On stretching the interval simplex-permutohedron

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Abstract
A family of polytopes introduced by E.M. Feichtner, A. Postnikov and B. Sturmfels, which were named nestohedra, consists in each dimension of an interval of polytopes starting with a simplex and ending with a permutohedron. This paper investigates a problem of changing and extending the boundaries of these intervals. An iterative application of Feichtner-Kozlov procedure of forming complexes of nested sets is a solution of this problem. By using a simple algebraic presentation of members of nested sets it is possible to avoid the problem of increasing the complexity of the structure of nested curly braces in elements of the produced simplicial complexes.

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1 Introduction

A history of the family of polytopes that appear in the title of this paper under the name “the interval simplex-permutohedron”, may be traced to Appendix B of [24], where Stasheff and Shnider gave a procedure of truncation of an \((n-2)\)-dimensional simplex, which realizes the associahedron \(K_n\) as a convex polytope. This procedure implicitly uses the fact that \(K_n\), as an abstract polytope, is induced by (constructions of) the graph whose edges connect the neighbour symbols of binary operations in a term with \(n\) variables. It is also noted in [24] that this procedure could be modified in order
to obtain cyclohedra out of simplices; the edges of graphs that induce
cyclohedra would connect the neighbour symbols of binary operations in cyclic
terms.

Carr and Devadoss used graphs in [2] more explicitly to define a family
of polytopes that includes simplices, associahedra, cyclohedra, permutohedra
and many others. This approach, which is based on the concept of tubings
is further developed in [6], [5] and [15].

Feichtner and Kozlov introduced in [11] the notions of building set and
nested set in order to give an abstract framework for the incidence combinatorics
occurring in some constructions in algebraic geometry (see for example
[4]). Feichtner and Sturmfels in [13] and Postnikov in [22] used building sets
independently, in a more narrow context, to describe the face lattices of a
family of simple polytopes named nestohedra in [23].

An alternative, inductive, approach that leads to the same family of polytope
is given in [10]. This was an independent discovery motivated by some
related work done in [7], [8] and [9].

For a fixed dimension $n$, the family of polytopes in question may be con-
sidered as an interval starting with an $n$-dimensional simplex and ending with
an $n$-dimensional permutohedron, while the partial ordering of the interval
is induced by the relation $\subseteq$ on the corresponding building sets. Such inter-
vals contain some polytopes, like for example associahedra and cyclohedra,
which are widely used in topology and category theory. This is the reason
why we give the advantage to the families of simple polytopes (though the
corresponding families of simplicial polytopes are easier to handle). How-
ever, if one prefers to simplify everything a bit, the mathematical content of
the results would not change if we switch to the intervals starting with an $n$-
dimensional simplex and ending with the polytope polar to an $n$-dimensional
permutohedron.

This paper gives a solution to the problem of changing and extending
these intervals. We find the Feichtner-Kozlov concept of building sets of
arbitrary finite-meet semilattices, applied to the case of simplicial complexes,
appropriate for this purpose.

In particular, we are interested in iterating the procedure of generating
complexes of nested sets. The main obstacle is that every iteration of this
procedure increases the complexity of the structure of nested curly braces in
elements of the produced simplicial complex. This makes it almost impossible
to write down the result of the procedure after just a couple of iterations. We
show here how to avoid this problem by using a simple algebraic presentation
of the members of nested sets as elements of a freely generated commutative
semigroup.

Such an iterated procedure is applied to the boundary complex of a sim-
plicial polytope. This provides an opportunity to replace the simplex at the beginning of the interval simplex-permutohedron by an arbitrary simple polytope, which automatically replaces the permutohedron at the end of the interval by some other simple polytope. By iterating the procedure, one obtains interesting families of simplicial complexes.

Truncation of a polytope in its proper face is the operation that guarantees that all these simplicial complexes may be realized as the face lattices (with $\emptyset$ removed) of some simple polytopes. In dimension 3, locally at each vertex, one may find one of the following six types of truncation (cf. the examples of Section 10).

![Diagram showing six types of truncation: no truncation, vertex, edge, vertex + edge, vertex + 2 edges, vertex + 3 edges.]

2 Nested complexes and abstract polytopes of hypergraphs

In this section we compare the three approaches given in [13], [22] and [10], which all lead to the same family of polytopes. We start with some preliminary notions.

Let $\alpha_1, \ldots, \alpha_n$ for $n \geq 1$ be finite and incomparable (with respect to $\subseteq$) sets. Then $C = P(\alpha_1) \cup \ldots \cup P(\alpha_n)$, where $P(\alpha)$ is the power set of $\alpha$, is a finite abstract simplicial complex based on the set $\{\alpha_1, \ldots, \alpha_n\}$ of its base\footnote{The term “facet” is reserved here for a face of a polytope whose codimension is 1.}. Since we deal here only with finite abstract simplicial complexes, we call them just simplicial complexes. For $C$ a simplicial complex, we have that $\langle C, \cap \rangle$ is a meet-semilattice that induces the poset $\langle C, \subseteq \rangle$. Depending on a context, by a simplicial complex we usually mean such a meet-semilattice or a poset.

For $B$ a subset of the domain of a function $f$, we denote by $f[B]$ the set $\{f(b) \mid b \in B\}$. It is easy to see that two simplicial complexes $C$ and $D$ are isomorphic as meet-semilattices (or as posets) if and only if there is
a bijection \( \varphi : \bigcup C \to \bigcup D \) such that \( \alpha \in C \) iff \( \varphi[\alpha] \in D \) (or equivalently, such that \( \alpha \) is a basis of \( C \) iff \( \varphi[\alpha] \) is a basis of \( D \)). If \( \psi \) is a function such that the above \( \varphi \) is its restriction to \( \bigcup C \), then we say that \( \psi \) underlies an isomorphism between \( C \) and \( D \). So, we have the following remarks.

**Remark 2.0.1.** If \( C \) is a simplicial complex and \( \psi \) is a function such that its restriction to \( \bigcup C \) is one-one, then \( \psi \) underlies an isomorphism between \( C \) and the simplicial complex \( \{\psi[\alpha] \mid \alpha \in C\} \).

**Remark 2.0.2.** If \( \psi \) underlies isomorphisms between \( C \) and \( D_1 \) and between \( C \) and \( D_2 \), then \( D_1 = D_2 \).

Let \( \alpha \) be an arbitrary finite set. According to Definition 7.1 of [22], a set \( B \) of nonempty subsets of \( \alpha \) is a building set\(^2\) of \( P(\alpha) \) when the following conditions hold:

- (B1) If \( \beta, \gamma \in B \) and \( \beta \cap \gamma \neq \emptyset \), then \( \beta \cup \gamma \in B \);
- (B2) \( B \) contains all singletons \( \{a\} \), for \( a \in \alpha \).

In the terminology of [10] (Sections 3-4), this notion corresponds to the notion of atomic saturated hypergraph and it is easy to see that this is a restriction to the case of power sets of the more general notion of building sets of arbitrary finite-meet semilattices given in [11] (Definition 2.2).

Let \( N \) be a family of sets. As in [10], we say that \( \{\beta_1, \ldots, \beta_t\} \subseteq N \) is an \( N \)-antichain when \( t \geq 2 \) and \( \beta_1, \ldots, \beta_t \) are incomparable with respect to \( \subseteq \). Also, for a family of sets \( B \), we say that an \( N \)-antichain \( \{\beta_1, \ldots, \beta_t\} \) misses \( B \) when the union \( \beta_1 \cup \ldots \cup \beta_t \) does not belong to \( B \).

### 2.1 Nested set complexes of Feichtner and Sturmfels

Let \( B \) be a building set of \( P(\alpha) \), containing \( \alpha \). A subset \( N \) of \( B \) is nested when every \( N \)-antichain misses \( B \). It is easy to see that for \( M \subseteq N \subseteq B \), if \( N \) is nested, then \( M \) is nested too. So, the nested subsets of \( B \) form a simplicial complex whose bases are the maximal nested subsets of \( B \).

In [13], for \( B \) as above, this simplicial complex would be denoted by \( \mathcal{N}(P(\alpha), B) \), and the link of \( \alpha \) in \( \mathcal{N}(P(\alpha), B) \) would be denoted by \( \mathcal{N}(P(\alpha), B) \) and called the nested set complex of \( P(\alpha) \) with respect to \( B \). (Nested set complexes of [13] are defined not only for power sets but for arbitrary finite lattices.)

\(^2\)Originally “a building set on \( \alpha \)” in [22], but I prefer to keep to the terminology of [11] where the building sets of semilattices are introduced.
Unfortunately, the name nested set complex and the symbol $N$ were previously used in [12] (see also [3]) for the result and the name of the operation that generalizes $\tilde{N}$ from above.

### 2.2 Nested complexes of Postnikov

Let $B$ be a building set of $P(\alpha)$ not necessarily containing $\alpha$. According to Definition 7.3 of [22], a subset $N$ of $B$ is a nested set when it satisfies the following:

1. If $\beta, \gamma \in N$ then $(\beta \subseteq \gamma$ or $\gamma \subseteq \beta$ or $\beta \cap \gamma = \emptyset)$;
2. If $\beta_1, \ldots, \beta_t$ for $t \geq 2$ are mutually disjoint elements of $N$, then $\beta_1 \cup \ldots \cup \beta_t$ does not belong to $B$;
3. $N$ contains all maximal elements of $B$.

The nested complex $N(B)$ is a poset of all nested sets ordered by inclusion. Let $N^*(B)$ be obtained from $N(B)$ by removing every maximal element of $B$ from nested sets. Then $N^*(B)$ is a simplicial complex whose bases are maximal nested sets with maximal elements of $B$ removed.

### 2.3 Abstract polytopes of hypergraphs

This is an alternative, inductive approach to the same matters, which is given in [10]. For $\alpha$ a finite set, let $H \subseteq P(\alpha)$ be such that $\emptyset \not\in H$ and $\alpha = \bigcup H$. Then $H$ is a hypergraph on $\alpha$ (see [1], Section 1.1). A hypergraph $H$ is atomic when for every $x \in \bigcup H$ we have that $\{x\} \in H$ (see [10], Section 3).

A hypergraph partition of a hypergraph $H$ is a partition $\{H_1, \ldots, H_n\}$, with $n \geq 1$, of $H$ such that $\{\bigcup H_1, \ldots, \bigcup H_n\}$ is a partition of $\bigcup H$. A hypergraph $H$ is connected when it has only one hypergraph partition. (If $H$ is nonempty, then this hypergraph partition is $\{H\}$.) For example, $H = \{\{x\}, \{y\}, \{z\}, \{x, y, z\}\}$ is connected.

A hypergraph partition $\{H_1, \ldots, H_n\}$ of $H$ is finest when every $H_i$ is a connected hypergraph on $\bigcup H_i$. We say that $\bigcup H_i$ is a connected component of $\bigcup H$. For $H \subseteq P(\alpha)$ and $\beta \subseteq \alpha$ let

$$H_\beta =_{df} \{\gamma \in H \mid \gamma \subseteq \beta\} = H \cap P(\beta).$$

Let $H$ be an atomic hypergraph. By induction on the cardinality $|\bigcup H|$ of $\bigcup H$ the constructions of $H$ are defined as follows

1. if $|\bigcup H| = 0$, then $H$ is the empty hypergraph $\emptyset$, and $\emptyset$ is the only construction of $\emptyset$;
(1) if $|\bigcup H| \geq 1$, and $H$ is connected, and $K$ is a construction of $H_{\bigcup H - \{x\}}$ for $x \in \bigcup H$, then $K \cup \{\bigcup H\}$ is a construction of $H$;

(2) if $|\bigcup H| \geq 2$, and $H$ is not connected, and $\{H_1, \ldots, H_n\}$, where $n \geq 2$, is the finest hypergraph partition of $H$, and for every $i \in \{1, \ldots, n\}$ we have that $K_i$ is a construction of $H_i$, then $K_1 \cup \ldots \cup K_n$ is a construction of $H$.

A subset of a construction is called a **construct** when it contains every connected component of $\bigcup H$. The abstract polytope of $H$, denoted by $\mathcal{A}(H)$, is a poset of all the constructs of $H$ ordered by $\supseteq$, plus the bottom element.

A hypergraph $H$ is **saturated** when it satisfies the condition (B1) of the definition of building set with $B$ replaced by $H$. So, the notions of atomic saturated hypergraph on $\alpha$ and of building set of $P(\alpha)$ coincide. It follows from Proposition 4.7 of [10] that for $\bar{H}$ being the intersection of all the atomic saturated hypergraphs, i.e. building sets, containing an atomic hypergraph $H$, we have $\mathcal{A}(H) = \mathcal{A}(\bar{H})$.

