Using Cosmology to constrain the Topology of Hidden Dimensions

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A four-dimensional universe, arising from a flux compactification of Type IIB string theory, contains scalar fields with a potential determined by topological and geometric parameters of the internal–hidden–dimensions. We show that inflation can be realized via rolling towards the large internal volume minima that are generic in these scenarios, and we give explicit formulae relating the microscopic parameters (e.g., the Euler number of the internal space) to the cosmological observables (e.g., the spectral index). We find that the tensor-to-scalar ratio, the running of the spectral index, and the potential energy density at the minimum are related by consistency relations and are exponentially small in the number of e-foldings. Further, requiring that these models arise as low-energy limits of string theory eliminates most of them, even if they are phenomenologically valid. In this context, this approach provides a strategy for systematically falsifying stringy inflation models.

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Introduction.—A central question in high-energy physics is whether top-down models such as string theory can be tested or at least constrained by experimental data. Traditionally, we look at particle spectra and interactions as measured in accelerator experiments. Here we ask whether the precision astrophysical data that is becoming available can play a similar role.

In string theory, the four observable dimensions of the world are usually accompanied by an additional six-dimensional compact space, whose geometry is described by a set of scalar fields. The potentials for these scalar fields, known as “moduli”, are parametrized by topological and geometric quantities derived from the structure of the internal space, i.e., the hidden dimensions. In a scenario where inflation \[1\] is realized by rolling moduli scalar fields, the primordial cosmological perturbations as observed e.g., in the properties of Cosmic Microwave Background radiation (CMB) and of the large-scale structures, will be affected by the shape of the potential, and hence by the topology and geometry of the hidden dimensions. Progress towards realizing inflation in string theory includes \[2, 3, 4\], for a review see e.g., \[5\].

Here, we study four-dimensional universes that arise from Type IIB string theory \[6, 7\]. As usual, the internal six-dimensional space is chosen as a Calabi–Yau manifold, in order to solve the equations of motion of string theory while achieving realistic particle physics. Generically in such settings, the scalar potential has a minimum at a point where the internal space has a volume that is much larger than the Planck scale \[8\]. In these models, particle physics and inflation has been discussed in, e.g., \[4, 8\]. Here, working with a simple two–scalar example, we show that inflation can be realized as the scalar fields roll towards the large–volume minimum. We derive explicit formulae for the topological and geometric parameters of the internal manifold in terms of the observables.

These models predict specific consistency relations between the cosmological observables, and are thus falsifiable. In particular the tensor-to-scalar ratio, the running of the spectral index and the cosmological constant value are related and exponentially small in the number of e-foldings. Requiring that these scenarios arise as low–energy limits of string theory eliminates many phenomenologically viable models. This provides a strategy for systematically falsifying stringy inflation models of this kind. Our methods and conclusions generalize easily from the example given here to more complex compactifications \[12\].

Inflation in large-volume Calabi–Yau string compactifications.—In designing a string model, one chooses the topology of the internal compact manifold and the strengths of the magnetic fluxes threading it. The shapes and sizes of surfaces within the internal space are described by four-dimensional scalar fields, the moduli. As shown in \[8, 9\], the scalar fields describing the shape of the hidden dimensions can be treated as constants for our purposes. Likewise, the dilaton, a special scalar field whose expectation value sets the string coupling constant \(g_s\), is also fixed to a constant. By contrast, the scalar fields describing the sizes of the hidden dimension, \(\{\tau_i\}\), acquire a potential from quantum effects in string theory \[8, 9, 10\] and could roll to realize inflation \[4\]. While inflation can also arise from rolling of the axion partners of \(\tau_i\), as studied in \[8\], here we integrate these axions out of the effective potential, and study rolling in \(\tau_i\).

