Twisted vertex operators and Bernoulli polynomials

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Abstract

Using general principles in the theory of vertex operator algebras and their twisted modules, we obtain a bosonic, twisted construction of a certain central extension of a Lie algebra of differential operators on the circle, for an arbitrary twisting automorphism. The construction involves the Bernoulli polynomials in a fundamental way. We develop new identities and principles in the theory of vertex operator algebras and their twisted modules, and explain the construction by applying general results, including an identity that we call modified weak associativity, to the Heisenberg vertex operator algebra. This paper gives proofs and further explanations of results announced earlier. It is a generalization to twisted vertex operators of work announced by the second author some time ago, and includes as a special case the proof of the main results of that work.

Keywords: Vertex operator algebras; twisted modules; Bernoulli polynomials

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1 Introduction

1.1 Twisted vertex operators and twisted modules

The theory of vertex operator algebras and their twisted modules is a powerful theory with applications in many areas of mathematics and physics. In mathematics, vertex operators, twisted and untwisted, are involved in a fundamental way in the construction of modules for many infinite-dimensional algebras, including affine Lie algebras and central extensions of algebras of differential operators, for instance, the Virasoro algebra. In particular, twisted vertex operators were discovered and used in \[LV\], where a formal–differential–operator construction of the affine Lie algebra \(A_1^{(1)}\) was found. Moreover, various types of vertex operators, including twisted vertex operators, and relations between them, were used in \[FLM3\] to construct the “moonshine module” for the Monster group, the largest sporadic finite simple group, in such a way that the Monster was realized as the automorphism group of this “algebra of vertex operators.” In physics, vertex operator algebras form the structure at the foundation of conformal field theory \[BPZ\]. What came to be understood as twisted modules for vertex operator algebras are the main building blocks for the so-called orbifold models in conformal field theory (see \[DHVW1, DHVW2, DFMS\] and \[DVVV\]). In particular, the moonshine module \[FLM1, PLM3\] was the first example of an orbifold construction – it appeared before the concept of orbifold compactification was developed in string theory.

The precise notion of vertex operator algebra arose from formalizing the natural algebraic structure underlying (untwisted) vertex operators, which are objects based on a lattice (cf. \[Bo, FLM3\]), at least when the lattice is even. For the basic theory, based on formal calculus and the Jacobi identity, of vertex operator algebras, including lattice vertex operator algebras, the reader can consult \[PLMS, LL\]. Twisted vertex operators, parametrized by the vectors in a space having a structure of vertex operator algebra together with a lattice isometry of finite order, led to the notion of twisted module for a vertex operator algebra. A systematic treatment of certain “twisted formal calculus” underlying various vertex operator constructions associated to an even lattice
was first carried out in [L1]. A more complete study of twisted vertex operators was then carried out in [FLM2], where in particular the formal operator $e^{Ax}$ was introduced as an essential part of the definition and as the main addition compared to the construction of untwisted vertex operators (see also [FLM3], [DL2]). The definition of “twisted module” for vertex operator algebras was introduced in [FFR] (see also [D]); this notion summarizes the key properties that had been discovered for twisted vertex operators, in particular, the twisted Jacobi identity. A large part of the theory of twisted modules was further developed in [Li2]. More recently, a “coordinate-free” approach was developed in [FrS] along the lines of [FrB] (cf. [H2]). It is important to mention here the work [BDM] where, by using the theory of twisted modules, a conceptual explanation of the so-called cyclic orbifold construction in conformal field theory (cf. [BHS]) was given. In addition to [BHS], we would like to mention the program developed in [dBHO], [HO], [GHHO] and [HH] including studies of twisted vertex operators associated with twisted abelian current algebras.

The present paper contributes to the general theory of twisted modules and to the deeper study of twisted vertex operators in important special settings as well; see Section 1.4 below. New general principles in vertex operator algebra theory are introduced and developed, and then are applied to twisted modules and twisted vertex operators associated with lattice vertex operator algebras, to shed light on the relations between certain values of the Riemann zeta function and certain algebras of differential operators on the circle. The results in this paper were announced some time ago in [L4] and [L5] in the untwisted case, and recently in [DLM] in the more general, and more subtle, twisted case. In particular, the present paper is the first giving the full treatment of even the untwisted case of these ideas, methods, results and proofs. It also happens that the new ideas introduced here (and announced in [DLM]) enable us to further elucidate even the earlier untwisted case of [L4] and [L5]. Our main results are the following: theorems on the new and useful concept of “resolving factors” (Theorems 3.4 and 3.5 in the untwisted case and Theorems 3.8, 3.9 and 4.7 in the more general twisted setting; see Remarks 3.2 and 4.2); the new general commutator formula Theorem 6.1; and the main result in the special context incorporating differential operators on the circle, Theorem 7.3.

Throughout this paper, we extensively use the framework and technology of formal calculus (cf. [FLM3]), and in fact it is necessary to use formal calculus both to express and to prove our results. From the point of view of a physicist, this formal-calculus framework for vertex operator algebra theory is revealing itself as the most appropriate way of both rigorously and effectively studying the chiral sector in conformal field theory. Our results and proofs, in both the general and special settings in this paper, in fact include new applications of formal calculus; this serves to emphasize its efficiency as a tool in uncovering surprising structures in conformal field theory and in addition, exploits the intuition that one gets from studying conformal field theory as a physicist. Specifically, for instance, our twisted construction in the special context mentioned above is highly nontrivial, even though it involves only a small part of the general identities that we establish; what we call the “correction” terms (involving Bernoulli
polynomials), crucial ingredients of our constructions (as we explain below), are in general very hard to compute, and our formal-calculus methods provide an extremely efficient way of computing them.

We have tried to be careful to make this paper reasonably self-contained and accessible even to non-experts in vertex operator algebra theory. In the literature, there are a number of useful variants of certain basic definitions and concepts, and because of the subtle nature of the results and proofs presented here, we have found it appropriate to record the precise background and the exact versions of a number of basic notions that we need.

1.2 Twisted vertex operators and Bernoulli polynomials

In this subsection, we shall present some background and motivation for this work and in particular, we shall discuss how new ideas in vertex operator algebra theory enable one to shed light on properties of zeta values and Bernoulli polynomials. Some of the following motivation is extracted from our earlier research announcements [L4], [L5] and [DLM] of the present work. We have decided to recall this motivation here in order to make the present paper more self-contained, since it is the first paper to supply all the proofs of the announced results.

In order to understand the basic ideas behind our work, consider the famous classical “formula”

\[ 1 + 2 + 3 + \cdots = \frac{-1}{12}, \]  

which has the rigorous meaning

\[ \zeta(-1) = -\frac{1}{12}. \]  

Here \( \zeta \) is the Riemann zeta function

\[ \zeta(s) = \sum_{n>0} n^{-s} \]  

(analytically continued), and (1.1) is classically generalized by the formal equality

\[ \sum_{n>0} n^s = \zeta(-s) \]  

for \( s = 0, 1, 2, \ldots \). It is well known that the classical number theory underlying this analytic continuation is related to the issue of regularizing certain infinities in quantum field theory, in particular, in conformal field theory. One of the goals of the present work is to develop some general principles of vertex operator algebra theory that elucidate the passage from the unrigorous but suggestive formula (1.1) to formula (1.2), and the generalization (1.3), as announced in [L4] and [L5]. This passage was noticed by S. Bloch [Bl] to have simplifying effects on certain infinite dimensional Lie algebras of formal differential operators on the circle, and his work was the main motivation behind the ideas initially
announced in [L4] and [L5]. Developing further the theory of twisted vertex operators, as announced in [DLM1], we will also generalize this to analogous formal relations involving the Hurwitz zeta function

\[ \zeta(s, v) = \sum_{n \geq 0} (n + v)^{-s}, \]  

that is, to formal unrigorous relations of the type

\[ \sum_{n \geq 0} (n + v)^{s} = \zeta(-s, v) \]  

for \( v \) certain rational numbers and for \( s = 0, 1, 2, \ldots \). In terms of the Lie algebras studied by Bloch, this turns out to be related to representations of these Lie algebras on certain twisted spaces that we discuss in this paper. We will hence “explain” and considerably generalize this work of Bloch’s, and in the process, discover new structures in the theory of vertex operator algebras. In fact, in [DLM1] new developments were announced that are of interest for this “explanation” of Bloch’s work and more generally in the theory of vertex operator algebras both in the twisted and untwisted settings. In the following we will review this work of Bloch’s and then explain the meaning of our results in this context.

Before reviewing this work, though, let us recall, from [L4], a variant of Euler’s heuristic interpretation of (1.1) and its generalization (1.4). Consider the following formal expansion (in powers of \( x \)), which defines the Bernoulli numbers \( B_k \) for \( k \geq 0 \):

\[ \frac{1}{1 - e^x} = - \sum_{k \geq 0} \frac{B_k}{k!} x^{k-1}. \]  

Expand the left–hand side unrigorously as the formal geometric series

\[ 1 + e^x + e^{2x} + \cdots = 1 + \sum_{k \geq 0} \frac{1^k}{k!} x^k + \sum_{k \geq 0} \frac{2^k}{k!} x^k + \cdots . \]  

For \( k > 1 \), the coefficient of \( x^{k-1} \) in this formal expression is

\[ \frac{1}{(k-1)!} (1^{k-1} + 2^{k-1} + \cdots), \]

which formally resembles \( \frac{1}{(k-1)!} \zeta(-k + 1) \). Also, the coefficient of \( x^0 \) in (1.8) is formally

\[ 1 + \frac{1}{0!} (1^0 + 2^0 + \cdots), \]

which we formally view as \( 1 + \zeta(0) \) (and not as \( \zeta(0) \)). Thus, formally equating the coefficients of \( x^l \) for \( l \geq 0 \) in (1.8) tells us what (1.4) means and “explains” the well–known relation between Bernoulli numbers and the zeta function

\[ \zeta(-k + 1) = -\frac{B_k}{k} \]  

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for $k > 1$, and the fact that

$$
\zeta(0) = -B_1 - 1 \quad \left( = -\frac{1}{2} \right).
$$

(1.12)

Note, incidentally, that the coefficient of $x^{-1}$ in $\zeta(0)$, which is $-B_0 = -1$, does not appear in (1.11) – (1.12); our unrigorous equating of coefficients of powers of $x$ in the two series (1.8) and (1.7) only applies to nonnegative powers.

Similar considerations apply to the formal relation (1.6): Let $v$ be a complex number and consider the formal expansion (again in powers of $x$) which defines the Bernoulli polynomials $B_k(v)$ for $k \geq 0$:

$$
e^{vx} + e^{(v+1)x} + e^{(v+2)x} + \cdots = \sum_{k \geq 0} \frac{v^k}{k!} x^k + \sum_{k \geq 0} \frac{(v + 1)^k}{k!} x^k + \sum_{k \geq 0} \frac{(v + 2)^k}{k!} x^k + \cdots.
$$

(1.13)

Now, for $k \geq 1$, the coefficient of $x^{k-1}$ in this formal expression is

$$
\frac{1}{(k-1)!} (v^{k-1} + (v+1)^{k-1} + \cdots)
$$

(1.15)

which agrees with (1.11) when $k = 1$ and $v = 0$, and this is formally $\frac{1}{(k-1)!} \zeta(-k+1, v)$ (so that $\zeta(0, 0) = 1 + \zeta(0)$, for example). Thus, formally equating the coefficients of $x^l$ for $l \geq 0$ in (1.13) gives an interpretation of (1.6) and “explains” the well-known relation between Bernoulli numbers and the Hurwitz zeta function

$$
\zeta(-k+1, v) = -\frac{B_k(v)}{k}
$$

(1.17)

for $k \geq 1$. Note that setting $v = 1$ in (1.16) gives the Riemann zeta function:

$$
\zeta(s, 1) = \zeta(s).
$$

(1.18)

Comparing (1.11) with (1.17) at $v = 1$ then gives the well-known relations

$$
B_k(1) = B_k \quad (= B_k(0))
$$

(1.19)

for $k > 1$ and

$$
B_1(1) = B_1 + 1 \quad (= B_1(0) + 1).
$$

(1.20)

Note that as above, considering the pole in (1.13) gives $B_0(v) = 1$ as a polynomial, but this pole does not arise in (1.15) – (1.20).
The key point in these formal unrigorous arguments is the interplay between the formal geometric series expansion (in powers of $e^x$) and the expansion in powers of $x$.

Bloch [Bl] found very interesting phenomena relating the values $\zeta(-n)$, $n = 1, 3, 5, \ldots$, of the zeta function at negative odd integers to the commutators of certain formal differential operators on the circle, and used (1.4) in order to interpret them. We now sketch a part of his work and interpretation. Consider the Lie algebra

$$\mathfrak{d} = \text{Der} \; \mathbb{C}[t, t^{-1}]$$

(1.21)
of formal vector fields on the circle, with basis $\{ t^n D | n \in \mathbb{Z} \}$, where

$$D = D_t = t \frac{d}{dt}.$$  

(1.22)

and recall the Virasoro algebra $\mathfrak{v}$, the central extension

$$0 \to \mathbb{C}c \to \mathfrak{v} \to \mathfrak{d} \to 0,$$

(1.23)

where $\mathfrak{v}$ has basis $\{ L(n) | n \in \mathbb{Z} \}$ together with a central element $c$; the bracket relations are given by

$$[L(m), L(n)] = (m - n)L(m + n) + \frac{1}{12} (m^3 - m)\delta_{m+n,0}c$$

(1.24)

and $L(n)$ maps to $-nD$ in (1.23). This Lie algebra has the following well-known realization: Start with the Heisenberg Lie algebra with basis consisting of the symbols $h(n)$ for $n \in \mathbb{Z}$, $n \neq 0$ and a central element $1$, with the bracket relations

$$[h(m), h(n)] = m \delta_{m+n,0}1.$$  

(1.25)

For convenience we adjoin an additional central basis element $h(0)$, so that the relations (1.25) hold for all $m, n \in \mathbb{Z}$. This Lie algebra acts irreducibly on the polynomial algebra

$$S = \mathbb{C}[h(-1), h(-2), h(-3), \ldots]$$

(1.26)
as follows: For $n < 0$, $h(n)$ acts by multiplication; for $n > 0$, $h(n)$ acts as $n \frac{\partial}{\partial h(-n)}$; $h(0)$ acts as $0$; and $1$ acts as the identity operator. Then $\mathfrak{v}$ acts on $S$ via:

$$L(n) \mapsto \frac{1}{2} : \sum_{j \in \mathbb{Z}} h(j)h(n - j) :$$

(1.27)

for all $n \in \mathbb{Z}$. Here the operators (1.27) are in “normal-ordered form,” denoted by the colons, that is, the $h(n)$ for $n > 0$ act to the right of the $h(n)$ for $n < 0$. It is well known that the operators (1.27) indeed satisfy the bracket relations (1.24). (This exercise and the related constructions are presented in [FLM2], for example, where the standard generalization of this construction of $\mathfrak{v}$ using a Heisenberg algebra based on a finite-dimensional space of operators $h(n)$ for each $n$ is also carried out.)
As is well known, removing the normal ordering in the definition of $L(0)$ introduces an infinity which formally equals $\frac{1}{2}\zeta(-1)$, since the unrigorous expression
\[
\bar{L}(0) = \frac{1}{2} \sum_{j \in \mathbb{Z}} h(-j)h(j)
\] (1.28)
formally equals (by (1.25))
\[
L(0) + \frac{1}{2}(1 + 2 + 3 + \cdots),
\] (1.29)
which itself formally equals, according to our considerations above,
\[
L(0) + \frac{1}{2}\zeta(-1) = L(0) - \frac{1}{24}. (1.30)
\]
Rigorizing $\bar{L}(0)$ by defining it as
\[
\bar{L}(0) = L(0) + \frac{1}{2}\zeta(-1), (1.31)
\]
we set
\[
\bar{L}(n) = L(n) \text{ for } n \neq 0, (1.32)
\]
to get a new basis of $\mathfrak{v}$. (We are identifying the elements of $\mathfrak{v}$ with operators on $\mathcal{S}$.) The brackets become:
\[
[\bar{L}(m), \bar{L}(n)] = (m - n)\bar{L}(m + n) + \frac{1}{12}m^3 \delta_{m+n,0}; (1.33)
\]
that is, $m^3 - m$ in (1.24) has become the pure monomial $m^3$.

In [Bl], Bloch considered the larger Lie algebra of formal differential operators, spanned by
\[
\{t^n D^m | n \in \mathbb{Z}, m \geq 0\} (1.34)
\]
or more precisely, we restrict to $m > 0$ and further, to the Lie subalgebra $\mathcal{D}^+$, containing $\mathfrak{d}$, spanned by the differential operators of the form $D^r(t^n D)^r D^r$ for $r \geq 0$, $n \in \mathbb{Z}$. Then we can construct a central extension of $\mathcal{D}^+$ using generalizations of the operators (1.27):
\[
L^{(r)}(n) = \frac{1}{2} \sum_{j \in \mathbb{Z}} j^r h(j)(n-j)^r h(n-j); (1.35)
\]
for $n \in \mathbb{Z}$. These operators provide [Bl] a central extension of $\mathcal{D}^+$ such that
\[
L^{(r)}(n) \mapsto (-1)^{r+1} D^r(t^n D)^r D^r. (1.36)
\]
A central point of [Bl] is that the formal removal of the normal-ordering procedure in the definition (1.35) of $L^{(r)}(0)$ adds the infinity "sum $\sum_{n>0} n^{2r+1}$" $= (-1)^r \frac{1}{2}\zeta(-2r - 1)$ (generalizing (1.28)–(1.30)), and if we correspondingly define
\[
\bar{L}^{(r)}(0) = L^{(r)}(0) + (-1)^r \frac{1}{2}\zeta(-2r - 1) (1.37)
\]
and $L^{(r)}(n) = L^{(r)}(n)$ for $n \neq 0$ (generalizing (1.31) and (1.32)), the commutators simplify in a remarkable way: The complicated polynomial in the scalar term of $[L^{(r)}(m), L^{(s)}(-m)]$ reduces to a pure monomial in $m$, by analogy with, and generalizing, the passage from $m^3 - m$ to $m^3$ in (1.33); see [Bl] (and below) for the formulas and further results.

We now recast these results in the more natural language of formal calculus. This will make more apparent the generalization and interpretation stemming from the theory of vertex operator algebras.

Using a formal variable $x$, we form the generating functions

$$h(x) = \sum_{n \in \mathbb{Z}} h(n)x^{-n}$$

and

$$L^{(r)}(x) = \sum_{n \in \mathbb{Z}} L^{(r)}(n)x^{-n},$$

and using $D_x$ to denote the operator $x \frac{d}{dx}$, we observe that

$$L^{(r)}(x) = \frac{1}{2} : (D^r_x h(x))^2 :$$

where the colons, as always, denote normal ordering.

Then we introduce suitable generating functions over the number of derivatives, and we use the formal multiplicative analogue

$$e^{yD_x} f(x) = f(e^y x)$$

of the formal Taylor theorem

$$e^{y} f(x) = f(x + y),$$

where $f(x)$ is an arbitrary formal series of the form $\sum_n a_n x^n$; $n$ is allowed to range over something very general, like $\mathbb{Z}$ or even $\mathbb{C}$, say; and the $a_n$ lie in a fixed vector space (cf. [FLM2], Proposition 8.3.1). Although $:(D^r_x h(x))^2 :$ (recall (1.40)) is hard to put into a “good” generating function over $r$, we make the problem easier by making it more general: Consider independently many derivatives on each of the two factors $h(x)$ in $: h(x)^2 :$, use two new independent formal variables $y_1$ and $y_2$, and form the generating function

$$L^{(y_1,y_2)}(x) = \frac{1}{2} : (e^{y_1 D_x} h(x))(e^{y_2 D_x} h(x)) : = \frac{1}{2} : h(e^{y_1} x) h(e^{y_2} x) :$$

(where we use (1.41)), so that $L^{(r)}(x)$ is a “diagonal piece” of this generating function. Using formal vertex operator calculus techniques, we can calculate

$$[ : h(e^{y_1} x_1) h(e^{y_2} x_2) : : h(e^{y_3} x_1) h(e^{y_4} x_2) :].$$

Now, formally removing the normal ordering gives the formal expression $h(e^{y_1} x) h(e^{y_2} x)$, which is not rigorous, as we see by (for example) trying to compute the constant term in the variables $y_1$ and $y_2$ in this expression; the failure
of this expression to be defined in fact corresponds exactly to the occurrence of formal sums like $\sum_{n>0} n^r$ with $r > 0$, as we have been discussing. However, we have

$$h(x_1)h(x_2) = \mathcal{O}_x h(x_1)h(x_2) + x_2 \frac{\partial}{\partial x_2} \frac{1}{1 - x_2/x_1} \quad (1.45)$$

and it follows that

$$h(e^{y_1}x_1)h(e^{y_2}x_2) = \mathcal{O}_x h(e^{y_1}x_1)h(e^{y_2}x_2) + x_2 \frac{\partial}{\partial x_2} \frac{1}{1 - e^{y_2}x_2/e^{y_1}x_1} \quad (1.46)$$

note that $x_2 \frac{\partial}{\partial x_2}$ can be replaced by $-\frac{\partial}{\partial y_1}$ in the last expression. The expression $1 - e^{y_1}x_1/e^{y_2}x_1$ came from, and is, a geometric series expansion (recall \[.45\]).

