The real meaning of complex Minkowski-space world-lines

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Abstract
In connection with the study of shear-free null geodesics in Minkowski space, we investigate the real geometric effects in real Minkowski space that are induced by and associated with complex world-lines in complex Minkowski space. It was already known, in a formal manner, that complex analytic curves in complex Minkowski space induce shear-free null geodesic congruences. Here we look at the direct geometric connections of the complex line and the real structures. Among other items, we show, in particular, how a complex world-line projects into the real Minkowski space in the form of a real shear-free null geodesic congruence.

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1. Introduction
It has been known for several years that regular shear-free null geodesic congruences (NGCs) in Minkowski space are generated, in a formal manner, by four complex functions of a single complex variable, τ, often designated by \( \xi^a(\tau) = (\xi^0(\tau), \xi^1(\tau), \xi^2(\tau), \xi^3(\tau)) \) [1, 2]. It turns out that these four functions can be naturally interpreted as determining a complex world-line in complexified Minkowski space [3]. With complex Minkowski coordinates, \( (z^a) \), the world-line is described by \( z^a = \xi^a(\tau) \). It is the purpose of this paper to try to flesh out this interpretation in detail and in particular to see what direct real meaning can be assigned to these complex world-lines. (Though this discussion could be extended to asymptotically flat spacetimes and asymptotically shear-free NGC, we will restrict the discussion to shear-free null geodesic congruences in Minkowski spacetime \( M \) and leave the more general case for a later time.)

After the formal analytic discovery that regular shear-free NGCs could be generated by the four analytic functions \( \xi^a(\tau) \), the complex Minkowski space interpretation arose from the following observation: consider an arbitrary complex analytic world-line in complex Minkowski space, described by the four analytic functions \( \xi^a(\tau) \). The real geometric effects in real Minkowski space that are induced by and associated with this complex world-line are investigated in this paper.
Minkowski space, \( z^a = \hat{x}^a(\tau) \), and construct its future (surface forming) complex light-cone. Many of the rays (the null geodesics) of this complex cone 'pierce' the real Minkowski space. Of particular importance is the intersection region of these complex light-cones with complex null infinity, \( \mathcal{I}^+_C \). For each complex cone this intersection (a complex 'cut') is a complexified sphere. Restricting these cuts to real null infinity, namely to \( \mathcal{I}^r \), leads to real cuts, \( S^2 \). (Note that when \( \mathcal{I}^r \) is coordinatized by the standard Bondi coordinates \( (u, \zeta, \bar{\zeta}) \), these real cuts can be given by \( u = G_R(s, \zeta, \bar{\zeta}) \) with \( s \) parametrizing the family of cuts.) The (twisting) shear-free real null geodesic congruences are completely determined by the tangent directions \( (L, \bar{L}) \) of these real cuts. The determination of the real cuts forces a restriction on the range of the complex parameter \( \tau \). The allowed values of \( \tau \) are restricted to the values \( \tau = s + i\Lambda(s, \zeta, \bar{\zeta}) \), \( s \) and \( \Lambda(s, \zeta, \bar{\zeta}) \) real, with \( \Lambda(s, \zeta, \bar{\zeta}) \) determined by the complex world-line.

In section 2 we first outline our notation. This is then followed by a method for the description in Minkowski space of arbitrary NGCs and the specialization to shear-free NGCs. The relationships just mentioned, of the complex cuts on \( \mathcal{I}^+_C \) to the real shear-free NGCs, are revisited in detail. In this context we review the procedure for the restriction of these cuts to the real \( \mathcal{I}^r \) and the determination of both the cut functions \( G_R(s, \zeta, \bar{\zeta}) \) and \( \Lambda(s, \zeta, \bar{\zeta}) \). The section ends with a review of the so-called optical equations and parameters [4], which play an important role in the later discussions.

In section 3 we discuss some of the real geometric effects which emerge from the study of the shear-free NGCs once one has taken into account the reality constraint. We observe how the general complex world-line differs from a pure real world-line once the reality constraints on the cut function \( u = G(s, \zeta, \bar{\zeta}) \) have been used. In particular, we observe that a complex-world line generates distorted two-sphere cross-sections of \( \mathcal{I}^r \). This is in contrast to the undistorted (spherical) cross-sections generated by real world-lines. Further we note that the imaginary part of a complex world-line is a measure of the twist of the (real) NGC it describes. In addition, using both the tangent directions \( (L, \bar{L}) \) and an implicit relationship between the real parameter \( s \) and the retarded time, \( \sqrt{2u} = t - r \), we give an explicit Minkowski-space coordinate description of the entire associated shear-free NGC. The section concludes with some remarks on how the reality constraints on the complex cuts affect the CR structure of \( \mathcal{I}^r \) associated with the shear-free NGCs.

The main results of this work are contained in section 4 where we demonstrate how one can map or project the complex world-line (with the reality condition imposed on \( \tau \)) into the real Minkowski space. For each complex world-line this mapping directly leads to a complete regular shear-free NGC.

In section 5 the development of the caustics of shear-free NGCs (e.g. [5]) is studied. This involves finding the spacetime regions, in terms of the Minkowski coordinates, where the optical parameter \( \rho \) (i.e. the complex divergence of the shear-free NGC) 'blows up'. Generically the caustic, at any fixed time, appears as a topologically \( S^1 \) 'ring' moving through the Minkowski space in time, i.e. a real closed 'string'. The behavior of the caustic for certain non-generic cases is also investigated.

