A Comparison of Hofer’s Metrics on Hamiltonian Diffeomorphisms and Lagrangian Submanifolds

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Abstract

We compare Hofer’s geometries on two spaces associated with a closed symplectic manifold \((M,\omega)\). The first space is the group of Hamiltonian diffeomorphisms. The second space \(\mathcal{L}\) consists of all Lagrangian submanifolds of \(M \times M\) which are exact Lagrangian isotopic to the diagonal. We show that in the case of a closed symplectic manifold with \(\pi_2(M) = 0\), the canonical embedding of \(\text{Ham}(M)\) into \(\mathcal{L}\), \(f \mapsto \text{graph}(f)\) is not an isometric embedding, although it preserves Hofer’s length of smooth paths.

1 Introduction and Main Results

In this paper we compare Hofer’s geometries on two remarkable spaces associated with a closed symplectic manifold \((M,\omega)\). The first space \(\text{Ham}(M,\omega)\) is the group of Hamiltonian diffeomorphisms. The second consists of all Lagrangian submanifolds of \((M \times M, -\omega \oplus \omega)\) which are exact Lagrangian isotopic to the diagonal \(\triangle \subset M \times M\). Let us denote this second space by \(\mathcal{L}\). The canonical embedding

\[ j : \text{Ham}(M,\omega) \to \mathcal{L}, \quad f \mapsto \text{graph}(f) \]

preserves Hofer’s length of smooth paths. Thus, it naturally follows to ask whether \(j\) is an isometric embedding with respect to Hofer’s distance. Here, we provide a negative answer to this question for the case of a closed symplectic manifold with \(\pi_2(M) = 0\). In fact, our main result shows that the image of \(\text{Ham}(M,\omega)\) inside \(\mathcal{L}\) is “strongly distorted” (see Theorem 1.1 below).

Let us proceed with precise formulations. Given a path \(\alpha = \{f_t\}, \ t \in [0,1]\) of Hamiltonian diffeomorphisms of \((M,\omega)\), define its Hofer’s length (see [H]) as

\[ \text{length}(\alpha) = \int_0^1 \{ \max_{x \in M} F(x,t) - \min_{x \in M} F(x,t) \} \ dt \]

where \(F(x,t)\) is the Hamiltonian function generating \(\{f_t\}\). For two Hamiltonian diffeomorphisms \(\phi\) and \(\psi\), define the Hofer distance \(d(\phi,\psi) = \inf \text{length}(\alpha)\) where the infimum is taken over all smooth paths \(\alpha\) connecting \(\phi\) and \(\psi\). For further discussion see e.g. [LM1],[MS], and [P1].

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Hofer’s metric can be defined in a more general context of Lagrangian submanifolds (see [C]). Let \((P, \sigma)\) be a closed symplectic manifold, and let \(\Delta \subset P\) be a closed Lagrangian submanifold. Consider a smooth family \(\alpha = \{L_t\}, \ t \in [0, 1]\) of Lagrangian submanifolds, such that each \(L_t\) is diffeomorphic to \(\Delta\). We call \(\alpha\) an exact path connecting \(L_0\) and \(L_1\), if there exists a smooth map \(\Psi: \Delta \times [0, 1] \to P\) such that for every \(t\), \(\Psi(\Delta \times \{t\}) = L_t\), and in addition \(\Psi^* \sigma = dH_t \wedge dt\) for some smooth function \(H : \Delta \times [0, 1] \to \mathbb{R}\). The Hofer length of an exact path is defined by

\[
\text{length}(\alpha) = \int_0^1 \left\{ \max_{x \in \Delta} H(x, t) - \min_{x \in \Delta} H(x, t) \right\} \, dt.
\]

It is easy to check that the above notion of length is well-defined. Denote by \(L(P, \Delta)\) the space of all Lagrangian submanifolds of \(P\) which can be connected to \(\Delta\) by an exact path. For two Lagrangian submanifolds \(L_1\) and \(L_2\) in \(L(P, \Delta)\), define the Hofer distance \(\rho\) on \(L(P, \Delta)\) as follows:

\[
\rho(L_1, L_2) = \inf \text{length}(\alpha),
\]

where the infimum is taken over all exact paths on \(L(P, \Delta)\) that connect \(L_1\) and \(L_2\).

