Bayesian Quantile Regression with Multiple Proxy Variables

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Abstract

Data integration has become more challenging with the emerging availability of multiple data sources. This paper considers Bayesian quantile regression estimation when the key covariate is not directly observed, but the unobserved covariate has multiple proxies. In a unified estimation procedure, the proposed method incorporates these multiple proxies, which have various relationships with the unobserved covariate. The proposed approach allows the inference of both the quantile function and unobserved covariate. Moreover, it requires no linearity of the quantile function or parametric assumptions on the regression error distribution and simultaneously accommodates both linear and nonlinear proxies. The simulation studies show that this methodology successfully integrates multiple proxies and reveals the quantile relationship for a wide range of nonlinear data. The proposed method is applied to the administrative data obtained from the Survey of Household Finances and Living Conditions provided by Statistics Korea. The proposed Bayesian quantile regression is implemented to specify the relationship between assets and salary income in the presence of multiple income records.

Keywords: data integration; nonparametric regression; measurement error; natural cubic spline.
1 Introduction

Data integration has become more popular due to the availability of multiple data sources, such as administrative data, Web-collected data, and emerging unstructured big data. However, the data integration process, especially for model-based integration, is not straightforward due to selection bias in data collection, different observation times and frequencies, and multiple proxies for the same targets. Among these important technical issues, we address multiple proxies in this paper.

These proxies include various domains from the case in which covariates are measured with errors (Carroll et al., 2006; Fuller, 2009) when the true covariate is simply unobserved (Filmer and Pritchett, 2001), and when the variable of interest is a conceptual variable measured with a different definition (Solon, 1992; Zimmerman, 1992; Mazumder, 2001). In this research, we consider the problem of estimating the quantile function in the presence of multiple proxies of the true covariate. As in conventional linear regression, the inconsistency of the quantile regression estimator in the absence of the true covariate is a commonly discussed issue in the literature (Brown, 1982; He and Liang, 2000; Carroll et al., 2006; Wei and Carroll, 2009; Montes-Rojas et al., 2011; Hausman et al., 2021).

Several previous studies have incorporated proxy variables in quantile regression estimation. He and Liang (2000) considered the problem of estimating quantile regression coefficients in errors-in-variables models with a proxy variable and proposed an estimator in the context of linear and partially linear models. Wei and Carroll (2009) presented a nonparametric method for correcting bias caused by measurement error in the linear quantile regression model by constructing joint estimating equations that simultaneously hold for all quantile levels. However, this method assumes the existence of replicates of the proxy variable. Firpo et al. (2017) also proposed a semiparametric two-step estimator when repeated measures for the proxy are available. Schennach (2007, 2008) discussed identifying a nonparametric quantile function under various settings when an instrumental is variable measured on all sampling units. Wang et al. (2012) proposed a modification of the standard quantile regression objective function tailored to the Gaussian measurement error model. However, less attention has been paid to the proxy variable that varies from a conventional additive measurement error or the existence of multiple proxies with such deviation.

Regarding combining the information for proxy variables that do not correspond to the standard additive measurement error, Mauro (1995) pointed out that the conventional approach of putting all the proxies in regression may result in many insignificant individual coefficients. Filmer and Pritchett (2001) suggested using the first principal component to summarize the proxies in a single variable.
However, one problem with this approach is that there is no reason to suppose that this component maximizes the predictive power of the true covariate. Lubotsky and Wittenberg (2006) addressed this problem in the linear regression context with proxy variables linearly related to the true covariate and served the integrating method for minimizing the attenuation bias. Fuller (2009) presented the factor analysis approach assuming the structural distribution of the true covariate and proxy errors to estimate the linear regression coefficient in the presence of multiple proxies.

Nevertheless, most methods rely on linear parametric assumptions, which are very sensitive to mis-specification, making inferences based on this questionable. Motivated by the measurement error and factor analysis approach, we demonstrate a flexible combining method that can integrate information from a wide range of different proxies and simultaneously deal with the quantile regression model with inference about the unobserved covariate. With the assumption that the proxy error added to the proxy variable is independent of each other and true covariate, we describe the Bayesian inference regarding the nonparametric quantile regression function (Thompson et al., 2010) with multiply observed proxies.

The proposed framework has several distinctive advantages. First, we alleviate the form of proxy variables limited to the additive measurement error. It enables us to incorporate proxies with both linear and nonlinear relationships in a true covariate, provided that the proxy error is assumed to be additive. Second, the proposed method can incorporate an arbitrary number of multiple proxies and infer the true covariate and quantile regression function. Third, the proposed method does not make any assumption about the structure of the proxy variables, such as the repeated measurement or presence of an instrumental variable, which might be a strong assumption in practice. Fourth, the proposed framework does not assume the linearity of the quantile regression function or parametric assumptions on the regression error distribution, which restricts the model flexibility. We specifically used the nonparametric spline quantile function suggested by Thompson et al. (2010) to increase the model flexibility.

