A direct method for solving inverse Sturm–Liouville problems*

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Abstract

We consider two main inverse Sturm–Liouville problems: the problem of recovery of the potential and the boundary conditions from two spectra or from a spectral density function. A simple method for practical solution of such problems is developed, based on the transmutation operator approach, new Neumann series of Bessel functions representations for solutions and the Gelfand–Levitan equation. The method allows one to reduce the inverse Sturm–Liouville problem directly to a system of linear algebraic equations, such that the potential is recovered from the first element of the solution vector. We prove the stability of the method and show its numerical efficiency with several numerical examples.

Keywords: Sturm–Liouville spectral problem, Gelfand–Levitan equation, inverse spectral problem, transmutation operator, Neumann series of Bessel functions, Fourier–Legendre series

(Some figures may appear in colour only in the online journal)

1. Introduction

We consider classical inverse Sturm–Liouville problems on a finite interval. Several techniques have been proposed for their numerical solution (see [6, 8–10, 13, 14, 17, 23, 26–28, 30, 31]). However, usually the methods proposed require either the knowledge of additional parameters...
like, e.g., the mean of the potential or do not offer a complete solution of the problem, e.g., they may not be able to recover the boundary conditions. Often both mentioned drawbacks are present. For example, one of the most successful algorithms, proposed in [30], requires an estimate of the mean of the potential as well as the knowledge of the boundary conditions. Moreover, the authors write [30, p 177]: ‘as was mentioned earlier, if complete spectral data is available, then it is in theory possible to determine the boundary conditions as part of the solution of the problem. We do not believe, however, that this is numerically feasible in most cases.’ The boundary conditions are assumed to be known in [6, 10, 14, 28]. The method from [13] as many other approaches requires the knowledge of a parameter \( \omega \) related to the mean of the potential and participating in the second term of the asymptotics of the eigenvalues.

The approach developed in the present work allows one to obtain a complete solution of the inverse Sturm–Liouville problem without requiring any additional data. Moreover, we show that the parameter \( \omega \) can be obtained on a first step without using the asymptotics of the eigenvalues. The method is based on the idea presented in [17, 18] where it was shown that the inverse Sturm–Liouville problem of recovering the problem by its spectral density function (to which we refer as Problem 1, see its precise statement below, as Problem 2.1) can be reduced directly to a system of linear algebraic equations by making use of the Gelfand–Levitan integral equation and the Fourier–Legendre series expansion of the transmutation integral kernel obtained in [19]. Moreover, for recovering the potential and the boundary conditions there is no need to solve the Gelfand–Levitan equation and find the transmutation kernel. The first coefficient of the Fourier–Legendre series which corresponds to the first component of the solution vector of the linear algebraic system is sufficient for recovering the Sturm–Liouville problem.

In the present work we improve the approach from [17, 18] by using the representation for the Gelfand–Levitan kernel \( F(x,t) \) obtained recently in [16]. We prove the stability of the method and extend it onto the inverse Sturm–Liouville problem by two spectra (Problem 2, see its statement below as Problem 2.2).

The series representation for \( F(x,t) \) from [16] to the difference from the classical representation [11] does not present a jump discontinuity and, moreover, converges uniformly on the whole domain of definition. However, it requires the knowledge of the parameter \( \omega \). We show that as a consequence of the Fourier–Legendre series representation of the transmutation kernel the parameter \( \omega \) can be efficiently computed from first eigenvalues without exploiting their asymptotics. We obtain a system of linear algebraic equations for the coefficients of the Fourier–Legendre series based on the representation for \( F(x,t) \) from [16], show that its truncated version is uniquely solvable, and the numerical process is stable. Besides, for a still more accurate solution of Problem 1 some auxiliary techniques are presented. The first allows us to obtain the ‘flipped’ problem, that is the Problem 1 for the flipped potential \( q(\pi - x) \) with the boundary conditions being interchanged. We show how the norming constants of this problem are obtained from the spectral data of the original problem, and this result is also based on the Fourier–Legendre series representation of the transmutation kernel. The second technique serves for computing the second term of asymptotics of the norming constants as well as some further asymptotic terms of the eigenvalues.

Another result of the present work is the reduction of Problem 2 to Problem 1 which gives an excellent numerical method for solving the two spectra problem. The reduction uses again the Fourier–Legendre series expansion of the transmutation integral kernel.

The method developed here is accurate and fast and allows one to obtain a complete solution of the inverse Sturm–Liouville problem which includes not only the potential but also the boundary conditions. We illustrate its numerical performance by several examples including a smooth, non-smooth and a discontinuous potentials.
2. Preliminaries

2.1. Spectral data

Let \( q \in L^2(0, \pi) \) be real valued. Consider the Sturm–Liouville equation
\[
-\psi'' + q(x)\psi = \rho^2 \psi, \quad x \in (0, \pi),
\] (2.1)
where \( \rho \in \mathbb{C} \), and two sets of boundary conditions
\[
y'(0) - hy(0) = 0, \quad y'(\pi) + Hy(\pi) = 0,
\] (2.2)
and
\[
y'(0) - hy(0) = 0, \quad y(\pi) = 0,
\] (2.3)
where \( h \) and \( H \) are arbitrary real constants.

Denote
\[
\omega := h + H + \frac{1}{2} \int_0^\pi q(t) \, dt
\] (2.4)
and
\[
\omega_1 := h + \frac{1}{2} \int_0^\pi q(t) \, dt = \omega - H.
\] (2.5)

By \( \varphi(\rho, x) \) we denote a solution of (2.1) satisfying the initial conditions
\[
\varphi(\rho, 0) = 1 \quad \text{and} \quad \varphi'(\rho, 0) = h.
\] (2.6)

Obviously, for all \( \rho \in \mathbb{C} \) the function \( \varphi(\rho, x) \) fulfills the first boundary condition, \( \varphi'(\rho, 0) - h\varphi(\rho, 0) = 0 \), and thus, the spectrum of problem (2.1), (2.2) is the sequence of numbers \( \{ \lambda_n = \rho_n^2 \}_{n=0}^\infty \) such that
\[
\varphi'(\rho_n, \pi) + H\varphi(\rho_n, \pi) = 0
\]
while the spectrum of problem (2.1), (2.3) is the sequence of numbers \( \{ \nu_n = \mu_n^2 \}_{n=0}^\infty \) such that
\[
\varphi(\mu_n, \pi) = 0.
\]

Denote
\[
\alpha_n := \int_0^\pi \varphi^2(\rho_n, x) \, dx.
\] (2.7)

The set \( \{ \alpha_n \}_{n=0}^\infty \) is referred to as the sequence of norming constants of problem (2.1), (2.2).

Let us formulate two basic inverse Sturm–Liouville problems.

**Problem 2.1 (Recovery of a Sturm–Liouville problem from its spectral function).**

Given two sequences of real numbers \( \{ \lambda_n \}_{n=0}^\infty \) and \( \{ \alpha_n \}_{n=0}^\infty \), find a real valued \( q \in L^2(0, \pi) \), and the constants \( h, H \in \mathbb{R} \), such that \( \{ \lambda_n \}_{n=0}^\infty \) be the spectrum of problem (2.1), (2.2) and \( \{ \alpha_n \}_{n=0}^\infty \) its sequence of norming constants.

**Problem 2.2 (Recovery of a Sturm–Liouville problem from two spectra).**

Given two sequences of real numbers \( \{ \lambda_n \}_{n=0}^\infty \) and \( \{ \nu_n \}_{n=0}^\infty \), find a real valued \( q \in L^2(0, \pi) \), and the
constants $h, H \in \mathbb{R}$, such that $\{\lambda_n\}_{n=0}^{\infty}$ be the spectrum of problem (2.1), (2.2) and $\{\nu_n\}_{n=0}^{\infty}$ the spectrum of problem (2.1), (2.3).

In order to assure the existence of a solution, the sequences of numbers appearing in the formulation of these two problems can not be completely arbitrary. The following criteria are valid.

**Theorem 2.3 (see [37, theorem 1.3.2]).** The sequences of real numbers $\{\lambda_n\}_{n=0}^{\infty}$ and $\{\alpha_n\}_{n=0}^{\infty}$ represent spectral data of a Sturm–Liouville problem (2.1), (2.2) if and only if the following relations hold

$$\rho_n := \sqrt{\lambda_n} = n + \frac{\omega}{\pi n} + \frac{k_n}{n}, \quad \alpha_n = \frac{\pi}{2} + \frac{K_n}{n}, \quad \{k_n\}, \{K_n\} \in \ell_2, \quad (2.8)$$

and

$$\alpha_n > 0, \quad \lambda_n \neq \lambda_m \ (n \neq m).$$

**Theorem 2.4 (see [37, theorem 1.3.4] and [32] for a generalization).** The sequences of real numbers $\{\lambda_n\}_{n=0}^{\infty}$ and $\{\nu_n\}_{n=0}^{\infty}$ are spectra of the Sturm–Liouville problems (2.1), (2.2) and (2.1), (2.3), respectively, if and only if the following relations hold

$$\mu_n := \sqrt{\nu_n} = n + \frac{1}{2} + \frac{\omega_1}{\pi n} + \frac{K_n}{n}, \quad \{k_n\}, \{K_n\} \in \ell_2, \quad (2.9)$$

and

$$\lambda_n < \nu_n < \lambda_{n+1}, \quad n \geq 0.$$

**2.2. The transmutation integral kernel and the Gelfand–Levitan equation**

The solution $\varphi(\rho, x)$ admits the integral representation (see, e.g., [22, chapter 6], [24, chapter 1], [33, chapter 3])

$$\varphi(\rho, x) = \cos \rho x + \int_0^x G(x, t) \cos \rho t \, dt, \quad (2.10)$$

where the transmutation integral kernel $G$ is a continuous function of both arguments in the domain $0 \leq t \leq x \leq \pi$ and satisfies the equalities

$$G(x, x) = h + \frac{1}{2} \int_0^x q(t) \, dt \quad (2.11)$$

and

$$\frac{\partial}{\partial t} G(x, t) \bigg|_{t=0} = 0.$$ 

It is of crucial importance that $G(x, t)$ is independent of $\rho$.

Let

$$F(x, t) = \sum_{n=0}^{\infty} \left( \frac{\cos \rho_n x \cos \rho_n t}{\alpha_n} - \frac{\cos \rho_n x \cos \rho_n t}{\alpha_n^0} \right), \quad 0 \leq t, x < \pi \quad (2.12)$$
where

\[ \alpha_n^0 = \begin{cases} \pi/2, & n > 0, \\ \pi, & n = 0. \end{cases} \]

The functions \( G \) and \( F \) are related by the Gelfand–Levitan equation

\[
G(x, t) + F(x, t) + \int_0^x F(t, s)G(x, s)\, ds = 0, \quad 0 < t < x < \pi. \tag{2.13}
\]

Due to a slow convergence of the series (2.12) and the presence of a jump discontinuity of the series (2.12) at \( x = t = \pi \) (we refer to [16] for more details), it was proposed in [18, section 13.4] to work with the integrated version of the Gelfand–Levitan equation

\[
\int_0^t G(x, \tau)\, d\tau + \tilde{F}(x, t) + \int_0^x G(x, s)\tilde{F}(s, t)\, ds = 0, \quad 0 < t < x \tag{2.14}
\]

obtained from (2.13) by using the symmetry \( F(t, s) = F(s, t) \) of the function \( F \) and integrating it with respect to \( t \). Here

\[
\tilde{F}(x, t) = \int_0^t F(x, \tau)\, d\tau + \sum_{n=1}^{\infty} \left( \frac{\cos \rho_0 x \sin \rho_0 t}{\alpha_0 \rho_n} - \frac{\cos nx \sin nt}{\alpha_n \rho_n} \right). \tag{2.15}
\]

The advantage of (2.15) lies in a faster convergence of the series.