In what follows we choose \(P = M \times M, \sigma = -\omega \oplus \omega\) and take \(\Delta\) to be the diagonal of \(M \times M\). We abbreviate \(\mathcal{L} = \mathcal{L}(P, \Delta)\) as in the beginning of the paper. Based on a result by Banyaga [B], it can be shown that every smooth path on \(\mathcal{L}(P, \Delta)\) is necessarily exact. Our main result is the following:

**Theorem 1.1.** Let \((M, \omega)\) be a closed symplectic manifold with \(\pi_2(M) = 0\). Then there exist a family \(\{\varphi_t\}, \ t \in [0, \infty)\) in \(\text{Ham}(M, \omega)\) and a constant \(c\) such that:

1. \(d(\text{id}, \varphi_t) \to \infty\) as \(t \to \infty\).
2. \(\rho(\text{graph}(\text{id}), \text{graph}(\varphi_t)) = c\).

In fact, we construct the above family \(\{\varphi_t\}\) explicitly:

**Example 1.2.** Consider an open set \(B \subset M\). Suppose that there exists a Hamiltonian diffeomorphism \(h\) such that \(h(B) \cap \text{Closure} \ (B) = \emptyset\). By perturbing \(h\) slightly, we may assume that all the fixed points of \(h\) are non-degenerate. Let \(F(x, t)\), where \(x \in M, \ t \in [0, 1]\) be a Hamiltonian function such that \(F(x, t) = c_0 < 0\) for all \(x \in M \setminus B, \ t \in [0, 1]\). Assume that \(F(t, x)\) is normalized such that for every \(t\), \(\int_M F(t, \cdot) \omega^n = 0\). We define the family \(\{\varphi_t\}, \ t \in [0, \infty)\) by \(\varphi_t = h f_t\), where \(\{f_t\}\) is the Hamiltonian flow generated by \(F(t, x)\). As we’ll see below, the family \(\{\varphi_t\}\) satisfies the requirements of Theorem 1.1.

Theorem 1.1 has some corollaries:

1. The embedding of \(\text{Ham}(M, \omega)\) in \(\mathcal{L}\) is not isometric, rather, the image of \(\text{Ham}(M, \omega)\) in \(\mathcal{L}\) is highly distorted. The minimal path between two graphs of Hamiltonian diffeomorphisms in \(\mathcal{L}\), might pass through exact Lagrangian submanifolds which are not the graphs of any Hamiltonian diffeomorphisms. Compare with the situation described in [M], where it was proven that in the case of a compact manifold, the space of Hamiltonian deformations of the zero section in the cotangent bundle is locally flat in the Hofer metric.

2. The group of Hamiltonian diffeomorphisms of a closed symplectic manifold with \(\pi_2(M) = 0\) has an infinite diameter with respect to Hofer’s metric.
3. Hofer’s metric $d$ on $\text{Ham}(M, \omega)$ does not coincide with the Viterbo-type metric on $\text{Ham}(M, \omega)$ defined by Schwarz in [S].

As a by-product of our method we prove the following result (see Section 3 below):

**Theorem 1.3.** Let $(M, \omega)$ be a closed symplectic manifold with $\pi_2(M) = 0$. Then there exists an element $\varphi$ in $(\text{Ham}(M, \omega), d)$ which cannot be joined to the identity by a minimal geodesic.

The first example of this kind was established by Lalonde and McDuff [LM2] for the case of $S^2$.

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## 2 Proof of The Main Theorem

In this section we prove Theorem 1.1. Throughout this section let $(M, \omega)$ be a closed symplectic manifold with $\pi_2(M) = 0$. Let $\{\varphi_t\}$, $t \in [0, \infty)$ the family of Hamiltonian diffeomorphisms defined in Example 1.2. We begin with the following lemma which states that Hamiltonian diffeomorphisms act as isometries on the space $(\mathcal{L}, \rho)$. The proof of the lemma follows immediately from the definitions.

