VERY SIMPLE 2-ADIC REPRESENTATIONS AND HYPERELLIPTIC JACOBIANS

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1. Introduction

Throughout this paper\( K \) is a field of characteristic \( \neq 2 \) and \( K_a \) its algebraic closure. If \( f(x) \in K[x] \) is a separable polynomial of degree \( n \geq 5 \) then it gives rise to the hyperelliptic curve
\[ C = C_f : y^2 = f(x). \]
We write \( J(C) = J(C_f) \) for its jacobian; it is a \( g \)-dimensional abelian variety defined over \( K \), whose dimension \( g \) is equal to \( \frac{n-1}{2} \) when \( n \) is odd and equal to \( \frac{n-2}{2} \) when \( n \) is even. In this paper we deal with special (but somehow “generic”) case when the Galois group \( \text{Gal}(f) \) of \( f \) is either the symmetric group \( S_n \) or the alternating group \( A_n \). We study the \( \ell \)-adic Lie algebra \( g_{\ell,J(C_f)} \) attached to the Galois action on the \( \ell \)-torsion of \( J(C) \) and prove that this Lie algebra is “as large as possible” when \( K \) is a number field. As a corollary, we obtain the Tate conjecture and the Hodge conjecture for all self-products of \( J(C) \). In fact, we prove that all the Tate/Hodge classes involved can be presented as linear combinations of products of divisor classes.

Our approach is based on a study of the 2-adic image of the Galois group \( \text{Gal}(K) \) in the automorphism group \( \text{Aut}_{\mathbb{Z}_2}(T_2(J(C))) \) of the \( \mathbb{Z}_2 \)-Tate module \( T_2(J(C)) \) of \( J(C) \). We prove that the algebraic envelope of the image contains the symplectic group attached to the theta divisor on \( J(C_f) \). Our proof is based on known lower bounds for dimensions of nontrivial (projective) representations of \( A_n \) in characteristics 0 and 2 [11, 32, 33] and a notion of very simple representation introduced and studied in [41] and [43]. This allows us to prove that \( g_{\ell,J(C_f)} \) is “as large as possible” for \( \ell = 2 \). Now the rank independence on \( \ell \) for \( g_{\ell,J(C_f)} \) [37] allows us to extend this assertion to all primes \( \ell \), using a variant of a theorem of Borel - de Siebenthal [38].

The present paper is a natural continuation of our previous articles [30], [40], [41], [42] and [43].

2. Abelian varieties and \( \ell \)-adic Lie algebras

Let \( \ell \) be a prime and \( K \) be a field of characteristic different from \( \ell \). We fix its algebraic closure \( K_a \) and write \( \text{Gal}(K) \) for the absolute Galois group \( \text{Aut}(K_a/K) \). Let \( K(\ell) \) be the abelian extension of \( K \) obtained by adjoining to \( K \) all \( \ell \)-power roots of unity. We write
\[ \chi_\ell : \text{Gal}(K) \to \text{Gal}(K(\ell)/K) \subset \mathbb{Z}_\ell \]
for the corresponding cyclotomic character. If every finite algebraic extension of \( K \) contains only finitely many \( \ell \)-power roots of unity (e.g., \( K \) is finitely generated over

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its prime subfield) then the image \( \chi(\text{Gal}(K)) \) is infinite. We write \( \mathbb{Z}_\ell(1) \) for the Gal\((K)\)-module \( \mathbb{Z}_\ell \) with the Galois action defined by character \( \chi \). If \( X \) is an abelian variety defined over \( K \) and \( m \) is a positive integer not divisible by \( \text{char}(K) \) then we write \( X_m \) for the kernel of multiplication by \( m \) in \( X(K_a) \). It is well-known \cite{17} that \( X_m \) is a free \( \mathbb{Z}/m\mathbb{Z} \)-module of rank \( 2\dim(X) \) provided with a natural structure of Gal\((K)\)-module. Suppose \( \ell \) is a prime distinct from \( \text{char}(K) \). As usual, we write \( T_\ell(X) \) for the projective limit of Gal\((K)\)-modules \( X_\ell \) where the transition maps are multiplications by \( \ell \). It is well-known that \( T_\ell(X) \) is a free \( \mathbb{Z}_\ell \)-module of rank \( 2\dim(X) \) provided with natural continuous homomorphism \( (\ell\text{-adic representation}) \)

\[
\rho_{\ell,X} : \text{Gal}(K) \to \text{Aut}_{\mathbb{Z}_\ell}(T_\ell(X)) \cong \text{GL}(2\dim(X), \mathbb{Z}_\ell).
\]

In addition \( X_\ell = T_\ell(X)/\ell T_\ell(X) \) (as Galois module).

Recall \cite{17} that each polarization \( \lambda \) on \( X \) defined over \( K \) gives rise to Riemann form

\[
e_\lambda : T_\ell(X) \times T_\ell(X) \to \mathbb{Z}_\ell(1) \cong \mathbb{Z}_\ell;
\]

\( e_\lambda \) is a non-degenerate Gal\((K)\)-equivariant alternating \( \mathbb{Z}_\ell \)-bilinear form on \( T_\ell(X) \). It is perfect if and only if \( \ell \) does not divide \( \text{deg}(\lambda) \) (e.g., \( \lambda \) is principal). Clearly, the corresponding automorphism group

\[
\text{Aut}_{\mathbb{Z}_\ell}(T_\ell(X), e_\lambda) \cong \text{Sp}(2\dim(X), \mathbb{Z}_\ell).
\]

It is well-known that the image

\[
G_{\ell,X} := \rho_{\ell,X}(\text{Gal}(K)) \subset \text{Aut}_{\mathbb{Z}_\ell}(T_\ell(X))
\]

is contained in the group of symplectic similitudes

\[
\text{Gp}(T_\ell(X), e_\lambda) := \{ u \in \text{Aut}_{\mathbb{Z}_\ell}(T_\ell(X)) \mid \exists c = c_u \in \mathbb{Z}_\ell^* \text{ such that } e_\lambda(u x, u y) = c u(x, y) \quad \forall x, y \in T_\ell(X) \}.
\]

More precisely, if \( \sigma \in \text{Gal}(K) \) and \( u = \rho_{\ell,X}(\sigma) \) then \( c_u = \chi_\ell(\sigma) \). It is also well-known that the composition of \( \rho_{\ell,X} \) and the determinant map

\[
\det : \text{Aut}_{\mathbb{Z}_\ell}(T_\ell(X)) \to \mathbb{Z}_\ell^*
\]

coincides with \( \chi_{\ell}^{\dim(X)} \).

We write \( \tilde{\rho}_{\ell,X} \) for the corresponding modular representation

\[
\tilde{\rho}_{\ell,X} : \text{Gal}(K) \to \text{Aut}(X_\ell);
\]

we denote by \( \tilde{G}_{\ell,X} \) the image \( \tilde{\rho}_{\ell,X}(\text{Gal}(K)) \subset \text{Aut}(X_\ell) \). Clearly, \( \tilde{\rho}_{\ell,X} \) coincides with the composition of \( \rho_{\ell,X} \) and the reduction map modulo \( \ell \)

\[
\text{Aut}_{\mathbb{Z}_\ell}(T_\ell(X)) \to \text{Aut}(X_\ell);
\]

the subgroup \( \tilde{G}_{\ell,X} \) coincides with the image of \( G_{\ell,X} \) under the reduction map.

As usual, \( V_\ell(X) \) stands for the \( \mathbb{Q}_\ell \)-Tate module

\[
V_\ell(X) = T_\ell(X) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell;
\]

it is a \( 2\dim(X) \)-dimensional \( \mathbb{Q}_\ell \)-vector space provided with a natural structure of Gal\((K)\)-module and \( T_\ell(X) \) is identified with a Gal\((K)\)-stable \( \mathbb{Z}_\ell \)-lattice in \( V_\ell(X) \). The form \( e_\lambda \) extends by \( \mathbb{Q}_\ell \)-linearity to the non-degenerate alternating \( \mathbb{Q}_\ell \)-bilinear form

\[
e_\lambda : V_\ell(X) \times V_\ell(X) \to \mathbb{Q}_\ell.
\]
We write $\text{Sp}(V_\ell(X), e_\lambda)$ for the corresponding symplectic group viewed as $\mathbb{Q}_\ell$-linear algebraic subgroup of $\text{GL}(V_\ell(X))$. Let

$$\mathfrak{sp}(V_\ell(X), e_\lambda) \cong \mathfrak{sp}(2\dim(X), \mathbb{Q}_\ell)$$

be the Lie algebra of $\text{Sp}(V_\ell(X), e_\lambda)$; it is an absolutely irreducible $\mathbb{Q}_\ell$-linear subalgebra of $\text{End}_{\mathbb{Q}_\ell}(V_\ell(X))$.

It is well-known $\mathfrak{g}_\ell, \mathfrak{x}$ that $G_{\ell,\mathfrak{X}}$ is an $\ell$-adic Lie subgroup of $\text{Aut}_{\mathbb{Z}_2}(T_\ell(X))$. We write $g_{\ell,\mathfrak{x}}$ for the Lie algebra of of $G_{\ell,\mathfrak{X}}$; it is a $\mathbb{Q}_\ell$-Lie subalgebra of $\text{End}_{\mathbb{Q}_\ell}(V_\ell(X))$. The inclusion $G_{\ell,\mathfrak{X}} \subset \text{Gp}(T_\ell(X), e_\lambda)$ implies easily that

$$g_{\ell,\mathfrak{x}} \subset \mathbb{Q}_\ell \text{Id} \oplus \mathfrak{sp}(V_\ell(X), e_\lambda).$$

In addition, $g_{\ell,\mathfrak{x}} \subset \mathfrak{sp}(V_\ell(X), e_\lambda)$ if either $K = K(\ell)$ or there exists a finite algebraic extension of $K$ that contains all $\ell$-power roots of unity. Clearly, $g_{\ell,\mathfrak{x}} = \{0\}$ if and only if $G_{\ell,\mathfrak{x}}$ is finite. It is also clear that if every finite algebraic extension of $K$ contains only finitely many $\ell$-power roots of unity then $g_{\ell,\mathfrak{x}}$ does not lie in the Lie algebra $\mathfrak{g}(V_\ell(X))$ of linear operators in $V_\ell(X)$ with zero trace.

**Remark 2.1.** If $K$ is a field of characteristic zero finitely generated over $\mathbb{Q}$ then, by a theorem of Bogomolov $\mathbb{Q}_\ell$, $g_{\ell,\mathfrak{x}}$ contains $\mathbb{Q}_\ell \text{Id}$.

**Theorem 2.2.** Let us assume that $\dim(X) \geq 4$ and let us put $d = 2\dim(X)$. Let us assume that $\text{char}(K) \neq 2$ and let us put $\ell = 2$. Suppose $G_{2,\mathfrak{X}}$ contains a subgroup isomorphic to the alternating group $A_{d+1}$ (e.g., $G_{2,\mathfrak{X}} = A_{d+1}, S_{d+1}, S_{d+2}$ or $A_{d+2}$). Then either $g_{2,\mathfrak{x}} = \mathbb{Q}_2 \text{Id} \oplus \mathfrak{sp}(V_2(X), e_\lambda)$ or $g_{2,\mathfrak{x}} = \mathfrak{sp}(V_2(X), e_\lambda)$. In addition, the ring $\text{End}(X)$ of all $K_\mathfrak{x}$-endomorphisms of $X$ is $\mathbb{Z}$.

If every finite algebraic extension of $K$ contains only finitely many $2$-power roots of unity then $g_{2,\mathfrak{x}} = \mathbb{Q}_2 \text{Id} \oplus \mathfrak{sp}(V_2(X), e_\lambda)$. If there exists a finite algebraic extension of $K$ that contains all $\ell$-power roots of unity then $g_{2,\mathfrak{x}} = \mathfrak{sp}(V_2(X), e_\lambda)$.)

**Theorem 2.3.** Suppose $K$ is a field and $\text{char}(K) \neq 2$. Suppose $X$ is an abelian variety defined over $K$. Let us assume that $\dim(X) \geq 4$ and let us put $d = 2\dim(X)$. Let us assume that $G_{2,\mathfrak{X}}$ contains a subgroup isomorphic to the alternating group $A_{d+1}$ and $K$ enjoys one of the two following properties:

(i) $K$ is a field of characteristic zero finitely generated over $\mathbb{Q}$;

(ii) $p = \text{char}(K) > 0$ and $K$ is is a global field.

Then for all primes $\ell \neq \text{char}(K)$

$$g_{\ell,\mathfrak{x}} = \mathbb{Q}_\ell \text{Id} \oplus \mathfrak{sp}(V_\ell(X), e_\lambda).$$

**Theorem 2.4.** Let $K$ be a field with $\text{char}(K) \neq 2$, $K_\mathfrak{a}$ its algebraic closure, $f(x) \in K[\bar{x}]$ a separable polynomial of degree $n \geq 9$, whose Galois group $\text{Gal}(f)$ enjoys one of the following properties:

(i) $\text{Gal}(f)$ is either $S_n$ or $A_n$;

(ii) $n = 11$ and $\text{Gal}(f)$ is the Mathieu group $M_{11}$;

(iii) $n = 12$ and $\text{Gal}(f)$ is either the Mathieu group $M_{12}$ or $M_{11}$.