We will be especially interested in the case when \( \rho_0 = 0 \). Then

\[
\tilde{F}(x, t) = \left( \frac{1}{\alpha_0} - \frac{1}{\alpha_0^\pi} \right) t + \sum_{n=1}^{\infty} \left( \frac{\cos \rho_0 x \sin \rho_0 t}{\alpha_0 \rho_n} - \frac{\cos nx \sin nt}{\alpha_n \rho_n} \right). \tag{2.16}
\]

However, the integrated Gelfand–Levitan equation is no more a Fredholm equation of the second kind (with respect to the function \( G(x, \cdot) \)). As a result, it is unclear whether it is possible to obtain stability results for corresponding numerical schemes of approximate solution of (2.14). Recall that for Fredholm equations there is a well-developed general theory, see, e.g., [15, chapter 14, section 4].

In [16] another representation for the function \( F \) was derived, based on the following idea. The convergence of the series (2.12) can be improved by subtracting certain expressions termwise and eventually adding a closed form expression equal to the infinite sum of subtracted functions. Namely (see [16, (25)]),

\[
F(x, t) = \sum_{n=1}^{\infty} \left( \frac{\cos \rho_0 x \cos \rho_0 t}{\alpha_0} - \frac{\cos nx \cos nt}{\alpha_n} \right) \\
+ \frac{2\omega}{\pi^2 n} \left( x \sin nx \cos nt + t \sin nt \cos nx \right) \\
+ \frac{1}{\alpha_0} - \frac{1}{\alpha_0^\pi} \left( \omega \left( \pi \max\{x, t\} - x^2 - t^2 \right) \right), \tag{2.17}
\]

here \( \omega \) is the parameter given by (2.4).
The series in (2.17) has no jump discontinuity at \( x = t = \pi \) and converges uniformly and absolutely and much faster than (2.12), allowing one to work directly with the Gelfand–Levitan equation (2.13) without passing to its integrated version (2.14).

2.3. Fourier–Legendre series expansion of the transmutation kernel

The following result from [19] will be used.

**Theorem 2.5 ([19]).** The integral transmutation kernel \( G(x, t) \) admits the following Fourier–Legendre series representation

\[
G(x, t) = \sum_{n=0}^{\infty} \frac{g_n(x)}{x} P_{2n} \left( \frac{t}{x} \right), \quad 0 < t \leq x \leq \pi,
\]

where \( P_k \) stands for the Legendre polynomial of order \( k \).

For every \( x \in (0, \pi] \) the series converges in the norm of \( L_2(0, x) \). The first coefficient \( g_0(x) \) has the form

\[
g_0(x) = \phi(0, x) - 1,
\]

and the rest of the coefficients can be calculated following a simple recurrent integration procedure.

Notice that from (2.11) and (2.18) the equality follows

\[
G(x, x) = \sum_{n=0}^{\infty} \frac{g_n(x)}{x}.
\]

**Remark 2.6.** Equality (2.19) shows that the potential \( q(x) \) can be recovered from \( g_0(x) \). Indeed,

\[
q(x) = \frac{\phi''(0, x)}{\phi(0, x)} = \frac{g_0''(x)}{g_0(x) + 1}.
\]

Moreover, the constant \( h \) is also recovered directly from \( g_0(x) \), since

\[
h = g_0'(0).
\]

2.4. Neumann series of Bessel functions representations for solutions and their derivatives

With the aid of theorem 2.5 the following series representations for the solution \( \phi(\rho, x) \) and for its derivative are obtained.

**Theorem 2.7 ([19]).** The solution \( \phi(\rho, x) \) and its derivative with respect to \( x \) admit the following series representations

\[
\phi(\rho, x) = \cos \rho x + \sum_{n=0}^{\infty} (-1)^n g_n(x) j_{2n}(\rho x)
\]

and

\[
\phi'(\rho, x) = -\rho \sin \rho x + \left( h + \frac{1}{2} \int_0^x q(s) \, ds \right) \cos \rho x + \sum_{n=0}^{\infty} (-1)^n \gamma_n(x) j_{2n}(\rho x),
\]

(2.23)
where \( j_k(z) \) stands for the spherical Bessel function of order \( k \), defined as \( j_k(z) := \sqrt{\pi/2} I_{k + 1/2}(z) \) (and \( I_l \) stands for the Bessel function of order \( l \)) (see, e.g., [1]). The coefficients \( g_n(x) \) are those from theorem 2.5, and the coefficients \( \gamma_n(x) \) can be calculated following a simple recurrent integration procedure, starting with

\[
\gamma_0(x) = \varphi'(0, x) - \frac{1}{2} \int_0^x q(s) \, ds.
\]

(2.24)

For every \( \rho \in \mathbb{C} \) both series converge pointwise. For every \( x \in [0, \pi] \) the series converge uniformly on any compact set of the complex plane of the variable \( \rho \), and the remains of their partial sums admit estimates independent of \( \text{Re} \rho \).

We refer the reader to [18, 19] for the proof and additional details related to this result. The coefficients \( g_n \) and \( \gamma_n \) decay as \( n \to \infty \), and the decay rate is faster for smoother potentials. Namely, the following result is valid.

Proposition 2.8 ([20]). Let \( q \in W_p[0, \pi] \) for some \( p \in \mathbb{N}_0 \), i.e., the potential \( q \) is \( p \) times differentiable with the last derivative being bounded on \([0, \pi]\). Then there exist constants \( c \) and \( d \), independent of \( N \), such that

\[
|g_N(x)| \leq \frac{c x^{p+2}}{(2N - 1)^{p+1/2}} \quad \text{and} \quad |\gamma_N(x)| \leq \frac{d x^{p+1}}{(2N - 1)^{p-1/2}}, \quad 2N \geq p + 1.
\]

3. Method of solution of problem 2.1

The idea of the method was proposed in [17], and its modification involving the integrated Gelfand–Levitan equation (2.14) instead of (2.13) was developed in [18, section 13.4]. We begin by briefly recalling that result.

From now on and without loss of generality, we suppose that \( \rho_0 = 0 \). This always can be achieved by a simple shift of the potential. If originally \( \rho_0 \neq 0 \), then we can consider the new potential \( \tilde{q}(x) := q(x) - \rho_0^2 \). Obviously, the eigenvalues \( \{ \lambda_n \}_{n=0}^\infty \) and \( \{ \nu_n \}_{n=0}^\infty \) shift by the same amount, while the numbers \( h, H \) and \( \{ \alpha_n \}_{n=0}^\infty \) do not change. After recovering \( \tilde{q}(x) \) from the shifted eigenvalues, one gets the original potential \( q(x) \) by adding \( \rho_0^2 \) back.

3.1. Main system of linear algebraic equations

Denote

\[
C_{km}(x) = \begin{cases} c_{k,k}(x) + \frac{1}{(4k + 1)(4k + 3)}, & \text{when } m = k, \\ c_{k,k+1}(x) - \frac{1}{(4k + 3)(4k + 5)}, & \text{when } m = k + 1, \\ c_{k,m}(x), & \text{otherwise}, \end{cases}
\]

(3.1)

\[
c_{k,0,m}(x) = \left( \frac{1}{\alpha_0} - \frac{1}{\pi} \right) \sqrt{\frac{\pi}{3}} + (-1)^m \sum_{n=1}^\infty \left( \frac{j_{2m}(\rho_n x) j_1(\rho_n x)}{\alpha_n \rho_n} - \frac{2 j_{2m}(n x) j_1(n x)}{\pi n} \right),
\]

(3.2)

\[
c_{k,m}(x) = (-1)^m \sum_{n=1}^\infty \left( \frac{j_{2m}(\rho_n x) j_{2k+1}(\rho_n x)}{\alpha_n \rho_n} - \frac{2 j_{2m}(n x) j_{2k+1}(n x)}{\pi n} \right), \quad k = 1, 2, \ldots,
\]

(3.3)
and
\[
d_k(x) = \left(\frac{1}{\pi} - \frac{1}{\alpha_0}\right) \frac{x \delta_{0k} \pi}{3} - (-1)^k \
\times \sum_{n=1}^{\infty} \left(\frac{\cos \rho_n x j_{2k+1}(\rho_n x)}{\alpha_n \rho_n} - \frac{2 \cos nx j_{2k+1}(nx)}{\pi n}\right),
\]
(3.4)
where $\delta_{0k}$ stands for the Kronecker delta. The series in these definitions converge uniformly with respect to $x \in [0, \pi]$. This follows from (2.8) and asymptotics of spherical Bessel functions
\[
j_k(t) \sim \sin \left(t - \frac{k \pi}{2}\right) + O \left(\frac{1}{t^2}\right), \quad \text{when } t \in \mathbb{R} \text{ and } t \rightarrow \infty.
\]

**Theorem 3.1 ([18]).** The coefficients $g_m(x)$ satisfy the system of linear algebraic equations
\[
\sum_{m=0}^{\infty} g_m(x) C_{km}(x) = d_k(x), \quad \text{for all } k = 0, 1, \ldots
\]
(3.5)
For all $x \in [0, \pi]$ and $k = 0, 1, \ldots$ the series in (3.5) converges.

It is of crucial importance the fact that for recovering the potential $q$ as well as the constant $h$ it is not necessary to compute many coefficients $g_m(x)$ (that would be equivalent to approximate solution of the integrated Gelfand–Levitan equation) but instead the sole $g_0(x)$ is sufficient for this purpose, see remark 2.6 as well as example 5.2 where we illustrate with numerical results the advantage of recovering $q$ from $g_0$ and not from the kernel $G$. The system (3.5) is obtained by using the integrated Gelfand–Levitan equation (2.14). As we show below, the results are even better and the stability of the method can be proved if the system of linear algebraic equations for the coefficients $g_m(x)$ is obtained from the Gelfand–Levitan equation (2.13) where the representation (2.17) for the input Gelfand–Levitan kernel $F(x, t)$ is used. Since the representation (2.17) requires the knowledge of the parameter $\omega$ we show first how it can be computed.

### 3.2. Recovery of $\omega$ and of the characteristic function

For all $k = 0, 1, \ldots$ from (2.2) and theorem 2.7 we have that
\[
\varphi'(\rho_k, \pi) + H \varphi(\rho_k, \pi) = -\rho_k \sin \rho_k \pi + \omega \cos \rho_k \pi \\
+ \sum_{n=0}^{\infty} (-1)^n (\gamma_n(\pi) + H g_n(\pi)) j_{2n}(\rho_k \pi) = 0.
\]
(3.6)
Let us denote
\[
h_n := \gamma_n(\pi) + H g_n(\pi), \quad n \geq 0.
\]
Later on we show that the coefficients $h_n$ are necessary components for the solution of problem 2.2. Also they can be used to perform the “flipping of the potential”, i.e., to transform Problem 1 into the inverse problem for the potential $q(\pi - x)$ having the parameters $h$ and $H$ interchanged, see subsection 3.6 for details.

