Equi-invariability, bounded invariance complexity and L-stability for control systems

Xing-fu Zhong$^1$, Zhi-jing Chen$^2,^*$ & Yu Huang$^3$

$^1$School of Mathematics and Statistics, Guangdong University of Foreign Studies, Guangzhou 510006, China; $^2$School of Mathematics and Systems Science, Guangdong Polytechnic Normal University, Guangzhou 510665, China; $^3$School of Mathematics, Sun Yat-sen University, Guangzhou 510275, China
Email: xfzhong@gdufs.edu.cn, chzhjing@mail2.sysu.edu.cn, stshyu@mail.sysu.edu.cn

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Abstract In this paper, we introduce the notions of bounded invariance complexity, bounded invariance complexity in the mean and mean Lyapunov-stability for control systems. Then we characterize these notions by introducing six types of equi-invariability. As a by-product, two new dichotomy theorems for the control system on the control sets are established.

Keywords equi-invariability, invariance complexity, dichotomy theorem, control set, invariance entropy

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1 Introduction

In this paper, we mainly consider a discrete-time control system on a metric space $X$ of the following form:

$$x_{n+1} = F(x_n, u_n) =: F_{u_n}(x_n), \quad n \in \mathbb{N}_0 = \{0, 1, \ldots\},$$

where $F$ is a map from $X \times U$ to $X$, $U$ is a compact set, and $F_u(\cdot) \equiv F(\cdot, u)$ is continuous for every $u \in U$. Given a control sequence $\omega = (\omega_0, \omega_1, \ldots)$ in $U$, the solution of (1.1) can be written as

$$\phi(k, x, \omega) = F_{\omega_{k-1}} \circ \cdots \circ F_{\omega_0}(x).$$

For convenience, we denote the system (1.1) by

$$\Sigma = (\mathbb{N}_0, X, U, \mathcal{U}, \phi),$$

where $\mathcal{U} = U^{\mathbb{N}_0}$. Furthermore, we assume that $\phi: \mathbb{N}_0 \times X \times \mathcal{U} \to X$ is continuous.

Invariance entropy introduced by Colonius and Kawan [7] as well as topological feedback entropy introduced by Nair et al. [22] characterizes the minimal data rate for making a subset of the state space invariant. It is a very useful invariant to describe the exponential growth rate of the minimal number of
different control functions sufficient for orbits to stay in a given set when starting in a subset of this set. For controlled invariant sets with zero invariance entropy, it is useful to consider the invariance complexity function first studied by Wang et al. [24], which is an analogue in topological dynamical systems (see [15] and the references therein). We refer the readers to [1–8], [11–14,16,17,19,25] and [26,27] for more details about invariance entropy.

In 1993, Colonius and Kliemann [9] introduced a notion of a control set and obtained a beautiful result that control sets of a given control system coincide with maximal topologically mixing (transitive) sets of the control flow induced by the control system under some assumptions. We refer the readers to [9,10] for more connections between control properties for control systems and basic notions for dynamical systems. Recently, Wang et al. [24] introduced a notion of equi-invariability and showed that an equi-invariant compact set has bounded invariance complexity and the converse is not true in general. In particular, they established a dichotomy theorem that a control set with dense interior is either equi-invariant or unstable.

In this paper, we introduce six types of equi-invariability, which are the analogies to equi-continuity, equi-continuity in the mean, and mean equi-continuity in topological dynamical systems (see [15,20,23]). Then we discuss the relationships with each other. In particular, we use three versions of equi-invariability to characterize bounded invariance complexity, bounded invariance complexity in the mean and mean L-stability for control systems, respectively. As a by-product, we obtain two new dichotomy theorems for a control set with dense interior.

The rest of this paper is organized as follows. In Section 2, we first introduce six types of equi-invariability and discuss the relationships with each other. Then we respectively characterize bounded invariance complexity, bounded invariance complexity in the mean and mean L-stability by three types of equi-invariability. In Section 3, we obtain two new dichotomy theorems for control systems on control sets. All the counter-examples are given in Appendix A.

2 Finite equi-invariability and invariance complexity

Consider a control system \( \Sigma = (\mathbb{N}_0, X, U, \mathcal{U}, \phi) \), where \( X \) is a metric space with a metric \( d \). Recall that a subset \( Q \) of \( X \) is said to be controlled invariant if for any \( x \in Q \), there exists a control \( \omega_x \in \mathcal{U} \) such that

\[
\phi(\mathbb{N}_0, x, \omega_x) \subset Q.
\]

Our task is to keep a controlled invariant set \( Q \) invariant. It is almost impossible to realize this task in practice if the choice of the control \( \omega_x \) is sensitive to \( x \in Q \) because of the error caused in implementation. On the other hand, we can control such a point \( x \in Q \) if the associated control \( \omega_x \) keeps not only the orbit of \( x \) but also the orbits starting from some neighborhood of \( x \) in a neighborhood of \( Q \). Such a point is called an equi-invariant point of \( Q \) in [24]. From the viewpoint of control theory, the equi-invariant point \( x \) of \( Q \) means that \( x \) can be stabilized robustly to any neighborhood of \( Q \).

In this section, we introduce six types of equi-invariability and discuss their relationships with the corresponding bounded invariance complexity for a control system.

**Definition 2.1.** Let \( \Sigma = (\mathbb{N}_0, X, U, \mathcal{U}, \phi) \) be a system, \( Q \subset X \) be a nonempty set and \( x \in Q \).

1. \( x \) is called a finitely equi-invariant point of \( Q \), denoted by \( x \in \text{FEI}(Q) \), if for every \( \varepsilon > 0 \), there exist \( \delta > 0 \) and a finite set \( F \subset \mathcal{U} \) such that for every \( y \in B(x, \delta) \cap Q \) there exists \( \omega \in F \) with

\[
\phi(\mathbb{N}_0, y, \omega) \subset B_{\varepsilon}(Q).
\]

Q is called a finitely equi-invariant set if \( \text{FEI}(Q) = Q \).

2. \( x \) is called a finitely equi-invariant point in the mean of \( Q \), denoted by \( x \in \text{FEIM}(Q) \), if \( x \) satisfies the item (1) where the formula (2.1) is replaced by

\[
\frac{1}{n} \sum_{i=0}^{n-1} d(\phi(i, y, \omega), Q) < \varepsilon, \quad \forall n \in \mathbb{N}.
\]
Q is called a \emph{finitely equi-invariant set in the mean} if \( \text{FEIM}(Q) = Q \).

(3) \( x \) is called a \emph{finitely mean equi-invariant point} of \( Q \), denoted by \( x \in \text{FMEI}(Q) \), if \( x \) satisfies the item (1) where the formula (2.1) is replaced by

\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(\phi(i, y, \omega), Q) < \varepsilon.
\]

(2.3)

Q is called a \emph{finitely mean equi-invariant set} if \( \text{FMEI}(Q) = Q \).

Furthermore, \( x \) is said to be an \emph{equi-invariant point} of \( Q \), an \emph{equi-invariant point in the mean} of \( Q \) and a \emph{mean equi-invariant point} of \( Q \) if \( x \) satisfies (1)–(3), respectively, with the set \( F \) being a singleton.

We write \( x \in \text{EI}(Q) \), \( x \in \text{EIM}(Q) \) and \( x \in \text{MEI}(Q) \), respectively.

\textbf{Remark 2.2.}  
(i) The concept of equi-invariance was introduced in [24].

(ii) From the viewpoint of control theory, \( x \in \text{EI}(Q) \) means that \( x \) can be stabilized robustly to any neighborhood of \( Q \); \( x \in \text{EIM}(Q) \) means that \( x \) can be stabilized robustly to any neighborhood of \( Q \) in the mean; \( x \in \text{MEI}(Q) \) means that \( x \) can be stabilized robustly to any neighborhood of \( Q \) eventually in the mean.

(iii) There are the following implication relations among these six types of equi-invariability (see Figure 1).

We give examples in Appendix A to show the above seven “\( \Rightarrow \)” relations are possible. We establish in the next section the conditions under which EI \( \Leftrightarrow \) FEI, EIM \( \Leftrightarrow \) FEIM and MEI \( \Leftrightarrow \) FMEI, respectively. See Corollaries 3.2, 3.8 and 3.13.

Now let us discuss the relations between equi-invariability and the control complexity. It is well known that invariance entropy introduced by Colonius and Kawan [7] as well as topological feedback entropy introduced by Nair et al. [22] characterizes the minimal data rate for making a subset of the state space invariant. It is a very useful invariant to describe the exponential growth rate of the minimal number of different control functions sufficient for orbits to stay in a given set when starting in a subset of this set.

