THE ONGOING BINOMIAL REVOLUTION

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Abstract. The Binomial Theorem has long been essential in mathematics. In one form or another it was known to the ancients and, in the hands of Leibniz, Newton, Euler, Galois, and others, it became an essential tool in both algebra and analysis. Indeed, Newton early on developed certain binomial series (see Section 3) which played a role in his subsequent work on the calculus. From the work of Leibniz, Galois, Frobenius, and many others, we know of its essential role in algebra. In this paper we rapidly trace the history of the Binomial Theorem, binomial series, and binomial coefficients, with emphasis on their decisive role in function field arithmetic. We also explain conversely how function field arithmetic is now leading to new results in the binomial theory via insights into characteristic \( p \) \( L \)-series.

1. Introduction

The Binomial Theorem has played a crucial role in the development of mathematics, algebraic or analytic, pure or applied. It was very important in the development of the calculus, in a variety of ways, and has certainly been as important in the development of number theory. It plays a dominant role in function field arithmetic. In fact, it almost appears as if function field arithmetic (and a large chunk of arithmetic in general) is but a commentary on this amazing result. In turn, function field arithmetic has recently returned the favor by shedding new light on the Binomial Theorem. It is our purpose here to recall the history of the Binomial Theorem, with an eye on applications in characteristic \( p \), and finish by discussing these new results.

We obviously make no claims here to being encyclopedic. Indeed, to thoroughly cover the Binomial Theorem would take many volumes. Rather, we have chosen to walk a quick and fine line through the many relevant results.

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2. Early History

According to our current understanding, the Binomial Theorem can be traced to the 4-th century B.C. and Euclid where one finds the formula for \((a + b)^2\). In the 3-rd century B.C. the Indian mathematician Pingala presented what is now known as “Pascal’s triangle” giving binomial coefficients in a triangle. Much later, in the 10-th century A.D., the Indian mathematician Halayudha and the Persian mathematician al-Karaji derived similar results as did the 13-th century Chinese mathematician Yang Hui. It is remarkable that al-Karaji appears to have used mathematical induction in his studies.

Indeed, binomial coefficients, appearing in Pascal’s triangle, seem to have been widely known in antiquity. Besides the mathematicians mentioned above, Omar Khayyam (in the
11-th century), Tartaglia, Cardano, Vi`ete, Michael Stifel (in the 16-th century), and William Oughtred, John Wallis, Henry Briggs, and Father Marin Mersenne (in the 17-th century) knew of these numbers. In the 17-th century, Blaise Pascal gave the binomial coefficients their now commonly used form: for a nonnegative integer \( n \) one sets
\[
\binom{n}{k} := \frac{n!}{k!(n-k)!} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k(k-1)(k-2)\cdots1}.
\]

With this definition we have the very famous, and equally ubiquitous, Binomial Theorem:
\[
(x + y)^n = \sum_k \binom{n}{k} x^k y^{n-k}.
\]

And of course, we also deduce the first miracle giving the integrality of the binomial coefficients \( \binom{n}{k} \).

Replacing \( n \) by a variable \( s \) in Equation (1) gives the binomial polynomials
\[
\binom{s}{k} := \frac{s(s-1)(s-2)\cdots(s-k+1)}{k(k-1)(k-2)\cdots1} = \frac{s(s-1)(s-2)\cdots(s-k+1)}{k!}.
\]

3. Newton, Euler, Abel, and Gauss

We now come to Sir Isaac Newton and his contribution to the Binomial Theorem. His contributions evidently were discovered in the year 1665 (while sojourning in Woolsthorpe, England to avoid an outbreak of the plague) and discussed in a letter to Oldenburg in 1676. Newton was highly influenced by work of John Wallis who was able to calculate the area under the curves \((1-x^2)^n\), for \( n \) a nonnegative integer. Newton then considered fractional exponents \( s \) instead of \( n \). He realized that one could find the successive coefficients \( c_k \) of \((-x^2)^k\), in the expansion of \((1-x^2)^s\), by multiplying the previous coefficient by \( \frac{s-k+1}{k} \) exactly as in the integral case. In particular, Newton formally computed the Maclaurin series for \((1-x^2)^{1/2}\), \((1-x^2)^{3/2}\) and \((1-x^2)^{1/3}\).

(One can read about this in the paper [Co1] where the author believes that Newton’s contributions to the Binomial Theorem were relatively minor and that the credit for discussing fractional powers should go to James Gregory – who in 1670 wrote down the series for \( b \left(1 + \frac{4}{5}\right)^{a/c} \). This is a distinctly minority viewpoint.)

