Let $N$ be a complete manifold with bounded geometry, such that $\sec N \leq -\sigma < 0$ for some positive constant $\sigma$. We investigate the mean curvature flow of the graphs of smooth length-decreasing maps $f : \mathbb{R}^m \to N$. In this case, the solution exists for all times and the evolving submanifold stays the graph of a length-decreasing map $f_t$. We further prove uniform decay estimates for all derivatives of order $\geq 2$ of $f_t$ along the flow.

1. Introduction

Let $(M, g_M)$ and $(N, g_N)$ be complete Riemannian manifolds, and consider a smooth map $f : M \to N$. The map $f$ is called strictly length-decreasing or contraction, if there exists a fixed $\delta \in (0, 1]$, such that

$$\|df(v)\|_{g_N} \leq (1 - \delta)\|v\|_{g_M}$$

holds for all $v \in \Gamma(TM)$. In the present paper, we want to deform the map $f$ by deforming its graph

$$\Gamma(f) := \{(x, f(x)) \in M \times N : x \in M\}$$

via the mean curvature flow inside the product space $M \times N$. That is, we consider the system

$$\partial_t F_t(x) = \vec{H}(x, t), \quad F_0(M) = (x, f(x)),$$

where $\vec{H}(x, t)$ denotes the mean curvature vector of the submanifold $F_t(M)$ in $M \times N$ at $F_t(x)$. A smooth solution to the mean curvature flow for which $F_t(M)$ is a graph for $0 < T_g \leq \infty$ can be described completely in terms of a smooth family of maps $f_t : M \to N$, with $f_0 = f$. In the case of long-time existence of the graphical solution (i.e. $T_g = \infty$) and convergence, we would thus obtain a smooth homotopy from $f$ to
a minimal map \( f_\infty : M \to N \).

In the compact case, there are many results for length- and area-decreasing maps (see e.g. \([11,17–19,21,23,27,28]\) and references therein). For example, if \( f : M \to N \) is strictly area-decreasing, \( M \) and \( N \) are space forms with \( \dim M \geq 2 \) subject to the relations

\[
\sec M \geq \lvert \sec N \rvert, \quad \sec M + \sec N > 0,
\]

Wang and Tsui proved long-time existence of the graphical mean curvature flow and convergence of \( f_t \) to a constant map \([23]\). The curvature assumptions were then weakened by Lee and Lee \([11]\) and Savas-Halilaj and Smoczyk \([19]\).

In the non-compact case, Ecker and Huisken considered the flow of hypersurfaces in \( \mathbb{R}^{n+1} \) and entire graphs with at most linear growth and provided conditions under which the initial graphs asymptotically approach self-expanding solutions \([7]\). In fact, the growth assumption for the long-time existence theorem can be removed, so that only a Lipschitz condition on the initial graph is required \([6]\). In the higher-codimensional setting, however, due to the complexity of the normal bundle of the graph, the methods of Ecker and Huisken cannot be applied.

Nevertheless, by considering the Gauß map of the immersion, several results in this setting were obtained (see e.g. \([24,26]\)). When considering two-dimensional graphs, Chen, Li and Tian established long-time existence and convergence results by evaluating angle functions on the tangent bundle \([5]\). Further, there are some results showing long-time existence and convergence of the flow under smallness conditions on the differential of the defining map \([2,3,17]\).

For Lagrangian graphs \( \Gamma(f) \subset \mathbb{R}^m \times \mathbb{R}^m \) generated by Lipschitz continuous functions \( f : \mathbb{R}^m \to \mathbb{R}^m \), Chau, Chen and He showed short-time existence of solutions with bounded geometry, as well as decay estimates for the mean curvature vector and all higher-order derivatives of the defining map, which imply the long-time existence of the solution \([2]\). This result was generalized in \([3]\) by relaxing the length-decreasing condition and subsequently by the author to strictly length-decreasing maps between Euclidean spaces of arbitrary dimension \([12]\). A similar theorem also holds if one considers the mean curvature flow of strictly area-decreasing maps between two-dimensional Euclidean spaces \([13]\).

The aim of the present article is to prove estimates and long-time existence for strictly length-decreasing maps \( f : \mathbb{R}^m \to N \), where \((N, g_N)\) is a complete Riemannian manifold with bounded geometry, i.e. for any integer \( k \geq 0 \) we have

\[
\sup_{x \in N} \lVert \nabla^k R_N(x) \rVert < \infty
\]

and the injectivity radius satisfies \( \text{inj}(N) > 0 \). Similarly, a map \( f : M \to N \) between two complete Riemannian manifolds \((M, g_M)\) and \((N, g_N)\) has bounded geometry, if it satisfies

\[
\sup_{x \in M} \lVert \nabla^k df(x) \rVert < \infty \quad \text{for all} \quad k \geq 0.
\]

Let us remark that the length-decreasing condition \((1.1)\) is essentially measured by the difference

\[
s := g_{\mathbb{R}^m} - f^* g_N
\]
of the two metrics $g_{\mathbb{R}^m}$ and $g_N$. The estimate for the eigenvalues of $s$ in the following theorem is given in terms of an average, as given by
\[ \text{tr}(s) = \sum_{i,j=1}^{m} g_{ij}s_{ij}, \]
where $g$ is the induced metric on the graph $\Gamma(f) \subset \mathbb{R}^m \times N$.

**Theorem A.** Let $(N, g_N)$ be a complete Riemannian manifold with bounded geometry. Assume that $N$ has negative sectional curvature, i.e. there is $\sigma > 0$ with
\[ \sec N \leq -\sigma. \]
Further, let $f : \mathbb{R}^m \to N$ be a smooth strictly length-decreasing map with bounded geometry. Then the mean curvature flow with initial condition $F_0(x) := (x, f(x))$ has a long-time solution such that the following statements hold.

(i) The evolving submanifold stays the graph of a strictly length-decreasing map $f_t : \mathbb{R}^m \to N$ for all $t > 0$.

(ii) The trace $\text{tr}(s)$ is non-decreasing in time. If $m > 1$ and $\inf_{\mathbb{R}^m \times \{0\}} \text{tr}(s) < m - 1$, then the estimate
\[ \text{tr}(s) \geq \frac{C_1(m - 1) \exp (\frac{C_1}{m}t) - m}{C_1 \exp (\frac{C_1}{m}t) - 1} \]
holds, where
\[ C_1 := 1 + \frac{1}{(m - 1) - \inf_{\mathbb{R}^m \times \{0\}} \text{tr}(s)}. \]

(iii) The mean curvature vector of the graph stays bounded, i.e. there is a constant $C_2 \geq 0$, such that
\[ \| \vec{H} \|^2 \leq C_2. \]

(iv) All spatial derivatives of order $k \geq 2$ of $f_t$ satisfy the estimate
\[ t^{k-1} \sup_{x \in \mathbb{R}^m} \| \nabla^{k-1} df_t(x) \|^2 \leq C_{k,\delta} \quad \text{for all } k \geq 2 \]
and for some constants $C_{k,\delta} \geq 0$ depending only on $k$ and $\delta$.

Let us shortly comment on the strategy of the proofs. As in [2, 12], the idea is to identify suitable functions and symmetric tensors to which a maximum principle can be applied. Since Hamilton’s tensorial maximum principle [9] is not applicable in the non-compact case, we follow an idea from [2] in order to extend it to the setting at hand.

Before introducing the precise geometric setting for the proof, let us make a few remarks on the assumptions and consequences of the theorem.

**Remark 1.1** (Long-time Behavior). Theorem A implies that $f_t$ becomes stationary for $t \to \infty$. In particular, in this limit the map $f_t : (\mathbb{R}^m, g_{\mathbb{R}^m}) \to (N, g_N)$ becomes totally geodesic.

However, depending on the initial conditions, the map $f_t$ may exhibit different long-time behavior. To see this, let us consider maps $f_t : \mathbb{R} \to \mathbb{H}^2$ with initial datum $f_0$. Here, we use the disk model of $\mathbb{H}^2$ for illustration.

(i) If $f_0$ is a geodesic (up to scaling), it is $f_t(x) = f_0(x)$ for all $t \geq 0$. In particular, the height of $f_t$ as a graph over $f_0$ is zero (see example 6.3).
(ii) If \( f_0(\mathbb{R}) \) is a circle centered at the origin in \( \mathbb{H}^2 \), i.e. \( f_0(x) = r_0(\sin(x), \cos(x)) \) for \( 0 \leq r_0 < 1 \), the image \( f_0(\mathbb{R}) \) shrinks homothetically as a submanifold of \( \mathbb{H}^2 \). Thus, the height of \( f_t \) as a graph over \( f_0 \) remains finite (see example 6.2).

(iii) If \( f_0(\mathbb{R}) \) is a circle with one point at spatial infinity in \( \mathbb{H}^2 \), the image \( f_t(\mathbb{R}) \) moves out to spatial infinity. We observe that the height of \( f_t \) as a graph over \( f_0 \) is unbounded (see example 6.1).

Remark 1.2. If \( f_0 \) maps into a compact region \( K \subset N \), then the estimate (iii) on the mean curvature vector implies that \( f_t \) also maps into compact regions \( K_t \) for all finite times.

Remark 1.3. If \( \sigma = 0 \), (i), (iii) and (iv) of theorem A still hold. Thus, in this weaker formulation, the theorem can be applied e.g. to maps \( f : \mathbb{R}^m \to N_1 \times N_2 \), where \( N_1 \) and \( N_2 \) have non-positive sectional curvatures.

For example, let \( N \) be an arbitrary Riemannian manifold with non-positive sectional curvature, let \( p_0 \in N \) be fixed, \( m \geq k \), and let \( A : \mathbb{R}^m \to \mathbb{R}^k \) be a linear map satisfying \( \| Au \|_k^2 \leq (1 - \delta) \| u \|_m^2 \) for all \( u \in \mathbb{R}^m \) and some \( \delta \in (0,1] \). In particular, \( A \) may be chosen such that the map \( f : \mathbb{R}^m \to \mathbb{R}^k \times N \) given by \( f(x) := (Ax,p_0) \) satisfies \( \text{tr}(s) = c \) for any constant \( c \in (m-k,m] \). Note that the mean curvature flow with initial datum \( F(x) := (x,f(x)) \) is stationary.

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2. Maps between Euclidean Spaces

2.1. Geometry of Graphs. We recall the geometric quantities in a graphical setting, where we mostly follow the presentation in [17, Section 2].

Let \((M, g_M)\) and \((N, g_N)\) be Riemannian manifolds of dimensions \( m \) and \( n \), respectively. On the product manifold \((M \times N, g_{M \times N} := g_M \times g_N)\), the two natural projections

\[
\pi_M : M \times N \to M, \quad \pi_N : M \times N \to N,
\]

are submersions, that is they are smooth and have maximal rank. A smooth map \( f : M \to N \) defines an embedding \( F : M \to M \times N \) via

\[
F(x) := (x,f(x)), \quad x \in M.
\]
The graph of \( f \) is defined to be the submanifold

\[
\Gamma(f) := F(M) = \{(x,f(x)) : x \in M\} \subset M \times N.
\]
Since \( F \) is an embedding, it induces another Riemannian metric on \( M \), given by

\[
g := F^* g_{M \times N}.
\]
The four metrics \( g_M, g_N, g_{M \times N} \) and \( g \) are related by

\[
g_{M \times N} = \pi_M^* g_M + \pi_N^* g_N,
\]

\[
g = F^* g_{M \times N} = g_M + f^* g_N.
\]
Further, \( F \) defines an orthogonal splitting of the bundle

\[
F^* T(M \times N) = dF(TM) \oplus T^\perp M,
\]
which induces a splitting of a vector field \( v \in \Gamma(F^* T(M \times N)) \) as

\[
v = v^\top \oplus v^\perp.
\]
We call $v^\perp$ the tangential part of $v$ and $v^\top$ the normal part of $v$. The projection onto the normal part is denoted by $\text{pr}^\perp : F^*T(M \times N) \to T^\perp M$. Using a local $g$-orthonormal frame $\{e_1, \ldots, e_m\}$ of $TM$, $\text{pr}^\perp$ can be expressed as

$$\text{pr}^\perp(\xi) = \xi - \sum_{k=1}^m g_{M\times N}(\xi, dF(e_k)) \, dF(e_k).$$

As in [18, 19], let us introduce the symmetric 2-tensors

$$s_{M\times N} := \pi_M^* g_M - \pi_N^* g_N, \quad s := F^* s_{M\times N} = g_M - f^* g_N.$$ 

Note that $s_{M\times N}$ is a semi-Riemannian metric of signature $(m, n)$ on the manifold $M \times N$. Like in [17], let us also introduce

$$s^\perp(\xi, \eta) := s_{M\times N}(\text{pr}^\perp(\xi), \text{pr}^\perp(\eta)), \quad \xi, \eta \in \Gamma(F^*T(M \times N)).$$

We denote the restriction of $g_{M\times N}$ to the normal bundle by $g^\perp$.

The Levi-Civita connection on $M$ with respect to the induced metric $g$ is denoted by $\nabla$ and the corresponding curvature tensor by $R$. By restricting Levi-Civita connection $\nabla^{g_{M\times N}}$ of $M \times N$ to the normal bundle, we obtain the normal connection, given by

$$\nabla^\perp \xi := \text{pr}^\perp(\nabla^{g_{M\times N}} \xi), \quad v \in \Gamma(TM), \quad \xi \in \Gamma(F^*T(M \times N)).$$

2.2. Second Fundamental Form. The second fundamental tensor of the graph $\Gamma(f)$ is the section $A \in \Gamma(T^\perp M \otimes \text{Sym}(T^* M \otimes T^* M))$ defined as

$$A(v, w) := (\nabla dF)(v, w) := \nabla^{g_{M\times N}}(v) \, dF(w) - dF(\nabla_v w),$$

where $v, w \in \Gamma(TM)$ and where we denote the connection on $F^*T(M \times N) \otimes T^* M$ induced by the Levi-Civita connection also by $\nabla$. The trace of $A$ with respect to the metric $g$ is called the mean curvature vector field of $\Gamma(f)$ and it will be denoted by $\vec{H} := \text{tr} \, A$.

