Buckling analysis of Functionally Graded Material (FGM) square plates using Quadrilateral Element

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Abstract. Functionally Graded Material in one type of material that currently gets much attention in the civil engineering field because it is claimed as the material that can resist the high-temperature environment. FGM is consist of two or more material that continuously changed along the thickness direction of the structure. FGM is often formed by ceramic at the top of the structure that can resist the high-temperature environment and metal at the bottom of the structure that flexible and can resist the mechanical load. DKMQ Element gives a good convergence behavior in a thick and a thin plate problem. The purpose of this research is to study the convergence behavior of the DKMQ element in the buckling analysis of FGM plate under uniaxial compression. The result is a critical buckling load that will be compared to the reference. The results show that the DKMQ element gives a good result for buckling analysis on the FGM plate.

1. Introduction
The development in the civil engineering field is also required to be more advanced in material selection. In 1984 a research group in Sendai, Japan, discovered a new type of composite material, namely FGM (Functionally Graded Material), which is claimed to be a material that can adapt and resist the high-temperature environments (1–3). FGM is a part of a composite material consisting of two or more material components whose volume fraction continuously changed along the thickness direction, thus preventing delamination between layers (4–7).

In 1993, Katili discovered elements of DKMQ (Discrete Kirchhoff Mindlin Quadrilateral) whose formulation is free of shear locking and passed the patch test (8–11). This element also gives a good result for isotropic and orthotropic materials, both on thick plates and thin plates problems (12). The study of plates and shells using the DKMQ and DKMT are presented in [12]–[23].

In this paper, the formulation of the DKMQ element for buckling analysis FGM plate has been derived. Wong et al. had developed the formulation of the DKMQ element for buckling analysis isotropic bending plate (24). The study for buckling analysis on the plate are presented in (1, 25–33).

2. Problem formulation
2.1. Functionally Graded Materials
Figure 1 shows the geometry of the FGM plate composed of ceramic on the top of the structure and metal at the bottom of the structure. Its material properties are continuously changed by varying volume fractions (Figure 2). By using power-law from [1], we have:

\[
P(z) = (P_C - P_M) V_C(z) + P_M
\]  

(1)
\[ V_C(z) = \left( \frac{1}{2} + \frac{z}{h} \right)^n \]  

(2)

\( P \) denotes the volume fraction of FGM where subscripts \( M \) is metal, and \( C \) is ceramic.

\[ E_m = E_i = \int_{\gamma} E(z)dz = \left( \frac{E_C - E_M}{n+1} + E_M \right)h \]  

(3)

\[ E_n = \int_{\gamma} E(z)z^2dz = \left( \frac{n^2 + n + 2}{4(n+1)(n+2)(n+3)}(E_C - E_M) + \frac{E_M}{12} \right)h^3 \]  

(4)

\[ E_{nb} = \int_{\gamma} E(z)zdz = \left( \frac{n}{2(n+1)(n+2)}(E_C - E_M) \right)h^2 \]  

(5)

2.2. DKMQ Formulations for FGM Plates

DKMQ is a quadrilateral element that has 4 nodes at the edges of it sides. For FGM plates, each node of the DKMQ element has five degrees of freedom, as seen in Figure 3. Element DKMQ for FGM plate. There is the couple membrane effect (membrane-bending) on FGM material.
Figure 3. Element DKMQ for FGM plate.

The interpolation functions of the DKMQ element are:

\[ u = \sum_{i=1}^{n_1} N_i u_i; \quad v = \sum_{i=1}^{n_2} N_i v_i; \quad w = \sum_{i=1}^{n_2} N_i w_i \]

\[ \beta_x = \sum_{i=1}^{n_1} N_i \beta_{x_i} + \sum_{k=3}^{8} P_k C_k \Delta \beta_{x_k}; \quad \beta_y = \sum_{i=1}^{n_1} N_i \beta_{y_i} + \sum_{k=3}^{8} P_k S_k \Delta \beta_{y_k} \]

where, \( N_i \) and \( P_k \) are.

