Use of kinematic restrictions in case of parallel spur gearing design

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Abstract. Since 1984 the design of the involute parallel spur gear is normalized by the Romanian standard STAS 12268-84 (Gearing cylinders with evolving teeth). This method of calculation has some drawback and some of its items may even confuse the designer. The present paper analyses the standard design method attempting to establish a set of kinematic limits of the gearing, in order to eliminate the successive modifications of calculated values of some parameters initially determined by using the standard.

1. Introduction
For design of involutes spur gears based on the ISO recommendation and the international notation, the standardized method (STAS 12268-84) contains calculation steps, but the basic criteria used are not clearly formulated.

For beginner designers (as students) or for experienced engineers but who have forgot the basic theory, this type of design can cause confusion. They find that the calculation overlaps partially over the theoretical facts, but cannot explain why it is necessary to modify some of the results of the calculations [8], [9].

The steps of spur gear design according to the standardized method are the following, [3]:

- Centre distance determination (a) – some factors are calculated and approximated after the material selection; sometimes the value of centre distance is normalized and permits to determine pitch circle diameters of the gears (wheels)
- Module determination (mₙₚ) – the minimum value its normalization, this value is changed afterwards, without any explanation
- Number of teeth (zₐ and zₜ) – determination depending of centre distance (a), module (mₙₚ) and gear ratio (u = zₜ/zₐ); the decrease of the pinion teeth number (zₐ) (resulting in an increase of module), does not justify this action. “The choice” of teeth number is possible from indicated sets of values, depending of heat treatment of gear materials
- Recalculation the normal module, depending of the new adopted values of zₐ
- Addendum modification coefficient determination and its distribution on the two wheels
- Geometric an kinematic calculation of involute parallel gear pair
• Since the number of teeth initially calculated \(z_1\) is standardized by decreasing, the module has a higher value than the calculated value and it is necessary to recalculate the normal module, which creates some confusion for the designer, [10].

2. Standard design methodology
The standardized method of spur gear design does not use a criterion to find out what is the main and dangerous stress, [1], [12]:
• the formula for centre distance calculation is derived from Hertz contact stress relation;
• the module determination formula is derived from bending stress of the tooth (beam strength).

The main problem that may arise in designing is that through classical methods, a critical pin number is determined. If the actual tooth number \(z_1 \leq z_{perr}\), the problem that may occur will be the pitting, and the calculation is based on the Hertz contact stress.

As a first observation on the number of teeth and choice according to the standards, the values may not be convenient.

The standard says: “if the value of \(z_1\) results great, \(z_1\) is decreased and, consequently, \(m_n\) and \(s_F\) “grow”: but the STAS does not say what means “great”. The next explanatory note is confusing: “decreasing \(z_1\) and increasing module \(m_n\) is unfavorable (small precision, risk of size and wear, more manufactures etc)”.

At this stage of the project, the following choice is recommended for all \(z_1\) values obtained,[13],[14]:
• \(z_{lmin} = 10\)
• for hardened and tempered tooth gears \(z_1 \approx 10...17\) (21)
• for normalized tooth gear, \(z_1 \approx 25...35\)
• for induction hardening, flame hardening, nitriding surface, \(z_1 \approx 15...23\) (35).

The module not directly determined with the number of teeth initially chosen, the recalculation of module is, clearly necessary, after the number of teeth modification.

For the addendum modification coefficient determination, (sum value Xs), the standard uses the relation derived from circumferential backlash condition.

For determining “sum value Xs”, the standard uses the relationship derived from the circumferential backsum. Since STAS does not specify the limit of use although the formula is correct, there is the possibility that in some cases to be unacceptable values.

3. Kinematic optimization of spur gear design
Since in the analysis of the gearing of mechanisms there are many features of the involuntary profiles of the teeth, but these are not taken into account in the standardized method of calculation there is the possibility of elaborating an optimization mode. If we impose certain limits (or "restrictions") based on the tool manufacturing process, the type of instrument used, and the interference effect in the working process.

