Moody’s Correlated Binomial Default Distributions for Inhomogeneous Portfolios

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Abstract. This paper generalizes Moody’s correlated binomial default distribution for homogeneous (exchangeable) credit portfolio, which is introduced by Witt, to the case of inhomogeneous portfolios. As inhomogeneous portfolios, we consider two cases. In the first case, we treat a portfolio whose assets have uniform default correlation and non-uniform default probabilities. We obtain the default probability distribution and study the effect of the inhomogeneity on it. The second case corresponds to a portfolio with inhomogeneous default correlation. Assets are categorized in several different sectors and the inter-sector and intra-sector correlations are not the same. We construct the joint default probabilities and obtain the default probability distribution. We show that as the number of assets in each sector decreases, inter-sector correlation becomes more important than intra-sector correlation. We study the maximum values of the inter-sector default correlation. Our generalization method can be applied to any correlated binomial default distribution model which has explicit relations to the conditional default probabilities or conditional default correlations, e.g. Credit Risk †, implied default distributions. We also compare some popular CDO pricing models from the viewpoint of the range of the implied tranche correlation.

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1. Introduction

The modeling of portfolio credit risk and default correlation are hot topics and pose entirely new problems \[1, 2, 3, 4, 5\]. CDOs are financial innovations to securitize portfolios of defaultable assets. Many probabilistic models have been studied in order to price CDO tranches \[6, 7, 8, 9, 10, 11, 12, 13, 14, 15\]. Most of them are implemented with Monte Carlo simulations and as the number of names in a portfolio increases, the computational time increases. The Factor approach uses a small number of latent factors that induce the default dependency \[11\]. Conditionally on the latent variables values, default events are independent. It becomes easy to calculate the loss (default) distribution function. Along this line, some semi-explicit expressions of most relevant quantities were obtained \[16\].

On the other hand, correlated binomial models were also proposed to describe the default dependency structures. The first one is a one-factor exchangeable version of CreditRisk + \[17, 18, 19\]. The aggregate loss distribution function is given by the beta-binomial distribution (BBD). The second one is Moody’s correlated binomial default distribution model, which was introduced by Witt \[20\]. (hereafter the MCB model) The authors also consider the applicability of the long-range Ising model \[21, 22\]. These models use Bernoulli random variables. Differences stem from different definitions of the conditional correlations \[23\]. In the MCB model \[20\], the conditional default correlation between assets is set to be constant irrespective of the number of defaults. Those of BBD decay with an increase in default. We are able to adapt a suitable form for the conditional correlations. Recently it has become possible to calibrate \(\rho_n\) from implied default distributions \[24\]. By using the “implied correlated” binomial model, whose conditional correlations are those of the implied distribution, it may become easy to estimate hedge ratios and so forth.

The advantage of these correlated binomial models come from the fact that they are easier to evaluate than other more refined models. If a probabilistic model is implemented by a Monte Carlo simulation, the evaluation of the price of these derivatives consumes much computer time and the inverse process to obtain the model parameters becomes tedious work. With the above correlated binomial models, one can estimate the model parameters from the CDO premiums more easily. However, these models are formulated only for homogeneous portfolios, where the assets are exchangeable and they have the same default probability \(p\) and default correlation \(\rho\). Generalization to more realistic inhomogeneous portfolios where assets have different default probabilities and different default correlations should be done.

In this paper, we show how to generalize Moody’s correlated binomial default distribution (MCB) model to two types of inhomogeneous portfolios. Our generalization method can be applied to other correlated binomial models, including implied correlated binomial distributions, by changing the condition on \(\rho_n\). We obtain the default probability function \(P_{X}(n)\) and examine the dependence of the expected loss rates of the tranches on the inhomogeneities. With the proposed model, we also estimate the implied
values of the default correlation. Comparison of the range of the tranche correlations with those of the Gaussian copula model and BBD model are also performed.

About the organization in this paper, we start with a short review of Moody’s correlated binomial default distribution (MCB) model in Section 2. The dependence of \( \rho_n \) on the number of defaults \( n \) is compared with the BBD and Gaussian copula. Section 3 is the main part of the paper. We show how to couple multiple MCB models as a portfolios credit risk model for inhomogeneous portfolio. In the first subsection, we couple two Bernoulli random variables \( X, Y \) and recall on the limit of the correlation \( \rho_{xy} \) between them. In the next subsection, we couple an MCB model of \( N \) assets \( X_1, X_2, \cdots, X_N \) with a random variable \( Y \) and we study the maximum value of the correlation between \( X_i \) and \( Y \). Then we couple two MCB models with \( N \) and \( M \) assets. Choosing the model parameters properly, we construct an MCB model for an inhomogeneous default probability case and obtain the default distribution function. The last subsection is devoted to an inhomogeneous default correlation case. Assets are categorized in different sectors and inter-sector and intra-sector default correlations are not the same. We consider a portfolio with \( K \) sectors and \( k \)-th sector contains \( N_k \) assets. Within each sector, the portfolio is homogeneous and it has parameters as \( p_k \) and \( \rho_k \). The inter-sector default correlations are not the same and they depend on the choice of sector pairs. We construct the joint default probabilities and the default probability function \( P_N(n) \) for the portfolio explicitly. In Section 4 using the above results, we estimate the implied default correlation for each tranche from a CDO’s market quotes (iTraxx-CJ Series 2). We compare the range of the correlations of MCBs, BBD and the Gaussian copula model. We conclude with some remarks and future problems.

2. Moody’s correlated binomial default distribution

We review the definitions and some properties of Moody’s correlated binomial default distribution (MCB) model. We consider a homogeneous portfolio, which is composed of exchangeable \( N \) assets. Here the term “homogeneous” means that

\[
X_i \\
i=1,2,\ldots,N \\
p = \langle X_i \rangle \\
\rho : \text{Correlation}
\]

Figure 1. Homogeneous portfolio with \( N \) assets. The assets are exchangeable and the default probability is \( p \) and the default correlation is \( \rho \). The state of i-th asset is described by a Bernoulli random variable \( X_i \).
the constituent assets are exchangeable and their default probabilities and default correlations are uniform. We denote them as \( p \) and \( \rho \). Bernoulli random variables \( X_i \) show the states of the \( i \)-th assets. \( X_i = 1 \) means that the asset is defaulted and the non-default state is represented as \( X_i = 0 \). The joint default probabilities are denoted as

\[
P(x_1, x_2, \cdots, x_N) = \text{Prob}(X_1 = x_1, X_2 = x_2, \cdots, X_N = x_N).
\] (1)

In order to determine \( P(x_1, x_2, \cdots, x_N) \), we need \( 2^N - 1 \) conditions for them. Here \( 2^N \) corresponds to the number of possible configurations and \( -1 \) comes from the overall normalization condition for the joint probabilities. From the assumption of the homogeneity for the portfolio, the number of degrees of freedom of the joint probabilities are reduced. The probability for \( n \) defaults and \( N - n \) non-defaults is the same for any configuration \( (x_1, x_2, \cdots, x_N) \) with \( \sum_{i=1}^{N} x_i = n \). The number of defaults \( n \) is ranged from 0 to \( N \) and considering the overall normalization condition, remaining degrees of freedom are \( N \).

In the MCB model, the conditional default probabilities are introduced. We denote \( p_n \) as the default probability for any assets under the condition that any other \( n \) assets of the portfolio are defaulted. To exemplify the situation concretely, we take the \( n \) assets as the first \( n \) of \( N \) assets, we denote them with \( n' \) as \( n' = 1, 2, \cdots, n \). The condition that they are defaulted is written concisely as \( \prod_{n' = 1}^{n} X_{n'} = 1 \). The conditional default probability for \( n + 1 \)-th assets under the condition of \( n \) defaults can then be written as

\[
p_n = < X_{n+1} | \prod_{n' = 1}^{n} X_{n'} = 1 > .
\] (2)

Here, \( < A | C > \) means the expected value of random variable \( A \) under condition \( C \) is satisfied. \( X_n \) takes 1 for \( n \)-th asset default, \( < X_n | C > \) corresponds to its default probability under condition \( C \). Of course, any asset from \( k = n + 1, \cdots, N \) can be chosen in the evaluation of the expected value for \( p_n \) under the condition that \( \prod_{n' = 1}^{n} X_{n'} = 1 \). \( p_0 \) is nothing but the default probability \( p \).

