A note on the equivalence of a barotropic perfect fluid with a k-essence scalar field

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Abstract

In this brief report, we obtain the necessary and sufficient condition for a class of noncanonical single scalar field models to be exactly equivalent to barotropic perfect fluids, under the assumption of an irrotational fluid flow. An immediate consequence of this result is that the nonadiabatic pressure perturbation in this class of scalar field systems vanishes exactly at all orders in perturbation theory and on all scales. The Lagrangian for this general class of scalar field models depends on both the kinetic term and the value of the field. However, after a field redefinition, it can be effectively cast in the form of a purely kinetic k-essence model.

I. INTRODUCTION

Both scalar fields and perfect fluids are pervasive in our present models for the evolution of the Universe since its birth until the current epoch of accelerated expansion. Both (early time) inflation and the present dark energy epoch are commonly supposed to be supported by scalar fields. The intermediate epochs in the evolution of the Universe usually consist of a radiation-dominated era followed by a matter-dominated era. These can be well described by barotropic perfect fluids with equations of state $w$ equal to one third and zero, respectively. A barotropic perfect fluid can be defined as a perfect fluid where the pressure is a function of the energy density only (the same function for the background and the perturbations).

Because of the importance of scalar fields and perfect fluids in present-day cosmological models, the theory of cosmological perturbations in these models has been intensively studied and is well developed. For instance, cosmological perturbations for perfect fluids at linear order have been studied in [1,2], at second order in [3,4] and more recently at third order [5,6]. Some fully nonlinear results have also been obtained; see for example [7,10].

Similarly, linear cosmological perturbations in canonical scalar field systems have been studied in [2,3,11,12], and in [14,17] at second and third order in perturbations. More recently, cosmological perturbations in noncanonical scalar field models have attracted much attention. The reason for such interest is that it was realized that inflationary models in string theory [18,19] naturally contain scalar fields with nonstandard kinetic terms and these models can also give rise to large non-Gaussianity of the primordial curvature perturbation. Ref. [21] studied linear perturbations. For second-order perturbations, see for instance [22,23] and for third-order, see [24,27]. Perturbations up to third order of inflationary models with multiple noncanonical scalar fields have also been studied; see for instance [28,32].

Because of their importance in cosmology, it is natural to ask when, if in any case, can we describe a scalar field by a perfect fluid and vice versa. If there are some models where this equivalence is realized one may be able to use the known results of perturbation theory mentioned above to study models of perfect fluids or scalar fields where the calculation in one side of the duality has not been done. A recent explicit example of such a case can be found in [37]. When both results of perturbation theory for scalar fields and perfect fluids exist, in the dual models, one can use the equivalence as an extra consistency check for the calculations.

In this brief report, we will discuss the equivalence of a barotropic perfect fluid with a so-called k-essence/k-inflation scalar field [38].

This paper is organized as follows. In the next section, we will introduce a general class of k-essence/k-inflation models under study. We will also present the background equations of motion. In section III we will briefly discuss linear perturbations; we shall obtain and solve the second-order partial differential equation that gives us the class of scalar field models that is dual to a barotropic perfect fluid, under the assumption of an irrotational fluid flow. Finally, section IV is devoted to conclusion.

II. THE MODEL

We consider a fairly general class of single scalar field models described by the action

$$S = \frac{1}{2} \int d^4x \sqrt{-g} \left[ M_{Pl}^2 R + 2P(X,\phi) \right],$$

(1)

where $\phi$ is the scalar field, $M_{Pl}$ is the Planck mass that will be set to unity hereafter, $R$ is the Ricci scalar, and $X \equiv -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$ is the scalar field kinetic term. $g_{\mu\nu}$ is the metric tensor. We label the scalar field Lagrangian by $P$ and we assume that it is a well behaved function of two variables, the scalar field $\phi$ and its kinetic term $X$.

