ON THE GLOBAL REGULARITY OF TWO-DIMENSIONAL
GENERALIZED MAGNETOHYDRODYNAMICS SYSTEM

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ABSTRACT. We study the two-dimensional generalized magnetohydrodynamics system with dissipation and diffusion in terms of fractional Laplacians. In particular, we show that in case the diffusion term has the power $\beta = 1$, in contrast to the previous result of $\alpha \geq \frac{1}{2}$, we show that $\alpha > \frac{1}{3}$ suffices in order for the solution pair of velocity and magnetic fields to remain smooth for all time.

Keywords: Global regularity, magnetohydrodynamics system, Navier-Stokes system.

1. INTRODUCTION AND STATEMENT OF RESULTS

We study the generalized magnetohydrodynamics (MHD) system defined as follows:

$$\begin{cases}
\frac{\partial u}{\partial t} + (u \cdot \nabla)u - (b \cdot \nabla)b + \nabla \pi + \nu \Lambda^{2\alpha} u = 0, \\
\frac{\partial b}{\partial t} + (u \cdot \nabla)b - (b \cdot \nabla)u + \eta \Lambda^{2\beta} b = 0, \\
\nabla \cdot u = \nabla \cdot b = 0, \\
u(x,0) = u_0(x), b(x,0) = b_0(x),
\end{cases}$$

where $u : \mathbb{R}^N \times \mathbb{R}^+ \rightarrow \mathbb{R}^N$ is the velocity vector field, $b : \mathbb{R}^N \times \mathbb{R}^+ \rightarrow \mathbb{R}^N$ the magnetic vector field, $\pi : \mathbb{R}^N \times \mathbb{R}^+ \rightarrow \mathbb{R}$ the pressure scalar field and $\nu, \eta \geq 0$ are the kinematic viscosity and diffusivity constants respectively. We also denote by $\Lambda$ a fractional Laplacian operator defined via Fourier transform as $\hat{\Lambda}^{2\gamma} f(\xi) = |\xi|^{2\gamma} \hat{f}(\xi)$ for any $\gamma \in \mathbb{R}$.

In case $N = 2, 3$, $\nu, \eta > 0$, $\alpha = \beta = 1$, it is well-known that (1) possesses at least one global $L^2$ weak solution; in case $N = 2$, it is also unique (cf. [19]). Moreover, in any dimension $N \geq 2$, the case $\nu, \eta > 0$, the lower bounds on the powers of the fractional Laplacians at $\alpha \geq \frac{1}{2} + \frac{N}{4}$, $\beta \geq \frac{1}{2} + \frac{N}{4}$ imply the existence of the unique global strong solution pair (cf. [26]).

Some numerical study have shown that the velocity vector field may play relatively important role in regularizing effect (e.g. [8], [18]). Starting from the works of [9] and [34], we have seen various regularity criteria of the MHD system in terms of only the velocity vector field (e.g. [1], [4], [6], [7], [10], [25], [28], [33]). Moreover, motivated by the work of [20], the author in [24] showed that in case

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$N \geq 2, \nu, \eta > 0, \alpha \geq \frac{1}{2} + \frac{N}{4}, \beta > 0$ such that $\alpha + \beta \geq 1 + \frac{N}{2}$, the system (1) even in logarithmically super-critical case still admits a unique global strong solution pair. The endpoint case $\nu > 0, \eta = 0, \alpha = 1 + \frac{N}{2}$ was also completed recently in [23] and [29] (cf. [27] for further generalization).

On the other hand, in case $N = 2$, it is well-known that the Euler equations, the Navier-Stokes system with no dissipation, still admits a unique global strong solution. This is due to the conservation of vorticity which follows upon taking a curl on the system. In the case of the MHD system, upon taking a curl and then $L^2$-estimates of the resulting system, every non-linear term has $b$ involved. Exploiting this observation and divergence-free conditions, the authors in [2] showed that in case $N = 2$, full Laplacians in both dissipation and magnetic diffusion are not necessary for the solution pair to remain smooth; rather, only a mix of partial dissipation and diffusion in the order of two derivatives suffices.

Very recently, the authors in [22] have shown that in case $N = 2$, the solution pair remains smooth in any of the following three cases:

1. $\alpha \geq \frac{1}{2}, \beta \geq 1$,
2. $\alpha \geq 2, \beta = 0$,
3. $\frac{1}{2} > \alpha \geq 0, 2\alpha + \beta > 2$.

In particular, their result implies that in the range of $\alpha \in [0, \frac{1}{2})$, $\beta$ must satisfy

$$\beta > 2 - 2\alpha.$$  \hspace{1cm} (2)

These results implied that if $\alpha = 0$, then $\beta > 2$ was necessary to obtain global regularity result. This was improved in [31] to show that either of the following conditions suffices:

1. $\alpha = 0, \beta > \frac{3}{2}$,
2. $\frac{1}{2} > \alpha > 0, \frac{3}{2} \geq \beta > \frac{5}{4}, \alpha + 2\beta > 3$.

In particular, this implies that in the range of $\alpha \in (0, \frac{1}{2})$, $\beta$ must satisfy

$$\beta > \frac{3 - \alpha}{2}$$ \hspace{1cm} (3)

(cf. also [32]). In this paper we make further improvement in this direction. Let us present our results.

**Theorem 1.1.** Let $N = 2, \nu, \eta > 0, \alpha > \frac{1}{3}, \beta = 1$. Then for all initial data pair $(u_0, b_0) \in H^s(\mathbb{R}^2) \times H^s(\mathbb{R}^2), s \geq 3$, there exists a unique global strong solution pair $(u, b)$ to (1) such that

$$u \in C([0, \infty); H^s(\mathbb{R}^2)) \cap L^2([0, \infty); H^{s+\alpha}(\mathbb{R}^2)),$$

$$b \in C([0, \infty); H^s(\mathbb{R}^2)) \cap L^2([0, \infty); H^{s+1}(\mathbb{R}^2)).$$

**Theorem 1.2.** Let $N = 2, \nu, \eta > 0, \alpha \in (0, \frac{1}{4}], \beta \in (1, \frac{3}{2}]$ such that

$$3 < 2\beta + \frac{2\alpha}{1 - \alpha}.$$ \hspace{1cm} (4)

Then for all initial data pair $(u_0, b_0) \in H^s(\mathbb{R}^2) \times H^s(\mathbb{R}^2), s \geq 3$, there exists a unique global strong solution pair $(u, b)$ to (1) such that
Remark 1.1. (1) We observe that (4) is equivalent to
\[ \frac{3}{2} - \frac{\alpha}{1 - \alpha} < \beta, \]
and this is a better lower bound than that of (2) or (3) for \( \alpha \in (0, \frac{1}{3}] \).

