Dynamical systems

Exponential decay of correlations for a real-valued dynamical system embedded in $\mathbb{R}^2$

Décroissance des corrélations pour une récurrence à deux termes

Lisette Jager, Jules Maes, Alain Ninet

Laboratoire de mathématiques, FR CNRS 3399, EA 4535, Université de Reims Champagne-Ardenne, Moulin de la Housse, BP 1039, 51687 Reims, France

ABSTRACT

We study the real valued process $\{X_t, t \in \mathbb{N}\}$ defined by $X_{t+2} = \varphi(X_t, X_{t+1})$, where the $X_t$ are bounded. We aim at proving the decay of correlations for this model, under regularity assumptions on the transformation $\varphi$.

© 2015 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

RÉSUMÉ

On étudie le processus réel $\{X_t, t \in \mathbb{N}\}$ défini par $X_{t+2} = \varphi(X_t, X_{t+1})$, les $X_t$ étant bornés. Sous des hypothèses de régularité sur la transformation $\varphi$, on établit la décroissance des corrélations pour ce modèle.

© 2015 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

1. Introduction

Since the 1980s, the study by statisticians of nonlinear time series has allowed one to model a great number of phenomena in Physics, Economics, and Finance [5], [6]. Then, in the 1990s, the theory of Chaos became an essential axis of research for the study of these processes [5]. For an exhaustive review on this subject, one can consult Collet–Eckmann [2] about chaos theory and Chan–Tong [8,9] about nonlinear time series. Within this framework, a general model could be written as

$$X_{t+1} = \varphi(X_t, \ldots, X_{t-d+1}) + \epsilon_t,$$

where $\varphi$ is nonlinear and $\epsilon_t$ is a noise. We propose a first study of the "skeleton" of this model, as Tong calls it, beginning with $d = 2$ and, more precisely, of the dynamical system induced by this model. Indeed, we consider the model with bounded variables, $X_{t+2} = \varphi(X_t, X_{t+1})$, with $\varphi : U^2 \to U$ for $U = [-L, L]$ and $L \in \mathbb{R}^+$, $\varphi$ being defined piecewisely on $U^2$. This model gives rise to a dynamical system $(\Omega, \tau, \mu, T)$ where $\mu$ is a measure on the $\sigma$-algebra $\tau$, invariant under the transformation $T : \Omega \to \Omega$ and $\Omega$ is a compact subset of $\mathbb{R}^2$. Under hypotheses on $\varphi$, which imply that $T$ satisfies the hypotheses of Saussol [7], and if we suppose that $T$ is mixing, we obtain the exponential decay of correlations. More precisely, for well-chosen applications $f$ and $h$, there exist constants $C = C(f,h) > 0$, $0 < \rho < 1$ such that:

E-mail addresses: lisette.jager@univ-reims.fr (L. Jager), jules.maes@univ-reims.fr (J. Maes), alain.ninet@univ-reims.fr (A. Ninet).

http://dx.doi.org/10.1016/j.crma.2015.07.015
1631-073X/© 2015 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.
\[
\left| \int_\Omega f \circ T^k h \, d\mu - \int_\Omega f \, d\mu \int_\Omega h \, d\mu \right| \leq C \rho^k.
\]

This result can be seen as a covariance inequality of the following kind:

\[
|\text{Cov}(f(X_0), h(X_0))| \leq C \rho^k.
\]

Other ways could certainly be used to get the same result, under different hypotheses on the induced system, for example the method of Young’s towers [10]. To have a general view on these different technics, one can read the article of Alves–Freitas–Luzzatto–Vaiienti [1] and [3], [4], [6].

2. Hypotheses and results

Let \( L \in \mathbb{R}^+_* \). Let \( \varphi : [-L, L]^2 \to [-L, L] \) be piecewisely defined on \([-L, L]^2\). To study the process \( \{X_t, t \in \mathbb{N}\} \) defined by \( X_{t+2} = \varphi(X_t, X_{t+1}) \), there exist different ways of choosing the induced dynamical system \( Z_{t+1} = T(Z_t) \) with \( Z_t \in \mathbb{R}^2 \). We tried two different approaches, on the one hand the canonical method, setting \( T(x, y) = (y, \varphi(x, y)) \), and on the other hand a double iteration, which comes down to setting \( T(x, y) = (\varphi(x, y), \varphi(y, \varphi(x, y))) \). The first approach, up to a conjugation, is the most fruitful, the second one requiring stronger hypotheses and yielding weaker results. We therefore set \( T(x, y) = (\frac{\varphi}{2}, \gamma \varphi(x, \frac{\varphi}{2})) \) with \( Z_t = (X_t, \gamma X_{t+1}) \), for a suitable positive \( \gamma \). It then became possible to work in spaces similar to Saussol’s \( V_\alpha \) and to use his results.

