Outermost boundaries for star-connected components in percolation

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Abstract

Tile $\mathbb{R}^2$ into disjoint unit squares $\{S_k\}_{k \geq 0}$ with the origin being the centre of $S_0$ and say that $S_i$ and $S_j$ are star-adjacent if they share a corner and plus-adjacent if they share an edge. Every square is either vacant or occupied. If the occupied plus-connected component $C^+(0)$ containing the origin is finite, it is known that the outermost boundary $\partial^+_0$ of $C^+(0)$ is a unique cycle surrounding the origin. For the finite occupied star-connected component $C(0)$ containing the origin, we prove in this paper that the outermost boundary $\partial^+_0$ of $C^+(0)$ is a unique connected graph consisting of a union of cycles $\bigcup_{1 \leq i \leq n} C_i$ with mutually disjoint interiors. Moreover, we have that each pair of cycles in $\partial^+_0$ share at most one vertex in common and we provide an inductive procedure to obtain a circuit containing all the edges of $\bigcup_{1 \leq i \leq n} C_i$. This has applications for contour analysis of star-connected components in percolation.

Key words: Star connected components, outermost boundary, union of cycles.

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1 Introduction

Tile $\mathbb{R}^2$ into disjoint unit squares $\{S_k\}_{k \geq 0}$ with origin being the centre of $S_0$. We say $S_1$ and $S_2$ are adjacent or star-adjacent if they share a corner between them. We say that squares $S_1$ and $S_2$ are plus-adjacent, if they share an edge between them. Here we follow the notation of Penrose (2003). Suppose every square is assigned one of the two states: occupied or vacant. In many applications like for example, percolation, it is of interest to determine the outermost boundary of the plus-connected or star-connected components containing the origin. We make formal definitions below. The case of plus-connected components is well studied (Bollobas and Riordan (2006), Penrose (2003)) and in this case, the outermost boundary is simply a cycle containing the origin. Our main result is that the outermost boundary for the star-connected component is a connected union of cycles with disjoint interiors.

Let $C(0)$ denote the star-connected occupied component containing the origin and throughout we assume that $C(0)$ is finite. Thus if $S_0$ is vacant then $C(0) = \emptyset$. Else $S_0 \in C(0)$ and if $S_1, S_2 \in C(0)$ there exists a sequence of distinct occupied squares $(Y_1, Y_2, ..., Y_t)$ all belonging to $C(0)$, such that $Y_i$ is adjacent to $Y_{i+1}$ for all $i$ and $Y_1 = S_1$ and $Y_t = S_2$. Let $G_C$ be the graph with vertex set being the set of all corners of the squares $\{S_k\}_{k \in C(0)}$ and edge set consisting of the edges of the squares $\{S_k\}_{k \in C(0)}$.

Two vertices $u$ and $v$ are said to be adjacent in $G_C$ if they share an edge between them. We say that an edge $e$ in $G_C$ is adjacent to square $S_k$ if it is one of the edges of $S_k$. We say that $e$ is a boundary edge if it is adjacent to a vacant square and is also adjacent to an occupied square. A path $P$ in $G_C$ is a sequence of distinct vertices $(u_0, u_1, ..., u_t)$ such that $u_i$ and $u_{i+1}$ are adjacent for every $i$. A cycle $C$ in $G_C$ is a sequence of distinct vertices $(v_0, v_1, ..., v_m, v_0)$ starting and ending at the same point such that $v_i$ is adjacent to $v_{i+1}$ for all $0 \leq i \leq m - 1$ and $v_m$ is adjacent to $v_0$. A circuit $C'$ in $G_C$ is a sequence of vertices $(w_0, w_1, ..., w_r, w_0)$ starting and ending at the same point such that $w_i$ is adjacent to $w_{i+1}$ for all $0 \leq i \leq r - 1$, $w_r$ is adjacent to $w_0$ and no edge is repeated in $C'$. Thus vertices may be repeated in circuits and for more related definitions, we refer to Chapter 1, Bollobas (2001).

