On the level sets of the Takagi-van der Waerden functions

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May 13, 2014

Abstract

This paper examines the level sets of the continuous but nowhere differentiable functions

\[ f_r(x) = \sum_{n=0}^{\infty} r^{-n} \phi(r^n x), \]

where \( \phi(x) \) is the distance from \( x \) to the nearest integer, and \( r \) is an integer with \( r \geq 2 \). It is shown, by using properties of a symmetric correlated random walk, that almost all level sets of \( f_r \) are finite (with respect to Lebesgue measure on the range of \( f \)), but that for an abscissa \( x \) chosen at random from \([0, 1)\), the level set at level \( y = f_r(x) \) is uncountable almost surely. As a result, the occupation measure of \( f_r \) is singular.

AMS 2000 subject classification: 26A27 (primary), 60G50 (secondary)

Key words and phrases: Takagi function, Van der Waerden function, Nowhere-differentiable function, Level set, Correlated random walk

1 Introduction

The Takagi-van der Waerden functions are defined by

\[ f_r(x) := \sum_{n=0}^{\infty} r^{-n} \phi(r^n x), \quad r = 2, 3, \ldots, \]

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where $\phi(x) = \text{dist}(x, \mathbb{Z})$, the distance from $x$ to the nearest integer. The first two are shown in Figure 1. The case $r = 2$ has been studied extensively in the literature; see [2, 11]. It was introduced in 1903 by Takagi [15], who proved that it is continuous and nowhere differentiable, and $f_2$ is now generally known as the Takagi function. Apparently unaware of Takagi’s note, Van der Waerden [16] in 1930 proved the nondifferentiability of $f_{10}$, using an argument that works for any even $r \geq 4$, but not for odd $r$ or $r = 2$. Van der Waerden’s paper prompted Hildebrandt [9] and De Rham [13] to rediscover the function $f_2$. Billingsley [5] later gave the simplest proof of nondifferentiability for $r = 2$, and it is his argument that is easily extended to all $r$. A short proof is included in Section 2 below.

The main purpose of this note, however, is to prove the following theorem about the level sets of $f_r$, thereby solving Problem 7.6 of [2]. Let

$$L_r(y) := \{x \in [0, 1) : f_r(x) = y\}, \quad y \in \mathbb{R}.$$ 

In what follows, the phrase “almost every” is always meant in the Lebesgue sense, and $\lambda$ denotes Lebesgue measure on $\mathbb{R}$.

**Theorem 1.1.**

(i) For each $r \geq 2$ and for almost every $y$, $L_r(y)$ is finite.

(ii) For each $r \geq 2$ and for almost every $x \in [0, 1)$, $L_r(f_r(x))$ is uncountable.

**Corollary 1.2.** The occupation measure $\mu_{f_r}$ of $f_r$, defined by

$$\mu_{f_r}(A) := \lambda\{x \in [0, 1) : f_r(x) \in A\}$$

for Borel sets $A$, is singular with respect to Lebesgue measure $\lambda$.

**Proof.** Let $A := \{y \in \mathbb{R} : |L_r(y)| < \infty\}$. Then $\lambda(\mathbb{R} \setminus A) = 0$ by part (i) of the theorem, whereas $\mu_{f_r}(A) = \lambda\{x \in [0, 1) : |L_r(f_r(x))| < \infty\} = 0$ by part (ii). \qed
The above results were initially proved by Buczolich \cite{Buczolich} for the case \( r = 2 \) (i.e. the Takagi function); alternative proofs were given recently by the present author \cite{PresentAuthor} and Lagarias and Maddock \cite{LagariasMaddock}. For even \( r \geq 4 \), the proof is a fairly straightforward extension of the original arguments, but for odd \( r \), the proof involves some additional subtleties; we use results about recurrence of correlated random walks.

It is worth noting that in the Baire category sense, the typical level set of \( f_r \) is uncountably infinite. This follows since \( f_r \), being nowhere differentiable, is certainly monotone on no interval, and a general theorem of Garg \cite{Garg, Theorem 1} states that for any continuous function \( f \) which is monotone on no interval, the set \( \{ y : f^{-1}(y) \text{ is a perfect set} \} \) is residual in the range of \( f \).

The functions \( f_r \) were studied previously by Baba \cite{Baba}, who computed the maxima \( M_r := \max \{ f_r(x) : 0 \leq x \leq 1 \} \) and showed that the set \( \{ x \in [0, 1] : f_r(x) = M_r \} \) is a Cantor set of dimension 1/2 when \( r \) is even, but is the singleton \( \{ 1/2 \} \) when \( r \) is odd. This suggests that the even and odd cases are fundamentally different, an issue that will be encountered again in the proof of Theorem 1.1(i) below.

Finally, it was shown by De Amo et al. \cite{DeAmo} that the Hausdorff dimension of any level set of \( f_2 \) is at most 1/2. It is straightforward to extend this result to \( f_r \) for even \( r \), but whether it is true for odd \( r \) as well remains unclear, though it is relatively easy to see that \( f_r \) always has level sets of dimension at least 1/2.

The following notation is used throughout this paper: \( \mathbb{N} \) denotes the set of positive integers, and \( \mathbb{Z}_+ \) the set of nonnegative integers. For a set \( E \), \( |E| \) denotes the number of elements of \( E \).

\section{Proof of nondifferentiability}

For completeness, we first give a short proof of the nondifferentiability of \( f_r \).

\textbf{Theorem 2.1.} For each \( r \geq 2 \), \( f_r \) has a well-defined finite derivative at no point.

