A NEW PROOF OF THE NEW INTERSECTION THEOREM

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Abstract. In 1987 Roberts completed the proof of the New Intersection Theorem (NIT) by settling the mixed characteristic case using local Chern characters, as developed by Fulton and also by Roberts. His proof has been the only one recorded of the NIT in mixed characteristic.

This paper gives a new proof of this theorem, one which mostly parallels Roberts’ original proof, but avoids the use of local Chern characters. Instead, the proof here uses Adams operations on $K$-theory with supports as developed by Gillet-Soulé.

New Intersection Theorem. Let $A$ be a (commutative, Noetherian) local ring. If the complex of finite rank free $A$-modules

$$0 \to F_n \to \cdots \to F_1 \to F_0 \to 0$$

has non-zero homology of finite length, then $n \geq \dim(A)$.

In 1973, Peskine-Szpiro [7] proved the New Intersection Theorem (NIT) in prime characteristic $p > 0$ using the Frobenius map. Their work ushered characteristic $p$ methods to the forefront of commutative algebra; by 1975, Hochster’s work [3, 4] established a reduction to characteristic $p > 0$ from equicharacteristic zero to give a proof of the NIT in all equicharacteristic rings. In 1987, Roberts [9, 10] proved this theorem for mixed characteristic rings using local Chern characters.

In this paper, we give a new proof of the NIT in the mixed characteristic case. This proof parallels Roberts’ original proof in many respects, but differs in that it entirely avoids using local Chern theory. Instead, we use Adams operations on $K$-theory with supports, as developed by Gillet-Soulé [2]. The difference between this proof

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1Our proof also applies to the equicharacteristic $p > 0$ case, but it is considerably more complicated than the original argument of Peskine-Spiro.
and Roberts’ proof is much like the difference between his proof \[8\] of Serre’s Vanishing Conjecture and that of Gillet-Soulé [2].

We wish to thank Paul Roberts for telling us about the key difficulty in proving the NIT via Adams operations: the lack of a natural grading for Grothendieck groups.

1. $K$-theory, Adams operations, and the Frobenius

Schemes in this paper are assumed to be quasi-projective over the spectrum of a Noetherian ring. Let $X$ be a scheme and $Z \subseteq X$ be a closed subscheme. Define the Grothendieck group $K^Z_0(X)$ to be the abelian group generated by classes of bounded complexes of locally free coherent sheaves on $X$ with homology supported in $Z$, modulo the relations coming from short exact sequences and quasi-isomorphisms. Similarly define $G^Z_0(X)$ as the Grothendieck group on bounded complexes of coherent $\mathcal{O}_X$-modules with homology supported in $Z$.

If $X = \text{Spec}(A)$ and $Z = \text{Spec}(A/I)$, then generators of $G^Z_0(X)$ (and $K^Z_0(X)$) are complexes of finitely generated (and projective) $A$-modules.

When $Z = X$, the support conditions on homology are vacuous and the superscripts are omitted. In this case, $G_0(X)$ is the usual Grothendieck group of coherent sheaves on $X$. Similarly $K_0(X)$ is the usual Grothendieck group of locally free coherent sheaves on $X$.

If $Z'$ is a closed subscheme of $X$, then tensor product of complexes induces cup product and cap product pairings

$$K^Z_0(X) \otimes K^{Z'}_0(X) \longrightarrow K^{Z \cup Z'}_0(X) \quad \text{and} \quad K^Z_0(X) \otimes G^{Z'}_0(X) \longrightarrow G^{Z \cap Z'}_0(X).$$

If $X = \text{Spec}(A)$ for a local ring $A$ and $Z$ is the closed point, then the Euler characteristic $\chi$ induces an isomorphism $G^Z_0(X) \cong \mathbb{Z}$. Composing $\chi$ with cap product gives

$$\chi([\mathbb{P}] \cap [\mathbb{M}]) = \sum (-1)^i \text{length}(H_i(\mathbb{P} \otimes_A \mathbb{M})).$$

For $f: Y \to X$ a morphism of schemes, set $W = f^{-1}(Z)$. There is an induced pull-back map $f^*: K^Z_0(X) \to K^W_0(Y)$ in $K$-theory. When $f$ is projective (e.g., finite) there is a push-forward map $f_*: G^W_0(Y) \to G^Z_0(X)$ in $G$-theory. Push-forward along the inclusion $Z \hookrightarrow X$ induces an isomorphism $G_0(Z) \cong G^Z_0(X)$.