Any cycle $C$ divides the plane $\mathbb{R}^2$ into two disjoint connected regions. As in Bollobas and Riordan (2006), we denote the bounded region to be the interior of $C$ and the unbounded region to be the exterior of $C$. We have the following definition.
**Definition 1.** We say that edge $e$ in $G_C$ is an outermost boundary edge of the component $C(0)$ if the following holds true for every cycle $C$ in $G_C$: either $e$ is an edge in $C$ or $e$ belongs to the exterior of $C$.

We define the outermost boundary $\partial_0$ of $C(0)$ to be the set of all outermost boundary edges of $G_C$.

Thus outermost boundary edges cannot be contained in the interior of any cycle in $G_C$. Our main result is the following.

**Theorem 1.** Suppose $C(0)$ is finite. The outermost boundary $\partial_0$ of $C(0)$ is a unique set of cycles $C_1, C_2, \ldots, C_n$ in $G_C$ with the following properties:

(i) The graph $\bigcup_{1 \leq i \leq n} C_i$ is a connected subgraph of $G_C$.

(ii) If $i \neq j$, the cycles $C_i$ and $C_j$ have disjoint interiors and share at most one vertex.

(iii) Every square $S_k \in C(0)$ is contained in the interior of some cycle $C_j$.

(iv) If $e \in C_j$ for some $j$, then $e$ is a boundary edge of $C(0)$ adjacent to an occupied square of $C(0)$ in the interior of $C_j$ and also adjacent to a vacant square in the exterior.

Moreover, there exists a circuit $C_{out}$ containing every edge of $\bigcup_{1 \leq i \leq n} C_i$.

The outermost boundary $\partial_0$ is therefore also an Eulerian graph with $C_{out}$ denoting the corresponding Eulerian circuit (for definitions, we refer to Chapter 1, Bollobas (2001)). We remark that the above result also provides a more detailed justification of the statement made about the outermost boundary and the corresponding circuit in the proof of Lemma 3 of Ganesan (2013). Using the above result, we also obtain the outermost circuit that is used to construct the top-down crossing in oriented percolation in a rectangle in Ganesan (2015).

The proof of the above result also obtains the outermost boundary cycle in the case of plus-connected components. We recall that $S_1$ and $S_2$ are plus-adjacent if they share an edge between them. Analogous to the star-connected case, we define $C^+(0)$ to be the plus-connected component containing the origin and define the graph $G^+_C$ consisting of edges and corners of squares in $C^+(0)$. We have the following.

**Theorem 2.** Suppose $C^+(0)$ is finite. The outermost boundary $\partial_0^+$ of $C^+(0)$ is unique cycle $C_{out}^+$ in $G_C^+$ with the following property:

(i) All squares of $C^+(0)$ are contained in the interior of $C_{out}^+$.

(ii) Every edge in $C_{out}^+$ is a boundary edge adjacent to an occupied square of $C^+(0)$ in the interior of $C_{out}^+$ and a vacant square in the exterior.
This is in contrast to star-connected components which may contain multiple cycles in the outermost boundary.

To prove Theorem 1, we use the following intuitive result about merging cycles. Analogous to $G_C$, let $G$ be the graph with vertex set being the corners of the squares $\{S_k\}_k$ and edge set being the edges of the squares $\{S_k\}_k$.

**Theorem 3.** Let $C_1$ and $C_2$ be cycles in $G$ that have more than one vertex in common. There exists a unique cycle $C_3$ consisting only of edges of $C_1$ and $C_2$ with the following properties:

(i) the interior of $C_3$ contains the interior of both $C_1$ and $C_2$,

(ii) if an edge $e$ belongs to $C_1$ or $C_2$, then either $e$ belongs to $C_3$ or is contained in its interior.

Moreover, if $C_2$ contains at least one edge in the exterior of $C_1$, then the cycle $C_3$ also contains an edge of $C_2$ that lies in the exterior of $C_1$.

The above result essentially says that if two cycles intersect at more than one point, there is an innermost cycle containing both of them in its interior. We provide an iterative construction for obtaining the cycle $C_3$, analogous to Kesten (1980) for crossings, in Section 3.

The paper is organized as follows: In Section 2, we prove Theorem 1 and in Section 3, we prove Theorem 2 and Theorem 3.

