CONCENTRATING SOLUTIONS FOR A FRACTIONAL KIRCHHOFF EQUATION WITH CRITICAL GROWTH

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ABSTRACT. In this paper we consider the following class of fractional Kirchhoff equations with critical growth:

\[
\begin{cases}
\varepsilon^{2s} a + \varepsilon^{4s-3} b \int_{\mathbb{R}^3} |(-\Delta)^s y|^2 u^2 dx \left( (-\Delta)^s u + V(x)u = f(u) + |u|^{2^* - 2} u \right. & \text{in } \mathbb{R}^3, \\
\left. u \in H^s(\mathbb{R}^3), \quad u > 0 \right. & \text{in } \mathbb{R}^3,
\end{cases}
\]

where \( \varepsilon > 0 \) is a small parameter, \( a, b > 0 \) are constants, \( s \in (\frac{3}{4}, 1) \), \( 2^* = \frac{6}{3 - 2s} \) is the fractional critical exponent, \( (-\Delta)^s \) is the fractional Laplacian operator, \( V \) is a positive continuous potential and \( f \) is a superlinear continuous function with subcritical growth. Using penalization techniques and variational methods, we prove the existence of a positive solution \( u_\varepsilon \) which concentrates around a local minimum of \( V \) as \( \varepsilon \to 0 \).

1. Introduction

This paper is devoted to the existence and concentration of positive solutions for the following critical fractional Kirchhoff equation:

\[
\begin{cases}
\varepsilon^{2s} a + \varepsilon^{4s-3} b \int_{\mathbb{R}^3} |(-\Delta)^s y|^2 u^2 dx \left( (-\Delta)^s u + V(x)u = f(u) + |u|^{2^* - 2} u \right. & \text{in } \mathbb{R}^3, \\
\left. u \in H^s(\mathbb{R}^3), \quad u > 0 \right. & \text{in } \mathbb{R}^3,
\end{cases}
\]

where \( \varepsilon > 0 \) is a small parameter, \( a, b > 0 \) are constants, \( s \in (\frac{3}{4}, 1) \) is fixed, \( 2^* = \frac{6}{3 - 2s} \) is the fractional critical exponent, \( (-\Delta)^s \) is the fractional Laplacian operator, which (up to normalization factors) may be defined for smooth functions \( u : \mathbb{R}^3 \to \mathbb{R} \) as

\[
(-\Delta)^s u(x) = -\frac{1}{2} \int_{\mathbb{R}^3} \frac{u(x + y) + u(x - y) - 2u(x)}{|y|^{3+2s}} dy \quad (x \in \mathbb{R}^3),
\]

(see [18, 38] and the references therein for further details and applications).

The potential \( V : \mathbb{R}^3 \to \mathbb{R} \) is a continuous function satisfying the following conditions introduced by del Pino and Felmer in [17]:

(\( V_1 \)) there exists \( V_1 > 0 \) such that \( V_1 = \inf_{x \in \mathbb{R}^3} V(x) \),

(\( V_2 \)) there exists a bounded open set \( \Lambda \subset \mathbb{R}^3 \) such that

\[
\Lambda \, V_0 = \inf \Lambda \subset \mathbb{R}^3 \text{ such that } V_0 = \inf \Lambda \, V < \inf_{\partial \Lambda} V,
\]

while \( f : \mathbb{R} \to \mathbb{R} \) is a continuous function verifying the following hypotheses:

(\( f_1 \)) \( f(t) = o(t^3) \) as \( t \to 0^+ \),

(\( f_2 \)) there exist \( q, \sigma \in (4, 2^*_s), C_0 > 0 \) such that

\[
f(t) \geq C_0 t^{q - 1} \quad \forall t > 0, \quad \lim_{t \to 0} \frac{f(t)}{t^{\sigma - 1}} = 0,
\]

(\( f_3 \)) there exists \( \vartheta \in (4, 2^*_s) \) such that \( 0 < \vartheta F(t) \leq tf(t) \) for all \( t > 0 \),

(\( f_4 \)) the map \( t \mapsto \frac{f(t)}{t^{\sigma - 1}} \) is increasing in \((0, \infty)\).

Since we will look for positive solutions to (1.1), we assume that \( f(t) = 0 \) for \( t \leq 0 \).

We note that when \( a = 1, b = 0 \) and \( \mathbb{R}^3 \) is replaced by \( \mathbb{R}^N \), then (1.1) reduces to a fractional Schrödinger equation of the type

\[
\varepsilon^{2s}(-\Delta)^s u + V(x)u = h(x, u) \quad \text{in } \mathbb{R}^N,
\]

which has been introduced by Laskin [35] as a result of expanding the Feynman path integral, from the Brownian like to the Lévy like quantum mechanical paths. Equation (1.2) has received a great interest from many mathematicians, and several results have been obtained under different and suitable assumptions on \( V \) and \( h \); see for instance [6–8, 19, 22, 33, 47] and the references therein. In particular way, the existence
and concentration as $\varepsilon \to 0$ of positive solutions to (1.2) has been widely investigated in recent years. For instance, Dāvila et al. [16] showed via Lyapunov-Schmidt reduction, that if the potential $V$ satisfies

$$V \in C^{1,\alpha}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \text{ and } \inf_{x \in \mathbb{R}^N} V(x) > 0,$$

then (1.1) has multi-peak solutions. Shang et al. [49] used Ljusternik-Schnirelmann theory to obtain multiple positive solutions for a fractional Schrödinger equation with critical growth assuming that the potential $V : \mathbb{R}^N \to \mathbb{R}$ verifies the following assumption proposed by Rabinowitz [46]:

$$V_\infty = \liminf_{|x| \to \infty} V(x) > \inf_{x \in \mathbb{R}^N} V(x) = V_0, \text{ where } V_\infty \in (0, \infty].$$

Fall et al. [21] established necessary and sufficient conditions on the smooth potential $V$ in order to produce concentration of solutions of (1.1) when the parameter $\varepsilon$ converges to zero. Moreover, when $V$ is coercive and has a unique global minimum, then ground-states concentrate at this point. Alves and Miyagaki [4] studied the existence and concentration of positive solutions to (1.1), via a penalization approach, under assumptions $(V_1)$-$(V_2)$ and $f$ is a subcritical nonlinearity. Later, their result has been extended for critical and supercritical nonlinearities in [6, 31].

On the other hand, if we set $s = \varepsilon = 1$ and we replace $f(u) + |u|^{2^*_s - 2}u$ by a more general nonlinearity $h(x,u)$, then (1.1) becomes the well-known classical Kirchhoff equation

$$- \left( a + b \int_{\mathbb{R}^3} |\nabla u|^2 dx \right) \Delta u + V(x)u = h(x,u) \quad \text{in } \mathbb{R}^3,$$

which is related to the stationary analogue of the Kirchhoff equation

$$\rho u_{tt} - \left( \frac{p_0}{h} + \frac{E}{2L} \int_0^L |u_x|^2 dx \right) u_{xx} = 0,$$

introduced by Kirchhoff [34] in 1883 as an extension of the classical D’Alembert’s wave equation for describing the transversal oscillations of a stretched string. Here $L$ is the length of the string, $h$ is the area of the cross-section, $E$ is the young modulus (elastic modulus) of the material, $\rho$ is the mass density, and $p_0$ is the initial tension. We refer to [12, 43] for the early classical studies dedicated to (1.4). We also note that nonlocal boundary value problems like (1.3) model several physical and biological systems where $u$ describes a process which depends on the average of itself, as for example, the population density; see [2, 14]. However, only after the Lions’ work [36], where a functional analysis approach was proposed to attack a general Kirchhoff equation in arbitrary dimension with external force term, problem (1.3) began to catch the attention of several mathematicians; see [1, 13, 25, 29, 30, 42, 51] and the references therein. For instance, He and Zou [30] obtained existence and multiplicity results for small $\varepsilon > 0$ of the following perturbed Kirchhoff equation

$$- \left( a\varepsilon^2 + b\varepsilon \int_{\mathbb{R}^3} |\nabla u|^2 dx \right) \Delta u + V(x)u = g(u) \quad \text{in } \mathbb{R}^3,$$

where the potential $V$ satisfies condition (V) and $g$ is a subcritical nonlinearity. Wang et al. [51] studied the multiplicity and concentration phenomenon for (1.5) when $g(u) = \lambda f(u) + |u|^4 u$, $f$ is a continuous subcritical nonlinearity and $\lambda$ is large. Figueiredo and Santos Junior [25] used the generalized Nehari manifold method to obtain a multiplicity result for a subcritical Kirchhoff equation under conditions $(V_1)$-$(V_2)$. He et al. [29] dealt with the existence and multiplicity of solutions to (1.5), where $g(u) = f(u) + u^6$, $f \in C^1$ is a subcritical nonlinearity which does not verifies the Ambrosetti-Rabinowitz condition [5] and $V$ fulfills $(V_1)$-$(V_2)$.

In the nonlocal framework, Fiscella and Valdinoci [28] proposed for the first time a stationary fractional Kirchhoff variational model in a bounded domain $\Omega \subset \mathbb{R}^N$ with homogeneous Dirichlet boundary conditions and involving a critical nonlinearity:

$$\begin{cases}
M \left( \int_{\mathbb{R}^N} |(-\Delta)\tilde{s} u|^2 dx \right) (-\Delta)\tilde{s} u = \lambda f(x,u) + |u|^{2^*_s - 2}u & \text{in } \Omega, \\
u = 0 & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases}$$

where $M$ is a continuous Kirchhoff function whose model case is given by $M(t) = a + bt$. Their model takes care of the nonlocal aspect of the tension arising from nonlocal measurements of the fractional length of the string; see [28] for more details. After the pioneering work [28], several authors dealt with existence and multiplicity of solutions for (1.6); see [20, 24, 26, 38, 40] and their references. On the other hand, some interesting results for fractional Kirchhoff equations in $\mathbb{R}^N$ have been established in [9–11, 27, 37, 44, 45]. For instance, Autuori and Pucci [11] studied the existence and multiplicity of nontrivial solutions for a fractional Kirchhoff equation
involving suitable weights. Fiscella and Pucci [27] dealt with stationary fractional Kirchhoff $p$-Laplacian equations involving critical Hardy-Sobolev nonlinearities and nonnegative potentials. In [9] a multiplicity result for a fractional Kirchhoff equation involving a Berestycki-Lions type nonlinearity is proved. The author and Isernia [10] used penalization method and Lusternik-Schnirelmann category theory to study the existence and multiplicity of solutions for a fractional Schrödinger-Kirchhoff equation with subcritical nonlinearities; see also [32] where a subcritical version of (1.1) is considered. Liu et al. [37], via the monotonicity trick and the profile decomposition, proved the existence of ground states to a fractional Kirchhoff equation with critical nonlinearity in low dimension.

Motivated by the above works, in this paper we aim to study the existence and concentration behavior of solutions to (1.1) under assumptions $(V_1)$-$(V_2)$ and $(f_1)$-$(f_4)$. More precisely, our main result can be stated as follows:

**Theorem 1.1.** Assume that $(V_1)$-$(V_2)$ and $(f_1)$-$(f_4)$ hold. Then, there exists $\varepsilon_0 > 0$ such that (1.1) has a positive solution $u_\varepsilon$ for all $\varepsilon \in (0, \varepsilon_0)$. Moreover, if $\eta_\varepsilon$ denotes a global maximum point of $u_\varepsilon$, then we have

$$\lim_{\varepsilon \to 0} V(\eta_\varepsilon) = V_0,$$

and there exists a constant $C > 0$ such that

$$0 < u_\varepsilon(x) \leq \frac{C \varepsilon^{3+2s}}{\varepsilon^{3+2s} + |x - \eta_\varepsilon|^{3+2s}} \quad \text{for all } x \in \mathbb{R}^3.$$

The proof of Theorem 1.1 will be done via appropriate variational arguments. After considering the $\varepsilon$-rescaled problem associated to (1.1), we use a variant of the penalization technique introduced in [17] (see also [3, 23]) which consists in modifying in a suitable way the nonlinearity outside $\Lambda$, solving a modified problem and then check that, for $\varepsilon > 0$ small enough, the solutions of the modified problem are indeed solutions of the original one. These solutions will be obtained as critical points of the modified energy functional $J_\varepsilon$ which, in view of the growth assumptions on $f$ and the auxiliary nonlinearity, possesses a mountain pass geometry [5]. In order to recover some compactness properties for $J_\varepsilon$, we have to circumvent several difficulties which make our study rather delicate. The first one is related to the presence of the Kirchhoff term in (1.1) which does not permit to verify in a standard way that if $u$ is the weak limit of a Palais-Smale sequence ((PS) in short) $\{u_n\}_{n \in \mathbb{N}}$ for $J_\varepsilon$, then $u$ is a weak solution for the modified problem. The second one is due to the lack of compactness caused by the unboundedness of the domain $\mathbb{R}^3$ and the critical Sobolev exponent. Anyway, we will be able to overcome these problems looking for critical points of a suitable functional whose quadratic part involves the limit term of $(a + b|u_n|^2)$, and that, inspired by [37], the mountain pass level $c_\varepsilon$ of $J_\varepsilon$ is strictly less than a threshold value related to the best constant of the embedding $H^s(\mathbb{R}^3)$ in $L^{2s}(\mathbb{R}^3)$ (see [15]). Then, applying mountain pass lemma, we will deduce the existence of a positive solution for the modified problem. Finally, combining a compactness argument and a Moser iteration procedure [39], we prove that the solution of the modified problem is also a solution to the original one for $\varepsilon > 0$ small enough, and that it decays at zero at infinity with polynomial rate.

