Külshammer ideals and the scalar problem for blocks with dihedral defect groups

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Abstract. In by now classical work, K. Erdmann classified blocks of finite groups with dihedral defect groups (and more generally algebras of dihedral type) up to Morita equivalence. In the explicit description by quivers and relations of such algebras with two simple modules, several subtle problems about scalars occurring in relations remained unresolved. In particular, for the dihedral case it is a longstanding open question whether blocks of finite groups can occur for both possible scalars 0 and 1.

In this article, using Külshammer ideals (a.k.a. generalized Reynolds ideals), we provide the first examples of blocks where the scalar is 1, thus answering the above question to the affirmative. Our examples are the principal blocks of $\text{PGL}_2(F_q)$, the projective general linear group of $2 \times 2$-matrices with entries in the finite field $F_q$, where $q = p^n \equiv \pm 1 \mod 8$ with $p$ an odd prime number.

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1. Introduction

Since the pioneering work of J. Rickard ([27], [28], [29]) and of D. Happel ([13], [14]), derived equivalences and derived invariants have received much attention in representation theory. For the representation theory of finite groups, M. Broué’s abelian defect group conjecture ([3]) plays a most prominent role in these developments. Although quite a few derived invariants have been discovered so far, such as the center ([27]), Hochschild homology and cohomology ([27]), cyclic homology ([21]), $K$-theory ([30]), etc., most of these invariants are very difficult to compute. In the eighties of the last century, B. Külshammer ([22]) introduced a sequence of ideals in the center of a symmetric algebra defined over an algebraically closed field of positive characteristic and he called them generalized Reynolds ideals. L. Héthelyi, E. Horváth, B. Külshammer and J. Murray proved that these ideals are invariant under Morita equivalence ([15]). In 2005, A. Zimmermann ([31]) proved that these ideals are even invariant under derived equivalences.

A remarkable feature of Külshammer ideals is that they are in principal accessible for explicit computations. In particular, this makes these new derived invariants potentially useful for distinguishing algebras up to derived equivalence (which in general is a very hard problem, due to the lack of ‘computable’ derived invariants).

For the definition and more background on Külshammer ideals we refer the reader to Section 2 below, and for other recent developments around Külshammer ideals to the articles [1], [2], [18], [19], [32], [33].

The objective of this article is to present another application of Külshammer ideals to the scalar problem for blocks with dihedral defect groups which have two simple modules up to isomorphisms. Before stating our main result, we review some background.

Finite-dimensional algebras over an algebraically closed field are divided into three (mutually exclusive) representation types: finite, tame and wild. For blocks of group algebras of finite groups, the representation type is characterized by their defect groups. A block has finite representation type if and only if its defect groups are cyclic. These blocks are well understood, see [4]. Tame representation type only occurs when the ground field has characteristic 2 and when the defect groups are dihedral, semi-dihedral or generalized quaternion. In a series of seminal papers ([5], [6], [7], [8], [9], [10]) and the monograph [11], K. Erdmann introduced the larger classes of algebras of dihedral, semidihedral and quaternion type and classified these algebras up to Morita equivalence. Based on her work, the first named author later classified these algebras up to derived equivalence ([16], [17]). Nevertheless, several subtle problems remain unresolved, most of them connected to certain scalars occurring in relations. Let us explain in detail the situation for dihedral blocks with two simple modules. We recall the classification of K. Erdmann and the first named author.

**Theorem 1.1.** ([11], [16] Proposition (2.1)) Let $k$ be an algebraically closed field of characteristic 2 and $B$ a dihedral block of a finite group with dihedral defect groups,
of order $2^n$, and with two simple modules. Then there exists a scalar $c \in \{0,1\}$ such that $B$ is derived equivalent to the algebra $D(2A)^s(c)$ with $s = 2^{n-2}$ defined by the following quiver with relations:

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  α  ·  β  γ
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$\gamma\beta = 0$, $\alpha^2 = c(\alpha\beta\gamma)^s$, $(\alpha\beta\gamma)^s = (\beta\gamma\alpha)^s$

The following is a longstanding open problem.

**Question 1.2.** Can both values $c = 0$ and $c = 1$ occur for blocks of finite groups with dihedral defect groups and with two simple modules?

It is known that the above algebras for different scalars $c = 0$ and $c = 1$ are not derived equivalent ([20], see also [19]).

The following seems to be a complete list of dihedral blocks where the scalar could so far be determined; note that it is usually very hard to determine the value of the scalar directly, even for small examples.

**Example 1.3.** Let $k$ be an algebraically closed field of characteristic 2.

1. The group algebra $kS_4$ of the symmetric group $S_4$ is a block with dihedral defect groups and has two simple modules; for this block, the scalar is $c = 0$ ([11]).

2. In [12, Section 1.5], K. Erdmann constructed infinitely many dihedral blocks with two simple modules for which $c = 0$. These blocks are principal blocks of certain quotients of $G = \text{GU}_2(q)$, the general unitary group, where $q \equiv 3 \mod 4$. More precisely, let $\overline{G} := G/O_2'(G)$, then the principal 2-blocks of $G_1 := \overline{G}/Z(\overline{G})$ have dihedral defect groups and two simple module. Note that these blocks are not Morita equivalent to an algebra of the form $D(2A)^s(0)$ (the quivers of their basic algebras have two loops, see [12, Section 1.5]), but they are derived equivalent to some algebra $D(2A)^s(0)$ by [16, Proposition (2.1)].

In view of these examples, it was believed by experts that the scalar $c$ should be always 0 for blocks of finite groups. Surprisingly, we prove in this article the following theorem which gives the first examples for which the scalar $c = 1$ does occur.

**Theorem 1.4.** Let $k$ be an algebraically closed field of characteristic 2. Suppose that $q = p^m \equiv \pm 1 \mod 8$ for $p$ an odd prime number. Then the principal block of the group algebra $k\text{PGL}_2(q)$ of the projective general linear group is a dihedral block with two simple modules for which $c = 1$.

As a direct application we get that the dihedral blocks with two simple modules considered in [12] can not be derived equivalent to the principal blocks of the projective general linear groups considered here.
The main tool of the proof of our main theorem are Kulshammer ideals, a.k.a. generalized Reynolds ideals. These form a descending series of ideals of the center of a symmetric algebra in positive characteristic; for the definition and basic properties we refer to Section 2 below. The crucial fact we use is a recent result of A. Zimmermann and the first author [19, Thms 1.1 and 4.1], showing that for different scalars \( c = 0 \) and \( c = 1 \), the factor rings of the centre of the block modulo the first Kulshammer ideal are not isomorphic. More precisely, one can distinguish these factor rings by the dimension of the Jacobson radical modulo its square; this dimension is 3 when \( c = 0 \) and it is 2 if \( c = 1 \) (see [19, 4.5.2] for more details). Hence, given a block with dihedral defect group and two simple modules, these results allow in principle to decide whether the scalar is 0 or \( q \), at least if one is able to explicitly compute the first Kulshammer ideal for the block in question.

Remark 1.5. Our method cannot treat the case where \( q = p^m \equiv \pm 3 \mod 8 \). The main reason is that Kulshammer ideals are ideals of the center of the block in question and that when \( q = p^m \equiv \pm 3 \mod 8 \), the defect groups are of order 8 in which case the center is too small for applying the results of [19].

This article is organized as follows. We give a short introduction to Kulshammer ideals in Section 2 and some basic facts about the groups \( PGL_2(q) \) are collected in Section 3. Then Section 4 contains the proof of the theorem modulo a key proposition whose proof is given in Section 5. For simplicity, we concentrate on the case \( q \equiv 1 \mod 8 \) in these two sections. The other case \( q \equiv -1 \mod 8 \) is similar and we will state the corresponding results without proofs in the final section.

