On the role of dissipation in structure formation for dilute relativistic gases: the static background case

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Abstract

A correction to the Jeans stability criterion due to heat conduction is established for the case of high temperature gases. This effect is only relevant for relativistic fluids and includes an additional term due to a density gradient driven heat flux. The result is obtained by thoroughly analyzing the exponentially growing modes present in the dynamics of density fluctuations in the linearized relativistic Navier-Stokes regime. The corrections to the corresponding Jeans mass and wavenumber are explicitly obtained and are compared to the non-relativistic and non-dissipative values using the transport coefficients obtained in the BGK approximation.

Introduction

The establishment of conditions for the onset of gravitational instabilities in a dilute gas is one of the most important problems in astrophysics. Transport theory can be directly applied to tackle this problem using a simplified model in which the field of an isothermal self-gravitating gas surpasses its corresponding hydrostatic pressure in the absence of dissipation [1]. This basic model leads to the well-known critical parameters for gravitational structure formation namely, Jeans wave number and Jeans mass. These parameters are the standard benchmarks to understand the characteristic lengths and formation times corresponding to fundamental astrophysical objects such as stars and galaxies. It is natural to analyze the problem of the growth of density fluctuations due to gravity including several effects such as cosmological expansion. Pioneering work related to entropy production in the realm of cosmology was performed over 40 years ago by Weinberg [2] and was later included in classical textbooks [3]. The introduction of dissipation was first analyzed using numerical stability techniques by Corona-Galindo and Dehnen [4]; later on, useful algebraic representations for the dispersion relation governing the evolution of density fluctuations were developed, identifying a non-oscillating damped mode due to viscous processes [5].

The introduction of heat conduction in the analysis of Jeans instability in the case of a high temperature (relativistic) dilute fluid, was carefully studied in Ref. [6], identifying serious problems with Eckart’s constitutive equation (which proposes a coupling of the heat flux with the hydrodynamic acceleration of the fluid). Along with the development of relativistic kinetic theory, a relativistic constitutive equation up to first order in the gradients was successfully established [7] in order to eliminate the generic instabilities first identified by Hiscock and Lindblom [8] that were responsible of the pathological features mentioned in Ref [6]. Two damped non-oscillatory modes due to viscosity can also be found in the relativistic case, but the effects of heat conduction to the actual modified values of the Jeans wave number have not been addressed before in the case of first order in the gradients constitutive equations. It is the purpose of this work to fill this gap, showing that the first dissipative effects affecting real growing density modes in a simple relativistic fluid are those associated
to heat conduction, instead of viscosity. These rather surprising result is obtained as follows: In section 2 the relativistic transport equations for self-gravitating dilute fluids in the Navier-Stokes regime are reviewed. The linearized version of the transport system is obtained in section 3 leading to an algebraic dispersion relation which yields the relativistic Jeans wave-number in the non-dissipative limit. The conditions for gravitational collapse including heat and viscosity are thoroughly discussed in section 4, so that a simple dimensionless parameter involving the heat conductivity coefficient is established in order to evaluate the effect of heat conduction to the ordinary Jeans wave number. A discussion of this result and final remarks are included in the final section of this work.

The relativistic transport equations

As mentioned above, the system here addressed is a self gravitating dilute neutral gas in a relativistic, non-equilibrium situation. The relativistic nature of such system arises from the fact that the temperature is high enough such that the ratio of thermal to individual molecules’ rest energy is non-negligible, leading to relevant relativistic corrections in the evolution of state variables. In this sense, full relativistic effects are considered for the system, however since the density of the gas is assumed to be low, the gravitational potential is only considered within a linear approximation. Thus, in the case of a static background, the corresponding metric reads

\[ ds^2 = dr^2 + r^2 \left(d\theta^2 + \sin \theta d\varphi^2\right) - \left(1 - \frac{2\phi(r)}{c^2}\right) c^2 dt^2. \]  

where \( \phi \) is the gravitational potential and \( c \) the speed of light.

