Game-Theoretic Optimal Portfolios in Continuous Time

Alex Garivaltis†

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Abstract

We consider a two-person trading game in continuous time whereby each player chooses a constant rebalancing rule $b$ that he must adhere to over $[0,t]$. If $V_t(b)$ denotes the final wealth of the rebalancing rule $b$, then Player 1 (the “numerator player”) picks $b$ so as to maximize $E[V_t(b)/V_t(c)]$, while Player 2 (the “denominator player”) picks $c$ so as to minimize it. In the unique Nash equilibrium, both players use the continuous-time Kelly rule $b^* = c^* = \Sigma^{-1}(\mu - r\mathbf{1})$, where $\Sigma$ is the covariance of instantaneous returns per unit time, $\mu$ is the drift vector of the stock market, and $\mathbf{1}$ is a vector of ones. Thus, even over very short intervals of time $[0,t]$, the desire to perform well relative to other traders leads one to adopt the Kelly rule, which is ordinarily derived by maximizing the asymptotic exponential growth rate of wealth. Hence, we find agreement with Bell and Cover’s (1988) result in discrete time.

Keywords: Competitively Optimal Trading, Portfolio Choice, Continuously-Rebalanced Portfolios, Kelly Criterion, Asymptotic Capital Growth, Minimax

JEL Classification Codes: C44, C72, C73, D80, D81, G11
1 Introduction

1.1 Literature Review

Kelly (1956) obtained the eponymous Kelly rule ("Fortune’s Formula," Poundstone 2010) by maximizing the asymptotic growth rate of one’s capital when gambling on repeated horse races where the posted odds diverge from the true win probabilities. Famously (cf. with Thorp 2017), the Kelly rule was employed by card counter Edward O. Thorp to size his bets at the Nevada blackjack tables. Thorp went on to use the same principle (of the log-optimal constant-rebalanced portfolio) in money management on Wall Street. For the general discrete time portfolio problem, the Kelly investor willingly foregoes the tangency portfolio (of maximum Sharpe ratio) in exchange for the highest possible asymptotic capital growth rate. Breiman (1961) showed that a Kelly gambler will almost surely outperform any essentially different strategy (by an exponential factor), and he has the shortest mean waiting time to reach a distant wealth goal.

In a pair articles, Bell and Cover (1980, 1988) proved a short-term optimality property of the discrete time Kelly rule. They show that the Kelly criterion emerges as the solution of a wide class of "investment φ-games" where the goal is for one investor to outperform the other (in the sense of an increasing function φ(●) of the ratio of the two players’ final wealths). Both papers use an artifice whereby, before the game itself, each player is allowed make a “fair randomization” of his initial dollar, by exchanging it for any random variable distributed over [0,∞) whose mean is at most 1.
1.2 Contribution

This paper studies a similar game in continuous time, where each player commits to a rebalancing rule that must be used continuously over the interval $[0, t]$. The unique Nash equilibrium (that constitutes a saddle point of the expected ratio of wealths at $t$) is for both players to use the continuous time Kelly rule. This result, which matches that of Bell and Cover (1988), holds for the general market with $n$ correlated stocks ($i = 1, ..., n$) in geometric Brownian motion. This being done, we show that the continuous time Kelly rule is the basis for the solution of a “continuous time investment $\phi$-game” that is analogous to the discrete time version solved by Bell and Cover.

2 Model

We consider a continuous time trading game between two players. There is a risk-free bond whose price $B_t := e^{rt}$ evolves according to $dB_t = rB_t \, dt$ and a single stock whose price $S_t$ follows the geometric Brownian motion

$$dS_t := S_t(\mu \, dt + \sigma dW_t),$$

where $\mu$ is the drift, $\sigma$ is the volatility, and $W_t$ is a standard Brownian motion. At $t = 0$ each player chooses a constant rebalancing rule $b \in \mathbb{R}$ that he must adhere to for $0 \leq t \leq T$. A rebalancing rule $b$ is a fixed-fraction betting scheme that maintains the fraction $b$ of wealth in the stock and $1 - b$ in the bond at all times. Let $V_t(b)$ denote the wealth at $t$ of a $1$ deposit into the rebalancing rule $b$. At instant $t$, the trader holds $bV_t(b)/S_t$ shares of the stock and $(1 - b)V_t(b)e^{-rt}$ units of the bond. This portfolio will be held over the differential time step $[t, t + dt]$, after which point it
must be rebalanced again. The players are free to use any amount of leverage \((b > 1\) or \(b < 0\)), if desired.