More precisely, we suppose that the following hypotheses are fulfilled.

(H1) There exists \( d \in \mathbb{N}^* \) such that \([-L, L]^2 = \bigcup_{k=1}^d O_k \cup \mathcal{N} \), where the \( O_k \) are nonempty open sets, \( \mathcal{N} \) is negligible for the Lebesgue measure and the union is disjoint. The edges of the \( O_k \) can be split into a finite number of smooth components, each one included in a \( C^1 \), compact and one-dimensional submanifold of \( \mathbb{R}^2 \).

(H2) There exists \( \varepsilon_1 > 0 \) such that, for all \( k \in [1, \ldots, d] \), there exists an application \( \varphi_k \) defined on \( B_{\varepsilon_1}(\bar{O}_k) = \{(x, y) \in \mathbb{R}^2, d((x, y), \bar{O}_k) \leq \varepsilon_1\} \), with values in \( \mathbb{R} \), such that \( \varphi_k|_{\bar{O}_k} = \varphi|_{\bar{O}_k} \).

(H3) The application \( \varphi_k \) is bounded, belongs to the Hölder class \( C^{1,\alpha} \) on \( B_{\varepsilon_1}(\bar{O}_k) \) for a real \( \alpha \in [0, 1] \).

We moreover suppose that there exist \( A > 1 \) and \( M \in [0, A-1[ \) such that:

\[
\forall (u, v) \in B_{\varepsilon_1}(\bar{O}_k), \quad \left| \frac{\partial \varphi_k}{\partial u}(u, v) \right| \geq A, \quad \left| \frac{\partial \varphi_k}{\partial v}(u, v) \right| \leq M,
\]

to ensure the expanding properties.

(H4) The open sets \( O_k \) satisfy the following geometrical condition:

\[
\text{for all } (u, v) \text{ and } (u', v) \text{ in } B_{\varepsilon_1}(\bar{O}_k), \text{ there exists a } C^1 \text{ path } \Gamma = (\Gamma_1, \Gamma_2) : [0, 1] \to B_{\varepsilon_1}(\bar{O}_k) \text{ } C^1 \text{ joining } (u, v) \text{ and } (u', v), \text{ whose gradient does not vanish, and which satisfies}
\]

\[
\forall t \in ]0, 1[, \quad |\Gamma'(t)| > \frac{M}{A} |\Gamma_2'(t)|.
\]

(H5) Let \( Y \in \mathbb{N}^* \) be the maximal number of \( C^1 \) components of \( \mathcal{N} \) meeting at one point and set

\[
s = \left( \frac{2A + M^2 - M\sqrt{M^2 + 4A}}{2} \right)^{-1/2} < 1.
\]

One supposes that

\[
\eta := s^\alpha + \frac{8s}{\pi(1-s)} Y < 1.
\]

---

1. To get similar results on \([a, b]\) instead of \([-L, L]\), it suffices to conjugate by an affine application.

2. If \( \varphi_k \) is \( C^1 \) on \( B_{\varepsilon_1}(\bar{O}_k) \), it is \( C^{1,\alpha} \) on \( B_{\varepsilon_1}(\bar{O}_k) \) with \( \alpha = 1 \).

3. In suitable cases, this hypothesis can be replaced by a weaker but simpler one: for all points \((u, v)\) and \((u', v)\) in \( B_{\varepsilon_1}(\bar{O}_k) \), the segment \([ (u, v), (u', v) ] \) is included in \( B_{\varepsilon_1}(\bar{O}_k) \).
We set $\gamma = \frac{1}{\sqrt{\lambda}} < 1$ and, for all $k \in \{1, \ldots, d\}$, we denote by $U_k$ (resp. $W_k$, $N^\gamma$) the image of $O_k$ (resp. $B_{\epsilon_1}(\overline{O}_k)$, $N^\gamma$) under the compression that associates $(u, v)$ with each $(u, v) \in \mathbb{R}^2$. The set $\Omega = [-L, L] \times [-\gamma L, \gamma L]$, on which we shall be working, is the image of $[-L, L]^2$ under the same compression.