\textbf{Proof.} Following Billingsley \cite{Billingsley}, let \( \phi_k(x) := r^{-k}\phi(r^k x), k \in \mathbb{Z}_+ \), and let \( \phi_k^+(x) \) denote the right-hand derivative of \( \phi_k \) at \( x \). Then \( \phi_k^+ \) is well defined and \( \{-1, 1\}-\)valued. Fix \( x \in [0, 1) \), and for each \( n \in \mathbb{N} \), let \( u_n = j_n/2r^n \) and \( v_n = (j_n + 1)/2r^n \), where \( j_n \in \mathbb{Z}_+ \) is chosen so that \( u_n \leq x < v_n \). Note that for \( k \leq n \), \( \phi_k \) is linear on \( [u_n, v_n] \), so

\[ \frac{\phi_k(v_n) - \phi_k(u_n)}{v_n - u_n} = \phi_k^+(x), \quad 0 \leq k \leq n. \]

Suppose first that \( r \) is even; then \( \phi_k(v_n) - \phi_k(u_n) = 0 \) for \( k > n \), and so

\[ m_n := \frac{f_r(v_n) - f_r(u_n)}{v_n - u_n} = \sum_{k=0}^{\infty} \frac{\phi_k(v_n) - \phi_k(u_n)}{v_n - u_n} = \sum_{k=0}^{n} \phi_k^+(x). \]

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This implies $m_{n+1} - m_n = \pm 1$, and so $m_n$ cannot have a finite limit. Suppose next that $r$ is odd. Then for $k \geq n$,

$$\phi_k(v_n) - \phi_k(u_n) = \frac{(-1)^{j_n}}{2^r},$$

using the 1-periodicity of $\phi$. Thus, since $v_n - u_n = 1/2r^n$,

$$m_n := \frac{f_r(v_n) - f_r(u_n)}{v_n - u_n} = \sum_{k=0}^{n-1} \phi_k^+(x) + (-1)^{j_n} \sum_{k=n}^\infty \frac{1}{r^k} = \sum_{k=0}^{n-1} \phi_k^+(x) + (-1)^{j_n} \frac{r}{r-1}. $$

It is now clear that $m_{n+1} - m_n$ can take only finitely many values, zero not among them, and hence $m_n$ cannot have a finite limit as $n \to \infty$. Therefore, $f_r$ does not have a finite derivative at $x$.

**Remark 2.2.** The above argument is essentially the same as that given in the senior thesis of K. Spurrier [14], where a slightly more general result is proved. It is presented here only in a more concise form.

### 3 The correlated random walk

The proof of the main theorem is based on the fact that the slopes of the partial sums of the series (1) follow a correlated random walk when $x$ is chosen at random from $[0, 1)$. This is made precise below. For $n \in \mathbb{Z}_+$, define the partial sum

$$f_r^n(x) := \sum_{k=0}^{n-1} r^{-k} \phi(r^kx),$$

and let $s_n(x)$ denote the right-hand derivative of $f_r^n$ at $x$, for $x \in [0, 1)$.

**Definition 3.1.** A (symmetric) **correlated random walk** with parameter $p$ is a stochastic process $\{S_n\}_{n \in \mathbb{Z}_+}$ defined on a probability space $(\Omega, \mathcal{F}, P)$ such that $S_0 \equiv 0$ and $S_n = S_{n-1} + X_n$ for $n \geq 1$, where $X_1, X_2, \ldots$ are $\{-1, 1\}$-valued random variables satisfying $P(X_1 = 1) = P(X_1 = -1) = 1/2$, and

$$P(X_{n+1} = 1|X_n = 1) = P(X_{n+1} = -1|X_n = -1) = p, \quad n \in \mathbb{N}. $$

Observe that when $p = 1/2$, the correlated random walk is just a symmetric simple random walk. Correlated random walks have been used to model a wide variety of phenomena, and there is a substantial body of literature about them; a comprehensive list of references can be found in [7].
Lemma 3.2. Viewed as a stochastic process on the space $(\Omega, \mathcal{F}, P) = ([0, 1), \text{Borels, } \lambda)$, the sequence $\{s_n(x)\}_{n \in \mathbb{Z}^+}$ is a correlated random walk with parameter

$$p_r = \begin{cases} 
1/2, & \text{if } r \text{ is even} \\
\frac{r+1}{2r}, & \text{if } r \text{ is odd.}
\end{cases} \quad (2)$$

Proof. With the notation from the proof of Theorem 2.1, we have

$$s_{n+1}(x) = s_n(x) + \phi^+_n(x), \quad n \in \mathbb{Z}^+.$$ 

Note that $\phi^+_n(x) = 1$ if and only if $0 \leq r^n x \mod 1 < 1/2$, which is the case if and only if

$$2x \in \bigcup_{k \in \mathbb{Z}} \left[ \frac{2k}{r^n}, \frac{2k+1}{r^n} \right).$$

For fixed $k$, the interval $[2k/r^n, (2k + 1)/r^n)$ has length $r^{-n}$ and partitions into the $r$ subintervals

$$\left[ \frac{2kr}{r^n+1}, \frac{2kr+1}{r^n+1} \right), \left[ \frac{2kr+1}{r^n+1}, \frac{2kr+2}{r^n+1} \right), \ldots, \left[ \frac{(2k+1)r-1}{r^n+1}, \frac{(2k+1)r}{r^n+1} \right).$$

If $r$ is even, half of these are of the form $[2j/r^{n+1}, (2j+1)/r^{n+1})$ with $j \in \mathbb{Z}$; if $r$ is odd, $(r+1)/2$ of them are of this form. Thus, if $x$ is chosen at random from $[0, 1)$, we have

$$P(\phi^+_{n+1}(x) = 1|\phi^+_n(x) = 1) = p_r. \quad (3)$$

The equality

$$P(\phi^+_{n+1}(x) = -1|\phi^+_n(x) = -1) = p_r \quad (4)$$

can be established by a very similar argument. But it also follows from (3) and symmetry. More precisely, since $\phi_n$ is symmetric about $x = 1/2$, we have that

$$x \not\in \left\{ \frac{j}{2r^n} : j \in \mathbb{Z} \right\} \quad \Rightarrow \quad \phi^+_n(x) = -\phi^+_n(1-x). \quad (5)$$

Since the mapping $x \mapsto 1-x$ preserves Lebesgue measure and the set $\left\{ j/2r^n : j \in \mathbb{Z} \right\}$ is Lebesgue null for each $n$, (3) and (5) together imply (4).