Section 6 contains a discussion of these results.

2. Foundational material

2.1. Shear-free null geodesic congruences in Minkowski space

We begin with a description of the notation that is used. The coordinates of real and complex Minkowski space are denoted by \( x^a \) and \( z^a \), respectively. The Bondi coordinates on real \( \mathcal{I}^r \) are \( (u, \zeta, \bar{\zeta}) \). \( u \) labels slices or cuts while the complex stereographic coordinates \( (\zeta, \bar{\zeta}) \), with
\[ \zeta = e^{\theta \cot \theta}, \] label the individual null (generators) geodesics of \( \mathcal{I}^+ \). For the complexified \( \mathcal{I}^+ \), i.e. \( \mathcal{I}^+_c \), \( u \) takes on complex values and \( \zeta \rightarrow \tilde{\zeta} \) is allowed complex values close to \( \zeta \). At any spacetime point one can introduce a null tetrad \( \{ l, n, m, \bar{m} \} \) with standard inner product relations

\[ l^a n_a = -m^a \bar{m}_a = 1, \tag{1} \]

all other inner products vanishing. An arbitrary null geodesic starting at any arbitrary point \( x_0^a \) in the \( l^a \) direction is given by

\[ x^a = x_0^a + l^a r, \]

with \( r \) the affine parameter. We often make use of the explicit representation for the tetrad \( \{ \hat{l}, \hat{n}, \hat{m}, \bar{\hat{m}} \} \) given by

\[ \hat{l}^a = \frac{\sqrt{2}}{2} \left( 1, \frac{\zeta + \bar{\zeta}}{1 + \zeta \bar{\zeta}}, \frac{-\zeta - \bar{\zeta}}{1 + \zeta \bar{\zeta}}, \frac{-1 + \zeta \bar{\zeta}}{1 + \zeta \bar{\zeta}} \right), \]

\[ \hat{n}^a = \frac{\sqrt{2}}{2} \left( 1, \frac{-\zeta + \bar{\zeta}}{1 + \zeta \bar{\zeta}}, \frac{\zeta - \bar{\zeta}}{1 + \zeta \bar{\zeta}}, \frac{1 - \zeta \bar{\zeta}}{1 + \zeta \bar{\zeta}} \right), \]

\[ \hat{m}^a = \frac{\sqrt{2}}{2} \left( 0, \frac{1 - \bar{\zeta}^2}{1 + \zeta \bar{\zeta}}, \frac{-1 + \bar{\zeta}^2}{1 + \zeta \bar{\zeta}}, \frac{2 \bar{\zeta}}{1 + \zeta \bar{\zeta}} \right), \tag{2} \]

that is allowed to ‘swing’ around the entire sphere, \( (\zeta, \bar{\zeta}) \in S^2 \), of null directions, at any spacetime point.

A Notational Comment Since both \( \bar{\zeta} \) and \( \tilde{\zeta} \) were in principle to be used extensively throughout this work (with \( \bar{\zeta} \) being used when we are concerned with the real Minkowski space and \( \tilde{\zeta} \) when we are dealing with its complexification), it however becomes awkward to frequently point this out. We have thus taken the notational liberty of using the symbol \( \bar{\zeta} \) throughout the work and understand that it is independent of \( \zeta \) but it takes on the value of the complex conjugate of \( \zeta \) only when we refer to real Minkowski space.

Using these coordinates, tetrad and parameters \( (u, \zeta, \bar{\zeta}) \) and appropriate choice of \( L \), any NGC, in Minkowski space, can be expressed as \[ x^a = u \hat{l}^a - L \hat{m}^a - L \bar{\hat{m}}^a + (r - r_0)^a, \tag{3} \]

where \( L = L(u, \zeta, \bar{\zeta}) \) is an arbitrary holomorphic function, \( r_0(u, \zeta, \bar{\zeta}) \) is the arbitrary origin of the affine parameter \( r \) and

\[ \sqrt{2} \hat{t}^a = \hat{n}^a + \hat{l}^a = (\sqrt{2}, 0, 0, 0) \tag{4} \]

is a time-like vector. The parameters \( (u, \zeta, \bar{\zeta}) \) which label each null geodesic of the congruence are simply the Bondi coordinates of the points of \( \mathcal{I}^+ \) intersected by the geodesic.

Equation (3) can be interpreted in two ways: either as an expression for the NGCs or as a coordinate transformation between the standard Minkowski coordinates \( x^a \) and the null geodesic coordinates \( (u, r, \zeta, \bar{\zeta}) \).

It is known [6] that the complex shear of such a congruence (3) is given by

\[ \sigma(u, \zeta, \bar{\zeta}) = \partial L + LL', \tag{5} \]

where \( \partial \) is the usual spin-weighted operator on the two-sphere and \( L \equiv \partial_u L \). In the next subsection we return to \( \sigma \) and its complimentary optical parameter, the complex divergence \( \rho \).

