Octet Baryon Magnetic Moments from Lattice QCD: Approaching Experiment from the Three-Flavor Symmetric Point

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Lattice QCD calculations with background magnetic fields are used to determine the magnetic moments of the octet baryons. Computations are performed at the physical value of the strange quark mass, and two values of the light quark mass, one corresponding to the $SU(3)_F$ symmetric point, where the pion mass is $m_\pi \sim 800$ MeV, and the other corresponding to a pion mass of $m_\pi \sim 450$ MeV. The moments are found to exhibit only mild pion-mass dependence when expressed in terms of efficaciously chosen magneton units - the natural baryon magneton. Consequently simple extrapolations can be used to determine isovector magnetic moments at the physical point, which are found to agree with experiment within uncertainties. The isoscalar magnetic moments also extrapolate to physical values within uncertainties, which suggests that the omitted quark-disconnected contributions at $m_\pi \sim 450$ MeV are small. A curious pattern is revealed among the anomalous baryon magnetic moments which is linked to the constituent quark model, however, careful scrutiny paradoxically exposes additional features. Relations expected to hold in the large-$N_c$ of QCD are studied, and in one case, a clear preference for the quark model over the large-$N_c$ prediction is found. The more complex magnetically coupled Λ-Σ$^0$ system is treated in detail at the $SU(3)_F$ point, with the LQCD results comparing favorably with predictions based on $SU(3)_F$ symmetry. This analysis enables the first extraction of the isovector transition magnetic polarizability. The possibility that large magnetic fields stabilize strange matter is explored, but we find that such a scenario is unlikely.

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I. INTRODUCTION

The precisely measured values of the magnetic moments of the lowest-lying octet of $J^P = 1^+$ baryons, along with the rate of the radiative transition $\Sigma^0 \rightarrow \Lambda + \gamma$, have been essential in elucidating important aspects of the structure of hadrons. One of the major early successes in the phenomenological modeling of hadrons was recovering the pattern of these magnetic moments from the naïve nonrelativistic quark model (NRQM) [13]. In this model, baryons are comprised of three nonrelativistic constituent quarks with Dirac magnetic moments. When the three quark masses are fit to best reproduce the masses of the octet baryons, the magnetic moments predicted by this simple model reproduce those of nature surprisingly well. The predicted NRQM rate of $\Sigma^0 \rightarrow \Lambda + \gamma$ and of the radiative transitions from the lowest-lying decuplet of $J^P = 1^+$ baryons to the octet baryons are also in impressive agreement with experiment.
Making a closer connection to the underlying theory of the strong interactions, the global flavor symmetries of Quantum Chromodynamics (QCD), the two-flavor isospin SU(2) and chiral SU(2)\(_L\times SU(2)\)_R, as well as the three-flavor analogues SU(3)\(_F\) and SU(3)\(_L\times SU(3)\)_R, have been used to explore the magnetic moments \([6,9]\) and polarizabilities of the electrically neutral baryons \([23–31]\). Techniques developed in these works have also been used to determine the magnetic moments and polarizabilities of light nuclei \([32, 33]\) with LQCD. Further, the pion mass, which is consistent with the observed behavior, but only for a nucleon mass that depends linearly on the pion mass \([16\) and Martinelli et al.\(]\). Newer calculations of the magnetic moments and of the \(\Sigma^+\) magnetic dipole moment \(\mu\) have permitted determinations of both the magnetic moments and polarizabilities, as the effective mass of each spin state reaches a plateau after a modest (but usually different) number of time slices. The magnetic moments of the baryons correspond to just one kinematic point of the magnetic form factor, and the more general behavior of the form factor provides further insight into the distribution of charged currents within the baryon.

There are extensive studies of the baryon electromagnetic form factors, for example Refs. \([39–41]\), from which the magnetic moments and associated radii can be extracted. General nonuniform background fields have been recently proposed to extract higher electromagnetic moments, as well as charge radius from LQCD \([12, 13]\).

Previous LQCD calculations of the magnetic moments of the proton, neutron and light nuclei \([32, 33]\) have found that nearly all of their light-quark mass dependence is captured by the nucleon mass defining the unit of nuclear magnetons. In other words, \(M_N(m_\pi)|\mu_i(m_\pi)|\) is found to be approximately constant over a wide range of pion masses extending up to \(~1\,\text{GeV}\), and possibly beyond, where \(M_N(m_\pi)\) is the mass of the nucleon at a given pion mass, and \(\mu_i(m_\pi)\) is the magnetic moment of the nucleon or nucleus at that same pion mass. This behavior is quite intriguing for a number of reasons. Empirically, it is found that the nucleon mass is essentially linearly dependent on the pion mass for \(m_\pi \geq 250\,\text{MeV}\) with a coefficient very close to unity \([11, 14]\), but expected to tend towards the chiral behavior for smaller pion masses (see Ref. \([10]\) for recent progress). At even larger pion masses, this behavior is expected to evolve toward \(M_N \sim \frac{3}{2} m_\pi\). In the chiral expansion of the nucleon magnetic moments, the leading correction to the \(SU(3)\_F\)-symmetric predictions depends linearly on the pion mass, which is consistent with the observed behavior, but only for a nucleon mass that depends linearly on the pion masses. In the context of the NRQM, \(M_N|\mu_i|\) is required to be approximately independent of the pion mass as the nucleon magnetic moments result from combinations of quark spins, each normalized by the constituent quark mass which is \(~M_N/3\).

In this work, we extend our studies of the magnetic moments of the nucleons and light nuclei to baryons in the lowest-lying octet. Our calculations of the magnetic moments of baryons are accomplished by modifying the LQCD gauge link variables to include a background electromagnetic gauge potential in calculations of the quark propagators. The magnetic moment of a baryon is extracted from the component of the energy-splitting between its two spin states that depends linearly on the magnetic field. Essentially, we compute the Zeeman effect for each baryon. In particular, calculations of their magnetic moments and of the \(\Sigma^0 \rightarrow \Lambda + \gamma\) radiative decay matrix element are performed at the \(SU(3)\_F\) symmetric point with \(m_\pi \sim 800\,\text{MeV}\) at two lattice spacings with the physical strange-quark mass, and further, at a pion mass of \(m_\pi \sim 450\,\text{MeV}\) with the physical strange quark mass. These computations allow for a somewhat detailed
exploration of how the quark masses modify baryon magnetic moments. Disconnected diagram contributions to the magnetic moments are not included in these calculations, which impacts the magnetic moments obtained at \(m_q \sim 450 \text{MeV}\), but their omission is estimated to be small. The main results of this work are summarized as follows.

- Natural baryon magnetons \([nBM]\), where the mass of each baryon is used to define its magnetic moment, are found to capture the majority of the quark-mass dependence of magnetic moments for the entire multiplet, even away from the limit of \(SU(3)_F\) flavor symmetry. In particular, the isovector moments in units of \([nBM]\), which do not receive contributions from omitted disconnected diagrams, are found to exhibit only mild dependence on the pion mass.

- In such \([nBM]\) units, the anomalous moments of the proton and \(\Sigma^+\) are \(\delta\mu_B \sim +2\), of the neutron and \(\Xi^0\) are \(\delta\mu_B \sim -2\), and those of the \(\Sigma^-\) and \(\Xi^-\) are \(\delta\mu_B \sim 0\). Such values are consistent with the \(SU(3)_F\)-symmetric coefficients \([4][6]\) assuming values of \(\mu_D \sim +3\) and \(\mu_F \sim +2\).

- These values for \(SU(3)_F\)-symmetric couplings, and the mild quark-mass dependence they exhibit, are suggestive of the NRQM. The magnetic moment relations predicted by the NRQM are scrutinized, and interesting features are found in the LQCD results.

- Extracting the matrix element of the radiative transition \(\Sigma^0 \rightarrow \Lambda + \gamma\) is found to be more challenging because the \(\Sigma^0\) and \(\Lambda\) are close in mass in the absence of a background magnetic field. At the \(SU(3)_F\)-symmetric point, a matrix of correlation functions is diagonalized to reveal the closely-spaced energy eigenstates, from which this matrix element is determined.

- The predicted large-\(N_c\) relations between magnetic moments, and the parametric scaling of their corrections, are compared with the results of the LQCD calculations. In general, they are found to be compatible. However, one of the relations is violated at the \(\sim 40\%\)-level but, importantly, is found to be much more consistent with the predictions of the NRQM.

In addition to the magnetic moments, higher-order magnetic interactions, such as the magnetic polarizabilities, can also be determined from the LQCD calculations \([33]\). Moreover, the energy dependence of each spin state can be calculated over a range of magnetic fields, allowing for an exploration of the possibility that a large magnetic field could stabilize strange matter in dense astrophysical objects. Our results indicate that considerably larger baryon densities than are conceivably achieved in neutron stars are needed to stabilize strange matter.

The organization of this investigation is arranged as follows. In Sec. I an overview of the computational aspects of our work is given, including details about the QCD-gauge-field configurations employed, the background field implementation, and the expectations for baryon energy levels in magnetic fields. Technical analysis of baryon correlation functions computed with LQCD appear in Appendix A where the fits to ratios of spin-polarized correlators, the extraction of Zeeman splittings, and subsequent fits to the magnetic field dependence of Zeeman splittings, are discussed. Sec. II focuses on the results obtained from LQCD, beginning with a discussion of the units for octet baryon magnetic moments in Sec. II A. The pion-mass dependence of anomalous parts of isovector and isoscalar magnetic moments are investigated in Sec. II B where simple linear extrapolations are performed. In this section, a simple continuum extrapolation for a subset of the LQCD results is investigated. The interpretation of the values of magnetic moments obtained from LQCD and experiment is the subject of Sec. II C where magnetic moment relations are considered, and values of the Coleman-Glashow moments \([6]\), \(\mu_D\) and \(\mu_F\), are extracted. The values obtained are suggestive of the NRQM, however, careful scrutiny of quark-model predictions exposes a few new puzzles. Similarly, interesting features emerge from a comparison to relations between magnetic moments that are satisfied in the large-\(N_c\) limit of QCD. The magnetically coupled system of \(\Lambda\) and \(\Sigma^0\) baryons is studied in Sec. II D. At the \(SU(3)_F\) symmetric point, the breaking of the four-fold degeneracy of spin states as a function of the magnetic field is studied. Analytic results are obtained, and confirmed by LQCD computations. A few necessary technical matters relating to the transition correlation functions are discussed in Appendix A. The lattice-determined, spin-dependent energy levels of hyperons are used to explore the possibility that large magnetic fields might stabilize strange matter in Sec. V. Our presentation ends in Sec. VI with a summary of the main results.

II. COMPUTATIONAL OVERVIEW

In the present study, lattice calculations are performed using three ensembles of QCD gauge configurations. Each ensemble was generated using a Lüscher-Weisz gauge action \([47]\) with tadpole-improved \([48]\), clover-fermion action \([49]\). Configurations used in this work were taken at intervals of ten hybrid Monte Carlo trajectories. A summary of these gauge configurations is provided in Table 1.

Two of the gauge-field ensembles, which we label Ensembles I and II, feature \(N_f = 3\) degenerate, dynamical quark flavors with mass close to that of the physical strange quark. The resulting mass of non-singlet pseudoscalar mesons for these ensembles is found to be
TABLE I. Summary of the three ensembles of QCD gauge-field configurations used in this work. Further details regarding Ensemble I can be found in Ref. [50] [51], while Ensemble III has been detailed in Ref. [53].

| Name | \( L/a \) | \( T/a \) | \( \beta \) | \( a_m \) | \( m_{\pi}\) | \( \text{MeV} \) | \( N_{ch} \) |
|------|----------|----------|----------|--------|--------|----------|--------|
| I    | 32       | 48       | 6.1      | -0.2450 | 0.1453(16) | 806.9(8.9) | 1006   |
| II   | 48       | 64       | 6.3      | -0.2050 | 0.1030(11) | 766.89(8.1) | 94     |
| III  | 32       | 96       | 6.1      | -0.2800 | 0.1167(16) | 449.9(4.6)  | 544    |

\( \sim 800 \text{MeV} \) (Ensemble I) and \( \sim 760 \text{MeV} \) (Ensemble II). Ensemble III has been generated with \( N_f = 2+1 \) dynamical light-quark flavors, where the strange quark mass is taken at its physical value. The isospin-degenerate light-quark flavors, where the strange quark mass is determined using quarkonium hyperfine splittings. In Refs. [34, 53], while Ensemble III has been recently detailed in Refs. [54] [55].

The background magnetic fields are implemented by gauge links in fixed \( U(1) \) electromagnetic links, \( U_{\mu}^{(Q)}(x) \), having the form

\[
U_{1}^{(Q)}(x) = \begin{cases} 
1 & \text{for } x_1 \neq L - a \\
\exp \left( -iQ n_\Phi \frac{2\pi x_1}{L} \right) & \text{for } x_1 = L - a 
\end{cases},
\]

\[
U_{2}^{(Q)}(x) = \exp \left( iQ n_\Phi \frac{2\pi x_1}{L} \right),
\]

\[
U_{3}^{(Q)}(x) = U_{4}^{(Q)}(x) = 1,
\]

where the integer \( n_\Phi \) is the magnetic flux quantum of the torus which satisfies \( |n_\Phi| \leq \frac{1}{4}L^2/a^2 \), see Ref. [51].

Typically, this multiplication is carried out individually for each quark flavor due to flavor-symmetry breaking introduced by quark mass differences and quark electric charges, \( Q \) (which appear above in units of the magnitude of the electron’s charge, \( e > 0 \)). Using Eq. (1), the \( U(1) \) flux through an elementary plaquette in the (\( \mu, \nu \))-plane is identically equal to \( \exp(iQeF_{\mu\nu}) \), where

\[
QeB_z = \frac{2\pi}{L^2} n_\Phi,
\]

with \( B_z \) the \( z \)-component of the magnetic field, \( B_z = F_{12} = -F_{21} \), and all other components of the electromagnetic field-strength tensor vanish. Throughout this work, the flux quanta \( n_\Phi = 3, -6, \) and 12 are employed. The factors of three result from the fractional nature of quark charges in units of \( e \), and the doubling of flux is employed to economize on the computation of quark propagators. For example, the up-quark propagator with \( n_\Phi = 3 \) is the same as the down-quark propagator with \( n_\Phi = -6 \). On Ensembles I and II, the latter is identical to the strange-quark propagator due to mass degeneracy. This equality of down and strange propagators preserves an \( SU(2) \) symmetry, commonly called \( U \)-spin, and can be thought of as a rotation in three-dimensional flavor space about the up-quark axis.