\[ N_i = \frac{1}{4} (1 - \xi)(1 - \eta); \quad N_2 = \frac{1}{4} (1 + \xi)(1 - \eta); \quad N_3 = \frac{1}{4} (1 + \xi)(1 + \eta); \quad N_4 = \frac{1}{4} (1 - \xi)(1 + \eta) \]

\[ P_3 = \frac{1}{2} (1 - \xi^2)(1 - \eta); \quad P_6 = \frac{1}{2} (1 + \xi)(1 - \eta^2); \quad P_7 = \frac{1}{2} (1 - \xi^2)(1 + \eta); \quad P_8 = \frac{1}{2} (1 - \xi)(1 - \eta^2) \]

The formulation for the stiffness matrix of DKMQ element is derived from Total Potential Energy,

\[ \Pi = \Pi_{\text{int}} - \Pi_{\text{ext}} \]

The modification of the Hu-Washizu principle that related to the Reissner-Mindlin plate theory

\[ \Pi_{\text{ext}} = \int_A w f dA \]

\[ \Pi_{\text{int}} = \Pi_{\text{int}^m} + \Pi_{\text{int}^b} + \Pi_{\text{int}^s} \]

The membrane energy:

\[ \Pi_{\text{int}^m} = \frac{1}{2} \langle u_n \rangle \{ k_m \} \{ u_n \} \]

The bending energy:

\[ \Pi_{\text{int}^b} = \frac{1}{2} \langle u_n \rangle \{ k_b \} \{ u_n \} \]

The membrane bending energy:

\[ \Pi_{\text{int}^{mb}} = \frac{1}{2} \langle u_n \rangle \{ [k_{mb}] + [k_{mb}]^T \} \{ u_n \} \]

The shear energy:

\[ \Pi_{\text{int}^s} = \frac{1}{2} \langle u_n \rangle \{ k_s \} \{ u_n \} \]

The theoretical formulation for the geometric stiffness matrix:

\[ \Pi_{\sigma} = \frac{1}{2} h \langle \nabla u \rangle \{ \sigma_0 \} \{ \nabla u \} dA + \frac{1}{2} h \langle \nabla v \rangle \{ \sigma_0 \} \{ \nabla v \} dA + \frac{1}{2} h \langle \nabla w \rangle \{ \sigma_0 \} \{ \nabla w \} dA + \]

\[ \frac{1}{2} h^3 \int_A \{ \nabla \beta_x \} \{ \sigma_0 \} \{ \nabla \beta_x \} dA + \frac{1}{2} h^3 \int_A \{ \nabla \beta_y \} \{ \sigma_0 \} \{ \nabla \beta_y \} dA \]

\[ \frac{1}{2} h^4 \int_A \{ \nabla \beta_{x,y} \} \{ \sigma_0 \} \{ \nabla \beta_{x,y} \} dA \]
\[ \Pi_\sigma = \frac{1}{2} \langle \sigma, \sigma \rangle \]

The membrane stress matrix:

\[
[\sigma_0] = \begin{bmatrix}
\sigma_x^0 \\
\tau_{xy}^0 \\
\tau_{xy}^0 \\
\sigma_y^0
\end{bmatrix}
\]

\[ \sigma^0 = \frac{N^0}{h} \]

The critical buckling load \( N_{cr} \), from the eigenvalue equation:

\[
([k] - N_{cr} [k_G]) \{ u_n \} = 0
\]

3. Numerical result

Figure 4 illustrates the uniaxial compression on FGM Square Plate. The constituent component of FGM is Aluminium (Al) for the metal and Zirconia (ZrO2-1) for the ceramic. Material properties of FGM are \( E_m = 200000; E_c = 70000; \) Poisson’s ratio=0.3. Square FGM Plate with ratio \( L/h = 10 \) for thick plate and \( L/h = 100 \) for the thin plate. Figure 5 shows the various boundary condition that had been analyzed.