To our knowledge, this is the first time that concentration phenomenon for problem (1.1) is investigated in the literature.

The paper is organized as follows: in Section 2 we introduce the modified problem and we provide some technical results. In Section 3 we give the proof of Theorem 1.1.

## 2. The modified problem

### 2.1. Preliminaries.

Here, we fix the notations and we recall some useful preliminary results on fractional Sobolev spaces (see also [18, 38] for more details).

If $A \subset \mathbb{R}^3$, we denote by $|u|_{L^q(A)}$ the $L^q(A)$-norm of a function $u : \mathbb{R}^3 \to \mathbb{R}$, and by $|u|_q$ its $L^q(\mathbb{R}^3)$-norm. We denote by $B_r(x)$ the ball centered at $x \in \mathbb{R}^3$ with radius $r > 0$. When $x = 0$, we put $B_r = B_r(0)$. Let us define $D^{s,2}(\mathbb{R}^3)$ as the completion of $C_c^{\infty}(\mathbb{R}^3)$ with respect to the norm

$$[u]^2 = \iint_{\mathbb{R}^6} \frac{|u(x) - u(y)|^2}{|x - y|^{3+2s}} \, dx \, dy = \int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 \, dx,$$

where the second identity holds up to a constant; see [18]. Then we consider the fractional Sobolev space

$$H^s(\mathbb{R}^3) = \left\{ u \in L^2(\mathbb{R}^3) : [u] < \infty \right\}.$$
endowed with the norm
\[ \|u\|^2 = |u|^2 + |u|_2^2. \]
We recall the following main embeddings for the fractional Sobolev spaces:

**Theorem 2.1.** [18] Let \( s \in (0,1) \). Then there exists a sharp constant \( S_s = S_s(s) > 0 \) such that for any \( u \in D^{s,2}(\mathbb{R}^3) \)
\[ |u|_{2s}^2 \leq S_s^{-1}|u|^2. \]
Moreover, \( H^s(\mathbb{R}^3) \) is continuously embedded in \( L^p(\mathbb{R}^3) \) for any \( p \in [2,2^*_s] \) and compactly in \( L^p_{loc}(\mathbb{R}^3) \) for any \( p \in [1,2^*_s] \).

The following lemma is a version of the well-known Lions type result:

**Lemma 2.1.** [22] If \( \{u_n\}_{n \in \mathbb{N}} \) is a bounded sequence in \( H^s(\mathbb{R}^3) \) and if
\[ \lim_{n \to \infty} \sup_{y \in \mathbb{R}^3} \int_{B_R(y)} |u_n|^2 dx = 0 \]
where \( R > 0 \), then \( u_n \to 0 \) in \( L^r(\mathbb{R}^3) \) for all \( r \in (2,2^*_s) \).

We also recall the following useful technical result.

**Lemma 2.2.** [41] Let \( u \in D^{s,2}(\mathbb{R}^3) \). Let \( \varphi \in C_c^\infty(\mathbb{R}^3) \) and for each \( r > 0 \) we define \( \varphi_r(x) = \varphi(x/r) \). Then, \( [u \varphi_r] \to [u] \) as \( r \to 0 \). If in addition \( \varphi = 1 \) in a neighborhood of the origin, then \( [u \varphi_r] \to [u] \) as \( r \to \infty \).

### 2.2. Functional Setting.
In order to study (1.1), we use the change of variable \( x \mapsto \varepsilon x \) and we will look for solutions to

\[
\begin{cases}
(a + b|u|^2)(-\Delta)^s u + V(\varepsilon x)u = f(u) + |u|^{2^*_s - 2}u & \text{in} \mathbb{R}^3, \\
u \in H^s(\mathbb{R}^3), & u > 0.
\end{cases}
\]

(2.1)

Now, we introduce a penalization function in the spirit of [17] which will be fundamental to obtain our main result. First of all, without loss of generality, we will assume that

\[ 0 \in \Lambda \text{ and } V(0) = V_0 = \inf_{\Lambda} V. \]

Let \( K > 2 \) and \( a > 0 \) be such that
\[ f(a) + a^{2^*_s - 1} = \frac{V_1}{K} a \]
and we define
\[ \tilde{f}(t) = \begin{cases} f(t) + t^{2^*_s - 1} & \text{if } t \leq a \\ \frac{V_1}{K} t & \text{if } t > a, \end{cases} \]
and
\[ g(x,t) = \begin{cases} \chi_\Lambda(x)(f(t) + t^{2^*_s - 1}) + (1 - \chi_\Lambda(x))\tilde{f}(t) & \text{if } t > 0 \\ 0 & \text{if } t \leq 0. \end{cases} \]

It is easy to check that \( g \) satisfies the following properties:

\begin{itemize}
\item[(g_1)] \( \lim_{t \to 0^+} \theta \frac{g(x,t)}{t^2} = 0 \) uniformly with respect to \( x \in \mathbb{R}^3 \),
\item[(g_2)] \( g(x,t) \leq \tilde{f}(t) + t^{2^*_s - 1} \) for all \( x \in \mathbb{R}^3, t > 0 \),
\item[(g_3)] (i) \( 0 \leq \partial_t G(x,t) < g(x,t) \) for all \( x \in \Lambda \text{ and } t > 0 \),
\item[(ii)] \( 0 \leq 2G(x,t) < g(x,t) \) for all \( x \in \mathbb{R}^3 \setminus \Lambda \text{ and } t > 0 \),
\end{itemize}

(\( g_4 \) for each \( x \in \Lambda \) the function \( \frac{g(x,t)}{t^2} \) is increasing in \((0,\infty)\), and for each \( x \in \mathbb{R}^3 \setminus \Lambda \) the function \( \frac{g(x,t)}{t^2} \) is increasing in \((0,a)\).

Then, we consider the following modified problem

\[
\begin{cases}
(a + b|u|^2)(-\Delta)^s u + V(\varepsilon x)u = g(\varepsilon x, u) & \text{in} \mathbb{R}^3, \\
u \in H^s(\mathbb{R}^3), & u > 0.
\end{cases}
\]

(2.3)

The corresponding functional is given by
\[ J_\varepsilon(u) = \frac{1}{2} \|u\|_\varepsilon^2 + \frac{b}{4} |u|^4 - \int_{\mathbb{R}^3} G(\varepsilon x, u) dx, \]
which is well-defined on the space

$$\mathcal{H}_\varepsilon = \left\{ u \in H^s(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(\varepsilon x)u^2 \, dx < \infty \right\}$$

endowed with the norm

$$\|u\|_{\mathcal{H}_\varepsilon}^2 = a[u]^2 + \int_{\mathbb{R}^3} V(\varepsilon x)u^2 \, dx.$$ 

Clearly $\mathcal{H}_\varepsilon$ is a Hilbert space with inner product

$$(u,v)_{\mathcal{H}_\varepsilon} = a \int_{\mathbb{R}^6} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{3+2s}} \, dx dy + \int_{\mathbb{R}^3} V(\varepsilon x)uv \, dx.$$ 

It is standard to show that $J_\varepsilon \in C^1(\mathcal{H}_\varepsilon, \mathbb{R})$ and its differential is given by

$$\langle J_\varepsilon'(u),v \rangle = (u,v)_{\mathcal{H}_\varepsilon} + b[u]^2 \int_{\mathbb{R}^6} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{3+2s}} \, dx dy - \int_{\mathbb{R}^3} g(\varepsilon x,u)v \, dx$$

for any $u,v \in \mathcal{H}_\varepsilon$. Let us introduce the Nehari manifold associated to (2.3), that is,

$$\mathcal{N}_\varepsilon = \left\{ u \in \mathcal{H}_\varepsilon \setminus \{0\} : \langle J_\varepsilon'(u),u \rangle = 0 \right\}.$$ 

We begin proving that $J_\varepsilon$ possesses a nice geometric structure:

**Lemma 2.3.** The functional $J_\varepsilon$ has a Mountain-Pass geometry:

(a) there exist $\alpha, \rho > 0$ such that $J_\varepsilon(u) \geq \alpha$ with $\|u\|_{\mathcal{H}_\varepsilon} = \rho$;

(b) there exists $\varepsilon \in \mathcal{H}_\varepsilon$ with $\|\varepsilon\|_{\mathcal{H}_\varepsilon} > \rho$ such that $J_\varepsilon(\varepsilon) < 0$.

**Proof.** (a) By assumptions $(g_1)$ and $(g_2)$ we deduce that for any $\xi > 0$ there exists $C_\xi > 0$ such that

$$J_\varepsilon(u) \geq \frac{1}{2}\|u\|_{\mathcal{H}_\varepsilon}^2 - \int_{\mathbb{R}^3} G(\varepsilon x,u) \, dx \geq \frac{1}{2}\|u\|_{\mathcal{H}_\varepsilon}^2 - \xi C\|u\|_{\mathcal{H}_\varepsilon}^2 - C_\xi C\|u\|_{\mathcal{H}_\varepsilon}^2.$$

Then, there exist $\alpha, \rho > 0$ such that $J_\varepsilon(u) \geq \alpha$ with $\|u\|_{\mathcal{H}_\varepsilon} = \rho$.

(b) Using $(g_3)$-i), we deduce that for any $u \in C^\infty_c(\mathbb{R}^3) \setminus \{0\}$ such that $u \geq 0$ and $\text{supp}(u) \subset \Lambda_\varepsilon$, and for all $\tau > 0$ it holds

$$J_\varepsilon(\tau u) = \frac{\tau^2}{2}\|u\|_{\mathcal{H}_\varepsilon}^2 + b\frac{\tau^4}{4}[u]^4 - \int_{\Lambda_\varepsilon} G(\varepsilon x,\tau u) \, dx$$

$$\leq \frac{\tau^2}{2}\|u\|_{\mathcal{H}_\varepsilon}^2 + b\frac{\tau^4}{4}[u]^4 - C_1 \tau^\vartheta \int_{\Lambda_\varepsilon} u^\vartheta \, dx + C_2,$$

for some constants $C_1, C_2 > 0$. Recalling that $\vartheta \in (4,2^*_s)$ we can conclude that $J_\varepsilon(\tau u) \to -\infty$ as $\tau \to \infty$. $\square$

In view of Lemma 2.3, we can use a variant of the Mountain-Pass Theorem without $(PS)_c$ (see [52]) to deduce the existence of a Palais-Smale sequence $\{u_n\}_{n \in \mathbb{N}} \subset \mathcal{H}_\varepsilon$ such that

$$J_\varepsilon(u_n) = c_\varepsilon + o_n(1) \quad \text{and} \quad J_\varepsilon'(u_n) = o_n(1)$$

where

$$c_\varepsilon = \inf_{\gamma \in \Gamma_\varepsilon} \max_{t \in [0,1]} J_\varepsilon(\gamma(t)) \quad \text{and} \quad \Gamma_\varepsilon = \left\{ \gamma \in C([0,1],\mathcal{H}_\varepsilon) : \gamma(0) = 0, J_\varepsilon(\gamma(1)) \leq 0 \right\}.$$ 

As in [52], we can use the following equivalent characterization of $c_\varepsilon$ more appropriate for our aim:

$$c_\varepsilon = \inf_{\varepsilon \in \mathcal{H}_\varepsilon \setminus \{0\}} \max_{t \geq 0} J_\varepsilon(\varepsilon t).$$

Moreover, from the monotonicity of $g$, it is easy to see that for all $u \in \mathcal{H}_\varepsilon \setminus \{0\}$ there exists a unique $t_0 = t_0(u) > 0$ such that

$$J_\varepsilon(t_0 u) = \max_{t \geq 0} J_\varepsilon(t u).$$

In the next lemma, we will see that $c_\varepsilon$ is less then a threshold value involving the best constant $S_*$ of Sobolev embedding $\mathcal{D}^{s,2}(\mathbb{R}^3)$ in $L^{2^*_s}(\mathbb{R}^3)$ (see Theorem 2.1). More precisely:

**Lemma 2.4.** There exists $T > 0$ such that

$$c_\varepsilon < \frac{a}{2} S_* T^{3-2s} + \frac{b}{4} S_*^2 T^{6-4s} - \frac{1}{2s} T^3 =: c_*$$

for all $\varepsilon > 0$. 

