Convergence Analysis of the Summation of the Euler Series by Padé Approximants and the Delta Transformation

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Abstract

Sequence transformations are valuable numerical tools that have been used with considerable success for the acceleration of convergence and the summation of diverging series. However, our understanding of their theoretical properties is far from satisfactory. The Euler series $E(z) = \sum_{n=0}^{\infty} (-1)^n n! z^n$ is a very important model for the ubiquitous factorially divergent perturbation expansions in physics. In this article, we analyze the summation of the Euler series by Padé approximants and the delta transformation [E. J. Weniger, Comput. Phys. Rep. 10, 189 (1989), Eq. (8.4-4)] which is a powerful nonlinear Levin-type transformation that works very well in the case of strictly alternating convergent or divergent series. Our analysis is based on a new factorial series representation of the truncation error of the Euler series [R. Borghi, Appl. Num. Math. 60, 1242 (2010)]. We derive explicit expressions for the transformation errors of Padé approximants and of the delta transformation. A subsequent asymptotic analysis proves rigorously the convergence of both Padé and delta. Our asymptotic estimates clearly show the superiority of the delta transformation over Padé. This is in agreement with previous numerical results.

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1 Introduction

In mathematics, divergent series have been a highly controversial topic \[40, 61, 123\], and to some extent they still are (see for example \[141, Appendix D\] and references therein). Already Euler had frequently used divergent series, albeit not necessarily in a sufficiently rigorous way \[4, 5, 53, 56, 87, 125\]. But as soon as the concept of convergence had been better understood, Euler’s approach was criticized, and in the earlier part of the nineteenth century a strong tendency emerged to ban divergent series completely from rigorous mathematics.

Mathematical orthodoxy prevailed for a while, but at the end of the nineteenth century it had become clear that divergent series were simply too useful to be neglected. In addition, the work of mathematicians like Borel, Padé, Poincaré, Stieltjes, and others ultimately led in the later part of the nineteenth century to a mathematically rigorous theory of divergent series (for more details and additional references, see for example the book by Kline [85, Chapter 47], or the articles by Ferraro [60] and Tuciarone [123]).

The work of the mathematicians mentioned above also showed that divergent series can be used for computational purposes if they are combined with suitable summation techniques. This observation turned out to be extremely consequential since it later inspired a considerable amount of research on summation techniques. As a contemporary example of this research, see the proceedings of a recent conference published in \[37, 142\].

Rigorous convergence proofs of summation techniques are notoriously difficult, in particular in the case of nonlinear techniques like the ones considered in this article. This explains why relatively little had been achieved so far. However, physicist normally do not hesitate to employ numerical techniques in general and summation techniques in special, even if their theoretical properties are not fully understood. They are usually satisfied with convincing numerical evidence that a given technique produces correct results at least for some important special cases.

Generally speaking, the practical usefulness of divergent series in physics has always been more important than concerns about mathematical rigor. A classical example is Stirling’s series for the logarithm of the gamma function [101, Eq. (5.11.1)]. Stirling’s series diverges, but is semiconvergent (for a condensed review of the concept of semiconvergence, which had been introduced by Stieltjes [117] already in 1886, see [144, Appendix E]). Thus, the truncation of Stirling’s divergent series produces excellent approximations for large arguments. Those approximants played a major role in the formal development of statistical mechanics. Another example is Poincaré’s seminal work on asymptotic series [104], which was inspired by his work in astronomy where truncated asymptotic series turned out to be extremely useful.

In quantum physics, divergent series are indispensable. Already in 1952, Dyson [57] had argued that perturbation expansions in quantum electrodynamics must diverge factorially. Around 1970, Bender and Wu [9, 10, 11] showed in their pioneering work on anharmonic oscillators that factorially divergent perturbation expansions occur also in nonrelativistic quantum mechanics. Later, it was found that factorially divergent perturbation expansions are actually the rule in quantum physics rather than the exception (see for example the articles by Fischer [64, Table 1] and Suslov [119], or the articles reprinted in the book by Le Guillou and Zinn-Justin [88]). The observation, that the perturbation expansions of quantum physics diverge almost by default, caused a renaissance of divergent series in theoretical physics.

From the perspective of a theoretical physicist, a divergent series is a fully acceptable mathematical and computational tool, provided that this series is summable to a finite generalized limit. Of course, there remains the practically very important question how such a divergent series can be summed in an effective and numerically reliable way.

The most important summation techniques used by theoretical physicists are Borel sum-
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mation [12], which replaces a divergent perturbation expansion by a Laplace-type integral, and Padé approximants [102], which transform the partial sums of a (formal) power series to rational functions. There is an extensive literature on these summation techniques and their applications not only in theoretical physics. Borel summation is for example discussed in books by Costin [48], Shawyer and Watson [109], and Sternin and Shatalov [116], and the most complete treatment of Padé approximants can be found in the monograph by Baker and Graves-Morris [5]. Both Borel summation and Padé approximants have been remarkably successful, and there are far too many articles describing their applications to be cited here. Nevertheless, these summation methods have – like all other numerical techniques – certain shortcomings and limitations. Therefore, it is certainly justified to look for alternatives that are at least in some cases capable of producing better results.

In recent years, a lot of research has been done on so-called sequence transformations which turned out to be very useful numerical tools for the summation of divergent series. Of course, there is also an extended recent literature, in particular on non-linear and non-regular sequence transformations. Examples are the monographs by Brezinski [26, 27], Brezinski and Redivo Zaglia [33], Sidi [111], and Wimp [151], the review by Weniger [126], or the proceedings [37, 142] of a recent conference.

The majority of these references emphasize the theoretical properties of sequence transformations and are mainly of interest for mathematicians who want to work on sequence transformations, and not so much for physicists who usually want to work with sequence transformations. But recently, the situation has changed and several books have appeared that describe how nonlinear sequence transformations can be employed effectively as computational tools. Examples are books by Bornemann, Laurie, Wagon, and Waldvogel [25, Appendix A], by Gil, Segura, and Temme [70] Chapter 9 (compare also the related review articles by Gil, Segura, and Temme [71] and by Temme [121]), and the last (3rd) edition of Numerical Recipes [105] Chapter 5 (Weniger’s detailed review [140] of the treatment of sequence transformations in Numerical Recipes [105] can be downloaded from the Numerical Recipes web page). Last, but not least: Sequence transformations are also treated in the recently published NIST Handbook of Mathematical Functions [101, Chapter 3.9 Acceleration of Convergence] and in a very recent book by Trefethen [122, Chapter 28].

In this context, it may be of interest to note that Padé approximants, which transform the partial sums of a (formal) power series to a doubly indexed sequence of rational functions, can also be viewed to be nothing but a special class of sequence transformations. Levin-type transformations [89, 126, 138], whose most important features are reviewed in Section 4, were found to be a remarkably effective and powerful class of sequence transformations. They differ substantially from other, better known sequence transformations as for example Wynn’s epsilon algorithm [153], which will be discussed in Section 4.2. They use as input data not only finite substrings of a sequence $\{s_n\}_{n=0}^\infty$, whose elements may be the partial sums $s_n = \sum_{k=0}^n a_k$ of a slowly convergent or divergent infinite series, but also finite substrings of a sequence $\{\omega_n\}_{n=0}^\infty$ of explicit estimates of the truncation errors of $\{s_n\}_{n=0}^\infty$. The explicit incorporation of the information contained in the remainder estimates $\{\omega_n\}_{n=0}^\infty$ into the transformation process is to a large extend responsible for the frequently superior performance of Levin-type transformations. As discussed later in more detail, numerous successful applications of Levin-type transformations are described in the literature, and they are now also discussed in the recently published NIST Handbook [101, Chapter 3.9(v) Levin’s and Weniger’s Transformations].

From the perspective of somebody who only wants to employ Levin-type transformations as computational tools, the situation looks quite good. However, a numerical technique is only fully satisfactory if it is augmented by a sufficiently detailed and easily applicable convergence
theory. In that respect, the situation is not good at all. Our understanding of the convergence properties of Levin-type transformations is far from satisfactory, and it is particularly bad when it comes to the summation of factorially divergent series.

In the case of Padé approximants, the situation is much better. If the input data are the partial sums of a Stieltjes series, whose most important properties are reviewed in Appendix A, it can be proved rigorously that certain subsequences of the Padé table converge to the value of the corresponding Stieltjes function (compare our discussion in Section 3.1). This is very convenient. If we can prove that a factorially divergent power series is a Stieltjes series and that it also satisfies the Carleman condition (5.7), we immediately know that this series is Padé summable.

Because of the use of explicit remainder estimates and the exploitation of the information contained in them, Levin-type transformations often produce remarkably good convergence acceleration and summation results. But when it comes to a theoretical analysis of the properties of Levin-type transformations, remainder estimates cause additional complications that make rigorous proofs more difficult. Because of the specific features of Levin-type transformations, we can only hope to prove that a certain Levin-type transformation produces convergent results when applied to the elements of a given slowly convergent or divergent input sequence \( (s_n)_{n=0}^\infty \). In this article, we only consider the summation of the factorially divergent Euler series (2.1), which is – as discussed in more detail in Section 2 – the simplest prototype of the ubiquitous factorially divergent series expansions.

It will become clear later that the technical problems, which occur in our convergence studies, differ substantially among the various Levin-type transformations. In this article, we only consider the so-called delta transformation [126, Eq. (8.4-4)]. This transformation is known to be very effective in the case of both convergent and divergent alternating series. It sums the Euler series or related factorially divergent expansions much more effectively than for example Padé approximants. It will become clear later that this delta transformation is particularly suited for a theoretical convergence analysis of the summation of the Euler series.

The Euler series (2.1) is a Stieltjes series. Therefore, it is guaranteed that its partial sums are Padé summable to the Euler integral (2.2). It is, however, not so well known that Padé approximants to the Euler series can be expressed in closed form with the help of Drummond’s sequence transformation [55] which is another Levin-type transformation [126, Sections 9.5 and 13.2]. The resulting expression for this Padé approximant [126, Eq. (13.3-5)] can according to Eq. (6.2) be brought into a form which resembles the finite difference operator representation (5.6) of the delta transformation. Accordingly, our convergence analysis of the delta summation of the Euler series can easily be modified to cover the Padé summation of the Euler series. In this way, we were able to derive closed-form expressions for the transformation errors of both the Padé and the delta summation of the Euler series. Our subsequent asymptotic analysis proves rigorously that both summation processes converge. In addition, we obtain estimates for the rate of convergence of both processes, which are in agreement with previous numerical results and which demonstrate the superiority of the delta transformation.

Our article is organized as follows: In Section 2, we briefly review the Euler series. In Section 3.1, the for our purposes most important properties of Padé approximants are reviewed, and in Section 3.2, we briefly review Wynn’s epsilon algorithm. Section 4 discusses the special features of Levin-type transformations. This completes the introductory part which should enable nonspecialist readers to appreciate the core of our work. This is contained in Sections 5-7. Finally, we present in Section 8 some conclusions and some outlook on possible extensions of our work.
2 The Euler Series as a Paradigm for Factorial Divergence

The classic example of a factorially divergent power series is the so-called Euler series

\[ E(z) = \sum_{m=0}^{\infty} (-1)^m m! z^m = _2F_0(1, 1; -z), \quad z \to 0, \tag{2.1} \]

whose summation and interpretation had already been studied by Euler (see for example [38, pp. 323 - 324], [76, pp. 26 - 29] or [4, 5]). If \(|\arg(z)| < \pi\), the Euler series is asymptotic as \(z \to 0\) in the sense of Poincaré [104] to the so-called Euler integral

\[ E(z) = \int_0^\infty \frac{\exp(-t)}{1 + zt} \, dt, \quad |\arg(z)| < \pi. \tag{2.2} \]

If we set \(\Phi(t) = 1 - \exp(-t)\) in Eqs. (A.1), (A.2), and (A.3), we immediately see that the Euler series (2.1) is a Stieltjes series and that the associated Euler integral (2.2) is a Stieltjes function. It also follows from Eq. (A.5) that the partial sum

\[ E_n(z) = \sum_{\nu=0}^{n} (-1)^\nu \nu! z^\nu, \quad n \in \mathbb{N}_0, \tag{2.3} \]

of the Euler series can according to

\[ E(z) = E(z) + R_n(E; z), \quad n \in \mathbb{N}_0, \quad |\arg(z)| < \pi, \tag{2.4} \]

be expressed by the Euler integral plus a truncation error term

\[ R_n(E; z) = -(-z)^{n+1} \int_0^\infty \frac{t^{n+1} \exp(-t)dt}{1 + zt}, \quad n \in \mathbb{N}_0, \quad |\arg(z)| < \pi. \tag{2.5} \]

The integral in Eq. (2.5) is also a Stieltjes integral. Equation (A.6) implies that the truncation error term (2.5) is bounded in magnitude by the first term neglected in the partial sum (2.3):

\[ |R_n(E; z)| \leq \begin{cases} (n+1)! |z^{n+1}|, & |\arg(z)| \leq \pi/2, \\ (n+1)! |z^{n+1} \csc(\arg(z))|, & \pi/2 < |\arg(z)| < \pi. \end{cases} \tag{2.6} \]

The terms of the partial sum (2.3) decrease with increasing index \(n\) as long as \((n+1)|z| < 1\) holds. Thus, for \(z > 0\) the Euler series should be truncated at \(n \approx (1 - z)/z\). Accordingly, the accuracy, which can be obtained by optimally truncating the Euler series at its minimal term, depends strongly on the magnitude of the argument \(z\) (see for example [62, Section 2]). The inclusion of higher terms only leads to a deterioration of accuracy, and ultimately the Euler series diverges for every nonzero \(z \in \mathbb{C}\). This observation confirms once more that the radius of convergence of the Euler series \(_2F_0(1, 1; -z)\) is zero.

