A potential multipartite entanglement measure

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Abstract

In this Brief Report we discuss entanglement of multiparticle quantum systems. We propose a potential measure of a type of entanglement of pure states of $n$ qubits, the $n$-tangle. For a system of two qubits the $n$-tangle is equal to the square of the concurrence, and for systems of three qubits it is equal to the "residual entanglement". We show that the $n$-tangle, is also equal to the generalization of concurrence squared for even $n$, and use this fact to prove that the $n$-tangle is an entanglement monotone. However, the $n$-tangle is undefined for odd $n > 3$. Finally we propose a measure related to the $n$-tangle for mixed state systems of $n$ qubits, and find an analytical formula for this measure for even $n$.

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I. INTRODUCTION

The quantum phenomenon of entanglement is presently a subject of much active research and discussion. This comes from fundamental interest in quantum phenomena, and is also due to recent proposals for quantum computation \[1,2\]. Entanglement is the property that provides a quantum computer advantages over its classical counterpart. If one is designing a quantum computer then quantifying the entanglement of a large number of qubits is likely to be valuable. Quantum entanglement allows correlations between separated quantum particles that are not possible in classical systems \[3\]. Hence, entanglement measures should also prove valuable in the quantum applications of cloning, communication and encryption.

A method for classifying and quantifying the entanglement in a particular state would greatly increase our understanding of this phenomenon; there have been numerous studies into quantum entanglement, with equally numerable entanglement measures proposed \[4–14\]. There remain many open questions regarding quantification of entanglement. In particular, states with more than one subsystem have only just begun to be considered. While entanglement measures of pure states are essential, so is their applicability to mixed states. The presence of noise in a quantum channel \[15\] or the decoherence effects of qubits interacting with an environment \[16\] will transform an idealized pure state into a mixed one.

One type of multipartide entanglement is \(n\)-way or \(n\)-party entanglement, entanglement that critically involves all \(n\) particles. For example, a three qubit state with only three-way (or three-party) entanglement has the property that tracing out one of the qubits leaves the other two particles unentangled \[7\]. It was recently proven that states with \(n\)-way entanglement \((n > 2)\) cannot be reversibly distilled from two-way entanglement \[9\]. An example of a state with only three-way entanglement is the GHZ state: \(|GHZ> = (|000> + |111>)/\sqrt{2}\), for which case \(\tau_{ABC}(|GHZ>) = 1\). The \(W\) state, \(|W> = (|001> + |010> + |100>)/\sqrt{3}\), with \(\tau_{ABC}(|W>) = 0\), is an example of a state with two-way but no three-way entanglement: tracing out one of the particles leaves a partially entangled pair of qubits. In general, three-qubit states have both kinds of entanglement.
The concurrence has been shown to be a useful entanglement measure for pure and mixed states with two qubits, and can be related to the entanglement of formation \[5\]. A recent paper by Coffman, Kundu, and Wootters \[4\] using concurrence to examine three qubit quantum systems introduced the concept of “residual entanglement”, or the 3-tangle, \(\tau_{ABC}\). \(\tau_{ABC}(\ket{\psi})\) is a potential way to quantify the amount of three-way entanglement in the system ABC.

In this Brief Report we will show that a generalization of the 3-tangle for \(n\) qubits, the \(n\)-tangle \(\tau\), is related to a generalization of pure state concurrence for states with an even number of qubits. This allows us to prove that the \(n\)-tangle is an entanglement monotone for states with three or an even number of qubits. We also show that the \(n\)-tangle equals 1 for the \(n\) qubit generalization of GHZ state \[17\], and 0 for the \(n\) qubit generalization of the W state \[7\]. Lastly, we introduce a mixed state measure of entanglement related to \(n\)-tangle that is analogous to the entanglement of formation and find an analytical formula for this measure for states with an even number of qubits.

The Brief Report is organized as follows. In Sec. II we define the \(n\)-tangle and show that for states with even \(n\), \(\tau_{1...n}\) is equal to the square of a natural generalization of pure state concurrence. Since two qubit concurrence is related to entanglement and entanglement of formation \[3\], this suggests that \(n\)-tangle may have a physical interpretation. We prove that \(\tau_{1...n}\) is an entanglement monotone \[8\], which gives further evidence that the \(n\)-tangle measures a type of entanglement. We also consider the value of \(n\)-tangle for generalizations of the GHZ and W states and another example state. The extension of our pure state results to mixed states is shown in Sec. III. A mixed state version of the \(n\)-tangle, \(\tau_{\text{min}}^{\text{min}}\), is introduced, and an analytical formula for \(\tau_{1...n}^{\text{min}}\) for even \(n\) is presented. In Sec. IV we conclude with a discussion of our results.

