ENDPOINT RESULTS FOR FOURIER INTEGRAL OPERATORS ON NONCOMPACT SYMMETRIC SPACES

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Abstract. Let $X$ be a noncompact symmetric space of rank one and let $h^1(X)$ be a local atomic Hardy space. We prove the boundedness on $h^1(X)$ of a class of Fourier integral operators related with the wave equation associated with the Laplacian on $X$ and we estimate the growth of the norm of these operators on $h^1(X)$ depending on the time.

1. Introduction

Given a second order differential operator $\mathcal{L}$ on a manifold $\mathbb{M}$ consider the Cauchy problem for the associated wave equation

$$
\begin{cases}
\partial_t^2 u(t, x) + \mathcal{L}u(t, x) = 0, \\
u(0, x) = f(x), \\
\partial_t u(0, x) = g(x)
\end{cases}
$$

An interesting problem is to find $L^p$-bounds of the solution $u$ at a certain time $t$ in terms of Sobolev norms of the initial data $f$ and $g$. This problem is well understood for the standard Laplacian in $\mathbb{R}^n$ [Mi, P]. It was also studied for the Laplace–Beltrami operator on compact manifolds [SSS], for the subLaplacian on groups of Heisenberg type [MS, MSt] and for the Laplacian on compact Lie groups [CFS]. Ionescu [I] investigated the same problem on noncompact symmetric spaces of rank one. More precisely, let $X$ be a noncompact symmetric space of rank one and dimension $n$ and denote by $d$ the number $(n-1)/2$. Let $\Delta$ denote the Laplace–Beltrami operator on $X$, whose $L^2$-spectrum is the half-line $[\rho^2, \infty)$, and set $\mathcal{L} = \Delta - \rho^2$ (see Section 2 for the definition of $\rho$). The wave equation associated with $\mathcal{L}$ was considered in [AMPS, APV1, APV2, He2, Ia, Ta]. By the spectral theorem the solution of the Cauchy problem \ref{1.1} associated with $\mathcal{L}$ is given by

$$u(t, \cdot) = \cos(t\sqrt{\mathcal{L}})f + \frac{\sin t\sqrt{\mathcal{L}}}{\sqrt{\mathcal{L}}}g.$$
Finding $L^p$-bounds for $u$ amounts to prove the boundedness on $L^p(\mathcal{X})$ of the operators
\[
\mathcal{T}_{\pm t} = m(\sqrt{L})e^{\pm it\sqrt{L}},
\]
for $t > 0$ and for suitable symbols $m$, and estimate the growth of the norm of such operators on $L^p(\mathcal{X})$ depending on $t$.

In this paper we prove a sharp endpoint result for $p = 1$ for $\mathcal{T}_{\pm t}$. To state our result, we need some notation. For every $a \geq 0$ and $b \in \mathbb{R}$ let $S_a^b$ be the set of continuous functions $m$ on the complex tube $\{\lambda \in \mathbb{C} : |\text{Im } \lambda| \leq a\}$, analytic in the interior of the tube, infinitely differentiable on the two lines $|\text{Im } \lambda| = a$, which satisfy the symbol inequalities
\[
|\partial^\alpha m(\lambda)| \leq C (1 + |\text{Re } \lambda|)^{b-\alpha} \quad \forall \alpha \in \mathbb{N}, |\text{Im } \lambda| \leq a.
\]

Let $\mathfrak{h}^1(\mathcal{X})$ be a local atomic Hardy space of Goldberg type defined by Taylor [T] and Meda and Volpi [MV]. This is an atomic Hardy space where atoms are either functions supported in balls of radius less or equal 1, with vanishing integral and satisfying a $L^2$-size condition or functions supported in balls of radius 1, satisfying a $L^2$-size condition but without cancellation property (see Section 3 for more details on such space). Our main result is the following.

**Theorem 4.4.** If $m \in S_{\rho}^{-d}$ is an even symbol, then the operators $\mathcal{T}_{\pm t}$, $t > 0$, are bounded on $\mathfrak{h}^1(\mathcal{X})$ and
\[
\|\mathcal{T}_{\pm t} f\|_{\mathfrak{h}^1} \leq C e^{\rho t}.
\]

Ionescu [I] proved an endpoint result for $p = \infty$ related to Theorem 4.4: he showed that if $m \in S_{\rho}^{-d}$, then the operator $m(\sqrt{L})\cos(t\sqrt{L}) = (T_t + T_{-t})/2$ is bounded from $L^\infty(\mathcal{X})$ to a suitable $BMO(\mathcal{X})$ space. Starting from Ionescu’s result, one could easily prove an endpoint result for $p = 1$ and show that if $m \in S_{\rho}^{-d}$, then the operators $\mathcal{T}_{\pm t}$ are bounded from $H^1(\mathcal{X})$ to $L^1(\mathcal{X})$ and
\[
\|\mathcal{T}_{\pm t}\|_{H^1 \to L^1} \leq C e^{\rho t}.
\]

Here $H^1(\mathcal{X})$ is the atomic Hardy space introduced in [CMM] which is the predual of the space $BMO(\mathcal{X})$ introduced in [I]. Let us mention that previously Giulini and Meda [GM] proved $L^p$ estimates for $p \in (1, \infty)$ for oscillating multipliers of the form $\Delta^{-\beta/2} e^{i\Delta^{\alpha/2}}$, $\alpha > 0, \text{Re } \beta \geq 0$, which, for $\alpha = 1$ and $\beta = d$, are related to the operators $\mathcal{T}_{\pm 1}$. Let us mention that on a noncompact symmetric space one cannot deduce the growth in $t$ of the norm of the operators $\mathcal{T}_{\pm t}$ starting from the norm at $t = 1$, as one can do in different context, like the Euclidean setting or stratified nilpotent groups, using dilation arguments.

Theorem 4.4 is an endpoint result for $\mathcal{T}_{\pm t}$ for $p = 1$ which is more precise than (1.2) because the Hardy space $\mathfrak{h}^1(\mathcal{X})$ is larger than $H^1(\mathcal{X})$ and strictly contained in $L^1(\mathcal{X})$. Its proof requires pointwise estimates of both the convolution kernel $k_{\pm t}$ of $\mathcal{T}_{\pm t}$ and its derivative, which are singular on the sphere of radius $t$: we will cut.
the kernel $k_{\pm t}$ in different pieces supported in annuli and estimate the $h^1$-norm of the convolution of each piece with $h^1$-atoms. The main difficulty is to estimate the $h^1$-norm of the convolution of an atom with the singular part of $k_{\pm t}$ supported in an annulus around the sphere of radius $t$: to treat this problem the key idea is Lemma 3.6, where we are able to estimate the $h^1$-norm of an $L^2$-function with vanishing integral supported in an annulus in terms of the measure of the annulus and the $L^2$-norm of the function.

Theorem 4.4 shows that the natural Hardy space for endpoint results of Fourier integral operators in this context is the local Hardy space. We think that a similar approach could be used to obtain boundedness properties for Fourier integral operators on local Hardy spaces of Goldberg type $h^p(X)$, with $0 < p < 1$. We shall introduce such spaces and investigate this problem in a forthcoming paper. Moreover, our result paves the way to prove a similar result on noncompact symmetric spaces of arbitrary rank, where in [CoGiuMe] the $L^p$-boundedness of Fourier integral operators related with the wave equation was only studied at fixed time $t = 1$.

Our paper is organized as follows. In Section 2 we summarize the notation for noncompact symmetric space of rank one and the spherical analysis on them. In Section 3 we recall the definition of the local Hardy space $h^1(X)$ and we prove some technical lemma on the $h^1$-norm of functions. Section 4 is devoted to the pointwise estimates of the kernel and the derivative of the kernel of operators $T_{\pm t}$ and to the proof of Theorem 4.4.

Throughout the article the expression $A \lesssim B$ means that there exists a positive constant $C$ such that $A \leq C B$.

2. Notation

We shall use the same notation as in [1] and refer the reader to [AJ, GV, H] for more details on noncompact symmetric spaces and spherical analysis on them.

Let $G$ be a connected noncompact semisimple Lie group with finite centre, $\mathfrak{g}$ its Lie algebra, $\theta$ a Cartan involution of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the associated Cartan decomposition. Let $K$ be a maximal compact subgroup of $G$ and $X = G/K$ be the associated symmetric space of dimension $n$. Let $a$ be a maximal abelian subspace of $\mathfrak{p}$. We will assume that the dimension of $a$ is one, i.e. that the rank of $X$ is one. The Killing form on $\mathfrak{g}$ induces a $G$-invariant distance on $X$, which we shall denote by $d(\cdot, \cdot)$. For every $x \in X$ we denote by $|x|$ the distance $d(x, o)$, where $o = eK$ and $e$ is the identity of $G$. For every $r > 0$ and $x \in X$ we denote by $B(x, r)$ the ball centred at the point $x$ of radius $r$. Finally, for every $0 < r < R$ we denote by $A^R_r$ the annulus $A^R_r = \{ x \in X : r \leq |x| \leq R \}$. Let $a^*$ be the real dual of $a$ and for $\alpha \in a^*$ let $\mathfrak{g}_\alpha = \{ X \in \mathfrak{g} : [H, X] = \alpha(H)X \ \forall H \in a \}$. Let
\( \Sigma = \{ \alpha \in a^* \setminus \{0\} : \dim g_{\alpha} \neq 0 \} \) be the set of non-zero roots. It is well known that either \( \Sigma = \{-\alpha, \alpha\} \) or \( \Sigma = \{-2\alpha, -\alpha, 2\alpha\} \). Let \( m_1 = \dim g_{\alpha}, \ m_2 = \dim g_{2\alpha} \) and \( \rho = (m_1 + 2m_2)/2 \). Set \( n = g_{\alpha} + g_{2\alpha} \) and \( N = \exp n \).