The projection formula relates push-forward and pull-back; we visualize the formula as “commutativity” of the diagram:

$$
\begin{array}{ccc}
K^Z_0(X) \otimes G_0(X) & \longrightarrow & G^Z_0(X) \\
\downarrow f^* & & \uparrow f_* \\
K^W_0(Y) \otimes G_0(Y) & \longrightarrow & G^W_0(Y)
\end{array}
$$

(1)
Thus \( f_*(f^*\alpha \cap \beta) = \alpha \cap f_*\beta \) for all \( \alpha \in K^Z_0(X) \) and \( \beta \in G_0(Y) \). For \( f \) a finite map between spectra of commutative rings, the projection formula follows from associativity and cancellation of tensor products.

The main uses of the projection formula within this paper are
(i) when \( Y \) is a closed subscheme of \( X \) and \( f \) is the inclusion, and
(ii) when \( Y = X \) has characteristic \( p > 0 \) and \( f \) is the Frobenius (when it is a finite map).

In \([2]\), Gillet-Soulé establish Adams operations \( \Psi^k \) for \( k \geq 1 \) on the groups \( K^Z_0(X) \). These are natural endomorphisms extending the usual Adams operations on \( K^X_0(X) = K_0(X) \). These Adams operations have the following properties.

(A1) \( \Psi^k : K^Z_0(X) \to K^Z_0(X) \) is an abelian group endomorphism.

(A2) \( \Psi^k(\alpha \cup \beta) = \Psi^k(\alpha) \cup \Psi^k(\beta) \), for all \( \alpha \in K^Z_0(X) \) and \( \beta \in K^Z_0(X) \).

(A3) \( \Psi^k \) is functorial with respect to pull-back: \( \Psi^k f^* \alpha = f^* \Psi^k \alpha \).

(A4) On an affine scheme, if \( K(r) \) is the Koszul complex on one ring element \( r \), then \( \Psi^k(K(r)) = k(K(r)) \).

When \( Z = X = Z' \), then \( \Psi^k \) is a ring endomorphism of \( K_0(X) = K_0(X) \).

Our proof of the New Intersection Theorem involves passing between a mixed characteristic ring and its reduction modulo \( p \). In both contexts the Adams operations are available while the Frobenius map only exists in characteristic \( p \).

**Theorem 2.** Let \( A \) be a Noetherian ring of characteristic \( p \). Let \( X \) be quasi-projective over \( \text{Spec}(A) \) and let \( Z \) be a closed subscheme of \( X \). Write \( \phi : X \to X \) for the Frobenius endomorphism. Then the \( p \)-th Adams operation and the pull-back by Frobenius coincide upon capping with classes in \( G_0(X) \):

\[
\Psi^p \alpha \cap \beta = \phi^* \alpha \cap \beta \in G^Z_0(X)
\]

for all \( \alpha \in K^Z_0(X) \) and all \( \beta \in G_0(X) \).

**Remark 3.** In the remark following \([2] \text{ 4.13}\), Gillet-Soulé assert (without proof) that \( \Psi^p = \phi^* \) on \( K^Z_0(X) \). Our proof of Theorem 2 does not prove this stronger statement.

**Proof.** Let \( L_0 \) and \( L_1 \) be line bundles on \( X \). By definition, a complex of the form \( \cdots \to 0 \to L_0 \to 0 \to \cdots \) or \( \cdots \to 0 \to L_1 \to L_0 \to 0 \to \cdots \), where \( L_i \) lies in degree \( i \), is called elementary. When \( \alpha \) is the class of an elementary complex, the theorem holds because

\[
\phi^*(\alpha) = [\cdots \to 0 \to (L_0)^{\otimes p} \to 0 \to \cdots] = \Psi^p(\alpha)
\]

and

\[
\phi^*(\alpha) = [\cdots \to 0 \to (L_1)^{\otimes p} \to (L_0)^{\otimes p} \to 0 \to \cdots] = \Psi^p(\alpha);
\]
the last equality follows from [6, 3.2(c)].

The splitting principle for complexes [11, 18.3.12(4)] gives, for each \( \alpha \in K^*_0(X) \), a projective morphism of schemes \( \pi: Y \to X \) such that

(i) the map \( \pi_*: G_0(X') \to G_0(X) \) is surjective, and

(ii) \( \pi^*(\alpha) \) is a \( \mathbb{Z} \)-linear combination of elementary complexes supported in \( W := \pi^{-1}(Z) \).

