Response of distance measures to the equation of state

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ABSTRACT
We show that the distance measures (such as the luminosity and angular diameter distances) are linear functionals of the equation of state function $w(z)$ of the dark energy to a fair degree of accuracy in the regimes of interest. That is, the distance measures can be expressed as a sum of (i) a constant and (ii) an integral of a weighting function multiplied by the equation of state parameter $w(z)$. The existence of such an accurate linear response approximation has several important implications: (a) Fitting a constant-$w$ model to the data drawn from an evolving model has a simple interpretation as a weighted average of $w(z)$. (b) Any polynomial (or other expansion coefficients can also be expressed as weighted sums of the true $w$. (c) A replacement for the commonly used heuristic equation for the effective $w$, as determined by the CMB, can be derived and the result is found to be quite close to the heuristic expression commonly used. (d) The reconstruction of $w(z)$ by Huterer et al. (2002) can be expressed as a matrix inversion. In each case the limitations of the linear response approximation are explored and found to be surprisingly small.

Key words: cosmology: theory – methods: statistical – cosmological parameters.

1 INTRODUCTION
Current cosmological data suggests that the expansion of the universe is accelerating (e.g. Perlmutter et al. 1999; Riess et al. 1998; Efstathiou et al. 2002; Lewis & Bridle 2002; Melchiorri et al. 2002). A number of models have been proposed to explain this fact, the simplest of which is the cosmological constant. While a cosmological constant is enough to explain the current data, its constancy leads to a fine tuning problem (Sahni & Starobinski 2000; Peebles & Ratra, 2002; Padmanabhan, 2002b). A simple phenomenological generalization of the cosmological constant is to model the dark energy component that drives the acceleration as an ideal fluid with an equation of state given by $P = w\rho$ in which the equation of state parameter $w$ is allowed to vary with time. In this parameterization the cosmological constant corresponds to $w = -1$, while for other viable models $-0.6 \lesssim w \lesssim -1$ at the present epoch. Such a parameterization indeed arises naturally in several models, such as quintessence (Ratra & Peebles 1988; Wetterich 1988), K-essence (Armendariz-Picon et al. 1999) and a tachyonic scalar field (Gibbons 2002; Padmanabhan 2002a; Padmanabhan & Roy Choudhuri 2002; Bagla et al. 2003). However the precise form of $w(z)$ is model dependent.

In view of this there has been considerable interest in attempts to summarize the current (and future) data in terms of a few numbers. There are several ways of doing this, such as using a polynomial approximation for $w(z)$ (e.g. Weller & Albrecht, 2002) or in terms of derivatives of the expansion factor (called Stueffinders by Sahni et al. 2002). To be able to fit data using a polynomial form we have to choose a low order polynomial since realistic data contains only a finite amount of information. On the other hand we need to use enough polynomial orders so that the data is adequately described. Saini et al. (2003) investigate in some detail the issue of how to go about this and show that the current data and near future data seem to require at most a low order polynomial. (This, of course, does not imply that the true equation of state is also of a low order polynomial form.)

These methods attempt to capture the effect of an unknown function $w(z)$, which is equivalent to infinite number of parameters, in terms of a finite (and often small) number of parameters. In the limited redshift range probed by supernova data the true variations in $w(z)$ might be such that a low order polynomial fit is a reasonable approximation. Alternatively one can expand the function $w(z)$ in terms of a set of basis functions which are complete in the given redshift interval. If the basis functions are chosen judiciously only a small set of expansion coefficients will be required to describe the function $w(z)$ within the limits of the experimental accuracy. Owing to the random noise in cosmological data any of these choices could fit the data adequately in a maximum likelihood sense. Since the distance measures are integrals over nonlinear expressions involving $w(z)$, any parameter determined by such a maximum likelihood analysis represents some non-trivial average of the true $w(z)$. Our main aim in this paper is to show that the actual functional relation of the cosmological distance measures to $w(z)$ is not far from being linear. This allows us to deduce an approximate relation between the fitted coefficients and the true $w(z)$. This allows one to obtain a quantitative description of the averaging involved in reducing a function $w(z)$ to a finite set of numbers.