In the sequel we shall identify \( A = \exp a \) with \( \mathbb{R} \) choosing the unique element \( H_0 \) of \( a \) such that \( \alpha(H_0) = 1 \) and considering the diffeomorphism \( a : \mathbb{R} \to A \) defined by \( a(s) = \exp(sH_0) \). It is well known that \( G \) admits the Cartan decomposition \( G = KA^+K \), where \( A^+ = \{ a(s) : s \geq 0 \} \) and the Iwasawa decomposition \( G = NAK \). For every \( g \in G \) we denote by \( t(g) \) the unique element in \( \mathbb{R} \) such that \( g = \exp(t(g)H_0)k \), for some \( n \in N \) and \( k \in K \).

For every integrable function \( f \) on \( G \) we have

\[
\int_{G} f(g) \, dg = C \int_{K} \int_{\mathbb{R}^+} \int_{K} f(k_1a(s)k_2) \delta(s) \, dk_1 \, ds \, dk_2,
\]

where \( dg \) is the Haar measure of \( G \), \( dk \) is the Haar measure of \( K \) normalized in such a way that \( \int_{K} dk = 1 \)

\[
\delta(s) = C(\sinh s)^{m_1}(\sinh 2s)^{m_2} \leq C \begin{cases} s^{n-1} & s \leq 1 \\ e^{2ps} & s > 1. \end{cases}
\]

We identify right \( K \)-invariant functions on \( G \) with functions on \( X \) and \( K \)-biinvariant functions on \( G \) with \( K \)-invariant functions on \( X \) which can also be identified with functions depending only on the coordinate \( s \in \mathbb{R}^+ \). More precisely, if \( f \) is a \( K \)-biinvariant function on \( G \) we shall denote by \( F : \mathbb{R}^+ \to \mathbb{C} \) the function such that \( f(k_1a(s)k_2) = F(s) \) for every \( s \in \mathbb{R}^+, k_1, k_2 \in K \). We define the convolution of two reasonable functions \( f_1, f_2 \) on \( X \) as

\[
f_1 \ast f_2(x) = \int_{G} f_1(gh)f_2(h^{-1}) \, dh \quad \forall x = gK \in X.
\]

We denote by \( \mu \) the Riemannian measure on \( X \) and for every \( p \in [1, \infty) \) let \( L^p(X) \) be the space of measurable functions \( f \) such that \( \|f\|_p = \int_X |f|^p \, d\mu < \infty \). For every \( K \)-invariant function \( f \) on \( X \)

\[
\int_X f(x) \, d\mu(x) = \int_{\mathbb{R}^+} F(s) \delta(s) \, ds,
\]

where \( F \) is defined as above. The formula above and the left-invariance of the metric imply that

\[
\mu(B(x, r)) = \mu(B(a, r)) = \begin{cases} r^n & r \leq 1 \\ e^{2pr} & r > 1 \end{cases} \quad \forall r > 0, x \in X.
\]

We recall that a spherical Fourier transform on the symmetric space is defined, which associates to each left \( K \)-invariant function \( f \) on \( X \), i.e. to each radial function, its Fourier transform \( \hat{f} \), defined by

\[
\hat{f}(\lambda) = \int_{G} f(g) \phi_{\lambda}(g) \, dg \quad \lambda \in a^*_C,
\]
where the spherical functions are defined by
\[ \phi_\lambda(g) = \int_K \exp[(i\lambda + \rho)t(kg)] \, dk \quad g \in G, \lambda \in \mathfrak{a}_C^*. \]
It is well known that for every radial function in \( L^2(\mathbb{X}) \)
\begin{align*}
\|f\|_{L^2} &= C \int_0^\infty |\tilde{f}(\lambda)|^2 |c(\lambda)|^{-2} d\lambda, \\
f(x) &= C \int_0^\infty \tilde{f}(\lambda) \phi_\lambda(x) |c(\lambda)|^{-2} d\lambda,
\end{align*}
where \( c \) is the Harish-Chandra function. In particular, by the Plancherel and the inversion formulae above, any bounded function \( m : \mathbb{R}^+ \to \mathbb{C} \) defines a bounded operator on \( L^2(\mathbb{X}) \) given by \( \hat{\mathcal{F}}_\pm t = m(\sqrt{\mathcal{F}})e^{\pm it\sqrt{\mathcal{F}}} \) corresponding to the spherical Fourier multipliers \( \lambda \mapsto m(\lambda)e^{\pm it\lambda} \).

### 3. The local Hardy space \( h^1(\mathbb{X}) \)

We recall here the definition of the local atomic Hardy space \( h^1(\mathbb{X}) \), which can be thought as the analog in the context of noncompact symmetric space of the local Hardy space introduced by Goldberg in the Euclidean setting [G]. The space \( h^1(\mathbb{X}) \) was introduced and studied by Meda and Volpi [MV] and Taylor [T] in more general contexts. It is easy to see that noncompact symmetric spaces satisfy the geometric assumptions of [MV] and [T], so that the theory developed in those papers can be applied in our setting.

**Definition 3.1.** Let \( s > 0 \). A **standard atom** at scale \( s \) is a function \( a \) in \( L^1(\mathbb{X}) \) supported in a ball \( B \) of radius \( \leq s \) such that

1. \( \|a\|_{L^2} \leq \mu(B)^{-1/2} \) (size condition);
2. \( \int a \, d\mu = 0 \) (cancellation condition).

A **global atom** at scale \( s \) is a function \( a \) in \( L^1(\mathbb{X}) \) supported in a ball \( B \) of radius exactly \( s \) such that \( \|a\|_2 \leq \mu(B)^{-1/2} \).

Standard and global atoms at scale \( s \) will be referred to as **atoms** at scale \( s \).

**Definition 3.2.** Let \( s > 0 \). The local atomic Hardy space \( h^1_s(\mathbb{X}) \) is the space of functions \( f \) in \( L^1(\mathbb{X}) \) such that \( f = \sum c_j a_j \), where \( \sum_j |c_j| < \infty \) and \( a_j \) are atoms at scale \( s \). The norm \( \|f\|_{h^1_s} \) is defined as the infimum of \( \sum_j |c_j| < \infty \) over all atomic decompositions of \( f \).

Given any \( s > 0 \) [MV, Proposition 3.4] shows that \( h^1_s(\mathbb{X}) = h^1_1(\mathbb{X}) \) and the norms \( \| \cdot \|_{h^1_s} \) and \( \| \cdot \|_{h^1} \) are equivalent. We shall simply denote by \( h^1(\mathbb{X}) \) the space \( h^1_1(\mathbb{X}) \).
Thanks to the atomic structure of the Hardy space and to the following result, one can study the $h^1$-boundedness of a linear operator bounded on $L^2(X)$ testing the operator on atoms.

**Proposition 3.3.** Suppose that $\mathcal{U}$ is a $h^1(X)$-valued linear operator defined on finite linear combination of atoms at scale 1 such that

$$\sup \{ \| Ua \|_{h^1} : a \text{ atom at scale } 1 \} < \infty.$$ 

Then there exists a unique bounded operator $\tilde{U}$ from $h^1(X)$ to $h^1(X)$ which extends $\mathcal{U}$.

If $\mathcal{U}$ is also bounded on $L^2(X)$, then $\tilde{U}$ and $\mathcal{U}$ coincide on $h^1(X) \cap L^2(X)$.

**Proof.** The proof is an easy adaptation of the proof of [MV, Theorem 9.1, Proposition 9.2]. □

We collect here some technical lemmata where we estimate the $h^1$-norm of $L^2$-functions supported either in a ball or in an annulus, that will be useful in the proof of Theorem 4.4. We shall repeatedly use the notion of discretization of the space $X$, which we now recall.

For every $r \in (0, 1]$, we call $r/3$-discretization $\Sigma$ of $X$ a set of points which is maximal with respect to the properties

$$\min \left\{ d(z, w) : z, w \in \Sigma, z \neq w \right\} > \frac{r}{3},$$

$$d(x, \Sigma) \leq \frac{r}{3}, \quad \forall x \in X.$$ 

**Remark 3.4.** Let $\Sigma$ be a $r/3$-discretization of $X$, for some $r \in (0, 1]$. Then the family of balls $\mathcal{B} = \{ B(z, r) : z \in \Sigma \}$ is a uniformly locally finite covering of $X$. More precisely, there exists a constant $M$, independent of $r$, such that

$$1 \leq \sum_{B \in \mathcal{B}} \chi_B(x) \leq M, \quad \forall x \in X. \quad (3.1)$$

Indeed, given any point $x \in X$, if $x \in B(z, r)$, then $z \in B(x, r)$. Thus $\sum_{B \in \mathcal{B}} \chi_B(x) = M(x) = |\Sigma \cap B(x, r)|$. Let $\{ w_1, \ldots, w_{M(x)} \} = \Sigma \cap B(x, r)$. If $w_i, w_j \in \Sigma \cap B(x, r)$, with $w_i \neq w_j$, then $B(w_i, \frac{r}{6}) \cap B(w_j, \frac{r}{6}) = \emptyset$. Thus $\bigcup_{i=1}^{M(x)} B(w_i, \frac{r}{6}) \subseteq B(x, r + \frac{r}{6})$ and by (2.1)

$$CM(x)r^n \leq \mu \left( \bigcup_{i=1}^{M(x)} B(w_i, \frac{r}{6}) \right) \leq \mu \left( B(x, r + \frac{r}{6}) \right) \leq Cr^n.$$ 

Thus there exists a constant $M$ independent of $x$ and $r$ such that $M(x) \leq M$, which proves (3.1).

**Lemma 3.5.** Let $f$ be a function in $L^2(X)$ supported in a ball $B = B(o, R)$. 

Proof. To prove (i) it suffices to notice that 

\[ \|f\|_{b^1} \leq \mu(B)^{1/2} \|f\|_{L^2}. \]

(ii) if \( R \geq 1 \), then \( f \) is in \( b^1(\mathbb{X}) \) and

\[ \|f\|_{b^1} \leq C \mu(B)^{1/2} \|f\|_{L^2}. \]

We notice that \( \frac{f}{\mu(B)^{1/2} \|f\|_{L^2}} \) is a standard atom at scale 1.

