Instabilities of the Hubbard chain in a magnetic field

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We find and characterize the instabilities of the repulsive Hubbard chain in a magnetic field by studying all response functions at low frequency ω and arbitrary momentum. The instabilities occur at momenta which are simple combinations of the (U = 0) σ = ±↑, ↓ Fermi points, ±k_F. For finite values of the on-site repulsion U the instabilities occur for single σ electron adding or removing at momenta ±k_F, for transverse spin-density wave (SDW) at momenta ±2k_F (where 2k_F = k_F↑ + k_F↓), and for charge-density wave (CDW) and SDW at momenta ±2k_F and ±2k_F↑. While at zero magnetic field removing or adding single electrons is dominant, the presence of that field brings about a dominance for the transverse ±2k_F SDW over all the remaining instabilities for a large domain of U and density n values. We go beyond conformal-field theory and study divergences which occur at finite frequency in the one-electron Green function at half filling and in the transverse-spin response function in the fully-polarized ferromagnetic phase.

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I. INTRODUCTION

One-dimensional (1D) electronic quantum problems have peculiar instabilities which follow from their restrictive geometry. For instance, it is well known that the low-frequency charge and spin response functions of a gas of electrons in a 1D lattice show logarithmic divergences at twice the Fermi wavevector. This is the so called Peierls instability which in real quasi-one-dimensional materials occurs through coupling of the electrons to suitable phonon modes giving rise to lattice distortions and to phenomena like charge-density waves (CDW) or spin-density waves (SDW).

These instabilities also occur in the presence of a magnetic field and, in real materials there are electron - electron interactions which at low dimensions have a strong impact on the physical properties. In this paper we study the effects of the interplay between a magnetic field and many-electron interactions on the 1D quantum-problem instabilities. We consider an integrable many-electron problem, the Hubbard chain in a magnetic field. One of our goals is to use elsewhere the present results on the 1D instabilities as a starting point for the study of the corresponding instabilities of a system of weakly coupled chains. Therefore, we include in our present 1D study the one-particle Green function which involves excitations that change the electron numbers. In the coupled-chain system this 1D function will provide important information on the relevance of interchain electron hopping. In Sec. VI we will return to the problem of the Hubbard coupled-chain system in a magnetic field.

At zero onsite interaction, U = 0, the Hubbard chain reduces to a 1D tight-binding electron model with first-neighbor transfer integral t. In this case it is easy to evaluate the low-frequency response functions Reχσ(k, ω) (with σ being charge ρ, spin projection σ_z, and so on) and to characterize and detect the corresponding divergences. At zero magnetic field the charge function Reχρ(±2k_F, ω), the spin function Reχσ(±2k_F, ω), the spin-transverse function Reχσ⊥(±2k_F, ω), the singlet superconductivity functions Reχ^ss(0, ω) and Reχ^ss(±2k_F, ω), and the triplet σ superconductivity function Reχ^tsσ(0, ω) all diverge as −ln(ω). If we apply a magnetic field we find that the H = 0 instabilities are limiting cases of the finite-field situation – for 0 < H < 2µ_o sin(2πn)^2 (here H_c = 2µ_o sin(2πn)^2 is the U = 0 critical field for onset of fully-polarized ferromagnetism and µ_o is the Bohr magneton) the charge functions Reχρ(±2k_F↑, ω) and Reχρ(±2k_F↓, ω), the spin functions Reχσ↑(±2k_F↑, ω) and Reχσ↑(±2k_F↑, ω), the spin-transverse function Reχσ⊥(±2k_F, ω), the singlet superconductivity functions Reχ^ss(±2k_F↑ − k_F↓, ω) and Reχ^ss(±2k_F, ω), and the triplet σ superconductivity function Reχ^tsσ(0, ω) all diverge as −ln(ω). Also, both for H = 0 and H > 0 the single-electron Green function diverges at ±k_F as 1/ω.
On the other hand, the study of all divergences of the low-frequency response functions $\text{Re} \chi^\sigma(k, \omega)$ is for the corresponding many-electron quantum system a problem of some complexity. The BA solution and conformal-field theory (CFT) can be combined to evaluate the asymptotics of the correlation functions of the Hubbard chain in a magnetic field $\{2, 3\}$. However, the momentum-dependent expressions derived in Ref. $\{3\}$ (for zero magnetic field see also Ref. $\{1\}$) and some of the quantities studied in Ref. $\{2\}$ refer to the equal-time situation and we need here the low-frequency response functions. Moreover, Ref. $\{2\}$ evaluated and presented expressions for the response functions which refer to particular relations between the values of the low-frequency $\omega$ and small-momentum $(k - k_0)$. Here, $k_0$ are particular combinations of the $U = 0$ electronic Fermi momenta [see Eq. (12) below]. We find that the low-frequency correlation functions $\text{Re} \chi^\sigma(k, \omega)$ can only diverge at the particular values $k = k_0$. Unfortunately, Ref. $\{2\}$ has not solved the present problem of the instabilities because its expression (5.7) corresponds to a very particular relation between $\omega$ and $(k - k_0)$ which does not provide the relevant and general $\omega$ dependence at $k = k_0$.

The problem was solved for the particular case of the charge $\vartheta = \rho$ and spin $\vartheta = \sigma_z$ response functions in Ref. $\{1\}$. However, that paper has not considered all correlation functions. Although the charge and spin excitations studied in Ref. $\{1\}$ are relevant and correspond to real instabilities, they are never dominant, as we show in this paper.

Using the same methods as in Refs. $\{1, 2\}$, we find that all instabilities correspond at finite values of $U$ to power-law divergences (and in some singular points of the parameter space to logarithmic divergences) in the correlation functions. The corresponding exponents are non-classic combinations of two-pseudoparticle phase shifts. These pseudoparticles are the quantum objects introduced in Refs. $\{12, 16\}$ which refer to the operator algebra $\{17, 19\}$ which diagonalizes the quantum problem. Their zero-momentum forward-scattering interactions control the overlaps between the Hamiltonian eigenstates and the excitations associated with the correlation functions $\{10, 11\}$. It is the exotic character of such overlap which leads to the Luttinger-liquid $\{20\}$ non-classic exponents.

While at zero magnetic field removing or adding single $\sigma$ electrons at $k = \pm k_F, \sigma$ is dominant (in the weakly coupled-chain problem this is associated with inter-chain hopping) and, due to the spin $SU(2)$ symmetry, the transverse $\pm 2k_F$ SDW and the $\pm 2k_F$ SDW and CDW have the same $\omega$ dependence, the presence of a magnetic field brings about a dominance for the transverse $\pm 2k_F$ SDW over the remaining instabilities for a large domain of $U$ and $n$ values. There are no divergences in the superconductivity and other response functions.

We study the effects of the half-filled metal – insulator transition and fully-polarized ferromagnetic transition on the above correlation functions. Some low-energy power-law divergences are present in both sides of these transitions but there occur discontinuities in the corresponding non-classical critical exponents. Moreover, we go beyond CFT and study power-law divergences which occur for some correlation functions at finite energy for the half-filling and fully-polarized ferromagnetic phases.

The paper is organized as follows: the Hamiltonian and the pseudoparticle operator basis are introduced in Sec. II. In Sec III we characterize the instabilities of the quantum problem. The comparative study of these instabilities is presented in Sec. IV. In Sec. V we consider the half-filling and fully-polarized ferromagnetic phases and the effects of the half-filling metal - insulator transition and fully-polarized ferromagnetic transitions on the quantum-problem instabilities. The coupled-chain problem in the presence of a magnetic field is discussed in Sec. VI. Finally, Sec. VII presents the concluding remarks.

