An Adapted Frame on Indicatrix Bundle of a Finsler Manifold and its Geometric Properties

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Abstract

In this paper, a frame is introduced on tangent bundle of a Finsler manifold in a manner that it makes some simplicity to study the properties of the natural foliations in tangent bundle. Moreover, we show that the indicatrix bundle of a Finsler manifold with lifted sasaki metric and natural almost complex structure on tangent bundle cannot be a sasakian manifold.

Keywords: Foliation, indicatrix bundle, Sasakian manifold.

1 Introduction

First time Sasaki [9], construct a natural Riemannian metric $G$ on the tangent bundle $TM$ of the Riemannian manifold $(M, g)$. Then $G$ was called the Sasaki metric on $TM$ and it was the main tool in studying interrelations between the geometries of $(M, g)$ and $(TM, G)$. Later, this idea was used to construct a Riemannian metric on tangent bundle of a Finsler manifold. The geometric objects that occur in Finsler geometry depend on both point and direction, therefore, the tangent bundle of a Finsler manifold plays a major role in the study of Finslerian objects. To emphasize this, there are several studies of interrelations between the geometry of foliations on the tangent bundle of a Finsler manifold and the geometry of the Finsler manifold itself [3, 5]. Among the natural foliations of tangent bundle on a Finsler manifold indicatrix bundle and Liouville vector fields play more important roles.

Let $(M, F)$ be a $n$-dimensional Finsler manifold and $TM$ its tangent bundle with sasaki lifted metric $G$. In this paper to study the geometry of some foliations of $TM$ which they were introduced in [5], a new local frame
of vector fields in $TTM$ is fixed. These local frames shape in a manner that vertical and horizontal Liouville vector fields of Finsler manifold $M$ are two vector fields of them. In this frame $TTM$ can be written as direct sum of vertical Liouville vector field, horizontal Liouville vector field and their orthogonal distribution with respect to the metric $G$. By help of these bases new properties of some foliations on tangent bundle of a Finsler manifold is established.

Aiming at studying the properties of these foliations on tangent bundle of a Finsler manifold, this paper is organized in the following way. In section 2, a short review of Finsler manifolds [1, 2] is done and the notations which is needed in the followings are presented. In Section 3, a frame of local vector fields in $TTTM$ is introduced to study the indicatrix bundle and other natural foliations of tangent bundle of a Finsler manifold. By this local frame $TTM$ can be written as direct sum of vertical Liouville vector field and its orthogonal distribution with respect to sasaki lifted metric $G$ on $TM$. Moreover, local components of Levi-Civita connection of sasaki metric $G$ on $TM$ are calculated in this basis. In [5], six natural foliations of tangent bundle of a Finsler manifold is introduced and some properties of them such as totally geodesic and bundle-like with respect to metric $G$ are studied. In Section 4, by using the new frame introduced in Section 3 some more theorem about these foliations are proved. Finally in Section 5, it is proved that the indicatrix bundle with its contact structure given in [3] cannot be a Sasakian manifold [6]. Therefore, we should not have any expectation of properties of Sasakian manifolds in Riemannian geometry on indicatrix bundle as a Riemannian submanifold of $TM$ with restricted metric $G$ of sasaki metric $G$.

2 Preliminaries and Notations

Let $(M, F)$ be an $n$-dimensional smooth Finsler manifold and $TM$ be its tangent bundle. If $(x^i)$ be the local coordinate on $M$ then the local coordinate on $TM$ is shown by $(x^i, y^i)$ where $(y^i)$ are the fibre coordinate. With respect to local coordinate system induced on $TM$, the natural local frame fields on $TM$ are given by $\frac{\partial}{\partial y^i}$ and $\frac{\partial}{\partial x^i}$. The vertical distribution $VTM$ is locally spanned by $\{ \frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^n} \}$. Considering the fundamental function $F$, then the horizontal distribution $HTM$ as a complementary distribution of $VTM$ can be naturally defined. To define $HTM$, the followings are needed.
The spray coefficients \( G_i \) of fundamental function \( F \) are given by:

\[
G_i := \frac{g^{ij}}{4} \left( \frac{\partial^2 F^2}{\partial y^j \partial x^k} y^k - \frac{\partial F^2}{\partial x^j} \right)
\]

where \( (g^{ij}) \) is the inverse matrix of Hessian matrix \( F \) given as follows:

\[
g := (g_{ij}) = \left( \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j} \right)
\]

