Combinatorial Approach to the Ground-State Energy of Square and Triangular ±J Spin Glasses

Pawel Polaszek *

Technische Universität Dresden, Institut für Theoretische Physik, D-01062 Dresden, Germany

A new combinatorial, analytical approach to the ground-state energy problem of spin glasses with different concentrations of ±J interactions is developed. The energy \( e_0 \) is expressed in terms of the fraction of broken bonds \( \mu_0 \) and expanded into a fast converging series of the average length of a segment between two frustrated plaquettes of a minimum-weighted perfect matching \( \Lambda \). The concept of so called diagram stability \( s \) is introduced in order to calculate coefficients of this expansion. Finally, the fraction \( \mu_0 \) as a function of the concentration \( p \) of negative bonds is obtained for triangular and square infinite lattices in excellent accordance with numerical simulations of large adequate systems.

I. INTRODUCTION

The Ising model on a regular lattice with ±J interactions between nearest neighbours is one of the simplest which is able to describe some physical features of spin glasses appearing in nature. Even for such "binary" interactions there is no analytical approach to the ground-state energy apart from unfrustrated Mattis spin glasses [2,3] which can be transformed via gauge transformation into a ferromagnetic system.

Exact, numerical methods aimed at finding the ground-state energy can only be applied to relatively small systems (as, e.g., the branch and bound method [8]). Approximate methods are used for larger systems (as, e.g., matching algorithms [4,5]). However, the numerical effort increases at least polynomially with system size [6]. Thus the precision of an extrapolation to the thermodynamic limit remains unknown. Therefore, any analytical or semi-analytical approach to the problem is worth the effort. The purpose of the present paper is to propose such an approach and to apply it to the square and triangular lattices in order to find the fraction of unsatisfied bonds in the ground state of an Ising system with randomly mixed ±J bonds.

The immediate stimulation has come from the article of Liebmann and Schuster [7] who considered the average minimal distance \( \bar{d}(p) \) between a frustrated plaquette and the nearest other frustrated plaquette in the square lattice. They used the following expansion:

\[
\bar{\mu}(p) = \sum_{i=0}^{\infty} C_i(p) ,
\]

where \( C_i \) is the conditional probability, namely the probability that a plaquette is frustrated and all surrounding plaquettes up to the distance \( d = i \) are not frustrated divided by the concentration of frustrated plaquettes. The authors’ main finding was that the parameter \( \bar{\mu}(p) \) represent a continuous function of the concentration \( p \) of negative bonds. They stated that series (1) converged rapidly and found that an excellent approximation for \( \bar{d}(p) \) was obtained using only the first three (or even two) terms. This allows one to hope that a similar approximation should also work for an analogous series expansion of the energy.

The parameter \( \bar{\mu}(p) \) was also investigated by Bendisch [8]. He found that the maximum of \( \bar{\mu}(p) \) at \( p \approx 0.10 \) (\( p \approx 0.15 \)) for the square (triangular) lattice reflects an inflection point in the ground-state magnetisation.

In the present work, a new method of combinatorial approach to a minimum-weighted perfect matching of the graph of frustrated plaquettes has been developed in order to obtain the expansion of the ground-state energy as a function of the concentration of negative bonds.

II. BASIC DEFINITIONS

The Ising Hamiltonian without external magnetic field is given by:

\[
H = -\sum_{i,j} J_{ij} S_i S_j .
\]

The interaction between the \( i \)-th and \( j \)-th spins has a ferromagnetic character for \( J_{ij} > 0 \) and otherwise an antiferromagnetic one. We will use the term of a positive (negative) bond interchangeably with the term of a ferromagnetic (antiferromagnetic) interaction. We restrict ourselves to the following probability distribution of \( J_{ij} \):

\[
f(J_{ij}) = (1 - p)\delta(J_{ij} - 1) + p\delta(J_{ij} + 1)
\]

for nearest neighbours (otherwise \( f(J_{ij}) = 0 \)). The parameter \( p \in [0,1] \) has a clear interpretation as the concentration of negative bonds or the probability that an interaction is antiferromagnetic. We will consider the two dimensional triangular and square lattices (coordination numbers 6 and 4, respectively).

