ON THE EQUAL SURPLUS SHARING INTERVAL SOLUTIONS
AND AN APPLICATION

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ABSTRACT. In this paper, we focus on the equal surplus sharing interval solutions for cooperative games, where the set of players are finite and the coalition values are interval numbers. We consider the properties of a class of equal surplus sharing interval solutions consisting of all convex combinations of them. Moreover, an application based on transportation interval situations is given. Finally, we propose three solution concepts, namely the interval Shapley value, ICIS-value and IENSC-value, for this application and these solution concepts are compared.

1. Introduction. Interval uncertainty affects our decision making activities on a daily basis making the data structure of intervals of real numbers more and more popular in theoretical models and related software applications. There are many real-life situations where people or businesses face interval uncertainty in decision making regarding cooperation. A suitable game theoretic model and some real-life situations to support decision making under interval uncertainty of coalition values is that of cooperative interval games. Hence, interval solution concepts for cooperative interval games are a useful tool to settle cooperation within the grand coalition via such binding contracts. Cooperative interval games are a useful tool for modeling various economic and Operations Research (OR) situations, where payoffs for people or businesses are affected by interval uncertainty. In literature, cooperative games modeled by interval uncertainty can be found in [1, 2, 3]. We mention here minimal spanning tree networks, management applications such as funds’ allocation of firms among their divisions and cost allocation and/or surplus sharing in joint projects, sequencing situations, conflict resolution and bankruptcy

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situations, assignment of taxes, when there is interval uncertainty regarding the homogeneous good at stake.

There are many interval solution concepts for cooperative interval games in characteristic function form. The one important of these solution concepts is the interval Shapley value [16]. The existence of the relationship between these interval solution concepts is an important issue for the theory of cooperative interval games as well as the application of them to real-life problems. In the context of interval game theoretic applications, the one-point interval solution concepts are widely used [1, 2, 3]. Cooperative interval games have important real-world applications. It has been used in security applications, airport security, cybersecurity. In many cases, an interval model is a more natural and effective way to represent the game [11].

This paper focuses on a uniform treatment of a special type of one-point solutions for cooperative games, called the equal surplus sharing solutions which are the Centre-of-gravity of the Imputation-Set value, shortly denoted by CIS-value, Egalitarian Non-Separable Contribution value, shortly denoted by ENSC-value and the equal division solution, shortly denoted by ED-solution [17]. Our main objective in this paper is to extend these solutions by using interval uncertainty. The CIS-value is the individual worth of the player to participate in the game as a solitary player. The ENSC-value is based on the separable contributions which are one of the various marginal contributions. The ENCIS-value is the so-called separable (or marginal) contribution of the player to participate in the game as a member of the player set. These values are well known concepts in the literature of game theory [6, 7, 8, 9, 10, 12, 13]. The egalitarian division of the surplus of the overall profits gives rise to three one-point solution concepts of the same kind representing the CIS-value, the ENSC-value and the ED-solution.

The cooperative game theory is widely used on interesting sharing cost/profit problems in many areas of OR such as transportation, connection etc. (see [4] for a survey on OR Games). Transportation situations are examined in [15]. Cooperative interval games are a useful tool for modeling various economic and OR situations where payoffs for people or businesses are affected by interval uncertainty [1, 2, 3]. For example, transportation situations with interval uncertainty are modeled by [14].

In this paper, we introduce the equal surplus sharing interval solutions by using interval calculus. These solutions are defined on the special subclass of cooperative interval games. Since transportation interval games are defined in this class, we apply these solutions to transportation interval situations.

This paper is organized as follows. In Section 2, we recall basic notions and facts from the theory of cooperative interval games and transportation interval situations. In Section 3, we introduce a class for interval game-theoretical solutions. Section 4 examines a class of equal surplus sharing solutions consisting of all convex combinations of these interval solutions. Finally an application on transportation situations with interval data is given in Section 5.

2. Preliminaries. In this section some solution concepts from cooperative game theory and some preliminaries from interval calculus used in the whole paper are given [2, 5, 17].

In classical cooperative game theory payoffs to coalitions of players are known with certainty. A classical cooperative game is a pair < N, v > where N =
\{1,2,\ldots,n\} is the set of players and \(v : 2^N \to I(\mathbb{R})\) is a map, assigning to each coalition \(S \in 2^N\) a real number, such that \(v(\emptyset) = 0\). We denote by \(G^N\) the family of all classical cooperative games with player set \(N\). Often, we identify a game \(<N,v>\) with its characteristic function \(v\).