(2) Theorem 1.1 also represents the smaller lower bound for the sum of \( \alpha + \beta \) at \( 1 + \frac{1}{3} \) required for the solution pair to remain smooth for all time in comparison to the previous works such as [24] and [26] at \( \alpha + \beta \geq 1 + \frac{N}{2} \) in \( N \)-dimension and [22] at \( \alpha + \beta \geq \frac{3}{2} \) in two-dimension.

(3) There are various spaces of functions in which one may obtain local well-posedness of the MHD system. We chose to state above for simplicity. The local theory may be obtained by using mollifiers as done in [14] and we omit the details referring interested readers to [2] where the authors considered (1) in case \( N = 2, \nu = 0, \eta > 0, \beta = 1 \) and showed in particular the existence of its weak solution pair (cf. also [19] and [26]).

(4) After this work was completed, this direction of research has caught much attention from many mathematicians and a remarkable development with new results has been seen. In particular, we mention that in [3] and [11], the authors obtained the global regularity result in the case \( \alpha = 0, \beta > 1 \). We also mention numerical analysis results obtained in [21] concerning the interesting case \( \alpha = 0, \beta = 1 \).

In the following section, let us set up notations and summarize key lemmas that will be used repeatedly. Thereafter, we prove our theorems.

2. Preliminaries

Let us denote a constant that depends on \( a, b \) by \( c(a, b) \) and when the constant is not of significance, let us write \( A \lesssim B, A \approx B \) to imply that there exists some constant \( c \) such that \( A \leq cB, A = cB \) respectively. We also denote partial derivatives and vector components as follows:

\[ \frac{\partial}{\partial t} = \partial_t, \quad \frac{\partial}{\partial x} = \partial_1, \quad \frac{\partial}{\partial y} = \partial_2, \quad u = (u_1, u_2), \quad b = (b_1, b_2). \]

For simplicity we also set

\[ w = \nabla \times u, \quad j = \nabla \times b, \quad (5) \]

\[ X(t) = \|w(t)\|_{L^2}^2 + \|j(t)\|_{L^2}^2, \quad Y(t) = \|\nabla w(t)\|_{L^2}^2 + \|\nabla j(t)\|_{L^2}^2. \]

We use the following well-known inequalities:

Lemma 2.1. Let \( f \) be divergence-free vector field such that \( \nabla f \in L^p, p \in (1, \infty) \). Then the following inequality holds:

\[ \|\nabla f\|_{L^p} \leq c(p)\|\text{curl } f\|_{L^p}. \]
Lemma 2.2. (cf. [13]) Let \( f, g \) be smooth such that \( \nabla f \in L^{p_0}, \Lambda^{s-1}g \in L^{p_2}, \Lambda^{s}f \in L^{p_3}, g \in L^{p_4}, p \in (1, \infty) \). Let \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4} \), \( p_2, p_3 \in (1, \infty) \), \( s > 0 \). Then the following inequality holds:

\[
\|\Lambda^{s}(fg) - f\Lambda^{s}g\|_{L^{p}} \lesssim (\|\nabla f\|_{L^{p_1}}\|\Lambda^{s-1}g\|_{L^{p_2}} + \|\Lambda^{s}f\|_{L^{p_3}}\|g\|_{L^{p_4}}).
\]

Lemma 2.3. (cf. [5], [12]) For any \( \alpha \in [0, 1], x \in \mathbb{R}^N, T^N \) and \( f, \Lambda^{2\alpha}f \in L^{p}, p \geq 2, \)

\[
2 \int |\Lambda^{\alpha}(f^{\frac{p}{2}})|^2 dx \leq p \int |f|^{p-2}f\Lambda^{2\alpha}f dx.
\]

Finally, the following product estimate has proven to be useful (e.g. [15], [16], [17], [30]):

Lemma 2.4. Let \( \sigma_1, \sigma_2 < 1, \sigma_1 + \sigma_2 > 0 \). Then there exists a constant \( c(\sigma_1, \sigma_2) > 0 \) such that

\[
\|fg\|_{\dot{H}^{\sigma_1+\sigma_2-1}} \leq c(\sigma_1, \sigma_2)\|f\|_{\dot{H}^{\sigma_1}}\|g\|_{\dot{H}^{\sigma_2}},
\]

for \( f \in \dot{H}^{\sigma_1}(\mathbb{R}^2), g \in \dot{H}^{\sigma_2}(\mathbb{R}^2) \).

3. Proof of Theorem 1.1

Throughout this section, we assume \( \alpha \in (\frac{1}{2}, \sqrt{2} - 1) \) as the case \( \alpha \in [\sqrt{2} - 1, \frac{1}{2}) \) may be done via slight modification using Gagliardo-Nirenberg inequalities. We note that the restriction of this range of \( \alpha \) in particular becomes crucial at (9); we chose the statements of Propositions 3.1-3.3 for simplicity of presentation. We work on

\[
\begin{aligned}
\partial_t u + (u \cdot \nabla)u - (b \cdot \nabla)b + \nabla \pi + \Lambda^{2\alpha}u &= 0, \\
\partial_t b + (u \cdot \nabla)b - (b \cdot \nabla)u + \Lambda^2b &= 0.
\end{aligned}
\]

(6)

Taking \( L^2 \)-inner products of (6) with \( u \) and \( b \) respectively, we can get

\[
\sup_{t \in [0, T]} \|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 + \int_0^T \|\Lambda^{\alpha}u\|_{L^2}^2 + \|\Lambda b\|_{L^2}^2 \, d\tau \leq c(u_0, b_0, T). \tag{7}
\]

It has been shown that the following proposition can be attained as long as \( \beta \geq 1 \) (cf. [22], [31]). We sketch its proof for completeness.