2. Kulshammer ideals (a.k.a. Generalized Reynolds ideals)

Let \( k \) be an algebraically closed field of positive characteristic \( p > 0 \). Let \( A \) be a (finite-dimensional) symmetric algebra, i.e. there exists a non-degenerate bilinear form \((\ ,\ ) : A \times A \to k\) such that for \( a, b, c \in A\),
\[
(a, b) = (b, a) \quad \text{and} \quad (ab, c) = (a, bc).
\]
Denote by \( K(A) \) the vector space generated by all commutators \([a, b] = ab - ba\) with \( a, b \in A\). For \( n \geq 0 \), we define
\[
T_n(A) = \{ x \in A \mid x^{p^n} \in K(A) \}.
\]
The \( n \)-th Kulshammer ideal of \( A \) is defined as the orthogonal space (with respect to the symmetrizing form on \( A\),
\[
T_n^\perp(A) = \{ x \in A \mid (x, y) = 0 \text{ for all } y \in T_n(A) \}.
\]
We then have the following fundamental lemma (which is not difficult to prove).

Lemma 2.1 ([23, no.(36)]). The subspaces \( T_n^\perp(A) \) form a decreasing sequence of ideals of the center \( Z(A) \)
\[
Z(A) = K(A)^\perp = T_0^\perp(A) \supseteq T_1^\perp(A) \supseteq T_2^\perp(A) \supseteq \cdots.
\]
We will often consider the factor rings $\overline{Z}(A) := Z(A)/T^+_1(A)$, and their Jacobson radicals. For an algebra $B$ we denote by $J(B)$ the Jacobson radical.

We illustrate the above notions using the typical examples of group algebras. Let $G$ be a finite group and $A = kG$ the group algebra. Then $A$ is a symmetric algebra via the following paring:

$$(\ ,
) : kG \times kG \to k , \ (g, h) = \delta_{g,h^{-1}} = \begin{cases} 1 & \text{if } g = h^{-1} \\ 0 & \text{otherwise} \end{cases}$$

for $g, h \in G$ and extension by linearity. Let $X$ be a subset of $G$. We introduce the following notations:

$$X^+ = \sum_{x \in X} x , \text{ and } X^{g^n} = \{g \in G : g^n \in X\}.$$  

The Külshammer ideals $T^+_n(kG)$ admit a nice description ([23, no.(38))]: for every $n \geq 0$, the vector space $T^+_n(kG)$ has a basis $\{(C^{g^n})^+ : C \in \mathcal{C}(G)\}$ where $\mathcal{C}(G)$ denotes the set of conjugacy classes of $G$.

Example 2.2. Let $k$ be an algebraically closed field $k$ of characteristic 2. We are going to compute Külshammer ideals for a cyclic group $G = C_{2m} = \langle g \rangle$ of order $2^m$ with $m \geq 1$. In particular we shall look at the factor ring $\overline{Z}(kG) := Z(kG)/T^+_1(kG)$ and its Jacobson radical.

The first Külshammer ideal $T^+_1(kC_{2m})$ has a vector space basis of the form

$$\{(C^{2^{i-1}})^+ : C \in \mathcal{C}(G)\} = \{g^j + g^{2^{m-i}+j} | 0 \leq j \leq 2^{m-1} - 1\}.$$ 

In fact, for a conjugacy class $C = \{g^i\}$ of $G$, the set $C^{2^{i-1}} = \{y \in G : y^2 = g^i\}$ is empty for $i$ odd, and consists of $g^{i/2}$ and $g^{2^{m-i}+i/2}$ if $i$ is even. In particular,

$$\dim T^+_1(kC_{2m}) = 2^{m-1} \text{ and also } \dim \overline{Z}(kG) = 2^{m-1}.$$ 

For the Jacobson radical we then get

$$\dim J(\overline{Z}(kG)) = 2^{m-1} - 1 , \text{ dim } J^2(\overline{Z}(kG)) = \max(0, 2^{m-1} - 2),$$ 

and

$$\dim J(\overline{Z}(kG))/J^2(\overline{Z}(kG)) = \begin{cases} 1 & \text{if } m > 1 \\ 0 & \text{if } m = 1 \end{cases}.$$

We will use these calculations in the third section.

Remark 2.3. Külshammer ideals are known to have good multiplicative properties. Let $A$ and $B$ be two symmetric $k$-algebras. Then $A \times B$ is also symmetric via the obvious bilinear form and

$$Z(A \times B) \cong Z(A) \times Z(B),$$ 

$$T^+_n(A \times B) \cong T^+_n(A) \times T^+_n(B),$$ 

$$\overline{Z}(A \times B) \cong \overline{Z}(A) \times \overline{Z}(B),$$ 

$$J(\overline{Z}(A \times B)) \cong J(\overline{Z}(A)) \times J(\overline{Z}(B)),$$

$$J(\overline{Z}(A \times B))/J^2(\overline{Z}(A \times B)) \cong J(\overline{Z}(A))/J^2(\overline{Z}(A)) \times J(\overline{Z}(B))/J^2(\overline{Z}(B))$$.
We leave the proof of these easy facts to the reader.

We can now state the theorem of A. Zimmermann cited in the introduction saying that the Kulshammer ideals are derived invariants.

**Theorem 2.4.** ([31]) Let \( k \) be an algebraically closed field of characteristic \( p > 0 \). Let \( A \) and \( B \) be symmetric \( k \)-algebras. If \( A \) and \( B \) are derived equivalent (i.e., their derived module categories \( \mathcal{D}^b(A) \cong \mathcal{D}^b(B) \) are equivalent as triangulated categories), then there exists an isomorphism \( \varphi : Z(A) \to Z(B) \) such that \( \varphi(T^+_n(A)) = T^+_n(B) \) for any \( n \geq 0 \).

From now on, \( k \) denotes an algebraically closed field of characteristic 2.

For proving the main result of this paper it will be crucial to be able to decide, given a particular dihedral block, which scalar occurs in the relation. This can be read off from the factor rings modulo the first Kulshammer ideals, by the following recent result of the first named author and A. Zimmermann. Recall that all blocks with dihedral defect group of order \( 2^n \) occur among the algebras \( D(2A)^s(c) \) defined in the introduction. Note that the following result only applies for dihedral defect groups of order at least 16.

**Theorem 2.5.** ([19]) Let \( s = 2^n - 2 \) with \( n \geq 4 \). Denote \( A^s_c = D(2A)^s(c) \) and \( Z_c = Z(A^s_c)/T^+_1(A^s_c) \). Then

\[
\dim J(Z_0)/J^2(Z_0) = 3, \quad \text{and} \quad \dim J(Z_1)/J^2(Z_1) = 2
\]

The main step of the proof of Theorem 2.4 is to calculate in an undirect way the dimension of \( J(Z(B_0))/J^2(Z(B_0)) \) for the principal block \( B_0 \) of \( kPGL_2(q) \).

### 3. Some group-theoretic facts

In this section we collect some basic facts about projective general linear groups. Most of them are well known, so we only give some indications of proofs.

Let \( q = p^n \) be a prime power for \( p \) an odd prime number. The group \( PGL_2(q) \) is defined as the factor group of the general linear group \( GL_2(q) \) over the finite field \( F_q \) modulo the center, i.e., the normal subgroup of all scalar multiples of the identity matrix. In particular, the group \( PGL_2(q) \) has order \( q(q + 1)(q - 1) \).

Denote by \( \sigma \) a generator of the multiplicative group \( F_q^* \) of invertible elements of the finite field \( F_q \). We set \( \tau := \sigma^{q+1} \); note that \( \tau \) is a generator of \( F_q^* \). Moreover, we denote by \( \epsilon \) (resp. \( \eta \)) a \((q + 1)\)-th (resp. \((q - 1)\)-th) primitive root of 1 in \( \mathbb{C} \).

The first table gives the conjugacy classes of \( PGL_2(q) \), where \( \lambda_1 \) and \( \lambda_2 \) are two eigenvalues and where the last column gives the order of the centralizer of a representative of a conjugacy class.

