Abstract

Objectives: In this work, the approximate solution of non-linear third order Korteweg-de Vries equation has been studied.

Methods: The proposed numerical technique engages finite difference formulation for temporal discretization, whereas, the discretization in space direction is achieved by means of a new cubic B-spline approximation.

Findings: In order to corroborate this effort, three test problems have been considered and the computational outcomes are compared with the current methods. It is found that the proposed scheme involves straightforward computations and operates superior to the existing methods.

Novelty/Improvements: The proposed numerical scheme is novel for Korteweg-de Vries equation and has never been employed for this purpose before.

Keywords: Cubic B-spline Collocation Method, Cubic B-spline Functions, Finite Difference Formulation, Korteweg-de Vries Equation

1. Introduction

The third order non-linear Korteweg-de Vries (KdV) equation occurs in many physical applications such as non-linear plasma waves which exhibit certain dissipative effects, propagation of waves and propagation of bores in shallow water waves. The KdV equation is given by

\[ u_t + \alpha u u_x + \beta u_x + \gamma u_{xxx} = 0, \quad x \in [a, b], \quad t \in [0, T], \]

with conditions

\[ u(x,0) = g(x), \]

\[ u(a,t) = \phi_1(t), \quad u(b,t) = \phi_2(t), \quad u_x(b,t) = \phi_3(t), \]

where, \( u = u(x,t), \alpha, \beta, \gamma \) are constants and \( g(x), \phi_1(t), \phi_2(t), \phi_3(t) \) are known functions.

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2. Cubic B-spline Functions

We uniformly partition the spatial domain \([a,b]\) into \(n+1\) equidistant knots as \(x_i = x_0 + ih, i = 0(1)n\) with \(h = \frac{1}{n}(b-a)\). The \(p^{th}\) B-spline function of degree \(r\), order \(r+1\), is defined as \(^1\)

For \(r = 0\)

\[
B_{0,p}(x) = \begin{cases} 
1, & \text{if } x \in [x_p, x_{p+1}] \\
0, & \text{otherwise}
\end{cases}
\]  

(3)

For \(r > 0\) and \(x \in [x_p, x_{p+1}]\)

\[
B_{r,p}(x) = \frac{(x-x_p)}{(x_{p+r}-x_p)}B_{r-1,p}(x) + \frac{1}{x_{p+r}-x_p}B_{r-1,p+1}(x).
\]  

(4)

Using (4), the typical CBS functions are defined as \(^2\)

\[
B_r(x) = \frac{1}{6h^r}B_r(x) + \frac{1}{6h^r}B_{r-1}(x) + \frac{1}{6h^r}B_{r+1}(x),
\]  

(5)

where, \(p = -1(1)n+1\). For a sufficiently smooth function \(u(x,t)\) there always exists a unique third degree spline \(U(x,t)\) which satisfies the prescribed interpolating conditions such that

\[
U(x,t) = \sum_{p=1}^{n+1} c_p(t) B_p(x),
\]  

(6)

where, \(c_p(t)\)'s are, time dependent real constants, yet to be calculated. For simplicity, we express the CBS approximations \(U(x_i), U'(x_i), U''(x_i)\) and \(U'''(x_i)\) by \(U_i, m_i, M_i\) and \(T_i\) respectively. The third degree basis spline functions (5) together with (6) yield the following relations

\[
U_i = \sum_{p=i+1}^{i+1} c_p B_p(x) = \frac{1}{6}(c_{i+1} + 4c_i + c_{i+1})
\]  

(7)

\[
m_i = \sum_{p=i-1}^{i} c_p B_p(x) = \frac{1}{2h}(-c_{i-1} + c_{i+1})
\]  

(8)

Moreover, for second and third order derivatives, we shall use the following new CBS approximations \(^3\)

The roadways of this study is: In section 2, we shall discuss some preliminaries of ordinary CBS interpolation. The numerical method is presented in section 3 and experimental outcomes are given in section 4.
Using (7)-(10) in (15), for $i = 0, 1, 2, 3, \ldots, n - 1$, we obtain the following linear equations involving $n + 3$ unknowns.

