\textbf{\(\mathcal{PT}\)–symmetric Sturmians}

Miloslav Znojil

Nuclear Physics Institute ASCR, 250 68 Řež, Czech Republic
email: znojil@ujf.cas.cz

\section*{Abstract}

Sturmian bound states emerging at a fixed energy and numbered by a complete set of real eigencouplings \(\lambda_n\) are considered. For Sturm-Schrödinger equations which are manifestly non-Hermitian we outline the way along which the correct probabilistic interpretation of the system can constructively be re-established via a new formula for the metric. \(\mathcal{PT}\)–symmetrized Coulomb potential is chosen for illustration purposes.
1 Introduction

During the early developments of Quantum Mechanics, Schrödinger equation for the Coulomb bound-state problem

\[ -\frac{d^2}{dr^2} \psi_{n,\ell}(r) + \frac{\ell(\ell + 1)}{r^2} \psi_{n,\ell}(r) - \frac{\lambda}{r} \psi_{n,\ell}(r) = -\kappa^2 \psi_{n,\ell}(r), \tag{1} \]

played a role in the successful quantitative description of hydrogen-like atoms as well as in the various theoretical considerations verifying, e.g., the possibility of the coexistence of the discrete and continuous spectra. Although the purely phenomenological appeal of the oversimplified model (1) has perceptibly weakened with time, its mathematical and methodical relevance seems to survive in full strength. This is the reason why we choose here this example for illustrative purposes. Before we formulate our main results, let us recollect at least a part of the necessary terminology concerning, first of all, the so-called Sturmians and the so-called $\mathcal{PT}$-symmetry.

1.1 Sturmians

One of the most important reasons of the nondecreasing popularity of the Coulombic quantum model (1) lies in its exact solvability in terms of Laguerre polynomials [1]. Recently, Kelbert et al [2] emphasized that one of the particularly important aspects of this solvability should be seen in a certain symmetry between the roles played by the bound-state energies \( E = -\kappa^2 \) and by the related strengths of the Coulombic attraction or charge \( \lambda > 0 \).

Due to such a symmetry one can rearrange eq. (1) in such a way that the energy value \( E = -\kappa^2 \) is kept fixed while the “bound-state-supporting” eigencharges \( \lambda > 0 \) are allowed to vary,

\[ -\frac{d^2}{dr^2} \phi_{n,\ell}^{(\kappa)}(r) + \frac{\ell(\ell + 1)}{r^2} \phi_{n,\ell}^{(\kappa)}(r) + \kappa^2 \phi_{n,\ell}^{(\kappa)}(r) = \lambda_{n,\ell} W(r) \phi_{n,\ell}^{(\kappa)}(r). \tag{2} \]

The original nonconstant potential \( W(r) = W^{(\text{Coulombic})}(r) = 1/r \) now plays the role of a non-trivial factor accompanying the new eigenvalue \( \lambda > 0 \). On the level of terminology, equation (2) replaces the original bound states
ψ^{(\lambda)}_{n,\ell}(r) by another set of the \( \kappa \)-dependent bound states \( \phi^{(\kappa)}_{n,\ell}(r) \) called Sturmians.

The overall theory and applications of Sturmian wave functions have been reviewed by Rotenberg [3] who covered areas ranging from the electron-hydrogen scattering to the interatomic charge-transfer collisions and solutions of the Faddeev and Born-Oppenheimer equations. In mathematical physics the Coulombic as well as non-Coulombic Sturmians find typical applications in perturbation theory [4]. Their use has also been reported to improve the convergence of certain non-perturbative algorithms and calculations [2]. Besides the immediate determination of the energy-dependent couplings the phenomenological use of Sturmians involves constructions of the so called quasi-exactly solvable quantum models where, typically, people use \( W(r) \sim r^M \) with a positive (half)integer \( M \) [5] and where due attention must be paid to the questions of completeness [6]. Last but not least, in connection with the so called resonant internal boundary layers certain exact Sturmians emerging at \( W(r) = r^N \) with any \( N = -1, 0, 1, 2, \ldots \) even proved helpful in the context of classical physics [7].