Post-multiplication of the \( U(1) \) gauge links onto QCD gauge links is an approximation that ignores effects of the electromagnetic field on the sea quark and and, indirectly, the gluonic sector. In a complete calculation, the background electromagnetic field couples to sea-quark degrees of freedom through the fermionic determinant. The present computations should thus be thought of as partially quenched (PQ) due to the omission of such contributions. Because magnetic moments arise from a response that is linear in the external field, however, there are cases for which sea-quark contributions vanish. Computations of magnetic moments at a \( SU(3)_{F} \) symmetric point, for example, are complete. In this case, the sea-quark contributions arising from expanding the fermionic determinant to linear order in the external field are necessarily proportional to the sum \( \sum_f Q_f = Q_u + Q_d + Q_s \), which vanishes. Away from the \( SU(3)_{F} \) symmetric point, the sum of sea-quark current effects no longer vanishes because contributions from each flavor are no longer identical. Decomposing the electromagnetic current into isoscalar and isovector contributions allows for a separation of these matrix elements into sea-quark charge dependent and independent terms, respectively. Thus computations of magnetic moments on Ensembles I and II are complete, while only the isovector magnetic moments computed on Ensemble III are complete. Omitted contributions to the current from sea quarks on Ensemble III are nonetheless expected to be small, e.g. Ref. [56].

To determine QCD energy eigenstates in the presence of external magnetic fields, interpolating operators are chosen which have the quantum numbers of the octet baryons. In particular, interpolating operators that have been tuned to produce good overlap with ground-state baryons in vanishing magnetic fields are employed. The imposition of sufficiently weak magnetic fields should not affect the outcome of these calculations.

\[1 \] We thank Stefan Meinel for these determinations.

\[2 \] The electromagnetic gauge links in Eq. (1) also give rise to non-trivial holonomies, which however, are only relevant for quarks propagating around the torus. Due to confinement, such long-distance effects scale as \( \sim \exp(-m_{\pi}L) \) and are negligible in the present study of magnetic moments; see Ref. [53] [55] for further details.
alter the operator overlaps substantially. While field-strength dependent overlaps are observed in practice, diminished overlaps have not impeded the ground-state saturation of correlations functions. For a complete discussion of these points and further details concerning the smeared-smeared (SS) and smeared-point (SP) correlation functions computed in this work, see Ref. [33].

Consider an octet baryon, denoted by $B$, that is subject to a constant and uniform magnetic field oriented along the $z$-direction, $B = B_z \hat{z}$. The energy eigenvalues of this baryon with its spin polarized in the $z$-direction, with magnetic quantum number $s = \pm \frac{1}{2}$, and zero longitudinal momentum, $p_z = 0$, have the form

$$E_B^{(s)}(B_z) = \sqrt{M_B^2 + (2n + 1)Q_B e B_z} - 2\mu_B s B_z + \ldots,$$

where $M_B$ is its mass, $Q_B$ its charge in units of $e$, and $n$ is the quantum number of the Landau level that it occupies. For a spin-$\frac{1}{2}$ baryon, there is a structure-dependent contribution from the magnetic moment, $\mu_B$, that is linear in the magnetic field. The ellipses denote contributions that involve two or more powers of the magnetic field, such as the magnetic polarizability. The moments are determined from LQCD computations of Zeeman splittings, $\Delta E$. These are defined to be energy differences between eigenstates of differing spin projections

$$\Delta E \equiv E_B^{(+\frac{1}{2})}(B) - E_B^{(-\frac{1}{2})}(B),$$

where the baryon label and magnetic-field dependence of $\Delta E$ are suppressed for notational ease. Using the expected magnetic-field dependence of the energy eigenvalues in Eq. (3), this reduces to

$$\Delta E = -2\mu_B B_z + \ldots,$$

where the ellipsis represents contributions that are higher order in the magnetic field. The procedure used to determine the magnetic moments relies on the precise determination of the Zeeman splittings in Eq. (1) from ratios of spin-projected correlation functions, and subsequent extrapolation to vanishing magnetic field using the expectation in Eq. (5). A detailed description of the analysis is relegated to Appendix A.

### III. BARYON MAGNETIC MOMENTS

The magnetic moments of the octet baryons are determined from LQCD calculations performed in background magnetic fields, using the procedures detailed in Appendix A. As with any LQCD calculation, the results are dimensionless quantities, made so by compensating powers of the lattice spacing. In what follows, conversions from these results into units that can be compared with experiment are discussed, and various features, including the values of anomalous magnetic moments, pion-mass dependence, lattice-spacing dependence, and relations between magnetic moments, are discussed.

#### A. Units for Magnetic Moments

The magnetic moment of an octet baryon, $B$, can be described using units of baryon magnetons or natural baryon magnetons, which are defined by

$$[\text{BM}] = \frac{e}{2M_B}, \quad [\text{nBM}] = \frac{e}{2M_B(m_\pi)},$$

respectively, where $M_B$ is the experimentally-measured mass of the baryon, and $M_B(m_\pi)$ is the mass of the baryon computed with LQCD (which depends on the input light quark mass through the lattice-determined value of the pion mass, $m_\pi$). From a phenomenological point of view, it is conventional to use nuclear magnetons, $[\text{NM}]$, for all baryons, for which we also define the corresponding natural nuclear magnetons, $[\text{nNM}]$. These are simply the special cases with $B = N$ of the above units,

$$[\text{NM}] = \frac{e}{2M_N}, \quad [\text{nNM}] = \frac{e}{2M_N(m_\pi)}.$$

Such units proved advantageous in our studies of magnetic moments of light nuclei [32],[34].

To convert magnetic moments from lattice magneton units to nuclear magnetons, $[\text{NM}]$, they are multiplied by $aM_N$, requiring knowledge of the lattice spacing. The results of our LQCD calculations of the octet baryon magnetic moments, in units of $[\text{NM}]$ are given in Table VII in Appendix A and are shown in Fig. I along with their experimental values. In these units, considerable pion-mass dependence is generally observed but, curiously, the magnetic moments of the $\Xi$ baryons are relatively insensitive. This situation changes somewhat when they are instead converted into natural nuclear magnetons, $[\text{nNM}]$. The moments in these units are also shown in Fig. I and are obtained by multiplying the lattice magneton values by $aM_N(m_\pi)$ (which does not introduce scale-setting uncertainties). The situation clarifies even further when using units of natural baryon magnetons, $[\text{nBM}]$, for which the lattice magneton values are multiplied by $aM_B(m_\pi)$. These values are also shown in Fig. I and can be obtained from Table I. Magnetic moments expressed in $[\text{nBM}]$ show the mildest pion-mass dependence and, moreover, are close to the experimental values, even at heavy quark masses.

Another salient feature of the $[\text{nBM}]$ units is that the Dirac contribution to the magnetic moment can be readily subtracted, leaving the anomalous magnetic moment,
FIG. 1. Magnetic moments of the octet baryons determined by LQCD calculations at $m_\pi \sim 800$ MeV (Ensemble I) and $m_\pi \sim 450$ MeV (Ensemble III), along with their experimental values. The quark-disconnected contributions to the magnetic moments at $m_\pi \sim 450$ MeV are not included, and they are known to vanish by SU(3)$_F$ symmetry arguments at $m_\pi \sim 800$ MeV. Comparisons with the experimental values are made in units of BM in the upper panel, nBM in the middle panel, and nNM in the lower panel. “–B” indicates the negative value of the moment, i.e. $\mu_{-B} \equiv -\mu_B$, so that all displayed quantities are positive. The uncertainties of the LQCD results reflect quadrature-combined statistical and systematic uncertainties. The values of the moments in units of NM and nNM are given in Table VIII in Appendix A while those in units of nBM follow from applying Eq. (8) to the results appearing in Table II.

TABLE II. Baryon anomalous magnetic moments, $\delta\mu_B$, in units of natural baryon magnetons, [nBM], defined in Eq. (6). The first uncertainty is statistical, while the second is the fitting systematic including that from the choice of fit functions. Ensemble I necessarily maintains exact $U$-spin symmetry, leading to repeated entries. Experimental values derived from Ref. [58] are given in units of baryon magnetons, [BM].

| B   | $\delta\mu_B$ [nBM] | $\delta\mu_B$ [BM] | Experiment [58] |
|-----|---------------------|---------------------|-----------------|
| p   | 2.052(14)(34)       | 1.895(22)(51)       | 1.7929(0)       |
| $\Sigma^+$ | 2.052(14)(34)       | 2.087(18)(44)       | 2.116(13)       |
| n   | −1.982(03)(19)      | −1.908(08)(37)      | −1.9157(0)      |
| $\Sigma^0$ | −1.982(03)(19)      | −1.894(10)(33)      | −1.752(20)      |
| $\Sigma^−$ | −0.136(14)(32)     | −0.206(21)(43)      | −0.480(32)      |
| $\Xi^−$ | −0.136(14)(32)     | 0.049(16)(34)       | 0.0834(35)      |

\[
\delta\mu_B \ [\text{nBM}] = \mu_B \ [\text{nBM}] - Q_B , \quad (8)
\]

which vanishes for a point-like particle. The Dirac moment is a short-distance contribution to the magnetic moment, and in our LQCD calculations it is fixed through the implementation of the external field through link variables. This emerges from the lattice Ward-Takahashi identity, because the corresponding electromagnetic current is the conserved point-split current. Thus non-vanishing anomalous magnetic moments provide a more direct probe of bound-state structure. The values are given in Table II and are shown graphically in Fig. 2. On the scale of fractions of [nBM], we strikingly see anomalous magnetic moments only having values $\delta\mu_B \sim \pm 2$ and $\delta\mu_B \sim 0$, for all six baryons with $I_z \neq 0$ (the $\Lambda$ and $\Sigma^0$ will be discussed later). The latter value is approximately attained for both the $\Sigma^−$ and $\Xi^−$ baryons, and suggests that their magnetic structure deviates very little from point-like particles. These striking features in Fig. 2 will be subsequently linked to the NRQM and the large-$N_c$ limit of QCD.

B. Chiral and Continuum Extrapolation

Our LQCD results demonstrate a rather mild pion-mass dependence of the baryon magnetic moments when given in units of natural baryon magnetons. As a result, rudimentary extrapolations of the LQCD results to the physical pion mass are attempted. Due to missing sea-quark contributions on Ensemble III, the isovector and isoscalar magnetic moments are extrapolated separately, and as the Dirac contribution is free from pion-mass dependence in natural baryon magnetons, only the anomalous parts of the isovector and isoscalar magnetic moments are extrapolated. A simple linear dependence
The anomalous magnetic moments of the octet baryons in units of \([\text{nBM}]\) compared with experiment in \([\text{BM}]\). The shorthand notation \(\mu_{-B} \equiv -\mu_B\) is used for display purposes. The \(\Sigma^-\) and \(\Xi^-\) baryons magnetically behave close to point-like Dirac particles. The non point-like structure of the remaining baryons are approximately the same (up to clockwise versus counter-clockwise circulation of current).

TABLE III. Linear extrapolation of anomalous magnetic moments, \(\delta\mu_{B}\), to the physical pion mass. The first uncertainties are statistical, while the second uncertainties are from systematics. Extrapolated values are compared with the experimentally determined values, which are given in boldface.

| \(\mu\) | \(\delta\mu_{B} [\text{nBM}]\) | Extrapolation \([\text{BM}]\) |
|-------|----------------|----------------|
| \(p-n\) | 4.034(15)(40) 3.802(25)(67) | 3.60(18) |
| \(\Sigma^- - \Sigma^+\) | 2.188(21)(48) 2.293(28)(63) | 2.39(18) |
| \(\Xi^- - \Xi^0\) | -1.846(14)(35) -1.943(18)(46) | -2.03(13) |
| \(p+n\) | 0.071(14)(38) -0.013(22)(59) | -0.09(16) |
| \(\Sigma^+ + \Sigma^-\) | 1.917(19)(44) 1.881(27)(60) | 1.85(17) |
| \(\Xi^0 + \Xi^-\) | -2.117(15)(39) -1.845(19)(50) | -1.60(15) |

Despite missing quark-disconnected contributions at \(m_\pi \sim 450 \text{ MeV}\), the anomalous part of isoscalar baryon magnetic moments also extrapolate to values consistent with experiment. For the nucleon, however, the quark-line connected result is consistent with zero with an uncertainty that is comparable to the experimental result. A precise calculation of this quantity from LQCD will be challenging due to the dominant quark-line disconnected contributions. For the \(\Sigma\) and \(\Xi\) baryons, however, the quark-line connected contributions seem to give the bulk of the anomalous part of their isoscalar moments. Under the assumption that the discrepancy between the central value of the extrapolation and the experimental value arises entirely due to missing quark-connected contributions, these contributions enter at the level of \(\sim 15\%\) and \(\sim 5\%\), for the \(\Sigma\) and \(\Xi\), respectively. Notice that the pion-mass dependence of the anomalous part of the isoscalar magnetic moment of the \(\Xi\) is found to be larger than that of the anomalous part of its isovector moment.

The continuum limit of the magnetic moments can be investigated from the values computed on Ensembles I and II which have relatively close pion masses. Bearing in mind the reduced statistics on Ensemble II, the magnetic moments in units of \([\text{nBM}]\) are compared in Table XIV for Ensembles I and II. Notice that the extracted magnetic moments from Ensemble II have statistical uncertainties which are 2–4 times as large as those from Ensemble I. This scaling is consistent with the differing sizes of the ensembles.

While the fermion action has only been perturbatively improved, with corrections naively scaling as \(O(a^2)\), the value of the clover coefficient with tadpole improvement

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4 The isoscalar magnetic moments receive \(m_\pi^2\) corrections about the \(SU(2)\) chiral limit, for example Ref. [59], however, the relevant symmetry group for our LQCD calculation at \(m_\pi \sim 450 \text{ MeV}\) is \(SU(4)\) due to the vanishing electric charges of sea quarks. Expanding (valence) isoscalar magnetic moments about the \(SU(4)\) chiral limit, gives rise to a linear pion-mass dependence [60]. Nonetheless, the choice of a linear extrapolation is not well motivated by such considerations as the present calculations are far from the chiral limit.
is consistent with that obtained from non-perturbative $O(a)$ improvement. Since the vector current is implemented through the link fields, Eq. (1), it inherits the same level of discretization effects. Thus the magnetic moments are assumed to have quadratic dependence on the lattice spacing near the continuum limit, of the form

$$\delta \mu_B(a) = \delta \mu_B(0) + C_B a^2.$$  (10)

Results of continuum extrapolations using Eq. (10) are given in Table IV and shown in Fig. 4. The anomalous magnetic moments should be more sensitive to the lattice spacing, and it is found that the continuum-extrapolated values for the charged baryons are consistent with those computed on the coarse Ensemble I. The magnetic moment of the $U$-spin doublet consisting of the neutron and $\Sigma^0$, however, exhibits the strongest lattice-spacing dependence in absolute terms. The difference between its magnetic moment on the coarse ensemble and the continuum-extrapolated value is relatively large and the coarse result is more than $3\sigma$ from the extrapolated result. The anomalous magnetic moment of the $U$-spin doublet consisting of $\Sigma^-$ and $\Xi^-$ baryons, however, exhibits the greatest relative change because the values are quite small and the extrapolated result is consistent with zero. This is surprising because it suggests that the deviation from point-like magnetic moments computed on Ensembles I and II could just be a lattice-spacing artifact. Better statistics and computations at an additional lattice spacing are needed to support this conclusion.

