On the $t$–adic Littlewood Conjecture

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Abstract

The $p$–adic Littlewood Conjecture due to De Mathan and Teulié asserts that for any prime number $p$ and any real number $\alpha$, the equation

$$\inf_{|m| \geq 1} |m| \cdot |m|_p \cdot |\langle m\alpha \rangle| = 0$$

holds. Here, $|m|$ is the usual absolute value of the integer $m$, $|m|_p$ its $p$–adic absolute value and $|\langle x\rangle|$ denotes the distance from a real number $x$ to the set of integers. This still open conjecture stands as a variant of the well–known Littlewood Conjecture. In the same way as the latter, it admits a natural counterpart over the field of formal Laurent series $K((t^{-1}))$ of a ground field $K$. This is the so–called $t$–adic Littlewood Conjecture ($t$–LC).

It is known that $t$–LC fails when the ground field $K$ is infinite. This article is concerned with the much more difficult case when the latter field is finite. More precisely, a fully explicit counterexample is provided to show that $t$–LC does not hold in the case that $K$ is a finite field with characteristic 3. Generalizations to fields with characteristics different from 3 are also discussed.

The proof is computer assisted. It reduces to showing that an infinite matrix encoding Hankel determinants of the Paper–Folding sequence over $F_3$, the so–called Number Wall of this sequence, can be obtained as a two–dimensional automatic tiling satisfying a finite number of suitable local constraints.

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1 Introduction

Let $x$ be a real number. Denote by $|x|$ its usual absolute value and by $|\langle x \rangle|$ its distance to the set of integers. The famous Littlewood Conjecture from the 1930’s states that for any two real numbers $\alpha$ and $\beta$, the following equation holds:

$$\inf_{m \neq 0} |m| \cdot |\langle m\alpha \rangle| \cdot |\langle m\beta \rangle| = 0,$$

where the infimum is taken over all non–zero integers. The best–known result towards this conjecture is due to Einsiedler, Katok and Lindenstrauss [18] who established that the set of possible counterexamples has Hausdorff dimension zero. It is, however, not even known whether the pair of quadratic irrationalities $(\alpha, \beta) = (\sqrt{2}, \sqrt{3})$ satisfies (1.1).

De Mathan and Teulié [16] suggested a variant of the Littlewood Conjecture which has since then been known as the $p$–adic Littlewood Conjecture. According to the latter, given a prime number $p$ and a real number $\alpha$,

$$\inf_{m \neq 0} |m| \cdot |m|_p \cdot |\langle m\alpha \rangle| = 0.$$ (1.2)

Here, $|m|_p$ stands for the $p$–adic absolute value of the integer $m$. Upon writing $m = p^{\nu_p(m)}n$, where $\nu_p(m)$ denotes the $p$–adic valuation of $m$ and where $n$ is an integer, (1.2) amounts to the following one:

$$\inf_{n \neq 0, k \geq 0} |n| \cdot |\langle np^k\alpha \rangle| = 0.$$ (1.3)

(Note that it is not required in (1.3) that the integer $n$ should not be divisible by the prime $p$. It is easy to see that this does not affect the claimed equivalence between the two formulations of the problem.) Akin to the Littlewood Conjecture, it is known thanks to the work of Einsiedler and Kleinbock [19] that the set of possible exceptions to the $p$–adic Littlewood Conjecture has Hausdorff dimension zero.

For detailed accounts on the Littlewood and the $p$–adic Littlewood Conjectures, see [4, 7, 33] and the references therein.

Both of these conjectures admit natural counterparts over function fields, which have attracted much attention. In order to state them, some terminology and notation are first introduced.
Let \( K \) be a field. Denote by \( K[t] \) the ring of polynomials with coefficients in \( K \), and by \( K(t) \) the field of rational functions over \( K \). The valuation on \( K[t] \) given by the degree of a polynomial extends to a valuation on \( K(t) \) so as to provide an absolute value given by

\[
|\Theta| = c_K^{\deg \Theta}
\]

for any \( \Theta \in K(t) \), where

\[
c_K = \begin{cases} 
q & \text{if } K \text{ is the finite field with } q \text{ elements;} \\
e & \text{if } K \text{ is infinite.}
\end{cases}
\]

The completion of the field of rational functions is then the field of formal Laurent series, which will be denoted by \( K((t)) \). Explicitly, an element \( \Theta \in K((t)) \) can be uniquely expressed as a power series with at most finitely many non-zero coefficients corresponding to positive powers of \( t \); that is, it can be uniquely expressed as

\[
\Theta = \theta_{-h}t^h + \cdots + \theta_1t + \theta_0 + \theta_1t^{-1} + \theta_2t^{-2} + \ldots,
\]

where \( (\theta_i)_{i \geq -h} \) is a sequence in \( K \) such that \( \theta_{-h} \neq 0 \). The degree of the Laurent series \( \Theta \) is then the integer \( h \) and its absolute value the quantity \( |\Theta| = c_K^h \). Furthermore, one defines the fractional part of \( \Theta \) as

\[
\langle \Theta \rangle = \theta_1t^{-1} + \theta_2t^{-2} + \ldots;
\]

that is, as \( \Theta \) minus its polynomial part \( \theta_{-h}t^h + \cdots + \theta_1t + \theta_0 \).

With the above notation, the Littlewood Conjecture over Function Fields (LCFF), due to Davenport and Lewis [15], can be stated in complete analogy with the real case as follows: for any \( \Theta \) and \( \Phi \) in \( K((t^{-1})) \), the equation

\[
\inf_{N \neq 0} |N| \cdot |\langle N\Theta \rangle| \cdot |\langle N\Phi \rangle| = 0
\]

holds. Here, the infimum is taken over all non-zero elements in \( K[t] \). In the same vein, De Mathan and Teulié [16] enunciated the \( t \)-adic Littlewood Conjecture (\( t \)-LC), which is the analogue over function fields of the \( p \)-adic Littlewood Conjecture: for any \( \Theta \) in \( K((t^{-1})) \), the equation

\[
\inf_{N \neq 0, h \geq 0} |N| \cdot |\langle N_t^h \Theta \rangle| = 0
\]

holds. Note that in this statement the variable \( t \) plays the role of the prime number \( p \) in the real case, which is justified by the fact that it can be viewed as an irreducible element in the ring \( K[t] \).

The recent work of Einsiedler, Lindenstrauss and Mohammadi [17] implies that the possible set of exceptions to LCFF and to \( t \)-LC both have zero Hausdorff dimension, akin to the real case. In the case of LCFF, Davenport and Lewis [15] established that the set of exceptions is never empty when the ground field \( K \) is infinite. Their work was complemented by that of Baker [6] and several other authors [10, 13, 14, 15] who provided explicit counterexamples in this case. When \( K \) is finite, a recent progress is due to Kwon [24]: using ideas from homogeneous dynamics, he proved that the LCFF fails if and only if certain triangular tiling of the plane exist, but his proof falls short of providing such tilings. In the other direction, see [1] for explicit constructions of pairs of power series satisfying LCFF.

As for \( t \)-LC, De Mathan and Teulié [16] established the analogue of the above-cited result by Davenport and Lewis by proving that the conjecture fails when the ground field \( K \) is infinite. Bugeaud and De Mathan [8] later provided explicit counterexamples in this case (they also gave

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2In a more general version of the conjecture, one may replace \( t \) with any irreducible polynomial in \( K[t] \).
examples of power series satisfying the conjecture in any characteristic). However, no result seems
to be known in order to decide whether the set of exceptions to $t$–LC is empty or not when $K$
is finite.

The aim of the present work is to fill up this gap. More precisely, the following theorem shows
that $t$–LC fails over the field with three elements $F_3$. It provides an explicit
counterexample defined from the Paper–Folding sequence $(f_n)_{n \geq 1}$. Among many other ways, the latter sequence
(also known as the Dragon Curve Sequence) can be defined by setting

$$f_n = \begin{cases} 0 & \text{if } k \equiv 1 \pmod{4}; \\ 1 & \text{if } k \equiv 3 \pmod{4}, \end{cases} \quad (1.7)$$

where $n = 2^{\nu_2(n)}k$ is a non–zero integer. For accounts on some of the properties enjoyed by this
sequence see [3, §6.5] and [31].

**Theorem 1.1.** The $t$–adic Littlewood Conjecture fails over $F_3$. Indeed, the Laurent series

$$\Phi = \sum_{n=1}^{+\infty} f_n t^{-n},$$

where $(f_n)_{n \geq 1}$ is the Paper–Folding sequence seen as a sequence defined over $F_3$, is such that

$$\inf_{N \not= 0, k \geq 0} |N| \cdot |(N t^k \Phi)| = 3^{-4}.$$  

If $\Theta$ is a power series in $F_3 ((t^{-1}))$, it can also be seen as an element of $F_q ((t^{-1}))$, where
$q = 3^s$ with $s \geq 1$. Furthermore, it follows from (1.4) and (1.5) that its absolute value over the
latter field is the $s^\text{th}$ power of its absolute value over the former field. Combined with the result
of De Mathan and Teulié [16] for the case when the ground field is infinite, these two observations
lead one to the following corollary:

**Corollary 1.2.** The $t$–adic Littlewood Conjecture fails over any ground field with characteristic 3.

2 Reduction of the Problem to the Vanishing of certain Hankel Determinants

The results established in this section are valid over any ground field $K$. Let $\Theta \in K ((t^{-1}))$. Given
the formulation of $t$–LC, one may restrict oneself without loss of generality to the case when the
polynomial part of $\Theta$ vanishes. Write

$$\Theta = \sum_{n=1}^{+\infty} \theta_n t^{-n}, \quad (2.1)$$

where $(\theta_n)_{n \geq 1}$ is a sequence in $K$ which, in what follows, is identified with the power series $\Theta$
itself.

Define the infinite Hankel matrix $H_\Theta$ formed from the power series $\Theta$ as the matrix $H_\Theta =
(\theta_{i+j-1})_{i,j \geq 1}$; that is, as

$$H_\Theta = \begin{pmatrix} \theta_1 & \cdots & \theta_k & \cdots & \theta_n & \cdots \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots \\ \theta_k & \cdots & \theta_n & \cdots & \theta_p & \cdots \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots \\ \theta_n & \cdots & \theta_p & \cdots & \theta_q & \cdots \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots \end{pmatrix}.$$  

To be precise, the (regular) Paper–Folding sequence is more commonly defined in the literature as the sequence
$(1 - f_n)_{n \geq 1}$. This corresponds to the coding defined by $0 \mapsto 1$ and $1 \mapsto 0$ applied to $(f_n)_{n \geq 1}$. It will be slightly
more convenient for us to work with definition (1.7)
Given indices \( n \geq 1 \) and \( l, m \geq 0 \), let \( H_\Theta(n; l, m) \) denote the finite rectangular \((l + 1) \times (m + 1)\) truncation of the previous matrix with top–left entry \( \theta_n \); that is,

\[
H_\Theta(n; l, m) = \begin{pmatrix}
\theta_n & \cdots & \theta_{n+m} \\
\vdots & \ddots & \vdots \\
\theta_{n+m} & \cdots & \theta_{n+2m} \\
\vdots & \ddots & \vdots \\
\theta_{n+l-m} & \cdots & \theta_{n+l} \\
\vdots & \ddots & \vdots \\
\theta_{n+l} & \cdots & \theta_{n+l+m}
\end{pmatrix}
\]

(this representation corresponds to the case that \( l \geq m \) and is easily adapted to the case that \( m \geq l \)). When \( l = m \), one will more conveniently set

\[
H_\Theta(n; l, l) = H_\Theta(n; l).
\]

**Definition 2.1.** Let \( \delta \geq 2 \) be an integer. The sequence \((\theta_n)_{n \geq 1}\) of elements in \( \mathbb{K} \) is said to have deficiency \( \delta \) if there exists integers \( n \geq 1 \) and \( l \geq 0 \) such that the \( \delta - 1 \) matrices

\[
H_\Theta(n; l), H_\Theta(n; l + 1), \ldots, H_\Theta(n; l + \delta - 3), H_\Theta(n; l + \delta - 2)
\]

are singular but such that in any sequence of \( \delta \) matrices of the form

\[
H_\Theta(n; l), H_\Theta(n; l + 1), \ldots, H_\Theta(n; l + \delta - 2), H_\Theta(n; l + \delta - 1)
\]

(where \( n \geq 1 \) and \( l \geq 0 \)), at least one of them is non–singular.

If, for any \( n \geq 1 \) and \( l \geq 0 \), none of the matrices \( H_\Theta(n; l) \) is singular, the sequence \((\theta_n)_{n \geq 1}\) is said to have deficiency 1. It is said to have unbounded deficiency if for any integer \( \delta \geq 2 \), there exist indices \( n \geq 1 \) and \( l \geq 0 \) such that all the matrices \((2.2)\) are singular.

Say that two finite square submatrices of \( H_\Theta \) (corresponding to consecutive row and column indices) are nested if they share the same top left entry and if one can be obtained from the other by the addition of a row at the bottom and a column to the right. With this terminology, the sequence \((\theta_n)_{n \geq 1}\) having deficiency \( \delta \geq 2 \) means that one can find a sequence of \( \delta - 1 \) singular submatrices in \( H_\Theta \) such that any two successive elements in this sequence are nested; furthermore, there is no sequence of \( \delta \) singular submatrices \( H_\Theta \) enjoying this property. Unbounded deficiency means that arbitrarily long sequences of singular nested submatrices exist in \( H_\Theta \) and deficiency 1 just means that \( H_\Theta \) contains no singular submatrix.

The following theorem reduces t–LC to considerations of deficiency of sequences in \( \mathbb{K} \).