Proposition 3.1. Let \( N = 2, \nu, \eta > 0, \alpha \in (\frac{1}{2}, \sqrt{2} - 1), \beta = 1. \) Then for any solution pair \((u, b)\) to (1) in \([0, T]\) there exists a constant \( c(u_0, b_0, T) \) such that

\[
\sup_{t \in [0, T]} \|w(t)\|_{L^2}^2 + \|j(t)\|_{L^2}^2 + \int_0^T \|\Lambda^{\alpha}w\|_{L^2}^2 + \|\Lambda j\|_{L^2}^2 \, d\tau \leq c(u_0, b_0, T).
\]

Proof. Taking curls on (6), we obtain

\[
\begin{aligned}
\partial_t w + \Lambda^{2\alpha}w &= -(u \cdot \nabla)w + (b \cdot \nabla)j, \\
\partial_t j + \Lambda^2j &= -(u \cdot \nabla)j + (b \cdot \nabla)w + 2(\partial_1 b_1(\partial_1 u_2 + \partial_2 u_1) - \partial_1 u_1(\partial_1 b_2 + \partial_2 b_1)).
\end{aligned}
\]

(8)

Taking \( L^2 \)-inner products with \( w \) and \( j \) respectively and using incompressibility of \( u \) and \( b \), we estimate
We fix solution pair \( c \) by Hölder’s, Gagliardo-Nirenberg and Young’s inequalities. Absorbing diffusive term, (7) and Gronwall’s inequality complete the proof of Proposition 3.1.

Next two propositions are the keys to the improvement from previous results:

**Proposition 3.2.** Let \( N = 2, \nu, \eta > 0, \alpha \in \left( \frac{1}{2}, \sqrt{2} - 1 \right), \beta = 1. \) Then for any solution pair \((u, b)\) to (1) in \([0, T]\), for any \( \gamma \in (1, 1 + \alpha) \), there exists a constant \( c(u_0, b_0, T) \) such that

\[
\frac{1}{2} \partial_t (\|\Lambda^\gamma b\|_{L^2}^2 + \|j\|_{L^2}^2) + \|\Lambda^{1+\gamma} b\|_{L^2}^2 \\
= 2 \int [\partial_t b_1 (\partial_t u_2 + \partial_2 u_1) - \partial_t u_1 (\partial_t b_2 + \partial_2 b_1)] j \\
\lesssim \|\nabla b\|_{L^4} \|\nabla u\|_{L^2} \|j\|_{L^4} \\
\lesssim \|j\|_{L^2} \|\nabla j\|_{L^2} \|w\|_{L^2} \\
\leq \frac{1}{2} \|\Lambda^\gamma b\|_{L^2}^2 + c \|j\|_{L^2}^2 \|w\|_{L^2}^2
\]

by Hölder’s, Gagliardo-Nirenberg and Young’s inequalities. Now we use Lemmas 2.4 and 2.1 to estimate

\[
\|b \cdot \nabla\|_{H^{\gamma-1}} \lesssim \|b\|_{H^{\gamma-\alpha}} \|\nabla u\|_{H^{\alpha}} \lesssim \|b\|_{H^{\gamma-\alpha}} \|w\|_{H^{\alpha}}.
\]

We then use Gagliardo-Nirenberg inequality, (7) and Proposition 3.1 to further bound by

\[
\|b \cdot \nabla\|_{H^{\gamma-1}} \lesssim \left( \|b\|_{H^{2(\gamma-\alpha)}} \|\nabla b\|_{L^2}^{2(\gamma-\alpha)} \right) \|w\|_{H^{\alpha}} \lesssim \|w\|_{H^{\alpha}}.
\]

Next, we fix \( \epsilon \in (0, 1 - \alpha) \) and estimate

\[
\|\partial_t b_1 (\partial_t u_2 + \partial_2 u_1)\|_{H^{\gamma-1}} \lesssim \|\partial_t b_1\|_{H^{\gamma-1+\epsilon}} \|\nabla b\|_{H^{1+\epsilon}} \\
\lesssim \|\partial_t b_1\|_{L^2(2-\gamma-\epsilon)} \|\nabla b\|_{L^2}^{2(\gamma-1+\epsilon)} \|j\|_{H^{1-\epsilon}} \lesssim \|j\|_{L^2}^{2(1-\epsilon)} \|j\|_{H^{1+\epsilon}} \lesssim (1 + \|j\|_{H^{1+\epsilon}}^2)
\]

by Lemma 2.4, Gagliardo-Nirenberg inequalities, Lemma 2.1, Proposition 3.1 and Young’s inequality. Thus, absorbing diffusive term, we have

\[
\partial_t \|\Lambda^\gamma b\|_{L^2}^2 + \|\Lambda^{1+\gamma} b\|_{L^2}^2 \lesssim (\|w\|_{H^{\alpha}}^2 + 1 + \|j\|_{H^{1+\epsilon}}^2).
\]

Hence, by Proposition 3.1, integrating in time we obtain
Proposition 3.3. Let $N = 2, \nu, \eta > 0, \alpha \in \left(\frac{1}{3}, \sqrt{2} - 1\right), \beta = 1$. Then for any solution pair $(u, b)$ to (1) in $[0, T]$, for any $\gamma \in (1, 1 + \alpha)$, there exists a constant $c(u_0, b_0, T)$ such that

\[
\sup_{t \in [0, T]} \|w(t)\|^{2(1+\alpha)}_{L^{2(1+\alpha)}} + \int_0^T \|w\|^{2(1+\alpha)}_{L^{2(1+\alpha)}} \, d\tau \leq c(u_0, b_0, T).
\]

Proof. We fix $\gamma \in (1, 1 + \alpha)$ and denote by

\[
p = \frac{2(1+\alpha)}{2 - \gamma}.
\]

We estimate by multiplying the vorticity equation of (8) by $|w|^{p-2}w$ and integrating in space

\[
\frac{1}{p} \partial_t \|w\|^p_{L^p} + \int \Lambda^{2\alpha} |w|^{p-2} w dx = \int (b \cdot \nabla) j |w|^{p-2} w dx,
\]

where we used incompressibility of $u$. Using Lemma 2.3, because $p \geq 2$, and homogeneous Sobolev embedding $H^\alpha \hookrightarrow L^{\frac{2}{2-\alpha}}$ we can obtain

\[
\int |\Lambda^{2\alpha} w| |w|^{p-2} w dx \geq \frac{2}{p} \|w\|_{H^\alpha}^2 \geq c(p, \alpha) \|w\|^p_{L^{\frac{2}{2-\alpha}}}.
\]

