On the Douglas–Rachford algorithm for solving possibly inconsistent optimization problems

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June 21, 2021

Abstract

More than 40 years ago, Lions and Mercier introduced in a seminal paper the Douglas–Rachford algorithm. Today, this method is well recognized as a classical and highly successful splitting method to find minimizers of the sum of two (not necessarily smooth) convex functions. While the underlying theory has matured, one case remains a mystery: the behaviour of the shadow sequence when the given functions have disjoint domains.

Building on previous work, we establish for the first time weak and value convergence of the shadow sequence generated by the Douglas–Rachford algorithm in a setting of unprecedented generality. The weak limit point is shown to solve the associated normal problem which is a minimal perturbation of the original optimization problem. We also present new results on the geometry of the minimal displacement vector.

2020 Mathematics Subject Classification: 65K10, 90C25; Secondary 47H05.

Keywords: convex functions, convex optimization problem, Douglas–Rachford algorithm, inconsistent optimization problem, minimal displacement vector, normal problem, proximal mapping, resolvent.

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1 Introduction

1.1 Problem statement and contribution

Throughout, we assume that

$$X$$ is a real Hilbert space space with inner product \( \langle \cdot, \cdot \rangle : X \times X \rightarrow \mathbb{R} \),

and induced norm \( \| \cdot \| \), and that

$$A$$ and $$B$$ are maximally monotone operators on $$X$$.

We set \( \text{dom} \ A := \{ x \in X \mid Ax \neq \emptyset \} \) and \( \text{ran} \ A := A(X) := \bigcup_{x \in X} Ax \). Recall that the associated Douglas–Rachford operator is

$$T := T_{A,B} := \text{Id} - J_A + J_B R_A,$$

where \( J_A := (\text{Id} + A)^{-1} \) and \( R_A := 2J_A - \text{Id} \) are the resolvent and reflected resolvent of $$A$$, respectively. (See, e.g., [6] for background material and further references.) It is well known (see [35]) that if $$A + B$$ admits a zero, i.e., \( Z_0 := \text{zer}(A + B) := (A + B)^{-1}(0) \neq \emptyset \), then the sequence \( (J_AT^n x)_{n \in \mathbb{N}} \) converges weakly to a point in $$Z_0$$. This is the celebrated Douglas–Rachford algorithm which dates back to Douglas and Rachford [19] but whose importance to optimization was revealed in the seminal paper by Lions and Mercier [24]. In particular, if $$f$$ and $$g$$ are proper lower semicontinuous convex functions on $$X$$ and $$A = \partial f$$ and $$B = \partial g$$, then \( (J_AT^n x)_{n \in \mathbb{N}} \) converges weakly to a minimizer of $$f + g$$ when \( \text{zer}(\partial f + \partial g) \neq \emptyset \). This explains the importance of the Douglas–Rachford algorithm in optimization.

However, it is natural to inquire what the behaviour of the Douglas–Rachford algorithm is when $$Z_0 = \emptyset$$. In fact, this has been the focus of recent research on the Douglas–Rachford algorithm and the closely related alternating direction method of multipliers; see, e.g., [3], [15], [25], and [34]. A key quantity to study this general case is the minimal displacement vector

$$v := \text{P_{ran}(Id - T)}(0),$$

i.e., $$v$$ is the projection of $$0$$ onto the nonempty closed convex subset \( \text{ran}(\text{Id} - T) \) of $$X$$. This vector encodes the minimal perturbation of the original problem that makes the sum problem possibly feasible. Some of the results we shall prove involve the vectors

$$v_D := \text{P_{domA-domB}}(0) \quad \text{and} \quad v_R := \text{P_{ranA+ranB}}(0)$$

(4b)
which are also well defined: indeed, the two sets

\[ D := \text{dom } A - \text{dom } B \quad \text{and} \quad R := \text{ran } A + \text{ran } B \]  

have closures that are convex because the closures of the four sets \( \text{dom } A, \text{dom } B, \text{ran } A, \text{ran } B \) are already convex due to the maximal monotonicity of \( A \) and \( B \) (see [6, Corollary 21.14]).

The goal of this paper is to substantially advance the understanding of the Douglas–Rachford algorithm applied to convex optimization problems. Fortunately, our main result can be stated elegantly after we introduce some necessary notation: Suppose that \( f \) and \( g \) belong to \( \Gamma_0(X) \), i.e., that \( (A, B) = (\partial f, \partial g) \) and thus \( J_A \) and \( J_B \) turn into the proximal mappings

\[ P_f := (\text{Id} + \partial f)^{-1} \quad \text{and} \quad P_g := (\text{Id} + \partial g)^{-1}, \]

with corresponding reflected proximal mappings \( R_f := 2P_f - \text{Id} \) and \( R_g := 2P_g - \text{Id} \), respectively. Under appropriate assumptions, our main result (see Theorem 6.9 below) states that for every \( x \in X \), there exists a vector \( z \in X \) such that

- (minimizer of normal problem) \( f(z) + g(z - v) = \min_{y \in X} (f(y) + g(y - v)) \);
- (shadow convergence) \( P_f T^nx \rightharpoonup z \) and \( P_g R_f T^nx \rightharpoonup z - v \);
- (value convergence) \( f(P_f T^nx) \to f(z) \) and \( g(P_g R_f T^nx) \to g(z - v) \).

This beautifully captures the case of finding a minimizer of \( f + g \) when \( v = 0 \)! Moreover, to the best of our knowledge, this is the first time where weak convergence of the shadow sequence \( (P_f T^nx)_{n \in \mathbb{N}} \) is obtained along with value convergence in this generality. Along our journey to the proof of this result, we discover and present various substantial improvements of earlier results in this quite general case.

In the remainder of this section, we provide a brief history of previous results and also an outline of the rest of the paper.

### 1.2 Brief history for the inconsistent case

The story begins with the 2003 paper [7] concerning two nonempty closed convex subsets \( U, V \) of \( X \). The authors of [7] proved that when \( (f, g) = (\iota_U, \iota_V) \), then the shadow
sequence \((P_UT^n x)_{n \in \mathbb{N}}\) is bounded and its weak cluster points are minimizers of the function \(i_U + i_V (\cdot - v)\). In 2013, and motivated by the results in [7], the authors of [9] established a powerful static framework to cope with the inconsistent problem in the general setting of maximally monotone operators. Another major milestone was the 2014 paper [10] where the connection between the range of the displacement mapping associated with the Douglas–Rachford operator, namely \(Id - T\), and the domains and ranges of the individual operators was established.

Building on [7], [9], and [10], the present authors provided useful convergence results for the shadow sequence of the Douglas–Rachford algorithm in various instances. Indeed, the first proof of strong convergence of the shadow sequence was given when \((f, g) = (i_U, i_V)\) for two closed affine subspaces \(U, V\) of \(X\) in 2015 in [13]. Even more strikingly, they obtained linear rates of convergence with the rate being quantified in terms of the cosine of the Friedrichs angle between \(U\) and \(V\). In another 2015 paper [11], together with M.N. Dao, they extended the result to the setting when one set is a nonempty closed convex (but not necessarily affine) subset of \(X\). Another milestone is the work in the 2016 paper [14] where the authors presented a new Fejér monotonicity principle to prove the full weak convergence of the shadow sequence in the case of two nonempty closed convex (not necessarily intersecting) subsets of \(X\). This completed the analysis for two indicator functions that began in [7] 13 years earlier. We refer the interested reader to [28] for a detailed collection of the previously mentioned results. The latest breakthrough was the 2019 paper [15] which dealt with the case when \(f = i_U\), where \(U\) is a closed linear subspace of \(X\), and \(g \in \Gamma_0(X)\). (In passing, we point out that at the time of writing [15] an assumption was made there — namely \(v_R = 0\) — that was sufficient for convergence. In the present paper, we clarify this further by proving that \(v_R = 0\) is also a necessary condition for convergence — see Proposition 5.2 below.)

We now turn to related works that built on the previous results. In the 2017 paper [25] (and the 2018 paper [34]) Ryu, Lin, and Yin proposed a method based on the Douglas–Rachford algorithm that identifies, in certain situations, infeasible, unbounded, and pathological conic (and feasible and infeasible convex, respectively) optimization problems. In the 2018 paper, they translated the analysis to ADMM, which is an incarnation of the Douglas–Rachford algorithm (see, e.g., [22] or [29]). The analysis hinges upon identifying the range of \(Id - T\) and the notion and location of the minimal displacement vector defined in [10]. Closely related in spirit to the results in [25] and [34] is the 2018 paper by Banjac, Goulart, Stellato, and Boyd [3]. Indeed, these authors showed that for certain classes of convex optimization problems, ADMM can detect primal and dual infeasibility of the problem and they propose a termination criterion. In the recent 2020 papers [4] and [2] the authors extended some of the geometric properties of the minimal displacement vector established in [15]. In particular, the decomposition of \(v\) into the sum of orthogonal vectors \(v = v_D + v_R\) (see (4b)) was seen to be useful in studying certain structured optimization problems. Finally, our work has proven to be useful even in
nonconvex settings; indeed, Borwein, Lindstrom, Sims, Schneider, and Skerritt have used the main result of [14] to extend some of the convex theory to the non-convex case in their 2018 paper [17].

1.3 Organization of the paper

The remainder of this paper is organized as follows. In Section 2, we collect various results on the normal problem, the solution set, and the set of minimizers of the sum of two functions to make subsequent proofs easier to follow. Useful properties of \( v_D \) and \( v_R \) (which are defined in (4b)) are revealed in Section 3. In Section 4, we present results on the interplay between the vectors \( v, v_D, v_R \), the Douglas–Rachford operator \( T \), and the generalized solution set \( Z \). We analyze the shadow sequence \( (J_AT_nx)_{n\in\mathbb{N}} \) with regards to Fejér monotonicity and conditions necessary for convergence in Section 5. Finally, in Section 6, we prove the main result.

Any terminology and notation not explicitly defined here can be found in [6].

2 Background material and auxiliary results

In this section, we recall and record various results concerning the normal problem, the extended solution set, and the set of minimizers of two functions.

Recall first the well-known inverse resolvent identity (see, e.g., [6, Proposition 23.20])

\[
J_A + J_A^{-1} = \text{Id}
\]  

as well as the Minty parametrization (see [27])

\[
\text{gra} \ A = \{ (J_A x, x - J_A x) \mid x \in X \}.
\]  

The following resolvent identities will be useful later:

**Lemma 2.1.** Let \( (y, w) \in X \times X \). Then \( J_A y = J_{-w+A}(-w + y) \) and \( J_A^{-1}y = w + J_{(-w+A)^{-1}}(-w + y) \).

**Proof.** Indeed, recall that \( J_{-w+A} = J_A (\cdot + w) \) by, e.g., [6, Proposition 23.17]. Now, \( J_A y = J_A ((-w + y) + w) = J_{-w+A}(-w + y) \) and \( J_A^{-1}y = y - J_A y = y - J_{-w+A}(-w + y) = w + (-w+y) - J_{-w+A}(-w + y) = w + J_{(-w+A)^{-1}}(-w + y) \). ■
Turning now to the Douglas–Rachford operator \( T \) introduced in (3), we note that
\[
\text{Id} - T = J_A - J_BR_A = J_{A^{-1}} + J_{B^{-1}}R_A,
\]
and also the following:

**Lemma 2.2.** We have the following:

(i) \( J_A - J_AT = J_{A^{-1}}T + J_{B^{-1}}R_A \); hence, \( \text{ran} (J_A - J_AT) \subseteq \text{ran} A + \text{ran} B \).

(ii) \( J_{A^{-1}} - J_{A^{-1}}T = J_AT - J_BR_A \); hence, \( \text{ran} (J_{A^{-1}} - J_{A^{-1}}T) \subseteq \text{dom} A - \text{dom} B \).

