Finite generation of Cox rings

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The projective plane $\mathbb{P}^2$ is one of the most ubiquitous objects in geometry. It appears as a compactification of the plane by adding “a line at infinity” as proposed around the second half of the 15th century by Piero della Francesca in his *De prospectiva pingendi*. Painters started using perspective as a consequence of this construction.

One has to wait until the beginning of the XIX century for August Ferdinand Möbius to introduce homogeneous coordinates which provide an algebraic framework for projective planes. Homogeneous coordinates appear as soon as one defines the $n$-dimensional complex projective space as

$$
\mathbb{P}^n := \left( \mathbb{C}^{n+1} \setminus \{0\} \right) / \sim
$$

where $\sim$ is the equivalence relation that identifies $x = (x_0, \ldots, x_n)$ and $\lambda x = (\lambda x_0, \ldots, \lambda x_n)$ for all $x \in \mathbb{C}^{n+1}$ and $\lambda \in \mathbb{C}^\ast := \mathbb{C} \setminus \{0\}$. Cox rings can be thought of as a generalization of homogeneous coordinates to a wider class of algebraic varieties, like the cartesian products of projective spaces. The main idea behind Cox rings is that they allow us to define the quotient construction of the projective space in an intrinsic way: where is the space $\mathbb{C}^{n+1}$ coming from? where is the equivalence relation coming from? why does one remove the zero vector? We provide answers to these questions. The projective space $\mathbb{P}^n$ is the disjoint union of the affine space $\mathbb{C}^n$ with the “hyperplane at infinity” $H$, given by the horizon line in Figure 1. The coordinate ring of $\mathbb{C}^n$ is the polynomial ring $S := \mathbb{C}[x_1, \ldots, x_n]$ in $n$ variables. Given an integer $d \geq 0$ one can form the vector space $S_d \subseteq S$ of polynomials of degree at most $d$. The Cox ring of $\mathbb{P}^n$ is the direct sum $\oplus_{d \geq 0} S_d$ which is a graded algebra isomorphic to the polynomial ring $\mathbb{C}[x_0, \ldots, x_n]$ in $n+1$ variables. This is the coordinate ring of $\mathbb{C}^{n+1}$ and its grading assigning degree one to each variable corresponds to the $\mathbb{C}^\ast$-action $\lambda \cdot x := \lambda x$, which induces the previous equivalence relation. Finally, the point $0 \in \mathbb{C}^{n+1}$ is in the closure of any orbit for this action, then one has to remove it in order to obtain a quotient which is not just one point.

From the construction of the Cox ring of the projective space, we can see that the degree of a homogeneous polynomial is a central notion. This degree can be replaced with the order of the pole of a polynomial in $S$ along the “hyperplane at infinity” $H$. In both cases, the notion depends on the choice of an open subset of the projective space: the affine space $\mathbb{C}^n$. The reason for this choice is that the complement $H = \mathbb{P}^n \setminus \mathbb{C}^n$ generates the divisor class group of $\mathbb{P}^n$, an invariant which is hidden behind the whole construction of the Cox ring. So, more generally, when one starts with an algebraic variety $X$ which admits

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an affine Zariski open subset \( U \subseteq X \) such that the codimension-one subvarieties in \( X \setminus U \) generate the divisor class group of \( X \) one can define the Cox ring of \( X \) in a similar way as for the Cox ring of \( \mathbb{P}^n \). The Cox ring is not necessarily finitely generated as an algebra over the complex numbers, but when it is finitely generated, it is the coordinate ring of an affine algebraic variety \( \overline{X} \). The grading of the Cox ring corresponds to an action of an abelian group \( H_X \) over \( \overline{X} \). One can show that, as in the case of the projective space, there is an open invariant subset \( \overline{X} \) of \( \overline{X} \) and a quotient map

\[
p_X : \overline{X} \to X
\]

by the action of \( H_X \).

Understanding all morphisms from a projective variety \( X \) to other projective varieties is a fundamental problem. One would like to decompose each map into simple steps and parametrize the possibilities. Morphisms between projective varieties can be factored as the composition of a morphism with connected fibers followed by one with finite fibers. Then, we would like to understand the decompositions of maps with connected fibers. A natural setting for this question is to consider not only morphisms but also the so-called rational contractions, that is, compositions of rational maps which are isomorphisms in codimension one and surjective morphisms with connected fibers, see [HK00, Definition 1.0]. If the Cox ring of \( X \) is finitely generated each such rational contraction is uniquely determined by the choice of a certain invariant open subset of \( \overline{X} \), which gives rise to another quotient. The collection of these subsets is in bijection with the set of cones, called Mori chambers, of a fan supported on the effective cone of \( X \), see Section 5.

The question of the finite generation of the Cox ring is thus central and we devote the second part of this note to it, ending it with some recent examples of finitely and non-finitely generated Cox rings. Along the way, we discuss some natural questions, including when are Cox rings polynomial rings? when do two algebraic varieties have isomorphic Cox rings? The answer to the first question leads us to toric varieties and the work of D. Cox [Cox95], while the answer to the second question allows us to introduce a fascinating geometric object: Mori dream spaces, following the work of Y. Hu and S. Keel [HK00].

A comprehensive reference on Cox rings is [ADHL15], which also discusses the history of the subject. We highlight J.-L. Colliot-Thélène and J.-J. Sansuc who introduced universal torsors in arithmetic geometry in the 1970s. David Cox introduced the homogeneous coordinate rings of toric varieties in [Cox95]. Y. Hu and S. Keel proposed the name Cox ring and showed how the finite generation of this ring is connected to the geometry of the variety. Finally, J. Hausen in [Hau08] gave a definition of Cox rings which generalizes the previous ones and is the one described in this article.

1 Graded algebras

Our first step towards the definition of Cox rings is to introduce graded algebras. Given a finitely generated abelian group \( A \), a \( \mathbb{C} \)-algebra \( R \) is \( A \)-graded if

\[
R := \bigoplus_{a \in A} R_a,
\]

where each \( R_a \) is a complex vector space and \( R_a \cdot R_b \subseteq R_{a+b} \) for any \( a, b \in A \). The elements of \( R_a \) are called homogeneous of degree \( a \). One says that \( R \) is finitely generated if there is a surjective homomorphism of \( \mathbb{C} \)-algebras from the polynomial ring \( \mathbb{C}[x_1, \ldots, x_r] \) to \( R \). Moreover, without loss of generality, one can assume the images of the variables are homogeneous. The kernel is the ideal of an affine algebraic set \( X \subseteq \mathbb{C}^r \) which is uniquely determined by \( R \) up to isomorphism. One can recover \( R \) as the coordinate ring of \( X \), and this ring is a domain precisely when \( X \) is an affine variety (i.e., an irreducible affine algebraic set). The group \( A \) determines the monoid algebra

\[
\mathbb{C}[A] := \bigoplus_{a \in A} \mathbb{C} \cdot \chi^a
\]

with product \( \chi^a \cdot \chi^b = \chi^{a+b} \) (see [ADHL15, Constr. 1.1.1.5]). One can show that the affine variety \( G \) defined by this algebra is isomorphic to \( \text{Hom}(A, \mathbb{C}^*) \) as a group. If \( A \) is isomorphic to \( \mathbb{Z}^n \oplus A_{\text{tor}} \), where the second summand denotes the torsion part, then we have \( G \cong (\mathbb{C}^*)^n \oplus A_{\text{tor}} \). The latter group is a
quasitorus which acts on $X$ in the following way. For any $a \in A$ denote by $\chi^a : G \to \mathbb{C}^*$ the character $g \mapsto \chi^a(g) := g(a)$. If we denote by $a_1, \ldots, a_r \in A$ the degrees of the above $r$ homogeneous generators of $R$, then there is an action of $G$ on $\mathbb{C}^r$ given by

$$g \cdot (x_1, \ldots, x_r) := (\chi^{a_1}(g)x_1, \ldots, \chi^{a_r}(g)x_r),$$

which induces an action of $G$ on $X$. In more abstract terms, this group action is induced by the homomorphism $R \to R \otimes \mathbb{C}[A]$ defined by $R_a \ni f_a \mapsto f_a \otimes \chi^a$. The above action can be defined even when $R$ is not finitely generated.

**Example 1.1.** Two examples, useful in the sequel, are given by two different $\mathbb{Z}^2$-gradings of the polynomial ring $\mathbb{C}[x_1, x_2, x_3, x_4]$. Each grading is defined by the columns of one of the following matrices (the degree of $x_i$ is the $i$-th column):

$$\begin{bmatrix}
1 & 1 & n & 0 \\
0 & 0 & 1 & 1
\end{bmatrix}, \quad \begin{bmatrix}
1 & 1 & 1 & 0 \\
0 & 1 & 1 & 1
\end{bmatrix},$$

where $n$ is a nonnegative integer. In each case $G \simeq (\mathbb{C}^*)^2$ acts on $\mathbb{C}^3$. If we denote by $t_1 := \chi^{(1,0)}(t)$ and $t_2 := \chi^{(0,1)}(t)$ then, for example, the second action is $t \cdot (x_1, \ldots, x_4) = (t_1 x_1, t_1 t_2 x_2, t_1 t_2 x_3, t_2 x_4)$.