**Theorem 2.2.** Let \((\theta_n)_{n \geq 1}\) be a sequence in \( \mathbb{K} \) identified with the power series \( \Theta \in \mathbb{K}((t^{-1})) \) as in \([2.1]\). Then \( \Theta \) satisfies equation \([1.6]\) (that is, the \( t \)-adic Littlewood Conjecture over \( \mathbb{K} \) is true for \( \Theta \)) if and only if the sequence \((\theta_n)_{n \geq 1}\) has unbounded deficiency.

Furthermore, \( \Theta \) has deficiency \( \delta \geq 1 \) if and only if

\[
\inf_{N \neq 0, k \geq 0} \frac{|N| \cdot |\langle N^k \Theta \rangle|}{|\langle N^k \Theta \rangle|} = c_\Theta^{-\delta},
\]

(2.3)

where \( c_\Theta \) is the constant defined in \([1.5]\).

**Proof.** Let \( N = a_h t^h + \ldots + a_1 t + a_0 \) be a non–zero polynomial of degree \( h \geq 0 \) with coefficients in \( \mathbb{K} \). Let \( k \geq 0 \) and \( l \geq 1 \) be integers. Clearly,

\[
|N| \cdot |\langle N^k \Theta \rangle| < c_\Theta^{-l} \quad \text{if and only if} \quad |\langle N^k \Theta \rangle| < c_\Theta^{-(l+h)}.
\]

(2.4)
The fractional part on the left-hand side of the latter inequality can be expanded as follows:

\[
\langle N t^k \theta \rangle = \left( a_0 t^{k+1} + a_{h-1} t^{k+1-1} + \ldots + a_1 t^{k+1} + a_0 t^k \right) \times
\left( \theta_1 t^{-1} + \theta_2 t^{-2} + \ldots + \theta_m t^{-m} + \ldots \right)
\]

\[
= t^{-1} \cdot (a_0 \theta_{k+1} + a_1 \theta_{k+2} + a_2 \theta_{k+3} + \ldots + a_h \theta_{k+h+1}) +
\]

\[
t^{-2} \cdot (a_0 \theta_{k+2} + a_1 \theta_{k+3} + \ldots + a_{h-1} \theta_{k+h+1} + a_h \theta_{k+h+2}) +
\]

\[
\ldots +
\]

\[
t^{-u} \cdot (a_0 \theta_{k+u} + \ldots + a_h \theta_{k+h+u}) +
\]

\[
\ldots
\]

The second inequality in (2.4) means that the coefficients of \(t^{-1}, \ldots, t^{-(l+h)}\) in the above expansion all vanish. Defining \(\mathbf{a}\) as the transpose of the row vector \((a_0, \ldots, a_h)\), this can be restated as follows:

\[
H_\Theta (k + 1; h + l - 1, h) \cdot \mathbf{a} = 0. \quad \text{(2.5)}
\]

Note that \(H_\Theta (k + 1; h + l - 1, h)\) is a rectangular matrix with dimensions \((h + l) \times (h + 1)\). Therefore, equation (2.5) holds for some non-zero vector \(\mathbf{a}\) if and only if this matrix does not have maximal rank \(h + 1\). This shows that

\[
(2.4) \iff \text{rank}(H_\Theta (k + 1; h + l - 1, h)) < h + 1. \quad \text{(2.6)}
\]

The remainder of the proof is split into two parts in order to establish the following claim: the first inequality in (2.4) holds for some integers \(l \geq 1\) and \(k \geq 0\) and some non-zero polynomial \(N\) if and only if the sequence \((\theta_n)_{n \geq 1}\) has deficiency at least \(l + 1\). The statements in Theorem 2.2 then clearly follow from this equivalence and from the definition of the deficiency of a sequence.

To begin with, assume that the first inequality in (2.4) holds for some integers \(l \geq 1\) and \(k \geq 0\) and some non-zero polynomial \(N\) of degree \(h \geq 0\). The argument to prove that the sequence \((\theta_n)_{n \geq 1}\) has deficiency at least \(l + 1\) is straightforward: the rank condition (2.6) means the first \(h + 1\) columns of the \((h + l) \times (h + 1)\) matrix \(H_\Theta (k + 1; h + l - 1, h)\) are linearly dependent. Extend this matrix to the square matrix \(H_\Theta (k + 1; h + l - 1)\) with dimensions \((h + l) \times (h + l)\). Then consider the submatrices \(H_\Theta (k + 1; m)\) with \(m = h, \ldots, h + l - 1\) (that is, the matrices obtained as the successive upper-left square submatrices of \(H_\Theta (k + 1; h + l - 1)\) starting from the one with dimensions \((h + 1) \times (h + 1)\)). These submatrices are all singular as their first \(h + 1\) columns satisfy the same relation of linear dependency as the columns of the initial rectangular matrix \(H_\Theta (k + 1; h + l - 1, h)\). As they are nested and as there are \(l\) of them, this proves the claim concerning the deficiency of the sequence \((\theta_n)_{n \geq 1}\).

Conversely, assume that the sequence \((\theta_n)_{n \geq 1}\) has deficiency at least \(l + 1\). The goal is to prove that the first inequality in (2.4) holds for some integer \(k \geq 0\) and some non-zero polynomial \(N\). This amounts to proving the existence of integers \(k, h \geq 0\) such that the rank condition (2.6) holds for the given integer \(l \geq 1\). The argument is more involved and requires the following lemma, which can be found in [20, §10]. The proof is reproduced here for the sake of completeness as it is rather short.

**Lemma 2.3.** Let \(H = (c_{i+j-1})_{1 \leq i, j \leq n}\) be an \(n \times n\) Hankel matrix with entries in \(\mathbb{K}\). Assume that the first \(r\) columns of \(H\) are linearly independent but that the first \(r + 1\) columns are linearly dependent (here, \(1 \leq r \leq n - 1\)). Then the principal minor of order \(r\), that is, \(\det \left( (c_{i+j-1})_{1 \leq i, j \leq r} \right)\), does not vanish.
Proof. Denote by $C_1, \ldots, C_n$ the columns of the matrix $H$ under consideration. By assumption, $C_1, \ldots, C_r$ are linearly independent whereas $C_{r+1}$ can be expressed as a linear combination of the latter:

$$C_{r+1} = \sum_{s=1}^{r} \alpha_s C_s,$$

where $\alpha_1, \ldots, \alpha_s$ are coefficients in the field $\mathbb{K}$. From the Hankel structure of the matrix, this implies that the entries of the matrix $H$ satisfy the recurrence relation

$$c_k = \sum_{s=1}^{r} \alpha_s c_{k-s}$$

valid for all $k = r+1, \ldots, r+n$. Write

$$(C_1, \ldots, C_r) = \begin{pmatrix} c_1 & c_2 & c_3 & \cdots & c_r \\ c_2 & c_3 & c_4 & \cdots & c_{r+1} \\ c_3 & c_4 & c_5 & \cdots & c_{r+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{n-1} & c_n & c_{n+1} & \cdots & c_{r+n-2} \\ c_n & c_{n+1} & c_{n+2} & \cdots & c_{r+n-1} \end{pmatrix},$$

which is a matrix of rank $r$ by assumption. It follows from the recurrence relations \[2.7\] that each of the rows of this matrix depends linearly on the preceding $r$ rows, hence on the first $r$ ones. Since the matrix has rank $r$, this implies that the first $r$ rows must be linearly independent; that is, that $\det \left( (c_{i+j-1})_{1 \leq i,j \leq r} \right)$ does not vanish, as was to be proved.

Since it is assumed that the sequence $(\theta_n)_{n \geq 1}$ has deficiency at least $l+1$, its infinite Hankel matrix $H_\theta$ contains $l$ nested singular submatrices, say $H_\theta (k+1; h+m)$, where $k, h \geq 0$ and where $m = 0, \ldots, l-1$. Since the largest of these matrices, viz. $H := H_\theta (k+1; h+l-1)$, is singular, its $h+l$ columns are linearly dependent. Since the second largest matrix $H_\theta (k+1; h+l-2)$ is also singular, Lemma 2.3 implies that the first $h+l-1$ columns of the initial matrix $H$ cannot be linearly independent. Furthermore, since the third largest matrix $H_\theta (k+1; h+l-3)$ is also singular, the same lemma implies that the first $h+l-2$ columns of the initial matrix $H$ cannot be linearly independent. An easy induction then shows that the first $h+1$ columns of the matrix $H$ are linearly dependent; that is, that rank $(H_\theta (k+1; h+l-1, h)) < h+1$. The equivalence stated in \[2.6\] then shows that the first inequality in \[2.4\] holds, as was to be established.

This completes the proof of Theorem 2.2.

Remark 2.4. It is worthwhile to mention another shorter but less self-contained proof of Theorem 2.2. A best approximation degree of $\Theta \in \mathbb{K} ((t^{-1}))$ is an integer $h$ such that there exists a polynomial $N$ of degree $h$ with

$$|\langle N \Theta \rangle| < \inf_{\deg M < h} |\langle M \Theta \rangle|.$$

Let $(h_m)_{m=0}^\infty$ be the sequence of best approximation degrees of $\Theta$ and let $(N_m)_{m=0}^\infty$ be the associated best approximation polynomials. It is known \[37, 89\] that

$$|\langle N_m \Theta \rangle| = c^K_{h_{m+1}}.$$

Therefore,

$$|N_m| |\langle N_m \Theta \rangle| = c^K_{(h_{m+1}-h_m)}.$$
Given $\Theta \in \mathbb{K}\left((t^{-1})\right)$, let $h_{k,m}$ be the $m$th best approximation degree of $t^k\Theta$. Then
\[
\inf_{N \neq 0, k \geq 0} |N| \left| \langle N t^k \Theta \rangle \right| = \inf_{m \geq 0, k \geq 0} |N_{k,m}| \left| \langle N_{k,m} t^k \Theta \rangle \right| = c_k^{-\alpha(\Theta)},
\]
where
\[
\alpha(\Theta) = \sup_{m \geq 0, k \geq 0} (h_{k,m+1} - h_{k,m}).
\]
On the other hand, a normal index for $\Theta$ is an integer $h$ such that the matrix $H_\Theta(1; h)$ is invertible. It is a standard fact in the theory of Padé approximation that easily follows from \([2,8]\) that $h$ is a normal index if and only if $h$ is a best approximation degree (see, e.g., \([21, Proposition 2]\) for details). Therefore, by Definition \([2.1]\) the deficiency of $\Theta$ is precisely the quantity $\alpha(\Theta)$ defined above.

**Remark 2.5.** Continuing the previous remark, note that $h_{k,m+1} - h_{k,m}$ is precisely the degree of the rational fraction $N_{k,m+1}/N_{k,m}$, which is also the degree of the $m$th polynomial in the continued fraction expansion of $t^k\Theta$ (see \([37, \S9]\) for an account on the theory of continued fractions in $\mathbb{K}\left((t^{-1})\right)$). Theorem \([2.2]\) may then be rephrased in this language as follows: equation \([2.3]\) holds if and only if the maximal degree of the polynomials in the continued fraction expansions of all the Laurent series $\Theta, t^0\Theta, t^1\Theta, \ldots$ is $\delta$.

In view of Theorem \([2.2]\) Theorem \([1.1]\) becomes an immediate corollary of the following statement, which will be established in the next sections:

**Theorem 2.6.** The Paper–Folding sequence $(f_n)_{n \geq 1}$ has deficiency 4 over $\mathbb{F}_3$.

Considerations of deficiency are ubiquitous in the literature due to their connections with linear recurrence sequences, Padé approximations and problems of irrationality. The results known in this topic are nevertheless rather limited.

When $\mathbb{K} = \mathbb{R}$, it is not hard to construct a sequence with deficiency 1 by requiring that it should increase sufficiently fast. The situation turns out to be much more complicated in the case that one has to determine the deficiency of a given sequence. The fundamental work by Allouche, Peyrière, Wen and Wen \([2]\) establishes, with the help of sixteen recurrence relations, that the principal minors of the infinite Hankel matrix of the Thue–Morse sequence never vanish. Coons \([11]\) obtained a result with a similar flavour in the case of sequences defined from the sum of the reciprocals of the Fermat numbers. A combinatorial and simpler proof of both of these results was later provided by Bugeaud and Han \([9]\). Coons and Vrbik \([12]\) also considered the case of the Paper–Folding sequence over $\mathbb{R}$ and established computationally that a large (finite) number of the principal minors of its Hankel matrix does not vanish.

The problem is even less understood over finite fields. The only reasonably complete result seems to be \([2]\), where it is shown that some sequences defined as Hankel determinants of the Thue–Morse sequence over $\mathbb{F}_2$ are 2-automatic. In the same paper are also considered the determinants of the successive nested submatrices sharing a common top left entry in the Hankel matrix of the Thue–Morse sequence. It is proved that the sequence thus defined is periodic and not constantly equal to zero modulo 2. This, however, does not mean that the Thue–Morse sequence has bounded deficiency over $\mathbb{F}_2$, as the period may vary with the choice of the top–left entry. In fact, as the generating function of the Thue–Morse sequence is quadratic over $\mathbb{F}_2\left(\left(t^{-1}\right)\right)$ (see, e.g., \([5, Equation (9)]\)), the arguments presented in Section \([7.2]\) below together with Theorem \([2.2]\) above imply that the Thue–Morse sequence has unbounded deficiency over $\mathbb{F}_2$.

