On cylindrically bounded $H$-Hypersurfaces of $\mathbb{H}^n \times \mathbb{R}$

G. Pacelli Bessa* M. Silvana Costa

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Abstract

We show that $H$-hypersurfaces of $\mathbb{H}^n \times \mathbb{R}$ contained in a vertical cylinder and with Ricci curvature with strong quadratic decay have mean curvature $|H| > (n-1)/n$.

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1 Introduction

Barbosa-Kenmotsu-Oshikiri in [3] and Salavessa in [16] independently proved that if an entire graph of a smooth function $f : \mathbb{H}^n \to \mathbb{R}$ has constant mean curvature $H$ then $|H| \leq (n-1)/n$. On the other hand, Bessa-Montenegro in [1] showed that the mean curvature of any compact $H$-hypersurface immersed in $\mathbb{H}^n \times \mathbb{R}$ satisfies $|H| > (n-1)/n$. It should be remarked that this result for embedded $H$-surfaces ($n = 2$) was proved by Nelli-Rosenberg in [14] and it was implicit in Hsiang-Hsiang’s paper [10]. The case of embedded $H$-hypersurfaces ($n \geq 2$) follows from Salavessa’s work [16], [17]. The purpose of this paper is to extend Bessa-Montenegro’s result to cylindrically bounded complete hypersurfaces $M$ of $\mathbb{H}^n \times \mathbb{R}$ with Ricci curvature with strong quadratic decay. We say that a complete Riemannian manifold $M$ has Ricci curvature $\text{Ric}_M$ with strong quadratic decay if

$$\text{Ric}_M(x) \geq -c^2 \left[ 1 + \rho_M^2(x) \log^2(\rho_M(x) + 2) \right],$$

where $\rho_M$ is the distance function on $M$ to a fixed point $x_0$ and $c = c(x_0) > 0$ is a constant depending on $x_0$. An immersed submanifold $M \subset N \times \mathbb{R}$ is said to be cylindrically bounded if $p(M)$ is a bounded subset of $N$, where $p : N \times \mathbb{R} \to N$ is the projection on the first factor $p(x, t) = x$. Our main result is the following theorem.

**Theorem 1.1** Let $M$ be a complete hypersurface immersed in $\mathbb{H}^n \times \mathbb{R}$ with Ricci curvature with strong quadratic decay. If $M$ is cylindrically bounded then $\sup_M |H| \geq (n-1)/n$. 

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Calabi, [2] in the sixties asked whether there were complete bounded minimal surfaces in \( \mathbb{R}^3 \). It is well known that this question was completely answered. See [15], [12] for non-existence of complete bounded minimal surfaces with bounded sectional curvature, see [6] for non-existence of complete bounded minimal surfaces with sectional curvature with strong quadratic decay and see [13] for the first example of a complete bounded minimal surface in \( \mathbb{R}^3 \). Naturally, one can ask whether there are complete bounded minimal surfaces in \( \mathbb{H}^2 \times \mathbb{R} \). The answer is not if the minimal surface has sectional curvature with strong quadratic decay. This is implied by Theorem (1.1). In fact, it can be restated as the following corollary.

**Corollary 1.2**  Let \( M \) be a complete \( H \)-hypersurface immersed in \( \mathbb{H}^n \times \mathbb{R} \) with Ricci curvature with strong quadratic decay. If \( |H| < (n - 1)/n \) then \( M \) is not cylindrically bounded.

It is well known that the coordinate functions of any minimal surface of \( \mathbb{R}^3 \) are harmonic, thus if one of the coordinates is bounded then the minimal surface is non-parabolic. In our second result, we show that cylindrically bounded \( H \)-hypersurfaces of \( \mathbb{H}^n \times \mathbb{R} \) with \( |H| < (n - 1)/n \) are non-parabolic.

**Theorem 1.3**  Let \( M \) be a complete immersed hypersurface in \( \mathbb{H}^n \times \mathbb{R} \) with bounded mean curvature \( \sup_M |H| \leq (n - 1)/n \) contained in a vertical cylinder. Then \( M \) is non-parabolic.

Any hypersurface of \( \mathbb{S}^n \times \mathbb{R} \) is cylindrically bounded, nevertheless, the Theorems (1.1) and (1.3) have appropriate versions for \( \mathbb{S}^n \times \mathbb{R} \).

**Theorem 1.4**  Let \( M \) be a complete hypersurface immersed in \( \mathbb{S}^n \times \mathbb{R} \) with Ricci curvature with strong quadratic decay. If \( p_1(M) \subset B_{\mathbb{S}^n}(r), r < \pi/2 \) then \( \sup_M |H| \geq (n - 1) \cot(r)/n \).

**Theorem 1.5**  Let \( M \) be a complete hypersurface immersed in \( \mathbb{S}^n \times \mathbb{R} \) with bounded mean curvature \( \sup_M |H| \leq (n - 1) \cot(r)/n, r \leq \pi/2 \). Then \( M \) is non-parabolic.

