Quantum hydrodynamic theory of quantum fluctuations in dipolar Bose-Einstein condensate

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Traditional quantum hydrodynamics of Bose-Einstein condensates (BECs) is restricted by the continuity and Euler equations. It corresponds to the well-known Gross-Pitaevskii equation. However, the quantum Bohm potential, which is a part of the momentum flux, has a nontrivial part with can evolve under the quantum fluctuations. To cover this phenomenon in terms of hydrodynamic methods we need to derive equations for the second rank tensor (the momentum flux), and the third rank tensor. In atomic BECs the interaction is the short-range interaction. In all equations we consider the main contribution of the short-range interaction which appears in the first order by the interaction radius. Derived hydrodynamics consists of four hydrodynamic equations. However, two equations contain interaction. The Euler equation contains interaction in the Gross-Pitaevskii approximation. The third moment evolution equation contains interaction leading to the quantum fluctuations. It is proportional to new interaction constant. The Gross-Pitaevskii interaction constant is the integral of potential, but the second interaction constant is the integral of second derivative of potential. If we have dipolar BECs we deal with a long-range interaction. Its contribution is proportional to the potential of dipole-dipole interaction (DDI) in the mean field regime. The Euler equation contains the derivative of the potential. The third rank tensor evolution equation contains the third derivative of the potential which is also proportional to the square of the Plank constant. It is responsible for the dipolar part of quantum fluctuations. Higher derivatives correspond to the small scale contributions of the DDI. The quantum fluctuations lead to existence of the second wave solution. Moreover, the quantum fluctuations introduce the instability of the BECs. If the dipole-dipole interaction is attractive, but being smaller then the repulsive SRI presented by the first interaction constant, there is the long-wavelength instability. This scenario can be realized for dysprosium at $a = 70a_B$, where $a_B$ is the Borh radius. For the repulsive DDI these is more complex picture. There is the small area with the long-wavelength instability which transits into stability interval, where two waves exist. And there is the short-wavelength instability, which is stronger then the long-wavelength instability. These results are found for the DDI strength comparable with the Gross-Pitaevskii short-range interaction, while the dimensionless second interaction constant is $10 \div 100$ times smaller then the Gross-Pitaevskii interaction constant.

Hydrodynamics is a method of description of classic and quantum phenomena. Mostly, it is applied to collective phenomena, but it can be applied to the single quantum particle [1, 2, 3, 4]. Different forms of the force fields distinguish the hydrodynamics of different physical systems. The Navier-Stokes momentum equation describes the water flows and classic liquids and classic gases and some atmospheric phenomena [2]. Hydrodynamics with the Euler equation containing the Lorentz force describes the waves, instabilities and other collective phenomena in plasmas [6, 7]. The quantum Bohm potential and spin-effects give an extension of hydrodynamics suitable for the quantum plasmas [8, 9, 10, 11]. The Gross-Pitaevskii equation for the Bose-Einstein condensate (BEC) of neutral atoms can be presented as the set of two hydrodynamic equations [12, 13]. Spinor BECs can be also modeled by corresponding hydrodynamics [14, 15, 16].

Keywords: quantum hydrodynamics, pressure evolution equation, extended hydrodynamics, quantum fluctuations, dipolar BEC.

The described examples are based on different hydrodynamics, which have same feature. They are the composition of two equations: the continuity equation and the Euler equation. The spin introduces additional equations, but our conclusion is about spinless part of dynamics.

There are examples of extended hydrodynamics, as an example mention the classic plasmas [17], where higher moments of the distribution function are adopted, including the second order tensor evolution.

However, if we want to capture the quantum fluctuations in BECs purely from hydrodynamics we need to derive two additional hydrodynamic equations. They are the pressure $p^{\alpha\beta}$ evolution equation, which is the second rank tensor existing in the Euler equation, and the third rank tensor $Q^{\alpha\beta\gamma}$ evolution equation.

The BECs is the collection of bosons being in the lowest energy level. Hence, we expect that the pressure $p^{\alpha\beta}$ and the third rank tensor $Q^{\alpha\beta\gamma}$ are equal to zero since their definitions explicitly refer to the presence of particles in excited states. However, these tensors have a source in the quantum theory. It is located in equation for

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tensor $Q^{\alpha\beta\gamma}$ which proportional to square of the Planck constant $\hbar^2$ and the interaction potential $g_2 \sim \int dr U''(r)$ (for the short-range interaction). Corresponding model is presented in this paper for the dipolar BECs. Therefore, it covers specifics related to the short-range interaction and the long-range interaction presented by the interaction of dipoles.

Interest of researchers to the dipolar BEC [18], [19], [20] started few years after experimental realization of BECs in vapors of alkaline atoms. The experimental realization of dipolar BECs happened in 2005 in chromium atoms [21], [22]. In 2016 experiments show that dipolar BECs of rare-earth elements demonstrates the quantum droplets formation [23], [24]. It is related to the large scale instability of dipolar BEC, which is stabilized at smaller scales.