### 2.4 The interval simplex-permutohedron

Our task now is to compare the posets $\mathcal{N}(P(\alpha), B), \mathcal{N}(B)$ and $\mathcal{A}(B)$ for $B$ a building set of $P(\alpha)$. We start with some abbreviations. Let $L_-$, for a finite lattice $L$ be the poset obtained from $L$ by removing the bottom element and, analogously, let $L^-$ be obtained from $L$ by removing the top element. Let $P^{\text{op}}$, for a poset $P$ be the poset obtained by reversing the order of $P$.

Let $B$ be a building set of $P(\alpha)$ and let $N \subseteq B$. Then it is easy to verify that

$$(\ast) \quad \text{(N1) and (N2) of Section 2.2 hold iff every $N$-antichain misses $B$.}$$

From $(\ast)$ we infer that for $B$ containing $\alpha$, we have the isomorphism between $\mathcal{N}(B)$ and $\mathcal{N}(P(\alpha), B)$ obtained by removing $\alpha$ from the elements of $\mathcal{N}(B)$.

Although Proposition 6.12 of [10] is formulated for atomic, saturated and connected hypergraphs, it is easy to verify that connectedness is not essential, i.e. that the following proposition holds.

**Proposition 2.4.1.** **For every building set $H$ and every $N \subseteq H$ we have:**

*every $N$-antichain misses $H$ iff for some construction $K$ of $H$ we have that $N \subseteq K$.**

From Proposition 2.4.1 and $(\ast)$ we infer that $(\mathcal{N}(B))^{\text{op}}$ is equal to $(\mathcal{A}(B))_-$. Since $\mathcal{N}(B)$ and $\mathcal{A}(B)$ for $B$ not containing $\alpha$ are easily defined in terms of $\mathcal{N}(H_1), \ldots, \mathcal{N}(H_n)$, for $\{H_1, \ldots, H_n\}$ being the finest hypergraph partition of $B$ (see the definition of $\otimes$ given in [10], Section 5) we may consider
only the building sets of $P(\alpha)$ that contain $\alpha$ (if $\alpha$ is nonempty). Such a building set is called in [10] atomic saturated connected hypergraph, or shortly ASC-hypergraph. For $H$ being an ASC-hypergraph, all the three posets $(\mathcal{N}(P(\alpha), H))^\text{op}$, $(\mathcal{N}(H))^\text{op}$ and $(\mathcal{A}(H))_\bot$ are isomorphic.

Let $\alpha$ be finite, nonempty set. The ASC-hypergraphs on $\alpha$, i.e. the building sets of $P(\alpha)$ containing $\alpha$, make a meet-semilattice $H$ with $\cap$ being the meet operation. We have that $H_\bot = \{\{a\} \mid a \in \alpha\} \cup \{\alpha\}$ and $H_\top = P(\alpha) - \{\emptyset\}$ are respectively the bottom and the top element of $\mathcal{H}$. (Since every pair of elements of $\mathcal{H}$ has the upper bound, it is easy to define the lattice structure on $\mathcal{H}$.)

Let $n = |\alpha| - 1$. We have that $\mathcal{A}(H_\bot)$ is isomorphic to the face lattice of an $n$-dimensional simplex. On the other hand, $\mathcal{A}(H_\top)$ is isomorphic to the face lattice of an $n$-dimensional permutohedron. For every $H \in \mathcal{H}$ we have that $\mathcal{A}(H)$ is isomorphic to the face lattice of some $n$-dimensional polytope $\mathcal{G}(H)$ (see [10], Section 9 and Appendix B). For a fixed dimension $n \geq 0$, we define the interval simplex-permutohedron to be the set $\{\mathcal{G}(H) \mid H \in \mathcal{H}\}$.

Since the function $\mathcal{G}$ defined in [10], Section 9, is one-one (although some members of the family are combinatorially equivalent), the interval simplex-permutohedron may be enriched with a poset or a lattice structure induced by the structure of $\mathcal{H}$. However, if we consider, as usual, polytopes only up to combinatorial equivalence, then the interval simplex-permutohedron is just a family of polytopes. We refer to [10], appendix B, where one may find the complete intervals simplex-permutohedron in dimension 3 and lower, and a chart of a fragment of $\mathcal{H}$ tied to dimension 3.

### 3 Complexes of nested sets for simplicial complexes

In this section we define the notion of a building set of a simplicial complex and the notion of a corresponding complex of nested sets, which are just restrictions of the notions defined in [11] for arbitrary finite-meet semilattices.

Let $C$ be a simplicial complex based on the set $\{\alpha_1, \ldots, \alpha_n\}$ (cf. Section 2). A set $B \subseteq C$ is a building set of $C$ when for every $i \in \{1, \ldots, n\}$, we have that $B_{\alpha_i} = B \cap P(\alpha_i)$ is a building set of $P(\alpha_i)$ in the sense of the definition given at the beginning of Section 2. It is easy to verify that this is the notion to which the notion of a building set of a finite-meet semilattice introduced in [11] (Definition 2.2) is reduced to, when the semilattice is a simplicial complex.

Let $\text{min} X$, for $X$ a family of sets, be the set of minimal (with respect
to \( \subseteq \) members of \( X \), and let \( \max X \) be defined analogously. Then we can prove the following proposition in a straightforward manner.

**Proposition 3.1.** For a finite, nonempty set \( \alpha \) and a simplicial complex \( C \), we have that:

1. \( B \) is a building set of \( C \) iff \( B_{\gamma} \) is a building set of \( P(\gamma) \) for every \( \gamma \in C \);
2. if \( B \) is a building set of \( P(\alpha) \), then \( B - \{ \alpha \} \) is a building set of \( P(\alpha) - \{ \alpha \} \);
3. \( B_{\perp} = \{ \{ a \} \mid a \in \bigcup C \} \) is a building set of \( C \);
4. if \( \alpha \in C \), then \( B_{\perp} \cup \{ \alpha \} \) is a building set of \( C \);
5. if \( B \) is a building set of \( C \) and \( \beta \in \min(B - B_{\perp}) \), then \( B - \{ \beta \} \) is a building set of \( C \);
6. if \( B \) is a building set of \( C \) and \( \beta \in \max(C - B) \), then \( B \cup \{ \beta \} \) is a building set of \( C \).

(Note that if \( \alpha \) is finite and nonempty, then \( P(\alpha) - \{ \alpha \} \) is a simplicial complex based on the set \( \{ \alpha - \{ a \} \mid a \in \alpha \} \).

Let \( B \) be a building set of a simplicial complex \( C \). A subset \( N \) of \( B \) is nested when for every \( N \)-antichain \( \{ \beta_1, \ldots, \beta_t \} \), the union \( \beta_1 \cup \ldots \cup \beta_t \) belongs to \( C - B \). This is again in accordance with Definition 2.7 of [11].

It is easy to see that for \( M \subseteq N \subseteq B \), if \( N \) is nested, then \( M \) is nested too. So, the nested subsets of \( B \) form again a simplicial complex whose bases are the maximal nested subsets of \( B \). We denote this simplicial complex by \( \tilde{\mathcal{N}}(C, B) \). (According to [11] it would be just \( \mathcal{N}(B) \), but since we want to make a comparison with the notions given in Section 2.1, we find \( \tilde{\mathcal{N}}(C, B) \) more appropriate since it is reduced to \( \tilde{\mathcal{N}}(P(\alpha), B_{\gamma}) \) when \( C = P(\alpha) \) and \( B \) contains \( \alpha \).) We have that \( \bigcup \tilde{\mathcal{N}}(C, B) = B \), since for every \( \beta \in B \) we have that \( \{ \beta \} \) is nested. Then we can prove the following proposition in a straightforward manner.

**Proposition 3.2.** For \( B \) a building set of a simplicial complex \( C \) the following statements are equivalent:

1. \( N \in \tilde{\mathcal{N}}(C, B) \);
2. \( N \subseteq B, \bigcup N \in C \) and every \( N \)-antichain misses \( B \);
3. there is \( \gamma \in C \) such that \( N \in \tilde{\mathcal{N}}(P(\gamma), B_{\gamma}) \);
4. there is a basis \( \alpha \) of \( C \) such that \( N \in \tilde{\mathcal{N}}(P(\alpha), B_{\alpha}) \).

(Note that for (3) and (4) above, we do not require that \( B_{\gamma} \) contains \( \gamma \) and
Proposition 3.3. We have that \( N \in \tilde{\mathcal{N}}(C,B) \) iff there is \( \gamma \in C \) such that \( N \) is a subset of a construction of the hypergraph \( B_\gamma \). Moreover, \( N \in \tilde{\mathcal{N}}(C,B) \) is a basis of \( \tilde{\mathcal{N}}(C,B) \) iff there is a basis \( \alpha \) of \( C \) such that \( N \) is a construction of the hypergraph \( B_\alpha \).

The following proposition defines \( \mathcal{N} \) of Section 2.1 in terms of \( \tilde{\mathcal{N}} \) from above.

Proposition 3.4. If \( B \) is a building set of \( P(\alpha) \), containing \( \alpha \), then
\[
\mathcal{N}(P(\alpha), B) = \tilde{\mathcal{N}}(\alpha, P(\alpha) - \{\alpha\}, B - \{\alpha\}).
\]

Proof. Since \( \alpha \in B \) we have that \( \alpha \) is nonempty. By (2) of Proposition 3.1, \( B - \{\alpha\} \) is a building set of the simplicial complex \( P(\alpha) - \{\alpha\} \).

For the proof of \( \subseteq \) direction, let \( N \in \mathcal{N}(P(\alpha), B) \). By the definition of \( \mathcal{N} \) (cf. Section 2.1), we have \( N \in \tilde{\mathcal{N}}(P(\alpha), B) \) and \( N \) is in the link of \( \alpha \). Hence, \( N \subseteq B - \{\alpha\} \). We have that \( \bigcup N \) is either empty, or it is equal to \( \beta \) for some \( \beta \in N \), or it is equal to \( \beta_1 \cup \ldots \cup \beta_t \) for some \( N \)-antichain \( \{\beta_1, \ldots, \beta_t\} \). In all the three cases it follows that \( \bigcup N \in P(\alpha) - \{\alpha\} \), since \( \alpha \notin N \), \( \alpha \in B \) and \( N \in \tilde{\mathcal{N}}(P(\alpha), B) \).

Let \( \{\beta_1, \ldots, \beta_t\} \) be an \( N \)-antichain. From \( N \in \tilde{\mathcal{N}}(P(\alpha), B) \) we conclude that \( \beta_1 \cup \ldots \cup \beta_t \notin B \), and hence \( \beta_1 \cup \ldots \cup \beta_t \notin B - \{\alpha\} \). So, by (2) of Proposition 3.2, we have that \( N \in \tilde{\mathcal{N}}(P(\alpha) - \{\alpha\}, B - \{\alpha\}) \).

For the proof of \( \supseteq \) direction, let \( N \in \tilde{\mathcal{N}}(P(\alpha) - \{\alpha\}, B - \{\alpha\}) \). So, \( N \subseteq B - \{\alpha\} \) and hence \( N \subseteq B \). If \( \{\beta_1, \ldots, \beta_t\} \) is an \( N \)-antichain, then
\[
\beta_1 \cup \ldots \cup \beta_t \in (P(\alpha) - \{\alpha\}) - (B - \{\alpha\}) = P(\alpha) - B.
\]

Hence, \( N \in \tilde{\mathcal{N}}(P(\alpha), B) \) and analogously, since every \( (N \cup \{\alpha\}) \)-antichain is an \( N \)-antichain, we have that \( N \cup \{\alpha\} \in \tilde{\mathcal{N}}(P(\alpha), B) \). From this and the fact that \( \alpha \notin N \), we conclude that \( N \) belongs to the link of \( \alpha \) in \( \tilde{\mathcal{N}}(P(\alpha), B) \). Hence \( N \in \mathcal{N}(P(\alpha), B) \).

This proposition sheds a new light on the interval simplex-permutohedron and it points out a way how to change the boundaries of this interval. We will discuss this later in Section 10.

4 Commutative semigroup notation

We show how to use the elements of commutative semigroups freely generated by sets to interpret the families of families...of sets. We deal with
“formal sums” of elements of some set and with functions mapping finite, nonempty subsets of this set to the formal sums, or mapping families of finite, nonempty subsets to families of formal sums etc. For example, if we take the set \( \{x, y, z, u\} \), then we are interested in mappings of the following form:

\[
\begin{align*}
\{x, y, z\} & \mapsto x + y + z, \\
\{\{x\}, \{x, y\}, \{z\}\} & \mapsto \{x, x + y, z\}, \text{ and} \\
\{\{\{x\}\}, \{x\}, \{y, u\}\}\} & \mapsto \{x\}, \{x, y + u\}\}.
\end{align*}
\]

For the sake of precision, we use some basic notions from category theory, which all may be found in [21]. However, no result from category theory is needed in the sequel. We try to illustrate all the notions we use by examples so that a reader not familiar with category theory may deal with these notions intuitively.