In any case, Newton’s work on the Binomial Theorem played a role in his subsequent work on calculus. However, Newton did not consider issues of convergence. This was discussed by Euler, Abel, and Gauss. Gauss gave the first satisfactory proof of convergence of such series in 1812. Later Abel gave a treatment that would work for general complex numbers. The theorem on binomial series can now be stated.

**Theorem 1.** Let \( s \in \mathbb{C} \). Then the series \( \sum_{k=0}^{\infty} \binom{s}{k} x^k \) converges to \((1+x)^s\) for all complex \( x \) with \(|x| < 1\).

**Remark 1.** Let \( s = n \) be any integer, positive or negative. Then for all complex \( x \) and \( y \) with \(|x/y| < 1\) one readily deduces from Theorem 1 a convergent expansion
\[
(x + y)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^k y^{n-k}.
\]
It is worth noting that Gauss’ work on the convergence of the binomial series marks the first time convergence involving any infinite series was satisfactorily treated!

Now let \( f(x) \) be a polynomial with coefficients in an extension of \( \mathbb{Q} \). The degree of \( \binom{x}{k} \) as a polynomial in \( x \) is \( k \). As such one can always expand \( f(x) \) as a linear combination of \( \binom{x}{k} \). Such an expansion is called the \textit{Newton series} and can be traced back to his \textit{Principia Mathematica} (1687). The coefficients of such an expansion are given as follows.

**Definition 1.** Set \((\Delta f)(x) := f(x + 1) - f(x)\).

**Proposition 1.** We have
\[
f(x) = \sum_{k}(\Delta^k f)(0)\binom{x}{k}.
\] (5)

4. **The \( p \)-th Power Mapping**

Let \( p \) be a prime number and let \( \mathbb{F}_p \) be the field with \( p \)-elements. The following elementary theorem is then absolutely fundamental for number theory and arithmetic geometry. Indeed its importance cannot be overstated.

**Theorem 2.** Let \( R \) be any \( \mathbb{F}_p \)-algebra. Then the mapping \( x \mapsto x^p \) is a homomorphism from \( R \) to itself.

As is universally known, the proof amounts to expanding by the Binomial Theorem and noting that for \( 0 < i < p \), one has \( \binom{p}{i} \equiv 0 \pmod{p} \) as the denominator of Equation [1] is prime to \( p \).

According to Leonard Dickson’s history (Chapter III of [Di1]), the first person to establish (a form of) Theorem 2 was Gottfried Leibniz on September 20, 1680. One can then rapidly deduce a proof of Fermat’s Little Theorem (i.e., \( a^p \equiv a \pmod{p} \) for all integers \( a \) and primes \( p \)). Around 1830 Galois used iterates of the \( p \)-th power mapping to construct general finite fields.

It was 216 years after Leibniz (1896) that the equally essential \textit{Frobenius automorphism} (or \textit{Frobenius substitution}) in the Galois theory of fields was born. Much of modern number theory and algebraic geometry consists of computing invariants of the \( p \)-th power mapping/Frobenius map.

Drinfeld modules are subrings of the algebra (under composition!) of “polynomials” in the \( p \)-th power mapping; thus their very existence depends on the Binomial Theorem.

**Remark 2.** With regard to the proof of Theorem 2 it should also be noted that Kummer in 1852 established that the exact power of a prime \( p \) dividing \( \binom{n}{k} \) is precisely the number of “carries” involved in adding \( n - k \) and \( k \) when they are expressed in their canonical \( p \)-adic expansion.

5. **The Theorem of Lucas**

Basic for us, and general arithmetic in finite characteristic, is the famous Theorem of Lucas from 1878 [Lu1]. Let \( n \) and \( k \) be two nonnegative integers and \( p \) a prime. Write \( n \) and \( k \) \( p \)-adically as \( n = \sum_i n_i p^i, 0 \leq n_i < p \) and \( k = \sum_i k_i p^i, 0 \leq k_i < p \).

**Theorem 3.** (Lucas) We have
\[
\binom{n}{k} \equiv \prod_i \binom{n_i}{k_i} \pmod{p}.
\] (6)
Proof. We have \((1 + x)^n = (1 + x)^{\sum n_i p^i} = \prod_i (1 + x)^{n_i p^i}\). Modulo \(p\), Theorem \ref{thm:binomial} implies that \((1 + x)^n = \prod_i (1 + x^{p^i})^{n_i}\). The result then follows by expressing both sides by the Binomial Theorem and the uniqueness of \(p\)-adic expansions. \qed

6. The Theorem of Mahler

The binomial polynomials \(\binom{x^s}{k}\) (given in Equation \ref{eq:binomial}) obviously have coefficients in \(\mathbb{Q}\) and thus also can be considered in the \(p\)-adic numbers \(\mathbb{Q}_p\).

