Some Properties of $m$-th Root Finsler Metrics

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Abstract

In this paper, we prove that every $m$-th root metric with isotropic mean Berwald curvature reduces to a weakly Berwald metric. Then we show that an $m$-th root metric with isotropic mean Landsberg curvature is a weakly Landsberg metric. We find necessary and sufficient condition under which conformal $\beta$-change of an $m$-th root metric be locally dually flat. Finally, we prove that the conformal $\beta$-change of locally projectively flat $m$-th root metrics are locally Minkowskian.

Keywords: Conformal change, $m$-th root metric, $\beta$-change, Locally dually flat metric, projectively flat metric.

1 Introduction

Let $(M, F)$ be a Finsler manifold of dimension $n$, $TM$ its tangent bundle and $(x^i, y^i)$ the coordinates in a local chart on $TM$. Let $F$ be the following function on $M$, by $F = \sqrt[m]{A}$, where $A$ is given by $A := a_{i_1...i_m}(x)y^{i_1}y^{i_2}...y^{i_m}$ with $a_{i_1...i_m}$ symmetric in all its indices [4][9][14][15][16]. Then $F$ is called an $m$-th root Finsler metric. The theory of $m$-th root metric has been developed by Shimada [14], and applied to Biology as an ecological metric [2]. It is regarded as a direct generalization of Riemannian metric in a sense, i.e., the second root metric is a Riemannian metric.

Let $(M, F)$ be a Finsler manifold of dimension $n$. Denote by $\tau(x, y)$ the distortion of the Minkowski norm $F_x$ on $T_xM_0$, let $\sigma(t)$ be the geodesic with $\sigma(0) = x$ and $\sigma(0) = y$. The rate of change of $\tau(x, y)$ along Finslerian geodesics $\sigma(t)$ called $S$-curvature. $F$ is said to have isotropic $S$-curvature and almost isotropic $S$-curvature if $S = (n+1)cF$ and $S = (n+1)cF + dh$, respectively, where $c = c(x)$ and $h = h(x)$ are scalar functions on $M$ and $dh = h_{ij}(x)dy^idy^j$ is the differential of $h$ [19]. Taking twice vertical covariant derivatives of the $S$-curvature gives rise the $E$-curvature. The Finsler metric $F$ is called weakly Berwald metric if $E = 0$ and is said to have isotropic mean Berwald curvature if $E = \frac{c}{n+1}cFh$, where $c = c(x)$ is a scalar function on $M$ and $h = h_{ij}dx^idy^j$ is the angular metric.

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Theorem 1.1. Let $F = \sqrt[n]{A}$ be an $m$-th root Finsler metric on an open subset $U \subseteq \mathbb{R}^n$.

(i) For a scalar function $c = c(x)$ on $M$, the following are equivalent:

(a) $S = (n+1)cF + \eta$;

(b) $S = \eta$.

(ii) For a scalar function $c = c(x)$ on $M$, the following are equivalent:

(a) $E = \frac{n+1}{2}cF^2$;

(b) $E = 0$.

Let $(M, F)$ be a Finsler manifold. There are two basic tensors on Finsler manifolds: fundamental metric tensor $g_{ij}$ and the Cartan torsion $C_{ij}$, which are second and third order derivatives of $\frac{1}{2}F^2$ at $y \in T_x M$, respectively. Taking a trace of Cartan torsion $C_{ij}$ give us the mean Cartan torsion $I_{ij}$. The rate of change of the Cartan torsion along Finslerian geodesics, $L_{ij}$, is said to be Landsberg curvature \[17\] \[18\]. Taking a trace of Landsberg curvature $L_{ij}$, yields the mean Landsberg curvature $J_{ij}$. $F$ is called isotropic mean Landsberg curvature if $J_{ij} = cF I_{ij}$, where $c = c(x)$ is a scalar function on $M$.

Theorem 1.2. Let $(M, F)$ be a non-Riemannian $m$-th root Finsler manifold. For a scalar function $c = c(x)$ on $M$, the following are equivalent:

(i) $J + cFI = 0$;

(ii) $J = 0$.

There are two important transformation in Finsler geometry: conformal change and $\beta$-change. Two metric functions $F$ and $\tilde{F}$ on a manifold $M$ are called conformal if the length of an arbitrary vector in the one is proportional to the length in the other, that is if $\tilde{g}_{ij} = \varphi g_{ij}$. The length of vector $\varepsilon$ means here the fact that $\varphi g_{ij}$, as well as $g_{ij}$, must be Finsler metric tensor. He showed that $\varphi$ falls into a point function. A change of Finsler metric $F \to \tilde{F}$ is called a $\beta$-change of $F$, if $\tilde{F}(x, y) = F(x, y) + \beta(x, y)$, where $\beta(x, y) = b_i(x)y^i$ is a 1-form on a smooth manifold $M$. It is easy to see that, if $\sup_{F(x,y)=1} |b_i(x)y^i| < 1$, then $\tilde{F}$ is again a Finsler metric. The notion of a $\beta$-change has been proposed by Matsumoto, named by Hashiguchi-Ichijyo and studied in detail by Shibata \[6\] \[8\] \[13\]. If the Finsler metric $F$ reduces to a Riemannian metric then $\tilde{F}$ reduces to a Randers metric. Due to this reason, the $\beta$-change has been called the Randers change of Finsler metric, also.

