We observe that the existence of sequential and parallel composition supermaps in higher order physics can be formalised using enriched category theory. Encouraged by physically relevant examples such as unitary supermaps and layers within higher order causal categories (HOCCs), we treat the modelling of higher order physical theories with enriched monoidal categories in analogy with the modelling of physical theories are with monoidal categories. We use the enriched monoidal setting to construct a suitable definition of structure preserving map between higher order physical theories via the Grothendieck construction. We then show that the convenient feature of currying in higher order physical theories can be seen as a consequence of combining the primitive assumption of the existence of parallel and sequential composition supermaps with an additional feature of linking. In a second application we use our definition of structure preserving map to show that categories containing infinite towers of enriched monoidal categories with full and faithful structure preserving maps between them inevitably lead to closed monoidal structures. The aim of the proposed definitions is to step towards providing a broad framework for the study and comparison of novel causal structures in quantum theory, and, more broadly, a paradigm of physical theory where static and dynamical features are treated in a unified way.
1 Introduction

Traditionally, physical theories have been concerned with the laws governing the evolution of certain physical systems, such as particles or fields. In the ontology of a theory, the physical systems are regarded as fundamental objects, while their evolution is regarded as a tool for predicting relations among objects in different regions of space and time. Over the past decade, a series of works in quantum information theory started exploring the idea that processes themselves could be regarded as objects, which can be acted upon by a kind of higher order physical transformations, known as quantum supermaps [1–7]. Quantum supermaps have found a wide range of applications to quantum information and computation [8–23], and to the study of new types of causal structures arising in quantum mechanics [3, 24–26]. In addition, higher order transformations provide a broad framework for general physical theories with dynamical causal structure, and, eventually, are expected contribute to the formulation of a complete theory of quantum gravity, as originally suggested by Hardy [27]. Complementary to this research direction is the development of programming languages which permit higher order types whilst retaining compatibility with quantum theory (by forbidding cloning [28], the signature of the cartesian monoidal structure underlying the standard lambda calculus), such as linear or quantum lambda calculi [29–34].

A compositional foundation for the study of physical theories, including quantum and classical theory, is provided by the process theory framework [35]. The framework is built on the notion of a symmetric monoidal category, which captures some basic structures present in a broad class of physical theories. Such structures include a notion of system, a notion of processes between systems, and, crucially, a notion of the sequential and parallel composition of processes, diagrammatically represented as

\[ \begin{array}{c}
\begin{array}{cc}
A & \rightarrow & B \\
\downarrow & & \downarrow \\
C & & D \\
\end{array}
\end{array} \quad \begin{array}{c}
\begin{array}{cc}
A & \rightarrow & B \\
\downarrow & & \downarrow \\
C & & D \\
\end{array}
\end{array} \]
The tensor unit $I$ represents the trivial system, and, for every object $A$, the processes of type $I \to A$ are viewed as states of system $A$. The process theoretic treatment has contributed new insights into quantum foundations and the general structure of physical theories [36–51].

In a recent work [52], a process theoretic framework for supermaps was developed for the purpose of providing a categorical language for causal structures. In this framework, causal structures are represented by the objects of a $*$-autonomous category of higher order processes $\text{Caus}(C)$ built on from a “pre-causal” category $C$. This construction reveals deep relations between $*$-autonomy and the structure of higher order transformations in quantum theory, in particular producing a convenient type system for reasoning about causal structures. However, there is a sense that the notion of a raw-material pre-causal (and so compact closed) category may be too restrictive a requirement in the study of infinite dimensional systems such as those encountered in quantum field theory, and ultimately quantum theories of gravity.

In this work we aim to pin down a notion of a higher order theory as a mathematical structure in its own right, independently of the study of causality, and independently of the notion of a raw material category from which a theory might be constructed. Our main motivations for formalisation of structural features of higher order physical theories are the following:

- Quantum resource theories were recently captured in the abstract by construction from the notion of a sub-symmetric monoidal category [53], at the level of identifying quantum theory as forming a symmetric monoidal category, this essentially phrases the definition of a resource theory as simply the definition of a sub-theory. To build on the approaches of [54–56] and fully extend resource theories in a satisfactory way to higher order quantum theory, we need to develop a notion of higher order sub-theory.

- Identification of structural features of higher order physics will allow for the study of higher order principles for the axiomatization of physical theories. Preliminary investigations in this direction, which build from the results of this paper by beginning from the assumption of closed monoidal structure and then imposing causality principles, are presented in [57].

- Once enough structural features are identified it may be possible to characterize supermap definitions in terms of universal properties. Indeed, the generally accepted definition of first-order quantum process can be motivated in this way, as arising from a universal property in the category of affine monoidal categories [58].

The contributions of this paper can be summarised in three parts, first, enrichment is used to formalize in categorical terms the existence of sequential and parallel composition supermaps. Second, suitable structure preserving maps with respect to enrichment are defined and identified as the result of the Grothendieck construction for the change of base for enriched symmetric monoidal categories. Third, closed monoidal structure of higher order quantum theories is motivated in terms of extra axioms on top of monoidal enrichment, in doing so the previously developed notion of structure-preserving map is applied. We now expand on each of these points in more detail.

**Process Manipulation:** Any symmetric monoidal category $C$, can be interpreted as theory of processes which can be composed in sequence or in parallel. A feature common to theories of transformations of processes is the existence of higher order transformations which actually
perform these composition rules. Namely a key feature is the existence of higher order processes which put processes together in sequence or in parallel

This feature of higher order processes can be identified with the categorical definition of a $V\simeq$-smc $C$. This equips the higher theory $V$ with types $[A, A']$ representing the space of processes of type $A \to A'$ in a lower order theory $C$, and a tensor $\otimes_V$ so that $[A, A'] \otimes [B, B']$ represents the space of bipartite processes which can be plugged together in either order, being treated as freely manipulable.

**Structure Preserving Maps:** Our second contribution is to show that one can define structure preserving maps between theories of higher order transformations. These are defined to be functors $F^C : C \to C'$, $F^V : V \to V'$ on the lower and higher order parts of the theory along with morphisms of type $F_{AB} : F^V[A, B] \to [F^C A, F^C B]$ which encode the preservation of enriching structure.

These functors are then identified as the result of the Grothendieck construction applied to the change of base functor for symmetric monoidal enriched categories. This introduced notion of structure preserving map allows us to formalise the following, given the statements (a): $C^3$ is a theory of higher order transformations for $C^2$ and (b): $C^2$ is a theory of higher order transformations for $C^1$, then there is a morphism $\Gamma : (b) \to (a)$ embedding the latter statement into the former.

A key application of identifying structure preserving maps as the result of the Grothendieck construction is that such maps are immediately guaranteed to be composable forming a category of higher-order theories.

