Computing the Strong Sphaleron Rate

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Abstract

We measure the diffusion constant for Chern-Simons number for classical, lattice SU(3) Yang-Mills theory, using a generalization of the topological definition of Chern-Simons number developed recently by Moore and Turok. The diffusion constant is much larger than that for SU(2), even before the ratio of coupling constants has been accounted for, which implies that chiral quark number is efficiently destroyed by strong processes during the electroweak phase transition. For the physical value of $\alpha_s$ we estimate the decay time for chiral quark number to be about $80/T$, although various systematics make this number uncertain by about a factor of 2.

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1 Introduction

Baryon number is violated in the standard model and there has been a growing interest in trying to understand mechanisms which might use this violation to generate the baryon asymmetry of the universe during the cosmological electroweak phase transition.

All the needed ingredients for generating a baryon asymmetry are present; the violation of baryon number shuts off abruptly while the plasma is out of equilibrium due to the motion of a bubble wall (phase boundary), and if there is $C$ and $CP$ violation then these conditions can give rise to a net baryon number generation. Interest in this scenario has focused especially on a particularly efficient mechanism in which the $CP$ violation, in the form of spatially varying Higgs condensate phases on the bubble wall surface, generates a chiral top quark asymmetry. This can be transported by particle diffusion into the symmetric electroweak phase; the left handed quark number then biases SU(2) winding number changing transitions (“sphalerons”) which generates a net baryon number.

A complication to this scenario is that chiral quark number is damped by SU(3) color winding number changing events, similar to the phenomenon responsible for the spontaneous breaking of chiral symmetry in QCD. To be more concrete, if there is a chemical potential $\mu$ for chiral quark number, then the total chiral quark number density (left handed

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quarks minus their antiparticles minus righthanded quarks, plus the ir antiparticles) is

\[ Q_5 \simeq \sum_{\text{species}} \frac{\mu T^2}{12} = 8N_cN_F\frac{\mu T^2}{12}, \quad (1) \]

where the 8 comes from summing over up and down types, particles and antiparticles, and the two chiralities, and the \( T^2/12 \) is the (leading order in \( \mu \) and \( g^2 \)) change in number density due to the chemical potential for an ultrarelativistic fermionic species. (\( N_c = 3 \) is the number of colors and \( N_F = 3 \) is the number of fermion generations.) The free energy liberated by a strong sphaleron event is \( \mu \) for each left handed particle destroyed or righthanded particle created, which equals \( 4N_F \), since in each generation there is a creation of a righthanded particle and a destruction of a lefthanded particle, both for up and down type quarks. The rate per unit volume of strong sphaleron transitions will then be \( 4N_F(\mu/T)\Gamma_{ss} \), where \( \Gamma_{ss} \), the linear response coefficient of strong sphalerons to a chemical potential, equals half the diffusion constant per unit volume of SU(3) Chern-Simons number, by a fluctuation dissipation relation \( \square \ [8, 9] \). Since each transition changes chiral quark number by \( -4N_F \), one finds \[ \frac{dQ_5}{dt} = (4N_F)^2 \frac{\mu}{T} \Gamma_{ss} = -\frac{24N_F}{N_c}Q_5 \frac{\Gamma_{ss}}{T^3}. \quad (2) \]

The time constant for the decay of \( Q_5 \) is therefore \[ \tau = \frac{T^3}{24\Gamma_{ss}}. \quad (3) \]

Even without calculation we know that \( \Gamma_{ss} \) is much larger than the corresponding weak sphaleron rate \( \Gamma_{ws} \), because SU(3) contains SU(2) and the strong coupling is larger (so nonperturbative physics sets in on a shorter length scale). Hence chiral quark number in the symmetric electroweak phase decays mainly through strong phenomena. If the time constant is shorter than the typical time a quark reflected from or moving off of the bubble wall spends in the symmetric phase before the wall catches up with it, then the strong sphaleron rate will be relevant and will reduce the baryon number generation. In this case we may only need to know the ratio of the strong and weak sphaleron rates to determine the baryon number generated. If \( \tau \) is shorter than the time it takes a particle to get from the middle of the bubble wall to the symmetric phase, then the chiral quark number may be destroyed before it reaches an environment where it can generate baryon number, and the strong sphalerons will qualitatively reduce the production rate for baryon number. (It is important to treat all relevant processes, such as the difference in diffusion constants between right and left handed quarks, and the conversion between chiral top quark number and Higgs particle abundance (net Higgs particle hypercharge) due to the top quark Yukawa coupling \[ \square \]; these might change this picture somewhat.) In any case it seems that the investigation of \( \Gamma_{ss} \) is well motivated.

It is conventional to write \( \Gamma \) in terms of a dimensionless constant \( \kappa \),

\[ \Gamma_{ss} = \kappa_{ss}\alpha_s^4T^4/2, \quad (4) \]

\footnote{The derivation here follows that in \[ \square \] except that they miss the factor of \( N_c \) in the relation between \( Q_5 \) and \( \mu \), so their rate constant for the decay of \( Q_5 \) is 3 times too large.}
where the factor of 2 is because in conventional usage $\kappa_{ss}$ denotes the diffusion constant for $N_{CS}$, not the response coefficient. $\kappa_{ss}$ may depend nontrivially on couplings and the particle content. One should take $\alpha_s$ at a renormalization point on order the temperature scale. Beyond leading order, $\kappa$ must have a logarithmic renormalization point dependence; we will discuss the “best” renormalization point in the next section.