**Divisor class group**

Our second step is to introduce a fundamental invariant of an algebraic variety: the *divisor class group*. Let $X$ be an algebraic variety. A rational map or rational function $X \dashrightarrow \mathbb{C}$ on $X$ is an equivalence class of morphisms from nonempty open subsets of $X$ to $\mathbb{C}$, where two such morphisms are identified if they agree on a nonempty open set. The set of all rational functions on $X$ is a field under pointwise sum and product called the *field of rational functions* of $X$.

The free abelian group generated by the codimension-one subvarieties is the *group of Weil divisors* of $X$. Given a Weil divisor $D = \sum_i a_i D_i$, its *support* is the union $\bigcup_i D_i$. A codimension-one subvariety is called a *prime divisor*. To any nonzero rational function $f \in \mathbb{C}(X)$ one associates a Weil divisor

$$\text{div}(f) := \sum_{D \text{ prime divisor}} \text{ord}_D(f)D,$$

where $\text{ord}_D(f)$ is an integer which intuitively represents the order of vanishing of $f$ along the prime divisor $D \subseteq X$. It turns out that the sum above is finite and that the assignment $f \mapsto \text{div}(f)$ is a homomorphism of abelian groups. Its image is the group of *principal divisors* and finally one defines the divisor class group $\text{Cl}(X)$ as the quotient of the group of Weil divisors modulo the subgroup of principal divisors. From now on we assume that this group is finitely generated, which is the case for example when $X$ is a rational, a Fano or a Calabi-Yau variety. Two Weil divisors $D, D'$ are *linearly equivalent* if their difference $D - D'$ is principal. We now recall the definition of a normal variety.

**Definition 1.2.** A normal affine variety $X$ is an affine variety which has singularities in codimension two or more and that for any open subset $U \subseteq X$, with complement of codimension two or more, the regular functions of $U$ are restrictions of regular functions of $X$. More generally a normal variety is one which is covered by normal affine ones.

**Example 1.3.** An example of a non-normal affine variety is given by any singular curve, since the first condition for normality is not satisfied.

**Example 1.4.** An example where the second condition is not satisfied is the affine variety $X$ defined by the subalgebra $A \subseteq \mathbb{C}[x, y]$ generated by all the monomials but $x$. It is easy to see that $A$ is in fact generated by $x^2, x^3, y, xy$, so that $X$ is the image in $\mathbb{C}^4$ of the map $(x, y) \mapsto (x^2, x^3, y, xy)$. If $u_1, u_2, u_3, u_4$ are coordinates of $\mathbb{C}^4$ then on the open subset $U := X \setminus V(u_1, u_3)$ the function $u_2/u_1 = u_4/u_3$ is regular but it is not restriction of a regular function on $X$.

Our next definition is that of a *Cartier divisor*. The usual definition is different from the one given in the following lines, but one can show that on a normal algebraic variety the two definitions coincide. A Weil divisor $D$ is *Cartier* if it is locally principal, that is
X admits an open covering and over each such open
subset \( U \) one has \( D|_U = \text{div}(f)|_U \) for some rational
function \( f \in \mathbb{C}(X) \), where \( D|_U \) means that one re-
moves all the prime divisors in the support of \( D \) that
do not intersect \( U \). A variety is \( \mathbb{Q} \)-factorial if any
Weil divisor admits a nonzero integer multiple which
is a Cartier divisor.

As a preliminary to the definition of Cox rings we
briefly recall the notion of a sheaf.

**Definition 1.5.** A presheaf \( \mathcal{F} \) of groups over
a topological space \( X \) is a contravariant func-
tor from the category of open subsets of \( X \)
with inclusions to the category of groups with
homomorphisms. To any inclusion of open
sets \( V \subseteq U \) one associates a homomorphism
\( \mathcal{F}(U) \to \mathcal{F}(V) \) usually denoted by \( f \mapsto f|_V \).
The set \( \mathcal{F}(U) \) is also denoted by \( \Gamma(\mathcal{F}, U) \) and
its elements are called the sections of \( \mathcal{F} \) over
\( U \). A presheaf is a sheaf if for any open cov-
ering \( \{U_i\} \) of an open subset \( U \) and any collection of
\( f_i \in \mathcal{F}(U_i) \), such that \( f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j} \) for all \( i, j \), there exists a unique \( f \in \mathcal{F}(U) \) such that \( f|_{U_i} = f_i \) for all \( i \). Presheaves
and sheaves of rings or algebras are defined
analogously.

**Example 1.6.** A typical example to keep
in mind is the presheaf of constant functions
with real values on a topological space. It
is not a sheaf in general since one can have
locally constant functions over disjoint con-
ected components which do not form a con-
stant function on the whole space (here the
sheaf condition is automatic being the inter-
sections \( U_i \cap U_j \) all empty). The presheaf of
locally constant functions is a sheaf.

**Example 1.7.** Two important examples are
the sheaves that assign to each open subset \( U \)
of a variety \( X \) the regular functions on \( U \) and
the rational functions on \( U \), with restrictions
given by the restriction of functions.

A Weil divisor \( D = \sum a_i D_i \) is effective if all of its
coefficients \( a_i \) are nonnegative. Any Weil divisor \( D \)
defines a sheaf \( \mathcal{O}_X(D) \) on \( X \) whose space of sections
on the open subset \( U \subseteq X \) is the vector space gen-
erated by the nonzero rational functions \( f \in \mathbb{C}(X) \) such
that \( (\text{div}(f) + D)|_U \) is effective. A basis \( \{f_0, \ldots, f_N\} \)
of the space of global sections \( \Gamma(X, \mathcal{O}_X(D)) \) induces
a rational map

\[
\psi_D: X \to \mathbb{P}^N, \quad p \mapsto [f_0(p) : \cdots : f_N(p)],
\]

and choosing a different basis changes \( \psi_D \) with \( \sigma \circ
\psi_D \), where \( \sigma \) is an automorphism of \( \mathbb{P}^N \). The divisor
\( D \) is base point free if \( \psi_D \) is a morphism; it is very
ample if \( \psi_D \) is an embedding; it is ample, respectively
semiaample, if there exists positive integer \( n \) such that
\( \psi_{nD} \) is very ample, respectively \( \psi_{nD} \) is a morphism.

**Cox rings**

To any finitely generated subgroup \( K \) of Weil divisors
on a normal algebraic variety \( X \) one can associate the
following sheaf of algebras

\[
\mathcal{S} := \bigoplus_{D \in K} \mathcal{O}_X(D).
\]

The sheaf \( \mathcal{S} \) is graded by \( K \) and it is possible to show
that its ring of global sections \( \Gamma(X, \mathcal{S}) \) is a factorial
ring whenever the class map \( \text{cl}|_K: K \to \text{Cl}(X) \) is a
surjection [ADHL15, Thm. 1.3.3.3]. If the divisor
class group is torsion-free then one can choose \( K \)
such that \( \text{cl}|_K \) is an isomorphism and define a Cox ring of
\( X \) as \( \Gamma(X, \mathcal{S}) \). It is not difficult to show that different
choices for \( K \) lead to isomorphic Cox rings. However
things get more complicated when the divisor class
group has torsion. In order to provide a definition of
Cox ring which also includes this possibility one starts
with a finitely generated subgroup of Weil divisors \( K \)
such that \( \text{cl}|_K \) is surjective and let \( K_0 \) be its kernel.
One can define a homomorphism \( \chi: K^0 \to \mathbb{C}(X)^* \)
such that \( \text{div} \circ \chi = \text{id} \) because \( K^0 \) is free abelian
and its elements are principal divisors. Given a principal
divisor \( D \), a rational function \( f \) such that \( \text{div}(f) = D \)
is defined only up to scalar multiplication, provided
that the only global invertible regular functions of
\( X \) are constants, which we are going to assume from
now on. Introducing \( \chi \) allows one to make a coherent
choice for all these scalars. Define the sheaf of ide-
als \( \mathcal{I} \subseteq \mathcal{S} \) which is locally generated by elements of
the form $1 - \chi(D)$, for $D \in K^0$. Given two divisors $D, D' \in K$ whose difference is in $K^0$ it is easy to see that the map $\mathcal{O}_X(D) \to \mathcal{O}_X(D')$ defined by multiplication by $\chi(D - D')$ is an isomorphism. Taking the quotient by $\mathcal{I}$ has the effect of identifying these sheaves keeping one copy for each divisor class.

**Definition 1.8.** Let $X$ be a normal algebraic variety with a finitely generated divisor class group and whose global invertible regular functions are constants. Given a choice of $K$ and $\chi$ as before one defines a *Cox sheaf* and a *Cox ring* of $X$ as

$$\mathcal{R} := \mathcal{S}/\mathcal{I} \quad \text{and} \quad \mathcal{R}(X) := \Gamma(X, \mathcal{R}),$$

respectively. It is possible to show that any two such Cox rings for $X$ are isomorphic, in other words the isomorphism class does not depend on the choice of $K$ and $\chi$.

