Integrating the Chirally Split Diffeomorphism Anomaly on a Compact Riemann Surface

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Abstract

A well-defined chirally split functional integrating the 2D chirally split diffeomorphism anomaly is exhibited on an arbitrary compact Riemann surface without boundary. The construction requires both the use of the Beltrami parametrisation of complex structures and the introduction of a background metric possibly subject to a Liouville equation. This formula reproduces in the flat case the so-called Polyakov action. Although it works on the torus (genus 1), the proposed functional still remains to be related to a Wess-Zumino action for diffeomorphisms.

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1 Introduction

In the last decade various attempts have been made in order to find out a chirally split Polyakov action emerging from the study on the quantization of 2-dimensional gravity which is conformally covariant on a compact Riemann surface without boundary.

Since another celebrated paper by Polyakov some partial results have been disseminated throughout several relevant papers. Quite recently there has been a revival interest in the subject of Liouville theory fifteen years later and in the relationship with the chirally split Polyakov effective action for induced 2-d gravity as well. As shown in, the existence of a class of such functionals has been proved by using the local index theorem for families of elliptic operators in the form given by Bismut et al., and as well as their supersymmetric version. It is emphasized that the author’s viewpoint falls into the perturbative approach to the chiral splitting property of bidimensional conformal models at non vanishing central charge as a local strengthened form of the Belavin-Knizhnik holomorphic factorization theorem.

In this Letter a globally defined ‘Polyakov formula’ is proposed although it is naively derived from the Liouville action proposed in. In doing so by using the Beltrami parametrization of complex structures, it will be found a well-defined holomorphically factorized functional integrating the globally defined factorized anomaly for diffeomorphims on an arbitrary compact Riemann surface without boundary. The related energy-momentum tensor will be discussed.

2 The smooth change of complex coordinates

Throughout the paper we will be concerned with the euclidean framework. We shall right away introduce a complex analytic atlas with local coordinates corresponding to the reference complex structure defined by a background metric, namely one has the following equality of quadratic forms,

\[ ds^2 = \hat{g}_{\alpha\beta} \, dx^\alpha dx^\beta = \hat{\rho}_{z\bar{z}} |dz|^2, \tag{2.1} \]

where \( \hat{\rho}_{z\bar{z}} \equiv \hat{\rho} > 0 \) is the coefficient of a positive real valued type (1,1) conformal background field.

We then parametrize locally the metric according to,

\[ ds^2 = g_{\alpha\beta} \, dx^\alpha dx^\beta = \rho_{z\bar{z}} |dz + \mu_{z\bar{z}} d\bar{z}|^2, \tag{2.2} \]

where \( \mu_{z\bar{z}} \equiv \mu \) is the local representative of the Beltrami differential \( (|\mu| < 1) \) seen as a \((-1,1)\) conformal field, which parametrizes the conformal class of the metric \( g \) and \( \rho_{z\bar{z}} \equiv \rho > 0 \) is the coefficient of a positive real valued type (1,1) conformal field.

Our compact Riemann surface \( \Sigma \) without boundary being now endowed with the analytic atlas with local coordinates \( (z, \bar{z}) \), let \( (Z, \bar{Z}) \) be the local coordinates of another holomorphic atlas.

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1 The author apologizes to all those who have been only implicitly referred to, in the bibliography.

2 We shall omit the index denoting the open set where these coordinates are locally defined since all formulae will glue through holomorphic changes of coordinates.
corresponding to the Beltrami differential $\mu$,
\[
dZ = \lambda(dz + \mu d\bar{z}) \quad \implies \quad \partial_Z = \frac{\bar{\partial} - \mu \partial}{\lambda(1 - \mu \overline{\partial})} \quad (2.3)
\]
where $\lambda = \partial_Z \overset{\text{def}}{=} \partial Z$ is an integrating factor fulfilling\[1\]
\[
(d^2 = 0) \iff (\bar{\partial} - \mu \partial) \ln \lambda = \partial \mu . \quad (2.4)
\]
Solving the above Pfaff system (2.3) is equivalent to solving locally the so-called Beltrami equation,
\[
(\bar{\partial} - \mu \partial) Z = 0 . \quad (2.5)
\]
According to Bers, see e.g. [20], the Beltrami equation (2.3) always admits as a solution a quasiconformal mapping with dilatation coefficient $\mu$. One thus remarks that $Z$ is a (non-local) holomorphic functional of $\mu$ as well as the integrating factor $\lambda$, and will be seen to serve as a parametrization of a family of global diffeomorphisms connected to the identity [21] in analogy to what it is thought about the “light-cone” gauge together with the “chiral diffeomorphisms” [1, 22]. However, the solution of the Beltrami equation define a smooth diffeomorphism $(z, \bar{z}) \rightarrow (Z, \overline{Z})$ which preserves the orientation (the latter condition secures the requirement $|\mu| < 1$, e.g. [9]), so that $(Z, \overline{Z})$ defines a new system of complex coordinates with $Z \rightarrow z$ when $|\mu| \rightarrow 0$.