The nonlinear connection coefficients \( G_{i}^{j} \) of \( F \) are defined by:

\[
G_{i}^{j} = \frac{\partial G_{i}^{j}}{\partial y^{i}}.
\]

Then:

\[
H T M = \langle \frac{\delta}{\delta x^1}, \ldots, \frac{\delta}{\delta x^n} \rangle
\]

where \( \frac{\delta}{\delta x^i} = \partial_{x^i} - G_{i}^{j} \frac{\partial}{\partial y^j} \). The lie brackets of these basis are given as follows:

\[
\left[ \frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j} \right] = R_{ij}^{k} \frac{\partial}{\partial y^k}, \quad \left[ \frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j} \right] = G_{ij}^{k} \frac{\partial}{\partial y^k}
\]

where \( R_{ij}^{k} = \frac{\partial G_{k}^{j}}{\partial x^i} - \frac{\partial G_{k}^{i}}{\partial x^j} \) and \( G_{ij}^{k} = \frac{\partial G_{i}^{k}}{\partial y^j} \). The dual local 1-forms of \( \frac{\delta}{\delta x^i} \) and \( \frac{\partial}{\partial y^i} \) are denoted by \( dx^i \) and \( dy^i \), respectively, where \( \delta y^i := dy^i + G_{i}^{j} dx^j \).

Then, the lifted sasaki metric \( G \) on \( T M \) in these local frames is given as follows:

\[
G := g_{ij} dx^i \otimes dx^j + g_{ij} dy^i \otimes dy^j
\]

which it is a Riemannian metric on \( T M \). The natural almost complex structure on \( T T M \) which is compatible with metric \( G \) is defined by:

\[
J := \frac{\delta}{\delta y^i} \otimes dy^i - \frac{\partial}{\partial y^i} \otimes dx^i
\]

The indicatrix bundle \( IM \) of Finsler manifold \( (M, F) \) is defined by:

\[
IM := \{ (x, y) \in TM | F(x, y) = 1 \}
\]

It is proved in [3] that \( IM \) with \( (\varphi, \eta, \xi, \bar{G}) \) is a contact metric manifold, where

\[
\begin{align*}
\eta &:= y^i g_{ij} dx^j, \quad \xi := y^i \frac{\delta}{\delta x^i} \\
\varphi &:= J|_{D}, \quad \varphi(\xi) := 0 \\
D &:= \{ X \in TTM | \eta(X) = \eta(JX) = 0 \}
\end{align*}
\]
and $\bar{G}$ is restriction of sasaki metric $G$ to indicatrix bundle. Distribution $D$ defined in (2) is called contact distribution of indicatrix bundle.

In addition, throughout the paper the Einstein convention, that is, repeated indices with one upper index and one lower index denote summation over their range is used. The indices $i, j, k, ...$ are used for range $1, \ldots, n$ if not stated otherwise.

3 A Frame on Indicatrix Bundle of a Finsler Manifold

Supposed $(M, F)$ as an $n$-dimensional Finsler manifold. Since the vertical distribution $VTM$ is integrable in $TTM$ and vertical Liouville vector field $L := y^i \frac{\partial}{\partial y^i}$ is a foliation in $TTM$ which belongs to $VTM$, therefore, orthogonal distribution $V'TM$ to $L$ in $VTM$ with respect to the metric $G$ is a foliation and a local frame can be set on it as follows:

$$V'TM =< \bar{\partial} \bar{\partial} y^1, \ldots, \bar{\partial} \bar{\partial} y^{n-1}>$$

(3)

where $\bar{\partial} \bar{\partial} y^a = E^i_a \frac{\partial}{\partial y^i}$ $\forall a = 1, \ldots, n - 1$ and $E^i_a$ is the $(n - 1) \times n$ matrix of maximum rank. The first property of this matrix is $E^i_a g_{ij} y^j = 0$ achieved by the feature $G(\bar{\partial} \bar{\partial} y^a, L) = 0$. Now, using the natural almost complex structure $J$ on $TTM$, the new local vector field frame in $TTM$ is introduced as follows:

$$< \frac{\delta}{\delta x^a}, \xi, \bar{\partial} \bar{\partial} y^a, L >$$

(4)

where $\frac{\delta}{\delta x^a} := J \frac{\partial}{\partial y^a}$. In [5], it was shown that $L$ is orthogonal to level surfaces of fundamental function $F$ in $TM$. Therefore, $\frac{\delta}{\delta x^a}, \xi$ and $\bar{\partial} \bar{\partial} y^a$ are tangent to these hyper-surfaces in $TM$. The Sasakian metric $G$ on $TM$ can be shown in the new frame field (4) as follows:

$$G := \begin{pmatrix} g_{ab} & 0 & 0 & 0 \\ 0 & F^2 & 0 & 0 \\ 0 & 0 & g_{ab} & 0 \\ 0 & 0 & 0 & F^2 \end{pmatrix}$$