The second table is the ordinary character table of \( PGL_2(q) \), where the characters in the last row come from cuspidal representations. The character table of \( GL_2(q) \) can be found in [25] and we only need to find out all ordinary characters which factor through the subgroup formed by scalar matrices. Note that our
Table 1. Conjugacy classes of $PGL_2(q)$ with $q = p^n$ odd

| conjugacy class $K$ | representative $x_K$ | $|C_G(x_K)|$ |
|---------------------|-----------------------|-------------|
| $\lambda_1 = \lambda_2 \in \mathbb{F}_q^*$ semisimple | $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ | $q(q + 1)(q - 1)$ |
| $\lambda_1 = \lambda_2 \in \mathbb{F}_q^*$ nonsemisimple | $A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ | $q$ |
| $\lambda_1 \neq \pm \lambda_2$, $\lambda_1, \lambda_2 \in \mathbb{F}_q^*$ | $A_{3,i} = \begin{bmatrix} 1 & 0 \\ 0 & \tau^i \end{bmatrix}$, $1 \leq i \leq \frac{q - 3}{2}$ | $q - 1$ |
| $\lambda_1 = -\lambda_2 \in \mathbb{F}_q^*$ | $A_{3,q-1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ | $2(q - 1)$ |
| $\lambda_1 \neq \pm \lambda_2$, $\lambda_1, \lambda_2 \in \mathbb{F}_{q^2} - \mathbb{F}_q$ | $A_{4,j} = \begin{bmatrix} 0 & -\sigma^j(q+1) \\ 1 & \sigma^j + \sigma^jq \end{bmatrix}$, $1 \leq j \leq \frac{q-1}{2}$ | $q + 1$ |
| $\lambda_1 = -\lambda_2 \in \mathbb{F}_{q^2} - \mathbb{F}_q$ | $A_{4,\frac{q-1}{2}} = \begin{bmatrix} 0 & \tau \\ 1 & 0 \end{bmatrix}$ | $2(q + 1)$ |

Notations may differ slightly from the notations in [25]. In total there are $q + 2$ ordinary irreducible characters for $PGL_2(q)$ (the same number as conjugacy classes, of course).

The 2-Sylow subgroups of $PGL_2(q)$ are known to be dihedral groups. As $|PGL_2(q)| = q(q + 1)(q - 1)$, if 2-Sylow subgroups are of order $2^n$ with $n \geq 2$ (i.e. the principal block is of defect $n$), then we can write $q - 1 = 2^{n-1}q'$ with $q'$ odd. Table 3 gives all 2-regular conjugacy classes (i.e. conjugacy classes consisting of elements whose order is not divisible by 2).

One can then use ordinary characters to determine the block structure of the group algebra $kPGL_2(q)$. We will have to distribute the ordinary irreducible characters into 2-blocks. This can be done using the following well-known criterion...
Table 2. Character table of $PGL_2(q)$ with $q = p^n$ odd

|          | $A_1$ | $A_2$ | $A_{3,i}$ $1 \leq i \leq \frac{q-3}{2}$ | $A_{3,\frac{q-1}{2}}$ | $A_{4,j}$ $1 \leq j \leq \frac{q-1}{2}$ | $A_{4,\frac{q+1}{2}}$ |
|----------|-------|-------|------------------------------------------|------------------------|------------------------------------------|------------------------|
| $1_G$    | 1     | 1     | 1                                       | 1                      | 1                                        | 1                      |
| $\theta$ | $q$   | 0     | 1                                       | 1                      | -1                                       | -1                     |
| $\text{sgn}$ | 1     | 1     | $(-1)^i$                                | $(-1)^{\frac{i-1}{2}}$ | $(-1)^j$                                 | $(-1)^{\frac{j-1}{2}}$ |
| $\theta \otimes \text{sgn}$ | $q$   | 0     | $(-1)^i$                                | $(-1)^{\frac{i-1}{2}}$ | $(-1)^j$                                 | $(-1)^{\frac{j-1}{2}}$ |
| $1 \leq s \leq \frac{q-3}{2}$ | $q+1$ | 1     | $\eta^{si} + \eta^{-si}$                | $2(-1)^s$              | 0                                        | 0                      |
| $1 \leq k \leq \frac{q-1}{2}$ | $q-1$ | -1    | 0                                       | 0                      | $-\epsilon^k - \epsilon^{-k}$            | $-2(-1)^k$             |

Table 3. 2-regular conjugacy classes of $PGL_2(q)$ with $q = p^n \equiv 1 \mod 8$

| conjugacy class $K$ | representative $x_K$ |
|---------------------|-----------------------|
| $\lambda_1 = \lambda_2 \in \mathbb{F}_q^*$, semisimple | $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ |
| $\lambda_1 = \lambda_2 \in \mathbb{F}_q^*$, nonsemisimple | $A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ |
| $\lambda_1 \neq \lambda_2 \in \mathbb{F}_q^*$ | $A_{3,2^{(n-1)i'}} = \begin{bmatrix} 1 & 0 \\ 0 & 2^{n-1} \end{bmatrix}$, $1 \leq i' \leq \frac{q-1}{2}$ |
| $\lambda_1 \neq \lambda_2 \in \mathbb{F}_q^{*2} - \mathbb{F}_q$ | $A_{4,2^{j'}} = \begin{bmatrix} 0 & -\sigma^{2^{j'+(q+1)}} \\ 1 & \sigma^{2^{j'+1}} + \sigma^{2^{j'}q} \end{bmatrix}$, $1 \leq j' \leq \frac{q-1}{4}$ |

(see for instance [24] (3.19 THEOREM)): Two ordinary characters $\chi$ and $\psi$ lie in
Table 4. 2-blocks for $PGL_2(q)$ with $q = p^n \equiv 1 \mod 8$

| 2-block | description | ordinary characters |
|---------|-------------|---------------------|
| principal block $B_0$ | dihedral block of defect $n$ | $1_G, \theta, sgn, \theta \otimes sgn, \mu_{q't}, 1 \leq t \leq 2^{n-2} - 1,$ |
| $B_{3,s} \sim kC_{2^{n-1}}$ | cyclic block of defect $n-1$ | $\mu_{q't-s}, 1 \leq t \leq 2^{n-2}$ |
| $1 \leq s \leq \frac{q-1}{2}$ | | $\mu_{q't+s}, 0 \leq t \leq 2^{n-2} - 1$ |
| $B_{4,u} \sim kC_2$ | cyclic block of defect 1 | $\chi_u, \chi_{q+1}$ |
| $1 \leq u \leq \frac{q-1}{2}$ | | $\chi_{u+1}$ |

the same 2-block if

$$\sum_{x \in G^0} \chi(x)\psi(x^{-1}) \neq 0$$

where $G^0$ is the set of all 2-regular elements.

Furthermore, a block with dihedral defect group of order $2^n$ has precisely $2^{n-2} + 3$ ordinary irreducible characters (see for example, [11, V.5.10 COROLLARY]).

Table 4 gives the list of all 2-blocks. Recall that the number $q'$ comes from the factorization $q - 1 = 2^{n-1}q'$. In this table and in the sequel, $\sim$ means Morita equivalence. These Morita equivalences can be deduced from a theorem of L. Puig [26]; in fact the cyclic blocks occurring have only one simple module (i.e. one modular irreducible Brauer character), hence are nilpotent, and then by Puig’s theorem they are Morita equivalent to the group algebra of their defect groups.

4. Proof of the main theorem

In this section we shall closely look at the Külshammer ideals for the principal block of the group algebra of $PGL_2(q)$. In particular, we shall prove our main result Theorem 1.4. However, the proof of a key result, which is of a technical nature, is postponed to the next section.

By abuse of notation, we will freely use the notations for representatives of the conjugacy classes (like e.g. $A_{3,i}$) now also for the entire conjugacy class. Therefore, we denote conjugacy class sums by e.g. $A_{3,i}^+$, and also its image in the quotient $\bar{Z}(kG) = Z(kG)/T_1^+(kG)$.

Recall Külshammer’s nice description ([23] no.(38)) of the ideals $T_1^+(kG)$ for group algebras $kG$. This has been mentioned already in Example 2.2; we restate
it here for the special case we are needing: \( T^\perp_1(kG) \) is the vector space with basis \( \{(K^{2^{-i}})^+ : K \in Cl(G)\} \) where \( Cl(G) \) the set of conjugacy classes of \( G \), and where
\[
K^{2^{-i}} = \{g \in G | g^2 \in K\}.
\]
Note that the sets \( K^{2^{-i}} \) are closed under conjugation in \( G \), i.e. it suffices to consider the representatives of the conjugacy classes from Table II.

