dual diagram to \( X_N^{(r)} \). \( c \) is characterized as the positive combination of \( h_i, i \in I \), for which \( \{h \in P^* \mid \langle h, \alpha_j \rangle = 0 \text{ for all } j \in I\} = \mathbb{Z}c \). Let \( \delta \) be the unique element \( \delta = \sum_{i \in I} a_i\alpha_i \) \( (a_i \in \mathbb{Z}_{\geq 0}) \), where \( a_i \) are the numerical labels of \( X_N^{(r)} \), and give a linear dependence between the columns of \( a_{ij} \) satisfying \( \{\lambda \in Q \mid \langle h_i, \lambda \rangle = 0 \text{ for all } i \in I\} = \mathbb{Z}\delta \). We denote by \( h \) the Coxeter number \( \sum_{i \in I} a_i \) and by \( h^\vee \) the dual Coxeter number \( \sum_{i \in I} a_i^\vee \).
Denote the affine Weyl group by $\tilde{W} \subset O(h^*) (= the orthogonal group of h^* with respect to ( , ))$ generated by the simple reflections $s_i(\lambda) = \lambda - (h_i, \lambda)\alpha_i, \lambda \in h, i \in I$. Note that $w(\delta) = \delta$ for $w \in W$. Denote by $( , )$ the non-degenerate symmetric bilinear form on $h^*$ invariant under the Weyl group action, uniquely characterized by $(c, \lambda) = (\delta, \lambda)$, for $\lambda \in h^*$. Note that $(\alpha_i, \alpha_j) = a_{i,j}^{-1}$ for $i, j \in I$

Let $\text{cl}: h^* \to h^*/Q\delta$ be the canonical projection. Let $h^*_{\text{cl}} = \{\lambda \in h^* \mid (c, \lambda) = 0\}$, and define the level zero weight lattice to be $P_0 = P \cap h^*_{\text{cl}}$. Let $h^*_{\text{cl}} = \text{cl}(h^*_{\text{cl}})$ and $P_0 = \text{cl}(P^0)$. Denote $\Delta_{\text{cl}} = \text{cl}(\Delta^{\text{re}})$. Since $w(\delta) = \delta$ and $(c, \lambda) = (\delta, \lambda)$, the image of $W \subset O(h_{\text{cl}})$ is well-defined and denoted $W_{\text{cl}}$. Then $W_{\text{cl}}$ is the Weyl group of the root system $(\Delta_{\text{cl}}, h^*_{\text{cl}})$, which is reduced, except in type $A^{(2)}_{2n}$ where it is of type $BC_n$. The bilinear form $( , )$ on $h^*$ descends to a bilinear form on $h^*_{\text{cl}}$, which is denoted also by $( , )$. It is nondegenerate.

We fix $0 \in I$ so that $W_{\text{cl}}$ is generated by $\{s_i; i \in I_0\}$, where $I_0 = I \setminus \{0\} = \{1, 2, \ldots, n\}$. If $g$ is not of type $A^{(2)}_{2n}$, the choice of 0 is unique up to a Dynkin diagram automorphism. In the case of $A^{(2)}_{2n}$, there are two choices of 0 (either of the two extremal vertices of the Dynkin diagram), and $(\alpha_0, \alpha_0) = 1$ or 4, and accordingly $a_0 = 2$ or 1, $a'_0 = 1$ or 2. Our choice of 0 is such that $(\alpha_0, \alpha_0) = 4$. As mentioned above, this is opposite the numbering convention in [13], but is natural when constructing $g = g(A^{(2)}_{2n})$ as a (twisted) loop algebra. We take $\{\text{cl}(\alpha_i) \mid i \in I_0\}$ as a set of simple roots of $\Delta_{\text{cl}}$, and the corresponding set $\Delta^+_\text{cl}$ of positive roots.

Let $\alpha^\vee = 2\alpha/(\alpha, \alpha)$. Let $Q^\vee = \sum_{\alpha \in \Delta^{\text{re}}} Z\alpha^\vee$. We set $Q_{\text{cl}} = \text{cl}(Q), Q'_{\text{cl}} = \text{cl}(Q'^\vee)$, $\bar{Q} = Q_{\text{cl}} \cap Q'_{\text{cl}}$. We have an exact sequence

\[
1 \to \bar{Q} \xrightarrow{\iota} \tilde{W} \xrightarrow{\text{cl}} W_{\text{cl}} \to 1,
\]

where $\iota$ is the translation operator given by [13] (6.5.2)], and $\text{cl}$ is the above projection $\tilde{W} \to W_{\text{cl}}$. By abuse of notation we denote $t(\xi)$ simply by $\xi$. For any $\alpha \in \Delta^{\text{re}}$, let $\tilde{\alpha}$ be the element in $\bar{Q} \cap Q_{\geq 0}\text{cl}(\alpha)$ with the smallest length. We set

\[
\tilde{\Delta} = \{\tilde{\alpha} : \alpha \in \Delta^{\text{re}}\}.
\]

Then $\tilde{\Delta}$ is a reduced root system, and $\bar{Q}$ is the root lattice of $\tilde{\Delta}$.

An affine Lie algebra $g$ is either untwisted or the dual of an untwisted affine Lie algebra or $A^{(2)}_{2n}$.

(i) If $g$ is untwisted, then $2/(\alpha, \alpha) \in Z$, $\bar{Q} = Q'_{\text{cl}} \subset Q_{\text{cl}}, \tilde{\alpha} = \text{cl}(\alpha^\vee)$, $\tilde{\Delta} = \text{cl}(\Delta^{\text{re}})$.

(ii) If $g$ is the dual of an untwisted algebra, then $(\alpha, \alpha)/2 \in Z$, $\bar{Q} = Q_{\text{cl}} \subset Q'_{\text{cl}}$, $\tilde{\alpha} = \text{cl}(\alpha)$, $\tilde{\Delta} = \text{cl}(\Delta^{\text{re}})$.

(iii) If $g = g(A^{(2)}_{2n})$, then $(\alpha, \alpha)/2 = 1/2, 1, \text{ or } 2$, $\bar{Q} = Q_{\text{cl}} = Q'_{\text{cl}}$, and

\[
\tilde{\alpha} = \begin{cases} 
\text{cl}(\alpha) & \text{if } (\alpha, \alpha) \neq 4, \\
\text{cl}(\alpha)/2 & \text{if } (\alpha, \alpha) = 4.
\end{cases}
\]

Note that $(\delta - \alpha)/2 \in \Delta^{\text{re}}$ if $(\alpha, \alpha) = 4$. $\tilde{\Delta}$ is of type $B_n$.

For $\alpha \in \Delta^{\text{re}}$ or $\alpha \in \Delta_{\text{cl}}$, we set

\[
d_\alpha = \max(1, \frac{(\alpha, \alpha)}{2}),
\]
and \(d_k = d_{\alpha_i}\). We have \(m \delta + \alpha \in \Delta^\text{re} \iff d_{\alpha_i} | m\). If \(X^\text{(r)}_N \neq A_2^{(2)}\), then \(\hat{\alpha} = d_{\alpha_i} \text{cl}(\alpha_i)\).

Let \(P^0_{cl}\) (resp. \(P^0_{cl}\)) be the dual of \(Q^\vee_{cl}\) (resp. \(Q_{cl}\)), considered as a lattice of \(h^0_{cl}\) via \((, )\). Set \(\tilde{P} = P^0_{cl} + P^0_{cl} = (Q_{cl} + Q_{cl})^*\). The sets \(\Delta\), \(\Delta^\vee\) are invariant under the translation by an element of \(\tilde{P}\). We define the extended affine Weyl group by \(\tilde{W} = \tilde{P} \times W_{cl}\). Let \(\mathcal{T} = \{w \in \tilde{W} | w(\Delta^+) \subset \Delta^+\}\). It is a subgroup of the group of Dynkin diagram automorphisms. We have \(\tilde{W} = \mathcal{T} \times W\). The length function \(\ell: \tilde{W} \to \mathbb{N}\) extends to \(\ell: W \to \mathbb{N}\) where \(\ell(\tau w) = \ell(w)\) for \(\tau \in \mathcal{T}, w \in \tilde{W}\).

Let us denote by \(\omega^\vee_i\) (\(i \in I_0\)) the fundamental coweights of the root system \((\Delta_{cl}, h^0_{cl})\), i.e., \((\text{cl}(\alpha_i), \omega^\vee_i) = \delta_{ij}\) for \(i, j \in I_0\). Let \(\tilde{\omega}_i = d_i \omega^\vee_i\). Then \(\{\tilde{\omega}_i\}_{i \in I_0}\) is a basis of \(\tilde{P}\). We have

\[
s_j \tilde{\omega}_i = \tilde{\omega}_j s_j (i \neq j),
\]

\[
s_i \tilde{\omega}_i s_i \tilde{\omega}_i^{-1} = t(-d_i \alpha^\vee_i) = \begin{cases} 
\tilde{\alpha}_i^{-1} & \text{if } (X^\text{(r)}_N, i) \neq (A_2^{(2)}, n), \\
\tilde{\alpha}_i^{-2} & \text{if } (X^\text{(r)}_N, i) = (A_2^{(2)}, n).
\end{cases}
\]

Set

\[
\mathcal{R}_> = \{\alpha \in \Delta^+ | \text{cl}(\alpha) \in \Delta^+_{cl}\}, \quad \mathcal{R}_< = \{\alpha \in \Delta^+ | \text{cl}(\alpha) \in -\Delta^+_{cl}\},
\]

\[
\mathcal{R}_0 = \{(m \delta, i) \in \mathbb{Z} \delta \times I_0 | m > 0, d_i | m\},
\]

\[
\mathcal{R} = \mathcal{R}_0 \cup \mathcal{R}_0 \cup \mathcal{R}_<.
\]

These are sets of positive roots counted with multiplicities.

2.2. Quantum affine algebras. We define the quantum affine algebra \(U = U(q)\) following the normalization in [10, 3]. Let \(q\) be an indeterminate. For nonnegative integers \(n \geq r\), define

\[
[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [n]_q! = \begin{cases} 
[n]_q[n - 1]_q! & (n > 0), \\
1 & (n = 0),
\end{cases}
\]

\[
[n]_q = \frac{[n]!}{[r]_q! [n - r]_q!}.
\]

We fix the smallest positive integer \(d\) such that \(d(\alpha_i, \alpha_i)/2 \in \mathbb{Z}\) for any \(i \in I\). We set \(q_s = q^{1/d}\).

Define the quantum affine algebra \(U\) to be the associative algebra with 1 over \(\mathbb{Q}(q_s)\) generated by elements \(E_i, F_i (i \in I), q^h (h \in d^{-1}P^+)\), with defining relations.

\[
q^0 = 1, \quad q^h q^{h'} = q^{h+h'},
\]

\[
q^h E_i q^{-h} = q^{(h, \alpha_i)} E_i, \quad q^h F_i q^{-h} = q^{-(h, \alpha_i)} F_i,
\]

\[
E_i F_j - F_j E_i = \delta_{ij} \frac{t_i - t_i^{-1}}{q_i - q_i^{-1}}
\]

\[
\sum_{p=0}^{b} (-1)^p E_i^{(p)} E_j F_i F_j^{(b-p)} = \sum_{p=0}^{b} (-1)^p F_i^{(p)} F_j E_i E_j^{(b-p)} = 0 \quad \text{for } i \neq j,
\]

where \(q_i = q^{(\alpha_i, \alpha_i)/2}, t_i = q^{(\alpha_i, \alpha_i)h_i/2}, b = 1 - (h_i, \alpha_j), E_i^{(p)} = E_i^p / [p]_q!, F_i^{(p)} = F_i^p / [p]_q!\).

Let \(U'\) be the quantized enveloping algebra with \(P_{cl} = \text{cl}(P)\) as a weight lattice. It is the subalgebra of \(U\) generated by \(E_i, F_i (i \in I), q^h (h \in d^{-1} \bigoplus_i \mathbb{Z} h_i)\).
Let $\mathbf{U}^+$ (resp. $\mathbf{U}^-$) be the $\mathbf{Q}(q_s)$-subalgebra of $\mathbf{U}$ generated by elements $E_i$'s (resp. $F_i$'s). Let $\mathbf{U}^0$ be the $\mathbf{Q}(q_s)$-subalgebra generated by elements $q^h$ ($h \in d^{-1} P^*$).

We have the triangular decomposition $\mathbf{U} \cong \mathbf{U}^+ \otimes \mathbf{U}^0 \otimes \mathbf{U}^-$. For $\xi \in \mathbf{Q}$, we define the root space $\mathbf{U}_\xi$ by

$$\mathbf{U}_\xi = \{ x \in \mathbf{U} \mid q^h x q^{-h} = q^{(h, \xi)} x \text{ for all } h \in P^* \}.$$ 

Let $\mathbf{A} = \mathbf{Z}[q_s, q_s^{-1}]$. Let $\mathbf{A}\mathbf{U}$ be the $\mathbf{A}$-subalgebra of $\mathbf{U}$ generated by elements $E_i^{(n)}$, $F_i^{(n)}$, $q^h$ for $i \in I$, $n \in \mathbf{Z}_{>0}$, $h \in d^{-1} P^*$.

Let us introduce a $\mathbf{Q}(q_s)$-algebra involutive automorphism ∨ and $\mathbf{Q}(q_s)$-algebra involutive anti-automorphisms * and $\psi$ of $\mathbf{U}$ by

$$E_i^\vee = F_i, \quad F_i^\vee = E_i, \quad (q^h)^\vee = q^{-h},$$

$$E_i^* = E_i, \quad F_i^* = F_i, \quad (q^h)^* = q^h,$$

$$\psi(E_i) = q_i^{-i} t_i^{-1} F_i, \quad \psi(F_i) = q_i^{-1} t_i E_i, \quad \psi(q^h) = q^h.$$ 

We define a $\mathbf{Q}$-algebra involutive automorphism $\overline{\cdot}$ of $\mathbf{U}$ by

$$\overline{E_i} = E_i, \quad \overline{F_i} = F_i, \quad \overline{q^h} = q^{-h},$$

$$\overline{a(q_s) u} = a(q_s^{-1}) \overline{u} \quad \text{for } a(q_s) \in \mathbf{Q}(q_s) \text{ and } u \in \mathbf{U}.$$ 

We define the coproduct $\Delta$ on $\mathbf{U}$ by

$$\Delta q^h = q^h \otimes q^h, \quad \Delta E_i = E_i \otimes t_i^{-1} + 1 \otimes E_i,$$

$$\Delta F_i = F_i \otimes 1 + t_i \otimes F_i.$$ 

(2.4) 

Let us denote by $\Omega$ the $\mathbf{Q}$-algebra anti-automorphism $* \circ \overline{\cdot} \circ \vee$ of $\mathbf{U}$. We have

$$\Omega(E_i) = F_i, \quad \Omega(F_i) = E_i, \quad \Omega(q^h) = q^{-h}, \quad \Omega(q_s) = q_s^{-1}.$$ 

A $\mathbf{U}$-module $M$ is called **integrable** if

(1) all $E_i$, $F_i$ ($i \in I$) are locally nilpotent, and

(2) it admits a weight space decomposition:

$$M = \bigoplus_{\lambda \in P} M_{\lambda}, \quad \text{where } M_{\lambda} = \{ u \in M \mid q^h u = q^{(h, \lambda)} u \text{ for all } h \in P^* \}.$$ 

Let $\tilde{\mathbf{U}}$ be the modified enveloping algebra [23, Part IV]. It is defined by

$$\tilde{\mathbf{U}} = \bigoplus_{\lambda \in P} \mathbf{U} a_{\lambda}, \quad \mathbf{U} a_{\lambda} = \mathbf{U} / \sum_{h \in P^*} \mathbf{U}(q^h - q^{(h, \lambda)}) .$$ 

Here the multiplication is given by

$$a_{\lambda} x = x a_{\lambda - \xi} \quad \text{for } \xi \in \mathbf{U}_\xi, \quad a_{\lambda} a_{\mu} = \delta_{\lambda \mu} a_{\lambda},$$

where $a_{\lambda}$ is considered as the image of 1 in the above definition of $\mathbf{U} a_{\lambda}$.

Let $\lambda$, $\mu \in P_+$. Let $V(\lambda)$ (resp. $V(-\mu)$) be the irreducible highest (resp. lowest) weight module of weight $\lambda$ (resp. $-\mu$) [23, §3.5]. Then there is a surjective homomorphism

$$\mathbf{U} a_{\lambda - \mu} \ni u \mapsto u(1_{\lambda} \otimes 1_{-\mu}) \in V(\lambda) \otimes V(-\mu),$$

where $1_{\lambda}$ (resp. $1_{-\mu}$) is a highest (resp. lowest) weight vector of $V(\lambda)$ (resp. $V(-\mu)$).
2.3. Bilinear Form. In constructing our crystal base a key component is a variant of a bilinear form introduced by Drinfeld which characterizes the global crystal basis of $\mathcal{A}U^+$. To introduce the form, first define an algebra structure on $U^+ \otimes U^+$ by

$$(x_1 \otimes x_2)(y_1 \otimes y_2) = q_i^{(w_1,w_2)}x_1y_1 \otimes x_2y_2,$$

where $x_t, y_t \ (t = 1, 2)$ are homogeneous. Let $r: U^+ \rightarrow U^+ \otimes U^+$ be the $Q(q_i)$-algebra homomorphism defined by extending $r(e_i) = E_i \otimes 1 + 1 \otimes E_i \ (i \in I)$. By $[23, 1.2.5]$, the algebra $U^+$ has a unique symmetric bilinear form $(\ , \ )$: $U^+ \times U^+ \rightarrow Q(q_i)$ satisfying $(1, 1) = 1$ and

$$(E_i, E_j) = \delta_{ij} \frac{1}{(1 - q_i^2)},$$

$$(x, y') = (r(x), y \otimes y'), \quad (xx', y) = (x \otimes x', r(y)),$$

where the form on $U^+ \otimes U^+$ is defined by $(x_1 \otimes y_1, x_2 \otimes y_2) = (x_1, x_2)(y_1, y_2)$.

For $i \in I$, introduce the unique $Q(q_i)$-linear map $r_i: U^+ \rightarrow U^+$ given by $r_i(1) = 0$, $r_i(E_j) = \delta_{ij}$ for $j \in I$, and satisfying $r_i(xy) = q_i^{(w_1,w_2)}r_i(x)y + x r_i(y)$ for all homogeneous $x, y \in U^+$ ([L3, 1.2.13]). Similarly, introduce the unique $Q(q_i)$-linear map $r_i: U^+ \rightarrow U^+$ given by $r_i(1) = 0$, $r_i(E_j) = \delta_{ij}$, and satisfying $r_i(xy) = r_i(x)y + q_i^{(w_1,w_2)}x r_i(y)$ for all homogeneous $x, y \in U^+$.

From the definition the form satisfies

$$(E_i y, x) = (1 - q_i^{-2})^{-1}(y, r_i(x)),$$

$$(y E_i, x) = (1 - q_i^{-2})^{-1}(y, r_i(x)).$$

2.4. Braid group action. For each $w \in \tilde{W}$, there exists an $Q(q_i)$-algebra automorphism $T_w$ ([23, §39]) (denoted there by $T_w'). Also, for any integrable $U$-module $M$, there exists $Q(q)$-linear map $T_w: M \rightarrow M$ satisfying $T_w(xu) = T_w(x)T_w(u)$ for $x \in U, u \in M$. We denote $T_{x_i}$ by $T_i$ hereafter. By [23, 39.4.5] we have

$$(2.6) \quad \Omega \circ T_w \circ \Omega = T_w.$$

The definition of the automorphism $T_w$ of $U$ can be extended to the case $w \in \tilde{W}$ by setting

$$\tau E_i = E_{\tau(i)}, \quad \tau F_i = F_{\tau(i)}, \quad \tau q^h_i = q^{h_{\tau(i)}}, \quad \tau q^d = q^d.$$

2.5. Crystal bases. We briefly recall the notion of crystal bases. For the notion of (abstract) crystals and more details, we refer to [17, 3].

For $n \in Z$ and $i \in I$, let us define an operator acting on any integrable $U$-module $M$ by

$$\tilde{F}_i^{(n)}(t_i) = \sum_{k \geq \max(0,-n)} F_i^{(n+k)}E_i^{(k)}a_k^n(t_i),$$

where $a_k^n(t_i) = (-1)^k q_i^{k(1-n)} t_i^{k-1} \prod_{v=1}^{k-1} (1 - q_i^{n+2v})$.

And we set $\tilde{e}_i = F_i^{(-1)}$, $\tilde{f}_i = F_i^{(1)}$.

These operators are different from those used for the definition of crystal bases in [18], but give us the same crystal bases by [18, Proposition 6.1].
Let $A_0 = \{ f(q_s) \in Q(q_s) \mid f \text{ is regular at } q_s = 0 \}$. Let $A_\infty = \overline{A_0}$ be the image of $A_0$ under $\overline{\cdot}$, that is, the subring of $Q(q_s)$ consisting of rational functions regular at $q_s = \infty$.

**Definition 2.7.** Let $M$ be an integrable $U$-module. A pair $(\mathcal{L}, B)$ is called a crystal basis of $M$ if it satisfies

1. $\mathcal{L}$ is a free $A_0$-submodule of $M$ such that $M \cong Q(q_s) \otimes_{A_0} \mathcal{L}$,
2. $\mathcal{L} = \bigoplus_{\lambda \in P} \mathcal{L}_\lambda$ where $\mathcal{L}_\lambda = \mathcal{L} \cap M_\lambda$ for $\lambda \in P$,
3. $B$ is a $Q$-basis of $\mathcal{L}/q \mathcal{L} \cong Q \otimes_{A_0} \mathcal{L}$,
4. $\bar{e}_i \mathcal{L} \subset \mathcal{L}$, $\bar{f}_i \mathcal{L} \subset \mathcal{L}$ for all $i \in I$,
5. if we denote operators on $\mathcal{L}/q \mathcal{L}$ induced by $\bar{e}_i$, $\bar{f}_i$ by the same symbols, we have $\bar{e}_i B \subset B \cup \{0\}$, $\bar{f}_i B \subset B \cup \{0\}$,
6. for any $b, b' \in B$ and $i \in I$, we have $b' = \bar{f}_i b$ if and only if $b = \bar{e}_i b'$.

We define functions $\varepsilon_i, \varphi_i : B \to Z_{\geq 0}$ by $\varepsilon_i(b) = \max\{ n \geq 0 \mid \bar{e}_i^n b \neq 0 \}$, $\varphi_i(b) = \max\{ n \geq 0 \mid \bar{f}_i^n b \neq 0 \}$. We set $\varepsilon_i^{\max} = \varepsilon_i(b)$, $\varphi_i^{\max} = \varphi_i(b)$.

Let $M$ be an integrable $U$-module with a crystal basis $(\mathcal{L}, B)$. Let $\overline{\mathcal{L}}$ be an involution of an integrable $U$-module $M$ satisfying $\overline{\mathcal{L}} = PU = PU$ for any $x \in U, u \in M$. Let $A_M$ be an $A_{\mathcal{L}}$-submodule of $M$ such that $A_M = \mathcal{L} M$, $u - \mu \in \{ q_s - 1 \} A M$ for $u \in A_M$. We say that $M$ has a global crystal basis $(\mathcal{L}, B, A_M)$ if the following conditions are satisfied

1. $M \cong Q(q_s) \otimes_{Z[q_s,q_s^{-1}]} A_M \cong Q(q_s) \otimes_{A_0} \mathcal{L}$,
2. $\mathcal{L} \cap \overline{\mathcal{L}} \cap A_M \to \mathcal{L} \cap A_M/q_s(\mathcal{L} \cap A_M)$ is an isomorphism,
3. $B \subset \mathcal{L} \cap A_M/q_s(\mathcal{L} \cap A_M)$.

As a consequence of the definition, natural homomorphisms

$$A_0 \otimes_{\mathbb{Z}} (\mathcal{L} \cap \overline{\mathcal{L}} \cap A_M) \to \mathcal{L}, \quad A_\infty \otimes_{\mathbb{Z}} (\mathcal{L} \cap \overline{\mathcal{L}} \cap A_M) \to \overline{\mathcal{L}},$$

are isomorphisms. We call the triple $(\mathcal{L}, \overline{\mathcal{L}}, A_M)$ a balanced triple.

Let $G$ be the inverse isomorphism $\mathcal{L} \cap A_M/q_s(\mathcal{L} \cap A_M) \to \mathcal{L} \cap \overline{\mathcal{L}} \cap A_M$. Then $\{ G(b) \mid b \in B \}$ is a basis of $M$. It is called a global crystal basis of $M$. The above conditions imply $G(b) = G(b)$.

$U^-$ (resp. $U^+$) has a global crystal basis $(\mathcal{L}(\infty), B(\infty), A_U^+)$ (resp. $(\mathcal{L}(-\infty), B(-\infty), A_U^+)$). (It is not an integrable $U$-module. But the above definitions has a modification.) For a dominant weight $\lambda \in P_+$, the irreducible highest weight module $V(\lambda)$ has a global crystal basis $(\mathcal{L}(\lambda), B(\lambda), A_V(\lambda))$. If $\lambda, \mu \in P_+$, then the tensor product $V(\lambda) \otimes V(-\mu)$ also has a global crystal basis $(\mathcal{L}(\lambda) \otimes \mathcal{L}(-\mu), B(\lambda) \otimes B(-\mu), A_{V}(\lambda, A_{V}(-\mu))$, where the bar involution is defined by using the quasi $R$-matrix [23, Part IV]. Moreover, $\hat{U}$ has a global crystal basis $(\mathcal{L}(\hat{U}), B(\hat{U}), A_{\hat{U}})$ such that the homomorphism [25] maps a global basis of $\hat{U}$ to the union of that of $V(\lambda) \otimes V(-\mu)$ and $0$ ([17, Theorem 2.1.2] and [23, Part IV]). Note that $\hat{U}$ is not an integrable $U$-module, and operators $\hat{e}_i, \hat{f}_i$ are defined only on $\mathcal{L}(\hat{U})/q_s \mathcal{L}(\hat{U})$. In fact, they are defined so that $B(\lambda) \otimes B(-\mu) \to B(\hat{U} Q_{\lambda, -\mu}) \subset B(\hat{U})$ is a strict embedding. Furthermore, the global basis is invariant under $* [17, 4.3.2]$. The proof given there also shows $V$ leaves the global basis invariant. We have $B(\hat{U}) = \bigoplus_{\lambda \in P} B(\hat{U} Q_\lambda)$. We define $\varepsilon^*, \varphi^*, \varepsilon_i^*, \varphi_i^*$ by $\varepsilon^*(b) = \varepsilon(b^*)$, $\varphi^*(b) = \varphi(b^*)$, $\varepsilon_i^*(b) = (\varepsilon_i(b^*))^*$ and $\varphi_i^*(b) = (\varphi_i(b^*))^*$. This
another crystal structure on $\tilde{U}$ is called the *star crystal structure*. Occasionally we denote $B(\tilde{U})$ simply by $\tilde{B}$.

### 2.6. Braid group action and global crystal bases.

We will recall results of [23]. Let

$$U^+[i] = \{ x \in U^+ \mid T_i(x) \in U^+ \}, \quad \ast U^+[i] = \{ x \in U^+ \mid T_i^{-1}(x) \in U^+ \}.$$ 

By [23, 38.1] we have direct sum decompositions of vector spaces

$$U^+ = U^+[i] \oplus E_i U^+, \quad \ast U^+ = \ast U^+[i] \oplus U^+ E_i.$$

Let $i^\pi \colon U^+ \to U^+[i]$, $\pi^i \colon U^+ \to \ast U^+[i]$ be the natural projections. We have the algebra isomorphism $T_i \colon U^+[i] \to \ast U^+[i]$. Then [23, Proposition 1.9] says

$$(2.8) \quad T_i(\pi^i(x)) = \pi^i(T_i(x))$$

for $x \in U^+[i]$.