Notice that due to the supposition $\rho_0 = 0$, we have that
\[
\varphi'(0, \pi) + H \varphi(0, \pi) = 0.
\]
From theorem 2.7 it follows that
\[
\varphi'(0, \pi) + H\varphi(0, \pi) = \gamma_0(\pi) + \omega_1 + H(g_0(\pi) + 1)
\]
\[
= \gamma_0(\pi) + Hg_0(\pi) + \omega_1 + H = h_0 + \omega.
\]

Thus,
\[
\omega = -h_0. \quad (3.7)
\]

This equality coincides with (3.6) for \( k = 0 \), while for \( k = 1, 2, \ldots \) equality (3.6) can be written in the form
\[
h_0 \cos \rho_k \pi - \sum_{n=0}^{\infty} (-1)^n h_n j_{2n}(\rho_k \pi) = -\rho_k \sin \rho_k \pi, \quad k = 1, 2, \ldots. \quad (3.8)
\]

Hence the numbers \( h_n \), including \( h_0 = -\omega \), can be approximated by solving the system
\[
h_0 \cos \rho_k \pi - \sum_{n=0}^{N_1-1} (-1)^n h_n j_{2n}(\rho_k \pi) = -\rho_k \sin \rho_k \pi, \quad k = 1, \ldots, N_1. \quad (3.9)
\]

Notice that the recovery of the coefficients \( h_n \) leads to the recovery of the characteristic function of the Sturm–Liouville problem (2.1), (2.2), \( \Phi(\rho) = \varphi'(\rho, \pi) + H\varphi(\rho, \pi) \) for all \( \rho \):
\[
\Phi(\rho) = -\rho \sin \rho \pi + \omega \cos \rho \pi + \sum_{n=0}^{\infty} (-1)^n h_n j_{2n}(\rho \pi).
\]

and for its calculation the knowledge of \( H \) is superfluous.

**Remark 3.2.** Solution of the system (3.9) leads to the recovery of the parameter \( \omega \), due to (3.7). Notice that this way of obtaining \( \omega \) does not involve the asymptotics of the eigenvalues.

The system (3.9) can be ill-conditioned for large values of \( N_1 \) (say, for more than 30 eigenvalues). Of course, when solving an inverse spectral problem one expects to make use of all known eigenvalues. Thus discarding some of them is not an option. To avoid the problem of the ill-conditioning, one can look for less coefficients \( h_n \) than the number of the given eigenvalues \( \rho_k \), i.e., to consider an overdetermined system
\[
h_0 \cos \rho_k \pi - \sum_{n=0}^{N_1} (-1)^n h_n j_{2n}(\rho_k \pi) = -\rho_k \sin \rho_k \pi, \quad k = 1, \ldots, N_1. \quad (3.10)
\]

choosing for the number of coefficients \( N_1 + 1 \) such value that the condition number of the coefficient matrix remains relatively small. The system can be solved using the Moore–Penrose pseudoinverse. Note that the exact coefficients \( h_n \) decay rather fast (see proposition 2.8), so taking less approximate coefficients does not present any problem.

**Remark 3.3.** As it follows from (2.2) and (2.10),
\[
0 = \varphi'(\rho_k, \pi) + H\varphi(\rho_k, \pi) = -\rho_k \sin \rho_k \pi + \omega \cos \rho_k \pi
\]
\[
+ \int_0^\pi \left( G'\pi(t) + HG(\pi, t) \right) \cos \rho_k t \, dt. \quad (3.11)
\]
Let \( G \colon [-\pi, \pi] \to \mathbb{R} \) be an even function such that \( G(t) = G'_0(\pi, t) + HG(\pi, t) \) for \( t \in [0, \pi] \). Then (3.11) can be written as
\[
\int_{-\pi}^{\pi} G(t) e^{i\rho_k t} \, dt = 2\rho_k \sin \rho_k \pi - 2\omega \cos \rho_k \pi, \quad \text{for} \ t \in [0, \pi].
\]
and equality (3.12) holds for all values \( \pm \rho_k, \ k \geq 0 \). Such problem is known as a non-harmonic moments problem \([4, 29, 36]\). Searching the solution of the problem (3.12) using the basis of Legendre polynomials results in the system (3.7), (3.8).

One can think of another way of solving the problem (3.12) using the fact that the system \( \{1, t\} \cup \{e^{\pm i\rho_k t}\}_{k=1}^{\infty} \) is a Riesz basis in \( L^2(-\pi, \pi) \). So the function \( G \) can be approximately recovered by the least squares method, see \([25, \text{section 5}]\). Taking into account the parity of the function \( G \), the minimization problem reduces to
\[
\min_{a_0, \ldots, a_n} \| G(t) - \sum_{k=0}^{n} a_k \cos \rho_k t \|_{L^2(0,\pi)},
\]
and its solution can be found from the linear system
\[
\sum_{k=0}^{n} a_k (\cos \rho_k t, \cos \rho_m t)_{L^2(0,\pi)} = (G(t), \cos \rho_m t)_{L^2(0,\pi)} \quad m = 0, \ldots, n. \quad (3.13)
\]
The system (3.13) is numerically stable, possessing a small and uniformly bounded with respect to \( n \) condition number, and its solution provides an approximation \( G_n(t) = \sum_{k=0}^{n} a_k \cos \rho_k t \) known to converge to the exact function \( G(t) \) in the \( L^2(0, \pi) \) space. A similar idea was implemented in \([30]\). Once the coefficients \( a_k \) are found, one can obtain approximate values of the coefficients \( h_m \) as well,
\[
h_m = (4m + 1) \int_{0}^{\pi} G(t) P_{2m} \left( \frac{t}{\pi} \right) \, dt \approx (4m + 1) \times \sum_{k=0}^{n} a_k \int_{0}^{\pi} P_{2m} \left( \frac{t}{\pi} \right) \cos \rho_k t \, dt \quad m = 0, 1, \ldots.
\]
Even though such scheme is numerically stable and approximate coefficients \( h_m \) are guaranteed to converge to the exact ones, the convergence is slow and in practice the approach based directly on the system (3.10) produces more accurate results. The main reason is that the coefficients \( a_k \) decay slowly, as \( 1/k^2 \), resulting in a large error in the approximate coefficients \( h_m \).

In the next section we develop an additional possibility for recovering the parameter \( \omega \) based on a more standard approach assuming the knowledge of a sufficient amount of spectral data allowing an efficient use of their asymptotics. It can be applied for calculating together with \( \omega \) some other asymptotic parameters of the spectral data.

3.3. Recovery of the parameter \( \omega \) and coefficients of the asymptotic expansions of eigenvalues \( \rho_n \) and norming constants \( \alpha_n \)

As it is well known (see, e.g., \([30]\)), the parameter \( \omega \), in principle, can be recovered using the asymptotics of the eigenvalues. Indeed, it immediately follows from (2.8) that
\[
\omega = \pi \lim_{n \to \infty} n(\rho_n - n).
\]
Note that due to (2.8) the sequence \( \{ n(\rho_n - n) - \frac{\omega}{\pi} \}_{n=0}^{\infty} = \{ k_n \}_{n=0}^{\infty} \) belongs to \( \ell_2 \). Suppose that only a finite set of eigenvalues \( \rho_0, \ldots, \rho_{N_1} \) is given. Then an approximate value of the parameter \( \omega \) can be found by minimizing the \( \ell_2 \) norm of the sequence \( \{ n(\rho_n - n) - \frac{\omega}{\pi} \}_{n=0}^{N_1} \). Here \( N_1 \) is some parameter between 1 and \( N_1 \) chosen to skip several first eigenvalues which can differ a lot from the asymptotic formula. One can take, e.g., \( N_s = \lceil N_1/2 \rceil \), and obtain that

\[
\omega \approx \arg \min_{\omega} \sum_{n=[N_1/2]}^{N_1} \left( n(\rho_n - n) - \frac{\omega}{\pi} \right)^2.
\]

Better accuracy can be achieved though by taking into account higher order asymptotic terms for the eigenvalues. Recall [37, remark 1.1.1] that for \( q \in W^1_2 \), \( N \geq 1 \), one has

\[
\rho_n = n + \sum_{j=1}^{N+1} \frac{\omega^{(j)}}{n^j} + \frac{\kappa_n}{n^{N+1} \rho} \quad \text{and} \quad \alpha_n = \frac{\pi}{2} + \sum_{j=1}^{N+1} \frac{\omega^{(j-1)}}{n^j} + \frac{\chi_n}{n^{N+1} \rho},
\]

where \( \omega^{(1)} = \frac{\omega}{\pi} \), \( \omega^{(2k)} = 0 \), \( \omega^{(2k-1)} = 0 \), \( k > 0 \) and \( \{ \kappa_n \}_{n=0}^{\infty} \in \ell_2 \), \( \{ \chi_n \}_{n=0}^{\infty} \in \ell_2 \). The first eigenvalues and norming constants can be rather distant from the asymptotic formulas (obtained by discarding the remainder terms \( \kappa_n / n^{N+1} \) and \( \chi_n / n^{N+1} \)), but higher index eigenvalues and norming constants are quite close. So one can recover the parameter \( \omega \) by finding approximate coefficients \( \omega^{(j-1)}, 1 \leq j \leq K \) from the best fit problem

\[
\sum_{j=1}^{K} \frac{\omega^{(j-1)}}{n^{j-1}} = \rho_n - n, \quad N_s \leq n \leq N_1.
\]

Since the smoothness of the potential is unknown beforehand, the parameter \( K < N_1 - N_s \) is chosen such that the least squares error of the fit for \( K \) is significantly better than the least squares error for \( K - 1 \). In practice, depending on the potential, an optimal value results to be 2 or 3. For the parameter \( N_s \), again, one can choose \( \lceil N_1/2 \rceil \).

Suppose that also a finite set of norming constants \( \alpha_0, \ldots, \alpha_{N_1} \) is given. Similarly, the approximate coefficients \( \omega^{(j-1)}, 1 \leq j \leq K - 1 \) can be found from the best fit problem

\[
\sum_{j=1}^{K-1} \frac{\omega^{(j-1)}}{n^j} = \alpha_n - \frac{\pi}{2}, \quad N_s \leq n \leq N_1.
\]

\textbf{Remark 3.4.} Taking into account that the residual sequences \( \{ \kappa_n \}_{n=0}^{\infty} \) and \( \{ \chi_n \}_{n=0}^{\infty} \) belong to the \( \ell_2 \) space, one can obtain approximate asymptotic coefficients \( \omega^{(j)}, 1 \leq j \leq K \) and \( \omega^{(j)}, 1 \leq j \leq K - 1 \) by minimizing the \( \ell_2 \) norms of the tails of these sequences, i.e., from the least squares problems

\[
\min_{\omega^{(1)}, \ldots, \omega^{(2K-1)}} \sum_{n=N_s}^{N_1} n^{4K-2} \left[ \rho_n - n - \sum_{j=1}^{K} \frac{\omega^{(j-1)}}{n^{j-1}} \right]^2
\]

and

\[
\min_{\omega^{(2)}, \ldots, \omega^{(2K-2)}} \sum_{n=N_s}^{N_1} n^{4K-2} \left[ \alpha_n - \frac{\pi}{2} - \sum_{j=1}^{K-1} \frac{\omega^{(j-1)}}{n^j} \right]^2.
\]
However, the numerical performance of the method based on (3.15) and (3.16) was slightly better.

**Remark 3.5.** The coefficients $\omega^{(2j-1)}$, $1 \leq j \leq K$ and $\omega^{(2j+1)}$, $1 \leq j \leq K - 1$, if computed on the first step, as described above, are not required but turn out useful when computing the coefficients of the system (3.5). Indeed, suppose we are given a finite set of the spectral data $\{\rho_n, \alpha_n\}_{n=0}^{N_1}$. Truncating the series (3.1)–(3.4) at $n = N_1$ is equivalent to taking the missing eigenvalues and norming constants to be $\rho_n = n$ and $\alpha_n = \pi/2$, $n > N_1$.