For a control system \( \Sigma = (\mathbb{N}_0, X, U, \mathcal{W}, \phi) \), let a subset \( Q \subset X \) be controlled invariant. For \( \omega \in \mathcal{W}, n \in \mathbb{N}, \varepsilon > 0 \), define

\[
Q_{n,\omega}^\varepsilon = \{ x \in Q : \phi([0, n), x, \omega) \subset B_\varepsilon(Q) \}.
\]

A subset \( F \subset \mathcal{W} \) is called an \((n, \varepsilon, Q)\)-spanning set if

\[
Q = \bigcup_{\omega \in F} Q_{n,\omega}^\varepsilon.
\]

Let

\[
r_{\text{inv}}(n, \varepsilon, Q) = \inf \{ \sharp F : F \text{ is an } (n, \varepsilon, Q)\text{-spanning set} \},
\]

where \( \sharp F \) denotes the cardinality of \( F \).

\begin{align*}
\text{EI} & \quad \Rightarrow \quad \Leftrightarrow \quad \text{EIM} & \quad \Rightarrow \quad \Leftrightarrow \quad \text{MEI} \\
\downarrow & & & & & \downarrow \\
\& (\text{Example A.6}) & & & \& (\text{Example A.8}) & & \& (\text{Example A.1}) & \& (\text{Example A.1}) & \& (\text{Example A.11}) \\
\text{FEI} & \quad \Rightarrow \quad \Leftrightarrow \quad \text{FEIM} & \quad \Rightarrow \quad \Leftrightarrow \quad \text{FMEI} \\
(\text{Example A.7}) & & & (\text{Example A.8}) & & & & \\
\end{align*}

\textbf{Figure 1}  Six types of equi-invariability
Recall that the outer invariance entropy of $Q$ is defined by
\[
    h_{\text{inv, out}}(Q) := \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{\log r_{\text{inv}}(n, \varepsilon, Q)}{n}.
\]  
(2.4)

See the monograph [18] for more details on invariance entropy.

If a control invariant set $Q$ has positive outer invariance entropy, i.e., $h_{\text{inv, out}}(Q) > 0$, then the numbers of controls needed to keep $Q$ in any neighborhood of $Q$ in $[0, n)$ grow exponentially with respect to $n$. Thus it is more difficult to realize such a control task in this case. On the contrary, such a control task is simple if $h_{\text{inv, out}}(Q) = 0$. A particular case of $h_{\text{inv, out}}(Q) = 0$ is the following definition.

**Definition 2.3** (See [24, Definition 3.3]). We say that $Q$ has bounded invariance complexity if for any $\varepsilon > 0$, there exists $C := C(\varepsilon) > 0$ such that $r_{\text{inv}}(n, \varepsilon, Q) \leq C$ for all $n \in \mathbb{N}$.

It is easy to see that if $Q$ has bounded invariance complexity then its outer invariance entropy is zero. For some $A \subset Q$, we denote the closure of $A$ in $Q$ with respect to the subspace topology on $Q$ by $\text{cl}_Q A$.

**Proposition 2.4** (See [24, Proposition 3.5]). Let $\Sigma = (\mathbb{N}_0, X, U, \mathcal{F}, \phi)$ be a control system and $Q$ be a compact subset of $X$.

(i) If $Q$ is equi-invariant, then $Q$ has bounded topological invariance complexity.

(ii) Conversely, if $Q$ has bounded topological invariance complexity then $\text{cl}_Q \text{EI}(Q) = Q$.

**Theorem 2.5.** Let $\Sigma = (\mathbb{N}_0, X, U, \mathcal{F}, \phi)$ be a system and $Q \subset X$ be a nonempty compact set. Then $Q$ is finitely equi-invariant if and only if $Q$ has bounded invariance complexity.

**Proof.** ($\Rightarrow$) Suppose that $Q$ is finitely equi-invariant. Then for any $\varepsilon > 0$ and $x \in Q$, there are $\delta_x > 0$ and $F_x \subset \mathcal{F}$ such that for every $y \in B(x, \delta_x) \cap Q$, there exists $\omega_y \in F_x$ such that
\[
    \phi(\mathbb{N}_0, y, \omega_y) \subset B_{\varepsilon}(Q).
\]

Since $Q$ is compact, we can find a finite open cover
\[
    \mathcal{C} := \{B(x_i, \delta_{x_i}), i = 1, \ldots, p\}
\]
of $Q$. Let $F = \bigcup_{i=1}^{p} \{F_{x_i}\}$. Then $F$ is finite and is an $(n, \varepsilon, Q)$-spanning set for every $n \in \mathbb{N}$. Hence $Q$ has bounded invariance complexity.

($\Leftarrow$) Given $\varepsilon > 0$, there exists $C$ such that $r_{\text{inv}}(n, \varepsilon, Q) \leq C$ for all $n \in \mathbb{N}$, i.e., for any $n \in \mathbb{N}$ there exists $F_n \subset \mathcal{F}$ such that
\[
    \sharp F_n \leq C \quad \text{and} \quad Q = \bigcup_{\omega \in F_n} Q_{n, \omega}^{\varepsilon/3}.
\]

By the compactness of $2^\mathcal{F}$ (the hyperspace of $\mathcal{F}$ [21]), we can pick a convergent subsequence $\{F_{n_i}\} \subset \{F_n\}$. We denote its limit by $F$, i.e.,
\[
    \lim_{i \to \infty} F_{n_i} = F.
\]

Therefore, we have $\sharp F \leq C$ by the fact that $\{A \in 2^\mathcal{F} : \sharp A \leq C\}$ is closed. For every $i \in \mathbb{N}$ and any $x \in Q$ there exists $\omega_{n_i} \in F_{n_i}$ such that
\[
    \phi([0, n_i), x, \omega_{n_i}) \subset B_{\varepsilon/3}(Q).
\]

Thus we get
\[
    \phi([0, n_i), x, \omega_{n_j}) \subset B_{\varepsilon/3}(Q)
\]
for any $j \geq i$. Suppose that $\omega_{n_i} \to \omega$. Then $\omega \in F$. Letting $j \to \infty$, we have, by the continuity of $\phi$,
\[
    \phi([0, n_i), x, \omega) \subset B_{\varepsilon/2}(Q).
\]

Since $n_i \to \infty$ as $i \to \infty$, we obtain
\[
    \phi(\mathbb{N}_0, x, \omega) \subset B_{\varepsilon/2}(Q).
\]
This implies that
\[ Q \subset \bigcup_{\omega \in F} \bigcap_{n=1}^{\infty} Q_{n,\omega}^{\varepsilon/2}. \]

It follows that \( Q \) is finitely equi-invariant.

Next, we discuss the relations between finite equi-invariance in the mean and bounded invariance complexity in the mean.

Given \( \omega \in \mathcal{U}, \ n \in \mathbb{N} \) and \( \varepsilon > 0 \), let
\[ \hat{Q}_{n,\omega}^{\varepsilon} = \left\{ x \in Q : \max_{1 \leq k \leq n} \left\{ \frac{1}{k} \sum_{i=0}^{k-1} d[\phi(i, x, \omega)], Q \right\} < \varepsilon \right\}. \]

A subset \( F \subset \mathcal{U} \) is called an \((n, \varepsilon, Q)\)-spanning set in the mean if
\[ Q = \bigcup_{\omega \in F} \hat{Q}_{n,\omega}^{\varepsilon/3}. \]

Let
\[ \hat{r}_{\text{inv}}(n, \varepsilon, Q) = \inf \{ \sharp F : F \text{ is an } (n, \varepsilon, Q)-\text{spanning set in the mean} \}. \]

**Definition 2.6.** We say that \( Q \) has bounded invariance complexity in the mean if for any \( \varepsilon > 0 \), there exists \( C = C(\varepsilon) > 0 \) such that \( \hat{r}_{\text{inv}}(n, \varepsilon, Q) \leq C \) for all \( n \in \mathbb{N} \).

**Theorem 2.7.** Let \( \Sigma = (\mathcal{N}_0, X, U, \mathcal{U}, \phi) \) be a system and \( Q \subset X \) be a nonempty compact set. Then \( Q \) is finitely equi-invariant in the mean if and only if \( Q \) has bounded invariance complexity in the mean.