Let us denote the evaluation of the second fundamental form (resp. mean curvature vector) in the direction of a vector $\xi \in \Gamma(F^*T(M \times N))$ by

$$A_\xi(v, w) := g_{M\times N}(A(v, w), \xi) \quad \text{resp.} \quad \vec{H}_\xi := g_{M\times N}(\vec{H}, \xi).$$

Note that $\vec{H}$ is a section in the normal bundle of the graph. If $\vec{H}$ vanishes identically, the graph is said to be minimal. A smooth map $f : M \to N$ is called minimal, if its graph $\Gamma(f)$ is a minimal submanifold of the product space $(M \times N, g_{M\times N})$.

On the submanifold, the Gauß equation

$$(R - F^* R_{M \times N})(u_1, v_1, u_2, v_2) = g_{M\times N}(A(u_1, u_2), A(v_1, v_2)) - g_{M\times N}(A(u_1, v_2), A(v_1, u_2))$$

(2.1) and the Codazzi equation

$$(\nabla_u A)(v, w) - (\nabla_v A)(u, w) = R_{M\times N}(dF(u), dF(v)) \, dF(w) - dF(R(u, v)w)$$

hold, where the induced connection on the bundle $F^*T(M \times N) \otimes T^* M \otimes T^* M$ is defined as

$$(\nabla_u A)(v, w) := D_{dF(u)}(A(v, w)) - A(\nabla_v w) - A(v, \nabla_u w).$$
2.3. Singular Value Decomposition. We recall the singular value decomposition theorem and closely follow [18, Section 3.2].

Fix a point $x \in M$, and let 
\[ \lambda_1^2(x) \leq \lambda_2^2 \leq \cdots \leq \lambda_m^2(x) \]
be the eigenvalues of $f^*g_N$ with respect to $g_M$. The corresponding values $\lambda_i \geq 0$, $i \in \{1, \ldots, m\}$, are called the singular values of the differential $df$ of $f$ and give rise to continuous functions on $M$. Let 
\[ r = r(x) := \text{rank } df(x). \]
Obviously, $r \leq \min\{m, n\}$ and $\lambda_1(x) = \cdots = \lambda_{n-r}(x) = 0$. At the point $x$ consider an orthonormal basis $\{\alpha_1, \ldots, \alpha_{m-r}; \alpha_{m-r+1}, \ldots, \alpha_m\}$ with respect to $g_M$ which diagonalizes $f^*g_N$. Moreover, at $f(x)$ consider a basis $\{\beta_1, \ldots, \beta_{n-r}; \beta_{n-r+1}, \ldots, \beta_n\}$ that is orthonormal with respect to $g_N$, such that 
\[ df(\alpha_i) = \lambda_i(x)\beta_{n-m+i} \]
for any $i \in \{m-r+1, \ldots, m\}$. This procedure is called the singular value decomposition of the differential $df$.

Now let us construct a special basis for the tangent and the normal space of the graph in terms of the singular values. The vectors 
\[ \tilde{e}_i := \begin{cases} 
\alpha_i \oplus 0, & 1 \leq i \leq m-r, \\
\frac{1}{\sqrt{1 + \lambda_i^2(x)}} (\alpha_i \oplus \lambda_i(x)\beta_{n-m+i}), & m-r+1 \leq i \leq m,
\end{cases} \]
form an orthonormal basis with respect to the metric $g_{M \times N}$ of the tangent space $dF(T_x M)$ of the graph $\Gamma(f)$ at $x$. It follows that with respect to the induced metric $g$, the vectors 
\[ e_i := \frac{1}{\sqrt{1 + \lambda_i^2(x)}} \alpha_i \]
form an orthonormal basis of $T_x M$. Moreover, the vectors 
\[ \xi_i := \begin{cases} 
0 \oplus \beta_i, & 1 \leq i \leq n-r, \\
\frac{1}{\sqrt{1 + \lambda_{i+m-n}(x)}}(-\lambda_{i+m-n}(x)\alpha_{i+m-n} \oplus \beta_i), & n-r+1 \leq i \leq n,
\end{cases} \]
form an orthonormal basis with respect to $g_{M \times N}$ of the normal space $T_{x}^\perp M$ of the graph $\Gamma(f)$ at the point $x$. From the formulae above, we deduce that 
\[ s_{M \times N}(\tilde{e}_i, \tilde{e}_j) = s(e_i, e_j) = \frac{1 - \lambda_i^2(x)}{1 + \lambda_i^2(x)} \delta_{ij}, \quad 1 \leq i, j \leq m. \]

Therefore, the eigenvalues of the 2-tensor $s$ with respect to $g$ are given by 
\[ \frac{1 - \lambda_1^2(x)}{1 + \lambda_1^2(x)} \geq \cdots \geq \frac{1 - \lambda_{m-1}^2(x)}{1 + \lambda_{m-1}^2(x)} \geq \frac{1 - \lambda_m^2(x)}{1 + \lambda_m^2(x)}. \quad (2.2) \]

Moreover, 
\[ s_{M \times N}(\xi_i, \xi_j) = \begin{cases} 
-\delta_{ij}, & 1 \leq i \leq n-r, \\
\frac{-1 - \lambda_{i+m-n}^2(x)}{1 + \lambda_{i+m-n}^2(x)} \delta_{ij}, & n-r+1 \leq i \leq n.
\end{cases} \quad (2.3) \]
Thus, if there exists a positive constant $\varepsilon$ such that $s \geq \varepsilon g$, then $s^\perp \leq -\varepsilon g^\perp$. Furthermore,

$$s_{m\times N}(\hat{e}_{m-r+i}, \xi_{n-r+j}) = -\frac{2\lambda_{m-r+i}(x)}{1 + \lambda_{m-r+i}(x)^2} \delta_{ij}, \quad 1 \leq i, j \leq r.$$  

### 3. Mean Curvature Flow

Let us consider the case where $M = \mathbb{R}^m$ and $N$ is a complete, non-compact Riemannian manifold with bounded geometry satisfying $\sec_M \leq -\sigma < 0$ for some $\sigma > 0$. Further, let $f : \mathbb{R}^m \to N$ be a smooth map and $T > 0$. We say that a family of maps $F : \mathbb{R}^m \times [0, T) \to \mathbb{R}^m \times N$ evolves under the mean curvature flow, if for all $x \in \mathbb{R}^m$

$$\begin{aligned}
\partial_t F(x, t) &= \hat{H}(x, t), \\
F(x, 0) &= (x, f(x)).
\end{aligned} \tag{3.1}$$

#### 3.1. Short-time Existence. Using that $N$ has bounded geometry, there is a neighborhood $U \subset \mathbb{R}^m \times N$ of $\Gamma(f_0)$ and $\Omega \subset \mathbb{R}^n$ such that $U$ is simply-connected and such that there is a diffeomorphism $\psi : U \to \mathbb{R}^m \times N$. Let us denote the local coordinates induced by $\psi$ on $N$ (depending on the point $x \in \mathbb{R}^m$) by $\{y^1, \ldots, y^m\}$. With this identification, if $F : \mathbb{R}^m \to \mathbb{R}^m \times N$ is a graph over $\mathbb{R}^m$ and $F(\mathbb{R}^m) \subset \mathbb{R}^m \times U$, we may equivalently consider $\psi \circ F : \mathbb{R}^m \to \mathbb{R}^m \times N$, which then has the form

$$\psi \circ F(x) = (x, f(x))$$

for some map $f : \mathbb{R}^m \to \Omega$.

Using standard coordinates $\{x^1, \ldots, x^m\}$ on $\mathbb{R}^m$ and denoting the local coordinates on $\mathbb{R}^m \times U$ collectively by $\{z^1, \ldots, z^{m+n}\} = \{x^1, \ldots, x^m, y^1, \ldots, y^n\}$, the evolution equation for the mean curvature flow in this chart is given by

$$\partial_t F_t = \sum_{i,j=1}^m g^{ij} \left( \partial_{ij}^2 F_t + \sum_{l=1}^m R_{ij}^l \partial_l F_t + \sum_{a,b,c=1}^{m+n} (\Gamma^{ab} \times N)_{ab}^c (\partial_i F_t^a)(\partial_j F_t^b) \frac{\partial}{\partial z^c} \right). \tag{3.2}$$

Assuming a graphical solution exists up to time $T > 0$, we may choose a time-dependent diffeomorphism $\varphi : \mathbb{R}^m \times [0, T) \to \mathbb{R}^m$, such that $F_t \circ \varphi_t(x) = (x, f_t(x))$ for each $t \in [0, T)$. Also using $\nabla^{\mathbb{R}^m \times N} = D \oplus \nabla_N$ (where $D$ denotes the flat connection on $\mathbb{R}^m$), the system (3.2) with initial condition $F_0(x) = (x, f(x))$ reduces to

$$\begin{aligned}
\partial_t f_t &= \sum_{i,j=1}^m \hat{g}^{ij} \left( \partial_{ij}^2 f_t + \sum_{a,b=1}^{m+n} (\Gamma^{ab} \times N)_{ab}^c (\partial_i f_t^a)(\partial_j f_t^b) \right), \\
f_0(x) &= f(x),
\end{aligned} \tag{3.3}$$

where here $\hat{g}^{ij}$ are the components of the inverse of $\hat{g} := g_{m} + f_t^* g_N$ and $\Gamma^{ab} \times N$ are the Christoffel symbols of the metric $g_N$. Since $\Gamma^{ab} \times N$ only depends on the geometry of $(N, g_N)$, the second term only contributes to lower orders.

If (3.3) has a smooth solution $f : \mathbb{R}^m \times [0, T) \to N$, then the mean curvature flow (3.1) has a smooth solution $F : \mathbb{R}^m \times [0, T) \to \mathbb{R}^m \times N$ given by the family of graphs

$$\Gamma(f(\cdot, t)) = \{(x, f(x, t)) : x \in \mathbb{R}^m\},$$

up to tangential diffeomorphisms (see e.g. [1, Chapter 3.1]).

For (3.3), we have the following short-time existence result.

#### Theorem 3.1. Let $(N, g_N)$ be a complete Riemannian manifold with bounded geometry. Further, let $f_0 : \mathbb{R}^m \to N$ be a smooth function, such that for each $k \geq 0$ we have

$$\sup_{x \in \mathbb{R}^m} \|\nabla^k f_0(x)\| \leq C_k$$

for some finite constants $C_k$. Then (3.3) has a short-time smooth solution $f$ on $\mathbb{R}^m \times [0, T)$ for some $T > 0$ with initial condition $f_0$, such that

$$\sup_{x \in \mathbb{R}^m} \|\nabla^k f_t(x)\| < \infty \text{ for every } k \geq 0 \text{ and } t \in [0, T).$$
Proof. By the above construction, we obtain $U \subset \mathbb{R}^m \times N$, $\Omega \subset \mathbb{R}^n$ and a diffeomorphism $\psi: U \to \mathbb{R}^m \times \Omega$. We may choose $\psi$, such that the coordinates induced on $N$ are normal coordinates. By the bounded geometry assumption on $N$, we may further assume that $U$ is chosen such that the Christoffel symbols $\Gamma_{\mathbb{R}^n}$ are uniformly bounded in $\{p\} \times \Omega$ for all $p \in \mathbb{R}^m$ [8, 10]. Thus, it is sufficient to obtain a short-time solution to \eqref{3.3} in $\mathbb{R}^m \times \Omega$. Since equation \eqref{3.3} is strongly parabolic and only differs by bounded lower-order terms from the mean curvature flow system in flat space, the claim follows in the same way as [2, Proposition 5.1]. In particular, for short times the solution stays inside $\Omega$, so that it maps to a solution to the mean curvature flow in $U$. \hfill \box

In the sequel, we will consider a special kind of solution to \eqref{3.1}. 

Definition 3.2. Let $F_t(x)$ be a smooth solution to the system \eqref{3.1} on $\mathbb{R}^m \times [0, T)$ for some $0 < T \leq \infty$, such that for each $t \in [0, T)$ and non-negative integer $k$, the submanifold $F_t(\mathbb{R}^m) \subset \mathbb{R}^m \times N$ satisfies

$$\sup_{x \in \mathbb{R}^m} \|\nabla^k A(x, t)\| < \infty, \quad (3.4)$$

$$C_1(t)g_{\mathbb{R}^m} \leq g \leq C_2(t)g_{\mathbb{R}^m}, \quad (3.5)$$

where $C_1(t)$ and $C_2(t)$ for each $t \in [0, T)$ are finite, positive constants depending only on $t$. Then we will say that the family of embeddings $\{F_t\}_{t \in \mathbb{T}}$ has bounded geometry.

Definition 3.3. Let $f_t(x)$ be a smooth solution to the system \eqref{3.3} on $\mathbb{R}^m \times [0, T)$ for some $0 < T \leq \infty$, such that for each $t \in [0, T)$ and positive integer $k$ the estimate

$$\sup_{x \in \mathbb{R}^m} \|\nabla^{k-1} df_t(x)\| < \infty$$

holds. Then we will say that $f_t(x)$ has bounded geometry for every $t \in [0, T)$.