1.2 \( \mathcal{PT} \)–symmetry

The recent growth of interest in manifestly non-Hermitian quantum Hamiltonians \( H \) with real spectra [8] has originally been motivated by the Bessis’ imaginary cubic example \( H = \hat{p}^2 + i\hat{x}^3 \neq H^\dagger \) and by its “symmetry” with respect to the antilinear product \( \mathcal{PT} \) of the space and time reflections [9]. In the framework of the resulting \( \mathcal{PT} \)–symmetric Quantum Mechanics (PTSQM, [8]) it has been explained, in ref. [10], why the well known and much appreciated exact solvability of the standard Hermitian version of the Coulomb problem (1) survives its non-Hermitian, \( \mathcal{PT} \)–symmetric deformation.

In what follows we intend to complement the \( \mathcal{PT} \)–symmetrized version of eq. (1) (and/or of its various generalizations) by a transfer of the key ideas and methodical conclusions of refs. [8, 10] to the parallel non-Hermitian Sturmian problems.

Returning once more to the guiding \( \mathcal{PT} \)–symmetrized Coulomb bound-
state example of ref. [10], let us remind the readers that the real potential \( W^{(\text{Coulombic})}(r) = 1/r \) has been replaced there by its purely imaginary alternative \( W^{(\text{PT})}(r) = i/r \). In the resulting eq. (1), viz.,
\[
- \frac{d^2}{dr^2} \tilde{\psi}_{n,\ell}(r) + \frac{\ell(\ell + 1)}{r^2} \tilde{\psi}_{n,\ell}(r) - \frac{i}{r} \tilde{\psi}_{n,\ell}(r) = -\kappa_{n,\ell}^2 \tilde{\psi}_{n,\ell}(r)
\]
the manifest loss of the Hermiticity of the model has partially been compensated by its \( \mathcal{PT} \)-symmetrization. This has been achieved by the replacement of the most usual real half-line of coordinates \( r \in (0, \infty) \) by a left-right symmetric complex parabola, typically, of the one-parametric form using a new real coordinate \( x \),
\[
r = r(x) = 2x + i (x^2 - 1), \quad x \in (-\infty, \infty).
\]
The original concept of the partial waves had to be abandoned so that the kinematical centrifugal force \( \ell(\ell + 1)/r^2 \) acquired an immediate dynamical meaning and the choice of the parameter \( \ell \) was not restricted to (half)integers, anymore.

The main, slightly unexpected result of ref. [10] was that the spectrum of the bound-state energies was positive,
\[
E_{(n,q)} = \frac{\lambda^2}{(2n + 1 - q - 2q\ell)^2}, \quad \ell \in \mathbb{IR}, \quad n = 0, 1, \ldots
\]
and that there emerged a new quantum number \( q = \pm 1 \) called “quasi-parity”. In a way encouraged by such an amazing structure of levels offered by the model, our present letter will try to extend the perspective of ref. [10] towards an analysis of the family of its Sturmian eigenstates. In this spirit we shall parallel the above-described transition from the bound states of eq. (1) to the Sturmians of eq. (2) under the additional assumption that one relaxes the Hermiticity of the equations.

1.3 \( \mathcal{PT} \)-symmetric Sturmians

Our transition to the \( \mathcal{PT} \)-symmetric Sturmian problem will start directly from the \( \mathcal{PT} \)-symmetric Schrödinger bound-state equation (3). In the way
guided by the Hermitian case we shall interchange the role of the energy and coupling, $\kappa^2 \leftrightarrow \lambda$. This leads to the generalized eigenvalue problem

$$
- \frac{d^2}{dr^2} \tilde{\phi}_{n,\ell}^{(\kappa)}(r) + \frac{\ell(\ell + 1)}{r^2} \tilde{\phi}_{n,\ell}^{(\kappa)}(r) + \kappa^2 \tilde{\phi}_{n,\ell}^{(\kappa)}(r) = \lambda_{n,\ell} W^{(PT)}(r) \tilde{\phi}_{n,\ell}^{(\kappa)}(r)
$$

(4)

where, as we already specified earlier, we may choose $W^{(PT)}(r) = i/r$ for the sake of definiteness.