**Lemma 2.1.** Let $\Gamma : \triangle \times [0, 1] \to M \times M$ be an exact Lagrangian isotopy in $\mathcal{L}$ and let $\Phi : M \times M \to M \times M$ be a Hamiltonian diffeomorphism. Then

$$\text{length}\{\Gamma\} = \text{length}\{\Phi \circ \Gamma\}.$$ 

In particular, $\rho(L_1, L_2) = \rho(\Phi(L_1), \Phi(L_2))$ for every $L_1, L_2 \in \mathcal{L}$.

Next, consider the following exact isotopy of the Lagrangian embeddings $\Psi : \triangle \times [0, \infty) \to M \times M$, $\Psi(x, t) = (x, \varphi_t(x))$. We denote by $L_t = \Psi(\triangle \times \{t\})$ the graph of $\varphi_t = h f_t$ in $M \times M$. The following proposition will be proved in Section 5 below.

**Proposition 2.2.** For every $t \in [0, \infty)$ there exists a Hamiltonian isotopy $\{\Phi_s\}$, $s \in [0, t]$ of $M \times M$, such that $\Phi_s(L_0) = L_s$ and such that for every $s$, $\Phi_s(\triangle) = \triangle$.

Hence, it follows from Proposition 2.2 and Lemma 2.1, that the family $\{\varphi_t\}$, $t \in [0, \infty)$ satisfies the second conclusion of Theorem 1.1 with constant $c = \rho(\triangle, L_0)$.

Let us now verify the first statement of Theorem 1.1. For this purpose we will use a theorem by Schwarz [S] stated below. First, recall the definitions of the action functional and the action spectrum. Consider a closed symplectic manifold $(M, \omega)$ with $\pi_2(M) = 0$. Let $\{f_t\}$ be a Hamiltonian path generated by a Hamiltonian function $F : [0, 1] \times M \to \mathbb{R}$. We denote by $\text{Fix}^0(f_1)$ the set of fixed points, $x$, of the time-1-map $f_1$ whose orbits $\gamma = \{f_t(x)\}$, $t \in [0, 1]$ are contractible. For $x \in \text{Fix}^0(f_1)$, take any 2-disc $\Sigma \subset M$ with $\partial \Sigma = \gamma$, and define the symplectic action functional by

$$A(F, x) = \int_{\Sigma} \omega - \int_0^1 F(t, f_t(x)) dt.$$ 

The assumption $\pi_2(M) = 0$ ensures that the integral $\int_{\Sigma} \omega$ does not depend on the choice of $\Sigma$. 

Remark 2.3. In the case of a closed symplectic manifold with \( \pi_2(M) = 0 \), a result by Schwarz [S], implies that for a Hamiltonian path \( \{f_t\} \) with \( f_t \neq \mathbb{I} \) there exist two fixed points \( x, y \in \text{Fix}^\circ(f_1) \) with \( \mathcal{A}(F, x) \neq \mathcal{A}(F, y) \). Moreover, the action functional does not depend on the choice of the Hamiltonian path generating \( f_1 \). Therefore, we can speak about the action of a fixed point of a Hamiltonian diffeomorphism, regardless of the Hamiltonian function used to define it.

Definition 2.4. For each \( f \) in \( \text{Ham}(M, \omega) \) we define the action spectrum

\[
\Sigma_f = \{ \mathcal{A}(f, x) \mid x \in \text{Fix}^\circ(f) \} \subset \mathbb{R}.
\]

The action spectrum \( \Sigma_f \) is a compact subset of \( \mathbb{R} \) (see e.g. [S],[HZ]).

Theorem 2.5. [S]. Let \((M, \omega)\) be a closed symplectic manifold with \( \pi_2(M) = 0 \). Then, for every \( f \) in \( \text{Ham}(M, \omega) \)

\[
d(\mathbb{I}, f) \geq \min \Sigma_f.
\]

Next, consider the family \( \{\varphi_t\} = \{hf_t\}, t \in [0, \infty) \). Note that \( \text{Fix}^\circ(h) = \text{Fix}^\circ(\varphi_t) \) for every \( t \).

The following proposition shows that the action spectrum of \( \varphi_t \) is a linear translation of the action spectrum of \( h \). Its proof is carried out in Section 4.