Proof. We argue as in [37]. Let \( \eta \in C_c^\infty(\mathbb{R}^3) \) be a cut-off function such that \( \eta = 1 \) in \( \mathcal{B}_\rho, \) \( \text{supp}(\eta) \subset \mathcal{B}_{2\rho} \) and \( 0 \leq \eta \leq 1, \) where \( \mathcal{B}_{2\rho} \subset \Lambda_\epsilon. \) For simplicity we assume \( \rho = 1. \) We know (see [15]) that \( S_\epsilon \) is achieved by \( U(x) = \kappa(\mu^2 + |x - x_0|^2)^{-\frac{\mu^2}{2}}, \) with \( \kappa \in \mathbb{R}, \mu > 0 \) and \( x_0 \in \mathbb{R}^3. \) Taking \( x_0 = 0, \) as in [48], we can define
\[
v_h(x) = \eta(x)u_h(x) \quad \forall h > 0,
\]
where
\[
u_h(x) = h^{-\frac{3}{2}}u^*(x/h) \quad \text{and} \quad u^*(x) = \frac{U(x/S_\epsilon^3)}{|U|_{S_\epsilon^3}}.
\]
Then \((-\Delta)^s u_h = |u_h|^{2s-2}u_h \) in \( \mathbb{R}^3 \) and \( [u_h]^2 = |u_h|_{2s}^2 = S_\epsilon^3. \) We recall the following useful estimates
\[
A_h = |v_h|^2 = S_\epsilon^3 + O(h^{3-2s}) \quad \text{(2.7)}
\]
\[
B_h = |v_h|^2 = O(h^{3-2s}) \quad \text{(2.8)}
\]
\[
C_h = |v_h|^2 \geq \begin{cases} O(h^{\frac{3}{2}}) & \text{if } q > \frac{3}{3-2s} \\ O(\log(h))h^{\frac{3}{2}} & \text{if } q = \frac{3}{3-2s} \\ O(h^{\frac{3}{2}}) & \text{if } q < \frac{3}{3-2s} \end{cases} \quad \text{(2.9)}
\]
\[
D_h = |v_h|^2 = S_\epsilon^3 + O(h^3). \quad \text{(2.10)}
\]
Let us note that for all \( h > 0 \) there exists \( t_0 > 0 \) such that \( J_\epsilon(\gamma_h(t_0)) < 0, \) where \( \gamma_h(t) = v_h(\cdot/t). \) Indeed, setting \( V_2 = \max_{x \in \mathbb{T}} V(x), \) we have
\[
J_\epsilon(\gamma_h(t)) \leq \frac{a}{2}t^{3-3s}[v_h]^2 + \frac{V_2}{2}t^3|v_h|^2 + \frac{b}{4}t^{6-4s}|v_h|^4 - \frac{t^3}{2s}|v_h|^2 - \frac{t^3}{q}|v_h|^q
\]
\[
= \frac{a}{2}t^{3-2s}A_h + \frac{b}{4}A_h^2t^{6-4s} + \left( V_2 \frac{B_h}{2} - \frac{D_h}{2s} - C_0C_h \right) t^3. \quad \text{(2.11)}
\]
Since \( 0 < 6 - 4s < 3, \) we can use (2.8) to deduce that
\[
V_2 \frac{B_h}{2} - \frac{D_h}{2s} \to \frac{1}{2s}S_\epsilon^3
\]
as \( h \to 0. \) Hence, using (2.7), we can see that for all \( h > 0 \) sufficiently small \( J_\epsilon(\gamma_h(t)) \to -\infty \) as \( t \to \infty, \) that is there exists \( t_0 > 0 \) such that \( J_\epsilon(\gamma_h(t_0)) < 0. \)

Now, as \( t \to 0^+ \) we have
\[
|\gamma_h(t)|^2 + |\gamma_h(t)|^2 \leq t^{3-2s}A_h + t^2B_h \to 0 \text{ uniformly for } h > 0 \text{ small.}
\]
We set \( \gamma_h(0) = 0. \) Then \( \gamma_h(t_0^+) \in \Gamma_\epsilon, \) where \( \Gamma_\epsilon \) is defined as in (2.6) and we infer that
\[
c_\epsilon \leq \sup_{t \geq 0} J_\epsilon(\gamma_h(t)).
\]
Taking into account that \( c_\epsilon > 0, \) by (2.11) there exists \( t_h > 0 \) such that
\[
\sup_{t \geq 0} J_\epsilon(\gamma_h(t)) = J_\epsilon(\gamma_h(t_h)).
\]
In the light of (2.7), (2.9) and (2.11) we deduce that \( J_\epsilon(\gamma_h(t)) \to 0^+ \) as \( t \to 0^+ \) and \( J_\epsilon(\gamma_h(t_h)) \to -\infty \) as \( t \to \infty \) uniformly for \( h > 0 \) small. Then there exist \( t_1, t_2 \) independent of \( h > 0 \) verifying \( 0 < t_1 \leq t_h \leq t_2. \)

Set
\[
H_h(t) := \frac{aA_h}{2}t^{3-2s} + \frac{bA_h^2}{4}t^{6-4s} - \frac{D_h}{2s}t^3.
\]
Therefore,
\[
c_\epsilon \leq \sup_{t \geq 0} J_\epsilon(t) + \left( V_2 \frac{B_h}{2} - \frac{C_0C_h}{q} \right) t^3.
\]
Using (2.9), for any \( q \in (2, 2_\epsilon) \) we have \( C_h \geq O(h^{\frac{3-2s}{2}}) \) and exploiting (2.11) we can infer
\[
c_\epsilon \leq \sup_{t \geq 0} J_\epsilon(t) + O(h^{3-2s}) - O(C_0h^{\frac{3-2s}{2}}).
\]
Since $3 - 2s > 0$ and $3 - \frac{(3-2s)q}{2} > 0$, we obtain
$$\sup_{t \geq 0} J_\varepsilon(t) \geq \frac{c_\varepsilon}{2} \quad \text{uniformly for } h > 0 \text{ small.}$$

Arguing as above, there exist $t_3, t_4 > 0$ independent of $h > 0$ such that
$$\sup_{t \geq 0} J_\varepsilon(t) = \sup_{t \in [t_3, t_4]} J_\varepsilon(t).$$

By (2.7) we deduce
$$c_\varepsilon \leq \sup_{t \geq 0} K(S^\frac{1}{\vartheta} t) + O(h^{3-2s}) - O(C_0 h^{3-\frac{(3-2s)q}{2}}), \quad (2.12)$$
where
$$K(t) = \frac{aS_s}{2} t^{3-2s} + \frac{bS_s^2}{4} t^{6-4s} - \frac{1}{2_s t^3}.$$

Let us note that
$$\begin{align*}
K'(t) &= \frac{3 - 2s}{2} aS_s t^{2-2s} + \frac{3 - 2s}{2} bS_s^2 t^{5-4s} - \frac{3 - 2s}{2} t^2 \\
&= (3 - 2s) t^{2-2s} + (aS_s + bS_s^2 t^{3-2s} - t^2) =: \frac{3 - 2s}{2} t^{2-2s} K(t).
\end{align*}$$

Moreover,
$$\tilde{K}'(t) = bS_s (3 - 2s) t^{2-2s} - 2st^{2s-1} = t^{2-2s} [bS_s^2 (3 - 2s) - 2st^{4s-3}].$$

Since $4s > 3$, there exists a unique $T > 0$ such that $K(t) > 0$ for $t \in (0, T)$ and $K(t) < 0$ for $t > T$. Thus $T$ is the unique maximum point of $K$. In virtue of (2.12) we have
$$c_\varepsilon \leq K(T) + O(h^{3-2s}) - O(C_0 h^{3-\frac{(3-2s)q}{2}}). \quad (2.13)$$

If $q > \frac{4s}{3-2s}$, then $0 < 3 - \frac{(3-2s)q}{2} < 3 - 2s$, and by (2.13), for any fixed $C_0 > 0$, it holds $c_\varepsilon < K(T)$ for $h > 0$ small.

If $2 < q < \frac{4s}{3-2s}$, then, for $h > 0$ small and $C_0 > h^{\frac{(3-2s)q}{2}-2s-1}$, we also have $c_\varepsilon < K(T).$ \hfill \Box

**Lemma 2.5.** Every sequence $\{u_n\}_{n \in \mathbb{N}}$ satisfying (2.5) is bounded in $\mathcal{H}_\varepsilon$.

**Proof.** In view of (g3) we can deduce that
$$c_\varepsilon + o_n(1) \|u_n\|_\varepsilon \geq J_\varepsilon(u_n) - \frac{1}{\vartheta} (J'_\varepsilon(u_n), u_n)
\overset{(2.14)}{=} \left( \frac{\vartheta - 2}{2\vartheta} \right) \|u_n\|^2_\varepsilon + b \left( \frac{\vartheta - 4}{4\vartheta} \right) [u_n]^4 + \frac{1}{\vartheta} \int_{\mathbb{R}^3 \setminus A_\varepsilon} [g(\varepsilon x, u_n) u_n - \vartheta G(\varepsilon x, u_n)] dx
$$
$$+ \frac{1}{\vartheta} \int_{A_\varepsilon} [g(\varepsilon x, u_n) u_n - \vartheta G(\varepsilon x, u_n)] dx
\geq \left( \frac{\vartheta - 2}{2\vartheta} \right) \|u_n\|^2_\varepsilon + \frac{1}{\vartheta} \int_{\mathbb{R}^3 \setminus A_\varepsilon} [g(\varepsilon x, u_n) u_n - \vartheta G(\varepsilon x, u_n)] dx
$$
$$\geq \left( \frac{\vartheta - 2}{2\vartheta} \right) \|u_n\|^2_\varepsilon - \left( \frac{\vartheta - 2}{2\vartheta} \right) \frac{1}{K} \int_{\mathbb{R}^3 \setminus A_\varepsilon} V(\varepsilon x) u_n^2 dx
$$
$$\geq \left( \frac{\vartheta - 2}{2\vartheta} \right) \left( 1 - \frac{1}{K} \right) \|u_n\|^2_\varepsilon. \quad (2.15)$$

Since $\vartheta > 4$ and $K > 2$, we can conclude that $\{u_n\}_{n \in \mathbb{N}}$ is bounded in $\mathcal{H}_\varepsilon$. \hfill \Box

**Lemma 2.6.** There is a sequence $\{z_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^3$ and $R, \beta > 0$ such that
$$\int_{B_R(z_n)} u_n^2 dx \geq \beta.$$

Moreover, $\{z_n\}_{n \in \mathbb{N}}$ is bounded in $\mathbb{R}^3$. 