Using this, we further estimate

\[
\frac{1}{p} \partial_t \|w\|^p_{L^p} + c(p, \alpha) \|w\|^p_{L^{\frac{2}{2-\alpha}}} \leq \|b\|_{L^\infty} \|\nabla j\|_{L^{\frac{2}{2-\alpha}}} \|w\|^{p-2}_{L^p} \|w\|_{L^{\frac{2}{2-\alpha}}},
\]

where we used the Hölder’s inequality. Now we use the homogeneous Sobolev embedding of $H^{\gamma-1} \hookrightarrow L^{\frac{2}{2-\alpha}}$ and Gagliardo-Nirenberg inequality to obtain

\[
\frac{1}{p} \partial_t \|w\|^p_{L^p} + c(p, \alpha) \|w\|^p_{L^{\frac{2}{2-\alpha}}} \lesssim \|b\|_{L^\infty} \|\nabla^j\|_{L^{\frac{2}{2-\alpha}}} \|\Lambda^{\gamma \jmath} w\|^{p-2}_{L^p} \|w\|_{L^{\frac{2}{2-\alpha}}}
\]

\[
\lesssim \|b\|_{L^\infty} \|\Lambda^{\gamma \jmath} w\|_{L^{\frac{2}{2-\alpha}}} \|\Lambda^{\gamma \jmath} w\|^{p-2}_{L^p} \|w\|_{L^{\frac{2}{2-\alpha}}}.
\]

We further bound by (7) and Proposition 3.2 to obtain

\[
\frac{1}{p} \partial_t \|w\|^p_{L^p} + c(p, \alpha) \|w\|^p_{L^{\frac{2}{2-\alpha}}}
\]

\[
\leq \frac{c(p, \alpha)}{2} \|w\|^p_{L^{\frac{2}{2-\alpha}}} + c \|\Lambda^{\gamma \jmath} w\|_{L^p}^\gamma \|w\|^{(p-2)(\frac{p}{p'})}_{L^p}
\]

\[
\leq \frac{c(p, \alpha)}{2} \|w\|^p_{L^{\frac{2}{2-\alpha}}} + c(1 + \|\Lambda^{\gamma \jmath} w\|^2_{L^p})(1 + \|w\|_{L^p})
\]

by Young’s inequalities. After absorbing the dissipative term, integrating in time gives
\[
\sup_{t \in [0,T]} \|w(t)\|_{L^P}^p + \int_0^T \|w\|_{L^{p\gamma}}^p \, dt \leq c(u_0, b_0, T)
\]
due to Proposition 3.2. This completes the proof of Proposition 3.3.

**Proposition 3.4.** Let \( N = 2, \nu, \eta > 0, \alpha \in \left( \frac{1}{3}, \sqrt{2} - 1 \right), \beta = 1. \) Then for any solution pair \((u, b)\) to (1) in \([0,T]\), there exists a constant \(c(u_0, b_0, T)\) such that

\[
\sup_{t \in [0,T]} \|\nabla w(t)\|_{L^2}^2 + \|\nabla j(t)\|_{L^2}^2 + \int_0^T \|\Lambda^\alpha \nabla w\|_{L^2}^2 + \|\Lambda \nabla j\|_{L^2}^2 \, dt \leq c(u_0, b_0, T).
\]

**Proof.** We apply \(\nabla\) on (8) and take \(L^2\)-inner products with \(\nabla w\) and \(\nabla j\) respectively to estimate

\[
\frac{1}{2} \partial_t (\|\nabla w(t)\|_{L^2}^2 + \|\nabla j(t)\|_{L^2}^2) + \|\Lambda^\alpha \nabla w\|_{L^2}^2 + \|\Lambda \nabla j\|_{L^2}^2
\]

\[
= - \int \nabla w \cdot \nabla u \cdot \nabla w \, dx - \int \nabla j \cdot \nabla u \cdot \nabla j \, dx
\]

\[
+ \int \nabla ((b \cdot \nabla) j) \cdot \nabla w + \nabla ((b \cdot \nabla) w) \cdot \nabla j \, dx
\]

\[
+ 2 \int \nabla [\partial_1 b_1 (\partial_1 u_2 + \partial_2 u_1)] \cdot \nabla j \, dx - 2 \int \nabla [\partial_1 u_1 (\partial_1 b_2 + \partial_2 b_1)] \cdot \nabla j \, dx = \sum_{i=1}^5 I_i.
\]

We estimate separately:

\[
I_1 \leq \|\nabla w\|_{L^2} \|\nabla u\|_{L^{\frac{\gamma}{\gamma-2}}} \|\nabla w\|_{L^{\frac{\gamma}{\gamma-2}}}
\]

\[
\lesssim \|\nabla w\|_{L^2} \|\nabla \|_{L^{\frac{\gamma}{\gamma-2}}} \|\Lambda^\alpha \nabla w\|_{L^2} \leq \frac{1}{8} \|\Lambda^\alpha \nabla w\|_{L^2}^2 + cY(t)\|w\|_{L^2}^2
\]

by Hölder’s inequality, homogeneous Sobolev embedding of \(\dot{H}^\alpha \hookrightarrow L^{\frac{2}{2\alpha}}, \) Lemma 2.1 and Young’s inequality. Next,

\[
I_2 \leq \|\nabla j\|_{L^4}^2 \|\nabla u\|_{L^2} \lesssim \|\nabla j\|_{L^2} \|\Delta j\|_{L^2} \|w\|_{L^2} \leq \frac{1}{8} \|\Delta j\|_{L^2}^2 + cY(t)
\]

by Hölder’s, Gagliardo-Nirenberg and Young’s inequalities. Next, we first integrate by parts and use the incompressibility conditions to obtain

\[
I_3 = \int \nabla ((b \cdot \nabla) j) \cdot \nabla w \, dx + \int \nabla ((b \cdot \nabla) w) \cdot \nabla j \, dx
\]

\[
= \int \Delta b \cdot (\nabla j) w + 2 \nabla b \cdot (\nabla \nabla j) w \, dx.
\]

We now estimate this by
\begin{align*}
I_3 & \lesssim (\|\nabla j\|_{L^2}^2 \|w\|_{L^2} + \|\nabla b\|_{L^4} \|\Delta j\|_{L^2} \|w\|_{L^4}) \\
& \lesssim (\|\nabla j\|_{L^2} \|\Delta j\|_{L^2} \|w\|_{L^2} + \|j\|_{L^2}^2 \|\nabla j\|_{L^2}^2 \|\Delta j\|_{L^2} \|w\|_{L^2}^2 \|\nabla w\|_{L^2}^2) \\
& \leq \frac{1}{8} \|\Delta j\|_{L^2}^2 + cY(t)
\end{align*}

due to the Hölder’s inequalities, Lemma 2.1, Gagliardo-Nirenberg inequalities, Proposition 3.1 and Young’s inequalities. Finally, after integrating by parts again,

\begin{align*}
I_4 + I_5 & \lesssim \int \nabla b \nabla u \Delta j \, dx \lesssim \|\nabla b\|_{L^4} \|w\|_{L^4} \|\Delta j\|_{L^2}
\end{align*}

by Hölder’s inequality and Lemma 2.1. Note this is same as the second term of \(I_3\) and hence its identical estimate suffices.