Factorially divergent inverse power series occur abundantly in special function theory. By means of some elementary operations, it can be shown that the Euler integral can be expressed as an exponential integral \(E_1\) with argument \(1/z\) [101, Eq. (6.2.2)]:

\[ E(z) = \frac{\exp(1/z)}{z} \int_z^\infty \frac{\exp(-t)}{t} \, dt = \frac{\exp(1/z)}{z} E_1(1/z). \tag{2.7} \]
Accordingly, the Euler series with argument $1/z$ corresponds to the asymptotic expansion of the exponential integral as $z \to \infty$ [101, Eq. (6.12.1)]:

$$z \exp(z) E_1(z) \sim \sum_{m=0}^{\infty} (-1)^m m! z^m = 2F_0(1, 1; -1/z), \quad z \to \infty, \quad |\arg(z)| < 3\pi/2. \quad (2.8)$$

Asymptotic series involving a factorially divergent generalized hypergeometric series $2F_0$ occur also among other special functions. Examples are the asymptotic expansion of the modified Bessel function of the second kind [101, Eq. (10.40.2)],

$$K_\nu(z) \sim \left[\pi/(2z)\right]^{1/2} \exp(-z) 2F_0(1/2 + \nu, 1/2 - \nu; -1/(2z)), \quad |z| \to \infty, \quad |\arg(z)| < 3\pi/2, \quad (2.9)$$

or the asymptotic expansion of the Whittaker function of the second kind [101, Eq. (13.19.3)],

$$W_{\kappa,\mu}(z) \sim \exp(-z/2) z^\kappa 2F_0(\mu - \kappa + 1/2, -\mu - \kappa + 1/2; -1/z), \quad |z| \to \infty, \quad |\arg(z)| < 3\pi/2. \quad (2.10)$$

These expansions have essentially the same features as the Euler series (2.1) or rather the asymptotic series (2.8) for $E_1(z)$. By optimally truncating them in the vicinity of their minimal terms, approximations to the special functions can be obtained whose accuracies depend on the magnitude of the effective argument $1/z$. However, these asymptotic expansions diverge factorially for every $|z| < \infty$ if additional terms beyond the minimal term are included.

For very large arguments, suitably truncated partial sums of the divergent asymptotic inverse power series mentioned above provide sufficiently accurate approximations to the special functions they represent. These approximations can be extremely useful numerically, but it can also happen that these approximations are already for only moderately large arguments too inaccurate to be practically useful.

However, the partial sums of such a divergent inverse power series can be used to construct either Padé approximants or other rational approximants based on sequence transformations (see for example [138, Section VI]). These rational approximants are often able to provide remarkably accurate results even for relatively small arguments. In [126, Section 13.3] it was shown that the transformations $d^{(0)}_k(\beta, s_n)$ and $d^{(1)}_k(\beta, s_n)$ defined in Eqs. (5.3) and (5.4), respectively, sum the divergent asymptotic inverse power series (2.8) for the exponential integral $E_1(z)$ much more effectively than Padé approximants. Qualitatively similar summation results were also obtained in the case of divergent asymptotic inverse power series (2.9) for the modified Bessel function $K_\nu(z)$ [145], and in the case of divergent asymptotic inverse power series (2.10) for the Whittaker function $W_{\kappa,\mu}(z)$ [133].

3 Padé Approximation and Wynn’s Epsilon Algorithm

3.1 A Review of Padé Approximation

The rational approximants, which are named after Padé [102] although they are actually much older (see for example [30, Chapters 4.5, 5.2.5, and 6.4] or the more compact treatment in [31, Section 2]), are extremely important computational tools. There are far too many books as well as countless articles describing work both on and with Padé approximants to be cited in this article. Our major source is the monograph by Baker and Graves-Morris [3] – occasionally nicknamed Red Bible – which extends the older book by Baker [2] and which provides
the currently most detailed treatment of Padé approximants. Of course, [3] also contains a wealth of useful references.

In this Section, we only review those algebraic and convergence properties of Padé approximants that are later needed to understand the summation of the factorially divergent Euler series (2.1). We also try to explain comprehensively in which respect Padé approximants resemble Levin-type transformations and how they differ.

Let us assume that a function \( f : \mathbb{C} \to \mathbb{C} \) possesses a (formal) power series

\[
f(z) = \sum_{n=0}^{\infty} \gamma_n z^n, \tag{3.1}\]

which may or may not converge, and that

\[
f_n(z) = \sum_{y=0}^{n} \gamma_y z^y, \quad n \in \mathbb{N}_0, \tag{3.2}\]

stands for a partial sum of this power series.

The **Padé approximant** \([m/n]_f(z)\) to \( f(z) \) is the ratio of two polynomials \( P^{[m/n]}(z) \) and \( Q^{[m/n]}(z) \) of degrees \( m \) and \( n \) in \( z \):

\[
[m/n]_f(z) = \frac{P^{[m/n]}(z)}{Q^{[m/n]}(z)}, \quad m, n \in \mathbb{N}_0, \tag{3.3a}\]

\[
P^{[m/n]}(z) = p_0 + p_1 z + \cdots + p_m z^m = \sum_{\mu=0}^{m} p_\mu z^\mu, \tag{3.3b}\]

\[
Q^{[m/n]}(z) = q_0 + q_1 z + \cdots + q_n z^n = \sum_{v=0}^{n} q_v z^v. \tag{3.3c}\]

Only \( m + n + 1 \) of the \( m + n + 2 \) unspecified polynomial coefficients in Eq. (3.3) are independent since multiplication of \( P^{[m/n]}(z) \) and \( Q^{[m/n]}(z) \) by a common nonzero constant does not change the value of the ratio \( P^{[m/n]}(z)/Q^{[m/n]}(z) \). Normally, one chooses \( q_0 = 1 \) – the so-called Baker condition – in order to remove this indeterminacy and to guarantee the analyticity of \([m/n]_f(z)\) at \( z = 0 \). The remaining \( m + n + 1 \) unspecified coefficients \( p_0, p_1, \ldots, p_m \) and \( q_1, q_2, \ldots, q_n \) are determined via the requirement that the Maclaurin series of \([m/n]_f(z)\) agrees with the power series (3.1) for \( f(z) \) as far as possible:

\[
f(z) - \frac{P^{[m/n]}(z)}{Q^{[m/n]}(z)} = O(z^{m+n+1}), \quad z \to 0. \tag{3.4}\]

This **accuracy-through-order** relationship or rather the equivalent condition

\[
Q^{[m/n]}(z) f(z) - P^{[m/n]}(z) = O(z^{m+n+1}), \quad z \to 0. \tag{3.5}\]

leads to a system of \( l + m + 1 \) linear equations for the polynomial coefficients [3, Eqs. (1.5) and (1.7) on pp. 2 and 3]. If this system of equations possesses a solution, which will be tacitly assumed henceforth, it follows from Cramer’s rule that a Padé approximant can be defined by

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a ratio of two determinants (see for example [3, Eq. (1.27)]):

\[
[m/n]_f(z) = \begin{vmatrix}
\gamma_{m-n+1} & \gamma_{m-n+2} & \cdots & \gamma_m \\
\gamma_{m-n+2} & \gamma_{m-n+3} & \cdots & \gamma_{m+1} \\
\vdots & \vdots & \ddots & \vdots \\
\gamma_m & \gamma_{m+1} & \cdots & \gamma_{m+n} \\
\sum_{j=0}^m \gamma_{j-n} z^j & \sum_{j=0}^m \gamma_{j-n+1} z^j & \cdots & \sum_{j=0}^m \gamma_{j+n} z^j
\end{vmatrix}.
\]

(3.6)

To the best of our knowledge, this determinantal expression was first derived in 1881 by Frobenius [69, Eqs. (5) and (6)].

For numerical purposes, neither systems of coupled linear equations nor the determinantal expression (3.6) are particularly appealing. Fortunately, numerous other more effective computational algorithms are known (see for example [49, Chapter II.3]). The probably most important and also most convenient recursive scheme for Padé approximants is Wynn’s epsilon algorithm [3, Chapter 6].

But in this article, we concentrate on the convergence of Padé approximants of Stieltjes series satisfying Eqs. (A.2) and (A.3) to their corresponding Stieltjes functions satisfying Eq. (A.1). This theory is summarized below in a highly condensed way.

The poles of the Padé approximants \([m + j/m]_f(z)\) with \(j \geq -1\) to a Stieltjes series of the type of Eq. (A.3) are simple, lie on the negative real semi-axis, and have positive residues [3, Theorem 5.2.1]. Thus, it is guaranteed that the poles of \([m + j/m]_f(z)\) mimic the cut of the corresponding Stieltjes function along the negative real semi-axis.

The Euler series is a Stieltjes series with a zero radius of convergence. In such a case, there is a dual problem if we try to retrieve the corresponding Stieltjes function \(f(z)\) from its divergent series with the help of a summation technique. Firstly, a Maclaurin series does not necessarily determine the corresponding function uniquely. Just consider the function \(g(z) = \exp(-1/z)\) whose derivatives at \(z = 0\) all vanish. This implies that \(f(z)\) satisfying Eq. (3.1) and \(f(z) + g(z)\) possess the same Maclaurin expansion. Therefore, we need a criterion that rules out the occurrence of a function of the type of \(g(z)\).

Secondly, the uniqueness of the integral representation (A.1) – or equivalently the determinacy of the corresponding Stieltjes moment problem – is not guaranteed if the Stieltjes moments \(\mu_n\) increase too rapidly with increasing index [72]. Fortunately, there is a comparatively simple sufficient condition due to Carleman [42] that solves these problems. If the Stieltjes moments \(\mu_n\) defined by Eq. (A.2) satisfy the so-called Carleman condition

\[
\sum_{k=0}^{\infty} \mu_k^{-1/(2k)} = \infty,
\]

(3.7)

then the Padé approximants \([m + j/m]_f(z)\) with \(j \geq -1\) constructed from the partial sums of the moment expansion (A.3) converge as \(m \to \infty\) to the corresponding Stieltjes function \(f(z)\) [3, Theorem 5.5.1].
It can be shown that the Carleman condition (3.7) is satisfied if the Stieltjes moments \( \mu_n \) do not grow faster than \( C^{n+1}(2n)! \) as \( n \rightarrow \infty \), where \( C \) is a suitable positive constant [1,12, Theorem 1.3]. Thus, the Euler series (3.1) satisfies the Carleman condition. An explicit proof, that the Euler series satisfies the Carleman condition, was given in [3, pp. 239 - 240].

Padé approximants to a Stieltjes function \( f(z) \) with \( z > 0 \) and \( j \geq -1 \) satisfy several very useful inequalities [2, Theorem 15.2]:

\[
\begin{align*}
(-1)^j & \left\lfloor \frac{m+j+1}{m+1} f(z) - \left\lfloor \frac{m+j}{m} f(z) \right\rfloor \right\rfloor \geq 0, \quad (3.8a) \\
(-1)^j & \left\lfloor \frac{m+j}{m} f(z) - \left\lfloor \frac{m+j+1}{m+1} f(z) \right\rfloor \right\rfloor \geq 0, \quad (3.8b) \\
\frac{m/m_j(z)}{f(z)} & \geq \left\lfloor \frac{m-1}{m} f(z) \right\rfloor. \quad (3.8c)
\end{align*}
\]

It follows from Eq. (3.8a) that for \( z > 0 \) the Padé sequence \( \left\lfloor \frac{m+j}{m} f(z) \right\rfloor_{m=0}^\infty \) is increasing if \( j \) is odd, and it is decreasing if \( j \) is even. In particular, the Padé sequences \( \left\lfloor \frac{m+1}{m} f(z) \right\rfloor_{m=0}^\infty \) and \( \left\lfloor \frac{m+m+1}{m} f(z) \right\rfloor_{m=0}^\infty \) are increasing, and the Padé sequence \( \left\lfloor \frac{m/m}{f(z)} \right\rfloor_{m=0}^\infty \) is decreasing.

If we set \( j = -1 \) in Eq. (3.8b) and replace \( m \) by \( m+1 \), we obtain for \( z > 0 \) the inequality

\[
\frac{m/m+1}{f(z)} \geq \left\lfloor \frac{m+1}{m} f(z) \right\rfloor, \quad m \in \mathbb{N}_0. \quad (3.9)
\]

Thus, the nesting sequences \( \left\lfloor \frac{m}{m+1} f(z) \right\rfloor_{m=0}^\infty \) and \( \left\lfloor \frac{m/m}{f(z)} \right\rfloor_{m=0}^\infty \) provide the best lower and upper bounds to a Stieltjes function \( f(z) \) with positive argument. The fact that there are nesting sequences of lower and upper bounds is of course extremely helpful in actual computations since we immediately have an estimate of the transformation error. These examples should suffice to show that Padé approximants possess a highly developed and practically useful convergence theory in the case of Stieltjes series.

It is the purpose of this article to construct in the case of the Euler series explicit expressions for the Padé transformation error \( F^{[m/n]}(z) \) defined by

\[
\left\lfloor \frac{m}{n} \right\rfloor_f(z) = f(z) + F^{[m/n]}(z), \quad m, n \in \mathbb{N}_0, \quad (3.10)
\]
as well as analogous expressions for the delta transformation (5.4). Explicit expressions of the type of Eq. (3.10) are apparently extremely rare in the theory of Padé approximants (some counterexamples can be found in articles by Allen, Chui, Madych, Narcowich, and Smith [1], Elliott [58], Karlsson and von Sydow [84], and Luke [94]).

To be practically useful, the transformation error \( F^{[m/n]}(z) \) in Eq. (3.10) should possess an explicit expression of manageable complexity as a function of \( m, n, \) and \( z \). This is very important, because we usually have to construct simple asymptotic approximations to \( F^{[m/n]}(z) \) in order to understand how the rate of convergence behaves if \( m \) and \( n \) become large. In this way, we will not only be able to prove convergence of both Padé and delta in the case of the Euler series, but we will also obtain estimates for the respective rates of convergence that hold in the limit of large transformation orders.

The rarity of explicitly known Padé transformation errors \( F^{[m/n]}(z) \) may well be a consequence of the fact that explicit expressions for Padé approximants are also very rare. In the books by Cuyt, Brevik Petersen, Verdonk, Waadeland, and Jones [54], Gil, Segura, and Temme [70], Luke [93], and Sidi [111], explicit expressions for the Padé approximants of some particularly convenient functions are listed. But in the vast majority of all functions \( f(z) \) possessing (formal) power series expansions, neither the solutions to the systems of coupled linear equations defining Padé approximants [8, Eqs. (1.5) and (1.7) on pp. 2 and 3] nor their determinantal expressions (3.6) can be expressed in closed form. Accordingly, Padé approximants are – as indicated by their name – normally numerical expressions that belong to the realm of approximation theory.

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3.2 Wynn’s Epsilon Algorithm

It is generally agreed that the modern era of sequence transformations started with two seminal articles by Shanks [108] and Wynn [133], respectively. Shanks introduced in 1955 a powerful sequence transformation that computes Padé approximants. Wynn showed in 1956 that not only the Shanks transformation but also Padé approximants can be computed effectively by means of his celebrated recursive epsilon algorithm. Of course, there is a very extensive literature on these transformations. They are not only treated in books on sequence transformations, but also in books on Padé approximants. In addition, there are countless articles describing applications. These transformations are also treated in the recently published NIST Handbook [101], §3.9(iv) Shanks’ Transformation.