3.2. Graphs. We recall some important notions in the graphic case, where we follow the presentation in [19, Section 3.1].

Let $f_0 : \mathbb{R}^m \to N$ denote a smooth map, such that $F_0(x) := (x, f_0(x))$ has bounded geometry. Then theorem 3.1 ensures that the system \eqref{3.3} has a short-time solution with initial data $f_0(x)$ on a time interval $[0, T)$ for some positive maximal time $T > 0$. Further, there exists a diffeomorphism $\varphi_t : \mathbb{R}^m \to \mathbb{R}^m$, such that

$$F_t \circ \varphi_t(x) = (x, f_t(x)), \quad (3.6)$$

where $F_t(x)$ is a solution of \eqref{3.1}.

To obtain the converse of this statement, let $\Omega_{\mathbb{R}^m}$ be the volume form on $\mathbb{R}^m$ and extend it to a parallel $m$-form on $\mathbb{R}^m \times N$ by pulling it back via the natural projection $\pi_{\mathbb{R}^m} : \mathbb{R}^m \times N \to \mathbb{R}^m$, that is, consider the $m$-form $\pi_{\mathbb{R}^m}^* \Omega_{\mathbb{R}^m}$. Define the time-dependent smooth function $u : \mathbb{R}^m \times [0, T) \to \mathbb{R}$ by

$$u := \# \Omega_t,$$

where $\#$ is the Hodge star operator with respect to the induced metric $g$ and

$$\Omega_t := F_t^* (\pi_{\mathbb{R}^m}^* \Omega_{\mathbb{R}^m}) = (\pi_{\mathbb{R}^m} \circ F_t)^* \Omega_{\mathbb{R}^m}.$$

The function $u$ is the Jacobian of the projection map from $F_t(\mathbb{R}^m)$ to $\mathbb{R}^m$. From the implicit mapping theorem it follows that $u > 0$ if and only if there exists a diffeomorphism $\varphi_t : \mathbb{R}^m \to \mathbb{R}^m$ and a map $f_t : \mathbb{R}^m \to N$, such that \eqref{3.6} holds, i.e. $u$ is positive precisely if the solution of the mean curvature flow remains a graph. By theorem 3.1, the solution will stay a graph at least in a short time interval $[0, T)$. 


3.3. Evolution Equations. Let us consider the evolution of the tensors defined in section 2.1 under the mean curvature flow. The evolution of the tensor $s$ is essentially calculated in [19, Lemma 3.1] and is given by the following statement.

**Lemma 3.4.** The evolution of the tensor $s$ for $t \in [0, T)$ is given by the formula

$$(\nabla \partial_t s - \Delta s)(v, w) = - s(Ric v, w) - s(w, Ric v)$$

$$- 2 \sum_{k=1}^{m} s_{R^{m} \times N}(A(e_k, v), A(e_k, w))$$

$$- 2 \sum_{k=1}^{m} f_{*}^{N}(e_k, v, e_k, w),$$

where $\{e_1, \ldots, e_m\}$ is any orthonormal frame with respect to $g$ and the Ricci operator is given by

$$Ric v := - \sum_{k=1}^{m} R(e_k, v)e_k.$$

For the tensor $s^\perp$, from [17, Lemma 3.3] we have the following evolution equation.

**Lemma 3.5.** Let $\xi$ be a unit vector normal to the evolving submanifold at a fixed point $(x_0, t_0)$ in space-time. Then

$$(\nabla \partial_t s^\perp - \Delta s^\perp)(\xi, \xi)$$

$$= 2 \sum_{i,j=1}^{m} A_{\xi}(e_i, e_j)s_{R^{m} \times N}(A(e_i, e_j), \xi) - 2 \sum_{i,j,k=1}^{m} A_{\xi}(e_i, e_j)A_{\xi}(e_i, e_k)s(e_j, e_k)$$

$$- 2 \sum_{i,j=1}^{m} R_{R^{m} \times N}(dF(e_i), dF(e_j), dF(e_i), \xi)s_{R^{m} \times N}(dF(e_j), \xi)$$

for any $g$-orthonormal basis $\{e_1, \ldots, e_m\}$ of $T_{x_0} \mathbb{R}^m$.

Let us define the symmetric 2-tensor $\vartheta \in \text{Sym}(F^{*}T^*(\mathbb{R}^{m} \times N) \otimes F^{*}T^*(\mathbb{R}^{m} \times N))$ by setting

$$\vartheta(\xi, \eta) := \overline{H}_{pr^{+}(\xi)} \overline{H}_{pr^{+}(\eta)}.$$

By [17, Lemma 3.4], this tensor satisfies the following evolution equation.

**Lemma 3.6.** The symmetric 2-tensor $\vartheta$ evolves under the mean curvature flow according to the formula

$$(\nabla \partial_t \vartheta - \Delta \vartheta)(\xi, \xi)$$

$$= 2 \sum_{i,j=1}^{m} A_{\vartheta}(e_i, e_j)A_{\vartheta}(e_i, e_j)\overline{H}_{\xi} - 2 \sum_{i=1}^{m}(\nabla_{e_i} \overline{H}, \xi)^2$$

$$- 2 \sum_{i=1}^{m} R_{R^{m} \times N}(\overline{H}, dF(e_i), dF(e_i), \xi)\overline{H}_{\xi}$$

for any vector $\xi$ in the normal bundle of the submanifold.

4. A Priori Estimates

4.1. Preserved Quantities. Following the idea in [2], we will need the function

$$\phi_{R}(x) := 1 + \frac{||x||_{\mathbb{R}^m}^2}{R^2},$$

where $|| \cdot ||_{\mathbb{R}^m}$ denotes the Euclidean norm on $\mathbb{R}^m$ and $R > 0$ is a constant which will be chosen later.
Lemma 4.1 ([12, Lemma 4.1]). Let $F(x,t)$ be a smooth solution to (3.1) with bounded geometry and assume there exists $\varepsilon > 0$, such that $s - \varepsilon g \geq 0$ for any $t \in [0,T]$. Fix any $T' \in [0,T)$ and $(x_0, t_0) \in \mathbb{R}^m \times [0,T')$. Then for any tangent vector $v$ and any normal vector $\xi$ at $(x_0, t_0)$, the following estimates hold,

\begin{align}
-c(T') \|x_0\|_{\mathbb{R}^m} s(v,v) &\leq \langle \nabla \phi_R, (\nabla s)(v,v) \rangle \leq c(T') \|x_0\|_{\mathbb{R}^m} s(v,v), \tag{4.2} \\
c(T') \|x_0\|_{\mathbb{R}^m} s(\xi, \xi) &\leq \langle \nabla \phi_R, (\nabla s^\perp)(\xi, \xi) \rangle \leq -c(T') \|x_0\|_{\mathbb{R}^m} s(\xi, \xi), \tag{4.3} \\
|\Delta \phi_R| &\leq c(T') \left( \frac{1}{R^2} + \|x_0\|_{\mathbb{R}^m} \right), \tag{4.4}
\end{align}

where $c(T') \geq 0$ is a constant depending only on $T'$.

To obtain the preservation of the length-decreasing property, for any $R, \eta > 0$ let us further set

$$\psi_{(x,t)} := e^{\eta t} \phi_R(x)s_{(x,t)} - \varepsilon g_{(x,t)}.$$ 

Lemma 4.2. Under the mean curvature flow, the tensor $\psi$ evolves according to the equation

\begin{align}
(\nabla_{A_t} \psi - \Delta \psi)(u_1, u_2) &\quad = -\psi(Ric u_1, u_2) - \psi(u_1, Ric u_2) \\
&\quad + 2\varepsilon \sum_{k=1}^m \langle A(u_1, e_k), A(u_2, e_k) \rangle \\
&\quad - 2e^{\eta t} \phi_R \sum_{k=1}^m s_{\mathbb{R}^m \times N}(A(u_1, e_k), A(u_2, e_k)) \\
&\quad - 2e^{\eta t} \phi_R \sum_{k=1}^m f^*_T R_N(e_k, u_1, e_k, u_2) \\
&\quad - e^{\eta t} \left\{ (\Delta \phi_R)s(u_1, u_2) + 2(\nabla \phi_R, (\nabla s)(u_1, u_2)) - \eta \phi_R s(u_1, u_2) \right\}
\end{align}

for any $u_1, u_2 \in \Gamma(T \mathbb{R}^m)$ and any local frame $\{e_1, \ldots, e_m\}$ which is orthonormal with respect to $g$.

Proof. The proof is the same as the proof of [12, Lemma 4.2], but one needs to take the additional curvature term occuring in lemma 3.4 into consideration. \hfill \Box

Lemma 4.3. Let $F(x,t)$ be a smooth solution to (3.1) with bounded geometry. Assume there exists $\varepsilon > 0$ with $s - \varepsilon g \geq 0$ at $t = 0$. Then it is $s - \varepsilon g \geq 0$ for all $t \in [0,T)$.

Proof. We first assume that also $s - \frac{\varepsilon}{2} \geq 0$ on $[0,T') \subset [0,T)$. By assumption, the curvature of $N$ satisfies $\sec N \leq 0$, so that the curvature term in the evolution equation of $\psi$ with respect to any $v \in \Gamma(T \mathbb{R}^m)$ satisfies

$$-2e^{\eta t} \phi_R \sum_{k=1}^m f^*_T R_N(e_k, v, e_k, v) \geq 0.$$

Thus, we may argue in exactly the same way as in the proof of [12, Lemma 4.3], which shows $s - \varepsilon g \geq 0$ is preserved on $[0,T')$.

Finally, the additionl assumption on $s$ is removed as in [12, Lemma 4.4], and the claim follows. \hfill \Box
It immediately follows that a smooth length-decreasing map \( f : \mathbb{R}^m \to N \) evolves through length-decreasing maps \( f_t : \mathbb{R}^m \to N \) under the mean curvature flow [12, Lemma 4.5].

The next step will be to show that the mean curvature vector of the graph remains bounded under the mean curvature flow. For this, let us set
\[
\chi := -e^{\eta t} \phi_R s^+ - \varepsilon_2 \Theta.
\]

**Lemma 4.4.** For any unit vector \( \xi \) normal to the evolving submanifold at a fixed point \( (x_0, t_0) \in \mathbb{R}^m \times [0, T) \), the tensor \( \chi \) satisfies the equation
\[
(\nabla_{\delta_i} \chi - \Delta^+ \chi)(\xi, \xi) = e^{\eta t}\left\{-\eta \phi_R s^+(\xi, \xi) + (\Delta \phi_R) s^+(\xi, \xi) + 2(\nabla \phi_R, (\nabla^+ s^+)(\xi, \xi))\right\}
- 2e^{\eta t} \phi_R \sum_{i,j=1}^m A_\xi(e_i, e_j) s_{\mathbb{R}^m \times N}(A(e_i, e_j), \xi)
- \sum_{i,j,k=1}^m A_\xi(e_i, e_j) A_\xi(e_i, e_k) s(e_j, e_k)
- 2\varepsilon_2 \left\{\sum_{i,j=1}^m A \eta (e_i, e_j) A_\xi(e_i, e_j) H_\xi - \sum_{i=1}^m (\nabla^+ H_\xi, \xi)^2\right\}
+ 2e^{\eta t} \phi_R \sum_{i,j=1}^m R_{\mathbb{R}^m \times N}(dF(e_i), dF(e_j), dF(e_i), \xi) s_{\mathbb{R}^m \times N}(dF(e_j), \xi)
+ 2\varepsilon_2 \sum_{i=1}^m R_{\mathbb{R}^m \times N}(\overline{H}, dF(e_i), dF(e_i), \xi) \overline{H}_\xi
\]
under the mean curvature flow.

**Proof.** We calculate
\[
(\nabla_{\delta_i} \chi - \Delta^+ \chi)(\xi, \xi) = -e^{\eta t} \phi_R (\nabla_{\delta_i}^+ s^+ - \Delta^+ s^+)(\xi, \xi) - \varepsilon_2 (\nabla_{\delta_i} \Theta - \Delta^+ \Theta)(\xi, \xi)
- \eta e^{\eta t} \phi_R s^+(\xi, \xi) + e^{\eta t} (\Delta \phi_R) s^+(\xi, \xi)
+ 2e^{\eta t} (\nabla \phi_R, (\nabla^+ s^+)(\xi, \xi)).
\]
The claim follows from the evolution equations for \( s^+ \) and \( \Theta \) in lemmas 3.5 and 3.6. \( \square \)

**Lemma 4.5.** Let \( F(x, t) \) be a smooth solution to (3.1) with bounded geometry and suppose \( s - \varepsilon_1 g \geq 0 \) on \([0, T] \) for some \( \varepsilon_1 > 0 \). Then there exists a constant \( \varepsilon_2 > 0 \) depending on \( \varepsilon_1 \), the dimension \( m = \text{dim} \mathbb{R}^m \) and the geometry of \( N \), such that
\[ s^+ + \varepsilon_2 \Theta \leq 0 \]
on \( \mathbb{R}^m \times [0, T) \).

**Proof.** We follow the same strategy as in the proof of [12, Lemma 5.2]. Fix any \( T' \in [0, T) \). We will first show that we can choose \( R_0 > 0 \), such that \( \chi \geq 0 \) on \( \mathbb{R}^m \times [0, T') \) for all \( R \geq R_0 \).

