A NEW CONJECTURE ON RATIONAL POINTS IN FAMILIES

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ABSTRACT. We prove asymptotics for Serre’s problem on the number of diagonal planar conics with a rational point and use this to put forward a new conjecture on counting the number of varieties in a family which are everywhere locally soluble.

1. Introduction

The fundamental problem in Diophantine Geometry is to determine solubility of Diophantine equations over the rationals. Given the difficulty of this problem, one can instead ask this for “random” equations as done by Bhargava for random ternary cubic curves [7] or Browning–Le Boudec–Sawin [18] in their resolution of the Poonen–Voloch conjecture [59]. A key step in these questions is to understand the probability of a random equation being everywhere locally soluble, i.e. soluble over \( \mathbb{R} \) and \( \mathbb{Q}_p \) for all \( p \). The naive guess would be to consider the product over all places of the probabilities that a random equation in the family is locally soluble; for example, as in the work of Bhargava–Cremona–Fisher–Jones–Keating for quadrics [8]. However, this guess can fail even for simple-looking families. As an example, define

\[
N(B) = \# \left\{ \begin{array}{l}
t \in (\mathbb{Z} \setminus \{0\})^3 : \\
gcd(t_0, t_1, t_2) = 1, \\
\max_i |t_i| \leq B, \\
\sum_{i=0}^3 t_i x_i^2 = 0 \text{ has a } \mathbb{Q}\text{-point}
\end{array} \right\}.
\]  

The problem of proving asymptotics for \( N(B) \) was originally raised by Manin in a letter to Serre, who proved in [66, page 399] that there exists a positive constant \( c_1 \) such that for all \( B \geq 2 \), one has \( N(B) \leq c_1 B^3 (\log B)^{-3/2} \). Subsequently, Hooley [45] and Guo [40] independently proved that there exists a positive constant \( c_2 \) such that for all \( B \geq 2 \), one has \( N(B) \geq c_2 B^3 (\log B)^{-3/2} \). Using different arguments, we prove an asymptotic:

**Theorem 1.1.** For all \( B \geq 2 \), we have

\[
N(B) = \frac{2 \cdot c_{\infty} \prod_p c_p}{\pi^{3/2}} \frac{B^3}{(\log B)^{3/2}} + O \left( \frac{B^3}{(\log B)^{5/2}} \right),
\]

where the product is over all primes \( p \) and

\[
c_{\infty} = 6, \quad c_2 = \frac{49/48}{(1-1/2)^{1/2}}, \quad c_p = \left( 1 + \frac{1 + \frac{1}{p+1}}{p^2} \right) \frac{(p^2 + p/2 + 1)}{(p+1)^2(1-1/p)^{1/2}}, \quad p \neq 2.
\]

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The factors $c_\infty$ and $c_p$ admit interpretations as local densities (for example $c_\infty = 6$ as six choices of the signs of the $t_i$ give a conic with a real point). However, the extra prefactors 2 and $\pi^{-3/2}$ make clear that merely taking the product of local densities fails to give the right constant even for simple equations. Tschinkel asked in [73, §2.1] whether the leading constant in Theorem 1.1 has an algebro-geometric explanation, in the context of Guo’s result [40]. The answer is as follows: The factors $c_\infty$ and $c_p$ can be interpreted using Peyre’s Tamagawa measure [57], with the terms $(1 - 1/p)^{1/2}$ being convergence factors. The factor 2 is the order of a subordinate Brauer group and $\pi^{3/2} = \Gamma(1/2)^3$.

An asymptotic for a function similar to $N(B)$ without any coprimality conditions on the $t$ is given in Remark 1.25.

**Remark 1.2 (New constants).** Products of Gamma function values have not appeared before in problems of this kind nor in the neighbouring area of the Batyrev–Manin conjecture. Tauberian theorems (e.g. Delange’s Tauberian theorem [29]) give only a single Gamma factor. Loosely speaking, it will transpire that there are certain families of equations where each singular equation gives its own Gamma factor in the product and it would be desirable to have Tauberian theorems for completely general multivariate Dirichlet series; special cases have been dealt with by la Bretèche [13] and Essouabri [31]. An unexpected feature is that there are also families of Diophantine equations where one instead has a single Gamma value despite having multiple singular equations. This distinction is decided by the geometry of the family, see the definition of $\Gamma(X, f)$ in Conjecture 2.14 for a precise statement. It would be useful to have more results for fibrations over other bases to get a better understanding of this distinction.

**Remark 1.3 (Previous methods).** Serre [66] proved $N(B) \ll B^3/(\log B)^{3/2}$ by employing the large sieve. He also restricted to three prime coefficients to prove the non-matching lower bound $N(B) \gg B^3/(\log B)^{3}$. Subsequently, Hooley [45] used Burgess’s estimate for character sums and the Brun sieve to prove a lower bound of the same order of magnitude as the upper bound proved by Serre. Hooley’s method starts from restricting attention to diagonal coefficients of special shape: odd, square-free and coprime in pairs. Guo [40], using similar arguments, proved an asymptotic for the modified version of $N(B)$, where each $t_i$ is square-free and all pairs $t_i, t_j$ are coprime for all $i \neq j$.

**Remark 1.4 (Comparison to Guo’s work).** Absorbing square factors into the diagonal coefficients leads to square-free coefficients, as in the setting of Guo. However, this causes technical complications as the change of variables alter the height and one ends up with counting in lopsided boxes. We thus chose to follow a treatment that is different than the one used by Guo and Hooley, namely, we replaced the use of Burgess’s character sum bounds and all sieve theoretic arguments by bilinear sum estimates for quadratic Kronecker symbols. Our results improve the error term in Guo’s work, see Remark 1.2.

There is a rapidly developing industry in this area:

| Type of random varieties       | Authors                                      |
|-------------------------------|----------------------------------------------|
| Thue and Fermat equations     | Akhtari–Bhargava [1], Browning–Dietmann [17], Dietmann–Marmon [30] |
| Conic bundles over elliptic curves | Bhakta–Loughran–Rydin Myerson–Nakahara [6] |
| Plane cubics                  | Bhargava [7]                                  |
| Quadrics                      | Bhargava–Cremona–Fisher–Jones–Keating [8]    |
| Negative Pell equation        | Blomer [10], Fouvry–Klüners [32], Koymans–Pagano [48] |
There is, however, no conjecture regarding the leading constant in the literature. In this paper we put forward a general conjecture on the leading constant for problems of this type (Conjecture 2.14). Our conjecture is based on that of Peyre [57] on the leading constant in the Batyrev–Manin conjecture, however, some key features are different, for example, multiple Gamma factors and Fujita invariants. The setting will not require explicit equations nor the Hasse principle to hold for the equations in the family; it will apply to fairly general fibrations over any projective space \( \mathbb{P}^n \).

We now explain our new conjecture by working in a more geometric framework.

1.1. A new conjecture. Let \( f : Y \to X \) be a dominant morphism of smooth projective varieties over a number field \( k \) with geometrically integral generic fibre. In this paper we are interested in the function

\[
N_{\text{loc}}(f, B) = \sharp\{x \in X(k) : H(k) \leq B, x \in f(Y(\mathbb{A}_k))\}
\]

which counts the number of everywhere locally soluble fibres of the morphism \( f \) and \( H \) is a height function on \( X \). Under suitable assumptions on \( f : Y \to X \) and \( H \), and possibly after removing a thin set of rational points, we conjecture the asymptotic formula

\[
c_{f,H} B(\log B)^{\rho(X)-\Delta(f)-1}.
\]  

Here \( \rho(X) = \text{rank Pic} X \). The invariant \( \Delta(f) \) was originally introduced in [50]; we recall its definition in §2.1. The main new part of the conjecture is the constant \( c_{f,H} \), which is a product of the following (see Conjecture 2.14 for precise statements):

1. Modified definition of Peyre’s effective cone constant.
(2) Products of values of the Gamma function.
(3) Fujita invariants.
(4) Order of a subordinate Brauer group.
(5) Adelic Tamagawa volume.

Factors (2) and (3) are completely new and the main challenge was to identity these factors. For (1), Peyre’s original definition [57, Def. 2.4] of the effective cone constant $\alpha(X)$ contains an extraneous factor $(\rho(X) - 1)!$, which we view as a value of the Gamma function and treat separately from the effective cone. The subordinate Brauer group and Tamagawa measures in (4) and (5) appeared in some form in [51]; our new contribution is a definition of the subordinate Brauer group for more general $f$, and to construct virtual Artin $L$-function associated to the Picard group and $f$ which give convergence factors for the product of local Tamagawa measures. Our conjecture is greatly inspired by the conjectures of Manin [33], Batyrev–Manin [4], Peyre [57], and Loughran–Smeets [50], as well as the works of Batyrev–Tschinkel [5], Salberger [61], and Loughran [51].

We verify that the conjecture holds in the following cases, for suitable $f$:

(i) Families with $\Delta(f) = 0$.
(ii) Equivariant compactifications of anisotropic tori [51].
(iii) Wonderful compactifications of adjoint semi-simple algebraic groups [54].
(iv) Compatibility with the circle method.
(v) The family of all plane diagonal conics (Theorem 1.1).

Cases (i)–(iii) involve showing that asymptotics in the literature agree with our conjecture. For (iv) we explain how the conjecture agrees with the prediction from the circle method for representing a polynomial as a sum of two squares.

Case (v) is the new case obtained in our paper. It was worked out by the second and third-named authors. The first named-author was working independently on formulating a conjecture for the leading constant for general fibrations and this case was the key missing example required for the formulation. The known cases in the literature do not have the following features, whereas the fibration in Theorem 1.1 does: the smooth locus admits non-constant invertible functions and the family corresponds to a transcendental Brauer group element. This made clear that it was still Peyre’s effective cone constant which should appear, gave the formula for the relevant special values of the Gamma function, and led to the Fujita invariants through change-of-height considerations.

### 1.2. Open cases of the conjecture and other examples.

We conjecture the asymptotic (1.2) under specific geometric assumptions of $f$. Firstly we assume that the fibre over each codimension 1 point of $X$ admits an irreducible component of multiplicity 1; this is necessary in general, as otherwise the set of everywhere locally soluble fibres can be finite [52, Thm. 1.4]. We also want $X$ to have lots of rational points, so we assume that $X$ is almost Fano. More restrictively, let $U \subseteq X$ denote the open subset given by the complement of those divisors lying below the non-split fibres. Then we assume that

\[
\text{either } \rho(X) = 1 \text{ or } k[U]^x = k^x, \tag{1.3}
\]

where $k[U]^x$ denotes the group of invertible regular functions on $U$. If either of these conditions are not satisfied, then in fact the stated asymptotic (1.2) need not hold.

**Example 1.5.** One can show without difficulty (see §3.3) that

\[
\#\{t \in \mathbb{Q} : H(t) \leq B, t \text{ is a sum of two squares}\} = cB/(\log B) + O(B/(\log B)^2)
\]

where the leading constant $c$ agrees with the conjecture and $H(t)$ denotes the anticanonical height on $\mathbb{P}^1$, which is the square of the usual naive height. (Here the relevant family...
of varieties is given by a smooth proper model of the conic bundle $t = x_1^2 + x_2^2$ over $\mathbb{P}^1$. Using this and Dirichlet’s hyperbola method, one shows that

$$\sharp\left\{ t_1, t_2 \in \mathbb{Q}^2 : H(t_1)H(t_2) \leq B, \text{ each } t_i \text{ is a sum of two squares} \right\} \sim 2c^2B(\log \log B)/(\log B).$$

This counting problem takes place on $\mathbb{P}^1 \times \mathbb{P}^1$ with $\rho(\mathbb{P}^1 \times \mathbb{P}^1) = 2$ and $\Delta(f) = 2$; this gives the correct factor of $\log B$ in (1.2), but the factor $\log \log B$ is completely unexpected! This example defines a split toric variety, hence shows that the asymptotic proved in $\text{[31]}$ does not hold for non-anisotropic toric varieties in general. Note that the assumption (1.3) does not hold here since $\rho(\mathbb{P}^1 \times \mathbb{P}^1) = 2 > 1$ and $k[G_m]^\times/k^\times = \mathbb{Z}^2 \neq 0$.

More exotic asymptotic formulae can be obtained by taking other suitable products, for example one can show that

$$\sharp\left\{ t_0, t_1, t_2 \in \mathbb{Q}^3 : H(t_0)H(t_1)H(t_2) \leq B, \text{ both } t_1, t_2 \text{ are a sum of two squares} \right\} \sim B(\log \log B)^2.$$

Given these outlandish asymptotic formulae with currently little geometric explanation, we focus on examples where we can give precise predictions, with the key geometric assumption being (1.3). This assumption holds when $X = \mathbb{P}^n$, which is our primary base of interest and the case studied by Serre $\text{[66]}$. Given the lack of understanding of the Hasse principle in general, we also focus our attention on counting only everywhere locally soluble varieties. In important special cases, for example conic bundles or more generally families of products of Brauer–Severi varieties, the Hasse principle holds and the conjecture indeed concerns the number of varieties in the family with a rational point.

**Example 1.6.** A natural next case to consider after Theorem 1.1 would be Fermat curves, i.e. for $d \geq 3$ the counting problem

$$\sharp\{ t \in \mathbb{P}^2(\mathbb{Q}) : H(t) \leq B, t_0x_0^d + t_1x_1^d + t_2x_2^d = 0 \text{ is everywhere locally soluble} \}.$$  

Conjecture $\text{[2.14]}$ predicts a precise asymptotic formula for this counting problem of the shape $c_dB^3(\log B)^{\Delta(d)}$. The value of $\Delta(d)$ can be found in $\text{[51]}$ §5.4. (For $p$ prime we have $\Delta(p) = 3/p$.) An explicit value for $c_d$ can be obtained using a similar method to the one used in §3.7. Proving this for any $d \geq 3$ seems very challenging. A more difficult problem still is to count the number which have a rational point. A naive guess would be that for $d \geq 4$ only $0\%$ of the everywhere locally soluble members have a rational point, whereas for $d = 3$ a positive proportion of the everywhere locally soluble members has a rational point. (The case $d = 3$ is a famous family studied by Selmer $\text{[64]}$).

**Example 1.7.** Another natural case to consider for the conjecture is conic bundles. The correct lower bound for the conjecture was proven in $\text{[70]}$ in the case of conic bundles over the projective line with three singular fibres over the algebraic closure. An explicit family of such surfaces is given by

$$y_0Q_0(x) + y_1Q_1(x) = 0 \subset \mathbb{P}^1 \times \mathbb{P}^2$$

where each $Q_i$ is a ternary quadratic form. Providing it is smooth, such an equation defines a del Pezzo surface of degree five.

**Remark 1.8.** The asymptotic in Example 1.5 gives a counter-example to the statement of $\text{[33]}$ Prop. 2; this claims that if two counting functions grow like $B(\log B)^r$ and $B(\log B)^s$ then the counting function of the product grows like $B(\log B)^{r+s+1}$. An inspection of the proof of $\text{loc. cit.}$ reveals that the authors are using the integral representative of the Beta function which is only valid for $r > -1$ and $s > -1$, therefore the result is in fact only proved under this additional assumption.
1.3. **Structure of the paper.** In §2, we set-up and state our new conjecture (Conjecture 2.14). We also put forward an alternative version of the conjecture (Conjecture 2.23) in the case of zero-loci of Brauer groups which is often easier to work with in practice. In §3, we verify that our conjecture agrees with results in the literature and the circle method, and also consider some simple new cases. We also verify that the statement of Theorem 1.1 agrees with Conjecture 2.14. The final §4 is dedicated to our main new result (Theorem 1.1).

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## 2. The conjecture

### 2.1. Set-up.** We first provide the setting for the conjecture and introduce some notation.

**Definition 2.1.** A smooth projective geometrically integral variety $X$ over a field $k$ is called **almost Fano** if

- $H^1(X, O_X) = H^2(X, O_X) = 0$;
- The geometric Picard group $\text{Pic } X$ is torsion free;
- The anticanonical divisor $-K_X$ is big and nef.

Let $X$ be an almost Fano variety over a number field $k$. We denote by $\rho(X) = \text{rank } \text{Pic } X$. Choose an adelic metric $(\| \cdot \|_v)_{v \in \text{Val}(k)}$ on the anticanonical bundle of $X$ and denote the associated height function by $H$. We let $Y$ be a smooth projective variety and $f : Y \to X$ a dominant morphism with geometrically integral generic fibre.

**Definition 2.2.** For each point $x \in X$, we choose some finite group $\Gamma_x$ through which the absolute Galois group $\text{Gal}(\kappa(x)/\kappa(x))$ acts on the irreducible components of $f^{-1}(x) \kappa(x) := f^{-1}(x) \otimes_{\kappa(x)} \kappa(x)$. We define

$$\delta_x(f) = \sharp \left\{ \gamma \in \Gamma_x : \gamma \text{ fixes an irreducible component of } f^{-1}(x) \kappa(x) \text{ of multiplicity 1} \right\} \cdot \sharp \Gamma_x.$$

Let $X^{(1)}$ denote the set of codimension 1 points of $X$. Then we let

$$\Delta(f) = \sum_{D \in X^{(1)}} (1 - \delta_D(f)).$$

**Definition 2.3.** A subset $\Omega \subseteq X(k)$ is called **thin** if it is a finite union of subsets which are either contained in a proper closed subvariety of $X$, or contained in some $\pi(Y(k))$ where $\pi : Y \to X$ is a generically finite dominant morphism of degree exceeding 1, with $Y$ an integral variety over $k$. 
Our conjecture concerns the counting function
\[ N_{\text{loc}}(f, B) = \sharp \{ x \in X(k) : H(x) \leq B, x \in f(Y(\mathbb{A}_k)) \}. \]
(Note in the conjecture it may be necessary to remove some thin subset from \( X(k) \) to obtain the correct asymptotic formula.)

2.2. Peyre’s constant. Our approach is greatly inspired by that of Peyre [57], who formulated a conjectural expression for the leading constant in the classical case of Manin’s conjecture. Peyre’s constant takes the shape
\[ \alpha(X) \beta(X) \tau(X(\mathbb{A}_k)^{\text{Br}}) \] (2.1)
Here \( \alpha(X) \) is a certain rational number defined in terms of the cone of effective divisors of \( X \), \( \beta(X) = H^1(k, \text{Pic} \tilde{X}) \), and \( \tau \) is Peyre’s Tamagawa measure (the factor \( \beta(X) \) first appeared in [5] and was given a geometric interpretation in [61]).

2.3. Effective cone constant. In [57, Def. 2.4] Peyre introduced his effective cone constant, denoted by \( \alpha(X) \). We require a renormalisation of Peyre’s constant; this is because \( \alpha(X) \) contains the factor \( (\rho(X) - 1)! \) on the denominator, which we will interpret as a special value of the Gamma function and treat separately. This renormalisation already appeared in [5], and we take the definition from there, albeit with different notation.

Let \( \Lambda_{\text{eff}}(X) \subset \text{Pic}(X)_{\mathbb{R}} := \text{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R} \) denote the closure of the cone of effective divisors of \( X \). Denote by \( \Lambda_{\text{eff}}(X)^{\wedge} \) the dual cone and by \( \text{Pic}(X)^{\wedge} \subset \text{Pic}(X)_{\mathbb{R}}^{\wedge} \) the dual lattice. Equip \( \text{Pic}(X)_{\mathbb{R}}^{\wedge} \) with the Haar measure so that \( \text{Pic}(X)^{\wedge} \) has covolume 1. We then define
\[ \theta(X) = \int_{v \in \Lambda_{\text{eff}}(X)^{\wedge}} e^{-(-KX,v)} dv. \]
By [5, Rem. 2.4.8], we have
\[ \theta(X) = \alpha(X) / (\rho(X) - 1)! . \] (2.2)
One advantage of this definition is that it is compatible with products, i.e. \( \theta(X_1 \times X_2) = \theta(X_1) \theta(X_2) \), as follows from [222] and [57, Lem. 4.2].

2.4. Virtual Artin L-functions. We shall use the formalism of virtual Artin L-functions. A virtual Artin representation over \( k \) is a formal finite sum \( V = \sum_{i=1}^n z_i V_i \) where \( z_i \in \mathbb{C} \) and the \( V_i \) are Artin representations of \( G_k = \text{Gal}(\bar{k}/k) \). We define \( \dim V = \sum_{i=1}^n z_i \dim V_i \) and let \( V^{G_k} = \sum_{i=1}^n z_i V_i^{G_k} \). The \( L \)-function of \( V \) is defined to be
\[ L(V, s) = \prod_{i=1}^n L(V_i, s)^{z_i}, \]
where \( L(V_i, s) \) is the usual Artin \( L \)-function associated to \( V_i \). Standard properties of Artin \( L \)-functions imply that \( L(V, s) \) admits a holomorphic continuation with no zeros to the region \( \text{Re} \ s \geq 1 \), apart from possibly at \( s = 1 \). We have
\[ L(V, s) = \frac{c_V}{(s-1)^r} + O \left( \frac{1}{(s-1)^{r-1}} \right) , \]
as \( s \to 1 \), where \( r = \dim V^{G_k} \) and \( c_V \neq 0 \). In this notation we shall write
\[ L^*(V, 1) = c_V . \]
Denote by \( L_v(V, s) \) the corresponding local Euler factor at a non-archimedean place \( v \).
2.5. Tamagawa measures. We now define our Tamagawa measure. The first step is the same as in Peyre’s paper [57, §2.2], but the key difference is that we require different convergence factors. We choose Haar measures on $k_v$ such that $O_v$ has measure 1 for almost all $v$ and such that the induced adelic measure satisfies $\text{vol}(A_k/k) = 1$. We let $\tau_v$ denote Peyre’s local Tamagawa measure associated to our choice of adelic metric $\| \cdot \|_v$ and Haar measure on $k_v$. To define the convergence factors, consider the following virtual Artin representation

$$\text{Pic}_f(\overline{X})_\mathbb{C} = \text{Pic}(\overline{X})_\mathbb{C} - \sum_{D \in X^{(1)}} (1 - \delta_D(f)) \text{Ind}_{k_D} \mathbb{C}. \quad (2.3)$$

Here $\text{Pic}(\overline{X})_\mathbb{C} = \text{Pic}(\overline{X}) \otimes \mathbb{C}$, Ind denotes the induced Galois representation, and $k_D$ denotes the algebraic closure of $k$ in $\kappa(D)$. Next let $\text{Pic}_f(X)_\mathbb{C} = \text{Pic}(\overline{X})_\mathbb{C}^{CF}$. The corresponding virtual Artin $L$-function is

$$L(\text{Pic}_f(\overline{X})_\mathbb{C}, s) = \frac{L(\text{Pic}(\overline{X})_\mathbb{C}, s)}{\prod_{D \in X^{(1)}} \zeta_{k_D}(s)^{1-\delta_D(f)}}. \quad (2.4)$$

For each place $v \in \text{Val}(k)$ we define

$$\lambda_v = \begin{cases} L_v(\text{Pic}(\overline{X})_\mathbb{C}, 1), & v \text{ non-archimedean}, \\ 1, & v \text{ archimedean}. \end{cases}$$

Our Tamagawa measure is now defined to be

$$\tau_f = L^*(\text{Pic}(\overline{X})_\mathbb{C}, 1) \prod_v \tau_v \lambda_v^{-1}. \quad (2.5)$$

We have not included a discriminant factor as in Peyre [57, Def. 2.1], since we have normalised our Haar measures so that $\text{vol}(A_k/k) = 1$. We now show that these $\lambda_v$ are indeed a family of convergence factors in our case.