This general field Lagrangian includes as particular cases the common canonical scalar field model, Dirac-Born-Infeld inflation [18,39] and k-inflation/k-essence [38].
The energy-momentum tensor is defined as
\[ T_{\mu\nu} = -2 \frac{\partial P}{\partial g_{\mu\nu}} + g_{\mu\nu} P, \]
which gives
\[ T_{\mu\nu} = P_X \partial_\mu \phi \partial_\nu \phi + g_{\mu\nu} P. \]
If \( \partial_\mu \phi \) is timelike (i.e. \( X > 0 \)), then this is of the perfect fluid form
\[ T_{\mu\nu} = (\rho + P) u_\mu u_\nu + g_{\mu\nu} P, \]
with the pressure \( P \) and the energy density
\[ \rho = 2X P_X - P, \]
where \( P_X \) denotes the derivative of \( P \) with respect to \( X \). The four-velocity\(^1\) is subject to the constraint \( u_\mu u^\mu = -1 \) and reads
\[ u_\mu = \frac{\partial_\mu \phi}{\sqrt{2X}}. \]
The scalar field equation of motion is
\[ \nabla^\nu (P_X \nabla_\nu \phi) + P, \phi = 0, \]
where \( \nabla_\nu \) denotes a covariant derivative.

We are interested in flat, homogeneous, and isotropic Friedmann-Lemaître-Robertson-Walker background universes described by the line element
\[ ds^2 = -dt^2 + a^2(t) \delta_{ij} dx^i dx^j, \]
where \( a(t) \) is the scale factor. The Friedmann equation and the continuity equation read
\[ 3H^2 = \rho_0, \quad \dot{\rho}_0 = -3H (\rho_0 + P_0), \]
where the Hubble rate is \( H = \dot{a}/a \), and \( \rho_0 \) is the background energy density of the scalar field given by
\[ \rho_0 = 2X_0 P_{0,X} - P_0, \]
where \( P_{0,X} \) denotes the derivative of \( P \) with respect to \( X \) evaluated at the background value \( X = X_0 \).

III. PERTURBATIONS AND THE DUAL MODELS

In this section, we will first discuss linear perturbations and the condition for the equivalence between a scalar field model and a barotropic fluid. Then we will obtain and solve the second-order partial differential equation that determines the class of scalar field models dual to an irrotational barotropic perfect fluid. At the end of the section, we will briefly discuss the behavior of the nonadiabatic pressure perturbation.

On the comoving time-slices, the scalar field fluctuations vanish, \( \delta \phi = 0 \), and the three-dimensional spatial metric \( h_{ij} \) is perturbed as
\[ h_{ij} = a^2 \left[ (1 + 2\mathcal{R}_c) \delta_{ij} + 2\partial_i \partial_j E + 2\partial_i F_j \right], \]
where tensor perturbations have been neglected and \( \mathcal{R}_c \) denotes the curvature perturbation on comoving slices. \( E \) and \( F_i \) can be put to zero by an appropriate choice of spatial coordinates, where \( F_i \) denotes an intrinsic vector perturbation. The linear equation of motion for \( \mathcal{R}_c \) is
\[ \frac{\partial}{\partial t} \left( \frac{a^3 \epsilon}{c^2_{ph}} \frac{\partial}{\partial t} \mathcal{R}_c \right) - a \epsilon \frac{\partial^2}{\partial x^i \partial x^j} \mathcal{R}_c = 0, \]
where \( \epsilon = -\dot{H}/H^2 \) and \( c_{ph} \) is the speed of propagation of scalar perturbations (“speed of sound”) given by
\[ c^2_{ph} = \frac{P_{0,X}}{\rho_{0,X}} = \frac{P_{0,X}}{P_{0,X} + 2X_0 P_{0,XX}}, \]

In the case of a barotropic perfect fluid, scalar perturbations propagate with speed \( c_s \) given by
\[ c^2_s = \frac{\dot{P}_0}{\dot{\rho}_0}, \]
which is often called the adiabatic sound speed. The comoving slicing condition for a fluid is \( T^0_0 = 0 \), and the curvature perturbation on comoving slices \( \mathcal{R}_c \) follows Eq. \( 12 \) with \( c^2_{ph} \) replaced by \( c^2_s \).

In general these two speeds are different \[40\]. But one can consider under which conditions they are the same. Requiring \( c^2_{ph} = c^2_s \), and after using \( \dot{P}_0 = P_{0,X} \dot{X} + P_{0,\phi} \dot{\phi} \) and the background equations of motion, one finds the following equation:\(^2\)
\[ P_{0,\phi} - X_0 P_{0,X\phi} + X_0 P_{0,\phi} \frac{P_{0,XX}}{P_{0,X}} = 0. \]

This equation can be integrated once to give
\[ X_0 P_{0,X} = A(\phi) P_{0,\phi}, \]
where \( A(\phi) \) is an arbitrary function of \( \phi \). Using the method of characteristics, one can further integrate

\(^1\) This is for a potential flow only. In general it could have a vector part.