Quantum droplets formation in the dipolar BECs demonstrates crucial role of the quantum fluctuations [23], [20], [21], [28], [29]. Traditionally the condensate depletion is studied in terms of Bogoliubov-de Gennes theory [30], [31], [32], where the depletion is presented via the Gross-Pitaevskii interaction constant. The depletion is found in literature, as a correction to the GP equation and the corresponding Euler equation [24], [23], [27], while in our analysis is comes via additional hydrodynamic equations. Hence, the structure of Gross-Pitaevskii equation includes the fourth order nonlinearity. The Bogoliubov-de Gennes theory of the BEC depletion is generalized in 2012 to include the dipole-dipole interaction [51].

Essential role of the quantum fluctuations in description of the quantum droplets formation is an essential part of motivation for this work. However, obtained results give more general picture of hydrodynamics of sound waves.

General classic analysis of hydrodynamic models shows that accurate description of the velocities of acoustic waves requires the account of the pressure evolution equation [33], [34], where the acoustic waves are waves with linear spectrum $\omega^2 = k^2 v^2$, with $\omega$ is the frequency of wave, $k$ is the wave vector, $v$ is the speed of sound. Classical evolution of higher rank tensors leads to terms proportional to higher degrees of the wave vector $k$.

It is obtained that there are quantum sources in equations for the higher rank tensors, like the quantum fluctuations for BECs, which gives contribution in the sound velocity.

Start our analysis with the microscopic Hamiltonian and present found extended set of hydrodynamic equations. We use the many-particle quantum hydrodynamics method, where evolution of functions describing the collective dynamics is found from the microscopic many-particle Schrodinger equation $i\hbar \partial_t \Psi(R, t) = \hat{H} \Psi(R, t)$ in the coordinate representation, where $R$ is the collection of $3N$ coordinates of $N$ particles. Hence, collective motion is governed by the exact microscopic dynamics. Dipolar BECs with the short-range interaction between atoms is modeled by the following Hamiltonian

$$\hat{H} = \sum_{i=1}^{N} \left( \frac{\hat{p}_i^2}{2m_i} + V_{ext}(\mathbf{r}_i, t) \right) + \frac{1}{2} \sum_{i \neq j} U_{ij} + \frac{1}{2} \sum_{i \neq j} U_{ij}^d,$$

where $m_i$ is the mass of $i$-th particle, $\hat{p}_i = -i\hbar \nabla_{\mathbf{r}_i}$ is the momentum of $i$-th particle. The short-range part of boson-boson interaction is presented via potential $U_{ij} = U_{SR}(\mathbf{r}_i - \mathbf{r}_j)$. The last term describes the long-range dipole-dipole interaction (DDI) of align dipoles [35] $U_{ij}^d = \mu^2 1 - 3r_{ij}^2/r_{ij}^4$. It is assumed that all dipoles are aligned parallel to the $z$-direction.

Definitely, Hamiltonian (1) does not contain information about kinds of particles (bosons or fermions). It does not include information about distribution of particles on quantum states. However, its application to the bosons and specification of temperature (which is a measure of the distribution on quantum states) at the macroscopic stage of description lead to the equations for BECs dynamics.

Transition to description of the collective motion of bosons is made via introduction of the concentration [36], [37], [38], [39]:

$$n = \int dR \sum_{i=1}^{N} \delta(\mathbf{r} - \mathbf{r}_i) \Psi^*(R, t) \Psi(R, t), \quad (2)$$

which is the first collective variable in our model. Other collective variables appear during the derivation. Equation (2) contains the following notations $dR = \prod_{i=1}^{N} d\mathbf{r}_i$ is the element of volume in $3N$ dimensional configurational space, with $N$ is the number of bosons.

The derivation shows that concentration (2) obeys the continuity equation

$$\partial_t n + \nabla \cdot (nv) = 0. \quad (3)$$

The velocity field $\mathbf{v}$ presented in the continuity equation obeys the Euler equation, which has the following form for bosons in the BEC state

$$mn\partial_v v_\alpha + mn(\mathbf{v} \cdot \nabla) v_\alpha + \partial_\beta T^{\alpha\beta} + n\delta^\beta V_{ext} = -gn\delta^\alpha n - n\delta^\alpha \Phi_d. \quad (4)$$

The Euler equation contains the short-range interaction (the first term on the right-hand side), and the dipole-dipole interaction (the last term on the right-hand side) in accordance with the Hamiltonian (1). The short-range interaction (SRI) contribution is obtained in the first order by the interaction radius. Therefore, The Euler equation contains the following interaction constant

$$g = \int dR U(r), \quad (5)$$
which is traditionally presented in the Gross-Pitaevskii equation [12]. The dipole-dipole interaction is presented via the macroscopic potential of dipole-dipole interaction

$$\Phi_d = \mu^2 \int dr' \frac{1}{|\mathbf{r} - \mathbf{r}'|^3} \left( 1 - 3 \frac{(z - z')^2}{|\mathbf{r} - \mathbf{r}'|^2} \right) n(\mathbf{r}', t),$$

(6)
since we consider it in the mean-field approximation due to its long-range nature.