**Theorem 2.4.** The infinite product measure $\prod_v \tau_v \lambda_v^{-1}$ converges on $f(Y(A_k)) \subseteq X(A_k)$.

**Proof.** It suffices to show that the product $\prod_v \tau_v(\text{Pic}_f(\overline{X}(k_v))) \lambda_v^{-1}$ converges. Recalling (2.4), we can rewrite this as

$$\prod_v \frac{\tau_v(X(k_v))}{L_v(\text{Pic}(X)_\mathbb{C}, 1)} \prod_v \left(1 - \frac{\tau_v(X(k_v))}{\text{Pic}_f(\overline{X}(k_v))} \right) \prod_{D \in X^{(1)}} \zeta_{k_D,v}(1)^{1-\delta_D(f)}. \quad (2.7)$$

The first Euler product is convergent by a result of Peyre [57, Prop. 2.2.2]. So it suffices to consider the second Euler product. To do so, we choose a model $f : Y \to X$ for the morphism $f$ over the ring of integers $O_k$ of $k$. We choose a sufficiently large set of primes $S$ of $k$ and let $p \not\in S$. We first note that

$$X(k_p) \setminus f(Y(k_p)) \subseteq \{ x \in X(k_p) : f^{-1}(x) \text{ mod } p \text{ is non-split} \}. \quad (2.6)$$

Indeed if $f^{-1}(x) \text{ mod } p$ were split then $f^{-1}(x) \text{ mod } p$ would have a smooth $\mathbb{F}_p$-point by the Lang–Weil estimates, providing $S$ is sufficiently large, which would give rise to a $k_p$-point by Hensel’s lemma. We now show a partial reverse inclusion using [30, Thm 2.8]. This says the following: Let $T \subset X$ be a reduced divisor which contains the non-smooth locus of $f$ and $\mathcal{T}$ its closure in $\mathcal{X}$. Then there exists a closed subset $Z \subset \mathcal{T}$ of codimension 2 in $\mathcal{X}$ which contains the singular locus of $\mathcal{T}$ such that

$$\{ x \in X(k_p) : f^{-1}(x) \text{ mod } p \text{ is non-split}, x \text{ mod } p^2 \text{ meets } \mathcal{T} \text{ transversely outside of } Z \} \subseteq X(k_p) \setminus f(Y(k_p)). \quad (2.7)$$
Combining (2.6) and (2.7) with formulae for Tamagawa measures \[19\] Lem. 3.2 yields
\[
\frac{1}{N} \tau_p(n) \leq \sum_{x \in \mathcal{X}(\mathbb{Q}_p / \mathbb{F}_p)} f^{-1}(x) \text{ mod } \mathfrak{p}^n
\]
where \( n = \dim X \). However by \[19\] Prop. 2.3] and the Lang-Weil estimates we have
\[
\sum_{x \in \mathcal{X}(\mathbb{Q}_p / \mathbb{F}_p)} f^{-1}(x) \text{ mod } \mathfrak{p}^n
\]
Using \( \tau_p(X(\mathbb{Q}_p)) = \sum_{f \in \mathcal{X}(\mathbb{Q}_p / \mathbb{F}_p)} f^{-1}(x) \text{ mod } \mathfrak{p}^n \), we have reduced to proving that
\[
\prod_{p \not\in S} \left( 1 - \sum_{x \in \mathcal{X}(\mathbb{Q}_p / \mathbb{F}_p)} f^{-1}(x) \text{ mod } \mathfrak{p}^n \right)
\]
converges. To do so we use that if \( \sum_{k=1}^{\infty} a_k \) converges then \( \prod_{k=1}^{\infty} (1 + a_k) \) converges. For a finite field extension \( k \subset K \), write
\[\zeta_{K,p}(1) = 1 + a_{K,p} / \mathbb{N} \mathfrak{p} + O(1) / \mathbb{N} \mathfrak{p} \].
Then the prime ideal theorem implies that
\[
\sum_{\mathfrak{p} \not\in \mathcal{X}} \frac{a_{K,p}}{\mathbb{N} \mathfrak{p}} = \log \log X + O(1)
\]
(Mertens for number fields). We conclude that
\[
\sum_{D \in \mathcal{X}(\mathbb{Q}_p)} (1 - \zeta_{K,p}(1)^{1 - \delta_D(f)}) = \sum_{D \in \mathcal{X}(\mathbb{Q}_p)} (1 - \delta_D(f)) \log \log X + O(1) = \Delta(f) \log \log X + O(1).
\]
However it follows from \[50\] Prop. 3.10] that
\[
\sum_{\mathfrak{p} \not\in \mathcal{X}} \frac{\sharp\{x \in \mathcal{X}(\mathbb{F}_p) : f^{-1}(x) \text{ is non-split} \}}{\mathbb{N} \mathfrak{p}^n} = \Delta(f) \sum_{\mathfrak{p} \not\in \mathcal{X}} \mathbb{N} \mathfrak{p}^{n-1} + O(X^n / (\log X)^2).
\]
(The result loc. cit. is stated without an error term, but a minor modification of the argument gives the stated error term on using Serre’s version of the Chebotarev density theorem \[68\] Thm. 9.11]). An easy application of partial summation and the prime ideal theorem then shows that
\[
\sum_{\mathfrak{p} \not\in \mathcal{X}} \frac{\sharp\{x \in \mathcal{X}(\mathbb{F}_p) : f^{-1}(x) \text{ is non-split} \}}{\mathbb{N} \mathfrak{p}^n} = \Delta(f) \log \log X + O(1).
\]
Thus the leading terms in the asymptotics cancel each other out, which shows convergence of (2.8), as required. \(\square\)

**Remark 2.5.** The product in Theorem 2.4 may be only conditionally convergent: for an example, see Proposition 3.3. This is in contrast to Peyre’s Tamagawa measure in Manin’s conjecture, which is always absolutely convergent.

2.6. **Subordinate Brauer groups.**
2.6.1. Brauer groups recap. The canonical reference for Brauer groups is now the book [26]. For a scheme $X$ we denote its Brauer group by $\text{Br} X := H^2(X, \mathbb{G}_m)$. Let $X$ be a variety over a field $k$ of characteristic 0. We denote by $\text{Br}_1 X := \ker(\text{Br} X \to \text{Br} X_\mathbb{K})$ its algebraic Brauer group. If $X$ is smooth then Grothendieck’s purity theorem [26, Thm. 3.7.2] yields an exact sequence
\[
0 \to \text{Br} X \to \text{Br} k(X) \to \bigoplus_{D \in X^{(1)}} H^1(k(D), \mathbb{Q}/\mathbb{Z})
\] where $\partial_D : \text{Br} k(X) \to H^1(k(D), \mathbb{Q}/\mathbb{Z})$ is the residue map at $D$. If $\partial_D(b) = 0$ then we say that $b$ is unramified along $D$.

If $k$ is a number field, there is the fundamental exact sequence from class field theory
\[
0 \to \text{Br} k \to \text{Br} \mathbb{P}^n_k = \text{Br} \mathbb{A}^n_k = \text{Br} k [26, \S 5.1].
\]

2.6.2. Subordinate Brauer groups. The terminology “subordinate Brauer group” was first used by Serre in the appendix of [67, Ch. II] in the case were $X = \mathbb{P}^1$ and working with an element of $\text{Br} k(\mathbb{P}^1)$. In [51, §2.6] this was generalised to other $X$ and a finite collection of Brauer group elements. We put this into an even more general setting and temporarily forgo the more stringent restrictions imposed in [27]. Let $k$ be a field of characteristic 0.

**Definition 2.6.** Let $f : Y \to X$ be a proper morphism of regular integral schemes over $k$ with geometrically integral generic fibre. We define
\[
\text{Br}_{\text{Sub}}(X, f) = \bigcap_{D \subset X^{(1)}} \{ \alpha \in \text{Br} k(X) : f^* \alpha \in \text{Br} f^{-1}(\mathcal{O}_{X,D}) \}.
\]

In the statement $\mathcal{O}_{X,D}$ denotes the local ring at $D$, which is a discrete valuation ring as $X$ is regular. The geometric interpretation of this definition is as follows: An element $\alpha \in \text{Br}_{\text{Sub}}(X, f)$ when pulled-back to $\text{Br} k(Y)$ becomes unramified at all those codimension 1 points of $Y$ which lies above codimension 1 points of $X$. It follows immediately that
\[
\text{Br}_{\text{vert}}(Y/X) \subseteq f^* \text{Br}_{\text{Sub}}(X, f) = \bigcap_{D \subset X^{(1)}} \text{Br}_{\text{vert}}(f^{-1}(\mathcal{O}_{X,D})/\mathcal{O}_{X,D}),
\] where $\text{Br}_{\text{vert}}(Y/X) = f^* \text{Br} k(X) \cap \text{Br} Y$ denotes the vertical Brauer group of $Y/X$.

**Lemma 2.7.** If $f : Y \to X$ is flat then $\text{Br}_{\text{vert}}(Y/X) = f^* \text{Br}_{\text{Sub}}(X, f)$.

**Proof.** As $f$ is flat, any codimension 1 point of $Y$ must lie over a codimension 1 point of $X$. Therefore any element of $f^* \text{Br}_{\text{Sub}}(X, f)$ is unramified at all codimension 1 points of $Y$, hence lies in $\text{Br} Y$ by purity (2.9). \qed

$\text{Br}_{\text{Sub}}(X, f)$ is a birational invariant, only depending on $X$ and the generic fibre of $f$.

**Lemma 2.8.** Let $f : Y \to X$ and $g : Z \to X$ be proper morphisms of regular integral schemes over $k$ with geometrically integral generic fibre and let $Y \to Z$ a birational map over $X$. Then
\[
\text{Br}_{\text{Sub}}(X, f) = \text{Br}_{\text{Sub}}(X, g).
\]

**Proof.** To prove the result we may assume that $X = \text{Spec} R$ is the spectrum of a discrete valuation ring. For $\alpha \in k(X)$, it suffices to show that $f^* \alpha \in \text{Br} Y$ if and only if $f^* \alpha \in \text{Br} Z$. This is a consequence of purity, specifically [26, Prop. 3.7.9]. \qed
We next give a formula to calculate the subordinate Brauer group.

**Lemma 2.9.** For each $D \in X^{(i)}$ and $E \in f^{-1}(D)$, denote by $m_E$ the multiplicity of $E$ as a divisor and by $\kappa_E$ the algebraic closure of $k(D)$ inside the function field of $E$. Then

$$\text{Br}_{\text{Sub}}(X, f) = \left\{ b \in \text{Br} k(X) : \delta_D(b) \in \ker(m_E \text{res} : H^1(k(D), \mathbb{Q}/\mathbb{Z}) \to H^1(\kappa_E, \mathbb{Q}/\mathbb{Z})) \right\}$$

where res denotes the usual restriction map on Galois cohomology.

**Proof.** This follows by simply writing down and comparing the residue exact sequences for $X$ and $Y$ (cf. [20 Prop. 10.1.4]). $\square$

**Lemma 2.10.** If the fibre over every codimension 1 point of $X$ contains an irreducible component of multiplicity 1, then $\text{Br}_{\text{Sub}}(X, f)/\text{Br} X$ is finite.

**Proof.** For any finite separable field extension $K \subset L$, the kernel of the restriction map $H^1(K, \mathbb{Q}/\mathbb{Z}) \to H^1(L, \mathbb{Q}/\mathbb{Z})$ is finite. Thus by Lemma 2.9, for any $b \in \text{Br}_{\text{Sub}}(X, f)$ there are only finitely possibilities for $\delta_D(b)$ as $D$ varies over the finitely many $D \in X^{(i)}$ such that $f^{-1}(D)$ is non-split. Since the subgroup $\text{Br} X \subset \text{Br}_{\text{Sub}}(X, f)$ consists of exactly those elements with trivial residue at all $D$ by (2.9), the result follows. $\square$

We finish with an example where $\text{Br}_{\text{vert}}(Y/X) \neq f^* \text{Br}_{\text{Sub}}(X, f)$. This example concerns families of conics over a surface, and shows that the subordinate Brauer group is a useful geometric invariant for proving non-existence of flat models. It also shows that, even when studying the basic problem of counting the number of conics in a family with a rational point, one cannot assume that the family is flat, i.e. one has to allow fibres of dimension $> 1$. This example first appeared in [51 Ex. 2.9].

**Lemma 2.11.** Let $S = \mathbb{P}^1 \times \mathbb{P}^1$ over $\mathbb{Q}$ and let $Y$ be a smooth projective variety over $\mathbb{Q}$ with a dominant morphism $f : Y \to S$ whose generic fibre is isomorphic to the conic $u_1u_2t_0^2 + t_1^2 = t_2^2$ over $k(S) = k(u_1, u_2)$. Then

1. $\text{Br}_{\text{Sub}}(S, f)/\text{Br} \mathbb{Q} \cong (\mathbb{Z}/2\mathbb{Z})^2$, generated by the quaternion algebras $(u_1, -1)$ and $(u_2, -1)$. Thus $f^* \text{Br}_{\text{Sub}}(S, f)/\text{Br} \mathbb{Q} \cong \mathbb{Z}/2\mathbb{Z}$ generated by $(u_1, -1)$.
2. $\text{Br}_{\text{vert}}(Y/S) = \text{Br} \mathbb{Q}$, hence $\text{Br}_{\text{vert}}(Y/S) \neq f^* \text{Br}_{\text{Sub}}(S, f)$
3. There is no smooth projective variety $Y$ over $\mathbb{Q}$ equipped with a flat morphism $\psi : Y \to S$ whose generic fibre is isomorphic to $u_1u_2t_0^2 + t_1^2 = t_2^2$.

**Proof.** (1) We will use the formula for the subordinate Brauer group given in Definition 2.22 (see Proposition 2.24), since this is much easier to use than Lemma 2.9 which requires an explicit smooth projective model. We use coordinates $(x_1, y_1) \times (x_2, y_2)$ on $S$, so that $u_i = x_i/y_i$. Let $\beta = (u_1u_2, -1)$. Then $\beta$ is ramified along the 4 lines

$$L_1 : x_1 = 0, \quad L_2 : x_2 = 0, \quad L_3 : y_1 = 0, \quad L_4 : y_2 = 0$$

each with residue $-1$. Let $\beta_1 = (u_1, -1)$, which is ramified along $L_1$ and $L_3$ with residue $-1$. Let $b \in \text{Br}_{\text{Sub}}(S, f)$. By Proposition 2.24, we know that $b$ must be ramified along some subset of the $L_i$ with residue $-1$. If $b$ is unramified then $b \in \text{Br} S = \text{Br} \mathbb{Q}$ is constant. If $b$ is ramified along only one of the $L_i$ then $b \in \text{Br} \mathbb{A}^2 = \text{Br} \mathbb{Q}$ is constant; a contradiction. Similarly if $b$ is ramified along at least 3 of the $L_i$, then $b - \beta$ is constant. So consider the case where $b$ is ramified along exactly 2 of the $L_i$. Using $S \setminus \{x_1 = 0, x_2 = 0\} \cong S \setminus \{x_1 = 0, y_2 = 0\} \cong S \setminus \{x_2 = 0, y_1 = 0\} \cong S \setminus \{y_1 = 0, y_2 = 0\} \cong \mathbb{A}^2$ we see that $b$ must be ramified along either $L_1 \cup L_3$ or $L_2 \cup L_4$; this shows that $b$ must differ from one of $\beta_1, \beta$, or $\beta_1 + \beta$ by a constant, as required. When pulling-back to $Y$, the kernel is generated by $\beta$. 11
(2) Blow-up \( S \) in the intersection of \( L_1 \) and \( L_2 \). Let \( E \) be the resulting exceptional divisor. Then \( u_1 \) and \( u_2 \) both have valuation 1 along \( E \). It follows that \( \beta \) is unramified along \( E \), but \( \beta_1 \) is ramified along \( E \), thus \( f^*\beta_1 \notin \text{Br}_{\text{vert}}(Y/S) \). The result now follows from (1) and (2).

Remark 2.12. In all known cases of the conjecture, there is a model for \( f : Y \to X \) where the fibre over each codimension 1 point of \( X \) is reduced, so that no multiplicities appear in the formula given in Lemma 2.9. It is thus conceivable that one needs to modify Definition 2.6 by e.g. restricting to only those irreducible components of multiplicity 1. More examples are required to clarify this.

Remark 2.13. The notation we use here \( \text{Br}_{\text{Sub}}(X,f) \) is different to that of [51, §2.6] which instead used \( \text{Sub}(X,\mathcal{B}) \) (in the context of zero-loci of Brauer groups). We decided to change to the former as it mirrors existing notation for the vertical Brauer group. Warning however: \( \text{Br}_{\text{Sub}}(X,f) \notin \text{Br}_X \) in general!

An additional kind of subordinate group was introduced in [51, §2.6] and denoted by \( \text{Sub}(k(X),\mathcal{B}) \); this is a birational invariant of the generic fibre of \( f \) (cf. Lemma 2.7) and pulls back to exactly the vertical Brauer group of the corresponding family of products of Brauer–Severi varieties [51 Thm. 2.11]. Despite this having better birational properties, in the literature relevant to Conjecture 2.14 it is the group \( \text{Br}_{\text{Sub}}(X,f) \) which appears.

2.7. The conjecture. We take the set-up of §2.1. Let \( U \subseteq X \) be the open subset of \( X \) given by removing the closure of those codimension 1 points of \( X \) which lie below a non-split fibre. We take \( V = f^{-1}(U) \). For a divisor \( D \subset X \), we denote by 
\[
a(D) = \inf\{t > 0 : K_X + tD \in \Lambda_{\text{eff}}(X)\}
\]
the Fujita invariant of \( D \). This appears in the Batyrev–Manin conjecture as the exponent of \( B \) when counting with respect to height functions associated to \( D \) (see [11 Conj. 3.2]). It arises for us in a completely different way as part of the leading constant.

Conjecture 2.14. Assume that \( \text{Br}_X = \text{Br}_1 \) and that \( f \) admits a smooth fibre over a rational point which is everywhere locally soluble. Assume that the fibre over every codimension 1 point of \( X \) contains an irreducible component of multiplicity 1. Assume that either \( \rho(X) = 1 \) or \( k[U]^\times = k^\times \). Then there exists a thin subset \( \Omega \subset X(k) \) such that 
\[
\#\{x \in X(k) \cap f(Y(A_k)) : H(x) \leq B, x \notin \Omega\} \sim c_{f,H} B(\log B)^{\rho(X) - \Delta(f) - 1}
\]
where 
\[
c_{f,H} = \frac{\theta(X) \cdot |\text{Br}_{\text{Sub}}(X,f)/\text{Br}_k| \cdot \tau_f(V(A_k))^{\text{Br}_{\text{Sub}}(X,f)}}{\Gamma(X,f)} \cdot \prod_{D \in X(1)} a(D)^{1-\delta_D(f)},
\]
\[
\Gamma(X,f) = \begin{cases} 
\prod_{D \in X(1)} \Gamma(\delta_D(f)), & \text{if } \rho(X) = 1, \\
\Gamma(\rho(X) - \Delta(f)), & \text{if } k[U]^\times = k^\times.
\end{cases}
\]

Note that \( \text{Br}_{\text{Sub}}(X,f)/\text{Br}_k \) is finite by Lemma 2.10 and the finiteness of \( \text{Br}_X/\text{Br}_k \).

Remark 2.15. One could interpret the factors \( a(D)^{1-\delta_D(f)} \) measure-theoretically and combined with the effective cone constant \( \theta(X) \), as the volume of a certain integral on the virtual Picard group \( \text{Pic}_f(X)_{\mathbb{R}} \) (see (2.3)).

Remark 2.16. Partial motivation for the appearance of \( \tau_f(V(A_k))^{\text{Br}_{\text{Sub}}(X,f)} \) is as follows. Consider the case where \( X = \mathbb{P}^1 \). Here \( f \) is flat, so \( f^*\text{Br}_{\text{Sub}}(X,f) = \text{Br}_{\text{vert}}(Y/X) \) by Lemma 2.7. Then the Harpaz–Wittenberg conjecture implies that \( U(k) \cap f(Y(A_k)) \subseteq \)}
Remark 2.17. When \( f : X \to X \) is the identity, one recovers Peyre’s constant \( 2.2 \) on using \( 2.2 \) and \( | \text{Br}_{\text{Sub}}(X, f)/\text{Br} k| = |\text{Br} X/\text{Br} k| = H^1(k, \text{Pic} \tilde X) \) since \( \text{Br} X = \text{Br}_1 X \).

Remark 2.18. When there is a product of Gamma factors it seems highly doubtful that multiple Dirichlet series will be required if trying to use height zeta function methods.

Remark 2.19. Conjecture 2.14 only depends on \( X \) and the generic fibre of \( f \), i.e. is a birational invariant (providing \( Y \) is smooth and projective). The birational invariance of \( \text{Br}_{\text{Sub}}(X, f) \) is Lemma 2.8 and the birational invariance of \( \delta_D(f) \) is [50] Lem. 3.11.

Remark 2.20. The thin set \( \Omega \) can depend on both \( f \) and \( X \) in general; see [20] for an example of this form.

We finish with a discussion of the Tamagawa volume \( \tau_f(f(V(A_k)))^{\text{Br}_{\text{Sub}}(X,f)} \). One would like to write this as a product of local densities, at least away from some finite set of places. This is not possible in general, for the following reason: since \( \text{Br}_{\text{Sub}}(X, f) \subseteq \text{Br} U \), the Brauer–Manin pairing \( 2.11 \) is only well-defined on \( U(A_k) \) and not on \( \prod_v U(k_v) \). However it is possible in the following situation. (By Lemma 2.7 this applies if \( f \) is flat.)

**Lemma 2.21.** Assume that \( \text{Br}_{\text{vert}}(Y/X) = f^* \text{Br}_{\text{Sub}}(X, f) \). Let \( \mathcal{A} \subset \text{Br}_{\text{vert}}(X, f) \) be a finite set of representatives of \( \text{Br}_{\text{vert}}(X, f)/\text{Br} k \) and \( S \) a finite set of places of \( k \) such that every element of \( \mathcal{A} \) evaluates trivially on \( Y(k_v) \) for \( v \notin S \). Then

\[
\tau_f(f(V(A_k)))^{\text{Br}_{\text{Sub}}(X,f)} = \tau_f(f(Y(\prod_{v \in S} k_v))^{\mathcal{A}}) \times \prod_{v \notin S} f(Y(k_v)).
\]

**Proof.** We have \( f(V(A_k))^{\text{Br}_{\text{Sub}}(X,f)} = f(V(A_k))^{f^* \text{Br}_{\text{Sub}}(X,f)} \) by functoriality of the Brauer–Manin pairing \( 2.11 \). Since \( \text{Br}_{\text{vert}}(Y/X) = f^* \text{Br}_{\text{Sub}}(X, f) \), the elements of the subordinate Brauer group become unramified when pulled-back to \( Y \), thus are well-defined on all of \( Y(A_k) = \prod_v Y(k_v) \). The set \( \mathcal{A} \) is finite, which is why it evaluates trivially outside a finite set of places. To finish we note that the complement \( f(Y(k_v)) \backslash f(V(k_v)) \) has measure zero since it is supported on a proper closed subscheme. \( \square \)

### 2.8. Specialisation of Brauer group elements

Serre’s original paper [66], as well as the papers [51] [54], are written in terms of the language of specialisation of Brauer group elements. Working with Brauer group elements instead of a family of varieties can be advantageous as one does not have to worry about constructing models, which can be problematic (see Lemma 2.11). We explain here explicitly how to apply Conjecture 2.14 to this problem via the resulting family of products of Brauer–Severi varieties.

#### 2.8.1. Set-up

Let \( X \) be as in §2.1. Let \( U \subseteq X \) be an open subset and \( \mathcal{B} \subset \text{Br} U \) a finite subgroup. We are interested in counting rational points in the zero-locus

\[
U(k)_{\mathcal{B}} := \{ x \in U(k) : b(x) = 0 \text{ for all } b \in \mathcal{B} \}.
\]

There is an analogous way to define a Tamagawa measure in this setting: In place of \( 2.3 \), one takes the virtual Artin representation

\[
\text{Pic}_{\mathcal{B}}(X)_{\mathbb{C}} = \text{Pic}(X)_{\mathbb{C}} - \sum_{D \in X^{(1)}} (1 - 1/|\partial_D(\mathcal{B})|) \text{Ind}_{k_D} \mathbb{C}^k.
\]
The local factors $\lambda_v$ of the corresponding virtual $L$-function are then taken as the convergence factors for a Tamagawa measure $\tau_{\mathfrak{B}} = L^*(\text{Pic}_{\mathfrak{B}}(X)_{\mathbb{C}}, 1) \prod_v \tau_v/\lambda_v$. This is well-defined on the adelic zero-locus

$$U(A_k)_{\mathfrak{B}} := \{(x_v) \in U(A_k) : b((x_v)) = 0 \text{ for all } b \in \mathfrak{B}\}.$$ 

One can show this using an analogous argument to Theorem 2.4, or alternatively it also follows from Proposition 2.24 below. We have the following version of the subordinate Brauer group in this setting.