Wynn’s epsilon algorithm corresponds to the following nonlinear recursive scheme [153, Eq. (4)] which is amazingly simple:

\[
\begin{align*}
\varepsilon_{-1}^{(n)} &= 0, \quad \varepsilon_{0}^{(n)} = s_n, \quad n \in \mathbb{N}_0, \\
\varepsilon_{k+1}^{(n)} &= \varepsilon_{k-1}^{(n+1)} + \frac{1}{\varepsilon_{k}^{(n+1)} - \varepsilon_{k}^{(n)}}, \quad k, n \in \mathbb{N}_0.
\end{align*}
\]

If the elements of the input sequence \(\{s_n\}_{n=0}^{\infty}\) are the partial sums (3.2) of a (formal) power series \(f(z)\), then the epsilon algorithm (3.11) produces Padé approximants to \(f(z)\):

\[
\varepsilon_{2k}^{(n)} = [k + n/k](z), \quad k, n \in \mathbb{N}_0.
\]

It should be noted that Wynn’s epsilon algorithm is not restricted to input data that are the partial sums of a (formal) power series. Therefore, it is actually more general and more widely applicable than Padé approximants. As a recent review we recommend [73].

If Wynn’s epsilon algorithm is used to convert the partial sums of the Stieltjes moment expansion (3.3) to Padé approximants according to Eq. (3.12), it makes sense to approximate the corresponding Stieltjes function \(f(z)\) by the following staircase sequence in the Padé table [126, Eq. (4.3-7)]:

\[
\begin{align*}
\varepsilon_{2}^{(\nu+2/\nu)} & = \left\lfloor \left\lfloor \frac{\nu + 1}{\nu} \right\rfloor \nu \right\rfloor_{\nu=0}^{\infty} \\
& = [0/0], [1/0], [1/1], \ldots, [\nu/\nu], [\nu + 1/\nu], [\nu + 1/\nu + 1], \ldots.
\end{align*}
\]

Here, \(\lfloor x \rfloor\) is the integral part of the real number \(x\), i.e., the largest integer \(m\) satisfying \(m \leq x\).

It follows from Eqs. (3.8c) and (3.9) that the Padé approximants in Eq. (3.13) satisfy for \(z > 0\) the following inequality [8, Eq. (27)]:

\[
[m + 1/m]_j(z) \leq f(z) \leq [m + 1/m + 1]_j(z), \quad m \in \mathbb{N}_0.
\]

Thus, the staircase sequence (3.13) contains in the case of a Stieltjes function \(f(z)\) with \(z > 0\) two nesting sequences \(e_{2m}^{(1)} = [m + 1/m]_j(z)\) and \(e_{2m}^{(0)} = [m/m]_j(z)\) of lower and upper bounds. Because of inequality (3.14), these bounds are less tight than those provided by the nesting sequences \([m/m + 1]_j(z)_{m=0}^{\infty}\) and \([m/m]_j(z)_{m=0}^{\infty}\) discussed in Section 3.1.

With the help of some techniques developed in [133], inequality (3.14) was together with some other theoretical properties of Stieltjes series used in [8] to investigate numerically whether the factorially divergent perturbation expansion for an energy eigenvalue of the \(\mathcal{PT}\)-symmetric Hamiltonian \(H(x) = p^2 + 1/ax^2 + i\lambda x^3\) is a Stieltjes series, which would imply its Padé summability. Recently, the conjecture formulated in [8] was proven rigorously by Grecchi, Maioli, and Martinez [73] (see also [74]).
4 Levin-Type Transformations

4.1 A General Construction Principle for Sequence Transformations

It is a typical feature of Wynn’s epsilon algorithm (3.11) and of many other sequence transformations that only the input of the numerical values of a finite substring of a sequence \(\{s_n\}_{n=0}^{\infty}\) is required. No other information is needed to compute an approximation to the (generalized) limit \(s\) of the input sequence \(\{s_n\}_{n=0}^{\infty}\). This is also true in the case of Padé approximants. For the computation of the Padé approximant \([m/n](z)\), only the numerical values of the partial sums \(f_\nu(z)\) with \(0 \leq \nu \leq m + n\) are needed.

Very often, this it is a highly advantageous feature. However, in some cases additional information on the index dependence of the truncation errors \(r_n = s_n - s\) is available. For example, the truncation errors of Stieltjes series are according to Eqs. (A.5) and (A.6) bounded in magnitude by the first term neglected in the partial sum, and they also have the same sign patterns as the first terms neglected. It is obviously desirable to utilize such a potentially very valuable structural information in order to enhance the efficiency of the transformation process. Unfortunately, there is no obvious way of incorporating such an information into Wynn’s recursive epsilon algorithm (3.11). This applies also to many other sequence transformations.

In 1973, Levin [89] introduced a sequence transformation which overcame these limitations. It uses as input data not only a sequence \(\{s_n\}_{n=0}^{\infty}\), which is to be transformed, but also a sequence \(\{\omega_n\}_{n=0}^{\infty}\) of explicit estimates of the remainders \(r_n = s_n - s\). Levin’s transformation, which is also discussed in [101], Chapter 3.9(v) Levin’s and Weniger’s Transformations, is generally considered to be one of the most powerful as well as most very versatile sequence transformations currently known [33, 111, 113, 114, 126, 138]. This explains why Levin’s \(\alpha\) transformation [126, Eq. (7.3-5)] is used internally in the computer algebra system Maple to overcome convergence problems (see for example [47, pp. 51 and 125] or [77, p. 258]).

The derivation of Levin’s sequence transformation becomes almost trivially simple if we use a general construction principle for sequence transformations introduced in [126, Section 3]. Here, we present a slightly upgraded and formally improved version. Our starting point is the model sequence

\[
\{s_n\}_{n=0}^{\infty} = \{s + \omega_n z_n^{(k)}\}, \quad k, n \in \mathbb{N}_0.
\]

(4.1)

Obviously, this ansatz can only be useful if the products \(\omega_n z_n^{(k)}\) provide sufficiently accurate approximations to the remainders \(r_n = s_n - s\) of the sequence \(\{s_n\}_{n=0}^{\infty}\) to be transformed.

The key quantities in Eq. (4.1) are the correction terms \(z_n^{(k)}\). The superscript \(k\) characterizes their complexity. In all examples considered here, \(k\) corresponds to the number of unspecified parameters occurring linearly in \(z_n^{(k)}\). Since the remainder estimates \(\omega_n\) are assumed to be known, the approach based on Eq. (4.1) conceptually boils down to the determination of the unspecified parameters in \(z_n^{(k)}\) and the subsequent elimination of \(\omega_n z_n^{(k)}\) from \(s_n\). Often, this approach leads to clearly better results than the construction and elimination of other approximations to \(r_n\).

As described in [126, Section 3], there is a systematic approach for the construction of a sequence transformation, which is exact for the model sequence (4.1). Let us assume that a class of linear operators \(\hat{T}_k\) can be found that annihilate the correction terms \(z_n^{(k)}\) according to \(\hat{T}_k(z_n^{(k)}) = 0\) for fixed \(k\) and for all \(n \in \mathbb{N}_0\). We then obtain a sequence transformation, which is exact for the model sequence (4.1), by applying \(\hat{T}_k\) to the ratio \([s_n - s]/\omega_n = z_n^{(k)}\). Since \(\hat{T}_k\) annihilates \(z_n^{(k)}\) and is by assumption also linear, the following sequence transformation

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\[ T_k(s_n, \omega_n) \] [126, Eq. (3.2-11)] is exact for the model sequence (4.1):

\[ T_k(s_n, \omega_n) = \frac{T_k(s_n/\omega_n)}{T_k(1/\omega_n)} = s. \] (4.2)

Originally, this approach was used for an almost trivially simple derivation of Levin’s transformation and for the construction of some other, closely related sequence transformations [126, Sections 7 - 9]. Later, Brezinski and Redivo Zaglia [34, 35] and Brezinski and Matos [32] showed that this approach is actually much more general than originally anticipated by Weniger, and that the majority of the currently known sequence transformations can be derived in this way. For further references on this topic, see [138, p. 1214].

As shown in [126, Sections 7 - 9] or in [138], simple and yet very powerful sequence transformations are obtained if the annihilation operator \( \hat{T}_k \) in Eq. (4.2) is based upon the finite difference operator \( \Delta \) defined by \( \Delta f(n) = f(n + 1) - f(n) \). Alternative annihilation operators based on divided differences were discussed in [126, Section 7.4].

As is well known, \( \Delta^k \) annihilates arbitrary polynomials \( \mathcal{P}_{k-1}(n) \) of degree \( k - 1 \) in \( n \) according to \( \Delta^k \mathcal{P}_{k-1}(n) = 0 \). Consequently, we choose \( \zeta_n^{(k)} \) in Eq. (4.1) in such a way that multiplication of \( \zeta_n^{(k)} \) by some suitable quantity \( W_k(n) \) yields a polynomial \( \mathcal{P}_{k-1}(n) \) of degree \( k - 1 \) in \( n \). If such pairs \( \zeta_n^{(k)} \) and \( W_k(n) \) can be found, we obviously have

\[ \Delta^k [W_k(n) \zeta_n^{(k)}] = \Delta^k \mathcal{P}_{k-1}(n) = 0. \] (4.3)

Thus, the weighted difference operator \( \hat{T}_k = \Delta^k W_k(n) \) annihilates \( \zeta_n^{(k)} \).

As a further restriction, we consider here as in [138, Section II] exclusively weights satisfying \( W_k(n) = P_{k-1}(n) \), where \( P_{k-1}(n) \) is another polynomial of degree \( k - 1 \) in \( n \). Thus, all Levin-type transformations considered here have the following general structure:

\[ T_k^{(n)}(s_n, \omega_n) = \frac{\Delta^k[P_{k-1}(n) s_n/\omega_n]}{\Delta^k[P_{k-1}(n)/\omega_n]}, \quad k \in \mathbb{N}, \quad n \in \mathbb{N}_0. \] (4.4)

With the help of [101, Eq. (25.1.1)]

\[ \Delta^k f(n) = (-1)^k \sum_{j=0}^{k} (-1)^j \binom{k}{j} f(n + j), \] (4.5)

the right-hand side of Eq. (4.4) can be expressed as the ratio of two binomial sums:

\[ T_k^{(n)}(s_n, \omega_n) = \frac{\sum_{j=0}^{k} (-1)^j \binom{k}{j} P_{k-1}(n + j) s_{n+j}}{\omega_{n+j}}, \quad k \in \mathbb{N}, \quad n \in \mathbb{N}_0, \] (4.6)

This is how Levin-type transformations are usually presented in the literature (see for example [138, Section II]). But for our convergence studies, the finite difference operator representation (4.4) is more useful than the explicit expression (4.6). Nevertheless, the ratio (4.6) has the undeniable advantage that we immediately obtain an explicit expression for \( T_k^{(n)}(s_n, \omega_n) \) if explicit expressions for the input sequence \( \{s_n\}_n \) and the remainder estimates \( \{\omega_n\}_{n=0}^\infty \) are known. For most other transformations, explicit expressions are normally out of reach.

In [138, Section II], several different polynomials \( P_{k-1} \) were discussed which lead to different Levin-type transformations. For these polynomials, the numerator and denominator

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Sums in Eq. (4.10) can also be computed recursively [138, Section III]. This is usually the best way of numerically computing a Levin-type transformation.

The most important polynomials discussed in [138, Section II] are the powers \( P_{k-1}(n) = (\beta + n)^{k-1} \) with \( \beta > 0 \), which yield Levin’s sequence transformation [59] in the notation of [124, Eq. (7.1-7)], and the Pochhammer symbols \( P_{k-1}(n) = (\beta + n)_{k-1} = \Gamma(\beta + n + k - 1)/\Gamma(\beta + n) \) with \( \beta > 0 \), which yield Weniger’s transformation [126, Eq. (8.2-7)].

Let us assume that \( z_n \) is the (generalized) limit of the sequence \( \{s_n\}_{n=0}^{\infty} \). Since the weighted difference operator \( \Delta^k \) is linear, the sequence transformation (4.4) can for all \( k \in \mathbb{N} \) and for all \( n \in \mathbb{N}_0 \) be expressed as follows:

\[
T_k^{(\omega)}(s_n, \omega_n) = s + \frac{\Delta^k [P_{k-1}(n) s_n - s \omega_n]}{\Delta^k [P_{k-1}(n) / \omega_n]} = s + \frac{\Delta^k [P_{k-1}(n) s_n - s \omega_n]}{\Delta^k [P_{k-1}(n) / \omega_n]}.
\]

This expression immediately tells how we can analyze the convergence of convergence acceleration and summation processes. We have to investigate whether and how fast the numerators and denominators of the ratio vanish for fixed \( n \) and as \( k \to \infty \). Obviously, \( T_k^{(\omega)}(s_n, \omega_n) \) converges to \( s \) if \( \Delta^k P_{k-1}(n) \) annihilates \( [s_n - s]/\omega_n = r_n/\omega_n \) more effectively than \( 1/\omega_n \). Thus, we can expect good transformation results if \( [s_n - s]/\omega_n \) depends on \( n \) less strongly than \( 1/\omega_n \).

4.2 Levin’s and Weniger’s Transformation

Inverse power series play a prominent role in pure and applied mathematics and also in the mathematical treatment of scientific and engineering problems. Accordingly, it is an obvious idea to assume that \( z_n^{(k)} \) can be expressed as a truncated inverse power series in \( \beta + n \):

\[
z^{(k)}_n = \sum_{j=0}^{k-1} \frac{c_j^{(k)}}{(\beta + n)^j}, \quad k \in \mathbb{N}, \quad n \in \mathbb{N}_0, \quad \beta > 0.
\]

Here and in analogous expressions occurring later it makes sense to assume \( c_0^{(k)} \neq 0 \), but otherwise the unspecified coefficients \( c_j^{(k)} \) are in principle completely arbitrary.