In terms of the $(Z, \overline{Z})$ complex coordinates which by virtue of (2.3) turns out to be isothermal coordinates for the metric $g$ by defining the non-local metric,
\[
\rho_{ZZ} \equiv \frac{\rho}{\lambda \overline{\lambda}} . \quad (2.6)
\]
In particular, the quadratic form (2.2), the volume form, the scalar curvature and the scalar Laplacian, respectively write,
\[
ds^2 = g_{\alpha \beta} dx^\alpha dx^\beta = \rho_{ZZ} |dZ|^2 , \quad \sqrt{g} = \rho (1 - \mu \overline{\partial}) ,
\]
\[
d^2x \sqrt{g} = \frac{dZ \wedge d\bar{Z}}{2i} \rho_{ZZ}^2 , \quad R(g) = \frac{-4}{\rho_{ZZ}^2} \partial_Z \partial_{\bar{Z}} \ln \rho_{ZZ} , \quad \Delta(g) = \frac{-4}{\rho_{ZZ}^2} \partial_Z \partial_{\bar{Z}} . \quad (2.7)
\]
\[3\]From now on, we shall reserve $\partial$’s for $\partial \equiv \partial_z$, $\bar{\partial} \equiv \partial_{\bar{z}}$. 

The locally defined Weyl×Diff BRS algebra and the WZ consistency conditions

The corresponding locally defined BRS algebra is by now well-known \[23, 24, 19, 25\],

\[
s \ln \rho = \Omega + \nabla c + \nabla \bar{c} + \mu \partial c + \bar{\mu} \partial \bar{c} , \quad s \rho = 0 ,
\]

\[
s \Omega = (c \cdot \partial) \Omega , \quad sc = (c \cdot \partial) c , \quad \text{and c.c.}
\]

\[
s \mu = \bar{\partial} C + C \partial \mu - \mu \partial C , \quad \text{and c.c.}
\]

\[
s \ln \lambda = \partial C + C \partial \ln \lambda \equiv D C , \quad \text{and c.c.}
\]

\[
s C = C \partial C , \quad \text{and c.c.}
\]

where \( C = c + \mu \overline{c} \) is the combination of the diffeomorphism ghost \((c, \overline{c})\) appropriate to the complex analytic structure of \( \text{Diff}_0(\Sigma) \) (the connected component to the identity) and where,

\[
\nabla c = \partial c + c \partial \ln \rho ,
\]

is the covariant derivative related to the Weyl factor \( \rho \) of the metric \( g \). Of course \( s^2 = 0 \).

One may extend the \( s \)-operation in the \((Z, \bar{Z})\) of complex coordinates system by, see e.g. \[26\],

\[
s Z = c^Z \equiv \lambda C , \quad s \lambda = \partial c^Z , \quad \& \ c.c. \quad s \ln \rho^{Z \bar{Z}} = \Omega + (c^Z \partial_Z + c^Z \partial_{\bar{Z}}) \ln \rho^{Z \bar{Z}} ,
\]

which will be used throughout to simplify some computational steps.

On the one hand, the standard Weyl anomaly \[11\] reflecting the famous trace anomaly \[27\] writes

\[
A(\Omega, \rho, \mu, \bar{\mu}) = -\frac{k}{12\pi} \int_\Sigma \frac{dZ \wedge d\bar{Z}}{2i} \Omega \partial_Z \partial_{\bar{Z}} \ln \rho^{Z \bar{Z}} ,
\]

and on the other hand, the chirally split form of the diffeomorphism anomaly reads \[28, 25\]

\[
A(C, \mu; \mathcal{R}) + \overline{A(C, \mu; \mathcal{R})} ,
\]

with

\[
A(C, \mu; \mathcal{R}) = \frac{k}{12\pi} \int_\Sigma \frac{d\bar{Z} \wedge dz}{2i} C \left( \partial^3 + 2\mathcal{R} \partial + \partial \mathcal{R} \right) \mu ,
\]

where \( \mathcal{R} \) is the representative of a background holomorphic projective connection \[29\], and the Wess-Zumino consistency conditions \[30\] for these two forms of the same anomaly phenomenon now reads,

\[
s A(\Omega, \rho, \mu, \bar{\mu}) = s A(C, \mu; \mathcal{R}) = 0.
\]
4 Towards the holomorphic factorization of the gravitational action