(5)
where \( a, b \in \{1, \ldots, n-1\} \) and \( g_{ab} = G(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b}) = G(\frac{\partial}{\partial y^a}, \frac{\partial}{\partial y^b}) = g_{ij}E_a^iE_b^j \).

Now, Lie brackets of vector fields (4) are presented as follows:

\[
\begin{align*}
(1) & \quad [\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b}] = (\frac{\delta E_b^j}{\delta x^a} - \frac{\delta E_a^j}{\delta x^b}) \frac{\partial}{\partial x^a} + E_a^i E_b^j R_{ij}^k \frac{\partial}{\partial y^k}, \\
(2) & \quad [\frac{\partial}{\partial x^a}, \frac{\partial}{\partial y^b}] = \left(\frac{\partial E_b^k}{\partial y^a} + E_a^i E_b^k \frac{\partial G_{ij}}{\partial y^a} - \frac{\partial E_a^k}{\partial y^b} \frac{\partial}{\partial y^b}\right), \\
(3) & \quad [\frac{\partial}{\partial y^a}, \frac{\partial}{\partial y^b}] = \left(\frac{\partial E_b^k}{\partial y^a} - \frac{\partial E_a^k}{\partial y^b}\right) \frac{\partial}{\partial y^b}, \\
(4) & \quad \left[\frac{\delta}{\delta x^a}, \xi \right] = -(E_a^i G_i^j + \xi(E_a^i)) \frac{\partial}{\partial x^a} + E_a^i y^j R^k_{ij} \frac{\partial}{\partial y^k}, \\
(5) & \quad \left[\frac{\delta}{\delta x^a}, \xi \right] = \frac{\delta G}{\delta x^a} - (\xi(E_a^i) + E_a^i G_i^j) \frac{\partial}{\partial y^a}, \\
(6) & \quad \left[\frac{\delta}{\delta x^a}, L \right] = -L(E_a^i \frac{\partial}{\partial x^a}), \\
(7) & \quad \left[\frac{\delta}{\delta y^a}, L \right] = \frac{\delta L}{\delta y^a} - L(E_a^i \frac{\partial}{\partial y^a}), \\
(8) & \quad [\xi, \xi] = [L, L] = [\xi, L] + \xi = 0.
\end{align*}
\]

The local components of Levi-Civita connection \( \nabla \) given by:

\[
\begin{align*}
2G(\nabla_X Y, Z) &= XG(Y, Z) + YG(X, Z) - ZG(X, Y) \\
-G([X, Z], Y) - G([Y, Z], X) + G([X, Y], Z)
\end{align*}
\]

for vector fields (4) and metric \( G \) are expressed as follows:

\[
\begin{align*}
\nabla_{\frac{\delta}{\delta x^a}} \frac{\delta}{\delta x^a} &= \left(\Gamma_{ab}^c + \frac{\delta E_a^i}{\delta x^a} E_d^k g_{kj} g^{de}\right) \frac{\delta}{\delta x^a} + \left(-g^{c}_{ab} + \frac{1}{2} R^{c}_{ab} \frac{\partial}{\partial y^c} + \frac{1}{2\pi} \tilde{R}_{ab} \xi, \\
\nabla_{\frac{\delta}{\delta x^a}} \frac{\partial}{\partial y^a} &= \left(\frac{1}{2} E_a^i E_b^k \frac{\delta G_{ik}}{\delta x^a} - G_a^h g_{hj} + G_b^h g_{hk}\right) \frac{\partial}{\partial y^a} \xi + \left(g^{c}_{ab} - \frac{1}{2} R_{bad}^{c} g^{ce} \frac{\partial}{\partial y^c} \frac{\delta}{\delta x^a} \xi, \\
\nabla_{\frac{\delta}{\delta y^a}} \frac{\partial}{\partial y^a} &= \left(g^{c}_{ab} - \frac{1}{2} R_{bad}^{c} g^{ce} \frac{\partial}{\partial y^c} \frac{\delta}{\delta x^a} \xi, \\
\nabla_{\frac{\delta}{\delta x^a}} \frac{\delta}{\delta y^a} &= \frac{1}{2} E_a^i E_b^k \frac{\delta G_{ik}}{\delta x^a} \frac{\delta g_{ij}}{\delta y^a} + G_a^h g_{hj} + G_b^h g_{hk}\right) \frac{\partial}{\partial y^a} \xi + \frac{1}{2} E_a^i E_b^k \frac{\delta G_{ik}}{\delta x^a} \frac{\delta g_{ij}}{\delta y^a} \xi, \\
\nabla_{\frac{\delta}{\delta x^a}} \xi &= \frac{1}{2} \tilde{R}_{bad} g^{de} \frac{\delta}{\delta x^a} \xi + \frac{1}{2} R_{bad} g^{de} \frac{\partial}{\partial y^c} \xi, \\
\nabla_{\frac{\delta}{\delta y^a}} \xi &= \frac{1}{2} \tilde{R}_{bad} g^{de} \frac{\delta}{\delta y^a} \xi + \frac{1}{2} R_{bad} g^{de} \frac{\partial}{\partial y^c} \xi, \\
\nabla_{\frac{\delta}{\delta x^a}} \frac{\partial}{\partial y^a} &= \frac{1}{2} \tilde{R}_{bad} g^{de} \frac{\delta}{\delta x^a} \xi + \frac{1}{2} R_{bad} g^{de} \frac{\partial}{\partial y^c} \xi, \\
\nabla_{\frac{\delta}{\delta y^a}} \frac{\partial}{\partial y^a} &= \frac{1}{2} \tilde{R}_{bad} g^{de} \frac{\delta}{\delta y^a} \xi + \frac{1}{2} R_{bad} g^{de} \frac{\partial}{\partial y^c} \xi, \\
\nabla_{\frac{\delta}{\delta x^a}} \frac{\delta}{\delta y^a} &= \frac{1}{2} \tilde{R}_{bad} g^{de} \frac{\delta}{\delta x^a} \xi + \frac{1}{2} R_{bad} g^{de} \frac{\partial}{\partial y^c} \xi, \\
\nabla_{\frac{\delta}{\delta y^a}} \frac{\delta}{\delta y^a} &= \frac{1}{2} \tilde{R}_{bad} g^{de} \frac{\delta}{\delta y^a} \xi + \frac{1}{2} R_{bad} g^{de} \frac{\partial}{\partial y^c} \xi, \\
\nabla_{\frac{\delta}{\delta x^a}} L &= \nabla L \frac{\delta}{\delta x^a} - L(E_a^i) E_d^k g_{kj} g^{de} \frac{\delta}{\delta x^a} = 0, \\
\nabla_{\frac{\delta}{\delta y^a}} L &= \nabla L \frac{\delta}{\delta y^a} - L(E_a^i) E_d^k g_{kj} g^{de} \frac{\partial}{\partial y^c} \xi = 0, \\
\n\nabla_{\frac{\delta}{\delta x^a}} \xi &= \nabla L \frac{\delta}{\delta x^a} - \xi = \nabla L \xi - L = 0.
\end{align*}
\]
where
\[ g^c_{ab} = g_{abcd}g^{dc} = \frac{1}{2} E^c_a E^d_b E^k_d g_{ij} g^{dc}, \quad \Gamma^c_{ab} = E^c_a E^d_b \Gamma^{h}_{ij} g^k h g^{dc} \quad (9) \]
\[ R^c_{ab} = R^c_{dabc} g^{dc} = E^c_a E^d_b E^k_d R^h_{ij} g_{jk} g^{dc}, \]
\[ \bar{R}_{ab} = \left( \frac{\delta E^c_a}{\delta x^a} - \frac{\delta E^c_b}{\delta x^b} \right) g_{ij} y^j, \quad R_{ab} = E^c_a E^d_b R_{ij} \]
and, \( (g_{ab}) \) is the inverse matrix of \( (g_{ab}) \).