Easy calculations then give Table 5 below, describing the sets \( K^{2^{-i}} \) for the groups \( PGL_2(q) \).

**Table 5. Computing \( T^\perp_1(kG) \)**

| Conjugacy class \( K \) | \( K^{2^{-i}} \) |
|------------------------|----------------|
| \( A_1 \)              | \( A_1, A_3, \frac{q-1}{2}, A_4, \frac{q+1}{2} \) |
| \( A_2 \)              | \( A_2 \) |
| \( A_{3,i}, 1 \leq i \leq \frac{q-3}{2}, \text{even} \) | \( A_{3,i}, A_{3, \frac{q-1-i}{2}} \) |
| \( A_{3,i}, 1 \leq i \leq \frac{q-3}{2}, \text{odd} \) | \( \emptyset \) |
| \( A_{3, \frac{q-1}{2}} \) | \( A_{3, \frac{q-1}{2}} \) |
| \( A_{4,j}, 1 \leq j \leq \frac{q-1}{2}, \text{even} \) | \( A_{4,j}, A_{4, \frac{q+1-j}{2}} \) |
| \( A_{4,j}, 1 \leq j \leq \frac{q-1}{2}, \text{odd} \) | \( \emptyset \) |
| \( A_{4, \frac{q+1}{2}} \) | \( \emptyset \) |

From the results listed in Table 5 and Külshammer’s description of the ideals \( T^\perp_1(kG) \) above, it is easy to deduce the following result. Note that part (i) is just the well-known fact that the conjugacy class sums form a basis of the center of a group algebra.

**Lemma 4.1.** Let \( G = PGL_2(q) \) where \( q \equiv 1 \mod 8 \).
(i) The center has \( \dim Z(kG) = q + 2 \) and a basis of it is given by
\[
\{ A_1^+, A_2^+; A_3^+, 1 \leq i \leq \frac{q-3}{2}; A_3^{+, \frac{q-1}{2}}; A_4^+, 1 \leq j \leq \frac{q-1}{2}; A_4^{+, \frac{q+1}{2}} \}.
\]

(ii) The first Kulshammer ideal has \( \dim T_1^\perp(kG) = \frac{q+3}{2} \) and a basis of it is given by
\[
\{ A_1^+, A_3^+, A_4^+, 1 \leq i \leq \frac{q-5}{4}; A_3^{+, \frac{q-1}{4}}; A_4^{+, \frac{q+1}{4}}, 1 \leq j \leq \frac{q-1}{4} \}.
\]

(iii) The factor ring has \( \dim Z(kG)/T_1^\perp(kG) = \frac{q+1}{2} \) and a basis of it is given by
\[
\{ A_1^+, A_3^+, A_4^+, 1 \leq i \leq \frac{q-5}{4}; A_3^{+, \frac{q-1}{4}}, 1 \leq j \leq \frac{q-1}{4} \}.
\]

\[ \square \]

Before proceeding to the proof of our main result Theorem 1.4, we consider a small example.

Example 4.2. Let \( q = 3^2 = 9 \). Then \( n = 4, q' = 1, q'-1 = 0 \) and
\[
kG = kPGL_2(9) \sim B_0 \oplus (kC_2)^{\oplus 2}.
\]

From the preceding lemma, we can read off the entries of the following Table 6. For the last column on cyclic blocks the entries have been computed in Example 2.2.

Table 6. \( PGL_2(9) \)

| \( kG \) | \( B_0 \) | \( (kC_2)^2 \) |
|-------|-------|-------|
| center \( Z \) | 11 | ? | 4 |
| \( T_1^\perp \) | 6 | ? | 2 |
| \( \overline{Z} = Z/T_1^\perp \) | 5 | ? | 2 |

One obtains that \( \dim Z(B_0)/T_1^\perp(B_0) = 5 - 2 = 3 \) and then in particular
\[
\dim J(\overline{Z(B_0)})/J^2(\overline{Z(B_0)}) \leq 2.
\]
This means that the principal 2-block of \( PGL_2(9) \) has scalar \( c = 1 \), by Theorem 2.5.

For proving Theorem 1.4 one needs to compute the Jacobson radical and its square of the factor ring \( \overline{Z} := Z(kG)/T_1^\perp(kG) \) for the group \( G = PGL_2(q) \). The key step is the following proposition whose (technical) proof is postponed to the next section (see Corollary 5.2 and Proposition 5.6).
**Proposition 4.3 (Key step).** With the above notations we have
\[
\dim J(Z) = \frac{q-1}{4} - \frac{q'-1}{2} \quad \text{and} \quad \dim J^2(Z) = \frac{q-1}{4} - (q' + 1).
\]

We now give the proof of theorem 1.4 using the above key proposition. Recall from Table 4 the block decomposition, up to Morita equivalence:
\[
kPGL_2(q) \sim B_0 \oplus (kC_{2n-1})^{\frac{q-1}{2}} \oplus (kC_2)^{\frac{q-1}{4}}.
\]

We collect the necessary information on the blocks and their Külshammer ideals in the following Table 7; the numbers in the table are dimensions. The entries for \(kG\) are from the above key proposition and Lemma 4.1, the entries in the last two columns on cyclic blocks have been computed in Example 2.2.

| \(kG\) | \(B_0\) | \((kC_{2n-1})^{\frac{q-1}{2}}\) | \((kC_2)^{\frac{q-1}{4}}\) |
|-------|-------|-----------------|-----------------|
| center \(Z\) | \(q+2\) | \(2^{n-2} + 3\) | \(2^{n-1} \times \frac{q-1}{2}\) |
| \(Z = \frac{Z}{T_1}\) | \(\frac{q+1}{2}\) | ? | \(2^{n-2} \times \frac{q-1}{2}\) |
| \(J(Z)\) | \(\frac{q-1}{2} - \frac{q-1}{2}\) | ? | \((2^{n-2}-1) \times \frac{q-1}{2}\) |
| \(J^2(Z)\) | \(\frac{q}{2} - (q' + 1)\) | ? | \((2^{n-2} - 2) \times \frac{q-1}{2}\) |
| \(J(Z)/J^2(Z)\) | \(\frac{q+3}{2}\) | ? | \(1 \times \frac{q-1}{2}\) |

From this table, one easily computes that
\[
\dim J(\frac{Z(B_0)}{J^2(Z(B_0))}) = \frac{q' + 3}{2} - \frac{q' - 1}{2} = 2.
\]

Now using Theorem 2.5 we can deduce that the scalar for the principal 2-block of \(PGL_2(q)\) is indeed \(c = 1\), thus proving the main result Theorem 1.4.

### 5. Proof of the key proposition

Throughout this section, we work in the quotient algebra \(\overline{Z}(kG) = Z(kG)/T_1{^-1}(kG)\). We will exhibit explicit basis for the radical \(J(\overline{Z}(kG))\) and the radical square \(J^2(\overline{Z}(kG))\). For notational convenience, we allow the index \(i\) in \(A_{3,i}\) to be an arbitrary integer (note that by definition of the representative \(A_{3,i}\), the index can be taken modulo \(q - 1\), and moreover in \(PGL_2(q)\) we have that \(A_{3,i} = A_{3,q-1 \pm 1}\)). The same convention applies to the index \(j\) in \(A_{4,j}\).

For the computations in this section, always keep in mind that we are working in characteristic 2.