Proposition 2. The functions \(\binom{s}{k}\), \(k = 0, 1, \ldots\), map \(\mathbb{Z}_p\) to itself.

Proof. Indeed, \(\binom{s}{k}\) takes the nonnegative integers to themselves. As these are dense in \(\mathbb{Z}_p\), and \(\mathbb{Z}_p\) is closed, the result follows. \qed

Let \(y \in \mathbb{Z}_p\), and formally set \(f_y(x) := (1 + x)^y\). By the above proposition, \(f_y(x) \in \mathbb{Z}_p[[x]]\). As such, we can consider \(f_y(x)\) in any non-Archimedean field of any characteristic where it will converge on the open unit disc.

Let \(\{a_k\}\) be a collection of \(p\)-adic numbers approaching 0 as \(k \to \infty\) and put \(g(s) = \sum a_k \binom{s}{k}\); it is easy to see that this series converges to a continuous function from \(\mathbb{Z}_p\) to \(\mathbb{Q}_p\). Moreover, given a continuous function \(f : \mathbb{Z}_p \to \mathbb{Q}_p\), the Newton series (Equation \ref{eq:newton}) certainly makes sense formally.

Theorem 4. (Mahler) The Newton series of a continuous function \(f : \mathbb{Z}_p \to \mathbb{Q}_p\) uniformly converges to it.

The proof can be found in \cite{Mahler} (1958). The Mahler expansion of a continuous \(p\)-adic function is obviously unique.

Mahler’s Theorem can readily be extended to continuous functions of \(\mathbb{Z}_p\) into complete fields of characteristic \(p\). One can also find analogs of it that work for functions on the maximal compact subrings of arbitrary local fields. In characteristic \(p\), an especially important analog of the binomial polynomials was constructed by L. Carlitz as a byproduct of his construction of the Carlitz module (see, e.g., \cite{Carlitz}).

Carlitz’s construction can be readily described. Let \(e_k(x) := \prod (x - \alpha)\) where \(\alpha\) runs over elements of \(\mathbb{F}_q[t]\), \(q = p^{m_0}\), of degree \(< k\). As these elements form a finite dimensional \(\mathbb{F}_q\)-vector space, the functions \(e_k(x)\) are readily seen to be \(\mathbb{F}_q\)-linear. Set \(D_k := e_k(t^k) = \prod f\) where \(f(t)\) runs through the monic polynomials of degree \(k\). Carlitz then establishes that \(e_k(g)/D_k\) is integral for \(g \in \mathbb{F}_q[t]\).

Remark 3. The binomial coefficients \(\binom{s}{k}\) appear in the power series expansion of \((1 + x)^s\). It is very important to note that the the polynomials \(e_k(x)/D_k\) appear in a completely similar fashion in terms of the expansion of the Carlitz module – an \(\mathbb{F}_q[t]\)-analogue of \(G_m\); see, e.g., Corollary 3.5.3 of \cite{Goss}.

Now let \(k\) be any nonnegative integer written \(q\)-adically as \(\sum_t k_t q^t\), \(0 \leq k_t < q\) for all \(t\).

Definition 2. We set

\[
G_k(x) := \prod_t \left( \frac{e_t(x)}{D_t} \right)^{k_t}.
\] (7)


The set \( \{G_k(x)\} \) is then an excellent characteristic \( p \)-replacement for \( \{ \binom{s}{k} \} \) in terms of analogs of Mahler’s Theorem, etc, see [Wa1]. In 2000 K. Conrad [Con1] showed that Carlitz’s use of digits in constructing analogs of \( \binom{s}{k} \) can be applied quite generally.

In a very important refinement of Mahler’s result, in 1964 Y. Amice [Am1] gave necessary and sufficient conditions on the Mahler coefficients guaranteeing that a function can be locally expanded in power series. In fact, Amice’s results work for arbitrary local fields and are also essential for the function field theory. Indeed, as the function \((1 + x)^y, y \in \mathbb{Z}_p\), is clearly locally analytic, Amice’s results show that its expansion coefficients tend to 0 very quickly, thus allowing for general analytic continuation of \( L \)-series and partial \( L \)-series [Go2].

In 2009, S. Jeong [Je1] established that the functions \( u \mapsto u^y, y \in \mathbb{Z}_p \) precisely comprise the group of \textit{locally-analytic} endomorphisms of the \( 1 \)-units in a local field of finite characteristic.