**Linking and Process Manipulation $\to$ Closed Monoidal Structure:** The third contribution is to show that the possibility to carry processes, that is, the existence of closed monoidal
structure, can be derived from appending an additional notion to the above primitive operational
principles observed in higher order physical theories. We show that currying can be viewed as a
consequence of (i) The possibility to compose processes in sequence, (ii) The possibility to compose
processes in parallel, (iii) The possibility to translate between an object \( A \) and the space \([I, A]\)
of states on \( A \). These principles are combined together in the definition of a linked, and faithful,
enriched monoidal category, which is shown to be equivalent to the definition of closed monoidal
category. The crux of the proof can be conveyed intuitively using the following picture

Here we see links used to convert between first and second-order systems, combined with sequential
composition supermaps to construct an evaluation process of type \( A \otimes [A, B] \rightarrow B \). This result
allows us to state, a series of reasonable physical principles which motivate working with closed
monoidal structure without directly assuming the possibility to curry processes.

By using the introduced notion of structure preserving map \( \Gamma : (b) \rightarrow (a) \) between layers within
sequences of higher order theories: we proceed to generalise this result by showing that any infinite
sequence of enriched monoidal categories, with well behaved structure preserving maps between
the layers of the sequence, leads to closed monoidal structure

Closed monoidal structure is a simple to state and easy to interpret mathematical structure on
top of monoidal structure, the aim of these results is to show that deductions made by combining
other physical principles with closed monoidal structure are likely to be general statements about
higher order physical theories, first steps in the direction of combining closed monoidal structure
with other standard physical principles such as causality and determinism are given in [57].
2 Notation and basic definitions

We will use the abbreviation SMC for symmetric monoidal category, the definition of which may be found in [59]. An SMC consists of objects $A, B, \ldots$ morphisms $f: A \to B$ and composition rules $(\circ, \otimes)$. A morphism $f: A \to B$ can be represented by a box with input wire $A$ and output wire $B$. The parallel composition $f \otimes g$ of morphisms is written by placing $f$ next to $g$, the sequential composition $g \circ f$ of $f: A \to B$ and $g: B \to C$ is written by connecting boxes along wire $B$ as in the following pictures:

There is furthermore a unit object $I$ which is not explicitly written, interpreted as representing only empty space. Similarly associativity of sequential composition and associativity of parallel composition up to natural isomorphism are absorbed into the graphical language, neither being explicitly written. The categorical notion of one monoidal category living inside another is that of a monoidal functor. In general, there may be more than one way of representing a monoidal category $C$ inside another monoidal category $D$. The notion of equivalence between two representations is that of a monoidal natural isomorphism. We will often refer to a full subcategory of a category $C$ with objects given by combining all objects in some set $S \subseteq \text{ob}(C)$ iteratively using some family of functions $\boxtimes_k: \text{ob}(C) \times \text{ob}(C) \to \text{ob}(C)$. As a shorthand for such a set generated by $S$ and functions $\boxtimes_k$ we use the symbol $S|\boxtimes_1|\ldots|\boxtimes_n$.

Closed Monoidal Categories

Closed monoidal structure [59] is a standard categorical structure defined with the purpose of abstracting the notion of currying found in the category Set of functions between sets. Currying for functions is the property that for every pair of sets $A, B$ there exists a function $\text{eval}_{A, B}: A \times \text{Set}(A, B) \to B$ which on elements is defined by $\text{eval}_{A, B}(a, f) := f(a)$. In the general monoidal setting then closed monoidal structure of a monoidal category $C$ is given by requiring a co-universal arrow of type $\text{eval}_{A, B}: A \otimes [A, B] \to B$. Explicitly such a co-universal arrow specifies for each morphism $f: A \otimes C \to B$ a unique morphism $\hat{f}: C \to [A, B]$ such that

This generalises currying by giving a natural isomorphism between $C(A \otimes C, B)$ and $C(C, [A, B])$. The evaluation morphism can intuitively be understood as an open hole into which a process can be inserted.

Higher Order Causal Categories

Moving beyond first-order process theories in which objects are interpreted as representing state spaces and processes are interpreted as transformations of
states, is the higher order framework for quantum theory, in which transformations of processes are considered, intuitively represented as

\[
S \approx \frac{s}{s}.
\]

To define iterated higher order transformations of quantum processes a construction of higher order quantum theory (HOQT) is provided in [7] which takes advantage of the Choi isomorphism. The Choi isomorphism can be viewed as a particular consequence of compact closure, and using this observation the deterministic part of HOQT can be generalised to a wider variety or racematerial physical theories. For any pre-causal (and so compact closed) category \(C\) a category \(\text{Caus}[C]\), which we now for notational convenience refer to as \([C]\), can be constructed. \([C]\) includes lower order types \(A, B, \ldots\), two tensor products \((\otimes, \&))\), and methods for constructing higher order types given by closed monoidal structure. To each pair of objects \(A, B\) an object \([A, B]\) is specified representing the space of transformations from \(A\) to \(B\). Higher order processes can then be represented as those acting on higher order types, for instance \(S : [A, A'] \to [B, B']\) represents a quantum supermap from processes of type \(A \to A'\) to processes of type \(B \to B'\). In this paper we will refer to the subcategory \([C]_1\) as the category with objects given by first order types as defined in [52]. For \(C := \text{CPM}[\text{FHilb}]\) then \([C]_1\) is equivalent to the category of CPTP quantum processes. We similarly refer to objects in \([A_1, B_1] \otimes [A_1, B_1]\) first order types as second order types and define \([C]_2\) to be the full subcategory with objects given by second order types. Iterating this we can define \([C]_{n+1}\) with objects given by \([A_n, B_n] \otimes [A_n, B_n]\). Each such category is symmetric monoidal with unit given by \(I_n := [I, [I, [\ldots]]]\).