If we try to set $x_1 = x_2 (= x)$ in \[.45\], the result is unrigorous on the left–hand side, but the result has rigorous meaning on the right–hand side, because the normal-ordered product $\mathcal{O}_x h(e^{y_1}x)h(e^{y_2}x)$ is certainly well defined, and the expression $-\frac{\partial}{\partial y_1} \frac{1}{1 - e^{-y_1+y_2}}$ can be interpreted rigorously as in \[.41\]; more precisely, we take $\frac{1}{1 - e^{-y_1+y_2}}$ to mean the formal (Laurent) series in $y_1$ and $y_2$ of the shape

$$\frac{1}{1 - e^{-y_1+y_2}} = (y_1 - y_2)^{-1}F(y_1, y_2), \quad (1.47)$$

where $(y_1 - y_2)^{-1}$ is understood as the expansion $y_1^{-1} \sum_{j \geq 0} y_2^j y_1^{-j}$ in nonnegative powers of $y_2$ and negative powers of $y_1$, and $F(y_1, y_2)$ is an (obvious) formal power series in (nonnegative powers of) $y_1$ and $y_2$. This motivates us to define a new “normal-ordering” procedure

$$\mathcal{O}_x h(e^{y_1}x)h(e^{y_2}x) = \mathcal{O}_x h(e^{y_1}x)h(e^{y_2}x) + \mathcal{O}_x \frac{\partial}{\partial y_1} \frac{1}{1 - e^{-y_1+y_2}}, \quad (1.48)$$

with the last part of the right–hand side being understood as we just indicated. This gives us a natural “explanation” of the zeta–function–modified operators defined in \[.43\]: We use \[.48\] to define the following analogues of the operators \[.43\]:

$$\mathcal{O}_x \mathcal{L}(y_1,y_2)(x) = \mathcal{O}_x h(e^{y_1}x)h(e^{y_2}x) \quad (1.49)$$

and the operator $\mathcal{L}^{(r)}(n)$ (with $r \geq 0$, $n \in \mathbb{Z}$) is exactly $(r!)^2$ times the coefficient of $y_1^j y_2^j x_0^n$ in \[.49\]; the significant case is the case $n = 0$.

**Remark 1.1.** This “rigorization” of the undefined formal expression $h(e^{y_1}x)h(e^{y_2}x)$ corresponds exactly to Euler’s heuristic interpretation of \[.41\] discussed above (as in \[.41\]). Note that the formal series on the right-hand side of \[.45\] involves terms with negative powers of $y_1$, and therefore, so does the expression \[.49\]. However, these terms singular in $y_1$ are not involved in the definition of the operators $\mathcal{L}^{(r)}(n)$, which, as we just mentioned, are related to the nonnegative powers of $y_1$ in $\mathcal{L}(y_1,y_2)(x)$. This omission of the singularities in $y_1$ corresponds to the fact that the Bernoulli number $B_0$ did not appear in our unrigorous considerations above. In calculating commutators of normal-ordered
bilinear expressions \((1.48)\), one obtains, as in Proposition \((7.2)\) below, for instance, similar normal-ordered bilinear expressions, in such a way that these singularities exactly cancel out, as they must. This cancellation process (see proof of Proposition \((7.2)\)) is quite subtle. Actually, Proposition \((7.2)\) deals with the general twisted case, and as such involves Bernoulli polynomials and is related to our discussion of the Hurwitz zeta function above (recall \((1.13) - (1.17)\)), not just to our discussion of the Riemann zeta function.

With the new normal ordering \((1.48)\) replacing the old one, remarkable cancellation occurs in the commutator \((1.44)\). This gives rise to the simple form of the central term in the bracket relations involving the new basis elements \(\bar{\mathcal{L}}(r)(n)\), noted by Bloch.

The removal of the normal-ordering operation, interpreted rigorously as above by means of the zeta regularization, has a simple but considerable generalization in the context of the theory of vertex operator algebras. It is recalled below (see \((5.1)\)) that the normal-ordering operation in vertex operator algebra theory has the form

\[
\mathcal{Y}(u, x^2 + x_0)\mathcal{Y}(v, x^2) = \text{regular part in } x_0 \text{ of } \mathcal{Y}(\mathcal{Y}(u, x_0) v, x^2).
\]

(1.50)

Here \(u\) lies in a vector space, say \(V\), that forms a vertex operator algebra, and \(\mathcal{Y}(\cdot, x)\) is the vertex operator map (see our short review in Section 3). In fact, formal series like \((1.38)\) are examples of what we call homogeneous vertex operators \(X(u, x)\) (see \((4.2)\)), for which the normal-ordering operation takes the form

\[
\mathcal{X}(u, e^y x^2)\mathcal{X}(v, x^2) = \text{regular part in } x_0 \text{ of } \mathcal{X}(\mathcal{Y}(u, y) v, x^2).
\]

(1.51)

where on both sides one replaces the formal variable \(y\) by the formal series \(\log(1 + \frac{x_0}{x^2})\), and where the map \(\mathcal{Y}[:, y]\) (see \([Z1], [Z2];\) see \((4.4)\) below) defines on \(V\) a vertex operator algebra (in “cylindrical coordinates”) isomorphic to that defined by the map \(\mathcal{Y}(\cdot, x)\). Now, the generalization of the zeta-regularized removal of the normal-ordering, as in \((1.48)\), to arbitrary vertex operators takes the extremely simple form (see \((5.8)\))

\[
\mathcal{X}(u, e^y x^2)\mathcal{X}(v, x^2) = \mathcal{X}(\mathcal{Y}(u, y) v, x^2),
\]

(1.52)

where this time, \(y\) does not need to be replaced by any formal series and no regular part needs to be extracted. This simple but extremely general operation is the generalization to vertex operator algebras of the heuristic argument of Euler justifying formal unrigorous relations like \((1.1)\) and \((1.4)\). Note that the expression \((1.52)\) generically contains singularities in \(y\), as we have remarked above in our simple example, and as we have also remarked, this phenomenon corresponds to the presence of the pole (in \(x\)) in \((1.7)\) and in \((1.13)\).

In this paper we introduce and develop an extension to twisted vertex operators of the operation \((1.52)\) (see \((5.1)\)), an extension that generalizes the heuristic argument leading to \((1.6)\). Also, we establish a new relation in the
theory of vertex operator algebras and twisted modules, which we call “mod-
ified weak associativity.” (See our announcement [DLM1].) This new relation
essentially expresses (1.52) as a formal limit applied to an ordinary product,
without any normal ordering, of homogeneous vertex operators multiplied by
an appropriate “resolving factor” (see Theorems 3.5, 3.9 and 4.7 below). This
relation gives a simple way of calculating brackets of \( \hat{\cdot} : \hat{\cdot} : \) normal-ordered bilin-
ear expressions like those in (1.52), (see Theorem 6.1), which we use in turn to
elucidate the simplification of the central term observed by Bloch (see Theorem
7.3).

Now that we have put our results in context, let us outline a few important
and interesting consequences and potential applications.

Our general commutator formula (Theorem 6.1) can be used to obtain non-
trivial identities involving Bernoulli polynomials and Bernoulli numbers. For
instance, formula (3.12) in the proof of the commutator formula Proposition 3.4
in [Bl], a formula relating the coefficients of powers of \( m \) in \( \sum_{k=1}^{m-1} (m-k)^n k^n \)
to Bernoulli numbers, becomes a consequence of our commutator formula, after
we plug in appropriate vectors and equate coefficients of certain powers on both
sides of the formula. This classical fact is just a sample; we anticipate interesting
results on properties of values of Bernoulli polynomials by comparing different
twistings in different realizations of Lie algebras of differential operators on the
circle.

It is interesting that our work also sheds some light on the representation
theory of Lie algebras of differential operators on the circle. It is known that
irreducible highest weight modules of Lie algebras of differential operators on
the circle are classified in terms of values of certain Bernoulli polynomials (see
for instance [KR]). Our Corollary 7.4 (a consequence of our construction) il-
lustrates how bosonic twisted constructions of untwisted modules offer a new
understanding of this phenomenon.

In addition, [L4] and [L5] were in retrospect a starting point for a study
of certain multi-point trace functions carried out by the third author in [L3].
These multi-point trace functions have several interesting properties (such as
quasi-modularity) and have been computed in a special case by Bloch and Okounkov
[BO]. A fascinating fact is that these functions also yield stationary Gromov-Witten invariants of an elliptic curve (after Okounkov and Pand-
haripande [OP]). Thus it is of great interest to understand these multi-point
trace functions in the general, twisted setting. Our present work, together with
[DLM], serves as a natural framework for such a study.

1.3 The Virasoro algebra and cylindrical coordinates

In our work two notions will be of particular importance: the notion of twisted
module for a vertex operator algebra, and, in a less predominant but still impor-
tant way, the algebra isomorphism (Z1, Z2) corresponding geometrically
to a change to cylindrical coordinates. The second notion appeared briefly (but
crucially) above in (1.51) and (1.52), through the vertex operator map denoted
\( Y[\cdot, x] \). In order to motivate this appearance, let us briefly explain how the
change to cylindrical coordinates gives a well-known simple geometrical interpretation of the passage from the usual bracket relations (1.24) for the Virasoro algebra to (1.24), discussed above in terms of the zeta regularization.

Consider the Virasoro algebra \( \mathfrak{v} \) (1.23) with basis \( \{ L(n) | n \in \mathbb{Z}, c \} \), \( c \) central and with the bracket relations (1.24). Consider

\[
T(z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2},
\]

where \( z \) is a complex variable and the generators \( L(n) \) act on a certain module for the Virasoro algebra. Let us assume that the formula (1.53) makes sense for \( z \neq 0 \) (this can be made rigorous if we use matrix coefficients [FLM2]). We are interested in how \( T(z) \) (the “stress–energy tensor”) transforms under a change of coordinates \( z \mapsto z' = f(z) \), where \( f(z) \) is a holomorphic function. That is, we would like to compute \( T'(z') \) as an expansion in \( z \) with coefficients expressed in terms of the generators \( L(n), n \in \mathbb{Z} \). This can be achieved by using the well–known transformation formula for \( T(z) \) (cf. [4]):

\[
\left( \frac{dz'}{dz} \right)^2 T'(z') = T(z) - \frac{c}{12} \{ z', z \}
\]

with

\[
\{ z', z \} = \left( \frac{f'''(z)}{f'(z)} - \frac{3}{2} \left( \frac{f''(z)}{f'(z)} \right)^2 \right).
\]

Now, consider an infinite cylinder constructed by identifying \( z' \) and \( z' + 2k\pi i \) for every \( z' \in \mathbb{C} \) and \( k \in \mathbb{Z} \). Then the map

\[
z \mapsto z' = \ln(z)
\]

defines a holomorphic map from the punctured plane \( \mathbb{C} \setminus \{ 0 \} \) onto the cylinder. The transformation of the stress–energy tensor under \( z' = \ln(z) \) is given by

\[
T'(z') = z^2 T(z) - \frac{c}{24},
\]

and if we write \( T'(z') = \sum_{n \in \mathbb{Z}} \tilde{L}(n) z^{-n} \) we have

\[
\tilde{L}(n) = L(n) - \frac{c}{24} \delta_{n,0}.
\]

Notice how the modes \( \tilde{L}(n) \) are defined: there is no shift of \(-2\) in the powers of \( z \). This is closely related to the concept of homogeneous vertex operator alluded to above, which will play an important role in our twisted construction. The new modes \( \tilde{L}(n) \) do not satisfy the standard bracket relations (1.24) of the Virasoro algebra; instead, the central term in the commutator is a pure monomial:

\[
[\tilde{L}(m), \tilde{L}(n)] = (m - n) \tilde{L}(m + n) + \frac{m^3}{12} \delta_{m+n,0} c.
\]
This well-known fact is the same phenomenon as the one described above in the context of Bloch’s results; see (1.33). This cylindrical coordinate transformation brings important modular invariance properties to objects constructed out of the new modes $\hat{L}(n)$; in particular, the graded dimension $\text{Tr}_{M(1)} \left( q^{\hat{L}(0)} \right)$, closely related to the partition function in statistical systems, is $1/\eta(q)$ where $\eta(q)$ is Dedekind’s eta–function, which has important modular properties when viewed as a function of the variable $\tau$ with $q = e^{2\pi i \tau}$. The transformation to cylindrical coordinates will play an important role in our considerations. From the vertex operator algebra point of view, this transformation is an isomorphism between two vertex operator algebras. This isomorphism is well understood [Z1], [Z2], [H1], [H2], and in [Z1] and [Z2] is used in the course of explaining modular invariance phenomena. In this paper we will obtain additional results that will put the previous description of its effects on the Virasoro algebra into the general framework of vertex operator algebra theory.

1.4 The present work

As we have been saying, work of Bloch’s [Bl] revealed a connection between values of the (analytically continued) Riemann zeta function at the negative integers and a central extension $\hat{D}^+$ of a certain classical Lie algebra of differential operators on the circle, a subalgebra of the central extension of the Lie algebra of formal differential operators on the circle of all nonnegative orders (denoted $W_{1+\infty}$ in the physics literature). In addition to the work [L4], [L5] discussed above (and in this paper), in [M1]–[M3], the general theory of vertex operator algebras was used in order to extend all these results to the context of central extensions of classical Lie superalgebras of differential operators on the circle, in connection with values of the Hurwitz zeta functions [M2].

This paper is a continuation of these works and also an extension in several directions, giving proofs and further explanation of results announced in [L4], [L5] and, for the generalization to the twisted setting, [DLM4]. The present goal is, first, to develop new concepts and identities in the general theory of vertex operator algebras and their twisted modules (this is carried out in Sections 3, 4 and 5), in particular to obtain a new general Jacobi identity for twisted operators (Theorem 4.5) and a general commutator formula for related iterates of such operators (Theorem 6.1). These new identities will then be specialized to the Heisenberg vertex operator algebra (cf. [LL]) and used to obtain proofs of results announced in [L4], [L5] and, mainly, to obtain a twisted construction, announced in [DLM4], of the algebra $\hat{D}^+$ studied in [Bl] (cf. Proposition 7.2 and Theorem 7.3), combining and extending methods from [L4], [L5], [M1]–[M3], [FLM2], [FLM3] and [DL2].

In these earlier papers, we used vertex operator techniques to analyze untwisted actions of the Lie algebra $\hat{D}^+$ on a module for a Heisenberg Lie algebra of a certain standard type, based on a finite-dimensional vector space equipped with a nondegenerate symmetric bilinear form. Now consider an arbitrary isometry $\nu$ of period say $p$, that is, with $\nu^p = 1$. We prove that the corresponding
\( \nu \)-twisted modules carry an action of the Lie algebra \( \hat{D}^+ \) in terms of twisted vertex operators, parametrized by certain quadratic vectors in the untwisted module. In particular, we extend a result from [FLM2], [FLM3], [DL2] where actions of the Virasoro algebra were constructed using twisted vertex operators. In addition, we explicitly compute certain “correction” terms for the generators of the “Cartan subalgebra” of \( \hat{D}^+ \) that naturally appear in any twisted construction. These correction terms are expressed in terms of special values of certain Bernoulli polynomials. They can in principle be generated, in the theory of vertex operator algebras, by the formal operator \( e^{\Delta x} \) involved in the construction of a twisted action for a certain type of vertex operator algebra. We generate these correction terms in an easier way, using a new \textit{modified weak associativity} (see Theorem 4.7) that is a consequence of the twisted Jacobi identity.

In [KR] Kac and Radul established a relationship between the Lie algebra of differential operators on the circle and the Lie algebra \( \hat{gl}(\infty) \). We believe that their work can be modified for purposes of classification of quasi–finite highest weight representations of \( \hat{D}^+ \) as well (for related constructions, generalizations and a relationship with dual pairs see [AFOQ], [AFMO], [FKRW], [KWY]). Our new methods and motivation for studying Lie algebras of differential operators, based on vertex operator algebras, are of a different scope, so we do not pursue their direction (however, see Corollary 7.4).

Various research directions are suggested by the present work. These include understanding multi–point correlation functions [M3] [BO] (certain graded \( q \)-series) built from the twisted vertex operators considered in this paper, as well as investigating relationships between \( W \)-algebras (in the sense of [FKRW]) and the present work. Also, the relation between the present work and [BDM] is interesting. We shall investigate these directions in future publications.

2 The Lie algebra \( \hat{D}^+ \) and its untwisted construction

Let \( D \) be the Lie algebra of formal differential operators on \( \mathbb{C}^\times \) spanned by \( t^n D^r \), where \( D = t \frac{d}{dt} \) and \( n \in \mathbb{Z}, r \in \mathbb{N} \) (the nonnegative integers). This Lie algebra has an essentially unique one-dimensional central extension \( \hat{D} = \mathbb{C}c \oplus D \) (denoted in the physics literature by \( \mathcal{W}_{1+\infty} \)).

The representation theory of the highest weight modules of \( \hat{D} \) was initiated in [KR], where, among other things, the complete classification problem of the so-called \textit{quasi-finite} representations\(^1\) was settled. The detailed study of the representation theory of certain subalgebras of \( D \) having properties related to those of certain infinite–rank “classical” Lie algebras was initiated in [KWY] along the lines of [KR]. In [B] and [M2], related Lie algebras (and superalgebras) are considered from different viewpoints. We will follow these lines and concentrate on the Lie subalgebra \( \hat{D}^+ \) described in [B] and recalled below.

\(^1\)These are representations with finite-dimensional homogeneous subspaces.
View the elements $t^n D^r$ ($n \in \mathbb{Z}$, $r \in \mathbb{N}$) as generators of the central extension $\mathcal{D}$. They can be taken to satisfy the following commutation relations (cf. [KR]):

\[
[t^m f(D), t^n g(D)] = t^{m+n}(f(D+n)g(D) - g(D+m)f(D)) + \Psi(t^m f(D), t^n g(D))c,
\]

where $f$ and $g$ are polynomials and $\Psi$ is the 2–cocycle (cf. [KR]) determined by

\[
\Psi(t^m f(D), t^n g(D)) = -\Psi(t^n g(D), t^m f(D)) = \delta_{m+n,0} \sum_{i=1}^{m} f(-i)g(m-i), \quad m > 0.
\]

We consider the Lie subalgebra $\mathcal{D}^+$ of $\mathcal{D}$ generated by the formal differential operators

\[
L_n^{(r)} = (-1)^{r+1} D^r (t^n D) D^r,
\]

where $n \in \mathbb{Z}$, $r \in \mathbb{N}$ [BI]. The subalgebra $\mathcal{D}^+$ has an essentially unique central extension (cf. [N]) and this extension may be obtained by restriction of the 2–cocycle $\Psi$ to $\mathcal{D}^+$. Let $\mathcal{D}^+ = \mathbb{C}c \oplus \mathcal{D}^+$ be the nontrivial central extension defined via the slightly normalized 2–cocycle $-\frac{1}{2} \Psi$, and view the elements $L_n^{(r)}$ as elements of $\bar{\mathcal{D}}^+$. This normalization gives, in particular, the usual Virasoro algebra bracket relations

\[
[L_m^{(0)}, L_n^{(0)}] = (m-n)L_m^{(0)} + \frac{m^3 - m}{12} \delta_{m+n,0} c. \quad (2.3)
\]

In [BI] Bloch discovered that the Lie algebra $\bar{\mathcal{D}}^+$ can be defined in terms of generators that lead to a simplification of the central term in the Lie bracket relations. Consider generators $L_n^{(r)}$ ($n \in \mathbb{Z}$, $r \in \mathbb{N}$) for the algebra $\bar{\mathcal{D}}^+$, defined by

\[
\bar{L}_n^{(r)} = L_n^{(r)} + \frac{(-1)^{r+1}}{2} c. \quad (2.4)
\]

Bloch found the following commutation relations for these generators:

\[
[\bar{L}_m^{(r)}, \bar{L}_n^{(s)}] = \sum_{i=\min(r,s)}^{r+s} a_i^{(r,s)} (m,n) \bar{L}_i^{(i)} + \frac{(r+s+1)^2}{2(2r+s+3)!} m^{2(r+s)+3} \delta_{m+n,0} c. \quad (2.5)
\]

The structure constants $a_i^{(r,s)} (m,n)$ are consequences of the bracket relation (2.4) and can be defined as follows: Consider the following symmetric polynomial in two variables $x_1, x_2$, with parameters $n \in \mathbb{Z}$, $r, s \in \mathbb{N}$:

\[
f^{(r,s)}(n; x_1, x_2) = (-1)^{s+1} \left( (x_2 + n)^{r+s+1} x_1^{r+s+1} + (x_1 + n)^{r+s+1} x_1^{r+s+1} x_1 x_2 \right).
\]

In general, any symmetric polynomial in two variables $x_1, x_2$ can be written in a unique way as a polynomial in $x_1 + x_2$ and $x_1 x_2$. The polynomial above is obviously symmetric, so can be written in the following form:

\[
f^{(r,s)}(n; x_1, x_2) = \sum_{i=\min(r,s)}^{r+s} f_i^{(r,s)} (n; x_1 + x_2)(x_1 x_2)^i
\]
where \( f^{(r,s)}(n,x) \) are homogeneous polynomials in \( n \) and \( x \) of total degree \( 2(r + s - i) + 1 \). Then the coefficients \( a^{(r,s)}(m,n) \) are defined by

\[
a^{(r,s)}(m,n) = f^{(r,s)}(n; -m - n).
\]

Notice that, oddly enough, the central term in the commutator is a pure monomial in \( m \), in contrast to the central term in \( (2.3) \) and in other bracket relations that can be found from \( (2.1) \). As was announced in \([L4], [L5]\) and reviewed in the Introduction above, in order to conceptualize this simplification (especially the appearance of zeta-values) one can construct certain infinite-dimensional projective representations of \( \mathcal{D}^+ \) using vertex operators. But first, let us explain Bloch's construction and heuristic conceptualization \([Bl]\).