### C. Magnetic Moment Relations

The curious pattern of baryon anomalous magnetic moments exhibited in Fig. 2 suggests that a further investigation of the relations between them is warranted. An examination of the deviations from the Coleman-Glashow relations leads to a consideration of relations

![Diagram](https://example.com/diagram.png)

**Fig. 3.** Pion-mass dependence of the anomalous part of the isovector and isoscalar magnetic moments in units of [nBM]. Isovector magnetic moments are free of quark-disconnected contributions; subtracting the Dirac part does not change this because it arises solely from valence quarks. Away from the $SU(3)_F$ symmetric point, the isovector moments require disconnected contributions that have not been determined. Removing the Dirac part, moreover, makes the resulting moments more sensitive to these missing contributions. The shorthand $\delta \mu_{A \pm B} \equiv \mu_A \pm \mu_B$ for sums and differences of the baryon magnetic moments is used. Experimental values are given in [BM], and have not been included in these fits.

**Table IV.** Lattice-spacing dependence of baryon anomalous magnetic moments, $\delta \mu_B$, determined in [nBM], Eq. (1), at a pion mass of $m_\pi \sim 800\,\text{MeV}$. The first uncertainty quoted is statistical, while the second is systematic, while the uncertainty on the extrapolated values combines the statistical and systematic uncertainties in quadrature.

| $B$  | $\delta \mu_B$ [nBM]  |
|------|----------------------|
|      | I                    |
| $p, \Sigma^+$ | 2.052(14)(34) |
| $n, \Xi^0$    | −1.982(03)(19)   |
| $\Sigma^-, \Xi$ | −0.136(14)(32) |
|      | II                   |
| $p, \Sigma^+$ | 1.86(07)(13)    |
| $n, \Xi^0$    | −1.840(10)(19)   |
| $\Sigma^-, \Xi$ | −0.056(28)(67) |
|      | Extrapolation       |
| $p, \Sigma^+$ | 1.67(34)     |
| $n, \Xi^0$    | −1.705(76)     |
| $\Sigma^-, \Xi$ | −0.02(19)    |

**Fig. 4.** Illustration of the lattice-spacing dependence of baryon anomalous magnetic moments from Table IV.
between the magnetic moments that hold in the NRQM and/or in the large-$N_c$ limit of QCD, and deviations therefrom.

1. SU(3)$_F$ Symmetry and the Coleman-Glashow Relations

In the limit of SU(3)$_F$ flavor symmetry, the lightest spin-half baryons form an octet, where the states are conventionally embedded as

$$B_i^j = \begin{pmatrix} \frac{1}{\sqrt{2}} \Sigma^0 + \frac{i}{\sqrt{6}} \Lambda & \Sigma^+ & p \\ \Sigma^- & -\frac{1}{\sqrt{2}} \Sigma^0 + \frac{i}{\sqrt{6}} \Lambda & n \\ \Xi^- & -\frac{1}{\sqrt{6}} \Lambda & \Xi^0 \end{pmatrix}_i^j,$$

which transforms as $B \rightarrow VB \gamma^\dagger$ under a transformation parametrized by $V \in SU(3)_F$. Electromagnetic interactions break the SU(3)$_F$ symmetry due to the different quark electric charges, which appear in the matrix

$$Q_i^j = \text{diag} \begin{pmatrix} 2/3, -1/3, -1/3 \end{pmatrix}_i^j. \quad (12)$$

As a result, the leading order (LO) baryon magnetic moment operators, which contain one insertion of $Q$, are not SU(3)$_F$-invariant. Such symmetry breaking is most easily accounted for by promoting the charge matrix to a spurion field transforming as $Q \rightarrow VQ \gamma^\dagger$, forming invariant operators using this field, and then allowing $Q$ to pick up the vacuum expectation value in Eq. (12).

As is well known, in the limit of exact SU(3)$_F$ symmetry there are only two independent magnetic moment operators in the Hamiltonian density,

$$\mathcal{H} = -\frac{e \, \sigma \cdot B}{2MB} \left[ \mu_D \langle \overline{B}(Q,B) \rangle + \mu_F \langle \overline{B}(Q,B) \rangle \right]. \quad (13)$$

where the angled brackets denote the trace over SU(3)$_F$ indices, namely $\langle A \rangle \equiv A_i^i$. For the six octet baryons with $I_3 \neq 0$, there are four relations among their magnetic moments resulting from this Hamiltonian density. The remaining two baryons with $I_3 = 0$ will be discussed in Sec. IV. Magnetic moment relations which emerge from Eq. (13) were first obtained by Coleman and Glashow [8], and describe the LQCD results obtained on Ensembles I and II. From Eq. (13), there are three $U$-spin symmetry relations which dictate the equalities,

$$\mu_p = \mu_{\Sigma^+}, \quad \mu_n = \mu_{\Xi^0}, \quad \text{and} \quad \mu_{\Sigma^-} = \mu_{\Xi^-}. \quad (14)$$

The correlation functions from which these moments are extracted satisfy analogous relations configuration-by-configuration on Ensembles I and II. Additionally, there is the non-trivial constraint

$$\mu_p + \mu_n + \mu_{\Sigma^-} = 0, \quad (15)$$

that emerges on Ensembles I and II after averaging over gauge configurations and is a useful check of our results.

As there is additional SU(3)$_F$ breaking due to quark mass differences on Ensemble III, as well as in nature, we investigate the size of deviations from the Coleman-Glashow relations by computing sums and differences of magnetic moments that vanish in the SU(3)$_F$ symmetric limit. Results are tabulated in Table V.

The Coleman-Glashow relations obviously also emerge in the SU(3)$_F$ chiral limit in which $m_u = m_d = m_s = 0$, with corrections occurring at next-to-leading order (NLO) in the chiral expansion. Such NLO corrections can be eliminated in forming the smaller set of so-called Caldi-Pagels relations [10]. Of interest here is the sum of all six $I_3 \neq 0$ baryon magnetic moments

$$\mu_{C-P} = \frac{1}{2} \left[ \mu_p + \mu_n + \mu_{\Sigma^+} + \mu_{\Sigma^-} + \mu_{\Xi^0} + \mu_{\Xi^-} \right]. \quad (16)$$

This sum vanishes up to next-to-next-to-leading order (NNLO) in the chiral expansion, which scale parametrically as $m_K^2/\Lambda^2 \sim 15\%$. The factor of $\frac{1}{2}$ has been chosen so that $\mu_{C-P}$ reduces to the relation in Eq. (15) with unit normalization in the limit of $U$-spin symmetry. Results for $\mu_{C-P}$ are also given in Table V. For each magnetic moment relation, the results are given in units of [nNM] and [nBM]. The first uncertainty is statistical, while the second is systematic. “C-P” indicates the sum of the six baryon magnetic moments appearing in Eq. (16).

| TABLE V. The sums and differences of magnetic moment in units of [nNM] and [nBM]. |  
| \hline | \hline | \mu_B [nNM] | \mu_B [nBM] | \mu_B [BM] |
| --- | --- | --- | --- | --- |
| $p - \Sigma^+$ | 0 | 0.081(15)(34) | 0.33(1) |
| $\Xi^0 - n$ | 0 | 0.264(10)(41) | 0.66(14) |
| $\Xi^- - \Sigma^-$ | 0 | 0.274(20)(42) | 0.599(26) |
| $p + n + \Sigma^-$ | -0.065(20)(49) | -0.112(29)(72) | -0.280(25) |
| C-P | -0.065(20)(49) | 0.116(22)(54) | 0.139(26) |
| $p - \Sigma^+$ | 0 | -0.192(15)(34) | -0.323(13) |
| $\Xi^0 - n$ | 0 | 0.014(11)(43) | 0.164(20) |
| $\Xi^- - \Sigma^-$ | 0 | 0.255(23)(47) | 0.564(35) |
| $p + n + \Sigma^-$ | -0.065(20)(49) | -0.219(31)(74) | -0.603(32) |
| C-P | -0.065(20)(49) | 0.011(24)(58) | -0.078(34) |

As there are additional SU(3)$_F$ breaking due to quark mass differences on Ensemble III, as well as in nature, we investigate the size of deviations from the Coleman-Glashow relations by computing sums and differences of magnetic moments that vanish in the SU(3)$_F$ symmetric limit. Results are tabulated in Table V.

The Coleman-Glashow relations obviously also emerge in the SU(3)$_F$ chiral limit in which $m_u = m_d = m_s = 0$, with corrections occurring at next-to-leading order (NLO) in the chiral expansion. Such NLO corrections can be eliminated in forming the smaller set of so-called Caldi-Pagels relations [10]. Of interest here is the sum of all six $I_3 \neq 0$ baryon magnetic moments

$$\mu_{C-P} = \frac{1}{2} \left[ \mu_p + \mu_n + \mu_{\Sigma^+} + \mu_{\Sigma^-} + \mu_{\Xi^0} + \mu_{\Xi^-} \right]. \quad (16)$$

This sum vanishes up to next-to-next-to-leading order (NNLO) in the chiral expansion, which scale parametrically as $m_K^2/\Lambda^2 \sim 15\%$. The factor of $\frac{1}{2}$ has been chosen so that $\mu_{C-P}$ reduces to the relation in Eq. (15) with unit normalization in the limit of $U$-spin symmetry. Results for $\mu_{C-P}$ are also given in Table V. For each magnetic moment relation, the results are given in units of [nNM] and [nBM]. The first uncertainty is statistical, while the second is systematic. “C-P” indicates the sum of the six baryon magnetic moments appearing in Eq. (16).

| TABLE V. The sums and differences of magnetic moment in units of [nNM] and [nBM]. |  
| \hline | \hline | \mu_B [nNM] | \mu_B [nBM] | \mu_B [BM] |
| --- | --- | --- | --- | --- |
| $p - \Sigma^+$ | 0 | 0.081(15)(34) | 0.33(1) |
| $\Xi^0 - n$ | 0 | 0.264(10)(41) | 0.66(14) |
| $\Xi^- - \Sigma^-$ | 0 | 0.274(20)(42) | 0.599(26) |
| $p + n + \Sigma^-$ | -0.065(20)(49) | -0.112(29)(72) | -0.280(25) |
| C-P | -0.065(20)(49) | 0.116(22)(54) | 0.139(26) |
| $p - \Sigma^+$ | 0 | -0.192(15)(34) | -0.323(13) |
| $\Xi^0 - n$ | 0 | 0.014(11)(43) | 0.164(20) |
| $\Xi^- - \Sigma^-$ | 0 | 0.255(23)(47) | 0.564(35) |
| $p + n + \Sigma^-$ | -0.065(20)(49) | -0.219(31)(74) | -0.603(32) |
| C-P | -0.065(20)(49) | 0.011(24)(58) | -0.078(34) |
density in Eq. (13), the baryon magnetic moments are,

\[
\mu_p = \left( \frac{1}{3} \mu_D + \mu_F \right) \, [nBM], \\
\mu_n = -\frac{2}{3} \mu_D \, [nBM], \\
\mu_{\Sigma^-} = \left( \frac{1}{3} \mu_D - \mu_F \right) \, [nBM].
\]

(17)

A correlated fit to these magnetic moments results leads to

\[
\mu_D(m_\pi = 800 \text{MeV}) = 2.958(35) \, [nNM], \\
\mu_F(m_\pi = 800 \text{MeV}) = 2.095(34) \, [nNM],
\]

where the quoted uncertainties are quadrature-combined statistical and systematic uncertainties. These parameter values and their correlated uncertainties are shown in Fig. [6].

The values of \(\mu_D\) and \(\mu_F\) can be estimated at \(m_\pi \sim 450 \text{MeV}\) using LQCD results from Ensemble III, along with additional assumptions. While \(SU(3)_F\) symmetry is not exact, there are only small deviations from the Coleman-Glashow relations on Ensemble III. To estimate the couplings, the proton and neutron magnetic moments are sufficient. At the level of effective field theory, it is convenient to consider the PQ extension of the flavor symmetry group of this calculation. At \(m_\pi \sim 450 \text{MeV}\), there are two degenerate light quarks, \(l = u, d\) and one heavier strange quark, \(s\). The theory has an infinite set of operators resulting from arbitrary insertions of the quark mass matrix, each of which introduces additional flavor structures into the magnetic moments, and changes the relations between the nucleon magnetic moments, and those of other members of the octet. Introducing three additional quarks, \(i, j, k\) and their associated ghosts, \(\tilde{i}, \tilde{j}, \tilde{k}\), that are degenerate with the \(l = u, d\) quarks and with charge assignments that are the same as the \(u, d, s\), enhances the \(SU(3)_F\) group to the \(SU(6)_F\) graded group when \(m_s = m_u = m_d\), and \(SU(2)_F \otimes U(1)_s\) group to the \(SU(5)_F \otimes U(1)_s\) graded group when \(m_s \neq m_u = m_d\). With these electric charge assignments, the super-charge matrix has vanishing supertrace, i.e., a singlet component is not introduced. It is then helpful to consider the proton and neutron of the present LQCD calculations as members of an octet in the \(i, j, k\) sector which has exact \(SU(3)_F\) symmetry. Thus, \(\mu_D\) and \(\mu_F\) can be extracted from the proton and neutron magnetic moments up to PQ effects, but this interpretation does not apply to the other hyperons, which all have valence strange quarks. The PQ effect is proportional to \((m_s - m_l)/N_c\), where \(N_c = 3\) is the number of colors, and the expected size is \(\sim 6\%\). With this PQ “machinery” in place, carrying out the extraction of the Coleman-Glashow moments, \(\mu_D\) and \(\mu_F\), from Ensemble III using the proton and neutron magnetic moments, results in

\[
\mu_D(m_\pi = 450 \text{MeV}) = 2.86(08)(22) \, [nNM], \\
\mu_F(m_\pi = 450 \text{MeV}) = 1.94(08)(15) \, [nNM],
\]

(19)

where the first uncertainty reflects the quadrature-combined statistical and systematic uncertainty, and the second estimates the PQ effect. These values have been included in Fig. [6].

A similar analysis of the experimentally measured proton and neutron magnetic moments, but with a slightly different super-charge matrix, leads to,

\[
\mu_D(m_\pi = 135 \text{MeV}) = 2.87(32) \, [nNM], \\
\mu_F(m_\pi = 135 \text{MeV}) = 1.84(20) \, [nNM],
\]

(20)

where the PQ effects are estimated to be \(\sim 11\%\), which dominate the uncertainty ellipse shown in Fig. [6]. Interestingly, the values of the Coleman-Glashow moments, \(\mu_D\) and \(\mu_F\), are found to exhibit only a mild quark-mass dependence. The nearness of these coefficients to integer values, moreover, is intriguing and highly suggestive of the NRQM.