The second and third terms on the right-hand side of equation (4) appear as the different parts of the momentum flux $\Pi^{\alpha\beta}$. Equations (3) and (4) contain no explicit contribution of the quantum fluctuations. If we need to find the contribution of additional effects, like the quantum fluctuations, in the hydrodynamic model, we should extend set of hydrodynamic equations. Therefore, we derive equations for the momentum flux second rank tensor evolution and the third rank tensor evolution.

The left-hand side of the Euler equation contains the tensor associated with the quantum Bohm potential $T_{0}^{\alpha\beta}$. It can be split on two parts $T_{0}^{\alpha\beta} = T_{0}^{\alpha\beta} + T_{qf}^{\alpha\beta}$. The noninteracting part of the quantum Bohm potential is given by equation

$$T_{0}^{\alpha\beta} = -\frac{\hbar^2}{4m^2} \left[ \partial_\alpha \partial_\beta n - \frac{\partial_\alpha n \cdot \partial_\beta n}{n} \right].$$

(7)

It corresponds to the Gross-Pitaevskii equation.

Analysis of further equations in the chain of hydrodynamic equations shows that $T_{qf}^{\alpha\beta}$ has nonzero value. Subindexes "qf" refers to the quantum fluctuations in BEC. As it is shown below, the equation for the third rank tensor evolution $Q_{qf}^{\alpha\beta\gamma}$ has a contribution both the short-range interaction and the DD1, where the interaction terms are also proportional to $\hbar^2$. It provides an additional contribution to Bohm potential $T_{qf}^{\alpha\beta}$. Functions $T_{qf}^{\alpha\beta}$ and $Q_{qf}^{\alpha\beta\gamma}$ should be equal to zero at zero temperatures (if there is no particles in the excited states). However, the quantum terms caused by interaction leads to their nonzero value, so some particles occupy the excited states, we associate this contribution with the quantum fluctuations [30], [31]. This phenomenon is well-known in physics of quantum gases. However, for the first time it is derived in terms of hydrodynamics model straight from microscopic quantum motion.

Equation for the nontrivial part of the momentum flux tensor $\Pi^{\alpha\beta}$, which is the part of the quantum Bohm potential caused by the quantum fluctuations, appears with no contribution of interaction:

$$\partial_t T_{qf}^{\alpha\beta} + \partial_\nu (v^\gamma T_{qf}^{\alpha\gamma\beta}) + \partial_\gamma T_{qf}^{\alpha\beta\gamma} + \partial_\gamma v^\beta + \partial_\gamma Q_{qf}^{\alpha\beta\gamma} = 0.$$  

(8)

Purely quantum terms like $T_{0}^{\alpha\beta}$ cancel each other in equation (5). It can be expected that $Q_{qf}^{\alpha\beta\gamma}$ is equal to zero, but equation for its evolution shows that it is not equal to zero even for the BECs.

Equation for the evolution of quantum-thermal part of the third rank tensor is:

$$\partial_t Q_{qf}^{\alpha\beta\gamma} + \partial_\beta (v^\nu Q_{qf}^{\nu\beta\gamma}) + Q_{qf}^{\alpha\nu\beta} \partial_\nu v^\beta + Q_{qf}^{\alpha\beta\nu} \partial_\nu v^\alpha + Q_{qf}^{\alpha\beta\gamma} \partial_\nu v^\gamma =$$

$$+ \partial_\delta P_{qf}^{\alpha\beta\gamma\delta} = \frac{\hbar^2}{4m^2} n \left( g_2 T_{0}^{\alpha\beta\gamma\delta} \partial_\nu n + \partial_\nu \partial_\beta \partial_\gamma \Phi_d \right)$$

$$+ \frac{1}{mn} (T_{qf}^{\alpha\beta} \partial_\delta T_{qf}^{\gamma\delta} + T_{qf}^{\alpha\gamma} \partial_\delta T_{qf}^{\beta\delta} + T_{qf}^{\beta\gamma} \partial_\delta T_{qf}^{\alpha\delta}),$$

(9)

where

$$T_{0}^{\alpha\beta\gamma\delta} = \delta^{a\beta} \delta^{\gamma\delta} + \delta^{\alpha\gamma} \delta^{\beta\delta} + \delta^{\alpha\beta} \delta^{\gamma\delta}.$$  

(10)

Equation (9) is the reduction of the third rank tensor evolution equation for the BECs. Equation (10) contains the second interaction constant for the short-range interaction

$$g_2 = \frac{2}{3} \int dr U''(r).$$

(11)

This interaction constant is proportional to the zeroth order moment of the second derivative of the potential of the short-range interaction, while the first interaction constant is the zeroth order moment of the potential of the short-range interaction. The second interaction constant appears in the first order by the interaction radius like the interaction in the Gross-Pitaevskii, but for the evolution of physical function of higher tensor rank.