7. Measure Theory

Given a local field \( K \) with maximal compact \( R \), one is able to describe a theory of integration for all continuous \( K \)-valued functions on \( R \). A \textit{measure on \( R \) with values in \( R \)} is a finitely-additive, \( R \)-valued function on the compact open subsets of \( R \). Given a measure \( \mu \) and a continuous \( K \)-valued function \( f \) on \( R \), the Riemann sums for \( f \) (in terms of compact open subsets of \( R \)) are easily seen to converge to an element of \( K \) naturally denoted \( \int_R f(z) \, d\mu(z) \).

Given two measures \( \mu_1 \) and \( \mu_2 \), we are able to form their convolution \( \mu_1 \ast \mu_2 \) in exactly the same fashion as in classical analysis. In this way, the space of measures forms a commutative \( K \)-algebra.

In the case of \( \mathbb{Q}_p \) and \( \mathbb{Z}_p \) one is able to use Mahler’s Theorem (Theorem 4 above) to express integrals of general continuous functions in terms of the integrals of binomial coefficients.

Now \((1 + z)^{x+y} = (1 + z)^x (1 + z)^y \) giving an \textit{addition formula} for the binomial coefficients. Using this in the convolution allows one to establish that the convolution algebra of measures (the \textit{Iwasawa algebra}) is isomorphic to \( \mathbb{Z}_p[[X]] \).

In finite characteristic, we obtain a \textit{dual} characterization of measures that is still highly mysterious and \textit{also} depends crucially on the Binomial Theorem. So let \( q \) be a power of a prime \( p \) as above. Let \( n \) be a nonnegative integer written \( q \)-adically as \( \sum n_k q^k \). Thus, in characteristic \( p \), we deduce

\[
(x + y)^n = \prod_k (x + y)^{n_k q^k} = \prod_k (x^{q^k} + y^{q^k})^{n_k}
\]

Now recall the definition of the functions \( G_n(x) \) (Definition 2 above) via digit expansions. As the functions \( e_j(x) \) are also \textit{additive} we immediately deduce from Equation 8 the next result.

\textbf{Theorem 5.} We have

\[
G_n(x + y) = \sum_{j=0}^{n} \binom{n}{j} G_j(x)G_{n-j}(x). \tag{9}
\]

In other words, the functions \( \{G_n(x)\} \), \textit{also} satisfy the Binomial Theorem!

Let \( \mathcal{D}_j \) be the hyperdifferential (= “divided derivative”) operator given by \( \mathcal{D}_j z^i := \binom{i}{j} z^{i-j} \). Let \( R\{\{\mathcal{D}\}\} \) be the algebra of formal power-series in the \( \mathcal{D}_i \) with the above multiplication rule where \( R \) is any commutative ring. Note further that this definition makes sense for all \( R \) precisely since \( \binom{i}{j} \) is always integral.
Let \( A = \mathbb{F}_q[t] \) and let \( f \in A \) be irreducible; set \( R := A_f \), the completion of \( A \) at \( f \). Using the Binomial Theorem for the Carlitz polynomials we have the next result \([Go3]\).

**Theorem 6.** The convolution algebra of \( R \)-valued measures on \( R \) is isomorphic to \( R\{\{D\}\} \).

**Remark 4.** The history of Theorem 6 is amusing. I had calculated the algebra of measures using the Binomial Theorem and then showed the calculation to Greg Anderson who, rather quickly(!), recognized it as the ring of hyperderivatives/divided power series.

**Remark 5.** One can ask why we represent the algebra of measures as operators as opposed to divided power series. Let \( \mu \) be a measure on \( \mathbb{R} \) (\( \mathbb{R} \) as above) and let \( f \) be a continuous function; one can then obtain a new continuous function \( \mu(f) \) by
\[
\mu(f)(x) := \int_R f(x + y) \, d\mu(y).
\]

The operation of passing from the expansion of \( f \) (in the Carlitz polynomials) to the expansion of \( \mu(f) \) formally appears as if the differential operator attached to \( \mu \) acted on the expansion. This explains our choice.

8. **The group \( S(p) \) and binomial symmetries in finite characteristic**

Let \( q = p^{m_0} \), \( p \) prime, as above, and let \( y \in \mathbb{Z}_p \). Write \( y \) \( q \)-adically as
\[
y = \sum_{k=0}^{\infty} y_k q^k
\]
where \( 0 \leq y_k < q \) for all \( k \). If \( y \) is a nonnegative integer (so that the sum in Equation 11 is obviously finite), then we set \( \ell_q(y) = \sum_k y_k \).