**Proof.** (i): Indeed, \( J_A - J_{B^{-1}}R_A = J_A - R_A + J_BR_A = J_A - 2J_A + \text{Id} + J_BR_A = \text{Id} - J_A + J_BR_A = T = J_AT + J_{A^{-1}}T \). Rearranging yields the desired result. (ii): Indeed, \( J_{A^{-1}} + J_BR_A = \text{Id} - J_A + J_BR_A = T = J_{A^{-1}}T + J_AT \). Rearranging yields the desired result. \( \blacksquare \)

Recalling the definitions of the sets \( D \) and \( R \) from (5), we have (see [21, Proposition 4.1] or [9, Corollary 2.14])
\[
\text{ran} (\text{Id} - T) = \{ a - b \mid (a, a^*) \in \text{gra} A, (b, b^*) \in \text{gra} B, a - b = a^* + b^* \} \subseteq D \cap R. \tag{11}
\]
It follows that \( \overline{\text{ran}} (\text{Id} - T) \subseteq \overline{D} \cap \overline{R} \). From this point onwards, we will assume that
\[
\overline{\text{ran}} (\text{Id} - T) = \overline{D} \cap \overline{R}. \tag{12}
\]
For applications, this assumption is rather mild as can been seen in the following:

**Remark 2.3.** It follows from [10, Corollary 6.5] that (12) holds if \( X \) is finite-dimensional and \( A \) and \( B \) are subdifferential operators of proper lower semicontinuous convex functions. See also [10, Theorem 5.2] for more general settings.

Following [9], recall that the normal problem associated with the ordered pair \((A, B)\) is
\[
\text{find } x \in X \text{ such that } 0 \in -v + Ax + B(x - v), \tag{13}
\]
where \( v \) is as in (4a). Next, the Attouch–Théra dual pair (see [1] and [26, page 40]) of the primal pair \((-v + A, B(\cdot - v))\) is \((-v + A, B(\cdot - v))^\circ := ((-v + A)^{-1}, (B(\cdot - v))^{-\circ})\), where \((B(\cdot - v))^{-\circ} := (-\text{Id}) \circ B(\cdot - v) \circ (-\text{Id})\) and \(B^{-\circ} := ((B(\cdot - v))^{-1})^{-\circ} = ((B(\cdot - v))^\circ)^{-1}\).

We will make use of the notation
\[
Z := Z_{(-v + A, B(\cdot - v))} = (-v + A + B(\cdot - v))^{-1}(0) \tag{14}
\]
and
\[
K := K_{(-v + A, B(\cdot - v))} = ((-v + A)^{-1} + (B(\cdot - v))^{-\circ})^{-1}(0) \tag{15}
\]
to denote the primal and dual solutions of the normal problem (13), respectively (see, e.g., [5]). It follows from [9, Proposition 3.2] that
\[ T_{-v+A,B(-v)} = T(\cdot + v); \tag{16} \]

moreover, [9, Proposition 2.24 and Proposition 3.3] imply
\[ Z \neq \emptyset \iff \text{Fix}(\cdot + v) \neq \emptyset \iff v \in \text{ran}(\text{Id} - T). \tag{17} \]

We now recall that the extended solution set associated with the normal problem (13) (see Eckstein and Svaiter’s [20, Section 2.1] and also [5, Section 3]) is defined by
\[ S := S_{(-v+A,B(-v))} := \{(z,k) \in X \times X \mid -k \in B(z-v), k \in -v + Az\} \subseteq Z \times K. \tag{18} \]

The usefulness of \( S \) becomes apparent in the next two results:

**Fact 2.4.** Recalling (9) and (16), we have
\[ S = \{(J_{-v+A} \times J_{(-v+A)^{-1}})(y,y) \mid y \in \text{Fix}(\cdot + v)\}. \tag{19} \]

If \( A \) and \( B \) are paramonotone\(^1\), then we additionally have:

(i) \( S = Z \times K \).

(ii) \( \text{Fix}(\cdot + v) = Z + K \).

*Proof.* The identity (19) is [5, Theorem 4.5]. (i)&(ii): See [5, Corollary 5.5(ii)&(iii)]. \( \blacksquare \)

**Lemma 2.5.** The following hold:

(i) \( \text{Fix}(\cdot + v) = -v + \text{Fix}(v + T) \).

(ii) \[ S = (0,-v) + \{(J_A \times J_{A^{-1}})(f,f) \mid f \in \text{Fix}(v + T)\} . \]

*Proof.* (i): Let \( f \in X \). Then \( f \in \text{Fix}(v + T) \iff f = v + Tf \iff f - v = T(f - v + v) \iff f - v \in \text{Fix}(\cdot + v) \iff f \in v + \text{Fix}(\cdot + v) \).

(ii): Combine (i) and Lemma 2.1 to learn that \( \{(J_A \times J_{A^{-1}})(f,f) \mid f \in \text{Fix}(v + T)\} = (0,v) + \{(J_{-v+A} \times J_{(-v+A)^{-1}})(f,f) \mid f \in \text{Fix}(\cdot + v)\} \). Now invoke (19). \( \blacksquare \)

We conclude this section with the following useful results concerning the minimizers of the sum of two functions.

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\(^1\)Let \( C : X \rightrightarrows X \) be monotone. Then \( C \) is paramonotone if \( \{(x,u),(y,v)\} \subseteq \text{gra} C \) and \( \langle x - y, u - v \rangle = 0 \) \( \Rightarrow \) \( \{(x,v),(y,u)\} \subseteq \text{gra} C \). (For a more detailed discussion and examples of paramonotone operators, we refer the reader to [23].)
Lemma 2.6. Let $f \in \Gamma_0(X)$ and let $g \in \Gamma_0(X)$ be such that $\text{argmin}(f + g) \neq \emptyset$. Let $x \in X$ and let $y \in \text{argmin}(f + g)$. Suppose that $x^* \in (-\partial f(x)) \cap \partial g(x)$. Then

$$f(y) = f(x) + \langle -x^*, y - x \rangle,$$  \hspace{1cm} (20a)

$$g(y) = g(x) + \langle x^*, y - x \rangle,$$  \hspace{1cm} (20b)

and

$$x^* \in (-\partial f(y)) \cap \partial g(y).$$ \hspace{1cm} (21)

If $\iota_C \in \{f, g\}$, where $C$ is nonempty closed convex subset of $X$, then we also have the following:

(i) $\langle x^*, y - x \rangle = 0$.

(ii) $K \perp (Z - Z)$.

(iii) $J_A P_{\text{Fix}T} = P_Z$.

Proof. Observe that $0 = -x^* + x^* \in \partial f(x) + \partial g(x) \subseteq \partial(f + g)(x)$. Hence $x$ is a minimizer of $f + g$. Consequently $f(x) + g(x) = f(y) + g(y)$; equivalently,

$$f(x) - f(y) = g(y) - g(x).$$ \hspace{1cm} (22)

The subgradient inequalities for $f$ and $g$ yield $(\forall z \in X)$

$$f(z) \geq f(x) + \langle -x^*, z - x \rangle,$$ \hspace{1cm} (23a)

$$g(z) \geq g(x) + \langle x^*, z - x \rangle.$$ \hspace{1cm} (23b)

In particular, we learn that

$$f(y) \geq f(x) - \langle x^*, y - x \rangle,$$ \hspace{1cm} (24a)

$$g(y) \geq g(x) + \langle x^*, y - x \rangle.$$ \hspace{1cm} (24b)

Hence

$$f(x) - f(y) \leq \langle x^*, y - x \rangle \leq g(y) - g(x).$$ \hspace{1cm} (25)

Combining the above inequality with (22) yields

$$f(x) - f(y) = \langle x^*, y - x \rangle = g(y) - g(x).$$ \hspace{1cm} (26)

(21): Let $z \in X$. Then (23), (26), and (20) yield $(\forall z \in X)$

$$f(z) \geq f(x) + \langle -x^*, z - x \rangle = \underbrace{f(x) + \langle -x^*, y - x \rangle + \langle -x^*, z - y \rangle}_{=f(y)} = f(y) + \langle -x^*, z - y \rangle,$$ \hspace{1cm} (27a)

$$g(z) \geq g(x) + \langle x^*, z - x \rangle = \underbrace{g(x) + \langle x^*, y - x \rangle + \langle x^*, z - y \rangle}_{=g(y)} = g(y) + \langle x^*, z - y \rangle.$$ \hspace{1cm} (27b)
Consequently we learn that \( x^* \in -\partial f(y) \) and \( x^* \in \partial g(y) \). (i): Suppose first that \( g = \iota_C \). Because \( x \) and \( y \) are minimizers of \( f + \iota_C \) we must have \( \{x, y\} \subseteq C \), hence \( \iota_C(x) = \iota_C(y) = g(x) = g(y) = 0 \). Now combine with (26). (ii): It follows from [5, Remark 5.4] that \( K = (-\partial f(x)) \cap \partial g(x) \). Now combine with (i). The proof when \( f = \iota_C \) is similar. (iii): Combine (ii) and [5, Theorem 6.7(ii) and Corollary 5.5(iii)].

**Proposition 2.7.** Let \( f \in \Gamma_0(X) \) and let \( g \in \Gamma_0(X) \) be such that \( \text{zer}(\partial f + \partial g) \neq \emptyset \). Then

\[
\text{argmin}(f + g) = \text{zer}(f + g) = \text{zer}(\partial f + \partial g). \tag{28}
\]

**Proof.** Observe that \( \text{zer}(\partial f + \partial g) \subseteq \text{zer}(f + g) = \text{argmin}(f + g) \) by, e.g., [6, Theorem 16.3 & Proposition 16.6(ii)]. It remains to establish the inclusion \( \text{zer}(f + g) \subseteq \text{zer}(\partial f + \partial g) \). To this end, let \( y \in \text{zer}(f + g) = \text{argmin}(f + g) \) and let \( x \in \text{zer}(\partial f + \partial g) \). Then \( \exists x^* \in (-\partial f(x)) \cap \partial g(x) \). Using (21), we learn that \( x^* \in (-\partial f(y)) \cap \partial g(y) \), hence \( 0 \in \partial f(y) + \partial g(y) \). Consequently, \( y \in \text{zer}(\partial f + \partial g) \). \( \blacksquare \)

### 3 \( v_D \) and \( v_R \)

In this section, we shall derive various results on the vectors \( v_D \) and \( v_R \) (see (4b)). Our analysis depends on the following two results.

**Fact 3.1.** Let \( U \) and \( V \) be nonempty closed convex subsets of \( X \). Then

\[
P_{\overline{U}}(0) \in (P_U - \text{Id})(V) \cap (\text{Id} - P_V)(U) \subseteq (-\text{rec } U) \cap (\text{rec } V) \circ. \tag{29}
\]

**Proof.** This follows from [7, Corollary 2.7] and [36, Theorem 3.1]. \( \blacksquare \)

**Lemma 3.2.** The following hold for \( A \) and \( B \) (see (2)):

(i) \((\text{rec } \overline{\text{dom } A}) \circ \subseteq \text{rec } (\overline{\text{ran } A}) \circ \subseteq \text{rec } (\text{ran } A) \circ \) and \((\text{rec } \overline{\text{dom } B}) \circ \subseteq \text{rec } (\text{ran } B) \circ \).

(ii) \((\text{rec } \overline{\text{ran } A}) \circ \subseteq \text{rec } (\overline{\text{dom } A}) \circ \subseteq \text{rec } (\overline{\text{dom } B}) \circ \).