For truncation of obtained set of equations we assume that $P_{qf}^{\alpha\beta\gamma\delta} = 0$.

The developed model contains the unknown parameter $g_2$. This is a parameter independent from interaction constant $g$. It would be methodologically incorrect to give an estimation of $g_2$ via $g$. Moreover, the second interaction constant $g_2$ is not related to interaction constants introduced in Refs. [36], [37], [38], [39], [40], [41], where the additional constants appear in the Euler equation at more detailed description of the force field in the third order by the interaction radius $gr \sim \int dr^2 \dot{U}(r)$. Some arbitrary values of interaction constant $g_2$ are used below for estimation of its contribution in the spectrum.

It can be useful to mention that the account of the short-range interaction in the Euler equation constant leads to additional interaction constant, which is the second order moment of the potential of the short-range interaction [35]. Hence, extension of the BEC models beyond the Gross-Pitaevskii approximation gives additional characteristics of potential which can be measured so the potential can be found with some accuracy.

Obtained structure of hydrodynamic equations is correct for the fermions as well. However, the first-order by the interaction radius is equal to zero. However, terms proportional to $g_2$ should appear in the next orders. Moreover, the form of dipole-dipole interaction is the same for fermions.

Appearance of the interaction constants [3] and [11] is not related to the scattering problem or the application of Bohm approximation. They are direct consequence of the small radius nature of the interaction given by potential $\dot{U}(r)$. They appear before we make any judgement about strength of interaction, while explicit form of interaction
terms proportional to derivative of concentration square $\nabla n^2$ are consequences of the weak interaction limit.

Calculate spectrum of bulk collective excitations. It appears as the generalization of the well-known Bogoliubov spectrum, where the generalization is caused by the quantum fluctuations. Consider small amplitude perturbations of the equilibrium state while the equilibrium state is described by the constant nonzero concentration $n_0$, zero value velocity $v_0 = 0$, and the zero value quantum Bohm tensor $T_0^{\alpha\beta} = 0$. Small perturbation of each function is considered as plane waves propagating parallel to the $x$-direction, for instance for concentration $\delta n = Ne^{-\omega t + ik_xx + k_z z}$, where $N$ is the amplitude of perturbation.

First, we stress our attention on the quantum fluctuations caused by the short-range interaction. Consider BECs of vapors of alkaline atoms, where the dipole moments gives no noticeable contribution. Change the interaction constant to zero value by the Feshbach resonance $g = 0$ and drop the contribution of the noninteracting part of the quantum Bohm potential $T_0^{\alpha\beta}$. Equation (9) simplifies to

$$\partial_t \delta Q^{\alpha\beta\gamma} = \frac{\hbar^2}{4m^3} g_2 \Gamma_n^{\alpha\beta\gamma\delta} n_0 \partial^3 \delta n. \tag{12}$$

The Euler equation (9) has zero right-hand side in this limit. Therefore, the spectrum of bulk excitation shows linear dependence of the frequency on the wave vector

$$\omega^2 = \frac{3(-g_2)n_0\hbar}{2 m^2} k^2. \tag{13}$$

The second interaction constant $g_2$ defines the speed of sound. Moreover, it should be negative to get a stable solution.

However, for repulsive interaction $U > 0$, we normally have $U'' > 0$. It gives positive second interaction constant. Hence, equation (13) shows an instability.

Next, we present the dispersion equation in general regime.

$$\omega^4 - \left[\frac{n_0}{m} \left( g + \mu^2 (\cos^2 \theta - 1/3) \right) k^2 + \frac{\hbar^2 k^4}{4m^2} \right] \omega^2$$

$$- \frac{\hbar^2 k^2 n_0}{m^3} \left( \mu^2 k^2 (\cos^2 \theta - 1/3) - 3g_2 \right) = 0, \tag{14}$$

where $k^2 = k_x^2 + k_z^2$.

First two terms in equation (14) are the source of the traditional Bogoliubov spectrum of dipolar BECs. The last term presents the quantum fluctuations.

Equation (14) has two solutions. One solution is the generalization Bogoliubov spectrum. The second solution is novel solution which exists if the last term in equation (14) is positive.

In the second term, the repulsive SRI combines with the repulsive DDI to increase the coefficient. However, the last term demonstrates different relation between the SRI and the DDI. There is the competition of these interactions if both interactions are repulsive interactions. Reason for such difference is in the following. The SRI is proportional to the first derivative of concentration in both equations containing the SRI. It is the Euler equation and the equation for evolution of the third rank tensor $Q^{\alpha\beta\gamma}_{ef}$. The DDI shows different picture since it is the long-range interaction. The DDI is always proportional to the macroscopic potential $\phi$. However, the Euler equation contains the first derivative of the potential, while the third rank tensor evolution equation includes the third derivative of the potential. At the transition to the plane waves two additional derivatives gives minus. In the SRI term extra derivatives are hidden is the second interaction constant.