4 Level sets: Proof of the main theorem

For $n \in \mathbb{N}$ and $j = 0, 1, \ldots, 2r^{n-1} - 1$, let $I_{n,j}$ denote the interval

$$I_{n,j} := \left[ \frac{j}{2r^{n-1}}, \frac{j+1}{2r^{n-1}} \right).$$
and let \( s_{n,j} \) denote the slope of \( f_r^n \) on \( I_{n,j} \). In other words, \( s_{n,j} = s_n(x) \) if \( x \in I_{n,j} \). Note that \( s_{n,j} \) is well defined since \( f_r^n \) is linear on \( I_{n,j} \). For \( x \in [0, 1) \) and \( n \in \mathbb{N} \), let \( I_n(x) \) denote that interval \( I_{n,j} \) which contains \( x \).

A fundamental difference between the case of even \( r \) and the case of odd \( r \) is the following: \( f_r(j/2^{n-1}) = f_r^n(j/2^{n-1}) \) for all \( j \in \mathbb{Z} \) when \( r \) is even, but this last equality holds only for even \( j \) when \( r \) is odd. Hence, \( f_r \) corresponds with \( f_r^n \) at both endpoints of \( I_{n,j} \) in the case of even \( r \), but at only one of the endpoints in the case of odd \( r \). In fact, when \( r \) and \( j \) are both odd, \( f_r(j/2^{n-1}) > f_r^n(j/2^{n-1}) \) for every \( k \geq 0 \), so the value of \( f_r \) at \( j/2^{n-1} \) does not get “fixed” after a finite number of steps in the construction of \( f_r \). This is the main reason why understanding the level sets of \( f_r \) is harder when \( r \) is odd.

Let

\[
G_l := \{(x, f_r(x)) : 0 \leq x \leq 1/2\},
\]

so \( G_l \) is the “left half” of the graph of \( f_r \) over \([0, 1]\). Our first goal is to prove a kind of self-similarity result, namely that the graph of \( f_r \) above any interval \( I_{n,j} \) with \( s_{n,j} = 0 \) consists of \( r \) nonoverlapping similar copies of \( G_l \), scaled by \( r^{-n} \) and with alternating orientations (see Corollary 4.2 below). This will follow from an analogous statement about the partial sums \( f_r^n \) in Lemma 4.1 below. Observe that an \( n \)th level interval \( I_{n,j} \) decomposes into \((n + 1)\)th level intervals as

\[
I_{n,j} = \bigcup_{i=0}^{r-1} I_{n+1,rj+i}.
\]

**Lemma 4.1.** If \( s_{n,j} = 0 \), then for each \( m > n \), for \( i = 0, 1, \ldots, r - 1 \) and for \( x \in \bar{I}_{n+1,rj+i} \) (where \( \bar{I} \) denotes the closure of \( I \)),

\[
f_r^m(x) - f_r^n \left( \frac{j}{2^{n-1}} \right) = \begin{cases} r^{-n} f_r^{m-n}(x'), & \text{if } rj + i \text{ is even} \\ r^{-n} f_r^{m-n} \left( \frac{1}{2} - x' \right), & \text{if } rj + i \text{ is odd} \end{cases}
\]

where \( x' \) is the point in \([0, 1/2]\) such that

\[
x = \frac{rj + i}{2^{n+1}} + x'.
\]

**Proof.** If \( s_{n,j} = 0 \), then \( f_r^n \) is constant on \( I_{n,j} \), and so

\[
f_r^m(x) - f_r^n \left( \frac{j}{2^{n-1}} \right) = f_r^m(x) - f_r^n(x) = \sum_{k=n}^{m-1} r^{-k} \phi(r^k x).
\]

If \( rj + i \) is even, then \( k \geq n \) implies, by the 1-periodicity of \( \phi \),

\[
\phi(r^k x) = \phi \left( r^{k-n} \left( \frac{rj + i}{2} + x' \right) \right) = \phi(r^{k-n} x'),
\]
so that
\[ \sum_{k=n}^{m-1} r^{-k} \phi(r^k x) = \sum_{k=n}^{m-1} r^{-k} \phi(r^{k-n} x') = r^{-n} f_r^{m-n}(x'). \]

Assume then that \( rj + i \) is odd. If \( r \) is odd as well, then the symmetry of \( \phi \) gives, for all \( k \geq n \),
\[ \phi \left( r^{k-n} \left( \frac{rj + i}{2} + x' \right) \right) = \phi \left( \frac{1}{2} + r^{k-n} x' \right) = \phi \left( \frac{1}{2} - r^{k-n} x' \right) = \phi \left( r^{k-n} \left( \frac{1}{2} - x' \right) \right), \]
and so
\[ \sum_{k=n}^{m-1} r^{-k} \phi(r^k x) = r^{-n} f_r^{m-n} \left( \frac{1}{2} - x' \right). \]

On the other hand, if \( r \) is even, then \( r^{k-n}/2 \in \mathbb{Z} \) for all \( k > n \), and we get (with the convention that the empty sum takes the value zero):
\[ \sum_{k=n}^{m-1} r^{-k} \phi(r^k x) = r^{-n} f_r^{m-n} \left( \frac{1}{2} - x' \right). \]

This completes the proof.

