A turn based traffic allocation game

Youyuan Dong
Chongqing foreign language school, P. R China
dobgyouyuan@gmail.com

Abstract: In this paper, we consider a game in which players needing to transport goods through road A or B, make decisions in turn how much goods they transport using each road. The cost per unit goods transported on road A is constant and on road B is proportional to the total goods on that road. The objective of each player is to minimize its own cost while considering other players’ choices. We present a partial solution to the n-player case and a full solution to the two player case.

1. Introduction
In this paper, we consider a version of congested transportation network with two roads where players take turns to make their decisions about how much traffic they allocate on each road. There is a wide range of literature on congested transportation network, see for example [1], [2], [3], [4], [5] and [6]. Suppose, there are n players, and for i = 1, 2,..., n, player i needs to transport Ti unit of goods using road A and B. Players make decisions in turn from 1 to n. Let ai be the amount of goods player i allocates on road A and bi = Ti − ai be that on road B. Assume the cost per unit of goods transported on road A is 1 and that on road B is \( \sum_{i=1}^{n} b_i \). So, the total cost for player i is \( a_i + b_i \sum_{i=1}^{n} b_i \). Each player in the game wants to minimize his cost while knowing that other players also want to do so. We investigate the optimal strategy of this game.

In this paper, we make two contributions. First, we show that when Ti’s are sufficiently large, if everyone plays according to the optimal strategy then \( b_i = 2^{-i} \) and \( a_i = T_i - 2^{-i} \). In particular, player i’s decision does not depend on n. Secondly, we found the full solution in the case when n = 2.

2. Multiplayer game with large traffic needs
In this section, we investigate the solution of the game when Ti is sufficiently large. We assume for simplicity that \( T_i \geq \frac{1}{2} \) for all i.

2.1. Theorem When \( T_i > \frac{1}{2} \) for all i and each player plays according to the optimal strategy, then \( b_i = 2^{-i} \) and \( a_i = T_i - 2^{-i} \).

Proof. We first consider player n’s decision. As he makes the latest decision, he can observe all the other players’ choices. He simply needs to minimize

\[
\sum_{i=1}^{n-1} a_i + b_n(Xb_i + b_n) \]

subject to \( a_n = T_n - b_n \) and \( a_n, b_n \geq 0 \). Substituting \( a_n = T_n - b_n \) we obtain that player n’s cost is

\[
T_n - b_n + b_n(Xb_i + b_n) = T_n + b_n(Xb_i - 1) + b_{2n} \]

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which is a quadratic equation in \( b_n \) and its unconditional optimum is \( b_n = \frac{1 - \frac{\sum_{i=1}^{n-1} b_i}{2}}{2} \). The condition \( T_n \geq \frac{1}{2} \) guarantees that \( \frac{1 - \frac{\sum_{i=1}^{n-1} b_i}{2}}{2} \leq T_n \) and thus \( b_n = \max(0, \frac{1 - \frac{\sum_{i=1}^{n-1} b_i}{2}}{2}) \) is the optimum conditional on \( a_n b_n \geq 0 \).

Now, we make the hypothesis that for player \( k \), knowing that each player after him plays according to the optimal strategy, than player \( k \) should let \( b_k = \max(0, \frac{1 - \frac{\sum_{i=1}^{k-1} b_i}{2}}{2}) \). We will show this using backward induction. We know that the hypothesis is true for \( k = n \). Assume it is true for all \( n \geq k > j \), we show that it is true for \( k = j \).

First, we consider the case when \( \sum_{i=1}^{j} b_i \geq 1 \), then by induction hypothesis, \( b_k = 0 \) for all \( n \geq k > j \), and the cost for player \( j \) would be

\[
\sum_{i=1}^{j-1} T_i - b_j + b_j(Xb_i + b_j).
\]

To minimize the cost, we would choose \( b_j = \max(0, 1 - \sum_{i=1}^{j-1} b_i) \).

