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FORMAL GEOMETRIC QUANTIZATION II

PAUL-EMILE PARADAN

Abstract. In this paper we pursue the study of formal geometric quantization of non-compact Hamiltonian manifolds. Our main result is the proof that two quantization processes coincide. This fact was obtained by Ma and Zhang in the preprint Arxiv:0812.3989 by completely different means.

Contents

1. Introduction and statement of results 1
2. Quantizations of non-compact manifolds 4
3. Proof of Theorem 1.4 19
4. Other properties of $Q^\Phi$ 25
5. Example: the cotangent bundle of an orbit 28
References 32

In the previous article [21], we have studied some functorial properties of the “formal geometric quantization” process $Q^{-\infty}$, which is defined on proper Hamiltonian manifolds, e.g. non-compact Hamiltonian manifolds with proper moment map.

There is another way, denoted $Q^\Phi$, of quantizing proper Hamiltonian manifolds by localizing the index of the Dolbeault Dirac operator on the critical points of the square of the moment map [15, 19, 20].

The main purpose of this paper is to provide a geometric proof that the quantization processes $Q^{-\infty}$ and $Q^\Phi$ coincide. This fact was proved by Ma and Zhang in the recent preprint [15] by completely different means.

Keywords: moment map; symplectic reduction; geometric quantization; transversally elliptic symbol.

1. Introduction and statement of results

Let us first recall the definition of the geometric quantization of a smooth and compact Hamiltonian manifold. Then we show two ways of extending the notion of geometric quantization to the case of a non-compact Hamiltonian manifold.

Let $K$ be a compact connected Lie group, with Lie algebra $\mathfrak{k}$. In the Kostant-Souriau framework, a Hamiltonian $K$-manifold $(M, \Omega, \Phi)$ is pre-quantized if there

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is an equivariant Hermitian line bundle $L$ with an invariant Hermitian connection $\nabla$ such that
\[(1.1)\quad L(X) - \nabla_{X_M} = i\langle \Phi, X \rangle \quad \text{and} \quad \nabla^2 = -i\Omega,\]
for every $X \in \mathfrak{t}$. Here $X_M$ is the vector field on $M$ defined by $X_M(m) = \frac{d}{dt}e^{itX}m|_{t=0}$.

The data $(L, \nabla)$ is also called a Kostant-Souriau line bundle, and $\Phi : M \to \mathfrak{t}^*$ is the moment map. Remark that conditions (1.1) imply via the equivariant Bianchi formula the relation
\[(1.2)\quad i\langle X_M, \Omega \rangle = -d\langle \Phi, X \rangle, \quad X \in \mathfrak{t}.\]

Let us recall the notion of geometric quantization when $M$ is compact. Choose a $K$-invariant almost complex structure $J$ on $M$ which is compatible with $\Omega$ in the sense that the symmetric bilinear form $\Omega(\cdot, J\cdot)$ is a Riemannian metric. Let $\mathcal{D}_L$ be the Dolbeault-Dirac operator with coefficients in $L$, and let $\mathcal{D}_L^*$ be its (formal) adjoint. The Dolbeault-Dirac operator on $M$ with coefficients in $L$ is $D_L = \mathcal{D}_L + \mathcal{D}_L^*$, considered as an elliptic operator from $\mathcal{A}^{0,\text{even}}(M, L)$ to $\mathcal{A}^{0,\text{odd}}(M, L)$. Let $R(K)$ be the representation ring of $K$.

**Definition 1.1.** The geometric quantization of a compact Hamiltonian $K$-manifold $(M, \Omega, \Phi)$ is the element $\mathcal{Q}_K(M) \in R(K)$ defined as the equivariant index of the Dolbeault-Dirac operator $D_L$.

Let us consider the case of a proper Hamiltonian $K$-manifold $M$: the manifold is (perhaps) non-compact but the moment map $\Phi : M \to \mathfrak{t}^*$ is supposed to be proper. Under this properness assumption, one defines the formal geometric quantization of $M$ as an element $\mathcal{Q}_K^\infty(M)$ that belongs to $R^{-\infty}(K)$ [33, 31]. Let us recall the definition.

Let $T$ be a maximal torus of $K$. Let $\mathfrak{t}^*$ be the dual of the Lie algebra of $T$ containing the weight lattice $\Lambda^* : \alpha \in \Lambda^*$ if $i\alpha : \mathfrak{t} \to i\mathbb{R}$ is the differential of a character of $T$. Let $C_K \subset \mathfrak{t}^*$ be a Weyl chamber, and let $\tilde{K} := \Lambda^* \cap C_K$ be the set of dominant weights. The ring of characters $R(K)$ has a $\mathbb{Z}$-basis $V^K_\mu, \mu \in \tilde{K} : V^K_\mu$ is the irreducible representation of $K$ with highest weight $\mu$.

A representation $E$ of $K$ is admissible if it has finite $K$-multiplicities : $\dim(\text{hom}_K(V^K_\mu, E)) < \infty$ for every $\mu \in \tilde{K}$. Let $R^{-\infty}(K)$ be the Grothendieck group associated to the $K$-admissible representations. We have an inclusion map $R(K) \hookrightarrow R^{-\infty}(K)$ and $R^{-\infty}(K)$ is canonically identified with $\text{hom}_\mathbb{Z}(R(K), \mathbb{Z})$.

For any $\mu \in \tilde{K}$ which is a regular value of moment map $\Phi$, the reduced space (or symplectic quotient) $M^\mu := \Phi^{-1}(K \cdot \mu)/K$ is a compact orbifold equipped with a symplectic structure $\Omega_\mu$. Moreover $L_\mu := (L|_{\Phi^{-1}(\mu)} \otimes \mathcal{C}_{-\mu})/K_\mu$ is a Kostant-Souriau line orbibundle over $(M_\mu, \Omega_\mu)$. The definition of the index of the Dolbeault-Dirac operator carries over to the orbifold case, hence $\mathcal{Q}(M_\mu) \in \mathbb{Z}$ is defined. In Section 2.3, we explain how this notion of geometric quantization extends further to the case of singular symplectic quotients. So the integer $\mathcal{Q}(M_\mu) \in \mathbb{Z}$ is well defined for every $\mu \in \tilde{K}$; in particular $\mathcal{Q}(M_\mu) = 0$ if $\mu \notin \Phi(M)$.

**Definition 1.2.** Let $(M, \Omega, \Phi)$ be a proper Hamiltonian $K$-manifold which is prequantized by a Kostant-Souriau line bundle $L$. The formal quantization of $(M, \Omega, \Phi)$
is the element of $R^{-\infty}(K)$ defined by

$$Q^{-\infty}_K(M) = \sum_{\mu \in K} Q(M_{\mu}) V^K_{\mu}.$$ 

When $M$ is compact, the fact that

$$(1.3) \quad Q_K(M) = Q^{-\infty}_K(M)$$

is known as the “quantization commutes with reduction Theorem”. This was conjectured by Guillemin-Sternberg in [9] and was first proved by Meinrenken [17] and Meinrenken-Sjamaar [18]. Other proofs of (1.3) were also given by Tian-Zhang [26] and the author [19]. For complete references on the subject the reader should consult [25, 28].

One of the main features of the formal geometric quantization $Q^{-\infty}$ is its stability relatively to the restriction to subgroups.

**Theorem 1.3** ([21]). Let $M$ be a pre-quantized Hamiltonian $K$-manifold which is proper. Let $H \subset K$ be a closed connected Lie subgroup such that $M$ is still proper as a Hamiltonian $H$-manifold. Then $Q^{-\infty}_K(M)$ is $H$-admissible and we have $Q^{-\infty}_K(M)|_H = Q^{-\infty}_H(M)$ in $R^{-\infty}(H)$.

When $M$ is a proper Hamiltonian $K$-manifold, we can also define another “formal geometric quantization”, denoted

$$(1.4) \quad Q^\Phi_K(M) \in R^{-\infty}(K),$$

by localizing the index of the Dolbeault-Dirac operator $D_L$ on the set $Cr(||\Phi||^2)$ of critical points of the square of the moment map (see Section 2.2 for the precise definition). We proved in previous papers [20, 21, 23] that

$$(1.5) \quad Q^{-\infty}_K(M) = Q^\Phi_K(M).$$

in some situations:

- $M$ is a coadjoint orbit of a semi-simple Lie group $S$ that parametrizes a representation of the discrete series of $S$,
- $M$ is a Hermitian vector space.

In her ICM 2006 plenary lecture [29], Vergne conjectured that (1.5) holds when $Cr(||\Phi||^2)$ is compact. Recently, Ma and Zhang [15] prove the following generalization of this conjecture.

**Theorem 1.4.** The equality (1.5) holds for any proper Hamiltonian $K$-manifold.

This article is dedicated to the study of the quantization map $Q^\Phi$:

- In Section 2.2, we give the precise definition of the quantization process $Q^\Phi$.
- In particular, we refine the constant $c_\gamma$ appearing in [15][Theorem 0.1].
- In Section 2.3, we explain how to compute the quantization of a point.
- We give in Section 3 another proof of Theorem 1.4 by using the technique of symplectic cutting developed in [21].
- In Section 4, we consider the case where $K = K_1 \times K_2$ acts on $M$ in a way that the symplectic reduction $M / /_0 K_1$ is a smooth proper $K_2$-Hamiltonian manifold. We show then that the $K_1$-invariant part of $Q^\Phi_{K_1 \times K_2}(M)$ is equal to $Q^\Phi_{K_2}(M / /_0 K_1)$. 

In Section 1, we study the example where $M$ is the cotangent bundle of a homogeneous space: $M = T^*(K/H)$ where $H$ is a closed subgroup of $K$. We see that $T^*(K/H)$ is a proper Hamiltonian $K$-manifold prequantized by the trivial line bundle. A direct computation gives

$$Q^\Phi_K(T^*(K/H)) = L^2(K/H) \text{ in } R^{-\infty}(K).$$

Let us denote $[T^*(K/H)]_{\mu,K}$ the symplectic reduction at $\mu \in \hat{K}$ of the K-Hamiltonian manifold $T^*(K/H)$. Theorem 2.4 together with (1.4) give

$$Q([T^*(K/H)]_{\mu,K}) = \dim[V^K_{\mu}]^H,$$

for any $\mu \in \hat{K}$. Here $[V^K_{\mu}]^H \subset V^K$ is the subspace of $H$-invariant vectors.

Then we consider the action of a closed connected subgroup $G \subset K$ on $T^*(K/H)$. We first check that $T^*(K/H)$ is a proper Hamiltonian $G$-manifold if and only if the restriction $L^2(K/H)|_G$ is an admissible $G$-representation. Then, using Theorem 1.3, we get that

$$Q^\infty_G(T^*(K/H)) = L^2(K/H)|_G \text{ in } R^{-\infty}(G).$$

In other words, the multiplicity of $V^G_\lambda$ in $L^2(K/H)$ is equal to the quantization of the reduced space $[T^*(K/H)]_{\lambda,G}$.

2. Quantizations of non-compact manifolds

In this section we define the quantization process $Q^\Phi$, and we give another definition of the quantization process $Q^{-\infty}$ that uses the notion of symplectic cutting [21].

2.1. Transversally elliptic symbols. Here we give the basic definitions from the theory of transversally elliptic symbols (or operators) defined by Atiyah-Singer in [1]. For an axiomatic treatment of the index morphism see Berline-Vergne [1, 2] and Paradan-Vergne [22]. For a short introduction see [14].

Let $\mathcal{X}$ be a compact $K$-manifold. Let $p: T\mathcal{X} \to \mathcal{X}$ be the projection, and let $(-, -)_\mathcal{X}$ be a $K$-invariant Riemannian metric. If $E^0, E^1$ are $K$-equivariant complex vector bundles over $\mathcal{X}$, a $K$-equivariant morphism $\sigma \in \Gamma(T\mathcal{X}, \text{hom}(p^*E^0, p^*E^1))$ is called a symbol on $\mathcal{X}$. The subset of all $(x, v) \in T\mathcal{X}$ where $\sigma(x, v) : E^0_x \to E^1_x$ is not invertible is called the characteristic set of $\sigma$, and is denoted by $\text{Char}(\sigma)$.

In the following, the product of a symbol $\sigma$ by a complex vector bundle $F \to M$, is the symbol

$$\sigma \otimes F$$

defined by $\sigma \otimes F(x, v) = \sigma(x, v) \otimes \text{Id}_{F_x}$ from $E^0_x \otimes F_x$ to $E^1_x \otimes F_x$. Note that $\text{Char}(\sigma \otimes F) = \text{Char}(\sigma)$.

Let $T_K\mathcal{X}$ be the following subset of $T\mathcal{X}$:

$$T_K\mathcal{X} = \{(x, v) \in T\mathcal{X}, (v, X_\mathcal{X}(x))_x = 0 \text{ for all } X \in \mathfrak{t}\}.$$

A symbol $\sigma$ is elliptic if $\sigma$ is invertible outside a compact subset of $T\mathcal{X}$ (i.e. $\text{Char}(\sigma)$ is compact), and is $K$-transversally elliptic if the restriction of $\sigma$ to $T_K\mathcal{X}$ is invertible outside a compact subset of $T_K\mathcal{X}$ (i.e. $\text{Char}(\sigma) \cap T_K\mathcal{X}$ is compact). An elliptic symbol $\sigma$ defines an element in the equivariant $K$-theory of $T\mathcal{X}$ with compact support, which is denoted by $\text{K}_K(T\mathcal{X})$, and the index of $\sigma$ is a virtual finite dimensional representation of $K$, that we denote $\text{Index}_K^\mathcal{X}(\sigma) \in R(K)$ [7, 3, 3, 3, 3].

---

1The map $\sigma(x, v)$ will be also denote $\sigma_x(v)$
Let

\[ R_{tc}^{-\infty}(K) \subset R^{-\infty}(K) \]

be the \( R(K) \)-submodule formed by all the infinite sum \( \sum_{\mu \in K} m_\mu V^K_\mu \) where the map \( \mu \in K \mapsto m_\mu \in \mathbb{Z} \) has at most a polynomial growth. The \( R(K) \)-module \( R_{tc}^{-\infty}(K) \) is the Grothendieck group associated to the trace class virtual \( K \)-representations: we can associate to any \( V \in R_{tc}^{-\infty}(K) \), its trace \( k \mapsto \text{Tr}(k, V) \) which is a generalized function on \( K \) invariant by conjugation. Then the trace defines a morphism of \( R(K) \)-module

\[ R_{tc}^{-\infty}(K) \mapsto C^{-\infty}(K)^K. \]

A \( K \)-transversally elliptic symbol \( \sigma \) defines an element of \( K_K(T_KX) \), and the index of \( \sigma \) is defined as a trace class virtual representation of \( K \), that we still denote \( \text{Index}^K_K(\sigma) \in R_{tc}^{-\infty}(K) \).

Remark that any elliptic symbol of \( TX \) is \( K \)-transversally elliptic, hence we have a restriction map \( K_K(TX) \to K_K(T_KX) \), and a commutative diagram

\[
\begin{array}{ccc}
K_K(TX) & \longrightarrow & K_K(T_KX) \\
\text{Index}_X^K & \downarrow & \text{Index}_X^K \\
R(K) & \longrightarrow & R_{tc}^{-\infty}(K).
\end{array}
\]

Using the excision property, one can easily show that the index map \( \text{Index}^K_U : K_K(T_KU) \to R_{tc}^{-\infty}(K) \) is still defined when \( U \) is a \( K \)-invariant relatively compact open subset of a \( K \)-manifold (see [19][section 3.1]).

Suppose now that the group \( K \) is equal to the product \( K_1 \times K_2 \). When a symbol \( \sigma \) is \( K_1 \times K_2 \)-transversally elliptic we will be interested in the \( K_1 \)-invariant part of its index, that we denote

\[ \left[ \text{Index}^{K_1 \times K_2}_X(\sigma) \right]^{K_1}_{K_2} \in R_{tc}^{-\infty}(K_2). \]

An intermediate notion between the “ellipticity” and “\( K_1 \times K_2 \)-transversal ellipticity” is the “\( K_1 \)-transversal ellipticity”. When a \( K_1 \times K_2 \)-equivariant morphism \( \sigma \) is \( K_1 \)-transversally elliptic, its index \( \text{Index}^{K_1 \times K_2}_X(\sigma) \in R_{tc}^{-\infty}(K_1 \times K_2) \), viewed as a generalized function on \( K_1 \times K_2 \), is smooth relatively to the variable in \( K_2 \). It implies that \( \text{Index}^{K_1 \times K_2}_X(\sigma) = \sum_\lambda \theta(\lambda) \otimes V^{K_1}_\lambda \) with

\[ \theta(\lambda) \in R(K_2), \quad \forall \lambda \in K_1. \]

In particular, we know that

\[ \left[ \text{Index}^{K_1 \times K_2}_X(\sigma) \right]^{K_1}_K(0) = \theta(0) \]

belongs to \( R(K_2) \).