Intuitively, each of these categories \([C]_{n+1}\) can be interpreted as a theory of higher order transformations of the processes in \([C]^n\). In this paper we will argue the a key feature of \([C]_{n+1}\) which allows it to be considered in this way is that it contains morphisms which implement the sequential and parallel composition of morphisms of \([C]^n\). Let us now see how such morphisms can be seen to be present within the layers of HOCCs. For first order types \(A, B, A', B'\) then \([A, A'] \otimes [B, B']\) represents the space of no-signalling channels from \(AB\) to \(A'B'\). I.E those which forbid signalling from \(A\) to \(B'\) and from \(B\) to \(A'\):

For any \(A, B, C\) in \([C]_1\) there exists a morphism \(\circ_{ABC} : [A, B] \otimes [B, C] \to [A, C]\) in \([C]_2\) given by plugging together wires of non-signalling channels

\[
\begin{array}{c}
\text{A} \\
\hline
\text{B} \\
\end{array}
\begin{array}{c}
\text{B'} \\
\hline
\text{C} \\
\end{array}
\]

Indeed, one can see that \(\circ_{ABC}\) is a morphism by checking it preserves first order processes, this is
verified by noting that every deterministic non-signalling transformation factorises as:
\[
\begin{array}{ccc}
A & A' \\
\downarrow & \downarrow \\
B & B' \\
\end{array} \quad \mapsto \quad \begin{array}{ccc}
A & A' \\
\downarrow & \downarrow \\
B' & B \\
\end{array} \\
\]
and so applying the morphism \( \circ_{ABC} \) gives:

\[
\begin{array}{ccc}
A & B \\
\downarrow & \downarrow \\
C & B \\
\end{array} \quad \mapsto \quad \begin{array}{ccc}
A & C \\
\downarrow & \downarrow \\
B & B \\
\end{array}
\]
which is a morphism of \([C]^1\) since \([C]^1\) is an smc. This observation, that there is a composition process which can be applied to tensor products of types generalises to \([C]^i\) and can be seen as a consequence of the fact that \([C]\) is a closed monoidal category. In this sense the existence of a sequential composition process can be seen as a generalisation of the non-signalling property of quantum channels to higher order processes.

The \(Caus[C]\) construction can be generalised to \(Caus_{\leq}[C]\) to include non-deterministic events as are present in HOQT whenever the pre-causal category \(C\) is a locally posetal monoidal 2-category. Intuitively a locally posetal monoidal 2-category is one for which every homset \(C(A,B)\) is equipped with a partial-order \(\leq\) such that \(f \leq f' \land g \leq g'\) implies both that \(f \circ g \leq f' \circ g'\) and \(f \otimes g \leq f' \otimes g'\). The objects of \(Caus_{\leq}[C]\) (from now on denoted \([C]_{\leq}\) for convenience) are taken to be those of \([C]\) and the morphisms from \((A,c_A)\) to \((B,c_B)\) are taken to be those \(f\)'s such that there exists \(f'\) s.t. \(f \leq f' \in [C]((A,c_A),(B,c_B))\). Note that \(Caus_{\leq}[C] = Caus[C]\), we however keep in mind the preorder on positive operators inherited from the natural ordering relation on the positive numbers, in this case the non-deterministic events are those of HOQT. From now on we denote \(Caus_{\leq}[C]\) by \([C]_{\leq}\), and identically to the construction given for \([C]\) one can construct for each \(i\) an \(i^{th}\)-order category \([C]_{\leq i}\).

3 Main Observation: Monoidal Enrichment

In this section we aim to highlight and formalize a prominent feature of higher order physical theories, the existence of primitive higher order sequential and parallel composition processes. In short, we make the following observation

\(Higher Order Theories of Transformations of Processes are Enriched Monoidal Categories.\)

The use of enrichment as a semantics for higher-order manipulation of functions has been previously observed in [60], where the higher order theory was taken to be cartesian, we will however be interested in theories with non-trivial correlations between processes being manipulated such as those present between the two halves of a non-signalling channel. When we use the term "enriched monoidal category" we mean a slight generalisation of it’s standard usage, rather than the notion of a \(V\)-enriched symmetric monoidal category \(C\) (from now on termed \(V\)-smc \(C\) for short) we use the notion of a \(V_{\mathbb{R}}\)- smc \(C\) (defined explicitly in the appendix). In a \(V_{\mathbb{R}}\)- enriched symmetric
monoidal category $\mathbf{C}$, for each pair of objects $A, B$ of $\mathbf{C}$ there exists an object $[A, B]$ in $\mathbf{V}$ whose states represent processes in the underlying category $\mathbf{C}$ via a bijection

$$\kappa : \mathbf{C}(A, B) \cong \mathbf{V}(I, [A, B])$$

In string diagrams of $\mathbf{V}$ and $\mathbf{C}$ respectively the isomorphism can be represented by:

$$[A, B] \cong [ ]$$ \hspace{1cm} (4)

From now on we omit from explicitly writing $\kappa$ whenever its presence is clear. In the standard definition of a $\mathbf{V}$-smc $\mathbf{C}$ this bijection would be required to be an equality and so would not so easily incorporate standard constructions of higher order physical theories in which variants of the Choi isomorphism are used $[7, 52, 61]$. For each $[A, B]$ and $[B, C]$ in a $\mathbf{V}_\mathbb{N}$-smc $\mathbf{C}$ there is a morphism in $\mathbf{V}$ which allows to plug their underlying processes together:

$$\bigcirc : [A, B] \otimes [B, C] \rightarrow [A, C]$$

represented formally on the left as a string diagram in $\mathbf{V}$ and informally on the right to show intuitively its action on the underlying category $\mathbf{C}$:

$$[A, C] \cong [ ]$$ \hspace{1cm} (5)

Associativity and unitality of the sequential composition process in $\mathbf{V}$ and the guarantee that it actually implements sequential composition for $\mathbf{C}$, are represented by:

$$\bigcirc \bigcirc \bigcirc \bigcirc \bigcirc$$ \hspace{1cm} (6)

Monoidal enrichment $[62]$ provides furthermore a parallel composition process in $\mathbf{V}$: of type

$$\otimes_{ABA'B'} : [A, A'] \otimes [B, B'] \rightarrow [A \otimes B, A' \otimes B']$$

which can be represented formally and intuitively respectively by:

$$[A \otimes B', A' \otimes B] \cong [ ]$$ \hspace{1cm} (7)

Again in a $\mathbf{V}_\mathbb{N}$-smc $\mathbf{C}$ the following conditions are required, guaranteeing that $\otimes_{ABA'B'}$ really does behave like a parallel composition process:

$$\bigcirc \bigcirc \bigcirc \bigcirc \bigcirc$$ \hspace{1cm} (8)
Which represents, associativity of parallel composition, parallel composition with empty space having no effect, and finally that the morphism indeed implements the parallel composition of processes respectively. Lastly the condition:

\[
\circ \otimes \otimes \otimes \circ \quad (9)
\]
is required, which represents the interchange law between sequential and parallel composition. The above conditions are technically not well typed without introducing the $V$-natural isomorphisms which carry no operational relevance, the formal conditions which include these manipulations of empty space are taken care of in the appendix.

3.1 Examples

We now present a series of examples of theories of supermaps, and observe that each is indeed example of an enriched monoidal category.

Example 1 (Higher Order Causal Categories). For any $i$ there is a $[C]_{i}^{\text{smc}} |\otimes|$ and similarly for any $i, i + 1$ there is a $[C]_{i+1}^{\text{smc}} |\otimes|$. In each case enrichment is ensured by the closed monoidal structure of $[C]$ [52]. The same can be said for $[C] \otimes$ and each $[C]_{\leq i}$.