A first attempt to compute $\Gamma_{ss}$ was made in [9], using the classical approximation for the infrared dynamics first suggested in [10] and a numerical implementation of Yang-Mills theory and of Chern-Simons number developed and used in [11, 12]. Unfortunately, this definition of Chern-Simons number suffers from lattice artifacts, leading to a spurious ultraviolet signal and an incorrect normalization of the response to the real, infrared winding number change. The lattice implementation of Yang-Mills theory also requires perturbative corrections to the tree level match between lattice and physical length scales [13]. It is known in the case of SU(2) that both corrections are numerically important. The SU(2) diffusion constant without these corrections gives a lattice spacing independent $\kappa$ [12], which is qualitatively different from what is expected on theoretical grounds. Arnold, Son, and Yaffe have argued that the interaction of the infrared modes responsible for winding number change with “hard” short wavelength excitations is essential, and because of it $\kappa$ should depend on the product of the plasma frequency $\omega_{pl}$ and the inverse nonperturbative length scale $l_{np} \sim 1/g^2 T$ as

$$\kappa \propto (\omega_{pl} l_{np})^{-2}, \quad (5)$$

at least when $\omega_{pl} l_{np} \gg 1$ [14]. Since $\omega_{pl}$ on the lattice depends on the lattice spacing $a$ as $a^{-1/2}$ [15], this implies that $\kappa$ should be proportional to $a$; and for the physical quantum system $\omega_{pl} \sim gT$, so $\kappa_{ws} \propto \alpha_w$.

Recently a topological technique for measuring winding number change on the lattice has been developed for classical SU(2) gauge theory [16]. Using it, and applying the corrected matching between lattice and physical length scales, reveals that $\Gamma_{ws}$ does depend on lattice spacing, verifying that previous results were contaminated with lattice artifacts. The coarsest lattices used there were insufficient to achieve $(\omega_{pl} l_{np})^2 \gg 1$, and the dependence was (therefore?) weaker than linear with $a$; but we will assume here that the reasoning of Arnold, Son, and Yaffe is correct and that for sufficiently large hard thermal loop effects the rate does scale with $(\omega_{pl} l_{np})^{-2}$.

Because $\Gamma$ depends on the physics of hard modes, which is definitely not reproduced correctly on the lattice [17], we do not know how precisely to convert $\Gamma$ measured there into $\Gamma$ in the quantum theory. This problem was addressed recently by Arnold, who argues that a reasonably accurate conversion is possible [13]. The small $a$ limit of the ratio of $\Gamma$ for SU(2) and SU(3) should also be insensitive to this problem, since the distortion of the hard thermal loops is common to the two lattice theories. Hence we can compute the ratio $\Gamma_{ss}/\Gamma_{ws}$ on the lattice with smaller systematics. The purpose of this letter is to extend the results of [13] and the technique of [16] to SU(3), and to use them to compute $\Gamma_{ss}$ on the lattice, and hence to find $\Gamma_{ss}/\Gamma_{ws}$ and to estimate $\Gamma_{ss}$. 

3
2 Diffusion constant: General discussion

Most analytic work on $\Gamma$ has discussed the broken phase of Yang-Mills Higgs (YMH) theory. In this case the gauge connection is usually close to some topological vacuum, and the diffusion rate is controlled by the free energy of configurations midway between vacua, “sphalerons.” This way of viewing things can also be useful in the symmetric phase or in pure Yang-Mills (YM) theory, where the diffusion rate will also depend on how often the system is straddling between vacua.

Consider classical YM or YMH theory in a fixed finite volume and regulated at a length scale much smaller than $1/g^2 T$. (We are thinking of lattice regulation, but we will use continuum notation here for convenience.) We can define the closest vacuum to the spatial connection $\vec{A}$ at time $t$ by “cooling” [18]. That is, consider evolving $\vec{A}$ under (gauge invariant) straight dissipative dynamics,

$$\frac{\partial \vec{A}(x, t, \tau)}{\partial \tau} = -\frac{\partial H(A(t, \tau))}{\partial \vec{A}(x, t, \tau)}, \quad A(x, t, 0) = A(x, t),$$

with $H$ the Hamiltonian. (The cooling time $\tau$ is not to be confused with the decay rate for chiral quark number discussed in the introduction.) Such cooling was recently used as a technique to improve the local operator method for tracking $N_{CS}$ [19]. At sufficiently large $\tau$ the connection will settle into a vacuum configuration, and we define this to be the nearest vacuum.

It will sometimes occur that there is a time $t_{sph}$ such that the nearest vacuum at time $t_{sph} + \epsilon$ has a different winding number than the vacuum at time $t_{sph} - \epsilon$. At time $t_{sph}$ the dissipative evolution will never get to a vacuum state, and just before and after, it will take a very long time. We can define the system to be “in a sphaleron” if the cooling time $\tau$ required to get close to vacuum exceeds some threshold $\tau_{thresh}$. Here “close to vacuum” can be given a rigorous definition, eg the total action of the remaining cooling path to the vacuum is less than $(\pi/g)^2$.