Let \( \rho \) be a permutation of the set \( \{0, 1, 2, \ldots\} \).

**Definition 3.** We define \( \rho_\ast(y) \), \( y \in \mathbb{Z}_p \), by
\[
\rho_\ast(y) := \sum_{i=0}^{\infty} y_k q^{\rho(i)}.
\]

Clearly \( y \mapsto \rho_\ast(y) \) is a bijection of \( \mathbb{Z}_p \). Let \( S(q) \) be the group of bijections of \( \mathbb{Z}_p \) obtained this way. Note that if \( q_0 \) and \( q_1 \) are powers of \( p \), and \( q_0 \mid q_1 \), then \( S(q_1) \) is naturally realized as a subgroup of \( S(q_0) \).

**Proposition 3.** Let \( \rho_\ast(y) \) be defined as above.
1. The mapping \( y \mapsto \rho_\ast(y) \) is a homeomorphism of \( \mathbb{Z}_p \).
2. ("Semi-additivity") Let \( x, y, z \) be three \( p \)-adic integers with \( z = x + y \) and where there is no carry over of \( q \)-adic digits. Then \( \rho_\ast(z) = \rho_\ast(x) + \rho_\ast(y) \).
3. The mapping \( \rho_\ast(y) \) stabilizes both the nonnegative and nonpositive integers.
4. Let \( n \) be a nonnegative integer. Then \( \ell_q(n) = \ell_q(\rho_\ast(n)) \).
5. Let \( n \) be an integer. Then \( n \equiv \rho_\ast(n) \pmod{q-1} \).

For the proof, see [Go4].

**Proposition 4.** Let \( \sigma \in S(p) \), \( y \in \mathbb{Z}_p \), and \( k \) a nonnegative integer. Then we have
\[
\binom{y}{k} \equiv \binom{\sigma y}{\sigma k} \pmod{p}.
\]
Proof. This follows immediately from the Theorem of Lucas (Theorem 3).

**Corollary 1.** Modulo $p$, we have $\left(\frac{\sigma y}{y}\right)_k = \left(\frac{y}{y-\sigma}\right)_k$.

**Corollary 2.** We have $p \mid \left(\frac{y}{y-\sigma}\right)_k$ if and only if $p \mid \left(\frac{\sigma y}{y}\right)_k$.

**Proposition 5.** Let $i$ and $j$ be two nonnegative integers. Let $\sigma \in S(p)$. Then

$$\left(\frac{i+j}{i}\right) \equiv \left(\frac{\sigma i + \sigma j}{\sigma i}\right) \pmod{p}. \quad (14)$$

**Proof.** The theorems of Lucas and Kummer show that if there is any carry over of $p$-adic digits in the addition of $i$ and $j$, then $\left(\frac{i+j}{i}\right)$ is 0 modulo $p$. However, there is carry over of the $p$-adic digits in the sum of $i$ and $j$ if and only if there is carry over in the sum of $\sigma i$ and $\sigma j$; in this case both sums are 0 modulo $p$. If there is no carry over, then the result follows from Part 2 of Proposition 3 and Proposition 4. \qed

Let $R$ be as in the previous section.

**Corollary 3.** The mapping $\mathcal{D}_i \mapsto \mathcal{D}_{\sigma i}$ is an automorphism of $R\{\{\mathcal{D}\}\}$.

It is quite remarkable that the group $S(q)$ very much appears to be a symmetry group of characteristic $p$ $L$-series. Indeed, in examples, this group preserves the orders of trivial zeroes as well as the denominators of special zeta values (the “Bernoulli-Carlitz” elements). Moreover, given a nonnegative integer $i$, one has the “special polynomials” of characteristic $p$ $L$-series arising at $-i$. It is absolutely remarkable, and highly nontrivial to show, that the degrees of these special polynomials are invariant of the action of $S(q)$ on $i$. Finally, the action of $S(q)$ even appears to extend to the zeroes themselves of these characteristic $p$ functions. See [Go4] for all this.

9. The Future

We have seen how the Binomial Theorem has impacted the development of both algebra and analysis. In turn these developments have provided the foundations for characteristic $p$ arithmetic. Furthermore, as in Section 8, characteristic $p$ arithmetic has contributed results relating to the Binomial Theorem of both an algebraic (automorphisms of $\mathcal{Z}_p$ and binomial coefficients) and analytic (automorphisms of algebras of divided derivatives) nature. Future research should lead to a deeper understanding of these recent offshoots of the Binomial Theorem as well as add many, as yet undiscovered, new ones.

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