Suppose \( \chi \) is not positive on \( \mathbb{R}^m \times [0, T'] \) for some \( R \geq R_0 \). Then, as \( \chi > 0 \) on \( \mathbb{R}^m \times \{0\}, s - \varepsilon_1 g \geq 0 \) (and thus \( s^+ + \varepsilon_1 g^+ \leq 0 \) on \([0, T), \phi_R(x) \to \infty \) as \( \|x\| \to \infty \) and by the bounded geometry condition (3.4), it follows that \( \chi > 0 \) outside some compact set \( K \subset \mathbb{R}^m \) and all \( t \in [0, T'] \). We conclude that there exists \((x_0, t_0) \in K \times [0, T']\), such that \( \chi \) has a zero eigenvalue at \((x_0, t_0)\) and that \( t_0 \) is the first such time. In
other words, we have $\chi_{(x_0,t_0)}(\xi,\eta) = 0$ for some nonzero vector $\xi$ and all vectors $\eta$, and $\chi > 0$ on $\mathbb{R}^m \times [0, t_0)$. Extend $\xi$ to a local smooth vector field. By the second derivative criterion, at $(x_0, t_0)$ we have

$$\chi(\xi, \eta) = 0, \quad (\nabla^\perp \chi)(\xi, \xi) = 0, \quad (\nabla^\perp_{\delta^i} \chi)(\xi, \xi) \leq 0 \quad \text{and} \quad (\Delta^\perp \chi)(\xi, \xi) \geq 0$$

(4.5)

for all $\eta \in T_{x_0} \mathbb{R}^m$. Let us set

$$A := e^{\eta t} \{ - \eta \phi_R s^\perp(\xi, \xi) + (\Delta \phi_R) s^\perp(\xi, \xi) + 2(\nabla \phi_R, (\nabla^\perp s^\perp)(\xi, \xi)) \},$$

$$B := -2e^{\eta t} \phi_R \sum_{i,j=1}^m A\xi(e_i, e_j)s_{\mathbb{R}^m \times N}(A(e_i, e_j), \xi)$$

$$+ 2e^{\eta t} \phi_R \sum_{i,j,k=1}^m A\xi(e_i, e_j)A\xi(e_i, e_k)s(e_j, e_k)$$

$$- 2\varepsilon_2 \left\{ \sum_{i,j=1}^m A\xi(e_i, e_j)A\xi(e_i, e_j)\xi_{\xi_{\xi}} - \sum_{i=1}^m (\nabla^\perp_{e_i} \xi_{\xi_{\xi}})^2 \right\},$$

$$C := 2e^{\eta t} \phi_R \sum_{i,j=1}^m R_{\mathbb{R}^m \times N}(dF(e_i), dF(e_j), dF(e_i), \xi)s_{\mathbb{R}^m \times N}(dF(e_j), \xi)$$

$$+ 2\varepsilon_2 \sum_{i=1}^m R_{\mathbb{R}^m \times N}(\xi_{\xi_{\xi}}, dF(e_i), dF(e_i), \xi)\xi_{\xi_{\xi}}.$$ 

Then at $(x_0, t_0)$ it is

$$0 \geq (\nabla^\perp_{\delta^i} \chi)(\xi, \xi) - (\Delta^\perp \chi)(\xi, \xi) = A + B + C.$$

The proof of [12, Lemma 5.2] yields that we can choose $R_0 > 0$ (depending on $\eta$ and $T'$) large enough, so that

$$\text{Eq. (4.5)}$$

$$A > 0$$

for any $x_0$ and for all $R \geq R_0$,

and, furthermore, if $\varepsilon_2$ satisfies $0 < \varepsilon_2 \leq \frac{2m}{m}$, we obtain

$$B \geq 2e^{\eta t_0} \phi_R \varepsilon_1 \sum_{i,j=1}^m A^2\xi(e_i, e_j) \geq 2e^{\eta t_0} \phi_R \frac{\varepsilon_1}{m} \xi_{\xi_{\xi}} \xi_{\xi_{\xi}} \geq 2e^{\eta t_0} \frac{\varepsilon_1}{m} \xi_{\xi_{\xi}} \xi_{\xi_{\xi}}.$$ 

It remains to show that $B + C \geq 0$. Eq. (4.5) implies $e^{\eta t_0} \phi_R s^\perp(\xi, \eta) = -\varepsilon_2 \Theta(\xi, \eta)$ at $(x_0, t_0)$, and we calculate

$$C = 2e^{\eta t_0} \phi_R \sum_{i,j=1}^m R_{\mathbb{R}^m \times N}(dF(e_i), dF(e_j), dF(e_i), \xi)s_{\mathbb{R}^m \times N}(dF(e_j), \xi)$$

$$+ 2\varepsilon_2 \sum_{i=1}^m \sum_{k=1}^n R_{\mathbb{R}^m \times N}(\xi_k, dF(e_i), dF(e_i), \xi)\xi_{\xi_{\xi}} \xi_{\xi_{\xi}}$$

$$= 2e^{\eta t_0} \phi_R \sum_{i,j=1}^m R_{\mathbb{R}^m \times N}(dF(e_i), dF(e_j), dF(e_i), \xi)s_{\mathbb{R}^m \times N}(dF(e_j), \xi)$$

$$- 2e^{\eta t_0} \phi_R \sum_{i=1}^m \sum_{k=1}^n R_{\mathbb{R}^m \times N}(\xi_k, dF(e_i), dF(e_i), \xi)\xi_{\xi_{\xi}}(\xi_k, \xi).$$
From the bounded geometry of $N$ and the boundedness of the singular values of the map $f_t$ we infer that there is a finite constant $C > 0$ depending only on the geometry of $N$ and the singular values, such that $C$ may be estimated as

$$C \geq -2C_{\phi_R} \phi_R.$$ 

Again using Eq. (4.5) yields

$$\varepsilon_2 H_\xi^2 = -e^{\eta t_0} \phi_R s^\perp (\xi, \xi) \geq e^{\eta t_0} \phi_R \varepsilon_1 > 0,$$

so that

$$\frac{\varepsilon_2}{\varepsilon_1} H_\xi^2 \geq e^{\eta t_0} \phi_R > 0$$

and accordingly

$$C \geq -2C_{\phi_R} \phi_R \varepsilon_2 \varepsilon_1.$$ 

From the estimate for $B$, we then get

$$B + C \geq 2e^{\eta t_0} \left( \frac{\varepsilon_1}{m} - C_{\phi_R} \varepsilon_2 \right) H_\xi^2.$$

Now choose $\varepsilon_2$, such that $0 < \varepsilon_2 < \min\{2\varepsilon_1/m, \varepsilon_1^2\}$. Then it is $B + C > 0$ and furthermore $A + B + C > 0$, which contradicts Eq. (4.5), so that $\chi < 0$ along the flow.

The claim of the lemma follows by first sending $R \to \infty$, then $\eta \to 0$, and finally $T' \to T$. $\square$

**Corollary 4.6.** Under the mean curvature flow, the mean curvature vector of the graph of a length-decreasing map $f_t: \mathbb{R}^m \to N$ stays bounded, i.e. there is a constant $C > 0$, such that

$$\|\vec{H}\|^2 \leq C.$$

**Proof.** Let $\{\xi_1, \ldots, \xi_n\}$ be an orthonormal basis of $T_{x_0}^\perp \mathbb{R}^m$. Using Lemma 4.5, we obtain

$$\varepsilon_2 \|\vec{H}\|^2 = \varepsilon_2 \sum_{k=1}^n H_\xi^2 \leq -\sum_{k=1}^n s^\perp (\xi_k, \xi_k) \leq n,$$

which establishes the claim. $\square$

**4.2. Improved First-Order Estimate.** In some cases, namely if $m > 1$ and $\inf_{\mathbb{R}^m \times \{0\}} \text{tr}(s) < m - 1$, the estimate on the singular values of $d f_t$ can be improved.

**Lemma 4.7.** Assume $N$ satisfies the curvature bound

$$\sec_N \leq -\sigma < 0$$

for some constant $\sigma > 0$. If $f_t: \mathbb{R}^m \to N$ is a family of strictly length-decreasing maps that evolve under the mean curvature flow, the trace of the tensor $s$ satisfies

$$(\partial_t - \Delta) \text{tr}(s) \geq 2\sigma \sum_{k=1}^m \frac{2}{1 + \lambda_k^2} \left( \sum_{l=1}^m \frac{\lambda_l^2}{1 + \lambda_l^2} - \frac{\lambda_k^2}{1 + \lambda_k^2} \right).$$

**Proof.** Using lemma 3.4 and the Gauß equation (2.1), we obtain the evolution equation for the trace of the tensor $s$,

$$(\partial_t - \Delta) \text{tr}(s) = -2 \sum_{k,l=1}^m \left( s_{\mathbb{R}^m \times N} - 1 - \frac{\lambda_k^2}{1 + \lambda_k^2} \right) (A(e_k, e_l), A(e_k, e_l)).$$
Proof. Using $2\, tr(\cdot) \geq 0$ and the mean curvature flow, the trace of the tensor $s$ satisfies

$$ -2 \sum_{k,l=1}^{m} \frac{2}{1 + \lambda_k^2} f^* R_N(e_k, e_l) $$

Since the maps $f_t$ are length-decreasing, it is $s_{R^m \times N}(A(u, v), A(u, v)) \leq 0$ for any $u, v \in \Gamma(TR^m)$ and also $\lambda_k^2 \leq 1$. Consequently,

$$ (\partial_t - \Delta) tr(s) \geq -2 \sum_{k,l=1}^{m} \frac{2}{1 + \lambda_k^2} f^* R_N(e_k, e_l) $$

$$ = -2 \sum_{k,l=1}^{m} \frac{2}{1 + \lambda_k^2} sec_N(df(e_k) \wedge df(e_l)) f^* g_N(e_k, e_l) f^* g_N(e_l, e_l) $$

$$ \geq 2\sigma \sum_{k=1}^{m} \frac{2}{1 + \lambda_k^2} \left( \sum_{l=1}^{m} \frac{\lambda_l^2}{1 + \lambda_l^2} - \frac{\lambda_k^2}{1 + \lambda_k^2} \right) . \quad \Box $$

Corollary 4.8. Assume $N$ satisfies the curvature bound

$$ sec_N \leq -\sigma < 0 $$

for some constant $\sigma > 0$. If $f_t : R^m \rightarrow N$ is a family of strictly length-decreasing maps, then under the mean curvature flow, the trace of the tensor $s$ satisfies

$$ (\partial_t - \Delta) tr(s) \geq \frac{\sigma}{2} (m - tr(s)) (m - 1 - tr(s)) . $$

Proof. Using $2\, tr(f^* g_N) = m - tr(s)$ and $\lambda_k^2 \leq 1$, we calculate

$$ (\partial_t - \Delta) tr(s) \geq 2\sigma \sum_{k=1}^{m} \frac{2}{1 + \lambda_k^2} \lambda_k^2 \left( \sum_{l=1}^{m} \frac{\lambda_l^2}{1 + \lambda_l^2} - \frac{\lambda_k^2}{1 + \lambda_k^2} \right) $$

$$ \geq 2\sigma \sum_{k=1}^{m} \lambda_k^2 \left( \sum_{l=1}^{m} \frac{\lambda_l^2}{1 + \lambda_l^2} - \lambda_k^2 \right) $$

$$ = \sigma \sum_{k=1}^{m} \lambda_k^2 \left( \sum_{l=1}^{m} \frac{\lambda_l^2}{1 + \lambda_l^2} - \lambda_k^2 \right) $$

$$ = \sigma \left( (m - tr(s))^2 - 4 \sum_{k=1}^{m} \left( \frac{\lambda_k^2}{1 + \lambda_k^2} \right)^2 \right) $$

$$ \geq \frac{\sigma}{2} \left( (m - tr(s))^2 - (m - tr(s)) \right) $$

$$ \geq \frac{\sigma}{2} (m - tr(s)) (m - 1 - tr(s)) . \quad \Box $$

In the following, we assume $\inf_{x \in R^m} tr(s) < m - 1$. Let us set

$$ \gamma(x, t) := e^{\phi_R(x)} tr(s) - \frac{c_1 (m - 1) \exp \left( \frac{\sigma}{2} t \right) - m}{c_1 \exp \left( \frac{\sigma}{2} t \right) - 1} , $$

$$ (\partial_t - \Delta) \gamma(x, t) \leq \frac{\sigma}{2} (m - tr(s)) (m - 1 - tr(s)) . \quad \Box $$
where 
\[ c_1 := 1 + \frac{1}{(m-1) - \inf_{\mathbb{R}^m \times \{0\}} \text{tr}(s)} > 1. \]

**Lemma 4.9.** The function \( \gamma \) satisfies the evolution inequality
\[ (\partial_t - \Delta) \gamma \geq e^{\eta t} \phi_R \frac{\sigma}{2} (m - \text{tr}(s))(m - 1 - \text{tr}(s)) - \frac{\sigma}{2} \frac{c_1 \exp \left( \frac{\sigma}{2} t \right)}{(c_1 \exp \left( \frac{\sigma}{2} t \right) - 1)^2} + e^{\eta t} \{ \eta \phi_R \text{tr}(s) - 2 \langle \nabla \phi_R, \nabla \text{tr}(s) \rangle - (\Delta \phi_R) \text{tr}(s) \}. \]

**Proof.** We calculate
\[
(\partial_t - \Delta) \gamma = e^{\eta t} \phi_R (\partial_t - \Delta) \text{tr}(s) - \frac{\sigma}{2} \frac{c_1 \exp \left( \frac{\sigma}{2} t \right)}{(c_1 \exp \left( \frac{\sigma}{2} t \right) - 1)^2} + e^{\eta t} \{ \eta \phi_R \text{tr}(s) - 2 \langle \nabla \phi_R, \nabla \text{tr}(s) \rangle - (\Delta \phi_R) \text{tr}(s) \}.
\]
The claim follows from corollary 4.8. \( \square \)

**Lemma 4.10.** Let \( F(x, t) \) be a smooth solution to (3.1) with bounded geometry and assume there exists \( \varepsilon > 0 \), such that \( s - c \varepsilon \geq 0 \) for any \( t \in [0, T] \). Fix any \( T' \in [0, T] \) and \( (x_0, t_0) \in \mathbb{R}^m \times [0, T'] \). Then the following estimate holds at \((x_0, t_0)\),
\[
-c(T') \frac{\|x_0\|_{\mathbb{R}^m}}{R^2} \text{tr}(s) \leq \langle \nabla \phi_R, \nabla \text{tr}(s) \rangle \leq c(T') \frac{\|x_0\|_{\mathbb{R}^m}}{R^2} \text{tr}(s),
\]
where \( c(T') \geq 0 \) is a constant depending only on \( T' \).