In order to address the response of the gas to a density fluctuation, the relativistic transport equations need to be analyzed under such conditions. Based on relativistic kinetic theory, it has been shown that such equations, considering the number density \( n \), the internal energy \( \varepsilon \) and the hydrodynamic four-velocity \( u^\nu \) as state variables, are given by two conservation statements in the corresponding four-dimensional spacetime, namely [9, 10]

\[ N_{\mu}^{\nu} = 0 \]  

and

\[ T_{\mu\nu} = 0, \]  

which represent particle and energy-momentum balances, respectively. Here \( N_{\mu}^{\nu} = nu_{\mu}^{\nu} \) is the particle four-flow and

\[ T_{\mu\nu} = \frac{n\varepsilon}{c^2} u_{\mu} u_{\nu} + p h_{\mu\nu}^{\nu} + \pi_{\mu\nu} + \frac{1}{c^2} q_{\mu\nu}, \]  

is the stress-energy tensor, where \( p \) is the hydrostatic pressure, \( q_{\nu} \) is the heat flux, \( \pi_{\nu} \) is the Navier tensor and \( h^{\mu\nu} \) is the usual spatial projector. By substituting \( N_{\mu}^{\nu} \) and \( T_{\mu\nu} \) in Eqs. \( 2 \) and \( 3 \), the set of transport equations can be shown to be explicitly given by

\[ \dot{n} + n\theta = 0, \]  

\[ \left(\frac{n\varepsilon}{c^2} + \frac{p}{c^2}\right) \dot{u}_{\nu} + \left(\frac{n\varepsilon}{c^2} + \frac{p}{c^2}\theta\right) u_{\nu} + p_{\mu\nu} h_{\mu\nu} + \pi_{\mu\nu} + \frac{1}{c^2} \left(q_{\mu\nu} u_{\nu} + q_{\nu\mu} u_{\nu\mu} + \theta q_{\nu} + u_{\mu} q_{\nu\mu}\right) = 0, \]  

\[ nC_n \dot{T} + p\theta + u_{\nu\mu} \pi_{\nu\mu} + q_{\mu\nu} + \frac{1}{c^2} \dot{u}_{\nu} q_{\nu} = 0, \]  

where a semicolon denotes a covariant derivative and a dot a proper time derivative \( \dot{A}_{\nu} = u_{\mu} A_{\nu\mu} \). Also, \( \theta = u_{\nu\mu} \) and \( C_n \) is the heat capacity at constant particle density. Notice that here the gravitational field acts on the fluid through space-time curvature and thus, the gravitational potential is present in Eq. \( 5 \) through the covariant derivatives

\[ A_{\nu\mu} = A_{\nu\mu} - \Gamma_{\mu\nu}^{\lambda} A_{\lambda}, \]
where $\Gamma^\lambda_{\mu\nu}$ are the usual Christoffel symbols.

As is well known, the set of transport equations given by Eqs. (4)-(6) requires closure relations for the dissipative fluxes $Q^\nu$ and $\pi^\mu_{\nu}$ and an additional equation for the gravitational potential. The constitutive equations that relate the dissipative fluxes with the state variables’ gradients are given by the expressions obtained from relativistic kinetic theory, namely [7, 11]

$$
\pi^\mu_{\nu} = -2\eta h^\alpha_{\mu} h^\beta_{\nu} r^\gamma_{\alpha\beta} - \zeta \delta^\mu_{\nu},
$$

(8)

and

$$
Q^\nu = -h^\nu_{\mu} \left( L^\mu_{TT} \frac{T^\mu}{T} + L^\mu_{nT} \frac{n^\mu}{n} \right).
$$

(9)

The explicit expressions for the transport coefficients $\eta, \zeta, L^\mu_{TT}, L^\mu_{nT}$ in a relaxation time approximation can be found in Refs. [7, 9, 11] (also see the Appendix). Meanwhile, for the gravitational field, a Poisson equation

$$
\nabla^2 \phi = -4\pi G \rho
$$

(10)

is considered. Introducing Eqs. (8)-(10) in Eqs. (4)-(6) closes the set for the variables $n, u^\nu$ and $T$. In the next section, the effects of the relativistic terms contained such set in the conditions for a gravitational collapse will be explored within the linear approximation.

