Fuzzy Core Equivalence in Large Economies: A Role for the Infinite-Dimensional Lyapunov Theorem*

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Abstract

We present the equivalence between the fuzzy core and the core under minimal assumptions. Due to the exact version of the Lyapunov convexity theorem in Banach spaces, we clarify that the additional structure of commodity spaces and preferences is unnecessary whenever the measure space of agents is “saturated”. As a spin-off of the above equivalence, we obtain the coincidence of the core, the fuzzy core, and the Schmeidler’s restricted core under minimal assumptions. The coincidence of the fuzzy core and the restricted core has not been articulated anywhere.

Key words: large economy; fuzzy core; core; restricted core; infinite-dimensional commodity space; Lyapunov’s theorem; saturated measure space.

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1 Introduction

Classical core theory in exchange economies deals with the situation where agents have only one of two alternative possibilities: whether to join or not to join a coalition – there is no option for varying degrees of commitment to, and participation in, more than a single one coalition. On the contrary, fuzzy core theory proposed by Aubin (1981, 1982) allow for the partial participation of agents in coalitions where the attainable outcomes of the allocations of goods depend on the degree of commitment and participation of the agents, thereby modelling a more pluralistic conception of decision-making and identity formation as is evident from even the casual observations of group behavior.

In particular, the significant observation Aubin made is the equivalence between the fuzzy core and the set of Walrasian allocations (fuzzy core–Walras equivalence) in finite agent economies with finite-dimensional commodity spaces; see Hüsseinov (1994) for the case of large economies. This is another remarkable coincidence result on the core since the well-known Debreu–Scarf limit theorem for Edgeworth replica economies; see Debreu and Scarf (1963).

Motivated by Aubin’s formulation, Noguchi (2000); Bhowmik and Graziano (2014) demonstrated the equivalence between the core and fuzzy core in atomless economies. Although the presence of price systems is irrelevant to the direct verification of such equivalence, they need to assume the infinite-dimensional commodity space to be an ordered Banach space whose positive cone has a nonempty interior and preferences of each agent to be
strictly monotone. This is a standard assumption to establish the core–Walras equivalence with infinite-dimensional commodity spaces via the use of the separation theorem; see Bewley (1973); Evren and Hüsseinov (2008); Rustichini and Yannelis (1993).

The purpose of this paper is twofold. First, we present the equivalence between the fuzzy core and the core under minimal assumptions unlike Noguchi (2000); Bhowmik and Graziano (2014). Due to the exact version of the Lyapunov convexity theorem in Banach spaces established in Khan and Sagara (2013), we clarify that the additional structure of commodity spaces and preferences is unnecessary whenever the measure space of agents is “saturated” in the sense of Keisler and Sun (2009). Although Bhowmik and Graziano (2014) removed the convexity of preferences from Noguchi (2000) whenever the measure space of agents is nonatomic, they must impose the requirement mentioned above because of the inevitable use of the approximate version of the Lyapunov convexity theorem; see also Evren and Hüsseinov (2008); Rustichini and Yannelis (1993).

Second, as a spin-off of the above equivalence, we obtain the coincidence of the core, the fuzzy core, and the restricted core under minimal assumptions. The equivalence between the core and the restricted core in large economies along the lines of Schmeidler (1972) was extended by Khan and Sagara (2013) to Banach commodity spaces under saturation, but the coincidence of the fuzzy core and the restricted core has not been articulated anywhere. As a matter of course, with the aforementioned additional assumption, the classical result on the core–Walras equivalence in large economies along the lines of Aumann (1964); Hildenbrand (1974) leads to the restricted core–Walras equivalence as well as the coincidence of the fuzzy core and the restricted core in infinite-dimensional commodity space whenever the measure space of agents is nonatomic.