\( N \) independent conditional default probabilities \( p_n (i = 0, \cdots, N - 1) \) are determined by the following condition on the default correlations.

\[
\text{Cor}(X_{n+1}, X_{n+2} | \prod_{n' = 1}^{n} X_{n'} = 1) = \rho.
\] (3)

Here, \( \text{Cor}(X, Y | C) \) is defined as

\[
\text{Cor}(X, Y | C) = \frac{< XY | C > - < X | C > < Y | C >}{\sqrt{< X | C > (1 - < X | C >) < Y | C > (1 - < Y | C >)}}.
\] (4)

The conditions on the default correlations give us the following recursion relations for \( p_n \) as

\[
p_{n+1} = p_n + (1 - p_n) \rho.
\] (5)

These recursion relations can be solved to give \( p_n \) as

\[
p_n = 1 - (1 - p)(1 - \rho)^n.
\] (6)
Moody’s Correlated Binomial Default Distribution

$p_n$ increases with $n$ and $p_n \to 1$ as $n \to \infty$ for $\rho > 0$.

From these conditional default probabilities $p_n(n = 0, \ldots, N - 1)$, the joint default probabilities for the configuration $\bar{x} = (x_1, x_2, \ldots, x_N)$ are given as

$$P(\bar{x}) = P(x_1, x_2, \ldots, x_N) = \prod_{n=1}^{N} X_n^x (1 - X_n)^{1-x_n}.$$  \hspace{1cm} (7)

The normalization condition for $P(\bar{x})$ is guaranteed by the following decomposition of unity.

$$1 = < 1 > = \prod_{n=1}^{N} X_n + (1-X_n) > \prod_{n=1}^{N} \left( \sum_{x_j=0}^{1} \right) < \prod_{n=1}^{N} X_n^x (1 - X_n)^{1-x_n} \hspace{1cm} (8)$$

The probability for $n$ defaults is

$$P_N(n) = N C_n \times P(1, 1, \ldots, 1, 0, \ldots, 0) = N C_n \prod_{i=1}^{n} X_i \prod_{i=n+1}^{N} (1 - X_i) >$$

$$= N C_n \sum_{k=0}^{N-n} N-n C_k (-1)^k \left( \prod_{n'=0}^{n+k-1} p_{n'} \right). \hspace{1cm} (9)$$

We modify the above MCB model as follows. In the MCB model, the default correlation is set to be constant irrespective of the number of default (see(3)). We change the condition as

$$\text{Cor}(X_{n+1}, X_{n+2} | \prod_{n'=1}^{n} X_{n'} = 1) = \rho \exp(-n \lambda). \hspace{1cm} (10)$$

Here, we introduce a parameter $\lambda > 0$ and the default correlation under $n$ defaults decay as $\exp(-n \lambda)$. If we set $\lambda = 0$, the modified model reduces to the original MCB model.

There are two motivations for the modification. The first one is that it is mathematically necessary to couple multiple MCB models. We discuss the mechanism in greater detail in the next section. Here, we only comment on the limit value of $p_n$ as $n \to \infty$. The modification changes the recursive relation for $p_n$ to

$$p_{n+1} = p_n + (1 - p_n) \rho \exp(-n \lambda). \hspace{1cm} (11)$$

$p_n$ is calculated as

$$p_n = 1 - (1 - p) \prod_{n'=0}^{n-1} (1 - \rho_{n'}). \hspace{1cm} (12)$$

Here $\rho_n$ is defined as $\rho_n = \rho \exp(-n \lambda)$.

$p_n$ increases with $n$, however the increase is reduced by the decay of the correlation with $n$. The limit value of $p_n$ with $n \to \infty$ is roughly estimated as

$$p_\infty = \lim_{n \to \infty} p_n = 1 - (1 - p) (1 - \rho) \exp(-\rho e^{-\lambda} e^{-\lambda}). \hspace{1cm} (13)$$

For $\lambda = 0$, $p_\infty = 1$ and $p_\infty = p + (1 - p) \rho = p_i$ for $\lambda = \infty$. In Figure 2 we show the enumerated data for $p_\infty$ and the results from eq.(13). As $\lambda$ increases, $p_\infty$ decreases and the $\lambda$ dependence is well described by eq.(13).
The second motivation is that popular CDO pricing models have decaying correlation with \( n \). The BBD model’s \( \rho_n \) is given as \[ \rho_n = \frac{\rho}{1 + n\rho}. \] (14)

In the Gaussian copula model, we do not have the explicit form for \( \rho_n \). From its aggregate loss distribution function, it is possible to estimate them. In Figure 3, we show \( \rho_n \) for the MCB, BBD and Gaussian copula models. We set \( p = \rho = 0.1 \) and \( N = 30 \). The Gaussian copula’s \( \rho_n \) does not show monotonic dependence on \( n \). After a small peak, it decays to zero. In order to mimic the Gaussian copula model within the framework of a correlated binomial model, such a dependence should be incorporated in the assumption on \( \rho_n \).

Figure 4 depicts \( p_n \) in the same setting. With the same \( p \) and \( \rho \), all \( p_n \) curves pass through \( p_0 = p \) at \( n = 0 \) and \( p_1 = p + (1-p)\rho \) at \( n = 1 \). After \( n > 1 \), the behaviors of \( p_n \) depend on the models’ definitions on \( \rho_n \). \( p_n \) saturate to about 0.4 for MCB(\( \lambda = 0.3 \)), which means that a large scale avalanche does not occur and the loss distribution function has a short tail. In MCB with \( \lambda = 0.0 \) and Gaussian copula models, their \( p_n \) saturate to 1. The behaviors are reflected in the fat and long tails in their loss distribution.

Figure 5 shows the semi-log plot of \( P_{30}(n) \) for the MCB, BBD and Gaussian copula models. We also plot the binomial distribution Bin(30, 0.1). The default correlation shifts the peak of the binomial distribution to \( n = 0 \) and \( P_{30}(n) \) comes to have a long tail. MCB, BBD and Gaussian copula have almost the same bulk shape. In particular, in MCB, even if we change \( \lambda \), \( P_{30}(n) \) has almost the same shape for \( n \leq 15 \). The bulk shape of \( P_N(n) \) is mainly determined by \( p_n \) with small \( n \). \( p_n \)'s with large \( n \) comes from very rare events \( \prod_{n'=1}^{n} X_{n'} = 1 \) and contains information about the tails of the
distributions. They do not affect the bulk part significantly.

There are differences in their tails. One sorts the models in the order of thinnest tail to fatest tail, we have

$$\text{MCB}(\lambda = 0.3) < \text{BBD} < \text{Gaussian copula} < \text{MCB}(\lambda = 0.0).$$

MCB($\lambda = 0$) has almost the same shape as the Gaussian copula. However, it has a bigger tail than the gaussian coupla at $n = 30$. The tail of MCB($\lambda = 0.3$) is short compared with other models. We can understand this behavior from the behavior of $p_n$.

We also note another role of the damping parameter $\lambda$. In the calculation $P_N(n)$,
Moody's Correlated Binomial Default Distribution

Figure 5. Loss Distribution $P_{30}(n)$ for $p = 0.1$ and $\rho = 0.1$. $\lambda = 0.0, 0.3$ and $p = \rho = 0.1$. We also plot the profile for the binomial distribution.

there are many cancellations $\sum_{k=0}^{N-n} N-n C_k (-1)^k$ in (9) from the decomposition of $\prod_{i=n+1}^{N}(1 - X_i)$. This causes numerical errors in the evaluation and it is difficult to get $P_N(n)$ for $N \geq 40$, even if we use long double precision variables in the numerical implementation. When we set $\lambda = 0.3$, the numerical error diminishes greatly and we can obtain $P_N(n)$ even for $N = 100$. This point is important when one uses the MCB model for analysis of the actual CDOs that have at least 50 assets. In addition, with $\lambda > 0$, we can take $\rho$ to be negatively large enough. In S&P’s data, a negative default correlation of 0.1% or so has been reported [25]. We think that this point is also an advantage of the modified model.

Hereafter, we mainly focus on the generalization of the MCB model. However, the same method and reasoning should be applicable to other correlated binomial models with any assumption on $\rho_n$. If we set $\rho_n$ as in (14), we have Beta-Binomial default distribution models for inhomogeneous portfolios.