**Proof.** (\( \Rightarrow \)) Similar to the proof of (\( \Rightarrow \)) for Theorem 2.5, we can prove “only if” for Theorem 2.7 by replacing \( \phi(N_0, y, \omega y) \subset B_{\varepsilon}(Q) \) by
\[ \frac{1}{n} \sum_{i=0}^{n-1} d[\phi(i, y, \omega y)], Q] < \varepsilon, \quad \forall n \in \mathbb{N}. \]

(\( \Leftarrow \)) Given \( \varepsilon > 0 \), there exists \( C \) such that \( \hat{r}_{\text{inv}}(n, \varepsilon, Q) \leq C \) for all \( n \in \mathbb{N} \), i.e., for any \( n \in \mathbb{N} \) there exists \( F_n \subset \mathcal{U} \) such that
\[ \sharp F_n \leq C \quad \text{and} \quad Q = \bigcup_{\omega \in F_n} \hat{Q}_{n,\omega}^{\varepsilon/3}. \]

By the compactness of \( 2^\mathcal{U} \), we can pick a convergent subsequence \( \{F_{n_i}\} \) in \( \{F_n\} \). We denote its limit by \( F \). Therefore, we have \( \sharp F \leq C \) by the fact that \( \{A \in 2^\mathcal{U} : \sharp A \leq C \} \) is closed. For every \( i \in \mathbb{N} \) and any \( x \in Q \) there exists \( \omega_{n_i} \in F_{n_i} \) such that
\[ \max_{1 \leq k \leq n_i} \left\{ \frac{1}{k} \sum_{i=0}^{k-1} d[\phi(i, x, \omega_{n_i})], Q \right\} < \frac{\varepsilon}{3}. \]

Thus we get
\[ \max_{1 \leq k \leq n_i} \left\{ \frac{1}{k} \sum_{i=0}^{k-1} d[\phi(i, x, \omega_{n_i})], Q \right\} < \frac{\varepsilon}{3} \]
for any \( j \geq i \). Suppose that \( \omega_{n_i} \to \omega \). Then \( \omega \in F \). Letting \( j \to \infty \), we have, by the continuity of \( \phi \),
\[ \max_{1 \leq k \leq n_i} \left\{ \frac{1}{k} \sum_{i=0}^{k-1} d[\phi(i, x, \omega)], Q \right\} < \frac{\varepsilon}{2}. \]

Since \( n_i \to \infty \) as \( i \to \infty \), we obtain
\[ \max_{1 \leq k \leq n_i} \left\{ \frac{1}{k} \sum_{i=0}^{k-1} d[\phi(i, x, \omega)], Q \right\} < \frac{\varepsilon}{2}, \quad \forall n \in \mathbb{N}. \]
This implies that
\[ Q \subset \bigcup_{\omega \in F} \bigcap_{n=1}^{\infty} \hat{Q}^2_{n,\omega}. \]

It follows that \( Q \) is finitely equi-invariant in the mean.

Finally in this section, we characterize the concept of finitely mean equi-invariance by finitely mean stability of \( Q \) in the sense of Lyapunov.

Let \( E \subset \mathbb{N}_0 \). We define the upper density \( \overline{D}(E) \) of \( E \) by
\[
\overline{D}(E) = \limsup_{n \to \infty} \frac{\sharp(E \cap [0, n-1])}{n}.
\]

**Definition 2.8.** Let \( \Sigma = (\mathbb{N}_0, X, \mathcal{U}, \phi) \) be a system and \( Q \subset X \) be a nonempty set. A point \( x \in Q \) is said to be a finitely mean stable point of \( Q \) in the sense of Lyapunov (abbreviated as finitely mean-L-stable point of \( Q \)) if for every \( \epsilon > 0 \) there exist \( \delta > 0 \) and a finite subset \( F \subset \mathcal{U} \) such that for every \( y \in B(x, \delta) \cap Q \) implies \( d(\phi(n, y, \omega), Q) < \epsilon \) for some \( \omega \in F \) and all \( n \in \mathbb{N}_0 \) except a set of upper density less than \( \epsilon \). We call \( Q \) finitely mean-L-stable if every \( x \in Q \) is a finitely mean-L-stable point of \( Q \).

**Theorem 2.9.** Let \( \Sigma = (\mathbb{N}_0, X, \mathcal{U}, \phi) \) be a system with \( \text{diam}(X) < \infty \) and \( Q \subset X \) be a nonempty compact set. Then \( Q \) is finitely mean-L-stable if and only if it is finitely mean equi-invariant.

**Proof.** (\( \Leftarrow \)) Suppose that \( Q \) is finitely mean equi-invariant. Then for any \( x \in Q \) and \( \epsilon > 0 \), there exist \( \delta > 0 \) and a finite subset \( F \subset \mathcal{U} \) such that for every \( y \in B(x, \delta) \cap Q \) with \( d(x, y) < \delta \), we have
\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(\phi(i, y, \omega), Q) < \epsilon^2
\]
for some \( \omega \in F \). Let
\[
E = \{ k \in \mathbb{N}_0 : d(\phi(k, y, \omega), Q) \geq \epsilon \}.
\]
Thus
\[
\epsilon^2 > \limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(\phi(i, y, \omega), Q) \geq \limsup_{n \to \infty} \frac{1}{n} (\epsilon \cdot \sharp([0, n-1] \cap E)) = \epsilon \cdot \overline{D}(E).
\]
It follows that \( \overline{D}(E) < \epsilon \). Therefore, \( Q \) is finitely mean-L-stable.

(\( \Rightarrow \)) Assume that \( Q \) is finitely mean-L-stable. For any \( x \in Q \) and \( \epsilon > 0 \), let
\[
\eta = \frac{\epsilon}{2(\text{diam}(X) + 1)}.
\]
Then there exist \( \delta > 0 \) and a finite subset \( F \subset \mathcal{U} \) such that for any \( y \in B(x, \delta) \cap Q \), \( d(\phi(n, y, \omega), Q) < \eta \) for some \( \omega \in F \) and all \( n \in \mathbb{N}_0 \) except a set of upper density less than \( \eta \). Let
\[
E = \{ k \in \mathbb{N}_0 : d(\phi(n, y, \omega), Q) \geq \eta \}.
\]
Thus \( \overline{D}(E) < \eta \) and
\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(\phi(i, y, \omega), Q) \leq \limsup_{n \to \infty} \frac{1}{n} (\text{diam}(X) \cdot \sharp([0, n-1] \cap E) + n \eta) \leq \text{diam}(X) \overline{D}(E) + \eta \leq \text{diam}(X) \eta + \eta \leq \frac{\epsilon}{2}.
\]
This implies that \( Q \) is finitely mean equi-invariant.

\( \square \)
3 Dichotomy theorems for control sets

In this section, we will discuss three types of dichotomy theorems for control sets.

First, let us recall some basic notions. Let \( \Sigma = (\mathbb{N}_0, X, \mathcal{U}, \phi) \) be a control system. For \( x \in X \) and \( n \in \mathbb{N} \), the set of points reachable from \( x \) up to time \( n \) is defined by

\[ O_{\leq n}^+(x) := \{ y \in X : \exists m \in [0, n], \omega \in \mathcal{U} \text{ with } y = \phi(m, x, \omega) \}. \]

The positive orbit of \( x \) is defined by

\[ O^+(x) = \bigcup_{n \in \mathbb{N}} O_{\leq n}^+(x). \]

**Definition 3.1** (See [9, Definition 3.1] or [18, Definition 1.12]). Let \( \Sigma = (\mathbb{N}_0, X, \mathcal{U}, \phi) \) be a system. A set \( D \subset X \) is called a control set of system \( \Sigma \) if the following conditions hold:

1. \( D \) is controlled invariant, i.e., for every \( x \in D \) there exists \( \omega \in \mathcal{U} \) such that \( \phi([0, n], x, \omega) \subset D \).
2. For all \( x \in D \) one has \( D \subset \text{cl} O^+(x) \), where \( \text{cl} O^+(x) \) denotes the closure of \( O^+(x) \).
3. \( D \) is maximal with these properties, i.e., if \( D' \supset D \) satisfies the conditions (1) and (2), then \( D' = D \).

### 3.1 The first type of dichotomy theorem

By Theorem 2.5 and [24, Corollary 3.7], we have the following corollary.

**Corollary 3.2.** Let \( \Sigma = (\mathbb{N}_0, X, \mathcal{U}, \phi) \) and \( Q \subset X \) be a compact control set with nonempty interior. Then the following conditions are equivalent:

1. \( Q \) is equi-invariant;
2. \( Q \) is finitely equi-invariant;
3. \( Q \) has bounded invariance complexity.

**Definition 3.3.** Let \( \Sigma = (\mathbb{N}_0, X, \mathcal{U}, \phi) \) be a system and \( Q \subset X \) be a nonempty set. We say that \( Q \) is an unstable set if there exists \( \varepsilon > 0 \) such that for any \( x \in Q \), \( \delta > 0 \) and \( \omega \in \mathcal{U} \), we have

\[ d[\phi(m, y, \omega), Q] \geq \varepsilon \]

for some \( y \in B(x, \delta) \cap Q \) and \( m \in \mathbb{N}_0 \).

In [24], Wang et al. showed the following dichotomy theorem for control sets.

**Theorem 3.4** (See [24, Theorem 3.13]). Let \( \Sigma = (\mathbb{N}_0, X, \mathcal{U}, \phi) \) be a system and \( Q \subset X \) be a control set with \( \text{cl} \text{Int}(Q) = \text{cl} Q \). Then \( Q \) is either equi-invariant or unstable.

### 3.2 The second type of dichotomy theorem

Let

\[ \text{EIM}_k(Q) = \left\{ x \in Q : \exists \delta > 0 \text{ and } \omega \in \mathcal{U} \text{ s.t. } \frac{1}{n} \sum_{i=0}^{n-1} d[\phi(i, y, \omega), Q] \leq \frac{1}{k}, \quad \forall n \in \mathbb{N} \text{ and } y \in B(x, \delta) \cap Q \right\}. \]

Then \( \text{EIM}_k(Q) \) is an open subset of \( Q \) and \( \text{EIM}(Q) = \bigcap_{k=1}^\infty \text{EIM}_k(Q) \).

