$K$-theory Schubert calculus of the affine Grassmannian

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Abstract

We construct the Schubert basis of the torus-equivariant $K$-homology of the affine Grassmannian of a simple algebraic group $G$, using the $K$-theoretic NilHecke ring of Kostant and Kumar. This is the $K$-theoretic analogue of a construction of Peterson in equivariant homology. For the case where $G = \text{SL}_n$, the $K$-homology of the affine Grassmannian is identified with a sub-Hopf algebra of the ring of symmetric functions. The Schubert basis is represented by inhomogeneous symmetric functions, called $K$-$k$-Schur functions, whose highest-degree term is a $k$-Schur function. The dual basis in $K$-cohomology is given by the affine stable Grothendieck polynomials, verifying a conjecture of Lam. In addition, we give a Pieri rule in $K$-homology. Many of our constructions have geometric interpretations by means of Kashiwara’s thick affine flag manifold.

Contents

1 Introduction 812
  1.1 Kostant and Kumar’s $K$-NilHecke ring 813
  1.2 The affine Grassmannian and the small-torus GKM condition 813
  1.3 The $K$-theoretic Peterson subalgebra and affine Fomin–Stanley subalgebra 813
  1.4 $G = \text{SL}_n$ and Grothendieck polynomials for the affine Grassmannian 814
  1.5 Related work 814
  1.6 Organization of the paper 814

2 The Kostant–Kumar $K$-NilHecke ring 815
  2.1 Kac–Moody algebras 815
  2.2 The rational form 815
  2.3 The 0-Hecke ring and integral form 816
  2.4 Duality and function ‘basis’ 817
  2.5 The GKM condition 819
  2.6 Structure constants and coproduct 819
  2.7 Explicit localization formulae 820

3 Equivariant $K$-cohomology of Kac–Moody flag manifolds 821
  3.1 Kac–Moody thick flags 821
  3.2 Equivariant $K$-cohomology 822
1. Introduction

Let $G$ be a simple simply connected complex algebraic group and $T \subset G$ the maximal torus. Let $\text{Gr}_G$ denote the affine Grassmannian of $G$. The $T$-equivariant $K$-cohomology $K^T(\text{Gr}_G)$ and $K$-homology $K_T(\text{Gr}_G)$ are equipped with distinguished $K^T(\text{pt})$-bases (denoted by $\{[O_{X_L'}]\}$ and $\{\xi_w\}$), called Schubert bases. Our first main result is a description of the $K$-homology $K_T(\text{Gr}_G)$ as a subalgebra $L$ of the affine $K$-NilHecke algebra of Kostant and Kumar [KK90]. This generalizes work of Peterson [Pet97] in homology. Our second main result is the identification, in the case where $G = SL_n(\mathbb{C})$, of the Schubert bases of the non-equivariant $K$-homology $K_*(\text{Gr}_G)$ and $K$-cohomology $K^*(\text{Gr}_G)$ with explicit symmetric functions called $K$-$k$-Schur functions and affine stable Grothendieck polynomials [Lam06]. This generalizes work of Lam [Lam08] in (co)homology.
1.1 Kostant and Kumar’s $K$-NilHecke ring

Let $\mathfrak{g}$ be a Kac–Moody algebra and let $X$ be the flag variety of $\mathfrak{g}$. Kostant and Kumar [KK90] studied the equivariant $K$-theory $K^T(X)$ via a dual algebra $K$ called the $K$-NilHecke ring. The ring $K$ acts on $K^T(X)$ by Demazure divided difference operators and scalar multiplication by $K^T(pt)$. In particular, they used $K$ to define a ‘basis’ $\{\psi^w_{K, K}\}$ of $K^T(X)$ (elements of $K^T(X)$ are infinite $K^T(pt)$-linear combinations of the ‘basis’).

Kostant and Kumar use the ind-scheme $X_{\text{ind}}$, which is an inductive limit of finite-dimensional schemes. Because of this, classes in $K^T(X_{\text{ind}})$ do not have an immediate geometric interpretation but, rather, are defined via duality in terms of geometric classes in $K^T(X_{\text{ind}})$. We use instead the ‘thick’ flag variety $X$ of Kashiwara [Kas89], which is an infinite-dimensional scheme. This allows us to interpret the $K$-NilHecke ring operations geometrically and to describe (in Theorem 3.2) the Schubert ‘basis’ of $K^T(X)$, representing coherent sheaves $O_w$ of finite-codimensional Schubert varieties. Our basis is different from that of Kostant and Kumar. On the other hand, in our treatment the $K$-homology $K^T(X)$ is now defined via duality.

1.2 The affine Grassmannian and the small-torus GKM condition

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra, and let $\mathfrak{g}_{af}$ be the untwisted affine algebra. Instead of using the affine torus $T_{af}$, we use the torus $T \subset G$ of the finite-dimensional algebraic group and study the equivariant $K$-cohomology $K^T(X_{af})$ and $K^T(Gr_G)$ of the affine flag variety and affine Grassmannian. We use the affine $K$-NilHecke ring for $\mathfrak{g}$, still denoted by $K$, rather than the slightly larger Kostant–Kumar $K$-NilHecke ring for $\mathfrak{g}_{af}$. The corresponding affine NilHecke ring in cohomology was considered by Peterson [Pet97].

We describe (in Theorem 4.3) the image of $K^T(X_{af})$ and $K^T(Gr_G)$ in $\prod_{w \in W_{af}} K^T(pt)$ under localization at the fixed points, where $W_{af}$ denotes the affine Weyl group. This is the $K$-theoretic analogue of a result of Goresky et al. [GKM04] in homology. We call the corresponding condition the small-torus GKM condition. It is significantly more complicated than the usual condition for GKM spaces [GKM98], which would apply if we had used the larger torus $T_{af}$. This description gives an algebraic proof of the existence of a crucial ‘wrong way’ map $K^T(X_{af}) \rightarrow K^T(Gr_G)$, which corresponds in the topological category to $\Omega K \leftarrow LK \rightarrow LK/T_{\mathbb{R}}$ where $K \subset G$ is a maximal compact subgroup, $T_{\mathbb{R}} = T \cap K$, and $\Omega K$ and $LK$ denote the spaces of based and unbased loops, respectively. The space of based loops $\Omega K$ is a topological model for the affine Grassmannian [PS86].

Another description of the $K$-homology of the affine Grassmannian is given by Bezrukavnikov et al. [BFM05], although the methods there do not appear to be particularly suited to the study of Schubert calculus.

1.3 The $K$-theoretic Peterson subalgebra and affine Fomin–Stanley subalgebra

We let $L = Z_{\mathbb{R}}(R(T))$ denote the centralizer in $K$ of the scalars $R(T) = K^T(pt)$ and call it the $K$-Peterson subalgebra. (This centralizer would be uninteresting if we had used $T_{af}$ instead of $T$.) We generalize (in Theorem 5.3) the following result of Peterson [Pet97] (see also [Lam08]) in homology.

**Theorem.** There is a Hopf-isomorphism $k : K^T(Gr_G) \rightarrow L$.

The Hopf-structure of $K^T(Gr_G)$ is derived from $\Omega K$. We also give a description of the images $k(\xi_w)$ of the Schubert bases under this isomorphism (Theorem 5.4).
Next, we consider a subalgebra $\mathbb{L}_0 \subset \mathbb{K}_0$, called the $K$-affine Fomin–Stanley subalgebra, of the affine 0-Hecke algebra. We shall show that $\mathbb{L}_0$ is the evaluation of $\mathbb{L}$ at zero, and that it is a model for the non-equivariant homology $K_*(\mathrm{Gr}_G)$.

1.4 $G = \mathrm{SL}_n$ and Grothendieck polynomials for the affine Grassmannian

We now focus on the case where $G = \mathrm{SL}_n$. In [Lam06], the affine stable Grothendieck polynomials $G_w(x)$ were introduced, where $w \in W_{af}$ is an affine permutation. The symmetric functions $G_w(x)$ lie in a completion $\hat{\Lambda}^{(n)}$ of a quotient of the ring of symmetric functions. A subset of the \{G_w(x)\} form a basis of $\hat{\Lambda}^{(n)}$, and the dual basis elements $g_w(x)$, called $K$-theoretic $k$-Schur functions, form a basis of a subalgebra $\Lambda^{(n)}$ of the ring of symmetric functions.

The symmetric functions $G_w(x)$ are $K$-theoretic analogues of the affine Stanley symmetric functions in [Lam06] and, on the other hand, affine analogues of the stable Grothendieck polynomials in [Buc02, FK94]. The symmetric functions $g_w(x)$ are $K$-theoretic analogues of the $k$-Schur functions $s_w(x)$ (see [LLM03, LM05, LM07]) and, on the other hand, affine (or $k$-) analogues of the dual stable Grothendieck polynomials [LP07, Len00].

Using the technology of the $K$-affine Fomin–Stanley subalgebra, we confirm a conjecture of Lam [Lam06] by proving the following result (see Theorem 7.17).

**Theorem.** There are Hopf-isomorphisms $K_*(\mathrm{Gr}_G) \cong \Lambda^{(n)}$ and $K^*(\mathrm{Gr}_G) \cong \hat{\Lambda}^{(n)}$ that identify the homology Schubert basis with the $K$-$k$-Schur functions $g_w(x)$ and the cohomology Schubert basis with the affine stable Grothendieck polynomials $G_w(x)$.

This generalizes the main result of [Lam08], and the general idea of the proof is the same.

We also obtain a Pieri rule (Corollary 7.6) for $K_*(\mathrm{Gr}_G)$. We give in Theorem 7.19 a geometric interpretation of $G_w(x)$ for any $w \in W_{af}$ as a pullback of a Schubert class from the affine flag variety to the affine Grassmannian. We conjecture that the symmetric functions $G_w(x)$ and $g_w(x)$ satisfy many positivity properties (Conjectures 7.20 and 7.21).

1.5 Related work

Morse [Mor] gives a combinatorial definition of the affine stable Grothendieck polynomials $G_w(x)$ in terms of affine set-valued tableaux and also proves the Pieri rule for $G_w$. The original $k$-Schur functions $s_w(x; t)$ in [LLM03, LM05], which arose in the study of Macdonald polynomials, involve a parameter $t$. It appears that a $t$-analogue $g_w(x; t)$ of $g_w(x)$ exists, defined in a similar manner to [LLMS, Conjecture 9.11]. The connection between $g_w(x; t)$ and Macdonald theory is explored in [BM].

Kashiwara and Shimozono [KS09] constructed polynomials, called affine Grothendieck polynomials, which represent Schubert classes in the $K$-theory of the affine flag manifold. It is unclear how affine Grothendieck polynomials compare with our symmetric functions.

1.6 Organization of the paper

In §2 we review the constructions of the $K$-NilHecke ring $\mathbb{K}$ and define our function ‘basis’ $\{\psi^w\}$. In §3 we introduce Kashiwara’s geometry of ‘thick’ Kac–Moody flag varieties $X$ and the corresponding equivariant $K$-cohomologies; we show how $\mathbb{K}$ corresponds to the geometry of $X$. Section 4 is devoted to equivariant $K$-theory for affine flags and affine Grassmannians with the small torus $T$ acting by means of the level-zero action of the affine Weyl group. In §5 we introduce the affine $K$-NilHecke ring and the $K$-Peterson subalgebra $\mathbb{L}$, and we prove that the latter is
isomorphic to $K_T(\text{Gr}_G)$. In §6 we study the $K$-affine Fomin–Stanley algebra. In §7 we restrict to $G = \text{SL}_n$ and describe explicitly the Hopf-algebra isomorphisms between $K_*(\text{Gr}_G)$ and $K^*(\text{Gr}_G)$ and symmetric functions.

A review of the cohomological NilHecke ring of Kostant and Kumar and the affine NilHecke algebra $\mathbb{A}$ as well as some tables of the symmetric functions $g_w$ and $G_w$ are provided in Appendix A.

2. The Kostant–Kumar $K$-NilHecke ring

One of the themes of [KK90] is that the Schubert calculus of the torus-equivariant $K$-theory $K^T(X)$ of a Kac–Moody flag manifold $X$ is encoded by the $K$-NilHecke ring $\mathbb{K}$, which acts on $K^T(X)$ as Demazure operators. We review the constructions of [KK90] but use a different ‘basis’ for $K^T(X)$, namely, the classes of equivariant structure sheaves of finite-codimensional Schubert varieties in the thick flag manifold of [Kas89].

For a statement $S$, we write $\chi(S) = 1$ if $S$ is true and $\chi(S) = 0$ if $S$ is false.

2.1 Kac–Moody algebras

Let $\mathfrak{g}$ be the Kac–Moody algebra over $\mathbb{C}$ associated with the following data: a Dynkin node set $I$, a symmetrizable generalized Cartan matrix $(a_{ij})_{i,j \in I}$, a free $\mathbb{Z}$-module $P$, linearly independent simple roots $\{\alpha_i | i \in I\} \subset P$, and the dual lattice $P^* = \text{Hom}_\mathbb{Z}(P, \mathbb{Z})$ with simple coroots $\{\alpha_i^\vee | i \in I\} \subset P^*$ such that $\langle \alpha_i^\vee, \alpha_j \rangle = a_{ij}$ where $\langle \cdot, \cdot \rangle : P^* \times P \to \mathbb{Z}$ is the pairing, with the additional property that there exist fundamental weights $\{\Lambda_i | i \in I\} \subset P$ satisfying $\langle \alpha_i^\vee, \Lambda_j \rangle = \delta_{ij}$. Let $Q = \bigoplus_{i \in I} \mathbb{Z}\alpha_i \subset P$ be the root lattice and $Q^\vee = \bigoplus_{i \in I} \mathbb{Z}\alpha_i^\vee \subset P^*$ the coroot lattice. Let $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{t} \oplus \mathfrak{n}_-$ be the triangular decomposition, with $\mathfrak{t} \supseteq P^* \otimes \mathbb{C}$. Let $\Phi$ be the set of roots and $\Phi^\vee$ the sets of positive and negative roots, and let $\mathfrak{g} = \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$ be the root space decomposition. Let $W \subset \text{Aut}(\mathfrak{t}^*)$ be the Weyl group, with involutive generators $r_i$ for $i \in I$ defined by $r_i \cdot \lambda = \lambda - \langle \alpha_i^\vee, \lambda \rangle \alpha_i$. For $i, j \in I$ with $i \neq j$, let $m_{ij}$ be 2, 3, 4, or $\infty$ according to whether $a_{ij}a_{ji}$ is 0, 1, 2, 3 or at least 4. Then $W$ has involutive generators $\{r_i | i \in I\}$ which satisfy the braid relations $(r_ir_j)^{m_{ij}} = \text{id}$. Let $\Phi^{re} = \{w\alpha_i | w \in W, i \in I\} \subset Q$ be the set of real roots, and for $\alpha = w\alpha_i$, let $r_\alpha = wr_iw^{-1}$ be the associated reflection and $\alpha^\vee = w\alpha_i^\vee$ the associated coroot. Let $\Phi^{+re} = \Phi^{re} \cap \Phi^+$ be the set of positive real roots.

2.2 The rational form

Let $T$ be the algebraic torus with character group $P$. The Weyl group $W$ acts on $P$ and therefore on $R(T)$ and $Q(T) = \text{Frac}(R(T))$, where

$$R(T) \cong \mathbb{Z}[P] = \bigoplus_{\lambda \in P} \mathbb{Z}e^\lambda$$

is the Grothendieck group of the category of finite-dimensional $T$-modules. Here $e^\lambda$, for $\lambda \in P$, is the class of the one-dimensional $T$-module with character $\lambda$.

Let $\mathbb{K}_{Q(T)}$ be the smash product of the group algebra $\mathbb{Q}[W]$ and $Q(T)$, defined by $\mathbb{K}_{Q(T)} = Q(T) \otimes_{\mathbb{Q}} \mathbb{Q}[W]$ with multiplication

$$(q \otimes w)(p \otimes v) = q(w \cdot p) \otimes vw$$

815
for \( p, q \in Q(T) \) and \( v, w \in W \). We write \( qw \) instead of \( q \otimes w \). For \( i \in I \), define the Demazure operator [Dem74] \( y_i \in \mathbb{K}_{Q(T)} \) by

\[
y_i = (1 - e^{-\alpha_i})^{-1}(1 - e^{-\alpha_i}r_i).
\]

The \( y_i \) are idempotent and satisfy the braid relations

\[
y_i^2 = y_i \quad \text{and} \quad y_i y_j \cdots = y_j y_i \cdots \quad \text{for} \ \text{all} \ \text{indices} \ \text{satisfying} \ m_{ij} \text{times} \ n_{ij} \text{times}.
\]

Define the elements \( T_i \in \mathbb{K}_{Q(T)} \) by

\[
T_i = y_i - 1 = (1 - e^{\alpha_i})^{-1}(r_i - 1).
\]

We have

\[
r_i = 1 + (1 - e^{\alpha_i})T_i.
\]

The \( T_i \) satisfy

\[
T_i^2 = -T_i \quad \text{and} \quad T_i T_j \cdots = T_j T_i \cdots \quad \text{for} \ \text{all} \ \text{indices} \ \text{satisfying} \ m_{ij} \text{times} \ n_{ij} \text{times}.
\]

Let \( T_w = T_{i_1} T_{i_2} \cdots T_{i_N} \in \mathbb{K}_{Q(T)} \), where \( w = r_{i_1} r_{i_2} \cdots r_{i_N} \) is a reduced decomposition; it is well-defined by (2.3). One can easily verify that

\[
T_i T_w = \begin{cases} T_{r_i w} & \text{if} \ r_i w > w, \\ -T_w & \text{if} \ r_i w < w, \end{cases} \quad \text{and} \quad T_w T_i = \begin{cases} T_{w r_i} & \text{if} \ w r_i > w, \\ -T_w & \text{if} \ w r_i < w, \end{cases}
\]

where \( < \) denotes the Bruhat order on \( W \). For \( \alpha \in \Phi^+ \), define \( T_\alpha = (1 - e^{\alpha})^{-1}(r_\alpha - 1) \). Let \( w \in W \) and \( i \in I \) be such that \( \alpha = w \alpha_i \). Then

\[
T_\alpha = w T_i w^{-1}.
\]

Note that \( \mathbb{K}_{Q(T)} \) acts naturally on \( Q(T) \); in particular, one has

\[
T_i \cdot (q q') = (T_i \cdot q) q' + (r_i \cdot q) T_i \cdot q' \quad \text{for} \ q, q \in Q(T).
\]

Therefore, in \( \mathbb{K}_{Q(T)} \) we have

\[
T_i q = (T_i \cdot q) + (r_i \cdot q) T_i \quad \text{for} \ q \in Q(T).
\]

### 2.3 The 0-Hecke ring and integral form

The 0-Hecke ring \( \mathbb{K}_0 \) is the subring of \( \mathbb{K}_{Q(T)} \) generated by the \( T_i \). It can also be defined by generators \( \{ T_i \mid i \in I \} \) and the relations (2.3). We have \( \mathbb{K}_0 = \bigoplus_{w \in W} 
\mathbb{Z} T_w \).

**Lemma 2.1.** \( \mathbb{K}_0 \) acts on \( R(T) \).

**Proof.** We have that \( \mathbb{K}_0 \) acts on \( Q(T) \); also, each \( T_i \) preserves \( R(T) \) by (2.5) and the following formulae for \( \lambda \in P \):

\[
T_i \cdot e^\lambda = \begin{cases} e^\lambda(1 + e^{\alpha_i} + \cdots + e^{(\alpha_i^\vee, \lambda) - 1}) & \text{if} \ \langle \alpha_i^\vee, \lambda \rangle > 0, \\ 0 & \text{if} \ \langle \alpha_i^\vee, \lambda \rangle = 0, \\ -e^\lambda(1 + e^{\alpha_i} + \cdots + e^{-(\alpha_i^\vee, \lambda) - 1}) & \text{if} \ \langle \alpha_i^\vee, \lambda \rangle < 0. \end{cases}
\]

The assertion follows. \( \square \)
Define the $K$-NilHecke ring $K$ to be the subring of $K_{Q(T)}$ generated by $K_0$ and $R(T)$. We have $K_{Q(T)} \cong Q(T) \otimes_{R(T)} K$. By (2.6),

$$K = \bigoplus_{w \in W} R(T)T_w.$$  \hspace{1cm} (2.7)

### 2.4 Duality and function ‘basis’

Let $\text{Fun}(W, Q(T))$ be the right $Q(T)$-algebra of functions from $W$ to $Q(T)$ under pointwise multiplication and scalar multiplication $(\psi \cdot q)(w) = q\psi(w)$ for $q \in Q(T)$, $\psi \in \text{Fun}(W, Q(T))$ and $w \in W$. By linearity, we identify $\text{Fun}(W, Q(T))$ with left $Q(T)$-linear maps $K_{Q(T)} \to Q(T)$ such that

$$\psi\left(\sum_{w \in W} a_w w\right) = \sum_{w \in W} a_w \psi(w).$$

Note that $\text{Fun}(W, Q(T))$ is a $K_{Q(T)}$-$Q(T)$-bimodule via

$$(a \cdot \psi \cdot q)(b) = \psi(qba) = q\psi(ba)$$  \hspace{1cm} (2.8)

for $\psi \in \text{Fun}(W, Q(T))$, $q \in Q(T)$ and $a, b \in K_{Q(T)}$.

