ON GROUP THEORETICAL HOPF ALGEBRAS AND EXACT FACTORIZATIONS OF FINITE GROUPS

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Abstract. We show that a semisimple Hopf algebra $A$ is group theoretical if and only if its Drinfeld double is a twisting of the Dijkgraaf-Pasquier-Roche quasi-Hopf algebra $D^\omega(\Sigma)$, for some finite group $\Sigma$ and some $\omega \in Z^3(\Sigma, k^\times)$. We show that semisimple Hopf algebras obtained as bicrossed products from an exact factorization of a finite group $\Sigma$ are group theoretical. We also describe their Drinfeld double as a twisting of $D^\omega(\Sigma)$, for an appropriate 3-cocycle $\omega$ coming from the Kac exact sequence.

1. Introduction

We shall work over an algebraically closed field $k$ of characteristic zero. Let $\Sigma$ be a finite group, and let $\omega \in Z^3(\Sigma, k^\times)$. Consider the category $\text{Vec}^\Sigma_\omega$ of $\Sigma$-graded vector spaces, with associativity constraint given by $\omega$. In the paper [10], for every pair of subgroups $F$ and $G$ of $\Sigma$, endowed with 2-cocycles $\alpha \in Z^2(F, k^\times)$ and $\beta \in Z^2(G, k^\times)$, satisfying certain conditions, a semisimple Hopf algebra $A = A_{\alpha, \beta}^\Sigma(\omega, F, G)$ is associated, in such a way that the category $\text{Rep} A$ is monoidally equivalent to the category $C$ of $k_\alpha F$-bimodules in $\text{Vec}^\Sigma_\omega$. Paraphrasing the terminology introduced by Etingof, Nikshych and Ostrik in [5], we shall use the name group theoretical to refer to a Hopf algebra arising from this construction.

Question 1.1. Is every semisimple Hopf algebra over $k$ group theoretical?

Remark 1.1. The answer to the analogous question for finite dimensional semisimple quasi-Hopf algebras is negative, as Remark 8.48 in [5] shows.

In this paper we shall prove the following characterization of group theoretical Hopf algebras. See Section 2.

Theorem 1.2. Let $A$ be a semisimple Hopf algebra over $k$. The following statements are equivalent:

(i) $A$ is group theoretical.

(ii) there exist a finite group $\Sigma$ and a 3-cocycle $\omega \in Z^3(\Sigma, k^\times)$ such that $D(A)$ is twist equivalent to $D^\omega(\Sigma)$.

Here, $D^\omega(\Sigma)$ is the quasi-Hopf algebra of Dijkgraaf, Pasquier and Roche [3]. Note that the group $\Sigma$ is not uniquely determined. The proof of Theorem 1.2 relies on a description of the category $\text{Rep} D^\omega(\Sigma)$ given in [8] and a result of Schauenburg [11] on the center of certain monoidal categories.

The theorem implies that a group theoretical Hopf algebra appears as a Hopf subalgebra of a Hopf algebra which can be constructed from group algebras and their duals, by means of the operations of taking bismash products, associators and twists; see [1].

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The following theorem will be proved in Section 4. Part (ii) generalizes the result in [2, Section 5].

**Theorem 1.3.** Let \( \Sigma = FG \) be an exact factorization of a finite group \( \Sigma \). Suppose that \( A \) is a Hopf algebra fitting into the abelian extension

\[
1 \to k^G \to A \to kF \to 1,
\]

associated to this factorization. Then we have:

(i) \( A \) is group theoretical.

(ii) Let \( [\tau, \sigma] \) denote the element of \( \text{Opext}(k^G, kF) \) corresponding to the extension \((1.2)\). Then \( \mathcal{D}(A) \cong \mathcal{D}(\omega)(\Sigma) \phi \), for some invertible \( \phi \in \mathcal{D}(\omega)(\Sigma) \otimes \mathcal{D}(\omega)(\Sigma) \), where the class of \( \omega \) is the 3-cocycle associated to \( [\tau, \sigma] \) in the Kac exact sequence [6].

See 3.2 for a discussion on \( \omega \). Observe that part (i) implies that all Hopf subalgebras and quotients of \( A \), \( A^* \) and their twistings are also group-theoretical; see [5]. The proof of Theorem 1.3 is done by explicitly constructing a monoidal equivalence \( \text{Rep} A \sim F(\text{Vec}_\Sigma^\omega) \). This equivalence is a special case of a result of Schauenburg; see [12, Theorem 3.3.5].

As a corollary, we obtain that a semisimple Hopf algebra whose category of representations is isomorphic to one of the categories described by Tambara and Yamagami in [16], is always group theoretical. In other words, these categories are group theoretical whenever they possess a fiber functor to the category of \( k \)-vector spaces. A special case of this fact appears in [5, Remark 8.48].