Example 2 (Unitary Supermaps). For any $M_{1} \subseteq [C]$ such that

\[
\forall \phi \in M_{1} \quad \exists L, M \in M \quad \therefore
\]
one can construct a category $M_{2} \subseteq [C]$ of completely-$M$-preserving supermaps and a corresponding $M_{2}^{\text{smc}} M_{1}$. $M_{2}$ is defined by taking objects in the algebra $[C, C]$ $\otimes$ and morphisms given by $S \in M_{2}([A, A'], [B, B']) \iff \forall \phi \in M_{1}(A, A')$ then $p^{-1} \circ (S \otimes I_{E, E'}) \circ \hat{\phi} \in M_{1}(B \otimes E, B' \otimes E')$. $M_{2}$ inherits monoidal structure from $[C]$ since for every $S, T$ then $(S \otimes T) \circ \hat{\phi}$ is in $M_{1}$ from the fact that $S \otimes I_{E, E'}$ preserves $M_{1}$ as does $T \otimes I_{E, E'}$, the braid. Similarly $(S \otimes T) \otimes I_{E, E'}$ preserves $M_{1}$ as a consequence of the embedding $X \otimes Y \rightarrow X' \otimes Y'$ in $[C]$. To show that what remains is still a monoidal enriched category, one must check that the composition morphisms $\otimes$ and $\circ$ completely preserve unitarity. Indeed this can be checked, first noting that for every type $([A, A'] \otimes [B, B']) \otimes [C, C']$ in the given algebra there is an embedding into $([A, A'] \otimes [B, B']) \otimes [C, C']$, this entails that every process of the former type embeds as a tripartite process satisfying

\[
\forall \sigma, \rho : \quad \therefore
\]
and so satisfies

\[
\therefore
\]
Using the decomposition theorem for non signalling relations then gives:
Finally, noting that the sequential composition morphism of $[C]$ is constructed from the underlying compact closed structure of $C$, we find that the sequential composition morphism is a morphism of $M^2$, since it sends morphisms of $M^1$ to morphisms of $M^1$ in the following sense: $M^1$:

![Diagram](image)

An important special case of subcategories captured by this construction is given by taking $C := CPM[FHilb]$ and taking $M^1 \subseteq [C]^1$ to be the category of unitary channels. In this case $M^2$ is a monoidal category of unitary supermaps, such supermaps have previously been defined and are of particular interest in quantum causal modelling [63]. The monoidal and enrichment structure whilst natural, had not yet been defined to the authors knowledge.

**Example 3** (Idempotent completion preserves enriched monoidal categories). Any enriched monoidal category can be completed to include idempotents as types, meaning that decoherences in lower order theories can be inherited to construct a higher order theory with classical types. For any $V$-smc $C$ the idempotent completions $K(C)$ of $C$ and $K(V)$ of $V$ define a $K(V)$-smc $K(C)$. The idempotent completion of a category has for each object a pair $(X,x)$ of an object $X$ of $C$ and an idempotent $x : X \to X$. The enriched structure is given by the functor $[(X,x),(Y,y)] = ([X,Y],[x,y])$ and the corresponding composition morphisms $\circ_{xyz}$ are given by $[x,z] \circ \circ_{YXZ} \circ ([x,y] \otimes [y,z])$ and similarly for the parallel composition morphisms. All required coherences follow from coherences of the $V$-smc $C$. This second example, when applied to the $[C]^2$-smc $[C]^1$ with $C = CPM[FHilb]$ of quantum supermaps over completely positive trace preserving quantum channels, produces a theory which includes classical channels in $C$ as those of type $(X,dec_X) \to (Y,dec_Y)$.

**Example 4.** For any $V \sim$-smc $C$ one can construct the $V_{\text{comb}}$-smc $C$ which is generated by the structural morphisms $V(I,[A,B]) \cup \{\circ_{ABC}\} \cup \{\otimes_{A'B'B'}\} \cup \{\text{coherences}\}$ of $V$, meaning that all parallel composition and sequential composition supermaps are kept along with all states. This category is the category of combs of processes from $C$, Indeed a comb drawn intuitively as:

![Diagram](image)

\begin{center}
\begin{tikzpicture}
  \node at (0,0) {\text{comb}};
\end{tikzpicture}
\end{center}

\begin{center}
\begin{tikzpicture}
  \node at (0,0) {\text{comb}};
\end{tikzpicture}
\end{center}

Can be formally represented as:

\begin{center}
\begin{tikzpicture}
  \node at (0,0) {\text{comb}};
\end{tikzpicture}
\end{center}

\begin{center}
\begin{tikzpicture}
  \node at (0,0) {\text{comb}};
\end{tikzpicture}
\end{center}

Consequently all combs of processes in $C$ have to exist as processes in $V$ for any $V \sim$-smc $C$.

3.2 Consequences

Some basic important consequences of monoidal enrichment are the following: There always exists a family of morphisms which represent partial insertion and a family functions which represent
usage of the output of a transformation.

**Partial Insertion**  The partial insertion morphism \( \Delta : [A, X] \otimes [Y \otimes X, Z] \to [Y \otimes A, Z] \) takes a valid sub-input of a process and inserts a pre-processing there, leaving the rest of the inputs unchanged. Formally it is defined by:

\[
\Delta \circ \otimes \kappa(i),
\]

up to unitors and associators, where \( \otimes : [Y, Y] \otimes [A, X] \to [YA, YX] \) and \( \circ : [YA, YX] \otimes [YX, Z] \to [YA, Z] \). The partial insertion can be intuitively be understood as representing the following picture:

\[
\approx \Delta Z Y \Delta X A X Y A Z.
\]

Crucially for \( A = I \) then \( \Delta \) satisfies:

\[
\text{(12)}
\]

Intuitively the above represents the equality between

\[
\text{(13)}
\]

and

\[
\text{(14)}
\]

**Usage**  The usage transformation is a particular natural transformation \( \theta : \mathcal{V}(-,[A,-]) \Rightarrow \mathcal{V}([I,A] \otimes -, [I,-]) \), a family of functions \( \theta_{BX} \) given by for each \( S : X \to [A,B] \) taking \( \theta_{BX}(S) \)
to be:

\[ X \]

\[ S \]

\[ I,A \]

\[ I,B \]

Intuitively, \( \theta \) place \( S \) into one of the two holes of a sequential composition supermap:

\[ \theta(S) = \]

\[ [I,B] \]

\[ [I,A] \]

\[ [I,A] \]

\[ [I,B] \]

\[ [I,B] \]

\[ [I,A] \]

\[ [I,A] \]

\[ [I,B] \]

\[ [I,B] \]

We will find that injectivity of the function \( \theta \) is crucial for results about embeddings between layers of higher order theories.