Strong role of quantum fluctuations is demonstrated
in $^{164}$Dy BEC, where the quantum fluctuations cause the quantum droplets formation [23]. Atoms of $^{164}$Dy have relatively large magnetic moment $\mu = 10\mu_B$, with $\mu_B$ is the Bohr magneton. SRI in this is characterized by the following value of the scattering length $a = 70a_B$, with $a_B$ is the Bohr radius, and $g = 4\pi\hbar^2a/m$.

We study the spectrum and mechanisms for instabilities for the uniform dipolar BEC. The equilibrium concentration is chosen to be $n_0 = 10^{14}$ cm$^{-3}$, which corresponds to the average concentrations of the trapped BECs in existing experiments. Numerical analysis of spectra is made in terms of the dimensionless parameters for the wave vector $\kappa = k/n_0^{1/3}$, frequency $\xi = m\omega/\hbar n_0^{2/3}$, dipole-dipole interaction strength $D = m\mu^2n_0^{1/3}/\hbar^2$, and the short-range interaction constants $g = G\hbar^2/\hbar n_0^{1/3}$, $g_2 = G_2\hbar^2n_0^{1/3}/m$. The first interaction constant can be represented via the scattering length $G = 4\pi an_0^{1/3}$.

We have the following values for the $^{164}$Dy BEC: $G = 0.22$ for $a = 70a_B$, $D = 0.09$ for $a_{dd} = 131a_B$ corresponding to $\mu = 10\mu_B$, since $a_{dd} = m\mu^2/3\hbar^2$, and $D = 3a_{dd}G_0^{1/3}$.

Unstable regimes given by equation (14) are demonstrated in Figs. 1, 2, 3.

The well-known instability causing collapse can appear if we have large DDI in the attractive regime. However, we have $G = 0.22$, $D = 0.09$, and $D(\cos^2\theta - 1/3)|_{\theta = \pi/3} = -0.0075 < G$. Hence, the Bogoliubov spectrum is stable. There is no large competition between $G$ and $D$ in the traditional spectrum for used parameters.

Another example of competition in presented by the last term in equation (14) which is caused by the quantum fluctuations.

The stability of the spectrum depends on the sign of the determinant $\Delta$ of dispersion equation (14). If the last term in equation (14) is negative it makes the determinant $\Delta$ positive. However, the square of frequency for the second solution of equation (14) is negative in this case (the first solution is associated with the Bogoliubov spectrum). It gives condition for the instability demonstrated in Fig. 1 and the small wave vector area of Fig. 3.

If the last term in equation (14) is positive we can have positive square of frequency for the second solution, but determinant $\Delta$ can become negative for the large quantum fluctuations presented by the last term in equation (14). Therefore, there is the mechanism for the instability.

The dipolar part of quantum fluctuations is not a constant, but it suppressed by the small dimensionless wave vectors $\kappa < 1$. However, we expect that $G_2 \ll G$. Hence, the dipolar contribution can overcome $G_2$ at $\kappa < 1$. For repulsive interaction we have $G_2 > 0$. It means that the attractive DDI increases the contribution of the SRI. We have competition of two terms if the DDI is repulsive. For instance, if $D = 0.09$, $\theta = \pi/6$, $(\cos^2\theta - 1/3) = 5/12$, and $G_2 = 0.01$. Hence, the critical wave vector is $\kappa = 0.89$ (see corresponding point in Fig. 3).

To conclude we mention that the extended hydrodynamic model of dipolar BECs has been developed to give a purely hydrodynamic description of quantum fluctuations. It has been found that the short-range interaction proportional to the zeroth moment of the second derivatives of the interaction potential and the third derivative of the macroscopic potential of dipole-dipole interaction are responsible for the quantum fluctuation appearance. These terms are also proportional to the square of the Planck constant. These terms are presented in the third rank tensor evolution equation, while the second rank tensor (superposition of the pressure and the quantum Bohm potential) evolution equation has no contribution of interaction. Therefore, found extended hydrodynamics consists of four equations for material fields of different tensor ranks: the continuity equation for the concentration, the Euler equation for the velocity vector field, the pressure second rank tensor evolution equation (the quantum pressure or the quantum Bohm potential) evolution equation has no contribution of interaction. The quantum fluctuations cause the depletion of BEC, so some excited states are occupied. The contribution of excited states has been mainly modeled by the nontrivial part of the quantum Bohm potential and the third rank tensor. Developed model has been applied to study the bulk excitations in the uniform BECs. Hence, a generalization of the Bogoliubov spectrum has been obtained.