**Proof.** It suffices to prove the statements for \( A \). (i): Observe that using, e.g., [6, Corollary 21.14 and Example 25.14] \( \overline{\text{dom } A} \) and \( \overline{\text{ran } A} \) are nonempty closed and convex subsets of \( X \), that \( N_{\overline{\text{dom } A}} \) is \( 3^* \) monotone and maximally monotone and that \( A = A + N_{\overline{\text{dom } A}} \). On the one hand, it follows from Brezis–Haraux theorem (see, e.g., [6, Theorem 25.24(ii)]) applied to \( A \) and \( N_{\overline{\text{dom } A}} \) that

\[
\overline{\text{ran } A} + \overline{\text{ran } N_{\overline{\text{dom } A}}} \subseteq \overline{\text{ran } A} + \overline{\text{ran } N_{\overline{\text{dom } A}}} = \overline{\text{ran } (A + N_{\overline{\text{dom } A}})} = \overline{\text{ran } A}. \tag{30}
\]
Hence
\[ \overline{\text{ran}} N_{\text{dom} A} \subseteq \text{rec} \overline{\text{ran}} A. \] (31)

On the other hand, it follows from [36, Theorem 3.1] that
\[ \overline{\text{ran}} N_{\text{dom} A} = \overline{\text{ran}} (\text{Id} - P_{\text{dom} A}) = (\text{rec} \overline{\text{dom}} A)\oplus. \] (32)

Now combine (31) and (32). (ii): Apply (i) to \( A^{-1} \).

We are now able to derive new information about the location of \( v_D \) and \( v_R \). When we specialize to subdifferential operators, then one obtains a recent result (see [2, Proposition 3.2]) that was proved differently by using recession functions (which are unavailable in general). These results in turn generalize [15, Proposition 2.3].

**Proposition 3.3.** The following hold:

(i) \( v_D \in (- \text{rec} \overline{\text{dom}} A)\ominus \cap (\text{rec} \overline{\text{dom}} B)\ominus = (- \text{rec} \overline{\text{dom}} A)\ominus \cap (\text{rec} \overline{\text{dom}} B)\ominus. \)

(ii) \( v_D \in (- \overline{\text{ran}} A) \cap (\overline{\text{rec}} \overline{\text{ran}} B). \)

(iii) \( v_R \in (- \overline{\text{ran}} A)\ominus \cap (- \overline{\text{ran}} B)\ominus = (- \text{rec} \overline{\text{dom}} A)\ominus \cap (\text{rec} \overline{\text{ran}} B)\ominus). \)

(iv) \( v_R \in (- \text{rec} \overline{\text{dom}} A) \cap (- \text{rec} \overline{\text{dom}} B) = (- \text{rec} \overline{\text{dom}} A \cap \text{rec} \overline{\text{dom}} B). \)

(v) \( \langle v_D, v_R \rangle = 0. \)

(vi) \( v_D + v_R \in \overline{\text{dom}} A - \overline{\text{dom}} B \cap \overline{\text{ran}} A + \overline{\text{ran}} B. \)

(vii) \( v = v_D + v_R. \)

(viii) \( \|v\|^2 = \|v_D\|^2 + \|v_R\|^2 = \|(v_R, v_D)\|^2. \)

**Proof.** (i): Apply Fact 3.1 with \((U, V)\) replaced by \((\overline{\text{dom}} A, \overline{\text{dom}} B)\).

(ii): Combine (i) and Lemma 3.2(i).

(iii): Apply Fact 3.1 with \((U, V)\) replaced by \((\overline{\text{ran}} A, -\overline{\text{ran}} B)\).

(iv): Combine (iii) and Lemma 3.2(ii).

(v): It follows from (i) and (iv) that \((-v_D, -v_R) \in (\text{rec} \overline{\text{dom}} A)\ominus \times \text{rec} \overline{\text{dom}} A. \) Hence \( \langle v_D, v_R \rangle = \langle -v_D, -v_R \rangle \leq 0. \) Similarly, (ii) and (iii) imply that \((v_D, -v_R) \in \text{rec} \overline{\text{ran}} B \times (\text{rec} \overline{\text{ran}} B)\ominus. \) Hence, \(-\langle v_D, v_R \rangle = \langle v_D, -v_R \rangle \leq 0. \) Altogether, \( \langle v_D, v_R \rangle = 0. \)

(vi): Indeed, in view of (iv) we have \(-v_R \in \text{rec} \overline{\text{dom}} B. \) Therefore, \( v_D + v_R \in \overline{\text{dom}} A - \overline{\text{dom}} B + v_R = \overline{\text{dom}} A - (-v_R + \overline{\text{dom}} B) \subseteq \overline{\text{dom}} A - \overline{\text{dom}} B = \overline{\text{dom}} A - \overline{\text{dom}} B. \) Similarly, in view of (ii) we have \( v_D \in \text{rec} \overline{\text{ran}} B. \) Therefore \( v_D + v_R \in \overline{\text{ran}} A + v_D + \overline{\text{ran}} B \subseteq \overline{\text{ran}} A + \overline{\text{ran}} B = \overline{\text{ran}} A + \overline{\text{ran}} B. \)

(vii): Observe that (vi) and (12) imply that \( \|v\| \leq \|v_D + v_R\|. \) It follows from (v), the definition of \( v \) and \( v_D \) that \( \|v_D\|^2 \leq \langle v_D, \overline{\text{D}} \rangle, \) hence \( \|v_D\|^2 \leq \langle v_D, \overline{\text{D}} \cap \overline{\text{R}} \rangle. \) Similarly,
(v), the definition of $v$ and and $v_R$ implies $\|v_R\|^2 \leq \langle v_R, R \rangle$, hence $\|v_R\|^2 \leq \langle v_R, \overline{D} \cap \overline{R} \rangle$. Therefore using Cauchy-Schwarz we learn that $\|v_D + v_R\|^2 = \|v_D\|^2 + \|v_R\|^2 \leq \langle v, v_D \rangle + \langle v, v_D + v_R \rangle \leq \|v\| \|v_D + v_R\|$. Hence, $\|v_D + v_R\| \leq \|v\|$. Altogether, $\|v\| = \|v_D + v_R\|$. In view of (12), (vi) and [30, Lemma 2], we learn that $v = v_D + v_R$.

(viii): Combine (v) and (vii).

\[\text{Remark 3.4 (The real line case). }\text{Suppose that }X = \mathbb{R}.\text{ It follows from Proposition 3.3(v) that }\quad v_D v_R = 0\text{ which implies that }\quad 0 \in \{v_D, v_R\}.\]

This conclusion is no longer true when $X \neq \mathbb{R}$ as we illustrate in Example 4.5 and Example 4.6 below.

The proof of the last result in this section requires the fact that if $C$ is a nonempty closed convex subset of $X$ and $x \in X$, then (see, e.g., [6, Proposition 29.1(iii)])

\[P_C(x) = -P_C(-x).\]  

\[\text{Corollary 3.5. The following hold:}\]

(i) $v_D = P_{\overline{\text{rec dom } A}} v$.

(ii) $v_R = P_{\overline{\text{ran } A}} v$.

(iii) $v_D = P_{\overline{\text{ran } A}} v$.

(iv) $v_R = P_{\overline{\text{rec dom } A}} v$.

\[\text{Proof. Observe that }\overline{\text{rec dom } A}\text{ and }\overline{\text{ran } A}\text{ are closed by, e.g., [6, Proposition 6.49(v)].}\]

(i)&(ii): It follows from Proposition 3.3(i),(iv),(v)&(vii) that $(-v_D, -v_R) \in (\overline{\text{rec dom } A})^\oplus \times \overline{\text{rec dom } A}$, that $v_D \perp v_R$ and that $-v_D = -v - (-v_R)$. Now combine with [6, Proposition 6.28] to learn that $-v_D = P_{\overline{\text{rec dom } A}}^\oplus (-v)$; equivalently, $v_D = -P_{\overline{\text{rec dom } A}}^\oplus (-v) = P_{\overline{\text{rec dom } A}}^\oplus v$, where the third identity followed from applying (34) with $(x, C)$ replaced by $(v, \overline{\text{rec dom } A})^\oplus).$ Similarly, $-v_R = P_{\overline{\text{rec dom } A}} (-v)$; equivalently, $v_R = -P_{\overline{\text{rec dom } A}} (v) = P_{-\overline{\text{rec dom } A}} v$, where the last identity followed from applying (34) with $(x, C)$ replaced by $(v, \overline{\text{rec dom } A})$.

(iii)&(iv): It follows from Proposition 3.3(ii),(iii),(v)&(vii) that $(-v_D, -v_R) \in \overline{\text{ran } A} \times (\overline{\text{rec dom } A})^\oplus$, that $v_D \perp v_R$, and that $-v_D = -v - (-v_R)$. Now proceed similar to the proof of (i)&(ii).
4 Static consequences

In this section, we present results on the interplay between the vectors \( v, v_D, v_R \) (see (4)), the Douglas–Rachford operator \( T \) (see (3)), and the generalized solution set \( Z \) (see (14)). Working in the product space \( X \times X \), we restate (4b) (see, e.g., [6, Proposition 29.4]) as:

\[
P_{\text{dom}A-\text{dom}B \times \text{ran}A+\text{ran}B}(0) = (v_D, v_R).
\]

(35)

The next result relates \((v_D, v_R)\) to the Douglas–Rachford operator \( T \) defined in (3):

**Lemma 4.1.** The following hold:

(i) Suppose that \( f \in \text{Fix}(v + T) \). Then

\[
Tf = f - v,
\]

\[
J_ATf = J_Af - v_R,
\]

\[
J_A^{-1}Tf = J_A^{-1}f - v_D.
\]

Moreover,

\[
\langle J_Af - J_ATf, J_A^{-1}f - J_A^{-1}Tf \rangle = \langle J_BR_Af - J_BR_ATf, J_B^{-1}R_Af - J_B^{-1}R_ATf \rangle = 0.
\]

(37a)

(37b)

(ii) \( v \in \text{ran} (\text{Id} - T) \Leftrightarrow \text{Fix}(v + T) \neq \emptyset \Rightarrow (v_D, v_R) \in D \times R \) (which were defined in (5)).

**Proof.** (i): (36a) is clear. Note that Lemma 2.2(i) & (ii) applied with \( x \) replaced by \( f \) yields

\[
J_Af - J_ATf \in \text{ran} A + \text{ran} B \quad \text{and} \quad J_A^{-1}f - J_A^{-1}Tf \in \text{dom} A - \text{dom} B.
\]

(38)

In view of Proposition 3.3(viii), the Minty parametrization of \( \text{gra} A \) (9), and (36a) imply

\[
\|v_R\|^2 + \|v_D\|^2 = \|J_Af - J_ATf\|^2 + \|J_A^{-1}f - J_A^{-1}Tf\|^2
\]

\[
\leq \|J_Af - J_ATf\|^2 + \|J_A^{-1}f - J_A^{-1}Tf\|^2
\]

\[
+ 2 \langle J_Af - J_ATf, J_A^{-1}f - J_A^{-1}Tf \rangle \geq 0
\]

\[
= \|f - Tf\|^2 = \|f - (f - v)\|^2 = \|v\|^2 = \|v_R\|^2 + \|v_D\|^2.
\]

(39a)

(39b)

(39c)

Hence all inequalities become equalities and therefore by definition of \( v_D \) and \( v_R \), in view of (35) and (37), we must have

\[
(J_Af - J_ATf, J_A^{-1}f - J_A^{-1}Tf) = (v_R, v_D).
\]

(40)

This proves (36b) and (36c).
On the one hand, \( v = f - Tf = J_A f - J_B R_A f = (\text{Id} - T)f = J_A T f - J_B R_A T f \), hence \( J_A f - J_A T f = J_B R_A f - J_B R_A T f \). On the other hand, we similarly get \( v = f - Tf = J_{A^{-1}} f + J_{B^{-1}} R_A f = (\text{Id} - T)f = J_{A^{-1}} T f + J_{B^{-1}} R_A T f \), hence \( J_{A^{-1}} f - J_{A^{-1}} T f = J_{B^{-1}} R_A f - J_{B^{-1}} R_A T f \). Altogether, (37a) holds. Moreover, (37b) follows from (37a), (40), and Proposition 3.3(v).