**Definition 2.22.** We say that $b \in \text{Br} k(X)$ is subordinate to $\mathfrak{B}$ with respect to $X$, if for each $D \in X^{(1)}$ the residue $\partial_D(b)$ lies in $\partial_D(\mathfrak{B})$. We let

$$\text{Br}_{\text{Sub}}(X, \mathfrak{B}) = \{b \in \text{Br} k(X) : \partial_D(b) \in \partial_D(\mathfrak{B}) \text{ for all } D \in X^{(1)}\},$$

denote the group of all such elements.

**2.8.2. The conjecture.** Our conjecture is now as follows.

**Conjecture 2.23.** Assume that $\text{Br} X = \text{Br}_1 X$, that $U(k)_{\mathfrak{B}} \neq \emptyset$, and that either $\rho(X) = 1$ or $k[U]^\times = k^\times$. Then there exists a thin subset $\Omega \subset X(k)$ such that

$$\sharp\{x \in U(k)_{\mathfrak{B}} : H(x) \leq B, x \notin \Omega\} \sim c_{\mathfrak{B}, H} B(\log B)^{\rho(X) - \Delta(\mathfrak{B}) - 1}$$

where

$$\Delta(\mathfrak{B}) = \sum_{D \in X^{(1)}} (1 - 1/|\partial_D(\mathfrak{B})|),$$

$$c_{\mathfrak{B}, H} = \frac{\theta(X) \cdot |\text{Br}_{\text{Sub}}(X, \mathfrak{B})/\text{Br} k| \cdot \tau_{\mathfrak{B}}(U(A_k)_{\mathfrak{B}}/\text{Br}_{\text{Sub}}(X, \mathfrak{B}))}{\Gamma(X, \mathfrak{B})} \cdot \prod_{D \in X^{(1)}} a(D)^{1 - 1/|\partial_D(\mathfrak{B})|},$$

$$\Gamma(X, \mathfrak{B}) = \begin{cases} \prod_{D \in X^{(1)}} \Gamma(1/|\partial_D(\mathfrak{B})|), & \text{if } \rho(X) = 1, \\ \Gamma(\rho(X) - \Delta(\mathfrak{B})), & \text{if } k[U]^\times = k^\times. \end{cases}$$

We relate this to Conjecture 2.14 as follows. As $X$ is quasi-projective a theorem of Gabber [23, Thm. 3.3.2] implies that every element $b \in \text{Br} U$ is the Brauer class of some Brauer–Severi scheme $V_b$ over $U$. (Recall that a Brauer–Severi scheme over $U$ is a scheme over $U$ which is étale locally isomorphic to $\mathbb{P}^n_U$ for some $n$.) We take $f : Y \to X$ to be a smooth projective model of the fibre product $\times_{b \in \mathfrak{B}} V_b \to U$ over $U$; this exists by Hironaka’s theorem on resolution of singularities. Take $V = f^{-1}(U)$.

**Proposition 2.24.**

$$U(k)_{\mathfrak{B}} = f(V(A_k)), \quad U(A_k)_{\mathfrak{B}} = f(V(A_k)) \quad (2.13)$$

$$1/|\partial_D(\mathfrak{B})| = \delta_D(f) \text{ for all } D \in X^{(1)}, \quad (2.14)$$

$$\text{Br}_{\text{Sub}}(X, \mathfrak{B}) = \text{Br}_{\text{Sub}}(X, f). \quad (2.15)$$

**Proof.** We have to be slightly careful, since the fibre product of smooth projective models of each $V_b$ will not give a smooth projective model of $\times_{b \in \mathfrak{B}} V_b$ in general, e.g. the fibre product of two conic bundles over $\mathbb{P}^1$ which have a singular fibre over a common point is non-regular. This introduces some technical aspects into our proof.

That (2.13) holds follows from combining Lang–Nishimura with the following: a Brauer–Severi variety over a field has a rational point if and only if the associated Brauer group element is trivial, and an element of $\text{Br} k$ is trivial if and only if it is trivial everywhere locally (2.10). (Here we take $\Omega$ sufficiently large so that $X(k) \setminus U(k) \subseteq \Omega$.)

For the remaining points, we may assume that $X = \text{Spec} R$ for a discrete valuation ring $R$. We will use some of the arguments and constructions in [51, §2.4, §2.6]. We first
construct an explicit model $Y$ (this is permissible by Remark \ref{rem:regularity}). Artin \cite[Thm. 1.4]{Artin1} has constructed regular flat proper integral schemes $V_b \to \text{Spec } R$ whose generic fibres are isomorphic to $V_b$ and whose special fibres are integral, for each $b \in \mathcal{B}$. Frossard \cite[Prop. 2.3]{Frossard} has shown that the algebraic closure of $k$ inside the function field of the special fibre of $V_b$ is exactly the cyclic field extension $k_b$ of $k$ determined by the residue $\partial_D(b)$.

We let $K/k$ be the compositum of the $k_b$. We take $V = \prod_{b \in \mathcal{B}} V_b \to X$ and take $Y$ to be a desingularisation of $V$.

Now $V$ need not be regular; however it is “almost smooth” in the terminology of \cite[Def. 2.1]{Peyre}. Explicitly, this means that for a flat discrete valuation ring $R \subseteq R'$ of ramification degree 1, any $R'$-point of $V$ lies in the smooth locus of $V$. By \cite[Lem. 3.11]{Peyre} we may calculate $\delta_D(f)$ on an almost smooth model. We thus use $V$ and find that

$$\delta_D(f) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \left\{ \gamma \text{ fixes an irreducible component of } \text{Spec}(\prod_{b \in \mathcal{B}} k_b) \right\} = \frac{1}{|\partial_D(\mathcal{B})|},$$

where $\Gamma = \text{Gal}(K/k)$, as required for (2.14). For (2.15), it suffices to show that

$$\text{Br } Y = \{ f^* \alpha : \alpha \in \text{Br } k(X), \partial_D(\alpha) \in \partial_D(\mathcal{B}) \}. \tag{2.16}$$

Our proof is based upon the proof of \cite[Thm. 2.11]{Peyre}. First $\text{Br } Y \subseteq f^* \text{Br } k(X)$ by a theorem of Amitsur \cite[Thm. 5.4.1]{Amitsur}. We show the right side of (2.16) is included in the left. Let $\alpha \in \text{Br } k(X)$ with $\partial_D(\alpha) \in \partial_D(\mathcal{B})$. Given this we may write $\partial_D(\alpha) = \sum_{b \in \mathcal{B}} r_b \partial_D(b)$ for some $r_b \in \mathbb{Z}$. Then the element $\alpha - \sum_{b \in \mathcal{B}} r_b b$ is unramified on $X$ hence unramified when pulled-back to $Y$. However $f^* b = 0 \in \text{Br } Y$ for all $b \in \mathcal{B}$ by Amitsur \cite[Thm. 5.4.1]{Amitsur}, thus $f^* \alpha \in \text{Br } Y$ as required. Now let $\alpha \in \text{Br } k(X)$ with $\partial_D(\alpha) \notin \partial_D(\mathcal{B})$. We pull-back $\alpha$ to the regular locus of $V$ and use functoriality of residues \cite[Prop. 1.4.6]{Peyre}, recalling the above description of the irreducible components of the special fibre, which all have multiplicity one. This shows that $\alpha$ is ramified on the regular locus of $V$, hence clearly ramified when pulled-back to $Y$. This shows (2.16) and completes the proof. \hfill $\Box$

By comparing conjectures and matching up relevant factors, we deduce the following.

**Corollary 2.25.** $Y \to X$ satisfies Conjecture \ref{conj:main} if and only if $\mathcal{B}$ satisfies Conjecture \ref{conj:main}.

Formula (2.15) from Proposition \ref{prop:formula} gives a more useful way than Lemma \ref{lem:calculation} to calculate the subordinate Brauer group for families of Brauer–Severi varieties (e.g. conic bundles) and their products, since it avoids the need to construct a regular proper model, which can be problematic (see Lemma \ref{lem:calculation}).

### 3. Verifying the Conjecture

In this section we gather various known results from the literature and some other new results and show that they are compatible with Conjecture \ref{conj:main}. Our main new result (Theorem \ref{thm:main}) will be proved in later sections.

#### 3.1. Local densities for projective space.

To assist with later calculations, we give some formulae for calculating local densities when the base variety $X = \mathbb{P}^n_Q$. For a place $v$ of $Q$ we let $\tau_v$ denote Peyre’s Tamagawa measure on $\mathbb{P}^n(Q_v)$. For a subset $\Omega_v \subset \mathbb{P}^n(Q_v)$ we denote by $\hat{\Omega}_v \subset \mathbb{Q}^{n+1}_v$ its affine cone. We let $\mu_v$ be the usual Haar measure on $\mathbb{Q}^{n+1}_v$.

**Proposition 3.1.** Let $v$ be a place of $Q$ and let $\Omega_v \subset \mathbb{P}^n(Q_v)$ be measurable.

1. $\tau_\infty(\Omega_\infty) = ((n+1)/2) \cdot \mu_\infty(\hat{\Omega}_v \cap [-1,1]^{n+1})$.
2. $\tau_p(\Omega_p) = (1 + 1/p + \cdots + 1/p^n) \cdot \mu_p(\hat{\Omega}_p \cap \mathbb{Z}_p^{n+1})$. 

15
\[
(3) \quad \tau_p(\Omega_p) = (1 - 1/p)^{-1} \cdot \mu_p\{x \in \hat{\Omega}_p \cap \mathbb{Z}_p^{n+1} : p \nmid x\}.
\]

**Proof.** For (1), let \(1 : \mathbb{R}^{n+1} \to \{0,1\}\) denote the indicator function of \(\hat{\Omega}_\infty\). Then by definition we have
\[
\tau_\infty(\Omega_\infty) = \int_{u \in \mathbb{R}^n} \frac{1(1, u_1, \ldots, u_n)}{\max\{1, |u_1|, \ldots, |u_n|\}^{n+1}} du
= -\frac{1}{2} \int_{u \in \mathbb{R}^{n+1}} \frac{1(1, u_1, \ldots, u_n)}{u_0^2} du
= \frac{n+1}{2} \int_{x \in \mathbb{R}^{n+1}} \frac{1(x) dx}{\max\{|x_0|, \ldots, |x_n|\} \leq 1}
\]
where we make the change of variables \((u_0, u_1, \ldots, u_n) = (1/x_{n+1}^0, x_1/x_0, \ldots, x_n/x_0)\), which has Jacobian determinant \(-(n+1)x_0^{-2(n+2)}\). This shows (1).

Now let \(p\) be a prime. The \(\sigma\)-algebra on \(\mathbb{P}^n(\mathbb{Q}_p)\) is generated by the residue discs \(D_y := \{x \in \mathbb{P}^n(\mathbb{Z}_p) : x \equiv y \mod p^r, y \in \mathbb{P}^n(\mathbb{Z}/p^r\mathbb{Z})\}\). These have volume
\[
\tau_p(D_y) = \frac{\# \mathbb{P}^n(\mathbb{F}_p)}{p^r \# \mathbb{P}^n(\mathbb{Z}/p^r\mathbb{Z})} = \frac{1}{p^{nr}}.
\]
It suffices to prove the result for the residue discs. Writing \(y = (y_0 : \cdots : y_n)\), without loss of generality each \(y_i \in \mathbb{Z}_p\) and \(y_0 = 1\). Then we have
\[
\sim\hat{\sim}D_y \cap \mathbb{Z}_p^{n+1} = \left\{x \in \mathbb{Z}_p^{n+1} : \left|\frac{x_i}{p^{r_p(x_0)}} - \frac{y_i x_0}{p^{r_p(x_0)}}\right|_p \leq p^{-r}, i = 1, \ldots, n\right\}.
\]
For (2) we have
\[
\mu_p(\sim\hat{\sim}D_y \cap \mathbb{Z}_p^{n+1}) = \sum_{m=0}^{\infty} \left\{x \in \mathbb{Z}_p^{n+1} : v_p(x_0) = m, |x_i - y_i x_0|_p \leq p^{-r-m}, i = 1, \ldots, n\right\}
= \sum_{m=0}^{\infty} \frac{1}{p^m} \left(1 - \frac{1}{p}\right) \left(\frac{1}{p^{r+m}}\right)^n = \frac{1}{p^{nr}} \cdot \frac{1 - 1/p}{1 - 1/p^{n+1}}
\]
as required. For (3) we have
\[
\mu_p\{x \in \sim\hat{\sim}D_y \cap \mathbb{Z}_p^{n+1} : p \nmid x\} = \{x \in \mathbb{Z}_p^{n+1} : v_p(x_0) = 0, |x_i - y_i x_0|_p \leq p^{-r}, i = 1, \ldots, n\}
= \left(1 - \frac{1}{p}\right) \frac{1}{p^{nr}}
\]
as required. \(\Box\)

3.2. **The case** \(\Delta(f) = 0\). Here Conjecture [2.14] is known in numerous cases by work of Loughran–Smeets [50], Huang [47], and Browning–Heath-Brown [21]. These are all proved via suitable applications of the geometric sieve (also called the sieve of Ekedahl). Poonen and Voloch’s [59] paper was one of the earliest applications of this sieve.

**Proposition 3.2.** Conjecture [2.14] holds when \(\Delta(f) = 0, k = \mathbb{Q}\), and \(X\) is either \(\mathbb{P}^n\), a split toric variety, or a smooth quadric hypersurface of dimension at least 3.

**Proof.** The results [50, Thm. 1.3], [47, Thm. 1.5], and [21, Cor. 1.6] show that the limit
\[
\lim_{B \to \infty} \frac{\#\{x \in X(\mathbb{Q}) : H(x) \leq B, x \in f(Y(A_0)), x \notin \Omega\}}{\#\{x \in X(\mathbb{Q}) : H(x) \leq B, x \notin \Omega\}}
\]

exists and equals the correct product of local densities. Here \( \Omega \) is the set of points in the complement of the open torus when \( X \) is a toric variety, and empty otherwise. Since \( \Delta(f) = 0 \) we have \( \delta_D(f) = 1 \) for all \( D \). It therefore suffices to show that

\[
\text{Br}_{\text{Sub}}(X, f) = \text{Br}_X. \tag{3.1}
\]

(Note that in all these cases \( \text{Br}_X = \text{Br}_Q \) since \( X \) is rational). We prove this via Lemma 2.9. Let \( D \in X^{(1)} \). The fibre over \( D \) is a so-called pseudo-split scheme, which means that every element of the absolute Galois group of \( k(D) \) fixes some irreducible component of \( f^{-1}(D) \) of multiplicity 1. It easily follows that

\[
\bigcap_E \ker \left( \text{res} : H^1(k(D), \mathbb{Q}/\mathbb{Z}) \to H^1(k_E, \mathbb{Q}/\mathbb{Z}) \right)
\]

is trivial, where the intersection is over all irreducible components \( E \) of \( f^{-1}(D) \) of multiplicity 1. The claim (3.1) now follows from Lemma 2.9 and (2.9). \( \square \)

3.3. **Rational numbers as sum of two squares.** Our first example with \( \Delta(f) > 0 \) is the counting problem

\[
N(B) := \# \{ t \in \mathbb{Q} : H(t) \leq B, t = x^2 + y^2 \text{ for some } x, y \in \mathbb{Q} \}.
\]

This is an elementary warm up for some of the more difficult examples which will be treated in the paper. It is included for completeness as it appears in Example 1.5. It also highlights an important subtle point which will occur later on; namely, it is essential that one uses the anticanonical height in Conjecture 2.14! Changing the height function will change the leading constant due to the appearance of \( \log B \) factors; the Fujita invariants were introduced to balance out these additional factors coming from changing the height function. We begin with the following asymptotic.

**Proposition 3.3.** We have

\[
N(B) = \frac{3}{2\pi \log B} \prod_{p \equiv 1 \mod 4} \left( 1 + \frac{1}{p} \right) \prod_{p \equiv 3 \mod 4} \left( 1 - \frac{p - 1}{p(p + 1)} \right) + O \left( \frac{B^2}{(\log B)^2} \right).
\]

**Proof.** Choosing coprime integers \( a, b \) to represent \( t \), we can write

\[
N(B) = O(1) + \frac{1}{2} \sum_{0 < |a|, |b| \leq B} 1(aX^2 = bY^2 + bZ^2 \text{ soluble in } \mathbb{Q}).
\]

To ensure real solubility, both \( a \) and \( b \) must have the same sign (we choose them both to be positive). By the Hasse principle and Hilbert reciprocity, solubility is equivalent to \( v_p(ab) \equiv 0 \pmod{2} \) for all \( p \equiv 3 \pmod{4} \). Hence,

\[
N(B) = O(1) + \sum_{0 < a, b \leq B} \prod_{p \equiv 1 \pmod{4} \text{ and } p|ab} 1(v_p(ab) \equiv 0 \pmod{2}).
\]

Write \( a = 2^\alpha s, b = 2^\beta t \) for odd integers \( s, t \). As \( \gcd(a, b) = 1 \), we have

\[
N(B) = O(1) + \sum_{0 \leq \alpha, \beta \leq 2 \log B} N(B2^{-\alpha}, B2^{-\beta}),
\]

where

\[
N(S, T) = \sum_{0 < s \leq S} f(s) \sum_{0 < t \leq T/2^\beta} f(t) \quad \text{and} \quad f(k) := \prod_{p \equiv 3 \pmod{4}} 1(v_p(k) \equiv 0 \pmod{2}).
\]

\[
N(S, T) = \sum_{0 < s \leq S} f(s) \sum_{0 < t \leq T/2^\beta} f(t) \quad \text{and} \quad f(k) := \prod_{p \equiv 3 \pmod{4}} 1(v_p(k) \equiv 0 \pmod{2}).
\]
Note that $f$ is multiplicative. By the Selberg–Delange method \cite[Ch. II.5, Thm. 5.2]{72}, we get
\[
\sum_{0 < t \leq T, \gcd(t, s) = 1} f(t) = c(s) \frac{T}{\sqrt{\log T}} + O \left( \frac{\tau(s)}{(\log T)^{3/2}} \right),
\]
where $\tau$ is the divisor function, the implied constant is absolute, and
\[
c(s) := \frac{1}{\pi^{1/2}} \prod_p \left( 1 + 1 \sum_{k \geq 1} \frac{f(p^k)}{p^k} \right) \left( 1 - \frac{1}{p} \right)^{1/2} = c_0 G(s),
\]
where
\[
c_0 = \prod_{p \equiv 1 \mod 4} \left( 1 - \frac{1}{p} \right) \prod_{p \equiv 3 \mod 4} \left( 1 - \frac{1}{p} \right)^{1/2}, G(s) = \prod_{p \mid s} \frac{p - 1}{p} \prod_{p \equiv 1 \mod 4} \frac{p^2 - 1}{p^2}.
\]
The second sum which needs to be considered then is
\[
\sum_{0 < s \leq S, s \text{ odd}} f(s) G(s),
\]
which another application of Selberg–Delange shows is equal to
\[
\frac{1}{\sqrt{2\pi}} \frac{S}{\sqrt{\log S}} \prod_{p \neq 2} \left( 1 - \frac{1}{p} \right)^{1/2} \left( 1 + \sum_{k \geq 1} \frac{f(p^k) G(p^k)}{p^k} \right) + O \left( \frac{S}{(\log S)^{3/2}} \right).
\]
Note that by the same approach, summing the error term in (3.2) will yield an error term of size $O \left( \frac{S T}{\sqrt{\log S (\log T)^3}} \right)$. This Euler product in the above equation simplifies to
\[
\prod_{p \equiv 1 \mod 4} \left( 1 - \frac{1}{p} \right)^{1/2} \left( 1 + \frac{1}{p} \right) \prod_{p \equiv 3 \mod 4} \left( 1 - \frac{1}{p} \right)^{1/2} \left( 1 + \frac{1}{p^2} \right).
\]
Multiplying this by $c_0$ gives
\[
\kappa := \prod_{p \equiv 1 \mod 4} \left( 1 + \frac{1}{p} \right) \prod_{p \equiv 3 \mod 4} \left( 1 - \frac{1}{p} \right) \left( 1 + \frac{1}{p} \right) = \prod_{p \equiv 1 \mod 4} \left( 1 + \frac{1}{p} \right) \prod_{p \equiv 3 \mod 4} \left( 1 - \frac{p - 1}{p(p + 1)} \right).
\]
Therefore $N(B)$ is asymptotic to
\[
\frac{\kappa}{(\sqrt{2\pi})^2} \sum_{\alpha, \beta = 0} B^2 = \frac{B^2}{\log B} \frac{3}{2\pi} \prod_{p \equiv 1 \mod 4} \left( 1 + \frac{1}{p} \right) \prod_{p \equiv 3 \mod 4} \left( 1 - \frac{p - 1}{p(p + 1)} \right).
\]
The total error is bounded by
\[
\sum_{\alpha, \beta \neq 0} B^2 2^{\alpha + \beta} \left( \log(B(2^{\alpha - 1})) \right)^{1/2} \left( \log(B(2^{\beta - 1})) \right)^{1/2} < \frac{B^2}{(\log B)^2}.
\]
We now explain how Proposition 3.3 is compatible with Conjecture 2.14. The base variety is $\mathbb{P}^1$. The open set is $U = G_m$ and the boundary divisors are 0 and $\infty$. We have
\[
\theta(\mathbb{P}^1) = 1/2, \quad | Br_{\text{sub}}(X, f)/Br k | = 2, \quad a(0) = a(\infty) = 2, \quad \Gamma(X, f) = \Gamma(1/2)^2 = \pi.
\]
This calculation of the subordinate Brauer group is relatively easy using Faddeev reciprocity \cite[Thm. 1.5.2]{20}, and the non-trivial representative is given simply by the quaternion algebra $(t, -1)$. For the local densities, we use Lemma 2.21 with $S = \emptyset$. 

18
Lemma 3.4. We have
\[
\tau_p(f(Y(\mathbb{Q}_p))) = \begin{cases} 
2, & p = \infty, \\
3/4, & p = 2, \\
1 + \frac{1}{p}, & p \equiv 1 \mod 4, \\
1 - \frac{p-1}{p(p+1)}, & p \equiv 3 \mod 4.
\end{cases}
\]

Proof. We use Proposition 3.1. This gives
\[
\tau_\infty(f(Y(\mathbb{R}))) = \mu_\infty\{(t_0, t_1) \in [-1, 1]^2 : t_0/t_1 > 0\} = 2.
\]
For \( p \equiv 1 \mod 4 \) every conic in the family has a \( \mathbb{Q}_p \)-point. So let \( p \equiv 3 \mod 4 \). The relevant measure is
\[
(1 - 1/p)^{-1} \mu_p\{(t_0, t_1) \in \mathbb{Z}_p^2 : p \nmid t, v_p(t_0t_1) \in 2\mathbb{Z}\} = (1 - 1/p)^{-1}(2(1 - 1/p)/(1 + 1/p) - (1 - 1/p)^2)
\]
as expected. (Here we used \( \mu_p\{x \in \mathbb{Z}_p : v_p(x) \in 2\mathbb{Z}\} = (1 + 1/p)^{-1} \) and inclusion-exclusion.) For \( p = 2 \) write \( t_i = 2^n u_i \) where \( u_i \) is a 2-adic unit. Then we have
\[
(t_0t_1, -1) = (-1)^{(u_1u_2-1)/2},
\]
thus the relevant volume is
\[
(1 + 1/2)^{-1} \mu_2\{(t_0, t_1) \in \mathbb{Z}_2^2 : (t_0t_1)/2^{v_2(t_0t_1)} \equiv 1 \mod 4\} = (1 + 1/2) \cdot 1/2 = 3/4.
\]

The virtual Picard group here is \( \mathbb{Z} - (1/2)\mathbb{Z} - (1/2)\mathbb{Z} \), so no convergence factors are required for the Tamagawa measure. Combining all the above with Proposition 3.3 one sees that we are off by a factor of 2 in Conjecture 2.13. This is because we used the naive height in Proposition 3.3 and Conjecture 2.13 uses the anticanonical height! Therefore it is the counting function \( N(B^{1/2}) \) which actually gives the correct constant, and this missing factor of 2 comes from the product of Fujita constants \( a(0)^{1/2} a(\infty)^{1/2} = 2 \).