Equation (4) and its physical meaning will be the main targets of our present analysis. We shall identify the key problem as lying in the correct specification of the concept of observables. In the language of mathematics this means that we shall describe a constructive recipe giving the metric operator $\Theta$ in the Hilbert space $\mathcal{H}^{(\text{physical})}$ of states of our model. This will guarantee the existence as well as the form of the standard probabilistic interpretation of its predictions.

In section 2 we shall start our considerations by briefly reviewing the well known basic theory and methods in the simpler, non-Sturmian case where $W = I$. We shall emphasize that the complete solvability/solution of the underlying Schrödinger equation in an auxiliary, unphysical but technically strongly preferred Hilbert space $\mathcal{H}^{(\text{user-friendly})}$ is usually assumed [8, 11]. This key methodical assumption enables us to make use of the general and explicit infinite-series formula for the necessary metric operator $\Theta$ at $W = I$ which is attributed, usually, to Mostafazadeh [12].

In the next section 3 the Sturmians with $W \neq I$ will be addressed and we shall describe the related generalization of the PTSQM formalism. We shall slightly reorder the flow of the arguments of section 2 and, after the introduction of this generalized eigenvalue problem in the space $\mathcal{H}^{(\text{user-friendly})}$ we shall jump immediately to the abstract analysis performed in another, “inaccessible” Hilbert space $\mathcal{H}^{(\text{third})}$ (cf. section 2 and paragraph 3.1 for definitions). Only then, employing the experience gained in section 2 we shall be able to extract all the consequences of the nontriviality of the weight operator $W \neq I$ in our Sturmian Schrödinger eq. (4) and its generalizations. In particular, our main result, viz., the spectral-series formula for the Sturmian-related metric $\Theta$ will be derived in paragraph 3.2.
Finally, section 4 will offer a brief summary of our non-Hermitian extension of the generalized eigenvalue Sturmian problem in Quantum Mechanics where, in the terminology which varies from author to author, the Hamiltonians $H$ and weights $W$ are admitted to be $\mathcal{PT}$-symmetric (in the sense of the so-called unbroken $\mathcal{PT}$-symmetry, \cite{8, 13}), quasi-Hermitian (i.e., Hermitian after the inner product is adapted, \cite{11, 14}), Hermitian (in a suitable space, \cite{15}) or cryptohermitian (i.e., Hermitian in a space which we are not necessarily going to specify, \cite{16}).

2 Physics of non-Hermitian bound states

Non-Hermitian Coulombic eq. (3) can be understood as a non-selfadjoint pair of the eigenvalue problems for $H$ and for $H^\dagger \neq H$ with a shared real spectrum $\{\lambda\}$,

$$H \mid \lambda \rangle = \lambda \mid \lambda \rangle , \\
H^\dagger \mid \lambda' \rangle = \lambda' \mid \lambda' \rangle .$$

The doubling of the ket symbol in some elements of our, by assumption, user-friendly Hilbert space $\mathcal{H}^{\text{user-friendly}}$ merely indicates their difference at the same real eigenvalue $\lambda$, $\mid \lambda \rangle \neq \mid \lambda \rangle$. There is nothing exotic in the latter Hilbert space since it can be visualized as the current space $L^2(\mathbb{R})$ of the quadratically integrable complex functions $f(x) \equiv \langle x \mid f \rangle$ for which the conjugate elements of the dual space of the functionals are obtained by the mere transposition plus complex conjugation, $\langle f \mid x \rangle \equiv f^*(x)$. At the same time, we must keep in mind that the symbol $\mathcal{H}^{\text{user-friendly}}$ denotes an unphysical Hilbert space since we have $H^\dagger \neq H$ in it.