As in \([FLM3]\), consider the (infinite-dimensional) Lie algebra \( \hat{\mathfrak{h}} \), the affinization of an abelian Lie algebra \( \mathfrak{h} \) of dimension \( d \) (over \( \mathbb{C} \)) with nondegenerate symmetric bilinear form \( \langle \cdot, \cdot \rangle \):

\[
\hat{\mathfrak{h}} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{h} \otimes t^n \oplus \mathbb{C},
\]

with the commutation relations

\[
[\alpha \otimes t^m, \beta \otimes t^n] = \langle \alpha, \beta \rangle m \delta_{m+n,0} \mathbb{C} \quad (\alpha, \beta \in \mathfrak{h}, \ m, n \in \mathbb{Z})
\]

\[
[C, \hat{\mathfrak{h}}] = 0.
\]

Set

\[
\hat{\mathfrak{h}}^+ = \bigoplus_{n > 0} \mathfrak{h} \otimes t^n, \quad \hat{\mathfrak{h}}^- = \bigoplus_{n < 0} \mathfrak{h} \otimes t^n.
\]

The subalgebra

\[
\hat{\mathfrak{h}}^+ \oplus \hat{\mathfrak{h}}^- \oplus \mathbb{C}
\]

is a Heisenberg Lie algebra. Form the induced (level–one) \( \hat{\mathfrak{h}} \)–module

\[
S = \mathcal{U}(\hat{\mathfrak{h}}) \otimes_{\mathcal{U}(\hat{\mathfrak{h}}^+ \oplus \hat{\mathfrak{h}}^- \oplus \mathbb{C})} \mathbb{C} \simeq S(\hat{\mathfrak{h}}^-) \quad \text{(linearly)},
\]

where \( \hat{\mathfrak{h}}^+ \oplus \mathfrak{h} \) acts trivially on \( \mathbb{C} \) and \( C \) acts as 1; \( \mathcal{U}(\cdot) \) denotes universal enveloping algebra and \( S(\cdot) \) denotes the symmetric algebra. Then \( S \) is irreducible under the Heisenberg algebra \( \hat{\mathfrak{h}}^+ \oplus \hat{\mathfrak{h}}^- \oplus \mathbb{C} \). We will use the notation \( \alpha(n) \) \( (\alpha \in \mathfrak{h}, \ n \in \mathbb{Z}) \) for the action of \( \alpha \otimes t^n \in \mathfrak{h} \) on \( S \). Then the correspondence

\[
L^{(r)}_n \mapsto \frac{1}{2} \sum_{q=1}^d \sum_{j \in \mathbb{Z}} \langle n - j \rangle^r :\alpha_q(j)\alpha_q(n - j) : \quad (n \in \mathbb{Z}, \ c \mapsto d), \quad (2.6)
\]

where \( \{\alpha_q\} \) is an orthonormal basis of \( \mathfrak{h} \), and where \( : \cdot : \) is the usual normal ordering, which brings \( \alpha(n) \) with \( n > 0 \) to the right, gives a representation of \( \mathcal{D}^+ \). Let us denote the operator on the right–hand side of \( (2.6) \) by \( L^{(r)}(n) \). In
particular, the operators $L^{(0)}(m)$ ($m \in \mathbb{Z}$) give a well-known representation of the Virasoro algebra with central charge $c \mapsto d$,

$$[L^{(0)}(m), L^{(0)}(n)] = (m - n)L^{(0)}(m + n) + d \frac{m^3 - m}{12} \delta_{m+n,0},$$

and the construction (2.6) for these operators is the standard realization of the Virasoro algebra on a module for a Heisenberg Lie algebra (cf. [FLM3]).

As we explained in the Introduction, the appearance of zeta-values in (2.4) can be conceptualized by noting (see [Bl]) that if we remove the normal ordering in (2.6) and use the relation $[\alpha_q(m), \alpha_q(-m)] = m$ to rewrite $\alpha_q(m)\alpha_q(-m)$, with $m \geq 0$, as $\alpha_q(-m)\alpha_q(m) + m$, then the resulting expression contains an infinite formal divergent series of the form

$$1^{2r+1} + 2^{2r+1} + 3^{2r+1} + \cdots.$$

The procedures discussed and analyzed in the Introduction lead to the operators (2.4), and these operators satisfy the bracket relations (2.5). In the rest of this paper, we proceed to supply the details and proofs of the results announced in [L4], [L5] (for a different proof of certain of these results, see [M2]) and [DLM3]. We remark that our results in the theory of twisted modules for vertex operator algebras are much more general than their consequences in the representation theory of the algebra $\hat{D}^+$; they can be applied to other examples of vertex operator algebras, and we expect them to have other interesting consequences.

3 Twisted modules for vertex operator algebras

and commutativity and associativity relations

In this section, we recall the definition of vertex operator algebra, (untwisted) module and twisted module. We derive important commutativity and associativity relations, some of which are known relations for vertex operator algebras and their untwisted modules, and some of which are new identities even when specialized to untwisted modules. For the basic theory of vertex operator algebras and modules, we will use the viewpoint of [LL].

In the theory of vertex operator algebras, formal calculus plays a fundamental role. Here we recall some basic elements of formal calculus (cf. [LL]). Formal calculus is the calculus of formal doubly–infinite series of formal variables, denoted below by $x$, $y$, and by $x_1, x_2, \ldots, y_1, y_2, \ldots$. The central object of formal calculus is the formal delta–function

$$\delta(x) = \sum_{n \in \mathbb{Z}} x^n$$

which has the property

$$\delta \left( \frac{x_1}{x_2} \right) f(x_1) = \delta \left( \frac{x_1}{x_2} \right) f(x_2)$$

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for any formal series \( f(x_1) \). The formal delta–function enjoys many other properties, two of which are:

\[
x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right) = x_1^{-1} \delta \left( \frac{x_1 + x_0}{x_1} \right)
\]

and

\[
x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) + x_0^{-1} \delta \left( \frac{x_2 - x_1}{-x_0} \right) = x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right).
\]

In these equations, binomial expressions of the type \((x_1 - x_2)^n, n \in \mathbb{Z}\) appear. Their meaning as formal series in \( x_1 \) and \( x_2 \), as well as the meaning of powers of more complicated formal series, is summarized in the “binomial expansion convention” – the notational device according to which binomial expressions are understood to be expanded in nonnegative integral powers of the second variable. When more elements of formal calculus are needed below, we shall recall them.

### 3.1 Vertex operator algebras and untwisted modules

We recall from [FLM3] the definition of the notion of vertex operator algebra, a variant of Borcherds' notion [Bo] of vertex algebra:

**Definition 3.1.** A vertex operator algebra \((V, Y, 1, \omega)\), or \(V\) for short, is a \( \mathbb{Z} \)-graded vector space

\[
V = \coprod_{n \in \mathbb{Z}} V(n); \text{ for } v \in V(n), \text{ wt } v = n,
\]

such that

\[
V(n) = 0 \text{ for } n \text{ sufficiently negative},
\]

\[
\dim V(n) < \infty \text{ for } n \in \mathbb{Z},
\]

equipped with a linear map \( Y(\cdot, x) \):

\[
Y(\cdot, x) : V \to (\text{End } V)[[x, x^{-1}]]
\]

\[
v \mapsto Y(v, x) = \sum_{n \in \mathbb{Z}} v_n x^{-n-1}, \quad v_n \in \text{End } V,
\]

where \( Y(v, x) \) is called the vertex operator associated with \( v \), and two particular vectors, \( 1, \omega \in V \), called respectively the vacuum vector and the conformal vector, with the following properties:

- truncation condition: For every \( v, w \in V \)
  \[
v_n w = 0
\]
  for \( n \in \mathbb{Z} \) sufficiently large;

- vacuum property:
  \[
  Y(1, x) = 1_V \quad (1_V \text{ is the identity on } V);
  \]
creation property:

\[ Y(v, x)1 \in V[[x]] \quad \text{and} \quad \lim_{x \to 0} Y(v, x)1 = v; \quad (3.6) \]

Virasoro algebra conditions: Let

\[ L(n) = \omega_{n+1} \quad \text{for} \quad n \in \mathbb{Z}, \quad \text{i.e.,} \quad Y(\omega, x) = \sum_{n \in \mathbb{Z}} L(n) x^{-n-2}. \quad (3.7) \]

Then

\[ [L(m), L(n)] = (m - n)L(m + n) + c_V \frac{m^3 - m}{12} \delta_{n+m,0} 1_V \]

for \( m, n \in \mathbb{Z} \), where \( c_V \in \mathbb{C} \) is the central charge (also called “rank” of \( V \)),

\[ L(0)v = (\text{wt } v)v \]

for every homogeneous element \( v \), and we have the \( L(-1) \)-derivative property:

\[ Y(L(-1)u, x) = \frac{d}{dx} Y(u, x); \quad (3.8) \]

Jacobi identity:

\[
x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) Y(u, x_1)Y(v, x_2) - x_0^{-1} \delta \left( \frac{x_2 - x_1}{-x_0} \right) Y(v, x_2)Y(u, x_1)
= x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right) Y(Y(u, x_0)v, x_2) . \quad (3.9)
\]

An important property of vertex operators is skew-symmetry, which is an easy consequence of the Jacobi identity (cf. [FHL]):

\[ Y(u, x)v = e^{xL(-1)}Y(v, -x)u. \quad (3.10) \]

Another easy consequence of the Jacobi identity is the \( L(-1) \)-bracket formula:

\[ [L(-1), Y(u, x)] = Y(L(-1)u, x). \quad (3.11) \]

Fix a vertex operator algebra \( (V, Y, 1, \omega) \), with central charge \( c_V \).

**Definition 3.2.** A \((\mathbb{Q}-\text{graded})\) module \( W \) for the vertex operator algebra \( V \) (or \( V \)-module) is a \( \mathbb{Q} \)-graded vector space,

\[ W = \coprod_{n \in \mathbb{Q}} W_{(n)}; \quad \text{for} \quad v \in W_{(n)}, \quad \text{wt } v = n, \]

such that

\[ W_{(n)} = 0 \quad \text{for} \quad n \text{ sufficiently negative,} \]

\[ \dim W_{(n)} < \infty \quad \text{for} \quad n \in \mathbb{Q}, \]
equipped with a linear map
\[ Y_W(v, x) = \sum_{n \in \mathbb{Z}} v_n^W \cdot x^{-n-1}, \quad v_n^W \in \text{End } W, \quad (3.12) \]

where \( Y_W(v, x) \) is still called the vertex operator associated with \( v \), such that the following conditions hold:

- **truncation condition:** For every \( v \in V \) and \( w \in W \)
  \[ v_n^W w = 0 \quad (3.13) \]
  for \( n \in \mathbb{Z} \) sufficiently large;

- **vacuum property:** \( Y_W(1, x) = 1_W \); \( 3.14 \)

- **Virasoro algebra conditions:** Let \( L_W(n) = \omega_W^{n+1} \) for \( n \in \mathbb{Z} \), i.e., \( Y_W(\omega, x) = \sum_{n \in \mathbb{Z}} L_W(n)x^{-n-2} \).
  We have
  \[ [L_W(m), L_W(n)] = (m - n)L_W(m + n) + c_V \frac{m^3 - m}{12} \delta_{m+n,0} 1_W, \]
  \[ L_W(0)v = (\text{wt } v)v \]
  for every homogeneous element \( v \in W \), and

- **Jacobi identity:**
  \[ Y_W(L(-1)u, x) = \frac{d}{dx}Y_W(u, x); \quad (3.15) \]

Jacobi identity:
\[ x_0^{-1}\delta \left( \frac{x_1 - x_2}{x_0} \right) Y_W(u, x_1)Y_W(v, x_2) - x_0^{-1}\delta \left( \frac{x_2 - x_1}{-x_0} \right) Y_W(v, x_2)Y_W(u, x_1) = x_2^{-1}\delta \left( \frac{x_1 - x_0}{x_2} \right) Y_W(Y(u, x_0)v, x_2). \quad (3.16) \]

We now recall the main commutativity and associativity properties of vertex operators in the context of modules ([FLMB, FHL, DL1, L1]; cf. [LL]), and then we will derive new identities somewhat analogous to these. All these identities will be generalized to twisted modules below. Note that taking the module to be the vertex operator algebra \( V \) itself, the relations below specialize to commutativity and associativity properties in vertex operator algebras.

From the Jacobi identity (3.16), one can derive the weak commutativity and weak associativity relations, respectively:
\[ (x_1 - x_2)^{k(u,v)}Y_W(u, x_1)Y_W(v, x_2) = (x_1 - x_2)^{k(u,v)}Y_W(v, x_2)Y_W(u, x_1) \quad (3.17) \]
\[ (x_0 + x_2)^{(u,w)}Y_W(u, x_0 + x_2)Y_W(v, x_2)w = (x_0 + x_2)^{(u,w)}Y_W(Y(u, x_0)v, x_2)w, \quad (3.18) \]
where \( u, v \in V \) and \( w \in W \), valid for large enough integers \( k(u,v) \) and \( l(u,w) \), their minimum value depending respectively on \( u, v \) and on \( u, w \). For definiteness, we will pick the integers \( k(u,v) \) and \( l(u,w) \) to be the smallest integers for which the relations above are valid. These relations imply the main “formal” commutativity and associativity properties of vertex operators, which, along with the fact that these properties are equivalent to the Jacobi identity, can be formulated as follows (see [LL]):

**Theorem 3.3.** Let \( W \) be a vector space (not assumed to be graded) equipped with a linear map \( Y_W(\cdot,x) \) such that the truncation condition (3.13) and the Jacobi identity (3.16) hold. Then for \( u, v \in V \) and \( w \in W \), there exist \( k(u,v) \in \mathbb{N} \) and \( l(u,w) \in \mathbb{N} \) and a (nonunique) element \( F(u,v,w;x_0,x_1,x_2) \) of \( W((x_0,x_1,x_2)) \) such that

\[
x_0^{k(u,v)}F(u,v,w;x_0,x_1,x_2) \in W[[x_0]]((x_1,x_2)),
\]

\[
x_1^{l(u,w)}F(u,v,w;x_0,x_1,x_2) \in W[[x_1]]((x_0,x_2))
\]

and

\[
Y_W(u,x_1)Y_W(v,x_2)w = F(u,v,w;x_1-x_2,x_1,x_2),
\]

\[
Y_W(v,x_2)Y_W(u,x_1)w = F(u,v,w;-x_2+x_1,x_1,x_2),
\]

\[
Y_W(Y(u,x_0)v,x_2)w = F(u,v,w;x_0,x_2+x_0,x_2)
\]

(where we are using the binomial expansion convention). Conversely, let \( W \) be a vector space equipped with a linear map \( Y_W(\cdot,x) \) such that the truncation condition (3.13) and the statement above hold, except that \( k(u,v) \in \mathbb{N} \) and \( l(u,w) \in \mathbb{N} \) may depend on all three of \( u, v \) and \( w \). Then the Jacobi identity (3.16) holds.

It is important to note that since \( k(u,v) \) can be (and typically is) greater than 0, the formal series \( F(u,v,w;x_1-x_2,x_1,x_2) \) and \( F(u,v,w;-x_2+x_1,x_1,x_2) \) are not in general equal. Along with (3.19), the first two equations of (3.20) represent formal commutativity, while the first and last equations of (3.20) represent formal associativity, as formulated in [LL] (see also [FLM3] and [FHL]). We will not prove this theorem; instead we will prove its twisted generalization below.

From the equations in Theorem 3.3, we can derive a number of relations similar to weak commutativity and weak associativity but involving formal limit procedures (the meaning of such formal limit procedures is recalled below). Even though only one of these will be of use in the following sections, we state here for completeness of the discussion the two relations that are not “easy” consequences of weak commutativity and weak associativity.

The first relation can be expressed as follows:

**Theorem 3.4.** With \( W \) as in Theorem 3.3,

\[
\lim_{x_0 \to -x_2+x_1} ((x_0+x_2)^{(u,w)}Y_W(Y(u,x_0)v,x_2)w) = x_1^{l(u,w)}Y_W(v,x_2)Y_W(u,x_1)w
\]

(3.21)
for \( u, v \in V \).

The meaning of the formal limit

\[
\lim_{x_0 \to -x_2 + x_1} \left( (x_0 + x_2)^{l(u,w)} Y_W(Y(u, x_0)v, x_2)w \right) \tag{3.22}
\]

is that one replaces each power of the formal variable \( x_0 \) in the formal series

\[(x_0 + x_2)^{l(u,w)} Y_W(Y(u, x_0)v, x_2)w \]

by the corresponding power of the formal series \(-x_2 + x_1\) (defined using the binomial expansion convention). Notice again that the order of \(-x_2\) and \(x_1\) is important in \(-x_2 + x_1\), according to the binomial expansion convention.

**Proof of Theorem 3.4:**

Apply the limit \( \lim_{x_0 \to -x_2 + x_1} \) to the expression

\[(x_0 + x_2)^{l(u,w)} Y_W(Y(u, x_0)v, x_2)w \]

written as in the right–hand side of the third equation of (3.20). This limit is well defined; indeed, the only possible problems are the negative powers of \( x_2 + x_0 \) in \( F(u, v, w, x_0, x_2) \), but they are cancelled out by the factor \((x_0 + x_2)^{l(u,w)}\). The resulting expression is read off the second relation of (3.20).

**Remark 3.1.** It is instructive to consider the following relation, deceptively similar to (3.21), but that is in fact an immediate consequence of weak associativity (3.18):

\[
\lim_{x_0 \to -x_2 + x_1} \left( (x_0 + x_2)^{l(u,w)} Y_W(Y(u, x_0)v, x_2)w \right) = x_1^{l(u,w)} Y_W(Y(u, x_1)v, x_2)w. \tag{3.23}
\]

More precisely, it can be obtained by noticing that the replacement of \( x_0 \) by \( x_1 - x_2 \) independently in each factor in the expression as written on the left–hand side of (3.22) is well defined. We emphasize that, by contrast, the relation (3.22) cannot be obtained in such a manner. Indeed, although the formal limit procedure \( \lim_{x_0 \to -x_2 + x_1} \) is of course well defined on the series on both sides of (3.22), one cannot replace \( x_0 \) by \( -x_2 + x_1 \) either in the factor \( Y_W(Y(u, x_0 + x_2)v, x_2)w \) on the left–hand side or in the factor \( Y_W(Y(u, x_0)v, x_2)w \) on the right–hand side of (3.22).

The second nontrivial relation, which we call modified weak associativity, will be important when generalized to twisted modules. It is stated as:

**Theorem 3.5.** With \( W \) as in Theorem 3.3,

\[
\lim_{x_1 \to x_2 + x_0} \left( (x_1 - x_2)^{k(u,v)} Y_W(Y(u, x_1)v, x_2) \right) = x_0^{k(u,v)} Y_W(Y(u, x_0)v, x_2) \tag{3.24}
\]

for \( u, v \in V \).

**Proof.** Apply the limit \( \lim_{x_1 \to x_2 + x_0} \) to the expression

\[(x_1 - x_2)^{k(u,v)} Y_W(Y(u, x_1)v, x_2)\]
written as in the right–hand side of the first equation of (3.20). This limit is well defined, since negative powers of \( x_1 - x_2 \) in \( F(u,v,w,x_1-x_2,x_1,x_2) \) are cancelled out by the factor \((x_1-x_2)^{k(u,v)}\). The resulting expression is read off the third relation of (3.20).

**Remark 3.2.** Equation (3.24) can be written in the following form:

\[
\lim_{x_1 \rightarrow x_2 + x_0} \left( \frac{x_1 - x_2}{x_0} \right)^{k(u,v)} Y_W(u, x_1) Y_W(v, x_2) = Y_W(Y(u, x_0)v, x_2).
\]

(3.25)

The factor \( \left( \frac{x_1 - x_2}{x_0} \right)^{k(u,v)} \) appearing in front of the product of two vertex operators on the left–hand side is crucial in giving a well–defined limit, but when the limit is applied to this factor without the product of vertex operators, the result is simply 1. We will call such a factor a “resolving factor”. Its power will be apparent, in particular, in the proof of the commutator formula (6.1); as will be explained, it allows one to evaluate nontrivial limits of sums of terms with cancelling “singularities” in a straightforward fashion, evaluating the limit of each term independently.

### 3.2 Twisted modules for vertex operator algebras

The notion of twisted module for a vertex operator algebra was formalized in [FFR] and [D] (see also the geometric formulation in [STS]; see also [DLM]), summarizing the basic properties of the actions of twisted vertex operators discovered in [FLM2, FLM3 and L2], the main nontrivial axiom in this notion is the twisted Jacobi identity of [FLM3 (and L2); cf. [FLM2].

A critical ingredient in formal calculus needed in the theory of twisted modules is the appearance of fractional powers of formal variables, like \( x^{1/p} \), \( p \in \mathbb{Z}_+ \) (the positive integers). For the purpose of formal calculus, the object \( x^{1/p} \) is to be treated as a new formal variable whose \( p \)-th power is \( x \). The binomial expansion convention is applied as stated at the beginning of Section 3 to binomials of the type \((x_1 + x_2)^{1/p}\). From a geometrical point of view, these rules correspond to choosing a branch in the “orbifold structure” described (locally) by the twisted vertex operator algebra module.