## 2 Proof of Theorem 1

**Proof of Theorem 1**. The first step is to obtain large cycles surrounding each occupied square in $C(0)$. We have the following Lemma.

**Lemma 4.** For every $S_k \in C(0)$, there exists a unique cycle $D_k$ satisfying the following properties:

(a) $S_k$ is contained in the interior of $D_k$,

(b) every edge in the cycle $D_k$ is a boundary edge adjacent to one occupied square of $C(0)$ in the interior and one vacant square in the exterior and

(c) if $C$ is any cycle in $G_C$ that contains $S_k$ in the interior, then every edge in $C$ either belongs to $D_k$ or is contained in the interior.

We denote $D_k$ to be the outermost boundary cycle containing the square $S_k$. We prove all statements at the end.

We claim that the set of distinct cycles in the set $\mathcal{D} \coloneqq \cup_{S_k \in C(0)} \{D_k\}$ is the desired outermost boundary $\partial_0$ and satisfies the conditions (i)-(iv)
mentioned in the statement of the theorem. By construction, we have that (iii) and (iv) are satisfied. To see that (ii) holds, we suppose that \( D_{k_1} \neq D_{k_2} \) and that \( D_{k_1} \) and \( D_{k_2} \) meet at more than one vertex. We know that \( D_{k_2} \) is not completely contained in \( D_{k_1} \). Thus \( D_{k_2} \) contains at least one edge in the exterior of \( D_{k_1} \). From Theorem 3, we obtain a cycle \( D'_{12} \) containing both \( D_{k_1} \) and \( D_{k_2} \) in the interior and containing an edge \( e \) present in \( D_{k_2} \) but not in \( D_{k_1} \) or its interior. The cycle \( D'_{12} \) satisfies condition (a) in Lemma above and thus contradicts the assumption that \( D_{k_1} \) satisfies (c). Thus \( D_{k_1} \) and \( D_{k_2} \) cannot meet at more than one vertex.

Also (i) holds, because of the following reason. First we note that by construction \( G_C \) is connected; let \( u_1 \) and \( u_2 \) be vertices in \( G_C \). Each \( u_i; i = 1, 2 \) is a corner of an occupied square \( S_i \in C(0) \) and by definition, \( S_1 \) and \( S_2 \) are star-connected via squares in \( C(0) \). Thus there exists a path in \( G_C \) from \( u_1 \) to \( u_2 \).

To see that \( \mathcal{D} \) is a connected subgraph of \( G_C \), we let \( v_1 \) and \( v_2 \) be vertices in \( \mathcal{D} \) that belong to cycles \( D_{r_1} \) and \( D_{r_2} \), respectively, for some \( r_1 \) and \( r_2 \). If \( r_1 = r_2 \), then \( v_1 \) and \( v_2 \) are connected by a path in \( D_{r_1} = (z_1 = v_1, z_2, ..., z_n, z_1) \). If \( r_1 \neq r_2 \), let \( \bar{P}_{12} = (w_1 = v_1, w_2, ..., w_{i-1}, w_i = v_2) \) be a path from \( v_1 \) to \( v_2 \) in \( G_C \). We iteratively construct a path \( P'_{12} \) from \( P_{12} \) using only edges of cycles in \( \mathcal{D} \). We first note that since (iii) holds, every edge in \( P_{12} \) either belongs to a cycle in \( \mathcal{D} \) or is contained in the interior of some cycle in \( \mathcal{D} \). Let \( i_1 \) be the first time \( P_{12} \) leaves \( D_{r_1} \); i.e., let \( i_1 = \min\{i \geq 1 : w_{i+1} \) belongs to exterior of \( D_{r_1} \}\).