(iv) There exist an odd power prime $q$ and an integer $m \geq 3$ such that $(q, m) \neq (3, 4)$, $n = \frac{q^m - 1}{q - 1}$ and $\text{Gal}(f)$ contains a subgroup isomorphic to the projective special linear group $\mathbb{L}_m(q) := \text{PSL}_m(F_q)$. (E.g., $\text{Gal}(f)$ is isomorphic either to the projective linear group $\text{PGL}_m(F_q)$ or to $\mathbb{L}_m(q)$.)
Let \( C_f \) be the hyperelliptic curve \( y^2 = f(x) \). Let \( X = J(C_f) \) be its jacobian, \( \lambda \) the principal polarization on \( J(C_f) \) attached to the theta divisor. Then \( p_{2,J(C_f)}(\text{Gal}(K)) \) contains an open subgroup of \( \text{Aut}_{\mathbb{Z}_2}(T_2(X), e_\lambda) \). In addition, the ring \( \text{End}(X) \) of all \( K_a \)-endomorphisms of \( X \) is \( \mathbb{Z} \).

**Theorem 2.5.** Let \( f(x) \in K[x] \) be a separable polynomial of degree \( n \geq 9 \), whose Galois group \( \text{Gal}(f) \) enjoys one of the following properties:

(i) \( \text{Gal}(f) \) is either \( S_n \) or \( A_n \);
(ii) \( n = 11 \) and \( \text{Gal}(f) \) is the Mathieu group \( M_{11} \);
(iii) \( n = 12 \) and \( \text{Gal}(f) \) is either the Mathieu group \( M_{12} \) or \( M_{11} \).
(iv) There exist an odd power prime \( q \) and an integer \( m \geq 3 \) such that \( (q, m) \neq (3, 4), n = \frac{q^m - 1}{q - 1} \) and \( \text{Gal}(f) \) contains a subgroup isomorphic to \( L_m(q) \). (E.g.,
\[ \text{Gal}(f) \] is isomorphic either to the projective linear group \( \text{PGL}_m(\mathbb{F}_q) \) or to \( L_m(q) \).)

Let \( C_f \) be the hyperelliptic curve \( y^2 = f(x) \). Let \( X = J(C_f) \) be its jacobian. Assume that either \( K \) is a field of characteristic zero finitely generated over \( \mathbb{Q} \) or \( K \) is a global field of odd characteristic. Then for all primes \( \ell \neq \text{char}(K) \)
\[ g_{\ell, X} = \mathbb{Q}_\ell \text{Id} \oplus \mathfrak{sp}(V_\ell(X), e_\lambda). \]

**Theorem 2.6.** Let \( K \) be a field of characteristic zero finitely generated over \( \mathbb{Q} \), \( f(x) \in K[x] \) a separable polynomial of degree \( n \geq 5 \), whose Galois group \( \text{Gal}(f) \) enjoys one of the following properties:

(i) \( \text{Gal}(f) \) is either \( S_n \) or \( A_n \);
(ii) \( n = 11 \) and \( \text{Gal}(f) \) is the Mathieu group \( M_{11} \);
(iii) \( n = 12 \) and \( \text{Gal}(f) \) is either the Mathieu group \( M_{12} \) or \( M_{11} \).
(iv) There exist an odd power prime \( q \) and an integer \( m \geq 3 \) such that \( (q, m) \neq (3, 4), n = \frac{q^m - 1}{q - 1} \) and \( \text{Gal}(f) \) contains a subgroup isomorphic to \( L_m(q) \). (E.g.,
\[ \text{Gal}(f) \] is isomorphic either to the projective linear group \( \text{PGL}_m(\mathbb{F}_q) \) or to \( L_m(q) \).)

Let \( C_f \) be the hyperelliptic curve \( y^2 = f(x) \). Let \( X = J(C_f) \) be its jacobian. Then for all primes \( \ell \)
\[ g_{\ell, X} = \mathbb{Q}_\ell \text{Id} \oplus \mathfrak{sp}(V_\ell(X), e_\lambda). \]

We prove Theorems 2.2, 2.3, 2.4 and 2.6 in Section 3.

### 3. Group theory

Throughout the paper we will freely use the following elementary observation.

**Proposition 3.1.** Suppose \( \ell \) is a prime, \( F \) is field which is a finite algebraic extension of \( \mathbb{Q}_\ell \). Suppose \( W \) is a finite-dimensional \( F \)-vector space and \( G \) is a compact subgroup of \( \text{Aut}_F(W) \) (in \( \ell \)-adic topology).

(i) Suppose \( M \) is a periodic group of finite exponent provided with discrete topology (e.g., a finite group). Then every homomorphism \( \pi : G \to M \) is continuous and its kernel is an open subgroup of finite index in \( G \).
(ii) Suppose \( H \) is an open normal subgroup of finite index in \( G \) and the quotient \( G/H \) is a finite simple non-abelian group. Then:
(a) $H' := H \cap \ker(\pi)$ is an open normal subgroup of finite index in $G' := \ker(\pi)$. If $H$ contains $\ker(\pi)$ then $G/H$ is a homomorphic image of $\pi(M)$, i.e., there is a surjective continuous homomorphism $\pi(M) \to G/H$. If $H$ does not contain $\ker(\pi)$ then $H' := H \cap \ker(\pi)$ is an open subgroup of finite index in $G' := \ker(\pi)$ and $G'/H' = G/H$. In other words, $G/H$ is a homomorphic image either of $\ker(\pi)$ or of the image of $\pi$.

(b) If $M$ is solvable then $H$ does not contain $\ker(\pi)$. In particular, $G'/H' = G/H$.

(c) Suppose that either $\ell = 2$ or $G/H$ is either a simple group of Lie type in odd characteristic or one of 26 simple sporadic groups. (In other words, modulo the classification, if $\ell$ is odd then $G/H$ is not a simple group of Lie type in characteristic 2.) Suppose also that $M$ is finite and for some positive integer $m$ there exists an embedding

$$M \hookrightarrow \text{PGL}_m(\mathbb{F}_\ell).$$

Then either $G'/H' = G/H$ or there exists an embedding

$$G/H \hookrightarrow \text{PGL}_m(\mathbb{F}_\ell).$$

(d) Suppose $\pi' : G \to M'$ is a continuous group homomorphism from $G$ to an $\ell$-adic Lie group $M'$. Then:

(d1) $\pi'(H)$ is an open subgroup of finite index in the compact group $\pi'(G)$ and $H' := H \cap \ker(\pi')$ is an open subgroup of finite index in the compact group $G' := \ker(\pi')$. In addition, the (super)orders of $\pi'(H)$ and $H'$ both divide the (super)order of $H$;

(d2) Either $H$ contains $\ker(\pi')$ and $\pi'(G)/\pi'(H) = G/H$ or $H$ does not contain $\ker(\pi')$ and $G'/H' = G/H$. In other words, $G/H$ is a homomorphic image either of $\ker(\pi)$ or of the image of $\pi$.

(d3) Suppose $G$ is a closed subgroup of a product $S_1 \times \cdots \times S_m$ of finitely many $\ell$-adic Lie groups $S_i$. Then there exist a factor $S_j$, a compact subgroup $G \subset R_j$ and an open normal subgroup $\bar{H}$ of $G$ such that the quotient $G/\bar{H}$ is isomorphic to $G/H$. In addition, one may choose $\bar{H}$ in such a way that the (super)order of $\bar{H}$ divides the (super)order of $H$.

(e) Suppose $E$ is field which is a finite algebraic extension of $\mathbb{Q}_\ell$. Suppose $\alpha : \mathfrak{R}_1 \to \mathfrak{R}_2$ is a central isogeny between two semisimple $E$-algebraic groups $\mathfrak{R}_1$ and $\mathfrak{R}_2$. Suppose $\gamma : G \to \mathfrak{R}_2(E)$ is a continuous group homomorphism (in $\ell$-adic topology), whose kernel is a finite commutative group. Let us put

$$G_\alpha := \gamma^{-1}(\alpha(\mathfrak{R}_1(E))) = \{ g \in G \mid \gamma(g) \in \alpha(\mathfrak{R}_1(E)) \subset \mathfrak{R}_2(E) \}, \quad H_\alpha := H \cap G_\alpha;
$$

$$G_{-1,\alpha} := \{ u \in \mathfrak{R}_1(E) \mid \alpha(u) \in \gamma(G_\alpha) \subset \mathfrak{R}_2(E) \},$$

$$H_{-1,\alpha} := \{ u \in \mathfrak{R}_1(E) \mid \alpha(u) \in \gamma(H_\alpha) \subset \mathfrak{R}_2(E) \}.$$

Then:

(e1) $G_\alpha$ is an open normal subgroup of finite index in $G$, $H_\alpha$ is an open normal subgroup of finite index in $G_\alpha$ and $G_\alpha/H_\alpha = G/H$.

(e2) $G_{-1,\alpha}$ is a compact group, $H_{-1,\alpha}$ is an open subgroup of finite index in $G_{-1,\alpha}$ and $G_{-1,\alpha}/H_{-1,\alpha} = G/H$. 
(e3) Assume that $H$ is a pro-$\ell$-group. Then the (super)orders of $H_\alpha$ and $\gamma(H_\alpha)$ divide the (super)order of $H$. If $q$ is a prime divisor of the (super)order of $H_{-1,\alpha}$ then either $q$ divides the (super)order of $H$ or $q$ divides $\deg(\alpha)$.

Proof. Let us do the case (i). It suffices to check that $\ker(\pi)$ is an open subgroup and therefore is closed. If $n$ is the exponent of $M$ then $\ker(\pi)$ contains $G^n := \{x^n \mid x \in G\}$. Let us consider $W$ as finite-dimensional $\mathbb{Q}_\ell$-vector space. Then $G$ is a compact $\ell$-adic Lie group, thanks to $\ell$-adic “Cartan Theorem” ([21], Part II, Sect. 9). Now basic properties of the exponential map imply that $G^n$ contains an open neighborhood of the identity. This implies that $G^n$ contains an open subgroup and therefore $\ker(\pi)$ also contains an open subgroup. Since every subgroup containing an open subgroup is also open, $\ker(\pi)$ is an open subgroup of $G$. The compactness of $G$ implies easily that the index is finite.

Now let us do the case (ii).

(a) If $\ker(\pi) \subset H$ then we have

$$\pi(G) \cong G/\ker(\pi) \to G/H.$$  

If $H$ does not contain $\ker(\pi)$ then $G'/H'$ is a nontrivial normal subgroup of $G/H$. Now the simplicity of $G/H$ implies that $G'/H' = G/H$.

(b) Clearly, $\pi(G) \subset M$ is solvable. If $H$ contains $\ker(\pi)$ then the simple non-abelian $G/H$ becomes a homomorphic image of solvable $\pi(G)$. Contradiction.

(c) Replacing $M$ by $\pi(G)$, we may assume that $\pi$ is surjective. We may also assume that $H$ contains $\ker(\pi)$ and therefore there is a surjective homomorphism $\alpha : M \to G/H$. Since $M$ is isomorphic to a subgroup of $\text{PGL}_m(\mathbb{F}_\ell)$, it follows from a theorem of Feit-Tits ([8]; see also [13]) that $G/H$ is also isomorphic to a subgroup of $\text{PGL}_m(\mathbb{F}_\ell)$.

(d) The assertion (d1) is obvious. If $H$ contains $\ker(\pi')$ then we have

$$G/H = (G/\ker(\pi'))/(H/\ker(\pi')) = \pi'(G)/\pi'(H).$$  

If $H$ does not contain $\ker(\pi')$ then $G'/H'$ is a nontrivial normal subgroup of $G/H$. Now the simplicity of $G/H$ implies that $G'/H' = G/H$. This proves (d2). In order to prove (d3), let us assume first that $m = 2$ and therefore $G \subset S_1 \times S_2$. Let us define

$$\phi' : G \subset S_1 \times S_2 \to S_2$$  

as the restriction to $G$ of the projection map $S_1 \times S_2 \to S_2$. Applying (d2) to $\pi' = \phi'$, we conclude that either $\phi'(G)/\phi'(H) \cong G/H$ and one could put

$$\tilde{H} := \phi'(H) \subset \tilde{G} := \phi'(G) \subset S_2$$  

or

$$H = \ker(\phi') \cap \tilde{H} \subset \tilde{G} := \ker(\phi') \subset S_1 \times \{1\} \cong S_1.$$  

In order to do the case of $m > 2$ one has only to write down

$$S_1 \times \cdots S_m = S_1 \times (S_2 \times \cdots \times S_m)$$  

and apply induction.

(e) It is known ([14], Remarks 1–2 on pp. 41-42; see also [27], Ex. 16.3.9(1) on p. 277) that $\alpha(\mathfrak{h}_1(E))$ is a normal subgroup in $\mathfrak{h}_2(E)$ and the quotient...
Proposition 3.2. Suppose $V$ is a finite-dimensional vector space over a finite field $k$ of characteristic $\ell$ and $G$ is a subgroup of $\Aut(V)$ enjoying the following properties:

(i) $G$ is perfect, i.e. $G = [G,G]$;

(ii) $G$ contains a normal abelian subgroup $Z$ such that the quotient $\Gamma := G/Z$ is a simple non-abelian group.

(iii) There exists a positive integer $d \geq \dim_k(V)$ such that every nontrivial projective representation of $\Gamma$ in characteristic $\ell$ has dimension $\geq d$.

Then:

(a) The $G$-module $V$ is absolutely simple, $\dim_k(V) = d$ and $Z$ is the center of $G$.

(b) Every subgroup of $G$ (except $G$ itself) has index $\geq \max(5,d+1)$;

(c) For each finite field $k'$ of characteristic $\ell$ and each positive integer $a < d$ every homomorphism $G \to \PGL_a(k')$ is trivial.

