Deformed Boson Scheme including Conventional $q$-Deformation in Time-Dependent Variational Method. II

Deformation of the $su(2)$- and the $su(1,1)$-Algebras in the Schwinger Boson Representation

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In this paper as a continuation of Part I, the case of two kinds of boson operators is treated. The deformation of the coherent states for the $su(2)$- and the $su(1,1)$-algebras and their related deformed algebras are discussed in various form including the most popular form.

§1. Introduction

The present paper, Part (II), the continuation of a previous work referred to as (I), 1 is concerned with the deformed boson scheme for the case of two kinds of boson operators. There exist two reasons why we intend to investigate this case. We mentioned in §6 of (I) that the deformed boson scheme realizes its real ability in the present case. In the case of two kinds of boson operators, we know two spin systems which obey the $su(2)$- and $su(1,1)$-algebras in the Schwinger boson representations. 2 As was reviewed in Ref. 3, we can construct coherent states which are suitable for obtaining the classical counterparts of these two spin systems. These counterparts are useful for describing the time-evolution of these spin systems in the framework of the time-dependent variational method. For example, with the help of the $su(1,1)$-spin, damped and amplified oscillation can be described in a conservative form. 4 Therefore, we are naturally led to an investigation of the deformation of the coherent states. This is the first reason. The second reason is related to the deformed algebras. Quantum mechanically, the $su(2)_q$- and the $su(1,1)_q$-algebras are quite interesting and various aspects of the $su(2)_q$-algebra have been investigated. In particular, the investigation based on the form $[x]_q = (q^x - q^{-x})/(q - q^{-1})$ is well known. 5 The present authors also presented an idea regarding a method of deriving the Holstein-Primakoff representation for the $su(2)_q$- and the $su(1,1)_q$-algebras 6 in the framework of the MYT boson mapping. 7 Then, it is also a quite natural course to investigate the $su(2)_q$- and $su(1,1)_q$-algebras in our deformed boson scheme. The above two are the main reasons why we present Part (II).

In (I), the boson coherent state was deformed by a function $f(x)$ which determines $[x]_q$. The boson system treated in (II) consists of two kinds of boson operators...
and under a certain principle, the coherent states for the $su(2)$- and the $su(1,1)$-spins in the Schwinger boson representation are deformed by maximally three independent functions. Algebraically, it may be sufficient to employ two independent functions. Therefore, by changing the forms of these three functions, we are able to derive various types of the deformations including the most popular form for the $su(2)_q$-algebra. An interesting point is found in the fact that the Holstein-Primakoff boson representations for the $su(2)$- and the $su(1,1)$-spins are possible deformations of the $su(2)$- and the $su(1,1)$-algebras in the Schwinger representation. This form has already been used for describing the damped and amplified oscillational motion in the $su(2)$-spin system.

In §2, the $su(2)$- and the $su(1,1)$-spin systems expressed in terms of two kinds of boson operators and their classical counterparts are derived. The starting coherent states are of the same forms as those appearing in Ref. 3). Section 3 is devoted to obtaining the deformed coherent states. Its basic idea is the use of three, algebraically, two independent functions. Further, the classical counterparts are discussed. In §4, the $su(2)_q$- and the $su(1,1)_q$-algebras are obtained in our deformed boson scheme and various forms are discussed. In §5, the deformations referred to as the pseudo $su(2)$- and the pseudo $su(1,1)$-algebras are discussed and the Holstein-Primakoff representation is derived in the Schwinger representation by appropriately selecting the functions characterizing the deformation. Finally, in §6, some concluding remarks, together with a statement of a subsequent problem, are given.

§2. The $su(2)$- and the $su(1,1)$-spin systems and their classical counterparts

In Part I, we investigated the deformed boson scheme in many-body systems composed of one kind of boson operator ($c, c^*$). In the present paper (Part II), we will treat the case of many-body systems consisting of two kinds of boson operators ($\hat{a}, \hat{a}^*$) and ($\hat{b}, \hat{b}^*$). The operators ($\hat{a}, \hat{a}^*$) and ($\hat{b}, \hat{b}^*$) obey the same relations as those of ($\hat{c}, \hat{c}^*$), which are shown in Eqs. (I-2.1)-(I-2.6). For the present system, we know two spin systems, which obey the $su(2)$- and the $su(1,1)$-algebras. Hereafter, various relations for these systems will be presented together, an “a” and “b” of the same equations, namely, (Eq. a) and (Eq. b). The generators ($\hat{S}_0^0, \hat{S}_0$) for the $su(2)$-spin and ($\hat{T}_0^0, \hat{T}_0$) for the $su(1,1)$-spin are written down in the forms

$$\hat{S}_+^0 = \hat{a}^* \hat{b}, \quad \hat{S}_-^0 = \hat{b}^* \hat{a}, \quad \hat{S}_0 = (\hat{a}^* \hat{a} - \hat{b}^* \hat{b})/2 ,$$

$$\hat{T}_+^0 = \hat{a}^* \hat{b}^*, \quad \hat{T}_-^0 = \hat{b} \hat{a}, \quad \hat{T}_0 = (\hat{a}^* \hat{a} + \hat{b}^* \hat{b})/2 .$$

Further, the following operators $\hat{S}$ and $\hat{T}$ are introduced in each spin system:

$$\hat{S} = (\hat{b}^* \hat{b} + \hat{a}^* \hat{a})/2 ,$$

$$\hat{T} = (\hat{b} \hat{b}^* - \hat{a} \hat{a}^*)/2 .$$

The commutation relations are given in the form

$$[\hat{S}_+^0, \hat{S}_-^0] = +2\hat{S}_0 , \quad [\hat{S}_0, \hat{S}_\pm^0] = \pm \hat{S}_\pm^0 , \quad [\hat{S}_\pm^0, \hat{S}] = 0 ,$$

$$(2.3a)$$
\[
[T^0_+, T^0_-] = -2\hat{T}_0, \quad [\hat{T}_0, \hat{T}^0_\pm] = \pm \hat{T}^0_\pm, \quad [\hat{T}^0_\pm, \hat{T}] = 0. \quad (2.3b)
\]

In some papers written by the present authors, we investigated two forms of normalized wave packets, which are expressed in the following forms:
\[
|c^0_+\rangle = \left(\sqrt{T_0}\right)^{-1} \exp(\gamma \hat{S}^0_+) \exp(\delta \hat{b}^*)|0\rangle \\
= \left(\sqrt{T_0}\right)^{-1} \exp(\gamma \hat{a}^* \hat{b}) \exp(\delta \hat{b}^*)|0\rangle , \\
T_0 = \exp(|\delta|^2(1 + |\gamma|^2)) , \\
|c^0_-\rangle = \left(\sqrt{T_0}\right)^{-1} \exp(\gamma \hat{T}^0_+) \exp(\delta \hat{b}^*)|0\rangle \\
= \left(\sqrt{T_0}\right)^{-1} \exp(\gamma \hat{a}^* \hat{b}^*) \exp(\delta \hat{b}^*)|0\rangle , \\
T_0 = (1 - |\gamma|^2)^{-1} \exp(|\delta|^2/(1 - |\gamma|^2)) . \quad (2.4a)
\]

\[
|c^0_+\rangle = \left(\sqrt{T_0}\right)^{-1} \exp(\gamma \hat{S}^0_+) \exp(\delta \hat{b}^*)|0\rangle \\
= \left(\sqrt{T_0}\right)^{-1} \exp(\gamma \hat{a}^* \hat{b}) \exp(\delta \hat{b}^*)|0\rangle , \\
T_0 = \exp(|\delta|^2(1 + |\gamma|^2)) , \\
|c^0_-\rangle = \left(\sqrt{T_0}\right)^{-1} \exp(\gamma \hat{T}^0_+) \exp(\delta \hat{b}^*)|0\rangle \\
= \left(\sqrt{T_0}\right)^{-1} \exp(\gamma \hat{a}^* \hat{b}^*) \exp(\delta \hat{b}^*)|0\rangle , \\
T_0 = (1 - |\gamma|^2)^{-1} \exp(|\delta|^2/(1 - |\gamma|^2)) . \quad (2.4b)
\]