By [23, 14.3] $i^\pi$ (resp. $\pi^i$) maps the global basis to the union of a basis of $U^+[i]$ (resp. $\ast U^+[i]$) and $0$. Then [23, Theorem 1.2] says that $T_i(\pi(G(b)))$, if it is nonzero, is equal to $\pi^i(G(b'))$ for some $b'$, and the map $b \mapsto b'$ gives a bijection between $\{ b \in B(\infty) \mid \pi^i(G(b)) \neq 0 \}$ and $\{ b' \in B(\infty) \mid \pi^i(G(b')) \neq 0 \}$. This result is based in part on an earlier result [32].

#### 2.7. Affinization.

Let $M$ be an integrable $U'$–module, and let $M = \bigoplus_{\lambda \in P} M_\lambda$ be its weight decomposition. We define a $U$–module $M_{aff}$ by

$$M_{aff} = \bigoplus_{\lambda \in P} M_{cl(\lambda)}.$$ 

The action of $e_i$ and $f_i$ are defined by restricting to each summand, so that the canonical homomorphism $cl \colon M_{aff} \to M$ is $U'$–linear. We define the $U'$–linear automorphism $z$ of $M_{aff}$ with weight $\delta$ by $(M_{aff})_\lambda \overset{z}{\to} M_{cl(\lambda)} = M_{cl(\lambda + \delta)} \overset{\sim}{\to} (M_{aff})_{\lambda + \delta}.$

Choose a section $s \colon P_{cl} \to P$ of $cl \colon P \to P_{cl}$ such that $s(cl(\alpha_i)) = \alpha_i$ for any $i \in I_0 = I \setminus \{0\}$. Then $M$ is embedded into $M_{aff}$ by $s$ as a vector space. We have an isomorphism of $U'$–modules

$$(2.9) \quad M_{aff} \simeq Q(q_s)[z, z^{-1}] \otimes M.$$ 

Here and $e_i \in U'$ and $f_i \in U'$ act on the right hand side by $z^{\delta_{i0}} \otimes e_i$ and $z^{-\delta_{i0}} \otimes f_i$.

Similarly, for a crystal with weights in $P_{cl}$, we can define its affinization $B_{aff}$ by

$$(2.10) \quad B_{aff} = \bigsqcup_{\lambda \in P} B_{cl(\lambda)}.$$ 

If an integrable $U'$–module $M$ has a crystal basis $(L, B)$, then its affinization $M_{aff}$ has a crystal basis $(L_{aff}, B_{aff})$.

For $a \in Q(q_s)$, we define the $U'$–module $M_a$ by

$$(2.11) \quad M_a = M_{aff}/(z - a)M_{aff}.$$
2.8. **Extremal weight modules.** A crystal \( B \) over \( U \) is called regular if, for any \( J \subseteq I \), \( B \) is isomorphic (as a crystal over \( U(\mathfrak{g}_J) \)) to the crystal associated with an integrable \( U(\mathfrak{g}_J) \)-module. (It was called normal in [17].) Here \( U(\mathfrak{g}_J) \) is the subalgebra generated by \( E_j, F_j \) \((j \in J)\), \( q^h \) \((h \in d^{-1}P^*)\). By [13], the affine Weyl group \( \hat{W} \) acts on any regular crystal. The action \( S \) is given by

\[
S_{s_i}b = \begin{cases} 
 f_i^{(h_i, wt b)}b & \text{if } \langle h_i, wt b \rangle \geq 0, \\
 e_i^{(h_i, wt b)}b & \text{if } \langle h_i, wt b \rangle \leq 0
\end{cases}
\]

for the simple reflection \( s_i \). We denote \( S_{s_i} \) by \( S_i \) hereafter.

**Definition 2.12.** Let \( M \) be an integrable \( U \)-module. A vector \( u \in M \) with weight \( \lambda \in P \) is called extremal, if the following holds for all \( w \in \hat{W} \):

\[
\begin{align*}
E_i T_w u = 0 & \text{ if } \langle h_i, w\lambda \rangle \geq 0, \\
F_i T_w u = 0 & \text{ if } \langle h_i, w\lambda \rangle \leq 0.
\end{align*}
\]

In this case, we define \( S_w u \) so that

\[
S_i S_w u = \begin{cases} 
 f_i^{(h_i, w\lambda)} u & \text{if } \langle h_i, w\lambda \rangle \geq 0, \\
 e_i^{(h_i, w\lambda)} u & \text{if } \langle h_i, w\lambda \rangle \leq 0.
\end{cases}
\]

This is well-defined, i.e., \( S_w u \) depends only on \( w \).

Similarly, for a vector \( b \) of a regular crystal \( B \) with weight \( \lambda \), we say that \( b \) is extremal if it satisfies

\[
\begin{align*}
\hat{e}_i S_w b = 0 & \text{ if } \langle h_i, w\lambda \rangle \geq 0, \\
\hat{f}_i S_w b = 0 & \text{ if } \langle h_i, w\lambda \rangle \leq 0.
\end{align*}
\]

**Lemma 2.14 ([10, Lemma 2.11]).** Suppose that an integrable \( U \)-module \( M \) has a crystal basis \( (\mathcal{L}, \mathcal{B}) \). If \( u \in \mathcal{L} \subset M \) is an extremal vector of weight \( \lambda \) satisfying \( b = u \mod q\mathcal{L} \in \mathcal{B} \), then \( b \) is an extremal vector, and we have

\[
S_w u = (-1)^{N_+} q^{\sum_{\alpha \in \mathbb{R}_+} \max((\alpha, \lambda), 0)} T_w u, \quad S_w b = S_w u \mod q\mathcal{L} \quad \text{for all } w \in \hat{W},
\]

where \( N_+ = \sum_{\alpha \in \mathbb{R}_+} \max((\alpha, \lambda), 0) \), and \( N_+^\vee \) is given by replacing \( \alpha \) by \( \alpha^\vee \).

For \( \lambda \in P \), Kashiwara defined the \( U \)-module \( V(\lambda) \) generated by \( u_\lambda \) with the defining relation that \( u_\lambda \) is an extremal vector of weight \( \lambda \) ([17]). It has a presentation

\[
V(\lambda) = U a_\lambda / I_\lambda, \quad I_\lambda = \bigoplus_{b \in \mathcal{B}(Ua_\lambda) \setminus \mathcal{B}(\lambda)} \mathbb{Q}(q) G(b),
\]

where \( \mathcal{B}(\lambda) = \{ b \in \mathcal{B}(Ua_\lambda) \mid b^* \text{ is extremal}\} \). (Recall that \( \mathcal{B}(U) \) is regular [2,3], so extremal vectors make sense.) \( I_\lambda \) is a left \( U \)-module and \( V(\lambda) \) has a crystal base \( (\mathcal{L}(\lambda), \mathcal{B}(\lambda)) \) together with a \( \mathcal{A} \)-module submodule \( \mathcal{A} V(\lambda) \) with a global crystal basis, naturally induced from those of \( U a_\lambda \). We have \( \mathcal{A} V(\lambda) = \bigoplus_{b \in \mathcal{B}(\lambda)} \mathcal{A} G(b) \mod I_\lambda \).

By the construction of \( V(\lambda) \),

\[
\{ G(b) u_\lambda \mid b \in \mathcal{B}(U) \} \setminus \{0\} = \{ G(b) \ mod \ I_\lambda \mid b \in \mathcal{B}(\lambda) \}.
\]

By abuse of notation \( \mathcal{B}(\lambda) \) is considered both as the crystal basis of \( V(\lambda) \) and as the subset of the crystal basis of \( U a_\lambda \).
For any \( w \in W \), \( u_\lambda \mapsto S_{w^{-1}u_\lambda} \) gives an isomorphism of \( U \)-modules:

\[
V(\lambda) \sim \sim V(w\lambda).
\]

This isomorphism sends the global basis to the global basis. Similarly, we have an isomorphism of crystals

\[ S^*_w : B(\lambda) \sim \sim B(w\lambda). \]

Here we regard \( B(\lambda) \) as a subcrystal of \( B(\hat{U}) \), and \( S^*_w \) is the Weyl group action on \( B(\hat{U}) \) with respect to the star crystal structure. Since \( B(\lambda) \subset B(U_{a_\lambda}) \cong B(\infty) \otimes T_\lambda \otimes B(-\infty) \) where \( T_\lambda \) is the crystal with one element of weight \( \lambda \), we can consider \( B(\lambda) \) as a subcrystal of \( B(\infty) \otimes T_\lambda \otimes B(-\infty) \).

If \( \lambda \) is dominant or anti–dominant, then \( V(\lambda) \) is isomorphic to the highest weight module or the lowest weight module of weight \( \lambda \), so in this case the notation is consistent. For \( \lambda \not\in P^0 \), \( \lambda \) is in the Tits cone \( \bigcup_w w(P_+^0) \) and so \( V(\lambda) \) is isomorphic to a representation with a dominant or anti–dominant weight.

**Theorem 2.15** ([18, Theorem 5.1, Corollary 5.2]). (i) For \( \lambda \in P^0 \), the weight of any extremal vector of \( B(\lambda) \) is contained in \( \text{cl}(\text{cl}(W\lambda)) \).

(ii) For any \( \lambda \in P \), the weight of any vector of \( B(\lambda) \) is contained in the convex hull of \( W\lambda \).

When \( X^{(r)}_N \neq A_{2n}^{(2)} \), we define the “fundamental weights of level zero” by setting

\[
\varpi_i = \Lambda_i - a_i^\vee \Lambda_0, \quad i \in I_0.
\]

When \( X^{(r)}_N = A_{2n}^{(2)} \), we set

\[
\varpi_n = 2\Lambda_n - \Lambda_0 \in P^0, \\
\varpi_i = \Lambda_i - \Lambda_0 \in P^0, \quad i = 1, \ldots, n - 1.
\]

We have \( P^0_{\text{cl}} = \text{cl}(P^0) = \bigoplus_{i \in I_0} \mathbb{Z}\text{cl}(\varpi_i) \). We say that \( \lambda \) is a “basic weight” if \( \text{cl}(\lambda) \) is \( W_{\text{cl}} \)-conjugate to \( \text{cl}(\varpi_i) \) for some \( i \in I_0 \).

In the \( A_{2n}^{(2)} \) case, our choice of fundamental level zero weights is different than that of [18]. It is a simple check that both choices span the same \( \mathbb{Z} \)-lattice in \( h \). It follows that the image of this lattice under \( \text{cl} \) is independent of the choice. Choosing a basis of \( P^0_{\text{cl}} \) amounts to fixing a Weyl chamber of \( (\Delta_{\text{cl}}, h^0_{\text{cl}}) \). Since any two Weyl chambers are conjugate under the Weyl group action, it follows that our choice of \( \varpi_i, i \in I_0 \) give the same set of basic weights as that in [18].

In [18, §5], Kashiwara describes the structure of level zero fundamental representations corresponding the basic weights \( \varpi_i \) (\( i \in I_0 \)). Let us note

\[
(\varpi_i, \tilde{a}_j) = \delta_{ij}d_i, \quad \{ n \in \mathbb{Z} \mid \varpi_i + n\delta \in \hat{W}\varpi_i \} = \mathbb{Z}d_i,
\]

where \( d_i \) is as in (2.2). We obtain a \( U' \)-linear automorphism \( z_i \) of \( V(\varpi_i) \) of weight \( d_i\delta \), which sends \( u_{\varpi_i} \), to \( u_{\varpi_i+d_i}\delta \). We define the “fundamental level zero representation” \( U' \)-module \( W(\varpi_i) \) by:

\[
W(\varpi_i) = V(\varpi_i)/(z_i - 1)V(\varpi_i).
\]

**Theorem 2.16** ([18, Theorem 5.15]). (i) \( W(\varpi_i) \) is a finite–dimensional irreducible integrable \( U' \)-module.

(ii) \( W(\varpi_i) \) has a global crystal basis with a simple crystal, i.e., the weight of any extremal vector of \( B(W(\varpi_i)) \) is contained in \( W_{\text{cl}} \cdot \text{cl}(\varpi_i) \) and \( \#B(W(\varpi_i))_{\text{cl}(\varpi_i)} = 1. \)
(iii) For any \( \mu \in \text{wt}(V(\varpi)) \),
\[ W(\varpi)_{\text{cl}(\mu)} \simeq V(\varpi)_{\mu}. \]

(iv) \( \dim W(\varpi)_{\text{cl}(\varpi)} = 1 \).

(v) The weight of any extremal vector of \( W(\varpi) \) belongs to \( \hat{W} \cdot \text{cl}(\varpi) \).

(vi) \( \text{wt} (W(\varpi)) \) is the intersection of \( \text{cl}(\varpi) + Q_{cl} \) and the convex hull of \( \hat{W} \cdot \text{cl}(\varpi) \).

(vii) \( Q(q_{\varpi})[z^{1/d_i}] \otimes Q(q_{\varpi})[z] V(\varpi) \simeq W(\varpi)_{\text{aff}}, \) where the action of \( z^{1/d_i} \) on the left hand side corresponds to the action of \( z \) on the right hand side as defined in \( \text{§2.2}. \)

(viii) \( V(\varpi) \) is isomorphic to the submodule \( Q(q_{\varpi})[z^{d_i}, z^{-d_i}] \otimes W(\varpi) \) of \( W(\varpi)_{\text{aff}} \) as a \( U \)-module. Here we identify \( W(\varpi)_{\text{aff}} \) with \( Q(q_{\varpi})[z, z^{-1}] \otimes W(\varpi) \) as in \( \text{[2.1]} \).

(ix) Any irreducible finite-dimensional integrable \( U' \)-module with \( \text{cl}(\varpi) \) as an extremal weight is isomorphic to \( W(\varpi)_{\alpha} \) for some \( \alpha \in Q(q_{\varpi}) \setminus \{0\} \).

2.9. Bilinear form characterizing \( V(\lambda) \). We recall the following properties of \( V(\lambda) \) with respect to a natural bilinear form.

**Proposition 2.17** ([31, Proposition 4.1]). The extremal weight module \( V(\lambda) \) has a unique bilinear form \( (\quad , \quad) \) satisfying

\[
(u_{\lambda}, G(b)) = \begin{cases} \ 1 & \text{if } G(b) = u_{\lambda}, \\ \ 0 & \text{otherwise} \end{cases}
\]

\[
(\tau u, v) = (u, \psi(\tau)v) \quad \text{for } x \in \mathcal{A}U, \ u, v \in V(\lambda).
\]

The following theorem gives a characterization of the global basis of \( V(\lambda) \) with respect to the form:

**Theorem 2.20** ([31, Theorem 3]). (i) \( \{G(b) \mid b \in \mathcal{B}(\lambda)\} \) is almost orthonormal for \( (\quad , \quad) \), that is, \( (G(b), G(b')) \equiv \delta_{bb'} \mod qZ[q] \).

(ii) \( \{ \pm G(b) \mid b \in \mathcal{B}(\lambda)\} = \{ u \in \mathcal{A}V(\lambda) \mid \pi = u, \ (u, u) \equiv 1 \mod qZ[q] \} \).

For fundamental representations this result is due to [33].

**Remark 2.21.** By the ensuing discussion in §3, Theorem 2.20 holds in the non-symmetric case also.

3. An integral crystal base for \( \mathcal{A}U^+ \)

In this section we construct an integral basis \( B \) of \( \mathcal{A}U^+ \) such that \( B \subset \mathcal{L}(-\infty) \) and under the natural map \( \pi : \mathcal{L}(-\infty) \to \mathcal{L}(-\infty)/q^{-1}\mathcal{L}(-\infty), \pi(B) = \mathcal{B}(-\infty) \). (In fact, we construct a series of bases parametrized by \( p \in \mathbb{Z} \) with these properties.)

3.1. Root Vectors. We introduce root vectors in \( \mathcal{A}U^+ \). Recall that we chose \( \tilde{\omega}_i \in \tilde{P} \triangleleft \tilde{W} \) for each \( i \in I_0 \) (see §2.1). We choose \( \tau_i \in \mathcal{T} \) so that \( \tilde{\omega}_i \tau_i^{-1} \in \tilde{W} \). Choosing a reduced expression for \( \tilde{\omega}_i \tau_i^{-1} \) for each \( i \in I_0 \) we fix a reduced expression of \( \tilde{\omega}_n \tilde{\omega}_{n-1} \ldots \tilde{\omega}_1 \): \( \tilde{\omega}_n \tilde{\omega}_{n-1} \ldots \tilde{\omega}_1 = s_{i_1} s_{i_2} \ldots s_{i_N} \tau, \quad (\tau = \tau_0 \ldots \tau_1). \)

We define a doubly infinite sequence
\[
h = (\ldots, i_{-1}, i_0, i_1, \ldots)
\]
by setting $i_{k+N} = \tau(i_k)$ for $k \in \mathbb{Z}$. Note that for any integers $m < p$, the product $s_i m_s i_{m+1} \ldots s_p \in \hat{W}$ is a reduced expression. We have
\[
R_\geq = \{\alpha_{i_0}, s_{i_0}(\alpha_{i_1}), s_{i_0}s_{i_1}(\alpha_{i_2}), \ldots\},
\]
\[
R_\leq = \{\alpha_{i_1}, s_{i_1}(\alpha_{i_2}), s_{i_1}s_{i_2}(\alpha_{i_3}), \ldots\}.
\]

\textbf{Remark 3.2}. Our definition of the PBW basis will depend on the sequence $\mathbf{h}$. In particular, it depends on the choice of the numbering $I_0 = \{1, 2, \ldots, n\}$. Almost all of the results in [12] which we will use are independent of the numbering. But when $X_N^{(r)} = A_{2n}^{(2)}$, Corollary 4.2.6 depends on our choice such that $\hat{\omega}_n$ corresponding to the short root $\alpha_n$ appears first in (3.1). We choose our $\mathbf{h}$ to agree with that in [12]. (Our vertex $n$ is labeled by 1 there.)

Set
\begin{equation}
\beta_k = \begin{cases} 
s_{i_0}s_{i_1} \ldots s_{i_{k-1}}(\alpha_{i_k}), & \text{if } k \leq 0, 
s_{i_1}s_{i_2} \ldots s_{i_{k-1}}(\alpha_{i_k}), & \text{if } k > 0.
\end{cases}
\end{equation}

Define a total order on $R$ by setting
\begin{equation}
\beta_0 < \beta_{-1} < \beta_{-2} \cdots < \delta^{(1)} < \cdots < \delta^{(n)} < 2\delta^{(1)} < \cdots < \beta_3 < \beta_2 < \beta_1,
\end{equation}
where $k\delta^{(i)}$ denotes $(k\delta, i) \in R_\leq$.

We now define root vectors for each element of $R_\geq \cup R_\leq$.
\begin{equation}
E^{\beta_k} = \begin{cases} 
T_{i_0}^{-1}T_{i_1}^{-1} \cdots T_{i_{k-1}}^{-1}(E_{i_k}) & \text{if } k \leq 0, 
T_{i_1}T_{i_2} \cdots T_{i_{k-1}}(E_{i_k}) & \text{if } k > 0.
\end{cases}
\end{equation}

By [23, 40.1.3] these are in $U^+$. As usual, set $F_\beta = \Omega(E_\beta)$ for all $\beta \in R_\geq \cup R_\leq$.

\textbf{Remark 3.6}. We note that the root vectors $E_{d_i,k\delta,\pm\alpha_i}$ are described explicitly by:
\begin{equation}
E_{d_i,k\delta,\pm\alpha_i} = T_{\hat{\omega}_i}^{-k}E_i \quad (k \geq 0), \quad E_{d_i,k\delta,-\alpha_i} = T_{\hat{\omega}_i}^kT_{i}^{-1}E_i \quad (k > 0).
\end{equation}
These are the Drinfeld generators for $U$.

Having defined real root vectors, we define the imaginary root vectors. For $k > 0$, $i \in I_0$, set
\[
\tilde{\psi}_{i,kd_i} = E_{kd_i,\delta,-\alpha_i} - q_i^{-2}E_{\alpha_i}E_{kd_i,\delta,-\alpha_i},
\]
and define elements $E_{i,kd_i,\delta} \in U^+$ by the functional equation
\[
(q_i - q_i^{-1}) \sum_{k=1}^{\infty} E_{i,kd_i,\delta} u^k = \log \left(1 + \sum_{k=1}^{\infty} (q_i - q_i^{-1}) \tilde{\psi}_{i,kd_i} u^k\right).
\]
We have
\[
[E_{i,kd_i,\delta}, E_{j,ld_j,\delta}] = 0.
\]
(For the untwisted case see [21]. For general case see [22, Theorem 5.3.2].)
For each $i \in I_0$ we introduce the “integral” imaginary root vectors $(\hat{\psi}, E_{i,kd_i,\delta})$ by
\begin{equation}
\sum_{k \geq 0} \hat{P}_{i,kd_i} u^k = \begin{cases} 
\exp \left(\sum_{k \geq 1} \frac{E_{i,kd_i,\delta} u^k}{[kd_i,\delta]}\right) & \text{if } (X_N^{(r)}, i) = (A_{2n}^{(2)}, n),
\exp \left(\sum_{k \geq 1} \frac{E_{i,kd_i,\delta} u^k}{[kd_i,\delta]}\right) & \text{otherwise}.
\end{cases}
\end{equation}
They satisfy the following recursive identity:

\[
\tilde{P}_{i, kd} = \begin{cases} 
\frac{1}{[2k]_n} \sum_{s=1}^{k} q_n^{2(s-k)} \psi_{n,s} \tilde{P}_{n,k-s} & \text{if } (X_n^i, i) = (A_{2n}^2, n), \\
\frac{1}{[k]_n} \sum_{s=1}^{k} q_n^{s-k} \psi_{1, sd} \tilde{P}_{1,(k-s)d} & \text{otherwise}.
\end{cases}
\]  

(3.8)

These definitions are based on the definition of imaginary root vectors which appeared for type \( A_1^{(1)} \) in [3]. The \( A_{2n}^2 \) case is generalized from [3] where it appears in the \( A_2^2 \) case.

Let \( \mathbf{U}^+ (>) \) (resp. \( \mathbf{U}^+ (<) \), \( \mathbf{U}^+ (0) \)) be the \( \mathbb{Q}(q_n) \)-subalgebra of \( \mathbf{U}^+ \) generated by the \( E_{\beta_k} \) for \( k \leq 0 \) (resp. \( E_{\beta_k} \) for \( k > 0 \), \( E_{i,kd,\delta} \) for \( k > 0 \)).

The \( \tilde{P}_{i, kd} \) (\( i \in I_0, k > 0 \)) are used to construct a basis of the imaginary part of \( \mathbf{U}^+ \) as follows. Let \( \rho_0 = (\rho^{(1)}, \rho^{(2)}, \ldots, \rho^{(n)}) \) be an \( n \)-tuple of partitions where each \( \rho^{(i)} = (\rho^{(i)}_1 \geq \rho^{(i)}_2 \geq \cdots) \). For a partition \( \rho \), we denote its transpose by \( \rho^t \). For each \( \rho^{(i)} \), define the Schur function in the \( \tilde{P}_{i, kd} \) by

\[
S_{\rho^{(i)}} = \det(\tilde{P}_{i,(\rho^{(i)}_k - k + m)d})_{1 \leq k, m \leq t},
\]

where \( t \geq l(\rho^{(i)} t) \). This puts the \( \tilde{P}_{i, kd} \) in the role of elementary symmetric functions. Note that in [3] the transpose of \( S_{\rho^{(i)}} \) is considered, but this make no difference. Denote the product over \( i \in I_0 \) of \( S_{\rho^{(i)}} \) by

\[
S_{\rho_0} = \prod_{i=1}^{n} S_{\rho^{(i)}}.
\]

**Definition 3.9.** Let \( c_+ \in \mathbb{N}^{\mathbb{Z}_{\leq 0}} \) and \( c_- \in \mathbb{N}^{\mathbb{Z}_{> 0}} \) be functions which are almost everywhere 0. Let \( \rho_0 \) as above. Let \( \mathcal{C} \) denote the set of all such triples \( c = (c_+, c_0, c_-) \).

For each \( p \) and \( c \in \mathcal{C} \) we define:

\[
c_{+p} = (c(p), c(p-1), c(p-2), \ldots)
\]

and \( c_{-p} = (c(p+1), c(p+2), c(p+3), \ldots) \).

where the components \( c(j) \) are actually \( c_+(j) \) (resp. \( c_-(j) \)) when \( j \leq 0 \) (resp. \( j > 0 \)).

Define

\[
E_{c_{+p}} = E_{t_p}^{c(p)} T_{t_p}^{-1}(E_{t_{p-1}}^{c(p-1)}) T_{t_p}^{-1}(E_{t_{p-2}}^{c(p-2)}) \cdots \\
E_{c_{-p}} = \cdots T_{t_{p+1}}(E_{t_{p+2}}^{c(p+2)}) T_{t_{p+1}}(E_{t_{p+2}}^{c(p+3)}) T_{t_p}^{-1}(E_{t_{p+2}}^{c(p+2)}) \cdots
\]

where the exponents above are written \( c(j) \) when they should be \( c_+(j) \) or \( c_-(j) \) for \( j \leq 0 \) or \( j > 0 \) respectively.

Note that when \( p = 0 \), the monomials \( E_{c_{\pm 0}} \) are formed by multiplying the \( (E_{\beta_k})^{c_{\pm 0}(k)} \) for real roots \( \beta_k \in \mathcal{R}_2 \) in the order [3,4]. Also, in this case we will omit the subindex and write \( c_{\pm 0} \) instead of \( c_{\pm 0} \)—there should be no confusion with the \( c_{\pm} \) above.
For \( c \in \mathbb{C}, p \in \mathbb{Z} \) we define (cf. [23, 40.2.3]):

\[
L(c, p) = \left( E_{1p}(c(p)) T_{1p}^{-1}(E_{1p-1}(c(p-1))) T_{1p}^{-1}(E_{1p+1}(c(p-2))) \cdots \right) \\
\times T_{1p+1} T_{1p+2} \cdots T_{10}(S_{c_{0}}) \\
\times \left( \cdots T_{1p+1} T_{1p+2} E_{1p+3}(c(p+3)) T_{1p+1}(E_{1p+2}(c(p+2)) E_{1p+1}(c(p+1)) \right) \\
= E_{c_{+}}(T_{1p+1} T_{1p+2} \cdots T_{10}(S_{c_{0}})) E_{c_{-}},
\]

for \( p \leq 0 \). For \( p \geq 1 \), we replace the middle part by \( T_{1_p}^{-1} T_{1_p-1} \cdots T_{1_2}^{-1} T_{1_1}^{-1}(S_{c_{0}}) \).

Note that

\[
L(c, p - 1) = T_{1p} L(c, p) \text{ if } c(p) = 0,
\]

\[
L(c, p + 1) = T_{1p}^{-1} L(c, p) \text{ if } c(p + 1) = 0,
\]

When \( p = 0 \) we will denote \( L(c, 0) \) by \( B_{c} \). In this case, each \( c \in \mathbb{C} \) indexes an element which we will call of “PBW-type” in \( U^{+} \):

\[
B_{c} = E_{c_{+}} \cdot S_{c_{0}} \cdot E_{c_{-}}.
\]

We will call \( B_{c} \) for \( c \in \mathbb{C} \) purely imaginary if \( c_{+} = c_{-} = 0 \). In this case we may write \( c_{0} \) instead of \( c = (0, c_{0}, 0) \).

For each \( p \in \mathbb{Z} \), we define a partial ordering \( \prec_{p} \) on \( \{ c \mid c \in \mathbb{C} \} \) by letting \( c \prec_{p} c' \) if and only if

\[
E_{c_{+}} \preceq E_{c'_{+}} \quad \text{and} \quad E_{c_{-}} \preceq E_{c'_{-}},
\]

and one of these is strict.

Here both \( \preceq \) are the lexicographic ordering from left to right. For example, the first inequality means either \( c = c' \) or \( c(p) = c'(p), c(p-1) = c'(p-1), \ldots, c(p-k+1) = c'(p-k+1) \) and \( c(p-k) \prec c'(p-k) \) for some \( k \geq 0 \). Note that

\[
\text{if } c(p) = c'(p) = 0, \text{ then } c \prec_{p} c' \Leftrightarrow c \prec_{p-1} c'.
\]

We now state the main theorem of this section.

**Theorem 3.13.** (i) For each \( p \in \mathbb{Z} \), \( \{ L(c, p) \mid c \in \mathbb{C} \} \) is an almost orthonormal basis of the \( Q(q_{s}) \)-vector space \( U^{+} \), i.e., \( (L(c, p), L(c', p)) \in \delta_{c,c'} + q_{s}^{-1}Z[[q_{s}^{-1}]] \cap Q(q_{s}) \).