In the spirit of the genuine Wess-Zumino action, the smooth change of coordinates (2.3) might be seen as already said before like a parametrisation of a family of global diffeomorphisms connected to the identity [21]. Afterwards, roughly speaking, one will argue since the effective action has to break the diffeomorphism invariance, i.e. passing from the \((z, \bar{z})\) system of complex coordinates to the \((Z, \bar{Z})\) one is not covariantly performed. The important feature will be that the effective action remains well-defined in the \((z, \bar{z})\) background coordinates. Instead of using the zweiben formalism [31, 23], the covariant effective action for 2d gravity [11, 1]

\[
\Gamma[g] = \frac{k}{4\pi} \int_{\Sigma} d^{2}x \sqrt{g} R(g)(-\Delta(g))^{-1} R(g) ,
\]

is promoted in the \((Z, \bar{Z})\) coordinate system, by using formally the replacements given in (2.7). One formally has

\[
\Gamma[\mu, \bar{\mu}, \rho] = -\frac{k}{12\pi} \int_{\Sigma} dZ \wedge d\bar{Z} \frac{1}{2i} \partial_{Z} \partial_{\bar{Z}} \ln \rho_{Z\bar{Z}} \frac{1}{\partial_{Z} \partial_{\bar{Z}}} \partial_{Z} \partial_{\bar{Z}} \ln \rho_{Z\bar{Z}} .
\]

Of course, the Goldstone scalar field \(\Phi = (-\Delta(g))^{-1} R(g)\) is expected to be a non-local functional in the metric \(g\) and thus in the Beltrami coefficient \(\mu\). Firs of all, recall that on a compact Riemann surface the operator \(\partial_{Z} \partial_{\bar{Z}}\) maps scalar fields to conformal fields of weight \((1,1)\) with

\[
\partial_{Z} \partial_{\bar{Z}} : \Omega^{0,0}/R \rightarrow \mathfrak{S} \equiv \{\omega \in \Omega^{1,1}, \int_{\Sigma} \omega = 0\}.
\]

the constants being the zero modes, and the image is so defined in order to insure the inverse operator. It is noticed that \(\partial_{Z} \partial_{\bar{Z}} \ln \rho_{Z\bar{Z}}\) does not belong to the image of \(\partial_{Z} \partial_{\bar{Z}}\). However, in order to have an idea of this non-local dependence let us be now very naive by writing

\[
\frac{1}{\partial_{Z} \partial_{\bar{Z}}} \partial_{Z} \partial_{\bar{Z}} \ln \rho_{Z\bar{Z}} \big|_{Z} = \ln \rho_{Z\bar{Z}} ,
\]

such that the functional (4.2) turns out to be nothing but the usual Liouville action with no cosmological term. The latter is manifestly not covariantly written it the \((Z, \bar{Z})\) system of complex coordinates and where the related Liouville field as introduced in [11]

\[
\phi(Z, \bar{Z}) = \ln \rho_{Z\bar{Z}}(Z, \bar{Z}), \quad \phi(Z, \bar{Z}) = \phi(W, \bar{W}) + \ln \left| \frac{dW}{dZ} \right|^{2}, \quad W = W(Z) ,
\]

is a non-local functional in the complex structure \(\mu\). In some sense, in this very naive approach the ill-definiteness in the \((Z, \bar{Z})\) complex coordinates of the action (1.2) into which (4.4) has been plugged does not matter because, fortunately, it will be conformally covariant in the background \((z, \bar{z})\) system of complex coordinates. On the one hand, one has that

\[
\phi(z, \bar{z}) = \left( \ln \rho - \ln \lambda - \ln \bar{\lambda} \right)(z, \bar{z})
\]
is a well-defined chirally split non-local scalar field in the background complex structure with

$$s \phi(z, \overline{z}) = \Omega(z, \overline{z}) + ((c \cdot \partial) \Phi)(z, \overline{z}),$$  \hspace{1cm} (4.7)$$

by virtue of eq.(2.4). And on the other hand, formula (4.2) now rewrites in a globally defined way on a compact Riemann surface without boundary,

$$\Gamma[\mu, \overline{\nu}, \rho] = \frac{k}{24\pi} \int_{\Sigma} \frac{dz \wedge d\overline{z}}{2i} \ln \left( \frac{\rho}{\lambda^\Lambda} \right) \left( \partial \overline{\sigma} - (\partial - \overline{\sigma}) (1 - \mu \overline{\nu})^{-1} \mu \partial - (\overline{\sigma} - \partial \mu) (1 - \mu \overline{\nu})^{-1} \partial \overline{\sigma} \right),$$  \hspace{1cm} (4.8)$$
after using the following decomposition for the symmetric scalar Laplacian \[4\],

$$dZ \wedge dZ = \frac{dz \wedge d\overline{z}}{2i} \left( \partial \overline{\sigma} - (\partial - \overline{\sigma}) (1 - \mu \overline{\nu})^{-1} \mu \partial - (\overline{\sigma} - \partial \mu) (1 - \mu \overline{\nu})^{-1} \partial \overline{\sigma} \right),$$  \hspace{1cm} (4.9)$$
and with the covariant derivative \[3.2\], \[\nabla \mu = \partial \mu + \mu \partial \ln \rho\], acting on \[\mu\]. In the course of the computation it is crucial to use eq.(2.4) for the integrating factor. Since it is just a change of coordinates, one easily checks, using the algebra \[3.1\] (or \[3.3\] combined with \[4.9\]), that (up to globally defined surface terms),

$$s \Gamma[\mu, \overline{\nu}, \rho] = A(\Omega, \rho, \mu, \overline{\nu})$$
the Weyl anomaly \[3.4\] in the \((Z, \overline{Z})\) complex coordinates.