The product \( \beta + n \) \( \sum_{j=0}^{k-1} c_j^{(k)}/(\beta + n)^j \) is a polynomial of degree \( k - 1 \) in \( n \). Thus, it is annihilated by \( \Delta^k \), and we obtain in the notation of [124, Eqs. (7.1-6) and (7.1-7)] the following expressions for Levin’s sequence transformation [59]:

\[
L_k^{(\omega)}(\beta, s_n, \omega_n) = \frac{\Delta^k [\beta + n]^{k-1} s_n / \omega_n}{\Delta^k [\beta + n]^{k-1} / \omega_n}.
\]

\[
= \sum_{j=0}^{k} (-1)^j \frac{k}{j} \frac{\beta + n + j}{(\beta + n + k)^{j-1}} \frac{s_n}{\omega_n+j}, \quad k, n \in \mathbb{N}_0, \quad \beta > 0.
\]

Different choices for \( z_n^{(k)} \) lead to different sequence transformations. We can for instance assume that \( z_n^{(k)} \) can be expressed as a truncated factorial series in \( \beta + n \):

\[
z^{(k)}_n = \sum_{j=0}^{k-1} \frac{c_j^{(k)}}{(\beta + n)^j}, \quad k \in \mathbb{N}, \quad n \in \mathbb{N}_0, \quad \beta > 0.
\]
Here, \((\beta + n)\) = \(\Gamma(\beta + n + j)/\Gamma(\beta + n)\) is a Pochhammer symbol [101], Eqs. (5.2.4) - (5.2.5)].

At first sight, the ansatz \((4.11)\) may look somewhat strange. However, as briefly discussed in Appendix B or in more detail in [143], there is a highly developed, but also largely forgotten theory of factorial series. They play a similar role in the theory of difference equations as inverse power series in the theory of differential equations. Factorial series were fairly popular in the early twentieth century, but now there is a deplorable lack of public awareness about factorial series and their practical usefulness. Only a relatively small group of specialists still uses them. However, we believe that this widespread neglect of factorial series is not justified and that they can be extremely useful mathematical tools.

Since the product \(\prod_{j=0}^{k-1} \frac{(\beta + n + j)}{\beta + n} j\) is a polynomial of degree \(k - 1\) in \(n\), it is annihilated by \(\Delta^k\). We then obtain the following expressions for the so-called \(S\) transformation [126], Eqs. (8.2-6) and (8.2-7)]:

\[
S_k^{(\alpha)}(\beta, s_n, \omega_n) = \frac{\Delta^k[\prod_{j=0}^{k-1} (\beta + n + j)/\beta + n]}{\Delta^k[\prod_{j=0}^{k-1} \omega_n + j]} \tag{4.12}
\]

\[
= \sum_{j=0}^{k} (-1)^j \frac{k!}{j!} \frac{\prod_{j=0}^{k-1} (\beta + n + j)/\beta + n}{\prod_{j=0}^{k-1} \omega_n + j} \quad \text{for } k, n \in \mathbb{N}_0, \quad \beta > 0. \tag{4.13}
\]

A highly condensed review of the historical development leading to the derivation of this transformation was given in [143, Section 2]). It is now common to call \(S_k^{(\alpha)}(\beta, s_n, \omega_n)\) and its variants the Weniger transformation (see for example [13, 16, 17, 51, 71, 90, 121] or [70, Eq. (9.53) on p. 287]). This terminology was also used in the recently published NIST Handbook of Mathematical Functions [101], Chapter 3.9(v) Levin’s and Weniger’s Transformations].

The transformation \(S_k^{(\alpha)}(\beta, s_n, \omega_n)\) was first used for the evaluation of auxiliary functions in molecular electronic structure calculations [146]. Later, predominantly the delta variant \((5.6)\), which will be discussed later, was used with considerable success for the evaluation of special functions and related objects [79, 80, 83, 126, 127, 128, 129, 133, 137, 141, 145], the summation of divergent perturbation expansions [41, 44, 45, 46, 81, 82, 127, 128, 129, 131, 132, 134, 135, 138, 147, 148], and the prediction of unknown perturbation series coefficients [8, 81, 82, 133]. More recently, the delta transformation had also been employed in optics in the study of nonparaxial free-space propagation of optical wavefields [23, 24, 52, 90, 91] as well as in the numerical evaluation of diffraction catastrophes [13, 14, 15, 16, 17, 18, 20, 21, 22].

### 4.3 Drummond’s Transformation

In the examples considered before, it was tacitly assumed that the truncation errors \(r_n = s - s_n\) of a slowly convergent or divergent sequence can be approximated by the product \(\omega_n z_n^{(k)}\), and that the correction term \(z_n^{(k)}\) approaches a constant as \(n \to \infty\). It is, however, possible to proceed differently.

The essential feature, which had been utilized in the previous examples, is that polynomials of degree \(k - 1\) in \(n\) are annihilated by \(\Delta^k\). Accordingly, we can also make the following ansatz:

\[
z_n^{(k)} = \sum_{j=0}^{k-1} \omega_j^{(k)} (\beta + n)^j = \sum_{j=0}^{k-1} \omega_j^{(k)} n^j, \quad k \in \mathbb{N}, \quad n \in \mathbb{N}_0, \quad \beta > 0. \tag{4.14}
\]
5: Convergence Analysis of the Delta Summation

5.1 Preliminaries

Starting from the decomposition (2.4), we first have to choose the remainder estimates \( \{\omega_n\}_{n=0}^{\infty} \) of the Euler series. We recall that the remainder estimates \( \{\omega_n\}_{n=0}^{\infty} \) have to be chosen in such a way that \( \omega_n \) is proportional to the dominant term of the asymptotic expansion of \( r_n \) \cite{126, Eq. (7.3-1)}, which implies here

\[
\mathcal{R}_n(\varepsilon, z) = \omega_n \left[ c + O(1/n) \right], \quad n \to \infty.
\]

The best simple estimate for the truncation error of a strictly alternating convergent series is the first term not included in the partial sum \cite{86, p. 259}. Since the Euler series is a Stieltjes series, its truncation error \( \mathcal{R}_n(\varepsilon; z) \) is according to Eq. (2.6) bounded in magnitude by the first term neglected in the partial sum. Moreover, the first term neglected is also an estimate of the truncation error of a divergent hypergeometric series \( z F_0(1; \mu; z) \) (see also \cite{110, Eq. 1.17}) for the construction of an explicit expression for the Padé approximant to the divergent hypergeometric series \( z F_0(1; \mu; z) \) (see also \cite{126, Section 13.3}).

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Levin’s $d$ transformation (5.3) and the delta transformation (5.4) possess the following finite difference operator representations [126, Eqs. (7.1-6) and (8.2-6)]:

$$d_k^{(n)}(\beta, s_n) = \frac{\Delta^k((\beta + n)^{k-1}/s_n)}{\Delta^k((\beta + n)^{k-1}/s_n)}, \quad k, n \in \mathbb{N}_0,$$

$$\delta_k^{(n)}(\beta, s_n) = \frac{\Delta^k((\beta + n)^{k-1}/s_n)}{\Delta^k((\beta + n)^{k-1}/s_n)}, \quad k, n \in \mathbb{N}_0.$$

If the elements of the input sequence $\{s_n\}_{n=0}^{\infty}$ are the partial sums of a (formal) power series, then $d_k^{(n)}(\beta, f_n(z))$ and $\delta_k^{(n)}(\beta, f_n(z))$ can be expressed as ratios of two polynomials in $z$ of degrees $k + n$ and $k$, respectively [148, Eqs. (4.26) and (4.27)]:

$$d_k^{(n)}(\beta, f_n(z)) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{(\beta + n + j)^{k-1} z^{j-k} f_{n+j}(z)}{(\beta + n + k)^{k-1} \gamma_{n+j+1}}, \quad k, n \in \mathbb{N}_0,$$

$$\delta_k^{(n)}(\beta, f_n(z)) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{(\beta + n + j)^{k-1} z^{j-k} f_{n+j}(z)}{(\beta + n + k)^{k-1} \gamma_{n+j+1}}, \quad k, n \in \mathbb{N}_0.$$

The rational approximants $d_k^{(n)}(\beta, f_n(z))$ and $\delta_k^{(n)}(\beta, f_n(z))$ satisfy the accuracy-through-order relationships $f(z) - d_k^{(n)}(\beta, f_n(z)) = O(z^{k+n+1})$ and $f(z) - \delta_k^{(n)}(\beta, f_n(z)) = O(z^{k+n+2})$ as $z \to 0$ [148, Eqs. (4.28) and (4.29)], which resemble the accuracy-through-order relationship (4.3) of Padé approximants. In [138, Section 6], it was shown that $d_k^{(n)}(\beta, f_n(z))$ and $\delta_k^{(n)}(\beta, f_n(z))$ can also be viewed as Padé-type approximants which are generalizations of Padé approximants that were introduced by Brezinski in his article [23] and fully developed in his book [24).

As previously mentioned, the positive scaling parameter $\beta$ occurring in $d_k^{(n)}(\beta, s_n, \omega_n)$ and $S_k^{(n)}(\beta, s_n, \omega_n)$ and their variants is customarily set to 1 in most practical applications. Thus, in the following text we shall consider exclusively $\beta = 1$.

It is the aim of our article to show that the delta transformation applied to the partial sums of the Euler series converges for fixed $n \in \mathbb{N}_0$ and for $z \in \mathbb{C} \setminus (-\infty, 0]$ to the Euler integral:

$$E(z) = \lim_{k \to \infty} \delta_k^{(n)}(1, E_n(z)) = \lim_{k \to \infty} \frac{\sum_{j=0}^{k} \binom{k}{j} \frac{(n + j + 1)^{k-1} z^{j-k}}{(n + j + 1)!} \sum_{\nu=0}^{n+j} (-1)^\nu z^{\nu} \nu!}{\sum_{j=0}^{k} \binom{k}{j} \frac{(n + j + 1)^{k-1} z^{j-k}}{(n + j + 1)!} z^{j-k}}.$$

Our main technical problem is that both the numerators and denominators of the delta transformation (5.6) are binomial sums of the type of $\Delta^k F_n = \sum_{j=0}^{\nu} (-1)^j \binom{k}{j} F_{n+j}$. Unfortunately, it is in general extremely difficult to estimate the asymptotics of such a binomial sum $\Delta^k F_n$ with fixed $n$ as $k \to \infty$ [67, 68], let alone to find an explicit expression for $\Delta^k F_n$.

These technical problems can be overcome with the help of Borghi’s factorial series rep-
presentation for the remainder of the Euler series \[19, \text{Eq. (52)}\]:

\[
\mathcal{R}_k(\varepsilon; z) \frac{(-1)^{n+1}(n+1)!z^{\varepsilon+1}}{(-1)^{n+1}(n+1)!z^{\varepsilon+1}} = - \sum_{k=0}^{\infty} \frac{L_k^{(-1)}(1/z)}{z(n+1)_{k+1}}.
\] (5.10)

Here, \(L_k^{(a)}(x)\) is a generalized Laguerre polynomial \[101, \text{Eq. (18.5.12)}\]. This expansion, which extends and perfects the purely symbolic approach of \[139, \text{Section 7}\], is our central mathematical tool. With the help of the factorial series (5.10), we will eventually arrive at a function which is, with the exception of the zeros of the denominator polynomial, analytic for all \(L \in \mathbb{C}\).

As shown in Eq. (5.8), the delta transformation \(\delta_k^{(0)}\) of two polynomials of degrees \(\kappa_1 + \kappa_2\) cut along the negative real semi-axis. Obviously, a rational function like \(\delta_k^{(0)}\) is somehow able to mimic the cut \((-\infty, 0]\). But we can only hope for good approximations to the Euler integral, if delta approximation artifacts, can occur in regions where convergence is to be expected. Obviously, not all poles of the rational functions necessarily correspond to the singularities of the function have to be approximated somehow by the poles of the rational function. However, not all poles of the rational functions necessarily correspond to the singularities of the function which is to be approximated. These spurious poles, which are essentially approximation artifacts, can occur in regions where convergence is to be expected. Obviously, spurious poles can easily frustrate any attempt of formulating a rigorous convergence theory.

In the case of Padé approximants \([M + J/M]\) with \(J \geq -1\) to a Stieltjes series, it is known that all their poles are simple, that they lie on the negative real semi-axis, and that they have positive residues (see for example \[115, \text{Theorem 5.2.1 on p. 201}\]). In this way, a Padé approximant simulates the cut of a Stieltjes function on the negative real semi-axis. Therefore, we have to prove in the case of delta that all zeros of the denominator polynomials on the r.h.s. of Eq. (5.11) are real and negative.

It is comparatively easy to show that the denominator in Eq. (5.11) can be expressed as a terminating hypergeometric series \(2F_2[101, \text{Eq. (16.2.1)}]\). We only have to use Eq. (5.3) together with \([-1] = (-1)^{\nu}(-\nu)/\nu!\) to obtain

\[
\Delta^k \frac{(n+1)_{k-1}}{(-1)^{n+1}(n+1)!z^{\varepsilon+1}} = \frac{(-1)^{k+\nu+1}}{(n+1)! \varepsilon^{\nu+1}} 2F_2 \left( \frac{-k, k+n}{n+1, n+2}; -\frac{1}{\varepsilon} \right).
\] (5.12)
We now have to prove that all zeros of this terminating $_2F_2$ are real and negative. Since all coefficients of this $_2F_2$ are positive, it suffices to prove that all zeros are real.

This aim can be accomplished by considering some special generalized hypergeometric series that are called Pólya frequency functions [106,107]. In [106, Theorem 4.1] it was shown that the generalized hypergeometric series

$$pF_q\left(\frac{\alpha_1 + k_1, \ldots, \alpha_p + k_p}{\alpha_1, \ldots, \alpha_q}; x\right)$$  \hspace{1cm} (5.13)

with $p \leq q$, $\alpha_1, \ldots, \alpha_q > 0$, and $k_1, \ldots, k_p \in \mathbb{N}$ is a Pólya frequency function, which according to [54, Lemma 5] implies that the associated terminating generalized hypergeometric series

$$p+1F_q\left(-m, \alpha_1 + k_1, \ldots, \alpha_p + k_p; -x\right)$$  \hspace{1cm} (5.14)

with $m \in \mathbb{N}$ has only real zeros. If we now choose $p = 1$, $q = 2$, $m = k$, $\alpha_1 = n + 1$, $k_1 = k - 1$, $\alpha_2 = n + 2$, and $x = 1/z$, we see that for all $n \in \mathbb{N}_0$ and for all $k \geq 2$ the terminating $_2F_2$ in Eq. (5.12) has only negative zeros. Direct inspection shows that this is also true for $k = 1$. Thus, the denominator in (5.11) simulates for all $k \in \mathbb{N}$ and for all $n \in \mathbb{N}_0$ the cut $(-\infty, 0]$ of the Euler integral. We can also rule out spurious poles of delta in $C \setminus (-\infty, 0]$.