II. THE HUBBARD CHAIN AND THE PSEUDOPARTICLE OPERATOR BASIS

We consider the Hubbard chain $\{3, 5\}$ with a finite chemical potential $\mu$ and in the presence of a magnetic field $\hat{H} = -t \sum_{j, \sigma} \left[c_{j, \sigma}^\dagger c_{j+1, \sigma} + c_{j+1, \sigma}^\dagger c_{j, \sigma}\right] + U \sum_j \left[c_{j, \uparrow}^\dagger c_{j, \uparrow} - 1/2\right]\left[c_{j, \downarrow}^\dagger c_{j, \downarrow} - 1/2\right] + 2\mu \hat{n}_z + 2\mu_0 H \hat{S}_z , \tag{1.1}

where $c_{j, \sigma}^\dagger$ and $c_{j, \sigma}$ are the creation and annihilation operators, respectively, for electrons at the site $j$ with spin projection $\sigma = \uparrow, \downarrow$ and $\hat{n}_z = -\frac{1}{2}[N_\sigma - \sum_\sigma \hat{N}_\sigma]$ and $\hat{S}_z = -\frac{1}{2} \sum_\sigma \sigma \hat{N}_\sigma$. Here $\hat{N}_\sigma$ is the $\sigma$-electron number operator. We introduce the $U = 0$ Fermi points $k_{F\sigma} = \pi n_\sigma$ (and $k_F = [k_{F\uparrow} + k_{F\downarrow}]/2 = \pi n/2$) where $n_\sigma = N_\sigma/N$ and $n = N/N_\sigma$, and $N_\sigma$ is the number of lattice sites. The total-electron number is $N = \sum_\sigma N_\sigma$ and the spin density is given by $m = n_\uparrow - n_\downarrow$.

The critical exponents which determine the response-function properties and the associate instabilities of the many-electron problem (1.1) are fully controlled by the pseudoparticle interactions. Let us then provide some basic information on the low-energy pseudoparticle operator basis which diagonalizes the quantum problem (1.1). This diagonalization uses the BA $\{3, 5\}$. We consider all finite values of $U$, electron densities $0 < n < 1$, and spin densities $0 < m < n$. (In Sec. V we also consider the $n = 1$ half-filling and $m = n$ fully-polarized ferromagnetic phases.) For this parameter
space, the low-energy physics is dominated by the lowest-weight states (LWS's) of the spin and $\eta$-spin algebras which in Refs. [17–19] were called of type I – in this paper we call them for LWS's I. These are described by real BA rapidities, whereas all or some of the BA rapidities which describe the LWS's II are complex and non-real [19]. Both the LWS's II and the non-LWS's of out the BA solution have energy gaps relative to each canonical ensemble ground state [17–18] and do not contribute to the quantum problem instabilities studied in this paper.

In the Hilbert subspace spanned by the LWS’s I the BA solution was shown to refer to an operator algebra which involves two types of pseudoparticle creation (annihilation) operators $b_{q,\alpha}^\dagger$ ($b_{q,\alpha}$). These obey the usual anti-commuting algebra [17–19]

$$\{b_{q,\alpha}^\dagger, b_{q',\alpha'}\} = \delta_{q,q'}\delta_{\alpha,\alpha'}, \quad \{b_{q,\alpha}^\dagger, b_{q',\alpha'}^\dagger\} = 0, \quad \{b_{q,\alpha}, b_{q',\alpha'}\} = 0.$$  

Here $\alpha$ refers to the two pseudoparticle colors $c$ and $s \, [17–19]$. The discrete pseudomomentum takes on values $q_j = \frac{2\pi j}{N_\eta}$, where in contrast to the usual momentum, $I_\eta^0$ are consecutive integers or half-odd integers. It can be shown that in spite of the above anti-commuting algebra the $\alpha$ pseudoparticles have neither a fermionic nor a bosonic statistics [21]. However, provided we introduce the topological-momentum shift operators studied in Refs. [17–19] the second-quantization pseudoparticle representation can be used. There are $d_{F,\alpha}$ values of $I_\eta^0$ ($d_{F,\alpha}$ is the $\alpha$-band Fock-space dimension), i.e. $j = 1, ..., d_{F,\alpha}$. (The $\alpha$ topological-momentum shift operators produce shifts of $\pm \frac{\pi}{N_\eta}$ in the $\alpha$-band pseudomomentum numbers $I_\eta^0 \, [17–19]$.) A LWS I is specified by the distribution of $N_\alpha$ occupied values, which we call $\alpha$ pseudoparticles, over the $d_{F,\alpha}$ available values. There are $d_{F,\alpha} - N_\alpha$ corresponding empty values, which we call $\alpha$ pseudoholes [19]. These are good quantum numbers such that $d_{F,c} = N_c$, $d_{F,s} = N_s$, $d_{F,s} = N_s + N_c$, and $N_s = N_c$.

The numbers $I_\eta^0$ are integers (or half-odd integers) for $N_s$ even (or odd), and $I_j^s$ are integers (or half-odd integers) for $d_{F,s}$ odd (or even) [17]. Therefore, all excitations which change $N_s = N_s' \pm 1$ and $d_{F,s} = d_{F,s}' \pm 1$ by an odd integer always involve topological-momentum shifts [17–19]. All the LWS’s I can be generated by acting onto the vacuum $|V\rangle$ (zero-electron density) suitable combinations of pseudoparticle operators [17–18] and are Slatter determinants of pseudoparticle levels of the simple form

$$|\phi; N_c, N_s\rangle = \prod_{\alpha=c,s} \prod_{q=0}^{d_{F,\alpha}} |b_{q,\alpha}^\dagger||V\rangle.$$  

We emphasize that all LWS’s I are characterized by fixed values of $N_c$ and $d_{F,s}$ and therefore the pseudomomentum takes on numbers $I_\eta^0$ of well defined integer or half-odd integer character. The ground state (GS) of a canonical ensemble with $N_c$ and $N_s$ pseudoparticle numbers has the particular form

$$|\text{GS}; N_c, N_s\rangle = \prod_{\alpha=c,s} \prod_{q=0}^{d_{F,\alpha}} |b_{q,\alpha}^\dagger||V\rangle.$$  

When $N_\alpha$ is odd (even) and the numbers $I_\eta^0$ are integers (half-odd integers) the pseudo-Fermi points are symmetric and given by $q_{F,\alpha} = -q_{F,\alpha} = q_{F,\alpha} - \frac{\pi}{N_\alpha}$ where

$$q_{F,\alpha} = \frac{\pi N_\alpha}{N_\alpha}.$$  

On the other hand, when $N_\alpha$ is odd (even) and $I_\eta^0$ are half-odd integers (integers) we have that $q_{F,\alpha} = q_{F,\alpha}$ and $q_{F,\alpha} = q_{F,\alpha} - \frac{2\pi}{N_\alpha}$ or $q_{F,\alpha} = q_{F,\alpha} - \frac{2\pi}{N_\alpha}$ and $q_{F,\alpha} = q_{F,\alpha}$. Similar expressions are obtained for the pseudo-Brioullin zones limits $q^\pm_{\alpha}$ if we replace in the above expressions $N_c$ by $N_s$. On the other hand, the pseudo-Brioullin zones limits $q^\pm_{\alpha}$ are always symmetric and given by $q^\pm_{\alpha} = -q^\pm_{\alpha} = \frac{\pi}{N_{\alpha}}[d_{F,s} - 1]$.

In the pseudoparticle basis spanned by the LWS’s I and in normal order relatively to the GS (4) the Hamiltonian (1) has the following form [17–18]

$$\hat{H} := \sum_{i=1}^\infty \hat{H}^{(i)},$$  

where, to second pseudoparticle scattering order
Here (7) are the Hamiltonian terms which are relevant at low energy. Furthermore, at low energy and small momentum the only relevant term is the non-interacting term \( \hat{H}^{(1)} \). Therefore, the \( c \) and \( s \) pseudoparticles are non-interacting at the small-momentum and low-energy fixed point and the spectrum is described in terms of the bands \( \epsilon_n(q) \) studied in Ref. [3].