Now, we discuss some interesting solution solution concepts from cooperative game theory. Examples of such solutions are the \textit{CIS-value}, the \textit{ENSC-value}, the \textit{ED-solution} and the \textit{ENCIS-value}.

We give the formal definitions of these values. The \textit{CIS-value} assigns to every player its individual worth, and distributes the remainder of \(v(N)\) equally among all players and defined by

\[
\text{CIS}_i(v) = v(\{i\}) + \frac{1}{|N|}(v(N) - \sum_{j \in N} v(\{j\})) \quad \text{for all } i \in N.
\]

The \textit{ENSC-value} which introduced by [17] assigns to every game \(v\) the \textit{CIS-value} of its dual game and defined by

\[
\text{CIS}_i(v^*) = \text{ENSC}_i(v) = -v(N \setminus \{i\}) + \frac{1}{|N|}(v(N) - \sum_{j \in N} v(N \setminus \{j\})),
\]

for all \(i \in N\). Here \(v^*\) is the dual game of the game \(v \in G^N\) and defined by

\[
v^*(S) = v(N) - v(N \setminus S).
\]

The \textit{ED-solution} just distributes \(v(N)\) equally among all players and defined by

\[
\text{ED}_i(v) = \frac{v(N)}{|N|}
\]

for all \(i \in N\).

The \textit{ENCIS-value} is defined by the convex combination of \textit{CIS-value} and \textit{ENSC-value} [17]. The \textit{ENCIS-value} is defined by for \(\beta \in [0,1]\),

\[
\text{ENCIS}^\beta(v) = \beta \text{CIS}(v) + (1 - \beta) \text{ENSC}(v)
\]  \hspace{1cm} (1)

2.1. \textbf{Cooperative interval game.} In cooperative interval game theory payoffs to coalitions of players are known with uncertainty. A \textit{cooperative interval game in coalitional form} (see [2]) is an ordered pair \(<N,w>\), where \(N = \{1,2,\ldots,n\}\) is the set of players, and \(w : 2^N \to I(\mathbb{R})\) is the characteristic function such that \(w(\emptyset) = [0,0]\), where \(I(\mathbb{R})\) is the set of all nonempty, compact intervals in \(\mathbb{R}\). For each \(S \in 2^N\), the worth set \(w(S)\) of the coalition \(S\) in the interval game \(<N,w>\) is of the form \([w(S),\bar{w}(S)]\). The family of all interval games with player set \(N\) is denoted by \(IG^N\). Similarly, we identify an interval game \(<N,w>\) with its characteristic function \(w\).

Let \(I,J \in I(\mathbb{R})\) with \(I = [I,\bar{I}], J = [J,\bar{J}], |I| = \bar{I} - I\) and \(\alpha \in \mathbb{R}_+.\) Then,

\begin{align*}
(I + J) &= [I,J] + [J,\bar{J}] = [I + J,\bar{J} + J]; \\
(\alpha I) &= [\alpha I,\alpha \bar{I}] = \alpha [I,\bar{I}].
\end{align*}

By (i) and (ii) we see that \(I(\mathbb{R})\) has a cone structure.

In this paper we also need a partial substraction operator. We define \(I - J\), only if \(|I| \geq |J|\), by \(I - J := [\bar{J} - I, \bar{I} - J]\). Let us note that \(I - J \leq \bar{I} - J\).

We call a game \(<N,w>\) size monotonic if \(<N,|w|>\) is monotonic, i.e., \(|w|(S) \leq |w|(T)\) for all \(S,T \in 2^N\) with \(S \subseteq T\). For further use we denote by \(SMIG^N\) the class of \textit{size monotonic interval games} with player set \(N\).
The interval marginal operators and the interval Shapley value were defined on $SMIG^N$ in [2] as follows.

Denote by $\Pi(N)$ the set of permutations $\sigma : N \to N$ of $N = \{1, 2, \ldots, n\}$. The interval marginal operator $m^\sigma : SMIG^N \to I(\mathbb{R})^N$ corresponding to $\sigma$, associates with each $w \in SMIG^N$ the interval marginal vector $m^\sigma(w)$ of $w$ with respect to $\sigma$ defined by $m_i^\sigma(w) = w(P^\sigma(i) \cup \{i\}) - w(P^\sigma(i))$ for each $i \in N$, where $P^\sigma(i) := \{ r \in N | \sigma^{-1}(r) < \sigma^{-1}(i) \}$, and $\sigma^{-1}(i)$ denotes the entrance number of player $i$.

The interval Shapley value is $\Phi : SMIG^N \to I(\mathbb{R})^N$ is defined by

$$\Phi(w) := \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^\sigma(w), \text{ for each } w \in SMIG^N.$$

Further, we use the notation $I(\mathbb{R}_+)$ for the set of all closed nonnegative intervals in $\mathbb{R}$.