Let us recall the multiplicative property of the index map for the product of manifolds that was proved by Atiyah-Singer in [1]. Consider a compact Lie group \( K_2 \) acting on two manifolds \( X_1 \) and \( X_2 \), and assume that another compact Lie group \( K_1 \) acts on \( X_1 \) commuting with the action of \( K_2 \).

The external product of complexes on \( TX_1 \) and \( TX_2 \) induces a multiplication (see [1][22]):

\[ \circledast : K_{K_1 \times K_2}(T_{K_1}X_1) \times K_{K_2}(T_{K_2}X_2) \longrightarrow K_{K_1 \times K_2}(T_{K_1 \times K_2}(X_1 \times X_2)). \]
The following property will be used frequently in the paper.

**Theorem 2.1** (Multiplicative property). For any \([\sigma_1] \in K_{K_1 \times K_2}(T_K,\mathcal{X}_1)\) and any \([\sigma_2] \in K_{K_3}(T_{K_2,\mathcal{X}_2})\) we have

\[
\text{Index}^{K_1 \times K_2}_{\mathcal{X}_1 \times \mathcal{X}_2}([\sigma_1] \odot [\sigma_2]) = \text{Index}^{K_1}_{\mathcal{X}_1}([\sigma_1]) \otimes \text{Index}^{K_2}_{\mathcal{X}_2}([\sigma_2]).
\]

We will use in this article the notion of support of a generalized character.

**Definition 2.2.** The support of \(\chi := \sum_{\mu \in \hat{K}} a_{\mu} V^{K}_{\mu} \in R^{-\infty}(K)\) is the set of \(\mu \in \hat{K}\) such that \(a_{\mu} \neq 0\).

We will say that \(\chi \in R^{-\infty}(K)\) is supported outside \(B \subset \mathfrak{t}^*\) if the support of \(\chi\) does not intersect \(B\). Note that an infinite sum \(\sum_{i \in I} \chi_i\) converges in \(R^{-\infty}(K)\) if for each ball

\[B_r = \{\xi \in \mathfrak{t}^* \mid \|\xi\| < r\}\]

the set \(\{i \in I \mid \text{support}(\chi_i) \cap B_r \neq \emptyset\}\) is finite.

**Definition 2.3.** We denote by \(O(r)\) any character of \(R^{-\infty}(K)\) which is supported outside the ball \(B_r\).

### 2.2. Definition and first properties of \(\mathcal{Q}^\Phi\)

Let \((M, \Omega, \Phi)\) be a proper Hamiltonian \(K\)-manifold prequantized by an equivariant line bundle \(L\). Let \(J\) be an invariant almost complex structure compatible with \(\Omega\). Let \(p : TM \to M\) be the projection.

Let us first describe the principal symbol of the Dolbeault-Dirac operator \(\overline{\Omega}_L + \overline{\mathcal{J}}_L\). The complex vector bundle \((T^*M)^{0,1}\) is \(K\)-equivariantly identified with the tangent bundle \(TM\) equipped with the complex structure \(J\). Let \(h\) be the Hermitian structure on \((TM, J)\) defined by : \(h(v, w) = \Omega(v, Jw) - i\Omega(v, w)\) for \(v, w \in TM\). The symbol

\[
\text{Thom}(M, J) \in \Gamma \left( M, \text{hom}(p^*(\wedge^{\text{even}}_CM), p^*(\wedge^{\text{odd}}_CM)) \right)
\]

at \((m, v) \in TM\) is equal to the Clifford map

\[
(2.10) \quad c_m(v) : \wedge^{\text{even}}_CM_m M \longrightarrow \wedge^{\text{odd}}_CM_m M,
\]

where \(c_m(v).w = v \wedge w - i(v)w\) for \(w \in \wedge^{\text{even}}_CM_m M\). Here \(i(v) : \wedge^{\text{odd}}_CM_m M \to \wedge^{\text{odd}}_CM_m M\) denotes the contraction map relative to \(h\). Since \(c_m(v)^2 = -\|v\|^2 \text{Id}\), the map \(c_m(v)\) is invertible for all \(v \neq 0\). Hence the characteristic set of \(\text{Thom}(M, J)\) corresponds to the 0-section of \(TM\).

It is a classical fact that the principal symbol of the Dolbeault-Dirac operator \(\overline{\Omega}_L + \overline{\mathcal{J}}_L\) is equal to

\[
(2.11) \quad \text{Thom}(M, J) \otimes L,
\]

see \[\] . Here also we have \(\text{Char}(\text{Thom}(M, J) \otimes L) = 0\) – section of \(TM\).

**Remark 2.4.** When the manifold \(M\) is a product \(M_1 \times M_2\) the symbol \(\text{Thom}(M, J) \otimes L\) is equal to the product \(\sigma_1 \odot \sigma_2\) where \(\sigma_k = \text{Thom}(M_k, J_k) \otimes L_k\).

\[\text{Here we use an identification } T^*M \simeq TM \text{ given by an invariant Riemannian metric.}\]
When \( M \) is compact, the symbol \( \text{Thom}(M, J) \otimes L \) is elliptic and then defines an element of the equivariant \( K \)-group of \( TM \). The topological index of \( \text{Thom}(M, J) \otimes L \in K_K(TM) \) is equal to the analytical index of the Dolbeault-Dirac operator \( \overline{\partial}_L + \overline{J}_L \):

\[
Q_K(M) = \text{Index}_K^\text{analytic}(\text{Thom}(M, J) \otimes L) \quad \text{in} \quad R(K). 
\]

When \( M \) is not compact the topological index of \( \text{Thom}(M, J) \otimes L \) is not defined. In order to extend the notion of geometric quantization to this setting we deform the symbol \( \text{Thom}(M, J) \otimes L \) in the “Witten” way \([13, 14]\). Consider the identification \( \xi \mapsto \xi, V^* \mapsto V \) defined by a \( K \)-invariant scalar product on \( V^* \). We define the Kirwan vector field on \( M \):

\[
\kappa_m = \left( \frac{\Phi(m)}{M} \right)(m), \quad m \in M.
\]

**Definition 2.5.** The symbol \( \text{Thom}(M, J) \otimes L \) pushed by the vector field \( \kappa \) is the symbol \( c^\kappa \) defined by the relation

\[
c^\kappa|_m(v) = \text{Thom}(M, J) \otimes L|_m(v - \kappa_m)
\]

for any \((m, v) \in TM\).

Note that \( c^\kappa|_m(v) \) is invertible except if \( v = \kappa_m \). If furthermore \( v \) belongs to the subset \( T_KM \) of tangent vectors orthogonal to the \( K \)-orbits, then \( v = 0 \) and \( \kappa_m = 0 \). Indeed \( \kappa_m \) is tangent to \( K \) while \( v \) is orthogonal.

Since \( \kappa \) is the Hamiltonian vector field of the function \( \frac{1}{2} \| \Phi \|^2 \), the set of zeros of \( \kappa \) coincides with the set \( \text{Cr}(\| \Phi \|^2) \) of critical points of \( \| \Phi \|^2 \). Finally we have

\[
\text{Char}(c^\kappa) \cap T_KM \simeq \text{Cr}(\| \Phi \|^2).
\]

In general \( \text{Cr}(\| \Phi \|^2) \) is not compact, so \( c^\kappa \) does not define a transversally elliptic symbol on \( M \). In order to define a kind of index of \( c^\kappa \), we proceed as follows. For any invariant open relatively compact subset \( U \subset M \) the set \( \text{Char}(c^\kappa|_U) \cap T_KU \simeq \text{Cr}(\| \Phi \|^2) \cap U \) is compact when

\[
\partial U \cap \text{Cr}(\| \Phi \|^2) = \emptyset.
\]

When \( (2.13) \) holds we denote

\[
Q^K_K(U) := \text{Index}_U^K(c^\kappa|_U) \in R_{\text{ic}}^\infty(K)
\]

the equivariant index of the transversally elliptic symbol \( c^\kappa|_U \).

It will be useful to understand the dependence of the generalized character \( Q^K_K(U) \) relatively to the data \((U, \Omega, L)\). So we consider two proper Hamiltonian \( K \)-manifolds \((M, \Omega, \Phi)\) and \((M', \Omega', \Phi')\) respectively prequantized by the line bundles \( L \) and \( L' \). Let \( V \subset M \) and \( V' \subset M' \) two invariant open subsets.

**Proposition 2.6.** • The generalized character \( Q^K_K(U) \) does not depend of the choice of an invariant almost complex structure on \( U \) which is compatible with \( \Omega|_U \).

• Suppose that there exists an equivariant diffeomorphism \( \Psi : V \to V' \) such that

1. \( \Psi^*(\Phi') = \Phi \),
2. \( \Psi^*(L') = L \),
3. there exists an homotopy of symplectic forms taking \( \Psi^*(\Omega'|_{V'}) \) to \( \Omega|_V \).
Let \( U' \subset \bigcap U \subset V' \) be an invariant open relatively compact subset such that \( \partial U' \) satisfies (2.14), and \( U = \Psi^{-1}(U') \). Then \( \partial U \) satisfies (2.14) and
\[
\mathcal{Q}^U_{K}(U') = \mathcal{Q}^U_{K}(U) \in R^{-\infty}(K).
\]

Proof. Let us prove the first point. Let \( c^i_U, i = 0, 1 \) be the transversally elliptic symbols defined with the compatible almost complex structure \( J_i,i = 0, 1 \). Since the space of compatible almost complex structure is contractible, there exist an invertible bundle map \( U, \Gamma(U) \). We use Lemma 2.2 in [19], we know that there exists an invertible bundle map \( A \in \Gamma(U, \text{End}(TU)) \), homotopic to the identity, such that \( A \circ J_0 = J_1 \circ A \). With the help of \( A \) we prove then that the symbols \( c^i_U \) and \( c^i_U \) define the same class in \( K_K(T_KU) \) (see [19] Lemma 2.2). Hence their equivariant index coincide.

Let us prove the second point. The characters \( \mathcal{Q}^0_{K}(U) \) and \( \mathcal{Q}^1_{K}(U') \) are computed as the equivariant index of the symbols \( c^i|_U \) and \( c^i|_U \). Let \( \tilde{c}^i_U \) the pull back of \( c^i|_U \), by \( \Psi \). Thanks to the point (1) and (2), the only thing which differs in the definitions of the symbols \( c^i|_U \) and \( \tilde{c}^i|_U \) are almost complex structures \( J \) and \( \tilde{J} = \Psi^*(J') \) : the first one is compatible with \( \Omega \) and the second one with \( \Psi^*(\Omega') \). Since these two symplectic structure are homotopic, one sees that the almost complex structures \( J \) and \( \tilde{J} \) are also homotopic. So we can conclude like in the first point.

Let us recall the basic fact concerning the singular values of \( \|\Phi\|^2 \).

**Lemma 2.7.** The set of singular values of \( \|\Phi\|^2 : M \to \mathbb{R} \) forms a sequence \( 0 \leq r_1 < r_2 < \ldots < r_k < \ldots \) which is finite iff \( \text{Cr}(\|\Phi\|^2) \) is compact. In the other case \( \lim_{k \to \infty} r_k = \infty \).

At each regular value \( R \) of \( \text{Cr}(\|\Phi\|^2) \), we associate the invariant open subset \( M_{<R} := \{\|\Phi\|^2 < R\} \) which satisfies (2.14). The restriction \( c^i|M_{<R} \) defines then a transversally elliptic symbol on \( M_{<R} \). Let \( \mathcal{Q}^i_{K}(M_{<R}) \) be its equivariant index.

Let us show that \( \mathcal{Q}^i_{K}(M_{<R}) \) has a limit when \( R \to \infty \). The set \( \text{Cr}(\|\Phi\|^2) \) has the following decomposition
\[
\text{Cr}(\|\Phi\|^2) = \bigcup_{\beta \in B} K \cdot (M^\beta \cap \Phi^{-1}(\beta)) \cap E_m = Z_\beta
\]
where the \( B \) is a subset of the Weyl chamber \( \mathfrak{t}_m^* \). Note that each part \( Z_\beta \) is compact, hence \( B \) is finite only if \( \text{Cr}(\|\Phi\|^2) \) is compact. When \( \text{Cr}(\|\Phi\|^2) \) is non-compact, the set \( B \) is infinite, but it is easy to see that \( B \cap B_r \) is finite for any \( r \geq 0 \). For any \( \beta \in B \), we consider a relatively compact open invariant neighborhood \( U_\beta \) of \( Z_\beta \) such that \( \text{Cr}(\|\Phi\|^2) \cap \partial U_\beta = Z_\beta \).

**Definition 2.8.** We denote
\[
\mathcal{Q}^0_{\beta}(M) \in R^{-\infty}_{\|\xi\|}(K)
\]
the index of the transversally elliptic symbol \( c^i|_{U_\beta} \).

A simple application of the excision property \( \mathfrak{R} \) gives that
\[
\mathcal{Q}^i_{K}(M_{<R}) = \sum_{\|\beta\| < R} \mathcal{Q}^i_{\beta}(M).
\]

\( ^3 \) The index of \( c^i|_{U_\beta} \) was denoted \( RH^K_\beta(M, L) \) in [19].
Theorem 2.9. The generalized character $Q^β_K(M)$ is supported outside the open ball $B_{∥β∥}$. 

Proof. Proposition 2.9 follows directly from the computations done in [19]. First consider the case where $β \neq 0$ is a $K$-invariant element of $B$. Let $i : Tβ → T$ be the compact torus generated by $β$. If $F$ is a $Z$-module we denote $F ⊗ R^{-∞}(Tβ)$ the $Z$-module formed by the infinite formal sums $∑_a E_a h^a$ taken over the set of weights of $Tβ$, where $E_a \in F$ for every $a$.

Since $Tβ$ lies in the center of $K$, the morphism $π : (k, t) ∈ K × Tβ → kt ∈ K$ induces a map $π^∗ : R^{-∞}(K) → R^{-∞}(K) ⊗ R^{-∞}(Tβ)$.

The normal bundle $N$ of $Mβ$ in $M$ inherits a canonical complex structure $J_N$ on the fibers. We denote by $N$ → $Mβ$ the complex vector bundle with the opposite complex structure. The torus $Tβ$ is included in the center of $K$, so the bundle $N$ and the virtual bundle $N^* : \sum C^jN^j \oplus N_{odd}$ carry a $K \times Tβ$-action: they can be considered as elements of $K_K(\sum C^jN^j) = K_K(\sum C^jN^j) \otimes R(Tβ)$.

We prove in [19] the following localization formula:

$π^∗[Q^β_K(M)] = R\hat{R}_β^{K \times Tβ} \left( Mβ, L_{|Mβ} \otimes N^* \right)$,

as an equality in $R^{-∞}(K) \otimes R^{-∞}(Tβ)$. With (2.18) in hand, it is easy to see that $V^β_{iμ}$ occurs in the character $Q^β_K(M)$ only if $(μ, β) \geq ∥β∥^2$ (See Lemma 9.4 in [19]).

Now we consider the case were $β ∈ B$ is not a $K$-invariant element. Let $σ$ be the unique open face of the Weyl chamber $R^*_σ$ which contains $β$. Let $K_σ$ be the corresponding stabilizer subgroup. We consider the symplectic slice $Y_σ ⊂ M$: it is a $K_σ$ invariant Hamiltonian submanifold of $M$ which is prequantized by the line bundle $L_{|Y_σ}$. The restriction of $Φ$ to $Y_σ$ is a moment map $Φ_σ : Y_σ → R^*_σ$ which is proper in a neighborhood of $β ∈ Y_σ$. The set

$K_σ \cdot (Y^β_σ \cap Φ_σ^{-1}(β)) = Mβ \cap Φ^{-1}(β)$

is a component of $Cr([Φ_σ]^2)$. Let $Q^β_{K_σ}(Y_σ) ∈ R_{K_σ^{-∞}}(K_σ)$ be the corresponding character (see Definition 2.8).

We prove in [19] [Section 7], the following induction formula:

$Q^β_{K_σ}(M) = \text{Hol}^K_{K_σ} \left( Q^β_{K_σ}(Y_σ) \right)$

where $\text{Hol}^K_{K_σ} : R_{K_σ^{-∞}}(K_σ) → R_{K^{-∞}}(K)$ is the holomorphic induction map. See the Appendix in [19] for the definition and properties of these induction maps.