Taking limits as \( m \to \infty \) in (6), we immediately obtain:

**Corollary 4.2.** If \( s_{n,j} = 0 \), then for \( i = 0, 1, \ldots, r - 1 \) and \( x \in \bar{I}_{n+1,rj+i} \),
\[ f_r(x) - f_r^n \left( \frac{j}{2r^{n-1}} \right) = \begin{cases} r^{-n} f_r(x'), & \text{if } rj + i \text{ is even} \\ r^{-n} f_r \left( \frac{1}{2} - x' \right), & \text{if } rj + i \text{ is odd} \end{cases} \]
where \( x' \) is defined by (7).

**Proof of Theorem 1.1(ii).** Let
\[ \Omega_0 := \{ x \in [0, 1) : s_n(x) = 0 \text{ for infinitely many } n \}. \]

We claim that for each \( x \in \Omega_0 \), the level set \( L_r(f_r(x)) \) is uncountably infinite. Since the symmetric correlated random walk is recurrent (see, for instance, Example 1 on
To prove the claim, the following notation is helpful. Let \( \Sigma := \{1, 2, \ldots, r\} \), let \( \Sigma^* := \bigcup_{k=0}^{\infty} \Sigma^k \) where \( \Sigma^0 := \{\emptyset\} \), and denote by \( \Sigma^N \) the set of all infinite sequences \((i_1, i_2, \ldots)\) such that \( i_k \in \Sigma \) for all \( k \in \mathbb{N} \). Let \( x \in \Omega_0 \), and let \( \{n_k : k = 0, 1, \ldots\} \) be a strictly increasing sequence of positive integers such that \( s_{n_k}(x) = 0 \) for all \( k \).

Repeated application of Lemma 4.1 shows that there is an \( r \)-ary tree of intervals \( I_i, i \in \Sigma^* \) with the following properties:

1. \( I_{\emptyset} = I_{n_0}(x) \);
2. if \( i, j \in \Sigma^k \) with \( i \neq j \), then \( I_i \cap I_j = \emptyset \);
3. for each \( i \in \Sigma^k \), \( f_{\sigma_i}^{n_k} \) is constant on \( I_i \) with value \( y_k := f_{\sigma_i}^{n_k}(x) \);
4. for each \( i \in \Sigma^k \), there is \( j \in \mathbb{Z} \) such that \( I_i = I_{n_k,j} \);
5. if \( i = (i_1, \ldots, i_k) \in \Sigma^k \) and \( j = (i_1, \ldots, i_k, j) \) for any \( j \in \Sigma \), then \( I_j \subset I_i \).

For any sequence \( i = (i_1, i_2, \ldots) \in \Sigma^N \), the intersection \( \bigcap_{k=1}^{\infty} I_{i_1, \ldots, i_k} \) is a single point \( x_i \), and if \( i \) and \( j \) are different members of \( \Sigma^N \), then \( x_i \neq x_j \). Moreover, for each \( i \in \Sigma^N \), \( f_i(x) = \lim_{k \to \infty} y_k = f_i(x) \). Thus, \( L_r(f_i(x)) \) is uncountable, and the claim is established.

For the Takagi function (i.e. the case \( r = 2 \)), the sequence \( \{s_n(x)\}_n \) uniquely determines \( x \), and the sequence \( \{|s_n(x)|\}_n \) determines \( f_2(x) \) (see, for instance, [1, Lemma 2.1]). However, neither statement holds when \( r \geq 3 \). Instead of the simple Lemma 2.1 of [1], we need the carefully constructed mapping \( \rho \) in Lemma 4.3 below. This requires some additional notation. Let

\[
C := \left\{ \frac{j}{2^{r^n}} : n \in \mathbb{Z}_+, j = 0, 1, \ldots, 2^{r^n} - 1 \right\},
\]

and

\[
D := [0, 1) \setminus C,
\]

so \( C \) is the set of “corner” points of the partial sums \( f_r^n \), and \( D \), its complement, is the set of points at which each partial sum \( f_r^n \) has a well-defined two-sided derivative. For \( x \in [0, 1) \), define

\[
n_+(x) := \inf\{n : s_n(x) < 0\} - 1 = \sup\{n : s_1(x) \geq 0, \ldots, s_n(x) \geq 0\}.
\]

Let

\[
D_{fin} := \{x \in D : s_n(x) = 0 \text{ for only finitely many } n\},
\]

\[
D^+_{fin} := \{x \in D_{fin} : s_n(x) \geq 0 \text{ for every } n\}.
\]
Lemma 4.3. There is a mapping $\rho : D \to D$ such that:

(i) $f_r(\rho(x)) = f_r(x)$ for all $x$;

(ii) $|s_n(\rho(x))| = |s_n(x)|$ for all $n$ and all $x$;

(iii) $\rho(x) = x$ if and only if $s_n(x) \geq 0$ for all $n$;

(iv) if $s_n(x) < 0$ for some $n$, then $n_+(\rho(x)) > n_+(x)$;

(v) $\rho(D_{\text{fin}}) \subset D_{\text{fin}}$;

(vi) the restriction of $\rho$ to $D_{\text{fin}}$ is finite-to-one;

(vii) the limit $\rho^\infty(x) := \lim_{n \to \infty} \rho^n(x)$ exists for every $x$, and $s_n(\rho^\infty(x)) \geq 0$ for all $n$ and all $x$; here $\rho^n$ denotes $n$-fold iteration of $\rho$.