**Proposition 5.1.** For all \(i \in \mathbb{Z}\) the following equation holds in \(\overline{Z}(kG)\):
\[
(A_{3,i}^\pm)^2 = \begin{cases} 
A_{3,2i}^\pm, & \text{if } \frac{q-1}{2} \mid i \\
0, & \text{if } i = u\frac{q-1}{2} \text{ with } u \text{ odd} \\
A_{1}^\pm, & \text{if } i = u\frac{q-1}{4} \text{ with } u \text{ even}
\end{cases}
\]
Proof. For all $i \in \mathbb{Z}$ we have
\[
(A_{3, i}^{+})^2 = \left( \sum_{g \in G/C_G(A_{3, i})} g A_{3, i} g^{-1} \right)^2 = \sum_{g \in G/C_G(A_{3, i})} g A_{3, i}^2 g^{-1} = \sum_{g \in G/C_G(A_{3, i})} g A_{3, 2i} g^{-1}.
\]
If $\frac{q - 1}{2} \mid i$, then it is easy to see that $C_G(A_{3, i}) = C_G(A_{3, 2i})$, so we have
\[
(A_{3, i}^{+})^2 = \sum_{g \in G/C_G(A_{3, i})} g A_{3, 2i} g^{-1} = \sum_{g \in G/C_G(A_{3, 2i})} g A_{3, 2i} g^{-1} = A_{3, 2i}^+.
\]
If $i = u \frac{q - 1}{4}$ with $u$ even, then $|C_G(A_{3, i})/C_G(A_{3, 2i})| = 2$, and we have
\[
(A_{3, i}^{+})^2 = 2 \sum_{g \in G/C_G(A_{3, 2i})} g A_{3, 2i} g^{-1} = 2A_{3, 2i}^+ = 0.
\]
If $i = u \frac{q - 1}{4}$ with $u$ even, then
\[
(A_{3, i}^{+})^2 = \sum_{g \in G/C_G(A_{3, i})} g A_{3, 2i} g^{-1} = |G/C_G(A_{3, i})|A_1^+ = \frac{q(q + 1)}{2} A_1^+ = A_1^+,
\]
where for the last equality we use that $q \equiv 1 \mod 8$. \hfill \Box

As a consequence, the next result provides a basis of the radical of the factor ring $\mathbb{Z}(kG)$, in terms of the basis of $\mathbb{Z}(kG)$ given in Lemma 4.1(iii). Before stating the result, let us introduce some more notations which will also be useful in the sequel. For $0 \leq s \leq \frac{q - 1}{2}$, denote
\[
I_s = \{i : 1 \leq i \leq \frac{q - 5}{4} = 2^{n-3}q' - 1 \text{ and } i \equiv s \mod q'\}.
\]
Let $I_{s\text{even}}$ (resp. $I_{s\text{odd}}$) be the set of even (resp. odd) numbers in $I_s$. Note that $|I_0| = 2^{n-3} - 1$ and $|I_s| = 2^{n-2}$ for $1 \leq s \leq \frac{q - 1}{2}$.

Proposition 5.2. A basis of $J(\mathbb{Z}(kG))$ is given by
\[
\{A_1^+, A_{3, i}^{+}; A_{3, i}^{+}, i \in I_0; A_{3, i}^{+} + A_{3, s}^{+}, i \in I_s, i \neq s, 1 \leq s \leq \frac{q' - 1}{2}\}.
\]

As a consequence, $\dim J(\mathbb{Z}(kG)) = \frac{q - 1}{2} - \frac{q' - 1}{2}$.

Proof. We will first prove that these elements are nilpotent, thus are contained in the radical. In fact, by Proposition 5.1 (and because we are working in characteristic 2) we have
\[
(A_1^+ + A_{3, i}^+)^2 = (A_1^+)^2 + (A_{3, i}^+)^2 = A_1^+ + A_{3, i}^+ = 0.
\]
Next, let $i \in I_0$, i.e. $i = uq'$ with $1 \leq u \leq 2^{n-3} - 1$; write $u = 2^t u'$ with $u'$ odd. Then by Proposition 5.1 we have
\[
(A_{3, i}^+)^{2^{n-2-t}} = (A_{3, 2^{n-3}u'q'2^t}^+) = (A_{3, 2^{n-3}u'q'}^+)^2 = (A_{3, u'q'}^+)^2 = 0,
\]
Lemma 5.4. For $1 \leq i \leq \frac{q-3}{2}$ we have

$$A_{3,i}^+ = \sum_{\alpha, \beta \in \mathbb{F}_q} \left( \begin{array}{cc} 1 + \alpha\beta - \alpha \beta \tau^i & -\beta + \beta \tau^i \\ \alpha(1 + \alpha\beta)(1 - \tau^i) & -\alpha\beta + (1 + \alpha\beta)\tau^i \end{array} \right) + \sum_{\gamma \in \mathbb{F}_q} \left( \begin{array}{cc} \gamma(\tau^i - 1) & 0 \\ \tau^i & 1 \end{array} \right)$$

where the first equality is obtained by iteration of Proposition 5.1 since $\frac{q-1}{2} = 2^{n-3}q'$ doesn’t divide $2^{n-4}(q'2^iu') = 2^{n-4}q'u'$.

Finally, let $i \in I_s$ for $1 \leq s \leq \frac{q-1}{2}$, and write $i = uq' \pm s$. Note that here $q' > 1$ (otherwise such $s$ don’t exist). Then, again by Proposition 5.1

$$\left( A_{3,i}^+ + A_{3,s}^+ \right)^{2n-1} = \left( A_{3,i}^+ \right)^{2n-1} + \left( A_{3,s}^+ \right)^{2n-1} + A_{3,2n-1}^{+1} q'u_{s \pm 2n-1} + A_{3,2n-1}^{+1} q'u'_{s \pm 2n-1} + A_{3,2n-1}^{+1} q'u'_{s \pm 2n-1}$$

$$= A_{3,i}^+ q'u_{s \pm 2n-1} + A_{3,s}^+ q'u_{s \pm 2n-1} + A_{3,2n-1}^{+1} q'u'_{s \pm 2n-1} + A_{3,2n-1}^{+1} q'u'_{s \pm 2n-1}$$

$$= 0.$$

For the second equality above note that we are indeed always in the first case of Proposition 5.1 since $2^{n-2}i \equiv \pm 2^{n-2}s \mod q' = \frac{q-1}{2}$, and the representative satisfies $| \pm 2^{n-2}| \leq 2^{n-3}(q' - 1)$ which can not become 0 modulo $\frac{q-1}{2} = 2^{n-3}q'$, since $q' > 1$. For the fourth equality recall the definition of $A_{3,j}$: indices can be taken modulo $q - 1$, and $A_{3,j}^+ = A_{3,-j}^+$ since the matrices $A_{3,j}$ and $A_{3,-j}$ are conjugate in $PGL_2(q)$.

The elements listed in the statement of the proposition are thus all in the radical and they are evidently linearly independent. In total we have

$$1 + (2^{n-3} - 1) + \frac{q' - 1}{4} \times (2^{n-2} - 1) = \frac{q - 1}{4} - \frac{q' - 1}{2}$$

elements. But on the other hand, from the block structure, the radical has at most the dimension

$$\frac{q + 1}{2} - 1 - \frac{q' - 1}{2} - \frac{q - 1}{4} = \frac{q - 1}{4} - \frac{q' - 1}{2}$$

which is the dimension of $\mathbb{Z}(kG)$ minus the number of blocks (cf. Table 4). So the result follows.

Remark 5.3. It is easy to see that for $1 \leq s \leq \frac{q-1}{2}$, there is only one number in $I_s$ which is divisible by $2^{n-2}$, denoted by $\varphi(s)$ (not to be confused with Euler’s totient function). We can replace the element $A_{3,i}^+ + A_{3,s}^+$, $i \in I_s$, $i \neq s$ by $A_{3,i}^+ + A_{3,\varphi(s)}^+$, $i \in I_s$, $i \neq \varphi(s)$ in the basis. We will use this point in the proof of Proposition 5.6.

Furthermore, if $i, j \in I_s$ for $1 \leq s \leq \frac{q-1}{2}$, then

$$A_{3,i}^+ + A_{3,j}^+ = A_{3,i}^+ + A_{3,s}^+ + A_{3,j}^+ + A_{3,s}^+ \in J(\mathbb{Z}(kG)).$$

We now turn to studying the square of the radical. To this end we first need some preparations.