In following, the Levi-Civita connection and metric on indicatrix bundle are denoted by \( \bar{\nabla} \) and \( \bar{G} \), respectively, which \( \bar{G} \) is the restriction of metric \( (5) \). In order to compute the components of Levi-Civita connection \( \bar{\nabla} \) on indicatrix bundle \( IM \) from \( (8) \) the Guass Formula \( [7] \):
\[ \bar{\nabla}_X Y = \bar{\nabla}_X Y + H(X, Y) \quad (10) \]
where \( H \) is the second fundamental form of \( IM \) in \( TM \) is needed. It is obvious that \( \bar{\nabla} \) is tangent to \( IM \) for all combinations of \( \frac{\delta}{\delta x^a}, \xi \) and \( \frac{\partial}{\partial y^a} \) except \( \bar{\nabla} \frac{\partial}{\partial y^a} \). Therefore, \( \bar{\nabla} \) is equal to \( \nabla \) for the other combinations of \( \frac{\delta}{\delta x^a}, \xi \) and \( \frac{\partial}{\partial y^a} \) by using the Gauss formula \( (10) \). The curvature tensor \( R \) of \( \bar{\nabla} \) defined by \( R(X, Y)Z = \bar{\nabla}_X \bar{\nabla}_Y Z - \bar{\nabla}_Y \bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]}Z \) is related to the curvature tensor \( \bar{R} \) of \( \bar{\nabla} \) in following equations:
\[
\begin{align*}
R\left( \frac{\delta}{\delta x^a}, \frac{\delta}{\delta y^b} \right) \frac{\partial}{\partial y^c} &= \bar{R}\left( \frac{\delta}{\delta x^a}, \frac{\delta}{\delta y^b} \right) \frac{\partial}{\partial y^c} + \frac{1}{2} \bar{R}_{cab} L \\
R\left( \frac{\delta}{\delta x^a}, \frac{\partial}{\partial y^b} \right) \frac{\partial}{\partial x^c} &= \bar{R}\left( \frac{\delta}{\delta x^a}, \frac{\partial}{\partial y^b} \right) \frac{\partial}{\partial x^c} + \frac{1}{2 \sqrt{2}} \left( R_{bac} - 2 g_{abc} \right) L \\
R\left( \frac{\partial}{\partial y^a}, \frac{\partial}{\partial y^b} \right) \frac{\partial}{\partial y^c} &= \bar{R}\left( \frac{\partial}{\partial y^a}, \frac{\partial}{\partial y^b} \right) \frac{\partial}{\partial y^c} - \frac{1}{2 \sqrt{2}} g_{abe} \frac{\partial}{\partial y^e} + \frac{1}{\sqrt{2}} g_{ac} \frac{\partial}{\partial y^c} \\
R\left( \frac{\partial}{\partial y^a}, \frac{\delta}{\delta x^b} \right) \frac{\partial}{\partial y^c} &= \bar{R}\left( \frac{\partial}{\partial y^a}, \frac{\delta}{\delta x^b} \right) \frac{\partial}{\partial y^c} + \frac{1}{2} E^c_a E^d_b E^k_d \left( G^h_{ijk} g_{hi} + G^h_{ikj} g_{hj} - \delta g_{ij} \right) \frac{\partial}{\partial x^k} L \\
R\left( \frac{\delta}{\delta x^a}, \xi \right) \frac{\delta}{\delta x^b} &= \bar{R}\left( \frac{\delta}{\delta x^a}, \xi \right) \frac{\delta}{\delta x^b} - \frac{1}{2 \sqrt{2}} \bar{R}_{ab} L \\
R\left( \frac{\partial}{\partial y^a}, \xi \right) \frac{\delta}{\delta x^b} &= \bar{R}\left( \frac{\partial}{\partial y^a}, \xi \right) \frac{\delta}{\delta x^b} - \frac{1}{2 \sqrt{2}} \bar{R}_{ab} L \\
\end{align*}
\]
(11)

For the other combinations of \( \frac{\delta}{\delta x^a}, \frac{\partial}{\partial y^a} \) and \( \xi \), tensor fields \( R \) and \( \bar{R} \) coincide with each other.

4 Foliations on \((TM, G)\)

In this section, the local frame on \( TM \) which it was introduced in previous section is used to study some properties of natural foliations on tangent bundle of a Finsler manifold. In [5], a comprehensive study was done on six foliations of \( TM \) presented as follows:
1. $L$: vertical Liouville vector field.

2. $\xi$: horizontal Liouville vector field.

3. $L \oplus \xi$.

4. $V^TM$: defined by $V^TM = \langle \frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^n} \rangle$.

5. $V'TM$: defined in (3).

6. $V'_{\perp}TM$: which is perpendicular to $L$ in $T^TM$ with respect to the metric $G$.