$$
\begin{align*}
\frac{w^i}{6} & \left( c_{i+1} + 4c_{i} + c_{i-1} \right) + \gamma \frac{y^i}{2k} \left( -c_{i+1} + c_{i-1} \right) \\
+ \frac{\beta}{12h} & \left( 14c_{i+3} + 33c_{i+1} + 28c_{i} - 14c_{i-1} \right) \\
+ \frac{\gamma}{72h} & \left( -68c_{i+3} + 249c_{i+1} - 351c_{i} + 238c_{i-1} - 78c_{i-3} + 9c_{i-5} + c_{i-7} \right) = z_1^i, \\
\frac{w^i}{6} & \left( c_{i+1} + 4c_{i} + c_{i-1} \right) + \gamma \frac{y^i}{2k} \left( -c_{i+1} + c_{i-1} \right) \\
+ \frac{\beta}{12h} & \left( c_{i+1} + 8c_{i} - 18c_{i-1} + 8c_{i-3} + c_{i-5} \right) \\
+ \frac{\gamma}{72h} & \left( 10c_{i+1} - 213c_{i} + 378c_{i-1} - 55c_{i-2} - 450c_{i-3} + 225c_{i-4} - 3c_{i-5} \right) = z_2^i, \\
\frac{w^i}{6} & \left( c_{i+1} + 4c_{i} + c_{i-1} \right) + \gamma \frac{y^i}{2k} \left( -c_{i+1} + c_{i-1} \right) \\
+ \frac{\beta}{12h} & \left( c_{i+1} + 8c_{i} - 18c_{i-1} + 8c_{i-3} + c_{i-5} \right) \\
+ \frac{\gamma}{72h} & \left( 10c_{i+1} - 213c_{i} + 378c_{i-1} - 55c_{i-2} - 450c_{i-3} + 225c_{i-4} - 3c_{i-5} \right) = z_3^i.
\end{align*}
$$

Three more equations are obtained from the boundary conditions (2) as

$$
\begin{align*}
U_0^{j+1} &= \phi_1 \left( t_{j+1} \right), \\
U_n^{j+1} &= \phi_2 \left( t_{j+1} \right), \\
m_{n+1}^{j+1} &= \phi_3 \left( t_{j+1} \right).
\end{align*}
$$

The set of equations (16)–(24) can be written in matrix form as

$$
AC^{j+1} = B,
$$

where $A$ denotes the coefficient matrix of order $n + 3$, $B$ is column matrix of order $n + 3$ and $C^{j+1} = \left[ c_{i-1} c_{i} c_{i+1} \cdots c_{n+1} \right]^T$ is the set of control points at the $(j+1)^{th}$ time level.
Before starting any computation using (25), we obtain the following three equations from initial condition (2)

\[ m^0_b = g'(x_0), \]
\[ U^0_i = g(x_i), \quad i = 0, 1, \ldots, n, \]
\[ m^0_a = g'(x_n). \]

Using (7)–(8), we get

\[ -c^0_{i-1} + c^1_{i-1} = 2h g'(x_0), \]
\[ c^1_{i-1} + 4c^1_i + c^1_{i+1} = 6g(x_i), \quad i = 1, 2, \ldots, n, \]
\[ -c^1_{n+1} + c^0_{n+1} = 2h g'(x_n). \]

The above system can be expressed in matrix form as

\[ AC^0 = B. \]

The unknown column vector \( C^0 \) is determined by well-known Thomas algorithm. The numerical computations are executed in Mathematica 9.

### 4. Numerical Results

In this section, the approximate solution to (1)–(2) is presented. The accuracy and validity of the proposed numerical method is tested by three error norms \( L_\infty, L_2 \) and Root Mean Square (RMS), which are calculated as

\[ L_\infty = \max_i |U_i - u_i|, \quad L_2 = \sqrt{\sum_{i=0}^{n} (U_i - u_i)^2}, \quad \text{RMS} = \sqrt{\frac{\sum_{i=0}^{n} (U_i - u_i)^2}{n+1}} \]

where, \( U_i \) and \( u_i \) represent the numerical and exact solutions at the \( i^{th} \) knot respectively. The approximate results are compared with Multi-Quadratic Radial Basis Functions (MQRBF), Multi-Quadric (MQ) and Inverse Multi-Quadric (IMQ) radial basis functions method, Multi-Quadric Quasi-Interpolation (MQQI) approach and integrated multi-quadric quasi-interpolation (IMQQI) method.

**Example 1:**

Consider the following KdV equation

\[ u_t + 6uu_x + u_{xxx} = 0, \quad x \in [a, b], \quad t \in [0, T] \]

\[ u(x, 0) = \frac{\lambda}{2} \text{sech}^2\left( \frac{\sqrt{\lambda}}{2} x - \mu \right) \]

The exact solution is \( u(x, t) = \frac{\lambda}{2} \text{sech}^2\left( \frac{\sqrt{\lambda}}{2} (x - \lambda t) - \mu \right) \).