Evaluation gives a perfect pairing $\langle \cdot, \cdot \rangle : K_{Q(T)} \times \text{Fun}(W, Q(T)) \to Q(T)$ defined by

$$\langle a, \psi \rangle = \psi(a),$$

which is $Q(T)$-bilinear in the sense that

$$\langle qa, \psi \rangle = q\langle a, \psi \rangle = \langle a, \psi \cdot q \rangle.$$  \hspace{1cm} (2.9)

Define the subring $\Psi \subset \text{Fun}(W, Q(T))$ by

$$\Psi = \{ \psi \in \text{Fun}(W, Q(T)) \mid \psi(K) \subset R(T) \} = \{ \psi \in \text{Fun}(W, Q(T)) \mid \psi(T_w) \in R(T) \text{ for all } w \in W \}. \hspace{1cm} (2.10)$$

Clearly, $\Psi$ is a $K$–$R(T)$-bimodule. By (2.7), for $v \in W$ there are unique elements $\psi^v \in \Psi$ such that

$$\psi^v(T_w) = \delta_{v,w}$$

for all $w \in W$. We have $\Psi = \prod_{v \in W} R(T)\psi^v$.

**Remark 2.1.** In § 3 we show that $\psi^v(w)$ is the restriction of the equivariant structure sheaf $[\mathcal{O}_{X_v}]$ of the finite-codimensional Schubert variety $X_v \subset X$ of the thick Kac–Moody flag manifold $X$ to the $T$-fixed point $w$. See Appendix A.2 for the relationship between our functions $\psi^v$ and those of [KK90].

Letting $w = \text{id}$, we have

$$\delta_{v,\text{id}} = \psi^v(T_{\text{id}}) = \psi^v(\text{id}). \hspace{1cm} (2.11)$$

**Lemma 2.2.** For $v \in W$ and $i \in I$,

$$y_i \cdot \psi^v = \begin{cases} 
\psi^{vr_i} & \text{if } vr_i < v, \\
\psi^v & \text{if } vr_i > v.
\end{cases}$$
Proof. For \( w \in W \) we have \( T_{wy_i} = T_w(1 + T_i) = \chi(wr_i > w)(T_w + T_{wr_i}) \). Therefore, by (2.8),
\[
\langle T_w, y_i \cdot \psi^v \rangle = \langle T_{wy_i}, \psi^v \rangle = \chi(wr_i > w)(T_w + T_{wr_i}, \psi^v) = \chi(wr_i > w)(\delta_{v,w} + \delta_{v,wr_i}) = \chi(vr_i > v)\delta_{v,w} + \chi(v > vr_i)\delta_{vr_i,w},
\]
from which the lemma follows. \( \square \)

Remark 2.2. From (2.11) and Lemma 2.2 we obtain the following ‘right-hand’ recurrence for \( \psi^v(w) \).

(i) If \( w = \text{id} \), then \( \psi^v(\text{id}) = \delta_{v,\text{id}} \).

(ii) Otherwise, let \( i \in I \) be such that \( wr_i < w \). Then
\[
\psi^v(w) = \begin{cases} 
\psi^v(wr_i) & \text{if } v < vr_i, \\
(1 - e^{-w(\alpha_i)})\psi^{vr_i}(w) + e^{-w(\alpha_i)}\psi^v(wr_i) & \text{if } vr_i < v.
\end{cases}
\]

(2.12)

This rewrites \( \psi^v(w) \) in terms of \( \psi^{v'}(w') \) for \( (v', w') \) such that either \( w' < w \) or both \( w' = w \) and \( v' < v \).

Lemma 2.3. We have \( \psi^v(w) = 0 \) unless \( v \leq w \).

Proof. The statement is true for \( w = \text{id} \) by (2.11). Otherwise, let \( i \in I \) be such that \( wr_i < w \) and suppose that \( v \not\in w \). Then \( v \not\in wr_i \) and \( vr_i \not\in w \) (see [Hum90]). The assertion is proved by induction using (2.12). \( \square \)

The next result follows from the definitions.

Proposition 2.4. For all \( v, w \in W \), we have \( w = \sum_{v \in W} \psi^v(w)T_v \).

Remark 2.3. Proposition 2.4 leads to a ‘left-hand’ recurrence for \( \psi^v(w) \) as follows.

(i) For \( w = \text{id} \) we have (2.11).

(ii) Otherwise, let \( i \in I \) be such that \( r_iw < w \). By induction on length, we obtain
\[
w = r_i(r_iw)
\]
\[
= (1 + (1 - e^{\alpha_i})T_i)\left(\sum_u \psi^u(r_iw)T_u\right)
\]
\[
= \sum_u \psi^u(r_iw)T_u + (1 - e^{\alpha_i})\sum_u T_i\psi^u(r_iw)T_u
\]
\[
= \sum_u \psi^u(r_iw)T_u + (1 - e^{\alpha_i})\sum_u ((T_i \cdot \psi^u(r_iw))T_u + (r_i \cdot \psi^u(r_iw))T_iT_u).
\]

Taking the coefficient of \( T_v \), we see that
\[
\psi^v(w) = \psi^v(r_iw) + (1 - e^{\alpha_i})((T_i \cdot \psi^v(r_iw)) + \chi(r_iw < v)r_i \cdot (\psi^{r_iw}(r_iw) - \psi^v(r_iw))).
\]

Therefore, for \( r_iw < w \) we have
\[
\psi^v(w) = \begin{cases} 
r_i \cdot \psi^v(r_iw) & \text{if } r_iw > v, \\
e^{\alpha_i} r_i \cdot \psi^v(r_iw) + (1 - e^{\alpha_i}) r_i \cdot \psi^{r_iw}(r_iw) & \text{if } r_iw < v.
\end{cases}
\]

818
Define the inversion set of $v \in W$ by
\[
\text{Inv}(v) = \{ \alpha \in \Phi^+ \mid r_\alpha v < v \}.
\]

**Lemma 2.5.** For all $v \in W$, we have $\psi^v(v) = \prod_{\alpha \in \text{Inv}(v)} (1 - e^\alpha)$.

**Proof.** This follows directly from Remark 2.3 and Lemma 2.3.

**Remark 2.4.** We have $\psi^v(w) = \eta(w \cdot \psi^{v^{-1}}(w^{-1}))$, where $\eta : Z[P] \rightarrow Z[P]$ is given by $\eta(e^\lambda) = e^{-\lambda}$.

**2.5 The GKM condition**

We recall the $K$-theoretic Groesky–Kottwitz–Macpherson (GKM) condition as a criterion for membership in $\Psi$. This condition and the associated geometry is discussed in §3.3.

**Proposition 2.6.** $\Psi$ is the set of $\psi \in \text{Fun}(W, Q(T))$ such that
\[
\psi(r_\alpha w - \psi(w) (1 - e^\alpha) R(T) \quad \text{for all } \alpha \in \Phi^+ \text{ and } w \in W. \tag{2.13}
\]

**Proof.** Let $\beta = w^{-1} \alpha$ and $\psi \in \Psi$. Then $r_\alpha w = w \beta$ and
\[
\frac{\psi(r_\alpha w) - \psi(w)}{1 - e^\alpha} = \psi((1 - e^\alpha)^{-1}(w \beta - w)) = \psi(w T_\beta),
\]
which is in $R(T)$ since $wT_\beta \in K$, by using (2.4) and (2.2).

For the converse, let $\psi \in \text{Fun}(W, Q(T))$ satisfy (2.13) and suppose that $\psi \neq 0$. Let $v \in \text{Supp}(\psi)$ be a minimal element. For every $\alpha \in \Phi^+$ such that $r_\alpha v < v$, we have $\psi(v) \in (1 - e^\alpha) R(T)$ by (2.13), Lemma 2.3 and the minimality of $v$. Since the factors $(1 - e^\alpha)$ are relatively prime by [Kac90, Proposition 6.3], $\psi(v) \in \text{Supp}(\psi) R(T)$ by Lemma 2.5. Then $\psi' \in \Psi$, where $\psi'(w) = \psi(w) - (\psi(v) / \psi^v(w)) \psi^v(w)$ for $w \in W$. Moreover, $v \notin \text{Supp}(\psi')$ and $\text{Supp}(\psi') \setminus \text{Supp}(\psi)$ consists of elements strictly greater than $v$. Repeating the argument for $\psi'$ and so on, we see that $\psi$ is in $\prod_{v \in W} R(T) \psi^v$. \qed

**2.6 Structure constants and coproduct**

The proof of the following result is straightforward but lengthy.

**Proposition 2.7.** Let $M$ and $N$ be left $K$-modules. Define
\[
M \otimes_{R(T)} N = (M \otimes_Z N) / \langle qm \otimes n - m \otimes qn \mid q \in R(T), m \in M, n \in N \rangle.
\]

Then $K$ acts on $M \otimes_{R(T)} N$ by
\[
q \cdot (m \otimes n) = qm \otimes n, \quad T_i \cdot (m \otimes n) = T_i \cdot m \otimes n + m \otimes T_i \cdot n + (1 - e^{\alpha_i}) T_i \cdot m \otimes T_i \cdot n.
\]

Under this action we have
\[
w \cdot (m \otimes n) = w m \otimes w n. \tag{2.14}
\]

Consider the case where $M = N = K$. By Proposition 2.7 there is a left $R(T)$-module homomorphism $\Delta : K \rightarrow K \otimes_{R(T)} K$ defined by $\Delta(a) = a \cdot (1 \otimes 1)$. It satisfies
\[
\Delta(q) = q \otimes 1 \quad \text{for } q \in R(T), \tag{2.15}
\]
\[
\Delta(T_i) = 1 \otimes T_i + T_i \otimes 1 + (1 - e^{\alpha_i}) T_i \otimes T_i \quad \text{for } i \in I. \tag{2.16}
\]

819
Let \( a \in \mathbb{K} \) and \( \Delta(a) = \sum_{v,w} a_{v,w} T_v \otimes T_w \) with \( a_{v,w} \in R(T) \). It follows from Proposition 2.7 that the action of \( a \) on \( M \otimes R(T) N \) can be computed in the following simple ‘componentwise’ fashion: \( a \cdot (m \otimes n) = \sum_{v,w} a_{v,w} T_v m \otimes T_w n \). In particular, if \( b \in \mathbb{K} \) and \( \Delta(b) = \sum_{v' , w'} b_{v',w'} T_{v'} \otimes T_{w'} \), then

\[
\Delta(ab) = \Delta(a) \cdot \Delta(b) := \sum_{v,v',w,w'} a_{v,w} b_{v',w'} T_v T_{v'} \otimes T_w T_{w'}.
\]  

(2.17)

**Remark 2.5.** The naive componentwise product is ill-defined on all of \( \mathbb{K} \otimes R(T) \mathbb{K} \), for if it were well-defined, then \( (T_i \otimes 1)(q \otimes 1) = (T_i \otimes 1)(1 \otimes q) \) or, equivalently, \( T_i q \otimes 1 = T_i \otimes q = q(T_i \otimes 1) = qT_i \otimes 1 \), which is false for \( q = e_{\alpha_i} \).

There is a left \( R(T) \)-bilinear pairing \( \langle \cdot , \cdot \rangle : (\mathbb{K} \otimes R(T) \mathbb{K}) \times (\Psi \otimes R(T) \Psi) \to R(T) \) given by

\[
\langle a \otimes b, \phi \otimes \psi \rangle = \langle a, \phi \rangle \langle b, \psi \rangle.
\]

**Lemma 2.8.** For all \( a \in \mathbb{K} \) and \( \phi , \psi \in \Psi \), we have \( \langle a, \phi \psi \rangle = \langle \Delta(a), \phi \otimes \psi \rangle \).

**Proof.** First, extend the definitions in the obvious manner to \( \mathbb{K}_{Q(T)} \) and \( \text{Fun}(W, Q(T)) \). Using left \( Q(T) \)-linearity, we may then take \( a = w \). Then \( \langle \Delta(w), \phi \otimes \psi \rangle = \langle w \otimes w, \phi \otimes \psi \rangle = \phi(w) \psi(w) = \langle w, \phi \psi \rangle \).

Define the structure ‘constants’ \( c_{uv}^{aw} \in R(T) \) by \( \psi^u \psi^v = \sum_{w \in W} c_{uv}^{aw} \psi^w \). The structure constants of \( \Psi \) are recovered by the map \( \Delta \).

**Proposition 2.9.** We have \( \Delta(T_w) = \sum_{u,v} c_{uv}^{aw} T_u \otimes T_v \) for all \( w \in W \).

**Proof.** This follows from Lemma 2.8 and the fact that \( \psi^u(T_v) = \delta_{uv} \).

### 2.7 Explicit localization formulae

For the sake of completeness, we give an explicit formula for the values \( \psi^v(w) \). It is a variant of a formula due independently to Graham \([\text{Gra02}]\) and Willems \([\text{Wil04}]\). Let \( \varepsilon : \mathbb{K}_{Q(T)} \to Q(T) \) be the left \( Q(T) \)-module homomorphism defined by \( \varepsilon(w) = 1 \) for all \( w \in W \).

**Proposition 2.10.** Let \( v, w \in W \) and let \( w = r_{i_1} r_{i_2} \cdots r_{i_N} \) be any reduced decomposition of \( w \). For \( b_1 b_2 \cdots b_N \in \{0, 1\}^N \), let \( |b| = \sum_{i=1}^N b_i \). Then

\[
\psi^v(w) = \varepsilon \sum_{b \in B(\bullet,v)} (-1)^{|b|} \prod_{k=1}^N \left\{ (1 - e^{\alpha_i}) r_{i_k} \right\} \text{ if } b_k = 1,
\]

\[
\text{if } b_k = 0,
\]

(2.18)

where the sum runs over

\[
B(\bullet,v) = \left\{ b = (b_1, b_2, \ldots, b_N) \in \{0, 1\}^N \left| \prod_{k=1}^N T_{i_k} = \pm T_v \right. \right\}.
\]

(2.19)

Formula (2.18) is the \( K \)-theoretic analogue of the formula \([\text{AJS94}, \text{Bil99}, \ (1.2)]\) for the restriction of a \( T \)-equivariant Schubert cohomology class \([X_v]\) to a \( T \)-fixed point \( w \).
Example 2.6. Let $G = SL_3$, $v = r_1$ and $w = r_1 r_2 f_1$. Then there are three possible binary words $b$, namely $(1, 0, 0)$, $(0, 0, 1)$ and $(1, 0, 1)$, yielding

$$\psi^v(w) = \varepsilon((1 - e^{a_1})r_1 r_2 r_1 + r_1 r_2 (1 - e^{a_1}) r_1 - (1 - e^{a_1}) r_1 r_2 (1 - e^{a_1}) r_1)$$

$$= (1 - e^{a_1}) + (1 - e^{r_1 r_2 a_1}) - (1 - e^{a_1}) (1 - e^{r_1 r_2 a_1})$$

$$= (1 - e^{a_1}) + (1 - e^{a_2}) - (1 - e^{a_1}) (1 - e^{a_2})$$

$$= 1 - e^{a_1 + a_2}.$$  

Using the reduced decomposition $w = r_2 r_1 r_2$ instead, there is only one summand $b = (0, 1, 0)$, and we obtain

$$\psi^v(w) = \varepsilon(r_2 (1 - e^{a_1}) r_1 r_2) = 1 - e^{2 a_1} = 1 - e^{a_1 + a_2}.$$  

Proof of Proposition 2.10. Let the right-hand side of (2.18) be denoted by $\phi^v(w)$. We prove that $\psi^v(w) = \phi^v(w)$ by induction on $w$ and then on $v$. Let $i = i_N$. If $v < v_N$, then in any summand $b$ we have $b_N = 0$, so that $\phi^v(w) = \phi^v(w r_i) = \psi^v(w r_i) = \psi^v(w)$ by induction on the length $\ell(w)$ of $w$ and (2.12). Otherwise, let $v > v_N$. The part of $\phi^v(w)$ with $b_N = 0$ is given by $\phi^v(w r_i) = \psi^v(w r_i)$. The rest of $\phi^v(w)$ consists of summands with $b_N = 1$. Consider the left-to-right product $\pm T_u$ of $T_u$ for which $b_k = 1$ except that the term $b_N = 1$ is omitted. For the $b$ such that $u = v$, the last factor $T_i N = T_i$ produces an additional negative sign, and we obtain $-(1 - e^{\alpha_i})) \phi^v(w r_i) = (e^{-w(\alpha_i)} - 1) \psi^v(w r_i)$ because the product $r_i \cdots r_{i-1}$ of the reflections is $w r_i$. For the $b$ with $u = v r_i$, we obtain that $(w r_i (1 - e^{\alpha_i})) \phi^v(r_i) = (1 - e^{-w(\alpha_i)}) \psi^v(w r_i).$ In total, we obtain the right-hand side of (2.12), which equals $\psi^v(w)$.

3. Equivariant K-cohomology of Kac–Moody flag manifolds

Kostant and Kumar [KK90] use the ‘thin’ Kac–Moody flag manifold $X_{\text{ind}}$, which is an ind-scheme with finite-dimensional Schubert varieties [Kum02]. In contrast, we employ the larger ‘thick’ Kac–Moody flag manifold $X$ [Kas89], which is a scheme of infinite type with finite-codimensional Schubert varieties. Using the thick Kac–Moody flag manifold, we give natural geometric interpretations to the constructions in the $K$-NilHecke ring.

3.1 Kac–Moody thick flags

For the following discussion see [Kas89]. Let $T$ be the algebraic torus with character group $P$, $U_\pm$ the group scheme with $\text{Lie}(U_\pm) = n_\pm$, and $B_\pm$ the Borel subgroups with $\text{Lie}(B_\pm) = t \oplus n_\pm$. For $i \in I$, let $P_i^\pm$ be the parabolic group with $\text{Lie}(P_i^\pm) = \text{Lie}(B_\pm) \oplus \mathfrak{g}_{\pm \alpha_i}$. These groups are all contained in an affine scheme $G_\infty$ of infinite type that contains a canonical ‘identity’ point $e$. Let $G \subset G_\infty$ be the open subset defined by $G = P_{i_1} \cdots P_{i_m} e P_{j_1}^- P_{j_2}^- \cdots P_{j_m}^-$.

Given the above, $X = G/B_-$ is then the thick Kac–Moody flag manifold; it is a scheme of infinite type over $\mathbb{C}$. For each subset $J \subset I$, let $P_J^- \subset G$ be the group generated by $B_-$ and $P_i^-$ for $i \in J$.

Write $X^J = G/P_J^-$. Let $W_J \subset W$ be the subgroup generated by $r_i$ for $i \in J$, and let $W^J$ be the set of minimal-length coset representatives in $W/W_J$. For $w \in W^J$, let $X^J_w = B w P_J^- / P_J^-$ where $B = B_+$; it is locally closed in $X^J$. We have the $B$-orbit decomposition

$$X^J = \bigsqcup_{w \in W^J} X^J_w.$$  

821
Let \( X_w^J = \overline{X_w^J} \) be the Schubert variety. It has codimension \( \ell(w) \) in \( X^J \) and coherent structure sheaf \( \mathcal{O}_{X_w^J} \). We have
\[
X_w^J = \bigcup_{v \in W^J \atop v \geq w} \hat{X}_v^J.
\] (3.1)

Let \( S \) be a finite Bruhat order ideal of \( W^J \) (a finite subset \( S \subset W^J \) such that if \( v \in S, u \in W^J \) and \( u \leq v \), then \( u \in S \)). Let \( \Omega_S^J = \bigcup_{w \in S} X_w^J = \bigcup_{w \in S} wBP_j^J / P_j^J \) be a \( B \)-stable finite union of translations of the big cell \( \hat{X}^J := BP_j^J / P_j^J \), which is open in \( X^J \). The big cell is an affine space of countable dimension (finite if \( g \) is finite-dimensional): \( \hat{X}^J \cong \text{Spec}(\mathbb{C}[[x_1, x_2, \ldots]]) \).

### 3.2 Equivariant \( K \)-cohomology

Denote by \( \text{Coh}^T(\Omega_S^J) \) the category of coherent \( T \)-equivariant \( \mathcal{O}_{\Omega_S^J} \)-modules, and let \( K^T(\Omega_S^J) \) be the Grothendieck group of \( \text{Coh}^T(\Omega_S^J) \). For each \( w \in S \), \( \mathcal{O}_{X_w^J} \) belongs to \( \text{Coh}^T(\Omega_S^J) \) and therefore defines a class \( [\mathcal{O}_{X_w^J}] \in K^T(\Omega_S^J) \). Define
\[
K^T(X^J) = \lim_{S} K^T(\Omega_S^J).
\]

One may show (as in [KS09] for the case where \( J = \emptyset \)) that
\[
K^T(X^J) \cong \prod_{w \in W^J} K^T(\mathcal{O}_{X_w^J}).
\] (3.2)

Recall that \( K^T(pt) \cong R(T) \cong \mathbb{Z}[P] = \bigoplus_{\lambda \in P} \mathbb{Z}^{\lambda} \). Elements of \( K^T(X^J) \) are possibly infinite \( K^T(pt)\)-linear combinations of equivariant Schubert classes \( [\mathcal{O}_{X_w^J}] \).