We now fix a positive integer \( p \) and a primitive \( p \)-th root of unity

\[ \omega_p \in \mathbb{C}. \]  

(3.26)

We record here two important properties of the formal delta–function involving fractional powers of formal variables:

\[
\delta(x) = \frac{1}{p} \sum_{r=0}^{p-1} \delta(\omega_p^r x^{1/p})
\]

(3.27)

and

\[
x_2^{-1} \delta \left( \omega_p^r \left( \frac{x_1 - x_0}{x_2} \right)^{1/p} \right) = x_1^{-1} \delta \left( \omega_p^{-r} \left( \frac{x_2 + x_0}{x_1} \right)^{1/p} \right). 
\]

(3.28)
Recall the vertex operator algebra \((V, Y, 1, \omega)\) with central charge \(c_V\) of the previous subsection. Fix an automorphism \(\nu\) of period \(p\) of the vertex operator algebra \(V\), that is, a linear automorphism of the vector space \(V\) preserving \(\omega\) and \(1\) such that
\[
\nu Y(v, x) \nu^{-1} = Y(\nu v, x) \quad \text{for } v \in V, \tag{3.29}
\]
and
\[
\nu^p = 1_V. \tag{3.30}
\]

**Definition 3.6.** A \((\mathbb{Q}\text{-graded})\) \(\nu\)-twisted \(V\)-module \(M\) is a \(\mathbb{Q}\)-graded vector space, 
\[
M = \coprod_{n \in \mathbb{Q}} M(n); \quad \text{for } v \in M(n), \quad \text{wt } v = n,
\]
such that
\[
M(n) = 0 \text{ for } n \text{ sufficiently negative}, \quad \dim M(n) < \infty \text{ for } n \in \mathbb{Q},
\]
equipped with a linear map
\[
Y_M(\cdot, x) : V \to (\text{End } M)[[x^{1/p}, x^{-1/p}]] \quad v \mapsto Y_M(v, x) = \sum_{n \in \frac{1}{p}\mathbb{Z}} v_n^\nu x^{-n-1}, \quad v_n^\nu \in \text{End } M, \tag{3.31}
\]
where \(Y_M(v, x)\) is called the twisted vertex operator associated with \(v\), such that the following conditions hold:

- **truncation condition:** For every \(v \in V\) and \(w \in M\)
  \[
v_n^\nu w = 0 \quad \text{for } n \in \frac{1}{p}\mathbb{Z} \text{ sufficiently large};
\]
- **vacuum property:**
  \[
  Y_M(1, x) = 1_M; \tag{3.33}
\]

**Virasoro algebra conditions:** Let
\[
L_M(n) = \omega_{n+1}^\nu \quad \text{for } n \in \mathbb{Z}, \quad \text{i.e., } Y_M(\omega, x) = \sum_{n \in \mathbb{Z}} L_M(n)x^{-n-2}.
\]
We have
\[
[L_M(m), L_M(n)] = (m-n)L_M(m+n) + c_V \frac{m^3 - m}{12} \delta_{m+n,0} 1_M,
\]
and
\[
L_M(0)v = (\text{wt } v)v
\]
for every homogeneous element \(v\), and
\[
Y_M(L(-1)u, x) = \frac{d}{dx} Y_M(u, x); \tag{3.34}
\]

25
Jacobi identity:

\[
x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) Y_M(u, x_1) Y_M(v, x_2) - x_0^{-1} \delta \left( \frac{x_2 - x_1}{-x_0} \right) Y_M(v, x_2) Y_M(u, x_1)
\]

\[= \frac{1}{p} \sum_{r=0}^{p-1} \delta \left( \omega_p^r \left( \frac{x_1 - x_0}{x_2} \right)^{1/p} \right) Y_M(Y(u^r u, x_0)v, x_2).
\]

(3.35)

Note that when restricted to the fixed–point subalgebra \( \{ v \in V \mid \nu v = v \} \), a twisted module becomes a true module: the twisted Jacobi identity reduces to the untwisted one (3.16), by (3.27). This will enable us to construct natural representations of the algebra \( D^+ \) on suitable twisted modules.

We derive below various commutativity and associativity properties of twisted vertex operators. In order to express some of these properties, we need one more element of formal calculus: a certain projection operator. Consider the operator \( P_{[x_0, x_0^{-1}]} \) acting on the space \( \mathbb{C}\{x_0\} \) of formal series with any complex powers of \( x_0 \), which projects to the formal series with integral powers of \( x_0 \):

\[P_{[x_0, x_0^{-1}]} : \mathbb{C}\{x_0\} \to \mathbb{C}[[x_0, x_0^{-1}]]. \quad (3.36)\]

We will extend the meaning of this notation in the obvious way to projections acting on formal series with coefficients lying in vector spaces other than \( \mathbb{C} \), vector spaces which might themselves be spaces of formal series in other formal variables. Notice that when this projection operator acts on a formal series in \( x_0 \) with powers that are in \( \frac{1}{p} \mathbb{Z} \), for instance on \( f(x_0) \in \mathbb{C}[[x_0^{1/p}, x_0^{-1/p}]] \), it can be described by an explicit formula:

\[P_{[x_0, x_0^{-1}]} f(x_0) = \frac{1}{p} \sum_{r=0}^{p-1} \left( \lim_{x_0^{1/p} \to \omega_p^r x_0^{1/p}} f(x) \right). \]

(See Remark 3.3 below for the meaning of formal limit procedures involving fractional powers of formal variables.) We will also extend this projection notation to different kinds of formal series in obvious ways. For instance,

\[P_{x_0^{q/p} [x_0, x_0^{-1}]} : \mathbb{C}\{x_0\} \to \mathbb{C} x_0^{q/p}[[x_0, x_0^{-1}]]. \]

Again, of course, we will extend the meaning of this notation to formal series with coefficients in vector spaces other than \( \mathbb{C} \).

The twisted Jacobi identity (3.35) implies twisted versions of weak commutativity and weak associativity, proved below (\( u, v \in V, w \in M \)):

\[(x_2 - x_1)^k Y_M(v, x_2) Y_M(u, x_1) = (x_2 - x_1)^k Y_M(u, x_1) Y_M(v, x_2) \]

\[P_{[x_0, x_0^{-1}]} \left( (x_0 + x_2)^l Y_M(u, x_0 + x_2) Y_M(v, x_2) w \right)
\]

\[= (x_2 + x_0)^l \frac{1}{p} \sum_{r=0}^{p-1} \omega_p^{-lr} Y_M(Y(u^r u, x_0) v, x_2) w. \]

(3.38)
These relations are valid for all large enough \( k \in \mathbb{N} \) and \( l \in \frac{1}{p} \mathbb{N} \), their minimum value depending respectively on \( u, v \) and on \( u, w \). For definiteness, we will denote these minimum values by \( k(u, v) \) and \( l(u, w) \), respectively (they depend also on the module \( M \); in particular, they differ from the integer numbers \( k(u, v) \) and \( l(u, w) \) used in the previous subsection in connection with the module \( W \)).

As in the untwisted case, these relations imply the main “formal” commutativity and associativity properties of twisted vertex operators \([\text{Li2}]\), which, along with the fact that these properties are equivalent to the Jacobi identity, can be formulated as follows:

**Theorem 3.7.** Let \( M \) be a vector space (not assumed to be graded) equipped with a linear map \( Y_M(\cdot, x) \) \((3.31)\) such that the truncation condition \((3.32)\) and the Jacobi identity \((3.35)\) hold. Then for \( u, v \in V \) and \( w \in M \), there exist \( k(u, v) \in \mathbb{N} \) and \( l(u, w) \in \frac{1}{p} \mathbb{N} \) and a (nonunique) element \( F(u, v, w; x_0, x_1, x_2) \) of \( M((x_0, x_1^{1/p}, x_2^{1/p})) \) such that

\[
\begin{align*}
x_0^{k(u,v)}F(u, v, w; x_0, x_1, x_2) & \in M[[x_0]]((x_1^{1/p}, x_2^{1/p})), \\
x_1^{l(u,w)}F(u, v, w; x_0, x_1, x_2) & \in M[[x_1^{1/p}]]((x_0, x_2^{1/p})),
\end{align*}
\]

and

\[
\begin{align*}
Y_M(u, x_1)Y_M(v, x_2)w & = F(u, v, w; x_0, x_1, x_2), \\
Y_M(v, x_2)Y_M(u, x_1)w & = F(u, v, w; x_0, x_1, x_2), \\
Y_M(Y(\nu^{-s}u, x_0)v, x_2)w & = \lim_{x_1^{1/p} \to \omega_p^{s}(x_0 + x_2)^{1/p}} F(u, v, w; x_0, x_1, x_2)
\end{align*}
\]

(3.39)

for \( s \in \mathbb{Z} \) (where we are using the binomial expansion convention). Conversely, let \( M \) be a vector space equipped with a linear map \( Y_M(\cdot, x) \) \((3.31)\) such that the truncation condition \((3.32)\) and the statement above hold, except that \( k(u, v) \) \((\in \mathbb{N})\) and \( l(u, w) \) \((\in \frac{1}{p} \mathbb{N})\) may depend on all three of \( u, v \) and \( w \). Then the Jacobi identity \((3.35)\) holds.

**Remark 3.3.** Formal limit procedures involving fractional powers of formal variables like \( x_1^{1/p} \) have the same meaning as in \((3.32)\), but with \( x_1^{1/p} \) being treated as a formal variable by itself. For instance, the formal limit procedure

\[
\lim_{x_1^{1/p} \to \omega_p^{s}(x_0 + x_2)^{1/p}} F(u, v, w; x_0, x_1, x_2)
\]

above means that one replaces each integral power of the formal variable \( x_1^{1/p} \) in the formal series \( F(u, v, w; x_0, x_1, x_2) \) by the corresponding power of the formal series \( \omega_p^{s}(x_2 + x_0)^{1/p} \) (defined using the binomial expansion convention).

**Proof of Theorem 3.7.** By applying \( \text{Res}_{x_0} x_0^{k(u,v)} \) to the twisted Jacobi identity \(3.35\) for \( k(u, v) \in \mathbb{N} \) minimal such that

\[
x_0^{k(u,v)}Y_M(\nu^{-s}u, x_0)v \in M[[x_0]] \quad \text{for all } r \in \mathbb{Z}
\]

(3.41)
(which exists by the truncation condition), we find

\[(x_1 - x_2)^{l(u,v)} [Y_M(u, x_1), Y_M(v, x_2)] = 0 \quad (u, v \in V). \tag{3.42}\]

This relation is the expression of weak commutativity \(\text{(3.37)}\), which is the same as for untwisted modules \(\text{(3.17)}\). It immediately implies, using the truncation condition, that exists by the truncation condition, we find

\[x \text{ term which still satisfies the first condition of (3.39) and equations (3.43), where the}\]

\[F_1 + 1/p\text{ relation (3.28) and applying Res}_{x}\text{ (which exists by the truncation condition). Using the formal delta–}\]

\[\text{function itself applied to a vector } w \text{ such that for each } M \text{ which we denote } F(u, v, w; x_0, x_1, x_2), \text{satisfying the first condition of (3.39)}\]

and such that the product of two twisted vertex operators can be written

\[Y_M(u, x_1)Y_M(v, x_2)w = F(u, v, w; x_1 - x_2, x_1, x_2)\]

\[Y_M(v, x_2)Y_M(u, x_1)w = F(u, v, w; -x_2 + x_1, x_1, x_2). \tag{3.43}\]

Many series \(F(u, v, w; x_0, x_1, x_2)\) can be chosen to satisfy these relations. In such a series \(F(u, v, w; x_0, x_1, x_2)\), consider the coefficient \(c_n(x_0, x_2) \in M((x_0, x_2^{1/p}))\) of the monomial \(x_1^n\) for \(n \in \frac{1}{p}\mathbb{Z}\). Suppose that \((x_0 + x_2)^{-1}c_n(x_0, x_2) \in M((x_0, x_2^{1/p}))\) for some positive integer \(j\). Then one can define a new series \(\tilde{F}(u, v, w; x_0, x_1, x_2)\), which still satisfies the first condition of (3.39) and equations (3.38), where the term \(x_1^n c_n(x_0, x_2)\) is replaced by \(x_1^{n-j} (x_0 + x_2)^{-j} c_n(x_0, x_2)\). In view of such transformations, it is always possible to choose \(F(u, v, w; x_0, x_1, x_2)\) in the form

\[F(u, v, w; x_0, x_1, x_2) = \sum_{n \in \frac{1}{p}\mathbb{Z}, \ n \geq n_0} x_1^n c_n(x_0, x_2) \text{ for some } n_0 \in \frac{1}{p}\mathbb{Z},\]

such that for each \(n\), the maximal integer \(j_n\) giving \((x_0 + x_2)^{-j_n} c_n(x_0, x_2) \in M((x_0, x_2^{1/p}))\) is \(j_n = 0\). Let us choose such a formal series \(F(u, v, w; x_0, x_1, x_2)\).

Now we pick \(l \in \mathbb{Z}\) and \(q \in \mathbb{N}, 0 \leq q \leq p - 1\) such that \(l + q/p = l(u, w) + 1 + 1/p, \) where \(l(u, w) \in \frac{1}{p}\mathbb{N}\) is the minimal rational number satisfying

\[x_1^{l(u, w)} Y_M(u, x_1) w \in M[[x_1^{1/p}]] \tag{3.44}\]

(which exists by the truncation condition). Using the formal delta–function relation (3.28) and applying \(\text{Res}_{x_1} x_1^{l+q/p}\) to the twisted Jacobi identity (3.35), itself applied to a vector \(w\), we find

\[P_{[[x_0, x_0^{-1}]]} \left( (x_0 + x_2)^{l+q/p} Y_M(u, x_0 + x_2) Y_M(v, x_2) w \right)\]

\[= (x_2 + x_0)^{l+q/p} \frac{1}{p} \sum_{r=0}^{p-1} \omega_p^{-qr} Y_M(Y(u^r u, x_0) v, x_2) w. \tag{3.45}\]

(Here \(P_{[[x_0, x_0^{-1}]]}\) is the projection defined by (3.36).) From (3.43) and (3.45), we obtain

\[\lim_{x_1 \to x_0 + x_2} P_{[[x_1, x_1^{-1}]]} \left( x_1^{l+q/p} F(u, v, w; x_0, x_1, x_2) \right) \tag{3.46}\]

\[= (x_2 + x_0)^{l+q/p} \frac{1}{p} \sum_{r=0}^{p-1} \omega_p^{-qr} Y_M(Y(u^r u, x_0) v, x_2) w. \]
The right–hand side of (3.46) contains only finitely many negative powers of $x_0$. In view of the comment after (3.43), this implies that the expression inside the limit on the left–hand side does not contain negative powers of $x_1$. That is, the part of the series $F(u, v, w; x_0, x_1, x_2)$ for which the powers of $x_1$ have a fractional part equal to $-q/p$, is of the form $x_1^{-l-q/p} M[[x_1]]((x_0, x_2^{1/p}))$.

Now, the argument is left unchanged, and formula (3.46) stays valid, if we choose $l \in \mathbb{Z}$ and $q \in \mathbb{N}$, $0 \leq q \leq p - 1$ such that $l + q/p = l(u, w) - 1 + t/p$ for all $t \in \mathbb{Z}_+$ (recall that $l(u, w) = 1/p \mathbb{Z}$ is the minimal rational number satisfying (3.43)). In particular, repeating the argument for the cases $1 \leq t \leq p$ gives, from Equation (3.45), the twisted version of weak associativity (3.38), and gives, from the discussion after (3.46), the second condition of (3.39). Other values of $t \in \mathbb{Z}_+$ do not give new information.

Then, for any $l + q/p = l(u, w) - 1 + t/p$ with $t \in \mathbb{Z}_+$, we can change the limit to $\lim_{x_1 \to x_2 + x_0}$ on the left–hand side of (3.46), and we obtain:

$$
\lim_{x_1 \to x_2 + x_0} \left( P_{x_1^{-q/p}([x_1, x_1^{-1}])} F(u, v, w; x_0, x_1, x_2) \right)
= \frac{1}{p} \sum_{r=0}^{p-1} \omega^{-qr} Y_M(Y(\nu^{r} u, x_0)v, x_2)w.
$$

Choosing values of $t$ large enough such that $l \geq l(u, w)$, one can see that the formula above is valid for all $q \in \mathbb{Z}$. Hence we can apply the summation $\sum_{q=0}^{p-1} \omega^{-qs}$ for any $s \in \mathbb{Z}$ to this formula, obtaining

$$
\lim_{x_1^{1/p} \to \omega_s(x_2 + x_0)^{1/p}} F(u, v, w; x_0, x_1, x_2) = Y_M(Y(\nu^{-s} u, x_0)v, x_2)w \quad (s \in \mathbb{Z}).
$$

Next, we prove the converse. Assume (3.39), (3.40) and the truncation condition. In fact, the truncation condition is essential for the statements (3.40) to be valid. Using, in the last term of (3.2), the relation (3.27), and applying the resulting identity to $F(u, v, w; x_0, x_1, x_2)$, we obtain (3.39).

**Remark 3.4.** Note that this proof illustrates the phenomenon, which arises again and again throughout the theory of vertex operator algebras, that formal calculus inherently involves just as much “analysis” as “algebra”: in many relations there are integers that can be left unspecified, except for their minimum values, and the proof involves taking these integers “large enough”. Recall that essentially the same issues arose for example in the use of formal calculus for the proof of the Jacobi identity for (twisted) vertex operators in [FLM3] (see Chapters 8 and 9). This is certainly not surprising, since we are using the Jacobi identity (for all twisting automorphisms) in order to prove, in a different approach, properties of (twisted) vertex operators.

Along with (3.39), the first two equations of (3.40) represent what we call formal commutativity for twisted vertex operators, while the first and last equations of (3.40) represent formal associativity for twisted vertex operators.
specialized to the untwisted case $p = 1$ ($\nu = 1$), these two relations lead respectively to the usual formal commutativity and formal associativity for vertex operators, as described in (3.21).

As in the case of ordinary vertex operators, one can write other relations involving formal limit procedures. Among them, two cannot be directly obtained from weak commutativity and weak associativity. One of these, the relation generalizing (3.21), is stated as follows:

**Theorem 3.8.** With $M$ as in Theorem 3.7,

$$\lim_{x_0 \to -x_2 + x_1} \left( (x_2 + x_0)^l \frac{1}{p} \sum_{r=0}^{p-1} \omega^{-lrp} Y_M(Y(\nu' u, x_0)v, x_2)w \right) = P_{[[x_1, x_1^{-1}]]} \left( x_1^l Y_M(u, x_2)Y_M(v, x_1)w \right),$$

(3.49)

for all $l \in \frac{1}{p} \mathbb{Z}$, $l \geq l(u, w)$.

**Proof.** This is proved along the lines of the proof of Theorem 3.4, with some additions due to the fractional powers. One uses the third equation of (3.40) in order to rewrite the left–hand side of (3.49) as

$$\lim_{x_0 \to -x_2 + x_1} \left( (x_2 + x_0)^l \frac{1}{p} \sum_{r=0}^{p-1} \omega^{-lrp} \lim_{x_1/p \to (x_2 + x_0)/p} F(u, v, w; x_0, x_3, x_2) \right).$$

The sum over $r$ keeps only the terms in which $x_2 + x_0$ is raised to a power which has a fractional part equal to the negative of the fractional part of $l$. Multiplying by $(x_2 + x_0)^l$, for any $l \in \frac{1}{p} \mathbb{Z}$, $l \geq l(u, w)$, brings the remaining series to a series with finitely many negative powers of $x_2$ (as well as $x_0$), to which it is possible to apply the limit $\lim_{x_0 \to -x_2 + x_1}$. This limit of course brings only integer powers of $x_1$, and the right–hand side of (3.49) can be obtained from the second equation of (3.40).

**Remark 3.5.** A relation similar to the last one, but that is a direct consequence of weak associativity (3.28), is

$$\lim_{x_0 \to -x_2 + x_1} \left( (x_2 + x_0)^l \frac{1}{p} \sum_{r=0}^{p-1} \omega^{-lrp} Y_M(Y(\nu' u, x_0)v, x_2)w \right) = P_{[[x_1, x_1^{-1}]]} \left( x_1^l Y_M(u, x_1)Y_M(v, x_2)w \right)$$

(3.50)

for all $l \in \frac{1}{p} \mathbb{Z}$, $l \geq l(u, w)$. This generalizes (3.23) (see the comments in Remark 3.1). It can be obtained by applying the formal limit involved in the left–hand side to both sides of (3.38).

The most important relation for our purposes, generalizing (3.24) and which we call modified weak associativity for twisted vertex operators, is given by the following theorem:
Theorem 3.9. With $M$ as in Theorem 3.7,
\[
\lim_{x_1^{1/p} \to \omega_\nu^{s} x_1^{1/p}} \left( (x_1 - x_2)^k(u,v) Y_M(u,x_1) Y_M(v,x_2) \right) = x_0^k(u,v) Y_M(Y(\nu^{-s} u, x_0) v, x_2)
\] (3.51)
for $u, v \in V$ and $s \in \mathbb{Z}$.

**Proof.** The proof is a straightforward generalization of the proof of Theorem 3.5. □

Remark 3.6. The specialization of Theorems 3.7 and 3.9 to the untwisted case $p = 1$ and $M = V$ gives, respectively, Theorems 3.3 and 3.5.

Finally, we derive a simple relation that specifies the structure of the formal series $Y_M(u, x)$ (see [DL2]).

Theorem 3.10. With $M$ as in Theorem 3.7,
\[
\lim_{x_1^{1/p} \to \omega_\nu^{s} x_1^{1/p}} Y_M(\nu^s u, x_1) = Y_M(u, x)
\] (3.52)
for $u \in V$ and $s \in \mathbb{Z}$.