The error norms \( L_\infty, L_2 \) and RMS are listed in Tables 1–3, when \( n = 200 \) and \( \Delta t = 0.01 \). It is revealed that the proposed numerical scheme produces more reliable and accurate results as compared to MQRBF, MQ, IMQ, MQQI and IMQQI. Figure 1 shows a very close agreement of the numerical solution with closed form solution for \( t = 1, 3, 5 \). Three dimensional plots of exact and approximate solutions are shown in Figures 2 and 3. The absolute computational error using \( n = 200, \Delta t = 0.01 \) is displayed in Figure 4.

| Table 1. Absolute numerical error for Example 1, when \( 0 \leq x \leq 40 \), \( 0 \leq t \leq 5 \), \( \lambda = 0.5 \), \( \mu = 7 \) |
|---|---|---|---|---|
| \( t \) | MQ\(^0\) \( \Delta t = 0.001 \) | MQ\(^0\) \( \Delta t = 0.001 \) | MQ\(^0\) \( \Delta t = 0.001 \) | Proposed method \( \Delta t = 0.01 \) |
| 1 | \( 1.79 \times 10^{-5} \) | \( 6.96 \times 10^{-5} \) | \( 1.53 \times 10^{-3} \) | \( 8.63 \times 10^{-6} \) |
| 2 | \( 3.01 \times 10^{-5} \) | \( 1.96 \times 10^{-4} \) | \( 2.87 \times 10^{-3} \) | \( 1.11 \times 10^{-5} \) |
| 3 | \( 3.98 \times 10^{-5} \) | \( 3.83 \times 10^{-3} \) | \( 4.14 \times 10^{-3} \) | \( 1.26 \times 10^{-5} \) |
| 4 | \( 4.78 \times 10^{-5} \) | \( 5.91 \times 10^{-3} \) | \( 5.39 \times 10^{-3} \) | \( 1.36 \times 10^{-5} \) |
| 5 | \( 5.46 \times 10^{-5} \) | \( 8.37 \times 10^{-3} \) | \( 6.81 \times 10^{-3} \) | \( 1.45 \times 10^{-5} \) |
Table 2. Error norms for Example 1, when $0 \leq x \leq 40$, $0 \leq t \leq 5$, $\lambda = 0.5$, $\mu = 7$

| $t$ | Proposed method | $L_\infty$ | $L_2$ | RMS |
|-----|----------------|------------|-------|-----|
| 1   | IMQIQI$^{12}$  | $8.63 \times 10^{-6}$ | $3.22 \times 10^{-5}$ | $2.27 \times 10^{-6}$ |
|     |                | $1.67 \times 10^{-5}$ | $6.00 \times 10^{-4}$ | $4.23 \times 10^{-5}$ |
| 2   | IMQIQI$^{12}$  | $1.11 \times 10^{-5}$ | $4.33 \times 10^{-5}$ | $3.06 \times 10^{-6}$ |
|     |                | $2.38 \times 10^{-4}$ | $9.22 \times 10^{-4}$ | $6.51 \times 10^{-5}$ |
| 3   | IMQIQI$^{12}$  | $1.26 \times 10^{-5}$ | $4.94 \times 10^{-5}$ | $3.48 \times 10^{-6}$ |
|     |                | $2.38 \times 10^{-4}$ | $1.13 \times 10^{-4}$ | $8.00 \times 10^{-5}$ |
| 4   | IMQIQI$^{12}$  | $1.36 \times 10^{-5}$ | $5.33 \times 10^{-5}$ | $3.76 \times 10^{-6}$ |
|     |                | $3.14 \times 10^{-4}$ | $1.29 \times 10^{-3}$ | $9.12 \times 10^{-5}$ |
| 5   | IMQIQI$^{12}$  | $1.45 \times 10^{-5}$ | $5.64 \times 10^{-5}$ | $3.98 \times 10^{-6}$ |
|     |                | $3.41 \times 10^{-4}$ | $1.42 \times 10^{-3}$ | $1.00 \times 10^{-4}$ |