We now prove (ii) following the proof of [MV, Lemma 3.3] with slight modifications.

Let \( \Sigma \) be a \( 1/3 \)-discretization of \( \mathbb{X} \). Denote by \( z_1, \ldots, z_N \) the points in \( \Sigma \) such that \( B(z_j, 1) \cap B \neq \emptyset \). Note that \( N \leq C \mu(B) \). Denote by \( B_j \) the ball \( B(z_j, 1) \) and define

\[ \psi_j = \frac{\chi_{B_j}}{\sum_{k=1}^{N} \chi_{B_k}}. \]

We have \( f = \sum_{j=1}^{N} f_j \), where \( f_j = f \psi_j \). We notice that \( \frac{f_j}{\mu(B_j)^{1/2} \|f_j\|_{L^2}} \) is a global atom at scale 1. Then

\[ \|f\|_{b^1} \leq \sum_{j=1}^{N} \mu(B_j)^{1/2} \|f_j\|_{L^2} \leq C \sum_{j=1}^{N} \|f_j\|_{L^2} \]

\[ \leq C N^{1/2} \left( \sum_{j=1}^{N} \mu(B_j)^{1/2} \|f_j\|_{L^2}^2 \right)^{1/2} \leq C \mu(B)^{1/2} \|f\|_{L^2}, \]

where we used Schwarz’s inequality and the fact that \( N \leq C \mu(B) \). \( \square \)

Lemma 3.6. Let \( f \) be a function in \( L^2(\mathbb{X}) \) with vanishing integral supported in an annulus \( A^{R+r}_{R-r}, \ r \in (0, 1], R > r \). Then \( f \) is in \( b^1(\mathbb{X}) \) and

\[ \|f\|_{b^1} \leq C \|A^{R+r}_{R-r}\|^{1/2} \|f\|_{L^2}. \]

Proof. We take a \( r/3 \)-discretization \( \Sigma \) of \( \mathbb{X} \). The set \( A^{R+r}_{R-r} \cap \Sigma \) has at most \( N \) elements \( z_1, \ldots, z_N \). Then \( A^{R+r}_{R-r} \subseteq \bigcup_{j=1}^{N} B_j \subseteq A^{R+2r}_{R-2r} \). Thus \( N \leq C r^{-n} \mu(A^{R+2r}_{R-2r}) \).

Let \( B_j \) be the ball \( B(z_j, r) \). We define

\[ \psi_j = \frac{\chi_{B_j}}{\sum_{k=1}^{N} \chi_{B_k}}, \]

and \( f_j^0 = f \psi_j \). Obviously \( f = \sum_{j=1}^{N} f_j^0 \).

Let \( \phi_j^0 \) be a nonnegative function in \( C_c^\infty(B_j) \) that has integral 1. It is clear that we may choose the functions \( \phi_j^0 \) so that there exists a constant \( A \) such that \( \|\phi_j^0\|_{L^2} \leq A \) for all \( j \). Next, define

\[ f_j^1 = f_j^0 - \phi_j^0 \int f_j^0 \, d\mu \quad \text{and} \quad f_j^2 = \phi_j^0 \int f_j^0 \, d\mu. \]
Then, the support of \( f_j^1 \) is contained in \( B_j \), the integral of \( f_j^1 \) vanishes and
\[
\| f_j^1 \|_{L^2} \leq \| f_j^0 \|_{L^2} + A \int |f_j^0| \, d\mu \\
\leq \| f_j^0 \|_{L^2} + A \| f_j^0 \|_{L^2 \mu(B_j)}^{1/2} \\
\leq C \| f_j^0 \|_{L^2}.
\]
Hence \( \frac{f_j^1}{\| f_j^1 \|_{L^2 \mu(B_j)}^{1/2}} \) is a standard atom at scale \( r \) and there exists a constant \( C \), independent of \( j \), such that
\[
\| f_j^1 \|_{H^1} \leq \| f_j^1 \|_{L^2 \mu(B_j)}^{1/2} \leq C \| f_j^0 \|_{L^2} r^{n/2}.
\]

The function \( f_j^2 \) is supported in \( B_j \subset B(z_j, 1) \), and
\[
\| f_j^2 \|_{L^2} \leq A \int |f_j^0| \, d\mu \lesssim \| f_j^0 \|_{L^2 \mu(B_j)}^{1/2} \leq C \| f_j^0 \|_{L^2} r^{n/2}.
\]
Hence \( \frac{f_j^2}{\| f_j^2 \|_{L^2 \mu(B(z_j, 1))}^{1/2}} \) is a global atom at scale 1 and there exists a constant \( C \), independent of \( j \), such that
\[
\| f_j^2 \|_{H^1} \leq \| f_j^2 \|_{L^2 \mu(B(z_j, 1))}^{1/2} \leq C \| f_j^0 \|_{L^2} r^{n/2}.
\]

It follows that
\[
f = \sum_{j=1}^N f_j^0 = \sum_{j=1}^N (f_j^1 - f_j^2)
\]
and
\[
\| f \|_{H^1} \leq C \sum_{j=1}^N \| f_j^1 \|_{H^1} + \sum_{j=1}^N \| f_j^2 \|_{H^1} \\
\leq C \sum_{j=1}^N \| f_j^0 \|_{L^2} r^{n/2}.
\]

Then we use Schwarz’s inequality and the fact that \( N \leq C r^{-n} \mu(A_{R-2r}^{R+2r}) \), and obtain that
\[
\| f \|_{H^1} \leq C N^{1/2} \left( \sum_{j=1}^N \| f_j^0 \|_{L^2}^2 \right)^{1/2} r^{n/2} \\
\leq C r^{-n/2} \mu(A_{R-2r}^{R+2r})^{1/2} \| f \|_{L^2} r^{n/2} \\
\leq C \mu(A_{R-2r}^{R+2r})^{1/2} \| f \|_{L^2}.
\]

In the second line we have used the fact that \( \sum_{j=1}^N \| f_j^0 \|_{L^2}^2 \leq M \| f \|_{L^2} ^2 \), where \( M \) is the constant in Remark 3.4. This completes the proof of the lemma.

\[\square\]

**Lemma 3.7.** Let \( \gamma \) be a radial function supported in \( B(o, \beta) \).

(i) If \( a \) is a global atom at scale 1 supported in \( B(o, 1) \), then
\[
\| a \ast \gamma \|_{H^1} \leq C \mu(B(o, 1 + \beta))^{1/2} \| \gamma \|_{L^2};
\]
(ii) if \( a \) is a standard atom supported in \( B(o, r) \), \( r \in (0, 1] \), then
\[
\| a \ast \gamma \|_{h^1} \leq C \mu(B(o, r + \beta))^{1/2} \min(\| \gamma \|_{L^2}, r \| \nabla \gamma \|_{L^2}),
\]
where \( \nabla \) is the Riemannian gradient.

Proof. To prove (i), if \( a \) is a global atom supported in \( B(o, 1) \), then \( a \ast \gamma \) is supported
in \( B(o, 1 + \beta) \) and
\[
\| a \ast \gamma \|_2 \leq \| a \|_{L^1} \| \gamma \|_{L^2} \leq \| \gamma \|_{L^2}.
\]
Thus, (i) follows from Lemma 3.5.

To prove (ii), if \( a \) is a standard atom supported in \( B(o, r) \), \( r \leq 1 \), then \( a \ast \gamma \) is
supported in \( B(o, r + \beta) \) and
\[
\| a \ast \gamma \|_2 \leq \| a \|_{L^1} \| \gamma \|_{L^2} \leq \| \gamma \|_{L^2}.
\]
By arguing as in [MMV, Lemma 2.9] and using the cancellation of the atom we
obtain that
\[
\| a \ast \gamma \|_2 \leq r \| \nabla \gamma \|_{L^2}.
\]
Thus, (ii) follows from Lemma 3.5. \( \square \)

The following lemma is an easy consequence of the triangular inequality.

Lemma 3.8. If \( \gamma \) is a function supported in \( A_{\gamma_1}^2 \) and \( a \) is an atom supported in
\( B(o, r) \), then \( a \ast \gamma \) is supported in \( A_{\gamma_1}^{2+r} \).

4. BOUNDEDNESS OF \( \mathcal{F}_{\pm t} \) ON \( h^1(\mathbb{X}) \)

We first recall a result about the behavior of the Harish-Chandra function and
of spherical functions on noncompact symmetric spaces of rank one. It follows from
\cite[Propositions A.1, A.2]{I} and is based on various results in \cite{ST}. Denote by \( \rho' \) the
number \( \rho + \frac{1}{10} \).

Lemma 4.1. The Harish-Chandra function \( c \) satisfies the following:

(i) for all \( \lambda \in \mathbb{R} \)
\[
|c(\lambda)|^{-2} = c(\lambda)^{-1} c(-\lambda)^{-1};
\]
(ii) the function \( \lambda \mapsto \lambda^{-1} c(-\lambda)^{-1} \) is analytic inside the region \( \text{Im} \lambda \geq 0 \) and
for all \( \alpha \geq 0 \) there exists a positive constant \( C_\alpha \) such that
\[
\left| \partial_\lambda^\alpha \left( \lambda^{-1} c(-\lambda)^{-1} \right) \right| \leq C_\alpha (1 + |\text{Re} \lambda|)^{d-1-\alpha} \quad \forall 0 \leq \text{Im} \lambda \leq \rho';
\]
(iii) the function \( \lambda \mapsto \lambda c(\lambda) \) is analytic in a neighborhood of the real axis and
for all \( \alpha \geq 0 \) there exists a positive constant \( C_\alpha \) such that
\[
\left| \partial_\lambda^\alpha (\lambda c(\lambda)) \right| \leq C_\alpha (1 + |\text{Re} \lambda|)^{1-d-\alpha} \quad \forall \lambda \in \mathbb{R}.
\]
The spherical functions \( \phi_\lambda \) satisfy the following properties:
(a) \(|\partial^\alpha_{\lambda} \phi (s)| \leq C e^{-\rho s} (1 + |\lambda|)^\ell \quad \forall \lambda, s \in \mathbb{R}, \ell \in \mathbb{N}.