Let \( \text{Cs} \) be the category of commutative semigroups and let \( G : \text{Cs} \to \text{Set} \) be the forgetful functor, which maps a commutative semigroup to its underlying set. Let \( F : \text{Set} \to \text{Cs} \) be its left adjoint assigning to every set the free commutative semigroup generated by this set. (If \( X \) is empty, then \( FX \) is the empty semigroup.) Consider the monad \( \langle T, \eta, \mu \rangle \) defined by this adjunction \( (T = G \circ F, \eta \text{ is the unit of this adjunction and } \mu = G \varepsilon F, \text{ where } \varepsilon \text{ is the counit of this adjunction}) \).

For every set \( X \) we have that \( TX \) is the underlying set of the free commutative semigroup generated by \( X \). We can take the elements of \( TX \) to be the formal sums of elements of \( X \), so that if \( X = \{x, y, z, u\} \), then

\[
TX = \{x, y, z, u, 2x, x + y, \ldots, 3x + y + 5z + u, \ldots\}.
\]

We use \(+^2\) to denote the formal addition of \( T^2X = TTX \). Analogously, \(+^3\) denotes the formal addition of \( T^3X \) etc. So, if \( X \) is as above, then

\[
T^2X = \{x, \ldots, 3x + y + 5z + u, \ldots, x +^2 2x, \ldots, z +^2 (3x + y + 5z + u), \ldots\}.
\]

For the natural transformations \( \eta \) and \( \mu \) of the monad \( \langle T, \eta, \mu \rangle \), we have that \( \eta_X : X \to TX \) is such that \( \eta_X(x) = x \), for every \( x \in X \), and \( \mu_X : T^2X \to TX \) evaluates \(+^2\) as \(+\), so that, for example, \( \mu_X(z +^2 (3x + y + 5z + u)) = 3x + y + 6z + u \).

Besides the endofunctor \( T : \text{Set} \to \text{Set} \) we are interested in the power set functor \( P : \text{Set} \to \text{Set} \) (see [21], Section I.3) and its modification \( P_F : \text{Set} \to \text{Set} \), which assigns to a set \( X \) the set of finite, nonempty subsets of \( X \). On arrows, \( P_F \) is defined as the power set functor, so that for \( f : X \to Y \) and \( A \in P_FX \), we have that \( P_Ff(A) = f[A] \).
For every set \( X \), let \( \sigma_X : P_fX \to TX \) be the function such that for \( A = \{a_1, \ldots, a_n\} \subseteq X \), where \( n \geq 1 \), we have \( \sigma_X(A) = a_1 + \ldots + a_n \). It follows that \( \sigma_X \) is one-one. (Note that \( \sigma \) is not a natural transformation from \( P_f \) to \( T \).) For example, if \( X \) is as above, then \( \sigma_X(\{x\}) = x \), and for \( N = \{\{x\}, \{x, y\}, \{z\}\} \), we have that \( P\sigma_X(N) = \{x, x+y, z\} \).

5 Iterating \( \tilde{N} \)

If \( B_0 \) is a building set of a simplicial complex \( C \), then \( \tilde{N}(C, B_0) \) is a simplicial complex, and if \( B_1 \) is a building set of \( \tilde{N}(C, B_0) \), then \( \tilde{N}(\tilde{N}(C, B_0), B_1) \) is again a simplicial complex and we may iterate this procedure. So, for \( N = (s) \) and \( \tilde{N}(C, B_0, \ldots, B_{n+1}) =_{df} \tilde{N}(N(C, B_0, \ldots, B_n), B_{n+1}) \).

Example 5.1 Let \( C \) be a simplicial complex whose bases are \( \{x, y, z\}, \{x, y, u\}, \{x, z, u\} \) and \( \{y, z, u\} \), i.e. \( C = P(X) - \{X\} \) for \( X = \{x, y, z, u\} \), and let \( B_0 = \{\{x\}, \{y\}, \{z\}, \{u\}, \{x, y\}\} \). By Proposition 3.3, the bases of \( \tilde{N}(C, B_0) \) are the constructions of \( (B_0)_\alpha \) for all the bases \( \alpha \) of \( C \). So, \( \tilde{N}(C, B_0) \) has the following six bases:

- \( \{\{x\}, \{x, y\}, \{z\}\} \) and \( \{\{y\}, \{x, y\}, \{z\}\} \) derived from \( \{x, y, z\} \),
- \( \{\{x\}, \{x, y\}, \{u\}\} \) and \( \{\{y\}, \{x, y\}, \{u\}\} \) derived from \( \{x, y, u\} \),
- \( \{\{x\}, \{z\}, \{u\}\} \) derived from \( \{x, z, u\} \),
- \( \{\{y\}, \{z\}, \{u\}\} \) derived from \( \{y, z, u\} \).

For \( B_1 = \{\{y\}, \{x, y\}, \{z\}, \{u\}, \{x, y\}, \{x, z\}, \{x, y\}, \{x, y\}\} \), again by Proposition 3.3, we have that \( \tilde{N}(C, B_0, B_1) \) has the following eight bases:

- \( \{\{x\}, \{x, y\}, \{x, y\}\} \), \( \{\{x, y\}, \{x, y\}, \{z\}\} \)
- \( \{\{x\}, \{x, y\}, \{x, y\}\} \), \( \{\{x, z\}, \{x, y\}, \{u\}\} \)
- \( \{\{y\}, \{x, y\}, \{z\}\} \), \( \{\{y\}, \{x, y\}, \{u\}\} \)
- \( \{\{x\}, \{z\}, \{u\}\} \), \( \{\{y\}, \{z\}, \{u\}\} \)

This example illustrates how the iterated application of \( \tilde{N} \) increases complexity of the structure of nested curly braces in the elements of the resulting simplicial complex. Our intention is to make these matters simpler.

For \( C \) a simplicial complex such that \( \cup C \subseteq X \), \( B \) a building set of \( C \), and \( \sigma_X : P_fX \to TX \) defined in the preceding section, let

\[ \Sigma_X(C, B) =_{df} \{P\sigma_X(N) \mid N \in \tilde{N}(C, B)\} \]
It is easy to see that $\tilde{\Sigma}_X(C, B)$ is indeed a simplicial complex. For $C$, $X$ and $B_0$ as in Example 5.1, we have that $\tilde{\Sigma}_X(C, B_0)$ is a simplicial complex with the bases $\{x, x + y, z\}, \{y, x + y, z\}, \{x, x + y, u\}, \{y, x + y, u\}, \{x, z, u\}$ and $\{y, z, u\}$. We have the following proposition.

**Proposition 5.2.** The function $\sigma_X$ underlies an isomorphism between $\tilde{N}(C, B)$ and $\tilde{\Sigma}_X(C, B)$.

**Proof.** Since $\sigma_X$ is one-one, we just apply Remark 2.0.1. $\dashv$

Let $\sigma_X^1 =_{df} \sigma_X$, and for $n \geq 1$, let $\sigma_X^{n+1} =_{df} \sigma_{T^nX} \circ P_F \sigma_X^{n}$. It is easy to see that $\sigma_X^n : P^n_F X \to T^n X$ satisfies the following inductive clauses:

$$
\sigma_X^1(\{a_1, \ldots, a_k\}) = a_1 + \ldots + a_k,
$$

$$
\sigma_X^{n+1}(\{\alpha_1, \ldots, \alpha_m\}) = \sigma_X^n(\alpha_1) +^{n+1} \ldots +^{n+1} \sigma_X^n(\alpha_m).
$$

For example, we have that $\sigma_X^2(\{x\}) = x, \sigma_X^2(\{\{x\}, \{x, y\}\}) = x +^2 (x + y)$, and $\sigma_X^3(\{\{\{x\}\}, \{\{x\}, \{x, y\}\}\}) = x +^3 (x +^2 (x + y))$.

For $n \geq 1$, $C$ a simplicial complex such that $\cup C \subseteq X$, $B_0$ a building set of $C$, and $B_i$, for $1 \leq i \leq n$ a building set of $\tilde{N}(C, B_0, \ldots, B_{n-1})$, let

$$
\tilde{\Sigma}_X(C, B_0, \ldots, B_n) =_{df} \{P \sigma_X^{n+1}(N) \mid N \in \tilde{N}(C, B_0, \ldots, B_n)\}.
$$

We easily compute that $\tilde{\Sigma}_X(C, B_0, B_1)$, for $C$, $X$, $B_0$ and $B_1$ as in Example 5.1, has the following eight bases $\{x, x +^2 (x + y), z\}, \{x + y, x +^2 (x + y), z\}, \{x, x +^2 (x + y), u\}, \{x + y, x +^2 (x + y), u\}, \{y, x + y, z\}, \{y, x + y, u\}, \{x, z, u\}, \{y, z, u\}$. The following proposition is a generalization of Proposition 5.2.

**Proposition 5.3.** The function $\sigma_X^{n+1}$ underlies an isomorphism between $\tilde{N}(C, B_0, \ldots, B_n)$ and $\tilde{\Sigma}_X(C, B_0, \ldots, B_n)$.

**Proof.** Since $\sigma_X$ is one-one and $P_F$ preserves this property, we conclude that $\sigma_X^{n+1}$ is one-one. Then we apply Remark 2.0.1. $\dashv$

**Remark 5.4.** If $\psi$ is a function that underlies an isomorphism between simplicial complexes $C$ and $D$, and $B$ is a building set of $C$, then $P^n_F \psi(B)$ is a building set of $D$ and the function $P_F \psi$ underlies an isomorphism between $\tilde{N}(C, B)$ and $\tilde{N}(D, P^n_F \psi(B))$.

**Proposition 5.5.** For $n \geq 1$, the function $\sigma_X^{n+1}$ underlies an isomorphism between $\tilde{N}(C, B_0, \ldots, B_n)$ and $\tilde{\Sigma}_X(C, B_0, \ldots, B_{n-1}), P^n_F \sigma_X^{n}(B_n))$.

**Proof.** By Proposition 5.3 and Remark 5.4 we have that $P_F \sigma_X^n$ underlies an isomorphism between $\tilde{N}(\tilde{N}(C, B_0, \ldots, B_{n-1}), B_n)$, which is equal to
\( \hat{N}(C, B_0, \ldots, B_n) \) and \( \tilde{N}(\Sigma_X(C, B_0, \ldots, B_{n-1}), P^2\sigma^n_X(B_n)) \). By Proposition 5.2, since \( \cup \Sigma_X(C, B_0, \ldots, B_{n-1}) \subseteq T^n X \), we have that \( \sigma^n_{T^n X} \) underlies an isomorphism between \( \hat{N}(\Sigma_X(C, B_0, \ldots, B_{n-1}), P^2\sigma^n_X(B_n)) \) and \( \tilde{N}(\Sigma_X(C, B_0, \ldots, B_{n-1}), P^2\sigma^n_X(B_n)) \). So, \( \sigma_{T^n X} \circ P^\sigma X \), which is equal to \( \sigma_X^{n+1} \) underlies the desired isomorphism.

For \( n \geq 1 \), by Propositions 5.3, 5.5 and Remark 2.0.2 we have that

\[
\tilde{\Sigma}_X(C, B_0, \ldots, B_n) = \tilde{N}(\Sigma_X(C, B_0, \ldots, B_{n-1}), P^2\sigma^n_X(B_n)).
\]

Proposition 5.3 enables us to simplify a bit the notation—we just codify complicated curly braces notation using +, +^2, +^3, etc., leaving the original nested structure untouched. However, we will show that the necessity of such a nested structure for describing these simplicial complexes is an illusion.

Let again \( C, X, B_0 \) and \( B_1 \) be as in Example 5.1. It is easy to check that \( \Sigma_X(C, B_0, B_1) \) (and hence \( \hat{N}(C, B_0, B_1) \)) is isomorphic to the simplicial complex \( K \) whose bases are \{\( x, 2x + y, z \}, \{x + y, 2x + y, z \}, \{x, 2x + y, u \}, \{x + y, 2x + y, u \}, \{y, x + y, u \}, \{x, z, u \} \) and \{\( y, z, u \)\}. (To obtain \( K \) from \( \Sigma_X(C, B_0, B_1) \), we have just “evaluated” +^2 as + in the elements of its bases.) We show that this will always be the case.

### 6 Flattening the nested structure

For a component \( \mu_X : T^2 X \to TX \) of the multiplication of the monad \( (T, \eta, \mu) \) (cf. Section 4), we define inductively the function \( \mu^n_X : T^{n+1} X \to TX \) in the following way. For \( n = 0 \), let \( \mu^n_X \) be the identity function on \( TX \), and for \( n \geq 1 \), let \( \mu^n_X = \mu_X \circ T \mu^{n-1}_X \). (For \( n \geq 1 \), by the associativity law of the monad, we have that \( \mu^n_X = \mu_X \circ T \mu^n_\mu \circ \cdots \circ T^{n-1}_\mu X \).) For example, \( \mu_X^2(x +^2 (x + y)) = 2x + y \) and \( \mu_X^3(x +^3 (x + (x + y))) = 3x + y \).