The plus (+) and the minus (−) signs indicate the wave packets for the \(su(2)\) and the \(su(1,1)\)-spin systems, respectively. The quantities \(\gamma\) and \(\delta\) denote complex parameters and our aim is to investigate the behavior of these parameters. The forms (2.4) show that the states \(|c^0_+\rangle\) are coherently superposed in terms of various states obtained by operating with \(\hat{S}^0_+\) and \(\hat{T}^0_+\) successively on the state which is a linear combination of the states with the minimum weights \((\hat{S}^0_+ \exp(\delta \hat{b}^*)|0\rangle = \hat{T}^0_+ \exp(\delta \hat{b}^*)|0\rangle = 0)\). For the states (2.4a) and (2.4b), we can prove the following relations:
\[
\hat{\gamma}^0|c^0_+\rangle = \gamma(1 - \epsilon(\hat{N}_b + \epsilon)^{-1})|c^0_+\rangle , \quad \hat{\delta}^0|c^0_+\rangle = \delta|c^0_+\rangle \quad (2.5a)
\]
for (2.4a) and
\[
\hat{\gamma}^0|c^0_-\rangle = \gamma|c^0_-\rangle , \quad \hat{\delta}^0|c^0_-\rangle = \delta|c^0_-\rangle \quad (2.5b)
\]
for (2.4b), where \(\hat{\gamma}^0\) and \(\hat{\delta}^0\) are defined as
\[
\hat{\gamma}^0 = \hat{S}^0_0(\hat{N}_b + 1 + \epsilon)^{-1} , \quad \hat{\delta}^0 = \hat{b}
\]
for (2.4a) and
\[
\hat{\gamma}^0 = (\hat{N}_b + 1 + \epsilon)^{-1}\hat{T}^0_0 , \quad \hat{\delta}^0 = [1 - \hat{N}_a(\hat{N}_b + 1 + \epsilon)^{-1}]\hat{b}
\]
for (2.4b), respectively. Of course, \(\hat{N}_a\) and \(\hat{N}_b\) denote the boson number operators:
\[
\hat{N}_a = \hat{a}^* \hat{a} , \quad \hat{N}_b = \hat{b}^* \hat{b} . \quad (2.7)
\]
The symbol \(\epsilon\) denotes an infinitesimal parameter and \(\epsilon(\hat{N}_b + \epsilon)^{-1}\) in Eq. (2.5a) plays the role of the projection operator for the states \(|n\rangle = (\sqrt{n!})^{-1}(\hat{b}^*)^n|0\rangle ; n = 0, 1, 2, \ldots\):
\[
\epsilon(\hat{N}_b + \epsilon)^{-1}|n\rangle = \delta_{n,0}|n\rangle . \quad (\epsilon \rightarrow 0) \quad (2.8)
\]
Then, for a large value of \(|\delta|\), we can regard \(|c^0_+\rangle\) as \(|c^0_+\rangle \sim (1 - \epsilon(\hat{N}_b + \epsilon)^{-1})|c^0_+\rangle\). Thus, the states (2.4a) and (2.4b) are regarded as the eigenstates of \(\hat{\gamma}^0\) and \(\hat{\delta}^0\) defined in the relations (2.6a) and (2.6b), respectively.
The time-dependent variational method for the states $|c^0\rangle$ starts with the following relation:

$$\delta \int \langle c^0_+ | i \partial_t - \hat{H} | c^0_+ \rangle dt = 0.$$  \hspace{1cm} (2.9)

The states $|c^0\rangle$ satisfy the relation

$$\langle c^0_+ | i \partial_t | c^0_+ \rangle = \frac{i}{2} (\gamma^* \dot{\gamma} - \dot{\gamma}^* \gamma) \frac{\partial \Gamma_0}{\partial |\gamma|^2} \cdot \Gamma_0^{-1} + \frac{i}{2} (\delta^* \dot{\delta} - \dot{\delta}^* \delta) \frac{\partial \Gamma_0}{\partial |\delta|^2} \cdot \Gamma_0^{-1}.$$  \hspace{1cm} (2.10)

Then, let us define the following quantities:

$$x = \gamma \sqrt{(\partial \Gamma_0 / \partial |\gamma|^2) \cdot \Gamma_0^{-1}}, \quad y = \delta \sqrt{(\partial \Gamma_0 / \partial |\delta|^2) \cdot \Gamma_0^{-1}}.$$  \hspace{1cm} (2.11)

With the use of the new parameters $x$ and $y$, the relation (2.10) can be rewritten as

$$\langle c^0_+ | i \partial_t | c^0_+ \rangle = (i/2) \cdot (x^* \dot{x} - \dot{x}^* x) + (i/2) \cdot (y^* \dot{y} - \dot{y}^* y).$$  \hspace{1cm} (2.12)

The above is called the canonicity condition and in the sense of the time-dependent variational method, $(x, x^*)$ and $(y, y^*)$ can be regarded as the boson-type canonical variables in classical mechanics. The relation (2.11) tells us that $x$ and $y$ can be expressed in terms of $\gamma$, $\delta$, $|\gamma|^2$ and $|\delta|^2$. In the present case, with the use of the forms (2.4a) and (2.4b) for $\Gamma_0$, $\gamma$ and $\delta$ can be expressed inversely in terms of $x$, $y$, $|x|^2$ and $|y|^2$. For the $su(2)$-spin system, we have

$$\gamma = x \left( \sqrt{|y|^2 - |x|^2} \right)^{-1}, \quad \delta = y \sqrt{|y|^2 - |x|^2} \cdot |y|^{-2}.$$  \hspace{1cm} (2.13a)

In the case of the $su(1,1)$-spin, we have

$$\gamma = x \left( \sqrt{|y|^2 + 1 + |x|^2} \right)^{-1}, \quad \delta = y \sqrt{|y|^2 + 1} \cdot \left( \sqrt{|y|^2 + 1 + |x|^2} \right)^{-1}.$$  \hspace{1cm} (2.13b)

Next, we investigate the classical counterpart of the operators $\hat{\gamma}^0$ and $\hat{\delta}^0$. First, we note the relations

$$\langle c^0_+ | \hat{\gamma}^0 | c^0_+ \rangle = \gamma, \quad \langle c^0_+ | \hat{\delta}^0 | c^0_+ \rangle = \delta.$$  \hspace{1cm} (2.14)

From the argument given in the relation (2.8), the first relation in (2.14) is approximated. Further, we define the difference of any function $F(\hat{N}_a, \hat{N}_b)$ in the form

$$\Delta_{\hat{N}_a}^{(+)} F(\hat{N}_a, \hat{N}_b) = \pm \left[ F(\hat{N}_a + 1, \hat{N}_b) - F(\hat{N}_a, \hat{N}_b) \right],$$

$$\Delta_{\hat{N}_b}^{(+)} F(\hat{N}_a, \hat{N}_b) = \pm \left[ F(\hat{N}_a, \hat{N}_b + 1) - F(\hat{N}_a, \hat{N}_b) \right].$$  \hspace{1cm} (2.15)

With the use of the differences (2.15), the commutation relations of $(\hat{\gamma}^0, \hat{\gamma}^{0*})$ and $(\hat{\delta}^0, \hat{\delta}^{0*})$ for the $su(2)$-spin system are given as

$$[\hat{\gamma}^0, \hat{\gamma}^{0*}] = (\Delta_{\hat{N}_a}^{(+)} - \Delta_{\hat{N}_b}^{(-)} - \Delta_{\hat{N}_b}^{(+)} \Delta_{\hat{N}_a}^{(-)}) (\hat{\gamma}^{0*} \hat{\gamma}^0),$$

$$[\hat{\delta}^0, \hat{\delta}^{0*}] = (\Delta_{\hat{N}_a}^{(-)} - \Delta_{\hat{N}_b}^{(+)} - \Delta_{\hat{N}_b}^{(-)} \Delta_{\hat{N}_a}^{(+)}) (\hat{\delta}^{0*} \hat{\delta}^0).$$
The relation (2.16a) contains the infinitesimal parameter $\epsilon$. After operating with this relation on any state, we should take the limit $\epsilon \to 0$. The relations (2.13a) and (2.13b) tell us that $\gamma$ and $\delta$ are expressed in terms of the canonical variables $(x, x^*)$ and $(y, y^*)$. Then, we can calculate the Poisson bracket for these variables defined in the form

$$[A, B] = \partial_x A \cdot \partial_{x^*} B - \partial_{x^*} A \cdot \partial_x B + (\partial_y A \cdot \partial_{y^*} B - \partial_{y^*} A \cdot \partial_y B).$$