In what follows, we rely on the methodology and some of the results of \[12\]. The full potential given in \[8\] simplifies when the sizes of internal surfaces, as parameterized by \(\tau_i\), are large compared to the string scale. In this case, there is a systematic expansion in powers and exponentials of these fields \[12\]. For a generic choice of internal manifold and magnetic fluxes in it, the scalar potential
is minimized at a point where the overall volume is large and there is a hierarchy $\tau_1 \gg \tau_2, \cdots, \tau_N \gg 1$. Thus we will work in the large–volume approximation of the full potential. To facilitate explicit analysis we will study a simple model with two scalar fields $\tau_1, \tau_2$, in terms of which the volume of the internal space is

$$V = \mu_1^{3/2} - \lambda_1^{3/2} = \frac{1}{x} (1 - \frac{\lambda x^{3/2}}{\mu_1^{3/2}}); x = \frac{1}{\mu_1^{3/2}}.$$

(1)

Here $\lambda$ and $\mu$ are related to topological quantities ("intersection numbers") of the internal manifold.

In this large–volume limit, the scalar potential given in [7, 11] becomes [12],

$$V_{\text{tot}} = V_{\text{up}} + V_2 + V_3 + V_4$$

(2)

$$V_{\text{up}} = \gamma x^2$$

where, as natural in the large–volume regime, we have assumed

$$\lambda x^{3/2} \ll 1, \quad \xi x \ll 1, \quad A e^{-\frac{\pi}{\sqrt{2}}}/|W_0| \ll 1;$$

(6)

and, for string theory to be weakly coupled, $g_s < 1$. The parameter $\gamma$ in the uplift term, $V_{\text{up}}$, measures the strength of supersymmetry breaking effects [7, 11]. The classical superpotential, $|W_0|$, has its origin in the magnetic fluxes in the internal space, and results in the fixing of the shape scalars and the string coupling; in string theory $W_0$ can be finely tuned; $M_P = 1/\sqrt{8\pi G} = 2.4 \times 10^{18}\text{GeV}$ is the reduced Planck mass. $A$ and $a$ arise quantum mechanically in string theory, these are computable and fixed in a given model, but below we will allow them to be free parameters. Finally, $\xi$ is proportional to the Euler number $\chi$ of the internal space: $\xi = -\frac{\chi(M_P)}{2x^{3/2}}$, with $\chi(3) \approx 0.909$ the Riemann zeta function at 3. The minimum of this potential has $\tau_1 \gg \tau_2 \ll 3$, and hence $V \approx 1/x$. In other words, our analysis is valid in the large region of the seven dimensional parameter space $\{\gamma, \lambda, \xi, A, a, g_s, W_0\}$ and the two dimensional field space $\{\tau, x\}$ where [16] applies.

In general, given an arbitrary initial condition, both scalar fields could roll simultaneously. For simplicity, we will study conditions under which it is a good approximation to ignore rolling in one of the field directions. Ref. [12] shows that in the two-field case the potential along the $\tau$ direction is not flat enough to yield inflation. This will be important as a mechanism to end inflation in a manner consistent with observational constraints. Thus we search for slow roll inflationary conditions in $x$ in a region of the potential in which any motion in $\tau$ is negligible. One sufficient condition to restrict rolling to the $x$ direction only is $\partial V/\partial \tau = 0$; this can be shown from the magnitude hierarchy between the two fields. Thus, at the start of inflation, the initial condition $(x_0, \tau_0)$ satisfies

$$A e^{-y_0}/|W_0| = 3\lambda x_0^{3/2} \frac{1 - y_0}{y_0(1 - 4y_0)}; \quad y = \alpha/\gamma_a$$

(7)

The scalar potential must have a local minimum towards which the inflating field evolves and where reheating can occur. Ref. [12] show that at such a minimum $y > 1$.

**Slow Roll Inflation in the $x$ direction.**—To compute the slow roll parameters (e.g., [14])

$$\epsilon_V \equiv \frac{M_P^2}{2} \left(\frac{V'}{V}\right)^2; \eta_V \equiv \frac{M_P^2}{2} \left(\frac{V''}{V}\right) ; \xi_V \equiv \frac{M_P^2}{2} \left(\frac{V'}{V}\right)^2,$$

we must take derivatives with respect to canonically normalized fields. The transformation between $x$ and the canonical normalized field $\tau_1$ is $\frac{dx}{d\tau_1} = -\sqrt{3/2}/M_P x + \mathcal{O}(\lambda x^{3/2}, \xi x)$ [12]. This includes the transformation between $x$ and $\tau_1$ as well as the standard non-canonical kinetic term of $\tau_1$ (see, e.g., [3]).