(ii): Indeed, \( \text{Fix}(v + T) \neq \emptyset \iff (\exists x \in X) x - Tx = v \). Now combine (38) and (40).

In the following result, we relate \( v \) to \((v_D, v_R)\):

**Proposition 4.2.** Let \( \alpha \geq 0 \), let \( \beta \leq 0 \) and let \((x, x^*) \in X \times X\). Suppose that \( x^* \in (v - A x) \cap B(x - v) \). Then the following hold:

(i) \( x^* + \alpha v_D \in (v - Ax) \cap B(x - v) \).

(ii) \( x + \beta v_R \in A^{-1}(-x^* + v) \cap (v + B^{-1} x^*) \).

(iii) \( x^* \in (v - \alpha v_D - Ax) \cap B(x + \beta v_R - v) \).

**Proof.** Let \((a, a^*) \in \text{gra} A\) and let \((b, b^*) \in \text{gra} B\). (i) On the one hand, observe that \( v - x^* \in Ax \). Moreover,

\[
\langle a - x, a^* - (v - x^* - \alpha v_D) \rangle = \langle a - x, a^* - (v - x^*) \rangle + \langle a - x, \alpha v_D \rangle 
\geq \langle a - x, \alpha v_D \rangle = \alpha \langle v - (a - (x - v)), 0 - v_D \rangle 
= \alpha \langle v_D - (a - (x - v)), 0 - v_D \rangle \geq 0.
\]

Here (41b) follows from the monotonicity of \( A \) and (41c) follows from combining Proposition 3.3(vii)&(v) and (4b) by observing that \((a, x - v) \in \text{dom} A \times \text{dom} B\). The maximality of \( A \) implies that

\[
v - x^* - \alpha v_D \in Ax.
\]

On the other hand, observe that \( x^* \in B(x - v) \). Moreover,

\[
\langle b - (x - v), b^* - (x^* + \alpha v_D) \rangle = \langle b - (x - v), b^* - x^* \rangle + \alpha \langle b - (x - v), 0 - v_D \rangle 
\geq \alpha \langle b - (x - v), 0 - v_D \rangle 
= \alpha \langle v_D - (a - b), 0 - v_D \rangle \geq 0.
\]

Here (43b) follows from the monotonicity of \( B \) and (43c) follows from combining Proposition 3.3(vii)&(v) and (4b) by observing that \((x, b) \in \text{dom} A \times \text{dom} B\). The maximality of \( B \) implies that

\[
x^* + \alpha v_D \in B(x - v).
\]

Altogether, we conclude that \( x^* + \alpha v_D \in (v - A x) \cap B(x - v) \).

(ii): On the one hand, observe that \( x \in A^{-1}(v - x^*) \cap (v + B^{-1} x^*) \). Therefore

\[
\langle a^* - (v - x^*), a - (x + \beta v_R) \rangle = \langle a^* - (v - x^*), a - x \rangle + \beta \langle a^* - (v - x^*), 0 - v_R \rangle
\]

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by observing that follows from the monotonicity of to be follows from the monotonicity of follows from combining (45d)

(47b)

implies that and that (47d)

(14)

&

(4b)

implies that

and 

Proposition 3.3(vii)

Proposition 3.3(vii)

immaximality of

A

Theorem 4.3. Set

Then the following hold:

(i) 

(ii) Suppose that \( v_R = 0 \). Then \( \tilde{Z} = Z \).

Proof. (i): Suppose that \( x \in \tilde{Z} \). Then (\( \exists x^* \in X \)) such that \( x^* \in (-Ax) \cap (-v_R + B(x - v_D)) \). Let \((b, b^*) \in \text{gra} B \). We have

\[
\langle b - (x - v), b^* - (x^* + v) \rangle = \langle b - (x - v_D - v_R), b^* - (x^* + v_D + v_R) \rangle
\]
\[ = \langle b - (x - v_D), b^* - (x^* + v_R) \rangle \quad (51b) \]
\[ + \langle b - (x - v_D), 0 - v_D \rangle + \langle v_R, b^* - (x^* + v_R) \rangle \]
\[ \geq \langle v_D - (x - b), 0 - v_D \rangle + \langle v_R, b^* - x^* - v_R \rangle \quad (51c) \]
\[ \geq 0 \quad (51d) \]

where (51a) follows from Proposition 3.3(vii), where (51b) follows from Proposition 3.3(v), where (51c) follows from the monotonicity of B and Proposition 3.3(vii), and where (51d) follows from definitions of \( v_D \) and \( v_R \) by observing that \((x, b) \in \text{dom} \, A \times \text{dom} \, B \) and \((-x^*, b^*) \in \text{ran} \, B \times \text{ran} \, A \). The maximality of \( B \) yields \( x^* + v \in B(x - v) \). Recalling that \(-x^* \in A x \) we learn that \( x \in Z \).

(ii): In view of (i) it is sufficient to prove the inclusion \( Z \subseteq \tilde{Z} \). To this end, let \( x \in Z \). Observe that Proposition 3.3(vii) implies that \( v = v_D \). Then \( \exists x^* \in X \) such that \( x^* \in (v - Ax) \cap B(x - v) \). It follows from Proposition 4.2(iii) applied with \( \alpha = 1 \) and Proposition 3.3(vii) that \( x^* \in (v - v_D - Ax) \cap B(x - v) = (0 - Ax) \cap B(x - v_D) \). We conclude that \( x \in \tilde{Z} \).

**Remark 4.4.** Some comments on Theorem 4.3 are in order.

(i) The assumption \( v_R = 0 \) is critical in the conclusion of Theorem 4.3(ii) as we illustrate in Example 4.5 below. Example 4.5 also shows that the inclusion Theorem 4.3(i) cannot be improved to equality in general.

(ii) The converse of Theorem 4.3(ii) is not true as we illustrate in Example 4.6 below.

Before we present the limiting examples announced in Remark 4.4, we recall that if \( C \) is a nonempty closed convex subset of \( X \) and \( a \in X \), then

\[ N_{a + C} = N_C(\cdot - a). \quad (52) \]

**Example 4.5.** Suppose that \( X = \mathbb{R}^2 \), let \( \gamma < 0 \), and \((a, \beta, \delta) \in \mathbb{R}^3 \). Set \( a = (a, \beta) \), \( b = (\gamma, \delta) \), \( K = \mathbb{R}_+ \times \{0\} \), \((A, B) = (N_{a+K}, b + N_K) \), and \( \tilde{Z} = \{x \in X \mid 0 \in -v_R + Ax + B(x - v_D)\} \). Then the following hold:

(i) \( \text{dom} \, A - \text{dom} \, B = \mathbb{R} \times \{\beta\} \).
(ii) \( \text{ran} \, A + \text{ran} \, B = ]-\infty, \gamma[ \times \mathbb{R} \).
(iii) \( v_D = (0, \beta) \).
(iv) \( v_R = (\gamma, 0) \neq (0, 0) \).
(v) \( v = (\gamma, \beta) \).
(vi) \( Z = [\max\{\gamma, a\}, +\infty[ \times \{\beta\} \).
(vii) \( \tilde{Z} = [\max\{0, a\}, +\infty[ \times \{\beta\} \).
(viii) \( \tilde{Z} \subseteq Z \Leftrightarrow a < 0 \).
Proof. (i) Indeed, dom $A - \text{dom } B = a + (K - K) = a + \mathbb{R} \times \{0\} = \mathbb{R} \times \{\beta\}$.

(ii): It follows from [36, Theorem 3.1] that ran $A + \text{ran } B = (\text{rec } K) \ominus + b + (\text{rec } K) \ominus = b + K \ominus = b + \mathbb{R}_- \times \mathbb{R}$.

(iii): It follows from (i) and (4b) that $v_D = P_{\mathbb{R} \times \{\beta\}}(0, 0) = (0, \beta)$. (iv): It follows from (ii) and the assumption that $\gamma < 0$ that $v_R = P_{[-\infty, \gamma] \times \mathbb{R}}(0, 0) = (\gamma, 0)$.

(v): Combine (iii), (iv) and Proposition 3.3(vii).

(vi): Indeed, let $x \in \mathbb{R}^2$. Then (v) and (52) applied with $C$ replaced by $K$ yield
\[
x \in Z \Leftrightarrow (0, 0) \in (-\gamma, -\beta) + N_{(a, \beta) + K} x + (\gamma, \delta) + N_K (x - (\gamma, \beta)) \quad (53a)
\]
\[
\Leftrightarrow (0, 0) \in (0, \delta - \beta) + N_K (x - (a, \beta)) + N_K (x - (\gamma, \beta)). \quad (53b)
\]

Set $Y = \{(x_1, \beta) \in \mathbb{R}^2 \mid x_1 \geq \max\{\gamma, \alpha\}\}$. On the one hand, in view of (53), we learn that $(\forall (x_1, x_2) \in Z)$ we must have $x_1 \geq \max\{\gamma, \alpha\}$ and $x_2 = \beta$. Hence, $Z \subseteq Y$. On the other hand, $(\forall x = (x_1, x_2) \in Y)$ we have
\[
(0, 0) \in (0, \delta - \beta) + \{0\} \times \mathbb{R} + \{0\} \times \mathbb{R}
\]
\[
\subseteq (0, \delta - \beta) + N_K (x - (a, \beta)) + N_K (x - (\gamma, \beta)). \quad (54b)
\]
Hence (53) implies that $Y \subseteq Z$. Altogether, we conclude that (vi) holds.

(vii): Let $x \in \mathbb{R}^2$. Then (52) applied with $C$ replaced by $K$, (iv) and (iii) yield
\[
x \in \tilde{Z} \Leftrightarrow (0, 0) \in (-\gamma, 0) + N_{(a, \beta) + K} x + (\gamma, \delta) + N_K (x - (0, \beta)) \quad (55a)
\]
\[
\Leftrightarrow (0, 0) \in (0, \delta) + N_K (x - (a, \beta)) + N_K (x - (0, \beta)). \quad (55b)
\]
Set $\tilde{Y} = \{(x_1, \beta) \in \mathbb{R}^2 \mid x_1 \geq \max\{0, \alpha\}\}$. On the one hand, in view of (55) $(\forall (x_1, x_2) \in \tilde{Z})$ we must have $x_1 \geq \max\{0, \alpha\}$ and $x_2 = \beta$. Hence $\tilde{Z} \subseteq \tilde{Y}$. On the other hand, $(\forall x = (x_1, x_2) \in \tilde{Y})$ we have
\[
(0, 0) \in (0, \delta) + \{0\} \times \mathbb{R} + \{0\} \times \mathbb{R} \subseteq (0, \delta) + N_K (x - (a, \beta)) + N_K (x - (0, \beta)). \quad (56)
\]
Hence (55) yields $\tilde{Y} \subseteq \tilde{Z}$. Altogether, we conclude that (vii) holds.

(viii): Indeed, (vi) and (vii) imply that $\tilde{Z} = Z \Leftrightarrow \max\{\gamma, \alpha\} = \max\{0, \alpha\} \Leftrightarrow \alpha \geq 0$. ■

Example 4.6. Let $U$ be a closed linear subspace of $X$ and let $(a, b) \in U^\perp \times U$. Suppose that $(A, B) = (N_{a + U}, b + N_U)$ and set $\tilde{Z} := \{x \in X \mid 0 \in -v_R + Ax + B(x - v_D)\}$. Then the following hold:

(i) dom $A - \text{dom } B = a + U$. 

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(ii) \( \text{ran } A + \text{ran } B = b + U\perp \).

(iii) \((v_D, v_R) = (a, b) \in U\perp \times U \).

(iv) \( v = a + b \).

(v) \( Z = a + U \).

