Compact shell solitons in $K$ field theories

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Abstract

Some models providing shell-shaped static solutions with compact support (compactons) in 3+1 and 4+1 dimensions are introduced, and the corresponding exact solutions are calculated analytically. These solutions turn out to be topological solitons, and may be classified as maps $S^3 \to S^3$ and suspended Hopf maps, respectively. The Lagrangian of these models is given by a scalar field with a non-standard kinetic term ($K$ field) coupled to a pure Skyrme term restricted to $S^2$, raised to the appropriate power to avoid the Derrick scaling argument. Further, the existence of infinitely many exact shell solitons is explained using the generalized integrability approach. Finally, similar models allowing for non-topological compactons of the ball type in 3+1 dimensions are briefly discussed.

Key words: Compact solitons, Hopf maps

1. Introduction

Recently, some effort has been invested into the investigation of compactons, that is, soliton solutions of non-linear field theories with compact support. By now, two established classes of scalar field theories are known which give rise to the existence of compacton solutions. One may either choose potentials in the Lagrangian (or energy density) which have a non-continuous first derivative at (some of) their minima [1]-[9], or one may employ a non-standard kinetic term ($K$ field theory) [10], [11]. Concretely, the kinetic term has to contain higher than second powers in the first derivatives of the fields. For non-relativistic field theories, compactons were first discovered and studied for some generalizations of the KdV equation in [12], [13]. Compactons can be realized directly in some mechanical systems [1], and, in addition, they have been applied recently to brane cosmology [14], [15], [16]. Most of these investigations have dealt with topological compactons, where the existence and stability of the compacton solutions is related to a nontrivial vacuum manifold and some nonzero topological charge. Also in this letter we shall mainly deal with this case of topological compactons, where the existence and stability of the compacton solutions is related to a nontrivial vacuum manifold and some nonzero topological charge. Also in this letter we shall mainly deal with this case of topological compactons, where the existence and stability of the compacton solutions is related to a nontrivial vacuum manifold and some nonzero topological charge.

As is typical in soliton theory in general, it is easier to find systems with compacton solutions in low (that is 1+1) dimensions. The simplest, most obvious generalization of topological compacton systems with a non-standard kinetic term to higher dimen-
sions meets the same obstacles as in the case of conventional solitons, and also the remedy to circumvent the obstacle is the same, namely the introduction of a gauge field in addition to the scalar fields, like in the case of vortices and monopoles [17]. In this letter, we shall pursue a different path for the construction of higher-dimensional topological compactons. Concretely, we couple a real scalar field $\xi$ with non-standard kinetic term to a complex scalar field $u$ where $u$ maps from a compact submanifold of the base space ($S^2$ or $S^3$ for $\mathbb{R}^3$ or $\mathbb{R}^4$, respectively) to an $S^2$ target space, thereby providing an important contribution to the non-trivial topology of the compacton solutions. The non-trivial topology is completed by the boundary conditions on $\xi$, where $\xi$ describes a map $\Sigma$ with a non-standard kinetic term coupled to the complex scalar field $\bar{u}$.

In order to derive static solutions we introduce the coordinates

$$
\bar{X}_{\mu} = \begin{pmatrix} r \sqrt{2} \cos \phi_2 \\ r \sqrt{2} \sin \phi_2 \\ r \sqrt{1 - z \cos \phi_1} \\ r \sqrt{1 - z \sin \phi_1} \end{pmatrix},
$$

where $z \in [0, 1], \phi_1 \in [0, 2\pi], \phi_2 \in [0, 2\pi]$ are coordinates on $S^3$, and $r \in \mathbb{R}_+$ gives the extension to $\mathbb{R}^4$. Moreover, we assume the ansatz

$$
u = f(z)e^{i(n_1 \phi_1 + n_2 \phi_2)}
$$

and

$$\xi = \xi(r).
$$

This ansatz provides that

$$\nabla \xi \nabla \nu = 0
$$

and, as a consequence, one can remove the coupling function $\sigma$ from equation (5). Equation (5) for the field $\xi$ simplifies, in fact, to the field equation for the Lagrangian $H_{\mu \nu}$ with base space $S^3$. Solutions to this model have been constructed in [20], [21], and below we just review the results which we need in the sequel.