Therefore, absorbing dissipative and diffusive terms, we have

\begin{align*}
\partial_t Y(t) + \|\Lambda^\alpha \nabla w\|_{L^2}^2 + \|\Lambda \nabla j\|_{L^2}^2 \lesssim Y(t)(1 + \|w\|_{L^2}^2).
\end{align*}

Now it can be checked that

\begin{align*}
1 < 2 - \frac{\alpha(1 + \alpha)}{1 - \alpha} < 1 + \alpha \quad \forall \alpha \in \left(\frac{1}{3}, \sqrt{2} - 1\right) \quad (9)
\end{align*}

and hence we can choose \(\gamma = 2 - \frac{\alpha(1 + \alpha)}{2(1 - \alpha)}\) so that by Hölder’s inequality and Proposition 3.3,

\begin{align*}
\int_0^T \|w\|_{L^2}^2 \, d\tau = \int_0^T \|w\|_{L^{2(1 + \alpha)}(1 - \alpha)}^2 \, d\tau \leq T^{\frac{2 - 1 + \alpha}{1 + \alpha}} \left(\int \|w\|_{L^{2(1 + \alpha)}(1 - \alpha)}^2 \right)^{\frac{2 - 1 + \alpha}{1 + \alpha}} \leq c(u_0, b_0, T).
\end{align*}

Therefore by Gronwall’s inequality,

\begin{align*}
\sup_{t \in [0, T]} \|\nabla w(t)\|_{L^2}^2 + \|\nabla j(t)\|_{L^2}^2 + \int_0^T \|\Lambda^\alpha \nabla w\|_{L^2}^2 + \|\Lambda \nabla j\|_{L^2}^2 \, d\eta \leq c(u_0, b_0, T).
\end{align*}

This completes the proof of Proposition 3.4.

Proof of Theorem 1.1

We now prove Theorem 1.1. We apply \(\Lambda^s, s \in \mathbb{R}^+\) on (6) and take \(L^2\)-inner products with \(\Lambda^s u\) and \(\Lambda^s b\) respectively to estimate using Lemma 2.2 and incompressibility conditions to estimate
\begin{align*}
\partial_t (||\Lambda^* u||_{L^2}^2 + ||\Lambda^* b||_{L^2}^2) & + ||\Lambda^{*+\alpha} u||_{L^2}^2 + ||\Lambda^{*+1} b||_{L^2}^2 \\
& = - \int \Lambda^*[ (u \cdot \nabla) u] \cdot \Lambda^* u - u \cdot \nabla \Lambda^* u \cdot \Lambda^* u \, dx \\
& - \int \Lambda^*[ (u \cdot \nabla) b] \cdot \Lambda^* b - u \cdot \nabla \Lambda^* b \cdot \Lambda^* b \, dx \\
& + \int \Lambda^*[ (b \cdot \nabla) b] \cdot \Lambda^* u - b \cdot \nabla \Lambda^* b \cdot \Lambda^* u \, dx \\
& + \int \Lambda^*[ (b \cdot \nabla) u] \cdot \Lambda^* b - b \cdot \nabla \Lambda^* u \cdot \Lambda^* b \, dx \\
& \lesssim (||\nabla u||_{L^\infty} ||\Lambda^{*-1} \nabla u||_{L^2} + ||\Lambda^* u||_{L^2} ||\nabla u||_{L^\infty}) ||\Lambda^* u||_{L^2} \\
& + (||\nabla u||_{L^2} ||\Lambda^{*-1} \nabla b||_{L^2} + ||\Lambda^* u||_{L^2} ||\nabla b||_{L^2}) ||\Lambda^* b||_{L^2} \\
& + (||\nabla b||_{L^2} ||\Lambda^{*-1} \nabla b||_{L^2} + ||\Lambda^* b||_{L^2} ||\nabla b||_{L^2}) ||\Lambda^* u||_{L^2} \\
& + (||\nabla b||_{L^2} ||\Lambda^{*-1} \nabla u||_{L^2} + ||\Lambda^* b||_{L^2} ||\nabla u||_{L^2}) ||\Lambda^* b||_{L^2} \\
& \lesssim (||w||_{L^2} \|\nabla w||_{L^2}^{1-\alpha} ||\Lambda^* u||_{L^2} ||\Lambda^{*+\alpha} u||_{L^2} + \|\Lambda^* b||_{L^2} + \|\Lambda^* u||_{L^2} ||\nabla j||_{L^2} ||\Lambda^* b||_{L^2} ||\Lambda^{*+1} b||_{L^2} \\
& + (||j||_{L^2} \|\nabla j||_{L^2} ||\Lambda^* u||_{L^2} + ||\Lambda^* b||_{L^2} ||\nabla w||_{L^2} ||\Lambda^* b||_{L^2} ||\Lambda^{*+1} b||_{L^2}) \\
& \cdot ||\Lambda^* u||_{L^2} \\
& \leq \frac{1}{2} (||\Lambda^{*+\alpha} u||_{L^2}^2 + ||\Lambda^{*+1} b||_{L^2}^2) + c(||\Lambda^* b||_{L^2}^2 + ||\Lambda^* u||_{L^2}^2). \\
\end{align*}

by Hölder’s and Gagliardo-Nirenberg inequalities, homogeneous Sobolev embedding of $H^\alpha \hookrightarrow L^{2\alpha}$. Due to Propositions 3.1 and 3.4 and Young’s inequalities we have

\begin{align*}
\partial_t (||\Lambda^* u||_{L^2}^2 + ||\Lambda^* b||_{L^2}^2) & + ||\Lambda^{*+\alpha} u||_{L^2}^2 + ||\Lambda^{*+1} b||_{L^2}^2 \\
& \leq \frac{1}{2} (||\Lambda^{*+\alpha} u||_{L^2}^2 + ||\Lambda^{*+1} b||_{L^2}^2) + c(||\Lambda^* b||_{L^2}^2 + ||\Lambda^* u||_{L^2}^2).
\end{align*}

Absorbing the dissipative and diffusive terms, Gronwall’s inequality implies the desired result.