Lemma 5.4. For $1 \leq i \leq \frac{q-3}{2}$ we have

$$A_{3,i}^+ = \sum_{\alpha, \beta \in \mathbb{F}_q} \left( \begin{array}{cc} 1 + \alpha\beta - \alpha \beta \tau^i & -\beta + \beta \tau^i \\ \alpha(1 + \alpha\beta)(1 - \tau^i) & -\alpha\beta + (1 + \alpha\beta)\tau^i \end{array} \right) + \sum_{\gamma \in \mathbb{F}_q} \left( \begin{array}{cc} \gamma(\tau^i - 1) & 0 \\ \tau^i & 1 \end{array} \right)$$
Proof. As usual, let $G := PGL_2(q) = GL_2(q)/\mathbb{F}_q^*$. Consider the following subgroups of $GL_2(q)$

$$T = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, a, b \in \mathbb{F}_q^* \right\}$$
$$B = \left\{ \begin{pmatrix} a & c \\ 0 & b \end{pmatrix}, a, b \in \mathbb{F}_q^*, c \in \mathbb{F}_q \right\}.$$

Then for $1 \leq i \leq \frac{q^2-3}{2}$ we have $C_G(A_3,i) = T/\mathbb{F}_q^*$, hence $G/C_G(A_3,i) \cong GL_2(q)/T$.

Representatives of $GL_2(q)/B$ can be chosen as

$$\left\{ \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}, \alpha \in \mathbb{F}_q \right\}; \quad \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\},$$

and a set of representatives of $B/T$ can be chosen as

$$\left\{ \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}, \beta \in \mathbb{F}_q \right\}.$$

So a set of representatives of $G/C_G(A_3,i)$ can be chosen as

$$\left\{ \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix}, \alpha, \beta, \gamma \in \mathbb{F}_q \right\}.$$

The result then follows easily from direct calculations. \hfill \square

Proposition 5.5. Let $i, j \in \mathbb{Z}$ such that $q-1 \nmid i, j, i \pm j$. Then in $\mathbb{Z}(kG)$ we have

$$A_{3,i}^+A_{3,j}^+ = A_{3,i+j}^+ + A_{3,i-j}^+.$$

Proof. First consider the product in the center $\mathbb{Z}(kG)$; then $A_{3,i}^+A_{3,j}^+ = \sum_K a_K K^+$, where $K$ runs through the set of conjugacy classes. Then

$$A_{3,i}^+A_{3,j}^+ = \sum_{g,h \in G/C_G(A_3,i) = G/C_G(A_3,j)} gA_{3,i}g^{-1}hA_{3,j}h^{-1}$$
$$= \sum_{g \in G/C_G(A_3,i)} gA_{3,i}(\sum_{h \in G/C_G(A_3,j)} g^{-1}hA_{3,j}h^{-1}g)g^{-1}$$
$$= \sum_{g \in G/C_G(A_3,i)} gA_{3,i}A_{3,j}^+g^{-1}.$$

Therefore,

$$a_K = |G/C_G(A_3,i)| \times |\text{elements of } K \text{ in } A_{3,i}A_{3,j}^+|/|K|.$$

One can use this simple counting principle to compute $a_K$. We will treat the most difficult case where $K = A_{3,u}$ for $1 \leq u \leq \frac{q^2-3}{2}$ and leave the other cases to the reader. The above formula for the coefficient $a_K$ now reads for $K = A_{3,u}$ as

$$a_K = q(q+1) \times |\text{elements of } K \text{ in } A_{3,i}A_{3,j}^+|/q(q+1) = |\text{elements of } K \text{ in } A_{3,i}A_{3,j}^+|.$$
Now by Lemma 5.4
\[
A_{3,i}A_{3,j}^+ = \sum_{\alpha, \beta \in \mathbb{F}_q} \left( \begin{array}{cc} 1 & 0 \\ 0 & \tau^i \end{array} \right) \left( \begin{array}{cc} 1 + \alpha\beta - \alpha\beta\tau^j & -\beta + \beta\tau^j \\ \alpha(1 + \alpha\beta)(1 - \tau^j) & -\alpha\beta + (1 + \alpha\beta)\tau^j \end{array} \right) \\
+ \sum_{\gamma \in \mathbb{F}_q} \left( \begin{array}{cc} 1 & 0 \\ 0 & \tau^i \end{array} \right) \left( \begin{array}{cc} \tau^j & 0 \\ -\gamma + \gamma\tau^j & 1 \end{array} \right)
\]
\]
\[
= \sum_{\alpha, \beta \in \mathbb{F}_q} \left( \begin{array}{cc} 1 + \alpha\beta - \alpha\beta\tau^j & -\beta + \beta\tau^j \\ \alpha(1 + \alpha\beta)\tau^i(1 - \tau^j) & -\alpha\beta\tau^i + (1 + \alpha\beta)\tau^i\tau^j \end{array} \right) \\
+ \sum_{\gamma \in \mathbb{F}_q} \left( \tau^i\gamma(\tau^j - 1) \right) \left( \begin{array}{cc} \tau^j & 0 \\ \tau^i & 1 \end{array} \right).
\]

Denote by \( B \) the first matrix in the preceding formula and by \( C \) the second matrix. If \( B \) represents the same coset as \( A_{3,u} \) in \( PGL_2(q) \), then there exists \( \lambda \in \mathbb{F}_q^* \) which satisfies, by considering the determinant and the trace
\[
\lambda^2\tau^u = \tau^{i+j} \quad \text{and} \quad \lambda(1 + \tau^u) = 1 + \tau^{i+j} + \alpha\beta(1 - \tau^i)(1 - \tau^j).
\]
The case that \( i + j - u \) is odd is impossible, as \( \lambda \in \mathbb{F}_q^* \) and \( \tau \) is a generator of \( \mathbb{F}_q^* \), i.e. the squares in \( \mathbb{F}_q^* \) are given by the even powers of \( \tau \). So consider now the case where \( i + j - u \) is even, then \( \lambda = \pm \tau^{\frac{i+j-u}{2}} \) and
\[
\pm(1 + \tau^u)\tau^{\frac{i+j-u}{2}} = 1 + \tau^{i+j} + \alpha\beta(1 - \tau^i)(1 - \tau^j).
\]
If \( \pm(1 + \tau^u)\tau^{\frac{i+j-u}{2}} = 1 + \tau^{i+j}, \) i.e. \( u = \pm(i + j) \mod q - 1, \) then \( \alpha\beta = 0. \) We have \( 2q - 1 \) possibilities for the pair \( (\alpha, \beta) \), namely \( (0, 0); (0, \beta), \beta \in \mathbb{F}_q^*; (\alpha, 0), \alpha \in \mathbb{F}_q^* \).
If \( \pm(1 + \tau^u)\tau^{\frac{i+j-u}{2}} \neq 1 + \tau^{i+j} \), then
\[
\alpha\beta = \pm(1 + \tau^u)\tau^{\frac{i+j-u}{2}} - 1 - \tau^{i+j} \quad \text{and} \quad \frac{(1 - \tau^i)(1 - \tau^j)}{(1 - \tau^i)(1 - \tau^j)} \neq 0
\]
and we have \( q - 1 \) possibilities for the pair \( (\alpha, \beta) \).

It is not difficult to see that \( C \) is similar to \( A_{3,i-j} \) in \( PGL_2(q) \) and so there are \( q \) possibilities for \( \gamma \). The counting principle gives \( a_K = 1 \) for \( K = A_{3,i,j}^+ \) and \( a_K = 0 \) for \( K = A_{3,u} \) with \( \pm u \neq i \pm j \mod q - 1. \) \hfill \( \square \)

The following crucial result determines a basis of the square of the radical, in terms of the basis of the radical obtained in Proposition 5.2.

**Proposition 5.6.** A basis of \( J^2(\mathbb{Z}/(kG)) \) is given by the union \( B_0 \cup \bigcup_{s=1}^{q-1} B_s \) where
\[
B_0 = \{ A_{3,i}^+, i \in \mathcal{I}_0^{even}; A_{3,q}^+; A_{3,i}^+, i \in \mathcal{I}_0^{odd} - \{ q' \} \}
\]
and for \( 1 \leq s \leq \frac{q-1}{2} \), if \( s \) is odd,
\[
B_s = \{ A_{3,s}^+, A_{3,i}^+, i \in \mathcal{I}_s^{odd} - \{ s \}; A_{3,q'+s}^+; A_{3,i}^+, i \in \mathcal{I}_s^{even} - \{ q' + s \} \}
\]
and if $s$ is even,

$$B_s = \{ A_{3,s}^+ + A_{3,i}^+, i \in T_{s}^{\text{even}} - \{s\}; A_{3,q'+s}^+ + A_{3,i}^+, i \in T_{s}^{\text{odd}} - \{q' + s\} \}.$$ 

As a consequence, $\dim J^2(\mathbb{Z}(kG)) = \frac{2s - 1}{4} - (q' + 1)$

**Proof.** We will write $\mathbb{Z} = \mathbb{Z}(kG)$ in the following. We will first prove that the elements listed above are indeed in the square of the radical.