In order to compute the coefficients $C_{lm}(x)$ and $d_l(x)$ more accurately one can add to this set an arbitrarily large number of the ‘asymptotic spectral data’

$$\rho_n = n + \sum_{j=1}^{K} \frac{\omega^{(2j-1)}}{n^{2j-1}}, \quad \alpha_n = \frac{\pi}{2} + \sum_{j=1}^{K-1} \frac{\omega^{(2j+1)}}{n^{2j}}, \quad n = N_1 + 1,$$

see (3.14). The value of $M$ can be chosen very large.

Now, disposing of two alternative procedures for computing the parameter $\omega$ we are in a position to obtain the new system of linear algebraic equations for the coefficients $g_n(x)$ and prove the stability of the numerical method based on it.

### 3.4. Alternative main system of linear algebraic equations

A system similar to (3.5) can be obtained using the representation (2.17) instead of (2.16). An advantage of (2.17) is in preserving the Fredholm form of the Gelfand–Levitan equation and at the same time resolving the problem of slow convergence of the series (2.12) near $x = t = \pi$.

Moreover, the structure of the representation (2.17) extends further the idea discussed in remark 3.5 by taking into account the whole infinite series of ‘asymptotic spectral data’ and not just some (large) number of terms. In (2.17) only the first order approximations are used, so still some number of ‘asymptotic spectral data’ may be added explicitly. Nevertheless, the proposed modification based on (2.17) allows one to save the computational cost of dealing with thousands of terms in (3.1)–(3.4) which are necessary to achieve a comparable accuracy when using the truncated system (3.5). And working with the original Gelfand–Levitan equation allows us to prove the stability results for the approximate solutions obtained by truncating the infinite system.

Let us denote

$$\tilde{c}_{lm}(x) = \frac{\omega x}{8\pi} \left( \frac{\delta_{mk-1}}{(2k - 3/2)_3} - \frac{2\delta_{mk}}{(2k - 1/2)_3} + \frac{\delta_{mk+1}}{(2k + 1/2)_3} \right)$$

$$+ (-1)^{k+m} \sum_{n=1}^{\infty} \left[ \frac{j_{2k}(\rho_n x)j_{2m}(\rho_n x)}{\alpha_n} - \frac{2j_{2k}(nx)j_{2m}(nx)}{\pi} \right]$$

$$+ \frac{2\pi}{\pi + \rho_n} \left( xj_{2k}(nx)j_{2m+1}(nx) + xj_{2k+1}(nx)j_{2m}(nx) \right)$$

$$- \frac{2(k + m)}{n} \frac{\delta_{0n}}{\pi^n} \frac{2\omega x^2}{15\pi^2} \delta_{1n} \tilde{c}_{0n}(x),$$

$$\tilde{C}_{0n}(x) = \left( \frac{1}{\alpha_0} - \frac{1}{\pi} + \frac{2\omega x^2}{3\pi^2} \right) \delta_{0n} + \frac{2\omega x^2}{15\pi^2} \delta_{1n} \tilde{c}_{0n}(x),$$

(3.17)
\[ \tilde{C}_{1m}(x) = \frac{2\omega x^2}{15\pi^2} \delta_{0m} + \tilde{c}_{1m}(x), \]  
(3.19) 
\[ \tilde{C}_{km}(x) = \tilde{c}_{km}(x), \quad k = 2, 3, \ldots, m \in \mathbb{N}_0, \]  
(3.20)

And

\[ \tilde{d}_k(x) = -\left( \frac{1}{\alpha_0} - \frac{1}{\pi} + \frac{4\omega x^2}{3\pi^2} - \frac{\omega x}{\pi} \right) \delta_0 - \frac{2\omega x^2}{15\pi^2} \delta_k 
- (-1)^k \sum_{n=1}^{\infty} \left[ \cos \rho_n \frac{x}{\alpha_n} j_{2k}(\rho_n x) - \frac{2 \cos nx}{n} j_{2k}(nx) \right] 
+ 2\omega \left( x \sin \frac{nxj_{2k}(nx)}{n} - x \cos \frac{nxj_{2k+1}(nx)}{n} \right) \right] . \]  
(3.21)

Where \( \delta_{km} \) stands for the Kronecker delta and \( (k)_m \) is the Pochhammer symbol.

Then the following result is valid.

**Theorem 3.6.** The coefficients \( g_m(x) \) satisfy the system of linear algebraic equations

\[ g_m(x) = \frac{g_1(x)}{(4k + 1)x} + \sum_{m=0}^{\infty} g_m(x) \tilde{C}_{km}(x) = \tilde{d}_k(x), \quad k = 0, 1, \ldots . \]  
(3.22)

For all \( x \in [0, \pi] \) and \( k = 0, 1, \ldots \) the series in (3.22) converges.

**Proof.** The idea is to substitute expressions (2.17) and (2.18) into (2.13). Consider first

\[ \int_0^x P_{2m} \left( \frac{s}{x} \right) F(s,t) \frac{ds}{x} \]
\[ = \left( \frac{1}{\alpha_0} - \frac{1}{\pi} + \frac{\omega t^2}{\pi^2} \right) \int_0^x P_{2m} \left( \frac{s}{x} \right) \frac{ds}{x} + \frac{\omega}{\pi} \int_0^x s^2 P_{2m} \left( \frac{s}{x} \right) \frac{ds}{x} \]
\[ - \omega \int_0^x \frac{P_{2m} \left( \frac{s}{x} \right) \max \left( s, t \right) \frac{ds}{x}}{\alpha_n} + \sum_{n=1}^{\infty} \left[ \cos \rho_n t \frac{x}{\alpha_n} \int_0^x P_{2m} \left( \frac{s}{x} \right) \cos nx \frac{ds}{x} + \frac{2 \omega}{\pi} t \int_0^x \frac{P_{2m} \left( \frac{s}{x} \right) \sin nx \frac{ds}{x}}{\cos nx} \right] \right] . \]  
(3.23)

The change of the sum and of the integration is possible due to uniform convergence of the series for \( F \). Let us calculate the integrals appearing in (3.23). Using the orthogonality of the Legendre polynomials we obtain

\[ \int_0^x P_{2m} \left( \frac{s}{x} \right) \frac{ds}{x} = \delta_{0m}, \]
\[ \int_0^x s^2 P_{2m} \left( \frac{s}{x} \right) \frac{ds}{x} = x^2 \int_0^1 \tau^2 P_{2m}(\tau) d\tau = \begin{cases} x^2/3, & m = 0, \\ 2x^2/15, & m = 1, \\ 0, & m > 1. \end{cases} \]
Recall that
\[ \int_0^x P_{2m} \left( \frac{s}{x} \right) \cos bs \, \frac{ds}{x} = (-1)^m j_{2m}(bx). \]
Differentiating both sides with respect to \( b \) we obtain that
\[ \int_0^x P_{2m} \left( \frac{s}{x} \right) s \sin bs \, \frac{ds}{x} = (-1)^m \left( x j_{2m+1}(bx) - \frac{2m}{b} j_{2m}(bx) \right). \]
Now,
\[ \int_0^x \max \{s, t\} P_{2m} \left( \frac{s}{x} \right) \, \frac{ds}{x} = t \int_0^t P_{2m} \left( \frac{s}{x} \right) \, \frac{ds}{x} + \int_t^x \frac{s}{x} P_{2m} \left( \frac{s}{x} \right) \, ds. \]
Let \( m > 0 \). Using the identities (note that \( P_{2m+1}(0) = 0 \))
\[ \int_0^t P_{2m} \left( \frac{s}{x} \right) \, ds = \frac{t}{4m+1} \left( P_{2m+1} \left( \frac{t}{x} \right) - P_{2m-1} \left( \frac{t}{x} \right) \right) \]
and
\[ x P_n(x) = \frac{n+1}{2n+1} P_{n+1}(x) + \frac{n}{2n+1} P_{n-1}(x) \]
we obtain that
\[ \int_0^t \frac{t}{4m+1} \left( P_{2m+1} \left( \frac{t}{x} \right) - P_{2m-1} \left( \frac{t}{x} \right) \right) \]
\[ = \frac{(2m+2)x P_{2m+2}(t/x)}{(4m+1)(4m+3)} - \frac{x P_{2m}(t/x)}{(4m-1)(4m+3)} \]
\[ = \frac{(2m-1)x P_{2m-2}(t/x)}{(4m-1)(4m+1)}. \]
Similarly,
\[ \int_0^x \frac{s}{x} P_{2m} \left( \frac{s}{x} \right) \, ds = \int_x \frac{2m+1}{4m+1} P_{2m+1} \left( \frac{s}{x} \right) + \frac{2m}{4m+1} P_{2m-1} \left( \frac{s}{x} \right) \]
\[ = \frac{(2m+1)x}{(4m+1)(4m+3)} \left( P_{2m+2} \left( \frac{s}{x} \right) - P_{2m} \left( \frac{s}{x} \right) \right) \]
\[ + \frac{2mx}{(4m-1)(4m+1)} \left( P_{2m} \left( \frac{s}{x} \right) - P_{2m-2} \left( \frac{s}{x} \right) \right) \]
\[ = \frac{(2m+1)x P_{2m+2}(t/x)}{(4m+1)(4m+3)} - \frac{x P_{2m}(t/x)}{(4m-1)(4m+3)} \]
\[ + \frac{2mx P_{2m-2}(t/x)}{(4m-1)(4m+1)}. \]
Hence,
\[ \int_0^x \max \{s, t\} P_{2m} \left( \frac{s}{x} \right) \, ds = \frac{x P_{2m+2}(t/x)}{(4m+1)(4m+3)} - \frac{2x P_{2m}(t/x)}{(4m-1)(4m+3)} + \frac{x P_{2m-2}(t/x)}{(4m-1)(4m+1)}. \]
(3.24)
One can easily check that equality (3.24) is valid also for \( m = 0 \) if one takes \( P_{-2} \equiv 0 \). Thus, the Gelfand–Levitan equation (2.13) takes the form

\[
-F(x, t) = \sum_{n=0}^{\infty} \frac{g_n(x)}{x} P_{2n} \left( \frac{t}{x} \right) + \sum_{m=0}^{\infty} \tilde{g}_m(x) \left\{ \left( \frac{1}{\alpha_0} - \frac{1}{\pi} + \frac{\omega x^2}{3\pi^2} + t^2 \right) \delta_{0m} \\
+ \frac{2\omega x^2}{15\pi^2} \delta_{m1} - \frac{\omega x}{\pi} \left( \frac{P_{2m+2}(t/x)}{(4m+1)(4m+3)} - \frac{2P_{2m}(t/x)}{(4m-1)(4m+1)} \right) \\
+ \frac{P_{2m-2}(t/x)}{(4m-1)(4m+1)} \right\} (-1)^m \sum_{n=1}^{\infty} \left[ \frac{\cos \rho_0 t}{\alpha_n} j_{2m}(\rho_0 x) \\
- \frac{2\cos nt}{\pi n} j_{2m}(nx) + \frac{2m}{\pi n} (x \cos nt j_{2m+1}(nx)) \right] \}
\]