3.4. Polynomial represented by a binary quadratic form – compatibility with the circle method. We now consider a more difficult example. Let \( g \in \mathbb{Q}[x_0, \ldots, x_n] \) be an irreducible homogeneous polynomial of even degree \( d \) and \( a \in \mathbb{Z} \) an element which is not a square in the function field of the divisor \( g(x) = 0 \).

Lemma 3.5. Conjecture 2.14 predicts that
\[
\# \left\{ x \in \mathbb{P}^n(\mathbb{Q}) : H_{-K_{\mathbb{Q}}}(x) \leq B, \quad g(x) = t_0^2 - at_1^2 \text{ for some } t_0, t_1 \in \mathbb{Q} \right\} \sim \frac{2 \cdot \prod_p \omega_p}{\sqrt{\pi d(n+1) (\log B)^{1/2}}} B,
\]
where
\[
\omega_\infty = \begin{cases} 
(n+1)2^n, & a > 0, \\
(n+1)/2 \cdot \vol \{ x \in [-1, 1]^{n+1} : g(x) > 0 \}, & a < 0,
\end{cases}
\]
and \((\cdot, \cdot)_p\) denotes the Hilbert symbol.

Note we expect that no thin set \( \Omega \) is required in this case, i.e. that one may take \( \Omega = \emptyset \). Moreover no non-singularity hypothesis is required on \( g \) (\( g \) may even be geometrically reducible). In the above volume, we are implicitly ignoring the subset \( g(x) = 0 \), which has measure zero. Note also that \((g(x), a)_p = 1 \) whenever \( a \in \mathbb{Q}_p^\times \).
Proof. By Corollary 2.25 we can use Conjecture 2.23 instead. As $g$ has even degree and is irreducible, the quaternion algebra $\alpha = (g, a)$ is ramified exactly along the divisor $D = \{g(x) = 0\}$ with residue $a \in \kappa(D)$, which is non-trivial by assumption. We have

$$\theta(\mathbb{P}^n) = 1/(n + 1), \quad a(D) = (n + 1)/d, \quad \Gamma(1/2) = \sqrt{\pi}.$$ 

This gives the factors on the denominator. We next claim that

$$\text{Br}_{\text{Sub}}(\mathbb{P}^n, \alpha)/\text{Br} \mathbb{Q} \cong \mathbb{Z}/2\mathbb{Z}$$

generated by the image of $\alpha$ (this gives the factor 2 on the numerator). To see this, let $\beta \in \text{Br}_{\text{Sub}}(\mathbb{P}^n, \alpha)$ be non-constant. Then $\alpha - \beta$ is everywhere unramified, hence is constant as $\text{Br} \mathbb{P}^n = \text{Br} \mathbb{Q}$. This proves the claim.

For the local densities, we use Lemma 2.21 with $S = \emptyset$. To calculate these we use Proposition 3.1. For the real density, when $a > 0$ there is always a real point so we recover the usual real density of $(n + 1)2^n$. When $a < 0$ there is a real solution if and only if $g(x)$ is positive. For the $p$-adic densities, the condition that there is a $\mathbb{Q}_p$-point is exactly that $(g(x), a)_p = 1$. To finish it suffices to note that the convergence factors are given by the local Euler factors of $\zeta(s)^{1/2}$, since the virtual representation here is just $\mathbb{Z} - (1/2)\mathbb{Z}$, and that $\lim_{s \to 1} (s - 1)^{1/2} \zeta(s)^{1/2} = 1$.

We now prove that the formula in Lemma 3.5 is compatible with predictions from the Hardy–Littlewood circle method, at least when $a = -1$. Letting $H$ be the naive Weil height in $\mathbb{P}^n$ we shall study

$$N(B) := \sharp \{x \in \mathbb{P}^n(\mathbb{Q}) : H(x) \leq B, t_0^2 + t_1^2 = g(x) \text{ has a Q-point}\}.$$ 

By Lemma 3.5 our result verifies Conjecture 2.14 after considering the anticanonical counting function $N(B^{1/(n+1)})$.

**Theorem 3.6.** Let $g \in \mathbb{Q}[x_0, \ldots, x_n]$ be an irreducible homogeneous polynomial of even degree $d$. Assuming that $g$ is non-singular and that $n + 1 > (d - 1)2^d$ we have

$$\lim_{B \to \infty} \frac{N(B)B^{-n-1}}{(\log B)^{-1/2}} = \frac{\text{vol}(x \in [-1, 1]^{n+1} : g(x) > 0)}{(\pi d)^{1/2}} \prod_p \left(1 - \frac{1}{p}\right)^{1/2} \left(1 + \frac{1}{p} + \cdots + \frac{1}{p^n}\right) \ell_p,$$

where for any prime $p$ we let

$$\ell_p := \text{vol} \{x \in \mathbb{Z}_p^{n+1} : t_0^2 + t_1^2 = g(x) \text{ has a Q}_p\text{-point}\}.$$ 

**Proof.** Our method of proof is based upon [71], as well as Birch’s seminal work [9].

**Step 1 (using the circle method.)** By Möbius inversion we may use arguments akin to [71] Lemma 2.1 to write

$$N(B) = \frac{1}{2} \sum_{l \in \mathbb{N} \cap [1, \log(2B)]} \mu(l) \Theta(B/l) + O(B^{n+1}(\log 2B)^{-1}),$$

where $\Theta(P) := \sharp \{x \in \mathbb{Z}^{n+1} : |x| \leq P, g(x) \neq 0, t_0^2 + t_1^2 = g(x) \text{ has a Q-point}\}$ and $|\cdot|$ denotes the supremum norm in $\mathbb{R}^{n+1}$. Letting $e(z) := e^{2\pi iz}$ and using the standard circle method integral $\int_0^1 e(\alpha(g(x) - m))d\alpha$ to detect whether $g(x)$ equals a given integer $m$ we can show as in [71] Equation (2.2) that

$$\Theta(P) = \int_0^1 S(\alpha)E_Q(\alpha)d\alpha,$$

where $S(\alpha) := \sum_{|x| \leq P} e(\alpha g(x))$ is the standard exponential sum associated to $g$ and

$$E_Q(\alpha) = \sum_{0 < m \leq \max \{g([-1, 1]^{n+1})\} \text{ has a Q-point}} e(\alpha m).$$
Step 2 (bounding away the minor arcs.) We next show that the ‘minor arcs’ contribute to the error term. The trivial bound $E_Q(\alpha) = O(P^d)$ shows that for any set $A \subset [0,1)$ one has

$$\left| \int_A S(\alpha) E_Q(\alpha) \, d\alpha \right| \ll P^d \int_A |S(\alpha)| \, d\alpha. \quad (3.4)$$

Birch [9, Equations (10)-(11), Lemma 4.3] showed that there exist $\eta > 0$ such that

$$Birch showed in [9, Lemma 5.1] that there exists constants such that when $A$ is defined as the complement of

$$\bigcup_{1 \leq q \leq P(d-1) \theta_0} \bigcup_{a \in (\mathbb{Z}/q\mathbb{Z})^\times} \left[ \frac{a}{q} - \frac{P^{-d+(d-1) \theta_0}}{2q}, \frac{a}{q} + \frac{P^{-d+(d-1) \theta_0}}{2q} \right]$$

one has $\int_A |S(\alpha)| \, d\alpha = O(P^{n+1-d-\delta})$. As shown in [9, Lemma 4.1] these intervals are disjoint, hence, by (3.4) we obtain

$$\Theta(P) = \sum_{1 \leq q \leq P(d-1) \theta_0} \sum_{a \in (\mathbb{Z}/q\mathbb{Z})^\times} \int_{|\beta| < P^{-d+(d-1) \theta_0}} S\left( \frac{a}{q} + \beta \right) \frac{E_Q\left( \frac{a}{q} + \beta \right)}{\gamma} \, d\beta + O(P^{n+1-\delta}). \quad (3.5)$$

Note that we have replaced the condition $|\beta| < \frac{P^{-d+(d-1) \theta_0}}{2q}$ by $|\beta| < P^{-d+(d-1) \theta_0}$; this can be done owing to the remarks in [9, page 253].

Step 3 (first approximation in the major arcs.) Now that we have obtained a representation of $\Theta(P)$ as an integral over the ‘major arcs’ we proceed to approximate $S(a/q + \beta)$ by a product of an exponential sum and an exponential integral in each arc. Namely, letting

$$S_{a,q} = \sum_{t \in (\mathbb{Z}/q\mathbb{Z})^{n+1}} e(af(t)/q) \text{ and } I(\gamma) = \int_{\zeta \in \mathbb{R}^{n+1} \cap [-1,1]^{n+1}} e(\gamma g(\zeta)) \, d\zeta,$$

Birch showed in [9, Lemma 5.1] that there exists $\eta > 0$ that depends at most on $g$ such that

$$S\left( \frac{a}{q} + \beta \right) = q^{-n-1} P^{n+1} S_{a,q} I(P^d \beta) + O(P^{n+2q})$$

whenever $|\beta| < P^{-d+(d-1) \theta_0}$. Injecting this into (3.5) yields constants $\theta_i > 0$ that depend only on $g$ such that

$$\Theta(P) = P^{n+1} \sum_{1 \leq q \leq P^{\theta_1}} \sum_{a \in (\mathbb{Z}/q\mathbb{Z})^\times} \frac{S_{a,q}}{q^{n+1}} \int_{|\gamma| < P^{\theta_2}} \left| I(\gamma) \frac{E_Q\left( \frac{a}{q} + \frac{\gamma}{P^d} \right)}{\gamma \frac{P^d}{P^d}} \right| \, d\gamma + O(P^{n+1-\theta_1}), \quad (3.6)$$

where we used the change of variables $\gamma = P^d \beta$.

Step 4 (second approximation in the major arcs.) To estimate the term $E_Q$ inside the integral we use [71, Lemma 3.6]. This result is essentially related to the density of integers $m$ in an arithmetic progression that are a sum of two integer squares. It will therefore only be useful when $q$ and $\gamma$ have smaller size than in (3.6) and we thus continue by restricting the summation and integration range in (3.6). To do so we recall that Birch [9, Lemmas 5.2 & 5.4] showed that there exist constants $\epsilon_i > 0$ depending only on $g$ such that

$$|I(\gamma)| \ll \min\{1, |\gamma|^{-1-\epsilon_1}\} \text{ and } \frac{|S_{a,q}|}{q^{n+1}} \ll q^{-1-\epsilon_2}. \quad (3.7)$$

Combining them with the standard result $\max \{|E_Q(\alpha)| : \alpha \in \mathbb{R}\} \ll P^d (\log P)^{-1/2}$ we can use a similar argument as in [71, Lemma 4.2] to conclude that for each constants
\[ A_1, A_2 > 0 \] there exists a positive constant \( A_3 \) depending on \( A_1, A_2 \) and \( g \) such that
\[
\frac{\Theta(P)}{P^{n+1}} = \sum_{\substack{1 \leq q \leq (\log P)^{A_1} \\ a \in (\mathbb{Z}/q\mathbb{Z})^*}} S_{a,q} \int_{|\gamma| \leq (\log P)^{A_2}} I(\gamma) \frac{E_{\gamma} \left( \frac{a}{q} + \frac{\gamma}{P^d} \right)}{P^d} \frac{d\gamma}{P^d} + O \left( (\log P)^{-1/2 - A_3} \right).
\]

Lemma 3.6 in [71] states that for all constants \( A > 0 \) and \( q \leq (\log P)^A \) one has
\[
E_{\gamma} \left( \frac{a}{q} + \frac{\gamma}{P^d} \right) = 2^{1/2} C_0 \mathfrak{f}(a, q) \int_{2}^{\max \{|g([-1, 1]^{n+1})|P^d\}} \frac{e(\gamma P^{-d} t)}{\sqrt{\log t}} dt + O \left( \frac{q^3 (1 + |\gamma|) P^d}{(\log P)^{1/2 + 1/7}} \right),
\]
where \( \mathfrak{f}(a, q) \) is defined in [71, Equation (3.7)] and
\[
C_0 := \prod_{p \equiv 3 (\text{mod } 4)} (1 - p^{-2})^{1/2}.
\]

Taking \( A_1 \) and \( A_2 \) appropriately small and using [3.7] shows that there exist \( A_i > 0 \) depending only on \( g \) such that
\[
\frac{\Theta(P)}{P^{n+1}} = 2^{1/2} C_0 \sum_{\substack{q \leq (\log P)^{A_1} \\ a \in (\mathbb{Z}/q\mathbb{Z})^*}} S_{a,q} \frac{\mathfrak{f}(a, q)}{q^{n+1}} \left( \int_{|\gamma| \leq (\log P)^{A_2}} I(\gamma) \mathcal{H}_P(\gamma) \frac{d\gamma}{P^d} \right) + o \left( (\log P)^{-1/2} \right),
\]
where
\[
\mathcal{H}_P(\gamma) := \int_{\max \{|g([-1, 1]^{n+1})|P^d\}}^{\infty} \frac{e(\gamma P^{-d} t)}{\sqrt{\log t}} dt.
\]

By [71, Equation (3.11)] the function \( \mathfrak{f}(a, q) \) is bounded independently of \( a \) and \( q \), hence, by [3.7] we get
\[
\sum_{\substack{1 \leq q \leq (\log P)^{A_1} \\ a \in (\mathbb{Z}/q\mathbb{Z})^*}} \frac{S_{a,q}}{q^{n+1}} \mathfrak{f}(a, q) = \sum_{\substack{q \in \mathbb{N} \\ a \in (\mathbb{Z}/q\mathbb{Z})^*}} \frac{S_{a,q}}{q^{n+1}} \mathfrak{f}(a, q) + o(1).
\]

Similarly, the trivial bound \( \mathcal{H}_P(\gamma) \ll P^d (\log P)^{-1/2} \) combined with [3.7] shows that
\[
\int_{|\gamma| \leq (\log P)^{A_2}} I(\gamma) \mathcal{H}_P(\gamma) \frac{d\gamma}{P^d} = \int_{-\infty}^{+\infty} I(\gamma) \mathcal{H}_P(\gamma) \frac{d\gamma}{P^d} + o((\log P)^{-1/2}).
\]

Injecting (3.10)-(3.11) into (3.9) then yields
\[
\frac{\Theta(P)}{P^{n+1}} = \mathfrak{S} \int_{\mathbb{R}} I(\gamma) \mathcal{H}_P(\gamma) \frac{d\gamma}{P^d} + o \left( (\log P)^{-1/2} \right),
\]
where is the analogue of the classical singular series in our setting and given by
\[
\mathfrak{S} := \sum_{q \in \mathbb{N}} \sum_{a \in (\mathbb{Z}/q\mathbb{Z})^*} \frac{S_{a,q}}{q^{n+1}} \left( 2^{1/2} C_0 \mathfrak{f}(a, q) \right).
\]

One can simplify the integral over \( \gamma \) by using the Fourier analysis approach in [71] Lemmas 4.3-4.4. This will result in
\[
\int_{\mathbb{R}} I(\gamma) \frac{\mathcal{H}_P(\gamma)}{P^d} d\gamma = \frac{\text{vol}(t \in [-1, 1]^{n+1} : g(t) > 0)}{\sqrt{\log(P^d)}} + O((\log P)^{-3/2}),
\]
which, when combined with (3.12), results in
\[
\lim_{B \to \infty} \frac{\Theta(P)}{P^{n+1} (\log(P^d))^{-1/2}} = \text{vol}(t \in [-1, 1]^{n+1} : g(t) > 0) \mathfrak{S}.
\]

**Step 5 (The singular series.)** One may now use the multiplicative properties of \( S_{a,q} \) to express \( \mathfrak{S} \) as an Euler product; this is a standard procedure albeit somewhat awkward in the presence of the terms \( \mathfrak{f}(a, q) \). One can deal with these terms similarly as
in [71] Lemmas 5.1] using nothing more than Ramanujan exponential sums. Successively, the individual terms of the Euler product can be written as a limit of densities as in [74 §3.5.2–3.5.3]; this will result in the decomposition

\[
\sum_{q \in \mathbb{N}} \sum_{\alpha \in (\mathbb{Z}/q\mathbb{Z})^*} \frac{S_{\alpha,q}}{q^{\nu+1}} \delta(a,q) = \ell_2^* \prod_{p \equiv 3 \pmod{4}} (1 - 1/p)^{-1} \ell_p^*,
\]

where

\[
\ell_p^* := \lim_{N \to \infty} \# \{x \in (\mathbb{Z} \cap [0,p^N])^{n+1} : \epsilon_0^2 + \epsilon_1^2 = g(x) \text{ has a } \mathbb{Q}_p \text{-point} \}.
\]

A straightforward argument involving Haar measures shows that \( \ell_p^* = \ell_p \). In light of (3.14) we conclude that

\[
\lim_{B \to \infty} \frac{\Theta(P) (\log P)^{1/2} / P^{n+1}}{d^{1/2}} = \frac{\text{vol}([t] \leq 1 : g(t) > 0) 2^{1/2} C_0 \ell_2}{2 \zeta(n+1) \prod_{p \equiv 3 \pmod{4}} (1 - 1/p)}.
\]

**Step 6 (Conclusion of the proof.)** Fix any \( \epsilon > 0 \). Denoting the right-hand side of (3.15) by \( c \) and using (3.3) and (3.15) we infer that for all sufficiently large \( B \) one has

\[
\frac{N(B)}{B^{n+1}} - \frac{c}{2} \sum_{l \leq \log(2B)} \frac{\mu(l)^{l-n-1}}{(\log(B/l))^{1/2}} \leq \epsilon \sum_{l \leq \log(2B)} \frac{l^{-n-1}}{(\log(B/l))^{1/2}} + O \left( \frac{1}{\log B} \right).
\]

For \( l \leq \log(2B) \) one has

\[
(\log(B/l))^{-1/2} = (\log B)^{-1/2} \left( 1 + O \left( \frac{\log \log B}{\log B} \right) \right),
\]

hence, the right-hand side of (3.16) becomes

\[
\epsilon \sum_{l \leq \log(2B)} \frac{l^{-n-1}}{(\log(B/l))^{1/2}} + O \left( \frac{1}{\log B} \right) \ll \frac{1}{(\log(B/l))^{1/2}} \sum_{l \leq \log(2B)} \frac{1}{l^{n+1}} + \frac{1}{\log B} \ll \frac{\epsilon}{(\log B)^{1/2}}.
\]

Furthermore, by (3.17) the sum in the left-hand side of (3.16) is

\[
\sum_{l \leq \log(2B)} \frac{\mu(l)^{l-n-1}}{(\log(B/l))^{1/2}} = 1 + o(1) \sum_{l \leq \log(2B)} \frac{\mu(l)}{l^{n+1}} = 1/\zeta(n+1) + o(1).
\]

Putting these estimates together shows that

\[
\lim_{B \to \infty} \frac{N(B)(\log B)^{1/2}}{B^{n+1}} = \frac{\text{vol}([t] \leq 1 : g(t) > 0) 2^{1/2} C_0 \ell_2}{2 \zeta(n+1) \prod_{p \equiv 3 \pmod{4}} (1 - 1/p)}.
\]

Alluding to (3.3) and the standard identity

\[
\frac{1}{\zeta(n+1)} = \prod_p \left( 1 - \frac{1}{p} \right) \left( 1 + \frac{1}{p} + \cdots + \frac{1}{p^n} \right)
\]

shows that

\[
\frac{C_0}{\zeta(n+1)} \prod_{p \equiv 3 \pmod{4}} \frac{\ell_p}{1 - 1/p} = \frac{1}{\zeta(n+1)} \prod_{p \equiv 3 \pmod{4}} \left( 1 - \frac{1}{p} \right)^{-1/2} \left( 1 + \frac{1}{p} \right)^{1/2} \ell_p
\]

\[
= \left( 1 - \frac{1}{2^{n+1}} \right) \prod_{p \equiv 1 \pmod{4}} \left( 1 - \frac{1}{p} \right)^{1/2} \left( 1 + \frac{1}{p} + \cdots + \frac{1}{p^n} \right) \left( 1 - \frac{1}{p} \right)^{1/2} \ell_p,
\]

\[
\times \prod_{p \equiv 3 \pmod{4}} \left( 1 - \frac{1}{p} \right)^{1/2} \left( 1 + \frac{1}{p} + \cdots + \frac{1}{p^n} \right) \left( 1 + \frac{1}{p} \right)^{1/2} \ell_p.
\]
Upon using the standard fact
\[
\frac{2}{\pi^{1/2}} = \prod_{p \equiv 1\,(\text{mod }4)} \left(1 - \frac{1}{p}\right)^{1/2} \prod_{p \equiv 3\,(\text{mod }4)} \left(1 + \frac{1}{p}\right)^{1/2}
\]
the previous expression becomes
\[
\frac{2}{\pi^{1/2}} \left(1 - \frac{1}{2^{n+1}}\right) \prod_{p \equiv 1\,(\text{mod }4)} \left(1 - \frac{1}{p}\right)^{1/2} \left(1 + \frac{1}{p} + \cdots + \frac{1}{p^n}\right) \prod_{p \equiv 3\,(\text{mod }4)} \left(1 - \frac{1}{p}\right)^{1/2} \left(1 + \frac{1}{p} + \cdots + \frac{1}{p^n}\right) \ell_p,
\]
which, owing to \(\ell_p\) being 1 for \(p \equiv 1\,(\text{mod }4)\), simplifies to
\[
\frac{2}{\pi^{1/2}} \left(1 - \frac{1}{2^{n+1}}\right) \prod_{p \equiv 2\,(\text{mod }4)} \left(1 - \frac{1}{p}\right)^{1/2} \left(1 + \frac{1}{p} + \cdots + \frac{1}{p^n}\right) \ell_p.
\]
This shows that as \(B \to \infty\), the function \(N(B)B^{-n-1}(\log B)^{1/2}\) has limit
\[
\frac{\text{vol}(|t| \leq 1 : g(t) > 0)}{(\pi d)^{1/2}} \cdot 2^{1/2} \ell_2 \left(1 - \frac{1}{2^{n+1}}\right) \prod_{p \equiv 2\,(\text{mod }4)} \left(1 - \frac{1}{p}\right)^{1/2} \left(1 + \frac{1}{p} + \cdots + \frac{1}{p^n}\right) \ell_p,
\]
which equals
\[
\frac{\text{vol}(|t| \leq 1 : g(t) > 0)}{(\pi d)^{1/2}} \prod_{p} \left(1 - \frac{1}{p}\right)^{1/2} \left(1 + \frac{1}{p} + \cdots + \frac{1}{p^n}\right) \ell_p,
\]
thereby concluding the proof. \(\square\)

**Remark 3.7.** In Theorem 3.6 the Fujita invariant is \((n+1)/d\). Using the anticanonical height allows us to see that the term \(d^{1/2}\) in the constant in Theorem 3.6 is essentially coming from this invariant. In the proof of Theorem 3.6 this constant comes from the singular integral as can be seen from (3.13). The archimedean density \(\text{vol}(t \in [-1,1]^{n+1} : g(t) > 0)\) also emanates from the singular integral. Finally, the Gamma factor \(\Gamma(1/2) = \pi^{1/2}\) and the product of non-archimedean densities come out of the singular series.

### 3.5. Anisotropic tori.
Let \(T \subset X\) be a smooth equivariant compactification of an anisotropic torus \(T\). Let \(H\) be the associated Batyrev-Tschinkel height and \(\mathcal{B} \subset \text{Br}_1\) a finite subgroup. Here as \(T\) is anisotropic we have \(k[T]^* = k^*\), so we are in the second case of Conjecture 2.23.