Player 1 (the “numerator player”) chooses the rebalancing rule \(b \in \mathbb{R}\) and Player 2 (the “denominator player”) chooses a rebalancing rule \(c \in \mathbb{R}\). We consider the two-person, zero-sum game with payoff kernel \(\pi(b, c) := \mathbb{E}[V_T(b)/V_T(c)]\). The numerator player seeks to maximize the expected ratio of his final wealth to that of the opponent’s. The denominator player seeks to minimize this quantity.

### 2.1 Payoff Computation

Each player’s wealth follows a geometric Brownian motion

\[
\frac{dV_t(b)}{V_t(b)} = b \frac{dS_t}{S_t} + (1 - b) \frac{dB_t}{B_t} = [r + b(\mu - r)]dt + b\sigma dW_t. \tag{2}
\]

Solving, we obtain

\[
V_t(b) = \exp\{[r + b(\mu - r) - \sigma^2 b^2/2]t + b\sigma W_t\}. \tag{3}
\]

The ratio of final wealths is

\[
\frac{V_t(b)}{V_t(c)} = \exp\{[(\mu - r)(b - c) + (c^2 - b^2)\sigma^2/2]t + (b - c)\sigma W_t\}. \tag{4}
\]

Thus, since the ratio of final wealths is log-normally distributed (cf. Shonkwiler 2013), we have, after simplification,

\[
\mathbb{E}\left[\frac{V_t(b)}{V_t(c)}\right] = \exp\{(\mu - r - \sigma^2 c)(b - c)t\}. \tag{5}
\]
After a monotonic transformation, we may re-write the payoff kernel as

\[ \pi(b, c) := (\mu - r - \sigma^2 c)(b - c), \]  

which is the exponential growth rate of \( \mathbb{E}[V_t(b)/V_t(c)] \).

### 2.2 Equilibrium

Player 1’s best response correspondence is

\[ b^*(c) = \begin{cases} 
+\infty & \text{if } c < (\mu - r)/\sigma^2 \\
\mathbb{R} & \text{if } c = (\mu - r)/\sigma^2 \\
-\infty & \text{if } c > (\mu - r)/\sigma^2 
\end{cases} \]

Player 2’s best response function is

\[ c^*(b) = \frac{1}{2} \left( b + \frac{\mu - r}{\sigma^2} \right). \]

Thus, the unique Nash equilibrium is \( b^* = c^* = (\mu - r)/\sigma^2 \), which happens to be the continuous time Kelly rule (cf. Luenberger 1998). Ordinarily, the Kelly (1956) rule is derived by maximizing the asymptotic continuously-compounded capital growth rate

\[ \text{Growth Rate}(b) := \lim_{t \to \infty} \frac{1}{t} \log V_t(b) = r + (\mu - r)b - \frac{\sigma^2 b^2}{2}. \]

Hence, even over very short intervals of time \([0, t]\), the desire to outperform other traders in the market dictates the use of the Kelly rule \( b^* := (\mu - r)/\sigma^2 \). We have thus derived a short-term optimality property of the continuous time Kelly rule that matches the results obtained by Bell and Cover (1988) in discrete time.
2.3 Several Correlated Stocks

We extend the above result to the general stock market with \( n \) correlated stocks \((i = 1, \ldots, n)\) whose prices \( S_{it} \) follow the geometric Brownian motions (cf. Björk 1998)

\[
dS_{it} := S_{it}(\mu_i \, dt + \sigma_i \, dW_{it}),
\]