The unit cell of the 2D lattice will be called a plaquette. The term of frustration is reserved for systems: the
smallest system is a plaquette, so we will say that a plaquette is frustrated (unfrustrated) when an odd (even) number of plaquette bonds is negative. This term is not adequate for bonds because there always exist two configurations of connected spins which locally minimize the energy. Thus, we will say that a bond is unsatisfied or broken when the configuration of connected spins does not minimize the energy locally.

A useful formulation of the ground-state problem can be obtained using the duality concept [6], [9], as a problem of finding a minimum-weighted perfect matching in the graph of frustrated plaquettes. Thus, the ground-state energy can be expressed as:

$$E_0 = -\frac{Nc}{2} + 2w_0 ,$$

where $N$ is the number of spins, $c$ is the coordination number and $w_0$ is the weight of a minimum-weighted perfect matching which equals to the total number of broken bonds in the real lattice. The ground-state energy per bond $e_0$ reads:

$$e_0 = -1 + 2\mu_0 ,$$

where the parameter

$$\mu_0 = \frac{w_0}{Nc/2}$$

has a clear interpretation as the fraction of bonds unsatisfied in a ground state. The parameter $\mu_0$ admits an interesting generalization in [10] where the so called “misfit parameter” is defined as:

$$\mu_0 = \frac{E_0 - E_{id\min}}{E_{id\max} - E_{id\min}} .$$

$E_{id\min}$ and $E_{id\max}$ are minimal and maximal energies, respectively, of an adequate “ideal” system which is analogous to the original one but it is not frustrated. The advantage of definition (8) is that it allows a comparison of frustrated systems of different nature. In our case, the ”ideal” system is a Mattis spin glass, thus $E_{id\min} = -Nc$, $E_{id\max} = -E_{id\max}$ and simple algebra leads again to formula (8).

The notion of the fraction of bonds broken in a ground state appeared in the literature earlier [11]. For convenience justified by expression (5) for $e_0$ as a linear function of $\mu_0$, from now on we will be concerned with calculating the fraction $\mu_0$, i.e., the misfit parameter. Also final results will be presented in terms of this quantity.

III. SERIES EXPANSION OF THE MISFIT PARAMETER

The weight of a perfect matching in a ground state is equal to the total sum of lengths of segments each of them connecting a pair of frustrated plaquettes:

$$w_0 = \sum_{\Lambda=1}^{\infty} \Lambda N\Lambda ,$$

where $N\Lambda$ is the number of segments of length $\Lambda$ or

$$w_0 = N_f \overline{\Lambda} ,$$

with $N_f$ as the number of frustrated plaquettes and $\overline{\Lambda}$ as the average segment length:

$$\overline{\Lambda} = \sum_{\Lambda=1}^{\infty} \Lambda \rho(\Lambda) .$$

$P(\Lambda)$ is the probability that a randomly chosen segment has a length $\Lambda$. Clearly:

$$P(\Lambda) = \frac{N\Lambda}{\sum_{N=1}^{\infty} N\Lambda}$$

and $N_f/2 = \sum N\Lambda$, so formula (8) holds. The misfit parameter can be written:

$$\mu_0 = \frac{N_f}{Nc} \overline{\Lambda} .$$

Elementary, geometrical considerations lead to a relation between the coordination number and the total number of plaquettes in a regular lattice (neglecting boundary effects):

$$N_f = \frac{N(c-2)}{2} P_F(p) ,$$

where $P_F(p)$ is the probability that a plaquette is frustrated in the system with a concentration $p$ of negative bonds. From now on, we will omit in the notation $P_F(p)$ the explicit dependence on $p$, and we will write just $P_F$. Finally, we obtain a useful formula:

$$\mu_0 = \frac{c-2}{2c} P_F .$$

At this point, we stress that the statistics $P(\Lambda)$ depends on the choice of one of the ground states. For example, in Fig. 4 are presented two possible minimum-weighted perfect matchings: one of them provides two segments of length 2 and the other one gives one segment of length 1 and one segment of length 3. Thus, there is no universal distribution of $P(\Lambda)$, but this distribution may vary from one ground state to another. The common feature of these distributions is that they must give the same mean length of segments $\overline{\Lambda}$. To keep symbols as simple as possible we will not introduce a new index like $P_m(\Lambda)$, but the reader should keep in mind that calculations are performed for a specific ground state. The choice of this ground state will be discussed later.
IV. Method

Referring to the discussion in section [3], we restrict ourselves to segments with lengths 1, 2 or 3. Hence, approximately, \( P(\Lambda \geq 4) = 0 \). It follows:

\[
\Xi = \frac{3P(3) + 2P(2) + P(1)}{P(3) + P(2) + P(1)}
\]

or

\[
\Xi = \frac{3R_{i,2}R_{j,1} + 2R_{i,1} + 1}{R_{i,2}R_{j,1} + R_{i,1} + 1}
\]

where \( R_{i,j} = P(i)/P(j) \).

