Marginally stable circular orbit of a test body in spherically symmetric and static spacetimes

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We study a marginally stable circular orbit (MSCO) such as the innermost stable circular orbit (ISCO) of a timelike geodesic in any spherically symmetric and static spacetime. It turns out that the metric components are separable from the constants of motion along geodesics. We show also that a metric component $g_{rr}$ with a radial coordinate $r$ does not affect MSCOs. This suggests that, as a test of gravity, any ISCO measurement may be put into the same category as gravitational redshift experiments. MSCOs for exact solutions to the Einstein’s equation are also mentioned.

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I. INTRODUCTION

In Newtonian gravity, any radius of a circular orbit for a test body is possible and stable around a massive object that is spherically symmetric. In general relativity, on the other hand, the orbital radius has a lower bound that is called the innermost stable circular orbit (ISCO). The existence of the ISCO in the Schwarzschild solution for the Einstein’s equation is fascinating [1]. Moreover, ISCOs may play key roles in astrophysics as well as in gravity theory. For instance, ISCOs are of great importance in gravitational waves astronomy [2], because ISCOs are thought to be the location at the transition from the inspiralling phase to the merging one, especially when a compact object is orbiting around a massive black hole probably located at a galactic center. Furthermore, in high energy astrophysics, ISCOs are related to the existence for the inner edge of an accretion disk around a black hole [3]. It is thus expected that a measurement of the ISCO radius will bring us important information on the strong gravity, especially the nonlinear spacetime geometry that is beyond the solar-system tests. For near-future astrophysical tests of the no-hair theorem for black holes, therefore, it is intriguing to study the nature of the ISCOs in non-Schwarzschild spacetimes such as black holes with electric charges and/or scalar fields [4] and black holes in modified gravity theories [5].

It has been thought that unstable circular orbits would be generally produced by strong gravity in general relativity. In order to discuss the existence of such stable (or unstable) circular orbits, we must study not only the orbit conditions following the timelike geodesic equation but also the orbit stability. For the Schwarzschild case, there is no upper bound on a radius of a stable circular orbit. If the cosmological constant is added into the Einstein’s equation, the situation is changed. Stuchlik and Hledik [6] have pointed out that the outermost stable circular orbit (OSCO) of a test body is possible in the Kottler (often called the Schwarzschild-de Sitter) spacetime [7]. The ISCO and OSCO are a boundary between a stable region and an unstable one. Hereafter, we call it a marginally stable circular orbit (MSCO). If MSCOs are two and only two in a spacetime, one MSCO may correspond to the ISCO and the other may correspond to the OSCO. Note that the number of MSCOs might be larger than two, if such a spacetime geometry is described by a very complicated form of the metric. For this case, the smallest MSCO may be called the ISCO and the largest one may be the OSCO.

Calculations of the ISCO radius in the Schwarzschild spacetime are known in textbooks [1]. They introduce the effective potential $V_{eff}(r)$ with the constants of motion such as the specific energy and the specific angular momentum of the test body. This method works pretty well for the Schwarzschild spacetime, because the problem is expressed by using a quadratic equation [1]. Even for the Kottler case, however, this method does not work so well, because a quintic equation cannot be solved generally by hand (algebraically) [8]. Stuchlik and Hledik [6] have nicely reduced this quintic equation for the Kottler case to a quartic one and they have shown that the ISCO radius remains the same as that for the Schwarzschild case. Yet, except for such particular cases, the effective-potential method is not so useful, mostly because, in the effective potential, the metric components and the constants of motion for the test body such as the specific energy and the specific angular momentum are tightly coupled to each other.

The main purpose of this paper is to study MSCOs of a timelike geodesic in any spherically symmetric and static spacetime that may have a deficit angle. We show that the metric components in the equations under study are separable from the parameters of the test body. In particular, it turns out that a metric component $g_{rr}$ does not affect the radius of a MSCO, where $r$ denotes a radial coordinate. Furthermore, we derive a single equation for the MSCO radius. This equation is used for investigating the existence of a MSCO (and its radius if it exists) for a spherically symmetric and static spacetime. Several examples are discussed: Schwarzschild, Kottler [7], Reissner-Nordström (RN) [9], and Janis-Newman-Winicour (JNW) spacetimes [10].