Proof. Step 0. Each normal subgroup $H$ of $G$ either lies in $Z$ or coincides with $G$. Indeed, if $H$ contains $Z$ then the simplicity of $G/Z$ implies that either $H = Z$ or $H = G$. Assume that $H$ neither contains nor is contained in $Z$. Let us put $D := H \cap Z \neq Z$. Then $G_0 = G/D$ is perfect, $Z_0 = Z/D$ is a nontrivial abelian normal subgroup in $G/D = G_0$ and $G_0/H_0 = G/Z = \Gamma$ is simple non-abelian and $H_0 := H/D$ is a nontrivial normal subgroup of $G_0$ which meets $Z_0$ only at the identity element. The simplicity of $G_0/Z_0$ implies that $G_0 = Z_0 \times H_0$ which contradicts the perfectness of $G_0$.

We write $Z'$ for the Sylow-$\ell$-subgroup of $Z$. Clearly, $Z'$ is normal in $G$. It is also clear that if $Z' \neq \{1\}$ then every semisimple faithful finite-dimensional representation of $Z'$ in characteristic $\ell$ must have dimension 1. On the other hand, since $G$ is non-abelian, $\dim_k(V) > 1$. This implies that the $Z'$-module $V$ is semisimple if and
only if $Z' = \{1\}$. Taking into account that $Z'$ is normal in $G$ and applying Clifford’s theorem ([3], §49, Th. 49.2), we conclude that if the $G$-module $V$ is semisimple then $Z' = \{1\}$ and therefore the order of $Z$ is not divisible by $\ell$.

**Step 1.** Assume that the $G$-module $V$ is absolutely simple. Then $\#(Z)$ is prime to $\ell$. Let $k_1$ be the finite field obtained by adjoining to $k$ all $\#(Z)$th roots of unity and consider the $k_1$-vector space $V_1 = V \otimes_k k_1$. Clearly, $V_1$ carries a natural structure of absolutely simple faithful $G$-module and

$$\dim_{k_1}(V_1) = \dim_k(V) \leq d.$$  

For each character $\chi : Z \to k_1^*$ we write $V^\chi$ for the subspace

$$V^\chi := \{ v \in V_1 \mid zv = \chi(z)v \ \forall z \in Z \in G\}.$$  

Clearly,

$$V_1 = \oplus_{\chi} V^\chi,$$

$G$ permutes all $V^\chi$’s and this action factors through $G/Z = \Gamma$. It is also clear that the set of non-zero $V^\chi$’s consists, at most, of $\dim_{k_1}(V_1)$ elements. This implies that if the action of $G$ on all $V^\chi$’s is non-trivial then $G/Z = \Gamma$ contains a subgroup $S' \neq S$ with index $r \leq \dim_{k_1}(V_1) \leq d$. This gives us a nontrivial homomorphism $G \to \mathfrak{S}_r$ which must be an embedding, in light of simplicity of $\Gamma$. This implies that $r \geq 5$ and therefore $\mathfrak{S}_r$ is isomorphic to a subgroup of $\text{PGL}_{r-1}(\mathbb{F}_\ell)$ and therefore $\Gamma$ is isomorphic to a subgroup of $\text{PGL}_{r-1}(\mathbb{F}_\ell)$. Since $r \leq d$, we get a contradiction to property (iii). Hence, $G$ maps each $V^\chi$ into itself and therefore each $V^\chi$ is a $G$-invariant subspace of $V_1$. The absolute simplicity of $V_1$ implies that $V_1 = V^\chi$ for some $\chi$. This implies that $Z \subset k_1^*\text{Id}$; in particular, $Z$ is a central cyclic subgroup of $G$. Since $G/Z$ is simple non-abelian, $Z$ coincides with the center of $G$. Now the absolute simplicity of $V$ implies that $Z \subset k^*\text{Id} \subset \text{Aut}_k(V)$ and we get an embedding

$$\Gamma = G/Z \hookrightarrow \text{PGL}(V).$$

This implies that $d \leq \dim_k(V)$. Since $d \geq \dim_k(V)$, we conclude that $d = \dim_k(V)$. This ends the proof of (a) in the case of absolutely simple $G$-module $V$.

**Step 2.** Assume that the $G$-module $V$ is simple (but not necessarily absolutely simple). Let us put $\kappa = \text{End}_G(V)$. Clearly, $\kappa \supset k$ is a finite field of characteristic $\ell$, $V$ carries a natural structure of absolutely simple $\kappa[G]$-module and

$$\dim_{\kappa}(V) = \frac{\dim_k(V)}{[\kappa : k]}.$$  

Applying the (special case of the) assertion (a) (proven in Step 1) to the absolutely simple $\kappa[G]$-module $V$, we conclude that

$$d = \dim_{\kappa}(V) = \frac{\dim_k(V)}{[\kappa : k]} \leq \dim_k(V) \leq d.$$  

We conclude that $\dim_k(V) = \dim_{\kappa}(V)$ and therefore $[\kappa : k] = 1$. This implies that $\kappa = k$, i.e., the $G$-module $V$ is absolutely simple. This ends the proof of (a) in the case of simple $G$-module $V$.

**Step 3.** Assume that the $G$-module $V$ is semisimple (but not necessarily simple). Since $V$ is faithful, there is a simple non-trivial $G$-submodule $W$ of $V$. We have

$$\dim_k(W) \leq \dim_k(V) \leq d.$$  

Let us denote by $G_0 \neq \{1\}$ the image of $G$ in $\text{Aut}_k(W)$. Since $G$ is perfect, its homomorphic image $G_0$ is also perfect. Since $G_0 \neq \{1\}$, the kernel of $G \to G_0$
lies in \( Z \). We write \( Z_0 \) for the image of \( Z \) in \( G_0 \). Clearly, \( Z_0 \) is an abelian normal subgroup of \( G_0 \) and \( G_0/Z_0 = G/Z = \Gamma \). Applying the (special case of the) assertion (a) (proven in Step 2) to the faithful simple \( G_0 \)-module \( V \), we conclude that

\[
d = \dim_k(W) \leq \dim_k(V) \leq d.
\]

This implies that \( \dim_k(W) = \dim_k(V) \) and therefore \( V = W \) is a simple \( G \)-module. This ends the proof of (a) in the case of semisimple \( G \).

**Step 4.** End of the proof of (a). Assume that the \( G \)-module \( V \) is not semisimple. Let \( V^{ss} \) be its semisimplification. Clearly, the \( G \)-module \( V^{ss} \) is semisimple but not simple. Clearly, the kernel of the natural homomorphism \( G \to \text{Aut}_k(V^{ss}) \) consists of unipotent matrices and therefore is a finite normal \( \ell \)-group. This implies that this kernel lies in \( Z \). Let \( G_1 \) be the image of \( G \) in \( \text{Aut}_k(V^{ss}) \) and \( Z_1 \) be the image of \( Z \) in \( G_1 \). Clearly, \( Z_1 \) is an abelian normal subgroup of \( G_1 \) and \( G_0/Z_0 = G/Z = \Gamma \). It is also clear that the \( G \)-module \( V^{ss} \) is faithful semisimple but not simple. Applying the (special case of the) assertion (a) (proven in Step 3) to the faithful semisimple \( G_1 \)-module \( V^{ss} \), we conclude that \( V^{ss} \) is simple. We get a contradiction which proves that \( V \) is semisimple. This ends the proof of (a).

**Step 5.** Proof of (c). Suppose \( a < d \) is a positive integer, \( k' \) is a finite field of characteristic \( \ell \) and \( \phi : G \to \text{PGL}_a(k') \) is a nontrivial group homomorphism. By Step 0, \( \ker(\phi) \subset Z \). Let us put \( G_2 := \phi(G) \subset \text{PGL}_a(k') \) and

\[
Z_2 := \phi(Z) \subset G_2 = \phi(G) \subset \text{PGL}_a(k').
\]

Clearly, \( G_2 \) is perfect, \( Z_2 \) is a central cyclic subgroup of \( G_2 \) and \( G_2/Z_2 = \Gamma \). Let us denote by \( G_3 \) (resp. \( Z_3 \)) the preimage of \( G_2 \) (resp. of \( Z_2 \)) in \( \text{GL}_a(k') \) with respect to the projectivization map \( \text{GL}_a(k') \to \text{PGL}_a(k') \). Clearly,

\[
k_2^* \text{Id} \subset Z_3 \subset G_3 \subset \text{GL}_a(k')
\]

and

\[
Z_3/k_2^* \text{Id} = Z_2 \subset G_2 = G_3/k_3^* \text{Id}, \quad G_3/Z_3 = G_2/Z_2 = \Gamma.
\]

Since \( k_2^* \text{Id} \) lies in the center of \( Z_3 \) (and of \( G_3 \)) and the quotient \( Z_3/k_2^* \text{Id} = Z_2 \) is cyclic, \( Z_3 \) is abelian. If \( G_3 \) were perfect then we could apply the already proven assertion (a) to the faithful \( k'[G_3] \)-module \( k'^a \) and conclude that \( a = \dim_k k'^a = d \). This would lead us to contradiction, since \( a < d \) and we conclude that \( \phi \) is trivial and we are done. However, there is no reason to believe that \( G_3 \) is perfect. So, let us choose a minimal subgroup \( G' \) of \( G_3 \) which maps onto \( G_3/Z_3 = \Gamma \). Such a choice is possible in light of finiteness of \( G_3 \). Let us denote by \( Z' \) the intersection of \( G' \) and \( Z_3 \). Clearly, \( Z' \) is an abelian normal subgroup in \( G' \) and \( G'/Z' = G_3/Z_3 = \Gamma \). The minimality of \( G' \) and perfectness of \( \Gamma \) imply that \( G' \) is perfect. Now, applying the assertion (a) to the faithful \( k'[G'] \)-module \( k'^a \), we conclude that \( a = \dim_k k'^a = d \).

Since \( a < d \) our assumption that \( \phi \) is non-trivial was wrong. This proves (c).

**Step 6.** We still have to prove (b). Assume that \( G \) contains a subgroup of index \( < 5 \). This gives us a non-trivial homomorphism of \( G \) into the solvable group \( S_4 \). Since its kernel must lie in the abelian group \( Z \), we conclude that \( G \) is solvable which is not the case. So, if \( d < 5 \) then we are done.

Assume that \( d \geq 5 \) and \( G \) contains a subgroup of index \( r \leq d \). This gives us a non-trivial homomorphism of \( G \) into \( S_r \subset S_d \). Since \( S_d \) is isomorphic to a subgroup \( \text{PGL}_{d-1}(\mathbb{F}_q) \), we conclude that there exists a nontrivial homomorphism \( G \to \text{PGL}_{d-1}(\mathbb{F}_q) \). But this contradicts (c). \( \square \)
Remark 3.3. Let $d \geq 8$ be an even integer.

(a) In characteristic 2 all nontrivial projective representations of $A_{d+1}$ and of $A_{d+2}$ have dimension $\geq d$. Indeed, it is well-known that the groups $A_{d+1}$ and $A_{d+2}$ are perfect and their Schur multipliers coincide and equal to 2. This implies that all irreducible projective linear representation of $A_{d+1}$ and of $A_{d+2}$ in characteristic 2 are, in fact, linear representations. By a theorem of Wagner [32], all nontrivial linear representation of $A_{d+1}$ and of $A_{d+2}$ in characteristic 2 have dimension $\geq d$. This implies that all irreducible projective representation of $A_{d+1}$ and of $A_{d+2}$ in characteristic 2 have dimension $\geq d$. Taking into account that $A_{d+1}$ and $A_{d+2}$ are simple non-abelian groups, we conclude that all nontrivial projective representation of $A_{d+1}$ and of $A_{d+2}$ in characteristic 2 have dimension $\geq d$.

(b) Each nontrivial projective representation of $A_{d+2}$ in characteristic zero has dimension $\neq d$. Indeed, $d+2$ is an even integer $\geq 10$ and the desired assertion about representations in characteristic zero was proven in [12].

(c) There does not exist a faithful symplectic $d$-dimensional representation of $A_{d+1}$ in characteristic 0. Indeed, there exists exactly one (up to an isomorphism) faithful $d$-dimensional representation of $A_{d+1}$ in characteristic 0 (Th. 2.5.15 on p. 71 of [11]) and this representation is absolutely irreducible and orthogonal; hence it could not be symplectic.

(d) Suppose a short exact sequence

$$1 \to \mathbb{Z}/2\mathbb{Z} \to A_{d+1}' \to A_{d+1} \to 1$$

defines a non-splitting central extension of $A_{d+1}$ (i.e., $A_{d+1}'$ is the universal central extension of $A_{d+1}$). If $d \geq 10$ then every faithful representation of $A_{d+1}'$ in characteristic 0 has dimension $\neq d$. Indeed, for $10 \leq d \leq 12$ this assertion follows from the character tables in [10]. So, further we assume that $d \geq 14$.

We start with an elementary discussion of the dyadic expansion $d+1 = 2^w_1 + \cdots + 2^w_s$ of $d+1$. Here $w_i$’s are distinct nonnegative integers with $w_1 < \cdots < w_s$ and $s$ is the exact number of terms (non-zero digits) in the dyadic expansion of $n$. Since $d+1$ is odd, $w_0 = 0$ and each $w_i \geq i-1$. This implies that $d+1 \geq 2^s - 1$ and therefore

$$s \leq \log_2 (d+2).$$

By a theorem of Wagner (Th. 1.3(ii) on pp. 583–584 of [13]), each proper projective representation of $A_{d+1}$ (i.e., a nontrivial linear representation of $A_{d+1}'$) in characteristic $\neq 2$ has dimension divisible by $N := 2^\lceil \log_2 (d+1) \rceil = 2^\lceil \log_2 (d+2) \rceil$. So, in order to prove (b), it suffices to check that $d$ is not divisible by $N$ for all even $d \geq 14$.

