COMPACT SPACES WITH A P-BASE

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Abstract. In the paper, we investigate (scattered) compact spaces with a $P$-base for some poset $P$. More specifically, we prove that, under the assumption $\omega_1 < b$, any compact space with an $\omega^\omega$-base is first-countable and any scattered compact space with an $\omega^\omega$-base is countable. These give positive solutions to Problems 8.6.9 and 8.7.7 in [1]. Using forcing, we also prove that in a model of $\omega_1 < b$, there is a non-first-countable compact space with a $P$-base for some poset $P$ with calibre $\omega_1$.

1. Introduction

Let $P$ be a partially ordered set. A topological space $X$ is defined to have a neighborhood $P$-base at $x \in X$ if there exists a neighborhood base $(U_p[x])_{p \in P}$ at $x$ such that $U_p[x] \subset U_{p'}[x]$ for all $p \geq p'$ in $P$. We say that a topological space has a $P$-base if it has a neighborhood $P$-base at each $x \in X$. All topological spaces in this paper are regular.

We will use Tukey order to compare the cofinal complexity of posets. The Tukey order [19] was originally introduced, early in the 20th century, as a tool to understand convergence in general topological spaces, however it was quickly seen to have broad applicability in comparing partial orders. Given two directed sets $P$ and $Q$, we say $Q$ is a Tukey quotient of $P$, denoted by $P \geq T Q$, if there is a map $\phi : P \to Q$ carrying cofinal subsets of $P$ to cofinal subsets of $Q$. In our context, where $P$ and $Q$ are both Dedekind complete (every bounded subset has the least upper bound), we have $P \geq T Q$ if and only if there is a map $\phi : P \to Q$ which is order-preserving and such that $\phi(P)$ is cofinal in $Q$. If $P$ and $Q$ are mutually Tukey quotients, we say that $P$ and $Q$ are Tukey equivalent, denoted by $P = T Q$. It is straightforward to see that a topological space $X$ has a $P$-base if and only if $T_x(X) \leq T P$ for each $x \in X$, here, $T_x(X) = \{U : U$ is an open neighborhood of $x\}$.

Topological spaces and function spaces with an $\omega^\omega$-base were systematically studied in [1]. Lots of work about the $\omega^\omega$-base in topological groups have been done in [2], [8], [15], and [18]. In this paper we investigate the Tukey reduction of a $P$-base in some (scattered) compact spaces with $P$ satisfying some Calibre conditions. This paper is organized in the following way.

In Section 3, we show that if $P$ has Calibre $\omega_1$, then any compact space with a $P$-base is countable tight. Furthermore, we prove that if a compact space with countable tightness has a $K(M)$-base for some separable metric space $M$, then it is first-countable. As a corollary, any compact space with an $\omega^\omega$-base is first-countable under the assumption $\omega_1 < b$. This gives a positive answer to Problem

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8.7.7 in [1]. In Section 4 we address Problem 8.6.9 in [1] positively by showing that any scattered compact space with an $\omega^\omega$-base is countable under the assumption $\omega_1 < b$. It is natural to ask whether under the assumption $\omega_1 < b$ any compact with a $P$-base is first-countable if $P$ satisfies some Calibre properties, for example, Calibre $\omega_1$. In Section 5 we prove that in a model of Martin’s Axiom in which $\omega_1 < b$, there is a non-first-countable compact space with a $P$-base for some poset $P$ with calibre $\omega_1$.

2. Preliminaries

For any separable metric space $M$, $\mathcal{K}(M)$ is the collection of compact subsets of $M$ ordered by set-inclusion. Fremlin observed that if a separable metric space $M$ is locally compact, then $\mathcal{K}(M) = _T \omega$. Its unique successor under Tukey order is the class of Polish but not locally compact spaces. For $M$ in this class, $\mathcal{K}(M) = _T \omega^\omega$ where $\omega^\omega$ is ordered by $f \leq g$ if $f(n) \leq g(n)$ for each $n \in \omega$. In [2], Gartside and Mamataleshvili constructed a $2^\omega$-sized antichain in $\mathcal{K}(M) = \{ \mathcal{K}(M) : M \in \mathcal{M} \}$ where $\mathcal{M}$ is the set of separable metric spaces.

Let $P$ be a directed poset, i.e. for any points $p, p' \in P$, there exists a point $q \in P$ such that $p \leq q$ and $p' \leq q$. A subset $C$ of $P$ is cofinal in $P$ if for any $p \in P$, there exists a $q \in C$ such that $p \leq q$. Then $\text{cof}(P) = \min \{|C| : C \text{ is cofinal in } P\}$. We also define $\text{add}(P) = \min \{|Q| : Q \text{ is unbounded in } P\}$. For any $f, g \in \omega^\omega$, we say that $f \leq^* g$ if the set $\{ n \in \omega : f(n) > g(n) \}$ is finite. Then $b = \text{add}(\omega^\omega, \leq^*)$ and $d = \text{cof}(\omega^\omega, \leq^*)$. See [3] for more information about small cardinals.

Let $\kappa \geq \mu \geq \lambda$ be cardinals. We say that a poset $P$ has calibre $(\kappa, \mu, \lambda)$ if for every $\kappa$-sized subset $S$ of $P$ there is a $\mu$-sized subset $S_0$ such that every $\lambda$-sized subset of $S_0$ has an upper bound in $P$. We write calibre $(\kappa, \mu, \mu)$ as calibre $(\kappa, \mu)$ and calibre $(\kappa, \kappa, \kappa)$ as calibre $\kappa$. It is known that $\mathcal{K}(M)$ has Calibre $(\omega_1, \omega)$ for any separable metric space $M$, hence so does $\omega^\omega$. Under the assumption $\omega_1 = b$, $\omega_1$ is a Tukey quotient of $\omega^\omega$. Furthermore, under the assumption $\omega_1 < b$, the poset $\omega^\omega$ has Calibre $(\omega_1, \omega_1, \omega_1)$, i.e. Calibre $\omega_1$. We will use this fact in several places of this paper.

It is clear that if $P \leq_T Q$ and $Q \leq_T R$ then $P \leq_T R$ for any posets $P, Q,$ and $R$. So we get the following proposition.

**Proposition 2.1.** Let $P$ and $Q$ be posets such that $P \leq_T Q$. Then if a space $X$ has a neighborhood $P$-base at $x \in X$, then $X$ also has a neighborhood $Q$-base at $x$. Hence, any space with a $P$-base also has a $Q$-base.

**Proposition 2.2.** If $X$ has a $P$-base, then any subspace of $X$ also has a $P$-base.

**Proposition 2.3.** Let $P$ be a poset with $\omega_1 \leq_T P$ and $P =_T \omega \times P$. Then the space $\omega_1 + 1$ has a $P$-base.

**Proof.** For each $\alpha < \omega_1$, the space $\omega_1 + 1$ has a countable local base at $\alpha$. Hence $T_\alpha(\omega_1 + 1) \leq_T P$ due to the fact that $P =_T \omega \times P$.

Let $\phi$ be a map from $P$ to $\omega_1$ which carries confinal subsets of $P$ to confinal subsets of $\omega_1$. Then we define a map $\psi$ from $P$ to $T_{\omega_1}(\omega_1 + 1)$ by $\psi(p) = (\phi(p), \omega_1)$ for each $p \in P$. Clearly $\psi$ carries confinal subsets of $P$ to confinal subset of $T_{\omega_1}(\omega_1 + 1)$.