### 3.3 Restriction to \( T \)-fixed points

For \( w \in W^J \), let \( i_w^J : \{pt\} \to W^J \cong (X^J)^T \subset X^J \) be the inclusion with image \( \{wP_j^J / P_j^J\} \). Restriction to the \( T \)-fixed points induces an injective \( R(T) \)-algebra homomorphism
\[
K^T(X^J) \xrightarrow{\text{res}^J} K^T((X^J)^T) \cong K^T(W^J) \cong \text{Fun}(W^J, R(T)),
\] (3.3)

(see [HHH05, KK90]), where \( \text{Fun}(W^J, R(T)) \) is the \( R(T) \)-algebra of functions \( W^J \to R(T) \) with pointwise multiplication and \( R(T) \)-action
\[
(q \psi)(w) = q \psi(w)
\]
for \( q \in R(T), \ psi \in \text{Fun}(W^J, R(T)) \) and \( w \in W^J \). Let \( \iota_J : \text{Fun}(W^J, R(T)) \to \text{Fun}(W, R(T)) \) be defined by extending functions to be constant on cosets in \( W/W_J \):
\[
\iota_J(\psi)(w) = \psi(w')
\]
for \( \psi \in \text{Fun}(W^J, R(T)) \), where \( w' \in W^J \) is such that \( w'w_J = wW_J \).

Define \( \Psi^J \subset \text{Fun}(W, R(T)) \) by \( \psi \in \Psi^J \) if and only if \( \psi \) is in the image of \( \iota_J \) and
\[
\psi(r_\alpha w) - \psi(w) \in (1 - e^\alpha)R(T) \quad \text{for all } w, r_\alpha w \in W, \alpha \in \Phi^+. \]

We call this the GKM condition\(^1\) for \( K^T(X^J) \).

\(^1\)The corresponding criterion was proved in [GKM98] for equivariant cohomology for more general spaces, commonly called GKM spaces. For Kac–Moody flag ind-schemes the criterion follows directly from results in [KK90]. See also [HHH05] for more general cohomology theories and spaces.
Theorem 3.1 [HHH05, KK90].

\[ K^T(X^J) \cong \Psi^J. \]

For the sake of completeness, we include a proof of Theorem 3.1. For \( v \in W^J \), define \( \psi_v^w \in \text{Fun}(W^J, R(T)) \) to be the image of \([\mathcal{O}_{X^J}]\):

\[ \psi_v^w (w) = i_w^J(\mathcal{O}_{X^J}) \quad \text{for} \quad v, w \in W^J. \] (3.5)

When \( J = \emptyset \) we shall write \( X = X^\emptyset, \Psi = \Psi^\emptyset \) and so on, suppressing \( \emptyset \) in the notation. Observe that the definition of \( \Psi = \Psi^\emptyset \) in this section agrees with the definition (2.9) by Proposition 2.6. Provisionally, for \( J = \emptyset \), we write \( \psi^v_0 \) for the functions defined by (3.5) and show that they agree with the functions defined by (2.10) using the \( K \)-NilHecke ring.

**Theorem 3.2.** For all \( v \in W \), we have \( \psi^v = \psi^v_0 \).

### 3.4 Push–pull and \( y_i \)

Fix \( i \in I \). For the singleton \( J = \{i\} \), let \( P_i^- = P_i^J \) and \( X^i = X^J \), and let \( p_i : X \to X^i \) be the projection, which is a \( \mathbb{P}^1 \)-bundle. In [KS09] it is shown that

\[ p_i^* p_i^* (\mathcal{O}_{X^i}) = \begin{cases} [\mathcal{O}_{X, v}] & \text{if } v r_i < v, \\ [\mathcal{O}_{X^i}] & \text{if } v r_i > v. \end{cases} \] (3.6)

**Proposition 3.3.** The map \( \psi \mapsto y_i : \psi \) is an \( (R_T) \)-module endomorphism of \( \text{Fun}(W, R(T)) \) such that the following diagram commutes.

\[
\begin{array}{ccc}
K^T(X) & \xrightarrow{\text{res}} & \text{Fun}(W, R(T)) \\
p_i^* p_i^* & & y_i & & \\
\downarrow & & \downarrow & & \\
K^T(X) & \xrightarrow{\text{res}} & \text{Fun}(W, R(T))
\end{array}
\]

**Proof.** Let \( x_i^0 \in X_i \) be the point \( P_i^- / P_i^- \). Let \( G_i \supset G_i^+ \supset T \) be the subgroups with \( \text{Lie}(G_i) = t \oplus g_{+i} \oplus g_{-i}, \) and \( \text{Lie}(G_i^+) = t \oplus g_{+i} \).

Now let \( w \in W \) with \( w r_i < w \). Then \( \text{Ad}(w) G_i^+ \subset B \), and it stabilizes \( x_i^0 \) and \( p_i^{-1}(x_i^0) \). Let \( j : p_i^{-1}(x_i^0) \to X \) be the inclusion. Then for \( F \in \text{Coh}^B(X) \), the first left derived functor \( L_1 j^* F \) is \( \text{Ad}(w) G_i^+ \)-equivariant. For \( x \in \{w, w r_i\} \), let \( i_x \) be the inclusion \( \{\text{pt}\} \to x x_i^0 \subset p_i^{-1}(x_i^0) \). Hence we have the commutative diagram

\[
\begin{array}{ccc}
K^B(X) & \xrightarrow{i_x^*} & K^T(pt) \\
\downarrow j^* & & \downarrow i_x^* \\
K^{\text{Ad}(w)} G_i^+(p_i^{-1}(x_i^0)) & &
\end{array}
\]

and the following isomorphisms that forget down to the Levi:

\[ K^B(X) \cong K^T(X), \]

\[ K^{\text{Ad}(w)} G_i^+(p_i^{-1}(x_i^0)) \cong K^T(p_i^{-1}(w x_i^0)). \]

This allows reduction to the case of \( p_i^{-1}(w x_i^0) \cong \mathbb{P}^1 \), where the result is standard; see [CG97, Corollary 6.1.17]. \( \Box \)
We show that the functions \( \psi^v_0 \) satisfy the recurrence in Remark 2.2.

Since \( X_\text{id} = X \), it follows that \( \psi^\text{id}_0 (\text{id}) = 1 \).

Let \( v \in W \). By (3.1), we have \( w \in X_v \) if and only if \( w \geq v \). Therefore
\[
\psi^v_0 (w) = 0 \quad \text{unless} \quad v \leq w.
\]
In particular, \( \psi^v_0 (\text{id}) = 0 \) if \( v \neq \text{id} \). Therefore the values \( \psi^v_0 (w) \) satisfy the base case of the recurrence. Proving (2.12) holds for \( \psi^v_0 \) is equivalent to showing that
\[
y_i \cdot \psi^v_0 = \begin{cases} 
    \psi^{vr_i}_0 & \text{if } vr_i < v, \\
    \psi^v_0 & \text{if } vr_i > v.
\end{cases}
\]
But this holds by Proposition 3.3 and (3.6).

Let \( p_J : X \to X^J \) be the projection. Then there is the following commutative diagram of injective \( R(T) \)-algebra maps, where the horizontal maps are restriction maps as in (3.3).

\[
\begin{array}{ccc}
K^T (X^J) & \xrightarrow{\text{res}_J} & \text{Fun}(W^J, R(T)) \\
\downarrow{p_J} & & \downarrow{\iota_J} \\
K^T (X) & \xrightarrow{\text{res}} & \text{Fun}(W, R(T))
\end{array}
\]

We have \( p_J^*([O_{X_\text{id}}]) = [O_{X_v}] \) and \( \iota_J(\psi_v) = \psi^v \) for \( v \in W^J \).

**Proof of Theorem 3.1.** For the \( J = \emptyset \) case, the result follows from (3.2), Theorem 3.2 and Proposition 2.6. For \( J \neq \emptyset \), let \( \psi = \sum w a_w \psi^w \) be in the image of \( \iota_J \). It suffices to show that \( a_w = 0 \) for \( w \notin W^J \). Since \( T_i \) is a \( Q(T) \)-multiple of \( r_i - 1 \), we have \( T_i \cdot \psi = 0 \) whenever \( i \in J \). Suppose that \( a_w \neq 0 \) for some \( w \notin W^J \). Pick such a \( w \) with minimal length, and let \( i \in J \) be such that \( wr_i < w \). From Lemma 2.2 and \( T_i = y_i - 1 \), we deduce that the coefficient of \( \psi^{wr_i} \) in \( T_i \cdot \psi \) is non-zero, which is a contradiction.

\[ \square \]

4. The affine flag manifold and affine Grassmannian

We now specialize our constructions to the case of an affine root system, and consider the thick affine flag manifold \( X_\text{af} \) and thick affine Grassmannian \( \text{Gr}_G \) and their equivariant \( K \)-cohomology. However, instead of using the full affine torus \( T_\text{af} \subset G_\text{af} \), we shall use the torus \( T \subset G \) and consider \( K^T (\text{Gr}_G) \). We give a small-torus GKM condition, which is the \( K \)-theoretic analogue of a result of Goresky et al. [GKM04] in cohomology.

4.1 The affine flag manifold

We fix notation specific to affine root systems and their associated finite root systems.

Let \( g \supset b \supset t \) such that \( g \) is a simple Lie algebra over \( \mathbb{C} \), \( b \) is a Borel subalgebra, and \( t \) is a Cartan subalgebra. Also, take a Dynkin node set \( I \), a finite Weyl group \( W \), simple reflections \( \{ r_i \mid i \in I \} \), a weight lattice \( P = \bigoplus_{i \in I} \mathbb{Z} \omega_i \subset t^\ast \) with fundamental weights \( \omega_i \), a root lattice \( Q = \bigoplus_{i \in I} \mathbb{Z} \alpha_i \subset t^\ast \) with simple roots \( \alpha_i \), and a coroot lattice \( Q^\vee = \bigoplus_{i \in I} \mathbb{Z} \alpha_i^\vee \subset t^\ast \).

Let \( G \supset B \supset T \) such that \( G \) is a simple and simply connected algebraic group over \( \mathbb{C} \) with \( \text{Lie}(G) = g \), \( B \) is a Borel subgroup, and \( T \) is a maximal algebraic torus.

Let \( g_\text{af} = (\mathbb{C}[t, t^{-1}] \otimes g) \oplus \mathbb{C} c \oplus \mathbb{C} d \) be the untwisted affine Kac–Moody algebra with canonical simple subalgebra \( g \), canonical central element \( c \) and degree derivation \( d \). Let \( g_\text{af} = n_\text{af}^+ \oplus t_\text{af} \oplus n_\text{af}^- \)
be the triangular decomposition with affine Cartan subalgebra \( t_{af} \). Let \( I_{af} = \{ 0 \} \cup I \) be the affine Dynkin node set. Let \( \{ \alpha_i \in \mathbb{Z}_{>0} \mid i \in I_{af} \} \) be the unique collection of relatively prime positive integers giving a dependency for the columns of the affine Cartan matrix \( (a_{ij})_{i,j \in I_{af}} \). Then \( \delta = \sum_{i \in I_{af}} a_i \alpha_i \) is the null root. The affine weight lattice is given by \( P_{af} = \mathbb{Z} \delta + \bigoplus_{i \in I_{af}} \mathbb{Z} \Lambda_i \subset t_{af}^* \) where \( \{ \Lambda_i \mid i \in I_{af} \} \) are the affine fundamental weights. Let \( Q_{af} \) and \( Q_{af}^\vee \) be the affine root and coroot lattices. Let \( W_{af} \) be the affine Weyl group, with simple reflections \( r_i \) for \( i \in I_{af} \). Considering the subset \( I \) of \( I_{af} \), \( W = W_I \subset W_{af} \) and \( W_I^{af} \) is the set of minimal-length coset representatives in \( W_{af}/W \). We have \( W_{af} \cong W \rtimes Q^\vee \) with \( \lambda \in Q^\vee \) written as \( \lambda = t_\lambda \in W_{af} \). There is a bijection \( W_I^{af} \to Q^\vee \) sending \( w \in W_{af} \) to \( \lambda \in Q^\vee \), where \( \lambda \) is defined by \( wW = t_\lambda W \). Let \( \lambda^- \in W \cdot \lambda \) be antidominant and \( u \in W \) the shortest element such that \( u\lambda^- = \lambda \). Then \( w = t_\lambda u \) and \( \ell(w) = \ell(t_\lambda) - \ell(u) \).

Let \( \Phi_{af} = \mathbb{Z} \delta \cup (\mathbb{Z} \delta + \Phi) \) be the set of affine roots. The affine real roots are given by \( \Phi_{af}^{re} = \mathbb{Z} \delta + \Phi \). Let \( \Phi_{af}^\pm \) be the sets of positive and negative affine roots. The set of positive affine real roots is defined by \( \Phi_{af}^{+re} = \Phi_{af}^+ \cap \Phi_{af}^{re} = \Phi^+ \cup (\mathbb{Z}_{>0} \delta + \Phi) \). A typical real root \( \alpha + m\delta \in \Phi_{af}^{re} \), with \( \alpha \in \Phi \) and \( m \in \mathbb{Z} \), has associated reflection \( r_{\alpha+ m\delta} = r_{\alpha} t_{m\alpha} \in W_{af} \).

Let \( G_{af} \supset P_I^- \supset B_I^- \supset T_{af} \) be the schemes of \( \S 3.1 \) associated with \( g_{af} \), where \( P_I^- \) is the maximal parabolic group scheme for the subset of Dynkin nodes \( I \subset I_{af} \) and \( B_I^- \) is the negative affine Borel group; then \( X_{af} = G_{af}/B_{af}^- \) is the thick affine flag manifold and \( \text{Gr}_G = X_I^I = G_{af}/P_I^- \) the thick affine Grassmannian.

### 4.2 Equivariant K-theory for affine flags with small-torus action

Following Peterson [Pet97] and Goresky et al. [GKM04] in the cohomology case, we consider the action of the smaller torus \( T = T_{af} \cap G \). The goal is to formulate and prove the analogue of Theorem 3.1 for \( K^T(X_{af}) \) and \( K^T(\text{Gr}_G) \). We let \( \Psi'_{af} \subset \text{Fun}(W_{af}, R(T_{af})) \) denote the ring defined by (2.9) for the affine Lie algebra \( g_{af} \), so that \( \Psi'_{af} \cong K^{T_{af}}(X_{af}) \).

The natural projection \( P_{af} \to P \) of weight lattices is surjective with kernel \( \mathbb{Z} \delta \oplus \mathbb{Z} \Lambda_0 \). It induces the projections

\[
\mathbb{Z}[P_{af}] \xrightarrow{\phi} \mathbb{Z}[P] \xrightarrow{\phi_0} \mathbb{Z}
\]

and a commutative diagram

\[
\begin{align*}
K^T_{af}(X_{af}) & \xrightarrow{\text{res}'} \Psi'_{af} \quad \text{For} \quad \xrightarrow{\phi'} K^T_{X_{af}} \xrightarrow{\text{res}} \text{Fun}(W_{af}, R(T))
\end{align*}
\]

where the horizontal maps are restrictions to \( W_{af} = X_{af}^{T_{af}} \subset X_{af}^T \) and the vertical map For regards a \( T_{af} \)-equivariant \( O_{X_{af}} \)-module as a \( T \)-equivariant one. We change notation slightly, writing the Schubert classes as \( \psi^{re} \in \Psi'_{af} \) and defining \( \psi^v := \phi \circ \psi^{re} \in \text{Fun}(W_{af}, R(T)) \).

The following definition is inspired by the analogous cohomological condition in [GKM04, Theorem 9.2]. A function \( \psi \in \text{Fun}(W_{af}, R(T)) \) can be extended by linearity to give a function \( \psi' \in \text{Fun}(\bigoplus_{w \in W_{af}} R(T) \cdot w, R(T)) \). In the following definition we abuse notation by identifying \( \psi \) with \( \psi' \).

**Definition 4.1.** We say that \( \psi \in \text{Fun}(W_{af}, R(T)) \) satisfies the small-torus Grassmannian GKM condition if

\[
\psi((1 - t_{\alpha} \alpha)^d w) \in (1 - e^{\alpha})^d R(T) \quad \text{for all } d \in \mathbb{Z}_{>0}, w \in W_{af} \text{ and } \alpha \in \Phi.
\]
We say that $\psi \in \operatorname{Fun}(W_{af}, R(T))$ satisfies the \textit{small-torus GKM condition} if, in addition to (4.2), we have
\[ \psi((1 - t_{\alpha^\vee})^{d-1}(1 - r_\alpha)w) \in (1 - e^{\alpha})dR(T) \quad \text{for all } d \in \mathbb{Z}_{>0}, w \in W_{af} \text{ and } \alpha \in \Phi. \quad (4.3) \]

Let $\Psi_{af}$ be the set of $\psi \in \operatorname{Fun}(W_{af}, R(T))$ that satisfy the small-torus GKM condition, and let $\Psi_{af}^I$ be the set of $\psi \in \operatorname{Fun}(W_{af}, R(T))$ that are constant on cosets $wW$ for $w \in W_{af}$ and satisfy the small-torus Grassmannian GKM condition.

**Lemma 4.2.** Suppose $\psi$ satisfies (4.2) and let $J := (1 - e^\alpha)^dR(T)$. Then
\[ \psi((1 - t_{\alpha^\vee})^{d-1}w) \quad \text{and} \quad \psi((1 - t_{\alpha^\vee})^{d-1}t_{p\alpha^\vee}w) \]
are congruent modulo the ideal $J$ for all $p \in \mathbb{Z}$.

**Proof.** By (4.2),
\[ J \ni \psi((1 - t_{\alpha^\vee})^dw) = \psi((1 - t_{\alpha^\vee})^{d-1}w) - \psi((1 - t_{\alpha^\vee})^{d-1}t_{p\alpha^\vee}w) \]
so that the assertion holds for $p = 1$. Repeating the same argument for $\psi((1 - t_{\alpha^\vee})^{d-1}t_{p\alpha^\vee}w)$ yields the lemma for all $p \in \mathbb{Z}_{>0}$. Replacing $w$ by $t_{-p\alpha^\vee}w$ gives the statement for all $p \in \mathbb{Z}$. \hfill $\square$

**Theorem 4.3.**

(i) $K^T(X_{af}) \cong \Psi_{af} = \prod_{v \in W_{af}} R(T) \psi^v$.

(ii) $K^T(\operatorname{Gr}_G) \cong \Psi_{af}^I = \prod_{v \in W_{af}^I} R(T) \psi^v$.

**Proof.** Owing to Lemmata 2.3 and 2.5, the set $\{ \psi^v \mid v \in W_{af} \}$ is independent over $R(T)$. Arguing as in [KS09], one can show that $K^T(X_{af})$ consists of possibly infinite $R(T)$-linear combinations of the $[O_{X_v}]$. By the commutativity of the diagram (4.1), we conclude that
\[ K^T(X_{af}) \cong \prod_{v \in W_{af}} R(T)[O_{X_v}], \]
that the map $F$ is surjective, and that $\operatorname{res}'$ is injective with image $\prod_{v \in W_{af}} R(T) \psi^v$. For (i) it remains to show that $\Psi_{af} = \prod_{v \in W_{af}} R(T) \psi^v$. Let $v \in W_{af}$. We first show that $\psi^v \in \Psi_{af}$. Let $w \in W_{af}$, $\alpha \in \Phi$ and $d \in \mathbb{Z}_{>0}$. Let $W^C \subset W_{af}$ be the subgroup generated by $t_{\alpha^\vee}$ and $r_\alpha$; it is isomorphic to the affine Weyl group of $\operatorname{SL}_2$. Define the function $f : W' \to R(T)$ by $f(x) = \psi^v(xw)$. Since $\psi^v$ satisfies the big-torus GKM condition (3.4) for $X_{af}$, $f$ satisfies (3.4) for a copy of the $\operatorname{SL}_2$ affine flag variety $X'$. Therefore $f$ is a possibly infinite $R(T)$-linear combination of Schubert classes in $X'$. By Propositions 4.4 and 4.5, proved below, $\phi \circ f$ satisfies the small-torus GKM condition for $X'$. It follows that $\psi^v \in \Psi_{af}$.

Conversely, suppose that $\psi \in \Psi_{af}$. We show that $\psi \in \prod_{v \in W_{af}} R(T) \psi^v$. Let $x = t_\lambda u \in \operatorname{Supp}(\psi)$ be of minimal length, with $u \in W$ and $\lambda \in Q^\vee$. It suffices to show that
\[ \psi(x) \in \psi^v(x)R(T) \]
because, upon defining $\psi' \in \Psi_{af}$ by $\psi' = \psi - (\psi(x)/\psi^v(x))\psi^x$, we have $\operatorname{Supp}(\psi') \subseteq \operatorname{Supp}(\psi)$, and by repeating this we may write $\psi$ as a $R(T)$-linear combination of the $\psi^x$.