3. Generalization to Inhomogeneous Portfolios

In this section we couple multiple MCB models and construct the joint default probabilities and $P_N(n)$ for inhomogeneous portfolios. In addition, we show that the inter-sector default correlation can be set to be large enough by choosing $\lambda$ and other
parameters. We think that it is possible to use the model as a model for portfolio credit risk.

3.1. Coupling of X and Y: 1+1 MCB model

Before proceeding to the coupling of multiple MCB models, we recall some results for the coupling of two random variables \( X \) and \( Y \). The default probability is \( p_x \) (\( p_y \)) for \( X \) (respectively for \( Y \)) and the default correlation between them is \( \rho_{xy} \).

\[
\text{Corr}(X, Y) = \rho_{xy}. \tag{15}
\]

As in the MCB model, we introduce the conditional default probabilities as

\[
p_0 = p_x = \langle X \rangle \quad \text{and} \quad p_1 = \langle X | Y = 1 \rangle.
\]

\[
q_0 = p_y = \langle Y \rangle \quad \text{and} \quad q_1 = \langle Y | X = 1 \rangle. \tag{16}
\]

From the default correlation \( \rho_{xy} \), \( p_1 \) and \( q_1 \) are calculated as

\[
p_1 = p_0 + (1 - p_0) \sqrt{\frac{p_0(1 - q_0)}{(1 - p_0)q_0} \rho_{xy}}
\]

\[
q_1 = q_0 + (1 - q_0) \sqrt{\frac{q_0(1 - p_0)}{(1 - q_0)p_0} \rho_{xy}}. \tag{17}
\]

In the symmetric (homogeneous) case \( p_x = p_y \), the equality \( p_1 = q_1 \) holds and they are given as

\[
p_1 = q_1 = p_x + (1 - p_x)\rho_{xy}.
\]

The correlation \( \rho_{xy} \) can be set to be 1 and in the limit \( p_1 = q_1 \to 1 \). The maximum value of \( \rho_{xy} \) is 1 in the symmetric case. Conversely in the asymmetric case \( (p_x \neq p_y) \), \( \rho_{xy} \) cannot set to be 1. The maximum value of \( \rho_{xy} \) is determined by the condition that \( p_1 \leq 1 \) and \( q_1 \leq 1 \). From these conditions, we derive the following conditions \([26]\) on \( \rho_{xy} \) as

\[
\rho_{xy} \leq \sqrt{\frac{q_0(1 - p_0)}{(1 - q_0)p_0}} \quad \text{and} \quad \rho_{xy} \leq \sqrt{\frac{p_0(1 - q_0)}{(1 - p_0)q_0}}. \tag{18}
\]

We introduce an asymmetric parameter \( r \) as

\[
r = \frac{p_y}{p_x} \tag{19}
\]

and a function \( f(r, p) \) as

\[
f(r, p) = \sqrt{\frac{(1 - p)r}{1 - rm}}. \tag{20}
\]

The maximum value of \( \rho_{xy} \) is then given as

\[
\text{Max}(\rho_{xy}) = \text{Min}(f(r, p_0), f(r, p_0)^{-1}). \tag{21}
\]

Here \( \text{Max}(\rho_{xy}) \) represents the maximum value of \( \rho_{xy} \) and \( \text{Min}(A, B) \) means that the smaller value of \( A \) and \( B \) is taken. Figure 6 shows \( \text{Max}(\rho_{xy}) \) as a function of the
asymmetric (inhomogeneity) parameter $r$. We show two curves, the solid one for $p_x = 0.1$ and the dotted one for $p_x = 0.5$. As the inhomogeneity $r$ increases, that is $r$ departs from $r = 1$, $\text{Max}(\rho_{xy})$ decreases. For fixed $r$, as $p_x$ becomes large, $\text{Max}(\rho_{xy})$ becomes small. The reason is that the condition $p_1 \leq 1$ becomes more difficult to satisfy as $p_x = p_0$ increase. $p_1$ is a monotonous increasing function of $p_0$. In the previous section, the conditional default probability $p_n$ becomes smaller as we set $\lambda$ larger. When we set a large $\lambda$, we show that it is possible to couple multiple MCB models with strong default correlation.

3.2. $N + 1$ MCB model

For the second step, we couple an $N$ assets MCB model with one two-valued random variable $Y$. We introduce $N$ random variables $X_n(n = 1, \ldots, N)$ and the default probability and the default correlation for them is $< X_n > = p_x$ and $\rho$. The default probability for $Y$ is $< Y > = p_y$ and the default correlation between $Y$ and $X_n$ is written as $\rho_{xy}$. We assume homogeneity for the $N$ assets MCB model and the default correlation between $X_n$ and $Y$ is independent of the asset index $n$. As in the previous cases, we introduce conditional default probabilities as

$$p_{n,0} = < X_{n+1} | \prod_{n'=1}^{n} X_{n'} = 1 > \quad \text{and} \quad p_{n,1} = < X_{n+1} | \prod_{n'=1}^{n} X_{n'} \times Y = 1 >$$

$$q_n = < Y | \prod_{n'=1}^{n} X_{n'} = 1 > . \quad (22)$$
The joint default probabilities \( P(\vec{x}, y) = P(\vec{X} = \vec{x}, y) \) are calculated by decomposing the following expression with these conditional default probabilities

\[
P(\vec{x}, y) = < \prod_{n=1}^{N} X_n^{x_n} (1 - X_n)^{1-x_n} \times Y^y (1 - Y)^{1-y} >
\]

(23)

The joint default probabilities with the condition \( Y = 1 \) are

\[
P(\vec{x} | Y = 1) = \frac{P(\vec{x}) - P(\vec{x} | Y = 1)p_y}{1 - p_y}.
\]

(26)

The probability for \( n \) defaults with \( Y = 1 \) is

\[
P_N(n | Y = 1) = NC_n \sum_{k=0}^{N-n} (-1)^k (\prod_{n'=0}^{n+k-1} \ p_{n', 0})
\]

(28)

Using the same argument as for the joint probabilities with \( Y = 0 \), \( P_N(n | Y = 0) \) is written as

\[
P_N(n | Y = 0) = \frac{P_N(n) - P_N(n | 1)p_y}{1 - p_y}.
\]

(29)

About the conditional default probabilities \( p_{n,0} \), we impose the same conditions as with the homogeneous \( N \) assets MCB model.

\[
\text{Cor}(X_{n+1}, X_{n+2} | \prod_{n'=1}^{n} X_{n'} = 1) = \rho e^{-n\lambda}
\]

(30)

The same recursive relation (31) for \( p_{n,0} \) is obtained and the \( p_{n,0} \) is given by

\[
p_{n,0} = 1 - (1 - p_x) \prod_{n'=0}^{n-1} (1 - p_{n'}).
\]

(31)

Here \( \rho_n \) is defined as before. For the conditions on \( p_{n,1} \) and \( q_n \), there are two possible ways to realize a strong correlation. The only way to realize a strong correlation \( \rho_{xy} \) between \( X_{n+1} \) and \( Y \) is

\[
\text{Cor}(X_{n+1}, Y | \prod_{n'=1}^{n} X_{n'} = 1) = \rho_{xy} e^{-n\lambda}.
\]

(32)

In this case, the relations for \( p_{n,1} \) and \( q_n \) are

\[
p_{n,1}q_n = q_{n+1}p_{n,0} = p_{n,0}q_n + \rho_{xy} \sqrt{p_{n,0}(1 - p_{n,0})q_n(1 - q_n)} e^{-n\lambda}.
\]

(33)
The recursive relations are
\[
p_{n,1} = p_{n,0} + \rho_{xy} e^{-n\lambda}(1 - p_{n,0}) \sqrt{\frac{p_{n,0}(1 - q_n)}{(1 - p_{n,0})q_n}}
\]
(34)

\[
q_{n+1} = q_n + \rho_{xy} e^{-n\lambda}(1 - q_n) \sqrt{\frac{(1 - p_{n,0})q_n}{p_{n,0}(1 - q_n)}}
\]
(35)

If we set \( p_y = p_x \) and \( \rho_{xy} = \rho \), these relations reduce to \( p_{n,1} = p_{n+1,0} \) and \( q_n = p_{n,0} \) and this coupled model is nothing but the \((N+1)\) assets MCB model.