Proof. Assume by contradiction that the first conclusion of lemma is not true. From Lemma 2.1 we have
\[ u_n \to 0 \text{ in } L^q(\mathbb{R}^3) \text{ for any } q \in (2, 2^*_s), \]
which, together with \((f_1)\) and \((f_2)\), yield
\[ \int_{\mathbb{R}^3} F(u_n) \, dx = \int_{\mathbb{R}^3} f(u_n) u_n \, dx = o_n(1) \text{ as } n \to \infty. \]
Since \(\{u_n\}_{n \in \mathbb{N}}\) is bounded in \(\mathcal{H}_\varepsilon\), we may assume that \(u_n \to u\) in \(\mathcal{H}_\varepsilon\).

Now, we can observe that
\[ \int_{\mathbb{R}^3} G(\varepsilon \, x, u_n) \, dx \leq \frac{1}{2s} \int_{\Lambda_\varepsilon \cup \{u_n \leq a\}} (u_n^+)^{2s} \, dx + \frac{V_1}{2k} \int_{(\mathbb{R}^3 \setminus \Lambda_\varepsilon) \cap \{u_n > a\}} u_n^2 \, dx + o_n(1) \quad (2.16) \]
and
\[ \int_{\mathbb{R}^3} g(\varepsilon \, x, u_n) u_n \, dx = \int_{\Lambda_\varepsilon \cup \{u_n \leq a\}} (u_n^+)^{2s} \, dx + \frac{V_1}{k} \int_{(\mathbb{R}^3 \setminus \Lambda_\varepsilon) \cap \{u_n > a\}} u_n^2 \, dx + o_n(1). \quad (2.17) \]
Using \(J'_\varepsilon(u_n), u_n = o_n(1)\) and \((2.17)\), we have
\[ \|u_n\|_\varepsilon^2 - \frac{V_1}{k} \int_{(\mathbb{R}^3 \setminus \Lambda_\varepsilon) \cap \{u_n > a\}} u_n^2 \, dx + b|u_n|^4 = \int_{\Lambda_\varepsilon \cup \{u_n \leq a\}} (u_n^+)^{2s} \, dx + o_n(1). \quad (2.18) \]
Assume that
\[ \int_{\Lambda_\varepsilon \cup \{u_n \leq a\}} (u_n^+)^{2s} \, dx \to \ell^3 \geq 0 \]
and
\[ [u_n]^2 \to B^2. \]
Note that \(\ell > 0\), otherwise \((2.18)\) yields \(\|u_n\|_\varepsilon \to 0\) as \(n \to \infty\) which implies that \(J'_\varepsilon(u_n) \to 0\), and this is impossible because \(c_\varepsilon > 0\). Then, from \((2.18)\) and Sobolev inequality we obtain
\[ aS_s \left( \int_{\Lambda_\varepsilon \cup \{u_n \leq a\}} (u_n^+)^{2s} \, dx \right)^{\frac{2}{2s}} + bS_s^2 \left( \int_{\Lambda_\varepsilon \cup \{u_n \leq a\}} (u_n^+)^{2s} \, dx \right)^{\frac{1}{2s}} \leq \int_{\Lambda_\varepsilon \cup \{u_n \leq a\}} (u_n^+)^{2s} \, dx + o_n(1). \quad (2.19) \]
Since \(\ell > 0\), it follows from \((2.19)\) that
\[ K'(\ell) = \frac{3 - 2s}{2} \ell^{-1}(aS_s \ell^{3-2s} + bS_s^2 \ell^{6-4s} - \ell^3) \leq 0 \]
so we can deduce that \(\ell \geq T\), where \(T\) is the unique maximum of \(K\) defined in Lemma 2.4.

Let us consider the following functional:
\[ \mathcal{I}_\varepsilon(u) := \frac{(a + bB^2)}{2} [u]^2 + \frac{1}{2} \int_{\mathbb{R}^3} V(\varepsilon \, x) u^2 \, dx - \int_{\mathbb{R}^3} G(\varepsilon \, x, u) \, dx \]
\[ = J_\varepsilon(u) - \frac{b}{4} [u]^4 + \frac{b}{2} B^2 [u]^2, \quad (2.20) \]
and we note that \(\{u_n\}_{n \in \mathbb{N}}\) is a \((PS)_{c_\varepsilon + \frac{b}{4} B^4}\) sequence for \(\mathcal{I}_\varepsilon\), that is
\[ \mathcal{I}_\varepsilon(u_n) = c_\varepsilon + \frac{b}{4} B^4 + o_n(1), \quad \mathcal{I}'_\varepsilon(u_n) = o_n(1). \quad (2.21) \]

Then, combining (2.16), (2.21), \( \ell \geq T \) and Sobolev inequality we can infer

\[
c_\varepsilon = I_\varepsilon(u_n) - \frac{b}{4} B^4 + o_n(1)
\]

\[
\geq \frac{a}{2} [u_n]^2 + \frac{b B^2}{2} |u_n|^2 - \frac{b}{4} B^4 + \frac{1}{2} \int_{\mathbb{R}^3} V(\varepsilon x)u_n^2 dx - \frac{V_1}{2k} \int_{(\mathbb{R}^3 \setminus \Lambda_\varepsilon) \cap \{u_n > \rho\}} u_n^2 dx
\]

\[
- \frac{1}{2} \int_{\Lambda_\varepsilon \cup \{u_n \leq \rho\}} (u_n^+)_{2s}^2 dx + o_n(1)
\]

\[
\geq \frac{a}{2} [u_n]^2 + \frac{b}{4} [u_n]^4 - \frac{1}{2} \int_{\Lambda_\varepsilon \cup \{u_n \leq \rho\}} (u_n^+)_{2s}^2 dx + o_n(1)
\]

\[
\geq \frac{a}{2} \frac{S_2}{2s} \left( \int_{\Lambda_\varepsilon \cup \{u_n \leq \rho\}} (u_n^+)_{2s}^2 dx \right)^{\frac{1}{2s}} + \frac{b}{4} \frac{S_2}{2s} \left( \int_{\Lambda_\varepsilon \cup \{u_n \leq \rho\}} (u_n^+)_{2s}^2 dx \right)^{\frac{1}{2s}} - \frac{1}{2} \int_{\Lambda_\varepsilon \cup \{u_n \leq \rho\}} (u_n^+)_{2s}^2 dx + o_n(1)
\]

\[
= \frac{a}{2} \frac{S_2}{2s} \left( \int_{\Lambda_\varepsilon \cup \{u_n \leq \rho\}} (u_n^+)_{2s}^2 dx \right)^{\frac{1}{2s}} + \frac{b}{4} \frac{S_2}{2s} \left( \int_{\Lambda_\varepsilon \cup \{u_n \leq \rho\}} (u_n^+)_{2s}^2 dx \right)^{\frac{1}{2s}} - \frac{1}{2} \int_{\Lambda_\varepsilon \cup \{u_n \leq \rho\}} (u_n^+)_{2s}^2 dx + o_n(1)
\]

\[
= \frac{a}{2} S_2 \varepsilon^{3-2s} + \frac{b}{4} S_2 \varepsilon^{6-4s} - \frac{1}{2s} \varepsilon^3 = c_\varepsilon,
\]

and this gives a contradiction by Lemma 2.4.

Now, we show that \( \{z_n\}_{n \in \mathbb{N}} \) is bounded in \( \mathbb{R}^3 \). For any \( \rho > 0 \), let \( \psi_\rho \in C(\mathbb{R}^3) \) be such that \( \psi_\rho = 0 \) in \( B_\rho \) and \( \psi_\rho = 1 \) in \( B_{2\rho}^C \), with \( 0 \leq \psi_\rho \leq 1 \) and \( |\nabla \psi_\rho| \leq \frac{C}{\rho} \), where \( C \) is a constant independent of \( \rho \). Since \( \{\psi_\rho u_n\}_{n \in \mathbb{N}} \) is bounded in \( H_\varepsilon \), it follows that \( \langle J'_\varepsilon(u_n), \psi_\rho u_n \rangle = o_n(1) \), that is

\[
(a + b[u_n]^2) \int_{\mathbb{R}^6} \frac{|u_n(x) - u_n(y)|^2}{|x - y|^{3+2s}} \psi_\rho(x) dx dy + \int_{\mathbb{R}^3} V(\varepsilon x)u_n^2 \psi_\rho dx
\]

\[
= o_n(1) + \int_{\mathbb{R}^3} g(\varepsilon x, u_n) u_n \psi_\rho dx - (a + b[u_n]^2) \int_{\mathbb{R}^6} \frac{(\psi_\rho(x) - \psi_\rho(y))(u_n(x) - u_n(y)) |u_n(y)| dy dx.
\]

Take \( \rho > 0 \) such that \( \Lambda_\varepsilon \subset B_\rho \). Then, using (g3)-(ii) we get

\[
\int_{\mathbb{R}^6} a \frac{|u_n(x) - u_n(y)|^2}{|x - y|^{3+2s}} \psi_\rho(x) dx dy + \int_{\mathbb{R}^3} V(\varepsilon x)u_n^2 \psi_\rho dx
\]

\[
\leq \int_{\mathbb{R}^6} \frac{1}{K} V(\varepsilon x) u_n^2 \psi_\rho dx - (a + b[u_n]^2) \int_{\mathbb{R}^6} \frac{(\psi_\rho(x) - \psi_\rho(y))(u_n(x) - u_n(y)) |u_n(y)| dy dx + o_n(1)
\]

which implies that

\[
\left(1 - \frac{1}{K}\right) V_1 \int_{B_{2\rho}^C} u_n^2 dx
\]

\[
\leq -(a + b[u_n]^2) \int_{\mathbb{R}^6} \frac{(\psi_\rho(x) - \psi_\rho(y))(u_n(x) - u_n(y)) |u_n(x)| dy dx + o_n(1).
\]  

(2.22)

At this point, we verify that

\[
\lim_{\rho \to \infty} \limsup_{n \to \infty} \int_{\mathbb{R}^6} \frac{|\psi_\rho(x) - \psi_\rho(y)|^2}{|x - y|^{3+2s}} |u_n(x)|^2 dy dx = 0.
\]  

(2.23)
Indeed, recalling that $0 \leq \psi_\rho \leq 1$ and $|\nabla \psi_\rho|_{\infty} \leq C/\rho$ and using polar coordinates, we obtain

$$
\int_{\mathbb{R}^3} \frac{|\psi_\rho(x) - \psi_\rho(y)|^2}{|x-y|^{3+2s}} |u_n(x)|^2 dx dy
\leq C \int_{\mathbb{R}^3} |u_n(x)|^2 \left( \int_{|y-x|>\rho} \frac{dy}{|x-y|^{3+2s}} \right) dx + \frac{C}{\rho^2} \int_{\mathbb{R}^3} |u_n(x)|^2 \left( \int_{|y-x|\leq\rho} \frac{dy}{|x-y|^{3+2s-2}} \right) dx
\leq C \int_{\mathbb{R}^3} |u_n(x)|^2 \left( \int_{|y-x|>\rho} \frac{dz}{|z|^{3+2s}} \right) dx + \frac{C}{\rho^2} \int_{\mathbb{R}^3} |u_n(x)|^2 \left( \int_{|z|\leq\rho} \frac{dz}{|z|^{1+2s}} \right) dx
\leq C \int_{\mathbb{R}^3} |u_n(x)|^2 dx + \frac{C}{\rho^2} \int_{\mathbb{R}^3} |u_n(x)|^2 dx
\leq C \int_{\mathbb{R}^3} |u_n(x)|^2 dx \leq \frac{C}{\rho^2s}
$$

where in the last passage we use the boundedness of $\{u_n\}_{n\in\mathbb{N}}$ in $\mathcal{H}_\varepsilon$. Taking the limit as $n \to \infty$ and then $\rho \to \infty$ we can deduce that (2.23) holds true.

Now, if $\{z_n\}_{n\in\mathbb{N}}$ is unbounded, we can use Lemma 2.6, $[\tilde{u}_n]^2 \to B^2 \in [0,\infty)$, (2.22) and (2.23) to deduce that $0 < \beta \left(1 - \frac{1}{R}\right) V_1 \leq 0$, which gives a contradiction.

We conclude this section giving the proof of the main result of this section:

**Theorem 2.2.** Assume that (V1)-(V2) and (f1)-(f4) hold. Then, problem (2.3) admits a positive ground state for all $\varepsilon > 0$.