For \( \tilde{\Sigma}_X(C, B_0, \ldots, B_n) \) defined in the preceding section, we define

\[
\tilde{S}_X(C, B_0, \ldots, B_n) = \{ P \mu^n_X(\alpha) \mid \alpha \in \tilde{\Sigma}_X(C, B_0, \ldots, B_n) \}.
\]

In our example above, we have that \( K = \tilde{S}_X(C, B_0, B_1) \) for \( C, X, B_0 \) and \( B_1 \) as in Example 5.1. Since the function \( \mu^n_X \) is not one-one when \( X \) is nonempty and \( n \geq 1 \), it is not so obvious that \( \mu^n_X \) underlies an isomorphism between \( \tilde{\Sigma}_X(C, B_0, \ldots, B_n) \) and \( \tilde{S}_X(C, B_0, \ldots, B_n) \). The rest of this section is devoted to a proof of this result.

For \( C \) a simplicial complex such that \( \cup C \subseteq TX \) and \( B \) a building set of \( C \), we formulate some conditions, which guarantee that \( \mu_X \) restricted to
$\bigcup \tilde{\Sigma}_T X(C, B)$ is one-one. That this is not always the case is shown by the following example.

Let $X = \{x, y\}$ and $C = P(\{x, x + y, 2x + y\})$. For a building set $B = \{\{x\}, \{x + y\}, \{2x + y\}, \{x, x + y\}\}$ of $C$, we have that $\bigcup \tilde{\Sigma}_T X(C, B) = \{x, x + y, 2x + y, x + 2(x + y)\}$ and $\mu_X(2x + y) = \mu_X(x + 2(x + y)) = 2x + y$.

Let $f$ be a function, and let $C$ be a simplicial complex such that for every $a \in \bigcup C$ we have that $f(a) \in \mathbb{R}^n$. We say that $f$ faithfully realizes $C$ when for every $\alpha, \beta \in C$, every $\{k_a > 0 \mid a \in \alpha\}$ and every $\{l_b > 0 \mid b \in \beta\}$, we have that

$$\sum_{a \in \alpha} k_a \cdot f(a) = \sum_{b \in \beta} l_b \cdot f(b),$$

implies that $\alpha = \beta$ and $k_a = l_a$ for every $a \in \alpha$.

For $f$ and $C$ as above, let

$$f(C) = \{\text{cone}(\{f(a) \mid a \in \alpha\}) \mid \alpha \in C\}.$$

(See [25] for the notions of cone and simplicial fan.) It is easy to conclude that if $f$ faithfully realizes $C$, then $f(C)$ is a simplicial fan whose face lattice is isomorphic to $C$. On the other hand, if $f(C)$ is a simplicial fan whose face lattice is isomorphic to $C$ via the mapping that extends $\{a\} \to \text{cone}(\{f(a)\})$ for $a \in \bigcup C$, then $f$ faithfully realizes $C$. We can now prove the following.

**Proposition 6.1.** Suppose $f : A \to \mathbb{R}_n$ and $g : B \to \mathbb{R}_n$ faithfully realize $C$ and $D$ respectively, and let $\bigcup g(D) \subseteq \bigcup f(C)$. If $L : \mathbb{R}_n \to \mathbb{R}_n$ is a linear transformation such that $Lf$ faithfully realizes $C$, then $Lg$ faithfully realizes $D$.

**Proof.** Suppose $\gamma, \delta \in D$, $p_c, q_d > 0$ for $c \in \gamma$, $d \in \delta$, and

$$\sum_{c \in \gamma} p_c \cdot Lg(c) = \sum_{d \in \delta} q_d \cdot Lg(d).$$

For $x = \sum_{c \in \gamma} p_c \cdot g(c)$ and $y = \sum_{d \in \delta} q_d \cdot g(d)$, we have $x, y \in \bigcup g(D) \subseteq \bigcup f(C)$. So, there are $\alpha, \beta \in C$, $k_a, l_b > 0$ for $a \in \alpha$ and $b \in \beta$, such that $x = \sum_{a \in \alpha} k_a \cdot f(a)$ and $y = \sum_{b \in \beta} l_b \cdot f(b)$.

From $L(x) = L(y)$ we obtain

$$\sum_{a \in \alpha} k_a \cdot Lf(a) = \sum_{b \in \beta} l_b \cdot Lf(b),$$

and since $Lf$ faithfully realizes $C$, we have $\alpha = \beta$ and $k_a = l_a$ for every $a \in \alpha$. Hence, $x = y$ and since $g$ faithfully realizes $D$ we obtain $\gamma = \delta$ and $p_c = q_c$ for every $c \in \gamma$. \qed
Let $X$ be a finite and nonempty set. We always assume that $X$ is linearly ordered. In our examples, if $X = \{x, y, z, u\}$ then we assume $x < y < z < u$. There is a natural identification of $TX$ with $\mathbb{N}^{|X|} - \{(0, \ldots, 0)\}$ when $X$ is finite and linearly ordered. If $X$ is as above, then we identify $2x$ with $(2, 0, 0, 0)$, $3x + y + 5z + u$ with $(3, 1, 5, 1)$ etc. Let $\kappa_X : TX \to \mathbb{R}^{|X|}$ be defined by this identification.

Lemma 6.2. Let $A, B \subseteq TX$ be such that $\sigma_X(X) \subseteq A \cap B$ and that there are $x, y \in X$ (not necessarily distinct) such that $A - \{\sigma_X(X)\} \subseteq T(X - \{x\})$ and $B - \{\sigma_X(X)\} \subseteq T(X - \{y\})$. If

\begin{equation}
(*) \quad \sum_{a \in A} k_a \cdot \kappa_X(a) = \sum_{b \in B} l_b \cdot \kappa_X(b), \text{ for } k_a, l_b > 0,
\end{equation}

then $A - \{\sigma_X(X)\}, B - \{\sigma_X(X)\} \subseteq T(X - (\{x\} \cup \{y\}))$ and $k_{\sigma_X(X)} = l_{\sigma_X(X)}$.

Proof. By restricting $(*)$ to the $x$th and $y$th coordinate respectively, we obtain $k_{\sigma_X(X)} = l + l_{\sigma_X(X)}$ and $k + k_{\sigma_X(X)} = l_{\sigma_X(X)}$, for $l = \sum_{b \in B - \{\sigma_X(X)\}} l_b \cdot (\kappa_X(b))_x$, where $(\kappa_X(b))_x$ is the $x$th coordinate of $\kappa_X(b)$, and analogously, $k = \sum_{a \in A - \{\sigma_X(X)\}} k_a \cdot (\kappa_X(a))_y$.

From this we infer $l + k = 0$. Since $l, k \geq 0$, we obtain $l = k = 0$, which implies that $k_{\sigma_X(X)} = l_{\sigma_X(X)}$ and $(\kappa_X(b))_x = (\kappa_X(a))_y = 0$, for $b \in B - \{\sigma_X(X)\}$ and $a \in A - \{\sigma_X(X)\}$. So, $x$ and $y$ do not occur in the elements of $B - \{\sigma_X(X)\}$ and $A - \{\sigma_X(X)\}$.

\[
\square
\]

Lemma 6.3. If $C$ is a simplicial complex such that $X = \bigcup C$, and $B$ is a building set of $C$, then $\kappa_X : TX \to \mathbb{R}^{|X|}$ faithfully realizes $C$ and $\tilde{\Sigma}_X(C, B)$. Moreover, we have $\bigcup \kappa_X(\tilde{\Sigma}_X(C, B)) \subseteq \bigcup \kappa_X(C)$.

Proof. It is obvious that $\kappa_X$ faithfully realizes $C$ since the set $\{\kappa_X(a) \mid a \in X\}$ is linearly independent. For the rest of the proof we may rely on comments given in [14] (Section 5) immediately after the definition of $\Sigma(\mathcal{L}, \mathcal{G})$. Here is how we prove these results.

Suppose $(*)$ $\sum_{a \in \alpha} k_a \cdot \kappa_X(a) = \sum_{b \in \beta} l_b \cdot \kappa_X(b)$ for $\alpha, \beta \in \tilde{\Sigma}_X(C, B)$ and $k_a, l_b > 0$. Let $\alpha', \beta' \in \tilde{\mathcal{N}}(C, B)$ be such that $\alpha = P \sigma_X(\alpha')$ and $\beta = P \sigma_X(\beta')$. From $(*)$ it follows that $x \in X$ occurs in some $a \in \alpha$ iff it occurs in some $b \in \beta$. Hence, by Proposition 3.2 we conclude that for

\[
\gamma = \{x \in X \mid x \text{ occurs in some } a \in \alpha\} = \{x \in X \mid x \text{ occurs in some } b \in \beta\}
\]

we have that $\alpha', \beta' \in \tilde{\mathcal{N}}(P(\gamma), B_\gamma)$, and hence $\alpha, \beta \in \tilde{\Sigma}_X(P(\gamma), B_\gamma)$.

To prove that $(*)$ implies that $\alpha = \beta$ and that $k_a = l_a$ for every $a \in \alpha$, we proceed by induction on the cardinality $|\gamma|$ of $\gamma$ by relying on the inductive
definition of a construction given in Section 2.3. (By Proposition 3.3 we have that \( \alpha' \) and \( \beta' \) are subsets of some constructions of \( B_\gamma \).) If \( |\gamma| = 0 \), then we are done.

Suppose \( |\gamma| \geq 1 \) and \( B_\gamma \) is connected, hence \( \gamma \in B_\gamma \). By the choice of \( \gamma \), we have \( \cup \alpha' = \cup \beta' = \gamma \), and since every \( \alpha' \)-antichain and every \( \beta' \)-antichain misses \( B_\gamma \), we must have that \( \gamma \in \alpha' \cap \beta' \), and hence \( \sigma_X(\gamma) \in \alpha \cap \beta \).

By the inductive definition of a construction we have that \( \alpha - \{\sigma_X(\gamma)\} \in \Sigma_X(P(\gamma - \{x\}), B_{\gamma - \{x\}}) \) and \( \beta - \{\sigma_X(\gamma)\} \in \Sigma_X(P(\gamma - \{y\}), B_{\gamma - \{y\}}) \) for some \( x, y \in \gamma \).

From Lemma 6.2 and Proposition 3.2, it follows that

\[
\alpha - \{\sigma_X(\gamma)\}, \beta - \{\sigma_X(\gamma)\} \in \Sigma_X(P(\gamma - (\{x\} \cup \{y\})), B_{\gamma - (\{x\} \cup \{y\}))
\]

and that \( k_{\sigma_X(\gamma)} = l_{\sigma_X(\gamma)} \). Hence, we may cancel \( k_{\sigma_X(\gamma)} \cdot \kappa_X(\sigma_X(\gamma)) \) on both sides of (*) and apply the induction hypothesis to \( \alpha - \{\sigma_X(\gamma)\}, \beta - \{\sigma_X(\gamma)\} \) and \( \gamma - (\{x\} \cup \{y\}) \). (Note that \( |\gamma - (\{x\} \cup \{y\})| \leq |\gamma| - 1 \).

Suppose \( |\gamma| \geq 2 \) and \( B_\gamma \) is not connected. Then for \( \{B_{\gamma_1}, \ldots, B_{\gamma_n}\} \), \( n \geq 2 \), being the finest hypergraph partition of \( B_\gamma \), we have that \( \alpha = \alpha_1 \cup \ldots \cup \alpha_n \) and \( \beta = \beta_1 \cup \ldots \cup \beta_n \), where \( \emptyset \neq \alpha_i, \beta_i \subseteq B_{\gamma_i} \). Since \( \kappa_X(x) \mid x \in \gamma \) is linearly independent, we have that (*) breaks into \( n \) equations and we may apply the induction hypothesis to each \( \alpha_i, \beta_i \) and \( \gamma_i \). This concludes the first part of the proof.

To prove that \( \bigcup \kappa_X(\Sigma_X(C, B)) \subseteq \bigcup \kappa_X(C) \), by Proposition 3.2, it is sufficient to show that for every \( \gamma \in C \), we have

\[
\bigcup \kappa_X(\Sigma_X(P(\gamma), B_\gamma)) \subseteq \text{cone}(\{\kappa_X(x) \mid x \in \gamma\})
\]

which is trivial. Actually, we may prove that

\[
\bigcup \kappa_X(\Sigma_X(P(\gamma), B_\gamma)) = \text{cone}(\{\kappa_X(x) \mid x \in \gamma\})
\]

relying on the inductive definition of a construction. This would deliver that \( \bigcup \kappa_X(\Sigma_X(C, B)) = \bigcup \kappa_X(C) \) but we don’t need this stronger result here.

**Remark 6.4.** If \( \psi \) underlies an isomorphism between \( C \) and \( D \) and \( f \circ \psi \) faithfully realizes \( C \), then \( f \) faithfully realizes \( D \).