If $d \geq 14$ then $2^{d-1} > (d+2)d^2$. Then $2^{d-1} > d^2$ and therefore $2^{d-s-1} > d^2$. Taking square roots at both sides, we get $2^{d-s-1} > d$. Then we see easily that $N = 2^\lceil \log_2 (d+2) \rceil > d$. This finishes the proof.

(d1) If $d = 8$ then all faithful absolutely irreducible representations of $A_{d+1}' = A_9'$ in characteristic 0 have dimension $\geq 8$; among them all the 8-dimensional representations are orthogonal ([1], p. 37). As above, this implies that there does not exist a faithful symplectic $d$-dimensional representation of $A_{d+1}'$ in characteristic 0.
Remark 3.4. Let us put \( d = 10 \). Recall [3] that the Schur multiplier of the Mathieu group \( M_{11} \) is 1 and therefore all projective representations of \( M_{11} \) are, in fact, linear. It is known [3] that all faithful irreducible representation of \( M_{11} \) in characteristic 2 have dimension \( \geq 10 \). Since \( M_{11} \) is perfect, all nontrivial representation of \( M_{11} \) in characteristic 2 have dimension \( \geq 10 \). It is also known [3] that in characteristic 0 all faithful irreducible representation of \( M_{11} \) have dimension \( \geq 10 \) and none of 10-dimensional absolutely irreducible representations of \( M_{11} \) is symplectic. This implies that in characteristic 0 none of faithful 10-dimensional representations of \( M_{11} \) is symplectic.

Remark 3.5. Let \( q \) be an odd power prime, \( m \geq 3 \) a positive integer, \( B = \mathbb{P}^{m-1}(\mathbb{F}_q) \) the \((m - 1)\)-dimensional projective space over \( \mathbb{F}_q \). Clearly, the cardinality of \( B \) is \( \frac{q^m - 1}{q - 1} \) which is odd (resp. even) if \( m \) is odd (resp. even). The projective special linear group \( L_m(q) := \text{PSL}(m, \mathbb{F}_q) \) acts naturally, faithfully and doubly transitively on \( B = \mathbb{P}^{m-1}(\mathbb{F}_q) \). It is well known that this action gives rise to the deleted permutation representation: a certain faithful absolutely irreducible \( \mathbb{Q}[L_m(q)] \)-module \((\mathbb{Q}B)^0 \) of \( \mathbb{Q} \)-dimension \( \frac{q^m - 1}{q - 1} - 1 \) (see for instance [11]). Clearly, the representation of \( L_m(q) \) in \((\mathbb{Q}B)^0 \) is orthogonal (since it is defined over \( \mathbb{Q} \)) and therefore is not symplectic, because it is absolutely irreducible. It is known ([3], Th. 1.1) that if \((q, n) \neq (3, 4)\) then in characteristic 0 all nontrivial irreducible projective representations of \( L_m(q) \) have dimension \( \geq \frac{q^m - 1}{q - 1} - 1 \) and there is exactly one (up to an isomorphism) a nontrivial irreducible projective representation of \( L_m(q) \) of dimension \( \frac{q^m - 1}{q - 1} - 1 \). This implies easily that if a short exact sequence

\[ 1 \to \mathbb{Z}/2\mathbb{Z} \to L_m(q)^\prime \to L_m(q) \to 1 \]

defines a central extension of \( L_m(q) \) then in characteristic 0 there does not exist a faithful symplectic absolutely irreducible representation of \( L_m(q)^\prime \) of dimension \( \leq \frac{q^m - 1}{q - 1} - 1 \).

Guralnich proved that if \((q, n) \neq (3, 4)\) then the dimension of each nontrivial projective irreducible representation of \( L_m(q) \) in characteristic 2 is greater than or equal to

\[ 2\left(\frac{q^m - 1}{q - 1} - 1\right)/2 \]

(see Th. 1.1 and Table III in [3]). Since \( L_m(q) \), this implies easily that in in characteristic 2 the dimension of every nontrivial projective representation of \( L_m(q) \) in characteristic 2 is greater than or equal to

\[ 2\left(\frac{q^m - 1}{q - 1} - 1\right)/2. \]

4. VERY SIMPLE REPRESENTATIONS

The following notion was introduced by the author in [11].

Definition 4.1. Let \( V \) be a vector space over a field \( k \), let \( G \) be a group and \( \rho : G \to \text{Aut}_k(V) \) a linear representation of \( G \) in \( V \). We say that the \( G \)-module \( V \) is very simple if it enjoys the following property:

If \( R \subset \text{End}_k(V) \) is an \( k \)-subalgebra containing the identity operator \( \text{Id} \) such that

\[ \rho(\sigma)R\rho(\sigma)^{-1} \subset R \quad \forall \sigma \in G \]
then either $R = k \cdot \text{Id}$ or $R = \text{End}_k(V)$.

Here is (obviously) an equivalent definition: if $R \subset \text{End}_k(V)$ is an $k$-subalgebra containing the identity operator $\text{Id}$ and stable under the conjugations by all $\rho(\sigma)$ then either $\dim_k(R) = 1$ or $\dim_k(R) = (\dim_k(V))^2$.

Remarks 4.2.  
(i) Clearly, the $G$-module $V$ is very simple if and only if the corresponding $\rho(G)$-module $V$ is very simple.
(ii) Clearly, if $V$ is very simple then the corresponding algebra homomorphism $k[G] \to \text{End}_k(V)$ is surjective. Here $k[G]$ stands for the group algebra of $G$. In particular, a very simple module is absolutely simple.
(iii) If $G'$ is a subgroup of $G$ and the $G'$-module $V$ is very simple then the $G$-module $V$ is also very simple.
(iv) Let $G'$ be a normal subgroup of $G$. If $V$ is a very simple $G$-module then either $\rho(G') \subset \text{Aut}_k(V)$ consists of scalars (i.e., lies in $k \cdot \text{Id}$) or the $G'$-module $V$ is absolutely simple. Indeed, let $R' \subset \text{End}_k(V)$ be the image of the natural homomorphism $k[G'] \to \text{End}_k(V)$. Clearly, $R'$ is stable under the conjugation by elements of $G$. Hence either $R'$ consists of scalars and therefore $\rho(G') \subset R'$ consists of scalars or $R' = \text{End}_k(V)$ and therefore the $G'$-module $V$ is absolutely simple.
(v) Suppose $F$ is a discrete valuation field with valuation ring $O_F$, maximal ideal $m_F$ and residue field $k = O_F/m_F$. Suppose $V_F$ a finite-dimensional $F$-vector space, $\rho_F : G \to \text{Aut}_F(V_F)$ a $F$-linear representation of $G$. Suppose $T$ is a $G$-stable $O_F$-lattice in $V_F$ and the corresponding $k[G]$-module $T/m_FT$ is isomorphic to $V$. Assume that the $G$-module $V$ is very simple. Then:

(a) The $G$-module $V_F$ is also very simple. In other words, a lifting of a very simple module is also very simple. Indeed, let $R_F \subset \text{End}_F(V_F)$ be an $F$-subalgebra containing the identity operator and stable under the conjugation by elements of $G$. Let us put

$$R_O = R \cap \text{End}_{O_F}(T) \subset \text{End}_{O_F}(T).$$

Clearly, $R_O$ is a free $O_F$-module, whose rank coincides with $\dim_F(R_F)$. It is also clear that $R_O$ is a pure $O_F$-submodule of $\text{End}_{O_F}(T)$. This implies that

$$R_k = R_O/m_FR_O = R_O \otimes_{O_F} k \subset \text{End}_{O_F}(T) \otimes_{O_F} k = \text{End}_k(T/m_FT) = \text{End}_k(V)$$

is an $k$-subalgebra of $\text{End}_k(V)$ of dimension $\dim_F(R_F)$, contains the identity operator and is stable under the conjugation by elements of $G$. Now the very simplicity of $V$ implies that either $\dim_k(R_k) = 1$ or $\dim_k(R_k) = (\dim_k(V))^2$. Since $\dim_k(V) = \dim_F(V_F)$, we conclude that either $\dim_F(R_F) = 1$ or $\dim_F(R_F) = \dim_F(W)^2$. Clearly, in the former case $R_F$ consists of scalars and in the latter one $R_F = \text{End}_F(V_F)$.

(b) Suppose that $\text{char}(F) = 0$ and $\rho_F$ is an embedding. Further we identify $G$ with its image $\rho_F(G) \subset \text{Aut}(V)$.

As usual, we write $\text{GL}_{V_F}$ for the $F$-algebraic group of automorphisms of $V$. In particular, $\text{GL}_{V_F}(F) = \text{Aut}(V)$. We write $\text{SL}_{V_F}$ for the $F$-algebraic
group of automorphisms of $V$ with determinant 1. In particular, $\text{SL}_V(F)$ coincides with $\text{SL}(V)$.

Let $G_{\text{alg}}$ be the \textit{algebraic envelope of} $G$, i.e., the smallest algebraic subgroup of $\text{GL}_V$, whose group of $F$-points contains $G$. Since $G \subset G_{\text{alg}}(F)$, the $G_{\text{alg}}(F)$-module $V_F$ is also very simple. Clearly, if $G \subset \text{SL}(V_F)$ then $G_{\text{alg}} \subset \text{SL}_V(F)$.

Let $G^0_{\text{alg}}$ be the identity component of $G_{\text{alg}}$. Clearly, $G^0_{\text{alg}}(F)$ is a normal subgroup of finite index in $G_{\text{alg}}(F)$. By (iv), either $G^0_{\text{alg}}(F)$ consists of scalars or the $G^0_{\text{alg}}(F)$-module $V_F$ is absolutely simple.

Assume, in addition, that if $G \subset \text{Aut}_F(V_F)$ is infinite and lies in $\text{SL}(V_F)$ then $G_{\text{alg}}(F)$ is also infinite and lies in $\text{SL}(V_F)$. Since $G^0_{\text{alg}}(F)$ has finite index in $G_{\text{alg}}(F)$, we conclude that $G^0_{\text{alg}}(F)$ is also infinite and lies in $\text{SL}(V_F)$. This implies that $G^0_{\text{alg}}(F)$ could not consist of scalars and therefore the $G^0_{\text{alg}}(F)$-module $V_F$ is absolutely simple. This implies easily that the $F$-algebraic group $G^0_{\text{alg}}$ is semisimple.

(c) Suppose $\ell$ is a prime, $F = \mathbb{Q}_\ell, O_F = \mathbb{Z}_\ell, k = \mathbb{F}_\ell$. We write $V_\ell$ for $V_F = V_{\mathbb{Q}_\ell}$. Assume that $G \subset \text{Aut}(V_\ell)$ is a compact subgroup. Then $G$ is a compact $\ell$-adic Lie subgroup of $\text{Aut}(V_\ell)$. We write $\mathfrak{g}$ for its Lie algebra; it is a $Q_\ell$-Lie subalgebra of $\text{End}(V_\ell)$. The absolute simplicity of the $G$-module $V_\ell$ implies that the $\mathfrak{g}$-module $V_\ell$ is semisimple (Prop. 1 of [23]) and therefore $\mathfrak{g}$ is reductive, i.e.,

$$\mathfrak{g} = \mathfrak{c} \times \mathfrak{s}$$

where $\mathfrak{c}$ is the center of $\mathfrak{g}$ and $\mathfrak{s} = [\mathfrak{g}, \mathfrak{g}]$ is a semisimple $\mathbb{Q}_\ell$-Lie algebra. This implies also that the $F$-algebraic group $G^0_{\text{alg}}$ is reductive (Prop. 2 of [23]) and its Lie algebra $\text{Lie}(G^0_{\text{alg}})$ coincides with $\mathfrak{c}_{\text{alg}} \times \mathfrak{s}$ where $\mathfrak{c}_{\text{alg}}$ is the algebraic envelope of $\mathfrak{c}$ and coincides with the center of $\text{Lie}(G_{\text{alg}})^0$.

We keep the assumption that $G \subset \text{Aut}_{\mathbb{Q}_\ell}(V_\ell)$ is infinite and lies in $\text{SL}(V_\ell)$. It follows that that $\text{Lie}(G^0_{\text{alg}})$ lies in the Lie algebra $\text{sl}(V_\ell)$ of linear operators in $V_\ell$ with zero trace. Since the $G^0_{\text{alg}}(F)$-module $V_\ell$ is absolutely simple (by (b)), the natural representation of the connected $\mathbb{Q}_\ell$-algebraic group $G^0_{\text{alg}}$ in $V_\ell$ is also absolutely irreducible. This implies, in turn, that the natural representation of the Lie algebra $\text{Lie}(G^0_{\text{alg}})$ in $V_\ell$ is also absolutely irreducible and therefore its center $\mathfrak{c}_{\text{alg}}$ is either zero or consists of scalars. Since

$$\mathfrak{c}_{\text{alg}} \subset \mathfrak{c}_{\text{alg}} \times \mathfrak{s} = \text{Lie}(G^0_{\text{alg}}) \subset \text{sl}(V_\ell),$$

we conclude that there are no non-zero scalars in $\mathfrak{c}_{\text{alg}}$ and therefore $\mathfrak{c}_{\text{alg}} = \{0\}$. This implies that $\mathfrak{c} = \{0\}$ and therefore

$$\text{Lie}(G^0_{\text{alg}}) = \mathfrak{s} = \mathfrak{g}.$$  

In particular, $\mathfrak{g}$ is a semisimple algebraic $\mathbb{Q}_\ell$-Lie algebra. This implies that $G$ is open in $G_{\text{alg}}(\mathbb{Q}_\ell)$ in $\ell$-adic topology and meets all the components of $G_{\text{alg}}$ (23), Prop. 2 and its Corollary).