\(^2\) While this work was being prepared for publication, the paper \[41\] appeared on the arXiv. They independently found the same equation, however, they do not solve it in general as we do here and simply give some very particular solutions of it.
the speed of sound, is given by

\[ c_{\text{ph}}^2 = f(Xg(\phi)) \]

where \( f \) and \( g \) are arbitrary functions. For this Lagrangian, the adiabatic sound speed, which is equal to the speed of sound, is given by

\[ c_{\text{ad}}^2 = \left(1 + 2Y \frac{f_{,Y}}{f_{,Y}}\right)^{-1} \]

where we define \( Y = Xg(\phi) \). This is the most general class of scalar field models \( P(X, \phi) \) that is exactly equivalent to a barotropic perfect fluid, under the assumption of the velocity being described by a single scalar potential, i.e. the fluid flow is irrotational. Figure 1 is a graphical description of this result. The fact that any solution to the Einstein equations in the presence of a barotropic perfect fluid can be interpreted as a solution in the presence of a purely kinetic k-essence scalar field is known in the literature \cite{25} (see also for example \cite{26, 27}). Here we show that all the other k-essence models that depend on the values of the scalar field and speed of sound and thus are also equivalent to barotropic perfect fluids. The Lagrangian of these other models has to be of the form \( \phi \phi \)

A standard canonic scalar field, i.e. \( P(X, \phi) = X - V(\phi) \), where \( V(\phi) \) is the potential, is not included in this class of models. The speed of sound is \( c_{\text{ph}}^2 = 1 \) and the adiabatic sound speed is

\[ c_{\text{ad}}^2 = 1 - \frac{2\epsilon}{3}, \quad \text{with} \quad \epsilon = -\frac{\dot{H}}{H^2}, \quad \eta = \frac{\dot{\epsilon}}{\epsilon H}. \]

All purely kinetic k-essence models, i.e. the field’s Lagrangian is of the form \( P(X) \), are included in this class of models. These fields have been used to drive inflation \cite{37} and to be the dark energy \cite{38}.

If \( P(X, \phi) = f(Y) \), where \( Y = Xg(\phi) \), then Eq. \( 15 \) can be written as

\[ \rho = 2Yf_{,Y}(Y) - f(Y), \]

and one has that the energy density \( \rho \) is a function of \( Y \) only. So if \( \rho(Y) \) is invertible one can in principle find the equation of state \( P(\rho) \). Assuming \( g > 0 \) so that \( Y > 0 \), the previous equation can also be put in the form

\[ P(Y) = \frac{1}{2} \sqrt{Y} \int^Y \frac{\rho(Y')}{\sqrt{Y'^2}} dY' + C\sqrt{Y}, \]

where \( C \) is an integration constant.

For a given invertible equation of state one has \( \rho + P = F(\rho) \), and if \( P(X, \phi) = f(Y) \), Eq. \( 15 \) can be written as

\[ 2f_{,Y}Y = F(f(Y)), \]

and integrated to give

\[ 2 \int \frac{df}{F(f)} = \ln Y, \]

from where one finds the Lagrangian given an equation of state \( F \).

It may be worth mentioning the expression for the conserved (“baryon”) number density \( n \). For \( P = f(Y) \) and \( \rho = F(\rho) - P = 2f_{,Y}Y - f(Y) \), it is determined by

\[ \frac{dn}{n} = \frac{dp}{\rho + P} = \frac{f_{,Y} + 2f_{,Y}Y}{2f_{,Y}}dY. \]

This may be readily integrated to give

\[ n = Kf_{,Y}Y^{-1/2}, \]

where \( K \) is a constant of integration, which would be a function of entropy for a perfect fluid.