![Figure 7](image_url)

**Figure 7.** \( \text{Max}(\rho_{xy})/\rho \) vs \( r = \frac{p_y}{p_x} \). \( N = 30, \rho = 0.1, p_x = 0.1 \). The solid line \( \lambda = 0.0 \) and the dotted line \( \lambda = 0.3 \)

We write the ratio \( \frac{q_n}{p_{n,0}} \) as \( r_n \) and the conditions that \( p_{n,1} \leq 1 \) and \( q_{n+1} \leq 1 \) are summarized as
\[
\rho_{xy} \leq e^{n\lambda} \times \text{Min}(f(r_n, p_{n,0}), f(r_n, p_{n,0})^{-1}).
\]
(36)

\( \text{Min}(f(r_n, p_{n,0}), f(r_n, p_{n,0})^{-1}) \) is nothing but the condition for \( \text{Max}(\rho_{xy}) \) of the two random variables \( X, Y \) with default probabilities \( (p_x, p_y) = (p_{n,0}, r_n p_{n,0}) \) (see eq.(21)).

As explained above, \( \text{Min}(f(r_n, p_{n,0}), f(r_n, p_{n,0})^{-1}) \) takes maximum value 1 at \( r_n = 1 \) for any value of \( p_n \). It also decrease with the increase of \( p_n \) for fixed \( r_n \).

The necessary condition for the model to be self-consistent is that \( p_{n,1} \leq 1 \) and \( q_n \leq 1 \) for all \( n \). We discuss \( \lambda = 0 \) and \( \lambda > 0 \) cases separately.

- \( \lambda = 0.0 \): \( p_{n,0} \) increases with \( n \) and \( p_{n,0} \to 1 \) as \( n \to \infty \). The range of \( p_{n,0} \) is \( [p_x, p_{N-1}] \approx [p_x, 1] \) and it is difficult to choose \( p_y \) such that \( r_n \simeq 1 \) for all \( n \). If \( r_n \) departs much from 1 for some \( n, \) \( \text{Max}(\rho_{xy}) \) decreases. The choice \( p_x = p_y \) and \( \rho_{xy} = \rho \) is possible, we anticipate that \( \text{Max}(\rho_{xy}) \) decreases from \( \rho \) as \( p_y \) departs from \( p_x \).
• \( \lambda > 0 \): The limit value of \( p_n \) becomes small (see eq.(13)) and the range of \( p_n \) is narrow as compared with the \( \lambda = 0.0 \) case. It may be possible to choose \( p_y \) such that the asymmetric parameter \( r_n \) is small for small \( n \). In addition, for large \( n \), it is easy to satisfy the condition (36) because of the prefactor \( \exp(n\lambda) \). We think that \( \text{Max}(\rho_{xy}) \) is large in this case.

We have checked numerically the values of the joint probabilities for all configurations \((x, y)\) with \( N = 30 \) and \( p_x = \rho = 0.1 \). In figure 7 we show the data of \( \text{Max}(\rho_{xy})/\rho \) for the case \( \lambda = 0.0 \) (solid line) and \( \lambda = 0.3 \) (dotted line). In the \( \lambda = 0.0 \) case, \( \text{Max}(\rho_{xy}) \approx \rho \) near \( r = 1 \) and as \( r \) departs from 1, \( \text{Max}(\rho_{xy}) \) decreases from \( \rho \). The data for \( \lambda = 0.3 \) case show that it is possible to set a large \( \text{Max}(\rho_{xy}) \) if we use a large \( r \). We can set \( \rho_{xy} \) as strong as several times of \( \rho \).

We also point out that the above \( N + 1 \) MCB model can be used to describe a credit portfolio where one obligor has great exposure. Such an obligor is described by \( Y \) and other obligors are by \( X \). If \( p_y \) is quite different from \( p_x \), we can couple \( Y \) and \( X \) with strong \( \rho_{xy} \approx \rho_x \) by setting a sufficiently large \( \lambda \).

3.3. \( N + M \) MCB Model : Coupled MCB model

Next we consider a portfolio with two sectors. The first sector has \( N \) assets and the second has \( M \) assets. To construct the joint default probabilities for the portfolio, we try to couple two MCB models. The former model’s \( N \) assets are described by \( X_n (n = 1, \cdots, N) \) and the states of the latter model’s assets are described by \( Y_m (m = 1, \cdots, M) \). The default probability and the default correlation in each sector are \((p_x, \rho_x)\) and \((p_y, \rho_y)\). The default correlation between the assets in different sectors is denoted as \( \rho_{xy} \) (see Figure 8).

Introducing the conditional default probabilities \( p_{n,m} \) and \( q_{n,m} \) as

\[
p_{n,m} = < X_{n+1} | \prod_{n' = 1}^{n} X_{n'} \prod_{m' = 1}^{m} Y_{m'} = 1 >, \quad p_{0,0} = p_x
\]  

(37)
Moody’s Correlated Binomial Default Distribution

\[ q_{n,m} = \langle Y_{m+1} \prod_{n'=1}^{n} X_{n'} \prod_{m'=1}^{m} Y_{m'} = 1 \rangle, \quad q_{0,0} = p_y, \]

we impose the following conditions on \( p \) and \( q_{0,m} \)

\[
\text{Cor}(X_{n+1}X_{n+2} | \prod_{n'=1}^{n} X_{n'} = 1) = \rho_x \exp(-n\lambda_x)
\]

\[
\text{Cor}(Y_{m+1}Y_{m+2} | \prod_{m'=1}^{m} Y_{m'} = 1) = \rho_y \exp(-m\lambda_y).
\]

The recursive relations for \( p_{n,0} \) and \( q_{0,m} \) are

\[
p_{n+1,0} = p_{n,0} + \rho_x \exp(-n\lambda_x)(1 - p_{n,0})
\]

\[
q_{0,m+1} = q_{0,n} + \rho_y \exp(-m\lambda_y)(1 - q_{0,m}).
\]

Their solutions are, by denoting \( \rho_{x,n} = \rho_x \exp(-n\lambda_x) \) and \( \rho_{y,m} = \rho_y \exp(-m\lambda_y) \),

\[
p_{n,0} = 1 - (1 - p_x) \prod_{n'=0}^{n-1} (1 - \rho_{x,n'})
\]

\[
q_{0,m} = 1 - (1 - p_y) \prod_{m'=0}^{m-1} (1 - \rho_{y,m'}).
\]

For the inter-sector correlation, we impose the next conditions on \( p_{n,m} \) and \( q_{n,m} \), which is a natural generalization of \( N + 1 \) case (see eq. (32)).

\[
\text{Cor}(X_{n+1}Y_{m+1} | \prod_{n'=1}^{n} X_{n'} \prod_{m'=1}^{m} Y_{m'} = 1) = \rho_{xy} e^{-n\lambda_x + m\lambda_y}.
\]

We obtain the following recursive relations,

\[
p_{n,m+1}q_{n,m} = q_{n+1,m}p_{n,m}
\]

\[
= p_{n,m}q_{n,m} + \rho_{xy} e^{-(n\lambda_x + m\lambda_y)} \sqrt{p_{n,m}(1 - p_{n,m})q_{n,m}(1 - q_{n,m})}.
\]

Using these relations, we are able to calculate \( p_{n,m} \) and \( q_{n,m} \) iteratively starting from \( p_{n,0} \) and \( q_{0,m} \).

