Generating subgraphs in chordal graphs

Vadim E. Levit\textsuperscript{a,}\textsuperscript{*}, David Tankus\textsuperscript{b}

\textsuperscript{a}Department of Computer Science, Ariel University, ISRAEL
\textsuperscript{b}Department of Software Engineering, Sami Shamoon College of Engineering, ISRAEL

Abstract

A graph $G$ is \textit{well-covered} if all its maximal independent sets are of the same cardinality. Assume that a weight function $w$ is defined on its vertices. Then $G$ is $w$-\textit{well-covered} if all maximal independent sets are of the same weight. For every graph $G$, the set of weight functions $w$ such that $G$ is $w$-well-covered is a vector space, denoted $WCW(G)$.

Let $B$ be a complete bipartite induced subgraph of $G$ on vertex sets of bipartition $B_X$ and $B_Y$. Then $B$ is \textit{generating} if there exists an independent set $S$ such that $S \cup B_X$ and $S \cup B_Y$ are both maximal independent sets of $G$. In the restricted case that a generating subgraph $B$ is isomorphic to $K_{1,1}$, the unique edge in $B$ is called a \textit{relating edge}. Generating subgraphs play an important role in finding $WCW(G)$.

Deciding whether an input graph $G$ is well-covered is \textit{co-NP}-complete. Hence, finding $WCW(G)$ is \textit{co-NP}-hard. Deciding whether an edge is relating is \textit{NP}-complete. Therefore, deciding whether a subgraph is generating is \textit{NP}-complete as well.

A graph is chordal if every induced cycle is a triangle. It is known that finding $WCW(G)$ can be done polynomially in the restricted case that $G$ is chordal. Thus recognizing well-covered chordal graphs is a polynomial problem. We present a polynomial algorithm for recognizing relating edges and generating subgraphs in chordal graphs.

\textit{Keywords:} weighted well-covered graph, maximal independent set, relating edge, generating subgraph, chordal graphs.

\textsuperscript{*}Corresponding Author
E-mail addresses: levitv@ariel.ac.il (V. E. Levit), davidt@sce.ac.il (D. Tankus).
1. Introduction

1.1. Basic definitions and notation

Throughout this paper $G$ is a simple (i.e., a finite, undirected, loopless and without multiple edges) graph with vertex set $V(G)$ and edge set $E(G)$. Cycles of $k$ vertices are denoted by $C_k$. When we say that $G$ does not contain $C_k$ for some $k \geq 3$, we mean that $G$ does not admit subgraphs isomorphic to $C_k$. Note that these subgraphs are not necessarily induced.

Let $u$ and $v$ be two vertices in $G$. The distance between $u$ and $v$, denoted $d(u, v)$, is the length of a shortest path between $u$ and $v$, where the length of a path is the number of its edges. If $S$ is a non-empty set of vertices, then the distance between $u$ and $S$, is defined as $d(u, S) = \min \{d(u, s) : s \in S\}$.

For every positive integer $i$, denote $N_i(S) = \{x \in V(G) : d(x, S) = i\}$, and $N_i[S] = \{x \in V(G) : d(x, S) \leq i\}$.

If $S$ contains a single vertex, $v$, then we abbreviate $N_i(\{v\}), N_i[\{v\}]$ to be $N_i(v), N_i[v]$, respectively. We denote by $G[S]$ the subgraph of $G$ induced by $S$. For every two sets, $S$ and $T$, of vertices of $G$, we say that $S$ dominates $T$ if $T \subseteq N_1[S]$.

1.2. Well-covered graphs

Let $G$ be a graph. A set of vertices $S$ is independent if its elements are pairwise nonadjacent. An independent set of vertices is maximal if it is not a subset of another independent set. An independent set of vertices is maximum if the graph does not contain an independent set of a higher cardinality.

The graph $G$ is well-covered if every maximal independent set is maximum \[15\]. Assume that a weight function $w : V(G) \rightarrow \mathbb{R}$ is defined on the vertices of $G$. For every set $S \subseteq V(G)$, define $w(S) = \sum_{s \in S} w(s)$. Then $G$ is $w$-well-covered if all maximal independent sets of $G$ are of the same weight.