Throughout this paper, we use the unit of $G = c = 1$. 
II. MARGINALLY STABLE CIRCULAR ORBIT

We consider spherically symmetric and static spacetimes. A general form of the line element for these spacetimes is

\[ ds^2 = -A(r)dt^2 + B(r)dr^2 + C(r)(d\theta^2 + \sin^2 \theta d\phi^2), \]

where we assume \( g_{tt} = -A(r) < 0 \), \( g_{rr} = B(r) > 0 \), \( g_{\theta\theta} = C(r) > 0 \), namely the spacetime signature as \(-,+,+,+\). Note that putting an ansatz as \( C(r) = r^2 \) would prevent us from properly describing a spacetime with a deficit angle such as the Barriola-Vilenkin monopole that is closely related to a static string [11].

Without loss of the generality, we can focus on the equatorial plane \( \theta = \pi/2 \), because the spacetime under study is spherically symmetric. For this equatorial case, the orbit of a test particle in this spacetime is determined by

\[ -d\tau^2 = -A(r)dt^2 + B(r)dr^2 + C(r)d\phi^2, \]

where \( \tau \) denotes the proper time along a timelike geodesic. One can define the Lagrangian of the test body as

\[ L = -A(r)t^2 + B(r)r^2 + C(r)\dot{\phi}^2, \]

where the dot denotes the derivative with respect to the proper time. Because the spacetime under study is static and spherically symmetric, there exist two constants of motion for the test body, associated with two Killing vectors, as

\[ E = \frac{1}{2} \frac{\partial L}{\partial \dot{t}} = -A(r)\dot{t}, \]
\[ L = \frac{1}{2} \frac{\partial L}{\partial \dot{\phi}} = C(r)\dot{\phi}, \]

where \( E \) and \( L \) are corresponding to the specific energy and the specific angular momentum, respectively.

With the help of the constants of motion, the time derivatives as \( \dot{t} \) and \( \dot{\phi} \) are written as

\[ \dot{t} = -\frac{E}{A(r)}, \]
\[ \dot{\phi} = \frac{L}{C(r)}. \]

Substituting Eqs. (6) and (7) into Eq. (2) leads to the orbit equation for the particle as

\[ \ddot{r}^2 = \frac{1}{B(r)} \left( \frac{E^2}{A(r)} - \frac{L^2}{C(r)} - 1 \right) \equiv -V(r), \]

where the unity in the right-hand side comes from the rest mass energy of the test body. Note that the last term \(-1\) does not appear, if we consider a light-like particle. It is known that the impact parameter as a ratio \( L/E \) determines completely the photon orbit, while the timelike particle orbit is described by two parameters as the energy and the angular momentum. Note that \( V(r) \) is different from the so-called effective potential.

By taking the derivative of Eq. (5) with respect to the proper time, we obtain the radial acceleration of the test body as

\[ \ddot{r} = -\frac{1}{2} \frac{dV(r)}{dr}. \]

This equation is nothing but the \( r \) component of the time-like geodesic of the test particle as

\[ \ddot{r} + \Gamma_{\alpha\beta}^{r} u_{\alpha} u_{\beta} = 0, \]

where \( u^\alpha \) denotes the four velocity of the test particle.

Up to this point, the particle orbit is arbitrary. Henceforward, let us focus on a circular orbit. The condition for the circular orbit is equivalent to \( \dot{r} = 0 \) (momentarily circular condition) and \( \ddot{r} = 0 \) (permanently circular condition). Note that \( \dot{r} = 0 \) is not sufficient, because it is satisfied at a particular location on the particle orbit like the periastron in Kepler orbits.

Next, let us discuss the linear stability of the circular orbit. We assume an infinitesimal displacement \( \delta r \) around the circular orbit as

\[ r = r_C + \delta r, \]

where \( r_C \) is the radius of the circular orbit (\( \dot{r}_C = \ddot{r}_C = 0 \)). This decomposition of \( r \) is substituted into Eq. (4) to obtain the equation for the displacement at the linear order as

\[ \frac{d^2}{dr^2}(\delta r) = -\frac{1}{2} \frac{d^2V(r_C)}{dr_C^2} \delta r, \]

where we used \( dV(r_C)/dr_C = 0 \). The stable (or unstable) condition is thus expressed by \( d^2V(r_C)/dr_C^2 > 0 \) (or \( d^2V(r_C)/dr_C^2 < 0 \)).