A remarkable exception to the current poor understanding of the structure of determinants formed from square submatrices in the infinite Hankel matrix of given sequences is due to Kamae, Tamura and Wen \([21]\). These authors consider the Fibonacci word over an alphabet $\{a, b\}$. Substituting $(a, b) = (1, 0)$ or $(a, b) = (0, 1)$ determines two real sequences and the corresponding infinite Hankel matrices. Rather explicit formulae are provided in \([21]\) for the determinant of any square submatrix sitting in the latter matrices. These formulae imply, in particular, that the Fibonacci sequences defined this way have unbounded deficiency. To the best of the authors’ knowledge, this,
together with the results of the present paper, constitutes the only cases when the determinantal structure of infinite Hankel matrices is completely elucidated for a given eventually non-periodic sequence.

In what follows, Theorem 2.6 will be proved by introducing the concept of a Number Wall, which is an array containing information about the Hankel determinants formed from a given sequence.

3 The Number Wall of a Sequence

3.1 Definition and Properties

Let $\Theta = (\theta_n)_{n \in \mathbb{Z}}$ be a doubly-infinite sequence defined over a field $K$. The Number Wall of this sequence is a two-dimensional array $S(\Theta) = (S_{m,n}(\Theta))_{m,n \in \mathbb{Z}}$ defined as follows: for any $m, n \in \mathbb{Z}$,

$$S_{m,n}(\Theta) = \left| \begin{array}{cccc} \theta_n & \theta_{n+1} & \cdots & \theta_{n+m-1} & \theta_{n+m} \\ \theta_{n-1} & \theta_n & \cdots & \theta_{n+m-2} & \theta_{n+m-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \theta_{n-m+1} & \theta_{n-m+2} & \cdots & \theta_n & \theta_{n+1} \\ \theta_{n-m} & \theta_{n-m+1} & \cdots & \theta_{n-1} & \theta_n \end{array} \right|$$

(3.1)

when $m \geq 0$ and $n \in \mathbb{Z}$ (each diagonal contains the same entry); $S_{m,n}(\Theta) = 1$ when $m = -1$ and $n \in \mathbb{Z}$, and $S_{m,n}(\Theta) = 0$ when $m < -1$ and $n \in \mathbb{Z}$. In the Number Wall $S(\Theta)$, rows and columns are indexed as in standard matrix notation;

that is, the first index $m$ records the row number and increases towards page bottom and the second index $n$ records the column number and increases towards page right. This convention will be adopted throughout the paper for any array of numbers in the plane.

A Number Wall records Toeplitz determinants of a sequence rather than its Hankel determinants. This enables one to express the properties of such a Wall (see below) in a symmetrical way (this insight is owed to John Conway). Since a Toeplitz determinant as above is obtained under reflection and sign change $(-1)^{m(m+1)/2}$ from the corresponding Hankel determinant, one of these two determinants vanishes if and only if so does the other.

For the sake of simplicity of notation, set from now on

$$S = S(\Theta) \quad \text{and} \quad S_{m,n} = S_{m,n}(\Theta)$$

for any $m, n \in \mathbb{Z}$. Properties of Number Walls have been extensively studied in [28]. The most fundamental of them, which turns out to be a particular case of the Desnanot–Jacobi identity for determinants, can be stated as follows:

**Theorem 3.1.** For any $m, n \in \mathbb{Z}$,

$$S_{m,n}^2 = S_{m+1,n}S_{m-1,n} + S_{m,n+1}S_{m,n-1}.$$

**Proof.** See [28, p.8].

It readily follows from this theorem that the entry $S_{m,n}$ in row $m$ in the Number Wall can be computed from entries in rows $m-1$ and $m-2$ provided that $S_{m-2,n}$ does not vanish. In the case, however, that the latter quantity vanishes, the above formula cannot be used anymore. A remarkable feature of Number Walls is that such zero entries can only occur in very specific shapes:

**Theorem 3.2.** Zero entries in a Number Wall can only occur within windows; that is, within square regions with horizontal and vertical edges.
Proof. See [28, p.9].

For the sake of brevity, a window in a Number Wall containing only zero entries will from now on be referred to as a window. In what follows, it will be convenient to define (with a slight abuse of terminology) the deficiency of a window in a Number Wall as being equal to $\delta \geq 1$ if the window has side length $\delta - 1$.

Figure 1 below depicts such window. The entries surrounding such a region (here corresponding to the sequences $A_k, B_k, C_k$ and $D_k$) will be referred to as the inner frame of the window. The entries surrounding the inner frame (here corresponding to the sequences $E_k, F_k, G_k$ and $H_k$) will be referred to as the outer frame of the window.

![Figure 1: A Number Wall Window.](image)

Extending in the natural way the definition of the deficiency of a one-sided sequence (see Definition 2.1) to a doubly infinite sequence $\Theta$, this concept can be reinterpreted in terms of some properties satisfied by the Number Wall $S$ of $\Theta$.

To this end, note first that, with the notation of Section 2, the determinant of the Hankel matrix $H_\Theta(n;l)$ (where $n \in \mathbb{Z}$ and $l \geq 0$) is, up to a possible change of sign, the entry $S_{n+l,l}$ of the Number Wall $S$. It thus follows from Definition 2.1 that $\Theta$ has deficiency $\delta \geq 2$ if and only if its Number Wall admits a diagonal with $\delta - 1$ zero entries (that is, if and only if there exist integers $n \in \mathbb{Z}$ and $l \geq 0$ such that $S_{n+l+k,l+k} = 0$ for all $0 \leq k \leq \delta - 2$) but no diagonal with zero entries of any longer size (the deficiency is equal to 1 if there is no zero entry in the Number Wall). In view of Theorem 3.2, this amounts to claiming that the Number Wall of the sequence $\Theta$ admits a window with side length $\delta - 1$ but no zero window of any larger size (in the case that the deficiency is 1, this amounts to claiming that there are no zeros at all in the Number Wall).

**Theorem 3.3.** The inner frame of a window with finite deficiency $\delta \geq 2$ comprises four geometric sequences, along top, left, right and bottom edge with ratios $P, Q, R$ and $S$ respectively, from origins $P$, $Q$, $R$, $S$. 

\[
\begin{array}{cccccccccc}
E_0 & E_1 & E_2 & \ldots & E_k & \ldots & E_{\delta-1} & E_\delta \\
F_0 & B_0, A_0 & A_1 & A_2 & \ldots & A_k & \ldots & A_{\delta-1} & A_\delta, C_\delta & G_\delta \\
F_1 & B_1 & 0 & 0 & \ldots & 0 & \ldots & 0 & C_{\delta-1} & G_{\delta-1} \\
F_2 & B_2 & 0 & \ddots & \ddots & (P) & \rightarrow & \vdots & \vdots & \vdots \\
& \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
F_k & B_k & 0 & \downarrow & \ddots & (R) & \vdots & \vdots & \vdots & \vdots \\
& \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
F_{\delta-1} & B_{\delta-1} & 0 & \ldots & 0 & \ldots & 0 & C_2 & G_2 \\
F_\delta & B_\delta, D_\delta & D_{\delta-1} & \ldots & D_k & \ldots & D_2 & D_1 & D_0, C_0 & G_0 \\
H_\delta & H_{\delta-1} & \ldots & H_k & \ldots & H_2 & H_1 & H_0 \\
\end{array}
\]
at top–left and bottom–right corner (see Figure [7]). Furthermore, these ratios satisfy the relation
\[
\frac{PS}{QR} = (-1)^{\delta - 1}.
\]

**Proof.** See [28, p.11]. □

**Corollary 3.4.** With the notation of Figure [7] and Theorem 3.3, the inner frame sequences (denoted by \(A_k, B_k, C_k\) and \(D_k\)) satisfy the relation
\[
\frac{A_kD_k}{B_kC_k} = (-1)^{(\delta - 1)k}
\]
for any \(0 \leq k \leq \delta\).

**Proof.** See [28, p.11]. □

**Theorem 3.5.** With the notation of Figure [7] and Theorem 3.3, the outer frame sequences (denoted by \(E_k, F_k, G_k\) and \(H_k\)) lying immediately outside the inner frame sequences (denoted by \(A_k, B_k, C_k\) and \(D_k\) respectively) and aligned with them satisfy the relation
\[
\frac{QE_k}{A_k} + (-1)^k \frac{PF_k}{B_k} = \frac{RH_k}{D_k} + (-1)^k \frac{SG_k}{C_k}
\]
for \(0 \leq k \leq \delta\).

**Proof.** See [28, p.11]. □

The relations above constitute the set of *frame constraints* of a Number Wall. They can be rephrased all together as follows:

**Corollary 3.6 (Frame Constraints).** Given a doubly infinite sequence \((\theta_n)_{n \in \mathbb{Z}}\) over a ground field \(\mathbb{K}\), its Number Wall \(S = (S_{m,n})_{m,n \in \mathbb{Z}}\) is generated by a recurrence expressing the non–zero elements in row \(m \in \mathbb{Z}\) in terms of the previous rows. More precisely, with the notation of Figure [7] and Theorem 3.3, given \(m,n \in \mathbb{Z}\),

\[
S_{m,n} = \begin{cases} 
0 & \text{if } m < -1; \\
1 & \text{if } m = -1; \\
\theta_n & \text{if } m = 0; \\
\frac{S_{m-1,n} - S_{m-1,n+1}S_{m-1,n-1}}{S_{m-2,n}} & \text{if } m > 0 \& S_{m-2,n} \neq 0; \\
D_k = \frac{(-1)^{(\delta - 1)k}B_kC_k}{(-1)^{k}S_{m-2,n}} & \text{if } m > 0 \& S_{m-2,n} = 0 \& S_{m-1,n} = 0; \\
H_k = \frac{AE_k/\theta_n + (-1)^kPF_k/B_k - (-1)^kSG_k/C_k}{R/D_k} & \text{if } m > 0 \& S_{m-2,n} = 0 \& S_{m-1,n} \neq 0.
\end{cases}
\]

(In the last two equations above, the index \(k\) is determined in the natural way from \(m, n\) and \(\delta\).)

Conversely, an array satisfying this recurrence is the Number Wall of the sequence determined by \(\theta_n = S_{0,n}\) for all \(n \in \mathbb{Z}\).

**Proof.** This follows immediately from Theorem 3.3 Corollary 3.3 and Theorem 3.3. Note that all denominators are guaranteed to be non–zero, and also that the parameters \(k, \delta, P, Q, R\) and \(S\) required for these equations to be a recurrence relation are well–determined. For example, a string of consecutive zeros appearing in a row by application of the equation in the fourth line provides the value of the deficiency \(\delta\) of the window and thus of the parameters \(P, Q\) and \(R\) (e.g., by application of the same equation to compute the vertical entries surrounding the window). Theorem 3.3 then gives the value of the parameter \(S\). □
The Frame Constraints expressed in Corollary 3.6 provide a necessary and sufficient condition for an infinite array to be the Number Wall of a sequence. They also give a Wall Builder Algorithm for generating finite segments of a given Number Wall as well as an alternative method for verifying its properties. This algorithm is more efficient than a direct use of the definition of a Number Wall from Toeplitz determinants as in (3.1). Indeed, a key feature of the Frame Constraints is that in the case of sequences with bounded deficiency, they are local. This means that if the sequence in the zeroth row \( \{S_{0,n}\}_{n \in \mathbb{Z}} \) has deficiency \( \delta \geq 1 \), then the entry \( S_{m,n} \) on row \( m \) and column \( n \) can be obtained from at most \( O(\delta^2) \) other entries sitting in previous rows, which number of entries is independent of \( m \) and \( n \) (see Section 5.1 below for further details). By contrast, Definition 3.1 is global in the sense that \( S_{m,n} \) depends on \( 2m + 1 \) entries on row 0; namely, on \( S_{0,n'} \) with \( |n' - n| \leq m \).

### 3.2 On the Number Wall of the Paper–Folding Sequence over \( \mathbb{F}_3 \)

Extend the Paper–Folding sequence to a doubly infinite sequence \( (f_n)_{n \in \mathbb{Z}} \) by setting \( f_0 = 0 \) and by defining \( f_n \) for \( n < 0 \) via formula (1.7), where the integer \( k \) is then negative (in other words, \( f_{-n} = 1 - f_n \) for any \( n \neq 0 \)).

Note that \( f_0 = f_1 = f_2 = 0 \). Consider then the \( 3 \times 3 \) Hankel matrix formed from these three values together with \( f_3 = 1 \) and \( f_4 = 0 \); that is, consider the matrix

\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0 \\
\end{pmatrix}.
\]

Clearly, all three principal minors of this matrix vanish, which shows that the Paper–Folding sequence has deficiency at least 4. The remainder of the proof will thus consist in proving that the Number Wall of this sequence contains no \( 4 \times 4 \) zero block.

Evidence for this conjecture is provided by Figure 2 below, which represents the portion of the Number Wall \( (S_{m,n})_{m,n \in \mathbb{Z}} \) under consideration in the ranges of indices \( -2 \leq m \leq 39 \) and \( -41 \leq n \leq 41 \) (the squares and circles in the figure will be interpreted later).

Figure 2: The Paper–Folding Number Wall over \( \mathbb{F}_3 \) in the ranges \(-2 \leq m \leq 39 \) and \(-41 \leq n \leq 41 \). Yellow (resp. blue, pink) entries equal 0 (resp. 1, 2).
The Paper–Folding sequence is well–known to be 2–automatic. From a theorem by Cobham, this amounts to claiming that it is the image, under a coding, of a fixed point of a 2–uniform substitution (see [3] §6.3 for proofs and definitions). The idea of the proof of Theorem 2.6 is that such a rich structure in the sequence should be reflected in its Number Wall. With this in mind, the strategy to establish that the Number Wall of the Paper–Folding sequence over $\mathbb{Z}_3$ admits no window with deficiency bigger than 4 can be described as follows:

- Build a suitably large segment of the Paper–Folding sequence and a large portion of its Number Wall;
- Construct a 2–dimensional substitution and a coding such that the generated tiles cover the portion of the Number Wall under consideration;
- Consider the infinite tiling obtained by these substitution and coding, and show that it is a valid Number Wall (that is, that it satisfies the Frame Constraints);
- Check that the sequence generating the Number Wall thus obtained is the Paper–Folding sequence by showing that this sequence sits in row $m = 0$.

