ON THE TYPE OF THE VON NEUMANN ALGEBRA OF AN OPEN SUBGROUP OF THE NERETIN GROUP

RYOYA ARIMOTO

Abstract. The Neretin group $N_{d,k}$ is the totally disconnected locally compact group consisting of almost automorphisms of the tree $T_{d,k}$. This group has a distinguished open subgroup $O_{d,k}$. We prove that this open subgroup is not of type I. This gives an alternative proof of the recent result of P.-E. Caprace, A. Le Boudec and N. Matte Bon which states that the Neretin group is not of type I, and answers their question whether $O_{d,k}$ is of type I or not.

1. Introduction

The Neretin group $N_{d,k}$ was introduced by Yu. A. Neretin in [Ne] as an analogue of the diffeomorphism group of the circle. This group $N_{d,k}$ consists of almost automorphisms of the tree $T_{d,k}$ and is a totally disconnected locally compact Hausdorff group. It has a distinguished open subgroup $O_{d,k}$; for an accurate definition, see Section 3.1. Recently, P.-E. Caprace, A. Le Boudec and N. Matte Bon proved that the Neretin group $N_{d,k}$ is not of type I by constructing two weakly equivalent but inequivalent irreducible representations of $N_{d,k}$ ([CBMB]). In their paper, they conjectured that the subgroup $O_{d,k}$ of the Neretin group $N_{d,k}$ is not type I either ([CBMB, Remark 4.8]). Our main theorem answers their question.

Theorem. The group von Neumann algebra of $L(O_{d,k})$ of the open subgroup $O_{d,k}$ of the Neretin group $N_{d,k}$ is of type II. In particular, the open subgroup $O_{d,k}$ of the Neretin group $N_{d,k}$ is not of type I.

This theorem gives an alternative proof of the fact that the Neretin group $N_{d,k}$ is not of type I, since the type I property is inherited to open subgroups. In the proof of our main theorem, we construct a nontrivial central sequence in the corner of the group von Neumann algebra $L(O_{d,k})$.

2020 Mathematics Subject Classification. 20E08, 22D10, 46L10.

Key words and phrases. Neretin group, type I group, group von Neumann algebra.
Acknowledgements. The author would like to express his deep gratitude to his supervisor, Professor Narutaka Ozawa, for his support and providing many insightful comments. This work was supported by JST SPRING, Grant Number JPMJSP2110 and by JSPS KAKENHI, Grant Number 20H01806.

2. Preliminaries

2.1. Von Neumann algebras. We refer the reader to [Di] for basics about von Neumann algebras. We review several topologies we use. Let $H$ be a separable Hilbert space. For $\xi \in H$, seminorms $p_\xi , p^*_\xi$ on $B(H)$ are defined by $p_\xi (x) = \| x\xi \|$ and $p^*_\xi (x) = \| x^*\xi \|$. The topology defined by these seminorms $\{ p_\xi \mid \xi \in H \} \cup \{ p^*_\xi \mid \xi \in H \}$ on $B(H)$ is called strong-* operator topology. For $\{ \xi_n \} \in \ell^2 \otimes H = \{ \{ \xi_n \} \mid \xi_n \in H , \sum_{n=1}^\infty \| \xi_n \|^2 < \infty \}$, seminorms $q_{\{\xi_n\}} , q^*_{\{\xi_n\}}$ are defined by $q_{\{\xi_n\}}(x) = (\sum_{n=1}^\infty \| x\xi_n \|^2)^{1/2}$ and $q^*_{\{\xi_n\}}(x) = (\sum_{n=1}^\infty \| x^*\xi_n \|^2)^{1/2}$. The topology defined by these seminorms $\{ q_{\{\xi_n\}} \mid \{ \xi_n \} \in \ell^2 \otimes H \} \cup \{ q^*_{\{\xi_n\}} \mid \{ \xi_n \} \in \ell^2 \otimes H \}$ on $B(H)$ is called ultrastrong-* topology. Note that these two topologies coincide on bounded subsets of $B(H)$.