In the case when \( \sum_{i=1}^{j} b_i \leq 1 \), the induction hypothesis tells us that

\[
b_{j+1} = \frac{1 - \sum_{i=1}^{j} b_i}{2},
\]

\[
b_{j+2} = \frac{1 - \sum_{i=1}^{j+1} b_i}{2} = \frac{1 - \sum_{i=1}^{j} b_i}{4},
\]

and so on. In general, we would have \( b_k = \frac{1 - \sum_{i=1}^{j} b_i}{2^{k-j}} \) for \( n \geq k > j \) and the cost for player \( j \) would be

\[
T_j - b_j + b_j(\sum_{i=1}^{n} b_i) = T_j - b_j + b_j(\sum_{i=1}^{j} b_i + \sum_{i=j+1}^{n} b_i)
\]

\[
= T_j - b_j + b_j(\sum_{i=1}^{j} b_i + (1 - \sum_{i=1}^{j} b_i) \sum_{k=j+1}^{n} \frac{1}{2^{k-j}})
\]

\[
= T_j - b_j + b_j\sum_{i=1}^{j} b_i + (1 - \sum_{i=1}^{j} b_i)(1 - \sum_{i=1}^{j} b_i))
\]

\[
= T_j - b_j + b_j(1 - \frac{1}{2^{n-j}} + \sum_{i=1}^{j} b_i)
\]

\[
= T_j - \frac{1 - \sum_{i=1}^{j-1} b_i}{2^{n-j}} b_j + (1 - \sum_{i=1}^{j} b_i) b_j.
\]

To minimize the cost, we would thus take \( b_j = \max(0, 1 - \sum_{i=1}^{j-1} b_i) \). Therefore, in general, we would have either \( b_j = \max(0, 1 - \sum_{i=1}^{j-1} b_i) \) or \( b_j = \max(0, 1 - \sum_{i=1}^{j-1} b_i) \). Note that in the case when \( \sum_{i=1}^{j-1} b_i \leq 1 \), choosing \( b_j = 1 - \sum_{i=1}^{j-1} b_i \) would make \( \sum_{i=1}^{j} b_i \leq 1 \), and thus by optionality condition, the cost would be smaller or equal when choosing \( \frac{1 - \sum_{i=1}^{j-1} b_i}{2} \) instead. So, we have \( b_j = \)
max(0, 1 - \sum_{j=1}^{n-1} b_j) \text{ as desired. Note that the condition } T_n \geq \frac{1}{2} \text{ guarantees } a_i \geq 0. \text{ This proves our hypothesis. As a result, } b_1 = \frac{1}{2}, b_2 = \frac{1-b_1}{2} = \frac{1}{4}, b_3 = \frac{1-b_1-b_2}{2} = \frac{1}{8} \text{ and so on and in general we have } b_i = 2^{-i} \text{ and } a_i = T_i - 2^{-i} \text{ as desired.}

In particular, we see that the values of } a_i \text{ and } b_i \text{ do not depend on } n. \text{ In addition, as } n \to \infty, \sum_{i=1}^{n} b_i \to 1.

3 The two player case

In this section, we investigate the case where there are only two players

3.1. Theorem In the case there are only two players, Player 1 should choose

\[
b_1 = \begin{cases} 
\min(\frac{1-T_2}{2}, T_1), & T_2 \leq 1/4, \\
\min(\frac{1-T_2}{2}, T_1), & 1/4 \leq T_2 \leq 1/3 \\
\min(\frac{1}{2}, T_1), & \text{Otherwise.}
\end{cases}
\]

subject to \( a_1 + b_2(b_1 + b_2) \geq \frac{(1-T_2)^2}{4} \geq \frac{\min(\frac{1}{2}, T_1)}{2} - \frac{\min(\frac{1}{2}, T_1)^2}{2} \). \quad (1)

Proof. Similar as in the proof of Theorem 2.1, we first consider the decision process of player 2. He needs to minimize

\[ a_2 + b_2(b_1 + b_2), \]

subject to \( a_2 = T_2 - b_2 \) and \( a_2, b_2 \geq 0 \). Substituting \( a_2 = T_2 - b_2 \), the problem can be transformed into minimizing

\[ b_2^2 + b_2(b_1 - 1), \]

subject to \( 0 \leq b_2 \leq T_2 \). We solve this optimization problem and obtain

\[
b_2 = \begin{cases} 
0, \\
T_2, \\
\frac{1-b_1}{2}, & \text{otherwise.}
\end{cases}
\]

Now, we consider player 1’s strategy. He will try to minimize his cost

\[ a_1 + b_1(b_1 + b_2), \]

subject to \( a_1, b_1 \geq 0 \) and \( a_1 + b_1 = T_1 \), where \( b_2 \) can be thought as a function of \( b_1 \) described in Equation 4.

Substitute \( a_1 = T_1 - b_1 \), denote \( c(b_1) \) the cost of player 1 after choosing \( b_1 \), we have
We now investigate the optimal $b_1$ in each case. We note that $T_1 - b_1 + b_1^2$ is strictly increasing in $b_1$ when $b_1 \geq 1$ and $c(b_1)$ is a continuous for $b_1 \geq 0$. Therefore, player 1 will not choose $b_1 > 1$.