In this paper, $n$ tuples of intervals $I = (I_1, \ldots, I_n)$ where $I_i \in I(\mathbb{R})$ for each $i \in N$, will play a key role. For further use we denote by $I(\mathbb{R})^N$ the set of all $n$ dimensional vectors whose components are elements in $I(\mathbb{R})$. In the sequel if $I_i = [L_i, T_i]$ is the interval payoff of player $i$, then $I = (I_1, \ldots, I_n)$ is an interval payoff vector for the players of the game constructed.

Finally, we recall the properties of interval games.

**Definition 2.1.** An interval game $w \in IG^N$ is called interval zero-normalized if for all $i \in N$ we have $w(i) = [0, 0]$.

**Definition 2.2.** An interval game $w \in IG^N$ is said to be superadditive if for all $S, T \subset N$ with $S \cap T = \emptyset$ the following two conditions hold:

1) $w(S \cup T) \geq w(S) + w(T)$,
2) $|w|(S \cup T) \geq |w|(S) + |w|(T)$

**Definition 2.3.** We call a game $w \in IG^N$ convex if

$$w(S) + w(T) \leq w(S \cup T) + w(S \cap T)$$

and $|w|(S) + |w|(T) \leq |w|(S \cup T) + |w|(S \cap T)$ for all $S, T \in 2^N$.

3. The interval game-theoretic solutions. In this section, we introduce some game-theoretic solutions by using interval calculus which are inspired by [17].

The interval CIS-value (ICIS-value) assigns every player to its individual interval worth, and distributes the remainder of the interval worth of the grand coalition $N$ equally among all players.

The ICIS-value is defined by

$$ICIS : SMIG^N \to I(\mathbb{R})^N$$

$$ICIS_i(w) = w(\{i\}) + \frac{1}{|N|} (w(N) - \sum_{j \in N} w(\{j\}))$$

such that

$$|w(N)| \geq \sum_{j \in N} |w(\{j\})|$$

for all $i \in N$ and for all $w \in SMIG^N$. 
Example 3.1. Let \( w \in SMIG^N \) and \( N = \{1, 2, 3\} \). The coalitional values are as follows:

| \( S \) | \( \emptyset \) | \( \{1\} \) | \( \{2\} \) | \( \{3\} \) | \( \{1, 2\} \) | \( \{1, 3\} \) | \( \{2, 3\} \) | \( N \) |
|---|---|---|---|---|---|---|---|---|
| \( w(S) \) | \( [0, 0] \) | \( [1, 2] \) | \( [1, 2] \) | \( [0, 0] \) | \( [4, 7] \) | \( [1, 4] \) | \( [3, 5] \) | \( [5, 9] \) |

The ICIS-value of the game is illustrated as

\[
ICIS_1(w) = w(1) + \frac{1}{3}(w(N) - (w(1) + w(2) + w(3)))
\]

\[
= [1, 2] + \frac{1}{3}([5, 9] - [2, 4])
\]

\[
= [1, 2] + \frac{1}{3}[3, 5]
\]

\[
= [2, \frac{3}{2}]
\]

\[
ICIS_2(w) = w(2) + \frac{1}{3}(w(N) - (w(1) + w(2) + w(3)))
\]

\[
= [1, 2] + \frac{1}{3}[3, 5]
\]

\[
= [2, \frac{3}{2}]
\]

\[
ICIS_3(w) = w(3) + \frac{1}{3}(w(N) - (w(1) + w(2) + w(3)))
\]

\[
= [0, 0] + \frac{1}{3}[3, 5]
\]

\[
= [1, \frac{2}{3}]
\]

Then, the ICIS-value is

\[
ICSI(w) = ([2, \frac{3}{2}], [2, \frac{3}{2}], [1, \frac{2}{3}]).
\]

The dual \( w^* \in SMIG^N \) of the interval game \( w \) is the game that assigns to each coalition \( S \subseteq N \) the interval worth that is lost by the grand coalition \( N \) if coalition \( S \) leaves \( N \), i.e.

\[
w^*(S) = w(N) - w(N \setminus S) \text{ for all } S \subseteq N.
\]

The interval ENSC-value (IENSC-value) assigns to every game \( w \) the ICIS-value of its dual game, i.e.

\[
IENSC : SMIG^N \rightarrow I(\mathbb{R})^N
\]

\[
IENSC_i(w) = ICIS_i(w^*)
\]

\[
= \frac{1}{|N|}(w(N) + \sum_{j \in N} w(N \setminus \{j\})) - w(N \setminus \{i\})
\]

such that

\[
\left| w(N) + \sum_{j \in N} w(N \setminus \{j\}) \right| \geq |N||w(N \setminus \{i\})|
\]

for all \( i \in N \) and for all \( w \in SMIG^N \).