(b) If \( s \leq 1, \lambda \in \mathbb{R} \) and \( s|\lambda| \geq 1 \), for every \( N \in \mathbb{N} \), \( \phi_{\lambda} \) can be written as

\[
\phi_{\lambda}(s) = e^{i\lambda s} a_1(\lambda, s) + e^{-i\lambda s} a_1(-\lambda, s) + O(\lambda, s),
\]

where the functions \( a_1, O : \{ \mathbb{R} \times [0, 1] : s|\lambda| \geq 1 \} \rightarrow \mathbb{C} \) satisfy

\[
\left| \partial^\alpha_{\lambda} a_1(\lambda, s) \right| \leq C (s(1 + |\lambda|))^{-d - \ell - \alpha} \quad \forall \ell \in \{0, 1\}, s \geq \frac{1}{10}, 0 \leq \text{Im} \lambda \leq \rho'.
\]

(c) If \( s \geq 1/10 \), then

\[
\phi_{\lambda}(s) = e^{-\rho s} \left( e^{i\lambda s} e(\lambda) a_2(\lambda, s) + e^{-i\lambda s} e(-\lambda) a_2(-\lambda, s) \right),
\]

where the function \( a_2 \) is such that for all \( \alpha \geq 0 \) there exist positive constants \( C_\alpha \) such that

\[
\left| \partial^\alpha_{\lambda} a_2(\lambda, s) \right| \leq C_\alpha (1 + |\lambda|)^{-\alpha} \quad \forall \ell \in \{0, 1\}, s \geq \frac{1}{10}, 0 \leq \text{Im} \lambda \leq \rho'.
\]

**Proof.** The properties of the Harish–Chandra function were given in [I]. See also [AJi, Formula (2.2.5)].

Formula (a) follows from [GV, Formula 5.1.18].

The proof of (b) follows the same outline of the proof of [I, Proposition A.2 (b)]. The only difference is that following the same arguments it is possible to estimate the derivatives of the term \( O(\lambda, s) \) which were not estimated in [I].

The proof of (c) is given in [I, Proposition A.2 (c)]. \( \square \)

In the following proposition we shall prove pointwise estimates of the kernel and the derivative of the kernel of the operators \( T_{\pm t} \). We will distinguish the case when \( t \) is either large or small. Notice that the kernel and its derivative behave in the same way far from the singularities, i.e., far from the point \( o \) and the sphere of radius \( t \), while they have a different behavior near \( o \) and near the sphere of radius \( t \). Let us mention that Ionescu [I] estimated the kernels of the operators \( T_{\pm t} \) (but not their derivatives) far from the sphere of radius \( t \), while he gave estimates of the derivatives of the kernels (but not of the kernels) near the sphere of radius \( t \).

**Proposition 4.2.** Let \( m \in S^d_\rho \) be an even symbol. Let \( k_{\pm t} \) be the radial kernel of the operator \( T_{\pm t} \), and \( K_{\pm t} \) be the function on \((0, \infty)\) such that \( k_{\pm t}(x) = K_{\pm t}(d(x, o)) \).
If $t \geq \frac{1}{2}$, then

$$|K_{\pm t}(s)| \lesssim \begin{cases} s^{-d-1} & \text{if } s \leq \frac{1}{10} \\ e^{-\rho s} & \frac{1}{10} \leq s \leq t - \frac{2}{10} \\ e^{-\rho t |t-s|^{-1}} & t - \frac{2}{10} \leq s \leq t + \frac{2}{10} \\ e^{\rho t} e^{-2\rho s} |t-s|^{-2} & s \geq t + \frac{2}{10} \end{cases}$$

(4.1)

$$|K'_{\pm t}(s)| \lesssim \begin{cases} e^{-\rho s} & \frac{1}{10} \leq s \leq t - \frac{2}{10} \\ e^{-\rho t |t-s|^{-1}} & t - \frac{2}{10} \leq s \leq t + \frac{2}{10} \\ e^{\rho t} e^{-2\rho s} |t-s|^{-2} & s \geq t + \frac{2}{10} \end{cases}$$

(4.2)

If $t < \frac{1}{2}$, then

$$|K_{\pm t}(s)| \lesssim \begin{cases} e^{-\rho s} |t-s|^{-2} & s \geq 1 \\ s^{-d-1} + s^{-d} |t-s|^{-1} & s \leq 1 \end{cases}$$

(4.3)

$$|K'_{\pm t}(s)| \lesssim \begin{cases} e^{-\rho s} |t-s|^{-2} & s \geq 1 \\ s^{-d-2} + s^{-d} |t-s|^{-2} + s^{-d-1} |t-s|^{-1} & s \leq 1 \end{cases}$$

(4.4)

**Proof.** We shall first give the proof for the operator $\mathcal{T}_-$ following the line of the proof of [I, Proposition 4].

By the inversion formula for the spherical transform (2.3) we get

$$K_{-t}(s) = C \int_{\mathbb{R}} m(\lambda) e^{-i\lambda t} \phi(\lambda) |c(\lambda)|^{-2} d\lambda.$$  

(4.5)

Case $t \geq \frac{1}{2}$. Let $\psi_t$ be a smooth cutoff function such that $\psi_t(s) = 1$ if $|s-t| \leq \frac{1}{10}$ and $\psi_t(s) = 0$ if $|s-t| \geq \frac{2}{10}$. Let $A_t(s) = \psi_t(s) K_{-t}(s)$ and $B_t(s) = (1 - \psi_t(s)) K_{-t}(s)$. To prove (4.1) and (4.2) it is enough to estimate $A_t$ and $B_t$ and their derivatives.

From (4.5) and Proposition 4.1 we deduce that

$$A_t(s) = C \psi_t(s) e^{-\rho s} \int_{\mathbb{R}} m(\lambda) e^{-i\lambda t} e^{i\lambda s} a_2(\lambda, s) c(-\lambda)^{-1} d\lambda.$$  

Since by Proposition 4.1 (c) the function $\lambda \mapsto m(\lambda) a_2(\lambda, s) c(-\lambda)^{-1}$ is a symbol on the real line of order 0, for $|t-s| \leq \frac{2}{10}$

$$|A_t(s)| \leq C e^{-\rho t |t-s|^{-1}}.$$  

Moreover, for $|t-s| \leq \frac{2}{10}$, $|A'_t(s)| \leq C e^{-\rho t |t-s|^{-2}}$ (see [I, p.287]).

The function $B_t$ can be estimated as in [I, Formula (3.9)]. To estimate the derivative of $B_t$ we distinguish different cases.
We first consider the case when \( s \leq \frac{1}{10} \). We choose a smooth cutoff function \( \eta \) such that \( \eta(v) = 1 \) if \( |v| \leq 1 \), \( \eta(v) = 0 \) if \( |v| \geq 2 \). By Proposition 4.1 (b) we write

\[
B_t(s) = C \left( 1 - \psi_t(s) \right) \int \eta(\lambda s) \phi_\lambda(s) m(\lambda) e^{-i\lambda t} |c(\lambda)|^{-2} d\lambda \\
+ C \left( 1 - \psi_t(s) \right) \int (1 - \eta(\lambda s)) O(\lambda, s) m(\lambda) e^{-i\lambda t} |c(\lambda)|^{-2} d\lambda \\
+ C \left( 1 - \psi_t(s) \right) \int (1 - \eta(\lambda s)) m(\lambda) e^{-i\lambda t} e^{i\lambda s} a_1(\lambda, s) |c(\lambda)|^{-2} d\lambda.
\]

Then

\[
(4.6)
B_t'(s) = C \int_{\mathbb{R}} \left[ - \psi_t'(s) \eta(\lambda s) \phi_\lambda(s) + (1 - \psi_t(s)) \lambda \eta'(\lambda s) \phi_\lambda(s) \\
+ (1 - \psi_t(s)) \eta(\lambda s) \partial_s \phi_\lambda(s) \right] m(\lambda) e^{-i\lambda t} |c(\lambda)|^{-2} d\lambda \\
+ C \int_{\mathbb{R}} \left[ - \psi_t'(s) (1 - \eta(\lambda s)) O(\lambda, s) - (1 - \psi_t(s)) \lambda \eta'(\lambda s) O(\lambda, s) \\
+ (1 - \psi_t(s)) (1 - \eta(\lambda s)) \partial_s O(\lambda, s) \right] m(\lambda) e^{-i\lambda t} |c(\lambda)|^{-2} d\lambda \\
+ C \int_{\mathbb{R}} \left[ - \psi_t'(s) (1 - \eta(\lambda s)) a_1(\lambda, s) - (1 - \psi_t(s)) \lambda \eta'(\lambda s) a_1(\lambda, s) \\
+ (1 - \psi_t(s)) (1 - \eta(\lambda s)) i\lambda a_1(\lambda, s) + (1 - \psi_t(s)) (1 - \eta(\lambda s)) \partial_s a_1(\lambda, s) \right] \\
\times e^{i\lambda s} m(\lambda) e^{-i\lambda t} |c(\lambda)|^{-2} d\lambda \\
= \beta_t^1(s) + \beta_t^2(s) + \beta_t^3(s).
\]

By Proposition 4.1 (a),

\[
|\beta_t^1(s)| \lesssim e^{-\rho s} (1 + s) \int_0^1 (1 + \lambda)^{d-2} \lambda^2 d\lambda + \int_1^{2/s} (1 + \lambda)^{d-1} \lambda^2 d\lambda \\
\lesssim 1 + \int_1^{2/s} \lambda^{d+1} d\lambda \\
\lesssim s^{-d-2}.
\]

Similarly, by Proposition 4.1 (b), for any integer \( N \)

\[
|\beta_t^2(s)| \lesssim \int_{1/s}^{\infty} s^{-d-N-1} \lambda^{-d-N-1+d} d\lambda \\
\lesssim s^{-d-1}.
\]

To estimate \( \beta_t^3 \) we write \( \cos(t\lambda) = (e^{it\lambda} + e^{-it\lambda})/2 \) and integrate by parts 2 times:

\[
|\beta_t^3(s)| \lesssim \frac{1}{|s - t|^2} \int_{\mathbb{R}} \partial_\lambda^2 \left[ - \psi_t'(s) (1 - \eta(\lambda s)) a_1(\lambda, s) - (1 - \psi_t(s)) \lambda \eta'(\lambda s) a_1(\lambda, s) \\
+ (1 - \psi_t(s)) (1 - \eta(\lambda s)) \partial_s a_1(\lambda, s) \right] \\
\times m(\lambda) |c(\lambda)|^{-2} d\lambda.
\]
By applying Proposition 4.1 (b), we can easily show that $|\beta_i(s)| \lesssim s^{-d-2}$.