### 5.3 Integral Representation for the Delta Transformation Error

Next, we analyze the numerator in Eq. (5.11). Our starting point is the factorial series expansion (5.10) for the remainder of the Euler series, which – as explained in Appendix B – leads to the following integral representation of $\mathcal{R}_n(\mathcal{E}; z)$:

$$\frac{\mathcal{R}_n(\mathcal{E}; z)}{(-1)^{n+1}(n+1)!z^{n+1}} = -\frac{1}{z} \int_0^1 t^{n} \varphi(t) \, dt,$$

$$\varphi(t) = \sum_{k=0}^\infty L_k^{(-1)}(1/z)(1-t)^k.$$  \hspace{1cm} (5.15b)

Equation (5.15) alone would not be such a great achievement. However, an explicit expression for the infinite series in Eq. (5.15b) can be obtained by using the generating function of the generalized Laguerre polynomials (see for example [101, Eq. (18.12.13)]) which yields

$$\frac{\mathcal{R}_n(\mathcal{E}; z)}{(-1)^{n+1}(n+1)!z^{n+1}} = -\frac{1}{z} \int_0^1 \exp\left(-\frac{1-t}{zt}\right) t^n \, dt.$$  \hspace{1cm} (5.16)

If we substitute Eq. (5.16) into the numerator of Eq. (5.11) and use Eq. (4.5), we obtain

$$\Delta^k \left\{ \frac{(n+1)k-1\mathcal{R}_n(\mathcal{E}; z)}{(-1)^{n+1}(n+1)!z^{n+1}} \right\} = -\frac{1}{z} \int_0^1 \exp\left(-\frac{1-t}{zt}\right) \Delta^k [(n+1)k-1] t^n \, dt$$

$$= -\frac{(-1)^k}{z} \int_0^1 \exp\left(-\frac{1-t}{zt}\right) \sum_{j=0}^k (-1)^j \binom{k}{j} (n+j+1)k-1 t^{n+j} \, dt.$$  \hspace{1cm} (5.17)

The finite sum in the integral can be expressed as a terminating hypergeometric series $_2F_1$:

$$\sum_{j=0}^k (-1)^j \binom{k}{j} (n+j+1)k-1 t^{n+j} = (n+1)k-1 t^n \cdot {}_2F_1\left(\begin{array}{c} \frac{-k,k+n+1}{} \\
{ }\end{array}; z\right).$$  \hspace{1cm} (5.18)
Accordingly, Eq. (5.17) can be expressed as follows:
\[
\Delta^k \left\{ \frac{(n+1)_k}{(n+1)!} \mathcal{R}_n(E; z) \right\} = \frac{(-1)^k}{z} \frac{(n+1)_k}{(n+1)!} \int_0^1 t^n \exp \left( -\frac{1-t}{zt} \right) 2F_1 \left( \begin{array}{c} -k, k+n \\ n+1 \end{array} ; t \right) dt.
\] (5.19)

If we now combine Eqs. (5.12) and (5.19), we obtain the following explicit expression for the transformation error for the delta summation of the Euler series:
\[
\delta_k^{(0)}(1, E_0(z)) - E(z) = -(-z)^n (n+1)! \int_0^1 t^n \exp \left( -\frac{1-t}{zt} \right) 2F_1 \left( \begin{array}{c} -k, k+n \\ n+1 \end{array} ; t \right) dt.
\] (5.20)

### 5.4 Asymptotic Estimate of the Delta Convergence Rate

To the best of our knowledge, the integral representation (5.20) is new and we consider it to be a major achievement. Unfortunately, it does not solve our basic problem of proving that the delta transformation sums the Euler series to the Euler integral according to Eq. (5.9). We were not able to find an explicit expression for the integral in Eq. (5.20). Therefore, it is by no means obvious how the right-hand side of Eq. (5.20) behaves if the transformation order \( k \) becomes large. As a remedy, we try to construct asymptotic approximations to the numerators and denominators in Eq. (5.20) which hold for fixed \( n \) as \( k \to \infty \).

In order to simplify notation and to make the resulting mathematical expressions more readable, we now set \( n = 0 \). This does not sacrifice too much generality since it corresponds to the most important choice in practical applications. We then obtain for the integral representation (5.20):
\[
\delta_k^{(0)}(1, E_0(z)) - E(z) = -\int_0^1 t^n \exp \left( -\frac{1-t}{zt} \right) 2F_1 \left( \begin{array}{c} -k, k+n \\ n+1 \end{array} ; t \right) dt.
\] (5.21)

For technical reasons, we make the variable substitution \( t = (1+x)/2 \), which yields:
\[
\delta_k^{(0)}(1, E_0(z)) - E(z) = -\frac{1}{2} \exp \left( \frac{1}{z} \right) \int_{-1}^1 \exp \left( -\frac{2z}{1+x} \right) 2F_1 \left( \begin{array}{c} -k, k+n \\ n+1 \end{array} ; \frac{1+x}{2} \right) dx.
\] (5.22)

In the following text, we will analyze the asymptotic behavior of the transformation error \( \delta_k^{(0)}(1, E_0(z)) - E(z) \) as \( k \to \infty \) by analyzing separately the asymptotic behavior of the numerators and denominators in Eq. (5.22). We accomplish this with the help asymptotic approximations to hypergeometric functions that were introduced by Fields [63] and by Fields and Luke [64, 65], respectively, and later discussed in a very detailed way in Luke’s book [92, Chapter 7.4].
For an asymptotic analysis of the denominator in Eq. (5.22) as $k \to \infty$, we start from the asymptotic expansion [92, Eq. 7.4.5(6) on p. 263]:

$$
P_{\lambda}^{(n)} \sim \frac{1}{\rho_{\lambda}} \left( \frac{n+\lambda}{n+1} \right)^{\lambda/2} F_{\lambda+1/2,1/2}\left( \frac{-n}{\rho_{\lambda}} ; -z \right)
$$

$$
+ \frac{(2\pi)^{(1/2)}}{\beta^{1/2} \Gamma(\sigma_p)} (N^\theta z)^\varphi \exp \left( \frac{Nz^{1/\beta} - (az/3) - \Omega(-z)/Nz^{1/\beta}}{O(N^{-2})} \right),
\]}

where $|\arg(z)| \leq \pi - \epsilon$, $\epsilon > 0$, $\delta = +(-)$ if $\arg(z) \leq (>) 0$. (5.23)

The asymptotic variable $N \to \infty$ is defined by $N^\theta = n(n+\lambda)$ with $\beta = q-p+1$. The remaining unspecified quantities occurring here are defined in [92, Eqs. 7.4.5(4) and 7.4.5(5) on p. 262].

For $p = 0$, the finite sum in Eq. (5.23) is an empty sum which vanishes. By setting $q = 2$, $n = k$, $\lambda = 0$, $\rho_1 = 1$ and $\rho_2 = 2$, we obtain $\beta = 3$, $N = k^{2/3}$, $a = 1$, and $y = -2/3$.

In the exponential factor in Eq. (5.23), we discard all contributions that vanish at least like $O(1/N)$ as $N \to \infty$. We obtain after some simple algebra:

$$
2 F_2 \left( \frac{-k,k}{1,2} ; -\frac{1}{z} \right) \sim \frac{\left( z/k^{2/3} \right)^{2/3}}{2\pi \sqrt{3}} \exp \left( \frac{3k^{2/3}}{z^{1/3}} - \frac{1}{3z} \right), \quad k \to \infty.
\]}

The integral in the numerator of Eq. (5.22) is more challenging. We first assume $z > 0$. This is necessary to justify the validity of the intermediate steps of our derivation. As is well known, manipulations based on the stationary phase approach or related techniques, which we will employ in Eqs. (5.28) - (5.37), tend to be very restrictive with respect to the ranges of arguments and parameters for which their validity can be guaranteed. Fortunately, it is often possible to show via analytic continuation that final results obtained in this way are less demanding and are valid in larger domains (compare also our discussion of Fejér’s formula 6.1(6) in Section 6.2).

The asymptotic variable $k$ occurs only in the hypergeometric function $2 F_1$ in the integral. Thus, we employ the following asymptotic approximation [92, Eq. 7.4.2(8) on p. 250] that holds as $N \to \infty$:

$$
P_{\lambda}^{(n)} \sim \frac{1}{\rho_{\lambda}} \sum_{i=1}^{\infty} \frac{(n+\lambda)^{i-\lambda/2}}{(n+1)^{i-\lambda/2}} \frac{\Gamma(\rho_{\lambda+1})}{\Gamma(\rho_{\lambda})} \frac{\Gamma(1/2)}{\Gamma(1/2)} \frac{|\sin(\theta/2)|^{2\gamma}}{\cos(\theta/2)^{2\gamma+2}}
\]}

$$
\times \exp \left( \frac{\varphi_1(\theta) + \varphi_2}{N^2} + O \left( \frac{1}{N^2} \right) \right) \cos \left( \frac{N\theta + \pi\gamma + \varphi_1(\theta)}{N} + \frac{\varphi_2(\theta)}{N^3} + O \left( \frac{1}{N^3} \right) \right),
\]}

where $|\arg(z)| \leq \pi - \epsilon$, $|\arg(1-z)| \leq \pi - \epsilon$, $\epsilon > 0$. (5.25)

The asymptotic variable $N \to \infty$ is defined by $N^2 = n(n+\lambda)$. In addition, we have $\cos(\theta) = 1 - 2z$ or equivalently $z = \sin^2(\theta/2)$. The remaining unspecified quantities are defined in [92, Eq. 7.4.2(9) on pages 251 and 252]. As in Eq. (5.23), the finite sum is for $p = 0$ an empty sum which vanishes.

If we set in Eq. (5.25) $p = 0$, $\lambda = 0$, $n = k$, $\rho_{\lambda+1} = \{1\}$, $z = (1+x)/2$, and if we neglect all contributions that vanish at least like $O(1/N)$ as $N \to \infty$, we obtain:

$$
2 F_1 \left( \frac{-k,k}{1,1} ; -\frac{1}{2} \right) \sim \frac{1}{\sqrt{\pi k}} \left[ \tan(\theta/2) \right]^{-1/2} \cos(k\theta - \pi/4), \quad k \to \infty.
\]}

Here $\cos(\theta) = -x$. If we now take into account that $x = \pi - \arccos(\theta)$, we obtain:

$$
2 F_1 \left( \frac{-k,k}{1,1} ; -\frac{1}{2} \right) \sim \frac{(-1)^k}{\sqrt{\pi k}} \left( \frac{1-x}{1+x} \right)^{1/4} \cos \left( k \arccos(x) + \frac{\pi}{4} \right), \quad k \to \infty.
\]}

(Riccardo Borghi and Ernst Joachim Weniger: Summation of the Euler Series)
For technical reasons it is advantageous to express the cosine in terms of exponentials:

\[
\text{2F}_1\left(-k, k, 1 + x; \frac{1}{2}\right) \sim \frac{(-1)^k}{\sqrt{\pi k}} \text{Re}\left(\left(1 - x\right)^{1/4} \exp\left(i k \arccos(x) + \frac{i\pi}{4}\right)\right), \quad k \to \infty, 
\]

(5.28)

If we insert this asymptotic approximation into the integral in Eq. (5.22), we obtain:

\[
\int_{-1}^{1} \exp\left(-\frac{2}{z} \frac{x + 1}{x + 1}\right) \text{2F}_1\left(-k, k, 1 + x; \frac{1}{2}\right) dx \sim \frac{(-1)^k}{\sqrt{\pi k}} \text{Re}\left(\exp\left(i\pi/4\right) J_k(1/z)\right), \quad k \to \infty, 
\]

(5.29a)

To estimate the integral \(J_k(\alpha)\) as \(k \to \infty\), we first set \(x = \cos(t)\), which yields

\[
J_k(\alpha) = \int_{-1}^{1} \exp\left(-\frac{2\alpha}{x + 1}\right) \left(1 - x\right)^{1/4} \exp\left(i k \arccos(x)\right) dx.
\]

(5.29b)

In this form, the large \(k\) behavior of the integral can be estimated with the help of the stationary phase approach (see for example [152, Chapter II.3]). For that purpose, we rewrite Eq. (5.31) as follows:

\[
J_k(\alpha) = 4 \exp(-\alpha) \int_{0}^{\infty} \frac{\tau^{3/2}}{(1 + \tau^2)^2} \exp(-\alpha \tau^2) \exp(i2k \arctan(\tau)) d\tau.
\]

(5.31)

The four saddles are solutions of the equation \(f'(\tau) = 0:\)

\[
4\alpha \tau^4 + (4\alpha - 3) \tau^2 - 4i\kappa + 3 = 0.
\]

(5.33)

If we now make in Eq. (5.33) the ansatz \(\tau = Ak^\gamma\), we obtain the following equation:

\[
4\alpha A^4 k^{4\gamma} + (4\alpha - 3) A^2 k^{2\gamma} - 4iA k^{2\gamma + 1} + 3 = 0.
\]

(5.34)

In the asymptotic limit \(k \to \infty\), we obtain the following approximations to the roots \(\tau_0, \ldots, \tau_3:\)

\[
\tau_m = \left(\frac{k}{A}\right)^{1/3} \exp\left(i\pi/6 + \frac{i2\pi m}{3}\right), \quad m = 0, \ldots, 2,
\]

(5.35a)

\[
\tau_3 \approx \frac{3i}{4k}.
\]

(5.35b)
Extensive numerical tests showed that it is sufficient to include in Eq. (5.31) only the contribution from $r_0$. Since we have

\[
 f_0 = f(r_0) = \frac{3}{2} \log \left( (ik/\alpha)^{1/3} \right) - a(ik/\alpha)^{2/3} + i2k \arctan \left( (ik/\alpha)^{1/3} \right) \\
 \sim i[k + 1/4] \pi + \frac{1}{2} \log(k/\alpha) + \frac{2}{3} - 3k^{2/3}(-\alpha)^{1/3}, \quad k \to \infty ,
\]

\[
 f''_0 = f''(r_0) = -\frac{2}{3r_0} - 2a - \frac{4i k r_0}{[1 + r_0]^2} \sim -6a , \quad k \to \infty ,
\]

\[
 g_0 = g(r_0) \sim (ik/\alpha)^{-4/3}, \quad k \to \infty ,
\]

we obtain after some long but in principle straightforward algebra:

\[
 \int_0^1 t^n \exp \left( -\frac{1-t}{z} \right) _2F_1 \left( -k, k + n ; n + 1 ; t \right) dt \\
 \sim 4(-1)^k (2/k)^{1/2} \exp(-\alpha) \Re \left[ \exp(i\pi/4) g_0 \exp(f_0) \right]^{-1/2} \\
 \sim \frac{4}{3^{1/2} \zeta^{1/3} k^{4/3}} \exp \left( -\frac{1}{3z} - \frac{3}{2} k^{2/3} \right) \cos \left( \frac{3}{2} k^{2/3} \pi \right), \quad k \to \infty .
\]

As remarked above, this result was derived under the assumption $z > 0$. We nevertheless expect this approximation to work for complex $z \in \mathbb{C} \setminus \{0\}$ thanks to analytic continuation. As shown in Section 7, numerical tests confirm our conjecture.