The rest of the paper proceeds as follows. In Section 2, we describe the basic setup for investigation. In Section 3, we introduce the method of combining multiple proxies and make inferences regarding the nonparametric quantile regression likelihood. The related prior and detailed Gibbs sampler steps are also followed. Section 4 presents the simulation results on various simulation data. In Section 5, we apply the methodology to the administrative data to study the quantile relationship between assets and the actual salary income. Section 6 provides concluding remarks.
2 Basic Setup

Let \( \{y_i, x_i\}_{i=1}^{n} \) be a random sample of size \( n \) where \( y_i \) is the outcome variable and \( x_i \) is the explanatory variable. Let \( g_p(x_i) \) be the \( p \)-th quantile of the conditional distribution of \( y_i \) given \( x_i \) such that

\[
P(y_i \leq g_p(x_i)) = p, \quad 0 < p < 1, \tag{1}
\]

Suppose that the covariate \( x_i \) is not directly observed, but multiple proxies are observed. These proxies are denoted as \( w_{ki}, k = 1, \ldots, K \). We can use the existing popular methods (e.g., Li and Vuong, 1998; Carroll et al., 2006; Delaigle et al., 2008) to estimate the quantile function \( g_p(x_i) \) when only a single proxy exists. However, these methods cannot be directly extended when multiple proxies exist for the unobserved \( x \).

To account for multiple proxies simultaneously, we first write a set of regression models such that

\[
y_i = g_p(x_i) + e_i \tag{2}
\]
\[
w_{1i} = x_i + u_{1i} \tag{3}
\]
\[
w_{ki} = h_k(x_i) + u_{ki}, \quad k = 2, \ldots, K \tag{4}
\]

where the residual \( e_i \) in Eq (2) follows an unspecified distribution satisfying Eq (1), and the measurement errors \( u_{ki} \) in Eqs (3) and (4) are assumed to have a zero mean \( E(u_{ki}) = 0 \) and a constant variance \( Var(u_{ki}) = \sigma_k^2 \) and are further assumed to be independent of each other and distributed independently of \( x_i \) and \( y_i \) conditional on \( x_i \). In Eq (3), we suppose that \( w_1 \) is observed by an adding additive error to the unobserved \( x \). Furthermore, in Eq (4), we leave the other proxies \( w_k (k \geq 2) \) with an arbitrary function of \( x \), denoted by \( h_k(x) \).

Lubotsky and Wittenberg (2006) used similar assumptions with Eqs (2) to (4) to estimate the regression coefficient for the regression of \( y_i \) on \( x_i \) with multiple proxies for \( x_i \). They allowed a nonzero covariance between the measurement errors; however, they assumed a linear relationship with all proxy variables \( w_k \) for \( k \geq 2 \) in Eq (4) and offered a lower bound on the regression coefficient of \( y_i \) on \( x_i \). The variable \( w_1 \) with the additive error was called the benchmark variable. The assumption of the existence of a benchmark variable is prevalent and necessary because it amounts to fixing the scale of the unobserved \( x \).

An additional assumption is required to combine the proxies and identify the unobserved covariate \( x \). (Aigner et al., 1984; Lubotsky and Wittenberg, 2006). In the proposed method, we treat the unobserved \( x \) as a latent variable with its probability distribution and assume a parametric distribution for the corresponding errors \( u_1, \ldots, u_K \) to treat the problem from a Bayesian point of view.
For a better understanding of the proposed method, henceforth we consider the model setup with $K = 2$ with each distribution assumed to follow normal distribution. However, the proposed multiple proxies model can be straightforwardly extended into general cases with $K \geq 3$ with an arbitrary distribution. The simplified settings can be expressed as follows:

\[ y_i = g_p(x_i) + \epsilon_i, \]
\[ w_{1i} = x_i + u_{1i}, \]
\[ w_{2i} = \alpha_0 + \alpha_1 x_i + u_{2i}, \]

where $x_i \sim N(\mu_x, \sigma^2_x)$ and $u_{ki} \sim N(0, \sigma^2_k)$, $k = 1, 2$.

In Eq (5), the proxy variables are not required to be independent of each other or $x_i$. However, the proxies are conditionally independent given the true covariate $x_i$ and can be specified as follows:

\[ w_{1i}|x_i, w_{2i} \sim N\left(x_i, \sigma^2_1\right), \]
\[ w_{2i}|x_i, w_{1i} \sim N\left(\alpha_0 + \alpha_1 x_i, \sigma^2_2\right). \]

This model setup can fully specify all conditional distribution between observed proxy variables and true latent covariate, which suggests that the likelihood of multiple proxy variables can be separately specified with the mixture representation based on conditional distribution given $x_i$ and their prior probability distribution. All the conditional distribution related to $x_i$ can be specified; therefore, the conditional posterior distribution of $x_i$ can be specified up to the normalizing constant. This makes Bayesian Gibbs sampling reasonable for the estimating method when $x_i$ is treated as an auxiliary variable and contributes to the generation of observed data $(y_i, w_{1i}, w_{2i})$.