The elements $\{1 - e^\alpha \mid \alpha \in \Phi^+\}$ are relatively prime in $R(T)$. Letting $\alpha \in \Phi^+$, by Lemma 2.5 it suffices to show that
\[ \psi(x) \in J := (1 - e^\alpha)^dR(T), \]

826
where \( d = |\text{Inv}_\alpha(x)| \) with Inv\(_\alpha(x)\) being the set of roots in Inv\(_x\) of the form \( \pm \alpha + k\delta \) for some \( k \in \mathbb{Z}_{\geq 0} \). Note that for \( \beta \in \Phi^\text{re,}A, \beta \in \text{Inv}(x) \) if and only if \( x^{-1} \cdot \beta \in -\Phi^\text{re,}A \). We have

\[
x^{-1} \cdot (\pm \alpha + k\delta) = u^{-1}t_{-\lambda} \cdot (\pm \alpha + k\delta) = \pm u^{-1} \alpha + (k \pm \langle \lambda, \alpha \rangle)\delta.
\]

Hence

\[
\text{Inv}_\alpha(x) = \begin{cases} \{\alpha, \alpha + \delta, \ldots, \alpha - (\langle \lambda, \alpha \rangle + \chi(\alpha \not\in \text{Inv}(u))\delta)\} & \text{if } \langle \lambda, \alpha \rangle \leq 0, \\
\{-\alpha + \delta, -\alpha + 2\delta, \ldots, -\alpha + (\langle \lambda, \alpha \rangle - \chi(\alpha \in \text{Inv}(u))\delta)\} & \text{if } \langle \lambda, \alpha \rangle > 0.
\end{cases}
\]

Suppose first that \( \langle \lambda, \alpha \rangle > 0 \). Then \( d = \langle \lambda, \alpha \rangle - \chi(\alpha \in \text{Inv}(u)) \). Applying (4.3) to \( y = t_{(1-d)\alpha \vee x} \), we get \( Z_1 \in J \) where

\[
Z_1 = \psi((1 - t_{\alpha \vee})^{d-1}(1 - r_\alpha)y) = (-1)^{d-1}\psi((1 - t_{-\alpha \vee})^{d-1}x) - \psi((1 - t_{\alpha \vee})^{d-1}r_\alpha y) = (-1)^{d-1}\psi(x) - \psi((1 - t_{\alpha \vee})^{d-1}r_\alpha y).
\]

The last equality holds by the assumption on \( \text{Supp}(\psi) \) and a calculation of Inv\(_\alpha(r_{\alpha + d\delta}x)\), giving \( x > r_{\alpha + d\delta}x > t_{-\alpha \vee}x \) for all \( k \in [1, d-1] \). By Lemma 4.2, we have \( Z_2 \in J \) where

\[
Z_2 = \psi((1 - t_{\alpha \vee})^{d-1}r_\alpha y) - \psi((1 - t_{\alpha \vee})^{d-1}t_{p\alpha \vee} r_\alpha y)
\]

for any \( p \in \mathbb{Z} \). Thus

\[
Z_1 + Z_2 = (-1)^{d-1}\psi(x) - \psi((1 - t_{\alpha \vee})^{d-1}t_{p\alpha \vee} r_\alpha y) \in J.
\]

By the assumption on \( \text{Supp}(\psi) \) and the calculation of \( \text{Inv}_\alpha(x) \),

\[
\psi((1 - t_{\alpha \vee})^{d-1}t_{p\alpha \vee} r_\alpha y) = 0
\]

for \( p = 2 - d \). It follows that \( \psi(x) \in J \).

Now suppose \( \langle \lambda, \alpha \rangle \leq 0 \). By the previous case, we may assume that \( t_{d\alpha \vee}x \not\in \text{Supp}(\psi) \). Thus

\[
\psi(x) = \psi((1 - t_{\alpha \vee})^{d}x) \in J
\]

by induction on \( \text{Supp}(\psi) \) and (4.2). This proves (i).

For (ii), it suffices to show that \( \psi \in \Psi^I_{af} \) if and only if \( \psi \in \prod_{v \in W^I_{af}} R(T)\psi^v \). First, let \( \psi \in \Psi^I_{af} \). Let \( w = t_{\lambda \vee} u \in W_{af} \) with \( \lambda \in Q^\vee \) and \( u \in W \), and take \( \alpha \in \Phi \). We shall verify that \( \psi \) satisfies the small-torus GKM condition. We have

\[
r_\alpha w = r_\alpha t_{\lambda \vee} u = t_{r_\alpha(\lambda)\alpha} r_\alpha u = t_{-\langle \lambda, \alpha \rangle \alpha} t_{\lambda} r_\alpha u.
\]

Since by assumption \( \psi \) is constant on cosets \( W_{af}/W \), we have

\[
\psi((1 - t_{\alpha \vee})^{d-1}(1 - r_\alpha)w) = \psi((1 - t_{\alpha \vee})^{d-1}(1 - t_{-\langle \lambda, \alpha \rangle \alpha})t_{\lambda}).
\]

But \( 1 - t_{k\alpha \vee} \) is divisible by \( 1 - t_{\alpha \vee} \) for any \( k \in \mathbb{Z} \). Therefore \( \psi \) satisfies the small-torus GKM condition because it satisfies the Grassmannian one. Part (i) and the fact that \( \psi \) is constant on cosets \( W_{af}/W \) implies that \( \psi \in \prod_{v \in W^I_{af}} R(T)\psi^v \).

Conversely, it suffices to show that for every \( v \in W^I_{af} \) we have \( \psi^v \in \Psi^I_{af} \). But this follows from part (i) and the fact that for such \( v, \psi^v \) is constant on cosets \( W_{af}/W \).

\[\square\]

4.3 The small-torus GKM condition for \( \widehat{\mathfrak{sl}}_2 \)

In this section we prove that the Schubert classes \( \psi^v \) for \( \widehat{\mathfrak{sl}}_2 \) satisfy the small-torus GKM condition of Definition 4.1. To this end, we first derive explicit expressions for the \( \psi^v \).
For $i \in \mathbb{Z}_{\geq 0}$, let
\begin{align*}
\sigma_{2i} &= (r_1 r_0)^i, \quad \sigma_{-2i} = (r_0 r_1)^i, \\
\sigma_{2i+1} &= r_0 \sigma_{2i}, \quad \sigma_{-(2i+1)} = r_1 \sigma_{-2i}.
\end{align*}
Then $\ell(\sigma_j) = |j|$ for $j \in \mathbb{Z}$, $W_{\text{af}}^I = \{\sigma_j \mid j \in \mathbb{Z}_{\geq 0}\}$, and
\[\sigma_{2i} = t_{-i}^a \sigma_i \quad \text{for } i \in \mathbb{Z}.
\]

Let $\psi^i_j := \psi^a_i(\sigma_j)$ for $i, j \in \mathbb{Z}$, where we have set $\delta = 0$. We set $x = e^a$ and let $S_{\leq a}$ be the sum $h_0 + h_1 + \cdots + h_a$ of homogeneous symmetric functions. We write $S_{\leq a}^r(x)$ to mean $S_{\leq a}[x, x, \ldots, x]$ where there are $i$ copies of $x$. For $i, a \in \mathbb{Z}$ such that $a, m \geq 0$, we have
\begin{align*}
\psi^{m^2}_{2i+2a} &= (1 - x)^m S^{m}_{\leq a}(x) = \psi^{m}_{-2i-2a-1} \quad \text{for } m = 2i \text{ or } 2i - 1, \\
\psi^{m}_{2i+2a+1} &= (1 - x^{-1})^m S^{m}_{\leq a}(x^{-1}) = \psi^{m}_{-2i-2a-2} \quad \text{for } m = 2i \text{ or } 2i + 1,
\end{align*}
and zero otherwise. Furthermore,
\[\psi^{-m}_{-i} = \psi_{i}^{m}(x^{-1}).\]
These relations are easily proved by induction using the left- and right-hand recurrences for the localization of Schubert classes together with the recurrence
\[S^{i}_{\leq a}(x) = x S^{i}_{\leq a-1}(x) + S^{i-1}_{\leq a}(x).
\]
We also have the explicit formula
\[S^{i}_{\leq a}(x) = \sum_{j=0}^{a} x^j \binom{j + i - 1}{i - 1}.
\]

**Proposition 4.4.** For all $d \geq 1$, $m \in \mathbb{Z}$ and $w \in W_{\text{af}}$ we have
\[\psi^{m}((1 - t_{-i}^{a} \sigma_i)^d w) \in (1 - x)^d \mathbb{Z}[x^\pm].\]

**Proof.** We prove the claim for $m = 2i$ and for the ranges $t_{-(i-a)\sigma_i}$ to $t_{(i+1+b)\sigma_i}$ for $a, b \in \mathbb{Z}_{\geq 0}$. The other cases are similar. Let $d = (i + a) + (i + 1 + b) = 2i + a + b + 1$. We must show that
\[Z := \sum_{k=0}^{d} (-1)^k \binom{d}{k} \psi^{2i}_{-2i-2-2b+2k} \in (1 - x)^d \mathbb{Z}[x^\pm].
\]
Since $\psi^{2i}_{2p} = 0$ for $-2i - 2 < 2p < 2i$,
\[Z = \left( \sum_{k=0}^{b} + \sum_{k=2i+1+b}^{d} \right) (-1)^k \binom{d}{k} \psi^{2i}_{-2i-2-2b+2k}
\]
\[= \sum_{k=0}^{b} (-1)^k \binom{d}{k} \psi^{2i}_{-2i-2-2b+2k} + \sum_{k=0}^{a} (-1)^{k+2i+1+b} \binom{d}{2i+1+b+k} \psi^{2i+2k}_{2i+2k}
\]
\[= (-1)^b \sum_{k=0}^{b} (-1)^k \binom{d}{b-k} \psi^{2i}_{-2i-2-2k} - (-1)^b \sum_{k=0}^{a} (-1)^k \binom{d}{a-k} \psi^{2i}_{2i+2k}.
\]
Substituting (4.5), (4.6) and (4.7) gives
\[(-1)^b Z = \sum_{k=0}^{b} (-1)^k \binom{d}{b-k} (1 - x^{b-k}) \sum_{j=0}^{k} x^{-j} \binom{j + 2i - 1}{2i - 1}\]
\[-\sum_{k=0}^{a} (-1)^k \binom{d}{a-k} (1 - x) \sum_{j=0}^{k} x^{j} \binom{j + 2i - 1}{2i - 1}.\]

Therefore we must show that \(Z' := (-1)^b Z(1 - x)^{-2i}\) is divisible by \((1 - x)^{a+b+1}\). Regarding \(Z'\) as a function of \(x\), we need to show that its \(r\)th derivative at \(x = 1\) vanishes, for \(0 \leq r \leq a + b\). This yields the identities
\[(-1)^r \sum_{k=0}^{b} (-1)^k \binom{d}{b-k} \sum_{j=0}^{k} \frac{(j + 2i + r - 1)!}{j!} = \sum_{k=0}^{a} (-1)^k \binom{d}{a-k} \sum_{j=r}^{k} \frac{(j + 2i - 1)!}{(j-r)!}.\]

Upon shifting the sums on the right-hand side using \(j' = j - r\) and \(k' = k - r\) and dividing both sides by \((-1)^r(2i + r - 1)!\), the inner sums simplify and we obtain
\[\sum_{k=0}^{b} (-1)^k \binom{d}{b-k} \binom{2i + r + k}{k} = \sum_{k=0}^{a-r} (-1)^k \binom{d}{a-r-k} \binom{2i + r + k}{k}.\]

Setting \(a' = a - r\), we claim that this sum is equal to \(\binom{a' + b}{b}\), which is symmetric in \(a'\) and \(b\) and hence implies equality of the two sides. This can be seen as follows. The coefficient of \(x^b\) in \((1 + x)^{a' + b}\) is \(\binom{a' + b}{b}\). Alternatively, we can calculate
\[[x^b](1 + x)^{a'}(1 + x)^b(1 + x)^c(1 + x)^{-c},\]
where \(c = 2i + r + 1\) and \((1 + x)^{-c}\) is meant to be expanded as a power series in \(x\). Then
\[[x^b](1 + x)^{a'}(1 + x)^b(1 + x)^c(1 + x)^{-c} = \sum_{k=0}^{b} [x^{b-k}](1 + x)^{a' + b + c}[x^k](1 + x)^{-c},\]
which is exactly the sum we wanted to evaluate. \(\square\)

**Proposition 4.5.** For all \(d \geq 1\), \(m \in \mathbb{Z}\) and \(w \in W_{af}\) we have
\[\psi^m((1 - t_{\alpha^\vee})^{d-1}(1 - r_{\alpha}w)) \in (1 - x)^d \mathbb{Z}[x^\pm].\]

**Proof.** Note that
\[\psi^m((1 - t_{\alpha^\vee})^{d-1}(1 - r_{\alpha}w)) = \psi^m((1 - t_{\alpha^\vee})^{d-1}w) - \psi^m((1 - t_{\alpha^\vee})^{d-1}r_{\alpha}w).\] (4.8)

Furthermore, by Proposition 4.4, \(\psi^m\) satisfies the small-torus Grassmannian GKM condition. Hence, applying Lemma 4.2 to \(\psi^m((1 - t_{\alpha^\vee})^{d-1}r_{\alpha}w)\), we can shift the argument \(r_{\alpha}w\) so that the equalities (4.5) and (4.6) can be used. This implies that (4.8) is zero modulo the ideal \((1 - x)^d \mathbb{Z}[x^\pm]\). \(\square\)

**4.4 The wrong-way map**

There is a natural inclusion map \(i_T : \Psi_{af} \rightarrow \Psi_{af}\). In the case at hand, there is a map \(\overline{\iota} : \Psi_{af} \rightarrow \Psi_{af}^I\), of which \(i_T\) is a section. This map is specific to the case of the affine Grassmannian; it also does not exist if one uses the larger torus \(T_{af}\).
Lemma 4.6. There is an $R(T)$-module homomorphism $\varpi : \Psi_{af} \to \Psi^I_{af}$, defined by $\varpi(\psi)(w) = \psi(t_\lambda)$ for $w \in W_{af}$, where $\lambda \in \mathcal{Q}^\vee$ is such that $wW = t_\lambda W$.

Proof. Let $\psi \in \Psi_{af}$, $w = t_\chi u \in W_{af}$ with $\chi \in \mathcal{Q}^\vee$, and $u \in W$, $\alpha \in \Phi$ and $d \in \mathbb{Z}_{>0}$. Then we have

$$\varpi(\psi)((1 - t_\chi)^d w) = \varpi(\psi)((1 - t_\chi)^d t_\lambda u) = \psi((1 - t_\chi)^d t_\lambda) \in (1 - e^\alpha)^d R(T). \quad \square$$

5. $K$-homology of the affine Grassmannian and the $K$-Peterson subalgebra

Let $\mathbb{K}'$ be the $K$-NilHecke ring for the affine Lie algebra $\mathfrak{g}_{af}$ defined via the general construction in §2. In this section we use the affine $K$-NilHecke ring $\mathbb{K}$, which differs from the use of $R(T)$ instead of $R(T_{af})$. Our main result, generalizing work of Peterson [Pet97], gives a Hopf-isomorphism of $K_T(\text{Gr}_G)$ with a commutative subalgebra $\mathbb{L} \subset \mathbb{K}$.

5.1 $K$-homology of the affine Grassmannian

We define the equivariant $K$-homology $K_T(\text{Gr}_G)$ of the affine Grassmannian to be the continuous dual $K_T(\text{Gr}_G) = \text{Hom}_{R(T)}(K^T(\text{Gr}_G), R(T))$, so that $K_T(\text{Gr}_G)$ is a free $R(T)$-module with basis comprising the Schubert classes $\xi_w$, dual to $[\mathcal{O}_{\mathcal{X}_L}] \in K^T(\text{Gr}_G)$.

The $K$-homology $K_T(\text{Gr}_G)$ and $K$-cohomology $K^T(\text{Gr}_G)$ are equipped with dual Hopf structures, which we now explain, focusing on $K^T(\text{Gr}_G)$ first. Let $K \subset G$ be the maximal compact form, $LK$ the space of continuous loops $S^1 \to K$, and $\Omega K$ the space of based loops $(S^1, 1) \to (K, 1)$. Let $T_\mathbb{R} = T \cap K$. We denote by $K^{T_\mathbb{R}}(\Omega K)$ the equivariant topological $K$-theory of $\Omega K$. By an (unpublished) well-known result of Quillen (see [HHH05, PS86]), the space $\Omega K$ is (equivariantly) weak-homotopy-equivalent to the ind-scheme affine Grassmannian $G(\mathbb{C}((t))) / G(\mathbb{C}[t])$. Thus we have $K^{T_\mathbb{R}}(\Omega K) = K^{T_\mathbb{R}}(G(\mathbb{C}((t))) / G(\mathbb{C}[t]))$, where $K^{T_\mathbb{R}}(G(\mathbb{C}((t))) / G(\mathbb{C}[t]))$ denotes the topological $K$-theory of the topological space underlying the ind-scheme $G(\mathbb{C}((t))) / G(\mathbb{C}[t])$.

The topological $K$-theory $K^{T_\mathbb{R}}(\Omega K) \cong K^{T_\mathbb{R}}(G(\mathbb{C}((t))) / G(\mathbb{C}[t]))$ is studied in [KK90], where it is identified with the ring $\Psi^I_{af}$. More precisely, Kostant and Kumar studied the equivariance with respect to the larger torus $T_{af}$, but the same argument as in our Theorem 4.3 gives $K^{T_\mathbb{R}}(G(\mathbb{C}((t))) / G(\mathbb{C}[t])) \cong \Psi^I_{af}$. Thus we obtain the sequence of isomorphisms

$$K^T(\text{Gr}_G) \cong \Psi^I_{af} \cong K^{T_\mathbb{R}}(G(\mathbb{C}((t))) / G(\mathbb{C}[t])) \cong K^{T_\mathbb{R}}(\Omega K),$$

and all the isomorphisms are compatible with restrictions to fixed points.

The composite map $r$ given by

$$\Omega K \hookrightarrow LK \twoheadrightarrow LK/T_\mathbb{R}$$

induces the map

$$K^T(X_{af}) \cong K^{T_\mathbb{R}}(LK/T_\mathbb{R}) \xrightarrow{r^*} K^{T_\mathbb{R}}(\Omega K) \cong K^T(\text{Gr}_G).$$

One can check, using a fixed-point calculation, that the map $\varpi$ of Lemma 4.6 is related to $r^*$ via the isomorphisms of Theorem 4.3.

The based loop group $\Omega K$ has a $T_\mathbb{R}$-equivariant multiplication map $\Omega K \times \Omega K \to \Omega K$ given by pointwise multiplication on $K$, and this induces the structure of a commutative and cocommutative Hopf algebra on $K^{T_\mathbb{R}}(\Omega K) \cong K^T(\text{Gr}_G)$. The co-commutativity of $K^{T_\mathbb{R}}(\Omega K)$ follows from the fact that it is a homotopy double-loop space ($K$ being already a homotopy loop space).
Via duality, we obtain a dual Hopf-algebra structure on $K_T(\text{Gr}_G)$. For the next result, we label the $T_{af}$-fixed points of $\text{Gr}_G$ by translation elements $t_\lambda$.

**Lemma 5.1.** Let $\lambda, \mu \in Q^\vee$, and consider the maps $i^*_\lambda, i^*_\mu : K^T(\text{Gr}_G) \to R(T)$ as elements of $K_T(\text{Gr}_G)$. Then in $K_T(\text{Gr}_G)$ we have

$$i^*_\lambda i^*_\mu = i^*_{\lambda + \mu}.$$  

**Proof.** It suffices to argue in $K^T_{af}(\Omega K)$. The map $i^*_\lambda i^*_\mu$ is induced by the map $pt \to \Omega K \times \Omega K \to \Omega K$, where the image of the first map is the pair $(t_\lambda, t_\mu) \in \Omega K \times \Omega K$ of fixed points and the second map is multiplication. Treating $t_\lambda, t_\mu : S^1 \to K$ as homomorphisms into $K$, we see that pointwise multiplication of $t_\lambda$ and $t_\mu$ gives $t_{\lambda + \mu}$. Thus $i^*_\lambda i^*_\mu = i^*_{\lambda + \mu}$. □

The antipode of $K_T(\text{Gr}_G)$ is given by $S(i^*_\lambda) = i^*_{-\lambda}$, since the fixed points satisfy $t_{\lambda}^{-1} = t_{-\lambda}$ in $\Omega K$.

5.2 The affine $K$-NilHecke ring and $K$-Peterson subalgebra

Let $W_{af}$ act on the finite-weight lattice $P$ by the (non-faithful) level-zero action $(ut_\lambda \cdot \mu) = u \cdot \mu$ for $u \in W$, $\lambda \in Q^\vee$ and $\mu \in P$.

Let $K$ be the smash product of the affine 0-Hecke ring $K_0$ with $R(T)$ (rather than $R(T_{af})$) using the commutation relations (2.6). We call this the affine $K$-NilHecke ring. The cohomological analogue of $K$ was studied by Peterson [Pet97]. We have $K = \bigoplus_{w \in W_{af}} R(T) T_w$.