At higher energies and (or) large momenta the pseudoparticles start to interact via zero-momentum transfer forward-scattering processes of the Hamiltonian (6) – (7). As in a Fermi liquid, these are associated with \( f \) functions [14,17] which read

\[
f_{\alpha\alpha'}(q, q') = 2\pi v_\alpha(q)\Phi_{\alpha\alpha'}(q, q') + 2\pi v_{\alpha'}(q')\Phi_{\alpha'\alpha}(q', q) + \sum_{j=\pm1} \sum_{\alpha'=c,s} 2\pi v_{\alpha'} \Phi_{\alpha'\alpha}(jq_{F\alpha'}, q)\Phi_{\alpha'\alpha}(jq_{F\alpha'}, q'),
\]

where the pseudoparticle group velocities are given by \( v_\alpha(q) = \frac{d\epsilon_n(q)}{dq} \) and \( v_\alpha = v_\alpha(q_{F\alpha}) \) are the pseudo-Fermi group velocities. In expression (8) \( \Phi_{\alpha\alpha'}(q, q') \) measures the phase shift of the \( \alpha' \) pseudoparticle of pseudomomentum \( q' \) due to the zero-momentum forward-scattering collision with the \( \alpha \) pseudoparticle of pseudomomentum \( q \). These phase shifts describe the pseudoparticle interactions and are defined in Ref. [14]. They control the low-energy physics. For instance, the related parameters

\[
\xi_{\alpha\alpha'}^j = \delta_{\alpha\alpha'} + \Phi_{\alpha\alpha'}(q_{F\alpha}, q_{F\alpha'}) + (-1)^j \Phi_{\alpha\alpha'}(q_{F\alpha}, q_{F\alpha'}) + (-1)^j \Phi_{\alpha\alpha'}(q_{F\alpha}, q_{F\alpha'}) ,
\]

play a determining role in the expressions we evaluate in this paper. There is a simple and direct relation between the pseudoparticle-interaction parameters and the quantities of CFT. For instance, the anti-symmetric combinations of two-pseudoparticle phase shifts (9), \( \xi_{\alpha\alpha'}^j \), are nothing but the entries of the transpose of the dressed-charge matrix [5]. Both the symmetric \([j=0 \text{ in Eq. (9)}]\) and anti-symmetric \([j=1 \text{ in Eq. (9)}]\) combinations of two-pseudoparticle phase shifts (9) fully control the instabilities of the quantum problem, as we find in the next sections.

### III. THE INSTABILITIES OF THE QUANTUM PROBLEM

In order to study the divergences of the low-frequency response functions \( \text{Re} \chi^\alpha(k, \omega) \) we start by characterizing the relevant GS transitions which determine these divergences.

Following Ref. [13], at low energy there are independent right \( N^+_\alpha \) and left \( N^-_\alpha \) pseudoparticle conservation laws in the Hubbard chain. Here \( N^+_\alpha \) (or \( N^-_\alpha \)) are the number of \( \alpha \) pseudoparticles, with \( q > 0 \) (or \( q < 0 \)). While fixed values of \( N_c \) and \( N_s \) (and then of \( N^+_\alpha \) and \( N^-_\alpha \) electron numbers) define a canonical ensemble, we introduce the concept of sub canonical ensemble which refers to fixed values of \( N^+_c, N^-_c, N^+_s, \) and \( N^-_s \). Obviously, the same canonical ensemble is realized by several sub-canonical ensembles.

Since all LWS’s I of form (3) are characterized by fixed values of the pseudoparticle numbers \( N^+_\alpha, N^-_\alpha, N^+_\alpha, \) and \( N^-_\alpha \), we can associate with each sub-canonical ensemble a low-energy Hilbert subspace spanned by these LWS’s I which have the same pseudoparticle numbers. We call pseudo-ground state (PGS) the LWS I (or LWS’s I) of minimal energy in each of these Hilbert sub spaces. All PGS’s are of the form

\[
|PGS; D_c, D_s \rangle = \prod_{\alpha=c,s} q_{F\alpha}^{(+)\alpha} + D_\alpha \tilde{\alpha}^\dagger_{\alpha} \prod_{q=q_{F\alpha}^-} \tilde{b}_{q,\alpha} \prod_{\alpha=c,s} b_{q,\alpha}^\dagger \left| V \right> ,
\]

where

\[
D_\alpha = \Delta N^+_\alpha = -\Delta N^-_\alpha
\]

refers to the \( \Delta N^\pm_\alpha \) changes relative to the GS (4) with the same \( N_c \) and \( N_s \) numbers.

An important point is that the transitions between a GS and any LWS I can be separated into two types of excitations: (a) a GS - GS or GS - PGS transition which involves changes in the pseudoparticle numbers and (b) a
Landau-liquid excitation associated with pseudoparticle-pseudohole processes relative to the final GS or PGS which do not change the right and left pseudoparticle numbers.

A remarkable property is that the low-frequency correlation-function divergences are fully determined by the above transitions (a), followed by transitions (b). This is because these divergences are determined by the quantum overlap between GS’s and PGS’s and the correlation-function excitations. The momentum of these transitions (a) is given by

$$k_0 = \sum_\alpha D_\alpha 2q_F \alpha,$$

(12)

where for the pure GS - PGS transitions (by pure GS - PGS transitions we mean those which do not change the $N_c$ and $N_s$ pseudoparticle numbers) $D_\alpha = 0, \pm 1, \pm 2, \ldots$ are always integers, whereas for the GS - GS transitions $D_\alpha$ can either be integers or half-odd integers.

The pseudoparticle-pseudohole processes (b) increase the value of the critical exponents which control the response-function divergences by positive integers. This leads to positive exponents associated with irrelevant correlation-function terms. Therefore, the divergences occur always precisely at momentum values $k = k_0$ (see Eq. (12)) which correspond to GS - GS or GS - PGS transitions (a).

In order to find the divergences of all low-frequency response functions $\text{Re} \chi^\theta(k, \omega)$ we have followed the steps of Refs. [6-8] and combined our suitable generator pseudoparticle analysis [6,7] with the results obtained from CFT [9,10]. The asymptotic expression of the correlation function $\chi^\theta(x, t)$ in $x$ and $t$ space is given by the summation of many terms of form (3.13) of Ref. [3] with dimensions of the fields suitable to that function. For small energy the corresponding correlation functions in $k$ and $\omega$ space are obtained by the summation of the Fourier transforms of these terms, which are of the form given by Eq. (5.2) of Ref. [3]. However, the results of Refs. [9,10] do not provide the specific expression at $k = k_0$ and small positive $\omega$. In this case the above summation is equivalent to a summation in the suitable final GS’s or PGS’s which correspond to different values for the dimensions of the fields.

As was already referred in Sec. I, we emphasize that expression (5.7) of Ref. [3] is not valid in our case. While we need the general response-function $\omega$ dependence at $k = k_0$, expression (5.7) of Ref. [3] is only valid for a very particular relation between $(k - k_0)$ and $\omega$. Moreover, the limit $(k - k_0) \to 0$ of that expression does not provide the general low-$\omega$ behavior at $k = k_0$.