We know from the previous case that

$Q^β_{K_σ}(Y_σ) = \sum_{μ ∈ K_σ} m_μ V^{K_σ}_μ$
where $m_\mu \neq 0 \implies (\mu, \beta) \geq \|\beta\|^2$. Then, with (2.17), we get

\[
Q^K_{\mu}(V_{\sigma}) = \sum_{(\mu, \beta) \geq \|\beta\|^2} m_\mu \text{Hol}^K_{\mu_{\sigma}}(V^K_{\mu_{\sigma}})
\]

\[
= \sum_{(\mu, \beta) \geq \|\beta\|^2} m_\mu \text{Hol}^K_{\tau}(t^\mu),
\]

where $\text{Hol}^K_{\tau} : R^{-\infty}(T) \rightarrow R^{-\infty}(K)$ is the holomorphic induction map.

Let $\rho$ be half the sum of the positive roots. The term $\text{Hol}^K_{\tau}(t^\mu)$ is equal to 0 when $\mu + \rho$ is not a regular element of $t^\ast$. When $\mu + \rho$ is a regular element of $t^\ast$, we have $\text{Hol}^K_{\tau}(t^\mu) = (-1)^{|\omega|} V^K_{\mu_{\omega}}$ where $\mu_{\omega} = \omega(\mu + \rho) - \rho$ is dominant for a unique $\omega \in W$.

Finally, a representation $V^K_{\lambda}$ appears in the character $Q^K(\Phi)$ only if $\lambda = \mu_{\omega}$ for a weight $\mu$ satisfying $(\mu, \beta) \geq \|\beta\|^2$. Hence, for such $\lambda$, we have

\[
\|\lambda\| = \|\mu + \rho - \omega^{-1}\rho\|
\]

\[
\geq (\mu + \rho - \omega^{-1}\rho, \frac{\beta}{\|\beta\|})
\]

\[
\geq \|\beta\|.
\]

In the last inequality we use that $(\rho - \omega^{-1}\rho, \beta) \geq 0$ since $\rho - \omega^{-1}\rho$ is a sum of positive roots, and $\beta \in t^\ast_+$.

\[\square\]

With the help of Theorem 2.9 and decomposition (2.17), we see that the multiplicity of $V^K_{\gamma}$ in $Q^K(\Phi_{\mu})$ does not depend on the regular value $R > \|\gamma\|^2$. We can refine the constant $c_\gamma$ appearing in [15] [Theorem 0.1]: take $c_\gamma$ equal to $\|\gamma\|^2$ instead of $\|\gamma + \rho\|^2 - \|\rho\|^2 \geq \|\gamma\|^2$.

**Definition 2.10.** The generalized character $Q^K(\Phi)$ is defined as the limit in $R^{-\infty}(K)$ of $Q^K_{\mu}(M_{<R})$ when $R$ goes to infinity. In other words

\[
(2.20)
Q^K_K(M) = \sum_{\beta \in \mathcal{B}} Q^K_{\beta}(M).
\]

Note that for any regular value $R$ of $\|\Phi\|^2$ we have the useful relation

\[
(2.21)
Q^K_K(M) = Q^K_K(M_{<R}) + O(\sqrt{R}).
\]

### 2.3. Quantization of a symplectic quotient

We will now explain how we define the geometric quantization of singular compact Hamiltonian manifolds: here “singular” means that the manifold is obtained by symplectic reduction.

Let $(N, \Omega)$ be a smooth symplectic manifold equipped with a Hamiltonian action of $K_1 \times K_2$ : we denote $(\Phi_1, \Phi_2) : N \rightarrow t^\ast_1 \times t^\ast_2$ the corresponding moment map. We assume that $N$ is pre-quantized by a $K_1 \times K_2$-equivariant line bundle $L$ and we

\[\footnote{Here $\rho$ is half the sum of the positive roots. Hence $\|\gamma + \rho\|^2 - \|\rho\|^2 - \|\gamma\|^2 = 2(\rho, \gamma) \geq 0$ and $(\rho, \gamma) = 0$ only if the weight $\gamma$ belongs to the center of $t \simeq t^\ast$.}
suppose that the map $\Phi_1$ is \textbf{proper}. One wants to define the geometric quantization of the (compact) symplectic quotient

$$N\sslash_0 K_1 := \Phi_1^{-1}(0)/K_1.$$  

Let $\kappa_1$ be the Kirwan vector field attached to the moment map $\Phi_1$. We denote by $c^{\kappa_1}$ the symbol $\text{Thom}(N,J) \otimes L$ pushed by the vector field $\kappa_1$. For any regular value $R_1$ of $\|\Phi_1\|^2$, we consider the restriction $c^{\kappa_1}|_{N_{<R_1}}$ to the invariant, open subset $N_{<R_1} := \{\|\Phi_1\|^2 < R_1\}$. The symbol $c^{\kappa_1}|_{N_{<R_1}}$ is $K_1 \times K_2$-equivariant and $K_1$-transversally elliptic, hence we can consider its index

$$\text{Index}_{N_{<R_1}}(c^{\kappa_1}|_{N_{<R_1}}) \in \mathbb{R}^{-\infty}(K_1 \times K_2),$$

which is smooth relatively to the parameter in $K_2$. We consider the following extension of Definition 2.10.

\textbf{Definition 2.11.} The \textbf{generalized character} $Q_{K_1 \times K_2}^{\phi_1}(N)$ is defined as the limit in $\mathbb{R}^{-\infty}(K_1 \times K_2)$ of $\text{Index}_{N_{<R_1}}(c^{\kappa_1}|_{N_{<R_1}})$ when $R_1$ goes to infinity.

Here $\text{Cr}(\|\Phi_1\|^2)$ is equal to the disjoint union of the compact $K_1 \times K_2$-invariant subsets $Z_{\beta_1} := K_1 \cdot (M^{\beta_1} \cap \Phi_1^{-1}(\beta_1))$, $\beta_1 \in B_1$. For $\beta_1 \in B_1$, we consider an invariant relatively compact open subset $U_{\beta_1}$ such that: $Z_{\beta_1} \subset U_{\beta_1}$ and $Z_{\beta_1} = \text{Cr}(\|\Phi_1\|^2) \cap \overline{U_{\beta_1}}$. Let $Q_{K_1 \times K_2}^{\beta_1}(N) \in \mathbb{R}^{-\infty}(K_1 \times K_2)$ be the equivariant index of the $K_1$-transversally elliptic symbol $c^{\kappa_1}_{\beta_1}|_{U_{\beta_1}}$. The $K_1$-transversality condition imposes that $Q_{K_1 \times K_2}^{\beta_1}(N) = \sum_{\lambda} \theta^{\beta_1}(\lambda) \otimes V^{K_1}_{\lambda}$ with

$$\theta^{\beta_1}(\lambda) \in R(K_2), \quad \forall \lambda \in K_1.$$

We have the following extension of Theorem 2.9.

\textbf{Theorem 2.12.} We have $Q_{K_1 \times K_2}^{\beta_1}(N) = \sum_{\lambda \in K_1} \theta^{\beta_1}(\lambda) \otimes V^{K_1}_{\lambda}$ where $\theta^{\beta_1}(\lambda) \neq 0$ only if $|\lambda| \geq |\beta_1|$. \hfill \qed

\textbf{Proof.} The proof works exactly like the one of Theorem 2.3.

Let us explain the “quantization commutes with reduction theorem”, or why we can consider the geometric quantization of

$$N\sslash_0 K_1 := \Phi_1^{-1}(0)/K_1$$

as the $K_1$-invariant part of $Q_{K_1 \times K_2}^{\phi_1}(N)$.

Let us first suppose that 0 is a regular value of $\Phi_1$. Then $N\sslash_0 K_1$ is a compact symplectic orbifold equipped with a Hamiltonian action of $K_2$ : the corresponding moment map is induced by the restriction of $\Phi_2$ to $\Phi_1^{-1}(0)$. The symplectic quotient $N\sslash_0 K_1$ is pre-quantized by the line orbibundle

$$L_0 := (L|_{\Phi_1^{-1}(0)})/K_1.$$  

Definition 2.1 extends to the orbifold case. We can still define the geometric quantization of $N\sslash_0 K_1$ as the index of an elliptic operator : we denote it by $Q_{K_2}(N\sslash_0 K_1) \in R(K_2)$. We have

\textbf{Theorem 2.13.} If 0 is a regular value of $\Phi_1$, the $K_1$-invariant part of $Q_{K_1 \times K_2}^{\phi_1}(N)$ is equal to $Q_{K_2}(N\sslash_0 K_1) \in R(K_2)$.
Suppose now that 0 is not a regular value of \( \Phi_1 \). Let \( T_1 \) be a maximal torus of \( K_1 \), and let \( C_1 \subset T_1^* \) be a Weyl chamber. Since \( \Phi_1 \) is proper, the convexity Theorem says that the image of \( \Phi_1 \) intersects \( C_1 \) in a closed locally polyhedral convex set, that we denote \( \Delta_{K_1}(N) \) \[ \square \).

We consider an element \( a \in \Delta_{K_1}(N) \) which is generic and sufficiently close to 0 \( \in \Delta_{K_1}(N) \) : we denote \((K_1)_a\) the subgroup of \( K_1 \) which stabilizes \( a \). When \( a \in \Delta_{K_1}(N) \) is generic, one can show (see \[ \square \)) that

\[
N/_{a}K_1 := \Phi_{K_1}^{-1}(a)/(K_1)_a
\]

is a compact Hamiltonian \( K_2 \)-orbifold, and that

\[
L_a := \left( L_{|\Phi_{K_1}^{-1}(a)} \right) / (K_1)_a.
\]

is a \( K_2 \)-equivariant line orbibundle over \( N/_{a}K_1 \) : we can then define, like in Definition 1.1, the element \( Q_{K_2}(N/_{a}K_1) \in R(K_2) \) as the equivariant index of the Dolbeault-Dirac operator on \( N/_{a}K_1 \) (with coefficients in \( L_a \)).

**Theorem 2.14.** The \( K_1 \)-invariant part of \( Q_{K_1 	imes K_2}^\Phi(M) \) is equal to \( Q_{K_2}(N/_{a}K_1) \in R(K_2) \). In particular, the elements \( Q_{K_2}(N/_{a}K_1) \) do not depend on the choice of the generic element \( a \in \Delta_{K_1}(N) \), when \( a \) is sufficiently close to 0.

**Proofs of Theorem 2.13 An Theorem 2.14.** When \( N \) is compact and \( K_2 = \{e\} \), the proofs can be found in \[ \square \] and in \[ \square \]. Let us explain briefly how the \( K \)-theoretic proof of \[ \square \] extends naturally to our case. Like in Definition 2.10, we have the following decomposition

\[
Q_{K_1 \times K_2}^\Phi(N) = \sum_{\beta \in \mathcal{B}_1} Q_{K_1 \times K_2}^{\beta}(N),
\]

And Theorem 2.12 tells us that \( \left[ Q_{K_1 \times K_2}^{\beta_1}(N) \right]^{K_1} = 0 \) if \( \beta_1 \neq 0 \). We have proved the first step:

\[
\left[ Q_{K_1 \times K_2}^{\Phi}(N) \right]^{K_1} = \left[ Q_{K_1 \times K_2}^{0}(N) \right]^{K_1}.
\]

The analysis of the term \( \left[ Q_{K_1 \times K_2}^{0}(N) \right]^{K_1} \) is undertaken in \[ \square \] when \( K_2 = \{e\} \); we explain that this term is equal either to \( Q(N/_{a}K_1) \) when \( 0 \) is a regular value, or to \( Q(N/_{a}K_1) \) with a generic. It work similarly with an action of a compact Lie group \( K_2 \). \( \square \)

**Definition 2.15.** The geometric quantization of \( N/_{0}K_1 := \Phi_{1}^{-1}(0)/K_1 \) is taken as the \( K_1 \)-invariant part of \( Q_{K_1 \times K_2}^\Phi(M) \). We denote it \( Q_{K_2}(N/_{0}K_1) \).

### 2.4. Quantization of points

Let \((M, \Omega, \Phi)\) be a proper Hamiltonian \( K \)-manifold prequantized by a Kostant-Souriau line bundle \( L \). Let \( \mu \in \widehat{K} \) be a dominant weight such that \( \Phi^{-1}(K \cdot \mu) \) is a \( K \)-orbit in \( M \). Let \( m^0 \in \Phi^{-1}(\mu) \) so that

\[
\Phi^{-1}(K \cdot \mu) = K \cdot m^0.
\]

Then the reduced space \( M_{\mu} := \Phi^{-1}(K \cdot \mu)/K \) is a point. The aim of this section is to compute the quantization of \( M_{\mu} \) : \( Q(M_{\mu}) \in \mathbb{Z} \).

Let \( H \) be the stabilizer subgroup of \( m^0 \). We have a linear action of \( H \) on the 1-dimensional vector space \( L_{m^0} \subset L \). We have \( H \subset K_{\mu} \) where \( K_{\mu} \) is the connected
subgroup of $K$ that fixes $\mu \in T^*$. Let $\mathbb{C}_{-\mu}$ be the 1-dimensional representation of $K$, associated to the infinitesimal character $-i\mu$.

Let us denote $\chi$ be the character of $H$ defined by the 1-dimensional representation $\mathbb{C}_\chi := L_{m^\alpha} \otimes \mathbb{C}_{-\mu}$. We know from the Kostant formula (1.1) that $\chi = 1$ on the identity component $H^0 \subset H$.

**Theorem 2.16.** We have

\begin{equation}
(2.22)
Q(M_\mu) = \begin{cases} 1 & \text{if } \chi = 1 \text{ on } H \\ 0 & \text{in the other case.} \end{cases}
\end{equation}

This Theorem tells us in particular that $Q(M_\mu) = 1$ when the stabiliser subgroup $H \subset K$ of a point $m^\omega \in \Phi^{-1}(\mu)$ is connected.

**Proof.** Let $N = M \times K \cdot \mu$ be the proper Hamiltonian $K$-manifold which is prequantized by the line bundle $L_N := L \otimes [\mathbb{C}_{-\mu}]$. Let us denote $\Phi_N$ the moment map on $N$. Since $\Phi^{-1}(K \cdot \mu)$ is a $K$-orbit in $M$, we see that $\Phi_N^{-1}(0)$ is the $K$-orbit through $n^\alpha := (m^\alpha, \mu)$ where $m^\alpha \in \Phi^{-1}(\mu)$. Note that $H$ is the stabilizer subgroups of $n^\alpha$.

Let $Q_K^{\Phi_N}(N) \in R^{-\infty}(K)$ be the formal quantization of $N$ through the proper map $\Phi_N$. By definition

$$Q(M_\mu) = \left[Q_K^{\Phi_N}(N)\right]^K = \left[Q_K^0(N)\right]^K,$$

where $Q_K^0(N)$ depends only of a neighborhood of $\Phi_N^{-1}(0)$.

The orbit $K \cdot n^\alpha \hookrightarrow N$ is an isotropic embedding since it is the 0-level of the moment map $\Phi_N$. Then to describe a $K$-invariant neighborhood of $K \cdot n^\alpha$ in $N$ we can use the normal-form recipe of Marle, Guillemin and Sternberg.

First we consider, following Weinstein (see [11, 31]), the symplectic normal bundle

\begin{equation}
(2.23)
V := T(K \cdot n^\alpha)^{\perp,\Omega}/T(K \cdot n^\alpha),
\end{equation}

where the orthogonal $(\perp,\Omega)$ is taken relatively to the symplectic 2-form. We have

$$V = K \times_H V$$

where the vector space $V := T_{n^\alpha}(K \cdot n^\alpha)^{\perp,\Omega}/T_{n^\alpha}(K \cdot n^\alpha)$ inherits a symplectic structure and an Hamiltonian action of the group $H$: we denote $\Phi_H : V \to \mathfrak{h}^*$ the corresponding moment map.

Consider now the following symplectic manifold

\begin{equation}
(2.24)
\bar{N} := V \oplus T^*(K/H) = K \times_H \left( (\mathfrak{t}/\mathfrak{h})^* \oplus V \right).
\end{equation}

The action of $H$ on $\bar{N}$ is Hamiltonian and the moment map $\Phi_{\bar{N}} : \bar{N} \to \mathfrak{t}^*$ is given by the equation

\begin{equation}
(2.25)
\Phi_{\bar{N}}([k; \xi, v]) = k \cdot (\xi + \Phi_H(v)) \quad k \in K, \ \xi \in (\mathfrak{t}/\mathfrak{h})^*, \ v \in V.
\end{equation}

The Hamiltonian $K$-manifold $\bar{N}$ is prequantized by the line bundle $L_{\bar{N}} := K \times_H \mathbb{C}_\chi$. 

The local normal form Theorem (see \[10, 21\] Proposition 2.5) tells us that there exists a $K$-Hamiltonian isomorphism $\Upsilon : \mathcal{U}_1 \sim \mathcal{U}_2$ between a $K$-invariant neighborhood $\mathcal{U}_1$ of $K \cdot n^o$ in $N$, and a $K$-invariant neighborhood $\mathcal{U}_2$ of $K/H$ in $N$. This isomorphism $\Upsilon$, when restricted to $K \cdot n^o$, corresponds to the natural isomorphism $K \cdot n^o \sim K/H$.