It has been obtained that the quantum fluctuations can cause the long-wavelength instability. Moreover, in the stability regime there are two wave solutions, where the second wave is caused by the quantum fluctuations. The second wave can go unstable at the small wavelengths.

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I. SUPPLEMENTERLY MATERIALS

A. Definitions of basic hydrodynamic variables

After derivation of the continuity equation \( \frac{\partial n}{\partial t} + \nabla \cdot (n \mathbf{v}) = 0 \) for concentration \( n \) from the Schrodinger equation with Hamiltonian \( H \), the current appears as the following integral of the wave function

\[
\mathbf{v} = \frac{1}{n} \int d\mathbf{r} \sum_{i=1}^{N} \delta(\mathbf{r} - \mathbf{r}_i) \times \frac{1}{2m_i} \left( \Psi^*(\mathbf{r}, t) \hat{\mathbf{p}}_i \Psi(\mathbf{r}, t) + c.c. \right),
\]

with c.c. is the complex conjugation.

Definition of current \( j^\alpha \) allows to derive the Euler equation for the current (momentum density) evolution

\[
\partial_t j^\alpha + \partial_\beta \Pi^{\alpha\beta} = -\frac{1}{m} n \partial_\alpha V_{\text{ext}} + \frac{1}{m} F^\alpha_{\text{int}},
\]

where

\[
\Pi^{\alpha\beta} = \int d\mathbf{r} \sum_{i=1}^{N} \delta(\mathbf{r} - \mathbf{r}_i) \frac{1}{4m_i^2} [\Psi^*(\mathbf{r}, t) \hat{\mathbf{p}}^\alpha_i \hat{\mathbf{p}}^\beta_i \Psi(\mathbf{r}, t)
\]

\[+ \hat{\mathbf{p}}^\alpha_i \Psi^*(\mathbf{r}, t) \hat{\mathbf{p}}^\beta_i \Psi(\mathbf{r}, t) + c.c.] \tag{17}
\]

is the momentum flux, and

\[
F^\alpha_{\text{int}} = -\int (\partial^\alpha U(\mathbf{r} - \mathbf{r}')) n_2(\mathbf{r}, \mathbf{r}', t) d\mathbf{r}',
\]

with the two-particle concentration

\[
n_2(\mathbf{r}, \mathbf{r}', t) = \int d\mathbf{r} \sum_{i,j=1,j\neq i}^{N} \delta(\mathbf{r} - \mathbf{r}_i) \delta(\mathbf{r}' - \mathbf{r}_j) \Psi^*(\mathbf{r}, t) \Psi(\mathbf{r}, t). \tag{19}
\]

It gives the general structure of the Euler equation and the definition of the momentum flux.

B. General structure of equation for the second order tensor

Extending the set of hydrodynamic equations we can derive the equation for the momentum flux evolution. Consider the time evolution of the momentum flux \( \Pi^{\alpha\beta} \) using the Schrodinger equation with Hamiltonian \( H \) and derive the momentum flux evolution equation

\[
\partial_t \Pi^{\alpha\beta} + \partial_\gamma M^{\alpha\beta\gamma} = -\frac{1}{m} j^\beta \partial_\alpha V_{\text{ext}} + \frac{1}{m} (F^{\alpha\beta} + F^{\beta\alpha}) + \frac{1}{m} F^{\alpha\beta}_{\text{int}}, \tag{20}
\]

where \( \Pi^{\alpha\beta} = \Pi^{\alpha\beta}_n + \Pi^{\alpha\beta}_b \),

\[
F^{\alpha\beta} = -\int (\partial^\alpha U(\mathbf{r} - \mathbf{r}')) j^\beta(\mathbf{r}, \mathbf{r}', t) d\mathbf{r}', \tag{21}
\]

\[
M^{\alpha\beta\gamma} = \int d\mathbf{r} \sum_{i=1}^{N} \delta(\mathbf{r} - \mathbf{r}_i) \frac{1}{8m_i^4} \left[ \Psi^*(\mathbf{r}, t) \hat{\mathbf{p}}^\alpha_i \hat{\mathbf{p}}^\beta_i \hat{\mathbf{p}}^\gamma_i \Psi(\mathbf{r}, t)
\]

\[+ \hat{\mathbf{p}}^\alpha_i \Psi^*(\mathbf{r}, t) \hat{\mathbf{p}}^\beta_i \hat{\mathbf{p}}^\gamma_i \Psi(\mathbf{r}, t) + \hat{\mathbf{p}}^\alpha_i \hat{\mathbf{p}}^\beta_i \Psi^*(\mathbf{r}, t) \hat{\mathbf{p}}^\gamma_i \Psi(\mathbf{r}, t) + c.c. \right], \tag{22}
\]

and

\[
j^\beta(\mathbf{r}, \mathbf{r}', t) = \int d\mathbf{r} \sum_{i,j=1,j\neq i}^{N} \delta(\mathbf{r} - \mathbf{r}_i) \delta(\mathbf{r}' - \mathbf{r}_j) \times \frac{1}{2m_i} (\Psi^*(\mathbf{r}, t) \hat{\mathbf{p}}_i \Psi(\mathbf{r}, t) + c.c.). \tag{23}
\]

If quantum correlations are dropped function \( j^\alpha(\mathbf{r}, \mathbf{r}', t) \) splits on product of the current \( j^\alpha(\mathbf{r}, t) \) and the concentration \( n(\mathbf{r}', t) \). Tensor \( M^{\alpha\beta\gamma} \) is the flux of the momentum flux. Interaction in the momentum flux evolution equation \( 20 \) is presented by symmetrized combinations of tensors \( F^{\alpha\beta} \), which is the flux or current of force.