Each of these steps will be detailed in the next sections. Beforehand, some lemmata related to the theory of tilings are proved. They will be needed to implement the above–described strategy.

## 4 Tilings of $\mathbb{Z}^d$

Let $\Sigma$ be a set. Its elements will be referred to as tiles. Fix once and for all an integer $d \geq 1$. A tiling of $\mathbb{Z}^d$ (resp. of $\mathbb{N}^d$) over $\Sigma$ is a function $T : \mathbb{Z}^d \rightarrow \Sigma$ (resp. a function $T : \mathbb{N}^d \rightarrow \Sigma$). Referring to a tiling without referencing $\mathbb{Z}^d$ or $\mathbb{N}^d$ will mean either.

This section partially follows [5] in order to recall a special type of tilings that are generated by a finite set of tiles, a substitution rule that allows one to replace tiles with squares of tiles, and a coding function from the set of tiles to a possibly different set of tiles. Several equivalent characterisations of such tilings are proved in [34]. In particular, it is shown therein that this may be taken as a definition for an automatic tiling, some properties of which are established in this section. These properties will enable one to show in Section 5 that the Number Wall of the Paper–Folding sequence is an automatic $\mathbb{Z}^2$–tiling, which will be used to establish Theorem 2.6

Some notation is first introduced. Boldface letters such as $\mathbf{m}$ will denote vectors whose coordinates $(n_1, \ldots, n_d)$ are integers or positive integers, as should be clear from the context. Given $\mathbf{m} = (m_1, \ldots, m_d) \in \mathbb{Z}^d$, $\mathbf{j} = (j_1, \ldots, j_d) \in \mathbb{Z}^d$ and an integer $r$, vectorial and scalar operations are defined coordinate–wise in the natural way:

\[
\mathbf{j} + \mathbf{m} = (j_1 + m_1, \ldots, j_d + m_d),
\]
\[
\gamma \mathbf{m} = (\gamma m_1, \ldots, \gamma m_d),
\]
\[
\frac{\mathbf{m}}{r} = \left(\frac{m_1}{r}, \ldots, \frac{m_d}{r}\right) \quad \text{when } r \neq 0.
\]

Let $[x]$ denote the ceiling of a real number $x$ and set $[x]_+ = \max \{0, [x]\}$. Given integers $n \in \mathbb{Z}$ and $l \geq 1$, let $[n]_l$ be the representative in $\{1, \ldots, l\}$ of $n$ modulo $l$. These pieces of notation are extended in the natural way to any integer vector $\mathbf{n} = (n_1, \ldots, n_d)$, viz.

$\mathbf{[n]} = ([n_1], \ldots, [n_d])$, $\mathbf{[n]}_+ = ([n_1]_+, \ldots, [n_d]_+)$ and $\mathbf{[n]}_l = ([n_1]_l, \ldots, [n_d]_l)$.

Also, the integer vector $\mathbf{0}$ (resp. $\mathbf{1}, \mathbf{2}, \mathbf{3}$) will stand for the vector all of whose components are equal to 0 (resp. equal to 1, to 2, to 3). Lastly, the notation $\mathbf{l}$ (resp. $\mathbf{r}, \mathbf{r}'$) will be reserved to denote the vector all of whose components are equal to a given integer $l$ (resp. to a given integer $r, r'$). In each of these cases, the dimension of the vectors under consideration will be clear from the context.
4.1 Substitution Tilings

Only tilings arising from a special type of substitutions (sometimes called uniform) will be required.

**Definition 4.1.** Let $\Sigma$ be a set of tiles and let $k \geq 2$ be an integer. A $k$–substitution is a map $\varphi : \Sigma \to \Sigma^{(1, \ldots, k)^d}$.

In the above definition, the set $\Sigma^{(1, \ldots, k)^d}$ is the set of mappings from the set of $d$–tuples with integer entries between 1 to $k$ to $\Sigma$. Thus, the $k$–substitution $\varphi$ maps each tile to a collection of $k^d$ tiles which can be seen as being arranged in the shape of a $d$–dimensional hypercube.

Given a substitution, one can construct a tiling by applying it again and again and “stacking up” shifts of the outcome. One way of doing so is introduced with the help of the following definition:

**Definition 4.2.** Assume that $\varphi$ is a $k$–substitution on $\Sigma$ (where $k \geq 2$). A tile $s \in \Sigma$ is said to be prolongable if $s = \varphi(s)(1)$.

Given a $2^d$–tuple $s = (s \in \Sigma : o \in \{0, 1\}^d)$ such that $s_o$ is $o$–prolongable for $\varphi$ for any $o \in \{0, 1\}^d$, define recursively a $Z^d$–tiling $T$ with the help of (4.2) and of the following extension of the initial condition (4.1) for every $o \in \{0, 1\}^d$, let

$$T(o) = s_o. \quad (4.3)$$

The $Z^d$–tiling thus obtained will be denoted by $T_{(\varphi, s)}$.

**Example 4.4** (The Thue–Morse $\mathbb{N}^2$–tiling). To get used to the above definitions and notation, consider the tiling introduced in [2, p.20]. It is a substitution tiling of $\mathbb{N}^2$ over $\Sigma = \{0, 1\}$ given by the $2$–substitution

$$0 \mapsto 0 \ 1, \quad 1 \mapsto 1 \ 0 \quad (4.1)$$

applied to the $(1, 1)$–prolongable tile 0. A finite portion of the tiling $T_{(\varphi, 0)}$ can be generated as follows:

$$\begin{align*}
0 & \mapsto 0 \ 1 \\
1 & \mapsto 1 \ 0 \\
0 & \mapsto 0 \ 1 \\
1 & \mapsto 1 \ 0 \\
1 & \mapsto 0 \ 0 \ 1 \\
1 & \mapsto 1 \ 0 \ 0 \ 1 \\
0 & \mapsto 1 \ 1 \ 0 \ 0 \ 1 \\
1 & \mapsto 0 \ 1 \ 0 \ 0 \ 1 \\
0 & \mapsto 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 1 \ 0 \\
1 & \mapsto 1 \ 0 \ 0 \ 1 \ 0 \ 1 \ 1 \ 0 \\
0 & \mapsto 1 \ 1 \ 0 \ 0 \ 1 \ 0 \ 1 \ 1 \ 0 \\
1 & \mapsto 0 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0 \ 1 \ 1 \ 0 \\
0 & \mapsto 1 \ 1 \ 0 \ 0 \ 1 \ 0 \ 1 \ 1 \ 0 \\
1 & \mapsto 0 \ 1 \ 0 \ 1 \ 0 \ 0 \ 1 \\
0 & \mapsto 0 \ 1 \ 0 \ 1 \ 1 \ 0 \ 0 \ 1 \\
1 & \mapsto 1 \ 0 \ 0 \ 1 \ 0 \ 1 \ 1 \ 0 \\
0 & \mapsto 1 \ 1 \ 0 \ 0 \ 1 \ 0 \ 1 \ 1 \ 0
\end{align*}$$

applied to the $(1, 1)$–prolongable tile 0. A finite portion of the tiling $T_{(\varphi, 0)}$ can be generated as follows:
Recall here the convention that the positive directions for the vertical and horizontal axes are as for matrices; that is, downwards and rightwards, respectively.

4.2 Coding of a Tiling

Given two sets of tiles $\Sigma$ and $\Delta$, a coding from $\Sigma$ to $\Delta$ is a map $\tau: \Sigma \rightarrow \Delta$. It will be convenient to reinterpret this definition as a 1-coding so as to fit a more general concept:

**Definition 4.5.** Let $\Sigma$ and $\Delta$ be sets and let $l \geq 1$ be an integer. An $l$-coding from $\Sigma$ to $\Delta$ is a map $\tau: \Sigma \rightarrow \Delta^{(1, \ldots , l)^d}$.

Given a tiling and a coding, another tiling can be generated by applying the coding to each of the tiles. Let $\Sigma$ and $\Delta$ be sets. Let $\tau$ be an $l$-coding from $\Sigma$ to $\Delta$, and let $T$ be any tiling over $\Sigma$. The image of $T$ under $\tau$ is the tiling over $\Delta$ denoted by $\tau(T)$ and defined as follows: for any integer vector $n$ in the domain of $T$,

$$ \tau(T)(n) = \tau(T\left(\left\lfloor \frac{n}{l} \right\rfloor \right))(\lfloor n_l \rfloor). $$

A tiling which is the image of a uniform substitution tiling under a 1-coding is said to be an automatic tiling. It can be shown that one does not alter the set of tilings one gets by allowing oneself the use of a coding as in Definition 4.5 instead of a 1-coding (this can be proved, e.g., via Lemma 6.9.1). The reason for introducing Definition 4.5 is that it will enable the use of an efficient algorithm that detects automaticity.

**Example 4.6.** The doubly–infinite Paper–Folding sequence introduced in §3.2 is well–known to be generated by the 2–substitution $\psi$ and the 1–coding $\rho$ defined in Figure 3 below, applied to the 0–prolongable and 1–prolongable tiles 2 and 0, respectively (see Example 10.3.3 for further details). With the previous notation, the Paper–Folding sequence is then $\rho(T(\psi, (2, 0)))$.

| Tile | Substitution $\psi$ | Coding $\rho$ |
|------|---------------------|---------------|
| 0    | 0 2                 | 0             |
| 1    | 0 3                 | 1             |
| 2    | 1 2                 | 0             |
| 3    | 1 3                 | 1             |

Figure 3: A standard substitution and coding that generate the Paper–Folding sequence.

4.3 Consistent Overlaps

In this subsection are proved two statements standing at the heart of our approach. To this end, some additional notation and definitions are first introduced. Given $j, m \in \mathbb{Z}^d$, denote by

$$ [j, m] = \{ n \in \mathbb{Z}^d : j_i \leq n_i \leq m_i \text{ for every } 1 \leq i \leq d \} $$

the rectangular shape with edges parallel to the coordinate axes and with opposite vertices $j$ and $m$. It will also be convenient to introduce the following notation to exclude some border values of such a parallelepiped:

$$ (j, m) = \{ n \in \mathbb{Z}^d : j_i < n_i \leq m_i \text{ for every } 1 \leq i \leq d \}. $$

Finally, define the multiplication of the latter set by a positive integer $r$ in the natural way:

$$ r \cdot (j, m) = (rj, rm). $$
Definition 4.7. Given a set of tiles \( \Sigma \), a rectangular pattern over \( \Sigma \) is a map \( P : [1, m] \to \Sigma \) for some \( m \in \mathbb{N}^d \). In the case that \( [1, m] = \{1, \ldots, r\}^d \) for some \( r \in \mathbb{N} \), the map \( P : \{1, \ldots, r\}^d \to \Sigma \) is referred to as an \( r \)-pattern.

The rectangular pattern \( P : [1, m] \to \Sigma \) is contained in \( T \), which stands either for a tiling or for another rectangular pattern, if there exists \( j \) such that

\[
P(n) = T(j + n)
\]

for every \( n \in [1, m] \).

Definition 4.8. Given a tiling or a rectangular pattern \( T \), given an \( l \)-coding \( \tau \) and an integer \( 0 \leq r < l \), the coding \( \tau \) is said to be \( r \)-consistent with respect to an overlap of \( r \) (or, for short, \( r \)-consistent) if for every integer vector \( n \), every index \( 1 \leq i \leq d \) and every integer \( l - r < r' \leq l \), one has that

\[
\tau(T(n))(\cdot, \ldots, \cdot, r', \cdot, \ldots) = \tau(T(n + e_i))(\cdot, \ldots, \cdot, r' - (l - r), \cdot, \ldots)
\]

whenever \( n \) and \( n + e_i \) are in the domain of \( T \). In this equation, \( r' \) and \( r' - (l - r) \) appear in the \( i \)th coordinate, the equality is as functions of the remaining \( d - 1 \) variables, and \( e_i \) stands for the \( i \)th vector of the standard basis of \( \mathbb{R}^d \).

Figure 4 illustrates the concept of a coding with overlap.

![Figure 4](image)

Figure 4: Schematic representation in the plane of an \( l \)-coding \( \tau \) which is \( r \)-consistent with respect to an underlying tiling \( T \). The circles represent four adjacent tiles in the tiling \( T \) to which the coding \( \tau \) is applied. The shaded regions correspond to the areas of overlap between two adjacent \( l \times l \) squares representing the images under \( \tau \) of these tiles.

If \( T \) is a tiling over \( \Sigma \) and \( \tau \) an \( l \)-coding which is \( r \)-consistent, define an \((l - r)\)-coding \( \tau_r \) by setting

\[
\tau_r(s) = \tau(s)|_{\{1, \ldots, l-r\}^d}
\]

for every \( s \in \Sigma \). The resulting tiling \( \tau_r(T) \) will be referred to as a tiling with \( l \)-coding and overlap \( r \).

For any \( j \in \{0, \ldots, r\}^d \), let \( \tau_{r,j} \) be the coding defined for a given \( s \in \Sigma \) as a map

\[
\tau_{r,j}(s) : \{1, \ldots, l-r\}^d \to \Delta
\]

such that

\[
\tau_{r,j}(s)(m) = \tau(s)(j + m)
\]

for all \( m \in \{1, \ldots, l-r\}^d \).