We can write the Cox ring making explicit its grading by the divisor class group as follows:

$$\mathcal{R}(X) = \bigoplus_{\text{Cl}(X)} \Gamma(X, \mathcal{O}_X(D)).$$

**Remark 1.9.** A *big open subset* of an algebraic variety is an open subset whose complement has codimension greater than or equal to 2. A direct consequence of Definition 1.8 is that if $U \subseteq X$ is a big open subset, then the inclusion induces an isomorphism between their Cox rings. Indeed, each prime divisor of $U$ restricts to a prime divisor of $X$ and conversely a divisor of $U$ has a unique closure in $X$. In particular, if $X$ and $Y$ are birationally equivalent varieties with isomorphic big open subsets then the Cox rings of $X$ and $Y$ are isomorphic.

**Example 1.10.** Consider $X = \mathbb{P}^n$. The divisor class group is freely generated by the class of a hyperplane $H$, so let $K = \langle H \rangle$. If $x_0, \ldots, x_n$ are homogeneous coordinates and $H = V(x_0)$, then the space of global sections of $\mathcal{O}_{\mathbb{P}^n}(dH)$ is generated by polynomials in the variables $\tilde{x}_i = \frac{x_i}{x_0}$ of degree up to $d$. This space is isomorphic to the space of degree $d$ homogeneous polynomials in $n+1$ variables. It follows that the Cox ring is isomorphic to the polynomial ring $\mathbb{C}[x_0, \ldots, x_n]$.

We conclude this section with an algebraic property of Cox rings. If the divisor class group is torsion-free then the Cox ring is a factorial ring. More generally one has the following, see [ADHL15, Thm. 1.5.3.7, Thm. 1.3.3.3] and [BH03, EKW04, Arz09].

**Theorem 1.11.** Every nonzero non-unit homogeneous element in the Cox ring can be written as a product of a finite number of irreducible homogeneous elements, uniquely up to order and units.

### 2 Quotient construction

Whenever the Cox ring of an algebraic variety $X$ is finitely generated it defines an affine variety named the *total coordinate space* of $X$ and denoted by $\hat{X}$. It follows, by our previous discussion, that the monoid algebra $\mathbb{C}[\text{Cl}(X)]$ defines a quasitorus $H_X$ which is isomorphic, as a group, to $\text{Hom}(\text{Cl}(X), \mathbb{C}^*)$. The $\text{Cl}(X)$-grading of the Cox ring induces an action of $H_X$ on $\hat{X}$. The Cox sheaf determines an invariant affine subvariety $\hat{X} \subseteq \hat{X}$ which admits $X$ as a good quotient. We briefly describe how this invariant open subset is constructed. Let $U_1, \ldots, U_s$ be affine open subsets whose union is $X$. The algebra of global sections $\Gamma(U, \mathcal{R})$ defines an invariant affine subvariety $\overline{U}_i \subseteq \hat{X}$ and one defines $\hat{X} := \overline{U}_1 \cup \cdots \cup \overline{U}_s$. The inclusion homomorphism $\Gamma(U, \mathcal{R})_0 \subseteq \Gamma(U, \mathcal{R})$ of the subring generated by the degree-zero homogeneous elements induces a quotient map $\overline{U}_i \rightarrow U_i$. All these maps glue together producing the quotient map $p_X : \hat{X} \rightarrow X$. Thus one has the following diagram

$$\begin{array}{ccc}
\hat{X} & \xrightarrow{p_X} & X \\
\downarrow & & \downarrow \\
\hat{X} & \subseteq & \hat{X}
\end{array}$$

**Example 2.1.** When $X$ is the projective space $\mathbb{P}^n$ we have already seen that the Cox ring is the polynomial ring $\mathbb{C}[x_0, \ldots, x_n]$ with the $\mathbb{Z}$-grading which assigns degree 1 to each variable. One can cover the projective space with the $n+1$ affine spaces $U_0, \ldots, U_n$, where $U_i := \mathbb{P}^n \setminus V(x_i)$. Over each such subset $\Gamma(U, \mathcal{R}) \nearlyequal \mathbb{C}[x_0, \ldots, x_n]_{x_i}$, where the right-hand side is the localization of the polynomial ring with respect to $x_i$. The inclusion homomorphism $\Gamma(U_i, \mathcal{R})_0 \subseteq \Gamma(U_i, \mathcal{R})$ induces a quotient map $\overline{U}_i \rightarrow U_i$. All these maps glue together producing the quotient map $p_X : \hat{X} \rightarrow X$. Thus one has the following diagram

$$\begin{array}{ccc}
\hat{X} & \xrightarrow{p_X} & X \\
\downarrow & & \downarrow \\
\hat{X} & \subseteq & \hat{X}
\end{array}$$
to the multiplicative subset generated by the powers of \(x_i\). In other words, one takes Laurent polynomials in the \(i\)-th variable. Thus, for example, \(U_0\) is the open subset of \(\mathbb{C}^{n+1}\) where the first variable does not vanish. The quotient morphism \(\overline{U}_0 \to U_0\) is given by \((x_0, \ldots, x_n) \mapsto [1 : x_1/x_0 : \cdots : x_n/x_0]\). All these morphisms glue together to give the usual quotient construction of the projective space.

Remark 2.2. The morphism \(p_X\) is a good quotient in the following sense: it is invariant, which means that \(p_X(h \cdot x) = p_X(x)\) for any \(h \in H_X\) and \(x \in X\); it is affine, which means that the preimage of an affine subset \(U\) is affine; and finally each invariant regular function on the affine subset \(p_X^{-1}(U)\) is the pullback of a regular function on \(U\).

Example 2.3. If we go back to Example 1.1 the two gradings on \(R := \mathbb{C}[x_1, x_2, x_3, x_4]\) define two actions of \((\mathbb{C}^*)^2\) on \(\mathbb{C}^4\). Let \(U_{ij} := \mathbb{C}^4 \setminus V(x_ix_j)\) for \(i \in \{1, 2\}\) and \(j \in \{3, 4\}\) and let \(U\) be the union of these four open affine subsets. In both cases the grading induces a quotient map of tori \((\mathbb{C}^*)^4 \to (\mathbb{C}^*)^2\). In the first case it is \((x_1, x_2, x_3, x_4) \mapsto (x_2/x_1, x_1^n x_4/x_3)\) and the four quotient maps glue together to give the good quotient \(U \to \mathbb{F}_n\), where \(\mathbb{F}_n\) is the \(n\)-th Hirzebruch surface. In the second case the morphism is \((x_1, x_2, x_3, x_4) \mapsto (x_2/x_3, x_1 x_4/x_3)\) and the images of \(U_{13}\) and \(U_{34}\) coincide, as a consequence the quotient morphism on \(U\) is not affine.

Example 2.3 shows that not every graded polynomial ring is a Cox ring. A complete characterization is given by the next proposition, where \(A_Q\) denotes the rational vector space \(A \otimes_{\mathbb{Z}} \mathbb{Q}\) and \(\text{cone}(w_i : i \in I)\) is the convex cone of \(A_Q\) generated by the vectors \(w_i\) indexed by \(I\). A reference for the following result is [ADHL15, Exer. 2.11].

Proposition 2.4. Let \(\mathbb{C}[x_1, \ldots, x_r]\) be an \(A\)-graded polynomial ring with homogeneous variables and let \(w_i := \text{deg}(x_i)\) for all \(i\). This polynomial ring is a Cox ring if the following conditions hold:

1. \(A\) is generated by any \(r - 1\) elements of \(\{w_1, \ldots, w_r\}\);
2. for each \(1 \leq i, j \leq r\) the interior of the cones \(\text{cone}(w_k : k \neq i)\) and \(\text{cone}(w_k : k \neq j)\) have non-empty intersection.

The first condition is equivalent to the triviality of the stabilizer of a general point of the divisor \(V(x_i)\) for all \(i\). The second condition is used to guarantee that the quotient is good. As an application, the first algebra in Example 1.1 is a Cox ring, while the second one is not.

3 Toric varieties

Proposition 2.4 characterizes the graded polynomial rings that are Cox rings. Now we would like to identify the varieties with such polynomial Cox rings. These are the toric varieties and we describe them in this section. The affine variety \(T := (\mathbb{C}^*)^n\) is called the \(n\)-dimensional algebraic torus or simply the torus. The function \(T \times T \to T\) given by coordinatewise multiplication is a morphism and it makes a torus \(T\) into an algebraic group.

Definition 3.1. A toric variety is a normal variety that contains a torus \(T\) as an open subset, and which has an action of \(T\) via a morphism \(T \times X \to X\) that restricted to \(T \subseteq X\) is the \(T \times T \to T\) coordinatewise multiplication action of \(T\) on itself.

Example 3.2. Projective space \(\mathbb{P}^n\) is a toric variety with the \(T\)-action given by \(t \cdot x = [x_0 : t x_1 : \cdots : t^n x_n]\) for any \(t = (t_1, \ldots, t_n) \in T\) and \(x = [x_0 : x_1 : \cdots : x_n] \in \mathbb{P}^n\). The coordinate chart \(U_0 := \mathbb{P}^n \setminus V(x_0)\) is an affine space invariant under the same action, so that \(\mathbb{A}^n\) is also a toric variety.