4.1 A well-defined chirally split action integrating the diffeomorphism anomaly on a compact Riemann surface

Using the well-defined Liouville action proposed in \[23\],

$$\Gamma_{\text{Liouv}}(\varphi, g) = - \frac{k}{12\pi} \int_{\Sigma} \frac{dz \wedge d\overline{z}}{2i} \left( \frac{1}{2} \varphi \partial \overline{\varphi} \partial \varphi + \varphi \partial \overline{\varphi} \partial \varphi \ln \rho \right),$$  \hspace{1cm} (4.10)$$
where the new scalar Liouville field is \[23\],

$$\varphi \equiv \ln \hat{\rho} \overline{Z} - \ln \rho \overline{Z} = \ln \hat{\rho} - \ln \rho, \quad \hat{\rho} \overline{Z} \equiv \frac{\rho}{\lambda^\Lambda}. $$  \hspace{1cm} (4.11)$$
The sum \[\Gamma[\mu, \overline{\nu}, \rho] + \Gamma_{\text{Liouv}}(\varphi, g)\] corresponding to an effective action for 2d gravity in a background, yields (after dropping a well-defined surface term),

$$\Gamma[\mu, \overline{\nu}, \hat{\rho}] = - \frac{k}{12\pi} \int_{\Sigma} \frac{dz \wedge d\overline{z}}{2i} \left( \frac{1}{2} \ln \hat{\rho} \overline{Z} \partial \overline{\varphi} \partial \varphi \ln \hat{\rho} \overline{Z} \right),$$  \hspace{1cm} (4.12)$$
for which the same comments made for \[4.3\] apply and where now the related Liouville field depending in the background \[\hat{\rho}\],

$$\phi(Z, \overline{Z}) = \ln \hat{\rho} \overline{Z}(Z, \overline{Z}),$$  \hspace{1cm} (4.13)$$
is a non-local functional in the complex structure $\mu$. We stress again that the ill-definiteness in the $(Z, \bar{Z})$ complex coordinates does not matter because, fortunately, one can assert that (4.12) is conformally covariant in the background $(z, \bar{z})$ system of complex coordinates in which now, on the one hand\(^4\),

$$\phi(z, \bar{z}) = \left( \ln \hat{\rho} - \ln \lambda - \ln \bar{\lambda} \right)(z, \bar{z})$$  \hspace{1cm} (4.14)

is a well-defined chirally split, but non-local, scalar field in the $(z, \bar{z})$ complex structure, and on the other hand, as above, formula (4.12) now rewrites in a globally defined way on a compact Riemann surface without boundary,

$$\Gamma[\mu, \bar{\mu}; \hat{\rho}] = \frac{k}{24\pi} \int_{\Sigma} \frac{d\bar{Z} \wedge dz}{2i} \ln \left( \frac{\hat{\rho}}{\lambda \bar{\lambda}} \right) \left( (\partial - \partial_{\bar{\mu}}) \frac{\nabla_{\mu}}{1 - \mu \bar{\mu}} + (\bar{\partial} - \partial_{\mu}) \frac{\nabla_{\bar{\mu}}}{1 - \mu \bar{\mu}} - \partial \bar{\partial} \ln \hat{\rho} \right),$$  \hspace{1cm} (4.15)

after using once more (4.13), and where the covariant derivative $\nabla_{\mu} = \partial_{\mu} + \mu \partial \ln \hat{\rho}$ related to the background metric has been introduced. As before it is crucial to use eq. (2.4) for the integrating factor. This functional may be considered as the coupling of the “diffeomorphism-Goldstone scalar boson” $\phi$ to the gauge fields $\mu$ and $\bar{\mu}$. One also checks that

$$s \Gamma[\mu, \bar{\mu}; \hat{\rho}] = \Omega(c, \bar{c}, \mu, \bar{\mu}; \hat{\rho}) \equiv \frac{-k}{12\pi} \int_{\Sigma} \frac{d\bar{Z} \wedge dz}{2i} \Omega_{\text{Comp}} \partial_{\bar{\mu}} \rho \nabla_{\mu} \ln \hat{\rho}$$,

the non-chirally split diffeomorphism anomaly given in [25] depending on the background metric $\hat{\rho}$ and with the compensating ghost $\Omega_{\text{Comp}} \equiv - \left( \nabla_{\bar{c}} c + \nabla_{\bar{\mu}} c + \mu \partial c + \bar{\mu} \partial \bar{c} \right)$.