We also review definitions of types of von Neumann algebras. A von Neumann algebra $M$ is of type I if it is isomorphic to $\prod_{j \in J} A_j \otimes B(H_j)$ for some set $J$ of cardinal numbers, where $A_j$ is an abelian von Neumann algebra and $H_j$ is a Hilbert space of dimension $j$. A von Neumann algebra $M$ is of type II$_1$ if it has no nonzero summand of type I and there exists a separating family of normal tracial states. A von Neumann algebra $M$ is of type II$_\infty$ if it has no nonzero summand of type I or II$_1$ but there exists an increasing net of projections $\{ p_i \}_{i \in I} \subset M$ converging strongly to $1_M$ such that $p_i M p_i$ is of type II$_1$ for every $i \in I$. A von Neumann algebra $M$ is of type II if it is a direct sum of a type II$_1$ and a type II$_\infty$ von Neumann algebra. A von Neumann algebra $M$ is of type III if it has no nonzero summand of type I, II$_1$ or II$_\infty$. Every von Neumann algebra $M$ has a unique decomposition $M \cong M_I \oplus M_{II} \oplus M_{III}$ where $M_I, M_{II}, M_{III}$ are of type I, type II, type III respectively.

We review types of von Neumann algebras from the perspective of central sequences and obtain a criterion of having no nonzero type I summand.

Definition. Let $M$ be a separable von Neumann algebra. A central sequence of $M$ is a sequence $\{ u_n \}$ of unitary elements in $M$ such that $[x, u_n]$ converges to 0 in the ultrastrong-* topology for all $x \in M$. A central sequence $\{ u_n \}$ of $M$ is trivial if there exists a sequence $\{ z_n \}$ of unitary elements of the center of $M$ such that $u_n - z_n$ converges to 0 in the ultrastrong-* topology.
Remark. A sequence \( \{u_n\} \) of unitary elements in \( M \) is a central sequence if and only if there exists \( M_0 \subset M \) such that \( M_0'' = M \) and for all \( x \in M_0 \), \( [x, u_n] \to 0 \) in the ultrastrong-* topology.

A. Connes showed that any type I factor has no nontrivial central sequence ([Co Corollary 3.10]) and this fact can be easily extended to type I von Neumann algebras.

**Lemma 2.1.** Let \( M \) be a separable von Neumann algebra. If \( M \) is of type I, then every central sequence of \( M \) is trivial.

**Proof.** We may assume that \( M \) is isomorphic to \( \mathcal{A} \otimes B(H) \) for some separable abelian von Neumann algebra \( \mathcal{A} \) and some separable Hilbert space \( H \). Let \( \{u_n\} \) be a central sequence in \( M \). Take some unit vector \( \eta_0 \in H \) and let \( p \in B(H) \) be the projection onto \( \mathbb{C}\eta_0 \). Then there exist \( a_n \in \mathcal{A} \) such that \( (1 \otimes p)u_n(1 \otimes p) = a_n \otimes p \in \mathcal{A} \otimes pB(H)p \approx \mathcal{A} \otimes \mathbb{C}p \). Since \( \mathcal{A} \) is abelian, there exists a unitary element \( v_n \in \mathcal{A} \) such that \( a_n = v_n|a_n| \). We will show \( u_n - v_n \otimes 1 \to 0 \) in the strong-* topology. First, we will show \( u_n - a_n \otimes 1 \to 0 \) in the strong-* topology. Fix a faithful representation \( \mathcal{A} \subset B(K) \) and take \( \xi \in K, \eta \in H \) arbitrarily. Then, for sufficiently large \( n \),

\[
\begin{align*}
    u_n(\xi \otimes \eta) &\approx (1 \otimes (\eta \otimes \eta_0^*))u_n(\xi \otimes \eta_0) \\
    &= (1 \otimes (\eta \otimes \eta_0^*))((a_n \otimes p)(\xi \otimes \eta_0)) \\
    &= (a_n \otimes 1)(\xi \otimes \eta)
\end{align*}
\]