(vi) \( \tilde{Z} = Z \).

Proof. (i)&(ii): This is clear.

(iii): It follows from (i), (ii) and, e.g., [6, Proposition 3.19 and Corollary 3.24(iii)] that 
\( v_D = P_{a+U}(0) = a - Pu a = P_{U\perp} a = a \), and 
\( v_R = P_{b+U\perp}(0) = b - Pu b = Pu b = b \).

(iv): Combine (iii) and Proposition 3.3(vii). (v): Indeed, let \( x \in X \). Then (iv), (52), and the assumption that \( (a, b) \in U\perp \times U \) imply that
\[
x \in Z \iff 0 \in -a - b + N_{a+U}(x) + b + N_U(x-a-b) \tag{57a}
\]
\[
\iff a \in N_U(x-a) + N_U(x-a-b) \tag{57b}
\]
\[
\iff a \in U\perp, x-a \in U, \text{ and } x-a-b \in U \tag{57c}
\]
\[
\iff x \in a + U. \tag{57d}
\]

(vi): Let \( x \in Z = a + U \) by (v). Then
\[
-v_R + Ax + B(x - v_D) = -b + N_{a+U}(x) + b + N_U(x-a) = U\perp + U\perp = U\perp \ni 0.
\]
Hence, \( Z \subseteq \tilde{Z} \). The opposite inclusion follows from Theorem 4.3(i). □

5 Dynamic consequences

In this section, we analyze the shadow sequence \((J_A T^n x)_{n\in \mathbb{N}}\) with regards to Fejér monotonicity and conditions necessary for convergence. Recall that if \( x \in X \), then
\[
J_A T^n x - J_B R_A T^n x = J_{A^{-1}} T^n x + J_{B^{-1}} R_A T^n x = T^n x - T^{n+1} x \to v \tag{58}
\]
where the identities are consequences of (10) and the limit follows from, e.g., [18, Corollary 1.5].

Proposition 5.1. Let \( x \in X \). Then the following hold:

(i) \( ||J_A T^n x - J_A T^{n+1} x||^2 + ||J_{A^{-1}} T^n x - J_{A^{-1}} T^{n+1} x||^2 \to ||v||^2 = ||v_D||^2 + ||v_R||^2 \).

(ii) \( J_A T^n x - J_A T^{n+1} x \to v_R \).

(iii) \( J_{A^{-1}} T^n x - J_{A^{-1}} T^{n+1} x \to v_D \).
Proof. (i): Indeed, we have
\[ \left\| T^n x - T^{n+1} x \right\|^2 - \left\| R_A T^n x - R_A T^{n+1} x \right\|^2 \]
\[ = \left\| J_A T^n x - J_A T^{n+1} x \right\|^2 + \left\| J_{A^{-1}} T^n x - J_{A^{-1}} T^{n+1} x \right\|^2 \]
\[ + 2 \langle J_A T^n x - J_A T^{n+1} x, J_{A^{-1}} T^n x - J_{A^{-1}} T^{n+1} x \rangle \]
\[ - \left( \left\| J_A T^n x - J_A T^{n+1} x \right\|^2 + \left\| J_{A^{-1}} T^n x - J_{A^{-1}} T^{n+1} x \right\|^2 \right. \]
\[ \left. - 2 \langle J_A T^n x - J_A T^{n+1} x, J_{A^{-1}} T^n x - J_{A^{-1}} T^{n+1} x \rangle \right) \]
\[ = 4 \langle J_A T^n x - J_A T^{n+1} x, J_{A^{-1}} T^n x - J_{A^{-1}} T^{n+1} x \rangle \]
\[ \to 0, \]
where (59c) follows from [14, Corollary 4.2]. (59) and (58) imply that
\[ \left\| R_A T^n x - R_A T^{n+1} x \right\| \to \| v \|. \]

Next,
\[ \left\| J_A T^n x - J_A T^{n+1} x \right\|^2 + \left\| J_{A^{-1}} T^n x - J_{A^{-1}} T^{n+1} x \right\|^2 \]
\[ = \frac{1}{2} \left( \left\| J_A T^n x - J_A T^{n+1} x \right\|^2 + \left\| J_{A^{-1}} A^{-1} T^n x - J_{A^{-1}} T^{n+1} x \right\|^2 \right. \]
\[ + 2 \langle J_A T^n x - J_A T^{n+1} x, J_{A^{-1}} T^n x - J_{A^{-1}} T^{n+1} x \rangle \]
\[ \left. + \left\| J_A T^n x - J_A T^{n+1} x \right\|^2 + \left\| J_{A^{-1}} T^n x - J_{A^{-1}} T^{n+1} x \right\|^2 \right) \]
\[ \left. - 2 \langle J_A T^n x - J_A T^{n+1} x, J_{A^{-1}} T^n x - J_{A^{-1}} T^{n+1} x \rangle \right) \]
\[ = \frac{1}{2} \left( \left\| T^n x - T^{n+1} x \right\|^2 + \left\| R_A T^n x - R_A T^{n+1} x \right\|^2 \right) \]
\[ \to \frac{1}{2} \left( \| v \|^2 + \| v \|^2 \right) = \| v \|^2, \]
where (61d) follows from (58), (60), and Proposition 3.3(viii).

(ii)&(iii): In view of Proposition 3.3(viii), we rewrite (i) as
\[ \left\| (J_A T^n x - J_A T^{n+1} x, J_{A^{-1}} T^n x - J_{A^{-1}} T^{n+1} x) \right\|^2 \to \left\| (v_R, v_D) \right\|^2. \]

Lemma 2.2(i)&(ii) implies that the sequence \( (J_A T^n x - J_A T^{n+1} x, J_{A^{-1}} T^n x - J_{A^{-1}} T^{n+1} x) \)\( n \in \mathbb{N} \) lies in \( \text{ran } A + \text{ran } A \times (\text{dom } A - \text{dom } B) \subseteq (\text{ran } A + \text{ran } A) \times (\text{dom } A - \text{dom } B) \). Using, e.g., [30, Lemma 2] in view of (35) we learn that \( (J_A T^n x - J_A T^{n+1} x, J_{A^{-1}} T^n x - \)
Proposition 5.2 (shadow convergence: necessary condition). Let \( x \in X \). Then the following hold:

(i) \((J AT^n x)_{n \in \mathbb{N}}\) is asymptotically regular \(\iff v_R = 0\).

(ii) \((J A^{-1} T^n x)_{n \in \mathbb{N}}\) is asymptotically regular \(\iff v_D = 0\).

Proof. (i)&(ii): This is a direct consequence of Proposition 5.1(ii)&(iii).

Proposition 5.3. Suppose that \( f \in \text{Fix}(v + T) \), let \( x \in X \), and let \( n \in \mathbb{N} \). Then the following hold:

(i) \( T^n f = f - nv \).

(ii) \( J AT^n f = J Af - nv_R \).

(iii) \( J A^{-1} T^n f = J A^{-1} f - nv_D \).

(iv) \( J B R_A T^n f = J B R_A f - nv_R \).

(v) \( J B^{-1} R_A T^n f = J B^{-1} R_A f + nv_D \).

Proof. (i): See [13, Proposition 2.5(iv)].

(ii)&(iii): We proceed by induction. When \( n = 0 \) the statement is trivial and for \( n = 1 \) use (36b) and (36c). Now suppose that for some \( n \geq 1 \) it holds that \((J AT^n f, J A^{-1} T^n f) = (J Af - nv_R, J A^{-1} f - nv_D)\). Lemma 2.2(i)&(ii) applied with \( x \) replaced by \( T^n f \) yields that \( J AT^n f - J A^{-1} T^n f \in \text{ran } A + \text{ran } B \) and \( J A^{-1} T^n f - J A^{-1} T^n f \in \text{dom } A - \text{dom } B \). Now

\[
\| (v_R, v_D) \|^2 
\leq \| J AT^n f - J A T^n+1 f, J A^{-1} T^n f - J A^{-1} T^n+1 f \|^2
\]
\[= \| J AT^n f - J A T^n+1 f \|^2 + \| J A^{-1} T^n f - J A^{-1} T^n+1 f \|^2
\]
\[\leq \| J AT^n f - J A T^n+1 f \|^2 + \| J A^{-1} T^n f - J A^{-1} T^n+1 f \|^2
\]
\[+ 2 \langle J AT^n f - J A T^n+1 f, J A^{-1} T^n f - J A^{-1} T^n+1 f \rangle
\]
\[= \| T^n f - T^n+1 f \|^2 = \| f - nv - (f - (n+1)v) \|^2
\]
\[= \| f \|^2 = \| (v_D, v_R) \|^2.
\]

Thus, by definition of \( v_D \) and \( v_R \), we have \((J AT^n f - J A T^n+1 f, J A^{-1} T^n f - J A^{-1} T^n+1 f) = (v_R, v_D)\). Recalling the inductive hypothesis, we learn that

\[
(J A T^n+1 f, J A^{-1} T^n+1 f) = (J A T^n f, J A^{-1} T^n f) - (v_R, v_D)
\]
= (J_A f - (n + 1)v_R, J_{A^{-1}} f - (n + 1)v_D). \quad (64b)

(iv): Using (i) we have \( v = f - T f = (\text{Id} - T) f = J_A f - J_B R_A f \), hence \( J_A f = v + J_B R_A f \). Let \( n \in \mathbb{N} \). (i) and (ii) imply that \( v = T^n f - T^{n+1} f = (\text{Id} - T) T^n f = J_A T^n f - J_B R_A T^n f \). Hence, \( J_B R_A T^n f = -v + J_A T^n f = -v + J_A f - n v_R = -v + v + J_B R_A f - n v_R = J_B R_A f - n v_R. \)

(v): Using (i) we have \( v = f - T f = (\text{Id} - T) f = J_{A^{-1}} f + J_B^{-1} R_A f \), hence \( J_{A^{-1}} f = v - J_B^{-1} R_A f \). Let \( n \in \mathbb{N} \). (i) and (iii) imply that \( v = T^n f - T^{n+1} f = (\text{Id} - T) T^n f = J_{A^{-1}} T^n f + J_B^{-1} R_A T^n f \). Hence, \( J_B^{-1} R_A T^n f = v - J_{A^{-1}} T^n f = v - (J_{A^{-1}} f - n v_D) = v - v + J_B^{-1} R_A f + n v_D = J_B^{-1} R_A f + n v_D. \)

We omit the simple proof of the following result.

**Lemma 5.4.** Let \( s \in X \) and let \( C \) be a nonempty subset of \( X \). Suppose that \( (x_n)_{n \in \mathbb{N}} \) is Fejér monotone with respect to \( C \). Then \( (s + x_n)_{n \in \mathbb{N}} \) is Fejér monotone with respect to \( s + C \).

We are now ready for the main result in this section.

**Theorem 5.5.** Suppose that \( f \in \text{Fix}(v + T) \), let \( x \in X \), and let \( n \in \mathbb{N} \). Then the following hold:

(i) The sequence \( (J_A T^n x + n v_R, J_{A^{-1}} T^n x + n v_D)_{n \in \mathbb{N}} \) is Fejér monotone with respect to the set \( \{ (J_A \times J_{A^{-1}})(f, f) \mid f \in \text{Fix}(v + T) \} \).

(ii) The sequence \( (J_A T^n x + n v_R, J_{A^{-1}} T^n x + n v_D)_{n \in \mathbb{N}} \) is bounded.

(iii) The sequence \( (J_B R_A T^n x + n v_R, J_{B^{-1}} R_A T^n x - n v_D)_{n \in \mathbb{N}} \) is bounded.

(iv) The sequence \( ((0, -v) + (J_A T^n x + n v_R, J_{A^{-1}} T^n x + n v_D))_{n \in \mathbb{N}} \) is Fejér monotone with respect to the set \( S \).