The joint default probability for the portfolio configuration \((\vec{x}, \vec{y})\) is calculated by decomposing the following expression with \( p_{n,m} \) and \( q_{n,m} \)

\[
P(\vec{x}, \vec{y}) = P(x_1, x_2, \ldots, x_N, y_1, y_2, \ldots, y_M)
\]

\[
= \langle \prod_{n=1}^{N} X_{n}^{x_n} (1 - X_{n})^{1-x_n} \prod_{m=1}^{M} Y_{m}^{y_m} (1 - Y_{m})^{1-y_m} \rangle.
\]

In particular, the probability for \( n, m \) defaults in each sector, which is denoted as \( P_{N,M}(n,m) \), is

\[
P_{N,M}(n,m) = N C_m \times M C_m \times P(1, 1, \ldots, 1, 0, \ldots, 0, 1, 1, \ldots, 1, 0, \ldots, 0)
\]

\[
= N C_m \times M C_m \times \langle \prod_{n'=1}^{n} X_{n'} \prod_{k=n+1}^{N} (1 - X_{k}) \prod_{m'=1}^{m} Y_{m'} \prod_{l=m+1}^{M} (1 - Y_{l}) \rangle
\]

\[
= N C_m \times M C_m \times \sum_{k=0}^{N-n} \sum_{l=0}^{M-m} (-1)^{k+l} N-n C_k \times M-m C_l
\]

\[
\times \langle \prod_{n'=1}^{n+k} X_{n'} \prod_{m'=1}^{m+l} Y_{m'} \rangle.
\]
Moody’s Correlated Binomial Default Distribution

$P_{N+M}(n)$ is easily calculated from $P_{N,M}(n, m)$ as

$$P_{N+M}(n) = \sum_{n' = 0}^{n} P_{N,M}(n', n - n').$$

In decomposing $\langle \prod_{n=1}^{k} X_n \prod_{m=1}^{l} Y_m \rangle$, one can do it in any order. The independence of the order of the decomposition of $\langle \prod_{n=1}^{k} X_n \prod_{m=1}^{l} Y_m \rangle$ is guaranteed by (45) and (46). We decompose it as

$$\langle \prod_{n=1}^{k} X_n \prod_{m=1}^{l} Y_m \rangle = \prod_{n=0}^{k-1} p_{n,0} \times \prod_{m=0}^{l-1} q_{k,m}.$$  (50)

For the maximum value of $\rho_{xy}$, it is necessary to check all values of the joint probabilities. However, $N + M$ model is reduced to $N + 1$ or $1 + M$ model by choosing $M = 1$ or $N = 1$ respectively. From the discussions and the results in the previous subsection for the $N + 1$ model, we can anticipate as follows.

(i) $\lambda_x = \lambda_y = 0$ : If $p_x = p_y$ and $\rho_x = \rho_y$, two MCB models are the same and we can set $\rho_{xy} = \rho_x = \rho_y$. Two models merge completely and we have a $(N + M)$ assets MCB model. As the asymmetry between the two models becomes large ($p_x \neq p_y$ or $\rho_x \neq \rho_y$), $\text{Max}(\rho_{xy})$ decreases from $\rho_x, \rho_y$.

(ii) $\lambda_x, \lambda_y > 0$: As $\lambda_x, \lambda_y$ increase, $\text{Max}(\rho_{xy})$ becomes large. The asymmetry in $p_x, p_y$ and $\rho_x, \rho_y$ diminishes $\text{Max}(\rho_{xy})$.

Using the above coupled $N + M$ MCB model, we study the effect of the dispersion of the default probability on $P_N(n)$ and on the evaluation of tranches. More complete analysis about the difference between the usage of individual spreads and of portfolio average spreads in CDO pricing has been performed in [5]. There, the usage of the average spread results in the lower estimation of the equity tranche. We consider a portfolio with $N + N$ assets. The assets in each sector have default probabilities $p \pm \Delta P_d$ and an intra-sector default correlation $\rho$. We set the inter-sector default correlation $\rho_{xy}$ also as $\rho$. The inhomogeneity in the default probability is controlled by $\Delta P_d$. If we set $\Delta P_d = 0$, the two sectors are completely merged to one sector and we have a homogeneous $2N$ MCB model with $p$ and $\rho$. In figure 9, we shows the default probability difference $\Delta P(n)$ between the inhomogeneous case $P_{N+N}(n)$ with $\Delta P_d \neq 0$ and homogeneous case $P_{2N}(n)$. $\Delta P(n)$ is defined as

$$\Delta P(n) = P_{N+N}(n) - P_{2N}(n).$$

We set $N = 20$, $p = \rho = 0.03$ and $\lambda = 0.3$. The solid curve represents the data for $\Delta P_d = 0.01$ and the dotted curve stands for the case $\Delta P_d = 0.02$. We see that $\Delta P(n)$ is large only for small $n$. We also plot the results for the $N + M$ BBD model. To construct the loss distribution function, it is necessary to change (39) and (40) to

$$\text{Cor}(X_{n+1}X_{n+2} \mid \prod_{n'=1}^{n} X_{n'} = 1) = \rho/(1 + n\rho) \quad (52)$$

$$\text{Cor}(Y_{m+1}Y_{m+2} \mid \prod_{m'=1}^{m} Y_{m'} = 1) = \rho/(1 + m\rho). \quad (53)$$
Figure 9. $\Delta P(n)$ vs $n$. We set $N = 20$, $p = 0.03$, $\rho = 0.03$ and $\lambda_x = \lambda_y = 0.3$. MCB in solid and dotted lines, and BBD in + and × symbols.

Eq. (45) is also changed as

$$
\text{Cor}(X_{n+1} Y_{m+1} \prod_{n'=1}^{n} X_{n'} \prod_{m'=1}^{m} Y_{m'} = 1) = \rho/(1 + (n + m)\rho).
$$

The recursive relations are also changed, but the remaining procedures are the same as for the $N + M$ MCB model. As we have shown in the previous section, the bulk shapes of the loss distribution of MCB and BBD models are almost the same, and the effects of $\Delta P_d$ on them are also similar.

To see the effect of $\Delta P_d$ on the evaluations of tranches, we need to study the change in the cumulative distribution functions $D(i)$. $D(i)$ is defined as

$$
D(i) = \sum_{n=i}^{N} P_N(n).
$$

$D(i)$ represents the expected loss rate of the $i$–th tranche, which is damaged if more than $i$ assets default. One of the important properties of $D(i)$ is

$$
\frac{1}{N} \sum_{i=1}^{N} D(i) = p.
$$

This identity means that the tranches distribute the portfolio credit risk between them. Another important property of $D(i)$ is that the expected loss rate $D[i, j]$ of the layer
Figure 10. Plots of $\Delta D(i)/D(i)$ vs $\Delta P_d$ and $\Delta D(i)$ vs $\Delta P_d$. $N = 20, p = \rho = 0.03$ and $\Delta P_d = 0.01$. We plot for $1 \leq n \leq 15$.

We see that the lower tranches' $(i \leq 2)$ expected losses increase by $\Delta P_d$, which is reasonable. The default probability of half the assets of the portfolio increases $p \rightarrow p + \Delta P_d$ and the expected losses of the subordinated tranches increase. For the senior tranches, the absolute value of $\Delta D(i)$ decreases with $i$, however this does not mean that $\Delta P_d$ does not have a small effect on them. The absolute value of $D(i)$ also decreases with $i$. From Figure 10, we see that the ratio $\Delta D(i)/D(i)$ does not necessarily decrease with $i$.

3.4. Multi Sector Case

To couple two or more MCB models, we consider a portfolio with $N$ assets, which are categorized in $K$ different sectors. Figure 11 sketches the structure of the portfolio. The $k$-th sector contains $N_k$ assets and the relation $\sum_{k=1}^{K} N_k = N$ holds. The states of the assets in the $k$-th sector is described by $X_{nk}^k (n_k = 1, \ldots, N_k)$ and the default rate and default correlation are denoted as $p_k$ and $\rho_k$. For the inter-sector default correlation, we
denote this as $\rho_{ij}$ for the default correlation between the $i$–th and $j$–th sector. The intra-sector default correlation and inter-sector default correlation are different and the former is larger than the latter in general [27].

Figure 11. Structure of the portfolio. There are $K$ sectors and $k$-th sector contains $N_k$ assets. The default probability and default correlation for the assets in the $k$-th sector is $p_k, \rho_k$. The inter-sector default correlation between the $i$–th and $j$–th sector is $\rho_{ij}$.

We have not yet succeeded in the coupling of three or more MCB models by generalizing the result for the coupled $N + M$ MCB model. The reason is that the self-consistency relations are very rigid restrictions on the MCB model. These relations, for which the two-sectors version is given by [16], assure the independence of the order of the decomposition in the estimation of the expected values of the product of random variables. It is difficult to impose any simple relations on the conditional default probabilities that satisfy the self-consistency relations.