**Lemma 3.5** (See [18, Corollary 1.1]). A control set \( D \) with nonempty interior has the no-return property, i.e., if \( x \in D, n \in \mathbb{N}_0 \) and \( \omega \in \mathcal{U} \) with \( \phi([0, n], x, \omega) \in D \) implies \( \phi([0, n], x, \omega) \subset D \).

**Lemma 3.6.** Let \( \Sigma = (\mathbb{N}_0, X, \mathcal{U}, \phi) \) be a system and \( Q \subset X \) be a control set. If

\[ \text{EIM}_k(Q) \cap \text{Int}(Q) \neq \emptyset \]

for some \( k \in \mathbb{N} \), then \( \text{EIM}_k(Q) = Q \).
Proof. Pick \( x \in \text{EIM}_k(Q) \cap \text{Int}(Q) \). Then there exist \( \delta > 0 \) and \( \omega \in \mathcal{U} \) such that for every \( y \in B(x, \delta) \subset \text{Int}(Q) \), we have
\[
\frac{1}{n} \sum_{i=0}^{n-1} d[\phi(i, y, \omega), Q] < \frac{1}{k}, \quad \forall n \in \mathbb{N}.
\]
For any \( x' \in Q \), there exist \( m \in \mathbb{N}_0 \), \( \omega' \in \mathcal{U} \) and \( \delta' > 0 \) such that
\[
\phi(m, B(x', \delta'), \omega') \subset B(x, \delta) \subset \text{Int}(Q).
\]
By the no-return property (see Lemma 3.5),
\[
\phi([0, m], B(x', \delta'), \omega') \subset Q.
\]
Let \( \tilde{\omega} = \omega^m \). Then for any \( y \in B(x', \delta') \),
\[
\frac{1}{n} \sum_{i=0}^{n-1} d[\phi(i, y, \tilde{\omega}), Q] \begin{cases} = 0, & 0 \leq n \leq m, \\ < \frac{1}{k}, & n > m. \end{cases}
\]
So \( x' \in \text{EIM}_k(Q) \) and \( \text{EIM}_k(Q) = Q \). \( \square \)

The following lemma comes from [24].

**Lemma 3.7** (See [24, Lemma 3.4]). Let \( X \) be a topological space and \( Q \subset X \) be a closed set. Suppose that \( Q = \bigcup_{i=1}^N Q_i \), where \( N \in \mathbb{N} \) and \( Q_i \) is a closed subset of \( Q \) for every \( 1 \leq i \leq N \). Then there exist closed subsets \( Q_1, \ldots, Q_N \) of \( Q \) such that
\[
Q = \bigcup_{i=1}^N \hat{Q}_i, \quad \bigcup_{i=1}^N \hat{Q}_i \setminus \bigcup_{j \not\in i} \hat{Q}_j = Q.
\]

By Theorem 2.7, we have the following corollary.

**Corollary 3.8.** Let \( \Sigma = (\mathbb{N}_0, X, U, \mathcal{U}, \phi) \) and \( Q \subset X \) be a compact control set with nonempty interior. Then the following conditions are equivalent:
1. \( Q \) is equi-invariant in the mean;
2. \( Q \) is finitely equi-invariant in the mean;
3. \( Q \) has bounded invariance complexity in the mean.

**Proof.** We have shown that (2) and (3) are equivalent in Theorem 2.7. It is clear that (1) implies (2) and we only need to prove (3) implies (1). For every \( k \in \mathbb{N} \), it follows from the proof of Theorem 2.7 that
\[
Q \subset \bigcup_{\omega \in F} \bigcap_{n=1}^\infty \hat{Q}_{n, \omega}^{1/2k},
\]
where \( F \) is a finite set of \( \mathcal{U} \) and
\[
\hat{Q}_{n, \omega}^{1/2k} = \left\{ x \in Q : \max_{1 \leq i \leq n} \left\{ \frac{1}{j} \sum_{i=0}^{j-1} d[\phi(i, x, \omega), Q] \right\} < \frac{1}{2k} \right\}.
\]
Let \( F = \{ \omega_i : 1 \leq i \leq \#F \} \) and
\[
Q_i = \bigcap_{n=1}^{\infty} \left\{ x \in Q : \max_{1 \leq i \leq n} \left\{ \frac{1}{j} \sum_{i=0}^{j-1} d[\phi(i, x, \omega_i), Q] \right\} \leq \frac{1}{2k} \right\}, \quad i = 1, \ldots, \#F.
\]
Then \( Q_i \) is closed in \( Q \) for \( i = 1, \ldots, \#F \) and \( Q = \bigcup_{i=1}^{\#F} Q_i \). Let \( Q'_i := Q_1 \) and \( Q'_i := cl_Q(Q_i \setminus \bigcup_{j=1}^{i-1} Q_j) \) for \( 2 \leq i \leq \#F \). Using Lemma 3.7, we have
\[
\bigcup_{i=1}^{\#F} Q'_i = Q, \quad \bigcup_{i=1}^{\#F} cl_Q(Q'_i \setminus \bigcup_{j \not\in i} Q'_j) = Q.
\]
For any $i \in \{1, \ldots, \#F\}$ and $x \in Q'_i \setminus \bigcup_{j \neq i} Q'_j$, there exists $\delta > 0$ such that
\[ B(x, \delta) \cap Q = B(x, \delta) \cap Q'_i. \]
Therefore, we get $B(x, \delta) \cap Q \subset Q'_i$, which implies that
\[ \frac{1}{n} \sum_{i=0}^{n-1} d[\phi(r, y, \omega_i), Q] < \frac{1}{k}, \quad \forall n \in \mathbb{N} \text{ and } y \in B(x, \delta) \cap Q. \]
Thus $Q'_i \setminus \bigcup_{j \neq i} Q'_j \subset \text{EIM}_k(Q)$. It follows that $c\text{EIM}_k(Q) = Q$. According to the Baire category theorem, we see that
\[ \text{EIM}(Q) = \bigcap_{k=1}^{\infty} \text{EIM}_k(Q) \]
is a dense $G_\delta$ subset of $Q$. Hence
\[ \text{EIM}(Q) \cap \text{Int}(Q) \neq \emptyset. \]

Pick $x \in \text{EIM}(Q) \cap \text{Int}(Q)$. Using Lemma 3.6, we obtain $\text{EIM}(Q) = Q$. □

**Definition 3.9.** Let $\Sigma = (\mathbb{N}_0, X, \mathcal{U}, \phi)$ be a system and $Q \subset X$ be a nonempty set. We say that $Q$ is an unstable set in the mean if there exists $\varepsilon > 0$ such that for any $x \in Q$, $\delta > 0$ and $\omega \in \mathcal{U}$ there exist $y \in B(x, \delta) \cap Q$ and $m \in \mathbb{N}_0$, satisfying
\[ \frac{1}{m} \sum_{i=0}^{m-1} d[\phi(i, y, \omega), Q] \geq \varepsilon. \]

By a direct observation, we get the following lemma.

**Lemma 3.10.** Let $\Sigma = (\mathbb{N}_0, X, \mathcal{U}, \phi)$ be a system. Then $Q \subset X$ is an unstable set in the mean if and only if there exists $k \in \mathbb{N}$ such that $\text{EIM}_k(Q) = \emptyset$.

**Theorem 3.11.** Let $\Sigma = (\mathbb{N}_0, X, \mathcal{U}, \phi)$ be a system and $Q \subset X$ be a control set with $c\text{Int}(Q) = cQ$. Then $Q$ is either equi-invariant in the mean or unstable in the mean.

**Proof.** If $Q = \text{EIM}(Q)$, then $Q$ is equi-invariant in the mean. If $Q \neq \text{EIM}(Q)$, then there exists $k \in \mathbb{N}$ such that $\text{EIM}_k(Q) \cap \text{Int}(Q) = \emptyset$ by Lemma 3.6. To obtain a contradiction, we suppose that $Q$ is not unstable in the mean. By Lemma 3.10, we have $\text{EIM}_k(Q) \neq \emptyset$ for all $k \in \mathbb{N}$. Fix any $k \in \mathbb{N}$ and pick $x \in \text{EIM}_k(Q)$. Then there exist $\delta > 0$ and $\omega \in \mathcal{U}$ such that
\[ \frac{1}{n} \sum_{i=0}^{n-1} d[\phi(i, y, \omega), Q] < \frac{1}{k}, \quad \forall n \in \mathbb{N} \text{ and } y \in B(x, \delta) \cap Q. \]
Noting that $c\text{Int}(Q) = cQ$, we have $B(x, \delta) \cap \text{Int}(Q) \neq \emptyset$. Hence there exist $y \in B(x, \delta) \cap \text{Int}(Q)$ and $\delta' > 0$ such that $B(y, \delta') \subset B(x, \delta) \cap \text{Int}(Q)$. So
\[ \frac{1}{n} \sum_{i=0}^{n-1} d[\phi(i, y, \omega), Q] < \frac{1}{k}, \quad \forall n \in \mathbb{N} \text{ and } y \in B(y, \delta') \cap Q \subset B(x, \delta) \cap Q. \]
This implies that $y \in \text{EIM}_k(Q) \cap \text{Int}(Q)$ for all $k \in \mathbb{N}$, which leads to a contradiction. □

### 3.3 The third type of dichotomy theorem

Finally, we discuss the dichotomy theorem based on mean equi-invariability.