(ii) Let \( p \in \mathbb{Z} \). The transition matrix between \( \{ L(c, p) \mid c \in \mathbb{C} \} \) and the global crystal basis of \( U^{+} \) is upper-triangular with 1’s on the diagonal and with above diagonal entries in \( q_{s}^{-1}Z[q_{s}^{-1}] \).

The property (i) follows from (ii) and the almost orthonormality of the global crystal base [23, 14.2.3]. However, we will use (i) during the proof of (ii) and we prove it independently. Let us also remark that (ii) implies that \( \{ L(c, p) \mid c \in \mathbb{C} \} \) is a basis of the integral form of \( U^{+} \). The proof of (i) will be given in §3.3. We postpone that of (ii) until Sect. 3.

### 3.2. Vertex subalgebras.

A key part of describing the PBW basis is its reduction to “vertex subalgebras”. The following proposition describes the \( n \) “vertex” subalgebras of \( U = U(X^{(k)}_{N}) \):

**Proposition 3.14.** Let \( i \in I_{0} \). Let \( U^{(i)} \) be the \( Q(q_{s}) \)-subalgebra of \( U = U(X^{(r)}_{N}) \) generated by

\[
\{ E_{i}, E_{i,d_{i} - d_{j}}, F_{i}, F_{i,d_{i} - d_{j}}, K_{i}^{\pm 1}, K_{d_{i}d_{j}}^{\pm 1} \}.
\]
Then $U^{(i)}$ is isomorphic as a $Q(q)$-algebra to $U(A^{(1)}_1)$ in all cases except when $X^{(r)}_N = A^{(2)}_{2n}$ and $i = n$, i.e., $(\alpha_i, \alpha_i) = 1$. The isomorphism $U(A^{(1)}_1) \to U^{(i)}$ is given by $E_{i} \mapsto E_i, E_0 \mapsto E_{d, \delta-\alpha_i}, F_1 \mapsto F_i, F_0 \mapsto F_{d, \delta-\alpha_i}, q \mapsto q_i$. For $(X^{(r)}_N, i) = (A^{(2)}_{2n}, n)$, $U^{(i)}$ is isomorphic as a $Q(q^{1/2})$-algebra to $A^{(2)}_2$, and the isomorphism $A^{(2)}_2 \to U^{(i)}$ is given by $E_0 \mapsto E_{\delta-2\alpha_n}, F_0 \mapsto F_{\delta-2\alpha_n}, q^{1/2} \to q^{1/2}$. (In particular, $E_{\delta-2\alpha_n}, F_{\delta-2\alpha_n} \in U^{(i)}$.)

Proof. For the untwisted case see [3]. In the twisted case, the result is due to [12]. Note that in [12] the quantum algebra of type $A^{(2)}_{2n}$ is normalized differently (the invariant bilinear form on $h^*$ is 2 times the one here). □

Next we have:

Proposition 3.15. For $i \in I_0, k > 0$ we have $\tilde{P}_{i,k,d_i} \in \mathcal{A}U^+$. Proof. This is proved as in [3, Corollary 2.2] for the symmetric case. In the $(X^{(r)}_N, i) = (A^{(2)}_{2n}, n)$ case this follows from Proposition 3.14 and [3, Corollary 8.6]. □

Proposition 3.16. Let $c \in C$. We have $L(c, p) \in \mathcal{A}U^+$. Proof. By [3, 6.3.2] (see [3] for the untwisted case) we have $T_{i_{p+1}}T_{i_{p+2}}\cdots T_{i_n}(S_0), T_{i_{p}}^{-1}T_{i_{p-1}}^{-1}\cdots T_{i_1}^{-1}(S_0) \in \mathcal{A}U$. Since $T_w$ preserves $\mathcal{A}U$, they are contained in $\mathcal{A}U \cap U^+ = \mathcal{A}U^+$ by Proposition 3.13. We also have $E_{c,\pm p} \in \mathcal{A}U^+$ by [3, 41.1.3]. □

Next we cite a key property of the $\tilde{P}_{i,k,d_i}$:

Proposition 3.17. Let $k > 0$.

\begin{equation}
\tilde{P}_{i,k,d_i} = \begin{cases} E^{(k)}_{i, \delta-\alpha_i} + q^{-1}_s x & \text{if } (X^{(r)}_N, i) \neq (A^{(2)}_{2n}, n), \\
E^{(k)}_{i, -2\alpha_n} E^{(2k)}_i + q^{-1}_s x & \text{if } (X^{(r)}_N, i) = (A^{(2)}_{2n}, n), \end{cases}
\end{equation}

where $x$ is a sum of terms $B c$ with coefficients in $Z[q_s^{-1}]$ where for each term $c_+, c_- \neq 0$. Furthermore, for each such $c \in C$, only imaginary root vectors $P_{i,t,d_i}$ with $0 < t < k$ appear in $S_0$.

Proof. This is derived as in [3, Proposition 2.2 and eq. (4.9)], where it appears for the untwisted affine case. For the $A^{(2)}_{2n}$ this appears as a special case of [3, Theorem 8.5]. □

3.3. Almost orthonormality. The following proposition is central to our calculations.

Proposition 3.19. For $i, j \in I_0, k, k' > 0$ we have

\[
(P_{i,k,d_i} P_{j,k',d_j}) \equiv \delta_{k,k'} \delta_{ij} \mod q^{-1}_s A_\infty.
\]

Proof. This result appears in the symmetric case in [3] and in the $A^{(2)}_2$ case it appears in [3]. In general, the proof is analogous to one of the previous cases. In the non–symmetric case where $i \neq j$, we may assume $\alpha_{ij} = -1$ since the condition is symmetric in $i$ and $j$. The result then follows from the following identity:

\begin{equation}
r_j(E_{i,k,d_i}) = \begin{cases} (-1)^{k+1} q_j^{-1} (1 - q_j^{-2}) E_{kd, \delta-\alpha_i} & \text{if } d_j | k d_i \\
0 & \text{otherwise.} \end{cases}
\end{equation}
which is derived from the following special case of [12, Theorem 5.2] (just as in the symmetric case the previous identity is derived in [8]):

$$
[E_{i,kd}, F_j] = \begin{cases} 
(-1)^{k+1} \frac{[r]}{r} K_{\alpha_j} E_{kd, \delta - \alpha_j} & \text{if } d_j | kd_i \\
0 & \text{otherwise}.
\end{cases}
$$

Next we need the following result regarding the coproduct formula for the imaginary root vectors. For any algebra $A$, let $A_+$ denote its augmentation ideal.

**Proposition 3.22.** Let $k > 0$, $i \in I_0$. Then

$$
\rho(\tilde{p}_{i,kd}) = \sum_{s=0}^{k} \tilde{p}_{i,sd} \otimes \tilde{p}_{i,(k-s)d_i} + \text{terms in } U^+(<)_+ U^+(0) \otimes U^+(0) U^+(>)_+.
$$

*Proof.* The proof of this follows from the relation between the braid group action and the coproduct (see [23, 3.1.5, 37.3.2]) by using Remark 3.6. In the untwisted case, the argument is given in [1]. In general, the argument is identical.

We have the following result which is proved as in [3] (see [3] for the $A_2^{(2)}$ case):

**Proposition 3.23.** Let $c_0, c'_0$ be two $n$–tuples of partitions as above. Then

$$
(S_{c_0}, S_{c'_0}) = \delta_{c_0, c'_0} \pmod{q_s^{-1} A_\infty}.
$$

*Proof of Theorem 3.23(i).* Let $c, c' \in C$. Suppose $p \geq 1$. We have

$$
T_{ip_1}^{-1} T_{ip_2}^{-1} \cdots T_{ip_k}^{-1} (S_{c_0}) \cap T_{ip_1}^{-1} (U^+),
$$

as we already remarked in the proof of Proposition 3.16. By [23, 38.2.1] we have

$$
\left( T_{ip_1}^{-1} T_{ip_2}^{-1} \cdots T_{ip_k}^{-1} (S_{c_0}), T_{ip_1}^{-1} T_{ip_2}^{-1} \cdots T_{ip_k}^{-1} (S_{c'_0}) \right)
= \left( T_{ip_1}^{-1} T_{ip_2}^{-1} \cdots T_{ip_k}^{-1} (S_{c_0}), T_{ip_1}^{-1} T_{ip_2}^{-1} \cdots T_{ip_k}^{-1} (S_{c'_0}) \right).
$$

By the induction and Proposition 3.22 these are equal to $\delta_{c, c'}$ modulo $q_s^{-1} A_\infty$.

We have by [23, 40.2.4]

$$
(L(c, p), L(c', p)) = (S_{c_0}, S_{c'_0}) \prod_{s \in \mathbb{Z}} (E^{(c(s))}, E^{(c'(s))}) \equiv \delta_{c, c'} \pmod{q_s^{-1} A_\infty}.
$$

Moreover, we know $(L(c, p), L(c', p)) \in \mathbb{Z}[q_s^{-1}]$ since $L(c, p), L(c', p) \in A U^+$ by [23, 14.2.6]. Hence we get the almost orthonormality.

For $p < 0$ we have the same result thanks to

$$
T_{ip_1}^{-1} T_{ip_2}^{-1} \cdots T_{ip_k}^{-1} (S_{c_0}) \in U^+ \cap T_{ip}^{-1} (U^+).
$$

By (3.25) \{L(c, p)\} is linearly independent. The PBW theorem says that the dimension of a weight space of $U^+$ is equal to the number of $c$’s with the given weight. Thus it is a basis.

**Corollary 3.26.** Let $L(-\infty) = \{x \in U^+ | (x, x) \in A_0\}$. Then it is an $A_0$-submodule of $U^+$, and \{\(L(c, p)|c \in C\} is its $A_0$-basis for each $p \in \mathbb{Z}$.

*Proof.* The assertion follows from [23, 14.2.2].
Note that our proof of this statement, as well as that of Theorem 3.13(i), is independent of the existence of the global crystal basis. The above definition of \( \mathcal{L}(\infty) \) coincides with one in \([17, 5.1.4]\), while it was a characterizing property in \([16, 5.1.4]\).

**Proposition 3.27.** For every \( c \in \mathcal{C}, p \in \mathbb{Z} \), there exist \( b = b(c, p) \in \mathcal{B}(\infty) \) and \( \text{sgn}(c, p) = \pm \) such that

\[
L(c, p) \equiv \text{sgn}(c, p)G(b(c, p)) \mod q_s^{-1}\mathcal{L}(\infty).
\]

**Proof.** By \([17, 14.2.2]\), we know that if \( x \in \mathcal{A}U^+ \) satisfies \( (x, x) \in 1 + q_s^{-1}A_\infty \), then \( x \in \mathcal{L}(\infty) \) and there exists \( b \in \mathcal{B}(\infty) \) such that \( x \equiv \pm G(b) \mod q_s^{-1}\mathcal{L}(\infty) \). The assertion follows from Theorem 3.13(i). \( \square \)

In Sect. 3 we will show \( \text{sgn}(c, p) = 1 \).

Note that the map \( \mathcal{C} \to \mathcal{B}(\infty) \) given by \( c \mapsto b(c, p) \) is bijective for any \( p \), since both are bases.

**Remark 3.29.** For any \( p, q \in \mathbb{Z} \) there exists a bijection \( c \in \mathcal{C} \leftrightarrow c' \in \mathcal{C} \) given by \( b(c, p) = b(c', q) \) by Proposition 3.27. It is extremely interesting problem to give an explicit description of the bijection. For finite type \( g \), the same construction gives the piecewise linear bijections \([12, 42.1.3]\), which have been studied by various peoples. Let \( b = b(c, p - 1) \) with \( c(p) = 0 \). By \([12]\), we have \( b(c, p) = c_i^{e_i(b)} f_i^{e_i(b)} b \), where \( i = i_p \). This is a first step towards this problem.

### 3.4. Upper triangular property of the bar involution

This subsection is a small detour. We give a proof that the bar involution is upper triangular with respect to our basis \( \{L(c, p)\} \). For symmetric or type \( A_1^{(2)} \), this together with \([12]\) gives us the elementary algebraic definition of the global crystal basis as explained in the Introduction. The reader in a hurry may skip the rest of this section.

We will need a “reordering lemma” to prove the upper-triangularity property of the bar action with respect to the ordering \( \prec_p \).

**Lemma 3.30.** Let \( p \in \mathbb{Z} \). Let \( c, c' \in \mathcal{C} \). Write

\[
L(c, p)L(c', p) = \sum_{c''} a_{c,c''}^p L(c'', p),
\]

where \( a_{c,c''}^p \in \mathbb{Q}(q_s) \).

(i) Then for each \( c'' \) in the above sum, \( c''_{+p} \geq c_{+p} \) and \( c''_{-p} \geq c'_{-p} \) with respect to the lexicographical ordering.

(ii) Further, if \( L(c, p) = E_{c_{+p}} \) (resp. \( E_{c_{-p}} \)) and \( L(c', p) = E_{c'_{+p}} \) (resp. \( E_{c'_{-p}} \)), then \( c''_{+p} > c_{+p} \) (resp. \( c''_{-p} > c'_{-p} \)) for each summand.

**Proof.** We prove this for \( p = 0 \), noting that for \( p \neq 0 \) the proof is identical. Assume \( L(c, 0) = E_{c_{+}}S_{c_{-}}E_{c_{-}} \) and \( L(c', 0) = E_{c'_{+}}S_{c'_{-}}E_{c'_{-}} \). Write the expression

\[
S_{c_{0}}E_{c_{-}}E_{c'_{+}}S_{c'_{0}}E_{c'_{-}} = \sum_{b \in \mathcal{C}} a_{c,c'}^b E_{b_{+}}S_{b_{0}}E_{b_{-}}.
\]

Here and for the remainder of the section any structure constants (such as \( a_{c,c'}^b \)) are assumed to be in \( \mathbb{Q}(q_s) \) unless otherwise stated. We have

\[
L(c, 0)L(c', 0) = \sum_{b} a_{c,c'}^b E_{c_{+}}E_{b_{+}}S_{b_{0}}E_{b_{-}}.
\]
For a given summand, if \( c_+ = 0 \) then clearly \( c''_+ = b_+ \geq c_+ \) in the Lemma. Assume \( c_+ > 0 \). For any given \( b \) in the sum, we may assume \( b_+ > 0 \) or else that summand also fulfills the requirement of the Lemma. Under the assumption that \( b_+ > 0 \) and \( c_+ > 0 \), the \( c''_+ \geq c_+ \) part in (i) and the \( c''_+ > c_+ \) part in (ii) follow if we check that for a fixed \( b_+ \) in (3.33),

\[
E_{c_+} E_{b_+} = \sum_{d_+ \in \mathbb{N}^3 > d_+ > c_+} a_{c_+, b_+} E_{d_+}.
\]

(3.34)

Let \( k > k' > 0 \). We have the following useful identity \[\text{(3.35)}\] (it was proved there in the finite type case, but the same proof works here):

\[
E_{\beta_{-k}} E_{\beta_{-k'}} = q^{(\beta_k, \beta_{k'})} E_{\beta_{-k}} E_{\beta_{-k'}} + \sum_{d'_+} a_{d'_+} E_{d'_+},
\]

where \( d'_+ (j) = 0 \) if \( j \geq k \) or \( j \leq k' \). Note that this condition is equivalent to saying that \( e_{-k'} > d'_+ > e_{-k} \), where \( e_k \) is the tuple whose \( j \)-th position is \( \delta_{jk} \).

Consider the set \( S \) of all monomials of weight \( \gamma = \text{wt} (E_{c_+} E_{b_+}) \in \mathbb{Q}_i \) formed from the real root vectors \( E_{\beta_{-k}}, k \geq 0 \). \( S \) is a finite set. We will order this set by a lexicographic order on the monomials, where \( E_{\beta_{-k'}} > E_{\beta_{-k}} \) if \( k > k' \geq 0 \). In this ordering, a monomial \( M \) is in the PBW order if and only if it is maximal among all monomials where \( E_{\beta_{-k}} \) appears for all \( k \) the same number of times as in \( M \).

On the left hand side of (3.34), moving from left to right take the first root vector in \( E_{c_+} E_{b_+} \), which is out of PBW order, i.e., the first root vector which is larger than in its immediate predecessor. Use (3.35) to reorder these two root vectors. By (3.35), we obtain a linear combination of monomials from \( S \), each greater in lexicographic order than \( E_{c_+} E_{b_+} \). Repeating this for each summand, and taking into account that \( S \) is a finite set, we ultimately obtain a linear combination of elements of \( S \), each of which is maximal among monomials formed from the same root vectors. Thus, each is in PBW form and also larger than \( E_{c_+} \) in lexicographic order, so that we obtain \( E_{c_+} E_{b_+} \) in the form of (3.34) as required.

\[\square\]

**Proposition 3.36.** Let \( c \in \mathcal{C}, p \in \mathbb{Z} \). Then

\[
\overline{L(c, p)} = L(c, p) + \sum_{c < c', p} a_{c, c'} L(c', p),
\]

where \( a_{c, c'} \in \mathbb{Q}(q_s) \).

**Proof.** We first prove

- **(a):** Fix \( c_0 \) and \( q \). Suppose that (3.37) is true for \( T_{i_0} T_{i_0+1} \cdots T_{i_0} S_{c_0} \) \((q \leq 0)\), or \( T_{i_0}^{-1} T_{i_0+1}^{-1} \cdots T_{i_0}^{-1} S_{c_0} \) \((q \geq 1)\). Let \( p \geq q \). If \( c \) satisfies \( c_{-p} = 0 \), \( c_{+p} = 0 \) and \( c_0 \) is the given one, (3.37) is also true for \( L(c, p) \). Furthermore, the condition \( c < c', p \) can be replaced by the stronger condition \( c_{+p} < c'_{+p} \).

(The other condition \((0 =) c_{-p} \leq c'_{-p} \) is trivially satisfied.)

We are considering

\[
L(c, p) = E_{p}^{(c(p))} \times T_{i_p}^{-1} E_{i_p-1}^{(c(p-1))} \times T_{i_p-1}^{-1} E_{i_p-2}^{(c(p-2))} \times \cdots \times T_{i_p}^{-1} E_{i_p+1}^{(c(q+1))} \times T_{i_p+1} T_{i_p+2} \cdots T_{i_0} S_{c_0}.
\]
Since \( T_i^{-1} T_i^{-1} \cdots T_i^{-1} (S_{c_0}) \). We prove the assertion by the induction on \( p \). When \( p = q \), \( (3.37) \) is true by the assumption.

First assume \( c(p) = 0 \). We consider \( L(c, p - 1) = T_i L(c, p) \) (see (3.10)). By the induction hypothesis, we have
\[
\sum_{\pi < p} a_{\pi, p} L(c', p - 1) = L(c, p - 1) + \sum_{\pi < p} a_{\pi, p} L(c', p - 1).
\]
(We put the superscript \( p - 1 \) in order to clarify its dependence on \( p - 1 \) in this part. For the other part, \( p \) will be fixed, and there will be no confusion.) We apply the composition \( T_i^{-1} \circ \pi^+ \) to both sides. By (2.8), the left hand side becomes \( \pi^+(L(c, p)) \), which is equal to \( L(c, p) \) modulo \( E_{i, p} U^+ \). For the right hand side, we use (3.10). We get
\[
L(c, p) = L(c, p) + \sum_{\pi < p} a_{\pi, p} L(c', p - 1) + E_{i, p} U^+.
\]

The condition \( c'(p) = 0 \) comes from \( \pi^+(L(c', p - 1)) \neq 0 \). (Otherwise, \( L(c', p - 1) \in U^+ E_{i, p} \) and \( \pi^+(L(c', p - 1)) = 0 \).) The part in \( E_{i, p} U^+ \) is a linear combination of \( L(c'', p) \) with \( c''(p) > 0 \). They satisfy \( c_{< p} < c''_{< p} \) since \( c(p) = 0 \). The summation in the second term can be replaced as \( \sum_{\pi < p} a_{\pi, p} L(c', p - 1) \) by (3.12). Thus we have the assertion under the assumption \( c(p) = 0 \).

Next we assume \( c(p) > 0 \). Let us define \( \hat{c} \) by setting \( \hat{c}(p) = 0 \) and all other entries are the same as \( c \). We have
\[
L(c, p) = E_{i, p}^{(c(p))} L(\hat{c}, p).
\]
Since \( \hat{c}(p) = 0 \), we have just proved
\[
L(\hat{c}, p) = L(\hat{c}, p) + \sum_{\pi < p} a_{\pi, p} L(\hat{c}', p).
\]
Therefore
\[
L(c, p) = E_{i, p}^{(c(p))} L(\hat{c}, p) = L(c, p) + \sum_{\pi < p} a_{\pi, p} \left[ \frac{c(p) + \hat{c}'(p)}{c(p)} \right] L(c', p),
\]
where \( \hat{c}' \) is defined by setting \( \hat{c}'(p) = \hat{c}'(p) + c(p) \) and other entries are the same as \( \hat{c}' \). We have \( \hat{c}_{< p} < \hat{c}'_{< p} \iff c_{< p} < c_{< p} \), and therefore the assertion.

Similarly we have
\[
(a'):\text{ Fix } c_0 \text{ and } q. \text{ Suppose that } (3.37) \text{ is true for } T_i q_{i+1} T_i q_{i+2} \cdots T_i (S_{c_0}) \text{ } (q \leq 0), \text{ or } T_i^{-1} T_i^{-1} \cdots T_i^{-1} (S_{c_0}) \text{ } (q \geq 1). \text{ Let } p \leq q. \text{ If } c \text{ satisfies } c_{< p} = 0, \text{ } c_{-q} = 0 \text{ and } c_0 \text{ is the given one, } (3.37) \text{ is also true for } L(c, p). \text{ Furthermore, the condition } c_{< p} \text{ can be replaced by the stronger condition } c_{< p} < c_{< p}. \text{ (The other condition } (0 =) c_{< p} \leq c_{< p} \text{ is trivially satisfied.)}
\]

Our next task is to show
\[
(b): \text{ Fix } c_0. \text{ Suppose that } (3.37) \text{ is true for } S_{c_0}. \text{ Then it is also true for } L(c, p) \text{ with } c_0 \text{ is the given one.}
\]

By \( (a), (a') \), if \( (3.37) \) is true for \( S_{c_0} \), then it is also true for \( T_i q_{i+1} T_i q_{i+2} \cdots T_i (S_{c_0}) \) \( (p \leq 0) \), or \( T_i^{-1} T_i^{-1} \cdots T_i^{-1} (S_{c_0}) \) \( (p \geq 1) \). (The conditions on \( c_{\pm p}, c_{\pm q} \) are vacuous
since $c_\pm = 0$.) This is a special case of (b). We return back to general $L(c, p)$ as in (b). We decompose it as $L(c, p) = L(c_+, p) L(c_0, p) L(c_-, p)$ where $c_+, c_0$ are understood as elements of $\mathcal{C}$ by setting other entries as 0. By (a) and the above special case of (b), we have

$$L(c_+, p) L(c_0, p) = L(c_+, p) L(c_0, p) + \sum_{c_+ < c'} a_{c_+, c'} L(c', p).$$

By (a') we have

$$L(c-, p) = L(c_-, p) + \sum_{c_- < c''} a_{c-, c''} L(c'', p).$$

Note that the assumption of (a') is trivially satisfied since $c_0 = 0$ in this case. Therefore

$$L(c, p) = L(c_+, p) + \sum_{c_+ < c'} a_{c_+, c'} L(c', p)L(c_-, p) + \sum_{c_- < c''} a_{c-, c''} L(c_+, p) L(c_-, p) L(c'', p) + \sum_{c_- < c''} a_{c-, c''} a_{c-, c''} L(c', p) L(c', p).$$

By Lemma 3.30, $L(c_+, p) L(c_-, p) = \sum a_{c', c_-} L(d, p)$ where the summation is over $d$ satisfying $c_+ \leq d_+, c_- \leq d_-$. Since $c_+ < c'_+, c_- < c''$, we have $c_+ < c_+, c_- < c_-$, and hence $c < c'$. (Recall that one of the inequalities must be strict in the definition of $\prec_p$.) The other two summations can be handled in the same way, and we have the assertion (b).

Next we replace the inequality $c \prec_p c'$ by a further stronger inequality in a special case.

(c): Consider the case $p = 0$. Let $c_k \in \mathcal{C}$ be such that $L(c_k, 0)$ is equal to $E_N^{(2)}(c_k - \alpha)$ if $(X_N^{(r)}, i) \neq (A_2^{(2)}, n)$, and $E_N^{(2)}(c_k - 2\alpha)$ if $(X_N^{(r)}, i) = (A_2^{(2)}, n)$. Then (3.37) is true for $L(c_k, 0)$ with $c_k \prec_p c'$ replaced by $(0 =) c_{k, -} < c'_+$ and $c_{k, -} < c_-'.

Here we have written $c_{k, -}$ instead of $(c_k)_{-}$, etc.

We check

$$L(c_k, 0) = L(c_k, 0) + \sum_{c_{k, -} < c'_, c_{k, -} < c'_{-}} a_{c_k, c'} L(c', 0).$$

By (a'), we already know $c_{k, -} < c_-'$. Thus we only need to show $c_+' \neq 0$. Suppose that $c_+' = 0$, and hence $L(c', 0) = S_{c_k} E_{c'}$. Then wt $(L(c', 0)) = \text{wt}(S_{c_k}) + \text{wt}(E_{c'})$, which is equal to $k(d, \delta - \alpha)$ if $(X_N^{(r)}, i) \neq (A_2^{(2)}, n)$, and $k(\delta - 2\alpha)$ if $(X_N^{(r)}, i) = (A_2^{(2)}, n)$. Since wt $(S_{c_k}) \in \mathbb{N}_0$, $E_{c'}$ must be a product of $E_N^{(c_k)} E_N^{(d_+ - \alpha)}$ when $(X_N^{(r)}, i) \neq (A_2^{(2)}, n)$, and $E_N^{(c_k)} E_N^{(d_+ - \alpha)} E_N^{(2\alpha - \alpha)}$ when $(X_N^{(r)}, i) = (A_2^{(2)}, n)$ (in an appropriate order). When $(X_N^{(r)}, i) \neq (A_2^{(2)}, n)$, we have $\sum c_l = k$ and wt $(S_{c_k}) + \sum l c_l = k$. These equations simultaneously hold only if $c_1 = k$, $c_l = 0 (l \neq 1)$, and wt $(S_{c_k}) = 0$ (i.e., $c_0 = 0$). When $(X_N^{(r)}, i) = (A_2^{(2)}, n)$, we have $\sum 2c_l + \sum d_m = 2k$, and...
\[\sum (2l-1)c_l + \sum md_m + \text{wt } (S_{c_0}) = k. \text{ Thus } \sum (2l-2)c_l + \sum (m-\frac{1}{2})d_m + \text{wt } (S_{c_0}) = 0,\]
and hence \(c_l = 0 \ (l \geq 2), \ d_m = 0, \ \text{wt } (S_{c_0}) = 0. \) We get \(c_1 = k. \) In both cases, we have \(L(c', 0) = E_{e_{k-1}}, \) which is impossible.

The following together with (b) completes the proof.