The problem of finding a maximum independent set is \textbf{NP}-complete. However, if the input is restricted to well-covered graphs, then a maximum independent set can be found in polynomial time using the greedy algorithm. Similarly, if a weight function $w : V(G) \rightarrow \mathbb{R}$ is defined on the vertices...
of $G$, and $G$ is $w$-well-covered, then finding a maximum weight independent set is a polynomial problem. There is an interesting application, where well-covered graphs are investigated in the context of distributed $k$-mutual exclusion algorithms [21].

The recognition of well-covered graphs is known to be co-NP-complete. This is proved independently in [6] and [18]. In [5] it is proven that the problem remains co-NP-complete even when the input is restricted to $K_{1,4}$-free graphs. However, the problem can be solved in polynomial time for $K_{1,3}$-free graphs [18, 20], for graphs with girth 5 at least [8], for graphs with a bounded maximal degree [4], for chordal graphs [16], and for graphs without cycles of lengths 4 and 5 [9].

For every graph $G$, the set of weight functions $w$ for which $G$ is $w$-well-covered is a vector space [4]. That vector space is denoted $WCW(G)$ [3]. Since recognizing well-covered graphs is co-NP-complete, finding the vector space $WCW(G)$ of an input graph $G$ is co-NP-hard. However, finding $WCW(G)$ can be done in polynomial time when the input is restricted to graphs with a bounded maximal degree [4], to graphs without cycles of lengths 4, 5 and 6 [13], and to chordal graphs [2].

1.3. Generating subgraphs and relating edges

Further we make use of the following notions, which have been introduced in [11]. Let $B$ be an induced complete bipartite subgraph of $G$ on vertex sets of bipartition $B_X$ and $B_Y$. Assume that there exists an independent set $S$ such that each of $S \cup B_X$ and $S \cup B_Y$ is a maximal independent set of $G$. Then $B$ is a generating subgraph of $G$, and the set $S$ is a witness that $B$ is generating. We observe that every weight function $w$ such that $G$ is $w$-well-covered must satisfy the restriction $w(B_X) = w(B_Y)$.

If the generating subgraph $B$ contains only one edge, say $xy$, it is called a relating edge [3]. In such a case, the equality $w(x) = w(y)$ is valid for every weight function $w$ such that $G$ is $w$-well-covered.

Recognizing relating edges is known to be NP-complete [3], and it remains NP-complete even when the input is restricted to graphs without cycles of lengths 4 and 5 [12]. Therefore, recognizing generating subgraphs is also NP-complete when the input is restricted to graphs without cycles of lengths 4 and 5. However, recognizing relating edges can be done in polynomial time if the input is restricted to graphs without cycles of lengths 4 and 6 [12], and to graphs without cycles of lengths 5 and 6 [13].
It is also known that recognizing generating subgraphs is a polynomial problem when the input is restricted to graphs without cycles of lengths 4, 6 and 7 \cite{11}, to graphs without cycles of lengths 4, 5 and 6 \cite{13}, and to graphs without cycles of lengths 5, 6 and 7 \cite{13}.

1.4. Chordal graphs

A graph is chordal (triangulated) if its every induced cycle is a triangle \cite{1}. Finding a maximum weight independent set in a chordal graph is a polynomial task \cite{10}. Deciding whether a chordal graph is well-covered can be done polynomially \cite{16}. Finding $W_{CW}(G)$ can be completed polynomially if $G$ is chordal \cite{2}. We present a polynomial time algorithm, which receives as input a chordal graph $G$ and an induced complete bipartite subgraph $B$. The algorithm decides whether $B$ is generating.

2. Polynomial results for chordal graphs

2.1. The vector space $W_{CW}(G)$

A vertex $x$ in a graph $G$ is simplicial if $N_1[x]$ is a clique. A simplicial clique is a maximal clique containing a simplicial vertex.

**Theorem 2.1.** \cite{16} Let $G$ be a chordal graph. Then $G$ is well-covered if and only if every vertex of $G$ belongs to exactly one simplicial clique.

In \cite{2}, a polynomial characterization of $W_{CW}(G)$ is presented, for the case that $G$ is chordal. The following definitions and notation are used.

Let $C(G)$ be the set of all simplicial cliques and $sc(G) = |C(G)|$. Let $C \in C(G)$ be a simplicial clique. The associated weighting function, denoted $f_C : V(G) \rightarrow \mathbb{R}$, is defined as follows. If $v \in C$ then $f_C(v) = 1$, otherwise $f_C(v) = 0$.