where \( \eta \otimes \eta_0^* \) is a Schatten form; \( \eta \otimes \eta_0^*(\xi) = (\xi, \eta_0)\eta \). Similarly, one has \( u_n^*(\xi \otimes \eta) \approx (a_n^* \otimes 1)(\xi \otimes \eta) \) for sufficiently large \( n \). Finally, we should prove \( |a_n| \to 1 \) in \( \mathcal{A} \) in the ultrastrong-* topology; if this holds, then \( a_n \otimes 1 - v_n \otimes 1 = v_n((|a_n| - 1) \otimes 1) \to 0 \) in the ultrastrong-* topology. Since \( t \mapsto \sqrt{t} \vee 0 \) is a linear growth function, it suffices to prove \( a_n^*a_n \to 1 \) in the strong-* topology. For arbitrary \( \xi \in K \),

\[
\begin{align*}
    \|a_n^*a_n\xi - \xi\| &= \|(a_n^*a_n \otimes p)\xi \otimes \eta_0 - \xi \otimes \eta_0\| \\
    &= \|((1 \otimes p)u_n^*(1 \otimes p)u_n(1 \otimes p)\xi \otimes \eta_0 - \xi \otimes \eta_0\| \\
    &\to 0.
\end{align*}
\]

Therefore, a central sequence \( \{u_n\} \) in \( M \) is trivial. \( \square \)

**Lemma 2.2.** Let \( M \) be a separable von Neumann algebra. Suppose there exists a faithful normal state \( \varphi \) and two central sequences \( \{u_n\}, \{v_n\} \) such that \( \varphi((u_nv_nu_n^*v_n^*)^k) \) converges to 0 for every \( k \in \mathbb{Z} \setminus \{0\} \). Then \( M \) has no nonzero type I summand.

**Proof.** For simplicity, we write \( u_nv_nv_n^*v_n^* \) as \( w_n \). Note that for every \( f \in C(\mathbb{T}) \), \( \varphi(f(w_n)) \to \int_{\mathbb{T}} f(z) \, dz \) where \( \mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\} \), since trigonometric polynomials are dense in \( C(\mathbb{T}) \).
Let \( p \in M \) be a central projection such that \( pM \) is of type I. Since every central sequence in a type I von Neumann algebra is trivial and \( \{p u_n\} \) and \( \{p v_n\} \) are central sequences in \( pM \), \( p w_n \) converges to \( p \) in the ultraweak* topology. Then for every \( f \in C(\mathbb{T}) \), \( \varphi(p f(w_n)) \to \varphi(p)f(1) \). Take \( \varepsilon > 0 \) arbitrarily and \( f \in C(\mathbb{T}) \) such that \( f \geq 0 \), \( f(1) = 1 \) and \( \int_T f(z) \, dz < \varepsilon \). Then \( \varphi(f(w_n)) \geq \varphi(p f(w_n)) \), so \( \varphi(p) \leq \int_T f(z) \, dz < \varepsilon \). Since \( \varepsilon \) is arbitrary, \( \varphi(p) = 0 \), i.e., \( p = 0 \). Therefore \( M \) has no nonzero type I summand. \( \Box \)

2.2. Hecke algebras. We refer the reader to [KLQ] and [LLN] for definitions and basic properties of Hecke algebras.

Suppose \((G, H)\) is a Hecke pair and \( H \backslash G \) is a discrete space. Then the Hecke algebra \( \mathcal{H}(G, H) \) acts on \( \ell^2(H \backslash G) \) from left; define \( \lambda : \mathcal{H}(G, H) \to B(\ell^2(H \backslash G)) \) by

\[
[\lambda(f)\xi](Hx) = \sum_{H y \in H \backslash G} f(Hx y^{-1}) \xi(H y)
\]

for \( f \in \mathcal{H}(G, H) \) and \( \xi \in \ell^2(H \backslash G) \). We may omit \( \lambda \) and write \( \mathcal{H}(G, H) \subset B(\ell^2(H \backslash G)) \).