**Proof.** In the Jacobi identity (3.35), replace $u$ by $\nu^s u$ and $x_1^{1/p}$ by $\omega_\nu^{s} x_1^{1/p}$. The right–hand side becomes
\[
\frac{1}{p} x_2^{-1} \sum_{r=0}^{p-1} \delta \left( \omega_\nu^{s} \left( \frac{x - x_0}{x_2} \right)^{1/p} \right) Y_M(Y(\nu^{r+s} u, x_0) v, x_2),
\]
which is independent of $s$, as is apparent if we make the shift in the summation variable $r \mapsto r - s$. Hence the left–hand side is also independent of $s$. Choosing $v = 1$ and using the vacuum property (3.2), this gives
\[
\left( x_0^{-1} \delta \left( \frac{x - x_2}{x_0} \right) - x_0^{-1} \delta \left( \frac{x_2 - x}{-x_0} \right) \right) \lim_{x_1^{1/p} \to \omega_\nu^{s} x_1^{1/p}} Y_M(\nu^s u, x_1) (3.53)
\]
\[
= \left( x_0^{-1} \delta \left( \frac{x - x_2}{x_0} \right) - x_0^{-1} \delta \left( \frac{x_2 - x}{-x_0} \right) \right) Y_M(u, x)
\]
which, upon using (3.32) and taking Res$_{x_1}$, gives (3.52). □

From Theorem 3.10, we directly have the following corollary:

**Corollary 3.11.** With $M$ as in Theorem 3.7
\[
Y_M(u, x) = \sum_{n \in \mathbb{Z} + q/p} u_n x^{-n-1} \quad \text{for} \quad u \in V, \ \nu u = \omega_\nu^{q} u, \ q \in \mathbb{Z}.
\]
4 Homogeneous twisted vertex operators and associated relations

4.1 Homogeneous untwisted vertex operators

Our construction of $\mathcal{D}^+$ from a particular vertex operator algebra, and our explanation of its connection with the Riemann zeta function and with Bernoulli polynomials (in the twisted case), involve homogeneous vertex operators. It is natural to consider homogeneous vertex operators acting on a $V$-module $W$ as defined in Definition 3.2. The homogeneous vertex operator associated to $v \in V$ will be denoted $X_W(v, x)$. It is defined mainly by the property

$$\text{wt} \left( \text{Res}_x x^{-1} X_W(v, x) \right) = 0.$$  \hspace{1cm} (4.1)

This can be achieved through:

$$X_W(v, x) = Y_W(x^{L(0)} v, x)$$  \hspace{1cm} (4.2)

for $v \in V$ (see [FLM3]). Homogeneous vertex operators clearly inherit the vacuum property (3.14) of ordinary vertex operators, and their main properties can be expressed through the equivalent of the Jacobi identity (3.16), a rewriting of Theorem 4.2 of [L4] (see also [L5], [M2]):

**Theorem 4.1.** For a $V$-module $W$ as in Definition 3.2 and for $u, v \in V$, we have

$$x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) X_W(u, x_1) X_W(v, x_2) - x_0^{-1} \delta \left( \frac{x_2 - x_1}{x_0} \right) X_W(v, x_2) X_W(u, x_1)$$

$$= x_1^{-1} \delta \left( e^y \frac{x_2}{x_1} \right) X_W(Y[u, y] v, x_2)$$  \hspace{1cm} (4.3)

where

$$y = \log \left( 1 + \frac{x_0}{x_2} \right)$$

and

$$Y[u, y] = Y(e^{y L(0)} u, e^y - 1).$$  \hspace{1cm} (4.4)

**Proof.** Consider the Jacobi identity (3.16) with the replacement $u \mapsto x_1^{L(0)} u$ and $v \mapsto x_2^{L(0)} v$. The left–hand side of (3.16) is then directly the left–hand side of (4.1). After using the identity (3.1), the right–hand side becomes

$$x_1^{-1} \delta \left( \frac{x_2 + x_0}{x_1} \right) Y_W(Y(x_1^{L(0)} u, x_0) x_2^{L(0)} v, x_2).$$

Using the property

$$x_2^{-L(0)} Y(u, x_0) x_2^{L(0)} = Y \left( x_2^{-L(0)} u, \frac{x_0}{x_2} \right),$$  \hspace{1cm} (4.5)
the right–hand side is
\[
x_1^{-1} \delta \left( \frac{x_2 + x_0}{x_1} \right) X_W \left( Y \left( \frac{x_1}{x_2}^{L(0)} u, \frac{x_0}{x_2} \right) v, x_2 \right).
\]

Making the replacement \( x_1 \mapsto x_2 + x_0 \) inside the \( Y \) vertex operator, allowed from the properties of the formal delta–function, one obtains the right–hand side of (4.3).

The expressions \( \log(1 + x) \) and \( e^y \) in (4.3), where \( x \) and \( y \) are formal variables, are defined by their series expansions in nonnegative powers of \( x \) and \( y \), respectively. The use of the formal variable \( y \) is natural here in particular because of the appearance of the vertex operator \( Y[u, y] \) defined and studied in [Z1], [Z2]. These operators give a new vertex operator algebra isomorphic to \( V \), and the isomorphism corresponds geometrically to a change–of–coordinates transformation expressed formally by \( y \mapsto e^y - 1 \) (Z1, Z2; see H1, H2 for the generalization to arbitrary coordinate changes).

Remark 4.1. The Jacobi identity (4.3) thus suggests that there is a close relationship between homogeneous vertex operators and the change to “cylindrical coordinates” mentioned in our Introduction. More precisely, see the right–hand side of the Jacobi identity (4.3) as a generating function in \( x_2 \) of endomorphisms of \( W \) associated to some elements of \( V \). These elements are computed using the vertex operator algebra structure given by the “cylindrical coordinates” vertex operator map \( Y[u, y] \). Hence, the endomorphisms thus generated generalize in some sense the modes \( \bar{L}_n \) introduced after formula (1.54). Indeed, similar iterates of vertex operators will be used below in Section 7 to generate a basis for the algebra \( \hat{D}^+ \), along with its realizations on certain twisted spaces, with properties similar to those for the basis \( \bar{L}_n \).

From the Jacobi identity (4.3), one can obtain the following commutator formula (4.4, 4.5):

Corollary 4.2. With \( X_W(u, x) \) defined by (4.2) and \( u, v \in V \), we have
\[
[X_W(u,x_1), X_W(v,x_2)] = \text{Res}_y \left( e^y \frac{x_2}{x_1} \right) X_W(Y[u,y]v,x_2).
\] (4.6)

Proof. Consider the following general fact concerning formal series:

\[
\text{Res}_x h(x) = \text{Res}_y \left( h(F(y)) \frac{d}{dy} F(y) \right) \quad \text{for} \quad h(x) \in A((x)), F(y) \in yA[[y]]
\]

where \( A \) is a commutative associative algebra (or more generally, a module for it) and where the coefficient of \( y^1 \) in \( F(y) \) is invertible. Apply \( \text{Res}_{x_0} \) on the Jacobi identity (4.3). On the left–hand side this gives the commutator, and on the right–hand side use the general fact above with \( F(y) = x_2(e^y - 1) \). The commutator formula follows.

An important property that will be used below is what we call the \( L[-1]–derivative \) property for homogeneous vertex operators:

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Theorem 4.3. With $X_W(u, x)$ defined by (4.2), $u \in V$ and $L[-1] = L(-1) + L(0)$, we have

$$X_W(e^{L[-1]y}u, x) = X_W(u, e^y x).$$

(4.7)

Proof. Using the $L(-1)$–derivative property (3.15) and the identity $L(-1)x^{L(0)} = x^{L(0)-1}L(-1)$, which is a consequence of the Virasoro algebra commutation relations, we have in general

$$x \frac{d}{dx} X_W(u, x) = X_W((L(0) + L(-1))u, x),$$

which proves Theorem 4.3.

As is suggested by the notation $L[-1]$, the combination $L(-1) + L(0)$ is the mode of the conformal vector representing the Virasoro element $L_{-1}$ in the vertex operator algebra $V$ endowed with the vertex operator map defined by (4.4) [Z1, Z2].

We will also use below some properties of Zhu’s vertex operators $Y[u, y]$ defined by (4.4): their skew–symmetry property, the equivalent of (3.10), and their $L[-1]$–derivative and $L[-1]$–bracket properties, equivalent to, respectively, (3.8) and (3.11). These properties are consequences of the work of Zhu [Z1, Z2] (see also the work of Huang [H1, H2]), that is, of the fact that the transformation (4.4) is a vertex operator algebra isomorphism. In fact, more simply, they are consequences of the Jacobi identity for the operators $Y[u, y]$ (which was proven from their definition and from the Jacobi identity for the operators $Y(u, x)$ using formal variable techniques in [L3]). However, for completeness, we give here simple direct proofs. We have:

Theorem 4.4. With $Y[u, y]$ defined by (4.4), $u, v \in V$ and $L[-1] = L(-1) + L(0)$, we have

$$Y[u, y]v = e^{L[-1]y}Y[v, -y]u,$$

(4.8)

and

$$Y[L[-1]u, y] = \frac{d}{dy} Y[u, y].$$

(4.9)

Proof. First, note that the left–hand side of (4.3) is invariant if we replace $(x_0, x_1, x_2)$ by $(-x_0, x_2, x_1)$ and $(u, v)$ by $(v, u)$. Require this invariance on the right–hand side of (4.3). Under this transformation the delta–function $x_1^{-1}\delta \left(e^y \frac{x_2}{x_1}\right) = x_1^{-1}\delta \left(\frac{x_2 + x_0}{x_1}\right)$ is invariant, and using this delta–function, $y$ transforms as $y \mapsto -y$ and $x_2$ transforms as $x_2 \mapsto e^y x_2$. Hence we have

$$X_W(Y[u, y]v, x_2) = X_W(Y[v, -y]u, e^y x_2).$$

(4.11)

Then

$$X_W(Y[u, -y]v, e^y x_2) = X_W(e^{y(L(-1)+L(0))}Y[v, -y]u, x_2),$$
and the injectivity of the vertex operator map $X_W(\cdot, x_2)$ along with (4.7) gives (4.8).

Second, using the $L(-1)$–derivative property (3.8) and again the identity $L(-1)x^{L(0)} = x^{L(0)} - L(-1)$, it is a simple matter to obtain (4.9).

Third, from the Jacobi identity (3.9), one can obtain the bracket relation

$$[L(0), Y(u, x)] = Y((xL(-1) + L(0))u, x).$$

From this and from the $L(-1)$–bracket relation (3.11), equation (4.10) follows.

Homogeneous vertex operators satisfy other important properties, similar to the formal commutativity and associativity properties of vertex operators. We will not state them here, rather we will state and prove their twisted generalization below, which can be easily specialized to the untwisted case.

### 4.2 Homogeneous twisted vertex operators

From now on we fix a $\mathbb{Q}$-graded $\nu$-twisted $V$-module $M$, as in Definition 3.6. We define homogeneous twisted vertex operators by a simple twisted generalization of (4.2):

$$X_M(u, x) = Y_M(x^{L(0)}u, x),$$

as in [FLM3]. We now state and prove the following twisted generalizations of the results above: a Jacobi identity, a commutator formula similar to (4.6), and formal commutativity and associativity properties, including modified weak associativity, for these homogeneous twisted vertex operators.

Recalling the definition (4.4), we have the following twisted generalization of the Jacobi identity (4.3) for homogeneous vertex operators:

**Theorem 4.5.** For a $\nu$-twisted $V$-module $M$ as in Definition 3.6, for $u, v \in V$ and with $Y[u, y]$ defined by (4.4), we have

$$x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) X_M(u, x_1) X_M(v, x_2) - x_0^{-1} \delta \left( \frac{x_2 - x_1}{x_0} \right) X_M(v, x_2) X_M(u, x_1)$$

$$= \frac{1}{p} \sum_{r=0}^{p-1} \delta \left( \omega_p^{-r} \left( e^{y} \frac{x_2}{x_1} \right)^{1/p} \right) X_M(Y[\nu^r u, y]v, x_2).$$

where

$$y = \log \left( 1 + \frac{x_0}{x_2} \right).$$

**Proof.** Consider the twisted Jacobi identity (3.35) with the replacement $u \mapsto x_1^{L(0)}u$ and $v \mapsto x_2^{L(0)}v$. The left-hand side of (3.35) is then directly the left-hand side of (4.13). After using the identity (3.28), the right-hand side becomes

$$\frac{1}{p} \sum_{r=0}^{p-1} \delta \left( \omega_p^{-r} \left( \frac{x_2 + x_0}{x_1} \right)^{1/p} \right) Y_M(Y(\nu^{x_1^{L(0)}u}, x_0)x_2^{L(0)}v, x_2).$$
Using the property (4.5) and the fact that \( \nu L(0) \nu^{-1} = L(0) \), the right-hand side is
\[
\frac{1}{p} x_1^{-1} \sum_{r=0}^{p-1} \delta \left( \omega_p^{-r} \frac{x_2 + x_0}{x_1} \right) X_M \left( Y \left( \frac{x_1}{x_2} L(0) \nu^r u, \frac{x_0}{x_2} v, x_2 \right) \right).
\]
Doing the replacement \( x_1 \mapsto x_2 + x_0 \) inside the \( Y \) vertex operator, allowed from the properties of the formal delta-function, one obtains the right-hand side of (4.13).

This Jacobi identity leads to the following commutator formula:

**Corollary 4.6.** With \( X_M(u, x) \) defined by (4.12) and \( u, v \in V \), we have
\[
[X_M(u, x_1), X_M(v, x_2)] = \text{Res}_y \frac{1}{p} \sum_{r=0}^{p-1} \delta \left( \omega_p^{-r} \left( e^y \frac{x_2}{x_1} \right)^{1/p} \right) X_M(Y(\nu^r u, y, v), x_2).
\]

**Proof.** The proof is a straightforward generalization of the proof of (4.6). The modified weak associativity relation (3.51) yields modified weak associativity for homogeneous twisted vertex operators:

**Theorem 4.7.** With \( X_M(u, x) \) defined by (4.12), \( u, v \in V \) and \( s \in \mathbb{Z} \), we have
\[
\lim_{x_0 \to (e^y - 1)x_2} \frac{1}{p} \omega_p^{(e^y x_2)\frac{1}{p}} x_0^{-1} \frac{x_2 - 1}{x_2} X_M(u, x_1) X_M(v, x_2) = (e^y - 1)^{k(u, v)} X_M(Y(\nu^{-s} u, y, v), x_2).
\]

**Proof.** Apply \( \lim_{x_0 \to (e^y - 1)x_2} \) to both sides of the modified weak associativity relation (3.51). Since \( (e^y - 1) \in y\mathbb{C}[[y]] \), this limit is applicable to any series in \( x_0 \) with finitely many negative powers of \( x_0 \), which is obviously the case of the left-hand side and independently of both factors \( X_M(u, x_1) \) and \( Y_M(Y(\nu^{-s} u, x_0)v, x_2) \) on the right-hand side of (3.51). On the left-hand side, use the formula
\[
\lim_{x_0 \to (e^y - 1)x_2} x_2^{-1} = (e^y x_2)^{1/p}.
\]
Then make the replacement \( u \mapsto x_1^{L(0)} u \) and \( v \mapsto x_2^{L(0)} v \) (and keep unchanged the integer number \( k(u, v) \)), and recall techniques used in the proof of Theorem 4.6 to obtain homogeneous vertex operators from ordinary vertex operators. Equation (4.16) is obtained by multiplying through the result by the factor \( x_2^{-k(u, v)} \).

**Remark 4.2.** Remark 3.2 concerning resolving factors generalizes to the homogeneous twisted case upon rewriting (4.10) in the form
\[
\lim_{x_1^{1/p} \to \omega_p^{(e^y x_2)\frac{1}{p}}} \left( \frac{x_1/x_2 - 1}{e^y - 1} \right)^{k(u, v)} X_M(u, x_1) X_M(v, x_2) = X_M(Y(\nu^{-s} u, y, v), x_2).
\]
The factor \( \left( \frac{x_1/x_2^{1/p}}{x_2^{1/p} - 1} \right)^{k(u,v)} \) appearing in front of the product of two vertex operators on the left-hand side is crucial in having a well defined limit, but when the limit is applied to this factor without the product of vertex operators, the result is simply 1.

Remark 4.3. Another important concept that becomes apparent from the rewriting (4.17) of Equation (4.16) is the concept of generalized normal ordering. In the next section, normal orderings in vertex operator algebra will be recalled and generalized. A normal ordering is essentially an operation that regularizes, in some well-defined way, the product of two (or more) vertex operators with the same formal variable (that is, “at the same point”). The usual one, denoted \( \downarrow \cdot \downarrow \), has the effect, in the Heisenberg algebra, of putting modes \( \alpha(n) \) with positive \( n \) to the right of modes with negative \( n \). This normal ordering also has a general definition in the theory of vertex operator algebras. On the other hand, the resolving factor on the left-hand side of Equation (4.17) allows one, in particular, to make the replacement of \( x_1 \) by \( x_2 \) in the formal series consisting of this resolving factor multiplied by a product of two homogeneous twisted vertex operators with formal variables \( x_1 \) and \( x_2 \). This is in the spirit of a normal ordering operation, and such a limit involving a resolving factor will be identified as a normal ordering operation of a generalized type in the next section. In particular, this is a reinterpretation of the normal ordering denoted \( \downarrow \cdot \downarrow \) in \( \mathcal{L} \)\( ^7 \), \( \mathcal{L} \)\( ^8 \).

We also have the homogeneous counterpart of the relations (3.40):

**Theorem 4.8.** For \( u, v \in V \) and \( w \in M \), there exists \( k(u,v) \in \mathbb{N} \) and \( l'(u, w) \in \frac{1}{p} \mathbb{N} \) and an element \( G(u, v, w; x_0, x_1, x_2) \) of \( M((x_0, x_1^{1/p}, x_2^{1/p})) \) such that

\[
\begin{align*}
\lim_{x_0 \to x_0 + \zeta_0^{(e^y - 1)x_1^2 + x_1}} x_0^{k(u,v)}G(u, v, w; x_0, x_1, x_2) &\in M[[x_0]]((x_1^{1/p}, x_2^{1/p})), \\
x_1^{l'(u,w)}G(u, v, w; x_0, x_1, x_2) &\in M[[x_1^{1/p}]]((x_0, x_2^{1/p})),
\end{align*}
\]

(4.18)

and

\[
\begin{align*}
X_M(u, x_1)X_M(v, x_2)w &= G(u, v, w; x_1 - x_2, x_1, x_2), \\
X_M(v, x_2)X_M(u, x_1)w &= G(u, v, w; x_1, x_1, x_2), \\
X_M(Y[v^{-s}u, y]v, x_2)w &= \lim_{x_0 \to x_0 + \zeta_0^{(e^y - 1)x_1^2 + x_1}} G(u, v, w; (e^y - 1)x_1^2 + x_1, x_1, x_2),
\end{align*}
\]

(4.19)

for \( s \in \mathbb{Z} \). Here \( k(u,v) \) can be taken to be the same as in Theorem 3.17, and \( l'(u, w) \) can be taken to be \( l(u, w) - \text{wt } u \) if \( u \) is homogeneous (that is, if \( L(0)u = (\text{wt } u)u \)).

**Proof.** Writing the first two equations of (3.40) with \( u \) and \( v \) replaced, respectively, by \( x_1^{L(0)}u \) and \( x_2^{L(0)}v \), and using the definition (4.12), gives the first two equations of (4.19), with \( G(u, v, w; x_0, x_1, x_2) = F(x_1^{L(0)}u, x_2^{L(0)}v, w; x_0, x_1, x_2) \). This last identification in particular gives both conditions in (4.18), along with
the fact that we can take $k(u, v)$ as being the same as in Theorem 3.7 and
$l'(u, w)$ as being equal to $l(u, w) - wt u$ if $u$ is homogeneous. The homogeneous
counterpart of the last equation of (3.40) can be obtained by using modified
weak associativity for homogeneous twisted vertex operators (4.16). Multiply
through the first equation of (4.19) by the factor $(x_1/x_2 - 1)^{k(u,v)}$, and apply
to both sides of the resulting equation the limit \( \lim_{x_1/x_2 \to \omega} \).

On the right–hand side, the limit can be taken directly, giving
\[
(e^y - 1)^{k(u,v)} \lim_{x_1/x_2 \to \omega} G(u, v, w; (e^y - 1)x_2, x_1, x_2).
\]

On the left–hand side, use (4.16), giving
\[
(e^y - 1)^{k(u,v)} X_M(Y[u, y]v, x_2).
\]

Every factor in both resulting expressions can be multiplied by \((e^y - 1)^{-k(u,v)} \in \mathbb{C}(y)\) independently. One can then cancel out the factors \((e^y - 1)^{k(u,v)}\) and obtain the last equation of (4.19).

Again, along with (4.18), the first two equations of (4.19) represent what we
call formal commutativity for homogeneous twisted vertex operators, while the
first and last equations of (4.19) represent formal associativity for homogeneous
twisted vertex operators.

Finally, we mention two straightforward consequences of results established
above: the \(L[-1]\)–derivative property of homogeneous twisted vertex operators
and the transformation properties of twisted vertex operators under the auto-
morphism \(\nu\). First, using the \(L(-1)\)–derivative property (3.34), we have, as in
the untwisted case (4.7):
\[
X_M(e^{L[-1]}y u, x) = X_M(u, e^y x).
\]

In particular, with the skew–symmetry of Zhu’s vertex operators (4.8), this
gives:
\[
X_M(Y[u, y]v, x) = X_M(Y[v, -y]u, e^y x).
\]

Second, since weights of elements of \(V\) are integers, Equation (3.52) can directly
be written in terms of homogeneous twisted vertex operators:
\[
\lim_{x_1/x_2 \to \omega} X_M(\nu^s u, x_1) = X_M(u, x).
\]

Remark 4.4. Taking \(p = 1\) in both Theorem 4.7 and Theorem 4.8 gives, re-
spectively, modified weak associativity, and formal commutativity and formal
associativity for homogeneous untwisted vertex operators.

5 Normal orderings in vertex operator algebras
and generalizations

This section is not essential for establishing our main results, but it extends
important concepts in the theory of vertex operator algebras and provides in-
teresting interpretations of some of our results. The concept of normal ordering appears naturally in the construction of vertex operator algebras. A new type of normal ordering denoted $\mathcal{D}^+$ was introduced in [L4], [L5], which, as we mentioned in Remark 4.3, can be reinterpreted as a limit process. This reinterpretation gives a direct link between a natural definition of this normal ordering and general vertex operator algebra principles. As was stated in [L4], [L5], this new normal ordering is the most natural one to use in order to define operators $\tilde{L}_n^{(k)}$ generating the algebra $\hat{D}^+$ and having the particular properties mentioned in Section 2. We will see in the next section that a proper generalization of this normal ordering is also most naturally related to the definition of the action of these particular generators on a twisted space. We now generalize and unify various normal-ordering operations in the general framework of vertex operator algebras.