The probability that the system is “in a sphaleron” depends only on $\tau_{thresh}$ and the thermodynamics of the spatial connections. Further, if we define

$$E_i^a(x, t, \tau) = [D_0, D_i]_a^a(x, t, \tau)$$

(with $D_0$ the covariant $t$ derivative), then since the distribution of values for $E(t, \tau = 0)$ is a thermodynamic property, and since $\partial E/\partial \tau$ depends only on the connections and on $E$, [19]

$$\frac{\partial E_i^a(x, t, \tau)}{\partial \tau} = E_j^b(x, t, \tau) \frac{\partial^2 H(A(t, \tau))}{\partial A_i^a(x, t, \tau) \partial A_j^b(x, t, \tau)},$$

then $E(t, \tau)$ is also distributed according to thermodynamics. If we define the sphaleron narrowly enough that the system typically remains “in a sphaleron” for a time short compared to the inverse plasma frequency, so $E(t, 0)$ does not change much from the beginning.
to the end of the sphaleron event, then the length of a sphaleron event will also depend only on $\tau_{\text{thresh}}$ and on thermodynamics, and the total spacetime density of sphaleron events will be a thermodynamic property, depending on the thermodynamics of $\vec{A}$ and $\vec{E}$ alone. We have not shown here that this spacetime density has a good large volume limit, but we will assume this to be the case.

Since the spacetime density of sphaleron events depends only on thermodynamics, we know that the quantum theory value is reproduced by the classical theory in the $a \to 0$ limit, with corrections due to the thermodynamics of the $\vec{A}$ fields which are $O(\alpha^2)$ if we use the dimensional reduction calculation \cite{21} to establish the value of the coupling constant of the 3-D theory. There may also be $O(\alpha)$ corrections in the thermodynamics of the $\vec{E}$ fields, i.e., in the relation between time scales \cite{22}, which have not been calculated. Further, since the only length scale in the thermodynamics of the $\vec{A}$ and $\vec{E}$ fields is $1/(g^2 T)$, the spacetime density of sphaleron events can be written as $\kappa_1(N_c) \times (\alpha T)^4$, with $\kappa_1(N_c)$ a pure number. In YM theory, $\kappa_1$ also depends on the Higgs potential parameters $x$ and $y$. It should approach the YM theory value in the limit of large positive $y$, deep in the symmetric phase, and it becomes exponentially small at large negative $y$, deep in the broken phase.

Each sphaleron event changes the winding number of the nearest vacuum by $\pm 1$. If the signs of each change were independent, then $\Gamma$ would equal $\kappa_1(N_c)\alpha^4 T^4 / 2$. But the signs will in general be correlated, $\Gamma = \kappa_2 \kappa_1(N_c)\alpha^4 T^4 / 2$, where $\kappa_2$ describes the degree of correlation in the signs of sphaleron events, and depends on the dynamics. In particular, Arnold, Son, and Yaffe argue that plasma oscillations will make the system go back and forth through sphalerons of opposite sign. On short time scales the motion of infrared magnetic fields will be oscillatory, and on long time scales it will be overdamped. They conclude that the system will go through on order $\left( \frac{\omega_{\text{pl}}}{g^2 T} \right)^2 \sim 1/(g^2 \hbar)$ sphaleron events per permanent winding number change, so $\kappa_2 \sim g^2 \hbar$ \cite{14}. In the classical theory, the role of $\hbar$ is played by the regulator scale, and $\kappa_2 \propto g^2 a T / 4 \equiv \beta_L^{-1}$, at least for large $\beta_L$ \cite{13}.

To determine $\Gamma$ correctly, it is necessary to count winding number changes correctly; to get the thermodynamics, and hence $\kappa_1$, right; and to get the dynamics, and hence $\kappa_2$, right. The first two problems are separate from the third, and we deal with them in the next section. Getting the dynamics right is harder, and in our opinion this problem has not been solved. However, the work of Arnold \cite{14} suggests that lattice results can be converted to continuum results with fairly modest systematic error, and we will use his matching here.

3 From SU(2) to SU($N_c$)

There are no complications in extending the standard Kogut-Susskind implementation of 3+1 dimensional SU(2) Yang Mills theory \cite{23} to SU(3), and the thermalization algorithm for the SU(3) case was developed in \cite{3}. What remains is to extend the one loop matching of the thermodynamics of lattice and continuum systems, and to extend the topological tracking of winding number, from SU(2) to SU(3) (or SU($N_c$)).
3.1 thermodynamics

We deal first with the thermodynamics. As we discussed in the last section, it is only the thermodynamics of the spatial connections (and of the E fields) which are important; so our goal is to make sure that the thermodynamics of spatial connections at finite lattice spacing $a$ are as close as possible to the continuum thermodynamics. Here and throughout we will use the notation of [12, 13]. We will not attempt to make this section self-contained; the reader is referred to [13] for details on the approach. All we do here is generalize to SU($N_c$) the gauge field part of the SU(2) calculation done there. The details of this section are not important in what follows, only the final result, so the uninterested reader can skip to the next subsection.