**Proof.** Note that
\[
\nabla_u \text{tr}(s) = \sum_{k=1}^m (\nabla_u s)(e_k, e_k) = 2 \sum_{k=1}^m s^N_{(u, e_k)} \langle A(u, e_k), dF(e_k) \rangle.
\]
The bounded geometry assumptions (3.4) and (3.5) imply that \( s, \nabla s \) and therefore \( \nabla \text{tr}(s) \) are uniformly bounded on \( \mathbb{R}^m \times [0, T'] \) by a constant depending only on \( T' \). Since also \( \text{tr}(s) \geq m \varepsilon \) by assumption, at \((x_0, t_0)\) we have
\[
-c(T') \frac{\|x_0\|_{\mathbb{R}^m}}{R^2} \text{tr}(s) \leq \langle \nabla \phi_R, \nabla \text{tr}(s) \rangle \leq c(T') \frac{\|x_0\|_{\mathbb{R}^m}}{R^2} \text{tr}(s). \quad \square
\]

**Lemma 4.11.** Assume \( m > 1 \) and that \( N \) satisfies the curvature bound
\[
\sec_N \leq -\sigma < 0
\]
for some constant \( \sigma > 0 \). If \( f_t : \mathbb{R}^m \to N \) is a family of strictly length-decreasing maps that evolve under the mean curvature flow and \( \inf_{\mathbb{R}^m \times \{0\}} \text{tr}(s) < m - 1 \), then the trace of the tensor \( s \) satisfies the estimate
\[
\text{tr}(s) \geq \frac{c_1 (m - 1) \exp \left( \frac{\sigma}{2} t \right) - m}{c_1 \exp \left( \frac{\sigma}{2} t \right) - 1},
\]
where
\[
c_1 := 1 + \frac{1}{(m-1) - \inf_{\mathbb{R}^m \times \{0\}} \text{tr}(s)}.
\]

**Proof.** We will show that for any fixed \( T' \in [0, T] \) and \( \eta > 0 \), there is \( R_0 \) depending only on \( \eta \) and \( T' \), such that \( \gamma > 0 \) on \( \mathbb{R}^m \times [0, T'] \) for all \( R \geq R_0 \).

On the contrary, suppose \( \gamma \) is not positive on \( \mathbb{R}^m \times [0, T] \) for some \( R \geq R_0 \). Then as \( \gamma \geq 0 \) on \( \mathbb{R}^m \times \{0\} \), \( \text{tr}(s) \geq m \varepsilon \) on \( \mathbb{R}^m \times [0, T] \) and \( \phi_R(x) \to \infty \) as \( \|x\| \to \infty \), it follows that \( \gamma > 0 \) outside some compact set \( K \subset \mathbb{R}^m \) for all \( t \in [0, T] \). We conclude
that there is \((x_0, t_0) \in K \times [0, T']\) such that \(\gamma(x_0, t_0) = 0\) and that \(t_0\) is the first such time. According to the second derivative criterion, at the point \((x_0, t_0)\) we have

\[
\gamma = 0, \quad \partial_t \gamma \leq 0, \quad \nabla \gamma = 0 \quad \text{and} \quad \Delta \gamma \geq 0.
\]

(4.6)

On the other hand, using lemma 4.9, we estimate the terms in the evolution equation for \(\gamma\)

\[
(\partial_t - \Delta) \gamma \geq A + B,
\]

where

\[
A := e^{\eta t} \phi_R \frac{\sigma}{2} (m - \text{tr}(s)) (m - 1 - \text{tr}(s)) - \frac{\sigma}{2} \frac{c_1 \exp \left( \frac{\sigma t}{2} \right)}{(c_1 \exp \left( \frac{\sigma t}{2} \right) - 1)^2},
\]

\[
B := e^{\eta t} \left\{ \eta \phi_R \text{tr}(s) - 2 (\nabla \phi_R, \nabla \text{tr}(s)) - (\Delta \phi_R) \text{tr}(s) \right\}.
\]

Further, using Eq. (4.6), at \((x_0, t_0)\) we obtain

\[
e^{\eta t_0} \phi_R \text{tr}(s) = \frac{c_1 (m - 1) \exp \left( \frac{\sigma t_0}{2} \right) - m}{c_1 \exp \left( \frac{\sigma t_0}{2} \right) - 1}.
\]

Since \(c_1 > 1\), the right-hand side is monotonically increasing in \(t\) for \(t \geq 0\) and bounded from above by \(m - 1\), so that

\[
\text{tr}(s) \leq e^{\eta t_0} \phi_R \text{tr}(s) = \frac{c_1 (m - 1) \exp \left( \frac{\sigma t_0}{2} \right) - m}{c_1 \exp \left( \frac{\sigma t_0}{2} \right) - 1} \leq m - 1.
\]

Consequently, we estimate

\[
A \geq \frac{\sigma}{2} \left( m - \frac{c_1 (m - 1) \exp \left( \frac{\sigma t_0}{2} \right) - m}{c_1 \exp \left( \frac{\sigma t_0}{2} \right) - 1} \right) (m - 1 - \text{tr}(s))
\]

\[
- \frac{\sigma}{2} \frac{c_1 \exp \left( \frac{\sigma t_0}{2} \right)}{(c_1 \exp \left( \frac{\sigma t_0}{2} \right) - 1)^2}
\]

\[
\geq \frac{\sigma}{2} \frac{c_1 \exp \left( \frac{\sigma t_0}{2} \right)}{c_1 \exp \left( \frac{\sigma t_0}{2} \right) - 1} \left( m - 1 - \frac{c_1 (m - 1) \exp \left( \frac{\sigma t_0}{2} \right) - m}{c_1 \exp \left( \frac{\sigma t_0}{2} \right) - 1} \right)
\]

\[
- \frac{\sigma}{2} \frac{c_1 \exp \left( \frac{\sigma t_0}{2} \right)}{(c_1 \exp \left( \frac{\sigma t_0}{2} \right) - 1)^2}
\]

\[
= \frac{\sigma}{2} \frac{c_1 \exp \left( \frac{\sigma t_0}{2} \right)}{c_1 \exp \left( \frac{\sigma t_0}{2} \right) - 1} \frac{1}{c_1 \exp \left( \frac{\sigma t_0}{2} \right) - 1} - \frac{\sigma}{2} \frac{c_1 \exp \left( \frac{\sigma t_0}{2} \right)}{(c_1 \exp \left( \frac{\sigma t_0}{2} \right) - 1)^2}
\]

\[
= 0.
\]

For the terms in \(B\), we use lemma 4.10 to calculate

\[
B \geq e^{\eta t_0} \left\{ \eta + \frac{\|x_0\|_m^2}{R^2} - 3 c(T') \frac{\|x_0\|_m}{R} - \frac{c(T')}{R^2} \right\} \text{tr}(s).
\]

Now choosing \(R_0 > 0\) (depending on \(\eta\) and \(T'\)) large enough, the term

\[
\frac{\eta}{2} + \frac{\|x_0\|^2}{R^2} - 3 c(T') \frac{\|x_0\|_m}{R^2} - \frac{c(T')}{R^2}
\]

is strictly positive for any \(R \geq R_0\) and any \(\|x_0\|_m\). This yields

\[
(\partial_t - \Delta) \gamma(x_0, t_0) = A + B > e^{\eta t_0} \frac{\eta}{2} \text{tr}(s) \geq e^{\eta t_0} \frac{\eta}{2} m \varepsilon > 0,
\]

which contradicts (4.6) and thus shows the claim.

The statement of the lemma follows by first letting \(R \to \infty\), then \(\eta \to 0\) and finally \(T' \to T\). □
Remark 4.12. We observe that the estimate in lemma 4.7 holds for any $\sigma \geq 0$, i.e. 
$$(\partial_t - \Delta) \text{tr}(s) \geq 0.$$ 

Arguing in the same way as in the proof of lemma 4.11, we obtain that tr(s) is non-decreasing in time.

5. Higher-Order Estimates

In this section we prove decay estimates for derivatives of order $n \geq 2$ of the function $f_t$ defining the graph. We begin by recalling some definitions, where we follow [16, Section 5] (see also [4]).

Definition 5.1 ($C^\infty$-convergence). Let $(E, \pi, M)$ be a vector bundle endowed with a Riemannian metric $g$ and a metric connection $\nabla$ and suppose that $\{\xi_k\}_{k \in \mathbb{N}}$ is a sequence of sections of $E$. Let $U$ be an open subset of $M$ with compact closure $\overline{U}$ in $M$. Fix a natural number $p \geq 0$. We say that $\{\xi_k\}_{k \in \mathbb{N}}$ converges in $C^p$ to $\xi_\infty \in \Gamma(E|\overline{U})$, if for every $\varepsilon > 0$ there exists $k_0 = k_0(\varepsilon)$, such that

$$\sup_{0 \leq \alpha \leq p, x \in U} |\nabla^\alpha (\xi_k - \xi_\infty)| < \varepsilon$$

whenever $k \geq k_0$. We say that $\{\xi_k\}_{k \in \mathbb{N}}$ converges in $C^\infty$ to $\xi_\infty \in \Gamma(E|\overline{U})$ if $\{\xi_k\}_{k \in \mathbb{N}}$ converges in $C^p$ to $\xi_\infty \in \Gamma(E|\overline{U})$ for any $p \geq 0$.

Definition 5.2 ($C^\infty$-convergence on compact sets). Let $(E, \pi, M)$ be a vector bundle endowed with a Riemannian metric $g$ and a metric connection $\nabla$. Let $\{U_n\}_{n \in \mathbb{N}}$ be an exhaustion of $M$ and $\{\xi_k\}_{k \in \mathbb{N}}$ be a sequence of sections of $E$ defined on open sets $A_k$ of $M$. We say that $\{\xi_k\}_{k \in \mathbb{N}}$ converges smoothly on compact sets to $\xi_\infty \in \Gamma(E)$ if:

(i) For every $n \in \mathbb{N}$ there exists $k_0$ such that $U_n \subset A_k$ for all natural numbers $k \geq k_0$.

(ii) The sequence $\{\xi|_{U_n}\}_{k \geq k_0}$ converges in $C^\infty$ to the restriction of the section $\xi_\infty$ on $U_n$.

Definition 5.3 (Pointed Manifold). A pointed Riemannian manifold $(M, g, x)$ is a Riemannian manifold $(M, g)$ with a choice of base point $x \in M$. If the metric $g$ is complete, we say that $(M, g, x)$ is a complete pointed Riemannian manifold.

Definition 5.4 (Cheeger-Gromov smooth convergence). A sequence of complete pointed Riemannian manifolds $\{(M_k, g_k, x_k)\}_{k \in \mathbb{N}}$ smoothly converges in the sense of Cheeger-Gromov to a complete pointed Riemannian manifold $(M_\infty, g_\infty, x_\infty)$, if there exists:

(i) An exhaustion $\{U_k\}_{k \in \mathbb{N}}$ of $M_\infty$ with $x_\infty \in U_k$, for all $k \in \mathbb{N}$.

(ii) A sequence of diffeomorphisms $\Phi_k : U_k \to \Phi_k(U_k) \subset M_k$ with $\Phi_k(x_\infty) = x_k$ and such that $\{\Phi_k^* g_k\}_{k \in \mathbb{N}}$ smoothly converges in $C^\infty$ to $g_\infty$ on compact sets in $M_\infty$.

The family $\{(U_k, \Phi_k)\}_{k \in \mathbb{N}}$ is called a family of convergence pairs of the sequence $\{(M_k, g_k, x_k)\}_{k \in \mathbb{N}}$ with respect to the limit $(M_\infty, g_\infty, x_\infty)$.

In the sequel, when we say smooth convergence, we will always mean smooth convergence in the sense of Cheeger-Gromov.

The family of convergence pairs is not unique. However, two families of convergence pairs $\{(U_k, \Phi_k)\}_{k \in \mathbb{N}}$ and $\{(W_k, \Psi_k)\}_{k \in \mathbb{N}}$ are equivalent in the sense that there exists an isometry $\mathcal{I}$ of the limit $(M_\infty, g_\infty, x_\infty)$ such that, for every compact subset $K$ of $M_\infty$ there exists a natural number $k_0$ such that for any $k \geq k_0$: 
(i) the mapping $\Phi_k^{-1} \circ \Psi_k$ is well-defined over $K$ and
(ii) the sequence $\{\Phi_k^{-1} \circ \Psi_k\}_{k \geq k_0}$ smoothly converges to $I$ on $K$.

The following proposition is quite standard.

**Proposition 5.5.** Let $(M, g)$ be a complete Riemannian manifold with bounded geometry. Suppose that $\{a_k\}_{k \in \mathbb{N}}$ is an increasing sequence of real numbers that tends to $\infty$ and let $\{x_k\}_{k \in \mathbb{N}}$ be a sequence of points on $M$. Then the sequence $(M, a_k^2 g, x_k)$ smoothly subconverges to the standard Euclidean space $(\mathbb{R}^m, g_{\text{eucl}}, 0)$.