5.1 Untwisted vertex operators

In order to make sense of the product of two vertex operators at the same “point” (that is, with the same formal variable), normal-ordering operations have to be introduced. We now recall the standard definition of normal ordering in vertex operator algebras (cf. [FLM3], [LL]). Split the vertex operator into two parts: $Y(u, x) = Y^+(u, x) + Y^-(u, x)$, where

$$Y^+(u, x) = \sum_{n<0} u_n x^{-n-1},$$

and

$$Y^-(u, x) = \sum_{n\geq0} u_n x^{-n-1}.$$  

The normal ordering is defined by:

$$\mathcal{O}Y(u, x)Y(v, x)\mathcal{O} = Y^+(u, x)Y(v, x) + Y(v, x)Y^-(u, x),$$

and more generally:

$$\mathcal{O}Y(u, x_1)Y(v, x_2)\mathcal{O} = Y^+(u, x_1)Y(v, x_2) + Y(v, x_2)Y^-(u, x_1).$$

It is not hard to see directly from the Jacobi identity that

$$Y(u_{-1}v, x) = \mathcal{O}Y(u, x)Y(v, x)\mathcal{O}.$$  

From the formulas (cf. [FHL])

$$Y(e^{x_0L(-1)}u, x_2) = e^{x_0\frac{u}{2}}Y(u, x_2),$$

$$e^{x_0L(-1)}Y(u, x_2)e^{-x_0L(-1)} = Y(u, x_2 + x_0)$$

and

$$Y^+(u, x_0) = e^{x_0L(-1)}u_{-1}e^{-x_0L(-1)},$$

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it follows that
\[
Y(Y^+(u, x_0)v, x_2) =
Y(e^{x_0L(-1)}u^{-}\nuL(-1)v, x_2) = e^{x_0/2}Y(u^{-}\nuL(-1)v, x_2)
\]
\[
= e^{x_0/2} \left( Y^+(u, x_2)Y(e^{-x_0L(-1)}v, x_2) + Y(e^{-x_0L(-1)}v, x_2)Y^-(u, x_2) \right)
\]
\[
= Y^+(u, x_2 + x_0)Y(v, x_2) + Y(v, x_2)Y^-(u, x_2 + x_0)
\]
\[
= :Y(u, x_2 + x_0)Y(v, x_2): \quad (u, v \in V). 
\] (5.1)

In other words, the \( \cdot \cdot \cdot \) normal ordering gives the regular part in \( x_0 \) (the part with nonnegative powers of \( x_0 \)) of the iterate \( Y(Y(u, x_0)v, x_2) \). It is convenient to have a normal-ordering notion that is equal to \( Y(Y(u, x_0)v, x_2) \): define \( \cdot \cdot \cdot \cdot \cdot \) by
\[
\cdot\cdot\cdot Y(u, x_2 + x_0)Y(v, x_2)\cdot\cdot\cdot = Y(Y(u, x_0)v, x_2). \] (5.2)

If we recall the modified weak associativity relation (3.24), this is equivalent to
\[
\cdot\cdot\cdot Y(u, x_2 + x_0)Y(v, x_2)\cdot\cdot\cdot = \lim_{x_1+2x_0 \to \omega} \left( \frac{x_1}{x_0} \right)^k Y(u, x_1)Y(v, x_2). \] (5.3)

Note that on the right-hand side, we have a limit applied to an ordinary (that is, not normal-ordered) product of vertex operators. Below, we will find similar expressions for homogeneous twisted vertex operators, recovering and generalizing the definition of the \( \cdot\cdot\cdot \cdot\cdot \) normal ordering introduced in [L4], [L5].

5.2 Twisted vertex operators

We want to generalize the normal-ordering operations \( \cdot\cdot\cdot \cdot\cdot \) and \( \cdot\cdot\cdot \cdot\cdot \cdot\cdot \) to twisted vertex operators. As we already mentioned, more general untwisted vertex operators can be obtained by taking \( \cdot\cdot\cdot \cdot\cdot \cdot\cdot \) normal-ordered product of several “generating” vertex operators. For example,
\[
Y(u^{-}_1v, x) = :Y(u, x)Y(u, x):. 
\]

On the other hand, for twisted vertex operators there is no simple formula of this sort (cf. [FLM3], [DL2]) and it is a nontrivial matter, as we explained in the introduction, to construct general twisted operators.

For our purposes, it is natural to define the normal-ordering operation \( \cdot\cdot\cdot \cdot\cdot \cdot\cdot \) in the following way, simply generalizing a similar formula for untwisted vertex operators (and motivated by modified weak associativity (3.51)):
\[
\lim_{x_1^{1/p} \to \omega(x_2+x_0)^{1/p}} :Y_M(u, x_1)Y_M(v, x_2): = Y_M(Y^+(\nu^{-}_s u, x_0)v, x_2), \] (5.4)
The homogeneous counterpart of the normal ordering defined by (5.3) will be denoted \( \cdot \cdot \cdot \). At the same point, as does the usual normal ordering, they truly regularize the product of two vertex operators (or their derivatives) for \( s \in \mathbb{Z} \) and \( u, v \in V \). Also, generalizing the \( \cdot \cdot \cdot \) normal ordering, we define:

\[
\lim_{x_1^{1/p} \to \omega(p)(x_2 + x_0)^{1/p}} Y_M(u, x_1)Y_M(v, x_2) \cdot \cdot \cdot \\
= \lim_{x_1^{1/p} \to \omega(p)(x_2 + x_0)^{1/p}} \left( \frac{x_1 - x_2}{x_0} \right)^{k(u,v)} Y_M(u, x_1)Y_M(v, x_2) \\
= Y_M(Y(\nu^{-s}u, x_0)v, x_2)
\]

for \( s \in \mathbb{Z} \) and \( u, v \in V \).

The definitions (5.4), (5.5) can be generalized to more general changes of variables in the following way: Let \( f(y) \in \mathbb{C}[[y]] \) be a formal power series such that \( f(y) = 1 + a_1y + a_2y^2 + \cdots \). Then

\[
\lim_{x_1^{1/p} \to \omega(p)f(y)^{1/p}x_2^{1/p}} Y_M(u, x_1)Y_M(v, x_2) \cdot \cdot \cdot \\
= \lim_{x_1^{1/p} \to \omega(p)f(y)^{1/p}x_2^{1/p}} \left( \frac{x_1/x_2 - 1}{f(y) - 1} \right)^{k(u,v)} Y_M(u, x_1)Y_M(v, x_2) \\
= Y_M(Y(\nu^{-s}u, x_2(f(y) - 1))v, x_2).
\]

The last formal expression is well defined because \( (f(y) - 1)^k, k \in \mathbb{Z} \) is a well-defined element inside \( \mathbb{C}[[y]] \). From this, one can take the regular part in \( y \) in order to define new normal-ordering operations \( \cdot \cdot \cdot f \) for every \( f(y) \) as specified:

\[
\lim_{x_1^{1/p} \to \omega(p)f(y)^{1/p}x_2^{1/p}} Y_M(u, x_1)Y_M(v, x_2) \cdot \cdot \cdot f \\
= \text{Reg}_y Y_M(Y(\nu^{-s}u, x_2(f(y) - 1))v, x_2),
\]

where \( \text{Reg}_y \) stands for the operator taking the regular part of a formal series in \( y \). Those normal-ordering operations are well defined when one sets \( y = 0 \), so that they truly regularize the product of two vertex operators (or their derivatives) at the same point, as does the usual \( \cdot \cdot \cdot \). The special case where \( y = x_0/x_2 \) and \( f(y) = 1 + y \) is the case of this usual normal ordering.

### 5.3 Homogeneous untwisted vertex operators

The homogeneous counterpart of the normal ordering defined by (5.3) will be denoted \( \cdot \cdot \cdot \). At the same point, as does the usual normal ordering, they truly regularize the product of two vertex operators (or their derivatives) for \( s \in \mathbb{Z} \) and \( u, v \in V \). Also, generalizing the \( \cdot \cdot \cdot \) normal ordering, we define:

\[
\cdot \cdot \cdot X_W(u, e^y x_2)X_W(v, x_2) \cdot \cdot \cdot \\
= \lim_{x_1 \to e^y x_2} \left( \frac{x_1/x_2 - 1}{e^y - 1} \right)^{k(u,v)} X_W(u, x_1)X_W(v, x_2) \\
= X_W(Y[u, y]v, x_2),
\]

for \( u, v \in V \). This corresponds essentially to the formulas (2.25), (2.26) of [L5]. There is also a homogeneous counterpart for the usual normal ordering \( \cdot \cdot \cdot \) in
a vertex operator algebra: one just takes the regular part in $y$ of the previous expression. We denote it $\hat{\cdot}$, and define it by

$$\hat{\cdot}X_W(u, e^y x_2)X_W(v, x_2)\hat{\cdot} = X_W(Y^+[\nu^{-s} u, y]v, x_2),$$

(5.9)

where $Y^+[u, y]$ is the regular part in $y$ of $Y[u, y]$.

### 5.4 Homogeneous twisted vertex operators

The previous construction can naturally be extended to the twisted setting. From (5.5), it is natural to define the $\hat{\cdot} \cdot \hat{\cdot}$ normal ordering by

$$\lim_{x_1^{-1/p} \rightarrow \omega_p^*(x_2 + x_0)^{1/p}} \hat{\cdot}X_M(u, x_1)X_M(v, x_2)\hat{\cdot}$$

$$= \lim_{x_1^{-1/p} \rightarrow \omega_p^*(x_2 + x_0)^{1/p}} \left(\frac{x_1 - x_2}{x_0}\right)^{k(u,v)} X_M(u, x_1)X_M(v, x_2)$$

$$= (x_2 + x_0)^{\text{wt}(v)}x_2^{\text{wt}(v)}Y_M(Y^{[\nu^{-s} u]}v, x_2),$$

(5.10)

for $s \in \mathbb{Z}$ and $u, v$ homogeneous in $V$. As in the previous section (see (5.6) and (5.7)), we can generalize this to a construction using an arbitrary change of variable parametrized by a formal power series $f(y) \in \mathbb{C}[y]$ such that $f(y) = 1 + a_1 y + a_2 y^2 + \cdots$, that is:

$$\lim_{x_1^{-1/p} \rightarrow \omega_p^*(f(y))^{1/p}x_2^{-1/p}} \hat{\cdot}X_M(u, x_1)X_M(v, x_2)\hat{\cdot}$$

$$= \lim_{x_1^{-1/p} \rightarrow \omega_p^*(f(y))^{1/p}x_2^{-1/p}} \left(\frac{x_1/f(y) - 1}{f(y) - 1}\right)^{k(u,v)} X_M(u, x_1)X_M(v, x_2)$$

$$= (x_2 f(y))^\text{wt}(v)x_2^{\text{wt}(v)}Y_M(Y^{[\nu^{-s} u]}x_2(f(y) - 1)v, x_2).$$

(5.11)

Again, taking the regular part in $y$ gives a family of normal-ordering operations, which we denote $\hat{\cdot} \cdot \hat{\cdot}_f$, that regularize products of homogeneous vertex operators and their derivatives at the same point:

$$\lim_{x_1^{-1/p} \rightarrow \omega_p^*(f(y))^{1/p}x_2^{-1/p}} \hat{\cdot}X_M(u, x_1)X_M(v, x_2)\hat{\cdot}_f$$

$$= \text{Reg}_y \left((x_2 f(y))^\text{wt}(v)x_2^{\text{wt}(v)}Y_M(Y^{[\nu^{-s} u]}x_2(f(y) - 1)v, x_2)\right).$$

(5.12)

Specializing to the case $f(y) = e^y$, we get the twisted generalization of the normal ordering $\hat{\cdot} \cdot \hat{\cdot}$ introduced in (5.5):

$$\lim_{x_1^{-1/p} \rightarrow \omega_p^*(e^y x_2)^{1/p}} \hat{\cdot}X_M(u, x_1)X_M(v, x_2)\hat{\cdot}$$

$$= \text{Reg}_y \left((x_2 e^y)^\text{wt}(v)x_2^{\text{wt}(v)}X_M(Y^{[\nu^{-s} u]}x_2(e^y - 1)v, x_2)\right)$$

$$= X_M(Y^+[\nu^{-s} u, y]v, x_2).$$

(5.13)
for $s \in \mathbb{Z}$ and $u, v \in V$ (not necessarily homogeneous), where $Y^+[u, y]$ is the regular part in $y$ of $Y[u, y]$. Concerning the $\frac{1}{s} \cdot \frac{1}{t}$ normal ordering, from (4.16) and (5.11) and taking again $f(y) = e^y$ it easily follows that

$$
\lim_{x^1/p \to \omega_0^p(e^x x^2)^1/p} \frac{1}{s} X_M(u, x_1) X_M(v, x_2) \frac{1}{t} = \frac{1}{t} \lim_{x^1/p \to \omega_0^p(e^x x^2)^1/p} \left( \frac{x_1/x_2 - 1}{e^y - 1} \right)^{k(u, v)} X_M(u, x_1) X_M(v, x_2) = X_M(Y[\nu^{-s}u, y]v, x_2)
$$

(5.14)

which is the normal-ordering operation that will be the most directly related to our construction of the algebra $\mathcal{D}^+$. This is the generalization to the twisted setting of the formulas (2.25), (2.26) of [L5].

6 Commutator formula for iterates on twisted modules

Modified weak associativity for homogeneous twisted vertex operators turns out to be a very useful calculational tool. Formal limit operations respect products, under suitable conditions, and using this principle, one can compute, for instance, commutators of certain iterates in a natural way. For our applications, an important commutator is

$$[X_M(Y[u_1, y_1]v_1, x_1), X_M(Y[u_2, y_2]v_2, x_2)],$$

which we would like to express in terms of similar iterates. Using modified weak associativity (4.14) and the commutator formula (4.15), we find a generalization of the main commutator formula of [L5] (it is also related to similar commutator formulas in [M1]–[M3]):

Theorem 6.1. For $X_M(\cdot, x)$ defined by (4.12) and $u_1, v_1, u_2, v_2 \in V$,

$$[X_M(Y[u_1, y_1]v_1, x_1), X_M(Y[u_2, y_2]v_2, x_2)] =$$

$$\text{Res}_y \frac{1}{p} \sum_{r=0}^{p-1} \left\{ \delta \left( \omega_p^{-r} (e^{-y} x_1/x_2) \right)^{1/p} X_M(Y[u_2, y_2]Y[\nu^{-r}v_1, -y_1 + y]Y[\nu^{-r}u_1, y]v_2, x_2) + \delta \left( \omega_p^{-r} (e^{-y} x_1/x_2) \right)^{1/p} X_M(Y[u_2, y_2]Y[\nu^{-r}u_1, -y_1 + y]Y[\nu^{-r}v_1, y]v_2, x_2) + \delta \left( \omega_p^{-r} (e^{-y} x_1/x_2) \right)^{1/p} X_M(Y[\nu^{-r}v_1, -y_1]Y[u_2, -y]Y[\nu^{-r}u_1, y_2 + y]v_2, x_2) + \delta \left( \omega_p^{-r} (e^{-y} x_1/x_2) \right)^{1/p} X_M(Y[\nu^{-r}u_1, y_1]Y[u_2, -y]Y[\nu^{-r}v_1, y_2 + y]v_2, x_2) \right\}.
$$
Remark 6.1. In order to obtain a form which more directly specializes to the commutator formula of [10], one has to modify the third term on the right-hand side of (10). Using skew-symmetry (4.5) to change the sign of $y_1$ in the vertex operator $Y[\nu^{-r}v_1, -y_1]$ and using the $L[-1]$-derivative property (4.4), one finds

$$[X_M(Y[u_1, y_1]v_1, x_1), X_M(Y[u_2, y_2]v_2, x_2)] =$$

$$\text{Res}_y \frac{1}{p} \sum_{r=0}^{p-1} \left\{ \delta \left( \omega_p^{-r} \left( e^y - e^{\frac{x_1}{x_2}} \right)^{1/p} \right) X_M(Y[u_2, y_2]Y[\nu^{-r}u_1, -y_1 + y]Y[\nu^{-r}u_1, y]v_2, x_2) + \delta \left( \omega_p^{-r} \left( e^{-y} \frac{x_1}{x_2} \right)^{1/p} \right) X_M(Y[u_2, y_2]Y[\nu^{-r}u_1, y_1 + y]Y[\nu^{-r}v_1, y]v_2, x_2) + \delta \left( \omega_p^{-r} \left( e^{-y + y_1 - y} \frac{x_1}{x_2} \right)^{1/p} \right) X_M(Y[Y[u_2, -y]Y[\nu^{-r}u_1, y_1]Y[\nu^{-r}v_1, y_2 - y_1 + y]v_2, x_2) + \delta \left( \omega_p^{-r} \left( e^{-y - y_1} \frac{x_1}{x_2} \right)^{1/p} \right) X_M(Y[Y[\nu^{-r}u_1, y_1]Y[u_2, -y]Y[\nu^{-r}v_1, y_2 + y]v_2, x_2) \right\}$$

which specializes to the formula in Theorem 3.4 of [10] when $p = 1$.

Proof of Theorem 10. Using (11.10) multiplied through by the factor $(e^y - 1)^{-k(u, v)}$, we rewrite the commutator as a commutator of quadratics:

$$[X_M(Y[u_1, y_1]v_1, x_1), X_M(Y[u_2, y_2]v_2, x_2)] =$$

$$\lim_{x_3 \to e^{y_1} x_1} \lim_{x_4 \to e^{y_2} x_2} \left\{ \left( \frac{x_3}{x_1} - 1 \right)^{k(u_1, v_1)} \left( \frac{x_4}{x_2} - 1 \right)^{k_2} \right\} [X_M(u_1, x_3)X_M(v_1, x_1), X_M(u_2, x_4)X_M(v_2, x_2)]$$

where we take $k_2 \geq k(u_2, v_2)$, otherwise unspecified for now. Then we use (11.14) to write the commutator as a sum of four terms:

$$[X_M(Y[u_1, y_1]v_1, x_1), X_M(Y[u_2, y_2]v_2, x_2)] =$$

$$\text{Res}_y \frac{1}{p} \sum_{r=0}^{p-1} \lim_{x_3 \to e^{y_1} x_1} \lim_{x_4 \to e^{y_2} x_2} \left\{ \left( \frac{x_3}{x_1} - 1 \right)^{k(u_1, v_1)} \left( \frac{x_4}{x_2} - 1 \right)^{k_2} \right\} \left\{ \delta \left( \omega_p^{-r} \left( e^{y} \frac{x_4}{x_1} \right)^{1/p} \right) X_M(u_1, x_3)X_M(Y[\nu^{-r}v_1, y]u_2, x_4)X_M(v_2, x_2) + \delta \left( \omega_p^{-r} \left( e^{-y} \frac{x_4}{x_1} \right)^{1/p} \right) X_M(u_1, x_3)X_M(u_2, x_4)X_M(Y[\nu^{-r}v_1, y]v_2, x_2) + \delta \left( \omega_p^{-r} \left( e^{y} \frac{x_4}{x_3} \right)^{1/p} \right) X_M(Y[\nu^{-r}u_1, y]u_2, x_4)X_M(v_2, x_2)X_M(v_1, x_1) + \delta \left( \omega_p^{-r} \left( e^{-y} \frac{x_4}{x_3} \right)^{1/p} \right) X_M(u_2, x_4)X_M(Y[\nu^{-r}u_1, y]v_2, x_2)X_M(v_1, x_1) \right\}.$$
By the truncation property, formal series of the form $Y[u, y]v$ have finitely many negative powers of $y$ for any $u, v \in V$. Also, in the expression inside the braces on the right–hand side above, other factors involving $y$ only contain nonnegative powers of $y$. Hence, we conclude that when evaluating the residue in $y$, only a finite number of terms are to be added, each term of course being a doubly–infinite series in the variables $x_1^{1/p}, x_2^{1/p}, x_3^{1/p}, x_4^{1/p}$ with coefficients in $\text{End} \ M$. Each such term can be expressed as a product of three homogeneous twisted vertex operators. In fact, each term contains a factor of the form $X_M(u, x_4)X_M(v, x_2)$ for some $u \in V$ and $v \in V$. Since there is a finite number of these terms, it is possible to choose a value of $k_2$ which is greater than or equal to $k(u, v)$ for all of the vectors $u, v$ involved in these factors (as well, of course, as greater than or equal to $k(u_2, v_2)$). Doing so, we can evaluate the limit $\lim_{x_4 \to e^{y_2}x_2}$ of each term independently using (4.16). This justifies the name “resolving factor” for the factor $(\frac{x_4/x_2-1}{e^{y_2}-1})^{k_2}$ on the right–hand side of (6.3), for $k_2$ large enough (see Remark 4.12). This gives:

$$[X_M(Y[u_1, y_1]v_1, x_1), X_M(Y[u_2, y_2]v_2, x_2)] =$$

$$\text{Res}_y \frac{1}{p} \sum_{r=0}^{p-1} \lim_{x_3 \to e^{y_1}x_1} \left\{ \left( \frac{x_3/x_1 - 1}{e^{y_1} - 1} \right)^{k(u_1, v_1)} \left( \delta \left( \omega_p^{-r} (e^{y_1} + \frac{x_2}{x_1})^{1/p} X_M(u_1, x_3)X_M(Y[^{\nu^r}y_1, y]u_2, y_2)v_2, x_2) 
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+ \delta \left( \omega_p^{-r} (e^{y_1} \frac{x_2}{x_1})^{1/p} X_M(u_1, x_3)X_M(Y[u_2, y_2]Y[^{\nu^r}y_1, y]v_2, x_2) 

+ \delta \left( \omega_p^{-r} (e^{y_1} + \frac{y_2}{x_3})^{1/p} X_M(Y[^{\nu^r}u_1, y]u_2, y_2)v_2, x_2)X_M(v_1, x_1) 

+ \delta \left( \omega_p^{-r} (e^{y_1} \frac{y_2}{x_3})^{1/p} X_M(Y[u_2, y_2]Y[^{\nu^r}u_1, y]v_2, x_2)X_M(v_1, x_1) \right) \right\}. $$

In order to evaluate the limit $\lim_{x_3 \to e^{y_1}x_3}$ of each term independently, one could replace $k(u_1, v_1)$ by an integer $k_1 \geq k(u_1, v_1)$ large enough. To make the procedure more transparent, however, we will use extra resolving factors. First, we apply (4.21) to the first and third terms on the right–hand side of the
previous equation and use the delta-function properties:

\[ [X_M(Y[u_1, y_1]v_1, x_1), X_M(Y[u_2, y_2]v_2, x_2)] = \]

\[
\text{Res}_y \frac{1}{p} \sum_{r=0}^{p-1} \lim_{x_3 \to e^{y_1}x_1} \left\{ \left( \frac{x_3/x_1 - 1}{e^{y_1} - 1} \right)^{k(u_1, v_1)} \right. \\
\left( \omega_p^{-r} \left( e^{y_2+y} \frac{x_3}{x_1} \right)^{1/p} \right) X_M(u_1, x_3) \quad \lim_{x_1^{1/p} \to x_3^{1/p} \omega_p^r} X_M(Y[v_2, -y_2]Y[u^r v_1, y]v_2, e^{-y}x_1) \\
\left. + \delta \left( \omega_p^{-r} \left( e^{y_2+y} \frac{x_3}{x_1} \right)^{1/p} \right) X_M(u_1, x_3) \quad \lim_{x_1^{1/p} \to x_3^{1/p} \omega_p^r} X_M(Y[u_2, y_2]Y[u^r v_1, y]v_2, e^{-y}x_1) \\
\right. \\
\left. + \delta \left( \omega_p^{-r} \left( e^{y_2+y} \frac{x_3}{x_1} \right)^{1/p} \right) X_M(Y[v_2, -y_2]Y[u^r u_1, y]u_2, e^{-y}x_3)X_M(v_1, x_1) \\
\right. \\
\left. + \delta \left( \omega_p^{-r} \left( e^{y_2+y} \frac{x_3}{x_1} \right)^{1/p} \right) X_M(Y[u_2, y_2]Y[u^r u_1, y]u_2, e^{-y}x_3)X_M(v_1, x_1) \right\}.
\]

Then, inside the braces, we insert the resolving factor

\[
\left( \frac{e^{y_3}x_3/x_1 - 1}{e^{y_1+y} - 1} \right)^{k_3} \left( \frac{e^{-y}x_3/x_1 - 1}{e^{y_1-y} - 1} \right)^{k_3}
\]

where \( k_3 \geq 0 \) is an integer large enough, yet unspecified. Note that this factor gives 1 under \( \lim_{x_3 \to e^{y_1}x_1} \), so it doesn’t change the result, but it allows us to calculate it easily. Indeed, looking at any fixed power of \( y_2 \), we can use an argument similar to what we explained after (6.5) and choose \( k_3 \) large enough (its minimum value depending on the power of \( y_2 \)) in order to evaluate the limit of each term inside the braces independently using (4.10). Since \( k_3 \) clearly does not appear in the result for any power of \( y_2 \), this procedure can be apply to the whole series in \( y_2 \).