The edge formed by the vertices \( w_{i_1} \) and \( w_{i_1+1} \) belongs to some cycle \( D_{s_1} = (x_1 = w_{i_1}, x_2, ..., x_r, x_1) \) or is contained in its interior. Since the cycles \( D_{r_1} \) and \( D_{s_1} \) have disjoint interiors, this necessarily means \( D_{s_1} \) and \( D_{r_1} \) meet at \( w_{i_1} \). Defining \( T_1 = (z_1 = v_1, z_2, ..., z_{j_1} = w_{i_1}) \), we note that \( T_1 \) is a path consisting only of edges in the cycle \( D_{r_1} \) and containing the vertex \( z_1 = v_1 \). Repeating the same procedure above, we obtain another path \( T_2 = (w_{i_1} = x_1, x_2, ..., x_{j_2} = w_{i_2}) \) contained in \( D_{s_1} \), where, as before, \( i_2 = \min\{i \geq i_1 + 1 : w_{i+1} \) belongs to exterior of \( D_{s_1} \}\) denotes the first time \( P_{12} \) leaves \( D_{s_1} \). We continue this procedure for a finite number of steps \( m \), until we reach \( v_2 \). By construction, the path \( T_i \) obtained at step \( i \), \( 2 \leq i \leq m \) is connected to \( \bigcup_{1 \leq j \leq i-1} T_j \). The final union of paths \( \bigcup_{1 \leq i \leq m} T_i \) is therefore a connected graph containing only edges in \( \mathcal{D} \) and contains \( v_1 \) and \( v_2 \).

It remains to see that an edge \( e \) belongs to the outermost boundary if and only if it belongs to some cycle in \( \mathcal{D} \). If \( e \) is an edge in a cycle \( D_k \in \mathcal{D} \) we have that \( e \) is adjacent to an occupied square \( S_e \) contained in the interior
of $D_k$ and a vacant square $S_e'$ in the exterior. If there exists a cycle $C$ in $G_C$ that contains $e$ in the interior, we then have that both $S_e$ and $S_e'$ are contained in the interior of $C$. Since $S_e'$ is exterior to $D_k$, the cycle $C$ contains at least one edge in the exterior of $D_k$. But if $D_e$ denotes the outermost cycle containing $S_e$, then by the discussion in the first paragraph, we must have that $D_e = D_k$. And thus every edge of $C$ either belongs to $D_e$ or is contained in the interior of $D_e$ which leads to a contradiction.

We also see that no other edge apart from edges of cycles in $D$ can belong to the outermost boundary since if $e_1 \notin D$, then $e_1$ is necessarily contained in the interior of some cycle $D_r \in D$.

Finally, to obtain the circuit we compute the cycle graph $H_{cyc}$ as follows: let $E_1, E_2, ..., E_n$ be the distinct outermost boundary cycles in $D$. Represent $E_i$ by a vertex $i$ in $H_{cyc}$. If $E_i$ and $E_j$ share a corner, we draw an edge between $i$ and $j$. We have the following lemma.

**Lemma 5.** We have that the graph $H_{cyc}$ described above is a tree.

We provide the proof of the above at the end.

We then obtain the circuit via induction on the number of vertices $n$ of $H_{cyc}$. For $n = 1$, it is a single cycle. Suppose we obtain the circuit of all cycle graphs containing at most $k$ vertices and let $H_{cyc}$ be a cycle graph containing $k + 1$ vertices. To obtain the circuit for $H_{cyc}$, we pick a leaf $q$ of $H_{cyc}$ and apply induction assumption on the cycle graph $H'_{cyc} = H_{cyc} \setminus q$. To fix a procedure, we choose $q$ such that the corresponding boundary cycle $E_q$ contains a square $S_j$ of least index $j$ in its interior. We have that $H'_{cyc}$ is connected and has $k$ vertices and thus has a circuit $C_k = (c_1, c_2, ..., c_r, c_1)$ containing all edges of every cycle in $H'_{cyc}$. Let $C_k$ meet the cycle $E_q = (d_1, d_2, ..., d_t, d_1)$ at $d_t = c_1$. We then form the new circuit $C_{k+1} = (d_1, d_2, ..., d_t = c_1, c_2, ..., c_r, c_1 = d_t, d_{t+1}, ..., d_i, d_1)$, which contains all edges of every cycle in $H_{cyc}$.