A MSCO is a transition point between stable circular orbits and unstable ones. Namely, the circular orbit \( r = r_C \) is marginally stable, if and only if \( V(r_C) = 0 \), \( dV(r_C)/dr = 0 \) and \( d^2V(r_C)/dr^2 = 0 \). This set of the equations can be equivalently rearranged as

\[ \frac{E^2}{A(r)} - \frac{L^2}{C(r)} - 1 = 0, \]

\[ E^2 \frac{d}{dr} \left( \frac{1}{A(r)} \right) - \frac{L^2}{C(r)} \frac{d}{dr} \left( \frac{1}{C(r)} \right) = 0, \]

\[ E^2 \frac{d^2}{dr^2} \left( \frac{1}{A(r)} \right) - \frac{L^2}{C(r)} \frac{d^2}{dr^2} \left( \frac{1}{C(r)} \right) = 0, \]

where the terms including \( dB(r)/dr \) and \( d^2B(r)/dr^2 \) vanish owing to Eq. (13). Therefore, \( B(r) \) makes no contribution to MSCOs. Moreover, any circular orbit is not affected by \( B(r) \). See [12] for more detail.
The present problem is rephrased geometrically as how to find a common intersection of three lines in $E^2-L^2$ plane. Eqs. (13)-14 are rearranged as
\[
\begin{pmatrix}
\frac{1}{A(r)} & \frac{-1}{C(r)} & -1 \\
\frac{d}{dr} \left( \frac{1}{A(r)} \right) & \frac{-d}{dr} \left( \frac{1}{C(r)} \right) & 0 \\
\frac{d^2}{dr^2} \left( \frac{1}{A(r)} \right) & \frac{-d^2}{dr^2} \left( \frac{1}{C(r)} \right) & 0
\end{pmatrix}
\begin{pmatrix}
E^2 \\
L^2 \\
1
\end{pmatrix}
= \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
\tag{16}
\]
This is in a very interesting form, in the sense that the geometrical part including $A(r)$ and $C(r)$ is separated from the particle parameters as $E$ and $L$. Namely, the $3 \times 3$ matrix in Eq. (16) depends only on the spacetime geometry without including any particle parameter. The determinant of the matrix vanishes, if $(E^2, L^2, 1)$ satisfy Eq. (16). Here, we should remember that the term of the unity is due to a timelike geodesic.

Therefore, a necessary condition for the existence of a MSCO is that the determinant of the matrix in Eq. (16) vanishes. This is explicitly written as
\[
dr \left( \frac{1}{A(r)} \right) \frac{d^2}{dr^2} \left( \frac{1}{C(r)} \right) - \frac{d}{dr} \left( \frac{1}{C(r)} \right) \frac{d^2}{dr^2} \left( \frac{1}{A(r)} \right) = 0.
\tag{17}
\]
This equation determines the radius of the MSCO, if the MSCO exists. Hereafter, we call Eq. (17) MSCO equation. Note that the MSCO equation does not contain a metric component $B(r)$ nor the constants of motion as $E$ and $L$. When Eq. (17) is satisfied by some $r$, Eq. (14) coincides with Eq. (15). This means geometrically that the two lines are identical.

It seems that Eq. (17) is the same as what has been recently derived by Rezzolla and Zhidenko as Eq. (41) in Ref. [13]. However, we stress that the present paper considers MSCOs, while they focused on ISCOs and they assumed also the asymptotic flatness and $C(r) = r^2$ at the very beginning of their calculations. Moreover, the above results, especially on $B(r)$, cannot be reached by using the effective-potential formulation of Ref. [13], mainly because their metric ansatz for $g_{rr}$ contains the inverse of $g_{rr} = -N(r)^2$ in their notation.