Let $(M, F)$ be a Finsler manifold. In this paper, we are consider the conformal $\beta$-changes of Finsler metrics

$\tilde{F} = e^{\alpha(x)}F + \beta$,

where $\beta(x, y) = b_i(x)y^i$ is a 1-form on a smooth manifold $M$ and $\alpha = \alpha(x)$ is the conformal factor. It is easy to see that, if $\sup_{F(x,y)=1} \|\beta\| < 1$, then $\tilde{F}$ is again a Finsler metric.
Let $F = \sqrt[2]{A}$ be an $m$-th root Finsler metric on an open subset $U \subset \mathbb{R}^n$. Put
\[ A_i = \frac{\partial A}{\partial y^i}, \quad A_{ij} = \frac{\partial^2 A}{\partial y^i \partial y^j}, \quad A_{x^i} = \frac{\partial A}{\partial x^i}, \quad A_0 = A_{x^i} y^i. \]

Suppose that $A_{ij}$ define a positive definite tensor and $A^{ij}$ denotes its inverse. The following hold
\[ g_{ij} = \frac{A^{\frac{2}{m^2}}}{m^2} [mA_{ij} + (2 - m) A_i A_j], \]
\[ g^{ij} = A^{-\frac{2}{m}} [mA^{ij} + \frac{m - 2}{m - 1} y^i y^j], \]
\[ y^i A_i = mA, \quad y^i A_{ij} = (m - 1) A_j, \quad A^{ij} A_i = \frac{1}{m - 1} y^j, \]
\[ y_i = \frac{1}{m} A^{\frac{2}{m} - 1} A_i, \quad A_i A_j A^{ij} = \frac{m}{m - 1} A. \]

In [1], Amari-Nagaoka introduced the notion of dually flat Riemannian metrics when they study the information geometry on Riemannian manifolds. A Finsler metric $F$ on an open subset $U \subset \mathbb{R}^n$ is called dually flat if it satisfies \((F^2)_{x^i y^j} y^k = 2(F^2)_{x^i}^{x^j} x^k\) [12][19].

In this paper, we consider conformal $\beta$-change of locally dually flat $m$-th root Finsler metrics and prove the following.

**Theorem 1.3.** Let $F = \sqrt[2]{A}$ be an $m$-th root Finsler metric on an open subset $U \subset \mathbb{R}^n$, where $A$ is irreducible. Suppose that $\bar{F} = e^\alpha F + \beta$ be conformal $\beta$-change of $F$ where $\beta = b_i(x)y^i$ and $\alpha = \alpha(x)$. Then $\bar{F}$ is locally dually flat if and only if there exists a 1-form $\theta = \theta_i(x)y^i$ on $U$ such that the following hold
\[ \beta_0 \beta + \beta_i \beta_0 = 2\beta_{x^i}, \quad (1) \]
\[ A_{x^i} = \frac{1}{3m} [mA_{\theta_1} + 2\theta A_i + 2(\alpha_0 A_i - \alpha_{x^i} A)], \quad (2) \]
\[ \beta[(\frac{1}{m} - 2) A_i A^{-1} A_0 - 4A_{x^i} + \alpha_0 A_i] + 2[A_i \beta_0 + (A_0 \beta)_i] = -2me^\alpha A \Psi, \quad (3) \]
where $\beta_0 = \beta_{x^i} y^k$, $\alpha_0 = \alpha_{x^i} y^i$, $\beta_{x^i} = \beta_i = (b_i)(x^j) y^i$, $\beta_0 = \beta_{x^i} y^i$, $\beta_0 = (b_i)_{x^i}$ and $\Psi = \alpha_0 \beta_i + \beta_0 - 2\beta_{x^i} - 2\alpha_{x^i} \beta$.

A Finsler metric is said to be locally projectively flat if at any point there is a local coordinate system in which the geodesics are straight lines as point sets. It is known that a Finsler metric $F(x, y)$ on an open domain $U \subset \mathbb{R}^n$ is locally projectively flat if and only if $G^i = P y^i$, where $P(x, \lambda y) = \lambda P(x, y)$, $\lambda > 0$ [7]. Finally, we study conformal $\beta$-change of locally projectively flat $m$-th root metrics and prove the following.