Slow-roll inflation, giving a nearly scale-invariant spectrum, occurs when $\epsilon_V, \eta_V, \xi_V \ll 1$. To first order (sufficiently accurate in the slow roll regime), the observables are (e.g., [13]):

$$r \simeq 16 \epsilon_V; \quad n_s \simeq 1 - 6 \epsilon_V + 2 \eta_V,$$

(9)

$$\frac{d n_s}{d \ln k} \simeq 16 \epsilon_V \eta_V - 24 \epsilon_V^2 - 2 \xi_V,$$

(10)

$$\Delta R^2 \simeq \frac{V_{\text{tot}}/M_P^4}{24\pi^2 \epsilon_V},$$

(11)

where $r$ is the tensor to scalar ratio, $n_s$ is the slope of the spectrum of scalar primordial fluctuations, $\frac{d n_s}{d \ln k}$ is the "running" of the spectrum, i.e. its deviation from a power law as a function of scale, and $\Delta R^2$ is the amplitude of the curvature perturbation spectrum.

In terms of the variables

$$z_1 = 3 \left(\frac{\lambda x_0^{3/2}}{\gamma_4 \pi 4}\right)|W_0|^2 g_s^4 M_P^4 \frac{1 - \alpha \tau_0 / g_s}{1 - 4 \alpha \tau_0 / g_s},$$

(12)

$$z_2 = \frac{1 - \alpha \tau_0 / g_s}{1 - 4 \alpha \tau_0 / g_s} z_1; \quad z_3 = \frac{(\xi x_0)|W_0|^2 g_s^4 M_P^4}{8 \gamma_4 \pi},$$

(13)

the potential at $(x, \tau_0)$ is

$$V = \gamma x \left[3(x - 2z_1) + 4x_0 z_2 + 3z_3 x_0^{-1} x^2\right]$$

(14)
and the three slow-roll parameters are
\[ \epsilon_V = \frac{3}{4} \left( \frac{2 - 4z_1 + 4z_2 + 9z_3}{1 - 2z_1 + 4z_2 + 3z_3} \right)^2, \quad (15) \]
\[ \eta_V = \frac{3}{2} \left( \frac{1}{1 - 2z_1 + 4z_2 + 3z_3} \right)^2, \quad (16) \]
\[ \xi_V = \frac{9}{4} \left( \frac{8 - 16z_1 + 4z_2 + 8z_3}{1 - 2z_1 + 4z_2 + 3z_3} \right)^2. \quad (17) \]

The first two equations can then be inverted giving
\[ z_1 = \frac{1}{2} \left( 2 \sqrt{3 + \sqrt{\eta_V - 24s\sqrt{\epsilon}}} \right) \quad (18) \]
\[ z_3 = \frac{4}{3} \sqrt{3 + \sqrt{\eta_V - 9s\sqrt{\epsilon}}} \quad (19) \]

but the third parameter is given by the other two
\[ \xi_V = -6(3\sqrt{3 + 2\sqrt{\eta_V}} - 11s\sqrt{\epsilon})s\sqrt{\epsilon}. \quad (20) \]

Here \( s \) stands for a sign correlated with the direction of rolling: \( s < 0 \) when \( \eta \) decreases and \( s > 0 \) when it increases.

This dependence between \( \epsilon_V, \eta_V, \xi_V \) implies that we can pick \( \{n_s, r\} \), which are linear in the slow roll parameters, as independent observables and \( dn_s/d\ln k \) is determined from them. Eq. (20) gives a consistency relation for the running of the spectral index:
\[ \frac{dn_s}{d\log k} = \frac{24\sqrt{3} (2 + n_s) \sqrt{\tau} - 9\sqrt{3} r^{3/2} + 2(25 + 8n_s) sr + 3 sr^2}{16s} \quad (21) \]