Table 3. Error norms for Example 1, when $30 \leq x \leq 80$, $0 \leq t \leq 10$, $\lambda = 0.14$, $\mu = 10$

| $t$ | $\Delta t$ | $L_\infty$ | $L_2$ | RMS |
|-----|-------------|------------|-------|-----|
| 1   | 0.01        | $2.00 \times 10^{-7}$ | $7.24 \times 10^{-7}$ | $5.11 \times 10^{-8}$ |
|     | 0.001       | $6.89 \times 10^{-6}$ | $2.14 \times 10^{-5}$ | $1.35 \times 10^{-6}$ |
| 2   | 0.01        | $4.43 \times 10^{-7}$ | $1.84 \times 10^{-6}$ | $1.30 \times 10^{-7}$ |
|     | 0.001       | $8.60 \times 10^{-6}$ | $3.50 \times 10^{-5}$ | $2.21 \times 10^{-6}$ |
| 3   | 0.01        | $5.84 \times 10^{-7}$ | $2.60 \times 10^{-6}$ | $1.83 \times 10^{-7}$ |
|     | 0.001       | $8.40 \times 10^{-6}$ | $4.10 \times 10^{-5}$ | $2.59 \times 10^{-6}$ |
| 4   | 0.01        | $6.84 \times 10^{-7}$ | $3.14 \times 10^{-6}$ | $2.21 \times 10^{-7}$ |
|     | 0.001       | $9.21 \times 10^{-6}$ | $4.28 \times 10^{-5}$ | $2.70 \times 10^{-6}$ |
| 5   | 0.01        | $7.87 \times 10^{-7}$ | $3.71 \times 10^{-6}$ | $2.61 \times 10^{-6}$ |
|     | 0.001       | $8.56 \times 10^{-6}$ | $4.55 \times 10^{-5}$ | $2.87 \times 10^{-6}$ |
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Figure 1. Numerical and exact solution for Example 1, when \( t = 1,3,5 \) and \( n = 200, \Delta t = 0.01, 0 \leq x \leq 40, \lambda = 0.5, \mu = 7 \).

Figure 2. Exact solution for Example 1, when \( 0 \leq x \leq 40, 0 \leq t \leq 1, \lambda = 0.5, \mu = 7 \).

Figure 3. Approximate solution for Example 1, with \( 0 \leq x \leq 40, 0 \leq t \leq 1, \lambda = 0.5, \mu = 7, n = 200, \Delta t = 0.01 \).

Figure 4. Absolute error for Example 1, with \( 0 \leq x \leq 40, 0 \leq T \leq 1, \lambda = 0.5, \mu = 7, n = 200, \Delta t = 0.01 \).

Example 2:
Consider the following KdV equation:
\[
\frac{\partial u}{\partial t} + 6u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0, \quad x \in [a,b], \quad t \in [0,T],
\]
\[
u(x,0) = 2 \text{sech}^2(x + 4).
\]

The exact solution is \( u(x,t) = 2 \text{sech}^2(x - 4t + 4) \). The computational error norms \( L_{\infty} \), \( L_2 \), and RMS are listed in Table 4 when \( n = 200 \) and \( \Delta t = 0.01 \). Figure 5 shows the approximate and exact solution at \( t = 0.2,0.4,0.6,0.8,1 \). The three dimensional plots of analytical and approximate solutions are displayed in Figures 6 and 7. The absolute computational error is portrayed in Figure 8 using \( n = 200 \) and \( \Delta t = 0.01 \).

Table 4. Error norms for Example 2, when \( -10 \leq x \leq 0 \), \( 0 \leq t \leq 1 \), \( \lambda = 0.5 \)

| \( t \) | \( L_{\infty} \) | \( L_2 \) | RMS |
|---|---|---|---|
| 0.2 | \( 3.39 \times 10^{-5} \) | \( 1.82 \times 10^{-4} \) | \( 1.29 \times 10^{-5} \) |
| 0.4 | \( 3.49 \times 10^{-5} \) | \( 2.40 \times 10^{-4} \) | \( 1.69 \times 10^{-5} \) |
| 0.6 | \( 5.12 \times 10^{-5} \) | \( 3.41 \times 10^{-4} \) | \( 2.40 \times 10^{-5} \) |
| 0.8 | \( 8.21 \times 10^{-5} \) | \( 4.76 \times 10^{-4} \) | \( 3.36 \times 10^{-5} \) |
| 1.0 | \( 8.09 \times 10^{-5} \) | \( 4.30 \times 10^{-4} \) | \( 3.03 \times 10^{-5} \) |
5. Conclusion

In this paper, numerical solution of non-linear third order KdV equation has been explored. We conclude the outcomes of this research as:

1. The presented algorithm is based on usual finite difference scheme and CBS collocation method.
2. The proposed technique is novel for third order non-linear KdV equation.
3. Usual finite difference scheme has been employed for temporal discretization.
4. The new CBS approximations have been used to interpolate the solution in space direction.
5. Due to straightforward and simple application, it outperforms the MQRBF, MQ, IMQ, MQQI and IMQQI approaches.

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