In this case, with \( \beta = \pi_*\beta' \), use the projection formula and functoriality of \( \Psi^p \) and \( \phi^* \) to get

\[
\Psi^p(\alpha) \cap \beta = \Psi^p(\alpha) \cap \pi_*\beta' = \pi_* (\pi^* \Psi^p(\alpha) \cap \beta') = \pi_* (\pi^* \pi^* \alpha \cap \beta') = \pi_* (\phi^* \pi^* \alpha \cap \beta') = \phi^* (\alpha) \cap \pi_* \beta' = \phi^* (\alpha) \cap \beta,
\]

where the middle equality holds because \( \pi^*(\alpha) \) decomposes into a \( \mathbb{Z} \)-linear combination of elementary complexes. \( \square \)

We replace the use of local Chern characters in Roberts’ proof of the NIT with the use of the Gillet-Soulé Adams operations. Local Chern characters take values in the graded Chow group of a scheme. By contrast, generally the Grothendieck group \( G_0(X) \) is ungraded. However, in the equicharacteristic \( p > 0 \) case, the action of Frobenius provides a grading on \( G_0(X) \).

**Lemma 4** (see [5, §2]). Let \((B, \mathfrak{m})\) be a local ring of characteristic \( p > 0 \) and dimension \( d \). Assume the residue field \( B/\mathfrak{m} \) is perfect and the Frobenius map on \( B \) is finite. Set \( Y = \text{Spec} \, B \) and let \( \phi \) be the scheme map induced by Frobenius. Then the action of \( \phi_* \) on the \( \mathbb{Q} \)-vector space \( G_0(Y)_\mathbb{Q} := G_0(Y) \otimes_{\mathbb{Z}} \mathbb{Q} \) is diagonalizable and its eigenvalues are a subset of \( \{ p^i \mid i = 0, \ldots, d \} \). That is, \( G_0(Y)_\mathbb{Q} \) decomposes as

\[
G_0(Y)_\mathbb{Q} = \bigoplus_{j=0}^{d} V_j
\]

where the action of \( \phi_* \) on \( V_j \) is multiplication by \( p^i \). Moreover, if \( B \) is a domain, then \( V_d \) is one-dimensional, spanned by \([B]\).

**Proof.** If \( q_1, \ldots, q_m \) are the minimal primes of \( B \) and \( Y_i = \text{Spec} \, B/q_i \), then the map \( \bigoplus_i G_0(Y_i) \to G_0(Y) \) is onto and commutes with \( \phi_* \). Since the quotient of a diagonalizable endomorphism of a vector space is diagonalizable, we may additionally assume that \( B \) is a domain, say with quotient field \( E \). We proceed by induction on \( d \).

The Localization Theorem for \( G \)-theory gives an exact sequence

\[
\bigoplus_{Z} G_0(Z) \to G_0(Y) \to G_0(\text{Spec}(E)) \to 0
\]

where \( Z \) ranges over all codimension one integral closed subschemes of \( Y \); these correspond to height one prime ideals of \( B \). Both maps in this sequence commute with \( \phi_* \).
By induction, the endomorphism $\phi_*$ of $\bigoplus_Z G_0(Z)_Q$ is diagonalizable and its eigenvalues are a subset of $\{p^i \mid i = 0, \ldots, d-1\}$. Hence the same holds for its image in $G_0(Y)_Q$. By [11, p. 125], the action of $\phi_*$ on $G_0(E)_Q \cong Q$ is multiplication by $p^d$. The result now follows, since an extension of diagonalizable endomorphisms of vector spaces is again diagonalizable provided the two sets of eigenvalues are disjoint. □

So every $\beta \in G_0(Y)_Q$ uniquely decomposes as $\beta = \beta_0 + \cdots + \beta_d$ such that $\phi_*(\beta_j) = p^j \beta_j$ for $j = 0, \ldots, \dim(Y)$. Define polynomials $q_j(t) = \sum_0^d a_i t^i$ by $q_j(t) = (\Pi t - p^i) / (\Pi p^j - p^i) \in Q[t]$ where both products run over the set $\{0, \ldots, d\} \setminus \{j\}$. Then the $\beta_j$’s are found by

$$\beta_j = q_j(\phi_*) \beta = a_0 + a_1 \phi_* \beta + \cdots + a_d \phi_*^d \beta.$$  

2. Dutta multiplicity vanishes on reduced complexes

**Theorem 6.** Let $(A, m)$ be a local domain with perfect residue field. Take a nonzero $x \in m$ and set $B = A/\langle x \rangle$. Assume that $B$ is of characteristic $p > 0$ and that the Frobenius map is finite on $B$. If $\mathbb{F}$ is a bounded complex of finitely generated free $A$-modules whose homology has finite length, then the Dutta multiplicity of $\mathbb{F} \otimes_A B$ is zero.