**Definition 1.** A \( \mathbb{V} \)-smc \( C \) will be called faithful if the usage transformation

\[
\theta : \mathbb{V}(-, [A, -]) \rightarrow \mathbb{V}([I, A] \otimes -, [I, -])
\]

is a monomorphism in the functor category \( \text{Cat}(\mathbb{V}^{op} \otimes C, \text{Set}) \).

Faithful usage when present says that two higher order processes \( S, T : X \rightarrow [A, B] \) should only be distinguishable if they are distinguishable when their outputs are applied to the space of states on \( A \). Stated formally faithful usage is the requirement of injectivity, that for all \( I, A, B \) the composition process \( \circ_{IAB} \) satisfies

\[
\theta(S) \neq \theta(T) \]

\[
[I,B] \]

\[ [I,A] \]

\[ [I,A] \]

This has the additional consequence of entailing that the functor \([I, -]\) be faithful. All of the previously stated examples have faithful usage transformations.

### 3.3 Structure preserving maps

The standard given notion of subcategory given by the definition of an enriched monoidal category is that of a faithful monoidal \( \mathbb{V} \) enriched functor, which leaves the higher order category \( \mathbb{V} \) unchanged. Our goal is to develop a language which allows us to compare theories of supermaps, the ready cooked notion of morphism of enriched monoidal categories will not be fit for our purposes since an enriched monoidal functor is one which allow comparisons of the following type:

\[
(\mathbb{V}, C) \rightarrow (\mathbb{V}, C'),
\]

that is, those in which the enriching category is left untouched. Even in the basic case of the inclusions between unitary combs unitary supermaps quantum combs and quantum supermaps,
functors which which fix the enriching category are unsuitable, it is after-all the enriching categories that we are varying:

$$\begin{array}{c}
(U^2, comb, U^1) \\
\downarrow \\
(U^1, U^1) \\
\downarrow \\
([C]^2, [C]^1) \\
\downarrow \\
([C], [C])
\end{array}$$

In fact these inclusions can be seen as instances of the change of base for enriched monodial categories. Change of base alone however, will be insufficient for our purposes, we will find under additional conditions of faithful usage, that for any sequence in which $C^3$ enriches $C^2$ which in turn enriches $C^1$ then there exists a notion of embedding

$$\begin{array}{c}
(C^2, C^1) \\
\downarrow \\
(C^3, C^2)
\end{array}$$

which cannot be understood as purely a change of base. An explicit example is the embedding of the $[C]^2_{-\text{smc}} [C]$ into the $[C]^3_{-\text{smc}} [C]^2$ for any pre-causal category $C$. To address these problems and provide a suitable notion of structure preserving map, we will construct a notion of functor which allows us to vary both the lower and higher categories at the same time using combinations of (vertical) change of base functors and (horizontal) enriched monoidal functors:

$$\begin{array}{c}
(C^2, C^1) \\
\downarrow \text{Change of base} \\
(C^3, C^2)
\end{array}$$

This pair of functors along with their expected compatibilities, can be viewed as an instance of what we call a pm-functor, a morphism in $\mathbf{PM} := \mathcal{G}(\mathcal{M}_{\text{sym}})$. Here $\mathcal{G}(\mathcal{M}_{\text{sym}})$ denotes the result of the Grothendieck construction of the symmetric monoidal change of base functor $\mathcal{M}_{\text{sym}}$, the functor $\mathcal{M}_{\text{sym}}$ is given formally in the appendix building on the results of [64]. The key components of a pm functor are laid out explicitly here to demonstrate how they incorporate combinations of functors $F^V : V \to V'$ and functors $F^C : C \to C'$ along with compatibilities between them.

**Definition 2.** A pm-functor is a morphism in $\mathbf{PM} := \mathcal{G}(\mathcal{M}_{\text{sym}})$, that is:

- A symmetric monoidal functor $F^V : V \to V'$
- A symmetric monoidal functor $F^C : C \to C'$
- A family of morphisms $F_{AB} : F^V[A, B] \to [F^C A, F^C B]$
Which together satisfy:

\[
\begin{align*}
\begin{array}{c}
\text{\(F_{BC}\)}
\end{array}
\end{align*}
\]

where for readability any structural isomorphisms and symmetry conditions for monoidal functors \(F_{C,V}\) have been omitted and any coherences required for an enriched symmetric monoidal functor from \(F^V[-,-]\) to \([F^C, F^C\)]\) have been omitted as if strict.

We will refer to a morphism \(F\) of \(PM\) as fully faithful if \(F^V, F^C\) are full and faithful and \(F_{AB}\) are isomorphisms. Indeed as a consequence we have constructed a notion of sub-enriched monoidal category, as an embedding which provides a faithful pm-morphism.

**Example 5.** There are pm-functors between the examples of unitary combs, general unitary supermaps, 2\textsuperscript{nd}-order combs and 2\textsuperscript{nd} order transformations built from \([C]\), as depicted in the preamble to this section. In each case \(F^C\) and \(F^V\) are given by inclusions and each \(F_{AB}\) can as a result be taken as the identity.

Another example with trivial \(F_{AB}\) is given by embedding into the Karoubi envelope.

**Example 6.** There is a pm-functor from any enriched monoidal category to its Karoubi envelope given by the embeddings \(C \rightarrow K[C]\) and \(V \rightarrow K[V]\). Taking again \(F_{AB}\) to be the identity the embeddings directly preserve all structural morphisms. This functor is faithful, meaning that it represents an instance of the invariant notion of subcategory.

An example with non-trivial \(F_{AB}\) is given by considering embeddings between different layers \([C]_i\) of \([C]\). From now on we will refer to the functor \([I, -] : C \rightarrow V\) as the raising functor \(R^V\).

**Lemma 1.** Given any \(C^3_{2}\)-sme \(C^2\) and \(C^2_{2}\)-sme \(C^1\) there exists a morphism \(\Gamma\) between them in \(PM\) given by given by \(\Gamma := (R^3_2, R^1_2, \gamma)\) where \(\gamma_{A,B} : R^3_2([A, B]^2) \rightarrow [R^1_2(A), R^1_2(B)]^3\) is given by

\[
(17)
\]

that is, applying partial insertion in \(C^3\) to the composition map in \(C^2\).

**Proof.** All the required coherences for pm-functors are here inherited by the coherence’s of \(C^3\) and \(C^2\), an explicit proof is given in the appendix.

Note that here \(O_{IAB} : [I, A] \otimes [A, B] \rightarrow [I, B]\) and we omit unitors so that \(\Delta\) has type \(\Delta : [I, [A, B]] \otimes [I, A] \otimes [A, B], [I, B]] \rightarrow [I, A] \otimes [I, B]\) \(\cong [I, A], [I, B]\).