3 Bayesian Estimation

3.1 Nonparametric Quantile Regression with Multiple Proxy Variables

This section describes how to combine the multiple proxies in Bayesian quantile regression. The conventional approach for quantile regression is based on the polynomial function of covariate $x_i$ (Koenker and Bassett, 1978; Yu and Moyeed, 2001; Yu et al., 2003). However, polynomial quantile regression is often restricted because the degree of the polynomial must be chosen in advance, and data might have a limited local effect on the shape of the polynomial regression curve, especially for high quantiles (Thompson et al., 2010). We consider the nonparametric cubic spline function $g_p(x)$ to alleviate the parametric assumption and secure the model flexibility. The natural cubic spline is a piecewise cubic
polynomial function with continuous first and second derivatives at each knot and is linear beyond the boundary knots. (Green and Silverman, 1993)

To describe the estimation procedures, we suppose that \( \tau_1, \ldots, \tau_N \) are the \( N \) fixed knots covering a range of \( x \). We also assume that \( g = (g_1, \ldots, g_N)^T \) denotes the values of natural cubic spline \( g_p(x) \) at knots \( \tau_1, \ldots, \tau_N \). As a desirable property of the natural cubic spline, there is a unique natural cubic spline function \( g_p(x) \) with knots \( \tau_1, \ldots, \tau_N \) satisfying \( g(\tau_i) = g_i, i = 1, \ldots, N \) for any given values \( g_1, \ldots, g_N \). Therefore we can handle the function \( g(x) \) by its finite-length surrogate \( g \). In terms of Bayesian inference, we can model \( g_p(x) \) by giving the prior to \( g \), not \( g_p(x) \). Following Green and Silverman (1993)'s discussion, the prior for \( g \) is defined by multivariate normal distribution:

\[
\pi(g|\lambda) \propto \exp\left(-\frac{1}{2}\lambda g^T K g\right),
\]

where \( K \) is the matrix made up of the function of the difference between knots, which is specified later, and \( \lambda \) contributes to the smoothness of curve \( g \) and have standard conjugate gamma prior:

\[
\pi(\lambda) = \frac{\lambda^{\alpha-1}\exp(-\frac{\lambda}{2})}{\Gamma(\alpha)\beta^\alpha}, \lambda > 0.
\]

The quadratic term \( g^T K g \) in the exponential kernel is equivalent to the roughness penalty \( \int_a^b g''_p(t)^2 dt = g^T K g \) (Green and Silverman, 1993). This type of quadratic prior, \( \exp\left(-\frac{1}{2}\lambda \int g''_p(t)^2 dt\right) \), measuring the complexity of the parameter is a natural choice because it corresponds to the penalized maximum likelihood. With this prior, the posterior log density of the function \( g_p(x_i) \) is equal to the loss function in the regression context with the roughness penalty added to the kernel of the log-likelihood function. (Hoerl and Kennard, 1970; Green and Silverman, 1993; Tibshirani, 1996)

To define the matrix \( K \), we start by defining the difference between each knot. We let \( h_i = \tau_{i+1} - \tau_i, \quad i = 1, \ldots, N - 1 \), the difference between each knot, and \( Q \) be the \( N \times (N - 2) \) banded matrix with entries \( q_{ij} \), for \( i = 1, \ldots, n \) and \( j = 2, \ldots, n - 1 \), given by

\[
q_{i-1,j} = \frac{1}{h_{j-1}}, \quad q_{j,j} = -\left(\frac{1}{h_{j-1}} - \frac{1}{h_j}\right), \quad q_{j+1,j} = \frac{1}{h_j},
\]

where \( q_{ij} = 0 \) for \( |i - j| \geq 2 \). Let \( R \) be another positive definite banded matrix and \( (N - 2) \times (N - 2) \) symmetric, with its elements given by

\[
r_{ii} = \frac{h_{i-1} + h_i}{3}, \quad i = 2, \ldots, N - 1, \quad r_{i,i+1} = r_{i+1,i} = \frac{h_i}{6}, \quad i = 2, \ldots, N - 2
\]

where \( r_{ij} = 0 \) for \( |i - j| \geq 2 \). Then the \( N \times N \) symmetric matrix \( K \) is defined as \( K = QR^{-1}Q^T \) with \( \text{rank}(K) = N - 2 \).
The final step for the Bayesian approach is to define the likelihood of the natural cubic spline function by changing the conventional polynomial part of standard quantile regression likelihood to the natural cubic spline form (Yu and Moyeed, 2001; Thompson et al., 2010). The resulting likelihood takes the following form:

\[ L(y \mid g, x) = p^n(1 - p)^n \exp \left\{ - \sum_{i=1}^{n} \rho_p(y_i - g_p(x_i)) \right\} \]  \hspace{1cm} (8)

Note that we explicitly specify \( x \) in likelihood \( L(y \mid g, x) \), as \( x \) is also an unknown variable with its prior specified in Eq (5). The proxy variables \( w_k \) are generated from true covariate \( x \) and its corresponding variance parameter \( \sigma_k^2 \) in Eq (5); thus the likelihood does not depend on other parameters if \( x \) is given.