We now define the map $k : K_T(\text{Gr}_G) \to K$ by the formula

$$\langle k(\xi), \psi \rangle = \langle \xi, \varpi(\psi) \rangle,$$

where $\psi \in \Psi_{af}$ and $\varpi$ is the wrong-way map of Lemma 4.6. We have used Theorem 4.3(ii) to obtain the pairing on the right-hand side. By letting $\psi$ vary over $\{\psi^w \in \Psi_{af}\}$, it is clear that (5.1) defines $k(\xi)$ uniquely in $K$.

We define the $K$-Peterson subalgebra $L := Z_K(R(T))$ of $K$ to be the centralizer of $R(T)$ inside $K$.

**Lemma 5.2.** We have $\text{Im}(k) = \bigoplus_{\lambda \in Q^\vee} Q(T)t_\lambda \cap K = L$.

**Proof.** For $\lambda \in Q^\vee$, we have $\langle i^*_\lambda, \varpi(\psi) \rangle = \psi(t_\lambda)$, so $k(i^*_\lambda) = t_\lambda \in K$. Since $i^*_\lambda$ spans $K_T(\text{Gr}_G)$ (over $Q(T)$), we have thus established the first equality. For the second equality, $\bigoplus_{\lambda \in Q^\vee} Q(T)t_\lambda \cap K \subseteq Z_K(R(T))$ holds because under the level-zero action $t_\lambda$ acts on $P$ trivially for all $\lambda \in Q^\vee$. For the other direction, let $a = \sum_{w \in W_{af}} a_w w \in Z_K(R(T))$ for $a_w \in Q(T)$. Then for all $\mu \in P$ we have

$$0 = e^\mu a - ae^\mu = \sum_{w \in W_{af}} a_w (e^\mu - e^{w\mu})w.$$

Therefore, for all $w \in W_{af}$, either $a_w = 0$ or $w\mu = \mu$ for all $\mu \in P$. Taking $\mu$ to be $W$-regular, we see that the latter holds only for $w = t_\lambda$ with some $\lambda \in Q^\vee$. □

The algebra $L$ inherits a coproduct $\Delta : L \to L \otimes_{R(T)} L$ from the coproduct of $K$. (In §2.6 the coproduct of $K'$ is given, and it specializes easily to a coproduct for $K$.) That $\Delta(L) \subseteq L \otimes_{R(T)} L$ follows from (2.14) and the equality $L = \bigoplus_{\lambda \in Q^\vee} Q(T)t_\lambda \cap K$. We make $L$ a Hopf algebra by defining $S(t_\lambda) = t_{-\lambda}$.

The following results generalize properties of Peterson’s $j$-map in the homology case; see [Lam08, Theorem 4.4].
Theorem 5.3. The map \( k : K_T(Gr_G) \to \mathbb{L} \) is a Hopf-isomorphism.

Proof. To check that a map is a Hopf-morphism, it suffices to check that it is a bialgebra morphism, since the compatibility with antipodes follows as a consequence.

It is clear from the definition that \( k \) is injective. Since \( k \) is \( R(T) \)-linear, to check that \( k \) is compatible with the Hopf-structure we check the product and coproduct structures on the basis \( \{ t_\lambda \mid \lambda \in Q^\vee \} \). By Lemma 5.1 we have \( k(t^*_\lambda + t^*_\mu) = k(t^*_\lambda t_\mu) = t_{\lambda + \mu} \), so \( k \) is an algebra morphism. That \( k \) is a coalgebra morphism follows from an argument similar to that used in proving Lemma 2.8 and Proposition 2.9. Thus \( k : K_T(Gr_G) \to \mathbb{L} \) is a Hopf-isomorphism. \( \square \)

Theorem 5.4. For each \( w \in W^I_{af} \), there is a unique element \( k_w \in \mathbb{L} \) of the form
\[
k_w = T_w + \sum_{v \in W^I_{af} \setminus W^I_{af}} k^v_w T_v
\]
for \( k^v_w \in R(T) \). Furthermore, \( k_w = k(\xi_w) \) and \( \mathbb{L} = \bigoplus_{w \in W^I_{af}} R(T) k_w \).

Proof. Since the Schubert basis \( \{ \xi_w \mid w \in W^I_{af} \} \) is a \( R(T) \)-basis of \( K_T(Gr_G) \), upon setting \( k_w = k(\xi_w) \) we obtain, by Theorem 5.3, a \( R(T) \)-basis of \( \mathbb{L} \). By (5.1) and the fact that \( \varpi(\psi^u) = \psi^v \) for \( v \in W^I_{af} \), we obtain (5.2). Finally, the element \( k_w \in \mathbb{L} \) is unique because the set \( \{ T_w \mid w \in W^I_{af} \} \) is linearly independent.

Define the \( T \)-equivariant \( K \)-homological Schubert structure constants \( d^w_{uv} \in R(T) \) for \( K_T(Gr_G) \) by
\[
k_u k_v = \sum_{w \in W^I_{af}} d^w_{uv} k_w
\]
where \( u, v \in W^I_{af} \). Since \( k_u \in Z_K(R(T)) \), we have
\[
k_u k_v = k_u \sum_{y \in W_{af}} k^y_u T_y = \sum_{y \in W_{af}} k^y_u k_u T_y = \sum_{x, y \in W_{af}} k^y_u k^x_u T_x T_y.
\]
Applying \( \psi^w \) with \( w \in W^I_{af} \) and using (5.2) gives
\[
d^w_{uv} = \sum_{x, y \in W_{af}} k^y_u k^x_u \psi^w(T_x T_y).
\]
Since \( w \in W^I_{af} \), \( \psi^w(T_x T_y) = 0 \) unless \( y \in W^I_{af} \). But, for \( y \in W^I_{af} \), we have \( k^y_u = \delta_{yu} \) by (5.2). Therefore
\[
d^w_{uv} = \sum_{x \in W_{af}} (-1)^{\ell(w) - \ell(v) - \ell(x)} k^x_u.
\]

6. The \( K \)-affine Fomin–Stanley algebra and \( K \)-homology of the affine Grassmannian

In this section we reduce to the non-equivariant setting. Our main result (Theorem 6.4) describes the specialization at zero of \( \mathbb{L} \). We will rely on the corresponding known statements from the cohomological setting, in particular [Lam08, Proposition 5.3].
6.1 $K$-affine Fomin–Stanley algebra

Define $\phi_0 : R(T) \to \mathbb{Z}$ by setting $\phi_0(\varepsilon^\lambda) = 1$ and extending by linearity. Define $\phi : \mathbb{K} \to \mathbb{K}_0$ by $\phi(a) = \sum_{w \in W} \phi(a_w)T_w$, where $a = \sum_{w \in W} a_wT_w$ with $a_w \in R(T)$.

The $K$-affine Fomin–Stanley algebra is defined as

$$\mathbb{L}_0 = \{ b \in \mathbb{K}_0 \mid \phi_0(bq) = \phi_0(q)b \text{ for all } q \in R(T) \} \subset \mathbb{K}_0.$$ 

The cohomological analogue of $\mathbb{L}_0$ was defined in [Lam08].

**Lemma 6.1.** Suppose that $a \in \mathbb{L}$. Then $\phi_0(a) \in \mathbb{L}_0$.

**Proof.** $\phi_0(a\varepsilon^\lambda) = \phi_0(\varepsilon^\lambda a) = \phi_0(a)$.

In what follows, we shall use the notation (such as $A_w$) for the cohomological nilHecke ring. We refer the reader to Appendix A for a review of this notation. Let $\preceq$ denote the covering relation in Bruhat order.

**Lemma 6.2.** Let $v \preceq w$ in $W_{af}$. Then for each $\lambda \in P$ we have

$$\phi_0(T_w e^\lambda, \psi^v) = \phi_0(A_w \lambda, \xi^v) = \langle A_w \lambda, \xi^v \rangle.$$ 

**Proof.** Write $v = wr_a$. By [Hum90] there exists a length-additive factorization of the form $w = u_1r_1u_2$ for some $i \in I_{af}$ such that $v = u_1u_2$ and $\alpha = u_2^{-1}a_i$. We have

$$\phi_0(T_w e^\lambda, \psi^v) = \phi_0(\psi^v(T_w e^\lambda) = \phi_0(u_1 \cdot T_i \cdot e^{u_2 \lambda}) = \phi_0\left(\frac{e^{r_iu_2 \lambda} - e^{u_2 \lambda}}{1 - e^{a_i}}\right),$$

since $\phi_0(wq) = \phi_0(q)$ for all $w \in W_{af}$ and $q \in R(T)$. Therefore

$$\phi_0(\psi^v(T_w e^\lambda) = \langle \alpha_i^\vee , u_2 \lambda \rangle = \langle \alpha_i^\vee , \lambda \rangle = \xi^v(A_w \lambda),$$

where we have used (2.1) acting on an exponential for the first equality, $W_{af}$-equivariance of $\langle \cdot , \cdot \rangle$ for the second equality, and Lemma A.1 for the third.

**Lemma 6.3.** Suppose that $a = \sum_{w \in W_{af}} a_wT_w \in \mathbb{L}_0$ where $a_w \in \mathbb{Z}$. Let $\ell$ be maximal so that $a_w \neq 0$ for some $w$ with $\ell(w) = \ell$. Then $a' = \sum_{\ell(w) = \ell} a_w A_w \in \mathbb{B}_0$.

**Proof.** We note that for $v \in W_{af}$ with $\ell(v) = \ell - 1$, we have for each $\lambda \in P$ that

$$\phi_0(\psi^v(a(\varepsilon^\lambda - 1)) = \phi_0(\psi^v\left(\sum_{\ell(w) = \ell} a_wT_w(\varepsilon^\lambda - 1)\right) = \phi_0(\xi^v(a' \lambda),$$

using Lemma 6.2. Since $a \in \mathbb{L}_0$, we have $\phi_0(a(\varepsilon^\lambda - 1)) = 0$ for all $\lambda$. Thus $a' \in \mathbb{B}_0$, as claimed.

**Theorem 6.4.** We have $\mathbb{L}_0 = \phi_0(\mathbb{L})$. Furthermore, $\mathbb{L}_0 = \bigoplus_{w \in W_{af}} \mathbb{Z} \phi_0(k_w)$ and $\phi_0(k_w)$ is the unique element in $\mathbb{L}_0 \cap (T_w + \bigoplus_{v \in W_{af}} \mathbb{Z} T_v)$.

**Proof.** For $a \in \mathbb{L}$, we have $\phi_0(a\varepsilon^\lambda) = \phi_0(\varepsilon^\lambda a) = \phi_0(a)$. Thus $\phi_0(\mathbb{L}) \subset \mathbb{L}_0$. Now suppose that $a = \sum_{w \in W_{af}} a_wT_w \in \mathbb{L}_0$. Define the support of $a$ to be the $w \in W_{af}$ such that $a_w \neq 0$. If the support of $a$ contains a Grassmannian element $w \in W_{af}$, then $a - a_w \phi_0(k_w) \in \mathbb{L}_0$, but by Theorem 5.4 its support has fewer Grassmannian elements than does $a$. So we may suppose that $a$ has no Grassmannian element in its support. By Lemma 6.3, the element $a'$ (as defined in the lemma) lies in $\mathbb{B}_0$ and has no Grassmannian support. By [Lam08, Proposition 5.3], we must have $a' = 0$. Thus $a = 0$. We conclude that $\mathbb{L}_0 = \phi_0(\mathbb{L})$. 

833
Since the $\phi_0(k_w)$, $w \in W_{af}^I$, are clearly linearly independent, it follows that they form a basis. The last statement follows from Theorem 5.4.

Some examples of the elements $\phi_0(k_w)$, illustrating Theorem 6.4, are presented in Appendix A.3.3.

**Corollary 6.5.** The ring $L_0$ is commutative.

**Proof.** Let $a, b \in L_0$. By Theorem 6.4, we have $a + a' \in L$ and $b + b' \in L$ for some elements $a'$ and $b'$ satisfying $\phi_0(a') = 0 = \phi_0(b')$. Since $L$ is commutative, we have

$$ab = \phi_0((a + a')(b + b')) = \phi_0((b + b')(a + a')) = ba.$$ 



### 6.2 Structure constants

We now consider the structure constants in $L_0$. The next lemma follows from either a direct calculation or Theorem 6.8 below.

**Lemma 6.6.** For $a, b \in L$, we have $\phi_0(ab) = \phi_0(a)\phi_0(b)$.

Using Lemma 6.6, apply $\phi_0$ to (5.3) to get that for $u, v \in W_{af}^I$,

$$\phi_0(k_u)\phi_0(k_v) = \sum_{w \in W_{af}^I} \phi_0(d_{uv}^w)\phi_0(k_w).$$  \hspace{1cm} (6.1)

In other words, $\phi_0(d_{uv}^w) \in \mathbb{Z}$ are the structure constants for the basis $\{\phi_0(k_v) \mid v \in W_{af}^I\}$ of $L_0$.

**Conjecture 6.7.** For $u, v, w \in W_{af}^I$ and $x \in W_{af}$,

$$(-1)^{\ell(w)}\phi_0(x_{uv}) \geq 0,$$

$$(-1)^{\ell(x)}\phi_0(k_u) \geq 0.$$  

By (5.4), the second statement implies the first.

The tables of $\phi_0(k_w)$ in Appendix A.3.3 support Conjecture 6.7.

### 6.3 Non-equivariant $K$-homology

One defines the non-equivariant $K$-cohomology $K^*(Gr_G)$ by considering non-equivariant coherent sheaves in the natural way. We have $K^*(Gr_G) = \bigoplus_{w \in W_{af}^I} \mathbb{Z}[O_{X_w}^I]_0$, where $[O_{X_w}^I]_0$ denotes a non-equivariant class. The non-equivariant $K$-homology $K_*(Gr_G)$, defined as the continuous $\mathbb{Z}$-dual to $K^*(Gr_G)$, has Schubert basis $\{\xi_0^w \mid w \in W_{af}^I\}$. We have the following commutative diagram.

$$\begin{array}{ccc}
K_T(Gr_G) & \phi_0 & \rightarrow & K_*(Gr_G) \\
\downarrow & & \downarrow & \\
\mathbb{L} & \phi_0 & \rightarrow & L_0
\end{array}$$

The subalgebra $\mathbb{L}_0$ is a Hopf algebra, with coproduct $\phi_0 \circ \Delta$. The following result generalizes [Lam08, Theorem 5.5] to $K$-homology.

**Theorem 6.8.** There is a Hopf-isomorphism $k_0 : K_*(Gr_G) \rightarrow \mathbb{L}_0$ such that $k_0(\xi_0^w) = \phi_0(k_w)$.
7. Grothendieck polynomials for the affine Grassmannian

In this section we specialize to affine type $A_{n-1}^{(1)}$ and $G = SL_n(\mathbb{C})$. We first introduce elements $k_i \in \mathbb{L}_0$ which, under a Hopf-algebra isomorphism $\mathbb{L}_0 \cong \Lambda_0 := \mathbb{Z} \langle h_1, \ldots, h_{n-1} \rangle$ between the $K$-affine Fomin–Stanley algebra and a subspace of symmetric functions, correspond to the homogeneous symmetric functions $h_i$. For $w \in W^I_{af}$, the image $g_w$ of $\phi_0(k_w)$ in $\Lambda_0$ is the $K$-theoretic $k$-Schur function $g_w$ which contains the $k$-Schur function (see [LLM03, LM07]) as the highest-degree term. The symmetric functions $g_w$ are related to the affine stable Grothendieck polynomials $\{G_w \mid w \in W^I_{af}\}$ of [Lam06] by duality.

7.1 Cyclically decreasing permutations and the elements $k_i$

For $G = SL_n$, we have $I = \{1, 2, \ldots, n - 1\}$ and $I_{af} = \{0\} \cup I$. For $i \in I$ we wish to compute the elements $\phi_0(k_{\sigma_i}) \in \mathbb{L}_0$ where $\sigma_i = r_{i-1} r_{i-2} \cdots r_1 r_0 \in W_{af}$.

A cyclically decreasing element $w \in W_{af}$ is one that has a reduced decomposition $w = r_{i_1} r_{i_2} \cdots r_{i_N}$ such that the indices $i_1, \ldots, i_N \in I_{af}$ are all distinct and a reflection $r_i$ never occurs somewhere to the left of a reflection $r_{i+1}$; here $I_{af}$ is identified with $\mathbb{Z}/n\mathbb{Z}$, so that indices are computed mod $n$). One can show that $w$ is cyclically decreasing if and only if all of its reduced decompositions have the above property. Since no non-commuting braid relations can occur, all the reduced words of $w$ also have the same indices $i_1, \ldots, i_N$.

For $i \in I$, let $k_i \in \mathbb{K}_0$ be defined by

$$k_i = \sum_w T_w$$

(7.1)

where $w$ runs over the cyclically decreasing elements of $W_{af}$ of length $i$. We set $k_0 = 1$. These elements were considered in [Lam06].

We define coordinates for the weight lattice $P$ of $\mathfrak{sl}_n$. Let $P \subset \mathbb{Z}^n = \bigoplus_{i=1}^n \mathbb{Z}e_i$, with fundamental weights $\omega_i = e_1 + e_2 + \cdots + e_i$ and $\alpha_i = e_i - e_{i+1}$ for $i \in I$. For a subset $J \subset \{1, \ldots, n\}$, let us write $e_J = \sum_{i \in J} e_i \in P$ for the 01-vector with 1s in the positions corresponding to elements of $J$. The $e_J$ with $|J| = k$ form the set of weights for the $k$th fundamental representation of $SL_n(\mathbb{C})$ with highest weight $\omega_k$, which is multiplicity-free. We have $r_i \cdot e_J = e_{r_iJ}$, where indices are taken mod $n$.

**Lemma 7.1.** We have

$$T_i \cdot e_J = \begin{cases} 0 & \text{if both } i, i + 1 \in J \text{ or both } i, i + 1 \notin J, \\ e_J & \text{if } i \in J \text{ and } i + 1 \notin J, \\ -e_J & \text{if } i \notin J \text{ and } i + 1 \in J. \end{cases}$$

Let $J$ and $K$ be disjoint subsets of $\mathbb{Z}/n\mathbb{Z}$ such that $J \cup K \neq \mathbb{Z}/n\mathbb{Z}$. We write $S_{J,K} \cdot e^\lambda$ for the action of $\{r_j \mid j \in J\}$ and $\{T_k \mid k \in K\}$ on $e^\lambda$, where the operators act in cyclically decreasing order (for example, $r_1$ would act before $T_2$).

**Lemma 7.2.** Let $J$ and $K$ be as above.

(i) If $|0, k - 1 \cap K| \geq 2$, then $S_{J,K} \cdot e^{\omega_k} = 0$.

(ii) Suppose $|0, k - 1 \cap K| = 1$. Let $a \in [0, k - 1] \cap K$. Then $S_{J,K} \cdot e^{\omega_k} = 0$ unless $[0, a - 1] \subset J$.

(iii) Suppose $|0, k - 1 \cap K| = 1$. Then $S_{J,K} \cdot e^{\omega_k} = 0$ if $[k, -1] \subset (J \cup K)$.
(iv) Suppose that \( S_{J,K} \cdot e^{\omega_k} \neq 0 \) and \([k, -1] \cap K \neq \emptyset\). For each \( a \in ([k, -1] \cap K) \), we have \([k, a] \subset (J \cup K)\).

(v) Suppose that \( S_{J,K} \cdot e^{\omega_k} \neq 0 \) and \([k, -1] \subset (J \cup K)\). Then \( [0, k - 1] \cap K = \emptyset\).

The following lemma is general (not just for affine type \( A_{n-1}^{(1)} \)).

**Lemma 7.3.** Suppose \( a \in \mathbb{K}_0\). If \( \lambda, \mu \in P \) are such that \( \phi_0(ae^\lambda) = a \) and \( \phi_0(ae^\mu) = a \), then \( \phi_0(ae^{\lambda+\mu}) = a \).

**Proof.** Write \( a(e^\lambda - 1) = \sum_w a_w T_w \) and \( T_w e^\mu = \sum_v b_v^{w,\mu} T_v \) for \( a_w, b_v^{w,\mu} \in R(T)\). We have \( \phi_0(a_w) = 0 \) for all \( w \) and \( \phi_0(a e^\mu - 1) = 0 \). Then

\[
\phi_0(a(e^{\lambda+\mu} - 1)) = \phi_0(a(e^\lambda - 1)e^\mu) + \phi_0(a(e^\mu - 1))
= \phi_0 \left( \sum_w a_w T_w e^\mu \right)
= \sum_v \phi_0 \left( \sum_w a_w b_v^{w,\mu} \right) T_v = 0
\]

since \( \phi_0 : R(T) \to \mathbb{Z} \) is a ring homomorphism. \( \square \)

**Proposition 7.4.** We have \( k_i \in \mathbb{L}_0 \).

**Proof.** By Lemma 7.3, it is enough to prove that \( \phi_0(k_i e^{\lambda}) = k_i \) for \( \lambda \) being either a fundamental weight or the negative of a fundamental weight. We deal with the case where \( \lambda = \omega_k \); negative fundamental weights are treated similarly.