Let
\[ \text{MEI}_k(Q) = \left\{ x \in Q : \exists \delta > 0 \text{ and } \omega \in \mathcal{U} \text{ s.t. } \limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d[\phi(i, y, \omega), Q] < \frac{1}{k}, \right. \]
\[ \forall y \in B(x, \delta) \cap Q \}. \]
Then $\text{MEI}_k(Q)$ is an open subset of $Q$ and $\text{MEI}(Q) = \bigcap_{k=1}^{\infty} \text{MEI}_k(Q)$. 
Lemma 3.12. Let $\Sigma = (N_0, X, U, \mathcal{U}, \phi)$ be a system and $Q \subset X$ be a control set. If
\[ \text{MEI}_k(Q) \cap \text{Int}(Q) \neq \emptyset \]
for some $k \in \mathbb{N}$, then $\text{MEI}_k(Q) = Q$.

Proof. Pick $x \in \text{MEI}_k(Q) \cap \text{Int}(Q)$. Then there exist $\delta > 0$ and $\omega \in \mathcal{U}$ such that for every $y \in B(x, \delta) \subset \text{Int}(Q)$, we have
\[ \limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d[\phi(i, y, \omega), Q] < \frac{1}{k}. \]

For any $x' \in Q$, there exist $m \in \mathbb{N}$, $\omega' \in \mathcal{U}$ and $\delta' > 0$ such that
\[ \phi(m, B(x', \delta'), \omega') \subset B(x, \delta) \subset \text{Int}(Q). \]

Applying the no-return property, we have
\[ \phi([0, m], B(x', \delta'), \omega') \subset Q. \]

Let $\hat{\omega} = \omega' \omega^m$. Then for any $y \in B(x', \delta')$,
\[
\limsup_{n \to \infty, n > m} \frac{1}{n} \sum_{i=0}^{n-1} d[\phi(i, y, \hat{\omega}), Q] = \limsup_{n \to \infty, n > m} \frac{1}{n} \left( \sum_{i=0}^{m} d[\phi(i, y, \hat{\omega}), Q] + \sum_{i=m+1}^{n-1} d[\phi(i, y, \hat{\omega}), Q] \right)
\leq \frac{1}{k}.
\]

So $x' \in \text{MEI}_k(Q)$ and $\text{MEI}_k(Q) = Q$. \hfill \Box

Corollary 3.13. Let $\Sigma = (N_0, X, U, \mathcal{U}, \phi)$, $\phi$ be uniformly continuous and $Q \subset X$ be a compact control set with nonempty interior. Then the following conditions are equivalent:

1. $Q$ is mean equi-invariant;
2. $Q$ is finitely mean equi-invariant;
3. $Q$ is finitely mean L-stable.

Proof. We have shown that (2) and (3) are equivalent in Theorem 2.9. It is clear that (1) implies (2) and we only need to prove (2) implies (1).

For every $\epsilon > 0$ and $\omega \in \mathcal{U}$, define
\[ \hat{Q}^\omega_\epsilon := \left\{ x \in Q : \limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d[\phi(i, x, \omega), Q] \leq \frac{\epsilon}{2} \right\}. \]

Let $\epsilon > 0$. For every $x \in Q$, by finite mean equi-invariance of $Q$, there exist $\delta_x > 0$ and $F_x$ such that for any $y \in B(x, \delta_x) \cap Q$, it holds that
\[ \limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d[\phi(i, x, \omega), Q] < \frac{\epsilon}{2} \]
for some $\omega \in F_x$. Take $\mathcal{C} := \{ B(x, \delta_x) \}_{x \in Q}$. Then $\mathcal{C}$ is an open cover of $Q$. By compactness of $Q$, there exist finite open balls $B(x_1, \delta_{x_1}), B(x_2, \delta_{x_2}), \ldots, B(x_r, \delta_{x_r})$ such that $Q \subset \bigcup_{i=1}^{r} B(x_i, \delta_{x_i})$. Put
\[ F = \bigcup_{i=1}^{r} F_{x_i} := \{ \omega^{(i)} : 1 \leq i \leq 2F \}. \]
Then $\bar{Q}^c_{\omega(i)}$ is closed in $Q$ for $i = 1, \ldots, \sharp F$ and $Q = \bigcup_{i=1}^{\sharp F} \bar{Q}^c_{\omega(i)}$. Let

$$Q'_1 := \bar{Q}^c_{\omega(1)} \quad \text{and} \quad Q'_i := \text{cl}_Q\left(\bar{Q}^c_{\omega(i)} \setminus \bigcup_{j=1}^{i-1} \bar{Q}^c_{\omega(j)}\right)$$

for $2 \leq i \leq \sharp F$. Using [24, Lemma 3.7], we have

$$\bigcup_{i=1}^{\sharp F} Q'_i = Q, \quad \bigcup_{i=1}^{\sharp F} \text{cl}_Q\left(Q'_i \setminus \bigcup_{j\neq i} Q'_j\right) = Q.$$

For each $i \in \{1, \ldots, \sharp F\}$ and $x \in Q'_i \setminus \bigcup_{j\neq i} Q'_j$, there exists $\delta > 0$ such that

$$B(x, \delta) \cap Q = B(x, \delta) \cap Q'_i.$$

Therefore, we get $B(x, \delta) \cap Q \subset Q'_i \subset \bar{Q}^c_{\omega(i)}$, which implies that

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{r=0}^{n-1} d[\phi(r, y, \omega^{(i)}), Q] < \epsilon \quad \text{for all } y \in B(x, \delta) \cap Q.$$

Thus $Q'_i \setminus \bigcup_{j\neq i} Q'_j \subset \text{MEI}_k(Q)$. It follows that $\text{cl}_Q\text{MEI}_k(Q) = Q$. According to the Baire category theorem, we see that

$$\text{MEI}(Q) = \bigcap_{k=1}^{\infty} \text{MEI}_k(Q)$$

is a dense $G_\delta$ subset of $Q$. Hence $\text{MEI}(Q) \cap \text{Int}(Q) \neq \emptyset$. Pick $x \in \text{MEI}(Q) \cap \text{Int}(Q)$. Using Lemma 3.12, we obtain $\text{MEI}(Q) = Q$. 

**Definition 3.14.** Let $\Sigma = (\mathbb{N}_0, X, U, \mathcal{W}, \phi)$ be a system and $Q \subset X$ be a nonempty set. We say that $Q$ is a *mean unstable set* if there exists $\varepsilon > 0$ such that for any $x \in Q$, $\delta > 0$ and $\omega \in \mathcal{W}$, there exists $y \in B(x, \delta) \cap Q$ with

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d[\phi(i, y, \omega), Q] \geq \varepsilon.$$

The following lemma is obvious.

**Lemma 3.15.** Let $\Sigma = (\mathbb{N}_0, X, U, \mathcal{W}, \phi)$ be a system. Then $Q \subset X$ is a mean unstable set if and only if there exists $k \in \mathbb{N}$ such that $\text{MEI}_k(Q) = \emptyset$.

**Theorem 3.16.** Let $\Sigma = (\mathbb{N}_0, X, U, \mathcal{W}, \phi)$ be a system and $Q \subset X$ be a control set with $c\text{Int}(Q) = \text{cl}Q$. Then $Q$ is either mean equi-invariant or mean unstable.

**Proof.** If $Q = \text{MEI}(Q)$, then $Q$ is mean equi-invariant. If $Q \neq \text{MEI}(Q)$, then there exists $k \in \mathbb{N}$ such that

$$\text{MEI}_k(Q) \cap \text{Int}(Q) = \emptyset$$

by Lemma 3.12. Suppose in contrast that $Q$ is not mean unstable. By Lemma 3.15, we have $\text{MEI}_k(Q) \neq \emptyset$ for all $k \in \mathbb{N}$. Fix any $k \in \mathbb{N}$ and pick $x \in \text{MEI}_k(Q)$. Then there exist $\delta > 0$ and $\omega \in \mathcal{W}$ such that

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d[\phi(i, y, \omega), Q] < \frac{1}{k}, \quad \forall y \in B(x, \delta) \cap Q.$$

By $c\text{Int}(Q) = \text{cl}Q$, it follows that

$$B(x, \delta) \cap \text{Int}(Q) \neq \emptyset.$$

Consequently, there exist $y \in B(x, \delta) \cap \text{Int}(Q)$ and $\delta' > 0$ such that

$$B(y, \delta') \subset B(x, \delta) \cap \text{Int}(Q).$$
So
\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(\phi(i, y, \omega), Q) < \frac{1}{k}, \quad \forall y \in B(y, \delta') \cap Q \subset B(x, \delta) \cap Q,
\]
which implies that \( y \in \text{MEI}_k(Q) \cap \text{Int}(Q) \) for all \( k \in \mathbb{N} \). This leads to a contradiction. \( \square \)