From the model setup above, we set \( \Theta = (g, \lambda, x, \mu_x, \sigma_x^2, \sigma_1^2, \sigma_2^2, \alpha) \) as the unknown parameters to be estimated, including the spline-related parameters \( g, \lambda \) and \( (\mu_x, \sigma_x^2, \sigma_1^2, \sigma_2^2, \alpha) \) associated with the unobserved covariate \( x \). Let \( \pi(\mu_x, \sigma_x^2, \sigma_1^2, \sigma_2^2, \alpha) = \pi(\mu_x)\pi(\sigma_x^2)\pi(\sigma_1^2)\pi(\sigma_2^2)\pi(\alpha) \) be an independent prior of parameters associated with \( x \). For the priors, we used the conjugate normal prior for \( \alpha \) and \( \mu_x \), and the conjugate inverse-gamma prior for \( \sigma_x^2, \sigma_1^2 \) and \( \sigma_2^2 \). Then, the posterior distribution of \( \Theta \) can be written as:

\[
\pi\left(\Theta \mid y\right) \propto L\left(y \mid g, x\right)\pi\left(w_1 \mid w_2, x\right)\pi\left(w_2 \mid x\right) \\
\times \pi\left(x \mid \mu_x, \sigma_x^2\right)\pi\left(g \mid \lambda\right)\pi\left(\lambda\right)\pi\left(\sigma_x^2\right)\pi\left(\sigma_1^2\right)\pi\left(\sigma_2^2\right)\pi\left(\alpha\right).
\]

We use the following conjugate priors for \( \alpha, \mu_x, \sigma_x^2, \sigma_1^2 \) and \( \sigma_2^2 \):

\[
\alpha \sim \mathcal{N}\left(0, \Sigma_\alpha\right), \quad \mu_x \sim \mathcal{N}\left(0, \sigma_\mu^2\right), \quad \sigma_x^2 \sim \mathcal{IG}\left(a_x, b_x\right), \quad \sigma_1^2 \sim \mathcal{IG}\left(a_1, b_1\right), \quad \sigma_2^2 \sim \mathcal{IG}\left(a_2, b_2\right).
\]

Moreover, Eq (8) implies that given auxiliary variable \( x \), we can directly apply the mentioned non-parametric quantile likelihood in Bayesian inference. Further, the dependency of \( x \) is only through the prespecified formula of \( y, w_1, w_2 \), and its own distribution parameters, \( \mu_x, \sigma_x^2 \) which is observed or can be fully specified in terms of the conditional distribution. Therefore, given other parameters, the conditional posterior distribution of \( x \) up to normalizing constant can be specified as follows:

\[
\pi\left(x \mid \Theta_{x-}\right) \propto L\left(y \mid g, x\right)\pi\left(w_1 \mid x, \sigma_1^2\right)\pi\left(w_2 \mid x, \alpha, \sigma_2^2\right)\pi\left(x \mid \mu_x, \sigma_x^2\right) \\
\]

where \( \pi\left(w_1 \mid x, \sigma_1^2\right)\pi\left(w_2 \mid x, \alpha, \sigma_2^2\right)\pi\left(x \mid \mu_x, \sigma_x^2\right) \) represent the normal distributions derived in Eq (6). \( L(y \mid g, x) \) denotes the likelihood in Eq (8), and \( \Theta_{x-} \) denotes all the other parameters except \( x \). This result implies that auxiliary variable \( x \) can be sampled using observed data \( (y, w_1, w_2)^T \) and the related parameter \( \mu_x \) and \( \sigma_x^2 \).
On the other hand, the conditional posterior of $x$ depends on the information from the proxy variables and their associated parameters $(\mu_x, \sigma_x^2, \sigma_1^2, \sigma_2^2, \alpha)$. Given the auxiliary variable $x$, observed data $w_2$ and related parameter $\sigma_2^2$, the conditional posterior of the parameter $\alpha$ that specifies the relationship between $x$ and $w_2$ can be specified up to the normalizing constant as follows:

$$
\pi(\alpha | \Theta_{\alpha-}) \propto \pi(w_2 | x, \alpha, \sigma_2^2) \pi(\alpha),
$$

where $\Theta_{\alpha-}$ denotes all other parameters except $\alpha$. The parameter sampled from this conditional distribution contributes to the conditional posterior distribution of $x$ at the following iteration.

**Remark 1** Although we used a linear function for the relationship $h_k(x)$ between $w_k$ and $x$, $h_k(x)$ can only contribute to its expectation of the conditional distribution, and the family of the distribution remains the same. Therefore, the generalization of $h_k(x)$ to an arbitrary functional form specified in the likelihood form is possible. For example, the natural cubic spline can be used for $h_k(x)$.

The sampling steps for the other parameters are similar. Given $x$ sampled with Eq (9), the spline-related parameters $g$ and $\lambda$ are modeled with the sampled $x$ and observed outcome variable $y$. The parameters associated with the proxies $(\mu_x, \sigma_x^2, \sigma_1^2, \sigma_2^2, \alpha)$ are updated with the sampled $x$ and observed proxies $w_1$ and $w_2$. The updated parameters $\Theta_{x-}$ contribute to the sampling of $x$ in the next iteration. We provide the details of the Gibbs sampling procedure in the following subsection.

### 3.2 Gibbs Sampling Step

We formulate the entire problem in the Bayesian framework and present the Metropolis-Hastings steps. (Gamerman and Lopes, 2006) Following Thompson et al. (2010), we model a quantile function of covariate $x$ using the natural cubic spline with $N$ evenly spaced fixed knots covering a range of $x$.

The parameter $\Theta$ should be simulated from the posterior density. A Gibbs sampling algorithm for the quantile regression model is constructed by sampling each component of $\Theta$ from the full conditional distributions. To implement the Gibbs sampling algorithm, we initialize the parameter $\Theta^{(0)}$. Following Thompson et al. (2010)’s initialization, $g^{(0)}$ is set as the posterior mean values of the quantile regression curve (Yu and Moyeed, 2001) at $\tau_1, \ldots, \tau_N$ and $\lambda^{(0)}$ is obtained by applying the generalized cross-validation of the usual smoothing spline (Green and Silverman, 1993). In addition, $x^{(0)}$ is set as one of the multiple proxies $w_1$, as it is a more reliable proxy in the initialization step with no information about $\alpha$.