The pressure is the average of the square of the thermal velocity, when tensor \( Q^{\alpha\beta\gamma} \) is the average of the product of three projections of the thermal velocity. For the BEC we have \( p^{\alpha\beta}_B = 0 \), and \( Q^{\alpha\beta\gamma}_B = 0 \). Function \( Q^{\alpha\beta\gamma}_f \equiv L^{\alpha\beta\gamma}_B \) is the thermal-quantum term, where both contributions are intertwine together (the general structure of \( L^{\alpha\beta\gamma} \) is introduced in Ref. [34]). The notion "thermal" refers to the presence of particles in the excited states, while nature of the excitation can be arbitrary. In our case, the reason of excitation is the interaction existing in third rank tensor evolution equation.

The equation for evolution of the third rank tensor is derived for \( M^{\alpha\beta\gamma} \). It contains some contribution of the interaction, similarly to the right-hand side of equation \( 20 \). The structure of equation changes after the introduction of the velocity field \( \mathbf{v} \) (compare for instance equations \( 8 \) and \( 20 \)). The contribution of interaction partially cancels via the time derivatives of the velocity field given by the Euler equation \( 4 \).

Methods of calculation of the terms containing the short-range interaction are presented in Refs. [36, 37, 38]. Refs. [39, 67] are focused on ultracold fermions, but methodology is the same.
is also obtained. There is no interaction contribution in this equation. We have general tendency that equations for evolution of the even rank tensors have no contribution of interaction. However, equations for evolution of the odd rank tensors have contribution of interaction. This interaction has nonzero contribution in the first order by the interaction radius. New interaction constants appear in each equation. To illustrate the last statement we present a part of the fifth rank tensor evolution equation.

The divergence of the quantum-thermal part of the fifth rank tensor is dropped. Its thermal part has zero value in the equilibrium state, so it can be used as an equation of state. The mixed quantum-thermal part $R^{\alpha\beta\gamma\delta\mu}_f$ is also assumed to be equal to zero. However, there is $R^{\alpha\beta\gamma\delta\mu}_q$ caused by the quantum fluctuations of higher order, but being obtained in the first order by the interaction radius. For the SRI, the quantum fluctuations is proportional to $\nabla n$, and the the third interaction constant $g_3 \sim \int dr U^{(4)}(r)$, where $U^{(4)}(r)$ is the fourth derivative of the SRI potential. For the DDI we have that the time derivative of $R^{\alpha\beta\gamma\delta\mu}_q$ is caused by the fifth space derivative of the macroscopic potential of DDI $\Phi_d$.

So, the evolution each tensor of uneven rank $A^{2n-1}_q$ gives the contribution of higher order quantum fluctuations via new interaction constant $g_n \sim \int dr [2n-2U^{(r)}]$ for the SRI. Its evolution is proportional to the space derivative of $2n-1$ order of $\Phi_d$ for the DDI.

### D. Linearized hydrodynamic equations

The linear approximation of the hydrodynamic equations (8)-(10) has the following form:

\begin{align}
\omega \delta n &= n_0 (k_x \delta v^x + k_z \delta v^z), \\
\omega m n_0 \delta v^x &= k_x \delta T^{xx}_{qf} - k_z \delta T^{xz}_{qf} \\
-\frac{\hbar^2 k^2}{4m} k_x \delta n &= gn_0 k_x \delta n + n_0 k_x \delta \Phi_d, \\
\omega m n_0 \delta v^z &= k_z \delta T^{zz}_{qf} - k_x \delta T^{xz}_{qf} \\
-\frac{\hbar^2 k^2}{4m} k_z \delta n &= gn_0 k_z \delta n + n_0 k_z \delta \Phi_d,
\end{align}

\begin{align}
\omega \delta T^{xx}_{qf} &= k_x \delta Q^{xx}_{qf} + k_z \delta Q^{xz}_{qf}, \\
\omega \delta T^{xz}_{qf} &= k_x \delta Q^{xx}_{qf} + k_z \delta Q^{zz}_{qf}, \\
\omega \delta T^{zz}_{qf} &= k_x \delta Q^{xz}_{qf} + k_z \delta Q^{zz}_{qf}.
\end{align}

### C. Equation for evolution of the fourth rank tensor

To understand the approximation given by equations (8)-(10) we need to consider equations for the higher rank tensors.