3.1. Rack type cutter restriction (limit) is imposed by undercutting (manufacture interference) process avoiding.

If we note \(r_{u1}\) the usable flank circle radius and \(r_{b1}\) the base circle radius of pinion, the condition to avoid the undercutting is, [10], equation (1):

\[
r_{u1} \geq r_{b1} \cdot \cos \alpha
\]

where \(\alpha = 20^\circ\) is nominal pressure angle with \(r_{b1}\) the base circle radius of pinion, equation (2):
and $r_{u1}$ is the usable flank circle radius, equation (3):

$$r_{u1} = r_{f1} + c_1 = \frac{m}{2} (z_1 - 2 - 2x_1)$$

If we replace the equation (2) and (3) in equation (1) we get, equation (4):

$$z_1 - 2 + 2x_1 \geq z_1 \cdot \cos^2 \alpha$$

Where $\sin \alpha = \sin 20^\circ \approx 0,342 \Rightarrow \sin^2 20^\circ \approx 0,117$, equation (5):

$$\Rightarrow x_1 \geq 1 - 0,5 \cdot z_1 \cdot \sin^2 \alpha \approx 1 - 0,06 \cdot z_1$$

The form $x_1 \geq 0,06 \cdot z_1$ is the ISO TC-60 recommendation but not mentioned in STAS, equation (6):

$$\Rightarrow x_1 \geq 1 - \frac{z_1}{17}$$

In order to avoid interference in manufacturing process, we convert formula (5) and find the minimum number of teeth of the pinion, meaning 17.

If we impose the condition, equation (7) and (8):

$$r_{u1} = r_{b1}$$

$$\Rightarrow z_1 - 2 + 2x_1 = z_1 \cdot \cos \alpha$$

It results, equation (9):

$$x_1 = 1 + 0,5 \cdot z_1 (\cos \alpha - 1) = 1 - 0,03 \cdot z_1$$

We convert equation (9) and find the pinion with minimum 33 teeth.

$$x_1 = 1 - \frac{z_1}{33}$$

If $x_1 = 0$ we have a complete involute profile.

3.2. Minimum thickness of teeth on addendum circle restriction

The general condition is, [11], equation (11):
\[ s_\alpha \geq k \cdot m \]  \hspace{2cm} (11)

where \( s_\alpha \) is the thickness of tooth on addendum circle, \( m \) is the module \( k = 4 \ldots 5 \) for hardened steels and \( k = 2 \ldots 3 \) for ductile materials.

The value of \( s_\alpha \) is:

\[ s_\alpha = \frac{\cos \alpha}{\cos \alpha_\alpha} \cdot \left[ \frac{\pi}{2} + 2x_1 \tan \alpha - z_1 (\text{inv}\alpha_\alpha - \text{inv}\alpha) \right] \]  \hspace{2cm} (12)

where

\[ \text{inv}\alpha_\alpha = \tan \alpha_\alpha - \alpha_\alpha \]  \hspace{2cm} (13)

To solve equations (11) and (12) is very difficult.

We must admit an approximate value in equation (14).

\[ x_1 = 0.02 \cdot z_1 \]  \hspace{2cm} (14)

\[ \begin{align} \text{Figure 1.} & \quad \text{Is representing the restrictions of equation (6) and (10).} \\
& \quad \text{• For the values of } x_1 \text{ determined by relation (6), the undercutting (manufacture interference) does not exist if the number of teeth is } z_1 \geq 17. \\
& \quad \text{• If } z_1 < 17 \text{ in order to avoid undercutting, it is necessary to adopt an addendum modification coefficient } (x_1) \text{ calculated with formula (6).} \\
& \quad \text{• The } x \text{ plus wheel has a reduced thickness of teeth on the addendum circle;} \\
& \quad \text{⇒ for } z_1 = 11, m : s_\alpha = 0.3m \text{ and} \\
& \quad \text{⇒ for } z_1 = 8 \text{ } m : s_\alpha = 0.1m \end{align} \]
• The wheel with $z_1 \geq 33$ teeth has an entirely involute shape profile;
• If $z_1 < 33$ an involute profile results if the $x_1$ is calculated by formula (10).
• The x plus wheel has a reduced thickness of teeth on the addendum circle,[5];
  \[ \Rightarrow \text{ for } z_1 = 13, m; s_a = 0.3m \]

### 3.3. Limiting the thickness of the teeth

The limitation of the teeth thickness or the “zero return” condition result from the equal value of the teeth thickness and the width of the tooth space, [8], [9], [10].