One iteration of the Gibbs sampling algorithm at the iteration $t$ is given as follows:
1. Generate candidate $g^*$ from the multivariate normal distribution,

$$g^*|g^{(t-1)} \sim \text{MVN}(g^{(t-1)}, \Sigma),$$

and accept $g^*$ with probability,

$$r = \min \left\{ 1, \frac{L(y|g^*, x^{(t-1)})\pi(g^*|\lambda)q(g^{(t-1)}|g^*)}{L(y|g^{(t-1)}, x^{(t-1)})\pi(g^{(t-1)}|\lambda)q(g^*|g^{(t-1)})} \right\},$$

where $q$ is the proposal density function.

2. Generate candidate $\lambda^*$ from the log-normal distribution,

$$\eta^* \sim N(\log(\lambda^{(t-1)}), \sigma^2_{\lambda}),$$

where $\lambda^* = \exp(\eta^*)$, and accept $\lambda^*$ with probability,

$$r = \min \left\{ 1, \frac{\pi(g^{(t)}|\lambda^*)\pi(\lambda^*)q(\lambda^{(t-1)}|\lambda^*)}{\pi(g^{(t)}|\lambda^{(t-1)})\pi(\lambda^{(t-1)}|\lambda^*)q(\lambda^*|\lambda^{(t-1)})} \right\}.$$

3. Generate $x^*$ from the multivariate normal distribution,

$$x^*|x^{(t-1)} \sim \text{MVN}(x^{(t-1)}, \Sigma_{xx}),$$

and accept $x^*$ with probability,

$$r = \min \left\{ 1, \frac{L(y|g^{(t)}), x^*)\pi(w_1|w_2, x^*)\pi(w_2|x^*)\pi(x^*|\mu^{(t-1)}, (\sigma^2_{x})^{(t-1)})q(x^{(t-1)}|x^*)}{L(y|g^{(t)}, x^{(t-1)})\pi(w_1|w_2, x^{(t-1)})\pi(w_2|x^{(t-1)})\pi(x^{(t-1)}|\mu^{(t-1)}, (\sigma^2_{x})^{(t-1)})q(x^*|x^{(t-1)})} \right\}.$$
7. Sample $\mu_x^{(t)} \sim N(M_*, V_*)$, where
$$V_* = \left( \frac{n}{\sigma_x^2} \right)^{-1} \text{ and } M_* = \left( \frac{\sum_{i=1}^n x_i^{(t)}}{\sigma_x^2} + \frac{M_\mu}{V_\mu} \right) / \left( \frac{n}{\sigma_x^2} + \frac{1}{V_\mu} \right)$$
with $M_\mu$ and $V_\mu$ the prior mean and variance for $\mu_x$.

8. Sample $\alpha^{(t)} \sim \text{MVN}(M_*, V_*)$, where
$$V_* = \left( \frac{X^T X}{\sigma_1^2} + V_\alpha^{-1} \right)^{-1}, M_* = \left( \frac{X^T X}{\sigma_1^2} + V_\alpha^{-1} \right)^{-1} \left( \frac{X^T W_1}{\sigma_1^2} + V_\alpha^{-1} M_\alpha \right)$$
with $M_\alpha$ and $V_\alpha$ as the prior mean vector and covariance matrix for $\alpha$ and $X$ as the vector of $x_i^{(t)}$'s, $i = 1, \ldots, n$.

Steps 1 to 3 require the Metropolis-Hastings algorithm, and the other steps can be easily sampled from the conjugate distribution. Inference about the unobserved regressor, quantile spline function, or regression coefficient is based on these posterior samples.

4 Simulation

A simulation study is conducted to evaluate the performance of the proposed method on various simulation datasets. We consider three scenarios studied in Altman (1992); Friedman et al. (2001), and Oh et al. (2011). To match the scale between each dataset, we scale the range of $x$ to $[-5, 5]$ for each simulation with the appropriate affine transformation.

We use three error distributions to evaluate the proposed model’s performance in various experimental settings: the first is a normal distribution, the second is the $t$-distribution with a larger tail, and the third comprises heterogeneous errors dependent on the covariate $x_i$. We further suppose that the actual covariates $x_i$ are not directly observed and that various proxies are observed in the manner described in (5); that is, we have
$$w_{1i} = x_i + u_{1i}$$
$$w_{2i} = \alpha_0 + \alpha_1 x_i + u_{2i}$$
with $\alpha = (4, 3)$ and $u_{1i}, u_{2i} \sim N(0, 1)$.

In each simulation, three estimators are considered for the comparison:

1. Model without the measurement error (woME): the benchmark estimator calculated using Thompson et al. (2010)’s model, with true covariate $x$ observed without measurement error.
2. Naïve estimator (Naïve): the naïve estimator that replaces $x$ with $w_1$;

3. Bayesian estimator with multiply proxies (BEMP): the proposed Bayesian estimator which incorporates two multiple proxies $w_1$ and $w_2$.