As we are interested in shear-free NGCs in \( M \), we look for those NGCs given by (3) where \( \sigma(u, \zeta, \bar{\zeta}) = 0 \), i.e. where \( L \) satisfies

\[ \partial L + LL = 0. \tag{6} \]
Solutions to the shear-free equation (6) are found by first introducing a complex potential function $\tau = T(u, \zeta, \bar{\zeta})$, which satisfies the PDE:

$$\partial T + LT = 0.$$  \hspace{1cm} (7)

Note that at this point we must consider the complexification of $\mathcal{J}^+$ by allowing $u$ to take on complex values and free $\bar{\zeta}$ from being the complex conjugate of $\zeta$. We also assume that $L$ is complex analytic in the three independent arguments $(u, \zeta, \bar{\zeta})$.

As explained later, equation (7) is actually the CR equation that assigns a CR structure to $\mathcal{J}^+$ that is associated with any shear-free NGC structure. For now it can just be viewed as a relation that the potential function $\tau = T(u, \zeta, \bar{\zeta})$ must satisfy if $L(u, \zeta, \bar{\zeta})$ is known.

Assuming that this potential function can be inverted to give a complex ‘cut function’ of $\mathcal{J}_C^+$

$$u = G(\tau, \zeta, \bar{\zeta}) \Leftrightarrow \tau = T(u, \zeta, \bar{\zeta}),$$  \hspace{1cm} (8)

replace $T(u, \zeta, \bar{\zeta})$ by $G(\tau, \zeta, \bar{\zeta})$ as an independent variable. Repeated implicit differentiations of $G$ transform the shear-free condition, equations (6) and the expression for $L$ (7) to the relations 

$$\partial^2_{(\tau)} G(\tau, \zeta, \bar{\zeta}) = 0,$$

$$L(u, \zeta, \bar{\zeta}) = \partial_{(\tau)} G.$$  \hspace{1cm} (9, 10)

Here, $\partial_{(\tau)}$ indicates the application of the $\partial$-operator while the variable $\tau$ is held constant. The $u$-dependence of $L$ is recovered in equation (10) by first applying $\partial_{(\tau)}$ to the solution $u = G(\tau, \zeta, \bar{\zeta})$, and then eliminating $\tau$, using $\tau = T(u, \zeta, \bar{\zeta})$.

Using the known properties of the tensorial spin-$s$ spherical harmonics and the $\partial$-operator \cite{8}, it follows that regular solutions to (9) are of the form

$$u = za\hat{l}_a = \sqrt{\frac{2}{2}} z^0 - \frac{1}{2} z^i Y_0^i,$$

where $z^a$ are four arbitrary complex constants. This, for each choice of the four constants, gives only one complex cross-section of $\mathcal{J}_C^+$. The entire family of cut functions may be obtained by letting $z^a$ become $\tau$ dependent, so that the four arbitrary parameters become an arbitrary complex world-line, $z^a = \xi^a(\tau)$:

$$u = G(\tau, \zeta, \bar{\zeta}) = \xi^a(\tau)\hat{l}_a(\zeta, \bar{\zeta})$$

$$= \frac{\sqrt{2}}{2} \xi^0(\tau) - \frac{\xi^i(\tau)}{2} Y_0^i(\zeta, \bar{\zeta}).$$  \hspace{1cm} (11)

It thus follows that the function $L$ is given parametrically by

$$u = \xi^a(\tau)\hat{l}_a(\zeta, \bar{\zeta})$$  \hspace{1cm} (12)

$$L(u, \zeta, \bar{\zeta}) = \xi^a(\tau)\hat{m}_a(\zeta, \bar{\zeta}) = \xi^i(\tau)Y_0^i(\zeta, \bar{\zeta}).$$  \hspace{1cm} (13)

By a regular solution, we mean that the function $L(u, \zeta, \bar{\zeta})$ should remain finite over the $(\zeta, \bar{\zeta})$ sphere. Geometrically this means that all rays of the NGC intersect $\mathcal{J}^+$ i.e. none lie on $\mathcal{J}^+$ itself.

Before this parametric description of $L(u, \zeta, \bar{\zeta})$ can be implemented, we must find the restriction on $\tau$ to yield real $u$. This is discussed later.

There are three different ways of interpreting the function $L$. From (3), we have that $L$ is simply a function which labels NGCs in $\mathcal{M}$; by choosing different holomorphic complex functions, we select different null geodesic congruences in the spacetime. One can also
interpret $L$ in equation (6) as the function (rotation angle) used to transform from a Bondi tetrad frame at $\mathcal{I}^+$ to a null tetrad frame where $l^a$ is tangent to a shear-free NGC; any $L$ which satisfies the PDE (6) will result in such a transformation. Finally, relation (10), the complex tangent directions to the cut, allows us to view $L$ as a complex stereographic angle field on $\mathcal{I}^+$ describing the past null cone at each point $(u, \xi, \bar{\xi})$, the direction at which the outgoing null geodesic intersects with the asymptotic boundary $\mathcal{I}^+$ [3]. All three of these views are equally viable, and we can apply each of them to $L$ as best suits our purposes in what follows.