For the scalar field Lagrangian \( \phi \phi \), because both the pressure and the energy density are functions of one variable only, i.e. \( Y \), it can be shown that the nonadiabatic pressure perturbation defined for instance in \( \phi \phi \) vanishes exactly to all orders in perturbations and on all scales. This is not surprising because all models of the form \( \phi \phi \) are dual to barotropic (i.e. adiabatic) perfect fluids where the nonadiabatic pressure perturbation vanishes by definition. In other words, the equivalence between the dual models is independent of the background because the derived second-order differential equation \( 15 \) is independent of the background but dependent only on the form of \( P \) as a function of \( X \) and \( \phi \), despite the fact that it was derived from the condition for the equivalence in linear perturbation theory.

\[ \hat{\phi} \]

\[ \text{Fluid} \]

\[ P(X, \phi) = f(Xg(\phi)) \]

\[ P(X, \phi) = f(Xg(\phi)) \]

\[ \text{Barotropic} \]

\[ \text{Non-Barotropic} \]

\[ \text{Fig. 1: The left ellipse represents the set of all the models with a general Lagrangian } P(X, \phi) \text{ while the right ellipse represents the set of all the perfect fluids. We have shown that the intersection of these two sets corresponds to barotropic perfect fluids or scalar field models with Lagrangian } P(X, \phi) = f(Xg(\phi)) \]
IV. CONCLUSION

In this brief report, we have studied under which conditions a general k-essence scalar field is equivalent to an irrotational barotropic perfect fluid. We have found that the condition can be written as a second-order partial differential equation for the Lagrangian of the field, Eq. (15), that simply states that the sound speed $c_{ph}$ at which scalar perturbations propagate has to be equal to the adiabatic sound speed $c_a$. We have found the most general solution for that equation, Eq. (17). The Lagrangian of the solutions we found depends explicitly on both the kinetic term $X$ and the scalar field value $\phi$. However, this dependence is of the restrictive form $P(X, \phi) = f(X g(\phi))$, where $f$ and $g$ are arbitrary functions. After a suitable field redefinition, $Y = gX = -\frac{1}{2} g^{\mu \nu} \partial_\mu \phi \partial_\nu \phi$ where $d\varphi = \sqrt{g} d\phi$, the Lagrangian can be cast in the form of a purely kinetic k-essence model. We note, however, that in reality there may exist plural fields or fluids that interact with each other, which may lead to small violations of the perfect fluid/adiabaticity condition. In such a case, the field redefinition may or may not be useful depending on the form of interactions as well as on the situation of interest.

One can show that, for the Lagrangian (17), the nonadiabatic pressure perturbation vanishes exactly to all others in perturbations and on all scales. This is because both the pressure and the energy density are functions of one variable only. This result is less surprising if we note that these scalar field models are equivalent to barotropic perfect fluids where the nonadiabatic pressure perturbation vanishes by definition.