**Definition 5.6.** We say the a sequence $\{(M_k, g_k, x_k)\}_{k \in \mathbb{N}}$ of complete pointed Riemannian manifolds has uniformly bounded geometry if the following conditions are satisfied:

(i) For any $j \geq 0$ there exists a uniform constant $C_j \geq 0$, such that for each $k \in \mathbb{N}$ it holds $\|\nabla^2 R_{M_k}\| \leq C_j$.

(ii) There exists a uniform constant $c_0$ such that $\text{inj}(M_k) \geq c_0 > 0$.

**Theorem 5.7** (Cheeger-Gromov compactness). Let $\{(M_k, g_k, x_k)\}_{k \in \mathbb{N}}$ be a sequence of complete pointed Riemannian manifolds with uniformly bounded geometry. Then the sequence $\{(M_k, g_k, x_k)\}_{k \in \mathbb{N}}$ subconverges smoothly to a complete pointed Riemannian manifold $(M_\infty, g_\infty, x_\infty)$.

**Definition 5.8** (Convergence of isometric immersions). Let $F_k : (M_k, g_k, x_k) \to (L_k, h_k, y_k)$ be a sequence of isometric immersions, such that $F_k(x_k) = y_k$ for any $k \in \mathbb{N}$. We say that the sequence $\{F_k\}_{k \in \mathbb{N}}$ converges smoothly to an isometric immersion $F_\infty : (M_\infty, g_\infty, x_\infty) \to (L_\infty, h_\infty, y_\infty)$ if the following conditions are satisfied:

(i) The sequence $\{(M_k, g_k, x_k)\}_{k \in \mathbb{N}}$ smoothly converges to $(M_\infty, g_\infty, x_\infty)$.

(ii) The sequence $\{(L_k, h_k, y_k)\}_{k \in \mathbb{N}}$ smoothly converges to $(L_\infty, h_\infty, y_\infty)$.

(iii) If $\{(U_k, \Phi_k)\}_{k \in \mathbb{N}}$ is a family of convergence pairs of $\{(M_k, g_k, x_k)\}_{k \in \mathbb{N}}$ and $\{(W_k, \Psi_k)\}_{k \in \mathbb{N}}$ is a family of convergence pairs of $\{(L_k, h_k, y_k)\}_{k \in \mathbb{N}}$, then for each $k \in \mathbb{N}$ the relation $F_k \circ \Phi_k(U_k) \subset \Psi_k(W_k)$ holds and $\Psi_k^{-1} F \circ \Phi_k$ smoothly converges to $F_\infty$ on compact sets.

Assume that $(M, g_M)$ and $(N, g_N)$ are complete manifolds with bounded geometry and that $F : M \times [0, T) \to N$ is a solution of the mean curvature flow with bounded geometry. For any $\tau > 0$ and $(x_0, t_0) \in M \times [0, T)$, let us define the parabolic scaling by $\tau$ at $(x_0, t_0)$ by setting

$$F^\tau_I : (M, x_0) \to (N, \tau^2 g_N, F^\tau_I(x_0)), \quad F^\tau_I(x) := F_{t_0+t/\tau^2}(x).$$

**Remark 5.9.** The parabolic scaling preserves the graph property. More precisely, assume that the family of immersions $F : M \times [0, T) \to M \times N$ are graphs with induced metric metric $g = g_{M \times N}$. Then there is a family of diffeomorphisms $\varphi : M \times [0, T) \to M$ and a function $f_t : M \to N$, such that $F_t \circ \varphi_t(x) = (x, f_t(x))$. Let the parabolic scaling by $\tau$ at $(x_0, t_0)$ be given by $F^\tau_I(x) = F_{t_0+t/\tau^2}(x)$, and let $\varphi^\tau_I(x) := \varphi_{t_0+t/\tau^2}(x)$. We calculate

$$F^\tau_I \circ \varphi^\tau_I(x) = F_{t_0+t/\tau^2}(\varphi_{t_0+t/\tau^2}(x)) = (x, f_{t_0+t/\tau^2}(x)).$$

Furthermore, the metric induced on the graph is given by

$$g_\tau := (F^\tau_I)^* g_{M \times N} = \tau^2 g.$$
Lemma 5.10. Let $F : \mathbb{R}^m \times [0, T) \to \mathbb{R}^m \times N$ be a smooth, graphic solution to (3.1) with bounded geometry. Suppose the corresponding maps $f_t : \mathbb{R}^m \to N$ satisfy $\|df_t\|^2 \leq C_1$ and $\|\nabla df_t\|^2 \leq C_2$ on $\mathbb{R}^m \times [0, T)$ for some constants $C_1, C_2 \geq 0$. Then for every $l \geq 3$ there exists a constant $C_l$, such that

$$\sup_{x \in \mathbb{R}^m} \|\nabla^{l-1} df_t\|^2 \leq C_l$$

for all $t \in [0, T)$.

Proof. If $\|\nabla^2 df_t\|^2 \leq C_3$ in $[0, T)$, then a parabolic bootstrapping argument for the quasilinear Eq. (3.3) gives $\|\nabla^{l-1} df_t\|^2 \leq C_l$ for $l \geq 3$. It is therefore sufficient to prove the claim for $l = 3$.

Suppose $\|\nabla^2 df_t\|^2$ was not bounded on $\mathbb{R}^m \times [0, T)$. By the bounded geometry assumption on $F$ (resp. on $f$), there would be a sequence $t_k \to T$, such that

$$2\mu_k := \sup_{x \in \mathbb{R}^m} \|\nabla^2 df_{t_k}(x)\|^2 \to \infty \quad \text{and} \quad \sup_{x \in \mathbb{R}^m} \|\nabla^2 df_{t_k}(x)\|^2 \leq 2\mu_k < \infty .$$

This implies there is a sequence $\{x_k\}$, such that $\|\nabla^2 df_{t_k}(x_k)\|^2 \geq \mu_k \to \infty$ for $t_k \to T$.

Set $\tau_k := \mu_k^{1/4}$ and consider the sequence

$$F^\tau_k : (\mathbb{R}^m, \tau^2_k g_{\mathbb{R}^m \times N}, x_k) \to (\mathbb{R}^m \times N, \tau^2_k \bar{g}_{\mathbb{R}^m \times N}, F^\tau_k(x_k))$$

of immersions of pointed Riemannian manifolds, where

$$F^\tau_k(x, r) := F \left( x, \frac{r}{\tau_k} + t_k \right) .$$

Then $\{F^\tau_k\}$ is a sequence of mean curvature flows for $r \in [-\tau^2_k t_k, 0]$.

Since $N$ has bounded geometry, $(\mathbb{R}^m \times N, \tau^2_k \bar{g}_{\mathbb{R}^m \times N}, F^\tau_k(x_k))$ converges on every compact subset to the Euclidean space $(\mathbb{R}^{m+n}, \bar{g}_{\mathbb{R}^m \times \mathbb{R}^n}, 0)$ with its standard metric. The sequence $(\mathbb{R}^m, F^\tau_k \ast (\tau^2_k \bar{g}_{\mathbb{R}^m \times N}), x_k)$ converges smoothly to a geometric limit $(M_\infty, g_\infty, x_\infty)$ in the Cheeger-Gromov sense, and every manifold of the sequence is the graph of a function $\tilde{f}_k$. In particular, the limiting manifold $M_\infty$ is the graph of a function $\tilde{f}_\infty : \mathbb{R}^m \to \mathbb{R}^n$. Since $M_\infty$ satisfies the mean curvature flow equation in $\mathbb{R}^{m+n}$, the function $\tilde{f}_\infty$ satisfies the quasilinear parabolic equation

$$\partial_t \tilde{f}_\infty(x) = \sum_{i,j=1}^m g^i_\infty \frac{\partial^2 \tilde{f}_\infty}{\partial x^i \partial x^j} .$$

Using the assumptions $\|df_t\|^2 \leq C_1$ and $\|\nabla df_t\|^2 \leq C_2$ in $\mathbb{R}^m \times [-\tau^2_k t_k, 0]$, we calculate

$$\|d\tilde{f}_{t_k}(x, r)\|^2 \leq C_1 ,$$

$$\|\nabla d\tilde{f}_{t_k}(x, r)\|^2 \leq \frac{\|\nabla df(x, t)\|^2}{\tau_k^2} \leq \frac{C_2}{\tau_k^2} \quad \text{as} \quad k \to \infty ,$$

where $\|\cdot\|_{t_k}$ denotes the norms with respect to the rescaled metrics $\tau^2_k \bar{g}_N$ and $\bar{g}_{t_k}$. Using the definition of $\tau_k$ and the definition of the sequence $(x_k, t_k)$, we obtain

$$\|\nabla^2 d\tilde{f}_{t_k}(x, r)\|^2 \leq \frac{\|\nabla^2 df(x, t)\|^2}{\tau_k^4} = \frac{\|\nabla^2 df(x, t)\|^2}{\mu_k} \leq 2$$

and

$$\|\nabla^2 d\tilde{f}_{t_k}(x_k, 0)\|^2 \geq \frac{\|\nabla^2 df(x_k, t_k)\|^2}{\mu_k} \geq \frac{1}{2} .$$
Since the sub-convergence to \((M_\infty, g_\infty, x_\infty)\) was smooth, the limit \(\tilde{f}_\infty\) is smooth and satisfies
\[
\|\nabla d\tilde{f}_\infty\| = 0 \quad \text{and} \quad \|\nabla^2 d\tilde{f}_\infty(x_\infty, 0)\| \geq 1,
\]
which is a contradiction, so that \(\|\nabla^2 df_t\|\) has to be bounded.

The estimates for the higher-order derivatives follow by differentiating equation (3.3) and repeating the above argument. We thus obtain that the spatial derivatives as well as the time derivatives of \(f_t\) of any positive order are uniformly bounded. □

**Lemma 5.11.** Let \(F : \mathbb{R}^m \times [0, T) \to \mathbb{R}^m \times N\) be a smooth, graphic solution to (3.1) with bounded geometry and denote by \(f_1 : \mathbb{R}^m \to N\) the corresponding maps. Assume \(f_0\) is strictly length-decreasing and further assume that \(\|\tilde{H}\| \leq C\) on \(\mathbb{R}^m \times [0, T)\) for some constant \(C \geq 0\). Then for every \(k \geq 1\) there exists a constant \(C_k \geq 0\), such that
\[
\sup_{x \in \mathbb{R}^m} \|\nabla^k dF(t)(x)\|^2 \leq C_k
\]
for all \(t \in [0, T)\).

**Proof.** By lemma 4.3, the length-decreasing condition is preserved in \([0, T)\), so that the relation \(f^*_k g_N \leq (1 - \delta) g_{\mathbb{R}^m}\) holds in \([0, T)\). This shows the claim for \(l = 1\). By lemma 5.10, we only need to prove the case \(l = 2\). Suppose the claim was false for \(l = 2\). Let
\[
\eta(t) := \sup_{x \in \mathbb{R}^m} \|\nabla df(x, t')\|.
\]
Then there is a sequence \((x_k,t_k)\) along which we have \(\|\nabla df(x_k, t_k)\| \geq \eta(t_k)/2\) while \(\eta(t_k) \to \infty\) as \(t_k \to T\). Let \(\tau_k := \eta(t_k)\). Consider the sequence
\[
F_{\tau_k}^k : (\mathbb{R}^m, F_{\tau_k}^k * (\tau_k^2 g_{\mathbb{R}^m \times N}, x_k)) \to (\mathbb{R}^m \times N, \tau_k^2 g_{\mathbb{R}^m \times N}, F_{\tau_k}^k(x_k))
\]
of immersions of pointed Riemannian manifolds, where
\[
F_{\tau_k}^k(x, r) := F\left(x, \frac{r}{\tau_k} + t_k\right).
\]
Then \(\{F_{\tau_k}^k\}\) is a sequence of mean curvature flows for \(s \in [-\tau_k^2 t_k, 0]\).