**Proof.**

First note that \(c_0 \prec_p c' \) is equivalent to the above inequalities since \(\text{wt } (L(c', 0)) \in \mathbb{Z}_\delta.\)

We prove (d) by the induction on the length of \(c_0. \) When \(\ell(c_0) = 0, \) we understand \(S_{c_0} = 1, \) and the assertion is trivial. We assume \((X_N^{(r)}, i) \neq (A_{2n}^{(2)}, n) \) now, but the argument works even when \((X_N^{(r)}, i) = (A_{2n}^{(2)}, n) \) with obvious modifications. By Proposition 3.17 we have

\[
\tilde{P}_{t,kd} = E^{(k)}_{\tilde{d},\tilde{a}} - E^{(k)}_{\tilde{a}} + \sum_{0 < e'_\pm} a_{c'} L(c', 0).
\]

Taking \(- \) of both sides of this equality we have

\[
\tilde{P}_{t,kd} = E^{(k)}_{\tilde{d},\tilde{a}} - E^{(k)}_{\tilde{a}} + \sum_{0 < e'_\pm} a_{c'} L(c', 0)
\]
\[
+ \sum_{k',<e'_\pm} a_{c',c''} L(c'', 0) E^{(k)}_{\tilde{a}} + \sum_{0 < e'_\pm} a_{c'} L(c', 0),
\]

where we have used \((c)\) in the second equality. By Lemma 3.30 \(L(c'', 0) E^{(k)}_{\tilde{a}} \) is a sum of \(L(d, 0) \) with \(c''_\pm \leq d_\pm. \) (The other inequality \(0 \leq d_- \) is trivially satisfied.) Since \((0 =) c_{k,+} < c''_+, \) we have \(0 < d_+. \)

Recall that by Proposition 3.17 the purely imaginary part of \(L(c', 0) \) is a polynomial in \(P_{t,td}, \) with \(0 < t < k. \) By the induction hypothesis, (3.37) is true for this polynomial. Then it is also true for \(L(c', 0) \) by (b). Thus (3.37) is true for \(\tilde{P}_{t,kd}. \)

By Lemma 3.30 the assertion is also true for \(S_{c_0} \) if it is a polynomial in \(P_{t,td}, \) with \(t \leq k. \)

□

When \(g\) is symmetric or type \(A_2^{(2)} \) we know that the set \(\{L(c, 0) \mid c \in \mathcal{C}\} \) is an \(A\)-basis of \(\mathcal{A}U^+\) from [1] and [3]. Using this result, we can obtain the more general

**Lemma 3.39.** For each \(p \in \mathbb{Z}, \) the set \(\{L(c, p) \mid c \in \mathcal{C}\} \) is an \(A\)-basis of \(\mathcal{A}U^+. \)

**Proof.**

First note that

\[
(3.40) \quad \{L(c, 0) \mid c \in \mathcal{C}, c(0) = 0\}
\]

is an \(A\)-basis of \(\mathcal{A}U^+ \cap U^+[i_0]. \) To see this take any \(x \in \mathcal{A}U^+ \cap U^+[i_0]. \) Since the lemma holds for \(p = 0, \) we have \(x = \sum_{c \in \mathcal{C}} a_c L(c', 0) \) with \(a_c \in \mathcal{A}. \) Now \(x \in U^+[i_0] \iff \) each \(L(c', 0) \in U^+[i_0] \) since \(U^+ = U^+[i_0] \oplus E_{i_0} U^+ \) and clearly each \(L(c', 0) \) is in one of these direct summands. Consider the image of (3.40) under the two maps \(T_{i_0}\) and \(*\) respectively. We have

\[
(3.41) \quad T_{i_0} : \{L(c, 0) \mid c \in \mathcal{C}, c(0) = 0\} \xrightarrow{1-1} \{L(c, -1) \mid c \in \mathcal{C}, c(0) = 0\},
\]
as well as

\[
(3.42) \quad * : \{L(c, 0) \mid c \in \mathcal{C}, c(0) = 0\} \xrightarrow{1-1} \{L(c, 0)^* \mid c \in \mathcal{C}, c(0) = 0\}.
\]
Since both $*$ and $T_{i_0}$ leave $\mathcal{A}U^+$ invariant, and both take $U^+[i_0]$ isomorphically to $U^+[\bar{i}_0]$ we obtain that the two sets
\begin{equation}
(3.43) \quad \{ L(c,-1) \mid c \in \mathcal{C}, c(0) = 0 \}, \quad \{ L(c,0)^\ast \mid c \in \mathcal{C}, c(0) = 0 \},
\end{equation}
are both $\mathcal{A}$-bases of $U^+[i_0] \cap \mathcal{A}U^+$. This implies that the two sets
\begin{equation}
(3.44) \quad \{ L(c,-1)E_{i_0}^{(k)} \mid c \in \mathcal{C}, c(0) = 0, k \in \mathbb{N} \}, \quad \{ L(c,0)^\ast E_{i_0}^{(k)} \mid c \in \mathcal{C}, c(0) = 0, k \in \mathbb{N} \},
\end{equation}
span the same $\mathcal{A}$-submodules of $\mathcal{A}U^+$. The right hand set is a $\mathcal{A}$-basis of $\mathcal{A}U^+$ since it’s the image under $*$ of a $\mathcal{A}$-basis of $\mathcal{A}U^+$. Therefore the left hand set is a $\mathcal{A}$-basis of $\mathcal{A}U^+$, but this set is exactly $\{ L(c,-1) \mid c \in \mathcal{C} \}$. With the same reasoning, an induction gives the lemma for all $p < 0$. A similar argument works for $p > 0$.

From Proposition 3.34 and Lemma 3.33 we deduce using [23, 24.2.1]:

**Theorem 3.45.** Let $\mathfrak{g}$ be affine of symmetric or of type $A_2^{(2)}$.

(i) For any $p \in \mathbb{Z}, c \in \mathcal{C}$ there is a unique $b(c,p) \in \mathcal{A}U^+$ such that
\begin{itemize}
  \item[(a)] $b(c,p) = b(c,p)$,
  \item[(b)] $b(c,p) = L(c,p) + \sum_{c < c', c' \in \mathcal{C}} a_{c,c'}L(c',p)$, where $a_{c,c'} \in q^{-1}\mathbb{Z}[q^{-1}]$.
\end{itemize}

(ii) The $\mathbb{Z}$-homomorphism
\begin{equation*}
\mathcal{L}(-\infty) \cap \overline{\mathcal{L}(-\infty)} \cap \mathcal{A}U^+ \to \mathcal{L}(-\infty) \cap \mathcal{A}U^+/q^{-1}\mathcal{L}(-\infty) \cap \mathcal{A}U^+
\end{equation*}
is an isomorphism.

For $p, q \in \mathbb{Z}$, there exists a bijection $c \in \mathcal{C} \leftrightarrow c' \in \mathcal{C}$ such that $b(c,p) = \pm b(c',q)$ by the proof of [23, 14.2.3]. We will show $b(c,p) = G(b)$ for some $b \in \mathcal{B}(-\infty)$ in Sect. 3 but it is desirable to have a proof of $+$-sign in the above equality, independent of the existence of the global crystal basis.

4. Crystal structure of $\overline{\mathcal{U}}$

Let $P_\mathcal{C} = \sum_{\alpha \in \Phi_+} N_{\alpha} \mathcal{C}_\alpha$. Let $\lambda = \sum_{\alpha \in \Phi_+} \lambda_\alpha \mathcal{C}_\alpha \in P_\mathcal{C}^\circ$. Let $G_\lambda = \prod \mathbb{GL}_\lambda(C)$ and $\text{Irr} G_\lambda$ the set of irreducible representations of $G_\lambda$. Let $\hat{B}(\lambda)$ be the crystal of $\bar{C}_\lambda$, $V(\mathcal{C}_\alpha)^{\otimes \lambda_\alpha}$.

In [18, §13] Kashiwara conjectures a description of the crystal structure of $V(\mathcal{C})$ in terms of $\text{Irr} G_\lambda$ and $\hat{B}(\lambda)$. This conjecture was proved in [24, 33] in the symmetric affine case. It can now be checked for arbitrary type modulo sign using the results of the previous section. The modification is straightforward, but we recall the proof for the sake of reader. We also give a Peter–Weyl type description of $\mathcal{B}(\overline{\mathcal{U}})$ which is conjectured in [18, §13] (see also [31]), but not proved in [24, 33].

The sign ambiguity will be removed in Sect. 3 based on results in this section. Thus returning back to this section again, we get Kashiwara’s conjecture.

**Remark 4.1.** In the previous section we constructed an integral basis of $\mathcal{A}U^+$. In this section, for the purposes of calculation we replace the basis elements $B_c, c \in \mathcal{C}$ with $B_c^+ = \overline{B_c}, c \in \mathcal{C}$. We also replace the $F_\alpha = \Omega(E_\alpha), \alpha \in \mathcal{R}^+$ which appeared in the previous section, with $F_\alpha^- = \overline{E_\alpha}, \alpha \in \mathcal{R}^+$. Additionally, we define the integral basis of $\mathcal{A}U^-$ given by $B_c^- = \overline{B_c}$. Let $P_{-kd_\alpha} = P_{kd_\alpha}^-$. There are two reasons to do this. First, applying the $\ast$ operator allows us to work in $\mathcal{L}(\pm \infty)$ rather than $\overline{\mathcal{L}(\pm \infty)}$. Second, the operators $\ast$ and $\forall$ reverse the root orderings in $B_c^\pm$. 
and so we are able to work with highest weight (relative to the map $\cl$) level zero representations instead of lowest weight level zero representations.

**Definition 4.2.** For $\lambda = \sum \lambda_i \varpi_i \in P_+^0$, let

$$N^{R_0}(\lambda) = \{ c_0 = (\rho(1), \ldots, \rho(n)) \mid \rho(i) \text{ a partition, } \ell(\rho(i)) \leq \lambda_i, i = 1, \ldots, n \}.$$ 

This is identified with the set of irreducible polynomial representations of $G_\lambda$, and the set of $n$-tuples $(s_{\rho(1)}, \ldots, s_{\rho(n)})$, where $s_{\rho(i)}$ is a Schur function in variables $z_{i,1}, \ldots, z_{i,\lambda_i}$.

4.1. Preparatory results.

**Proposition 4.3.** (i) For any $\lambda \in P_+^0$, any vector in $B(\lambda)$ is connected to an extremal weight vector of the form $b_1 \otimes t_\lambda \otimes u_{-\infty}$, where $b_1$ is purely imaginary with respect to the crystal base.

(ii) Furthermore, all such possible $b_1 \in B(\infty)$ are given by $\sgn(c_0,0) S_{c_0}^- u_\infty \mod q_\lambda \mathcal{L}(\infty)$ where $c_0 \in N^{R_0}(\lambda)$.

**Proof.** (i) By [18 Proof of Theorem 5.1] any vector is connected to an extremal weight vector of the form $b_1 \otimes t_\lambda \otimes u_{-\infty}$, where $\text{wt}(b_1) = -k \delta$. Using the basis of §3 to express $b_1$, we take $\sgn(c,0) B_c^- = b_1 \mod q_\lambda \mathcal{L}(\infty)$, for some $c$. Assume that $B_c^-$ isn’t purely imaginary. Since $\text{wt}(B_c^-) = -k \delta$, and $B_c^-$ (resp. $B_c^-$) consists only of terms in root vectors with positive real part (resp. negative real part), it follows $c_- \neq 0$. By Theorem 2.17 we have $B_c^- u_\lambda = 0$. However, by assumption $B_c^- u_\lambda \in \mathcal{L}(\lambda)$ such that $B_c^- u_\lambda \neq 0 \mod q_\lambda \mathcal{L}(\lambda)$. This is a contradiction.

(ii) From Proposition 3.17 we have

$$\tilde{P}_i, -kd, u_\lambda = \begin{cases} F_{d_i, \delta - \alpha_i}^{-k} F_{i}^{(k)} u_\lambda & \text{if } (X_N^{(r)}, i) \neq (A_{2n}^{(2)}, n), \\ F_{\delta - 2n}^{(2k)} F_{n}^{(2k)} u_\lambda & \text{if } (X_N^{(r)}, i) = (A_{2n}^{(2)}, n). \end{cases}$$

Here $u_\lambda$ generates $V(\lambda)$. Since the weights of $V(\lambda)$ are in the convex hull of $W \lambda$ (see Theorem 2.13), this implies that $\tilde{P}_i, -kd, u_\lambda = 0$ for $k > \lambda_i$. Note that for any $i$, $\ell(\rho(i)_1) \leq \lambda_i \iff \rho(i)_1 \leq \lambda_i$. Since the $\tilde{P}_i, -kd$ all commute, considering the top row of the determinant $S_{c_0}^-$, we have $S_{c_0}^- u_\lambda = 0$ for $c_0 \notin N^{R_0}(\lambda)$. \hfill $\square$

Let $\tilde{\alpha}_i \in \tilde{Q}$ as in §2.1 and let $S_{\tilde{\alpha}_i}$ be the corresponding operator in Definition 2.12.

**Lemma 4.5.** Let $\lambda = \sum \lambda_i \varpi_i \in P_+^0$. Let $V(\lambda)$ be the extremal weight module generated by $u_\lambda$.

$$S_{\tilde{\alpha}_i} u_\lambda = \begin{cases} F_{d_i, \delta - \alpha_i}^{-\lambda_i} F_{\alpha_i}^{(\lambda_i)} u_\lambda & \text{if } (X_N^{(r)}, i) \neq (A_{2n}^{(2)}, n), \\ F_{\delta - 2n}^{(2\lambda_i)} F_{\alpha_i}^{(2\lambda_i)} u_\lambda & \text{if } (X_N^{(r)}, i) = (A_{2n}^{(2)}, n) \end{cases} = \tilde{P}_i, -kd, u_\lambda$$

**Proof.** The second equality follows from the first equality and (4.4). We prove the first equality after applying $\circ \lor$. (See the above remark.)

First consider the case $(X_N^{(r)}, i) \neq (A_{2n}^{(2)}, n)$. We have an identity $\tilde{\alpha}_i = s_{d_i, \delta - \alpha_i} s_i = \tilde{\omega}_i s_\omega i^{-1} s_i$ in the affine Weyl group (see (2.3)), where $s_{d_i, \delta - \alpha_i}$ is the reflection with
respect to $d_\delta - \alpha_i$. Then

$$S_{\delta_i}u_{\lambda} = S\omega_i S_i S\omega_i^{-1} S_{i\delta}u_{\lambda} = S\omega_i S_i S\omega_i^{-1} E(\lambda^{(i)})u_{\lambda - \delta}$$

where $\omega_i' = \tilde{\omega}_i s_i$

$$= S\omega_i E(\lambda^{(i)} S_i S\omega_i^{-1} E(\lambda^{(i)})u_{\lambda} = q^{-n} (-1)^{N_+} T\omega_i (E(\lambda^{(i)})) T\omega_i (S\omega_i^{-1} E(\lambda^{(i)})u_{\lambda})$$

$$= q^{M_+} (-1) T\omega_i q^{-n} (-1)^{N_+} T\omega_i (E(\lambda^{(i)})) S\omega_i S\omega_i^{-1} E(\lambda^{(i)})u_{\lambda}$$

$$= q^{M_+} (-1) T\omega_i q^{-n} (-1)^{N_+} E(\lambda^{(i)}) E(\lambda^{(i)})u_{\lambda - \delta}$$

by Lemma 2.14. Here

$$M_+ = \sum_{\alpha \in \Delta + \cap \omega_i^{-1} (\Delta -)} \max (\alpha', \omega_i' - \delta, 0)$$

and $N_+ = \sum_{\alpha \in \Delta + \cap \omega_i^{-1} (\Delta -)} \max (\alpha, \omega_i - \delta, 0)$

and $M_+, N_+$ are defined by replacing $\alpha$ by $\alpha'$.

We have used Remark 3.6 in the last equality.

We set $\alpha' = -\tilde{\omega}_i \alpha$. Then $\alpha' \in \Delta^+ \cap \omega_i^\prime \Delta^-$ and

$$- (\alpha, s_i \omega_i'^{-1} s_i (\lambda)) = (\alpha', \alpha + \lambda) = (\alpha', \lambda),$$

$$- (\alpha, \omega_i'^{-1} s_i (\lambda)) = (\alpha', s_i \lambda).$$

For $\beta \in \Delta_\delta$, let us denote by $\beta'$ the unique element of $\Delta^+$ such that $\text{cl}(\beta') = \beta$ and $\beta' - \delta \notin \Delta^+$ for any $n > 0$. We have

$$\Delta^+ \cap \omega_i^{-1} (\Delta^-) = \{ \beta' + n d_\beta \delta \mid \beta \in \Delta_\delta, n \in \mathbb{Z}, 0 \leq n < (\omega_i, \beta)/d_\delta \}.$$ 

Therefore,

$$\Delta^+ \cap \omega_i^{-1} (\Delta^-) = \Delta^+ \cap \omega_i^{-1} (\Delta^-) \setminus \{ \omega_i (\alpha_i) \}$$

$$= \{ \beta' + n d_\beta \delta \mid \beta \in \Delta_\delta, n \in \mathbb{Z}, 0 \leq n < (\omega_i, \beta)/d_\delta \} \setminus \{ d_\delta - \alpha_i \},$$

where we have used $(\omega_i, \text{cl}(\alpha_i)) = d_i$. If $\beta \in \Delta_\delta^+$, then $(\omega_i, \beta) \geq 0$ and there are no corresponding terms in the summation of $M_+, N_+$. If $\beta \in \Delta_\delta^-$, then $(\alpha', \lambda) \leq 0$ for $\alpha' = \beta' + n d_\beta \delta$ and $(\alpha', s_i \lambda) \leq 0$ except possibly when $\beta = -\text{cl}(\alpha_i)$, i.e., $\alpha' = -\alpha_i + n d_\delta \delta$. However, there are no such roots in $\Delta^+ \cap \omega_i^{-1} (\Delta^-)$. Therefore $M_+ = N_+ = 0$. By the same reason, we also have $M_+^\prime = N_+^\prime = 0$.

Next consider the case $(X_N^{(r)}(i), (A_{2n}^{(2)}))$. We have an identity $\alpha_n = s_{2n_2} s_n$. Following [12, §4.2], we set

$$w = s_0 s_1 \ldots s_n \in \hat{W}.$$

(Our numbering is different from one in [loc. cit.]) We have $w^{n-1}(\alpha_0) = \delta = 2 \alpha_n$. We can repeat the above calculation replacing the identity by $\tilde{\alpha}_n = w^{n-1} s_0 w^{1-n} s_n$. We have

$$S_{\tilde{\alpha}_n} u_{\lambda} = q^{M_+} (-1) T\omega_i q^{-n} (-1)^{N_+} T\omega_i (E(\lambda^{(i)})) E(\lambda^{(i)})u_{\lambda - \delta}$$

where

$$M_+ = \sum_{\alpha' \in \Delta + \cap \omega_i^{-1} (\Delta -)} \max (\alpha', s_n (\lambda)), 0),$$

$$N_+ = \sum_{\alpha' \in \Delta + \cap \omega_i^{-1} (\Delta -)} \max (\alpha', \lambda), 0),$$

and $N_+^\prime, M_+^\prime$ are similar.
We have \( \Delta^+ \cap w^{n-1}(\Delta^-) \cap \text{cl}^{-1}(\Delta^+_0) = \emptyset \). Therefore the above summations are over \( \alpha' \in \Delta^+ \cap w^{n-1}(\Delta^-) \cap \text{cl}^{-1}(\Delta^+_0) \). Then \((\alpha', \lambda) \leq 0 \) for any such \( \alpha' \), and \((\alpha', s_n(\lambda)) \leq 0 \) except possibly when \( \text{cl}(\alpha') = -\text{cl}(\alpha_n) \) or \(-2\text{cl}(\alpha_n)\), i.e., \( \alpha' = m\delta - \alpha_n \) or \((2m - 1)\delta - 2\alpha_n \) for some \( m \in \mathbb{Z}_{>0} \). However we have

\[
w^{1-n}(\alpha_n) = \alpha_1 + \cdots + \alpha_n
\]

by \([12, \text{4.2.3}]\), which means that such \( \alpha' \) cannot be in \( \Delta^+ \cap w^{n-1}(\Delta^-) \). Therefore \( M_+ = M_+^c = N_+ = N_+^c = 0 \).

Finally we have \( T_{w^{n-1}}(E_{\alpha_n}) = E_{d - 2\alpha_n} \) by \([12, \text{4.2.6}]\). \( \square \)

Let \( z_i \) be the \( U' \)-module automorphism of \( V(\varpi_i) \) defined in \( \text{\S}2.6 \).

**Lemma 4.6.** Let \( i \in I_0 \). Then on \( V(\varpi_i) \):

\[
\tilde{P}_{i,-d_i}u_{\varpi_i} = \tilde{P}_{i,-d_i}u_{\varpi_i} = z_i^{-1}u_{\varpi_i}, \quad \tilde{P}_{i,-kd_i}u_{\varpi_i} = 0, \quad \text{for } k > 1.
\]

**Proof.** The statement for \( k > 1 \) was already observed in the proof of Proposition \([13, \text{ii}]\). Lemma \([13] \) (with \( \lambda = \varpi_i, \lambda_i = 1 \)) says \( \tilde{P}_{i,-d_i}u_{\varpi_i} = S_{\alpha_i}u_{\varpi_i} \). In particular, \( \tilde{P}_{i,-d_i}u_{\varpi_i} \) is an element of the global basis. On the other hand, \( V(\varpi_i) \) has a unique global basis element of weight \( \varpi_i - d_i\delta \) (see Theorem \([2.16] \)), which by definition equals \( S_wu_{\varpi_i} = z_i^{-1}u_{\varpi_i} \), where \( w = \tilde{\alpha}_i \). The assertion follows. \( \square \)

### 4.2. The map \( \Phi_\lambda \) and Kashiwara’s conjecture.

Let \( \lambda = \sum_{i \in I_0} \lambda_i \varpi_i \in P^0_+ \). The module

\[
\tilde{V}(\lambda) = \bigotimes_{i \in I_0} V(\varpi_i)^{\otimes \lambda_i}
\]

has a crystal base \( (\otimes_i \mathcal{L}(\varpi_i)^{\otimes \lambda_i}, \otimes_i \mathcal{B}(\varpi_i)^{\otimes \lambda_i}) \). Let \( \tilde{u}_\lambda = \bigotimes_i u_{\varpi_i}^{\otimes \lambda_i} \).

For each \( i \), and each of the \( \nu = 1, \ldots, \lambda_i \) factors of \( V(\varpi_i)^{\otimes \lambda_i} \), we let \( z_{i,\nu} \) be the commuting automorphisms defined in \([18, \text{\S}4.2] \). By \([18, \text{Theorem 8.5}] \), the submodule

\[
\tilde{V}(\lambda) = \mathcal{A} U_z \bigotimes_{1 \leq i \leq n, 1 \leq \nu \leq \lambda_i} \tilde{u}_\lambda \subset \tilde{V}(\lambda)
\]

has a global crystal basis \( (\tilde{\mathcal{L}}, \tilde{\mathcal{B}}) \) such that \( \tilde{\mathcal{L}} \subset \bigotimes_i \mathcal{L}(\varpi_i)^{\otimes \lambda_i}, \tilde{\mathcal{B}} = \bigotimes_i \mathcal{B}(\varpi_i)^{\otimes \lambda_i} \). Since \( \tilde{V}(\lambda) \) contains the extremal vector \( \tilde{u}_\lambda \) of weight \( \lambda \) we have a unique \( U \)-linear morphism

\[
\Phi_\lambda : V(\lambda) \to \tilde{V}(\lambda),
\]

sending \( u_\lambda \) to \( \tilde{u}_\lambda \), and which commutes with the crystal operators \( \tilde{e}_i, \tilde{f}_i \).

For each \( n \)-tuple of partitions \( \varpi_0 = (\rho^{(1)}, \rho^{(2)}, \ldots, \rho^{(n)}) \) we consider the product of Schur functions in the variables \( z_{i,\nu}^{\pm 1} \) (see \([20] \)):

\[
s_{\varpi_0}(z_{i,\nu}^{\pm 1}) = \prod_{i \in I_0} s_{\rho^{(i)}}(z_{i,1}^{\pm 1}, \ldots, z_{i,\lambda_i}^{\pm 1}).
\]

Note that for each \( i \), \( s_{\rho^{(i)}}(z_{i,\nu}^{\pm 1}) \) acts as the 0 map if \( \lambda_i < \ell(\rho^{(i)}) \). We will omit the indices \( i, \nu \) and write \( s_{\varpi_0}(z^{\pm 1}) \).

Using Lemma \([13] \) we have:

**Proposition 4.10.** Let \( \varpi_0 = (\rho^{(1)}, \rho^{(2)}, \ldots, \rho^{(n)}) \) be an \( n \)-tuple of partitions:

\[
\Phi_\lambda(S_{\varpi_0}^{-} u_\lambda) = s_{\varpi_0}(z^{-1}) \tilde{u}_\lambda.
\]
Proof. Note that $\sigma \circ (\gamma \times \gamma) \circ \Delta(a) = \Delta(a^\gamma)$ for $a \in U$. Here $\sigma$ is the exchange of two factors of the tensor product. Since our $\tilde{P}_{i, -kd_i}$ ($k > 0$) are those given in Sect. 3 after applying $- \circ \psi$ we have by Proposition 3.22 and 3.1.5] (after noting the difference between our coproduct and the one there)

$$
\Delta(\tilde{P}_{i, -kd_i}) = \sum_{s = 0}^{k} \tilde{P}_{i, -sd_i} \otimes \tilde{P}_{i, (s - k)d_i},
$$

This and Lemma 3.6 imply that $\Delta^{\lambda}(\tilde{P}_{i, -kd_i})$ acts as $e_k(z_{i, 1}^{-1}, \ldots, z_{i, \lambda_i}^{-1})$ on $\tilde{V}(\lambda)$ where $e_k$ is the $k$-th elementary symmetric function. Since polynomials in the $\tilde{P}_{i, -kd_i}$ (resp. elementary symmetric functions) generate the Schur functions $S_{c_0}$ (resp. $s_{c_0}(z^{-1})$), we have $\Phi_\lambda(S_{c_0} \otimes u \otimes u_{-\infty}) = s_{c_0}(z^{-1}) \tilde{u}_\lambda$.

Next we consider the image of $B(\lambda)$ under $\Phi_\lambda$. By Proposition 4.13 every element of $B(\lambda)$ is connected to an extremal vector of the form $\text{sgn}(c_0, 0)S_{c_0} \otimes u \otimes u_{-\infty} \otimes t_\lambda \otimes u_{-\infty} \mod q_\lambda \mathcal{L}(\lambda)$. Therefore we have,

$$(4.12) \quad B(\lambda) = \{X_1 X_2 \cdots X_n (\text{sgn}(c_0, 0)S_{c_0} \otimes u \otimes u_{-\infty} \mod q_\lambda \mathcal{L}(\lambda)) | X_i = \tilde{c}_i \text{ or } \tilde{f}_i, c_0 \in N^{\mathcal{R}_0}(\lambda) \} \setminus \{0\}.$$ 

Since $\Phi_\lambda$ commutes with crystal operators, and the $z_{i, \nu}$ induce automorphisms of the $U'$-crystal of $V(\varpi_i)$, we have $\Phi_\lambda(\mathcal{L}(\lambda)) \subset \tilde{\mathcal{L}}$ by Proposition 4.10. Denote by $\Phi_{\lambda|q=0}$ the induced map $\mathcal{L}(\lambda)/q_\lambda \mathcal{L}(\lambda) \to \tilde{\mathcal{L}}/q_\lambda \mathcal{L}(\lambda)$.

**Proposition 4.13.** Let $B_0(\tilde{V}(\lambda))$ be the connected component of $\tilde{B}$ containing $\tilde{u}_\lambda$. Then

$$\Phi_{\lambda|q=0}: \{b \in B(\lambda) \} \to \{\text{sgn}(c_0, 0)S_{c_0} \otimes u \otimes u_{-\infty} \mod q_\lambda \mathcal{L}(\lambda) \} \times N^{\mathcal{R}_0}(\lambda), \ b' \in B_0(\tilde{V}(\lambda))\}$$

is a bijection.