**Proof.** Using Lemma 2.3 and a variant of the Mountain Pass Theorem without (PS) condition (see [52]), we know that there exists a Palais-Smale sequence $\{u_n\}_{n\in\mathbb{N}}$ for $\mathcal{J}_\varepsilon$ at the level $c_\varepsilon$, where $c_\varepsilon < c_s$ by Lemma 2.4.

Taking into account Lemma 2.5, we can see that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{H}_\varepsilon$, so we may assume that $u_n \rightharpoonup u$ in $\mathcal{H}_\varepsilon$ and $u_n \to u$ in $L^q_{loc}(\mathbb{R}^3)$ for all $q \in [1,2^*_s)$. From Lemma 2.6, we can deduce that $u$ nontrivial. Since $\langle \mathcal{J}_\varepsilon'(u), \varphi \rangle = o_n(1)$ for all $\varphi \in C_0^\infty(\mathbb{R}^3)$, we can see that

$$
\int_{\mathbb{R}^3} a(-\Delta)^{\frac{s}{2}} u(-\Delta)^{\frac{s}{2}} \varphi + V(\varepsilon x) u \varphi dx + b B^2 \left( \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} u(-\Delta)^{\frac{s}{2}} \varphi dx \right) = \int_{\mathbb{R}^3} g(\varepsilon x, u) \varphi dx,
$$

where $B^2 = \lim_{n \to \infty} |u_n|^2$. Let us note that $B^2 \geq |u|^2$ by Fatou’s Lemma. If by contradiction $B^2 > |u|^2$, we may use (2.24) to deduce that $\langle \mathcal{J}_\varepsilon'(u), u \rangle < 0$. Moreover, conditions (g1)-(g2) imply that $\langle \mathcal{J}_\varepsilon'(tu), tu \rangle > 0$ for small $t > 0$. Then there exists $t_0 \in (0,1)$ such that $t_0 u \in \mathcal{N}_\varepsilon$ and $\langle \mathcal{J}_\varepsilon'(t_0 u), t_0 u \rangle = 0$. Using Fatou’s Lemma, $t_0 \in (0,1)$, $s \in (\frac{3}{4},1)$ and (g3) we get

$$
c_\varepsilon \leq \mathcal{J}_\varepsilon(t_0 u) - \frac{1}{4} \langle \mathcal{J}_\varepsilon'(t_0 u), t_0 u \rangle
\leq \mathcal{J}_\varepsilon(u) - \frac{1}{4} \langle \mathcal{J}_\varepsilon'(u), u \rangle
\leq \liminf_{n \to \infty} \mathcal{J}_\varepsilon(u_n) - \frac{1}{4} \langle \mathcal{J}_\varepsilon'(u_n), u_n \rangle
= c_\varepsilon
$$

which gives a contradiction. Therefore $B^2 = |u|^2$ and we deduce that $\mathcal{J}_\varepsilon'(u) = 0$. Hence, $\mathcal{J}_\varepsilon$ admits a nontrivial critical point $u \in \mathcal{H}_\varepsilon$. Since $\langle \mathcal{J}_\varepsilon'(u), u \rangle = 0$ and $g(x,t) = 0$ for $t \leq 0$, where $u^- = \min\{u,0\}$, it is easy to check that $u \geq 0$ in $\mathbb{R}^3$. Moreover, proceeding as in the proof of Lemma 3.2 below, we can see that $u \in L^\infty(\mathbb{R}^3)$. Using Proposition 2.9 in [50] and $s > \frac{3}{4}$ we deduce that $u \in C^{1,\alpha}(\mathbb{R}^3)$, and applying the maximum principle [50] we can conclude that $u > 0$ in $\mathbb{R}^3$. Finally, arguing as in (2.25) with $t_0 = 1$, we can show that $u$ is a ground state solution to (2.3).
2.3. The autonomous problem.
Let us consider the following family of limit problems related to (2.3), that is, for \( \mu > 0 \)
\[
\begin{align*}
\{ (a + \beta |u|^{2s})(-\Delta)^s u + \mu u &= f(u) + |u|^{2^*-2} u & \text{in } \mathbb{R}^3, \\
\mu &> 0 \\n\end{align*}
\]
whose corresponded Euler-Lagrange functional is given by
\[
\mathcal{I}_\mu(u) = \frac{1}{2} \left( a \int_{\mathbb{R}^6} \frac{|u(x) - u(y)|^2}{|x - y|^{3+2s}} \, dx \, dy + \mu \int_{\mathbb{R}^3} u^2 \, dx \right) + \frac{b}{4} \int_{\mathbb{R}^3} |u|^4 \, dx - \int_{\mathbb{R}^3} F(u) + \frac{1}{2^*} \int_{\mathbb{R}^3} (u^+)^{2^*} \, dx 
\]
which is well defined on the Hilbert space \( \mathcal{H}_\mu := H^s(\mathbb{R}^3) \) endowed with the inner product
\[
\langle u, \varphi \rangle_\mu = a \int_{\mathbb{R}^6} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{3+2s}} \, dx \, dy + \mu \int_{\mathbb{R}^3} u \varphi \, dx.
\]
The norm induced by the above inner product is given by
\[
\|u\|^2_\mu = a \int_{\mathbb{R}^6} \frac{|u(x) - u(y)|^2}{|x - y|^{3+2s}} \, dx \, dy + \mu \int_{\mathbb{R}^3} u^2 \, dx.
\]
We denote by \( \mathcal{N}_\mu \) the Nehari manifold associated to \( \mathcal{I}_\mu \), that is
\[
\mathcal{N}_\mu = \left\{ u \in \mathcal{H}_\mu \setminus \{0\} : \langle \mathcal{I}_\mu'(u), u \rangle = 0 \right\},
\]
and
\[
c_\mu = \inf_{u \in \mathcal{N}_\mu} \mathcal{I}_\mu(u),
\]
or equivalently
\[
c_\mu = \inf_{u \in \mathcal{H}_\mu \setminus \{0\}} \max_{t \geq 0} \mathcal{I}_\mu(tu).
\]
Arguing as in Theorem 2.2, it is easy to deduce that:

**Theorem 2.3.** For all \( \mu > 0 \), problem (2.26) admits a positive ground state solution.

Let us prove the following useful relation between \( c_\varepsilon \) and \( c_{\varepsilon_0} \):

**Lemma 2.7.** It holds \( \limsup_{\varepsilon \to 0} c_\varepsilon \leq c_{\varepsilon_0} \).

**Proof.** For any \( R > 0 \) we set \( u_R(x) = \psi_R(x) u_0(x) \), where \( u_0 \) is positive ground state given by Theorem 2.3 with \( \mu = V_0 \), and \( \psi_R(x) = \psi(x/R) \) with \( \psi \in C_c^\infty(\mathbb{R}^3) \), \( \psi \in [0, 1] \), \( \psi = 1 \) if \( |x| \leq \frac{1}{2} \) and \( \psi = 0 \) if \( |x| \geq 1 \). For simplicity, we assume that \( \text{supp}(\psi) \subset B_1 \subset \Omega \). Using Lemma 2.2 and the Dominated Convergence Theorem we can see that
\[
u_R \to u_0 \text{ in } H^s(\mathbb{R}^3) \quad \text{as } R \to \infty.
\]
For each \( \varepsilon, R > 0 \) there exists \( t_{\varepsilon,R} > 0 \) such that
\[
\mathcal{J}_\varepsilon(t_{\varepsilon,R} u_R) = \max_{t \geq 0} \mathcal{J}_\varepsilon(t u_R).
\]
Then \( \mathcal{J}_\varepsilon'(t_{\varepsilon,R} u_R) = 0 \) and this implies that
\[
\frac{1}{t_{\varepsilon,R}^2} \int_{\mathbb{R}^3} a|(-\Delta)^{\frac{\varepsilon}{2}} u_R|^2 + V_0 u_R^2 \, dx + b \left( \int_{\mathbb{R}^3} |(-\Delta)^{\frac{\varepsilon}{2}} u_R|^2 \, dx \right)^2 \\
= \int_{\mathbb{R}^3} f(t_{\varepsilon,R} u_R) u_R \, dx + t_{\varepsilon,R}^{2^*-4} \int_{\mathbb{R}^3} u_R^{2^*} \, dx.
\]
From the last equality, we can deduce that for any \( R > 0 \) we have
\[
0 < \lim_{\varepsilon \to 0} t_{\varepsilon,R} = t_R < \infty.
\]
Taking the limit as \( \varepsilon \to 0 \) in (2.28) we get
\[
\frac{1}{t_R^2} \int_{\mathbb{R}^3} a|(-\Delta)^{\frac{\varepsilon}{2}} u_R|^2 + V_0 u_R^2 \, dx + b \left( \int_{\mathbb{R}^3} |(-\Delta)^{\frac{\varepsilon}{2}} u_R|^2 \, dx \right)^2 \\
= \int_{\mathbb{R}^3} f(t_R u_R) u_R \, dx + t_R^{2^*-4} \int_{\mathbb{R}^3} u_R^{2^*} \, dx.
\]
Putting together (2.28) and (2.29) we deduce that \( t_R = 1 \) and \( J_0(t_R u_R) = \max_{t \geq 0} J_0(t u_R) \). Consequently, we have
\[
c_\varepsilon \leq \max_{t \geq 0} J_\varepsilon(t u_R) = J_\varepsilon(t_R u_R)
\]
which implies that
\[
\limsup_{\varepsilon \to 0} c_\varepsilon \leq I_{V_0}(t_R u_R).
\]
Taking the limit as \( R \to \infty \) and using (2.27) we get
\[
\limsup_{\varepsilon \to 0} c_\varepsilon \leq c_{V_0}.
\]
\[\square\]

3. Proof of Theorem 1.1

This last section is devoted to the proof of the main result of this work. Firstly, we prove the following compactness result which will be fundamental to show that the solutions of (2.3) are also solutions to (2.1) for \( \varepsilon > 0 \) small enough.

**Lemma 3.1.** Let \( \varepsilon_n \to 0 \) and \( \{u_n\}_{n \in \mathbb{N}} \subset \mathcal{H}_{\varepsilon_n} \) be such that \( J_{\varepsilon_n}(u_n) = c_{\varepsilon_n} \) and \( J'_{\varepsilon_n}(u_n) = 0 \). Then there exists \( \{\tilde{y}_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^3 \) such that the translated sequence
\[
\tilde{u}_n(x) := u_n(x + \tilde{y}_n)
\]
has a subsequence which converges in \( H^s(\mathbb{R}^3) \). Moreover, up to a subsequence, \( \{y_n\}_{n \in \mathbb{N}} := \{\varepsilon_n \tilde{y}_n\}_{n \in \mathbb{N}} \) is such that \( y_n \to y_0 \) for some \( y_0 \in \Lambda \) such that \( V(y_0) = V_0 \).

**Proof.** Using \( \langle J'_{\varepsilon_n}(u_n), u_n \rangle = 0 \) and \( (g_1), (g_2) \), it is easy to see that there is \( \gamma > 0 \) (independent of \( \varepsilon_n \)) such that
\[
\|u_n\|_{\varepsilon_n} \geq \gamma > 0 \quad \forall n \in \mathbb{N}.
\]
Taking into account \( J_{\varepsilon_n}(u_n) = c_{\varepsilon_n}, \langle J'_{\varepsilon_n}(u_n), u_n \rangle = 0 \) and Lemma 2.7, we can argue as in the proof of Lemma 2.5 to deduce that \( \{u_n\}_{n \in \mathbb{N}} \) is bounded in \( \mathcal{H}_{\varepsilon_n} \). Therefore, proceeding as in Lemma 2.6, we can find a sequence \( \{\tilde{y}_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^3 \) and constants \( R, \alpha > 0 \) such that
\[
\liminf_{n \to \infty} \int_{B_R(\tilde{y}_n)} |u_n|^2 dx \geq \alpha.
\]
Set \( \tilde{u}_n(x) := u_n(x + \tilde{y}_n) \). Then \( \{\tilde{u}_n\}_{n \in \mathbb{N}} \) is bounded in \( H^s(\mathbb{R}^3) \), and we may assume that
\[
\tilde{u}_n \rightharpoonup \tilde{u} \text{ weakly in } H^s(\mathbb{R}^3).
\]
Moreover, \( \tilde{u} \neq 0 \) in view of
\[
\int_{B_R} |\tilde{u}|^2 dx \geq \alpha.
\]
Now, we set \( y_n := \varepsilon_n \tilde{y}_n \). Firstly, we show that \( \{y_n\}_{n \in \mathbb{N}} \) is bounded. To achieve our purpose, we prove the following claim:
**Claim 1** \( \lim_{n \to \infty} \text{dist}(y_n, \Lambda) = 0 \).