**Lemma 3.8.** Conjecture 2.23 holds in this case.

**Proof.** The result [51] Thm. 5.15] gives the expression
\[
C_{X,\mathcal{B},H} = \frac{\theta(X) \cdot |\text{Br}_{\text{Sub}}(X,\mathcal{B})/\text{Br} F| \cdot \tau_{\mathcal{B}}(T(A_F)^{\text{Br}_{\text{Sub}}(X,\mathcal{B})})}{\Gamma(\rho(X) - \Delta_X(\mathcal{B}))}.
\]
This is seen to agree with the conjecture on noting that \(a(D) = 1\) for any divisor \(D\) on the boundary, since \(-K_X\) is the sum of the boundary divisors and the boundary divisors freely generate the cone of effective divisors [3] Prop. 1.2.12, Prop. 1.3.11. \(\square\)

**Remark 3.9.** As a warning, we remind the reader that (1.2) need not hold for smooth equivariant compactifications of split tori (see Example 1.5). It would however be very interesting to prove Conjecture 2.23 for an anisotropic torus \(T\) and an arbitrary finite subgroup \(\mathcal{B} \subset \text{Br} T\), i.e. possibly consisting of transcendental Brauer group elements.
3.6. Wonderful compactifications of adjoint semisimple algebraic groups. Let $G \subset X$ be the wonderful compactification of an adjoint semi-simple algebraic group over a number field $k$. Let $H$ be an anticanonical height function associated to a smooth adelic metric and $\mathcal{B} \subset Br\_G$ a finite subgroup. Here as $G$ is semi-simple we have $k[G]^\times = k^\times$, so we are in the second case of Conjecture 2.23.

**Lemma 3.10.** Conjecture 2.23 holds in this case.

**Proof.** The result [54 Thm. 4.20] gives the expression

$$C_{X,\mathcal{B},H} = \frac{|Br_{\text{sub}}(X, \mathcal{B})/Br\_k| \cdot \tau_{\mathcal{B}}(G(A_k)^{Br_{\text{sub}}(X, \mathcal{B})})}{|Pic\_G| \cdot \Gamma(\rho(X) - \Delta_X(\mathcal{B})) \cdot \prod_{\alpha \in A_1}(1 + \kappa_\alpha)^{-1/2D_\alpha(\mathcal{B})}|}$$

where $A$ denotes the set of boundary components of $X \setminus G$. Here $1 + \kappa_\alpha$ is exactly the coefficient of $D_\alpha$ in $K_X$ [54 Prop. 3.3], thus $a(\alpha) = 1 + \kappa_\alpha$. Moreover by [54 Thm. 4.21] we have $\theta(X) = (|Pic\_G| \cdot \prod_{\alpha \in A}(1 + \kappa_\alpha)^{-1}$. This agrees with the conjecture. \[\square\]

3.7. Diagonal plane conics. In this final section, we explain how the formula given in Theorem 1.1 agrees with Conjecture 2.23 (this is equivalent to Conjecture 2.14 here by Corollary 2.23). The proof of Theorem 1.1 will occur later in the paper.

Consider the variety $V \subset \mathbb{P}^2 \times \mathbb{P}^2$ given by the family of diagonal plane conics. We have the projection $f : V \rightarrow \mathbb{P}^2$ which is a conic bundle over $\mathbb{P}^2$. We are interested in the counting function

$$N(f, B) = \sharp\{x \in \mathbb{P}^2(k) : H(k) \leq B, x \in f(V(\mathbb{Q} ))\}$$

where $H$ is the naive height on $\mathbb{P}^2$. We view $G_m^2 \subset \mathbb{P}^2$ as the complement of the coordinate axis and use coordinates $x, y$ on $G_m^2$. Then the conic bundle corresponds to the quaternion algebra $\alpha = (x, y) \in Br\_G^2$. This is ramified along the three coordinate axes $D_i : x_i = 0$ with residue of order 2. We have $\rho(\mathbb{P}^2) = 1$ and $-K_{\mathbb{P}^2} = 3D_i$ in the Picard group. We take $\mathcal{B} = \langle \alpha \rangle$. In the notation of Conjecture 2.23 this altogether gives

$$\Delta(\mathcal{B}) = 3/2, \quad \theta(\mathbb{P}^2) = 1/3, \quad \Gamma(\mathbb{P}^2, \mathcal{B}) = \Gamma(1/2)^3 = \pi^{3/2}, \quad a(D_i) = 3$$

The relevant virtual Artin $L$-function is $\zeta(s)^{-1/2}$, which gives the convergence factors $(1 - 1/p)^{1/2}$ and $\lim_{s \rightarrow 1}(s - 1)^{-1/2}\zeta(s)^{-1/2} = 1$.

**Lemma 3.11.** $Br_{\text{sub}}(\mathbb{P}^2, \alpha)/Br\_Q \cong \mathbb{Z}/2\mathbb{Z}$, with generator given by the image of $\alpha$.

**Proof.** Let $\beta$ be subordinate to $\alpha$. Then $\beta$ is ramified along some subset of the three coordinate axes. It must be ramified along at least two axes since $Br\_A^2 = Br\_Q$. But then $\alpha - \beta$ is ramified along at most one axis, thus constant. \[\square\]

Hence, Conjecture 2.23 predicts that

$$\lim_{B \rightarrow \infty} \frac{N(f, B^{1/3})}{B(\log B)^{-3/2}} = (1/3) \cdot 2 \cdot \tau_f(U(A_Q)^{\mathcal{B}}) \cdot \frac{\pi^{3/2}}{3^{3/2}}. \quad (3.18)$$

Note that we considered $N(f, B^{1/3})$ so as to work with the anticanonical height function on $\mathbb{P}^2$. For the 2-adic density we shall use the following standard fact which follows from [23] pg 79-80, Theorem 4.1, Lemma 4.3].

**Lemma 3.12.** Let $r \in (\mathbb{Z}_2 \setminus \{0\})^3$ with $v_2(r_0r_1r_2) \in \{0, 1\}$. Then $\sum_{i=0}^3 r_i x_i^2 = 0$ has a $\mathbb{Q}_2$-point if and only if

$$\begin{cases} r_i + r_j \equiv 0 \mod 4, & \text{if } v_2(r_0r_1r_2) = 0, \text{ for some } i \neq j, \\ r_i + r_j + sr_k \equiv 0 \mod 8, & \text{if } v_2(r_k) = 1, \text{ for } \{i, j, k\} = \{0, 1, 2\} \text{ and some } s \in \{0, 1\}. \end{cases}$$
Lemma 3.13. We have

$$
\tau_p(f(V(Q_p))) = \begin{cases} 
9, & p = \infty, \\
49/48, & p = 2, \\
(1 + \frac{1}{p} + \frac{1}{p^2}) \frac{(2p^2+p+2)}{2(p+1)^2}, & p \text{ odd}.
\end{cases}
$$

Proof. We use Proposition 3.11. For $p = \infty$ one uses that $\sum_{i=0}^2 t_i x_i^2 = 0$ has a real solution exactly when not all signs of the $t_i$ coincide; this gives $(3/2) \cdot 6 = 9$. So consider the case where $p$ is a prime. We need to calculate

$$
\mu_p \left( t \in \mathbb{Z}_p^3 : (-t_0 t_1, -t_0 t_2)_p = 1 \right)
$$

where $(\cdot, \cdot)_p$ denotes the Hilbert symbol. (In the above we are implicitly removing the subset $t_0 t_1 t_2 = 0$ of measure zero where the Hilbert symbol is undefined.) Writing $t_i = p^{r_i(t_i)} c_i$ (so that $c_i$ is a unit) and absorbing squares we find that this volume is

$$
\sum_{c \in (\mathbb{Z}/p\mathbb{Z})^3} p^{v_0 v_1 v_2} \mu_p \left( c \in \mathbb{Z}_p^3 : (-p^{\lambda_0 + \lambda_1} c_0 c_1, -p^{\lambda_0 + \lambda_2} c_0 c_2)_p = 1 \right).
$$

For $\lambda \in \{0,1\}$ we have $\sum_{\nu = 0, \nu \equiv \lambda (\text{mod } 2)} p^{-\nu} = p^{-\lambda} (1 - p^{-2})^{-1}$, hence the above becomes

$$
(1 - p^{-2})^{-3} \sum_{\lambda \in \{0,1\}} p^{-\lambda_0 - \lambda_1 - \lambda_2} \mu_p \left( c \in \mathbb{Z}_p^3 : (-p^{\lambda_0 + \lambda_1} c_0 c_1, -p^{\lambda_0 + \lambda_2} c_0 c_2)_p = 1 \right).
$$

By symmetry this is

$$
(1 - p^{-2})^{-3} \left\{ (1 + 1/p^3) \beta'_{p} + 3(1/p + 1/p^2) \beta''_{p} \right\}, \tag{3.19}
$$

where

$$
\beta'_{p} := \mu_p \left( c \in \mathbb{Z}_p^3 : (-c_0 c_1, -c_0 c_2)_p = 1 \right), \quad \beta''_{p} := \mu_p \left( c \in \mathbb{Z}_p^3 : (-p c_0 c_1, -c_0 c_2)_p = 1 \right).
$$

We now assume that $p$ is odd. Here $(u_1, u_2)_p = 1$ for units $u_1, u_2$ thus

$$
\beta'_{p} = \mu_p(\mathbb{Z}_p^3) = (1 - 1/p)^3.
$$

Furthermore $(-c_0 c_1, -c_0 c_2)_p = 1$ if and only $-c_0 c_2$ is a square modulo $p$. Thus,

$$
\beta''_{p} = \left( 1 - \frac{1}{p} \right) \mu_p \left( c \in \mathbb{Z}_p^2 : \left( \frac{-c_0 c_2}{p} \right) = 1 \right) = \left( 1 - \frac{1}{p} \right) \frac{1}{2} \left( 1 - \frac{1}{p} \right)^2.
$$

The proof for odd primes concludes by using (3.19) and simplifying. By Lemma 3.12 we can write $\beta'_{p}$ as

$$
\sum_{s \in \{1,3\}^3 \setminus \{1,1,1\}} \mu_2 \left( c \in \mathbb{Z}_2^3 : c \equiv s \pmod{4} \right) = \frac{1}{43} \sum_{s \in \{1,3\}^3 \setminus \{1,1,1\}} 1 = \frac{3}{32}.
$$

Similarly, by Lemma 3.12 we see that $\beta''_{p}$ equals

$$
\sum_{s \in \{1,3,5,7\}^3 \setminus \{1,1,1,3,3,3\}} \mu_2 \left( c \equiv s \pmod{8} \right) = \frac{2}{8^3} \sum_{s \in \{1,3,5,7\}^3 : s_1 + s_2 \equiv 0, -s_0 \pmod{8}} \left\{ s_1 + s_2 \in \{0, -s_0\} \right\} = \frac{2}{8^3} \cdot \frac{3}{32} = \frac{3}{16}.
$$

which equals $1/16$, since fixing $s_0$ and $s_1$ uniquely determines two distinct values for $s_2$. Alluding to (3.19) shows that

$$
\tau_2(f(Q_2)) = \frac{(1 + 1/2 + 1/4)}{(1 - 2^{-2})^3} \left( 1 + \frac{1}{2^3} \right) \frac{3}{32} + 3 \left( 1 + \frac{1}{2^2} \right) \frac{1}{16} = \frac{49}{48}. \quad \square
$$
Therefore
\[
\tau_f(\mathbb{A}_\mathbb{Q}, \mathfrak{s}) = \tau_\infty(\mathbb{U}(\mathbb{R}, \mathfrak{s})) \prod_{p \text{ prime}} \frac{\tau_p(\mathbb{U}(\mathbb{Q}_p, \mathfrak{s}))(1 - 1/p)^{1/2}}{1 - 1/p} = 9 \cdot \frac{49/48}{(1 - 1/2)^{1/2}} \prod_{p \text{ prime}} \left( 1 + \frac{1}{p} + \frac{1}{p^2} \right) \frac{(p^2 + p/2 + 1)}{(p + 1)^2(1 - 1/p)^{1/2}},
\]
which equals \(3/2 \cdot c_\infty \prod_p c_p\) in the notation of Theorem 2.23. Hence, by (3.18), Conjecture 2.23 predicts
\[
\lim_{B \to \infty} c_\infty \prod_p c_p B(\log B)^{-3/2} = \frac{3^{3/2}}{\pi^{3/2}}.
\]
This agrees with the leading constant in Theorem 1.1, taking into account the sign discrepancy between \(N(f, B^{1/3})\) and \(N(B)\) (see (1.1)). This shows that Theorem 1.1 implies Conjecture 2.23, hence Conjecture 2.14, in the case of diagonal plane conics.

**Remark 3.14.** For expositional purposes we decided to pull the factor 3/2 out of the archimedean density in Theorem 1.1 so as to use a Lebesgue measure instead of a Tamagawa measure (see Proposition 3.1). The reciprocal factor 2/3 exactly corresponds to the choice of sign multiplied by \(\alpha(\mathbb{P}^2) = 1/3\).

### 4. Diagonal Plane Conics

In this section we prove Theorem 1.1. The underpinning force behind it is Theorem 4.1 which we state and prove in §4.1. It generalises the result of Guo [40] by allowing for more general equations and for lopsided height conditions. Theorem 1.1 is then proved as an application of Theorem 4.1 in §4.8.

#### 4.1. Statement of Theorem 4.1

For fixed non-zero integers \(m_{12}, m_{13}, m_{23}\) we are interested in the frequency with which
\[
\frac{1}{m_{23}} n_1 x_1^2 + \frac{1}{m_{13}} n_2 x_2^2 = \frac{1}{m_{12}} n_3 x_3^2
\]
is soluble in \(\mathbb{Q}\), as the three square-free integers \((n_1, n_2, n_3)\) range over a box of the form \(\prod_{i=1}^3 [-X_i, X_i]\). Guo’s work corresponds to the case when each \(m_{ij}\) equals 1 and \(X_1 = X_2 = X_3\). For the proof of Theorem 1.1 we need to allow the coefficients \(m_{ij}\) to tend to infinity with \(X_i\). For the application it is necessary to add more flexible assumptions: for \(X_1, X_2, X_3 \geq 1\) and \(b = (b_1, b_2, b_3), m = (m_{12}, m_{13}, m_{23}) \in \mathbb{N}^3\) we let
\[
\mathcal{N}_{b, m}(X) = \sum_{n \in \mathbb{N}^3} \mu^2(n_1 n_2 n_3) \begin{cases} 1, & \text{if } (4.1) \text{ has a } \mathbb{Q}\text{-point}, \\ 0, & \text{otherwise}, \end{cases}
\]
where the conditions in the summation are
\[
n_1 \leq X_1, \quad n_2 \leq X_2, \quad n_3 \leq X_3, \quad \gcd(n_1 n_2 n_3, m_{12} m_{13} m_{23}) = \gcd(n_1, b_2, b_3) = \gcd(n_2, b_1, b_3) = \gcd(n_3, b_1, b_2) = 1. \quad (4.3)
\]
Define
\[
\beta(b, m) = \prod_{p \neq 2} \left( 1 - \frac{1}{p} \right)^{2} \left( 1 + \frac{\#\{1 \leq i < j \leq 3 : p \nmid m_{12} m_{13} m_{23} \cdot \gcd(b_i, b_j)\}}{2p} \right) \quad (4.5)
\]
and
\[
c(b, m) = \begin{cases} 2, & \text{if } 2 \text{ divides } m_{12} m_{13} m_{23}, \\ 3 + \#\{1 \leq i < j \leq 3 : \gcd(b_i, b_j)\}, & \text{otherwise}. \end{cases}
\]
For a positive integer $m$ let $m_{\text{odd}} := m2^{-v_2(m)}$, where $v_2$ is the standard 2-adic valuation. We denote by $\tau(m)$ the standard divisor function.

**Theorem 4.1.** Fix any $0 < \eta < 1$ and any $A > 0$. Then for all $b, m \in \mathbb{N}^3$ with $m_{12}m_{13}m_{23}$ square-free,

$$
gcd(b_1, b_2, b_3) = \gcd(m_{12}, b_3) = \gcd(m_{13}, b_2) = \gcd(m_{23}, b_1) = 1,$$

and all $X_1, X_2, X_3 \geq 2$ with

$$\min_{i=1,2,3} X_i \geq \left( \max_{i=1,2,3} X_i \right)^{\eta}$$

we have

$$\mathcal{N}_{b,m}(X) = \frac{(2\pi)^{-3/2}\beta(b, m)c(b, m)}{2\tau((m_{12}m_{13}m_{23})_{\text{odd}})} \prod_{i=1}^{3} \frac{X_i}{(\log X_i)^{1/2}} + O \left( \frac{X_1X_2X_3(\log \log 3B)^{3/2}}{(\log X_1)^{5/2}} \right),$$

where $B = \max\{b_1, b_2, b_3, m_{12}, m_{13}, m_{23}\}$ and the implied constant depends at most on $A$ and $\eta$.

The structure of the proof of Theorem 4.1 is as follows: we transform the counting function into averages of explicit arithmetic functions in §4.2. Asymptotics for averages of certain auxiliary three-dimensional arithmetic functions are given in §4.3. The main term contribution is then studied in §4.4. The error term treatment comprises two parts: characters of large conductor (in §4.5) and characters of small conductor (in §4.6). The final step in the proof of Theorem 4.1 is given in §4.7.

**Remark 4.2.** Theorem 4.1 generalises the result of Guo [40, Theorem 1.1]. Indeed, taking $A = \eta = 1/2$, $m = b = (1, 1, 1)$ and $X = (N, N, N)$ and multiplying by 6 to account for positive and negative signs yields the leading constant

$$6 \frac{(2\pi)^{-3/2}6}{2\tau(3)} \beta(1, 1) = \frac{1}{(\sqrt{\pi/2})^3} \prod_{p} \left( 1 - \frac{1}{p} \right)^{3/2} \left( 1 + \frac{3}{2p} \right),$$

which coincides with the one in [40 Theorem 1.1]. Furthermore, our error term is $O(N^3 \log N)^{-5/2}$ which improves the error term $O(N^3(\log N)^{-2})$ in [40 Theorem 1.1]. We believe our error term is best possible.

### 4.2. Using the Hasse principle to find detector functions.

We now start the proof of Theorem 4.1. The opening move is to find detector functions for solubility over $\mathbb{Q}$. Although it seems unusual, we separate right from the start those terms that will later give rise to the main term in Theorem 4.1.

For a prime $p$ and $a, b \in \mathbb{Q}_p$ we denote by $(a, b)_p$ the standard Hilbert symbol (see, for example, [55, Chapter III].)

**Lemma 4.3.** Let $a, b, c \in \mathbb{N}$ be such that $abc$ is divisible by at least one odd prime. Then

$$\frac{(1 + (ac, bc)_2)}{2\tau(r)} \left( 1 + \sum_{\delta \in \mathbb{N}, r \mid \delta} \prod_{\delta \in \{1, r\}} (ac, bc)_p \right) = \begin{cases} 1, & \text{if } aX^2 + bY^2 = cZ^2 \text{ has a } \mathbb{Q}-\text{point}, \\ 0, & \text{otherwise}, \end{cases}$$

where $r$ is the square-free integer given by the product of all odd prime divisors of $abc$.

**Proof.** By the Hasse–Minkowski theorem we write the indicator function as

$$\prod_{p \text{ prime}} \left( 1 + \frac{(ac, bc)_p}{2} \right).$$
because the conic has a point in \( \mathbb{R} \). Recall that \( p \nmid 2r \) implies that \((ac, bc)_p = 1\), hence
\[
\prod_{p \text{ prime}} \left( \frac{1 + (ac, bc)_p}{2} \right) = \prod_{p \nmid 2r} \left( \frac{1 + (ac, bc)_p}{2} \right) = \frac{1 + (ac, bc)_2}{2} \prod_{p \mid r} \left( \frac{1 + (ac, bc)_p}{2} \right).
\]
The product over \( p \mid r \) in the outmost right-hand side becomes
\[
\sum_{\delta \in \mathbb{N}} \prod_{p \mid \delta} (ac, bc)_p = 1 + \left( \sum_{\delta \in \mathbb{N}, \delta \mid r} \prod_{p \mid \delta} (ac, bc)_p \right) + \prod_{p \mid r} (ac, bc)_p.
\]
By Hilbert’s product formula we then obtain
\[
\prod_{p \mid r} (ac, bc)_p = \prod_{p \text{ prime} \atop p \not\mid 2} (ac, bc)_p = (ac, bc)_2.
\]
Here we implicitly used our assumption that \( r > 1 \). This concludes the proof. \( \square \)

**Lemma 4.4.** Assume that \( a, b, c \) are as in Lemma 4.3 and that \( abc \) is square-free. Then the sum over \( \delta \) in Lemma 4.3 equals
\[
\sum_{\delta, \tilde{\delta} \in \mathbb{N}^3: \delta_1 \delta_2 \delta_3 \neq 1, \tilde{\delta}_1 \tilde{\delta}_2 \tilde{\delta}_3 \neq 1, \delta_1 \delta_1 = a_{\text{odd}}, \delta_2 \delta_2 = b_{\text{odd}}, \delta_3 \delta_3 = c_{\text{odd}}} \left( \frac{bc}{\delta_1} \right) \left( \frac{ac}{\delta_2} \right) \left( \frac{-ab}{\delta_3} \right).
\]

**Proof.** Since \( a, b, c \) are pairwise coprime we can write every \( \delta \mid abc \) uniquely as \( \delta_1 \delta_2 \delta_3 \), where
\[
\delta_1 \mid a_{\text{odd}}, \delta_2 \mid \delta_1 \text{ odd} \quad \text{and} \quad \delta_3 \mid c_{\text{odd}}.
\]
Defining \( \tilde{\delta}_1 = a_{\text{odd}}/\delta_1 \) and similarly for \( \tilde{\delta}_2, \tilde{\delta}_3 \), we can then write the sum as
\[
\sum_{\delta, \tilde{\delta} \in \mathbb{N}^3: \delta_1 \delta_2 \delta_3 \neq 1, \tilde{\delta}_1 \tilde{\delta}_2 \tilde{\delta}_3 \neq 1, \delta_1 \tilde{\delta}_1 = a_{\text{odd}}, \delta_2 \tilde{\delta}_2 = b_{\text{odd}}, \delta_3 \tilde{\delta}_3 = c_{\text{odd}}} \prod_{p \mid \delta_1 \delta_2 \delta_3} (ac, bc)_p.
\]
For any integers \( s, t \) and an odd prime \( p \) with \( v_p(s) = 1, v_p(t) = 0 \) we can write \((s, t)_p = (\frac{s}{p})\), where \((\cdot)_p\) is the Jacobi symbol. Using that \( \delta_1 \delta_2 \delta_3 \) is square-free we get
\[
\prod_{p \mid \delta_1} (ac, bc)_p = \left( \frac{bc}{\delta_1} \right) \quad \text{and} \quad \prod_{p \mid \delta_2} (ac, bc)_p = \left( \frac{ac}{\delta_2} \right).
\]
Recalling that \((s, t)_p = (\frac{stp^{-2}}{p})\) for \( s, t \in \mathbb{Z}, p \neq 2 \) with \( v_p(s) = 1 = v_p(t) \), yields
\[
\prod_{p \mid \delta_3} (ac, bc)_p = \left( \frac{-ab}{\delta_3} \right).
\]
\( \square \)