**Corollary 1.** There is a morphism in \(PM\) from the \([C]_i^{i+1}\)-sme \([C]_i^i\) to the \([C]_i^{i+2}\)-sme \([C]_i^{i+1}\) for every \(i\). The same story holds for \([C]_i^\leq\).
4 Self-Contained Higher Order Theories

Current examples of higher-order quantum-like theories have a further common feature of self-containment. We mean by this the idea that all processes, no matter their higher or lower order status, exist in the same theory, that is, the same category. Explicit examples of theories which are self-contained in this sense are higher order quantum theory and higher order causal categories both of which are conveniently closed monoidal categories. Those familiar with classical category theory may imagine that closed monoidal structure is the appropriate mathematical formalization of the idea of self-containment. In this section rather than taking as an axiom that currying (the key feature of closed monoidal categories) is the appropriate formalization of a self-contained theory of higher order transformations, we will treat this as a statement to be proven, by combining enrichment with a more relaxed formalization of self-containment. The three operational features which we will use to derive currying and closed monoidal structure will at the intuitive level be the following:

• All processes in $C$ have higher order representations in $C$ (Self-enrichment).
• There is an equivalence $A \cong [I, A]$ between $A$ and the higher order system $[I, A]$ representing the states of $A$ (linking).
• The usage transformation is faithful

Conceptually, the first condition models the assumption that it is the same agents that can perform processes, super-processes, and so on. We model this with the notion of a $C$-enriched monoidal Category $C$. The second condition is captured by the following:

Definition 3. A Linked Monoidal Category is a $C$ Monoidal Category $C$ equipped with a monoidal natural isomorphism $\eta_A : A \to [I, A]$.

We furthermore say that a linked monoidal category $C$ is faithful if it is faithful as a $C_{\text{smc}}$ C. Intuitively, linked categories have enough structure to define canonical evaluation morphisms (the structural feature of closed monoidal categories), mixed-order morphisms of type $\text{eval}_{AB} : A \otimes [A, B] \to B$ which apply processes to lower order objects, by using link-morphisms and sequential composition morphisms:

![Diagram of linked monoidal category](image-url)
Intuitively in the above diagram the available inputs are the bottom wire $A$ and the dotted process input of type $[A, B]$, the output wire is the top wire of type $B$. Indeed linked categories will turn out to be closed monoidal if they satisfy one additional condition - that they have faithful usage, this is proven by constructing evaluation morphisms in the above way.

**Lemma 2.** A category $C$ is closed smc if and only if $C$ is a faithful linked monoidal category.

**Proof.** A full proof is given in the appendix, here we show for reference the formal construction of evaluations analogous to the above intuitive picture:

$$\text{eval} := \begin{array}{c}
\text{eval}_{A,B}
\end{array} - \begin{array}{c}
\eta
\end{array}$$

(18)

the requirement of being faithful ensures uniqueness/universality.

In this section we considered theories which are from the start assumed to be self contained, in the next we generalise this result to infinite towers of enriched monoidal categories, using along the way the developed notion of structure $[\text{preserving map as pm-functor. Given that linked faithful categories are exactly closed symmetric monoidal categories they give an alternative way to view a familiar categorical structure. For instance the closed monoidal structure of Set can be viewed as a consequence of the fact that (i) Set is monoidal (ii) Trivially Set is monoidally enriched in Set (iii) The bijection $\kappa$ for enrichment provides a function $\eta : A \rightarrow [I, A]$ which is monoidal (iv) The composition function is faithful. As noted in the preliminary section $[C]$ has closed monoidal structure for any $C.$

5 Towers of Higher Order Theories

In a second application of our framework we show that theories resulting from the gluing together of a suitably well-behaved tower of higher order theories, are again inevitably closed monoidal. The essence of the proof will be that of the previous section, the formal tools used will be the notions of enriched monoidal category along with properties of previously constructed pm-functors (structure preserving maps) within towers of enriched categories. We begin by presenting the notion of a tower of theories over a base theory, each a theory of supermaps over the theory that precedes it. The base theory represents a given physical theory, such as quantum or classical probability theory. The second layer represents a theory of supermaps, the third layer a theory of super-supermaps, and so on:

$$C_1 \rightarrow C_2 \rightarrow C_3 \rightarrow \ldots$$

(19)

The ultimate goal of introducing this construction is for the specification of a unified higher order theory, into which such a sequence will embed. Mathematically, a hierarchy of higher order physical processes is represented by an ascending sequence of enriched monoidal Categories

**Definition 4.** An ascending sequence of enriched monoidal Categories $C_1, C_2, \ldots, C_N$ is a specification for each $i \in [N-1]$ of a $C_{i+1}^{\text{smc}}$
In any such sequence, the category \( \mathcal{C}_i \) is “encoded” into the higher level \( \mathcal{C}_{i+1} \) by the monoidal raising functor \( R_{i+1}^i(\_):= [I_i, \_]^{i+1} \). It will be convenient to define the following compact notation for the induced encoding (full faithful braided monoidal functor) from level \( i \) to level \( j > 1 \).

\[
R_j^i: \mathcal{C}^i \rightarrow \mathcal{C}^j \quad R_j^i := R_{j-1}^j \circ R_{j-1}^i.
\]

There is a sense in which the agents inhabiting layer \( \mathcal{C}_j \) are strictly more powerful than the inhabitants of \( \mathcal{C}_{i<j} \), in the sense that each \( \mathcal{C}_{i<j} \) may be embedded into \( \mathcal{C}_j \). For any finite sequence \( \mathcal{C}^i \) of length \( n \), the final category \( \mathcal{C}_n \) may be seen as the arena in which agents may manipulate processes from any category in the preceding sequence. In fact as observed in section 3 the embeddings can be run in parallel and phrased as pm-morphisms.

**Lemma 3.** Let \( \mathcal{C}^1, \mathcal{C}^2, \ldots, \mathcal{C}^N \) be a ascending sequence of monoidal enriched categories: then for every \( 0 < i < N - 1 \) there exists a morphism \( \Gamma_i \) from the \( \mathcal{C}^{i+1}\text{-smc} \) to the \( \mathcal{C}_i^{i+2}\text{-smc} \) in \( \text{PM} \).

**Proof.** Direct consequence of lemma 1.

We will find that closed monoidal structure arises when the \( \Gamma_i \) are fully faithful.

**Definition 5.** An ascending sequence of monoidal enrichments is fully faithful if each \( \mathcal{C}_i^{i+1}\text{-smc} \) is faithful and each \( \Gamma_i \) is fully faithful.

We will discover that when fully coherent sequences surjectively embed into a monoidal category, a merger for the sequence, it is guaranteed that the merger will in turn be closed monoidal.