Let us multiply this equality by \( \frac{1}{x} P_{2k} \left( \frac{t}{x} \right) \) and integrate with respect to \( t \) from 0 to \( x \). Note that now \( \max\{x, t\} = x \), hence

\[
\int_0^x \max\{x, t\} P_{2k} \left( \frac{t}{x} \right) \frac{dt}{x} = \int_0^x P_{2k} \left( \frac{t}{x} \right) dt = x \delta_{0k},
\]

and one easily obtains that \( \int_0^x F(x, t)P_{2k} \left( \frac{t}{x} \right) \frac{dt}{x} = -\delta_k(x) \), while the right-hand side reduces to \( \frac{g_n(x)}{x} \sum_{m=0}^{\infty} \tilde{g}_m(x) \tilde{C}_{m,n}(x) \). In order to verify the convergence of the series in (3.22) it is sufficient to notice that for every \( x \in (0, \pi] \) the series \( \sum_{m=0}^{\infty} g_n(x) \tilde{C}_{m,n}(x) \) is the inner product of two \( L_2 \)-sequences. Indeed, note that the set of functions

\[
\{p_m(x)\}_{m=0}^{\infty} := \left\{ \frac{\sqrt{4m+1}P_{2m}(t/x)}{\sqrt{x}} \right\}_{m=0}^{\infty}
\]

forms an orthonormal basis in \( L_2(0, x) \). Hence the numbers \( \frac{g_n(x)}{\sqrt{4m+1}} \) are the Fourier coefficients of the function \( G(x, t) \) with respect to the basis \( \{p_m(t)\}_{m=0}^{\infty} \), while the numbers

\[
\sqrt{x} \sqrt{4m+1} \tilde{C}_{m,n}(x) = \sqrt{x} \sqrt{4m+1} \int_0^t \int_0^x F(s, \tau)P_{2m} \left( \frac{\tau}{x} \right) P_{2k} \left( \frac{s}{x} \right) \frac{ds}{x} d\tau - \int_0^x \int_0^1 F(s, x\tau)P_{2k}(\tau) d\tau \left[ \frac{\sin nx}{nx} \right]_{\alpha_n}^{\alpha_n}
\]

are the Fourier coefficients of the function \( F_k(x, t) := \int_0^1 F(x, \tau)P_{2k}(\tau) d\tau \) with respect to the basis \( \{p_m(t)\}_{m=0}^{\infty} \).

**Remark 3.7.** Note that the expressions \( -\frac{2\omega}{\pi n} (s \sin nx \cos nt + t \sin nt \cos nx) \), subtracted termwise from the series (2.12) in order to obtain the representation (2.17), are only the first order approximations to the terms \( \frac{\cos nx \cos nt \cos \rho_0 x}{\alpha_n} \). Hence, given a finite set of spectral data, one can still benefit from adding some number of ‘asymptotic spectral data’ as it was discussed in remark 3.5. However, much fewer terms are needed to achieve a similar accuracy when comparing to the original representation (2.12).
Remark 3.8. One can consider the integrated version of the representation (2.17) and substitute it into the integrated Gelfand–Levitan equation (3.5). This results in a system similar to (3.22), with more complicated expressions for the coefficients. Although one could expect a faster convergence of the series involved, our numerical experiments showed that in fact this does not lead to a significantly better accuracy.

3.5. Truncated main system of equations, its condition number, existence and stability of solution

For the numerical solution of the system (3.22) it is natural to truncate the infinite system, i.e., to consider $m, k \leq N$. We are going to prove that the solution of this truncated system converges to the exact one when $N \to \infty$, and that the process is stable, i.e., the condition number of the coefficient matrix is uniformly bounded with respect to $N$, and, as a result, small errors in coefficients and in the right-hand side of the system lead to small errors in the approximate solution. In order to apply the general theory, let us multiply each equation in (3.22) by $\sqrt{4k+1}x$ and rewrite the system as

$$
\frac{g_k(x)}{\sqrt{4k+1}x} + \sum_{m=0}^{\infty} \frac{g_m(x)}{\sqrt{4m+1}x} \cdot x \sqrt{4k+1} \sqrt{4m+1} \tilde{C}_{k,m}(x) = \sqrt{4k+1} \sqrt{x} d_k(x), \quad k = 0, 1, \ldots
$$

(3.26)

Denote

$$
\xi_k = \frac{g_k(x)}{\sqrt{4k+1}x}, \quad b_k = \sqrt{4k+1} \sqrt{x} d_k(x), \quad a_{km} = x \sqrt{4k+1} \sqrt{4m+1} \tilde{C}_{k,m}(x).
$$

Then (3.26) can be written as

$$
\xi_k + \sum_{k=0}^{\infty} a_{km} \xi_m = b_k, \quad k = 0, 1, \ldots
$$

(3.27)

As was mentioned in the proof of theorem 3.6, one has

$$
\xi_k = \int_0^x G(x, t)p_k(t) \, dt,
$$

(3.28)

$$
b_k = -\sqrt{4k+1} x \int_0^x F(x, t) P_{2k} \left( \frac{1}{x} \right) \frac{dt}{x} = -\int_0^x F(x, t) p_k(t) \, dt,
$$

(3.29)

$$
a_{km} = \sqrt{x} \sqrt{4k+1} \int_0^x \int_0^x F(x, s) P_{2k} \left( \frac{1}{x} \right) \frac{dt}{x} \cdot p_m(s) \, ds
$$

$$
= \int_0^x \int_0^x F(s, t) p_k(t) p_m(s) \, dt \, ds,
$$

(3.30)

where $\{p_n(t)\}_{m=1}^{\infty}$ is the orthonormal basis of $L_2(0, x)$ given by (3.25). Hence $\{\xi_k\}_{k=0}^{\infty} \in l_2$, $\{b_k\}_{k=0}^{\infty} \in l_2$, $\{a_{km}\}_{k,m=0}^{\infty} \in l_2 \otimes l_2$, and the system (3.27) is of the type studied in [15, chapters 14, section 3]. Moreover, it follows from (3.28)–(3.30) that the truncated system

$$
\xi_k + \sum_{k=0}^{N} a_{km} \xi_m = b_k, \quad k = 0, 1, \ldots, N,
$$

(3.31)
coincides with the system obtained by applying the Bubnov–Galerkin process to equation (2.13) with respect to the system (3.25), see [25, section 14]. However, in our approach we do not need to solve the complete system, only the first solution component $\xi_0$ is necessary to recover the potential. Below, in section 5 we illustrate the importance of this fact by a numerical example. Also we point out that the special form of the function $F$ and the function system (3.25) allowed us to avoid the necessity to compute numerically the scalar products arising in the Bubnov–Galerkin process and to reduce them to the sums (3.17) and (3.21).

Let $I_N$ be an $(N + 1) \times (N + 1)$ identity matrix, $L_N = (a_{km})_{k,m=0}^N$ the coefficient matrix of the truncated system and $R_N = (b_k)_{k=0}^N$ the truncated right-hand side. Let

$$U_N = \{ g_N^k(x) \}_{k=0}^N = \left\{ \frac{g_N^k(x)}{\sqrt{4k+1}} \right\}_{k=0}^N \quad (3.32)$$

denote the solution of the truncated system (3.31). Following [25, section 9] consider a system (called non-exact system)

$$(I_N + L_N + \Gamma_N)v = R_N + \Delta_N,$$

where $\Gamma_N$ is an $(N + 1) \times (N + 1)$ matrix representing errors in the coefficients $a_{km}$, and $\Delta_N$ is a column-vector representing errors in the coefficients $b_k$. Let $V_N$ denote the solution of the non-exact system. The solution of the Bubnov–Galerkin process is called stable if there exist constants $c_1, c_2$ and $r$ independent of $N$ such that for $\|\Gamma_N\| \leq r$ and arbitrary $\Delta_N$ the non-exact system is solvable and the following inequality holds

$$\|U_N - V_N\| \leq c_1\|\Gamma_N\| + c_2\|\Delta_N\|.$$  

Application of the general theory from [15, chapter 14, section 3] and [25, theorems 14.1 and 14.2] leads to the following result.

**Proposition 3.9.** Let $x > 0$ be fixed. Then for a sufficiently large $N$ the truncated system (3.31) has a unique solution denoted in (3.32) by $U_N$ and

$$\sum_{k=0}^N \frac{|g_N^k(x) - g_k(x)|^2}{(4k+1)} + \sum_{k=N+1}^\infty \frac{|g_k(x)|^2}{(4k+1)} \to 0, \quad N \to \infty,$$

from which it also follows that

$$g_N^0(x) \to g_0(x), \quad N \to \infty.$$ 

There exists such constant $r > 0$ that for arbitrary error matrices $\Gamma_n$ satisfying $\|\Gamma_n\| < r$ for all $n \in \mathbb{N}$ the condition numbers of the approximate coefficient matrices $I_N + L_N + \Gamma_N$ are uniformly bounded with respect to $N$ and the solution $U_N$ is stable.

3.6. Flipping the interval

A motivation for considering the integrated Gelfand–Levitan equation (2.14) and the alternative representation (2.17) is to solve the problem of the jump discontinuity of the series (2.12) at $x = t = \pi$ (for $\omega \neq 0$) and the resulting slow convergence of the series near this critical point. An additional improvement can be achieved by recovering the potential only on the first half of the interval, while on the second half recovering the flipped potential $q(\pi - x)$. 


Another motivation to consider the flipped problem consists in the following observation. Suppose the potential \( q \) is piecewise smooth, that is, there are several ‘special’ points inside \([0, \pi]\), where some derivative of \( q \) is discontinuous, while on the rest of the segment the potential is infinitely differentiable. It is known [19] that the series (2.18) converges fast (super polynomially) up to the first ‘special point’, and afterward the convergence slows down to a polynomial one. For the inverse problem this phenomenon results in a higher accuracy of the recovered potential on the segment from 0 to the first ‘special point’. By performing the flipping of the problem one can assure a higher accuracy of the recovered potential also on the segment from the last ‘special point’ to \( \pi \). The greatest advantage is achieved in the case when there is only one ‘special point’.

In this subsection we show how the spectral data for the flipped potential can be obtained. Consider the following ‘flipped’ spectral problem

\[
\begin{align*}
-\gamma'' + q(\pi - x)\gamma &= \rho^2\gamma, & x \in (0, \pi), \\
\gamma'(0) - H\gamma(0) &= 0, \quad \gamma'(\pi) + h\gamma(\pi) &= 0.
\end{align*}
\] (3.33)

Obviously it has the same eigenvalues as the problem (2.1, (2.2), and the functions \( \varphi_r^{\rho_n, x} \): \( \varphi_r^{\rho_n, \pi - x} \) are the eigenfunctions of (3.33), (3.34). But the functions \( \varphi_r^{\rho_n, x} \) do not necessarily satisfy (2.6), so the norming constants (let us denote them by \( \alpha_r^{\rho_n} \)) change.

Actually, \( \varphi_r^{\rho_n, 0} = \varphi_r^{\rho_n, \pi} \), so it can be seen from (2.6) and (2.7) that

\[
\alpha_r^{\rho_n} = \frac{\alpha_n}{\varphi_r^{\rho_n, \pi}}. \quad (3.35)
\]

The formula [37, formula (1.1.35)] states that

\[
\alpha_n = -\frac{\varphi'(\rho_n, \pi)}{2\rho_n} \left( \varphi'(\rho_n, \pi) + H\varphi(\rho_n, \pi) \right) \bigg|_{\rho = \rho_n}. \quad (3.36)
\]

Notice that with the aid of theorem 2.7 equality (3.36) can be written in the form

\[
\alpha_n = \frac{\varphi(\rho_n, \pi)}{2\rho_n} \left( (1 + \pi\omega) \sin \rho_n\pi + \pi\rho_n\cos \rho_n\pi \right.
\]
\[
+ \sum_{k=0}^{\infty} (-1)^k h_k \left( \frac{1}{\rho_n} \sqrt{\pi} j_{2k+1}(\rho_n\pi) - \frac{2k}{\rho_n} j_{2k}(\rho_n\pi) \right) \bigg) \right)
\] (3.37)

for \( n = 1, 2, \ldots \), and

\[
\alpha_0 = \varphi(0, \pi) \left( \pi + \omega \frac{\pi^2}{3} + h_1 \frac{\pi^2}{15} \right) , \quad (3.38)
\]

here \( h_k \) are the constants introduced in subsection 3.2.