II. THE \(N\)-TANGLE

For three qubits the “residual entanglement”, or \(\tau_{ABC}\), is given by
\[
\tau_{ABC}(|\psi>) = 2 \left| \sum a_{\alpha_1 \alpha_2 \alpha_3} a_{\beta_1 \beta_2 \beta_3} a_{\gamma_1 \gamma_2 \gamma_3} a_{\delta_1 \delta_2 \delta_3} \epsilon_{\alpha_1 \beta_1} \epsilon_{\beta_2 \gamma_1} \epsilon_{\gamma_2 \delta_1} \epsilon_{\alpha_3 \beta_3} \epsilon_{\beta_3 \gamma_3} \epsilon_{\gamma_3 \delta_3} \right|
\]

where the \(a\) terms are the coefficients in the standard basis defined by

\[|\psi> = \sum_{i_1, i_2, \ldots, i_n} a_{i_1, i_2, \ldots, i_n} |i_1 i_2 \ldots i_n>\]

and \(\epsilon_{01} = -\epsilon_{10} = 1\) and \(\epsilon_{00} = -\epsilon_{11} = 0\). We define the \(n\)-tangle by

\[
\tau_{1\ldots n} = 2 \left| \sum a_{\alpha_1 \ldots \alpha_n} a_{\beta_1 \ldots \beta_n} a_{\gamma_1 \ldots \gamma_n} a_{\delta_1 \ldots \delta_n} \epsilon_{\alpha_1 \beta_1} \epsilon_{\alpha_2 \beta_2} \ldots \epsilon_{\alpha_{n-1} \beta_{n-1}} \epsilon_{\gamma_1 \delta_1} \epsilon_{\gamma_2 \delta_2} \ldots \epsilon_{\gamma_{n-1} \delta_{n-1}} \epsilon_{\alpha_n \gamma_n} \epsilon_{\beta_n \delta_n} \right|
\]

for all even \(n\) and \(n = 3\). By reasoning similar to that used for \(n = 3\), the \(n\)-tangle is invariant under local unitarities. We show below that the \(n\)-tangle is invariant under permutations of the qubits. However, the above formula is not invariant under permutations of qubits for general odd \(n\) over 3, and hence is not a viable measure of odd-way entanglement (aside from \(n = 3\)).

There is a relationship that can be shown between \(\tau\) and pure state concurrence. Pure state concurrence is defined for states of two qubits in [5] by \(C(\psi) = |<\psi|\tilde{\psi}>|^2\), where \(|\tilde{\psi}> = \sigma_y^\otimes n |\psi^*>\) is the “spin flip” of \(|\psi>\) in terms of the Pauli spin matrix \(\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\). \(C\) is defined only for states of two qubits, but the obvious generalization uses the same equation, \(C_{1\ldots n}(\psi) = |<\psi|\tilde{\psi}>|^2\), where \(|\tilde{\psi}>\) now is for an \(n\)-qubit state. Note that for the two qubit case, \(\tau_{12} = C^2\). We will prove that the analogous equation, \(\tau_{1\ldots n} = C_{1\ldots n}^2\) is true for all even \(n\).

We shall find an expression for \(C_{1\ldots n}^2\) in terms of the coefficients in the standard basis. One can express an \(n\) qubit state \(|\psi>\) as a vector in the standard basis indexed by \(|\psi>_{i_1 \ldots i_n}\), where each \(i\) indexes one of the qubits. Then \(|\psi>_{i_1 \ldots i_n} = a_{i_1 \ldots i_n}\).

Note that \(\sigma_{y_{i_1 \ldots i_n}} = \epsilon_{i_1 j_1} \epsilon_{i_2 j_2} \ldots \epsilon_{i_n j_n} e^{i\theta}\) for some real \(\theta\) because \(\sigma_{y_{i,j}} = -i \epsilon_{ij}\). Therefore, \(|\tilde{\psi}> = \sigma_{y_{i_1 \ldots i_n}} |\psi^*>\) implies \(|\tilde{\psi}>_{i_1 \ldots i_n} = \sum_{\beta_1 \beta_2 \ldots \beta_n} a_{\beta_1 \beta_2 \ldots \beta_n}^* \epsilon_{i_1 \beta_1} \epsilon_{i_2 \beta_2} \ldots \epsilon_{i_n \beta_n} e^{i\theta}\) so

\[
<\psi|\tilde{\psi}> = \sum_{\alpha \beta} a_{\alpha_1 \ldots \alpha_n}^* a_{\beta_1 \beta_2 \ldots \beta_n} \epsilon_{\alpha_1 \beta_1} \epsilon_{\alpha_2 \beta_2} \ldots \epsilon_{\alpha_n \beta_n} e^{i\theta}.
\]