**Theorem 1.4.** Let $F = \sqrt[2]{A}$ be an $m$-th root Finsler metric on an open subset $U \subset \mathbb{R}^n$, where $A$ is irreducible. Suppose that $\bar{F} = e^\alpha F + \beta$ be conformal $\beta$-change of $F$ where $\beta = b_i(x)y^i$ and $\alpha = \alpha(x)$. Then $\bar{F}$ is locally projectively flat if and only if it is locally Minkowskian.
2 Preliminaries

Let $M$ be a $n$-dimensional $C^\infty$ manifold. Denote by $T_x M$ the tangent space at $x \in M$, by $TM = \bigcup_{x \in M} T_x M$ the tangent bundle of $M$ and by $TM_0 = TM \setminus \{0\}$ the slit tangent bundle. A Finsler metric on $M$ is a function $F : TM \to [0, \infty)$ which has the following properties: (i) $F$ is $C^\infty$ on $TM_0$; (ii) $F$ is positively 1-homogeneous on the fibers of tangent bundle $TM$; (iii) for each $y \in T_x M$, the following quadratic form $g_y$ on $T_x M$ is positive definite,

$$
    g_y(u, v) := \frac{1}{2} \left[ \frac{\partial^2}{\partial s \partial t} \left[ F^2(y + su + tv) \right] \right]_{s, t = 0}, \quad u, v \in T_x M.
$$

Let $x \in M$ and $F_x := F|_{T_x M}$. To measure the non-Euclidean feature of $F_x$, define $C_{x} : T_x M \otimes T_x M \otimes T_x M \to \mathbb{R}$ by

$$
    C_{x}(u, v, w) := \frac{1}{2} \frac{d}{dt} \left[ g_{y + tw}(u, v) \right]_{t = 0}, \quad u, v, w \in T_x M.
$$

The family $C := \{C_{x}\}_{x \in TM_0}$ is called the Cartan torsion. It is well known that $C=0$ if and only if $F$ is Riemannian.

Given a Finsler manifold $(M, F)$, then a global vector field $G$ is induced by $F$ on $TM_0$, which in a standard coordinate $(x^i, y^i)$ for $TM_0$ is given by $G = y^i \frac{\partial}{\partial y^i} - 2G^i(x, y) \frac{\partial}{\partial x^i}$, where $G^i(y)$ are local functions on $TM$. $G$ is called the associated spray to $(M, F)$. The projection of an integral curve of $G$ is called a geodesic in $M$. In local coordinates, a curve $c(t)$ is a geodesic if and only if its coordinates $(c^i(t))$ satisfy $\ddot{c}^i + 2G^i(\dot{c}) = 0$.

Define $B_y : T_x M \otimes T_x M \otimes T_x M \to T_x M$ and $E_y : T_x M \otimes T_x M \to \mathbb{R}$ by $B_y(u, v, w) := B^j_{\ jkl}(y) u^i v^k w^l |_x$, $E_y(u, v) := E_{jk}(y) w^j u^k |_x$, where

$$
    B^j_{\ jkl}(y) := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}(y), \quad E_{jk}(y) := \frac{1}{2} B^m_{\ jkm}(y),
$$

$u = u^i \frac{\partial}{\partial x^i}$, $v = v^i \frac{\partial}{\partial x^i}$ and $w = w^i \frac{\partial}{\partial x^i}$. $B$ and $E$ are called the Berwald curvature and mean Berwald curvature, respectively. A Finsler metric is called a Berwald metric and mean Berwald metric if $B = 0$ or $E = 0$, respectively.

Let

$$
    \tau(x, y) := \ln \left[ \frac{\sqrt{\det \left( g_{ij}(x, y) \right)}}{\text{Vol}(B^n(1))} \cdot \text{Vol}\left\{ (y^i) \in \mathbb{R}^n \mid F\left( y^i \frac{\partial}{\partial x^i} \right) < 1 \right\} \right],
$$

$\tau = \tau(x, y)$ is a scalar function on $TM \setminus \{0\}$, which is called the distortion.

Let

$$
    S(x, y) := \frac{d}{dt} \left[ \tau(\sigma(t), \dot{\sigma}(t)) \right]_{t=0},
$$

where $\sigma(t)$ is the geodesic with $\sigma(0) = x$ and $\dot{\sigma}(0) = y$. $S$ is called the S-curvature. $S$ said to be isotropic if there is a scalar functions $c(x)$ on $M$ such that

$$
    S(x, y) = (n + 1)c(x)F(x, y).
$$
3 Proof of the Theorem 1.1

In local coordinates \((x^i, y^i)\), the vector field \( \mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i} \) is a global vector field on \( TM_0 \), where \( G^i = G^i(x, y) \) are local functions on \( TM_0 \) given by following

\[
G^i := \frac{1}{4} g^{il} \left[ \frac{\partial^2 F^2}{\partial x^k \partial y^l} y^k - \frac{\partial F^2}{\partial x^l} \right], \quad y \in T_x M.
\]

By a simple calculation, we have the following.

**Lemma 3.1.** Let \( F = \sqrt[2]{A} \) be an \( m \)-th root Finsler metric on an open subset \( U \subseteq \mathbb{R}^n \). Then the spray coefficients of \( F \) are given by:

\[
G^i = \frac{1}{2} (A_{0j} - A_{x^j}) A^{ij}.
\]

Thus the spray coefficients of an \( m \)-th root Finsler metric are rational functions with respect to \( y \).