Chiral perturbation theory can be used to estimate the \(SU(3)_F\) chiral-limit values of the Coleman-Glashow moments. Beyond LO, however, these calculations subsume quark-mass dependence into the couplings, which consequently become scale and scheme dependent. This can
be remedied with future LQCD computations in which the pion-mass dependence is accounted for, however, phenomenological analyses cannot resolve this dependence. As a result, we use the determinations of \( \mu_D \) and \( \mu_F \) from Ref. [61], which employs a scheme in which the extracted parameters are relatively stable between the NLO and NNLO calculations, and estimates of the nucleon mass in the three-flavor chiral limit from the BMW collaboration [62]. We use values obtained from their NNLO calculation [61], including the decuplet degrees of freedom,

\[
\begin{align*}
\mu_D(\pi = 0) &= 3.8(1.1) \text{[nNM]}, \\
\mu_F(\pi = 0) &= 2.5(0.6) \text{[nNM]},
\end{align*}
\]

where the quoted uncertainty contains the difference between the NNLO and NLO values and also the uncertainty in the three-flavor chiral limit value of the nucleon mass. These chiral-limit estimates are consistent with those obtained from a similar analysis treating only the kaon mass dependence [63], as well as computations without explicit decuplet baryons, e.g. Ref. [64]. The relatively large uncertainties, unfortunately, preclude definite conclusions to be drawn about their values, other than that they are consistent with those at the \( SU(3)_F \) symmetric point. LQCD calculations at very light pion and kaon masses of both the nucleon mass and their magnetic moments would greatly reduce these uncertainties, and it appears that it is the only reliable tool with which to make such a refinement.

2. The NRQM

The baryon magnetic moments can be compared to those in the NRQM. Assuming strong isospin symmetry, there are two constituent quark masses, \( M_Q \) for the light quarks and \( M_S \) for the strange quark, and hence there are only two independent magnetic moments. These moments can be utilized to write the NRQM predictions in terms of a constituent quark magneton unit, which we define by

\[
[cQM] = \frac{e}{2M_Q},
\]

and the ratio of constituent quark masses

\[
\lambda = M_Q/M_S.
\]

By virtue of their fractional electric charges, the magnetic moments of the up and down constituent quarks are written in terms of the constituent quark magneton simply as \( \mu_u = \frac{2}{3} [cQM] \) and \( \mu_d = -\frac{1}{3} [cQM] \), while for the strange constituent quark, \( \mu_s = -\frac{2}{3} \lambda [cQM] \). With these definitions, the NRQM predictions for the nucleon magnetic moments take the form

\[
\begin{align*}
\mu_p &= \frac{4}{3} \mu_u - \frac{1}{3} \mu_d = 1 [cQM], \\
\mu_n &= -\frac{1}{3} \mu_u + \frac{4}{3} \mu_d = -\frac{2}{3} [cQM],
\end{align*}
\]

FIG. 6. Extracted values for the Coleman-Glashow moments, \( \mu_D \) and \( \mu_F \), from Ensembles I and III, and experiment. The error ellipses represent the uncertainty in these extractions due to both statistical and systematic uncertainties. For Ensemble III and experiment, the latter includes an estimate of PQ effects due the lack of \( SU(3)_F \) flavor symmetry in the quark sea. Integer values that are suggestive of the NRQM are shown with the dashed lines.

independent of \( \lambda \). Comparing these expressions with the Coleman-Glashow expression for the neutron magnetic moment, one can identify,

\[
[cQM] = \mu_D \text{[BM]},
\]

and the NRQM gives rise to \( \mu_D = [cQM]/[BM] = M_B/M_Q \), which is simply the ratio of the baryon mass to the constituent quark mass. Using the proton magnetic moment in the NRQM then gives rise to \( \mu_D = \frac{1}{3} \mu_D + \mu_F \) which yields \( \mu_F = 2M_B/(3M_Q) \). Assuming that the constituent quarks in the NRQM are noninteracting, this leads to \( \mu_D = +3 \) and \( \mu_F = +2 \), which are indicated in Fig. 6.

Moreover, the hyperon magnetic moments in the NRQM are given by the expressions

\[
\begin{align*}
\mu_{\Sigma^+} &= \frac{4}{3} \mu_u - \frac{1}{3} \mu_d = \left( \frac{8}{9} + \frac{1}{9} \lambda \right) [cQM], \\
\mu_{\Sigma^-} &= \frac{4}{3} \mu_d - \frac{1}{3} \mu_u = -\left( \frac{4}{9} - \frac{1}{9} \lambda \right) [cQM], \\
\mu_{\Xi^0} &= -\frac{1}{3} \mu_u + \frac{4}{3} \mu_d = \left( \frac{2}{9} + \frac{4}{9} \lambda \right) [cQM], \\
\mu_{\Xi^-} &= -\frac{1}{3} \mu_d + \frac{4}{3} \mu_u = \left( \frac{1}{9} - \frac{4}{9} \lambda \right) [cQM],
\end{align*}
\]

which depend on the strange constituent quark mass through \( \lambda \). Confronting LQCD results with the NRQM
TABLE VI. Constituent quark masses and ratios extracted from the NRQM predictions of baryon magnetic moments, based upon the relations in Eqs. (27), (30), and (31). To obtain these model parameters, the magnetic moments determined on Ensembles I and III, as well as the experimental values are used.

|      | $M_Q/M_N$  | $M_Q/M_B$  | $\lambda = M_Q/M_S$ |
|------|------------|------------|---------------------|
|      | I          | III        | Experiment          |
|      | $\frac{4}{5} \Delta \mu_N^{-1}$ | $0.3311(10)(26)$ | $0.3471(18)(48)$ | $0.3542(0)$ |
|      | $\frac{4}{5} \Delta \mu_{\Sigma}^{-1}$ | $0.3184(16)(37)$ | $0.3409(23)(50)$ | $0.3685(36)$ |
|      | $\frac{4}{5} \Delta \mu_{\Xi}^{-1}$ | $0.395(07)(17)$ | $0.408(08)(20)$ | $0.556(15)$ |
|      | $\frac{4}{5} \Delta \mu_N^{-1}$ | $0.3311(10)(26)$ | $0.3471(18)(48)$ | $0.3542(0)$ |
|      | $\frac{4}{5} \Delta \mu_{\Sigma}^{-1}$ | $0.3184(16)(37)$ | $0.3107(21)(46)$ | $0.2900(28)$ |
|      | $\frac{4}{5} \Delta \mu_{\Xi}^{-1}$ | $0.395(07)(17)$ | $0.354(07)(17)$ | $0.396(11)$ |

|      | $R_X$      | I          | III        | Experiment | NRQM |
|------|------------|------------|------------|------------|------|
| $R_N$ | $1.027(05)(15)$ | $1.012(08)(25)$ | $0.9732(0)$ | $1.00$ |
| $R_{NS}$ | $1.057(05)(15)$ | $1.026(07)(22)$ | $0.9456(91)$ | $1.00$ |
| $R_{NS}$ | $0.854(14)(33)$ | $0.858(17)(44)$ | $0.627(17)$ | $1.00$ |
| $R_{SS}$ | $0.808(17)(40)$ | $0.837(18)(43)$ | $0.663(25)$ | $1.00$ |
| $R_S$ | $0.736(33)(75)$ | $0.75(05)(11)$ | $0.216(96)$ | $1.00$ |

FIG. 7. The ratios of constituent quark mass to baryon mass determined from the isovector magnetic moment relations in Eq. (27). Magnetic moments expressed in [nNM] permit an extraction of $M_Q/M_N$ (upper panel), while moments expressed in [nBM] permit an extraction of $M_Q/M_B$ (lower panel). The dashed horizontal line indicates a mass ratio of $\frac{1}{3}$ that is expected in the NRQM.

predictions enables determinations of the constituent quark masses. For example, the three isovector combinations of magnetic moments

$$
\Delta \mu_N = \mu_p - \mu_n ,
\Delta \mu_{\Sigma} = \mu_{\Sigma^+} - \mu_{\Sigma^-} ,
\Delta \mu_{\Xi} = \mu_{\Xi^-} - \mu_{\Xi^0} ,
$$

are independent of the strange constituent quark mass, from which the constituent quark magneton unit can be extracted,

$$
\frac{e}{2M_Q} = \frac{3}{5} \Delta \mu_N = \frac{3}{4} \Delta \mu_{\Sigma} = 3 \Delta \mu_{\Xi} .
$$

Values of these quantities in units of [nNM] allow for various determinations of the mass ratio $M_N/M_Q$, while examining them in units of [nBM] provide extractions of $M_B/M_Q$. The LQCD results for these quantities, along with their experimental values, are collected in Table VI and displayed in Fig. 7. The $M_Q$ obtained from these isovector relations involving the nucleon and $\Sigma$'s are similar in both units, but show a modest systematic trend. However, the corresponding ratios obtained from the isovector magnetic moment of the $\Xi$'s exhibit much greater pion-mass dependence. The value $M_Q/M_N \sim 0.55$ inferred from the experimental determination of $\Delta \mu_{\Xi}$ suggests additional $SU(3)_F$ breaking beyond the NRQM. As expected, the level of $SU(3)_F$ breaking can be reduced by employing [nBM] units.

$M_S$ can be isolated from linear combinations of hyperon magnetic moments. From Eq. (26), there are two such possibilities,

$$
\frac{e}{2M_S} = 3 (\mu_{\Sigma^+} + 2\mu_{\Sigma^-}) = -\frac{3}{4} (\mu_{\Xi^0} + 2\mu_{\Xi^-}) .
$$

Ratios of these quantities to those appearing in Eq. (28) permit determinations of the constituent quark mass ra-
The deviation of the $SU(3)_F$ symmetric LQCD results from $\lambda = 1$, as shown in Fig. 8, indicates values of the magnetic moments that, while consistent with $SU(3)_F$ symmetry, are inconsistent with the NRQM at some level. While the relations, $\lambda_{\Sigma \Sigma}$ and $\lambda_{\Xi \Xi}$, equal unity for any values of the $SU(3)$ coefficients, $\mu_D$ and $\mu_F$, the others can take a range of values, as seen in Fig. 8. It is only for the integer values, $\mu_D = 3$ and $\mu_F = 2$ that all relations equal unity. The small deviations of $\mu_D$ and $\mu_F$ from these integer values, as shown in Fig. 9, are giving rise to these noticeable deviations from unity.

Ratios of magnetic moments can be compared to NRQM predictions, and as they are insensitive to the overall choice of units, they provide complementary information to the previously considered relations. Normalizing them so that predicted values in the NRQM are unity, leads to the relevant combinations,

$$R_N = -\frac{2}{3} \frac{\mu_p}{\mu_n},$$
$$R_{N\Sigma} = -\frac{1}{2} \frac{\mu_{\Sigma^+} - \mu_{\Sigma^-}}{\mu_n},$$
$$R_{N\Xi} = -2 \frac{\mu_{\Xi^-} - \mu_{\Xi^0}}{\mu_n},$$
$$R_{\Sigma \Sigma} = 4 \frac{\mu_{\Sigma^-} - \mu_{\Xi^0}}{\mu_{\Sigma^+} - \mu_{\Sigma^-}}. \quad (31)$$

These ratios compare two determinations of $M_Q$ (note that one of the three ratios $R_{N\Sigma}$, $R_{N\Xi}$ and $R_{\Sigma \Sigma}$ is redundant). A further ratio

$$R_S = -4 \frac{\mu_{\Sigma^+} + 2 \mu_{\Sigma^-}}{\mu_{\Xi^0} + 2 \mu_{\Xi^-}}, \quad (32)$$

compares two possible determinations of $M_S$. Results for the ratios in Eqs. (31) and (32) are given in Table VII and shown in Fig. 9. The ratio $R_N$ is close to unity, and is often touted as a success of the NRQM. The same applies equally well to the lesser-known ratio $R_{N\Sigma}$. When the $\Sigma$ magnetic moments enter into the determination of $M_Q$, however, the situation is less clear. Finally, the ratio $R_S$ highlights the inadequacy of the NRQM in explaining the experimentally measured magnetic moments of the $\Xi$ baryons. Notice that the LQCD results on Ensembles I and III generally seem to agree better with the NRQM than experiment does. The suggestive pattern among baryon isospin multiplets, however, is a puzzling feature that remains to be understood.
TABLE VII. Ratios of combinations of the magnetic moments that are predicted to be unity in the large-$N_c$ limit [33], as given in Eqs. (34), (35), (36) and (37). The uncertainties in the LQCD results correspond to the statistical and systematic, respectively. The order in the large-$N_c$ expansion is shown, denoted by $O$, at which deviations from unity are expected, with $\Delta m_q$ denoting corrections that vanish in the SU(3)$_F$-symmetry limit. The two "*" relations correspond to ratios that are predicted to be unity in the NRQM.

| $X$ | $\mathcal{R}_X$ | $\text{I}$ | $\text{II}$ | $\text{III}$ | $\text{Exp}$ | $O$ |
|-----|----------------|------------|------------|-------------|------------|------|
| $S^7$ | $1.105(10)(24)$ | $1.080(17)(41)$ | $1.213(36)$ | $1/N_c$ | |
| $V_{10}^1$ | $1.202(04)(10)$ | $1.228(07)(16)$ | $1.301(13)$ | $1/N_c$ | |
| $V_{10}^2$ | $0.8014(28)(66)$ | $0.8180(05)(11)$ | $0.8671(84)$ | $\Delta m_q/N_c$ | |
| $V_{10}^3$ | $0.9016(31)(75)$ | $0.9210(05)(12)$ | $0.9755(94)$ | $\Delta m_q/N_c$ | |
| $S/V_{11}$ | $0.6369(16)(42)$ | $0.7100(06)(13)$ | $0.8931(17)$ | $\Delta m_q, \Delta m_{q'}/N_c$ | |
| $V_1^*$ | $0.9940(08)(29)$ | $0.9280(08)(20)$ | $0.8991(16)$ | $1/N^2_c$ | |

3. The Large-$N_c$ Limit of QCD

Various relations between magnetic moments of the baryon octet emerge in the limit of a large number of colors, $N_c$. A comprehensive set of large-$N_c$ relations between moments was derived in Ref. [33], and includes relations valid to different orders in the $1/N_c$ expansion. Using the experimentally measured magnetic moments, these large-$N_c$ relations generally exhibit the expected pattern. Moreover, this expansion seems to indicate why certain NRQM predictions work better than others.