Thanks to $\Upsilon$, we know that the fiber $\Phi^{-1}_N(0) \subset V$ is reduced to $\{0\}$. This last point is equivalent to the fact that $\Phi_H$ (and then $\Phi_S$) is proper map (see [21]). We check easily that the set of critical points of $|\Phi_N|^2$ is reduced to $\Phi^{-1}_N(0) = K/H$. Then, thank to the isomorphism $\Upsilon$, we have that

\[(2.26) \quad Q^H_K(N) = Q^H_K(\tilde{N}) = Q^H_K(\tilde{N}).\]

Let $\text{Ind}_H^K : R^{-\infty}(H) \rightarrow R^{-\infty}(K)$ be the induction map that is defined by the relation $\langle \text{Ind}_H^K(\phi), E \rangle = \langle \phi, E|_H \rangle$ for any $\phi \in R^{-\infty}(H)$ and $E \in R(K)$. Note that

\[\text{Ind}_H^K(\phi)[K] = (\text{Ind}_H^K(\phi), C) = \langle \phi, C \rangle = [\phi]^H.\]

Since $\Phi_H : V \rightarrow \mathfrak{h}^*$ is proper one can consider the quantization of the vector space $V$ through the map $\Phi_H$: $Q^H_K(V) \in R^{-\infty}(H)$.

**Proposition 2.17.** • We have

\[(2.27) \quad Q^H_K(\tilde{N}) = \text{Ind}_H^K \left( Q^H_K(V) \otimes C_\chi \right) \]

• The formal quantization $Q^H_K(V)$ coincides, as a generalized $H$-module, to the $H$-module $S(V^*)$ of polynomial function on $V$.

• The set $[S(V^*)]^H$ of polynomials invariant by the connected component $H^o$ is reduced to the scalars.

With the last Proposition we can finish the proof of Theorem 2.16 as follows. We have

\[Q(M_\mu) = [Q^*_K(N)]^K = \left[ Q^*_K(\tilde{N}) \right]^K = \left[ Q^H_K(V) \otimes C_\chi \right]^H = [S(V^*) \otimes C_\chi]^H = [C_\chi]^H.\]

**Proof.** The first point of Proposition 2.17 follows from the property of induction defined by Atiyah (see Section 3.4 in [10]). Let us explain the arguments. We work with the $H$-manifold $\mathcal{Y} = (\mathfrak{t}/\mathfrak{h})^* \oplus V$ and the $H$-equivariant map $j : \mathcal{Y} \hookrightarrow \tilde{N} := K \times_H \mathcal{Y}, y \mapsto [e, y]$.

We notice\(^5\) that $T\tilde{N} \simeq K \times_H (\mathfrak{t}/\mathfrak{h} \oplus T\mathcal{Y})$, and that $T_K\tilde{N} \simeq K \times_H (T_H\mathcal{Y})$. Hence the map $j$ induces an isomorphism $j_* : K_H(T_H\mathcal{Y}) \rightarrow K_K(T_K\tilde{N})$. Theorem 4.1 of

\(^5\) These identities come from the following $K \times H$-equivariant isomorphism of vector bundles over $K \times \mathcal{Y}$: $T_H(\tilde{N}) \rightarrow K \times (\mathfrak{t}/\mathfrak{h} \oplus T\mathcal{Y}), (k, m, \dot{X}) \mapsto (k, m, pr_{\mathfrak{t}/\mathfrak{h}}(X) + v_m)$. Here $pr_{\mathfrak{t}/\mathfrak{h}} : \mathfrak{t} \rightarrow \mathfrak{t}/\mathfrak{h}$ is the orthogonal projection.
Atiyah [1] tells us that the following diagram
\begin{equation}
\begin{array}{ccc}
K_H(T_H Y) & \xrightarrow{j_*} & K_K(T_K \tilde{N}) \\
\text{Index}_H^\kappa & & \text{Index}_K^\kappa \\
R^{-\infty}(H) & \xrightarrow{\Ind_K^H} & R^{-\infty}(K)
\end{array}
\end{equation}
is commutative.

The tangent bundle $T\tilde{N}$ is equivariantly diffeomorphic to
\[ K \times_H [\mathfrak{t}/\mathfrak{h}] \oplus (\mathfrak{t}/\mathfrak{h})^* \oplus TV \approx K \times_H [(\mathfrak{t}/\mathfrak{h})_C \oplus TV] \]
where $(\mathfrak{t}/\mathfrak{h})_C$ is the complexification of the real vector space $\mathfrak{t}/\mathfrak{h}$. We consider on $\tilde{N}$ the almost complex structure $J_{\tilde{N}} = (i, J_V)$ where $i$ is the complex structure on $(\mathfrak{t}/\mathfrak{h})_C$ and $J_V$ is a compatible (constant) complex structure on the symplectic vector space $V$. Note that $J_{\tilde{N}}$ is compatible with the symplectic structure on a neighborhood $U$ of the 0-section of the bundle $\tilde{N} \to K/H$.

Let $\kappa_{\tilde{N}}$ be the Kirwan vector field on $\tilde{N}$:
\[ \kappa_{\tilde{N}}(\xi, \eta) = -\xi + i \{\xi, \Phi_H(v)\} \oplus \kappa_V(\eta) \in (\mathfrak{t}/\mathfrak{h})_C \oplus V. \]
Here $\kappa_V$ is the Kirwan vector field relative to the Hamiltonian action of $H$ on the symplectic vector space $V$. Note that $\kappa_{\tilde{N}}$ vanishes exactly on the 0-section of the bundle $\tilde{N} \to K/H$.

Let $c^{\kappa_{\tilde{N}}}$ be the symbol $\text{Thom}(\tilde{N}, J_{\tilde{N}}) \otimes L_{\tilde{N}}$ pushed by the vector field $\kappa_{\tilde{N}}$. The generalized character $\mathcal{Q}^{\Phi_{\tilde{N}}}_{\Spin}(\tilde{N})$ is either computed as the equivariant index of the symbols $c^{\kappa_{\tilde{N}}}$ or $c^{\kappa^+_{\tilde{N}}} |_U$.

**Remark 2.18.** The fact that $J_{\tilde{N}}$ is not compatible on the entire manifold $\tilde{N}$ is not problematic, since $J_{\tilde{N}}$ is compatible in a neighborhood $U$ of the set where $\kappa_{\tilde{N}}$ vanishes. See the first point of Lemma 2.6.

For $X + i\eta \oplus w \in T_{[\xi, \eta]} \tilde{N} \approx (\mathfrak{t}/\mathfrak{h})_C \oplus V$, the map
\begin{equation}
c^{\kappa_{\tilde{N}}}(X + i\eta \oplus w) = c \left( X + \xi + i(\eta - \{\xi, \Phi_H(v)\}) \right) \circ c \left( w - \kappa_V(\eta) \right)
\end{equation}
acts on the vector space $\wedge (\mathfrak{t}/\mathfrak{h})_C \otimes \wedge J_V V \otimes C_\chi$.

Let $\text{Bott}(\mathfrak{t}/\mathfrak{h})$ be the Bott morphism of the vector space $\mathfrak{t}/\mathfrak{h}$. It is an elliptic morphism defined by
\[ \text{Bott}(\mathfrak{t}/\mathfrak{h})|_{\xi}(\eta) = c(\xi + i\eta) \] acting on $\wedge (\mathfrak{t}/\mathfrak{h})_C$, for $\eta \in T_{\xi}(\mathfrak{t}/\mathfrak{h})$. Let $c^{\kappa_{V}}$ be the symbol $\text{Thom}(V, J_V)$ pushed by the vector field $\kappa_V$.

**Lemma 2.19.** We have
\[ c^{\kappa_{\tilde{N}}} = j_*(\text{Bott}(\mathfrak{t}/\mathfrak{h}) \oplus c^{\kappa_{V}} \otimes C_\chi). \]

**Proof.** We work with the symbol
\[ \sigma^T|_{\xi, \eta}(\eta) = c(\xi + i\eta - iT \{\xi, \Phi_H(v)\}) \]
acting on $\wedge (\mathfrak{t}/\mathfrak{h})_C$. Note that $\text{Bott}(\mathfrak{t}/\mathfrak{h}) = \sigma^0$. From (2.29), we see that $c^{\kappa_{\tilde{N}}} = j_*\left( \sigma^1 \oplus c^{\kappa_{V}} \otimes C_\chi \right)$. It is now easy to check that $\sigma^T \circ c^{\kappa_{V}} \otimes C_\chi, T \in [0, 1]$ is an homotopy of transversally elliptic symbols on $\mathfrak{t}/\mathfrak{h} \times V$. \qed
The commutative diagram (2.28) and the last Lemma gives
\[
\mathcal{Q}_K^\Phi (\widetilde{N}) = \text{Index}_N^K (c^\kappa V)
\]
\[
= \text{Ind}_H^K \left( \text{Index}_{\eta}^{\mathfrak{h} \times V} \left( \text{Bott}(\mathfrak{t}/\mathfrak{h}) \otimes c^\kappa V \right) \otimes C_\chi \right)
\]
\[
= \text{Ind}_H^K \left( \text{Index}_{\eta}^{\mathfrak{h}} \left( \text{Bott}(\mathfrak{t}/\mathfrak{h}) \right) \otimes \text{Index}_V^K (c^\kappa V) \otimes C_\chi \right)
\]
\[
= \text{Ind}_H^K \left( \mathcal{Q}_H^\Phi (V) \otimes C_\chi \right).
\]
We have used here that the equivariant index of Bott($\mathfrak{t}/\mathfrak{h}$) is equal to 1 (e.g., the trivial representation).

Let us proved now the second point of Proposition 2.17. The Kirwan vector field $\kappa^V$ satisfies the simple rule:
\[
(2.30) \quad (\kappa^V(v), J_V v) = -\Omega(\kappa^V(v), v) = \frac{1}{2} \|\Phi_H(v)\|^2, \quad v \in \mathcal{V}.
\]
It shows in particular that $\kappa^V(v) = 0 \Leftrightarrow \Phi_H(v) = 0$. Since the moment map $\Phi_H : V \to \mathfrak{h}^*$ is quadratic, the fact that $\Phi_H$ is proper is equivalent to the fact that $\Phi_H^T(0) = 0$.

We consider on $V$ the family of symbol $\sigma^s$:
\[
\sigma^s|_\nu(w) = c \left( w - sv^V(v) - (1 - s)J_V v \right)
\]
viewed as a map from $\wedge^\text{even}_V$ to $\wedge^\text{odd}_V$. Thanks to (2.30), one sees that $\sigma^s$ is a family of $K$-transversally elliptic symbol on $V$. Hence $\sigma^1 = c^\kappa V$ and $\sigma^0 = c(w - J_V v)$ defines the same class in the group $K_K(T_K V)$. The symbol $\sigma^0$ was first studied by Atiyah [10] when $\dim_{\mathbb{C}} V = 1$. The author considered the general case in [19]. We have
\[
\text{Index}_V^K(\sigma^0) = S(V^*) \quad \text{in} \quad R^{-\infty}(K).
\]

The last point of Proposition 2.17 is a consequence of the properness of the moment map $\Phi_H$ (see Section 5 of [21]).

\[\square\]

**Example 2.20** [21]. We consider the action of the unitary group $U_n$ on $\mathbb{C}^n$. The symplectic form on $\mathbb{C}^n$ is defined by $\Omega(v, w) = \frac{i}{2} \sum_k v_k \overline{w}_k - \overline{v}_k w_k$. Let us identify the Lie algebra $\mathfrak{u}_n$ with its dual through the trace map. The moment map $\Phi : \mathbb{C}^n \to \mathfrak{u}_n$ is defined by $\Phi(v) = \frac{1}{2} v \otimes v^*$ where $v \otimes v^* : \mathbb{C}^n \to \mathbb{C}^n$ is the linear map $w \mapsto (\sum_k \overline{v}_k w_k) v$. One checks easily that the pull-back by $\Phi$ of a $U_n$-orbit in $\mathbb{C}^n$ is either empty or a $U_n$-orbit in $\mathbb{C}^n$. We knows also that the stabiliser subgroup of a non-zero vector of $\mathbb{C}^n$ is connected since it is diffeomorphic to $U_{n-1}$. Finally we have
\[
(2.31) \quad \mathcal{Q}(\mathbb{C}^n)_\mu = \begin{cases} 1 & \text{if } \mu \in \widetilde{U}_n \text{ belongs to the image of } \Phi \\ 0 & \text{if } \mu \in \widetilde{U}_n \text{ does not belongs to the image of } \Phi. \end{cases}
\]
Then one checks that $\mathcal{Q}_{U_n}^{-\infty}(\mathbb{C}^n)$ coincides in $R^{-\infty}(U_n)$ with the algebra $S((\mathbb{C}^n)^*)$ of polynomial function on $\mathbb{C}^n$. 

\[\square\]
Example 2.21 (E). We consider the Lie group \( SL_2(\mathbb{R}) \) and its compact torus of dimension 1 denoted by \( T \). The Lie algebra \( sl_2(\mathbb{R}) \) is identified with its dual through the trace map, and the Lie algebra \( t \) is naturally identified with \( sl_2(\mathbb{R})^T \).

For \( l \in \mathbb{Z} \setminus \{0\} \), we consider the character \( \chi_l \) of \( T \) defined by

\[
\chi_l \left( \begin{array}{cc}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{array} \right) = e^{i\theta}.
\]

Its differential \( i\omega_{\chi_l} \in t^* \) correspond (through the trace map) to the matrix

\[
X_l = \left( \begin{array}{cc}
0 & \frac{l}{2} \\
-l/2 & 0
\end{array} \right).
\]

Let \( O_l \) be the coadjoint orbit of the group \( SL_2(\mathbb{R}) \) through the matrix \( X_l \). It is a Hamiltonian \( SL_2(\mathbb{R}) \)-manifold prequantized by the \( SL_2(\mathbb{R}) \)-equivariant line bundle \( L_l \simeq SL_2(\mathbb{R}) \times_T \mathbb{C}_l \), where \( \mathbb{C}_l \) is the \( T \)-module associated to the character \( \chi_l \). We look at the Hamiltonian action of \( T \) on \( O_l \). Let \( \Phi_T : O_l \to t^* \) be the corresponding moment map. One checks that the moment map \( \Phi_T \) is proper and that its image is equal to the half-line \( \{ aX_l, a \geq 1 \} \subset t^* \).

We check that for each \( \xi \in \{ aX_l, a \geq 1 \} \) the fiber \( \Phi_T^{-1}(\xi) \) is equal to a \( T \)-orbit in \( O_l \). For \( k \in \mathbb{Z} \), let us denote \( (O_l)_k \) the symplectic reduction of \( O_l \) at the level \( X_k \). We know that \( (O_l)_k = \emptyset \) if \( k \notin \{ al, a \geq 1 \} \), and that \( (O_l)_k \) is a point if \( k \in \{ al, a \geq 1 \} \).

In order to compute \( Q((O_l)_k) \) we look at the stabilizer subgroup \( T_m := \{ t \in T \mid t \cdot m = m \} \) for each point \( m \in O_l \). One sees that \( T_m = T \) if \( m = X_l \) and \( T_m \) is equal to the center \( \{ \pm Id \} \) of \( SL_2(\mathbb{R}) \), when \( m \neq X_l \).

Theorem 2.14 gives in this setting that, for \( k \in \{ al, a \geq 1 \} \),

\[
(2.32) \quad Q((O_l)_k) = \begin{cases} 1 & \text{if } l - k \text{ is even} \\ 0 & \text{if } l - k \text{ is odd}. \end{cases}
\]

Hence the formal geometric quantization of the proper \( T \)-manifold \( O_l \) is

\[
(2.33) \quad Q_T^{-\infty}(O_l) = \begin{cases} \mathbb{C}_l : \sum_{p \geq 0} \mathbb{C}_{2p} & \text{if } l > 0 \\ \mathbb{C}_l : \sum_{p \geq 0} \mathbb{C}_{-2p} & \text{if } l < 0. \end{cases}
\]

Here we recognize that \( Q_T^{-\infty}(O_l) \) coincides with the restriction of the holomorphic (resp. anti-holomorphic) discrete series representation \( \Theta_l \) to the group \( T \) when \( l > 0 \) (resp. \( l < 0 \)).

2.5. Wonderful compactifications and symplectic cuts. Another equivalent definition of the quantization \( Q^{-\infty} \) uses a generalisation of the technique of symplectic cutting (originally due to Lerman [E]) that was introduced in [2] and was motivated by the wonderful compactifications of De Concini and Procesi. Let us recall the method.