In order to construct the joint default probabilities for the assets states $(\vec{x}^1, \vec{x}^2, \ldots, \vec{x}^K)$, we do not glue together $K$ MCB models directly. Instead, as depicted in figure 12, we glue multiple MCB models through one random variable $Y$. More concretely, we prepare $K$ sets of $N_k + 1$ MCB models. $N_k + 1$ MCB model is the $N_k$ MCB model coupled with $Y$. The probability of $P(Y = 1)$ is written as $p_y$. We introduce the following conditional default probabilities

$$p_{n_k,0}^k = \langle X_{n_k+1}^k | \prod_{n_{k,1}^+ = 1}^{n_k} X_{n_{k,1}}^k = 1 \rangle \quad \text{and} \quad p_{n_k,1}^k = \langle X_{n_k+1}^k | \prod_{n_{k,1}^+ = 1}^{n_k} X_{n_{k,1}}^k \times Y = 1 \rangle$$
Moody’s Correlated Binomial Default Distribution

\[ q_{nk}^k = < Y | \prod_{n' = 1}^{n_k} X_{n_k}^{k} = 1 >. \] (58)

We also impose the following conditions on \( p_{nk,0}^k, p_{nk,1}^k \) and \( q_{nk}^k \) as

\[
\text{Cor}(X_{nk+1}^k, X_{nk+2}^k | \prod_{n' = 1}^{n_k} X_{n_k'}^{k}) = \rho_k \exp(-n_k \lambda) \] (59)

\[
\text{Cor}(X_{nk+1}^k, Y | \prod_{n' = 1}^{n_k} X_{n_k'}^{k}) = \rho_{ky} \exp(-n_k \lambda). \] (60)

The joint default probabilities \( P(\vec{x}, y) \) and the conditional joint default probabilities \( P_k(\vec{x}^k | y) \) are constructed as before.

\[
P(\vec{x}, y) = \prod_{n_k = 0}^{N_k} (X_{nk}^k)^{X_{nk}^k} (1 - X_{nk}^k)^{1 - X_{nk}^k} \times Y^y (1 - Y)^{1-y} \] (61)

\[
P(\vec{x}, y) = P_k(\vec{x}^k | y) P(y). \] (62)

Packing these conditional default probabilities \( P_k(\vec{x}^k | y) \) into a bundle, we construct the joint default probabilities for the total portfolio as

\[
P(\vec{x}^1, \vec{x}^2, \cdots, \vec{x}^K) = \sum_{y=0,1} p(y) \times \prod_{k=1}^{K} P(x_1^k, x_2^k, \cdots, x_{N_k}^k | y). \] (63)

We also obtain the default probability function \( P_N(n_1, n_2, \cdots, n_K) \) for \( n_k \) default in the \( k \)-th sector as

\[
P_N(n_1, n_2, \cdots, n_K) = \sum_{y=0,1} p(y) \times \prod_{k=1}^{K} P_{N_k}(n_k | y) \] (64)

From the expression, it it easy to calculate the probability for \( n \) defaults and we write it as \( P_N(n) \).

For the default correlation between the different sectors, we can show the next relations.

\[
\rho_{ij} = \rho_{iy} \times \rho_{jy}. \] (65)

More generally, the conditional inter-sector default correlations obey the following relations.

\[
\text{Cor}(X_{n_i+1}^i, X_{n_j+1}^j | \prod_{n_i' = 1}^{n_i} X_{n_i'}^i \prod_{n_j' = 1}^{n_j} X_{n_j'}^j = 1) = \rho_{ij} \exp(-(n_i + n_j) \lambda) \] (66)

These relations mean that our construction procedure is natural from the viewpoint of the original MCB model. In particular, in the \( K = 2 \) case, these relations are completely equivalent with those of the \( N + M \) coupled MCB model. See (45) and (66). The conditional default probabilities obey the same conditions.

Furthermore, from the results on \( \text{Max}(\rho_{xy}) \) of the \( N + 1 \) MCB model, we see that the model can induce a realistic magnitude of the inter-sector default correlation. By choosing \( \lambda \) and \( p_y \) properly, it is possible to set \( \rho_{ky} \) as large as several times of \( \rho_k \).
Figure 12. Gluing multiple MCB models with $Y$. The correlation between $X^k_{n_k}$ and $Y$ is $\rho_{ky}$. The default correlation between $X^i_{n_i}$ and $X^j_{n_j}$ is given as $\rho_{ij} = \rho_{iy} \times \rho_{jy}$.

Figure 13 plots $\max(\rho_{xy}) = \max(\rho_{ky})$ as functions of $\rho = \rho_k$. We set parameters as depicted in the figure. The solid line depicts the data for $r = \frac{p_y}{p_x} = 3.0$. The other two curves correspond to $r = 1.0$ (+) and $r = 15.0$ (×). We see that by setting $r = 3.0$, it is possible to set $\rho_{ky}$ as large as several times of $\rho_k$. If the intra-sector correlation $\rho_i = \rho_j$ is 10%, we can set $\rho_{iy} = \rho_{jy} = 20\%$. The inter-sector correlations is then $\rho_{ij} = \rho_{iy} \times \rho_{jy} = 4\%$. In general, $\rho_{\text{inter}}$ is smaller that $\rho_{\text{intra}}$, we think that the present model can incorporate a strong enough inter-sector default correlation.

To prove the relations (65) and (66), in order to calculate the correlation, we need to estimate the next expression.

$$< X^i_{n_i+1} X^j_{n_j+1} | \prod_{i' = 1}^{n_i} X^i_{i'} \prod_{j' = 1}^{n_j} X^j_{j'} = 1 >. \quad (67) $$

If we fix the random variable $Y$, $X^i_{n_i}$ and $X^j_{n_j}$ are independent. They are coupled by $Y$ and the above equation is estimated by the average over $Y = 0$ and $Y = 1$ as

$$< X^i_{n_i+1} X^j_{n_j+1} | \prod_{i' = 1}^{n_i} X^i_{i'} \prod_{j' = 1}^{n_j} X^j_{j'} = 1 > = < X^i_{n_i+1} | \prod_{i' = 1}^{n_i} X^i_{i'} = 1, Y = 1 > < X^j_{n_j+1} | \prod_{j' = 1}^{n_j} X^j_{j'} = 1, Y = 1 > p_y $$
Moody’s Correlated Binomial Default Distribution

$$\max(\rho_{xy})$$

Maximum value of $\rho_{xy}$ vs $\rho$.

Figure 13. Plot of $\max(\rho_{xy})$ vs $\rho$. $N = 30, p_x = 0.03$ and $\lambda = 0.3$.

$$< X_{n_i+1}^i | \prod_{i' = 1}^{n_i} X_{i'}^i = 1, Y = 0 > < X_{n_j+1}^j | \prod_{j' = 1}^{n_j} X_{j'}^j = 1, Y = 0 > (1 - p_y)$$

$$= p_{i,1}^i p_{n,1}^j p_y + \tilde{p}_{i,0}^i \tilde{p}_{n,0}^j (1 - p_y).$$  \hspace{1cm} (68)

Here, we denote the conditional default probabilities with the condition $Y = 0$ as $\tilde{p}_{n,0}^i$.

$$\tilde{p}_{n,0}^i = < X_{n_i+1}^i | \prod_{i' = 1}^{n_i} X_{i'}^i, (1 - Y) = 1 >.$$  \hspace{1cm} (69)

Between $p_{n,0}^i$ and $\tilde{p}_{n,0}^i$, the next relation holds.

$$p_{n,0}^i = p_{i,1}^i p_y + \tilde{p}_{n,0}^i (1 - p_y).$$  \hspace{1cm} (70)

In addition, from the correlation between $X_{n_i}^i$ and $Y$, we also have the next relations.

$$\tilde{p}_{n,0}^i p_y = p_{i,0} p_y + \rho_{xy} \sqrt{p_{i,0}(1 - p_{i,0}) p_y (1 - p_y)}.$$  \hspace{1cm} (71)

Putting these relations into (68), we can prove the next equations.

$$< X_{n_i+1}^i X_{n_j+1}^j | \prod_{i' = 1}^{n_i} X_{i'}^i \prod_{j' = 1}^{n_j} X_{j'}^j = 1 >$$

$$= p_{n,0}^i p_{n,0}^j + \rho_{xy} \sqrt{p_{n,0}^i (1 - p_{n,0}^i) p_{n,0}^j (1 - p_{n,0}^j)} \times e^{-\lambda (n_i + n_j)}.$$  \hspace{1cm} (72)

Using these relations, we calculate the inter-sector correlation and prove (66).