(v) The sequence \( (J_A T^n x)_{n \in \mathbb{N}} \) is bounded \( \iff v_R = 0 \).

(vi) \( v_R \neq 0 \implies \|J_A T^n x\| \to +\infty \).

(vii) The sequence \( (J_{A^{-1}} T^n x)_{n \in \mathbb{N}} \) is bounded \( \iff v_D = 0 \).

(viii) \( v_D \neq 0 \implies \|J_{A^{-1}} T^n x\| \to +\infty \).

(ix) Suppose that \( A \) and \( B \) are paramonotone. Then the sequence \( ((0, -v) + (J_A T^n x + n v_R, J_{A^{-1}} T^n x + n v_D))_{n \in \mathbb{N}} \) is Fejér monotone with respect to \( Z \times K \).

**Proof.** (i)&(ii): Indeed, it follows from [14, Theorem 2.7(v)] that \( (\forall (x, y) \in X \times X) \)

\[
\|J_A T^{n+1} x - J_{A^{-1}} T^{n+1} y\|^2 + \|J_{A^{-1}} T^{n+1} x - J_A^{-1} T^{n+1} y\|^2 \\
\leq \|J_A T^n x - J_A T^n y\|^2 + \|J_A^{-1} T^n x - J_{A^{-1}} T^n y\|^2.
\]
Applying the above equation with $y$ replaced by $f \in \text{Fix}(v + T)$ and recalling Proposition 5.3(ii)&(iii) yields
\[
\|J_A T^{n+1}x - (J_A f - (n+1)v_R)\|^2 + \|J_{A^{-1}} T^{n+1}x - (J_{A^{-1}} f - (n+1)v_D)\|^2 \\
\leq \|J_A T^n x - (J_A f - nv_R)\|^2 + \|J_{A^{-1}} T^n x - (J_{A^{-1}} f - nv_D)\|^2. \tag{66}
\]
Rearranging yields
\[
\|J_A T^{n+1}x + (n+1)v_R - J_A f\|^2 + \|J_{A^{-1}} T^{n+1}x + (n+1)v_D - J_{A^{-1}} f\|^2 \\
\leq \|J_A T^n x + nv_R - J_A f\|^2 + \|J_{A^{-1}} T^n x + nv_D - J_{A^{-1}} f\|^2, \tag{67}
\]
and the conclusion follows.

(iii): Combine (ii) and (58).

(iv): Combine (i), Lemma 2.5(ii) and Lemma 5.4 applied with $s$ replaced by $(0, -v)$ and $C$ replaced by $\{(J_A \times J_{A^{-1}})(f, f) \mid f \in \text{Fix}(v + T)\}$.

(v)&(vi): "⇒": Indeed, observe that \(\|J_A T^n x\| \geq n\|v\| - \|J_A T^n x + nv_R\|\). "⇐": This is a direct consequence of (ii) applied with $v_R = 0$.

(vii)&(viii): "⇒": Indeed, observe that \(\|J_{A^{-1}} T^n x\| \geq n\|v\| - \|J_{A^{-1}} T^n x + nv_D\|\). "⇐": This is a direct consequence of (ii) applied with $v_D = 0$.

(ix): Combine (iv) and Fact 2.4(i). \hfill \blacksquare

Remark 5.6 (unbounded shadows). In view of Theorem 5.5(ii) and Proposition 3.3(viii), we learn that if \(\text{Fix} T = \emptyset\) (in particular, if $v \neq 0$), then at least one of the sequences $(J_A T^n x)_{n \in \mathbb{N}}$, $(J_{A^{-1}} T^n x)_{n \in \mathbb{N}}$ is unbounded.

Corollary 5.7 (boundedness of the primal shadows). Suppose that $v_R = 0$. Suppose that $f \in \text{Fix}(v + T)$, and let $x \in X$. Then the following hold:

(i) \((\forall n \in \mathbb{N}) \ J_A T^n f = J_A(f - nv_D) = J_A f\).

(ii) $(J_A T^n x)_{n \in \mathbb{N}}$ is bounded.

(iii) $(J_{A^{-1}} T^n x)_{n \in \mathbb{N}}$ is bounded.

Proof. (i): Combine Proposition 3.3(vii) and Proposition 5.3(i)&(ii) with $v_R = 0$. (ii): Apply Theorem 5.5(ii) with $v_R = 0$. (iii): Combine (ii) and (58). \hfill \blacksquare

Corollary 5.8 (boundedness of the dual shadows). Suppose that $v_D = 0$. Suppose that $f \in \text{Fix}(v + T)$, and let $x \in X$. Then the following hold:

(i) \((\forall n \in \mathbb{N}) \ J_{A^{-1}} T^n f = J_{A^{-1}}(f - nv_R) = J_{A^{-1}} f\).
(ii) \((J_{A^{-1}}T^n x)_{n \in \mathbb{N}}\) is bounded.

(iii) \((J_{B^{-1}}R_A T^n x)_{n \in \mathbb{N}}\) is bounded.

Proof. (i): Combine Proposition 3.3(vii) and Proposition 5.3(i)\&(iii) with \(v_D = 0\). (ii): Apply Theorem 5.5(ii) with \(v_D = 0\). (iii): Combine (ii) and (58).  

Corollary 5.9. Let \(x \in X\). Then the following hold:

(i) \(\frac{1}{n}(J_A T^n x) \to -v_R\).

(ii) \(\frac{1}{n}(J_{A^{-1}}T^n x) \to -v_D\).

Proof. This is a direct consequence of Theorem 5.5(ii).  

6 Shadow convergence!

In this section, we shall prove the main result announced in Section 1.1. We assume throughout that

\[ f \in \Gamma_0(X) \text{ and } g \in \Gamma_0(X), \]

that \((A, B) = (\partial f, \partial g)\), and (see Proposition 3.3(vii)) that

\[ v_R = 0 \iff v = P_{\text{ran}(\text{Id}-T)}(0) = P_{\text{dom}f - \text{dom}g}(0) = v_D. \]  

(68)

We remind the reader on our abbreviations

\[ (P_f, P_f^*, P_g, R_f) = (\text{Prox}_f, \text{Prox}_{f^*}, \text{Prox}_g, 2 \text{Prox}_f - \text{Id}). \]

Then

\[ T = T_{(\partial f, \partial g)} = \text{Id} - P_f + P_g R_f. \]  

(69)

Remark 6.1. Let \(x \in X\). In view of Proposition 5.2(i), applied with \((A, B)\) replaced by \((\partial f, \partial g)\), we learn that the assumption \(v_R = 0\) is necessary for the convergence of the shadow sequence \((P_f T^n x)_{n \in \mathbb{N}}\).

Because \(v = v_D\), the definition of \(v_D\) implies

\[ (\forall (a, b) \in \text{dom} f \times \text{dom} g) \quad \langle v, v - (a - b) \rangle \leq 0. \]  

(70)

The following result provides several descriptions of \(Z\) in the optimization setting:
Proposition 6.2. Recalling (14), we have:

(i) \( Z = \{ x \in X \mid 0 \in \partial f(x) + \partial g(x - v) \} \).
(ii) \( Z \neq \emptyset \Rightarrow Z = \arg\min_{x \in X} (-\langle x, v \rangle + f(x) + g(x - v)) = \arg\min_{x \in X} (f(x) + g(x - v)) \).

Proof. (i): Apply Theorem 4.3(ii), with \((A, B)\) replaced by \((\partial f, \partial g)\), and use (68). (ii): Combine (14), (i) and Proposition 2.7 applied twice with \((f, g)\) replaced by \((-\langle v, \cdot \rangle + f, g(\cdot - v))\) and by \((f, g(\cdot - v))\) respectively.

The next result is a key step towards our main result:

Lemma 6.3 (the prox lemma). Let \( x \in X \) and let \( y \in \text{dom } f \cap (v + \text{dom } g) \). Then the following hold:

\[
0 \geq \langle y - P_f x, v \rangle, \\
f(y) \geq f(P_f x) + \langle y - P_f x, v + P_f x \rangle, \\
g(y - v) \geq g(P_g R_f x) + \langle y - P_g R_f x - v, P_f x - P_g R_f x - v \rangle \\
+ \langle -y + v + P_g R_f x, v + P_f x \rangle.
\]

Proof. Applying (70) with \((a, b)\) replaced by \((P_f x, y - v) \in \text{dom } \partial f \times \text{dom } g \) yields \( \langle P_f x - (y - v) - v, 0 - v \rangle \leq 0 \); equivalently, \( \langle y - P_f x, v \rangle \leq 0 \), which is (71a). We now prove (71b). Indeed, the characterization of \( P_f \) and (71a) yield

\[
f(y) \geq f(P_f x) + \langle y - P_f x, x - P_f x \rangle \\
\geq f(P_f x) + \langle y - P_f x, P_f x \rangle + \langle y - P_f x, v \rangle \\
= f(P_f x) + \langle y - P_f x, v + P_f x \rangle.
\]

Finally, we turn to (71c). Indeed, the characterization of \( P_g \) and (70) applied with \((a, b)\) replaced by \((y, P_g R_f x) \) in (73c) yields

\[
g(y - v) \geq g(P_g R_f x) + \langle y - v - P_g R_f x, R_f x - P_g R_f x \rangle \\
= g(P_g R_f x) + 2 \langle y - P_g R_f x - v, v \rangle \\
+ \langle y - P_g R_f x - v, P_f x - P_g R_f x - v \rangle \\
- \langle y - P_g R_f x - v, v + P_f x \rangle \\
\geq g(P_g R_f x) + \langle y - P_g R_f x - v, P_f x - P_g R_f x - v \rangle \\
- \langle y - P_g R_f T x - v, v + P_f x \rangle.
\]

The proof is complete.
To make further progress, we recall that $v = v_D$ and we assume additionally from now on (see (17) and Lemma 4.1(ii)) that

\[ Z \neq \emptyset; \text{ thus, } v \in \text{ran}(\text{Id} - T) \cap (\text{dom } \partial f - \text{dom } \partial g). \]  

(74)

We recall that (58), applied with $(A, B)$ replaced by $(\partial f, \partial g)$, and of [6, Example 23.3] imply

\[ (\forall x \in X) \quad P_f T^n x - P g R_f T^n x = T^n x - T^{n+1} x \to v. \]  

(75)

**Proposition 6.4.** Let $y \in \text{dom } f \cap (v + \text{dom } g)$ and let $x \in X$. Set $(\forall n \in \mathbb{N})$

\[ \epsilon_n = \langle y - P g R_f T^n x - v, P_f T^n x - P g R_f T^n x - v \rangle, \]  

(76a)

\[ \delta_n = \langle P_f T^n x - P g R_f T^n x - v, P_f T^n x - (T^n x + nv) \rangle. \]  

(76b)

Then

\[ \epsilon_n \to 0 \quad \text{and} \quad \delta_n \to 0. \]  

(77)

Moreover,

\[ (\forall n \geq 1) \quad f(y) + g(y - v) \geq f(P_f T^n x) + g(P g R_f T^n x) + \epsilon_n + \delta_n. \]  

(78)

**Proof.** Recall that $(T^n x + nv)_{n \in \mathbb{N}}$ is a bounded sequence by [13, Proposition 2.5(vi)], that $(P_f T^n x)_{n \in \mathbb{N}}$ and $(P g R_f T^n x)_{n \in \mathbb{N}}$ are bounded sequences by Corollary 5.7(ii)&(iii) applied with $(A, B) = (\partial f, \partial g)$. Combining this with (75) proves (77).