Let \( \rho : G \to B(\ell^2(H \backslash G)) \) be the right quasi-regular representation defined by \( [\rho, \xi](x) = \xi(x s) \). One can easily check that \( \mathcal{H}(G, H) \subset \rho(G)' \). Moreover, one has \( \mathcal{H}(G, H)^\sim = \rho(G)' \) (see [AD] Theorem 1.4.). The unit vector \( \delta_H \in \ell^2(H \backslash G) \) is a trace on \( \mathcal{H}(G, H) \). Since all \( \delta_H \) is a \( \rho(G) \)-cyclic vector. Moreover, if \( R(x) = R(x^{-1}) \) for every \( x \in G \), then it is not hard to see that \( \delta_H \) is a tracial vector, i.e., the vector state associated with \( \delta_H \) is a trace on \( \lambda(\mathcal{H}(G, H)) \). In particular, the vector state \( x \mapsto \langle x \delta_H, \delta_H \rangle \) is a faithful tracial state of \( \mathcal{H}(G, H) \) for a unimodular locally compact group \( G \) and its compact open subgroup \( H \).

For a finite group \( G \) and its subgroup \( H \leq G \), note that the Hecke algebra \( \mathcal{H}(G, H) \) is identical to \( p_H \mathbb{C}[G] p_H \) where \( p_H = \frac{1}{|H|} \sum_{h \in H} h \in \mathbb{C}[G] \) is a projection (see [KLQ] Corollary 4.4]).

The next proposition is a special case of [LLN] Proposition 1.3].

**Proposition 2.3.** Let \( G \) be a finite group acting on a finite group \( V \), and let \( \Gamma \) be a subgroup of \( G \) leaving a subgroup \( V_0 \) of \( V \) invariant. Then we have a canonical embedding \( \mathcal{H}(V, V_0)^\Gamma \hookrightarrow \mathcal{H}(V \times G, V_0 \times \Gamma) \). Moreover, the canonical traces are consistent with this embedding.

**Proof.** We will prove that there exists a canonical, trace preserving embedding \( (p_{V_0} \mathbb{C}[V] p_{V_0})^\Gamma \hookrightarrow p_{V_0 \times \Gamma} \mathbb{C}[V \times G] p_{V_0 \times \Gamma} \) where \( p_H = \frac{1}{|H|} \sum_{h \in H} h \) for a subgroup \( H \). Since \( \Gamma \) leaves \( V_0 \) invariant, \( p_{V_0} \) commutes with every element of \( \Gamma \) in \( \mathbb{C}[V_0 \times \Gamma] \). In particular, \( p_{V_0} \) commutes with \( p_T \) and \( p_{V_0} p_T = p_T p_{V_0} \). Note that \( p_T \) commutes with every element in \( \mathbb{C}[V]^\Gamma \). Therefore, multiplication with \( p_T \) is a *-homomorphism from \( (p_{V_0} \mathbb{C}[V] p_{V_0})^\Gamma \approx p_{V_0} \mathbb{C}[V]^\Gamma p_{V_0} \) to \( p_{V_0 \times \Gamma} \mathbb{C}[V \times G] p_{V_0 \times \Gamma} \). This map preserves the canonical trace, since it
is spatially implemented by the canonical isometry $W: \ell^2(V_0 \setminus V) \to \ell^2((V_0 \rtimes \Gamma) \setminus (V \rtimes G))$, and $W^* \delta_{V_0 \rtimes \Gamma} = \delta_{V_0}$. Since the canonical traces are faithful, this $*$-homomorphism is an embedding. □

**Corollary 2.4.** In addition to the assumptions of Proposition 2.3, suppose $G$ leaves $V_0$ invariant. Then there is a canonical trace preserving embedding $\mathcal{H}(G, \Gamma) \hookrightarrow \mathcal{H}(V \rtimes G, V_0 \rtimes \Gamma)$ and $\mathcal{H}(V, V_0)^G \subset \mathcal{H}(G, \Gamma)'$ in $\mathcal{H}(V \rtimes G, V_0 \rtimes \Gamma)$.