Hence the space $\omega_1 + 1$ has a neighborhood $P$-base at $\omega_1$. This finishes the proof. □
As a result of \( b \leq \omega \), the space \( \omega_1 + 1 \) has an \( \omega^\omega \)-base under the assumption \( \omega_1 = b \). Gartside and Mamateleashvili in [10] proved that \( \omega^\omega \times \omega_1 \leq \omega \cdot K(\mathbb{Q}) \leq \omega \).

\( \omega \omega \times |\omega_1|^\omega \), here \( \mathbb{Q} \) is the space of rationals. Hence, we have the following result.

**Corollary 2.4.** The space \( \omega_1 + 1 \) has a \( K(\mathbb{Q}) \)-base.

A generalization of \( G_\beta \)-diagonals is \( P \)-diagonals for some poset \( P \). A collection \( C \) of subsets of a space \( X \) is \( P \)-directed if \( C \) can be represented as \( \{ C_p : p \in P \} \) such that \( C_p \subseteq C_{p'} \) whenever \( p \leq p' \). We say \( X \) has a \( P \)-diagonal if \( X^2 \setminus \Delta \) has a \( P \)-directed compact cover, where \( \Delta = \{ (x,x) : x \in X \} \). The second author showed that any compact space with a \( K(\mathbb{Q}) \)-diagonal is metrizable in [7] and Sánchez proved that the same result holds for any compact space with a \( K(\mathbb{M}) \)-diagonal for some separable metric space \( M \) in [17]. Here, we include two results about spaces with (or without) \( P \)-diagonal giving that \( P \) satisfies some Calibre properties.

**Proposition 2.5.** Let \( P \) be a poset with Calibre \( (\omega_1, \omega) \). The space \( \omega_1 + 1 \) doesn’t have a \( P \)-diagonal.

**Proof.** Suppose that \( \omega_1 + 1 \) has a \( P \)-diagonal, i.e., a \( P \)-ordered compact covering \( \{ K_p : p \in P \} \) of \( (\omega_1 + 1)^2 \setminus \Delta \). Choose \( \alpha, \beta \) in \( \omega_1 \) for \( \gamma \in \omega_1 \) such that

\[ \alpha_0 < \beta_0 < \alpha_1 < \beta_1 < \cdots < \alpha_\gamma < \beta_\gamma < \cdots. \]

Let \( p_0 \) in \( P \) be such that \( (\alpha_\gamma, \beta_\gamma) \in K_{p_0} \), for each \( \gamma \in \omega_1 \). By Calibre \( (\omega_1, \omega) \), there are \( p \) in \( P \) and \( \gamma \) in \( \omega_1 \) such that \( \gamma_0 < \gamma_1 < \cdots < \gamma_\gamma < \gamma_{\gamma+1} < \cdots \) for each \( n \in \omega \). Then \( (\delta, \delta) \in K_p \), where \( \delta = \sup \{ \alpha_n : n \in \omega \} \). This contradiction finishes the proof. \( \square \)

**Proposition 2.6.** Let \( P \) be a poset with Calibre \( \omega_1 \). Any compact space with a \( P \)-diagonal has countable tightness.

**Proof.** Let \( \{ K_p : p \in P \} \) be a \( P \)-ordered compact covering of \( X^2 \setminus \Delta \). Suppose that \( X \) has uncountable tightness. Then, \( X \) has a free sequence of length \( \omega_1 \), hence a convergence free sequence of length \( \omega_1 \) by [12]. Let \( \{ x_\alpha : \alpha < \omega_1 \} \) be such a sequence and \( x^* \) the limit point.

Choose \( \alpha_\gamma \) and \( \beta_\gamma \) in \( \omega_1 \) for \( \gamma \in \omega_1 \) such that \( \alpha_0 < \beta_0 < \alpha_1 < \beta_1 < \cdots < \alpha_\gamma < \beta_\gamma < \cdots \). For each \( \gamma \), fix \( p_\gamma \) in \( P \) such that \( (x_{\alpha_\gamma}, x_{\beta_\gamma}) \in K_{p_\gamma} \). Since \( P \) has Calibre \( \omega_1 \), there is an uncountable subset \( \gamma_\tau \) of \( \omega_1 \) with \( \tau < \omega_1 \) such that \( p_{\gamma_\tau} \) is bounded above by \( p^* \in P \). Hence, \( K_{p_{\gamma_\tau}} \subseteq K_{p^*} \), for each \( \tau < \omega_1 \), furthermore, \( (x_{\alpha_\gamma}, x_{\beta_\gamma}) \in K_{p^*} \). Since \( \{ x_\alpha : \alpha < \omega_1 \} \) is a convergent sequence with limit point \( x^* \), the subsequences \( x_{\alpha_\gamma} \) and \( x_{\beta_\gamma} \) both converge to \( x^* \), hence, \( (x^*, x^*) \in K_{p^*} \), which contradicts the fact that \( K_{p^*} \subseteq X^2 \setminus \Delta \). This contradiction finishes the proof. \( \square \)

3. **Compact Spaces**

In this section, we study compact spaces possessing a \( P \)-base with \( P \) satisfying some Calibre property. Mainly, we investigate the problem whether each compact Hausdorff space with an \( \omega^\omega \)-base first countable under the assumption \( \omega_1 < b \). We start with some ZFC result about tightness of the spaces with a \( P \)-base.

**Theorem 3.1.** Let \( \kappa \) be an uncountable regular cardinal and \( P \) be a poset with Calibre \( \kappa \). If \( X \) is a compact Hausdorff space with a \( P \)-base, then \( t(X) < \kappa \).
Proof. Assume, for contradiction, that \( t(X) \geq \kappa \). Then, \( X \) has a free sequence of length \( \kappa \). Hence, by [12], \( X \) has a convergent sequence of length \( \kappa \). Let \( \{ x_\alpha : \alpha < \kappa \} \) be such a sequence and \( x^* \) be the limit point. Let \( S = \{ x_\alpha : \alpha < \kappa \} \cup \{ x^* \} \). Notice that for any unbounded subset \( \{ \alpha, \gamma : \gamma < \kappa \} \), \( x^* \) is the limit point of \( \{ x_\alpha : \gamma < \kappa \} \). Fix a neighborhood base \( \{ B_p : p \in P \} \) at \( x^* \). It is straightforward to see that \( S \setminus B_p \) has size \( < \kappa \) for each \( p \in P \).

For each \( \alpha < \kappa \), choose \( p_\alpha \in P \) such that \( x_\alpha \notin B_{p_\alpha} \). Let \( P' = \{ p_\alpha : \alpha \in \kappa \} \). If the cardinality of \( P' \) is \( < \kappa \), there exists a \( p_\alpha \in P' \) such that \( S \setminus B_{p_\alpha} \) has size \( \kappa \) which is a contradiction. Hence, \( P' \) have cardinality \( \kappa \). Since \( P \) has Calibre \( \kappa \), there is a \( \kappa \)-sized subset \( P'' \) of \( P' \) which is bounded above. List \( P'' \) as \( \{ p_{\alpha_i} : \gamma < \kappa \} \) and pick \( p^* \) to be the upper bound of \( P'' \). Then, \( S \setminus B_{p^*} = \{ x_\alpha : \gamma < \kappa \} \) which is a contradiction. This finishes the proof.