**Proof.** It is clear that $\Phi_{\lambda|q=0}(B(\lambda)) \setminus \{0\}$ is equal to the right hand side of the above. Using (4.12), we check that $\Phi_{\lambda|q=0}(B(\lambda))$ does not contain 0 and is injective. Let $b \in B(\lambda)$ such that $\Phi_{\lambda|q=0}(b) = 0$. Since $b$ is connected by crystal operators to $b_1 \otimes \mathcal{L} \otimes u_{-\infty}$, where $b_1 = \text{sgn}(c_0, 0)S_{c_0} \otimes u \otimes u_{-\infty} \mod q_\lambda \mathcal{L}(\lambda), c_0 \in N^{\mathcal{R}_0}(\lambda)$, this implies $\Phi_{\lambda|q=0}(S_{c_0} \otimes u \otimes u_{-\infty} \mod q_\lambda \mathcal{L}(\lambda)) = 0$. This contradicts Proposition 4.11. Now let $b_1, b_2 \in B(\lambda), b_1 \neq b_2$ and assume $\Phi_{\lambda|q=0}(b_1) = \Phi_{\lambda|q=0}(b_2)$. By applying crystal operators we may assume $b_1 = \text{sgn}(c_0, 0)S_{c_0} \otimes u \otimes u_{-\infty} \mod q_\lambda \mathcal{L}(\lambda)$ is extremal and purely imaginary, and where $b_2$ is of the same weight as $b_1$. For $w \in W$, we have

$$\Phi_{\lambda|q=0}(X_i S_{w} b_1) = X_i S_{w} \Phi_{\lambda|q=0}(b_1) = X_i S_{w} \Phi_{\lambda|q=0}(b_2) = \Phi_{\lambda|q=0}(X_i S_{w} b_2) = 0,$$

where $X_i$ is $\tilde{c}_i$ or $\tilde{f}_i$. Since Ker $\Phi_{\lambda|q=0} \cap B(\lambda) = \emptyset$, we have $X_i S_{w} b_2 = 0$. Therefore, $b_2$ is also extremal. Applying a sequence of $\tilde{f}_i^{\text{max}}$s (since $b_2$ is extremal, this is equivalent to the Weyl group action) we may assume $b_2 = \text{sgn}(c'_0, 0)S_{c'_0} \otimes u \otimes u_{-\infty} \mod q_\lambda \mathcal{L}(\lambda)$ (as in [18, Theorem 5.1]). Then we have that $\Phi_{\lambda}(b_1) = \Phi_{\lambda}(b_2) \neq 0$ and also that $b_1 = \text{sgn}(c''_0, 0)S_{c''_0} \otimes u \otimes u_{-\infty} \mod q_\lambda \mathcal{L}(\lambda)$ for some $c''_0$ purely imaginary in $\mathfrak{C}$ by Proposition 4.3(ii). But by Proposition 4.14 it is clear that $\Phi_{\lambda}(\text{sgn}(c''_0, 0)S_{c''_0} \otimes u \otimes u_{-\infty}) = \Phi_{\lambda}(\text{sgn}(c'_0, 0)S_{c'_0} \otimes u \otimes u_{-\infty})$ only if $S_{c'_0} \otimes u \otimes u_{-\infty} = S_{c''_0} \otimes u \otimes u_{-\infty}$. \quad $\square$
Lemma 4.14. (i) Let \( c_0, c'_0 \in \mathbb{N}^{\mathfrak{g}_0}(\lambda) \) and \( b, b' \in B_0(\mathcal{V}(\lambda)) \). Then \( s_{c_0}(z^{-1})b = s_{c'_0}(z^{-1})b' \) if and only if \( s_{c_0}(z^{-1}) = s_{c'_0}(z^{-1})p(z) \) and \( b' = p(z)^{-1}b \) for some \( p(z) = \prod_i (z_i, \cdots, z_i, \lambda_i)^{r_i} (r_i \in \mathbb{Z}) \).

(ii) For \( b \in B_0(\mathcal{V}(\lambda)) \) and \( p(z) \) as above, we have \( p(z)b \in B_0(\mathcal{V}(\lambda)) \).

Proof. (i) Recall that for fundamental representations \( V(\varpi_i) \cong Q(q_i)[z_i^\pm] \otimes W(\varpi_i) \), where \( W(\varpi_i) \) is the finite dimensional representation (see \([2,8]\)). For general \( \lambda = \sum_i \lambda_i \varpi_i \in \mathbb{P}^0 \), we put \( W(\lambda) = \bigotimes_{i \in I_0} W(\varpi_i)^{\otimes \lambda_i} \).

This is irreducible and has a global crystal basis \( B_0(\lambda) \). We have
\[
\mathcal{V}(\lambda) \cong Q(q_i)[z_i^\pm] \otimes W(\varpi_i) \otimes W(\lambda),
\]
\( \mathcal{B} \cong \{ \text{monomials in } z_i^\pm \} \times B_0(\lambda) \).

Therefore
\[
s_{c_0}(z^{-1})b = s_{c_0}(z^{-1})m(z)b_W, \quad s_{c'_0}(z^{-1})b' = s_{c'_0}(z^{-1})m'(z)b'_W,
\]
for some monomials \( m, m' \) and \( b_W, b'_W \in B_0(\lambda) \). These two are equal if and only if \( b_W = b'_W \) and \( s_{c_0}(z^{-1})m(z) = s_{c'_0}(z^{-1})m'(z) \). Then \( p(z) = m(z)/m'(z) \) is a symmetric function which is also a monomial. Therefore it must be of the above form.

(ii) Since \( p(z) \) commutes with \( \tilde{e}_i, \tilde{f}_i \), we may assume \( b = \tilde{u}_\lambda \). For \( w \in \mathcal{W} \), let \( S_w \) denote the corresponding crystal operator. Then we have
\[
S_w \tilde{u}_\lambda = \bigotimes_i (S_w u_{\varpi_i})^{\otimes \lambda_i}
\]
by \([2]\) Lemma 1.6 since \( B(V(\varpi_i)) \) is the affineization of a simple crystal. Let us take \( w = t(-\sum_i r_i \alpha_i) \). By Lemmas 4.3, 4.6 we get
\[
S_w \tilde{u}_\lambda = \bigotimes_{i,v} (z_{i,v})^{r_i} \tilde{u}_\lambda = p(z)\tilde{u}_\lambda.
\]
Thus \( p(z)\tilde{u}_\lambda \) is connected to \( \tilde{u}_\lambda \) in the crystal graph. \( \square \)

Let
\[
\mathbb{N}^{\mathfrak{g}_0}(\lambda)' = \{(\rho(1), \ldots, \rho(n)) \mid \rho(i) \text{ a partition, } \ell(\rho(i)) < \lambda_i, i = 1, \ldots, n\}.
\]
This can be identified with the set of irreducible representations of \( \prod_i SL_\lambda(\mathbb{C}) \).

Any \( c_0 \in \mathbb{N}^{\mathfrak{g}_0}(\lambda) \) decomposes uniquely as \( s_{c_0}(z^{-1}) = s_{c'_0}(z^{-1})p(z) \) with \( c'_0 \in \mathbb{N}^{\mathfrak{g}_0}(\lambda)' \) and a monomial \( p(z) \) as in the above lemma. Therefore Proposition 4.13 can be strengthened as
\[
\Phi_{\lambda;q=0} : B(\lambda) \cong \{ \text{sgn}(c_0,0)s_{c_0}(z^{-1})b' \mid c_0 \in \mathbb{N}^{\mathfrak{g}_0}(\lambda)', b' \in B_0(\mathcal{V}(\lambda)) \}
\cong \mathbb{N}^{\mathfrak{g}_0}(\lambda)' \times B_0(\mathcal{V}(\lambda)).
\]
A connected component of \( B(\lambda) \) is mapped to \( \{c_0 \times B_0(\mathcal{V}(\lambda)) \) for some \( c_0 \in \mathbb{N}^{\mathfrak{g}_0}(\lambda)' \). In particular, each connected component is isomorphic to each other as a \( P_\lambda \)-crystal.

Corollary 4.15. The map \( \Phi_\lambda : V(\lambda) \to \mathcal{V}(\lambda) \) is injective.
Proof. Since $\Phi_{\lambda|q=0}: \mathcal{L}(\lambda)/q_{s}\mathcal{L}(\lambda) \rightarrow \tilde{\mathcal{L}}/q_{s}\tilde{\mathcal{L}}$ maps the crystal basis $\mathcal{B}(\lambda)$ bijectively to \{sgn($c_{0},0$)$s_{c_{0}}(z^{-1})b'$ | $c_{0} \in N^{\mathcal{R}_{0}}(\lambda)'$, $b' \in \mathcal{B}_{0}(\tilde{V}(\lambda))$\}, which is linearly independent. Therefore $\Phi_{\lambda|q=0}$ is injective. Write an element $v \in \ker \Phi_{\lambda}, v \neq 0$, in terms of the global basis \{G(b) | $b \in \mathcal{B}(\lambda)$\} as $v = \sum c_{k}(q_{s})G(b)$. Multiplying by a power of $q$ we may assume that each $c_{k}(q_{s})$ is regular at $q = 0$, so that $v \mod q_{s}\mathcal{L}(\lambda) \neq 0$. This implies $\Phi_{\lambda|q=0}(v \mod q_{s}\mathcal{L}(\lambda)) \neq 0$, which is a contradiction.

We now state a main result in this subsection:

**Theorem 4.16.** (i) We have an isomorphism of crystals

$$\Phi_{\lambda|q=0}: \mathcal{B}(\lambda) \rightarrow \{\text{sgn}(c_{0},0)s_{c_{0}}(z^{-1})b' \mid c_{0} \in N^{\mathcal{R}_{0}}(\lambda)', b' \in \mathcal{B}_{0}(\tilde{V}(\lambda))\} \cong N^{\mathcal{R}_{0}}(\lambda)' \times \mathcal{B}_{0}(\tilde{V}(\lambda)).$$

A connected component of $\mathcal{B}(\lambda)$ is mapped to \{c_{0} \times \mathcal{B}_{0}(\tilde{V}(\lambda))\} for some $c_{0} \in N^{\mathcal{R}_{0}}(\lambda)'$. Also, any two connected components are isomorphic to each other as $P_{\lambda}$-crystals.

(ii) $\Phi_{\lambda}$ induces a bijection between the sets

$$\Phi_{\lambda}: G(\mathcal{B}(\lambda)) \rightarrow \{\text{sgn}(c_{0},0)s_{c_{0}}(z^{-1})G(b') \mid c_{0} \in N^{\mathcal{R}_{0}}(\lambda)', b' \in \mathcal{B}_{0}(\tilde{V}(\lambda))\}.$$ 

(iii) A vector $b \in \mathcal{B}(\lambda)$ is extremal if and only if

$$\Phi_{\lambda|q=0}(b) = \text{sgn}(c_{0},0)s_{c_{0}}(z^{-1})S_{w}\tilde{u}_{\lambda}$$

for some $c_{0} \in N^{\mathcal{R}_{0}}(\lambda)'$, $w \in \hat{W}$.

**Proof.** (i) is proved already. Let us show (ii). By Proposition 4.13, for each $b \in \mathcal{B}(\lambda)$ there exist $b' \in \mathcal{B}_{0}(\tilde{V}(\lambda))$ and $s_{c_{0}}(z^{-1})$ such that $\Phi_{\lambda}(b) = s_{c_{0}}(z^{-1})b' \mod q_{s}\tilde{\mathcal{L}}$. Let $G(b)$, $G(b')$ be the respective globalizations of $b$ and $b'$. Then $\Phi_{\lambda}(G(b)) \equiv s_{c_{0}}(z^{-1})G(b') \mod q_{s}\tilde{\mathcal{L}}$. Since $\Phi_{\lambda}$ commutes with the bar involution, we have $\Phi_{\lambda}(G(b)) \equiv s_{c_{0}}(z^{-1})G(b') \mod q_{s}^{-1}\tilde{\mathcal{L}}$. We get the assertion.

Let us prove (iii). It is enough to consider the case when $\Phi_{\lambda|q=0}(b) \in \mathcal{B}_{0}(\tilde{V}(\lambda))$. As in the proof of Lemma 4.14, we can write $\Phi_{\lambda|q=0}(b) = p(z)b_{W}$ where $p(z) = \prod(\xi_{1}, \cdots, \xi_{\lambda})^{r_{i}}(r_{i} \in \mathbb{Z})$ and $b_{W} \in \mathcal{B}_{W}(\lambda)$. Then $b$ is extremal if and only if $b_{W}$ is so. Furthermore, $b_{W}$ is extremal if and only if $b_{W} = S_{w}\tilde{u}_{\lambda}$ for $w_{0} \in W_{cl}$. This follows from [2 Lemma 1.6], after noting that ([18 Theorem 5.15]) $W(\varpi_{i})$ has a simple crystal such that the weight of any extremal vector belongs to $W_{cl}(\varpi_{i})$. Now we have $\Phi_{\lambda|q=0}(b) = p(z)b_{W} = S_{w}\tilde{u}_{\lambda}$ for some $w \in \hat{W}$ as in the proof of Lemma 4.14.

**Remark 4.17.** Taken together the results of this section and $\text{sgn}(c_{0},0) = 1$, which will be proved in the next section, give the conjectures [18 13.1, 13.2]. To obtain 13.1 (iii), consider that the crystal $\bigotimes_{i \in I_{0}}\mathcal{B}(\lambda, \varpi_{i})$ is by Proposition 4.13 in bijective correspondence with \{s_{c_{0}}(z^{-1})\mathcal{B}_{0}(\tilde{V}(\lambda))\}, and note that $\Phi_{\lambda}$ factors through $\bigotimes_{i \in I_{0}}V(\lambda, \varpi_{i})$.

4.3. A Peter–Weyl type decomposition theorem. Let $\mathcal{B}_{0}(\lambda)$ be the connected component of $\mathcal{B}(\lambda)$ containing $u_{\lambda}$. Consider $\bigsqcup_{\lambda \in P}\mathcal{B}_{0}(\lambda) \times \mathcal{B}(\lambda)$ as a crystal over $\mathfrak{g} \oplus \mathfrak{g}$. Here for $u \otimes v \in \mathcal{B}_{0}(\lambda) \otimes \mathcal{B}(\lambda)$, $X_{i}(u \otimes v) = X_{i}u \otimes v$ and $X_{i}^{\dagger}(u \otimes v) = u \otimes X_{i}v$ where $(X_{i} = \tilde{e}_{i}, \tilde{f}_{i})$. The Weyl group $\hat{W}$ acts on $\bigsqcup_{\lambda \in P}\mathcal{B}_{0}(\lambda) \times \mathcal{B}(\lambda)$ by $S_{w}^{*} \times S_{w}^{*}: \mathcal{B}_{0}(\lambda) \times \mathcal{B}(\lambda) \rightarrow \mathcal{B}_{0}(w\lambda) \times \mathcal{B}(\lambda - \omega_{\lambda}).$
Consider the map \( \mathcal{B}_0(\lambda) \times \mathcal{B}(-\lambda) \to \mathcal{B}(U_{\alpha,\lambda}) \), which sends \( u_\lambda \otimes b \in \mathcal{B}_0(\lambda) \times \mathcal{B}(-\lambda) \) to \( b^* \in \mathcal{B}(U_{\alpha,\lambda}) \). We will show that this is well-defined later. It is a map between crystals over \( \mathfrak{g} \oplus \mathfrak{g} \) where the usual crystal structure on \( \mathcal{B}(\tilde{U}) \) corresponds to the one on \( \mathcal{B}_0(\lambda) \) and the star crystal structure on \( \mathcal{B}(\tilde{U}) \) corresponds to the one on \( \mathcal{B}(-\lambda) \). We have

**Theorem 4.18.** \( \left( \mathcal{B}_0(\lambda) \times \mathcal{B}(-\lambda) \right) / \tilde{W} \cong \mathcal{B}(\tilde{U}) \) as crystals over \( \mathfrak{g} \oplus \mathfrak{g} \).

**Proof.** When \( \langle c, \lambda \rangle \neq 0 \), the \( \mathcal{B}(U_{\alpha,\lambda}) \) part of the crystal decomposition in Theorem 4.18 appears as \([17, \text{Proposition 10.2.2}]\). Therefore, it is sufficient to check the map in Theorem 4.18 for \( \lambda \in P^0 \), where the image on the right hand side is in \( \mathcal{B}(U_{\alpha,\lambda}) \).

The following proof is a modification of a result of Nakashima \([31, \text{Proposition 4.4}]\). We give it for the sake of the reader.

We first prove

\( (\text{C1}): \) For any extremal vector \( b \in \mathcal{B}^0(\tilde{U}) \), there exists a crystal embedding:

\[ \mathcal{B}_0(\text{wt}(b)) \hookrightarrow \mathcal{B}(\tilde{U}) \]

that is surjective. We now define a map \( \phi: \bigsqcup_\lambda \mathcal{B}_0(\lambda) \times \mathcal{B}(-\lambda) \to \mathcal{B}^0(\tilde{U}) \) by

\[ \phi(X_1 X_2 \cdots X_N u_\lambda \otimes b) = X_1 X_2 \cdots X_N b^* \]

for \( X_j = e_i \) or \( f_i \). By (C1) this is well-defined, i.e., (a) if \( X_1 X_2 \cdots X_N u_\lambda = 0 \), then \( X_1 X_2 \cdots X_N b^* = 0 \), and (b) if \( X_1 X_2 \cdots X_N u_\lambda = X'_1 X'_2 \cdots X'_{N'}, u_{\lambda'} \), then \( X_1 X_2 \cdots X_N b^* = X'_1 X'_2 \cdots X'_{N'} b^* \). (C1) is applicable since \( b^* \) is extremal of weight \( \lambda \) for \( b \in \mathcal{B}(\tilde{U}) \).

Since \( \tilde{e}_i, \tilde{f}_i \) commute with \( \tilde{e}_j, \tilde{f}_j \) on \( \mathcal{B}(\tilde{U}) \) \([17, \text{Theorem 5.1.1}]\), \( \phi \) is a morphism of bi-crystal. Since any connected component of \( \mathcal{B}(\tilde{U}) \) contains an extremal vector \( [17, \text{Corollary 9.3.3}] \), the map \( \phi \) is surjective.

Finally we show that \( \phi \) becomes injective if we divide \( \bigsqcup_\lambda \mathcal{B}_0(\lambda) \times \mathcal{B}(-\lambda) \) by \( \tilde{W} \). Suppose \( X_1 X_2 \cdots X_N b^* = X'_1 X'_2 \cdots X'_{N'} b'^* \) where each \( X_j, X'_j = e_i \) or \( f_i \), \( b, b'^* \in \mathcal{B}(\tilde{U}) \). Then \( b^* b'^* = S_w b^* \) for some \( w \in \tilde{W} \). In particular, \( b^* = S_w b^* \). We have \( X_1 X_2 \cdots X_N b^* = X'_1 X'_2 \cdots X'_{N'} S_w b^* \). (C1) implies \( X_1 X_2 \cdots X_N u_\lambda = X'_1 X'_2 \cdots X'_{N'} S_w u_\lambda \). The isomorphism \( S_w^*: \mathcal{B}_0(\lambda) \to \mathcal{B}_0(w \lambda) \) sends \( u_\lambda \) to \( S_w^{-1} u_{w \lambda} = S_w^{-1} u_{N} \). Therefore

\[ X'_1 X'_2 \cdots X'_{N'} u_\lambda \otimes b' = (S_w^* \times S_w^*)(X_1 X_2 \cdots X_N u_\lambda \otimes b) \]

□
Remark 4.19. In [31] a condition labeled \((C2)\) requiring \(B_0(\lambda_\lambda) = \{u_\lambda\}\) is used. This is false in general. A counter-example for type \(A_2^{(1)}\) can be found in [13, 5.10].

5. The proof of Theorem 3.13(ii)

In this section we will complete the proof of Theorem 3.13. The proof uses results in the previous section, in particular extremal weight modules.

Remark 5.1. Since we are working with generators in \(U^+\) we will need to consider the extremal weight modules \(V(-\lambda), \lambda \in P^+_0\).

Lemma 5.2. Let \(\lambda \in P^0_0\) and \(c_0 \in \mathbb{N}^{\mathfrak{h}_i}()\). Then \(S_{c_0}u_{-\lambda} \in G(\mathcal{B}(-\lambda))\).

Proof. We have \(sgn(c_0, 0)S_{c_0}u_{-\lambda} \in G(\mathcal{B}(-\lambda))\) by Proposition 4.10 and Theorem 4.16. So we only need to show \(sgn(c_0, 0) = 1\). Therefore it is enough to show \(S_{c_0}u_{-\mu} \in G(\mathcal{B}(-\mu))\) for some \(\mu\). We check this by induction on \(\sum \rho(\mu)\). Take \(\mu = \sum \rho(\mu)\aleph_0\). Fix an \(i\) and consider \(\rho(\mu)\) obtained by removing the last box in each row of \(\rho(\mu)\). Let \(c_0\) be the set of partitions we get by replacing \(\rho(\mu)\) by \(\rho(\mu)\).

By Lemma 4.5 and the Pieri rules for symmetric functions [26, p. 73],

\[
S_{c_0}S_{\alpha_i}u_{-\mu} = S_{c_0}P_{1, \mu_i, \delta, u_{-\mu}} = S_{c_0}u_{-\mu},
\]

where \(\mu_i = \ell(\rho(\mu))\). A priori, the right hand side is a sum over all \(S_{c_0}u_{-\mu}\) such that \(c_0, \aleph_0\) is a \(n\)-tuple of partitions which are identical to \(c_0, \aleph_0\) in factors different than \(i\), and for \(i\) we have \(\rho(\mu) \backslash \rho(\mu)\) is a vertical \(\mu_i\) strip. But there is only one such summand. (Under the identification between Schur functions and irreducible representations of \(G^*, \hat{P}_{1, \mu_i, \delta}\) is identified with a power of the determinant representation. So the tensor product of it with an irreducible representation remains irreducible.)

Let us note the following fact: If \(\lambda^* \in P_0^0\) and \(\lambda = \sum \lambda_i \omega_i \in P_0^0\) satisfy \(\text{cl}(\lambda^* \omega_i) = \text{cl}(\lambda)\) then \(u_{\infty} \otimes \alpha_{-\lambda} \otimes b_2 \in \mathcal{B}(\lambda) \iff u_{\infty} \otimes \alpha_{-\lambda} \otimes b_2 \in \mathcal{B}(\lambda^*)\) (see [13, Appendix]). In other words, \(S_{c_0}u_{-\lambda} \mod q_\lambda \mathcal{L}(\lambda) \in \mathcal{B}(\lambda) \iff S_{c_0}u_{-\lambda} \mod q_\lambda \mathcal{L}(\lambda) \in \mathcal{B}(\lambda^*)\). Then by the inductive assumption, \(S_{c_0}u_{\mu_i, \delta, \mu_{-\mu}}\) is an element of the global crystal basis. Therefore \(S_{c_0}u_{-\mu} = S_{c_0}S_{\alpha_i}u_{-\mu}\) is also in the global basis, since the isomorphism \(V(\mu_{\delta}, \delta, -\mu) \sim V(-\mu)\) sends the global basis to the global basis (see [7, Prop. 8.2.2 (iv)]).

Proof of \(sgn(c, p) = 1\) in Proposition 3.27. Let \(c_0 \in \mathfrak{c}\) be purely imaginary. In the proof of Lemma 5.2 we have shown \(sgn(c_0, 0) = 1\). From [24, Proposition 8.3(a)] we have \(sgn(c, 0) = 1\) for an arbitrary \(c\). The general case now follows from [25, Theorem 1.2] (see [2, 2.6]).

Let \((c, p)\) as in Proposition 3.27. We denote \(G(b(c, p))\) by \(G(c, p)\). By the same argument in the proof of [24, 42.1.12], we have

\[
\varphi_p(b(c, p)) = c(p) \quad \text{(hence } G(c, p) \in E_{p}(c(p)U^+),
\]

\[
\hat{e}_p(b(c, p)) = b(c^+, p)
\]

\[
\hat{f}_p(b(c, p)) = b(c^-, p) \quad \text{if } c(p) > 0,
\]

where \(c^+\) (resp. \(c^-\)) is defined by setting \(c^+(p) = c(p) + 1\) (resp. \(c^-(p) = c(p) - 1\)) and other entries are the same as \(c\). We have similar formulas for \(\varphi_p, \hat{e}_p, \hat{f}_p\).

Let \(U^+[i]\), \(U^+[i]^+\), \(\pi, \pi^+\) be as in [24, 5.3]. By (5.3) we have

\[
\pi^p(G(c, p)) \neq 0 \iff c(p) = 0 \iff \pi^p(G(c, p) - 1) \neq 0,
\]
The latter follows from \(3.10\) (see \([25, \S 6.2]\) for detail).

Let us define \(a_{c,c'}^p \in Q(q_i)\) by

\[
L(c, p) = \sum_{c'} a_{c,c'}^p G(c', p).
\]

The sum is over a finite set of \(c' \in C\) indexing all elements \(G(c, p) \in U^+\) having the same weight with \(L(c, p)\). Since the global basis is an integral base of \(A U^+\) and \(L(-\infty)\) we have \(a_{c,c'} \in A \cap A_{\infty} = Z[q_i^{-1}]\). We also have \(a_{c,c'}|q=\infty = \delta_{c,c'}\) by Proposition 3.27 together with \(\text{sgn}(c, p) = 1\). Theorem 6.2\] follows from

**Lemma 5.6.** The transition matrix \((a_{c,c'})\) between the global basis and the \(L(c, p)\), \(c \in C, p \in Z\) is upper triangular with respect to the ordering \(\prec_p\), and the diagonal entries are 1.

**Proof.** We first consider the case \(c_+ = 0 = c_-\), so that

\[
L(c, p) = T_{i_{p+1}} T_{i_{p+2}} \cdots T_{i_0}(S_{c_0}) \quad \text{or} \quad T_{i_p}^{-1} T_{i_{p-1}}^{-1} \cdots T_{i_2}^{-1} T_{i_1}^{-1}(S_{c_0}).
\]

In this case, by Lemma 5.2, we know that \(L(c, p) u_{-\lambda}\) is in the global basis of \(V(-\lambda)\) for any sufficiently large \(\lambda \in P_{\infty}^0\). Therefore \((L(c, p) - G(c, p)) u_{-\lambda} = 0\) for any sufficiently large \(\lambda \in P_{\infty}^0\), i.e.,

\[
\sum_{c' \neq c} (a_{c,c'}^p - \delta_{c,c'}) G(c', p) u_{-\lambda} = 0.
\]

By the construction of \(V(-\lambda)\), \(\{G(c', p) u_{-\lambda} \mid c' \in C\}\) is mapped to the union of the global basis of \(V(-\lambda)\) and 0. Hence

\[
a_{c,c'}^p = \delta_{c,c'} \quad \text{or} \quad G(c', p) u_{-\lambda} = 0.
\]

If \(c'_+ = 0 = c'_-\), then

\[
G(c', p) u_{-\lambda} = T_{i_{p+1}} T_{i_{p+2}} \cdots T_{i_0}(S_{c_0}) u_{-\lambda} \quad \text{or} \quad T_{i_p}^{-1} T_{i_{p-1}}^{-1} \cdots T_{i_2}^{-1} T_{i_1}^{-1}(S_{c_0}) u_{-\lambda}
\]

is nonzero for sufficiently large \(\lambda\). This means that \(a_{c,c'}^p = \delta_{c,c'}\) for such \(c'\). By the definition of the ordering, the remaining \(c'\)'s have \(c' \succ_p c\). We have the assertion in this case.

Next consider the case \(c(p+1) = c(p+2) = \cdots = 0\), i.e.,

\[
L(c, p) = \left( T_{i_p} E_{i_{p-1}}(c(p-1)) T_{i_p}^{-1} T_{i_{p-1}}^{-1}(c(p-2)) \cdots \right) \times T_{i_{p+1}} T_{i_{p+2}} \cdots T_{i_0}(S_{c_0}).
\]

We prove the assertion by the induction on \(q\) such that \(c(p-q) = c(p-q-1) = \cdots = 0\). When \(q = 0\), we have \(c_+ = 0 = c_-\), which we have checked already.

First assume \(c(p) = 0\). We consider \(L(c, p - 1) = T_{i_p} L(c, p)\) (see \(3.10\)). By the induction hypothesis, we have

\[
L(c, p - 1) = G(c, p - 1) + \sum_{c \prec p} a_{c,c'}^{p-1} G(c', p - 1).
\]
We apply the composition $T_{i_p}^{-1} \circ \pi^p$ to both sides. We have $L(c, p - 1) \in \mathfrak{U}^+[i_p]$, so the left hand side becomes $L(c, p)$. Therefore

$$L(c, p) = T_{i_p}^{-1}(\pi^p(G(c, p - 1))) + \sum_{c < p^{-1}c'} a^p_{c, c'} T_{i_p}^{-1}(\pi^p(G(c', p - 1))).$$

By [5.3], the right hand side is contained in

$$G(c, p) + \sum_{c < p^{-1}c'} a^p_{c, c'} G(c', p) + E_{i_p} \mathfrak{U}^+.$$

The condition $c'(p) = 0$ comes from $\pi^p(G(c', p - 1)) \neq 0$ (see [5.4]). The part in $E_{i_p} \mathfrak{U}^+$ is a linear combination of $G(c'', p)$'s with $c''(p) > 0$. Each such $c''$ is greater than $c$ with respect to $<_p$. The summation in the second term can be replaced as $\sum_{c < p^{-1}c'} G(c, p) = 0$. Thus we have the assertion under the assumption $c(p) = 0$.