**Proof of Lemma 5** We note that if there exists such a $D_k$, then it is unique by definition. Let $\mathcal{E}$ be the set of all cycles in $G_C$ satisfying condition (a); i.e., if $C$ is a cycle containing $S_k$ in its interior then $C \in \mathcal{E}$. The set $\mathcal{E}$ is not empty since the cycle formed by the four edges of $S_k$ belongs to $\mathcal{E}$. We merge cycles in $\mathcal{E}$ two by two using Theorem 3 to obtain the desired cycle $D_k$. We first pick a cycle $F_1$ in $\mathcal{E}$ using a fixed procedure; for example, using an analogous iterative procedure as described in Section 1 of Ganesan (2014) for choosing paths.
We again use the same procedure to pick a cycle $F_2$ in $E \setminus F_1$ and from Theorem 3 obtain a cycle $F'_1$ consisting of only edges of $F_1$ and $F_2$ and containing both $F_1$ and $F_2$ in its interior. The cycle $F'_1$ also satisfies (a) and thus belongs to $E$. Therefore, if $E$ has $t$ cycles, then $E_1 := (E \setminus \{F_1, F_2\}) \cup F'_1$ has at most $t - 1$ cycles; if $F_1$ contains an edge in the exterior of $F_2$ and the cycle $F_2$ also contains an edge in the exterior of $F_1$, then $E_1$ has $t - 2$ cycles. Else $F'_1$ is either $F_1$ or $F_2$ and the set $E_1$ therefore contains $t - 1$ cycles.

By construction, every cycle in $E$ is either a cycle in $E_1$ or is contained in the interior of a cycle in $E_1$. Therefore, if $E_1$ contains one cycle, it is the desired outermost boundary cycle $D_k$. Else we repeat the above procedure with $E_1$ and obtain another set $E_2$ containing at most $t - 2$ cycles and again with the property that every cycle in $E$ is either a cycle in $E_2$ or is contained in the interior of a cycle in $E_2$. Continuing this process, we are finally left with a single cycle $C_{fin}$. By construction it satisfies (a) and (c). It only remains to see that (b) is true.

Suppose there exists an edge $e$ of $C_{fin}$ that is not a boundary edge. Since $e$ is an edge of $G_C$, we then have that $e$ is adjacent to two occupied squares $S_1$ and $S_2$, with one of the squares, say $S_1$, contained in the interior of $C_{fin}$ and the other square $S_2$, contained in the exterior. The cycle $C_2$ containing the four edges of the square $S_2$ and the cycle $C_{fin}$ have the edge $e$ in common and thus more than one vertex in common. Since $C_2$ contains at least one edge in the exterior of $C_{fin}$, we use Theorem 3 to obtain a larger cycle $C'_2$ containing both $C_{fin}$ and $C_2$ in the interior. The cycle $C'_2$ contains at least one edge not in $C_{fin}$. But since $C_{fin}$ satisfies (c), this is a contradiction. Thus every edge $e$ of $C_{fin}$ is a boundary edge.

By the same argument above, we also see that the edge $e$ cannot be adjacent to an occupied square in the exterior of $C_{fin}$. Thus $e$ is adjacent to an occupied square in the interior and a vacant square in the exterior.

Proof of Lemma[3]: We already have that $H_{cyc}$ is connected. It is enough to see that it is acyclic. Before we prove that, we make the following observation. Consider a path $P = (i_1, i_2, ..., i_m)$ in $H_{cyc}$. We see that any vertex in $E_{i_1}$ and any vertex in $E_{i_m}$ is connected by a path consisting only of edges of the cycles $\{E_{i_k}\}_{1 \leq k \leq m}$.

Suppose $H_{cyc}$ contains a cycle $C = (r_1, r_2, ..., r_s, r_1)$. Let the boundary cycle $E_{r_1} = (u_1, u_2, ..., u_m, u_1)$ meet $E_{r_2}$ at $u_1$ and $E_{r_s}$ at $u_j$. We have that $j \neq 1$ since three boundary cycles cannot meet at a point. This is illustrated in Figure 1(b). The occupied square $S_1$ belongs to $E_{r_2}$ and the occupied

7
Figure 1: (a) Merging cycle ABCDA with the segment AEC. (b) Only two cycles can meet at a single point.

square $S_2$ belongs to $E_{r_s}$. It is necessary that the squares $S_3$ and $S_4$ are vacant and thus cannot be on the boundary of any other cycle.