Note that in the context of a discussion of $L(u, \xi, \bar{\xi})$ given parametrically by equations (12) and (13), we must point out, and be very aware, that there are three other functions that resemble $L(u, \xi, \bar{\xi})$ but should not be confused with it. From equations (12) and (13) we had

(a) $$u = \xi^a(\tau) \hat{l}_a(\xi, \bar{\xi})$$
$$L(u, \xi, \bar{\xi}) = \xi^a(\tau) \hat{m}_a(\xi, \bar{\xi}).$$ (14)

The similar ones are

(b) $$\pi = \pi^a(\tau) \hat{l}_a(\xi, \bar{\xi}),$$
$$\bar{L}(u, \xi, \bar{\xi}) = \pi^a(\tau) \hat{m}_a(\xi, \bar{\xi}).$$ (15)

(c) $$u = \xi^a(\tau) \hat{L}_a(\xi, \bar{\xi}),$$
$$L^*(u, \xi, \bar{\xi}) = \xi^a(\tau) \hat{m}_a(\xi, \bar{\xi}).$$ (16)

and

(d) $$\bar{\pi} = \bar{\pi}^a(\tau) \hat{l}_a(\xi, \bar{\xi}),$$
$$\bar{L}^*(u, \xi, \bar{\xi}) = \bar{\pi}^a(\tau) \hat{m}_a(\xi, \bar{\xi}).$$ (17)

The quantities $(L, \pi)$ and $(L^*, \bar{L}^*)$ are complex conjugate pairs. Only the first pair is basic to our discussion; nevertheless, the second pair does play a role later in section 4.

To end this section we emphasize that from the known $L(u, \xi, \bar{\xi})$ and its complex conjugate, when $\tau$ is taken so that $u$ is real, we have the regular shear-free null geodesic given by equation (3):

$$x^a = u\hat{t}^a - \hat{L}m^a - L\bar{m}^a + (r - r_0)\hat{t}^a.$$ (18)

2.2. The optical parameters

When describing a NGC in flat spacetime, two of the Newman–Penrose spin-coefficients are of particular importance; these are $\rho$ and $\sigma$, known as the optical parameters. In a null tetrad frame, where the vector $l$ is tangent to a NGC, these spin-coefficients are defined by [4]

$$\rho = m^a \bar{m}_b \nabla_a l_b = \frac{1}{2} \left( -\nabla_a l^a + i \text{curl} l^a \right),$$ (19)
$$\text{curl} l^a = \sqrt{\nabla_{[a} l_b]} \nabla^a l^b,$$
$$\sigma = m^a m_b \nabla_a l_b.$$ (20)
The parameter $\rho$ is referred to as the complex divergence (often simply as the divergence) and $\sigma$ is the complex shear of the null geodesic congruence. The ‘radial’ behavior of the optical equations (i.e. their $r$-dependence) is governed by the coupled set of equations known as the Sachs optical equations:

$$
\frac{\partial \rho}{\partial r} = \rho^2 + \sigma \bar{\sigma}, \quad \frac{\partial \sigma}{\partial r} = (\rho + \bar{\rho})\sigma.
$$

(21)

The optical equations can be written in the form of a matrix Riccati equation

$$
DP = P^2,
$$

(22)

where the differential operator is $D \equiv \partial_r$, and the matrix $P$ is

$$
P = \begin{pmatrix}
\rho & \sigma \\
\bar{\sigma} & \bar{\rho}
\end{pmatrix}.
$$

(23)

Integrating equation (22) and fixing the affine parameter origin $r_0$ as

$$
r_0 = -\frac{1}{2} \left( \bar{\delta L} + \delta L \cdot \bar{\delta L} + \delta L \cdot \bar{\delta L} \right),
$$

(24)

yields the flat-space solutions [3, 5]

$$
\rho = \frac{i \Sigma - r}{r^2 + \Sigma^2 - \sigma^0 \bar{\sigma}^0},
$$

(25)

$$
\sigma = \frac{\sigma^0}{r^2 + \Sigma^2 - \sigma^0 \bar{\sigma}^0},
$$

(26)

where $\Sigma = \Sigma(u, \zeta, \bar{\zeta}) \in \mathbb{R}$ and $\sigma^0 = \sigma^0(u, \zeta, \bar{\zeta}) \in \mathbb{C}$ are functions of integration, called the twist and asymptotic shear of the NGC, respectively. It can be shown, via equation (3), that the twist $\Sigma$ is related to the arbitrary complex function $L$ which selects the NGC in equation (3) by [5]

$$
2i \Sigma = \bar{\delta L} + L \bar{\delta L} - \delta L - L \delta L.
$$

(27)

As we are interested in shear-free NGCs in $\mathcal{M}$, we consider those solutions to (21) for which $\sigma^0 = 0$; automatically we see that all such NGCs are everywhere shear-free:

$$
\rho = \frac{1}{r + i \Sigma}, \quad \sigma = 0.
$$

(28)

The twist $\Sigma$ is essentially the imaginary part of the complex divergence $\rho$ of a shear-free NGC in $\mathcal{M}$. The caustics of such a NGC (those points at which $\rho \to \infty$) are hence given by

$$
r = 0, \quad \Sigma(u, \zeta, \bar{\zeta}) = 0,
$$

and are on occasion referred to as the ‘source’ of the NGC.

From equations (10), (12) and (27), it should be noted that if the world-line $\xi^a$ generating a shear-free NGC is taken to be real, then the twist of the congruence vanishes ($\Sigma = 0$). It is in this sense that we may think of the twist of a shear-free NGC as a measure of how far into $\mathcal{M}_C$ the associated complex world-line is displaced. If we interpret the complex world-line as the ‘source’ of the congruence (see section 4), then twist-free and shear-free NGCs have their source in $\mathcal{M}$ (a real world-line) while twisting shear-free NGCs appear (to an observer on $\mathcal{I}^+$) to have their source in $\mathcal{M}_C$. 