**Lemma 3.2.** Let \( F = \sqrt[2]{A} \) be an \( m \)-th root Finsler metric on an open subset \( U \subseteq \mathbb{R}^n \). Then the following are equivalent

a) \( S = (n + 1)cF + \eta \);

b) \( S = \eta \);

where \( c = c(x) \) is a scalar function and \( \eta = \eta_i(x)y^i \) is a 1-form on \( M \).

**Proof.** By Lemma 3.1, the \( E \)-curvature of an \( m \)-th root metric is a rational function in \( y \). On the other hand, by taking twice vertical covariant derivatives of the \( S \)-curvature, we get the \( E \)-curvature. Thus \( S \)-curvature is a rational function in \( y \). Suppose that \( F \) has almost isotropic \( S \)-curvature, \( S = (n + 1)c(x)F + \eta \), where \( c = c(x) \) is a scalar function and \( \eta = \eta_i(x)y^i \) is a 1-form on \( M \). Then the left hand side of \( S - \eta = (n + 1)c(x)F \) is a rational function in \( y \) while the right hand is irrational function. Thus \( c = 0 \) and \( S = \eta \).

**Lemma 3.3.** Let \( F = \sqrt[2]{A} \) be an \( m \)-th root Finsler metric on an open subset \( U \subseteq \mathbb{R}^n \). Then the following are equivalent

a) \( E = \frac{n+1}{2}cFh \);

b) \( E = 0 \);

where \( c = c(x) \) is a scalar function on \( M \).

**Proof.** Suppose that \( F = \sqrt[2]{A} \) has isotropic mean Berwald curvature

\[
E = \frac{n+1}{2} cFh,
\]

where \( c = c(x) \) is a scalar function on \( M \). The left hand side of \( E = \frac{n+1}{2}cFh \) is a rational function in \( y \) while the right hand is irrational function. Thus \( c = 0 \) and \( E = 0 \).
Proof of Theorem 1.1 By Lemmas 3.2 and 3.3 we get the proof.

By the Theorem 1.1 we have the following:

**Corollary 3.4.** Let $F = \sqrt[2m]{}A$ be an $m$-th root Finsler metric on an open subset $U \subseteq \mathbb{R}^n$. Suppose that $F$ has isotropic $S$-curvature $S = (n + 1)cF$, for some scalar function $c = c(x)$ on $M$. Then $S = 0$.

A Finsler metric $F$ satisfying $F_{x^k} = FF_{y^k}$ is called a Funk metric. The standard Funk metric on the Euclidean unit ball $B^n(1)$ is denoted by $\Theta$ and defined by

$$\Theta(x, y) := \sqrt{|y|^2 - (|x|^2|y|^2 < x, y >)^2} + < x, y >, \quad y \in T_x B^n(1) \simeq \mathbb{R}^n,$$

where $<, >$ and $|.|$ denote the Euclidean inner product and norm on $\mathbb{R}^n$, respectively. In [5], Chen-Shen introduce the notion of isotropic Berwald metrics. A Finsler metric $F$ is said to be isotropic Berwald metric if its Berwald curvature is in the following form

$$B_{ijkl}^i = c\{F_{y^i y^k} \delta^i_l + F_{y^i y^l} \delta^i_k + F_{y^i y^k y^l} \delta^i_j + F_{y^i y^k y^l y^j}\}, \quad (4)$$

for some scalar function $c = c(x)$ on $M$. Berwald metrics are trivially isotropic Berwald metrics. In (4), putting $i = l$ yields

$$E_{ij} = \frac{n + 1}{2} cF^{-1} h_{ij}.$$

Plugging it in (4) implies that

$$B_{ijkl}^i = \frac{2}{n + 1} \{E_{jk} \delta^i_l + E_{ki} \delta^i_j + E_{lj} \delta^i_k + E_{ik} \delta^i_l\}. \quad (5)$$

This means that every isotropic Berwald metric is a Douglas metric. For the definition of Douglas metrics see [3].

Now, let $F = \sqrt[2m]{}A$ be an $m$-th root Finsler metric on an open subset $U \subseteq \mathbb{R}^n$. Suppose that $F$ has isotropic Berwald curvature (4). By Lemma 3.1 the left hand side of (4) is a rational function in $y$ while the right hand is irrational function. Thus $c = 0$ and we have the following.

**Theorem 3.5.** Let $F = \sqrt[2m]{}A$ be an $m$-th root Finsler metric on an open subset $U \subseteq \mathbb{R}^n$. Suppose that $F$ has isotropic Berwald curvature. Then $F$ is a Berwald metric.

In [21], Tayebi-Rafie Rad proved that every isotropic Berwald metric (4) on a manifold $M$ has isotropic $S$-curvature $S = (n + 1)cF$, for some scalar function $c = c(x)$ on $M$. Thus by the Theorem 3.5 we have the following.

**Corollary 3.6.** Let $F = \sqrt[2m]{}A$ be an $m$-th root Finsler metric on an open subset $U \subseteq \mathbb{R}^n$. Suppose that $F$ has isotropic Berwald curvature. Then $S = 0$.  