The plan of this paper is as follows. In Section 2 we derive the
The linear response approximation relating the coordinate distance and the equation of state of the dark energy, and show that within a reasonable range of parameters for the dark energy it works to better than $\sim 2\%$. In Section 3 we use this approximation to relate the fitted coefficients of a polynomial form to the true equation of state for the case of fitting to the luminosity distance, and show that they are related through a weighted integral of the true equation of state.

We then generalize this result to the case of an arbitrary functional form which is linear in the parameters. In Section 4 we derive a similar relationship for CMB where, in the simplest case, only one measurement of the angular distance to the last scattering surface is available. In Section 5 we show that the linear approximation also enables a non-parametric estimation of $w(z)$. Our conclusions are presented in Section 6.

2 RESPONSE OF GEOMETRY TO THE EQUATION OF STATE

The luminosity and angular diameter distances depend on the dark energy through the coordinate distance $r[w, z]$ according to the equations: $D_L(z) = (1 + z)r[w, z]$ and $D_A(z) = r[w, z]/(1 + z)$. The square brackets in this notation explicitly show that at any redshift $z$ the coordinate distance requires full knowledge of the function $w(z)$ in a flat universe the coordinate distance $r[w, z]$ is given by

$$r[w, z] = (1 + g)^{1/2} \int_1^{1 + z} dx x^{-3/2} [g + Q[w, x]]^{-1/2}$$

where $x = 1 + z$, $g = \Omega_m/(1 - \Omega_m)$,

$$Q[w, z] = \exp[3 \int_1^{1 + z} dx w(x)/x],$$

and we have set $c$ and $H_0$ equal to unity. To explore the dependence of $r[w, z]$ on parameter $w(z)$ we evaluate the sensitivity of $r$ to $w(z)$. This can be characterized by the functional derivative of $r$ with respect to $w(z)$. Since this is not a routine weapon in the arsenal of the astronomer, we shall briefly introduce the concept before applying it.

In the case of a real function $f(x)$, the sensitivity of the function to the independent variable $x$ can be characterized by the derivative $df/dx = f'(x)$. Broadly speaking, a large value for $f'(a)$ indicates that $f$ is relatively more sensitive to the independent variable around the point $x = a$; and a small value for $f'(a)$ will indicate relative insensitivity of $f$ to the independent variable around $x = a$. This follows directly from the definition of derivative of a function

$$f'(a) = \lim_{\epsilon \to 0} \frac{f(a + \epsilon) - f(a)}{\epsilon}.$$  

In the case of $f$ depending not on a single variable but on a function $p(x)$ we need to study how $f$ changes if the function $p(x)$ is changed slightly around a point $x = b$. This is best done by changing the function $p(x)$ to a new function $p_1(x) = p(x) + \epsilon \delta p(x - b)$ which adds a “spike” at $x = b$ with a strength proportional to $\epsilon$. We can now evaluate the value of $f$ for both $p(x)$ and $p_1(x)$. The difference in the numerical values of $f$ in the limit of $\epsilon \to 0$ is a good measure of the sensitivity of $f$ to the functional form of $p(x)$ around $x = b$. More formally, this functional derivative is defined by

$$\frac{\delta f[p, x]}{\delta p(b)} = \lim_{\epsilon \to 0} \frac{f[p + \epsilon \delta p(x - b), x] - f[p, x]}{\epsilon}.$$  

Just as the ordinary derivative of a function depends on the location at which it is evaluated, the functional derivative depends both on the form of $p(x)$ around which the sensitivity is measured, $x = \alpha$, and as well as the point $x = b$ at which the input function is perturbed.

For the purposes of this paper we need the response of the coordinate distance $r[w, z]$ at a redshift $z$ to a change in the equation of state at a different redshift, $z'$. This can be computed around a given fiducial $w(z) = w^{fid}(z)$ from the functional derivative

$$\frac{\delta r[w^{fid}, z]}{\delta w(z')} = \lim_{\epsilon \to 0} \frac{r[w^{fid} + \epsilon \delta w(z - z'), z] - r[w^{fid}, z]}{\epsilon}$$

defined exactly as in equation 4 (also see Huterer & Turner, 2001 for a similar application of this idea).