4. Proof of Theorem 1.2

Throughout this section, we let $\alpha, \beta$ satisfy (4) and in particular we assume

\begin{equation}
2\beta + \frac{\alpha(1+\alpha)}{1-\alpha} < 3 < \alpha + 2\beta + \frac{\alpha(1+\alpha)}{1-\alpha}
\end{equation}

as the other case can be done similarly. We work on

\begin{equation}
\begin{cases}
\partial_t u + (u \cdot \nabla) u - (b \cdot \nabla) b + \nabla \pi + \Lambda^{2\alpha} u = 0, \\
\partial_t b + (u \cdot \nabla) b - (b \cdot \nabla) u + \Lambda^{2\beta} b = 0.
\end{cases}
\end{equation}

As before, taking $L^2$-inner products of (11) with $u$ and $b$ respectively, we immediately obtain

\begin{equation}
\sup_{t \in [0,T]} ||u(t)||_{L^2}^2 + ||b(t)||_{L^2}^2 + \int_0^T \|\Lambda^\alpha u||_{L^2}^2 + ||\Lambda^\beta b||_{L^2}^2 d\tau \leq c(u_0, b_0, T).
\end{equation}
Since $\beta \geq 1$, it is clear from the proof of Proposition 3.1 that its slight modification applied to the following system

$$
\begin{align*}
\partial_t w + \Lambda^{2\alpha} w &= -(u \cdot \nabla) w + (b \cdot \nabla) j \\
\partial_t j + \Lambda^{2\beta} j &= -(u \cdot \nabla) j + (b \cdot \nabla) w + 2[\partial_1 b_1 (\partial_1 u_2 + \partial_2 u_1) - \partial_1 u_1 (\partial_1 b_2 + \partial_2 b_1)]
\end{align*}
$$

(13)

leads to the following result:

**Proposition 4.1.** Let $N = 2, \nu, \eta > 0, \alpha \in (0, \frac{1}{4}], \beta \in (1, \frac{3}{2}]$ satisfy (10). Then for any solution pair $(u, b)$ to (1) in $[0, T]$, there exists a constant $c(u_0, b_0, T)$ such that

$$
\sup_{t \in [0, T]} \|w(t)\|_{L^2}^2 + \|j(t)\|_{L^2}^2 + \int_0^T \|\Lambda^\alpha w\|_{L^2}^2 + \|\Lambda^\beta j\|_{L^2}^2 \, dt \leq c(u_0, b_0, T).
$$

Now we prove the following proposition:

**Proposition 4.2.** Let $N = 2, \nu, \eta > 0, \alpha \in (0, \frac{1}{4}], \beta \in (1, \frac{3}{2}]$ satisfy (10). Then for any solution pair $(u, b)$ to (1) in $[0, T]$, for any $\gamma \in (\beta, \alpha + \beta)$, there exists a constant $c(u_0, b_0, T)$ such that

$$
\sup_{t \in [0, T]} \|\Lambda^\gamma b(t)\|_{L^2}^2 + \int_0^T \|\Lambda^{\beta+\gamma} b\|_{L^2}^2 \, dt \leq c(u_0, b_0, T).
$$

**Proof.** We fix $\gamma \in (\beta, \alpha + \beta)$. From the magnetic field equation of (11), we estimate after multiplying by $\Lambda^{\beta+\gamma} b$ and integrating in space

$$
\frac{1}{2} \partial_t \|\Lambda^\gamma b\|_{L^2}^2 + \|\Lambda^{\beta+\gamma} b\|_{L^2}^2 \\
\leq \left(\| (u \cdot \nabla) b \|_{H^{\gamma-\beta}} \|\Lambda^{\beta+\gamma} b\|_{L^2} + \| (b \cdot \nabla) u \|_{H^{\gamma-\beta}} \|\Lambda^{\beta+\gamma} b\|_{L^2}\right) \\
\leq \frac{1}{2} \|\Lambda^{\beta+\gamma} b\|_{L^2}^2 + c\left(\| (u \cdot \nabla) b \|^2_{H^{\gamma-\beta}} + \| (b \cdot \nabla) u \|^2_{H^{\gamma-\beta}}\right)
$$

by H"older’s and Young’s inequalities. Now by Lemma 2.4 and Gagliardo-Nirenberg inequality, Lemma 2.1 and Proposition 4.1, we estimate

$$
\| (u \cdot \nabla) u \|_{H^{\gamma-\beta}} \lesssim \| b \|^2_{H^{\beta+1-2\beta-\gamma}} \| \nabla u \|^2_{H^{\alpha}} \lesssim \| b \|^2_{L^2} \beta^{-2\gamma} \| \nabla b \|^2_{L^2} \beta^{-2\beta-\gamma} \| \nabla u \|^2_{L^2} \lesssim \| u \|^2_{H^{\alpha}} \lesssim \| u \|^2_{H^{\alpha}}.
$$

Next, we fix $\epsilon \in (\beta - 1, \beta - \alpha)$ and estimate using Lemma 2.4 and Gagliardo-Nirenberg inequalities, (12), Proposition 4.1 and Young’s inequality as follows:

$$
\| (u \cdot \nabla) b \|^2_{H^{\gamma-\beta}} \lesssim \| u \|^2_{H^{\beta+1-2\beta-\gamma}} \| \nabla b \|^2_{H^{\beta-\epsilon}} \\
\lesssim \| u \|^2_{L^2} \beta^{-2(\beta-\epsilon)-\gamma} \| \nabla u \|^2_{L^2} \beta^{-(2(\beta-\epsilon)-\gamma)} \| j \|^2_{H^{\beta-\epsilon}} \\
\lesssim \| j \|^2_{L^2} \beta^{-(2(\beta-\epsilon)-\gamma)} \beta^2 (1-\beta) \lesssim 1 + \| j \|^2_{H^{\beta}}.
$$