where \( \mu := (\mu_1, \ldots, \mu_n)' \) is the drift vector, \( \sigma := (\sigma_1, \ldots, \sigma_n)' \) is the vector of volatilities, and \( \Sigma \) is the covariance matrix of instantaneous returns per unit time, e.g. \( \Sigma_{ij} = \text{Cov}(dS_{it}/S_{it}, dS_{jt}/S_{jt})/dt \). The \( W_{it} \) are correlated standard Brownian motions, with \( \rho_{ij} := \text{Corr}(dW_{it}, dW_{jt}) \) and \( \Sigma_{ij} = \rho_{ij} \sigma_i \sigma_j \). We assume that \( \Sigma \) is invertible. In this context, a rebalancing rule is a vector \( b := (b_1, \ldots, b_n)' \in \mathbb{R}^n \), where the gambler continuously maintains the fixed fraction \( b_i \) of wealth in stock \( i \) at all times. He keeps the fraction \( 1 - \sum_{i=1}^{n} b_i \) of wealth in bonds. As in the univariate case, this permits the freest possible use of leverage, if desired.

Each player’s final wealth \( V_t(b) \) follows the geometric Brownian motion

\[
\frac{dV_t(b)}{V_t(b)} = \sum_{i=1}^{n} b_i \frac{dS_{it}}{S_{it}} + \left(1 - \sum_{i=1}^{n} b_i \right) \frac{dB_t}{B_t} = \left[r + (\mu - r1)'b \right]dt + \sum_{i=1}^{n} b_i \sigma_i dW_{it}.
\]

The solution of this stochastic differential equation is

\[
V_t(b) = \exp \left\{ [r + (\mu - r1)'b - b'\Sigma b/2]t + \sum_{i=1}^{n} b_i \sigma_i W_{it} \right\}.
\]

This can be verified directly by applying Itô’s Lemma for several diffusion processes (cf. Wilmott 2001) to the function \( F(W_1, \ldots, W_n, t) := \exp\{[r + (\mu - r1)'b - b'\Sigma b/2]t + \sum_{i=1}^{n} b_i \sigma_i W_{it} \} \).
\[ \sum_{i=1}^{n} b_i \sigma_i W_i \}. \] The ratio of final wealths is

\[
\frac{V_t(b)}{V_t(c)} = \exp \left\{ (\mu - r \mathbf{1})'(b - c) + (c' \Sigma c - b' \Sigma b)/2 |t + \sum_{i=1}^{n} (b_i - c_i) \sigma_i W_{it} \right\}. \tag{12}
\]

Thus, the ratio of final wealths is log-normally distributed, with

\[
\mathbb{E} \left[ \frac{V_t(b)}{V_t(c)} \right] = \exp \{ (\mu - r \mathbf{1} - \Sigma c)'(b - c) t \}. \tag{13}
\]

After monotonic transformation, we obtain the simplified payoff kernel

\[
\pi(b, c) := (\mu - r \mathbf{1} - \Sigma c)'(b - c). \tag{14}
\]

Player 1’s best response correspondence is

\[
b_i^*(c) = \begin{cases} +\infty & \text{if } (\Sigma c)_i < \mu_i - r \\ \mathbb{R} & \text{if } (\Sigma c)_i = \mu_i - r \\ -\infty & \text{if } (\Sigma c)_i > \mu_i - r, \end{cases}
\]

where \((\Sigma c)_i := \sum_{j=1}^{n} \rho_{ij} \sigma_i \sigma_j c_j\) is the \(i^{th}\) coordinate of the vector \(\Sigma c\). Assuming that \(\Sigma\) is invertible, Player 2’s best response function is

\[
c^*(b) = \frac{1}{2}[b + \Sigma^{-1}(\mu - r \mathbf{1})]. \tag{15}
\]

Intersecting the best responses, we find that the unique Nash equilibrium is \(b^* = c^* = \Sigma^{-1}(\mu - r \mathbf{1})\), which is the multivariate Kelly rule in continuous time. We thus have
the identity
\[
\max_{b \in \mathbb{R}} \min_{c \in \mathbb{R}} \mathbb{E} \left[ \frac{V_t(b)}{V_t(c)} \right] = \min_{c \in \mathbb{R}} \max_{b \in \mathbb{R}} \mathbb{E} \left[ \frac{V_t(b)}{V_t(c)} \right] = 1. \tag{16}
\]

Thus, since the Kelly rule \( b^* \) is Player 1’s maximin strategy, we have \( \mathbb{E}[V_t(b^*)/V_t(c)] \geq 1 \) for all \( c \), and since the Kelly rule \( c^* \) is Player 2’s minimax strategy, we have \( \mathbb{E}[V_t(b)/V_t(c^*)] \leq 1 \) for all \( b \).