LQCD computations of magnetic moments allow for further tests of these large-$N_c$ relations, as is done with the experimental values. We focus on the large-$N_c$ relations for the $I_3 \neq 0$ octet baryons, for which there are six relations appearing in Ref. [33]. These are re-expressed in terms of a ratio that is predicted to be unity in the large-$N_c$ limit, using a naming convention for the relation from which it is derived. The simplest ratios involve the isovector nucleon and $\Sigma$ magnetic moments,

$$\mathcal{R}_{V_{10}^1} = \frac{\mu_p - \mu_n}{\mu_{\Sigma^+} - \mu_{\Sigma^-}} = 1 + O(1/N_c) ,$$

$$\mathcal{R}_{V_{10}^2} = \frac{(1 - \frac{1}{N_c}) (\mu_p - \mu_n)}{\mu_{\Sigma^+} - \mu_{\Sigma^-}} = 1 + O(\Delta m_q/N_c) ,$$

$$\mathcal{R}_{V_{10}^3} = \frac{\mu_p - \mu_n}{(1 + \frac{1}{N_c}) (\mu_{\Sigma^+} - \mu_{\Sigma^-})} = 1 + O(\Delta m_q/N_c) ,$$

where $\Delta m_q = m_q - m$ denotes $SU(3)_F$-symmetry breaking due to different quark masses. In the SU(3)$_F$ limit, the second and third ratios have corrections which scale as $1/N^2_c$. Notice that the difference between these two ratios is also of $O(1/N^2_c)$. Experimentally the second two relations, $V_{10}^2$ and $V_{10}^3$, are satisfied better than the first, $V_{10}^1$, and in a way which is suggestive of $1/N^2_c$ corrections versus $1/N_c$ corrections, respectively. The remarkable proximity of the ratio $\mathcal{R}_{V_{10}^2}$ to unity appears to be accidental, due to a fortuitous choice of including high-order terms in $1/N_c$. From the LQCD results collected in Table VII and shown in Fig. 10, their pattern appears consistent with large-$N_c$ predictions. Results for $\mathcal{R}_{V_{10}^2}$ and $\mathcal{R}_{V_{10}^3}$ on Ensemble I each deviate from unity by ~20%, while the comparatively reduced quark mass on Ensemble III pulls $\mathcal{R}_{V_{10}^2}$ and $\mathcal{R}_{V_{10}^3}$ closer to unity, consistent with the predicted scaling.

This pattern is also observed in another large-$N_c$ magnetic moment ratio. Combining all of the isoscalar moments and the nucleon isovector moment, leads to

$$\mathcal{R}_{S/V_{11}} = \frac{1}{2} (\mu_p + \mu_n) + 3 \left( \frac{1}{N_c} - \frac{2}{N^2_c} \right) (\mu_p - \mu_n)$$

$$= 1 + O(\Delta m_q) + O(\Delta m_{q'/N_c}) ,$$

which is predicted to be unity in the SU(3)$_F$ limit up to $1/N^2_c$ corrections. This expectation is well met by forming the ratio from the experimentally measured moments. However, moving toward the SU(3)$_F$ point introduces significant deviations, arriving at a value consistent with the NRQM from Ensemble I. We suggest that the good agreement at the physical point is an accident due to a cancellation between higher order contributions SU(3)$_F$-breaking and those that are purely $1/N_c$. This is the first relation we are aware of that strongly favors the NRQM relation at the expense of the large-$N_c$. Care must be taken in extrapolating the trend to the physical point, because computations on Ensemble III have omitted the quark-disconnected contributions. These missing contributions scale with SU(3)$_F$ breaking as $O(\Delta m_{q/N_c})$.

The remaining two ratios are predicted to be unity in the large-$N_c$ limit independent of SU(3)$_F$ breaking, and are also unity in the NRQM. The first such ratio is formed from the isoscalar magnetic moments

$$\mathcal{R}_{S^7} = \frac{5 (\mu_p + \mu_n) - (\mu_{\Xi^0} + \mu_{\Xi^-})}{4 (\mu_{\Sigma^+} - \mu_{\Sigma^-})}$$

$$= 1 + O(1/N_c) .$$

The values for this relation, given in Table VII and shown in Fig. 10, appear consistent with $1/N_c$ scaling, and the LQCD results are slightly closer to unity. However, the result from Ensemble III is missing the $O(\Delta m_{q/N_c})$ quark-disconnected contribution.

The ratio formed from the isovector magnetic moments,

$$\mathcal{R}_{V_{11}} = \frac{\mu_p - \mu_n - 3 (\mu_{\Xi^0} - \mu_{\Xi^-})}{2 (\mu_{\Sigma^+} - \mu_{\Sigma^-})} = 1 + O(1/N^2_c) ,$$

for which there are no disconnected contributions and the LQCD calculations are complete, are also consistent with having corrections scale as $1/N^2_c$, and are insensitive to the pion mass.
IV. COUPLED \( \Lambda–\Sigma^0 \) SYSTEM

The two \( I_3 = 0 \) octet baryons, \( \Lambda \) and \( \Sigma^0 \), mix in the presence of a magnetic field as the quark charge assignments break isospin symmetry. As a result, the energy eigenstates in a background magnetic field are linear combinations of these flavor eigenstates, and a couple-channels analysis of the corresponding LQCD results is required.

A. \( SU(3)_F \) Symmetric Limit

In the basis defined by \( \left( \begin{array}{c} \Sigma^0 \\ \Lambda \end{array} \right) \), the \( 2 \times 2 \) Hamiltonian density resulting from the Coleman-Glashow effective interactions in Eq. \((33)\) becomes,

\[
H_{I_3=0} = \frac{e}{2M_B} \sigma \cdot B \left( \begin{array}{cc} \frac{1}{\sqrt{3}} & -1 \\ 1 & -\frac{1}{\sqrt{3}} \end{array} \right) ,
\]

where \( \mu_n = -\frac{2}{3} \mu_D \) is the magnetic moment of the neutron in baryon magnetons. In terms of these flavor eigenstates, the magnetic moments and dipole transition moment are given by

\[
\mu_\Lambda = \frac{1}{2} \mu_n , \quad \mu_{\Sigma^0} = -\frac{1}{2} \mu_n \quad \text{and} \quad \mu_{\Lambda \Sigma} = -\frac{\sqrt{3}}{2} \mu_n . \quad (39)
\]

In non-vanishing magnetic fields, the eigenstates, \( \lambda_\pm \), are linear combinations of the flavor eigenstates, and can be written in the form

\[
\left( \begin{array}{c} \lambda_+ \\ \lambda_- \end{array} \right) = \left( \begin{array}{cc} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{array} \right) \left( \begin{array}{c} \Sigma^0 \\ \Lambda \end{array} \right) . \quad (40)
\]

Diagonalizing the Hamiltonian density in Eq. \((38)\), leads to a mixing angle and eigenstate magnetic moments of

\[
\theta = 30^\circ , \quad \mu_{\lambda_\pm} = \mp \mu_n . \quad (41)
\]

Notice that the \( \lambda_+ \) eigenstate has a positive magnetic moment as \( \mu_n < 0 \), and that the moments of the eigenstates are twice the flavor diagonal moments. Consequently, weak magnetic fields only partially lift the degeneracy between the \( \Lambda \) and \( \Sigma^0 \) baryons because the opposite spin states come in nearly degenerate pairs, split only by their magnetic polarizabilities that enter at \( \mathcal{O}(B^2) \) and higher,

\[
E^{(+)}_{\lambda_+}(B) = E^{(-)}_{\lambda_-}(B) + \mathcal{O}(B^2) . \quad (42)
\]

The magnetic polarizabilities arise from operators involving two insertions of the electric charge matrix \( Q \). In the limit of \( SU(3)_F \) symmetry, there are four such independent operators that appear in the effective Hamiltonian density in the form,

\[
\Delta H = -\frac{1}{2} 4\pi B^2 \sum_{j=1}^{4} \beta_j O_j , \quad (43)
\]

where the \( \beta_j \) are numerical coefficients, and the operators \( O_j \) can be conveniently taken to be

\[
\begin{align*}
O_1 &= \langle \bar{B}B \rangle \langle Q^2 \rangle , \\
O_2 &= \langle \bar{B} \{ Q, \{ Q, B \} \} \rangle , \\
O_3 &= \langle \bar{B} \{ Q, [Q, B] \} \rangle \equiv \langle \bar{B} \{ Q, [Q, B] \} \rangle , \\
O_4 &= \langle \bar{B} [Q, [Q, B]] \rangle .
\end{align*}
\]

(44)

This basis has been chosen because the operators \( O_3 \) and \( O_4 \) do not contribute to the magnetic polarizabilities of electrically neutral octet baryons. The contribution from \( O_1 \) is the same for all octet baryons, and therefore does not contribute to the electromagnetic mixing of \( \Lambda \) and \( \Sigma^0 \) baryons. The resulting contributions to the effective Hamiltonian in the \( I_3 = 0 \) sector are,

\[
\Delta H_{I_3=0} = -\frac{1}{2} 4\pi B^2 \left[ \frac{2}{3} \beta_1 1 + \frac{2}{9} \beta_2 \left( \frac{5}{\sqrt{3}} \frac{\sqrt{3}}{3} \right) \right] . \quad (45)
\]

The magnetic polarizabilities of flavor basis states are readily identified as the linear combinations,

\[
\beta_\Lambda = \frac{2}{3} \beta_1 + \frac{2}{3} \beta_2 , \quad \beta_{\Sigma^0} = \frac{2}{3} \beta_1 + \frac{10}{9} \beta_2 , \quad (46)
\]

along with their magnetic transition polarizability

\[
\beta_{\Lambda \Sigma} = \frac{2\sqrt{3}}{9} \beta_2 . \quad (47)
\]

Notice that the additional operator \( \langle \bar{B}Q \rangle \langle BQ \rangle \) is redundant because of the Cayley-Hamilton identity.
Due to the structure of $\Delta H_{I=0}$, the eigenstates $\lambda_{\pm}$ necessarily remain eigenstates in its presence, and have magnetic polarizabilities given by

$$\beta_{\lambda_{\pm}} = \beta_n + \frac{4}{\sqrt{3}} \beta_{\Lambda \Sigma}, \quad \text{and} \quad \beta_{\lambda_{\pm}} = \beta_n,$$  

(48)

where these results are expressed in terms of the magnetic polarizability of the neutron, which, from Eq. (44), is $\beta_n = \frac{2}{3} \beta_1 + \frac{4}{3} \beta_2$. Therefore, the four nearly-degenerate eigenstates have energies,

- $E_{\lambda_{-\frac{1}{2}}}(B_z) = M_B + \mu_n \frac{eB_z}{2MB} - 2\pi \beta_n B_z^2$,
- $E_{\lambda_{+\frac{3}{2}}}(B_z) = M_B + \mu_n \frac{eB_z}{2MB} - 2\pi \left( \beta_n + \frac{4}{\sqrt{3}} \beta_{\Lambda \Sigma} \right) B_z^2$,
- $E_{\lambda_{-\frac{3}{2}}}(B_z) = M_B - \mu_n \frac{eB_z}{2MB} - 2\pi \beta_n B_z^2$,
- $E_{\lambda_{+\frac{1}{2}}}(B_z) = M_B - \mu_n \frac{eB_z}{2MB} - 2\pi \left( \beta_n + \frac{4}{\sqrt{3}} \beta_{\Lambda \Sigma} \right) B_z^2$,

(49)

up to quadratic order in the magnetic field. These have been listed in order of increasing energy, from the ground state upwards, with the assumption that the magnetic field points along the positive z-axis, i.e., $B_z > 0$, and that the transition polarizability $\beta_{\Lambda \Sigma}$ is negative leading to the inequality $\beta_{\lambda_{-\frac{3}{2}}} > \beta_{\lambda_{+\frac{1}{2}}}$.

These results must be adapted to our LQCD calculations, in which the electric charges of sea quarks vanish. As shown in Appendix B, the only modification required to Eq. (49) is the replacement of the neutron’s magnetic polarizability by its quark-connected part, $\beta_n \rightarrow \beta_n^{(c)}$. The magnetic transition polarizability is unchanged because it arises only from quark-connected contributions, $\beta_{\Lambda \Sigma} = \beta_{\Lambda \Sigma}^{(c)}$. Thus, the ordering of energy eigenstates depends on the sign of $\beta_{\Lambda \Sigma}$, which shall be seen to be negative.

### B. Coupled-Channel Analysis

At the level of baryon two-point correlation functions, the coupled-channel $\Lambda - \Sigma^0$ system requires a matrix of correlation functions.

$$G^{(s)}(t, n_{\Phi}) = \begin{pmatrix} G_{\Sigma \Sigma}^{(s)}(t, n_{\Phi}) & G_{\Sigma \Lambda}^{(s)}(t, n_{\Phi}) \\ G_{\Lambda \Sigma}^{(s)}(t, n_{\Phi}) & G_{\Lambda \Lambda}^{(s)}(t, n_{\Phi}) \end{pmatrix},$$  

(50)

for each value of spin, $s = \pm \frac{1}{2}$. The matrix of correlation functions is diagonal when the magnetic field vanishes, i.e. $n_\Phi = 0$; and, when the magnetic field is non-vanishing, the principal correlators obtained from diagonalization contain information about the energy eigenstates of the coupled system. In our production, the transition correlators $G_{\Sigma \Sigma}^{(s)}(t, n_{\Phi})$ and $G_{\Sigma \Lambda}^{(s)}(t, n_{\Phi})$ were not computed, however, they can be obtained on Ensembles.

FIG. 11. Energy eigenvalues, $\Delta E = E(B_z) - E(0)$, of the $\Lambda - \Sigma^0$ system as a function of magnetic field, $eB_z$ calculated with LQCD on ensemble I. The spectrum is consistent with the analytic expectation given in Eq. (49) with $\beta_{\Lambda \Sigma} < 0$. Fits to the magnetic-field dependence of each eigenstate are shown as bands, and include linear and quadratic magnetic field terms.

I and II by appealing to $SU(3)_F$ symmetry. In this limit, the transition correlation functions can be shown to be

$$G_{\Sigma \Lambda}^{(s)}(t, n_{\Phi}) = G_{\Lambda \Sigma}^{(s)}(t, n_{\Phi}) = \frac{\sqrt{3}}{2} \left[ G_{\Sigma \Sigma}(t, n_{\Phi}) - G_{\Lambda \Lambda}(t, n_{\Phi}) \right],$$  

(51)

as derived in Appendix B by capitalizing on the mass degeneracy of the $\Lambda$ and $\Sigma^0$ baryons. This coupled-channel system can then be solved by obtaining the principal correlators, $G_{\lambda_{\pm}}^{(s)}(t, n_{\Phi})$, which are solutions to the generalized eigenvalue problem [66]

$$G_{\lambda_{\pm}}^{(s)}(t, n_{\Phi}) |\lambda\rangle = G_{\lambda_{\pm}}^{(s)}(t, n_{\Phi}) G_{\lambda_{\pm}}^{(s)}(t_0, n_{\Phi}) |\lambda\rangle.$$  

(52)

A time-offset parameter $t_0$ has been introduced, and can be varied to stabilize extraction of the ground-state contribution to the principal correlators, which appears in the long-time limit as

$$G_{\lambda_{\pm}}^{(s)}(t, n_{\Phi}) \sim e^{-E_{\lambda_{\pm}}^{(s)}(B_z) t} + \cdots.$$  

(53)

Using a bootstrap ensemble of SS correlation functions for the $\Lambda$ and $\Sigma^0$ baryons, the matrix of correlations functions, Eq. (50), needed to solve the generalized eigenvalue problem posed in Eq. (52) is formed. For each spin, $s$, this is then used to extract the ground-state energy of

---

6 As the exact solution is known, i.e. the electromagnetic mixing angle is $\theta = 30^\circ$, the principal correlators are readily found to be

$$G_{\lambda_{\pm}}^{(s)}(t, n_{\Phi}) = \frac{1}{2} \left[ 3G_{\Sigma \Sigma}^{(s)}(t, n_{\Phi}) - G_{\Lambda \Lambda}^{(s)}(t, n_{\Phi}) \right],$$  

$$G_{\lambda_{\pm}}^{(s)}(t, n_{\Phi}) = \frac{1}{2} \left[ 3G_{\Lambda \Lambda}^{(s)}(t, n_{\Phi}) - G_{\Sigma \Sigma}^{(s)}(t, n_{\Phi}) \right].$$  

These solutions compare favorably with the numerically determined solution of the generalized eigenvalue problem, Eq. (52).
the two principal correlation functions by inspecting the plateau region of their effective masses. Results for the spectrum are shown in Fig. 11. The effect of varying \( t_0 \) is numerically insignificant on the extraction of ground-state energies. Further details about these fits appear in Appendix 2, along with the tabulated energies. Superposed on the numerically determined spectrum are fits to the magnetic-field strength dependence of the energies, from which the magnetic moments are determined. These are found to be consistent with \( \mp \mu_\Lambda \) for \( \lambda_\pm \). More precise values are obtained, however, by determining the Zeeman splittings for each principal correlator from taking ratios of the two spin-projections. These results are also given in Appendix 2.