Let \( X \) be a finite, nonempty set and let \( C \) be a simplicial complex such that \( \bigcup C \subseteq TX \). Let \( B \) be a building set of \( C \). We define

\[
\hat{S}_X(C, B) = \{P\mu_X(\alpha) \mid \alpha \in \Sigma_{TX}(C, B)\}.
\]

(It is easy to see that if \( \bigcup C \subseteq X \), then \( \hat{S}_X(C, B) = \Sigma_X(C, B) = \hat{S}_X(C, B) \).)

The following proposition shows when \( \mu_X \) underlies an isomorphism between \( \Sigma_{TX}(C, B) \) and \( \hat{S}_X(C, B) \).
Proposition 6.5. If $\kappa_X$ faithfully realizes the simplicial complex $C$, then $\mu_X$ restricted to $\bigcup \Sigma_{TX}(C, B)$ is one-one and $\kappa_X$ faithfully realizes $\hat{S}_X(C, B)$.

Proof. Let $Y = \bigcup C$ and assume that $Y$ is linearly ordered in an arbitrary way. By Lemma 6.3 we have that $\kappa_Y$ faithfully realizes $C$ and $\Sigma_Y(C, B)$. To simplify the notation, we assume that $+^2$ is the formal addition of $TY$. Otherwise, we would carry an extra isomorphism converting $+^2$ into the formal addition of $TY$ at some places of this proof. By this assumption we have that $\Sigma_Y(C, B) = \Sigma_{TX}(C, B)$, and hence $\kappa_Y$ faithfully realizes $\Sigma_{TX}(C, B)$.

Let $L : \mathbb{R}^{|Y|} \to \mathbb{R}^{|X|}$ be a linear transformation defined by $L\kappa_Y(a) = \kappa_X(a)$ for $a \in Y$. Since $\kappa_X$ faithfully realizes $C$, we have that $L\kappa_Y$ faithfully realizes $C$. By Lemma 6.3, it follows that $\bigcup \kappa_Y(\Sigma_{TX}(C, B)) \subseteq \bigcup \kappa_Y(C)$. Hence, the assumptions of Proposition 6.1 are satisfied for $f = g = \kappa_Y$, $C$ and $D = \Sigma_{TX}(C, B)$. So, we conclude that $L\kappa_Y$ faithfully realizes $\Sigma_{TX}(C, B)$.

By the definition of $L$ it follows that $L\kappa_Y(c) = \kappa_X(\mu_X(c))$, for every $c \in TY$, and since $\bigcup \Sigma_{TX}(C, B) \subseteq TY$, this holds for every $c \in \bigcup \Sigma_{TX}(C, B)$. Hence, for $c, d \in \bigcup \Sigma_{TX}(C, B)$, we have that $\mu_X(c) = \mu_X(d)$ implies $L\kappa_Y(c) = L\kappa_Y(d)$, which implies $c = d$ since $L\kappa_Y$ faithfully realizes $\Sigma_{TX}(C, B)$. We conclude that $\mu_X$ restricted to $\bigcup \Sigma_{TX}(C, B)$ is one-one.

By the above we have that $L\kappa_Y = \kappa_X \circ \mu_X$ (restricted to $TY$), that $\mu_X$ underlies an isomorphism between $\Sigma_{TX}(C, B)$ and $\hat{S}_X(C, B)$, and that $L\kappa_Y$ faithfully realizes $\Sigma_{TX}(C, B)$. Hence, by Remark 6.4, $\kappa_X$ faithfully realizes $\hat{S}_X(C, B)$.

The following remark is a corollary of Remark 5.4.

Remark 6.6. If $\psi : X \to Y$ is a function that underlies an isomorphism between simplicial complexes $C$ and $D$, and $B$ is a building set of $C$, then the function $T\psi$ underlies an isomorphism between $\hat{S}_X(C, B)$ and $\Sigma_Y(D, P^2\psi(B))$.

For $C$ a simplicial complex such that $\bigcup C \subseteq X$, and $B_0, \ldots, B_n$ as in the definition of $\hat{S}_X(C, B_0, \ldots, B_n)$ (cf. Section 5), we have the following.

Proposition 6.7. The restriction of $\mu_X^n$ to $\bigcup \hat{S}_X(C, B_0, \ldots, B_n)$ is one-one and $\kappa_X$ faithfully realizes $\hat{S}_X(C, B_0, \ldots, B_n)$.

Proof. We proceed by induction on $n$. In the basis, when $n = 0$, we have that $\mu_X^0$ is the identity function on $TX$, and by Lemma 6.3, $\kappa_X$ faithfully realizes $\hat{S}_X(C, B_0)$ which is equal to $\hat{S}_X(C, B_0)$. For the induction step we use that $\hat{S}_X(C, B_0, \ldots, B_n) = \Sigma_{TX}(\Sigma_X(C, B_0, \ldots, B_{n-1}), P^2\sigma_X^n(B_n))$.
Remark 2.0.1, we have that $\mu^{n-1}_X : T^n X \to TX$ underlies an isomorphism between $\Sigma_X(C, B_0, \ldots, B_{n-1})$ and $C' = \tilde{\Sigma}_X(C, B_0, \ldots, B_{n-1})$. Hence, by Remark 6.6, $T \mu^{n-1}_X$ underlies an isomorphism between $\Sigma_X(C, B_0, \ldots, B_n)$ and $\tilde{\Sigma}_{TX}(C', B)$, for $B = P^2(\mu^{n-1}_X \circ \sigma^n_X)(B_n)$. Since by the induction hypothesis we have that $\kappa_X$ faithfully realizes $C'$, we may apply Proposition 6.5 to conclude that $\mu_X$ restricted to $\bigcup \tilde{\Sigma}_{TX}(C', B)$ is one-one. So, the restriction of $\mu^n_X = \mu_X \circ T \mu^{n-1}_X$ to $\bigcup \tilde{\Sigma}_X(C, B_0, \ldots, B_n)$ is one-one.

By Proposition 6.5 we have that $\kappa_X$ faithfully realizes the simplicial complex $\tilde{\Sigma}_X(C', B)$. By Remark 2.0.1 we have that $\mu^n_X$ underlies an isomorphism between $\Sigma_X(C, B_0, \ldots, B_n)$ and $\tilde{\Sigma}_X(C', B)$, and between $\Sigma_X(C, B_0, \ldots, B_n)$ and $\tilde{\Sigma}_X(C, B_0, \ldots, B_n)$. Hence, by Remark 2.0.2, we have that $\tilde{\Sigma}_X(C', B) = \tilde{\Sigma}_X(C, B_0, \ldots, B_n)$. So, we obtain that $\kappa_X$ faithfully realizes the simplicial complex $\tilde{\Sigma}_X(C, B_0, \ldots, B_n)$.

As an immediate corollary of Propositions 5.3 and 6.7 we have the following for $C$, $X$ and $B_0, \ldots, B_n$ as above.

**Theorem 6.8.** The function $\mu^n_X \circ \sigma^{n+1}_X$ underlies an isomorphism between $\tilde{N}(C, B_0, \ldots, B_n)$ and $\tilde{\Sigma}_X(C, B_0, \ldots, B_n)$.

### 7 An alternative approach

Although Theorem 6.8 provides us with an efficient notation for the result of the iteration of $\tilde{N}$, still we have to calculate first $\tilde{N}(C, B_0, \ldots, B_n)$ in order to obtain $\tilde{\Sigma}_X(C, B_0, \ldots, B_n)$. Also, it is not easy to write down $B_n$ for large $n$, and since we always have that $\{\{\beta\} \mid \beta \in B_i\} \subseteq B_{i+1}$, there is some superfluous information carried in this notation. The aim of this section is to formulate a direct procedure of calculating these simplicial complexes and to give a simple, more economical notation for building sets. We will just formulate here the results without going into the proofs, which are straightforward but tedious.

Let $X$ be a finite and nonempty set. For $\beta \in P_F(TX)$, let $\beta^+ =_{df} \mu_X \sigma_{TX}(\beta)$. For example, if $\beta = \{x, x + y\}$, then $\beta^+ = 2x + y$. For $B \subseteq P_F(TX)$, let $[B]^+ = P(\mu_X \sigma_{TX})(B) = \{\beta^+ \mid \beta \in B\}$.

Let $\alpha \subseteq TX$ be a finite set such that $\kappa_X$ faithfully realizes the simplicial complex $P(\alpha)$. This means that $\{\kappa_X(\alpha) \mid \alpha \in \alpha\}$ is linearly independent. Hence, $\mu_X \sigma_{TX}$ is one-one on $P_F(\alpha)$ and $P(\mu_X \sigma_{TX})$ is one-one on $P(P_F(\alpha))$.

We say that $D \subseteq [P_F(\alpha)]^+ = [P(\alpha) - \{\emptyset\}]^+$ is a flat building set of $P(\alpha)$ when the following conditions hold:
(D1) If $\gamma, \delta \subseteq \alpha$ and $\gamma \cap \delta \neq \emptyset$ and $\gamma^+, \delta^+ \in D$, then $(\gamma \cup \delta)^+ \in D$;

(D2) $D \cap \alpha = \emptyset$.

This means that there is a building set $B$ of $P(\alpha)$ such that $[B]^+ - \alpha = D$. For example, if $\alpha = \{x, x+y, x+y+z\}$, then $\{2x+y\}$ is a flat building set of $P(\alpha)$.

For $D$ a flat building set of $P(\alpha)$ and $\beta \subseteq \alpha$, let $D_\beta = D \cap [P_F(\beta)]^+$ (cf. Section 2.3). If $B$ is a building set of $P(\alpha)$ such that $[B]^+ - \alpha = D$, then, since $\mu_X \sigma_{TX}$ is one-one, we have that $D_\beta = [B_\beta]^+ - \beta$. So, $D_\beta$ is a flat building set of $P(\beta)$. By induction on the cardinality $n \geq 0$ of $D$, we define the set $C_X(\alpha, D) \subseteq P_F(TX)$ of flat constructions of a flat building set $D$ of $P(\alpha)$ as follows

(0) if $n = 0$, i.e. $D = \emptyset$, then $C_X(\alpha, D) = \{\alpha\}$;

(1) if $n \geq 1$ and $\alpha^+ \in D$, then

$$C_X(\alpha, D) = \{\{\alpha^+\} \cup \gamma \mid (\exists x \in \alpha) \gamma \in C_X(\alpha - \{x\}, D_{\alpha - \{x\}})\};$$

(2) if $n \geq 1$ and $\alpha^+ \notin D$, then, for $M = \max\{\beta \mid \beta \subseteq \alpha \& \beta^+ \in D \cup \alpha\}$,

$$C_X(\alpha, D) = \{\gamma_1 \cup \ldots \cup \gamma_n \mid (\gamma_1, \ldots, \gamma_n) \in \prod_{\beta \in M} C_X(\beta, D_\beta)\}.$$

If $\alpha = \{x, x+y, x+y+z\}$ and $D = \{2x+y\}$, then $C_X(\alpha, D) = \{\{x, 2x+y, x+y+z\}, \{x+y, 2x+y, x+y+z\}\}$.

For a building set $B$ of $P(\alpha)$, by comparing the definition of a construction given in Section 2.3 and the definition of a flat construction given above we have the following.

**Proposition 7.1.** $K$ is a construction of $B$ iff $[K]^+$ is a flat construction of $[B]^+ - \alpha$.

Let now $C$ be a simplicial complex such that $\bigcup C \subseteq TX$ and suppose that $\kappa_X$ faithfully realizes $C$. We say that $D \subseteq \bigcup\{[P_F(\alpha)]^+ \mid \alpha \in C\}$ is a flat building set of $C$, when for every $\alpha \in C$ we have that $D_\alpha = D \cap [P_F(\alpha)]^+$ is a flat building set of $P(\alpha)$. This means that there is a building set $B$ of $C$ such that $[B]^+ - \bigcup C = D$. Let $\hat{S}_X(C, D)$ be the simplicial complex based on $\bigcup\{C_X(\alpha, D_\alpha) \mid \alpha$ is a basis of $C\}$.

For $C$ and $D$ as above, let $B$ be a building set of $C$ such that $[B]^+ - \bigcup C = D$. From the definition given after Remark 6.4, we conclude that $\hat{S}_X(C, B) =$
\[\{\mathcal{N}^+ \mid N \in \tilde{\mathcal{N}}(C, B)\}\], and by relying on Proposition 7.1, we have the following.