**Lemma 2.2.** \cite{2} $f_C \in W_{CW}(G)$ for every graph $G$, and for each simplicial clique $C \in C(G)$. Moreover, $\{f_C : C \in C(G)\}$ is an independent set of vectors, and $wcdim(G) \geq sc(G)$.

**Theorem 2.3.** \cite{2} Let $G$ be a chordal graph. Then $wcdim(G) = sc(G)$. 
By Lemma 2.2 for every graph $G$, the vector space spanned by $\{f_C : C \in C(G)\}$ is a subspace of $WCW(G)$. Moreover, if $G$ is chordal then, by Theorem 2.3 the vector space spanned by $\{f_C : C \in C(G)\}$ coincides with $WCW(G)$. Let $T$ be the set of all simplicial vertices in a chordal graph $G$, and let $S$ be a maximal independent set of $G[T]$. Clearly, $C(G) = \{N_1(v) : v \in S\}$. In order to construct a function $w \in WCW(G)$, the following algorithm can be implemented. For every $s \in S$, define $w(s)$ arbitrarily, while for each vertex $v \in V(G) \setminus S$, let $w(v) = w(N_1(v) \cap S)$. In other words, assigning 1 to one vertex in $S$ and 0 to all others, we describe a basis of $WCW(G)$. Clearly, this procedure is polynomial.

### 2.2. Generating subgraphs

The main result of this subsection is a polynomial time algorithm for recognizing generating subgraphs in chordal graphs. Let $G$ be a chordal graph, and let $B$ be an induced complete bipartite subgraph of $G$ on vertex sets of the bipartition $B_X = \{x_1, ..., x_l\}$ and $B_Y = \{y_1, ..., y_k\}$, where $l \leq k$.

**Lemma 2.4.** $l = 1$.

**Proof.** If $l \geq 2$ then $G[\{x_1, x_2, y_1, y_2\}]$ is isomorphic to $K_{2,2} = C_4$, which contradicts the fact that $G$ is chordal. ■

Since $l = 1$, we denote $B_X = \{x\}$. For each $V \in \{X, Y\}$ let $S \subseteq B_V$ and $U \in \{X, Y\} \setminus \{V\}$. Define $M_1(S) = N_1(S) \cap N_2(B_U)$ and $M_2(S) = N_1(M_1(S)) \cap N_2(B_Y)$. If $S$ contains a single vertex, $v$, abbreviate $M_1(\{v\})$ and $M_2(\{v\})$ to $M_1(v)$ and $M_2(v)$, respectively. Define $M_1(B) = M_1(B_X) \cup M_1(B_Y)$ and $M_2(B) = M_2(B_X) \cup M_2(B_Y)$.

**Lemma 2.5.** $M_1(B_X)$ and $M_1(B_Y)$ are disjoint and nonadjacent.

**Proof.** The fact that $M_1(B_X)$ and $M_1(B_Y)$ are disjoint follows immediately from the definition of $M_1(B_X)$ and $M_1(B_Y)$.

Let $x' \in M_1(B_X)$ and $y' \in M_1(B_Y)$, and assume on the contrary that $x' \in N(y')$. There exists $1 \leq i \leq k$ such that $y' \in N_1(y_i)$. Therefore, $C = (x, y_i, y', x')$ is a copy of $C_4$. Obviously, $xy' \notin E(G)$ and $y_i x' \notin E(G)$. Hence, $C$ is an induced $C_4$, which is a contradiction. ■

**Lemma 2.6.** $M_2(B_X)$ and $M_2(B_Y)$ are disjoint and nonadjacent.
Proof. Assume on the contrary that there exists a vertex \( v \in M_2(B_X) \cap M_2(B_Y) \). There exist two vertices \( x' \in M_1(B_X) \cap N_1(v) \) and \( y' \in M_1(B_Y) \cap N_1(v) \). By Lemma 2.7, \( x' \) and \( y' \) are distinct and nonadjacent. There exists \( y \in B_Y \) such that \( G[\{x, y, y', v, x'\}] \) is isomorphic to \( C_5 \), which is a contradiction.