**Theories consisting of fully faithful sequences are closed monoidal** We now present operational conditions on a theory \( \mathcal{C} \) in terms of an embedded ascending sequence \( \mathcal{C}^{(i)} \) which will lead to closed monoidal structure for \( \mathcal{C} \). Intuitively the conditions are the following

- \( \mathcal{C} \) contains nothing more and nothing less than a fully faithful sequence \( \mathcal{C}^i \) of enriched monoidal categories
- \( \mathcal{C} \) provides links between the layers of \( \mathcal{C}^i \)

Such an embedding for a generic sequence \( \mathcal{C}^{(i)} \) is captured categorically by a sequence of full and faithful functors \( F_i: \mathcal{C}^{(i)} \rightarrow \mathcal{C} \). To capture that there is *nothing more* in \( \mathcal{C} \) we require a further condition of essential surjectivity on the union (co-product) functor

\[
\coprod_i F_i: \mathcal{C}^{(i)} \rightarrow \mathcal{C}.
\]

Finally we impose the condition that there be a link between layers of the theory. The most basic notion of a linking between levels is via an isomorphism \( A \cong [I, A] \). Formally this equivalence when consistent with the monoidal embeddings, is captured by the existence of a monoidal natural isomorphism

\[
\eta_{i-1}^i: F_{i-1}(\_) \rightarrow F_i \circ R_{i-1}^i(\_),
\]

In short, \( \eta_{i-1}^i \) provides a witness for the equivalence between \( A \) and \( [I, A] \) inside \( \mathcal{C} \). For ease of notation we will denote the inverse \( (\eta_{i-1}^i)^{-1} \) by \( \eta_{i-1}^{-1} \) when needed. The existence of a natural isomorphism \( \eta_{i-1}^i \) for each \( i \) is can be concisely phrased in the language of 2-Categories, it is precisely the requirement that \( \mathcal{C} \) be a 2-Cone in the 2-Category \( \text{ffSymCat} \) of
• Symmetric monoidal categories
• Full faithful strong symmetric monoidal functors
• Symmetric monoidal natural transformations

For a diagram $D$ in $\mathbf{fSymCat}$ given by a fully faithful sequence of enriched monoidal categories, and the monoidal functors $R_{i+1}^i : C^i \to C^{i+1}$ between them, a cone over $D$ is an “apex” category $C$ equipped with a family of functors $F_i : C^i \to C$ such that each of the following triangles commutes up to a monoidal natural isomorphism $\eta_{i+1}^i$.

\[
\begin{array}{ccc}
C^i & \xrightarrow{R_{i+1}^i} & C^{i+1} \\
\downarrow & \searrow \eta_{i+1}^i & \\
F_i & \downarrow & C \\
\end{array}
\]

The above discussion culminates in the following definition, that of a Merger.

**Definition 6.** A Merger for a fully faithful ascending sequence of enriched monoidal categories (”merger” for short) $C^{(i)}$ is a 2-Cone $(F_i : C^i \to C)$ over the diagram

\[
\ldots \quad C_{i-1} \xrightarrow{R_{i-1}^{i-1}} C^i \xrightarrow{R_{i+1}^i} C^{i+1} \xrightarrow{R_{i+2}^{i+1}} \ldots
\]

in $\mathbf{fSymCat}$ such that

• $\coprod_i F_i$ is essentially surjective

A Merger is furthermore termed “n-th order” if the sequence has length $n$.

For any sequence of finite order the notion of a merger is essentially trivial, given a sequence of order $n$ one can simply construct a cone of the above type by taking $C = C^N$ and taking $F_i := R_{i}^n$. The primary technical contribution of this manuscript is the observation that the apex of any $\infty$-Order Merger possesses a simple categorical property, it must be a closed monoidal category.

**Theorem 1.** The apex $C$ of any Merger of infinite order is a closed symmetric monoidal category.

This result provides an operational justification for using closed symmetric monoidal categories to study higher order physics. From this position the consequences of basic physical principles in higher order physics can be explored within a simple mathematical addition to symmetric monoidal categories. First steps in this direction of research are taken in [65], in which an interaction between the strength of spatial correlations, determinism, and the possibility of signalling between parties is observed.

### 6 Conclusion

Presented in this manuscript is a proposed beginning of a mathematical framework for higher order physical theories, centred around monoidal enrichment. This framework is analogous in
attitude to the process theory framework for standard physics based on the notion of a symmetric monoidal category. The definitions proposed are easily iterated to define towers of theories, after which currying in higher order theories is understood through two results: Linked faithful enriched categories are exactly closed monoidal categories, and categories into which infinite towers of higher order theories are suitably embedded, are always similarly always closed monoidal. Many open questions then follow from this framework:

- Do quantum supermaps, quantum combs, higher order quantum theory, and higher order causal categories, satisfy universal properties with respect to the above defined structure preserving maps, in analogy to the universal properties satisfied by CPTP maps [66]? 

- How can we freely construct theories of supermaps over arbitrary circuit theories?

- What other properties should be expected of theories of supermaps, in particular, how can another key feature of local-applicability [67, 68], be combined with enriched monoidal structures.

- How can we model supermaps on constrained spaces such as one-way signalling channels and routed quantum supermaps [69–71]?

- What does the view of higher order physics as enriched monoidal structure have to say about resource theories of higher order processes [54–56]?

This paper aims to lay a basic starting point from which the above questions can be formulated and answered.

Acknowledgments

MW would like to thank A Vanrietvelde, J Hefford, V Wang and G Boisseau for useful conversations, in particular MW is grateful to J Hefford for numerous helpful suggestions based on an earlier draft. This work is supported by the Hong Kong Research Grant Council through grant 17300918 and through the Senior Research Fellowship Scheme SRFS2021- 7S02, by the Croucher Foundation, by the John Templeton Foundation through grant 61466, The Quantum Information Structure of Spacetime (qiss.fr). Research at the Perimeter Institute is supported by the Government of Canada through the Department of Innovation, Science and Economic Development Canada and by the Province of Ontario through the Ministry of Research, Innovation and Science. The opinions expressed in this publication are those of the authors and do not necessarily reflect the views of the John Templeton Foundation. MW gratefully acknowledges support by University College London and the EPSRC Doctoral Training Centre for Delivering Quantum Technologies.
References

[1] G. Chiribella, G. M. D’Ariano, and P. Perinotti. Transforming quantum operations: Quantum supermaps. *EPL (Europhysics Letters)*, 83(3):30004, 7 2008.

[2] Giulio Chiribella, Giacomo Mauro D’Ariano, and Paolo Perinotti. Theoretical framework for quantum networks. *Physical Review A*, 80(2):022339, 2009.

[3] Giulio Chiribella, Giacomo Mauro D’Ariano, Paolo Perinotti, and Benoit Valiron. Quantum computations without definite causal structure. *Physical Review A - Atomic, Molecular, and Optical Physics*, 88(2):022318, 8 2013.