For every non-negligible Borel set $S$ of $\mathbb{R}^2$, for every $f \in L^1_m(\mathbb{R}^2, \mathbb{R})$, set

$$Osc(f, S) = \text{Essup}_S f - \text{Essinf}_S f,$$

where $\text{Essup}_S$ and $\text{Essinf}_S$ are the essential supremum and infimum with respect to the Lebesgue measure $m$. One then defines:

$$|f|_\alpha = \sup_{0 < \varepsilon < \varepsilon_1} \varepsilon^{-\alpha} \int_{\mathbb{R}^2} Osc(f, B_\varepsilon(x, y)) \, dx \, dy, \quad \|f\|_\alpha = \|f\|_{L^1_m} + |f|_\alpha$$

and the set $V_\alpha = \{f \in L^1_m(\mathbb{R}^2, \mathbb{R}) \mid \|f\|_\alpha < +\infty\}$.

Let us introduce similar notions on $\Omega$: for every $0 < \varepsilon_0 < \gamma \varepsilon_1$, for every $g \in L^\infty_m(\Omega, \mathbb{R})$, one defines

$$N(g, \alpha, L) = \sup_{0 < \varepsilon < \varepsilon_0} \varepsilon^{-\alpha} \int_{\Omega} Osc(g, B_\varepsilon(x, y) \cap \Omega) \, dx \, dy.$$

One then sets:

$$\|g\|_{\alpha, L} = N(g, \alpha, L) + 16(1 + \gamma) \varepsilon_0^{1-\alpha} L \|g\|_{L^\infty} + \|g\|_{L^1_m}.$$

The function $g$ is said to belong to $V_\alpha(\Omega)$ if the above expression is finite. The set $V_\alpha(\Omega)$ does not depend on the choice of $\varepsilon_0$, whereas $N$ and $\|\cdot\|_{\alpha, L}$ do.

There exist relationships between these two sets. Indeed, thanks to Proposition 3.4 of [7], one can prove the following result.

**Proposition 2.1.**

(i) If $g \in V_\alpha(\Omega)$ and if one extends $g$ as a function denoted by $f$, setting $f(x, y) = 0$ if $(x, y) \notin \Omega$, then $f \in V_\alpha$ and

$$\|f\|_\alpha \leq \|g\|_{\alpha, L}.$$

(ii) Let $f$ be in $V_\alpha$. Set $g = f 1_\Omega$. Then $g \in V_\alpha(\Omega)$ and one has

$$\|g\|_{\alpha, L} \leq \left( 1 + 16(1 + \gamma) L \frac{\max(1, \varepsilon_0^{\alpha})}{\pi \varepsilon_0^{1+\alpha}} \right) \|f\|_\alpha.$$

Under the above hypotheses (H1) to (H5), one obtains a first result.

**Theorem 2.2.** Let $T$ be the transformation defined on $\Omega$ by: $\forall(x, y) \in U_k$:

$$T(x, y) = T_k(x, y) = \left( \frac{y}{\gamma'}, \gamma' \varphi_k(x), \frac{y}{\gamma'} \right).$$

Keeping the same formula, one extends the definition of $T_k$ to $W_k$. Then

(i) the Frobenius–Perron operator $P : L^1_m(\Omega) \to L^1_m(\Omega)$ associated with $T$ has a finite number of eigenvalues $\lambda_1, \ldots, \lambda_r$ of modulus one;

(ii) for each $i \in \{1, \ldots, r\}$, the eigenspace $E_i = \{f \in L^1_m(\Omega) : Pf = \lambda_i f\}$ associated with the eigenvalue $\lambda_i$ is finite dimensional and included in $V_\alpha(\Omega)$;