In $\mathcal{H}^{\text{user-friendly}}$ we may and should assume the validity of the standard biorthogonality, biorthonormality and bicompleteness relations,

$$\langle \lambda' \mid \lambda \rangle = \langle \lambda' \mid \lambda \rangle = \delta_{\lambda'\lambda} , \\
I = \sum_\lambda \mid \lambda \rangle \langle \lambda \mid = \sum_\lambda \mid \lambda \rangle \langle \lambda \|. $$

We can define the metric operator $\Theta = \Theta^\dagger > 0$ and “innovate” the inner product between the two elements $\mid \psi \rangle$ and $\mid \psi' \rangle$ of $\mathcal{H}^{\text{user-friendly}}$ \cite{11, 13, 17},

$$\mid \psi' \rangle \circ \mid \psi \rangle = \langle \psi' \mid \Theta \mid \psi \rangle \equiv \langle \psi' \mid \psi \rangle , \\
\Theta = \sum_\lambda \mid \lambda \rangle \langle \lambda \|. $$
The transition to this new product changes our Hilbert space into another, *unitarily non-equivalent* “correct” and “physical” Hilbert space $\mathcal{H}^{(\text{physical})}$ of the states of the system. Formally one could speak about an updated Hermitian conjugation operation

$$
\mathcal{T}^{(\text{physical})} : |\psi\rangle \rightarrow \langle \psi| \Theta = \langle \langle \psi|
$$

but its explicit use in $\mathcal{H}^{(\text{physical})}$, albeit mathematically correct, would be both unnecessary in practice and potentially very strongly misleading.

It is clear that all the physics described in $\mathcal{H}^{(\text{physical})}$ can equally well be described in another, unitarily equivalent, third Hilbert space $\mathcal{H}^{(\text{third})}$. With its elements and conjugate functionals marked by the slightly modified kets $|\chi\rangle$ and bras $\langle \chi|$, respectively, we reveal that we may require the unitary equivalence between $\mathcal{H}^{(\text{physical})}$ and $\mathcal{H}^{(\text{third})}$,

$$
\langle \langle \chi'|\chi \rangle = \langle \langle \chi'|\Theta|\chi \rangle = \langle \chi'|\chi \rangle . \tag{8}
$$

For this purpose it suffices that we choose *any* invertible map $\Omega$ and postulate, for all the elements of the respective vector spaces, that

$$
|\psi\rangle = \Omega |\psi\rangle , \quad \langle \psi'| = \langle \langle \psi'|\Omega^{-1}.
$$

Moreover, once we set $\Theta = \Omega^{\dagger}\Omega$, we may complete our list of mappings by the formula $\langle \langle \psi'|\Omega^{-1} = \langle \psi'|\Omega^{\dagger}$. This confirms an overall consistency of the setup where our initial Hamiltonian $H$ is non-Hermitian in $\mathcal{H}^{(\text{user-friendly})}$.

In $\mathcal{H}^{(\text{third})}$ the isospectral image of $H$

$$
\hat{h} = \Omega \ H \Omega^{-1} \tag{9}
$$

must be Hermitian, $\hat{h} = \hat{h}^{\dagger}$. This guarantees its observability which is, in its turn, represented by the relation

$$
H^{\dagger} = \Theta \ H \Theta^{-1} \tag{10}
$$

i.e., in our working space $\mathcal{H}^{(\text{user-friendly})}$, by the quasi-Hermiticity of $H$. 
3 Physics of non-Hermitian Sturmians

In the Hamiltonian-independent Hilbert space $\mathcal{H}^{(\text{user-friendly})}$, we may abbreviate eq. (11) and its conjugate version as follows,

$$H |\lambda\rangle = \lambda W |\lambda\rangle, \quad H^\dagger |\lambda'\rangle = \lambda' W^\dagger |\lambda'\rangle$$

(11)

These Sturmian equations differ from their non-Sturmian predecessors only solely by the presence of a nontrivial and non-Hermitian weight $W \neq I$. Nevertheless, in a complete parallel with section 2 we shall still assume that all the solutions of both of these equations are fully at our disposal.

