BEM solutions to BVPs governed by the anisotropic modified Helmholtz equation for quadratically graded media

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Abstract. A Boundary Element Method (BEM) is used for obtaining solutions to anisotropic quadratically graded media (FGM) boundary value problems (BVPs) governed by the modified Helmholtz type equation. A technique of transforming the variable coefficient governing equation to a constant coefficient equation is utilized for deriving a boundary integral equation. Some particular problems are considered to illustrate the application of the BEM. The results show the convergence, consistency, and accuracy of the BEM solutions.

1. Introduction
The BEM has been successfully used for solving many types of problems of homogeneous media. Some works on homogeneous media problems have been recently done by Azis et. al [1, 2] and Haddade et. al [3] in which the authors considered pollutant transport problems governed by 2D diffusion-convection, and also Azis [4] in which numerical solutions for the Helmholtz boundary value problems were obtained.

However, this is generally not the case for FGM. There are two techniques usually used to deal with problems of FGM. The first one uses a technique of deriving a relevant Green function or fundamental solution to the FGM problem. Cheng [5] had applied this technique. The second technique is by transforming the variable coefficient governing equation to a constant coefficient equation. Some progress on using the second technique has been made. For examples, Clements and Azis [6] considered the case for isotropic FGM. For anisotropic FGM some works have been studied by Azis and Clements [7, 8, 9] and Azis et. al [10]. In addition to this, recently Salam et. al [11] have been working on a class of elliptic problems for anisotropic FGM.

This paper discusses derivation of a boundary integral equation for numerically solving 2D boundary value problems governed by the dimensionless modified Helmholtz type equation of the form

$$\frac{\partial}{\partial x_i} \left[ \lambda_{ij} \phi(x_1, x_2) \right] - \beta^2 \phi(x_1, x_2) = 0$$

where the coefficients $\lambda_{ij}$ and $\beta^2$ depend on $x_1$ and $x_2$ and the repeated summation convention (summing from 1 to 2) is employed.
For (1) to be a second order elliptic partial differential equation, we assume that the matrix of coefficients \([\lambda_{ij}]\) is a real symmetric positive definite matrix. Explicitly equation (1) may be written as

\[
\frac{\partial}{\partial x_1} \left( \lambda_{11} \frac{\partial \phi}{\partial x_1} \right) + 2 \frac{\partial}{\partial x_1} \left( \lambda_{12} \frac{\partial \phi}{\partial x_2} \right) + \frac{\partial}{\partial x_2} \left( \lambda_{22} \frac{\partial \phi}{\partial x_2} \right) - \beta^2 \phi = 0
\]

Further, the coefficients \(\lambda_{ij}\) and \(\beta^2\) are required to be twice differentiable functions of the two independent variables \(x_1\) and \(x_2\). The analysis is also relevant for isotropic media as a special case of anisotropic media which occurs when \(\lambda_{11} = \lambda_{22}\) and \(\lambda_{12} = 0\).

A variety of problems of both isotropic and anisotropic inhomogeneous media are usually modeled with equation (1). Infiltration problems (when \(\beta^2 > 0\)), and antiplane strain in elastostatics and plane thermostatic problems (when \(\beta^2 = 0\)) are the areas for which the governing equation is of the type (1). Recently some works on the modified Helmholtz equation for isotropic and/or homogeneous media have been done. For example, Clements and Lobo in [12] and Solekhudin and Ang in [13] considered the modified Helmholtz equation as the governing equation for isotropic and homogeneous media which is a special case of the equation (1).

2. The boundary value problem

Referred to a Cartesian frame \(Ox_1x_2\) a solution to (1) is sought which is valid in a region \(\Omega\) in \(\mathbb{R}^2\) with boundary \(\partial \Omega\) which consists of a finite number of piecewise smooth closed curves. On \(\partial \Omega_1\) the dependent variable \(\phi(x)\) \((x = (x_1, x_2))\) is specified and on \(\partial \Omega_2\)

\[
P(x) = \lambda_{ij} \left( \frac{\partial \phi}{\partial x_j} \right) n_i
\]

is specified where \(\partial \Omega = \partial \Omega_1 \cup \partial \Omega_2\) and \(n = (n_1, n_2)\) denotes the outward pointing normal to \(\partial \Omega\).