Next we need to check the validity of the above gluing process. We consider the two sector case $K = 2$ and their intra-sector parameters are set to be the same as $N$, $p$ and $\rho$ in each sector. For the inter-sector default correlation $\rho_{12}$, we set $\rho_{1y}^2 = \rho_{2y}^2 = \rho_{12}$ in the above multi-sector model. If the gluing process of the multi-sector model works
well, $P_{2N}(n)$ should coincide with $P_{N+N}(n)$ of the coupled $N + N$ MCB model in the previous subsection. Figure 14 shows $\Delta P(n) = P_{2N}(n) - P_{N+N}(n)$ with $N = 5, 10, 20$ and $\lambda = 0.3$. We set $p = \rho = 0.03$ and $\lambda = 0.3$. In addition we also plot the data for $\rho = 0.02$ and $N = 20$. As the system size $N$ becomes large, the discrepancy $\Delta P(n)$ increases. With the same system size $N = 20$, as the inter-sector correlation increases, the discrepancy also increases. As we have stated previously, these models obey the same conditions on the conditional default probabilities, however the default probability profile does not coincide. The glueing process by the auxiliary random variable $Y$ may cause changes to the joint probabilities. We have not yet fully understood this point.

With the present model, we study the effects of the inhomogeneous default correlation $\rho_{\text{inter}} \neq \rho_{\text{intra}}$ on the default distribution function $P_N(n)$ and the loss rates $D(i)$. We consider ideal portfolios which have the same default probability $p_k = p$ and default correlation $\rho_k = \rho$. The inter-sector default correlations are also set to be the same value as $\rho_{ij} = \rho_{\text{inter}}$. $N_k$ are also set to be the same $N_k = N_s$. We fix the total number of assets as $N$ and compare $P_N(n)$ and $D(i)$ between the portfolios with different number of sectors $K$. Of course, between $K$ and $N_s$ the relation $N = N_s \times K$ holds. We set $\rho_{\text{inter}} < p$. As the number $K$ increases, the average default correlation is
Moody's Correlated Binomial Default Distribution

governed by the inter-sector correlation $\rho_{\text{inter}}$ and becomes weak.

At the extreme limit $K = N$ case, each sector contains only one asset. In the inter-sector default correlation, $P_N(n)$ is given by the superposition of $P_N(n|Y = 1)$ and $P_N(n|Y = 0)$. The conditional default probabilities $p^k_{1,0}, p^k_{1,1}$ and $\tilde{p}^k_{1,0}$ are given as

\begin{align}
p^k_{1,0} &= p \quad \text{and} \quad p^k_{1,1} = p + \sqrt{\rho_{\text{inter}} (1 - p)} \frac{p}{1 - p} (1 - p_y)
\end{align}

(73)

\begin{align}
\tilde{p}^k_{1,0} &= p - \sqrt{\rho_{\text{inter}} p} \sqrt{\frac{1 - p}{p}} \frac{p_y}{(1 - p_y)}.
\end{align}

(74)

$P_N(n|Y = 1)$ and $P_N(n|Y = 0)$ are the binomial distributions $\text{Bin}(N, p^k_{1,1})$ and $\text{Bin}(N, \tilde{p}^k_{1,0})$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{k_dependence_of_loss_distribution.png}
\caption{Semi-log plot of $P_N(n)$ vs $n$. $N = 100$, $K = 2, 10, 100$, $p = \rho = 0.03$, $\rho_{\text{inter}} = 0.01$ and $\lambda = 0.3$. We set $p_y = 0.5$. Solid line represents $K = 1$ and $\rho = 0.03$ case and the dotted line shows the curve for $K = 1$ and $\rho = 0.01$.}
\end{figure}

Figure 15 shows the semi-log plot of the default probability $P_N(n)$ for $K = 1, 2, 10, 50, 100$. We set the model parameters as $p = \rho = 0.03$, $N = 100$, $\rho_{\text{inter}} = 0.01$ and $p_y = 0.5$. The solid curve plots the data for $K = 1$ and $\rho = 0.03$ and the dotted line shows the data for $K = 1$ and $\rho = 0.01$. As $K$ increases, the data for each $K$ departs from the solid line. At $K = 10$, the data almost shrinks on the dotted line ($K = 1$ and $\rho = 0.01$). The data for $K = 50$ almost coincide with those of $K = 100$, whose $P_N(n)$ is given by the superposition of the two binomial distributions. This point is also a drawback of the present model. If the glueing process works perfectly, these data
should coincide with the homogeneous portfolio case \( K = 1 \) and \( \rho = 0.01 \). However, this discrepancy is inherent property of the model. Contrary to the discrepancy in the two-sector case, the conditions on the conditional default probabilities \( p_{nk,1}^k \) are different from those of the \( K = 1 \) homogeneous portfolio. They could cause the difference in \( P_N(n) \).

![Sensitivity of D(i) w.r.t. \( \rho_{\text{intra}} \)](image)

**Figure 16.** Plot of \( |\Delta D(i)| \) and \( D(i) \) vs \( i \). \( \Delta D(i) \) are between \( \rho = 0.01, 0.05 \) and \( \rho = 0.03 \). \( N = 100, K = 2, 10, 100, p = 0.03, \rho_{\text{inter}} = 0.01 \) and \( \lambda = 0.3 \). We set \( p_y = 0.5 \). The solid curve shows the data of \( D(i) \) with \( \rho = 0.03 \).

From the above discussions on \( P_N(n) \) with different \( K \), we think that the inter-sector default correlation \( \rho_{\text{inter}} \) is more important than the intra-sector default correlation \( \rho \) in cases of a large \( K \). In the \( K = 10 \) case, the \( P_N(n) \) are roughly given by those of the homogeneous portfolio with \( \rho = \rho_{\text{inter}} \). If one estimates the implied values of \( \rho \) and \( \rho_{\text{inter}} \) from the premium (or \( D(i) \)) of the portfolio with large \( K \), this point is crucial. In Figure 16, we show \( |\Delta D(i)| \) for \( K = 10 \) between \( \rho = 0.01, 0.05 \) and \( \rho = 0.03 \). We set the model parameters as in the previous figure. For comparison, we also plot \( D(i) \) for \( \rho = 0.03 \). \( D(i) \) represents the expected loss rate of the \( i \)-th tranche, the magnitude of \( |\Delta D(i)| \) is important when one estimates the implied default correlation \( \rho \) from the premium of the tranche. We see that \( |\Delta D(i)| \) with small \( i \) is small as compared with \( D(i) \). If we change \( \rho \) from \( \rho = 0.03 \), \( D(i) \) does not change significantly. It is difficult to obtain the implied values of \( \rho \) from the premium of the tranche with lower seniority. In contrast with medium values of \( i \approx 13 \), the magnitudes of \( D(i) \) and \( \Delta D(i) \) are almost comparable. \( D(i) \) is sensitive to the change in \( \rho \) and it is not difficult to derive the implied value of \( \rho \).
4. Implied Default Correlation

In the last section, as a concrete example, we try to estimate the implied values of the default correlation from the premium of a synthetic CDO. We treat iTraxx-CJ (Series 2), which is an equally weighted portfolio of 50 CDSs on Japanese companies. The standard attachment points and detachment points are \( \{0, 3\%\}, \{3\%, 6\%\}, \{6\%, 9\%\}, \{9\%, 12\%\} \) and \( \{12\%, 22\%\} \). Table 1 shows quotes on July 5, 2005. The quote for the \( \{0\%, 3\%\} \) tranche shows the upfront payment (as a percent of principal) that must be paid in addition to 300 basis points per year. The other quotes for the other tranches are the annual payment rates in basis points per year. The index indicates the cost of entering into a CDS on all 50 companies underlying the index. The recovery rate \( R \) is 0.35.