Concretely, the static equations of motion may be rewritten in the form

$$\frac{1}{r^3} \partial_r \left( r^3 \xi^2 \right) +
$$

where $\xi$ is a complex scalar providing via the stereographic projection a parametrisation of $S^2$. Here, minimal means that only one of the degrees of freedom is of the $K$ type. For different, non-minimal generalizations see e.g. [18], [19]. Observe that the above Lagrangian can also be derived by a reduction of a scalar $K$ field coupled to the standard $SU(2)$ non-abelian gauge field i.e., of $K$-dilaton Yang–Mills theory. The coupling between both fields is of the non-minimal type and governed by the coupling function $\sigma$ which is chosen in the form

$$\sigma(\xi) = \lambda(1 - \xi^2)^2.
$$

Notice that the coupling function is semipositive definite and vanishes for two values of the scalar field $\xi_{1,2} = \pm 1$. Therefore, it plays the role of an effective potential with two effective vacua for $\xi$. The constant $\lambda$ is a free parameter of the model.

The pertinent field equations read

$$\partial_\mu (|\xi_\nu \xi^\nu| \xi^\mu) - \lambda H_{\mu \nu}^2 \xi (1 - \xi^2) = 0,
$$

$$\partial_\mu \left( \frac{\sigma}{(1 + |\xi|^2)^2} K^\nu \right) = 0
$$

where

$$K_{\mu \nu} = (u_\nu \bar{u}^\nu) u_\mu - u_\nu \bar{u}_\mu.
$$

In order to derive static solutions we introduce the coordinates.

2. 4-dimensional compactons

The specific 4 + 1 dimensional model we are going to consider is given by the following expression

$$L = |\xi_\nu \xi^\nu| \xi_\mu \xi^\mu - \sigma(\xi) H^2_{\mu \nu},
$$

where $\xi_\mu \equiv \partial_\mu \xi$ etc. It describes a real scalar field $\xi$ with a non-standard kinetic term coupled to the pure Skyrme term constrained to $S^2$ target space

$$H^2_{\mu \nu} = \frac{1}{(1 + |\nu|^2)^4} \left( (u_\mu \bar{u}^\mu)^2 - u_\mu \bar{u}_\nu (u_\nu \bar{u}^\nu) \right).
$$

Here $u$ is a complex scalar providing via the stereographic projection a parametrisation of $S^2$. This Lagrangian provides a simple, minimal generalization of the concept of $K$ fields to a model with $S^3$ target space. Here, minimal means that only one of the degrees of freedom is of the $K$ type. For different, non-minimal generalizations see e.g. [18], [19]. Observe that the above Lagrangian can also be derived by a reduction of a scalar $K$ field coupled to the standard $SU(2)$ non-abelian gauge field i.e., of...
\[ + \frac{16\lambda f_x^2 f^2}{r^4(1 + f^2)^2}(n_1^2 z + n_2^2(1 - z))\xi(1 - \xi^2) = 0 \]  
(12)

\[ \partial_z \left( (n_1^2 z + n_2^2(1 - z)) \frac{f_x^2 f}{(1 + f^2)^2} \right) - \]

\[ -(n_1^2 z + n_2^2(1 - z)) \frac{f x^2}{(1 + f^2)^2} = 0. \]
(14)

The last expression may be simplified

\[ \partial_z \ln \left( (n_1^2 z + n_2^2(1 - z)) \frac{f x^2}{(1 + f^2)^2} \right) = 0. \]
(15)

Thus,

\[ \frac{f x^2}{(1 + f^2)^2} = \frac{c_1}{(n_1^2 z + n_2^2(1 - z))}, \]

where \(c_1\) is an integration constant. One can proceed further and solve this equation. However, for the topologically nontrivial configurations the complex field \(u\) should cover the whole target space \(S^2\) at least once. This requirement gives a condition for the integration constants leading to the solutions

\[ f = \sqrt{\frac{\ln n_1^2 - \ln n_2^2}{\ln(n_1^2 z + n_2^2(1 - z)) - \ln n_2^2}} - 1. \]
(17)

In the case when \(n_1 = \pm n_2\) we arrive at the very simply formula

\[ f = \sqrt{\frac{1}{z}} - 1. \]
(18)

Moreover, such a complex field being a map from \(S^3\) (the base space parameterized by \(z, \phi_1, \phi_2\) coordinates) to the target \(S^2\) can be classified by a topological invariant known as the Hopf index. In fact, solution (8), with (17), (18) is known to carry a non-vanishing Hopf index

\[ Q = n_1 n_2. \]
(19)