Therefore, we have shown

$$
\frac{1}{2} \partial_t \|\Lambda^\gamma b\|_{L^2}^2 + \|\Lambda^{\beta+\gamma} b\|_{L^2}^2 \leq \frac{1}{2} \|\Lambda^{\beta+\gamma} b\|_{L^2}^2 + c\left(\| u \|^2_{H^{\alpha}} + \| j \|^2_{H^{\beta}}\right).
$$

Integrating in time and using Proposition 4.1 complete the proof of Proposition 4.2.
Proposition 4.3. Let $N = 2, \nu, \eta > 0, \alpha \in (0, \frac{1}{3}], \beta \in (1, \frac{3}{2}]$ satisfy (10). Then for any solution pair $(u, b)$ to (1) in $[0, T]$, for any $\gamma \in (\beta, \alpha + \beta)$, there exists a constant $c(u_0, b_0, T)$ such that

$$\sup_{t \in [0, T]} \|w(t)\|_{L^{\frac{2(1+\alpha)}{2-\beta-\gamma}(1+\alpha)}}^\frac{2(1+\alpha)}{2-\beta-\gamma} + \int_0^T \|w\|_{L^{\frac{2(1+\alpha)}{2-\beta-\gamma}(1+\alpha)}}^\frac{2(1+\alpha)}{2-\beta-\gamma} d\tau \leq c(u_0, b_0, T).$$

Proof. We fix $\gamma \in (\beta, \alpha + \beta)$ and denote

$$p = \frac{2(1+\alpha)}{3-\beta-\gamma}.$$

Note due to (10), we have $3 - \beta - \gamma > 0$. We estimate by multiplying the vorticity equation of (13) by $|w|^{p-2}w$ and integrating in space, using Lemma 2.3 and the same homogeneous Sobolev embedding of $H^\alpha \hookrightarrow L^{\frac{2}{1+\alpha}}$ as before to obtain

$$\frac{1}{p} \partial_t \|w\|_{L^p}^p + c(p, \alpha) \|w\|_{L^\frac{2p}{p-2}}^p \leq \|b\|_{L^\infty} \|\nabla j\|_{L^\frac{p}{p-2}} \|w\|_{L^\frac{p}{p-2}}^p \|w\|_{L^\frac{p}{p-2}}^p$$

by Hölder’s inequality. By our choice of $p$, we see that we may continue our estimate by

$$\frac{1}{p} \partial_t \|w\|_{L^p}^p + c(p, \alpha) \|w\|_{L^\frac{2p}{p-2}}^p \leq \|b\|_{L^\infty} \|\nabla j\|_{L^\frac{2p}{p-2}} \|w\|_{L^\frac{p}{p-2}}^p \|w\|_{L^\frac{p}{p-2}}^p \leq \|b\|_{L^\frac{2p}{p-2}} \|\Lambda^\gamma b\|_{L^2} \|\Lambda^{\beta+\gamma} b\|_{L^2} \|w\|_{L^\frac{p}{p-2}}^p \|w\|_{L^\frac{p}{p-2}}^p \leq \frac{c(p, \alpha)}{2} \|w\|_{L^\frac{p}{p-2}}^p + c(\|\Lambda^{\beta+\gamma} b\|_{L^2} + 1)(\|w\|_{L^p}^p + 1)$$

where we used the Gagliardo-Nirenberg inequality, homogeneous Sobolev embedding of $H^{\beta+\gamma-2} \hookrightarrow L^{\frac{2}{\beta+\gamma-2}}$, Propositions 4.1 and 4.2, Young’s inequalities.

Absorbing dissipative term, Gronwall’s inequality and Proposition 4.2 complete the proof of Proposition 4.3.

Proposition 4.4. Let $N = 2, \nu, \eta > 0, \alpha \in (0, \frac{1}{3}], \beta \in (1, \frac{3}{2}]$ satisfy (10). Then for any solution pair $(u, b)$ to (1) in $[0, T]$, there exists a constant $c(u_0, b_0, T)$ such that

$$\sup_{t \in [0, T]} \|\nabla w(t)\|_{L^2}^2 + \|\nabla j(t)\|_{L^2}^2 + \int_0^T \|\Lambda^\alpha \nabla w\|_{L^2}^2 + \|\Lambda^\beta \nabla j\|_{L^2}^2 d\tau \leq c(u_0, b_0, T).$$

Proof. Similarly as before, we apply $\nabla$ on (13), take $L^2$-inner products with $\nabla w, \nabla j$ respectively to estimate
In sum, after absorbing dissipative and diffusive terms, we have

\[
\frac{1}{2} \partial_t (\| \nabla w(t) \|^2_{L^2} + \| \nabla j(t) \|^2_{L^2} + \| \Lambda^\alpha \nabla w \|^2_{L^2} + \| \Lambda^\beta \nabla j \|^2_{L^2}) = - \int \nabla w \cdot \nabla u \cdot \nabla w \, dx - \int \nabla j \cdot \nabla u \cdot \nabla j \, dx + \int \nabla ((b \cdot \nabla) j) \cdot \nabla w + \nabla ((b \cdot \nabla) w) \cdot \nabla j \, dx + 2 \int \nabla \partial_t b_1 (\partial_t u_2 + \partial_2 u_1) \cdot \nabla j \, dx - 2 \int \nabla [\partial_t u_1 (\partial_t b_2 + \partial_2 b_1)] \cdot \nabla j \, dx = \sum_{i=1}^5 I_i.
\]