We recall that \( T \) is a maximal torus in the compact connected Lie group \( K \), and \( W \) is the Weyl group. We define a \( K \)-adapted polytope in \( t^* \) to be a \( W \)-invariant Delzant polytope \( P \) in \( t^* \) whose vertices are regular elements of the weight lattice \( \Lambda^\ast \). If \( \{ \lambda_1, \ldots, \lambda_N \} \) are the dominant weights lying in the union of all the closed one-dimensional faces of \( P \), then there is a \( G \times G \)-equivariant embedding of \( G = K_C \) into

\[
\bigoplus_{i=1}^N \mathbb{P}(V_{\lambda_i}^* \otimes V_{\lambda_i})
\]
We can work with the dilated polytopes \( P \). We have explained how was defined their geometric quantization. Let \( \Omega_{X_p} \) be the symplectic 2-form on \( X_p \) which given by the Kahler structure. We recall briefly the different properties of \((X_p, \Omega_{X_p})\): all the details can be found in \[21\].

1. \( X_p \) is equipped with an Hamiltonian action of \( K \times K \). Let \( \Phi = (\Phi_1, \Phi_r) : M \to \mathfrak{t}^* \times \mathfrak{t}^* \) be the corresponding moment map.
2. The image of \( \Phi := (\Phi_1, \Phi_r) \) is equal to \( \{(k \cdot \xi, -k' \cdot \xi) \mid \xi \in K \text{ and } k, k' \in K \} \).
3. The Hamiltonian manifold \((X_p, K \times K)\) has no multiplicities: the pull-back by \( \Phi \) of a \( K \times K \)-orbit in the image is a \( K \times K \)-orbit in \( X_p \).

Let \( U_P := K \cdot P^\circ \) where \( P^\circ \) is the interior of \( P \). We define
\[
X_p^\circ := \Phi_l^{-1}(U_P)
\]
which is an invariant, open and dense subset of \( X_p \). We have the following important property concerning \( X_p^\circ \).

4. There exists an equivariant diffeomorphism \( \Upsilon : K \times U_P \to X_p^\circ \) such that \( \Upsilon^*(\Phi_1)(k, \xi) = k \cdot \xi \) and \( \Upsilon^*(\Phi_r)(k, \xi) = -\xi \).
5. This diffeomorphism \( \Upsilon \) is a quasi-symplectomorphism in the sense that there is a homotopy of symplectic forms taking the symplectic form on the open subset \( K \times U_P \) of the cotangent bundle \( T^*K \) to the pullback of the symplectic form \( \Omega_{X_p} \) on \( X_p^\circ \).
6. The symplectic manifold \((X_p, \Omega_{X_p})\) is prequantized by the restriction of the hyperplane line bundle \( \mathcal{O}(1) \to P(\bigoplus_{i=1}^N V_\lambda^* \otimes V_\lambda) \) to \( X_P \): let us denoted \( L_P \) the corresponding \( K \times K \)-equivariant line bundle.
7. The pull-back of the line bundle \( L_P \) by the map \( \Upsilon : K \times U_P \hookrightarrow X_p \) is trivial.

Let \((M, \Omega_M, \Phi_M)\) be a proper Hamiltonian \( K \)-manifold. We also consider the Hamiltonian \( K \times K \)-manifold \( X_P \) associated to a \( K \)-adapted polytope \( P \). We consider now the product \( M \times X_P \) with the following \( K \times K \) action:

- the action \( k \cdot (m, x) = (k \cdot m, x \cdot k^{-1}) \): the corresponding moment map is \( \Phi_1(m, x) = \Phi_M(m) + \Phi_r(x) \),
- the action \( k \cdot (m, x) = (m, k \cdot x) \): the corresponding moment map is \( \Phi_2(m, x) = \Phi_l(x) \).

**Definition 2.22.** We denote \( M_P \) the symplectic reduction at 0 of \( M \times X_P \) for the action \( \cdot : M_P := (\Phi_1)^{-1}(0)/(K, 1) \).

Then \( M_P \) inherits a Hamiltonian \( K \)-action with moment map \( \Phi_{M_P} : M_P \to \mathfrak{t}^* \) whose image is \( \Phi(M) \cap K \cdot P \).

One checks that \( M_P \) contains an open and dense subset of smooth points which quasi-symplectomorphic to the open subset \((\Phi_M)^{-1}(U_P)\). If the polytope \( P \) is fixed, we can work with the dilated polytopes \( nP \) for \( n \geq 1 \). We have then the family of compact, perhaps singular, \( K \)-hamiltonian manifolds \( M_{nP}, n \geq 1 \): in Section\[2.3\] we have explained how was defined their geometric quantization \( Q_K(M_{nP}) \in \mathcal{R}(K) \). We have a convenient definition for \( Q^{-\infty} \).
Proposition 2.23 \((2.34)\). We have the following equality in \(R^{-\infty}(K)\):

\[Q_{K}^{-\infty}(M) = \lim_{n \to \infty} Q_{K}(M_{nP}).\]

3. Proof of Theorem 1.4

The main result of this section is Theorem 3.1. Let \(r_{P} := \inf_{\xi \in \partial P} \|\xi\|\). The generalized character

\[Q_{K}^{\Phi}(M) - Q_{K}(M_{P}) \in R^{-\infty}(K)\]

is supported outside the ball \(B_{r_{P}}\).

Then, for the dilated polytope \(nP, n \geq 1\), the character \(Q_{K}^{\Phi}(M) - Q_{K}(M_{nP})\) is supported outside the ball \(B_{nr_{P}}\). Taking the limit when \(n\) goes to infinity gives

\[Q_{K}^{{\Phi}}(M) = \lim_{n \to \infty} Q_{K}(M_{nP}).\]

Finally, the identity of Theorem 1.4,

\[Q_{K}^{{\Phi}}(M) = Q_{K}^{-\infty}(M),\]

is a direct consequence of (2.34) and (3.35).

Recall that \(O(r) \in R^{-\infty}(K)\) denoted any generalized character supported outside the ball \(B_{r}\).

Theorem 3.1 follows from the comparison of three different geometrical situations. All of them concern Hamiltonian actions of \(K_{1} \times K_{2}\), where \(K_{1}\) and \(K_{2}\) are two copies of \(K\).

First setting. We work with the Hamiltonian \(K_{1} \times K_{2}\)-manifold \(M \times X_{P}\): here \(K_{1}\) acts both on \(M\) and on \(X_{P}\). Since the moment map \(\Phi_{1}\) (relative to the \(K_{1}\)-action) is proper we may “quantize” \(M \times X_{P}\) via the map \(\|\Phi_{1}\|^{2}\) : let

\[Q_{K_{1} \times K_{2}}^{\Phi_{1}}(M \times X_{P}) \in R^{-\infty}(K_{1} \times K_{2})\]

be the corresponding generalized character. Recall that \(Q_{K_{1}}(M_{P})\) is equal to \([Q_{K_{1} \times K_{2}}^{\Phi_{1}}(M \times X_{P})]^{K_{1}}\).

Second setting. We consider the same setting than before: the Hamiltonian action of \(K_{1} \times K_{2}\) on \(M \times X_{P}\). But we “quantize” \(M \times X_{P}\) through the global moment map \(\Phi = (\Phi_{1}, \Phi_{2})\). Here we have some liberty in the choice of the scalar product on \(\mathfrak{t}_{1}^{*} \times \mathfrak{t}_{2}^{*}\). If \(\|\xi\|^{2}\) is an invariant Euclidean norm on \(\mathfrak{t}^{*}\), we take on \(\mathfrak{t}_{1}^{*} \times \mathfrak{t}_{2}^{*}\) the Euclidean norm

\[(\|\xi_{1}, \xi_{2}\|^{2}) = \|\xi_{1}\|^{2} + \rho \|\xi_{2}\|^{2}\]

depending on a parameter \(\rho > 0\). Let us consider the quantization of \(M \times X_{P}\) via the map \(\|\Phi\|^{2}_{\rho}\):

\[Q_{K_{1} \times K_{2}}^{\Phi_{1}, \rho}(M \times X_{P}) \in R^{-\infty}(K_{1} \times K_{2}).\]

Third setting. We consider the cotangent bundle \(T^{*}K\) with the Hamiltonian action of \(K_{1} \times K_{2}\): \(K_{1}\) acts by right translations, and \(K_{2}\) by left translations. We consider the Hamiltonian action of \(K_{1} \times K_{2}\) on \(M \times T^{*}K\): here \(K_{1}\) acts both on
$M$ and on $T^*K$. Let $\Phi = (\Phi_1, \Phi_2)$ be the global moment map on $M \times T^*K$. Since the moment map $\Phi$ is proper we can "quantize" $M \times T^*K$ via the map $\|\Phi\|_\rho^2$: let $Q_{K_1 \times K_2}^\rho(M \times T^*K) \in R^{-\infty}(K_1 \times K_2)$

be the corresponding generalized character.

Theorem 3.1 is a consequence of the following propositions.

First we compare $Q_{K_1}^\rho(M)$ with the $K_1$-invariant part of $Q_{K_1 \times K_2}^\rho(M \times T^*K)$.

**Proposition 3.2.** For any $\rho \in [0, 1]$, we have

$$(3.37) \quad \left[ Q_{K_1 \times K_2}^\rho(M \times T^*K) \right]_{K_1}^K = Q_{K_2}^\rho(M) \text{ in } R^{-\infty}(K_2).$$

Then we compare the $K_1$-invariant part of the generalized characters $Q_{K_1 \times K_2}^\rho(M \times T^*K)$ and $Q_{K_1 \times K_2}^\rho(M \times \chi P)$.

**Proposition 3.3.** For any $\rho \in [0, 1]$, we have the following relation in $R^{-\infty}(K_2)$

$$(3.38) \quad \left[ Q_{K_1 \times K_2}^\rho(M \times \chi P) \right]_{K_1}^K - \left[ Q_{K_1 \times K_2}^\rho(M \times T^*K) \right]_{K_1}^K = O(r_P)$$

Finally we compare the $K_1$-invariant part of the generalized characters $Q_{K_1 \times K_2}^\rho(M \times \chi P)$ and $Q_{K_1 \times K_2}^\rho(M \times \chi P)$.

**Proposition 3.4.** There exists $\epsilon > 0$ such that

$$(3.39) \quad Q_{K_2}(M_P) - \left[ Q_{K_1 \times K_2}^\rho(M \times \chi P) \right]_{K_1}^K = O((\epsilon/\rho)^{1/2}) \text{ in } R^{-\infty}(K_2)$$

if $\rho > 0$ is small enough.

If we sum the relations (3.37), (3.38) and (3.39) we get

$Q_{K_2}^\rho(M) = Q_{K_2}^\rho(M_P) + O(r_P) + O((\epsilon/\rho)^{1/2})$

if $\rho$ is small enough. So Theorem 3.1 follows by taking $(\epsilon/\rho)^{1/2} \geq r_P$.

3.1. **Proof of Proposition 3.2.** The cotangent bundle $T^*K$ is identified with $K \times \ast^*$. The data is then (see Section 2): 

- the Liouville 1-form $\lambda = \sum_j \omega_j \otimes E_j$. Here $(E_j)$ is a basis of $\mathfrak{k}$ with dual basis $(E^*_j)$, and $\omega_j$ is the left invariant 1-form on $K$ defined by $\omega_j(a e^{[X]}_{[0]} = \langle E^*_j, X \rangle$.
- the symplectic form $\Omega := -d\lambda$.
- the action of $K_1 \times K_2$ on $K \times \ast^*$ is $(k_1, k_2) \cdot (a, \xi) = (k_2 a k_1^{-1}, k_1 \cdot \xi)$.
- the moment map relative to the $K_1$-action is $\Phi_1(a, \xi) = -\xi$.
- the moment map relative to the $K_2$-action is $\Phi_2(a, \xi) = a \cdot \xi$.

We work now with the Hamiltonian action of $K_1 \times K_2$ on $M \times T^*K$ given by

$$(k_1, k_2) \cdot (m, a, \xi) = (k_1 \cdot m, k_2 a k_1^{-1}, k_1 \cdot \xi).$$

The corresponding moment map is $\Phi = (\Phi_1, \Phi_2)$: $\Phi_1(m, a, \xi) = \Phi_M(m) - \xi$ and $\Phi_2(m, a, \xi) = a \cdot \xi$.

Let $c_1$ be a symbol $\text{Thom}(M, J_1) \otimes L$ attached to the prequantized Hamiltonian $K_1$-manifold $(M, \Omega)$. The cotangent bundle $T^*K$ is prequantized by the trivial line bundle: let $c_2$ be the symbol $\text{Thom}(T^*K, J_2)$ attached to the prequantized
Hamiltonian $K_1 \times K_2$-manifold $T^*K$. The product $c = c_1 \otimes c_2$ corresponds to the symbol Thom($N, J$) $\otimes L$ on $N = M \times T^*K$.

Let $\kappa_\rho$ be the Kirwan vector field associated to the map $\|\Phi\|^2 : M \times T^*K \to \mathbb{R}$.

We check that $\|\Phi\|^2(m, k, \xi) = \|\Phi_M(m) - \xi\|^2 + \rho\|\xi\|^2$, and

$$\kappa_\rho(m, k, \xi) = \left(\left(\Phi_M(m) - \xi\right) \cdot m; \left(\Phi_M(m) - (1 + \rho)\tilde{c}; -|\Phi_M(m)|\tilde{c}\right)\right).$$

Here $T_{(m, k, \xi)}(M \times T^*K) \simeq T_m M \times T \times T$.

We have

$$\text{Cr}(\|\Phi\|^2_\rho) = \{\kappa_\rho = 0\} = \bigcup_{\beta \in B} K_1 \times K_2 \cdot \left(M^{\beta} \cap \Phi_M^{-1}(\beta) \times \{1\} \times \left\{\frac{\beta}{\rho + 1}\right\}\right)$$

where $B$ parametrizes $\text{Cr}(\|\Phi_M\|^2)$. Hence one checks that the critical values of $\|\Phi\|^2_\rho$ are $\frac{\beta}{\rho + 1}\|\xi\|^2$, $\beta \in B$.

Let $c^{s_\rho}$ be the symbol $c$ pushed by the vector field $\kappa_\rho$: we have

$$c^{s_\rho}(v; X; Y) = c_1(v - \kappa_I) \otimes c_2(X - \kappa_{I, \rho}; Y - \kappa_M)$$

for $(v; X; Y) \in T_{(m, k, \xi)}(M \times T^*K) \simeq T_m M \times T \times T$.

For a real $R > 0$ we define the open invariant subsets of $M \times T^*K$

$$U_R := \{\|\Phi\|^2 < R\}$$

$$V_R := \{\|\Phi_M\|^2 < R\} \times T^*K.$$

By definition the generalized equivariant index $Q_{K_1 \times K_2}^{\Phi, \rho}(M \times T^*K)$ is defined as the limit of the equivariant index

$$Q_{K_1 \times K_2}^{\Phi, \rho}(U_R) := \text{Index}_{U_R}^{K_1 \times K_2}(c^{s_\rho}|_{U_R}),$$

when $R$ goes to infinity (and stays outside the critical values of $\|\Phi\|^2_\rho$).

In the other hand, when $R'$ is a regular value of $\|\Phi_M\|^2$, we see that the symbol $c_\rho|_{V_{R'}}$ is $K_1 \times K_2$-transversally elliptic. Let

$$\text{Index}_{V_{R'}}^{K_1 \times K_2}(c^{s_\rho}|_{V_{R'}})$$

be its equivariant index. Notice that the index map is well-defined on $V_R = \{\|\Phi_M\|^2 < R\} \times T^*K$ since $T^*K$ can be seen as a open subset of a compact manifold.

It is easy to check that for any $R > 0$ there exists $R' > R$ such that $U_R \subset V_{R'}$. It implies that $Q_{K_1 \times K_2}^{\Phi, \rho}(M \times T^*K)$ is also defined as the limit of (3.40) when $R'$ goes to infinity.

We look now to the deformation $\kappa_\rho(s) = (\kappa_I^s; \kappa_{I, \rho}^s; s\kappa_M)$, $s \in [0, 1]$ where

$$\kappa_I^s(m, \xi) = (\Phi_M(m) - s\xi) \cdot m \quad \text{and} \quad \kappa_{I, \rho}^s(m, \xi) = s\tilde{\Phi}_M(m) - (1 + s\rho)\tilde{c}.$$  

Let $c^{s_\rho(s)}$ be the symbol $c$ pushed by the vector field $\kappa_\rho(s)$.

**Lemma 3.5.** Let $R'$ be a regular value of $\|\Phi_M\|^2$.

- The family $c^{s_\rho(s)}|_{V_{R'}}$, $s \in [0, 1]$, defines an homotopy of $K_1 \times K_2$-transversally elliptic symbols on $V_{R'}$.