The organization of the paper is as follows. As preliminaries, Section 2 collects some terminologies on Bochner integrals and vector measures in Banach spaces, provides the definition of saturation of measure spaces, and then presents the infinite-dimensional Lyapunov convexity theorem under saturation. Section 3 explores the main result, the fuzzy core equivalence as is mentioned above.
2 Preliminaries

2.1 Bochner Integrals and Vector Measures

Let \((T, \Sigma, \mu)\) be a complete finite measure space and \((E, \| \cdot \|)\) be a Banach space. A function \(f : T \to E\) is said to be **strongly measurable** if there exists a sequence of simple (or finitely valued) functions \(f_n : T \to E\) such that \(\| f(t) - f_n(t) \| \to 0\) a.e. \(t \in T\); \(f\) is said to be **Bochner integrable** if it is strongly measurable and \(\int_T \| f(t) \| d\mu < \infty\), where the Bochner integral of \(f\) over \(A \in \Sigma\) is defined by \(\int_A f d\mu = \lim_n \int_A f_n d\mu\). By the Pettis measurability theorem (see Diestel and Uhl (1977, Theorem II.1.2)), \(f\) is strongly measurable if and only if it is Borel measurable with respect to the norm topology of \(E\) whenever \(E\) is separable. Denote by \(L^1(\mu, E)\) the space of \((\mu\)-equivalence classes of) \(E\)-valued Bochner integrable functions on \(T\) such that \(\| f(\cdot) \| \in L^1(\mu)\), normed by \(\| f \|_1 = \int_T \| f(t) \| d\mu\).

A countably additive set function from \(\Sigma\) into \(E\) is called a **vector measure**. For a vector measure \(m : \Sigma \to E\), a set \(N \in \Sigma\) is said to be \(m\)-null if \(m(A \cap N) = 0\) for every \(A \in \Sigma\). A vector measure \(m : \Sigma \to E\) is said to be **\(\mu\)-continuous** (or **absolutely continuous** with respect to \(\mu\)) if every \(\mu\)-null set is \(m\)-null. For a scalar valued simple function \(\varphi : T \to \mathbb{R}\) on \(T\) with \(\varphi = \sum_{i=1}^n \alpha_i \chi_{A_i}\), where \(\alpha_1, \ldots, \alpha_n\) are nonzero scalars, \(A_1, \ldots, A_n \in \Sigma\) are pairwise disjoint, and \(\chi_{A_i}\) is the indicator function of \(A_i\), define \(\Psi_m(\varphi) := \sum_{i=1}^n \alpha_i m(A_i)\). Then \(\Psi_m\) is a continuous linear operator from the space of simple functions of the above form into \(E\); see Diestel and Uhl (1977, p.6). Thus, \(\Psi_m\) has a unique continuous extension (still denoted by \(\Psi_m\)) to \(L^\infty(\mu)\). Hence, it is legitimate to define the integral of \(\varphi \in L^\infty(\mu)\) with respect to the vector measure \(m\) via \(\int_T \varphi(t) d\mu := \Psi_m(\varphi)\); see Diestel and Uhl (1977, Theorem I.1.13). This integral is called the **Bartle integral** of \(\varphi\).

2.2 Lyapunov Convexity Theorem in Banach Spaces

A finite measure space \((T, \Sigma, \mu)\) is said to be **essentially countably generated** if its \(\sigma\)-algebra can be generated by a countable number of subsets together with the null sets; \((T, \Sigma, \mu)\) is said to be **essentially uncountably generated** whenever it is not essentially countably generated. Let \(\Sigma_S = \{ A \cap S \mid A \in \Sigma \}\) be the \(\sigma\)-algebra restricted to \(S \in \Sigma\). Denote by \(L^1_S(\mu)\) the space of \(\mu\)-integrable functions on the measurable space \((S, \Sigma_S)\) whose elements are restrictions of functions in \(L^1(\mu)\) to \(S\). An equivalence relation \(\sim\) on \(\Sigma\) is given
by \( A \sim B \iff \mu(A \Delta B) = 0 \), where \( A \Delta B \) is the symmetric difference of \( A \) and \( B \) in \( \Sigma \). The collection of equivalence classes is denoted by \( \Sigma(\mu) = \Sigma/\sim \) and its generic element \( \hat{A} \) is the equivalence class of \( A \in \Sigma \). We define the metric \( \rho \) on \( \Sigma(\mu) \) by \( \rho(\hat{A}, \hat{B}) = \mu(A \Delta B) \). Then \((\Sigma(\mu), \rho)\) is a complete metric space (see Aliprantis and Border (2006, Lemma 13.13)) and \((\Sigma(\mu), \rho)\) is separable if and only if \( L^1(\mu) \) is separable; see Aliprantis and Border (2006, Lemma 13.14). The density of \((\Sigma(\mu), \rho)\) is the smallest cardinal number of the form \(|U|\), where \( U \) is a dense subset of \( \Sigma(\mu) \).