Let us now turn to the field equation for the real scalar \(\xi\). First of all one can observe that this expression leads to an ordinary differential equation for \(\xi = \xi(r)\) only if \(n_1^2 = n_2^2 = n^2\) in the solution for the \(u\). Therefore, only the solution (18) is admissible. Then, the \(z\)-dependence in the second term of (12) cancels and we get

\[ \frac{1}{r^3} \partial_r \left( r^3 \xi^3 \right) + \frac{4\lambda n^2}{r^4} \xi(1 - \xi^2) = 0. \]
(20)

Introducing the new variable \(x = \ln r\) we find that

\[ \xi^2 \xi_{xx} + \frac{4\lambda n^2}{3} \xi(1 - \xi^2) = 0. \]
(21)

This equation has been recently analyzed in the context of compact domain walls [10]. The corresponding compacton solution located at \(x_0\) reads

\[ \xi(x) = \begin{cases} 
-1 & \alpha x \leq \alpha x_0 - \frac{\pi}{2} \\
\sin (\alpha (x - x_0)) & \alpha x \in [\alpha x_0 - \frac{\pi}{2}, \alpha x_0 + \frac{\pi}{2}] \\
1 & \alpha x \geq \alpha x_0 + \frac{\pi}{2} 
\end{cases} \],

(22)

where

\[ \alpha = \left( \frac{4\lambda n^2}{3} \right)^{1/4}. \]
(23)

Finally, the 4 dimensional compacton solution is

\[ u(z, \phi_1, \phi_2) = \sqrt{\frac{1}{z}} - 1 \exp(i(\phi_1 + \phi_2)), \]
(25)

with the Hopf index of the underlying Hopf maps equal to \(n^2\).

The size of the compact soliton, if treated as an object living in the original 4 dimensional space, varies as one changes its position. The inner and outer compacton boundary points \((r_1, r_2)\) are

\[ r_1 = r_0 \exp \left( \frac{\pi}{2\alpha} \right), \quad r_2 = r_0 \exp \left( \frac{-\pi}{2\alpha} \right), \]
(26)

where \(x_0 = \ln r_0\) gives a parametrisation of the center of the solution. Thus the shell radius is

\[ R = r_2 - r_1 = 2r_0 \sinh \frac{\pi}{2\alpha}. \]
(27)

As we see, the compacton is getting narrower as it approaches the origin. On the other hand its radius grows while it moves in the opposite direction.

The energy of the solution is given as follows

\[ E = \int dV \left( \xi^4 + (1 - \xi^2)^2 \frac{16\lambda n^2 f_x^2 f^2}{r^4(1 + f^2)^2} \right). \]
(28)

Thus,

\[ E = \frac{(2\pi)^2}{2} \int_0^\infty dr \ r^3 \left( \xi^4 + \frac{4\lambda n^2}{r^4} (1 - \xi^2)^2 \right) \]

\[ = \frac{(2\pi)^2}{2} \int_{-\infty}^\infty dx \left( \xi^4 + 4\lambda n^2 (1 - \xi^2)^2 \right). \]
(30)
It is clearly visible that the compact solution for the real scalar field is of the Bogomolny type, satisfying a first order differential equation, which may be easily derived from (30) using the standard Bogomolny trick.

Specifically, for the one-compacton configuration we find

$$E = 3 \left( \frac{3}{4} \right)^{3/4} \pi^2 \lambda^{3/4} Q^{3/4}. \quad (31)$$

Interestingly, the energy depends on a non-integer power of the Hopf charge of the underlying \( u \) field, like in the Vakulenko-Kapitansky formula [22]. It should be mentioned, however, that this relation between energy and Hopf charge does not provide an energy bound for general Hopf charge, because the full field configuration is a suspended Hopf map, and its topological classification is therefore given by the homotopy group \( \pi_4(S^3) \simeq \mathbb{Z}_2 \), as we demonstrate below.