![Figure 4](image)

Figure 4. Uniaxial Compression on Square Plate.

![Figure 5](image)

Figure 5. Boundary Condition FGM Square Plate (a) SSSS, (b) CCCC, (c) SCSC, (d) SFSF.
Table 1 and Table 2 presented the non-dimensional critical buckling load given by the DKMQ element. And the result is DKMQ Element gives a good result compared to the REF, with 0.3% difference from Wong et al. (24), and 0.05% difference from Meshfree Method (28) for the thick plate in Table 1. For the thin plate in Table 2, the result is DKMQ Element also gives a good result compared to the REF, with 0.07% difference from Wong et al. (24).

Table 1. Non-dimensional Critical Buckling Load Square Plate $\overline{N_{cr}} = L^2 N_{cr} / \pi^2 D_b$, $D_b = Eh^3 / (12(1-\nu^2))$, $L/h=10$.

| Boundary Condition | Method       | $n = 0$ | $n = 0.5$ | $n = 1$ | $n = 2$ | $n = 5$ | $n = 10$ | $n = \infty$ |
|-------------------|--------------|---------|-----------|---------|---------|---------|---------|-------------|
| SSSS              | Present      | 3.7291  | 2.7297    | 2.3293  | 2.0431  | 1.8411  | 1.7013  | 1.3052      |
|                   | REF (Wong)   |         |           |         |         |         |         |             |
|                   | Meshfree(28) |         |           |         |         |         |         | 3.727       |
| CCCC              | Present      | 8.0068  | 5.9047    | 5.0399  | 4.3949  | 3.9085  | 3.5979  | 2.8024      |
| SCSC              | Present      | 7.2892  | 5.3662    | 4.5824  | 4.0048  | 3.5747  | 3.2927  | 2.5512      |
| SFSF              | Present      | 0.9147  | 0.6686    | 0.5706  | 0.5011  | 0.4528  | 0.4187  | 0.3201      |

Table 2. Non-dimensional Critical Buckling Load Square Plate $\overline{N_{cr}} = L^2 N_{cr} / \pi^2 D_b$, $D_b = Eh^3 / (12(1-\nu^2))$, $L/h=100$.

| Boundary Condition | Method       | $n = 0$ | $n = 0.5$ | $n = 1$ | $n = 2$ | $n = 5$ | $n = 10$ | $n = \infty$ |
|-------------------|--------------|---------|-----------|---------|---------|---------|---------|-------------|
| SSSS              | REF (Wong)   | 3.9971  | 2.9172    | 2.4897  | 2.1899  | 1.9845  | 1.8365  | 1.3990      |
|                   | Present      | 4.0000  |           |         |         |         |         |             |
| CCCC              | Present      | 10.0477 | 7.3339    | 6.2590  | 5.5050  | 4.9878  | 4.6155  | 3.5167      |
| SCSC              | Present      | 8.5885  | 6.2686    | 5.3499  | 4.7055  | 4.2637  | 3.9455  | 3.0060      |
| SFSF              | Present      | 0.9515  | 0.6944    | 0.5926  | 0.5213  | 0.4724  | 0.4372  | 0.3330      |

Figure 6 shows that DKMQ Element is convergence compared to the reference. Figure 7 shows that the critical buckling load value from the largest is CCCC-SCSC-SSSS-SFSF. And Figure 8 shows that if the power-law index value is increase, then the critical buckling load value will be decreased.
Figure 7. Critical Buckling Load FGM Square Plate with various boundary condition.

Figure 8. Critical Buckling Load FGM Square Plate with various power-law index.

4. Conclusion

The buckling analysis of DKMQ Element for FGM Plate has been presented. The buckling case of the FGM square plate using the DKMQ element has been analyzed to establish the accuracy of the element. The result is the DKMQ Element is convergence to the reference for both thick and thin plates. Furthermore, it is seen that the DKMQ Element shows excellent performance compared to the reference solution.

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