### Density fluctuations

In dilute self-gravitating gases, a gravitational collapse may be initiated provided the conditions for the exponential growth of density fluctuations are met for the particular wavelength of such perturbations. The standard approach to this problem consists in assuming small deviations from local equilibrium values for the state variables and analyzing their dynamics in a linear approximation. Thus, we assume a given state variable $X$ can be written as

$$
X(r, t) = X_0 + \delta X(r, t),
$$

(11)

where $X_0$ is its equilibrium value and $\delta X$ a fluctuation given by

$$
\delta X(r, t) = \hat{X} e^{-i\mathbf{q} \cdot \mathbf{r} + st},
$$

with $\mathbf{k}$ being the wave number vector, $k$ its magnitude, $s$ the (real) angular frequency and $\hat{X}$ a constant amplitude. Introducing such assumption in the set of transport equations and considering up to linear terms in fluctuations, the following set is obtained

$$
\frac{\partial (\delta n)}{\partial t} + n_0 \delta \theta = 0,
$$

(12)

$$
\rho_0 \frac{\partial (\delta \theta)}{\partial t} + p_0 \left( \frac{\nabla^2 (\delta T)}{T_0} + \frac{\nabla^2 (\delta n)}{n_0} \right) - A \nabla^2 (\delta \theta) - \frac{1}{c^2} \left( L^\mu_{TT} \frac{\nabla^2 (\delta T)}{T_0} + L^\mu_{nT} \frac{\nabla^2 (\delta n)}{n_0} \right) = -4\pi G m \rho_0 \delta n,
$$

(13)

$$
C_n n_0 \frac{\partial (\delta T)}{\partial t} + p_0 \delta \theta - \frac{L^\mu_{TT}}{T_0} \nabla^2 (\delta T) - \frac{L^\mu_{nT}}{n_0} \nabla^2 (\delta n) = 0,
$$

(14)

where $p_0 = n_0 k T_0$. Also

$$
\rho_0 = \frac{n_0 \varepsilon_0 + p_0}{c^2} = n_0 m \mathcal{G} \left( \frac{1}{z} \right),
$$

where $\mathcal{G} (1/z) = K_3 (1/z) / K_2 (1/z)$ with $K_n$ being the $n$-th modified Bessel function of the second kind and $A = \eta + 4\xi/3$. Notice that Eq. (12) corresponds to the divergence of the momentum balance equation. Since both the continuity and energy balance equations are coupled with the velocity only
through its divergence $\delta \theta$, and all the source terms in Eq. (5) are given by gradients, the evolution equation for the curl of the velocity field is not coupled to the system and a set of three scalar equations is obtained. This transverse mode decays exponentially with time [7].

The set of linearized equations [12]-[14], following the standard procedure, is analyzed in Laplace-Fourier space where they can be written in matrix form, where the inverse susceptibility matrix $M$, is given by

$$M = \begin{pmatrix} -kT_0q^2 + L_{nT}sq^2/n_0e^2 + 4\pi Gm\rho_0 & Aq^2 + \tilde{\rho}_0s & 0 \\ L_{nT}q^2/n_0 & n_0kT_0 & -n_0kq^2 + L_{TT}sq^2/T_0e^2 \\ 0 & C_nn_0s + L_{TT}q^2/T_0 & 0 \end{pmatrix}.$$ 

The corresponding dispersion relation can be written as a third order polynomial as follows

$$s^3 + \left(\frac{A}{\rho_0} + \frac{L_{TT}}{n_0TC_n} - D_{TR}\right)s^2 + \left(\frac{AL_{TT}k}{\rho_0n_0C_n}q^4 + \frac{p_0}{\rho_0q^2}\left(1 + \frac{k}{C_n}\right) - 4\pi Gmn_0\right)s + \frac{k}{C_n\rho_0T_0L_{TT}}(L_{TT} - L_{nT})q^4 - \frac{4\pi GmL_{TT}}{T_0C_n}q^2 = 0,$$

where

$$D_{TR} = \frac{1}{\rho_0c^2}\left(L_{nT} + \frac{k}{C_n}L_{TT}\right),$$

is a generalized thermal diffusivity. In order to obtain approximate solutions for $s$ it is assumed, as in previous works, that equation (15) has three different roots and can thus be factored as

$$(s - \gamma q^2)(s^2 + \mu s + \nu) = 0.$$  

(17)