Following Ref. [3], we have solved the following general integral

$$\tilde{g}(k_0, \omega) = \int_0^\infty dx \int_{-\infty}^\infty dt e^{i\omega t} \prod_{\alpha, t} \frac{1}{(x + i\nu_\alpha t)^{2\Delta_\alpha}},$$

(13)

which is the integral of expression (5.2) of Ref. [3] at $k = k_0$. In Eq. (13) $\Delta_\alpha^\pm$ with $\iota = +1$ and $\iota = -1$ for right and left $\alpha$ pseudoparticles, respectively, are the GS - GS or GS - PGS suitable dimensions of the fields [6,7] of the general form

$$2\Delta_\alpha^\iota = \left[ \sum_{\alpha'} \xi^1_{\alpha\alpha'} D_\alpha^\iota + \iota \sum_{\alpha'} \xi^0_{\alpha\alpha'} \frac{\Delta N_\alpha}{2} \right]^2,$$

(14)

where $\xi^j_{\alpha\alpha'}$ (with $j = 0, 1$) are the combinations of two-pseudoparticle phase shifts (9) and $\Delta N_\alpha$ refers to the changes in the numbers $N_\alpha$ associated with the transition. We find for the integral (13), $\tilde{g}(k_0, \omega) \propto \omega^{\varsigma_\theta}$. Here

$$\varsigma_\theta = 2 \sum_{\alpha, t} \Delta_\alpha^\iota - 2.$$  

(15)

Comparing our expression with expression (5.7) of Ref. [3] we confirm these expressions are different. This is because these expressions correspond to different limiting cases which do not commute. We find that at the momenta $k = k_0$ (given by Eq. (12)) and low frequency $\omega$ all correlation functions behave as $\text{Im} \chi^\theta(k_0, \omega) \propto \omega^{\varsigma_\theta}$ with the corresponding real part given by

$$\text{Re} \chi^\theta(k_0, \omega) \propto \omega^{\varsigma_\theta},$$

(16)

for $\varsigma_\theta \neq 0$ and by $\text{Re} \chi^\theta(k_0, \omega) \propto -\ln(\omega)$ for $\varsigma_\theta = 0$, respectively. Given this related expressions for the imaginary and real parts, below we consider only the latter part. Our task is detecting all response functions and momenta values $k = k_0$ for which the corresponding exponent $\varsigma_\theta$, Eq. (15), is zero or negative. We have evaluated it for all correlation functions and possible values of $k_0$. All divergences occur only for some of the correlation functions which show logarithmic divergences at $U = 0$, ie all divergences occurring at finite values of $U$ correspond to logarithmic
divergences at \( U = 0 \). However, we find that the onsite electron-electron interactions supress such divergences in the case of the the superconductivity functions and for some values of \( U \) in the case of the charge and spin functions \( \text{Re} \chi^{\sigma}(\pm 2k_F, \omega) \) and \( \text{Re} \chi^{\sigma_{\nu}}(\pm 2k_F, \omega) \), respectively. In general the effects of electron correlations replace the logarithmic divergence by a Luttinger-liquid\(^\text{[20]}\) power-law divergence.

The use in Eqs. (15) and (16) of suitable dimensions of the fields (14) for the different correlation functions leads to the following results. For the transverse-spin function we find

\[
\text{Re} \chi^{\sigma_{\perp}}(\pm 2k_F, \omega) \propto \omega^{\varsigma_{\perp}},
\]

where the exponent is such that \(-1 < \varsigma_{\perp} < 0\) and given by

\[
\varsigma_{\perp} = -2 + 2 \sum_{\alpha} \left[ (\frac{\xi_{\alpha c}}{2})^2 + (\frac{\xi_{\alpha s}^0}{2})^2 \right].
\]

(18)

The divergences of the charge and spin functions were already studied in Ref. \([10]\) and are controlled by the same exponents. The expressions read

\[
\text{Re} \chi^{\sigma}(\pm 2k_F, \omega) \propto \omega^{\varsigma_{\sigma}},
\]

where the exponents are such that \(-\frac{1}{2} < \varsigma_{c\downarrow} < 0\) and \(-\frac{1}{2} < \varsigma_{c\uparrow} < 2\) and are given by

\[
\varsigma_{c\downarrow} = -2 + 2 \sum_{\alpha} (\xi_{\alpha c}^1)^2,
\]

\[
\varsigma_{c\uparrow} = -2 + 2 \sum_{\alpha} (\xi_{\alpha c}^1 - \xi_{\alpha s}^1)^2.
\]

(20)

(We emphasize that at the parameter-space points which correspond to \( \varsigma_{c\uparrow} = 0 \) the expressions of Eq. (19) are not valid and should be replaced by logarithmic \( \omega \) dependences.)

In the case of the singlet-superconductivity function we find

\[
\text{Re} \chi^{s\sigma}(\pm 2k_F, \omega) \propto \omega^{\varsigma_{s\sigma}},
\]

\[
\text{Re} \chi^{s\sigma}(\pm |k_F - k_{F1}|, \omega) \propto \omega^{\varsigma_{s\sigma}},
\]

where the exponents are such that \( 0 < \varsigma_{s\uparrow} < 1 \) and \( 0 < \varsigma_{s\downarrow} < 3 \) and are given by

\[
\varsigma_{s\uparrow} = -2 + 2 \sum_{\alpha} \left[ (\frac{\xi_{\alpha c}^1}{2})^2 + (\xi_{\alpha c}^0 + \frac{\xi_{\alpha s}^0}{2})^2 \right]
\]

\[
\varsigma_{s\downarrow} = -2 + 2 \sum_{\alpha} \left[ (\frac{\xi_{\alpha c}^1}{2} - \xi_{\alpha s}^1)^2 + (\xi_{\alpha c}^1 - \xi_{\alpha s}^0)^2 \right].
\]

(22)

Since for \( U > 0 \) these exponents are positive (as the figures of next section confirm) it follows that the electron-electron interactions remove the \( U = 0 \) singlet-superconductivity instability. The same occurs with the triplet \( \sigma \) superconductivity function which reads

\[
\text{Re} \chi^{ts\sigma}(0, \omega) \propto \omega^{\varsigma_{ts\sigma}},
\]

where the exponent is such that \( 0 < \varsigma_{ts\downarrow} < 2 \) and \( 0 < \varsigma_{ts\uparrow} < 1 \) and is given by

\[
\varsigma_{ts\downarrow} = -2 + 2 \sum_{\alpha} (\xi_{\alpha c}^0 + \frac{\xi_{\alpha s}^0}{2})^2,
\]

\[
\varsigma_{ts\uparrow} = -2 + 2 \sum_{\alpha} (\xi_{\alpha s}^0)^2.
\]

(24)

The \( \sigma \) one-electron Green function was already studied in Ref. \([11]\) and for small \( \omega \) and \( U > 0 \) is such that

\[
\text{Re} G_{\sigma}(\pm k_F, \omega) \propto \omega^{\varsigma_{\sigma}},
\]

where the exponent \(-1 < \varsigma_{\sigma} < -1/2\) is given by

\[
\varsigma_{\sigma} = -2 + \sum_{\alpha} \frac{1}{2} [(\xi_{\alpha c}^1 - \xi_{\alpha s}^1)^2 + (\xi_{\alpha s}^0)^2],
\]

\[
\varsigma_{\perp} = -2 + \sum_{\alpha} \frac{1}{2} [(\xi_{\alpha s}^0)^2 + (\xi_{\alpha c}^0 + \xi_{\alpha s}^0)^2].
\]

(26)

All exponents (18), (20), (22), (24), and (26) have a Luttinger-liquid non-classical character being \( U, n, \) and \( m \) dependent functions. They are fully controlled by the two-pseudoparticle forward-scattering collisions [see Eqs. (9) and (16)].
In this section we present and discuss the $U$, electronic density, and magnetic-field dependence of the exponents (18), (20), (22), (24), and (26) which control the quantum-problem instabilities. Our comparative study of these exponents allows finding the dominant instabilities in the different regions of parameter space.