Proof. We first construct $\rho$. If $s_n(x) \geq 0$ for every $n$, set $\rho(x) := x$. Properties (i)-(iv) are trivially satisfied for such a point $x$. Assume now that $s_n(x) < 0$ for some $n$; then $n_0 := n_+(x) < \infty$, and $s_{n_0}(x) = 0$. If $n_0 = 0$, then $s_1(x) < 0$ and so $x \in (1/2, 1)$, since $x \not\in C$; in this case we put $\rho(x) := 1 - x$. Properties (i)-(iv) are easily checked in this case as well.

Suppose then that $n_0 \geq 1$. Let $j_0$ be the integer such that $I_{n_0}(x) = I_{n_0,j_0}$. If $x \in I_{n_0+1,r_{j_0}}$, put

$$\rho(x) := \frac{r_{j_0} + 1}{r_{n_0}} - x,$$

the reflection of $x$ across the right endpoint of $I_{n_0+1}(x)$. Otherwise, $x \in I_{n_0+1,r_{j_0}+l}$ for some $l \in \{1, \ldots, r - 1\}$, and we put

$$\rho(x) := \frac{r_{j_0} + l}{r_{n_0}} - x,$$

the reflection of $x$ across the left endpoint of $I_{n_0+1}(x)$.

The construction of $\rho(x)$ is illustrated in Figure 2 in which $x \in I_{n_0+1,r_{j_0}}$, whereas $x' \in I_{n_0+1,r_{j_0}+2}$ and $x'' \in I_{n_0+1,r_{j_0}+4}$. Note that in all cases when $n_0 < \infty$, we have that $\rho(x) \in I_{n_0}(x)$ and $s_{n_0+1}(\rho(x)) > 0$, which gives property (iv). Since $x \not\in C$, it is clear that $\rho(x) \neq x$, so (iii) holds. Property (i) follows by Corollary 4.2 and property (ii) follows by Lemma 11.

The construction of $\rho$ is complete, and it remains to verify properties (v), (vi) and (vii). Note that (v) follows immediately from (ii). To see (vi), let $z \in D_{\text{fin}}$, and let $n(z) := \max\{n \geq 0 : s_n(z) = 0\}$. Suppose $x \in D_{\text{fin}}$ with $\rho(x) = z$, and assume $x \neq z$. Then $n_+(x) \leq n(z)$, so there are only finitely many possible values for the number $n_0$ in the construction of $\rho(x)$ above. If $n_0 = 0$, there is only one $x$ with
n_+ (x) = n_0 such that \( \rho (x) = z \), since \( \rho (x) = 1 - x \) in this case. If \( n_0 \geq 1 \), then the set \( \{ x : n_+ (x) = n_0 \text{ and } \rho (x) = z \} \) can consist of two points (one reflected from the left of \( z \), the other reflected from the right; see Figure 2), but no more. Thus, \( \rho \) is finite-to-one on \( D_{\text{fin}} \).

Property (vii) is obvious if there exists \( k \) such that \( s_n (\rho^k (x)) \geq 0 \) for all \( n \). If there is no such \( k \), then \( n_+ (\rho^k (x)) \) is strictly increasing in \( k \) by (iv), and since \( \rho (x) \in I_{n_+ (x)} (x) \) for every \( x \), the intervals

\[
I_{n_+ (\rho^k (x))} (\rho^k (x)) , \quad k \in \mathbb{N}
\]

are nested and shrink to a single point. This point must be the limit of \( \rho^k (x) \).

**Corollary 4.4.** There is a finite-to-one mapping \( \pi : D_{\text{fin}} \to D_{\text{fin}}^+ \) such that \( f_r (\pi (x)) = f_r (x) \) for every \( x \in D_{\text{fin}} \).

**Proof.** Let \( \rho \) be as in Lemma 4.3. For \( x \in D_{\text{fin}} \), there is by properties (ii)-(v) an integer \( k \) such that \( \rho^{k+1} (x) = \rho^k (x) \), and we put \( \pi (x) := \rho^k (x) \). By property (i), \( f_r (\pi (x)) = f_r (x) \). Now fix \( z \in D_{\text{fin}}^+ \), and let \( m (z) := \# \{ n \geq 0 : s_n (z) = 0 \} \). If \( \pi (x) = z \), then \( z = \rho^k (x) \) for some \( k \), and by property (iv) this \( k \) can be taken to be no greater than \( m (z) \). Since \( \rho \) is finite-to-one on \( D_{\text{fin}} \), so is \( \rho^k \) for each \( k \). Thus, there are at most finitely points \( x \) such that \( \pi (x) = z \), so \( \pi \) is finite-to-one. \( \square \)
**Corollary 4.5.** For each $x \in D$, there is a point $x' \in D$ such that $s_n(x') = |s_n(x)|$ for every $n$, and $f_r(x') = f_r(x)$.

**Proof.** Take $x' := \rho^\infty(x)$. The result follows from properties (i), (ii) and (vii) of Lemma 4.3 and the continuity of $f_r$. \qed

The following lemma underlines the additional subtleties that need to be addressed in case $r$ is odd. Let

$$D^* := \{ x \in D : s_n(x) > 0 \text{ for every } n \geq 1 \}.$$ (10)

For two points $x$ and $x'$ with $x < x'$, say an interval $I = [a, b)$ separates $x$ and $x'$ if $x < a < b < x'$.

**Lemma 4.6.** Let $x, x' \in D^*$ with $x < x'$, and suppose that $f_r(x) = f_r(x')$. Then there exists a pair of integers $(n, j)$ such that (i) $I_{n,j}$ separates $x$ and $x'$; (ii) $s_{n,j} = 0$; and (iii) $f_r(x) \in f_r(I_{n,j})$.