To perform calculations we choose a specific ground state, namely a state with as few as possible matching intersections. Fig. [4] illustrates the rule of thumb for such a choice (sometimes it is not possible to avoid intersections - Fig. [5]). Next we perform a transformation of the system such that in the transformed system only broken bonds are negative (Fig. [6]). This transformation is defined as the mapping: \( J_{ij} \rightarrow |J_{ij}| \) for satisfied bonds and \( J_{ij} \rightarrow -|J_{ij}| \) for broken bonds. This mapping does not change the configuration of frustration of the system. Thus, both systems are equivalent with respect to the gauge symmetry in the meaning of [12]. In particular, they have the same ground-state energy. Furthermore, in the transformed system the concentration of negative bonds is equal to the fraction \( \mu_0 \) of bonds broken in the ground state, i.e., to the misfit parameter.

A. \( R_{2,1} \) for the triangular lattice

In Fig. [7], we show the three diagrams which contain segments of length 1 and the six diagrams with segments of length 2. In the present approximation, diagrams with matching intersections, such as shown in Fig. [8]; are omitted. This is justified by our choice of the ground state with as few intersections as possible. Because of the rotational symmetry between diagrams considered one can write:

\[
R_{2,1} = \frac{6}{3} \frac{n_{2a}}{n_{1a}} = \frac{2n_{2a}}{n_{1a}}
\]

where \( n_{1a} \) and \( n_{2a} \) are the probabilities of finding diagrams of type 1a and 2a, respectively, in the lattice.

To calculate the ratio \( n_{2a}/n_{1a} \) we add a plaquette to the diagram 1a in order to have the same number of plaquettes and bonds as in the diagram 2a. This additional plaquette may be either frustrated or not and we have three new diagrams: 1a’, 1a”, 1a”’ (Fig. [9]). If the lattice were a set of isolated diagrams (which is not the case) like diagrams 1a’, 1a”, 1a”’ and 2a, we could write:

\[
\frac{n_{2a}}{n_{1a}} = \frac{P_F^2(1 - P_F)\mu_0^2(1 - \mu_0)}{P_F^4(1 - P_F)\mu_0(1 - \mu_0)^3} = \frac{\mu_0}{1 - \mu_0}
\]

The equations above contain information on isolated diagrams. Diagrams 1a’ (or 1a”’), and 2a have the same number of frustrated and unfrustrated plaquettes but different numbers of satisfied and broken bonds. Diagrams 1a” and 2a have the same number of satisfied and broken bonds but differ in the numbers of frustrated and unfrustrated plaquettes. All these diagrams have the same total number of plaquettes and bonds. Hence, for isolated diagrams (i.e., independent of any influence of the outer part of the infinite lattice) [13] would hold. However, the diagrams are situated inside the lattice and feel the presence of other frustrations. Therefore, for diagrams within the lattice, we introduce a new parameter, called stability \( s_\alpha \) of diagram \( \alpha \), such that:

\[
\frac{n_{2a}}{n_{1a}''} = \frac{\mu_0 s_{2a}}{(1 - \mu_0) s_{1a}'}, \quad \frac{n_{2a}}{n_{1a}'''} = \frac{1}{\mu_0}, \quad \frac{n_{2a}}{n_{1a}'} = \frac{P_F^2(1 - P_F)\mu_0^2(1 - \mu_0)^5}{P_F^4(1 - P_F)\mu_0(1 - \mu_0)^4} = \frac{\mu_0}{1 - \mu_0}
\]

What is the meaning of the stability parameter? It measures the probability of the occurrence of such a frustration in the outer part of the lattice which destroys the matching of a diagram, i.e., some of the segments vanish and others appear. Fig. [10] shows an example of a destruction of the matching in diagram 2a. Destruction of the matching appears if a loop exists containing a given diagram and for which an alternative matching is energetically more profitable. Accordingly, the stability parameter can be expressed as a loop series:

\[
s_{\text{diag.}} = 1 - \sum_{\text{loops}} D_{\text{loop}},
\]

where the sum goes over all loops containing a given diagram and \( D_{\text{loop}} \) is the probability that there appears a frustration destroying the matching of a diagram inside a loop.