4 Proof of the Theorem 1.2

The quotient \( J/I \) is regarded as the relative rate of change of mean Cartan torsion \( I \) along Finslerian geodesics. Then \( F \) is said to be isotropic mean Landsberg metric if \( J = cF I \), where \( c = c(x) \) is a scalar function on \( M \). In this section, we are going to prove the Theorem 1.2. More precisely, we show that every \( m \)-th root isotropic mean Landsberg metric reduces to a weakly Landsberg metric.

**Proof of Theorem 1.2** The mean Cartan tensor of \( F \) is given by following

\[
I_{ij} = g^{jk} C_{ijk} = \frac{1}{m} A^{-3} \left[ m A A^{jk} + \frac{m-2}{m-1} y^j y^k \right] \times \left[ A^2 A_{ijk} + \left( \frac{2}{m} - 1 \right) \left\{ \left( \frac{2}{m} - 2 \right) A_i A_j A_k + A \left[ A_i A_j A_k + A_i A_k A_j + A_k A_j \right] \right\} \right].
\]

The mean Landsberg curvature of \( F \) is given by

\[
J_i = g^{jk} L_{ijk} = \frac{1}{2m} A^{-2} \left[ m A A^{jk} + \frac{m-2}{m-1} y^j y^k \right] \left[ -\frac{1}{2m} A^{k=1} A_s G_{ijk}^s \right] = -\frac{1}{2m} A^{-2} A_s G_{ijk}^s \left[ m A A^{jk} + \frac{m-2}{m-1} y^j y^k \right].
\]

Since \( J = cF I \), then we have

\[
A_s G_{ijk}^s = -2c A^{k=2} \left[ A^2 A_{ijk} + \left( \frac{2}{m} - 1 \right) \left\{ \left( \frac{2}{m} - 2 \right) A_i A_j A_k + A \left[ A_i A_j A_k + A_i A_k A_j + A_k A_j \right] \right\} \right].
\]

By the Lemma 3.1, the left hand side is a rational function in \( y \), while its right-hand side is an irrational function in \( y \). Thus, either \( c = 0 \) or \( A \) satisfies the following PDE:

\[
A^2 A_{ijk} + \left( \frac{2}{m} - 1 \right) \left( \frac{2}{m} - 2 \right) A_i A_j A_k + \left( \frac{m-2}{m} - 1 \right) A \left[ A_i A_j A_k + A_i A_k A_j + A_k A_j \right] = 0.
\]

That implies that \( C_{ijk} = 0 \). Hence, by Deike’s theorem, \( F \) is Riemannian metric, which contradicts our assumption. Therefore, \( c = 0 \). This completes the proof.

By the similarly method, we have the following.

**Theorem 4.1.** Let \( F = \sqrt[n]{A} \) be an non-Riemannian \( m \)-th root Finsler metric on an open subset \( U \subseteq \mathbb{R}^n \). Suppose that \( F \) has isotropic Landsberg curvature, i.e., \( L = cFC \), where \( c = c(x) \) is a scalar function on \( M \). Then \( F \) reduces to a Landsberg metric.
5 Proof of the Theorem 1.3

A Finsler metric \( F = F(x, y) \) on a manifold \( M \) is said to be locally dually flat if at any point there is a coordinate system \((x^i)\) in which the spray coefficients are in the following form

\[
G^i = -\frac{1}{2}g^{ij}H_{y^j},
\]

where \( H = H(x, y) \) is a \( C^\infty \) scalar function on \( TM_0 = TM \setminus \{ 0 \} \) satisfying \( H(x, \lambda y) = \lambda^2 H(x, y) \) for all \( \lambda > 0 \). Such a coordinate system is called an adapted coordinate system \([15]\). Recently, Shen proved that the Finsler metric \( F \) is in the following form

\[
F = \sum_{i,j=1}^n a_{ij}(x)y^iy^j,
\]

if at any point there is a coordinate system \((x^i)\) in which the spray coefficients are given by

\[
G^i = -\frac{1}{2}g^{ij}H_{y^j}.
\]

In this case, \( H = -\frac{1}{2}(F_2)_{xy}y^m \).

In this section, we will prove a generalized version of Theorem 1.3. Indeed we find necessary and sufficient condition under which a conformal \( \beta \)-change of an generalized \( m \)-th root Finsler metric be locally dually flat. Let \( F \) be a scalar function on \( TM \) defined by \( F = \sqrt{A^2/m + B} \), where \( A \) and \( B \) are given by

\[
A := a_{i_1...i_m}(x)y^{i_1}...y^{i_m}, \quad B := b_{ij}(x)y^iy^j.
\]

Then \( F \) is called generalized \( m \)-th root Finsler metric. Suppose that the matrix \((A_{ij})\) defines a positive definite tensor and \((A^{ij})\) denotes its inverse.