Then performing well-defined integrations by parts and dropping terms of the form $d\chi$, with $\chi$ a globally defined non-local one form on the compact Riemann surface, it is straightforward to find

$$\Gamma[\mu, \bar{\mu}; \hat{\rho}] = \frac{k}{24\pi} \int_{\Sigma} \frac{d\bar{Z} \wedge dz}{2i} \left( \ln \left( \frac{\hat{\rho}}{\lambda \bar{\lambda}} \right) \partial \bar{\partial} \ln \hat{\rho} + (\partial \ln \hat{\rho}) \nabla_{\mu} + (\bar{\partial} \ln \hat{\rho} - \partial \ln \lambda) \nabla_{\bar{\mu}} \right)$$

$$+ \frac{k}{12\pi} \int_{\Sigma} \frac{d\bar{Z} \wedge dz}{2i} \left( 1 - \mu \bar{\mu} \right) + \frac{1}{\Sigma} \left( \nabla_{\bar{\mu}} \nabla_{\mu} - \frac{1}{\Sigma} \partial \bar{\partial} \ln \hat{\rho} \right)(\nabla_{\mu})^2 - \frac{1}{\Sigma} \partial \bar{\partial} \ln \hat{\rho}) \right) \right) \left. \right|_{\Sigma}$$

$$\equiv -\Gamma_{\text{III}}(\hat{\rho}, \mu, \bar{\mu})$$

(4.17)

The last line exactly provides the globally defined local counterterm $\Gamma_{\text{III}}(\mu, \bar{\mu}; \hat{\rho})$ already found in [25] which is the appropriate form of that proposed in [31, 32, 33], insuring its global definition on an arbitrary compact Riemann surface without boundary. It is a necessary counterterm in order to obtain a chirally split form of the diffeomorphism anomaly. Once more eq. (2.4) for the integrating factor has been used. One can directly write out an holomorphically factorized effective action with a background metric by modifying (up to well-defined surface terms) the first integral in (4.17) as

$$\Gamma_{\text{Chiral}}[\mu, \bar{\mu}; \hat{\rho}] \equiv \Gamma[\mu, \bar{\mu}; \hat{\rho}] + \Gamma_{\text{III}}(\mu, \bar{\mu}; \hat{\rho}) = \frac{k}{12\pi} \int_{\Sigma} \frac{d\bar{Z} \wedge dz}{2i} \left( \frac{1}{2} \ln \left( \frac{\hat{\rho}}{\lambda \bar{\lambda}} \right) \partial \bar{\partial} \ln \hat{\rho} \right)$$

\(^4\)K. Yoshida in [7] used $\hat{\rho} = 1$. 

where both the curvature, \( \hat{\mathcal{R}} = \partial^2 \ln \hat{\rho} - \frac{1}{2} (\partial \ln \hat{\rho})^2 \), and the Schwarzian derivative of \( Z \) with respect to \( z \), \( \{ Z, z \} = \partial^2 \ln \lambda - \frac{1}{2} (\partial \ln \lambda)^2 \), explicitly appear. Remark that due to the first term in the integrand of (4.18), \( \Gamma_{\text{Chiral}}[\mu, \overline{\mu}; \hat{\rho}] \) can not be chirally split into the sum of two well-defined functionals, namely,

\[
\Gamma_{\text{Chiral}}[\mu, \overline{\mu}; \hat{\rho}] \neq \Gamma_{\text{Chiral}}[\mu, 0; \hat{\rho}] + \Gamma_{\text{Chiral}}[0, \overline{\mu}; \hat{\rho}] ,
\]

where clearly \( \delta_{\mu} \delta_{\overline{\mu}} \Gamma_{\text{Chiral}}[\mu, \overline{\mu}; \hat{\rho}] = 0 \). The former is related to the Liouville action for the field \( \phi \) as will be shown down below, (see formula (5.1)).

One has thus exhibited a class of Polyakov-like formulae as non-local functionals in the complex structure \( \mu \) and depending on a background metric \( \hat{\rho} \). The usual formula in the flat case \( [1] \) is recovered by setting \( \hat{\rho} = 1 \).

Furthermore, a direct comparison with the other proposal for a Polyakov action given by R. Zucchini shows that the integrating factor \( \lambda \) plays the role of the \( \mu \)-holomorphic \((1, 0)\)-differential introduced in \([8, 9]\). But it has the great advantage to never vanish since it is half of the Jacobian of the smooth change of complex variables \( z \rightarrow Z \). Moreover the holomorphic background \((1, 0)\)-differential is replaced by a \textit{non-vanishing} background \((1, 1)\)-differential, namely the background metric \( \hat{\rho} > 0 \). However the Polyakov action as proposed by Zucchini can be written in a chirally split form (equals sign in (4.19)).