3. The boundary integral equation

The boundary integral equation is derived by transforming the variable coefficient equation (1) to a constant coefficient equation. The coefficients \(\lambda_{ij}\) and \(\beta\) are required to take the form

\[
\lambda_{ij}(x) = \overline{\lambda}_{ij} g(x)
\]

\[
\beta^2(x) = \overline{\beta}^2 g(x)
\]

where the \(\overline{\lambda}_{ij}\) and \(\overline{\beta}\) are constants and \(g\) is an exponential function of \(x\). Use of (3) and (4) and in (1) yields

\[
\overline{\lambda}_{ij} \frac{\partial}{\partial x_i} \left( g \frac{\partial \phi}{\partial x_j} \right) - \overline{\beta}^2 g \phi = 0
\]

Let

\[
\phi(x) = g^{-1/2}(x) \psi(x)
\]

so that (5) may be written in the form

\[
\overline{\lambda}_{ij} \frac{\partial}{\partial x_i} \left[g \frac{\partial (g^{-1/2} \psi)}{\partial x_j} \right] - \overline{\beta}^2 g^{1/2} \psi = 0
\]

That is

\[
\overline{\lambda}_{ij} \left[ \frac{1}{4} g^{-3/2} \frac{\partial g}{\partial x_i} \frac{\partial g}{\partial x_j} - \frac{1}{2} g^{-1/2} \frac{\partial^2 g}{\partial x_i \partial x_j} \right] \psi + g^{1/2} \frac{\partial^2 \psi}{\partial x_i \partial x_j} \right] - \overline{\beta}^2 g^{1/2} \psi = 0
\]
Use of the identity
\[
\frac{\partial^2 g^{1/2}}{\partial x_i \partial x_j} = -\frac{1}{4}g^{-3/2} \frac{\partial g}{\partial x_i} \frac{\partial g}{\partial x_j} + \frac{1}{2}g^{-1/2} \frac{\partial^2 g}{\partial x_i \partial x_j}
\]
permits (7) to be written in the form
\[
g^{1/2} \lambda_{ij} \frac{\partial^2 \psi}{\partial x_i \partial x_j} - \psi \lambda_{ij} \frac{\partial^2 g^{1/2}}{\partial x_i \partial x_j} - \beta^2 g^{1/2} \psi = 0 \tag{8}
\]
If we further restrict the function \(g(x)\) to take the exponential form
\[
g(x) = [A (\alpha_0 + \alpha_1 x_1 + \alpha_2 x_2)]^2 \tag{9}
\]
where \(\alpha_m\) are constant, then
\[
\lambda_{ij} \frac{\partial^2 g^{1/2}}{\partial x_i \partial x_j} = 0 \tag{10}
\]
Substitution (10) into (8) implies a constant coefficients equation
\[
\lambda_{ij} \frac{\partial^2 \psi}{\partial x_i \partial x_j} - \beta^2 \psi = 0 \tag{11}
\]
Also, substitution of (3) and (6) into (2) gives
\[
P = -P_g \psi + P_\psi g^{1/2} \tag{12}
\]
where
\[
P_g (x) = \lambda_{ij} \frac{\partial g^{1/2}}{\partial x_j} n_i \quad P_\psi (x) = \lambda_{ij} \frac{\partial \psi}{\partial x_j} n_i
\]
A boundary integral equation for the solution of (11) is given in the form
\[
\eta(x_0) \psi(x_0) = \int_{\partial \Omega} [\Gamma(x, x_0) \psi(x) - \Phi(x, x_0) P_\psi(x)] \, ds(x) \tag{13}
\]
where \(x_0 = (a, b)\), \(\eta = 0\) if \((a, b) \notin \Omega \cup \partial \Omega\), \(\eta = 1\) if \((a, b) \in \Omega\), \(\eta = \frac{1}{2}\) if \((a, b) \in \partial \Omega\) and \(\partial \Omega\) has a continuously turning tangent at \((a, b)\).