### 3 Investment \( \phi \)-Game

Based on the fact that the Kelly rule \( b^* = c^* \) guarantees \( \mathbb{E}[V_t(b^*)/V_t(c)] \geq 1 \) for all \( c \) and \( \mathbb{E}[V_t(b)/V_t(c^*)] \leq 1 \) for all \( b \), we can obtain a general result analogous to that of Bell and Cover (1988). First, we need some definitions.

**Definition 1.** By a “fair randomization” of the initial dollar is meant a random variable \( W \) with support \([0, \infty)\) and \( \mathbb{E}[W] \leq 1 \).

**Definition 2.** For any increasing function \( \phi(\bullet) \), the “primitive \( \phi \)-game,” with value \( v_\phi \), is the two-person, zero-sum game with payoff kernel \( \mathbb{E}[\phi(W_1/W_2)] \), where player 1 chooses a fair randomization \( W_1 \) and player 2 chooses a fair randomization \( W_2 \). The value of the primitive \( \phi \)-game is \( v_\phi := \sup_{W_1} \inf_{W_2} \mathbb{E}[\phi(W_1/W_2)] = \inf_{W_2} \sup_{W_1} \mathbb{E}[\phi(W_1/W_2)] \). The random wealths \( W_1 \) and \( W_2 \) are independent of each other.

**Definition 3.** For any increasing function \( \phi(\bullet) \), the “investment \( \phi \)-game” is the two-person, zero-sum game with payoff kernel \( \mathbb{E}[\phi(W_1V_t(b)/(W_2V_t(c)))] \), where player 1 chooses a rebalancing rule \( b \) and a fair randomization \( W_1 \) of the initial dollar, and player 2 chooses a rebalancing rule \( c \) and a fair randomization \( W_2 \) of his initial dollar. The random wealths \( W_1 \) and \( W_2 \) are independent of all stock prices and independent of each other.
Theorem 1. The investment $\phi$-game has the same value $v_\phi$ as the primitive $\phi$-game. In equilibrium, both players use the continuous-time Kelly rule $b^* := \Sigma^{-1}(\mu - r 1)$, and the players use the same minimax randomizations $(W_1^*, W_2^*)$ that solve the primitive $\phi$-game.

Proof. First, we show that $E[\phi\{W_1^*V_t(b^*)/(W_2V_t(c))\}] \geq v_\phi$ for any fair randomization $W_2$ and any rebalancing rule $c$, where $b^*$ is the Kelly rule. Note that the quantity $W_2V_t(c)/V_t(b^*) \geq 0$ is a fair randomization, since $E[V_t(c)/V_t(b^*)] \leq 1$. The inequality $E[V_t(c)/V_t(b^*)] \leq 1$ follows from direct substitution of $b^* := \Sigma^{-1}(\mu - r 1)$ into the expected wealth ratio. Thus, since $W_1^*$, is Player 1’s minimax solution in the primitive $\phi$-game, we must have $E[\phi\{W_1^*V_t(b^*)/(W_2V_t(c))\}] \geq v_\phi$.

Similarly, we show that $E[\phi\{W_1V_t(b)/(W_2^*V_t(c^*))\}] \leq v_\phi$ for any fair randomization $W_1$ and any rebalancing rule $b$, where $c^*$ is the Kelly rule. Note that the quantity $W_1V_t(b)/V_t(c^*) \geq 0$ is a fair randomization, since $E[V_t(b)/V_t(c^*)] \leq 1$. Thus, since $W_2^*$, is Player 2’s minimax solution of the primitive $\phi$-game, we must have $E[\phi\{W_1V_t(b)/(W_2^*V_t(c^*))\}] \leq v_\phi$.

Thus, we have shown that $(W_1^*, b^*)$ forces the payoff to be $\geq v_\phi$ and $(W_2^*, c^*)$ forces the payoff to be $\leq v_\phi$ when $b^*$ and $c^*$ are equal to the Kelly rule and $(W_1^*, W_2^*)$ are the minimax strategies from the primitive $\phi$-game. This proves the theorem. □

Example 1. As in Bell and Cover (1980), we let $\phi(x) := 1_{[1,\infty)}(x)$ be the indicator function of $[1, \infty)$. This turns the payoff kernel into $Prob\{W_1V_t(b) \geq W_2V_t(c)\}$. The equilibrium amounts to the Kelly rule $b^* = c^*$ and the fair exchange of the initial dollar for a uniform$(0, 2)$ variable. The value of the game is $1/2$. 