**Proof.** The same argument as above shows that the first assertion holds. To show the second assertion, we identify $\mathcal{H}(V, V_0)^G$ and $\mathcal{H}(G, \Gamma)$ with $p_{V_0}C[V]^G p_{V_0}$ and $p_T C[G] p_T$, respectively. The assertion follows from the fact that $p_{V_0} p_T = p_T p_{V_0}$ and $C[V]^G \subset C[G]'$. □

### 2.3. Locally compact groups.

In this paper, topological groups are assumed to be Hausdorff. Let $G$ be a locally compact second countable group and $\mu$ be its left Haar measure. The **left regular representation** of $G$ is a unitary representation $\lambda: G \to U(L^2(G))$ defined by $(\lambda_g f)(h) = f(g^{-1} h)$ for $f \in L^2(G)$ where $L^2(G)$ is a Haar square integrable functions on $G$. The von Neumann algebra $\{ \lambda_g | g \in G \}'' \subset B(L^2(G))$ is called the group von Neumann algebra. The representation $\lambda$ extends to a representation of $L^1(G)$: $\lambda(f) g = f * g$ for $f \in L^1(G)$ and $g \in L^2(G)$.

A unitary representation $(\pi, H)$ of $G$ is called of **type I** if the associated von Neumann algebra $\pi(G)'' \subset B(H)$ is of type I. A locally compact group $G$ is called of **type I** if all its unitary representations are of type I. See [BH] Chapter 6, 7 for more details and properties of type I groups.

### 3. Neretin groups

Let $d, k \geq 2$ be integers and $T_{d,k}$ be a rooted tree such that the root has $k$ adjacent vertices and the others have $d + 1$ adjacent vertices. An **almost automorphism** of $T_{d,k}$ is a triple $(A, B, \varphi)$ where $A, B \subset T_{d,k}$ are finite subtrees containing the root with $|\partial A| = |\partial B|$ and $\varphi: T_{d,k} \setminus A \to T_{d,k} \setminus B$ is an isomorphism. The **Neretin group** $\mathcal{N}_{d,k}$ is the quotient of the set of all almost automorphisms by the relation which identifies two almost automorphisms $(A_1, B_1, \varphi_1), (A_2, B_2, \varphi_2)$ if there exits a finite subtree $\tilde{A} \subset T_{d,k}$ containing the root such that $A_1, A_2 \subset \tilde{A}$ and $\varphi_1|_{T_{d,k} \setminus \tilde{A}} = \varphi_2|_{T_{d,k} \setminus \tilde{A}}$. One can easily check that $\mathcal{N}_{d,k}$ is a group.

Let $d$ be the graph metric on $T_{d,k}$, $v_0$ be the root of $T_{d,k}$ and $B_n := \{ v \in T_{d,k} | d(v_0, v) \leq n \}$ for $n \geq 0$. Every automorphism of $T_{d,k}$ leaves $B_n$ invariant. For each $n \geq 0$, $\mathcal{O}_{d,k}^{(n)}$ denotes the subgroup consisting of automorphisms on $T_{d,k} \setminus B_n$ and we write $\mathcal{O}_{d,k} := \bigcup_{n=0}^{\infty} \mathcal{O}_{d,k}^{(n)}$. Each $\mathcal{O}_{d,k}^{(n)}$ is a subgroup of $\mathcal{N}_{d,k}$ containing $K := \text{Aut} (T_{d,k})$. Let $V_n :=$
\[ \partial B_n = \{ v \in T_{d,k} \mid d(v, v_0) = n \}. \] Note that \( O_{d,k}^{(n)} \cong \text{Aut} (T_{d,d}) \wr S_{|V_n|} = \text{Aut} (T_{d,d})^{[V_n]} \rtimes S_{|V_n|} \) and \( O_{d,k}^{(n)} \wr S_{|V_n|} < O_{d,k}^{(n+1)}. \)

The Neretin group \( N_{d,k} \) admits a totally disconnected locally compact group topology such that the inclusion map \( K \hookrightarrow N_{d,k} \) is continuous and open ([GL, Theorem 4.4]). The Neretin group \( N_{d,k} \) is compactly generated and simple; see [GL].

The group \( O_{d,k} \) is an open subgroup of \( N_{d,k}. \) It is unimodular and amenable since \( O_{d,k} \) is an increasing union \( \bigcup_{n=1}^{\infty} O_{d,k}^{(n)} \) of its compact subgroups.