6
2.3. Real cuts from complex good cuts

It can be shown by a limiting process (taking $r \to \infty$) \cite{9} that the intersection of the complex light-cone from points on the complex world-line, $\xi^a(\tau)$, with the complex $\mathcal{I}^+_C$ is described by the complex cut function, equation (11),

$$u = G(\tau, \zeta, \bar{\zeta}) = \frac{\sqrt{2}}{2} \xi^0(\tau) - \frac{\xi^i(\tau)}{2} Y^0_{i}. \quad (29)$$

Our immediate task is to find the real values of $u$, i.e. the intersection of the complex cone with real $\mathcal{I}^+$. This entails finding restrictions on the parameter $\tau$. This \cite{3} involves first splitting $\tau$ into its real and imaginary parts:

$$\tau = s + i\lambda. \quad (30)$$

The cut function can be written as a function of two real variables $(s, \lambda)$ instead of the single complex $\tau$ and then decomposed as

$$u = G(s + i\lambda, \zeta, \bar{\zeta}) = G_R(s, \lambda, \zeta, \bar{\zeta}) + iG_I(s, \lambda, \zeta, \bar{\zeta}), \quad (31)$$

where the real and imaginary parts of the cut function can be found by

$$G_R(s, \lambda, \zeta, \bar{\zeta}) = \frac{1}{2} \left[ G(s + i\lambda, \zeta, \bar{\zeta}) + G(s - i\lambda, \zeta, \bar{\zeta}) \right], \quad (32)$$

$$G_I(s, \lambda, \zeta, \bar{\zeta}) = \frac{1}{2i} \left[ G(s + i\lambda, \zeta, \bar{\zeta}) - G(s - i\lambda, \zeta, \bar{\zeta}) \right]. \quad (33)$$

The real values of $u$ are then found by setting

$$G_I(s, \lambda, \zeta, \bar{\zeta}) = 0. \quad (34)$$

Solving equation (34) for $\lambda$ then yields

$$\lambda = \Lambda(s, \zeta, \bar{\zeta}). \quad (35)$$

Combining equations (35), (30) and (31) gives us a real-valued cut function $G : \mathbb{R} \times S^2 \to \mathbb{R}$ of the form

$$u = G(\tau^{(R)}, \zeta, \bar{\zeta}) \equiv G_R(s, \Lambda, \zeta, \bar{\zeta}), \quad (36)$$

with

$$\tau^{(R)} \equiv s + i\Lambda(s, \zeta, \bar{\zeta}). \quad (37)$$

Care must be taken as to when $\tau^{(R)}$ is inserted into the complex $G(\tau, \zeta, \bar{\zeta})$. For instance, the function $L$ is calculated by first applying $\partial_{(\tau)}$ to the complex $G(\tau, \zeta, \bar{\zeta})$ and then afterward inserting $\tau^{(R)}$:

$$L(u, \zeta, \bar{\zeta}) = (\partial_{(\tau)} G)|_{\tau=\tau^{(R)}}. \quad (38)$$

Taking $L = \partial G_R$ would be incorrect, since the additional angular dependence in $G_R$ coming from $\Lambda(s, \zeta, \bar{\zeta})$ would yield a different result upon application of the $\delta$-operator. In the next section, we will explore some more consequences of this additional angular dependence.

3. Real geometric structures from the complex world-lines

In this section, we investigate some of the real effects coming from the restriction of the complex cuts to real $\mathcal{I}^+$. We begin by considering the differences between the real cuts generated by a complex world-line and those generated by a real world-line in real $\mathbb{M}$. We then discuss the analytic description of the shear-free NGCs using the restriction of the range of $\tau$ to $\tau^{(R)} = s + i\Lambda(s, \zeta, \bar{\zeta})$. The section concludes with a discussion of the effects of these results on the associated CR structure induced on $\mathcal{I}^+$ by the shear-free NGCs.
3.1. Real world-lines and complex world-lines

Suppose that for our arbitrary world-line generating a shear-free NGC, we choose \( \xi^a(s) \in \mathbb{M} \); that is, we take a real world-line parametrized by the real variable \( s \). As pointed out earlier, such a NGC is twist free \( (\Sigma = 0) \). Regular NGCs generated by this method are just the null geodesics of the real light-cones with apex on the real world-line. The cut function generated this way is given as

\[
 u = G(s, \zeta, \bar{\zeta}) = \frac{\xi^0(s)}{\sqrt{2}} - \frac{\xi^i(s)}{2} Y_0^0.
\]

(38)

The real parameter \( s \) labels the real ‘cuts’ of \( I^+ \).

It is easy to see that the constant \( s \) cross-sections of \( I^+ \) are (undistorted) two-spheres since they involve only the \( l = 0, 1 \) harmonics. The question is how does this compare with the cross-sections produced by a general complex world-line?