(iii) the operator $P$ decomposes as

$$P = \sum_{i=1}^{r} \lambda_i P_i + Q,$$

where the $P_i$ are projections on the spaces $E_i$, $\|P_i\|_1 \leq 1$ and $Q$ is a linear operator defined on $L^1_m(\Omega)$, satisfying $Q(V_\alpha(\Omega)) \subset V_\alpha(\Omega)$, $\sup_{n \in \mathbb{N}} \|Q^n\|_{\alpha, 1} < 1$ and $\|Q^n\|_{\alpha, L} = O(q^n)$ when $n \to +\infty$ for an exponent $q \in ]0, 1[$. Moreover, $P_i P_j = 0$ if $i \neq j$, $P_i Q = Q P_i = 0$ for all $i$;

(iv) the number 1 is an eigenvalue of $P$. Set $\lambda_1 = 1$, let $h_\ast = P_1 1_{\Omega_2}$ and let $d\mu = h_\ast \, dm$. Then $\mu$ is the greatest absolutely continuous invariant measure (ACIM) of $T$, that is to say: if $v << m$ and if $v$ is $T$-invariant, then $v << \mu$;
Theorem (v) the support of $\mu$ can be decomposed into a finite number of disjoint measurable sets, on which a power of $T$ is mixing. More precisely, for all $j \in \{1, 2, \ldots, \dim(E_1)\}$, there exist an integer $L_j \in \mathbb{N}^*$ and $L_j$ disjoint sets $W_{j,l}$ ($0 \leq l \leq L_j - 1$) satisfying $T(W_{j,l}) = W_{j,l+1 \text{mod}(L_j)}$ and $T^{L_j}$ is mixing on every $W_{j,l}$. We denote by $\mu_{j,l}$ the normalized restriction of $\mu$ to $W_{j,l}$, defined by

$$
\mu_{j,l}(B) = \frac{\mu(B \cap W_{j,l})}{\mu(W_{j,l})}, \quad \text{d}\mu_{j,l} = \frac{h^n 1_{W_{j,l}}}{\mu(W_{j,l})} \text{d}m.
$$

The fact that $T^{L_j}$ is mixing on every $W_{j,l}$ means that, for all $f \in L^1(\mu_{j,l})$ and all $h \in L^\infty(\mu_{j,l})$,

$$
l \rightarrow +\infty \quad \lim_{l \rightarrow +\infty} < T^{L_j} f, h >_{\mu_{j,l}} = < f, 1 >_{\mu_{j,l}} < 1, h >_{\mu_{j,l}}
$$

with the notations (indifferently employed) $< f, g >_{\mu'} = \int f g \text{d}\mu'$.

(vi) moreover, there exist $C > 0$ and $0 < \rho < 1$ such that, for all $h$ in $V_\alpha(\Omega)$ and $f \in L^1(\mu)$, one has

$$
\left| \int \frac{h}{\Omega} T \otimes \text{pcm}(E_1) h \text{d}\mu - \sum_{j=1}^{\dim(E_1)} \sum_{l=0}^{L_j-1} \mu(W_{j,l}) \left( < f, 1 >_{\mu_{j,l}} < 1, h >_{\mu_{j,l}} \right) \right| \leq C \|h\|_{\alpha, \Omega} \|f\|_{L^1_\mu(\Omega)} \rho^k;
$$

(vii) if, moreover, $T$ is mixing, then the preceding result can be written as follows: there exist $C > 0$ and $0 < \rho < 1$ such that, for all $h$ in $V_\alpha(\Omega)$ and $f \in L^1(\mu)$, one has:

$$
\left| \int \frac{h}{\Omega} T h \text{d}\mu - \int \frac{h}{\Omega} \text{d}\mu \right| \leq C \|h\|_{\alpha, \Omega} \|f\|_{L^1_\mu(\Omega)} \rho^k.
$$

We now come back to the initial problem and deduce from this result the invariant law associated with $X_t$. If $(X_t)_t$ is defined by $X_0, X_1$ (valued in $[-L, L]$) and the recurrence relation $X_{t+2} = \psi(X_t, X_{t+1})$, one sets $Z_t = (X_t, \gamma X_{t+1})$. Then $(Z_t)_t$ satisfies the recurrence relation $Z_{t+1} = T(Z_t)$, which implies the following result (by comparing the marginal distributions):

**Theorem 2.3.** Suppose that the random variable $Z_0 = (X_0, \gamma X_1)$ has the density $h_s$. Then, for all $t \in \mathbb{N}$, $Z_t$ has the density $h_s$ and $X_t$ has the density

$$
f : x \mapsto \int_{[-L, L]} h_s(x, v) \text{dv} = \gamma \int_{[-L, L]} h_s(u, \gamma x) \text{du}.
$$

If $F$ is defined on $[-L, L]$, let $\text{Tr } F$ be the function defined on $\Omega$ by $\text{Tr } F(x, y) = F(x)$.