Thus,
\[ | \langle \psi \rangle \tilde{\psi} |^2 = \left| \sum \alpha_1 \ldots \alpha_n \alpha_{\beta_1} \ldots \beta_n \alpha_{\gamma_1} \ldots \gamma_n a_{\delta_1} \ldots \delta_n \epsilon_{\alpha_1} \epsilon_{\alpha_2} \epsilon_{\beta_1} \epsilon_{\beta_2} \ldots \epsilon_{\alpha_n} \epsilon_{\beta_n} \epsilon_{\gamma_1} \epsilon_{\gamma_2} \epsilon_{\delta_1} \epsilon_{\delta_2} \ldots \epsilon_{\delta_n} \right| \quad (3) \]

where the sum is over all indices. Expanding the last index of each \( a \) and using the fact that \( \epsilon_{i,j} = -\epsilon_{j,i} \) for even \( n \), one gets

\[ | \langle \psi \rangle \tilde{\psi} |^2 = 
2 \sum \alpha_1 \ldots \alpha_{n-1} \alpha_{\beta_1} \ldots \beta_{n-1} \alpha_{\gamma_1} \ldots \gamma_{n-1} a_{\delta_1} \ldots \delta_{n-1} \epsilon_{\alpha_1} \epsilon_{\alpha_2} \epsilon_{\beta_1} \epsilon_{\beta_2} \ldots \epsilon_{\alpha_{n-1}} \epsilon_{\beta_{n-1}} \epsilon_{\gamma_1} \epsilon_{\gamma_2} \epsilon_{\delta_1} \epsilon_{\delta_2} \ldots \epsilon_{\delta_{n-1}}
\]

\[ -\sum \alpha_1 \ldots \alpha_{n-1} \alpha_{\beta_1} \ldots \beta_{n-1} \alpha_{\gamma_1} \ldots \gamma_{n-1} a_{\delta_1} \ldots \delta_{n-1} \epsilon_{\alpha_1} \epsilon_{\alpha_2} \epsilon_{\beta_1} \epsilon_{\beta_2} \ldots \epsilon_{\alpha_{n-1}} \epsilon_{\beta_{n-1}} \epsilon_{\gamma_1} \epsilon_{\gamma_2} \epsilon_{\delta_1} \epsilon_{\delta_2} \ldots \epsilon_{\delta_{n-1}} \] \quad (4)

Now we turn our attention to the expression for \( \tau \). Eq. (2) can be expanded to

\[ \tau_{1 \ldots n} = \left| \sum \alpha_1 \ldots \alpha_{n-1} \alpha_{\beta_1} \ldots \beta_{n-1} \alpha_{\gamma_1} \ldots \gamma_{n-1} a_{\delta_1} \ldots \delta_{n-1} \epsilon_{\alpha_1} \epsilon_{\alpha_2} \epsilon_{\beta_1} \epsilon_{\beta_2} \ldots \epsilon_{\alpha_{n-1}} \epsilon_{\beta_{n-1}} \epsilon_{\gamma_1} \epsilon_{\gamma_2} \epsilon_{\delta_1} \epsilon_{\delta_2} \ldots \epsilon_{\delta_{n-1}} \right| \]

Consider some term in the fully expanded version of the first line of the above equation, \( a_{\mu_1 \ldots \mu_{n-1} 0} a_{\bar{\mu}_1 \ldots \bar{\mu}_{n-1} 0} a_{\nu_1 \ldots \nu_{n-1} 1} a_{\bar{\nu}_1 \ldots \bar{\nu}_{n-1} 1} \), where \( \bar{\mu} = 1 \) if \( \mu = 0 \) and \( \bar{\mu} = 0 \) if \( \mu = 1 \). This term can be positive or negative. The expansion of the first line of the above equation also contains the term

\[ a_{\bar{\mu}_1 \ldots \bar{\mu}_{n-1} 0} a_{\mu_1 \ldots \mu_{n-1} 0} a_{\nu_1 \ldots \nu_{n-1} 1} a_{\bar{\nu}_1 \ldots \bar{\nu}_{n-1} 1} \]. For even \( n \) the sign of this term will be opposite the sign of the original term since the signs of an odd number of \( \epsilon \)'s have been flipped. So the two above terms will add to zero, as will all other terms in the first line of Eq. (5). The second line of Eq. (5) also goes to zero by the same argument. Thus, \( \tau_{1 \ldots n} = | \langle \psi \rangle \tilde{\psi} |^2 \) for all even \( n \).