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## 2. Characterization via Drinfeld doubles

### 2.1. We first review the construction in [10, Section 3]. This construction, and its relationship with the structure of semisimple Hopf algebras, has also been explained by Ocneanu.

Let \( \Sigma \) be a finite group, and let \( \omega \in Z^3(\Sigma, k^\times) \) be a normalized 3-cocycle. Consider the category \( \text{Vec}_\Sigma^\omega \) of \( \Sigma \)-graded vector spaces, with associativity constraint given by \( \omega \): explicitly, for any three objects \( U, U', U'' \) of \( \text{Vec}_\Sigma^\omega \), we have

\[
a_{U, U', U''} : (U \otimes U') \otimes U'' \to U \otimes (U' \otimes U''),
\]

given by

\[
a_{U, U', U''}(u \otimes u' \otimes u'') = \omega(||u||, ||u'||, ||u''||) u \otimes (u' \otimes u''),
\]

on homogeneous elements \( u \in U, u' \in U', u'' \in U'' \), where we use the symbol \( || \) to denote the corresponding degree of homogeneity. In other words, \( \text{Vec}_\Sigma^\omega \) is the category of representations of the quasi-Hopf algebra \( k^\Sigma^\omega \), with associator \( \omega \in (k^\Sigma)^{\otimes 3} \).

Let also \( F \) and \( G \) be subgroups of \( \Sigma \), endowed with 2-cocycles \( \alpha \in Z^2(F, k^\times) \) and \( \beta \in Z^2(G, k^\times) \), such that the following conditions are satisfied:

- The classes \( \omega|_F \) and \( \omega|_G \) are both trivial; \hspace{1cm} (2.2)
- \( \Sigma = FG \); \hspace{1cm} (2.3)
- The class \( \alpha|_{F \cap G} \beta^{-1}|_{F \cap G} \) is non-degenerate. \hspace{1cm} (2.4)

Then there is an associated semisimple Hopf algebra \( A = A_{\alpha, \beta}(\omega, F, G) \), such that the category \( \text{Rep} A \) is monoidally equivalent to the semisimple monoidal category \( \mathcal{C} = C(\Sigma, \omega, F, \alpha) \) of \( k_\alpha F \)-bimodules in \( \text{Vec}_\omega^\Sigma \). By [10, Corollary 3.1], equivalence classes subgroups \( G \) of \( \Sigma \) satisfying \((2.2), (2.3)\) and \( (2.4) \), classify fiber functors \( \mathcal{C} \to \text{Vec}_k \); these correspond to the distinct Hopf algebras \( A \).
The categories of the form $\mathcal{C}(\Sigma, \omega, F, \alpha)$ are called group theoretical in $\mathfrak{3}$. This motivates the following definition.

**Definition 2.1.** We shall say that a semisimple Hopf algebra $A$ is group theoretical if the category $\text{Rep} A$ is group theoretical.

A semisimple Hopf algebra $A'$ is twist equivalent to $A$ if and only if $\text{Rep} A'$ is equivalent to $\text{Rep} A$; thus, if $A'$ is twist equivalent to $A$, then $A'$ is group theoretical if and only if $A$ is; indeed twisting the comultiplication in $A$ corresponds to changing the fiber functor $\mathcal{C} \to \text{Vec}_k$ and thus to changing the data $(G, \beta)$. It follows from [5, 8.8] that duals, opposites, Hopf subalgebras, quotient Hopf algebras, and tensor products of group theoretical Hopf algebras are also group theoretical. Also, by [5, Remark 8.47], if $(A)$ is group theoretical, then so is $A$; the converse is also true, since $(A)$ is a 2-cocycle twist of $(A^*)^{\text{cop}} \otimes A$.

2.2. **Proof of Theorem 1.2.** Recall from $\mathfrak{3}$, that as a braided monoidal category, $\text{Rep} \, D^\omega(\Sigma)$ is isomorphic to the Drinfeld center of the category $\text{Vec}_k^\Sigma$, $Z(\text{Vec}_k^\Sigma)$.

(ii) $\implies$ (i). The quasi-Hopf algebra $D^\omega(\Sigma)$ is group theoretical; indeed, from the proof of Theorem 3.2 in $[10]$, $\text{Rep} \, D^\omega(\Sigma)$ is equivalent to $\mathcal{C}(\Sigma \times \Sigma, \tilde{\omega}, \Delta(\Sigma), 1)$. Therefore, the assumption (ii) implies that $(D)$ is group theoretical. Hence $A$ also is, by $\mathfrak{3}$, Remark 8.47.