The so called fundamental solution \(\Phi\) in (13) is any solution of the equation
\[
\lambda_{ij} \frac{\partial^2 \Phi}{\partial x_i \partial x_j} - \beta^2 \Phi = \delta (x - x_0)
\]
and the \(\Gamma\) is given by
\[
\Gamma(x, x_0) = \lambda_{ij} \frac{\partial \Phi(x, x_0)}{\partial x_j} n_i
\]
where \(\delta\) is the Dirac delta function. Following Azis [14], for two-dimensional problems \(\Phi\) and \(\Gamma\) are given by
\[
\Phi(x, x_0) = \begin{cases} 
\frac{K}{2\pi} \ln R & \text{if } \beta^2 = 0 \\
\frac{K}{2\pi} K_0 (\omega R) & \text{if } \beta^2 > 0
\end{cases}
\]
\[
\Gamma(x, x_0) = \begin{cases} 
\frac{K}{2\pi} \frac{1}{R} \lambda_{ij} \frac{\partial R}{\partial x_j} n_i & \text{if } \beta^2 = 0 \\
\frac{K}{2\pi} K_1 (\omega R) \lambda_{ij} \frac{\partial R}{\partial x_j} n_i & \text{if } \beta^2 > 0
\end{cases}
\tag{14}
\]
where

\[
K = \frac{\ddot{\tau}}{\zeta} \\
\omega = \sqrt{\left| \beta \right|^2 / \zeta} \\
\zeta = \frac{[\lambda_{11} + 2\lambda_{12}\ddot{\tau} + \lambda_{22}(\ddot{\tau}^2 + \ddot{\tau}^2)]}{2} \\
R = \sqrt{(\dot{x}_1 - \dot{a})^2 + (\dot{x}_2 - \dot{b})^2} \\
\dot{x}_1 = x_1 + \dot{\tau} x_2 \\
\dot{a} = a + \dot{\tau} b \\
\dot{x}_2 = \ddot{\tau} x_2 \\
\dot{b} = \ddot{\tau} b
\]

where \(\dot{\tau}\) and \(\ddot{\tau}\) are respectively the real and the positive imaginary parts of the complex root \(\tau\) of the quadratic

\[
\lambda_{11} + 2\lambda_{12}\tau + \lambda_{22}\tau^2 = 0
\]

and \(K_0, K_1\) denote the modified Bessel function of order zero and order one respectively. The derivatives \(\partial R/\partial x_j\) needed for the calculation of the \(\Gamma\) in (14) are given by

\[
\frac{\partial R}{\partial x_1} = \frac{1}{R} (\dot{x}_1 - \dot{a}) \\
\frac{\partial R}{\partial x_2} = \dot{\tau} \left[ \frac{1}{R} (\dot{x}_1 - \dot{a}) \right] + \ddot{\tau} \left[ \frac{1}{R} (\dot{x}_2 - \dot{b}) \right]
\]

Use of (6) and (12) in (13) yields

\[
\eta(x_0) g^{1/2}(x_0) \phi(x_0) = \int_{\partial \Omega} \left\{ \left[ g^{1/2}(x) \Gamma(x, x_0) - P_g(x) \Phi(x, x_0) \right] \phi(x) - \left[ g^{-1/2}(x) \Phi(x, x_0) \right] P(x) \right\} ds(x) \tag{15}
\]

This equation provides a boundary integral equation for determining \(\phi\) and \(P\) at all points of \(\Omega\).

4. Numerical examples

Some particular boundary value problems will be solved numerically by employing the integral equation (15). The main aim is to show the validity of the analysis used above for deriving the boundary integral equation (15) and the appropriateness of the BEM in solving the problems through the derived boundary integral equation (15). Standard boundary element method is employed to obtain numerical results. The integrals in equation (15) are evaluated numerically using the Bode’s quadrature (see Abramowitz and Stegun [15]).