In this simulation, another potential naïve model, which replaces $x$ with $w_2$, is not considered because the systematic difference (i.e. $w_{2i} = \alpha_0 + \alpha_1 x_i + u_{2i}$) might degrade the model performance to be inferior to that for the Naïve estimator.

We run 500 Monte Carlo (MC) simulations with the size of $n = 1000$. We assume identical MCMC sampling settings for all three models. We set the number of iterations to 300,000 and take every 50th sample after discarding the first 50,000 steps as the burn-in period. The convergence is satisfactory, and the average of 5,000 posterior samples is used for the point estimation. Three popular metrics are used to compare the estimators (Härdle, 1986; Fan, 1992; Gelman et al., 1995):

- 95% highest posterior density (HPD) interval: we check whether the true quantile function values in each knot (i.e. $g$) are included in its 95% HPD intervals;

- Mean squared error (MSE):

\[
MSE(\hat{g}_p) = \frac{1}{n} \sum_{i=1}^{n} (g_p(x_i) - \hat{g}_p(x_i))^2,
\]

where $\hat{g}_p(x_i)$ is estimate of $g_p(x_i)$;

- Integrated squared error (ISE):

\[
ISE(\hat{g}_p) = \int_{-\infty}^{\infty} (g_p(x_i) - \hat{g}_p(x_i))^2 dx.
\]

4.1 Simulation 1

Referring to Altman (1992), we generate the first synthetic datasets in a nonlinear regression shape,

\[
y_i = x_i \sin(2.5\pi x_i) + 0.5e_i, \quad i = 1, \ldots, 1000
\]

where $e_i \sim N(0,1)$. The true function of the $p$th quantile given $x$ can be expressed explicitly:

\[
g_p(x_i) = x_i \sin(2.5\pi x_i) + 0.5\Phi^{-1}(p),
\]

where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution.

Table 1 presents the simulation results for 500 MC simulations. The 95% HPD interval of the BEMP method includes the true quantile function correctly with little variation in performance across
| Quantile | woME    | Naïve  | BEMP    |
|---------|---------|--------|---------|
| $p = 0.1$ | HPD 0.961 (0.015) | 0.673 (0.111) | **0.953 (0.025)** |
|         | ISE 0.159 (0.156) | 0.706 (0.223) | **0.164 (0.109)** |
|         | MSE 0.015 (0.013) | 0.070 (0.022) | **0.016 (0.010)** |
| $p = 0.25$ | HPD 0.966 (0.015) | 0.592 (0.158) | **0.962 (0.018)** |
|         | ISE 0.055 (0.032) | 0.477 (0.204) | **0.070 (0.043)** |
|         | MSE 0.005 (0.003) | 0.047 (0.020) | **0.007 (0.004)** |
| $p = 0.5$ | HPD 0.999 (0.007) | 0.559 (0.250) | **0.998 (0.014)** |
|         | ISE 0.033 (0.020) | 0.498 (0.393) | **0.058 (0.055)** |
|         | MSE 0.003 (0.002) | 0.049 (0.039) | **0.006 (0.005)** |
| $p = 0.75$ | HPD 0.968 (0.012) | 0.705 (0.153) | **0.965 (0.022)** |
|         | ISE 0.046 (0.025) | 0.316 (0.211) | **0.062 (0.041)** |
|         | MSE 0.004 (0.002) | 0.031 (0.021) | **0.006 (0.004)** |
| $p = 0.9$ | HPD 0.964 (0.010) | 0.757 (0.092) | **0.959 (0.022)** |
|         | ISE 0.120 (0.100) | 0.443 (0.154) | **0.129 (0.059)** |
|         | MSE 0.012 (0.009) | 0.044 (0.015) | **0.013 (0.006)** |

Table 1: Monte Carlo means (standard errors) of HSI, ISE, and MSE for Simulation 1
all quantiles, which is comparable to the woME method. The naïve approach exhibits inferior performance with the 95% HPD inclusion probability ranging from 0.559 to 0.757. In terms of ISE and MSE, the proposed method outperforms the naïve approach, especially for the case in which \( p = 0.5 \), with an MSE four to eight times larger than the BEMP. This result reveals that the proposed method has an efficient estimator with more information gained from combining proxy variables.

Figure 1: Fitted lines of the estimators in Simulation 1. (a) The fitted lines for \( p = 0.1 \), (b) The fitted lines for \( p = 0.5 \), (c) The fitted lines for \( p = 0.9 \). The dotted black line represents the true quantile function. The red, green, and blue lines are fitted curves of woME, BEMP, and naïve, respectively.

Figure 1 visualizes the average of the fitted lines in 500 MC simulations for each estimator. We can further confirm that the BEMP method works better than the naïve approach across all quantiles in the presence of measurement errors.

4.2 Simulation 2

The second example based on Friedman et al. (2001) is generated from the smoothing splines, 

\[ y_i = \sin(12(x_i + 0.2))/(x_i + 0.2) + e_i, \quad i = 1, \ldots, 1000 \]

where \( e_i \sim t(3) \). The error term follows a \( t \)-distribution with a heavier tail than the normal distribution. The true quantile function \( g_p(x_i) \) can be expressed explicitly:

\[ g_p(x_i) = \sin(12(x_i + 0.2))/(x_i + 0.2) + 0.5\Phi_{t}^{-1}(p) \]

where \( \Phi_{t}(\cdot) \) is the cumulative function of the \( t \)-distribution.