Let $P_1$ and $P'_1$ be the two segments of $E_{r_1}$ starting at $u_1$ and ending at $u_j$. Since $u_1 \in E_{r_2}$ and $u_j \in E_{r_s}$, we have by the observation made in the first paragraph that there exists a path $P_2$ from $u_1$ to $u_j$, consisting only of edges in $\{E_{r_i}\}_{2 \leq i \leq s}$. This path necessarily lies in the exterior of $E_{r_1}$ and is illustrated in Figure 1(a). Here ABCDA represents the cycle $E_{r_1}$, the path $P_1$ is the segment ADC and the path $P'_1$ is the segment ABC. The path $P_2$ is denoted by the exterior segment AEC.

Thus it is necessary that either the cycle $C_{12}$ formed by $P_1 \cup P_2$ contains $P'_1$ in the interior or the cycle $C'_{12}$ formed by $P'_1 \cup P_2$ contains $P_1$ in the interior. Suppose the former holds and let $S_a$ be any occupied square in the interior of $E_{r_1}$. We know that $E_{r_1} = D_a$ is the outermost boundary cycle containing $S_a$ and satisfies conditions (a), (b) and (c) mentioned in Lemma 4. The cycle $C_{12}$ also contains $S_a$ in the interior and thus satisfies condition (a). Moreover, it contains at least one edge in the exterior of $E_{r_1}$ contradicting the fact that $E_{r_1}$ satisfies (c). Thus $H_{cyc}$ is acyclic. 

\[\blacksquare\]
3 Proofs of Theorem 2 and Theorem 3

Proof of Theorem 2 Let \( D_0 \) be the outermost boundary cycle containing the square \( S_0 \) as in Lemma 4. It satisfies the conditions (i) and (ii) in the statement of the theorem and is unique and thus \( C_{\text{out}}^+ = D_0 \).

Proof of Theorem 3 If every edge of \( C_1 \) is either on \( C_2 \) or contained in the interior of \( C_2 \), then the desired cycle \( C_3 = C_2 \). If similarly, \( C_2 \) is completely contained in \( C_1 \), we set \( C_3 = C_1 \). So we suppose that \( C_1 \) contains at least one edge in the exterior of \( C_2 \) and \( C_2 \) also contains at least one edge in the exterior of \( C_1 \).

We start with cycle \( C_1 \) and in the first step, identify a path of \( C_2 \) contained in the exterior of \( C_1 \). Set \( C_{1,0} := C_1 = (u_0, u_1, \ldots, u_{t-1}, u_0) \) and \( C_2 = (v_0, v_1, \ldots, v_{m-1}, v_0) \). For later notation, we define \( u_k = u_k \mod t \) if \( k \leq 0 \) or \( k \geq t \) and \( v_k = v_k \mod m \) if \( k \leq 0 \) or \( k \geq m \).

Start from some vertex of \( C_{1,0} \), say \( u_0 \), and look for the first intersection point that contains an exterior edge of \( C_2 \); i.e., an edge of \( C_2 \) that lies in the exterior of \( C_1 \). Let \( j_1 = \min \{ j \geq 0 : u_j \in C_{1,0} \} \) and \( j_1 \) is an endvertex of an exterior edge of \( C_2 \) and let \( v_{i_1} = u_{j_1} \). We suppose that the edge of \( C_2 \) with endvertices \( v_{i_1} \) and \( v_{i_1+1} \) lies in the exterior of \( C_1 \). Let \( r_1 = \min \{ i \geq i_1 + 1 : v_i \in C_{1,0} \} \) be the next time the cycles meet and define \( P_1 = (v_{i_1}, v_{i_1+1}, \ldots, v_{r_1}) \).

We note that none of the vertices \( v_j, i_1 + 1 \leq j \leq r_1 - 1 \) belong to \( C_{1,0} \). If \( v_{i_1} = v_{r_1} \), then \( P_1 \) is a cycle containing the edges of \( C_2 \) and thus \( P_1 = C_2 \). Since \( C_1 \) and \( C_2 \) contain more than one vertex in common, this cannot happen. Thus \( P_1 \) is a path and all edges of \( P_1 \) are in the exterior of \( C_{1,0} \).