For the rest of this section we switch from discussing the coordinate distance to the luminosity distance and change the independent variable from redshift $z$ to $x = 1 + z$. Multiplying by $(1 + z)$ we obtain the response function for the luminosity distance $\delta D_L[w^{fid}, x]/\delta w(x') \equiv K_w(x, x')$. The subscript $w$ on the kernel denotes that it depends on $w^{fid}$, the fiducial equation of state around which the approximation holds. Evaluating the expression in Eq[4] we obtain

$$K_w(x, x') = \left\{ \begin{array}{ll}
\frac{3(1 + z) Q[w^{fid}, y]}{2x^2} & \int_{x'}^x \frac{dy}{y^{1/2}} \left[ Q[w^{fid}, y] \right]^{1/2} \\
0 & \text{(for } x < x')
\end{array} \right.$$  

The kernel $K_w$ is a function of two parameters, the redshift at which $w(z)$ is perturbed and the redshift at which we consider the change in the luminosity distance. A surface plot of this function is shown in Fig[1]. Since the effect of varying $w(z)$ at a redshift $z'$ is felt only at $z > z'$, the kernel is zero in half the plane. For small $\delta w$ we can use this result to approximate the calculation of the luminosity distance as

$$D_L[w^{fid} + \delta w, z] \approx D_L[w^{fid}, z] + \delta D_L$$

$$\equiv D_L[w^{fid}, z] + \int_0^z K_w(z, z') \delta w(z') dz'.$$

To be useful this linear response approximation should hold to a good accuracy for a reasonable range of $w(z)$. The SuperNova Acceleration Probe (SNAP) survey is expected to observe about 2000 Type 1a supernovae (SNe) up to a redshift $z \sim 1.7$, each year (Aldering et al. 2002). A single supernova will measure the luminosity distance with a relative error of $\sim 7\%$. If we bin the supernovae in redshifts interval of $0.02$, this will give a relative error
in the luminosity distance of about $\sim 1\%$. Saini et al. (2002) show that this gives a level of precision and given the present uncertainty in the value of $\Omega_m$, the data seems to require at the most a linear polynomial order in $w(z)$. To show how accurately Eq. 7 gives the luminosity distance in this restricted case, in Fig. 3 we plot in the $w_0$-$w_1$ plane the percentage difference between the true luminosity distance and the one computed through Eq. 7 at $z = 1.5$. The kernel $K$ was computed for $w_{\text{fid}}(z) = -0.5$ in this calculation. Within the region $-1 < w(z) < 0$ the approximation is better than 2%. Therefore, the accuracy of the linear approximation is quite acceptable with respect to the projected uncertainties in the luminosity distance obtained from a SNAP class experiment. The approximation works even better at smaller redshifts but less so at higher redshifts, but the measurement errors are expected to be larger here. Therefore, the conclusions drawn from this approximation are expected to remain valid until an unprecedented accuracy is achieved in the measurement of luminosity distance.

### 3 Interpretation of the fitting parameters

Since we have no fundamental understanding of the nature of dark energy, it is necessary to assume some suitable, versatile, functional form for $w(z)$ to fit the cosmological data. Since the luminosity distance depends on the equation of state through an integral relation, and the cosmological data is often fitted through maximum likelihood method, we can always add a small fast varying term to any well fitting $w(x)$, while still retaining a good fit. Due to this integral dependence any flexible parameterization eventually recovers only some integrated property of the underlying equation of state. The simple polynomial forms for $w(z)$ usually assumed for the purposes of fitting the cosmological data are often viewed as the first few terms in a series expansion of $w(z)$. Since the behaviour of $w(z)$ is not necessarily polynomial like, the recovered coefficients of the polynomial are not related in a simple manner to the true coefficients of the series expansion. In this Section we use the linear response approximation described above to compute the expectation value for the coefficients of the polynomial approximating the $w(z)$, and relate them to the “true” input $w(z)$. This will show that such fitting functions serve a useful purpose of measuring some integrated properties of the true equation of state of the dark energy.