**Definition 2.1.** A finite measure space \((T, \Sigma, \mu)\) is saturated if \( L^1_S(\mu) \) is nonseparable for every \( S \in \Sigma \) with \( \mu(S) > 0 \).

The saturation of finite measure spaces is also synonymous with the uncountability of the density of \( \Sigma_S(\mu) \) for every \( S \in \Sigma \) with \( \mu(S) > 0 \); see Fremlin (2012, 331Y(e) and 365X(p)). Saturation implies nonatomicity; in particular, a finite measure space \((T, \Sigma, \mu)\) is nonatomic if and only if the density of \( \Sigma_S(\mu) \) is greater than or equal to \( \aleph_0 \) for every \( S \in \Sigma \) with \( \mu(S) > 0 \). Several equivalent definitions for saturation are known; see Fajardo and Keisler (2002); Fremlin (2012); Hoover and Keisler (1984); Keisler and Sun (2009). One of the simple characterizations of the saturation property is as follows. A finite measure space \((T, \Sigma, \mu)\) is saturated if and only if \((S, \Sigma_S, \mu)\) is essentially uncountably generated for every \( S \in \Sigma \) with \( \mu(S) > 0 \). A germinal notion of saturation already appeared in Kakutani (1944); Maharam (1942).

For our purpose, the power of saturation is exemplified in the Lyapunov convexity theorem in infinite dimensions.

**Theorem 2.1** (Khan and Sagara (2013)). Let \((T, \Sigma, \mu)\) be a saturated finite measure space and \( E \) be a separable Banach space. If \( m : \Sigma \to E \) is a \( \mu \)-continuous vector measure, then \( m \) has weakly compact convex range with:

\[
m(\Sigma) = \left\{ \int_T \varphi(t)dm \in E \mid 0 \leq \varphi \leq 1, \varphi \in L^\infty(\mu) \right\}.
\]

Conversely, every \( \mu \)-continuous vector measure \( m : \Sigma \to E \) has weakly compact convex range, then \((T, \Sigma, \mu)\) is saturated whenever \( E \) is infinite dimensional.

**Remark 2.1.** The significance of the saturation property lies in the fact that it is necessary and sufficient for the Lyapunov convexity theorem (see Khan and Sagara (2013, 2015, 2016); Greinecker and Podczeck (2013)), the
weak compactness and convexity of the Bochner integral of a multifunction (see Podczeck (2008); Sun and Yannelis (2008)), the bang-bang principle (see Khan and Sagara (2014, 2016)), and Fatou’s lemma (see Khan and Sagara (2014); Khan, Sagara and Suzuki (2016)). For a further generalization of Theorem 2.1 to nonseparable locally convex spaces, see Greinecker and Podczeck (2013); Khan and Sagara (2015, 2016); Sagara (2017); Urbinati (2019). Another intriguing characterization of saturation in terms of the existence of Nash equilibria in large games is found in Keisler and Sun (2009).