Hence

\[
\alpha_r^{\rho_n} = \left( \frac{1 + \pi\omega}{4\alpha_n\rho_n^2} \right)^2 \left( \frac{\varphi(\rho_n, \pi)}{2\rho_n} \left( (1 + \pi\omega) \sin \rho_n\pi + \pi\rho_n\cos \rho_n\pi \right.
\]
\[
+ \sum_{k=0}^{\infty} (-1)^k h_k \left( \frac{1}{\rho_n} \sqrt{\pi} j_{2k+1}(\rho_n\pi) - \frac{2k}{\rho_n} j_{2k}(\rho_n\pi) \right) \bigg) \right)^2
\] (3.39)
for \( n = 1, 2, \ldots \) and
\[
\alpha_0' = \left( \pi + \frac{\omega}{2} h_{\frac{1}{15}} + h_{\frac{1}{17}} \right)^2.
\] (3.40)

### 3.7 Algorithm of solution of problem 2.1

Given a finite set of spectral data \( \{\rho_n, \alpha_n\}_{n=0}^{N_1} \), the following direct method for recovering the potential \( q \) and the numbers \( h \) and \( H \) is proposed.

1. If \( \rho_0 \neq 0 \), perform the shift of the eigenvalues
\[
\tilde{\rho}_n = \sqrt{\rho_n^2 - \rho_0^2}, \quad n \geq 0,
\]
so that 0 becomes the first eigenvalue of the spectral problem. Let us denote the shifted eigenvalues by the same expression \( \rho_n \).

2. Find the coefficients \( \omega(2j - 1), 1 \leq j \leq K \) and \( \omega(2j, +), 1 \leq j \leq K - 1 \) as described in subsection 3.3. The parameter \( \omega \) can be recovered as \( \omega = \pi \omega(1) \).

3. Complement the set of ‘exact’ spectral data \( \{\rho_n, \alpha_n\}_{n=0}^{N_1} \) with a set of ‘asymptotic’ spectral data
\[
\{\rho_n = n + \sum_{j=1}^{K} \frac{\omega(2j-1)}{n^{2j-1}}, \alpha_n = \frac{\pi}{2} + \sum_{j=1}^{K-1} \frac{\omega(2j+1)}{n^{2j+1}}\}_{n=N_1+1}^{M}.
\]

4. For a set of points \( \{x_j\} \) from \((0, a]\), where \( a \in (\pi/2, \pi]\), compute the approximate values of the coefficients \( \tilde{C}_{km}(x) \) and \( \tilde{d}_l(x) \) for \( k, m = 0, \ldots, N \) with the aid of the formulas (3.17)–(3.21) and solve the system (3.31) obtaining thus \( g_0(x) \).

5. Compute \( q \) on the segment \([0, a]\) from (2.21). Take into account that \( \varphi(0, x) \) is an eigenfunction associated with the first eigenvalue \( \lambda_0 \) and hence does not have zeros on \([0, \pi]\) (see, e. g., [2, theorem 8.4.5]). This justifies the division over \( \varphi(0, x) \).

6. Compute \( h \) from \( h = g_0(0) \).

7. Solving the (overdetermined) system of linear algebraic equation (3.10) (with all \( M + 1 \) exact and ‘asymptotic’ eigenvalues), compute the constants \( h_0, \ldots, h_{N_1} \).

8. Perform the flipping of the interval as explained in subsection 3.6 and repeating steps 4–5 compute \( q_{rev} \) on the segment \([\pi/2, \pi]\).

9. Combine \( q \) and \( q_{rev} \) to recover the potential on the whole segment \([0, \pi]\).

10. Compute \( H \) using (2.4). For this compute the mean of the potential \( \int_0^\pi q(t) \, dt \), and thus,
\[
H = \omega - h - \frac{1}{2} \int_0^\pi q(t) \, dt.
\]

11. Recall that one has to add the original eigenvalue \( \rho_0^2 \) back to the recovered potential to return to the original potential \( q \).

Below, in section 5 we illustrate the performance of this algorithm with several numerical examples.

**Remark 3.10.** The steps 7–9 are optional. As we show in section 5, for many applications a potential \( q \) recovered performing only the steps 1–5 can be sufficient.
4. Reduction of problem 2.2 to problem 2.1

Now we suppose that the eigenvalues \( \{\lambda_n\}_{n=0}^{\infty} \) and \( \{\nu_n\}_{n=0}^{\infty} \) of the Sturm–Liouville problems (2.1), (2.2) and (2.1), (2.3), respectively, are given. We remind that \( \lambda_0 = 0 \). Then on the first step we can repeat the procedure described in subsections 3.3 and 3.2 and obtain the parameter \( \omega \) together with the constants \( h_n \).

4.1. Recovery of the parameter \( \omega_1 \) and coefficients of the asymptotic expansion of eigenvalues \( \mu_n \)

The parameter \( \omega_1 \) can be recovered using the asymptotics of the eigenvalues. Indeed, it immediately follows from (2.9) that

\[
\omega_1 = \pi \lim_{n \to \infty} n \left( \mu_n - n - \frac{1}{2} \right).
\]

Suppose that only a finite set of eigenvalues \( \mu_0, \ldots, \mu_{N_2} \) is given. Then, similarly to subsection 3.3, one can recover the parameter \( \omega_1 \) using the asymptotics of the eigenvalues \( \mu_k \). One has for \( q \in W_2^N, N \geq 1 \),

\[
\mu_n = n + \frac{1}{2} + \sum_{j=1}^{N+1} \frac{\omega_1^{(j)}}{n^j} + \frac{\eta_n}{\rho n^{N+1}}, \tag{4.1}
\]

where \( \omega_1^{(1)} = \frac{\omega_1}{\pi} \) and \( \left\{ \eta_n \right\}_{n=0}^{\infty} \in \ell_2 \). Contrary to (3.14), the coefficients \( \omega_1^{(2j)} \) need not be zeros. Hence one can recover the parameter \( \omega_1 \) by finding coefficients \( \omega_1^{(j)}, 1 \leq j \leq K \) in the best fit problem

\[
\sum_{j=1}^{K} \frac{\omega_1^{(j)}}{n^j} = \mu_n - n - \frac{1}{2}, \quad N_s \leq n \leq N_2. \tag{4.2}
\]

Similarly, the parameter \( K < N_2 - N_s \) is taken to be such that the least squares error of the fit for \( K \) is significantly better than the least squares error for \( K - 1 \). In practice, depending on the potential, optimal value results to be up to 5.

From (2.5) we compute

\[
H = \omega - \omega_1. \tag{4.3}
\]

4.2. Recovery of the norming constants \( \alpha_n \)

Our goal is to calculate the norming constants with the aid of the formula (3.36) and thus to reduce the problem to problem 2.1.

Notice that with the aid of Theorem 2.7 equality (3.36) can be written in the form

\[
\alpha_n = \frac{1}{2\rho_n} \left( \cos \rho_n \pi + \sum_{k=0}^{\infty} (-1)^k g_k(\pi) j_{2k}(\rho_n \pi) \right) \times \left( (1 + \pi \omega) \sin \rho_n \pi + \pi \rho_n \cos \rho_n \pi + \sum_{k=0}^{\infty} (-1)^k h_k \left( \pi j_{2k+1}(\rho_n \pi) - \frac{2k}{\rho_n} j_{2k}(\rho_n \pi) \right) \right) \tag{4.4}
\]
for \( n = 1, 2, \ldots \) and
\[
\alpha_0 = (1 + g_0(\pi)) \left( \pi + \frac{\pi^2}{3} + h \frac{\pi^2}{15} \right).
\]

Hence to calculate the norming constants we only need to calculate the coefficients \( g_k(\pi) \). Since \( \nu_k = \mu_k^2 \) are eigenvalues of the problem (2.1), (2.3) we have that
\[
\varphi(\mu_k, \pi) = \cos \mu_k \pi + \sum_{n=0}^{\infty} (-1)^n g_n(\pi) j_{2n}(\mu_k \pi) = 0
\]
for all \( k = 0, 1, \ldots \). Thus, for the coefficients \( g_n(\pi) \) we have the system of equations
\[
\sum_{n=0}^{\infty} (-1)^n g_n(\pi) j_{2n}(\mu_k \pi) = -\cos \mu_k \pi, \quad k = 0, 1, \ldots .
\]
Solving it, we find the coefficients \( g_n(\pi) \) and hence can calculate the constants \( \alpha_n \) from (4.4) and (4.5).

Similarly to subsection 3.2, having finite number of eigenvalues \( \{\mu_n\}_{n=0}^{N_2} \) given, it is not necessary to look for the same number of the coefficients \( g_n(\pi) \). One can consider an overdetermined system
\[
\sum_{n=0}^{N_2} (-1)^n g_n(\pi) j_{2n}(\mu_k \pi) = -\cos \mu_k \pi, \quad k = 0, 1, \ldots , N_2,
\]
taking as \( N_2 \) a value for which the coefficient matrix has relatively small condition number.

Thus, we have the constants \( \{\lambda_n\}_{n=0}^{N_1}, \{\alpha_n\}_{n=0}^{N_2} \) together with the parameter \( \omega \) and the constant \( H \), and can apply the algorithm from subsection 3.7 for recovering \( q(x) \) and \( h \).

**Remark 4.1.** We obtain from (2.11) and (2.20) that
\[
\omega_1 = G(\pi, \pi) = \sum_{n=0}^{\infty} \frac{g_n(\pi)}{\pi},
\]
an alternative way to recover the parameter \( \omega_1 \).

### 4.3. Algorithm of solution of problem 2.2

Given two finite sets of eigenvalues \( \{\rho_n\}_{n=0}^{N_1} \) and \( \{\mu_n\}_{n=0}^{N_2} \) of the Sturm–Liouville problems (2.1), (2.2) and (2.1), (2.3), respectively, \( N_2 \) can differ from \( N_1 \). The following algorithm allows one to approximate \( q(x) \) and the constants \( h, H \).

1. Find the coefficients \( \omega^{(2j-1)}, 1 \leq j \leq K_1 \) and \( \omega^{(\ell)}, 1 \leq j \leq K_2 \) as described in subsections 3.3 and 4.1. The parameters \( \omega \) and \( \omega_1 \) can be recovered as \( \omega = \pi \omega^{(1)} \), \( \omega_1 = \pi \omega_1^{(1)} \).
2. Compute \( H \) from (4.3).
3. Complement the set of ‘exact’ spectral data \( \{\rho_n\}_{n=0}^{N_1}, \{\mu_n\}_{n=0}^{N_2} \) with a set of ‘asymptotic’ spectral data
\[
\left\{ \rho_n = n + \sum_{j=1}^{K_1} \omega^{(2j-1)} \frac{\omega^{(2j-1)}}{\pi^2}, \quad \{\mu_n = n + \frac{1}{2} + \sum_{j=1}^{K_2} \frac{\omega^{(\ell)}}{\pi^2} \right\}_{n=N_1+1}^{M}.
\]
4. Solving the (overdetermined) systems (3.10) and (4.7) (with both given and ‘asymptotic’ eigenvalues) compute the constants \( h_n \) for \( n = 0, \ldots , N_1^0 \) and \( g_n(\pi) \) for \( n = 0, \ldots , N_2^0 \).
5. Compute the norming constants \( \alpha_0 \) by (4.5) and
\[
\alpha_n = \frac{1}{2\rho_n} \left( \cos \rho_n \pi + \sum_{k=0}^{N_n} (-1)^k g_k(\pi) j_{2k}(\rho_n \pi) \right) \left( 1 + \pi \omega \right) \sin \rho_n \pi + \pi \rho_n \cos \rho_n \pi
\]
\[
+ \sum_{k=0}^{N_n} (-1)^k h_k \left( \pi j_{2k+1}(\rho_n \pi) - \frac{2k}{\rho_n} j_{2k}(\rho_n \pi) \right)
\]
for \( n = 1, \ldots, M \).