The following assertion is a special case of Th. 4.3 of [13].

**Theorem 4.3.** Suppose a finite field field $k$, a positive integer $N$ and a group $H$ enjoy the following properties:
• $H$ is perfect, i.e., $H = [H, H]$;
• Each homomorphism from $H$ to $S_N$ is trivial;
• Let $N = ab$ be a factorization of $N$ into a product of two positive integers $a$ and $b$. Then either each homomorphism from $H$ to $\text{PGL}_a(k)$ is trivial or each homomorphism from $H$ to $\text{PGL}_b(F)$ is trivial.

Then each absolutely simple $H$-module of $k$-dimension $N$ is very simple. In other words, in dimension $N$ the properties of absolute simplicity and very simplicity over $F$ are equivalent.

**Corollary 4.4.** Suppose $V$ is a finite-dimensional vector space over a finite field $k$ of characteristic $\ell$ and $G$ is a subgroup of $\text{Aut}(V)$ enjoying the following properties:

(i) $G$ is perfect, i.e. $G = [G, G]$;
(ii) $G$ contains a normal abelian subgroup $Z$ such that the quotient $\Gamma := G/Z$ is a simple non-abelian group.
(iii) There exists a positive integer $d \geq \dim_k(V)$ such that every nontrivial projective representation of $\Gamma$ in characteristic $\ell$ has dimension $\geq d$.

Then $Z$ is a cyclic central subgroup of $G$ and the $G$-module $V$ is very simple.

**Proof.** Replacing $G$ by its minimal subgroup which maps onto $\Gamma$ we may assume, in light of Remark 4.2(iii) that $G$ is perfect. Now the very simplicity follows readily from Prop. 3.2 combined with Th. 4.3. \qed

5. $\ell$-adic representations

**Theorem 5.1.** Suppose $\ell$ is a prime, $V_\ell$ is a $\mathbb{Q}_\ell$-vector space of finite dimension $d > 1$, $G \subset \text{SL}(V_\ell) \subset \text{Aut}(V_\ell)$ a compact linear group. We write $G_{\text{alg}}$ for the algebraic envelope of $G$ and $\mathfrak{g}$ for the Lie algebra of $G$.

Suppose $T \subset V_\ell$ is a $G$-stable $\mathbb{Z}_\ell$-lattice in $V_\ell$. Let us put $V(\ell) := T/\ell T$. Clearly, $V(\ell)$ is a $d$-dimensional vector space and carries a natural structure of $G$-module.

Let us denote by $\tilde{G}_\ell$ the image of the natural homomorphism

$$G \to \text{Aut}_{\mathbb{Z}_\ell}(T) \to \text{Aut}(T/\ell T) = \text{Aut}_{\mathbb{F}_\ell}(V(\ell)).$$

Suppose $\tilde{G}_\ell$ enjoys the following properties:

(a) $\tilde{G}_\ell$ is a simple non-abelian group;
(b) Every faithful projective representation of $\tilde{G}_\ell$ in characteristic $\ell$ has dimension $\geq d$;
(c) Either $\ell = 2$ or $\tilde{G}_\ell$ is a finite group of Lie type in odd characteristic or one of 26 known sporadic groups. (In other words, modulo the classification, either $\ell = 2$ or $\tilde{G}_\ell$ is not a group of Lie type in characteristic 2.)
(d) Either $G$ is infinite or one of the two following conditions holds:

(i) $\ell > 2$ and there does not exist a lifting of the $\mathbb{F}_\ell[\tilde{G}_\ell]$-module $V(\ell)$ to an absolutely simple $\mathbb{Q}_\ell[\tilde{G}_\ell]$-module $\mathbb{Q}_\ell^d$. (E.g., every homomorphism of $\tilde{G}_\ell \to \text{GL}_d(\mathbb{Q}_\ell)$ is trivial.)
(ii) $\ell = 2$ and for each central extension (short exact sequence)

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \tilde{G}' \to \tilde{G}_2 \to 1$$

there does not exist such a lifting of the $\mathbb{F}_2[\tilde{G}']$-module $V(2)$ to an absolutely simple $\mathbb{Q}_2[\tilde{G}']$-module $\mathbb{Q}_2^d$ that the distinguished central subgroup $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathbb{Q}_2^d$ via multiplications by $\pm 1$. (E.g., every homomorphism $\tilde{G}_2 \to \text{PGL}_d(\mathbb{Q}_2)$ is trivial.)
(iii) \( \ell = 2 \), there exists a non-degenerate \( G \)-invariant alternating bilinear form on \( V_\ell \) and for each central extension

\[
1 \to \mathbb{Z}/2\mathbb{Z} \to \tilde{G}' \to \tilde{G}_2 \to 1
\]

there does not exist such a lifting of the \( \mathbb{F}_2[\tilde{G}'] \)-module \( V(2) \) to an absolutely simple \( \mathbb{Q}_2[\tilde{G}'] \)-module \( \mathbb{Q}_2^d \) that the distinguished central subgroup \( \mathbb{Z}/2\mathbb{Z} \) acts on \( \mathbb{Q}_2^d \) via multiplications by \( \pm 1 \) and there exists a non-degenerate \( \tilde{G}' \)-invariant alternating bilinear form on \( \mathbb{Q}_2^d \).

Then \( G \) is an open subgroup of \( G_{\text{alg}}(\mathbb{Q}_\ell) \) and the \( \mathbb{Q}_\ell \)-Lie algebra \( g \) is absolutely simple. If \( E \) is a finite Galois extension of \( \mathbb{Q}_\ell \) such that the \( E \)-simple Lie algebra \( g_E := g \otimes_{\mathbb{Q}_\ell} E \) splits then the faithful \( g_E \)-module \( V_E = V \otimes_{\mathbb{Q}_\ell} E \) is a fundamental representation of minimum dimension.

**Proof.**

**Step 0.** It follows from Corollary 4.4 that the \( \tilde{G}_\ell \)-module \( V(\ell) \) is very simple. In turn, it follows from Remark 4.2(v) that the \( G \)-module \( V_\ell \) is very simple. Let us denote by \( H \) the kernel of \( G \to \tilde{G}_\ell \). Clearly, \( H \) is a closed normal subgroup \( G \) and, by Remark 4.2(iv), either \( H \) consists of scalars or the \( H \)-module \( V_\ell \) is absolutely simple. If \( G \) is finite then \( H \) is also finite. Notice also that \( H \) is a pro-\( \ell \)-group. We have

\[
G/H = \tilde{G}_\ell.
\]

The simplicity of \( \tilde{G}_\ell \) implies that \( H \) is the largest closed normal pro-\( \ell \)-subgroup in \( G \).

**Step 1.** \( G \) is infinite. Indeed, let us assume that \( G \) is finite. Then \( H \) is a finite group consisting of automorphisms of \( T \) congruent to 1 modulo \( \ell \).

If \( \ell = 2 \) then, thanks to Minkowski-Serre Lemma 28, all nontrivial elements of \( H \) have order 2 and therefore \( H \) is a finite abelian group. Since every absolutely irreducible representation of a finite abelian group must have dimension 1, we conclude that \( H \) consists of scalars and therefore either \( H = \{1\} \) or \( H = \{\pm 1\} \). In both cases if we denote by \( \tilde{G}' \) the subgroup of \( \text{Aut}_{\mathbb{Z}_2}(T) \) generated by \( G \) and \( \{\pm 1\} \) then the reduction map modulo 2 gives us a central extension

\[
1 \to \{\pm 1\} \to \tilde{G}' \to \tilde{G}_2 \to 1.
\]

Clearly, the \( \tilde{G}' \)-module \( V_2 = \mathbb{Q}_2^d \) is very simple and is a lifting of the \( \tilde{G}_2 \)-module \( V_2 \). It is also clear that the \( \tilde{G}' \)-module \( V_2 = \mathbb{Q}_2^d \) is symplectic if the \( G \)-module \( V_2 \) is symplectic. This contradicts (d)(ii) and (d)(iii) respectively and therefore \( G \) must be infinite.

If \( \ell > 2 \) then, thanks to Minkowski-Serre Lemma 28, \( H = \{1\} \) and the reduction map gives us an isomorphism \( G \cong \tilde{G}_\ell \). This implies easily that the \( G = \tilde{G}_\ell \)-module \( V_\ell \cong \mathbb{Q}_\ell^d \) is a lifting of the \( \tilde{G}_\ell \)-module \( V(\ell) \). This contradicts (d)(i) and therefore \( G \) must be infinite.

**Step 2.** Now we know that \( G \) is infinite and lies in \( \text{SL}(V_\ell) \). It follows from Remark 4.2(v)(c) that the identity component \( G_{\text{alg}}^0 \) of \( G_{\text{alg}} \) is a semisimple \( \mathbb{Q}_\ell \)-algebraic group, its Lie algebra coincides with \( g \) and \( G \) is an open subgroup of \( G_{\text{alg}}(\mathbb{Q}_\ell) \). In addition, \( g \) is a semisimple \( \mathbb{Q}_\ell \)-Lie algebra coinciding with the Lie algebra of \( G_{\text{alg}} \) and the natural representation of \( g \) in \( V_\ell \) is absolutely irreducible.

**Step 3.** There exists a finite Galois extension \( E \) of \( \mathbb{Q}_\ell \) such that the semisimple \( E \)-Lie algebra

\[
g_E := g \otimes_{\mathbb{Q}_\ell} E
\]
is split; in particular, $\mathfrak{g}_E$ splits into a direct sum
$$
\mathfrak{g}_E = \oplus_{i \in I} \mathfrak{g}_i
$$
of absolutely simple split $E$-Lie algebras $\mathfrak{g}_i$. Here $I$ is the set of minimal non-zero ideals $\mathfrak{g}_i$ in $\mathfrak{g}_E$. It is well-known that
$$
V_E := V_{\ell} \otimes_{\mathbb{Q}_{\ell}} E
$$becomes a faithful absolutely simple $\mathfrak{g}_E$-module and splits into a tensor product
$$
V_E = \otimes_{i \in I} W_i
$$of faithful absolutely simple $\mathfrak{g}_i$-modules $W_i$. Since each $g_i$ is simple and $W_i$ is faithful,
$$
\dim_{\mathbb{E}}(W_i) \geq 2 \quad \forall i \in I.
$$
This implies that the cardinality
$$
r := \#(I) \leq \log_2(\dim_{\mathbb{E}}(V_E)) = \log_2(d) < d.
$$
Let us consider the adjoint representation
$$
\text{Ad} : G \to \text{Aut}(\mathfrak{g}) \subset \text{Aut}(\mathfrak{g}_E).
$$
Since the $\mathfrak{g}$-module $V_{\ell}$ is absolutely simple, $\ker(\text{Ad})$ coincides with the finite subgroups of scalars in $G$. (The finiteness follows from the inclusion $G \subset \text{SL}(V_{\ell})$. It follows easily that $\ker(\text{Ad})$ coincides with the center $Z(G)$ of $G$.)

Clearly, $\text{Ad}$ permutes elements of $I$ and therefore gives rise to the continuous homomorphism (composition)
$$
\pi_1 : G \to \text{Aut}(\mathfrak{g}_E) \to \text{Perm}(I) \cong S_r.
$$
Clearly, one could embed $S_r$ into $\text{PGL}_r(\overline{\mathbb{F}}_{\ell})$. Since $r < d$, it follows from Proposition 3.1(ii)(c) that $\tilde{G}_\ell = G/H = G_1/H_1$ where
$$
G_1 = \ker(\pi_1) \subset \text{Aut}_{\mathbb{Z}_{\ell}}(T), \quad H_1 = \ker(\pi_1) \bigcap H.
$$
Clearly, $G_1$ is an open subgroup of finite index in $G$ and therefore its Lie algebra coincides with $\mathfrak{g}$. It is also clear that $H_1$ is a pro-$\ell$-group. By definition of $G_1$ the image of
$$
G_1 \subset G \to \text{Aut}(\mathfrak{g}) \subset \text{Aut}(\mathfrak{g}_E) = \text{Aut}(\oplus_{i \in I} \mathfrak{g}_i)
$$lies in $\prod_{i \in I} \text{Aut}(\mathfrak{g}_i)$. Let us put
$$
\mathfrak{G} = \mathfrak{G}_{\text{al}_E}.
$$
We write $\mathfrak{G}_E$ for the semisimple split $E$-algebraic subgroup of $\text{GL}_{V_E}$ obtained from $\mathfrak{G}$ by extension of scalars. Clearly, the $E$-Lie algebra of $\mathfrak{G}_E$ coincides with $\mathfrak{g}_E$.

We write $\mathfrak{G}_i$ for the simply connected absolutely simple split $E$-algebraic subgroup, whose Lie algebra coincides with $\mathfrak{g}_i$ ([27]).