Note the $x_s$ the sum between $x_1$ and $x_2$ equation (15):

\[ x_s = x_1 + x_2 = \frac{z_s}{2} \cdot \frac{\text{inva} - \text{invb}}{\text{tg} \alpha} \]  \hspace{0.5cm} (15)

where $z_s = z_1 + z_2$

• For the X gearing where $z_1 < 17$ and $x_1$ calculated by relation (6), we can write, equation (16) – (18):

\[ x_1 \geq 1 - 0.06 \cdot z_1 \]
\[ x_2 \geq 1 - 0.06 \cdot z_2 \]
\[ \Rightarrow x_s = x_1 + x_2 \geq 2 - 0.06 \cdot z_s \]  \hspace{0.5cm} (17)

\[ z_1 \geq \frac{2}{(u+1)(0.5 \frac{\text{inva} - \text{invb}}{\text{tg} \alpha} + 0.06)} \]  \hspace{0.5cm} (18)

where $u = \frac{z_2}{z_1}$

The formula (18) may be considered as a restriction for pinion teeth number, in order to avoid the rack type undercut.

• For the X gearing where $x_1$ calculated by equation (10), we can write, equation (19) – (21):

\[ x_1 \geq 1 - 0.03 \cdot z_1 \]  \hspace{0.5cm} (19)

\[ x_2 \geq 1 - 0.03 \cdot z_2 \]

\[ \Rightarrow x_s = x_1 + x_2 \geq 2 - 0.03 \cdot z_s \]  \hspace{0.5cm} (20)

\[ z_1 \geq \frac{2}{(u+1)(0.5 \frac{\text{inva} - \text{invb}}{\text{tg} \alpha} + 0.03)} \]  \hspace{0.5cm} (21)

The equation (21) may be considered as a restriction for pinion teeth number, in order to provide the minimum thickness of the teeth on the addendum circle.
3.4. Total contact ratio restriction

To ensure silent operation of the gear, a number of restrictions must be considered, [2], [7].

The contact ratio is, equation (22):

$$\varepsilon_a = \frac{d_{a1} - d_{a2}}{2m \cos \alpha} + \frac{d_{b1} - d_{b2}}{2m \cos \alpha}$$

(22)

where: \(d_{a1}, d_{a2}\) – are the addendum circle diameters, \(d_{b1}, d_{b2}\) – are the base circle diameters.

If in relation (22) we put the values of the addendum circle diameters, the base circle diameters, distance between axles in function of \(z_1\) and \(z_2\) and we represent the value of \(\varepsilon_a\) which also depends on \(z_1\) and \(z_2\), a lot of curves having a maximum for value of \(z_2 \approx 130\) teeth result.

The restriction for contact ratio may be written as, equation (23):

$$z_1 = \frac{130}{u}$$

(23)

This value of \(z_1\) provides a maximum value of total contact ratio.

3.5. Pressure angle restriction

Positive x value of addendum modification coefficient produces an increase of the pressure angle \((\alpha_w > \alpha = 20^\circ)\), [4]

- The upper value of the pressure angle results from equation (14):

$$x_1 \leq 0,02z_1$$

(24)

$$x_2 \leq 0,02z_2$$

and, for these values, the equation (15) is transformed as:

$$x_2 = x_1 + x_2 \leq 0,02(z_1 + z_2)$$

(25)

for \(\alpha = 20^\circ\), we have:

$$\frac{\text{invcos} - \text{inva}}{\text{tg} \alpha} \leq 0,04$$

(26)

$$\Rightarrow \alpha \leq 25^\circ 25'$$

- The lower value of pressure angle is determined by condition (6):

$$x_1 \geq 1 - 0,06z_1$$

$$x_2 \geq 1 - 0,06z_2$$

(27)
if we make the replacements, the relation (15) takes the next form, equation (29):

$$\frac{\text{inv}_{n_2}-\text{inv}_{n_1}}{\tan \alpha} \geq \frac{2(2-0.06z_2)}{z_n} = \frac{4}{z_n} \cdot 0.12$$ (29)

The unfavorable case is for $x_z = x_1 + x_2 = 40$, equation (30):

$$\frac{\text{inv}_{n_2}-\text{inv}_{n_1}}{\tan \alpha} \geq -0.02$$ (30)

$$\Rightarrow \alpha \geq 16^\circ$$

- In conclusion, in order to maintain the pressure angle within the limits of $16^\circ \leq \alpha \leq 25^\circ 25'$, the values of addendum modification coefficient $x$ must be limited.