Next we assume $c(p) > 0$. Let us define $\tilde{c}$ by setting $\tilde{c}(p) = 0$ and all other entries are the same as $c$. We have

$$L(c, p) = E_i^{(c(p))} L(\tilde{c}, p).$$

Since $\tilde{c}(p) = 0$, we have just proved

$$L(\tilde{c}, p) = G(\tilde{c}, p) + \sum_{c' > p \tilde{c}} a^p_{c, c'} G(c', p).$$

By [23.14.3], $E_i^{(c(p))} G(\tilde{c}, p) = G(c, p)$ plus an element in $E_i^{(p + 1)} \mathfrak{U}^+$. If we write the element in a linear combination of $G(c'', p)$'s, all elements appearing satisfy $c'' >_p c$ by [5.3]. Next consider $E_i^{(c(p))} G(c', p)$, obtained from [5.7] by multiplication by $E_i^{(c(p))}$. If $c'(p) > 0$, then this is an element in $E_i^{(p + 1)} \mathfrak{U}^+$. Otherwise, it is equal to $G(c', p)$ plus an element in $E_i^{(p + 1)} \mathfrak{U}^+$, where $c'$ is defined by setting $c'(p) = c(p)$ and all other entries are the same as $c'$. In either cases, it is a linear combination of elements $G(c''', p)$ with $c''' >_p c$. Thus we have the assertion for $G(c, p)$.

Finally we consider the general case. We first remark

$$c(p), c(p - 1), c(p - 2), \ldots \leq (c'(p), c'(p - 1), c'(p - 2), \ldots).$$

This is proved by the induction on $q$ such that $c(p - q) = c(p - q - 1) = \cdots = 0$. When $q = 0$, the left hand side is $(0, 0, \ldots)$, so the inequality trivially holds. The remaining argument of the induction is exactly the same as above. Now we can prove the assertion of the lemma by the induction on $q$ such that $c(p + q) = c(p + q + 1) = \cdots = 0$. When $q = 0$, we are reduced to the case studied above, and have the assertion. The remaining argument is almost the same as above. When $c(p + 1) = 0$, we apply the induction hypothesis to $L(c, p + 1)$ and consider $T_{i_{p+1}} \pi L(c, p + 1)$. We use [5.3] to get

$$L(c, p) \in G(c, p) + \sum_{c < p^{-1}c'} a^{p+1}_{c, c'} G(c', p) + \mathfrak{U}^+ E_{i_{p+1}}.$$

The summation in the second part can be replaced as $\sum_{c < p^{-1}c'} G(c, p) = 0$ by [5.12]. The part in $\mathfrak{U}^+ E_{i_{p+1}}$ is a linear combination of $G(c''', p)$'s with $c'''(p + 1) > 0$. Since we already have (5.8) (with $c'$ replaced by $c''$), we have $c <_p c''$. We have
the assertion in the case \( c(p + 1) = 0 \). The case \( c(p + 1) > 0 \) can be reduced to the case \( c(p + 1) = 0 \) as above with use of (5.8).

6. Cell structure of \( \tilde{U} \).

In this section we prove Lusztig’s conjecture \[24\] on two–sided cells of the modified quantum affine algebra of level 0. Our strategy follows the same line as his proof of the corresponding conjecture for finite type cases. However, here we need to show that a certain bi-module \( \tilde{U}[\lambda] \) has a global crystal basis, whereas for finite type cases this assertion is a direct consequence of the definition. In defining \( \tilde{U}[\lambda] \), our earlier results on extremal weight modules play a key role. Their role is analogous to the role that dominant highest weight representations play in the finite type case.

Let \( \tilde{U} \) be the level 0 modified quantum affine algebra, i.e., \( \tilde{U} = \bigoplus_{\lambda \in P_{cl}^0} U_{a\lambda} \). Unfortunately this contradicts the notation in Sect. \[4\] where \( \tilde{U} = \bigoplus_{\lambda \in P} U_{a\lambda} \), but we do not want to introduce new notation. Let \( \mathcal{L} \) be its crystal lattice, \( \tilde{B} \) be its crystal base.

Throughout this section, \( \lambda \) is a dominant classical level 0 weight, i.e., \( \lambda \in P_{cl,+}^0 = \sum_{i \in I_0} \mathbb{Z}_{\geq 0} cl(\varpi_i) \), except in the proof of Theorem 6.29. Let \( V(\lambda) \) be the extremal weight module of weight \( \lambda \). A priori, it is defined for \( \lambda \in P^0 \), but its \( \tilde{U} \)-module structure depends only on \( cl(\lambda) \).

6.1. A bi-module \( \tilde{U}[\lambda] \).

**Definition 6.1.** For \( b \in B(\lambda) \), we denote by \( G_\lambda(b) \) the corresponding element in the global basis of \( V(\lambda) \). If we consider \( b \) as an element of \( \tilde{B} \), we denote by \( G(b) \) the corresponding element in \( \tilde{U} \).

Denote \( \# = * \circ \vee \). It is an involutive anti-automorphism of \( \tilde{U} \). By \[17, 4.3.2\] it leaves the global crystal basis of \( \tilde{U} \) invariant. (The result was proved for \( * \), but the same proof works for \( \vee \).) We have

**Lemma 6.2** \([23, 19.1.1], [24, 3.7]\). Let \( x \in a_\lambda \tilde{U}a_{\lambda'} \) and \( \nu = \lambda - \lambda' \). Then

\[
\psi(x) = q^{-(\nu, \lambda + \lambda')/2} x^\#.
\]

We denote by \( \leq \) the dominance partial order on the level 0 classical weights relative to the fundamental level zero weights \( cl(\varpi_i) \in P_{cl,+}^0 \) defined in §2.6.

Let us give a slightly different parametrization of \( B(\lambda) \) from that of Sect. \[4\]. Let \( W(\lambda), B_W(\lambda) \) be as in the proof of Lemma \[4.14\]. We have maps

\[
B_0(\lambda) \xrightarrow{\Phi_{\lambda|_{\nu=0}}} \{ \text{monomials in } z_{i,\nu}^{\pm} \} \times B_W(\lambda) \xrightarrow{\text{projection}} B_W(\lambda).
\]

The composition is surjective since \( B_W(\lambda) \) is connected by \[4\] Lemmas 1.9, 1.10]. By Lemma \[4.14\] each fiber is identified with the set of monomials \( p(z) = \prod (z_{i,1} \cdots z_{i,\lambda_i})^{r_i} \) \( (r_i \in \mathbb{Z}) \). We choose and fix a section \( B_W(\lambda) \rightarrow B_0(\lambda) \). (We do not require that it respect the crystal structure.) Then we have an identification (of sets, not of crystals)

\[
B(\lambda) \cong \text{Irr } G_\lambda \times B_W(\lambda),
\]
where \( G_\lambda = \prod GL_{\lambda_i}(\mathbb{C}) \), and \( \text{Irr}_\lambda \) is the set of irreducible representations of \( G_\lambda \). We identify \( \text{Irr}_\lambda \) with the set of Schur Laurent polynomials in \( \{ z_i^\pm \} \). We denote by \( s(z) \) the polynomial corresponding to \( s \).

**Definition 6.5.** For \( s \in \text{Irr}_\lambda \) we define \( S \in G(B(\lambda)) \) by \( \Phi_\lambda(S u_\lambda) = s(z) \tilde{u}_\lambda \). The existence and uniqueness are guaranteed in Theorem 4.16(ii). If \( S \) is the dual of a polynomial representation and corresponds to \( c_0 \), we have \( S = S_{c_0} - u_\lambda \) by Proposition 4.10. We also regard \( S \) as an element in \( G(B(Ua_\lambda)) \).

We will use the correspondence between \( s \leftrightarrow S \) hereafter.

**Definition 6.6.** Let \( \tilde{U}[\geq \lambda] \) (resp. \( \tilde{U}[\lambda] \)) be the two–sided ideal of \( \tilde{U} \) consisting of all elements \( x \in \tilde{U} \) acting on \( V(\lambda') \) by \( 0 \) for any \( \lambda' \not\geq \lambda \) (resp. \( \lambda' \not\geq \lambda \)). Let \( \tilde{U}[\lambda] = \tilde{U}[\geq \lambda]/\tilde{U}[\geq \lambda] \). We define \( \Lambda \tilde{U}[\geq \lambda], \Lambda \tilde{U}[\lambda] \) and \( \Lambda \tilde{U}[\lambda] \) in an analogous manner.

For \( \xi, \xi' \in P_+ \) let \( \tilde{U}[\geq \lambda]_{\xi,\xi'} \) (resp. \( \tilde{U}[\lambda]_{\xi,\xi'} \)) be the image of \( \tilde{U}[\geq \lambda] \) (resp. \( \tilde{U}[\lambda] \)) under the natural map \( \tilde{U} \to V(\xi) \otimes V(-\xi') \) given by \( x \mapsto x(u_\xi \otimes u_{-\xi'}) \). Let \( \tilde{U}[\lambda]_{\xi,\xi'} = \tilde{U}[\geq \lambda]_{\xi,\xi'}/\tilde{U}[\lambda]_{\xi,\xi'} \).

**Lemma 6.7.** (i) \( \tilde{U}[\geq \lambda] \) is invariant under \( \overline{\cdot} \).

(ii) For any \( \xi, \xi' \in P_+ \), the \( \tilde{U} \)-submodules \( \tilde{U}[\geq \lambda]_{\xi,\xi'} \) and \( \tilde{U}[\lambda]_{\xi,\xi'} \) of \( V(\xi) \otimes V(-\xi') \) are invariant under \( \tilde{F}_i^{(n)} \) defined in §2.5.

**Proof.** (i) The bar involution \( \overline{\cdot} \) on \( V(\lambda) \) satisfies \( \pi \pi = \pi \pi \) for \( x \in \tilde{U} \), \( u \in V(\lambda) \).

The assertion follows.

(ii) Let \( x \in \tilde{U}[\geq \lambda] \). We have

\[
\tilde{F}_i^{(n)}(x)(u_\xi \otimes u_{-\xi'}) = \sum_k F_i^{(n+k)}(t_i) a_k^0(t_i) x(u_\xi \otimes u_{-\xi'}),
\]

where all but finitely many terms of the sum are 0. Since each \( F_i^{(n+k)}(t_i) a_k^0(t_i) x \) is contained in \( \tilde{U}[\geq \lambda] \), we have the assertion. \( \square \)

**Lemma 6.8** (cf. [24, 4.5]). (i) \( Ua_\lambda, a_\lambda U \subseteq \tilde{U}[\geq \lambda] \).

(ii) For \( b \in B(Ua_\lambda) \), \( G(b) \in \tilde{U}[\geq \lambda] \) if and only if \( b \notin B(\lambda) \).

**Proof.** (i) Let \( xa_\lambda \in Ua_\lambda \). If \( xa_\lambda \) acts on \( V(\lambda') \) by a nonzero map, then so does \( a_\lambda \).

Since \( a_\lambda \) is a projector to the weight space with weight \( \lambda \), \( \lambda \) is a weight of \( V(\lambda') \).

Therefore we have \( \lambda \leq \lambda' \). Similarly, we have \( a_\lambda U \subseteq \tilde{U}[\geq \lambda] \).

(ii) Let us first remark that \( V(\lambda) \) has a basis \( \{ SU_\lambda \mid s \in \text{Irr}_\lambda \} \). This follows from results in Sect. 4.1. Note that \( \lambda \in P^0_\lambda \) in this section, and the weight space is a direct sum of weight spaces \( V(\lambda)_{\mu} \) with \( \lambda(\mu) = \lambda \).

Let \( b \in B(Ua_\lambda) \). Then \( G(b) = 0 \) as an operator on \( V(\lambda) \) if and only if \( G(b)SU_\lambda = 0 \) for any \( s \in \text{Irr}_\lambda \). This is equivalent to \( \Phi_\lambda(G(b)SU_\lambda) = s(z) \Phi_\lambda(G(b)U_\lambda) = 0 \), and in particular \( G(b)U_\lambda = 0 \). If \( b \notin B(\lambda) \), then \( G(b)U_\lambda = 0 \) is nothing but \( G(b) \).

In particular, it is nonzero. If \( b \notin B(\lambda) \), then \( G(b)U_\lambda = 0 \) by the definition of \( V(\lambda) \). \( \square \)

Let \( \langle \ , \rangle_\lambda \) be the bilinear form on \( V(\lambda) \) satisfying \( (xu, v)_\lambda = (u, v(x)_0)_\lambda \) for \( u, v \in V(\lambda), x \in \tilde{U} \) and \( (u_\lambda, G_\lambda(b))_\lambda = 1 \) or 0 according to \( G_\lambda(b) = u_\lambda \) or not (see Proposition 2.17).

**Proposition 6.9.** \( \tilde{U}[\geq \lambda] \) is invariant under \#.
Proof. Since \( \tilde{U}[\geq \lambda] \) is a direct sum of its intersection with \( a_\lambda U a_\lambda' \), by (6.3) it is enough to show that \( x \in \tilde{U}[\geq \lambda] \) if and only if \( \psi(x) \in \tilde{U}[\geq \lambda] \). The global crystal basis is almost orthonormal, that is (Remark 6.10). In particular, it is non-degenerate on \( V(\lambda) \). Therefore, \( x \in \text{Ann} V(\lambda) \iff (xG_\lambda(b), G_\lambda(b')) \) for all \( b, b' \in B(\lambda) \iff (G_\lambda(b), \psi(x)G_\lambda(b')) = 0 \iff \psi(x) \in \text{Ann} V(\lambda) \). It follows \( x \in \tilde{U}[\geq \lambda] \) if and only if \( \psi(x) \in \tilde{U}[\geq \lambda] \). □

Remark 6.10. Since Proposition 6.8 implies \( x \in \text{Ann} V(\lambda) \) if and only if \( x' \in \text{Ann} V(\lambda) \), Lemma 6.8 implies that \( b \in B(\lambda) \) if and only if \( G(b) \# \in \tilde{U}[\geq \lambda] \setminus \tilde{U}[\geq \lambda] \) for \( b \in \tilde{B}(U a_\lambda) \).

Note that \( \tilde{U}[\lambda] = \tilde{U}[\geq \lambda]/\tilde{U}[\geq \lambda] \) is integrable as a \( \tilde{U} \)-module by Theorem 2.13. In particular, the operators \( T_w \) are defined.

Lemma 6.11. Let \( b \in B(\lambda) \). Then \( G(b)^\# \) mod \( \tilde{U}[\geq \lambda] \) is an extremal vector in the \( \tilde{U} \)-module \( \tilde{U}[\lambda] \).

Proof. We show that \( S_{i_1} \cdots S_{i_N} G(b)^\# \) mod \( \tilde{U}[\geq \lambda] \) is \( i \)-extremal for all \( i \in I \) by induction on \( N \).

If \( N = 0 \), we need to show that \( e_i G(b)^\# \) or \( f_i G(b)^\# \) acts on \( V(\lambda) \) by 0 for each \( i \in I \). In other words if and only if \( e_i G(b)^\# \) or \( f_i G(b)^\# \) are 0. Note that \( G(b)^\# = a_\lambda G(b)^\# \) and \( a_\lambda \) is a projector to the weight space of weight \( \lambda \). Since \( V(\lambda) \) has a basis \( \{Su_\lambda \mid s \in \text{Irr } G_\lambda \} \) consisting of extremal vectors, we are done. Moreover, we have \( S_G(b)^\# \) mod \( \tilde{U}[\geq \lambda] = a_{s_\lambda} S_G(b)^\# \) mod \( \tilde{U}[\geq \lambda] \).

We show the statement for \( N \) assuming the statement for \( N - 1 \). By the induction hypothesis and the last part of the above argument, \( S_{i_1} \cdots S_{i_N} G(b)^\# \) mod \( \tilde{U}[\geq \lambda] \) is \( i \)-extremal for \( i \in I \) and

\[
S_{i_1} \cdots S_{i_N} G(b)^\# \text{ mod } \tilde{U}[\geq \lambda] = a_{s_{i_1}} \cdots a_{s_{i_N}} S_{i_1} \cdots S_{i_N} G(b)^\# \text{ mod } \tilde{U}[\geq \lambda].
\]

Since \( V(\lambda) \sum_{i=1}^N a_{s_i} \) is spanned by extremal vectors, we are done. □

By Lemma 6.11 for each \( b \in B(\lambda) \) we have a unique \( \tilde{U} \)-homomorphism

\[
\Psi_b : V(\lambda) \to \tilde{U}[\lambda],
\]

which sends \( u_\lambda \) to \( G(b)^\# \) mod \( \tilde{U}[\geq \lambda] \). By definition, we have

\[
\Psi_b (G_\lambda(b_1)) = G(b_1)G(b)^\# \text{ mod } \tilde{U}[\geq \lambda].
\]

Lemma 6.12. Let \( b_1, b_2 \in B(\lambda) \). Then

\[
G(b_1)^\# G(b_2) \equiv q^n \sum_{s \in \text{Irr } G_\lambda} (G_\lambda(b_2), G(b_1)Su_\lambda) S_{u_\lambda} \text{ mod } \tilde{U}[\geq \lambda],
\]

where \( n = (\text{wt } b_1, 2\lambda + \text{wt } b_1)/2 \) and \( S \) is an element corresponding to \( s \) by Definition 6.3.

Proof. Since \( \tilde{U}[\geq \lambda] \) is a two-sided ideal of \( \tilde{U} \), both sides are in \( \tilde{U}[\geq \lambda] \). Therefore it is enough to show that both sides define the same operator on \( V(\lambda) \). Since we have \( G(b_1)^\# G(b_2)u_\lambda = a_{s_\lambda} G(b_1)^\# G(b_2)u_\lambda \), it is contained in the weight space \( V(\lambda) \). We have a basis \( \{Su_\lambda \} \) which is orthonormal by (6.3) Proposition 4.10. Hence we have

\[
G(b_1)^\# G(b_2)u_\lambda = \sum_{s \in \text{Irr } G_\lambda} (G(b_1)^\# G(b_2)u_\lambda, Su_\lambda) S_{u_\lambda}.
\]

Now the assertion follows from (6.3). □
Lemma 6.13. Let $S$ correspond to $s \in \text{Irr } G_{\lambda}$ as in Definition 6.3. We consider it as an element in $G(B(Ua_\lambda))$. Then $S^\#$ corresponds to the dual representation of $s$, which will be denoted by $s^\#$ hereafter.

Proof. By Lemma 6.2 we have $(S^\# u_\lambda, S_1 u_\lambda) = (u_\lambda, SS_1 u_\lambda)$ for any $s_1 \in \text{Irr } G_{\lambda}$. By (3.4.10) we have $(u_\lambda, SS_1 u_\lambda) = (\Phi_{\lambda}(S u_\lambda), \Phi_{\lambda}(SS_1 u_\lambda)) = (\tilde{u}_\lambda, s(z) s_1(z) \tilde{u}_\lambda)^\sim$, where $(\ , \ )^\sim$ is a bilinear form on $\tilde{V}(\lambda)$ defined in [loc.cit. §4]. By a property of $(\ , \ )^\sim$, we have

$$(\tilde{u}_\lambda, s(z) s_1(z) \tilde{u}_\lambda)^\sim = (s(z^{-1}) \tilde{u}_\lambda, s_1(z) \tilde{u}_\lambda)^\sim = \begin{cases} 1 & \text{if } s(z^{-1}) = s_1(z), \\ 0 & \text{otherwise}. \end{cases}$$

(See an equation in the middle of [loc.cit., Proof of 4.9] and [loc.cit., Proof of 4.10]). Since $\{S u_\lambda\}$ is an orthonormal basis of $\tilde{V}(\lambda)$, this means $S^\# u_\lambda$ corresponds to the Schur Laurent polynomial $s(z^{-1})$, which corresponds to the dual representation of $s$ in turn. \qed

Lemma 6.14. Let $(s, b) \in \text{Irr } G_{\lambda} \times B_{\tilde{W}}(\lambda)$ and let $\tilde{b}$ be the corresponding element in $B(\lambda)$ under the identification (6.4). We have

$$G(b) S \equiv G(\tilde{b}) \mod \tilde{U}[\lambda], \quad S^\# G(b)^\# \equiv G(\tilde{b})^\# \mod \tilde{U}[\lambda].$$

Proof. By the construction of (6.4), $\Phi_{\lambda}(G(\tilde{b}) u_\lambda) = s(z) \Phi_{\lambda}(G_{\lambda}(b)) = s(z) G(b) \tilde{u}_\lambda$. By Definition 6.3, this is equal to $G(\tilde{b}) \Phi_{\lambda}(S u_\lambda) = \Phi_{\lambda}(G(b) S u_\lambda)$. Therefore we have $G(b) S u_\lambda = G(\tilde{b}) u_\lambda$.

Let us consider the first equation of the assertion. Since both sides of the equation are in $U a_\lambda$, the result follows from Lemma 6.8(ii) if we prove that they define the same operators on $V(\lambda)$. Since $V(\lambda)$ is spanned by $\{S_1 u_\lambda \mid s_1 \in \text{Irr } G_{\lambda}\}$, it is enough to show $G(b) S_1 u_\lambda = G(\tilde{b}) S_1 u_\lambda$ for every $S_1$. Consider the embedding $\Phi_{\lambda} : V(\lambda) \hookrightarrow \tilde{V}(\lambda)$ as in Corollary 4.13. By Proposition 4.10 and the fact that $s_{\alpha_0}(z^{-1})$ is $U$-linear, we have

$$\Phi_{\lambda}(G(b) S_1 u_\lambda) = s_1(z^{-1}) \Phi_{\lambda}(G(b) S u_\lambda) = s_1(z^{-1}) \Phi_{\lambda}(G(\tilde{b}) u_\lambda) = \Phi_{\lambda}(G(\tilde{b}) S' u_\lambda).$$

We get the assertion by the injectivity of $\Phi_{\lambda}$.

The second equation follows from the first together with Proposition 6.3. \qed

Lemma 6.15. Let $s_1, s_2 \in \text{Irr } G_{\lambda}$. Let $S_1, S_2 \in G(B(Ua_\lambda))$ be the corresponding elements by Definition 6.3. We have

$$S_1 S_2 \equiv \sum_s c_{s_1 s_2}^s S \mod \tilde{U}[\lambda],$$

where $s_1 s_2 = \sum_s c_{s_1 s_2}^s s$.

Proof. Since both sides are in $\tilde{U}[\lambda]$, it is enough to show that they define the same operator on $V(\lambda)$. As in the proof of Lemma 6.14, it follows from $S_1 S_2 u_\lambda = \sum_s c_{s_1 s_2}^s S u_\lambda$. But this follows directly from Definition 6.3. \qed

Definition 6.16. For $b, b' \in B_{\tilde{W}}(\lambda), s \in \text{Irr } G_{\lambda}$ we set

$$G_{\lambda}(b, s, b') = G(b) S G(b')^\# \mod \tilde{U}[\lambda] = \Psi_{b'}(G(b) S u_\lambda) \in \tilde{U}[\lambda].$$

The following is a direct consequence of Lemma 6.12.
Lemma 6.17. We have
\[ G_\lambda(b_1, s_1, b_1')G_\lambda(b_2, s_2, b_2') = q^n \sum_{s'' \in \text{Irr } G_\lambda} (G(b_2)S_2u_{\lambda}, G(b_1')S''u_{\lambda})_\lambda G(b_1)S_1S''G(b_2') \mod \tilde{U}^{[\lambda]}, \]
where \( n = (\text{wt } b_1', 2\lambda + \text{wt } b_1')/2. \)

Lemma 6.18. \( \{G_\lambda(b, s, b') \mid b, b' \in B_\lambda(\lambda), s \in \text{Irr } G_\lambda\} \) is linearly independent.

Proof. Suppose that \( \sum_{b, b', s} a_{b, b', s}G(b)SG(b')^\# \) acts on \( V(\lambda) \) by 0. Then for any \( b_1 \in B(\lambda) \)
\[ 0 = \sum_{b, b', s} a_{b, b', s}G(b)SG(b')^\#G_\lambda(b_1) \]
\[ = \sum_{b, b', s} a_{b, b', s}G(b) (G(b')S')^\#G_\lambda(b_1) \]
\[ = \sum_{b, b', s, s'} a_{b, b', s}q^{n(b')} (G_\lambda(b_1), G(b')S' S'u_{\lambda})_\lambda G(b)S'u_{\lambda}, \]
where \( n(b') = (\text{wt } b', 2\lambda + \text{wt } b')/2. \) Here we have used Lemma 6.12 in the third equality. Since \( \{G(b)S'u_{\lambda} \mid b \in B_\lambda(\lambda), s' \in \text{Irr } G_\lambda\} \) is linearly independent, we have
\[ \sum_{b, b', s} a_{b, b', s}q^{n(b')} (G_\lambda(b_1), G(b')S' S'u_{\lambda})_\lambda = 0 \]
for any \( b, b_1, s'. \) This equality for \( s' = 1, \) together with the nondegeneracy of \( (\ , \ )_\lambda \) and linearly independence of \( \{G(b)S' u_{\lambda}\} \) imply \( a_{b, b', s} = 0. \)

Definition 6.19. For \( \xi, \xi' \in P_+ \), let \( \tilde{\Lambda}(\xi, \xi') = (\tilde{\Lambda}(\xi) \otimes \tilde{\Lambda}(\xi')) \cap \tilde{U}^{[\lambda]} \).
Let \( \tilde{\Lambda}[\lambda] \) be the \( A_0 \)-submodule of \( \tilde{U}[\lambda] \) consisting of elements whose images under \( \tilde{U}[\lambda] \rightarrow \tilde{U}[\xi, \xi'] \) are in \( \tilde{\Lambda}[\lambda] \).

Lemma 6.20. \( \Psi_{b'}(\lambda(\xi)) \subset \tilde{\Lambda}[\lambda] \) for \( b' \in B_\lambda(\lambda) \).

Proof. The homomorphism \( \Psi_{b'} : V(\lambda) \rightarrow \tilde{U}[\lambda] \) intertwines the operators \( \tilde{e}_i, \tilde{f}_i \).
Since \( \tilde{\Lambda}[\lambda] \) is invariant under \( \tilde{e}_i, \tilde{f}_i \) (see Lemma 6.7(ii)), it is enough to show that \( \Psi_{b'}(Su_{\lambda}) \in \tilde{\Lambda}[\lambda] \). This follows from Lemma 6.14 as \( \Psi_{b'}(Su_{\lambda}) = SG(b')^\# \mod \tilde{U}^{[\lambda]} \).

Most importantly, we have \( G_\lambda(b, s, b') \in \tilde{\Lambda}[\lambda] \). Also more generally, Lemma 6.20 holds for \( b' \in B(\lambda) \) thanks to Lemma 6.15.

Definition 6.21. Let \( \tilde{B}[\lambda] \) be the subset of \( \tilde{B} \) consisting of elements which are connected to \( B(\lambda)^\# \) in the crystal graph. It has an induced bi-crystal structure from \( \tilde{B} \). By Theorem 6.10 \( \tilde{B}[\lambda] = B[\mu] \) if \( \mu \in W\lambda, B[\lambda] \cap B[\mu] = \emptyset \) otherwise. In particular, \( \tilde{B}[\lambda] (\lambda \in P^{\text{cl}}_{\text{cl}}) \) are mutually disjoint.
Proposition 6.23. There exists an isomorphism of bi-crystals
\[ \tilde{B}[\lambda] \cong B_W(\lambda) \times \text{Irr} G_\lambda \times B_W(\lambda) \]
where \( \tilde{e}_i, \tilde{f}_i, \tilde{e}_i^#, \tilde{f}_i^# \) of the right hand side are defined by
\[ \tilde{e}_i(b, s, b') = (\tilde{e}_i(b, s), b'), \quad \tilde{f}_i(b, s, b') = (\tilde{f}_i(b, s), b'), \]
\[ \tilde{e}_i^#(b, s, b') = (b, (id \times #) \circ \tilde{e}_i \circ (id \times #)(b'), s), \]
\[ \tilde{f}_i^#(b, s, b') = (b, (id \times #) \circ \tilde{f}_i \circ (id \times #)(b'), s), \]
where we use the identification \( \tilde{B}(\lambda) \cong B_W(\lambda) \times \text{Irr} G_\lambda \) of \([14]\). Here \( s^# \) denotes the dual representation of \( s \).