Now, we are going to prove the following:

**Theorem 5.1.** Let \( F = \sqrt{A^2/m + B} \) be an generalized \( m \)-th root Finsler metric on an open subset \( U \subset \mathbb{R}^n \), where \( A \) is irreducible. Suppose that \( \bar{F} = e^\alpha F + \beta \) is conformal \( \beta \)-change of \( F \) where \( \beta = b_i(x)y^i, \alpha = \alpha(x) \). Then \( \bar{F} \) is locally dually flat if and only if there exists a 1-form \( \theta = \theta_i(x)y^i \) on \( U \) such that the following holds

\[
e^{2\alpha}[2B_{x^i} + 4\alpha_{x^i}B - 2\alpha_0B_0] = 2(\beta_1\beta_0 + \beta_0\beta_0 - 2\beta_{x^i}),
\]

\[
A_{x^i} = \frac{1}{3m}[mA\theta_1 + 2\theta A_1 + 2(\alpha_0A_1 - \alpha_{x^i}A)],
\]

\[
T_t\beta = 2Y[(\alpha_0\beta_0 + \alpha_0\beta T_t - 2Y_{x^i}\beta) + 2e^\alpha Y\Psi],
\]

where \( Y := A^{\frac{2}{m}} + B, \beta_0 = \beta_{x^i}y^k, \alpha_0 = \alpha_{x^i}y^i, \beta_{x^i} = (b_i)_{x^i}y^i, \beta_0 = (b_i)_{0}y^i, \beta_{0l} = (b_{il})_{0} \), and

\[
Y_p = \frac{2}{m}A^{\frac{2}{m}-1}A_p + B_p,
\]

\[
Y_{0p} = \frac{2}{m}A^{\frac{2}{m}-2}[(\frac{2}{m} - 1)A_pA_0 + AA_{0p}] + B_{0p},
\]

\[
\Psi = \alpha_0\beta_0 - 2\beta_{x^i} - 2\alpha_{x^i}\beta.
\]
To prove Theorem 5.1 we need the following.

**Lemma 5.2.** Suppose that the equation \( \Phi A^{\frac{1}{m^2}} + \Psi A^{\frac{1}{m}} + \Theta = 0 \) holds, where \( \Phi, \Psi, \Theta \) are polynomials in \( y \) and \( m > 2 \). Then \( \Phi = \Psi = \Theta = 0 \).

**Proof of Theorem 5.1** The following hold
\[
\tilde{F}^2 = e^{2\alpha}(A^{\frac{1}{m}} + B) + 2e^\alpha \beta (A^{\frac{1}{m}} + B)^{1/2} + \beta^2,
\]
\[
(F^2)_{x^k} = 2\alpha x^k e^{2\alpha}(A^{\frac{1}{m}} + B) + e^{2\alpha}(\frac{2}{m} A^{\frac{1}{m}} - 1 A_x + B_x) + 2\alpha x^k e^\alpha \beta (A^{\frac{1}{m}} + B)^{\frac{1}{2}}
+ e^\alpha (A^{\frac{1}{m}} + B)^{-1/2}(\frac{2}{m} A^{\frac{1}{m}} - 1 A_x - B_x)\beta + 2(A^{\frac{1}{m}} + B)^{1/2} \beta_x B^2.
\]

Then
\[
[F^2]_{x^k y^l} = 2\alpha_0 e^{2\alpha} Y_{0l} + e^{2\alpha} Y_{0l} + 2\alpha_0 e^\alpha \beta_i Y_{0l} + 2\alpha e^\alpha \beta Y_{x^l} Y_{0l} + 2e^\alpha \beta_0 Y_{x^l} Y_{0l}
+ 2\beta_i \beta_0 + 2\beta \beta_0.
\]

Since \( \tilde{F} \) be a locally dually flat metric, then
\[
e^\alpha Y^{-\frac{1}{2}} \left[-\frac{1}{2} \beta Y_{0l} + Y(\beta Y_{0l} + \beta_i Y_{0l} + \beta_0 Y_{0l} + 2\alpha_0 \beta_i Y_{0l} - 2\beta Y_{x^l})
+2e^\alpha Y^2(\alpha_0 \beta_i + \beta_0 - 2\alpha_x \beta - 2\beta_x)
+\frac{2}{m} e^{2\alpha} A^{\frac{1}{m}} - 2 \left[2\alpha_0 AA_{0l} + (\frac{2}{m} - 1) A_{0l} A_{0l} - 2\alpha_x A^2 - 2AA_{x^l}\right]
+e^{2\alpha} \left[2\alpha_0 B_{0l} + B_{0l} - 4\alpha_x B - 2B_{x^l}\right]
-4\beta_x A + 2\beta_0 + 2\beta_0 = 0.
\]

By Lemma 5.2, we have
\[
2\alpha_0 AA_{0l} + (\frac{2}{m} - 1) A_{0l} A_{0l} + AA_{0l} - 2\alpha_x A^2 = 2AA_{x^l}, \quad (9)
\]
\[
\frac{1}{2} \beta Y_{0l} Y_{0l} = Y_0 (\beta Y_{0l} + \beta_0 Y_{0l} + 2\alpha_0 \beta Y_{0l} - 2\beta Y_{x^l}) + 2e^\alpha Y \Psi, \quad (10)
\]
\[
e^{2\alpha} \left[2\alpha_0 B_{0l} + B_{0l} - 4\alpha_x B - 2B_{x^l}\right] = 2(2\beta_x - \beta_i \beta_0 - 2\beta_0). \quad (11)
\]