If by contradiction the claim is not true, then we can find \( \delta > 0 \) and a subsequence of \( \{y_n\}_{n \in \mathbb{N}} \), still denoted by itself, such that
\[
\text{dist}(y_n, \Lambda) \geq \delta \quad \forall n \in \mathbb{N}.
\]
Thus, there is \( r > 0 \) such that \( B_r(y_n) \subset \Lambda^c \) for all \( n \in \mathbb{N} \). Since \( \tilde{u} \geq 0 \) and \( C^\infty_c(\mathbb{R}^3) \) is dense in \( H^s(\mathbb{R}^3) \), we can approximate \( \tilde{u} \) by a sequence \( \{\psi_j\}_{j \in \mathbb{N}} \subset C^\infty_c(\mathbb{R}^3) \) such that \( \psi_j \geq 0 \), that is \( \psi_j \to \tilde{u} \) in \( H^s(\mathbb{R}^3) \). Fix \( j \in \mathbb{N} \) and use \( \psi = \psi_j \) as test function in \( \langle J'_{\varepsilon_n}(u_n), \psi \rangle = 0 \). Then
\[
(a + b|\tilde{u}_n|^2) \int_{\mathbb{R}^6} \frac{(\tilde{u}_n(x) - \tilde{u}_n(y))(\psi_j(x) - \psi_j(y))}{|x - y|^{3+2s}} \, dx \, dy + \int_{\mathbb{R}^3} V(\varepsilon_n x + \varepsilon_n \tilde{y}_n) \tilde{u}_n \psi_j \, dx
\]
\[
= \int_{\mathbb{R}^3} g(\varepsilon_n x + \varepsilon_n \tilde{y}_n, \tilde{u}_n) \psi_j \, dx.
\]
Since \( u_{\varepsilon_n}, \psi_j \geq 0 \) and using the definition of \( g \), we can see that

\[
\int_{\mathbb{R}^3} g(\varepsilon_n x + \varepsilon_n \tilde{y}_n, \tilde{u}_n)\psi_j \, dx = \int_{B_r / \varepsilon_n} g(\varepsilon_n x + \varepsilon_n \tilde{y}_n, \tilde{u}_n)\psi_j \, dx + \int_{\mathbb{R}^3 \setminus B_r / \varepsilon_n} g(\varepsilon_n x + \varepsilon_n \tilde{y}_n, \tilde{u}_n)\psi_j \, dx
\]

\[
= \int_{B_r / \varepsilon_n} \tilde{f}(\tilde{u}_n)\psi_j \, dx + \int_{\mathbb{R}^3 \setminus B_r / \varepsilon_n} \left( f(\tilde{u}_n)\psi_j + \tilde{u}_n^{2s-1}\psi_j \right) \, dx
\]

\[
\leq \frac{V_1}{K} \int_{B_r / \varepsilon_n} \tilde{u}_n\psi_j \, dx + \int_{\mathbb{R}^3 \setminus B_r / \varepsilon_n} \left( f(\tilde{u}_n)\psi_j + \tilde{u}_n^{2s-1}\psi_j \right) \, dx.
\]

This together with (3.3) gives

\[
(a + b[\tilde{u}_n]^2) \int_{\mathbb{R}^6} \frac{(\tilde{u}_n(x) - \tilde{u}_n(y))(\psi_j(x) - \psi_j(y))}{|x - y|^{3+2s}} \, dx \, dy + A \int_{\mathbb{R}^3} \tilde{u}_n\psi_j \, dx
\]

\[
\leq \int_{\mathbb{R}^3 \setminus B_r / \varepsilon_n} \left( f(\tilde{u}_n)\psi_j + \tilde{u}_n^{2s-1}\psi_j \right) \, dx (3.4)
\]

where \( A = V_1(1 - \frac{1}{R}) \). Taking into account (3.1), \( \psi_j \) has compact support in \( \mathbb{R}^3 \) and \( \varepsilon_n \to 0 \), we can infer that as \( n \to \infty \)

\[
\int_{\mathbb{R}^6} \frac{(\tilde{u}_n(x) - \tilde{u}_n(y))(\psi_j(x) - \psi_j(y))}{|x - y|^{3+2s}} \, dx \, dy
\]

\[
\to \int_{\mathbb{R}^6} \frac{(\tilde{u}(x) - \tilde{u}(y))(\psi_j(x) - \psi_j(y))}{|x - y|^{3+2s}} \, dx \, dy
\]

and

\[
\int_{\mathbb{R}^3 \setminus B_r / \varepsilon_n} \left( f(\tilde{u}_n)\psi_j + \tilde{u}_n^{2s-1}\psi_j \right) \, dx \to 0.
\]

The above limits, (3.4) and \([\tilde{u}_n]^2 \to B^2\) imply that

\[
(a + bB^2) \int_{\mathbb{R}^6} \frac{(\tilde{u}(x) - \tilde{u}(y))(\psi_j(x) - \psi_j(y))}{|x - y|^{3+2s}} \, dx \, dy + A \int_{\mathbb{R}^3} \tilde{u}\psi_j \, dx \leq 0
\]

and passing to the limit as \( j \to \infty \) we can infer that

\[
(a + bB^2)[\tilde{u}]^2 + A[\tilde{u}]^2_2 \leq 0.
\]

This gives a contradiction by (3.2). Hence, there exists a subsequence of \( \{y_n\}_{n \in \mathbb{N}} \) such that \( y_n \to y_0 \in \Lambda \).

Secondly, we prove the following claim:

Claim 2 \( y_0 \in \Lambda \).

In the light of (g2) and (3.3) we can deduce that

\[
(a + b[\tilde{u}_n]^2) \int_{\mathbb{R}^6} \frac{(\tilde{u}_n(x) - \tilde{u}_n(y))(\psi_j(x) - \psi_j(y))}{|x - y|^{3+2s}} \, dx \, dy + \int_{\mathbb{R}^3} V(\varepsilon_n x + \varepsilon_n \tilde{y}_n)\tilde{u}_n\psi_j \, dx
\]

\[
\leq \int_{\mathbb{R}^3} \left( f(\tilde{u}_n) + \tilde{u}_n^{2s-1}\right)\psi_j \, dx.
\]

Letting the limit as \( n \to \infty \) we find

\[
(a + bB^2) \int_{\mathbb{R}^6} \frac{(\tilde{u}(x) - \tilde{u}(y))(\psi_j(x) - \psi_j(y))}{|x - y|^{3+2s}} \, dx \, dy + \int_{\mathbb{R}^3} V(y_0)\tilde{u}\psi_j \, dx
\]

\[
\leq \int_{\mathbb{R}^3} (f(\tilde{u}) + \tilde{u}^{2s-1})\psi_j \, dx,
\]

and passing to the limit as \( j \to \infty \) we obtain

\[
(a + bB^2) \int_{\mathbb{R}^6} \frac{|\tilde{u}(x) - \tilde{u}(y)|^2}{|x - y|^{3+2s}} \, dx \, dy + \int_{\mathbb{R}^3} V(y_0)\tilde{u}^2 \, dx \leq \int_{\mathbb{R}^3} (f(\tilde{u}) + \tilde{u}^{2s-1})\tilde{u} \, dx.
\]

Since \( B^2 \geq [u]^2 \) (by Fatou’s Lemma), the above inequality yields

\[
(a + b[u]^2) \int_{\mathbb{R}^6} \frac{|\tilde{u}(x) - \tilde{u}(y)|^2}{|x - y|^{3+2s}} \, dx \, dy + \int_{\mathbb{R}^3} V(y_0)\tilde{u}^2 \, dx \leq \int_{\mathbb{R}^3} (f(\tilde{u}) + \tilde{u}^{2s-1})\tilde{u} \, dx.
\]
Therefore, we can find \( \tau \in (0, 1) \) such that \( \tau \tilde{u} \in \mathcal{N}_{V(y_0)} \). Then, by Lemma 2.7, we can see that
\[
c_{V(y_0)} \leq \mathcal{I}_{V(y_0)}(\tau \tilde{u}) \leq \liminf_{n \to \infty} \mathcal{J}_{\varepsilon_n}(u_{\varepsilon_n}) = \liminf_{n \to \infty} c_{\varepsilon_n} \leq c_{V_0}
\]
which implies that \( V(y_0) \leq V(0) = V_0 \). Since \( V_0 = \inf_\Lambda V \), we can deduce that \( V(y_0) = V_0 \), and using \( (V_2) \), we get \( y_0 \notin \partial \Lambda \). In conclusion, \( y_0 \in \Lambda \).

**Claim 3** \( \tilde{u}_n \to \tilde{u} \) in \( H^s(\mathbb{R}^3) \) as \( n \to \infty \).

Let us define
\[
\tilde{\Lambda}_n = \frac{\Lambda - \varepsilon_n \tilde{y}_n}{\varepsilon_n}
\]
and
\[
\tilde{\chi}_n^1(x) = \begin{cases} 1 & \text{if } x \in \tilde{\Lambda}_n \\ 0 & \text{if } x \in \mathbb{R}^3 \setminus \tilde{\Lambda}_n \end{cases}
\]
\[
\tilde{\chi}_n^2(x) = 1 - \tilde{\chi}_n^1(x).
\]
Let us also consider the following functions for all \( x \in \mathbb{R}^3 \)
\[
h_n^1(x) = \left( \frac{1}{2} - \frac{1}{d} \right) V(\varepsilon_n x + \varepsilon_n \tilde{y}_n) |\tilde{u}_n(x)|^2 \tilde{\chi}_n^1(x)
\]
\[
h_n^2(x) = \left( \frac{1}{2} - \frac{1}{d} \right) V(y_0) |\tilde{u}(x)|^2
\]
\[
h_n^3(x) = \frac{1}{d} \left( f(\tilde{u}_n(x)) \tilde{u}_n(x) + |\tilde{u}_n(x)|^{2^*_s} \right) - \left( F(\tilde{u}(x)) + \frac{1}{2^*_s} |\tilde{u}(x)|^{2^*_s} \right) \tilde{\chi}_n^1(x)
\]
\[
h^3(x) = \frac{1}{d} \left( f(\tilde{u}(x)) \tilde{u}(x) + |\tilde{u}(x)|^{2^*_s} \right) - \left( F(\tilde{u}(x)) + \frac{1}{2^*_s} |\tilde{u}(x)|^{2^*_s} \right)
\]
In view of \( (f_3) \) and \( (g_3) \), we can observe that the above functions are nonnegative. Moreover, using \( (3.1) \) and Claim 2, we know that
\[
\tilde{u}_n(x) \to \tilde{u}(x) \quad \text{a.e. } x \in \mathbb{R}^3,
\]
\[
\varepsilon_n \tilde{y}_n \to y_0 \in \Lambda,
\]
which imply that
\[
\tilde{\chi}_n^1(x) \to 1, \ h_n^1(x) \to h^1(x), \ h_n^2(x) \to 0 \quad \text{and} \quad h_n^3(x) \to h^3(x) \quad \text{a.e. } x \in \mathbb{R}^3.
\]
Hence, applying the Fatou’s Lemma and using the invariance of \( \mathbb{R}^3 \) by translation, we can see that
\[
c_{V_0} \geq \limsup_{n \to \infty} c_{\varepsilon_n} = \limsup_{n \to \infty} \left( \mathcal{J}_{\varepsilon_n}(u_n) - \frac{1}{d} \langle \mathcal{J}'_{\varepsilon_n}(u_n), u_n \rangle \right)
\]
\[
\geq \limsup_{n \to \infty} \left[ \left( \frac{1}{2} - \frac{1}{d} \right) |\tilde{u}_n|^2 + \left( \frac{1}{4} - \frac{1}{d} \right) b|\tilde{u}_n|^4 + \int_{\mathbb{R}^3} (h_n^1 + h_n^2 + h_n^3) \, dx \right]
\]
\[
\geq \liminf_{n \to \infty} \left[ \left( \frac{1}{2} - \frac{1}{d} \right) |\tilde{u}|^2 + \left( \frac{1}{4} - \frac{1}{d} \right) b|\tilde{u}|^4 + \int_{\mathbb{R}^3} (h^1 + h^2 + h^3) \, dx \right] \geq c_{V_0}.
\]
Accordingly
\[
\lim_{n \to \infty} |\tilde{u}_n|^2 = |\tilde{u}|^2
\]
Using \( \gamma \), we can deduce that
\[
\lim_{n \to \infty} \int_{\mathbb{R}^3} V(\varepsilon_n x + \varepsilon_n \tilde{y}_n) |\tilde{u}_n|^2 \, dx = \int_{\mathbb{R}^3} V(y_0) |\tilde{u}|^2 \, dx,
\]
and we can deduce that
\[
\lim_{n \to \infty} |\tilde{u}_n|_2^2 = |\tilde{u}|_2^2. \tag{3.6}
\]
Putting together (3.1), (3.5) and (3.6) and using that \( H^s(\mathbb{R}^3) \) is a Hilbert space we obtain
\[
\|\tilde{u}_n - \tilde{u}\|_{V_0} \to 0 \text{ as } n \to \infty.
\]
This fact ends the proof of lemma. \( \square \)

In the next lemma, we use a Moser iteration argument [39] to prove the following useful \( L^\infty \)-estimate for the solutions of the modified problem (2.3).