We then construct an outermost cycle from \( C_{1,0} \) and \( P_1 \) as follows. Split \( C_{1,0} \) into two segments based on intersection with \( P_1 \). Suppose \( P_1 \) meets \( C_{1,0} \) at \( u_{a_1} \) and \( u_{b_1} \). We let \( C_{1,0}^{'} = (u_{a_1}, u_{a_1+1}, \ldots, u_{b_1}) \) and \( C_{1,0}'' = (u_{a_1}, u_{a_1-1}, \ldots, u_{b_1}) \). If the interior of \( C_{1,0}^{'} \cup P_1 \) contains the interior of \( C_{1,0}'' \cup P_1 \) as in Figure 1(a), we set \( C_{1,1} = C_{1,0}^{'} \cup P_1 \) to be the cycle obtained in the first iteration by the concatenation of the paths \( C_{1,0}^{'} \) and \( P_1 \). Here \( C_{1,0}^{''} \) is the segment \( AEC \), the path \( C_{1,0}^{'} \) is the segment \( ABC \) and the path \( P_1 \) is denoted \( AEC \). Else necessarily we have that the interior of \( C_{1,0}^{''} \cup P_1 \) contains the interior of \( C_{1,0}^{'} \cup P_1 \) and we set \( C_{1,1} = C_{1,0}'' \cup P_1 \). Since \( P_1 \neq \emptyset \), we have that \( C_{1,1} \) contains at least one exterior edge.
We then perform the same procedure as above on the cycle $C_{1,1}$ and continue this process for a finite number of steps to obtain the final cycle $C_{1,n}$. For each $j, 1 \leq j \leq n$, we have that the cycle $C_{1,j}$ satisfies the following properties:

1) the cycle $C_{1,j}$ contains only edges from $C_1$ and $C_2$,
2) every edge of $C_1$ either belongs to $C_{1,j}$ or is contained in the interior of $C_{1,j}$,
3) the cycle $C_{1,j}$ contains at least one exterior edge of $C_2$ and
4) the interior of $C_1$ is contained in $C_{1,j}$.

In particular, the above properties hold true for the final cycle $C_{1,n}$. If there exists an edge $e$ of $C_2$ in the exterior of $C_{1,n}$, then the edge $e$ belongs to a path $P_e$ of $C_2$ containing edges exterior to $C_1$. The path $P_e$ must meet $C_1$ and thus there exists an edge of $C_2$ that lies in the exterior of $C_1$ and contains an endvertex of $C_1$. But then the above procedure would not have terminated and thus we also have:

5) every edge of $C_2$ either belongs to $C_{1,n}$ or is contained in the interior of $C_{1,n}$.

Thus property (ii) stated in the result holds true and we need to see that (i) holds. For that we first prove uniqueness of the cycle $C_{1,n}$ obtained above. Suppose there exists another cycle $D' \neq C_{1,n}$ satisfying properties (1), (2) and (5) above. If $D'$ contains an edge $e'$ (which must necessarily belong to $C_1$ or $C_2$) in the exterior of $C_{1,n}$, it contradicts the fact that $C_{1,n}$ satisfies (2) and (5).

If $D'$ is completely contained in the interior of $C_{1,n}$ and is not equal to $C_{1,n}$, then there is at least one edge of $C_{1,n}$ (which belongs to $C_1$ or $C_2$) that lies in the exterior of $D'$, contradicting the assumption that $D'$ satisfies (2) and (5).

Thus any cycle satisfying properties (1), (2) and (5) is unique. We recall that $C_1$ also contains an edge in the exterior of $C_2$. Suppose now we start from $C_{2,0} := C_2$ and identify segments of $C_1$ lying in the exterior of $C_2$ and perform the same iterative procedure as above to obtain a final cycle $C_{2,m}$. This cycle must also satisfy (1), (2) and (5) and hence $C_{2,m} = C_{1,n}$. Moreover, $C_{2,m}$ satisfies:

3') the cycle $C_{2,m}$ contains at least one exterior edge of $C_1$ and
4') the interior of $C_2$ is contained in $C_{2,m}$.

Thus the cycle $C_{1,n}$ is unique and satisfies properties (i) and (ii) stated in the result. ■
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