We now turn to (78). Adding (71b) and (71c) applied with $x$ replaced by $T^n x$ yields

\[
\begin{align*}
f(y) + g(y - v) &\geq f(P_f T^n x) + \langle y - P_f T^n x, v + P_f T^n x \rangle + g(P g R_f T^n x) \\
&\quad + \epsilon_n - \langle y - P g R_f T^n x - v, v + P_f T^n x \rangle \\
&= f(P_f T^n x) + g(P g R_f T^n x) + \epsilon_n \\
&\quad + \langle v + P g R_f T^n x - P_f T^n x, v + P_f T^n x \rangle \\
&= f(P_f T^n x) + g(P g R_f T^n x) + \epsilon_n \\
&\quad + \langle v + P g R_f T^n x - P_f T^n x, v + T^n x + nv - P_f T^n x - nv \rangle \\
&= f(P_f T^n x) + g(P g R_f T^n x) + \epsilon_n + \delta_n \\
&\quad + (1 - n) \langle v - (P_f T^n x - P g R_f T^n x), v - 0 \rangle \\
&\geq f(P_f T^n x) + g(P g R_f T^n x) + \epsilon_n + \delta_n,
\end{align*}
\]

(79a)

(79b)

(79c)

(79d)

(79e)

where (79e) follows from applying (70) with $(a, b)$ replaced by $(P_f T^n x, P g R_f T^n x) \in \text{dom } \partial f \times \text{dom } \partial g$. 

**Proposition 6.5.** Set $\mu := \min_{x \in X}(f(x) + g(x - v))$ and let $x \in X$. Then the following hold:
(i) \((P_f T^n x)_{n \in \mathbb{N}}\) is bounded and its weak cluster points are minimizers of \(f + g(\cdot - v)\).

(ii) \((P_g R_f T^n x)_{n \in \mathbb{N}}\) is bounded and its weak cluster points are minimizers of \(f(\cdot + v) + g\).

Now let \(\bar{z}\) be a weak cluster point of \((P_f T^n x)_{n \in \mathbb{N}}\). Then:

(iii) \(f(P_f T^n x) \to f(\bar{z})\).

(iv) \(g(P_g R_f T^n x) \to g(\bar{z} - v)\).

(v) \(f(P_f T^n x) + g(P_g R_f T^n x) \to \mu\).

Proof. Set \((\forall n \geq 1) (p_n, q_n) = (P_f T^n x, P_g R_f T^n x)\).

(i): Corollary 5.7(ii)\&(iii) applied with \((A, B)\) replaced by \((\partial f, \partial g)\) imply that \((p_n)_{n \in \mathbb{N}}\) and \((q_n)_{n \in \mathbb{N}}\) are bounded sequences. Let \(\bar{z}\) be a weak cluster point of \((p_n)_{n \in \mathbb{N}}\) and observe that by (75) \(\bar{z} - v\) is a weak cluster point of \((q_n)_{n \in \mathbb{N}}\). Let \(y \in \text{dom } f \cap \text{dom } g(\cdot - v)\). The (weak) lower semicontinuity of \(f\) and \(g\) in view of (78) yields

\[
f(y) + g(y - v) \geq \liminf (f(p_n) + g(q_n)) \geq \lim f(p_n) + \lim g(q_n) \geq f(\bar{z}) + g(\bar{z} - v).
\]

This implies that

\[
\bar{z} \in \text{dom } f \cap \text{dom } g(\cdot - v).
\]

Observe that (78) implies that \(f(y) + g(y - v) \geq f(p_n) + g(q_n) + \epsilon_n + \delta_n\). Pick \((k_n)_{n \in \mathbb{N}}\) such that \(f(p_{k_n}) \to \lim f(p_n)\). Then

\[
f(y) + g(y - v) \geq \lim f(p_{k_n}) + \lim g(q_{k_n}) \geq \lim f(p_n) + \lim g(q_{k_n}) \geq f(\bar{z}) + g(\bar{z} - v).
\]

Setting \(y = \bar{z}\) in (82) yields

\[
\lim f(p_{k_n}) = \lim f(p_n) = \lim f(p_n).
\]

Hence \((f(p_n))_{n \in \mathbb{N}}\) converges. Similarly, we conclude that \((g(q_n))_{n \in \mathbb{N}}\) converges. Setting \(y = \bar{z}\) in (80) yields

\[
f(p_n) + g(q_n) \to f(\bar{z}) + g(\bar{z} - v) \geq \mu.
\]

Choosing \(y\) so that \(f(y) + g(y - v)\) is as close to \(\mu\) as desired, we see that (80) yields

\[
f(p_n) + g(q_n) \to f(\bar{z}) + g(\bar{z} - v) = \mu.
\]

Therefore, we conclude that \(\bar{z}\) is a minimizer of \(f + g(\cdot - v)\).

(ii): Combine (i) and (75).
(iii) & (iv): Observe that the lower semicontinuity of $f$ and $g$ respectively implies

$$f(\bar{z}) \leq \lim f(p_n) \quad \text{(86a)}$$
$$g(\bar{z} - v) \leq \lim g(q_n). \quad \text{(86b)}$$

Suppose for eventual contradiction that $f(c) < \lim f(p_n)$. Then (84) implies that $\lim g(q_n) > g(\bar{z} - v)$ which, by (86b), is absurd.

(v): Combine (iii), (iv), and (i).

**Proposition 6.6.** Let $x \in X$. Then the following hold:

(i) The sequence $(P_f T^n x)_{n \in \mathbb{N}}$ converges weakly to a minimizer of $f + g(\cdot - v)$.

(ii) The sequence $(P_g R_f T^n x)_{n \in \mathbb{N}}$ converges weakly to a minimizer of $f(\cdot + v) + g$.

**Proof.** (i): Recalling (68), it follows from Theorem 5.5(ix) applied with $(A, B)$ replaced by $(\partial f, \partial g(\cdot - v))$ that the sequence $(P_f T^n x, -v + P_f T^n x + n v)_{n \in \mathbb{N}}$ is Fejér monotone with respect to $Z \times K$. Now let $z_1$ and $z_2$ be two weak cluster points of $(P_f T^n x)_{n \in \mathbb{N}}$. On the one hand, Proposition 6.5(i) and Proposition 6.2(ii) imply that

$$\{z_1, z_2\} \subseteq \text{argmin}_{x \in X}(f + g(\cdot - v)); \text{ hence, } z_1 - z_2 \in Z - Z. \quad (87)$$

On the other hand, [8, Lemma 2.2]] implies that $z_1 - z_2 \in (Z - Z)^{\perp}$. Combining with (87) we conclude that $z_1 - z_2 \in (Z - Z) \cap (Z - Z)^{\perp} = \{0\}$. Hence, $z_1 = z_2$.

(ii): Combine (i) and (75).

**Proposition 6.7.** Suppose that $f = \iota_C$, where $C$ is a nonempty closed convex subset of $X$. Let $x \in X$. Then there exists $\bar{z} \in Z \subseteq C$ such that the following hold:

(i) $P_C T^n x \rightharpoonup \bar{z}$ and $\bar{z}$ is a minimizer of $g(\cdot - v)$ over $C$.

(ii) $P_Z P_C T^n x \rightharpoonup \bar{z}$.

**Proof.** (i): Clearly, $P_f = P_C$. Now combine with Proposition 6.6(i).

(ii): Applying Theorem 5.5(ix) with $(A, B)$ replaced by $(N_C, \partial g)$ we learn that the sequence $((P_C T^n x, (\text{Id} - P_C) T^n x + (n - 1)v))_{n \in \mathbb{N}}$ is Fejér monotone with respect to $Z \times K$. Combining this with [6, Proposition 5.7] we learn that $(\exists z, k) \in Z \times K$ such that $P_Z P_C T^n x, P_K((\text{Id} - P_C) T^n x + (n - 1)v) \rightarrow (z, k)$. In particular, $P_Z P_C T^n x \rightharpoonup z$. On the other hand, [5, Corollary 5.8] yields $P_Z P_C T^n x \rightharpoonup \bar{z}$. ■

**Example 6.8.** Suppose that $X = \mathbb{R}^m$, where $m \geq 1$. Let $b \in \mathbb{R}^m$, let $u \in \mathbb{R}^m \setminus \{0\}$, let $\eta \in \mathbb{R}$, set $B := \{x \in \mathbb{R}^m \mid \langle x, u \rangle \leq \eta\}$, and set $C := \{x \in \mathbb{R}^m \mid -c \leq x \leq c\}$, where $c \in [0, +\infty)^m$. 26
Suppose that \( f = \iota_B \) and that 
\[
g : \mathbb{R}^m \to [-\infty, +\infty] : x \mapsto \begin{cases} 
\sum_{i=1}^m |x_i|, & \text{if } x \in C; \\
+\infty, & \text{otherwise.}
\end{cases}
\] (88)

Let \( x \in X \). Then the following hold:

(i) \( \partial f = N_B \).
(ii) \( g = \|\cdot\|_1 + \iota_C \).
(iii) \( \partial g = \partial \|\cdot\|_1 + N_C \).
(iv) \( \text{dom } f - \text{dom } g = \text{dom } \partial f - \text{dom } \partial g = B - C = \overline{B - C} \).
(v) \( \text{dom } f^* + \text{dom } g^* = \mathbb{R}^m \).
(vi) \( \text{ran } (\text{Id} - T) = B - C \).
(vii) \( v_D = P_{B-C}(0) \in B - C \).
(viii) \( v_D = 0 \iff B \cap C \neq \emptyset \).
(ix) \( v_R = 0 \).
(x) \( v = v_D \).
(xi) \( Z = \text{argmin}_{y \in \mathbb{R}^m} (\iota_B(y) + g(y - v)) \neq \emptyset \).

(xii) \( P_f x = P_B x = \begin{cases} 
x, & \text{if } \langle x, u \rangle \leq \eta; \\
x + (\eta - \langle x, \eta \rangle)u/\|u\|^2, & \text{if } \langle x, u \rangle > \eta.
\end{cases} \)

(xiii) \( P_s x = (\xi_i)_{i=1}^n \), where \( \xi_i = \min\{\max\{|x_i| - 1, 0\}, c_i\} \text{ sign}(x_i) \).

(xiv) \( T = \text{Id} - P_B + P_s(2P_B - \text{Id}) \).

(xv) \( P_B T^n x \to \lim P_Z P_B T^n x \in Z \).

Proof. (i)&(ii): This is clear. 

(iii): This follows from combining [6, Corollary 16.48(iii) and Example 16.13].

(iv): Clearly, \( \text{dom } \partial f = \text{dom } N_B = \text{dom } \iota_B = B \). Moreover, \( \text{dom } g = \text{dom } \|\cdot\|_1 \cap \text{dom } \iota_C = \text{dom } \partial \|\cdot\|_1 \cap \text{dom } N_C = C \). Finally, observe that \( B \) is compact and \( C \) is closed; therefore, \( B - C \) is closed.

(v): Indeed, it follows from, e.g., [6, Corollary 21.25], and [32, Remark on page 216] that 
\[
\mathbb{R}^m = \text{ran } \partial g = \text{dom } \partial g^* \subseteq \text{dom } g^* \subseteq \mathbb{R}^m.
\] (89)

Hence, \( \text{dom } g^* = \mathbb{R}^m \) and the conclusion follows.
(vi): Combine (iv), (v), and Remark 2.3.

(vii)&(viii): Combine (iv) and (4b).

(ix): This is a direct consequence of (4b) and (v).

(x): Combine (ix) and Proposition 3.3(vii).

(xi): The first identity follows from combining (i), (iii) and Proposition 6.2(i). Next, observe that \( f \) and \( g \) are polyhedral functions. Moreover, (iv) and (vii) imply that \( \text{dom} f \cap (v + \text{dom} g) = B \cap (v + C) \neq \emptyset \). Therefore, [31, Theorem 23.18] yields that \( \partial(f + g(\cdot - v)) = \partial f + \partial g(\cdot - v) \). We learn that \( \partial f + \partial g(\cdot - v) \) is maximally monotone and, in view of [12, Theorem 3.13], that \( \text{ri ran} (\partial f + \partial g(\cdot - v)) = \text{ri ran} \partial f + \text{ri ran} \partial g \subseteq \text{ran} \partial f + \text{ran} \partial g \).