(i) $\implies$ (ii). Suppose that $A = A^\Sigma_{\alpha, \beta}(\omega, F, G)$ is group theoretical. By definition, there is an equivalence of monoidal categories $\text{Rep} \, A \sim_B (\text{Vec}_k^\Sigma)_B$, where $B = k_\alpha F$. Therefore, the Drinfeld centers of these categories are equivalent.

In view of $\mathfrak{1}$, the center of the category $\mathcal{B} = (\text{Vec}_k^\Sigma)_B$ is equivalent to the center of $\text{Vec}_k^\Sigma$. This implies that $\text{Rep} \, D(A) \sim Z(\text{Vec}_k^\Sigma) \sim \text{Rep} \, D^\omega(\Sigma)$.

It follows from $\mathfrak{3}$, Theorem 6.1 that there exists an invertible element $\phi \in D^\omega(\Sigma) \otimes D^\omega(\Sigma)$ such that $D(A) \simeq D^\omega(\Sigma)_{\phi}$. This finishes the proof of the theorem.

One may also use the results in $\mathfrak{10}$ instead of $\mathfrak{1}$ in the proof of the implication (i) $\implies$ (ii): we have $\mathcal{C} = (\mathcal{C}(\Sigma, \omega, F, \alpha))^*$ with respect to the indecomposable module category of $k_\alpha F$-modules in $\text{Vec}_k^\Sigma$. By $\mathfrak{1}$, Corollary 2.1, the centers $Z(\mathcal{C})$ and $Z(\mathcal{C}^*)$ are equivalent.

**Remark 2.2.** (i) Let $H = D^\omega(\Sigma)$, and let $\Omega \in H^{\otimes 3}$ be the associator. Note that, since $D(A)$ is a Hopf algebra, $\phi$ must satisfy the following condition:

\begin{equation}
(1 \otimes \phi)(\text{id} \otimes \Delta)(\phi) \Omega (\Delta \otimes \text{id})(\phi^{-1})(1 \otimes \phi^{-1}) \in \Delta^{(2)}(H)',
\end{equation}

where $\Delta^{(2)}(H)' \subseteq H^{\otimes 3}$ denotes the centralizer of the subalgebra $(\Delta \otimes \text{id})\Delta(H)$.

(ii) Let $A$ be a finite dimensional quasi-Hopf algebra. Then the quantum double, $D(A)$, of $A$ has the property that the center of $\text{Rep} \, A$ is equivalent to $\text{Rep} \, D(A)$; see $\mathfrak{3}$. It turns out that the proof of Theorem 1.2 extends mutatis mutandis to the quasi-Hopf setting, implying that the characterization still holds true after replacing 'Hopf algebra' by 'quasi-Hopf algebra' in the statement of 1.2.

3. **Bicrossed products arising from exact factorizations**

We shall consider finite groups $F$ and $G$, together with a right action of $F$ on the set $G$, and a left action of $G$ on the set $F$

\[ \triangleleft : G \times F \to G, \quad \triangleright : G \times F \to F. \]
subject to the following conditions:
\[
(3.1) \quad s \triangleright xy = (s \triangleright x)(s \triangleleft x \triangleright y),
\]
\[
(3.2) \quad st \triangleleft x = (s \triangleleft (tb \triangleright x))(t \triangleleft x),
\]
for all \( s, t \in G, x, y \in F \). It follows that \( s \triangleright 1 = 1 \) and \( 1 \triangleleft x = 1 \), for all \( s \in G, x \in F \).

Such a data of groups and compatible actions is called a matched pair of groups. See [3]. Given finite groups \( F \) and \( G \), providing them with a pair of compatible actions is equivalent to finding a group \( \Sigma \) together with an exact factorization \( \Sigma = FG \): the actions \( \triangleright \) and \( \triangleleft \) are determined by the relations \( gx = (g \triangleright x)(g \triangleleft x) \), \( x \in F, g \in G \).

There are well defined maps \( p : F \xrightarrow{\pi} \Sigma \xrightarrow{\rho} G \), where
\[
(3.3) \quad p(xy) = g, \quad \pi(xg) = x, \quad x \in F, g \in G.
\]
Some of the properties of these maps are summarized in the next lemma.

**Lemma 3.1.** (i) \( \pi(ab) = \pi(a)(p(a) \triangleright \pi(b)) \), for all \( a, b \in \Sigma \).

(ii) \( p(ab) = (p(a) \triangleleft \pi(b))p(b) \), for all \( a, b \in \Sigma \).