Because a segment of length 1 is energetically the most profitable one, we admit:

\[
s_{1a'} = s_{1a''} = s_{1a'''} \approx 1.
\]

For \( s_{2a} \) we will consider only the first order loops in the expansion [24]. We are left with one type of smallest loops with six plaquettes (Fig. [11]). Frustration of plaquettes belonging to the diagram is fixed. Furthermore there are only two configurations of frustration of the remaining plaquettes which destroy the matching of diagram 2a. One of these configurations contains two frustrated and one unfrustrated plaquette and the other contains
three frustrated plaquettes. Thus, the contribution to the stability $s_{2a}$ up to first order is:

$$D_6 = \frac{1}{3}(1 - P_F)p_F^2 + p_F^3 = \frac{1}{3}p_F^2 + \frac{2}{3}p_F^3$$

and

$$s_{2a} \approx 1 - \frac{1}{3}p_F^2 - \frac{2}{3}p_F^3 .$$

Combining (17), (19) and (22) we obtain:

$$R_{2,1} = 2 \left( \frac{n_{1a'}}{n_{2a}} \right)^{-1} = 2 \left( \frac{1 - \mu_0}{\mu_0 s_{2a}} + 2 \frac{P_F}{(1 - P_F) s_{2a}} + \frac{\mu_0}{\mu_0} \right)^{-1} = \frac{2\mu_0(1 - \mu_0)(1 - P_F)s_{2a}}{(1 - \mu_0)^2(1 - P_F) + 2\mu_0(1 - \mu_0)P_F + \mu_0^2(1 - P_F)} ,$$

with $s_{2a}$ given by (23).

**B. $R_{3,2}$ for the triangular lattice**

There are six diagrams which contain segments of length 2 and twelve diagrams with segments of length 3 (Fig. 5). As before, we omit diagrams with matching intersections. A practical approximation is made:

$$R_{3,2} \approx \frac{12 n_{3a}}{6 n_{2a}} \approx \frac{n_{2a}}{n_{1a}} = R_{2,1} .$$

**C. Final formula for the triangular lattice**

Using (14), (16) and (22) we get:

$$2c\mu_0 - (c - 2)P_F \frac{3R_{2,1}^2 + 2R_{2,1} + 1}{R_{2,1}^2 + R_{2,1} + 1} = 0 ,$$

where for the triangular lattice the probability that the plaquette is frustrated amounts to:

$$P_F = p(4p^2 - 6p + 3) .$$

The last two equations combined with (22) for $R_{2,1}$ and (23) for $s_{2a}$ make it possible to calculate the misfit parameter $\mu_0$. To solve this system of equations numerically a simple bisection algorithm was applied. Results are presented in Fig. 10 together with the data of simulations of large samples [13, 14].

**D. $R_{2,1}$ for the square lattice**

As in the triangular lattice, we choose the ground state with as few matching intersections as possible and again we perform a gauge transformation such that in the transformed system only broken bonds are negative.

There are two diagrams which contain segments of length 1 and six diagrams which contain segments of length 2 (Fig. 7). As before, we omit diagrams with matching intersections in accordance with the foregoing choice of the ground state. Thus, for the square lattice we obtain a formula analogous to (17):

$$R_{2,1} = \frac{4 n_{2a}}{2 n_{1a}} + \frac{2 n_{2c}}{2 n_{1a}} \approx \frac{3 n_{2a}}{n_{1a}} .$$

However, not all segments of length 2 have the same symmetry in the square lattice (Fig. 5, diagrams 2a and 2c), so unlike (17), eq. (28) is only an approximation. We add a plaquette to the diagram 1a to have the same number of plaquettes and bonds as in diagram 2a. We have three new diagrams: 1a', 1a'' and 1a''' (Fig. 8). Applying the concept of the diagram stability for the square lattice we get an equation identical to (19). As before, the approximation (21) is made, since segments of length 1 are energetically the most profitable.