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Appendix A

Example A.1 (FEI but not EI; FEIM but not EIM). Consider a control system of the form (1.1), where

1. $X = [0, 1] = \{ x \in \mathbb{R} : 0 \leq x \leq 1 \}$;
2. $U = \{0, 1\}$;
3. $F_0, F_1 : X \rightarrow X$ are defined by

$$
F_0(x) = \begin{cases}
  x, & \text{if } 0 \leq x < \frac{3}{8}, \\
  5\left(x - \frac{1}{2}\right) + 1, & \text{if } \frac{3}{8} \leq x < \frac{1}{2}, \\
  1, & \text{if } \frac{1}{2} \leq x \leq 1
\end{cases}
$$

and

$$
F_1(x) = \begin{cases}
  1, & \text{if } 0 \leq x < \frac{1}{4}, \\
  -4\left(x - \frac{1}{4}\right) + 1, & \text{if } \frac{1}{4} \leq x < \frac{3}{8}, \\
  \frac{1}{2}, & \text{if } \frac{3}{8} \leq x < \frac{1}{2}, \\
  4\left(x - \frac{5}{8}\right) + 1, & \text{if } \frac{1}{2} \leq x < \frac{5}{8}, \\
  1, & \text{if } \frac{5}{8} \leq x \leq 1
\end{cases}
$$

Let $Q = \left[\frac{1}{4}, \frac{1}{2}\right]$. Then $Q$ is finitely equi-invariant but not equi-invariant (see Figure 2).

Proof. We divide our proof into three claims.

Claim A.2. $Q$ is finitely equi-invariant.

Figure 2 (Color online) The subinterval $[\frac{1}{4}, \frac{1}{2}]$ is finitely equi-invariant
Proof. Fix any \( x \in Q \). For any \( \epsilon > 0 \), by choosing \( \delta = \epsilon \) and \( F = \{ \omega := 0^\infty, \hat{\omega} := 1^\infty \} \), then by the definitions of \( F_0 \) and \( F_1 \), for any \( y \in B(x, \delta) \cap Q \) it holds that \( \phi(\mathbb{N}, y, \omega) = y \in Q \) if \( \frac{1}{4} \leq y \leq \frac{3}{8} \), and \( \phi(\mathbb{N}, y, \hat{\omega}) = \frac{1}{2} \in Q \) if \( \frac{3}{8} < y \leq \frac{1}{2} \). \( \square \)

Claim A.3. For any \( \frac{1}{4} \leq x < \frac{3}{8} \) and any control sequence \( \hat{\omega} \in \{ 0^n1^\infty : n \geq 1 \} \cup \{ \omega : \omega_{i+1} = 10 \text{ for some } i \geq 0 \} \cup \{ 1^\infty \} \), it holds that \( \lim_{n \to \infty} \phi(n, x, \hat{\omega}) = 1 \).

Proof. Let \( x \in [\frac{1}{4}, \frac{3}{8}) \) and a control sequence

\[
\hat{\omega} \in \{ \omega : \omega_{i+1} = 10 \text{ for some } i \geq 0 \} \cup \{ 1^\infty \} \cup \{ 0^n1^\infty : n \geq 1 \}.
\]

Then we have the following cases.

Case 1. \( \hat{\omega} = 1^\infty \). Then \( F_{\omega}(x) > \frac{1}{2} \). So by the definition of \( F_1 \), we have \( \lim_{n \to \infty} \phi(n, x, \hat{\omega}) = 1 \).

Case 2. \( \hat{\omega} \in \{ \omega : \omega_{i+1} = 10 \text{ for some } i \geq 0 \} \). Note that \( F_0 \circ F_1(z) > \frac{1}{2} \) for all \( z \in [0, 1] \). This implies that \( \lim_{n \to \infty} \phi(n, x, \hat{\omega}) = 1 \).

Case 3. \( \hat{\omega} \in \{ 0^n1^\infty : n \geq 1 \} \). Note that for any \( n \geq 1 \),

\[
F_1 = F_0 \circ \cdots \circ F_0(z) = F_1(z) > \frac{1}{2} \quad \text{for all } z \in \left[ \frac{1}{4}, \frac{3}{8} \right]_n.
\]

Thus \( \lim_{n \to \infty} \phi(n, x, \hat{\omega}) = 1 \). \( \square \)

Claim A.4. The point \( \frac{3}{8} \) is not equi-invariant.

Proof. Suppose in contrast that \( \frac{3}{8} \) is equi-invariant. Then for any \( \epsilon > 0 \) there exist \( \delta > 0 \) and \( \omega \in U^\mathbb{N} \) such that \( \phi(\mathbb{N}, y, \omega) \subset B(Q, \epsilon) \) for all \( y \in B(\frac{3}{8}, \delta) \). Since

\[
U^\mathbb{N} = \{ 0^n1^\infty : n \geq 1 \} \cup \{ \omega : \omega_{i+1} = 10 \text{ for some } i \geq 0 \} \cup \{ 0^\infty, 1^\infty \},
\]

by Claim A.3 and the proof of Claim A.2, we have \( \omega = 0^\infty \). However, for any \( z \in (\frac{3}{8}, \frac{1}{2}] \), we have \( \lim_{n \to \infty} F_0^n(z) = 1 \). This implies that \( \lim_{n \to \infty} \phi(n, z, \omega) = 1 \). This leads to a contradiction. Therefore, \( \frac{3}{8} \) is not equi-invariant. \( \square \)

By Claims A.2 and A.4, the set \( Q \) is finitely equi-invariant but not equi-invariant.

Remark A.5. By Claim A.4 and the proof of Claim A.2, one can see that every point in \( Q \) is equi-invariant except the point \( \frac{3}{8} \).

The following example shows that the equi-invariance in the mean is strictly weaker than the equi-invariance.

Example A.6 (EIM but not EI). Consider a control system of the form (1.1), where

1. \( X = [0, 1] = \{ x \in \mathbb{R} : 0 \leq x \leq 1 \} \);
2. \( U = \{ 0, 1, 2 \} \);
3. \( F_0, F_1 \) and \( F_2 : X \to X \) are defined by

\[
F_0(x) = \begin{cases} 
  x, & \text{if } 0 \leq x < \frac{3}{8}, \\
  \frac{1}{2} - \frac{3}{5} x + 1, & \text{if } \frac{3}{8} \leq x < \frac{1}{2}, \\
  1, & \text{if } \frac{1}{2} \leq x < \frac{5}{8}, \\
  \frac{3}{8}, & \text{if } \frac{3}{4} \leq x < 1,
\end{cases}
\]

(1) \( X = [0, 1] = \{ x \in \mathbb{R} : 0 \leq x \leq 1 \} \);
\( F_1(x) = \begin{cases} 
 1, & \text{if } 0 \leq x < \frac{1}{4}, \\
 -4 \left( x - \frac{1}{4} \right) + 1, & \text{if } \frac{1}{4} \leq x < \frac{3}{8}, \\
 \frac{1}{2}, & \text{if } \frac{3}{8} \leq x < \frac{1}{2}, \\
 4 \left( x - \frac{5}{8} \right) + 1, & \text{if } \frac{1}{2} \leq x < \frac{5}{8}, \\
 1, & \text{if } \frac{5}{8} \leq x \leq 1 
\end{cases} \)

and \( F_2(x) = 1 \) for all \( x \in [0, 1] \).

Let \( Q = \left[ \frac{1}{4}, \frac{1}{2} \right] \). Then the set \( Q \) is equi-invariant in the mean but not equi-invariant (see Figure 3).

**Proof.** Similar to the proof of Example A.1, almost all the points in \( Q \) are equi-invariant except \( \frac{3}{8} \). So it suffices to show that \( \frac{3}{8} \) is equi-invariant in the mean. Indeed, for any \( 0 < \epsilon < \frac{1}{8} \), choose \( N > 0 \) such that \( \frac{1}{2N} < \epsilon \) and \( 0 < \delta' < \frac{1}{8} \) with

- \( F_0^N \left( \frac{3}{8} + \delta' \right) = \frac{1}{2} \) and
- \( F_i^N \left( \frac{3}{8} + \delta' \right) < \frac{1}{2}, i = 0, 1, \ldots, N - 1 \).