3 Fuzzy Core Equivalence

3.1 Fuzzy Core

We can now turn to the substantive formulation of a large economy along the lines of Aumann (1964); Hildenbrand (1974). Let \((T, \Sigma, \mu)\) be a finite measure space of agents with its generic element denoted by \(t \in T\). A commodity space \(E\) is a Banach space. A consumption set \(X(t)\) of each agent is described by a multifunction \(X: T \rightarrow E\) such that \(X(t) \subset E\) for every \(t \in T\). A preference relation \(\succ(t) \subset X(t) \times X(t)\) of each agent is described by a multifunction \(\succ: T \rightarrow E \times E\) such that \(\succ(t)\) is a transitive and irreflexive binary relation on the consumption set \(X(t)\), where the relation \((x, y) \in \succ(t)\) is denoted by \(x \succ(t) y\). An initial endowment of each agent \(\omega(t) \in X(t)\) is provided by a Bochner integrable function \(\omega \in L^1(\mu, E)\). An economy \(E\) is a quadruple \(E = [(T, \Sigma, \mu), (X(t))_{t \in T}, (\succ(t))_{t \in T}, (\omega(t))_{t \in T}]\).

A Bochner integrable function \(f \in L^1(\mu, E)\) is called an assignment for an economy \(E\) if \(f(t) \in X(t)\) a.e. \(t \in T\). An assignment \(f\) is called an allocation if \(\int_T f(t) d\mu = \int_T \omega(t) d\mu\). A coalition is a nonempty set \(A\) in \(\Sigma\) with \(\mu(A) > 0\). A coalition \(A \in \Sigma\) blocks an allocation \(f\) if there exists an allocation \(g\) such that \(g(t) \succ(t) f(t)\) for every \(t \in A\) and \(\int_A g(t) d\mu = \int_A \omega(t) d\mu\). The set of all allocations that no coalition in \(\Sigma\) can block is called the core of the economy \(E\), denoted by \(\mathcal{C}(E)\).

A measurable function \(\alpha: T \rightarrow [0, 1]\) with \(\mu(\{\alpha > 0\}) > 0\) is called a fuzzy coalition. Following Aubin (1981, 1982), \(\alpha(t)\) is regarded as the rate of participation of agent \(t\) in the coalition \(\{\alpha > 0\} \in \Sigma\). If a fuzzy coalition \(\alpha\) is an indicator function of the form \(\chi_A\), then it is identified with a “crisp” coalition \(A\). A fuzzy coalition \(\alpha: T \rightarrow [0, 1]\) blocks an allocation \(f\) if there exists an allocation \(g\) such that \(g(t) \succ(t) f(t)\) for every \(t \in \{\alpha > 0\}\).
and \( \int_T \alpha(t)g(t)d\mu = \int_T \alpha(t)\omega(t)d\mu \). The set of all allocations that no fuzzy coalition can block is called the fuzzy core of the economy \( \mathcal{E} \), denoted by \( \mathcal{C}^F(\mathcal{E}) \). The inclusion \( \mathcal{C}^F(\mathcal{E}) \subseteq \mathcal{C}(\mathcal{E}) \) follows from the definitions.

Under the saturation of the measure space of agents, the coincidence of the core and fuzzy core can be established without any order structure on the Banach commodity space and strict monotonicity of the preferences unlike Noguchi (2000); Bhowmik and Graziano (2014).

Theorem 3.1. Let \( (T, \Sigma, \mu) \) be a saturated finite measure space and \( E \) be a Banach space. Then \( \mathcal{C}(\mathcal{E}) = \mathcal{C}^F(\mathcal{E}) \).

Proof. It suffices to show the inclusion \( \mathcal{C}(\mathcal{E}) \subseteq \mathcal{C}^F(\mathcal{E}) \) is obvious. Toward this end, we first treat the case where \( E \) is separable. Suppose to the contrary that there is \( f \in \mathcal{C}(\mathcal{E}) \) that does not belong to \( \mathcal{C}^F(\mathcal{E}) \). Then there exist a fuzzy coalition \( \alpha : T \to [0, 1] \) and an allocation \( g \in L^1(\mu, E) \) such that \( g(t) \succ_t f(t) \) for every \( t \in \{ \alpha > 0 \} \) and \( \int_T \alpha(t)g(t)d\mu = \int_T \alpha(t)\omega(t)d\mu \). Define the vector measure \( m : \Sigma \to E \) by

\[
m(S) := \int_S (g(t) - \omega(t))d\mu, \quad S \in \Sigma.
\]

(3.1)

Since \( m \) is absolutely continuous with respect to the saturated finite measure \( \mu \), in view of Theorem 2.1, there exists \( A \in \Sigma \) such that \( m(A) = \int_T \alpha(t)d\mu \), and hence, \( \int_A (g(t) - \omega(t))d\mu = \int_T \alpha(t)d\mu = \int_T \alpha(t)(g(t) - \omega(t))d\mu = 0 \). This means that coalition \( A \) blocks the allocation \( f \) in the core, a contradiction.