Proposition 2.6. For every \( t \in [0, \infty) \), and for every fixed point \( z \in \text{Fix}^\circ(\varphi_t) = \text{Fix}^\circ(h) \),

\[
\mathcal{A}(\varphi_t, z) = \mathcal{A}(h, z) - tc_0
\]

where \( c_0 \) is the negative (constant) value that \( F \) attains on \( M \setminus B \) (see Example 1.2).

We are now in a position to complete the proof of Theorem 1.1. Indeed, the action spectrum is a compact subset of \( \mathbb{R} \), hence its minimum is finite. By proposition 2.6 the minimum of \( \Sigma_{\varphi_t} \) tends to infinity as \( t \to \infty \). Thus,

\[
d(\mathbb{I}, \varphi_t) \to \infty \text{ as } t \to \infty
\]

as follows from Theorem 2.5. This completes the proof of Theorem 1.1. \( \square \)

3 Geodesics in \( \text{Ham}(M, \omega) \) and Proof of Theorem 1.3

In this section we describe our result about geodesics in the group of Hamiltonian diffeomorphisms endowed with the Hofer metric \( d \). We refer the reader to [BP], [LM1], [LM2], and [P2] for further details on this subject.

Let \( \gamma = \{\phi_t\}, \ t \in [0, 1] \) be a smooth regular path in \( \text{Ham}(M, \omega) \), i.e. \( \frac{d}{dt}\phi_t \neq 0 \) for every \( t \in [0, 1] \).

The path \( \gamma \) is called a minimal geodesic if it minimizes the distance between its end-points:

\[
\text{length} (\gamma) = d(\phi_0, \phi_1).
\]

The graph of a Hamiltonian path \( \gamma = \{\phi_t\} \) is the family of embedded images of \( M \) in \( M \times M \) defined by the map \( \Gamma : M \times [0, 1] \to M \times M, \ (x, t) \mapsto (x, \phi_t(x)) \). Next, consider the family \( \{\varphi_t\}, \ t \in [0, \infty) \) that was constructed in Example 1.2. We will show that there exists no minimal geodesic joining the identity and \( \varphi_{t_0} \), for some \( t_0 \).
Proof of Theorem 1.3. Assume (by contradiction) that for every \( t \), there exists a minimal geodesic in \( \text{Ham}(M,\omega) \) joining the identity with \( \varphi_t \). Fix \( t_0 \in [0, \infty) \). There exists a Hamiltonian path \( \alpha = \{ f_s \}, \ s \in [0, 1] \) in \( \text{Ham}(M,\omega) \) such that

\[
d_{t_0} := d(\mathbb{I}, \varphi_{t_0}) = \text{length}(\alpha).
\]

Expressed in Lagrangian submanifolds terms, \( \Psi = \{ \text{graph}(f_s) \}, \ s \in [0, 1] \) is an exact path in \( M \times M \) joining the diagonal with \( \text{graph}(\varphi_{t_0}) \). By Proposition 2.2, there exists a Hamiltonian isotopy \( \Phi \) such that for every \( t \), \( \Phi_t(\text{graph}(\varphi_{t_0})) = \text{graph}(\varphi_t) \), and \( \Phi_t(\Delta) = \Delta \). We will choose \( t_1 \) to be sufficiently close to \( t_0 \) so as to ensure that \( \{ \Phi_{t_1}(\text{graph}(f_s)) \}, \ s \in [0, 1] \) is the graph of some Hamiltonian path \( \gamma \) in \( \text{Ham}(M,\omega) \). We claim the following

\[
d_{t_1} \leq \text{length}(\gamma) = \text{length}\{\text{graph}(\gamma)\} = \text{length}\{\text{graph}(\alpha)\} = \text{length}(\alpha) = d_{t_0}.
\]

Indeed, a straightforward computation yields that the embedding \( f \mapsto \text{graph}(f) \) preserves Hofer’s length, and from Lemma 2.1, \( \text{length}\{\text{graph}(\alpha)\} = \text{length}\{\text{graph}(\gamma)\} \). We have shown that for every \( t_0 \) there exists \( \varepsilon > 0 \) such that if \( |t - t_0| \leq \varepsilon \) then \( d_t \leq d_{t_0} \). Since \( d_t \) is a continuous function, we conclude that \( d_t \) is a constant function. On the other hand, by Theorem 1.1, \( d_t = d(\mathbb{I}, \varphi_t) \to \infty \) as \( t \to \infty \). Hence there is a contradiction. \( \square \)