Thus from 4.6 and the estimates above, we deduce that for every $s \leq \frac{1}{10}$, $|B_i'(s)| \lesssim s^{-d-2}$.

Suppose now that $\frac{1}{10} \leq s$. By Proposition 4.1 (c) we have

$$B_i(s) = C \left(1 - \psi_t(s)\right) e^{-\rho s} \int_{\mathbb{R}} m(\lambda) a_2(\lambda, s) c(-\lambda)^{-1} e^{i\lambda s} e^{-i\lambda t} d\lambda.$$  

Suppose that $\frac{1}{10} \leq s \leq t - \frac{1}{10}$. We have that

$$B_i'(s) = -C\psi_t'(s) e^{-\rho s} \int_{\mathbb{R}} m(\lambda) a_2(\lambda, s) c(-\lambda)^{-1} e^{i\lambda s} e^{-i\lambda t} d\lambda$$

$$+ C \left(1 - \psi_t(s)\right) e^{-\rho s} \int_{\mathbb{R}} m(\lambda) \left(\partial_s a_2(\lambda, s) + (-\rho + i\lambda) a_2(\lambda, s)\right) c(-\lambda)^{-1} e^{i\lambda s} e^{-i\lambda t} d\lambda.$$  

Since $\lambda \mapsto m(\lambda) \left(a_2(\lambda, s) + \partial_s a_2(\lambda, s)\right) c(-\lambda)^{-1}$ is a symbol of order 0, and $\lambda \mapsto m(\lambda) i\lambda a_2(\lambda, s) c(-\lambda)^{-1}$ is a symbol of order 1, we obtain that $|B_i'(s)| \lesssim e^{-\rho s}$.

If $s \geq t + \frac{1}{10}$ we move the contour of integration in formula (4.7) to the line $\mathbb{R} + i\rho$ and obtain

$$B_i(s) = C \left(1 - \psi_t(s)\right) e^{-2\rho s} \int_{\mathbb{R}} m(\lambda + i\rho) a_2(\lambda + i\rho, s) c(-\lambda - i\rho)^{-1} e^{i\lambda s}$$

$$\times e^{-i(\lambda + i\rho)t} d\lambda.$$  

By taking the derivative we get

$$B_i'(s) = [-C\psi_t'(s) - C(1 - \psi_t(s))(\rho + \rho)]\beta_t(s)$$

$$+ C \left(1 - \psi_t(s)\right) e^{-2\rho s} \int_{\mathbb{R}} m(\lambda + i\rho) (a_2(\lambda + i\rho, s) c(-\lambda - i\rho)^{-1} i\lambda$$

$$+ \partial_s a_2(\lambda + i\rho, s) c(-\lambda - i\rho)^{-1}) e^{i\lambda s} e^{-i(\lambda + i\rho)t} d\lambda.$$  

The estimates of the derivatives of $a_2$ and $c^{-1}$ contained in Proposition 4.1 imply that

$$|B_i'(s)| \lesssim e^{-2\rho s} e^{\rho t} |t - s|^{-2}.$$  

By combining the estimates of $B_i$, $B_i'$, $A_i$ and $A_i'$ one deduces the required estimates of $K_{-1}$ and its first derivative for $t$ large.

Case $t \leq \frac{1}{2}$. Let $\psi_0$ be a smooth cutoff function such that such that $\psi_0(s) = 1$ if $|s - 1| \leq \frac{3}{4}$ and $\psi_0(s) = 0$ if $|s - 1| \geq 1$. Let $A_t(s) = \psi_0(s) K_t(s)$ and $B_t(s) = (1 - \psi_0(s)) K_t(s)$.

We first analyze $B_t$ and notice that $B_t(s) = B_t'(s) = 0$ if $s \leq 1/4$. If $s > 3/4$, then by Proposition 4.1 (c)

$$B_t(s) = C \left(1 - \psi_0(s)\right) \int_{\mathbb{R}} m(\lambda) \cos(\lambda t) e^{-\rho s} e^{i\lambda s} a_2(\lambda, s) c(-\lambda)^{-1} d\lambda,$$
which by moving the contour of integration from the real line to \( \mathbb{R} + i \rho \) becomes

\[
B_t(s) = C \left( 1 - \psi_0(s) \right) e^{-2ps} \times \int_{\mathbb{R}} m(\lambda + i \rho) e^{i\lambda s} a_2(\lambda + i \rho, s) e^{-i(\lambda + i \rho)t} c(-\lambda - i \rho)^{-1} d\lambda.
\]

The function \( B_t \) can be estimated as in [I, p. 289]. Since \( B_t \) is the Fourier transform at \( s + t \) of a symbol of order 0, \( s \geq 1/4 \) and \( t < 1/2 \), for every integer \( M \geq 1 \)

\[
|B'_t(s)| \lesssim e^{-2ps} |t - s|^{-M}.
\]

It remains to consider

\[
A_t(s) = \psi_0(s) \int \eta(\lambda s) \phi_\lambda(s) m(\lambda) e^{-i\lambda t} |c(\lambda)|^{-2} d\lambda + \psi_0(s) \left( 1 - \eta(\lambda s) \right) O(\lambda, s) m(\lambda) e^{-i\lambda t} |c(\lambda)|^{-2} d\lambda + \psi_0(s) \left( 1 - \eta(\lambda s) \right) e^{i\lambda s} a_1(\lambda, s) m(\lambda) e^{-i\lambda t} |c(\lambda)|^{-2} d\lambda = \alpha_1^t(s) + \alpha_2^t(s) + \alpha_3^t(s),
\]

where \( \eta \) is a smooth cutoff function such that \( \eta(v) = 1 \) if \( |v| \leq 1 \) and \( \eta(v) = 0 \) if \( |v| \geq 2 \). For every \( s \leq 1 \) we have

\[
|\alpha_1^t(s)| \lesssim |\psi_0(s)| \int |\eta(\lambda s) \phi_\lambda(s) m(\lambda) e^{-i\lambda t} |c(\lambda)|^{-2} d\lambda \lesssim \int_0^{2/s} e^{-ps}(1 + s) (1 + \lambda)^{-d} |c(\lambda)|^{-2} d\lambda \lesssim \int_0^1 \lambda^2 d\lambda + \int_1^{2/s} \lambda^{-d+n-1} d\lambda \lesssim s^{-d-1},
\]

and

\[
|\alpha_2^t(s)| \lesssim |\psi_0(s)| \int |(1 - \eta(\lambda s)) O(\lambda, s) m(\lambda) e^{-i\lambda t} |c(\lambda)|^{-2} d\lambda \lesssim \int_1^{\infty} \frac{(s\lambda)^{-d-n-1} \lambda^{-d+n-1} d\lambda}{s^{d-1}} \lesssim s^{-d-1}.
\]

Finally, \( \alpha_3^t \) is the inverse Fourier transform computed at \( s + t \) of the symbol \( \lambda \mapsto (1 - \eta(\lambda s)) a_1(\lambda, s) m(\lambda) |c(\lambda)|^{-2} \) of order 0. Then

\[
|\alpha_3^t(s)| \lesssim s^{-d} |s - t|^{-1}.
\]

It then follows that for every \( s \leq 1 \)

\[
|A_t(s)| \lesssim s^{-d-1} + s^{-d} |s - t|^{-1}.
\]

In a similar way, one can prove that for every \( s \leq 1 \)

\[
|A'_t(s)| \lesssim s^{-d-1} + s^{-d} |s - t|^{-2} + s^{-d-1} |t - s|^{-1}.
\]

By combining the estimates of \( B_t, B'_t, A_t \) and \( A'_t \) one deduces the required estimates of \( K_{-t} \) and its first derivative for \( t \) small.
The proof for $K_t$ follows exactly the same line, starting from the formula

$$K_t(s) = C \int_{\mathbb{R}} m(\lambda) e^{i\lambda t} \phi_\lambda(s) |c(\lambda)|^{-2} d\lambda,$$

and using the expansions of the spherical functions given in Lemma 4.1. Notice that this time the change of the contour of integration, whenever it is useful, will be done from $\mathbb{R}$ to the line $\mathbb{R} - i\rho$.

$$\square$$

We will need the following lemma.

**Lemma 4.3.** Let $m$ be an even symbol in $S^0_0$ and $\mathcal{U}_m$ be the operator defined by the Fourier multiplier $m$. The following hold:

(i) if $2 < q < \infty$ and $\frac{1}{q} = \frac{1}{2} + \frac{b}{n}$, then $\mathcal{U}_m$ is bounded from $L^2(X)$ to $L^q(X)$;

(ii) if $1 < s \leq 2$ and $\frac{1}{s} = \frac{1}{2} - \frac{b}{n}$, then $\mathcal{U}_m$ is bounded from $L^s(X)$ to $L^2(X)$.