#### 3.1 Interpreting the fit with a constant $w$

The simplest fit to the cosmological data is the constant $w$ model, $w_{\text{fit}}(z) = w_{0}^{\text{fit}}$. We are interested in relating this to the true $w(z) = w_{\text{true}}(z)$. For an arbitrary equation of state this relation is non-trivial and non-linear but given the approximations considered in § 2 we can construct the relationship analytically. Let us first consider fitting the luminosity distance to redshift. This requires a knowledge of the present day matter density $\Omega_m$, as well and, for simplicity, we shall assume that this is known. (When $\Omega_m$ is unknown, both $\Omega_m$ and $w$ become biased; see Maor et al. 2001). Our approach could be extended to the case in which $\Omega_m$ is not known, however this is beyond the scope of the present paper.

In the maximum likelihood reconstruction the quantity

$$\chi^2 = \int_0^{z_{\text{max}}} dz \left( \frac{D_L(z) - (D_L(\text{true}) + n(z))}{\sigma(z)} \right)^2$$

is minimized, where $n(z)$ is the noise in the measurement at redshift $z$ and $\sigma(z)$ is the variance. (We have replaced the conventional summation with an integral.) We now approximate the fitting function and the luminosity distance by

$$D_L(z) \approx D_L[w_{\text{fit}}, z] + \Delta w \int_0^{z_{\text{max}}} K_w(z, z') dz'$$

$$D_L(z) \approx D_L[w_{\text{fit}}, z] + \int_0^{z_{\text{max}}} K_w(z, z') \delta w(z') dz'$$

where we have assumed a constant $w_{\text{fit}}$ and defined $\Delta w = w_{0}^{\text{fit}} - w_{\text{true}}$ and $\delta w(z) = w_{\text{true}}(z) - w_{\text{fit}}$. Minimizing $\chi^2$ and taking the expectation value we obtain

$$\Delta w = \int_0^{z_{\text{max}}} \Phi_w(z') \delta w(z') dz'$$

where

$$\Phi_w(z) = \int K_w(x, x') \frac{K_w(x, 1 + z) + x'}{/ \sigma^2(x) dx dx'}$$

all where integrals are in the range $(0, z_{\text{max}})$. The noise term has dropped out since its expectation value is zero. Adding $w_{0}^{\text{fit}}$ to both sides of Eq. 11 we obtain

$$w_{0}^{\text{fit}} = \int_0^{z_{\text{max}}} \Phi_w(z') w_{\text{true}}(z') dz'$$

The fitted constant, $w_{\text{fit}}$, is just a weighted average of the true equation of state, with weighting function $\Phi_w$. This weighting function is shown in Fig. 3 for the case of supernovae distributed evenly from $z = 0$ to $z = 2.0$. It decreases steadily with redshift, as might be expected from the fact that all the supernovae are affected by the equation of state at $z = 0$, whereas the equation of state at $z > 2.0$ affects no supernovae. This result shows that the value of

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1 For example $w(x)$ and $w(x) + \theta \sin(x/\theta)$, where $\theta \to 0$, would both fit the data equally well though none of the derivatives agree. Although this is a contrived example this point has been well illustrated more realistically in Maor et al. (2000).
ever, quite cumbersome. We can simplify the notation a little by considering discretized expressions. Suppose the luminosity distance is known at a large number of, uniformly distributed redshifts $z_i$ where $i = 1, N$. We consider a model $w(z)$ which is given at redshifts $z''_k$ where $k = 1, M$.

After having chosen a $w^{\text{fid}}$, which for simplicity is taken to be a constant, we can define the vectors $\mathbf{d} \equiv \{\delta D_L(z_i)\}$ and $\mathbf{w} \equiv \{\delta w(z'_i)\}$ and matrix $\mathbf{K} \equiv \{K(z''_k, z'_j)\delta z\}$, where $\mathbf{K}$ is a $N \times M$ matrix and $\delta z$ denotes the redshift interval between the bins. With these definitions Eq. (11) becomes

$$\Delta w = \mathbf{u}^T \mathbf{M} \mathbf{w}^{\text{true}}; \quad \mathbf{M} = \mathbf{K}_w^T \mathbf{K}_w. \quad (17)$$