**Corollary 7.** The New Intersection Theorem holds if the residue characteristic is positive.

**Proof of Corollary.** Let $(A, m, k)$ be a local ring of dimension $d$ with $\text{char} \ k > 0$. Let $\mathbb{F}$ be a complex $0 \to F_n \to \cdots \to F_1 \to F_0 \to 0$ of finite rank free $A$-modules with nonzero homology of finite length. Assume $n < d$. Then for any morphism $A \to B$ with $B$ of characteristic $p$ and $n \leq \dim B$, the Dutta multiplicity of $\mathbb{F} \otimes_A B$ is positive; see [9] or [11, 7.3.5].

Our next aim is to employ the theorem. Use a faithfully flat extension to reduce to the case when $A$ is complete and $k$ is algebraically closed (and hence perfect). Kill a minimal prime $P$ with $\dim(A/P) = d$ to reduce to when $A$ is a domain; note $\mathbb{F} \otimes_A A/P$ remains non-exact by Nakayama’s Lemma.

If $A$ has mixed characteristic, then there is a prime integer $p$ in $m$, so take $x = p$. If $A$ is equicharacteristic, then take $x$ to be any nonzero element in $m$. (If $m = 0$, there was nothing to prove.) Set $B = A/\langle x \rangle$. By the Cohen Structure Theorem, $B \cong k[[Y_1, \ldots, Y_s]]/I$ and hence, in particular, the Frobenius map on $B$ is finite. Apply the theorem to arrive at the contradiction that the Dutta multiplicity is also zero. □
Proof of Theorem. Let $X = \text{Spec}(A)$, $Y = \text{Spec}(B)$ and $\iota : Y \hookrightarrow X$ be the canonical closed immersion. Set $d = \dim(Y)$ so that $\dim(X) = d + 1$. Take $\beta = [B] \in G_0(Y)_{\mathbb{Q}}$; observe $\mathbb{F} \cap \beta = [\mathbb{F} \otimes_A B] = \iota^*(\mathbb{F})$.

The complex $\mathbb{F}$ has homology supported in $Z = \text{Spec}(A/\mathfrak{m})$ and $\iota^*(\mathbb{F})$ is supported in $\iota^{-1}(Z) = W$. By definition, the Dutta multiplicity is

$$\chi_{\infty}(\iota^*\mathbb{F}) := \lim_{n \to \infty} p^{-dn} \cdot \chi((\phi^n)^*(\iota^*\mathbb{F}) = \lim_{n \to \infty} p^{-dn} \cdot \chi((\phi^n)^*(\iota^*\mathbb{F}) \cap \beta);$$

the last equality holds since $- \cap \beta$ is the identity on $G_0^W(Y)$. The element $(\phi^n)^*(\iota^*\mathbb{F}) \cap \beta$ is in $G_0(W)$ and $\chi : G_0^W(Y) \cong \mathbb{Z}$. Since the residue field is perfect, $\phi_*$ is the identity map on $G_0^W(Y) \cong G_0(W)$, so

$$\chi_{\infty}(\iota^*\mathbb{F}) = \lim_{n \to \infty} p^{-dn} \cdot \chi((\phi^n)^*(\iota^*\mathbb{F}) \cap \beta).$$

By Lemma 4 there is a decomposition $\beta = \sum_0^d \beta_j$ into eigenvectors for $\phi_*$. The projection formula (1) for $\phi^n$ gives

$$p^{-dn} \phi^n((\phi^n)^*(\iota^*\mathbb{F}) \cap \beta) = p^{-dn} \phi^n((\phi^n)^*(\iota^*\mathbb{F}) \cap (\beta_0 + \cdots + \beta_d)) = \iota^*(\mathbb{F}) \cap p^{-nd} \phi^n(\beta_0 + \beta_1 + \cdots + \beta_d) = \iota^*[\mathbb{F}] \cap p^{-nd}(\beta_0 + p^\beta_1 + \cdots + p^\beta_d) = (\iota^*[\mathbb{F}] \cap p^{-nd}\beta_0) + (\iota^*[\mathbb{F}] \cap p^{-nd}\beta_1) + \cdots + (\iota^*[\mathbb{F}] \cap \beta_d).$$