To order $q^2$ one can show that the real root is not affected by the gravitational field and is given by $\gamma = -L_{TT}/n_0T_0C_n$. This corresponds to a decaying mode and is associated with Rayleigh’s peak when a scattering spectrum is present. On the other hand, the coefficients $\mu$ and $\nu$ feature both relativistic and gravitational corrections:

$$\mu = \frac{q^2L_{TT}}{\rho_0}\left(A\frac{L_{nT}}{C_n} - \frac{k}{c^2L_{TT}} - \frac{k}{c^2C_n}\right)$$  

and

$$\nu = -4\pi Gmn_0 + \frac{p_0}{\rho_0}\left(1 + \frac{k}{C_n}\right)q^2 + \frac{k}{C_n}L_{TT}D_{TR}\frac{L_{TT}}{p_0n_0c^2}q^4.$$  

(19)

The values of $\mu$ and $\nu$ determine the dynamics of the fluctuations. In the non-dissipative case $\mu = 0$ and the fluctuations either oscillate or feature both purely imaginary values for $s$ or feature a combination of growing and decaying modes, depending on the value of $q$. In that case, it is straightforward to obtain that

$$q_J^2 = \frac{4\pi Gmn_0}{C_s^2},$$

where $C_s^2 = zc^2(1 + k/C_n)/\mathcal{G}(1/z)$ is the relativistic speed of sound, marks the boundary between purely oscillatory ($q^2 < \tilde{q}_J^2$) and exponential ($q^2 > \tilde{q}_J^2$) behavior. Notice that $\tilde{q}_J$ corresponds to the relativistic value of the Jean’s wave number corrected by relativistic the value of $C_s$.

Returning to Eq. (17), since $\mu \neq 0$, the purely oscillatory behavior is not present. This occurs also in the dissipative non-relativistic case [5]. If the quadratic expression in (17) has two real roots, the density fluctuations either grow or decay with time. On the other hand, if the roots have a non-vanishing imaginary part, and since $\mu > 0$, the dynamics of the fluctuations correspond to damped oscillations which can be identified with the Brillouin peaks in the corresponding spectrum [12].

The case of interest in this work is the one leading to an exponential growth of $\delta n$, which could initiate to a gravitational collapse of the cloud. The condition on the wave number, depending on
the dissipation parameters, for this phenomenon to be feasible in the system will be analyzed in the next section. It is very important to point out at this stage that the quartic order term in equation (19), which is usually neglected when comparing $\mu^2$ and $\nu$ and is here retained, arises from the purely relativistic term in Eq. (5), which indicates the presence of thermal dissipation of momentum for high temperature gases.

**Gravitational collapse**

In this section, a detailed analysis of the factor $(s^2 + \mu s + \nu)$ in the dispersion relation is carried out in order to obtain the criterion for a gravitational instability to occur. Clearly, the limiting value which separates exponential and oscillatory regimes for density fluctuations is determined by the sign of $\bar{s} = 1 - \frac{4\nu}{\mu^2}$ since the corresponding roots are given by

$$s = -\frac{\mu}{2} \left( 1 \pm \sqrt{\bar{s}} \right).$$

(20)

Imaginary solutions are obtained for $\bar{s} < 0$ which, since $\mu > 0$ (see Fig. 1), leads to damped oscillations. On the other hand, for $\bar{s} > 0$, one obtains real roots and thus an exponential time dependence for $\delta n$, which corresponds to either growing or decaying modes. It is important to notice that the criterion for exponential growth is not completely given by $\bar{s} > 0$, but also by the sign of $s$ since only real and positive values of $s$ may lead to a gravitational collapse. Using once again the fact that $\mu > 0$, it is clear that such requirement is met only if $\bar{s} > 1$ (considering the “−” sign in Eq. (20)), or equivalently $\nu < 0$.