In order to provide a description for the Cox ring of a toric variety we need to review the combinatorial language used to describe such varieties. To begin with, each one-parameter subgroup \(\mathbb{C}^* \to T\) is uniquely determined by a vector of exponents in a space \(N \simeq \mathbb{Z}^n\). Dually each character \(T \to \mathbb{C}^*\) is determined by a vector \(u\) in the dual \(M = \text{Hom}(N, \mathbb{Z}) \simeq \mathbb{Z}^n\) of \(N\). The duality is expressed by a unimodular bilinear pairing \(M \times N \to \mathbb{Z}, (u, v) \mapsto \langle u, v \rangle\). This extends to a duality of the corresponding rational vector spaces \(M_Q = M \otimes_{\mathbb{Z}} \mathbb{Q}, N_Q = N \otimes_{\mathbb{Z}} \mathbb{Q}\). If \(X\) is an affine toric variety and \(x_0 \in X\) is a point in the open torus.
Remark 3.3. A toric variety $X$ arises from a fan. For all the integer points $v \in X$ algebra is in the relative interior of $\sigma$ which implies that $S$ of a unique $T$-orbit, the set of one-parameter subgroups $t \mapsto t^v$ such that $\lim_{t \to 0} t^v \cdot x_0$ exists in $X$ form the set of integer points of a convex polyhedral (i.e., finitely generated) cone $\sigma \subseteq N_\mathbb{Q}$. This cone $\sigma$ is strictly convex (i.e., $\{0\} = \text{a face of } \sigma$), since $T$ is an invariant open subset of $X$. On the other hand, given such a cone, its dual $\sigma^\vee := \{ u \in M_\mathbb{Q} : \langle u, v \rangle \geq 0 \text{ for any } v \in \sigma \}$ defines the monoid algebra $\mathbb{C}[\sigma^\vee \cap M]$ which turns out to be finitely generated. The corresponding affine variety $X_\sigma$ is toric isomorphic to $X$, with torus action induced by the homomorphism $\mathbb{C}[\sigma^\vee \cap M] \to \mathbb{C}[\mathbb{R}] \otimes \mathbb{C}[\sigma^\vee \cap M]$ defined by $\chi^m \mapsto \chi^m \otimes \chi^m$.

More generally toric varieties are constructed by gluing affine toric varieties which share a common torus $T$. Any toric variety $X$ has an open cover by affine toric varieties, and thus it corresponds to a collection $\Sigma$ of strictly convex polyhedral cones in the space $N_\mathbb{Q}$ and the gluing condition implies that $\Sigma$ is a fan: each face of a cone in $\Sigma$ is again $\mathbb{R}$-invariant subvarieties, each arising as the closure of a unique $T$-orbit. The toric variety $X_{\Sigma}$ is toric isomorphic to $X$, with torus action induced by the homomorphism $\mathbb{C}[\sigma^\vee \cap M] \to \mathbb{C}[\mathbb{R}] \otimes \mathbb{C}[\sigma^\vee \cap M]$ defined by $\chi^m \mapsto \chi^m \otimes \chi^m$.

By Theorem 3.4, the Cox ring of a toric variety $X$ with torus invariant prime divisors $D_1, \ldots, D_r$ is the $\mathbb{C}$-graded polynomial ring $\mathbb{C}[x_1, \ldots, x_r]$, where $\deg(x_i) = [D_i] \in \text{Cl}(X)$ for each $i$. Since the Laurent polynomial ring is factorial, the divisor class group of a torus $T$ is trivial so that any divisor of a toric variety is linearly equivalent to a $T$-invariant one. As a consequence, if $K$ is the group of $T$-invariant Weil divisors then the class map $\text{cl}|_K : K \to \text{Cl}(X)$ is surjective. Its kernel $K^0$ consists of $T$-invariant principal divisors. Each such divisor $\text{div}(\chi)$ cannot intersect the torus $T$, so that $\chi$ is a regular function on $T$. Moreover $T$-invariance implies that $\chi|_T$ is a character of the torus. Vice versa, any character of $T$ gives a principal torus invariant divisor. From now on we also assume that the only global regular invertible functions of $X$ are constants, which is equivalent to the rays of the fan $\Sigma$ spanning $N_\mathbb{Q}$. It follows that the map $M \to K^0$ defined by $u \mapsto \text{div}(\chi^u)$ is an isomorphism of abelian groups. Now, let $\mathcal{S}$ be as in (1) and let $D_1, \ldots, D_r$ be the prime $T$-invariant divisors of $X$. The algebra $\mathbb{C}[x_1, \ldots, x_r] \otimes \mathbb{C}[M]$ is $K$-graded by $\deg(x_i) := D_i$ and $\deg(\chi^m) := \text{div}(\chi^{-m})$. The homomorphism of $K$-graded algebras

$$\mathbb{C}[x_1, \ldots, x_r] \otimes \mathbb{C}[M] \to \Gamma(X, \mathcal{S})$$

defined by $x_i \otimes 1 \mapsto 1 \in \Gamma(X, \mathcal{S})_{D_i}$ and $1 \otimes \chi^m \mapsto \chi^m \in \Gamma(X, \mathcal{S})_{\text{div}(\chi^{-m})}$, is an isomorphism. The surjectivity follows from the fact that for any $T$-invariant divisor $D$ the vector space $\Gamma(X, \mathcal{S})_D$ is generated by monomials of $\mathbb{C}[M]$. To prove the injectivity, first of all one shows that $\text{div}(\chi^m) = \sum \langle m, v_i \rangle D_i$, where each $v_i$ is the primitive generator of the one-dimensional cone of the fan corresponding to $D_i$. Thus, if $D = \sum a_i D_i$, the degree $D$ part of each of the above algebras is isomorphic to the vector space

$$\bigoplus_{m \in \Delta_D \cap M} \mathbb{C} \cdot \chi^m,$$

where $\Delta_D := \{ m \in M_\mathbb{Q} : \langle m, v_i \rangle + a_i \geq 0 \}$. By the definition of Cox ring, $R(X)$ is isomorphic to the quotient $\Gamma(X, \mathcal{S})/\text{I}(X, T)$, where $\mathcal{I}$ is the ideal sheaf locally generated by $1 - \chi^m$ for all $m \in M$. As a consequence we have the following.

Theorem 3.4. The Cox ring of a toric variety $X$ with torus invariant prime divisors $D_1, \ldots, D_r$ is the $\Gamma(X, \mathcal{S})$-graded polynomial ring $\mathbb{C}[x_1, \ldots, x_r]$, where $\deg(x_i) = [D_i] \in \text{Cl}(X)$ for each $i$. 

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Next, we describe the correspondence between full-dimensional lattice polytopes and pairs consisting of a projective toric variety together with an ample divisor. A lattice polytope $\Delta \subseteq \mathbb{Q}$ is a polytope with vertices in $\mathbb{M}$. Each maximal proper face $F$ of $\Delta$ corresponds to the generator $v_F \in \mathbb{N}$ of the monoid of points of $\mathbb{N}$ which lie on the inward normal to $F$. Now, each face $Q$ of $\Delta$ defines a cone $\sigma_Q$ of $\mathbb{N}$ generated by all the $v_F$ as $F$ runs over the maximal proper faces of $\Delta$ which contain $Q$. The set of such cones is the normal fan $\Sigma_\Delta$ to $\Delta$ and the corresponding toric variety is denoted by $X_\Delta$. The ample divisor is $H := -\sum F \min_{u \in \Delta} \langle u, v_F \rangle D_F$, where $F$ runs over all the maximal faces of $\Delta$ and $D_F$ denotes the prime divisor of $X_\Delta$ defined by the one-dimensional cone of the fan generated by $v_F$. Every pair consisting of a projective toric variety $X$ together with an ample $T$-invariant divisor $H$ of $X$ arises this way, where the polytope is $\Delta_H$ as defined above.

**Example 3.5.** Given at least two positive integers $a_0, \ldots, a_n$, such that any $n$ of them have greatest common divisor one, the weighted projective space $\mathbb{P}(a_0, \ldots, a_n)$ is the toric variety obtained as the good quotient of $\mathbb{C}^{n+1} \setminus \{0\}$ by the $\mathbb{C}^*$-action $t \cdot (x_0, \ldots, x_n) := (t^{a_0}x_0, \ldots, t^{a_n}x_n)$. Alternatively, if $l$ is the least common multiple of the $n+1$ numbers, then $\mathbb{P}(a_0, \ldots, a_n)$ is the toric variety defined by the lattice polytope $\Delta \subseteq \mathbb{Q}^{n+1}$ with vertices $(l/a_0, 0, \ldots, 0), (0, l/a_1, \ldots, 0), \ldots, (0, \ldots, 0, l/a_n)$ contained in the hyperplane with equation $a_0x_0 + \cdots + a_nx_n = l$. The divisor class group of the weighted projective space is free of rank one and the Cox ring is the polynomial ring $\mathbb{K}[x_0, \ldots, x_n]$, graded by $\deg(x_i) := a_i$.