Combining Eq. (5.31) with Eqs. (5.22) and (5.24) eventually yields:

\[
 \delta_k^{(0)}(1, E_0(z)) - \mathcal{E}(z) \sim \frac{4\pi}{z} \exp \left( \frac{1}{z} \right) \exp \left( -\frac{9 k^{2/3}}{2 z^{1/3}} \right) \cos \left( \frac{3}{2} k^{2/3} \pi \right), \quad k \to \infty .
\]

This is our main result. It not only proves for $n = 0$ that the delta transformation sums the factorially divergent Euler series to the Euler integral according to Eq. (5.39), but it also provides an asymptotic estimate of the rate of convergence as a function of the transformation order $k$. To the best of our knowledge, the transformation error estimate (5.38) is also the first transformation error estimate for any Levin-type transformation and any non-trivial input sequence that holds in the limit of infinite transformation orders $k$.

6 Convergence Analysis for Padé Approximants

6.1 An Explicit Expression for the Padé Transformation Error

As reviewed in Section 3.1, Padé approximants to Stieltjes series possess a highly developed convergence theory. The Euler series is a Stieltjes series, and the Euler integral is the corresponding Stieltjes function. Therefore, Padé approximants sum for fixed $n \geq -1$ the factorially divergent Euler series to the Euler integral [3, Theorem 5.5.1]:

\[
 \mathcal{E}(z) = \lim_{k \to \infty} [k + n/k] c(z), \quad |\arg(z)| < \pi .
\]

However, an explicit expression for the transformation error $[k + n/k] c(z) - \mathcal{E}(z)$ of the type of Eq. (5.20) seems to be unknown. We also want to derive suitable asymptotic approximations.
to \([k + n/k]_{E}(z) - \mathcal{E}(z)\) which make it possible to compare quantitatively the respective rates of convergence of Padé and delta.

Sidi [110, Eq. (2.18)] (compare also [111, Chapter 17.3]) could show that the Padé approximants \([k + n/k]_{E}(z)\) with \(k, n \in \mathbb{N}_0\) of the Euler series can be expressed by Drummond’s sequence transformation \(D^{(\mathcal{E})}_{k}(s_n, \omega_n)\) [58] with remainder estimates \(\omega_n = \Delta s_n\). This Levin-type transformation possesses according to Eq. (4.15) a finite difference operator representation, which is more convenient for our purposes, and according to Eq. (4.16) also a closed form expression as the ratio of finite sums. Thus, the Padé approximants \([k + n/k]_{E}(z)\) to the Euler series can be expressed in terms of Drummond’s sequence transformation as follows (see also [126, Eq. (13.3-5)]):

\[
[k + n/k]_{E}(z) = \mathcal{D}^{(\mathcal{E})}_{k}(E_n, \Delta E_n) = \frac{\Delta^{k}[E_n(z)/\Delta E_n(z)]}{\Delta^{k}[1/\Delta E_n(z)]} = \frac{\Delta^{k}\left\{ \sum_{v=0}^{n} (-1)^{v+1} v! z^{v-n-1} \right\}}{(-1)^{v+1} (n+1)!} \frac{1}{(n+1)!} \quad (6.2)
\]

\[
= \sum_{j=0}^{k} \binom{k}{j} z^{k-j} (n+j+1)! \sum_{v=0}^{n+j} (-1)^{v} v! z^{v}, \quad k, n \in \mathbb{N}_0. \quad (6.3)
\]

The fact, that the Padé approximants to the Euler series can be expressed by a certain Levin-type transformation, is very helpful for our purposes. We can employ the same mathematical technology as in Section 5.

Our first step is the following finite difference operator representation for the transformation error which closely resembles Eq. (5.11):

\[
[k + n/k]_{E}(z) - \mathcal{E}(z) = \frac{\Delta^{k}\left\{ \frac{\mathcal{R}_{E}(E; z)}{\Delta E_n(z) \Delta^{k}} \right\}}{\Delta^{k}\left\{ \frac{1}{\Delta E_n(z) \Delta^{k}} \right\}}. \quad (6.4)
\]

Next, we derive explicit expressions for the numerator and denominator on the right-hand side of Eq. (6.4). With the help of Eq. (5.5), the denominator can be expressed as a terminating confluent hypergeometric series \(\mathcal{F}_{1}\):

\[
\Delta^{k}\left\{ \frac{1}{(-1)^{v+1} (n+1)! z^{v+1}} \right\} = \frac{(-1)^{k}}{(-1)^{v+1} (n+1)!} \sum_{j=0}^{k} \binom{k}{j} (-k)_j (-z)^{j} (n+2)! j! \quad (6.5)
\]

\[
= \frac{(-1)^{k}}{(-z)^{v+1} (n+1)!} \mathcal{F}_{1}(-k; n+2; -1/z). \quad (6.5)
\]

We do not have to analyze whether all zeros of the \(\mathcal{F}_{1}\) are real and negative, since all poles of a Padé approximant to a Stieltjes series are simple and lie on the negative real semi-axis [3, Theorem 5.2.1]. But we could deduce this directly from the fact that the \(\mathcal{F}_{1}\) in Eq. (6.5) can be expressed as a generalized Laguerre polynomial [101, Eq. (18.5.12)]:

\[
\mathcal{F}_{1}(-k; n+2; -1/z) = \frac{k!}{(n+2)!} L^{(n+1)}_{k}(z) = \frac{(n+1)!}{(k+1)_{n+1}} L^{(n+1)}_{n+1}(z). \quad (6.6)
\]
6: Convergence Analysis for Padé Approximants

As is well known, all zeros of a generalized Laguerre polynomial $L^{(\alpha)}_n(x)$ with $\alpha > -1$ are real and positive (see for example [6, Theorem 4.5.1 on p. 116] or [59, Chapter 10.17 on p. 204]). Therefore, all zeros of $L^{(\alpha)}_k(-1/z)$ must be real and negative.

If we now combine (6.5) and (6.6), we obtain for the denominator in Eq. (6.4):

$$\Delta^k \left\{ \frac{1}{(1)^{n+1}(n+1)!z^{n+1}} \right\} = \frac{(-1)^k}{(-z)^{n+1}(k+1)_{n+1}} L^{(n+1)}_k(-1/z).$$

(6.7)

The derivation of an explicit expression for the numerator in Eq. (6.4) is more demanding. We achieve some simplification by reformulating the right-hand-side of Eq. (5.16) with the help of the variable transformation $\xi = (1-t)/(zt)$:

$$\frac{R_n(E; z)}{(-1)^{n+1}(n+1)!z^{n+1}} = -\int_0^\infty \frac{\exp(-\xi)}{(1+\xi)^n} \, d\xi.$$  

(6.8)

We now apply $\Delta$ under the integral sign and use Eq. (5.5). Then, we obtain with the help of the binomial theorem [101, Eq. (1.2.2)]:

$$\Delta^k \left\{ \frac{R_n(E; z)}{(-1)^{n+1}(n+1)!z^{n+1}} \right\} = -\int_0^\infty \frac{\exp(-\xi)}{(1+\xi)^n} \Delta^k \left\{ \frac{1}{(1+\xi)^n} \right\} \, d\xi$$

(6.9)

$$= (-1)^{k+1} \int_0^\infty \frac{\exp(-\xi)}{(1+\xi)^{n+2}} \sum_{j=0}^k (-1)^j \binom{k}{j} \frac{1}{(1+\xi)^j} \, d\xi$$

(6.10)

$$= (-1)^{k+1} \int_0^\infty \frac{\exp(-\xi)}{(1+\xi)^{n+2}} \left\{ 1 - \frac{1}{1+\xi} \right\}^k \, d\xi$$

(6.11)

$$= (-z)^k \int_0^\infty \frac{\xi^k \exp(-\xi)}{(1+\xi)^{n+2}} \, d\xi.$$  

(6.12)

The last integral can be expressed in closed form in terms of a Kummer $U$ function [101, Eq. (13.4.4)], yielding:

$$\Delta^k \left\{ \frac{R_n(E; z)}{(-1)^{n+1}(n+1)!z^{n+1}} \right\} = (-1)^{k+1} \frac{k!}{z} U(k+1, -n, 1/z).$$

(6.13)

If we now substitute Eqs. (6.7) and (6.13) into Eq. (6.4), we obtain the following closed form expression for the Padé transformation error:

$$[k + n/k]_E(z) - E(z) = (-z)^n (k+1)_{n+1}! \frac{U(k+1, -n, 1/z)}{L^{(n+1)}_k(-1/z)}. $$

(6.14)

With the help of $U(a, -n, z) = z^{n+1} U(a + n + 1, n + 2, z)$ [101, Eq. (13.2.11)], the Kummer function in Eq. (6.14) can be expressed by $z^{-n} U(k + n + 2, n + 2, 1/z)$, which yields

$$[k + n/k]_E(z) - E(z) = (-1)^n \frac{(k+1)_{n+1}!}{z} \frac{U(k + n + 2, n + 2, 1/z)}{L^{(n+1)}_k(-1/z)}. $$

(6.15)

To the best of our knowledge, the explicit expressions (6.14) and (6.15) for the Padé transformation error of the Euler series are new.
6.2 Asymptotic Analysis of the Padé Convergence Rate

We want to proceed as we did it in the case of the delta transformation in Section 5.4, i.e., we want to determine independently the leading order asymptotics of the numerators and denominators on the right-hand side of Eqs. (6.14) or (6.15) for fixed \( n \) as \( k \to \infty \).

For an analysis of the denominators in Eqs. (6.14) and (6.15), we can use the following leading order asymptotic approximation to a generalized Laguerre polynomial, which is commonly called Fejér’s formula (see for example [120, Theorem 8.22.1 on p. 198]):

\[
L_n^{(\alpha)}(x) = \frac{e^{i\pi/4}n^{\alpha/2-1/4}}{\pi^{1/4}2^{\alpha/2+1/4}} \left[ \cos \left( 2 \sqrt{\pi n} - \frac{\alpha \pi}{2} - \frac{\pi}{4} \right) + O \left( n^{-1/4} \right) \right], \quad n \to \infty, \quad x > 0. \quad (6.16)
\]

As remarked above, all zeros of a generalized Laguerre polynomial \( L_n^{(\alpha)}(x) \) with \( \alpha > -1 \) are located on the positive real semi-axis (see for example [6, Theorem 4.5.1 on p. 116] or [59, Chapter 10.17 on p. 204]). The side condition \( x > 0 \) implies that Fejér’s formula (6.16) is valid in the oscillatory region \([0, \infty)\) of \( L_n^{(\alpha)}(x) \). However, for \( x \in \mathbb{C} \setminus [0, \infty) \) \( L_n^{(\alpha)}(x) \) is a monotonic function without zeros.

In the numerators of Eqs. (6.14) and (6.15), there occurs a generalized Laguerre polynomial \( L_k^{(\alpha+1)}(-1/z) \) with \( \alpha \in \mathbb{C} \setminus (-\infty, 0] \). This raises the question whether Fejér’s formula (6.16) is really suited for our purposes if the argument of \( L_k^{(\alpha+1)}(-1/z) \) does not belong to the oscillatory region \((-\infty, 0] \). The behavior of \( L_k^{(\alpha+1)}(-1/z) \) in its oscillatory region \(-\infty < -1/z < 0 \) is important because it enables the Padé approximant \([k+n/k]\) to mimic the cut of the Euler integral, but for our convergence analysis it is of no interest.

With respect to these questions, the mathematical literature is contradictory. Beals and Wong [3, p. 116] emphasized that the convergence of Fejér’s formula (6.16) is uniform in \( x \) for any compact interval \( 0 < \delta \leq x \leq \delta^{-1} \), but they did not say anything about its behavior away from the positive real semi-axis. Similarly, Erdélyi, Magnus, Oberhettinger, and Tricomi [59, p. 199], Ismail [78, p. 118], and Szegö [120, p. 198] remarked that Fejér’s formula (6.16) is uniformly valid in any compact subset of \([0, \infty)\). In contrast, Buchholz [39, p. 137] stated that Fejér’s formula (6.16) is valid for \( 0 \leq \arg(x) \leq 2\pi \).

This apparent inconsistency can be resolved if we replace in Fejér’s formula (6.16) the cosine by exponentials via \( \cos(\varphi) = [\exp(i\varphi) + \exp(-i\varphi)]/2 \). This yields:

\[
L_n^{(\alpha)}(x) = \frac{e^{i\pi/4}n^{\alpha/2-1/4}}{\pi^{1/4}2^{\alpha/2+1/4}} \left[ \exp \left( i \left( 2 \sqrt{\pi n} - \frac{\alpha \pi}{2} - \frac{\pi}{4} \right) \right) + \exp \left( -i \left( 2 \sqrt{\pi n} - \frac{\alpha \pi}{2} - \frac{\pi}{4} \right) \right) + O \left( n^{-1/4} \right) \right],
\]

\[
n \to \infty, \quad |\arg(2 \sqrt{\pi n})| < \pi. \quad (6.17)
\]

For \( x > 0 \), this is simply Fejér’s formula (6.16) in disguise. But for \( x < 0 \), we get something new because now only the second exponential in (6.17) matters. The first one is exponentially subdominant as \( n \to \infty \) and can be neglected. For \( x < 0 \), the branches of \((-x)^{-\alpha/2-1/4}\) and \((-x)^{1/2}\) have to chosen in such a way that they are real and positive.

Thus, for an analysis of the large index asymptotics of \( L_k^{(\alpha+1)}(-1/z) \), which occurs in the denominators in Eqs. (6.14) and (6.15), we use (6.17) which provides the following leading order approximation:

\[
L_k^{(\alpha+1)}(-1/z) \sim \frac{(z/\pi)^{1/2}}{2} \exp \left( -\frac{1}{2z} + \frac{2k^{1/2}}{\pi z^{1/2}} \right) (z/k)^{(2\alpha+1)/4}, \quad k \to \infty. \quad (6.18)
\]
6: Convergence Analysis for Padé Approximants

This is essentially a special case of Perron’s formula [78, Theorem 4.8.9 on p. 118] (a more general expression was given in [124, Theorem 8.22.3 on p. 199]):

\[ L_n(\alpha) = \frac{e^{\alpha n} n^{2\alpha - 1/4}}{\pi^{1/2} (-1)^{2\alpha + 1/4}} \exp\left(2 \sqrt{n - \alpha}\right), \quad n \to \infty, \quad x \in \mathbb{C} \setminus (0, \infty). \] (6.19)

Consequently, the leading order asymptotic approximation (6.17) interpolates between Fejér’s formula (6.16) and Perron’s formula (6.19).