4.1. Problem 1

We will consider a problem with analytical solution. The aim is to verify the convergence and accuracy of the BEM. We choose an anisotropic medium with

\[
\lambda_{11} = 1, \lambda_{12} = 0.5, \lambda_{22} = 1
\]

and we take an inhomogeneity function \(g(x)\) satisfying (9) of the form

\[
g(x) = [2 (1 + 0.5x_1 + 0.75x_2)]^2
\]
Plot of $g(x)$ is shown in Figure 1. For the analytical solution $\phi$ we will take

$$\psi(x) = 3.5 \exp (0.75x_1 + 0.25x_2)$$

so that from (11) $\beta^2 = 0.8125$. Therefore from (6) the analytical solution

$$\phi(x) = \frac{3.5 \exp (0.75x_1 + 0.25x_2)} {2 (1 + 0.5x_1 + 0.75x_2)}$$

![Figure 1. A quadratical inhomogeneity function $g(x) = [2 (1 + 0.5x_1 + 0.75x_2)]^2$](image)

The geometry of the region $\Omega$ and the boundary conditions are as depicted in Figure 2.

![Figure 2. The geometry of Problem 1](image)

The results are shown in Table 1. The BEM solution converges to the analytical solution as the number of elements increases. The accuracy of the BEM solutions lie on the fourth figure of the decimals.
Table 1. BEM and analytical solutions for Problem 1

| \( (x_1, x_2) \) | \( \phi \) | \( \frac{\partial \phi}{\partial x_1} \) | \( \frac{\partial \phi}{\partial x_2} \) | \( \phi \) | \( \frac{\partial \phi}{\partial x_1} \) | \( \frac{\partial \phi}{\partial x_2} \) |
|-----------------|---|---|---|---|---|---|
| BEM 160 elements | BEM 320 elements |
| (0.1,0.5) | 1.5005 | 0.5981 | -0.4138 | 1.5002 | 0.5984 | -0.4141 |
| (0.3,0.5) | 1.6289 | 0.6873 | -0.3929 | 1.6287 | 0.6874 | -0.3933 |
| (0.5,0.5) | 1.7761 | 0.7858 | -0.3747 | 1.7758 | 0.7856 | -0.3751 |
| (0.7,0.5) | 1.9440 | 0.8954 | -0.3583 | 1.9437 | 0.8948 | -0.3587 |
| (0.9,0.5) | 2.1351 | 1.0179 | -0.3433 | 2.1346 | 1.0169 | -0.3433 |
| (0.5,0.1) | 1.9700 | 0.7343 | -0.6184 | 1.9702 | 0.7344 | -0.6207 |
| (0.5,0.3) | 1.8610 | 0.7653 | -0.4789 | 1.8608 | 0.7651 | -0.4799 |
| (0.5,0.7) | 1.7095 | 0.8007 | -0.2946 | 1.7092 | 0.8005 | -0.2947 |
| (0.5,0.9) | 1.6571 | 0.8123 | -0.2311 | 1.6568 | 0.8122 | -0.2313 |

BEM 640 elements Analytical

| (0.1,0.5) | 1.5001 | 0.5985 | -0.4143 | 1.5000 | 0.5987 | -0.4145 |
| (0.3,0.5) | 1.6286 | 0.6874 | -0.3935 | 1.6284 | 0.6874 | -0.3938 |
| (0.5,0.5) | 1.7757 | 0.7855 | -0.3754 | 1.7755 | 0.7853 | -0.3756 |
| (0.7,0.5) | 1.9435 | 0.8945 | -0.3589 | 1.9433 | 0.8942 | -0.3591 |
| (0.9,0.5) | 2.1343 | 1.0164 | -0.3434 | 2.1341 | 1.0159 | -0.3435 |
| (0.5,0.1) | 1.9702 | 0.7343 | -0.6217 | 1.9703 | 0.7342 | -0.6227 |
| (0.5,0.3) | 1.8608 | 0.7649 | -0.4804 | 1.8607 | 0.7648 | -0.4809 |
| (0.5,0.7) | 1.7090 | 0.8004 | -0.2948 | 1.7088 | 0.8003 | -0.2948 |
| (0.5,0.9) | 1.6566 | 0.8122 | -0.2312 | 1.6565 | 0.8121 | -0.2313 |

4.2. Problem 2

We consider a problem for a homogeneous isotropic medium with \( g(x) = 25 \), \( \lambda_{11} = \lambda_{22} = 1 \), \( \lambda_{12} = 0 \) and \( \beta^2 = 1 \). The geometry of the region \( \Omega \) and the boundary conditions are as depicted in Figure 3. As shown in Figure 3 the boundary conditions are symmetrical about the axes \( x_1 = 0 \). Table 2 shows the results of the BEM solution using 80, 160, 320 and 640 elements of equal length. As expected, the results converge as the number of elements increases and also they are symmetrical about the axes \( x_1 = 0.5 \).