The ordering of energy levels in this system follows that of Eq. (49), which anticipates a negative value for the transition magnetic polarizability. From fits to the spin-averaged principal correlators, the value of the transition magnetic polarizability is found to be

\[
\beta_{\Sigma\Lambda} = -1.82(06)(12)(02) \times 10^{-4} \text{fm}^3
\]  

(54)

Results for the quark-connected part of the neutron polarizability extracted from the \( \Lambda-\Sigma^0 \) system are consistent with the direct calculation of \( \beta^{(\Sigma)}_n \) in Ref. 33.

C. \( SU(3)_F \) Breaking and the Physical Point

Without off-diagonal \( \Sigma_0-\Lambda \) correlators, it is not possible to investigate the mixing of the \( I_3 = 0 \) baryons on Ensemble III. Nonetheless it is instructive to anticipate the behavior of this system with \( SU(3)_F \) breaking, which can be accomplished using the experimentally measured magnetic moments, from which we see that this system will be challenging to address in future LQCD computations. The LQCD calculations themselves will not be difficult, nor the extraction of energy levels and Zeeman splittings, but it will be the extraction of the moments and polarizabilities that will be challenging. Elements of the magnetic moment matrix,

\[
\mathbf{M} = \begin{pmatrix}
\mu_{\Sigma^0} & \mu_{\Lambda\Sigma} \\
\mu_{\Lambda\Sigma} & \mu_\Lambda
\end{pmatrix}
\]  

(55)

have not been completely determined experimentally. In particular, the sign of the transition moment is not known and the magnetic moment of the \( \Sigma^0 \) baryon has not been measured. The former only affects the magnetic mixing angle. Given the magnitude of the transition moment, \( |\mu_{\Lambda\Sigma}| = 1.61(8) \text{ [NM]} \), and the proximity of nature to the \( SU(3)_F \) symmetric limit where \( \mu_{\Lambda\Sigma} > 0 \), it is reasonable to assume that \( \mu_{\Lambda\Sigma} > 0 \) holds elsewhere. The value of \( \mu_{\Sigma^0} \) can be fixed from the assumption of isospin symmetry. In the limit of \( SU(2)_F \) symmetry, the \( \Sigma \) baryons form an isotriplet, and by considering the three LO magnetic moment operators that act on this triplet in the isospin limit, for example Ref. 67, \( \mu_{\Sigma^0} \) is found to be,

\[
\mu_{\Sigma^0} = \frac{1}{2} (\mu_{\Sigma^+} + \mu_{\Sigma^-}) + \mathcal{O}(\alpha, m_d - m_u) 
\]  

(56)

This is because a single insertion of the charge matrix, with isoscalar and isovector components, is unable to induce an isotensor magnetic moment. Using the experimentally measured magnetic moments of the \( \Sigma^\pm \) leads to \( \mu_{\Sigma^0} \), up to isospin-breaking corrections estimated to be \( \sim 1\% \), and therefore a magnetic moment matrix of

\[
\mathbf{M} = \begin{pmatrix}
0.649(14)(06) & 1.61(8) \\
1.61(8) & -0.613(4)
\end{pmatrix} \text{ [NM]}
\]  

(57)

The uncertainties quoted are experimental, with the exception of the second uncertainty given for \( \mu_{\Sigma^0} \), which is an estimate of isospin breaking effects. Diagonalizing this matrix gives,

\[
\theta = 34.30(62)(15) \degree, \\
\mu_+ = 1.747(88)(13) \text{ [NM]}, \\
\mu_- = -1.711(82)(06) \text{ [NM]} 
\]  

(58)

which are within \( \sim 15\% \) of their values in the \( SU(3)_F \) symmetry limit, Eq. (41).

Breaking of \( SU(3)_F \) symmetry by the baryon masses further complicates the \( \Lambda-\Sigma^0 \) system. The \( \lambda_\pm \), are no longer simply the linear combinations of states that diagonalize the magnetic moment matrix, and the Hamiltonian,

\[
H_{I_3=0} = \Delta_{\Lambda\Sigma} \begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix} - \sigma \cdot B \frac{e M_N}{2 M_N} \mathbf{M}
\]  

(59)
where $\Delta_{\Lambda\Sigma}$ is $\Delta_{\Lambda\Sigma} = M_{\Sigma^0} - M_\Lambda$, must be diagonalized. Using $\lambda^{(s)}_\pm$ to denote the eigenstates in the presence of a magnetic field, the corresponding energy eigenvalues are given by

\[
E_{\lambda^{(s)}_\pm} = \frac{1}{2} \left[ \Delta_{\Lambda\Sigma} - \frac{e s B_z}{M_N} (\mu_{\Sigma^0} + \mu_\Lambda) \pm \sqrt{\left( \Delta_{\Lambda\Sigma} - \frac{e s B_z}{M_N} (\mu_{\Sigma^0} - \mu_\Lambda) \right)^2 + \left( \mu_{\Lambda\Sigma} e B_z / M_N \right)^2} \right],
\]

with $s = \pm \frac{1}{2}$. Contributions from the magnetic polarizabilities are omitted because there is a lack of experimental information to constrain them. Using the experimentally measured mass splitting and the magnetic moments from Eq. (58), the magnetic-field dependence of the energy eigenstates are shown in Fig. 12 and should be contrasted with that in the $SU(3)_F$ symmetric case shown in Fig. 11. As Ensemble III is closer to the $SU(3)_F$ limit than it is to nature, we expect the $\Lambda-\Sigma^0$ system to also behave more like that found in the $SU(3)_F$ limit.

\section{Strange Matter in Large Magnetic Fields}

Up to this point, this study has focused on the patterns and scaling of the baryon magnetic moments, which, along with the electric charges, dominate the response of the baryons to small applied magnetic fields. The calculations presented herein allow exploration of the behavior of baryons in very large magnetic fields, up to field strengths, $B \lesssim 10^{19}$ Gauss, comparable to fields conjectured to exist in the interiors of magnetars \cite{18}. In particular, it is interesting to consider the neutron and hyperon states to address the possibility of a large magnetic field stabilizing strange baryons in dense matter, and consequently softening the nuclear equation of state.

The composition of dense matter is determined by an interplay between the hadron masses, Pauli-blocking, and conserved charges, and by the interactions between hadrons. In the presence of a magnetic field, the relative energies of the hadrons change, as do the interactions between them. Fully addressing the composition of dense matter in the presence of a magnetic field is a very complex problem beyond the scope of our work. Here, the way in which the hadron masses change in large magnetic fields is highlighted, focusing on the neutral baryons, as shown in Fig. 13.

In the $SU(3)_F$ symmetric limit and in the absence of electromagnetism, there are sixteen degenerate states corresponding to the two spin states associated with each octet baryon. Including the leading QED self-energy corrections to the baryon mass increases the masses of the charged baryons by $\sim 1\text{ MeV}$. Therefore, at zero density and in the absence of a background magnetic field, the lowest-lying state consists of the degenerate neutron, $\Lambda$, $\Sigma^0$, and $\Xi^0$, each with two spin degrees of freedom, while the proton, $\Sigma^\pm$, and $\Xi^-$, are degenerate, but higher in energy by $\sim 1\text{ MeV}$. With the addition of a background magnetic field, the neutron and $\Xi^0$ remain degenerate for all values of the magnetic field due to the $U$-spin symmetry, and their spin-down components are the lowest energy octet baryon states for $B \lesssim 10^{19}$ Gauss. While negatively shifted, the spin-down components of the coupled $\Lambda-\Sigma^0$ system are found to be higher in energy than those of the neutron and $\Xi^0$, see Fig. 13. Therefore, in the ab-

\begin{figure}
\centering
\includegraphics[width=\textwidth]{fig13}
\caption{Energies of the electrically neutral baryons in magnetic fields relative to the nucleon mass in zero field. The top panel shows energies at the $SU(3)_F$ symmetric point with a pion mass of $m_\pi \sim 800\text{ MeV}$, while the bottom panel shows energies for $m_\pi \sim 450\text{ MeV}$. The latter results do not include contributions from quark-disconnected diagrams. Not all relevant energy levels could be extracted from the LQCD calculations, which is responsible for the termination of two of the levels at $eB_z = 3 \times 10^{18}$ Gauss in the lower panel.}
\end{figure}
sence of strong interactions between baryons, the ground state of dense hadronic matter would have an equal number density of neutrons and $\Sigma^0$'s. Not enough is presently known about the interactions between baryons in dense magnetized matter to determine how this conclusion will be modified. The quadratic and higher shifts are compromised by the omission of quark-disconnected diagrams in these calculations. Given the general smallness of disconnected insertions, these contributions are also expected to be small.

For the $N_f = 2 + 1$ case with $m_\pi \sim 450$ MeV, the two neutron spin states remain the states of lowest energy for $B \lesssim 4 \times 10^{18}$ Gauss. Therefore, in the absence of interactions, spontaneous generation of magnetic fields of this size is unlikely to stabilize strange baryons in dense matter, continuing to prefer neutron matter. Above this critical value, however, one of the components of the coupled $\Lambda-\Sigma^0$ system becomes lighter than the spin-up neutron, see Fig. 13. Consequently, the lowest energy configuration in such large fields is likely to have non-zero strangeness. Na"ively extrapolating these results to the physical point, where the $SU(3)_F$-breaking differences between baryons masses are larger, and considering the energetics of such systems (the energy density in large magnetic fields and the energy recovered from lowering the energy of the baryon states), suggests that it is unlikely that a spontaneously generated magnetic field could stabilize strange matter.

VI. SUMMARY

The magnetic moments of the lowest-lying octet of baryons have been calculated with LQCD including uniform and constant background magnetic fields. This technique allows for determinations of the energies of each baryon spin state as a function of applied magnetic field, while the corresponding Zeeman splittings allow for extractions of the magnetic moments. Our calculations were performed on three ensembles of gauge configurations, from which the pion-mass and lattice-spacing dependence of the magnetic moments were explored. Several interesting observations are made based on these results.

- Baryon magnetic moments exhibit mild pion-mass dependence ($\lesssim 10\%$ from $m_\pi \sim 800$ MeV down to the physical point) when expressed in units of natural baryon magnetons, [$nBM$] defined in Eq. [6]. This feature is shown in Fig. 1.

- In such natural units, the baryon anomalous magnetic moments take essentially only three values: $\delta\mu_B \sim 0, \pm 2$, see Fig. 2. The vanishing anomalous moments imply nearly point-like magnetic structure for the $\Sigma^-$ and $\Xi^-$ hyperons.

- Values of baryon anomalous magnetic moments are consistent with the $SU(3)_F$ symmetric limit with Coleman-Glashow moments $\mu_D \sim +3$ and $\mu_F \sim +2$ over a wide range of pion masses, see Fig. 6.

- These particular values of the Coleman-Glashow moments are consistent with the NRQM, however, careful scrutiny of NRQM predictions reveals further features, see Figs. 7-9.

- In most cases, the magnetic moments are consistent with relations derived from the large-$N_c$ limit of QCD. There is one exception that, at the $SU(3)_F$-symmetric point, is more consistent with the NRQM than the large-$N_c$ limit.

- A coupled-channels analysis is required to extract magnetic moments and transition moments from the $\Lambda-\Sigma^0$ system because the magnetic field induces a mixing between the $\Lambda$ and $\Sigma^0$. At the $SU(3)_F$ symmetric point, such an analysis produced energy shifts in the $\Lambda-\Sigma^0$ sector consistent with expectations based upon $SU(3)_F$ symmetry, see Fig. 11, and permitted the first determination of the transition magnetic polarizability, given in Eq. [54].

- Spin-dependent energy shifts of the baryons in large magnetic fields are obtained, see Fig. 13, from which we conclude that it is unlikely that such fields stabilize strange matter in dense astrophysical objects at realistic densities.

After the decades that have passed since the discovery that nature is in close proximity to an exact flavor symmetry among the three lightest quarks, the magnetic moments of the lightest baryons continue to provide (increasingly subtle) glimpses into their structure. The ability to manipulate the structure of matter that is now, and increasingly in the future, provided by LQCD calculations, enabled by the largest supercomputers, is providing new insights that cannot be gained through laboratory experiments. On the basis of what we have found, another generation of more precise LQCD calculations over a broader range of light-quark masses is warranted. The scientific impact of such a series of LQCD calculations would be enhanced by one or more experiments to improve the precision of the strange baryon magnetic moments and, if possible, their polarizabilities.

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Appendix A: Analysis—Fits to Zeeman Splittings and Extraction of Magnetic Moments

Fits to the two-point correlation functions calculated with LQCD, that lead to extractions of the baryon Zeeman splittings as a function of the background magnetic field, are described in this Appendix. Subsequent fits to the field-strength dependence of these splittings, that enable the determination of baryon magnetic moments, are also described. While multiple independent analyses of the LQCD correlation functions have been performed in the present work, only one analysis is detailed, as the values extracted agree between analyses within the quoted uncertainties.

Calculating quark propagators on each QCD gauge configuration with background magnetic fields enables computation of spin-projected baryon two-point functions, and the SS and SP baryon interpolating operators employed in this work are discussed in Ref. [33]. With the exception of the coupled Λ-Σ system, which is detailed separately in Appendix B, the main analysis utilizes both SS and SP correlation functions in linear combinations chosen to minimize uncertainties on the extraction of energies. From each ensemble of $N_{\text{cfg}}$ correlation functions (see Table I), associated with a given baryon channel, the average correlation functions from blocks of size $N_{\text{block}}$, where $N_{\text{block}} = 7, 1, 6$, and 6, are calculated for Ensembles I, II, and III, respectively. These block-averaged correlation functions are labeled by $G_i(t, n, \Phi)$. Where $s$ denotes the projection of baryon spin along the $z$-axis, and $i$ is the blocked ensemble index, $i = 1, \ldots, N_{\text{cfg}}/N_{\text{block}}$. Such time-dependent correlators are computed for the various baryons, for each value of the magnetic flux quantum. We use the quanta $n \Phi = 0, 3, -6$, and 12 ($n \Phi = -6$ is treated as $n \Phi = 6$ by reversing the spin axis). The block-averaged correlation functions are used to create bootstrap ensembles of size $N_{\text{BS}} = N_{\text{boot}} N_{\text{cfg}}/N_{\text{block}}$, where each member of the bootstrap ensemble consists of an average of $M_{\text{BS}} = N_{\text{cfg}}/N_{\text{block}}$ random samples of the blocked data. The bootstrap factor, $N_{\text{boot}}$, has the value 4, 3, and 4 on Ensembles I, II, and III, respectively.