Table 2 reveals that similar to Simulation 1. In most cases, the proposed method includes the true quantile function (HPD 0.85 ~ 0.91), whereas the naïve approach hardly includes the true value (HPD 0.35 ~ 0.50). The inference of extreme quantiles (e.g., \( p = 0.1 \) and \( p = 0.9 \)) becomes worse because sufficient data for the inference is hard to obtain. The problem worsens, especially when the true covariate
| Quantile | woME       | Naïve      | BEMP       |
|----------|------------|------------|------------|
|          | HPD        | ISE        | MSE        |
| p = 0.1  | 0.929 (0.045) | 0.507 (0.083) | 0.865 (0.071) |
|          | 1.506 (0.539) | 11.338 (2.301) | 2.630 (0.690) |
|          | 0.147 (0.053) | 1.127 (0.226) | 0.260 (0.070) |
| p = 0.25 | 0.953 (0.024) | 0.400 (0.087) | 0.892 (0.058) |
|          | 0.844 (0.268) | 8.131 (1.229) | 2.044 (0.529) |
|          | 0.082 (0.024) | 0.806 (0.125) | 0.201 (0.054) |
| p = 0.5  | 0.958 (0.018) | 0.353 (0.084) | 0.907 (0.050) |
|          | 0.593 (0.158) | 6.635 (0.799) | 1.423 (0.373) |
|          | 0.057 (0.014) | 0.655 (0.089) | 0.138 (0.036) |
| p = 0.75 | 0.955 (0.022) | 0.406 (0.091) | 0.876 (0.043) |
|          | 0.705 (0.183) | 9.102 (1.792) | 2.040 (0.655) |
|          | 0.068 (0.017) | 0.895 (0.174) | 0.199 (0.065) |
| p = 0.9  | 0.941 (0.033) | 0.507 (0.081) | 0.859 (0.047) |
|          | 1.628 (0.587) | 13.617 (2.772) | 3.888 (1.238) |
|          | 0.158 (0.056) | 1.344 (0.272) | 0.382 (0.123) |

Table 2: Monte Carlo means (standard errors) of HSI, ISE, and MSE for Simulation 2
is the unknown variable. As a result, the relative difference between the ISE of this method and that of the woME is larger than in Simulation 1. Nevertheless, it exhibits superior performance with the ISE about four times smaller than the naïve approach.

Figure 2: Fitted lines for estimators in Simulation data 2. (a) The fitted lines for $p = 0.1$, (b) The fitted lines for $p = 0.5$, (c) The fitted lines for $p = 0.9$. The dotted black line represents true quantile function. The red, green, and blue lines are fitted curves of woME, BEMP, and naïve, respectively.

Figure 2 indicates that the naïve approach fails to capture the true underlying relationship compared to the BEMP and woME. For these data, the true quantile function is considerably curved, and the performance of the naïve approach suffers further.

4.3 Simulation 3

As the last example, we followed Oh et al. (2011) with heteroskedasticity error,

$$y_i = \sin(10x_i) + (x_i + 0.25)/(0.1)e_i, \quad i = 1, \ldots, 1000$$

where $e_i \sim N(0, 1)$. It follows that the actual quantile function of the $p$th quantile given $x$ is as follows,

$$g_p(x_i) = \sin(10x_i) + (x_i + 0.25)/(0.1)\Phi^{-1}(p)$$

Since $\Phi^{-1}(p)$ is multiplied by $x_i$, the spread of the quantile function depends on the value of $x$; therefore, heteroscedasticity exists. In other words, the spread of the quantile function becomes larger as the value of $x$ increases. The heterogeneous error is prevalent in real data.

Table 3 reveals that the proposed methodology performs better than the naïve approach for the three criteria in all quantiles. However, similar to the previous simulation results, the boundary effect causes performance deterioration in the extreme quantiles. Again, the fitted quantile function of the BEMP in Figure 3 follows the true quantile function relatively well, whereas the Naïve approach cannot capture the overall shape of the nonlinear relationship due to the measurement error in the proxy variable.
Table 3: Monte Carlo means (standard errors) of HSI, ISE, and MSE for Simulation 3

| Quantile | woME      | Naïve     | BEMP      |
|----------|-----------|-----------|-----------|
| $p = 0.1$ | HPD 0.982 (0.020) | 0.380 (0.061) | **0.723 (0.095)** |
|          | ISE 0.148 (0.144) | 2.5448 (0.450) | **0.945 (0.287)** |
|          | MSE 0.014 (0.014) | 0.254 (0.045) | **0.094 (0.029)** |
| $p = 0.25$ | HPD 0.981 (0.020) | 0.316 (0.161) | **0.867 (0.078)** |
|          | ISE 0.070 (0.045) | 2.1430 (1.383) | **0.483 (0.362)** |
|          | MSE 0.007 (0.004) | 0.214 (0.138) | **0.048 (0.036)** |
| $p = 0.5$ | HPD 0.993 (0.016) | 0.498 (0.085) | **0.948 (0.027)** |
|          | ISE 0.056 (0.035) | 0.772 (0.160) | **0.169 (0.071)** |
|          | MSE 0.005 (0.003) | 0.077 (0.016) | **0.017 (0.007)** |
| $p = 0.75$ | HPD 0.999 (0.008) | 0.519 (0.065) | **0.815 (0.107)** |
|          | ISE 0.062 (0.036) | 1.145 (0.217) | **0.428 (0.129)** |
|          | MSE 0.005 (0.004) | 0.114 (0.022) | **0.042 (0.013)** |
| $p = 0.9$ | HPD 0.998 (0.009) | 0.501 (0.054) | **0.641 (0.095)** |
|          | ISE 0.138 (0.136) | 2.144 (0.382) | **1.034 (0.270)** |
|          | MSE 0.013 (0.013) | 0.214 (0.038) | **0.103 (0.027)** |