We write $\mathfrak{G}_{i}^{\text{Ad}} \subset \text{GL}_{\mathfrak{g}_i}$ for the adjoint group of $G_{\mathfrak{g}_i}$. If $\mathfrak{G}_i^{\text{Ad}} \subset \text{GL}_{\mathfrak{g}_i}$ is the adjoint group of $\mathfrak{G}_i$ then
$$
\mathfrak{G}_{E}^{\text{Ad}} = \prod_{i \in I} \mathfrak{G}_i^{\text{Ad}}.
$$
It is well-known that for each $i \in I$ the group $\mathfrak{G}_i^{\text{Ad}}(E)$ is a closed (in $\ell$-adic topology) normal subgroup in $\text{Aut}(\mathfrak{g}_i)$ of finite index 1, 2 or 6; in the latter case the quotient
\[ \text{Aut}(g_i)/\mathfrak{G}^\text{Ad}(E) \text{ is isomorphic to } S_3. \] (Recall that \( g_j \) is split.) Let us consider the composition
\[
\pi_2 : G_1 \to \prod_{i \in I} \text{Aut}(g_i) \to \prod_{i \in I} \text{Aut}(g_i)/\prod_{i \in I} \mathfrak{G}^\text{Ad}(E) = \prod_{i \in I} \text{Aut}(g_i)/\mathfrak{G}^\text{Ad}(E).
\]

It follows from Proposition 3.1(ii)(b) that if we put
\[
G_2 := \ker(\pi_2) \subset G_1 \subset \text{Aut}_Z(T), \quad H_2 = H_1 \cap G_2
\]
then \( G_2 \) is compact, \( H_2 \) is an open normal pro-\( \ell \)-subgroup of finite index in \( G_2 \) and \( G_2/H_2 = \hat{G}_\ell \). By definition of \( G_2 \) the image of
\[
G_2 \subset G_1 \to \prod_{i \in I} \text{Aut}(g_i) \to \text{Aut}(g_j)
\]
lies in \( \mathfrak{G}^\text{Ad}(E) \) for all \( j \in I \).

**Step 4.** Let us consider the canonical central isogeny of semisimple \( E \)-algebraic groups
\[
\alpha := \prod_{i \in I} \text{Ad},_E : \prod_{i \in I} \mathfrak{G} \to \prod_{i \in I} \mathfrak{G}^\text{Ad} = \mathfrak{G}^\text{Ad}.
\]
We also have the continuous group homomorphism
\[
\pi_2 : G_2 \to \prod_{i \in I} \mathfrak{G}^\text{Ad}(E) = \mathfrak{G}^\text{Ad}(E),
\]
whose kernel is a finite commutative group (consisting of scalars). Applying Prop. 3.1(ii)(e2), we conclude that there exists a compact subgroup \( G_3 \subset \prod_{i \in I} \mathfrak{G}^\text{Ad}(E) = \mathfrak{G}^\text{Ad}(E) \) and an open normal subgroup \( H_3 \) of finite index in \( G_3 \) such that \( G_3/H_3 \cong \hat{G}_\ell \). By Prop. 3.1(ii)(e3), every prime divisor of the (super)order of \( H_3 \) is either \( \ell \) or a divisor of one of \( \text{deg} (\text{Ad},_E) \). Applying Prop. 3.1(ii)(d3), we conclude that there exist \( j \in I \), a compact subgroup \( G_4 \subset \mathfrak{G}^\text{Ad}(E) \) and an open normal subgroup \( H_4 \) of finite index in \( G_4 \) such that \( G_4/H_4 \cong \hat{G}_\ell \). In addition, every prime divisor of the (super)order of \( H_4 \) is either \( \ell \) or a divisor of one of \( \text{deg} (\text{Ad},_E) \).

**Step 5.** Let \( W \) be a finite-dimensional \( E \)-vector space which carries a structure of faithful absolutely simple \( g_j \)-module. I claim that
\[
d' := \dim_E(W) \geq d.
\]
In order to prove this inequality first, notice that there exists a \( E \)-rational representation
\[
\rho_W : \mathfrak{G} \to \text{GL}_W,
\]
whose kernel is a finite central subgroup of \( \mathfrak{G} \). Let us consider the continuous homomorphism \( \pi_4 \) defined as
\[
G_4 \to \mathfrak{G}^\text{Ad}(E) \xrightarrow{\rho_W} \text{Aut}(W).
\]
Clearly, \( \ker(\pi_4) \) is a finite commutative group. Applying Prop. 3.1(ii)(d2), we conclude that \( G_5 = \rho_W(G_4) \) is a compact subgroup of \( \text{Aut}(W) \) containing an open normal subgroup \( H_5 := \rho_W(H_4) \) and \( \hat{G}_\ell = G_5/H_5 \).

Since \( G_5 \) is compact, there is a \( G_5 \)-stable \( O_E \)-lattice \( T_O \) in \( W \). Notice that Here \( O_E \) stands for the ring of integers in \( E \). Let \( m_E \) be the maximal ideal in \( O_E \) and \( k = O_E/m_E \) be the corresponding residue field. Let us denote by \( \pi_5 \) the restriction
\[
G_5 \subset \text{Aut}_{O_E}(T) \to \text{Aut}_k(T/m_E T) \cong \text{GL}_{d'}(k).
\]
of the residue map to $G_5$. Clearly, ker($\pi_5$) lies in the kernel of the reduction map \(\text{Aut}_{O_k}(T) \to \text{Aut}_k(T/m_{E}T)\) and therefore is a pro-$\ell$-group. Hence $\bar{G}_\ell$ could not be a homomorphic image of ker($\pi_5$). Let us put
\[
G_6 = \pi_5(G_5) \subset \text{GL}_{d}(k), \quad H_6 = \pi_5(H_5) \subset \text{G}_6.
\]
Applying Proposition 3.1(ii)(d2) to $G_5$ and $H_5$, we conclude that
\[
\bar{G}_\ell = G_5/H_5 = G_6/H_6.
\]
Let us consider the projectivization map
\[
\pi_6 : G_6 \subset \text{GL}_{d}(k) \to \text{PGL}_{d}(k).
\]
Clearly, ker($\pi_6$) is a cyclic group and $\bar{G}_\ell$ could not be its homomorphic image. Let us put
\[
G_7 = \pi_6(G_6) \subset \text{PGL}_{d}(k), \quad H_7 = \pi_6(H_6) \subset G_7.
\]
Applying Proposition 3.1(ii)(d2) to $G_6$ and $H_5$, we conclude that
\[
\bar{G}_\ell = G_6/H_6 = G_7/H_7.
\]
Since $G_7$ is a subgroup of PGL$_d(k) \subset$ PGL$_d(F)$, it follows from a theorem of Feit-Tits ([8]; see also [13]) that $\bar{G}_\ell$ is also isomorphic to a subgroup of PGL$_d(F)$. In light of property (b), $d' \geq d$ and we are done.

In particular, if we consider the faithful $\mathfrak{g}_j$-module $W_j$ then we get
\[
\dim_E(W_j) \geq d = \dim_E(V_E) = \prod_{i \in \ell} \dim_E(W_i).
\]
Since each $W_i$ is a faithful $\mathfrak{g}_i$-module and therefore $\dim_E(W_i) > 1$, we conclude that the whole set coincides with singleton \{j\}. This means that $\mathfrak{g}_E = \mathfrak{g}_j$ is absolutely simple and therefore $\mathfrak{g}$ is also absolutely simple. We also conclude that $V_E = W_j$ has the minimum dimension $d$.

It follows from Weyl’s character formula [3] that every faithful simple $\mathfrak{g}_E = \mathfrak{g}_j$-module of minimum $E$-dimension $d_0$ is fundamental. This implies that $V_E = W_j$ is fundamental. \(\square\)

**Corollary 5.2.** We keep all the notations and assumptions of Theorem 5.1. Recall that $\mathfrak{g}^0$ is the identity component for $G_{\text{alg}}$. Assume, in addition that there exists a non-degenerate alternating bilinear form
\[
e : V_\ell \times V_\ell \to \mathbb{Q}_\ell
\]
such that $G \subset \text{Aut}(V_\ell, e)$. We write Sp$_{V_\ell, e}$ for the corresponding symplectic $\mathbb{Q}_\ell$-algebraic group. If $d \neq 56$ then $\mathfrak{g} = \text{Sp}_{V_\ell, e}$. If $d = 56$ then either $\mathfrak{g} = \text{Sp}_{V_\ell, e}$ or $\mathfrak{g}^0$ is a simply-connected absolutely simple $\mathbb{Q}_\ell$-algebraic group of type $E_7$.

**Proof.** Clearly, $\mathfrak{g} \subset \text{Sp}_{V_\ell, e}$. This implies that the $\mathfrak{g}$-module $V_\ell$ is symplectic and therefore the $\mathfrak{g}_E$-module $V_E$ is also symplectic. By Theorem 3.1, $\mathfrak{g}_E$ is absolutely simple and $V_E$ is fundamental of minimum dimension. It follows from Tables in [2] that if $V_E$ is a symplectic fundamental representation of minimum dimension of $\mathfrak{g}_E$ then either $\mathfrak{g}_E$ is the Lie algebra of the symplectic group of $V_E$ or $\dim_{\mathbb{Q}}(V_E) = 56$ and $\mathfrak{g}_E$ is a Lie algebra of type $E_7$ and the highest weight of $V_E$ is the only minuscule dominant weight. One has only to recall that
\[
d = \dim_{\mathbb{Q}_\ell}(V_\ell) = \dim_{E}(V_E).
\]
Corollary 5.3. We keep all the notations and assumptions of Theorem 5.4. Assume, in addition that there exists a non-degenerate alternating bilinear form
\[ e : V_\ell \times V_\ell \to \mathbb{Q}_\ell \]
such that \( G \subset \text{Aut}(V_\ell, e) \). We write \( \text{Sp}_{V_\ell, e} \) for the corresponding symplectic \( \mathbb{Q}_\ell \)-algebraic group. Assume also that \( \ell = 2 \) and \( \tilde{G}_2 \) contains a subgroup isomorphic to the alternating group \( A_{d+1} \) (e.g., \( \tilde{G}_2 \cong A_{d+1} \) or \( \tilde{G}_2 \cong A_{d+2} \)). Then \( \mathfrak{G} = \text{Sp}_{V_\ell, e} \).

Proof. In light of Cor. 5.2 we may assume that \( d = 56 \) and \( \mathfrak{G}^0 \) is a simply-connected absolutely simple \( \mathbb{Q}_2 \)-algebraic group of type \( E_7 \). We have to arrive to a contradiction. First, notice that in the notations of Step 3 (Proof of Th. 5.1) \( V_E = W_J \) and \( \mathfrak{G}^0 = \mathfrak{G} = \prod_{i \in I} \mathfrak{G}_i = \mathfrak{G}_j \). Also, \( \deg(\text{Ad}_{1,E}) = 2 \). By Step 5 of Proof of Th. 5.1 there are a compact subgroup \( G_4 \subset \mathfrak{G}_j(E) = \mathfrak{G}_0(E) \), an open normal subgroup \( H_4 \subset G_4 \) such that \( G_4/H_4 = \tilde{G}_2 \) and \( 2 \) is the only prime divisor of the (super)order of \( H_4 \). Let us put \( p = 5 \). Notice that \( p \) is not a torsion number for \( F_7 \) ([20], 1.3.6; [21]). Clearly, every Sylow-\( p \)-subgroup of \( G_4 \) is isomorphic to a Sylow-\( p \)-subgroup of \( \tilde{G}_2 \). By assumption, \( \tilde{G}_2 \) contains a subgroup isomorphic to \( A_{d+1} = A_{57} \). This implies that \( \tilde{G}_2 \) contains an elementary abelian \( p \)-subgroup \( (\mathbb{Z}/p\mathbb{Z})^8 \) and therefore \( G_4 \) also contains an elementary abelian \( p \)-subgroup \( (\mathbb{Z}/p\mathbb{Z})^8 \). Since \( G_4 \subset \mathfrak{G}(E) \), the group \( \mathfrak{G}^0(Q_2) \) also contains \( (\mathbb{Z}/p\mathbb{Z})^8 \). Since \( p \) is not a torsion number for \( \mathfrak{G}^0 \), a maximal torus of \( \mathfrak{G}^0 \) contains a subgroup isomorphic to \( (\mathbb{Z}/p\mathbb{Z})^8 \). Since the rank of \( \mathfrak{G}^0 \) is \( 7 < 8 \), we obtain the desired contradiction.
\[ \square \]

Theorem 5.4. Suppose \( V_2 \) is a \( \mathbb{Q}_2 \)-vector space of even finite dimension \( d \geq 8 \), \( G \subset \text{SL}(V_2) \subset \text{Aut}(V_2) \) a compact linear group. Suppose that there exists a non-degenerate alternating bilinear form
\[ e : V_\ell \times V_\ell \to \mathbb{Q}_2 \]
such that \( G \subset \text{Aut}(V_\ell, e) \). We write \( G_{d,9} \) for the algebraic envelope of \( G \) and \( \mathfrak{g} \) for the Lie algebra of \( G \).

Suppose \( T \subset V_2 \) is a \( G \)-stable \( \mathbb{Z}_2 \)-lattice in \( V_2 \). Let us put \( V(2) := T/2T \). Clearly, \( V(2) \) is a \( d \)-dimensional vector space and carries a natural structure of \( G \)-module. Let us denote by \( \tilde{G}_2 \) the image of the natural homomorphism
\[ G \to \text{Aut}_{\mathbb{Z}_2}(T) \to \text{Aut}(T/2T) = \text{Aut}_{\mathbb{F}_2}(V(2)). \]

Suppose that \( \tilde{G}_2 \) contains a subgroup isomorphic to \( V_{d+1} \).

Then \( \mathfrak{G} = \text{Sp}_{V_\ell, e} \) and \( \mathfrak{g} \) coincides with the Lie algebra \( \mathfrak{sp}_{V_\ell, e} \) of the symplectic group \( \text{Sp}_{V_\ell, e} \).