- The $K_1$-invariant part of $\text{Index}_{V_{R'}}^{K_1 \times K_2}(c^{s_\rho(s)}|_{V_{R'}})$ is equal to $Q_{K_2}^{\Phi, \rho}(M \times R')$. 

Proof. The first point follows from the fact that \( \text{Char}(c_{\kappa^q(s)}|_{V_{R'}}) \cap T_{K_1 \times K_2}(V_{R'}) \), which is equal to

\[
\{(m, k, \frac{s}{1 + s\rho} \Phi_M(m)), \ k \in K \text{ and } m \in \text{Cr}(\|\Phi_M\|^2) \cap \{\|\Phi_M\|^2 < R'\}\},
\]

stays in a compact set when \( s \in [0, 1] \).

The symbol \( c_{\kappa^q(0)}|_{V_{R'}} \) is equal to the product of the symbol \( c_{\kappa^q}|_{M < R'} \), which is \( K_1 \)-transversally elliptic, with the symbol

\[
c_{\kappa^q}(X; Y) = c_2(X + \xi; Y)
\]

which is a \( K_2 \)-transversally elliptic on \( T^*K \). A basic computation done in section 5.1.4 gives that

\[
\text{Index}_{T^*K}^K(c_{\kappa^q}) = L^2(K) = \sum_{\mu \in \mathcal{K}} (V_{\mu}^K)^* \otimes V_{\mu}^K
\]
in \( R^{-\infty}(K_1 \times K_2) \). Finally the “multiplicative property” (see Theorem 2.1) gives

\[
\text{Index}_{V_{R'}}^K(c_{\kappa^q}|_{V_{R'}}) = \text{Index}_{V_{R'}}^{K_1 \times K_2}(c_{\kappa^q}|_{V_{R'}}) \otimes \text{Index}_{V_{R'}}^{K_1 \times K_2}(c_{\kappa^q}|_{V_{R'}}) = \sum_{\mu \in \mathcal{K}} \mathcal{Q}_{K_1}(M < R') \otimes (V_{\mu}^{K_1})^* \otimes V_{\mu}^{K_2}
\]

Taking the \( K_1 \)-invariant completes the proof of the second point. \( \square \)

Finally we have proved that the generalized character \( [\text{Index}_{V_{R'}}^{K_1 \times K_2}(c_{\kappa^q}|_{V_{R'}})]^{K_1} \) is equal to \( \mathcal{Q}_{K_2}(M < R') \). Taking the limit \( R' \to \infty \) gives

\[
\left[ \mathcal{Q}_{K_1 \times K_2}(M \times T^*K) \right]^{K_1} = \lim_{R' \to \infty} \left[ \text{Index}_{V_{R'}}^{K_1 \times K_2}(c_{\kappa^q}|_{V_{R'}}) \right]^{K_1} = \lim_{R' \to \infty} \mathcal{Q}_{K_2}(M < R') = \mathcal{Q}_{K_2}(M).
\]

3.2. Proof of Proposition 3.3. We work here with the Hamiltonian action of \( K_1 \times K_2 \) on \( M \times X_P \). The action is \( (k_1, k_2) \cdot (m, x) = (k \cdot m, k_2 \cdot x \cdot k_1^{-1}) \) and the corresponding moment map is \( \Phi = (\Phi_1, \Phi_2) \) with \( \Phi_1(m, x) = \Phi_M(m) + \Phi_\rho(x) \) and \( \Phi_2(m, x) = \Phi_t(x) \). Let \( \|\Phi_1, \Phi_2\|^2 = \|\Phi_1\|^2 + \|\Phi_2\|^2 \) be the Euclidean norm \( t_1^* \times t_2^* \) attached to \( \rho > 0 \).

Let us consider the quantization of \( M \times X_P \) via the map \( \|\Phi\|^2 \):

\[
\mathcal{Q}_{K_1 \times K_2}^{\rho}(M \times X_P) \in R^{-\infty}(K_1 \times K_2)
\]

The critical set \( \text{Cr}(\|\Phi\|^2) \) admits the decomposition

\[
(3.41) \quad \text{Cr}(\|\Phi\|^2) = \bigcup_{\gamma \in B_\rho} K_1 \times K_2 \cdot C_{\gamma}
\]

where \( (m, x) \in C_{\gamma} \) if and only if \( \gamma = (\gamma_1, \gamma_2) \) with

\[
\begin{align*}
\Phi_M(m) + \Phi_\rho(x) &= \gamma_1 \\
\Phi_t(x) &= \gamma_2 \\
\gamma_1 \cdot m &= 0 \\
\gamma_1 \cdot x + \rho \gamma_2 \cdot x &= 0.
\end{align*}
\]

(3.42)
We have
\[ Q_{K_1 \times K_2}^\Phi(M \times \mathcal{X}_P) = \sum_{\gamma \in B_\rho} Q_{K_1 \times K_2}^\gamma(M \times \mathcal{X}_P) \]
where the generalized character \( Q_{K_1 \times K_2}^\gamma(M \times \mathcal{X}_P) \) is computed as an index of a transversally elliptic symbol in a neighborhood of \( K_1 \times K_2 \cdot C_\gamma \subset M \times \Phi^{-1}(K_2 \cdot \gamma_2) \).

Thanks to Theorem 2.9 we know that the support of the generalized character \( Q_{K_1 \times K_2}^\gamma(M \times \mathcal{X}_P) \) is contained in \( \{(a, b) \in K_1 \times K_2 \mid \|a\|^2 + \rho\|b\|^2 \geq \|\gamma\|^2_2\} \). Hence
\[ \text{support} \left( (Q_{K_1 \times K_2}^\gamma(M \times \mathcal{X}_P))^1 \right) = \left\{ b \in K_2 \mid \rho\|b\|^2 \geq \|\gamma\|^2_2 \right\} \]
Let \( r_P = \inf_{\xi \in \partial P} \|\xi\| \). We know then that
\[ \left[ Q_{K_1 \times K_2}^\gamma(M \times \mathcal{X}_P) \right]^{K_1} = \sum_{\gamma \in B_\rho} Q_{K_1 \times K_2}^\gamma(M \times \mathcal{X}_P) + O(r_P). \]
Let \( R_P < \rho r_2^2 \) be a regular value of \( \|\Phi\|^2_\rho : M \times \mathcal{X}_P \to \mathbb{R} \) such that for all \( \gamma \in B_\rho \) we have \( \|\gamma\|^2_2 < \rho r_2^2 \iff \|\gamma\|^2_\rho < R_P \). Then
\[ Q_{K_1 \times K_2}^\Phi(M \times \mathcal{X}_P) = \sum_{\gamma \in B_\rho} Q_{K_1 \times K_2}^\gamma(M \times \mathcal{X}_P) + O(r_P). \]
For the generalized index \( Q_{K_1 \times K_2}^\Phi(M \times T^*K) \) we have also a decomposition
\[ Q_{K_1 \times K_2}^\Phi(M \times T^*K) = \sum_{\gamma \in B_\rho} Q_{K_1 \times K_2}^\gamma(M \times T^*K) \]
where \( B_\rho \) parametrizes the critical set of \( \|\Phi\|^2_\rho : M \times T^*K \to \mathbb{R} \). Like before we get
\[ \left[ Q_{K_1 \times K_2}^\Phi(M \times T^*K) \right]^{K_1} = \sum_{\gamma \in B_\rho} Q_{K_1 \times K_2}^\gamma(M \times T^*K) + O(r_P). \]
Here \( R_P' < \rho r_2^2 \) is a regular value of \( \|\Phi\|^2_\rho : M \times T^*K \to \mathbb{R} \) such that for all \( \gamma \in B_\rho \) we have \( \|\gamma\|^2_2 < \rho r_2^2 \iff \|\gamma\|^2_\rho < R'_P \).

Lemma 3.6. We have
\[ Q_{K_1 \times K_2}^\Phi(M \times \mathcal{X}_P)_{R_P} = Q_{K_1 \times K_2}^\Phi(M \times T^*K)_{R'_P}. \]
Proof. The Lemma will follow from Proposition 2.6. We take here \( V' = M \times \mathcal{X}_P \), \( V = M \times K \times U_P \subset M \times T^*K \) and the equivariant diffeomorphism \( \Psi : V \to V' \) is equal to \( \text{Id} \times \hat{Y} \) where \( \hat{Y} \) was introduced in Section 2.3. Note that \( \Psi \) satisfies points (1) - (3) of Proposition 2.4
Note that \( \|\Phi(m, x)\|^2_\rho < \rho r_2^2 \) implies that \( \|\Phi_1(x)\| < r_P \) and then \( x \in \mathcal{X}_P \). Hence the open subset \( U' := (M \times \mathcal{X}_P)_{R_P} \) is contained in \( V' = M \times \mathcal{X}_P \). In the same way the open subset \( U := (M \times T^*K)_{R'_P} \) is contained in \( V \). We have \( \Psi(U) = U' \) if \( R_P = R'_P \).

We have proved that (3.46) is a consequence of Proposition 2.6.

Finally, if we take the difference between (3.44) and (3.45), we get
\[ \left[ Q_{K_1 \times K_2}^\Phi(M \times \mathcal{X}_P) \right]^{K_1} - \left[ Q_{K_1 \times K_2}^\Phi(M \times T^*K) \right]^{K_1} = O(r_P), \]
which is the relation of Proposition 3.3.
3.3. Proof of Proposition 3.4. Here we want to compare the $K_1$-invariant part of the characters $Q_{K_1 \times K_2}^{\Phi, \rho}(M \times X_P)$ and $Q_{K_1 \times K_2}^{\Phi_1}(M \times X_P)$.

We know after Theorem 2.14 that
\[
Q_{K_2}(M_P) = \left[ Q_{K_1 \times K_2}^{\Phi_1}(M \times X_P) \right]^{K_1}
\]
when $\epsilon > 0$ is any regular value of $\|\Phi_1\|^2$, and $U_{\epsilon} := \{ \|\Phi_1\|^2 < \epsilon \} \subset M \times X_P$.

In this section we fix once for all $\epsilon > 0$ small enough so that
\[
\text{Cr}(\|\Phi_1\|^2) \cap \{ \|\Phi_1\|^2 \leq \epsilon \} = \{ \Phi_1 = 0 \}.
\]

Let $c_1$ be the symbol $\text{Thom}(M, J_1) \otimes L$ attached to the prequantized Hamiltonian $K_1$-manifold $(M, \Omega)$. Let $c_3$ be the symbol $\text{Thom}(X_P, J_3) \otimes L_P$ attached to the prequantized Hamiltonian $K_1 \times K_2$-manifold $X_P$. The product $c = c_1 \otimes c_3$ corresponds to the symbol $\text{Thom}(N, J) \otimes L$ on $N = M \times X_P$.

Let $\kappa_0$ and $\kappa_\rho$ be the Kirwan vector fields associated to the functions $\|\Phi_1\|^2$ and $\|\Phi_1\|^2$ on $M \times X_P$:
\[
\kappa_0(m, x) = \left( \Phi_1(m, x) \cdot m ; \Phi_1(m, x) \cdot \eta \right), \quad \kappa_\rho(m, x) = \kappa_0(m, x) + \rho(0, \Phi_1(x) \cdot x).
\]

Let $e^{s_{\rho}}$ be the symbol $c$ pushed by the vector field $\kappa_\rho$: we have
\[
e^{s_{\rho}}(v; \eta) = c_1(v - \kappa_I) \odot c_3(\eta - \kappa_S - \rho \kappa_G)
\]
for $(v; \eta) \in T_{(m, x)}(M \times X_P)$.

The character $Q_{K_1 \times K_2}^{\Phi_1}(U_{\epsilon})$ is given by the index of the $K_1$-transversally elliptic symbol $e^{s_{\rho}}|_{U_{\epsilon}}$. The character $Q_{K_1 \times K_2}(M \times X_P)$ is given by the index of the $K_1 \times K_2$-transversally elliptic symbol $e^{s_{\rho}}$.

**Lemma 3.7.** There exists $\rho(\epsilon) > 0$ such that
\[
\text{Cr}(\|\Phi_1\|^2) \cap \{ \|\Phi_1\|^2 \leq \epsilon \} \subset \left\{ \|\Phi_1\|^2 \leq \frac{\epsilon}{2} \right\}
\]
for any $0 \leq \rho \leq \rho(\epsilon)$.

**Proof.** With the help of Riemannian metrics on $M$ and $X_P$ we define
\[
a(\epsilon) := \inf_{\epsilon/2 \leq \|\Phi_1(m, x)\| \leq \epsilon} \|e^{0}(m, x)\|
\]
and
\[
b := \sup_{x \in X_P} \|\Phi_1(x) \cdot x\|.
\]
We have $a(\epsilon) > 0$ thanks to (3.47), and $b < \infty$ since $X_P$ is compact. It is now easy to check that $\{ \kappa_\rho = 0 \} \cap \{ \epsilon/2 \leq \|\Phi_1\|^2 \leq \epsilon \} = \emptyset$ if $0 \leq \rho < \frac{a(\epsilon)}{b}$.

The symbols $e^{s_{\rho}}|_{U_{\epsilon}}$, $\rho \in [0, \rho(\epsilon)]$ are $K_1 \times K_2$-transversally elliptic, and they define the same class in $K_{K_1 \times K_2}(T_{K_1 \times K_2}(U_{\epsilon}))$. Hence $Q_{K_2}(M_P)$ can be computed as the $K_1$-invariant part of
\[
Q_{K_1 \times K_2}^{\Phi, \rho}(M_P) := \text{Index}_{K_1 \times K_2}^{U_{\epsilon}}(e^{s_{\rho}}|_{U_{\epsilon}}) \in R^{-\infty}(K_1 \times K_2)
\]
for $\rho \in [0, \rho(\epsilon)]$.\[\square\]
A component $K_1 \times K_2 \cdot \mathcal{C}$ of $\text{Cr}(\|\Phi\|_p^2)$ is contained in $U$, if and only $\|\gamma_1\| < \epsilon$: hence the decomposition (3.43) for the character $Q_{K_1 \times K_2}^\Phi (M \times X_P)$ gives

$$Q_{K_1 \times K_2}^\Phi (M \times X_P) = Q_{K_1 \times K_2}^\Phi (U) + \sum_{\gamma \in \mathcal{P}_p, \|\gamma_1\|^2 \geq \epsilon} Q_{K_1 \times K_2}^\gamma (M \times X_P).$$

where

$$Q_{K_1 \times K_2}^\phi (U) = \sum_{\|\gamma_1\|^2 < \epsilon} Q_{K_1 \times K_2}^\gamma (M \times X_P).$$

Taking the $K_1$-invariant gives

$$[Q_{K_1 \times K_2}^\Phi (M \times X_P)]^{K_1} = Q_{K_1} (M_P) + \sum_{\gamma \in \mathcal{P}_p, \|\gamma_1\|^2 \geq \epsilon} [Q_{K_1 \times K_2}^\gamma (M \times X_P)]^{K_1}.$$  

In general we know that the support of the generalized character $[Q_{K_1 \times K_2}^\gamma (M \times X_P)]^{K_1}$ is included in $\{b \in K_2 \mid \rho\|b\|^2 \geq \|\gamma_1\|^2 + \rho\|\gamma_2\|^2\}$. When $\|\gamma_1\|^2 \geq \epsilon$ we have then that the support of $[Q_{K_1 \times K_2}^\gamma (M \times X_P)]^{K_1}$ is contained in $\{b \in K_2 \mid \rho\|b\|^2 \geq \epsilon\}$.

Finally (3.43) imposes that

$$[Q_{K_1 \times K_2}^\Phi (M \times X_P)]^{K_1} = Q_{K_1} (M_P) + O((\epsilon/\rho)^{1/2}).$$

when $0 < \rho \leq \rho(\epsilon)$, which is the precise content of Proposition 3.4.

4. Other properties of $Q^\Phi$

Let $(M, \omega, \Phi)$ be a proper Hamiltonian $K$-manifold which is prequantized by a line bundle $L$. The character $Q_k^\Phi (M)$ is computed by means of a scalar product on $\mathfrak{k}^*$. The fact that $Q_k^\Phi (M) = Q_k^{\Phi^\infty} (M)$ gives the following

Proposition 4.1. The character $Q_k^\Phi (M)$ does not depend of the choice of a scalar product on $\mathfrak{k}^*$

In this section we work in the setting where $K = K_1 \times K_2$. Let $\Phi_1$ be the moment map relative to the $K_1$-action.

4.1. $\Phi_1$ is proper. In this subsection we suppose that the moment map $\Phi_1$ relative to the $K_1$-action is proper. We fix an invariant Euclidean norm $\|\cdot\|^2$ on $\mathfrak{k}$ in such a way that $\mathfrak{k}_1 = \mathfrak{k}_2$.