Corollary 3.2. Let \( P \) be a poset with Calibre \( \omega_1 \). Each compact Hausdorff space with a \( P \)-base is countable tight.

A poset \( P \) has Calibre \( (\omega_1, \omega) \) if it has Calibre \( \omega_1 \). It is showed in [16] that for a separable metric space \( M \) the poset \( K(M) \) has Calibre \( \omega_1 \) if it has Calibre \( (\omega_1, \omega_1, \omega) \). Hence it is natural to ask whether a compact space has countable tightness if it has a \( P \)-base with \( P \) having Calibre \( (\omega_1, \omega) \). The following example shows that the answer is negative. So the result above is ‘optimal’ in terms of the Calibre complexity of posets having the form \( K(M) \) with \( M \) being a separable metric space.

Example 3.3. There is a poset \( P \) with Calibre \( (\omega_1, \omega) \) and a compact space \( X \) with a \( P \)-base, but \( t(X) \geq \omega \).

Proof. Let \( P \) be \( K(\mathbb{Q}) \) which clearly has Calibre \( (\omega_1, \omega) \). From Proposition [24] the space \( \omega_1 + 1 \) has a \( P \)-base, but its tightness is uncountable. □

Again, since \( \omega^{\omega_1} \) has Calibre \( \omega_1 \) under the assumption \( \omega_1 < b \), we obtain the following result about spaces with an \( \omega^{\omega_1} \)-base.

Corollary 3.4. Assume that \( \omega_1 < b \). Each compact Hausdorff space with an \( \omega^{\omega_1} \)-base is countable tight.

It is folklore that any GO-space with countable tightness is first countable. Hence, applying Corollary [34] to compact GO-spaces, we obtain the following results.

Corollary 3.5. Let \( P \) be a poset with Calibre \( \omega_1 \). Each compact GO-space has a \( P \)-base if and only if it is first countable.

The following example shows that the result above doesn’t hold for general GO-spaces.

Example 3.6. There is a poset \( P \) with Calibre \( \omega_1 \) such that there exists a GO-space with a \( P \)-base and uncountable tightness.

Proof. Consider the set \( \omega_2 + 1 \) in the ordinal order. Let \( T \) be the topology on \( \omega_2 + 1 \) such that every point except \( \omega_2 \) is isolated and a base at \( \omega_2 \) is \( \{ (\alpha, \omega_2) : \alpha < \omega_2 \} \). So the space \( (\omega_2 + 1, T) \) is a non-first-countable GO-space and clearly has a neighborhood \( \omega_2 \)-base at \( \omega_2 \). It is straightforward to verify that the poset \( \omega_2 \) has Calibre \( \omega_1 \) since every \( \omega_1 \)-sized subset is bounded above. □
The result below was proved in [1] through a different approach. We obtain it here as a result of \( \omega^\omega \) having Calibre \( \omega_1 < \text{b} \).

**Corollary 3.7.** Assume that \( \omega_1 < \text{b} \). Each compact GO-space has an \( \omega^\omega \)-base if and only if it is first countable.

It is natural to ask as in [1] (Problem 8.7.7) whether the same result holds for any compact space. Next, we’ll give a positive answer to this problem by showing that any compact space with a \( P \)-base is first countable if \( P = \mathcal{K}(M) \) for some separable metric space \( M \) has Calibre \( \omega_1 \).

First, we show that any compact space with countable tightness is first countable if it has a \( P \)-base and \( P = \mathcal{K}(M) \) for some separable metric space. We use the ideas and techniques from [4].

**Theorem 3.8.** Let \( P = \mathcal{K}(M) \) for some separable metric space \( M \). If \( X \) is a compact space with countable tightness and has a \( P \)-base, then \( X \) is first-countable.

**Proof.** Fix \( x \in X \) and an open \( P \)-base \( \{U_p[x] : p \in P\} \) at \( x \). For each \( p \in P \), let \( K_p = X \setminus U_p[x] \). Then, \( \{K_p : p \in P\} \) is a \( P \)-directed compact cover of \( X \setminus \{x\} \).

For any separable metric space \( M \), the space \( P = \mathcal{K}(M) \) with the Hausdorff metric \( d^H \) is also separable, hence second countable. Also if \( \{p_n : n \in \omega\} \) is a sequence converging to \( p \in P \), then \( p^* = p \cup (\bigcup\{p_n : n \in \omega\}) \) is compact, hence it is an element in \( P \) with \( p_n \subseteq p^* \) and \( p \subseteq p^* \).

Fix a countable base \( \{B_n : n \in \omega\} \) of \( P \). For each \( n \in \omega \), define \( L(B_n) = \bigcup\{K_p : p \in B_n\} \). And for each \( p \in P \), define \( C(p) = \bigcap\{L(B_n) : p \in B_n\} \). For each \( p \in P \), we pick a decreasing local base \( \{B_{ni} : i \in \omega\} \subseteq \{B_n : n \in \omega\} \) at \( p \) such that for each \( i \in \omega \) there is a positive number \( \epsilon_i \) satisfying that \( B_{ni} \supseteq D^H(p, \epsilon_i) \supseteq \overline{B_{ni}^{b+1}} \), where \( D^H(p, \epsilon_i) \) is the open ball centered at \( p \) with radius \( \epsilon_i \). Define \( C'(p) = \bigcap\{L(B_{ni}) : i \in \omega\} \). It is straightforward to verify that \( C'(p) = C(p) \).

First, we claim that \( x \) is not in the closure of \( C(p) \) for all \( p \in P \). Fix \( p \in P \). By the countable tightness of \( X \), it suffices to show that \( x \) is not in the closure of any countable subset of \( C(p) \). Let \( \{y_i : i \in \omega\} \) be a countable subset of \( C(p) \). For each \( i \in \omega \), choose \( q_i \in B_{ni}^b \), with \( y_i \in K_{q_i} \). Clearly \( \{q_i : i \in \omega\} \) is a sequence converging to \( p \), hence \( p^* = p \cup (\bigcup\{q_i : i \in \omega\}) \) is an element in \( P \) with \( \{y_i : i \in \omega\} \subseteq K_{p^*} \), which implies that \( x \) is not in the closure of \( \{y_i : i \in \omega\} \).

Then, we claim that for each \( p \in P \), there is an \( i \in \omega \) such that \( x \) is not in the closure of \( L(B_{ni}^b) \). Fix \( p \in P \). Choose any open set \( U \) such that \( C(p) \subseteq U \) and \( x \notin \overline{U} \). It suffices to prove that there is an \( i \) so that \( L(B_{ni}^b) \subset U \). Suppose not. Choose \( y_i \in L(B_{ni}^b) \setminus U \) for each \( i \in \omega \). Then for each \( i \in \omega \), choose \( q_i \in B_{ni}^b \) so that \( y_i \in K_{q_i} \). Define \( p_i^* = p \cup (\bigcup\{q_j : j > i\}) \). Hence \( \{y_j : j > i\} \subseteq K_{p_i^*} \).