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Lemma 4.9. With the above notation and definitions, the tiling \( \tau_{(r,j)}(T) \) satisfies the relation
\[
\tau_{(r,j)}(T)(\mathbf{n}) = \tau_r(T)(\mathbf{j} + \mathbf{n})
\]
for any integer vector \( \mathbf{n} \) such that both of these quantities are well–defined.

Proof. Decompose the vectors \( \mathbf{n} \) and \( \mathbf{j} \) as
\[
\mathbf{n} = \mathbf{n}'(l-r) + \mathbf{n}'' \quad \text{and} \quad \mathbf{j} = \mathbf{j}'(l-r) + \mathbf{j}'',
\]
where each of the components of the integer vectors \( \mathbf{n}'' \) and \( \mathbf{j}'' \) lies between 1 and \( l-r \). Decompose also the vector \( \mathbf{n}'' + \mathbf{j}'' \) as
\[
\mathbf{n}'' + \mathbf{j}'' = \mathbf{u}(l-r) + \mathbf{v},
\]
where here also each of the components of the vector \( \mathbf{v} \) lies between 1 and \( l-r \).

Then,
\[
\tau_{(r,j)}(T)(\mathbf{n}) = \tau_{(r,j)}\left(T\left(\left\lfloor \frac{\mathbf{n}}{l-r} \right\rfloor\right)\left(\lfloor \mathbf{n}_{l-r} \rfloor\right)\right)
\]
\[
= \tau\left(T\left(\left\lfloor \frac{\mathbf{n}}{l-r} \right\rfloor\right)\left(\lfloor \mathbf{n}_{l-r} + \mathbf{j} \rfloor\right)\right)
\]
\[
= \tau\left(T\left(\mathbf{n}' + 1\right)\left(\mathbf{v} + \left(l-r\right)\left(\mathbf{j}' + \mathbf{u}\right)\right)\right)
\]
\[
= \tau\left(T\left(\mathbf{n}' + \mathbf{j}' + \mathbf{u} + 1\right)\right)\left(\mathbf{v}\right)
\]
\[
= \tau_r\left(T\left(\left\lfloor \frac{\mathbf{n} + \mathbf{j}}{l-r} \right\rfloor\right)\left(\left\lfloor \mathbf{j} + \mathbf{n}_{l-r} \right\rfloor\right)\right)
\]
\[
= \tau_r(T)(\mathbf{j} + \mathbf{n}). \quad \square
\]

Lastly, it will be useful to extend the definition of a coding with overlap to rectangular patterns:

Definition 4.10. Let \( P \) be a rectangular pattern on \([1, \mathbf{m}]\) and let \( \tau \) be an \( r \)-consistent \( l \)-coding (where \( 0 \leq r < l \)). The image of \( P \) under the coding \( \tau \) is the rectangular pattern \( \tau_r(P) \) defined on \([1, (l-r)\mathbf{m} + r]\) by
\[
\tau_r(P)(\mathbf{n}) = \tau\left(P\left(\min\left\lfloor \frac{\mathbf{n}}{l-r} \right\rfloor, \mathbf{m}\right)\right)\left(\lfloor \mathbf{n}_{l-r} + \left(l-r\right)\cdot \left\lfloor \frac{\mathbf{n} - (l-r)\mathbf{m}}{l-r} \right\rfloor \right)\right).
\]

(4.8)

Here, the \( \min \) function is applied to vectors coordinate–wise.

Note that if the integer vector \( \mathbf{n} \) is decomposed uniquely as
\[
\mathbf{n} = \mathbf{n}'(l-r) + \mathbf{n}''
\]
with \( 1 \leq n'' \leq l-r \), then formula (4.8) can be rewritten more compactly as
\[
\tau_r(P)(\mathbf{n}) = \tau\left(P\left(\min\left\lfloor \frac{\mathbf{n}'}{l-r} \right\rfloor, \mathbf{m}\right)\right)\left(\mathbf{n}'' + \left(l-r\right)\cdot \left\lfloor \mathbf{n}' - \mathbf{m} + 1 \right\rfloor \right). \quad (4.8)
\]

This definition is designed to satisfy the following property:

Lemma 4.11. If \( T \) is a tiling and \( \tau \) is an \( l \)-coding which is \( r \)-consistent, then for any rectangular pattern \( P \) in \( T \), it holds that \( \tau_r(P) \) is contained in \( \tau_r(T) \). More precisely, if \( P \) is a pattern on \([1, \mathbf{m}]\) and \( \mathbf{j} \) is an integer vector such that \( P(\mathbf{n}) = T(\mathbf{j} + \mathbf{n}) \) for all \( \mathbf{n} \in [1, \mathbf{m}] \), then
\[
\tau_r(P)(\mathbf{n}) = \tau_r(T)(\left(l-r\right)\mathbf{j} + \mathbf{n})
\]
for every \( \mathbf{n} \in [1, (l-r)\mathbf{m} + r] \).
Proof. Decompose any given \( n \in [1, (l - r)m + r] \) as

\[
    n = n'(l - r) + k
\]  \hspace{1cm} (4.9)

with \( 0 \leq n' \leq m_i - 1 \) and

\[
    1 \leq k_i \leq \begin{cases} 
    l - r & \text{if } n'_i < m_i - 1, \\
    l & \text{if } n'_i = m_i - 1.
    \end{cases}
\]

Furthermore, let

\[
    k = k'(l - r) + k''
\]  \hspace{1cm} (4.10)

with \( k'_i \geq 0 \) and \( 1 \leq k''_i \leq l - r \) for any \( 1 \leq i \leq d \). Then

\[
    \tau_r(T)((l - r)j + n) = \tau_r \left( T \left( j + \left\lceil \frac{n}{l - r} \right\rceil \right) \right) \left( [n]_{l - r} \right) = \tau_r(T(j + n' + 1))(k'') = \tau_r(P(n' + 1))(k) = \tau_r(P)(n). \]

In these relations, the third equality follows upon applying identity (4.5) \( k'_i \) times in the \( i \)th coordinate for each \( 1 \leq i \leq d \). The last equality follows from Definition 4.10 and a direct calculation upon using decomposition (4.10) in each coordinate where \( n'_i \leq m_i - 1 \) and upon using decompositions (4.9) and (4.10) in each coordinate where \( n'_i = m_i - 1 \). \hfill \Box

Lemma 4.11 enables one to state the first of the two fundamental results of this subsection:

**Lemma 4.12.** Assume that \( T \) is a tiling over \( \Sigma \) and that \( \tau : \Sigma \to \Delta^{(1...l)^d} \) is an \( l \)-coding which is \( r \)-consistent (where \( 0 \leq r < l \)). Then for every integer \( r' \geq 1 \), every \( r' \)-pattern in \( \tau_r(T) \) is contained in the image under the coding \( \tau_r \) of an \( s(l, r, r') \)-pattern in \( T \), where

\[
    s(l, r, r') = 1 + \left\lceil \frac{r' - (r + 1)}{l - r} \right\rceil +.
\]

Proof. Assume that \( P : \{1, \ldots, r'\}^d \to \Delta \) is an \( r' \)-pattern in \( \tau_r(T) \) and let \( j \) be an integer vector such that

\[
    P(n) = \tau_r(T)(j + n) \]  \hspace{1cm} (4.11)

for every \( n \in \{1, \ldots, r'\}^d \). Decompose \( j \) as

\[
    j = j'(l - r) + j''
\]

with \( 0 \leq j''_i < l - r \) for every \( 1 \leq i \leq d \). Let \( P' \) be the \( s(l, r, r') \)-pattern in \( T \) defined by

\[
    P'(n) = T(j' + n)
\]

for all \( n \in \{1, \ldots, s(l, r, r')\}^d \).

It then follows from Lemma 4.11 (applied with the \( d \)-tuple \( m \) all of whose components are equal to \( s(l, r, r') \)) that

\[
    \tau_r(P') \left( j'' + n \right) = \tau_r(T)(j + n) = P(n) \]  \hspace{1cm} (4.11)

whenever

\[
    j'' + n \in [1, (l - r) \cdot s(l, r, r') + r].
\]
This condition holds in the present case. Indeed, since \( n \in \{1, \ldots, r'\}^d \) and \( j''_i < l - r \) for every \( 1 \leq i \leq d \),
\[
j''_i + n_i \leq l - r - 1 + r' \leq l + (l - r) \cdot \left\lceil \frac{r' - (r + 1)}{l - r} \right\rceil + (l - r) \left( 1 + \left\lceil \frac{r' - (r + 1)}{l - r} \right\rceil \right) + r,
\]
whence the claim from the definition of the quantity \( s(l, r, r') \).

This concludes the proof of the lemma. \( \Box \)

Lemma 4.12 will be useful in the proof of Theorem 2.6. In order to apply it, one first needs to verify that a given coding is consistent. While this can be a hard task in general, the following statement provides an easy-to-check criterion in the case of substitution tilings:

**Lemma 4.13.** Let \( T \) be a \( k \)-substitution tiling (where \( k \geq 2 \)). Assume that there are integers \( m_i \leq 0 < M_i \) (\( 1 \leq i \leq d \)) such that \( m_i = 0 \) if \( T \) is an \( \mathbb{N}^d \)-tiling and such that \( m_i < 0 \) if \( T \) is a \( \mathbb{Z}^d \)-tiling verifying the following condition: every 2-pattern contained in \( T|_{k(m,M)} \) is already contained in \( T|_{(m,M)} \).

Then every 2-pattern in \( T \) is already contained in \( T|_{(m,M)} \).

Before proving the lemma, extend first the definition of a \( k \)-substitution \( \varphi \) over a set of tiles \( \Sigma \) to rectangular patterns. This can be done in the natural way by setting
\[
\varphi(P)(n) = \varphi \left( P \left( \left\lceil \frac{n}{k} \right\rceil \right) \right) \left( \left\lfloor \frac{n}{k} \right\rfloor \right) \tag{4.12}
\]
for all \( n \in \mathbb{N}^d \).

**Proof.** Let \( \Sigma \) be the set of tiles and let \( \varphi : \Sigma \to \Sigma^{\{1, \ldots, k\}^d} \) be the substitution defining \( T \). Let \( P : \{1, 2\}^d \to \Sigma \) be the 2-pattern in \( T \) defined by
\[
P(n) = T(j + n) \tag{4.13}
\]
for some integer vector \( j \). Assume that \( P \) is contained in \( T|_{k(m,M)} \) for some \( l \geq 1 \). It will be shown by induction on the integer \( l \) that \( P \) also lies in \( T|_{(m,M)} \). The claim being true by assumption when \( l = 1 \), assume that \( l \geq 2 \) and consider the rectangular pattern
\[
P' \colon \left[ 1, 1 + \left\lceil \frac{j + 2}{k} \right\rceil - \left\lceil \frac{j + 1}{k} \right\rceil \right] \to \Sigma
\]
defined by
\[
P'(n) = T \left( \left\lceil \frac{j + n}{k} \right\rceil \right). \tag{4.14}
\]
Then, \( P' \) is contained in a 2-pattern. Indeed, decompose the \( i^{th} \) component \( j_i \) of \( j \) as
\[
j_i = k u_i + v_i \quad \text{with} \quad 0 \leq v_i \leq k - 1
\]
and note that
\[
1 + \left\lceil \frac{j_i + 2}{k} \right\rceil - \left\lceil \frac{j_i + 1}{k} \right\rceil = \begin{cases} 2 & \text{if } v_i = k - 1; \\ 1 & \text{otherwise.} \end{cases} \tag{4.15}
\]
Furthermore, the pattern $P$ is contained in the image under $\varphi$ of the pattern $P'$. Indeed, on the one hand,

$$\varphi(P') ([j+1]_k + n - 1) = \varphi\left(P'\left(\left[\frac{j+1}{k}\right] + n - 1\right)\right) ([j+1]_k + n - 1)_k$$

$$= \varphi\left(T\left(\left[\frac{j}{k}\right] + n\right)\right) ([j+n]_k)$$

$$= T(j+n)$$

$$= P(n).$$

On the other hand, it easily follows from (4.15) that for any

$$n \in \left[1, 1 + \left\lfloor \frac{j+2}{k} \right\rfloor - \left\lfloor \frac{j+1}{k} \right\rfloor\right],$$

one has that

$$\left\lfloor \frac{j+1}{k} + n - 1\right\rfloor \in \left[1, 1 + \left\lfloor \frac{j+2}{k} \right\rfloor - \left\lfloor \frac{j+1}{k} \right\rfloor\right],$$

whence the claim.

By the induction assumption, the pattern $P'$, which is contained in $T|_{k-1}(m,M)$ from its definition, already appears in $T|_{(m,M)}$. Its image under $\varphi$ is therefore contained in the image of $T|_{(m,M)}$ under $\varphi$, which is clearly contained in $T|_{k(m,M)}$. Thus, from the above claim, the pattern $P$ also lies in $T|_{k(m,M)}$, and therefore in $T|_{(m,M)}$ by assumption.

This completes the proof of the lemma.

**Corollary 4.14.** Let $T$ be a tiling satisfying the assumptions of Lemma 4.13. Assume that $\tau$ is an $l$–coding satisfying this property: there exists an integer $r$, where $1 \leq r < l$, such that the consistency condition (4.5) holds for every $n$ with $m_i < n_i \leq M_i$.

Then the coding $\tau$ is $r$–consistent for the tiling $T$.

**Proof.** It immediately follows from Lemma 4.13 that the consistency condition (4.5) only needs to be checked in the region $(m,M)$ for it to hold everywhere.

**Remark 4.15.** Properties of substitution tilings with overlaps are also studied in [29], where they are referred to as C3DEL systems — an acronym representing “Deterministic Lindenmayer system with constant width, inflation, encoding and context”. This terminology extends the one presented in [3, §7.12].