Proof. There is a set-theoretical bijection between the right hand side and the subset
\[ \{ G_{\lambda}(b, s, b') \mod q_s \tilde{L}[\lambda] \mid b, b' \in B_W(\lambda), s \in \text{Irr} G_\lambda \} \subset \tilde{L}[\lambda]/q_s \tilde{L}[\lambda]. \]
The \( \tilde{U} \)-module structure on \( \tilde{U}[\lambda] \) defines operators \( \tilde{e}_i, \tilde{f}_i \) on \( \tilde{L}[\lambda]/q_s \tilde{L}[\lambda] \). If we write
\[ G_{\lambda}(b, s, b') = G(b) G(b')^# \mod \tilde{U}[\lambda] = \Psi_{b'}(G\lambda(b)), \]
with \( b = (b, s) \in B(\lambda) \) defined by \([6, 4]\), then
\[ X_i G_{\lambda}(b, s, b') = \Psi_{b'}(X_i G\lambda(b)) \equiv \Psi_{b'}(G\lambda(X_i b)) \mod q_s \tilde{L}[\lambda], \]
for \( X_i = \tilde{e}_i \) or \( \tilde{f}_i \). Here we have used Lemma \([6, 20]\) for the second equality. This shows that the bijection respects the crystal structure. The identification of \#-crystal structure follows from the above discussion and
\[ G_{\lambda}(b, s, b')^# = G(b') S^# G(b)^# \mod \tilde{U}[\lambda], \]
together with Lemma \([6, 13]\). We identify \( B_W(\lambda) \times \text{Irr} G_\lambda \times B_W(\lambda) \) with \([6, 24]\) hereafter.

Consider the natural \( \mathbb{Q} \)-linear map
\[ \pi: \tilde{L}[\lambda]/q_s \tilde{L}[\lambda] \to \tilde{L}[\lambda]/q_s \tilde{L}[\lambda], \]
induced by the projection \( \tilde{U}[\lambda] \to \tilde{U}[\lambda] \). Recall that the Kashiwara operators \( \tilde{e}_i, \tilde{f}_i \) on \( \tilde{L}/q_s \tilde{L} \) are defined so that they are compatible with projections \( \tilde{L}/q_s \tilde{L} \to \tilde{L}(\xi) \otimes \tilde{L}(-\xi') \) induced from \( \tilde{U} \to V(\xi) \otimes V(-\xi') \) for all \( \xi, \xi' \in P_+ \) (see \([17]\, Theorem 2.1.2\) and \([23]\, Part IV\)). Since \( \tilde{L}[\xi, \xi'] \equiv \tilde{L}[\lambda]/q_s \tilde{L}[\lambda] \) is invariant under \( \tilde{F}^{(\xi)} \) by Lemma \([6, 7]\, ii)\), the subspace \( \tilde{L}[\lambda]/q_s \tilde{L}[\lambda] \subset \tilde{L}/q_s \tilde{L} \) is invariant under \( \tilde{e}_i, \tilde{f}_i \). The map \( \pi \) intertwines \( \tilde{e}_i, \tilde{f}_i \) as \( \tilde{L}[\lambda]/q_s \tilde{L}[\lambda] \rightarrow \tilde{L}[\lambda]/q_s \tilde{L}[\lambda] \) does so. It also intertwines \#, and hence \( \tilde{e}_i^#, \tilde{f}_i^# \).

Since \( G(b)^# \in \tilde{U}[\lambda] \) for \( b \in B(\lambda) \), \( B[\lambda] \) is contained in \( \tilde{L}[\lambda]/q_s \tilde{L}[\lambda] \) as it is invariant under \( \tilde{e}_i, \tilde{f}_i \). We have \( \pi(b^#) = (u_{\lambda}, s, b') \), where \( b = (s^#, b') \) under \([6, 4]\). Therefore we have
\[ \pi(B[\lambda]) \subset B_W(\lambda) \times \text{Irr} G_\lambda \times B_W(\lambda) \cup \{0\}. \]
Consider $\text{Ker} \pi \cap \tilde{B}[\lambda]$. It is invariant under $\tilde{e}_i$, $\tilde{f}_i$, so every connected component contains an extremal vector, and in particular an element in $\mathcal{B}(\lambda)^\#$. But $\mathcal{B}(\lambda)^\#$ is mapped bijectively to $\{u_\lambda\} \times \text{Irr} G_\lambda \times \mathcal{B}_W(\lambda)$ as we mentioned already. So $\text{Ker} \pi \cap \tilde{B}[\lambda]$ is the empty set. Thus we have a map

$$\pi|_{\tilde{B}[\lambda]} : \tilde{B}[\lambda] \to \mathcal{B}_W(\lambda) \times \text{Irr} G_\lambda \times \mathcal{B}_W(\lambda).$$

It is enough to show that this map is bijective since it is clear that bi-crystal operators are intertwined.

Note that any element of $\mathcal{B}_W(\lambda) \times \text{Irr} G_\lambda \times \mathcal{B}_W(\lambda)$ can be connected to a point in $\{u_\lambda\} \times \text{Irr} G_\lambda \times \mathcal{B}_W(\lambda)$ in the crystal graph, since the assertion is so for $\mathcal{B}(\lambda)$ (Theorem 1.10). Since $\{u_\lambda\} \times \text{Irr} G_\lambda \times \mathcal{B}_W(\lambda)$ is equal to $\pi(\mathcal{B}(\lambda)^\#)$, the map $\pi|_{\tilde{B}[\lambda]}$ is surjective.

Suppose that $\tilde{b}, \tilde{b}' \in \tilde{B}[\lambda]$ satisfy $\pi(\tilde{b}) = \pi(\tilde{b}')$. We may assume that $\tilde{b} \in \mathcal{B}(\lambda)^\#$. The condition $\pi(\tilde{b}) = \pi(\tilde{b}')$ and $\text{Ker} \pi \cap \tilde{B}[\lambda] = \emptyset$ imply $\tilde{b}'$ is an extremal vector with the weight $\text{wt} \tilde{b}' = \text{wt} \tilde{b}$. This means $\tilde{b}' \in \mathcal{B}(\lambda)^\#$. Since $\pi$ is bijective on $\mathcal{B}(\lambda)^\#$, we have $\tilde{b} = \tilde{b}'$. □

Hereafter, we identify $\tilde{B}[\lambda]$ with $\mathcal{B}_W(\lambda) \times \text{Irr} G_\lambda \times \mathcal{B}_W(\lambda)$ by Proposition 6.23.

**Lemma 6.25.** $\tilde{B} = \bigsqcup_{\lambda \in P_{cl,+}^0} \tilde{B}[\lambda]$.

**Proof.** Each connected component of $\tilde{B}$ contains an extremal vector [4], 9.3.4]. By definition, the set of extremal vectors is equal to $\bigsqcup_{\lambda \in P_{cl,+}^0} \mathcal{B}(\lambda)^\# = \bigsqcup_{\lambda \in P_{cl,+}^0} \mathcal{B}(\lambda)^\#.$

Furthermore, we have an isomorphism of crystals $S_w^* : \mathcal{B}(\lambda) \xrightarrow{\cong} \mathcal{B}(w\lambda)$ for $w \in \hat{W}$. Therefore each component contains a vector in $\mathcal{B}(\lambda)^\#$ with $\lambda \in P_{cl,+}^0$. Therefore $\tilde{B} = \bigsqcup_{\lambda \in P_{cl,+}^0} \tilde{B}[\lambda]$. By the remark in Definition 6.21, this is a disjoint union. □

For $\xi, \xi' \in P_+$, the kernel of the surjective homomorphism $\tilde{U}_\alpha_{\xi-\xi'} \to V(\xi) \otimes V(-\xi')$ is the left ideal generated by $f_i^{[\xi, h_i] + 1} a_{\xi-\xi'}$ and $e_i^{[\xi', h_i] + 1} a_{\xi-\xi'}$. Let $P(\xi, \xi')$ be the set of $\mu \in P_{cl,+}^0$ such that all $f_i^{[\xi, h_i] + 1} a_{\xi-\xi'}$ and $e_i^{[\xi', h_i] + 1} a_{\xi-\xi'}$ act by 0 on $V(\mu')$ for any $\mu' \in P_{cl,+}^0$ satisfying $\mu' \leq \mu$. Therefore a homomorphism $V(\xi) \otimes V(-\xi') \to \text{End} V(\mu)$ is well-defined if $\mu \in P(\xi, \xi')$. Moreover, any $\mu \in P_{cl,+}^0$ is contained in $P(\xi, \xi')$ for sufficiently large $\xi, \xi'$.

Let $\tilde{B}[\mu|_{\xi, \xi'}] = \tilde{B}[\mu] \cap (\mathcal{B}(\xi) \otimes \mathcal{B}(-\xi))$. Then we have $\mathcal{B}(\xi) \otimes \mathcal{B}(-\xi') = \bigsqcup_{\mu} \tilde{B}[\mu|_{\xi, \xi'}]$ by Lemma 6.22.

Let $(b, s, b') \in \tilde{B}[\mu|_{\xi, \xi'}]$. We have $G_\mu(b, s, b')(u_\xi \otimes u_{-\xi'}) \in \tilde{L}[\mu|_{\xi, \xi'}]$ by Lemma 7.20. We choose lifts $\tilde{G}_\mu(b, s, b') \in \tilde{L}[\mu|_{\xi, \xi'}]$. (Note that we do not have $G(b)SG(b')^\# \in \mathcal{L}$ in general.) Then

$$\bigsqcup_{\mu \in P_{cl,+}^0} \{ \tilde{G}_\mu(b, s, b') \mid (b, s, b') \in \tilde{B}[\mu|_{\xi, \xi'}] \}$$

is a $A_{\mu}$-basis of $\mathcal{L}(\xi) \otimes \mathcal{L}(\xi')$, as it induces a $Q$-basis of $\mathcal{L}(\xi) \otimes \mathcal{L}(\xi')/q_\mu(\mathcal{L}(\xi) \otimes \mathcal{L}(\xi'))$. (The transition matrix between this basis and $\mathcal{B}(\xi) \otimes \mathcal{B}(-\xi')$ is upper triangular with 1’s on the diagonal with respect to the block decomposition induced by $\mu$ and the order $\leq$.)
Lemma 6.26. Let $\tilde{G}_\mu(b, s, b') \in \tilde{L}[\mu]_{\xi, \xi'}$ as above. If

$$x = \sum_{\mu \in P^{(+)\mu}_{\xi, \xi'}} \sum_{(b, s, b') \in \tilde{B}[\mu]_{\xi, \xi'}} a^\mu_{b, s, b'} \tilde{G}_\mu(b, s, b') \in \tilde{U}[\tilde{\mu}]_{\lambda, \xi', \xi},$$

then $a^\mu_{b, s, b'} = 0$ if $\mu \not\geq \lambda$ and $\mu \in P(\xi, \xi')$.

Proof. We take any minimal element $\mu_0$ with respect to $\preceq$ among $\mu$’s with $a^\mu_{b, s, b'} \neq 0$ for some $(b, s, b') \in \tilde{B}[\mu]$. Assume $\mu_0 \not\geq \lambda$ and $\mu_0 \in P(\xi, \xi')$. Then $V(\xi) \otimes V(-\xi') \rightarrow \text{End}V(\mu_0)$ is well-defined, and the image of $x$ is $0$ by the assumption. On the other hand, any $\mu$ with $a^\mu_{b, s, b'} \neq 0$ for some $(b, s, b') \in \tilde{B}[\mu]$ satisfies $\mu \not\leq \mu_0$ by the minimality condition. Therefore among terms in $x$, only those summands with $\mu = \mu_0$ act nontrivially on $V(\mu_0)$. Therefore, as operators on $V(\mu_0)$, we have

$$x|_{V(\mu_0)} = \sum_{(b, s, b') \in \tilde{B}[\mu_0]_{\xi, \xi'}} a^\mu_{b, s, b'} \tilde{G}_{\mu_0}(b, s, b')|_{V(\mu_0)} = 0.$$

However the operator $\tilde{G}_{\mu_0}(b, s, b')|_{V(\mu_0)}$ does not depend on the choice of the lift $\tilde{G}_{\mu_0}(b, s, b')$ of $G_{\mu_0}(b, s, b')$, and we have $a^\mu_{b, s, b'} = 0$ for $(b, s, b') \in \tilde{B}[\mu_0]_{\xi, \xi'}$, as $G_{\mu_0}(b, s, b')$ are linearly independent (Lemma 6.18). This is a contradiction. Thus we have $\mu_0 \geq \lambda$ or $\mu_0 \not\in P(\xi, \xi')$. Since $\mu_0$ is any minimal element, this completes the proof.

Proposition 6.27. Let $\beta = (b, s, b') \in \tilde{B}[\lambda]$. Considering $\beta$ as an element of $\tilde{B}$, let $G(\beta)$ be the corresponding element of the global crystal basis of $\tilde{U}$. Then we have

$$G(\beta) \in \tilde{U}[\tilde{\mu}], \quad G(\beta) \bmod \tilde{U}[\tilde{\mu}] = G_\lambda(b, s, b').$$

Proof. We will show the assertion by showing

(a) $G(\beta)(u_{\xi} \otimes u_{-\xi'})$ acts by $0$ on $V(\mu)$ if $\mu \not\geq \lambda$ and $\mu \in P(\xi, \xi')$.

(b) $G(\beta)(u_{\xi} \otimes u_{-\xi'})$ and $G_\lambda(b, s, b')(u_{\xi} \otimes u_{-\xi'})$ define the same operator on $V(\lambda)$ if $\lambda \in P(\xi, \xi')$.

for any $\xi, \xi' \in P_+$. We fix $\xi, \xi'$ hereafter.

Let $(b_1, s_1, b'_1) \in \tilde{B}[\mu]_{\xi, \xi'}$. We choose and fix a lift $\tilde{G}_\mu(b_1, s_1, b'_1) \in \tilde{L}[\mu]_{\xi, \xi'}$ of $G_\mu(b_1, s_1, b'_1)(u_{\xi} \otimes u_{-\xi'})$ as in Lemma 6.26. We write

$$G(\beta)(u_{\xi} \otimes u_{-\xi'}) = \sum_{\mu \in P^{(+)\mu}_{\xi, \xi'}} \sum_{(b, s, b') \in \tilde{B}[\mu]_{\xi, \xi'}} a^\mu_{b_1, s_1, b'_1} \tilde{G}_\mu(b_1, s_1, b'_1)$$

for some $a^\mu_{b_1, s_1, b'_1} \in \mathcal{A}_0$.

Claim. (i) If $\mu \not\geq \lambda$ and $\mu \in P(\xi, \xi')$, then $a^\mu_{b_1, s_1, b'_1} \in q_\lambda \mathcal{A}_0$ for $(b_1, s_1, b'_1) \in \tilde{B}[\mu]_{\xi, \xi'}$.

(ii) If $\lambda \in P(\xi, \xi')$, then $a^\lambda_{b_1, s_1, b'_1} \in \delta_{(b, s, b'), (b_1, s_1, b'_1)} + q_\lambda \mathcal{A}_0$ for $(b_1, s_1, b'_1) \in \tilde{B}[\lambda]_{\xi, \xi'}$.

We can replace $G(\beta)(u_{\xi} \otimes u_{-\xi'})$ by an element which is equal to it modulo $q_\lambda(L(\xi) \otimes L(-\xi'))$ in showing (i) and (ii). Let us write

$$(b, s) = X_1 \cdots X_N(u_\lambda, s_0)$$

for some $X_i = \tilde{e}_j$ or $\tilde{f}_j$. Then $G(\beta)(u_{\xi} \otimes u_{-\xi'})$ is equal to $X_1 \cdots X_N S_0 G(b')^\#(u_{\xi} \otimes u_{-\xi'})$ modulo $q_\lambda(L(\xi) \otimes L(-\xi'))$. We show the assertion for this element. By Lemma 6.7, it is contained in $\tilde{U}[\tilde{\mu}]_{\lambda, \xi', \xi}$. By Lemma 6.26 we have (i).
To show (ii), suppose $\lambda \in P(\xi, \xi')$ and $V(\xi) \otimes V(\xi') \to \text{End} V(\lambda)$ is well-defined. If $\mu \notin P(\xi, \xi')$, then $\lambda \not\preceq \mu$ by the assumption. We combine this with the above discussion to have $a_{b_1,s_1,b'_1}^\mu = 0$ unless $\lambda \not\preceq \mu$. Therefore among terms in (6.28), only those summands with $\mu = \lambda$ act nontrivially on $V(\lambda)$. Thus we have

$$X_1 \cdots X_N S_0 G(b')^\# (u_\xi \otimes u_{-\xi}) = \sum_{(b_1,s_1,b'_1) \in \mathcal{B}[\lambda]} a_{b_1,s_1,b'_1}^\lambda \tilde{G}_\lambda(b_1,s_1,b'_1) \mod \tilde{U}[\lambda].$$

On the other hand, we have

$$X_1 \cdots X_N S_0 G(b')^\# \mod \tilde{U}[\lambda] = \Psi_{b'}(X_1 \cdots X_N S_0 u_\lambda).$$

Since $\Psi_{b'}(\tilde{\xi}(\lambda)) \subset \tilde{\xi}[\lambda]$ by Lemma 6.20, this is equal to $G_\lambda(b,s,b')$ modulo $q_s \tilde{\xi}[\lambda]$. This completes the proof of the claim.

We take any minimal element $\mu_0$ with respect to $\preceq$ among $\mu$'s with $a_{b_1,s_1,b'_1}^\mu \not\equiv 0$ for some $(b_1,s_1,b'_1) \in \mathcal{B}[\mu]$. We obtain a contradiction under the assumption $\mu_0 \not\preceq \lambda$ and $\mu_0 \in P(\xi, \xi')$. As in the proof of Lemma 6.20, only summands with $\mu = \mu_0$ act nontrivially on $V(\mu_0)$. Hence, for $b'' \in \mathcal{B}(\mu_0)$ we have

$$G(\beta)G_{\mu_0}(b'') = \sum_{(b_1,s_1,b'_1) \in \mathcal{B}[\mu_0]} a_{b_1,s_1,b'_1}^\mu \tilde{G}_{\mu_0}(b_1,s_1,b'_1)G_{\mu_0}(b'')$$

$$= \sum_{(b_1,s_1,b'_1) \in \mathcal{B}[\mu_0]} a_{b_1,s_1,b'_1}^\mu G(b_1)S_1 G(b'_1)^\# G_{\mu_0}(b'').$$

From $G(\beta)G_{\mu_0}(b'') = G(\beta)G_{\mu_0}(b'')$, we have

$$0 = \sum_{(b_1,s_1,b'_1) \in \mathcal{B}[\mu_0]} \left(a_{b_1,s_1,b'_1}^\mu - a_{b_1,s_1,b'_1}^\mu \right) G(b_1)S_1 G(b'_1)^\# G_{\mu_0}(b'').$$

Arguing as in the proof of Lemma 6.18, we get

$$a_{b_1,s_1,b'_1}^\mu = a_{b_1,s_1,b'_1}^\mu .$$

However, by the assumption $\mu_0 \not\preceq \lambda$ and $\mu_0 \in P(\xi, \xi')$, we have $a_{b_1,s_1,b'_1}^\mu \in q_s A_0$ using Claim (i). Thus this equality is impossible unless $a_{b_1,s_1,b'_1}^\mu = 0$. This is a contradiction, and we get (a).

Similarly, if $\lambda \in P(\xi, \xi')$, we have $a_{b_1,s_1,b'_1}^\lambda = a_{b_1,s_1,b'_1}^\lambda$. Then Claim (ii) implies $a_{b_1,s_1,b'_1}^\lambda = \delta_{(b,s,b'),(b_1,s_1,b'_1)}$. We get (b).

Now we have

**Theorem 6.29.** (i) $\tilde{U}[\lambda]$ has a crystal base $\left(\tilde{\xi}[\lambda], \mathcal{B}[\lambda]\right)$, where $\mathcal{B}[\lambda]$ is identified with a subset of $\tilde{\xi}[\lambda]/q_s \tilde{\xi}[\lambda]$ as in the proof of Proposition 6.23.

(ii) $\left(\tilde{\xi}[\lambda], \mathcal{L}[\lambda], A \tilde{U}[\lambda]\right)$ is a balanced triple of $\tilde{U}[\lambda]$ and $\{G(b) \mod \tilde{U}[\lambda \mid b \in \mathcal{B}[\lambda]\}$ is its global crystal basis.

**Proof.** By Proposition 6.27, we can take $G(\beta)(u_\xi \otimes u_{-\xi})$ for the lift $\tilde{G}_\lambda(b,s,b')$ in Lemma 6.26. Therefore Lemma 6.26 implies that $\bigcup_{\beta \preceq \lambda} \{G(\beta) \mid \beta \in \mathcal{B}[\mu]\}$ is a basis of $\tilde{U}[\lambda]$. Hence $G(\mathcal{B}[\lambda]) \mod \tilde{U}[\lambda]$ is a basis of $\tilde{U}[\lambda]$. Other axioms of the global crystal basis obviously hold. \qed
6.2. **Cell structure of \( \bar{U} \).** We recall the definition of cells in \( \bar{U} \) with respect to the global basis \( G(\bar{B}) = \{ G(\beta) \mid \beta \in \bar{B} \} \). Let \( \mathcal{F} \) be the collection of all subsets \( K \subset \bar{B} \) with the property: the \( \mathbb{Q}(q_\lambda) \)-subspace of \( \bar{U} \) spanned by \( G(K) \) is a two-sided ideal of \( \bar{U} \). If \( \beta, \beta' \in \bar{B} \), we say that \( \beta \preceq \beta' \) if \( \beta \in \cap_{K \in \mathcal{F}, \beta' \in K} K \). We say that \( \beta \sim \beta' \) if \( \beta \preceq \beta' \) and \( \beta' \preceq \beta \). The equivalence classes of \( \sim \) are called two-sided cells. Similarly, by considering left ideals or right ideals above, we define \( \preceq_L \) or \( \preceq_R \). The equivalence classes are then called left cells or right cells respectively. Another equivalent definition of \( \preceq \) is as follows: Let 

\[
G(\beta)G(\beta') = \sum_{\beta''} c_{\beta\beta'}^{\beta''}(q_\lambda)G(\beta'')
\]

define the structure constants \( c_{\beta\beta'}^{\beta''}(q_\lambda) \in \mathcal{A} \) of \( \bar{U} \) with respect to the global basis. For \( \beta, \beta' \in \bar{B} \) we say \( \beta \preceq_L \beta' \) (resp. \( \preceq_R \)) if there is a sequence \( \beta_1 = \beta', \beta_2, \ldots, \beta_N = \beta \) in \( \bar{B} \) and a sequence \( \gamma_1, \ldots, \gamma_{N-1} \in \bar{B} \) such that \( c_{\beta_{i-1}\beta_i}^{\gamma_i} \neq 0 \) (resp. \( c_{\beta_i\gamma_i}^{\beta_i} \neq 0 \)) for \( s = 1, \ldots, N-1 \). We write \( \beta \preceq \beta' \) if either of the above structure constants is non-zero for all \( \beta_1, \gamma_1, \beta_{i+1}, s = 1, \ldots, N-1 \).

**Proposition 6.30.** (i) \( \bar{B}[\lambda] \) is a two-sided cell of \( \bar{B} \).

(ii) For any \( b_2 \in B_W(\lambda), \{ (b_1, s, b_2) \in \bar{B}[\lambda] \mid s \in \text{Irr} \, G_\lambda, b_1 \in B_W(\lambda) \} \) is a left cell.

(iii) For any \( b_1 \in B_W(\lambda), \{ (b_1, s, b_2) \in \bar{B}[\lambda] \mid s \in \text{Irr} \, G_\lambda, b_2 \in B_W(\lambda) \} \) is a right cell.

**Proof.** (i) Since \( \bar{U}[\geq \lambda], \bar{U}[\leq \lambda] \) are two-sided ideals of \( \bar{U} \), \( \bar{B}[\lambda] \) is a union of two-sided cells. Let \( b \in \bar{B} \). By [17, 6.4.3], \( f_iG(\beta) \) is equal to \([\varepsilon_i(\beta) + 1]G(\bar{f}_i\beta) \) plus a linear combination of elements in \( \bar{B} \) different from \( \bar{f}_i\beta \). Therefore \( G(\bar{f}_i\beta) \preceq_L G(\beta) \) if \( \bar{f}_i\beta \neq 0 \). Similarly we have \( G(\bar{e}_i\beta) \preceq_L G(\beta) \) if \( \bar{e}_i\beta \neq 0 \). Thus if \( \beta \) and \( \beta' \) are in the same connected component of \( \bar{B} \), we have \( G(\beta) \sim_L G(\beta') \). Taking \( \# \), we conclude that if \( \beta \) and \( \beta' \) are in the same connected component of \( \bar{B} \) as \( \# \)-crystal, we have \( G(\beta) \sim_R G(\beta') \). By Theorem 4.10 we have the assertion if \( a_\lambda \sim S_\lambda \) for \( s \in \text{Irr} \, G_\lambda \). This follows from a consideration of two-sided cells of the based ring \( \langle R(G_\lambda), \text{Irr} \, G_\lambda \rangle \), where \( R(G_\lambda) \) is the representation ring of \( G_\lambda \) and \( \text{Irr} \, G_\lambda \) is considered as its basis. By using the Pieri formula, we can easily check that it consists of a single two-sided cell, and hence our assertion. The proofs of (ii), (iii) are contained in the above proof.

Remark \( \bar{B} = \bigcup_{\lambda \in \mathcal{P}_c^{\geq 0}} \bar{B}[\lambda] \) by Lemma 5.23. Therefore the above proposition gives a complete description of two-sided, left, right cells of \( \bar{B} \).

Next we define a function on \( G(\bar{B}) \) which will allow us construct a limit algebra \( \bar{U}_0 = \bigoplus_{\lambda \in \mathcal{P}_c^{\geq 0}} \bar{U}[\lambda]_0 \) as \( q \to 0 \).

**Definition 6.31.** Let \( \beta = (b, s, b') \in \bar{B}[\lambda] \). Define \( a(\beta) = -\langle \text{wt}(b'), 2\lambda + \text{wt}(b') \rangle / 2 \).

**Remark 6.32.** \( a(\beta) \) is equal to the half of the dimension of the quiver variety \( \mathcal{M}(\nu, \omega) \) (or the dimension of the lagrangian subvariety \( \mathcal{L}(\nu, \omega) \)), where \( \nu = -\text{wt}(b'), \omega = \lambda \). (See e.g. [24, 2.3.2(4)]) The inner product \( (, , ) \) induced on \( \mathcal{P}_c^{\geq 0} \) is equal to the standard inner product for the untwisted \( ADE \) case (see [15, Corollary 6.4]), which is the only case for which quiver varieties are defined.
Lemma 6.33 (cf. [24, 1.4, 4.11]). Let $\beta = (b, s, b') \in B[\lambda]$. The following holds:

(i) $(q^{(\beta, \beta)}\tilde{L}[\lambda] \subset \tilde{L}[\lambda]$ and for each $\beta$, $a(\beta) \geq 0$ is smallest integer with this property,

(ii) for a two-sided cell $B[\lambda]$ and any $\lambda_1 \in \mathcal{P}_0^{cl,+}$, the restriction of $a$ to $\{ \beta \in B[\lambda] \mid | G(\beta) \in Ua_{\lambda_1} \}$ is constant.