![Figure 1: The quantity $\mu$ in Eq. (18) as a function of $z$, here we have used the explicit expressions form $L_{nT} (z), L_{TT} (z), \eta (z)$ and $\zeta (z)$ [7, 9, 11].](image)

Thus, considering $\nu = 0$ as the equation determining the Jeans criterion, the corresponding critical wave number can be written as

$$q_J^2 = \frac{C_s^2 n_0 T_0 C_n}{2 L_{TT} D_{TR}} \left( 1 + \frac{16\pi G m L_{TT} D_{TR}}{T_0 C_n} \frac{D_{TR}}{C_s^4} - 1 \right).$$

(21)

Imaginary solutions have been excluded in Eq. (21) since only temporal instabilities are relevant for this problem. The expression for $q_J$ above corresponds to the relativistic Jeans’ wave number in the case of dissipative fluids and constitutes the main contribution of this work. Notice that, eventhough the viscosity plays a role in separating imaginary and real solutions, the final criterion for growing modes is independent of $A$. Indeed, since $\bar{s} > 1$ is enforced, the condition that $\bar{s}$ be positive remains
weaker and is already satisfied independently of the values of the viscosities. The fact that viscous dissipation does not alter the value for $q_J$ was already pointed out, for non-relativistic systems ($z \ll 1$), in Ref. [5]. Thus, dissipation affects the Jeans criterion only for high temperature gases and in that case, it is only thermal dissipation that plays a role.

To get some insight on the nature of the relativistic corrections arising from dissipative effects we consider

$$\epsilon \equiv \frac{16\pi G m L \eta}{T_0 C_n C_s^2} \ll 1 \quad (22)$$

in Eq. (21), such that the Jeans’ number can be written as (see Figure 2)

$$q_J^2 \approx \tilde{q}_J^2 \left( 1 - \frac{\epsilon}{4} \right) \quad (23)$$

Notice that, since in the relaxation approximation all transport coefficients are proportional to the relaxation time $\tau$ [9, 7, 11], the correction $\epsilon$ results in a dimensionless parameter that depends on the ratio of the microscopic and gravitational characteristic times. Defining $R = (\tau / \tau_g)^2$, with $\tau_g = 1/\sqrt{4\pi G m n}$, being the gravitational time scale, Fig. (2) verifies the validity of the approximation in Eq. (23) for two values of $R$.

From these results one can also determine the Jeans’ mass

$$M_J = \frac{4}{3} \pi \rho \left( \frac{\pi}{q_J} \right)^3, \quad (24)$$

which under the assumption $\epsilon \ll 1$ can be written as

$$M_J \approx \tilde{M}_J \left( 1 + \frac{3}{8} \epsilon \right) \quad (25)$$

Here $\tilde{M}_J = \frac{4}{3} \pi \rho (\pi / \tilde{q}_J)^3$ is the relativistic Jeans mass in the absence of dissipation.

The magnitude of the correction to the Jeans’ mass and wavenumber, given essentially by the parameter $\epsilon$, can be appreciated in Fig. (2). The plot shows the dependence of $\epsilon$ with the relativistic parameter $z = kT/mc^2$, considering the explicit expressions for the transport coefficients $L_n$, $L_T$, $\eta$ and $\zeta$ obtained in [7, 9, 11] (see the Appendix) and for two different values of $R$. Notice that the correction vanishes in the non-relativistic limit and reaches a constant value for large $z$. Indeed, $\epsilon$ can be written as (see Appendix)

$$\epsilon \equiv \mathcal{F}(z) R$$

where $\lim_{z \to 0} \mathcal{F}(z) = 0$ and $\lim_{z \to 0} \mathcal{F}(z) = 3$.

![Figure 2: Jeans wave number in Eq. (21) and its approximation in Eq. (23) for $R=0.001$ at the left and $R = 0.00001$ at the right.](image-url)
Discussion and concluding remarks

The generalization to the Jeans’ criterion for high temperature dissipative systems was obtained as expressions for $q_J$ and $M_J$. These parameters depend on the value of the transport coefficients as well as the mass of the system. Eventhough they are closely related, one can try to separate the physical origin of the new values on the purely relativistic and the dissipative ones. The isolated effect of the high temperature impacts the values of the inertia coefficient $\rho_0 \rightarrow \tilde{\rho}_0$ which is actually rooted in the relativistic expression for the internal energy. Also the heat capacity $C_n$ and the speed of sound $C_s$ are modified for $z \gtrsim 1$.