In the Table we present various limiting values of the correlation-function exponents. There the $m \to n$ values are also valid when $H \to H_c$, $U \to U_c$, and $n \to n_c$, where $H_c$, $U_c$, and $n_c$ are the (interrelated) critical values for the onset of “ferromagnetism” (i.e., full spin polarization, $m \to n$) and are defined by Eq. (2) of Ref. [1]. Below we often use the normalized magnetic field $h = H/H_c$. In the limit $n \to 1$ (not presented in the Table) the exponent expressions also simplify because $\xi^0_{sc} = \xi^0_{ss} = 1$ and $\xi^0_{sc} = \xi^0_{ss} = 0$ and the remaining parameters change from $\xi^1_{sc} = 1/2$, $\xi^1_{ss} = 1/\sqrt{2}$, $\xi^0_{sc} = -1/\sqrt{2}$, and $\xi^0_{ss} = \sqrt{2}$ when $m \to 0$ to $\xi^1_{sc} = 0$, $\xi^1_{ss} = 1$, $\xi^0_{sc} = 0$, and $\xi^0_{ss} = 1$ as $m \to n$.

At finite values of $U$ the exponents $\varsigma_{\parallel s}$, $\varsigma_{\perp s}$, $\varsigma_{s \perp}$, and $\varsigma_{s \parallel}$ are always negative, whereas $\varsigma_{s \sigma}$ is only negative for some regions of parameter space. This exponent is negative in the limit of $m \to 0$ for all values of the density and $U$. For $0 < m < n$ it can have both negative and positive values. It is an increasing function of the magnetic field, becoming positive for a value of that field which depends on $n$ and $U$. In the limit $m \to n$ it becomes positive for all finite values of $U$ and all electronic densities $n$. For finite values of $U$ all the remaining exponents are positive. In the figures we use often the electronic density $n = 0.5$ because it is typical for many of the quasi-one-dimensional Beckgaard salts.

In Fig. 1 the exponents (a) $\varsigma_{s \perp}$, (b) $\varsigma_{s \parallel}$, and $\varsigma_{s \sigma}$, (c) $\varsigma_{\parallel}$, and (d) $\varsigma_{\sigma}$ and in Fig. 2 the superconductivity exponents (a) $\varsigma_{ss +}$, (b) $\varsigma_{ss -}$, (c) $\varsigma_{s t \perp}$, and (d) $\varsigma_{s t \parallel}$ are plotted vs $U$ for $n = 0.5$ and various values the of the magnetic field.

While the exponents $\varsigma_{s \perp}$ and $\varsigma_{s \parallel}$ are always a decreasing function of $U$ and the exponent $\varsigma_{s \sigma}$ a deacreasing and increasing function of $U$ for smaller and large values of $h$, respectively, all remaining above exponents are always an increasing function of $U$.

In Fig. 3 the exponents (a) $\varsigma_{s \perp}$, (b) $\varsigma_{s \parallel}$, and $\varsigma_{s \sigma}$, (c) $\varsigma_{\parallel}$, and (d) $\varsigma_{\sigma}$ and in Fig. 4 the superconductivity exponents (a) $\varsigma_{ss +}$, (b) $\varsigma_{ss -}$, (c) $\varsigma_{s t \perp}$, and (d) $\varsigma_{s t \parallel}$ are plotted vs the electronic density for $U = 10$ and three values of the magnetic field.

All the exponents are smooth functions of the density showing either a maximum or a minimum for an intermediate value of the density. While the exponents $\varsigma_{s \perp}$ and $\varsigma_{s \parallel}$ show a maximum value, the Green-function and superconductivity exponents show a minimum value.

The magnetic-field dependences of the correlation-function exponents are illustrated in Fig. 5 and 6 where these parameters are plotted as functions of $h$ for density $n = 0.5$ and various values of $U$. While the exponents $\varsigma_{s \perp}$ and $\varsigma_{s \parallel}$ are always a decreasing function of the magnetic field, all the remaining exponents plotted in the figures are in general an increasing function of $h$.

We emphasize that the exponent $\varsigma_{\sigma}$ is always smaller than or equal to the exponent $\varsigma_{\parallel}$. In order to understand the parameter-space critical line defined by the equation $\varsigma_{s \perp} = \varsigma_{\parallel}$, the exponents $\varsigma_{s \perp}$ and $\varsigma_{\parallel}$ associated with the two dominant instabilities are plotted in Figs. 7 (a)-(d) as functions of the magnetic field $h$ and for several values of the electronic density $n$ and onsite interaction (a) $U = 1$, (b) $U = 3$, (c) $U = 5$, and (d) $U = 20$. At zero magnetic field $\varsigma_{s \perp} = \varsigma_{\parallel}$, $\varsigma_{\sigma} < \varsigma_{s \perp}$, and the exponent $\varsigma_{s \perp}$ equals the exponents $\varsigma_{s \parallel}$ and $\varsigma_{s \sigma}$ which are equal in that limit. (The latter were already studied in Ref. [10].) Therefore, adding or removing of electrons is in this case the dominant instability. (In the case of the weakly coupled-chain Hubbard system this leads to a dominance for interchain electron hopping.) Also at small values of $U$ that instability remains dominant for all values of the electronic density and magnetic field.

However, the magnetic field removes the $SU(2)$ spin symmetry and for intermediate and large $U$ values induces a dominant character for the spin-flip excitation over adding or removing of single electrons. This occurs for $U > U^*_c$, where $U^*_c = U^*_c(n,h)$ is a critical value. Depending on the parameters that we keep constant, we can also define critical values $n^*_c(U,h)$ and $h^*_c(n,U)$. The critical values are obtained by solution of the above equation, $\varsigma_{\sigma} = \varsigma_{s \perp}$, upon suitable boundary conditions. Therefore, they correspond to the points of Figs. 7 (a)-(d) where the two exponent lines cross. As these figures confirm, $U^*_c$ changes from

$$ U^*_c = \frac{4t \sin(\pi n)}{\tan(\frac{\pi}{4}[\sqrt{2} - 1])}, $$

where $h = 0$ if $U^*_c = \infty$.

In Figs. 8 and 9 the functions $U^*_c(n)$ and $h^*_c(n)$ are presented for constant values of $h$ and $U$, respectively. These define the critical lines which divide the regions of parameter space where $\uparrow$-electron adding and removing on the one hand and spin-transverse SDW on the other hand are the dominant instability.

At finite values of the magnetic field $h < 1$, $\uparrow$-electron adding or removing remains dominant for $U < U^*_c$ (see Fig. 8). This region is larger for intermediate densities and for small values of the magnetic field and smaller for small
densities and densities close to half filling and for large values of \( h \). Therefore, the function \( U_0(n) \) plotted in Fig. 8 shows a maximum at intermediate electronic densities and is larger at smaller values of the magnetic field \( h \).

On the other hand, at intermediate and large values of \( U \), \( \uparrow \)-electron adding or removing remains dominant for \( h < h^*_c \) (see Fig. 9). This region is larger for intermediate densities and for small values of \( U \) and smaller for small densities and densities close to \( n = 1 \) and for large values of \( U \). It follows that the function \( h^*_c(n) \) plotted in Fig. 9 shows a maximum for intermediate values of the electronic density and is larger for smaller values of \( U \).

V. THE METAL - INSULATOR HALF-FILLING AND FULLY-POLARIZED FERROMAGNETIC TRANSITIONS

In this section we consider only these correlation functions whose exponents are negative for finite \( U \) and omit the study of the superconductivity correlation functions.

When we change the chemical potential \( \mu \) from \( \mu > \frac{\Delta_M}{2} \) to \( \mu < \frac{\Delta_M}{2} \), where \( \Delta_M \) is the half-filling Mott-Hubbard gap [6] which at finite magnetic field was studied in Ref. [13] (and thus the electronic density changes from \( n < 1 \) to \( n = 1 \)), there occurs in the many-electron system a metal - insulator transition [6]. Also when we change the magnetic field from \( H < H_c \) to \( H > H_c \) (and thus the spin density \( m \) changes from \( m < n \) to \( m = n \)), there occurs a transition to fully-polarized ferromagnetism. In this section we find that i)- some exponents show a discontinuity because they have different values in both sides of the above transitions and ii)- these transitions remove some of the low-energy power-law divergences in the correlation functions but in some cases these divergences are shifted to finite energies.