Thus, the IENSC-value assigns to every player in a game its interval marginal contribution to the “grand coalition” and distributes the remainder equally among the players.

Example 3.2. \( N = \{1, 2, 3\} \) is the set of players and the coalitional values are:

| \( S \) | \( \emptyset \) | \( \{1\} \) | \( \{2\} \) | \( \{3\} \) | \( \{1, 2\} \) | \( \{1, 3\} \) | \( \{2, 3\} \) | \( N \) |
|---|---|---|---|---|---|---|---|---|
| \( w(S) \) | \( [0, 0] \) | \( [1, 2] \) | \( [1, 2] \) | \( [0, 0] \) | \( [4, 7] \) | \( [1, 4] \) | \( [3, 5] \) | \( [5, 9] \) |
We calculate the \textit{IENSC-value} of this game as follows:

\[
\text{IENSC}_1(w) = \frac{1}{3} (w(N) + w(23) + w(12) + w(13)) - w(23) \\
= \frac{1}{3}([13, 25]) - [3, 5] \\
= [1 \frac{1}{3}, 3 \frac{1}{3}],
\]

\[
\text{IENSC}_2(w) = \frac{1}{3} (w(N) + w(23) + w(12) + w(13)) - w(13) \\
= \frac{1}{3}([13, 25]) - [1, 4] \\
= [3 \frac{1}{3}, 4 \frac{1}{3}],
\]

\[
\text{IENSC}_3(w) = \frac{1}{3} (w(N) + w(23) + w(12) + w(13)) - w(12) \\
= \frac{1}{3}([13, 25]) - [4, 7] \\
= [\frac{1}{3}, 1 \frac{1}{3}].
\]

Then, the \textit{IENSC-value} is given

\[
\text{IENSC}(w) = ([1 \frac{1}{3}, 3 \frac{1}{3}], [3 \frac{1}{3}, 4 \frac{1}{3}], [\frac{1}{3}, 1 \frac{1}{3}]).
\]

The \textit{IENCIS-value} is defined for \( \beta \in [0, 1] \) as follows:

\[
\text{IENCIS}_{SMIG}^N : I(\mathbb{R})^N \\
\text{IENCIS}_{\beta}^N (w) = \beta \text{ICIS}(w) + (1 - \beta) \text{IENSC}(w)
\]

such that

\[
|w|(i) = |w|(N \setminus \{i\}) = |w|(j)
\]

for all \( i, j \in N \) with \( i \neq j \).

Finally, the \textit{interval ED-solution (IED-solution)} is given by

\[
\text{IED} : IG^N \to I(\mathbb{R})^N \\
\text{IED}_i(w) = \frac{w(N)}{|N|} \text{ for all } i \in N.
\]

\textbf{Example 3.3.} Let \( w \in IG^N \) and \( N = \{1, 2, 3\} \). The coalitional values are as follows:

| \( S \) | \( \emptyset \) | \{1\} | \{2\} | \{3\} | \{1, 2\} | \{1, 3\} | \{2, 3\} | \{3\} | \{5, 9\} |
|-------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| \( w(S) \) | [0, 0] | [1, 2] | [1, 2] | [0, 0] | [4, 7] | [1, 4] | [3, 5] | [5, 9] | [5, 9] |

The \textit{IED-solution} of the game can be found as

\[
\text{IED}_1(w) = \text{IED}_2(w) = \text{IED}_3(w) = \frac{w(N)}{3} = \frac{[5, 9]}{3} = [1 \frac{2}{3}, 3]
\]

Then, the \textit{IED-solution} is obtained by

\[
\text{IED}(w) = ([1 \frac{2}{3}, 3], [1 \frac{2}{3}, 3], [1 \frac{2}{3}, 3]).
\]