To determine baryon masses, a bootstrap ensemble from block and spin-averaged correlation functions

$$G_i(t) = \frac{1}{2} \left[ G_i^{(+1/2)}(t, 0) + G_i^{(-1/2)}(t, 0) \right],$$

(A1)

in vanishing magnetic field, $n \Phi = 0$, is used. To aid in the analysis, the effective mass function for each member of the bootstrap ensemble

$$\{aM_{\text{eff}}(t)\}_i \equiv -\log \frac{G_i(t + a)}{G_i(t)} ,$$

(A2)

is formed, and the effective masses are shown in Fig. 14. Fits to $G_i(t)$ enable extraction of the baryon ground-state energies through the long-time behavior of the ensemble average $\sim Z \exp(-Et)$, and fit windows are chosen to maximize the correlated $\chi^2$-probability, also known as the integrated $\chi^2$ and often denoted by $Q = 1 - P$. The window is then varied over the eight adjacent fit windows obtained by adjusting the starting and ending time by one lattice unit in either direction. Values of the baryon masses extracted from the fits are collected in Table VIII.
FIG. 14. Baryon effective mass plots for Ensembles I–III and fits to the baryon masses, with bands depicting the quadrature-combined statistical and systematic uncertainties. The latter arise from both the fit and the choice of fit window.

TABLE VIII. The left panel shows baryon masses, $aM_B$, determined on the three ensembles. The first uncertainty shown on masses is statistical, while the second is the systematic due to the fit and choice of fit window. Notice that the octet baryons are degenerate on Ensembles I and II, consistent with $SU(3)_F$ symmetry. Additionally, in the left panel, Zeeman splittings, $a \Delta E$, computed on the three ensembles, for three values of the magnetic flux quantum, $n_M$, are shown. The first uncertainty is statistical, while the second is the systematic due to the choice of fit window. Zeeman splittings on Ensembles I and II are paired due to $U$-spin symmetry leading to repeated entries for magnetic moments.

|       | $aM_B$ |        |        |        |        |        |        |        |
|-------|--------|--------|--------|--------|--------|--------|--------|--------|
|       |        |        |        |        |        |        |        |        |
| I     |        |        |        |        |        |        |        |        |
|       |        |        |        |        |        |        |        |        |
| II    |        |        |        |        |        |        |        |        |
|       |        |        |        |        |        |        |        |        |
| III   |        |        |        |        |        |        |        |        |
|       |        |        |        |        |        |        |        |        |

|       | $aM_N$ | $aM_A$ | $aM_\Sigma$ | $aM_\Xi$ | $aM_\Lambda$ |
|-------|--------|--------|-------------|----------|-------------|
| I     |        |        |             |          |             |
| II    |        |        |             |          |             |
| III   | 0.72649(17)(82) | 0.77729(14)(69) | 0.79078(13)(71) | 0.83732(10)(53) |          |

|       | $a \Delta E$ | $n_\Phi = 3$ | $n_\Phi = 6$ | $n_\Phi = 12$ |        |        |        |        |
|-------|--------------|-------------|-------------|-------------|--------|--------|--------|--------|
| I     |              |             |             |             |        |        |        |        |
| II    |              |             |             |             |        |        |        |        |
| III   | $-0.0280(08)(21)$ | $-0.0559(08)(23)$ | $-0.0986(34)(88)$ | $-0.1393(08)(26)$ | $-0.2353(22)(64)$ |        |        |        |        |
|       |              |             |             |             |        |        |        |        |
| $p, \Sigma^+$ | $-0.04664(18)(50)$ | $-0.0933(04)(13)$ | $-0.1804(12)(52)$ | $-0.2033(22)(22)$ | $-0.3241(19)(50)$ | $-0.403(19)(34)$ | $-0.504(18)(39)$ | $-0.606(23)(73)$ |
| $n, \Xi^0$ | $0.029999(23)(77)$ | $0.057920(07)(23)$ | $0.10127(07)(37)$ | $0.1324(22)(22)$ | $0.1804(12)(52)$ | $0.2064(19)(34)$ | $0.241(19)(34)$ | $0.276(23)(73)$ |
| $\Sigma^-, \Xi^-$ | $0.017933(26)(70)$ | $0.03344(07)(15)$ | $0.06953(13)(26)$ | $0.1047(22)(22)$ | $0.1324(22)(22)$ | $0.1605(19)(34)$ | $0.1967(19)(34)$ | $0.233(23)(73)$ |

|       | $a \Delta E$ | $n_\Phi = 3$ | $n_\Phi = 6$ | $n_\Phi = 12$ |        |        |        |        |
|-------|--------------|-------------|-------------|-------------|--------|--------|--------|--------|
| I     |              |             |             |             |        |        |        |        |
| II    |              |             |             |             |        |        |        |        |
| III   |              |             |             |             |        |        |        |        |
|       |              |             |             |             |        |        |        |        |
| $p, \Sigma^+$ | $-0.07250(23)(94)$ | $-0.1393(08)(26)$ | $-0.2353(22)(64)$ | $-0.3241(19)(50)$ | $-0.403(19)(34)$ | $-0.504(18)(39)$ | $-0.606(23)(73)$ |        |
| $\Sigma^+$ | $-0.07058(20)(69)$ | $-0.1352(06)(20)$ | $-0.2314(19)(50)$ | $-0.3241(19)(50)$ | $-0.403(19)(34)$ | $-0.504(18)(39)$ | $-0.606(23)(73)$ |        |
| $n$ | $0.04746(09)(35)$ | $0.0993(03)(14)$ | $0.1399(08)(28)$ | $0.2064(19)(34)$ | $0.241(19)(34)$ | $0.276(23)(73)$ |        |        |
| $\Xi^0$ | $0.04099(10)(28)$ | $0.0771(05)(14)$ | $0.1224(05)(15)$ | $0.1605(19)(34)$ | $0.1967(19)(34)$ | $0.233(23)(73)$ |        |        |
| $\Sigma^-$ | $0.02836(29)(98)$ | $0.0538(04)(16)$ | $0.1054(18)(39)$ | $0.1605(19)(34)$ | $0.1967(19)(34)$ | $0.233(23)(73)$ |        |        |
| $\Xi^-$ | $0.02083(37)(97)$ | $0.0418(05)(19)$ | $0.0962(23)(73)$ | $0.1605(19)(34)$ | $0.1967(19)(34)$ | $0.233(23)(73)$ |        |        |
To determine the baryon Zeeman splittings, $a\Delta E$, double ratios of block-averaged, spin-projected baryon correlation functions are constructed. These are chosen to have the form
\[ R_i(t) = \frac{G_i^{(+\frac{1}{2})}(t, n_F)}{G_i^{(-\frac{1}{2})}(t, n_F)} \frac{G_i^{(+\frac{1}{2})}(t, 0)}{G_i^{(-\frac{1}{2})}(t, 0)}, \]  
where, for simplicity, the dependence on the flux quantum in $R_i(t)$ is implicit. The bootstrap ensemble of double ratios suppresses statistical fluctuations. These ratios are shown in Fig. 14 and the long-time behavior of their ensemble average, $\sim Z \exp (-\Delta E t)$, enables extraction of the Zeeman splitting $\Delta E$ defined in Eq. (4). The fitting strategy employed for $R_i(t)$ is the same as that utilized for $G_i(t)$. There is one important restriction on establishing fit windows for $R_i(t)$: windows must begin only after each of the individual correlators in the double ratio, Eq. (A3), exhibits ground-state saturation. The extracted Zeeman splittings are given in Table VIII. From the extracted values of Zeeman splittings, $a\Delta E$, as a function of the magnetic flux quantum, the magnetic moments can be extracted from the linear splitting in sufficiently weak magnetic fields, Eq. (5), and consequently fits of the Zeeman splittings to various functions of the magnetic field are considered. Three functional forms are fit to the bootstrap ensembles,
\[ F_1(B) = -\mu B, \quad F_2(B) = -\mu B + f_2 B|B|, \quad \text{and} \quad F_3(B) = -\mu B + f_3 B^3, \]  
where the second fit function is motivated by relativistic corrections to the Zeeman splittings due to Landau levels. Accordingly $F_2(B)$ is used only to fit the Zeeman splittings of charged particles. When lattice units are used for the magnetic field, $a^2 e B$, the subsequent extraction of magnetic moments $\mu$ are in lattice magnetons,
\[ [\text{LatM}] = \frac{1}{2} e a. \]  
The magnetic field dependence of the extracted Zeeman splittings is shown in Fig. 16 along with two representative fits: a fit to all three splittings using $F_3(B)$, and a fit using $F_1(B)$ omitting the largest magnetic field. To quantify the uncertainty due to the choice of fit function, we also use fits to all three splittings using $F_1(B)$, and $F_2(B)$ for charged particles. The values extracted for magnetic moments appear in Table VIII. In many cases, the dominant uncertainty in determining magnetic moments arises from the systematics of the fit. This can be remedied in the future by performing computations at additional magnetic field strengths.

Appendix B: Coupled $\Lambda-\Sigma^0$ System Analysis

Technical details related to the coupled $\Lambda-\Sigma^0$ system are contained in this Appendix. First, the PQ analysis of the magnetic polarizabilities of octet baryons, necessitated by the vanishing sea-quark electric charges in our LQCD calculations, is sketched. Next, determining the transition correlation function from diagonal baryon correlation functions is made explicit. Finally, the analysis of the principal correlation functions obtained from solving the generalized eigenvalue problem in Eq. (72) is detailed.

1. Partially Quenched Magnetic Polarizabilities of the Octet Baryons

The vanishing of sea-quark electric charges can be addressed using a PQ framework for baryons, developed first in the context of baryon chiral perturbation theory, see Refs. [70–72]. The sea quarks $u_{\text{sea}}$, $d_{\text{sea}}$, and $s_{\text{sea}}$ appear in the vector $\Psi_i = (u, d, s, u_{\text{sea}}, d_{\text{sea}}, s_{\text{sea}}, \bar{u}, \bar{d}, \bar{s})$, which transforms in the fundamental representation of the $SU(6)$ graded group. Accordingly $u$, $d$, and $s$ are valence quarks, while $\bar{u}$, $\bar{d}$, and $\bar{s}$ are their ghostly counterparts which are not Grassman valued. Invariant operators often require the explicit appearance of grading factors of the form $(-1)^\eta$, where $\eta_f = 1$ for fermionic indices and $\eta_b = 0$ for bosonic indices. The quark electric charge matrix can be written as
\[ Q_{ij} = \text{diag} (Q_u, Q_d, Q_s, Q_{u_{\text{sea}}}, Q_{d_{\text{sea}}}, Q_{s_{\text{sea}}}, Q_u, Q_d, Q_s), \]  
where the ghost quarks necessarily share the electric charges of their valence counterparts, and the condition $\text{str} \ Q = Q_{u_{\text{sea}}} + Q_{d_{\text{sea}}} + Q_{s_{\text{sea}}} = 0$ ensures that no unphysical singlet operators appear. While all sea-quark charges vanish in our computation, $Q_{u_{\text{sea}}} = Q_{d_{\text{sea}}} = Q_{s_{\text{sea}}} = 0$, it is nonetheless useful to treat quantities as functions of the valence- and sea-quark electric charges.
The lowest-lying spin-half baryons in $SU(6|3)$ are embedded in a $240$-dimensional supermultiplet $B_{ijk}$, and the octet baryons $B_i$ formed from three valence quarks are embedded in this supermultiplet as $B_{ijk} = \frac{1}{\sqrt{6}} (\epsilon_{ijk} B_k + \epsilon_{ikl} B_j)$, where the $|$ notation represents the restriction of all indices to the valence sector. Magnetic polarizability operators are $SU(6|3)$ invariants constructed from $\mathcal{B}, \mathcal{B}$, and two insertions of the PQ charge matrix $Q$. The effective Hamiltonian density describing the magnetic polarizabilities of the $240$-plet baryons has the
FIG. 16. The magnetic-field dependence of baryon Zeeman splittings computed on Ensembles I–III. Two representative fits are shown: the darker bands correspond to linear plus cubic fits to all three field values, $F_B$ in Eq. (44), while lighter bands correspond to linear fits that exclude the value at the largest magnetic field. For the positively charged baryons (appearing in the first column), the negative of their Zeeman splittings are shown.

Form

$$\Delta \mathcal{H}_{PQ} = -\frac{1}{2} 4\pi B^2 \left[ \beta^{(PQ)}_1 B^{kji} B_{ijk} \text{str} (Q^2) + \beta^{(PQ)}_2 (Q Q)_l B_{ijk} + \beta^{(PQ)}_3 (-1)^{(\eta_k + \eta_j) (\eta_k + \eta_i)} B^{kji} (Q Q)_k l B_{ijl} + \beta^{(PQ)}_4 (-1)^{(\eta_k + \eta_m) \sum Q_l Q_j m B_{lmk}} + \beta^{(PQ)}_5 (-1)^{(\eta_k + \eta_m + 1) \sum B^{kji} Q_l Q_j m B_{lmk}} \right], \quad (B2)$$

where we have used str $(Q) = 0$, and where the $\beta^{(PQ)}_j$ are numerical coefficients. Notice that the number of independent operators is one greater in the PQ theory compared to QCD, see Eq. (44). The relations between the five PQ coefficients and the four QCD coefficients in the QCD limit can easily be found from matching the two expressions, but the full result is not required here. In considering the magnetic polarizabilities of electrically neutral baryons, only the first operator in Eq. (B2) depends on the electric charges of sea quarks. Contributions to magnetic polarizabilities from this operator are identical for all baryons, and zero for the transition polarizability. Thus $\beta_{\Lambda \Sigma}$ is independent of sea-quark charges in the symmetric limit, i.e. $\beta_{\Lambda \Sigma} = \beta_{\Lambda \Sigma}^{(c)}$, where the superscript $(c)$ denotes the quark-connected.
In general, setting the sea-quark charges to zero corresponds to the retaining the quark-connected parts of magnetic polarizabilities. For example, the connected part of the neutron magnetic polarizability is denoted by $\beta_n^{(c)}$, and satisfies $\beta_n^{(c)} = \beta_n - \frac{2}{\sqrt{3}}\beta_n^{(PQ)}$, where $\beta_n$ is the magnetic polarizability of the neutron in QCD. Therefore, at LO in $SU(3)_F$ breaking, the PQ Hamiltonian for the $I_3 = 0$ baryons at $O(B^2)$ can be written in the simple form,

$$\Delta H_{PQ}^{I_3=0} = \frac{1}{4\pi B^2} \left[ \beta_n^{(PQ)} \text{str} \mathbf{Q}^2 \mathbf{1} + \left( \frac{\beta_n^{(c)}}{\beta_{3\Lambda}} + \sqrt{3} \beta_{\Lambda \Sigma} \beta_n^{(c)} + \frac{1}{\sqrt{3}} \beta_{3\Sigma} \right) \right].$$  \tag{B3}$$

Upon setting the sea-quark charges to zero, the connected parts of $\Lambda$ and $\Sigma^0$ magnetic polarizabilities can be identified, $\beta_\Lambda^{(c)} = \beta_n^{(c)} + \frac{1}{\sqrt{3}} \beta_{3\Lambda}$ and $\beta_{\Sigma^0}^{(c)} = \beta_n^{(c)} + \sqrt{3} \beta_{3\Sigma}$, respectively, along with the magnetic polarizabilities of the $\lambda_\pm$ eigenstates. These are given by $\beta_{\lambda_+} = \beta_n^{(c)} + \frac{4}{\sqrt{3}} \beta_{3\Lambda}$ and $\beta_{\lambda_-} = \beta_n^{(c)}$. This implies that the only modification necessary to account for vanishing sea-quark charges in Eq. \[49\] is the replacement $\beta_n \rightarrow \beta_n^{(c)}$.