5 Generating the Putative Number Wall of the Paper–Folding Sequence over $\mathbb{F}_3$

The goal of this section is to explain how to generate an automatic Number Wall that coincides with a large segment of the Paper–Folding Number Wall. Verifying that this automatic tiling is in fact equal to the Paper–Folding Number Wall, and that it has deficiency 4, is postponed to Section 6.
5.1 Building a Finite Portion of a Number Wall

The recursive formula in Corollary 3.6 gives a formula for \( S_{m,n} \) in terms of elements occupying previous rows. This might seem impractical at first glance due to infinitely many computations required to calculate each row. However, looking more carefully reveals that the same formula gives an algorithm for calculating an isosceles triangle contained in a trapezoidal portion of the wall of the form

\[
(S_{k,l})_{0 \leq k \leq m, \ n-m+k \leq l \leq n+m-k}
\]

given the finite portion \((S_{0,l})_{n-m \leq l \leq n+m}\) of the initial sequence \((S_{0,l})_{l \in \mathbb{Z}}\). The discovery of this algorithm is discussed in [28]. It is described in Figure 5 below.

5.2 Tiling a Finite Table with a Substitution, Coding and Overlap

Assume one is given a rectangular segment of an automatic tiling. The goal is to find a substitution and a coding which generate the entire tiling. If the number of tiles in the infinite tiling is bounded and if the given pattern is large enough, this then is possible. For an \( N \)-tiling, i.e. for sequences, this is done by Sutner and Tetruashvili [36] via finite automata. By a well–known process described in [26, Propositions 10.1.5 & 10.2.2], their algorithm also yields a substitution and a coding which generate the entire sequence. It is possible to extend this approach to higher dimensions. However, such an algorithm might not be efficient enough for our purposes, as the computational cost may be exponential in the number of tiles (see [35, p.16] for further details). In the case of the Paper–Folding Number Wall, this number is in the order of magnitude of several thousands.

A different approach will be used hereafter. It was initiated by the third–named author [28, 29] in order to provide an algorithm sufficiently efficient in the present context. The underlying method only allows one to find automatic tilings with an injective coding; however, this proves adequate for the Paper–Folding Number Wall. The overlap between coded tiles is used to ensure that an injective coding is possible. The algorithm is based on Lemmata 4.12 and 4.13 and is described in Figure 6.

5.3 The Paper–Folding Number Wall

The algorithms of the previous sections enable one to detect a structure in the Number Wall of the Paper–Folding sequence:

**Theorem 5.1.** The finite portion of the Paper–Folding Number Wall over \( \mathbb{F}_3 \)

\[
(S_{m,n})_{-55 \leq m \leq 2400, -5220 \leq n \leq 5220}
\]  

agrees with a tiling \( T' = \tau'(T) \). This tiling is the image under a coding \( \tau' \) of a substitution tiling \( T = T(\varphi, (1, 2, 3, 4)) \). Here,

1. \( \varphi \) is a 2–substitution over \( \Sigma = \{1, \ldots, 2353\} \) for which tiles 1, 2, 3 and 4 are respectively \((0, 0), (0, 1), (1, 0) \) and \((1, 1)\)–prolongable;
2. \( \tau' \) is an 8–coding defined with the help of a 13–coding \( \tau \) by the relation

\[
\tau'(s)(n) = \tau(s)(n + 3)
\]  

for every \( s \in \Sigma \) and every \( n \in \{1, \ldots, 8\}^2 \);
3. \( \tau \) is a 13–coding defined over \( \Sigma \) and taking values in \( \mathbb{F}_3 \). This coding is 5–consistent for the tiling \( T \) restricted to the region

\[
\mathcal{R} = [-3, 149] \times [-325, 325].
\]
Data: $S_n$ for $a \leq n \leq b$

Result: $S_{m,n}$ for $0 \leq m \leq \frac{b-a}{2}$ and $a + m \leq n \leq b - m$

while $a \leq n \leq b$

| $S_{-2,n} = 0$ and $S_{-1,n} = 1$
| while $0 \leq m \leq \frac{b-a}{2}$ and $a + m \leq n \leq b - m$
| if $m = 0$
| $S_{m,n} = S_n$
| else if $S_{m-2,n} \neq 0$
| $S_{m,n} = \frac{S_{m-1,n} - S_{m-1,n+1}S_{m-1,n-1}}{S_{m-2,n}}$
| else
| Set $p = q = k = 1$
| while $S_{m-p-2,n} = 0$
| $p = p + 1$
| end
| while $S_{m-p-1,n-q} = 0$ and $n - q \geq a + m - p - 1$
| $q = q + 1$
| end
| while $S_{m-p-1,n+k} = 0$ and $n + k \leq b - m - p - 1$
| $k = k + 1$
| end
| $\delta = k + q$
| if $\delta > p + 2$
| Inside a zero block let $S_{m,n} = 0$
| else if $\delta = p + 2$
| On the inner frame let

$$S_{m,n} = \frac{(-1)^{\delta-1} k S_{m-q,n-q} S_{m-k,n+k}}{S_{m-\delta,n-q+k}}$$

| else
| On the outer frame where:

$$P = \frac{S_{m-\delta-1,n+k-q}}{S_{m-\delta-1,n+k-q-1}}$$

$$Q = \frac{S_{m-q-1,n-q}}{S_{m-q-2,n-q}}$$

$$R = \frac{S_{m-\delta+q-1,n+k}}{S_{m-k,n+k}}$$

and $S = \frac{S_{m-1,n}}{S_{m-1,n+1}}$

| let $S_{m,n} =$

$$\frac{S_{m-1,n}}{R} \left( \frac{Q S_{m-\delta-2,n+k-q}}{S_{m-\delta-1,n+k-q}} + (-1)^k \left( \frac{P S_{m-q-1,n-q-1}}{S_{m-q-1,n-q}} - \frac{S_{m-k-1,n+k+1}}{S_{m-k-1,n+k}} \right) \right)$$

end

end

Figure 5: How to calculate a portion of the Number Wall given a portion of a sequence.

Remark 5.2. The additional translation included in the definition of the coding $\tau'$ in (5.2) is an adjustment of a technical nature (see Section 8.4 below for details). It enables one to match the zeroth row of the tiling with the finite portion of the Paper–Folding sequence used to build part of its Number Wall. This will allow later an easy comparison between the substitution that
Theorem 5.1 is obtained by application of the algorithm described in Figure 6 with the parameters

\[ k = 2, \quad l = 13, \quad r = 5, \quad a = -55, \quad b = 2400, \quad c = -5220 \quad \text{and} \quad \delta = 5220. \]

The region \( R \) defined in (5.3) is obtained from Pass 3 of this algorithm: it corresponds to the range of indices \((m,n) \in \mathbb{Z}^2\) such that

\[
-3.125 = \frac{-55 + 5}{(13 - 5) \cdot 2} < m \leq \frac{2400 - 5}{(13 - 5) \cdot 2} = 149.6875
\]

and

\[
-325.9375 = \frac{-5220 + 5}{(13 - 5) \cdot 2} < n \leq \frac{5220 - 5}{(13 - 5) \cdot 2} = 325.9375.
\]

Note that Pass 3 then guarantees that the tiling \( T \) satisfies the assumptions of Lemma 4.13 upon setting

\[
m = (-4, -326) \quad \text{and} \quad M = (149, 325). \quad (5.4)
\]

The resulting conclusion is stated as a proposition:

**Proposition 5.3.** With the notation of Theorem 5.1, every 2-pattern in \( T \) already appears in \( T|_{(m,M)} \).
Proving that the tiling $T' = \tau'(T)$ in Theorem 5.1 is the Paper–Folding Number Wall and determining its deficiency require a fine analysis of the pattern of zeros appearing in the portion $\tau'(T|_R)$. This, in turn, relies on some properties of the coding $\tau$ and of the $2$–substitution $\varphi$. These properties are stated in the next two propositions and depend on two specific subsets of tiles, namely
\[ S = \{1,2,6,7,12,13,20,29\} \quad \text{and} \quad S' = S \cup \{5\}. \tag{5.5} \]

**Proposition 5.4** (Properties of the $13$–coding $\tau$). The $13$–coding $\tau$ introduced in Theorem 5.1 satisfies the following property: if, for a given tile $s \in \Sigma$, the $13 \times 13$ square $\tau(s)$ contains a $4$–pattern comprising only zero entries, then $s \in S'$.

More precisely, under such an assumption,
- either $s = 5$ and $\tau(s)$ is identically zero;
- or else $s \in S$, in which case the top 9 rows of $\tau(s)$ are identically zero while all entries in the tenth row equal 1.

Furthermore, for every $s \in \Sigma$, the $13$–pattern $\tau(s)$ is contained in the rectangular pattern $\tau'(T|_R)$.\tag{5.1}

The last claim in Proposition 5.4 follows from the construction of the coding $\tau$ in Pass 1 of the algorithm described in Figure 6. The rest of the proposition is established by (computer) inspection of the coding $\tau : \Sigma \rightarrow \Sigma^{\{1,\ldots,13\}^2}$, which is available at \[30\] in the file $dragon_codes_B.dat$.

**Proposition 5.5** (Properties of the $2$–substitution $\varphi$). The $2$–substitution $\varphi$ introduced in Theorem 5.1 satisfies the following properties:

1. the image of tile 5 is the $2 \times 2$ square all of whose entries are 5;
2. for any tile $s \in S$,
   \[ \varphi(s) = \frac{5}{s_1} \quad \frac{5}{s_2} \quad \text{with} \quad s_1, s_2 \in S; \tag{5.6} \]
3. if the image $\varphi(s)$ of a tile $s \in \Sigma$ contains a tile in $S'$, then $s \in S'$.

Proposition 5.5 is established by (computer) inspection of the substitution $\varphi : \Sigma \rightarrow \Sigma^{\{1,2\}^2}$, which is explicit and available at \[30\] in the file $dragon_tetrads_B.dat$. This file actually contains the 6721 2–patterns that can be found in the tiling $T$ restricted to the region $R$. The first 2353 correspond to the successive images of tiles $1, \ldots, 2353 \in \Sigma$ under the $2$–substitution $\varphi$. Also, the values of $s_1$ and $s_2$ appearing in (5.6) are explicitly recorded in Figure 8 below as they will be needed later.

Together with the initial conditions
\[ T(0,0) = 1, \quad T(0,1) = 2, \quad T(1,0) = 3 \quad \text{and} \quad T(1,1) = 4, \tag{5.7} \]
the $2$–substitution $\varphi$ can be used to generate any finite portion of the tiling $T = T_{\varphi,(1,2,3,4)}$. An example of such a portion is represented in Figure 7. Note the presence of the initial pattern
\[
\begin{array}{cc}
1 & 2 \\
3 & 4 \\
\end{array}
\]

therein (cf. the second and third rows).

### 6 Verifying the Properties of the Generated Number Wall

Proving that the tiling described in the statement of Theorem 5.1 is the $\mathbb{Z}^2$–tiling given by the Paper–Folding Number Wall will be done by verifying that it satisfies the Frame Constraints. From Corollary 3.6, this will imply that it is the Number Wall of the sequence that sits in its zeroth row. The claim will then follow from verifying that this sequence is the Paper–Folding sequence, thereby establishing Theorem 2.6. Let then $T$, $T'$, $\tau$, $\tau'$ and $\varphi$ be as in Theorem 5.1.
Figure 7: Portion of the tiling $T$ as described in Theorem (5.1)

Theorem 6.1. The coding $\tau$ is 5–consistent for the tiling $T$.

Proof. The assumptions of Lemma (4.13) have been shown to hold in order to state Proposition 5.3. Corollary 4.14 then implies that the consistency condition needs only be verified in $T|_{(m,M]}$, where $m$ and $M$ are defined in (5.4). This verification has been done as part of Point 3 in Theorem 5.1.

Since $\tau$ is a 13–coding which is 5 consistent, equation (5.2) implies that, with the notation of Section 4.3,

$$\tau' = \tau_{(5,3)}.$$

It then follows from Lemma 4.9 that the images of the tiling $T$ under $\tau'$ and $\tau_5$ are the same up to a shift of 3, both vertically and horizontally; that is, for any $n \in \mathbb{Z}$,

$$\tau'(T)(n) = \tau_5(T)(n + 3). \quad (6.1)$$

This equation enables one to interpret the squares and circles in Figure 2 and to illustrate how the substitution, coding and overlap look like. The top circle marks the origin. The top-left $13 \times 13$ square is the image under $\tau$ of tile 17, with a circle in its center. The image of 17 under $\varphi$ is the 2–pattern given by the matrix

$$\begin{pmatrix}
35 & 45 \\
47 & 58
\end{pmatrix}$$

(this information is contained in Figure 7). The images under $\tau$ of these four tiles are represented by four squares with a circle around their respective centers. By definition, it then follows that $\tau_5$ maps tiles 17, 35, 45, 47 and 58 to the top-left $8 \times 8$ squares sitting in their respective images under $\tau$. Finally, equation (6.1) shows that the tiling $T' = \tau'(T)$ is obtained by translating all the $8 \times 8$ squares thus obtained by the vector $3 = (3, 3)$.

Theorem 6.2. The zero entries with non–negative row indices in the tiling $T' = \tau'(T)$ appear in the form of squares with side lengths at most 3 and with horizontal and vertical edges.

With the conventions and definitions adopted in Section 3.1, Theorem 6.2 is thus saying that a window in the part of the tiling $T'$ with non–negative row indices has deficiency at most 4.

The proof of Theorem 6.2 requires first the following lemma, which gives some properties of the tiling $T$:

Lemma 6.3. The tiling $T$ is such that:

1. all entries in rows with index at most -1 are identically equal to 5;
2. the zeroth row contains only tiles from $S$;
3. the rows with positive indices contain no entry in $S'$.