4. Proof of Theorem

We normalize the Haar measure \( \mu \) on \( O_{d,k} \) so that \( \mu (K) = 1. \) Let \( p = \lambda (\chi_K) \) be the projection onto the subspace of left \( K \)-invariant functions. This subspace can be identified with \( \ell^2 (K \backslash O_{d,k}). \) The Hecke algebra \( H (O_{d,k}, K) \subset B (\ell^2 (K \backslash O_{d,k})) \) is a dense subalgebra of the corner \( pL (O_{d,k})p \subset B (\ell^2 (K \backslash O_{d,k})) \) with respect to the weak operator topology. We will show that \( pL (O_{d,k})p \) is of type II.

Since \( K \) acts on \( V_n, \) there exists a canonical group homomorphism \( K \to \text{Aut} (V_n) \cong S_{|V_n|}. \) The range of this homomorphism is denoted by \( P_n = \text{Aut} (B_n) \subset S_{|V_n|}. \) Similarly, let \( Q_n \) be the range of the canonical group homomorphism \( \text{Aut} (T_{d,d}) \to \text{Aut} (W_n), \) where \( W_n \) is the subset \( \{ v \in T_{d,d} \mid d(v, v_0) = n \} \) of \( T_{d,d}. \) One has \( H (O_{d,k}, K) \cong \bigcup_{n=1}^{\infty} H (O_{d,k}^{(n)}, K) \) and \( H (O_{d,k}^{(n)}, K) \cong H (S_{|V_n|}, P_n). \) We use this identification freely. For finite groups \( G_1, G_2 \) and their subgroups \( H_i < G_i, \) \( H (G_1, H_1) \otimes H (G_2, H_2) \cong H (G_1 \times G_2, H_1 \times H_2). \) Proposition 2.3 for \( G = S_{|V_n|}, \Gamma = P_n, V = S_{|V_n|}^{(1)}, V_0 = Q_I^{(1)} \) implies

\[
( (H (O_{d,d}^{(l)}, \text{Aut} (T_{d,d})))^{\otimes |V_n|} )^{P_n} \cong (H (S_{d|V_n|}, Q_l^{(1)}))^{P_n} \\
\rightarrow H (S_{d|V_n|} \rtimes S_{|V_n|}, Q_l^{(1)} \rtimes P_n) \\
= H (S_{d|V_n|} \rtimes S_{|V_n|}, P_{n+l}) \\
< H (S_{|V_n|} \rtimes P_{n+l})
\]

for \( l \in \mathbb{N}. \) Moreover, Corollary 2.4 implies \( H (S_{d|V_n|} \rtimes Q_l^{(1)}, S_{|V_n|}^{(1)} \rtimes P_{n+l}) \subset H (S_{|V_n|} \rtimes P_{n+l}). \) Since \( H (O_{d,d}^{(l)}, \text{Aut} (T_{d,d})) \cong H (S_{d|V_n|} \rtimes Q_l, S_{|V_n|} \rtimes P_{n+l}) \) is not a Gelfand pair for \( l \geq 3 \) (see [GM, Theorem 1.2]), \( H (O_{d,d}^{(3)}, \text{Aut} (T_{d,d})) \) is noncommutative.