Proof. First note that using Lemmas 6.13, 6.17 and Proposition 6.27 we have the following equalities in $U[\lambda] = U[\varnothing \lambda]/U[\varnothing \lambda]$:

\[ G(b, s, b')G(b_1, s_1, b'_1) \mod U[\varnothing \lambda] = q^{-a(\beta)} \sum_{s' \in \text{Irr} \ G_{\lambda}} (G(b_1)S_1u_{\lambda_1}G(b')S'u_{\lambda_1}G(b)S'G(b'_1)) \mod U[\varnothing \lambda] \]

\[ = q^{-a(\beta)} \sum_{s', s'' \in \text{Irr} \ G_{\lambda}} c_{s',s''}(b, s, b') \mod U[\varnothing \lambda] \]

where $ss' = \sum_{s''} c_{s',s''}$ and $c_{s',s''} \in \mathbb{Z}_{\geq 0}$. By Theorem 2.20 and Lemma 6.14, this is contained in $q^{-a(\beta)}\tilde{L}[\lambda]$. (i) now follows by multiplying both sides by $q^{a(s)}$. Note that when $b'_1 = b_2$ the product is in $L[\lambda] \setminus q_\lambda L[\lambda]$ by Theorem 2.20 and so $a(\beta)$ is the smallest integer with this property. We check that $a(\beta)$ is positive. We have $G(b, s, b')G(b', 1, b) = q^{-a(\beta)}G(b)$ mod $(L[\lambda] + q_\lambda L[\lambda])$ by Theorem 2.20. It follows $c(q_\lambda) = c(b, s, b')$. Taking $b, b'$ of both sides and using the $\sim$-invariance of the global basis we also have $c(q_\lambda) = q^{a(s)} \mod q_\lambda^{-1}Z[q_\lambda^{-1}]$. This implies $a(\beta) \geq 0$. (ii) is clear from the definition: If $\beta = (b, s, b') \in B[\lambda]$ satisfies $G(\beta) \in Ua_{\lambda_1}$, we have $wt(b') = \lambda_1 - \lambda$.

Using Lemma 6.33 for any two-sided cell, we define the ring $\tilde{U}[\lambda]_0$ which is a “limit as $q \to 0$.” For $\beta \in B[\lambda]$ define $\hat{\beta} = q^{a(\beta)}G(\beta)$. Then $\{ \beta \mid \beta \in B[\lambda] \}$ is a $q(\lambda_1)$-basis of $U[\lambda]$. For $\beta, \beta' \in B[\lambda]$ such that $G(\beta') \in Ua_{\lambda_1}$, we have $\beta' = \sum_{\beta''} q^{a(\beta'')}c_{\beta,\beta''} \beta''$ in $U[\lambda]$. (This follows from $a(\beta'') = a(\beta'')$, which follows from Lemma 6.33 (ii) above, since the only non-zero coefficients $c_{\beta,\beta''}$ are those for which $G(\beta'') \in Ua_{\lambda_1}$.) Since $q^{a(\lambda_1)}c_{\beta,\beta''} \in Z[q_\lambda]$ by Lemma 6.33 (i), the $Z[q_\lambda]$-submodule $U[\lambda]_0$ generated by $\{ \beta \mid \beta \in G(\beta) \}$ is a $Z[q_\lambda]$-subalgebra of $U[\lambda]$. We define $U[\lambda]_0 = U[\lambda]^{-1}/q_\lambda U[\lambda]^{-1}$. Define $t_\beta$ to be the image of $\beta$ in $U[\lambda]_0$. Then $U[\lambda]_0$ is a ring with a $Z$-basis $\{ t_\beta \mid \beta \in G(\beta) \}$. If we denote by $\{ \gamma_{\beta,\beta'/(\beta,\beta',\beta'') \in G(\beta) \}$ the structure constants of $U[\lambda]_0$ then $q^{a(\beta)}c_{\beta,\beta''} \equiv \gamma_{\beta,\beta''}$ mod $q_\lambda Z[q_\lambda]$. We define $U_0 = \bigoplus U[\lambda]_0$ to be the direct sum of these rings.

Lemma 6.34. $G(b, s, b')\# = G(b', s', b) \mod (b, s, b') \in B[\lambda]$.

Proof. Note $(G(b)SG(b')\#)\# = G(b')SG(b)\#$. By Lemma 6.14 $S\#$ corresponds to the dual representation $s\#$ modulo $U[\lambda]$. We have $G_\lambda(b, s, b')\# = G_\lambda(b', s'\#, b)$ by the definition, then the assertion follows by Proposition 5.27.

Definition 6.35. Let $\lambda \in \mathcal{P}_0^{cl,+}$. Let $D_{B[\lambda]} = \{ (b, 1, b) \mid b \in B_\lambda \}$. Denote by $\tilde{U}(0)$ the subalgebra of $\tilde{U}$ generated by $a_{\lambda,xa_{\lambda}}$ for $x \in U$.

Remarks 6.36. (i) By the definition and Lemma 6.34, it is clear that $G(\beta)\# = G(\beta)$ for $\beta \in D_{B[\lambda]}$. It is also clear that $G(\beta)$ is in $U(0)$.
(ii) It is not true that $G(\beta)^c = G(\beta)$ implies $\beta \in D_{B[\lambda]}$. Consider $(b,s,b)$ with $s^c = s$ and $s \neq 1$, e.g. $s(z) = (z_1^2 + z_1 z_2 + z_2^2)/z_1 z_2$.

(iii) The identifications (6.4) depends on the choice of the section $B_W(\lambda) \to B_0(\lambda)$. But the ambiguity of monomials $p(z) = \prod_i (z_{i,1} \cdots z_{i,\lambda_i})^* = G(\beta) = G(b,1,b)$. Therefore the bijection $B_W(\lambda) \to D_{B[\lambda]}$ is independent of the section.

The following gives a characterization of $D_{B[\lambda]}$:

**Proposition 6.37** (cf. [24] Conj. 5.3(b)). Let $\lambda_1 \in P^0_{cl}$. If $G(\beta) \in \tilde{U}(0)a_{\lambda_1} \cap G(B[\lambda])$, then $q^{-a(\beta)}(a_{\lambda_1}, G(\beta)) \equiv 1 \mod q_sZ[q_s]$ if $\beta \in D_{B[\lambda]}$ and $\equiv 0$ otherwise.

**Proof.** By definition, $a$ takes the constant $(\lambda - \lambda_1, \lambda + \lambda_1)/2$ on $\tilde{U}(0)a_{\lambda_1} \cap G(B[\lambda])$. We denote this constant by $a_0$. If $\mu > \lambda$, then

$$\left(\mu - \lambda_1, \mu + \lambda_1 \right) - (\lambda - \lambda_1, \lambda + \lambda_1) = (\mu - \lambda, \mu + \lambda) > 0.$$ 

Thus our assertion is equivalent to saying that if $G(\beta) \in \tilde{U}(0)a_{\lambda_1} \cap \bigsqcup_{\mu \geq \lambda} G(B[\mu])$, then $q^{-a(\beta)}(a_{\lambda_1}, G(\beta)) \equiv 1 \mod q_sZ[q_s]$ if $\beta \in D_{B[\lambda]}$ and $\equiv 0$ otherwise. We will check this.

In this proof we replace $\tilde{U}$ by $\bigsqcup_{\lambda \in \mu} Ua_\lambda$ the modified quantum enveloping algebra defined for $P$. Then by [23] 26.2.3, $(\ ,\ )$ is the limit of the inner product on $V(\xi) \otimes V(-\xi')$ (denoted also by $(\ ,\ )$, where $\xi - \xi' = \lambda_1$ and $\xi, \xi'$ tend to $\infty$.

For each $G(\beta) = G(b,s,b') \in \tilde{U}(0)a_{\lambda_1} \cap \bigsqcup_{\mu \geq \lambda} G(B[\mu])$, we choose and fix an expression $(b,s) = X_1 X_2 \cdots X_N u_\lambda, s_0$ where $s_0 \in \text{Irr} G_{\lambda_1}$, $X_i = \tilde{e}_i$ or $\tilde{f}_i$. Then

$$q^{-a(\beta)}(u_\xi \otimes u_-\xi', X_1 X_2 \cdots X_N S_0 G(b')\#(u_\xi \otimes u_-\xi'))$$

$$= \left((X_1 X_2 \cdots X_N S_0)^\#(u_\xi \otimes u_-\xi'), G(b')\#(u_\xi \otimes u_-\xi')\right) \mod q_s Z[q_s]$$

$$= \delta_{(b,s), (\nu', 1)} \mod q_s Z[q_s],$$

where $\tilde{b} = (b,s)$ by (6.4). Here we have used (6.3) in the first equality, and the almost orthonomal property of the global crystal basis ([23] 26.3.1) in the second and third equalities. Since $\{G(\beta) = G(b,s,b')\} = \tilde{U}(0)a_{\lambda_1} \cap \bigsqcup_{\mu \geq \lambda} G(B[\mu])$ is a $A_{\lambda_1}$-basis of $\tilde{L}^{(\lambda)}$, the set of corresponding elements $\{X_1 X_2 \cdots X_N S_0 G(b')\#(u_\xi \otimes u_-\xi')\}$ spans $(\tilde{U}(0)a_{\lambda_1} \cap \tilde{L}^{(\lambda)}(u_\xi \otimes u_-\xi'))$. Therefore the above together with our previous remark $a(\beta) \geq a_0$ implies

$$q^{-a(\beta)}(u_\xi \otimes u_-\xi', (\tilde{U}(0)a_{\lambda_1} \cap \tilde{L}^{(\lambda)})(u_\xi \otimes u_-\xi')) \in Z[q_s].$$

Since $G(b,s,b')(u_\xi \otimes u_-\xi') = X_1 X_2 \cdots X_N S_0 G(b')\#(u_\xi \otimes u_-\xi')$ mod $\tilde{L}^{(\lambda)}(u_\xi \otimes u_-\xi')$, the above shows

$$q^{-a(\beta)}(u_\xi \otimes u_-\xi', G(b,s,b')(u_\xi \otimes u_-\xi')) \equiv \delta_{\lambda, \mu} \delta_{(b,s), (\nu', 1)} \mod q_sZ[q_s].$$

Taking limit $\xi, \xi' \to \infty$, we get the assertion. \hfill \Box

**Lemma 6.38** (cf. [24] 1.5). Let $d_1 = G(b_1, 1, b_1), d_2 = G(b_2, 1, b_2)$ be in $D_{B[\lambda]}$. Let $\beta = G(b_3, s, b'_3)$. Then in $\tilde{U}[\lambda]_{t_0}, t_0, t_3 t_3 t_3 t_3 = \delta_{b_1, b_2} \delta_{b_3, b_2} t_3 t_3 t_3$. 

Proof. By Lemma 6.17 we have
\[ G(b_1, 1, b_1)G(b_2, s, b_2') \equiv q^{-a(d_1)} \sum_{s'' \in \text{Irr} G_\lambda} (G(b_3)S_{u_\lambda}(G(b_1)S''_{u_\lambda})\lambda G(b_1)S''_{G(b_2')}) \mod \tilde{\mathcal{L}}[\gamma \lambda] \equiv q^{-a(d_1)}\delta_{b_1,b_2}G(b_1)SG(b_2') \mod (\tilde{\mathcal{L}}[\gamma \lambda] + q_2\tilde{\mathcal{L}}[\tilde{\gamma} \lambda]). \]

The second equivalence follows from Theorem 2.20. A similar calculation shows
\[ G_\lambda(b_3, s, b_3')G_\lambda(b_2, 1, b_2) \equiv q^{-a(\beta)}\delta_{b_1',b_2}G(b_3, s, b_3') \mod (\tilde{\mathcal{L}}[\gamma \lambda] + q_2\tilde{\mathcal{L}}[\tilde{\gamma} \lambda]). \]

Since \( a(\beta) = a(d_2) \) if \( b_3' = b_2 \) the result follows by multiplying both sides by \( q^{2a(d_2)}. \)

Remark 6.39. Lemma 6.38 says that the \( \mathbf{Z} \)-ring \( \tilde{U}[\lambda]_0 \) has a generalized unit which is compatible with the basis \( t_\beta, \beta \in G(\tilde{B}[\lambda]). \) In particular, the ring \( \tilde{U}[\lambda]_0 \) has an identity, \( 1 = \sum_{b_1 \in B_W(\lambda)} t_{G(b_1,1,b_1)} \).

\( \tilde{U}[\lambda] \) is both a left \( \tilde{U} \)-module and a right \( \tilde{U} \)-module where the respective \( \lambda \) structures are given by:
\[ G(\beta)G(\beta') = \sum_{\beta'' \in B[\lambda]} c_{\beta\beta'}^{\beta''}(q_s)G(\beta''), \]

when \( \beta \in \tilde{B}, \beta' \in \tilde{B}[\lambda] \) in the first case and when \( \beta \in \tilde{B}[\lambda], \beta' \in \tilde{B} \) in the second case.

We define \( M_{\tilde{U}[\lambda]} \) to be the vector space spanned over \( Q(q_s, q_s') \) by \( \{G(\beta) \mid \beta \in \tilde{B}[\lambda]\} \). \( M_{\tilde{U}[\lambda]} \) is a \( \tilde{U} \)-bimodule where the left action is given by (6.40) and the right action is given by (6.40) where \( c_{\beta\beta'}^{\beta''}(q_s) \) is replace by \( c_{\beta\beta'}^{\beta''}(q_s) \). We now show that the left and right module structures commute, i.e., \( G(\beta_1)G(\beta_2)G(\beta_3) = (G(\beta_1)G(\beta_2))G(\beta_3) \) where \( \beta_2 \in \tilde{B}[\lambda] \) and \( \beta_1, \beta_3 \in \tilde{B}. \) In terms of the structure constants this is equivalent to

Lemma 6.41 (cf. [24, 1.7, 4.15]). Let \( \beta_1, \beta_2 \in \tilde{B}[\lambda]. \) Let \( \beta_1, \beta_3 \in \tilde{B}. \) Then
\[ \sum_{\beta \in B[\lambda]} c_{\beta_1,\beta}(q_s)c_{\beta,\beta_3}(q_s') = \sum_{\beta \in B[\lambda]} c_{\beta_1,\beta_2}(q_s)c_{\beta_2,\beta_3}(q_s'). \]

Proof. We fix \( \beta_i = (b_i, s_i, b_i') \) for \( i = 1, 2. \) For \( \beta_j \in \tilde{B} (j = 1, 3) \) and \( b_i \in B_W(\lambda) \) \((i = 1, 2, 4), \) we define \( g^{(s,b)}_{\beta_i, b_i} \in \text{A} \) by
\[ G(\beta_j)G_\lambda(1, b_i) = \sum_{(s,b) \in B(\lambda)} g^{(s,b)}_{\beta_j, b_i}G_\lambda(s, b). \]

We check that
(a) \( G(\beta_j)G(b_i, s_i, b_i') = \sum_{(s,b) \in B(\lambda)} \sum_{s'' \in \text{Irr} G_\lambda} c_{\beta_j, b_i}^{s''}G(b_i, s, b_i') \mod \tilde{\mathcal{L}}[\gamma \lambda], \)
(b) \( G(b_i, s_i, b_i')G(\beta_j) = \sum_{(s,b) \in B(\lambda)} \sum_{s'' \in \text{Irr} G_\lambda} c_{s''}^{s}G(b_i, s', b) \mod \tilde{\mathcal{L}}[\gamma \lambda], \)
where \( c^{s}_{s} \) are structure constants of \((R(G_{\lambda}), \text{Irr} G_{\lambda})\) as before.

We consider the \( \bar{U} \)-homomorphism \( \Psi : V(\lambda) \to \bar{U}[\lambda] \) such that \( \Psi_{b_{i}}(u_{\lambda}) = S_{i}G(b^{\#}) \) mod \( \bar{U}[\lambda] \), where \( b_{i} = (s^{\#}, b_{j}) \) by (6.3). (The existence of \( \Psi_{b_{i}} \) is guaranteed by Lemma 6.11.) The image of \( \Psi_{b_{i}} \) gives

\[
G(\beta_{j})G(b_{i}, s, b'_{j}) = \sum_{(s,b)} g_{\beta_{j}, b_{i}}(s,b) SS_{i}G(\beta'_{j}).
\]

Now (a) follows. (b) follows from (a) by taking \( \# \).

By (a) and (b) we have

\[
c_{\beta_{j}, \beta_{j}}(q_{s}) = \sum_{s'} \frac{\delta^{\#}}{\frac{b_{j}}{b_{j}}} c^{s_{j}}_{s_{j}} g^{(s', b_{j})}_{\beta_{j}, b_{j}}(q_{s}),
\]

\[
c_{\beta_{j}, \beta_{j}}(q'_{s}) = \sum_{s'} \frac{\delta^{\#}}{\frac{b_{j}}{b_{j}}} c^{s_{j}}_{s_{j}} g^{(s', b_{j})}_{\beta_{j}, b_{j}}(q'_{s}),
\]

\[
c_{\beta_{j}, \beta_{j}}(q_{s}) = \sum_{s'} \frac{\delta^{\#}}{\frac{b_{j}}{b_{j}}} c^{s_{j}}_{s_{j}} g^{(s', b_{j})}_{\beta_{j}, b_{j}}(q_{s}),
\]

\[
c_{\beta_{j}, \beta_{j}}(q'_{s}) = \sum_{s'} \frac{\delta^{\#}}{\frac{b_{j}}{b_{j}}} c^{s_{j}}_{s_{j}} g^{(s', b_{j})}_{\beta_{j}, b_{j}}(q'_{s}),
\]

where \( \beta = G(b, s, b') \). This makes the identity of the lemma equivalent to

\[
\sum_{(b, s, b') \in B[\lambda]} \sum_{s', s'' \in \text{Irr} G_{\lambda}} \frac{\delta^{\#}}{\frac{b_{j}}{b_{j}}} c^{s_{j}}_{s_{j}} g^{(s', b_{j})}_{\beta_{j}, b_{j}}(q_{s})\delta_{b_{j}, b_{j}} c^{s_{j}}_{s_{j}} g^{(s''', b_{j})}_{\beta_{j}, b_{j}}(q'_{s})
\]

\[
= \sum_{(b, s, b') \in B[\lambda]} \sum_{s', s'' \in \text{Irr} G_{\lambda}} \frac{\delta^{\#}}{\frac{b_{j}}{b_{j}}} c^{s_{j}}_{s_{j}} g^{(s', b_{j})}_{\beta_{j}, b_{j}}(q_{s})\delta_{b_{j}, b_{j}} c^{s_{j}}_{s_{j}} g^{(s''', b_{j})}_{\beta_{j}, b_{j}}(q'_{s}).
\]

Note that

\[
\sum_{s} c^{s_{j}}_{s} c^{s_{j}}_{s} = \sum_{s} c^{s_{j}}_{s} c^{s_{j}}_{s} \quad \text{and} \quad \sum_{s} c^{s_{j}}_{s} c^{s_{j}}_{s} = \sum_{s} c^{s_{j}}_{s} c^{s_{j}}_{s}
\]

are equal, since both are the multiplicity of \( s_{4} \) in \( s' \otimes s_{2} \otimes s_{3} \otimes s'' \). Now the above identity is immediate.

Fix a two-sided cell \( G(B[\lambda]) \). We define a \( Q(q_{s}) \)-linear map \( \Phi : \bar{U} \to Q(q_{s}) \otimes \bar{U}[\lambda]_{0} \) by

\[
\Phi(G(\beta)) = \sum_{d \in D_{\bar{B}[\lambda]}} c_{\beta_{j}, \beta_{j}}^{d}(q_{s}) t_{\beta_{j}}, \quad (\beta \in \bar{B})
\]

which is well-defined since \( D_{\bar{B}[\lambda]} \) is finite and for a fixed \( \beta, d \) there are only finitely many \( c_{\beta_{j}, \beta_{j}}^{d} \neq 0 \).

We have the following description of \( \Phi \) due to [24] Prop. 1.9

**Proposition 6.43.** (i) \( \Phi \) is an algebra homomorphism.

(ii) Let \( P(B[\lambda]) \) be the set of \( \lambda \in P \) such that \( a_{\lambda}d = d \) for some \( d \in D_{\bar{B}[\lambda]} \). Then \( \Phi(\sum_{\lambda \in P(B[\lambda])} a_{\lambda}) = 1 \), and \( \Phi(a_{\lambda}) = 0 \) for \( \lambda \notin P(B[\lambda]) \).

Next we describe an explicit realization of the ring structure of \( \bar{U}[\lambda]_{0} \). As usual, \( \lambda = \sum_{i} \lambda_{i} v_{i} \). Let \( T_{\lambda} \) be the set of triples \( (d, d', s) \) where \( d, d' \in D_{\bar{B}[\lambda]} \) and \( s \in \text{Irr} G_{\lambda} \).
Let $J_{\lambda}$ be the free abelian group on $T_{\lambda}$ with a ring structure defined by

$$(d_1, d_1', s)(d_2, d_2', s') = \delta_{d_1, d_2} \sum_{s'' \in \text{Irr} G_{\lambda}} c_{s''}^{s'}(d_1, d_2', s''),$$

where $c_{s''}^{s'}$ is the multiplicity of $s''$ in the tensor product $s \otimes s'$ as above. We have:

**Theorem 6.44.** (i) There exists a ring isomorphism $\overline{U}[\lambda]_0 \overset{\sim}{\rightarrow} J_{\lambda}$ which gives a bijection between $\{ t_{\beta} \mid \beta \in \overline{B}[\lambda] \}$ and $\{(d, d', s) \mid d, d' \in D_{\overline{B}[\lambda]}, s \in \text{Irr} G_{\lambda} \}$.

(ii) For any $d_0 \in D_{\overline{B}[\lambda]}$, the subset of $\overline{B}[\lambda]$ corresponding to $\{(d, d', s) \mid d' = d_0 \}$ under the bijection in (i) is a left cell.

(iii) For any $d_0 \in D_{\overline{B}[\lambda]}$, the subset of $\overline{B}[\lambda]$ corresponding to $\{(d, d', s) \mid d = d_0 \}$ under the bijection in (i) is a right cell.

**Proof.** By definition, the elements $b \in B_W(\lambda)$ are in 1–1 correspondence with the elements of $t_b = t_{(b, 1, b)} \in D_{\overline{B}[\lambda]}$. The map which sends $t_{(b_1, s, b_1')} \mapsto (t_{b_1}, t_{b_1'}, s)$ is a bijection. Using Lemma 6.17 as in the proof of Lemma 6.33, we have

$$t_{(b_1, s, b_2)}(t_{b_2, s}, t_{b_2', s}) = \delta_{b_2, b_2'} \sum_{s_1 \in \text{Irr} G_{\lambda}} c_{s_1}^{s_2} t_{(b_2, s_1, b_2')},$$

This implies (i), (ii) and (iii) follow from Proposition 6.30. \qed

The following proposition partly explains why our approach to the limit algebra $J_{\lambda}$ via $V(\lambda)$ is natural.

**Proposition 6.46.** Let $V(\lambda)_0$ be the free $\mathbb{Z}$-module with basis $B(\lambda) \cong D_{\overline{B}[\lambda]} \times \text{Irr} G_{\lambda}$ endowed with a $J_{\lambda}$-module structure by

$$(d_1, d_2, s) \cdot (d', s') = \delta_{d_2, d'} \sum_{s'' \in \text{Irr} G_{\lambda}} c_{s''}^{s'}(d_1, s'').$$

Then $V(\lambda)_0 \otimes_{\mathbb{Z}} Q(q_s)$, pulled back by the composition of $\overline{U} \overset{\Phi}{\rightarrow} \overline{U}[\lambda]_0 \overset{\sim}{\rightarrow} J_{\lambda}$, is isomorphic to $V(\lambda)$.

**Proof.** For $b = (b_W, s) \in B(\lambda) \cong B_W(\lambda) \times \text{Irr} G_{\lambda}$, we define $\beta_b \in \overline{B}[\lambda]$ by $\beta_b = (b_W, s, u_{\lambda})$. For $\beta \in \overline{B}$, we have

$$G(\beta)G_{\lambda}(b) = \sum_{b' \in B(\lambda)} c_{\beta' \beta_b}^{\beta} G_{\lambda}(b').$$

In fact, $G_{\lambda}(b) = G(\beta_b)u_{\lambda}$ implies $G(\beta)G_{\lambda}(b) = \sum_{b' \in B} c_{\beta \beta_b}^{\beta'} G_{\lambda}(b')u_{\lambda}$ and $G(\beta')u_{\lambda}$ is $G_{\lambda}(\beta')$ if $\beta' \in B(\lambda)$ and 0 otherwise. On the other hand, the $\overline{U}[\lambda]_0$-module structure on $V(\lambda)_0$ is given by

$$t_{\beta'} b = \sum_{b' \in B(\lambda)} \gamma_{\beta' \beta_b}^{\beta} b' \quad \text{for} \ \beta' \in \overline{B}[\lambda].$$

We define a $Q(q_s)$-linear map $\Psi: V(\lambda) \rightarrow V(\lambda)_0 \otimes_{\mathbb{Z}} Q(q_s)$ by

$$\Psi(G_{\lambda}(b)) = \sum_{d \in D_{\overline{B}[\lambda]}, b' \in B(\lambda)} c_{\beta_b \beta}^{\beta'} b', \quad (b \in B(\lambda)).$$

Then we have $\Psi(G_{\lambda}(b)) = \Phi(G(\beta_b))u_{\lambda}$, since $t_{\beta'} u_{\lambda} = b'$ if $\beta' = \beta_b$ and 0 otherwise by Theorem 6.44. We get

$$\Psi(G(\beta)G_{\lambda}(b)) = \Phi(G(\beta)G(\beta_b))u_{\lambda} = \Phi(G(\beta))\Phi(G(\beta_b))u_{\lambda} = \Phi(G(\beta))\Psi(G_{\lambda}(b)).$$
This shows that $\Psi$ is $\bar{U}$–linear.

Let $L(\lambda)_0$ be the $A_0$-submodule of $V(\lambda)_0 \otimes \mathbb{Q}(q_s)$ generated by $q^{-\alpha(\beta_0)}b$ ($b \in B(\lambda)$). Note that $\alpha(\beta_0)$ is independent of $b$ by Lemma 6.33(ii). Since $q^{\alpha(\beta_0)}c_{\bar{\beta}_0d} = q^{\alpha(\beta_0)}c_{\bar{\beta}_0d} \in \mathbb{Z}[q_s]$ by Lemma 6.33(i), we have $\Psi(L(\lambda)) \subset L(\lambda)_0$. The induced homomorphism $\Psi_0 : L(\lambda)/q_sL(\lambda) \to L(\lambda)_0/q_sL(\lambda)_0$ is given by

$$\Psi_0(b) = \sum_{d \in D_{B[\lambda]}, b' \in B(\lambda)} \gamma^{\bar{\beta}_0d'}b'$$

where $b' \in B(\lambda)$ is identified with $q^{-\alpha(\beta_0)}b' \mod q_sL(\lambda)_0$. By Theorem 6.44 we have $\gamma^{\bar{\beta}_0d} = \delta_{bb'}\delta_{\beta_0d}$, hence the right hand side is equal to $b$. This shows that $\Psi$ is an isomorphism as in the proof of Corollary 4.13.

**Proposition 6.47.** For $\bar{U}(A_n)$ we have $\#D_{B[\lambda]} = \prod_{i=1}^n \binom{n+1}{i} \lambda_i$.

**Proof.** We check the dimension of $W(\lambda) = \bigotimes_i W(\omega_i)^{\lambda_i}$. The result follows if we show the dimension of $W(\omega_i)$ is $\binom{n+1}{i}$. By considering the Drinfeld polynomials of $W(\omega_i)$ [5, Remark 3.3], each is an evaluation module where the underlying finite dimensional representation of $U(A_n)$ is $V(\omega_i)$ where the $\omega_i$ are the fundamental weights of type $A_n$ [5, Prop. 12.2.13]. These are of dimension $\binom{n+1}{i}$ for $i = 1, \ldots, n$.

**Remark 6.48.** The above argument shows that the number of $D_{B[\lambda]}$ can be given if we know $\dim W(\omega_i)$ for all $i \in I_0$. They are known for untwisted affine Lie algebras for classical groups and some exceptional groups [6]. The second author writes a computer program for them (for untwisted cases) by using Frenkel-Mukhin’s algorithm [13]. The program gives us answers except one fundamental representation for $E_8$, corresponding to the triple node.

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