Here, the sets of tiles $S$ and $S'$ are those defined in (5.5).
Proof. Consider first the orthant comprising the point \((0, 0)\) with the associated initial condition \(T(0, 0) = 1\). Let \(s\) be a tile in the zeroth line of this orthant. The construction rule for the tiling \(T\) described in (4.2) implies that the bottom two entries in \(\varphi(s)\) determine two additional entries in the zeroth line of this orthant. Furthermore, all entries in the zeroth line are obtained this way. Also, all entries which are not in the zeroth row are determined, either from the top two entries in the image under \(\varphi\) of a tile lying in the zeroth row, or else from any entry in the image of any other tile. Since tile \(1\) belongs to the set \(S\), Points 1 & 2 in Proposition 5.5 show, on the one hand that the zeroth row in the orthant under consideration contains only tiles from \(S\) and on the other that all rows with index at most -1 are identically equal to 5.

A similar reasoning applies to the orthant comprising the point \((0, 1)\), which is mapped under \(T\) to tile 2, which is an element in \(S\). This establishes the first two claims in the lemma.

Consider now the orthant comprising the point \((1, 1)\) with the associated initial condition \(T(1, 1) = 4\). All elements in this orthant are obtained as entries in the image under \(\varphi\) of a tile \(s\) lying in this orthant. As tile 4 is not in \(S\), an easy induction based on Point 3 in Proposition 5.5 implies that no entry in this orthant is in \(S\).

A similar reasoning applies to the orthant comprising the point \((1, 0)\), which is mapped under \(T\) to tile 3, which is not an element in \(S\). This establishes the last claim in the lemma.

Proof of Theorem 6.2. The first step is to show that the part of the tiling \(T'\) with non–negative row indices cannot contain any \(4 \times 4\) zero window.

Lemma 6.4. Let \(P\) be a \(4\)–pattern contained in \(T'\) which is identically zero. Then \(P\) is contained in the portion of the tiling \(T'\) with negative row indices.

Proof. It is an immediate consequence of equation (6.1) that \(P\) also sits as a \(4\)–pattern in the tiling \(\tau_5(T)\). Lemma 4.12 applied with the parameters \(l = 13, r = 5\) and \(r' = 4\) then implies that \(P\) is contained in the image \(\tau_5(P')\) of a \(1\)–pattern \(P'\) sitting in \(T\). Let then \((m, n) \in \mathbb{Z}^2\) be such that \(P'(1) = T(m, n)\). From Definition 4.10 it holds that \(\tau_5(P') = \tau(T(m, n))\).

Since the zero \(4\)–pattern \(P\) is contained in \(\tau(T(m, n))\), Proposition 5.4 implies that \(T(m, n) \in S'\). Thus, from Lemma 6.3 it holds that \(m \leq 0\) and moreover that \(T(m, n) = 5\) when \(m < 0\) and \(T(m, n) \in S\) when \(m = 0\).

Note then these two observations: on the one hand, \(\tau'(T(m, n))\) is the \(8 \times 8\) subsquare in the \(13 \times 13\) square \(\tau(T(m, n))\) corresponding to the row and column indices between 4 and 11 (see equation (4.1)). On the other hand, it follows from the way the tiling \(\tau'(T)\) is defined in (4.4) that the \(8 \times 8\) square \(\tau'(T(m, n))\) sits between rows \(-7\) and 0 if \(m = 0\) and has maximal row index \(-8\) if \(m \leq -1\).

Since when \(m = 0\), the \(4\)–pattern \(P\) cannot overlap rows 10 to 13 in \(\tau(T(0, n))\) (cf. Proposition 5.4), the above two observations enable one to conclude that the pattern \(P\) is contained in the region of the tiling \(\tau'(T)\) with row index at most -2.

In order to complete the proof of Theorem 6.2 consider a \(4\)–pattern \(P\) in the region of the tiling \(T'\) with non–negative row indices. From Lemma 6.4 \(P\) cannot be identically zero. Furthermore, Lemma 4.12 implies here also that \(P\) is contained in \(\tau(T(m, n))\) for some \(m, n \in \mathbb{Z}\). From the last claim in Proposition 5.4, the pattern \(\tau(T(m, n))\) appears in the portion of the Number Wall (5.1), where all zero entries occur within squares with horizontal and vertical edges (this is Theorem 3.2).

Since there cannot be a zero \(4\)–pattern in the region \(m \geq 0\) of the tiling \(T'\), the zeros in the pattern \(P\) take the shape of a rectangle \(D\) obtained as the intersection between a square \(C\) with side length at most 3 and the \(4 \times 4\) square determined by \(P\).

Assume that \(C\) is not entirely contained in the pattern \(P\) as there is otherwise nothing more to prove. The rectangle \(D\) then admits a point, say \(A\), lying in the interior of the \(4 \times 4\) square determined by \(P\). Consider another \(4\)–pattern \(P'\) containing \(D\), one of which corners coincides with the point \(A\). The same argument as previously shows, on the one hand that \(P'\) is contained in the tiling \(T'\) and, on the other, that all zero entries in \(P'\) appear in the form of squares with side length at most 3. This is enough to conclude that the rectangle \(D\) can be extended to a square with side length at most 3 lying in the tiling \(T'\).
This concludes the proof of Theorem 6.2.

**Theorem 6.5.** The tiling $T'$ satisfies the Frame Constraints.

**Proof.** By Theorem 6.2 the maximal side length of a window lying the in part of $T'$ with non-negative row indices is 3. Therefore, by Corollary 3.6, the Frame Constraints are determined by patterns of size at most 7 (see also Figure 1). Applying Lemma 4.12 again with $l = 13$, $r = 5$ and $r' = 7$, every 7-pattern in $\tau_5(T)$ (and therefore in $T' = \tau'(T)$ from (6.1)) is contained in $\tau_5(P)$ for some 2-pattern $P$ in $T$. From Proposition 5.3, this pattern $P$ already appears in $T'|_{[m,M]}$.

Also, from Theorem 5.1, the image of $T'|_{[m,M]}$ under $\tau_5$ is, up to a translation by the vector $3 = (3,3)$, the restriction of the Number Wall (5.1) to the region $[8m,8M]$. Note that from the values of $m$ and $M$ in (5.4), this restriction followed by a translation by $3$ is clearly contained in (5.1). Since the Frame Constraints are satisfied in the Number Wall (5.1), this completes the proof of the theorem.

Theorems 6.2 and 6.5 can be rephrased as follows:

**Corollary 6.6.** The tiling $T'$ is a Number Wall with deficiency 4.

From the last claim in Corollary 3.6, $T'$ is the Number Wall of the sequence sitting in its zeroth row. This sequence is now determined:

**Theorem 6.7.** The tiling $T'$ has the Paper–Folding sequence in its zeroth row.

**Proof.** From Point 3 in Lemma 6.3, only the eight tiles in $S$ appear in the zeroth row of $T'$. Figure 8 tabulates these tiles, their images under the substitution $\varphi$ (in other words, the values of $s_1$ and $s_2$ in (5.6)) and also their images under the codings $\tau$ and $\tau'$ restricted to the zeroth line (which is the 11th row of $\tau$ and thus the 8th row of $\tau'$ — this follows immediately from the definition of $\tau'$ in (5.2) and from the fact that, given an integer $n$, $\tau'(T(0,n))$ sits between rows $-7$ and 0 in the tiling $T'$). For simplicity, the original eight tiles are mapped bijectively to \{0,...,7\} and the above restrictions of $\varphi$, $\tau$ and $\tau'$ are still denoted in the same way.

```
| Tile | Coded tile | Substitution $\varphi$ | Coding $\tau, \tau'$ |
|------|------------|-----------------------|----------------------|
| 2    | 0          | 0 2                   | 110 00100110 10      |
| 13   | 1          | 0 3                   | 110 00100111 30      |
| 7    | 2          | 1 6                   | 110 00110110 70      |
| 12   | 3          | 1 7                   | 110 00110111 70      |
| 20   | 4          | 4 2                   | 111 00100110 30      |
| 6    | 5          | 4 3                   | 111 00100111 30      |
| 1    | 6          | 5 6                   | 111 00110110 70      |
| 29   | 7          | 5 7                   | 111 00110111 70      |
```

Figure 8: The generated tiling restricted to the zeroth row: substitutions and codings (the original eight tiles are mapped bijectively to \{0,...,7\}).

Using coded tiles, the zeroth row of the tiling $T'$ is thus the Z–tiling $\tau'(T_{[\varphi,(0,0)]})$ (this follows from the first two initial conditions in (5.7), which are expressed in the language of non-coded tiles).

Looking at the boxed segments in Figure 8 that represent the coding $\tau'$ restricted to the zeroth line reveals that mapping the (coded) tiles 4, 5, 6, 7 to 0, 1, 2, 3 respectively in the substitution $\varphi$ is consistent with their definition on 0, 1, 2, 3 (for instance, $\varphi(4) = 4 2$ and mapping 4 to 0 transforms the tiling 4 2 to 0 2, which is indeed $\varphi(0)$). Furthermore, this mapping leaves unchanged the images of the tiles under $\tau'$ (for instance, $\tau'(4) = \tau'(0)$). This implies in particular that the zeroth row of $T'$ is also the Z–tiling $\tau'(T_{[\varphi,(2,0)]})$. 27
A further comparison with $\psi$ and $\rho$ introduced in Example 4.6 shows that, in fact,
$$
\tau'(\varphi(s)) = \rho(\psi^4(s))
$$
for each $s \in \{0, 1, 2, 3\}$, and therefore for all $s \in \{0, \ldots, 7\}$ upon identifying tiles as above when needed.

It is elementary to verify that
$$
T(\psi^4, (2, 0)) = T(\psi, (2, 0)).
$$
Since the Paper–Folding sequence is $\rho(T(\psi, (2, 0)))$ (cf. Example 4.6), this concludes the proof that the zeroth row of the generated Number Wall is the Paper–Folding sequence. \qed

7 The $t$–adic Littlewood Conjecture in other Characteristics

7.1 Sequences with Small Deficiency over Finite Fields

The aim of this section is to discuss to what extent the value of the deficiency appearing in Theorem 2.6 (viz. 4) can be improved and/or generalized to other characteristics.

It is easily seen that one cannot have a Number Wall over $\mathbb{F}_3$ with no zero entries. This amounts to saying that any infinite Hankel matrix over $\mathbb{F}_3$ admits a singular connected minor (that is, a singular square submatrix whose row and column indices are consecutive). This prompts the following more general open problem, which the authors have not been able to solve:

**Question 7.1.** Let $\mathbb{F}_q$ be a finite field with $q \geq 2$ elements. Does there exist an integer $n(q) \geq 1$ such that any square matrix with dimensions $n(q) \times n(q)$ and with entries in $\mathbb{F}_q$ admits a singular connected minor (as defined above)?

In other words, Question 7.1 amounts to asking how big a hyperinvertible matrix can be over a finite field, where hyperinvertibility of a matrix means that all connected square submatrices are invertible. Note that when the field is infinite, the well–known class of Cauchy matrices provides examples of arbitrarily large hyperinvertible matrices. Also, Question 7.1 is well–understood in the case that one does not restrict oneself to the case of connected minors: it is indeed proved in [25] that there is no $q \times (q - 1)$ matrix over $\mathbb{F}_q$ whose (non–necessarily connected) minors of order $q - 1$ and $q - 2$ are all different from zero.

On another front, Theorem 2.6 raises the question so as to whether there exists a sequence with deficiency smaller than 4 over $\mathbb{F}_3$ and, possibly, with optimal value 2. This is indeed the case, and the discovery of a sequence generating a Number Wall with only isolated zeros, the Pagoda sequence $\pi_n$ for $n \in \mathbb{Z}$, was made in [27] (see also [29] for a fuller account. The origin of the name of the sequence is also explained in the latter reference). It is defined from the Paper–Folding sequence as follows: for any $n \in \mathbb{Z}$,
$$
\pi_n = f_{n+1} - f_{n-1}.
$$

Tiling the Number Wall obtained from this sequence confirms the above claim:

**Theorem 7.2.** The Number Wall of the Pagoda sequence over $\mathbb{F}_3$ has only isolated zero entries.

The proof of Theorem 7.2 proceeds along the same lines as the proof of Theorem 5.1 and will not be detailed here. In view of Theorem 2.2 Theorem 7.2 implies the existence of a formal Laurent power series $\Xi$ such that
$$
\inf_{N \neq 0, k \geq 0} |N| \cdot |\langle Nt^k \Xi \rangle| = 3^{-2}.
$$
This equality corresponds to the “worst” possible case when $t$–LC fails over $\mathbb{F}_3$.

---

4in the rows with non–negative indices. This assumption will always be implicit in what follows.

5These details can be made explicit from the codes available at [30, 32] which deal, not only with the case of the Paper–Folding sequence, but also with that of the Pagoda sequence.
As a matter of fact, a generalisation to other characteristics is suggested by computer evidence:

**Conjecture 7.3.** The Paper–Folding and Pagoda sequences seen as sequences over a finite field \( \mathbb{F}_p \) have bounded deficiency 4 and 2 respectively for all prime \( p \equiv 3 \pmod{4} \), and unbounded deficiency for all other primes.

Conjecture 7.3 has been checked extensively by computer inspection of the Number Walls of the sequences under consideration. For instance, in the case of the Paper–Folding sequence, it has been verified in finite \( 3000 \times 3000 \) segments of its Number Walls over \( \mathbb{F}_p \) for all \( p \leq 101 \). The difficulty in validating this conjecture for a given characteristic is that the number of tiles needed to put in place the strategy exposed in Section 3.2 increases very quickly with the characteristic (for instance, preliminary attempts to construct a Pagoda Wall over \( \mathbb{F}_7 \) indicate the putative existence of a tiling made of 1.4 million tiles).