Assume, on the contrary, that there exist two adjacent vertices, \( x'' \in M_2(B_X) \) and \( y'' \in M_2(B_Y) \). There exist \( x' \in M_1(B_X) \cap N_1(x'') \) and \( y' \in M_1(B_Y) \cap N_1(y'') \). By Lemma 2.7, \( x' \) and \( y' \) are distinct and nonadjacent. There exists \( y \in B_Y \) such that \( G[\{x, y, y', y'', x'', x'\}] \) is isomorphic to \( C_6 \), which is a contradiction.

Lemma 2.7. Let \( 1 \leq i < j \leq k \). Then \( M_1(y_i) \) and \( M_1(y_j) \) are disjoint and nonadjacent.

Proof. If there existed a vertex \( v \in M_1(y_i) \cap M_1(y_j) \) then \( G[\{x, y_i, v, y_j\}] \) was isomorphic to \( C_4 \).

If \( y_i' \in M_1(y_i) \) and \( y_j' \in M_1(y_j) \) were adjacent then \( G[\{x, y_i, y_i', v, y_j', y_j\}] \) was isomorphic to \( C_5 \).

Lemma 2.8. Let \( 1 \leq i < j \leq k \). Then \( M_2(y_i) \) and \( M_2(y_j) \) are disjoint and nonadjacent.

Proof. Assume, on the contrary, that there exists a vertex \( v \in M_2(y_i) \cap M_2(y_j) \). There exist two vertices, \( y_i' \in M_1(y_i) \cap N_1(v) \) and \( y_j' \in M_1(y_j) \cap N_1(v) \). By Lemma 2.7, \( y_i' \) and \( y_j' \) are distinct and nonadjacent. Hence, \( G[\{x, y_i, y_i', v, y_j', y_j\}] \) is isomorphic to \( C_6 \), which is a contradiction.

Assume, on the contrary, that \( y_i'' \in M_2(y_i) \) and \( y_j'' \in M_2(y_j) \) are adjacent. There exist two vertices, \( y_i' \in M_1(y_i) \cap N_1(y_i'') \) and \( y_j' \in M_1(y_j) \cap N_1(y_j'') \). By Lemma 2.7, \( y_i' \) and \( y_j' \) are distinct and nonadjacent. Hence, \( G[\{x, y_i, y_i', y_i'', y_j', y_j\}] \) is isomorphic to \( C_7 \), which is a contradiction.

Define a function \( f : 2^{M_2(B)} \rightarrow 2^{M_1(B)} \) as \( f(S) = N_1(S) \cap (M_1(B)) \) for every \( S \subseteq M_2(B) \). In short, we write \( f(w) \) instead of \( f(\{w\}) \).

Lemma 2.9. Let \( w \in M_2(B) \). Then \( G[f(w)] \) is a clique.

Proof. There exists \( b \in B \) such that \( w \in M_2(b) \). It should be proved that \( N_1(w) \cap M_1(b) \) is a clique. Assume, on the contrary, that there exist two nonadjacent vertices, \( v_1 \) and \( v_2 \), in \( N_1(w) \cap M_1(b) \). Then \( G[\{b, v_1, w, v_2\}] \) is isomorphic to \( C_4 \), which is a contradiction.
**Lemma 2.10.** Let $w_1$ and $w_2$ be two adjacent vertices in $M_2(B_X) \cup M_2(B_Y)$. Then at least one of the following holds:

- $f(w_1) \subseteq f(w_2)$.
- $f(w_2) \subseteq f(w_1)$.

**Proof.** There exists $b \in B$ such that $\{w_1, w_2\} \subseteq M_2(b)$. It should be proved that at least one of the following inclusions holds:

- $N_1(w_1) \cap M_1(b) \subseteq N_1(w_2) \cap M_1(b)$.
- $N_1(w_2) \cap M_1(b) \subseteq N_1(w_1) \cap M_1(b)$.