On the other hand, the effects of dissipation arise from the relativistic heat terms in the momentum balance equation. In particular, these effects can be traced back to the last term on the left hand side of Eq. (5). It is worthwhile to mention that this same term was found to be initially the source of the so-called generic instabilities when a constitutive equation featuring an acceleration term instead of the density gradient term in Eq. (9) was considered [8]. A detailed discussion of this point can be found in Ref. [7]. One can thus argue that this particular term introduces significant modifications to the dynamics of the relativistic fluid, more so than the relativistic corrections to transport coefficients and other parameters. Moreover, as can be seen by inspecting the set of linearized transport equations (Eqs. (12)-(14)), thermal dissipation due to density and temperature gradients enhance the effects of the pressure gradient, thus increasing the mass required for the collapse. More precisely, as can be seen in Fig. (3), the parameter $D_{TR}$, which features a combination of both transport coefficients, is positive for all values of $z$. It is important to emphasize at this point that only in the relativistic regime, and due to the relativistic heat terms in Eq. (13), dissipation alters the Jeans’ criterion. For dissipative non-relativistic fluids ($z \ll 1$), $q_J$ and $M_J$ have the same values as in the non-dissipative case.

Acknowledgements

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Appendix

In this section we show explicitly the transport coefficients appearing in the closure relations (8) and (9). The quantities \( \eta(z) \), \( \xi(z) \), \( L_{TT}(z) \) and \( L_{\alpha T}(z) \) correspond to the bulk viscosity, shear viscosity and the conductivities associated to heat flux respectively, as functions of the relativistic parameter \( z \). These coefficients where obtained through the usual methods in kinetic theory based on a relaxation time approximation [7][9][11]. For the viscosities we have

\[
\eta(z) = \frac{1}{3} n m c^2 \tau \left[ \frac{z^2(2 - 20z^2)}{3z^2G \left( \frac{1}{z} \right)} + 13z^2G^2 \left( \frac{1}{z} \right) - 2zG^3 \left( \frac{1}{z} \right) - 3z^2 \right]
\]

(26)

and

\[
\xi(z) = n m c^2 \tau z G \left( \frac{1}{z} \right),
\]

(27)

where

\[
\alpha(z) = \frac{1}{z} \left[ -\frac{1}{z} G \left( \frac{1}{z} \right) + \frac{1}{zG \left( \frac{1}{z} \right)} + 5 \right] - \frac{1}{G \left( \frac{1}{z} \right)}.
\]

(28)

Considering that heat flux is written as in Eq. (9) we have

\[
L_{TT}(z) = n m c^4 z^3 \tau \left[ \frac{1}{zG \left( \frac{1}{z} \right)} - 1 \right] \left[ \alpha(z) + \frac{1}{G \left( \frac{1}{z} \right)} \right]
\]

(29)
and

\[ L_{\alpha T} (z) = -nmc^4 z^3 \tau \left[ \alpha (z) + \frac{1}{\mathcal{G} (\frac{1}{2})} \right], \]  

(30)

notice that \( L_{TT} \) and \( L_{\alpha T} \) have opposite sign.

On the other hand, the correction \( \epsilon \) introduced in equation (22) is a complicated function of the relativistic parameter \( z \) and \( R = \tau^2 / \tau_g^2 \), which relates the microscopic and gravitational time scales. In order to gain insight on the magnitude of the correction \( \epsilon \), one can write \( \epsilon (z, R) \). To do this we insert the values of \( L_{TT} \) and \( D_{TR} \), given in equations (29) and (16) respectively as functions of \( z \) and the relativistic values of \( C_n \) and \( C_s \) in equation (22) to obtain

\[ \epsilon (z, R) \equiv R \mathcal{F} (z), \]

where

\[ \mathcal{F} (z) = 4 z^3 \mathcal{G} \left( \frac{1}{z} \right) \left( \frac{1}{z} - \frac{1}{\mathcal{G} (\frac{1}{2})} \right) \left[ \left( \frac{1}{z} - \frac{1}{\mathcal{G} (\frac{1}{2})} \right) - \alpha (z) \right]. \]  

(31)