Proof. Clearly, \( \mathfrak{G} \subset \text{Sp}_{V_\ell, e} \). Replacing \( G \) by its open subgroup of finite index, we may assume that \( \tilde{G}_2 = A_{d+1} \). Taking into account Remark 5.3, we observe that \( \ell = 2 \), \( G \), \( d \) and \( G_2 \) satisfy the conditions of Theorem 5.1 and Corollary 5.3.
\[ \square \]

6. Applications to abelian varieties

Theorem 6.1. Suppose \( K \) is a field with \( \text{char}(K) \neq 2 \), suppose \( X \) is an abelian variety over a \( K \) and \( \lambda \) is a polarization on \( X \). Suppose \( \tilde{G}_{2,K} \) is the image of \( \text{Gal}(K) \) in \( \text{Aut}(X_2) \). Let us put \( d = 2\dim(X) \). Assume that \( \tilde{G}_{2,K} \) contains a simple non-abelian subgroup \( \mathfrak{G} \), enjoying the following properties.
Every faithful projective representation of $G$ in characteristic 2 has dimension $\geq d$. If $d = 56$ (i.e., $\text{dim}(X) = 28$), assume $G$ contains $A_{d+1}$. Then:

(i) Either $g_{2,X} = \text{sp}(V_2(X), e_\lambda)$ or $g_{2,X} = Q_2 \text{Id} \oplus \text{sp}(V_2(X), e_\lambda)$ or $g_{2,X} = \{0\}$ or $g_{2,X} = Q_2 \text{Id}$.

If every finite algebraic extension of $K$ contains only finitely many 2-power roots of unity then either $g_{2,X} = Q_2 \text{Id}$ or $g_{2,X} = Q_2 \text{Id} \oplus \text{sp}(V_2(X), e_\lambda)$. If there exists a finite algebraic extension of $K$ that contains all $\ell$-power roots of unity then either $g_{2,X} = \text{sp}(V_2(X), e_\lambda)$ or $g_{2,X} = \{0\}$.

(ii) Assume that for each central extension (short exact sequence)

$$1 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow G' \rightarrow G \rightarrow 1$$

there does not exist such a lifting of the $\mathbb{F}_2[G']$-module $X_2$ to an absolutely simple $Q_2[G']$-module $Q_2^2$ that the distinguished central subgroup $\mathbb{Z}/2\mathbb{Z}$ acts on $Q_2^2$ via multiplications by $\pm 1$ and there exists a non-degenerate $G'$-invariant alternating bilinear form on $Q_2^2$. Then:

(a) the ring $\text{End}(X)$ of all $K_2$-endomorphisms of $X$ is $\mathbb{Z}$.

(b) either $g_{2,X} = \text{sp}(V_2(X), e_\lambda)$ or $g_{2,X} = Q_2 \text{Id} \oplus \text{sp}(V_2(X), e_\lambda)$.

If every finite algebraic extension of $K$ contains only finitely many 2-power roots of unity then $g_{2,X} = Q_2 \text{Id} \oplus \text{sp}(V_2(X), e_\lambda)$. If there exists a finite algebraic extension of $K$ that contains all $\ell$-power roots of unity then $g_{2,X} = \text{sp}(V_2(X), e_\lambda)$.

Proof. Replacing if necessary $K$ by its suitable finite separable extension we may assume that $G_{2,X} = G$. Let us put $V_2 = V_2(X), T = T_2(X), e = e_\lambda$ and

$$G = \rho_{2,X}(\text{Gal}(K_2/K(2)) \subset G_{2,X} = \rho_{2,X}(\text{Gal}(K)) \subset \text{Aut} \mathbb{Z}_2(T) \subset \text{Aut}_{Q_2}(V_2).$$

Clearly, $G$ is a normal closed subgroup of $G_{2,X}$ and the quotient $G_{2,X}/G$ is a one-dimensional or zero-dimensional compact commutative $\ell$-adic Lie group. Since $G$ is simple non-abelian, the image of $G$ in $\text{Aut}(T/2T)$ coincides with $G_{2,X} = G$.

Assume that the condition (ii) holds. Applying Corollaries 5.2 and 5.3 we conclude that the Lie algebra of $G$ coincides with $\text{sp}(V_2(X), e_\lambda)$. Since $G$ is a closed subgroup of $G_{2,X}$,

$$\text{sp}(V_2(X), e_\lambda) \subset g_{2,X} \subset Q_2 \text{Id} \oplus \text{sp}(V_2(X), e_\lambda)$$

and therefore either $g_{2,X} = \text{sp}(V_2(X), e_\lambda)$ or $g_{2,X} = Q_2 \text{Id} \oplus \text{sp}(V_2(X), e_\lambda)$. In order to prove that $\text{End}(X) = \mathbb{Z}$, recall that $\text{End}(X) \otimes Q_2 \subset \text{End}_{Q_2}(V_2(X))$ commutes with $g_{2,X}$. Since the centralizer of $\text{sp}(V_2(X), e_\lambda)$ consists of scalars, $\text{End}(X) \otimes Q_2 = Q_2 \text{Id}$ and therefore $\text{End}(X) = \mathbb{Z}$.

Assume that the condition (i) holds. If $G$ is infinite then applying Theorem 5.4 we conclude by the same token that either $g_{2,X} = \text{sp}(V_2(X), e_\lambda)$ or $g_{2,X} = Q_2 \text{Id} \oplus \text{sp}(V_2(X), e_\lambda)$. Assume that $G$ is finite. Then the dimension of $G_{2,X}$ is either 0 or 1. If it is 0 then $g_{2,X} = \{0\}$. Assume that the dimension of $G_{2,X}$ is 1 and therefore $G_{2,X}$ is infinite and $g_{2,X}$ is a one-dimensional $Q_2$-vector subspace of $\text{End}_{Q_2}(V_2(X))$.

Our goal now is to prove that $g_{2,X}$ consists of scalars. Let $u$ be a non-zero element of $g_{2,X}$. Clearly, $g_{2,X} = Q_2 u$ and the conjugation by $G_{2,X}$ leaves $g_{2,X} = Q_2 u$ stable. This implies that the conjugation by $G_{2,X}$ also leaves stable the subalgebra $Q_2[u] \subset \text{End}_{Q_2}(V_2(X))$. Assume for a moment that the $G_{2,X}$-module $V_2(X)$ is very simple. Then either $Q_2[u] = \text{End}_{Q_2}(V_2(X))$ or $Q_2[u] = Q_2 \text{Id}$. The former equality could not be true, since $Q_2[u]$ is commutative while $\text{End}_{Q_2}(V_2(X))$ is not.
Hence $Q_2[u] = Q_2Id$, i.e., $u$ is a scalar and therefore $g_{2,X} = Q_2u = Q_2Id$ and we are done.

In order to finish the proof, recall that if every finite algebraic extension of $K$ contains only finitely many 2-power roots of unity then $g_{2,X}$ does not lie in $sl(V_2(X))$ and therefore $g_{2,X}$ is neither $\{0\}$ nor $sp(V_2(X),e_\lambda)$. If there exists a finite algebraic extension of $K$ that contains all $\ell$-power roots of unity then

$$g_{2,X} \subset sl(V_2(X)) \bigcap \{Q_2Id \oplus sp(V_2(X), e_\lambda)\} = sp(V_2(X), e_\lambda)$$

and therefore $g_{2,X}$ is neither $Q_2Id \oplus sp(V_2(X), e_\lambda)$ nor $Q_2Id$.

\[ \square \]

**Corollary 6.2.** Suppose $K$ is a field with char($K$) $\neq 2$, suppose $X$ is a 5-dimensional abelian variety over a $K$ and $\lambda$ is a polarization on $X$. Suppose $\tilde{G}_{2,X}$ is the image of $Gal(K)$ in $Aut(X_2)$. Assume that $\tilde{G}_{2,X}$ contains a subgroup isomorphic to $M_{11}$ (e.g., $\tilde{G}_{2,X} = M_{11}$ or $\tilde{G}_{2,X}$ is isomorphic to the Mathieu group $M_{12}$). Then:

(a) the ring $End(X)$ of all $K_\alpha$-endomorphisms of $X$ is $Z$.
(b) either $g_{2,X} = sp(V_2(X), e_\lambda)$ or $g_{2,X} = Q_2Id \oplus sp(V_2(X), e_\lambda)$.

If every finite algebraic extension of $K$ contains only finitely many 2-power roots of unity then $g_{2,X} = Q_2Id \oplus sp(V_2(X), e_\lambda)$. If there exists a finite algebraic extension of $K$ that contains all $\ell$-power roots of unity then $g_{2,X} = sp(V_2(X), e_\lambda)$.

**Proof.** We have $d := 2dim(X) = 10$. The proof follows readily from Theorem 6.1(ii) (applied to $\mathcal{G} = M_{11}$) combined with Remark 3.4.

\[ \square \]

**Corollary 6.3.** Suppose $K$ is a field with char($K$) $\neq 2$, suppose $X$ is an abelian variety over a $K$ and $\lambda$ is a polarization on $X$. Suppose $\tilde{G}_{2,X}$ is the image of $Gal(K)$ in $Aut(X_2)$. Let us put $d = 2dim(X)$. Assume that $\tilde{G}_{2,X}$ contains a simple non-abelian subgroup $\mathcal{G}$, enjoying the following properties.

There exist an odd power prime $q$ and an integer $m \geq 3$ such that $(q, m) \neq (3,4)$, $\frac{q^m - 1}{q - 1} = d + 1$ or $d + 2$ and $\mathcal{G} \cong L_m(q)$. Then:

(a) the ring $End(X)$ of all $K_\alpha$-endomorphisms of $X$ is $Z$.
(b) either $g_{2,X} = sp(V_2(X), e_\lambda)$ or $g_{2,X} = Q_2Id \oplus sp(V_2(X), e_\lambda)$.

If every finite algebraic extension of $K$ contains only finitely many 2-power roots of unity then $g_{2,X} = Q_2Id \oplus sp(V_2(X), e_\lambda)$. If there exists a finite algebraic extension of $K$ that contains all $\ell$-power roots of unity then $g_{2,X} = sp(V_2(X), e_\lambda)$.

**Proof.** The proof follows readily from Theorem 6.1(ii) (applied to $\mathcal{G} = L_m(q)$) combined with Remark 3.3.

\[ \square \]

7. PROOF OF MAIN RESULTS

**Proof of Theorem 2.2.** In order to get the assertions about $g_{2,X}$, one has only to combine Remark 6.3 and Theorem 5.1 (applied to $\mathcal{G} = A_{d+1}$).

\[ \square \]

**Remark 7.1.** Suppose $f(x) \in K[x]$ is a polynomial of degree $n \geq 5$ without multiple roots and $X = J(C_f)$ is the Jacobian of $C = C_f : y^2 = f(x)$. One may easily check that $d = 2dim(J(C)) = n - 1$ if $n$ is odd and $d = 2dim(J(C)) = n - 2$ when
n is even. It is well-known (see for instance Sect. 5 of [11]) that if $X = J(C_f)$ is the jacobian of
\[ C = C_f : y^2 = f(x) \]
where $f(x) \in K[x]$ has no multiple roots then $\tilde{G}_{2,X}$ is isomorphic to $\text{Gal}(f)$. Clearly, if $n \geq 9$ and $\text{Gal}(f)$ contains $A_n$ then $\text{Gal}(f)$ contains $A_{d+1}$. It is also clear that if $n = 12$ and $\text{Gal}(f)$ contains $M_{12}$ then it also contains $M_{11}$.

**Proof of Theorem 2.3.** By Th. 2.2, 
\[ \text{dim}(X_{\ell}) \geq \text{dim}(X) \]
for all primes $\ell$. It follows from Remark 7.1 combined with Theorem 6.1 and Corollaries 6.2 and 6.3 that $\text{End}(X) = \mathbb{Z}$ and $g_{2,X}$ contains $\text{sp}(V_2(X), e_\lambda)$. One has only to recall that $\text{sp}(V_2(X), e_\lambda)$ coincides with the Lie algebra of $\text{Aut}_{\mathbb{Z}_p}(T_2(X), e_\lambda)$ and $g_{2,X}$ is the Lie algebra of $\rho_{2,J(C_f)}(\text{Gal}(K))$.

**Proof of Theorem 2.4.** It follows from Remark 7.1 combined with Theorem 6.1 and Corollaries 6.2 and 6.3 that $\text{End}(X) = \mathbb{Z}$ and $g_{2,X}$ contains $\text{sp}(V_2(X), e_\lambda)$. One has only to recall that $\text{sp}(V_2(X), e_\lambda)$ coincides with the Lie algebra of $\text{Aut}_{\mathbb{Z}_p}(T_2(X), e_\lambda)$ and $g_{2,X}$ is the Lie algebra of $\rho_{2,J(C_f)}(\text{Gal}(K))$.

**Lemma 7.2.** Assume that the field $K$ is either finitely generated over $\mathbb{Q}$ or a global field of characteristic $> 2$. Let $X$ be an abelian variety defined over $K$ such that $g_{2,X} = \mathbb{Q}_2 \text{Id} \oplus \text{sp}(V_2(X), e_\lambda)$. Then $g_{2,X} = \mathbb{Q}_2 \text{Id} \oplus \text{sp}(V_2(X), e_\lambda)$ for all primes $\ell \neq \text{char}(K)$. This assertion follows readily from the next auxiliary statement.

**Proof of Lemma 7.2.** First, assume that $K$ is a global field. It follows easily (as in the proof of Th. 6.1(ii)) that $\text{End}(X) = \mathbb{Z}$. Now it follows from results of [53, 1] that $g_{2,X}$ is an absolutely irreducible reductive subalgebra of $\text{End}_{\mathbb{Q}_2}(V_2(X))$ for all primes $\ell \neq \text{char}(K)$. We also have

\[ g_{2,X} \subset \mathbb{Q}_2 \text{Id} \oplus \text{sp}(V_2(X), e_\lambda) \subset \text{End}_{\mathbb{Q}_2}(V_2(X)). \]

Notice that the rank of the reductive $\mathbb{Q}_2$-Lie algebra $g_{2,X}$ does not depend on the choice of $\ell$ [87]. This implies that $g_{2,X}$ and $\mathbb{Q}_2 \text{Id} \oplus \text{sp}(V_2(X), e_\lambda)$ have the same rank $\dim(X) + 1$. It follows from a variant of a theorem of Borel - de Siebenthal ([88], Key Lemma on p. 522) that $g_{2,X} = \mathbb{Q}_2 \text{Id} \oplus \text{sp}(V_2(X), e_\lambda)$. This proves Lemma 7.2 in the case of global $K$.