If instead we fit the data with a linear model $w^{\text{fit}} = w^{\text{fid}} + \Delta w_0 + \Delta w_1 z$ then, following the same procedure that led to Eq (11) we find

$$\Delta w_0 = \left[ (\mathbf{u}^T \mathbf{M} \mathbf{z}) \mathbf{u}^T \mathbf{M} - (\mathbf{u}^T \mathbf{M} \mathbf{z} \mathbf{z}^T) \mathbf{u}^T \right] \mathbf{w}^{\text{true}} \quad (18)$$

$$\Delta w_1 = \left[ (\mathbf{u}^T \mathbf{M} \mathbf{z} \mathbf{z}^T) \mathbf{u}^T - (\mathbf{u}^T \mathbf{M} \mathbf{z} \mathbf{z}^T) \mathbf{u}^T \mathbf{M} \mathbf{z} \mathbf{z}^T \right] \mathbf{w}^{\text{true}}, \quad (19)$$

where we have also defined the vector $\mathbf{z} = \{z_i\}$. This generalizes the concept of the weighting function to the linear case. It should be noted that the weighting function for $\Delta w_0$ is different than in Eq. (17), since we are also fitting for $w_1$. However, as expected, these results agree for the special case when $\mathbf{w}^{\text{true}} = \mathbf{0u}$. In the most general case we could approximate $w(z)$ as a linear combination of an arbitrary set of functions $F_i$ (e.g. Gerke & Efstathiou, 2002) as

$$w^{\text{fit}}(z) = w^{\text{fid}} + \sum_{i=1}^{N} c_i F_i(z) \quad (20)$$

In this case the coefficients $c_i$s are related to the true equation of state as follows. Defining the $N$ vectors $\mathbf{F}_i = \{F_i(z_k)\}$ we obtain $w^{\text{fit}} = \sum_{i=1}^{N} c_i \mathbf{F}_i$. On performing a maximum likelihood analysis we find an expression for the coefficients $c_i$ as follows

$$c_i = \sum_{i=1}^{N} a_{i,j}^{-1} y_j \quad (21)$$

where we have defined the two matrices

$$a_{i,j} = \mathbf{F}_i^T \mathbf{M} \mathbf{F}_j \quad (22)$$

$$y_i = \mathbf{F}_i^T \mathbf{M} \mathbf{w}^{\text{true}}. \quad (23)$$

## 4 Effective w Seen by the CMB

Current and near future supernova surveys will probe redshifts only up to around $z = 2$. We can also compute the effective equation of state probed by the Cosmic Microwave Background (CMB) at $z_{\text{cmb}} \sim 1000$. In the simplest case the CMB gives us the angular distance to the last scattering surface and therefore it does not give
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The weighting function which relates the effective equation of state $w_{\text{eff}}$ and the true equation of state $w(z)$ as described in the text. The solid line shows the weighting function computed from the linear response approximation using $w_{\text{fid}} = -1$ while the dotted line shows the weighting function inferred from $\Omega_Q$. The two functions agree to a good accuracy.

Figure 4. The weighting function which relates the effective equation of state $w_{\text{eff}}$ and the true equation of state $w(z)$ as described in the text. The solid line shows the weighting function computed from the linear response approximation using $w_{\text{fid}} = -1$ while the dotted line shows the weighting function inferred from $\Omega_Q$. The two functions agree to a good accuracy.

Figure 5. The weighting function for the CMB evaluated using $w_{\text{fid}} = 0$ (solid line). The dotted line again shows the weighting function inferred from $\Omega_Q$. The two functions do not agree to a good accuracy, as discussed in the text.

5 NON-PARAMETRIC RECONSTRUCTION OF THE EQUATION OF STATE PARAMETER

Although the fitting functions that we have considered so far provide useful information about the true equation of state of the dark energy, the linear approximation discussed above in principle allows a non-parametric reconstruction of the equation of state.

Consider a given SNe data set. To a first approximation we can fit it with a constant $w$, even though this may not be a good fit. We set the fiducial equation of state $w_{\text{fid}}$ to this best fit constant value in all that follows. We next compute the difference between the SNe distances and the $D_L$ computed for $w_{\text{fid}}$ to obtain the residuals $\delta D_L$. We use the notation described in the first paragraph of §3 to obtain $w_{\text{fid}}$. These are given by $d^*$, where the superscript $s$ signifies that these are obtained from measured supernovae distances. These are related to the $w(z)$ in bins through Eq. 16 where $w = \{w(z_b) - w_0\}$ with $w_0$ given by the constant $w_0$ fit. Since these are noisy estimates of distances we need to minimize the $\chi^2$ function with respect to $w$ to obtain the Maximum Likelihood estimator for $w$; that is, we minimize $\chi^2 = (d^* - K w)^T (d^* - K w)$ and obtain

$$w = (K^T K)^{-1} K^T d^* ,$$

where we have assumed the noise on $d^*$ is constant with redshift, although the result could be extended for the general case.