Figure 3. The geometry of Problem 2 and Problem 3
### Table 2. BEM solution for Problem 2

| \((x_1, x_2)\) | \(\phi\) | \(\partial \phi / \partial x_1\) | \(\partial \phi / \partial x_2\) | \(\phi\) | \(\partial \phi / \partial x_1\) | \(\partial \phi / \partial x_2\) |
|----------------|--------|-----------------|-----------------|--------|-----------------|-----------------|
| (0.1,0.5)      | 0.5060 | 0.0748          | 0.9118          | 0.5060 | 0.0750          | 0.9110          |
| (0.3,0.5)      | 0.5214 | 0.0647          | 0.7857          | 0.5214 | 0.0646          | 0.7845          |
| (0.5,0.5)      | 0.5282 | -0.0000         | 0.7435          | 0.5283 | -0.0000         | 0.7423          |
| (0.7,0.5)      | 0.5214 | -0.0647         | 0.7857          | 0.5214 | -0.0646         | 0.7845          |
| (0.9,0.5)      | 0.5060 | -0.0748         | 0.9118          | 0.5060 | -0.0750         | 0.9110          |
| (0.5,0.1)      | 0.3230 | -0.0000         | 0.2217          | 0.3235 | -0.0000         | 0.2207          |
| (0.5,0.3)      | 0.3996 | -0.0000         | 0.5291          | 0.3999 | -0.0000         | 0.5279          |
| (0.5,0.7)      | 0.6935 | 0.0000          | 0.9045          | 0.6933 | -0.0000         | 0.9032          |
| (0.5,0.9)      | 0.8901 | 0.0000          | 1.0658          | 0.8896 | -0.0000         | 1.0643          |
| (0.5,0.5)      | 0.5283 | -0.0000         | 0.7417          | 0.5283 | -0.0000         | 0.7414          |
| (0.5,0.7)      | 0.5214 | -0.0646         | 0.7840          | 0.5215 | -0.0646         | 0.7837          |
| (0.5,0.9)      | 0.5060 | -0.0751         | 0.9105          | 0.5061 | -0.0751         | 0.9103          |
| (0.5,0.1)      | 0.3238 | 0.0000          | 0.2203          | 0.3239 | 0.0000          | 0.2201          |
| (0.5,0.3)      | 0.4001 | 0.0000          | 0.5273          | 0.4001 | 0.0000          | 0.5270          |
| (0.5,0.7)      | 0.6932 | -0.0000         | 0.9026          | 0.6931 | -0.0000         | 0.9022          |
| (0.5,0.9)      | 0.8894 | 0.0000          | 1.0635          | 0.8893 | 0.0000          | 1.0631          |

### 4.3. Problem 3

Now we consider a problem for an exponentially graded isotropic material with \(g(x) = [2(1 + 0.5x_1 + 0.75x_2)]^2\), \(\lambda_{11} = \lambda_{22} = 1, \lambda_{12} = 0\) and \(\beta^2 = 1\). Again, the geometry of the region \(\Omega\) and the boundary conditions are as depicted in Figure 3. Table 3 shows the results of the BEM solution using 80, 160, 320 and 640 elements of equal length. As expected, the results converge as the number of elements increases but they are not symmetrical anymore as the material is not homogeneous.
The modified Helmholtz type governing equation (1) is sometimes used for modeling physical problems such as steady infiltration problems (when $\beta^2 > 0$), and antiplane strain in elastostatics and plane thermostatic problems (when $\beta^2 = 0$). The boundary integral equation (15) is derived from this governing equation (1) and then from (15) a BEM is constructed for calculation of numerical solutions to the problems for anisotropic quadratically graded media. The results show that the BEM solution gives a convergence, consistency, and accuracy. Therefore the results also prove that the analysis used for deriving the boundary integral equation (15) is valid. Together with its ease in implementation, it may be concluded that BEM is a useful numerical method for solving such kind of problems.

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