It is convenient to write the resulting expression for the commutator in a form which does not involve the factor \( e^{-y} \) in the argument of the homogeneous twisted vertex operators. This form can be obtained by using the \( L[-1] \)–derivative property for homogeneous twisted vertex operators \( \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \). Using further the identity \( e^{L[-1]}yY[u, y_1]e^{-L[-1]}y_2 = Y[u, y_1 + y_2] \) (which comes from the \( L[-1] \)–bracket property and the \( L[-1] \)–derivative property \( \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \) and \( \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \)
of Zhu’s vertex operators), we obtain:

\[
[X_M(Y[u_1, y_1]v_1, x_1), X_M(Y[u_2, y_2]v_2, x_2)] = \]

\[
\Res_y \frac{1}{p} \sum_{r=0}^{p-1} \left\{ \delta \left( \omega_p^r \left( e^{-x_2 y_1} \frac{x_{1}}{x_2} \right) \right)^{1/p} X_M(Y[u_1, y_1]Y[\nu^{-r} u_2, -y]Y[\nu^{-r} v_2, -y]v_1, x_1) + \delta \left( \omega_p^r \left( e^{y_1 - y_2 x_1} \frac{x_1}{x_2} \right) \right)^{1/p} X_M(Y[\nu^{-r} u_2, -y]Y[u_1, y]Y[\nu^{-r} v_2, y_1 - y]v_1, x_1) \right\}.
\]

This is completely equivalent to the commutator formula (6.1). Indeed, the commutator is unchanged if we make the transformation \( x_1 \leftrightarrow x_2, y_1 \leftrightarrow y_2, u_1 \leftrightarrow u_2 \) and \( v_1 \leftrightarrow v_2 \), and change the overall sign. Doing this operation on the right–hand side and absorbing the overall sign in a change of sign of the variable \( y \) on which we take the residue, we obtain (6.1).

\[ \square \]

**Remark 6.2.** It is important to note that although the derivation above does not make it apparent, there are many subtleties related to cancellation of “singularities” in evaluating the commutator by taking the limits in (6.3). Recall that the formula (6.3) is valid for any \( k_2 \geq k(u_2, v_2) \), but that the limit on \( x_3 \) was taken by assuming \( k_2 \) large enough; for a given vertex operator algebra, this can be (and typically is) larger than \( k(u_2, v_2) \). The result (6.3) is valid no matter the value of \( k_2 \geq k(u_2, v_2) \). What happens is that for \( k_2 = k(u_2, v_2) \), for instance, the limit cannot typically be taken independently on each of the four terms inside the braces on the right–hand side of (6.3), because they contain “singular terms” of the type \( (x_3 - x_2)^{-n} \) for \( n \) positive and large enough. However, the nontrivial statement is that all these singular terms cancel out inside the braces, since the limit must exist. In a particular vertex operator algebra, one could (and often does, see [FLM2] for instance) evaluate this limit explicitly by taking \( k_2 = k(u_2, v_2) \) and observing the cancellation of the remaining singular terms. A similar phenomenon happens when evaluating the limit on \( x_3 \) in (6.3).

\[ \square \]

## 7 Main results

We now obtain a representation of \( \tilde{D}^+ \) on a certain natural module for a twisted affine Lie algebra based on a finite-dimensional abelian Lie algebra (essentially a twisted Heisenberg Lie algebra), generalizing Bloch’s representation on the module \( S \simeq S(\hat{\mathfrak{h}}^-) \) constructed in Section 2. This is also a generalization of the twisted Virasoro algebra construction (see [FLM2], [FLM3], [DL2]).
7.1 Description of the twisted module

Let \( \mathfrak{h} \) be a finite-dimensional abelian Lie algebra (over \( \mathbb{C} \)) of dimension \( d \) on which there is a nondegenerate symmetric bilinear form \( \langle \cdot, \cdot \rangle \). Let \( \nu \) be an isometry of \( \mathfrak{h} \) of period \( p > 0 \):

\[
(\nu \alpha, \nu \beta) = \langle \alpha, \beta \rangle, \quad \nu^p \alpha = \alpha
\]

for all \( \alpha, \beta \in \mathfrak{h} \). Consider the affine Lie algebra \( \hat{\mathfrak{h}} \) and its abelian subalgebra \( \hat{\mathfrak{h}}^- \) recalled in Section 2. The induced (level–one) \( \hat{\mathfrak{h}} \)-module \( S \cong S(\hat{\mathfrak{h}}^-) \) (linearly) carries a natural structure of vertex operator algebra. This structure is constructed as follows (cf. [FLM3]). First, one identifies the vacuum vector as the element 1 in \( S \):

\[ 1 = 1. \]

Recalling the notation \( \alpha(n) (\alpha \in \mathfrak{h}, n \in \mathbb{Z}) \) for the action of \( \alpha \otimes t^n \in \hat{\mathfrak{h}} \) on \( S \), one constructs the following formal series acting on \( S \):

\[
\alpha(x) = \sum_{n \in \mathbb{Z}} \alpha(n)x^{-n-1} \quad (\alpha \in \mathfrak{h}).
\]

Then, the vertex operator map \( Y(\cdot, x) \) is given by

\[
Y(\alpha_1(-n_1) \cdots \alpha_j(-n_j)1, x) = \left( \frac{d}{dx} \right)^{n_1-1} \alpha_1(x) \cdots \left( \frac{d}{dx} \right)^{n_j-1} \alpha_j(x),
\]

for \( \alpha_k \in \mathfrak{h}, n_k \in \mathbb{Z}_+, \ k = 1, 2, \ldots, j \), for all \( j \in \mathbb{N} \), where \( \cdots \) is the usual normal ordering, which brings \( \alpha(n) \) with \( n > 0 \) to the right. Choosing an orthonormal basis \( \{ \alpha_q | q = 1, \ldots, d \} \) of \( \mathfrak{h} \), the conformal vector is \( \omega = \frac{1}{2} \sum_{q=1}^{d} \alpha(q)(-1)\alpha(q)(-1)1 \). This implies in particular that the weight of \( \alpha(n)1 \) is \( n \):

\[
L(0)\alpha(-n)1 = n\alpha(-n)1 \quad (\alpha \in \mathfrak{h}, n \in \mathbb{Z}_+).
\]

The isometry \( \nu \) on \( \mathfrak{h} \) lifts naturally to an automorphism of the vertex operator algebra \( S \), which we continue to call \( \nu \), of period \( p \).

We now proceed as in [L1], [FLM2], [FLM3] and [DL2] to construct a space \( S[\nu] \) that carries a natural structure of \( \nu \)-twisted module for the vertex operator algebra \( S \). In these papers, the twisted module structure was observed assuming that \( \nu \) preserves a rational lattice in \( \mathfrak{h} \). Since the properties of this twisted module will be essential in our argument below, we make the same assumption here.

Recalling our primitive \( p \)-th root of unity \( \omega_p \), for \( r \in \mathbb{Z} \) set

\[
\mathfrak{h}_{(r)} = \{ \alpha \in \mathfrak{h} \mid \nu \alpha = \omega_p^r \alpha \} \subset \mathfrak{h}.
\]

For \( \alpha \in \mathfrak{h} \), denote by \( \alpha(r) \), \( r \in \mathbb{Z} \), its projection on \( \mathfrak{h}_{(r)} \). Define the \( \nu \)-twisted affine Lie algebra \( \hat{\mathfrak{h}}[\nu] \) associated with the abelian Lie algebra \( \mathfrak{h} \) by

\[
\hat{\mathfrak{h}}[\nu] = \prod_{n \in \mathbb{Z}_+} \mathfrak{h}_{(pn)} \otimes t^n \oplus \mathbb{C} \mathbb{C}
\]

(7.2)
with

\[
[\alpha \otimes t^m, \beta \otimes t^n] = \langle \alpha, \beta \rangle m\delta_{m+n,0} C \quad (\alpha \in \mathfrak{h}_{(pn)}, \beta \in \mathfrak{h}_{(pn)}, \ m, n \in \mathbb{Z})
\]

\[
[C, \hat{\mathfrak{h}}[\nu]] = 0.
\] (7.3)

Set

\[
\hat{\mathfrak{h}}[\nu]^+ = \bigoplus_{n>0} \mathfrak{h}_{(pn)} \otimes t^n, \quad \hat{\mathfrak{h}}[\nu]^− = \bigoplus_{n<0} \mathfrak{h}_{(pn)} \otimes t^n.
\] (7.4)

The subalgebra

\[
\hat{\mathfrak{h}}[\nu]^+ \oplus \hat{\mathfrak{h}}[\nu]^− \oplus \mathbb{C}
\] (7.5)

is a Heisenberg Lie algebra. Form the induced (level-one) \( \hat{\mathfrak{h}}[\nu] \)-module

\[
S[\nu] = U(\hat{\mathfrak{h}}[\nu]) \otimes U(\hat{\mathfrak{h}}[\nu]^+ \oplus \mathfrak{h}(0) \oplus \mathbb{C}) \cong S(\hat{\mathfrak{h}}[\nu]^-) \quad \text{(linearly),}
\] (7.6)

where \( \hat{\mathfrak{h}}[\nu]^+ \oplus \mathfrak{h}(0) \) acts trivially on \( \mathbb{C} \) and \( \mathbb{C} \) acts as 1; \( U(\cdot) \) denotes universal enveloping algebra. Then \( S[\nu] \) is irreducible under the Heisenberg algebra \( \hat{\mathfrak{h}}[\nu]^+ \oplus \hat{\mathfrak{h}}[\nu]^− \oplus \mathbb{C} \mathbb{C} \). We will use the notation \( \alpha^\nu(n) \) for the action of \( \alpha \otimes t^n \in \hat{\mathfrak{h}}[\nu] \) on \( S[\nu] \).

\textbf{Remark 7.1.} The special case where \( p = 1 \ (\nu = 1_{\mathbb{h}}) \) corresponds to the \( \hat{\mathfrak{h}} \)-module \( S \) discussed in Section 2.

As we mentioned above, the \( \hat{\mathfrak{h}}[\nu] \)-module \( S[\nu] \) is naturally a \( \nu \)–twisted module for the vertex operator algebra \( S \). One first constructs the following formal series acting on \( S[\nu] \):

\[
\alpha^\nu(x) = \sum_{n \in \mathbb{Z}} \alpha^\nu(n)x^{-n-1},
\] (7.7)

as well as the formal series \( W(v, x) \) for all \( v \in S \):

\[
W(\alpha_1(-n_1) \cdots \alpha_j(-n_j)1, x) = \frac{1}{(n_1 - 1)!} \left( \frac{d}{dx} \right)^{n_1 - 1} \alpha^\nu_1(x) \cdots \frac{1}{(n_j - 1)!} \left( \frac{d}{dx} \right)^{n_j - 1} \alpha^\nu_j(x) \quad (7.8)
\]

where \( \alpha_k \in \mathfrak{h}, \ n_k \in \mathbb{Z}_+, \ k = 1, 2, \ldots, j, \) for all \( j \in \mathbb{N} \). The twisted vertex operator map \( Y_{S[\nu]}(\cdot, x) \) acting on \( S[\nu] \) is then given by

\[
Y_{S[\nu]}(v, x) = W(e^{\Delta x}v, x) \quad (v \in S)
\] (7.9)

where \( \Delta x \) is a certain formal operator involving the formal variable \( x \) \cite{FLM2, ELM3, DL2}. This operator is trivial on \( \alpha(-n)1 \in S \ (n \in \mathbb{Z}_+) \), so that one has in particular

\[
Y_{S[\nu]}(\alpha(-n)1, x) = \frac{1}{(n - 1)!} \left( \frac{d}{dx} \right)^{n - 1} \alpha^\nu(x).
\] (7.10)
One crucial role of the formal operator $\Delta_x$ is to make the fixed–point subalgebra \{ $u \mid \nu u = u$ \} act according to a true module action. This property will be essential below in constructing a representation of the algebra $\hat{D}^+$ on the twisted space $S[\nu]$. For instance, the conformal vector $\omega$ is in the fixed point subalgebra, so that the vertex operator $Y_{S[\nu]}(\omega, x)$ generates a representation of the Virasoro algebra on the space $S[\nu]$. This representation of the Virasoro algebra was explicitly constructed in [DL2]. As one can see in the results of [DL2] and as will become clear below, the resulting representation of the Virasoro generator $L_0$ is not an (infinite) sum of normal-ordered products the type $\sum_{n \in \mathbb{Z}^\times} :\alpha(n)\beta(-n):$; rather, there is an extra term proportional to the identity on $S[\nu]$, the so-called correction term, which appears because of the operator $\Delta_x$. The correction term was calculated in [DL2] using the explicit action of $e^{\Delta_x}$ on $\omega$. In the case of the period–2, $\nu = -1$ automorphism, this action is given by [FLM2], [FLM3]:

$$e^{\Delta_x} \omega = \omega + \frac{1}{16}(\dim h)x^{-2},$$

and for general automorphism, the calculation was carried out in [DL2] (see also [FFR] and [FLM3]). This is relevant, for instance, in the construction of the moonshine module [FLM3].

In order to have the correction term for the representation of the algebra $\hat{D}^+$ on the twisted space $S[\nu]$, one can calculate the action of $e^{\Delta_x}$ for a general automorphism on the vectors generating the representation of the whole algebra $\hat{D}^+$. This is a complicated problem, mainly because generators of $\hat{D}^+$ have arbitrary large weights. Below we will calculate the correction terms using the general theory of twisted modules for vertex operator algebras, in particular using the modified weak associativity relation for twisted operators, as well as the simple result (7.10). Hence in our argument, the explicit action of $\Delta_x$ on vectors generating the representation of the algebra $\hat{D}^+$ is not of importance; all we need to know is that there exists such an operator $\Delta_x$ giving to the space $S[\nu]$ the properties of a twisted module for the vertex operator algebra $S$.

### 7.2 Construction of the corresponding representation of $\hat{D}^+$

In order to construct a representation of $\hat{D}^+$ on $S[\nu]$ we need to consider certain homogeneous twisted vertex operators associated to $Y_{S[\nu]}(\cdot, x)$. For $\alpha \in \mathfrak{h}$ we define the following series acting on $S[\nu]$:

$$\alpha^\nu(x) = X_{S[\nu]}(\alpha(-1)1, x) = \sum_{n \in \mathbb{Z}} \alpha^\nu(n)x^{-n} \quad (7.11)$$

where we used (7.10) and the fact that $\alpha(-1)1$ has weight 1. Recalling the orthonormal basis \{ $\alpha_q | q = 1, \ldots, d$ \} of $\mathfrak{h}$, we define the following two formal
series acting on $S[\nu]$:

\[
L^{\nu,y_1,y_2}(x) = \frac{1}{2} \sum_{q=1}^{d} : \alpha_q^\nu(e^{y_1} x) \alpha_q^\nu(e^{y_2} x) : - \frac{1}{2} \frac{\partial}{\partial y_1} \left( \sum_{k=0}^{p-1} \frac{(e^{k(y_1+y_2)} - 1) \dim h(k)}{1 - e^{-y_1+y_2}} \right) \tag{7.12}
\]

and

\[
\tilde{L}^{\nu,y_1,y_2}(x) = \frac{1}{2} \sum_{q=1}^{d} : \alpha_q^\nu(e^{y_1} x) \alpha_q^\nu(e^{y_2} x) : - \frac{1}{2} \frac{\partial}{\partial y_1} \left( \sum_{k=0}^{p-1} \frac{(e^{k(y_1+y_2)} - 1) \dim h(k)}{1 - e^{-y_1+y_2}} \right) . \tag{7.13}
\]

**Remark 7.2.** In the special case $p = 1$ and $d = 1$, the operators $L^{\nu,y_1,y_2}(x)$ and $L^{\nu,y_1,y_2}(x)$, respectively, specialize to the operators $L^{(y_1,y_2)}(x)$ and $\tilde{L}^{(y_1,y_2)}(x)$ of [4], [5].

We now prove an important statement for our construction of the representation of $\tilde{D}^+$ on the twisted space $S[\nu]$:

**Proposition 7.1.** The series $\tilde{L}^{\nu,y_1,y_2}(x)$ as defined in (7.13) can be identified with the following iterate of vertex operators:

\[
\tilde{L}^{\nu,y_1,y_2}(x) = X_{S[\nu]} \left( \frac{1}{2} \sum_{q=1}^{d} Y[\alpha_q(-1)1, y_1-y_2]\alpha_q(-1)1, e^{y_2} x) . \tag{7.14}
\]

**Proof.** One first rewrites the formal series (7.13) in the form:

\[
\tilde{L}^{\nu,y_1,y_2}(x_2) = \frac{1}{2} \lim_{x_1 \to x_2} \sum_{q=1}^{d} \left( \frac{x_{1}}{x_{2}} e^{y_1-y_2} - 1 \right) \alpha_q^\nu(e^{y_1} x_1) \alpha_q^\nu(e^{y_2} x_2) \tag{7.15}
\]

for any fixed $k \in \mathbb{N}$, $k \geq 2$, which comes from

\[
\sum_{q=1}^{d} \alpha_q^\nu(e^{y_1} x_1) \alpha_q^\nu(e^{y_2} x_2) = \sum_{q=1}^{d} : \alpha_q^\nu(e^{y_1} x_1) \alpha_q^\nu(e^{y_2} x_2) : - \frac{\partial}{\partial y_1} \left( \sum_{k=0}^{p-1} \frac{e^{k(y_1+y_2)} - 1}{1 - e^{-y_1+y_2}} \dim h(k) \right) .
\]

Then one uses modified weak associativity (1.16) with $s = 0$, $y = y_1 - y_2$ and the replacements $x_2 \mapsto e^{y_2} x_2$, $x_1 \mapsto e^{y_1} x_1$, along with the definition (7.14).

**Remark 7.3.** From the expression (7.13) of the operator $\tilde{L}^{\nu,y_1,y_2}(x)$, we see that it can be written in terms of the normal ordering introduced in (5.13):

\[
\tilde{L}^{\nu,y_1,y_2}(x) = \frac{1}{2} + \sum_{q=1}^{d} \alpha_q^\nu(e^{y_1} x) \alpha_q^\nu(e^{y_2} x) . \tag{7.16}
\]
This generalizes formula (3.17) of [L4], (equivalently, formula (1.42) of [L5]), and the Proposition above in particular proves formulas (2.25), (2.26) of [L5].

Once Proposition [7.1] is established, we can use the general theory of twisted modules for vertex operator algebras in order to easily extend some results in the untwisted setting to the twisted setting. This is the way in which we chose to prove, next, the commutator formula for the formal series $\bar{L}^{\nu_1,\nu_2}(x)$ acting on $S[\nu]$, Proposition [7.2]. The corresponding commutator formula for the untwisted operators $\bar{L}^{(y_1,y_2)}(x)$ was announced in [L4] (a proof was given in [M2]). Below, we give an alternative proof of this untwisted commutator formula (the untwisted case of Proposition [7.2]) by specializing to the untwisted general commutator formula (6.1). Then we will use Proposition [7.1] to extend the proof to the twisted setting.