Observe that (89) implies that \( \text{ran} \partial g(\cdot - v) = \text{ran} \partial g = \mathbb{R}^m = \text{ri} \mathbb{R}^m = \text{ri ran} \partial g(\cdot - v) \).

Altogether, we learn that \( \text{ran} (\partial f + \partial g(\cdot - v)) = \mathbb{R}^m \), hence \( Z = \text{zer}(\partial f + \partial g(\cdot - v)) \neq \emptyset \). Now combine with Proposition 6.2(ii).

(xii): This follows from, e.g., [6, Proposition 29.20(iii)].

(xiii): This follows from [16, Example 6.23].

(xiv): This is (3) applied with \((A, B)\) replaced by \((\partial f, \partial g)\).

(xv): Apply Proposition 6.7 with \((C, g)\) replaced by \((B, \| \cdot \|_1 + t_C) \) $\blacksquare$

Having collected already all pieces required for its proof, we now summarize our work in the following:

**Theorem 6.9. (main result: primal shadows converge!)** Let \( f, g \) be in \( \Gamma_0(X) \). Suppose that \( 0 \in \text{dom} f^* + \text{dom} g^* \), that \( v \in \text{ran} (\text{Id} - T) \), and that \( \text{zer}(\partial f + \partial g(\cdot - v)) \neq \emptyset \). Set \( \mu := \min_{x \in X} (f(x + v) - g(x)) \). Let \( x \in X \). Then there exists a vector \( \overline{z} \in X \) such that the following hold:

(i) \( P_f T^n x \rightarrow \overline{z} \) and \( \overline{z} \) is a minimizer of \( f + g(\cdot - v) \).

(ii) \( P_g R_f T^n x \rightarrow \overline{z} - v \) and \( \overline{z} - v \) is a minimizer of \( f(\cdot + v) + g \).

(iii) \( f(P_f T^n x) + g(P_g R_f T^n x) \rightarrow f(\overline{z}) + g(\overline{z} - v) = \mu \).

(iv) \( f(P_f T^n x) \rightarrow f(\overline{z}) \).

(v) \( g(P_g R_f T^n x) \rightarrow g(\overline{z} - v) \).

**Proof.** Note that \( 0 \in \text{dom} f^* + \text{dom} g^* \subseteq \text{dom} f^* + \text{dom} g^* = \text{ran} \partial f + \text{ran} \partial g = \text{ran} \partial f + \text{ran} \partial g = \mathbb{R} \Rightarrow v_R = 0 \). Now combine with Proposition 6.2, Proposition 6.6, and Proposition 6.5(v)–(iv). $\blacksquare$

Finally, let us “dualize” Theorem 6.9 by applying it to \((f^*, g^*)\) to obtain that the dual
Corollary 6.10. (dual shadows converge) Let $f, g$ be in $\Gamma_0(X)$. Suppose that $0 \in \text{dom } f - \text{dom } g$, that $v \in \text{ran } (\text{Id} - T)$, and that $\text{zer}(\partial f^* + \partial (g^{*\vee})(\cdot - v)) \neq \emptyset$. Let $x^* \in X$. Then there exists a vector $\bar{k} \in X$ such that the following hold:

(i) $P_{f^*} T^n x \rightharpoonup \bar{k}$ and $\bar{k}$ is a minimizer of $f^* + g^{*\vee}(\cdot - v)$.

(ii) $P_{g^{*\vee}} R_{f^*} T^n x \rightharpoonup \bar{k} - v$ and $\bar{k} - v$ is a minimizer of $f^*(\cdot + v) + g^{*\vee}$.

(iii) $f^*(P_{f^*} T^n x) + g^{*\vee}(P_{g^{*\vee}} R_{f^*} T^n x) \to \min_{x^* \in X} (f^*(x^*) + g^{*\vee}(x^* - v))$.

(iv) $f^*(P_{f^*} T^n x) \to f^*(\bar{k})$.

(v) $g^*(- P_{g^{*\vee}} R_{f^*} T^n x) \to g^*(- \bar{k} + v)$.

Proof. Observe that, likewise $f$ and $g$, both $f^*$ and $g^{*\vee}$ are convex, lower semicontinuous and proper. Moreover, $(f^*)^* = f$, $(g^{*\vee})^* = g^{\vee}$. Hence

$$\text{dom}(f^*)^* + \text{dom}(g^{*\vee})^* = \text{dom } f - \text{dom } g.$$ (90)

Finally, observe that $(\partial f^*, \partial (g^{*\vee})) = ((\partial f)^{-1}, (\partial g)^{-\circ})$. Consequently, (69) and [21, Lemma 3.6 on page 133] implies that

$$T(\partial f^*, \partial g^{*\vee}) = T.$$ (91)

(i–v): Combine (90), (91) and Theorem 6.9 applied with $(f, g)$ replaced by $(f^*, g^{*\vee})$. ■

Acknowledgments

The research of HHB and WMM was partially supported by Discovery Grants of the Natural Sciences and Engineering Research Council of Canada.

References

[1] H. Attouch and M. Théra, A general duality principle for the sum of two operators, *Journal of Convex Analysis* 3 (1996), 1–24.

[2] G. Banjac, On the minimal displacement vector of the Douglas–Rachford operator, *Optimization Letters* 49 (2021), 197–200.

[3] G. Banjac, P. Goulart, B. Stellato, and S. Boyd, Infeasibility detection in the alternating direction method of multipliers for convex optimization, *Journal of Optimization Theory and Applications* 183 (2019), 490–519.
[4] G. Banjac and J. Lygeros, On the asymptotic behavior of the Douglas–Rachford and proximal-point algorithms for convex optimization, *Optimization Letters* (2021) https://doi.org/10.1007/s11590-021-01706-3

[5] H.H. Bauschke, R.I. Boţ, W.L. Hare, and W.M. Moursi, Attouch-Théra duality revisited: paramonotonicity and operator splitting, *Journal of Approximation Theory* 164 (2012), 1065–1084.

[6] H.H. Bauschke and P.L. Combettes, *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*, 2nd edition, Springer, 2017.

[7] H.H. Bauschke, P.L. Combettes, and D.R. Luke, Finding best approximation pairs relative to two closed convex sets in Hilbert spaces, *Journal of Approximation Theory* 127 (2004), 178–192.

[8] H.H. Bauschke, M.N. Dao, and W.M. Moursi, On Fejér monotone sequences and nonexpansive mappings, *Linear and Nonlinear Analysis* 1 (2015), 287–295.

[9] H.H. Bauschke, W.L. Hare, and W.M. Moursi, Generalized solutions for the sum of two maximally monotone operators, *SIAM Journal on Control and Optimization* 52 (2014), 1034–1047.

[10] H.H. Bauschke, W.L. Hare, and W.M. Moursi, On the range of the Douglas–Rachford operator, *Mathematics of Operation Research* 41 (2016), 884–897.

[11] H.H. Bauschke, M.M. Dao and W.M. Moursi, The Douglas–Rachford algorithm in the affine-convex case, *Operations research Letters* 44 (2016), 379–382.

[12] H.H. Bauschke, S.M. Moffat, and X. Wang, Near equality, near convexity, sums of maximally monotone operators, and averages of firmly nonexpansive mappings, *Mathematical Programming (Series B)* 139 (2013), 55–70.

[13] H.H. Bauschke and W.M. Moursi, The Douglas–Rachford algorithm for two (not necessarily intersecting) affine subspaces, *SIAM Journal on Optimization* 26 (2016), 968–985.

[14] H.H. Bauschke and W.M. Moursi, On the Douglas–Rachford algorithm, *Mathematical Programming (Series A)* 164 (2017), 263–284.

[15] H.H. Bauschke and W.M. Moursi, On the behaviour of the Douglas–Rachford algorithm for minimizing a convex function subject to a linear constraint, *SIAM Journal on Optimization* 30 (2020), 2559–2576.

[16] A. Beck, *First-Order Methods in Optimization*, SIAM 2017. https://doi.org/10.1137/1.9781611974997

[17] J.M. Borwein, S.B. Lindstrom, B. Sims, A. Schneider, and M.P. Skerritt, Dynamics of the Douglas–Rachford method for ellipses and $p$-spheres, *Set-Valued and Variational Analysis* 26 (2018), 385–403.

[18] R.E. Bruck and S. Reich, Nonexpansive projections and resolvents of accretive operators in Banach spaces, *Houston Journal of Mathematics* 3 (1977), 459–470.

[19] J. Douglas and H.H. Rachford, On the numerical solution of heat conduction problems in two and three space variables, *Transactions of the AMS* 82 (1956), 421–439.

[20] J. Eckstein and B.F. Svaiter, A family of projective splitting methods for the sum of two maximal monotone operators, *Mathematical Programming (Series B)* 111 (2008), 173–199.

[21] J. Eckstein, *Splitting Methods for Monotone Operators with Applications to Parallel Optimization*, Ph.D. thesis, MIT, 1989.
[22] D. Gabay, Applications of the method of multipliers to variational inequalities. In: *Augmented Lagrangian Methods: Applications to the Numerical Solution of Boundary-Value Problems* 15 (1983), 299–331, North-Holland, Amsterdam.

[23] A.N. Iusem, On some properties of paramonotone operators, *Journal of Convex Analysis* 5 (1998), 269–278.

[24] P.L. Lions and B. Mercier, Splitting algorithms for the sum of two nonlinear operators, *SIAM Journal on Numerical Analysis* 16 (1979), 964–979.

[25] Y. Liu, E.K. Ryu, and W. Yin, A new use of Douglas–Rachford splitting for identifying infeasible, unbounded, and pathological conic programs, *Mathematical Programming (Series A)* 177 (2019), 225–253.

[26] B. Mercier, *Inéquations Variationnelles de la Mécanique* (Publications Mathématiques d’Orsay, no. 80.01), Orsay, France: Université de Paris-XI, 1980. [http://portail.mathdoc.fr/PMO/PDF/M_MERCIER-87.pdf](http://portail.mathdoc.fr/PMO/PDF/M_MERCIER-87.pdf)

[27] G.J. Minty, Monotone (nonlinear) operators in Hilbert spaces, *Duke Mathematical Journal* 29 (1962), 341–346.

[28] W.M. Moursi, The Douglas–Rachford operator in the possibly inconsistent case: static properties and dynamic behaviour. Ph.D. thesis, University of British Columbia (2016). [https://doi.org/10.14288/1.0340501](https://doi.org/10.14288/1.0340501)

[29] W.M. Moursi and Y. Zinchenko, A note on the equivalence of operator splitting methods, in *Splitting Algorithms, Monotone Operator Theory, and Applications* (2019) 331–349, Springer.

[30] A. Pazy, Asymptotic behavior of contractions in Hilbert space, *Israel Journal of Mathematics* 9 (1971), 235–240.

[31] R.T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, 1970.

[32] R.T. Rockafellar, On the maximal monotonicity of subdifferential mappings, *Pacific Journal of Mathematics* 33 (1970), 209–216.

[33] R.T. Rockafellar and R.J-B Wets, *Variational Analysis*, Springer-Verlag, corrected 3rd printing, 2009.

[34] E.K. Ryu, Y. Liu, and W. Yin, Douglas–Rachford splitting and ADMM for pathological convex optimization, *Computational Optimization and Applications* 74 (2019), 747–778.

[35] B.F. Svaiter, On weak convergence of the Douglas-Rachford method, *SIAM Journal on Control and Optimization* 49 (2011), 280–287.

[36] E.H. Zarantonello, Projections on convex sets in Hilbert space and spectral theory, in *Contributions to Nonlinear Functional Analysis*, Academic Press, New York, 1971, pp. 237–424.