For the $su(2)$-spin system, the result is as follows:

$$[\gamma, \gamma^*] = (\partial_{N_a} - \partial_{N_b})|\gamma|^2,$$
$$[\delta, \delta^*] = \partial_{N_b} |\delta|^2,$$
$$[\gamma, \delta] = 0,$$
$$[\gamma, \delta^*] = \delta^* \gamma \cdot |\delta|^2 \partial_{N_b} |\delta|^{-2}.$$  

For the $su(1, 1)$-spin system, we have:

$$[\gamma, \gamma^*] = (\partial_{N_a} + \partial_{N_b})|\gamma|^2,$$
$$[\delta, \delta^*] = \partial_{N_b} |\delta|^2,$$
$$[\gamma, \delta] = 0,$$
$$[\gamma, \delta^*] = \delta^* \gamma \cdot |\gamma|^{-2} \partial_{N_b} |\gamma|^2.$$  

Here, $\partial_{N_a}$ and $\partial_{N_b}$ denote the differential with respect to $N_a$ and $N_b$, respectively. The variables $N_a$ and $N_b$ are the expectation values of $\hat{N}_a$ and $\hat{N}_b$, respectively:

$$N_a = \langle e_\pm|\hat{N}_a|e_\pm \rangle = |\gamma|^2 \partial \Gamma_0/\partial |\gamma|^2 \cdot \Gamma_0^{-1} = |x|^2,$$
$$N_b = \langle e_\pm|\hat{N}_b|e_\pm \rangle = |\delta|^2 \partial \Gamma_0/\partial |\delta|^2 \cdot \Gamma_0^{-1} \mp |\gamma|^2 \partial \Gamma_0/\partial |\gamma|^2 \cdot \Gamma_0^{-1} = |y|^2 \mp |x|^2.$$  

(2.19)
For the above derivation, the following formula is useful:

$$|\gamma|^2 \partial_{N_a} |\delta|^2 + |\delta|^2 \partial_{N_b} |\gamma|^2 - |\delta|^2 \partial_{N_b} |\gamma|^2 = 0 \, .$$ (2.20)

For the commutation relations (2.16a) and (2.16b), we perform the replacement

$$[ \, , ] \longrightarrow [ \, , ]_P \, , \quad \Delta \longrightarrow \partial \, .$$ (2.21)

Then, in the limits $\epsilon \to 0$ and $\partial_{N_a} \partial_{N_b} |\gamma|^2 \to 0$, the relations (2.16) are reduced to the relations (2.18). The limit $\partial_{N_a} \partial_{N_b} |\gamma|^2 \to 0$ means the neglect of quantal fluctuations around $(\partial_{N_a} \pm \partial_{N_b})|\gamma|^2$. Thus, for $\gamma$ and $\delta$ given in the relation (2.13), we have the following correspondence:

$$\hat{\gamma}^0 \sim \gamma \, , \quad \hat{\delta}^0 \sim \delta \, .$$ (2.22)

These relations imply that $\gamma$ and $\delta$ introduced as the variational parameters are the classical counterparts of the operators $\hat{\gamma}^0$ and $\hat{\delta}^0$, respectively.

The expectation values of $(\hat{S}^0_{\pm 0}, \hat{S})$ and $(\hat{T}^0_{\pm 0}, \hat{T})$ for the states $|c^0_{\pm}\rangle$ are expressed in the form

\begin{align*}
    \langle c^0_+ | \hat{S}^0_+ | c^0_+ \rangle &= \gamma^* |\delta|^2 = x^* \sqrt{2s - |x|^2} \, , \\
    \langle c^0_+ | \hat{S}^0_- | c^0_+ \rangle &= \gamma |\delta|^2 = x \sqrt{2s - |x|^2} \, , \\
    \langle c^0_+ | \hat{S}^0_0 | c^0_+ \rangle &= -(1 - |\gamma|^2) |\delta|^2 / 2 = |x|^2 - s \, , \\
    \langle c^0_+ | \hat{S}^0_+ | c^0_+ \rangle &= (1 + |\gamma|^2) |\delta|^2 / 2 = s \, , \\
    \langle c^0_+ | \hat{S}^0_0 | c^0_+ \rangle &= \gamma^* (1 - |\gamma|^2 + |\delta|^2)(1 - |\gamma|^2)^{-2} = x^* \sqrt{2t + |x|^2} \, , \\
    \langle c^0_+ | \hat{S}^0_0 | c^0_+ \rangle &= \gamma (1 - |\gamma|^2 + |\delta|^2)(1 - |\gamma|^2)^{-2} = x \sqrt{2t + |x|^2} \, , \\
    \langle c^0_0 | \hat{T}^0_0 | c^0_0 \rangle &= (1 + |\gamma|^2)(1 - |\gamma|^2 + |\delta|^2)(1 - |\gamma|^2)^{-2} / 2 = |x|^2 + t \, , \\
    \langle c^0_0 | \hat{T}^0_0 | c^0_0 \rangle &= (1 - |\gamma|^2 + |\delta|^2)(1 - |\gamma|^2)^{-1} / 2 = (|y|^2 + 1) / 2 = t \, .
\end{align*}

The above forms are the classical forms of the $su(2)$- and the $su(1,1)$-generators in the Holstein-Primakoff boson representation. The quantities $s$ and $t$ shown in the relations (2.23a) and (2.23b) denote the magnitudes of the $su(2)$- and the $su(1,1)$-spins, respectively, in the classical sense.

### §3. Deformation of the coherent state $|c^0_\pm\rangle$

We carry out the deformation of the state $|c^0_\pm\rangle$ by introducing three functions $\tilde{f}$, $\tilde{g}$ and $\tilde{h}$ which play the same role as that of $f$ appearing in the state (1.2-8). Our idea for the deformation starts with the following form:

$$|c_+\rangle = (\sqrt{T})^{-1} \exp \left( \gamma \hat{a}^* \tilde{f}(\hat{N}_a) \cdot \tilde{g}(\hat{N}_b) \hat{b} \cdot \exp \left( \hat{\delta}^* \tilde{h}(\hat{N}_b) \right) \right) |0\rangle \, , \quad (3.1a)$$

$$|c_-\rangle = (\sqrt{T})^{-1} \exp \left( \gamma \hat{a}^* \tilde{f}(\hat{N}_a) \cdot \hat{b}^* \tilde{g}(\hat{N}_b) \right) \cdot \exp \left( \hat{\delta}^* \tilde{h}(\hat{N}_b) \right) |0\rangle \, . \quad (3.1b)$$

We can see in the expressions (3.1a) and (3.1b) that two parts of the exponential forms in the states (2.4a) and (2.4b) are deformed by $\tilde{f}(\hat{N}_a)$, $\tilde{g}(\hat{N}_b)$ and $\tilde{h}(\hat{N}_b)$. The
quantity \( \Gamma \) denotes the normalization constant. The states \(|c_{\pm}\rangle\) can be rewritten in the form

\[
|c_{\pm}\rangle = \left(\sqrt{\Gamma}\right)^{-1} \sum_{m,n} \gamma^m \delta^n \left(\frac{n!}{(n \mp m)!}\right)^{\pm1} f(m) g(n \mp m)^{\mp1} (g(n) \cdot h(n)^{-1})^{\mp1} \\
\times \left(\frac{m!(n \mp m)!}{\sqrt{m!(n \mp m)!}}\right)^{-1} (\hat{a}^*)^m (\hat{b}^*)^n m|0\rangle.
\]

The state (3.2) can be rewritten as

\[
|c_{\pm}\rangle = \left(\sqrt{\Gamma}\right)^{-1} \sum_{m,n} \frac{(|\gamma|^2)^m (|\delta|^2)^n}{m!n!} \left(\frac{n!}{(n \mp m)!}\right)^{\pm1} f(m)^2 g(n \mp m)^{-2} (g(n) \cdot h(n)^{-1})^{\pm2}.
\]

The normalization constant \( \Gamma \) is obtained as

\[
\Gamma = \sum_{m,n} \left(\frac{(|\gamma|^2)^m (|\delta|^2)^n}{m!n!}\right)^{\pm1} f(m)^2 g(n \mp m)^{-2} (g(n) \cdot h(n)^{-1})^{\pm2}.
\]