Assume, on the contrary, that there exist $v_1 \in (N_1(w_1) \setminus N(w_2)) \cap M_1(b)$ and $v_2 \in (N_1(w_2) \setminus N_1(w_1)) \cap M_1(b)$. If $v_1v_2 \in E$ then $G[\{v_1, v_2, w_2, w_1\}]$ is isomorphic to $C_4$. Otherwise, $G[\{v_1, b, v_2, w_2, w_1\}]$ is isomorphic to $C_5$. In both cases we obtained a contradiction, which completes the proof. □

**Lemma 2.11.** Let $C$ be a connected component of $G[M_2(B)]$. Then $f(V(C))$ is a clique.

**Proof.** There exists $b \in B$ such that $C \subseteq M_2(b)$. Assume on the contrary that $f(V(C))$ is not a clique. Then there exist two nonadjacent vertices, $v_1$ and $v_2$, in $f(V(C))$.

If there existed a vertex $w \in N_1(v_1) \cap N_1(v_2) \cap M_2(b)$, then $\{v_1, v_2\} \subseteq f(w)$, which is a contradiction to Lemma 2.9. Hence, $N_1(v_1) \cap N_1(v_2) \cap M_2(b) = \emptyset$.

Let $P$ be a shortest path in $C$ between $N_1(v_1) \cap M_2(b)$ and $N_1(v_2) \cap M_2(b)$. It holds that $G[V(P) \cup \{v_1, b, v_2\}]$ is an induced cycle of length $3 + |V(P)| \geq 5$, which contradicts the fact that $G$ is chordal. Therefore, $f(V(C))$ is a clique. □

**Lemma 2.12.** Assume that $C$ is a connected component of $G[M_2(B)]$, $w_1, w_2 \in V(C)$, and $P = (w_1 = u_1, \ldots, u_r = w_2)$ is a shortest path in $C$ between $w_1$ and $w_2$.

(i) Suppose there exists a vertex $v \in f(w_1) \setminus f(w_2)$. Then there exists an index $1 \leq s < r$ such that $v \in f(u_i) \iff i \leq s$.

(ii) Suppose there exists a vertex $v \in f(w_2) \setminus f(w_1)$. Then there exists an index $1 < t \leq r$ such that $v \in f(u_i) \iff i \geq t$. 

7
Proof. (i) Define 
\[ s = \min \{ i : 1 \leq i < r, v \in f(u_i), v \not\in f(u_{i+1}) \}, \]
and assume on the contrary that there exists \( s + 2 \leq s' \leq r \) such that \( v \in f(u_{s'}) \). Then \( G[v, u_s, ..., u_{s'}] \) is an induced cycle of length \( s' - s + 2 \geq 4 \), which contradicts the fact that the graph is chordal.
(ii) Similar to Case (i). ■

Lemma 2.13. Let \( C \) be a connected component of \( G[M_2(B)] \), and let \( w_1 \) and \( w_2 \) be two vertices in \( C \). Then there exists a vertex \( w \in V(C) \) such that \( f(w_1) \cup f(w_2) \subseteq f(w) \).

Proof. If \( f(w_1) \subseteq f(w_2) \) then \( w = w_2 \). Similarly, if \( f(w_2) \subseteq f(w_1) \) then \( w = w_1 \). Hence, assume that \( f(w_1) \setminus f(w_2) \neq \emptyset \) and \( f(w_2) \setminus f(w_1) \neq \emptyset \).

Let \( P = (w_1 = u_1, ..., u_r = w_2) \) be a shortest path in \( C \) between \( w_1 \) and \( w_2 \). Define
\[ s = \max \{ i : 1 \leq i \leq r, f(u_i) \supseteq f(w_1) \}; t = \min \{ i : s \leq i \leq r, f(u_i) \supseteq f(w_2) \}. \]

If \( s = t \) then \( f(w_1) \cup f(w_2) \subseteq f(u_s) \). Therefore, one may assume that \( s < t \).

There exist \( v_1 \in f(u_s) \setminus f(u_{s+1}) \) and \( v_2 \in f(u_t) \setminus f(u_{t-1}) \). By Lemma 2.12, \( v_1 \) is not adjacent to \( u_{s+1}, ..., u_r \) and \( v_2 \) is not adjacent to \( u_1, ..., u_{t-1} \). By Lemma 2.11, \( v_1 v_2 \in E(G) \). Hence, \( G[v_1, u_s, ..., u_t, v_2] \) is an induced cycle of size \( t - s + 3 \geq 4 \), which is a contradiction. ■

Lemma 2.14. Let \( C \) be a connected component of \( G[M_2(B)] \). Then there exists a vertex \( w \in V(C) \) such that \( f(w) = f(V(C)) \).