Separability of \( E \) can be removed by the following procedure. Note that the vector measure \( m : \Sigma \to E \) defined in (3.1) has a relatively compact range because it has a Bochner integrable density; see Diestel and Uhl (1977, Corollary II.3.9). Since the closure \( \overline{m(\Sigma)} \) of the range \( m(\Sigma) \) is compact, it is also complete and separable; see Dunford and Schwartz (1958, Theorem I.6.15). Then take a countable dense subset \( D \) of \( \overline{m(\Sigma)} \) and consider its closed linear hull \( \tilde{E} := \text{span}(D) \) spanned by \( D \). Then \( \tilde{E} \) is separable (see Dunford and Schwartz (1958, Lemma II.1.5)), and hence, it is a closed separable vector subspace of \( E \) containing \( \overline{m(\Sigma)} \) due to the inclusion \( \text{span}(D) \subseteq \text{span}(D) \). Finally, apply Theorem 2.1 to the vector measure \( m : \Sigma \to \tilde{E} \) which is induced by restricting the range of \( m \) in the above argument. \( \square \)

Remark 3.1. The notion of fuzzy core explored in this paper is somewhat different from the one in Noguchi (2000); Bhowmik and Graziano (2014).
Specifically, the fuzzy core defined here is smaller than the one defined in the above references because they formulate a fuzzy coalition as a “simple” measurable function $\alpha : T \rightarrow [0, 1]$ with $\mu(\{\alpha > 0\}) > 0$, so that every fuzzy coalition in their sense takes only “finite” values in the closed unit interval. This means that blocking fuzzy coalitions to an allocation in their sense are automatically blocking fuzzy coalitions in our sense. Such a discrepancy of the two notions of fuzzy core, however, disappears in Theorem 3.1 whenever the measure space of fuzzy core is saturated.

### 3.2 Restricted Core

According to Schmeidler (1972) followed by Vind (1972), we introduce coalitions whose population size is restricted to an arbitrarily small real number $\varepsilon \in (0, \mu(T)]$. A coalition $A \in \Sigma$ is called an $\varepsilon$-coalition if $\mu(A) = \varepsilon$. The set of all allocations that no $\varepsilon$-coalition can block is called the $\varepsilon$-core of the economy $\mathcal{E}$, denoted by $C^\varepsilon(\mathcal{E})$. It follows from the definitions that $C(\mathcal{E}) \subset C^\varepsilon(\mathcal{E})$ and $C^F(\mathcal{E}) \subset C^\varepsilon(\mathcal{E})$ for every $\varepsilon \in (0, \mu(T)]$.

The following result due to Khan and Sagara (2013) is an infinite-dimensional analogue of Schmeidler (1972) under saturation. For completeness, we provide a more direct and simpler proof based on Theorem 2.1.

**Lemma 3.1** (Khan and Sagara (2013)). Let $(T, \Sigma, \mu)$ be a saturated finite measure space and $E$ be a Banach space. If $\int_A f(t)d\mu = 0$ for some $f \in L^1(\mu, E)$ and $A \in \Sigma$ with $\mu(A) > 0$, then for every $\theta \in [0, 1]$ there exists $B \in \Sigma$ with $B \subset A$ such that $\int_B f(t)d\mu = 0$ and $\mu(B) = \theta \mu(A)$.