**Lemma 3.2.** Let \( \varepsilon_n \to 0 \) and \( u_n \in \mathcal{H}_{\varepsilon} \) be a solution to (2.3). Then, up to a subsequence, \( v_n = u_n(\cdot + \tilde{y}_n) \in L^\infty(\mathbb{R}^3) \), and there exists \( C > 0 \) such that
\[
|v_n|_\infty \leq C \quad \text{for all } n \in \mathbb{N}.
\]

**Proof.** For any \( L > 0 \) and \( \beta > 1 \), let us define the function
\[
\gamma(v_n) = \gamma_{L,\beta}(v_n) = v_n v_{L,n}^{2(\beta - 1)} \in \mathcal{H}_{\varepsilon}
\]
where \( v_{L,n} = \min\{v_n, L\} \). Since \( \gamma \) is an increasing function, we have
\[
(a - b)(\gamma(a) - \gamma(b)) \geq 0 \quad \text{for any } a, b \in \mathbb{R}.
\]
Let us consider
\[
\mathcal{E}(t) = \frac{|\dot{t}|^2}{2} \quad \text{and} \quad \Gamma(t) = \int_0^t (\gamma'(\tau))^2 d\tau.
\]
Then, applying Jensen inequality we get for all \( a, b \in \mathbb{R} \) such that \( a > b \),
\[
\mathcal{E}'(a - b)(\gamma(a) - \gamma(b)) = (a - b)(\gamma(a) - \gamma(b)) = (a - b) \int_b^a \gamma'(t) dt
\]
\[
= (a - b) \int_b^a (\Gamma'(t))^2 dt \geq \left( \int_b^a (\Gamma'(t)) dt \right)^2.
\]
The same argument works when \( a \leq b \). Therefore
\[
\mathcal{E}'(a - b)(\gamma(a) - \gamma(b)) \geq |\Gamma(a) - \Gamma(b)|^2 \quad \text{for any } a, b \in \mathbb{R}. \tag{3.7}
\]
Using (3.7), we can see that
\[
|\Gamma(v_n)(x) - \Gamma(v_n)(y)|^2 \leq (v_n(x) - v_n(y))(v_n v_{L,n}^{2(\beta - 1)}(x) - v_n v_{L,n}^{2(\beta - 1)}(y)). \tag{3.8}
\]
Choosing \( \gamma(v_n) = v_n v_{L,n}^{2(\beta - 1)} \) as test function in (2.3) and using (3.8) we obtain
\[
a[\Gamma(v_n)]^2 + \int_{\mathbb{R}^3} V_n(x)|v_n|^{2(\beta - 1)} v_{L,n} \, dx
\]
\[
\leq (a + b|v_n|^2) \int_{\mathbb{R}^3} \frac{(v_n(x) - v_n(y))((v_n v_{L,n}^{2(\beta - 1)})(x) - (v_n v_{L,n}^{2(\beta - 1)})(y)) \, dx dy + \int_{\mathbb{R}^3} V_n(x)|v_n|^{2(\beta - 1)} v_{L,n} \, dx
\]
\[
\leq \int_{\mathbb{R}^3} g_n(v_n) v_n v_{L,n}^{2(\beta - 1)} \, dx. \tag{3.9}
\]
Since
\[
\Gamma(v_n) \geq \frac{1}{\beta} v_n v_{L,n}^{\beta - 1},
\]
and using Theorem 2.1, we have
\[
[\Gamma(v_n)]^2 \geq S_* |\Gamma(v_n)|_2^2 \geq \left( \frac{1}{\beta} \right)^2 S_* |v_n v_{L,n}^{\beta - 1}|_2^2. \tag{3.10}
\]
By assumptions \((g_1)\) and \((g_2)\), for any \(\xi > 0\) there exists \(C_\xi > 0\) such that
\[
|g_n(v_n)| \leq \xi |v_n| + C_\xi |v_n|^{2\beta - 1}.
\] (3.11)
Taking \(\xi \in (0, V_1)\), and from (3.10) and (3.11), we can see that (3.9) yields
\[
|w_{L,n}|^{\frac{2}{2+2\beta}} \leq C\beta^2 \int_{\mathbb{R}^3} |v_n|^{2\beta} v_n^{2(\beta - 1)} dx.
\] (3.12)
where \(w_{L,n} := v_n^{2\beta - 1} \). Now, we take \(\beta = \frac{2^*}{2}\) and fix \(R > 0\). Recalling that \(0 \leq v_{L,n} \leq v_n\), we have
\[
\int_{\mathbb{R}^3} v_n^{2\beta(n)} v_{L,n}^{2(\beta - 1)} dx = \int_{\mathbb{R}^3} v_n^{2\beta(n)} v_n^{2\beta(n)} v_{L,n}^{2\beta(n)} dx
\]
\[
= \int_{\mathbb{R}^3} v_n^{2\beta(n)} (v_n v_{L,n})^{2\beta(n)} dx
\]
\[
\leq \int_{\{v_n < R\}} R^{2\beta(n)} v_n^{2\beta(n)} dx + \int_{\{v_n > R\}} v_n^{2\beta(n)} (v_n v_{L,n})^{2\beta(n)} dx
\]
\[
\leq \int_{\{v_n < R\}} R^{2\beta(n)} v_n^{2\beta(n)} dx + \left( \int_{\{v_n > R\}} v_n^{2\beta(n)} dx \right) \left( \int_{\mathbb{R}^3} (v_n v_{L,n})^{2\beta(n)} dx \right)^{\frac{2\beta}{2+2\beta}}.
\] (3.13)
Since \(\{v_n\}_{n \in \mathbb{N}}\) is bounded in \(L^{2\beta(n)}(\mathbb{R}^3)\), we can see that for any \(R\) sufficiently large
\[
\left( \int_{\{v_n > R\}} v_n^{2\beta(n)} dx \right)^{\frac{2\beta}{2+2\beta}} \leq \varepsilon^{-2}.
\] (3.14)
Putting together (3.12), (3.13) and (3.14) we get
\[
\left( \int_{\mathbb{R}^3} (v_n v_{L,n})^{2\beta(n)} dx \right)^{\frac{2\beta}{2+2\beta}} \leq C\beta^2 \int_{\mathbb{R}^3} R^{2\beta(n)} v_n^{2\beta(n)} dx < \infty
\]
and taking the limit as \(L \to \infty\), we obtain \(v_n \in L^{2\beta(n)}(\mathbb{R}^3)\).
Now, noting \(0 \leq v_{L,n} \leq v_n\) and letting to the limit as \(L \to \infty\) in (3.12), we have
\[
|v_n|^{\frac{2\beta}{2+2\beta}} \leq C\beta^2 \int_{\mathbb{R}^3} v_n^{2\beta(n) + 2(\beta - 1)},
\]
from which we deduce that
\[
\left( \int_{\mathbb{R}^3} v_n^{2\beta(n)} dx \right)^{\frac{1}{2(\beta(n) - 1)}} \leq C\beta^2 \left( \int_{\mathbb{R}^3} v_n^{2\beta(n) + 2(\beta - 1)} dx \right)^{\frac{1}{2(\beta(n) - 1)}}.
\]
For \(m \geq 1\) we define \(\beta_{m+1}\) inductively so that \(2\beta + 2(\beta_{m+1} - 1) = 2\beta_m\) and \(\beta_1 = \frac{2^*}{2}\). Then we have
\[
\left( \int_{\mathbb{R}^3} v_n^{2\beta_{m+1} + 2(\beta_{m+1} - 1)} dx \right)^{\frac{1}{2(\beta_{m+1} - 1)}} \leq C\beta_{m+1}^{2\beta_{m+1} - 1} \left( \int_{\mathbb{R}^3} v_n^{2\beta_{m+1} + 2(\beta_{m+1} - 1)} dx \right)^{\frac{1}{2(\beta_{m+1} - 1)}}.
\]
Let us define
\[
D_m = \left( \int_{\mathbb{R}^3} v_n^{2\beta_{m+1} + 2(\beta_{m+1} - 1)} dx \right)^{\frac{1}{2(\beta_{m+1} - 1)}}.
\]
A standard iteration argument shows that we can find \(C_0 > 0\) independent of \(m\) such that
\[
D_{m+1} \leq \prod_{k=1}^{m} C\beta_{k+1}^{2\beta_{k+1} - 1} D_1 \leq C_0 D_1.
\]
Passing to the limit as \(m \to \infty\) we get \(|v_n|_{\infty} \leq K\) for all \(n \in \mathbb{N}\). \(\square\)