At half filling both the charge and the one-particle excitations associated with creation of one electron cost a minimal energy \( \Delta_M \). Therefore, both the charge response function and the one-electron Green function vanish for positive energy, \( \omega < \Delta_M \).

On the other hand, there are gapless spin excitations which lead to low-energy power-law behavior for the spin response functions. However, the half-filling metal - insulator transition leads to discontinuities in the corresponding exponents. We find

\[
\text{Re}\chi^\sigma \left( \pm \pi, \omega \right) \propto \omega^{\varsigma_{\sigma \perp}}, \quad \text{Re}\chi^\sigma \left( \pm 2k_{F\sigma}, \omega \right) \propto \omega^{\varsigma_{\sigma\uparrow\downarrow}},
\]

where the exponents \( \varsigma_{s\perp}, \varsigma_{cs\downarrow}, \text{ and } \varsigma_{cs\uparrow} \) are for \( n \rightarrow 1 \) and at \( n = 1 \) (and for \( \mu \rightarrow \frac{\Delta_M}{2} \) from above and for \( \mu < \frac{\Delta_M}{2} \)) given by

\[
\varsigma_{s\perp} = -2\left[\frac{3}{4} - \frac{1}{2\xi_{ss}^2}\right], \quad \varsigma_{cs\downarrow} = -2\left[1 - (\xi^1_{cs\downarrow})^2 - (\xi^1_{ss})^2\right], \quad \varsigma_{cs\uparrow} = -2\left[1 - (1 - \xi^1_{cs\downarrow})^2 - (\xi^1_{ss})^2\right],
\]

and

\[
\varsigma_{s\perp} = -2\left[1 - \frac{1}{2\xi^1_{ss}}\right], \quad \varsigma_{cs\downarrow} = -2\left[1 - (\xi^1_{ss})^2\right], \quad \varsigma_{cs\uparrow} = -2\left[1 - (\xi^1_{ss})^2\right],
\]

respectively. In the limit \( m \rightarrow 0 \) these exponents read

\[
\varsigma_{s\perp} = \varsigma_{cs\downarrow} = \varsigma_{cs\uparrow} = -1/2,
\]

for \( n \rightarrow 1 \) (and for \( \mu \rightarrow \frac{\Delta_M}{2} \) from above) and

\[
\varsigma_{s\perp} = \varsigma_{cs\downarrow} = \varsigma_{cs\uparrow} = -1,
\]

for \( n = 1 \) (and for \( \mu < \frac{\Delta_M}{2} \)).

On the other hand, in the limit \( m \rightarrow n \) these exponents are given by

\[
\varsigma_{s\perp} = -1, \quad \varsigma_{cs\downarrow} = 0, \quad \varsigma_{cs\uparrow} = 2,
\]

for \( n \rightarrow 1 \) (and for \( \mu \rightarrow \frac{\Delta_M}{2} \) from above) and

\[
\varsigma_{s\perp} = -3/2, \quad \varsigma_{cs\downarrow} = 0, \quad \varsigma_{cs\uparrow} = 0,
\]

for \( n \rightarrow 1 \) (and for \( \mu \rightarrow \frac{\Delta_M}{2} \) from above) and

\[
\varsigma_{s\perp} = 0, \quad \varsigma_{cs\downarrow} = 0, \quad \varsigma_{cs\uparrow} = 0.
\]
for \( n = 1 \) (and for \( \mu < \frac{\Delta_{MH}}{\mu} \)). Obviously, in the cases where the exponents vanish expressions (28) should be replaced by a logarithmic function. Note that the logarithmic behavior is only reached in the limit \( m \to n \) since otherwise the correspondent exponents are different from zero.

In the case of fully-polarized ferromagnetism there are no spin-down electrons in the system. In this case a spin flip costs a minimal energy given by

\[
\Delta_F = 2\mu_0[H - H_c],
\]

where \( H > H_c \) and \( \Delta_F \) is the fully-polarized ferromagnetism gap. Therefore, the transverse-spin response function and the down-spin one-electron Green function vanish for positive energy, \( \omega < \Delta_F \). Also, there is no \( k = k_{Fz} = 0 \) divergence in \( \text{Re} \chi^\sigma(k, \omega) \) and \( \text{Re} \chi^\sigma_z(k, \omega) \).

The charge response function, spin response function \( \text{Re} \chi^\sigma_z \), and up-spin one-electron Green function all have a low-energy power-law structure in both sides of the ferromagnetic transition. That transition leads to discontinuities in the exponents of these correlation functions, which read

\[
\text{Re} \chi^\sigma(\pm 2k_F, \omega) \propto \omega^{\varsigma_{\sigma \pm}}, \quad \text{Re} \chi^\sigma_z(\pm 2k_F, \omega) \propto \omega^{\varsigma_{\sigma z}}, \quad \text{Re} G^\uparrow(\pm 2k_F, \omega) \propto \omega^{\varsigma_{\uparrow}},
\]

where for \( m \to n \) (and \( H \to H_c \) from below) the exponents are given by

\[
\varsigma_{\sigma \pm} = 2[1 - \eta_0]^2, \quad \varsigma_{\uparrow} = -1 + \frac{1}{2}[1 - \eta_0]^2,
\]

and

\[
\eta_0 = \left(\frac{2}{\pi}\right)\tan^{-1}\left(\frac{4t\sin(\pi n)}{U}\right).
\]

On the other hand, at \( m = n \) (and \( H > H_c \)) the above charge and spin response functions have a non-interacting logarithmic behavior

\[
\text{Re} \chi^\sigma(\pm 2k_F, \omega) \propto -\ln(\omega), \quad \text{Re} \chi^\sigma_z(\pm 2k_F, \omega) \propto -\ln(\omega),
\]

because

\[
\varsigma_{\sigma \pm} = 0,
\]

and the up-spin Green function exponent reads

\[
\varsigma_{\uparrow} = -1.
\]

Based on the occurrence of only zero-momentum pseudoparticle forward scattering and on the pseudoparticle-number conservation laws [22] we find that in the case of half filling there are power-law divergences at finite energies, for instance, in the one-electron Green function. Finite-energy power-law divergences also occur, for example, in the transverse-spin response function in the fully-polarized ferromagnetic phase. (The above property of the pseudoparticle interactions is valid at all energy scales [22]. A complete study of the quantum problem for all energies involves an infinite number of pseudoparticle branches – fortunately, the finite-energy divergences we study in this paper only involve the \( c \) and \( s \) branches.)

By generalizing the pseudoparticle operator basis to half filling and the fully-polarized phase, we find an equivalent low-energy pseudoparticle theory where some of the low-energy \( \omega \) divergences of Sec. III are replaced by low-energy \( (\omega - \Delta_{MH}) \) and \( (\omega - \Delta_F) \), respectively, divergences. At these energies the half-filling and fully-polarized ferromagnetic pseudoparticle theories involve the same parameters as the general \( 0 < n < 1 \) problem studied in previous sections. For instance, for \( (\omega - \Delta_{MH}) > 0 \) and \( (\omega - \Delta_F) > 0 \) all pseudoparticle-Hamiltonian parameters are obtained by considering the limits \( n \to 1 \) and \( m \to n \), respectively, of the corresponding \( 0 < n < 1 \) and \( 0 < m < n \) expressions (and not the corresponding \( n = 1 \) and \( m = n \) expressions).