By the property of the Hausdorff metric \( d^H \), it is straightforward to verify that \( d^H(p_i^*, p) \leq \epsilon_i \), hence \( p_i^* \in L(B_{ni}^b) \), which implies that \( K_{p_i^*} \subseteq L(B_{ni}^b) \) for each \( i \in \omega \). Therefore, \( \bigcap\{K_{p_i^*} : i \in \omega\} \subseteq C(p) \). Then, all the limit points of \( \{y_i : i \in \omega\} \) are in \( C(p) \) which contradicts \( C(p) \subseteq U \) and the choices of \( \{y_i : i \in \omega\} \).

Finally, we prove that the family \( \mathcal{L} = \{L(B_n) : x \notin L(B_n)\} \) is a cover of \( X \setminus \{x\} \), furthermore, \( \{B : B = X \setminus S \text{ for some } S \in \mathcal{L}\} \) is a local base at \( x \). For each \( p \in P \), there is an \( i \in \omega \) such that \( x \) is not in the closure of \( L(B_{ni}^b) \). Hence \( L(B_{ni}^b) \in \mathcal{L} \). Since \( K_p \subseteq L(B_{ni}^b) \), this completes the proof.

\[\square\]
Theorem 3.9. Let $P = \mathcal{K}(M)$ for some separable metric space $M$ such that $P$ has Calibre $\omega_1$. Any compact space $X$ with a $P$-base is first-countable.

Proof. By Corollary 3.3, $X$ has countable tightness since $P$ has Calibre $\omega_1$. Then by Theorem 3.8, $X$ is first-countable. \qed

Then using the fact that $\omega^\omega$ has Calibre $\omega_1$ under the assumption $\omega_1 < b$, we get a positive answer to Problem 8.7.7 in [1].

Corollary 3.10. Assume $\omega_1 < b$. A compact space has an $\omega^\omega$-base if and only if it is first countable.

4. SCATTERED COMPACT SPACES

We recall that a topological space $X$ is scattered if each non-empty subspace of $X$ has an isolated point. The complexity of a scattered space can be determined by the scattered height.

For any subspace $A$ of a space $X$, let $A'$ be the set of all non-isolated points of $A$. It is straightforward to see that $A'$ is a closed subset of $A$. Let $X^{(0)} = X$ and define $X^{(\alpha)} = \bigcap_{\beta < \alpha} (X^{(\beta)})'$ for each $\alpha > 0$. Then a space $X$ is scattered if $X^{(\alpha)} = \emptyset$ for some ordinal $\alpha$. If $X$ is scattered, there exists a unique ordinal $h(x)$ such that $x \in X^{(h(x))} \setminus X^{(h(x) + 1)}$ for each $x \in X$. The ordinal $h(X) = \sup \{ h(x) : x \in X \}$ is called the scattered height of $X$ and is denoted by $h(X)$. It is known that any compact scattered space is zero-dimensional. Also, it is straightforward to show that for any compact scattered space $X$, $X^{(h(x))}$ is a non-empty finite subset.

Theorem 4.1. Let $P$ be a poset with Calibre $\omega_1$ and $X$ a scattered compact space with a $P$-base. Then $X$ is countable.

Proof. If $h(X) = 0$, then $X$ is countable because it is compact.

Assume $h(X) = \alpha$ and any compact scattered space with a $P$-base is countable if it has a scattered height $< \alpha$. Since $X$ is compact, $X^{(\alpha)}$ is a nonempty finite subset of $X$. List $X^{(\alpha)} = \{ x_1, \ldots, x_n \}$. For each $i \in \{ 1, \ldots, n \}$, take a closed and open neighborhood $U_i$ of $x_i$ with $U_i \cap X^{(\alpha)} = \{ x_i \}$. Then $X \setminus \bigcup \{ U_i : i = 1, \ldots, n \}$ is a scattered compact space with scattered height $< \alpha$, hence it is countable by the assumption. So it is sufficient to show that $U_i$ is countable for each $i = 1, \ldots, n$.

Fix $i \in \{ 1, \ldots, n \}$. Consider the subspace $Y = U_i \cap X$. By proposition 2.2, $Y$ has a neighborhood $P$-base $\{ B_p : p \in P \}$ at $\{ x_i \}$. For each $p \in P$, $Y \setminus B_p$ is a compact subspace with scattered height $< \alpha$, hence is countable by the inductive assumption.

Assume that $Y$ is uncountable. Take an uncountable subset $\{ y_\alpha : \alpha < \omega_1 \}$ of $Y \setminus \{ x_i \}$. For each $\alpha < \omega_1$, we choose $p_\alpha \in P$ such that $y_\alpha \notin B_{p_\alpha}$.

If $\{ p_\alpha : \alpha < \omega_1 \}$ is countable, there is a $p^* \in \{ p_\alpha : \alpha < \omega_1 \}$ such that there is an uncountable subset $D$ of $\{ y_\alpha : \alpha < \omega_1 \}$ such that $D \subset Y \setminus B_{p^*}$, which is a contradiction.

If $\{ p_\alpha : \alpha < \omega_1 \}$ is uncountable, then it has an uncountable subset $P'$ which is bounded above using the Calibre $\omega_1$ property of $P$. List $P' = \{ p_{\alpha_\gamma} : \gamma < \omega_1 \}$. Let $p^*$ be an upper bound of $P'$. Then we have that $y_{\alpha_\gamma} \notin B_{p^*}$ for each $\gamma < \omega_1$. This also contradicts with the fact that $Y \setminus B_{p^*}$ is countable. This finishes the proof. \qed

Using the same approach we obtain the following example.
Example 4.2. The one point Lindelöfication of uncountably many points doesn’t have a $P$-base if $P$ has Calibre $\omega_1$, hence, under the assumption $\omega_1 < \mathfrak{b}$, it doesn’t have an $\omega^{\omega}$-base.

The example above uses the fact that under the assumption $\omega_1 < \mathfrak{b}$, the poset $\omega^\omega$ has Calibre $\omega_1$. Furthermore, using Theorem 4.4, we obtain the following result which answers Problem 8.6.8 in [1] positively. This also gives a partial positive answer to Problem 8.6.8 in the same paper.

Corollary 4.3. Assume $\omega_1 < \mathfrak{b}$. Any scattered compact space with an $\omega^{\omega}$-base is countable, hence metrizable.

It was proved in [1] that any compact space with an $\omega^{\omega}$-base and finite scattered height is countable, hence metrizable. Next, we show that the same result holds for any compact space with a $P$-base and finite scattered height if $P$ has Calibre $(\omega_1, \omega)$.

Theorem 4.4. Let $P$ be a poset with Calibre $(\omega_1, \omega)$ and $X$ a compact Hausdorff space with a $P$-base and finite scattered height. Then $X$ is countable, hence metrizable.

Proof. Fix a natural number $n > 0$. Assume that any compact Hausdorff space with scattered height $< n$ is countable. Let $X$ be a compact Hausdorff space with scattered height $n$. We’ll show that $X$ is countable. As discussed in the proof of Theorem 4.4 we could assume that $X^{(n)}$ is a singleton, denoted by $x$, without loss of generality. Suppose, for contradiction, that $X$ is uncountable. Define $m$ to be the greatest natural number such that $X^{(m)}$ is uncountable and $X^{(m+1)}$ is countable. Then there are two cases: 1. $m = n - 1$; 2. $m < n - 1$. We will obtain contradictions in both cases.