**Proof.** Assume first that $E$ is separable. Suppose that $\int_A f(t)d\mu = 0$ for some $f \in L^1(\mu, E)$ and $A \in \Sigma$ with $\mu(A) > 0$. Define the $\mu$-continuous vector measure $m : \Sigma \rightarrow E \times \mathbb{R}$ by

$$m(S) := \left(\int_S f(t)d\mu, \mu(S)\right), \quad S \in \Sigma.$$ 

In view of Theorem 2.1, the range $m(\Sigma_A)$ is convex. Since $(0, \theta \mu(A)) = \theta m(A) + (1 - \theta)m(\emptyset) \in m(\Sigma_A)$ for every $\theta \in [0, 1]$, there exists $B \in \Sigma_A$ such that $(0, \theta \mu(A)) = m(B) = (0, \mu(B))$ by the convexity of $m(\Sigma_A)$. Separability of $E$ can be removed from the above argument by the same procedure as demonstrated in the proof of Theorem 2.1. \qed

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As demonstrated in Schmeidler (1972) for finite-dimensional commodity spaces, if an allocation \( f \) does not belong to \( C(\mathcal{E}) \), then arbitrarily small \( \varepsilon \)-coalitions can block \( f \). More specifically, we reproduce the following result from Khan and Sagara (2013).

**Theorem 3.2** (Khan and Sagara (2013)). Let \( (T, \Sigma, \mu) \) be a saturated finite measure space and \( E \) be a Banach space. If \( f \) is an allocation that is blocked by a coalition \( A \in \Sigma \) via an allocation \( g \), then for every \( \varepsilon \in (0, \mu(A)] \) there exists an \( \varepsilon \)-coalition \( B \in \Sigma \) with \( B \subset A \) such that \( B \) blocks \( f \) via \( g \).

**Proof.** Since \( A \in \Sigma \) is a blocking coalition to \( f \), there exists an allocation \( g \) such that \( \int_A (g(t) - \omega(t))d\mu = 0 \). Take any \( \varepsilon \in (0, \mu(A)] \). Applying Lemma 3.1 for the choice of \( \theta = \varepsilon/\mu(A) \in (0, 1] \) yields that there exists an \( \varepsilon \)-coalition \( B \in \Sigma_A \) such that \( \int_B (g(t) - \omega(t))d\mu = 0 \). Hence, \( f \) is blocked by the \( \varepsilon \)-coalition \( B \) with \( B \subset A \) via the allocation \( g \). \( \square \)

**Corollary 3.1.** There exists \( \delta \in (0, \mu(T)] \) such that \( C^\varepsilon(\mathcal{E}) = C(\mathcal{E}) = C^F(\mathcal{E}) \) for every \( \varepsilon \in (0, \delta] \).

**Proof.** In view of Theorem 3.1, it suffices to show the inclusion for some \( \delta \in (0, \mu(T)] \): \( C^\varepsilon(\mathcal{E}) \subset C(\mathcal{E}) \) for every \( \varepsilon \in (0, \delta] \). Suppose to the contrary that for every \( \delta \in (0, \mu(T)] \) there exist \( \varepsilon \in (0, \delta] \) and \( f \in C^\varepsilon(\mathcal{E}) \) such that \( f \) does not belong to \( C(\mathcal{E}) \). Then some coalition \( A \in \Sigma \) blocks \( f \). Choose here \( \delta = \mu(A) \). It follows from Theorem 3.2 that there exists an \( \varepsilon \)-coalition \( B \in \Sigma \) with \( B \subset A \) such that \( B \) blocks \( f \), a contradiction to the fact that \( f \in C^\varepsilon(\mathcal{E}) \) with \( \varepsilon \in (0, \mu(A)] \). \( \square \)

### 4 Concluding Remark

This is a sharply focused note on a role of the infinite-dimensional version of the Lyapunov convexity theorem on an equivalence theorem for Aubin’s fuzzy core, and we conclude it by drawing attention to the resemblance between Aubin’s construct and that of Aumann–Shitovitz on cores of measure-theoretic economies with atoms; see Gabszewicz and Shitovitz (1992) for a survey, and Anderson et al. (2021) for ongoing work. Zadeh’s insight has now turned into a mature subfield of applied mathematics (see, for example Bělohlávek et al. (2017) and we think it worthwhile to pursue this connection that has so far eluded the workers in either register, and both the disciplines of economics and of applied mathematics.
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