Inverse Problems in Integral Formulas

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It is well known that the values of an analytic function are closely interrelated, since its value along a closed contour \( \Gamma \) completely determines its value inside \( \Gamma \) (Cauchy formula); i.e., the values on \( \Gamma \) are boundary controls \([1]\).

A broad class of ill-posed problems arising in physics, engineering, and other fields consists of inverse problems (see, e.g., \([2–4]\)). An inverse problem for the Cauchy integral formula in the case of a circle was solved in \([5]\).

Let \( D \) be a star-shaped domain with respect to the origin in the plane of a complex variable \( z \), \( f(z) \) be a holomorphic function in \( D \), and

\[
L_{1,0}[f(z)] = f(z) + zf'(z).
\]

The following result hold true \([6]\).

**Theorem 1.** If \( f(z) \) is holomorphic in \( D \) and continuous, together with its derivative \( f'(z) \), in the closed domain \( \overline{D} \), then, for \( z \in D \),

\[
f(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{L_{1,0}[f(\xi)]}{\xi - \tau z} d\xi,
\]

and

\[
f(z) = f(0) + \frac{1}{2\pi i} \int_{\Gamma} \frac{f'(\xi)}{\xi - \tau z} d\xi
\]

(\( \tau \) is real), where the integrals are taken along the contour \( \Gamma \) in the positive direction.

In this paper, we solve inverse problems for integral formulas (1) and (2) (Theorem 2). Given the images of the operator \( L_{1,0}[f(z)] \) (the values of \( f'(z) \) up to the additive constant \( f(0) \)) on any circle \( C_r \) \(|z| = r, 0 < r < R\), lying in the disk \( K_R; |z| < R \), formula (3) (formula (4)) expresses its values at the other points of \( K_R \); moreover, for \( z \in K_r \) \((0 < r < R)\), this formula passes into (1) ((2), respectively).

Let \( K_r \) be the disk \(|z| < r \) and \( C_r \) be its boundary \(|z| = r \) \((0 < r < R)\).

**Theorem 2.** If \( f(z) \) is holomorphic in the disk \( K_R; |z| < R \), then, in \( K_R \setminus C_r \),

\[
f(z) = \frac{1}{2\pi i} \int_{0}^{+\infty} e^{-\zeta t} \left[ \frac{\zeta}{c_{1}^r} \left( e^{\xi \zeta} - 1 \right) \right] L_{1,0}[f(\xi)] d\xi,
\]

and

\[
f(z) = f(0) + \frac{1}{2\pi i} \int_{0}^{+\infty} e^{-\zeta t} \left[ \frac{\zeta}{c_{1}^r} \left( e^{\xi \zeta} - 1 \right) \right] f'(\xi) d\xi,
\]

where \( 0 < r < R \) and \( C_r \) is traversed in the positive direction.

**Proof.** Let \( z \) be an arbitrary point of \( K_R \) that does not belong to the circle \( C_r \), and let \( C_\rho \) be a circle centered at the origin of radius \( \rho \) \((\rho < R)\) such that \( z \) and \( C_\rho \) lie inside \( C_r \). Consider a closed doubly connected domain \( \partial \) whose boundary is the composite contour \( \Gamma = C_\rho + C_r \). In this domain, the function

\[
\frac{\zeta}{c_{1}^r} \left( e^{\xi \zeta} - 1 \right) L_{1,0}[f(\xi)]
\]

is an analytic function of \( \xi \) (with \( z \) and \( t \) being constant). Therefore, by the second mean value theorem,

\[
\int_{\Gamma} \frac{\zeta}{c_{1}^r} \left( e^{\xi \zeta} - 1 \right) L_{1,0}[f(\xi)] d\xi = 0
\]
or, equivalently, 
\[
\int_{c_p}^{c_t} \frac{\xi}{L_{1,0}[f(\xi)]} d\xi = \int_{c_p}^{c_t} \frac{\xi}{L_{1,0}[f(\xi)]} d\xi,
\]
whence
\[
\frac{1}{2\pi i} \int_{c_p}^{c_t} \frac{e^{-\xi}}{\xi} d\xi \int_{c_p}^{c_t} \frac{L_{1,0}[f(\xi)]}{\xi} d\xi = \frac{1}{2\pi i} \int_{c_p}^{c_t} \frac{e^{-\xi}}{\xi} d\xi \int_{c_p}^{c_t} \frac{L_{1,0}[f(\xi)]}{\xi} d\xi
\]
or
\[
\frac{1}{2\pi i} \int_{c_p}^{c_t} \frac{L_{1,0}[f(\xi)]}{\xi} d\xi \int_{c_p}^{c_t} \frac{e^{-\xi}}{\xi} d\xi = \frac{1}{2\pi i} \int_{c_p}^{c_t} \frac{e^{-\xi}}{\xi} d\xi \int_{c_p}^{c_t} \frac{L_{1,0}[f(\xi)]}{\xi} d\xi,
\]
where \(0 < \alpha < +\infty\).

However,
\[
\int_{c_p}^{c_t} \frac{e^{-\xi}}{\xi} d\xi = \int_{c_p}^{c_t} \frac{1}{\xi} \left( e^{-\xi} + \frac{1}{1!} e^{-\xi} + \frac{1}{2!} e^{-\xi} + \ldots + \frac{1}{n!} e^{-\xi} + \ldots \right) dt.
\]
Since \(\frac{t^n}{n!} < 1\) for \(0 \leq t < +\infty\), the series under the integral sign converges uniformly (for constant \(z\) and \(\xi\) such that \(|z| < |\xi|\) with respect to \(t \in [0, \alpha]\), where \(\alpha < +\infty\). Integrating this series with respect to \(t\) on the interval \([0, \alpha]\), we obtain
\[
\int_{c_p}^{c_t} \frac{e^{-\xi}}{\xi} d\xi = \int_{c_p}^{c_t} \frac{e^{-\xi}}{\xi} d\xi.
\]

The last series converges uniformly (for constant \(z\) and \(\xi\) such that \(|z| < |\xi|\)) with respect to \(\alpha\) for \(0 \leq \alpha \leq +\infty\), since \(\int_{c_p}^{c_t} \frac{t^n}{n!} \leq 1\) and, hence,
\[
\int_{c_p}^{c_t} \frac{e^{-\xi}}{\xi} d\xi = \sum_{n=1}^{+\infty} \frac{1}{n!} \left( e^{-\xi} + \frac{1}{1!} e^{-\xi} + \frac{1}{2!} e^{-\xi} + \ldots + \frac{1}{n!} e^{-\xi} + \ldots \right) dt.
\]

On the other hand, for \(|z| < |\xi| = \rho\), we have
\[
\int_{c_p}^{c_t} \frac{e^{-\xi}}{\xi} d\xi = \sum_{n=1}^{+\infty} \frac{1}{n!} \left( e^{-\xi} + \frac{1}{1!} e^{-\xi} + \frac{1}{2!} e^{-\xi} + \ldots + \frac{1}{n!} e^{-\xi} + \ldots \right) dt.
\]
since, for $|z| < |\xi| = R$, it is true that

$$\int_0^1 \frac{d\tau}{z - \tau z} = \int_0^\infty e^{\frac{zt}{\xi}} - 1 e^{-t} dt.$$  

**Corollary 2.** For $r = R$, formulas (3) and (4) become (1) and (2), respectively, for $z \in K_R$, where $\Gamma$ is $C_R$.

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