**Cox rings of $T$-varieties**

We conclude the section by discussing the Cox ring of a $T$-variety.

**Definition 3.6.** A $T$-variety is a normal variety that admits an effective action of a torus $T$. The difference $\dim(X) - \dim(T)$ is the complexity of $X$, so that the $T$-varieties of complexity zero are the toric varieties.

As before we assume that $X$ has only constant invertible global functions and finitely generated divisor class group. For any point $x \in X$, let $T_x \subseteq T$ denote its stabilizer. The complement of the open subset $X_0 := \{x \in X : T_x \text{ is finite}\}$ is union of a finite number of prime divisors $E_1, \ldots, E_m$. According to a result of Sumihiro [Sum74, Cor. 3], there is a geometric quotient $q : X_0 \to X_0/T$ with an irreducible normal but possibly non-separated orbit space $X_0/T$. As shown in [HS10, Proof of Thm. 1.2] the prevariety $X_0/T$ admits a separation, that is, a surjective rational map $\pi : X_0/T \dasharrow Y$ onto a normal variety $Y$, with only constant invertible global functions and a finitely generated divisor class group, which is a local isomorphism over a big open subset of $Y$. The same reference shows that there are prime divisors $C_0, \ldots, C_r$ of $Y$ such that $\pi$ is an isomorphism outside their union, that each $\pi^{-1}(C_i)$ is a disjoint union of prime divisors $C_{ij}$ and that all divisors with non-trivial finite generic isotropy occur among the closures $D_{ij}$ of the preimages of the $C_{ij}$ via $q$. For each $i$ one defines a monomial $x_i^{l_i} := \prod_j x_{ij}^{l_{ij}}$ where $l_{ij}$ is the order of the isotropy group of a general point of $D_{ij}$. If one denotes by $1_{C_i}$ a defining section for $C_i$ in Cox coordinates on $Y$, then, by [HS10, Thm. 1.2], the Cox ring of $X$ is isomorphic to

$$\mathcal{R}(Y)[x_{ij}, y_k]/(x_i^{l_i} - 1_{C_i} : 0 \leq i \leq r),$$

where each $x_{ij}$ is a variable corresponding to a defining section of $D_{ij}$ and each $y_k$ is a variable corresponding to a defining section of $E_k$, see [ADHL15, Thm. 4.4.1.3]. When the complexity of $X$ is zero, we again see that the Cox ring is a polynomial ring. Indeed, in this case only the variables $y_1, \ldots, y_m$ survive because there are no points with non-trivial finite isotropy. If $X$ has complexity one then $Y$ is the projective line $\mathbb{P}^1$ because of the finiteness of its divisor class group, and we see that in this case the Cox ring of $X$ is finitely generated. More generally, the Cox ring of $X$ is finitely generated if and only if the Cox ring of $Y$ is finitely generated.

**4. Finite generation**

We now discuss the finite generation of Cox rings from a geometric perspective. Given an $A$-graded
algebra $R$ and a submonoid $H \subseteq A$, one can form the corresponding Veronese subalgebra

$$R_H := \bigoplus_{a \in H} R_a$$

which inherits finite generation from $R$, if $H$ is finitely generated. To see this, fix a graded surjection $\pi: \mathbb{C}[x_1, \ldots, x_r] \to R_H$, where $x_i$ is homogeneous of degree $a_i := \deg(\pi(x_i))$ and let $Q: \mathbb{Z}^r \to A$ be the homomorphism defined by $e_i \mapsto a_i$. The monoid $M := Q^{-1}(H) \cap \mathbb{Z}_{\geq 0}^r$ is finitely generated, being the intersection of two finitely generated submonoids of $\mathbb{Z}^r$ [ADHL15, Prop. 1.1.2.2]. Thus the monoid algebra $\mathbb{C}[M]$ is finitely generated as well and $\pi$ maps it surjectively onto $R_H$. More generally, given a homomorphism $\phi: \mathbb{Z}^r \to A$, the $\mathbb{Z}^r$-graded algebra

$$S := \bigoplus_{u \in \mathbb{Z}^r} R_{\phi(u)}$$

is finitely generated. Indeed, the Veronese subalgebra $R_H$, where $H$ is the image of $\phi$, is finitely generated. Let $\pi: \mathbb{C}[u_1, \ldots, u_k] \to R_H$ be a surjection of graded algebras, where each $u_i$ is homogeneous of degree $b_i := \deg(\pi(u_i))$. Let $P: \mathbb{Z}^n \to \mathbb{Z}^r$ be the kernel of $\phi$ and let $\eta: H \to \mathbb{Z}^r$ be a map of sets such that $\phi \circ \eta = \text{id}$. The Laurent polynomial ring $\mathbb{C}[u_1, \ldots, u_k] \otimes_{\mathbb{C}} \mathbb{C}[\mathbb{Z}^n]$ is graded by $\mathbb{Z}^r$ by giving degree $\eta(b_i)$ to the variable $u_i$ and degree $P(m)$ to $x_m$. The homomorphism $\mathbb{C}[u_1, \ldots, u_k] \otimes_{\mathbb{C}} \mathbb{C}[\mathbb{Z}^n] \to S$, defined by $u_i \otimes 1 \mapsto \pi(u_i) \in S_{\eta(b_i)}$ and $1 \otimes x_m \mapsto 1 \in S_{P(m)}$, is a surjection of $\mathbb{Z}^r$-graded algebras. Observe that if one replaces $\phi$ with a homomorphism of monoids then the previous arguments still prove finite generation of the corresponding graded algebras. An immediate consequence of these observations is that the finite generation of the Cox ring is equivalent to that of the algebra

$$\bigoplus_{(m_1, \ldots, m_n) \in \mathbb{Z}^n} \Gamma(X, O_X(m_1D_1 + \cdots + m_nD_n))$$

whenever the classes of $D_1, \ldots, D_n$ generate $\text{Cl}(X)$. In fact, the above equivalence still holds when the classes of $D_1, \ldots, D_n$ generate a subgroup of finite index of $\text{Cl}(X)$, see [ADHL15, Prop. 1.1.2.5]. These algebras are frequently used when studying the finite generation of Cox rings because one can choose the $D_i$ according to the geometry of the variety. Some authors also call such an algebra a Cox ring of the variety. Here we follow the convention that the Cox ring of a variety is the algebra graded over the whole divisor class group as defined in Section 1. Our terminology agrees with the original presentation of the Cox ring of a toric variety as a polynomial ring in [Cox95, Section 1].

The section ring $R(X, D_1, \ldots, D_r)$ of a collection of Weil divisors $D_1, \ldots, D_r$ on a variety $X$ is defined as

$$\bigoplus_{(m_1, \ldots, m_r) \in (\mathbb{Z}_{\geq 0})^r} \Gamma(X, O_X(m_1D_1 + \cdots + m_rD_r)).$$

Example 4.1. We now give a combinatorial argument for the finite generation of the section ring $R(X, F_1, \ldots, F_r)$ for any Weil divisors $F_1, \ldots, F_r$ on a $\mathbb{Q}$-factorial toric variety. Let us first consider the case $r = 1$ and let $D := F_1$. Let us see that its section ring $R(X, D)$ is finitely generated. We may assume that $D$ is $T$-invariant and write $D = \sum a_iD_i$ where $D_1, \ldots, D_r$ are the torus invariant prime divisors and $a_1, \ldots, a_r$ are integers. Since $\Delta_{mD} = m\Delta_D$ for any positive integer $m$, the desired finite generation is equivalent to the finite generation of the monoid of integral points in $M_\mathbb{Q} \subseteq \mathbb{Q}$ which are inside the rational polyhedral cone generated by $\Delta_D \times \{1\}$ and $\Delta_0 \times \{0\}$. The monoid of integral points in a rational polyhedral cone is always finitely generated by Gordon’s Lemma, then the finite generation of $R(X, D)$ follows. The figure below helps visualize the idea behind this finite generation.