(6) Apply steps 4–9 from subsection 3.7 for computing \( q(x) \) and \( h \).

4.4. Modification for other spectral data

Though in the present work only two classical inverse Sturm–Liouville problems are considered, the method is applicable to some other inverse Sturm–Liouville problems. For example, it is well known (see, e.g., [7, p 103]) that the problem of recovery of a symmetric potential from one spectrum reduces to a two-spectrum problem.

The problem of recovery of the potential from one spectrum and endpoint data can be solved by our methods as well. Suppose that besides the spectrum of a problem (2.1), (2.3) a sequence of endpoint data is given, say, \( \{\phi'\left(\mu_n, \pi\right)\}_{n=0}^{\infty} \). Then the set of constants \( \{g_n(\pi)\} \) can be recovered from (4.6), and consequently the parameter \( \omega_1 \) from (4.8). The set of constants \( \{\gamma_n(\pi)\} \) can be recovered from the system

\[
\sum_{n=0}^{\infty} (-1)^n \gamma_n(\pi) j_{2n}(\mu_k \pi) = \phi'(\mu_k, \pi) + \mu_k \sin \mu_k \pi - \omega_1 \cos \mu_k \pi, \quad k = 0, 1, \ldots,
\]

obtained from (2.23). Due to theorem 2.7, having the sequences \( \{g_n(\pi)\} \) and \( \{\gamma_n(\pi)\} \), is equivalent to have \( \phi(\rho, \pi) \) and \( \phi'(\rho, \pi) \) for all \( \rho \). This means that a second spectrum (corresponding to an arbitrary \( H \)) can be computed, and hence the problem reduces to the two-spectra problem.

5. Numerical examples

Numerical implementation of the algorithms proposed in subsections 3.7 and 4.3 is quite simple and does not require more than build-in functions of, e.g., Matlab. Only several comments should be made.

First, we obtained the parameter \( \omega \) from the asymptotics (see subsection 3.3) and similarly to subsection 3.5, we considered \( \frac{g_n(\pi)}{\sqrt{4n + 1}} \) and \( \frac{h_n}{\sqrt{4n + 1}} \) as the new unknowns for the systems (3.10) and (4.7), rewriting them as

\[
\sum_{n=0}^{N_n} (-1)^n \frac{h_n}{\sqrt{4n + 1}} \sqrt{4n + 1} j_{2n}(\rho_k \pi) = -\omega \cos \rho_k \pi + \rho_k \sin \rho_k \pi, \quad k = 1, \ldots, M,
\]

\[
\sum_{n=0}^{N_n} (-1)^n \frac{g_n(\pi)}{\sqrt{4n + 1}} \sqrt{4n + 1} j_{2n}(\mu_k \pi) = -\cos \mu_k \pi, \quad k = 0, 1, \ldots, M.
\]
After such modification all (except the largest one which is close to 1 and possibly several smallest ones) singular values of the coefficient matrices become very close to \( \frac{\sqrt{2}}{2} \). Reducing the numbers \( N_1^q \) and \( N_2^q \), the smallest eigenvalues disappear, and all the eigenvalues become larger than \( \frac{1}{2} \). So one can take as a criterion for choosing the parameters \( N_1^q \) and \( N_2^q \), e.g., that the corresponding condition numbers be less than 10.

Second, in many formulas we need the values of the spherical Bessel functions \( j_k(t) \) for a list of indices \( k = 0, \ldots, N \) for the same argument \( t \). A considerable speedup is achieved by applying the backwards recursion formula

\[
j_{n-1}(t) = \frac{2n+1}{t} j_n(t) - j_{n+1}(t),
\]

see, e.g., [5, 12] for details. So only \( j_{N-1}(t) \) and \( j_{N}(t) \) have to be computed using, e.g., \( \text{besselj} \) function from Matlab. A special care should be taken only for small values of \( t \) and large values of \( N \), since the absolute values of \( j_{N-1}(t) \) and \( j_{N}(t) \) can be less than \( 10^{-307} \), that is zero in the double machine precision arithmetics.

Third, numerical differentiation is used to recover the potential \( q \) and the parameter \( h \) from the coefficient \( g_0 \). To obtain good results, the set of points \( \{x_i\} \) chosen on the step 4, should not be large, 100–200 points are sufficient. We refer the reader to [35, pp 111–114], where an explanation is given for a closely related problem of solution of the Volterra integral equation of the first kind. We have interpolated the obtained values \( g_0(x_i) \) by a 6th order spline and used Matlab’s function \( \text{der} \) to perform the differentiation. It should be noted that adding a large set of ‘asymptotic’ spectral data (on steps 3 of the proposed algorithms) greatly improves the stability of numerical differentiation in the sense that the obtained potential \( q \) changes very little whenever one takes 50, 100 or 500 points \( \{x_i\} \).

Another possibility to perform numerical differentiation is to approximate the coefficient \( g_0 \) by a polynomial on the whole segment \([0, \pi]\) and differentiate this polynomial. By taking \( \{x_i\} \) to be the Chebyshev nodes (adapted to the segment \([0, \pi]\)), coefficients of the Fourier–Chebyshev series of \( g_0 \) can be obtained quite easily. Higher-order oscillatory component in \( g_0 \) (appearing due to truncation of the series in (3.17)–(3.21)), leading to large error during numerical differentiation, can be filtered out by truncating the Fourier–Chebyshev series once the coefficients become relatively small. The remaining partial sum provides an approximating polynomial. We refer the reader to [3] for further details.

5.1. Numerical solution of problem 2.1

In this subsection we illustrate the algorithm from subsection 3.7 on two examples: of a smooth potential and of a potential possessing a discontinuous derivative. We have taken the same
Figure 1. Absolute error of the recovered coefficient \( g_0(x) \) for example 5.1. 4 different main systems of equations were used: (3.22) having \( \omega = 0 \) (original G–L system), (3.5) (integrated G–L system), (3.22) (modified G–L system) and the one which can be obtained if one integrates first the expression (2.17) (integrated modified G–L system). 8 equations left in the truncated system in all cases. On the left: 201 exact eigenvalues and norming constants were used to compute the coefficients of the systems. On the right: additionally 1800 AEVs and norming constants were added.

Figure 2. Absolute error of the recovered coefficient \( g_0(x) \) for example 5.1 obtained from the truncated system (3.22). On the left: different number of equations in the truncated system, 201 exact and 1800 AEVs and norming constants were used. One the right: 8 equations in the truncated system, 201 exact but different number of AEVs and norming constants were used.

potentials as those considered in [14, 17, 18], however having both parameters \( h \) and \( H \) different from zero to illustrate that proposed method works in the most general setting.

Example 5.1. Consider the spectral problem (2.1), (2.2) with \( q(x) = 2 + \sin 2x, h = 1 \) and \( H = 1/2 \). We have computed the spectral data \( \{\rho_n, \alpha_n\}_{n=0}^{N_1} \) for \( N_1 = 200 \) using the method proposed in [19].

The parameter \( \omega \) was recovered as explained in subsection 3.3 with \( K = 1, 2, 3 \). The following accuracy was obtained: for \( K = 1: 5.2 \times 10^{-5} \), for \( K = 2: 5.7 \times 10^{-9} \) and for \( K = 3: 7.7 \times 10^{-12} \). There was observed no further improvement when taking \( K > 3 \).

As was mentioned in remark 3.2, the parameter \( \omega \) also can be recovered from \( \omega = -H_0 \), without need to use eigenvalues asymptotics. In table 1 we present absolute errors for both...
methods for different number of eigenvalues used. As one can see, the accuracy of the both methods is comparable.

On figure 1 we compare the accuracy of the coefficient $g_0$ recovered using different truncated systems of equations: the original system from [17] (coinciding with (3.22) when one takes $\omega = 0$), the system (3.5) and the system (3.22). Additionally we considered the system which one can obtain integrating the expression (2.17) first. As one can see, the system (3.22) provides the most accurate approximation. On figure 2 we present the absolute error of the recovered coefficient $g_0$ depending on the number of equations in the truncated system and on the number of added asymptotic spectral data. Only eight equations were sufficient to stabilize the error, while for such smooth potentials one can observe constant accuracy improvement even adding tens of thousands of asymptotic eigenvalues (AEVs) and norming constants.

On figure 3 we present the absolute error of the recovered potential. As one can see, the effect of added asymptotic eigendata is much smaller, and it is more evident near the interval endpoints. Flipping the interval as was explained in subsection 3.6 improves the accuracy near
Figure 5. Recovered potential (on the left) and its absolute error (on the right, dashed blue line—obtained directly from (2.21), solid black line—improved near $x = \pi$ using the flipped interval) for example 5.2 obtained from the truncated system (3.22) with 8 equations. 201 exact and 4800 asymptotic eigendata pairs were used.

Figure 6. Recovered potential $q$ (on the right) for example 5.2 obtained from the truncated system (3.22) with 8 equations. On the left: 11 exact and 990 asymptotic eigendata pairs were used. On the right: 40 exact and 961 asymptotic eigendata pairs were used.

$x = \pi$ by one decimal digit. In all the experiments the coefficient $g_0$ was computed on a uniform mesh of 201 points.

Finally, we recovered the potential $q$ and the constants $h$ and $H$ using the truncated system (3.22) and adding 20000 AEVs and norming constants. The $L_1$ error of the recovered potential was $1.1 \times 10^{-8}$, absolute errors of the constants $h$ and $H$—$1.8 \times 10^{-9}$ and $2.3 \times 10^{-9}$, respectively. The computation time of the proposed algorithm, even with this huge amount of added AEVs, was less than 30 s on a laptop computer. By taking 5000 asymptotic eigendata pairs only, the $L_1$ error of the recovered potential increased to $5.0 \times 10^{-8}$, while the computation time decreased to 7 s.

In many practical situations only a few eigenvalues and norming constants are available. Similarly to examples from [18, section 13.3] we took only 11 pairs $\{\rho_n, \alpha_n\}$, complemented them with 1990 asymptotic values and solved the same inverse problem. On figure 4 we present the recovered potential and its absolute error.
Example 5.2. Consider the spectral problem (2.1), (2.2) with
\[ q(x) = \left| 3 - |x^2 - 3| \right|, \]  
\( h = 1 \) and \( H = 2 \). We have computed the spectral data \( \{\rho_n, \alpha_n\}_{n=0}^{N_1} \) for \( N_1 = 200 \) using the method proposed in [19].

On figure 5 we present the recovered potential \( q \) and its absolute error, only 8 equations were used. The potential \( q \) was recovered with \( L_1(0, \pi) \) error of \( 3.9 \times 10^{-4} \). The constants \( \omega, h \) and \( H \) were recovered with absolute errors of \( 3.7 \times 10^{-6}, 6 \times 10^{-8} \) and \( 4 \times 10^{-6} \), respectively. On figure 6 we show the result of the proposed algorithm when only 11 pairs of eigendata (to compare with [18]) or 40 pairs of eigendata (to compare with [14]) were used.