*Case $B_0.* Let $i \in T_0^{\text{even}},$ i.e. $i = uq'$ for some $1 \leq u \leq 2^{n-3} - 1$ with $u$ even. Then by Proposition 5.1

$$A_{3,i}^+ = A_{3,uq'}^+ = (A_{3,\frac{s}{2}q'})^2 \in J^2(\mathbb{Z}).$$

Now let $i \in T_0 \setminus \{q'\},$ i.e. $i = uq'$ for some $1 < u \leq 2^{n-3} - 1$ where $u$ is odd. Then by Proposition 5.3 we have

$$A_{3,q'+s}^+ + A_{3,i}^+ = (A_{3,\frac{s}{2}q'} + A_{3,uq'})^2 \in J^2(\mathbb{Z}).$$

*Case $B_s$ with $s$ odd. For $i \in T_{s}^{\text{even}} - \{q' + s\}$ we claim that

$$A_{3,q'+s}^+ + A_{3,i}^+ = (A_{3,\frac{s}{2}q'} + A_{3,uq'})^2 \in J^2(\mathbb{Z})$$

(where the first equality holds by Proposition 5.1). In fact, write $i = (2u + 1)q' + s$ with $u \in \mathbb{Z}.$ Then, if $i = (2u + 1)q' + s,$ we get

$$A_{3,\frac{s}{2}q'} + A_{3,\frac{s}{2}q'} + A_{3,uq'} \in J(\mathbb{Z})$$

by Remark 5.3 and we have

$$A_{3,q'+s}^+ + A_{3,i}^+ = (A_{3,\frac{s}{2}q'} + A_{3,uq'})^2 \in J^2(\mathbb{Z});$$

similarly, if $i = (2u + 1)q' - s,$

$$A_{3,\frac{s}{2}q'} + A_{3,\frac{s}{2}q'} + A_{3,(u+1)q'} \in J(\mathbb{Z})$$

and we have

$$A_{3,q'+s}^+ + A_{3,i}^+ = (A_{3,\frac{s}{2}q'} + A_{3,(u+1)q'}^2) \in J^2(\mathbb{Z}).$$

Suppose now that $i \in T_{s}^{\text{odd}} - \{s\}.$ Then write $i = 2uq' \pm s$ for $u \in \mathbb{Z}.$ Obviously $\frac{s}{2}$ doesn’t divide $u.$ If $u$ is even, then by Proposition 5.3 and using the notation from Remark 5.3 we get

$$J^2(\mathbb{Z}) \ni A_{3,uq'}^+ (A_{3,\frac{s}{2}q'}^+, A_{3,\varphi(s)}) = A_{3,uq'}^+ + A_{3,2uq'}^+ + A_{3,uq' + \varphi(s)} + A_{3,uq' - \varphi(s)}.$$ 

As $uq' \pm \varphi(s)$ are even,

$$A_{3,uq' + \varphi(s)} + A_{3,uq' - \varphi(s)} = (A_{3,\frac{s}{2}q'} + A_{3,\frac{s}{2}q'}) \in J^2(\mathbb{Z})$$

and therefore $A_{3,uq'}^+ + A_{3,2uq'}^+ \in J^2(\mathbb{Z}).$

If $u$ is odd, then

$$J^2(\mathbb{Z}) \ni A_{3,uq'}^+ (A_{3,\frac{s}{2}q'}^+, A_{3,\frac{s}{2}q'}) = A_{3,uq'}^+ + A_{3,2uq'}^+ + A_{3,uq'}^+ + A_{3,uq'}^+.$$
As $uq' \pm s$ are even, 
\[ A_{3, uq' + s}^{+} + A_{3, uq' - s}^{+} = (A_{3, \frac{uq'}{3} + \frac{s}{2}}^{+} + A_{3, \frac{uq'}{3} - \frac{s}{2}}^{+})^2 \in J^2(\mathbb{Z}) \]
and therefore $A_{3, s}^+ + A_{3, 2uq' + s}^+ \in J^2(\mathbb{Z})$.

Case $B_s$ with $s$ even. For $i \in I_{s}^{\text{even}} - \{s\}$, we have
\[ A_{3, i}^+ + A_{3, i}^+ = (A_{3, i}^+ + A_{3, i}^+)^2 \in J^2(\mathbb{Z}), \]
since if we write $i = 2uq' \pm s$, then $\frac{i}{2} = uq' \pm \frac{s}{2} \in I_{\frac{s}{2}}$.

Suppose now that $i \in I_{s}^{\text{odd}} - \{q' + s\}$. Then write $i = uq' \pm s$ for $u \in \mathbb{Z}$ with $u \neq 1$ odd. Then either $4 \mid u - 1$ or $4 \mid u + 1$. Suppose $4 \mid u - 1$, the other case being similar. For $i = uq' + s$ we have (since $u \neq 1$)
\[ J^2(\mathbb{Z}) \ni A_{3, uq' + s}^+ + A_{3, uq' + s}^+ + A_{3, uq' + s}^+, \]
as $\frac{uq'}{u} \pm \frac{s}{2}$ are even, $A_{3, \frac{uq'}{u} + \frac{s}{2}}^+ + A_{3, \frac{uq'}{u} - \frac{s}{2}}^+ \in J^2(\mathbb{Z})$ and therefore $A_{3, i}^+ + A_{3, q' + s}^+ = A_{3, uq' + s}^+ + A_{3, q' + s}^+ \in J^2(\mathbb{Z})$.

For the case $i = uq' - s$, we prove firstly that $A_{3, q' + s}^+ + A_{3, q' - s}^+ \in J^2(\mathbb{Z})$. In fact,
\[ J^2(\mathbb{Z}) \ni A_{3, q'}^+ + A_{3, q'}^+ = A_{3, q' + s}^+ + A_{3, q' - s}^+ + A_{3, 2q' + s}^+ + A_{3, s}^+, \]
and hence
\[ A_{3, 2q' + s}^+ + A_{3, s}^+ = (A_{3, q' + s}^+ + A_{3, s}^+)^2 \in J^2(\mathbb{Z}). \]

Now we consider
\[ J^2(\mathbb{Z}) \ni A_{3, q'}^+ + A_{3, q'}^+ = A_{3, q' + s}^+ + A_{3, q' - s}^+, \]
as above, $A_{3, \frac{uq'}{u} + \frac{s}{2}}^+ + A_{3, \frac{uq'}{u} - \frac{s}{2}}^+ \in J^2(\mathbb{Z})$, and then we also get $A_{3, uq' - s}^+ + A_{3, q' - s}^+ \in J^2(\mathbb{Z})$.