4 Proof of Proposition 2.6

We investigate the expression \( A(\varphi_t, z) \) for some fixed \( t \). Since the action functional does not depend on the choice of the Hamiltonian path generating the time-1-map (see Remark 2.3), we consider the following path generating \( \varphi_t \).

\[
\gamma(s) = \begin{cases} 
  f_{2st} & , \ s \in [0, \frac{1}{2}] \\
  h_{2s-1}f_t & , \ s \in (\frac{1}{2}, 1].
\end{cases}
\]

Note that since \( h(B) \cap B = \emptyset \) and \( f_t \) is supported in \( B \), then for \( z \in \text{Fix}^c(\varphi_t) = \text{Fix}^c(h) \) the path \( \{\gamma_s(z)\}, \ s \in [0, 1] \) coincides with the path \( \{h_s(z)\}, \ s \in [0, 1] \). Denote by \( \alpha \) the loop \( \{\gamma_s(z)\}, \ s \in [0, 1] \) and let \( \Sigma \) be any 2-disc with \( \partial \Sigma = \alpha \). The details of the calculation of \( A(\varphi_t, z) \) are as follows:

\[
A(\varphi_t, z) = \int_{\Sigma} \omega - \int_0^1 tF(s, z) ds - \int_0^1 H(s, h_s(z)) ds,
\]

where \( F \) and \( H \) are the Hamiltonian functions generating \( \{h_t\} \) and \( \{f_t\} \) respectively. Recall that by definition, \( F \) is equal to a constant \( c_0 \) in \( M \setminus B \). This implies that

\[
A(\varphi_t, z) = \int_{\Sigma} \omega - \int_0^1 H(s, h_s(z)) ds - tc_0.
\]

The right hand side is exactly \( A(h, z) - tc_0 \). Hence, the proof is complete.

5 Extending the Hamiltonian Isotopy

In this section we prove Proposition 2.2. Let us first recall some relevant notations. Let \( \{\varphi_t\}, \ t \in [0, \infty) \) the family of Hamiltonian diffeomorphisms defined in Example 1.2. Consider the following exact isotopy of Lagrangian embeddings \( \Psi : \Delta \times [0, \infty) \to M \times M, \ \Psi(x, t) = (x, \varphi_t(x)) \). We denote
by $L_t = \Psi(\triangle \times \{t\})$ the graph of $\varphi_t = h f_t$ in $M \times M$, and by $\triangle$ the diagonal in $M \times M$. It follows from the construction of the family $\{\varphi_t\}$, that for every $t$, $\text{Fix}(\varphi_t) = \text{Fix}(h)$. Hence, $L_t$ intersects the diagonal at the same set of points for every $t$. Moreover, we assumed that all the fixed points of $h$ are non-degenerate, therefore for every $t$, $L_t$ transversely intersect the diagonal. In order to prove Proposition 2.2, we first need the following lemma.

**Lemma 5.1.** Let $x, y \in \text{Fix}^c(\varphi_t) = \text{Fix}^c(h)$, i.e., intersection points of the family $\{L_t\}$ and the diagonal in $M \times M$. Take a smooth curve $\alpha : [0, 1] \to M$ with $\alpha(0) = x$ and $\alpha(1) = y$ and let $\Sigma : [0, 1] \times [0, 1] \to M$, $\Sigma(t, s) = \varphi_t(\alpha(s))$ be a 2-disc such that $\partial \Sigma_t = \varphi_t \alpha - h \alpha$. Then the symplectic area of $\Sigma_t = \Sigma(t, \cdot)$ vanishes for all $t$.