**Proposition 7.2.** \(\tilde{\mathcal{S}}_\mathcal{X}(C, B) = \tilde{\mathcal{E}}_\mathcal{X}(C, D)\).

From Proposition 6.5 we conclude that \(\kappa_X\) faithfully realizes \(\tilde{\mathcal{E}}_\mathcal{X}(C, D)\). Since \(\bigcup \tilde{\mathcal{E}}_\mathcal{X}(C, D) \subseteq TX\), we may iterate this procedure and for \(n \geq 0\), \(D_0\) a flat building set of \(C\), and \(D_{i+1}\), for \(0 \leq i \leq n\) a flat building set of \(\tilde{\mathcal{E}}_\mathcal{X}(C, D_0, \ldots, D_i)\), we define

\[
\tilde{\mathcal{E}}_\mathcal{X}(C, D_0, \ldots, D_{n+1}) = def \tilde{\mathcal{E}}_\mathcal{X}(\tilde{\mathcal{E}}_\mathcal{X}(C, D_0, \ldots, D_n), D_{n+1}).
\]

Let \(C\) be a simplicial complex such that \(\bigcup C \subseteq X\), and let \(B_0, \ldots, B_n\) be as in the definition of \(\tilde{\mathcal{S}}(C, B_0, \ldots, B_n)\). Let \(G_0 = B_0\), and for \(1 \leq i \leq n\), let \(G_i = P^2(\mu_{X}^{-1} \circ \sigma_{X})(B_i)\). Let \(D_0 = [G_0]^+ - \bigcup C\), and for \(1 \leq i \leq n\), let \(D_i = [G_i]^+ - [G_{i-1}]^+\). By using Proposition 7.2 and results of the preceding section, we have the following.

**Proposition 7.3.** \(\tilde{\mathcal{S}}_\mathcal{X}(C, B_0, \ldots, B_n) = \tilde{\mathcal{E}}_\mathcal{X}(C, D_0, \ldots, D_n)\).

In the table below we give, as an example, the bases of simplicial complexes \(C, \tilde{\mathcal{E}}_\mathcal{X}(C, D_0), \tilde{\mathcal{E}}_\mathcal{X}(C, D_0, D_1)\) and \(\tilde{\mathcal{E}}_\mathcal{X}(C, D_0, D_1, D_2)\), where \(X = \{x, y, z, u\}\), \(C = P(X) - \{X\}\), \(D_0 = \{x + y, x + y + z\}\), \(D_1 = \{2x + y, 2x + y + z, 2x + 2y + z, 3x + 2y + z\}\) and \(D_2 = \{6x + 5y + 3z, 3x + 3y + 2z\}\).

## 8 \(\tilde{\mathcal{S}}_\mathcal{X}\) and combinatorial blowups

Feichtner and Kozlov defined in [11] (Definition 3.1; see also [3], Definition 1.4) the poset \(Bil_\alpha\mathcal{L}\), called the *combinatorial blowup* of a finite-meet semilattice \(\mathcal{L}\) at its element \(\alpha\), in the following way. (Here the authors presumably assumed that \(\alpha\) is not the bottom element of \(\mathcal{L}\).) The set of elements of \(Bil_\alpha\mathcal{L}\) is

\[
\{\gamma \in \mathcal{L} \mid \gamma \not\leq \alpha\} \cup \{(\alpha, \gamma) \mid \gamma \in \mathcal{L} \text{ and } \gamma \not\geq \alpha \text{ and } (\gamma \vee \alpha)_\mathcal{L} \text{ exists}\},
\]

while for \(\beta, \gamma \not\leq \alpha\), the order relation is given by

1. \(\beta > \gamma\) in \(Bil_\alpha\mathcal{L}\) if \(\beta > \gamma\) in \(\mathcal{L}\);
2. \((\alpha, \beta) > (\alpha, \gamma)\) in \(Bil_\alpha\mathcal{L}\) if \(\beta > \gamma\) in \(\mathcal{L}\);
3. \((\alpha, \beta) > (\alpha, \gamma)\) in \(Bil_\alpha\mathcal{L}\) if \(\beta \geq \gamma\) in \(\mathcal{L}\).

Let \(X\) be finite, nonempty set, and let \(C\) be a simplicial complex such that \(Y = \bigcup C \subseteq TX\) and \(\kappa_X\) faithfully realizes \(C\). Note that for every \(\alpha \in C\)
| $C$          | $\mathcal{E}_X(C, D_0)$                      | $\mathcal{E}_X(C, D_0, D_1)$                      | $\mathcal{E}_X(C, D_0, D_1, D_2)$                      |
|-------------|---------------------------------------------|--------------------------------------------------|--------------------------------------------------|
| $\{x, y, z\}$ | $\{x, x + y, 3x + 2y + z\}$                | $\{x + y + z, 3x + 3y + 2z, 6x + 5y + 3z\}$     | $\{x + y + z, 3x + 2y + z, 6x + 5y + 3z\}$     |
|             |                                             | $\{2x + 2y + z, 3x + 3y + 2z, 6x + 5y + 3z\}$     |                                                  |
| $\{x, x + y, x + y + z\}$ | $\{x + y + z, 3x + 2y + z\}$                | $\{x + y + z, 3x + 2y + z, 6x + 5y + 3z\}$     |                                                  |
|             |                                             | $\{2x + 2y + z, 3x + 3y + 2z, 6x + 5y + 3z\}$     |                                                  |
|             |                                             |                                                  |                                                  |
| $\{x, y + z, 2x + y + z\}$ | $\{x + y + z, 3x + 2y + z\}$                | $\{x + y + z, 3x + 2y + z, 6x + 5y + 3z\}$     |                                                  |
|             |                                             | $\{2x + 2y + z, 3x + 3y + 2z, 6x + 5y + 3z\}$     |                                                  |
|             |                                             |                                                  |                                                  |
| $\{x, 2x + y + z\}$ | $\{x, 2x + y + z\}$                        | $\{x, 2x + y + z\}$                             |                                                  |
|             |                                             | $\{x, 2x + y + z\}$                             |                                                  |
|             |                                             | $\{x, 2x + y + z\}$                             |                                                  |
|             |                                             | $\{x, 2x + y + z\}$                             |                                                  |
|             |                                             | $\{x, 2x + y + z\}$                             |                                                  |
|             |                                             | $\{x, 2x + y + z\}$                             |                                                  |
|             |                                             | $\{x, 2x + y + z\}$                             |                                                  |
|             |                                             |                                                  |                                                  |
| $\{x + y + z\}$ | $\{y, x + y + z, 3x + 3y + 2z\}$           |                                                   |                                                  |
|             |                                             | $\{y, x + y + z, 3x + 3y + 2z\}$                 |                                                  |
|             |                                             | $\{y, x + y + z, 3x + 3y + 2z\}$                 |                                                  |
|             |                                             | $\{y, x + y + z, 3x + 3y + 2z\}$                 |                                                  |
| $\{x, z\}$ | $\{x + y + z\}$                            |                                                   |                                                  |
|             |                                             | $\{x + y + z\}$                                  |                                                  |
|             |                                             | $\{x + y + z\}$                                  |                                                  |
|             |                                             | $\{x + y + z\}$                                  |                                                  |
|             |                                             |                                                  |                                                  |
| $\{y, z\}$ | $\{x + y + z\}$                            |                                                   |                                                  |
|             |                                             | $\{x + y + z\}$                                  |                                                  |
|             |                                             | $\{x + y + z\}$                                  |                                                  |
|             |                                             | $\{x + y + z\}$                                  |                                                  |
|             |                                             |                                                  |                                                  |
| $\{x, u\}$ | $\{x, u + y\}$                             |                                                   |                                                  |
|             |                                             | $\{x, u, 2x + y\}$                               |                                                  |
|             |                                             | $\{x, u, 2x + y\}$                               |                                                  |
|             |                                             | $\{x, u, 2x + y\}$                               |                                                  |
|             |                                             | $\{x, u, 2x + y\}$                               |                                                  |
|             |                                             | $\{x, u, 2x + y\}$                               |                                                  |
|             |                                             |                                                  |                                                  |
| $\{x, z\}$ | $\{x, z\}$                                 |                                                   |                                                  |
|             |                                             | $\{x, z\}$                                       |                                                  |
|             |                                             | $\{x, z\}$                                       |                                                  |
|             |                                             | $\{x, z\}$                                       |                                                  |
|             |                                             | $\{x, z\}$                                       |                                                  |
| $\{y, z\}$ | $\{y, z\}$                                 |                                                   |                                                  |
|             |                                             | $\{y, z\}$                                       |                                                  |
|             |                                             | $\{y, z\}$                                       |                                                  |
|             |                                             | $\{y, z\}$                                       |                                                  |
|             |                                             | $\{y, z\}$                                       |                                                  |
|             |                                             | $\{y, z\}$                                       |                                                  |

21
that is neither empty nor a singleton, we have that $\alpha^+ = \mu_X \sigma_{TX}(\alpha) \notin Y$. It is easy to see that for such $C$, we have that $\text{Bl}_\alpha C$, for $\alpha \neq \emptyset$, is isomorphic to the simplicial complex

$$\{ \gamma \in C \mid \alpha \not\subseteq \gamma \} \cup \{ \gamma \cup \{ \alpha^+ \} \mid \alpha \not\subseteq \gamma \text{ and } \alpha \cup \gamma \in C \},$$

which we will also denote by $\text{Bl}_\alpha C$ and call the blowup of $C$ at $\alpha$. Note that if $\alpha$ is a singleton, then $\text{Bl}_\alpha C = C$.

For $X$ and $C$ as above we prove the following proposition whose content may be derived from the proof of Proposition 2.4 of [3].

**Proposition 8.1.** If $B = B \cup \{ \beta \}$ are building sets of $C$, then for $\alpha = [\max B_\beta]^+$ we have

$$\hat{S}_X(C, B \cup \{ \beta \}) = \text{Bl}_\alpha \hat{S}_X(C, B).$$

**Proof.** If the cardinality $| \max B_\beta|$ of $\max B_\beta$ is 1, then since $B$ is a building set of $C$, we have that $\beta \in B$. Hence, $\hat{S}_X(C, B \cup \{ \beta \}) = \hat{S}_X(C, B) = \text{Bl}_\alpha \hat{S}(C, B)$, since $\alpha$ is a singleton. So, we may proceed with the assumption that $| \max B_\beta | \geq 2$, i.e. $\beta \notin B$.

For the proof of $\subseteq$-direction, let $\gamma \in \hat{S}_X(C, B \cup \{ \beta \})$. Note that $\hat{S}_X(C, B \cup \{ \beta \}) = \{ [N]^+ \mid N \in \hat{N}(C, B \cup \{ \beta \}) \}$. According to Proposition 3.2, we have that $\gamma = [N]^+$ for some $N$ that satisfies $N \subseteq B \cup \{ \beta \}$, $\bigcup N \in C$ and every $N$-antichain misses $B \cup \{ \beta \}$. Since $B$ is a building set of $C$, we have that the members of $\max B_\beta$ are mutually disjoint and $\bigcup (\max B_\beta) = \beta$. We may conclude that $\alpha^+ = \beta^+$ and $\alpha \not\subseteq \gamma$ (otherwise, since $\kappa_X$ faithfully realizes $C$, $\max B_\beta$ would be an $N$-antichain that does not miss $B \cup \{ \beta \}$).

Suppose $\beta^+ \notin \gamma$, which is equivalent to $\beta \notin N$ since $\kappa_X$ faithfully realizes $C$. We have that $N \subseteq B$ and every $N$-antichain misses $B$ since it misses $B \cup \{ \beta \}$. By Proposition 3.2, we have that $\gamma \in \hat{S}_X(C, B)$, and since $\alpha \not\subseteq \gamma$, we conclude that $\gamma \in \text{Bl}_\alpha \hat{S}_X(C, B)$.

Suppose $\beta^+ \in \gamma$. We want to show that $\alpha \cup (\gamma - \{ \alpha^+ \}) \in \hat{S}_X(C, B)$. Since $\alpha^+ = \beta^+$, we have that $\gamma - \{ \alpha^+ \} = \gamma - \{ \beta^+ \}$, and hence $\alpha \cup (\gamma - \{ \alpha^+ \}) = [M]^+$ for $M = \max B_\beta \cup (N - \{ \beta \})$. We have that $M \subseteq B$ and $\bigcup M = \bigcup N \in C$. We have to show that every $M$-antichain misses $B$.

Suppose $S$ is an $M$-antichain that does not miss $B$. We have that $S \cap \max B_\beta \neq \emptyset$, since otherwise $S$ would be an $N$-antichain that does not miss $B \cup \{ \beta \}$. Let $\beta' \in S \cap \max B_\beta$ and let $S' = S - B_\beta$.