If we now use the asymptotic approximation (6.17), we obtain for the numerators in Eqs. (6.14) and (6.15):

\[ \frac{L_n^{(n+1)}(-1/z)}{(n+1)_{n+1}} - \frac{1}{2\sqrt{n}} \exp\left(-\frac{1}{2z} + \frac{2k^{1/2}}{z^{1/2}} \sqrt{\frac{2(n+3/4)}{k(n+3/4)}}\right) \quad k \to \infty. \] (6.20)

Concerning the numerator in Eq. (6.14), we start from the following leading order asymptotic approximation [101, Eq. (13.8.8)],

\[ U(a, b, x) \sim \frac{e^{\frac{1}{2\beta}}}{\Gamma(\alpha)} \left[ 2 \tan\left(\frac{w(2/2)}{\beta}\right) \left(1 - e^{-w}\right)^{-b} \right]^{1-b} K_{1-b}(2\beta a) + \frac{1}{a} \left(\frac{a^{-1/b} + 1}{1 + b}\right)^{-1} e^{-2\beta a} O(1), \quad a \to \infty, \] (6.21a)

\[ w = \arccosh\left(1 + \frac{x}{2a}\right), \quad \beta = \frac{w + \sinh(w)}{2}. \] (6.21b)

which holds uniformly with respect to \( x \) in \([0, \infty)\) for fixed \( b \leq 1\). Here, \( K_{1-b}(2\beta a) \) is a modified Bessel function of the second kind [101, Eq. (10.27.4)].

If we now set in Eq. (6.21) \( a = k + 1, b = -n, \) and \( x = 1/z\), we obtain with the help of [101, Eqs. (4.33.1) and (4.38.4)] the following asymptotic estimates:

\[ w = \arccosh\left(1 + \frac{1}{2kz}\right) \sim \frac{1}{\sqrt{k}}, \quad k \to \infty, \] (6.22a)

\[ \beta = \frac{w + w[1 + O(w)]}{2} = w[1+ O(w)], \quad w \to 0, \] \( k \to \infty. \) (6.22b)

If we now insert the asymptotic estimates in Eq. (6.22) into Eq. (6.21), we obtain:

\[ k! U(k+1, -n, 1/z) \sim 2 \sqrt{\pi} \exp\left(\frac{1}{2z}\right) (zk)^{-(a+1)/2} K_{a+1}\left(2\sqrt{k/z}\right), \quad k \to \infty. \] (6.23)

If we now approximate the modified Bessel function \( K_{a+1}\left(2\sqrt{k/z}\right) \) by the leading term of its asymptotic expansion (2.9), we obtain the following asymptotic estimate for the numerator in Eq. (6.14):

\[ k! U(k+1, -n, 1/z) \sim \sqrt{\pi} \exp\left(\frac{1}{2z} - \frac{2^{1/2}}{z^{1/2}}\right) \sim \left(2a+1\right)^{1/4} k^{-2(n+3)/4}, \quad k \to \infty. \] (6.24)
Finally, we insert Eqs. (6.20) and (6.24) into Eq. (6.14) to obtain the following leading order asymptotic approximation to the Padé transformation error:

\[ [k + n/k]_E(z) - E(z) \sim (-1)^n \frac{2\pi}{z} \exp\left(\frac{1}{z}\right) \exp\left(-\frac{4k^{1/2}}{z^{1/2}}\right), \quad k \to \infty. \]  

(6.25)

It is remarkable that the \(n\)-dependence of the leading order estimate (6.25) of the transformation error of the Padé approximant \([k + n/k]_E(z)\) occurs only in the form of the sign factor \((-1)^n\) which decides whether the Padé sequence \([k+n/k]_E(z)\) provides upper or lower bounds. This is in agreement with Stieltjes inequalities (3.8) discussed in Section 3.1. Moreover, we can conclude that all Padé sequences \([k + n/k]_E(z)\) and \([k + n'/k]_E(z)\), respectively, occur only in subdominant terms.

If we set in Eq. (6.25) \(n = 0\), we obtain the following leading order asymptotic approximation to the transformation error of diagonal Padé approximants:

\[ [k/k]_E(z) - E(z) \sim \frac{2\pi}{z} \exp\left(\frac{1}{z}\right) \exp\left(-\frac{4k^{1/2}}{z^{1/2}}\right), \quad k \to \infty. \]  

(6.26)

A comparison of our asymptotic estimates (5.38) and (6.26) shows that the rate of convergence of a diagonal Padé approximant \([k,k]_E(z)\), whose behavior is characterized by the decay term \(\exp(-4k^{1/2}/z^{1/2})\), is much lower than that of the delta transformation \(\delta_k^{(0)}(1, E_0(z))\) characterized by \(\exp(-9k^{2/3}/2z^{1/3})\). To the best of our knowledge, this result constitutes the first theoretical explanation of the well known superiority of the delta transformation over Padé approximants with respect to the summation of factorially divergent alternating series. In addition, Eqs. (5.38) and (6.26) also provide theoretical estimates of the respective rates of convergence. This information seems to be missing in the current literature.

7 Numerical Results

7.1 Transformation Errors for Positive Arguments

The derivation of our asymptotic transformation error estimates (5.38) and (6.26) required several partly drastic approximations whose validity is only guaranteed in the asymptotic domain of very large transformation orders \(k\). Therefore, it is by no means obvious whether our estimates produce meaningful results also for only moderately large or even small transformation orders \(k\). We check this question numerically.

In our analytical manipulations in Sections 5.4 and 6.2, we had always assumed \(z > 0\). But our subsequent numerical results will demonstrate that our estimates also work for complex arguments \(z\) as long as we are not too close to the cut \((-\infty, 0)\) of the Euler integral.

Figure 1 displays the transformation errors for Padé and delta (open circles) vs the transformation order \(k\) of the sequences \([\delta_k^{(0)}]\) (labeled as “delta”) and \([k/k]\) (labeled as “Padé”), for the summation of the Euler series at \(z = 10\) (since the Euler series is asymptotic to the Euler integral as \(z \to 0\), an argument \(z = 10\) represents a very challenging summation problem). The solid curves represent the corresponding asymptotic estimates (5.38) and (6.26), respectively.

The results in Fig. 1 show that our asymptotic estimates (5.38) and (6.26) also work quite well for small transformation orders \(k\), i.e., far away from the asymptotic regime \(k \to \infty\). Several other numerical experiments not shown here, which used different positive values of the argument \(z\) of the ES, confirmed our conclusion about the usefulness of the asymptotic estimates (5.38) and (6.26) away from the asymptotic regime.
7.2 Transformation Errors for Complex Arguments

The plots in Fig. 2 also display the observed transformation errors vs $k$ as in Fig. 1 but now for complex arguments $z = |z| \exp(i\varphi)$. The transformation errors corresponding to $z = 10 \exp(i\varphi)$ are plotted for several values of $\varphi \in (0, \pi)$. But in all plots, the superiority of the delta transformation over Padé approximants is evident.

The rates of convergence displayed in Fig. 2 assume their maxima for $z > 0$ ($\iff \varphi = 0$) and decrease monotonically in magnitude as $z = |z| \exp(i\varphi)$ approaches the cut $(-\infty, 0]$ ($\iff \varphi = \pm \pi$). Both for delta and for Padé, these observations can be explained by analyzing our asymptotic estimates (5.38) and (6.26). We distinguish three cases (i): $z > 0 \iff \varphi = 0$, (ii): $z \in \mathbb{C} \setminus (-\infty, 0] \iff \varphi \in (-\pi, \pi) \setminus \{0\}$, and (iii): $z \in (-\infty, 0] \iff \varphi = \pm \pi$.

If $z > 0$, the convergence of delta is guaranteed by the exponential factor in our asymptotic estimate (5.38). The cosine term in Eq. (5.38) only produces some modulation, which can be observed in Fig. 1 but which do not affect the essentially exponential decay. Similarly, the convergence of Padé is for $z > 0$ guaranteed by our asymptotic estimate (6.26).

If $z$ belongs to the cut complex plane $\mathbb{C} \setminus (-\infty, 0]$, the situation is much more complicated. Because of $\cos(x) = \frac{\exp(ix) + \exp(-ix)}{2}$, the cosine factor in Eq. (5.38) can for $\varphi \in (-\pi, \pi) \setminus \{0\}$ no longer be ignored in our convergence analysis. If we transform the cosine in Eq. (5.38) to complex exponentials and multiply it by the exponential in Eq. (5.38) – which is best done with the help of a computer algebra system like Maple or Mathematica – we obtain exponentially decaying terms modulated by sine and cosine terms which, however, guarantee the convergence of delta in the cut complex plane $\mathbb{C} \setminus (-\infty, 0]$. Similarly, our asymptotic estimate (6.26) implies that Padé converges in the cut complex plane $\mathbb{C} \setminus (-\infty, 0]$.
Figure 2: The same as in Fig. 1 but for complex z = 10 exp(i\varphi) with \varphi = \pi/4 (a), \varphi = \pi/2 (b), \varphi = 3\pi/4 (c), and \varphi = 9\pi/10 (d).

But if z lies on the cut (−∞, 0], the situation is quite different. If we again transform the cosine in Eq. (5.38) by complex exponentials and multiply it by the exponential in Eq. (5.38), we now obtain for \varphi = ±\pi in addition to exponentially decaying terms also sine and cosine terms which are not damped by exponentially decaying terms. Thus, delta does not converge if z lies on the cut (−∞, 0]. Such a divergence occurs also in the case of Padé.

This negative result is actually no surprise. As discussed before, rational approximants like delta or Padé can only mimic the cut (−∞, 0] of the Euler integral but not exactly reproduce it. In addition, the Euler series is for z ∈ (−∞, 0] an extremely demanding summation problem since the Euler integral is for z ∈ (−∞, 0] a double-valued function. This follows at once from the fact that the Euler integral is according to Eq. (2.2) essentially the exponential integral $E_1$ with argument 1/z. But for negative real argument the exponential integral $E_1(z)$ is also double-valued. It satisfies [101, Eq. (6.5.1)]:

$$E_1(-z \pm i0) = -Ei(z) \mp i\pi, \quad z > 0.$$  

(7.1)

Here, Ei is another exponential integral which for z > 0 is essentially a Cauchy principal value integral [101, Eqs. (6.2.5) and (6.5.2)]. If we replace in Eq. (7.1) z by 1/(-z ± i0) = -1/z ± i0 with z > 0 and combine the resulting expression with Eq. (2.2), we obtain [140, Eqs. (28) - (30)):

$$\int_0^\infty \frac{\exp(-t)dt}{1 + (-z \mp i0)t} = \frac{\exp(1/(-z \mp i0))}{-z \mp i0} E_1(1/(-z \mp i0))$$

$$= \frac{\exp(-1/z)}{z} [Ei(1/z) \mp i\pi], \quad z > 0.$$  

(7.2)
This expression shows that $E(-z \mp i0)$ is double-valued since it contains a nonzero imaginary part $\mp \pi \exp(-1/z)/z$, whose sign depends on how we approach the negative real axis and which is nonanalytic as $z \to 0$.

### 7.3 Summation of the Euler Series by Levin’s Transformation

In view of the similarity of Levin’s $\mathcal{L}$ and Weniger’s $\mathcal{S}$ transformation, a skeptical reader might wonder why we do not also consider the $d$-variant of the Levin’s transformation defined in Eq. (5.3) whose finite difference operator representation is $\Delta^k(\beta + n)^{k-1}/\Delta s_n$, $k, n \in \mathbb{N}_0$.

Unfortunately, differences between the transformations defined by Eqs. (5.6) and (7.3), respectively, are greater than they might seem at first sight. Our convergence analysis of the delta summation of the Euler series is based on the fact that the truncation error $R_k(E; z)$ of the Euler series possesses a factorial series representation according to Eq. (5.10). This greatly simplified our analytical manipulations since both Pochhammer symbols $(n + \beta)_{k-1}$ as well as factorial series are very convenient objects from the perspective of finite difference operators. In contrast, powers like $(n + \beta)^{k-1}$ occurring in Eq. (7.3) are extremely inconvenient objects.

On the basis of the available evidence we have to conclude that our theoretical analysis of the summation of the Euler series cannot be extended to Levin’s $d$ transformation (7.3) and related variants of Levin’s transformation in a straightforward way. One potential alternative, that could possibly supplement our approach, would be the use of so-called Nörlund-Rice integrals for the asymptotic evaluation of binomial sums $\Delta^k F_n = \sum_{j=0}^{k}(-1)^{j+1}F_{n+j}$ as suggested by Flajolet and Sedgewick. This may well be a promising idea for future investigations.

In spite of our deplorable lack of theoretical results, we can gain at least some insight by performing numerical experiments. In Fig. 3 we employ Levin’s $d$ transformation (7.3) for the summation of the Euler series. As in Figs. 1 and 2, we plot in Fig. 3 the transformation errors $d_k^0(1, E_0(z)) - E(z)$ (open circles) evaluated via Eq. (7.3) vs the transformation order $k$ for $z = 10$. The solid curve was obtained as a purely numerical fit of the exponential model $d_k^0(1, E_0(z)) - E(z) = A \exp(-\alpha k^2)$ which resembles our transformation error estimates (5.38) and (6.26) obtained for the delta transformations and diagonal Padé approximants.

Several numerical tests carried out for different positive arguments $z$ of the Euler series provide convincing evidence that the transformation errors $d_k^0(1, E_0(z)) - E(z)$ can for larger transformation orders $k$ and positive arguments $z$ be modeled by a single exponential $A \exp(-\alpha k^2)$ and that the exponent $\alpha$ is close to $3/4$. Moreover, we also observe in Fig. 3 some oscillatory modulations that could be caused by something resembling the cosine term in Eq. (5.38).

Our results indicate that at least for sufficiently large transformation orders $k$ the transformation errors $d_k^0(1, E_0(z)) - E(z)$ of Levin’s $d$ transformation should decay faster than the corresponding transformation errors $d_k^0(1, E_0(z)) - E(z)$ of the delta transformation, whose decay rate is according to Eq. (5.38) characterized by the exponential $\exp(-9k^{2/3}/[2z^{1/3}])$.