\textbf{Remark 3.4.} We note that the \textit{interval Shapley value, ICIS-value} and the \textit{IENSC-value} are defined in \( SMIG^N \), but the \textit{IED-solution} is defined in \( IG^N \).
4. A class of equal surplus sharing interval solutions. In this paper, we discuss the class of interval solutions that consists of all convex combinations of the IED-solution, the ICIS-value and the IENSC-value, i.e., for $\alpha, \beta \in [0, 1]$, we consider interval solutions $I_{\phi^{\alpha,\beta}}$ given by

$$I_{\phi^{\alpha,\beta}}(w) = \alpha I_{E^{\alpha}}(w) + (1 - \alpha) I_{E^{\beta}}(w),$$

where $I_{E^{\alpha}}(w)$ is given by (1). We denote the class of all interval solutions that are obtained in this way by $I\Phi := \{I_{\phi^{\alpha,\beta}} : \alpha, \beta \in [0, 1]\}$. Clearly, the interesting solutions in this class are the ICIS-value, which is obtained by taking $\alpha = \beta = 1$ (i.e. $ICIS(w) = \phi^{1,1}(w)$), the IENSC-value, which is obtained by taking $\alpha = 1$, $\beta = 0$ (i.e. $IENSC(w) = \phi^{1,0}(w)$) and the IED-solution, which is obtained by taking $\alpha = 0$ (i.e. $IED(w) = I_{\phi^{0,\beta}}$, $\beta \in [0, 1]$). We thus can write $I_{\phi^{\alpha,\beta}}$ as

$$I_{\phi^{\alpha,\beta}}(w) = \alpha I_{\phi^{1,\beta}}(w) + (1 - w) I_{\phi^{0,1}}(v)$$

$$= \alpha \beta I_{\phi^{1,1}}(w) + \alpha (1 - \beta) I_{\phi^{1,0}}(w) + (1 - \alpha) I_{\phi^{0,1}}(w)$$

for $\alpha, \beta \in [0, 1]$.

Next, we provide an expression of the solutions $I_{\phi^{\alpha,\beta}}$ showing that they have some egalitarian flavour in the sense that they give each player $i$ in an interval game $w$ some value $\lambda_i^{\alpha,\beta}(w)$, and the remainder of $w(N)$ is equally split among all players.

**Proposition 4.1.** For every $w \in SMIG^N$ and $\alpha, \beta \in [0, 1]$ it holds that

$$I_{\phi_i^{\alpha,\beta}}(w) = \lambda_i^{\alpha,\beta}(w) + \frac{1}{|N|}(w(N) - \sum_{j \in N} \lambda_j^{\alpha,\beta}(w)),$$

where $\lambda_i^{\alpha,\beta}(w) = \alpha (\beta w(\{i\}) - (1 - \beta) w(N \setminus \{i\}))$ for $i \in N$ such that $|w|(i) = |w|(N \setminus \{i\}) = |w|(j)$ for all $i, j \in N$ with $i \neq j$.

**Proof.** In this proof, we inspired by [17]. For $w \in SMIG^N$ and $\alpha, \beta \in [0, 1]$ we have

$$I_{\phi_i^{\alpha,\beta}}(w) = \alpha I_{E^{\alpha}}(w) + (1 - \alpha) I_{E^{\beta}}(w)$$

$$= \alpha \left( \beta w(\{i\}) - (1 - \beta) w(N \setminus \{i\}) \right)$$

$$+ \frac{1}{N} \left( w(N) - \sum_{j \in N} (\beta w(j) - (1 - \beta) w(N \setminus \{j\})) \right)$$

$$= \alpha (\beta w(\{i\}) - (1 - \beta) w(N \setminus \{i\}))$$

$$+ \frac{1}{|N|} \left( w(N) - \sum_{j \in N} \alpha (\beta w(j) - (1 - \beta) w(N \setminus \{j\}) \right)$$

$$= \lambda_i^{\alpha,\beta}(w) + \frac{1}{|N|} (w(N) - \sum_{j \in N} \lambda_j^{\alpha,\beta}(w)).$$

□

**Proposition 4.2.** For every $\alpha, \beta \in [0, 1]$ and $w \in SMIG^N$ it holds that $I_{\phi_i^{\alpha,\beta}}(w^*) = I_{\phi_i^{\alpha,\beta}}(w)$. 
Proof. In this proof, we inspired by [17]. For \( w \in SMIG^N \) and \( \alpha, \beta \in [0, 1] \) we have

\[
I_{\phi_{i}^{\alpha,\beta}} (w^*) = \chi_{i}^{\alpha,\beta} (w^*) + \frac{1}{|N|} (w^* (N) - \sum_{j \in N} \chi_{j}^{\alpha,\beta} (w^*))
\]

\[
= \alpha (\beta w^* (\{i\}) - (1 - \beta) w^* (N \setminus \{i\}))
\]

\[
+ \frac{1}{|N|} \left( w^* (N) - \sum_{j \in N} \alpha (\beta w^* (j) - (1 - \beta) w^* (N \setminus \{j\})) \right)
\]