Recall that $|I_0| = 2^{n-3} - 1$ and $|I_s| = 2^{n-2}$ for $s > 1$. Then the total number of elements in the square of the radical listed in the proposition is
\[ ((2^{n-3} - 1) - 1) + \frac{q' - 1}{2} \times (2^{n-2} - 2) = \frac{q - 1}{4} - (q' + 1) \]
and moreover, these elements are evidently linearly independent. So
\[ \dim J^2(\mathbb{Z}(kG)) \geq \frac{q - 1}{4} - (q' + 1) \]
and then we deduce from Proposition \[ \text{Proposition 5.2} \] that
\[ \dim J(\mathbb{Z}(kG))/J^2(\mathbb{Z}(kG)) \leq \left( \frac{q - 1}{4} - \frac{q' - 1}{2} \right) - \left( \frac{q - 1}{4} - (q' + 1) \right) = \frac{q' + 3}{2}. \]
Külshammer ideals and dihedral blocks

We thus have by the already proven information in Table 7
\[ \dim J(\mathbb{Z}(B_0))/J^2(\mathbb{Z}(B_0)) \leq \frac{q' + 3}{2} - 1 \times \frac{q' - 1}{2} = 2. \]

However, by Theorem 2.5 this dimension can only be 2 or 3, thus
\[ \dim J(\mathbb{Z}(B_0))/J^2(\mathbb{Z}(B_0)) = 2, \]
which implies
\[ \dim J(\mathbb{Z}(kG))/J^2(\mathbb{Z}(kG)) = \frac{q' + 3}{2} \]
and then also completes the proof of the key Proposition 4.3. □

6. The case \( q \equiv -1 \mod 8 \)

In this section, we state without proofs the corresponding results for the case \( q \equiv -1 \mod 8 \). Now \( q + 1 = 2^{n-1}q' \) with \( q' \) odd where \( n \) is the defect of the principal block. Table 8 gives the list of 2-regular conjugacy classes in this case and Table 9 describes the block structure.

| Table 8. 2-regular conjugacy classes of \( \text{PGL}_2(q) \) with \( q = p^n \equiv -1 \mod 8 \) |
|-----------------------------------------------|
| conjugacy class \( K \) | representative \( x_K \) |
| \( \lambda_1 = \lambda_2 \in \mathbb{F}_q^* \), semisimple | \( A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \) |
| \( \lambda_1 = \lambda_2 \in \mathbb{F}_q^* \), nonsemisimple | \( A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \) |
| \( \lambda_1 \neq \lambda_2 \in \mathbb{F}_q^* \) | \( A_{3,2^i'} = \begin{bmatrix} 1 & 0 \\ 0 & 2^{i'} \end{bmatrix} \), \( 1 \leq i' \leq \frac{q-3}{4} \) |
| \( \lambda_1 \neq \lambda_2 \in \mathbb{F}_{q^2} - \mathbb{F}_q \) | \( A_{4,2^{n-1}j'} = \begin{bmatrix} 0 & -\sigma^{2n-1}j'(q+1) \\ 1 & \sigma^{2n-1}j'+\sigma^{2n-1}j'q \end{bmatrix} \), \( 1 \leq j' \leq \frac{q-1}{2} \) |

**Proposition 6.1.** Let \( G = \text{PGL}_2(q) \) where \( q = p^n \equiv -1 \mod 8 \).

(i) \( \dim Z(kG) = q + 2 \) and a basis of it is
\[ \{ A_1^+; A_2^+; A_{3,i}^+; A_{4,j}^+; A_{3,\frac{q-3}{2}}^+; A_{4,\frac{q-1}{2}}^+ \}; \]

(ii) \( \dim J(\mathbb{Z}(kG))/J^2(\mathbb{Z}(kG)) = \frac{q' + 3}{2} \).
Table 9. 2-blocks for $PGL_2(q)$ with $q = p^n \equiv -1 \mod 8$

| 2-block            | description                                      | ordinary characters |
|--------------------|--------------------------------------------------|---------------------|
| principal block $B_0$ | dihedral block of defect $n$                    | $1_G, \theta, \text{sgn}, \theta \otimes \text{sgn}, \chi_{q^t}, 1 \leq t \leq 2^{n-2} - 1,$ |
| $B_{3,s} \sim kC_2$ $1 \leq s \leq \frac{q-3}{4}$ | cyclic block of defect 1                        | $\mu_s, \mu_{\frac{q+1}{2} - s}$ |
| $B_{4,u} \sim kC_{2n-1}$ $1 \leq u \leq \frac{q-1}{2}$ | cyclic block of defect $n-1$                    | $\chi_{q^t - u}, 1 \leq t \leq 2^{n-2}$ $\chi_{q^t + u}, 0 \leq t \leq 2^{n-2} - 1$ |

(ii) $\dim T^+_1(kG) = \frac{n^3}{2}$ and a basis of it is

$$\{A_1^+ + A_{3, u-1}^+, A_{4, u+1}^+; A_2^+, A_3^+, A_{4, u-1}^+, 1 \leq i \leq \frac{q-3}{4} ; A_1^+ + A_{4, u-1}^+, 1 \leq j \leq \frac{q-3}{4} ; A_4^+ \}. $$

(iii) $\dim Z(kG)/T^+_1(kG) = \frac{n+1}{2}$ and a basis of it is

$$\{A_1^+; A_{4, u+1}^+, A_{3, i}, 1 \leq i \leq \frac{q-3}{4} ; A_{3, j}^+, 1 \leq j \leq \frac{q-3}{4} \}. $$

(iv) $\dim J(Z(kG)) = \frac{n^3}{2} - \frac{q^3}{2} - (q' + 1)$ and a basis of it is given by

$$\{A_1^+, A_{4, u+1}^+, A_{4, j}, j \in J_0; A_1^+, A_{4, s}, j \in J_s \{s\}, 1 \leq s \leq \frac{q-1}{2} \}$$

where for $0 \leq s \leq \frac{q-1}{2}$,

$$J_s = \{j : 1 \leq j \leq \frac{q-3}{4} = 2^{n-3}q' - 1, j \equiv s \mod q' \}. $$

(v) $\dim J^2(Z(kG)) = \frac{n^3}{2} - (q' + 1)$ and a basis of it is given by the union

$$C_0 \cup (\cup_{s=1}^{\frac{q-1}{2}} C_s)$$

where

$$C_0 = \{A_{4, j}, j \in J_0^{\text{even}}; A_{4, q}^+, A_{4, s}^+, j \in J_0^{\text{odd}} \{q'\} \}$$

and for $1 \leq s \leq \frac{q-1}{2}$, if $s$ is odd,

$$C_s = \{A_{4, s}^+, A_{4, j}, j \in J_0^{\text{odd}} \{s\}; A_{4, q'+s}^+, A_{4, j}^+, j \in J_0^{\text{even}} \{q' + s\} \}$$

and if $s$ is even,

$$C_s = \{A_{4, s}^+, A_{4, j}, j \in J_0^{\text{even}} \{s\}; A_{4, q'+s}^+, A_{4, j}^+, j \in J_0^{\text{odd}} \{q' + s\} \}. $$
To prove (iv), (v) of the preceding proposition, one needs the following computational result.

**Proposition 6.2.** The following holds in \( \mathbb{Z}(kG) \):

(i) For all \( j \in \mathbb{Z} \)

\[
(A_{4,j}^+)^2 = \begin{cases} 
A_{1,2j}^+, & \text{if } \frac{q+1}{4} \mid j \\
0, & \text{if } j = u \frac{q+1}{4} \text{ with } u \text{ odd} \\
A_1^+, & \text{if } j = u \frac{q+1}{4} \text{ with } u \text{ even}
\end{cases}
\]

(ii) For \( i,j \in \mathbb{Z} \), if \( \frac{q+1}{4} \mid i,j, i \pm j \), then in \( \mathbb{Z}(kG) \)

\[
A_{4,i}^+ \times A_{4,j}^+ = A_{4,i+j}^+ + A_{4,i-j}^+
\]

The proof of the main theorem 1.4 in this case then follows from Table 10 whose entries can be deduced from Proposition 6.1.

Table 10.

|                | \( kG \) | \( B_0 \) | \( (kC_2)^{\frac{q+3}{2}} \) | \( (kC_{2n-1})^{\frac{q-1}{2}} \) |
|----------------|---------|---------|----------------|----------------|
| center \( Z \) | \( q + 2 \) | \( 2^{n-2} + 3 \) | \( 2 \times \frac{q-3}{2} \) | \( 2^{n-1} \times \frac{q-1}{2} \) |
| \( \mathbb{Z} = Z/T_1 \) | \( \frac{q+1}{2} \) | ? | \( 1 \times \frac{q-3}{2} \) | \( 2^{n-2} \times \frac{q-1}{2} \) |
| \( J(Z) \)    | \( \frac{q+1}{2} - \frac{q-1}{2} \) | ? | 0 | \( (2^{n-2} - 1) \times \frac{q-1}{2} \) |
| \( J^2(Z) \)  | \( \frac{q+1}{2} - (q' + 1) \) | ? | 0 | \( (2^{n-2} - 2) \times \frac{q-1}{2} \) |
| \( J(Z)/J^2(Z) \) | \( \frac{q+1}{2} \) | ? | 0 | \( 1 \times \frac{q-1}{2} \) |

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