So let us prove that the obtained configuration may be understood as a suspended Hopf map, i.e., a map from the \( S^3 \) base space onto the \( S^4 \) target space characterized by the nontrivial homotopy class \( \pi_4(S^3) \). It is convenient to combine the fields \( (\xi, u) \) into a \( SU(2) \) matrix \( U \)

$$U = \sin \frac{\pi}{2} \xi I + i \cos \frac{\pi}{2} \xi T; \quad (32)$$

where

$$T = \frac{1}{1 + |u|^2} \begin{pmatrix} |u|^2 - 1 & -2iu \\ 2i\bar{u} & 1 - |u|^2 \end{pmatrix} \quad (33)$$

and \( I \) is the unit matrix. Thus, the \( U \) field maps \( \mathbb{R}^4 \) onto the three dimensional target sphere. For every fixed value of \( \xi \neq \pm 1 \) the \( U \) field is just a Hopf map \( S^3 \to S^2 \) with the previously found nonvanishing topological charge. For \( \xi = \pm 1 \), representing the poles of \( S^3 \), we get the identity map. Therefore we get a full covering of the \( S^3 \). The boundary condition, \( U \to I \) as \( r \to \infty \), allows for compactification of the original \( \mathbb{R}^4 \) space to \( S^4 \). These facts render the \( U \) a representative of the nontrivial homotopy class [23]. We remark that topological solitons which may be classified as suspended Hopf maps (although not of the compacton type) have been studied recently, e.g., in [24] and in [25].

Interestingly, the compactons we found do not have the structure of a ball. Instead, they have the form of a shell, where the energy density is radially symmetric, and is zero both inside the inner compacton boundary and outside the outer boundary. Further, the one compacton solution may be easily extended to multi-compacton configurations by taking an alternating collection of sufficiently separated compactons (which interpolate from the vacuum value \( \xi = -1 \) to \( \xi = 1 \) with increasing radius) and anti-compactons (which interpolate from \( \xi = 1 \) to \( \xi = -1 \) with increasing radius), forming an onion-like structure with one compacton or anti-compacton as the innermost shell, surrounded by further compacton and anti-compacton shells. The energy of the solution equals just the sum of the energies of all \( N \) compact solitons. The corresponding topological charge is nontrivial if the number of compactons is not the same as the number of anticomactpons, whereas it is zero if the number of compactons and anti-compactons is equal. We remark that the Hopf charge of the \( u \) field within each (anti-)compacton may be chosen independently.

Let us also notice that the simplest compact Hopf map is stable as far as linear radial perturbations are considered. In this case the stability analysis of [10], [14], [15] holds.

The existence of infinitely many exact suspended hopfion solutions in our model makes it interesting to further investigate its integrability properties. First we observe that the model has infinitely many symmetries and, therefore, infinitely many conserved currents. The symmetries are just the area-preserving diffeomorphisms acting on the target space \( S^2 \) spanned by the complex scalar field \( u \), and the conserved currents are the corresponding Noether currents. One way to further analyse the integrability is provided by the generalized zero curvature condition of Refs. [26], [27], which gives a well-defined extension of the standard integrability criterion (Zakharov-Shabat zero curvature representation) to higher than two dimensions. The corresponding generalized zero curvature condition is the condition for the holonomy in higher loop space to be independent of the deformations of loops or, in other words, it is just a condition for the flatness of the connection in loop space. Moreover, assuming the reparametrization invariance of the holonomy, one gets local generalized zero curvature conditions,