Now, the first-order loop expansion (20) for $s_{2a}$ for the square lattice will be considered. We have two smallest loops with six plaquettes containing diagram 2a (Fig. 8). The contributions from these two loops are:

$$D_6 \approx \frac{2}{3}(1 - P_F)p_F^2 + 2p_F^3 = \frac{2}{3}P_F + \frac{4}{3}p_F^3 .$$

Thus, for the square lattice we have:

$$s_{2a} \approx 1 - \frac{2}{3}p_F^2 - \frac{4}{3}p_F^3 .$$

Formally, 0 $\leq P_F \leq 1$ as $P_F$ is a probability but for the square lattice $P_F \leq 0.5$, so there is no danger that the above approximation leads to an unphysical, negative stability. Combining (24), (25) and (31) we obtain:

$$R_{2,1} = \frac{3\mu_0}{\mu_0 s_{2a}} + 2 \frac{P_F}{(1 - P_F) s_{2a}} + 3 \frac{\mu_0}{\mu_0} \frac{1 - \mu_0}{(1 - P_F)s_{2a}} \frac{3\mu_0(1 - \mu_0)(1 - P_F)}{3\mu_0(1 - \mu_0)(1 - P_F)s_{2a}} ,$$

with $s_{2a}$ given by (23).

**E. $R_{3,2}$ for the square lattice**

There are six diagrams which contain segments of length 2 and fourteen diagrams with segments of length 3
We omit diagrams with matching intersections, as before, and make a practical approximation is made:

\[ R_{3,2} \approx \frac{14}{6} \frac{n_{3a}}{n_{2a}} \approx \frac{7}{3} \frac{n_{2a}}{n_{1a}} = \frac{7}{9} R_{2,1} \ . \quad (32) \]

**F. Final formula for the square lattice**

Combining (14), (16) and (32) we get:

\[ 2c\mu_0 - (c - 2)P_F \frac{7}{9} \frac{R_{2,1}}{R_{2,1}^2} + 2R_{2,1} + 1 = 0 \ , \quad (33) \]

where the probability that the plaquette is frustrated in the square lattice is:

\[ P_F = 4p(1 - p) \left( p^2 + (1 - p)^2 \right) . \quad (34) \]

The last two equations when combined with (31) for \( R_{2,1} \) and (30) for \( s_{2a} \) make it possible to calculate the misfit parameter \( \mu_0 \) numerically. Results are presented in Fig. 10b together with the data of simulations of large samples [13,14].

**V. CONCLUSIONS**

The ground-state energy \( e_0 \) per bond is a simple linear function (5) of the misfit parameter \( \mu_0 \) and the present approach provides values of the misfit parameter in the ground state of mixed Ising ferro-antiferro magnets in excellent agreement with numerical experiments [13] for large triangular and square lattices. This proves the effectiveness of the method and justifies a posteriori the approximations made. The results for the boundary values of the concentration \( p \) of negative bonds, e.g., for \( p = 0 \) or \( p = 1 \) are equal to the well known exact solutions in these cases. In fact, for the square lattice in both cases \( P_F = 0 \) and because \( R_{2,1} \) given by (31) has no singularities the only solution of (33) is \( \mu_0 = 0 \).

The situation for the triangular lattice is the same for \( p = 0 \). On the contrary, for \( p \) tending to unity \( P_F \) tends to 1 and (23) with (24) lead to \( R_{2,1} \to 0 \) and \( \mu_0 \to 1/3 \). The plaquettes in the triangular lattice contain an odd number of bonds. Thus, in the antiferromagnetic limit the system becomes a fully frustrated one and elementary considerations prove that indeed one third of the bonds must be broken in any ground state (\( \mu_0 = 1/3 \)).

The case of \( p = 0.5 \) is under particular considerations of many authors. Thus, the solutions for this concentration of negative bonds are specified: \( \mu_0^a = 0.2218 \) and \( \mu_0^s = 0.1546 \).

Each step of the foregoing approximations can be developed relatively easily up to the second and further orders of precision. The method is open to such improvements. At this point, one of possible directions of the development should be mentioned which is somewhat less evident. We approximated the probability of finding a diagram or a loop by the formula \( (1 - P_F)N_u P_F N_f \), where \( N_f \) is the number of frustrated plaquettes and \( N_u \) is the number of those unfrustrated plaquettes out of which a given diagram is built. It is not quite correct and hopefully could be replaced by an exact formula in future calculations.

The method can be applied to any two dimensional regular lattice. An extension on the case of three dimensional cubic lattice is under the consideration.