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4 Simulation of a Sample Play of the Game

To illustrate, we use the example of “Shannon’s Demon” in continuous time. In Shannon’s classic discrete time example, there is cash (that pays no interest) and a “hot stock” that each period either doubles or gets cut in half in price, each with 50% probability. The continuous time analog is to set $r := 0$ and

$$dS_t := \sigma S_t \left( \frac{\sigma}{2} dt + dW_t \right),$$

with $\sigma := \log 2 \approx 0.693$. The unique pure strategy Nash equilibrium of this game is for both players to use the rebalancing rule $b^* := 0.5$; the players’ best response correspondences are plotted in Figure 1. For the sake of argument, assume that Player 1 behaves correctly, but Player 2 (perhaps confused by the 24% annual drift rate) chooses to put all his money into the stock, and hold.

Player 1’s wealth at $t$ is $\exp(0.06t + 0.3465W_t)$, and Player 2’s wealth at $t$ is $\exp(0.693W_t)$. The expected wealth ratio is $\exp(0.12t)$. In Figure 2 we have simulated a single play of the game, with a horizon of $T := 300$. At time $t$, the probability that Player 1 has more wealth than Player 2 is $N(0.173\sqrt{t})$, where $N(\bullet)$ is the cumulative normal distribution function. At $t := 50$, there is an 89% chance that Player 1 has more wealth. At $t := 100$ this number rises to 96%.

5 The General Stochastic Differential Game

Finally, we show that the restriction to constant rebalancing rules entails no loss of generality. We do this below for the one-stock case; the proof for several stocks is similar. Let $M_1t$ and $M_2t$ be the wealths of the numerator and denominator player, respectively. We now allow the players’ portfolios to depend on the most general
state vector, which is \((S_t, t, M_{1t}, M_{2t})\). Player 1’s trading strategy is now denoted \(b(S, t, M_1, M_2)\), and Player 2’s strategy is \(c(S, t, M_1, M_2)\). We show that in equilibrium, both players still adhere to the constant rebalancing rule \(b(S, t, M_1, M_2) = c(S, t, M_1, M_2) := (\mu - r)/\sigma^2\).

First, assume that the denominator player uses the Kelly rule \(c := (\mu - r)/\sigma^2\). We show that the numerator player’s best response is to use the same control policy. Let \(J(S, t, M_1, M_2)\) be the numerator player’s maximum value function. His HJB equation is

\[
- \frac{\partial J}{\partial t} = \max_{b \in \mathbb{R}} \left\{ \mu S \frac{\partial J}{\partial S} + [r + b(\mu - r)]M_1 \frac{\partial J}{\partial M_1} + [r + c(\mu - r)]M_2 \frac{\partial J}{\partial M_2} ight.
\]

\[
+ \frac{\sigma^2}{2} S^2 \frac{\partial^2 J}{\partial S^2} + \frac{b^2 \sigma^2}{2} M_1^2 \frac{\partial^2 J}{\partial M_1^2} + \frac{c^2 \sigma^2}{2} M_2^2 \frac{\partial^2 J}{\partial M_2^2}
\]

\[
+ b \sigma^2 S M_1 \frac{\partial^2 J}{\partial S \partial M_1} + c \sigma^2 S M_2 \frac{\partial^2 J}{\partial S \partial M_2} + b c \sigma^2 M_1 M_2 \frac{\partial^2 J}{\partial M_1 \partial M_2} \right\}. \quad (18)
\]

The boundary condition is \(J(S, T, M_1, M_2) := M_1/M_2\). We guess that \(J(S, t, M_1, M_2) \equiv \)

Figure 1: The best response correspondences \(b^*(c)\) and \(c^*(b)\) that obtain for the parameter values \(r := 0, \sigma := \log 2,\) and \(\mu := \sigma^2/2\).
Figure 2: Simulation of one play of the game ($b := 0.5$ and $c := 1$), for the parameter values $r := 0$, $\sigma := \log 2$, $\mu := \sigma^2/2$, $T := 300$.