For \( r \ll 1 \) the sign of the running tells us the sign of the direction of rolling of the field. In terms of \( \{n_s, r\} \),
\[ z_1 = \frac{136\sqrt{3} + 8\sqrt{3} n_s + 3\sqrt{3} r + 60\sqrt{\tau} (4\frac{\alpha y_0}{g_s} - 1)}{6Q} \quad (22) \]
\[ z_3 = \frac{2(40\sqrt{3} + 8\sqrt{3} n_s + 3\sqrt{3} r + 36\sqrt{\tau}) (4\frac{\alpha y_0}{g_s} - 1)}{9Q} \quad (23) \]

with \( Q = (8\sqrt{3} n_s + 3\sqrt{3} r + 4\sqrt{\tau} (11 + 4\alpha y_0 / g_s) + 8\sqrt{3} (5 + 12\alpha y_0 / g_s)) \). These equations relate the cosmological observables to the topological and geometric parameters.

There are two possible branches to roll. Let \( x_f \) denote the value of \( x \) where inflation ends and \( z = x_f / x_0 = 1 + \delta \), where \( |\delta| \ll 1 \) and can be positive or negative. Where \( \delta > 0 \), \( \eta \) decreases and where \( \delta < 0 \), \( \eta \) increases. To compute the number of e-foldings, we must determine where inflation ends. In our model this happens because the force in the direction of the second field, \( \tau \), becomes larger than the force along \( x \). The steepness of the potential along the \( \tau \) direction means that after this point, the rolling becomes kinetic energy-dominated and inflation stops. It is shown in [12] that if inflation ends in this way via fast roll in \( \tau \), then \( n_s \leq 1 \).

| data set                  | min       | max       | in/out |
|---------------------------|-----------|-----------|--------|
| WMAP                      | -0.116    | 0.0098    | in     |
| WMAP+Bolometers           | -0.112    | 0.0027    | in     |
| WMAP+HEMP                 | -0.12     | -0.00807  | out    |
| WMAP+SDSS                 | -0.109    | -0.0066   | out    |
| WMAP+2dFGRS               | -0.11     | 0.0027    | in     |

TABLE I: 95% confidence region for the running of the spectral index for various data sets combinations. WMAP+Bolometers means WMAP three-year data combined with Boomerang and ACBAR, WMAP+HEMP means WMAP three-year data with CBI and VSA. This model is allowed for some data sets combinations, but is disfavored by other. Ranges have been obtained from the publicly available LCDM+running model Markov Chains on LAMBDA [17].

The number of e-foldings is:
\[ N_e = \int_{t_i}^{t_f} H dt \approx \frac{1}{M_P^2} \int_{\phi_0}^{\phi_1} \frac{V}{V^\prime} d\phi. \quad (24) \]

Solving this integral we obtain a relation between cosmological parameters and number of efoldings [12],
\[ e^{-9N_e} \approx r. \quad (25) \]

Thus \( r \) is undetectable if we impose the currently favored number of e-foldings, i.e. \( 55 - 70 \). Along with the consistency relation (Eq. 24), this implies that \( dn_s/d\ln k \) is also vanishingly small, but its sign depends on the direction of rolling. While this is allowed by current WMAP [13] data \(-0.11 \leq dn_s/d\ln k \leq 0.0098 \) at 95% confidence, other data combinations would give slightly different conclusions (see Table 1). Forcoming observations (e.g. ACT; Planck [18]) will improve the measurement precision by about an order of magnitude. Since we find that \( r \) is exponentially suppressed, any measurement of a non-zero running will eliminate our models.

Measuring the topology.—We can now express the topological and geometric parameters in terms of observables:
\[ \gamma = \frac{3\pi^2 r \Delta^2 M^2_{pl} \left( 5 + n_s + 12y_0 \right)}{8\frac{\alpha y_0}{g_s}} \quad (26) \]
\[ \lambda_{0/2}^{3/2} = \frac{\pi^3}{12} \frac{r \Delta^2 R}{x_0^3 g_s^4 W_0^2} \left( 17 + n_s \right) \left( \frac{4y_0 - 1}{y_0 - 1} \right)^2 \quad (27) \]
\[ \xi = \frac{8\pi^3}{3} \frac{r \Delta^2 R}{x_0^3 g_s^4 W_0^2} \left( 5 + n_s \right) \quad (28) \]
\[ A e^{-\frac{\Delta y_0}{g_s}} = \frac{\pi^3}{4} \frac{r \Delta^2 R}{x_0^3 g_s^4 W_0^2} \left( 17 + n_s \right) \frac{4y_0 - 1}{y_0 - 1} \quad (29) \]