$$F_{\mu\nu}(A) = 0, \quad D_{\mu}B^\mu = 0, \quad (34)$$

i.e., flatness of a connection \( A_\mu \in \mathcal{G} \) and covariant constancy of a vector field \( B_\mu \in \mathcal{P} \), where \( \mathcal{G} \) is a Lie algebra and \( \mathcal{P} \) an abelian ideal (a representation space of the Lie algebra). A model is said to be integrable if one can rewrite the field equations as
the generalized zero curvature conditions (34) and
if the abelian ideal used in the construction has in-
finite dimensions.
One can verify that the model (1) admits such a
generalized zero curvature formulation provided we
impose an additional constraint on the fields. There-
fore our model, although not integrable in this sense,
possesses an integrable sector defined by the follow-
ing integrability condition
\[ u_\mu \xi^\mu = 0. \] (35)
In particular, the generalized zero curvature for-
mulation of the submodel is given by
\[
A_\mu = \frac{1}{1 + |u|^2} (-i u_\mu T_+ - i \bar{u}_\mu T_- + (u \bar{u}_\mu - \bar{u} u_\mu) T_3) \] (36)
\[
B_\mu = 2i |\xi_\nu \xi^\nu| \xi_\mu \sqrt{j(j+1)} P_0^{(j)} + \sigma_{\xi}^j \left( \bar{K}_\mu P_1^{(j)} + K_\mu P_3^{(j)} \right), \] (37)
where \( T_\pm, T_3 \) are the generators of the \( sl(2) \) Lie al-
gebra and \( P_\nu^{(j)} \) transforms under the spin-\( j \) re-
presentation of \( sl(2) \). The equations of motion for the
submodel are given by GZC in any spin representa-
tion, which implies the generalized integrability. The
importance of this submodel emerges from the fact
that our hopfions belong to it. Indeed, they solve the
field equation together with the constraint.
One can compare the integrability properties of our
model with the most known example of \( S^3 \) target
space model i.e., the Skyrme model. The Skyrme
model does not have infinitely many symmetries,
but it also has an integrable sector. However, in this
case it is given by two integrability conditions
\[ u_\mu \xi^\mu = 0, \quad u_\mu^2 = 0. \] (39)
Thus, the integrable submodel is much more con-
strained, and this fact obviously affects the chance
for the existence of exact solitons. In particular, the
new condition, that is, the eikonal equation, is rather
restrictive. It leads to one exact, given solution once
the nontrivial topology (the Hopf charge) for \( u \)
fixed (more precisely, it leads to one fixed solution
up to Moebius transformations, that is, rotations of
the target space \( S^2 \) where \( u \) takes its values). This is
in contrast to the first integrability condition, which
is easily obeyed by a general separation of variables
ansatz. This can explain why our model as well as
its extention presented in the next section possess
infinitely many exact solutions based on infinitely
many Hopf maps (or infinitely many maps \( S^2 \to S^2 \)
in the next section).
The relation between the restrictions imposed by
the integrability conditions and existence of exact
solutions is also confirmed by recent investiga-
tions of the pure quartic (Euclidean) Skyrme model by
Speight [22]. In fact, in this model, which from the
point of view of the generalized integrability is ex-
actly the same as the Skyrme model, only one exact
soliton with topology of the suspended hopfion has
been found. It is not of the compacton type.

3. Some 3-dimensional compactons

Here we want to study briefly some models which
give rise to compacton solutions analogous to the
case discussed in the preceding section, but in 3+1
dimensions, which is the case more directly relevant
for physical applications. We shall discuss explicitly
two cases.

3.1. Shell-shaped compactons

In the first example, we observe that the La-
grangian (1) of Section 2 is quartic in first deriva-
tives, therefore it is scale invariant precisely in four
dimensions, which is one way to circumvent Der-
rick’s theorem and have static solutions in four
dimensions. If we want to have static solutions in
three dimensions, one possibility consists, therefore,
in choosing a Lagrangian cubic in first derivatives.
This implies, however, that the resulting Lagrangian
is non-polynomial. For models of this type the study
of time-dependent dynamics is problematic (e.g.
boundedness of the energy, or global hyperbolicity),
therefore we shall introduce the energy functional
for static configurations directly. Concretely, the
three-dimensional model we study has the following
energy functional for static configurations
\[
E = \int d^3x \left[ (\xi_k \xi_k)^{\frac{2}{j}} - (\sigma(\xi) H_{jk}^2)^{\frac{4}{j}} \right] \] (40)
where \( j, k = 1, 2, 3 \). It is related to the model (1)
of Section 2 such that both terms of the model
(1) are taken to the power \( \frac{2}{j} \). Further, we have
already reduced to the static case. If we now intro-
duce three-dimensional spherical polar coordinates
\((x_1, x_2, x_3) \to (r, \theta, \varphi)\) and use the ansatz \( \xi = \xi(r), u = u(\theta, \varphi) \), then the coupling function can again be
removed from the equation for \( u \), and this equation
can be written as
\[ \partial_j \left( \frac{K_{ij}(u \bar{u}_i)^2 - u_i^2 u_j^2}{1 + u \bar{u}} \right) = 0. \]  

(41)

This equation is just the field equation of the model of Aratyn, Ferreira and Zimmermann (AFZ)\(^1\). For the ansatz \( u(\theta, \varphi) \) it has the solutions

\[ u = \tan \frac{\theta}{2} e^{i \nu \varphi} \]

(42)

where \( n \) is an integer and these solutions \( u \) describe maps \( S^2 \rightarrow S^2 \) with winding number \( n \). The corresponding \( H^2_{jk} \) reads

\[ H^2_{jk} = \frac{n^2}{4r^4}. \]

(43)

The equation for \( \xi(r) \) for this ansatz is

\[ \frac{1}{r^2} \partial_r (r^2 \xi^2) + \frac{3}{4} (H^2_{jk})^{3/2} \sigma \xi^2 = 0 \]