Further, they obey the conditions

\[
f(k), \ g(k), \ h(k) > 0. \quad (k = 0, 1, 2, \ldots)
\]

The above conditions are derived through the following process: The states \(|c_{\pm}\rangle\) can be expressed in the form

\[
\sqrt{\Gamma}|c_{\pm}\rangle = (1 + \tilde{h}(0)^{-1} \delta \hat{b}^* + \tilde{f}(0) \tilde{g}(0) \tilde{h}(0)^{-1} \gamma \delta \hat{a}^* + \cdots)|0\rangle
\]

\[
= (f(0)h(0)^{-1} + f(0)h(1)^{-1} \delta \hat{b}^* + f(1)g(0)^{-1}g(1)h(1)^{-1} \gamma \delta \hat{a}^* + \cdots)|0\rangle,
\]

(3.8a)

\[
\sqrt{\Gamma}|c_{\pm}\rangle = (1 + \tilde{h}(0) \delta \hat{b}^* + \tilde{f}(0) \tilde{g}(0) \gamma \hat{a}^* \hat{b}^* + \cdots)|0\rangle
\]

\[
= (f(0)h(0) + f(0)h(1) \delta \hat{b}^* + f(1)g(0)^{-1}g(1)h(0) \gamma \hat{a}^* \hat{b}^* + \cdots)|0\rangle.
\]

(3.8b)

For the coefficients of \( \hat{b}^*|0\rangle \) and \( \hat{a}^*|0\rangle \) in the state (3.8a), it is permitted to set up the condition

\[
f(0)h(0)^{-1} = 1, \quad f(0)h(1)^{-1} = \tilde{h}(0)^{-1} = 1, \quad f(1)g(0)^{-1}g(1)h(1)^{-1} = \tilde{f}(0)\tilde{g}(0)\tilde{h}(0)^{-1} = 1.
\]

(3.9a)
In the same way, the state (3.8b) gives us

\[
\begin{align*}
  f(0)h(0) &= 1, \\
  f(0)h(1) &= \tilde{h}(0) = 1, \\
  f(1)g(0)^{-1}g(1)h(0) &= \tilde{f}(0)\tilde{g}(0) = 1. 
\end{align*}
\] (3.9b)

From the relations (3.9a) and (3.9b), we have the condition (3.7). As was shown in the above, the deformation is performed with, maximally, three independent functions \( f(x) \), \( g(x) \) and \( h(x) \). However, we pay an attention to the case mentioned below. As is clear from the relations (2.2) and (2.3), the terms \( g(\tilde{N}_b \pm \tilde{N}_a) \cdot h(\tilde{N}_b \pm \tilde{N}_a)^{-1} \) in \( \Omega_{\pm}(\tilde{N}_a, \tilde{N}_b) \) appearing in the relation (3.5) commute with all the generators of the \( su(2) \)- and the \( su(1,1) \)-algebras, respectively. Then, for the generators of the algebras, the terms \( g(\tilde{N}_b \pm \tilde{N}_a) \cdot h(\tilde{N}_b \pm \tilde{N}_a)^{-1} \) have no effect and, for this reason, we restrict ourselves to the case

\[
\begin{align*}
  g(\tilde{N}_b \pm \tilde{N}_a) \cdot h(\tilde{N}_b \pm \tilde{N}_a)^{-1} &= 1, \\
  \Omega_{\pm}(\tilde{N}_a, \tilde{N}_b) &= f(\tilde{N}_a) : g(\tilde{N}_b)^{-1}. 
\end{align*}
\] (3.10)

(3.11)

This means that the deformation of \( |c_{\pm}^0\rangle \) can be characterized by two functions \( f(x) \) and \( g(x) \).

Next, we define the operators \( \hat{\gamma} \) and \( \hat{\delta} \) in the form

\[
\begin{align*}
  \hat{\gamma} &= \Omega_{\pm}(\tilde{N}_a, \tilde{N}_b)\hat{\gamma}_0\Omega_{\pm}(\tilde{N}_a, \tilde{N}_b)^{-1}, \\
  \hat{\delta} &= \Omega_{\pm}(\tilde{N}_a, \tilde{N}_b)\hat{\delta}_0\Omega_{\pm}(\tilde{N}_a, \tilde{N}_b)^{-1}.
\end{align*}
\] (3.12)

(3.13)

With the use of the state (3.5), the relations (2.5a) and (2.5b) give us

\[
\begin{align*}
  \hat{\gamma}|c_+\rangle &= \gamma(1 - \epsilon(\tilde{N}_b + \epsilon^{-1}))|c_+\rangle, \\
  \hat{\delta}|c_+\rangle &= \delta|c_+\rangle, \\
  \hat{\gamma}|c_-\rangle &= \gamma|c_-\rangle, \\
  \hat{\delta}|c_-\rangle &= \delta|c_-\rangle.
\end{align*}
\] (3.14a)

(3.14b)

The commutation relations for \( \hat{\gamma} \), \( \hat{\gamma}^* \), \( \hat{\delta} \) and \( \hat{\delta}^* \) for the \( su(2) \)-spin system are given as

\[
\begin{align*}
  [\hat{\gamma}, \hat{\gamma}^*] &= \left( \Delta_{\tilde{N}_a}^{(+)} \right. - \Delta_{\tilde{N}_b}^{(-)} - \Delta_{\tilde{N}_a}^{(+)}\Delta_{\tilde{N}_b}^{(-)} \left( \Delta_{\tilde{N}_a}^{(\cdot)} \right) \right) \hat{\gamma}^* \hat{\gamma}, \\
  [\hat{\delta}, \hat{\delta}^*] &= \Delta_{\tilde{N}_b}^{(+)}(\hat{\delta}^* \hat{\delta}) , \\
  [\hat{\gamma}, \hat{\delta}] &= \hat{\delta} \hat{\gamma} \cdot \epsilon \left[ (1 - \epsilon(\tilde{N}_b + 1 + \epsilon)^{-1})^{-1} \Delta_{\tilde{N}_b}^{(+)}(\tilde{N}_b + \epsilon)^{-1} \right], \\
  [\hat{\gamma}, \hat{\delta}^*] &= \hat{\delta}^* \hat{\gamma} \cdot \left[ (1 - \epsilon(\tilde{N}_b + 1 + \epsilon)^{-1} - 1(\hat{\delta} \hat{\delta}^*) \right] \\
  &\quad \times \Delta_{\tilde{N}_b}^{(+)} \left[ (1 - \epsilon(\tilde{N}_b + 1 + \epsilon)^{-1}) \hat{\delta} \hat{\delta}^* \right].
\end{align*}
\] (3.15a)

In the case of the \( su(1,1) \)-spin system, we have

\[
\begin{align*}
  [\hat{\gamma}, \hat{\gamma}^*] &= \left( \Delta_{\tilde{N}_a}^{(+)} + \Delta_{\tilde{N}_b}^{(+)} + \Delta_{\tilde{N}_a}^{(+)}\Delta_{\tilde{N}_b}^{(+)} \right) \hat{\gamma}^* \hat{\gamma}, \\
  [\hat{\delta}, \hat{\delta}^*] &= \Delta_{\tilde{N}_b}^{(+)}(\hat{\delta}^* \hat{\delta}) , \\
  [\hat{\gamma}, \hat{\delta}] &= 0 , \\
  [\hat{\gamma}, \hat{\delta}^*] &= \hat{\delta} \hat{\gamma} \cdot \left[ (1 - \Delta_{\tilde{N}_b}^{(-)}(\hat{\gamma} \hat{\gamma}^*))^{-1} \Delta_{\tilde{N}_b}^{(-)} \left[ (1 - \Delta_{\tilde{N}_a}^{(-)}(\hat{\gamma} \hat{\gamma}^*)) \hat{\gamma} \hat{\gamma}^* \right] \right].
\end{align*}
\] (3.15b)
The relation (3·15a) contains the infinitesimal parameter $\epsilon$. After operating with this relation on any state, we should take the limit $\epsilon \to 0$. It is interesting to see that the commutation relations (3·15a) and (3·15b) are of the forms exactly the same as those of the relations (2·16a) and (2·16b).