Applying $\chi$ and taking $n \to \infty$ gives

$$\chi_{\infty}(\iota^*\mathbb{F}) = \chi(\iota^*[\mathbb{F}] \cap \beta).$$

Now use (5) to get $\beta_d = q_d(\phi_*)(\beta)$ where $q_d(t) = a_0 + a_1 t + \cdots + a_d t^d$ is a polynomial with rational coefficients. Hence

$$\chi_{\infty}(\iota^*\mathbb{F}) = \chi(\iota^*[\mathbb{F}] \cap q_d(\phi_*)(\beta)) = \chi(\iota^*[\mathbb{F}] \cap \sum_0^d a_j \phi^j(\beta)) = \sum_0^d a_j \chi(\phi^j((\phi^n)^*(\iota^*\mathbb{F}) \cap \beta)), $$

where the last equality uses the projection formula (1) for the $\phi^j$'s. By Theorem 2 and using (A3)

$$\chi_{\infty}(\iota^*\mathbb{F}) = \sum_0^d a_j \chi(\phi^j((\psi^n)^j(\iota^*\mathbb{F}) \cap \beta)) = \sum_0^d a_j \chi(\phi^j(\iota^*((\psi^n)^j\mathbb{F}) \cap \beta)).$$

For each $j$ the element $\iota^*((\psi^n)^j\mathbb{F}) \cap \beta$ belongs to $G_0^W(Y)$, and, since the residue field is perfect, $\phi_*$ is the identity on $G_0^W(Y)$. Also, under the identifications, via $\chi$, of $G_0^W(Y)$ and $G_0^Z(X)$ with $\mathbb{Z}$, the map
\( \tau : G_0(W) \to G_0(Z) \) is the identity map. Thus
\[
\chi_\infty(\tau^*F) = \sum_0^d a_j \chi\left(\tau^*\left((\Psi^p)^jF\right) \cap \beta\right) = \sum_0^d a_j \chi\left(\tau^*\left((\Psi^p)^jF\right) \cap \tau_*(\beta)\right)
\]
Apply the projection formula for \( \tau \) to get
\[
\chi_\infty(\tau^*F) = \sum_0^d a_j \chi\left(\left((\Psi^p)^jF\right) \cap \tau_*(\beta)\right)
\]
But \( \tau_*(\beta) = 0 \) in \( G_0(X) \) since there is the short exact sequence
\[
0 \to A \xrightarrow{x} A \to B \to 0
\]
of \( A \)-modules. \( \square \)

REFERENCES

[1] William Fulton. Intersection theory, volume 2 of Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Springer-Verlag, Berlin, second edition, 1998.

[2] H. Gillet and C. Soulé. Intersection theory using Adams operations. Invent. Math., 90(2):243–277, 1987.

[3] Melvin Hochster. The equicharacteristic case of some homological conjectures on local rings. Bull. Amer. Math. Soc., 80:683–686, 1974.

[4] Melvin Hochster. Topics in the homological theory of modules over commutative rings. Published for the Conference Board of the Mathematical Sciences by the American Mathematical Society, Providence, R.I., 1975. Expository lectures from the CBMS Regional Conference held at the University of Nebraska, Lincoln, Neb., June 24–28, 1974, Conference Board of the Mathematical Sciences Regional Conference Series in Mathematics, No. 24.

[5] Kazuhiko Kurano. A remark on the Riemann-Roch formula on affine schemes associated with Noetherian local rings. Tohoku Math. J. (2), 48(1):121–138, 1996.

[6] Kazuhiko Kurano and Paul C. Roberts. Adams operations, localized Chern characters, and the positivity of Dutta multiplicity in characteristic 0. Trans. Amer. Math. Soc., 352(7):3103–3116, 2000.

[7] C. Peskine and L. Szpiro. Dimension projective finie et cohomologie locale. Applications à la démonstration de conjectures de M. Auslander, H. Bass et A. Grothendieck. Inst. Hautes Études Sci. Publ. Math., (42):47–119, 1973.

[8] Paul Roberts. The vanishing of intersection multiplicities of perfect complexes. Bull. Amer. Math. Soc. (N.S.), 13(2):127–130, 1985.

[9] Paul Roberts. Le théorème d’intersection. C. R. Acad. Sci. Paris Sér. I Math., 304(7):177–180, 1987.

[10] Paul Roberts. Intersection theorems. In Commutative algebra (Berkeley, CA, 1987), volume 15 of Math. Sci. Res. Inst. Publ., pages 417–436. Springer, New York, 1989.

[11] Paul C. Roberts. Multiplicities and Chern classes in local algebra, volume 133 of Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge, 1998.