In the case of half filling the lack of charge gapless excitations and the associate opening of the Mott-Hubbard gap leads to the following expression for the one-electron Green function

\[
\text{Re} G^\sigma(\pm k_F, \omega) \propto [\omega - \Delta_{MH}]^{\varsigma_{\sigma}},
\]

where the exponent \( \varsigma_{\sigma} \) is given by

9
\[ \varsigma_\uparrow = -2 + \frac{1}{2} [(1 - \xi_{cs}^1)^2 + 1 + (\xi_{as}^1)^2 + (\xi_{ac}^0)^2], \]  
(43)

\[ \varsigma_\downarrow = -2 + \frac{1}{2} [(\xi_{cs}^1)^2 + 1 + (\xi_{as}^1)^2 + (\xi_{ac}^0 + \xi_{sc}^1)^2], \]  
(44)

and \((\omega - \Delta_{MH})\) is small and positive.

Both in the limits \(m \to 0\) and \(m \to n\) the exponents (43) and (44) for up-spin and down-spin electrons, respectively, are equal and read

\[ \varsigma_\uparrow = \varsigma_\downarrow = -\frac{7}{8}, \]  
(45)

and

\[ \varsigma_\uparrow = \varsigma_\downarrow = -\frac{1}{2}, \]  
(46)

respectively.

On the other hand, in the fully-polarized ferromagnetic phase there are no gapless excitations in the transverse-spin channel. We find the following expression

\[ \text{Re} \chi_\sigma^\perp (\pm 2k_F, \omega) \propto [\omega - \Delta_F]^{\varsigma_\sigma}, \]  
(47)

where the exponent is given by

\[ \varsigma_\perp = -1 + (\eta_0)^2. \]  
(48)

Obviously, this expression refers to small positive values of the energy \((\omega - \Delta_F)\) and is not valid in the limit of \(U \to 0\) where \(\varsigma_\perp \to 0\).

Equations (42) – (48) reveal that the opening of the Mott-Hubbard and ferromagnetic gaps associated with the electronic-correlation and magnetic effects, respectively, moves some of the \(\omega \to 0\) instabilities to \(\omega \to \Delta_{MH}\) and \(\omega \to \Delta_F\), respectively. These divergences occur at the same momenta values than the corresponding low-frequency \(0 < n < 1\) and \(0 < m < n\) divergences.

The finite-frequency expressions (42) – (48) cannot be derived by CFT. Their validity follows from the purely non dissipative character of the pseudoparticle interactions which survives at all energy scales of the quantum problem \([21]\).

**VI. COUPLED CHAINS**

The exponents calculated in the preceding sections give the scaling dimension of the corresponding operators, in the presence of the interaction. Once they are known, we can analyze the effects of perturbations added to (1) which can be written as a sum of terms containing the operators analyzed above. For simplicity, we first examine the case in which these operators couple two different chains. Let us write \(\chi^\sigma(x, t) = \langle O^\sigma(x, t) O^\sigma \rangle\). The Fourier transform of \(\chi^\sigma\) goes at low frequencies as \(\omega^{\varsigma_\sigma}\). An interchain coupling term which involves the operator \(O^\sigma\) can be written as

\[ \mathcal{H}_{1,2} = J_\sigma \int dx O^\sigma_1(x) O^\sigma_2(x), \]  
(49)

where the labels 1 and 2 refer to the two different chains. We now add this term to (1) and study the scaling of \(J_\sigma\) as the cutoff is reduced. We define the dimensionless coupling \(\tilde{J}_\sigma = J_\sigma / \omega_c\), where \(\omega_c\) is an upper cutoff below which the approximation of \(\chi^\sigma\) by a power law is approximately valid. Then, neglecting the influence of the other interchain couplings we find \([23, 24]\)

\[ \omega_c \frac{\partial \tilde{J}_\sigma}{\partial \omega_c} = (-1 - 2\varsigma_\sigma) \tilde{J}_\sigma. \]  
(50)

Thus, the relevance of \(O^\sigma\) depends on whether \(\varsigma_\sigma\) is greater or less than \(-1/2\). When \(\varsigma_\sigma < -1/2\), \(O^\sigma\) is a relevant operator which grows upon scaling. The Renormalization-Group scheme used to derive \([19]\) breaks down when \(\tilde{J}_\sigma \sim 1\). This defines a crossover scale,
Below this crossover the coupling \( J_\theta \) dominates the physics of the coupled-chain system and a new method needs to be applied. In the case of many-coupled chains, forming a 2D or 3D array, mean-field approaches are, most likely, adequate. Note that this analysis can be extended, in a straightforward way, to a many-chain system. This crossover has been studied for a variety of interactions in Ref. 27.

The most commonly studied perturbation has been the interchain hopping. Then the required exponents are \( \varsigma_\uparrow \) and \( \varsigma_\downarrow \) which are given in Eq. (26). These operators are always relevant except as \( H \to H_c \). The fixed point which describes an isolated Luttinger liquid is destabilized and interchain phase coherence develops. Note, however, that as in the presence of the magnetic field \( \varsigma_\uparrow \neq \varsigma_\downarrow \), the interchain hopping becomes spin dependent at low energies. This effect should influence the transport properties. It would be interesting to check this fact experimentally.

So far, he have ignored the influence of the possible couplings among themselves. To leading order in the strength of the couplings, this is, at least, a second-order process, while the effect which leads to \( \varsigma \) is first order. It may happen, however, that new couplings are generated, which were zero at the initial stages of the renormalization process, i. e., in the bare Hamiltonian. When these new couplings are more relevant than the ones initially present, they may change drastically the physics at low energies.

The coupling most likely to give rise to new interactions is the interchain hopping. It involves two electronic operators, and, to next order, it can generate four fermion operators, like interchain magnetic exchange, or Cooper pair hopping. In the following, we analyze the process by which an exchange term is generated if, initially, interchain hopping is present. We consider the transverse-spin exchange coupling, as this is the other most relevant operator in the presence of electron-electron repulsion, besides the hopping. From the calculations presented in the preceding sections, it may become more relevant than the hopping itself in the presence of a magnetic field. The full scaling equation for this coupling, \( J_{s,\perp} \), is

\[
\frac{\partial \tilde{J}_{s,\perp}}{\partial \omega_c} = (-1 - 2 \varsigma_{s,\perp}) \tilde{J}_{s,\perp} - \Gamma \tilde{J}_t \tilde{J}_\downarrow,
\]

where \( \Gamma \) is a constant of order unity. The second term in the right-hand side stands for the fact that a simultaneous hopping of an up-spin electron in one direction and a down-spin electron in the opposite generates a spin-exchange process.

We can integrate Eqs. (50,52) with the initial conditions: \( \tilde{J}_t = \tilde{J}_\downarrow = \tilde{0} \) and \( \tilde{J}_{s,\perp} = 0 \). The scale-dependent coupling \( \tilde{J}_{s,\perp} \) depends on \( l = \log(\omega/\omega_c) \) as

\[
\tilde{J}_{s,\perp}(l) \approx \frac{\Gamma \tilde{J}_t \tilde{J}_\downarrow (e^{(2+2\varsigma_\uparrow+2\varsigma_\downarrow)l} - e^{(1+2\varsigma_{s,\perp})l})}{1 + 2\varsigma_\uparrow + 2\varsigma_\downarrow - 2\varsigma_{s,\perp}}.
\]

The low-energy behavior of the coupled system shows two different regimes, depending on the value of the denominator in the right-hand side term in Eq. (53):

i) If the denominator is of order unity, the interchain hopping diverges first, at a scale given by Eq. (54). Below this scale, a system of coupled chains resembles a 2D or 3D anisotropic Fermi liquid, with a residual exchange interaction. A mean-field analysis of this model leads to the possibility of a SDW ground state.

ii) When the denominator is much less than one, the induced transverse-spin exchange grows faster than the hopping term. Then, before coherence between the single-electron wavefunction at the different chains is established, the exchange-term becomes of the order of the cutoff. The system probably develops a spin gap, as in the previous case, but the transport properties should be drastically different.