Modulo a well-defined surface term, one directly checks that the chirally split diffeomorphism anomaly depending on the background metric \( \hat{\rho} \) is re-obtained by,

\[
s \Gamma_{\text{Chiral}}[\mu, \overline{\mu}; \hat{\rho}] = A(C, \mu; \hat{\rho}) + \overline{A(C, \mu; \hat{\rho})} ,
\]

with

\[
A(C, \mu; \hat{\rho}) = \frac{k}{12\pi} \int_\Sigma \frac{dx \wedge dz}{2i} C \left( \partial^2 \mu - (\partial - \mu \partial - 2\partial \mu) \hat{\mathcal{R}} \right) .
\]

However note that this action integrating the chirally split diffeomorphism anomaly has the following variation with respect to the background metric \( \hat{\rho} \)

\[
\delta_{\hat{\rho}} \Gamma_{\text{Chiral}}[\mu, \overline{\mu}; \hat{\rho}] = \frac{k}{12\pi} \left( \partial \overline{\nabla} \mu + \overline{\partial} \nabla \overline{\mu} - \overline{\partial} \overline{\nabla} \overline{\mu} \right) \delta \ln \hat{\rho} .
\]

In concluding this section, one may say that the construction of a Polyakov action heavily relies on the use of complex coordinates pertaining to the complex structure defined by \( \mu \) together with the integrating factor as well introduced by R. Stora in \([23, 34]\). The logarithm of the integrating factor turns out to be deeply related to the renormalization problem, namely it plays a crucial role in the search of a resummation for the Feynman perturbative series. As conjectured in \([28, 3]\), it has a universal character as defining the renormalized UV-behaviour at coincident points of the Feynman propagator \( \ln |z - w|^2 \) for free fields in the plane.
5 On the energy-momentum tensor in 2-d conformal field theory

It is well known that in a Lagrangian approach the energy-momentum is related to the Schwarzian derivative \( \{ Z, z \} \) and quite recently recalled in [10]. However, in this Lagrangian viewpoint, \( \mu \) and \( \pi \) are sources for the two components \( \Theta \) (respectively \( \Theta \)) of the energy momentum-tensor and yield the holomorphic factorization property of the correlation functions of the latter [36, 23, 19]. Geometrically one ought to expect that \( \text{Theta} \) is a true (2,0)-conformal field, namely a quadratic differential. In order to compute \( \delta \mu \Gamma_{\text{Chiral}}[\mu, \pi; \hat{\rho}] \) it will be technically much more convenient to rewrite the effective action (4.18) as

\[
\Gamma_{\text{Chiral}}[\mu, \pi; \hat{\rho}] = \frac{k}{12\pi} \int_{\Sigma} \frac{d\bar{z} \wedge dz}{2i} \left( \frac{1}{2} \left( \ln \hat{\rho} - \ln(\lambda \bar{\lambda}) \right) \partial \bar{\partial} \left( \ln(\lambda \bar{\lambda}) - \ln \hat{\rho} \right) - \bar{\partial} \ln \lambda \partial \ln \bar{\lambda} \right) + \mu (\hat{R} - \{ Z, z \}) + \bar{\partial} \ln \lambda (\partial \ln \hat{\rho} - \partial \ln \lambda) + \frac{\partial}{\partial \ln \bar{\lambda}} \left( \bar{\partial} \ln \hat{\rho} - \bar{\partial} \ln \bar{\lambda} \right),
\]

(5.1)

after the use of the identity due to equation (2.4), \( \partial \ln \lambda = D\mu \),

\[
\left( \ln(\lambda \bar{\lambda}) - \ln \hat{\rho} \right) \partial \bar{\partial} \ln(\lambda \bar{\lambda}) = \partial \left[ D\mu \left( \ln(\lambda \bar{\lambda}) - \ln \hat{\rho} \right) \right] + \bar{\partial} \left[ D\pi \left( \ln(\lambda \bar{\lambda}) - \ln \hat{\rho} \right) \right]
\]

\[
+ (\partial \ln \lambda - \partial \ln \hat{\rho}) D\mu + (\bar{\partial} \ln \bar{\lambda} - \bar{\partial} \ln \hat{\rho}) D\pi - 2D\mu D\pi.
\]

(5.2)

Note that the first term in (5.1) is a Liouville-like piece for the Liouville (scalar) field (4.14) and connects the two chiral sectors through the background metric. Then, it is straightforward to show that (modulo a well-defined total differential)

\[
\delta \mu \Gamma_{\text{Chiral}}[\mu, \pi; \hat{\rho}] = \frac{k}{12\pi} \int_{\Sigma} d\bar{z} \wedge dz \left( \frac{1}{2} \left( \ln \hat{\rho} - \ln(\lambda \bar{\lambda}) \right) \partial \bar{\partial} \left( \ln(\lambda \bar{\lambda}) - \ln \hat{\rho} \right) - \bar{\partial} \ln \lambda \partial \ln \bar{\lambda} \right) \delta \mu,
\]