First assume that $m = n - 1$. Then we fix a neighborhood $P$-base $\{B_p : p \in P\}$ at $x$. For each $p \in P$, $X^{(m)} \setminus B_p$ is finite as $X$ is compact and $B_p$ is open. Pick an uncountable subset $\{x_\alpha : \alpha < \omega_1\}$ of $X^{(m)}$ with $x_\alpha \neq x$ for all $\alpha < \omega_1$. For each $\alpha < \omega_1$, there is a $p_\alpha \in P$ such that $x_\alpha \notin B_{p_\alpha}$. If $\{p_\alpha : \alpha < \omega_1\}$ is countable, then there exists $p^* \in \{p_\alpha : \alpha < \omega_1\}$ such that $X^{(m)} \setminus B_{p^*}$ is uncountable which is a contradiction. If $\{p_\alpha : \alpha < \omega_1\}$ is uncountable, we can find a countable bounded subset $\{p_{\alpha_n} : n \in \omega\}$ of $\{p_\alpha : \alpha < \omega_1\}$ using the Calibre $(\omega_1, \omega)$ property of $P$. Let the upper bound of $\{p_{\alpha_n} : n \in \omega\}$ be $p^*$. Then, $x_{\alpha_n} \notin B_{p^*}$ for each $n \in \omega$. This is a contradiction.

Now we assume that $m < n - 1$. Then $X^{(m+1)} \setminus \{x\}$ is countable which can be listed as $\{x_\ell : \ell \in \omega\}$. For each $\ell$, pick a closed and open neighborhood $U_\ell$ of $x_\ell$. Then for each $\ell < \omega$, $U_\ell$ is a compact subspace with scattered height $< n$, hence is countable. Therefore, $X^{(m)} \setminus \bigcup \{U_\ell : \ell \in \omega\}$ is uncountable. Pick an uncountable subset $S = \{x_\alpha : \alpha < \omega_1\}$ of $X^{(m)} \setminus \bigcup \{U_\ell : \ell \in \omega\}$). Fix a neighborhood $P$ base $\{B_p : p \in P\}$ at $x$. For each $p \in P$, $S \setminus B_p$ is finite. Similarly as in the proof of case 1, we could obtain a contradiction using the Calibre $(\omega_1, \omega)$ property of $P$.

The result above doesn’t hold for compact space with uncountable scattered height since the space $\omega_1 + 1$ has a $\mathcal{K}(\mathbb{Q})$-base. However, we don’t know the answer to the following problem.

Question 4.5. Assume that $\omega_1 = \mathfrak{b}$. Let $P$ be a poset with Calibre $(\omega_1, \omega)$ and $X$ be any compact Hausdorff space with a $P$-base and countable scattered height. Is $X$ countable?
5. Calibre $\omega_1$ and non-first-countable compact space

We prove that there is a model of Martin’s Axiom in which there is a compact space that has a $P$-base for a poset $P$ with Calibre $\omega_1$. This space will be the space constructed by Juhasz, Koszmider, and Soukup in the paper [11]. This article [11] shows there is a forcing notion that forces the existence of a first-countable initially $\omega_1$-compact locally compact space of cardinality $\omega_2$ whose one-point compactification has countable tightness. We must prove that there is a poset $P$ as above. We must also show that extra properties of the space ensure that we can perform a further forcing to obtain a model of Martin’s Axiom and that the desired properties of a space naturally generated from the original space possess these same properties in the final model. The reader may be interested to note that in this way we produce a model of Martin’s Axiom and $\mathfrak{c} = \omega_2$ in which there is a compact space of countable tightness that is not sequential. This is interesting because Balogh proved in [3] that the forcing axiom, PFA, implies that compact spaces of countable tightness are sequential. It was first shown in [6] that the celebrated Moore-Mrowka problem was independent of Martin’s Axiom plus $\mathfrak{c} = \omega_2$. The methods in [6] are indeed based on the paper [11] using the notion of T-algebras first formulated in [13]. The example in [11] is itself a space generated by a T-algebra but is not explicitly formulated as such because of its simpler structure.

To do all this, at minimum cost, we must explicitly reference a number of statements and proofs from [11]. The construction is modeled on the following natural properties in the final model. The well-ordering on the underlying set arises canonically from the fact that such spaces are right-separated and scattered. There are functions $H$ with domain $\mu$ and a function $i : [\mu]^2 \to [\mu]^{<\aleph_0}$ satisfying that for all $\alpha < \beta < \mu$:

(1) $\alpha \in H(\alpha) \subset \alpha + 1$ and $H(\alpha)$ is a compact open set (i.e. $H(\alpha) \in \tau$),

(2) $i(\alpha, \beta)$ is a finite subset of $\alpha$,

(3) if $\alpha \notin H(\beta)$, then $H(\alpha) \cap H(\beta) \subset \bigcup\{H(\xi) : \xi \in i(\alpha, \beta)\}$

(4) if $\alpha \in H(\beta)$, then $H(\alpha) \setminus H(\beta) \subset \bigcup\{H(\xi) : \xi \in i(\alpha, \beta)\}$.

Conversely if $H$ and $i$ are functions as in (1)-(4) where (1) is replaced by simply

(1') $\alpha \in H(\alpha) \subset \alpha + 1$ (i.e. no mention of topology)

then using the family $\{H(\alpha) : \alpha \in \mu\}$ as a clopen subbase generates a locally compact scattered topology on $\mu$ in which $H, i$ satisfy property (1)-(4).

Statements (3) and (4) are combined into a single statement in [11] by adopting the notation

$$H(\alpha) * H(\beta) = \begin{cases} H(\alpha) \cap H(\beta) & \alpha \notin H(\beta) \\ H(\alpha) \setminus H(\beta) & \alpha \in H(\beta) \end{cases}.$$

As noted in [11] a locally compact scattered space can not have the properties listed above, hence the construction must be generalized. Also it is shown above (and in [11] for $P = \omega_\omega$) that a compact scattered space with a $P$-base that has Calibre $\omega_1$ will be first countable.

The generalization from [11] will use almost the same terminology and ideas to generate a topology on the base set $\omega_2 \times \mathbb{C}$ where $\mathbb{C} = 2^\mathbb{N}$ is the usual Cantor set and, for each $\alpha < \omega_2$, $\{\alpha\} \times \mathbb{C}$ will be homeomorphic to $\mathbb{C}$. For $n \in \mathbb{N} = \omega \setminus \{0\}$ and $\epsilon \in 2$, the notation $[n, \epsilon]$ will denote the clopen subset $\{f \in 2^\mathbb{N} : f(n) = \epsilon\}$ in
A model in which there is a special function $f_\tau$ so that $f_\tau$ generates a locally compact Hausdorff topology. Proposition 5.2.