**Proof.** Part (i) is proved in [1, Lemma 3].

Part (ii) follows by a duality argument. Indeed, the adjoint of $\mathcal{U}_m$ is the operator $\mathcal{U}_m^*$. Since $m \in S^0_0$, also $\overline{m} \in S^0_0$. By (i) the operator $\mathcal{U}_m^*$ is bounded from $L^2(X)$ to $L^q(X)$, with $2 \leq q < \infty$ and $\frac{1}{q} = \frac{1}{2} + \frac{b}{n}$. Then $\mathcal{U}_m$ is bounded from $L^q(X)$ to $L^2(X)$. Let $s = q'$. Then $1 < s \leq 2$ and $\frac{1}{s} = 1 - \frac{1}{q} = 1 - \frac{1}{2} - \frac{b}{n} = \frac{1}{2} - \frac{b}{n}$, as required.

$$\square$$

In the proof of our main result we shall repeatedly use smooth cutoff radial functions to cut the kernel of the operators $T_{\pm t}$. We introduce such cut off functions below.

Take a nonnegative function $\phi \in C^\infty_c(\mathbb{R})$ supported in $(1/2, 2)$ such that $0 \leq \phi \leq 1$, $\phi = 1$ in $[1, 3/2]$ and $|\phi'| \leq C$. For every $i \in \mathbb{N}$, $r > 0$ and every $x \in X$ define

$$\phi_i(x) = \phi\left(\frac{|x|}{2^i r}\right).$$

It is easy to see that $\phi_i$ is supported in the annulus $A^{2^{i+1}r}_{2^{i-1}r}$, $0 \leq \phi_i \leq 1$ and $|\nabla \phi_i| \leq C (2^i r)^{-1}$.

For every $h \in \mathbb{N}$, $r > 0$ and every $x \in X$ define

$$\eta_h(x) = \phi\left(\frac{t - |x|}{2^h r}\right) \quad \text{and} \quad \omega_h(x) = \phi\left(\frac{|x| - t}{2^h r}\right).$$

The function $\eta_h$ is supported in $A^{t - 2^{h-1}r}_{t - 2^{h+1}r}$, $0 \leq \eta_h \leq 1$ and $|\nabla \eta_h| \leq C (2^h r)^{-1}$. Similarly, one can show that $\omega_h$ is supported in $A^{t + 2^{h-1}r}_{t + 2^{h+1}r}$, $0 \leq \omega_h \leq 1$ and $|\nabla \omega_h| \leq C (2^h r)^{-1}$. 
Finally, take a nonnegative function $\psi \in C^\infty_c(\mathbb{R})$ supported in $[0, 1]$ such that $0 \leq \psi \leq 1$, $\psi = 1$ in $[1/4, 3/4]$ and for every $j \geq 2$ and $x \in X$ define
\begin{equation}
\psi_j(x) = \psi(|x| - j + 1).
\end{equation}
Obviously $\psi_j$ is supported in $A_{j-1}^j$ and $0 \leq \psi_j \leq 1$.

We can now state and prove our main result.

**Theorem 4.4.** If $m \in S^{-d}_\rho$ is an even symbol, then the operators $\mathcal{T}_{\pm t}$, $t > 0$, are bounded on $h^1(X)$ and
\begin{equation}
\|\mathcal{T}_{\pm t} f\|_{h^1} \leq C e^{\rho t}.
\end{equation}

**Proof.** Given $t > 0$, for simplicity we denote by $\mathcal{T}_t$ either the operator $\mathcal{T}_{+ t}$ or $\mathcal{T}_{- t}$, whose kernels satisfy the same estimates (see Proposition 4.5) and we denote by $k_t$ its convolution kernel. By Proposition 3.3 and since $\mathcal{T}_t$ is left invariant it is enough to prove that
\[ \sup \{\|\mathcal{T}_t a\|_{h^1} : a \text{ atom supported in } B(o, r), r \leq 1\} \leq C e^{\rho t}. \]

Let $a$ be an atom supported in $B(o, r)$, $r \leq 1$. Notice that by Lemma 4.3 (ii) with $\frac{1}{s} = \frac{1}{2} - \left(-\frac{d}{n}\right) = \frac{1}{2} + \frac{n-1}{2n} = 1 - \frac{1}{2n}$, Hölder inequality and the size condition of the atom we get
\begin{equation}
\|\mathcal{T}_t a\|_{L^2} \leq C\|a\|_{L^s} \leq C \mu(B)^{-1 + 1/s} = C \mu(B)^{-1 + 1/s} = Cr^{-1/2}.
\end{equation}

To prove that $\mathcal{T}_t a$ is in $h^1(X)$ we shall write $\mathcal{T}_t a$ as the sum of a series of functions which is absolutely convergent in $h^1$, decomposing the kernel $k_t$ as a series of “small pieces” and convolving the atom $a$ with each of such pieces.

We distinguish the case when $t$ is either large or small. Moreover, the proof is more delicate when $a$ is a standard atom supported in a ball of small radius: in this case the cancellation condition of the atom is used and the estimates of the gradient of the kernel are involved. The proof is easier when the atom is either a global atom or a standard atom supported in a ball of radius not too small (compared with $t$ and $1$): in this case the cancellation of the atom is not used and only the estimates of the size of the kernel are involved.

**Case I:** $t \geq 1/2$.

Choose $J$ such that $J - 2 \leq t + \frac{1}{10} \leq J - 1$. Then for every $j \geq J$, the function $a \ast (\psi_j k_t)$ is supported in $B(o, j + r)$. By Lemma 3.7 and estimate 4.1 we obtain
\[ \|a \ast (\psi_j k_t)\|_{h^1} \lesssim (\mu(B(o, j + r)))^{1/2} \|\psi_j k_t\|_{L^2} \]
\[ \lesssim e^{\rho j} \left( \int_{j-1}^{j} e^{2\rho t} e^{-4\rho s} |t-s|^{-4} e^{2\rho s} ds \right)^{1/2} \lesssim e^{\rho j} e^{-\rho j} e^{\rho t} |t - j|^2. \]
Thus
\[
\sum_{j=j}^{\infty} \|a \ast (\psi_j k_t)\|_{b^1} \leq e^{pt} \sum_{j=j}^{\infty} (j-t)^{-2} \leq e^{pt} \int_{j}^{\infty} \frac{dx}{(x-t)^2} \leq e^{pt}
\]
where we have used the fact that \( J - 2 \leq t + \frac{2}{10} \leq J - 1 \).

**Subcase 1A:** \( r \leq \frac{1}{10} \).

Suppose that \( a \) is a standard atom supported in \( B(o,r) \), \( r \leq \frac{1}{10} \). Consider a smooth nonnegative function \( \phi_0 \) taking values in \([0,1]\) supported in \( B(o,3r) \) such that
\[
\phi_0 + \sum_{i=1}^{l_1} \phi_i + \sum_{i=l_1+1}^{l_2} \phi_i + \sum_{h=3}^{H_1} \eta_h + \sum_{h=3}^{H_2} \omega_h + \sum_{j=j}^{\infty} \psi_j = 1 \quad \text{in} \quad \mathbb{X} \setminus A_{t-10r}^{t+10r},
\]
where \( \phi_i, \eta_h, \omega_h, \psi_j \) are defined by formulae (4.9), (4.10), (4.11) and
\[
2^{l_1-1}r \leq \frac{1}{10} \leq 2^{l_1+1}r,
\]
\[
2^{l_2-1}r \leq t - \frac{2}{10} \leq 2^{l_2+1}r,
\]
\[
t - 2^{H_1+1}r \leq t - \frac{2}{10} \leq t - 2^{H_1-1}r,
\]
\[
t + 2^{H_2+1}r \leq t + \frac{2}{10} \leq t + 2^{H_2-1}r.
\]
Define
\[
\sigma_t = \left[ 1 - \phi_0 + \sum_{i=1}^{l_1} \phi_i + \sum_{i=l_1+1}^{l_2} \phi_i + \sum_{h=3}^{H_1} \eta_h + \sum_{h=3}^{H_2} \omega_h + \sum_{j=j}^{\infty} \psi_j \right] k_t.
\]
Thus
\[
\mathcal{S}_t a = a \ast \sigma_t + a \ast (\phi_0 k_t) + \sum_{i=l_1+1}^{l_2} a \ast (\phi_i k_t) + \sum_{h=3}^{H_1} a \ast (\eta_h k_t) + \sum_{h=3}^{H_2} a \ast (\omega_h k_t) + \sum_{j=j}^{\infty} a \ast (\psi_j k_t).
\]

The function \( a \ast (\phi_0 k_t) \) is supported in \( B(o,4r) \) and by Lemma 3.5
\[
\|a \ast (\phi_0 k_t)\|_{b^1} \leq \mu(B(o,4r))^{1/2} \|a \ast (\phi_0 k_t)\|_{L^2} 
\]
\[
\lesssim r^{n/2} \|\mathcal{S}_t\|_{L^2 \to L^2} \|a\|_{L^2} \lesssim r^{n/2} r^{-n/2} = 1,
\]
where we have used the size condition of the atom and the fact that the norm of the operator \( f \mapsto f \ast (\phi_0 k_t) \) on \( L^2(\mathbb{X}) \) is bounded by the norm of \( \mathcal{S}_t \) on \( L^2(\mathbb{X}) \).