Note, finally, that this analysis is valid for arbitrary strength of the repulsion, \( U \), and weak interchain couplings. It would be interesting to connect this picture to the behavior obtained in the opposite limit of strong-interchain hopping and weak-interchain coupling, studied recently 28,29.

\[\epsilon_\theta = J_\theta \left( \frac{J_0}{\omega_c} \right)^{-1 + \frac{1}{2 \varsigma_\downarrow}}.\]

VII. CONCLUDING REMARKS

In this paper we have presented a comparative study of the Hubbard chain instabilities. In contrast to the one-electron 1D quantum problem, the many-electron instabilities are not associated in general with logarithmic divergences in the low-frequency response functions. Instead these functions show Luttinger-liquid power laws which are controlled by non-classical exponents. Moreover, the singlet and triplet superconductivity instabilities are removed by the effects of the electron correlations.
We have studied in detail the $U$, electronic density $n$, and magnetic-field $H$ dependence of all the exponents which are associated with instabilities of the many-electron quantum problem. The interplay of the magnetic field and electron correlations renders the transverse-spin SDW instability at momenta $\pm 2k_F$ dominant for large regions of parameter space. By removing the $SU(2)$ spin symmetry the magnetic field also makes the exponents associated with the spin and transverse-spin response functions different.

We went beyond CFT and characterized divergences which occur at finite energy values in some correlation functions in the cases of the half-filling and fully-polarized ferromagnetic phases. We have also studied the discontinuities which occur in some power-law exponents which have different values in both sides of the transitions to or from the above phases.

Our exact results might be useful for the study of the electronic instabilities in real quasi-one dimensional materials. For instance, transverse SDW’s phases were often observed in real low-dimensional materials $[22,24]$. Following the discussion of Sec. VI, elsewhere we will use the present instabilities of the 1D problem to construct a suitable effective Hamiltonian for the description of a system of weakly coupled Hubbard chains.

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TABLE – Limiting values of the exponents whose expressions are defined in Eqs. (18), (20), (22), (24), and (26).

Here the \( m \to 0 \) parameter \( \xi_0 = \xi_{cc}^1 \) changes from \( \xi_0 = \sqrt{2} \) at \( U = 0 \) to \( \xi_0 = 1 \) as \( U \to \infty \) and the parameter \( \eta_0 \) is given in Eq. (38).

| \( U \to 0 \) | \( H \to 0 \) | \( m \to 0, U \to \infty \) | \( m \to n \) | \( H \to H_c, U \to \infty \) |
|------------|-----------|-----------------|-------|-----------------|
| \( s_{s+} \) | 0         | \(-1 + \frac{\xi_0^2}{2} \) | \(-\frac{1}{2} \) | \(-1 + (\eta_0)^2 \) | \(-1 \) |
| \( s_{s-} \) | 0         | \(-1 + \frac{\xi_0^2}{2} \) | \(-\frac{1}{2} \) | 0 | 0 |
| \( s_{c+} \) | 0         | \(-1 + \frac{\xi_0^2}{2} \) | \(-\frac{1}{2} \) | 2(1 - \eta_0)^2 | +2 |
| \( s_{c-} \) | 0         | \(-1 + \frac{\xi_0^2}{2} \) | \(-\frac{1}{2} \) | \( [1 - \eta_0]^2 \) | +1 |
| \( s_{s+} \) | 0         | \(-1 + \frac{\xi_0^2}{2} \) | \(-\frac{1}{2} \) | \(-1 + 4(1 - \frac{\eta_0}{2})^2 \) | +3 |
| \( s_{s-} \) | 0         | \(-1 + \frac{\xi_0^2}{2} \) | \(-\frac{1}{2} \) | \( [1 - \eta_0]^2 \) | +2 |
| \( s_{s+} \) | 0         | \(-1 + \frac{\xi_0^2}{2} \) | \(-\frac{1}{2} \) | 0 | 0 |
| \( s_{s-} \) | 0         | \(-1 + \frac{\xi_0^2}{2} \) | \(-\frac{1}{2} \) | 0 | 0 |
| \( \xi_{+} \) | -1        | \(-2 + \frac{1}{2} \sqrt{2} + \frac{1}{2} \frac{\xi_0}{\eta_0}^2 \) | \(-\frac{3}{2} \) | \(-1 + \frac{\xi_0}{2}(1 - \eta_0)^2 \) | \(-\frac{1}{2} \) |
| \( \xi_{-} \) | -1        | \(-2 + \frac{1}{2} \sqrt{2} + \frac{1}{2} \frac{\xi_0}{\eta_0}^2 \) | \(-\frac{3}{2} \) | \(-1 + \frac{\xi_0}{2}(1 - \eta_0)^2 \) | \(-\frac{1}{2} \) |
FIG. 1. The exponents (a) $\varsigma_{s\perp}$, (b) $\varsigma_{cs\downarrow}$ and $\varsigma_{cs\uparrow}$, (c) $\varsigma_{c\downarrow}$, and (d) $\varsigma_{\uparrow}$ defined by Eqs. (18), (20), and (26) as functions of $U$ for $n = 0.5$ and for values of the magnetic field $h = 0.3$ (full line), $h = 0.6$ (dashed line), and $h = 0.9$ (dashed-dotted line).

FIG. 2. The exponents (a) $\varsigma_{ss\downarrow}$, (b) $\varsigma_{ss\uparrow}$, (c) $\varsigma_{ts\downarrow}$, and (d) $\varsigma_{ts\uparrow}$ defined by Eqs. (22) and (24) as functions of $U$ for $n = 0.5$ and for values of the magnetic field $h = 0.3$ (full line), $h = 0.6$ (dashed line), and $h = 0.9$ (dashed-dotted line).

FIG. 3. The exponents (a) $\varsigma_{s\perp}$, (b) $\varsigma_{cs\downarrow}$ and $\varsigma_{cs\uparrow}$, (c) $\varsigma_{\downarrow}$, and (d) $\varsigma_{\uparrow}$ defined by Eqs. (18), (20), and (26) as functions of the electronic density $n$ for $U = 10$ and for values of the magnetic field $h = 0.1$ (full line), $h = 0.2$ (dashed line), and $h = 0.3$ (dashed-dotted line).

FIG. 4. The exponents (a) $\varsigma_{ss\downarrow}$, (b) $\varsigma_{ss\uparrow}$, (c) $\varsigma_{ts\downarrow}$, and (d) $\varsigma_{ts\uparrow}$ defined by Eqs. (22) and (24) as functions of the electronic density $n$ for $U = 10$ and for values of the magnetic field $h = 0.1$ (full line), $h = 0.2$ (dashed line), and $h = 0.3$ (dashed-dotted line).

FIG. 5. The exponents $\varsigma_{s\perp}$ and $\varsigma_{\uparrow}$ associated with the two dominant instabilities as functions of the magnetic field $h$ and for several values of the electronic density $n$ for (a) $U = 1$, (b) $U = 3$, (c) $U = 5$, and (d) $U = 20$. Comparison of the values of both exponents shows that for intermediate and large values of $U$ the transverse SDW becomes dominant for magnetic fields $h$ larger than about 0.75 to 0.8.

FIG. 6. The critical line for $U_c^*(n)$ in the $U - n$ plane for values of the magnetic field $h = 0.8$ (full line) and $h = 0.9$ (dashed line).

FIG. 7. The critical line for $h_c^*(n)$ in the $h - n$ plane for values $U = 10$ (full line) and $U = 20$ (dashed line).