**Lemma 4.5.** Under the assumptions of Theorem 4.1 we have
\[
\mathcal{N}_{b, m}(X) = \frac{2 \mathcal{M}_{b, m}(X)}{\tau((m_{12}m_{13}m_{23})_{\text{odd}})} + \frac{\mathcal{E}_{b, m}(X)}{\tau((m_{12}m_{13}m_{23})_{\text{odd}})} + O(1),
\]
where the implied constant is absolute,
\[
\mathcal{M}_{b, m}(X) := \sum_{\mathbf{n} \in \mathbb{N}^3, (n_{11}n_{21}n_{31}m_{12}m_{23}m_{13})_{2} = 1} \frac{\mu^2(n_{11}n_{21}n_{31})}{\tau((n_{11}n_{21}n_{31})_{\text{odd}})}. 
\]
and $\mathcal{E}_{b,m}(X)$ is given by

$$
\sum_{\substack{h, h' \in \mathbb{N}^3 \quad \sigma \in \{0,1\}^3 \quad d, d' \in \mathbb{N} \quad \delta \notin X^2 - \sigma_i}} \sum_{d, d' \leq X, 2^\sigma_i} \left( \frac{-1}{d_3m_{12}} \right) \frac{\mu^2(2d_1d_2d_3d_1d_2d_3)}{\tau(d_1d_2d_3d_1d_2d_3)} \times \left( \frac{2^{\sigma_2 + \sigma_3}d_2d_3d_5m_{12}m_{23}}{d_1h_{23}} \right) ^{2^{\sigma_3 + \sigma_3}d_1d_3d_3m_{12}m_{23}} \left( \frac{2^{\sigma_1 + \sigma_2}d_1d_3d_5m_{12}m_{23}}{d_2h_{13}} \right) \left( \frac{2^{\sigma_1 + \sigma_2}d_1d_3d_5m_{12}m_{23}}{d_3h_{12}} \right),
$$

where

$$
\begin{align*}
\sigma_1 + \sigma_2 + \sigma_3 &\leq 1, \quad \gcd(2^{\sigma_1 + \sigma_2 + \sigma_3}, m_{12}m_{13}m_{23}) = 1, \\
\gcd(2^{\sigma_1}, b_2, b_3) &= \gcd(2^{\sigma_2}, b_1, b_3) = \gcd(2^{\sigma_3}, b_1, b_2) = 1,
\end{align*}
$$

and

$$
\begin{align*}
&d_1d_2d_3h_{12}h_{23} \neq 1, \quad \tilde{d}_1d_2d_3h_{12}h_{13}h_{23} \neq 1, \\
&(2^{\sigma_1 + \sigma_2}d_1d_3d_5m_{12}m_{23}, 2^{\sigma_2 + \sigma_3}d_2d_3d_5m_{12}m_{23}) = 1,
\end{align*}
$$

$$
\begin{align*}
&\gcd(d_1d_1d_2d_3d_3, m_{12}m_{13}m_{23}) = 1.
\end{align*}
$$

**Proof.** To employ Lemma 4.3 with $a = n_1m_{23}, b = n_2m_{13}, c = n_3m_{12}$ we must ensure that $n_1n_2n_3$ is divisible by a prime $p > 2$. Owing to the term $\mu^2(n_1n_2n_3)$ in $\mathcal{N}_{b,m}(X)$, this is equivalent to $n_1n_2n_3 > 2$. Hence, ignoring all terms with $1 \leq n_1n_2n_3 \leq 2$ shows that $\mathcal{N}_{b,m}(X)$ equals

$$
\sum_{n \in \mathbb{N}^3, n_1n_2n_3 > 2, (n_1n_3m_{12}m_{23}, n_2n_3m_{13}m_{23}) \neq 1} \frac{\mu^2(n_1n_2n_3)}{\tau((n_1n_2n_3m_{12}m_{13}m_{23})_{\text{odd}})} \left( 2 + \sum_{\delta \in \mathbb{N}^3 \setminus \{1\}, \delta_1 = (n_1m_{23})_{\text{odd}}, \delta_2 = (n_2m_{13})_{\text{odd}}, \delta_3 = (n_3m_{12})_{\text{odd}}} \frac{\theta(\delta)}{\tau(\delta)} \right)
$$

up to $O(1)$, where

$$
\theta(\delta) = \left( \frac{n_2n_3m_{12}m_{13}}{\delta_1} \right) \left( \frac{n_1m_2n_3m_{12}}{\delta_2} \right) \left( \frac{-n_1m_2n_2m_{13}}{\delta_3} \right).
$$

We can ignore the condition $n_1n_2n_3 > 2$ at the cost of an error term that is $O(1)$. The factor 2 in the brackets then gives rise to $\mathcal{M}_{b,m}(X)$. To analyse the term containing $\theta$ we use the assumption that $n_i$ is coprime to $m_{jk}$ to write $\delta_i = d_ih_{jk}$ and $\tilde{d}_i = \tilde{d}_ih_{jk}$ for some $d_i, \tilde{d}_i, h_{jk}, \tilde{h}_{jk} \in \mathbb{N}$ satisfying

$$
d_i\tilde{d}_i = (n_i)_{\text{odd}} \quad \text{and} \quad h_{jk}\tilde{h}_{jk} = (m_{jk})_{\text{odd}}.
$$

Defining $\sigma_i = v_2(n_i)$ and using $n_i = 2^{\sigma_i}d_i\tilde{d}_i$ we infer that $\theta$ equals

$$
\left( \frac{-1}{d_3m_{12}} \right) \prod_{\{i,j,k\} = \{1,2,3\}} \left( \frac{2^{\sigma_j + \sigma_k}d_j\tilde{d}_j d_k\tilde{d}_k m_{ij} m_{jk}}{d_i h_{jk}} \right),
$$

which concludes the proof. \qed
4.3. Asymptotic averages of 3-dimensional arithmetic functions. We establish asymptotics for sums of the form

\[ \sum_{n \in \mathbb{N}^3} \mu^2(n_1n_2n_3)g_1(n_1)g_2(n_2)g_3(n_3), \]

where \( g_i \) are certain multiplicative functions that essentially behave like the inverse of the divisor function. There are multidimensional generalisations of the Selberg–Delange theory under the condition that the underlying Dirichlet series has poles of integral order [13]. In our application the series behaves as \( (\zeta(s_1)\zeta(s_2)\zeta(s_3))^{1/2} \), where \( \zeta \) denotes the Riemann zeta function. The poles of this series are half-integers orders, thus we cannot use the existing theory. We shall instead use the convolution identity \( \mu^2(n) = \sum_{d|n} \mu(d) \) to reduce to the 1-dimensional Selberg–Delange setting.

**Lemma 4.6.** Assume that \( g_1, g_2, g_3 : \mathbb{N} \to \mathbb{C} \) are multiplicative function with \( |g_i(n)| \leq 1 \) for all \( n \) and \( i \). Then for all \( X_1, X_2, X_3 > 1 \), \( 1 \leq z \leq (X_1X_2X_3)^{1/4} \) and \( q \in \mathbb{N}^3, a \in \mathbb{Z}^3 \) with \( \gcd(a_i, q_i) = 1 \) for all \( i \), we have

\[
\sum_{k \in \mathbb{N} \cap [1,z]} \mu(k) \sum_{n \in \prod_{i=1}^{3} (\mathbb{N} \cap [1,X_i]), k^2|n_1n_2n_3} g_1(n_1)g_2(n_2)g_3(n_3) = \sum_{n \in \prod_{i=1}^{3} (\mathbb{N} \cap [1,X_i])} \mu^2(n_1n_2n_3)g_1(n_1)g_2(n_2)g_3(n_3) + O \left( \frac{X_1X_2X_3}{z^{1/2}} \left( \log(X_1X_2X_3) \right)^3 \right),
\]

where the implied constant is absolute.

**Proof.** Using \( \mu^2(n_1n_2n_3) = \sum_{k^2|n_1n_2n_3} \mu(k) \) we can write the sum in the lemma as

\[
\sum_{k \in \mathbb{N}} \mu(k) \sum_{n \in \prod_{i=1}^{3} (\mathbb{N} \cap [1,X_i])} g_1(n_1)g_2(n_2)g_3(n_3).
\]

The contribution of \( k > z \) is trivially

\[
\ll \sum_{k > z} t \sum_{k^2|t} \tau_3(t) = \sum_{k > z} s \sum_{s \leq X_1X_2X_3/k^2} \tau_3(k^2s),
\]

where \( \tau_3(t) \) denotes the number of ways that \( t \) can be written as a product of three positive integers. Using the inequality \( \tau_3(ab) \leq \tau^2(ab) \leq \tau^2(a)\tau^2(b) \), valid for all \( a, b \in \mathbb{N} \), we obtain the bound

\[
\sum_{k > z} \tau^2(k^2) \sum_{s \leq X_1X_2X_3/k^2} \tau^2(s) \ll \sum_{k > z} \tau^2(k^2) \frac{X_1X_2X_3}{k^2} \left( \log(X_1X_2X_3) \right)^3,
\]

where we used the fact that the average of \( \tau^2(s) \) is \( \log^3 s \). The proof now concludes by using and the bound \( \tau^2(k^2) \ll k^{1/2} \).

**Lemma 4.7.** For all \( k \in \mathbb{N} \) and \( \varepsilon > 0 \) we have

\[
\sum_{n \in \mathbb{N} \setminus k^2 \mathbb{N}} n^{-1/2-\varepsilon} \ll_{\varepsilon} k^{-1-\varepsilon},
\]

where the implied constant depends at most on \( \varepsilon \).
Proof. Letting \( n = k^2m \) makes the sum equal to
\[
\sum_{m \in \mathbb{N}} \frac{k^{-1-2\epsilon} - 1}{m^{1/2+\epsilon}} \leq \sum_{m \in \mathbb{N}} \frac{k^{-1-2\epsilon}}{m^{1/2}} \leq \prod_{p|k} \frac{1}{1 - 1/\sqrt{p}} \leq k^{-1-2\epsilon} \prod_{p|k} \frac{1}{1 - 1/\sqrt{p}}
\]
Noting that \( 1 - 1/\sqrt{2} \geq 1/8 \) and that \( 8^{\tau(k)} \leq \tau(k)^3 \ll k^\epsilon \) concludes the proof. \( \square \)

**Lemma 4.8.** Keep the setting of Lemma 4.4. Then for all \( w \geq 1 \) we have
\[
\sum_{k \in \mathbb{N} \cap [1, \epsilon]} \mu(k) \sum_{n' \in (\mathbb{N} \cap [1, w])^3 : p|n', n' \not= p|k} \sum_{\gcd(n', q_i) = 1} g_1(n') g_2(n') g_3(n') \prod_{i=1}^{3} \left( \sum_{n'' \in \mathbb{N} \cap [1, X/n_i], \gcd(n'', q_i) = 1} g_i(n'') \right)
\]
where the implied constant is absolute.

**Proof.** Write each \( n_i \) as \( n'_i n''_i \), where \( n''_i \) is coprime to \( k \) and all prime factors of \( n'_i \) divide \( k \). The sum over \( k \) in Lemma 4.4 then becomes
\[
\sum_{k \in \mathbb{N} \cap [1, \epsilon]} \mu(k) \sum_{n' \in \mathbb{N}^3, k|n'_i, \gcd(n'_i, q_i) = 1} \sum_{n'' \in \mathbb{N} \cap [1, X/n'_i], \gcd(n'', q_i) = 1} g_1(n'_i) g_2(n'_i) g_3(n'_i) \prod_{i=1}^{3} \sum_{n'' \in \mathbb{N} \cap [1, X/n_i], \gcd(n'', q_i) = 1} g_i(n'').
\]

The contribution of terms with \( \max_i n'_i > w \) is
\[
\ll \sum_{k \in \mathbb{N} \cap [1, \epsilon]} \sum_{n' \in \mathbb{N}^3, k|n'_i, \gcd(n'_i, q_i) = 1} \frac{X^3}{n'_i n''_i} \ll \sum_{k \in \mathbb{N} \cap [1, \epsilon]} \sum_{n \in \mathbb{N} \cap [1, k^2 n], \gcd(n, p|k) \not= p|k} \frac{\tau_3(n)}{n} \ll \sum_{k \in \mathbb{N} \cap [1, \epsilon]} \sum_{n \in \mathbb{N} \cap [1, k^2 n], \gcd(n, p|k) \not= p|k} \frac{\tau_3(n)}{n^{2/3}}.
\]
The inequality \( \tau_3(n) \leq \tau^2(n) \ll n^{1/12} \) and Lemma 4.7 with \( \epsilon = 1/12 \) show that the sum over \( k \) is bounded. \( \square \)

We define
\[
t_0 = \frac{1}{\sqrt{\pi}} \prod_{p \text{ prime}} t_p \left( 1 - \frac{1}{p} \right)^{1/2}, \quad t_p := 1 + \sum_{k=1}^{\infty} \frac{1}{(k+1)p^k}.
\]

**Lemma 4.9.** Let \( a, d \) be odd integers and \( q \in \{4, 8\} \). Fix any \( C > 0 \). For all \( x \geq 2 \) we have
\[
\sum_{1 \leq n \leq x, \gcd(n, d) = 1} \frac{1}{\phi(q)(\prod_{p|d} t_p)} \frac{1}{\sqrt{\log x}} \left\{ 1 + O \left( \frac{[\log \log 3d]^3/2}{\log x} \right) \right\} + O \left( \frac{\tau(d)x}{(\log x)^C} \right),
\]
where the implied constant depends at most on \( C \).

**Proof.** A straightforward argument that uses the zero-free region for Dirichlet \( L \)-functions and a Hankel contour integral, as in the proof of [35, Lemma 1], for example, shows that
\[
\sum_{1 \leq n \leq x, \gcd(n, d) = 1} \frac{\chi(n)}{\tau(n)} = t_0 I (\chi = \chi_0) \frac{1}{\prod_{p|d} t_p} \frac{1}{\sqrt{\log x}} \left\{ 1 + O \left( \frac{[\log \log 3dq]^3/2}{\log x} \right) \right\} + O \left( \frac{\tau(d)x}{(\log x)^C} \right)
\]
holds for each Dirichlet character \( \chi \) modulo \( q \), where \( \chi_0 \) is the principal character. Using orthogonality of characters concludes the proof. \( \square \)
For \( d \in \mathbb{Z} \setminus \{0\} \) we define

\[
\gamma(d) = \prod_{p \nmid 2} \left( 1 - \frac{1}{p} \right)^{3/2} \left( 1 + \frac{\# \{ 1 \leq i \leq 3 : p \nmid d_i \} }{2p} \right).
\]  

(4.11)

**Lemma 4.10.** Assume that \( a_1, a_2, a_3, d_1, d_2, d_3 \) are odd integers and that \( q_1, q_2, q_3 \in \{4, 8\}^3 \). Fix any \( C > 0 \). Then for all \( X \in \mathbb{R}^3 \) with \( \min_i X_i \geq \max_i |d_i| \) we have

\[
\sum_{n \in \prod_{i=1}^{3} \mathbb{N}^*} \frac{\mu^2(n_1 n_2 n_3) \tau(n_1) \tau(n_2) \tau(n_3)}{\tau(n_1) \tau(n_2) \tau(n_3)} = \gamma(d) \left( \prod_{i \equiv 1}^{3} \phi(q_i) \sqrt{\log X_i} \right)
+ O \left( \frac{X_1 X_2 X_3 (\log \log \max |d_i|)^{3/2}}{(\prod_{i=1}^{3} (\log X_i))^{1/2} (\log \min X_i)} + \frac{X_1 X_2 X_3 \tau(d_1) \tau(d_2) \tau(d_3)}{(\log(\max X_i))^{2/3} (\log \min X_i)^{3C}} \right),
\]

where the implied constant depends at most on \( C \).

**Proof.** Fix any \( C > 4 \). We employ Lemmas 4.6, 4.8 with

\[
g_i(n) = \frac{1 (\gcd(n, d_i) = 1)}{\tau(n)}, \ z = w = (\log \min X_i)^{3C}.
\]

This shows that the sum over \( n \) in our lemma equals

\[
\sum_{k \in \mathbb{N}^*} \frac{1}{\tau(n)} \left( \sum_{n' \in \prod_{i=1}^{3} \mathbb{N}^*} \frac{1}{\tau(n'')} \right)^{3/2} \sum_{n'' \in \prod_{i=1}^{3} \mathbb{N}^*} \frac{\mu^2(n_1 n_2 n_3) \tau(n_1) \tau(n_2) \tau(n_3)}{(\log(\max X_i))^{3/2} (\log \min X_i)^{3C}}
\]

up to an error term of size

\[
\ll \frac{X_1 X_2 X_3 (\log(\max X_i))^{3/2}}{\tau(n)} + \frac{X_1 X_2 X_3 (\log(\max X_i))^{3/2}}{(\log \min X_i)^{3C}}.
\]

We now apply Lemma 4.9 to each of the sums over \( n'' \). When estimating the sum for \( i = 3 \) we introduce an error term originating in \( \tau(d_1 x (\log x)^{-C} \) of Lemma 4.9. This contribution is

\[
\ll \sum_{k \in \mathbb{N}^*} \frac{X_1 X_2 \tau(d_3 k) X_3 \tau(d_3)}{(\log(\min X_i))^C} \sum_{n' \in \prod_{i=1}^{3} \mathbb{N}^*} \frac{\mu^2(n_1 n_2 n_3) \tau(n_1) \tau(n_2) \tau(n_3)}{(\log(\max X_i))^C} \sum_{m \in \mathbb{N}} \frac{\tau(m) \tau_3(m)}{m} \sum_{p \mid m} \frac{\tau(p)}{\min(p, 2k)}
\]

This is satisfactory since one can use \( \tau(k) \tau(m) \tau_3(m) \ll (km)^{1/4} \) and Lemma 4.7 with \( \epsilon = 1/4 \). Similar arguments provide the same error term contribution for \( i \in \{1, 2\} \). This process leaves us with the main term

\[
\sum_{k \in \mathbb{N}^*} \frac{\mu^2(n_1 n_2 n_3) \tau(n_1) \tau(n_2) \tau(n_3)}{\tau(n_1) \tau(n_2) \tau(n_3)} \times \prod_{i=1}^{3} \frac{1}{\phi(q_i) \sqrt{\log(X_i/n'_i)}} \left( 1 + O \left( \frac{(\log 3k|d_i|)^{3/2}}{\log(X_i/n'_i)} \right) \right),
\]

where we used the fact that since each \( q_i \) is even, the conditions \( \gcd(n'_i, 2) = 1 \) are equivalent to \( 2 \nmid k \). Note that when \( n'_i \leq w = (\log X_i)^{3C} \) one has

\[
\frac{1}{\log(X_i/n'_i)} = \frac{1}{\log X_i} \left( 1 - \frac{\log n'_i}{\log X_i} \right) = \frac{1}{\log X_i} \left( 1 + O_C \left( \frac{\log n'_i}{\log X_i} \right) \right).
\]
Thus, the contribution of \( E \) where \( \tau \) which is satisfactory due to \( k \) and similarly

\[
\sum_{n \in \mathbb{Z}[1,w]} \frac{1}{n} \left( \frac{\log n \tau}{\log X_i} \right) = \frac{1}{\sqrt{\log(X_i/n')}} \left( 1 + O_C \left( \frac{\log n'}{\log X_i} \right) \right).
\]

Hence, rearranging the order of summation gives

\[
\frac{t_0^3}{t_2^3} \left\{ \prod_{i=1}^{3} \left( \frac{X_i}{\Phi(d_i)} \right) \right\} \sum_{k \in \mathbb{N} \cap [1,z]} \frac{\mu(k) \xi(k)}{\prod_{i=1}^{3} \left( \Pi_{p|k} t_p \right)}
\]

where \( \xi(k) \) is given by

\[
\sum_{n' \in (\mathbb{N} \cap [1,w])^3} \frac{1}{\tau(n') \prod_{i=1}^{3} 1^{\frac{1}{\tau(n') \prod_{i=1}^{3}}} \left( 1 + O \left( \frac{E_1 + E_2}{\log \min_x X_i} \right) \right)}.
\]

where \( E_1 = \log(n'_1 n'_2 n'_3) \) and \( E_2 = (\log \log(3k|d_1 d_2 d_3)|)^{3/2} \).

The contribution of \( E_1 \) towards the sum over \( k \) is

\[
\ll \frac{1}{\log \min_x X_i} \sum_{k \in \mathbb{N}} \sum_{n' \in [k^2]} \frac{\tau_3(n) \log n}{n},
\]

which is satisfactory as can be seen by using \( \tau_3(n) \log n \ll n^{1/4} \) and Lemma 4.7 with \( \epsilon = 1/4 \).

When both \( k, d \in \mathbb{N} \) are large enough, we have \( \log(kd) \ll 2 \log \max\{k, d\} \), hence,

\[
(\log \log(kd))^{3/2} \ll (\log k)^{3/2} + (\log d)^{3/2} \ll k^{1/4}(\log \log d)^{3/2}.
\]

Thus, the contribution of \( E_2 \) towards the sum over \( k \) is

\[
\ll \frac{(\log \log(d_1 d_2 d_3))^{3/2}}{\log \min_x X_i} \sum_{k \in \mathbb{N}} \sum_{n' \in [k^2]} \frac{\tau_3(n)}{n},
\]

which is satisfactory due to \( \tau_3(n) \ll n^{1/8} \) and taking \( \epsilon = 3/8 \) in Lemma 4.7.

We are now left with estimating

\[
\frac{t_0^3}{t_2^3} \left\{ \prod_{i=1}^{3} \left( \frac{1}{\Pi_{p|k} t_p} \right) \right\} \sum_{1 \leq k \leq z} \prod_{i=1}^{3} \left( \Pi_{p|k} t_p \right) \frac{\mu(k)}{\prod_{i=1}^{3} \left( \Pi_{p|k} t_p \right)}
\]

Ignoring the conditions \( k \leq z \) and \( \max n' \leq w \) can be done at the cost of negligible error terms using arguments similar to the ones in the proof of Lemmas 4.6.8. We obtain

\[
\sum_{k \in \mathbb{N} \cap [1,z]} \frac{\mu(k)}{\prod_{i=1}^{3} \left( \Pi_{p|k} t_p \right)} \sum_{m \in \mathbb{N}} \frac{1}{m} \sum_{n' \in [m]} \frac{1}{\tau(n') \prod_{i=1}^{3} 1^{\frac{1}{\tau(n') \prod_{i=1}^{3}}} \cdot \prod_{i=1}^{3} 1^{\frac{1}{\tau(n')}}.
\]

The sum over \( m \) equals \( \Pi_{p|k} c_p \), where \( c_p \) is given by

\[
\sum_{j=2}^{\infty} \frac{1}{p^j} \sum_{a \in (\mathbb{Z}/p^j \mathbb{Z})^3} \frac{1}{(1 + a_j)} = t_p^{\frac{1}{p} \{1 < i \leq 3 : p \nmid d_i \}} - \frac{1}{2p} \{1 < i \leq 3 : p \nmid d_i \}.
\]

hence, the sum over \( k \) equals

\[
\prod_{p \nmid 2} \left( 1 - \frac{c_p}{t_p^{\frac{1}{p} \{1 < i \leq 3 : p \nmid d_i \}}} \right) = \prod_{p \nmid 2} \left( 1 - \frac{1}{t_p^{\frac{1}{p} \{1 < i \leq 3 : p \nmid d_i \}}} \right) \left( 1 + \frac{1}{2p} \{1 < i \leq 3 : p \nmid d_i \} \right).
\]
Thus, we obtain the main term
\[
\left\{ \prod_{p \neq 2}^{1} \frac{1}{\tau_{p}(1 \leq i \leq 3 \mid p \mid d_{i})} \right\} \left( 1 + \sum_{1 \leq i \leq 3 : \ n \mid d_{i}} \frac{1}{2p} \right) \left\{ \frac{\tau_{d}}{\prod_{p \mid d_{i}}^{1} \tau_{p}} \right\} = \frac{\gamma(d)}{(2\pi)^{3/2}},
\]
where we used that each $d_{i}$ is odd. This completes the proof. \(\square\)

4.4. The main term in Theorem 4.1

Recall the definition of $\mathcal{M}_{b,m}$ in Lemma 4.1.