**Proof.** By a direct computation of the symplectic area of $\Sigma_t$, we obtain that

$$\int_{\Sigma_t} \omega = \int_{[0, t] \times [0, 1]} \Sigma_t^* \omega = -\int_0^t dt \int_0^1 dF_t(\frac{\partial}{\partial s} \varphi_t(\alpha(s))) \, ds = \int_0^t \tilde{F}_t(\varphi_t(x)) \, dt - \int_0^t \tilde{F}_t(\varphi_t(y)) \, dt,$$

where $\tilde{F}_t$ is the Hamiltonian function generating the flow $\{\varphi_t\}$. A straightforward computation shows that $\tilde{F}(t, x) = F(t, h^{-1}(x))$, where $F$ is the Hamiltonian function generating the flow $\{f_t\}$. Recall that by definition, $F(x, t)$ is equal to a constant $c_0$ outside the ball $B$. Moreover, since $x, y \in \text{Fix}^c(h)$ and $h(B) \cap B = \emptyset$, then $x, y \notin B$. Therefore, $\tilde{F}_t(\varphi_t(x)) = \tilde{F}_t(\varphi_t(y)) = c_0$ for every $t$. Thus, we conclude that for every $t$, the symplectic area of $\Sigma_t$ vanishes as required.

**Proof of Proposition 2.2.** We shall proceed along the following lines. By the Lagrangian tubular neighborhood theorem (see [W]), there exists a symplectic identification between a small tubular neighborhood $U_s$ of $L_s$ in $M \times M$ and a tubular neighborhood $V_s$ of the zero section in the cotangent bundle $T^*L_s$. Moreover, it follows from a standard compactness argument that there exists $\delta_s = \delta(s, U_s) > 0$ such that $L_{s'} \subset U_s$ for every $s'$ with $|s' - s| \leq \delta_s$. Next, denote $I_s = (s - \delta_s, s + \delta_s) \cap [0, t]$, and consider an open cover of the interval $[0, t]$ by the family $\{I_s\}$, that is $[0, t] = \bigcup_{s \in [0, t]} I_s$. By compactness we can choose a finite number of points $S = \{s_1 < \ldots < s_n\}$ such that $[0, t] = \bigcup_{i=1}^n I_{s_i}$. Without loss of generality we may assume that $I_{s_j} \cap I_{s_{j+1}} = \emptyset$. Now, for every $s \in S$, we will construct a Hamiltonian function $\tilde{H}_s : U_s \to \mathbb{R}$ such that the corresponding Hamiltonian flow will shift $L_s$ toward $L_{s'}$ for $s' \in I_s$, and will leave the diagonal invariant. Next, by smoothly patching together those Hamiltonian flows on the intersections $U_{s_i} \cap U_{s_{i+1}}$, we will achieve the required Hamiltonian isotopy $\Phi$.

We fix $s_0 \in S$. Let $(p, q)$ be canonical local coordinates on $T^*L_{s_0}$ (where $q$ is the coordinate on $L_{s_0}$ and $p$ is the coordinate on the fiber). Moreover, we fix a Riemannian metric on $L_{s_0}$, and denote by $\| \cdot \|_{s_0}$ the induced fiber norm on $T^*L_{s_0}$. Consider the aforementioned tubular neighborhood $U_{s_0}$ of $L_{s_0}$ in $M \times M$. For every $x \in L_{s_0} \cap \triangle$ denote by $\sigma_{s_0}(x)$ the component of the intersection of $U_{s_0}$ and $\triangle$ containing the point $x$. Note that we may choose $U_{s_0}$ small enough such that the sets $\{\sigma_{s_0}(x)\}, x \in L_{s_0} \cap \triangle$, are mutually disjoint. In what follows we shall denote the image of $\sigma_{s_0}(x)$ under the above identification between $U_{s_0}$ and $V_{s_0}$, by $\sigma_{s_0}(x)$ as well.

We first claim that there exists a Hamiltonian symplectomorphism $\tilde{\varphi} : V_{s_0} \to V_{s_0}$ which for every intersection point $x \in L_{s_0} \cap \triangle$ sends $\sigma_{s_0}(x)$ to the fiber over $x$ and which leaves $L_{s_0}$ invariant. Indeed, since $L_{s_0}$ transversely intersects the diagonal, and since $\sigma_{s_0}(x)$ is a Lagrangian submanifold, $\sigma_{s_0}(x)$ is the graph of a closed 1-form of $p$-variable i.e, $\sigma_{s_0}(x) = \{(p, \alpha(p))\}$ where $\alpha(p)$ is locally defined near the intersection point $x$, and $\alpha(0) = 0$. Define a family of local diffeomorphisms by $\varphi_t(p, q) = (p, q - t\alpha(p))$. Since the 1-form $\alpha(p)$ is closed, $\{\varphi_t\}$ is a Hamiltonian flow.
Denote by $K(p,q)$ the Hamiltonian function generating $\{\varphi_t\}$. A simple computation shows that $K(p,q) = -\int \alpha(p)dp$. Hence $K(p,q)$ is independent on the $q$-variable i.e, $K(p,q) = K(p)$. Furthermore, we may assume that $K(0) = 0$. Next, we cut off the Hamiltonian function $K(p)$ outside a neighborhood of the intersection point $x$. Let $\beta(r)$ be a smooth cut-off function that vanishes for $r \geq 2\varepsilon$ and equal to 1 when $r \leq \varepsilon$, for sufficiently small $\varepsilon$. Define