For an arbitrary world-line \( \xi^a(\tau) \in \mathbb{M}_C \), we have the (complex) cut function from equation (11):

\[
 G(\tau, \zeta, \bar{\zeta}) = \frac{\xi^0(\tau)}{\sqrt{2}} - \frac{\xi^i(\tau)}{2} Y_0^0.
\]

From the previous section, this cut function can be restricted to the real \( I^+ \) by replacing \( \tau \) with \( \tau(R) \), giving the real cut function

\[
 u = G(\tau(R), \zeta, \bar{\zeta}) = \frac{\xi^0(s + i\Lambda(s, \zeta, \bar{\zeta}))}{\sqrt{2}} - \frac{\xi^i(s + i\Lambda(s, \zeta, \bar{\zeta}))}{2} Y_0^0.
\]

(39)

Once again, we see that the one-parameter family of slicings of \( I^+ \) are labeled by the real parameter \( s \). However, for a fixed \( s \), the cross-sections of \( I^+ \) we obtain are not in general just \( S^2 \). This follows because of the additional angular dependence of the world-line coming from \( \Lambda(s, \zeta, \bar{\zeta}) \) in \( \tau(R) \). \( \Lambda(s, \zeta, \bar{\zeta}) \) has, in general, an arbitrary spherical harmonic decomposition, so that the spherical harmonic expansion of \( G(s + i\Lambda(s, \zeta, \bar{\zeta}), \zeta, \bar{\zeta}) \) will be distorted from just the \( l = 0, 1 \) harmonics of the real world-lines.

We thus have that a major difference between non-twisting and twisting shear-free NGCs in Minkowski space is that the former produce spherical slicings of \( I^+ \), while the latter have distorted sphere cross-sections. The key ingredient for this distinction is the angular dependence of \( \Lambda(s, \zeta, \bar{\zeta}) \) which is needed when we force the complex cut function to take on real values.

3.2. The CR structures and Levi forms

It is known that \( I^+ \) of Minkowski space (or even asymptotically flat spacetimes) has a realizable CR structure associated with each choice of a shear-free NGC (or equivalently with the choice of a complex world-line generating the congruences) \[3, 10\]. As a real three-dimensional manifold, \( I^+ \) is said to be a CR manifold (or have an associated CR structure) if there is a real one-form \( \mathcal{L} \) and complex one-form \( \mathcal{M} \) defined up to the CR gauge transformations

\[
 \mathcal{L} \rightarrow a \mathcal{L},
\]

\[
 \mathcal{M} \rightarrow f \mathcal{M} + g \mathcal{L},
\]

(40)

where \( (a, f, g) \) are functions on \( I^+ \), with \( a \) real and non-vanishing, \( f \) complex and non-vanishing and \( g \) complex. These one-forms must also be linearly independent in the sense that

\[
 \mathcal{L} \wedge \mathcal{M} \wedge \mathcal{M} \neq 0.
\]

(41)
On \( J^* \) of Minkowski space, using the complex function \( L(u, \zeta, \bar{\zeta}) \) describing a shear-free NGC, these one-forms (modulo gauge freedom) are

\[
\mathcal{L} = du - \frac{L}{1 + \zeta \bar{\zeta}} d\zeta - \frac{\bar{L}}{1 + \zeta \bar{\zeta}} d\bar{\zeta}, \\
\mathcal{M} = \frac{d\bar{\zeta}}{1 + \zeta \bar{\zeta}},
\]

with the associated vector duals

\[
l = \frac{\partial}{\partial u}, \\
m = (1 + \zeta \bar{\zeta}) \frac{\partial}{\partial \zeta} + L \frac{\partial}{\partial \bar{\zeta}}.
\]

This CR structure is said to be realizable if we can find an embedding of the form \( \iota : J^* \to \mathbb{C}^2 \), where the function \( \iota(u, \zeta, \bar{\zeta}) \) is given by the two linearly independent solutions \( K_1 \) and \( K_2 \) of the CR equation:

\[
mK_i = \partial K_i + LK_i = 0.
\]

The first solution is simply \( K_1(u, \zeta, \bar{\zeta}) = \bar{\zeta} \), while the second given by \( K_2(u, \zeta, \bar{\zeta}) = T(u, \zeta, \bar{\zeta}) \), the complex potential function introduced earlier. It is in this sense that \( (7) \) is a CR equation, and \( T \) a corresponding CR function. So we have that in \( \mathbb{M} \), shear-free NGCs generate realizable CR structures on \( J^* \) of the form

\[
\iota(u, \zeta, \bar{\zeta}) = (\bar{\zeta}, \tau),
\]

and that each choice of complex world-line \( \xi^a \) induces a different corresponding CR structure.

Every CR manifold of this type is endowed with a Hermitian two-form (the Levi form), which encodes information about the pseudo-convexity of the manifold [11]. For the CR structure on \( J^* \), we can calculate the Levi form as

\[
h = -2i[d\mathcal{L}][m \otimes \bar{m}]],
\]

where \( \mathcal{L} \) is taken from (42), \( m \) and \( \bar{m} \) are taken from (43) and \( ' \sqcup ' \) stands for contraction.

Now, writing \( P \equiv 1 + \zeta \bar{\zeta} \), we have that

\[
d\mathcal{L} = \frac{L}{P} d\zeta \wedge du - \frac{\bar{L}}{P} du \wedge d\bar{\zeta} + \left[ \frac{\partial}{\partial \zeta} \left( \frac{L}{P} \right) - \frac{\partial}{\partial \bar{\zeta}} \left( \frac{\bar{L}}{P} \right) \right] d\zeta \wedge d\bar{\zeta}.
\]

Denoting components as \( d\mathcal{L} \equiv \mathcal{L}_{ab} \) and \( m \otimes \bar{m} \equiv M^{ab} \) respectively, it is then a simple calculation to see that

\[
\mathcal{L}_{ab}M^{ab} = \partial L + \bar{L} \bar{L} - \partial \bar{L} - L \bar{L},
\]

and it thus follows that the Levi form for the shear-free CR structure on \( J^* \) is

\[
h = -2i(\partial L + \bar{L} \bar{L} - \partial \bar{L} - L \bar{L}).
\]

From equation (27) we see that \( h \) is proportional to \( \Sigma \), the twist of the NGC. Combining this with the discussion of the previous subsection yields the following result.