**Lemma 4.11.** Under the assumptions of Theorem 4.1 we have
\[
\mathcal{M}_{b,m}(X) = \frac{\beta(b,m) c(b,m)}{(2\pi)^{3/2}} \left( \prod_{i=1}^{3} \frac{X_{i}}{\log X_{i}} \right) + O \left( \frac{X_{1}X_{2}X_{3}(\log \log 3B)^{3/2}}{(\log X_{1})^{5/2}} \right).
\]

**Proof.** We first assume that $2 \mid m_{23}$. By (4.4) we see that $n_{1}n_{2}n_{3}m_{13}m_{13}$ in $\mathcal{M}_{b,m}$ is odd, hence, Lemma 3.12 gives
\[
\mathcal{M}_{b,m}(X) = \sum_{a \in A, n \equiv a \pmod{8}} \sum_{b \equiv 1 \pmod{8}} \frac{\mu^{2}(n_{1}n_{2}n_{3})}{\tau(n_{1}n_{2}n_{3})},
\]
where $A := \{ a \in (\mathbb{Z}/8\mathbb{Z})^{3} : a_{0}m_{13} = a_{3}m_{12} \text{ or } a_{1}m_{23} + a_{2}m_{13} = a_{3}m_{12} \}$. Note that the condition $n \equiv a \pmod{8}$ implies that each $n_{i}$ is odd, thus, we can equivalently express (4.4) as $\gcd(n_{i}, d_{i}) = 1$, where $d_{i}$ are respectively the odd parts of $m_{12}m_{13}m_{23}\gcd(b_{2}, b_{3}), m_{12}m_{13}m_{23}\gcd(b_{1}, b_{3}), m_{12}m_{13}m_{23}\gcd(b_{1}, b_{3})$.

With these values of $d_{i}$ and with $q_{1} = q_{2} = q_{3} = 8$ we apply Lemma 4.10 to infer that in the setting of Theorem 4.1 one has
\[
\mathcal{M}_{b,m}(X) = \frac{\beta(b,m)}{(2\pi)^{3/2}} \left( \prod_{i=1}^{3} \frac{X_{i}}{\log X_{i}} \right) + O \left( \frac{X_{1}X_{2}X_{3}(\log \log 3B)^{3/2}}{(\log X_{1})^{5/2}} \right),
\]
with an implied constant depending at most on $C$. By (4.7) we can replace each $\min X_{i}$ and $\max X_{i}$ with $X_{i}$ by allowing the implied constants to depend on $\eta$. We have $\tau(d_{i}) \leq d_{i} \leq (\max \log X_{i})^{5/4}$ by (4.6), hence, taking $C$ large enough compared to $A$ leads to an error term that agrees with the one claimed in Theorem 4.1. The proof in the case $2 \mid m_{23}$ concludes by noting that $A$ has 32 elements. The proof in the case $2 \nmid m_{12}m_{13}$ is similar; in each such case one obtains a set $A$ that has 32 elements again, therefore, the resulting leading constants is the same.

It remains to consider the case $2 \nmid m_{12}m_{13}m_{23}$. Using that $n_{1}n_{2}n_{3}m_{12}m_{13}m_{23}$ is square-free, we can write $\mathcal{M}_{b,m}(X)$ as $\sum_{i=0}^{3} \mathcal{M}_{i}$, where
\[
\mathcal{M}_{0} := \sum_{n \in \mathbb{N}^{3}, n_{1} \equiv 1 \pmod{2}, n_{2} \equiv 1 \pmod{2}, n_{3} \equiv 1 \pmod{2}} \frac{\mu^{2}(n_{1}n_{2}n_{3})}{\tau(n_{1}n_{2}n_{3})}
\]
and
\[
\mathcal{M}_{i} := \sum_{n \in \mathbb{N}^{3}, n_{1} \equiv 1 \pmod{2}, n_{2} \equiv 0 \pmod{2}, n_{3} \equiv 1 \pmod{2}} \frac{\mu^{2}(n_{1}n_{2}n_{3})}{\tau(n_{1}n_{2}n_{3})}, \quad (1 \leq i \leq 3).
\]
Note that $\mathcal{M}_{1}$ is non-zero only when $2 \mid \gcd(b_{2}, b_{3})$ (analogously for $\mathcal{M}_{2}$ and $\mathcal{M}_{3}$). Hence
\[
\mathcal{M}_{1} = \left( 2 \mid \gcd(b_{2}, b_{3}) \right) \sum_{n \in \mathbb{N}^{3}, n_{1} \equiv 1 \pmod{2}, n_{2} \equiv 0 \pmod{2}, n_{3} \equiv 1 \pmod{2}} \frac{\mu^{2}(n_{1}n_{2}n_{3})}{\tau(n_{1}n_{2}n_{3})}.
\]
By Lemma 3.12 the sum over \( n \) equals

\[
\sum_{a \in A'} \sum_{n_1 \leq X_1/2, n_2 \leq X_2, n_3 \leq X_3, n \equiv a \pmod{8}} \mu^2(n_1 n_2 n_3) / \tau(n_1 n_2 n_3),
\]

where \( A' := \{ a \in ((\mathbb{Z}/8\mathbb{Z})^3 : a_2 m_{13} = a_3 m_{12} \text{ or } 2a_1 m_{23} + a_2 m_{13} = a_3 m_{12} \} \). Invoking Lemma 4.10 as above leads to

\[
\sum_{i < j} \beta(b, m) \cdot \{ 1 / 2 \prod_{i=1}^{3} X_i / \sqrt{\log X_i} \} + O_{A, \eta} \left( \frac{X_1 X_2 X_3 (\log \log B)^{3/2}}{(\log X_1)^{5/2}} \right).
\]

One checks that \( \#A' = 32 \). A similar argument deals with \( M_2 \) and \( M_3 \), resulting in

\[
\sum_{i=1}^{3} M_i = \# \{ i < j : 2 \mid (b_i, b_j) \} \beta(b, m) \cdot \{ 3 / 4 \prod_{i=1}^{3} X_i / \sqrt{\log X_i} \} + O \left( \frac{X_1 X_2 X_3 (\log \log B)^{3/2}}{(\log X_1)^{5/2}} \right).
\]

Finally, following a facsimile argument shows that \( M_0 \) equals

\[
\#A'' \beta(b, m) \cdot \{ 3 / 2 \prod_{i=1}^{3} X_i / \sqrt{\log X_i} \} + O \left( \frac{X_1 X_2 X_3 (\log \log B)^{3/2}}{(\log X_1)^{5/2}} \right),
\]

where \( A'' := \{ a \in ((\mathbb{Z}/4\mathbb{Z})^3 : a_2 m_{13} = a_3 m_{12} \text{ or } a_1 m_{23} = a_3 m_{12} \text{ or } a_1 m_{23} + a_2 m_{13} = 0 \} \).

This set has 6 elements, an observation that concludes the proof of Lemma 4.11. \( \square \)

### 4.5. The error term in Theorem 4.1: large conductors.

Recall the function \( E_{b,m}(X) \) defined in Lemma 4.3. Our primary focus in this section will be to bound this function. The variables \( d \) and \( h \) may have an adversely large size for the subsequent analytic arguments, thus we start by using the (necessarily small) “dual” variables \( d', h' \). In particular, let \( \sigma_{ij} = v_2(m_{ij}) \) and substitute \( m_{ij} = 2^{\sigma_i} h_i h_j \) to obtain

\[
\left( \frac{2^{\sigma_2+\sigma_3} d_2 d_3 d_2 d_3 \hat{m}_{12} m_{13}}{d_1 h_23} \right) = \prod_{i=2,3} \left( \frac{d_i}{d_1} \right) \left( \frac{h_{1i}}{h_i} \right) \left( \frac{h_{1i}}{d_i} \right)
\]

\[
\times \left( \frac{2^{\sigma_2+\sigma_3+\sigma_12} h_23}{d_2 d_3 h_{23}} \right) \prod_{i=2,3} \left( \frac{d_i}{h_23} \right) \left( \frac{h_{1i}}{h_{23}} \right) \left( \frac{h_{1i}}{h_23} \right).
\]

We can similarly obtain analogous expressions for the other Jacobi symbols in the right hand side of the equation in Lemma 4.5. Putting together these expressions will give various terms of the form \( \binom{m}{n} \binom{m}{n} \) where \( n, m \) are odd positive integers. Using the reciprocity law these terms can then be written as \( (-1)^{(n-1)(m-1)} \) and one will thus get the following expression for the product of the Jacobi symbols in the definition of \( E_{b,m} \):

\[
(-1)^{G(b)+G(d)} \left( \frac{2}{d_1 h_{23}} \right)^{\sigma_{21}+\sigma_{23}} \left( \frac{2}{d_2 h_{13}} \right)^{\sigma_{12}+\sigma_{13}} \left( \frac{2}{d_3 h_{12}} \right)^{\sigma_{23}+\sigma_{12}}
\]

\[
\times \left( \frac{d_2 d_3 h_{12} h_{13}}{d_1 h_{23}} \right) \left( \frac{d_1 d_2 h_{12} h_{13}}{d_2 h_{13}} \right) \left( \frac{-d_1 d_2 h_{23} h_{13}}{d_3 h_{12}} \right),
\]

36
where \( \sigma = \sigma_1 + \sigma_2 + \sigma_3 + \sigma_{12} + \sigma_{13} + \sigma_{23} \) and

\[
G(h) := (h_{12} - 1)(h_{13} - 1) + (h_{12} - 1)(h_{23} - 1) + (h_{13} - 1)(h_{23} - 1),
\]

\[
G_h(d) := (d_{11} - 1)(d_{21} - 1) + (d_{11} - 1)(d_{31} - 1) + (d_{21} - 1)(d_{31} - 1)
\]

\[
+ (d_{11} - 1)(h_{12} - 1) + (d_{11} - 1)(h_{13} - 1)
\]

\[
+ (d_{21} - 1)(h_{12} - 1) + (d_{21} - 1)(h_{23} - 1)
\]

\[
+ (d_{31} - 1)(h_{13} - 1) + (d_{31} - 1)(h_{23} - 1).
\]

Injecting this into Lemma 4.1 and then into \( \mathcal{E}_{b,m}(X) \) gives the following result.

**Lemma 4.12.** For all \( X_1, X_2, X_3 \geq 2 \) and \( b, m \) as in Theorem 4.1, the quantity \( \mathcal{E}_{b,m}(X) \) equals

\[
\sum_{\sigma \in \{0,1\}^3} \sum_{\mathbf{h} \in \mathbb{N}^3} (-1)^{G_h(\mathbf{d})} \mathcal{E}_{b,\mathbf{h}}(X) \left( \frac{2}{h_{23}} \right)^{\sigma - \sigma_{23}} \left( \frac{2}{h_{13}} \right)^{\sigma - \sigma_{13}} \left( \frac{2}{h_{12}} \right)^{\sigma - \sigma_{12}},
\]

where

\[
\mathcal{E}_{b,\mathbf{h}}(X) = \sum_{d,d_1,d_2,d_3 \in \mathbb{N}} \frac{(-1)^{G_h(\mathbf{d})}}{\tau(d_1d_2d_3d_4)} \left( \frac{2}{d_1} \right)^{\sigma - \sigma_{12}} \left( \frac{2}{d_2} \right)^{\sigma - \sigma_{23}} \left( \frac{2}{d_3} \right)^{\sigma - \sigma_{13}}
\]

\[
\times \left( \frac{d_2d_3h_{12}h_{13}}{d_1h_{23}} \right) \left( \frac{d_1d_3h_{12}h_{23}}{d_2h_{13}} \right) \left( \frac{-d_1d_2h_{23}h_{13}}{d_3h_{12}} \right),
\]

with

\[
d_1d_2d_3 \leq X_1/2^{\sigma_1}, d_2d_3 \leq X_2/2^{\sigma_2}, d_3d_3 \leq X_3/2^{\sigma_3},
\]

(4.13)

and

\[
\begin{aligned}
&d_1d_2d_3h_{12}h_{13}h_{23} \neq 1, \ d_1d_2d_3h_{12}h_{13}h_{23} \neq 1 \\
&\mu^2(d_1d_2d_3d_4d_5d_6) = 1, \ 2 \nmid d_1d_2d_3d_4d_5d_6, \\
&gcd(d_1d_2d_3d_4d_5d_6, h_{12}h_{13}h_{14}h_{23}h_{24}h_{34}) = 1, \\
&gcd(d_1d_2d_3d_4d_5d_6, b_1b_2b_3) = gcd(d_2d_3b_1b_2b_3) = gcd(d_3d_4d_5d_6, b_1b_2) = 1.
\end{aligned}
\]

(4.14)

**Remark 4.13.** The sums \( \mathcal{E}_{b,\mathbf{h}} \) in Lemma 4.12 may be thought as

\[
\sum_{d,d_1,d_2,d_3 \in \mathbb{N}} \frac{d_2d_3}{d_1} \frac{d_1d_3}{d_2} \frac{d_1d_2}{d_3},
\]

where our variables in the summation are square-free, odd and coprime in pairs. If each \( d_i \) is small, one has averages of quadratic characters of small conductor. We will handle this directly with Selberg–Delange estimates in §4.4, the same approach applies if each \( d_i \) is small instead. The remaining cases occur when both \( d_i \) and \( d_j \) run over long intervals for some \( i \neq j \). In this case the Jacobi symbol \( \left( \frac{d_i}{n} \right) \) in the sum above will give rise to cancellation. To make this precise we use the following bound for bilinear sums taken from the work of Friedlander–Iwaniec [35, Lemma 2], (see also [43, Corollary 4] and [34, §21]): for all \( M, N > 1 \) and all sequences \( a_m, b_n \in \mathbb{C} \) of modulus at most 1 we have

\[
\sum_{1 \leq m \leq M, 1 \leq n \leq N} a_mb_n\mu(2mn)^2 \left( \frac{m}{n} \right) \ll (MN^{5/6} + M^{5/6}N)(\log MN)^2,
\]

(4.15)

where the implied constant is absolute.
Lemma 4.14. Keep the setting of Lemma 4.12 and fix any $C > 0$. The contribution towards $\mathcal{E}_{b,h,h}(X)$ of those $d, d$ for which $\min\{d_i, \tilde{d}_j\} > \min_{i=1,2,3}(\log X_i)^{100C}$ holds for some $i \neq j$ is bounded by $O_C(X_1 X_2 X_3 (\log X_1)^{4-10C})$, where the implied constant depends at most on $C$ and $\eta$.

Proof. Let $Z = \min_{i=1,2,3}(\log X_i)$. By symmetry we can assume $(i, j) = (2, 1)$. Using $d_2 > Z^{100C}$ and (4.13) we see that $\tilde{d}_2 \leq X_2 Z^{-100C}$. Similarly one has $d_1 \leq X_1 Z^{-100C}$. Thus, the contribution is

$$\ll \sum_{d_1 \leq X_1 Z^{-100C}} \sum_{d_2 \leq X_2 Z^{-100C}} a(d_2) b(d_2) \mu^2(d_1) \mu^2(2d_1 d_2) \left( \frac{d_1}{d_2} \right),$$

where $a, b$ are products of factors of the form $(-1)^{G_{n}(\cdot)/4}$, $(2/\cdot)$, and $1/\tau(\cdot)$, multiplied by indicator functions of the coprimality conditions inherited from (4.14). By (4.15) we get the bound

$$\ll \sum_{d_1 \leq X_1 Z^{-100C}} \sum_{d_2 \leq X_2 Z^{-100C}} \left( \frac{X_1}{d_1} \left( \frac{X_2}{d_2} \right)^{\frac{1}{2}} + \left( \frac{X_1}{d_1} \left( \frac{X_2}{d_2} \right)^{\frac{1}{2}} \right) \right) (\log X_1 X_2)^2$$

$$\ll X_1 X_2 X_3 (\log X_1 X_2 X_3)^{4} Z^{-100C/6},$$

owing to $\{d_3, \tilde{d}_3 \in \mathbb{N} : d_3 \tilde{d}_3 \leq X_3\} = \sum_{d \leq X_3} \tau(d) \ll X_3 \log X_3$. This is sufficient due to (4.7). \qed

Lemma 4.15. Keep the setting of Lemma 4.12 and fix any $C > 0$. The contribution towards $\mathcal{E}_{b,h,h}(X)$ of those $d, d$ for which $\max\{d_i, \tilde{d}_j\} \leq \min_{i=1,2,3}(\log X_i)^{100C}$ holds for some $i \neq j$ is bounded by $O_C(X_1 X_2 X_3 (\log X_1)^{4} \min_{i=1,2,3} X_i^{-1/10})$, where the implied constant depends at most on $C$ and $\eta$.

Proof. We can assume that $i = 1$ and $j = 2$ by symmetry. Rearranging the contribution as $\sum_{d_1, \tilde{d}_2, \tilde{d}_3} \sum_{d_2, \tilde{d}_1}$ and using arguments identical to those in the proof of Lemma 4.14 is sufficient. \qed

We now give the outcome of the last two lemmas:

Lemma 4.16. Keep the setting of Lemma 4.12 and fix any $C > 0$. We have

$$\mathcal{E}_{b,h,h}(X) = \sum_{\lambda \in \{1, 2\}} \mathcal{E}_{b,h,h}(X) = O_C \left( X_1 X_2 X_3 (\log X_1)^{4-10C} \right),$$

where

$$\mathcal{E}_{b,h,h}(X) = \sum_{d, \tilde{d} \in \mathbb{N}_0^3} \frac{(-1)^{G_{n}(d)}}{\tau(d_1 d_2 d_3) (\frac{2}{d_1})^{\sigma - \sigma_1 - \sigma_2} (\frac{2}{d_2})^{\sigma - \sigma_2 - \sigma_3} (\frac{2}{d_3})^{\sigma - \sigma_3 - \sigma_1}}$$

$$\times \left( \frac{d_2 d_3 h_{12} h_{13}}{d_1 h_{23}} \right) \left( \frac{d_1 d_3 h_{12} h_{23}}{d_2 h_{13}} \right) \left( \frac{-d_1 d_2 h_{23} h_{13}}{d_3 h_{12}} \right),$$

with

$$\left\{ \begin{array}{ll}
\max\{d_i : 1 \leq i \leq 3\} \leq \min_{i=1,2,3}(\log X_i)^{100C} \leq \min\{\tilde{d}_i : 1 \leq i \leq 3\}, & \text{if } \lambda = 1, \\
\max\{\tilde{d}_i : 1 \leq i \leq 3\} \leq \min_{i=1,2,3}(\log X_i)^{100C} \leq \min\{d_i : 1 \leq i \leq 3\}, & \text{if } \lambda = 2.
\end{array} \right.$$
The implied constant depends at most on $C$ and $\eta$.

**Proof.** Let $L = \min_{i=1,2,3} (\log X_i)^{100C}$. The ranges covered by Lemmas 4.14 and 4.15 are part of the error term of the present lemma. Out of the remaining terms we first consider the contribution of those with $d_1 \leq L$. By Lemma 4.15 one must have $\min\{d_2, d_3\} > L$, which, by Lemma 4.14 proves $\max\{d_2, d_3\} < L$ and therefore $d_1 > L$ holds due to Lemma 4.15. Hence, the condition $d_1 \leq L$ is equivalent to the case $\lambda = 1$ in (4.16). A similar reasoning shows that the condition $d_1 > L$ is equivalent to the case $\lambda = 2$ in (4.16). □

4.6. The error term in Theorem 4.1: small conductors. We will now focus on bounding $\mathcal{E}(\lambda)$ that is defined in Lemma 4.14. As explained in Remark 4.13, this can be done by invoking estimates about averages of arithmetic functions twisted by a Dirichlet character of small conductor.

**Lemma 4.17.** Fix any $C > 0$. Then for any non-zero $d, q, q_0, j \in \mathbb{Z}$ with $\gcd(q_0, jq) = 1$, any non-principal Dirichlet character $\chi \pmod{q}$, any function $f : \mathbb{N} \to \mathbb{C}$ with period $q_0$ and any $X, Y \geq 2$, we have

$$\sum_{Y \leq n \leq X, n \equiv j \pmod{q_0}} \frac{\chi(n)f(n)}{\gcd(n,d)} = O\left(\left(\frac{X}{\log X}\right)^C \tau(d)q_0^2 q \max_{n \in \mathbb{N}} \{|f(n)|\}\right),$$

where the implied constant depends at most on $C$.

**Proof.** First note that it is sufficient to consider the case $Y = 1$. The periodicity of $f$ allows us to write the sum as

$$\sum_{j \pmod{q_0}} f(j) \sum_{1 \leq n \leq X, n \equiv j \pmod{q_0}} \frac{\chi(n)}{\gcd(n,d)} = \tau(d)q_0^2 q \max_{n \in \mathbb{N}} \{|f(n)|\} \sum_{1 \leq n \leq X, n \equiv j \pmod{q_0}} \frac{\chi(n)}{\gcd(n,d)} = \tau(d)q_0^2 q \max_{n \in \mathbb{N}} \{|f(n)|\} \sum_{1 \leq n \leq X, n \equiv j \pmod{q_0}} \frac{\chi(n)}{\gcd(n,d)}.$$

It therefore suffices to bound each sum in the modulus signs by $\ll C X (\log X)^{-C} \tau(d)q_0q$. By orthogonality of Dirichlet characters we can each such sum as

$$\frac{1}{\phi(q_0)} \sum_{\psi \pmod{q_0}} \psi(j) \sum_{1 \leq n \leq X, \gcd(n,d) = 1} \frac{\psi(n)}{\gcd(n,d)} \frac{\mu(n)}{\tau(n)} = \frac{1}{\phi(q_0)} \sum_{\psi \pmod{q_0}} \psi(j) \sum_{1 \leq n \leq X, \gcd(n,d) = 1} \frac{\psi(n)}{\gcd(n,d)} \frac{\mu(n)}{\tau(n)}.$$ (4.17)

Let us now see why $\psi \chi$ is a non-principal character modulo $q_0q$. Using that $\chi \pmod{q}$ is non-principal we may find $t \in \mathbb{Z}/q\mathbb{Z}$ with $\chi(t) \notin \{0, 1\}$. Then by $\gcd(q_0, q) = 1$ we can find an integer $\ell$ such that $\ell \equiv 1 \pmod{q_0}$ and $\ell \equiv t \pmod{q}$. Since $\psi(1) = 1$ we obtain $(\psi \chi)(\ell) = \chi(\ell) = \chi(t) \notin \{0, 1\}$, thus $\psi \chi$ is non-principal. Now let $d_0$ be the square-free product of all primes dividing $d$ and coprime to $qq_0$ so that $\gcd(d_0, qq_0) = 1$ and the condition $\gcd(n, d) = 1$ in (4.17) can be replaced by $\gcd(n, d_0) = 1$. Using [35, Lem. 1] then shows that the sum in (4.17) is at most

$$\ll C \frac{1}{\phi(q_0)} \sum_{\psi \pmod{q_0}} \frac{\tau(d_0)q_0q}{\log X} \frac{X}{(\log X)^C} = \frac{\tau(d_0)q_0q}{(\log X)^C},$$

which is sufficient due to $\tau(d_0) \leq \tau(d)$. □
Lemma 4.18. Keep the setting of Lemma 4.16 and fix any $C' > 0$. We have

\[ \mathcal{E}_{b,h,h}^{(1)}(X) \ll b_1 b_2 b_3 \left( \prod_{i<j} h_{ij} \right) \frac{X_1 X_2 X_3}{(\log X_1)^{C'}}, \]

where the implied constant depends at most on $C, C'$ and $\eta$.

Proof. By (4.14) we know that $d_1 d_2 d_3 h_{13} h_{23} > 1$. By symmetry we may assume that one of $d_1, d_2, h_{13}, h_{23}$ exceeds 1. Thus,

\[ \mathcal{E}_{b,h,h}^{(1)}(X) \ll \sum_{\tilde{d}_1 \mid X_1, \tilde{d}_2 \leq X_2} \mu^2(d_1 d_2 h_{13} h_{23}) \left| \sum_{\{d_3 \mid \text{gcd}(d_3, d_1) = 1 \}} \frac{\mu^2(d_3)}{\tau(d_3)} \left( \frac{d_3}{q} \right) \right|, \]

where the outer summation is subject to the condition $d_1 d_2 h_{13} h_{23} > 1$ and $q = d_1 d_2 h_{13} h_{23}, \ d = 2d_3 \text{gcd}(b_1, b_2) \tilde{d}_1 \tilde{d}_2 \prod_{i<j} h_{ij} \tilde{h}_{ij}, \ b = d_3 2^{\sigma_3}$.