The thermodynamics of the real time system we are considering are determined by the path integral

$$
Z = \int \mathcal{D}A_iA_0 \exp(-\beta_L H_L),
$$

$$
H_L = \sum_{x,i<j} \left( \frac{N_c}{2} - \frac{1}{2} \text{Re} \text{Tr} U_{ij}(x) \right) + \\
+ \sum_{x,i} \frac{1}{2} (U_i(x)A_0(x+i)U_i^+(x) - A_0(x))^2 + \sum_x \frac{m^2_D}{2} A_0^2(x),
$$

(9) (10)

with the bare Debye mass $m^2_D = 0$ [12]. Here $U_{ij}(x)$ is the elementary plaquette which extends from $x$ in the $i, j$ directions. The gauge coupling has been absorbed into $\beta_L$ which (at tree level) equals $\beta_L = 4/(g^2 a T)$. We want to improve the Hamiltonian so the thermodynamics produced by this partition function match more accurately those of the continuum system. The idea is that, since the lattice and continuum theories only differ strongly in the ultraviolet, one should compute the influence of ultraviolet modes on the infrared physics perturbatively, in the lattice and continuum theories, and find the difference (which is free of infrared divergences). Because the infrared length scale is well separated from the length scale where the lattice and continuum theories significantly differ, the difference can be written as an operator product expansion, and only the super-renormalizable terms are needed. One compensates for these terms by making shifts in the wave functions and couplings of the theory, thereby correcting the lattice theory in the infrared for its ultraviolet differences from the continuum theory. Because the theory is super-renormalizable, one loop perturbative corrections are $O(a)$, leaving the infrared behavior of the lattice and continuum theories matching up to $O(a^2)$.

The exception to this rule is dimension 2 operators, where the one loop contribution is $O(1/a)$ in physical units, which in lattice units is $m^2 \sim \beta_L^{-1}$, and a full $O(a)$ correction requires a three loop calculation. One dimension 2 operator, the Debye mass, appears here.

However, the large value of $m^2_D$ makes the influence of the $A_0$ field on the thermodynamics of the gauge fields perturbative; to study their thermodynamics, we will want to integrate out the $A_0$ field, and the 2 and 3 loop corrections to $m^2_D$ and 1 loop wave function corrections to the $A_0$ field will only change the result of that integration at order $\beta_L^{-3/2}$, as we discuss

\footnote{For $N_c \neq 2$ this notation differs from that usually used in 4 dimensional lattice QCD, which uses $1 - (1/N)\text{Re} \text{Tr}$.}
below. Hence, we will not calculate these corrections here. But we will need to know the Debye mass. Since the bare value is zero, \( m_D^2 \) equals the counterterm, which is computed at one loop in [24]:

\[
m_D^2 = \frac{2N_c \Sigma}{\pi \beta_L} \text{ (lattice)} = \frac{N_c \Sigma \beta_L}{8\pi} g^4 T^2 \text{ (physical)}. \tag{11}
\]

It only remains to find the correction to the plaquette term, which must be multiplicative, i.e., the term \( \sum 1 - (1/2) \text{Tr} U \) above is multiplied by a wave function correction \( Z_A \). This can be absorbed into (or understood as) a shift in \( \beta_L \). Contributions to \( Z_A \) arise both from self-energy and vertex corrections, and are actually easiest to compute in the theory with \( N_s \) fundamental scalars added. The gauge field is properly normalized if the full effect of a gauge line propagating between scalar lines is the same in the lattice and continuum theories. The scalar wave function receives a renormalization which, by minimal coupling, changes the strength of the scalar-gauge vertex. There are also loop corrections to the vertex and to the propagator, illustrated in Figure [1]. What enters the calculation is the difference between the loop corrections to the gauge-scalar vertex and the scalar wave function, and the pure gauge and \( A_0 \) contributions to the gauge self-energy. The only differences between the SU(2) and SU(\( N_c \)) calculations are the group factors; no new diagrams or new momentum integrals appear. Examining [13], one finds that the fundamental scalar corrections to the self-energy depend on \( N_s \text{Tr} T_a T_b = (N_s/2) \delta_{ab} \), and that all but one of the other diagrams have group factors proportional to \( \frac{N_c}{(2\pi)} \). The diagrams needed and their values are tabulated there.

The one exception is a contribution to the gauge field tadpole diagram arising from a term in the gauge field 4-point interaction, which comes entirely from anticommutators of Lie algebra generators, and has a group factor of

\[
\frac{1}{3} \left( 2 \left[ \delta_{ab} \delta_{cd} + \delta_{ac} \delta_{bd} + \delta_{ad} \delta_{bc} \right] + d_{abc} d_{bcd} + d_{ace} d_{bde} + d_{ade} d_{bce} \right) \tag{12}
\]

and a Lorentz dependence of \( (1/2) \sum_{ij} F_{ij}^4 \). This term is purely nonrenormalizable operator and has no analog in the continuum theory; it also dominates the correction between lattice and continuum theories. To find the group factor for the contribution to \( Z_A \) one contracts against \( \delta_{cd} \); the \( \delta_{ab} \) type terms give \( (2/3)(N + 1/N) \) and the \( d_{abc} \) type terms give \( (2/3)(N - 4/N) \). The space integral gives \( 1/(3\beta_L) \).