This equality indicates that \( n \)-tangle is a more natural measure of entanglement than concurrence because, for odd \( n \), \( C_{1 \ldots n} = 0 \), while a meaning of \( n \)-tangle is already established.
for \( n = 3 \) \([1]\). From Eq. (3) one can determine that the quantity \( C^2_{1...n} = \tau_{1...n} \) for even \( n \) is invariant under permutations of the qubits, since changing the order of the indices (i.e. the numbering of the Greek letters) is only renaming the indices. This allows us to apply the method used in \([7]\) to prove that \( \tau_{ABC} \) is an entanglement monotone to prove that the \( n \)-tangle is an entanglement monotone, a property that good measures of entanglement must satisfy \([6]\). As in \([7]\) (we explicitly follow their form and proof outline), the invariance of the \( n \)-tangle under permutations of the parties lets us consider local positive operator valued measures (POVM’s) for one party only. Let \( A_1, A_2 \) be two POVM elements such that \( A_1^\dagger A_1 + A_2^\dagger A_2 = I \), then \( A_i = U_i D_i V \) with \( U_i \) and \( V \) being unitary matrices, and \( D_i \) being diagonal matrices with entries \((a,b)\) and \((\sqrt{1-a^2}, \sqrt{1-b^2})\) respectively. For some initial state \( |\psi> \) let \( |\hat{\phi}_i> = A_i |\psi> \) be the subnormalized states obtained after application of the POVM. Let \( |\phi_i> = |\hat{\phi}_i>/\sqrt{p_i} \), \( p_i = <\hat{\phi}_i|\hat{\phi}_i> \). Then

\[
<\tau> = p_1 \tau(\phi_1) + p_2 \tau(\phi_2)
\]

(6)

Since the \( n \)-tangle is invariant under local unitarities \([7]\) \( \tau(U_i D_i V \psi) = \tau(D_i V \psi) \). Now, noting that every term of Eq. (2) contains two \( a \)'s with subscripts starting with zeros and two \( a \)'s with subscripts starting with ones and that every term is quartic with respect to the \( a \)'s, it can be shown that

\[
\tau(\phi_1) = \frac{a^2 b^2}{p_1} \tau(\psi), \quad \tau(\phi_2) = \frac{(1-a^2)^2 (1-b^2)^2}{p_2^2} \tau(\psi)
\]

(7)

Defining \( P_0 \) to be the sum of the squared magnitudes of the first \( 2^{n-1} \) components of \( |\psi> \) in the standard basis, and \( P_1 \) to be the sum of the squared magnitudes of the last \( 2^{n-1} \) components of \( |\psi> \), we can say

\[
p_1 = a^2 P_0 + b^2 P_1 \quad \text{and} \quad p_2 = (1-a^2) P_0 + (1-b^2) P_1
\]

(8)

Combining Eqs. (6-8) with the fact that \( P_0 + P_1 = 1 \) some algebra shows that \( <\tau>/\tau(\psi) \leq 1 \) thus proving that \( n \)-tangle is an entanglement monotone.

Some examples provide further support for the \( n \)-tangle being a measure of some type of \( n \)-party entanglement. An \( n \)-qubit CAT state, \((|0^\otimes n> + |1^\otimes n>)/\sqrt{2}\) is a state with entirely
n-way entanglement; measuring any one of the qubits in the standard basis determines the value of all of the other qubits, however, if one of the qubits is traced out, the remaining qubits are unentangled. For these states, the n-tangle is 1; all terms in Eq. (2) go are 0 for an n-CAT state except for when $\alpha_i = 0, \beta_i = 1, \gamma_i = 1$ and $\delta_i = 0$ or $\alpha_i = 1, \beta_i = 0, \gamma_i = 0$ and $\delta_i = 1$, so $\tau_{1...n}(|CAT>) = 2 - 1/4 + -1/4| = 1$. Another interesting set of states are the n-qubit W states \[7\] $|0...01 > + |0...010 > +... + |10... >)/\sqrt{n}$. For these states, the equality \[4\]

$$C^2_{12} + C^2_{13} + ... + C^2_{1n} = C^2_{1(23...n)}$$

holds, thus tracing out all but two of the qubits leaves the two remaining qubits partially entangled. Note that $C^2_{1(23...n)} \neq 0$ and that the W states are symmetric. Note that the $\epsilon$’s assure that all terms in Eq. (2) are 0 for W states, and hence n-tangle is zero for W states (except for $n = 2$, where $\tau = 1$, since $\tau$ measures two way entanglement in this case).