The resulting sum over $\tilde{d}_3$ can be estimated via Lemma 4.17 with $f(n) = 1$ and $q_0 = 1$. The character $(\cdot/q)$ modulo $q$ is non-principal due to the fact that $q$ is an odd square-free integer exceeding 1. We obtain the bound

\[ \ll C' \sum_{\tilde{d}_1 \mid X_1, \tilde{d}_2 \leq X_2} \frac{\tau(d_1 d_2 h_{13} h_{23} X_3)}{(\log X_3)^{C'}} \ll \frac{h_{13} h_{23} X_1 X_2 X_3}{(\log X_3)^{C'}} \sum_{\{d_3 \mid \text{gcd}(d_3, d_1) = 1 \}} \frac{\tau(d_1 d_2)}{\tau(d_3)}, \]

for every fixed $C' > 0$. Using $\tau(mn) \leq \tau(m) \tau(n)$, which holds for all $m, n \in \mathbb{N}$, we obtain

\[ \tau(d) \leq d_3 \tau(d_1) \tau(d_2) b_1 b_2 \prod_{i<j} h_{ij} \tilde{h}_{ij}. \]

This gives

\[ \sum_{\tilde{d}_1 \mid X_1, \tilde{d}_2 \leq X_2} \frac{\tau(d)}{(\log X_3)^{C'}} \ll b_1 b_2 \left( \prod_{i<j} h_{ij} \tilde{h}_{ij} \right) \frac{X_1 X_2 X_3}{(\log X_1)^{C'}}, \]

Using (4.17) and taking $C'$ large enough compared to $C$ concludes the proof.

Lemma 4.19. We have

\[ \mathcal{E}_{b,h,h}^{(2)}(X) \ll b_1 b_2 b_3 \left( \prod_{i<j} h_{ij} \tilde{h}_{ij} \right) \frac{X_1 X_2 X_3}{(\log X_1)^{C'}}, \]

where the implied constant depends at most on $C, C'$ and $\eta$.

Proof. We first focus on the terms satisfying $\tilde{d}_1 \tilde{d}_2 h_{13} \tilde{h}_{23} > 1$. We rearrange their contribution as $\sum_{d_1, d_2, \tilde{d}_1, \tilde{d}_2} \left| \sum_{d_3} \right|$. Here the outer sum includes the condition that $2d_1 d_2 \tilde{d}_1 \tilde{d}_2 d_3$ is square-free and the inner sum is given by

\[ \sum_{\{d_3 \mid \text{gcd}(d_3, d_1) = 1 \}} \frac{f_0(d_3) \mu^2(d_3)}{\tau(d_3)} \left( \frac{q}{d_3} \right), \]

where $b = 2^{\sigma_3} \tilde{d}_3, \ d$ is similar to the analogous one in the proof of Lemma 4.18

\[ f_0(d_3) := (-1)^{(d_3 - 1)/2} \left( \frac{2}{d_3} \right)^{\beta} \left( \frac{-1}{d_3} \right), \]
\[ \beta = \sigma - \sigma_3 - \sigma_{12} \in \{0, 1\}, \quad q = d_1d_2h_{12}h_{23} \] is a positive odd square-free integer with \( q > 1 \) and \( m = d_1 + d_2 + h_{13} + h_{23} \) is an even integer. By quadratic reciprocity we can write
\[
\left( \frac{q}{d_i} \right) = (-1)^{\frac{(q-1)(d_i-1)}{4}} \left( \frac{d_i}{q} \right),
\]
hence, the function \( f(d_i) = f_0(d_i)(-1)^{\frac{(q-1)(d_i-1)}{4}} \) has period \( q_0 = 8 \) and the sum equals
\[
\sum_{(\log \min X_i)^{100c} < d_i \leq X_i/b} \frac{f(d_i) \mu^2(d_i)}{\tau(d_i)} \left( \frac{d_i}{q} \right).
\]

Since \( q \) is square-free and satisfies \( q > 1 \), one can employ Lemma 4.17 to bound the inner sum. Then the proof can be completed as the one of Lemma 4.18. The remaining cases \( \tilde{d}_3 \tilde{h}_{12} > 1 \) can be dealt with similarly by rearranging the sums as \( \sum_{d_1, d_3, \tilde{d}_1, \tilde{d}_2, \tilde{d}_3 \mid \sum d_i} \). \( \square \)

### 4.7. Final steps in the proof of Theorem 4.11

Injecting Lemmas 4.18, 4.19 into Lemma 4.16 shows that for every fixed \( N > 0 \) one has
\[
\mathcal{E}_{b, h, h}(X) \ll b_1b_2b_3X_1X_2X_3(\log X_1)^{-N} \prod_{i < j} h_{ij}h_{ij},
\]
where the implied constant depends at most on \( \eta \) and \( N \). Recalling that \( h_{ij}h_{ij} \leq m_{ij} \), using (4.16), and taking \( N \) large enough compared to \( A \), shows that for every fixed \( M > 0 \) one has
\[
\mathcal{E}_{b, h, h}(X) \ll X_1X_2X_3(\log X_1)^{-M}.
\]
When this is fed into Lemma 4.12 it yields \( \mathcal{E}_{b, m}(X) \ll X_1X_2X_3(\log X_1)^{-M} \). Combining this with Lemma 4.15 completes the proof by alluding to Lemma 4.11.

### 4.8. The proof of Theorem 1.1

Recall the definition (1.1) of \( N(B) \). Our first step is to introduce new variables which account for square factors and common factors of the coefficients \( t_i \). Recall (1.12).

#### Lemma 4.20

Fix any \( C > 0 \). Then for all \( B \geq 2 \) the quantity \( N(B) \) equals
\[
6 \sum_{b \in \mathbb{N}^3, \gcd(b_1, b_2, b_3) = 1} \sum_{(m_{12}, m_{13}, m_{23}) \in \mathbb{N} \left( \frac{m_{12}, m_{13}, m_{23}}{\gcd(m_{12}, m_{13}, m_{23})} \right) \subseteq \left( \frac{m_{12}, m_{13}, m_{23}}{\gcd(m_{12}, m_{13}, m_{23})} \right)} \mu^2(m_{12}m_{13}m_{23}) \times N_b,m \left( \frac{B}{b_1^2m_{12}m_{13}}, \frac{B}{b_2^2m_{12}m_{23}}, \frac{B}{b_3^2m_{13}m_{23}} \right) + O_C \left( \frac{B^3}{(\log B)^C} \right),
\]
where the implied constant depends only on \( C \) and the conditions in the summation are \( \gcd(m_{12}, b_3) = \gcd(m_{13}, b_2) = \gcd(m_{23}, b_1) = 1 \).

**Proof.** Imposing the condition \( (t_1 - t_2)(t_1 - t_3)(t_2 - t_3)t_1t_2t_3 \neq 0 \) in \( N(B) \) introduces an error of size \( O(B^2) \). For the remaining \( t_i \) having a point in \( \mathbb{R} \) is equivalent to not all \( t_i \) having the same sign. Hence, by permuting variables, we obtain
\[
\frac{N(B)}{6} + O(B^2) = \left\{ t \in (\mathbb{N} \cap [1, B])^3 : \gcd(t_0, t_1, t_2) = 1, \sum_{i=0}^2 t_iX_i^2 = t_2X_2^2 \right\}.
\]
Each integer \( t_i \) can be written uniquely as \( b_i^2c_i+1 \) for some \( b_i, c_i+1 \in \mathbb{N} \) with \( \mu^2(c_i+1) = 1 \). Since the solubility of the equation is independent of square factors of the coefficients
We now restrict the range of $b$. Ignoring solubility over $\mathbb{Q}$ and gcd conditions, we may bound the contribution of the terms with at least one of $b_j$ exceeding $(\log B)^C$ by

$$\sum_{\substack{b \in \mathbb{N}^3 \backslash \{1, B \}^3 \backslash (\log B)^C \\gcd(b_1, b_2, b_3) = 1 \gcd(c_1, c_2, c_3) = 1}} \mu^2(c_1) \mu^2(c_2) \mu^2(c_3) \begin{cases} 1, & \text{if } c_1 x_1^2 + c_2 x_2^2 = c_3 x_3^2 \text{ has a } \mathbb{Q}\text{-point,} \\ 0, & \text{otherwise.} \end{cases}$$

Now we restrict the range of $b$. Ignoring solubility over $\mathbb{Q}$ and gcd conditions, we may bound the contribution of the terms with at least one of $b_j$ exceeding $(\log B)^C$ by

$$\sum_{b \in \mathbb{N}^3, b_j > (\log B)^C} \prod_{i=1}^3 \frac{X}{b_i^2} \ll \frac{B^3}{(\log B)^C}.$$ 

We now introduce new variables to keep track of common factors between the $c_i$. Define $m_{ij} = \gcd(c_i, c_j)$ so that there exists $n \in \mathbb{N}^3$ such that $c_1 = m_{12} m_{13} n_1$, $c_2 = m_{12} m_{23} n_2$ and $c_3 = m_{13} m_{23} n_3$. By construction it is evident that any two of the 6 variables $n_i$, $m_{ij}$ are coprime, thus the sum over $c$ is

$$\sum_{(m_{12} m_{13} m_{23}) \in \mathbb{N}^3, i \neq j; m_{ij} \leq B} \mu^2(m_{12} m_{13} m_{23}) \sum_{n \in \mathbb{N}^3, m_{ij} n \leq B} \mu^2(n_1 n_2 n_3) F(n),$$

where $F(n)$ is the indicator function of the event that the following curve has a $\mathbb{Q}$-point:

$$m_{12} m_{13} n_1^2 + m_{12} m_{23} n_2 n_3 x_2^2 = m_{13} m_{23} n_3 x_3^2.$$ 

The change of variables $Y_1 = m_{12} m_{13} x_1$, $Y_2 = m_{12} m_{23} x_2$, $Y_3 = m_{13} m_{23} x_3$ is invertible over $\mathbb{Q}$ (and all its completions) and it transforms the curve into

$$n_1 m_{23} Y_1^2 + n_2 m_{13} Y_2^2 = n_3 m_{12} Y_3^2.$$ 

In particular, the sum over $n$ equals

$$\mathcal{N}_{b, m} \left( \frac{B}{b_1^2 m_{12} m_{13}}, \frac{B}{b_2^2 m_{12} m_{23}}, \frac{B}{b_3^2 m_{13} m_{23}} \right).$$

Using the bound $\mathcal{N}_{b, m}(B) \ll B_1 B_2 B_3$ we see that the contribution of the cases $m_{12} > (\log X)^C$ is

$$\ll \sum_{b \in \mathbb{N}^3} \sum_{(m_{12} m_{13} m_{23}) \in \mathbb{N}^3, m_{12} > (\log X)^C} \frac{X^3}{(b_1 b_2 b_3)^3 (m_{12} m_{13} m_{23})^3} \ll \frac{X^3}{(\log X)^C}$$

and a similar argument deals with the terms satisfying $\max\{m_{13}, m_{23}\} > (\log X)^C$. \hfill \Box

Recall (4.15) and note that ignoring the condition $p \nmid m_{12} m_{13} m_{23} \gcd(b_i, b_j)$ yields

$$\beta(b, m) \leq \prod_p \left( 1 - \frac{1}{p} \right)^{3/2} \left( 1 + \frac{3}{2p} \right). \quad (4.19)$$

Lemma 4.21. For $B \geq 2$ we have

$$N(B) = \frac{3c_0}{(2\pi)^{3/2} (\log B)^{3/2}} + O \left( \frac{B^3}{(\log B)^{3/2}} \right),$$

where

$$c_0 = \sum_{b \in \mathbb{N}^3} \sum_{\gcd(b_1, b_2, b_3) = 1} \mu^2(m_{12} m_{13} m_{23}) \beta(b, m) c(b, m).$$
Proof. We shall apply Theorem 4.1 with

\[ X_1 = \frac{B}{b_1^2m_1m_{13}}, \quad X_2 = \frac{B}{b_2^2m_{12}m_{23}}, \quad X_3 = \frac{B}{b_3^2m_{13}m_{23}} \]

and with \( \max\{b_i, m_{ij}\} \leq (\log B)^C \), where \( C \) is the constant from Lemma 4.20 that furthermore satisfies \( C > 5/2 \). Therefore, (4.6) is satisfied with \( A = C \) and (4.7) holds with \( \eta = 1/2 \) due to \( B^{1/2} \ll C \min X_i \leq \max X_i \leq B \). Thus,

\[ N(B)B^{-3} = 6N_1(B) + O(E(B) + (\log B)^{-5/2}), \]

where

\[ N_1(B) := \frac{(2\pi)^{-3/2}}{2} \sum_{b,m \in \mathbb{N}^3} \frac{\mu^2(m_{12}m_{13}m_{23})}{(m_{12}m_{13}m_{23})^2} \beta(b, m)c(b, m) \prod_{i=1}^{3} \frac{b_i^2}{(\log X_i)^{1/2}}, \]

\[ E(B) := \sum_{b,m \in \mathbb{N}^3} \frac{(\log X_1)^{-5/2}}{(b_1b_2b_3m_{12}m_{13}m_{23})^2} \ll (\log X_1)^{-5/2} \ll (\log(B^{1/2}))^{-5/2} \]

and the conditions in \( \sum^* \) are as in Lemma 4.20. Furthermore, as in (4.12) we can write

\[ \prod_{i=1}^{3} (\log X_i)^{-1/2} = (\log B)^{-3/2} \left( 1 + O \left( \frac{\log(b_1b_2b_3m_{12}m_{13}m_{23})}{\log B} \right) \right). \]

Using this along with (4.19) yields

\[ N_1(B) = \frac{(2\pi)^{-3/2}}{2(\log B)^{3/2}} \sum_{b,m \in \mathbb{N}^3} \frac{\mu^2(m_{12}m_{13}m_{23})}{(b_1b_2b_3m_{12}m_{13}m_{23})^2} \beta(b, m)c(b, m) \]

\[ + O \left( (\log B)^{-5/2} \sum_{b,m \in \mathbb{N}^3} \frac{\log(b_1b_2b_3m_{12}m_{13}m_{23})}{(b_1b_2b_3m_{12}m_{13}m_{23})^2} \right). \]

The error term is satisfactory since the sum is \( \sum_{n \in \mathbb{N}} \tau_6(n)(\log n)n^{-2} = O(1) \), where \( \tau_6 \) denotes the number of different decompositions as a product of 6 positive integers. The main term sum \( \sum^* \) contains the condition \( \max\{b_i, m_{ij}\} \leq (\log X)^C \) that can be safely ignored by using (4.19). \( \square \)

We now continue by factoring \( c_0 \) as an Euler product. This otherwise straightforward task is somewhat hampered by the fact that neither of \( \beta(b, m), c(b, m) \) has standard multiplicative properties.

Lemma 4.22. We have

\[ c_0 = \frac{49}{3} \sum_{b \in \mathbb{N}^3, 2 | b_1b_2b_3, \gcd(b_1, b_2, b_3) = 1} \frac{1}{(b_1b_2b_3)^2} \sum_{m \in \mathbb{N}^3, 2 | m_{12}m_{13}m_{23}, \gcd(m_{12}, b_3) = \gcd(m_{13}, b_2) = \gcd(m_{23}, b_1) = 1} \frac{\mu^2(m_{12}m_{13}m_{23})}{(m_{12}m_{13}m_{23})^2} \frac{\beta(b', m')}{\tau(m_{12}m_{13}m_{23})}. \]

Proof. Write \( b_i = 2^{\beta_i}b'_i \) and \( m_{ij} = 2^{\mu_{ij}}m'_{ij} \), where each \( b'_i, m'_{ij} \) is odd. Note that \( \beta(b, m) \) only depends on the odd parts of \( b_i, m_{ij} \) and that \( c(b, m) = c((2^{\beta_i}), (2^{\mu_{ij}})) \). Hence, \( c_0 = \gamma c_0' \), where

\[ \gamma = \sum_{\beta_1, \beta_2, \beta_3 \geq 2, \min \beta_i = 0} \frac{1}{4^{\beta_1+\beta_2+\beta_3}} \sum_{0 \leq \mu_{12}+\mu_{13}+\mu_{23} \leq 1, \min(\mu_{12}, \beta_3) = \min(\mu_{13}, \beta_2) = \min(\mu_{23}, \beta_1) = 0} \frac{c((2^{\beta_i}), (2^{\mu_{ij}}))}{4^{\mu_{12}+\mu_{13}+\mu_{23}}} \]
Lemma 4.24. \(\gamma\) equals 8/3 + 6 + 6 + 5/3 = 49/3.

Recall \((4.11)\) and let

\[
\kappa = \prod_{p \neq 2} \left(1 - \frac{1}{p}\right)^{3/2} \left(1 + \frac{3}{2p}\right).
\]

A function \(f : \mathbb{N}^3 \to \mathbb{C}\) is called multiplicative if \(f(a_1b_1, a_2b_2, a_3b_3) = f(a_1)f(b_1)\) whenever \(a_1a_2a_3\) and \(b_1b_2b_3\) are coprime.

**Lemma 4.23.** For \(d \in \mathbb{N}^3\) the function \(f(d) = \gamma(d)/\kappa\) is multiplicative and given by

\[
f(d) = \prod_{p \neq 2} \left(1 - \frac{\#\left\{1 \leq i \leq 3 : p \mid d_i\right\}}{2p + 3}\right).
\]

**Proof.** If \(p \nmid d_1d_2d_3\) then the \(p\)-th terms of \(\gamma(d)\) and \(\kappa\) coincide, hence, \(\gamma(d)/\kappa\) equals

\[
\prod_{p \neq 2} \left(1 - \frac{\#\left\{1 \leq i \leq 3 : p \nmid d_i\right\}}{2p + 3}\right) = \prod_{p \neq 2} \left(1 - \frac{\#\left\{1 \leq i \leq 3 : p \mid a_i\right\}}{2p + 3}\right).
\]

To prove multiplicativity, note that if \(\gcd(a_1a_2a_3, b_1b_2b_3) = 1\) then \(f(a_1b_1, a_2b_2, a_3b_3)\) splits as

\[
\prod_{p \neq 2} \left(1 - \frac{\#\left\{1 \leq i \leq 3 : p \mid a_i\right\}}{2p + 3}\right) \prod_{p \neq 2} \left(1 - \frac{\#\left\{1 \leq i \leq 3 : p \mid b_i\right\}}{2p + 3}\right).
\]

By coprimality the condition \(p \mid a_ib_i\) in the first product is equivalent to \(p \mid a_i\), thus, it equals \(f(a)\). Similarly, the second product equals \(f(b)\). \(\square\)

**Lemma 4.24.** The sum over \(b, m\) in Lemma 4.22 equals

\[
\prod_{p \neq 2} \left(1 - \frac{1}{p}\right)^{3/2} \frac{(p^2 + p + 1)(2p^2 + p + 2)}{2(p^2 - 1)^2}.
\]

**Proof.** By Lemma 4.23 one directly sees that the function \(g : \mathbb{N}^6 \to \mathbb{C}\) given by

\[
g(b, m) = f(m_{12}m_{13}m_{23}\gcd(b_2, b_3), m_{12}m_{13}m_{23}\gcd(b_1, b_3), m_{12}m_{13}m_{23}\gcd(b_1, b_2))
\]

and \(c'\) is the sum over \(b, m\) given in the present lemma. If one of \(\mu_{ij}\) is 1, then \(c((2^\beta), (2^\nu)) = 2\), hence, the contribution equals

\[
3 \sum_{\beta_1, \beta_2 \geq 0} \frac{1}{4^{\beta_1}} \cdot \frac{2}{3} = 3.
\]

If all \(\mu_{ij}\) are 0 and all \(\beta_i\) are 0, then \(c((2^\beta), (2^\nu)) = 6\), hence, the contribution equals 6. If all \(\mu_{ij}\) are 0 and exactly one of \(\beta_i\) is \(\geq 1\), then \(c((2^\beta), (2^\nu)) = 5\), hence, the contribution equals

\[
3 \sum_{\beta_1 \geq 1, \beta_2 \geq 1} \frac{1}{4^\beta_1 + \beta_2} = \frac{5}{3}.
\]

Adding the various contributions shows that \(\gamma = 8/3 + 6 + 6 + 5/3 = 49/3\). \(\square\)
is multiplicative. Thus, the sum over $b, m$ in Lemma 4.22 is $\kappa \prod_{p \neq 2} \kappa'_p$, where

$$
\kappa'_p = \sum_{\beta_1, \beta_2, \beta_3 \geq 0} g(p^\beta_1, p^\beta_2, p^\beta_3, p^{\mu_{12}}, p^{\mu_{13}}, p^{\mu_{23}}) p^{2(\beta_1 + \beta_2 + \beta_3 + \mu_{12} + \mu_{13} + \mu_{23})} p^{2(\beta_1 + \beta_2 + \beta_3 + \mu_{12} + \mu_{13} + \mu_{23})/7} (p^{\mu_{12} + \mu_{13} + \mu_{23}}).
$$

The terms with $\mu_{12} + \mu_{13} + \mu_{23} = 1$ contribute

$$
3 \sum_{\beta_1, \beta_2 \geq 0} \sum_{\beta_3 \geq 0} g(p^\beta_1, p^\beta_2, p^\beta_3, 1, p, 1, 1) = \frac{3 f(p, p, p)}{2p^2} = \frac{3 f(p, p, p)}{2p^2 (1 - 1/p^2)^2}.
$$

The terms with $\mu_{12} + \mu_{13} + \mu_{23} = 0$ contribute

$$
\sum_{\beta_1, \beta_2, \beta_3 \geq 0} g(p^\beta_1, p^\beta_2, p^\beta_3, 1, 1, 1) = \frac{3 f(p, 1, 1)}{p(p^2 - 1)^2} = \frac{3 f(p, 1, 1)}{p(p^2 - 1)^2}.
$$

In total the sum over $b, m$ becomes

$$
\kappa \prod_{p \neq 2} \left( \frac{3(1 - \frac{3}{3+2p})}{2p^2 (1 - 1/p^2)^2} + 1 + \frac{3(1 - \frac{1}{2p+3})}{p^2 - 1} \right).
$$

By Lemmas 4.21, 4.22, 4.24 we see that $N(B)$ equals

$$
\frac{3}{(2\pi)^{3/2}} \frac{49}{3} \left( \prod_{p \neq 2} \left( 1 - \frac{1}{p} \right)^{3/2} \left( \frac{p^2 + p + 1}{2(p^2 - 1)^2} \right)^{3/2} \right) \frac{B^3}{(\log B)^3/2} + O \left( \frac{B^3}{(\log B)^5/2} \right).
$$

Using $48 = 6 \cdot 8$, this is easily checked to agree with the expression in Theorem 1.1. □

**Remark 4.25.** An alternative to (1.1), also considered by Serre in [66], is

$$
N_0(B) = \left\{ t \in \mathbb{Z} \setminus \{0\}^3 : \max_i |t_i| \leq B, \sum_{i=0}^3 t_i x_i^2 = 0 \text{ has a } \mathbb{Q}\text{-point} \right\},
$$

i.e. counting all $t$ and not just primitive $t$. A simple application of Möbius inversion, Proposition 3.1, and Lemma 3.13 shows that Theorem 1.1 is equivalent to

$$
N_0(B) = \frac{2}{\pi^{3/2}} \left( \vartheta_\infty \prod_{p \text{ prime}} \vartheta_p \right) \left( \frac{B^3}{(\log B)^3/2} + O \left( \frac{B^3}{(\log B)^5/2} \right) \right),
$$

where $\vartheta_\infty$ is the Lebesgue measure of $t$ in $[-1,1]^3$ for which the conic $\sum_{i=0}^2 t_i x_i^2$ has an $\mathbb{R}$-point and $\vartheta_p$ is the $p$-adic Haar measure of $t \in \mathbb{Z}_p$ for which $\sum_{i=0}^2 t_i x_i^2$ has a $\mathbb{Q}_p$-point.
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