A. Bejancu, in [5], studied the properties of these foliations such as totally geodesic and bundle-like for sasaki metric $G$. Here, some more theorems are proved of these foliations about these properties by help of the frame which it was introduced in Section 3.

**Corollary 4.1.** The sasaki lifted metric $G$ is bundle-like for foliation $V'_{\perp}TM$.

**Proof.** By help of (8), it is obtained that:

$$G(\nabla_L L + \nabla_L L, X) = 0 \quad \forall X \in \Gamma(V'_{\perp}TM)$$

and this completes the proof. $\square$

**Theorem 4.1.** The metric $G$ is bundle-like for foliation $V'TM$ if and only if $(M, g)$ is a Riemannian manifold.

**Proof.** From (8), it is a straightforward calculation to obtain:

$$G(\nabla_X Y + \nabla_Y X, \frac{\partial}{\partial y^c}) = 0 \quad \forall c \in \{1, \ldots, n-1\}, X, Y \in \Gamma(HTM \oplus L)$$

except $G(\nabla \frac{\partial}{\partial x^a} \frac{\partial}{\partial x^b} + \nabla \frac{\partial}{\partial x^c} \frac{\partial}{\partial y^a}, \frac{\partial}{\partial y^b})$ which is equal to $-2g_{abc}$. Therefore, $G$ is bundle-like for foliation $V'TM$ if and only if $g_{abc} = 0$, and it leads to $g_{ijk} = 0$ by definition $g_{abc}$ in (9). This completes the proof. $\square$

**Theorem 4.2.** The foliations $V'TM$ and $V'_{\perp}TM$ are not totally geodesic with respect to the Levi-Civita connection of Riemannian metric $G$ on $TM$.

**Proof.** From (8), it is obtained that: $H(\frac{\partial}{\partial y^a}, \frac{\partial}{\partial y^b}) = -\frac{1}{F^2}g_{ab}L$ for both foliations $V'TM$ and $V'_{\perp}TM$, which it cannot be vanish. This completes the proof. $\square$
Corollary 4.2. The metric $G$ for foliations $L$ and $L \oplus \xi$ cannot be bundle-like.

Proof. From (8), it is obtained that:
\[
G(\nabla \frac{\partial}{\partial y^a} \nabla \frac{\partial}{\partial y^b} + \nabla \frac{\partial}{\partial y^a} \nabla \frac{\partial}{\partial y^b}, L) = -2g_{ab}
\]
Therefore, $g_{ab} = 0$ is a necessary condition to $G$ be bundle-like for $L$ or $L \oplus \xi$, which it is impossible.

5 Sasakian Structure and Indicatrix Bundle of a Finsler Manifold

Now, let $(\tilde{\mathcal{M}}, \tilde{\varphi}, \tilde{\eta}, \tilde{\xi}, \tilde{g})$ be a contact metric manifold [6]. In [4], the new connection $\tilde{\nabla}$ was presented on the contact metric manifold $\tilde{M}$ as follows:
\[
\tilde{\nabla}_X Y = \nabla_X Y - \tilde{\eta}(X)\nabla_Y \tilde{\xi} - \tilde{\eta}(Y)\nabla_X \tilde{\xi} + \tilde{g}(X, \tilde{\varphi}Y)\tilde{\xi} + \frac{1}{2}(\mathcal{L}_{\tilde{\xi}}\tilde{g})(X, Y)\tilde{\xi}
\]
where $\nabla$ is Levi-Civita connection of Riemannian metric $\tilde{g}$. It was proved in [4] that the contact metric manifold $\tilde{M}$ is a Sasakian manifold if and only if
\[
(\tilde{\nabla}_X \tilde{\varphi}) Y = 0 \quad \forall X, Y \in \Gamma(T\tilde{M})
\] (12)
Since the indicatrix bundle has the contact metric structure in Finslerian manifolds by Proposition 4.1 in [3], here it is tried to find an answer to the question that "Can the indicatrix bundle with contact structure given in (2) be a Sasakian manifold?”. First, the following Lemma is proved in order to reduce the number of calculations.