Next we rephrase [11, Lemma 2.5]:

Proposition 5.2. If $H,i$ is an $\omega_2$-suitable pair then the subbase

\[
\{U(\alpha, C) : \alpha \in \omega_2\} \cup \{U(\alpha, [n, e]) : \alpha, n, e \in 2\}
\]

generates a locally compact Hausdorff topology $\tau_H$ on $\omega_2 \times C$ satisfying that for all $\alpha \in \omega_2$, $n \in \mathbb{N}$, and $C \subseteq \mathbb{C}$,

1. $U(\alpha, C), U(\alpha, [n, 1])$ are compact,
2. the collection of finite intersections of members of the family

\[
\{U(\alpha, [n, r(n)]) \mid n \in \mathbb{N}, F \in [\alpha]^{<\aleph_0}\}
\]

is a local base at $(\alpha, \tau)$

Next, the authors of [11] have to work very hard to produce an $\omega_2$-suitable pair so that $\tau_H$ is first-countable and initially $\omega_1$-compact. The first step is to work in a model in which there is a special function $f : [\omega_2]^2 \rightarrow [\omega_2]^{<\aleph_0}$ called a strong $\Delta$-function. Since we will not need any properties of this function we omit the definition, but henceforth assume that $f$ is such a function. We record additional minor modifications of results from [11, 4.1, 4.2].

Proposition 5.3. There is a ccc poset $P_f$ consisting of quadruples $p = (a_p, h_p, i_p, n_p)$ that are finite approximations of an $\omega_2$-suitable pair where

1. $a_p \in [\omega_2]^{<\aleph_0}$, $n_p \in \omega$
2. $h_p : [a_p]^2 \times n_p \rightarrow \mathcal{P}(a_p)$,
3. $i_p : [a_p]^2 \times n_p \rightarrow [a_p]^{<\aleph_0}$,

and, for each $P_f$-generic filter $G$, the relations

\[
H = \bigcup \{h_p : p \in G\} \text{ and } i = \bigcup \{i_p : p \in G\}
\]
are functions that form an $\omega_2$-suitable pair. In particular, if $p \in G$, $\alpha \in h_p(\beta)$, and $i_p(\alpha, \beta, 0) = 0$, then (in $V[G]$) $U(\alpha, \mathcal{C}) \subset U(\beta, \mathcal{C})$.

The space $(\omega_2 \times \mathcal{C}, \tau_H)$ is shown to have these additional properties [11, 4.2]:

**Proposition 5.4.** If $G$ is $P_f$-generic and $H, i$ are defined as in Proposition 5.3, then the following hold in $V[G]$:

1. $X_H = (\omega_2 \times \mathcal{C}, \tau_H)$ is locally compact 0-dimensional of cardinality $\mathfrak{c} = 2^{\aleph_1} = \aleph_2$.
2. $X_H$ is first-countable.
3. For every $A \in [X_H]^\omega$, there is a $\lambda < \omega_2$ such that $A \cap U(\lambda, \mathcal{C})$ is uncountable.
4. For every countable $A \subset X_H$, either $\overline{A}$ is compact or there is an $\alpha < \omega_2$ such that $(\omega_2 \setminus \alpha) \times \mathcal{C}$ is contained in $\overline{A}$.

Consequently $X_H$ is a locally compact, 0-dimensional, normal, first-countable, initially $\omega_1$-compact but non-compact space.

Finally, we need the following strengthening of [11, Lemma 7.1] but which is actually proven.

**Proposition 5.5.** If $p = (a_p, h_p, i_p, n_p) \in P_f$ and $a_p \subset \lambda \in \omega_2$, then there is a $q < p$ in $P_f$ such that

1. $a_q = a_p \cup \{\lambda\}$ and $n_q = n_p$,
2. $a_p \subset h_q(\lambda, 0)$,
3. $i_q(\alpha, \lambda, j) = 0$ for all $\alpha \in a_p$ and $j < n_q$.

We note that for $p, q$ as in Lemma 5.5 if $q$ is in the generic filter $G$, then $U(\alpha, \mathcal{C})$ is a subset of $U(\lambda, \mathcal{C})$ for all $\alpha \in a_p$. One consequence of this is that the family $\{U(\alpha, \mathcal{C}) : \alpha \in \omega_2\}$ is finitely upwards directed. Equivalently, the family of complements of these sets in the one-point compactification of $X_H$ is a neighborhood base for the point at infinity.

Now we strengthen [11, Lemma 7.2] which will be used to prove that the one-point compactification of $X_H$ has Calibre $\omega_1$. Some of our proofs will require forcing arguments and we refer the reader to [14] for more details. However some remarks may be sufficient to assist many readers. The forcing extension, $V[G_Q]$ by a $Q$-generic filter $G_Q$ for a poset $Q$ is equal to the valuation, $\text{val}_{G_Q}(A)$ for the collection of all $Q$-names $\dot{A}$ that are sets from $V$. The notation $q \Vdash x \in \dot{A}$ can be read as the assertion that $x \in \text{val}_{G_Q}(A)$ for any generic filter with $q \in G_Q$. The forcing theorem ([14, VII 3.6]) ensures, for example, that if $\dot{A}$ is a $Q$-name of a subset of a ground model set $B$, then $b$ is an element of $\text{val}_{G_Q}(\dot{A})$ exactly when there is an element $q \in G_Q$ such that $q \Vdash b \in \text{val}_{G_Q}(\dot{A})$. Additionally, the set of $q \in Q$ that satisfy that $q \Vdash x \in \dot{A}$ is a set in the ground model, as is the set of $x$ for which there exists a $q$ with $q \Vdash x \in \dot{A}$. This justifies the first line of the next proof.

**Lemma 5.6.** In $V[G]$, for each uncountable $A \subset \omega_2$, there is a $\lambda < \omega_2$ such that $U(\alpha, \mathcal{C}) \subset U(\lambda, \mathcal{C})$ for uncountably many $\alpha \in A$.

**Proof.** Let $\dot{A}$ be a $P_f$-name for a subset of $\omega_2$. Fix any condition $p \in G$ and assume that $p$ forces that $\dot{A}$ has cardinality $\aleph_1$. We prove that there is a $q < p$ and a $\lambda \in a_q$ satisfying that if $q \in G$ then there are uncountably many $\alpha \in \text{val}_G(\dot{A})$ such that
$U(\alpha, C) \subset U(\lambda, C)$. It is a standard fact of forcing that this would then establish the
Lemma (i.e. that there is then necessarily such a $q \in G$).

Let $I$ denote the set of $\alpha \in \omega_2$ satisfying that there is some $p_{\alpha} < p$ (which we
choose) forcing that $\alpha \in \dot{A}$. Since $p$ forces that $\dot{A}$ is a subset of $I$ it follows that
$I$ has cardinality at least $\omega_1$. Since $P_I$ is ccc, it also follows that $I$ has cardinality
equal to $\omega$ but suffices for this argument to choose any $\lambda \in \omega_2$ such that $I \cap \lambda$
is uncountable. For each $\alpha \in I$, choose $q_\alpha < p_{\alpha}$ so that $a_{q_\alpha} = a_p \cup \{\lambda\}$ and the
properties of the pair $p_{\alpha}, q_\alpha$ are as stated in Proposition 5.5.

Just as in the proof of [11, Lemma 7.2], the fact that $P_I$ is ccc ensures that there is
some $q < p$ such that so long as $q \in G$, the set $\{\alpha \in I \cap \lambda : q_\alpha \in G\}$ is uncountable.
As remarked after Proposition 5.5 it follows, in $V[G]$, that $U(\alpha, C) \subset U(\lambda, C)$ for
all $\alpha \in \{\alpha \in I \cap \lambda : q_\alpha \in G\}$. \hfill $\square$

**Theorem 5.7.** If $G$ is a $P_I$-generic filter, then in $V[G]$, the one-point compactification
of the space $X_H$ has a P-base for a poset with Calibre $\omega_1$.