**Proof.** It follows from \((3.1)\) and \((3.2)\). \( \square \)

3.1. Consider the left action of \( F \) on \( k^G, x . \phi(g) = \phi(g \triangleleft x), \phi \in k^G \), and let \( \sigma : F \times F \to (k^x)^G \) be a normalized 2-cocycle; that is, writting \( \sigma = \sum_{g \in G} \sigma_g \delta_g \), we have
\[
(3.4) \quad \sigma_{g \triangleleft x}(y, z)\sigma_g(x, yz) = \sigma_g(xy, z)\sigma_g(x, y),
\]
\[
(3.5) \quad \sigma_g(x, 1) = 1 = \sigma_g(1, x), \quad g \in G, x, y, z \in F.
\]

Dually, we consider the right action of \( G \) on \( k^F, \psi(x).g = \psi(x \triangleright g), \psi \in k^F \), and let \( \tau = \sum_{x \in F} \tau_x \delta_x : F \times F \to (k^x)^G \) be a normalized 2-cocycle; i.e.,
\[
(3.6) \quad \tau_x(gh, k)\tau_{k \triangleright x}(g, h) = \tau_x(h, k)\tau_x(g, hk),
\]
\[
(3.7) \quad \tau_x(g, 1) = 1 = \tau_x(1, g), \quad g, h, k \in G, x \in F.
\]

We assume in addition that \( \sigma \) and \( \tau \) obey the following compatibility conditions:
\[
(3.8) \quad \sigma_{ts}(x, y)\tau_{xy}(t, s) = \tau_x(t, s)\sigma_t(s \triangleright x, s \triangleleft x)\sigma_t(s \triangleright x, (s \triangleleft x) \triangleright y)\sigma_s(x, y),
\]
\[
(3.9) \quad \sigma_1(s, t) = 1, \quad \tau_1(x, y) = 1,
\]
for all \( x, y \in F, s, t \in G \).

Therefore the vector space \( A = k^G \otimes k^F \) becomes a (semisimple) Hopf algebra with the crossed product algebra structure and the crossed coproduct coalgebra structure. We shall use the notations \( A = k^G \tau \#_\rho k^F \), and \( \delta_g x \) to indicate the element \( \delta_g \otimes x \in A \). Then the multiplication and comultiplication of \( A \) are determined by
\[
(3.10) \quad (\delta_g x)(\delta_h y) = \delta_{g \triangleleft x, h} \sigma_g(x, y)\delta_g xy,
\]
\[
(3.11) \quad \Delta(\delta_g x) = \sum_{st = g} \tau_x(s, t) \delta_s(t \triangleright x) \otimes \delta_t x,
\]
for all \( g, h \in G, x, y \in F \). There is an exact sequence of Hopf algebras \( 1 \to k^G \to A \to k^F \to 1 \), and conversely every Hopf algebra \( A \) fitting into an exact sequence of this form is isomorphic to \( k^G \tau \#_\rho k^F \) for appropriate actions \( \triangleright, \triangleleft \), and cocycles \( \sigma \) and \( \tau \). Instances of this construction can be found in [1], [7], [4]; see also [3].
3.2. Fix a matched pair of groups \( \triangleleft : G \times F \to G, \triangleright : G \times F \to F \). The set of equivalence classes of extensions \( 1 \to k^G \to A \to kF \to 1 \) giving rise to these actions is denoted by \( \text{Opext}(k^G, kF) \): it is a finite group under the Baer product of extensions.

The class of an element of \( \text{Opext}(k^G, kF) \) can be represented by pair \((\tau, \sigma)\), where \( \sigma : G^2 \times F \to k^\times \) and \( \tau : G \times F^2 \to k^\times \) are maps satisfying conditions \( (3.4), (3.5), (3.7), (3.8) \) and \( (3.9) \). The group \( \text{Opext}(k^G, kF) \) can also be described as the \( H^1 \)-group of a certain double complex \([3, \text{Proposition 5.2}]\).

By a result of G. I. Kac \([3]\), there is an exact sequence
\[
0 \to H^1(\Sigma, k^\times) \xrightarrow{\text{res}} H^1(F, k^\times) \oplus H^1(G, k^\times) \to \text{Aut}(k^G \# kF) \to H^2(\Sigma, k^\times) \xrightarrow{\text{res}} H^2(F, k^\times) \oplus H^2(G, k^\times) \\
\text{Opext}(k^G, kF) \to H^3(\Sigma, k^\times) \xrightarrow{\text{res}} H^3(F, k^\times) \oplus H^3(G, k^\times) \to \ldots
\]

In view of \([12, 6.4]\), the element \([\tau, \sigma] \in \text{Opext}(k^G, kF)\) is mapped under \( \varpi \) onto the class of the 3-cocycle \( \omega(\tau, \sigma) \in Z^3(\Sigma, k^\times) \), defined by
\[
(3.12) \quad \omega(\tau, \sigma)(a, b, c) = \tau_{\pi(c)}(p(a)\pi(b)\pi(c)), \quad a, b, c \in \Sigma.
\]

Remark 3.2. Consider the case of the split extension \( A = k^G \# kF \), i.e., where both \( \sigma \) and \( \tau \) are trivial; so that the corresponding 3-cocycle \( \omega = \omega(\tau, \sigma) \) is also trivial. It has been shown in \([2, \text{Section 5}]\) that the Drinfeld double of \( A \) is in this case isomorphic to a 2-cocycle twist of the Drinfeld double of \( \Sigma \); the 2-cocycle is explicitly described in loc. cit. In particular, it follows from Theorem \([12, \text{Theorem 4.1}]\) that the split extension is group theoretical. This fact has also been observed in \([10, \text{Example 3.1}]\).