Now, for the general case, we may assume that each $F_i$ is a Cartier divisor and that it is a torus invariant divisor, as before. In this case, the projectivization $\mathbb{P}(\mathcal{E})$ of the vector bundle $\mathcal{E} = O_X(F_1) \oplus \cdots \oplus O_X(F_r)$ is again a toric variety and the section ring $R(X, F_1, \ldots, F_r)$ can be identified with the section ring $R(\mathbb{P}(\mathcal{E}), H_\mathcal{E})$ where $-H_\mathcal{E}$ is a tautological divisor on $\mathbb{P}(\mathcal{E})$, and the restriction of $H_\mathcal{E}$ to each fiber of $\mathbb{P}(\mathcal{E})$ is linearly equivalent to a hyperplane. The finite generation of the section ring $R(X, F_1, \ldots, F_r)$ follows from the case $r = 1$ considered before.
ample Cartier divisor on a normal variety $X$, section linearly equivalent to the pullback of a hyperplane $m$-
the ring of semiample Cartier divisors). Zariski’s finite generation of section rings of semiample Cartier divisors), If $D$ is a semi-
ample Cartier divisor on a normal variety $X$, then $\psi_{mD} : X \to \mathbb{P}^N$ is a morphism for some positive integer $m$. Let $Y$ be the image of $\psi_{mD}$. Then $mD$ is linearly equivalent to the pullback of a hyperplane section $H \subseteq Y$ and moreover $(\psi_{mD})_*\mathcal{O}_X = \mathcal{O}_Y$. It follows that the algebra $R(X, mD)$ is isomorphic to $R(Y, H)$, which is finitely generated since $H$ is very ample on $Y$. The latter property depends on the fact that, given a hyperplane $H_0 \subseteq \mathbb{P}^n$ with $H_0 \cap Y = H$, the algebra $R(\mathbb{P}^N, mH_0)$ is finitely generated and the sections $H_0^0(\mathbb{P}^N, mH_0)$ surject onto $H^0(Y, mh)$ for $m$ sufficiently large. Then $R(X, D)$ is finitely generated as well since its Veronese subalgebra $R(X, mD)$ corresponding to a subgroup with finite index is finitely generated. Zariski proves more generally that for any finite collection of semiample Cartier divisors $D_1, \ldots, D_r$ on a normal projective variety $X$ their section ring $R(X, D_1, \ldots, D_r)$ is finitely generated. We can see why this holds from what we have already proved. Passing to a finite index Veronese subalgebra, we can assume that each of $D_1, \ldots, D_r$ is base point free. Let us denote by $\mathcal{E} := \mathcal{O}_X(D_1) \oplus \cdots \oplus \mathcal{O}_X(D_r)$ the direct sum vector bundle and notice that $H_\mathcal{E}$ is base point free, where $-H_\mathcal{E}$ is a tautological divisor. In particular $H_\mathcal{E}$ is semiample, and hence $R(\mathbb{P}(\mathcal{E}), H_\mathcal{E})$ is finitely generated. The claim now follows by noticing that the latter section ring is isomorphic to the section ring $R(X, D_1, \ldots, D_r)$.

**Example 4.3.** Fano varieties and more generally log-
Fano varieties have finitely generated Cox rings, see [BCHM10].

**Behavior of finite generation under morphisms**

How does the finite generation of Cox rings behave under maps? This is a wide open question and here we just discuss it in some special cases. In what follows we assume $X$ to be a normal variety that has only constant invertible global functions and a finitely generated divisor class group. The algebras of sections over open subsets of $X$ of the sheaves of algebras considered before are finitely generated, if the Cox ring is so. More precisely in [B¨ak11, Thm. 1.2] it is proved that if the Cox ring of $X$ is finitely generated, then for any finitely generated subgroup $K$ of Weil divisors and any open subset $U \subseteq X$, the algebra of sections $R(U, K)$ is also finitely generated (see also [ADHL15, Exer. 1.18] for a description of the Cox ring of an open subset in the case when the global invertible regular functions on $U$ are constants). As a consequence, if $f : X \to Y$ is a birational contraction then the Cox ring of $Y$ is finitely generated whenever that of $X$ is, being the former isomorphic to the Cox ring of the complement of the exceptional divisor. More generally, even if $R(X)$ is not finitely generated, if $E$ is the exceptional divisor (for simplicity assume it to be irreducible), then the pushforward induces a surjection $R(X) \to R(Y)$ with kernel generated by $x_E - 1$, where $x_E$ is a defining section for $E$ in Cox coordinates. A more general result about finite generation is the following.

**Theorem 4.4** ([Oka16]). If $f : X \to Y$ is a surjective morphism of normal projective $\mathbb{Q}$-factorial varieties and $X$ has finitely generated Cox ring, then the Cox ring of $Y$ is finitely generated.

When $f$ has connected fibers the statement is a consequence of the above observations on Veronese subalgebras. Indeed, in this case $f_*\mathcal{O}_X = \mathcal{O}_Y$, and hence given a Cartier divisor $D$ on $Y$, by the projection formula there is an isomorphism
\( \Gamma(X,\mathcal{O}_X(f^*D)) \simeq \Gamma(Y,\mathcal{O}_Y(D)). \) It follows that if \( K \) is a subgroup of Cartier divisors of \( Y \) whose image in \( \text{Cl}(Y) \) has finite index, there is an isomorphism between \( R(Y,K) \) and a Veronese subalgebra of the Cox ring of \( X \). Thus \( R(Y,K) \) is finitely generated and so the Cox ring of \( Y \) is also finitely generated as well.

The finite generation of the Cox ring is preserved by passing to a good quotient as shown in [Bäk11, Thm. 1.1]. More precisely, if \( X \) is a normal variety with finitely generated Cox ring, \( G \) is a reductive affine algebraic group acting on \( X \) and \( U \subseteq X \) is an open invariant subset admitting a good quotient \( f: U \to U/G \), then the Cox ring of \( U/G \) is finitely generated if this space has only constant invertible global functions.

5 Mori chambers

Let us go back to the question: “when do two varieties have isomorphic Cox rings?” By Remark 1.9 this is the case when the two varieties are isomorphic in codimension one. Here we show that for normal projective \( \mathbb{Q} \)-factorial varieties with finitely generated Cox ring. Given an effective Weil divisor \( D \), the \( \mathbb{Z}_{\geq 0} \)-graded algebra \( R(X,D) \) is finitely generated so that it defines an affine variety equipped with a \( \mathbb{C}^* \)-action. Fix a minimal set of homogeneous generators \( g_1,\ldots,g_s \) of degrees \( d_1,\ldots,d_s \) and define the rational map

\[ \varphi_D: X \dashrightarrow \mathbb{P}(d_1,\ldots,d_s) \quad p \mapsto [g_1(p):\cdots:g_s(p)], \]

where the codomain is a weighted projective space and denote the closure of the image by \( X(D) \).

**Definition 5.3.** Two effective divisors \( D_1, D_2 \) of \( X \) are **Mori equivalent** if the following holds:

1. \( B(D_1) = B(D_2); \)
2. there is an isomorphism \( \phi: X(D_1) \to X(D_2) \) such that the following diagram commutes:

\[
\begin{array}{ccc}
X(D_1) & \xrightarrow{\varphi_{D_1}} & X \\
\downarrow \phi & & \downarrow \varphi_{D_2} \\
X(D_2) & \xrightarrow{\varphi_{D_2}} & X
\end{array}
\]

Mori equivalence descends to an equivalence relation on the effective cone. Each equivalence class is the relative interior of a rational polyhedral cone called a **Mori chamber** of \( X \). The **nef cone** \( \text{Nef}(X) \), closure of the cone of ample classes, is one of these
chambers. In particular the nef cone is generated by finitely many classes and one can show that each nef class is semiample. To give an idea of this fact assume for simplicity that ${\rm Cl}(X)$ has rank 2. Let $w_1, \ldots, w_r$ be the degrees of a minimal set of generators $f_1, \ldots, f_r$ of the Cox ring. Notice that $w_1, \ldots, w_r$ span the effective cone $\text{Eff}(X) = \text{Eff}(X)$, because every homogeneous section $f \in R(X)_{w}$ is a linear combination of monomials $\prod_i f_i^{a_i}$ and thus $w = \sum_i a_i w_i$. Moreover, by the combinatorial description of Mori chambers (given below in this section) the nef cone of $X$ is generated by two of these classes, say $w_j$ and $w_k$, as in Figure 3. Within the nef cone there can be other $w_i$ represented by black dots. We choose the darker region as in the figure, such that in its interior there are no degrees $w_i$ of the generators of the Cox ring. Each class $w$ in this region is in the interior of the nef cone and hence semiample by Kleiman’s criterion. Then, the stable base locus of $w$ is empty. Observe that $w$ is in the cone generated by $\{w_i : i \in I\}$ if and only if there is an integer $n > 0$ and a monomial of degree $nw$ of the form $\prod_{i \in J} f_i^{a_i}$. Since the stable base locus $B(w)$ is the intersection of the zero loci of these monomials we deduce that $B(w_j) \subseteq B(w)$ because each cone in the $w_i$ which contains $w$ must contain also $w_j$. Thus $B(w_j) = \emptyset$ or equivalently $w_j$ is semiample.

**Definition 5.4.** A normal projective $\mathbb{Q}$-factorial variety $X$ with a finitely generated divisor class group is a Mori dream space if there are normal projective $\mathbb{Q}$-factorial varieties $X_1, \ldots, X_n$ and isomorphisms in codimension one $\phi_i : X \rightarrow X_i$ such that each nef cone $\text{Nef}(X_i)$ is generated by finitely many semiample classes and

$$\text{Mov}(X) = \bigcup_{i=1}^{n} \phi_i^* \text{Nef}(X_i).$$

The following theorem, one of the main results of [HK00], shows that the definition above fits perfectly into the context of this note.

**Theorem 5.5.** Let $X$ be a normal projective $\mathbb{Q}$-factorial variety with finitely generated divisor class group. Then $X$ is a Mori dream space if and only if the Cox ring of $X$ is finitely generated.