Proof. Follows immediately from Lemma 2.13. ■
Figure 1: $B = G[\{x, y_1, y_2, y_3\}]$ is a generating subgraph. By Lemmas 2.5 and 2.7, $M_1(x)$, $M_1(y_1)$, $M_1(y_2)$, $M_1(y_3)$ are mutually disjoint and nonadjacent. By Lemmas 2.6 and 2.8 also $M_2(x)$, $M_2(y_1)$, $M_2(y_2)$, $M_2(y_3)$ are mutually disjoint and nonadjacent. Moreover, $C = G[\{x_1'', \ldots, x_5''\}]$ is a connected component of $M_2(x)$, and $(x_1'', \ldots, x_5'')$ is a shortest path between $x_1''$ and $x_5''$ in $C$. $f(x_1'') = \{x_1'\}$, $f(x_2'') = \{x_1', x_2'\}$, $f(x_3'') = \{x_1', x_2'\}$, $f(x_4'') = \{x_1', x_2', x_3'\}$, $f(x_5'') = \{x_2', x_3\}$. By Lemma 2.11 $f(C)$ is a clique. By Lemma 2.14 $f(C) = f(x_4'')$. Note that there are vertices which are adjacent to both $B_X$ and $B_Y$, but they are not important for the algorithm.

The following algorithm receives as its input a chordal graph $G$ and an induced complete bipartite subgraph $B$ of $G$. The algorithm decides whether $B$ is generating.
Algorithm 1: Recognizing generating subgraphs in chordal graphs

1. find $M(B)$.
2. find $M_2(B)$.
3. find the connected components of $M_2(B)$.
4. foreach $w \in M_2(B)$ do
   5. calculate $|f(w)|$.
6. $S \leftarrow \emptyset$.
7. foreach connected component $C$ of $M_2(B)$ do
   8. find a vertex $w_C \in C$ such that $|f(w_C)|$ is maximal.
9. $S \leftarrow S \cup \{w_C\}$.
10. if $S$ dominates $M(B)$ then
    11. output “$B$ is generating”.
12. else
    13. output “$B$ is not generating”.

Correctness of Algorithm 1: The set $S$ is independent, because it contains one vertex from each connected component of $M_2(B)$. Let $S'$ be another independent set of $M_2(B)$. We prove that every vertex in $M_1(B)$ which is dominated by $S'$ is also dominated by $S$. Assume on the contrary that there exists a vertex $v \in M_1(B)$ which is dominated by $S'$, but not by $S$. Let $w' \in S' \cap N(v)$, let $C$ be the connected component of $M_2(B)$ which contains $w'$, and let $w$ be the vertex in $C$ which belongs to $S$. It follows from the construction of $S$ and Lemma 2.14 that $f(w') \subseteq f(w)$, which is a contradiction. Therefore, $N_1(S') \cap M_1(B) \subseteq N_1(S) \cap M_1(B)$.

If $S$ dominates $M_1(B)$, then let $S^*$ be any maximal independent set of $G[V(G) \setminus N_1[B]]$ which contains $S$. Clearly, $S^*$ is a witness that $B$ is generating. However, if $S$ does not dominate $M_1(B)$, then there does not exist an independent set in $M_2(B)$ which dominates $M_1(B)$, and therefore, $B$ is not generating.

Complexity of Algorithm 1: Each stage of the algorithm can be implemented in $O(|V|^2)$ time. Therefore, this goes in parallel with the time complexity of the whole algorithm.

Corollary 2.15. Recognizing relating edges and generating subgraphs in chordal graphs can be done polynomially.
3. Conclusions and future work

In [14] the following four problems have been defined.

- **WC problem:**
  
  *Input:* A graph $G$.
  
  *Question:* Is $G$ well-covered?

- **WCW problem:**
  
  *Input:* A graph $G$.
  
  *Output:* The vector space $WCW(G)$.

- **GS problem:**
  
  *Input:* A graph $G$, and an induced complete bipartite subgraph $B$ of $G$.
  
  *Question:* Is $B$ generating?

- **RE problem:**
  
  *Input:* A graph $G$, and an edge $xy \in E(G)$.
  
  *Question:* Is $xy$ relating?