Our summation results in Fig. 3 indeed look very impressive. However, this does not imply that Levin’s $d$ transformation necessarily sums all factorially divergent series more effectively than the delta transformation. Such a generalization looks tempting but can be badly misleading. For example, in the summation of the factorially divergent Rayleigh-Schrödinger perturbation series for the ground state energy of the quartic anharmonic oscillator with the
help of Levin’s $d$ transformation initially seemed to produce convergent results. But in the case of large transformation orders the summation results clearly diverged. In the case of the delta transformation or Padé approximants, no divergence was observed in [148].

The divergence of Levin’s transformation was also confirmed in [128, Table 2], where the summations were performed with a Levin-type transformation that – depending on the value of a continuous parameter – interpolates between Levin’s $d$ and the delta transformation. A similar divergence of Levin’s transformation was later observed by Čížek, Zamastil, and Skála [46, p. 965] in the case of the hydrogen atom in an external magnetic field. More detailed discussions of the divergence of Levin’s transformation can be found in [130, pp. 211 - 216], in [140, pp. 7 - 9], or in [144, pp. 57 - 58].

No completely satisfactory explanation of the divergence of Levin’s $d$ transformation in the case of the anharmonic oscillators is known. This divergence remains a mystery. Our inability of explaining this divergence indicates that our current understanding of the subtleties of summation processes is far from being satisfactory. There still remains a lot of work to be done.

8 Conclusions and Outlook

The factorially divergent Euler series defined by Eq. (2.1) is an asymptotic series as $z \to 0$ for the Euler integral defined by Eq. (2.2). It is a very important model problem not only in special function theory, where many asymptotic expansions diverge factorially, but even more so for physicists in connection with the ubiquitous factorially divergent perturbation expansions. The topic of this article is an in depth analysis of the summation of the Euler series to the Euler integral with the help of Padé approximants and the delta transformation.
which is known to be a very powerful Levin-type sequence transformation. According to experience delta outperforms the much better known Padé approximants in the case of factorially divergent series.

Our theoretical manipulations produced closed form expressions for the transformation errors of both Padé and delta. From them, we derived asymptotic estimates which holds in the case of large transformation orders. Our estimates clearly shows that the delta transformation is indeed able to sum the factorially divergent Euler series to the Euler integral. To the best of our knowledge, our estimate (5.38) is the first explicit theoretical error estimate for any Levin-type transformation that holds in the limit of infinite transformation orders. A comparison of our asymptotic transformation error estimate (5.38) for the delta transformation with the analogous estimate (6.26) for diagonal Padé approximants provides the first theoretical explanation of the observed superiority of the delta transformation in the case of the Euler series over the much better known Padé approximants (compare [126, Tables 13-1 and 13-2]). We believe that our theoretical results should give potential users of Levin-type transformations more confidence in the power and usefulness of these transformations.

In spite of its undeniable success, our approach has obvious limitations. We were only able to derive our explicit expressions and our asymptotic estimates because the truncation error (2.5) of the Euler series can be expressed by the factorial series expansion (5.10). Unfortunately, no analogous factorial series expansions are known for the truncation errors of other factorially divergent asymptotic series for special functions. These unknown factorial series expansions for truncation errors have to be derived first before one could try to extend our approach to other special functions. Thus, our article could also be viewed to be an invitation to mathematicians interested in special function theory to do something about these missing factorial series expansions.

Another problem is that our approach cannot be applied to the typical perturbation expansions of physics because their series coefficients are usually not known explicitly, but only numerically. For these problems we need a new approach based on some general principles. Ideal would be something like the highly developed convergence theory of Padé approximants to Stieltjes series which was briefly reviewed in Section 3.1. Therefore, we actually hope that the results of this article might constitute a first and probably preliminary step toward the development of a more general theory of the summation of factorially divergent series with the help of Levin-type sequence transformations. But we fear that this will be not easy, and that a lot of work has to be done before this goal can be achieved.

Our analytical approach also has the obvious shortcoming that it cannot be extended in a straightforward way to other Levin-type transformations. As discussed in Section 7.3, we are currently not able to extend our approach, that was so very successful in the case of the delta transformation, to the analogous and at least at first sight very similar d transformation of Levin defined by Eq. (7.3). It seems that new analytical techniques have to be developed before we can accomplish anything in this direction. We already mentioned the so-called Nörlund-Rice integrals advocated by Flajolet and Sedgewick [67, 68].

Factorially divergent asymptotic and perturbation expansions undoubtedly constitute challenging numerical problems. However, in quantum physics perturbation expansions occur whose series coefficients grow roughly like ($\nu n$)! with $\nu > 1$, and which therefore constitute even more challenging problems. Examples are the Rayleigh-Schrödinger perturbation expansions for the energy eigenvalues of the sextic and octic quantum anharmonic oscillators, whose series coefficients grow roughly like $(2n)!$ and $(3n)!$, respectively. Padé approximants are not powerful enough to achieve anything substantial in the sextic case, although they in principle converge, and in the even more challenging octic case, Graffi and Grecchi [72] showed rigorously that Padé approximants are not able to sum this violently divergent pertur-
bation expansion. In contrast, the delta transformation turned out to be successful even for these extremely violently divergent perturbation expansions [132, 147, 148]. There can be no doubt that a better theoretical understanding of the summation of hyper-factorially divergent expansions would be highly desirable.

Our article most likely provides a definite answer to the problem of the delta summation of the Euler series and how the delta summation compares to Padé summation. But otherwise, we would be happy if it could serve as a starting point for more detailed theoretical investigations on the summation of divergent series with the help of Levin-type transformations.

We are fully aware that the topic of our article is mathematical in nature. However, mathematics is the language of physics, and divergent series have been and to some extent still are highly controversial. At the same time, (factorially) divergent series are indispensable in theoretical physics. We believe that our theoretical results provide strong evidence that the summation techniques we considered are not only toys for experimental mathematicians, but safe and reliable numerical tools. It is a welcome side effect that our results highlight the power and usefulness of Levin-type transformations which have not yet gained the recognition they deserve.

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Appendices

A Stieltjes Functions and Stieltjes Series

A function \( f: \mathbb{C} \to \mathbb{C} \) is called a Stieltjes function if it can be expressed as a Stieltjes integral:

\[
f(z) = \int_0^\infty \frac{d\Phi(t)}{1+zt}, \quad |\text{arg}(z)| < \pi.
\]  

(A.1)

Here \( \Phi(t) \) is a bounded, nondecreasing function taking infinitely many different values on the interval \( 0 \leq t < \infty \). Moreover, the moment integrals

\[
\mu_n = \int_0^\infty t^n \, d\Phi(t), \quad n \in \mathbb{N}_0,
\]

(A.2)
must be positive and finite for all finite values of \( n \).

Whether this series converges or diverges depends on the behavior of the Stieltjes moments \( \mu_n \) as \( n \to \infty \).

Because of their highly developed convergence theory in the case of Padé approximants (compare also Section 3.1), Stieltjes series are of considerable importance in the theory of divergent series. Detailed discussions of Stieltjes series can be found in books by Bender and Orszag [7, Chapter 8.6] and Baker and Graves-Morris [3, Chapter 5]. Other good sources with a stronger emphasis on mathematical aspects are a review article [149] and a book [150] by Widder.

Let \( \{u_n\}_{n=0}^\infty \) be a sequence. The Hankel determinants \( H_k(u_n) \) of this sequence are defined as follows (see for example [33, pp. 78 and 80]):

\[
H_0(u_n) = 1, \quad H_k(u_n) = u_{n+k} - u_{n+k+1} u_{n+1}, \quad n \in \mathbb{N}_0,
\]

(A.4a)

\[
H_k(u_n) = \begin{vmatrix}
  u_n & u_{n+1} & \cdots & u_{n+k-1} \\
  u_{n+1} & u_{n+2} & \cdots & u_{n+k} \\
  \vdots & \vdots & \ddots & \vdots \\
  u_{n+k-1} & u_{n+k} & \cdots & u_{n+2k-2}
\end{vmatrix}, \quad k \in \mathbb{N}, \quad n \in \mathbb{N}_0.
\]

(A.4b)

Hankel determinants play a very important role in the theory of Padé approximants. A necessary condition, that a power series of the type of (A.3) is indeed a Stieltjes series, is that the Hankel determinants \( H_k(\mu_n) \) of the Stieltes moments are positive for all \( k, n \geq 0 \) [3, Theorem 5.1.2].

Insertion of the partial sum \( \sum_{n=0}^k (-z)^n/n! \) of the geometric series into Eq. (A.4) shows that a Stieltjes function can be expressed as a partial sum of the Stieltjes series plus a truncation error term which is also a Stieltjes integral (see for example [126, Theorem 13-1]):

\[
f(z) = \sum_{\nu=0}^n (-1)^\nu \mu_\nu z^\nu + (-z)^{n+1} \int_0^\infty \frac{\nu+1}{1+zt} d\Phi(t), \quad |\text{arg}(z)| < \pi.
\]

(A.5)
Moreover, the truncation error term in Eq. (A.5) satisfies – depending upon the value of \( \arg(z) \) – the following inequalities (see for example [126, Theorem 13-2]):

\[
\left| (-z)^{n+1} \int_{0}^{\infty} \frac{e^{t} \Phi(t)}{1 + z} dt \right| \leq \begin{cases} 
\mu_{n+1} |z^{n+1}|, & |\arg(z)| \leq \pi/2, \\
\mu_{n+1} |z^{n+1} \csc(\arg(z))|, & \pi/2 < |\arg(z)| < \pi.
\end{cases}
\] (A.6)

B Basic Properties of Factorial Series

Let \( \Omega(z) \) be a function that vanishes as \( z \to +\infty \). A factorial series for \( \Omega(z) \) is an expansion of the following type:

\[
\Omega(z) = \frac{b_{0}}{z} + \frac{b_{1}1!}{z(z+1)} + \frac{b_{2}2!}{z(z+1)(z+2)} + \cdots = \sum_{\nu=0}^{\infty} \frac{b_{\nu}\nu!}{(z)_{\nu+1}}.
\] (B.1)

Here, \( (z)_{\nu+1} = \Gamma(z + \nu + 1)/\Gamma(z) = z(z+1)\ldots(z+\nu) \) is a Pochhammer symbol. In general, \( \Omega(z) \) will have simple poles at \( z = -m \) with \( m \in \mathbb{N}_0 \).

The definition of a factorial series according to Eq. (B.1) is typical of the mathematical literature. The separation of the series coefficients into a factorial \( n! \) and a reduced coefficient \( b_n \) offers some formal advantages.

Factorial series have a long tradition in mathematics. They were discussed already in Stirling’s book [118], which was first published in 1730. Recently, a new annotated translation of Sterling’s book was published by Tweddle [124], who remarked that Stirling was not the inventor of factorial series. Apparently, Stirling became aware of factorial series by the work of the French mathematician Nicole [124, p. 174]. Nevertheless, Stirling used factorial series extensively and thus did a lot to popularize them.

In the nineteenth and the early twentieth century, the theory of factorial series was fully developed by a variety of authors. Fairly complete surveys of the older literature as well as thorough treatments of their fundamental properties can be found in classic books on difference equations by Milne-Thomson [95], Nielsen [97], and Nörlund [98, 99, 100]. Factorial series are also discussed in the books by Knopp [86] and Nielsen [96] on infinite series. Additional and in particular more recent references can be found in [143].

In this Appendix, predominantly those properties of factorial series are discussed that are of importance for an understanding of the convergence properties of Levin-type transformations. It is extremely easy to apply higher powers of the finite difference operator \( \Delta \) to a factorial series. In fact, on using

\[
\Delta^{k} \frac{n!}{(z)_{n+1}} = \frac{(-1)^{k}(n+k)!}{(z)_{n+k+1}}, \quad k, n \in \mathbb{N}_0,
\] (B.2)

which can be proved by complete induction, the application of \( \Delta^{k} \) to a factorial series produces a very simple result:

\[
\Delta^{k} \Omega(z) = \sum_{\nu=0}^{\infty} \Delta^{k} \frac{b_{\nu}\nu!}{(z)_{\nu+1}} = \frac{(-1)^{k}}{\sum_{\nu=0}^{\infty} \frac{b_{\nu}(\nu+k)!}{(z)_{\nu+k+1}}}.\] (B.3)
This expression can be rewritten as follows:

\[
\Delta^k \Omega(z) = (-1)^k \sum_{\kappa=0}^{\infty} b_k^{(k)} (z)_{k+1},
\]

\[
b_k^{(k)} = \begin{cases} 
0, & \kappa < k, \\
b_{k-k}, & \kappa \geq k.
\end{cases}
\]

Thus, factorial series play a similar role in the theory of difference equations as power series in the theory of differential equations. This explains why factorial series are treated in classic books on finite differences [93, 98, 99, 100].

Alternative expressions for factorial series can be derived easily. For example, the beta function is defined by \( B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \) [101, Eq. (5.12.1)]. Accordingly, \( B(z, n + 1) = n!/(z)_{n+1} \) and the factorial series in Eq. (B.1) can also be viewed as an expansion in terms of beta functions (compare for instance [95, p. 288] or [103, p. 175]):

\[
\Omega(z) = \sum_{n=0}^{\infty} b_n B(z, n + 1).
\]

The beta function possesses the integral representation [101, Eq. (5.12.1)]

\[
B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} \, dt, \quad \text{Re}(x), \text{Re}(y) > 0,
\]

which implies

\[
B(z, n + 1) = \frac{n!}{(z)_{n+1}} = \int_0^1 t^{z-1} (1-t)^n \, dt, \quad \text{Re}(z) > 0 \quad n \in \mathbb{N}_0.
\]

Inserting this into (B.5) yields the following integral representation [97, Satz I on p. 244],

\[
\Omega(z) = \int_0^1 t^{z-1} \varphi(t) \, dt, \quad \text{Re}(z) > 0,
\]

\[
\varphi(t) = \sum_{n=0}^{\infty} b_n (1-t)^n.
\]

Frequently, the properties of \( \Omega(z) \) can be studied more easily via this integral representation than via the defining factorial series (B.1) (see for example [97, Chapter XVII]).

The integral representation (B.8) can also be used for an alternative derivation of (B.3). For that purpose, we use

\[
\Delta \tilde{r} = \tilde{r}^{t+1} - \tilde{r} = (t-1) \tilde{r},
\]

or equivalently

\[
\Delta^k \tilde{r} = (-1)^k (1-t)^{k} \tilde{r}.
\]

Inserting this into the integral representation (B.8) yields with the help of (B.7):

\[
\Delta^k \Omega(z) = (-1)^k \sum_{n=0}^{\infty} b_n \int_0^1 t^{z-1} (1-t)^{k+n} \, dt
\]

\[
= \sum_{n=0}^{\infty} b_n (k+n)! (z)_{k+n+1}.
\]
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