Lemma 3.3. Let \( \Sigma = FG \) be an exact factorization. Let \( \omega \in Z^3(\Sigma, k^\times) \) such that the class of \( \omega \) belongs to the image of \( \varpi \). Then, for arbitrary \( \alpha \in Z^2(F, k^\times) \) and \( \beta \in Z^2(G, k^\times) \) there is an associated semisimple Hopf algebra \( A_{\alpha,\beta}^\Sigma(\omega, F, G) \).

Proof. Write \([\omega] = \varpi[\tau, \sigma] \), where \([\tau, \sigma] \in \text{Opext}(k^G, kF) \). The exactness of the Kac sequence in the term \( H^3(\Sigma, k^\times) \) implies that \( \omega \) belongs to the image of \( \varpi \) if and only if the classes of \( \omega|_F \) and \( \omega|_G \) are trivial. Thus conditions \( (2.2), (2.3) \) and \( (2.4) \) are verified with arbitrary \( \alpha \) and \( \beta \). This proves the lemma.

Our aim in the next section is to show that the semisimple Hopf algebras \( A_{\alpha,\beta}^\Sigma(\omega, F, G) \) are obtained from the bicrossed product \( k^G \rtimes \tau \# \sigma kF \) by twisting the multiplication and the comultiplication; here \( \sigma \) and \( \tau \) are such that the class \([\tau, \sigma] \in \text{Opext}(k^G, kF)\) is mapped onto the class of the 3-cocycle \( \omega \) under \( \varpi \).

4. A MONOIDAL EQUIVALENCE

Along this section, we shall fix a representative \((\tau, \sigma)\) of a class in \( \text{Opext}(k^G, kF) \), and \( \omega \) will denote the 3-cocycle given by \((3.12)\). We shall write \( A := k^G \rtimes \tau \# \sigma kF \).

The first goal of this section is to explicitly construct a monoidal equivalence between the categories \( \text{Rep} A \) and \( F(\text{Vec}_\varpi)^F \), of \( F \)-bimodules in \( \text{Vec}_\varpi \). See Proposition \([15]\). This equivalence is a particular case of the result in Theorem 3.3.5 of \([12]\).
4.1. **The category** \( \text{Rep} A \). This category is described in the following proposition. We shall consider right \( A \)-actions, instead of left. We follow the lines of the method in \[2\].

**Proposition 4.1.** The category \( \text{Rep} A \) can be identified with the category \( \text{Vec}^G_F(\sigma, \tau) \) of left \( G \)-graded vector spaces \( V \), endowed with a right map \( \alpha : V \times F \to V \), subject to the following conditions:

\[
\begin{align*}
(4.1) & \quad v \lt 1 = v, \quad (v \lt x) \lt y = \sigma_{|v|}(x, y) v \lt xy, \\
(4.2) & \quad |v \lt x| = |v| \lt x, 
\end{align*}
\]

for all \( x, y \in F \), and for all homogeneous \( v \in V \), where \( |v| \) denotes the degree of homogeneity of \( v \in V \).

The tensor product of two objects \( V \) and \( V' \) of \( \text{Vec}^G_F(\sigma, \tau) \) is \( V \otimes V' \) with \( G \)-grading and \( F \)-map defined by

\[
\begin{align*}
(4.3) & \quad |v \otimes v'| = |v| |v'|, \\
(4.4) & \quad (v \otimes v') \lt x = \tau_{x}(|v|, |v'|) v \lt (|v'| \lt x) \otimes v' \lt x,
\end{align*}
\]

on homogeneous elements \( v \in V, v' \in V' \).