Since \(N\) has bounded geometry, \((\mathbb{R}^m \times N, \tau_k^2 g_{\mathbb{R}^m \times N}, F_{\tau_k}^k(x_k))\) converges on every compact subset to the Euclidean space \((\mathbb{R}^{m+n}, g_{\mathbb{R}^m \times \mathbb{R}^n}, 0)\) with its standard metric. The sequence \((\mathbb{R}^m, F_{\tau_k}^k * (\tau_k^2 g_{\mathbb{R}^m \times N}, x_k))\) converges smoothly to a geometric limit \((M_\infty, g_\infty, x_\infty)\) in the Cheeger-Gromov sense, and every manifold of the sequence is the graph of a function \(\tilde{f}_k\). In particular, the limiting manifold \(M_\infty\) is the graph of a function \(\tilde{f}_\infty : \mathbb{R}^m \to \mathbb{R}^n\). Since \(M_\infty\) satisfies the mean curvature flow equation, the function \(\tilde{f}_\infty\) satisfies the quasilinear parabolic equation
\[
\partial_t \tilde{f}_\infty = \sum_{i,j=1}^m g_\infty^{ij} \frac{\partial^2 \tilde{f}_\infty}{\partial x^i \partial x^j}.
\]
Note that by the definition of \(\tau_k = \eta(t_k)\), it is
\[
\|d\tilde{f}_{\tau_k}(x, r)\|_{\tau_k} = \|df(x, t)\| \leq C_1,
\]
\[
\|\nabla d\tilde{f}_{\tau_k}(x, r)\|_{\tau_k} = \tau_k^{-1}\|\nabla df(x, t)\| \leq 1
\]
for all \((x, r) \in \mathbb{R}^m \times [-\tau_k^2 t_k, 0]\), where \(\|\cdot\|_{\tau_k}\) denotes the norms with respect to the rescaled metrics \(\tau_k^2 g_N\) and \(g_{\tau_k}\). Moreover, by the definition of the sequence \((x_k, t_k)\),
the estimate
\[ \| \nabla df_{\tau_k}(x_k, 0) \|_{\tau_k} = \frac{\| \nabla df(x_k, t_k) \|}{\eta(t_k)} = \frac{\| \nabla df(x_k, t_k) \|}{\eta(t_k)} \geq \frac{1}{2} \] (5.1)
holds. By lemma 5.10 we conclude that all higher-order derivatives of \( f_{\tau_k} \) are uniformly bounded on \( \mathbb{R}^m \times [-\tau_k^2, 0] \). Since \( \| \tilde{H} \| \leq C \) for the graphs \( (x, f(x, t)) \) by assumption, after rescaling we have
\[ \| \tilde{H}_{\tau_k} \|_{\tau_k} \leq \frac{C}{\tau_k} \]
for the graphs \( (x, \tilde{f}_{\tau_k}(x, r)) \). It follows that for each \( r \), the limiting graph
\[ \Gamma(\tilde{f}_\infty(\cdot, r)) \subset \mathbb{R}^m \times \mathbb{R}^n \]
must have \( \tilde{H}_\infty = 0 \) everywhere, as well as \( \tilde{\lambda}_i^2 := \tilde{f}_\infty \mathbb{g}_{\mathbb{R}^m}(e_i, e_i) \leq 1 - \delta \). This in turn implies bounds on the Jacobian of the projection \( \pi_1 \) from the graph \( (x, \tilde{f}_\infty(x, r)) \) to \( \mathbb{R}^m \),
\[ \frac{1}{2m/2} < \Omega_\infty = \frac{1}{\sqrt{\prod_{i=1}^m (1 + \tilde{\lambda}_i^2)}} \leq 1. \]

Thus, we can apply a Bernstein-type theorem of Wang [25, Theorem 1.1] to conclude that the graph \( (x, \tilde{f}_\infty(x, r)) \) is an affine subspace of \( \mathbb{R}^m \times \mathbb{R}^n \). Therefore, \( \tilde{f}_\infty \) has to be a linear map for each \( r \), but this contradicts (5.1), which (taking the limit \( k \to \infty \)) implies the estimate \( \| D^2 \tilde{f}_\infty(x, 0) \| \geq 1/2 \).

**Lemma 5.12.** Let \( f_0 : \mathbb{R}^m \to N \) be smooth and length-decreasing. Then Eq. (3.3) has a smooth solution \( f_k(x) \) on \( \mathbb{R}^m \times [0, \infty) \) with initial condition \( f_0(x) \), such that for all \( k \geq 2 \) we have the estimate
\[ t^{k-1} \sup_{x \in \mathbb{R}^m} \| \nabla^{k-1} df_k(x) \| \leq C_{k, \delta} \]
for some constant \( C_{k, \delta} \geq 0 \) depending only on \( k \) and \( \delta \).

**Proof.** By theorem 3.1 we know that (3.3) has a smooth short-time solution on \( [0, T] \) with initial condition \( f_0(x) \) and bounded geometry for any \( t \in [0, T] \). Assume \( T < \infty \). Then lemma 5.11 implies \( \| \nabla^{k-1} df_k \| \leq C_{k, \delta} \) in \( [0, T] \) for all integers \( k \geq 1 \), which by continuity extends to \( [0, T] \). By theorem 3.1, we can extend the solution beyond \( T \), which contradicts the definition of \( T \), so that \( T = \infty \).

For the second part, first consider \( k = 1 \), i.e. we show
\[ \sup_{x \in \mathbb{R}^m} \| \nabla df(x, t) \|^2 t \leq C_{1, \delta} \] (5.2)
for any \( t \). Assume this is not the case. Since \( f(x, t) \) satisfies the bounded geometry condition, it is
\[ \sup_{x \in \mathbb{R}^m} \| \nabla df(x, t) \| < \infty \]
for any \( t \). If Eq. (5.2) does not hold, there exists a sequence \( t_k \to \infty \) with
\[ \sup_{x \in \mathbb{R}^m} \| \nabla df(x, t_k) \|^2 t_k = \sup_{x \in \mathbb{R}^m} \| \nabla df(x, t_k) \|^2 t_k =: 2\mu_k \xrightarrow{k \to \infty} \infty. \] (5.3)
Further, for each \( t_k \), there exists \( x_k \in \mathbb{R}^m \), such that
\[ \| \nabla df(x_k, t_k) \|^2 t_k \geq \mu_k. \] (5.4)
Let \( \tau_k := \sqrt{\mu_k/t_k} \) and consider the sequence

\[
F_{k}^{\tau_k} : (\mathbb{R}^m, F_k^{\tau_k}(\tau_k^2 \mathbb{R}^m \times N), x_k) \to (\mathbb{R}^m \times N, \tau_k^2 \mathbb{R}^m \times N, F_k^{\tau_k}(x_k))
\]

of immersions of pointed Riemannian manifolds, where

\[
F_k^{\tau_k}(x, r) := F \left( x, \frac{r}{t_k} + t_k \right).
\]

Then \( \{F_k^{\tau_k}\} \) is a sequence of mean curvature flows for \( s \in [-\tau_k^2 t_k, 0] \).

Since \( N \) has bounded geometry, \( (\mathbb{R}^m \times N, \tau_k^2 \mathbb{R}^m \times N, F_k^{\tau_k}(x_k)) \) converges on every compact subset to the Euclidean space \( (\mathbb{R}^{m+n}, \mathbb{R}^{m+n}, 0) \) with its standard metric. The sequence \( (R_m, F_k^{\tau_k}(\tau_k^2 \mathbb{R}^m \times N), x_k) \) converges smoothly to a geometric limit \( (M_{\infty}, \mathbb{G}_{\infty}, x_{\infty}) \) in the Cheeger-Gromov sense, and every manifold of the sequence is the graph of a function \( \tilde{f}_k \). In particular, the limiting manifold \( M_{\infty} \) is the graph of a function \( \tilde{f}_{\infty} : \mathbb{R}^m \to \mathbb{R}^n \). Since \( M_{\infty} \) satisfies the mean curvature flow equation, the function \( \tilde{f}_{\infty} \) satisfies the quasilinear parabolic equation

\[
\partial_t \tilde{f}_{\infty} = \sum_{i,j=1}^{m} \frac{\partial^2 \tilde{f}_{\infty}}{\partial x^i \partial x^j}.
\]

Further, by the definition of \( \tau_k \) and \( (x_k, t_k) \), the relations

\[
|| d\tilde{f}_{\tau_k}(x, r) ||_{\tau_k} = || df(x, t) ||, \quad || \nabla d\tilde{f}_{\tau_k}(x, r) ||_{\tau_k} = \frac{|| \nabla df(x, t) ||}{\sqrt{\mu_k/t_k}} \quad \text{Eq. (5.3)}
\]

hold on \( \mathbb{R}^m \times [-\mu_k, 0] \). On the other hand, by the definition of \( \mu_k \), it is

\[
|| \nabla d\tilde{f}_{\infty}(x, 0) ||_{\tau_k} = \frac{1}{\mu_k/t_k} || \nabla df(x, t_k) || \quad \text{Eq. (5.4)} \geq 1. \quad (5.5)
\]

Note that for the graph of \( f(x, t) \), we have \( || \tilde{H} || \leq C \) for all \( t \) and some constant \( C \geq 0 \). Thus, for any \( 0 < \mu < \mu_k \) we conclude

\[
|| \tilde{H} ||_{\tau_k}^2 \leq \frac{C}{\mu_k} \quad \text{as} \quad k \to \infty \to 0
\]

uniformly on \( \mathbb{R}^m \times [-\mu, 0] \). It follows that \( \tilde{H}_{\infty} = 0 \), so that \( (y, \tilde{f}_{\infty}(y, r)) \) is a minimal graph in \( \mathbb{R}^n \times \mathbb{R}^m \). Since

\[
\tilde{f}_{\tau_k}(\tau_k^2 \mathbb{R}^m \times N)(v, v) = \tau_k^2 \tilde{f}_{\tau_k} \mathbb{G}^N(v, v) \leq (1 - \delta) \tau_k^2 \mathbb{G}^M(v, v),
\]

the maps \( \tilde{f}_{\tau_k} \) are strictly length-decreasing, so that in particular the limiting map \( \tilde{f}_{\infty} : \mathbb{R}^m \to \mathbb{R}^n \) is strictly length-decreasing. Thus, by applying the Bernstein-type theorem [25, Theorem 1.1], we conclude that \( \tilde{f}_{\infty} \) has to be an affine map. But this contradicts Eq. (5.5), so that our initial assumption was false, which eventually proves Eq. (5.2).

Now let \( l \geq 3 \) and suppose that \( || \nabla^{l-1} df ||_{l}^{2l-1} \) is not uniformly bounded. Then there exists a sequence \( (x_k, t_k) \), such that

\[
\sup_{x \in \mathbb{R}^m \atop t \leq t_k} || \nabla^{l-1} df(x, t) ||_{l}^{2l-1} =: 2\sigma_k \quad \text{as} \quad k \to \infty \to \infty
\]

and

\[
|| \nabla^{l-1} df(x_k, t_k) ||_{l}^{2l-1} \geq \sigma_k. \quad (5.6)
\]
Let
\[ \tau_k := \sqrt{\frac{\sigma_k^{1/(l-1)}}{t_k}} \]
and consider the sequence
\[ F_{\tau_k} : (\mathbb{R}^m, F_{\tau_k} \ast (k^2_{k^2} \mathbb{R} \times N), x_k) \to (\mathbb{R}^m \times N, \tau_k^2 \mathbb{R} \times N, F_{\tau_k}(x_k)) \]
of immersions of pointed Riemannian manifolds for \( t_k/2 \leq t \leq t_k \), where
\[ F_{\tau_k}(x, r) := F \left( x, \frac{r}{T_k} + t \right) . \]
Then \( \{ F_{\tau_k} \} \) is a sequence of mean curvature flows for \( r \in [-\sigma_k^{1/(l-1)}, 0] \). For the corresponding maps \( \tilde{f}_{\tau_k} \), we calculate
\[ \| \nabla d\tilde{f}_{\tau_k}(x, r) \|_2^2 = \tau_k^{-2}\| \nabla df(x, t) \|_2^2 = \frac{t_k}{\sigma_k^{1/(l-1)}}\| \nabla df(x, t) \|_2^2 \]
\[ \leq \frac{2t}{\sigma_k^{1/(l-1)}}\| \nabla df(x, t) \|_2^2 \leq \frac{2C_1,2}{\sigma_k^{1/(l-1)}} . \]
Since \( \sigma_k \to \infty \) for \( k \to \infty \) and \( l \geq 3 \), we deduce
\[ \| \nabla d\tilde{f}_{\tau_k}(y, r) \|_2^2 \xrightarrow{k \to \infty } 0 . \]
Now fix any \( \eta \) with \( \eta \in (0, \sigma_k^{1/(l-1)}) \) for all \( k \). By lemma 5.10, all higher order derivatives of \( \tilde{f}_{\tau_k} \) are uniformly bounded on \( \mathbb{R}^m \times [-\eta, 0] \). Thus, \( \tilde{f}_{\tau_k} \) sub-converges on compact sets to a smooth solution of the quasilinear parabolic equation
\[ \partial_t \tilde{f}_\infty = \sum_{i,j=1}^m \tilde{g}^\infty \frac{\partial^2 \tilde{f}_\infty}{\partial x^i \partial x^j} \]
on \( \mathbb{R}^m \times [-\eta, 0] \). But then \( \| \nabla d\tilde{f}_\infty \|_{\infty} = 0 \) on \( \mathbb{R}^m \times [-\eta, 0] \) contradicts Eq. (5.6), which implies \( \| \nabla^{l-1} d\tilde{f}_{\tau_k}(x_k, 0) \|_{\tau_k}^2 \geq 1 \) for all \( l \geq 3 \) and any \( k \). \( \square \)

### 6. Examples

**Example 6.1** (Hyperbolic space I). We give an example of the mean curvature flow of a strictly length-decreasing map \( f \), where the corresponding graphs \( \Gamma(t) \) diverge to spatial infinity as time tends to infinity.

(i) **Geometric Setup.** Let \( m = 1 \) and let \( N = \mathbb{H}^2 \) be the 2-dimensional hyperbolic space. In the upper half-plane model, \( \mathbb{H}^2 \) is identified with
\[ \mathcal{H} := \{(y^1, y^2) \in \mathbb{R}^2 : y^2 > 0 \} \]
with the metric
\[ g_\mathcal{H} := \frac{1}{(y^2)^2} (dy^1 \otimes dy^1 + dy^2 \otimes dy^2) . \]
The only non-vanishing Christoffel symbols are given by
\[ \Gamma^1_{12} = \Gamma^1_{21} = -\Gamma^2_{11} = \Gamma^2_{22} = -\frac{1}{y^2} . \]
The induced metric of the graph at the point \( (x, f(x)) \) is given by
\[ g = \left( 1 + \frac{(\partial_x f^1(x))^2 + (\partial_x f^2(x))^2}{(f^2(x))^2} \right) dx \otimes dx \]
with inverse
\[
g^{-1} = \frac{(f^2(x))^2}{(f^2(x))^2 + (\partial_x f^1(x))^2 + (\partial_x f^2(x))^2} \frac{\partial}{\partial x} \otimes \frac{\partial}{\partial x}.
\]
Then equation (3.3) reads
\[
\partial_t f^1(x) = g^{-1} \left( \partial_x^2 f^1(x) - 2 \frac{1}{f^2(x)} (\partial_x f^1(x))(\partial_x f^2(x)) \right), \tag{6.1}
\]
\[
\partial_t f^2(x) = g^{-1} \left( \partial_x^2 f^2(x) + \frac{1}{f^2(x)} ((\partial_x f^1(x))^2 - (\partial_x f^2(x))^2) \right). \tag{6.2}
\]
In these coordinates, \( f \) is (strictly) length-decreasing at \( x \in \mathbb{R} \) if
\[
\frac{1}{(f^2(x))^2} ((\partial_x f^1(x))^2 + (\partial_x f^2(x))^2) \leq 1 - \delta
\]
for some \( \delta \in (0, 1] \).