3.1 Sturmian Schrödinger equation in $\mathcal{H}^{(\text{third})}$

We intend to preserve as many analogies with the $W = I$ guide as possible. We shall work with a non-unitary though still invertible mapping $\Omega$ of our space $\mathcal{H}^{(\text{user-friendly})}$ (with elements denoted by the same Dirac’s ket symbols $|\psi\rangle$ as before) onto the intermediate and abstract, third Hilbert space $\mathcal{H}^{(\text{third})}$. Its elements and their duals will be denoted by the same deformed, curved Dirac’s bra and ket symbols as above,

$$|\psi\rangle = \Omega |\psi\rangle \in \mathcal{H}^{(\text{third})}, \quad \langle \psi | = \langle \psi | \Omega \dagger \in (\mathcal{H}^{(\text{third})})^\dagger.$$ 

(12)

The lower-case isospectral equivalent $h = \Omega H \Omega^{-1}$ of our original non-Hermitian upper-case Hamiltonians $H \neq H^\dagger$ as well as the parallel partner $w = \Omega W \Omega^{-1}$ of any original non-Hermitian specific “weight” operator $W \neq W^\dagger$ are both assumed self-adjoint in the third space. This means that we shall require that

$$h^\dagger = (\Omega^{-1})^\dagger H^\dagger \Omega^\dagger = h, \quad w^\dagger = (\Omega^{-1})^\dagger W^\dagger \Omega^\dagger = w,$$

or, after a trivial re-arrangement,

$$H^\dagger = \Theta H \Theta^{-1}, \quad W^\dagger = \Theta W \Theta^{-1}$$

(13)

where we abbreviated $\Theta = \Omega \dagger \Omega$. This is our first result showing how the concept of the quasi-Hermiticity translates to the Sturmian scenario.
After the above change of the Hilbert space we may represent our upper-case Sturmian problem (11) by its new, lower-case reincarnation

\[ h |\lambda\rangle = \lambda w |\lambda\rangle \]  

which is necessarily self-adjoint in \( H^{(third)} \). Its simplicity facilitates the derivation of the Sturmian orthogonality relations

\[ \langle \lambda | w | \lambda' \rangle = \langle \lambda | w | \lambda \rangle \cdot \delta_{\lambda,\lambda'} \]  

and of the Sturmian completeness relations,

\[ I = \sum_{\lambda} \frac{1}{\langle \lambda | w | \lambda \rangle} \langle \lambda | w | \lambda \rangle \delta_{\lambda,\lambda'} \]  

as well as of the Sturmian spectral-representation formula

\[ h = \sum_{\lambda} \frac{\lambda}{\langle \lambda | w | \lambda \rangle} \langle \lambda | w | \lambda \rangle \delta_{\lambda,\lambda'} \]  

for the Hamiltonian in \( H^{(third)} \).

Our original Hilbert space \( H^{(user-friendly)} \) was, by assumption, so simple that one must always transfer all the relevant formulae and recipes to this space at the end. In this spirit let us insert definitions (12) in eq. (15) and arrive at the orthogonality relations in \( H^{(physical)} \),

\[ \langle \lambda | \Omega^\dagger w \Omega | \lambda' \rangle = \langle \lambda | \Theta W | \lambda \rangle = \langle \lambda | \Theta W | \lambda \rangle \cdot \delta_{\lambda,\lambda'} \].

Similarly, the appropriately adapted version of the completeness is obtained,

\[ I = \sum_{\lambda} \frac{1}{\langle \lambda | \Theta W | \lambda \rangle} \langle \lambda | \Theta W | \lambda \rangle \delta_{\lambda,\lambda'} \]  

Finally, the spectral decomposition of the Hamiltonian acquires the following form in \( H^{(physical)} \),

\[ H = \sum_{\lambda} \frac{\lambda}{\langle \lambda | \Theta W | \lambda \rangle} \langle \lambda | \Theta W | \lambda \rangle \delta_{\lambda,\lambda'} \]  

This enables us to generalize the above conclusion that whenever the spectrum of \( \lambda \)s remains non-degenerate, even the generalized double-ket eigenstates \( | \lambda \rangle \rangle \) of \( H^\dagger \) will coincide with the elementary products \( \Theta | \lambda \rangle \).
3.2 Formula for the Sturmian metric $\Theta$

The key benefit of our pragmatic return to the space $\mathcal{H}^{(user\text{-}friendly)}$ is that we may evaluate the matrix elements $\langle \lambda \mid \Theta W \mid \lambda \rangle$ in eq. (18) and renormalize them to one whenever the Hermitian product $\Theta W$ is positive definite (which we assume).