$$\tilde{K}(p,q) = \beta(\|p\|) \cdot \beta(\|q\|) \cdot K(p).$$

A straightforward computation shows that, $\frac{\partial \tilde{K}}{\partial q}(0, \cdot) = \frac{\partial \tilde{K}}{\partial p}(0, \cdot) = 0$. Hence the time-1-map of the Hamiltonian flow corresponding to $\tilde{K}(p,q)$ is the required symplectomorphism. Therefore, we now can assume that $\sigma_{s_0}(x)$ coincide with the fiber over the point $x$.

Next, since $\Psi$ is an exact Lagrangian isotopy, we have that for every $s \in I_{s_0}$, $L_s$ is a graph of an exact 1-form $dG_s$ in the symplectic tubular neighborhood $V_{s_0}$ of $L_{s_0}$. Hence, in the above local coordinates $(p,q)$ on $T^*L_{s_0}$, $L_s$ takes the form $L_s = (dG_s(q), q)$. Moreover, note that $dG_s(0) = 0$.

Define

$$\tilde{H}_{s_0}(p,q) = \beta(\|p\|) \cdot G_s(q).$$

Consider the Hamiltonian vector field corresponding to $\tilde{H}_{s_0}$,

$$\tilde{\xi} = \begin{cases} \dot{p} = -\frac{\partial \tilde{H}}{\partial q} = -\beta(\|p\|) \cdot \frac{\partial G_s(q)}{\partial q} \\ \dot{q} = \frac{\partial \tilde{H}}{\partial p} = \frac{\partial}{\partial p} (\beta(\|p\|)) \cdot G_s(q) \end{cases}$$

It follows that for every $s \in I_{s_0}$ such that $L_s \subset \{(p,q) \mid \|p\| < \varepsilon\}$, the Hamiltonian flow is given by

$$(p,q) \to \left(p + \frac{\partial G_s(q)}{\partial q}, q\right)$$

Hence, locally, the Hamiltonian flow shift $L_{s_0}$ toward $L_s$ as required. It remains to prove that $\tilde{\xi}$ vanishes on the diagonal. First, since $dG_s(0) = 0$, it follows that $\dot{p} = 0$. Next, consider $x$ and $y$, two intersection points of the family $\{L_s\}$ and the diagonal. It follows from Lemma 5.1 that the symplectic area between $L_{s_0}$ and $L_s$ in $V_{s_0}$ vanishes for every $s \in I_{s_0}$. Hence, by the same argument as in Lemma 5.1, for every such $s$ we have

$$0 = \int_{\Sigma_s^1} \omega = \int_{[0,s] \times [0,1]} \Sigma_s^1 \omega = \int_0^s \left(G_s(x) - G_s(y)\right) ds$$

Thus, we get that $G_s(x) - G_s(y) = 0$. Note that by changing the functions $\{G_s\}$ by a summand depending only on $s$, we can assume that for every $s$, $G_s$ vanishes on $L_s \cap \Delta$. It now easily follows that $\xi_{|\Delta} = 0$. Therefore, we have that the diagonal is invariant under the Hamiltonian flow. Finally, by (smoothly) patching together all the Hamiltonian flows corresponding to the Hamiltonian functions $\tilde{H}_i$, for $i = 1, \ldots, n$, we conclude that there exists a Hamiltonian isotopy $\Phi$ such that $\Phi_s(L_0) = L_s$ and $\Phi_s(\Delta) = \Delta$. This completes the proof of the proposition. \[\square\]

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