**Proof.** The poset $P$ consists of the family $\{U(\alpha, C) : \alpha \in \omega_2\}$ ordered by inclusion.
To complete the proof we have to note that $\omega < T_P$. For this it is enough to prove that
there is a countable subset of $P$ that has no upper bound. It is relatively easy
to prove that $X_H$ is separable (indeed, that $\omega \times C$ is dense) but oddly enough this
is not stated in [11] and we can more easily simply note that $X_H$ is not $\sigma$-compact
because by Proposition 5.3 it is countably compact and non-compact. \hfill $\square$

An important feature of the construction of $X_H$ from the $\omega_2$-suitable pair $H,i$ is
that even in a forcing extension by a ccc poset $Q$ (in fact by any poset that preserves
that $\omega_1$ and $\omega_2$ are cardinals), the new interpretation of the space obtained using
$H,i$ (i.e. the base set $\omega_2 \times C$ may change because there can be new elements of
$C$) is still locally compact and 0-dimensional. This is similar to the fact that local compactness of scattered spaces is preserved by any forcing (a result by Kunen).
The other properties of $X_H$, such as first-countability and initial $\omega_1$-compactness,
as well as properties of its one-point extension are not immediate and will depend
on what subsets of $\omega_2$ have been added.

An unexpected feature of the $\omega_2$-suitable pair is that, in fact, the first countability
of $X_H$ is preserved by any forcing.

**Lemma 5.8.** For each poset $Q$ in $V[G]$ and $Q$-generic filter $G_Q$, the space $X_H$ is
first-countable in $V[G][G_Q]$.

**Proof.** Of course we will use the fact that, in $V[G]$, $X_H$ is first-countable (as stated
in Proposition 5.4). Fix any $\alpha \in \omega_2$ and recall from Proposition 5.2 that the
collection of all finite intersections of the family
$$\{U(\alpha, [n, r(n)]) \setminus U[F] : n \in \mathbb{N}, F \in [\alpha]^{<\mathbb{N}}\}$$
is a local base at $(\alpha, r) \in \{\alpha\} \times C$ (in any model). In $V[G]$, for each $r \in \mathbb{C}$, let
$Z(\alpha, r) = \bigcap_{n \in \mathbb{N}} U(\alpha, [n, r(n)])$ and let $K(\alpha, r) = \{\xi < \alpha : \{\xi\} \times C \subset Z(\alpha, r)\}$. Let
us recall that $\xi \in K(\alpha, r)$ if and only if $Z(\alpha, r) \cap (\{\xi\} \times C)$ is not empty. Similarly, by
the definition of $U(\alpha, [n, r(n)])$ given in Definition 5.4 $K(\alpha, r) = \bigcap\{H(\alpha, [n, r(n)]) : n \in \mathbb{N}\}$. Since, for all $n \in \mathbb{N}$, $\{H(\alpha, [n, 0]), H(\alpha, [n, 1])\}$ is a partition of $H(\alpha, 0)$,
it follows that $\{K(\alpha, r) : r \in \mathbb{C}\}$ is also a partition of $H(\alpha, 0)$. Since $X_H$ is first-countable (in $V[G]$), for each $r \in \mathbb{C}$, there is a countable $F_r \subset K(\alpha, r)$ such that
$K(\alpha, r) \times C \subset \bigcup\{U(\xi, 0) : \xi \in F_r\}$. \hfill $\square$
Now we are ready to show that, in $V[G][G_Q]$, each point of $\{\alpha\} \times \mathbb{C}$ is a $G_\delta$-point in $X_H$. For each $r \in \mathbb{C}$, we again define the $G_\delta$-set $Z(\alpha, r)$ and $K(\alpha, r) \subset H(\alpha, 0)$ as we did in $V[G]$ but as calculated in the new model $V[G][G_Q]$. It is immediate that $Z(\alpha, r) \cap \{(\alpha) \times \mathbb{C}\}$ is equal to $(\alpha, r)$. Since there are no changes to the values of $H(\alpha, [n, \varepsilon])$ for $(n, \varepsilon) \in \mathbb{N} \times 2$, the value of $K(\alpha, r)$ for each $r \in \mathbb{C} \cap V[G]$ is unchanged and the family $\{K(\alpha, r) : r \in \mathbb{C}\} \subset \mathbb{C}$ is a partition of $H(\alpha, 0)$. It clearly remains the case that, for $r \in \mathbb{C} \cap V[G]$, $K(\alpha, r) \times \mathbb{C}$ is a subset of $\bigcup \{U(\xi, 0) : \xi \in F_r\}$. This implies that $(\alpha, r)$ is a $G_\delta$-point for each $r \in \mathbb{C} \cap V[G]$. Now consider a point $s \in \mathbb{C}$ that is not an element of $V[G]$. But now we have that $K(\alpha, s)$ is empty since $H(\alpha, 0)$ is covered by the family $\{K(\alpha, r) : r \in \mathbb{C} \cap V[G]\}$. This implies that $Z_s$ is equal to the singleton set $\{(\alpha, s)\}$. □

Next we prove that we can extend the model $V[G]$ to obtain a model in which Martin’s Axiom holds (and $\varepsilon = \omega_2$). We do so using the following result from [14] VI 7.1, VIII 6.3 (i.e. the standard method to construct a model of Martin’s Axiom).

**Proposition 5.9.** In the model $V[G]$, there is an increasing chain $\{Q_\xi : \xi \leq \omega_2\}$ of partially ordered sets satisfying for each $\xi < \omega_2$

1. $Q_\xi$ is a ccc poset of cardinality at most $\aleph_1$,
2. each maximal antichain of $Q_\xi$ is a maximal antichain of $Q_{\omega_2}$,
3. if $G_2$ is a $Q_{\omega_2}$-generic filter, then in the model $V[G][G_2]$
   a. Martin’s Axiom holds and $\varepsilon = \omega_2$
   b. for each $A \subset \omega_2 \times \mathbb{C}$ of cardinality less than $\omega_2$, there is a $\xi < \omega_2$ such that $A$ is in the model $V[G][G_2 \cap Q_\xi]$.

For the remainder of this section let $\{Q_\xi : \xi \leq \omega_2\}$ be the poset as in this Proposition and let $G_2$ be a $Q_{\omega_2}$-generic filter. The model $V[G][G_2 \cap Q_\xi]$ is actually equal to the valuation by $G_2$ of all $Q_\xi$-names that are in $V[G]$.

First we prove that the poset of $P$ (consisting of the family $\{U(\alpha, \mathbb{C}) : \alpha \in \omega_2\}$ ordered by inclusion) still has Calibre $\omega_1$ in the forcing extension of $V[G]$ by $Q_{\omega_2}$. In fact, by Proposition 5.9 it suffices to prove that any ccc poset $Q$ of cardinality at most $\aleph_1$ preserves that $P$ has Calibre $\omega_1$.