**Proof.** There clearly exists a smallest $n$ for which there exists $j$ such that $I_{n,j}$ separates $x$ and $x'$. Fix this $n$ and such a $j$, and let $s := s_{n,j}$. Observe that $[x, x'] \subset I_{n-1}(x) \cup I_{n-1}(x')$, as otherwise there would exist $i$ such that $I_{n-1, i}$ separates $x$ and $x'$. Since $x$ and $x'$ lie in $D^*$, the slopes of $f_r^{i-1}$ are strictly positive on $I_{n-1}(x) \cup I_{n-1}(x')$, and hence, on $[x, x']$. As a result, $f_r^n$ is nondecreasing throughout $[x, x']$, and in particular, $s \geq 0$. We have

$$f_r(x) = f_r^n(x) + \sum_{k=n}^{\infty} r^{-k} \phi(r^k x) = f_r^n(x) + r^{-n} f_r(r^n x)$$

$$\leq f_r^n \left( \frac{j}{2r^{n-1}} \right) + r^{-n} \max_z f_r(z) < f_r^n \left( \frac{j}{2r^{n-1}} \right) + r^{-n}. \tag{11}$$

Now if $s \geq 1$, then

$$f_r^n \left( \frac{j+1}{2r^{n-1}} \right) - f_r^n \left( \frac{j}{2r^{n-1}} \right) = \frac{s}{2r^{n-1}} \geq \frac{1}{2r^{n-1}} \geq r^{-n},$$

so that

$$f_r(x) < f_r^n \left( \frac{j+1}{2r^{n-1}} \right) \leq f_r^n(x') \leq f_r(x').$$

This contradicts the hypothesis of the lemma, and hence, $s = 0$.

It remains to show that $f_r(x) \in f_r(I_{n,j})$. First, since $s = 0$, we have

$$\max_{u \in I_{n,j}} f_r(u) = f_r^n \left( \frac{j}{2r^{n-1}} \right) + r^{-n} \max_z f_r(z) \geq f_r(x),$$

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as in (11). On the other hand,

\[ f_r(x) = f_r(x') \geq f^n_r(x') \geq f^n_r \left( \frac{j + 1}{2^{r-1}} \right) = \min_{u \in I_{n,j}} f_r(u), \]

since the value of \( f_r \) at one or both endpoints of \( I_{n,j} \) must equal the (constant) value of \( f^n_r \) on \( I_{n,j} \), in view of Corollary 4.2. Thus, \( f_r(x) \in f_r(I_{n,j}) \), and the proof is complete.

**Remark 4.7.** The hypothesis of Lemma 4.6 is never satisfied when \( r \) is even; since this fact is not needed to prove Theorem 1.1 its (not too difficult) proof is omitted here. When \( r \) is odd, however, there may exist points \( x \) and \( x' \) satisfying the hypothesis of the lemma. For instance, when \( r = 3, x = 17/108 \) and \( x' = 37/108 \), a straightforward calculation yields that \( (s^n_n(x))_{n \geq 1} = (s^n_n(x'))_{n \geq 1} = (1, 2, 3, 4, 3, 4, \ldots) \), and \( f_3(x) = f_3(x') = 3/8 \).

**Remark 4.8.** In connection with the previous remark, it is worth noting that \( f_r(x) \) is rational when \( x \) is rational. This follows since if \( x \) is rational, it has a base-\( r \) expansion which is eventually periodic, and so \( r^n x \mod 1 \) is eventually periodic. By the 1-periodicity of \( \phi \), this implies that \( \phi(r^n x) \) is eventually periodic, and it is easy to deduce from this that \( f_r(x) \) is rational.

Before stating the next important preliminary result, we define the discrete sets

\[ \mathcal{A} := \{ (n, j) : n \in \mathbb{N}, 0 \leq j < 2r^{n-1}, \text{ and } s_{n,j} = 0 \}, \]
\[ \mathcal{A}^+ := \{ (n, j) \in \mathcal{A} : \text{ the slope of } f^k_r \text{ is nonnegative on } I_{n,j} \text{ for } k = 1, \ldots, n \}, \]

and finally,

\[ \mathcal{A}^+(y) := \{ (n, j) \in \mathcal{A}^+ : y \in f_r(I_{n,j}) \}, \quad y \in \mathbb{R}. \]

To visualize the set \( \mathcal{A}^+(y) \), it may help to recall from Corollary 4.2 that if \( s_{n,j} = 0 \), then the graph of \( f_r \) above \( I_{n,j} \) consists of \( r \) small-scale similar copies of the half graph \( G_t \), sitting side by side at the same height with alternating orientations. The set \( \mathcal{A}^+(y) \) consists of those pairs \( (n, j) \) in \( \mathcal{A} \) for which the horizontal line at level \( y \) intersects these \( r \) similar copies of \( G_t \), and, furthermore, the partial sum functions \( f^1_r, \ldots, f^n_r \) all have nonnegative slope throughout \( I_{n,j} \).

**Proposition 4.9.** Let \( y \in \mathbb{R} \setminus f_r(C) \). If \( |\mathcal{A}^+(y)| < \infty \), then \( |L_r(y)| < \infty \).

**Proof.** Recall the definitions of \( D_{fin} \) and \( D^+_{fin} \) from (8) and (9). Let

\[ D^+ := \{ x \in D : s_n(x) \geq 0 \text{ for every } n \}. \]
Let $y$ be as given, and suppose $|A^+(y)| < \infty$. Then $L_r(y) \subset D$, and

$$L_r(y) \cap D^+ \subset D_{\text{fin}}.$$  \hspace{1cm} (12)

To see (12), let $x \in L_r(y) \cap D^+$. Then $f_r(x) = y$ and $s_n(x) \geq 0$ for all $n$. Because $|A^+(y)| < \infty$, there are only finitely many pairs $(n, j)$ in $A^+$ such that $x \in I_{n,j}$. Since $s_n(x) \geq 0$ for all $n$, this last statement remains true if we replace $A^+$ with $A$. Thus, there are only finitely many $n$ such that $s_n(x) = 0$, which means $x \in D_{\text{fin}}$.