(44)

or, for the specific coupling function \( \sigma = \lambda(1 - \xi^2)^2 \) and the \( H^2_{jk} \) above,

\[ \frac{1}{r^2} \partial_r (r^2 \xi^2) + \left( \frac{\lambda n^2}{4} \right) \frac{1}{r^2} (1 - \xi^2)^{3/2} = 0. \]

(45)

Introducing again the variable \( x = \ln r \), this equation becomes

\[ \xi_x \xi_{xx} + 2^{-2} \left( \frac{\lambda n^2}{4} \right) (1 - \xi^2)^{3/2} = 0 \]

(46)

which has exactly the same compacton solution (22) as in Section 2, where now the constant \( \alpha \) is

\[ \alpha = 2^{-3/2} \left( \frac{\lambda n^2}{4} \right)^{3/2} \]

(47)

Therefore, this model has exactly the same shell-like spherically symmetric compacton solutions in three dimensions as the previous model of section 2 has in four dimensions.

The topology of the compacton solutions is now given by maps \( S^3 \rightarrow S^3 \). This topology may again be described by the SU(2) matrix \( U \) of Eq. (32), where, however, now the complex scalar \( u \) is a map \( S^2 \rightarrow S^2 \). Therefore, the solution field configuration takes its value at the north pole of the target space \( S^3 \) for \( \xi = -1 \) (inside the inner shell boundary), covers the full target \( S^3 \) while \( \xi \) varies from \( \xi = -1 \) to \( \xi = 1 \), and takes its value at the south pole for \( \xi = 1 \) (outside the outer shell boundary).

Also the integrability properties are exactly equivalent to the ones of the model of Section 2. The present model has infinitely many symmetries and infinitely many conserved currents, like the one of Section 2. Further, the generalized zero curvature representation does not exist for the full model, but only for a submodel defined by the additional condition (35), again in complete analogy with the model of Section 2.

3.2. Ball-shaped compactons

Another simple modification which allows for compactons in three spatial dimensions is given by the Lagrangian

\[ L = |\xi_\nu \xi^\nu| \xi_\mu \xi^\mu - \sigma(\xi) \bar{H}. \]

(48)

Here, the term

\[ \bar{H} = \frac{u_\mu \bar{u}^\mu}{(1 + u \bar{u})^2} \]

is just the Lagrangian of the \( \text{CP}^1 \) model. The above Lagrangian contains one quartic term and one quadratic term in first derivatives, and so may have finite energy solutions in three dimensions, according to the Derrick scaling argument. It is, however, not scale invariant nor does it have infinitely many symmetries, in contrast to the models studied above. Therefore, we do not expect to find fully analytical solutions in this case, see below.

We again use the ansatz \( \xi = \xi(r) \), \( u = u(\theta, \varphi) \) in spherical polar coordinates in three space dimensions. With this ansatz, the coupling function \( \sigma = \lambda(1 - \xi^2)^2 \) may again be eliminated from the field equation for \( u \), and this equation is, therefore, just the field equation of the \( \text{CP}^1 \) model on base space \( S^2 \). The simplest solution of this equation is

\[ u = \tan \frac{\theta}{2} e^{i \nu \varphi}. \]

(50)

The \( \text{CP}^1 \) energy density of this solution is

\[ - \bar{H} = \frac{\nabla u \cdot \nabla \bar{u}}{(1 + u \bar{u})^2} = \frac{1}{2r^2}. \]

(51)

There exist many more solutions of the \( \text{CP}^1 \) model like, e.g., higher powers of the simplest solution, but

\(^1\) The energy density \( (H^2_{jk})^{3/2} \) is, in fact, precisely the energy density of the AFZ model. In three-dimensional, Euclidean base space, the AFZ model has infinitely many soliton solutions of the knot type [28], [29], whose existence is related both to the conformal base space symmetry and to the infinitely many target space symmetries of this model. Here we are, however, interested in solutions on the base space \( S^2 \).
the corresponding energy densities are no longer independent of the angular coordinates. These higher solutions are, therefore, not compatible with our separation ansatz \( \xi = \xi(r) \), and we have to restrict to the simplest solution (50) in what follows. For this simplest CP\(^1\) solution, we find the following Euler–Lagrange equation for \( \xi(r) \)

\[
\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \xi^3 \right) + \frac{\lambda}{2r^2} \xi(1 - \xi^2) = 0
\]  

(52)

or, after the variable transformation \( s = r^\frac{1}{2} \)

\[
\frac{1}{27} \xi_s s^2 + \frac{\lambda}{2} s^2 \xi(1 - \xi^2) = 0.
\]  