All told, the contributions to the gauge field renormalization are (\( \xi = 0.152859, \Sigma = 3.17591 \))

\[
\beta_L(Z_A - 1) = \frac{2}{9} \left( 2N_c - \frac{3}{N_c} \right) + N_c \left( \frac{37\xi}{12\pi} - \frac{1}{9} \right) + N_c \left( \frac{\Sigma}{24\pi} - \frac{\xi}{6\pi} \right) + N_s \left( \frac{\Sigma}{48\pi} - \frac{\xi}{12\pi} \right), \tag{13}
\]

where the first term is from the contribution discussed above, the second term is from all other gauge field contributions, the third is from the (adjoint scalar) \( A_0 \) field contribution to the self-energy, and the last term is if there are fundamental scalars present. If one puts in a “naive” bare value of \( \beta_{L,\text{bare}} \) then the simulation is equivalent to one with the appropriate \( Z_A \) and \( \beta_{L,\text{imp}} = Z_A^{-1} \beta_{L,\text{bare}} \), which numerically is

\[
\beta_{L,\text{imp}} = \beta_{L,\text{bare}} - \left( \frac{4N_c}{9} - \frac{2}{3N_c} \right) - 0.0389 N_c - 0.0340 N_c - 0.0170 N_s \tag{14}
\]
Figure 1: Examples of diagrams contributing to the renormalization of the gauge field. The full effect of a gauge line propagating between scalar lines must match between theories; besides self-energy corrections, there are also corrections at the vertex, and corrections because the vertex is renormalized at the same time the scalar propagator is.

where again the contributions are from the anticommuting part of the tadpole diagram, other vector contributions, the $A_0$ field, and any scalars present. For $N_c = 3$ and no fundamental scalars, we find $\beta_{L,\text{imp}} = \beta_{L,\text{bare}} - 1.330$. This gives us the correction we need to convert from lattice to physical length scales.

Two comments are in order here. First, the correction is totally dominated by the tadpole term. This behavior is typical in lattice gauge theory. It is caused by the compact nature of the connections and it is the reason a perturbative matching between the lattice and continuum theories is so necessary. Second, the correction is larger for SU(3) than for SU(2). Since the diffusion constant depends on the fourth power of the conversion between length scales, one must ensure that the matching is quite good. If for instance we assume that there are unknown $O(a^2)$ tadpole type corrections of magnitude equal the square of the $O(a)$ correction, and we ask that these corrections be at the 1% level (which will still give 4% systematic errors), we need $\beta_L > 13$; we must work on quite fine lattices. Though this increases the numerical demands, it is not all bad, since it means that the elementary plaquettes will be quite close to the identity and it should be possible to make the connections quite smooth. We will need this property in the next subsection.

Now we have related the lattice theory to the continuum theory with $A_0$ field, and with the wrong Debye mass. We need to understand how the value of $m_D^2$ modifies the gauge field thermodynamics, and this is easiest done by integrating out the $A_0$ field. In terms of the natural 3-D length scale $1/g^2T$, the Debye mass is $m_D^2 \sim \beta_L(g^2T)^2$, so it is indeed heavy enough to integrate out. At one loop, the modification to the gauge coupling is

$$\bar{g}^2 = g^2 \left( 1 - \frac{N_c g^2 T}{48\pi m_D} \right).$$

This correction is formally $O(\beta_L^{-1/2})$. Now recall that $m_D^2$ receives two loop corrections which are $O(\beta_L^{-2})$ in lattice units, or $O(g^4 T^2)$ in physical units. One can see by plugging $m_D^2 = (A\beta_L + B) g^4 T^2$ into Eq. (17) that this correction leads to an $O(\beta_L^{-3/2})$ correction to $\bar{g}^2$. Similarly, the $O(\beta_L^{-1})$ correction to the $A_0$ wave function renormalization can be absorbed by a rescaling of $A_0$ into a shift in $m_D^2$ of order $O(m_D^2 \beta_L^{-1})$, also leading to an $O(\beta_L^{-3/2})$ correction. We will neglect these $O(\beta_L^{-3/2})$ corrections.
In addition to the $O(\beta^{-1}_L)$ correction we have just mentioned, the integration over the $A_0$ field gives an $O(\beta^{-3/2}_L)$ two loop correction, and induces nonrenormalizable operators which affect physics at the nonperturbative length scale by $O(\beta^{-3/2}_L)$. While the nonrenormalizable operators are ignorable, the two loop correction is parametrically as important as the $O(\beta^{-1}_L)$ correction to the lattice-continuum match which we have just calculated. However, we believe that it is numerically much smaller. This is because of the “tadpole” character of the dominant 1-loop effects we have studied. It can be argued that the expansion parameter for the integration over the $A_0$ field is $g^2 T/4\pi m_D \sim 0.1 \beta^{-1}_L$, whereas the “tadpole” corrections were $\sim 1 \times \beta^{-1}_L$. Numerically, at $\beta_L = 16$, the one loop, $O(\beta^{-1}_L)$ correction to integrating out the $A_0$ field, Eq. (15), is less than 1%, while the $O(\beta^{-1}_L)$ correction in the lattice-continuum match is $\sim 8\%$. Hence, although the two loop correction from integrating out the $A_0$ field is formally of the same parametric order as corrections we include, it should be numerically unimportant. Note that the numerical unimportance of this two loop effect is also important in the study of thermodynamic properties of SU(2) YMH theory using the dimensional reduction program if one integrates out the $A_0$ field in that program. We will only make corrections due to Eq. (14) and Eq. (15) in this paper.