Pick \( \delta = \min \{ \delta', \epsilon \} \) and two control sequences \( \omega = 0^N 20 \infty \). Then for any \( y \in B \left( \frac{3}{8}, \delta \right) \), it holds that

- \( F_0^N \circ \cdots \circ F_0^N (y) \in Q \) whenever \( n = 0, 1, \ldots, N - 1 \);
- \( F_0^N \circ \cdots \circ F_0^N (y) \equiv 1 \), which implies that

\[
\frac{1}{N} \sum_{i=0}^{N-1} d[\phi(i, y, \omega), Q] = \frac{1}{N} \sum_{i=0}^{N-1} d[\phi(i, y, \omega), Q] + \frac{d[\phi(N, y, \omega), Q]}{N} = 0 + \frac{1}{2N} < \epsilon;
\]

- \( F_0^N \circ \cdots \circ F_0^N (y) \equiv \frac{3}{8} \in Q \) whenever \( n > N \), which implies that

\[
\frac{1}{n} \sum_{i=0}^{n} d[\phi(i, y, \omega), Q] = \frac{1}{n} \sum_{i=0}^{N-1} d[\phi(i, y, \omega), Q] + \frac{d[\phi(N, y, \omega), Q]}{N} + \frac{1}{n} \sum_{i=N+1}^{n} d[\phi(i, y, \omega), Q]
\]

\[
= 0 + \frac{1}{2N} + 0 < \epsilon
\]

for all \( n > N \).

Thus the point \( \frac{3}{8} \) is equi-invariant in the mean. \( \square \)

**Figure 3** (Color online) The subinterval \( \left[ \frac{1}{4}, \frac{1}{2} \right] \) is equi-invariant in the mean.
Before we give the forthcoming example, we recall some notions. Let $I$ be a finite set. The one-sided symbolic space is

$$I^{N_0} = \{x = (x_0, x_1, \ldots) : x_i \in I \text{ for } i \in \mathbb{N}_0\}$$

with the distance

$$\rho(x, y) = \begin{cases} 0, & \text{if } x = y, \\ \frac{1}{i+1}, & \text{if } x \neq y \text{ and } i = \min\{j : x_j \neq y_j\}. \end{cases}$$

The shift map $\sigma : I^{N_0} \to I^{N_0}$ is defined as

$$x = (x_0, x_1, \ldots) \mapsto \sigma(x) = (x_1, x_2, \ldots).$$

Then $(I^{N_0}, \sigma)$ is a full shift. For $\omega \in I^n$, the length of $\omega$ is $l(\omega) = n$. A cylinder of $\omega$ is

$$[\omega] = \{x \in I^{N_0} : (x_0, \ldots, x_{n-1}) = \omega\}.$$

Now we give the following example to show that there exists a set which is finitely equi-invariant in the mean but not finitely equi-invariant.

**Example A.7 (FEIM but not FEI).** Let $I = \{a, b, c, d, e\}$, $A = ab$, $B = cde$ and $B = \{A, B\}$. Consider a control system of the form (1.1), where

1. $X = I^{N_0}$;
2. $U = \{0, 1, 2, 3\}$;
3. $F_0, F_1, F_2$ and $F_3 : X \to X$ are defined by $F_0 = \sigma^2$, $F_1 = \sigma^3$, $F_2 \equiv b^\infty$ and

$$F_3(x) = \begin{cases} (ab)^\infty, & \text{if } x \in [b], \\ b^\infty, & \text{otherwise}. \end{cases}$$

Define an injective map $\varphi$ from $B^{N_0}$ to $I^{N_0}$ by

$$\varphi(\mu)_{|\sum_{j=0}^{i-1} l(\mu_j), \sum_{j=0}^{i} l(\mu_j)) = \mu_i$$

for $\mu \in B^{N_0}$. Let $Q = \varphi(B^{N_0})$. Then the set $Q$ is equi-invariant in the mean but not finitely equi-invariant.

**Proof.** By the construction of $Q$ and the definitions of $F_0, F_1, F_2$ and $F_3$, the set $Q$ is compact and for any $x \in Q$ there exists a unique control sequence $\omega \in B$ such that $\phi(N_0, x, \omega) \subset Q$. This implies that $Q$ is not finitely equi-invariant.

Next, we show that $Q$ is equi-invariant in the mean. Let $x \in Q$. Fix any positive real number $\epsilon > 0$. Choose an integer with $n > \frac{1}{\epsilon}$. Since the topology of the subspace $Q$ is

$$\mathcal{B}_0 = \{[u] : u \in I^n, n \geq 0\} \cap Q$$

$$= \{([ab]^{n_1}(cde)^{m_1}(ab)^{n_2}(cde)^{m_2} \cdots (ab)^{n_k}(cde)^{m_k}) : n_1 + m_1 + \cdots + m_k + n_k \geq 0, k \geq 1\},$$

there exist $n_1, n_2, \ldots, n_k, m_1, m_2, \ldots, m_k$ such that $N := n_1 + m_1 + \cdots + m_k + n_k > n$, and

$$x \in ([ab]^{n_1}(cde)^{m_1}(ab)^{n_2}(cde)^{m_2} \cdots (ab)^{n_k}(cde)^{m_k}).$$

Pick a control sequence $\omega = 0^n 1^{m_1} 0^{n_2} 1^{m_2} \cdots 0^{n_k} 1^{m_k} 230^\infty$. Then for any

$$y \in ([ab]^{n_1}(cde)^{m_1}(ab)^{n_2}(cde)^{m_2} \cdots (ab)^{n_k}(cde)^{m_k}),$$

we have

$$\phi([0, N], y, \omega) \subset Q, \quad \phi(N + 1, y, \omega) = b^\infty \quad \text{and} \quad \phi([N + 2, \infty), y, \omega) = (ab)^\infty \in Q.$$
So
\[
\frac{1}{n} \sum_{i=0}^{n-1} d[\phi(i, y, \omega), Q] = \begin{cases} 
0, & \text{if } n \leq N + 1, \\
\frac{1}{n} d[\phi(N + 1, y, \omega), Q] = \frac{1}{n} d(h^\infty, Q) \leq \frac{1}{N} < \epsilon, & \text{if } n \geq N + 2.
\end{cases}
\]

Thus, the point \( x \) is equi-invariant in the mean and since \( x \) is arbitrary, we have \( Q \) is equi-invariant in the mean.

By the definition of \( F_1 \), it holds that

(a) \( F_1(x) > F_1^1(x) > F_1^2(x) > \cdots > F_1^n(x) > \cdots \) and \( \lim_{n \to \infty} F_1^n(x) = \frac{1}{4} \) for any \( x \in [0, \frac{1}{4}] \);

(b) \( F_1(x) > F_1(y) \) and \( F_1^n(x) < F_1^n(y) \) for any \( 0 \leq x < y \leq \frac{1}{4} \) and \( n > 1 \).

Next, we divide our proof into two claims.

Claim A.9. \( Q \) is mean equi-invariant.

Proof. Fix any \( x \in Q \) and \( 0 < \epsilon < \frac{1}{4} \). Since \( \lim_{n \to \infty} F_1^n(0) = \frac{1}{4} \), there exists \( N' > 0 \) such that

Let \( Q = [0, \frac{1}{4}] \). Then the set \( Q \) is mean equi-invariant but not finitely equi-invariant in the mean (see Figure 4).

Proof. Consider a control system of the form (1.1), where

(1) \( X = [0, 1] = \{ x \in \mathbb{R} : 0 \leq x \leq 1 \} \);

(2) \( U = \{ 0, 1 \} \);

(3) \( F_0 \) and \( F_1 : X \to X \) are defined by

\[
F_0(x) = \begin{cases} 
\frac{1}{2}, & \text{if } 0 \leq x < \frac{1}{4}, \\
2 \left( x - \frac{1}{2} \right) + 1, & \text{if } \frac{1}{4} \leq x < \frac{1}{2}, \\
1, & \text{if } \frac{1}{2} \leq x \leq 1 
\end{cases}
\]

and

\[
F_1(x) = \begin{cases} 
12 \left( x - \frac{1}{4} \right)^2 + \frac{1}{4}, & \text{if } 0 \leq x < \frac{1}{4}, \\
\left( x - \frac{1}{4} \right)^2 + \frac{1}{4}, & \text{if } \frac{1}{4} \leq x \leq 1 
\end{cases}
\]

Let \( Q = [0, \frac{1}{4}] \). Then the set \( Q \) is mean equi-invariant but not finitely equi-invariant in the mean (see Figure 4).
Proof. It suffices to show that 0 is not finitely equi-invariant in the mean, i.e., there exists a positive real number \( \epsilon > 0 \) such that for any \( n \geq N \),

\[
\limsup \frac{1}{n} \sum_{i=0}^{n-1} d(\phi(i, y, \omega), Q) = \frac{1}{n} \sum_{i=0}^{N'} d(\phi(i, y, \omega), Q) + \frac{1}{n} \sum_{i=N'+1}^{n-1} d(\phi(i, y, \omega), Q) \\
\leq \frac{1}{N} \sum_{i=0}^{N'} d(\phi(i, y, \omega), Q) + \frac{1}{n} \sum_{i=N'+1}^{n-1} d(\phi(i, y, \omega), Q) \\
\leq \frac{N' + 1}{N} + \frac{n - N' - 1}{n} \cdot \frac{\epsilon}{2} \\
< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,
\]

which implies that \( \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(\phi(i, y, \omega), Q) \leq \epsilon \). Thus every point in \( Q \) is finitely mean equi-invariant. \( \square \)