## 2 Preliminaries

Let \( \varphi : M \rightarrow N \) be an isometric immersion, where \( M \) and \( N \) are complete Riemannian \( m \) and \( n \) manifolds respectively. Consider a smooth function \( g : N \rightarrow \mathbb{R} \) and the composition \( f = g \circ \varphi : M \rightarrow \mathbb{R} \). Identifying \( X \) with \( d\varphi(X) \) we have at \( q \in M \) and for every \( X \in T_q M \) that

\[
\langle \nabla f, X \rangle = df(X) = dg(X) = \langle \nabla g, X \rangle,
\]

therefore

\[
\nabla g = \nabla f + (\nabla g)^\perp, \tag{1}
\]

where \( (\nabla g)^\perp \) is perpendicular to \( T_q M \). Let \( \nabla \) and \( \nabla \) be the Riemannian connections on \( M \) and \( N \) respectively, \( \alpha(q)(X,Y) \) and \( \text{Hess}(q)(X,Y) \) be respectively the second fundamental
form of the immersion $\varphi$ and the Hessian of $f$ at $q \in M$, $X, Y \in T_p M$. Using the Gauss equation we have that

$$\text{Hess } f(q)(X, Y) = \text{Hess } g(\varphi(q))(X, Y) + (\text{grad } g, \alpha(X, Y))_{\varphi(q)}. \tag{2}$$

Taking the trace in (2), with respect to an orthonormal basis $\{e_1, \ldots, e_m\}$ for $T_p M$, we have that

$$\Delta f(q) = \sum_{i=1}^m \text{Hess } g(\varphi(q))(e_i, e_i) + (\text{grad } g, \sum_{i=1}^m \alpha(e_i, e_i)). \tag{3}$$

These formulas (2) and (3) are well known in the literature, see [4], [5], [7], [8], [11]. Recall the Hessian Comparison Theorem.

**Theorem 2.1** Let $M$ be a complete Riemannian manifold and $x_0, x_1 \in M$. Let $\rho(x)$ be the distance function $\text{dist}_M(x_0, x)$ to $x_0$ and let $\gamma$ be a minimizing geodesic joining $x_0$ and $x_1$. Let $K_\gamma$ be the radial sectional curvatures of $M$ along $\gamma$ and let $\mu(\rho)$ be this function defined below.

$$\mu(\rho) = \begin{cases} 
  k \cdot \coth(k \cdot \rho(x)), & i f \ \sup K_\gamma = -k^2 \\
  \frac{1}{\rho(x)^2}, & i f \ \sup K_\gamma = 0 \\
  k \cdot \cot(k \cdot \rho(x)), & i f \ \sup K_\gamma = k^2 \ \text{and} \ \rho < \pi/2k.
\end{cases} \tag{4}$$

Then the Hessian of $\rho$ satisfies

$$\text{Hess } \rho(x)(X, X) \geq \mu(\rho(x)) \cdot \|X\|^2, \quad \text{Hess } \rho(x)(\gamma', \gamma') = 0 \tag{5}$$

Where $X$ is any vector in $T_x M$ perpendicular to $\gamma'(\rho(x))$.

The second main ingredient to the proof of our results is the Omori-Yau maximum principle [15], [18], [9] in the generalized version proved by Chen-Xin in [6].

**Theorem 2.2 (Omori-Yau Maximum Principle, [6])** Let $M$ be a complete Riemannian manifold with Ricci curvature with strong quadratic decay. Let $u$ be a $C^2$ function bounded above on $M$. Then for any sequence $\epsilon_k \to 0$ of positive numbers there exists a sequence of points $x_k \in M$ such that

i. $\lim_{k \to \infty} u(x_k) = \sup_M u$

ii. $|\text{grad } u|(x_k) < \epsilon_k$

iii. $\Delta u(x_k) < \epsilon_k$.

**Remark 2.3** It is clear in the proof of the Omori-Yau maximum principle that $u$ is allowed to be only $C^0$ in a measure zero subset of $M$. So that $u$ can be the distance function on $M$ to a fixed point. See page 360 of [6].
3 Proof of the Results

Our results (Thms. [1.1 1.3 1.4 1.5]) are particular case of a more general theorem that we present here. First, let \( \varphi : M \hookrightarrow N \times \mathbb{R} \) be an isometric immersion of a complete Riemannian \( m \)-manifold \( M \) into a complete Riemannian \( n \)-manifold \( N \) with a pole and radial sectional curvature \( K_N \leq -\kappa^2 < 0 \). Let \( \rho_N : N \rightarrow \mathbb{R} \) be the distance function to the pole \( x_0 \in N \). Set \( g : N \times \mathbb{R} \rightarrow \mathbb{R} \) and \( f = g \circ \varphi : M \rightarrow \mathbb{R} \). The Laplacian of \( f \) is given by

\[
\Delta f(q) = \sum_{i=1}^{m} \text{Hess}(g(q))(e_i, e_i) + \langle \text{grad} g, \vec{H} \rangle.
\]