Let us “quantize” $(M, \Omega)$ via the invariant proper function $\|\Phi_1\|^2$: let

$$Q_{K_1 \times K_2}^{\Phi_1} (M) \in R^{-\infty} (K_1 \times K_2)$$

be the corresponding generalized character.

Theorem 4.2. We have

$$Q_{K_1 \times K_2}^\Phi (M) = Q_{K_1 \times K_2}^{\Phi_1} (M) \in R^{-\infty} (K_1 \times K_2).$$
Proof. On $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ we may consider the family of invariant Euclidean norms: 

$$||X_1 \oplus X_2||_\rho^2 = ||X_1||^2 + \rho ||X_2||^2$$

for $X_j \in \mathfrak{k}_j$. Let 

$$Q^\Phi_{\rho, K_1 \times K_2}(M) \in R^{-\infty}(K_1 \times K_2)$$

be the quantization of $M$ computed via the map $||\Phi||^2_\rho = ||\Phi_1||^2 + \rho ||\Phi_2||^2$. By definition, $Q^\Phi_{\rho_1, K_1 \times K_2}(M)$ is equal to $Q^\Phi_{\rho_0, K_1 \times K_2}(M)$, and we know after Proposition [1.2] that $Q^\Phi_{\rho, K_1 \times K_2}(M)$ coincides with the generalized character $Q^\Phi_{\rho, K_1 \times K_2}(M) \in R^{-\infty}(K)$ for any $\rho > 0$.

Let us prove that prove that $Q^\Phi_{\rho, K_1 \times K_2}(M) = Q^\Phi_{\rho, K_1 \times K_2}(M)$. We denote $O(r) \in R^{-\infty}(K_1 \times K_2)$ any generalized character supported outside the ball

$$\{ \xi \in \mathfrak{t}_1^* \times \mathfrak{t}_2^* \mid ||\xi_1||^2 + ||\xi_2||^2 < r^2 \}.$$ 

And we denote $O_1(r) \in R^{-\infty}(K_1 \times K_2)$ any generalized character supported outside the

$$\{ \xi \in \mathfrak{t}_1^* \times \mathfrak{t}_2^* \mid ||\xi_1|| < r \}.$$ 

Let $R_1 > 0$ be a regular value of $||\Phi_1||^2$: the open subset $\{ ||\Phi_1||^2 < R_1 \}$ is denoted $M < R_1$. We know that

$$Q^\Phi_{\rho, K_1 \times K_2}(M) = Q^\Phi_{\rho, K_1 \times K_2}(M < R_1) + O_1(\sqrt{R_1}).$$

Like in the Lemma [1.7], we know that

$$\text{Cr}(||\Phi||_\rho^2) \cap \{ ||\Phi_1||^2 = R_1 \} = \emptyset.$$ 

for $\rho \geq 0$ small enough. The identity [1.5(1)] first implies that

$$Q^\Phi_{\rho, K_1 \times K_2}(M) = \sum_{\gamma \in \mathbb{G}_\rho} Q^\gamma_{\rho, K_1 \times K_2}(M) + \sum_{\gamma \in \mathbb{G}_\rho} Q^\gamma_{\rho, K_1 \times K_2}(M) \mid \frac{||\gamma||^2}{||\gamma||} < R_1$$

$$= Q^\Phi_{\rho, K_1 \times K_2}(M < R_1) + O(\sqrt{R_1}).$$

In the second equality we have used that $Q^\gamma_{\rho, K_1 \times K_2}(M) \sim O(\sqrt{R_1})$ when $||\gamma||^2 > R_1$ since the ball $\{ (\xi_1, \xi_2) \in \mathfrak{t}_1^* \times \mathfrak{t}_2^* \mid ||\xi_1||^2 + ||\xi_2||^2 < R_1 \}$ is contained in

$$\{ (\xi_1, \xi_2) \in \mathfrak{t}_1^* \times \mathfrak{t}_2^* \mid ||(\xi_1, \xi_2)||^2 < ||(\gamma_1, \gamma_2)||^2_\rho \}.$$ 

The identity [1.5(1)] shows also that the symbol $e^{\rho \gamma} \mid M < R_1$, are homotopic for $\rho \geq 0$ small enough. Hence

$$Q^\Phi_{\rho, K_1 \times K_2}(M < R_1) = Q^\Phi_{\rho, K_1 \times K_2}(M < R_1).$$

We get finally that $Q^\Phi_{\rho, K_1 \times K_2}(M) = Q^\Phi_{\rho, K_1 \times K_2}(M) = O(\sqrt{R_1}) + O_1(\sqrt{R_1})$ for any regular value $R_1$ of $||\Phi_1||^2$. We have proved that $Q^\Phi_{\rho, K_1 \times K_2}(M) = Q^\Phi_{\rho, K_1 \times K_2}(M) = 0$. \hfill \square

Let us explain how Theorem [1.2] contains the identity that we called “quantization commutes with reduction in the singular setting” in [2]. By definition the $K_1$-invariant part of the right hand side of (4.43) is equal to the geometric quantization of the (possibly singular) compact Hamiltonian $K_2$-manifold

$$M \parallel_0 K_1 := \Phi_1^{-1}(0)/K_1.$$
Using now the fact that the left hand side of (4.49) is equal to $Q_{K_1 \times K_2}(M)$, we see that the multiplicity of $V^K_{\mu}$ in $Q_{K_2}(M/_{0}K_1)$ is equal to the geometric quantization of the (possibly singular) compact manifold

$$M \times \overline{\mu/_{(0,\mu)}K_1 \times K_2}.$$ 

4.2. The symplectic reduction $M/_{0}K_1$ is smooth. Let $(M, \Omega)$ be an Hamiltonian $K_1 \times K_2$-manifold with a proper moment map $\Phi = (\Phi_1, \Phi_2)$. In this section we suppose that 0 is a regular value of $\Phi_1$ and that $K_1$ acts freely on $\Phi_1^{-1}(0)$. We work then with the (smooth) Hamiltonian $K_2$-manifold

$$N := \Phi_1^{-1}(0)/K_1.$$ 

We still denote by $\Phi_2 : N \to \mathfrak{t}_2^*$ the moment map relative to the $K_2$-action: note that this map is proper. Hence we can quantize the $K_2$-action on $N$ via the map $\Phi_2$. Let $Q_{K_2}(N) \in R^{-\infty}(K_2)$ be the corresponding character.

**Proposition 4.3.** We have

$$[Q_{K_1 \times K_2}(M)]^{K_1} = Q_{K_2}(N) \in R^{-\infty}(K_2).$$

**Proof.** When $\Phi_1$ is proper, the manifold $N$ is compact. Then the right hand side of (4.51) is equal to $Q_{K_2}(N)$, and we know from Theorem 4.2 that the left hand side of (4.51) is equal to $[Q_{K_1 \times K_2}(M)]^{K_1}$. In this case (4.51) becomes $[Q_{K_1 \times K_2}(M)]^{K_1} = Q_{K_2}(M/_{0}K_1)$ which is the content of Theorem 2.13.

Let us consider the general case where $\Phi_1$ is not proper. Thanks to Theorem 1.4 one knows that the multiplicities of $V^K_{\mu}$ in $[Q_{K_1 \times K_2}(M)]^{K_1}$ and $Q_{K_2}(N)$ are respectively equal to the quantization of the (possibly singular) symplectic reductions

$$\mathcal{M}_\mu := M \times \overline{\mu/_{(0,\mu)}K_1 \times K_2},$$

and

$$\mathcal{M}_\mu' := N \times \overline{\mu/_{0}K_2}, \text{ with } N = M/_{0}K_1.$$ 

Note that $\mathcal{M}_\mu$ and $\mathcal{M}_\mu'$ coincide as symplectic reduced space. Let us prove that their geometric quantization are identical also. The proof will be done for $\mu = 0$: the other case follows from the shifting trick.

Let $c$ be the $K_1 \times K_2$-equivariant symbol $\text{Thom}(M, J) \otimes L_M$. Let $\kappa$ be the Kirwan vector field attached to the moment map $\Phi = (\Phi_1, \Phi_2)$. Let $c^\kappa$ be the symbol $c$ pushed by $\kappa$. Let us denote $M_{<\epsilon}$ the open subset $\{||\Phi||^2 < \epsilon\}$. For $\epsilon > 0$ small enough, the symbol $c^\kappa|_{M_{<\epsilon}}$ is $K_1 \times K_2$-transversally elliptic, and $Q(M_0)$ is the $K_1 \times K_2$-invariant part of $\text{Index}_{\mu/_{0}K_1}(c^\kappa|_{M_{<\epsilon}})$.

Let $c_2$ be the $K_2$-equivariant symbol $\text{Thom}(N, J) \otimes L_N$. Let $\kappa_2$ be the Kirwan vector field attached to the moment map $\Phi_2$. Let $c_2^\kappa$ be the symbol $c_2$ pushed by $\kappa_2$. Let us denote $N_{<\epsilon}$ the open subset $\{||\Phi_2||^2 < \epsilon\}$. For $\epsilon > 0$ small enough, the symbol $c_2^\kappa|_{N_{<\epsilon}}$ is $K_2$-transversally elliptic, and $Q(M'_0)$ is the $K_2$-invariant part of $\text{Index}_{\mu/_{0}K_2}(c_2^\kappa|_{N_{<\epsilon}})$.

Our proof follows from the comparison of the classes

$$[c^\kappa|_{M_{<\epsilon}}] \in K_{K_1 \times K_2}(T_{K_1 \times K_2}M_{<\epsilon})$$

and

$$[c_2^\kappa|_{N_{<\epsilon}}] \in K_{K_2}(T_{K_2}N_{<\epsilon})$$

where $\kappa$ and $\kappa_2$ are the moment maps for $K_1 \times K_2$ and $K_2$, respectively.
A neighborhood of the smooth submanifold $Z := \Phi^{-1}_1(0)$ in $M$ is diffeomorphic to a neighborhood of the 0-section of the bundle $Z \times \mathfrak{t}_1^* \to Z$. Let $Z_{<\varepsilon} = Z \cap M_{<\varepsilon}$ so that $N_{<\varepsilon} = Z_{<\varepsilon}/K_1$. Hence $[c^a|_{M_{<\varepsilon}}]$ can be seen naturally a class in the $K$-group $K_{K_1 \times K_2}(T_{K_1 \times K_2}(Z_{<\varepsilon} \times \mathfrak{t}_1^*))$.

Following Atiyah [2, Theorem 4.3], the inclusion map $j : Z_{<\varepsilon} \hookrightarrow Z_{<\varepsilon} \times \mathfrak{t}_1^*$ induces the Thom isomorphism

$$j_! : K_{K_1 \times K_2}(T_{K_1 \times K_2}Z_{<\varepsilon}) \longrightarrow K_{K_1 \times K_2}(T_{K_1 \times K_2}(Z_{<\varepsilon} \times \mathfrak{t}_1^*)) ,$$

with the commutative diagram

$$\begin{array}{ccc}
K_{K_1 \times K_2}(T_{K_1 \times K_2}Z_{<\varepsilon}) & \xrightarrow{j_!} & K_{K_1 \times K_2}(T_{K_1 \times K_2}(Z_{<\varepsilon} \times \mathfrak{t}_1^*)) \\
\text{Index}_{Z_{<\varepsilon}}^{K_1 \times K_2} & \downarrow & \text{Index}_{Z_{<\varepsilon} \times \mathfrak{t}_1^*}^{K_1 \times K_2} \\
& R^{-\infty}(K_1 \times K_2) & \\
\end{array}$$

Let $\pi_1 : Z_{<\varepsilon} \to N_{<\varepsilon}$ be the quotient relative to the free action of $K_1$. The corresponding isomorphism

$$\pi_1^* : K_{K_2}(T_{K_2}N_{<\varepsilon}) \longrightarrow K_{K_1 \times K_2}(T_{K_1 \times K_2}Z_{<\varepsilon})$$

satisfies the following rule:

$$(4.53) \quad \left[ \text{Index}_{Z_{<\varepsilon}}^{K_1 \times K_2}(\pi_1^* \theta) \right]_{K_1} = \text{Index}_{K_2}^{N_{<\varepsilon}}(\theta)$$

for any $\theta \in K_{K_2}(T_{K_2}N_{<\varepsilon})$.

**Lemma 4.4** ([19]). We have

$$j_! \circ \pi_1^* \left( [c_2^a|_{N_{<\varepsilon}}] \right) = [c^a|_{M_{<\varepsilon}}]$$

in $K_{K_1 \times K_2}(T_{K_1 \times K_2}(Z_{<\varepsilon} \times \mathfrak{t}_1^*))$.

**Proof.** This Lemma is proven in [19, Section 6.2] when the group $K_2$ is trivial. It is easy to check that the proof extends naturally to our setting. \qed

If one uses Lemma 4.4 together with (1.53) and (4.53), we get that

$$Q(M_0) = \left[ \text{Index}_{Z_{<\varepsilon}}^{K_1 \times K_2}(c^a|_{M_{<\varepsilon}}) \right]^{K_1 \times K_2} = \left[ \text{Index}_{N_{<\varepsilon}}^{K_2}(c_2^a|_{N_{<\varepsilon}}) \right]^{K_2} = Q(M_0).$$

\qed

5. Example: the cotangent bundle of an orbit

5.1. **The formal quantization of $T^* K$.** Let $K$ be a compact connected Lie group equipped with the action of two copies of $K$: $(k_1, k_2) \cdot a = k_2ak_1^{-1}$. Then we have a Hamiltonian action of $K_1 \times K_2$ on the cotangent bundle $T^* K$. In this section, we check that each formal geometric quantization of $T^* K$, $Q^{-\infty}_{K_1 \times K_2}(T^* K)$ and $Q^\Phi_{K_1 \times K_2}(T^* K)$, are both equal to the $K_1 \times K_2$-module $L^2(K)$. 


The tangent bundle $TK$ is identified with $K \times \mathfrak{k}$ through the right translations: to $(a, X) \in K \times \mathfrak{k}$ we associate $\frac{d}{dt}ae^{tX}|_0$. The action of $K_1 \times K_2$ on the cotangent bundle $T^*K \simeq K \times \mathfrak{k}^*$ is then

$$(k_1, k_2) \cdot (a, \xi) = (k_2ak_1^{-1}, k_1 \cdot \xi).$$

The symplectic form on $T^*K$ is $\Omega := -d\lambda$, where $\lambda$ is the Liouville 1-form. Let us compute these two form in coordinates. The tangent bundle of $T^*K \simeq K \times \mathfrak{k}^*$ is identified with $T^*\mathfrak{k} \times \mathfrak{k} \times \mathfrak{k}^*$. For each $(a, \xi) \in T^*K$, we have a two form $\Omega(a, \xi)$ on $\mathfrak{k} \times \mathfrak{k}^*$. A direct computation gives

$$\Omega(a, \xi)(X, Y) = \langle \xi, [X, Y] \rangle, \quad \Omega(a, \xi)(\eta, \eta') = 0, \quad \Omega(a, \xi)(X, \eta) = \langle \eta, X \rangle$$

for $X, Y \in \mathfrak{k}$ and $\eta, \eta' \in \mathfrak{k}^*$. So $\Omega(a, \xi) = \Omega_0 + \pi_\xi$ where $\Omega_0$ is the canonical (constant) symplectic form on $\mathfrak{k} \times \mathfrak{k}^*$ and $\pi_\xi$ is the closed two form on $\mathfrak{k}$ defined by $\pi_\xi(X, Y) = \langle \xi, [X, Y] \rangle$.

If we identify $\mathfrak{k} \simeq \mathfrak{k}^*$ through an invariant Euclidean norm, the symplectic structure on $T(a, \xi)(T^*K) \simeq \mathfrak{k} \times \mathfrak{k}^*$ is given by a skew-symmetric matrix

$$A_\xi := \begin{pmatrix} \text{ad}(\xi) & I_n \\ -I_n & 0 \end{pmatrix}.$$ 

We will work with the following compatible almost complex structure on the tangent bundle of $T^*K$: $J_\xi = -A_\xi(-A_\xi^2)^{-1/2}$. When $\xi = 0$, the complex structure $J_0$ on $\mathfrak{k} \times \mathfrak{k}^*$ is defined by the matrix

$$J_0 := \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}.$$ 

Hence the complex $K$-module $(\mathfrak{k} \times \mathfrak{k}^*, J_0)$ is naturally identified with the complexification $\mathfrak{t} \mathfrak{k}$ of $\mathfrak{k}$.

One checks easily that the moment map relative to the $K_1 \times K_2$-action is the proper map $\Phi : T^*K \to \mathfrak{t}_1^* \times \mathfrak{t}_2^*$ defined by $\Phi(a, \xi) = (-\xi, a \cdot \xi)$.