$M_1/M_2$, which obviously satisfies the boundary condition. Under this guess, Player 1’s HJB equation simplifies to

$$\max_{b \in \mathbb{R}} (\mu - r - \sigma^2 c)(b - c) = 0,$$

where $c := (\mu - r)/\sigma^2$. This value of $c$ makes the maximand identically 0, so of course $b^* := c$ is a maximizer. Thus, substitution of $J \equiv M_1/M_2$ has turned the HJB equation into an identity. This proves that the numerator player’s best response to the Kelly rule is to play the Kelly rule himself.

We can repeat the above calculation, this time assuming that the numerator player’s policy is $b(S, t, M_1, M_2) \equiv (\mu - r)/\sigma^2$. Using $J$ again to denote the denominator player’s (minimum) value function, we get the same HJB equation and boundary condition, except that $\max_{b \in \mathbb{R}} \{\bullet\}$ is replaced by $\min_{c \in \mathbb{R}} \{\bullet\}$. We again make the guess
\( J \equiv M_1/M_2 \), which turns Player 2’s HJB equation into the identity

\[
\min_{c \in \mathbb{R}} (\mu - r - \sigma^2 c)(b - c) = 0. \tag{20}
\]

The unique minimizer is \( c = b = (\mu - r)/\sigma^2 \). This completes the proof that the constant control policies \( b(S, t, M_1, M_2) = c(S, t, M_1, M_2) = (\mu - r)/\sigma^2 \) are best responses to each other. The proof for several stocks is similar, except that \((\mu - r - \sigma^2 c)(b - c)\) is replaced by \((\mu - r1 - \Sigma c)'(b - c)\).

6 Conclusion

For the continuous time two-person trading game whereby Player 1 seeks to maximize the expected ratio of his wealth to that of Player 2 (and Player 2 seeks to minimize this ratio), the unique Nash equilibrium is for both players to use the (possibly leveraged) Kelly rebalancing rule \( b^* := \Sigma^{-1}(\mu - r1) \). More generally, we showed that the Kelly rule is the basis for the solution of a “continuous-time investment \( \phi \)-game” that is the analog of the discrete time version solved by Bell and Cover (1980, 1988). For practically any criterion \( \phi\{W_1 V_t(b)/(W_2 V_t(c))\} \) of short-term relative performance, the correct behavior is for both players to use the Kelly rule \( b^* = c^* \) in conjunction with appropriate fair randomizations \((W_1^*, W_2^*)\) of the initial dollar. Thus, the continuous time Kelly rule (which is renowned for its optimal asymptotic growth rate) is desirable even for a trader whose goal is to perform well relative to other traders over very short periods of time.
References

[1] Bell, R.M. and Cover, T.M., 1980. Competitive Optimality of Logarithmic Investment. *Mathematics of Operations Research, 5*(2), pp.161-166.

[2] Bell, R. and Cover, T.M., 1988. Game-Theoretic Optimal Portfolios. *Management Science, 34*(6), pp.724-733.

[3] Björk, T., 1998. *Arbitrage Theory in Continuous Time*. New York: Oxford University Press.

[4] Breiman, L., 1961. Optimal Gambling Systems for Favorable Games. *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, Volume 1: Contributions to the Theory of Statistics*. The Regents of the University of California.

[5] Kelly, J., 1956. A New Interpretation of Information Rate. *Bell Sys. Tech. Journal, 35*, pp.917-926.

[6] Luenberger, D.G., 1998. *Investment Science*. New York: Oxford University Press.

[7] Poundstone, W., 2010. *Fortune’s Formula: The Untold Story of the Scientific Betting System that Beat the Casinos and Wall Street*. New York: Hill and Wang.

[8] Shonkwiler, R.W., 2013. *Finance with Monte Carlo*. Berlin: Springer.

[9] Thorp, E.O., 2017. *A Man for All Markets: From Las Vegas to Wall Street, How I Beat the Dealer and the Market*. New York: Random House.

[10] Wilmott, P., 2001. *Paul Wilmott Introduces Quantitative Finance*. Chichester: John Wiley & Sons.