For arbitrary $x \in L_r(y)$, there is by Corollary 4.3 a point $x' \in L_r(y) \cap D^+$ such that $|s_n(x)| = s_n(x')$ for each $n$. But then $x' \in D_{\text{fin}}$ by (12), and so $x \in D_{\text{fin}}$. Therefore, $L_r(y) \subset D_{\text{fin}}$. By Corollary 4.4, it is now enough to show that $|L_r(y) \cap D_{\text{fin}}^+| < \infty$. To this end, define an equivalence relation $\sim$ on $D$ as follows. Let

$$n(x) := \sup\{n \geq 0 : s_n(x) = 0\},$$

and say $x \sim x'$ if either $x = x'$, or all of the following hold:

(i) $n(x) = n(x') =: n < \infty$;

(ii) $I_n(x) = I_n(x')$; and

(iii) $x - x' = m r^{-n}$ for some $m \in \mathbb{Z}$.

Note that, since $I_n(x)$ has length $1/(2r^{n-1})$, condition (ii) forces the number $m$ in (iii) to satisfy $|m| < r/2$, and therefore, $\sim$ has finite equivalence classes.

Let $E$ be the set of those points $x$ in $L_r(y) \cap D_{\text{fin}}^+$ which are the leftmost member of their equivalence class; that is,

$$x \in E \iff x \in L_r(y) \cap D_{\text{fin}}^+ \text{ and } x \leq x' \text{ for each } x' \text{ with } x \sim x'.$$

Since $\sim$ has finite equivalence classes, it suffices to show that $E$ is finite.

**Claim:** For any two points $x, x' \in E$ with $x < x'$, there is a pair $(n, j) \in A^+(y)$ such that $I_{n,j} \cap [x, x'] \neq \emptyset$, and $I_{n,j}$ contains at most one of $x$ and $x'$.

Assuming the Claim for now, we can finish the proof of the Proposition as follows. Suppose, by way of contradiction, that $E$ is infinite. Then we can find a strictly monotone sequence $\{x_\nu\}$ in $E$; say $\{x_\nu\}$ is increasing. (The argument when $\{x_\nu\}$ is decreasing is entirely similar.) The Claim implies the existence of pairs $\{(n_\nu, j_\nu) : \nu \in \mathbb{N}\}$ in $A^+(y)$ such that for each $\nu$, $I_{n_\nu,j_\nu} \cap [x_\nu, x_{\nu+1}] \neq \emptyset$ and $I_{n_\nu,j_\nu}$ contains at most one of $x_\nu$ and $x_{\nu+1}$. Consider an arbitrary interval $I_{n,j}$, and suppose $I_{n,j}$ intersects $k$ of the intervals $[x_\nu, x_{\nu+1}]$. Then $I_{n,j}$ must fully contain at least $k - 2$ of these, so by the hypothesis about $I_{n_\nu,j_\nu}$, $I_{n,j}$ can occur at most twice in the sequence.

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\( \{I_{n,j}\} \). Hence, this sequence contains infinitely many distinct intervals. But this implies \(|A^+(y)| = \infty\), contradicting the hypothesis of the proposition. Therefore, \(|E| < \infty\).

We now turn to the proof of the Claim. Let \( x, x' \in E \) with \( x < x' \), and consider three cases:

Case 1: \( s_n(x) > 0 \) and \( s_n(x') > 0 \) for all \( n \geq 1 \). By Lemma 4.6 there is a pair \( (n, j) \in A^+ \) such that \( I_{n,j} \) separates \( x \) and \( x' \), and \( y \in f_r(I_{n,j}) \). Thus, \( (n, j) \in A^+(y) \).

Case 2: There is a pair \( (n, j) \in A^+ \) such that \( x \in I_{n,j} \) but \( x' \notin I_{n,j} \), or vice versa. Since \( y = f_r(x) = f_r(x') \), we have in either case that \( y \in f_r(I_{n,j}) \), so \( (n, j) \in A^+(y) \).

Case 3: If neither Case 1 nor Case 2 holds, then there are integers \( n \) and \( j \) such that \( n(x) = n(x') = n \), and \( I_{n}(x) = I_{n}(x') = I_{n,j} \). We argue that in fact, \( I_{n+1}(x) = I_{n+1}(x') \). From the definition of \( n(x) \) we have \( s_n(x) = s_n(x') = 0 \). Since \( x \) and \( x' \) are the leftmost points of their respective equivalence classes under \( \sim \), their distance from the left endpoint of \( I_{n,j} \) is at most \( r^{-n} \). Furthermore, since both \( s_{n+1}(x) > 0 \) and \( s_{n+1}(x') > 0 \), \( x \) and \( x' \) must both lie in the leftmost subinterval \( I_{n+1,k} \) of \( I_{n,j} \) on which \( f_r^{n+1} \) has positive slope. Thus, \( I_{n+1}(x) = I_{n+1}(x') \).

Let \( z \) be the left endpoint of \( I_{n+1}(x) \), and put

\[ \xi := r^n(x - z), \quad \xi' := r^n(x' - z). \]

Then \( \xi, \xi' \in D \cap [0,1/2) \), and by Lemma 4.1 \( s_k(\xi) = s_{n+k}(x) \), and \( s_k(\xi') = s_{n+k}(x') \) for each \( k \in \mathbb{N} \). In particular, \( \xi, \xi' \in D^* \), where \( D^* \) was defined by (10). By Case 1 and Corollary 4.2 there is a pair \( (m, j) \in A^+ \) such that \( I_{m,j} \) separates \( \xi \) and \( \xi' \), and

\[ y' := r^n(y - f^n_r(z)) \in f_r(I_{m,j}). \tag{13} \]

Let \( I := z + r^{-n}I_{m,j} \). Then \( I = I_{n+m,k} \) for some \( k \), and \( I \) separates \( x \) and \( x' \). By Corollary 4.2,

\[ f_r(I) = f_r(z + r^{-n}I_{m,j}) = f^n_r(z) + r^{-n}f_r(I_{m,j}), \]

and hence, by (13),

\[ y = f^n_r(z) + r^{-n}y' \in f_r(I). \]

It follows that \( (n + m, k) \in A^+(y) \), and so \( I = I_{n+m,k} \) has the required property.