We see immediately that any CR structure generated by a real world-line \( \xi^a \in \mathbb{M} \) (i.e. twist-free) induces a Levi-flat CR structure on \( J^* \).
3.3. Parametrically describing the shear-free NGC

Earlier, in equation (3), we gave the general expression for any Minkowski space NGC where $L(u, \zeta, \bar{\zeta})$ was an arbitrary function. If the congruence was to be shear free, $L(u, \zeta, \bar{\zeta})$ had to satisfy the shear-free condition, equation (6). From the parametric relations,

$$L(u, \zeta, \bar{\zeta}) = G(s + i\Lambda(s, \zeta, \bar{\zeta}), \zeta, \bar{\zeta}),$$

we have the parametric description of the shear-free NGCs in terms of $(u, \zeta, \bar{\zeta})$

$$x^a = u\hat{t}^a - \hat{m}^a + (r - r_0)\hat{l}^a.$$

Equivalently, we can express the Minkowski space coordinates in terms of the real parameter $s$, which labels the leaves of the good cut foliation of $I^+$. To do this, we simply take $L$ before replacing $\tau$ by $u$ from (12), i.e.

$$L(u(\tau), \zeta, \bar{\zeta}) = \xi^i(\tau)Y_{1i}^\dagger(\zeta, \bar{\zeta}),$$

and replace $u$ in (3) by $G(\tau, \zeta, \bar{\zeta})$. The resulting expression when restricted to $\tau(\mathbb{R}) = s + i\Lambda(s, \zeta, \bar{\zeta})$ yields the explicit description of any flat space shear-free NGC in terms of $(s, \zeta, \bar{\zeta})$:

$$\xi^a(s + i\Lambda(s, \zeta, \bar{\zeta})) = \xi^b(\tau(\mathbb{R}))Y_{1b}^\dagger(\zeta, \bar{\zeta}).$$

Care must be taken here when calculating $r_0$, which involves terms of the form $\hat{L}$. Here, the $\hat{L}$-operator should first be applied to $L(u(\tau), \zeta, \bar{\zeta})$ and only then insert $\tau(\mathbb{R})$; this is again because of the added angular dependence coming from $\Lambda(s, \zeta, \bar{\zeta})$. In these new coordinates, $(s, \zeta, \bar{\zeta})$ label the geodesics of the congruence by naming them for the spot where they intersect $I^+$. $r$ is the affine parameter along each geodesic.

4. Shear-free congruence directly from the complex world-line

Our principle result is the demonstration that the real shear-free NGC can be (easily) found by mapping the complex world-line into the real Minkowski space.

**Theorem.** There exists a mapping from the complex world-line $\xi^a(\tau) \in \mathbb{M}_\mathbb{C}$ to the real shear-free NGC in $\mathbb{M}$ given by two complex null displacements.

Beginning with the world line, $\xi^a(\tau)$, written in terms of its components $(\xi^b, \xi^n_b, \xi^b_m, \xi^b_{\bar{m}})$ as

$$\xi^a(\tau) = \xi^b(\tau)\hat{n}^a_b + \xi^n(\tau)\hat{m}^a_n + \xi^b(\tau)\hat{m}^a_b - \xi^b(\tau)\hat{m}^a_{\bar{m}},$$

we replace $n^a$ by

$$\hat{n}^a = \sqrt{2}t^a - \bar{t}^a \quad t^a = \delta^a_0$$

yielding

$$\xi^a(\tau) = \sqrt{2}\xi^b(\tau)\hat{n}^a_b + \sqrt{2}\xi^n(\tau)\hat{m}^a_n + 2\xi^b(\tau)\hat{m}^a_m - \xi^b(\tau)\hat{m}^a_{\bar{m}}.$$
Remembering, from equations (12) and (16), that
\[ u = \xi^b (\tau) \hat{h}^b \]
\[ L^* (u, \zeta, \bar{\zeta}) = \xi^a (\tau) \hat{m}^a (\zeta, \bar{\zeta}), \]
equation (53) becomes
\[ \xi^a (\tau) = \sqrt{2} u^a + \sqrt{2} \xi^0 (\tau) \hat{h}^a - 2u \hat{h}^a - L \hat{m}^a - L^* \hat{m}^a, \quad (54) \]
or
\[ \xi^a (\tau) + 2u \hat{h}^a - \sqrt{2} \xi^0 (\tau) \hat{h}^a + L^* \hat{m}^a = \sqrt{2} u^a - L \hat{m}^a. \quad (55) \]
Finally subtracting \( L \hat{m}^a \) and adding \((r - r_0) \hat{l}^a\) to both sides of equation (55) we obtain
\[ x^a = \xi^a (\tau) + (2u - \sqrt{2} \xi^0 (\tau) \hat{h}^a + (L^* - \bar{L}) \hat{m}^a + (r - r_0) \hat{l}^a \]
\[ = \sqrt{2} u^a - L \hat{m}^a - L^* \hat{m}^a + (r - r_0) \hat{l}^a. \quad (56) \]
To complete our task we now restrict the values of \( \tau \) to those that produce a real \( u \), namely
\[ \tau \to \tau^{(R)} = s + i \Lambda (s, \zeta, \bar{\zeta}). \]

We see that by adding several null vectors proportional to \( \hat{m}^a \) and \( \hat{l}^a \) directly to the complex world-line \( \xi^a (\tau) \), we obtain a mapping of the world-line directly to the real shear-free NGC, equation (51).