The case of an arbitrary $K$ finitely generated over $\mathbb{Q}$ follows from the case of number field with the help of Serre’s variant of Hilbert irreducibility theorem for finite extensions in the case of characteristic zero ([2], Sect. 10.6; [27], Sect. 1; [18], Prop. 1.3). Indeed, let us fix an odd prime $\ell$. Then there exists a number field $K_0$ and an abelian variety $Y$ over $K_0$ such that $\dim(Y) = \dim(X) := g$ and

\[ g_{2,Y} \cong g_{2,X}, \quad g_{2,Y} \cong g_{2,Y} \]

as $\mathbb{Q}_2$-Lie algebra and $\mathbb{Q}_2$-Lie algebra respectively. This implies that

\[ g_{2,Y} \cong g_{2,X} = \mathbb{Q}_2 \text{Id} \oplus \text{sp}(V_2(X), e_\lambda) \cong \mathbb{Q}_2 \text{Id} \oplus \text{sp}(2g, \mathbb{Q}_2) \]

and easy dimension arguments imply that $Y$ over $K_0$ satisfies the conditions of Lemma 7.2. Since we already know that Lemma 7.2 is true in the case of a ground number field, we conclude that

\[ g_{2,Y} \cong \mathbb{Q}_2 \text{Id} \oplus \text{sp}(2g, \mathbb{Q}_2) \]

and therefore

\[ g_{2,Y} \cong \mathbb{Q}_2 \text{Id} \oplus \text{sp}(2g, \mathbb{Q}_2). \]

Again easy dimension arguments imply that

\[ g_{2,Y} = \mathbb{Q}_2 \text{Id} \oplus \text{sp}(V_2(X), e_\lambda). \]
This ends the proof of Theorem 2.3.

Proof of Theorem 2.5. The proof is a straightforward application of Theorem 2.3 combined with Remark 7.1.

Proof of Theorem 2.6. In light of Theorem 2.5, we may assume that $5 \leq n \leq 8$. Let us put $X = J(C_f)$. Clearly, $\dim(X) = 2$ or 3. Notice also that $\text{End}(X) = \mathbb{Z}$. Let us put $g := \dim(X)$.

First assume that $K$ is a number field. Then it follows from results of [36] (combined with the results of [6]) that $g_{\ell, X} \cong \mathbb{Q}_\ell \text{Id} \oplus \text{sp}(V_{\ell}(X), e_\lambda)$ for all primes $\ell$.

Now assume that $K$ is an arbitrary field of characteristic zero finitely generated over $\mathbb{Q}$. Let us fix a prime $\ell$. Thanks to Prop. 1.3 and Cor. 1.5 in [18], there exists a number field $K_0$ and an abelian variety $Y$ over $K_0$ such that $\dim(Y) = g_{\ell, X} \cong g_{\ell, Y}$. Since $\text{End}(X) = \mathbb{Z}$, we conclude that $\text{End}(Y) = \mathbb{Z}$.

Since we already know that Theorem 2.6 is true in the case of ground number field, we conclude that $g_{\ell, Y} \cong \mathbb{Q}_\ell \text{Id} \oplus \text{sp}(2g, \mathbb{Q}_\ell)$ and therefore

$g_{\ell, X} \cong \mathbb{Q}_\ell \text{Id} \oplus \text{sp}(2g, \mathbb{Q}_\ell)$.

Now easy dimension arguments imply that

$g_{\ell, X} = \mathbb{Q}_\ell \text{Id} \oplus \text{sp}(V_{\ell}(X), e_\lambda)$.

Remark 7.3. Concerning specializations of the endomorphism rings of abelian varieties see also [15].

8. Tate classes

Theorem 8.1. Let $K$ be a field with $\text{char}(K) \neq 2$, $K_s$ its separable algebraic closure, $f(x) \in K[x]$ a separable polynomial of degree $n \geq 5$, whose Galois group $\text{Gal}(f)$ enjoys one of the following properties:

(i) $\text{Gal}(f)$ is either $S_n$ or $A_n$;

(ii) $n = 11$ and $\text{Gal}(f)$ is the Mathieu group $M_{11}$;

(iii) $n = 12$ and $\text{Gal}(f)$ is either the Mathieu group $M_{12}$ or $M_{11}$.

(iv) There exist an odd power prime $q$ and an integer $m \geq 3$ such that $(q, m) \neq (3, 4)$, $n = \frac{q^m - 1}{q - 1}$ and $\text{Gal}(f)$ contains a subgroup isomorphic to the projective special linear group $L_m(q) := \text{PSL}_m(F_q)$. (E.g., $\text{Gal}(f)$ is isomorphic either to the projective linear group $\text{PGL}_m(F_q)$ or to $L_m(q)$.)

Let $C_f$ be the hyperelliptic curve $y^2 = f(x)$ and let $J(C_f)$ be its jacobian. Assume, in addition, that either $K$ is a global field of positive characteristic and $n \geq 9$ or $K$ is a field of characteristic zero finitely generated over $\mathbb{Q}$.

Let $K' \subset K_s$ be a separable finite algebraic extension of $K$. Then for all primes $\ell \neq \text{char}(K)$ and on each self-product $J(C_f)^M$ of $J(C_f)$ every $\ell$-adic Tate class
over $K'$ can be presented as a linear combination of products of divisor classes over $K_a$. In particular, the Tate conjecture is valid for all $J(C_f)^M$ over all $K'$.

Proof. Let us put $X := J(C_f)$. Recall that one may view $\ell$-adic Tate classes on self-products of $X$ as tensor invariants of $g_{\ell,X} \cap \sp(V_\ell(X), e_\lambda)$ \[30\].

Assume that $g_{\ell,X} = \Q_{\ell,\text{Id}} \oplus \sp(V_\ell(X), e_\lambda)$. Then

$$g_{\ell,X} \cap \sp(V_\ell(X), e_\lambda) = \sp(V_\ell(X), e_\lambda).$$

It follows with the help of results from the invariant theory for symplectic groups \([14, 39]\) that each $\ell$-adic Tate class on $X^M = J(C_f)^M$ could be presented as a linear combination of products of divisor classes and therefore is algebraic. \[ \]

Remark 8.2. The Tate conjecture is true in codimension 1 for arbitrary abelian varieties over $K$ \[34, 7\].

9. HODGE CLASSES

Theorem 9.1. Suppose $f(x) \in \C[x]$ is a polynomial of degree $n \geq 5$ and without multiple roots. Let $C_f$ be the hyperelliptic curve $y^2 = f(x)$ and $J(C_f)$ its jacobian. Assume that all the coefficients of $f$ lie in a subfield $K \subset \C$ and the Galois group $\text{Gal}(f)$ of $f$ over $K$ enjoys one of the following properties:

(i) $\text{Gal}(f)$ is either $S_n$ or $A_n$;
(ii) $n = 11$ and $\text{Gal}(f)$ is the Mathieu group $M_{11}$;
(iii) $n = 12$ and $\text{Gal}(f)$ is either the Mathieu group $M_{12}$ or $M_{11}$.
(iv) There exist an odd power prime $q$ and an integer $m \geq 3$ such that $(q, m) \neq (3, 4)$, $n = \frac{q^m - 1}{q - 1}$ and $\text{Gal}(f)$ contains a subgroup isomorphic to the projective special linear group $L_m(q) := \text{PSL}_m(F_q)$. (E.g., $\text{Gal}(f)$ is isomorphic either to the projective linear group $\text{PGL}_n(F_q)$ or to $L_m(q)$.)

Then each Hodge class on every self-product $J(C_f)^M$ of $J(C_f)$ can be presented as a linear combination of products of divisor classes. In particular, the Hodge conjecture is valid for all $J(C_f)^M$.

Proof. If $n \leq 8$ then $\dim(J(C_f)) \leq 3$. But it is well-known that the assertion of the theorem is true for all complex abelian varieties, whose dimension does not exceed 3 (see, for instance, \[18\]). Further, we assume that

$$n \geq 9.$$\]

Replacing $K$ by its subfield obtained by adjoining to $\Q$ all coefficients of $f$, we may assume that $K \subset \C$ is finitely generated over $\Q$. Let us put $X = J(C_f)$. Thanks to Theorem 2.3,

$$g_{\ell,X} = \Q_{\ell,\text{Id}} \oplus \sp(V_\ell(X), e_\lambda)$$

for all primes $\ell$.

The first rational homology group $\Pi_1 := H_1(X(\C), \Q)$ of the complex torus $X(\C)$ is a $2\dim(X)$-dimensional $\Q$-vector space provided with a natural structure of rational polarized Hodge structure of weight $-1$. We refer to \[13\] for the definition of its Mumford-Tate group $MT = MT_X \subset \text{GL}(\Pi_1)$. It is a reductive algebraic $\Q$-group. We write $\text{mt} = \text{mt}_X$ for the Lie algebra of $MT$; it is a reductive completely reducible $\Q$-Lie subalgebra of $\text{End}_\Q(\Pi_1)$. The polarization $\lambda$ on $X$ gives rise to a non-degenerate alternating bilinear (Riemann) form \[17\].
such that
\[ mt \subset \mathbb{Q} \text{Id} \oplus \text{sp}(\Pi_{\mathbb{Q}}, L_{\lambda}). \]

Let us choose by \( K_a \) the algebraic closure of \( K \) in \( \mathbb{C} \). For each prime \( \ell \) let us put
\[ \Pi_{\ell} := \Pi_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell. \]

Then there is the well-known natural isomorphism \([17], [38]\)
\[ \gamma_{\ell} : \Pi_{\ell} \cong V_{\ell}(X) \]
such that, by a theorem of Piatetski-Shapiro - Deligne - Borovoi \([3], [21]\),
\[ \gamma_{\ell} g_{\ell,X} \gamma_{\ell}^{-1} \subset mt \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \subset [\mathbb{Q} \text{Id} \oplus \text{sp}(\Pi_{\mathbb{Q}}, L_{\lambda})] \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \subset \text{End}_{\mathbb{Q}_\ell}(\Pi_{\ell}). \]

Now easy dimension arguments imply that
\[ \gamma_{\ell} g_{\ell,X} \gamma_{\ell}^{-1} = mt \otimes_{\mathbb{Q}} \mathbb{Q}_\ell = [\mathbb{Q} \text{Id} \oplus \text{sp}(\Pi_{\mathbb{Q}}, L_{\lambda})] \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \]
and therefore
\[ mt = \mathbb{Q} \text{Id} \oplus \text{sp}(\Pi_{\mathbb{Q}}, L_{\lambda}). \]

Recall that Hodge classes on self-products on \( X \) could be viewed as tensor invariants of \( mt \bigcap sp(\Pi_{\mathbb{Q}}, L_{\lambda}) = sp(\Pi_{\mathbb{Q}}, L_{\lambda}) \). As in the case of Tate classes, results from the invariant theory for symplectic groups \([13]\) imply that each Hodge class on a self-product of \( X \) can be presented a linear combination of products of divisor classes.

**Remark 9.2.** In the course of the proof we established the equality \( \gamma_{\ell} g_{\ell,X} \gamma_{\ell}^{-1} = mt \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \). This proves the Mumford-Tate conjecture \([21]\) for \( X = J(C_f) \) in the case of finitely generated \( K \) when \( n \geq 9 \).

### References

[1] F. A. Bogomolov, Sur l’algébricité des représentations \( l \)-adiques. C. R. Acad. Sci. Paris Ser. A-B 290 (1980), A701–A703.

[2] N. Bourbaki, Groupes et Algèbres de Lie, Chapitres 7–8, Hermann, Paris 1975.

[3] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, R. A. Wilson, Atlas of finite groups. Clarendon Press, Oxford, 1985.

[4] Ch. Curtis, I. Reiner, Representation theory of finite groups and associative algebras. Interscience Publishers, New York London 1962.

[5] P. Deligne, *Hodge cycles on Abelian varieties* (notes by J. S. Milne), Springer Lecture Notes in Math. 900(1982) 9 - 100.

[6] G. Faltings, *Endlichkeitssätze für abelsche Varietäten über Zähkörperf*. Inv. Math. 73 (1983), 349–366.

[7] G. Faltings, *Complements to Mordell*. Chapter VI in: G. Faltings, G. Wustholz et al., Rational points. Third edition. Aspects of Mathematics, E6. Friedr. Vieweg & Sohn, Braunschweig, 1992.

[8] W. Feit, J. Tits *Projective representations of minimum degree of group extensions*. Canad. J. Math. 30 (1978), 1092–1102.

[9] R. M. Guralnick, Pham Huu Tiep, *Low-dimensional representations of special linear groups in cross characteristic*. Proc. London Math. Soc. 78 (1999), 116–138.

[10] P.N. Hoffman, J. F. Humphreys, *Projective representations of the symmetric groups*. Clarendon Press, Oxford, 1992.

[11] G. James, A. Kerber, The representation theory of the symmetric group. Addison Wesley Publishing Company, Reading, MA 1981.

[12] Ch. Jansen, K. Lux, R. Parker, R. Wilson, An Atlas of Brauer characters. Clarendon Press, Oxford, 1995.
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