As before,

\[
I_1 \leq \| \nabla w \|_{L^2} \| \nabla u \|_{L^\infty} \| \nabla w \|_{L^\infty} \leq \frac{1}{8} \| \Lambda^\alpha \nabla w \|^2_{L^2} + c Y(t) \| w \|^2_{L^\infty}
\]

by Hölder’s inequality, homogeneous Sobolev embedding of $H^\alpha \hookrightarrow L^{\frac{2\alpha}{2\alpha - \beta}}$ and Young’s inequalities. Next,

\[
I_2 \leq \| \nabla j \|^2_{L^2} \| \nabla u \|_{L^2} \lesssim \| \nabla j \|^2_{L^2} \| \Lambda^\beta \nabla j \|^2_{L^2} \leq \frac{1}{8} \| \Lambda^\beta \nabla j \|^2_{L^2} + c Y(t)
\]

by Hölder’s inequality, Proposition 4.1, Gagliardo-Nirenberg and Young’s inequalities. Next, we estimate $I_3$ after same integration by parts in the proof of Proposition 3.4,

\[
I_3 \lesssim (\| \nabla j \|^2_{L^2} \| w \|_{L^2} + \| \nabla b \|_{L^{\frac{2\beta}{\beta - 1}}} \| \Delta j \|_{L^{\frac{2\beta}{\beta - 1}}} \| w \|_{L^2})
\]

\[
\lesssim \| \nabla j \|^{2(\frac{2\beta}{2\beta - 1})}_{L^2} \| \Lambda^\beta \nabla j \|^{2(\frac{2\beta}{2\beta - 1})}_{L^2} + \| \nabla b \|^{\frac{2\beta + \gamma - 3}{2\beta - 1}}_{L^2} \| \Lambda^{\beta + \gamma} b \|_{L^2}^{\frac{2\beta - \gamma}{2\beta - 1}} \| \Lambda^\beta \nabla j \|_{L^2}
\]

\[
\leq \frac{1}{8} \| \Lambda^\beta \nabla j \|^2_{L^2} + c (Y(t) + 1 + \| \Lambda^{\beta + \gamma} b \|^2_{L^2})
\]

by Hölder’s and Gagliardo-Nirenberg inequalities, homogeneous Sobolev’s embedding of $H^{\beta - 1} \hookrightarrow L^{\frac{2\beta}{\beta - 1}}$ and Proposition 4.1.

The estimates of $I_4$ and $I_5$ are simple: after the same integration by parts as before, we have

\[
I_4 + I_5 \lesssim \int |\nabla b| \| \nabla u \| \| \Delta j \| \, dx \lesssim \| \nabla b \|_{L^{\frac{2\beta}{\beta - 1}}} \| \Delta j \|_{L^{\frac{2\beta}{\beta - 1}}} \| w \|_{L^2}
\]

by Hölder’s inequality and hence the same estimate as the second term of $I_3$ suffices. In sum, after absorbing dissipative and diffusive terms, we have

\[
\partial_t Y(t) + \| \Lambda^\alpha \nabla w \|^2_{L^2} + \| \Lambda^\beta \nabla j \|^2_{L^2} \lesssim (Y(t) + 1) (1 + \| w \|^2_{L^\infty} + \| \Lambda^{\beta + \gamma} b \|^2_{L^2}).
\]

Now we see that we may choose $\gamma = 3 - \beta - \frac{\alpha (1 + \alpha)}{1 - \alpha}$ so that

\[
\beta < \gamma < \alpha + \beta
\]

due to (10) and therefore, by Hölder’s inequality we have...
\[
\int_0^T \|w\|^2_{L^2} \, dt \leq T^{2+\alpha+2-\beta} \left( \int_0^T \|w\|^{2+\alpha+2-\beta}_{L^{2+\alpha+2-\beta}} \, dt \right)^{\frac{2-\beta-\alpha}{2+\alpha}} \leq c(u_0, b_0, T)
\]
due to Proposition 4.3. Thus, Gronwall’s inequality and Proposition 4.2 complete the proof of Proposition 4.4.

**Proof of Theorem 1.2**

We are now ready to complete the proof of Theorem 1.2. Similarly as before we apply $\Lambda^s, s \in \mathbb{R}^+$ on (11) and take $L^2$-inner products with $\Lambda^s u$ and $\Lambda^s b$ respectively to estimate using Lemma 2.2

\[
\partial_t (\|\Lambda^s u\|^2_{L^2} + \|\Lambda^s b\|^2_{L^2}) + \|\Lambda^{s+\alpha} u\|^2_{L^2} + \|\Lambda^{s+\beta} b\|^2_{L^2}
\]

\[
\lesssim \|\nabla u\|_{L^6} \|\Lambda^s u\|_{L^2} \|\Lambda^s u\|_{L^2}^{\frac{1}{6}}
\]

\[
+ (\|\nabla u\|_{L^6} \|\Lambda^{s+\beta} u\|_{L^6}^{\frac{1}{6}} + \|\Lambda^{s+\beta} u\|_{L^6} \|\nabla b\|_{L^6}^{\frac{1}{6}}) \|\Lambda^s u\|_{L^2}
\]

\[
+ (\|\nabla b\|_{L^6} \|\Lambda^{s+\beta} b\|_{L^6}^{\frac{1}{6}} + \|\Lambda^{s+\beta} b\|_{L^6} \|\nabla u\|_{L^6}^{\frac{1}{6}}) \|\Lambda^s b\|_{L^2}
\]

\[
\lesssim \|\nabla u\|_{L^2} \|\Lambda^s u\|_{L^2} \|\Lambda^{s+\alpha} u\|_{L^2} + \|\Lambda^s b\|_{L^2} \|\Lambda^{s+\beta} b\|_{L^2}
\]

\[
+ (\|w\|_{L^2}^{1-\alpha} \|\nabla u\|_{L^2} \|\Lambda^{s+\alpha} u\|_{L^2}^{\frac{a+2\beta-1}{2}} \|\Lambda^{s+\beta} b\|_{L^2}^{\frac{1-\alpha}{2}} + \|\Lambda^{s+\beta} u\|_{L^2} \|\Lambda^s b\|_{L^2}
\]

\[
+ (\|\Lambda^s b\|_{L^2} \|\Lambda^{s+\alpha} u\|_{L^2}^{\frac{a+2\beta-1}{2}} \|\Lambda^{s+\beta} b\|_{L^2}^{\frac{1-\alpha}{2}} + \|\Lambda^s u\|_{L^2} \|\nabla w\|_{L^2}^{\frac{1-\alpha}{2}}) \|\Lambda^s u\|_{L^2}
\]

\[
\leq \frac{1}{2} (\|\Lambda^{s+\alpha} u\|^2_{L^2} + \|\Lambda^{s+\beta} b\|^2_{L^2}) + c(\|\Lambda^s u\|^2_{L^2} + \|\Lambda^s b\|^2_{L^2})
\]

by Hölder’s inequalities, Lemma 2.1, homogeneous Sobolev embedding of $\dot{H}^s \hookrightarrow L^{2-s}$ and Gagliardo-Nirenberg and Young’s inequalities. Absorbing the dissipative and diffusive terms, Gronwall’s inequality complete the proof of Theorem 1.2.

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