Consider now the cases \( i = 1, \ldots, l_1 \). The function \( a \ast (\phi_i k_t) \) is supported in \( B(o,(2i+1+1)r) \). By Lemma 3.4 and by estimates (4.11), (4.12) we obtain that
\[
\|a \ast (\phi_i k_t)\|_{b^1} \lesssim (\mu(B(o,(2i+1+1)r))^{1/2} \|a \ast (\phi_i k_t)\|_{L^2}
\]
\[
\lesssim (2^i r)^{n/2} r \left( \int_{2^i-1}^{2^{i+1}r} [\rho_i r]^{-2} \rho_i^{-2d-2} + \rho_i^{-2d-4} \right) ds \left[ \int_{2^i-1}^{2^{i+1}r} \rho_i^{-2} \right]^{1/2}
\]
\[
\lesssim 2^{in/2} r^{n/2+1} (2^i r)^{-4} 2^i r^{1/2} \lesssim (2^i)^{n/2-3/2} r^{n/2-1/2}.
\]
Thus, since \( I_1 \sim \log_2 \left( \frac{1}{10r} \right) \), we get
\[
\sum_{i=1}^{I_1} \| a \ast (\phi_i k_t) \|_{b^1} \lesssim \sum_{i=1}^{I_1} (2^i)^{n/2-3/2} r^{n/2-1/2} \lesssim C r^{n/2-1/2} \int_1^{\log_2 \left( \frac{1}{10r} \right)} (2^x)^{n/2-3/2} dx \lesssim r^{n/2-1/2} \frac{1}{r^{n/2-3/2}} \lesssim r.
\]

Consider now the cases when \( i = I_1 + 1, \ldots, I_2 \). The function \( a \ast (\phi_i k_t) \) is supported in \( B(o, (2^{i+1} + 1) r) \). By Lemma 3.7 and by estimate (4.1), (4.2), we obtain that
\[
\| a \ast (\phi_i k_t) \|_{b^1} \leq C \| (B(o, (2^{i+1} + 1) r))^{1/2} r \|_{L^2} \| \nabla (\phi_i k_t) \|_{L^2}
\leq C e^{\rho t} r^r \left( \int_{2^{i-1} r}^{2^i r} [(2^i r)^{-2} e^{-2\rho s} + e^{-2\rho s}] e^{2\rho s} ds \right)^{1/2}
\leq C e^{\rho t} r^{3/2} 2^{i/2}.
\]

Thus, since \( I_2 \sim \log_2 \left( (t - \frac{1}{10r})/r \right) \), we get
\[
\sum_{i=I_1+1}^{I_2} \| a \ast (\phi_i k_t) \|_{b^1} \lesssim r^{3/2} \sum_{i=I_1+1}^{I_2} e^{\rho^2 r} 2^{i/2} \lesssim r^{3/2} \int_{I_1+1}^{I_2} e^{\rho r^2} 2^{i/2} dx
\lesssim r^{3/2} 2^{i/2} \int_{2^{I_1+1}}^{2^2} \sqrt{e^{\rho^2 v^2}} \frac{dv}{v} \lesssim r^{3/2} e^{\rho r^2} r^{-1} \lesssim r^{1/2} e^{\rho t}.
\]

Consider now \( 3 \leq h \leq H_1 \). By Lemma 3.8, the function \( a \ast (\eta_h k_t) \) is supported in \( A_{t-2h^{-1} r + r} \), has vanishing integral and by Lemma 3.6 and estimate (4.1), (4.2),
\[
\| a \ast (\eta_h k_t) \|_{b^1} \lesssim \mu (A_{t-2h^{-1} r + r})^{1/2} r \| \nabla (\phi_i k_t) \|_{L^2}
\lesssim e^{\rho t} e^{-\rho^2 h} r^{2h/2} r^{3/2} \left( \int_{t-2h^{-1} r}^{t-2h^{-1} r} [(2^h r)^{-2} |t-s|^{-2} + |t-s|^{-4}] ds \right)^{1/2}
\lesssim e^{\rho t} e^{-\rho^2 h} r^{2h/2} r^{3/2} \left( (2^h r)^{-4} 2^h r \right)^{1/2} = e^{\rho t} e^{-\rho^2 h} (2^h)^{-1}.
\]

Thus using the fact that \( 2^{H+1} r \sim \frac{2}{10} \), changing variables \( v = 2^x \) and \( u = vr \), we obtain
\[
\sum_{h=3}^{H_1} \| a \ast (\eta_h k_t) \|_{b^1} \lesssim e^{\rho t} \sum_{h=3}^{H_1} e^{-\rho^2 h} (2^h)^{-1} \lesssim e^{\rho t} \int_{3}^{H_1} e^{-\rho^2 r} (2^x)^{-1} dx
\lesssim e^{\rho t} \int_{8}^{10r} e^{-\rho r u^{-1}} \frac{dv}{v} \lesssim e^{\rho t} \int_{8r}^{10r} e^{-\rho u} r^{-2} du \lesssim e^{\rho t} r \frac{1}{r} = e^{\rho t}.
\]

In the same way one can prove that
\[
\sum_{h=3}^{H_2} \| a \ast (\omega_h k_t) \|_{b^1} \lesssim e^{\rho t}.
\]
It remains to consider \( a \ast \sigma_t \), where \( \sigma_t \) is the singular part of the kernel supported in \( A_{t-10r}^{t+10r} \). By Lemma 3.5, the function \( a \ast \sigma_t \) is supported in \( A_{t-11r}^{t+11r} \). For every \( x \in A_{t-11r}^{t+11r} \), we have

\[
\mathcal{T}_t a(x) = a \ast \sigma_t(x) + a \ast (\eta_3 k_t)(x) + a \ast (\omega_3 k_t)(x),
\]

so that

\[
\|a \ast \sigma_t\|_{L^2} \leq \|\mathcal{T}_t a\|_{L^2} + \|a \ast (\eta_3 k_t)\|_{L^2} + \|a \ast (\omega_3 k_t)\|_{L^2} \lesssim r^{-1/2} + r \|\nabla(\eta_3 k_t)\|_{L^2} + r \|\nabla(\omega_3 k_t)\|_{L^2} \lesssim r^{-1/2}.
\]

The second line follows from (4.13) and last line follows from the computations we made before for \( \nabla(\eta_3 k_t) \) and a similar computation for \( \nabla(\omega_3 k_t) \).

We deduce from Lemma 3.6 and (4.20) that

\[
\|a \ast \sigma_t\|_{b^1} \lesssim \mu(A_{t-11r}^{t+11r})^{1/2} \|a \ast \sigma_t\|_{L^2} \lesssim e^{\rho t} r^{1/2} r^{-1/2} = e^{pt}.
\]

From (4.14), (4.15), (4.16), (4.17), (4.18), (4.19) and (4.21) we conclude that

\[
\|\mathcal{T}_t a\|_{b^1} \lesssim e^{pt}.
\]

Subcase IB: \( \frac{1}{10} < r \leq 1 \).

Suppose that \( a \) either is a global atom supported in \( B(o, 1) \) or a standard atom supported in \( B(o, r) \), with \( \frac{1}{10} < r \leq 1 \). Choose two smooth cutoff functions \( \phi_0 \) and \( \phi_t \) which take value in \([0, 1]\) such that

\[
\text{supp}(\phi_0) \subseteq B(o, 3), \quad \text{supp}(\phi_t) \subseteq A_{2}^{t-10r}.
\]

\[
\phi_0 + \phi_t + \sum_{j=J}^{\infty} \psi_j = 1 \quad \text{in} \quad \mathbb{R} \setminus A_{2}^{t+10r}.
\]

Define

\[
\sigma_t = \left[ 1 - \phi_0 - \phi_t - \sum_{j=J}^{\infty} \psi_j \right] k_t.
\]

The function \( a \ast (\phi_0 k_t) \) is supported in \( B(o, 3 + r) \) and by Lemma 3.5

\[
\|a \ast (\phi_0 k_t)\|_{b^1} \lesssim \mu(B(o, 4))^{1/2} \|\mathcal{T}_t\|_{L^2 \to L^2} \|a\|_{L^2} \lesssim 1,
\]

where we argued as in (4.15). By Lemma 3.7 and estimates (4.1) we get

\[
\|a \ast (\phi_t k_t)\|_{b^1} \lesssim \mu(B(o, t - \frac{1}{5} + r))^{1/2} \|\phi_t k_t\|_{L^2}
\]

\[
\lesssim e^{pt} \left( \int_{1/2}^{t-\frac{1}{5}} e^{-2\rho s} |t-s|^{-4} e^{2\rho s} ds \right)^{1/2} \lesssim e^{pt}.
\]

It remains to estimate the \( h^1 \)-norm of \( a \ast \sigma_t \), which is supported in \( A_{t-10r}^{t+10r+r} \). We have that for every \( x \in A_{t-10r}^{t+10r+r} \)

\[
\mathcal{T}_t a(x) = a \ast \sigma_t(x) + a \ast (\phi_t k_t)(x) + a \ast (\psi_j k_t)(x),
\]
so that
\[ \|a \ast \sigma_t\|_{L^2} \leq \|\mathcal{T}a\|_{L^2} + \|a \ast (\phi_t k_t)\|_{L^2} + \|a \ast (\psi_J k_t)\|_{L^2} \]
\[ \leq \|\mathcal{T}\|_{L^2 \to L^2} \|a\|_{L^2} + \|\phi_t k_t\|_{L^2} + \|\psi_J k_t\|_{L^2} \lesssim 1, \]
which follows from (4.13) and the computations we made in (4.23) and (4.14). We then have
\[ (4.24) \|a \ast \sigma_t\|_{b^1} \lesssim \mu(B(o, t + 1/10 + r))^{1/2} \|a \ast \sigma_t\|_{L^2} \lesssim e^{\rho t}. \]
From (4.14), (4.22), (4.23), (4.24) we deduce that
\[ \|T_t a\|_{b^1} \lesssim e^{\rho t}. \]

Case II: \( t < 1/2. \)

For every \( j \geq 2 \) by Lemma 3.7 and estimates (4.3) we get
\[ \|a \ast (\psi_j k_t)\|_{b^1} \lesssim \mu(B(o, j + 2))^{1/2} \|\psi_j k_t\|_{L^2} \]
\[ \lesssim e^{\rho j} \left( \int_{j=1}^j e^{-4\rho s}(1 + |t-s|)^2 e^{2\rho s} ds \right)^{1/2} \]
\[ \lesssim e^{\rho j} e^{-\rho j} (j-t)^{-2} \lesssim j^{-2}, \]
where functions \( \psi_j \) are defined in (4.11). Thus
\[ (4.25) \sum_{j=2}^{\infty} \|a \ast (\psi_j k_t)\|_{b^1} \lesssim \sum_{j=2}^{\infty} j^{-2} \lesssim 1. \]