**Lemma 5.10.** If $G_Q$ is $Q$-generic over $V[G]$ for a ccc poset, then $P$ has Calibre $\omega_1$ in the model $V[G][G_Q]$.

**Proof.** Let $\dot{A}$ be a $Q$-name of a subset of $\omega_2$ and let $q$ be any element of $Q$. Let $I$ be the set of $\alpha \in \omega_2$ such that there exists some $q_\alpha < q$ such that $q_\alpha \forces \alpha \in \dot{A}$. For each $\alpha \in I$ choose such a $q_\alpha < q$. Fix any $\lambda < \omega_2$ so that $I_\lambda = \{\alpha \in I : U(\alpha, \mathbb{C}) \subset U(\lambda, \mathbb{C})\}$ is uncountable. Choose $\bar{q} < q$ so that for all $Q$-generic $G_Q$ with $\bar{q} \in G_Q$, the set $\{\alpha \in I_\lambda : q_\alpha \in G_Q\}$ is uncountable. Since $\alpha \in \text{val}_{G_Q}(\dot{A})$ for all $\alpha \in I$ with $q_\alpha \in G_Q$, this completes the proof that $P$ retains the Calibre $\omega_1$ property. □

It follows from the results so far that, in the model $V[G][G_2]$, the one-point compactification of $X_H$ has a $P$-base and that $P$ has Calibre $\omega_1$. Also, $\{U(\alpha, 0) : \alpha \in \omega_2\}$ is an open cover of $X_H$ that has no countable subcover, so the one-point compactification is not first-countable. This completes the proof of the desired properties, but it is of independent interest to prove this next result because of the connection to the Moore-Mrowka problem.
Proof. We have already established items (1), (2), and (3). Item (3) implies that the one-point compactification of $X$ has countable tightness. Indeed, it follows from item (4) that for any two disjoint closed subsets of $X$, it is normal. Thus, it remains to prove item (4). This will require a forcing proof over the model $V[G]$. Before we begin, let us notice that:

Fact 1. In $V[G]$, if $S$ is an unbounded subset of $\omega$, then the closure of $S \times C$ will contain $(\omega_2 \setminus \alpha) \times C$ for some $\alpha \in \omega_2$.

This follows from the property in item (4) because of the facts that $S \times C$ does not have compact closure and that the one-point compactification of $X$ has countable tightness.

Recall, from Proposition 5.9, that, in $V[G]$, the closure in $X_H$ of each countable subset of $X$ is either compact or contains $(\omega_2 \setminus \alpha) \times C$ for some $\alpha \in \omega_2$. We will prove that this statement remains true in $V[G][G_2]$. Before doing so we note that item (4) is a consequence of this claim. It is immediate from (4) that $X_H$ is countably compact. The fact that then $X_H$ is initially $\omega_1$-compact follows from the fact that a compact $P$-space has no converging $\omega_1$-sequences. The fact that $X_H$ is normal is noted in [11, §8] and is similar to the proof that an Ostaszewski space is normal. Indeed, it follows from item (4) that for any two disjoint closed subsets of $X_H$ at least one of them is compact.

Let $\dot{A}$ be a $Q_{\omega_2}$-name of a countable subset of $X_H$. Assume there is a $\alpha \in G_2$ such that $q$ forces that the closure of $\dot{A}$ is not compact. Note that $q$ forces that for all finite $F \subset \omega_2$, $\dot{A} \setminus U[F]$ is not empty. Also, that the closure of $\dot{A}$ is forced to miss $\{\alpha\} \times C$ if and only if $\dot{A}$ is forced to miss $U(\alpha, 0) \setminus U[F]$ for some finite $F \subset \alpha$.

By Proposition 5.9, there is a $\xi < \omega_2$ and a $Q_\xi$-name $\dot{B}$ satisfying that $val_{G_2 \cap Q_\xi}(\dot{B})$ is equal to $val_{G_2}(\dot{A})$. By possibly choosing a larger value of $\xi$, we may assume that $q \in Q_\xi$. We first note that it suffices to work with $\dot{B}$ and the poset $Q_\xi$.

Fact 2. For each $\lambda \in \omega_2$, $k \in N$, and $t : \{1, \ldots, k\} \in 2$, and finite $F \subset \lambda$, the following are equivalent:

1. $val_{G_2}(\dot{A})$ misses $\{U(\lambda, [n, t(n)])[1 \leq n \leq k] \setminus U[F] \}$.
2. $val_{G_2 \cap Q_\xi}(\dot{B})$ misses $\{U(\lambda, [n, t(n)])[1 \leq n \leq k] \setminus U[F] \}$.

We must prove that the closure of $val_{G_2 \cap Q_\xi}(\dot{B})$ contains $\{\lambda\} \times C$ for a co-initial set of $\lambda \in \omega_2$. This means that we are interested in the set of $\lambda \in \omega_2$ such that $\{\lambda\} \times C$ is not contained in the closure of $\dot{B}$. For any such $\lambda$, there must be a $q \geq q_\xi$, an integer $k_\lambda$, and a function $t_\lambda : \{1, \ldots, k_\lambda\} \rightarrow 2$, and a finite $F_\lambda \subset \lambda$ such that $q_\xi$ forces that $\dot{B}$ is disjoint from $\{U(\lambda, [n, t_\lambda(n)])[1 \leq n \leq k_\lambda] \setminus U[F_\lambda] \}$. Let $S$ denote the set of all $\lambda$ such that such a sequence $\langle q_\lambda, k_\lambda, t_\lambda, F_\lambda \rangle$ exists.
If \( \alpha \notin S \), then \( q \) forces that \( \{ \alpha \} \times C \) is contained in the closure of \( \hat{B} \). Of course this also means that \( q \) forces that the closure of \( \hat{B} \) contains the closure of \( (\omega_2 \setminus S) \times C \). In this case, Fact 1 implies that there is an \( \alpha \in \omega_2 \) such that the closure of \( (\omega_2 \setminus S) \times C \), and therefore of \( \text{val}_{G_\tau \cap Q_\tau}(\hat{B}) \), will contain \( (\omega_2 \setminus \alpha) \times C \) as required. Therefore we conclude that if \( S \) is unbounded, then \( \omega_2 \setminus S \) is bounded.

We assume that \( S \) is unbounded and obtain a contradiction. Since \( Q_\tau \) has cardinality \( \aleph_1 \), it follows from the pressing down lemma that there is a stationary subset \( S_1 \) of \( S \), consisting of limits with cofinality \( \omega_1 \), and a tuple \( \langle \bar{q}, k, t, F \rangle \) such that, for all \( \lambda \in S_1 \)

1. \( \bar{q} = q_\lambda \), \( k = k_\lambda \), \( t = t_\lambda \), and
2. \( F = F_\lambda \).

Let \( W \) be the union of the family \( \{ \bigcap \{ U(\lambda, [n, t(n)]) : 1 \leq n \leq k \} \setminus U[F] : \lambda \in S_1 \} \). Since \( W \) is open and the property of item (4) holds in \( V[G] \), it follows that \( X_H \setminus W \) is compact. Choose any finite \( F_1 \subset \omega_1 \) so that \( X_H \subset W \cup U[F_1] \). It now follows that \( \bar{q} \) forces that \( \hat{B} \) is contained in \( U[F \cup F_1] \), which is a contradiction. \( \Box \)

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