In all three cases, the conclusion of the Claim follows. The proof is complete. \( \Box \)

Remark 4.10. In view of Remark 4.7 Cases 1 and 3 in the above proof cannot occur when \( r \) is even, so in that case the Proposition is (nearly) trivial. This illustrates once more the additional complications that arise when \( r \) is odd.
The final ingredient of the proof of Theorem 1.1(i) is the following lemma, for which we give a probabilistic proof based on Lemma 3.2. Recall that \( \mathcal{A}^+ \) is the set of pairs \((n, j)\) such that the slope of \( f^k_r \) is nonnegative on \( I_{n,j} \) for \( k = 1, \ldots, n \).

**Lemma 4.11.** It holds that

\[
\sum_{(n,j) \in \mathcal{A}^+} \lambda(f_r(I_{n,j})) < \infty.
\]

**Proof.** Assume that on some probability space \((\Omega, \mathcal{F}, P)\), a correlated random walk \( \{S_n\} \) with steps \( \{X_n\} \) and parameter \( p = p_r \) is defined as in Definition 3.1, where \( p_r \) is as in (2). Define a second probability measure \( P_+ \) on \((\Omega, \mathcal{F})\) by

\[
P_+(A) := 2p_r P(A \cap \{X_1 = 1\}) + 2(1 - p_r) P(A \cap \{X_1 = -1\}), \quad A \in \mathcal{F}.
\]

Under \( P_+ \), the process \( \{S_n\} \) is a CRW with a “flying start” in the upward direction: \( P_+(X_1 = 1) = p_r \), and \( P_+(X_n = 1|X_{n-1} = 1) = P(X_n = -1|X_{n-1} = -1) = p_r \) for \( n \geq 2 \).

Observe first that, for \((n, j) \in \mathcal{A}^+\),

\[
\lambda(f_r(I_{n,j})) \leq \lambda(I_{n,j}), \quad (14)
\]

because the graph of \( f_r \) above \( I_{n,j} \) consists of \( r \) similar copies of the half graph \( G_l \), and as such, its height is no greater than its width.

Define the sets

\[
\mathcal{A}_n^+ := \{ j : 0 \leq j < 2r^{n-1}, \ (n, j) \in \mathcal{A}^+ \}, \quad n \in \mathbb{N}.
\]

Note that for fixed \( n \), the intervals \( I_{n,j}, j \in \mathcal{A}_n^+ \) are disjoint. Thus, we have

\[
\sum_{(n,j) \in \mathcal{A}^+} \lambda(I_{n,j}) = \sum_{n=1}^{\infty} \sum_{j \in \mathcal{A}_n^+} \lambda(I_{n,j}) = \sum_{n=1}^{\infty} \lambda \left( \bigcup_{j \in \mathcal{A}_n^+} I_{n,j} \right) \nonumber \\
= \sum_{n=1}^{\infty} \lambda\{x \in [0, 1) : s_1(x) \geq 0, \ldots, s_{n-1}(x) \geq 0, s_n(x) = 0\} \\
= \sum_{n=1}^{\infty} P(S_1 \geq 0, \ldots, S_{n-1} \geq 0, S_n = 0),
\]

where the last equality follows by Lemma 3.2. Denote the \( n \)th term of the last series above by \( a_n \), that is,

\[
a_n := P(S_1 \geq 0, \ldots, S_{n-1} \geq 0, S_n = 0), \quad n \in \mathbb{N},
\]
and let
\[ b_n := P(S_1 > 0, \ldots, S_{n-1} > 0, S_n = 0), \quad n \in \mathbb{N}. \]
By a standard argument (see, for instance, [10, p. 345]),
\[ b_{n+2} = P(S_1 = 1, S_2 \geq 1, \ldots, S_n \geq 1, S_{n+2} = 0) \]
\[ = \frac{1}{2} P_+(S_1 \geq 0, \ldots, S_{n-1} \geq 0, S_n = 0) \cdot p_r \]
\[ = p_r^2 a_n. \]
Hence,
\[ \sum_{(n,j) \in A^+} \lambda(I_{n,j}) = \sum_{n=1}^\infty a_n = p_r^{-2} \sum_{n=1}^\infty b_{n+2} \leq p_r^{-2} P(S_n = 0 \text{ for some } n \geq 3) < \infty. \]
Together with (14), this completes the proof.

Proof of Theorem 1.1(i). Since \( f_r(C) \) is countable, we have, by Proposition 4.9, Lemma 4.11 and the Borel-Cantelli lemma,
\[ \lambda\{y : |L_r(y)| = \infty\} = \lambda\{y \in \mathbb{R} \setminus f_r(C) : |L_r(y)| = \infty\} \]
\[ \leq \lambda\{y : |A^+(y)| = \infty\} \]
\[ = \lambda\{y : y \in f_r(I_{n,j}) \text{ for infinitely many pairs } (n,j) \in A^+\} \]
\[ = 0. \]
Thus, \( L_r(y) \) is finite for almost every \( y \).

Acknowledgment

The author wishes to thank the anonymous referee for a careful reading of the paper and for suggesting several improvements to the presentation.

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