This restriction will percievably simplify our formulae. Firstly, in terms of the other two abbreviations

$$|\psi\{\rangle = W |\psi\rangle, \quad |\psi\} \rangle = W^\dagger |\psi\rangle$$

for elements of our working Hilbert space $\mathcal{H}^{(user\text{-}friendly)}$, the two alternative forms of the simplified orthogonality conditions will be obtained easily,

$$\langle \lambda \mid \Theta W \mid \lambda' \rangle = \{ \langle \lambda \mid \lambda' \rangle = \langle \langle \lambda \mid \lambda' \rangle = \delta_{\lambda,\lambda'} . \quad (21)$$

Next we get the two alternative forms of the completeness relations,

$$I = \sum_\lambda |\lambda\rangle \{ \langle \lambda | = \sum_\lambda |\lambda\} \langle \lambda | . \quad (22)$$

Finally, mixed but compact expressions will result for both the spectral-representation expansions

$$W = \sum_\lambda |\lambda\rangle \{ \langle \lambda | , \quad H = \sum_\lambda |\lambda\} \lambda \{ \langle \lambda | \quad (23)$$

of our weight and Hamiltonian operators.

At this moment one could decide to follow the Mostafazadeh’s $W = I$ method [12]. Its basic idea is that one inserts the spectral formulae (23) in the quasi-Hermiticity constraint $H^\dagger \Theta = \Theta H$,

$$\sum_\lambda |\lambda\} \lambda \{ \langle \lambda | \Theta = \sum_\lambda \Theta |\lambda\} \lambda \{ \langle \lambda | . \quad (24)$$

This relation strongly suggests that the Sturmian analogue of the single-series $W = I$ formula (7) should be sought via the double series ansatz,

$$\Theta = \sum_{\lambda,\lambda'} |\lambda\{ \} M_{\lambda,\lambda'} \{ \langle \lambda' | , \quad M_{\lambda,\lambda'} = \langle \lambda | \Theta | \lambda' \rangle . \quad (25)$$

Fortunately, there exists an alternative approach which could have been used, after all, also at $W = I$. Its main idea is based on the identity $|\psi\} \rangle = \Theta |\psi\rangle$
which would imply that the Mostafazadeh’s $W = I$ formula (7) immediately follows from the multiplication of the bicompleteness relation (6) by $\Theta$. At $W \neq I$ the analogous idea makes us to replace merely eq. (6) by eq. (22), giving the unexpected but by far simpler, single-series final result

$$\Theta = \sum_{\lambda} \langle \lambda \rangle \{ \lambda \} . \quad (26)$$

The impression of an apparent non-Hermiticity of this asymmetric formula is misleading and it is virtually trivial to verify that $\Theta = \Theta^\dagger$ in $\mathcal{H}^{(user-friendly)}$.

Marginally, we would like to add that as long as the alternative, double-series formula (25) for the metric is concerned, it might still find some applications (cf., e.g., [18] for a specific illustrative example). In similar situations our single-series formula will still offer two useful representations

$$M_{\lambda,\lambda'} = \langle \lambda | \Theta | \lambda' \rangle = \langle \langle \lambda | \lambda' \rangle = \{ \lambda | W^{-1} | \lambda' \rangle = \langle \langle \lambda | W^{-1} | \lambda' \rangle \} \quad (27)$$

of the necessary nondiagonal matrix of coefficients in the less economical but manifestly symmetric double-series expansion (25) of the Sturmian metric.

4 Summary

A return to solvable Schrödinger equation containing Coulomb potential helped us to establish connection between two conceptual issues which appeared (or reappeared) in the very recent literature on Quantum Mechanics.

The first issue reflects the well known concept of Sturmian bound states which already found numerous and fairly diversified applications in Quantum Mechanics over the years [3].

The second issue involves the concept of the so called non-Hermitian Hamiltonians with real spectra, the study of which has been made very popular by C. Bender and his coauthors [8].