Given a root for Eq. (17), Eq. (16) is solved for $E^2$ and $L^2$ as
\[
E^2 = -\frac{1}{\Delta} \frac{d}{dr} \left( \frac{1}{C(r)} \right),
\tag{18}
\]
\[
L^2 = -\frac{1}{\Delta} \frac{d}{dr} \left( \frac{1}{A(r)} \right),
\tag{19}
\]
where we define a determinant as
\[
\Delta = \left| \begin{array}{cc}
\frac{1}{A(r)} & -\frac{1}{C(r)} \\
\frac{d}{dr} \left( \frac{1}{A(r)} \right) & -\frac{d}{dr} \left( \frac{1}{C(r)} \right)
\end{array} \right|.
\tag{20}
\]
Any root $r$ for Eq. (17) can be substituted into Eqs. (18) and (19) in order to see whether the sufficient condition as $0 \leq E^2 < \infty$ and $0 \leq L^2 < \infty$ is satisfied. If the sufficient condition is satisfied, this $r$ is a MSCO radius, denoted as $r_{MSCO}$. If not, the root is unphysical and it must be discarded. Note that the expressions of $E^2$ and $L^2$ in Eqs. (18) and (19) are applicable not only to MSCOs but also to any circular orbit, because Eqs. (13) and (14) hold for arbitrary circular orbits regardless of their stability.

Before closing this section, we mention a circular orbit of a light-like geodesic. The null geodesic is fully characterized by the impact parameter of the light path. The equation for the photon sphere (light surface) was derived for spherically symmetric spacetimes by Claudel, Virbhadra and Ellis [14] and in a gravitational lensing formulation by Virbhadra and Ellis [15]. See Eq. (54) in Ref. [14] and Eq. (11) in Ref. [15]. They show that $B(r) = g_{rr}$ does not appear in the photon sphere radius, though it is not assumed to vanish.

III. APPLICATION TO EXACT SOLUTIONS FOR THE EINSTEIN’S EQUATION

In this section, we apply the above result to some of exact solutions of the Einstein’s equation.

A. Schwarzschild spacetime

Let us begin with the Schwarzschild spacetime as
\[
ds^2 = -\left( 1 - \frac{r_g}{r} \right) dt^2 + \frac{dr^2}{1 - \frac{r_g}{r}} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2),
\tag{21}
\]
where the Schwarzschild radius is defined as $r_g = 2M$ for the ADM mass $M$. For this spacetime metric, Eq. (17) becomes
\[
r - 3r_g = 0,
\tag{22}
\]
where we assume $r \neq 0$ because $r = 0$ is the spacetime singularity. Hence, we obtain $r_{MSCO} = 3r_g$ in agreement with the well-known fact.

B. Kottler (Schwarzschild-de Sitter) spacetime

Next, we consider the Kottler spacetime [7]. The line element is
\[
ds^2 = -\left( 1 - \frac{r_g}{r} - \frac{\Lambda}{3} r^2 \right) dt^2 + \frac{dr^2}{1 - \frac{r_g}{r} - \frac{\Lambda}{3} r^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2),
\tag{23}
\]
where $\Lambda$ denotes the cosmological constant. Note that this spacetime is not asymptotically flat. The timelike geodesics in the Kottler spacetime have been often studied by several authors [16], especially on the periastron
shift, though none of them has examined the allowed region of the orbits.

For this spacetime, the MSCO equation becomes

\[
\frac{8}{3} \lambda r^4 - 5r_g \lambda r^3 - r_g r + 3r_g^2 = 0,
\]

where we assume \( r \neq 0 \) and \( r \neq \infty \). It is convenient to use the normalized variables in terms of the Schwarzschild radius \( r_g \). We define \( x \equiv r/r_g \) and \( \lambda \equiv \lambda r_g^2/3 \). The above quartic equation becomes

\[
8\lambda x^4 - 15\lambda x^3 - x + 3 = 0.
\]

What we have to do is to investigate positive zeros for this quartic equation. Sturm’s theorem [8] expresses the number of distinct real roots of a polynomial \( p \) located in an interval in terms of the number of changes of signs of the values of the Sturm’s sequence at the bounds of the interval. This theorem tells that there are three cases: If \( 0 < \lambda < 16/16875 \), we have two positive zeros, namely two MSCOs, where one is corresponding to the ISCO and the other is the OSCO. If \( 16/16875 < \lambda \), there is no MSCO (after the ISCO and the OSCO merge at \( \lambda = 16/16875 \)). This implies that every circular orbit becomes unstable for this case.