**Proof.** Let \( V \in \text{Vec}^G_F(\sigma, \tau) \). The identification is done by defining a right action of \( A \) on \( V \) by the formula \( v.\delta_gx := \delta_{g(|v|)} v \lt x \), for all homogeneous \( v \in V \), and for all \( g \in G, x \in F \). It is straightforward to verify that this is indeed an action and that tensor products are preserved. \(\square\)

4.2. **The category** \( F(\text{Vec}^\Sigma)_{\Sigma} \). Let \( F(\text{Vec}^\Sigma)_{\Sigma} \) denote the category of \( F \)-bimodules in the monoidal category \( \text{Vec}^\Sigma_{\Sigma} \). Thus, in view of \[[3,12]\], the associativity constraint in this category is given by

\[
\begin{align*}
(4.5) & \quad a_{U,U',U''}((u \otimes u') \otimes u'') = \tau_{|u|(||u'|| |u''|)} (p||u|| \lt \sigma ||u'||, p||u'||) \sigma_{|u|} (p||u'||, p||u'|| \lt \sigma ||u''||) u \otimes (u' \otimes u''),
\end{align*}
\]

on homogeneous elements \( u \in U, u' \in U', u'' \in U'' \).

**Lemma 4.2.** Objects in the category \( F(\text{Vec}^\Sigma)_{\Sigma} \) are vector spaces \( U \), together with a left \( \Sigma \)-grading \( || \cdot || \), and maps \( \to : F \times U \to U \), \( \leftarrow : U \times F \to F \), subject to the following conditions:

(i) \( \to \) is a left action:

\[
\begin{align*}
(4.6) & \quad 1 \to u = u, \quad x \to (y \to u) = xy \to u, \quad \forall x, y \in F, u \in U; \\
(4.7) & \quad u \leftarrow 1 = u, \quad (u \leftarrow x) \leftarrow y = \sigma_{|u|}(x, y) u \leftarrow xy, \quad \forall x, y \in F, u \in U_{||u||}; \\
(4.8) & \quad x \to (u \leftarrow y) = (x \to u) \leftarrow y, \quad \forall x, y \in F, u \in U; \\
(4.9) & \quad ||x \to u \leftarrow y|| = x||u||, \quad x, y \in F, u \in U_{||u||}.
\end{align*}
\]

Tensor product \( U \otimes U' \) is defined on objects \( U \) and \( U' \) as follows: \( U \otimes U' = U \otimes_F U' \) as vector spaces, with

(v) \( \Sigma \)-grading

\[
||u \otimes u'|| = ||u|| ||u'||,
\]

on homogeneous elements \( u, u' \).
(vi) left $F$-action
\[(4.11) \quad x \mapsto (u \otimes u') = (x \mapsto u) \otimes u';\]

(vii) right twisted $F$-action
\[(4.12) \quad (u \otimes u') \mapsto x = \tau_x (p||u||\sigma p||u'||, p||u'||) \sigma p||u|| (p||u'||, p||u'||) u \otimes (u' \mapsto x),\]
for all $x \in F$, and for all homogeneous elements $u, u'$.

Proof. It follows from the definitions using (4.5). Conditions (i) and (ii) correspond, respectively, to the commutativity of the following diagrams
\[
\begin{array}{cccc}
(kF \otimes kF) \otimes U & \xrightarrow{\alpha_{kF,kF,U}} & kF \otimes (kF \otimes U) & \xrightarrow{\alpha_{U,kF,kF}} U \otimes (kF \otimes kF) \\
\downarrow{m \otimes \text{id}} & & \downarrow{\text{id} \otimes \sigma} & \downarrow{\text{id} \otimes m} \\
kF \otimes U & \rightarrow & kF \otimes U & \rightarrow U \otimes kF & \rightarrow U \otimes kF \\
\downarrow & & \downarrow & & \downarrow \\
U & = & U, & = & U.
\end{array}
\]
Conditions (iii) and (iv) and formula (v) are easy to see. The actions (vi) and (vii) correspond respectively, to the left and right actions in the category, which are obtained by factorizing the following maps:
\[(kF \otimes (U \otimes U')) \xrightarrow{\alpha^{-1}} (kF \otimes U) \otimes U' \xrightarrow{\text{id} \otimes \sigma} U \otimes U',\]
\[(U \otimes U') \otimes kF \xrightarrow{\alpha} U \otimes (U' \otimes kF) \xrightarrow{\text{id} \otimes \sigma} U \otimes U'.\]
This finishes the proof of the lemma.

4.3. We now define functors $\mathcal{F} : F(\text{Vec}_{\Sigma}^G)_F \rightarrow \text{Vec}_{\Sigma}^G(\sigma, \tau)$ and $\mathcal{G} : \text{Vec}_{\Sigma}^G(\sigma, \tau) \rightarrow F(\text{Vec}_{\Sigma}^G)_F$, in the form $\mathcal{F}(U) := \hat{F} U$, with left $G$-grading and right twisted $F$-action given by
\[(4.13) \quad |u| := p||u||,\]
\[(4.14) \quad u \triangleleft x := u \mapsto x, \quad x \in F,\]
for all homogeneous elements $u \in \hat{F} U$; and $\mathcal{G}(V) = kF \otimes V$, with left $\Sigma$-grading, left $kF$-action and right twisted $kF$-action given by
\[(4.15) \quad ||x \otimes v|| := x|v|,\]
\[(4.16) \quad x \mapsto (y \otimes x) = xy \otimes v,\]
\[(4.17) \quad y \otimes x \mapsto : = y(|v| \triangleright x) \otimes (v \triangleleft x),\]
for all $x, y \in F$, and homogeneous $v \in V$.