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[1] J. M. Bardeen, Phys. Rev. D22, 1882 (1980).
[2] H. Kodama and M. Sasaki, Prog. Theor. Phys. Suppl. 78, 1 (1984).
[3] V. F. Mukhanov, H. A. Feldman, and R. H. Brandenberger, Phys. Rept. 215, 203 (1992).
[4] N. Bartolo, E. Komatsu, S. Matarrese, and A. Riotto, Phys. Rept. 402, 103 (2004), astro-ph/0406398.
[5] K. A. Malik and D. Wands, Phys. Rept. 475, 1 (2009), 0809.4944.
[6] N. Bartolo, S. Matarrese, and A. Riotto, (2010), 1001.3957.
[7] G. D’Amico, N. Bartolo, S. Matarrese, and A. Riotto, JCAP 0801, 005 (2008), 0707.2894.
[8] A. J. Christopherson and K. A. Malik, JCAP 0911, 012 (2009), 0909.0942.
[9] D. Lyth, K. A. Malik, and M. Sasaki, JCAP 0505, 004 (2005), astro-ph/0411220.
[10] D. Langlois and F. Vernizzi, Phys. Rev. Lett. 95, 091303 (2005), astro-ph/0503416.
[11] V. F. Mukhanov and G. V. Chibisov, JETP Lett. 33, 532 (1981).
[12] M. Sasaki, Prog. Theor. Phys. 70, 394 (1983).
[13] M. Sasaki, Prog. Theor. Phys. 76, 1036 (1986).
[14] J. M. Maldacena, JHEP 05, 013 (2003), astro-ph/0210603.
[15] K. A. Malik, JCAP 0703, 004 (2007), astro-ph/0610864.
[16] D. Seery, J. E. Lidsey, and M. S. Sloth, JCAP 0701, 027 (2007), astro-ph/0610210.
[17] D. Seery, M. S. Sloth, and F. Vernizzi, JCAP 0903, 018 (2009), 0811.3934.
[18] E. Silverstein and D. Tong, Phys. Rev. D70, 103505 (2004), hep-th/0405221.
[19] X. Chen, Phys. Rev. D71, 063506 (2005), hep-th/0408084.
[20] X. Chen, JHEP 08, 045 (2005), hep-th/0501184.
[21] J. Garriga and V. F. Mukhanov, Phys. Lett. B458, 219 (1999), hep-th/9904176.
[22] D. Seery and J. E. Lidsey, JCAP 0506, 003 (2005), astro-ph/0503692.
[23] X. Chen, M.-x. Huang, S. Kachru, and G. Shiu, JCAP 0701, 002 (2007), hep-th/0605045.
[24] X. Chen, M.-x. Huang, and G. Shiu, Phys. Rev. D74, 121301 (2006), hep-th/0610235.
[25] F. Arroja and K. Koyama, Phys. Rev. D77, 083517 (2008), 0802.1167.
[26] X. Chen, B. Hu, M.-x. Huang, G. Shiu, and Y. Wang, JCAP 0908, 008 (2009), 0905.3494.
[27] F. Arroja, S. Mizuno, K. Koyama, and T. Tanaka, Phys. Rev. D80, 043527 (2009), 0903.3641.
[28] D. Seery and J. E. Lidsey, JCAP 0509, 011 (2005), astro-ph/0508096.
[29] D. Langlois and S. Renaux-Petel, JCAP 0804, 017 (2008), 0801.1085.
[30] S. Renaux-Petel, D. Renaux-Petel, D. A. Steer, and T. Tanaka, Phys. Rev. Lett. 101, 061301 (2008), 0804.3139.
[31] D. Langlois, S. Renaux-Petel, D. A. Steer, and T. Tanaka, Phys. Rev. D78, 063523 (2008), 0806.0336.
[32] F. Arroja, S. Mizuno, and K. Koyama, JCAP 0805, 015 (2008), 0806.0619.
[33] S. Renaux-Petel and G. Tasinato, JCAP 0901, 012 (2009), 0810.2405.
[34] S. Mizuno, F. Arroja, K. Koyama, and T. Tanaka, Phys. Rev. D80, 023530 (2009), 0905.4557.
[35] S. Mizuno, F. Arroja, and K. Koyama, Phys. Rev. D80, 083517 (2009), 0907.2439.
[36] S. Renaux-Petel, JCAP 0910, 012 (2009), 0907.2476.
[37] L. Boubekeur, P. Creminelli, J. Norena, and F. Vernizzi,
[38] C. Armendariz-Picon, T. Damour, and V. F. Mukhanov, Phys. Lett. B458, 209 (1999), hep-th/9904075.
[39] M. Alishahiha, E. Silverstein, and D. Tong, Phys. Rev. D70, 123505 (2004), hep-th/0404084.
[40] A. J. Christopherson and K. A. Malik, Phys. Lett. B675, 159 (2009), 0809.3518.
[41] S. Unnikrishnan and L. Sriramkumar, (2010), 1002.0820.
[42] R. Akhoury, C. S. Gauthier, and A. Vikman, JHEP 03, 082 (2009), 0811.1620.
[43] A. Diez-Tejedor and A. Feinstein, Int. J. Mod. Phys. D14, 1561 (2005), gr-qc/0501101.
[44] S. Matarrese, Proc. Roy. Soc. Lond. A401, 53 (1985).
[45] C. Quercellini, M. Bruni, and A. Balbi, Class. Quant. Grav. 24, 5413 (2007), 0706.3667.
[46] D. Bertacca, N. Bartolo, A. Diaferio, and S. Matarrese, JCAP 0810, 023 (2008), 0807.1020.
[47] E. J. Copeland, M. Sami, and S. Tsujikawa, Int. J. Mod. Phys. D15, 1753 (2006), hep-th/0603057.
[48] D. Wands, K. A. Malik, D. H. Lyth, and A. R. Liddle, Phys. Rev. D62, 043527 (2000), astro-ph/0003278.
[49] K. A. Malik and D. Wands, Class. Quant. Grav. 21, L65 (2004), astro-ph/0307055.