$T_1 + b_1(T_2 - 1) + b_1^2$ is a quadratic function in $b_1$ with minimum attained at $\frac{1 - T_2}{2}$ and $T_1 - b_1 + b_1^2$ is a quadratic function in $b_1$ with minimum attained at $\frac{1}{2}$. We now solve the constrained minimization problem in each case. First, we minimize

$$T_1 + b_1(T_2 - 1) + b_1^2$$

subject to $0 \leq b_1 \leq \min(1 - 2T_2, T_1)$. When $T_2 > \frac{1}{2}$, there is no feasible solution. When $T_2 < \frac{1}{2}$, the minimum is attained at $\min(T_1, 1 - 2T_2, \frac{1 - T_2}{2})$. Now, we minimize

$$T_1 - \frac{b_1}{2} + \frac{b_1^2}{2}$$

subject to $\max(0, 1 - 2T_2) \leq b_1 \leq T_1$. When $1 - 2T_2 > T_1$, there will be no feasible solution. Otherwise, the minimum attained at

$$b_1 = \begin{cases} 
T_1, & T_1 < \frac{1}{2} \\
1 - 2T_2, & 1 - 2T_2 > \frac{1}{2} \\
\frac{1}{2}, & \text{otherwise}
\end{cases}$$

(4)

In other words, the minimum is attained at $\text{Median}(1 - 2T_2, \frac{1}{2})$.

We now investigate the minimum of $c(b_1)$ in the case when $T_2 \leq \frac{1}{2}$ and $1 - 2T_2 \leq T_1$. In this case, $\min(T_1, 1 - 2T_2, \frac{1 - T_2}{2}) = \min(1 - 2T_2, \frac{1 - T_2}{2})$. As $c$ is continuous, if $T_2 < \frac{1}{2}$, then $1 - 2T_2 \leq \frac{1 - T_2}{2}$, the minimum of $c(b_1)$ will be attained at $b_1 = \text{Median}(T_1, 1 - 2T_2, \frac{1}{2})$. Similarly, when $T_2 \leq \frac{1}{4}$, then $\text{Median}(T_1, 1 - 2T_2, \frac{1}{2}) = 1 - 2T_2$ and the minimum of $c(b_1)$ will be attained at $\frac{1 - T_2}{2}$. Finally, when $\frac{1}{4} \leq T_2 \leq \frac{1}{3}$, we need to compare

$$c\left(\frac{1 - T_2}{2}\right) = T_1 - \frac{(1 - T_2)^2}{4}$$

and

$$c\left(\min\left(\frac{1}{2}, T_1\right)\right) = T_1 - \frac{\min\left(\frac{1}{2}, T_1\right)}{2} + \frac{\min\left(\frac{1}{2}, T_1\right)^2}{2}$$

and

$$c\left(\min\left(\frac{1}{2}, T_1\right)\right) = T_1 - \frac{\min\left(\frac{1}{2}, T_1\right)}{2} + \frac{\min\left(\frac{1}{2}, T_1\right)^2}{2}$$
and choose $b_1$ to between $\min\left(\frac{1}{2}, T_2\right)$ and $\frac{1-T_2}{2}$ whichever attains a smaller $c$.

Now, we summarize the result, we have player 1 should choose

$$b_1 = \begin{cases} \min\left(\frac{1-T_2}{2}, T_2\right), & T_2 \leq 1/4, \\ \min\left(\frac{1-T_2}{2}, T_1\right), & 1/4 \leq T_2 \leq 1/2, \\ \min\left(\frac{1}{2}, T_1\right), & \text{Otherwise}, \end{cases}$$

as desired.

Note that, when e.g. $T_1 \geq \frac{1}{2}$, we would have player 1 should choose

$$b_1 = \begin{cases} \frac{1-T_2}{2}, & T_2 \leq 1 - \frac{\sqrt{2}}{2}, \\ \frac{1}{2}, & \text{Otherwise}. \end{cases}$$

In particular, there is a discontinuity in the choice of $b_1$ as a function of $T_2$ at $T_2 = 1 - \frac{\sqrt{2}}{2}$. Moreover, when $T_2 = 0$, $b_1 = \frac{1}{2}$ and $b_1$ decreases as $T_2$ increases until $T_2 = 1 - \frac{\sqrt{2}}{2}$ and $b_1$ jumps back to $\frac{1}{2}$ afterwards.

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