**Claim A.10.** \( Q \) is not finitely equi-invariant in the mean.

**Proof.** It suffices to show that 0 is not finitely equi-invariant in the mean, i.e., there exists a positive real number \( \epsilon > 0 \) such that for any \( n \geq N \), the control sequences \( \omega^{(1)}, \ldots, \omega^{(k)} \in \mathcal{W} \) and \( y \in B(0, \delta) \cap [0, \frac{1}{4}] \), one can find some control sequence \( \omega^{(r)} \), 1 \( \leq r \leq k \), satisfying that \( \frac{1}{n} \sum_{i=0}^{n-1} d(\phi(i, y, \omega^{(r)}), Q) \geq \epsilon \) for some \( n \). Indeed, for any \( 0 < \delta < \frac{\sqrt{3} - 1}{4\sqrt{3}} \), the control sequence \( \omega \in \mathcal{W} \) and \( y \in B(0, \delta) \cap [0, \frac{1}{4}] \), it holds that

\[
\frac{1}{2} (d(\phi(0, y, \omega), Q) + d(\phi(1, y, \omega), Q)) \geq \frac{1}{2} d(\phi(1, y, \omega), Q) = \frac{1}{2} d(F_0(y), Q) = \frac{1}{2}
\]

whenever \( \omega_0 = 0 \), and

\[
\frac{1}{2} (d(\phi(0, y, \omega), Q) + d(\phi(1, y, \omega), Q)) \geq \frac{1}{2} d(\phi(1, y, \omega), Q) = \frac{1}{2} d(F_1(y), Q) > \frac{1}{2}
\]

whenever \( \omega_0 = 1 \). \( \square \)

This completes the proof of Example A.8. \( \square \)

Next, we provide an example which is finitely mean equi-invariant but not mean equi-invariant.

**Example A.11 (FMEI but not MEI).** Consider a control system of the form (1.1), where

1. \( X = [0, 1] = \{ x \in \mathbb{R} : 0 \leq x \leq 1 \} \);
2. \( U = \{ 0, 1 \} \);
3. \( F_0 \) and \( F_1 : X \to X \) are defined by

\[
F_0(x) = \begin{cases} 
\frac{1}{8}, & \text{if } 0 \leq x < \frac{1}{4}, \\
2x - \frac{3}{8}, & \text{if } \frac{1}{4} \leq x < \frac{3}{8}, \\
\left( x - \frac{3}{8} \right)^2 + \frac{3}{8}, & \text{if } \frac{3}{8} \leq x \leq 1
\end{cases}
\]

and

\[
F_1(x) = \begin{cases} 
0, & \text{if } 0 \leq x < \frac{1}{16}, \\
2x - \frac{1}{8}, & \text{if } \frac{1}{16} \leq x < \frac{1}{8}, \\
\frac{1}{4} \left( x - \frac{1}{8} \right)^2 + \frac{1}{8}, & \text{if } \frac{1}{8} \leq x < \frac{1}{4}, \\
x + \frac{1}{2}, & \text{if } \frac{1}{4} \leq x < \frac{1}{2}, \\
0, & \text{if } \frac{1}{2} \leq x \leq 1.
\end{cases}
\]
Claim A.13. The point $Q = \left[\frac{1}{4}, \frac{1}{2}\right]$. Then the set $Q$ is finitely mean equi-invariant but not mean equi-invariant (see Figure 5).

Proof. By the definitions of $F_0$ and $F_1$, we have the following properties:

(a) $F_0(x) \geq F_0^2(x) \geq \cdots \geq F_0^n(x) \geq \cdots$ for any $x \in \left[\frac{1}{4}, \frac{3}{8}\right] \cup \left(\frac{3}{8}, \frac{1}{2}\right]$;
(b) $\lim_{n \to \infty} F_0^n(x) = \frac{1}{8}$ for any $x \in \left[\frac{1}{4}, \frac{3}{8}\right]$ and $\lim_{n \to \infty} F_0^n(x) = \frac{3}{8}$ for any $x \in \left(\frac{3}{8}, \frac{1}{2}\right]$;
(c) $F_0^n(x) \leq F_0^n(y)$ for any $\frac{1}{4} \leq x \leq y \leq \frac{1}{2}$ and $n \geq 0$, $F_1^n(x) \leq F_1^n(y)$ for any $\frac{3}{8} \leq x \leq y \leq \frac{1}{4}$ and $n \geq 0$;
(d) $F_1(x) < F_1^2(x) < F_1^3(x) < \cdots < F_1^n(x) < \cdots$ and $\lim_{n \to \infty} F_1^n(x) = \frac{1}{4}$ for any $x \in \left(\frac{1}{8}, \frac{1}{4}\right)$, and $F_1(\frac{1}{4}) = \frac{1}{4}$;
(e) $F_1(x) \geq F_1^2(x) \geq F_1^3(x) \geq \cdots \geq F_1^n(x) \geq \cdots$ and $\lim_{n \to \infty} F_1^n(x) = 0$ for any $x \in \left(\frac{3}{8}, \frac{1}{2}\right]$;
(f) $F_0(x), F_1(x) \in [0, \frac{1}{4}]$ for all $x \in [0, \frac{1}{4}]$.

Next, we divide our proof into three claims.

Claim A.12. Every point in $Q \setminus \left\{\frac{3}{8}\right\}$ is mean equi-invariant.

Proof. Case 1. $x \in \left(\frac{3}{8}, \frac{1}{4}\right]$. Take $\omega = 0^\infty$. Then, by the properties (a)–(c), we have $d(\phi(n, x, \omega), Q) = 0$ for all $n \geq 0$. It follows that $x$ is mean equi-invariant.

Case 2. $x \in \left[\frac{1}{4}, \frac{3}{8}\right)$. Then $\frac{1}{8} < F_1(x) \leq \frac{3}{8}$. Take $\omega = 1^\infty$. Then for any $\epsilon > 0$, by the property (d), we have

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(\phi(i, x, \omega), Q) < \epsilon.$$ 

According to the property (c), we have $x$ is mean equi-invariant. \hfill \Box

Claim A.13. The point $\frac{3}{8}$ is finitely mean equi-invariant.

Proof. It comes directly from the proof of Claim A.12. \hfill \Box

Claim A.14. The point $\frac{3}{8}$ is not mean equi-invariant.

Proof. Suppose in contrast that $\frac{3}{8}$ is mean equi-invariant, i.e., for any $\epsilon > 0$ there exist $\delta > 0$ and $\omega \in \mathcal{W}$ such that

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(\phi(i, y, \omega), Q) < \epsilon$$

for all $y \in (\frac{3}{8} - \delta, \frac{3}{8} + \delta)$. Indeed, on one hand, for $x \in \left(\frac{3}{8}, \frac{1}{2}\right]$, let $\omega \in \mathcal{W}$. If $\omega \in \{1\omega' : \omega' \in \mathcal{W}\}$, then

$$\phi(1, x, \omega) = F_1(x) \in \left[0, \frac{1}{8}\right] \quad \text{and} \quad \phi(n, x, \omega) \in \left[0, \frac{1}{8}\right]$$
for all \( n > 1 \) by the property (f). This implies that
\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(\phi(i, y, \omega), Q) \geq \frac{1}{8}.
\]

If \( \omega = 0^n 1\omega' \) for some \( n \geq 1 \) and \( \omega' \in \mathcal{V} \), then \( \phi(n + 1, x, \omega) = F_1(x) \in [0, \frac{1}{8}] \) and consequently,
\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(\phi(i, y, \omega), Q) \geq \frac{1}{8}.
\]

Since
\[
\mathcal{V} = \{0^\infty\} \cup \{1\omega' : \omega' \in \mathcal{V}\} \cup \{0^n1\omega' : n \geq 1 \text{ and } \omega' \in \mathcal{V}\},
\]
by the proof of Claim A.12, there exists only one control sequence \( \omega = 0^\infty \) such that
\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(\phi(i, y, \omega), Q) < \frac{1}{8}.
\]

On the other hand, if \( x \in [\frac{1}{4}, \frac{3}{8}) \), take \( \omega = 0^\infty \). By the property (b), there exists \( N > 0 \) such that
\[
d\left[ \phi(n, y, \omega), \frac{1}{4}\right] = d(\phi(n, y, \omega), Q) \geq \frac{1}{16}
\]
for all \( n \geq N \). Thus
\[
\limsup_{n \to \infty, n > N} \frac{1}{n} \sum_{i=0}^{n-1} d(\phi(i, x, \omega), Q) = \limsup_{n \to \infty, n > N} \frac{1}{n} \left( \sum_{i=0}^{N-1} d(\phi(i, x, \omega), Q) + \sum_{i=N}^{n-1} d(\phi(i, x, \omega), Q) \right)
\]
\[
\geq \limsup_{n \to \infty, n > N} \frac{1}{n} \sum_{i=N}^{n-1} d(\phi(i, \phi(m, x, \omega), \omega), Q)
\]
\[
\geq \frac{1}{16}.
\]
This leads to a contradiction. So \( \frac{3}{8} \) is not mean equi-invariant.

By Claims A.12 and A.13, the set \( Q \) is finitely mean equi-invariant. By Claim A.14, it is not mean equi-invariant.

\[\square\]