(53)

This equation differs from the previous ones by the explicit presence of the factor \( s^2 \) (the independent variable) in the second term, that is, it is no longer an autonomous equation. As expected, we were not able to find analytic solutions to this equation, so we will resort to a qualitative analysis and to a numerical study in the sequel. Further, we shall find that its solutions are no longer topological and may, therefore, have arbitrarily small energies. Concretely, a numerical integration of Eq. (53) leads to the following results:

- There do not exist shell-type solutions. If one starts the integration at an inner boundary \( s_0 > 0 \) with \( \xi(s_0) \) taking one vacuum value (e.g., \( \xi(s_0) = -1 \)), and \( \xi'(s_0) = 0 \), then the integration into the direction \( s > s_0 \) never reaches the other vacuum value \( \xi = +1 \). Instead, a point \( s_1 > s_0 \) is reached where \( \xi'(s_1) = 0 \) and \( -1 < \xi(s_1) < 1 \), and at this point \( \xi(s) \) becomes singular (it is obvious from Eq. (53) that at a point where \( \xi^2 = 0 \), either \( \xi \) must take one of its vacuum values, or \( \xi'' \) becomes singular).

- There exist, however, solutions of the ball type. If one starts the numerical integration at an outer compacton boundary (e.g., with \( \xi(s_1) = +1 \) and \( \xi'(s_1) = 0 \)) and integrates towards \( s < s_1 \), then the integration will simply hit the point \( s = 0 \). In order to see that the resulting solution is an acceptable compacton, it is more useful to reverse the integration and to start at \( s = 0 \).

- Let us assume that we start the integration at \( s = 0 \) with some value \( 0 < \xi(0) < 1 \) and with \( \xi'(0) = k > 0 \). First, we observe that due to the suppression factor \( s^2 \) in the second term of Eq. (53), \( \xi(0) \) and \( \xi'(0) \) may take arbitrary values without making \( \xi''(0) \) singular (concretely, if \( \xi''(0) > 0 \) then \( \xi''(0) = 0 \)). For \( s > 0 \), we note that for \( 0 < \xi < 1 \) it holds that \( \xi'' < 0 \), whereas for \( \xi > 1 \) it holds that \( \xi'' > 0 \), as follows easily from Eq. (53).

- Therefore, with the initial conditions given above, the following picture emerges for an integration starting at \( s = 0 \). If \( k = \xi'(0) > 0 \) is too large, then the integration curve for \( \xi(s) \) will cross the line \( \xi = 1 \) and then grow forever, producing a formal solution with infinite energy. If \( k \) is too small, the integration curve will reach a point \( s_2 \) where \( \xi'(s_2) = 0 \) but still \( \xi(s_2) < 1 \). At this point the integration curve becomes singular, because \( \xi''(s_2) \) is singular. It follows that there exists a fine tuned value \( k_* \) for the integration constant \( k > 0 \) such that the integration curve touches the line \( \xi = 1 \) instead of crossing it, that is, it reaches the value \( \xi'(s_1) = 0 \) precisely at the point \( s_1 \) where \( \xi(s_1) = 1 \). This configuration is the compacton. The above qualitative discussion is completely confirmed by an explicit numerical integration.

In the above argument, we could start the integration at \( s = 0 \) for an arbitrary value \( 0 < \xi(0) < 1 \). By choosing a \( \xi(0) \) arbitrarily close to the value \( +1 \) we can, therefore, make the size and the energy of the compacton arbitrarily small. These compactons are, therefore, no longer topological. This makes their stability under time-dependent perturbations more problematic (a detailed stability analysis is beyond the scope of the present letter).

Remark: We chose the coupling function \( \sigma = \lambda(1 - \xi^2)^2 \) as the simplest representative of a class of coupling functions with (at least) two vacuum values in order to allow for topological compactons. In the last example, however, the compactons are not topological in any case, therefore the presence of more than one vacuum in the coupling function is not necessary in this case.