Finally, there are also $O(a)$ lattice corrections to Eq. (14), which can be understood as corrections in the conversion between lattice and continuum time units. These have not been calculated, but it was argued in [16] that they are dominated by tadpole effects which are the same as those occurring in the lattice-continuum match for the spatial gauge fields. We will use the prescription proposed there, leading to another $O(a)$ but probably modest error.

The thermodynamics are now under control, and while there will still be uncorrected $O(a)$ errors in $\kappa_1$, the large “tadpole” type corrections are taken care of.

### 3.2 winding number

Now we discuss the extension of the winding number tracking technique of [16] to SU($N_c$).

The idea of that paper is to keep track of a notional group valued scalar field $S$, with Hamiltonian

$$H_S = \sum_{x,i} \frac{N_c}{2} - \frac{1}{2} \text{Re Tr} S^t(x)U_i(x)S(x+i).$$

One then evolves $S$ dissipatively and aggressively to minimize $H_S$. The Chern-Simons number of a configuration with $S = I$ everywhere is approximated to be zero, and the total winding number change during an evolution is tracked by gauge transforming to the gauge $S = I$ whenever that gauge is everywhere smooth, ie there is no neighborhood where the connection matrices $U$ are far from the identity. When the winding number of the underlying gauge field configuration changes, then $S$ will go through a period where it is not smooth somewhere, as it adjusts to describe the new winding number state. When it has returned to being everywhere smooth, the gauge transform to $S = I$ is a large (but smooth) transformation; we find its winding number and use it to increment the cumulative winding number change to date. The winding number of a gauge transformation is determined with an algorithm which is essentially that of Woit [20].

Two things become more complicated when one goes to SU($N_c$). The first is the implementation of the dissipative algorithm for $S$. The basic element of the dissipative algorithm
is to minimize $H_S$ with respect to $S$ at one site $x$. The easiest way in SU(2) to find $S(x)$ which minimizes $H_S$ is to sum the parallel transports of nearest neighbors, which will be a constant times the desired element of SU(2), and to project the modulus to SU(2). For SU($N_c$) the sum of several group elements is not generally a multiple of a group element, and one must orthogonally project to SU($N_c$) by a more complicated algorithm. First, scale the matrix so the modulus of its determinant will be close to 1. Call the resulting matrix $M$. To project towards U($N_c$), one repeatedly replaces $M \rightarrow (3/2)M - (1/2)MM^\dagger M$; if this process converges it gives the closest element of U($N_c$) to the original matrix $M$. Then one makes a U(1) rotation by the angle $(-1/3)\arg \det M$ to get to SU($N_c$). If the projection to U($N_c$) fails to converge, for instance because the slave field is varying wildly around the point, then one does not update at this point; but in practice this essentially never happens. The algorithm to combine these elementary projections into an efficient quench is the same as in the SU(2) case.

The second complication is determining the winding number of a slave field configuration. For SU(2) there is a simple algorithm because the group has the same dimension as the space. Interpolating $S(x)$ between lattice sites by a geodesic rule, $S$ becomes a map from $T^3$ to SU(2)$\cong S^3$, which are of the same dimension, and the winding number is just the oriented sum of times some fixed point in SU(2) is covered. There is an extension of this idea to SU(3) by choosing a canonical map of SU(3) with a dimension 4 subspace cut out into $S^3 \times S^5$ such that the $S^3$ part carries the relevant topological information. The 4 dimensional excision will not generally be important since SU(3) is 8 dimensional and we are mapping from a 3 dimensional space.

However, because the underlying fields are quite smooth, it turns out there is an easier way to determine the winding number. One constructs the vacuum field obtained by gauge transforming the naive $U = I$ vacuum by $S$, and then estimates its $N_{CS}$ directly by integrating $\epsilon_{ijk} f_{abc} A_i^a A_j^b A_k^c$. Defining

$$A_i^a(x + i/2) = \frac{1}{2} \text{ReTr} - i\lambda^a U_i(x) = \frac{1}{2} \text{ReTr} S^\dagger(x + i)(-i\lambda^a)S(x)$$