As a consequence of the theorem, it makes sense to talk about Mori chambers of a Mori dream space. In this case one shows that the cones in the decomposition of $\text{Mov}(X)$ given in Definition 5.4 are all full-dimensional Mori chambers. A combinatorial description of these chambers in terms of a presentation of the Cox ring can be given as explained below. Given a subset $I \subseteq \{1, \ldots, r\}$, denote by $C_I$ the cone of $\text{Cl}_Q(X)$ generated by $\{w_i : i \in I\}$. The cone $C_I$ corresponds to the reduced monomial $\prod_{i \in I} f_i$ in the generators of the Cox ring. It is not difficult to show that the moving cone of $X$ is [ADHL15, Prop. 3.3.2.9]

$$\text{Mov}(X) = \bigcap_{i=1}^{r} C_{\{1, \ldots, r\} \setminus \{i\}}.$$

Indeed, for any class $w$ in the right-hand side and any $i \in \{1, \ldots, r\}$ there is an $n > 0$ and a monomial in $R(X)_{nw}$ which does not contain $f_i$. It is also possible to give a combinatorial description of Mori chambers: the Mori chamber which contains the class $w$ is

$$\lambda(w) = \bigcap_{w \in C_I} C_I.$$

Finally observe that Mori equivalence refines the equivalence relation $D_1 \sim D_2$ if $B(D_1) = B(D_2)$. The equivalence classes for this relation are the relative interiors of unions of some Mori chambers. The two equivalence relations can be distinct for a given Mori dream space, as shown by the following picture which displays the Mori chamber decomposition of the effective cone of a toric threefold. The degrees of generators of the Cox ring are the six black dots.
There are 17 full-dimensional Mori chambers, five of which are inside the moving cone. The nef cone is the black chamber, while the two gray chambers have the same stable base locus [LMR20].

Remark 5.6. We end-up with a remark about GIT chambers for the action of the quasitorus $H^X$ on the affine variety $X$. Given an effective class $w \in \text{Cl}_Q(X)$ one defines the open subset $X_{ss}(w) \subseteq X$ consisting of the $\bar{x} \in X$ such that $f(\bar{x}) \neq 0$ for some $n > 0$ and $f \in \mathcal{R}(X)_{nw}$. This is the subset of semistable points defined by $w$, which is an open $H^X$-invariant subset of $X$. The set of effective classes that give rise to the same subset of semistable points is the relative interior of a polyhedral cone. These cones are called GIT chambers and one of the main results of [HK00] states that GIT chambers are exactly Mori chambers.

6 Examples

We now illustrate some approaches that have been used in the literature to determine the finite or non-finite generation of the Cox rings of notable varieties. While the complete proofs frequently involve arguments specific to the geometry of the variety under consideration, we mention a few ideas with great applicability. We include references for the use of such ideas in concrete examples.

Given a normal projective $\mathbb{Q}$-factorial variety with a finitely generated Cox ring (i.e., a Mori dream space), the cones of nef, pseudoeffective and movable divisors are rational polyhedral. Hence, one can disprove the finite generation of the Cox ring by studying these cones. For example, the pseudoeffective cone of the moduli space $\overline{M}_{1,n}$ of stable genus-one curves with $n$ ordered marked points is not polyhedral for $n \geq 3$. Hence, the moduli space $\overline{M}_{1,n}$ does not have a finitely generated Cox ring for $n \geq 3$ [CC14].

Example 6.1. The finite or non-finite generation of the Cox ring has been decided for the blow up of $\mathbb{P}^{r-1}$ at $n > r \geq 2$ points in very general position: finite generation occurs if and only if (see [Muk04] and [CT06, Thm. 1.3])

$$\frac{1}{r} + \frac{1}{n-r} > \frac{1}{2}.$$ 

If instead the set $S$ of points lies on a hyperplane $H \subseteq \mathbb{P}^n$, with $n > 2$, then, the Cox ring of $\text{Bl}_S \mathbb{P}^n$ is isomorphic to a polynomial ring in one variable over the Cox ring of $\text{Bl}_S H$, see [GHPS12, Lem. 4.3].

Since nef divisors on a Mori dream space are semiample, one may establish that a Cox ring is non-finitely generated by exhibiting a nef divisor on the variety that is not semiample. This approach has been helpful when studying the Cox rings of blowups at a general point of weighted projective planes. These Cox rings are also studied in commutative algebra since their finite generation is equivalent to the Noetherianity of the extended saturated Rees algebra of a monomial prime ideal. For a concrete example using this approach see Example 6.2, which adapts a general argument from [GK16].

Example 6.2 (Bl $\mathbb{P}(12, 13, 17)$ is not a Mori dream space [GK16]). The triangle $\Delta$ with integral vertices $(11, -26), (50, 0), (-1, 34)$ defines a toric pair consisting of the weighted projective plane $\mathbb{P}_\Delta = \mathbb{P}(12, 13, 17)$ together with the ample class $H$ of self-intersection $H^2 = 52 \cdot 51$ equal to twice the area of $\Delta$. Let $\pi: X \to \mathbb{P}_\Delta$ be the blow up at the unit element of the torus $e \in \mathbb{P}_\Delta$ with exceptional divisor $E$. The binomial $1 - y$ defines a curve in $\mathbb{P}_\Delta$ which passes simply through $e$ so that its strict transform $C \subseteq X$ is an irreducible curve with $C \cdot E = 1$. Notice that the triangle $\Delta$ supports the Laurent polynomial $f(x,y) := x^{11}y^{-26}(1 - y)^{52}$, which has a nonzero coefficient over each edge of $\Delta$, and which vanishes to order 52 at $e$. Therefore the class of $C$ is
\[ \frac{1}{52} \pi^* H - E. \] Using the above intersection numbers, the self-intersection of \( C \) on \( X \) can be computed as

\[ C^2 = \left( \frac{1}{52^2} (\pi^* H)^2 + E^2 \right) = \frac{52 \cdot 51}{52^2} - 1 < 0. \]

Therefore, \( C \) is an irreducible negative curve on \( X \). Let us show that \( X \) is not a Mori dream space. Since \( \text{Cl}(X) \) has rank two, the pseudoeffective cone is generated by the classes of the two negative curves \( C \) and \( E \). Taking dual with respect to the intersection form, we see that the nef cone of \( X \) is generated by \( \pi^* H \) together with any class \( D \) satisfying \( D \cdot E > 0 \) and \( D \cdot C = 0 \). We can then take \( D := \pi^* H - 51E \). Notice that \( \Delta \) contains \((49, 0)\) and \((50, 0)\), but the remaining lattice points in \( \Delta \) have an \( x \)-coordinate with value at most 48. Then, the partial derivative \( \partial_x^a \partial_y^b |_{(x=1,y=1)} \) vanishes when evaluated on the monomial \( x^a y^b \) for each lattice point in \( \Delta \), except its left vertex \((-1, 34)\).

It follows that any Laurent polynomial \( g(x, y) \) supported on \( \Delta \) that vanishes to order 51 at \( e \) has the coefficient corresponding to the left vertex of \( \Delta \) equal to zero. Indeed, \( \partial_x^a \partial_y^b g(x, y) |_{(x=1,y=1)} \) is equal to a nonzero multiple of the coefficient of \( x^{-1} y^{34} \) in \( g(x, y) \), but on the other hand it is equal to zero since \( g(x, y) \) has order 51 at \( e \). Then every effective divisor linearly equivalent to \( D \) passes through the point in \( X \) that is mapped by \( \pi \) to the torus invariant point of \( X_\Delta \) corresponding to the left vertex \((-1, 34)\) of \( \Delta \). In particular, \( D \) is not base point free. Given any positive integer \( m \), we can translate the triangle \( m \Delta \) such that the left vertex has the form \((-1, a)\) and the right vertex is \((51m - 1, 0)\), and we can repeat the same argument using now the partial derivative \( \partial_x^{51m - 2a} \partial_y |_{(x=1,y=1)} \) to conclude that \( mD \) is not base point free. Then, \( D \) is a nef divisor that is not semiample on \( X \), and therefore \( X = \text{Bl}_e \mathbb{P}(12, 13, 17) \) is not a Mori dream space.

One of the most fruitful methods to decide the finite or non-finite generation of the Cox ring of one variety is to appropriately change the variety under consideration to another more suitable for the finite generation question. This is frequently done using the behavior of finite generation under isomorphisms in codimension one (Rmk. 1.9 and Rmk. 5.1) and surjective morphisms (Thm. 4.4 and Rmk. 5.2), as follows. Suppose that there is a chain of rational maps

\[ X_1 \rightarrow X_2 \rightarrow \cdots \rightarrow X_{n-1} \rightarrow X_n \] (2)

where each \( X_i \) is a normal projective \( \mathbb{Q} \)-factorial variety and each map is either a surjective morphism or an isomorphism in codimension one. If \( X_1 \) has a finitely generated Cox ring then the same holds for \( X_n \). To illustrate these ideas we now outline the construction of one such chain of rational maps, starting from the moduli space \( \overline{M}_{0,n} \) for any \( n \geq 10 \) and ending on the blown-up toric surface \( \text{Bl}_e \mathbb{P}(12, 13, 17) \) discussed in Example 6.2. This construction shows that the Cox ring of \( \overline{M}_{0,n} \) is not finitely generated for \( n \geq 10 \).