**Proposition 7.2.** The series $\bar{L}^{\nu_1,\nu_2}(x)$ as defined by (7.13) satisfies the following bracket relation:

$$\left[\bar{L}^{\nu_1,\nu_2}(x_1), \bar{L}^{\nu_3,\nu_4}(x_2)\right] = \frac{1}{2} \frac{\partial}{\partial \nu_1} \bar{L}^{\nu_1,y_1+y_2+y_3,\nu_4}(x_2) \delta \left( \frac{e^{y_1}x_1}{e^{y_3}x_2} \right) + \bar{L}^{\nu_1,y_1+y_2,y_3,\nu_4}(x_2) \delta \left( \frac{e^{y_1}x_1}{e^{y_3}x_2} \right)$$

$$- \frac{1}{2} \frac{\partial}{\partial \nu_2} \bar{L}^{\nu_1,y_1+y_2+y_3,\nu_4}(x_2) \delta \left( \frac{e^{y_2}x_1}{e^{y_3}x_2} \right) + \bar{L}^{\nu_1,y_1+y_2,y_3,\nu_4}(x_2) \delta \left( \frac{e^{y_2}x_1}{e^{y_3}x_2} \right).$$

**Proof.** Let us first prove this formula in the untwisted case $p = 1$. We specialize our general commutator formula (6.1) to the case $p = 1$, with $u_2 = v_2 = \alpha_q(-1)\mathbf{1}$, $u_1 = v_1 = \alpha_{q'}(-1)\mathbf{1}$ and the replacements $y_2 \mapsto y_3 - y_4$, $y_1 \mapsto y_1 - y_2$, $x_2 \mapsto e^{y_3}x_2$, $x_1 \mapsto e^{y_4}x_1$. We sum independently over $q$ and $q'$, from 1 to $d$, and multiply through by a factor of $1/4$. We recall that here $\alpha_q, q = 1, \ldots, d$, form an orthonormal basis for $\mathfrak{h}$. In fact, in order to directly obtain the form of the right–hand side as written in (7.17), it is preferable to use Equation (6.1) instead of Equation (6.4), although they are equivalent. Also, an important formula for our purposes is

$$Y[\alpha(-1)\mathbf{1}, y] = \langle \alpha, \beta \rangle y^{-2} \mathbf{1} + \text{series in nonnegative powers of } y$$

for $\alpha, \beta \in \mathfrak{h}$. Applying this to the first term on the right–hand side of the commutator formula (6.1), we find the term

$$\frac{1}{4} \sum_{q=1}^d \text{Res}_y \left\{ y^{-2} \delta \left( e^{y_2-y_3-y}x_1 \right) \right\}.$$
We use these two properties along with the \( L[-1] \)-derivative property of homogeneous vertex operators \( \Omega[7.1] \) to write the expression above in the form

\[
\frac{1}{2} \sum_{q=1}^{d} \text{Res}_y \left\{ y^{-2} \delta \left( e^{y_2-y_3-y \frac{x_1}{x_2}} \right) \right\}.
\]

\[
\cdot X_S(\alpha_q(-1)\mathbf{1}, y_1 - y_2 + y_3 - y_4 + y|\alpha_q(-1)\mathbf{1}, e^{y_2-y_3+y_4-y x_1}) \right\}.
\]

Using the main delta–function property and the fact that \( \text{Res}_y(y^{-2}f(y-y_2)) = -\frac{\partial}{\partial y_2} f(-y_2) \) for \( f(y) \) a formal series in nonnegative powers of \( y \), the expression above gives the first term in the second parentheses on the right–hand side of \( (7.17) \). Similar arguments applied to the second term on the right–hand side of the commutator formula \( (6.7) \) lead to the second term in the second parentheses on the right–hand side of \( (7.17) \).

For the third term on the right–hand side of the commutator formula \( (6.7) \), one first uses \( (4.21) \) to bring it to the form

\[
\delta \left( \omega^r_p \left( e^{y_1-y_2-y \frac{x_1}{x_2}} \right)^{1/p} \right) X_M(\alpha_q(-1)\mathbf{1}, y_1 - y_2 + y|\alpha_q(-1)\mathbf{1}, e^{y_1-y x_1}.
\]

Then similar arguments as above applied to this expression lead to the first term in the first parentheses on the right–hand side of \( (7.17) \). In a similar way, the last term on the right–hand side of the commutator formula \( (7.17) \) gives the second term in the first parentheses on the right–hand side of \( (7.17) \).

The same arguments could be used to prove the formula \( (7.17) \) in the twisted case. Instead, we will use a general and simple argument from the theory of vertex operator algebras. First, it is easy to see that the formal series \( \sum_{q=1}^{d} \alpha_q(-1)\mathbf{1}, y_1 - y_2 + y \) with coefficients in \( S \), appearing in the vector argument of the homogeneous twisted vertex operator \( X_{S[\nu]}(\cdot, x) \) in \( (7.17) \), is invariant under the automorphism \( \nu \). Hence by the properties of twisted modules for vertex operator algebras, the space \( S[\nu] \) is a true module for the algebra satisfied by the particular elements of \( S \) generated in \( y_1 \) and \( y_2 \) by the formal series \( \sum_{q=1}^{d} \alpha_q(-1)\mathbf{1}, y_1 - y_2 + y \). The homogeneous twisted vertex operator \( X_{S[\nu]}(\cdot, x) \) gives actions of these elements on \( S[\nu] \), which are then in agreement with their actions on \( S \) itself for all twisting automorphisms. Hence the form of the bracket relations for the formal series \( L^{\nu y_1,y_2}(x) \) acting on \( S[\nu] \) is independent of the twisting automorphism \( \nu \), which, combined with the proof of \( (7.17) \) in the untwisted case above, proves \( (7.17) \) in the twisted case.

Our twisted construction of \( \hat{D}^+ \) is then a simple consequence of the fact that the form of the previous commutator formula is independent of the twisting automorphism \( \nu \), which, as emphasized in the proof above, is due to Proposition \( (7.17) \) and to aspects of the general theory of twisted modules for vertex operator algebras. Moreover, as we will see, Proposition \( (7.17) \) immediately leads to the main properties of the generators \( L_n^{(\nu)} \) (see Section \( 2 \)) of the algebra \( \hat{D}^+ \), in particular to the monomial central term in the bracket relations.
The formal series \( \bar{L}^{\nu,y_1,y_2}(x) \) \( \text{(7.13)} \) generates a representation of the Lie algebra \( \hat{D}^+ \) on the \( h[\nu] \)-module \( S[\nu] \), generalizing the untwisted case studied in [12]. In fact, combined with what we said in Remark 4.1, Proposition 7.1 suggests that \( \bar{L}^{\nu,y_1,y_2}(x) \) should be a generating function for modes that generalize the Virasoro modes in “cylindrical coordinates” as defined after equation (1.54). More precisely, let

\[
L^{\nu,y_1,y_2}(x) = \sum_{n \in \mathbb{Z}, r_1, r_2 \in \mathbb{N}} L^{\nu,r_1,r_2}(n)x^{-n}\frac{y_1^{r_1}y_2^{r_2}}{r_1!r_2!}, \tag{7.18}
\]

\[
\bar{L}^{\nu,y_1,y_2}(x) = \frac{1}{2}d(y_1 - y_2)^2 + \sum_{n \in \mathbb{Z}, r_1, r_2 \in \mathbb{N}} \bar{L}^{\nu,r_1,r_2}(n)x^{-n}\frac{y_1^{r_1}y_2^{r_2}}{r_1!r_2!}. \tag{7.19}
\]

Then the following holds (recall the generators \( \hat{D}^+ \) of \( \hat{D}^+ \)):

**Theorem 7.3.** Let

\[
L^{\nu,r}(n) = L^{\nu,r,r}(n) \quad (n \in \mathbb{Z}, r \in \mathbb{N}),
\]

\[
\bar{L}^{\nu,r}(n) = \bar{L}^{\nu,r,r}(n) \quad (n \in \mathbb{Z}, r \in \mathbb{N}).
\]

(a) The assignment

\[
L^{(r)}_n \mapsto L^{\nu,r}(n), \quad c \mapsto d,
\]

defines a representation of the Lie algebra \( \hat{D}^+ \) on \( S[\nu] \).

(b) The assignment

\[
\bar{L}^{(r)}_n \mapsto \bar{L}^{\nu,r}(n), \quad c \mapsto d
\]

also defines a representation of the Lie algebra \( \hat{D}^+ \), with the central term being a pure monomial, as in \( \text{(2.4)} \).

**Proof.** We first prove assertion (a). The formal series \( L^{\nu,y_1,y_2}(x) \) defined by \( \text{(7.12)} \) specializes, in the untwisted case \( p = 1 \), to the following series acting on \( S \) (see also Remark 7.2):

\[
L^{(y_1,y_2)}(x) = \frac{1}{2}\sum_{q=1}^{d} \alpha_q(e^{y_1}x)\alpha_q(e^{y_2}x),
\]

where \( \alpha(x) = x\alpha(x) \) for \( \alpha \in h \). It is easy to see that an expansion in \( y_1 \) and \( y_2 \) as above:

\[
L^{(y_1,y_2)}(x) = \sum_{n \in \mathbb{Z}, r_1, r_2 \in \mathbb{N}} L^{(r_1,r_2)}(n)x^{-n}\frac{y_1^{r_1}y_2^{r_2}}{r_1!r_2!},
\]

leads to operators \( L^{(r)}_n = L^{(r,r)}(n) \) that represent on \( S \) generators \( L^{(r)}_n \) of the algebra \( \hat{D}^+ \), as in \( \text{(2.6)} \).
The other operators $L^{(r_1,r_2)}(n)$, $r_1 \neq r_2$ are linear combinations of the operators $L^{(r)}(n)$. Indeed, they can be written

$$L^{(r_1,r_2)}(n) = \frac{1}{2} \sum_{q=1}^{d} \sum_{j \in \mathbb{Z}} j^{r_1}(n-j)^{r_2} \cdot \alpha_q(j) \cdot \alpha_q(n-j) \quad (n \in \mathbb{Z}, r_1, r_2 \in \mathbb{N}).$$

(7.20)

To see that these are linear combinations of $L^{(r)}$, note that because the normal ordering is symmetric, the coefficients of the operators $\cdot \alpha_q(j) \cdot \alpha_q(n-j) \cdot$ can be symmetrized:

$$L^{(r_1,r_2)}(n) = \frac{1}{2} \sum_{q=1}^{d} \sum_{j \in \mathbb{Z}} (j^{r_1}(n-j)^{r_2} + (n-j)^{r_1}(j)^{r_2}) \cdot \alpha_q(j) \cdot \alpha_q(n-j) \cdot .$$

In general, any symmetric polynomial in two variables $z_1, z_2$ can be written in a unique way as a polynomial in $z_1 + z_2$ and $z_1 z_2$. Hence, the symmetric polynomial

$$\frac{1}{2} \left( j^{r_1}(n-j)^{r_2} + (n-j)^{r_1}(j)^{r_2} \right)$$

in $j, n-j$ can be written in a unique way as a polynomial in $n$ and $j(n-j)$. The coefficient (which is itself a polynomial in $n$) of $(j(n-j))^r$ in this polynomial is the coefficient of $L^{(r)}(n)$ in the linear combination representing $L^{(r_1,r_2)}(n)$. That is, denoting this coefficient by $C^{(r_1,r_2)}_n$, we have

$$L^{(r_1,r_2)}(n) = \sum_{0 \leq r \leq (r_1+r_2)/2} C^{(r_1,r_2)}_n L^{(r)}(n).$$

(7.21)

In fact, the operators $L^{(r,r)}(n)$ for all $r \in \mathbb{N}$, $n \in \mathbb{Z}$ form a basis in the linear space spanned by $L^{(r_1,r_2)}(n)$ for all $r_1, r_2 \in \mathbb{N}$, $n \in \mathbb{Z}$. We will not need the explicit form of these coefficients, except for $C^{(r_1,r_2)}_0$, which can be found easily:

$$C^{(r_1,r_2)}_0 = (-1)^{r_1} \delta_{r_1+r_2,2r}.$$  

(7.22)

In the twisted case, the operators $L^{(r_1,r_2)}(n)$ are also linear combinations of operators $L^{(r,r)}(n)$, with the same coefficients $C^{(r_1,r_2)}_n$. This is trivial for $n \neq 0$. For $n = 0$, we only need to verify that the relation (7.21) is still satisfied if one replaces the operators $L^{(r_1,r_2)}(0)$ and $L^{(r)}(0) = L^{(r)}(0)$ by the coefficients of $y_1^{r_1} y_2^{r_2} / (r_1! r_2 !)$ and of $(y_1 y_2)^r / (r! r!)$, respectively, in the following formal power series in $y_1 - y_2$:

$$f(y_1 - y_2) = -\frac{1}{2} \frac{\partial}{\partial y_1} \left( \sum_{k=0}^{p-1} \frac{e^{(-y_1+y_2)/p} - 1}{1 - e^{-y_1+y_2}} \right),$$

which appears in the definition of $L^{(r_1,r_2)}(x)$. This formal power series is even under change of sign of its argument: $f(y) = f(-y)$. It is easy to see that
this indeed gives
\[ f(y_1 - y_2) = \sum_{r_1, r_2 \geq 0} f^{(r_1, r_2)} \frac{y_1^{r_1} y_2^{r_2}}{r_1! r_2!} \]  
(7.23)
where the coefficients \( f^{(r_1, r_2)} \) are of the form
\[ f^{(r_1, r_2)} = \sum_{r \geq 0} c^{(r_1, r_2)} f^{(r)} \]  
(7.24)
with \((7.22)\).

Hence, the operators \( L^{\nu, r_1, r_2}(n) \) are linear combinations of operators \( L^{\nu, r}(n) \) with the same coefficients \( C_n^{(r_1, r_2)} \) as those appearing in \((7.21)\), the form of the bracket relations \((7.21)\) is independent of the twisting automorphism, and the difference \( \frac{d}{dx} \) between the formal series \( L^{\nu, y_1, y_2}(x) \) and \( L^{\nu, y_1, y_2}(x) \) is also independent of the twisting automorphism. This implies that the operators \( L^{\nu, r}(n) \) satisfy, for all twisting automorphisms \( \nu \), the same bracket relations as do the operators \( L^{(r)}(n) \) that generate the algebra \( \hat{D}^+ \). This proves assertion (a).

In order to prove assertion (b), we only need to prove that the central term in the commutator of generators \( L^{\nu, r}(n) \) is a monomial. This can be seen from the bracket formula \((7.17)\). The source of the central term in this bracket formula is the term \( \frac{1}{2} \frac{d}{dy_2} \) in \((7.13)\). As expected, it appears on the right–hand side of \((7.17)\) only when the power of \( x_1 \) is the same as that of \( x_2^{-1} \). On both sides of this formula, take then the term in \( (x_1/x_2)^m \) for some fixed \( m \in \mathbb{Z} \), and fix the powers of \( y_1, y_2, y_3 \) and \( y_4 \). This selects, on the left–hand side of \((7.17)\), two elements of the form \( L^{\nu, r_1, r_2}(0) \) and \( L^{\nu, r_3, r_4}(0) \) of which we take the bracket. On the right–hand side, in the first term, for instance, the part of \( L^{\nu, y_1, y_2+y_3, y_4}(0) \) relevant to the calculation of the central term, from the formula \((7.14)\), is the formal series \( \frac{1}{2} \frac{d}{dy_2} \) of the form \( f^{(r_1, r_2)}(0) \), which is a sum of terms in which the sum of the powers of \( y_1, y_2, y_3 \) and \( y_4 \) is fixed to -2. In \((7.14)\), this formal series is multiplied by the formal delta–function \( \delta(\epsilon y_1 - y_3, x_1/x_2) \), which gives a sum of contributions of the form \( m^k(y_1 - y_3)^k \) for all nonnegative values of \( k \). In a contribution \( (y_1 - y_3)^k \), the sum of the powers of \( y_1 \) and of \( y_3 \) is \( k \). But since the powers of \( y_1, y_2, y_3 \) and \( y_4 \) are fixed in \((7.17)\), the value of \( k \) is also fixed. Specifically, in the bracket of \( L^{\nu, r_1, r_2}(0) \) with \( L^{\nu, r_3, r_4}(0) \), it is given by \( k = r_1 + r_2 + r_3 + r_4 + 3 \). The other terms on the right–hand side of \((7.17)\) lead to the same value of \( k \). Hence, only one power of \( m \) appears; this proves assertion (b).

\[ \textbf{Remark 7.4.} \text{Equivalently, it is possible to represent the algebra } \hat{D}^+ \text{ using the operators } L^{(r, 0)}(n) \text{ instead of the operators } L^{(r, r)}(n). \text{ Indeed, expanding } j^{(r_1, r_2)}(n - j)^{r_2} \text{ in a polynomial in } j \text{ and } n \text{ in } (7.24), \text{ one can write any operator } L^{(r_1, r_2)}(n) \text{ as a linear combination of } L^{(r, 0)}(n). \text{ From these linear combinations, one can see that the operators } L^{(r, 0)}(n) \text{ for all } r \in \mathbb{N}, \ n \in \mathbb{Z} \text{ form a basis in the linear space spanned by } L^{(r_1, r_2)}(n) \text{ for all } r_1, r_2 \in \mathbb{N}, \ n \in \mathbb{Z}. \]
Explicit expressions for the operators $L^{\nu;r}(n)$ and $\bar{L}^{\nu;r}(n)$, involving Bernoulli polynomials, are easy to obtain from (7.12) and (7.13):

\[
L^{\nu;r}(n) = \frac{1}{2} \sum_{q=1}^{d} \sum_{j \in \mathbb{Z}} j^r (n-j)^r \alpha_q^{\nu}(j) \alpha_q^{\nu}(n-j) \cdot \nonumber \\
-\delta_{n,0} \frac{(-1)^r}{4(r+1)} \sum_{k=0}^{p-1} \dim h(k) (B_{2(r+1)}(k/p) - B_{2(r+1)}) \tag{7.25}
\]

and

\[
\bar{L}^{\nu;r}(n) = \frac{1}{2} \sum_{q=1}^{d} \sum_{j \in \mathbb{Z}} j^r (n-j)^r \alpha_q^{\nu}(j) \alpha_q^{\nu}(n-j) \cdot \nonumber \\
-\delta_{n,0} \frac{(-1)^r}{4(r+1)} \sum_{k=0}^{p-1} \dim h(k) B_{2(r+1)}(k/p). \tag{7.26}
\]

From our construction, the appearance of Bernoulli polynomials is seen to be directly related to general properties of homogeneous twisted vertex operators.

The next result is a simple consequence of Theorem 7.3. It describes the action of the “Cartan subalgebra” of $\hat{D}^+$ on a highest weight vector of a canonical quasi-finite $\hat{D}^+$–module; here we are using the terminology of [KR]. This corollary gives the “correction” terms referred to in the introduction.

**Corollary 7.4.** Given a highest weight $\hat{D}^+$–module $W$, let $\delta$ be the linear functional on the “Cartan subalgebra” of $\hat{D}^+$ (spanned by $L_0^{(k)}$ for $k \in \mathbb{N}$) defined by

\[
L_0^{(k)} \cdot w = (-1)^k \delta(L_0^{(k)}) w, \nonumber
\]

where $w$ is a generating highest weight vector of $W$, and let $\Delta(x)$ be the generating function

\[
\Delta(x) = \sum_{k \geq 1} \frac{\delta(L_0^{(k)}) x^{2k}}{(2k)!} \nonumber
\]

(cf. [KR]). Then for every automorphism $\nu$ of period $p$ as above,

\[
\mathcal{U}(\hat{D}^+) \cdot 1 \subset S[\nu] \nonumber
\]

is a quasi–finite highest weight $\hat{D}^+$–module satisfying

\[
\Delta(x) = \frac{1}{2} \frac{d}{dx} \sum_{k=0}^{p-1} \frac{(e^{\nu x} - 1) \dim h(k)}{1 - e^x}. \tag{7.27}
\]

**Proof.** Clearly $\mathcal{U}(\hat{D}^+) \cdot 1 \subset S[\nu]$ is a module for $\hat{D}^+$, submodule of $S[\nu]$. Its highest weight vector is 1, and the action of the algebra element $L_0^{(k)}$ on this
vector is given by the operator $L^{\nu;k,k}(0)$ in the formal series (7.12) defined by (7.14). This action comes entirely from the term proportional to the identity operator in (7.12), which immediately gives (7.26).

Finally, we have an additional result (for the untwisted bosonic case an equivalent result was obtained in [Bl] and for the spinor construction in [M2]):

**Corollary 7.5.** The generating function

$$X_S[\nu](\sum_{q=1}^d \alpha_q(-m-1)\alpha_q(-m-1)1, x),$$  \hspace{1cm} (7.28)

$m \in \mathbb{N}$, defines the same $\hat{D}^+$-module as in Theorem 7.3. That is, every operator $L^{(r)}(n)$ (or equivalently $L^{(r)}(n)$) can be expressed as a linear combination of the expansion coefficients of the operator (7.28), and vice versa.

**Proof.** Clearly, we can replace $X$–operators (7.28) by

$$Y_S[\nu](\sum_{q=1}^d \alpha_q(-m-1)\alpha_q(-m-1)1, x).$$

From [DL2] we have

$$Y_S[\nu](\alpha_q(-m-1)\alpha_q(-m-1)1, x) = W(e^{\Delta_x} \alpha_q(-m-1)\alpha_q(-m-1)1, x),$$

where $\Delta_x$ is given by formula (4.42) in [DL2] and

$$W(\alpha_q(-m-1)\alpha_q(-m-1)1, x) = \left( \frac{1}{m!} \left( \frac{d}{dx} \right)^m \alpha_q(x) \right) \left( \frac{1}{m!} \left( \frac{d}{dx} \right)^m \alpha_q(x) \right).$$

In addition it is not hard to see that

$$W(e^{\Delta_x} \alpha_q(-m-1)\alpha_q(-m-1)1, x) = W(\alpha_q(-m-1)\alpha_q(-m-1)1, x) + x^{-2m-2}f(q, m),$$ \hspace{1cm} (7.29)

where $f(q, m) \in \mathbb{C}$. Consider

$$\sum_{q=1}^d W(\alpha_q(-m-1)\alpha_q(-m-1)1, x).$$ \hspace{1cm} (7.30)

As in the untwisted case [M2], it follows that the space spanned by the expansion coefficients of (7.30) defines the same space of operators as the space spanned by the expansion coefficients of

$$\sum_{q=1}^d \alpha_q^{\nu}(e^{y_1}x)\alpha_q^{\nu}(e^{y_2}x).$$ \hspace{1cm} (7.31)
In other words every generator of $\hat{D}^+$ of degree $\neq 0$ (described in Theorem 7.3) is a linear combination of the Fourier coefficients in (7.28) and vice versa. In addition, because of the twisted Jacobi identity (4.13), the operators of the form (7.28) are closed with respect to the commutator; therefore they generate a Lie algebra. But the generators of nonzero degree uniquely determine the action of the whole Lie algebra $\hat{D}^+$ (this fact follows by induction). The result follows.

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