where \( y_0 = \alpha y_0 / g_s \). Eqs. (27-28) yield
\[ \xi = 32\lambda_{0/2}^{3/2} \left( 5 + n_s \right) \left( \frac{y_0 - 1}{4y_0 - 1} \right)^2 \sim 1/3\lambda_{0/2}^{3/2}. \quad (30) \]

In this setting, inflation begins and ends entirely within the large volume region and thus our analysis is within
the regime of validity of the approximations \([13]\). It is shown in \([12]\) that if vacuum energy after inflation ends is positive (i.e. \(V_{\text{min}} > 0\)), then \(1 < y_0 < 2\). Using \([20]\) one shows that \(V_{\text{min}} \sim M_{\text{Pl}}^4 r \Delta R^2\). Since the amplitude of primordial perturbations amplitude is \(\Delta R^2 \sim 10^{-9}\), \([20]\) implies that \(V_{\text{min}}\) is exponentially small in the number of e-foldings. Hence it is much smaller than the measured dark energy density \((\sim 10^{-12} M_{\text{Pl}}^4)\). The relation between \(r\) and the vacuum energy implies in our setting that phenomenologically viable inflation automatically leads to a tiny vacuum energy density.

However, to evade accelerator bounds on Kaluza-Klein particle masses, it turns out that we must have \(g_s \sim 10^{-13}\), and the internal space cannot be too large \((x_0 \gtrsim 10^{-19})\) \([12]\). Then for generic values of \(V_0\) (typically of \(O(1)\) \([10]\)), \([21, 28]\) imply that realizing inflation requires exponentially small \(\lambda\) and \(\xi\). While there is no phenomenologically impediment to choosing such small parameters, they cannot arise as topological invariants of a Calabi-Yau manifold (e.g., the Euler number is an integer!). Thus, for \(V_0 \sim 1\), requiring that our potential both gives rise to inflation and originate in string theory, eliminates it as models of nature.

To realize inflation with topological parameters in a reasonable range, we have to pick an exponentially small \(W_0^2 \sim r\) when \(\xi > \xi^* = (c(3)/(2\pi)^3 \sim 0.0036\), the required range of \(W_0\) is \([12]\)

\[
\frac{r \Delta R^2}{x_0 g_s} \ll |W_0|^2 \ll \frac{16\pi^5}{\xi^*} 10^{30} e^{-9 N_*}.
\]

While this kind of slow-roll model is statistically disfavored \([14]\), it is not ruled out. However, even if its parameters are realizable in string theory, if inflation required either small internal volumes or \(g_s > 1\), further corrections to the potential (Eq. 2-5) would be needed for consistency. Even if these checks are passed, there is a further test: given a Calabi-Yau manifold, the parameters \(A\) and \(a\) in the potential are determined by the theory, but must match \([20]\). This provides a strategy for systematically falsifying stringy inflation models of this kind.

Conclusions—Robust predictions of these models are \(i)\) the tensor to scalar ratio \(r \approx 0\); any future detection of primordial gravity waves would falsify this model, \(ii)\) \(dn_s/d\ln k \sim 0\); \(n_s \lesssim 1\); the sign of running is related to the direction of rolling, \(iii)\) generally, in these models there are consistency relations between observables, \(iv)\) specifically in this model the value of the cosmological constant, \(r\) and \(dn_s/d\ln k\) are related and exponentially small in the number of efoldings. Our results show that constraints can be imposed on the topology of the hidden dimensions from cosmological observations. Further, the requirement that the scalar potential arises from a string theory is a stringent limitation. This approach provides a strategy for systematically ruling out stringy inflation models.

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\[\text{References}\]

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