(ii) **Solution to the Mean Curvature Flow.** Let us make the ansatz
\[
f_t(x) = (f^1_t(x), f^2_t(x)) = (x, d(t))
\]
for some function \( d : \mathbb{R} \to \mathbb{R} \) which is to be determined. Since \( \partial_x f^1_t(x) = 1 \) and \( \partial_x f^2_t(x) = 0 \), the length-decreasing condition is equivalent to
\[
\frac{1}{d^2(t)} \leq 1 - \delta \quad \Leftrightarrow \quad d^2(t) \geq \frac{1}{1 - \delta}.
\]
Further, equation (6.1) is identically satisfied, while equation (6.2) evaluates to
\[
\partial_t d(t) = \frac{d(t)}{d^2(t) + 1}.
\]
Integrating this equation, we obtain
\[
t - t_0 = \ln \left[ d(t) \exp \left( \frac{1}{2} d^2(t) \right) \right] = \frac{1}{2} \ln \left[ d^2(t) \exp (d^2(t)) \right].
\]
The (increasing) solution for \( d(t) \) is then given by
\[
d(t) = \sqrt{W(\exp(2(t - t_0)))},
\]
where \( W \) denotes the principal branch of the Lambert \( W \) function. In particular, the lower bound from the length-decreasing condition is preserved, and in the limit \( t \to \infty \), we have \( f^*_tg_H \to 0 \). Using the asymptotic expansion
\[
\sqrt{W(\exp(2(t - t_0)))} \xrightarrow{t \to \infty} \frac{\ln \exp(2(t - t_0))}{2(t - t_0)},
\]
we also see that \( d(t) \to \infty \) as \( t \to \infty \).

The embedding corresponding to the graph of \( f_t \) is given by
\[
F_t(x) = (x, x, d(t)) = \left( x, x, \sqrt{W(\exp(2(t - t_0)))} \right) \in \mathbb{R} \times H.
\]
Since the induced metric on the graph
\[
g(\partial_x, \partial_x) = 1 + \frac{1}{d^2(t)} = 1 + \frac{1}{W(\exp(2(t - t_0)))}
\]
does not depend on \( x \), the Christoffel symbols of the graph vanish and the second fundamental form evaluates to
\[
\Lambda(\partial_x, \partial_x) = \nabla dF(\partial_x) dF(\partial_x) = \frac{1}{d(t)} \frac{\partial}{\partial y^2}.
\]
Consequently, the mean curvature vector is given by
\[ \vec{H} = g^{-1} A(\partial_x, \partial_x) = \frac{d^2(t)}{1 + d^2(t)} \frac{1}{d(t)} \frac{\partial}{\partial y} = \frac{d(t)}{1 + d^2(t)} \frac{\partial}{\partial y}. \]
From the differential equation satisfied by \( d \) we also obtain
\[ \partial_t F_t(x) = \frac{d(t)}{1 + d^2(t)} \frac{\partial}{\partial y}, \]
which shows that \( F_t \) is indeed a solution of the mean curvature flow.

(iii) **Decay Behavior.** Let us calculate the square norm of \( \vec{H} \),
\[ \| \vec{H} \|^2 = \frac{d^2(t)}{(f^2(x))^2} \frac{1}{(1 + d^2(t))^2} = \frac{d^2(t)}{(1 + d^2(t))^2} = \frac{1}{(1 + d^2(t))^2}. \]
In particular, since \( \| \vec{H} \| \approx \frac{1}{2(t - t_0) - \ln(2(t - t_0))} \) for large \( t \), we have
\[ c_1 \leq t \| \vec{H} \| \leq c_2 \]
for some \( 0 < c_1 < c_2 \) and large \( t \). Further, for sufficiently large \( t_1 < t_2 \), we calculate the length of the curve \( \gamma : [t_1, t_2] \to \mathbb{R} \times \mathcal{H} \) given by
\[ \gamma(x, t) := F_t(x) \]
to be
\[ L(\gamma) = \int_{t_1}^{t_2} \| \partial_t F_t(x) \| dt = \int_{t_1}^{t_2} \| \vec{H}(x, t) \| dt \]
\[ \geq \int_{t_1}^{t_2} \frac{c_1}{t} dt = c_1 (\ln t_2 - \ln t_1). \]
This diverges as \( t_2 \to \infty \) and agrees with the previous discussion.

**Example 6.2** (Hyperbolic space II). In this example, we construct an explicit solution to the graphical mean curvature flow of a map \( f : \mathbb{R} \to \mathbb{H}^2 \), where this time we use the Poincaré disk as a model for the hyperbolic space.

(i) **Geometric Setup.** The Poincaré disk model of hyperbolic space is the open unit disk \( D \subset \mathbb{R}^2 \) endowed with the metric
\[ g_D := \frac{4}{(1 - r^2)^2} (dx \otimes dx + dy \otimes dy), \]
where \( r^2 := x^2 + y^2 \). The inverse metric can be written as
\[ g^{-1} = \frac{1 - r^2}{4} (\partial_x \otimes \partial_x + \partial_y \otimes \partial_y) \]
and for the Christoffel symbols, we obtain
\[ \Gamma^1_{11} = \Gamma^2_{12} = \Gamma^2_{21} = -\Gamma^2_{22} = \frac{2x}{1 - r^2}, \]
\[ \Gamma^1_{12} = \Gamma^1_{21} = -\Gamma^1_{11} = \Gamma^1_{22} = \frac{2y}{1 - r^2}. \]
The metric induced on the graph of a function \( f : \mathbb{R} \to D \) is given by
\[ g = \left( 1 + 4 \frac{(\partial_x f^1)^2 + (\partial_x f^2)^2}{(1 - (f^1)^2 - (f^2)^2)^2} \right) dx \otimes dx. \]
The mean curvature flow system (3.3) reads
\[
\partial_t f^1 = g^{-1} \left( \frac{2}{1 - (f^1)^2 - (f^2)^2} \left( f^1 (\partial_x f^1)^2 - f^1 (\partial_x f^2)^2 \right) + 2 f^2 (\partial_x f^1)(\partial_x f^2) \right),
\]
\[
\partial_t f^2 = g^{-1} \left( \frac{2}{1 - (f^1)^2 - (f^2)^2} \left( -f^2 (\partial_x f^1)^2 + f^2 (\partial_x f^2)^2 \right) + 2 f^1 (\partial_x f^1)(\partial_x f^2) \right).
\]

The map \( f \) is strictly length-decreasing precisely if
\[
f^* g_D(v, v) = 4 \frac{\left( \partial_x f^1 \right)^2 + \left( \partial_x f^2 \right)^2}{1 - (f^1)^2 - (f^2)^2} \| v \|^2 \leq (1 - \delta) \| v \|^2
\]
for some \( \delta \in (0, 1] \) and all vector fields \( v \).

(ii) **Solution to the Mean Curvature Flow.** Let us make the ansatz
\[
f_t(x) = (r(t) \sin(x), r(t) \cos(x)).
\]
Then \( f_t \) is strictly length-decreasing precisely if
\[
4 \frac{r^2}{1 - r^2} \leq 1 - \delta \quad \iff \quad r \leq \sqrt{2} - 1 - \delta
\]
for some \( \delta \in (0, \sqrt{2} - 1) \) (depending on \( \delta \)). The induced metric simplifies to
\[
g = \left( 1 + 4 \frac{r^2}{1 - r^2} \right) dx \otimes dx = \left( \frac{1 + r^2}{1 - r^2} \right)^2 dx \otimes dx
\]
and its inverse is given by
\[
g^{-1} = \left( \frac{1 - r^2}{1 + r^2} \right)^2 \partial_x \otimes \partial_x.
\]

The mean curvature flow system reads
\[
\partial_t r(t) = -r(t) \frac{1 - r^2(t)}{1 + r^2(t)}
\]
and the solution for \( r(t) \leq 1 \) to this ordinary differential equation is given by
\[
r(t) = \frac{1}{2} \left( \sqrt{c_1^2 e^{2t} + 4} - c_1 e^t \right), \quad c_1 \geq 0.
\]
We remark that
\[
r(t) \to \frac{1}{\sqrt{c_1^2 e^{2t} + 4} + c_1 e^t} \quad \text{as} \quad t \to \infty \to 0.
\]

(iii) **Decay Behavior.** Since \( g \) only depends on \( r \), which does not depend on \( x \), the Christoffel symbols of the induced metric vanish, and the second fundamental form is given by
\[
A_{xx} = -r(t) \sin(x) \frac{1 + r^2(t)}{1 - r^2(t)} \frac{\partial}{\partial y^1} - r(t) \cos(x) \frac{1 + r^2(t)}{1 - r^2(t)} \frac{\partial}{\partial y^2}.
\]

The mean curvature vector thus is
\[
\vec{H} = g^{-1} A_{xx} = -r(t) \sin(x) \frac{1 - r^2(t)}{1 + r^2(t)} \frac{\partial}{\partial y^1} - r(t) \cos(x) \frac{1 - r^2(t)}{1 + r^2(t)} \frac{\partial}{\partial y^2}.
\]
Since
\[ \partial_t F_t(x) = (\partial_t r(t)) \sin(x) \frac{\partial}{\partial y} + (\partial_t r(t)) \cos(x) \frac{\partial}{\partial y^2}, \]
the differential equation for the mean curvature flow reduces to the ordinary differential equation considered above. Further, since
\[ r(t) \leq \frac{1}{2c_1} e^{-t}, \]
for the square norm of the mean curvature vector we obtain
\[ \|H\|^2 = \frac{4r^2(t)}{(1 + r^2(t))^2} \leq \frac{1}{c_1^2} e^{-2t}. \]

**Example 6.3** (Hyperbolic Space III). We give explicit examples of strictly length-decreasing maps \( f : \mathbb{R} \to \mathbb{H}^2 \) which are stationary points of the mean curvature flow.

(i) In the upper half-plane model \( \mathcal{H} \), consider the family of functions given by
\[ f : \mathbb{R} \times [0, T) \to \mathcal{H}, \quad f(x, t) := (x_0, \exp(cx)), \]
where \( 0 \leq c \leq 1 - \delta \) is fixed. Then for any \( v \in \Gamma(\mathcal{H} \mathbb{R}) \) we have
\[ f^* g_{\mathcal{H}}(v, v) = c^2 \|v\|^2_{\mathbb{R}} \leq (1 - \delta) \|v\|^2_{\mathbb{R}}, \]
so that \( f \) is a (family of) strictly length-decreasing maps. Further, since
\[ \partial_x f^1_t(x) + \frac{1}{f^2_t(x)} \left( (\partial_x f^1_t(x))^2 - (\partial_x f^2_t(x))^2 \right) = c^2 \exp(cx) + \frac{1}{\exp(cx)} (-c^2 (\exp(cx))^2) = 0, \]
which mean that \( f_t(x) \) is stationary under graphical mean curvature flow.

(ii) In the disk model, consider the family of functions given by
\[ f : \mathbb{R} \times [0, T) \to D, \quad f(x, t) := \left( \tanh \left( \frac{c}{2} x \right), 0 \right), \]
where \( 0 \leq c \leq 1 - \delta \) is fixed. Then for any \( v \in \Gamma(\mathcal{H} \mathbb{R}) \) we have
\[ f^* g_D(v, v) = c^2 \|v\|^2_{\mathbb{R}} \leq (1 - \delta) \|v\|^2_{\mathbb{R}}, \]
so that \( f \) is a (family of) strictly length-decreasing maps. Using the ansatz above, the equation for the first component of \( f_t(x) \) that needs to be satisfied for the graphical mean curvature flow is given by
\[ \partial_t f^1_t(x, t) = g^{-1} \partial_x^2 f^1_t(x, t) + g^{-1} \sum_{a,b=1}^2 I_{ab}^1 (\partial_x f^a_t(x, t)) (\partial_x f^b_t(x, t)) \]
\[ = g^{-1} \left( \partial_x^2 f^1_t(x, t) + \frac{2 f^1_t(x, t)}{1 - (f^1_t(x, t))^2} (\partial_x f^1_t(x, t))^2 \right). \]

Since
\[ \partial_x^2 f^1_t(x, t) = -\frac{c^2}{2} \tanh \left( \frac{c}{2} x \right) \left( 1 - \tanh^2 \left( \frac{c}{2} x \right) \right) \]
\[ = -c \tanh \left( \frac{c}{2} x \right) \partial_x f^1_t(x, t) = -c f^1_t(x, t) \partial_x f^1_t(x, t) \]
\[ = -\frac{c^2}{2 (1 - (f^1_t(x, t))^2)} (\partial_x f(x, t))^2, \]
the mean curvature flow equation is identically satisfied.
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