Table 1. Quotes for iTraxx-CJ Tranches on July 5, 2005. Quotes for the \( \{0\%, 3\%\} \) tranche are the percent of the principal that must be paid up front in addition to 300 basis points per year. Quotes for other tranches and the index are in basis points. Source: Morgan Stanley Japan Securities Co. and Bloomberg

| Tranche      | 5-year Quotes |
|--------------|---------------|
| \( \{0\%, 3\%\} \) | 15.75         |
| \( \{3\%, 6\%\} \) | 113.25        |
| \( \{6\%, 9\%\} \) | 42.0          |
| \( \{9\%, 12\%\} \) | 30.5          |
| \( \{12\%, 22\%\} \) | 15.5          |
| Index        | 24.55         |

In order to get the implied default correlation for each tranche, it is necessary to relate the loss distribution function \( P_N(n) \) to the premiums. The premiums are the present value of the expected cash flows. The calculation of this present value involves three terms. We denote by \( P_k(a_L, a_H) \) the remaining notional for the \( \{a_L, a_H\} \) tranche after \( k \) defaults. It is given as

\[
P_k(a_L, a_H) = \begin{cases} 
(a_H - a_L)N & k < \lfloor a_L N/(1 - R) \rfloor \\
 a_H N - k(1 - R) & \lfloor a_L N/(1 - R) \rfloor \leq k \leq \lfloor a_H N/(1 - R) \rfloor \\
0 & k \geq \lfloor a_H N/(1 - R) \rfloor 
\end{cases} \tag{75}
\]

Here, \( \lfloor x \rfloor \) means the smallest integer greater than \( x \). For simplicity, we treat the 5-year as one period. The three terms are written as

\[
A = 5.0 \times < P_k(a_L, a_H) > e^{-5.0r} \\
B = 2.5 \times ((a_H - a_L)N - < P_k(a_L, a_H) >) e^{-r \frac{a}{2}} \\
C = (\lfloor a_H - a_L \rfloor N - < P_k(a_L, a_H) >) e^{-r \frac{a}{2}} \tag{76}
\]

where \( r \) is the risk-free rate of interest and we set \( r = 0.01 \). By considering the total value of the contract, one can see that the break even spread is given as \( C/(A + B) \).

For the index, \( a_H = 1.0 \) and \( a_L = 0.0 \) and it is possible to estimate the average default probability \( p \). Instead, we use the CDS data for each company and estimate the average default probability \( p \) and its dispersion \( \Delta P_d \) as

\[
p = 1.8393\%/5\text{-year} \quad \Delta P_d = 1.131\%/5\text{-year}.
\]

With these parameters, the tranche correlations can be implied from the spreads quoted in the market for particular tranches. These correlations are known as tranche...
correlations or compound correlations. As a pricing model, we use MCB, BBD and Gaussian copula models. For MCB, we use the following candidates.

- Original MCB model (MCB1). $N = 50, K = 1$ and $\lambda = 0.0$.
- Short tail MCB model (MCB2). $N = 50, K = 1$ and $\lambda = 0.3$.
- Short tail MCB model (MCB3). $N = 50, K = 1$ and $\lambda = 0.6$.
- Disordered MCB model (MCB4). $N = 25 + 25, \lambda = 0.3, \rho_{xy} = \rho_x = \rho_y = \rho$ and $p_x = p + \Delta P_d, p_y = p - \Delta P_d$. $N + M$ MCB model with inhomogeneous default probability.
- Two-sector MCB model (MCB5). $K = 2, \rho_x = \rho_y = \rho, p_x = p_{y} = p$ and $\rho_{xy} = 0.0$. Assets are categorized in 2 sectors and $\rho_{inter} = 0.0$.

Table 2. Implied tranche correlation (%) for 5-year iTraxx-CJ on July 5, 2005.

| Tranches | MCB 1 | MCB 2 | MCB 3 | MCB 4 | MCB 5 | BBD | Gaussian |
|----------|-------|-------|-------|-------|-------|-----|----------|
| {0%, 3%} | 11.79 | 10.8  | 9.96  | 12.88 | 21.2  | 11.4| 13.8     |
| {3%, 6%} | 1.27  | 1.18  | 1.13  | 1.36  | 2.45  | 1.26| 1.35     |
| {6%, 9%} | 3.16  | 3.08  | 3.09  | 3.46  | 6.32  | 3.15| 3.23     |
| {9%, 12%}| 6.16  | 5.95  | 5.90  | 6.65  | 12.15 | 6.11| 6.31     |
| {12%, 22%}| 9.78 | 9.67  | 9.90  | 10.67 | 19.97 | 9.73| 9.46     |

Table 2 shows implied tranche correlations for the 5-year quotes in Table 1. We see a "Correlation Smile", which is a typical behavior of implied correlations across portfolio tranches [13].

We also find that the implied correlations are different among the models. For MCB, models with a larger $\lambda$ have a smaller correlation skew. The correlation for {0%, 3%} decreases with $\lambda$ and other correlations do not change significantly. If a probabilistic model describe the true default distribution, there should not exist any correlation skew. As a model approaches the true distribution, we can expect that the skew decreases. MCB3 is more faithful to the true default distribution. The skew of BBD is between MCB 1 and MCB 2, which is reasonable because the profile of BBD is between MCB1 and MCB2 (see figure 5). Gaussian copula’s skew range is larger than MCB 1, MCB 2, MCB 3 and BBD.

About the effect of $\Delta P_d$, the implied correlations are considerably different between $\Delta P_d \neq 0$ (MCB 4) and $\Delta P_d = 0.0$ (MCB 2). In the estimation of the implied correlation, we cannot neglect the fluctuation $\Delta P_d$. In particular, for the tranches {0%, 3%}, the implied value is affected greatly by $\Delta P_d$. As has been discussed in [5], the increase in the dispersion of the default probability increases the loss in the equity tranche. It is necessary to increase the implied correlation to match with the market quote.

In the case $K = 2$ (MCB5), the correlation level and correlation skew are very large. This means that if assets are categorized in many sectors and inter-sector correlation is
very weak, the loss distribution accumulates around the origin \( n = 0 \). In order to match with the market quote, a large intra-sector correlation is necessary.

For the estimation of the model parameter \( \lambda \), we think that the resulting loss distribution should look closer to the implied loss distribution. That is, the range of tranche correlation skew should be small. Figure 17 shows the tranche correlation vs \( \lambda \). As we increase \( \lambda \), the skew range becomes small. At \( \lambda \simeq 0.61 \), the range becomes minimal. We should calibrate \( \lambda \) to be \( \lambda = 0.61 \sim 0.62 \).

![Tranche Correlation vs \( \lambda \) for 5-year iTraxx-CJ on July 5, 2005](image)

**Figure 17.** Range of implied tranche correlation for 5-year iTraxx-CJ on July 5, 2005 vs \( \lambda \).

5. Concluding Remarks

In this paper, we generalize Moody’s correlated binomial default distribution to the inhomogeneous portfolio cases. As the inhomogeneity, we consider the non-uniformity in the default probability \( p \) and in the default correlation \( \rho \) and \( \rho_{\text{inter}} \). To treat the former case, we construct a coupled \( N + M \) MCB model and obtain the default probability function \( P_{N+M}(n) \). The inhomogeneity in \( p \) causes changes in the expected loss rates of the tranches with lower seniority.

In order to treat the inhomogeneity in the default correlation, we construct a multi-sector MCB model by glueing multiple MCB models by an auxiliary random variable \( Y \). We cannot take out the joining lines between the MCB models, for small portfolio and small \( \rho_{\text{inter}} \), the construction works well. For the inhomogeneity in \( \rho \), we divide a homogeneous portfolio into \( K \) sectors. We set \( \rho_{\text{inter}} < \rho = \rho_{\text{intra}} \) and see the effect of the increase in \( K \) on \( P_N(n) \). As the sector number \( K \) increases, the inter-sector
correlation $\rho_{\text{inter}}$ becomes more important than the intra-sector default correlation $\rho$. With large $K$, the default correlation is governed by $\rho_{\text{inter}}$ only. The CDOs, whose assets are categorized in many sectors, $\rho_{\text{inter}}$ should be treated more carefully than $\rho = \rho_{\text{intra}}$.

In order to check the validity of the MCB model and our generalization method, more careful treatment and calibration should be done. We assume that $\rho_n$ decays exponentially with $n$. With such a modification, the skew of the correlations diminishes, however, the skew remains significantly. Other models for $\rho_n$ should be considered. For this purpose, it is necessary to study the implied loss distribution directly. Recently, Hull and White [28, 29] developed a method to derive the implied loss distribution and to obtain the implied copula function from the market quotes of CDOs. The authors proposed a calibration method for $\rho_n$ from the implied loss function [24]. By incorporating this information in the MCB model’s framework, we may have a “Perfect” correlated binomial default distribution model which reflects market quotes completely. We think that our generalization method provides important information based directly on the market quotes.

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Moody’s Correlated Binomial Default Distribution

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