**Proposition 4.3.** The functors $\mathcal{F}$ and $\mathcal{G}$ are inverse equivalences of categories.

Proof. We first show that the functor $\mathcal{G}$ is well defined. Let $V \in \text{Vec}_{\Sigma}^G(\sigma, \tau)$. Conditions (4.6) and (4.8) follow easily. We compute, for all $x, y, z \in F$, and homogeneous $v \in V$,
\[
\sigma_{p(|v|)}(x, y) (z \otimes v) \mapsto z = \sigma_{p(|v|)}(x, y) z(|v| \triangleright xy) \otimes (v \triangleleft xy)
\]
\[
= \sigma_{|v|}(x, y) \sigma_{|v|}(x, y)^{-1} z(|v| \triangleright x) ((|v| \triangleleft x) \otimes (v \triangleleft x) \triangleleft y)
\]
\[
= z(|v| \triangleright x) (|v| \triangleleft x) \otimes (v \triangleleft x) \triangleleft y = ((z \otimes v) \mapsto x) \triangleleft y,
\]

the second and third equalities because of (4.1) and (4.2) and the compatibility condition (3.1). Condition $u\leftarrow 1 = u$ follows from $|v|\cdot 1 = 1$ and $v\cdot 1 = v$. This proves (4.7). Condition (4.9) is verified as follows:

$$||x\cdot (z \otimes v)\leftarrow y|| = ||xz(|v\cdot y|) \otimes v \cdot y|| = xz(|v\cdot y|) |v\cdot y| = xz(|v\cdot y|) (v \cdot y) = xz|v|,$$

for all $x, y, z \in F$ and homogeneous $v \in V$. We have shown that the functor $\mathcal{G}$ is well defined. The proof for $\mathcal{F}$ is similar and we omit it.

Next, let $V \in \text{Vec}_{\mathcal{F}}^G(\sigma, \tau)$. We have natural isomorphisms $\mathcal{F} \mathcal{G} V = kt \otimes V \simeq V$ as twisted right $F$-modules, where $t = \frac{1}{|F|} \sum_{x \in F} x$ is the normalized integral in $kF$. It is not difficult to check that this isomorphism preserves gradings. On the other hand, for $U \in F(\text{Vec}_{\mathcal{F}}^G)_F$, there is a natural isomorphism of left $\Sigma$-graded left $F$-modules $U \simeq kF \otimes F U$, which is compatible with the twisted right action. Therefore, the functors are inverse equivalences, as claimed.

4.4. Let $U$ and $U'$ be objects of $F(\text{Vec}_{\mathcal{F}}^G)_F$. We define natural isomorphisms $\xi : \mathcal{F}(U \otimes U') \to \mathcal{F}(U) \otimes \mathcal{F}(U')$ in the form

$$\xi(u \otimes u') = u\leftarrow \pi||u'|| \otimes t\rightarrow u',$$

for $u \in U$, and $u' \in U'$ homogeneous, where $t \in kF$ is the normalized integral.

Remark 4.4. That $\xi$ is indeed an isomorphism can be seen as a consequence of the structure theorem for Hopf modules [13].

**Proposition 4.5.** ($\mathcal{F}, \xi^{-1}$) is a monoidal equivalence of categories.

**Proof.** We first see that $\xi$ is indeed an isomorphism in $\text{Vec}_{\mathcal{F}}^G(\sigma, \tau)$. Let $u \in U$, $u' \in U'$ be homogeneous elements; we have $|u \otimes u'| = p(|u \otimes u'|) = p(||u||||u'||)$. On the other hand,

$$||\xi(u \otimes u')|| = ||u\leftarrow \pi||u'||||t\rightarrow u'|| = p(||u\leftarrow \pi||u'||||p||t\rightarrow u'||) = p(||u\leftarrow \pi||u'||||p||u'||)$$

$$= ||u\leftarrow \pi||u'||||p||u'|| = (p(||u||\cdot \pi||u'||)) p||u'|| = p(||u||||u'||).$$