### 3.6. Gear ration restriction

The best values of gear ratio can be deduced on minimum number of teeth, [6], [13]. We consider two gearing pairs:

- the first (I) having the transmission ratio, equation (31) and (32):

$$i_1 = \frac{n_2}{n_1} \quad (31)$$

$$u_1 = \frac{x_1}{z_n}$$ (32)

- and the second (II) having, equation (33) and (34):

$$i = i_1 \cdot i_2$$ (33)

$$u = u_1 \cdot u_2 = \frac{x_1}{z_1} \cdot \frac{z_2}{z_n}$$ (34)

The total number of teeth is, equation (35):

$$Z_1 = z_1 + z_2$$

$$Z_2 = z_1 + z'_2 + z_1 + z_2$$ (35)
The difference of teeth number is:

$$\Delta = Z_2 - Z_1 = z_1\left(1 + u_1 + u_2 - u\right) = z_1\left(1 + u_1 + \frac{u}{u_1} - u\right) =$$

$$= z_1\frac{u^2 - u(u-1) + u}{u_1} = f(u_1, u) \quad (36)$$

This function has a maximum for, equation (37) and (38):

$$u_1 = u_2 = \sqrt{u} \quad (37)$$

$$u \geq 3 + 2\sqrt{2} = 5.82 \approx 6 \quad (38)$$

The conclusion is that the spur involute gear is optimum for gear ratio $u = 1 \ldots 6$; for values higher than 6 we recommend to use two gearings, having $u_1 = u_2 = \sqrt{2}$.

3.7. Working interface restriction

The upper part of the drive pulley tends to penetrate the bottom of the driven pulley. Figure 2 shows the primary interference, defined by the existence of segment $AE$ inside segment $T_1T_2$. The condition is:

$$r_{a1} \leq O_1T_2$$

$$r_{a2} \leq O_2T_1$$

(39)

Figure 2. The primary interference.

Figure 3 shows the secondary interference, defined by the following relations:

$$r_{a1} \leq O_1U_2$$

$$r_{a2} \leq O_2U_1$$

(40)
This is the interference in the joining (connecting) area between the bottom of the tooth and wheel body. The gearing with $x_s < 0$ and $z_1 < 17$ always produces working interference.

4. Conclusions

- The restrictions imposed in Chapter 3, paragraphs 3.1 to 3.7 are the kinematic limits of the spur gear. If these are followed, an optimal transmission according to the kinematic design can be achieved.
- For designing gears the Romanian standard does not propose restrictions, it partially respects them. The STAS computation steps are outlined in the general overview in Chapter 1.
- The STAS considers that the principal stress of the teeth is Hertz contact stress and compels us to calculate the centre distance by a formula derived from this condition. But the formula is valid only for $z_1 \leq z_{1cr}$ (where $z_{1cr}$ is critical number of teeth).
- STAS recommends calculating the module in the first phase due to bending stress, but then changing the number of teeth changes the module value.
- The number of teeth are determined as a function of centre distance ($a$), module ($m$) and transmission rate ($u$). But the calculated value of is modified, decreasing to a “chosen” value - no explanation being given.
- The kinematic restriction 3.1 - 3.7 can be used as an initial condition in the design of a gear. They are well known in the theory of gears, but the standard ignores them.
- The proposal is that the designer will select, at first, the type of gearing (i.e. to select $z_1$ that respects 3.1-3.7 restrictions), and afterwards calculate the other characteristics of gearing.
- The logical order of the steps calculation may be:
  - Material selection for pinion and gear and their thermic treatment;
  - Type of gearing selection;
  - Pinion number of teeth selection, based on 3.1-3.7 kinematic restrictions; the designer will decide which of these are more important;
  - Critical number of teeth determination;
  - If $z_1 \leq z_{1cr}$, the principal stress is considered Hertz contact stress and the centre distance is calculated from this condition. The formula is the same as in the STAS.
  - For $z_1 > z_{1cr}$, the principal stress is considered the beam stress of teeth (bending stress).
  - Module determination;
  - Addendum modification coefficient determination;
Geometric and kinematic calculation.
- The above method avoids the confusions of standard and eliminates the calculation of module m followed by its modification and the $z_1$ calculation followed by its modification, without any explanation.

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