\[
= \alpha (\beta w (N)) - \beta w (N \setminus \{i\}) - (1 - \beta) w (N) + (1 - \beta) w (\{i\})
\]

\[
+ \frac{1}{|N|} \left( w (N) - \sum_{j \in N} \alpha (\beta w (N) - \beta w (N \setminus \{j\})) \right)
\]

\[
+ \frac{(1 - \beta)}{|N|} (w (j) - w (N))
\]

\[
= w (N) (\alpha \beta - \alpha (1 - \beta) + \frac{1}{|N|} - \frac{|N| \alpha \beta}{|N|} + \frac{|N| \alpha (1 - \beta)}{|N|})
\]

\[
+ \alpha ((1 - \beta) w (i) - \beta w (N \setminus \{i\}))
\]

\[
+ \frac{\alpha}{|N|} \sum_{j \in N} (\beta w (N \setminus \{j\}) - (1 - \beta) w (j))
\]

\[
= \frac{1}{|N|} w (N) + \alpha ((1 - \beta) w (\{i\}) - \beta w (N \setminus \{i\}))
\]

\[
- \frac{1}{|N|} \left( \sum_{j \in N} \alpha (1 - \beta) w (\{j\}) - \beta w (N \setminus \{j\}) \right)
\]

\[
= I_{\phi_{i}^{\alpha,1-\beta}} (w).
\]

5. **An application.** In this section some preliminaries from transportation interval situations are given [14].

In a transportation interval situation the set of players is partitioned into two disjoint subsets \( P \) and \( Q \), containing \( n \) and \( m \) players respectively. The members of \( P \) will be called producers, whereas the members of \( Q \) will be the retailers. Each origin player \( i \in P \) has a positive integer interval number of units of a certain indivisible good, \( p'_i \), and each destination player \( j \in Q \) demands a positive integer interval number of units of this good, \( q'_j \). The shipping of one unit from origin player \( i \) to destination player \( j \) produces a nonnegative interval real profit \( b'_{ij} \). \( x_{ij} \) is the integer number of trips taken to transport product from origin \( i \) to destination \( j \). Here, \( p'_i := [p'_i, \overline{p'}_i] \), \( q'_j := [q'_j, \overline{q'}_j] \) and \( b'_{ij} := [b'_{ij}, \overline{b'}_{ij}] \in I (\mathbb{R}) \).

A transportation interval situation is characterized by a 5-tuple \( (P, Q, B', p', q') \), where \( B' \) is the \( n \times m \) matrix of interval profits, \( p' \) is the \( n \)-dimensional vector of available interval units at the origins, and \( q' \) is the \( m \)-dimensional vector of interval demands.
For every transportation interval situation \((P, Q, B', p', q')\) and every coalition \(S \subseteq N := P \cup Q\), with producers \(S_P := S \cap P\) and retailers \(S_Q := S \cap Q\), and assuming that these sets are both non-empty, we can define the maximization problem of the pessimistic scenario by:

\[
\mathcal{T}(S) : \text{maximize } \sum_{i \in S_P} \sum_{j \in S_Q} b'_{ij} x_{ij}
\]

such that

\[
\begin{align*}
\sum_{j \in S_Q} x_{ij} & \leq p'_i, \quad i \in S_P, \\
\sum_{i \in S_P} x_{ij} & \leq q'_j, \quad j \in S_Q, \\
x_{ij} & \geq 0, \quad (i, j) \in S_P \times S_Q,
\end{align*}
\]

and the maximization problem of the optimistic scenario is stated as:

\[
\mathcal{T}(S) : \text{maximize } \sum_{i \in S_P} \sum_{j \in S_Q} b'_{ij} x_{ij}
\]

such that

\[
\begin{align*}
\sum_{j \in S_Q} x_{ij} & \leq p'_i, \quad i \in S_P, \\
\sum_{i \in S_P} x_{ij} & \leq q'_j, \quad j \in S_Q, \\
x_{ij} & \geq 0, \quad (i, j) \in S_P \times S_Q.
\end{align*}
\]

We denote by \(\vartheta(\mathcal{T}(S))\) the optimal interval value of the problem \(\mathcal{T}(S)\). Here, \(\vartheta(\mathcal{T}(S)) = [\vartheta(\mathcal{T}(S)), \vartheta(\mathcal{T}(S))] \in I(\mathbb{R})\) such that \(\vartheta(\mathcal{T}(S))\) is the optimal value of the maximization problem of the pessimistic scenario, \(\vartheta(\mathcal{T}(S))\) is the optimal value of the maximization problem of the optimistic scenario. Then, we can define a cooperative interval game associated with every transportation interval situation \((P, Q, B', p', q')\) in the following way:

- The set of players is \(N = P \cup Q\);
- The characteristic function \(<N, w>\) is given by:

\[
\begin{align*}
\vartheta(S) := \begin{cases} 
[0,0], & \text{if } S = \emptyset \text{ or } S \text{ is contained in } P \text{ or in } Q, \\
[\vartheta(\mathcal{T}(S)), \vartheta(\mathcal{T}(S))], & \text{in any other case},
\end{cases}
\end{align*}
\]

and \(\vartheta(S)\) satisfies the condition

\[
\vartheta(\mathcal{T}(S)) + \vartheta(\mathcal{T}(T)) \geq \vartheta(\mathcal{T}(T)) + \vartheta(\mathcal{T}(S)) \text{ for all } S \subset T.
\]

Now, we present the definition of a transportation interval game.

**Definition 5.1.** A transportation interval game is a cooperative interval game \(w \in SMIG^N\) arising from a transportation situation \((P, Q, B', p', q')\). Often, we identify a transportation interval situation \((P, Q, B', p', q')\) with its associated transportation game \(w\).

**Remark 5.2.** Since transportation interval games are defined on \(SMIG^N\), we choose a transportation interval situation application (see Theorem 4.1. in [14]).

**Remark 5.3.** We note that transportation interval games are superadditive [14].

Now, we apply these values to transportation interval situations.

**Example 5.4.** Consider the 3-person transportation interval situation \((P, Q, B', p', q')\) which has one producer and two retailers:

\[
P = \{1\}, \quad Q = \{2, 3\}, \quad B' = ([3, 5], [5, 6]), \quad p' = [3, 5], \quad q' = ([2, 4], [1, 3]).
\]
Now, we define a transportation interval game associated with a transportation interval situation \((P, Q, B', p', q')\). Here, \(N = \{1, 2, 3\}\) is the set of players and the characteristic functions of the transportation interval game are as follows:

| \(S\) | \(\emptyset\) | \(\{1\}\) | \(\{2\}\) | \(\{3\}\) | \(\{1, 2\}\) | \(\{1, 3\}\) | \(\{2, 3\}\) | \(N\) |
|-------|---------|---------|---------|---------|---------|---------|---------|-----|
| \(w (S)\) | \([0, 0]\) | \([0, 0]\) | \([0, 0]\) | \([0, 0]\) | \([6, 20]\) | \([5, 18]\) | \([0, 0]\) | \([11, 28]\) |

So, we find the transportation interval game \(< N, w >\) corresponding to a transportation interval situation. Now, we want to calculate \(\text{interval Shapley value}, \ ICIS\)-value and \(\text{IENSC-value}\). We calculate the interval \(\text{Shapley value}\) of this game as follows: Then, the interval marginal vectors are given in the Table 1. The set of permutations of \(N\) are

\[
\pi (N) = \left\{ \begin{array}{c}
s_1 = (1, 2, 3), s_2 = (1, 3, 2), s_3 = (2, 1, 3), \\
s_4 = (2, 3, 1), s_5 = (3, 1, 2), s_6 = (3, 2, 1) \end{array} \right\}
\]

Firstly, for \(s_3 = (2, 1, 3)\), we calculate the interval marginal vectors. Then,

\[
m^s_{\sigma_1} (w) = w(12) - w(2) = [6, 20] - [0, 0] = [6, 20],
\]

\[
m^s_{\sigma_2} (w) = w(2) = [0, 0],
\]

\[
m^s_{\sigma_3} (w) = w(N) - w(12) = [11, 28] - [6, 20] = [5, 8].
\]

The others can be calculated similarly, which is shown in Table 1.

**Table 1. Interval marginal vectors.**

| \(\sigma\)           | \(m^s_{\sigma_1} (w)\) | \(m^s_{\sigma_2} (w)\) | \(m^s_{\sigma_3} (w)\) |
|----------------------|------------------------|------------------------|------------------------|
| \(s_1 = (1, 2, 3)\)  | \([0, 0]\)             | \([6, 20]\)            | \([5, 8]\)            |
| \(s_2 = (1, 3, 2)\)  | \([0, 0]\)             | \([6, 10]\)            | \([5, 18]\)           |
| \(s_3 = (2, 1, 3)\)  | \([6, 20]\)            | \([0, 0]\)             | \([5, 8]\)            |
| \(s_4 = (2, 3, 1)\)  | \([11, 28]\)           | \([0, 0]\)             | \([0, 0]\)            |
| \(s_5 = (3, 1, 2)\)  | \([5, 18]\)            | \([6, 10]\)            | \([0, 0]\)            |
| \(s_6 = (3, 2, 1)\)  | \([11, 28]\)           | \([0, 0]\)             | \([0, 0]\)            |