Equation for the quantum-thermal part of the fourth rank tensor

\begin{align}
\partial_t P^{\alpha\beta\gamma\delta}_{qf} + \partial_\nu (\nu^\rho P^{\alpha\beta\gamma\delta}_{qf}) + P^{\alpha\beta\gamma\delta}_{qf} \partial_\nu \nu^\alpha + P^{\alpha\gamma\delta\nu}_{qf} \partial_\nu \nu^\gamma + P^{\alpha\beta\delta\nu}_{qf} \partial_\nu \nu^\beta \\
+ P^{\alpha\beta\delta\nu}_{qf} \partial_\nu \nu^\gamma + P^{\alpha\beta\gamma\nu}_{qf} \partial_\nu \nu^\delta &= \frac{1}{mn} \left[ Q^{\gamma\delta}_{qf} \delta^\nu T^{\alpha\nu} + Q^{\alpha\delta}_{qf} \delta^\nu T^{\beta\nu} + Q^{\alpha\gamma}_{qf} \delta^\nu T^{\beta\nu} \right]
\end{align}

\begin{align}
\partial_t qf^{\alpha\beta\gamma\delta} + \partial_\nu (\nu^\rho qf^{\alpha\beta\gamma\delta}) + qf^{\alpha\beta\gamma\delta} \partial_\nu \nu^\alpha + qf^{\alpha\gamma\delta\nu} \partial_\nu \nu^\gamma + qf^{\alpha\beta\delta\nu} \partial_\nu \nu^\beta \\
+ qf^{\alpha\beta\delta\nu} \partial_\nu \nu^\gamma + qf^{\alpha\beta\gamma\nu} \partial_\nu \nu^\delta &= \frac{1}{mn} \left[ Q^{\gamma\delta}_{qf} \delta^\nu T^{\alpha\nu} + Q^{\alpha\delta}_{qf} \delta^\nu T^{\beta\nu} + Q^{\alpha\gamma}_{qf} \delta^\nu T^{\beta\nu} \right].
\end{align}
\[ \omega \delta Q^{xxx} = \frac{\hbar^2}{4m^3} n_0 \left( k_z^3 \delta \Phi_d - 3k_z g_2 \delta n \right), \quad (31) \]

\[ \omega \delta Q^{zzz} = \frac{\hbar^2}{4m^3} n_0 \left( k_z^3 \delta \Phi_d - 3k_z g_2 \delta n \right), \quad (32) \]

\[ \omega \delta Q^{xzz} = \frac{\hbar^2}{4m^3} n_0 k_x \left( k_x^2 \delta \Phi_d - g_2 \delta n \right), \quad (33) \]

\[ \omega \delta Q^{xzx} = \frac{\hbar^2}{4m^3} n_0 k_x \left( k_x^2 \delta \Phi_d - g_2 \delta n \right), \quad (34) \]

where \( I^{xxxx} = I^{zzzz} = 3, I^{xzzz} = 1, I^{xxzz} = 0. \)

Linearized potential of dipole-dipole interaction is \[ \delta \Phi_d = \mu^2 (\cos^2 \theta - 1/3) \delta n. \quad (35) \]

Equations (25)-(35) are used to obtain spectra (13)-(14).

**E. Signature of the second interaction constant**

It is used in the text that for the repulsive interaction \( g_2 > 0. \) Fig. 4 demonstrates the simple example of repulsive potential. It shows that \( U'' > 0, \) hence \( g_2 > 0. \)

**F. Dimensionless form of dispersion equation**

Present the dispersion equation (14) in dimensionless form for zero dipole contribution

\[ \xi^4 - [G \kappa^2 + 0.25 \kappa^4] \xi^2 + 3 \kappa^4 G_2 = 0, \quad (36) \]

where dimensionless wave vector \( \kappa = k/n_0^{1/3}, \) dimensionless frequency \( \xi = m \omega / \hbar n_0^{2/3}, \) dimensionless interaction constants \( g = G \hbar^2 / mn_0^{1/3}, \) and \( g_2 = G_2 \hbar^2 / n_0^{1/3}, \) or \( G = 4 \pi a n_0^{1/3}. \)

Consider dimensionless form of equation (14) for dipolar BECs

\[ \xi^4 - [G \kappa^2 + D (\cos^2 \theta - 1/3) \kappa^2 + 0.25 \kappa^4] \xi^2 - [D (\cos^2 \theta - 1/3) \kappa^2 - 3G_2] \kappa^4 = 0, \quad (37) \]

where \( D = m \mu^2 n_0^{1/3} / \hbar^2. \)

**G. Spectrum: small quantum fluctuation limit**

No instability appears if quantum fluctuations are dominated by the attractive DDI. However, the second stable low frequency wave solution appears in this regime, as it is shown by the two lower lines in Fig. 5.