This equation also allows the trivial calculation of the Fisher matrix $\mathcal{F} = K^T K$ corresponding to the uncertainties on the reconstructed equation of state. It was noted by Huterer & Starkman (2002) that the Fisher matrix was a surprisingly weak function of the model parameters which matches with the result obtained above and shows that it reflects on the validity of the linear response approximation.

Although Eq.27 gives a formal solution of the problem, this estimation is, in general, very noisy. Huterer et al. show that even for SNAP-like data there are only a few principal components which are well determined by this method. It is clear that this lack of resolution is largely due to the fact that no constraints are imposed on the behaviour of $w$. We discuss some of the ways of rectifying this in a separate paper.
6 CONCLUSIONS

We have shown that in the relevant range of parameters expected for the dark energy the luminosity distance is a linear functional to the equation of state to a surprisingly high level of accuracy. This approximation allows us to find the relationship between the usual polynomial models for \( w(z) \) and the true underlying equation of state of the dark energy. Although the usual interpretation of these polynomial approximation is in terms of a series expansion of \( w(z) \), we show that the coefficients of the polynomial approximation are related to the true equation of state through an integral relation. Only in the exact polynomial-like equation of state would these approximations measure the real \( w(z) \). We show that the fitting of cosmological data with such forms is still useful since the parameters of such models measure certain, well-defined, integrated properties of the underlying equation of state. Finally, the approximation allows a formal, non-parametric way to measure the equation of state.

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REFERENCES

Aldering, G., et al. 2002, astro-ph/0209550
Armendariz-Picon, C., Damour, T., Mukhanov, V. 1999 Phys. Lett. B458, 209
Bagla, J., Jassal, H., & Padmanabhan, T. 2003 Phys. Rev. D in press astro-ph/0212198
Efstathiou, G. et al. 2002, MNRAS, 330, L29
Huey, G., Wang, L., Dave, R., Caldwell, R., & Steinhardt, P. J. 1999, Phys. Rev. D, 59, 63005
Huterer, D. & Turnier, M.-S., 2001, Phys. Rev. D, 64, 123527
Huterer, D. & Starkman, G. 2002, astro-ph/0207517
Gerke, B. F. & Efstathiou, G. 2002, MNRAS, 335, 33
Gibbons, G. W. 2002 hep-th/0204008
Lewis, A., & Bridle, S. 2002, Phys. Rev. D 66, 103511
Maor, I., Brustein, R., & Steinhardt, P. J. 2001, Phys. Rev. Lett. 86, 6
Maor, I., Brustein, R., McMahon, J., & Steinhardt, P. J. 2002, Phys. Rev. D, 65, 123003
Melchiorri, A., Mersini, L., Odman, C., & Trodden, M. 2002 astro-ph/0211522
Padmanabhan, T. 2002a Phys. Rev. D 66, 021301
Padmanabhan, T., 2002b hep-th/0212290
Padmanabhan, T., & Roy Choudhury, T. 2002 Phys. Rev. D66, 081301
Peebles, P.J.E., Ratra, B., 2002, astro-ph/0207347
Perlmutter, S. et al. 1999, ApJ, 517, 565
Ratra, B., & Peebles, P.J.E. 1988 Phys. Rev. D 37, 3406
Riess, A. G. et al. 1998, AJ, 116, 1009
Sahni, V. & Starobinsky, A. 2000, International Journal of Modern Physics D, 9, 373
Sahni, V., Saini, T. D., Starobinsky, A. A., & Alam, U. 2002 astro-ph/0201498
Saini, T. D., Weller, J, Bridle, S., 2003, in preparation
Weller, J. & Albrecht, A. 2002, Phys. Rev. D, 65, 103512
Wetterich, C. 1988 Nucl. Phys. B 302, 668