(5.3)

where from eq.(2.4) \( (\bar{\partial} - \mu \partial) \delta \ln \lambda = D\delta \mu \). The energy-momentum tensor (5.3) turns out to be holomorphically factorized as a non-local current in the complex structure \( \mu \). Geometrically, as expected, thanks to the introduction of a background metric, it is a smooth quadratic differential with respect to the background complex coordinates \( (z, \bar{z}) \) as a difference of two (smooth) projective connections. Accordingly in the background metric, one has the following anomalous conformal Ward identities for the diffeomorphism symmetry

\[
(\bar{\partial} - \mu \partial - 2\partial \mu) \frac{\Gamma_{\text{Chiral}}[\mu, \pi; \hat{\rho}]}{\delta \mu} = \frac{k}{12\pi} \left( \partial^3 \mu - (\bar{\partial} - \mu \partial - 2\partial \mu) \hat{R} \right),
\]

(5.4)

since \( (\bar{\partial} - \mu \partial - 2\partial \mu) \{ Z, z \} = \partial^3 \mu \), and where the r.h.s. of (5.4) is the integrand of the global anomaly (4.21).

6 Background Liouville field versus background holomorphic projective connection

The above Polyakov formula (5.1) is supposed to be a resummation of the perturbative expansion in the correlation functions. Remark that the vacuum expectation value (VEV) of \( \Theta \) depends
on the background metric by quantum action principle, one has

$$< \Theta > _{\hat{\rho}} = \frac{\Gamma_{\text{Chiral}}[\mu, \pi; \hat{\rho}]}{\delta \mu} \bigg|_{\mu = \pi = 0} = \frac{k}{12 \pi} \hat{R}, \quad (6.1)$$

where the expression on the r.h.s. $\hat{R} = \partial^2 \ln \hat{\rho} - \frac{1}{2} (\partial \ln \hat{\rho})^2$, has already been proposed by D. Friedan in [37]. The property of holomorphy, $\partial < \Theta > _{\hat{\rho}} = 0$, is achieved if and only if the background metric $\hat{\rho}$ fulfills the following globally defined Liouville equation

$$\partial \partial \ln \hat{\rho} - a \hat{\rho} = 0 \quad (6.2)$$

with real constant $a$, that is the background metric is of constant scalar curvature. Recall that $\hat{\rho}$ belongs to the conformal class of the flat metric. On an arbitrary compact Riemann surface, one can convince oneself that the Liouville equation (6.2) turns out to be equivalent to the holomorphy of the curvature $\hat{\rho}$, namely $\overline{\partial} \hat{\rho} = 0$ [38].

Therefore, in the case where the background metric is governed by the Liouville equation (6.2), the chirally split diffeomorphism anomaly (4.21) writes

$$a(C, \mu; \hat{\rho}) = k \frac{1}{12 \pi} \int_\Sigma d\hat{z} \wedge d\hat{z} \partial^3 + 2 \hat{R} \partial \hat{R} \mu \quad (6.3)$$

and the energy-momentum tensor has a holomorphic VEV as required.

However, if it is not the case, that is $\hat{R}$ is only smooth, it is customary to introduce a background holomorphic projective connection $\hat{\mathcal{R}}$ by adding a further globally defined chirally split local counterterm [25]

$$\Gamma_{\text{II}}(\mu, \mu; \hat{\rho}, \hat{\mathcal{R}}, \hat{\mathcal{R}}, \mu, \mu) = \frac{k}{12 \pi} \int_\Sigma d\hat{z} \wedge d\hat{z} \partial^3 \left( \hat{R} \partial \hat{R} - \hat{\mathcal{R}} \partial \hat{\mathcal{R}} \right)$$

(6.4)

to the chirally split functional. More precisely, in this latter situation formula (5.1) is replaced by

$$\Gamma_{\text{Chiral}}[\mu, \pi; \hat{\rho}, \hat{\mathcal{R}}, \hat{\mathcal{R}}] \equiv \Gamma_{\text{Chiral}}[\mu, \pi; \hat{\rho}] + \Gamma_{\text{II}}(\mu, \mu; \hat{\rho}, \hat{\mathcal{R}}, \hat{\mathcal{R}})$$

with the replacement of $\hat{\mathcal{R}}$ by $\hat{\mathcal{R}}$,

$$\Gamma_{\text{Chiral}}[\hat{\rho}, \hat{\mathcal{R}}, \hat{\mathcal{R}}, \mu, \pi] = \frac{k}{12 \pi} \int_\Sigma d\hat{z} \wedge d\hat{z} \left( \frac{1}{2} \left( \ln \hat{\rho} - \ln (\lambda \bar{\lambda}) \right) \partial \partial \left( \ln (\lambda \bar{\lambda}) - \ln \hat{\rho} \right) + \partial \partial \ln \lambda \partial \ln \bar{\lambda} + \mu(\hat{\mathcal{R}} - \{Z, z\}) + \overline{\partial} \ln \lambda \left( \partial \ln \hat{\rho} - \partial \ln \lambda \right) + \overline{\partial} \ln \bar{\lambda} \left( \partial \ln \hat{\rho} - \partial \ln \bar{\lambda} \right) \right),$$