From the above examples, it is tempting to hypothesize that n-tangle is a measure of n-way entanglement, but a counterexample shows otherwise: Consider the four qubit pure state that is the tensor product of two singlet states. A simple calculation shows that 4-tangle has a value of 1 for this state. If 4-tangle measured 4-way entanglement, its value should have been 0, since this state has no entanglement between the pairs of entangled qubits. Thus, while n-tangle appears to be related to some kind of multipartide entanglement, it is not by itself a measure of n-way entanglement.

**III. MIXED STATE GENERALIZATION OF N-TANGLE**

We would like to have a mixed state generalization of n-tangle. Such a quantity would enable us to classify and quantify even more types of entanglement. For example, a four qubit pure state would have 6 values of the mixed state 2-tangle between each of the pairs of qubits, 4 values for the mixed state 3-tangle between each set of 3 qubits, and a value for the 4-tangle. We suggest defining, for an n-qubit mixed state $\rho$, $\tau^{\text{min}}(\rho)$ to be the minimum of $\sum_i p_i \tau(\psi_i)$ for all pure state decompositions of $\rho$, given by $\rho = \sum_i p_i |\psi_i> <\psi_i|$. This is
analogous to the entanglement of formation \( [8] \), and in fact the entanglement of formation is a function of \( \tau_{12}^{\text{min}}(\rho_{12}) \) for states of two qubits \([5]\). This definition is also justified by the fact that Eq. (24) of \([4]\) can now be rewritten as

\[
C^2_{A(BC)} = \tau_{AB}^{\text{min}} + \tau_{AC}^{\text{min}} + \tau_{ABC}
\]

That is, for two qubits, \( \tau_{12}^{\text{min}}(\rho) \) already has physical significance, so it appears to be a natural way to define a mixed state \( \tau \). Now, in \([5]\), Wootters presents a proof that \( C^{\text{min}}(\rho) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\} \) where \( \lambda_i \) is the square root of the \( i^{\text{th}} \) eigenvalue, in decreasing order, of \( \rho \). This proof is generalizable to show that \( C^{\text{min}}(\rho) = \max\{0, \lambda_1 - \lambda_2... - \lambda_n\} \) for an \( n \)-qubit system, and therefore \( \tau^{\text{min}}(\rho) = C^2_{\text{min}}(\rho) = [\max\{0, \lambda_1 - \lambda_2... - \lambda_n\}]^2 \). This result is also a subset of a more general proof by Uhlmann \([18]\).

A large number of doubts remain about the meaning of \( n \)-tangle. In particular, we would like to have a physically meaningful definition of \( n \)-way entanglement so that we could compare \( n \)-tangle and other multipartite entanglement measures with meaningful values. It seems likely that \( n \)-tangle, in combination with other multipartite entanglement measures (most likely the \( n \)-tangles of smaller subsystems within a given state) will be related to a multipartite generalization of the 2-qubit entanglement, \( E \), related to the Shannon entropy. Unfortunately, no such generalization of entanglement is obvious. If a formula for \( \tau_{1...n}^{\text{min}} \) for \( n = 3 \) could be found, it might be possible to prove statements analogous to Eq. (10) which would lend more legitimacy to \( n \)-tangle. We would also like to have a generalization of \( n \)-tangle for states with subsystems larger than qubits.

**IV. DISCUSSION**

In summary, we have proposed a potential measure of a type of \( n \)-partide entanglement of pure and mixed states: for pure states, the \( n \)-tangle, and for mixed states the related \( \tau_{1...n}^{\text{min}} \). These measures show many signs of being useful ways to quantify a type of multipartite entanglement. For even \( n \), \( n \)-tangle and \( \tau_{1...n}^{\text{min}} \) are equal to the square of a generalization of
pure and mixed state concurrence, $n$-tangle is also an entanglement monotone. $n$-tangle has values of 1 for $n - CAT$ states and values of 0 for $W$ states where $n > 2$ but has a value of 1 for a product state of two singlets. Hopefully these measures will further our understanding of multi-partide entanglement. In particular, further exploration of their mixed state forms may lead to the discovery of relationships between different types of entanglement within a particular system.

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