Now, let us investigate the classical counterpart of the present case. In the same meaning as that in the case of $|c^0_{\pm}\rangle$, the parameters $\gamma$ and $\delta$ are the eigenvalues of the operators $\hat{\gamma}$ and $\hat{\delta}$, respectively. This fact suggests that we can treat the deformed wave packet $|c_{\pm}\rangle$ in the same way as that for the state $|c^0_{\pm}\rangle$ presented in §2. The state $|c_{\pm}\rangle$ satisfies the same relation as that shown in Eq. (2·10):

\[
\langle c_{\pm}|i\partial_t|c_{\pm}\rangle = (i/2)(\gamma^* \dot{\gamma} - \dot{\gamma}^* \gamma)(\partial \Gamma/\partial |\gamma|^2) \cdot \Gamma^{-1} + (i/2)(\delta^* \dot{\delta} - \dot{\delta}^* \delta)(\partial \Gamma/\partial |\delta|^2) \cdot \Gamma^{-1}.
\]

The relation (3·16) can be rewritten as

\[
\langle c_{\pm}|i\partial_t|c_{\pm}\rangle = (i/2)(x^*\dot{x} - \dot{x}^* x) + (i/2)(y^*\dot{y} - \dot{y}^* y).
\]

Here, $x$ and $y$ are defined as

\[
x = \gamma \sqrt{\partial \Gamma/\partial |\gamma|^2} \cdot \Gamma^{-1}, \quad y = \delta \sqrt{\partial \Gamma/\partial |\delta|^2} \cdot \Gamma^{-1}.
\]

The new parameters $(x, x^*)$ and $(y, y^*)$ can be regarded as canonical variables and, in principle, $\gamma$ and $\delta$ can be expressed in terms of those canonical variables by solving inversely Eq. (3·18). However, in general, it is impossible to solve this explicitly. In the case of $|c^0_{\pm}\rangle$, it is possible. The expectation values of $\hat{N}_a$ and $\hat{N}_b$ for $|c_{\pm}\rangle$ are given in the form

\[
\hat{N}_a = \langle c_{\pm}|\hat{N}_a|c_{\pm}\rangle = |\gamma|^2 \partial \Gamma/\partial |\gamma|^2 \cdot \Gamma^{-1} = |x|^2,
\]

\[
\hat{N}_b = \langle c_{\pm}|\hat{N}_b|c_{\pm}\rangle = |\delta|^2 \partial \Gamma/\partial |\delta|^2 \cdot \Gamma^{-1} = |y|^2 \mp |x|^2.
\]

Here, we have used the relation (3·18). The above are the same as those given in the relation (2·19). Therefore, we can prove that $(\gamma, \gamma^*)$ and $(\delta, \delta^*)$ in the present case obey the same relations as those for $(\gamma, \gamma^*)$ and $(\delta, \delta^*)$ in the case treated in §2. For example, the Poisson brackets for $(\gamma, \gamma^*)$ and $(\delta, \delta^*)$ in the present case have the same forms as those shown in the relations (2·18a) and (2·18b). Thus, under the replacement (2·21), the commutation relations (3·15a) and (3·15b) are reduced to the relations (2·18a) and (2·18b). Of course, the terms resulting from the quantal fluctuation are ignored. This is in the same situation as that in §2. Then, we have the correspondence

\[
\hat{\gamma} \sim \gamma, \quad \hat{\delta} \sim \delta.
\]

In this way, we have obtained the classical counterpart of the deformed boson scheme in parallel with the case of the conventional boson coherent state $|c^0_{\pm}\rangle$.

§4. The $su(2)_q$- and the $su(1, 1)_q$-algebras in the present deformed boson scheme

It is interesting to investigate the $su(2)_q$- and $su(1, 1)_q$-algebras in the present deformed boson scheme. Conventionally, the $su(2)_q$-algebra is formulated in terms
Following the conventional manner, we define the relations

\[ [\hat{S}_0, \hat{S}_\pm] = \pm \hat{S}_\pm, \quad [\hat{S}_+, \hat{S}_-] = + [2\hat{S}_0]_q. \]  

(4.1a)

In analogy with the above case, the \( su(1,1)_q \)-algebra is formulated by setting up the following relation for the operators \( (\hat{T}_\pm, \hat{T}_0, [2\hat{T}_0]_q) \):

\[ [\hat{T}_0, \hat{T}_\pm] = \pm \hat{T}_\pm, \quad [\hat{T}_+, \hat{T}_-] = - [2\hat{T}_0]_q. \]  

(4.1b)

For the operators \( [2\hat{S}_0]_q \) and \( [2\hat{T}_0]_q \), conventionally we have

\[ [x]_q = (q^x - q^{-x})/(q - q^{-1}) \quad \text{for} \quad (q : \text{real}) \]  

(4.2)

We investigate the above algebras in terms of the space composed of two kinds of boson operators.

For the above-mentioned purpose, let us define the operators

\[ \hat{\alpha} = \Omega_\pm(\hat{N}_a, \hat{N}_b)\hat{\alpha} \Omega_\pm(\hat{N}_a, \hat{N}_b)^{-1}, \]

\[ \hat{\beta} = \Omega_\pm(\hat{N}_a, \hat{N}_b)^{-1} \hat{\beta} \Omega_\pm(\hat{N}_a, \hat{N}_b)^{\pm1}. \]  

(4.3)

Here, \( \Omega_\pm(\hat{N}_a, \hat{N}_b) \) is defined in Eq. (3.11). Then, for \( \hat{S}_\pm \) and \( \hat{T}_\pm \), we have the forms

\[ \hat{S}_+ = \hat{\alpha}^* \hat{\beta}, \quad \hat{S}_- = \hat{\beta}^* \hat{\alpha}, \]  

(4.4a)

\[ \hat{T}_+ = \hat{\alpha}^* \hat{\beta}^*, \quad \hat{T}_- = \hat{\beta} \hat{\alpha}. \]  

(4.4b)

With the use of the operators in (4.3), \( \hat{S}_\pm \) and \( \hat{T}_\pm \) can be expressed as follows:

\[ \hat{S}_+ = f(\hat{N}_a)^{-1} g(\hat{N}_b) \hat{S}_+^0 f(\hat{N}_a) g(\hat{N}_b)^{-1}, \quad \hat{S}_+^0 = \hat{\alpha}^* \hat{b}, \]

\[ \hat{S}_- = f(\hat{N}_a) g(\hat{N}_b)^{-1} \hat{S}_-^0 f(\hat{N}_a)^{-1} g(\hat{N}_b), \quad \hat{S}_-^0 = \hat{\beta}^* \hat{a}, \]  

(4.5a)

\[ \hat{T}_+ = f(\hat{N}_a)^{-1} g(\hat{N}_b)^{-1} \hat{T}_+^0 f(\hat{N}_a) g(\hat{N}_b), \quad \hat{T}_+^0 = \hat{\alpha}^* \hat{b}^*, \]

\[ \hat{T}_- = f(\hat{N}_a) g(\hat{N}_b) \hat{T}_-^0 f(\hat{N}_a)^{-1} g(\hat{N}_b)^{-1}, \quad \hat{T}_-^0 = \hat{b} \hat{\alpha}. \]  

(4.5b)

Following the conventional manner, further, we define \( [2\hat{S}_0]_q \) and \( [2\hat{T}_0]_q \) through the relations

\[ [\hat{S}_+, \hat{S}_-] = + [2\hat{S}_0]_q, \]  

(4.6a)

\[ [\hat{T}_+, \hat{T}_-] = - [2\hat{T}_0]_q. \]  

(4.6b)

The definitions (4.6a) and (4.6b) give us the operators \( [2\hat{S}_0]_q \) and \( [2\hat{T}_0]_q \) in the forms

\[ [2\hat{S}_0]_q = [\hat{N}_a]_f [\hat{N}_b + 1]_g - [\hat{N}_a + 1]_f [\hat{N}_b]_g, \]  

(4.7a)

\[ [2\hat{T}_0]_q = [\hat{N}_a + 1]_f [\hat{N}_b + 1]_g - [\hat{N}_a]_f [\hat{N}_b]_g. \]  

(4.7b)