Figure 3: Fitted line for estimators in Simulation data 3. (a) The fitted line for $p = 0.1$, (b) The fitted line for $p = 0.5$, (c) The fitted line for $p = 0.9$. The dotted black line represents the true quantile function. The red, green and blue lines are fitted curves of the woME, BEMP and naïve, respectively.
5 Application to Administrative Data

We apply the proposed method to a real public dataset that includes assets and income variables. Statistics Korea released the microdata on the Survey of Household Finances and Living Conditions (SFLC), incorporating administrative data obtained from other government institutions. The released dataset includes basic demographic variables for 18,064 families and various economic features such as salary income, property income, asset, asset management plans, debt, and debt repayment capacity for each family unit, collected in 2020.

This application aims to determine the quantile relationship between assets and the true salary income. The administrative salary income is exposed to measurement error so that the direct use of this information can result in misleading inference. Therefore, we consider using the administrative salary income and property income as two types of proxies; one is exposed to additive error, and the other is a correlated proxy. It is worth using as a proxy since it makes sense to assume that the property income and salary income have a high correlation. In consequence, the model is described below:

\[
\text{asset}_i | \text{true salary income}_i = g_p(\text{true salary income}_i) + e_i \\
\text{administrative salary income}_i = \text{true salary income}_i + u_{1i} \\
\text{property income}_i = \alpha_0 + \alpha_1 \text{true salary income}_i + u_{2i}
\]

As a preprocessing process, we remove the abnormal data such as zero or extreme quantiles (i.e., 0.999 and 0.001 percentile in terms of each variable). After the preprocessing procedure, the data contains 11,317 family units. Further, we try log transform for asset variables to alleviate data skewness and improve model convergence. Following Thompson et al. (2010)’s suggestion, we take \( N = 30 \) knots equally spaced over the range of variables, which is log-transformed administrative salary income.

Following the simulation setup, we consider the five quantiles according to \( p \in \{0.1, 0.25, 0.5, 0.75, 0.9\} \). The resulting quantile lines are presented in Figure 4. The spread of fitted relation is not parallel, meaning a heterogenous effect exists. More importantly, we determine that the difference between each fitted line is not equal in the lower level of administrative salary income (i.e., administrative salary income \( \leq 10,000 \)). Especially for the higher quantile, such as \( p = 0.9 \), the fitted line in the lower level of the administrative salary income generally fluctuates upward with the fitted quantile asset, increasing in administrative salary income \( \simeq 3,000 \) and reducing again. This outcome might be due to various causes; for example, people with their assets not proportional to salary income structurally or those with high quantile asset values may intentionally misreport salary income. In either case, nonparametric quan-
Figure 4: Fitted line for income data application. The $x$-axis denotes the administrative salary income from the survey, but is suspected of having a survey measurement error. The $y$-axis represents the assets. The fitted line is based on the quantile $p = (0.1, 0.25, 0.5, 0.75, 0.9)$. 
tile regression can derive a new intuition that was impossible through parametric regression and more informative in combining proxy than the naïve approach.

6 Conclusion

This paper proposed a Bayesian quantile regression estimation method integrating multiple proxies of the unobserved covariate. The proposed method has two notable advantages compared to previous methods. First, the proposed approach handles multiple data sources in a Bayesian estimation procedure, whereas the previous methods have been developed for handling a single proxy with only an additive error or limited structure. Another strength is that the proposed method is not restricted to linear regression functions and does not require parametric assumptions about the distribution of regression errors. Provided that the proxy error is assumed to be additive, the proposed method can accommodate multiple proxies with linear and nonlinear relationships with the true covariate.

We confirmed the effectiveness of the proposed methodology using a set of three simulated datasets for which linear modelings might be inappropriate. We demonstrated that the proposed method is promising in incorporating multiple proxies simultaneously and capturing the underlying relationship compared with the method of using one proxy variable directly. We also presented an application of this methodology with the public dataset from the SFLC and provided the underlying relationship between the asset and salary income in the presence of multiple income records.

Some limitations exist in this study. We adopted a cubic spline function for quantile regression; thus, the behavior of the fitted regression line tends to be erratic near the boundaries, which is a well-known characteristic of the spline approach (Friedman et al., 2001). The current study still relies on the presence of a benchmark variable, which is necessary for estimating the scale of the unobserved covariate, and we left it for future research. Furthermore, the proposed method can be generalized to other regression problems, such as linear regression or the nonparametric approach that can be specified in the likelihood form by replacing the likelihood of the quantile regression with others. The extensions of the proposed data integration method and their further analysis, such as robustness, remain for future research.

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