Here the symplectic manifold $T^*K$ is prequantized by the trivial line bundle.

### 5.1.1. Computation of $Q_{K_1 \times K_2}^\infty(T^*K)$. Let $\mathcal{O}_1 \times \mathcal{O}_2$ be a coadjoint orbit of $K_1 \times K_2$ in $\mathfrak{t}_1^* \times \mathfrak{t}_2^*$. One checks that

$$\Phi^{-1}(\mathcal{O}_1 \times \mathcal{O}_2) = \begin{cases} \emptyset & \text{if } \mathcal{O}_1 \neq -\mathcal{O}_2 \\ \{ K_1 \times K_2 - \text{orbit} \} & \text{if } \mathcal{O}_1 = -\mathcal{O}_2. \end{cases}$$

We knows that the stabiliser subgroup $K_\xi$ of an element $\xi \in \mathfrak{k}^*$ is connected. Then the stabiliser subgroup $(K_1 \times K_2)(a, \xi) = \{(k_1, ak_1a^{-1}), k_1 \in K_\xi \}$ is also connected.

Let $(T^*K)(\mu, \lambda)$ be the symplectic reduction of $T^*K$ at the level $(\mu, \lambda) \in \widehat{K}^2$. For any $\mu \in \widehat{K}$, we define $\mu^* \in \widehat{K}$ by the relation $-K \cdot \mu = K \cdot \mu^*$: note that $V^*_{\mu^*} \simeq (V^*_{\mu})^*$. If one uses Theorem 2.14, one has

$$Q((T^*K)(\mu, \lambda)) = \begin{cases} 0 & \text{if } \lambda \neq \mu^* \\ 1 & \text{if } \lambda = \mu^*. \end{cases}$$
Finally
\[
Q_{K_1 \times K_2}^{-\infty} (T^* K) = \sum_{(\mu, \lambda) \in \tilde{K} \times \tilde{K}} Q \left( (T^* K)_{(\mu, \lambda)} \right) V_{\mu}^{K_1} \otimes V_{\lambda}^{K_2}
\]
\[
= \sum_{\mu \in \tilde{K}} V_{\mu}^{K_1} \otimes (V_{\mu}^{K_2})^* = L^2 (K).
\]

5.1.2. Computation of \( Q_{K_1 \times K_2}^{c} (T^* K) \). The Kirwan vector field on \( T^* K \) is
\[
\kappa (a, \xi) = -2 \xi \in \mathfrak{k}.
\]

Let \( \mathcal{C}^c \) be the symbol \( \text{Thom}(T^* K, J) \) pushed by the vector field \( \frac{1}{h} \mathcal{C} \). At each \((a, \xi) \in T^* K\), the map \( \mathcal{C}^c_{a, \xi} (X \oplus \eta) \) from \( \wedge^{even}_J (t \times t^*) \) to \( \wedge^{odd}_J (t \times t^*) \) is equal to the clifford map \( \mathcal{C}(X + \xi \oplus \eta) \). Note that \( \mathcal{C}^c \) is a \( K_2 \)-transversally elliptic symbol on \( T^* K \): we have \( \text{Char}(\mathcal{C}^c) \cap T_{K_2} (T^* K) = \{ (1, 0) \} \). We will now compute the equivariant index of \( \mathcal{C}^c \).

First we consider the homotopy \( t \in [0, 1] \rightarrow J_{t \xi} \) of symplectic structure on \( T^* K \). Let \( \tilde{\mathcal{C}}^c \) be the symbol acting on \( \wedge^{even}_\mathfrak{c} (t \times t^*) \) and \( \tilde{\mathcal{C}}^c \) define the same class in \( K_{K_1 \times K_2}(T_{K_2} (T^* K)) \).

The projection \( \pi : T^* K \rightarrow t^* \) corresponds to the quotient map relative to the free action of \( K_2 \). At the level of \( K \)-groups we get an isomorphism
\[
\pi^* : K_{K_1 \times K_2} (T_{K_2} (T^* K)) \rightarrow K_{K^1} (T^* t).
\]

Atiyah \( 1 \) proves that
\[
\text{Index}^{T^* K}_{K_1 \times K_2} (\sigma) = \sum_{\mu \in \tilde{K}} \text{Index}^{K_1}_{\mu} \left( \pi^* (\sigma \otimes V_{\mu}^{K_2}) \right) \otimes (V_{\mu}^{K_2})^*
\]
for any class \( \sigma \in K_{K_1 \times K_2}(T_{K_2} (T^* K)) \). In our case the symbol \( \pi^* (\tilde{\mathcal{C}}^c) \) is equal to the Bott symbol \( \text{Bott}(t^*) \), and for any \( K_2 \)-module \( E_2 \) we have
\[
\pi^* (\tilde{\mathcal{C}}^c \otimes E_2) = \text{Bott}(t^*) \otimes E_1
\]
where \( E_1 \) is the module \( E_2 \) with the action of \( K_1 \). Then
\[
Q_{K_1 \times K_2}^{c} (T^* K) = \text{Index}^{T^* K}_{K_1 \times K_2} (\tilde{\mathcal{C}}^c)
\]
\[
= \sum_{\mu \in \tilde{K}} \text{Index}^{K_1}_{\mu} \left( \text{Bott}(t^*) \otimes V_{\mu}^{K_1} \right) \otimes (V_{\mu}^{K_2})^*
\]
\[
= \sum_{\mu \in \tilde{K}} V_{\mu}^{K_1} \otimes (V_{\mu}^{K_2})^* = L^2 (K),
\]
since \( \text{Index}^{K_1}_{\mu} (\text{Bott}(t^*)) = 1 \).

5.2. The formal quantization of \( T^*(K/H) \). Let \( H \) be a closed connected subgroup of \( K \). Look at \( T^* K \) as a Hamiltonian manifold relatively to the action of \( H \times K \subset K_1 \times K_2 \). The moment map \( \Phi = (\Phi_H, \Phi_K) \) is defined by \( \Phi_H (a, \xi) = -\text{pr}(\xi) \) and \( \Phi_K (a, \xi) = a \cdot \xi \), where \( \text{pr} : t^* \rightarrow \mathfrak{h}^* \) is the projection. Note that \( \Phi \) is a proper map.

The cotangent bundle \( T^*(K/H) \), viewed as \( K \)-manifold, is equal to the symplectic reduction of \( T^* K \) relatively to the \( H \)-action: if the kernel of the projection \( \text{pr} \) is denoted \( \mathfrak{h}^* \), we have
\[
\Phi_H^{-1}(0)/H = K \times_H \mathfrak{h}^* = T^*(K/H).
\]
We are here in the setting of section 5.2. The reduction of the $H \times K$ proper Hamiltonian manifold $T^*K$ relatively to the $H$-action is smooth, then its formal quantization is computed as follows

$$Q^\mathcal{H}_K(T^*(K/H)) = \left[Q^\mathcal{H}_{H\times K}(T^*K)\right]^H = \left[Q^\mathcal{H}_{K_1\times K_2}(T^*K)|_{H\times K}\right]^H$$

(5.56)

Here the fact that $Q^\mathcal{H}_{H\times K}(T^*K)$ is equal to the restriction of $Q^\mathcal{H}_{K_1\times K_2}(T^*K) = L^2(K)$ to $H \times K$ is a consequence of Theorem 1.3.

Let us denoted $\left[T^*(K/H)\right]_\mu$ the symplectic reduction at $\mu \in \widehat{K}$ of the K-Hamiltonian manifold $T^*(K/H)$. Theorem 1.3 together with (1.4) gives

**Corollary 5.1.** For any $\mu \in \widehat{K}$, we have

$$Q\left[T^*(K/H)\right]_\mu = \dim \left[V^K_\mu\right]^H,$$

where $[V^K_\mu]^H$ is the subspace of $H$-invariant vector.

5.3. The formal quantization of $T^*(K/H)$ relatively to the action of $G$. Let $G$ be a closed connected subgroup of $K$. We look at the hamiltonian action of $G$ on $T^*(K/H)$. Let $\Phi_G : T^*(K/H) \to \mathfrak{g}^*$ be the moment map. We consider also the restriction of the K-module $L^2(K/H)$ to $G$.

We have

**Proposition 5.2.** The following statements are equivalent

1. The moment map $\Phi_G : T^*(K/H) \to \mathfrak{g}^*$ is proper.
2. $\Phi_G^{-1}(0)$ is equal to the zero section.
3. $k \cdot \mathfrak{g} + \mathfrak{h} = \mathfrak{k}$, for any $k \in K$.
4. $\mathfrak{g} + \mathfrak{h} = \mathfrak{k}$
5. $G$ acts transitively on $K/H$.
6. $L^2(K/H)^G \simeq \mathbb{C}$
7. $L^2(K/H)|_G$ is an admissible $G$-representation.

Proof. (1) $\implies$ (7) is a consequence of Theorem 1.3. Let us prove that (7) $\implies$ (6). Suppose now that

$$L^2(K/H)|_G = \sum_{\mu \in \widehat{K}} [V^K_\mu]^H \otimes \left(V^K_\mu\right)^*|_G$$

is an admissible $G$-representation. It means that for any $\lambda \in \widehat{G}$ the set

$$A_\lambda := \left\{ \mu \in \widehat{K} \mid [V^K_\mu]^H \neq \{0\} \text{ and } \left[(V^K_\lambda)^* \otimes \left(V^K_\mu\right)^*|_G\right] \neq \{0\} \right\}$$

is finite. Then the vector space $L^2(K/H)^G$ is equal to the finite dimensional vector space $\sum_{\mu \in A_0} [V^K_\mu]^H \otimes \left(V^K_\mu\right)^*G$. It is not difficult to check that if $\mu \in A_0$, then $k\mu \in A_0$ for $k >> 1$. Finally the fact that $A_0$ is finite implies that $A_0$ is reduced to $\mu = 0$. Hence the only $G$-invariant functions on $K/H$ are the scalars.

(6) $\iff$ (5) $\iff$ (4) $\iff$ (3) is a general fact concerning smooth actions of a compact connected Lie group $G$ on a compact connected manifold $M$. The manifold $M$ does not have $G$-invariant functions which are not scalar if and only if the action of $G$ on $M$ is transitive. And given a point $m \in M$, the orbit $G \cdot m$ is all of $M$ if
and only if tangent spaces $T_m(G \cdot m)$ and $T_mM$ are equal. If we take $m = \overline{k^{-1}}$ in $M = K/H$, the condition $T_m(G \cdot m) = T_mM$ is equivalent to $k \cdot g + h = f$.

Let us check $(3) \implies (2)$. Let $[k, \xi] \in K \times H \cdot h^\perp = \mathcal{T}^*(K/H)$. We have $\Phi_{G}([k, \xi]) = 0$ if and only if $k \cdot \xi \in g^\perp$. Hence the vector $\xi$ belongs to 

$$k^{-1} \cdot g^\perp \cap h^\perp = (k^{-1} \cdot g + h)^\perp.$$ 

Hence condition $(3)$ imposes that $\xi = 0$.

$(2) \iff (1)$ comes from the fact that $\Phi_{G}$ is a homogeneous map of degree one between the vector bundle $\mathcal{T}^*(K/H)$ and the vector space $g^\ast$. $\square$

Suppose now that the cotangent bundle $\mathcal{T}^*(K/H)$ is a proper Hamiltonian $G$-manifold. Let us denote $[\mathcal{T}^*(K/H)]_{\mu,G}$ the (compact) symplectic reduction at $\mu \in \hat{G}$ of the G-Hamiltonian manifold $\mathcal{T}^*(K/H)$. Then,

**Corollary 5.3.** The multiplicity of $V_{\mu}^G$ in $L^2(K/H)$ is equal to the quantization of the reduced space $[\mathcal{T}^*(K/H)]_{\mu,G}$.

**Proof.** Using Theorem 5.3, equality (5.56) gives then

$$Q_{\mu}^G(\mathcal{T}^*(K/H)) = Q_{\mu}^G(\mathcal{T}^*(K/H))_{\mu,G} = L^2(K/H)_{\mu,G}.$$ 

In other words, the multiplicity of $V_{\mu}^G$ in $L^2(K/H)$ is equal to the quantization of the reduced space $[\mathcal{T}^*(K/H)]_{\mu,G}$. $\square$

**References**

[1] M.F. Atiyah, Elliptic operators and compact groups, Springer, 1974. Lecture notes in Mathematics, 401.
[2] M.F. Atiyah, G.B. Segal, The index of elliptic operators II, Ann. Math. 87, 1968, p. 531-545.
[3] M.F. Atiyah, I.M. Singer, The index of elliptic operators I, Ann. Math. 87, 1968, p. 484-530.
[4] M.F. Atiyah, I.M. Singer, The index of elliptic operators III, Ann. Math. 87, 1968, p. 546-604.
[5] M.F. Atiyah, I.M. Singer, The index of elliptic operators IV, Ann. Math. 93, 1971, p. 139-141.
[6] N. Berline and M. Vergne, The Chern character of a transversally elliptic symbol and the equivariant index, Invent. Math., 124, 1996, p. 11-49.
[7] N. Berline and M. Vergne, L’indice équivariant des opérateurs transversalement elliptiques, Invent. Math., 124, 1996, p. 51-101.
[8] J. J. Duistermaat, The heat equation and the Lefschetz fixed point formula for the Spin$^c$-Dirac operator, Progress in Nonlinear Differential Equation and Their Applications, vol. 18, Birkhauser, Boston, 1996.
[9] V. Guillemin and S. Sternberg, Geometric quantization and multiplicities of group representations, Invent. Math., 67, 1982, p. 515-538.
[10] V. Guillemin and S. Sternberg, A normal form for the moment map, in Differential Geometric Methods in Mathematical Physics(S. Sternberg, ed.), Reidel Publishing Company, Dordrecht, 1984.
[11] V. Guillemin and S. Sternberg, Symplectic techniques in physics, Cambridge University Press, Cambridge, 1990.
[12] B. Kostant, Quantization and unitary representations, in Modern Analysis and Applications, Lecture Notes in Math., Vol. 170, Springer-Verlag, 1970, p. 87-207.
[13] E. Lerman, Symplectic cut, Math Res. Lett. 2, 1995, p. 247-258.
[14] E. Lerman, E. Meinrenken, S. Tolman and C. Woodward, Non-Abelian convexity by symplectic cuts, Topology, 37, 1998, p. 245-259.
[15] X. Ma, W. Zhang, Geometric quantization for proper moment map, Arxiv:0812.3989
[16] E. Meinrenken, On Riemann-Roch formulas for multiplicities, J. Amer. Math. Soc., 9, 1996, p. 373-389.

[17] E. Meinrenken, Symplectic surgery and the Spin$^c$-Dirac operator, Advances in Math., 134, 1998, p. 240-277.

[18] E. Meinrenken, R. Sjamaar, Singular reduction and quantization, Topology, 38, 1999, p. 699-762.

[19] P-E. Paradan, Localization of the Riemann-Roch character, J. Funct. Anal. 187, 2001, p. 442-509.

[20] P-E. Paradan, Spin$^c$ quantization and the K-multiplicities of the discrete series, Annales Scientifiques de l’E. N. S., 36, 2003, p. 805-845.

[21] P-E. Paradan, Formal geometric quantization, Ann. Inst. Fourier 59, 2009, p. 199-238.

[22] P-E. Paradan M. Vergne, Index of transversally elliptic operators, 40 pages, to appear in Astérisque, Soc. Math. Fr., Arxiv math/08041225.

[23] P-E. Paradan, Multiplicities of the discrete series, 38 pages. ArXiv:0812.0059.

[24] R. Sjamaar, E. Lerman, Stratified symplectic spaces and reduction, Annals of Math., 134, 1991, p. 375-422.

[25] R. Sjamaar, Symplectic reduction and Riemann-Roch formulas for multiplicities, Bull. Amer. Math. Soc. 33, 1996, p. 327-338.

[26] Y. Tian, W. Zhang, An analytic proof of the geometric quantization conjecture of Guillemin-Sternberg, Invent. Math, 132, 1998, p. 229-259.

[27] M. Vergne, Multiplicity formula for geometric quantization, Part I, Part II, and Part III, Duke Math. Journal, 82, 1996, p. 143-179, p 181-194, p 637-652.

[28] M. Vergne, Quantification géométrique et réduction symplectique, Séminaire Bourbaki 888, 2001.

[29] M. Vergne, Applications of Equivariant Cohomology, International Congress of Mathematicians 2006, Vol. I, Eur. Math. Soc., Zürich, 2007, p. 635-664. Arxiv: math/0607389.

[30] A. Weinstein, Lecture on symplectic manifold, CBMS Regional Conf. Series in Math., 29, 1983.

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