Subcase IIA: \( r < \frac{1}{20}. \)

Suppose that \( a \) is a standard atom supported in \( B(o, r) \) with \( r \leq \frac{1}{20}. \) We choose a smooth cutoff function \( \phi_0 \) supported in \( B(o, 3r) \) taking values in \([0, 1]\) such that
\[ \phi_0 + \sum_{i=2}^1 \phi_i + \sum_{i=1}^{l_2} \phi_i + \sum_{j=2}^{\infty} \psi_j = 1 \quad \text{in} \quad X \\setminus A^{t+10r}_{1-10r}, \]
where \( \phi_i \) are defined by (4.9) and
\[ 2^{l_1-1} r < t - 10r < 2^{l_1+1} r, \]
\[ 2^{l_2-1} r < t + 10r < 2^{l_2+1} r, \]
\[ 2^{l_2-1} r < 1 < 2^{l_2+1} r. \]
Define
\[ \sigma_t = [1 - \phi_0 - \sum_{i=3}^l \phi_i - \sum_{i=1}^{l_2} \phi_i - \sum_{j=1}^{\infty} \psi_j] k_t. \]
Then \( a \ast (\phi_0 k_t) \) is supported in \( B(o, 4r) \) and
\[ (4.26) \|a \ast (\phi_0 k_t)\|_{b^1} \lesssim \mu((B(o, 4r)))^{1/2} \|a \ast (\phi_0 k_t)\|_{L^2} \lesssim C r^{n/2} \|a\|_{L^2} \|\mathcal{T}\|_{L^2 \to L^2} \lesssim 1, \]
Thus by the change of variables (4.2) and by Lemma 3.4 and estimates (4.9), (4.11) for every \( i \in \{2, \ldots, I \} \) we have

\[
\|a \ast (\phi_i k_i)\|_{b^1} \lesssim \mu(B(o,2^{i+1}r + r))^{1/2} r \|\nabla (\phi_i k_i)\|_{L^2} \\
\lesssim (2^i r)^{n/2} r \left( \int_{2^{i+1}r}^{2^{i+1}r} [ (2^i r)^{-2} s^{-2} - 2^{-d} (2^i r)^{-2} s^{-2} t - s]^{-2} \\
+ s^{-2d-4} + s^{-2d} |t - s|^{-4} + s^{-2d} |t - s|^{-2} s^{2d} ds \right)^{1/2} \\
\lesssim (2^i r)^{n/2} r [(2^i r)^{-3} + (2^i r)^{-1} |t - 2^i r|^{-2} + 2^i r |t - 2^i r|^{-4}]^{1/2}.
\]

Thus

\[
\sum_{i=2}^{I} \|a \ast (\phi_i k_i)\|_{b^1} \lesssim r^{(n-1)/2} \sum_{i=2}^{I} (2^i)^{(n-3)/2} + r^{(n+1)/2} \sum_{i=2}^{I} (2^i)^{(n-1)/2} |t - 2^i r|^{-1} \\
+ r^{(n+3)/2} \sum_{i=2}^{I} (2^i)^{(n+1)/2} |t - 2^i r|^{-2} \\
\lesssim r^{(n-1)/2} \int_{2}^{2^I} (2^x)^{(n-3)/2} dx + r^{(n+1)/2} \int_{2}^{2^I} (2^x)^{(n-1)/2} |t - 2^x r|^{-1} dx \\
+ r^{(n+3)/2} \int_{2}^{2^I} (2^x)^{(n+1)/2} |t - 2^x r|^{-2} dx.
\]

By the change of variables \( 2^x v = u \) and \( u = v r \), since \( 2^I r \sim t - 10r \) we get

\[
\sum_{i=3}^{I} \|a \ast (\phi_i k_i)\|_{b^1} \lesssim r^{(n-1)/2} \int_{4}^{2^I} v^{(n-3)/2} \frac{dv}{v} + r^{(n+1)/2} \int_{4}^{2^I} v^{(n-1)/2} |t - v r|^{-1} \frac{dv}{v} \\
+ r^{(n+3)/2} \int_{4}^{2^I} v^{(n+1)/2} |t - v r|^{-2} \frac{dv}{v} \\
\lesssim r^{(n-1)/2} (2^I)^{(n-3)/2} + r^{(n+1)/2} \int_{4r}^{2^Ir} (u/r)^{(n-3)/2} |t - u|^{-1} \frac{du}{r} \\
+ r^{(n+3)/2} \int_{4r}^{2^Ir} (u/r)^{(n-1)/2} |t - u|^{-2} \frac{du}{r} \\
\lesssim r^{(n-1)/2} \left( \frac{t - 10r}{r} \right)^{(n-3)/2} \\
+ r^{(n+1)/2} r^{(1-n)/2} (2^I r)^{(n-3)/2} \left[ - \log(10r) + \log(t - 8r) \right] \\
+ r^{(n+3)/2} r^{(1-n)/2} (2^I r)^{(n-1)/2} \left[ - |t - 2^I r|^{-1} + |t - 4r|^{-1} \right] \\
\lesssim 1.
\]

Arguing as before, we can prove that

\[
(4.28) \sum_{i=I_2}^{I} \|a \ast (\phi_i k_i)\|_{b^1} \lesssim 1.
\]
It remains to consider $a * \sigma_t$, where $\sigma_t$ is the singular part of the kernel supported in $A_{t-10r}^{t+10r}$. By Lemma 3.6, $a * \sigma_t$ is supported in $A_{t-11r}^{t+11r}$. For every $x \in A_{t-11r}^{t+11r}$, we have

$$\mathcal{T}_t a(x) = a * \sigma_t(x) + a * (\phi_I k_I)(x) + a * (\phi_I k_I)(x),$$

so that

$$\|a * \sigma_t\|_{L^2} \leq \|\mathcal{T}_t a\|_{L^2} + \|a * (\phi_I k_I)\|_{L^2} + \|a * (\phi_I k_I)\|_{L^2} \lesssim r^{-1/2} + r \|\nabla (\phi_I k_I)\|_{L^2} + r \|\nabla (\phi_I k_I)\|_{L^2} \lesssim r^{-1/2},$$

where we have applied (4.13) and the computations we made above. Then by Lemma 3.6

$$\|a * \sigma_t\|_{b^1} \lesssim \mu(A_{t-11r}^{t+11r})^{1/2} \|a * \sigma_t\|_{L^2} \lesssim r^{1/2} r^{-1/2} = 1.$$

It follows from (4.26), (4.27), (4.28), (4.25), and (4.29) that

$$\|\mathcal{T}_t a\|_{b^1} \lesssim 1.$$

Subcase IIB: $\frac{t}{20} < r \leq 1$.

Suppose that $a$ is either a standard atom supported in $B(o, r)$ with $\frac{t}{20} < r \leq 1$ or a global atom supported in $B(o, 1)$. Notice that $t + 10r < 30r$. We choose a smooth cutoff function $\phi_0$ supported in $B(o, 31r)$ taking values in $[0, 1]$ such that

$$\phi_0 + \sum_{i=5}^I \phi_i + \sum_{j=2}^\infty \psi_j = 1 \quad \text{in} \quad X,$$

where $I$ is such that $2^{I-1}r < 1 < 2^{I+1}r$.

Then $a * (\phi_0 k_I)$ is supported in $B(o, 31r)$ and

$$\|a * (\phi_0 k_I)\|_{b^1} \lesssim \mu(B(o, 31r))^{1/2} \|a\|_{L^2} \|\mathcal{T}_t\|_{L^2 \to L^2} \lesssim 1,$$

where we argued as in (4.15). For every $i = 5, \ldots, I$ by Lemma 3.7 and estimates (4.4)

$$\|a * (\phi_i k_i)\|_{b^1} \lesssim \mu(B(o, (2^{i+1} + 1)r))^{1/2} r \|\nabla (\phi_i k_i)\|_{L^2} \lesssim (2^i r)^{n/2} r \left( \int_{2^{i-1}r}^{2^{i+1}r} [(2r)^{-2}s^{-2} + (2r)^{-2}|t - s|^{-2} \right.$$

$$+ s^{-4} + |t - s|^{-4} + s^{-2}|t - s|^{-2} ds)^{1/2} \lesssim (2^i r)^{n/2} r \left( \int_{2^{i-1}r}^{2^{i+1}r} s^{-4} ds \right)^{1/2} \lesssim (2^i r)^{n/2} r (2^i r)^{-3/2}.$$
Thus from (4.30), (4.25) and (4.31)

\[ \| T_t a \|_{h^1} \leq \| a \ast (\phi_0 k_t) \|_{h^1} + \sum_{i=5}^{I} \| a \ast (\phi_i k_t) \|_{h^1} + \sum_{j=2}^{\infty} \| a \ast (\psi_j k_t) \|_{h^1} \]

\[ \leq 1 + r^{(n-1)/2} + \int_{4}^{2^I} (2^x)^{(n-3)/2} \, dx \]

\[ \leq 1 + r^{(n-1)/2} \int_{2^I}^{2^{I+1}} v^{(n-3)/2-1} \, dv \leq 1 + r^{(n-1)/2} (2^I)^{(n-3)/2} \]

\[ \leq 1 + r^{(n-1)/2} \left( \frac{1}{r} \right)^{(n-3)/2} \lesssim 1, \]

where we used the fact that $2^{I+1} r < 1 < 2^{I+1} r$.

This conclude the proof of the theorem.

\[ \Box \]

Our result shows that the local Hardy space $h^1(\mathbb{X})$ is well adapted to obtain endpoint results for Fourier integral operators on symmetric spaces. As a natural development of this work, in the near future we shall study the boundedness on $h^1(\mathbb{X})$ of oscillating multipliers related both to the Schrödinger and the wave equations associated with the shifted Laplacian $\mathcal{L}$ and the nonshifted Laplacian $\Delta$ on a noncompact symmetric space of arbitrary rank and their consequences on regularity properties of the solutions of such equations.

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