It concluded in a table presenting complexity results on the above four problems for various graphs. The findings of the current paper may be considered as an extra line for this table. Specifically, every entry of this line claims that the corresponding problem is polynomial for chordal graphs. It was proved that the **GS** problem and the **RE** problem are **NPC** even for bipartite graphs [14]. On the other hand, for this family of graphs, it is known that the **WC** problem is polynomial [17], while the complexity status of the **WCW** problem is still open.

Since both chordal graphs and bipartite graphs are perfect, it seems natural to investigate perfect graphs with polynomially solvable **WC** and/or **WCW** problems. Some of such subclasses of graphs are known. For instance, those with bounded clique size and those with no induced $C_4$ [7].
References

[1] C. Begre, *Some classes of perfect graphs*, in “Graph Theory and Theoretical Physics” (F. Harary, ed.) Academic Press, New York, 1967, pp. 155–166.

[2] J. I. Brown, R. J. Nowakowski, *Well covered vector spaces of graphs*, SIAM Journal on Discrete Mathematics 19 (2006) 952–965.

[3] J. I. Brown, R. J. Nowakowski, I. E. Zverovich, *The structure of well-covered graphs with no cycles of length 4*, Discrete Mathematics 307 (2007) 2235-2245.

[4] Y. Caro, N. Ellingham, G. F. Ramey, *Local structure when all maximal independent sets have equal weight*, SIAM Journal on Discrete Mathematics 11 (1998) 644-654.

[5] Y. Caro, A. Sebő, M. Tarsi, *Recognizing greedy structures*, Journal of Algorithms 20 (1996) 137-156.

[6] V. Chvatal, P. J. Slater, *A note on well-covered graphs*, Quo Vadis, Graph Theory?, Annals of Discrete Mathematics 55, North Holland, Amsterdam (1993) 179-182.

[7] N. Dean, J. Zito, *Well-covered graphs and extendability*, Discrete Mathematics 126 (1994) 67-80.

[8] A. Finbow, B. Hartnell, R. Nowakowski, *A characterization of well-covered graphs of girth 5 or greater*, Journal of Combinatorial Theory B 57 (1993) 44-68.

[9] A. Finbow, B. Hartnell, R. Nowakowski, *A characterization of well-covered graphs that contain neither 4- nor 5-cycles*, Journal of Graph Theory 18 (1994) 713-721.

[10] A. Frank, *Some polynomial algorithms for certain graphs and hypergraphs*, Proc. 5th British Combinatorial Conference, (1975) Congressus Numerantium XV, Eds. C. Nash-Williams and J. Sheehan, 211–226.
[11] V. E. Levit, D. Tankus *Weighted well-covered graphs without $C_4$, $C_5$, $C_6$, $C_7$*, Discrete Applied Mathematics 159 (2011) 354-359.

[12] V. E. Levit, D. Tankus, *On relating edges in graphs without cycles of length 4*, Journal of Discrete Algorithms 26 (2014) 28-33.

[13] V. E. Levit, D. Tankus, *Well-covered graphs without cycles of lengths 4, 5 and 6*, Discrete Applied Mathematics 186 (2015) 158-167.

[14] V. E. Levit, D. Tankus, *Complexity results for generating subgraphs*, Algorithmica 80 (2018) 2384-2399.

[15] M. D. Plummer, *Some covering concepts in graphs*, Journal of Combinatorial Theory 8 (1970) 91-98.

[16] E. Prisner, J. Topp and P. D. Vestergaard, *Well-covered simplicial, chordal and circular arc graphs*, Journal of Graph Theory 21 (1996) 113–119.

[17] G. Ravindra, *Well-covered graphs*, Journal of Combinatorics, Information and System Sciences 2 (1977) 20-21.

[18] R. S. Sankaranarayana, L. K. Stewart, *Complexity results for well-covered graphs*, Networks 22 (1992) 247-262.

[19] D. Tankus, M. Tarsi, *Well-covered claw-free graphs*, Journal of Combinatorial Theory B 66 (1996) 293-302.

[20] D. Tankus, M. Tarsi, *The structure of well-covered graphs and the complexity of their recognition problems*, Journal of Combinatorial Theory B 69 (1997) 230-233.

[21] M. Yamashita, T. Kameda, *Modeling k-coteries by well-covered graphs*, Networks 34 (1999) 221–228.