**VI. ACKNOWLEDGEMENTS**

The author would like to thank Dresden research group: S. Kobe, T. Klotz and J. Weissbarth for fruitful discussions, advices and critical reading of the manuscript as well as for the warm hospitality extended to him when in Dresden. Special thanks are due to J. Bendisch for the permission of comparison with and of publishing the results of his numerical simulations. The author acknowledges also A. R. Ferchmin for the critical reading of the manuscript.

The work was supported by the DFG (project no. Ko 1416).
1 D. C. Mattis, Phys. Lett. A 56 421 (1976).
2 S. Kobe and A. Hartwig, Comp. Phys. Comm. 16 1 (1978); A. Hartwig, F. Daske and S. Kobe, Comp. Phys. Comm. 32 133 (1984).
3 M. Grötschel, M. Jünger and G. Reinelt, Heidelberg Colloquium on Glassy Dynamics edited by J. L. van Hemmen and I. Morgenstern, Lecture Notes in Physics 275 325 (1987).
4 M. Achilles, J. Bendisch, K. Cassirer and H. von Trotha, Gesellschaft für Mathematik und Datenverarbeitung, D-5205 St. Augustin, Germany, GMD-Studies 186 (1991).
5 H. Freund and P. Grassberger, J. Phys. A: Math. Gen. 22 4045 (1989).
6 F. Barahona, J. Phys. A: Math. Gen. 15 3241 (1982).
7 R. Liebmann and H. G. Schuster, J. Phys. C: Solid State Phys. 14 709 (1981).
8 J. Bendisch, Phys. A 202 48 (1994).
9 I. Bieche, R. Maynard, R. Rammal and J. P. Uhry, J. Phys. A: Math. Gen. 13 2553 (1980).
10 S. Kobe and T. Klotz, Phys. Rev. E 52 (1995).
11 D. L. Stein, G. Baskaran, S. Liang and M. N. Barber, Phys. Rev. B 36 5567 (1987).
12 R. Balian, J. M. Drouffe and C. Itzykson, Phys. Rev. D 11 2098 (1975).
13 J. Bendisch, Unpublished data obtained from numerical simulations of up to 200 spins (matching algorithm), private communication.
14 W. Lebrecht and E. E. Vogel, Average values of 500 samples for each concentration of negative bonds for square and triangular lattices of size 6 × 8 and 6 × 6, respectively (by enumerating the lower-energy portion of the configuration space), private communication.

FIG. 1. Two different ground states contributing to different probability distributions $P(\Lambda)$.

FIG. 2. a) One specific ground state without a matching intersection of segments representing the basis of the method. b) A ground state with a matching intersection that can be transformed to a) via an inversion of the proper spin (marked with ↓). c) A configuration of frustrated plaquettes necessitating intersection (disregarded in this work).

FIG. 3. The gauge transformation bound to a given ground state leading from the initial lattice (a) to the lattice with only broken bonds being negative (b).

FIG. 4. Diagrams in the triangular lattice taken into account during considerations. Diagrams with matching intersections were not considered.

FIG. 5. Diagrams 1a’, 1a”, 1a”’ and 2a taken into account in the calculation of the parameter $R_{2,1}$ of the triangular lattice. Note that there are two possibilities for diagram 1a”.

FIG. 6. Two frustration configurations inside the smallest loop destroying the matching of diagram 2a of the triangular lattice. Solid lines represent new segments, crossed lines - destroyed segments.

FIG. 7. Diagrams in the square lattice taken into account. Diagrams with matching intersections were not considered.

FIG. 8. Diagrams 1a’, 1a”, 1a”’ and 2a taken into account in the calculation of the parameter $R_{2,1}$ of the square lattice. Note that there are three possibilities for diagram 1a” and also three possibilities for diagram 1a”’.

FIG. 9. The two smallest loops containing the diagram 2a in the square lattice which contribute to the stability calculation.

FIG. 10. Misfit parameter $\mu_0$ as a function of the concentration $p$ of negative bonds for the triangular lattice (a) and for the square lattice (b), obtained in this work. Crosses denote results obtained by numerical simulations of large samples (up to 200 spins) [13]. Rhombs denote average values over 500 samples for each concentration of negative bonds for square and triangular lattices of size $6 \times 8$ and $6 \times 6$, respectively [14].