2. $\Lambda$--$\Sigma^0$ Transition Correlation Function and $U$-Spin Symmetry

The transition correlation function between $\Lambda$ and $\Sigma^0$ baryons can be obtained from the diagonal baryon two-point functions in the limit of exact $U$-spin symmetry. This result has been utilized in the analysis of $\Lambda$--$\Sigma^0$ mixing in Sec. IV B and the derivation is given here. For simplicity, the magnetic field-strength dependence is implicit below.

The neutron interpolating operator used in this work has the form $\chi_n^a(x) = \epsilon_{abc} [u^{aT}(x)C\gamma_5d^b(x)]d^c_\alpha(x)$, for which the neutron two-point function can be written in terms of the sum of two quark contractions. Using the spin-projection matrices $\mathcal{P}(\pm\frac{1}{2}) = \frac{1}{2}(1 \pm \Sigma_3)$, it can be expressed as

$$G^{(s)}_{nn}(x) = \langle 0|\mathcal{P}^{(s)}_{\alpha\beta}\chi_n^\alpha(x)\chi_n^\beta(0)|0\rangle = \langle S(U,S)\rangle + \langle S(U,S)\rangle,$$  \tag{B4}$$

making use of a short-hand notation for the quark-level contractions

$$\langle 1(2,3)\rangle = \epsilon_{abc}\epsilon_{a'b'c'}\mathcal{P}^{(s)}_{\gamma\delta}(C_{\gamma5})\alpha\beta(C_{\gamma5})\alpha\beta'\langle G^{(1)}(x,0)\rangle_{\alpha\alpha'}\langle G^{(2)}(x,0)\rangle_{\beta\beta'}\langle G^{(3)}(x,0)\rangle_{\gamma\gamma'},$$

$$\langle 1(2,3)\rangle = \epsilon_{abc}\epsilon_{a'b'c'}\mathcal{P}^{(s)}_{\gamma\delta}(C_{\gamma5})\alpha\beta(C_{\gamma5})\alpha\beta'\langle G^{(1)}(x,0)\rangle_{\alpha\alpha'}\langle G^{(2)}(x,0)\rangle_{\beta\beta'}\langle G^{(3)}(x,0)\rangle_{\gamma\gamma'}.$$  \tag{B5}$$

The quantities $G^{(j)}$, for $j = 1, \ldots, 3$, represent the propagators for three distinguishable quark flavors, and the angled bracket notation denotes the traces over spinor indices. One can easily demonstrate the identity $\langle 1(2,3)\rangle = \langle 1(3,2)\rangle$ for the second type of quark contraction. In the relevant $n$, $\Sigma^0$ and $\Lambda$ baryon correlation functions, $1 \rightarrow U$, $2 \rightarrow S$ on account of $U$-spin symmetry, i.e. because $m_d = m_s$ and $Q_d = Q_s$.

From the $\Sigma^0$ interpolating operator, $\chi_{\Sigma^0}^a(x) = \frac{1}{\sqrt{2}}\epsilon_{abc} \left[ (s^{aT}(x))C\gamma_5d^b(x)\right]u_\alpha(x) + (s^{aT}(x))C\gamma_5u^b(x)d_\alpha(x)$, the two-point function has the form

$$G^{(s)}_{\Sigma\Sigma}(x) = \langle 0|\mathcal{P}^{(s)}_{\alpha\beta}\chi_{\Sigma^0}^\alpha(x)\chi_{\Sigma^0}^\beta(0)|0\rangle = \frac{1}{2} \left[ \langle U(U,S)\rangle + \langle U(S,U)\rangle + \langle S(U,S)\rangle + \langle S(U,S)\rangle \right],$$  \tag{B6}$$

making explicit use of $U$-spin symmetry above by writing all down-quarks propagators as strange-quark propagators. Similarly, from the $\Lambda$ interpolating operator, which has the form $\chi_\Lambda^a(x) = \frac{1}{\sqrt{3}}\epsilon_{abc} \left[ 2(u^{aT}(x))C\gamma_5d^b(x)\right]c_\alpha(x) + (u^{aT}(x))C\gamma_5s^b(x)d_\alpha(x) - (d^{aT}(x))C\gamma_5s^b(x)c_\alpha(x)\right]$, the $\Lambda$ two point function is given by

$$G^{(s)}_{\Lambda\Lambda}(x) = \langle 0|\mathcal{P}^{(s)}_{\alpha\beta}\chi_\Lambda^\alpha(x)\chi_\Lambda^\beta(0)|0\rangle = \frac{1}{4} \left[ 5 \langle U(U,S)\rangle + 4 \langle U(S,U)\rangle + \langle U(U,S)\rangle + \langle S(U,S)\rangle \right].$$  \tag{B7}$$

Transition correlation functions between the $\Lambda$ and $\Sigma^0$ baryons are defined by $G^{(s)}_{\Lambda\Sigma}(x) = \langle 0|\mathcal{P}^{(s)}_{\alpha\beta}\chi_\Lambda^\alpha(x)\chi_{\Sigma^0}^\beta(0)|0\rangle$ and $G^{(s)}_{\Sigma\Lambda}(x) = \langle 0|\mathcal{P}^{(s)}_{\alpha\beta}\chi_{\Sigma^0}^\alpha(x)\chi_\Lambda^\beta(0)|0\rangle$. These can be written in terms of the following quark contractions

$$G^{(s)}_{\Lambda\Sigma}(x) = G^{(s)}_{\Sigma\Lambda}(x) = \frac{1}{2\sqrt{3}} \left[ \langle U(U,S)\rangle + \langle U(S,U)\rangle + \langle S(U,S)\rangle - \langle S(U,S)\rangle - 2 \langle S(U,S)\rangle \right].$$  \tag{B8}$$

From this expression, the desired relations, $G^{(s)}_{\Lambda\Sigma}(x) = G^{(s)}_{\Sigma\Lambda}(x) = \frac{1}{\sqrt{3}} \left[ G^{(s)}_{\Sigma\Sigma}(x) - G^{(s)}_{\Lambda\Lambda}(x) \right]$, follow, along with $G^{(s)}_{\Lambda\Lambda}(x) = \frac{1}{2} \left[ 3G^{(s)}_{\Lambda\Lambda}(x) - G^{(s)}_{\Sigma\Sigma}(x) \right]$. The latter holds configuration-by-configuration for each magnetic field in accordance with $U$-spin symmetry. It shows, moreover, that the neutron correlation function can be omitted from this discussion, in favor of $G^{(s)}_{\Lambda\Sigma}(x) = G^{(s)}_{\Sigma\Lambda}(x) = \frac{\sqrt{3}}{2} \left[ G^{(s)}_{\Sigma\Sigma}(x) - G^{(s)}_{\Lambda\Lambda}(x) \right]$. 
FIG. 17. Ratios formed from spin-projected, principal correlators in the $Λ–Σ^0$ system calculated with Ensemble I. The ratios $r(t)$ defined in Eq. (59) are used to obtain energy differences of a given spin state, whereas further ratios, $R(t)$ defined in Eq. (B10), are used to obtain the Zeeman splittings. Ratios of spin-averaged principal correlators, $R(t)$ in Eq. (B11), are used to obtain spin-averaged energy differences. For each ratio, results of exponential fits to the ratios are also shown. Shaded bands depict the uncertainty on the extracted energy, and include quadrature-combined statistical and systematic uncertainties, with the latter arising from the fit and choice of fit window. In cases where the extracted energy differences or Zeeman splittings are negative, inverse ratios are presented as indicated.

3. $Λ–Σ^0$ Fit Results

On the $U$-spin symmetric ensembles, the energy eigenstates of the $Λ–Σ^0$ system are calculated from principal correlators that are solutions to the generalized eigenvalue problem in Eq. (52), requiring the same baryon operators for the source and sink. The correlation functions used in this work are constructed from multiple different source locations on each configuration, making the smeared source interpolators the same as the zero-momentum projected
TABLE IX. Energy eigenvalues of the $\Lambda$-$\Sigma^0$ system obtained from principal correlation functions calculated with Ensemble I. Ratios have been normalized in Eq. (B9) to produce the energy differences $aE(B_z) - aE(0)$. Zeeman splittings, $a\Delta E$, computed from ratios of spin-projected principal correlators, Eq. (B10), as well as spin-averaged energy differences, $\bar{E}(B_z) - \bar{E}(0)$, are also obtained. The first uncertainty quoted is statistical, while the second is the systematic due to the fit and choice of fit window. Baryon magnetic moments, $\mu_B$, are determined in lattice magnetons [LatM], see Eq. (A5). The first uncertainty quoted on magnetic moments is statistical, while the second is the systematic due to the fit and choice of fit function. Magnetic polarizabilities are determined from fits to the magnetic-field dependence of spin-averaged energy differences, and the associated uncertainties are statistical, systematic, and additionally scale-setting for the case of the standard physical units, $[10^{-4} \text{fm}^3]$.

| $aE(B_z) - aE(0)$ | $a\Delta E$ |
|-------------------|-------------|
| $n_\Phi = 3$     | $n_\Phi = 6$ | $n_\Phi = 12$ |
| $\lambda^+_\pm$  | $-0.01461(02)(20) - 0.02801(27)(65) - 0.05162(06)(24)$ |
|                   | $0.1566(03)(19) + 0.03218(29)(71) + 0.06721(13)(41)$ |
|                   | $0.01294(02)(26) - 0.02079(36)(70) - 0.01924(40)(90)$ |
| $\lambda^-\pm$   | $-0.01710(03)(19) - 0.03706(27)(65) - 0.08266(21)(62)$ |

smeared sink interpolators within statistical uncertainties. Therefore, only the SS correlation functions are utilized in this part of the analysis.

For each of the spin-projected, principal correlators, $G^{(s)}_\lambda(t, n_\Phi)$, ratios of correlation functions

$$r^{(s)}_\lambda(t,t_0) = \frac{G^{(s)}_\lambda(t,n_\Phi)}{G^{(s)}_\lambda(t,0)/G^{(s)}_\lambda(t_0,0)}$$

are formed, where the magnetic-field dependence is treated as implicit. The time offset $t_0$ is the same parameter employed to solve Eq. (22), and the value $t_0/a = 3$ is used. (The effect of varying $t_0$ is found to be numerically insignificant in this particular analysis). For the normalization of the ratios, it is possible to divide the principal correlators by any linear combination of the diagonal $\Lambda$ and $\Sigma^0$ correlation functions in zero magnetic field due to their mass degeneracy (including the exact linear combinations for the $\lambda_\pm$ states which are known from the analytic solution). For the present study, the $\Sigma^0$ correlator is used for $\lambda_+$, and the $\Lambda$ correlator is used for $\lambda_-$. These would be the natural choices in the case of broken SU(3)$_F$. The correlation function ratios are shown in Fig. 17 and allow for the extraction of the energy differences, $E(B_z) - E(0)$. Results of exponential fits to these ratios are presented in Table IX.

To compute magnetic moments, it is efficacious to isolate them by taking further ratios

$$R(t) = r^{(+\frac{1}{2})}_\lambda(t)/r^{(-\frac{1}{2})}_\lambda(t),$$

whose long-time behavior leads to the Zeeman splittings $\Delta E$ (see also Eq. (A3)). There is one such ratio for the $\lambda_+$ eigenstates, and another for the $\lambda_-$ eigenstates. These double ratios and exponential fits to their time dependence are shown in Fig. 17 and given in Table IX. Values of the energy differences, $E(B_z) - E(0)$, and Zeeman splittings, $\Delta E$, allow determination of magnetic moments through fits to their magnetic-field dependence. For the spin-dependent energy differences, a linear plus quadratic fit function, namely $\tilde{F}^{(s)}_2(B) = -2\mu_B s B + \tilde{F}^{(s)}_3 B^2$, for $s = \pm \frac{1}{2}$ is utilized. The fits are shown in Fig. 11 and extracted values of magnetic moments are given in Table IX. For Zeeman splittings, the fit functions $\tilde{F}_1$ and $\tilde{F}_3$ appearing in Eq. (A4) are used. Representative fits are shown in Fig. 18.

The final part of the analysis concerning the $\Lambda$-$\Sigma^0$ system is the determination of magnetic polarizabilities, which are responsible for lifting the residual degeneracy of the different eigenstates of opposite spin. To determine the
polarizabilities, products of ratios of spin-projected principal correlators,

\[ R(t) = \sqrt{r_{\lambda_+}^{(+\frac{1}{2})}(t) r_{\lambda_-}^{(-\frac{1}{2})}(t)}, \]

are formed, whose long-time exponential behavior is governed by the spin-averaged energy differences, \( \Delta E \equiv \overline{E}(B_z) - \overline{E}(0) \), where \( \overline{E}(B) = \frac{1}{2} \left[ E^{(+\frac{1}{2})}(B) + E^{(-\frac{1}{2})}(B) \right] \). There is one such ratio for each of the two eigenstates \( \lambda_\pm \) and each has been plotted in Fig. 17 along with exponential fits. Results of fitting the \( R(t) \) ratios are provided in Table IX. Values of the spin-averaged energy differences are then fit as a function of the magnetic field to extract the magnetic polarizability, \( \beta \), using the two fit functions

\[ F_2(B) = -\beta B^2, \quad \text{and} \quad F_4(B) = -\beta B^2 + g_4 B^4. \]

Using values of the magnetic field in lattice units, \( a^2 eB_z \), leads to fit parameters \( \beta \) in lattice polarizability units, \( \text{[LatU]} = e^2 a^3 \). Table IX also provides values for \( \hat{\beta} \), which are polarizabilities in units of \( e^2 / M_B(M_T - M_B) \), where \( aM_T = 1.3321(10)(19) \) is the mass of the baryon decuplet on Ensemble I, and values for \( \beta \) in the conventional polarizability units of \( \text{[10^{-4} fm^3]} \) using the lattice spacing are given in Table [I].

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