One can rewrite (9) as follows
\[
A(2A_{x^l} - A_{0l} + 2\alpha_x A) = (\frac{2}{m} - 1) A_0 + 2\alpha_0 A A_{0l}.
\]

Irreducibility of \( A \) and \( \text{deg}(A_l) = m - 1 \) imply that there exists a 1-form \( \theta = \theta_l y^l \) on \( U \) such that
\[
A_0 = \theta A.
\]

By (13), we get
\[
A_{0l} = A \theta_l + \theta A_l - A_{x^l}.
\]

Substituting (13) and (14) into (12) yields (7). The converse yields by a direct computation. This completes the proof. \( \square \)
6 Proof of the Theorem 1.4

It is known that a Finsler metric $F(x, y)$ on $U$ is projective if and only if its geodesic coefficients $G^i$ are in the form

$$G^i(x, y) = P(x, y)y^i,$$

where $P :TU = U \times \mathbb{R}^n \rightarrow \mathbb{R}$ is positively homogeneous with degree one, $P(x, \lambda y) = \lambda P(x, y), \lambda > 0$. We call $P(x, y)$ the projective factor of $F(x, y)$.

The following lemma plays an important role.

Lemma 6.1. (Rapcsák) Let $F(x, y)$ be a Finsler metric on an open subset $U \subset \mathbb{R}^n$. $F(x, y)$ is projective on $U$ if and only if it satisfies

$$F_{x^k y^l}y^k = F_{x^l}.$$ (15)

In this case, the projective factor $P(x, y)$ is given by

$$P = \frac{F_{x^k y^k}}{2F}.$$ (16)

Much earlier, G. Hamel proved that a Finsler metric $F(x, y)$ on $U \subset \mathbb{R}^n$ is projective if and only if

$$F_{x^k y^l} = F_{x^l y^k}.$$ (17)

Thus (16) and (15) are equivalent.

In this section, we will prove a generalized version of Theorem 1.4. Indeed we study the conformal $\beta$-change of a generalized $m$-th root metric $F = \sqrt{A^m} + B$, where $A$ is irreducible. More precisely, we prove the following:

Theorem 6.2. Let $F = \sqrt{A^m} + B$ be an generalized $m$-th root Finsler metric on an open subset $U \subset \mathbb{R}^n$, where $A$ is irreducible. Suppose that $\bar{F} = e^\alpha F + \beta$ be conformal $\beta$-change of $F$ where $\beta = b_i(x)y^i, \alpha = \alpha(x)$. Then $\bar{F}$ is locally projectively flat if and only if it is locally Minkowskian.

To prove Theorem 1.4 we need the following.

Lemma 6.3. Let $(M, F)$ be a Finsler manifold. Suppose that $\bar{F} = e^\alpha F + \beta$ be a conformal $\beta$-change of $F$. Then $\bar{F}$ is a projectively flat Finsler metric if and only if the following holds

$$e^\alpha(F_{0l} - F_{xl}) = e^\alpha(\alpha_{xl}F - \alpha_{0}F_{l}) + (b_{i})_{xl}y^i - (b_{i})_{0}.$$ (18)

Proof. The following hold

$$\bar{F} = e^\alpha F + \beta,$$

$$\bar{F}_{x^k} = \alpha_{x^k}e^\alpha F + e^\alpha F_{x^k} + (b_{i})x^k y^i,$$

$$\bar{F}_{0l} = \alpha_{0}e^\alpha F + e^\alpha F_{0l} + (b_{i})_{0}y^i,$$

$$\bar{F}_{0l} = \alpha_{0}e^\alpha F_{l} + e^\alpha F_{0} + (b_{i})_{0}.$$ 

This completes the proof. \qed
By using the Lemma 6.3 we are going to prove the following.

**Proposition 6.4.** Let $F = \sqrt{A^{2/m} + B}$ be an generalized $m$-th root Finsler metric on an open subset $U \subset \mathbb{R}^n$, where $A$ is irreducible, $m > 4$ and $B \neq 0$. Suppose that $\tilde{F} = e^\alpha F + \beta$ be conformal $\beta$-change of $F$ where $\beta = b_i(x)y^i, \alpha = \alpha(x)$. In that case, if $\tilde{F}$ is projectively flat metric then $F$ reduces to a Berwald metric.