**The moduli space \( \overline{M}_{0,n} \)**

The moduli space \( \overline{M}_{0,n} \) parameterizes configurations of \( n \) distinct ordered points in \( \mathbb{P}^1 \) up to automorphisms. A family of such configurations may have as a limit a configuration where some of the points collide, and therefore \( \overline{M}_{0,n} \) is not compact. The group \( \text{Aut}(\mathbb{P}^1) = \text{PGL}(2) \) acts on triples of distinct ordered points in \( \mathbb{P}^1 \), transitively and with trivial stabilizers. Hence, for \( n \geq 4 \) we have the identification

\[ \overline{M}_{0,n} = (\mathbb{P}^1 \setminus \{0, 1, \infty\})^{n-3} \cup \bigcup_{i \neq j} \Delta_{i,j}, \]

where \( \Delta_{i,j} \) denotes the locus where the \( i \)-th and \( j \)-th components coincide. There is a well-known compactification of \( \overline{M}_{0,n} \) denoted by \( \overline{M}_{0,n} \) and called the Grothendieck-Knudsen compactification of \( M_{0,n} \). This compactification parameterizes proper connected curves with at worst nodal singularities, which are a union of irreducible components isomorphic to \( \mathbb{P}^1 \) with dual graph forming a tree, with \( n \) ordered marked points in the smooth locus, and such that each component contains at least 3 points that are either a node or a marked point. Kapranov showed that \( \overline{M}_{0,n} \) is isomorphic to the iterated blow up in a suitable order of \( \mathbb{P}^{n-3} \) along the strict transforms of all linear subspaces spanned by subsets of \( n-1 \) points in linearly general position.

If we denote the \( n-1 \) points in this construction by \( p_1, \ldots, p_{n-1} \in \mathbb{P}^{n-3} \) and we first perform the iterated
blow up of \( \mathbb{P}^{n-3} \) along the strict transforms of all linear subspaces spanned by \( p_1, \ldots, p_{n-2} \) in order of increasing dimension, we obtain a toric variety called the Losev-Manin moduli space which we denote by \( LM_n \). We can assume that \( p_{n-1} \) is the unit element \( e \) of the torus of \( LM_n \). If we now blow up \( LM_n \) along the strict transforms of the remaining linear subspaces, all of them containing \( p_{n-1} \), the resulting variety is \( \overline{M}_{0,n} \). In particular, there exist surjective morphisms

\[
\overline{M}_{0,n} \to \text{Bl}_e LM_n \to LM_n.
\]

From this construction we also see that \( \overline{M}_{0,n} \) is a nonsingular projective variety. For \( n \leq 6 \) the variety \( M_{0,n} \) is log-Fano and hence it is a Mori dream space. Hu and Keel asked in [HK00] whether \( \overline{M}_{0,n} \) was a Mori dream space in general. A negative answer for \( n = 10 \) in [HKL18]. Let us describe the main points of the proof that \( \overline{M}_{0,n} \) is not a Mori dream space for \( n \geq 10 \). For each \( n \) there exists a surjective morphism

\[
\overline{M}_{0,n+1} \to \overline{M}_{0,n}
\]

called the forgetful morphism which intuitively forgets one of the marked points (for example the last one). Hence, given positive integers \( n \geq m \geq 4 \) there exists a surjective morphism \( \overline{M}_{0,n} \to \text{Bl}_e LM_m \). By Okawa’s result on images of Mori dream spaces, it follows that if \( \text{Bl}_e LM_n \) is not a Mori dream space, then \( \overline{M}_{0,n} \) is not a Mori dream space for all \( n \geq m \).

**Example 6.3.** Suppose that we have a morphism of projective \( Q \)-factorial toric varieties \( \mathbb{P}_{\Sigma'} \to \mathbb{P}_{\Sigma} \) induced by a surjective map of lattices \( \pi: N' \to N \). Suppose also that \( \ker(\pi) \) is one-dimensional, generated by two rays of \( \Sigma' \), and that the rays of \( \Sigma \) are precisely the images of the rays in \( \Sigma' \). Then the rational map \( \text{Bl}_e \mathbb{P}_{\Sigma'} \to \text{Bl}_e \mathbb{P}_{\Sigma} \) factors as an isomorphism in codimension one followed by a surjection by [CT15, Prop. 3.1]. Then, the codomain \( \text{Bl}_e \mathbb{P}_{\Sigma} \) is a Mori dream space if the domain \( \text{Bl}_e \mathbb{P}_{\Sigma} \) is so.

The rays in the fan of \( LM_n \) are generated by the vectors \( x \) in \( N = \mathbb{Z}^{n-3} \) such that either \( v \) or \( -v \) has all entries in \( \{0, 1\} \). Let \( N' \subseteq N \) be a saturated sublattice of rank \( n-5 \), generated by a subset of \( \{0, 1\}^{n-3} \). Assume that there exist \( v_1, v_2, v_3 \in \{0, 1\}^{n-3} \) whose images under the quotient map \( \pi: N \to N/N' \) generate \( N/N' \cong \mathbb{Z}^2 \), and such that there are pairwise coprime positive integers \( a, b, c \) such that \( av_1 + bv_2 + cv_3 \in N' \). Applying Example 6.3 iteratively, it follows that if \( \text{Bl}_e LM_n \) is a Mori dream space, then \( \text{Bl}_e \mathbb{P}(a, b, c) \) is also a Mori dream space. We can apply this to the homomorphism \( \pi: N = \mathbb{Z}^7 \to \mathbb{Z}^2 \) given by the matrix

\[
\begin{bmatrix}
1 & 0 & 1 & -2 & -1 & 1 & 0 \\
0 & 1 & -1 & -3 & -2 & 1 & 1
\end{bmatrix},
\]

with \( N' = \ker(\pi) \) and with \( v_1 = e_1 + e_3 + e_4 + e_5 \), \( v_2 = -(e_4 + e_5) \) and \( v_3 = -(e_1 + e_3 + e_6) \), and \( a = 12, b = 13, c = 17 \). This shows that for each \( n \geq 10 \) there exists a chain of maps

\[
\overline{M}_{0,n} = X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_r = \text{Bl}_e \mathbb{P}(12, 13, 17)
\]

such that all varieties are normal, projective and \( Q \)-factorial, and such that each map is either a surjective morphism or an isomorphism in codimension one. In Example 6.2 we showed that \( \text{Bl}_e \mathbb{P}(12, 13, 17) \) is not a Mori dream space, and therefore \( \overline{M}_{0,n} \) is not a Mori dream space for each \( n \geq 10 \).

One can use the same projection \( \pi: \mathbb{Z}^7 \to \mathbb{Z}^2 \) to prove that \( \text{Eff}(\overline{M}_{0,n}) \) is not polyhedral for all \( n \geq 10 \), as done in [CLTU20, Thm. 1.3]. Indeed, one gets a chain of maps as in (2),

\[
\overline{M}_{0,n} = X_0 \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_r = \text{Bl}_e \mathbb{P}_\Delta,
\]

where the images of the primitive generators of the rays of \( LM_{10} \) via \( \pi \) are points (both black and white) in the following figure.

The white points in Figure 5 generate the rays of the normal fan of the polygon \( \Delta \) whose vertices are the columns of the matrix

\[
\begin{bmatrix}
-1 & -4 & -3 & -2 & -6 & -7 & 0 \\
6 & 5 & 1 & 8 & 0 & 0 & 3
\end{bmatrix}
\]

and \( \mathbb{P}_\Delta \) is the corresponding toric surface. The linear system of Laurent polynomials whose monomial exponents are in \( \Delta \) and have multiplicity 7 at \((1, 1)\)
contains a unique irreducible curve. The strict transform of the closure of this curve is a smooth elliptic curve $C$ of $X := \text{Bl}_e P_\Delta$. It has the minimal equation $y^2 + xy = x^3 - x^2 - 4x + 4$.

This is the curve labeled 446.a1 in the LMFDB database and it has Mordell-Weil group $\mathbb{Z}^2$. It is possible to show that $O_C(C)$ is a non-trivial element of the Mordell-Weil group and so it is not torsion because the Mordell-Weil group is torsion-free. As a consequence $h^0(X, nC) = 1$ for any $n > 0$ so that $[C]$ spans an extremal ray of $\text{Eff}(X)$. By the Riemann-Roch theorem $\text{Eff}(X)$ must contain the circular cone generated by the classes of divisors $D$ with $D^2 \geq 0$ and $D \cdot H \geq 0$, where $H$ is an ample class.

By a convexity argument one concludes that $\text{Eff}(X)$ cannot be polyhedral. By Remarks 5.1 and 5.2 one concludes that the effective cone of $\overline{M}_{0,10}$ is not polyhedral and thus, by the latter remark, the same holds for $\overline{M}_{0,n}$ for $n \geq 10$.

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