An immediate inspiration of our study stemmed from two sources. The first one was our study [10] where the Coulomb model (1) has been used as an exactly solvable illustrative example of the survival of the reality of the energy spectrum after a loss of manifest Hermiticity of the Hamiltonian.
The second source of inspiration can be seen in the recent new success of an application of the Sturmina bases in computational physics [2].

The main advantage of our present interpretation of the observability in the non-Hermitian Coulombic as well as more general Sturmians can be seen in its mathematical as well as physical consistency. We made it clear that once we succeeded in the explicit infinite-series construction of the metric $\Theta$, even the combined effect of the $\mathcal{PT}$–symmetrization and of the emergence of a nontrivial weight $W \neq I$ still enables us to treat the resulting, apparently non-Hermitian Sturm-Schrödinger equation as fully compatible with the standard principles of Quantum Mechanics.

In the future one could expect that the feasibility of the construction of the Sturmian metric via formula (26) could lead to its applications far beyond the present exceptional and exactly solvable illustrative Coulomb model with $\mathcal{PT}$–symmetric $W(r) \sim 1/r$. In particular, in the topologically nontrivial class of models called “quantum toboggans” [18, 19] there appeared a few new challenging open questions which could prove tractable by our present approach, especially (though not only) in their simplest and exactly solvable special “quantum knot” case of ref. [20] where $W(r) \sim 1/r^2$.

Acknowledgement

Work supported by GAČR, grant Nr. 202/07/1307, Institutional Research Plan AV0Z10480505 and by the MŠMT “Doppler Institute” project Nr. LC06002.
References

[1] S. Flügge, Practical Quantum Mechanics I, Spinger-Verlag, Berlin, 1971, p. 171.

[2] E. Kelbert, A. Hyder, F. Demir, Z. T. Hlousek and Z. Papp, J. Phys. A: Math. Theor. 40 (2007) 7721.

[3] M. Rotenberg, Adv. At. Mol. Phys. 6 (1970) 233.

[4] M. Znojil, J. Math. Phys. 38 (1997) 5087.

[5] A. Voros, J. Phys. A: Math. Gen. 33 (2000) 7423.

[6] R. Szymkowski and B. Zywicka-Mozejko, Phys. Rev. A 62 (2000) 022104.

[7] C. M. Bender and Q. Wang, J. Phys. A: Math. Gen. 34 (2001) 9835.

[8] C. M. Bender, Rep. Prog. Phys. 70 (2007) 947.

[9] V. Buslaev and V. Grecchi, J. Phys. A: Math. Gen. 26 (1993) 5541;
   C. M. Bender and S. Boettcher, Phys. Rev. Lett. 80 (1998) 5243.

[10] M. Znojil and G. Lévai, Phys. Lett. A 271 (2000) 327.

[11] F. G. Scholtz, H. B. Geyer and F. J. W. Hahne, Ann. Phys. (NY) 213 (1992) 74.

[12] A. Mostafazadeh, J. Math. Phys. 43 (2002) 2814 and ibid. 3944.

[13] C. M. Bender, D. C. Brody and H. F. Jones, Phys. Rev. Lett. 89 (2002) 270401 and
    ibid. 92 (2004) 119902 (erratum).

[14] J. Dieudonné, in Proc. Int. Symp. Lin. Spaces, Pergamon, Oxford, 1961, p. 115.

[15] J. P. Williams, Proc. Amer. Math. Soc. 20 (1969) 121.

[16] A. V. Smilga, arXiv.org: 0706.4064.

[17] B. Bagchi, C. Quesne and M. Znojil, Mod. Phys. Lett. A 16 (2001) 2047;
   A. Mostafazadeh, J. Math. Phys. 43 (2002), 205;
   M. Znojil, SIGMA 4 (2008) 001, arXiv overlay: 0710.4432v3 [math-ph].

[18] M. Znojil, J. Phys. A: Math. Theor., to appear [arXiv:0803.0403v1 [quant-ph] 4 Mar 2008 and 0803.0403v2 [quant-ph] 21 Apr 2008).

[19] M. Znojil, Phys. Lett. A 372 (2008) 584.

[20] M. Znojil, Phys. Lett. A, to appear (arXiv: 0802.1318v1 [quant-ph] 10 Feb 2008).