Here, \([x]_f\) and \([x]_g\) are given as

\[ [x]_f = x f(x)^{-2} f(x - 1)^2, \quad [x]_g = x g(x)^{-2} g(x - 1)^2. \]  

(4.8)
The above argument suggests that \( \hat{S}_\pm, [2\hat{S}_0]_q \) and \( \hat{T}_\pm, [2\hat{T}_0]_q \) defined in the relations (4.5a), (4.7a), (4.5b) and (4.7b) form the \( su(2)_q \) and the \( su(1,1)_q \)-algebras, respectively. The operators \( \hat{S}_\pm, [2\hat{S}_0]_q \) may be functions of \( \hat{S} \), which commutes with them. The situation for \( \hat{T}_\pm, [2\hat{T}_0]_q \) is the same as that above: These operators may be functions of \( \hat{T} \), which commutes with them. The forms (4.5a), (4.7a), (4.5b) and (4.7b) can be rewritten as follows:

\[
\hat{S}_+ = \hat{E}_a^* \hat{E}_b \sqrt{[\hat{S} + \hat{S}_0 + 1]_f [\hat{S} - \hat{S}_0]_g} = \sqrt{[\hat{S} + \hat{S}_0]_f [\hat{S} - \hat{S}_0 + 1]_g} \hat{E}_a^* \hat{E}_b ,
\]

\[
\hat{S}_- = \hat{E}_b^* \hat{E}_a \sqrt{[\hat{S} + \hat{S}_0]_f [\hat{S} - \hat{S}_0 + 1]_g} = \sqrt{[\hat{S} + \hat{S}_0 + 1]_f [\hat{S} - \hat{S}_0]_g} \hat{E}_b^* \hat{E}_a ,
\]

\[
[2\hat{S}_0]_q = [\hat{S} + \hat{S}_0]_f [\hat{S} - \hat{S}_0 + 1]_g - [\hat{S} + \hat{S}_0 + 1]_f [\hat{S} - \hat{S}_0]_g ,
\]

\[
[2\hat{T}_0]_q = [\hat{T} - \hat{T} + 1]_f [\hat{T}_0 + \hat{T} - 1]_g \hat{E}_a^* \hat{E}_b^* ,
\]

\[
\hat{T}_+ = \hat{E}_a^* \hat{E}_b \sqrt{[\hat{T} - \hat{T} - 1]_f [\hat{T}_0 + \hat{T} - 1]_g} = \sqrt{[\hat{T} - \hat{T} + 1]_f [\hat{T}_0 + \hat{T} - 1]_g} \hat{E}_a^* \hat{E}_b ,
\]

\[
[2\hat{T}_0]_q = [\hat{T} - \hat{T} + 1]_f [\hat{T}_0 + \hat{T} - 1]_g - [\hat{T} - \hat{T}]_f [\hat{T}_0 + \hat{T} - 1]_g .
\]

Here, \( \hat{S}, \hat{T}, \hat{S}_0 \) and \( \hat{T}_0 \) are given in the relations (2.2) and (2.1), respectively. The operators \( \hat{E}_c, \hat{E}_c^* \) for \( c = a, b \) are defined as

\[
\hat{E}_c = \left( \sqrt{\hat{N}_c + 1} \right)^{-1} \hat{c} , \quad \hat{E}_c^* = \hat{c}^* \left( \sqrt{\hat{N}_c + 1} \right)^{-1} . \quad (\hat{N}_c = \hat{c}^* \hat{c}) (4.11)
\]

The property of the above operators is as follows:

\[
\hat{E}_c \hat{E}_c^* = 1 , \quad \hat{N}_c \hat{E}_c^* \hat{E}_c = \hat{E}_c^* \hat{E}_c \hat{N}_c = \hat{N}_c . \quad (4.12)
\]

In (I), we showed some examples for the deformed boson. In this section, three of them are applied to the \( su(2)_q \)- and the \( su(1,1)_q \)-algebras.

(i) The most popular form (commonly used form):

\[
f(n) = \sqrt{n(q - q^{-1})/(q^n - q^{-n})} \; f(n - 1) ,
\]

\[
g(n) = \sqrt{n(q - q^{-1})/(q^n - q^{-n})} \; g(n - 1) . \quad (4.13)
\]

In this case, both the \( su(2)_q \)- and the \( su(1,1)_q \)-generators can be expressed in the forms

\[
\hat{S}^{(i)}_+ = \frac{1}{\sqrt{\hat{S} - \hat{S}_0 + 1}} \hat{S}_0 \frac{1}{\sqrt{\hat{S} + \hat{S}_0 + 1}} \sqrt{\frac{q(\hat{S} - \hat{S}_0) - q^{-1}(\hat{S} - \hat{S}_0)}{q - q^{-1}}} \sqrt{\frac{q(\hat{S} + \hat{S}_0 + 1) - q^{-1}(\hat{S} + \hat{S}_0 + 1)}{q - q^{-1}}},
\]
\[ \hat{S}_-^{(i)} = \sqrt{\frac{q^{(S-S_0)}}{q-q^{-1}}} \sqrt{\frac{q^{(S+S_0+1)}}{q-q^{-1}}} \frac{1}{\sqrt{\hat{S}+S_0+1}} \frac{1}{\sqrt{\hat{S}-S_0+1}}, \]
\[ [2\hat{S}_0]^{(i)} = \frac{q^{2\hat{S}_0} - q^{-2\hat{S}_0}}{q-q^{-1}}. \quad (4.14a) \]
\[ \hat{T}_+^{(i)} = \hat{T}_+^0 \frac{1}{\sqrt{(\hat{T}_0 + \hat{T})(\hat{T}_0 - \hat{T} + 1)}} \sqrt{\frac{q^{(T_0+T)} - q^{-T_0+T}}{q-q^{-1}}} \sqrt{\frac{q^{(T_0-T+1)} - q^{-T_0-T+1}}{q-q^{-1}}}, \]
\[ \hat{T}_-^{(i)} = \sqrt{\frac{q^{(T_0+T)} - q^{-T_0+T}}{q-q^{-1}}} \sqrt{\frac{q^{(T_0-T+1)} - q^{-T_0-T+1}}{q-q^{-1}}} \frac{1}{\sqrt{(\hat{T}_0 + \hat{T})(\hat{T}_0 - \hat{T} + 1)}} \hat{T}_0^0, \]
\[ [2\hat{T}_0]^{(i)} = \frac{q^{2\hat{T}_0} - q^{-2\hat{T}_0}}{q-q^{-1}}. \quad (4.14b) \]

As is clear from the form of \([2\hat{S}_0]_q\) shown in Eq. (4.14a), the functions (4.13) give us the most popular form for the \(su(2)_q\) and the \(su(1,1)_q\)-algebras.

(ii)\textsubscript{a} The form presented by Penson and Solomon\textsuperscript{9) for the \(su(2)_q\)-algebra:
\[ f(n) = q^{-(n-1)/2} f(n-1) , \quad g(n) = q^{-(n-1)/2} g(n-1) . \quad (4.15a) \]
In this case, we have
\[ \hat{S}_+^{(ii)} = q^{\hat{S}_{-1/2}} \cdot \hat{S}_0^0 , \quad \hat{S}_-^{(ii)} = q^{\hat{S}_{-1/2}} \cdot \hat{S}_-^0 , \]
\[ [2\hat{S}_0]^{(ii)} = q^{2(\hat{S}_{-1/2})} \cdot 2\hat{S}_0 . \quad (4.16a) \]

(ii)\textsubscript{b} The form presented by Penson and Solomon for the \(su(1,1)_q\)-algebra:
\[ f(n) = q^{+(n-1)/2} f(n-1) , \quad g(n) = q^{-(n-1)/2} g(n-1) . \quad (4.15b) \]
In this case, we have
\[ \hat{T}_+^{(ii)} = q^{\hat{T}_{-1/2}} \cdot \hat{T}_0^0 , \quad \hat{T}_-^{(ii)} = q^{\hat{T}_{-1/2}} \cdot \hat{T}_0^0 , \]
\[ [2\hat{T}_0]^{(ii)} = q^{2(\hat{T}_{-1/2})} \cdot 2\hat{T}_0 . \quad (4.16b) \]