Let \( \tau \) be the vector state associated with \( \delta_K \in \ell^2 (K \backslash O_{d,k}). \) This is a trace, since \( O_{d,k} \) is a unimodular locally compact group and \( K \) is its compact open subgroup. Note that \( \tau (x^{\otimes |V_n|}) = (\tau (x))^{V_n} \) for \( x \in H (O_{d,d}^{(3)}, \text{Aut} (T_{d,d})) \) where \( \tau \) also denotes the canonical trace on \( H (O_{d,d}^{(3)}, \text{Aut} (T_{d,d})). \) Since \( H (O_{d,d}^{(3)}, \text{Aut} (T_{d,d})) \) is a non-commutative finite dimensional algebra, there exist two unitaries \( u, v \in H (O_{d,d}^{(3)}, \text{Aut} (T_{d,d})) \) such that \( |\tau ((u^* v^* u v)^k)| < 1 \) for \( k \geq 1. \)
and $|\tau((v^*u^*v^*)^k)| < 1$ for all $k \in \mathbb{Z} \setminus \{0\}$. Set $u_n := u^{\otimes |V_n|} \in \mathcal{H}(\mathcal{O}^{(n)}_{d,k}, K)'$ and $v_n := v^{\otimes |V_n|} \in \mathcal{H}(\mathcal{O}^{(n)}_{d,k}, K)'$. Then for every $x \in \mathcal{H}(\mathcal{O}_{d,k}, K)' = pL(\mathcal{O}_{d,k})p$, $[x, u_n] \to 0$ and $[x, v_n] \to 0$ in the ultrastrong-$\ast$ topology. Thus $\{u_n\}$ and $\{v_n\}$ are central sequences. In addition, $\tau((u_n v_n u_n^* v_n^*)^k) = \tau((uvu^*v^*)^k) \to 0$ as $n \to \infty$ for every $k \in \mathbb{Z} \setminus \{0\}$. So by Lemma 2, $pL(\mathcal{O}_{d,k})p$ has no nonzero type I summand and it is of type II.

Let $K_n := \{\varphi \in K \mid \varphi|_{B_n} = \text{id}_{B_n}\}$ and $p_n := \frac{1}{\mu(K_n)} \lambda(\chi_{K_n}) \in L(\mathcal{O}_{d,k})$. Then $\{p_n\}$ converges $1_{L(\mathcal{O}_{d,k})}$ in the strong operator topology. Applying the same argument as above to $p_n L(\mathcal{O}_{d,k})p_n$, one finds that $p_n L(\mathcal{O}_{d,k})p_n$ is of type II. Therefore $L(\mathcal{O}_{d,k})$ is of type II.

References

[AD] C. Anantharaman-Delaroche; Approximation properties for coset spaces and their operator algebras. 24th International Conference on Operator Theory, 2015, 23–45.

[BH] B. Bekka and P. de la Harpe; Unitary Representations of Groups, Duals, and Characters. Mathematical Surveys and Monographs, 250. American Mathematical Society, Providence, RI, 2020.

[BO] N. Brown and N. Ozawa; $C^\ast$-algebras and Finite-Dimensional Approximations. Graduate Studies in Mathematics, 88. American Mathematical Society, Providence, RI, 2008.

[CBMB] P.-E. Caprace, A. Le Boudec and N. Matte Bon; Piecewise strongly proximal actions, free boundaries and the Neretin groups. preprint, arXiv:2107.07765v2.

[Co] A. Connes; Almost periodic states and factors of type III$_1$. Journal of Functional Analysis, 16.4 (1974), 415–445.

[Di] J. Dixmier; Von Neumann Algebras. translated from the second French edition by F. Jellett, North-Holland, Amsterdam, New York, Oxford, 1981.

[GL] L. Garncarek and N. Lazarovich; The Neretin groups. In P. Caprace and N. Monod (Eds.), New Directions in Locally Compact Groups (London Mathematical Society Lecture Note Series, pp. 131–144). Cambridge University Press, Cambridge, 2018.

[GM] C. Godsil and K. Meagher; Multiplicity-free permutation representations of the symmetric group. Annals of Combinatorics, 13.4 (2010), 463–490.

[KLQ] S. Kaliszewski, M. B. Landstad and J. Quigg; Hecke $C^\ast$-Algebras, Schlichting Completions and Morita Equivalence. Proceedings of the Edinburgh Mathematical Society, 51.3 (2008), 657–695.

[LLN] M. Laca, Nadia S. Larsen and S. Neshveyev; Hecke algebras of semidirect products and the finite part of the Connes-Marcolli $C^\ast$-algebra. Advances in Mathematics, 217.2 (2008), 449–488.

[Ne] Yu. A. Neretin; On Combinatorial Analogs of the Group of Diffeomorphisms of the Circle. Russian Academy of Sciences, Izvestiya Mathematics, 41.2 (1993), 337–349.

RIMS, Kyoto University, 606-8502 Japan

Email address: arimoto@kurims.kyoto-u.ac.jp