When \( \phi_0(k_i (e^{\omega_k} - 1)) \) is expanded in the \( T_w \) basis, only a term that involves cyclically decreasing \( w \) would occur with non-zero coefficient. Fix \( J \). Let us show that \( [T_J] \phi_0(k_i (e^{\omega_k} - 1)) = 0 \), where \( T_J \) is the product of \( T_j \) with \( j \in J \) in cyclically decreasing order and \( [T_J]a \) denotes the coefficient of \( T_J \) in \( a \in \mathbb{K}_0 \). This is clear if \( |J| = i \), by (2.6) and Lemma 7.1. So suppose that \( |J| < i \). Then

\[
[T_J] \phi_0(k_i (e^{\omega_k} - 1)) = \sum_{K : |K| = 1 - |J|} \phi_0(S_{J,K} e^{\omega_k}). \quad (7.2)
\]

We will say that a subset \( K \) in the above sum is good if \( S_{J,K} e^{\omega_k} \neq 0 \). Let us define an involution \( \iota \) on good subsets such that \( \phi_0(S_{J,K} e^{\omega_k}) = -\phi_0(S_{J,\iota(K)} e^{\omega_k}) \). By Lemma 7.2(i), we may write the set of good subsets as the disjoint union \( S_0 \sqcup S_1 \) where

\[
S_0 = \{ K : |K \cap [0, k - 1]| = 0 \} \quad \text{and} \quad S_1 = \{ K : |K \cap [0, k - 1]| = 1 \}.
\]

The involution satisfies \( \iota(S_a) = S_{1-a} \).

Suppose that \( K \in S_0 \). Then \( K \cap [k, -1] \neq \emptyset \) and we may set \( a \) to be the maximal element of \( K \cap [k, -1] \). Let \( \iota(K) = K \setminus \{a\} \cup \{b\} \) where \( b \in [0, k - 1] \) is minimal so that \( j \notin J \) and \( [0, j - 1] \subset J \). One can check directly that \( \iota(K) \) is also good. Using Lemma 7.1, one sees that \( \iota(K) \) and \( K \) contribute different signs in (7.2).

Suppose that \( K \in S_1 \). Let \( a \) be the unique element in \( K \cap [0, k - 1] \). Using Lemma 7.2(iii), we pick \( b \in [k, -1] \setminus (J \cup K) \) minimal in \([k, -1]\). Set \( \iota(K) = K \setminus \{a\} \cup \{b\} \). Again, one can check that \( \iota(K) \) is good and that \( K \) and \( \iota(K) \) contribute different signs.

Finally, it follows from Lemma 7.2(ii) and (iv) that \( \iota \) is an involution. \( \square \)

**Corollary 7.5.** For \( 1 \leq i \leq n - 1 \), \( k_i = \phi_0(k_{\sigma_i}) \).
**K-theory Schubert calculus of the affine Grassmannian**

*Proof.* By Proposition 7.4, \( k_i \in \mathbb{L}_0 \), and by definition it has unique Grassmannian term \( T_{\sigma_i} \). The result follows from Theorem 6.4. \( \square \)

By forgetting equivariance, we obtain a \( K \)-homology Pieri rule for \( K_*(Gr_{SL_n}) \); see [LLMS, LM05] for the homological version.

**Corollary 7.6.** For \( 1 \leq i \leq n-1 \), \( \phi_0(d_{w,v}^i) \) equals \((-1)^{\ell(w)-\ell(v)-i}\) times the number of cyclically decreasing elements \( x \in W_a \) with \( \ell(x) = i \) and \( \ell_x T_v = \pm T_w \).

*Proof.* This follows from (5.4), (7.1) and Corollary 7.5. \( \square \)

The \( K \)-cohomology Pieri rule is likely to be much more complicated; see [LLMS] for the cohomological version.

**7.2 Coproduct of the \( k_i \)**

In this section we determine the coproduct \( \phi_0(\Delta(k_i)) \) explicitly.

Let \( J \) and \( K \) be two subsets of \( \mathbb{Z}/n\mathbb{Z} \) with total size less than \( n-1 \). We define a sequence of non-negative integers \( \text{cd}_{J,K} = (\text{cd}(j) : j \in \mathbb{Z}/n\mathbb{Z}) \) by

\[
\text{cd}(j) = \max_{0 \leq t \leq n-1} \{|J \cap [j-t,j]| + |K \cap [j-t,j]| - t\}.
\]

(The intervals \([j-t,j]\) are to be considered as cyclic intervals.) It is then clear that \( \text{cd}(j+1) - \text{cd}(j) \in \{-1,0,1\} \).

We note that \( \text{cd}(j) \geq 0 \) for all \( j \in \mathbb{Z}/n\mathbb{Z} \).

**Lemma 7.7.** Let \( J \) and \( K \) be two subsets of \( \mathbb{Z}/n\mathbb{Z} \) with total size less than \( n-1 \).

(i) There exists \( j \) such that \( \text{cd}(j) = 0 \) and \( j \notin J \cup K \).

(ii) \( \text{cd} \) is the unique sequence such that \( \text{cd}(j+1) - \text{cd}(j) = |j \cap J| + |j \cap K| - 1 \), except when \( \text{cd}(j) = 0 \) and \( j \notin (J \cup K) \).

*Proof.* To prove (i), suppose that no such \( j \) exists. Then \( \text{cd}(j+1) - \text{cd}(j) = |j \cap J| + |j \cap K| - 1 \) for each \( j \). But \( 0 = (\text{cd}(j) - \text{cd}(j-1)) + \cdots + (\text{cd}(j+1) - \text{cd}(j)) \), so this is impossible because \( |J| + |K| \leq n-1 \). Now we prove (ii). Everything except uniqueness is clear. Let \( \text{cd}' \) be any sequence with the asserted properties. The same calculation as in (i) shows that there is \( j' \) such that \( \text{cd}'(j') = 0 = \text{cd}'(j'+1) \) and \( j' \notin (J \cup K) \). By recursively calculating \( \text{cd}(j'+1) \) and \( \text{cd}'(j'+1) \), then \( \text{cd}(j'+2) \) and \( \text{cd}'(j'+2) \), and so on, we find that \( \text{cd}(j) \geq \text{cd}'(j) \) for all \( j \). But a symmetric argument shows that \( \text{cd}(j) \leq \text{cd}'(j) \) for all \( j \). \( \square \)

Define \( t(J,K) = (t_i : i \in \mathbb{Z}/n\mathbb{Z}) \in \{L,R,B,E\}^n \) as follows (here \( E \) stands for ‘empty’, \( L \) for ‘left’, \( R \) for ‘right’, and \( B \) for ‘both’):

\[
t_j = \begin{cases} E & \text{if } \text{cd}(j) = 0 \text{ and } j \notin J \cup K, \\ L & \text{if } \text{cd}(j) = 0 \text{ and } j \in J \setminus K, \\ R & \text{if } j \notin J \text{ and } (\text{cd}(j) > 0 \text{ or } j \in K), \\ B & \text{otherwise}. \end{cases}
\]

We say that two sequences \( \text{cd} \) and \( t \) are **compatible** if:

(a) \( t_j \in \{E,L\} \) implies \( \text{cd}(j) = 0 \);

(b) \( \text{cd}(j+1) - \text{cd}(j) = 0 \) if \( t_j = L \);

(c) \( \text{cd}(j+1) - \text{cd}(j) \in \{-1,0\} \) if \( t_j = R \);
(d) \(\text{cd}(j + 1) - \text{cd}(j) \in \{0, 1\}\) if \(t_j = B\);
(e) \(t_j = E\) for some \(j \in [0, n - 1]\).

Define the support of \((\text{cd}, t)\) to be \(\{j \mid t_j \neq E\}\).

**Lemma 7.8.** The map \((J, K) \mapsto (\text{cd}_{J,K}, t(J, K))\) is a bijection between pairs of subsets of \(\mathbb{Z}/n\mathbb{Z}\) with total size \(k < n\) and pairs of compatible sequences with support of size \(k\).

**Proof.** It is easy to see that \((\text{cd}_{J,K}, t(J, K))\) is compatible with support of the correct size. We first check that the pair of sequences determines \(J\) and \(K\). By itself, \(t(J, K)\) completely determines \(J\): we have \(j \in J\) if and only if \(t_j \in \{L, B\}\). Also, \(j \in K\) if and only if either \(t_j = R\) and \(\text{cd}(j + 1) = \text{cd}(j)\) or \(t_j = B\) and \(\text{cd}(j + 1) = \text{cd}(j) + 1\). Thus \((\text{cd}_{J,K}, t(J, K))\) determines \((J, K)\).

Conversely, given compatible \((\text{cd}, t)\), we recursively construct \(J\) and \(K\) by starting at some value \(j\) such that \(t_j = E\). For such a value we have \(j \notin J \cup K\). We then decide whether \(j + 1 \in J\) and/or \(j + 1 \in K\), and so on. By construction, we obtain two subsets \(J\) and \(K\) such that \(\text{cd}(j + 1) - \text{cd}(j) = |j \cap J| + |j \cap K| - 1\), unless \(\text{cd}(j) = 0 = \text{cd}(j + 1)\) and \(j \notin J \cup K\). By Lemma 7.7, we have \(\text{cd}_{J,K} = \text{cd}\). Using compatibility, one can check that the size of the support of \((\text{cd}, t)\) is equal to \(|J| + |K|\). But then it follows that \(t(J, K) = t\). \(\square\)

**Proposition 7.9.** We have \(\phi_0(\Delta(k_r)) = \sum_{0 \leq j < r} k_j \otimes k_{r-j}\).

**Proof.** Our proof follows the strategy in [Lam08, §7.2]. Let \(J = \{i_1, \ldots, i_r\} \subset \mathbb{Z}/n\mathbb{Z}\). Using (2.16), we calculate \(\phi_0(\Delta(T_{i_1}) \cdot \Delta(T_{i_2}) \cdots \Delta(T_{i_r}))\), where \(r_1 \cdots r_\ell\) is a cyclically decreasing reduced expression and \(\cdot\) means the ‘componentwise’ product on \(\Delta(K)\) of (2.17).

Let us expand this product by picking, for each component, one of the three terms of (2.16). As usual, we write \(\alpha_{ij} = \alpha_i + \cdots + \alpha_{j-1}\) for any cyclic interval \([i, j]\). We first note that

\[
T_j (1 - e^{\alpha_{ij}}) = (1 - e^{\alpha_{i,j+1}})T_j + \frac{1 - e^{\alpha_{i,j+1}} - (1 - e^{\alpha_{ij}})}{1 - e^{\alpha_j}} = (1 - e^{\alpha_{i,j+1}})T_j + e^{\alpha_{ij}}.
\]

Because of the cyclically decreasing condition, whenever the above calculation is encountered, the coefficient \(e^{\alpha_{ij}}\) will always commute with any \(T_i\) which occurs to the left.

We shall now show by induction that the only terms in the expansion of \(\Delta(T_{i_k}) \cdots \Delta(T_{i_r})\) that contribute to \(D\) look like

\[
T_v \otimes \prod_{i \in S} (1 - e^{\alpha_{i,k+1}})qT_w
\]

where:

(i) either \(S\) is empty and \(\prod_{i \in S} (1 - e^{\alpha_{i,k+1}}) = 1\), or \(S \subseteq \{i_k, i_{k+1}, \ldots, i_\ell\}\);
(ii) \(q \in R(T)\) commutes with \(r_{i_1}, \ldots, r_{i_{k-1}}\) and satisfies \(\phi_0(q) = 1\).

Such a term contributes nothing to \(D\) if \(|S| > 0\) and \(i_k \notin \{i_1, \ldots, i_{k-1}\}\). To prove the inductive step, we assume that \(i_{k-1} = i_k + 1\) and calculate (using (2.5))

\[
T_{i_k+1} \prod_{i \in S} (1 - e^{\alpha_{i,k+1}})q = \prod_{i \in S} (1 - e^{\alpha_{i,k+2}})qT_{i_k+1} + \prod_{i \in S'} (1 - e^{\alpha_{i,k+2}}) + q',
\]

838
where \( S' \subset S \) and \( q' \in R(T) \) commutes with \( r_{i_1}, \ldots, r_{i_{k-2}} \) and satisfies \( \phi_0(q') = 0 \). Clearly, the term involving \( q' \) contributes nothing to \( D \), and the first two terms lead to expressions of the form (7.3).

Given a choice of one of the three terms in (2.16), we define a sequence \( t_j \) by:

(a) \( t_j = E \) if \( j \notin J' \);
(b) \( t_j = L \) if we pick \( T_j \otimes 1 \);
(c) \( t_j = R \) if we pick \( 1 \otimes T_j \);
(d) \( t_j = B \) if we pick \( T_j \otimes (1 - e^{a_j})T_j \).

Furthermore, let us make a choice of one of the two terms in (7.4), whenever we have such a choice. At each step of our calculation we are looking at a term of the form (7.3). We set \( \text{cd}(j) \) to be the size of \( S \) in the term just before \( \Delta(T_j) \) is applied. If \( j \notin J' \), then \( \text{cd}(j) = 0 \). If this entire process produces a non-zero term of \( D \), then \( (\text{cd}, t) \) is a compatible sequence: the sequence \( \text{cd} \) ‘wraps around’ properly because eventually the coefficient \( \prod_{j \in S}(1 - e^{a_j+b_j}) \) has to equal 1, otherwise it will vanish when \( \phi_0 \) is applied. Conversely, a compatible pair \( (\text{cd}, t) \) with support equal to \( J \) always arises in this fashion.

By Lemma 7.8, there is a bijection between compatible pairs \( (\text{cd}, t) \) with support of size \( r \) and pairs of subsets \( (J, K) \) with total size equal to \( r \). It is easy to check that the term in \( D \) corresponding to \( (\text{cd}, t) \) is exactly \( T_J \otimes T_K \). \( \square \)

7.3 Symmetric function realizations

Let \( \Lambda = \bigoplus \mathbb{Z}m_\lambda \) be the ring of symmetric functions over \( \mathbb{Z} \), where \( m_\lambda \) is the monomial symmetric function [Mac95] and \( \lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_t > 0) \) runs over all partitions. Let \( \hat{\Lambda} = \prod \mathbb{Z}m_\lambda \) be the graded completion of \( \Lambda \). Let \( |\lambda| = \lambda_1 + \cdots + \lambda_t \) denote the size of a partition.

Let \( \Lambda^{(n)} = \Lambda / (m_\lambda | \lambda_1 \geq n) \) denote the quotient by the ideal generated by monomial symmetric functions labeled by partitions with first part greater than \( n \). We write \( \hat{\Lambda}^{(n)} \) for the graded completion of \( \Lambda^{(n)} \). Now let \( \Lambda(n) = \mathbb{Z}[h_1, h_2, \ldots, h_{n-1}] \subset \Lambda \) denote the subalgebra generated by the first \( n - 1 \) homogeneous symmetric functions. Both \( \Lambda(n) \) and \( \hat{\Lambda}^{(n)} \) are Hopf algebras.

The Hall inner product \( \langle \cdot, \cdot \rangle : \Lambda \times \mathbb{Z} \Lambda \to \mathbb{Z} \) extends by linearity with respect to infinite graded linear combinations to a pairing \( \langle \cdot, \cdot \rangle : \Lambda \times \hat{\Lambda} \to \mathbb{Z} \), which in turn descends to a pairing \( \Lambda(n) \times \mathbb{Z} \Lambda^{(n)} \to \mathbb{Z} \). This pairing expresses \( \Lambda(n) \) as the continuous (Hopf-)dual of \( \hat{\Lambda}^{(n)} \) and \( \hat{\Lambda}^{(n)} \) as the graded completion of the graded (Hopf-)dual of \( \Lambda(n) \). For short, we will just say that \( \Lambda(n) \) and \( \hat{\Lambda}^{(n)} \) are dual. The basis \( \{h_\lambda | \lambda_1 < n\} \subset \Lambda(n) \) and ‘basis’ \( \{m_\lambda | \lambda_1 < n\} \subset \hat{\Lambda}^{(n)} \) are dual under the Hall inner product.

7.4 Affine stable Grothendieck polynomials

The affine stable Grothendieck polynomials \( G_v(x_1, x_2, \ldots) \) for \( v \in W_{af} \) are the formal power series defined by the identity [Lam06]

\[
\prod_{i \geq 1} \sum_{j=0}^{n-1} (x_i^j k_j) = \sum_{v \in W_{af}} G_v(x_1, x_2, \ldots) T_v, \tag{7.5}
\]

where the \( x_i \) are indeterminates that commute with the elements of \( \mathbb{K}_0 \) and \( k_j \in \mathbb{K}_0 \) are the elements defined in (7.1). Alternatively, for a composition \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_t) \), the coefficient of \( x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_t^{\alpha_t} \) in \( G_v(x) \) is equal to the coefficient of \( T_v \) in \( k_{\alpha_1} k_{\alpha_2} \cdots k_{\alpha_t} \). It is clear that
$G_v(x)$ is a sum of monomials such that no variable occurs with degree more than $n-1$ in any monomial. Examples of the $G_v(x)$ are given in Appendix A.3.6.

The following result was proved directly in [Lam06, Theorem 44].

**Proposition 7.10.** For each $v \in W_{af}$, we have $G_v(x) \in \hat{\Lambda}^{(n)}$.

**Proof.** By Proposition 7.4 and Corollary 6.5, the $k_i$ commute. This implies that $G_v(x)$ is a symmetric function. In addition, all monomial symmetric functions $m_\lambda$ which occur with non-zero coefficient in $G_v(x)$ satisfy $\lambda_1 < n$, so $G_v(x)$ can be naturally identified with its image in $\hat{\Lambda}^{(n)}$.

The next result follows from (2.3).

**Lemma 7.11.** The graded components of $G_v(x)$ are alternating; that is, the coefficient of $m_\lambda$ in $G_v(x)$ has sign equal to that of $(-1)^{|\lambda| - \ell(v)}$.

In [Lam06], for each $v \in W_{af}$, a homogeneous symmetric function $F_v(x)$, the affine Stanley symmetric function, is defined. The next result follows by inspection.

**Lemma 7.12.** Let $v \in W^I_{af}$. The lowest-degree component (of degree $\ell(v)$) of $G_v(x)$ is equal to $F_v(x)$.

(Readers not familiar with affine Stanley symmetric functions may take this as the definition of $F_v(x)$.) The $F_v(x)$, $v \in W^I_{af}$, were called affine Schur functions in [Lam06] and are equivalent to the dual $k$-Schur functions of [LM07].

**Proposition 7.13.** The set $\{G_v(x) \mid v \in W^I_{af}\}$ is a ‘basis’ of $\hat{\Lambda}^{(n)}$. In other words, $\hat{\Lambda}^{(n)} = \prod_{v \in W^I_{af}} \mathbb{Z} G_v(x)$.

**Proof.** This follows from the fact that $\{F_v(x) \mid v \in W^I_{af}\}$ is a basis of $\Lambda^{(n)}$; see [Lam06, LM07].

**Remark 7.1.** Suppose $w \in W_{af}$ is such that some (or, equivalently, every) reduced expression for $w$ does not involve all of the simple generators $r_0, r_1, \ldots, r_{n-1}$. It then follows from comparing the definitions that the stable affine Grothendieck polynomial $G_w(x)$ is equal to the usual stable Grothendieck polynomial [FK94] labeled by $u \in W = S_n$, where $u$ is obtained from $w$ by cyclically rotating the indices until $r_0$ is not present.

**Remark 7.2.** There is a bijection between $v \in W^I_{af}$ and $(n-1)$-bounded partitions $\{\lambda \mid \lambda_1 < n\}$; see [LM05, Lam06]. The partition $\lambda$ associated to $v$ can be obtained from the exponents of the dominant monomial term $x_1^{\lambda_1}x_2^{\lambda_2} \cdots$ in $F_v(x_1, x_2, \ldots)$. We may thus relabel $\{G_v(x) \mid v \in W^I_{af}\}$ as $\{G^{(k)}_\lambda(x) \mid \lambda_1 < n\}$, where $k = n - 1$ (owing to historical reasons). A table showing this correspondence is given in Appendix A.3.1. Remark 7.1 implies that $G^{(k)}_\lambda(x) = G_\lambda(x)$ whenever the largest hook length of $\lambda$ is less than or equal to $k$, where $G_\lambda(x)$ is the stable Grothendieck polynomial labeled by partitions and studied by Buch in [Buc02].

### 7.5 $K$-theoretic $k$-Schur functions

Since $\{G_v(x) \mid v \in W^I_{af}\}$ is a ‘basis’ of $\hat{\Lambda}^{(n)}$, there is a dual basis $\{g_v(x) \mid v \in W^I_{af}\}$ of $\Lambda^{(n)}$. This definition of $g_v(x)$ has been stated previously by Lam. We call the symmetric functions $g_v(x)$ affine dual stable Grothendieck polynomials or $K$-theoretic $k$-Schur functions. Examples of the $g_v(x)$ are given in Appendix A.3.4.