Remark: In the above discussion about the integration from the center \( s = 0 \) we restricted to the interval \( 0 < \xi(0) < 1 \) just for reasons of simplicity. It presents no difficulty to extend the discussion and to cover cases where \( \xi \) starts outside this interval at \( s = 0 \). For an adequately fine-tuned value of \( k = \xi'(0) \) there always exists a compacton. We plot two typical compact ball solutions in Figures 1 and 2. Here in Figure 1 \( \xi \) starts at \( \xi(0) = 0.465 \) whereas in Figure 2 it starts at \( \xi(0) = -0.833 \). The qualitative discussion above is completely confirmed by the numerical solutions shown in Figures 1 and 2.
4. Discussion

In this paper, we found analytic solutions for compact solitons of the shell type in 3+1 and 4+1-dimensional $K$ field theories. To our knowledge, these are the first examples of compactons in $K$ field theories in those dimensions. The topology of these shell type compactons is quite interesting, being a combination of the topology of the complex field $u$ which lives on a compact base space ($S^2$ or $S^3$, respectively) and is classified by the homotopy of maps from that base space to the target space $S^2$, and of the topology of the real field $\xi$ which lives on the positive real line (radial coordinate), and where the topology is induced by the boundary conditions implied by the finiteness of the energy. This situation is reminiscent of the Skyrme model, which has exactly the same field contents and the same topology. The main difference is the quartic kinetic term of the real scalar field $\xi$ in the models discussed in the present paper. Further, we were able to find an infinite number of exact compacton solutions in both cases, labelled by an integer $n$. The existence of these infinitely many analytical solutions is probably related to the infinitely many symmetries and infinitely many conserved currents, which are present in both models, as typically happens in models with infinitely many conservation laws. We also investigated the integrability properties of these two models and found that, although they have infinitely many conservation laws, they do not possess a generalized zero curvature representation, but a submodel defined by a further constraint equation does. In this sense, these models seem to be halfway between integrable and non-integrable. Finally we investigated a model without infinitely many symmetries, and without the scale symmetry of the models discussed above. This model does not allow for analytical solutions, but it is not difficult to find solutions numerically. It turns out that this model does not have solutions of the shell type. It does, however, possess compact solutions of the ball type, but these solutions are no longer topological solitons. The real scalar field $\xi$ may, in fact, take values on arbitrarily small intervals $[\xi_0, 1]$ of the real axis (here $\xi_0$ is the starting value at zero radius, and 1 is the vacuum value), so the full target space may be an arbitrary segment of $S^3$. Consequently, there exist compact balls with arbitrary size and energy. As stated, it has been the main purpose of the present paper to establish the existence of compactons in $K$ field theories in higher dimensions, and to study their properties. In the case of the shells the underlying theories are scale invariant, and the dynamical generation of a scale (the compacton size) is by itself an important result in relativistic field theory. Nevertheless, we want to discuss briefly some possible applications and further directions of investigation. As already mentioned, compactons in low (i.e. 1+1) dimensions have been applied to brane cosmology, where the compacton acts as a domain wall such that the confinement of both gravity and field fluctuations to the domain wall (“thick brane”) is an automatic consequence of the field equations. Recently, compact solutions of the $Q$-ball type and of the $Q$-shell type have been found in certain higher (i.e. 3+1 dimensional) gauge theories with normal kinetic terms and V-shaped potential (for the scalar field), [30]. Further, in Ref. [31] the coupling of these compact $Q$ balls and $Q$ shells to gravity has been investigated in detail by the use of numerical methods. It was found there that the $Q$ balls may form stable self-gravitating solutions, whereas the $Q$ shells may not only form self-gravitating $Q$ shells but may also dress black holes (i.e. may contain a Schwarzschild interior). This whole analysis is complicated by the fact that the basic solutions of Ref. [30] are (time dependent) $Q$ balls and $Q$ shells rather than static solitons (e.g. the asymptotic metric is Kerr rather than Schwarzschild). Therefore, the coupling of the solutions of the present paper (or of related solutions of comparable simplicity) to gravity should lead to a simpler analysis of the problem of gravitational self-coupling and probably even allow for mainly analytical calculations. This question is under current investigation.

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Fig. 1. The compact ball of Section 3.2. The starting values for the field \( \xi \) are \( \xi(0) = 0.465, \xi'(0) = 0.912 \), and the resulting compacton radius is \( s_1 = 0.8 \). We display both the compacton field \( \xi \), starting at 0.465 and ending at the vacuum value 1, and the first derivative \( \xi' \).

Fig. 2. The compact ball of Section 3.2. The starting values for the field \( \xi \) are \( \xi(0) = -0.833, \xi'(0) = 1.709 \), and the resulting compacton radius is \( s_1 = 1.3 \). We display both the compacton field \( \xi \), starting at -0.833 and ending at the vacuum value 1, and the first derivative \( \xi' \).