Thus $\xi$ preserves $G$-gradings. Let now $u \in U$, $u' \in U'$ be homogeneous, and let also $x \in F$. We compute

$$\xi((u \otimes u') \cdot x) = \tau_x (p||u||\cdot \pi||u'||, p||u'||) \sigma_{p||u||}(\pi||u'||, p||u'|||x) \xi (u \otimes (u' \leftarrow x))$$

$$= \tau_x (p||u||\cdot \pi||u'||, p||u'|||x) \sigma_{p||u||}(\pi||u'||, p||u'|||x) u\leftarrow \pi((||u'|||x) \otimes (t\rightarrow u') \leftarrow x)$$

$$= \tau_x (p||u||\cdot \pi||u'||, p||u'||) (u\leftarrow \pi(||u'|||x)) \otimes (t\rightarrow u') \leftarrow x = (u\leftarrow \pi||u'|||t\rightarrow u') \otimes x = \xi(u \otimes u') \cdot x;$$

the first equality because of (4.12) and (4.14), the second by (4.3), the third because of (4.7) and the relationship $p||t\rightarrow u'|| = tp||u'|| = p||u'||$, for all $u'$; the last equality by (4.4). Therefore $\xi$ preserves also right $F$-actions.

Finally, the compatibility of ($\mathcal{F}, \xi^{-1}$) with the monoidal structures is shown by straightforward computations. One can use for this the following claim, which is a consequence of Lemma 3.4 and the compatibility between $|||$, and $\leftarrow$.

**Claim 4.1.** $\pi||u\leftarrow \pi(||u'||)| = \pi||u||\pi||u'||$, for all homogeneous $u, u' \in U$. 

4.5. We are now ready to complete the proof of the main result of this section.

Proof of Theorem 1.3. Part (i) is the content of Proposition 4.5 plus Proposition 1.1. Part (ii) also follows from Proposition 4.5 and the results in 3 and 11; cf. 2.2.

Remark 4.6. Let $\alpha \in \mathbb{Z}^2(F,k^\times)$ and $\beta \in \mathbb{Z}^2(G,k^\times)$. Then there are exact sequences $1 \to (k^G)_{J(\beta)} \to A_{J(\beta)} \to kF \to 1$ and $1 \to k^G \to A^{\alpha} \to (kF)^{\alpha} \to 1$, where $A_{J(\beta)}$ and $A^{\alpha}$ are obtained from $A$ by twisting the comultiplication and the multiplication, respectively, by means of the obvious 2-cocycles $J(\beta)$ and $\alpha$. See [12, Lemma 6.3.1]. As shown in [12], this defines an action of $H^2(F,k^\times) \oplus H^2(F,k^\times)$ on $\text{Opext}(k^G, kF)$, which comes from the map $H^2(F,k^\times) \oplus H^2(F,k^\times) \to \text{Opext}(k^G, kF)$ of the Kac exact sequence; in particular, the corresponding extensions give the same three cocycle class on $\Sigma$. The group theoretical Hopf algebra $A^\Sigma_{\alpha,\beta}(\omega, F, G)$ arising from arbitrary $\alpha$ and $\beta$ as in Lemma 3.3, is precisely $A^\alpha_{J(\beta)}$.

4.6. Let $G$ be a finite group. In the paper [16] Tambara and Yamagami parametrize all monoidal structures in a semisimple category with simple objects $G \cup \{m\}$, satisfying $g \otimes h = gh, g \otimes m \simeq m \otimes g \simeq m$, for all $g \in G$, and $m \otimes m \simeq \bigoplus_{g \in G} g$. It turns out that $G$ must be abelian, and these categories are classified by pairs $(\chi, r)$, where $\chi$ is a non-degenerate symmetric bilinear form on $G$, and $r$ is a square root of $|G|$. Denote the corresponding category by $\mathcal{C}(G, \chi, r)$.

In the paper [15] the question of when one of these categories arises as the category of representations of a semisimple Hopf algebra is studied.

Corollary 4.7. Let $A$ be a semisimple Hopf algebra and suppose that there exist a data $(G, \chi, r)$, such that $\text{Rep } A$ is equivalent to $\mathcal{C}(G, \chi, r)$ as monoidal categories. Then $A$ fits into a central extension

$$0 \to k\mathbb{Z}_2 \to A \to kG \to 1.$$  \hspace{2cm} (4.19)

In particular, $A$ is group theoretical.

This generalizes the last statement in [3, Example 8.48].

Proof. It suffices to prove that $A$ fits into such an extension. The assumption implies that $\dim A = 2|G|$. On the other hand, the category $\mathcal{C}(G, \chi, r)$ contains $\text{Rep } kG$ as a full monoidal subcategory. Therefore, $k^G \simeq kG$ is embedded in $A^*$ as a Hopf subalgebra of index 2. Hence, $k^G$ is a normal Hopf subalgebra in $A^*$ and necessarily $A^*/A^*(k^G)^+ \simeq k\mathbb{Z}_2$. This completes the proof.

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