The proof of the following result is standard (see, for example, [Sta01, Lemma 7.9.2]).
Lemma 7.14. We have

\[ \sum_{\lambda_1 < n} h_\lambda(x) m_\lambda(y) = \sum_{v \in W_{af}^I} g_v(x) G_v(y). \]

Let \( k = n - 1 \). The \( k \)-\textit{Schur functions} \( \{ s_v^{(k)}(x) \mid v \in W_{af}^I \} \) (see [LLM03, LM07]) form the basis of \( \Lambda_{(n)} \) dual to \( \{ F_v(x) \mid v \in W_{af}^I \} \), and are usually labeled by the \( k \)-bounded partitions \( \{ \lambda \mid \lambda_1 < n \} \). (The \( k \)-Schur functions originally defined in [LLM03] depend on an additional parameter \( t \), and setting \( t = 1 \) conjecturally gives the \( k \)-Schur functions of [LM07], which are the ones used here.)

Lemma 7.15. Let \( v \in W_{af}^I \). Then the highest-degree homogeneous component of \( g_v(x) \) is equal to the \( k \)-\textit{Schur function} \( s_v^{(k)}(x) \).

Proof. We prove this by induction on \( \ell(v) \). The base case is clear: \( G_{id}(x) = F_{id}(x) = 1 = g_{id}(x) = s_{id}^{(k)}(x) \). Suppose the claim has been proven for all \( w \) satisfying \( \ell(w) < \ell \), and let \( \ell(v) = \ell \). One then checks that the symmetric function

\[ g_v(x) = s_v^{(k)}(x) - \sum_{w: \ell(w) < \ell} \langle s_v^{(k)}(x), G_w(x) \rangle g_w(x) \]

is a solution to the system of equations

\[ \langle g_v(x), G_u(x) \rangle = \delta_{vu} \quad \text{for all} \quad u \in W_{af}^I. \]

Remark 7.3. Relabel \( \{ g_v(x) \mid v \in W_{af}^I \} \) as \( \{ g_\lambda^{(k)}(x) \mid \lambda_1 \leq k \} \), as in Remark 7.2. Since \( \lim_{k \to \infty} G_\lambda^{(k)}(x) = G_\lambda(x) \), it follows that \( \lim_{k \to \infty} g_\lambda^{(k)}(x) = g_\lambda(x) \), where the \( g_\lambda(x) \) are the dual affine stable Grothendieck polynomials studied in [LP07, Len00].

7.6 Non-commutative \( K \)-theoretic \( k \)-\textit{Schur functions}

Define \( \varphi: \Lambda_{(n)} \to \mathbb{L}_0 \) by \( h_i \mapsto k_i \). This map is well-defined since the \( h_i \) are algebraically independent and \( \mathbb{L}_0 \) is commutative. The \textit{non-commutative} \( K \)-\textit{theoretic} \( k \)-\textit{Schur functions} are the elements \( \{ \varphi(g_v) \mid v \in W_{af}^I \} \subset \mathbb{L}_0 \).

Proposition 7.16. Let \( w \in W_{af} \) and \( v \in W_{af}^I \). The coefficient of \( T_w \) in \( \varphi(g_v) \) is equal to the coefficient of \( G_v(x) \) in \( G_w(x) \) when the latter is expanded in terms of \( \{ G_u(x) \mid u \in W_{af}^I \} \).

Proof. Applying \( \varphi \) to Lemma 7.14 and comparing with (7.5), we have

\[ \sum_{v \in W_{af}^I} G_v(y) \varphi(g_v) = \sum_{w \in W_{af}} G_w(y) T_w. \]

Now take the coefficient of \( T_w \) on both sides. \qed

7.7 Grothendieck polynomials for the affine Grassmannian

The following is our main theorem.

Theorem 7.17.

(i) The map \( \varphi: \Lambda_{(n)} \to \mathbb{L}_0 \) is a Hopf-isomorphism, sending \( g_v \) to \( \phi_0(k_v) \) for \( v \in W_{af}^I \).

(ii) We have a Hopf-algebra isomorphism \( k_0^{-1} \circ \varphi: \Lambda_{(n)} \to K_*(Gr_{SL_n}) \), sending \( g_v \) to \( \xi_v^0 \) for \( v \in W_{af}^I \).
(iii) There is a dual Hopf-algebra isomorphism $K^*(\text{Gr}_{SL_n}) \cong \hat{\Lambda}^{(n)}$, sending $[\mathcal{O}_{X_L}]_0$ to $G_v(x)$ for $v \in W_{af}^I$.

(iv) The following diagram commutes.

$$
\begin{array}{ccc}
K^*(\text{Gr}_{SL_n}) \times K^*(\text{Gr}_{SL_n}) & \longrightarrow & \mathbb{Z} \\
\Lambda^{(n)} \times \hat{\Lambda}^{(n)} & \longrightarrow & \mathbb{Z} \\
\text{id} & & \\
\end{array}
$$

**Proof.** Given the definitions and Theorem 6.8, all the statements follow from the first one. By Theorem 6.4 and Propositions 7.13 and 7.16, we deduce that $\varphi(g_v) = \phi_0(k_v)$. It follows that $\varphi$ is an isomorphism. Since $\Delta(h_i) = \sum_{0 \leq j \leq i} h_j \otimes h_{i-j}$ in $\Lambda^{(n)}$, it follows from Proposition 7.9 that $\varphi$ is a Hopf-morphism.

**Corollary 7.18.** For $1 \leq r \leq n-1$, we have $g_{sr}(x) = h_r(x)$.

Recall the map $r^*: K^T(X_{af}) \to K^T(\text{Gr}_G)$ defined in § 5.1. We use $r_0^*: K^*(X_{af}) \to K^*(\text{Gr}_{SL_n})$ to denote the evaluation of $r^*$ at zero.

**Theorem 7.19.** The image of $G_w(x)$ under the isomorphism $\hat{\Lambda}^{(n)} \cong K^*(\text{Gr}_{SL_n})$ is equal to $r_0^*([\mathcal{O}_{X_L}]_0)$.

**Proof.** As observed previously, the map $\varpi$ of Lemma 4.6 is related to $r^*$ via the isomorphisms of Theorem 4.3. By (5.1), $(k(\xi_w), \psi^w) = (\xi_v, \varpi(\psi^w))$. It follows that the coefficient of $T_w$ in $k_v$ is equal to the coefficient of $[\mathcal{O}_{X_L}]$ in $r^*([\mathcal{O}_{X_w}]_0)$. Applying $\phi_0$ to these coefficients and comparing with Proposition 7.16 gives the result.

### 7.8 Conjectural properties

In this section we list conjectural properties of the symmetric functions $g_w(x)$ and $G_w(x)$. When $w \in W_{af}^I$, we will use partitions to label these symmetric functions; see Remark 7.2. Recall also that $k = n-1$.

**Conjecture 7.20.** The basis $\{g^{(k)}_\lambda\}$ of $\Lambda^{(n)}$ has the following properties.

(i) Each $g^{(k)}_\lambda$ is a positive integer (necessarily finite) sum of $k$-Schur functions. (By Lemma 7.15, the top homogeneous component of $g^{(k)}_\lambda$ is the $k$-Schur function $s^{(k)}_\lambda$).

(ii) The coproduct structure constants $c^{(k)}_{\mu\nu}$ in $\Delta(g^{(k)}_\lambda) = \sum_{\mu,\nu} c^{(k)}_{\mu\nu} g^{(k)}_\mu \otimes g^{(k)}_\nu$ are alternating integers, that is, $(-1)^{|\lambda|-|\nu|-|\mu|} c^{(k)}_{\mu\nu} \in \mathbb{Z}_{\geq 0}$. Furthermore, $c^{(k)}_{\mu\nu} = 0$ unless $|\mu| + |\nu| \leq |\lambda|$.

(iii) The coefficients in the expansion $g^{(k)}_\lambda = \sum_\mu a^{(k+1)}_\mu g^{(k+1)}_\mu$ are alternating integers, that is, $(-1)^{|\lambda|-|\mu|} a^{(k)}_\mu \in \mathbb{Z}_{\geq 0}$.

Conjecture 7.20.(1) has been checked for $n = 2, 3, 4$ and 5 for $|\lambda| \leq 8$ using the software package Sage [Sag]; see also the tables in Appendix A.3.4. Data confirming Conjecture 7.20.(ii) can be found in Appendix A.3.5. Conjecture 7.20.(iii) has been checked for $n = 2, 3$ and 4 for $|\lambda| \leq 8$ using Sage. According to Conjecture 6.7, the product structure constants for $\{g^{(k)}_\lambda\}$ should be alternating.
**K-theory Schubert calculus of the affine Grassmannian**

Conjecture 7.21.

(i) Every affine stable Grothendieck polynomial $G_w$ for $w \in W_{af}$ is a finite alternating linear combination of $\{G^{(k)}_\lambda\}$.

(ii) Every $G^{(k)}_\lambda$ is an alternating integer linear combination of the affine Schur functions $\{F^{(k)}_\mu\}$.

(iii) The structure constants in the product $G^{(k)}_\mu G^{(k)}_\nu = \sum_\lambda c^{\mu\nu}_\lambda G^{(k)}_\lambda$ are alternating integers, that is, $(-1)^{|\lambda|-|\mu|-|\nu|} c^{\mu\nu}_\lambda \in \mathbb{Z}_{\geq 0}$. Furthermore, $c^{\mu\nu}_\lambda = 0$ unless $|\mu| + |\nu| \leq |\lambda|$.

(iv) The coefficients in the expansion $G^{(k+1)}_\mu = \sum_\lambda a^\mu_\lambda G^{(k)}_\lambda$ are alternating integers, that is, $(-1)^{|\lambda|-|\mu|} a^\mu_\lambda \in \mathbb{Z}_{\geq 0}$.

By Proposition 7.16, the ‘alternating’ part of Conjecture 7.21(i) is implied by Conjecture 6.7. Evidence for Conjecture 7.21(ii) is provided in the table of Appendix A.3.6. Conjecture 7.21(ii) is related to Conjecture 7.20(i) via a matrix inverse. Conjecture 7.21(iii) is equivalent to Conjecture 7.20(ii); indeed, the two sets of structure constants are identical. Conjecture 7.21(iv) is equivalent to Conjecture 7.20(iii).

Remark 7.4. The factorization of affine Grassmannian homology Schubert classes as described in [Mag] (see also [Lam08, LM07]) appears also to hold in some form in $K$-homology. Suppose that $w \in W^T_{af}$ has a length-additive factorization $w = vu$ where $u \in W^T_{af}$ is equal, modulo length-zero elements, to the translation $t_{-\omega'}$ by a negative fundamental coweight in the extended affine Weyl group [Mag] or, equivalently, that the partition $\lambda$ corresponding to $u$ is a rectangle of the form $\ell \times (k-\ell)$. Then it appears that $g_w$ is a multiple of $g_u$ in $\Lambda_{(n)}$.

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Appendix A. The affine NilHecke ring and tables

A.1 The (cohomological) affine NilHecke ring

A summary of the (notational) correspondence between (co)homology and $K$-(co)homology is given in Table A1.

Some of our notation differs from that in [Lam08].

We now recall the affine NilHecke ring $\mathbb{A}$. Let $S = \text{Sym}(P)$ where $P$ is the weight lattice of the finite-dimensional group $G$. Then $W_{af}$ acts on $P$ (and therefore on $S \cong H^T(\text{pt})$) by the level-zero action. The affine NilCoxeter algebra $\mathbb{A}_0$ is the ring with generators $\{A_i \mid i \in I_{af}\}$ and relations

$$A_i^2 = 0 \quad \text{and} \quad A_i A_j \cdots = A_j A_i \cdots,$$

with $m_{ij}$ times $m_{ij}$ times.

Define $A_w$ in the obvious way and define the NilCoxeter algebra by $\mathbb{A}_0 = \bigoplus_{w \in W_{af}} \mathbb{Z} A_w$. Then $\mathbb{A}_0$ acts on $S$ by

$$A_i \cdot \lambda = \langle \alpha_i^\vee, \lambda \rangle,$$

$$A_i \cdot (ss') = (r_i \cdot s)A_i \cdot s' + (A_i \cdot s)s'$$

for $i \in I_{af}$, $\lambda \in P$ and $s, s' \in S$.  

843
### Table A1. Terminology.

| (Co)homology | K-(co)homology | Terminology                      |
|---------------|----------------|----------------------------------|
| $A_i$         | $T_i$          | $(K$-)NilHecke generators        |
| $\mathbb{A} = \bigoplus_w S A_w$ | $\mathbb{K} = \bigoplus_w R(T) T_w$ | $(K$-)NilHecke ring              |
| $\mathbb{B} = \mathbb{Z}_A(S)$ | $\mathbb{L} = \mathbb{Z}_R(R(T))$ | Peterson’s subalgebra            |
| $\mathbb{B}_0 = \phi_0(\mathbb{B})$ | $L_0 = \phi_0(\mathbb{L})$ | Affine Fomin–Stanley subalgebra   |
| $\{j_w\} \subset \mathbb{B}$ | $\{k_w\} \subset \mathbb{L}$ | Schubert basis                   |
| $s_{w}^{(k)}(x)$ | $g_{w}(x)$ | $(K$-theoretic) $k$-Schur functions |
| $F_{w}(x)$ | $G_{w}(x)$ | Affine Stanley symmetric functions/stable affine Grothendieck polynomials |

The affine Kostant–Kumar NilHecke ring $\mathbb{A}$ (see [Pet97]) is the smash product of $\mathbb{A}_0$ and $S$. It has relations

$$A_i s = (r_i \cdot s) A_i + (A_i \cdot s)$$

for $i \in I_{af}$ and $s \in S$. Then

$$\mathbb{A} = \bigoplus_{w \in W_{af}} S A_w.$$ 

In $\mathbb{A}$ we have $r_i = 1 - \alpha_i A_i$, and $A$ acts on $\text{Fun}(W, S)$ by

$$(a \cdot \xi)(w) = \xi(w a),$$

viewing $\xi \in \text{Fun}(W, S)$ as an element of $\text{Hom}_Q(\mathbb{A}_Q, Q)$ (the left $Q$-module homomorphisms) where $Q = \text{Frac}(S)$.

**Lemma A.1** [KK86]. In $\mathbb{A}$, we have

$$A_w \lambda = (w \cdot \lambda) A_w + \sum_{v = w r_\alpha < w} \langle \alpha^\vee, \lambda \rangle A_v.$$ 

Let $\phi_0 : S \to \mathbb{Z}$ be defined by evaluation at zero. Let $\phi_0 : \mathbb{A} \to \mathbb{A}_0$ be the map defined by $\phi_0(\sum_w a_w A_w) = \sum_w \phi_0(a_w) A_w$ for $a_w \in S$. Let $\mathbb{B} = \mathbb{Z}_A(S)$ be the Peterson subalgebra [Pet97], the centralizer subalgebra of $S$ in $\mathbb{A}$, and let

$$\mathbb{B}_0 = \{b \in \mathbb{B} | \phi_0(bs) = \phi_0(s)b \text{ for all } s \in S\}$$

be the Fomin–Stanley subalgebra [Lam08].

**Theorem A.2.**

(i) [Pet97] For each $w \in W_{af}^I$, there is a unique element $j_w \in \mathbb{B}$ such that

$$j_w \in A_w + \bigoplus_{v \in W \setminus W_{af}^I} S A_v,$$

$$\mathbb{B} = \bigoplus_{w \in W_{af}^I} S j_w.$$
(ii) \[ \mathbb{B}_0 = \phi_0(\mathcal{B}). \]

(iii) For each \( w \in W_{af}^I \), \( \phi_0(j_w) \) is the unique element of \( \mathbb{B}_0 \) such that
\[
\phi_0(j_w) \in A_w + \bigoplus_{v \in W \setminus W_{af}} \mathbb{Z} A_v,
\]
\[
\mathbb{B}_0 = \bigoplus_{w \in W_{af}^I} \mathbb{Z} \phi_0(j_w).
\]

Compare these results with the \( K \)-theoretic analogues of Theorems 5.4 and 6.4.

For \( G = \text{SL}_{k+1} \), the element \( \phi_0(j_w) \) is called a non-commutative \( k \)-Schur function \cite{Lam08}.

A.2 Comparison with the fixed-point functions of \cite{KK90}

A.2.1 M"obius inversion for Bruhat order. The M"obius function for the Bruhat order on \( W \) is
\[
(v, w) \mapsto (-1)^{\ell(w) - \ell(v)} \chi(v \leq w)
\]
where \( \chi(P) = 1 \) if \( P \) is true and \( \chi(P) = 0 \) if \( P \) is false; see \cite{Deo77}. In other words, let \( M \) be the \( W \times W \) incidence matrix \( M_{vw} = \chi(v \leq w) \) of the Bruhat order, and let \( N \) be the M"obius matrix \( N_{vw} = \chi(v \leq w)(-1)^{\ell(w) - \ell(v)} \). Then \( M \) and \( N \) are inverse to each other:
\[
\sum_{u \leq v \leq w} (-1)^{\ell(w) - \ell(v)} = \delta_{uw} = \sum_{u \leq v \leq w} (-1)^{\ell(v) - \ell(u)}.
\]

A.2.2 Kostant and Kumar functions. We now return to the (\( K \)-theoretic) notation of \S2. The following lemma is standard.

**Lemma A.3.**
\[
y_w = \sum_{v \leq w} T_v.
\]

For \( v \in W \) define \( \psi_{KK}^v \in \text{Fun}(W, Q(T)) \) by
\[
\psi_{KK}^v(y_w) = \delta_{vw}.
\]

This is equivalent to
\[
w = \sum_v \psi_{KK}^v(w)y_v.
\]

By (A3) and (A2) we have
\[
w = \sum_v \psi_{KK}^v(w) \sum_{u \leq v} T_u = \sum_u T_u \sum_{v \geq u} \psi_{KK}^v(w).
\]

By Proposition 2.4, 2.7 and (A1) we have
\[
\psi^w = \sum_{v \geq u} \psi_{KK}^v,
\]
\[
\psi_{KK}^u = \sum_{v \geq u} (-1)^{\ell(v) - \ell(u)} \psi^v.
\]

\[\text{The functions denoted by } \psi^v(w) \text{ in \cite{KK90} are the same as the functions that we denote by } \psi_{KK}^{v^{-1}}(w^{-1}).\]
Recall the definition of $\eta$ from Remark 2.4. One can show the following. Let $\rho$ be the sum of fundamental weights.

**Lemma A.4.** For all $v, w \in W$,

$$\psi_{KK}^w(w) = (-1)^{\ell(v)} e^{\rho - w\rho} \eta(\psi^v(w)).$$

**Remark A.1.** Let $\partial X_v = X_v \setminus X_v^0$ be the boundary of the Schubert cell $X_v^0$ in the Schubert variety $X_v$. Then there is an exact sequence

$$0 \to I_{\partial X_v \subset X_v} \to \mathcal{O}_{X_v} \to \mathcal{O}_{\partial X_v} \to 0.$$

Since $[\mathcal{O}_{X_{u}}] \mapsto \psi^u$ under the isomorphism $K^T(X) \to \Psi$, $[I_{\partial X_v \subset X_v}] \mapsto \psi_{KK}^w$.

**A.3 Tables**

**A.3.1 Table on Grassmannians versus $k$-bounded partitions.** We list the correspondence between reduced words for Grassmannian elements and $k$-bounded partitions, where $k = n - 1$.

| $n$ | $k$-bounded partition | $w \in W_{af}$ |
|-----|-----------------------|-----------------|
| 2   | 1                     | 0               |
|     | 11                    | 10              |
|     | 111                   | 010             |
|     | 1111                  | 1010            |
|     | 11111                 | 01010           |
| 3   | 1                     | 0               |
|     | 2                     | 10              |
|     | 11                    | 20              |
|     | 21                    | 210             |
|     | 111                   | 120             |
|     | 22                    | 0210            |
|     | 211                   | 2120            |
|     | 1111                  | 0120            |
|     | 221                   | 10210           |
|     | 2111                  | 02120           |
|     | 11111                 | 20120           |

| $n$ | $k$-bounded partition | $w \in W_{af}$ |
|-----|-----------------------|-----------------|
| 4   | 1                     | 0               |
|     | 2                     | 10              |
|     | 11                    | 30              |
|     | 3                     | 210             |
|     | 21                   | 130             |
|     | 111                  | 230             |
|     | 31                   | 3210            |
|     | 22                   | 0130            |
|     | 211                  | 2130            |
|     | 1111                 | 1230            |
|     | 32                   | 03210           |
|     | 311                  | 32130           |
|     | 221                  | 20130           |
|     | 2111                 | 21230           |
|     | 11111                | 01230           |

**A.3.2 $SL_2$.** Set $\alpha = \alpha_1 = -\alpha_0$. We have $t_\alpha = r_0 r_1$ and $t_{-\alpha} = r_1 r_0$. Indexing $T_w$ and $k_w$ by reduced words, we have

$$t_\alpha = (1 - e^{-\alpha})^2 T_{01} + (1 - e^{-\alpha})(T_0 + T_1) + 1,$$

$$t_{-\alpha} = (1 - e^{\alpha})^2 T_{10} + (1 - e^{\alpha})(T_0 + T_1) + 1,$$

$$k_0 = 1,$$

$$k_0 = T_0 + T_1 + (1 - e^{-\alpha}) T_{01},$$

$$k_{10} = T_{10} + e^{-\alpha} T_{01}. $$
So

\[
\begin{align*}
\phi_0(k_\emptyset) &= 1, \\
\phi_0(k_0) &= T_0 + T_1, \\
\phi_0(k_{10}) &= T_{10} + T_{01}.
\end{align*}
\]

In general,

\[
\phi_0(k_{\sigma_r}) = T_{\sigma_r} + T_{\sigma_r},
\]

where \(\sigma_r\) are the elements in (4.4).