and

$$A_i^a(x) = \frac{A_i^a(x + i/2) + A_i^a(x - i/2)}{2},$$

the integral is

$$N_{CS} = \frac{1}{2\pi^2} \sum_x f_{abc} A_1^a(x)A_2^b(x)A_3^c(x).$$

The result is not gauge invariant and will not be an integer. However, if the slave field is smooth, then the result will be close to an integer; if the slave field used in a gauge transformation is always suitably smooth then it will be possible to unambiguously reconstruct what integer the above integral was “trying to give us”. We have tested this idea with SU(2), where it is possible to compare the value of the integral to the (integer) winding number determined topologically. This is illustrated in Figure 4 as seen there, the values from Eq. (19) always understate the winding number, but by a fairly consistent amount. The values obtained for gauge transformations of one winding number do not overlap those which arise from another winding number, so one can unambiguously reconstruct the winding number of gauge transformations from the values of Eq. (19). To implement the same idea in SU(3),
one just writes the time and value of Eq. (19) every time a gauge transformation is made. Plotting the values of the integral which occurred, one finds the appropriate breaks and can then reconstruct the winding number changes, also illustrated in Figure 2. There is never any difficulty in the reconstruction for the lattice spacing and volumes used in this paper, though presumably this technique should start to have problems on very coarse lattices or large volumes.

4 Numerical results

It appears from theoretical arguments [14, 15] and numerical results [16] that the $N_{CS}$ diffusion constant in SU(2), $\kappa_{ws}$, depends on the lattice spacing, because finer lattices have more hard modes contributing to hard thermal loops [17]. Since it is more numerically expensive to study SU(3) it makes the most sense to try to come to grips with this problem, and with the problem of trying to include hard thermal loops properly, in the SU(2) theory. Hence we will determine the lattice value of $\Gamma_{ss}$ at only one lattice spacing, albeit a fairly fine one. It should be possible to use this value to establish the ratio $\Gamma_{ss}/\Gamma_{ws}$, up to errors from how $\Gamma$ approaches the large $\omega_{pl}$ scaling regime. The absolute value of $\Gamma_{ss}$ can also be estimated using the arguments in [13], but the systematic errors will be bigger here.

It is not known what finite volume systematics may occur in the calculation of $\Gamma_{ss}$, so we measure it on a range of (cubic toroidal) lattices. Our results are for $\beta_{L,bare} = 16$ and for $8^3$, $12^3$, $16^3$, $24^3$, and $32^3$ lattices. The results are tabulated in Table 1, which presents $\kappa_{ss}$,
Table 1: Dependence of $\kappa_{ss}$ on lattice volume, showing the approach to an infinite volume limit.

| $\beta_{L,\text{bare}}$ | $N$ | $\kappa_{ss}$  |
|-------------------------|-----|---------------|
| 16                      | 8   | 0.00 $\pm$ 0.07 |
| 16                      | 12  | 4.0 $\pm$ 0.8  |
| 16                      | 16  | 8.3 $\pm$ 1.1  |
| 16                      | 24  | 8.8 $\pm$ 1.4  |
| 16                      | 32  | 8.0 $\pm$ 1.2  |

where $\kappa_{ss}$ is defined in Eq. (4). Naively, one would convert from lattice to continuum units by

$$\kappa_{ss} = \text{diffusion constant (lattice units)} \times (\pi \beta_{L,\text{bare}})^4,$$

but we have used the thermodynamic corrections derived in the last section, as described in [16].

The largest 3 lattices are statistically compatible, so we have achieved the large volume limit at least at the level of statistics obtained here. The winding number changes on the smallest lattice were all immediately followed by a winding number change of opposite sign; this might be the system getting almost up to a half integer $N_{CS}$ state and then turning back, and does not represent a permanent change to the underlying vacuum winding number. We have only an upper limit for the diffusion constant at that lattice spacing.

We would like to use this lattice rate to estimate the rate in the physical quantum theory. As we have argued, thermodynamic errors, ie errors in $\kappa_1$, are under control and should be smaller than our statistical errors. However, the same cannot be said of systematics in the dynamics, ie in $\kappa_2$. The problem of relating the classical lattice theory and the real, continuum quantum one has recently been studied by Arnold [15]. He argues that, in the large HTL effect regime, the evolution of infrared magnetic fields is overdamped on time scales longer than $1/g^2 T$, and the strength of the damping sets $\kappa_2$. The damping occurs because $\vec{E}$ fields set up currents of “hard” modes; the currents propagate; and they enter the soft mode equations of motion somewhere else. Hence, the infrared fields at one location “feel” the electric fields along the past light cone, due to interactions with hard modes. This is the physics of the hard thermal loops. However, the distribution of hard modes on the lattice is very anisotropic; so will be the conveying of information by the hard modes; and so will be the damping of the infrared fields. Arnold argues that one can make an approximate match between classical lattice and continuum quantum values of $\kappa_2$ by taking an angular average of the damping strength. There are systematics associated with this, which he estimates conservatively as being on order 30%. There is another systematic because the lattice system is probably not deep in the strong damping regime; neither is the continuum quantum theory at realistic $\alpha_s$.

Using Arnold’s proposed match of damping coefficients, we find that $\Gamma_{ss}$ for SU(3) with six flavors of quarks is (including the factor of $1/2$ to go from the diffusion constant to the response coefficient)

$$\Gamma_{ss} = (108 \pm 15_{\text{stat}} \times 2^{+1}_{\text{syst}}) \alpha_s^5 T^4 \quad (m_D^2 = 2 g_s^2 T^2).$$

(21)
Here I have made a conservative estimate of the systematic errors to be about a factor of 2.