This example can be used to demonstrate the advantage of recovering the coefficient \( g_0 \) only and not looking for the whole solution \( G(x, t) \) of the Gelfand–Levitan equation, recovering the potential \( q \) from the expression \( G(x, x) = \sum_{n=0}^{\infty} \frac{g_n(x)}{x} \), see (2.20) and (2.11). Indeed, the coefficients \( g_n(x) \) for this potential decay fast for \( x < \sqrt{3} \) and slowly for \( x > \sqrt{3} \) as \( n \to \infty \). Here \( \sqrt{3} \) is the first point where the derivative is discontinuous. For example, the value of \( g_{500}(x) \) for \( x > \sqrt{3} + 0.1 \) oscillates in sign reaching values larger than \( 10^{-6} \). We refer the reader to [19, (3.9) and (4.9)] for some estimates. For that reason, few coefficients \( g_n(x) \) are necessary.
Figure 8. Exact (blue lines) and recovered (black dots) potentials from example 5.3 together with the number of exact eigenvalue pairs used, average absolute errors and found values of the parameters $h$ and $H$.

To approximate the kernel $G(x, x)$ for $x < \sqrt{3}$, but even hundreds are not sufficient for a comparable approximation of the kernel $G(x, x)$ for $x > \sqrt{3}$. Additionally, we do not have exact values of the coefficients $g_n$, they are obtained by solving a truncated system (3.31), containing some errors in the coefficients and in the right-hand side due to round-off errors, series truncation etc. So while more coefficients $g_n(x)$ are necessary to be able to closely approximate the function $G(x, x)$, summing them up results in a still larger error due to imperfections in each computed coefficient $g_n(x)$. Take aside the amount of computation time required to calculate all the coefficients and solve the system (3.31) with hundreds of equations. On the contrary, the absolute error of the coefficient $g_0$ alone is rather small, compensating the necessity of one more numerical differentiation, and it can be obtained from the system of as few as 5–8 equations. We illustrate this on figure 7. As one can see, even using 11 coefficients $g_n(x)$ to
approximate $G(x, x)$ for $x < \sqrt{3}$ and 251 coefficients for $x > \sqrt{3}$ did not allow us to recover the potential $q$ with the same accuracy as in the case when only the coefficient $g_0$ was used, cf. figure 5.

5.2. Numerical solution of problem 2.2

In this subsection we illustrate the algorithm from subsection 4.3. The performance of the algorithm applied for the smooth potential from example 5.1 was similarly excellent, being able to recover both the potential and the constants $h$ and $H$ with accuracy better than $10^{-8}$. For that reason we decided to present only non-smooth potentials.

**Example 5.3.** Consider the spectral problems (2.1), (2.2) and (2.1), (2.3) with $h = 1$, $H = 2$ and potentials, possessing discontinuous derivatives. We take the same potential (5.1) and the following ‘sawtooth’ potential

$$q_2(x) = \int_0^x \text{sign} \left( \sin \left( \frac{10t}{4 - t} \right) \right) \, dt,$$

(5.2)

inspired by an excellent article [34]. For both potentials we considered 40 or 201 eigenvalue pairs, complemented them with AEVs to get in total 10 001 pairs and solved the system (3.22) truncated to eight equations. On figure 8 we present the recovered potentials together with $L_1(0, \pi)$ norm of the absolute errors and obtained values of the parameters $h$ and $H$.

Finally, on figure 9 on the left we present the original coefficients $g_0(\pi)$ and $h_n$ and the ones recovered by solving the systems (3.10) and (4.7) and on the right we present absolute errors of the recovered norming constants.

**Example 5.4.** Finally, we consider the spectral problems (2.1), (2.2) and (2.1), (2.3) with $h = 1$, $H = 2$ and discontinuous potential which we tried to match with those from [30].
Figure 10. Exact (blue dashed lines) and recovered (black lines) potential from example 5.4 together with the number of exact eigenvalue pairs used, average absolute error, error in the recovered parameter $\omega$ and found values of the parameters $h$ and $H$.

Namely,

\[
q_3(x) = \begin{cases} 
0, & x \in \left[0, \frac{\pi}{8}\right] \cup \left[\frac{3\pi}{8}, \frac{3\pi}{5}\right], \\
-\frac{12x}{\pi} + \frac{3}{2}, & x \in \left[\frac{\pi}{8}, \frac{\pi}{4}\right], \\
\frac{12x}{\pi} - \frac{9}{2}, & x \in \left[\frac{\pi}{4}, \frac{3\pi}{8}\right], \\
4, & x \in \left[\frac{3\pi}{8}, \frac{4\pi}{8}\right], \\
2, & x \in \left[\frac{4\pi}{8}, \frac{5\pi}{5}\right].
\end{cases}
\]  

(5.3)

On figure 10 we present the recovered potential together with $L_1(0, \pi)$ norm of the absolute error, absolute error of the recovered parameter $\omega$ and obtained values of the parameters $h$ and $H$ for 30 and 201 eigenvalue pairs. Again, 8 equations were used in the truncated system. The number of eigenvalues were complemented to 5000 with the AEVs.

6. Conclusions

The method presented here allows one to reduce the inverse spectral problem to a system of linear algebraic equations, and only the first component of the solution vector is sufficient for recovering the potential. This, together with the convergence and stability results, means that a reduced number of equations (say, five–eight) is enough to obtain an accurate result. Though in the present work only two classical inverse Sturm–Liouville problems are considered, we also show how some other inverse Sturm–Liouville problems can be solved by the
same method. The numerical realization of the method is simple and involves nothing but the build-in functions of a modern numerical computing environment, such as Matlab.

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References

[1] Abramovitz M and Stegun I A 1972 Handbook of Mathematical Functions (New York: Dover)
[2] Atkinson F V 1964 Discrete and Continuous Boundary Problems (New York: Academic)
[3] Auret J L and Trefethen L N 2017 Chopping a Chebyshev series ACM Trans. Math. Softw. 43 33
[4] Avdonin S A and Ivanov S A 1995 Families of Exponentials. The Method of Moments in Controllability Problems for Distributed Parameter Systems (Cambridge: Cambridge University Press)
[5] Barnett A R 1996 The calculation of spherical Bessel and Coulomb functions Computational Atomic Physics, Electron and Positron Collisions with Atoms and Ions (Berlin: Springer) pp 181–202
[6] Brown B M, Samko V S, Knowles I W and Marletta M 2003 Inverse spectral problem for the Sturm–Liouville equation Inverse Problems 19 235–52
[7] Chadan K, Colton D, Paivarinta L and Rundell W 1997 An Introduction to Inverse Scattering and Inverse Spectral Problems (Philadelphia, PA: SIAM)
[8] Drignei M-C 2015 A Newton-type method for solving an inverse Sturm–Liouville problem ACM Trans. Math. Softw. 43 33
[9] Gao Q, Cheng X and Huang Z 2013 Modified Numerov’s method for inverse Sturm–Liouville problems J. Comput. Appl. Math. 253 181–99
[10] Gao Q, Cheng X and Huang Z 2014 On a boundary value method for computing Sturm–Liouville potentials from two spectra Int. J. Comput. Math. 91 490–513
[11] Gel’fand I M and Levitan B M 1951 On the determination of a differential equation from its spectral function Izvestiya AN SSSR, Ser. matem. 15 309–60
[12] Gillman E and Fiebig H R 1988 Accurate recursive generation of spherical Bessel and Neumann functions for a large range of indices Comput. Phys. 2 62–72
[13] Ignatiev M and Yurko V 2008 Numerical methods for solving inverse Sturm–Liouville problems Result. Math. 52 63–74
[14] Kammanee A and Böckmann C 2009 Boundary value method for inverse Sturm–Liouville problems Appl. Math. Comput. 214 342–52
[15] Kantorovich L V and Akilov G P 1982 Functional Analysis 2nd edn (Oxford: Pergamon) Translated from the Russian by Howard L Silcock.
[16] Khmelnytskaya K V, Kravchenko V V and Torba S M 2019 A representation of the transmutation kernels for the Schrödinger operator in terms of eigenfunctions and applications Appl. Math. Comput. 353 274–81
[17] Kravchenko V V 2019 On a method for solving the inverse Sturm–Liouville problem J. Inverse Ill-posed Probl. 27 401–7
[18] Kravchenko V V 2020 Direct and Inverse Sturm–Liouville Problems: A Method of Solution (Basel: Birkhäuser)
[19] Kravchenko V V, Navarro L J and Torba S M 2017 Representation of solutions to the one-dimensional Schrödinger equation in terms of Neumann series of Bessel functions Appl. Math. Comput. 314 173–92
[20] Kravchenko V V and Torba S M 2018 A Neumann series of Bessel functions representation for solutions of Sturm–Liouville equations Calcolo 55 11
[21] Levitan B M 1987 Inverse Sturm–Liouville Problems (Zeist: VSP)
[22] Levitan B M and Sargsjan I S 1991 Sturm–Liouville and Dirac Operators (Dordrecht: Kluwer Academic Publishers)
[23] Lowe B D, Pilant M and Rundell W 1992 The recovery of potentials from finite spectral data SIAM J. Math. Anal. 23 482–504
[24] Marchenko V A 2011 *Sturm–Liouville Operators and Applications: Revised Edition* (Providence, Rhode Island: AMS Chelsea Publishing)
[25] Mihlin S G 1971 *The Numerical Performance of Variational Methods* (Groningen, The Netherlands: Wolters-Noordhoff Publishing)
[26] Neamaty A, Akbarpoor S and Yilmaz E 2018 Solving inverse Sturm–Liouville problem with separated boundary conditions by using two different input data *Int. J. Comput. Math.* 95 1992–2010
[27] Neamaty A, Akbarpoor S and Yilmaz E 2019 Solving symmetric inverse Sturm–Liouville problem using Chebyshev polynomials *Mediterr. J. Math.* 16 74
[28] Röhrl N 2005 A least-squares functional for solving inverse Sturm–Liouville problems *Inverse Problems* 21 2009–17
[29] Russell D L 1967 Nonharmonic Fourier series in the control theory of distributed parameter systems *J. Math. Anal. Appl.* 18 542–60
[30] Rundell W and Sacks P E 1992 Reconstruction techniques for classical inverse Sturm–Liouville problems *Math. Comp.* 58 161
[31] Sacks P E 1988 An iterative method for the inverse Dirichlet problem *Inverse Problems* 4 1055–69
[32] Savchuk A M and Shkalikov A A 2005 Inverse problem for Sturm–Liouville operators with distribution potentials: reconstruction from two spectra *Russ. J. Math. Phys.* 12 507–14
[33] Sitnik S M and Shishkina E L 2020 *Transmutations, Singular and Fractional Differential Equations with Applications to Mathematical Physics* (Amsterdam: Elsevier)
[34] Trefethen L N 2011 Six myths of polynomial interpolation and quadrature *Math. Today (Southend-on-Sea)* 47 184–8
[35] Verlan A F and Sizikov V S 1986 *Integral Equations: Methods, Algorithms, Programs* (Kiev: Naukova Dumka) (in Russian)
[36] Young R M 2001 *An Introduction to Nonharmonic Fourier Series* Revised 1st edn (New York: Academic)
[37] Yurko V A 2007 *Introduction to the Theory of Inverse Spectral Problems* (Moscow: Fizmatlit) (in Russian)