C. Reissner-Nordström spacetime

Here, we study Reissner-Nordström spacetime as [9]

\[
ds^2 = -\left(1 - \frac{r_g}{r} + \frac{e^2}{r^2}\right) dt^2 + \left(1 - \frac{r_g}{r} + \frac{e^2}{r^2}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2),
\]

which describes a charged black hole for \( 4e^2 < r_g^2 \), and a naked singularity for \( 4e^2 > r_g^2 \). For this spacetime, the MSCO equation becomes cubic as

\[
r_g r^3 - 3r_g^2 r^2 + 9e^2 r_g r - 8e^4 = 0.
\]

Eq. (27) is rewritten as

\[
x^3 - (3 - 9q^2)x - (2 - 9q^2 + 8q^4) = 0,
\]

where we define \( x \equiv (r/r_g) - 1 \) and \( q \equiv e/r_g \) (\( q^2 < 1/4 \) for a black hole and \( q^2 > 1/4 \) for a naked singularity). It can be shown that there is only the single MSCO if and only if \( 0 < e^2 < 5r_g^2/16 \), while there is no MSCO for \( 5r_g^2/16 < e^2 \).

By using the Vieta’s theorem for a cubic equation [3], the MSCO radius is obtained as

\[
r_{MSCO} = r_g + 2r_g \sqrt{1 - 3q^2} \cos \left[\frac{1}{3} \arccos P\right],
\]

where \( P \) denotes

\[
P = \frac{2 - 9q^2 + 8q^4}{2(1 - 3q^2)^{1/2}} \sqrt{1 - 3q^2}.
\]

Note that \( P \) approaches the unity as \( e \rightarrow 0 \) (\( q \rightarrow 0 \)). Hence, Eq. (20) recovers \( 3r_g \) in this limit.

D. Janis-Newman-Winicour spacetime

Finally, let us examine the JNW solution, where the JNW metric is expressed by [10]

\[
ds^2 = -\left(1 - \frac{r_g}{\gamma r}\right) dt^2 + \left(1 - \frac{r_g}{\gamma r}\right)^{-1} dr^2 + \left(1 - \frac{r_g}{\gamma r}\right)^{1-\gamma} r^2 (d\theta^2 + \sin^2 \theta d\phi^2).
\]

This solution is characterized by the ADM mass \( M = r_g/2 \) and the scalar charge \( q \). Here, the parameter \( \gamma \) is defined as

\[
\gamma = \frac{M}{\sqrt{M^2 + q^2}},
\]

where it satisfies \( 0 < \gamma \leq 1 \). Note that the JNW spacetime approaches the Schwarzschild one as \( q \rightarrow 0 \), namely \( \gamma \rightarrow 1 \).

The MSCO equation becomes quadratic as

\[
2\gamma^2 r^2 - 2(1 + 3\gamma)\gamma r_g r + (1 + \gamma)(1 + 2\gamma)r_g^2 = 0.
\]

Note that this equation agrees with that for the Schwarzschild case as \( \gamma \rightarrow 1 \).

Eq. (33) has two roots as

\[
r = \frac{(1 + 3\gamma) \pm \sqrt{-1 + 5\gamma^2 r_g^2}}{2\gamma}.
\]

Therefore, there are three cases. If \( 0 < \gamma < 1/\sqrt{5} \), no MSCO appears. If \( 1/\sqrt{5} < \gamma < 1/2 \), there exist two MSCOs (namely, one ISCO and one OSCO). If \( 1/2 < \gamma \leq 1 \), a single MSCO (namely, ISCO) is possible. It follows that, for the marginal case as \( \gamma = 1/\sqrt{5} \), the ISCO and OSCO merge into the single MSCO with a radius as \((\sqrt{5} + 3)r_g/2\).

IV. CONCLUSION

We studied the equations for MSCOs such as ISCOs of a timelike geodesic in any spherically symmetric and static spacetime that may have a deficit angle. We found that the metric components in the equations are separable from the constants of motion as the specific energy and angular momentum. We showed also that a metric component \( g_{rr} \) does not affect any MSCO radius, where \( r \) denotes a radial coordinate. This suggests that, as a
gravity test, any measurement of the ISCO may be put into the same category as gravitational redshift experiments, even in the strong field region.

Moreover, the MSCO equation for the radius was derived as a necessary condition for the existence of a MSCO. The sufficient condition is that \( 0 \leq E^2 < \infty \) and \( 0 \leq L^2 < \infty \) are satisfied.

Several examples were also mentioned: Schwarzschild, Kottler (often called Schwarzschild-de Sitter), Reissner-Nordström, and Janis-Newman-Winicour (JNW) spacetimes.

A generalization of the present formulation to a stationary and axially symmetric spacetime is left as a future work.

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