Lemma 1. If $(\tilde{\mathcal{M}}, \tilde{\varphi}, \tilde{\eta}, \tilde{\xi}, \tilde{g})$ be a contact metric manifold with contact distribution $\tilde{D}$, then $\tilde{M}$ is Sasakian manifold if and only if:
\[
(\tilde{\nabla}_X \tilde{\varphi}) Y = 0 \quad \forall X, Y \in \Gamma(\tilde{D})
\]
Proof. For all $\tilde{X} \in \Gamma(T\tilde{M})$, they can be written in the form $X + f\tilde{\xi}$ where $X \in \Gamma\tilde{D}$ and $f \in C^\infty(\tilde{M})$. Therefore:
\[
(\tilde{\nabla}_X \tilde{\varphi}) \tilde{Y} = (\tilde{\nabla}_{X+f\tilde{\xi}} \tilde{\varphi})(Y + h\tilde{\xi}) = (\tilde{\nabla}_X \tilde{\varphi}) Y + (\tilde{\nabla}_{f\tilde{\xi}} \tilde{\varphi}) Y + (\tilde{\nabla}_X \tilde{\varphi}) h\tilde{\xi}
\]
\[
+ (\tilde{\nabla}_{f\tilde{\xi}} h\tilde{\xi}) = (\tilde{\nabla}_X \tilde{\varphi}) Y + f(\tilde{\nabla}_{\tilde{\xi}} \tilde{\varphi} Y - \tilde{\varphi} \tilde{\nabla}_{\tilde{\xi}} Y) + \tilde{\nabla}_X h(\tilde{\varphi}) - \tilde{\varphi}(\tilde{\nabla}_X h)(\tilde{\xi})
\]
\[
+ f(\tilde{\nabla}_{\tilde{\xi}} h\tilde{\xi} - \tilde{\varphi} \tilde{\nabla}_{\tilde{\xi}} h)(\tilde{\xi}) = (\tilde{\nabla}_X \tilde{\varphi}) Y
\]
The lemma is proved using Theorem 3.2 in [4] and the last equation. \(\square\)
Now, the following Theorem can be expressed:

**Theorem 5.1.** Let \((M, F)\) be a Finsler manifold. Then, indicatrix bundle \(IM\) with its natural contact structure given in (2) can never be a Sasakian manifold.

**Proof.** From lemma 1, \(IM\) is a Sasakian manifold if and only if:

\[
\left(\tilde{\nabla}_{\delta} \varphi\right)_{\delta x^a} = \left(\tilde{\nabla}_{\delta} \varphi\right)_{\delta y^b} = \left(\tilde{\nabla}_{\delta} \varphi\right)_{\delta x^a} = \left(\tilde{\nabla}_{\delta} \varphi\right)_{\delta y^b} = 0
\]

Using (8), one of the components in above equations where it must be zero is \(g_{ab}\) which it is impossible and shows that the indicatrix bundle cannot have a Sasakian structure on contact structure given in (2).

**Another proof for Theorem 5.1**

The following argument was presented in Chapter 6 of [6]. We consider \(TM = IM \times \mathbb{R}\) for the Finslerian manifold \((M, F)\) and introduce the almost complex structure \(\tilde{J}\) by means of \(\varphi\) defined in (2) as follows:

\[
\tilde{J}(X + fL) = \varphi(X) - f\xi + \eta(X)L
\]

where \(X\) is a vector field tangent to indicatrix bundle and \(\xi, \eta\) were defined in Section 2. Using a straight calculation, it can be seen that \(\tilde{J}\) is equal to \(J\) defined in (1). The contact structure \((\varphi, \eta, \xi)\) will be a Sasakian structure if and only if \((\varphi, \eta, \xi)\) is normal, that is, \(\tilde{J}\) (or \(J\)) is integrable. Integrability of \(\tilde{J}\) (or \(J\)) is equivalent to vanishing the following equations

\[
N_J\left(\frac{\delta}{\delta x^a}, \frac{\delta}{\delta x^b}\right) = -N_J\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = -R_{ij}^k \frac{\partial}{\partial y^k}
\]

\[
N_J\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\right) = -R_{ij}^k \frac{\delta}{\delta x^k}
\]

Therefore, \(\tilde{J}\) (or \(J\)) is integrable if and only if \(M\) is a flat manifold. Up to now, it can be shown that \(IM\) is Sasakian if and only if \(M\) is flat. Furthermore, it was proved that \(\xi\) is a killing vector field if \(IM\) be a Sasakian manifold [6]. Therefore, \(M\) has constant curvature 1 using Theorem 3.4 in [5] and it is a contradiction to the previous result which shows that \(M\) is flat if \(IM\) is a Sasakian manifold. \(\blacksquare\)
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