One then obtains the following result, which is a direct consequence of the sixth point of Theorem 2.2, applied to $\text{Tr } F$ and $\text{Tr } H$:

**Theorem 2.4.** For every Borel set $B$ and every interval $I$, if $(X_0, X_1)$ has the invariant distribution, then

$$
\left| P \left( X_k \otimes \text{pcm}(I) \in B, X_0 \in I \right) - \sum_{j=1}^{\dim(E_1)} \sum_{l=0}^{L_j-1} \mu(W_{j,l}) \mu_{j,l}(B) \right| < 1, \mu_{j,l} >_{\mu_{j,l}} \right| \leq 16 (1 + \gamma) C L^3 \left( 10 \varepsilon_0^{1-\alpha} + L \right) \rho^k.
$$

More generally, let $F$, defined and measurable on $[-L, L]$, be such that $\text{Tr } F$ belongs to $L^1_\mu(\Omega)$. Let $H \in L^\infty([-L, L])$ be such that

$$
\sup_{0 < \varepsilon < \varepsilon_0} \varepsilon^{-\alpha} \int_{[-L, L]} \text{Osc}(H, |x - e, x + e| \cap [-L, L]) \text{dx} < +\infty.
$$

Then $\text{Tr } H \in V_\alpha(\Omega)$ and

$$
\left| E(F(X_k \otimes \text{pcm}(I))) H(X_0) - \sum_{j=1}^{\dim(E_1)} \sum_{l=0}^{L_j-1} \mu(W_{j,l}) \mu_{j,l}(\text{Tr } F \mu_{j,l} \mu_{j,l}(\text{Tr } H)) \right| \leq C(F, H) \rho^k
$$

with $\rho = \rho_k$.
\[ C(F, H) = C \| \text{Tr } F \|_{L^1} \left( 2 \gamma \sup_{0 < \varepsilon < \varepsilon_0} e^{-\alpha} \int_{[-L, L]} \text{Osc}(H, [x - \varepsilon, x + \varepsilon] \cap [-L, L]) \, dx \right) + 16(1 + \gamma) e^{1 - \alpha} \| H \|_{L^\infty([-L, L])} + 2 \gamma \| H \|_{L^1([-L, L])} \). \\

If, moreover, T is mixing, then:

\[ |\text{Cov}(F(X_k), H(X_0))| \leq C(F, H) \rho^k. \]

References

[1] J.F. Alves, J.M. Freitas, S. Luzatto, S. Vaienti, From rates of mixing to recurrence times via large deviations, Adv. Math. 228 (2) (2011) 1203–1236.
[2] P. Collet, J.-P. Eckmann, Concepts and Results in Chaotic Dynamics: A Short Course, Theoretical and Mathematical Physics, Springer-Verlag, Berlin, 2006.
[3] F. Hofbauer, G. Keller, Ergodic properties of invariant measures for piecewise monotonic transformations, Math. Z. 180 (1982) 119–140.
[4] C.T. Ionescu Tulcea, G. Marinescu, Théorie ergodique pour des classes d’opérations non complètement continues, Ann. of Math. (2) 52 (1950) 140–147.
[5] A. Lasota, M.C. Mackey, Chaos, Fractals and Noise: Stochastic Aspects of Dynamics, Springer Verlag, New York, 1998.
[6] C. Liverani, Multidimensional expanding maps with singularities: a pedestrian approach, Ergod. Theory Dyn. Syst. 33 (1) (2013) 168–182.
[7] B. Saussol, Absolutely continuous invariant measures for multidimensional expanding maps, Isr. J. Math. 116 (2000) 223–248.
[8] H. Tong, Nonlinear Time Series: A Dynamical System Approach (with an appendix by K.S. Chan), Oxford Statistical Science Series, vol. 6, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1990.
[9] H. Tong, Nonlinear time series analysis since 1990: some personal reflections, Acta Math. Appl. Sin. Engl. Ser. 18 (2) (2002) 177–184.
[10] L.-S. Young, Recurrence times and rates of mixing, Isr. J. Math. 110 (1999) 153–188.