### A.3.3 Table of \(\phi_0(k_w)\)

We index \(T_w\) by reduced words.

| \(n\) | \(w\) | \(\phi_0(k_w)\) |
|------|------|-----------------|
| 3    | \(\emptyset\) | 1 |
|      | 0    | \(T_0 + T_1 + T_2\) |
|      | 10   | \(T_{10} + T_{02} + T_{21}\) |
|      | 20   | \(T_{20} + T_{01} + T_{12}\) |
|      | 210  | \(T_{210} + T_{020} + T_{021} + T_{101} + T_{102} + T_{212}\) |
|      | 120  | \(T_{120} + T_{201} + T_{202} + T_{010} + T_{012} + T_{121}\) |
|      | 0210 | \(T_{0210} + T_{1021} + T_{2102}\) |
|      | 1210 | \(T_{1210} + T_{0201} + T_{0212} + T_{1020} + T_{1012} + T_{2101} - T_{020} - T_{101} - T_{212}\) |
|      | 0120 | \(T_{0120} + T_{2012} + T_{1201}\) |
| 4    | \(\emptyset\) | 1 |
|      | 0    | \(T_0 + T_1 + T_2 + T_3\) |
|      | 10   | \(T_{10} + T_{21} + T_{32} + T_{03} + T_{02} + T_{13}\) |
|      | 30   | \(T_{30} + T_{01} + T_{12} + T_{23} + T_{02} + T_{13}\) |
|      | 210  | \(T_{210} + T_{103} + T_{032} + T_{321}\) |
|      | 310  | \(T_{310} + T_{132} + T_{021} + T_{023} + T_{030} + T_{031} + T_{320} + T_{323} + T_{213} + T_{212}\) |
|      | 230  | \(T_{230} + T_{301} + T_{012} + T_{123}\) |
|      | 3210 | \(T_{3210} + T_{3201} + T_{3212} + T_{3213} + T_{2101} + T_{2102} + T_{2103} + T_{1030} + T_{1031} + T_{1032}\) |
|      | 0310 | \(T_{0310} + T_{0213} + T_{1302} + T_{1021} + T_{2132} + T_{3203}\) |
|      | 2310 | \(T_{2310} + T_{3201} + T_{3230} + T_{3231} + T_{0301} + T_{0302} + T_{0312} + T_{0210} + T_{0212}\) |
|      | 1230 | \(T_{1230} + T_{1232} + T_{1231} + T_{2303} + T_{2302} + T_{2301} + T_{3010} + T_{3013}\) |

|     |     |     |
|     |     |     |

| 847 |
### A.3.4 Table of $g^{(k)}_\lambda$

We index $g^{(k)}_\lambda$ by $k$-bounded partitions.

| $n$ | $\lambda$ | $g^{(n-1)}_\lambda$ in terms of $s_\lambda$ | $g_w$ in terms of $s^{(n-1)}_\lambda$ |
|-----|------------|---------------------------------------------|-----------------------------------|
| 2   | 1          | $s_1$                                       | $s_1^{(1)}$                       |
|     | 11         | $s_1 + s_{11} + s_2$                        | $s_1^{(1)} + s_{11}^{(1)}$        |
|     | 111        | $s_1 + 2s_{11} + s_{111} + 2s_2 + s_3 + 2s_{21}$ | $s_1^{(1)} + 2s_{11}^{(1)} + s_{111}^{(1)}$ |
|     | 1111       | $s_1 + 3s_{11} + 3s_{111} + s_{1111} + 3s_2 + 3s_3 + 6s_{21} + 3s_{31} + 2s_{22} + 3s_{211} + s_4$ | $s_1^{(1)} + 3s_{11}^{(1)} + 3s_{111}^{(1)} + s_{1111}$ |
|     | 11111      | $s_1 + 4s_{11} + 6s_{111} + 4s_{1111} + s_{11111} + 4s_2 + 6s_3 + 12s_{21} + 12s_{31} + 8s_{22} + 12s_{211} + 4s_4 + s_5 + 4s_{41} + 6s_{311} + 5s_{221} + 4s_{2111} + 5s_{32}$ | $s_1^{(1)} + 4s_{11}^{(1)} + 6s_{111}^{(1)} + 4s_{1111}^{(1)} + s_{11111}^{(1)}$ |
|     | 3          | $s_1$                                       | $s_1^{(2)}$                       |
|     | 2          | $s_2$                                       | $s_2^{(2)}$                       |
|     | 11         | $s_1 + s_{11}$                              | $s_1^{(2)} + s_{11}^{(2)}$        |
|     | 21         | $s_2 + s_{21} + s_3$                        | $s_2^{(2)} + s_{21}^{(2)}$        |
|     | 111        | $s_1 + s_2 + 2s_{11} + s_{21} + s_{111}$    | $s_1^{(2)} + 2s_{11}^{(2)} + s_{111}^{(2)} + s_2^{(2)}$ |
|     | 22         | $s_2 + s_{21} + s_{22} + s_3 + s_4 + s_{31}$ | $s_2^{(2)} + s_{21}^{(2)} + s_{22}^{(2)}$ |
|     | 211        | $s_{21} + s_{211} + s_3 + s_{31}$          | $s_{21}^{(2)} + s_{211}^{(2)}$    |
|     | 1111       | $s_1 + 2s_2 + 3s_{11} + 3s_{21} + 3s_{111} + s_{22} + s_{211} + s_{1111}$ | $s_1^{(2)} + 3s_{11}^{(2)} + 3s_{111}^{(2)} + s_{22}^{(2)} + s_{211}^{(2)}$ |
|     | 221        | $s_2 + 2s_{21} + 2s_{22} + s_{221} + s_{211} + 2s_3 + 2s_4 + 3s_{31} + s_{311} + 2s_{32} + s_5 + 2s_{41}$ | $s_2^{(2)} + 2s_{21}^{(2)} + s_{211}^{(2)} + 2s_{22}^{(2)} + s_{221}^{(2)}$ |
|     | 2111       | $2s_{21} + s_{22} + 3s_{211} + s_{221} + s_{2111} + 2s_3 + 4s_{31} + 3s_{311} + 3s_{32} + s_{41}$ | $2s_{21}^{(2)} + 3s_{211}^{(2)} + s_{2111}^{(2)} + s_{22}^{(2)}$ |
|     | 11111      | $s_1 + 3s_2 + 4s_{11} + 8s_{21} + 6s_{111} + 4s_{22} + 7s_{211} + 4s_{1111} + 2s_{221} + 2s_{2111} + s_{11111} + 2s_3 + 3s_{31} + s_{311} + s_{32}$ | $s_1^{(2)} + 4s_{11}^{(2)} + 6s_{111}^{(2)} + 4s_{1111}^{(2)} + s_{11111}^{(2)} + 3s_{21}^{(2)} + 2s_{21}^{(2)} + 3s_{211}^{(2)}$ |
| 4   | 1          | $s_1$                                       | $s_1^{(3)}$                       |
|     | 2          | $s_2$                                       | $s_2^{(3)}$                       |
|     | 11         | $s_1 + s_{11}$                              | $s_1^{(3)} + s_{11}^{(3)}$        |
|     | 3          | $s_3$                                       | $s_3^{(3)}$                       |
|     | 21         | $s_2 + s_{21}$                              | $s_2^{(3)} + s_{21}^{(3)}$        |
|     | 111        | $s_1 + 2s_{11} + s_{111}$                  | $s_1^{(3)} + 2s_{11}^{(3)} + s_{111}^{(3)}$ |
|     | 31         | $s_3 + s_{31} + s_4$                        | $s_3^{(3)} + s_{31}^{(3)}$        |
|     | 22         | $s_2 + s_{21} + s_{22}$                    | $s_2^{(3)} + s_{21}^{(3)} + s_{22}^{(3)}$ |
|     | 211        | $s_2 + 2s_{21} + s_{211} + s_3 + s_{31}$   | $s_2^{(3)} + 2s_{21}^{(3)} + s_{211}^{(3)} + s_3^{(3)}$ |
|     | 1111       | $s_1 + 3s_{11} + 3s_{111} + s_{1111} + s_2 + 2s_{21} + s_{211}$ | $s_1^{(3)} + 3s_{11}^{(3)} + 3s_{111}^{(3)} + s_{1111}^{(3)} + s_2^{(3)} + 2s_{21}^{(3)}$ |
**Table for the coproduct of** $g^{(k)}_\lambda$. The following table gives $\Delta(g^{(k)}_\lambda) = \sum_{\nu,\mu} c^{\nu\mu}_\lambda g^{(k)}_\nu \otimes g^{(k)}_\mu$, where we suppress the superscript `(k)' and write simply $g_\nu \otimes g_\mu$ for $g^{(k)}_\nu \otimes g^{(k)}_\mu$, with $\nu$ and $\mu$ being $k$-bounded partitions.

| $n$ | $\lambda$ | $\Delta(g^{(n-1)}_\lambda)$ |
|-----|------------|-------------------------------|
| 2   | 1          | $g_1 \otimes g_0 + g_0 \otimes g_1$ |
|     | 11         | $g_{11} \otimes g_0 + 2g_1 \otimes g_1 + g_0 \otimes g_{11}$ |
|     | 111        | $g_{111} \otimes g_0 + 3g_{11} \otimes g_1 + 3g_1 \otimes g_{11} + g_0 \otimes g_{111} - 2g_1 \otimes g_1$ |
|     | 1111       | $g_{1111} \otimes g_0 + 4g_{111} \otimes g_1 + 6g_{11} \otimes g_{111} + 4g_1 \otimes g_{1111} + g_0 \otimes g_{11111} - 5g_{11} \otimes g_1 - 5g_1 \otimes g_{111} + 2g_1 \otimes g_1$ |
|     | 11111      | $g_{11111} \otimes g_0 + 5g_{1111} \otimes g_1 + 10g_{111} \otimes g_{111} + 10g_{11} \otimes g_{1111} + 5g_1 \otimes g_{11111} + g_0 \otimes g_{111111} - 9g_{111} \otimes g_{111} - 9g_1 \otimes g_{11111} + 7g_{111} \otimes g_1 + 7g_1 \otimes g_{111} - 2g_1 \otimes g_1$ |
| 3   | 1          | $g_1 \otimes g_0 + g_0 \otimes g_1$ |
|     | 22         | $g_2 \otimes g_0 + g_1 \otimes g_1 + g_0 \otimes g_2$ |
|     | 11         | $g_{11} \otimes g_0 + g_1 \otimes g_1 + g_0 \otimes g_{11}$ |
|     | 21         | $g_{21} \otimes g_0 + g_{11} \otimes g_1 + 2g_2 \otimes g_1 + g_1 \otimes g_{11} + 2g_1 \otimes g_2 + g_0 \otimes g_{21} - g_1 \otimes g_1$ |
|     | 111        | $g_{111} \otimes g_0 + 2g_{11} \otimes g_1 + g_2 \otimes g_1 + 2g_1 \otimes g_{111} + g_0 \otimes g_{1111} + g_1 \otimes g_2 - g_1 \otimes g_1$ |
|     | 22         | $g_{22} \otimes g_0 + 2g_{21} \otimes g_1 + g_{11} \otimes g_{11} + g_2 \otimes g_{111} + g_{11} \otimes g_2 + 3g_2 \otimes g_2 + 2g_1 \otimes g_{21} + g_0 \otimes g_{22} - g_2 \otimes g_1 - g_1 \otimes g_2$ |
|     | 211        | $g_{211} \otimes g_0 + g_{111} \otimes g_{11} + g_{21} \otimes g_{11} + g_{21} \otimes g_{111} + 2g_2 \otimes g_{111} + g_1 \otimes g_{1111} + 2g_1 \otimes g_{211} + g_0 \otimes g_{2111} - 2g_{11} \otimes g_1 - 2g_2 \otimes g_1 - 2g_1 \otimes g_{111} + 2g_2 \otimes g_1$ |
|     | 1111       | $g_{1111} \otimes g_0 + 2g_{111} \otimes g_{11} + 2g_{11} \otimes g_{111} + 2g_1 \otimes g_{1111} + g_0 \otimes g_{11111} + g_1 \otimes g_2 - g_1 \otimes g_1$ |
| 4   | 1          | $g_1 \otimes g_0 + g_0 \otimes g_1$ |
|     | 2          | $g_2 \otimes g_0 + g_1 \otimes g_1 + g_0 \otimes g_2$ |
|     | 11         | $g_{11} \otimes g_0 + g_1 \otimes g_1 + g_0 \otimes g_{11}$ |
|     | 3          | $g_3 \otimes g_0 + g_2 \otimes g_1 + g_2 \otimes g_1 + g_0 \otimes g_3$ |
|     | 21         | $g_{21} \otimes g_0 + g_{11} \otimes g_1 + g_2 \otimes g_1 + g_1 \otimes g_{111} + g_1 \otimes g_2 + g_0 \otimes g_{21} - g_1 \otimes g_1$ |
|     | 111        | $g_{111} \otimes g_0 + g_{111} \otimes g_1 + g_1 \otimes g_{111} + g_0 \otimes g_{1111}$ |
|     | 31         | $g_{31} \otimes g_0 + g_{21} \otimes g_1 + 2g_3 \otimes g_1 + g_2 \otimes g_{111} + g_{11} \otimes g_2 + 2g_2 \otimes g_2 + g_1 \otimes g_{21} + 2g_1 \otimes g_3 + g_0 \otimes g_{31} - g_2 \otimes g_1 - g_1 \otimes g_2$ |
|     | 22         | $g_{22} \otimes g_0 + g_{21} \otimes g_1 + g_{11} \otimes g_{11} + g_2 \otimes g_{21} + g_0 \otimes g_{22}$ |
|     | 211        | $g_{211} \otimes g_0 + g_{111} \otimes g_{11} + 2g_{21} \otimes g_1 + g_3 \otimes g_1 + g_{11} \otimes g_{111} + 2g_2 \otimes g_{111} + g_1 \otimes g_{1111} + g_1 \otimes g_3 - g_{11} \otimes g_1 - g_2 \otimes g_1 - g_1 \otimes g_2 - g_1 \otimes g_{111}$ |
|     | 1111       | $g_{1111} \otimes g_0 + 2g_{1111} \otimes g_1 + g_{211} \otimes g_1 + 2g_{111} \otimes g_{111} + g_2 \otimes g_{1111} + 2g_1 \otimes g_{1111} + g_0 \otimes g_{11111} + g_1 \otimes g_{11111} + g_1 \otimes g_2 - g_1 \otimes g_1$ |

849
A.3.6 *Table of $G_{\lambda}^{(k)}$. We have suppressed the superscript ‘$(k)$’ on $F_{\mu}^{(k)}$ in the following table.*

| $n$ | $\lambda$ | $G_{\lambda}^{(n-1)}$ in terms of $s_{\lambda}$ | $G_{\lambda}^{(n-1)}$ in terms of $F_{\mu}^{(n-1)}$ |
|-----|-----------|--------------------------------------|--------------------------------------|
| 2   | 1         | $s_1 - s_{12} + s_{13} - s_{14} + s_{15} - s_{16} \pm \cdots$ | $F_{1} - F_{12} + F_{13} - F_{14}$  + $F_{15} - F_{16} \pm \cdots$ |
|     |           | $s_{12} - 2s_{13} + 3s_{14} - 4s_{15} + 5s_{16} - 6s_{17} + \cdots$ | $F_{12} - 2F_{13} + 3F_{14} - 4F_{15} + 5F_{16} - 6F_{17} + \cdots$ |
| 11  |           | $s_{13} - 3s_{14} + 6s_{15} - 10s_{16} + 15s_{17} - 21s_{18} + \cdots$ | $F_{13} - 3F_{14} + 6F_{15} - 10F_{16} + 15F_{17} - 21F_{18} + \cdots$ |
| 11  |           | $s_{14} - 4s_{15} + 10s_{16} - 20s_{17} + 35s_{18} - 56s_{19} + \cdots$ | $F_{14} - 4F_{15} + 10F_{16} - 20F_{17} + 35F_{18} - 56F_{19} + \cdots$ |
| 2   | 1         | $s_1 - s_{12} + s_{13} - s_{14} + s_{15} - s_{16} \pm \cdots$ | $F_{1} - F_{12} + F_{13} - F_{14}$  + $F_{15} - F_{16} \pm \cdots$ |
| 2   |           | $s_2 - s_{21} + s_{211} - s_{2111} + s_{21111} - s_{211111} \pm \cdots$ | $F_{2} - F_{21} + F_{211} - F_{2111} + F_{21111} - F_{211111} \pm \cdots$ |
| 11  |           | $s_{12} - 2s_{13} + 3s_{14} - 4s_{15} + 5s_{16} - 6s_{17} \pm \cdots$ | $F_{12} - 2F_{13} + 3F_{14} - 4F_{15} \pm \cdots$ |
|     |           | $s_{13} - 3s_{14} + 6s_{15} - 10s_{16} + 15s_{17} - 21s_{18} \pm \cdots$ | $F_{13} - 3F_{14} + 6F_{15} - 10F_{16} \pm \cdots$ |
| 21  |           | $s_{14} - 2s_{15} + s_{22} - 2s_{23} + s_{213} - 3s_{216}$  $s_{221} - 2s_{2213} - 2s_{2214} - 2s_{2215} + s_{2216} - 5s_{2217} + 6s_{2218} - 8s_{2219} \pm \cdots$ | $F_{22} - 2F_{221} + 3F_{2213} + F_{2214}$  $s_{2215} - 3F_{2216} + 4F_{2217} - 6F_{2218} \pm \cdots$ |
| 111 |           | $s_{13} - 3s_{14} + 6s_{15} - 10s_{16} + 15s_{17} - 21s_{18} \pm \cdots$ | $F_{13} - 3F_{14} + 6F_{15} - 10F_{16} + 15F_{17} - 21F_{18} \pm \cdots$ |
| 22  |           | $s_{14} - 2s_{15} + s_{22} - 2s_{23} + s_{213} - 3s_{216} + s_{222} - 2s_{2212} - 2s_{2214} + 4s_{2217} + 3s_{2218} - s_{2219} + s_{2221} - 2s_{2223} + 4s_{2224} + s_{2225} + s_{2226} - 5s_{2227} + 6s_{2228} - 8s_{2229} \pm \cdots$ | $F_{22} - 2F_{221} + 3F_{2213} + F_{2214}$  $s_{2215} - 3F_{2216} + 4F_{2217} - 6F_{2218} \pm \cdots$ |
| 211 |           | $s_{13} - 3s_{14} + 6s_{15} - 10s_{16} - 15s_{17} - 20s_{18} + 35s_{19} - 56s_{20} \pm \cdots$ | $F_{211} - 3F_{212} + 3F_{213} - 3F_{215} + F_{22} - 2F_{221} + 3F_{2213} + F_{2214}$  $s_{2215} - 3F_{2216} + 4F_{2217} - 6F_{2218} \pm \cdots$ |
| 111 |           | $s_{14} - 4s_{15} + 10s_{16} - 20s_{17} + 35s_{18} - 56s_{19} \pm \cdots$ | $F_{221} - 2F_{221} - 3F_{2213} + F_{2214}$  $s_{2215} - 3F_{2216} + 4F_{2217} - 6F_{2218} \pm \cdots$ |
| 221 |           | $s_{22} - 2s_{23} + s_{2212} + 3s_{2214} + 3s_{2215} - 5s_{2216} - 6s_{2217} - 8s_{2218} + 10s_{2219} + 3s_{2221} + s_{2224} - 15s_{2215} - 8s_{2216} - 3s_{2217} - 2s_{2218} + 3s_{2219} \pm \cdots$ | $F_{221} - 3F_{2212} + 3F_{2213} + F_{2214} + 3F_{2215} + F_{2216} \pm \cdots$ |
### K-theory Schubert calculus of the affine Grassmannian

\[
\begin{align*}
2111 & \quad -2s_{15} + s_{213} + 7s_{16} - s_{2212} - 2s_{214} - 16s_{17} \quad F_{211} - F_{2211} - 3F_{214} + F_{251} \\
& \quad + 3s_{215} - s_{21} + 2s_{23} - 4s_{216} + 30s_{18} \\
& \quad - 3s_{214} - s_{24} - 2s_{212} + 4s_{215} + 5s_{217} \\
& \quad - 50s_{19} + 3s_{213} + 2s_{241} + 77s_{110} - s_{25} \\
& \quad - 3s_{242} - 14s_{214} - 5s_{216} - 6s_{218} \pm \cdots \\
1111 & \quad s_{15} - 5s_{16} + 15s_{17} - 35s_{18} + 70s_{19} \quad F_{15} - 5F_{16} + 15F_{17} - 35F_{18} \pm \cdots \\
& \quad - 126s_{110} \pm \cdots
\end{align*}
\]

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851
T. Lam, A. Schilling and M. Shimozono

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