**Proof.** By Lemma 6.3 we get

$$F_{x^i} = \frac{2A^{2/m}A_{x^i} + mAB_{x^i}}{2mA\sqrt{A^2 + B}}.$$  

Then we have

$$F_{x^iy^k} = (A^\frac{\alpha}{4} + B)^{-1/2}\left[1\left(\frac{2A^{2/m}A_0}{mA} + B_0\right)\left(\frac{2A^{2/m}A_l}{mA} + B_l\right)\left(A^{\frac{\alpha}{4}} + B\right)^{-1} + \frac{1}{2}\left(\frac{4A^{2/m}A_0A_l}{m^2A^2} + \frac{2A^{2/m}A_0A_l}{mA} - \frac{2A^{2/m}A_0A_l}{mA^2} + B_0\right)\right].$$

Thus

$$F_{0l} - F_{x^i} = e^\alpha\left(A^{\frac{\alpha}{4}} + B\right)^{-1/2}\left[A^{\frac{\alpha}{4}}(mAA_l\alpha_0 + (1 - m)A_1A_0 + mAA_0l - mAA_0x^i) + \frac{1}{2}m^2A^2B_l\alpha_0 + (2m)A_1A_0B + mAA_0lB + \frac{1}{2}m^2A^2B_l\alpha_0 + (2m)A_1A_0B + mAA_0lB\right] + \frac{1}{2}m^2A^2B_l\alpha_0 + B_0B_l - \frac{1}{2}B_0B_l - BB_{x^i}).$$

By (18), we obtain the following:

$$\Phi A^{\frac{\alpha}{4}} + \Psi A^{\frac{\alpha}{4}} + \Theta = 0,$$

where

$$\Phi = -\frac{mA}{2}\left[A_0B_1 + B_0A_1 + 2B(A_{x^i} - A_1\alpha_0 - A_0l) + mA(B_{x^i} - B_l\alpha_0 - B_0)\right] - (m - 2)A_0A_1B,$$

$$\Psi = mA(A_{0l} + A_1\alpha_0 - A_{x^i}) - (m - 1)A_0A_l,$$

$$\Theta = \frac{1}{4}m^2A^2\left[2BB_0\alpha_0 - 2B_0B_l + B_0B_l + 2B_{x^i}B\right],$$

$$+ m^2A^2(A^{\frac{\alpha}{4}} + B)^{-1}e^{-\alpha}\left[(b_l)_0 - (b_i)_{x^i}y^i + e^\alpha(\alpha_{x^i}A^{\frac{\alpha}{4}} - \frac{1}{m}\alpha_0A^{\frac{\alpha}{4}} - 1A_0)\right].$$

By Lemma 5.2 we have

$$\Phi = 0, \quad (19)$$
$$\Psi = 0, \quad (20)$$
$$\Theta = 0. \quad (21)$$
By (20), it follows that
\[ mA(A_l \alpha_0 + A_0 l - A_x l) = (m - 1)A_0 A_l. \] (22)

Then irreducibility of \( A \) and \( \text{deg}(A_l) = m - 1 < \text{deg}(A) \) implies that \( A_0 \) is divisible by \( A \). This means that, there is a 1-form \( \theta = \theta_i y^i \) on \( U \) such that,
\[ A_0 = 2mA \theta. \] (23)

Substituting (23) into (22), yields
\[ A_0 l = A_x l - A_l \alpha_0 + 2(m - 1)A \theta. \] (24)

Plugging (23) and (24) into (19), we get
\[ mA(2 \theta B_l - B_0 l - B_l \alpha_0 + B_x l) = A_l (4B \theta - B_0). \] (25)

Clearly, the right side of (25) is divisible by \( A \). Since \( A \) is irreducible, \( \text{deg}(A_l) \) and \( \text{deg}(2 \theta B - \frac{1}{2} B) \) are both less than \( \text{deg}(A) \), then we have
\[ B_0 = 4B \theta. \] (26)

By (23) and (26), we get the spray coefficients \( G^i = P y^i \) with \( P = \theta \). Thus \( F \) is a Berwald metric.

The Riemann curvature \( K_y = R^i_k d x^k \otimes \frac{\partial}{\partial x^i}|_x : T_x M \rightarrow T_x M \) is a family of linear maps on tangent spaces, defined by
\[ R^i_k = 2 \frac{\partial G^i}{\partial x^k} - y^j \frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2G^i \frac{\partial^2 G^j}{\partial y^i \partial y^k} - \frac{\partial G^i}{\partial y^j} \frac{\partial G^j}{\partial y^k}. \]

For a flag \( P = \text{span}\{y, u\} \subset T_x M \) with flagpole \( y \), the flag curvature \( K = K(P, y) \) is defined by
\[ K(P, y) := \frac{g_y(u, K_y(u))}{g_y(y, y)g_y(u, u) - g_y(y, u)^2}. \]

When \( F \) is Riemannian, \( K = K(P) \) is independent of \( y \in P \), which is just the sectional curvature of \( P \) in Riemannian geometry. We say that a Finsler metric \( F \) is of scalar curvature if for any \( y \in T_x M \), the flag curvature \( K = K(x, y) \) is a scalar function on the slit tangent bundle \( TM_0 \). One of the important problems in Finsler geometry is to characterize Finsler manifolds of scalar flag curvature [10][11]. If \( K = \text{constant} \), then the Finsler metric \( F \) is said to be of constant flag curvature.

**Proof of Theorem 6.2** By Proposition 6.4, \( F \) is a Berwald metric. On the other hand, according to Numata’s Theorem every Berwald metric of non-zero scalar flag curvature \( K \) must be Riemannian. This is contradicts with our assumption. Then \( K = 0 \), and in this case \( F \) reduces to a locally Minkowskian metric.
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