Table 1 illustrates the interval marginal vectors of the cooperative transportation interval game. The average of the six interval marginal vectors is the interval Shapley value of this game, which can be written as:

\[
\Phi(w) = ([\frac{5}{2}, 15\frac{2}{3}], [3, 6\frac{7}{9}], [2\frac{1}{2}, 5\frac{2}{3}]).
\]

We calculate the \(\text{ICIS-value}\) of this game as follows:

\[
\text{ICIS}_1 (w) = w(1) + \frac{1}{3}(w(N) - (w(1) + w(2) + w(3)))
\]

\[
= [0, 0] + \frac{1}{3}([11, 28] - [0, 0])
\]

\[
= [3\frac{2}{3}, 9\frac{1}{3}],
\]

\[
\text{ICIS}_2 (w) = w(2) + \frac{1}{3}(w(N) - (w(1) + w(2) + w(3)))
\]

\[
= [3\frac{2}{3}, 9\frac{1}{3}],
\]

\[
\text{ICIS}_3 (w) = w(3) + \frac{1}{3}(w(N) - (w(1) + w(2) + w(3)))
\]

\[
= [3\frac{2}{3}, 9\frac{1}{3}],
\]

Then, the \(\text{ICIS-value}\) is obtained by

\[
\text{ICSI} (w) = ([3\frac{2}{3}, 9\frac{1}{3}], [3\frac{2}{3}, 9\frac{1}{3}], [3\frac{2}{3}, 9\frac{1}{3}]).
\]
Finally, we calculate the \textit{IENSC-value} of this game as follows:
\[
IENSC_1(w) = -w(23) + \frac{1}{3}(w(N) + w(23) + w(12) + w(13)) \\
= -[0, 0] + \frac{1}{3}([22, 66]) \\
= [\frac{7}{3}, 22],
\]
\[
IENSC_2(w) = -w(13) + \frac{1}{3}(w(N) + w(23) + w(12) + w(13)) \\
= -[5, 18] + \frac{1}{3}([22, 66]) \\
= [\frac{2}{3}, 4],
\]
\[
IENSC_3(w) = -w(12) + \frac{1}{3}(w(N) + w(23) + w(12) + w(13)) \\
= -[6, 20] + \frac{1}{3}([22, 66]) \\
= [\frac{1}{3}, 2].
\]

Then, the \textit{IENSC-value} is obtained by
\[
IENSC(w) = ([\frac{7}{3}, 22], [\frac{2}{3}, 4], [\frac{1}{3}, 2]).
\]

We note that this game is interval zero-normalized because for all \(i \in N\) we have \(w(i) = [0, 0]\). It is clear that this game is superadditive but not convex. For the coalitions \(S = (12)\) and \(T = (13)\), this game does not fulfill the condition of convexity:
\[
|w|(12) + |w|(13) \leq |w|(1) + |w|(123), \\
14 + 13 \leq 0 + 17.
\]

In this study, we propose three solution concepts, namely the \textit{interval Shapley value}, \textit{ICIS-value} and \textit{IENSC-value} belonging to our transportation model by using cooperative game theory. Table 2 illustrates our results of this application.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
\textbf{Interval Solutions} & \textbf{Player 1} & \textbf{Player 2} & \textbf{Player 3} \\
\hline
\textit{Interval Shapley value} & \(\frac{5}{2}, \frac{15}{2}\) & 3.6\(\frac{2}{3}\) & 2.4\(\frac{5}{7}\) \\
\hline
\textit{ICIS-value} & \(\frac{3}{2}, \frac{9}{2}\) & 3\(\frac{2}{3}\) & 3\(\frac{2}{3}\) \\
\hline
\textit{IENSC-value} & \(\frac{7}{3}, 22\) & 2\(\frac{1}{3}\) & 1\(\frac{1}{3}\) \\
\hline
\end{tabular}
\caption{The equal surplus sharing interval solutions of Example 5.4.}
\end{table}

If the total costs of the players for various values compare, it can be seen in Table 2 that; lowest total cost for Player 1 is in \textit{ICIS-value} and \(\textit{ICIS-value}\), lowest total cost for Player 2 is in \textit{IENSC-value}, the lowest total cost for Player 3 is in \textit{IENSC-value}. Take interval Shapley value into consideration, the total cost of the players are optimal (not minimum and not maximum). These values can be used in different application areas such as OR and economic situations. The results of this application are calculated by MAPLE software programme.

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