If $S' = \emptyset$, then $\bigcup S \subseteq \beta$. From $S' \in B$ we conclude that $\beta' \subseteq \bigcup S \in B_\beta$ and hence $\beta' = \bigcup S$, which together with $\beta' \in S$ contradicts the assumption that $S$ is an $M$-antichain.
If $S' \neq \emptyset$, then for every $\gamma \in S'$ we have $\beta \not\subseteq \gamma$. Otherwise, $\beta', \gamma \in S$ and $\beta' \subseteq \gamma$ which contradicts the assumption that $S$ is an $M$-antichain. Since $\gamma \not\subseteq \beta$ holds by the definition of $S'$, we have that $S' \cup \{\beta\}$ is an $N$-antichain. We have that $(\overline{\cup S'}) \cup \beta = (\cup S) \cup \beta \subseteq \cup N \in C$, and from $\emptyset \neq \beta' \subseteq (\cup S) \cap \beta$, $\cup S \in B \cup \{\beta\}$ and $\beta \in B \cup \{\beta\}$, since $B \cup \{\beta\}$ is a building set of $C$, we have that $(\cup S) \cup \beta \in B \cup \{\beta\}$. So, $S' \cup \{\beta\}$ is an $N$-antichain that does not miss $B \cup \{\beta\}$, which is a contradiction.

By Proposition 3.2 we have that $\alpha \cup (\gamma - \{\alpha^+\}) \in \mathcal{S}_X(C, B)$. Since $\alpha \not\subseteq \gamma$, we have that $\alpha \not\subseteq \gamma - \{\alpha^+\}$, and hence $(\gamma - \{\alpha^+\}) \cup \{\alpha^+\} = \gamma \in \text{Bl}_\gamma \mathcal{S}_X(C, B)$.

For the proof of $\exists$-direction, suppose first that $\gamma \in \text{Bl}_\gamma \mathcal{S}_X(C, B)$ is such that $\gamma \in \mathcal{S}_X(C, B)$ and $\alpha \not\subseteq \gamma$. By Proposition 3.2, we have that $\gamma = [N]^+$ for some $N$ that satisfies $N \subseteq B$, $\cup N \in C$ and every $N$-antichain misses $B$.

We show that

\[ (*) \text{ if } S \text{ is an } N\text{-antichain, then } \cup S \neq \beta. \]

Since $\alpha \not\subseteq \gamma$, we have that max $B_\beta \not\subseteq N$. So, there is $\beta' \in \max B_\beta$ such that $\beta' \not\subseteq N$. Suppose $\cup S = \beta$, then since all the elements of $\max B_\beta$ are mutually disjoint and $S \subseteq B_\beta$, there must be a subset $S'$ of $S$ such that $\cup S' = \beta'$. From $\beta' \not\subseteq N$ we conclude that $\beta' \not\subseteq S'$ and hence $|S'| \geq 2$. So, $S'$ is an $N$-antichain that does not miss $B$, which is a contradiction. Hence, $(*)$ holds and this guarantees that every $N$-antichain misses $B \cup \{\beta\}$.

By Proposition 3.2 we have that $\gamma \in \mathcal{S}_X(C, B \cup \{\beta\})$.

Suppose now that $\gamma \cup \{\alpha^+\} \in \text{Bl}_\alpha \mathcal{S}_X(C, B)$, where $\alpha \cup \gamma \in \mathcal{S}_X(C, B)$ and $\alpha \not\subseteq \gamma$. Since $\mathcal{S}_X(C, B)$ is a simplicial complex, we have $\gamma \in \mathcal{S}_X(C, B)$. By Proposition 3.2, we have that $\gamma = [N]^+$ and $\alpha \cup \gamma = [M]^+$ for $N$ and $M$ such that $N, M \subseteq B$, $\cup N, \cup M \in C$ and every $N$-antichain and every $M$-antichain misses $B$.

Let $K = N \cup \{\beta\}$. We have that $\gamma \cup \{\alpha^+\} = [K]^+$ and $K \subseteq B \cup \{\beta\}$ and $\cup K = (\cup N) \cup \beta = (\cup N) \cup \max B_\beta = \cup M \in C$. We have to show that every $K$-antichain misses $B \cup \{\beta\}$.

If $S$ is a $K$-antichain such that $\beta \not\subseteq S$, then $S$ is an $N$-antichain and it misses $B$. We can prove that $\cup S \neq \beta$ as we proved $(*)$ above, and hence $S$ is a $K$-antichain that misses $B \cup \{\beta\}$.

If $S$ is a $K$-antichain such that $\beta \subseteq S$, then $\cup S \neq \beta$ (otherwise, $S$ is not a $K$-antichain). If $\cup S \in B$, then for $S' = (S - \{\beta\}) \cup \max B_\beta$ we have $\cup S' = \cup S \in B$. By eliminating from $S'$ every member properly contained in some other member of $S'$, we obtain an $M$-antichain $S''$ such that $\cup S'' \in B$ (it is easy to see that $|S''| \geq 2$), which is a contradiction. So $S$ is a $K$-antichain that misses $B \cup \{\beta\}$.

Hence, every $K$-antichain misses $B \cup \{\beta\}$, and by Proposition 3.2, we have that $\gamma \cup \{\alpha^+\} \in \mathcal{S}_X(C, B \cup \{\beta\})$. 

23
Let $X$ be finite, nonempty set, and let $C$ be a simplicial complex such that $\emptyset \neq Y = \bigcup C \subseteq TX$ and $\kappa_X$ faithfully realizes $C$. Let $B$ be a building set of $C$. Then we have the following.

**Proposition 8.2.** There exist $\beta_1, \ldots, \beta_m \in B$ such that $\tilde{\mathcal{S}}_X(C, B) = Bl_{\beta_1}(\ldots Bl_{\beta_m}C)$. 

**Proof.** We proceed by induction on the cardinality of $B - B_\perp$, where $B_\perp = \{\{a\} \mid a \in Y\}$. If $B = B_\perp$, then $\tilde{\mathcal{S}}_X(C, B) = C = Bl_{\{a\}}C$ for arbitrary $a \in Y$. Otherwise, let $\beta_1 \in \min(B - B_\perp)$. By (5) of Proposition 3.1 we have that $B - \{\beta_1\}$ is a building set of $C$. By Proposition 8.1, since $[\max B_{\beta_1}]^+ = \beta_1$, we have that $\tilde{\mathcal{S}}_X(C, B) = Bl_{\beta_1} \tilde{\mathcal{S}}_X(C, B - \{\beta_1\})$ and we may apply the induction hypothesis. $\dashv$

For $C$ a simplicial complex such that $\emptyset \neq \bigcup C \subseteq X$, and $B_0, \ldots, B_n$ as in the definition of $\tilde{\Sigma}_X(C, B_0, \ldots, B_n)$ (cf. Section 5), we have the following.

**Proposition 8.3.** There exist $\beta_1, \ldots, \beta_m \in P_F(TX)$ such that $\tilde{\mathcal{S}}_X(C, B_0, \ldots, B_n) = Bl_{\beta_1}(\ldots Bl_{\beta_m}C)$. 

**Proof.** We proceed by induction on $n$. If $n = 0$, then $\tilde{\mathcal{S}}_X(C, B_0) = \tilde{\mathcal{S}}_X(C, B_0)$ and we may apply Proposition 8.2, since $\kappa_X$ faithfully realizes $C$. If $n \geq 1$, then as in the proof of Proposition 6.7, we have that $\tilde{\mathcal{S}}_X(C, B_0, \ldots, B_n) = \tilde{\mathcal{S}}_X(\tilde{\mathcal{S}}_X(C, B_0, \ldots, B_{n-1}), B)$, for $B = P^2(\mu_X^{-1} \cdot \sigma_X^n)(B_n)$, and $\kappa_X$ faithfully realizes $\tilde{\mathcal{S}}_X(C, B_0, \ldots, B_{n-1})$. By the induction hypothesis $\tilde{\mathcal{S}}_X(C, B_0, \ldots, B_{n-1}) = Bl_{\beta_1}(\ldots Bl_{\beta_k}C)$ and it remains just to apply Proposition 8.2 to $\tilde{\mathcal{S}}_X(Bl_{\beta_1}(\ldots Bl_{\beta_k}C), B)$. $\dashv$

## 9 Stellar subdivision and truncation of polytopes

In this section we elaborate two operations on polytopes, one dual to the other, which realize the operation of combinatorial blowup on the face lattices of polytopes. For a general reference to the theory of polytopes, we refer the reader to [17] and [25]. We also try to keep to the notation used in these two books.

According to [18] (see also [19]) a stellar subdivision of a polytope $P$ in a proper face $F$ is a polytope $\text{conv}(P \cup \{x^F\})$ where $x^F$ is a point of the form $y^F - \varepsilon(y^P - y^F)$, where $y^P$ is in the interior of $P$, $y^F$ is in the relative interior
of \( F \), and \( \varepsilon \) is small enough. We use \( \text{st}_F P \) to denote a stellar subdivision of \( P \) in \( F \). For example, if \( P \) is a cube \( A_1B_1C_1D_1A_2B_2C_2D_2 \) given on the left-hand side, then \( \text{st}_{B_1B_2} P \) is illustrated on the right-hand side:

To show that the stellar subdivision is well defined, i.e. that the face lattice of \( \text{st}_F P \) does not depend on the choice of \( x^F \), we need a more precise definition of this notion. For this we rely on some notions introduced in [17].

Let \( P \subseteq \mathbb{R}^d \) be a \( d \)-polytope, \( H \) a hyperplane such that \( H \cap \text{int} P = \emptyset \), and let \( V \in \mathbb{R}^d \). Then \( V \) is beneath, or beyond \( H \) (with respect to \( P \)), provided \( V \) belongs to the open halfspace determined by \( H \) which contains \( \text{int} P \), or does not meet \( P \), respectively. If \( V \in \mathbb{R}^d \) and \( F \) is a facet of the \( d \)-polytope \( P \subseteq \mathbb{R}^d \), then \( V \) is beneath \( F \) or beyond \( F \) provided \( V \) is beneath or beyond \( \text{aff} F \), respectively. A stellar subdivision \( \text{st}_F P \) of a polytope \( P \) in a proper face \( F \) is a polytope \( \text{conv}(P \cup \{x^F\}) \) where \( x^F \) is a point beneath every facet not containing \( F \) and beyond every facet containing \( F \).

Let \( \mathcal{F}(P) \) denote the face lattice of a polytope \( P \) and let \( \mathcal{F}^-(P) \) denote the meet-semilattice obtained from \( \mathcal{F}(P) \) by removing the top element \( P \). The following proposition connects the operation of combinatorial blowup on finite-meet semilattices and the operation of stellar subdivision on polytopes.

**Proposition 9.1.** For a proper face \( F \) of a polytope \( P \) we have that

\[
\mathcal{F}^-(\text{st}_F P) \cong \text{Bl}_F \mathcal{F}^-(P).
\]

**Proof.** Apply Theorem 1 of [17] (Section 5.2).

We define now an operation that is polar (dual) to stellar subdivision, which we will call truncation. This operation is mentioned in [17] under the name “cutting off” and it appears, under different names, in some recent publications (“blow-up” in [2], “shaving construction” in [23], etc.)
Let $P \subseteq \mathbb{R}^d$ be a $d$-polytope, $V \in P$, and let $\pi^+$ be a halfspace. We say that $\pi^+$ is beneath $V$ when $V$ belongs to int $\pi^+$, and we say that $\pi^+$ is beyond $V$ when $V$ does not belong to $\pi^+$.

A truncation $\text{tr}_F P$ of a polytope $P$ in a proper face $F$ is a polytope $P \cap \pi^+$ where $\pi^+$ is a halfspace beneath every vertex not contained in $F$ and beyond every vertex contained in $F$. This defines an operation dual to stellar subdivision and hence we have the following proposition.

**Proposition 9.2.** For a proper face $F$ of a polytope $P$ we have that (up to combinatorial equivalence)

$$(\text{st}_F P)^\Delta = \text{tr}_F P^\Delta.$$ 

For example, if $P$ is an octahedron $ABCDMN$ given on the left-hand side, then $\text{tr}_{AB} ABCDMN$ is illustrated on the right-hand side:

Let $\mathcal{F}_-(P)$ denote the join-semilattice obtained from $\mathcal{F}(P)$ by removing the bottom element $\emptyset$. Then we have the following proposition, which is analogous to Proposition 9.1 (the operation $^\text{op}$ reverses the order of a poset as in Section 2.4).

**Proposition 9.3.** For a proper face $F$ of a polytope $P$ we have that

$$(\mathcal{F}_-(\text{tr}_F P))^{^\text{op}} \cong \text{Bl}_F (\mathcal{F}_-(P))^{^\text{op}}.$$ 

Roughly speaking, a stellar subdivision makes a polytope more simplicial, while truncation makes it more simple. This is essential for the fact that the family of all simplicial $d$-polytopes is dense in the family of all $d$-polytopes (see [17], Section 5.5, Theorem 5), and analogously, that the family of all simple $d$-polytopes is dense in the family of all $d$-polytopes.
10 Stretching the interval

Let $P$ be a simple polytope with $X$ as the set of its facets. Then

$$C_P = \{ G^* \mid G \text{ is a nonempty face of } P \},$$

where $G^* = \{ F \in X \mid G \subseteq F \}$, is a simplicial complex isomorphic to $(\mathcal{F}_-(P))^\text{op}$. (Note that the bases of $C_P$ correspond here to the vertices of $P$.) So, for a simple polytope $P$, our Proposition 9.3 reads

$$C_{\text{tr} F} P \sim = \text{Bl}_{F^*} C_P.$$

By combining this with Proposition 8.3, for $B_0$ a building set of $C_P$, and $B_i$, for $1 \leq i \leq n$ a building set of $\tilde{N}(C_P, B_0, \ldots, B_{i-1})$, we easily obtain that

$$\tilde{S}_X(C_P, B_0, \ldots, B_n) \sim = C_{\text{tr} F_1(\ldots \text{tr} F_m P)},$$

for some $F_1, \ldots, F_m$ such that $F_m$ is a proper face of $P$, and $F_j, 1 \leq j \leq m-1$, is a proper face of $\text{tr} F_{j+1}(\ldots \text{tr} F_m P)$. So, we have the following.