If indeed true, Conjecture 7.3 would imply that the \( t \)-adic Littlewood Conjecture fails (at least) over any field with characteristic a prime congruent to 3 modulo 4.

### 7.2 On the Laurent Series of the Paper–Folding Sequence in a Field with Characteristic 2

The Hankel matrices of the Paper–Folding and Pagoda sequences are much better understood over \( \mathbb{K} = \mathbb{F}_2 \). This follows from the fact that the Laurent series they define in \( \mathbb{F}_2 ((t^{-1})) \) are both quadratic. Indeed, note that \( f_{2n+1} = n \) modulo 2 and \( f_{2n} = f_n \) for every \( n \in \mathbb{N} \). Therefore,

\[
\Phi(t) = \sum_{n>0} f_n t^{-n} \\
= \sum_{n>0} f_{2n} t^{-2n} + t^{-1} \sum_{n>0} f_{2n+1} t^{-2n} \\
= \sum_{n>0} f_n t^{-2n} + t^{-1} \sum_{n>0} nt^{-2n} \\
= \Phi(t^2) - t^{-3} \sum_{n \geq 1} t^{-4n} \\
= \Phi(t^2) - \frac{t^{-3}}{1 - t^{-4}}.
\]

This implies that the Laurent series \( \Phi(t) \) satisfies this quadratic equation over \( \mathbb{F}_2 (t) \):

\[
\Phi(t^2) + \Phi(t) + \frac{t}{1 + t^4} = 0.
\]

Similarly for \( \Pi(t) = \sum_{n>0} \pi_n t^{-n} \), one obtains from the relations \( f_0 = f_1 = 0 \) that

\[
\Pi(t) = \sum_{n \geq 0} (f_{n+1} - f_{n-1}) t^{-n} = (t-t^{-1}) \Phi(t).
\]

Thus, \( \Pi(t) \) satisfies the quadratic equation

\[
\Pi(t^2) + \frac{1 + t^2}{t} \Pi(t) + \frac{1}{t} = 0.
\]

In the same paper as where the \( p \)-adic Littlewood Conjecture was first stated [16], De Mathan and Teulié established that \( t \)-LC holds for quadratic irrational power series. This is therefore in particular the case for the series \( \Phi \) and \( \Pi \) above. Of course, this claim extends naturally to any field with characteristic 2.

It should also be noted that much more is currently known about the occurrence of windows in the Number Walls of quadratic series — see for further details the paper by Kemarsky, Paulin and Shapira [22], which uses dynamics on Bruhat–Tits tree. The connections to the present work are explained in page 5 therein.
8 Implementation and Code

It is worthwhile to mention several matters related to the implementation of the algorithms described in Figures 5 and 6. The complete Magma program is contained in the file `dragon_wall_B.mag` available at [30]. The results have been confirmed by an independent implementation in Sage, for which the program is available at [32] (6).

8.1 The Wall Builder

The wall builder described in Figure 5 is implemented as a Magma program `procedure NumberWall (~seq, mlo, nhi, nlo, nhi, ~wal)`, where `seq` and `wal` hold respectively the sequence $S_{0,n}$ and the wall entries $S_{m,n}$, and `mlo` and `mhi` (resp. `nlo` and `nhi`) the ranges of $m$ (resp. of $n$) in the output segment. Initial rows $m \leq 0$ are determined via Definition 3.1. The variable number `mlo` of rows of the top infinite zero part is asserted to be less than or equal to $-2$, and serves as a sentinel window that allows conveniently for subsequent tiling implementation requirements.

The natural boundary of a wall segment with, say, $0 \leq m \leq m_{hi}$ and $0 \leq n \leq n_{hi}$ is trapezoidal, descending from a sequence segment of length $2m_{hi} + n_{hi}$ at $m = 0$ to length $n_{hi}$ at $m = m_{hi}$ (see also §5.1 for details). To avoid complicated program logic where square window frames cross the boundary, two further effectively infinite sentinel windows are attached along left and right segment edges. This permits filling a rectangular boundary with lengths roughly $m_{hi} \times (2m_{hi} + n_{hi})$, partially containing spurious entries which can ultimately be ignored. Finally, after pruning to lengths $m_{hi} \times n_{hi}$, the rectangle contains only valid entries.

8.2 Counting Deficiencies in a Wall

A separate Magma program `procedure WallDeficiencies (~wal, mlo, nhi, nlo, nhi, ~mulset)` computes as a multiset `mulset` the number of windows of each deficiency $\delta = g + 1$ (where $g$ is the side length of a window). Broken windows (where the pane crosses the boundary of the segment) are represented temporarily by $\delta = -1$. Not all cases with $\delta \in \{-1, 1\}$ are recorded; the purpose is to accurately detect all visible windows occurring properly within the region. The algorithm utilises a subset of the method discussed in Section 8.1.

8.3 Finding Patterns in Sequences and Tilings

The tile builder described in Figure 6 is implemented as `procedure SquareTiling (~tab, mlo, nhi, nlo, nhi, ~tel, ~tetrads, ~stab)` 7 Here, with the notation of Theorem 5.1:

- `tab` holds the wall $S_{m,n}$;
- `codes` returns the images $\tau(s)$ of the tiles $s \in \Sigma$ under the 13–coding $\tau$;
- `stab` returns the tiling $T$ restricted to the region $R$;
- `tetrads` returns the images under the 2–substitution $\phi$ of the 2353 tiles $s \in \Sigma$ followed by the rest of the 2–patterns in `stab` that are not obtained as such images;
- `tel`, which stands for `tile edge length`, is the value taken by $l - 1$, where $l$ is the parameter used to refer to the coding $\tau$ as an $l$–coding (thus, `tel`=12 for the coding $\tau$ in Theorem 5.1). In practice, the parameter `tel` is restricted to even values;
- `cid` is the distance between the centers of two overlapping encoded tiles $\tau(s)$ and $\tau(s')$ in the tiling $T'$ (from Figure 2 for instance, `cid`=8 in the case under consideration). In practice, the parameter `cid` is also restricted to even values.

6It should be noted that both in [30] and in [32], the `Paper–Folding sequence` is referred to as the `Dragon sequence`.  
7In the codes [30] [32], a `tile` is referred to as a `state` and a `substitution` as an `inflation`.

30
Note that the value of the parameter $r$ when referring to the coding $\tau$ as being $r$–consistent is then $r = tel - cid + 1$ (hence, $r = 5$ in Theorem 5.1). It is the width of overlap between two adjacent encoded tiles.

8.4 Adjusting some Parameters

As the 2–dimensional substitution $\varphi$ is meant to generate the Paper–Folding sequence along the zeroth row of the Number Wall, and as the Paper–Folding sequence can be obtained as a one dimensional 2–substitution followed by a coding — see Example 4.6 —, it is natural to impose that $\varphi$ should be a $k$–substitution with $k = 2$.

The translation by the vector $3 = (3, 3)$ in the definition of the coding $\tau'$ in (5.2) is due to the fact that the algorithm described in Figure 6 is actually run in [30, 32] with a slightly different definition of the coding $\tau_r$ defined in (4.6); namely, with the coding

$$\tilde{\tau}_r(s) = \tau(s)|_{\{u, \ldots, v\}^d}.$$  \hfill (8.1)

Here, $s$ is a tile in $\Sigma$ and

$$u = \left\lfloor \frac{l + 1}{2} \right\rfloor - \left\lfloor \frac{l - r}{2} \right\rfloor + 1 \quad \text{and} \quad v = \left\lfloor \frac{l + 1}{2} \right\rfloor + \left\lfloor \frac{l - r}{2} \right\rfloor$$

are integers ($\lfloor x \rfloor$ denotes the integer part of a real number $x$). Note that when $l = 13$ and $r = 5$ as in Theorem 5.1, $u = 4$ and $v = 11$. Definition (8.1) ensures that the center of the square with side length $l - r$ determined by $\{u, \ldots, v\}^d$ coincides with a point nearest to the center of the square $\tau(s)$ it is contained in (which square has side length $l$). From Lemma 4.9, the resulting tiling $\tilde{\tau}_r(T)$ differs from the tiling $\tau_r(T)$ by a translation by the vector $j \in \mathbb{Z}^d$ all of whose components are equal to $u - 1$.

In the case of Theorem 5.1, the coding (8.1) presents the advantage of providing a tiling $\tilde{\tau}_r(T)$ which coincides exactly with the Number Wall under consideration. The theory developed in Section 4.3 with the coding $\tau_r$ can nevertheless be seen as more robust. Indeed, on the one hand, the choice of the top–left square considered in the definition of $\tau_r$ in (4.6) is more natural at least in the case of $N^2$–tilings. On the other, the choice of a point nearest to the center of a square is not necessarily unique depending on the parity of $l$ and $r$ whereas the top–left subsquare with a predefined side length is always unambiguously defined, given a square.

8.5 Canonical Order in a Two Dimensional Array

One matter of a purely algorithmic nature is the ordering of the tiles in the set $\Sigma = \{1, \ldots, 2353\}$. In Pass 1 of the algorithm described in Figure 6, not much importance is given to this ordering. However, in practice, imposing a certain canonical label to the elements of $\Sigma$ is of crucial importance in order to check the correctness of the algorithms and to interpret and exploit their outputs. Due to its theoretical irrelevance, this step has not been included in Figure 6 (it should otherwise be considered as the last Pass of the algorithm).

A general method of assigning a canonical index to a given tile in $\Sigma$ is based on the “matrix Manhattan metric” and is applicable to symmetric segments in any dimension. This method is described here in dimension two in the context of Theorem 5.1 with the notation introduced therein.

Imposing that the parameters $tel$ and $cid$ introduced in Section 8.3 should be even guarantees that the squares $\{\tau(s)\}_{s \in \Sigma}$ which, from Proposition 5.4, appear in the tiling $T' = \tau'(T)$, are centered at integer points in the plane (this follows from the definition of the coding $\tau'$ in (5.2) and of the construction rule (4.4)).

Consider the portion of the tiling lying in the region defined by the conditions

$$m^{lo} \leq m \leq m^{hi} \quad \text{and} \quad n^{lo} \leq n \leq n^{hi}$$  \hfill (8.2)
and let there be a square $C$ obtained as the image under $\tau$ of a tile in $\Sigma$. Note that from the injectivity of the coding $\tau$ (this is guaranteed by the algorithm in Figure 6 — see also Section 5.2), the underlying tile is uniquely determined by $C$. This tile is then assigned a canonical value defined as

$$\Delta = \min \text{dist}(m,n),$$

where

$$\text{dist}(m,n) = |m| + |n| + \frac{m}{10^b} + \frac{n}{10^c},$$

with

$$b = m_{hi} - m_{lo} \quad \text{and} \quad c = n_{hi} - n_{lo},$$

and where the minimum in (8.3) is taken over all centers $(m,n)$ of squares identical to $C$ lying in the region (8.2).

It is elementary to establish that $\Delta$ is independent of the region (8.2) provided this region should be sufficiently large along each axis. The values assigned in (8.3) allow reordering the tiles according to their “earliest” appearance in $T$, first by least Manhattan (or $\ell_1$) distance from the origin, then by row $m$, and finally by column $n$ (utilising floating–point arithmetic for this purpose avoids integer overflow for large tables). This ordering then defines the index of the tile in $\Sigma = \{1, \ldots, 2353\}$. Figure 7 shows an example.

### 8.6 Code Design

Primary design targets for the present program were robustness and portability, rather than optimum deployment of computer time or space, leading to several similar instances of deliberately inefficient consumption of resources. One is the preliminary rectangular wall created and then pruned by the wall builder in Section 8.1. In similar vein, the tile builder requires room for tiles involving the initial sequence to form correctly, which tiles are created by padding the wall with rows from the top sentinel window: in practice, the choice of a parameter

$$m_{lo} \leq -\frac{5}{2} \cdot (cid + tel)$$

proves adequate for this purpose. Another non–optimal step is the naive machinery selecting coordinates $(m,n)$ of centers of encoded tiles and converting those to Magma array addresses $(i,j)$ (which start from 1). The ensuing time penalty remains negligible, but the improved transparency is significant for maintainability, and for establishing the extent to which the program satisfies its specification.

Similar considerations prompt limitations on the Number Wall and tiling models employed above, mostly motivated by maintaining symmetry, but easily relaxable at the cost of some increase in complexity. As mentioned in Section 3 using Hankel determinants instead of Toeplitz determinants in the definition of a Number Wall would introduce asymmetry between $m,n$ into the Frame Theorem (Corollary 3.6), thereby complicating its statement and application. Also, allowing odd tile edge length $tel$ or odd centre separation $cid$ (see Section 8.3) would require processing half–integer coordinates for the centers of encoded tiles. Note also that the Paper–Folding sequence has been extended to a doubly–infinite sequence in Section 8.2 in view of the following observation: one–sided sequences would introduce boundaries in their Number Wall, which would require special consideration. In contrast, since the actual data objects involved are finite segments of walls, such boundaries are in practice automatically incorporated at the computational stage.

One case where this minimalist philosophy has been abandoned concerns the canonical ordering of tiles as described in Section 8.5. Assigning these required extra program to build a ‘mini–wall’ of tiles, as well as an extra stage to sort the output. But the enhanced tile builder no longer needs time–expensive searches for tile encodings when building the substitution $\varphi$ (cf. Pass 2 in Figure 6), nor does it now require simultaneous access to the entire Number Wall. The result is doubled speed and (in concert with a rolling row–by–row wall builder) a potentially significant reduction in space, besides the improvement in usability which constituted its primary motivation.
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