Here \( \vec{H} \) is the mean curvature vector with norm \( \| \vec{H} \| = n|H| \). We choose an orthonormal basis \( \{e_1, \ldots, e_n\} \) for \( T_q M \) in the following way. Start with an orthonormal basis (from polar coordinates) for \( T_{\rho_N(q)}N, \{\text{grad}\rho_N, \partial/\partial\theta_2, \ldots, \partial/\partial\theta_n\} \). We can choose be an orthonormal basis for \( T_q M \) as follows \( e_1 = \langle e_1, \partial/\partial\theta \rangle \partial/\partial\theta + \langle e_1, \text{grad}\rho_N \rangle \text{grad}\rho_N \) and \( e_j = \partial/\partial\theta_j, j = 2, \ldots, n \) (up to an re-ordination), where \( \partial/\partial\theta \) is tangent to the \( \mathbb{R} \)-factor. By the Hessian comparison theorem we have \( q \in M \) that

\[
\Delta f(q) \geq (n-1)\kappa \cdot \coth(\kappa \cdot \rho_N) - n|H|.
\]

Where the right hand side of the inequality (6) is computed at \( \varphi(q) \).

Suppose that \( \varphi(M) \) has mean curvature vector \( \| \vec{H} \| = n|H| < (n-1)\kappa \). This implies that the function \( f \) is subharmonic. If \( \varphi(M) \) is cylindrically bounded then \( f \) is a bounded subharmonic function and \( M \) is non-parabolic. Suppose in addition the \( M \) has Ricci curvature with strong quadratic decay. By the Omori-Yau Maximum principle there exist sequences \( \epsilon_k \rightarrow 0 \) and \( x_k \in M \) with \( \Delta f(x_k) < \epsilon_k \). The inequality (6) at \( x_k \) becomes

\[
\epsilon_k > \Delta f(x_k) \geq (n-1)\kappa - n \sup_M |H| > 0
\]

Letting \( \epsilon_k \rightarrow 0 \) we get a contradiction. This proves the following more general result from which Theorems (1.1 and 1.3) are corollaries.

**Theorem 3.1** Let \( N \) be a complete Riemannian \( n \)-manifold with a pole and radial sectional curvature bounded above, \( K_N \leq -\kappa^2 < 0 \). Let \( \varphi : M \hookrightarrow N \times \mathbb{R} \) be a complete immersed submanifold. Then

i. If \( \varphi(M) \) is cylindrically bounded and has bounded mean curvature vector \( \| \vec{H} \| < (n-1)\kappa \) then \( M \) is non-parabolic.

ii. If \( \varphi(M) \) is cylindrically bounded and \( M \) has Ricci curvature strong quadratic decay then \( \sup_M |H| \geq (n-1)\kappa/n \).

Now suppose that \( \varphi : M \hookrightarrow \mathbb{S}^n \times \mathbb{R} \) is a complete immersed submanifold. The inequality (6) becomes in this setting the following inequality. For any \( q \in M \)

\[
\Delta f(q) \geq [(n-1) \cdot \cot(\rho_{\mathbb{S}^n}(\cdot)) - n|H|] (\varphi(q)).
\]


If \( p(\varphi(M)) \subset B_{S^n}(r), \ r < \pi/2 \) then \( \cot(r) > 0 \). Thus if \( \sup_M |H| < (n - 1) \cot(r)/n \) then \( f \) is a bounded subharmonic function. And if in addition, \( M \) has Ricci curvature with strong quadratic decay then we have by Omori-Yau maximum principle that

\[
\epsilon_k > \Delta f(x_k) \geq (n - 1) \cdot \cot(r) - n \sup_M |H| = b^2 > 0
\]

Letting \( \epsilon_k \to 0 \) we get a contradiction. This proves the following theorem that implies Theorems (1.4 and 1.5) as consequences.

**Theorem 3.2** Let \( \varphi : M \hookrightarrow S^n \times \mathbb{R} \) be a complete immersed submanifold.

i. If \( p(\varphi(M)) \subset B_{S^n}(r), \ r < \pi/2 \) and \( \sup_M |H| < (n - 1) \cot(r)/n \) then \( M \) is non-parabolic.

ii. If \( p(\varphi(M)) \subset B_{S^n}(r), \ r < \pi/2 \) and \( M \) has Ricci curvature strong quadratic decay then \( \sup_M |H| \geq (n - 1) \cot(r)/n \).

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*Address of the authors:*
Departamento de Matematica
Campus do Pici, Bloco 914
Universidade Federal do Ceará-UFC
60455-760 Fortaleza-Ceará
Brazil

bessa@math.ufc.br & mscosta@mat.ufc.br