Now, we give the proof of Theorem 1.1.
Proof of Theorem 1.1. Firstly, we prove that there exists $\tilde{\varepsilon}_0 > 0$ such that for any $\varepsilon \in (0, \tilde{\varepsilon}_0)$ and any solution $u_{\varepsilon} \in \mathcal{H}_{\varepsilon}$ of (2.3), it results
\[ |u_{\varepsilon}|_{L^\infty(\mathbb{R}^3 \setminus \Lambda_{\varepsilon})} < a. \] (3.15)
Assume by contradiction that for some subsequence $\{\varepsilon_n\}_{n \in \mathbb{N}}$ such that $\varepsilon_n \to 0$, we can find $u_{\varepsilon_n} \in \mathcal{H}_{\varepsilon_n}$ such that $J_{\varepsilon_n}(u_{\varepsilon_n}) = \varepsilon_n, J'_{\varepsilon_n}(u_{\varepsilon_n}) = 0$ and
\[ |u_{\varepsilon_n}|_{L^\infty(\mathbb{R}^3 \setminus \Lambda_{\varepsilon_n})} \geq a. \] (3.16)
From Lemma 3.1, there exists $\{\tilde{y}_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^3$ such that $u_{\varepsilon_n} = u_{\varepsilon_n} \cdot \tilde{y}_n \to \tilde{u}$ in $H^s(\mathbb{R}^3)$ and $\varepsilon_n \tilde{y}_n \to y_0$ for some $y_0 \in \Lambda$ such that $V(y_0) = V_0$. Now, if we choose $r > 0$ such that $B_r(y_0) \subset 2B_r(y_0) \subset \Lambda$, we can see that $B_{\varepsilon_n}(\tilde{y}_n) \subset \Lambda_{\varepsilon_n}$. Then, for any $y \in B_{\varepsilon_n}(\tilde{y}_n)$ it holds
\[ |y - y_0| \leq |y - \tilde{y}_n| + |\tilde{y}_n - y_0| < \frac{1}{\varepsilon_n} (r + o_n(1)) \leq \frac{2r}{\varepsilon_n} \text{ for } n \text{ sufficiently large}. \]
Hence
\[ \mathbb{R}^3 \setminus \Lambda_{\varepsilon_n} \subset \mathbb{R}^3 \setminus B_{\varepsilon_n}(\tilde{y}_n) \] (3.17)
for any $n$ big enough.
Now, we observe that $\tilde{u}_n$ is a solution to
\[ (-\Delta)^s \tilde{u}_n + \tilde{u}_n = \xi_n \text{ in } \mathbb{R}^3, \]
where
\[ \xi_n(x) := (a + b(\tilde{u}_n)^2)^{-1}[g(\varepsilon_n x + \varepsilon_n \tilde{y}_n, \tilde{u}_n) - V_n(x)\tilde{u}_n] + \tilde{u}_n \]
and
\[ V_n(x) := V(\varepsilon_n x + \varepsilon_n \tilde{y}_n). \]
Put
\[ \xi(x) := (a + b(\tilde{u})^2)^{-1}[f(\tilde{u}) + |\tilde{u}|^{2^*_s - 2}\tilde{u} - V(y_0)\tilde{u}] + \tilde{u}. \]
Using Lemma 3.2, the interpolation in the $L^p$ spaces, $\tilde{u}_n \to \tilde{u}$ in $H^s(\mathbb{R}^3)$, assumptions $(g_1)$ and $(g_3)$ we can see that
\[ \xi_n \to \xi \text{ in } L^p(\mathbb{R}^3) \quad \forall p \in [2, \infty), \]
so there exists $C > 0$ such that
\[ |\xi_n|_\infty \leq C \text{ for any } n \in \mathbb{N}. \]
Consequently, $\tilde{u}_n(x) = (K \ast \xi_n)(x) = \int_{\mathbb{R}^3} K(x - z)\xi_n(z) \, dz$, where $K$ is the Bessel kernel and satisfies the following properties [22]:
(i) $K$ is positive, radially symmetric and smooth in $\mathbb{R}^3 \setminus \{0\}$,
(ii) there is $C > 0$ such that $K(x) \leq C/|x|^{3+2s}$ for any $x \in \mathbb{R}^3 \setminus \{0\}$,
(iii) $K \in L^r(\mathbb{R}^3)$ for any $r \in [1, \frac{3}{3+2s})$.
Hence, arguing as in Lemma 2.6 in [4], we can see that
\[ \tilde{u}_n(x) \to 0 \text{ as } |x| \to \infty \text{ uniformly in } n \in \mathbb{N}. \] (3.18)
Then we can find $R > 0$ such that
\[ \tilde{u}_n(x) < a \text{ for } |x| \geq R, n \in \mathbb{N}, \]
which yields $u_{\varepsilon_n}(x) < a$ for any $x \in \mathbb{R}^3 \setminus B_R(\tilde{y}_n)$ and $n \in \mathbb{N}$. This fact and (3.17) imply that there exists $\nu \in \mathbb{N}$ such that for any $n \geq \nu$ and $r/\varepsilon_n > R$ we have
\[ \mathbb{R}^3 \setminus \Lambda_{\varepsilon_n} \subset \mathbb{R}^3 \setminus B_{\varepsilon_n}(\tilde{y}_n) \subset \mathbb{R}^3 \setminus B_R(\tilde{y}_n), \]
which gives $u_{\varepsilon_n}(x) < a$ for any $x \in \mathbb{R}^3 \setminus \Lambda_{\varepsilon_n}$ and $n \geq \nu$. This fact contradicts (3.16), so (3.15) holds true.
Let us denote by $u_{\varepsilon}$ a solution to (2.3). Since $u_{\varepsilon}$ satisfies (3.15), from the definition of $g$ it follows that $u_{\varepsilon}$ is a solution of (2.1). Then $\tilde{u}(x) = u(x/\varepsilon)$ is a solution to (1.1).
Finally, we study the behavior of the maximum points of solutions to problem (2.1). Take $\varepsilon_n \to 0^+$ and consider a sequence $\{u_{\varepsilon_n}\}_{n \in \mathbb{N}} \subset \mathcal{H}_{\varepsilon_n}$ of solutions to (2.1). We first notice that, by $(g_1)$, there exists $\gamma \in (0, a)$ such that
\[ g(\varepsilon_n x, t) = f(t) t + t^{2^*_s} \leq \frac{V_1}{K} t^2 \text{ for any } x \in \mathbb{R}^3, t \leq \gamma. \] (3.19)
Arguing as before, we can find $R > 0$ such that
\[ |u_n|_{L^\infty(B_R(\tilde{y}_n))} < \gamma. \] (3.20)
Moreover, up to extract a subsequence, we may assume that
\[ |u_n|_{L^\infty(B_R(\tilde{y}_n))} \geq \gamma. \]  
(3.21)

Indeed, if (3.21) does not hold, we can see that (3.20) yields \(|u_n|_{L^\infty(\mathbb{R}^3)} < \gamma\). Then, in view of \(\langle J'_{\xi_n}(u_n), u_n \rangle = 0\) and (3.19), we can see that
\[ \|u_n\|_{\mathcal{E}_n^2}^2 \leq \|u_n\|^2_{\mathcal{E}_n^2} + b|u_n|^4 = \int_{\mathbb{R}^3} g(\varepsilon_n x, u_n)u_n \, dx \leq \frac{V_1}{R} \int_{\mathbb{R}^3} u_n^2 \, dx \]
which gives \(|u_n|_{\mathcal{E}_n^2} = 0\), that is a contradiction. Hence (3.21) holds true. In the light of (3.20) and (3.21), we can deduce that the maximum points \(p_n \in \mathbb{R}^3\) of \(u_n\) belong to \(B_R(\tilde{y}_n)\). Hence, \(p_n = \tilde{y}_n + q_n\) for some \(q_n \in B_R\). Recalling that the associated solution to (1.1) is of the form \(\hat{u}_n(x) = u_n(x/\varepsilon_n)\), we conclude that the maximum point \(\eta_n\) of \(\hat{u}_n\) is \(\eta_n := \varepsilon_n \tilde{y}_n + \varepsilon_n q_n\). Since \(\{q_n\}_{n \in \mathbb{N}} \subset B_R\) is bounded and \(\varepsilon_n \tilde{y}_n \to y_0\) with \(V(y_0) = V_0\), from the continuity of \(V\) we can infer that
\[ \lim_{n \to \infty} V(\eta_n) = V(y_0) = V_0. \]

Finally, we give an estimate of the decay of solutions to (1.1). Invoking Lemma 4.3 in [22], we know that there exists a positive function \(w\) such that
\[ 0 < w(x) \leq \frac{C}{1 + |x|^{3+2s}}, \]  
(3.22)
and
\[ (-\Delta)^s w + \frac{V_1}{2(a + bA_1^2)} w \geq 0 \quad \text{in } \mathbb{R}^3 \setminus B_{R_1}, \]  
(3.23)
for some suitable \(R_1 > 0\), and \(A_1 > 0\) is such that
\[ a + b|u_n|^2 \leq a + bA_1^2 \quad \forall n \in \mathbb{N}. \]

Using \((f_1)\), the definition of \(g\) and (3.18), we can find \(R_2 > 0\) sufficiently large such that
\[ (-\Delta)^s \tilde{u}_n + \frac{V_1}{2(a + b\tilde{u}_n^2)} \tilde{u}_n \leq (-\Delta)^s \tilde{u}_n + \frac{V_1}{2(a + b|\tilde{u}_n|^2)} \tilde{u}_n \]
\[ = \frac{1}{a + b|\tilde{u}_n|^2} \left[ g(\varepsilon_n x + \varepsilon_n y_n, \tilde{u}_n) - \left( V - \frac{V_1}{2} \right) \tilde{u}_n \right] \]
\[ \leq \frac{1}{a + b|\tilde{u}_n|^2} \left[ g(\varepsilon_n x + \varepsilon_n y_n, \tilde{u}_n) - \frac{V_1}{2} \tilde{u}_n \right] \leq 0 \quad \text{in } \mathbb{R}^3 \setminus B_{R_2}. \]  
(3.24)

Choose \(R_3 = \max\{R_1, R_2\}\), and we set
\[ c = \inf_{B_{R_3}} w > 0 \quad \text{and} \quad \tilde{w}_n \equiv (d + 1)w - c\tilde{u}_n, \]  
(3.25)
where \(d = \sup_{n \in \mathbb{N}} |\tilde{u}_n|_\infty < \infty\). In what follows, we show that
\[ \tilde{w}_n \geq 0 \quad \text{in } \mathbb{R}^3. \]  
(3.26)

Firstly, we can observe that (3.23), (3.24) and (3.25) yield
\[ \tilde{w}_n \geq cd + w - cd > 0 \quad \text{in } B_{R_1}, \]  
(3.27)
\[ (-\Delta)^s \tilde{w}_n + \frac{V_1}{2(a + b\tilde{u}_n^2)} \tilde{w}_n \geq 0 \quad \text{in } \mathbb{R}^3 \setminus B_{R_3}. \]  
(3.28)

Now, we argue by contradiction and we assume that there exists a sequence \(\{\tilde{x}_{n,k}\}_{k \in \mathbb{N}} \subset \mathbb{R}^3\) such that
\[ \inf_{x \in \mathbb{R}^3} \tilde{w}_n(x) = \lim_{k \to \infty} \tilde{w}_n(\tilde{x}_{n,k}) < 0. \]  
(3.29)

By (3.18), (3.22) and the definition of \(\tilde{w}_\varepsilon\), it is clear that \(|\tilde{w}_\varepsilon(x)| \to 0\) as \(|x| \to \infty\), uniformly in \(n \in \mathbb{N}\). Thus, \(\{\tilde{x}_{n,k}\}_{k \in \mathbb{N}}\) is bounded, and, up to subsequence, we may assume that there exists \(\tilde{x}_n \in \mathbb{R}^3\) such that \(\tilde{x}_{n,k} \to \tilde{x}_n\) as \(k \to \infty\). From (3.29) it follows that
\[ \inf_{x \in \mathbb{R}^3} \tilde{w}_n(x) = \tilde{w}_n(\tilde{x}_n) < 0. \]  
(3.30)
Using the minimality of $\bar{x}_n$ and the representation formula for the fractional Laplacian [18], we can see that
\begin{equation}
(-\Delta)^s \tilde{w}_n(\bar{x}_n) = \frac{C_{3s}}{2} \int_{\mathbb{R}^3} \frac{2\tilde{w}_n(\bar{x}_n) - \tilde{w}_n(\bar{x}_n + \xi) - \tilde{w}_n(\bar{x}_n - \xi)}{|\xi|^{3+2s}} \, d\xi \leq 0.
\end{equation}
Taking into account (3.27) and (3.29) we can infer that $\bar{x}_n \in \mathbb{R}^N \setminus \mathcal{B}_R$. This together with (3.30) and (3.31) implies
\begin{equation}
(-\Delta)^s \tilde{w}_n(\bar{x}_n) + \frac{V_1}{2(a + bA_1^2)} \tilde{w}_n(\bar{x}_n) < 0,
\end{equation}
which is impossible in view of (3.28). Hence, (3.26) holds true and using (3.22) we get
\begin{equation}
0 < \hat{u}_n(x) = u_n(\frac{x}{\varepsilon_n}) = \tilde{u}_n(\frac{x}{\varepsilon_n} - \check{y}_n)\quad \text{and} \quad \eta_{\varepsilon_n} = \varepsilon_n \check{y}_n + \varepsilon_n q_n,
\end{equation}
for some $\hat{C} > 0$. Since $\hat{u}_n(x) = u_n(\frac{x}{\varepsilon_n}) = \tilde{u}_n(\frac{x}{\varepsilon_n} - \check{y}_n)$ and $\eta_{\varepsilon_n} = \varepsilon_n \check{y}_n + \varepsilon_n q_n$, we can use (3.32) to deduce that
\begin{equation}
0 < \hat{u}_n(x) = u_n \left( \frac{x}{\varepsilon_n} \right) = \tilde{u}_n \left( \frac{x}{\varepsilon_n} - \check{y}_n \right)
\leq \frac{\hat{C}}{1 + \left| x - \check{y}_n \right|^{3+2s}}
\leq \frac{\hat{C} \varepsilon_n^{3+2s}}{\varepsilon_n^{3+2s} + |x - \varepsilon_n \check{y}_n|^{3+2s}}
\leq \frac{\hat{C} \varepsilon_n^{3+2s}}{\varepsilon_n^{3+2s} + |x - \eta_{\varepsilon_n}|^{3+2s}}, \quad \forall x \in \mathbb{R}^3.
\end{equation}
This ends the proof of the Theorem 1.1. \hfill $\Box$

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