**Proposition 10.1.** For $P$, $B_0, \ldots, B_n$ as above there is a simple polytope $Q$ such that $\tilde{S}_X(C_P, B_0, \ldots, B_n) \sim = C_Q$.

Let $\Delta$ be an $n$-dimensional simplex with $X$ as the set of its facets. If $B_\perp$ is the minimal building set of $C_\Delta = P(X) - \{X\}$, i.e. $B_\perp = \{ \{a\} \mid a \in X \}$, then $\tilde{S}_X(C_\Delta, B_\perp) = C_\Delta$. On the other hand, if $B_\tau$ is the maximal building set of $C_\Delta$, i.e. $B_\tau = C_\Delta - \{\emptyset\}$, then $\tilde{S}_X(C_\Delta, B_\tau) \sim = C_P$, where $P$ is an $n$-dimensional permutohedron.

By varying $B$ over all possible building sets of $C_\Delta$, according to the definition given at the end of Section 2.4, by relying on Proposition 3.4, we obtain the whole interval simplex-permuohedron in dimension $n$. Proposition 9.10 of [10] together with our Proposition 3.4 shows that for every building set $B$ of $C_\Delta$ there is a simple $n$-dimensional polytope $Q$ such that $\tilde{S}_X(C_\Delta, B) \sim = C_Q$. According to [10], $Q$ may be presented explicitly by a finite set of inequalities (halfspaces), easily derived from $B$. However, if one is satisfied with a less explicit construction of $Q$, we suggest just to rely on Proposition 10.1, which we will always do in the sequel.

It should be clear how we can modify now the interval simplex-permuohedron. If we replace $\Delta$ in $\tilde{S}_X(C_\Delta, B)$ by some other simple polytope $P$ with $X$ as the set of its facets, and by varying $B$ over all possible building sets of the simplicial complex $C_P$, then we obtain a new family of simplicial complexes tied to a new interval of simple polytopes. For example, in dimension
the interval simplex-permutohedron is the interval triangle-hexagon (triangle, quadrilateral, pentagon and hexagon). If we replace the triangle by a quadrilateral, then we obtain the interval quadrilateral-octagon.

For $B_{\bot} = \{\{a\} \mid a \in X\}$ being the minimal building set of $C_P$, we have, as before, that $\tilde{S}_X(C_P, B_{\bot}) = C_P$. So, $P$ is the initial polytope of the new interval. On the other hand, for $B_{\top} = C_P - \{\emptyset\}$ being the maximal building set of $C_P$, by Proposition 10.1, there is a simple polytope $Q$ such that $\tilde{S}_X(C_P, B_{\top}) \sim C_Q$. We may call $Q$ a $P$-based permutohedron, and in that case the ordinary permutohedron would be a $\Delta$-based permutohedron. So, $P$-based permutohedron is the terminal polytope of the new interval.

If we want to stretch the interval simplex-permutohedron and all the other intervals obtained by replacing $\Delta$ by some other simple polytope $P$, then it is sufficient to allow the iterated application of $\tilde{S}_X$. As an example, we may consider the following four polytopes, which correspond to the simplicial complexes described in the table of Section 7 ($X = \{x, y, z, u\}, D_0 = \{x + y, x + y + z\}, D_1 = \{2x + y, 2x + y + z, 2x + 2y + z, 3x + 2y + z\}$ and $D_2 = \{6x + 5y + 3z, 3x + 3y + 2z\}$).
In dimension 2, if we start with a triangle, this procedure delivers every polygon, i.e. every simple two-dimensional polytope. However, in dimension 3, if we start with a tetrahedron it is not the case that this procedure delivers every simple three-dimensional polytope. For example, dodecahedron would never correspond to $\tilde{\mathcal{S}}_X(C_P, B)\beta$ unless $P$ is itself a dodecahedron and $B = B_{\perp}$. This is because every truncation of a simple polytope in dimension 3 leaves at least one facet to be a triangle or a quadrilateral.

We have described above how to produce a $P$-based permutohedron for an arbitrary simple polytope $P$. We conclude this paper with an example of a family of polytopes, which we call permutohedron-based associahedra. The polytope $PA_n$ corresponds to $\tilde{\mathcal{S}}_X(C_{\Delta}, B_{\perp}, B)$, where $|X| = n + 1$, $C_{\Delta} = P(X) - \{X\}$, $B_{\perp} = C_{\Delta} - \{\emptyset\}$ and $B$ is

$$\{\{\{a_1, \ldots, a_k\}, \{a_1, \ldots, a_k, a_{k+1}\}, \ldots, \{a_1, \ldots, a_k, a_{k+1}, \ldots, a_l\}\ | a_1, \ldots, a_l \text{ are different elements of } X, \ 1 \leq k \leq \ell \leq n\}.$$  

Note that the bases of $\tilde{\mathcal{S}}_X(C_{\Delta}, B_{\perp})$ are of the form $\alpha = \{a_1, a_1 + a_2, \ldots, a_1 + a_2 + \ldots + a_n\}$, for $a_1, \ldots, a_n$ different elements of $X$ and $n = |X| - 1$. If we denote $a_1 + \ldots + a_k$ by $b_k$ for $1 \leq k \leq n$, then $\alpha = \{b_1, \ldots, b_n\}$ and $(P^2\sigma_X(B))_\alpha$ is of the form $\{\{b_k, b_{k+1}, \ldots, b_l\} | 1 \leq k \leq \ell \leq n\}$.

Sometimes it is much easier to present a building set by a graph whose saturated closure (see [10], Section 4) is this building set. For example,

$$\begin{array}{ccc}x & y & z \end{array}$$

is the graph whose saturated closure is the following building set of $P(\{x, y, z\})$

$$\{\{x\}, \{y\}, \{z\}, \{x, y\}, \{y, z\}, \{x, y, z\}\}.$$

(This building set of $P(\{x, y, z\})$ gives rise to a 2-dimensional associahedron, i.e. pentagon $K_4$.) In this sense, $(P^2\sigma_X(B))_\alpha$ may be presented by the following graph:
and, hence, $\tilde{S}_X(P(\alpha), (P^2\sigma_X(B))_n)$ corresponds to an $(n - 1)$-dimensional associahedron $K_{n+1}$. This means that every vertex of the permutohedron that corresponds to $\tilde{S}_X(C_\Delta, B_{\tau})$ expands into $K_{n+1}$ in the polytope that corresponds to $\tilde{S}_X(C_\Delta, B_\tau, B) = \tilde{S}(\tilde{S}_X(C_\Delta, B_\tau, P^2\sigma_X(B))).$

The bases of $\tilde{S}_X(C_\Delta, B_\tau)$ are in one to one correspondence with the permutations of $X$ in such a way that $a_1 \ldots a_n a_{n+1}$, for $\{a_{n+1}\} = X - \{a_1, \ldots, a_n\}$, is the permutation that corresponds to the above $\alpha$. The bases of $\tilde{S}_X(C_\Delta, B_\tau, B)$ derived from $\alpha$ may be interpreted as terms obtained from $a_1 \ldots a_n a_{n+1}$ by putting $n - 2$ pairs of brackets (the outermost brackets are omitted). This is done in such a way that, for example, if $n = 4$, the basis

$$\{b_1, b_4, b_3 + b_4, b_1 + b_2 + b_3 + b_4\}$$

of $\tilde{S}_X(C_\Delta, B_\tau, B)$, which is derived from $\alpha$, is interpreted as $(a_1 a_2)(a_3 (a_4 a_5))$.

Hence, $PA_n$ has the vertices, as an $n$-dimensional permuto-associahedron $K\Pi_n$ (see [25], Section 9, Example 9.14) labelled by the terms built out of $n + 1$ different letters with the help of one binary operation. Some edges of $PA_n$ correspond to associativity, i.e. they connect two terms such that one is obtained from the other by replacing a subterm of the form $A \cdot (B \cdot C)$ by $(A \cdot B) \cdot C$. The other edges correspond to transpositions of neighbours. These transpositions are such that in a term of the form $A \cdot B$, one may permute the rightmost letter in $A$ with the leftmost letter in $B$. So, they correspond to “most unexpected transposition of neighbours”. Eventually, our family of permutohedron-based associahedra may be taken as an alternative presentation of the symmetric monoidal category freely generated by a set of objects. Here is a picture of the 3-dimensional permutohedron-based associahedron.
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References

[1] C. Berge, Hypergraphs: Combinatorics of Finite Sets, North-Holland, Amsterdam, 1989.

[2] M. Carr, S.L. Devadoss, Coxeter complexes and graph-associahedra, Topology Appl. 153 (2006) 2155-2168.

[3] S.Lj. Ćukić, E. Delucchi, Simplicial shellable spheres via combinatorial blowups, Proc. Amer. Math. Soc. 135 (2007) 2403-2414.

[4] C. De Concini, C. Procesi, Wonderful models of subspace arrangements, Selecta Math. (N.S.) 1 (1995) 459494.

[5] S.L. Devadoss, A realization of graph-associahedra., Discrete Math. 309 (2009) 271-276.

[6] S.L. Devadoss, S. Forcey, Marked tubes and the graph multiplihedron, Algebr. Geom. Topol. 8 (2008) 2084-2108.

[7] K. Došen, Z. Petrič, Associativity as commutativity, J. Symbolic Logic 71 (2006) 217-226 (available at: arXiv).

[8] K. Došen, Z. Petrič, Shuffles and concatenations in constructing of graphs, preprint, 2010 (available at: arXiv).

[9] K. Došen, Z. Petrič, Weak Cat-operads, preprint, 2010 (available at: arXiv).

[10] K. Došen, Z. Petrič, Hypergraph polytopes, Topology Appl. 158 (2011) 1405-1444 (available at: arXiv).

[11] E.M. Feichtner, D.N. Kozlov, Incidence combinatorics of resolutions, Selecta Math. (N.S.) 10 (2004) 37-60.

[12] E.M. Feichtner, I. Müller, On the topology of nested set complexes, Proc. Amer. Math. Soc. 133 (2005) 999-1006.
[13] E.M. Feichtner, B. Sturmfels, Matroid polytopes, nested sets and Bergman fans, Port. Math. (N.S.) 62 (2005) 437-468.

[14] E.M. Feichtner, S. Yuzvinsky, Chow rings of toric varieties defined by atomic lattices, Invent. Math. 155 (2004) 515-536.

[15] S. Forcey, D. Springfield, Geometric combinatorial algebras: Cyclohedron and simplex, J. Algebraic Combin. 32 (2010) 597-627.

[16] E. Gawrilow, M. Joswig, Polymake: a framework for analyzing convex polytopes. Polytopescombinatorics and computation (Oberwolfach, 1997), DMV Sem., 29, Birkhuser, Basel, 2000, 43-73.

[17] B. Grünbaum, Convex Polytopes, second edition, Springer, New York, 2003.

[18] M. Henk, J. Richter-Gebert, G.M. Ziegler, Basic properties of convex polytopes, in J.E. Goodman, J. O’Rourke (Eds), Handbook of Discrete and Computational Geometry, second ed., Chapman & Hall/CRC, 2004, Section 16.

[19] D. Jojić, Extendable shelling, simplicial and toric h-vector of some polytopes, Publ. Inst. Math. (N.S.) 81(95) (2007) 85-93.

[20] J.-L. Loday et al. (Eds), Operads: Proceedings of Renaissance Conferences, Contemp. Math. 202, American Mathematical Society, Providence, 1997.

[21] S. Mac Lane, Categories for the Working Mathematician, expanded second edition, Springer, Berlin, 1998.

[22] A. Postnikov, Permutohedra, associahedra, and beyond, Int. Math. Res. Not. 2009 (2009) 1026-1106.

[23] A. Postnikov, V. Reiner, L. Williams, Faces of generalized permutohedra, Doc. Math. 13 (2008) 207-273.

[24] J.D. Stasheff, From operads to physically inspired theories (Appendix B co-authored with S. Shnider), in [20], pp. 53-81.

[25] G.M. Ziegler, Lectures on Polytopes, Springer, Berlin, 1995.