We thus have the explicit relationship of the complex world-line to the shear-free NGC.

5. Caustic geometry

As mentioned in section 2, the caustic set, where the complex divergence \( \rho \) of the NGC diverges, can be interpreted as the ‘source’ of the NGC itself. In this section, we will derive the coordinate expression of the caustics of the NGC in Minkowski space. In doing this, we assume certain generic behavior regarding the invertibility of several of the functions which appear. From this we see that the caustic appears as a closed string or loop that evolves with time in the Minkowski space. The non-generic special cases appear to have the same caustic structure.

5.1. Generic world-line caustics

Recall that for a shear-free NGC in \( \mathbb{M} \), the complex divergence has the form
\[ \rho = - \frac{1}{r + i \Sigma}, \quad (57) \]
and caustics occur (in the geodesic coordinates) where \( \rho \to \infty \). Since both \( r \) and \( \Sigma \) are real, it follows immediately that the caustic set is given by the relationships
\[ r = 0, \quad \Sigma (u, \zeta, \bar{\zeta}) = 0. \quad (58) \]
Putting \( r = 0 \) into the Minkowski space coordinate formula (3), we get
\[ x^a = u^a - L \hat{m}^a - L^* \hat{m}^a - r_0 \hat{l}^a, \quad (59) \]
where \( r_0 = - \frac{1}{4} (\partial L + \bar{L} \bar{L} + \partial L + LL) \) from (24). Everything on the right-hand side is a function of \((u, \zeta, \bar{\zeta})\). For a fixed time in Minkowski coordinates, say \( x^0 = t \), we then have from equations (59), (2) and (4)
\[ t = x^0 (u, \zeta, \bar{\zeta}) = \sqrt{2} u - \frac{\sqrt{2}}{2} r_0 (u, \zeta, \bar{\zeta}). \quad (60) \]
By the generic assumption, equation (60) can be inverted to give
\[ u = H(t, \zeta, \bar{\zeta}). \] (61)

Also assuming that
\[ \Sigma(u, \zeta, \bar{\zeta}) = -\frac{i}{2}(\bar{\delta}L + LL - \delta L - \bar{L}L) = 0 \] (62)
can be solved for \( u \) yields
\[ u = F(\zeta, \bar{\zeta}). \] (63)

Eliminating \( u \) from (61) and (63) gives
\[ H(t, \zeta, \bar{\zeta}) - F(\zeta, \bar{\zeta}) \equiv K(t, \zeta, \bar{\zeta}) = 0. \] (64)

Writing \( \zeta = x + iy, \bar{\zeta} = x - iy \) and assuming that (64) can indeed be solved for \( y \), we finally obtain
\[ y = Z(t, x). \] (65)

This allows us to combine (61), (63) and (65) in the spatial portion of (59) to get
\[ x' = X'(t, x). \] (66)

Note, from the stereographic projection, that the range of \( x \) is between plus and minus infinity with plus and minus infinity being identified with each other. This means that for our generic world-line the NGC’s caustics at a fixed Minkowski time \( t \) will be an \( S^1 \) worth of spatial points. The spacetime caustics of the generic shear-free NGCs in \( \mathbb{M} \) are thus composed of evolving closed strings, i.e. tube-like structures, where movement up or down the tube is in the Minkowski time \( t \).

**Comment.** We have tacitly assumed that in the generic situation, at least for portions of the curve, all the relevant functions can be inverted. Unfortunately to prove this is difficult. Nevertheless, in a recent Oxford PhD thesis by Rafal Swiderski, it was proved that for complex world-lines sufficiently close to real straight world-lines, the inversions were possible. Though it is hard to analyze all possible special cases where the above arguments obviously do not work, nevertheless it appears as if the above caustic structure is preserved in all cases. For example, if the twist \( \Sigma \) does not depend on \( u \) (see equations (62) and (63)), it is easy to see that we have the same closed curve at any instant of time. The Kerr congruence is a special case of this.

### 6. Conclusion

We have found a surprising real effect in Minkowski spacetime that originates with an arbitrary complex analytic world-line in complex Minkowski space, namely that it generates a regular shear-free NGC and that all such congruences have their origin in such a complex line. One first ‘sees’ the complex line by the intersection of the complex light-cone with real null infinity, \( \mathbb{I}^+ \), where the distortions of the cut come from the displacement of the world-line from the real into the complex. Likewise the twist of the NGC is a direct measure of this displacement. The most complete relationship between the complex world-line and the NGC is the mapping of the world-line via two complex null displacements directly into the congruence itself.

Although we have restricted our discussion to flat spacetime, several of the ideas developed here do generalize to asymptotically flat spacetimes. In such spacetimes, one still finds that asymptotically shear-free NGCs have an associated complex world-line ‘living’ in an auxiliary Minkowski space. The same restriction procedure is performed on \( \tau \) to produce real cuts [3]. The asymptotic twist is again a measure of the displacement of the line into the complex.
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