(iii)\textsubscript{a} The modified form for the \(su(2)_q\)-algebra:
\[ f(n) = \sqrt{n(1-q^2)/(1-q^{2n})} f(n-1) , \]
\[ g(n) = \sqrt{n(1-q^2)/(1-q^{2n})} g(n-1) . \quad (4.17a) \]
In this case, we have the form
\[ \hat{S}_+^{(iii)} = q^{\hat{S}_{-1/2}} \cdot \hat{S}_+^{(i)} , \quad \hat{S}_-^{(iii)} = q^{\hat{S}_{-1/2}} \cdot \hat{S}_-^{(i)} , \]
\[ [2\hat{S}_0]^{(iii)} = q^{2(\hat{S}_{-1/2})} \cdot [2\hat{S}_0]^{(i)}. \quad (4.18a) \]
The modified form for the $su(1,1)_q$-algebra:

\[
\begin{align*}
  f(n) &= \sqrt{n(1-q^{-2})/(1-q^{-2n})}f(n-1), \\
  g(n) &= \sqrt{n(1-q^2)/(1-q^{2n})}g(n-1). 
\end{align*}
\]

(4·17b)

In this case, we have the form

\[
\begin{align*}
  \hat{T}^{(iii)}_+ &= q^{T-1/2} \cdot \hat{T}^{(i)}_+, \\
  \hat{T}^{(iii)}_- &= q^{1-\hat{T}/2} \cdot \hat{T}^{(i)}_-, \\
  [2\hat{T}^0_q]^{(iii)} &= q^{2(T-1/2)} \cdot [2\hat{T}^0_q]^{(i)}. 
\end{align*}
\]

(4·18b)

The above form is proposed by the present authors in (I) and it is interesting to see that this form is in the intermediate situation between the forms of (i) and (ii).

§5. **The pseudo $su(2)$- and the pseudo $su(1,1)$-algebras**

With the aim of describing a boson system interacting with an external field, the present authors proposed three forms of coherent states in the $su(2)$-spin system. In this section, we reinvestigate these forms from the viewpoint of the deformed boson scheme presented in §4. For this purpose, let us consider the following case:

\[
\begin{align*}
  f(n) &= \left(\sqrt{1+q(n-1)}\right)^{-1}f(n-1), \\
  g(n) &= \sqrt{1+r(n-1)}g(n-1). 
\end{align*}
\]

(5·1)

Here, $q$ and $r$ denote real parameters. It should be noted that, as was shown in §4, there exist two functions $f(n)$ and $g(n)$ which can be chosen independently. Then, $\hat{S}_\pm$ and $\hat{T}_\pm$ given in the relations (4·5a) and (4·5b), respectively, can be written in the forms

\[
\begin{align*}
  \hat{S}_+ &= \hat{a}^* \sqrt{1+q\hat{N}_a} \left(\sqrt{1+r\hat{N}_b}\right)^{-1} \hat{b}, \\
  \hat{S}_- &= \hat{b}^* \left(\sqrt{1+r\hat{N}_b}\right)^{-1} \sqrt{1+q\hat{N}_a} \hat{a}, \\
  \hat{T}_+ &= \hat{a}^* \sqrt{1+q\hat{N}_a} \hat{b}^* \left(\sqrt{1+r\hat{N}_b}\right)^{-1}, \\
  \hat{T}_- &= \left(\sqrt{1+r\hat{N}_b}\right)^{-1} \hat{b} \sqrt{1+q\hat{N}_a} \hat{a}. 
\end{align*}
\]

(5·2)

Through the definitions (4·6a) and (4·6b), $[2\hat{S}_0]_q$ and $[2\hat{T}_0]_q$ can be expressed as follows:

\[
\begin{align*}
  [2\hat{S}_0]_q &= +2(\hat{S}_0)_q + 2(\hat{S}_0)_p \cdot \epsilon(\hat{N}_b + \epsilon)^{-1}, \\
  [2\hat{T}_0]_q &= -2(\hat{T}_0)_q - 2(\hat{T}_0)_p \cdot \epsilon(\hat{N}_b + \epsilon)^{-1}, \\
  (\hat{S}_0)_q &= (1-r)/2 \cdot \left[\hat{N}_a - \hat{N}_b - q\hat{N}_a(1 - \hat{N}_a + 2\hat{N}_b)\right] - r/2 \cdot \left[1 + 2q\hat{N}_a\right] 
\end{align*}
\]
\[-r(1 - r)/2 \cdot \left[ (1 + 2q\tilde{N}_a)\tilde{N}_b^2 (1 + r\tilde{N}_b)^{-1} \right. \\
\left. - (1 + \tilde{N}_a)(1 + q\tilde{N}_a)(1 + (r - 2)\tilde{N}_b - r\tilde{N}_b^2)(1 + r\tilde{N}_b)^{-1}(1 - r + r\tilde{N}_b)^{-1}\right], \tag{5.4a} \]

\[(\hat{T}_0)_q = (1 - r)/2 \cdot \left[ (\tilde{N}_a + \tilde{N}_b + 1) + q\tilde{N}_a(1 + \tilde{N}_a + 2\tilde{N}_b) \right] + r/2 \cdot \left[ 1 + 2q\tilde{N}_a \right] \\
\left. - r(1 - r)/2 \cdot \left[ (1 + 2q\tilde{N}_a)\tilde{N}_b^2 (1 + r\tilde{N}_b)^{-1} \right. \\
\left. - \tilde{N}_a(1 - q + q\tilde{N}_a)(1 + (r - 2)\tilde{N}_b - r\tilde{N}_b^2)(1 + r\tilde{N}_b)^{-1}(1 - r + r\tilde{N}_b)^{-1}\right], \tag{5.4b} \]

\[(\hat{S}_0)_p = r/2 \cdot (1 + \tilde{N}_a)(1 + q\tilde{N}_a) \\
\times \left[ 1 - (1 - r)(1 + (r - 2)\tilde{N}_b - r\tilde{N}_b^2)(1 + r\tilde{N}_b)^{-1}(1 - r + r\tilde{N}_b)^{-1}\right], \tag{5.5a} \]

\[(\hat{T}_0)_p = r/2 \cdot \tilde{N}_a(1 - q + q\tilde{N}_a) \\
\times \left[ 1 - (1 - r)(1 + (r - 2)\tilde{N}_b - r\tilde{N}_b^2)(1 + r\tilde{N}_b)^{-1}(1 - r + r\tilde{N}_b)^{-1}\right]. \tag{5.5b} \]

Here, \(\epsilon(\tilde{N}_b + \epsilon)^{-1}\) was already introduced in the relation (2.5a) with its property (2.8). For the relations (5.2a) and (5.3a), three cases \((q = 0, r = 0), (q = 0, r = 1)\) and \((q > 0, r = 1)\) are investigated in Ref. 8). The first is identically the Schwinger boson representation and the third enables us to describe a boson system interacting with an external field.

With the above-mentioned background, let us treat the cases \((q < 0, r = 1), (q = 0, r = 1)\) and \((q > 0, r = 1)\) more systematically than that in Ref. 8). First, we define the following operators:

\[\bar{c} = \hat{E}_b^* \hat{a}, \quad \bar{c}^* = \hat{a}^* \hat{E}_b, \tag{5.6a}\]

\[\tilde{c} = \hat{E}_b \hat{a}, \quad \tilde{c}^* = \hat{a}^* \hat{E}_b^*. \tag{5.6b}\]

Here, \((\hat{E}_b, \hat{E}_b^*)\) is defined in the relation (4.11). The properties of the operator in (5.6a) are given by

\[\left[ \bar{c}, \bar{c}^* \right] = 1 - (1 + \tilde{N}_a) \cdot \epsilon(\tilde{N}_b + \epsilon)^{-1}, \]

\[\bar{c}^* \bar{c} = \tilde{N} = \tilde{N}_a. \tag{5.7a}\]

For the operator (5.6b), we also have

\[\left[ \tilde{c}, \tilde{c}^* \right] = 1 + \tilde{N}_a \cdot \epsilon(\tilde{N}_b + \epsilon)^{-1}, \]

\[\tilde{c}^* \tilde{c} = \tilde{N} = \tilde{N}_a - \tilde{N}_a \cdot \epsilon(\tilde{N}_b + \epsilon)^{-1}. \tag{5.7b}\]

Then, if we restrict ourselves to the subspace which does not include the vacuum \(\left| 0 \right\rangle\) nor any state consisting only of the \(\tilde{a}\)-boson, the operators \((\bar{c}, \bar{c}^*)\) defined in the
relations (5.6a) and (5.6b) can be regarded as boson operators and \( \tilde{N} \) denotes the boson number operator:
\[
[\tilde{c}, \tilde{c}^*] = 1, \quad \tilde{c}^* \tilde{c} = \tilde{N} = \tilde{N}_a.
\] (5.8)

Hereafter, we treat the above subspace.