(6.5)

while a direct computation yields

$$\delta^s \Gamma_{\text{Chiral}}[\mu, \pi; \hat{\rho}, \hat{\mathcal{R}}, \hat{\mathcal{R}}] = \mathcal{A}(C, \mu; \hat{\mathcal{R}}) + \overline{\mathcal{A}}(C, \mu; \hat{\mathcal{R}}) \quad (6.6)$$

with the standard chirally split diffeomorphism anomaly independent of $\hat{\rho}$ recalled in (3.5), and for the variations, on the one hand,

$$\delta^s \Gamma_{\text{Chiral}}[\mu, \pi; \hat{\rho}, \hat{\mathcal{R}}, \hat{\mathcal{R}}] = \frac{k}{12 \pi} \partial \partial \ln \hat{\rho} \delta \ln \hat{\rho} = \frac{k}{48 \pi} R(\hat{\rho}) \delta \rho \quad (6.7)$$
is proportional to the scalar curvature \( R(\hat{g}) = -\frac{4}{\hat{\rho}} \partial \bar{\partial} \ln \hat{\rho} \) of the background metric \( \hat{g} \), and on the other hand,
\[
\delta_\mu \Gamma_{\text{Chiral}}[\mu, \bar{\tau}, \hat{\rho}, \bar{\mathcal{R}}] = \frac{k}{12\pi}(\mathcal{R} - \{Z, \bar{z}\})\delta_\mu.
\]
Hence the corresponding anomalous conformal Ward identities now write
\[
(\bar{\partial} - \mu \partial - 2\partial \mu) \frac{\Gamma_{\text{Chiral}}[\mu, \bar{\tau}, \hat{\rho}, \bar{\mathcal{R}}]}{\delta_\mu} = -\frac{k}{12\pi} \left( \partial^3 + 2\mathcal{R} \partial + \partial \mathcal{R} \right) \mu,
\]
while the VEV for the energy-momentum tensor now is
\[
< \Theta > = \frac{\Gamma_{\text{Chiral}}[\mu, \bar{\tau}, \hat{\rho}, \bar{\mathcal{R}}]}{\delta_\mu} \bigg|_{\mu=\bar{\tau}=0} = \frac{k}{12\pi} \mathcal{R},
\]
which does no longer depend on the background metric. Both the latter formulae can respectively be deduced from (5.4) and (6.1) by the direct substitution \( \hat{R} \rightarrow \mathcal{R} \).

However, note that the independence on the background of \( \Gamma_{\text{Chiral}} \) is no longer achieved. This is certainly the signature of the ill-treatment of the zero mode problem.

The background feature defined by the smooth metric \( \hat{\rho} \) has been partially shifted into the favour of a background holomorphic projective connection \( \mathcal{R} \). Recall either the latter survives at the level of the anomaly problem or the former occurs through the combination \( \hat{R} \), see (6.9) or (5.4) respectively. In any case, the background(s) \( \hat{\rho} \) (and \( \mathcal{R} \)) is (are) strongly related to the details of the model, besides the central charge \( k \), and encodes the global effects for instance in the VEV for the stress-energy tensor, say.

7 Conclusion and outlook

This very naive approach yields a well-defined holomorphically factorized formula valid on any higher genus compact Riemann surface without boundary where a background metric links the two chiral sectors. This formula reproduces the well-known Polyakov action in the plane. However, the starting point with the usual Liouville action was very wrong. Recall that in [12, 13] the independence in the Weyl factor is crucial and is achieved to the benefit of quantities depending on the complex analytic geometry only. Here the dependence on the background is the signature of the ill-treatment of the zero mode problem and especially the difficulty in defining a propagator on a curved space.

Performing directly in \((Z, \bar{Z})\) a holomorphic change of coordinates, \( F(Z) \), formula (4.18) receives non-trivial boundary terms depending on \( \ln F'(Z) \), the latter vanish if \( F \) is affine, i.e. the Riemann surface has null Euler characteristic (torus). For higher genus, the problem of finding a Polyakov action remains open and call for further investigation, especially in the use of the genuine Wess-Zumino action. In order to define the perturbative series, one should start with the difference of two such functionals integrating the Weyl anomaly
\[
\int_{\Sigma} \frac{dZ \wedge d\bar{Z}}{2i} \left( \frac{1}{2} \Phi \partial Z \partial \bar{Z} \Phi + \Phi \ln \rho \right) - \int_{\Sigma} \frac{d\bar{Z} \wedge dz}{2i} \left( \frac{1}{2} \Phi \partial \bar{Z} \partial \Phi + \Phi \ln \rho \right),
\]
and try to express the scalar field $\Phi$ as a non-local functional of the Beltrami coefficient $\mu$ through its equation of motion. In doing this difference, the problem of zero modes ought to be resolved \[39\]. Also, links with the WZNW approach as treated in \[3\] must be considered.

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