The ratio \( \Gamma_{ss}/\Gamma_{ws} \) in the formal small \( \alpha \) limit has smaller systematics because the problem from the anisotropy of the damping coefficient is common to SU(2) and SU(3). Corrections to the large HTL limit are probably of different magnitude for the two theories, though they are presumably of the same sign. We can estimate them by fitting the SU(2) data at different values of \( \beta_L \) to the functional form \( A\beta_L^{-1} \) (leading) + \( B\beta_L^{-2} \) (correction) and seeing how large the extrapolation from \( \beta_L = 16 \) to \( \beta_L = \infty \) is. From the data in [16], which has the same thermodynamic improvements as here and a topological definition of \( N_{CS} \), we find that the extrapolation is a 21% correction. (The fit is startlingly good: \( \chi^2 = 0.42 \) for 5 points and 2 fitting parameters.) If the (unknown) difference between the SU(2) and SU(3) extrapolations is on order the same size as the SU(2) extrapolation, then taking the ratio of \( \beta_L = 16 \) data and using the size of the extrapolation to give the systematic error gives

\[
\frac{\Gamma_{ss}}{\Gamma_{ws}} = (10.7 \pm 1.5_{\text{stat}} \pm 2.3_{\text{syst}}) \left( \frac{\alpha_s}{\alpha_w} \right)^5 \frac{m_{Dw}^2 g_w^2}{m_{Ds}^2 g_s^2}.
\]  

(The direct ratio of \( \beta_L = 16 \) data is 7.1, but remember that \( m_{DL}^2 \propto N_c \).)

To evaluate this we need \( \alpha_s \) and \( \alpha_w \) in the dimensionally reduced 3-D theory. Using \( T_c = 100 \text{GeV} \) and \( \alpha_s(M_Z, \overline{\text{MS}}) = 0.118 \), we ran \( \alpha_s \) to the renormalization point \( \mu = 7.06T \) using the two loop renormalization group equation, and used this value as input in Eq. (146) of [21], adapted to six flavor SU(3), to find \( \alpha_{s,DR} \). This procedure should minimize two loop errors. The result is \( \alpha_{s,DR} = 0.086 \). We took \( \alpha_w \) from Figure 7 of [21], at \( T_c = 100 \text{GeV} \) and \( m_H = 70 \text{GeV} \): \( \alpha_{w,DR} = 1/31.7 \). Both values and particularly the value for \( \alpha_{s,DR} \) are smaller than we are used to; this is because the dimensional reduction procedure sets the coupling roughly to the value at \( \mu(\overline{\text{MS}}) \approx 7T \). Using these values, \( m_{Dw}^2 = 11 g_w^2 T^2 / 6 \), and \( m_{Ds}^2 = 2 g_s^2 T^2 \), evaluating Eq. (22) gives \( \Gamma_{ss}/\Gamma_{ws} \sim 1500 \). This is much smaller than the value we would get using \( \alpha_s = \alpha_s(M_Z) \approx 0.118 \). This large renormalization point dependence makes one nervous that subleading contributions in \( \alpha_s \) may be non-negligible.

5 Conclusion

We have calculated the diffusion constant for SU(3) Chern-Simons number at weak coupling (ie, high temperatures) by classical, lattice methods, using a topological definition of \( N_{CS} \), and find that, at equal values of coupling and hard thermal loops, the diffusion constant is larger than in SU(2) by an order of magnitude. Since in the standard model above the electroweak phase transition, the strong coupling constant is quite a bit larger than the weak coupling constant, the actual ratio of winding number diffusion rates for SU(3) and SU(2) is very large.

Since the hard thermal loop effects induced by the hard lattice modes are different from those which would occur for ultrarelativistic particles [17], we do not know with certainty how to convert the diffusion constant of the classical lattice system into the diffusion constant for the physical quantum system, and cannot establish the time constant \( \tau \) with which a

\footnote{Technically we should not just use \( \alpha_{s,DR} \) but should also include the correction, Eq. (15). But even after taking the fifth power of \( \alpha_s \), this correction is less than 10%. This is another example of how well behaved the integration over the \( A_0 \) field turns out to be.}
chiral quark number is damped by strong sphaleron processes in the plasma. But using the estimate, Eq. (21), we get $\tau \sim 80/T$. This is slow enough to allow quarks to escape the bubble wall before chiral quark number is destroyed, but it is still over 3 orders of magnitude faster than the rate at which chiral quark number is converted into baryons through weak sphaleron processes.

We should mention an interesting case where strong sphalerons are less important, which is for theories with a stop squark light enough to develop a condensate just before the electroweak phase transition. The thermodynamics of this model have been considered recently [27] and it apparently provides an especially strong phase transition. Since color is broken from SU(3) to SU(2) in the symmetric phase, strong sphalerons only proceed at the SU(2) rate for two of the colors and will only erase chiral quark number in the third color at an exponentially small rate. Of course, strong processes will mix chiral quark number between the two unbroken and one broken color, but the suppression of the rate by a factor of 10 is significant when the ratio $\Gamma_{ss}/\Gamma_{ws}$ is important to the final baryon number abundance.

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