With the use of the operators \((\tilde{c}, \tilde{c}^*)\) and \(\tilde{N}\), we express \(\hat{S}_\pm\) in the forms (5.2a) and \((\hat{S}_0)_q\) in the form (5.4a). Also, in the case of \(\hat{T}_\pm\) in the form (5.2b) and \((\hat{T}_0)_q\) in the form (5.4b), we have the following expressions:

(i) \(q < 0\):
\[
|q|^{-1/2}\hat{S}_+ = \tilde{c}^* \sqrt{|q|^{-1} - \tilde{N}}, \quad |q|^{-1/2}\hat{S}_- = \sqrt{|q|^{-1} - \tilde{N}} \tilde{c},
\] (5.9a)

(ii) \(q < 0\):
\[
|q|^{-1/2}\hat{T}_+ = \tilde{c}^* \sqrt{|q|^{-1} - \tilde{N}}, \quad |q|^{-1/2}\hat{T}_- = \sqrt{|q|^{-1} - \tilde{N}} \tilde{c},
\] (5.9b)

(iii) \(q = 0\):
\[
\hat{S}_+ = \tilde{c}^*, \quad \hat{S}_- = \tilde{c}, \quad -2(\hat{S}_0)_q = 1,
\] (5.10a)

(iv) \(q = 0\):
\[
\hat{T}_+ = \tilde{c}^*, \quad \hat{T}_- = \tilde{c}, \quad 2(\hat{T}_0)_q = 1,
\] (5.10b)

(v) \(q > 0\):
\[
q^{-1/2}\hat{S}_+ = \tilde{c}^* \sqrt{q^{-1} + \tilde{N}}, \quad q^{-1/2}\hat{S}_- = q^{-1/2} + \tilde{N} \tilde{c},
\] (5.11a)

(vi) \(q > 0\):
\[
q^{-1/2}\hat{T}_+ = \tilde{c}^* \sqrt{q^{-1} + \tilde{N}}, \quad q^{-1/2}\hat{T}_- = q^{-1} + \tilde{N} \tilde{c},
\] (5.11b)

We can see that \((|q|^{-1/2}\hat{S}_\pm, |q|^{-1} (\hat{S}_0)_q)\) and \((|q|^{-1/2}\hat{T}_\pm, -|q|^{-1} (\hat{T}_0)_q)\) form, respectively, the \(su(2)\)-algebras in the Holstein-Primakoff representation if \((\tilde{c}, \tilde{c}^*)\) can be regarded as the boson operator strictly. However, as was already mentioned, \((\tilde{c}, \tilde{c}^*)\) can be regarded as the boson operator in a certain subspace. In this sense, it may be permitted to call the above algebra the pseudo \(su(2)\)-algebra. Further, the sets \((q^{-1/2}\hat{S}_\pm, -q^{-1} (\hat{S}_0)_q)\) and \((q^{-1/2}\hat{T}_\pm, q^{-1} (\hat{T}_0)_q)\) form, respectively, the \(su(1,1)\)-algebras in the Holstein-Primakoff representation if \((\tilde{c}, \tilde{c}^*)\) can be regarded as the boson operator strictly. In this sense, for the same reason as in the case of the \(su(2)\)-algebra, it may be permitted to call the above algebra the pseudo \(su(1,1)\)-algebra. The sets \((\hat{S}_\pm)\) and \((\hat{T}_\pm)\) for \(q = 0\), respectively, behave as the boson operator formally. The quantities \(|q|^{-1/2}\) and \(q^{-1/2}\) denote the magnitudes of the \(su(2)\) and the \(su(1,1)\)-spins.
We showed deformations of the \(su(2)\)- and the \(su(1,1)\)-algebras in the Schwinger boson representation for three forms. Then, let us investigate the states constructed from these deformations. First, we consider the cases given in the relations (5.9a), (5.10a) and (5.11a). The condition \(\hat{S}_-|\tilde{0}\rangle = 0\) gives us
\[
|\tilde{0}\rangle = (\hat{E}_b^*)^A|0\rangle \quad (A = 1, 2, 3, \cdots) \tag{5.12}
\]
Successive operation of \(\tilde{c}^*\) on the state \(|\tilde{0}\rangle\) gives us the states in which the number of \(\hat{b}\)-bosons decreases and the number of \(\hat{a}\)-bosons increases. Then, in order to cause the form (5.9a) to be the \(su(2)\)-spin in the Holstein-Primakoff representation in the subspace which does not include the vacuum \(|0\rangle\) and has no state consisting only of the \(\hat{a}\)-boson, the following relation should be set up:
\[
|q|^{-1} = 2\sigma, \quad \sigma = 0, 1/2, 1, 3/2, \cdots, (A - 1)/2, \sigma_0 = -\sigma, -\sigma + 1, \cdots, \sigma - 1, \sigma. \tag{5.13}
\]
In the case of the relation (5.10a), the number of the operation of \(\tilde{c}^*\) on the state \(|\tilde{0}\rangle\) is restricted to
\[
n = 0, 1, 2, \cdots, (A - 1). \tag{5.14}
\]
In order to cause the form (5.11a) to be the \(su(1,1)\)-algebra in the Holstein-Primakoff representation in the same subspace, the following relation is necessary:
\[
q^{-1} = 2\tau, \quad \tau = \text{positive but arbitrary}, \quad \tau_0 = \tau, \tau + 1, \cdots, \tau + (A - 1)/2. \tag{5.15}
\]
The relation (5.15) shows that \(\tau_0\) cannot run to infinity and, then, strictly speaking, the form (5.11a) does not compose the \(su(1,1)\)-algebra. However, if \(A\) is sufficiently large, we can approximately regard the form (5.11a) as the \(su(1,1)\)-spin. This fact enables us to describe the damped and amplified oscillation of a boson system interacting with an external field. Next, we consider the cases (5.9b), (5.10b) and (5.11b). In these cases, also, \(\hat{T}_-|\tilde{0}\rangle = 0\) gives us
\[
|\tilde{0}\rangle = (\hat{E}_b^*)^A|0\rangle \quad (A = 1, 2, 3, \cdots) \tag{5.16}
\]
Successive operation of \(\tilde{c}^*\) on the state \(|\tilde{0}\rangle\) gives us the states in which the numbers of the \(\hat{a}\)- and \(\hat{b}\)-bosons increase. Then, we have the following relations:
\[
|q|^{-1} = 2\sigma, \quad \sigma = 0, 1/2, 1, 3/2, \cdots, \sigma_0 = -\sigma, -\sigma + 1, \cdots, \sigma - 1, \sigma. \tag{5.17}
\]
\[
n = 0, 1, 2, \cdots, \tag{5.18}
\]
\[
q^{-1} = 2\tau, \quad \tau = \text{positive but arbitrary}, \quad \tau_0 = \tau, \tau + 1, \tau + 2, \cdots \tag{5.19}
\]
The \(su(2)\)-algebra is compact and the \(su(1,1)\)-algebra is non-compact. From this fact, the difference between the two cases appears.
§6. Concluding remarks

Following the basic idea presented in (I), in this paper, we investigated the deformation of the systems obeying the $su(2)$- and the $su(1,1)$-algebras. With the use of two independent functions $f(x)$ and $g(x)$, the deformation is performed. One of the interesting points presented in this paper may be that the $su(2)_q$-algebra in the most popular form is nothing but one type of the deformations. The situation for the $su(1,1)_q$-algebra is the same situation as the above. For example, the deformation discussed in §5 is interesting, because this type was already used by the present authors for describing the damped and amplified oscillational motion in the $su(2)$-spin system. However, this treatment does not enable us to describe a statistically mixed state in the system discussed in Ref. 8). For this problem, we applied the $su(2,1)$-algebra in three kinds of boson operators. In Part (III), we will discuss this case in the deformed boson scheme given in (I).

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