[4] Giulio Chiribella, Alessandro Toigo, and Veronica Umanità. Normal completely positive maps on the space of quantum operations. *Open Systems & Information Dynamics*, 20(01):1350003, 2013.

[5] Stefano Facchini and Simon Perdrix. Quantum circuits for the unitary permutation problem. In *International Conference on Theory and Applications of Models of Computation*, pages 324–331. Springer, 2015.

[6] Paolo Perinotti. Causal structures and the classification of higher order quantum computations. In *Time in physics*, pages 103–127. Springer, 2017.

[7] Alessandro Bisio and Paolo Perinotti. Theoretical framework for higher-order quantum theory. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 475(2225):20180706, 5 2019.

[8] Giulio Chiribella, Giacomo Mauro D’Ariano, and Paolo Perinotti. Optimal cloning of unitary transformation. *Physical review letters*, 101(18):180504, 2008.

[9] Alessandro Bisio, Giulio Chiribella, GM D’Ariano, Stefano Facchini, and Paolo Perinotti. Optimal quantum tomography of states, measurements, and transformations. *Physical review letters*, 102(1):010404, 2009.

[10] Alessandro Bisio, Giulio Chiribella, Giacomo Mauro D’Ariano, Stefano Facchini, and Paolo Perinotti. Optimal quantum learning of a unitary transformation. *Physical Review A*, 81(3):032324, 2010.

[11] Denis Rosset, Francesco Buscemi, and Yeong-Cherng Liang. Resource theory of quantum memories and their faithful verification with minimal assumptions. *Physical Review X*, 8(2):021033, 2018.

[12] Daniel Ebler, Sina Salek, and Giulio Chiribella. Enhanced Communication with the Assistance of Indefinite Causal Order. *Physical Review Letters*, 120(12):120502, 3 2018.

[13] Gilad Gour and Andreas Winter. How to quantify a dynamical quantum resource. *Physical Review Letters*, 123(15):150401, 2019.

[14] Xin Wang and Mark M Wilde. Resource theory of asymmetric distinguishability for quantum channels. *Physical Review Research*, 1(3):033169, 2019.

[15] Jianwei Xu. Coherence of quantum channels. *Physical Review A*, 100(5):052311, 2019.

[16] Thomas Theurer, Dario Egloff, Lijian Zhang, and Martin B Plenio. Quantifying operations with an application to coherence. *Physical Review Letters*, 122(19):190405, 2019.
[17] Jisho Miyazaki, Akihito Soeda, and Mio Murao. Complex conjugation supermap of unitary quantum maps and its universal implementation protocol. *Physical Review Research*, 1(1):013007, 2019.

[18] Marco Túlio Quintino, Qingxiuxiong Dong, Atsushi Shimbo, Akihito Soeda, and Mio Murao. Probabilistic exact universal quantum circuits for transforming unitary operations. *Physical Review A*, 100(6):062339, 2019.

[19] Yunchao Liu and Xiao Yuan. Operational resource theory of quantum channels. *Physical Review Research*, 2(1):012035(R), 2020.

[20] Michal Sedlák, Alessandro Bisio, and Mário Ziman. Optimal probabilistic storage and retrieval of unitary channels. *Physical review letters*, 122(17):170502, 2019.

[21] Ryuji Talagi, Kun Wang, and Masahito Hayashi. Application of the resource theory of channels to communication scenarios. *Physical Review Letters*, 124(12):120502, 2020.

[22] Hlér Kristjánsson, Giulio Chiribella, Sina Salek, Daniel Ebler, and Matthew Wilson. Resource theories of communication. *New Journal of Physics*, 2020.

[23] Gaurav Saxena, Eric Chitambar, and Gilad Gour. Dynamical resource theory of quantum coherence. *Physical Review Research*, 2(2):023298, 2020.

[24] Ognyan Oreshkov, Fabio Costa, and Časlav Brukner. Quantum correlations with no causal order. *Nature Communications*, 3, 2012.

[25] Cyril Branciard, Mateus Araújo, Adrien Feix, Fabio Costa, and Časlav Brukner. The simplest causal inequalities and their violation. Technical report, 2015.

[26] Esteban Castro-Ruiz, Flaminia Giacomini, and Časlav Brukner. Dynamics of quantum causal structures. Technical report.

[27] Lucien Hardy. Towards quantum gravity: a framework for probabilistic theories with non-fixed causal structure. *Journal of Physics A: Mathematical and Theoretical*, 40(12):3081, 2007.

[28] W. K. Wootters and W. H. Zurek. A single quantum cannot be cloned. *Nature*, 299(5886):802–803, 1982.

[29] Peter Selinger. Towards a quantum programming language. *Mathematical Structures in Computer Science*, 14(4):527–586, 2004.

[30] Peter Selinger and Benoît Valiron. A lambda calculus for quantum computation with classical control. In *Lecture Notes in Computer Science*, volume 3461, pages 354–368. Springer Verlag, 2005.

[31] André Van Tonder. A lambda calculus for quantum computation. *SIAM Journal on Computing*, 33(5):1109–1135, 1 2004.

[32] Nick Benton, Gavin Bierman, Valeria De Paiva, and Martin Hyland. Linear-Calculus and Categorical Models Revisited. Technical report.

[33] Michele Pagani, Peter Selinger, and Benoît Valiron. Applying Quantitative Semantics to Higher-Order Quantum Computing *. Technical report, 2013.

[34] Simon Ambler. *First order linear logic in symmetric monoidal closed categories*. 1991.
[35] Bob Coecke and Aleks Kissinger. *Picturing quantum processes: A first course in quantum theory and diagrammatic reasoning*. Cambridge University Press, 3 2017.

[36] John H. Selby, Carlo Maria Scandolo, and Bob Coecke. Reconstructing quantum theory from diagrammatic postulates. 2 2018.

[37] Jamie Vicary. Categorical Formulation of Finite-Dimensional Quantum Algebras. *Communications in Mathematical Physics*, 304(3):765–796, 6 2011.

[38] Bob Coecke and Dusko Pavlovic. Quantum measurements without sums. In *Mathematics of Quantum Computation and Quantum Technology*, pages 559–596. CRC Press, 1 2007.

[39] Samson Abramsky and Bob Coecke. Physical traces: Quantum vs. classical information processing. In *Electronic Notes in Theoretical Computer Science*, volume 69, pages 1–22. Elsevier B.V., 2 2003.

[40] Bob Coecke and Ross Duncan. Interacting Quantum Observables: Categorical Algebra and Diagrammatics. 6 2009.

[41] Bob Coecke and Aleks Kissinger. The compositional structure of multipartite quantum entanglement. In *Lecture Notes in Computer Science (including subseries Lecture Notes in Artificial Intelligence and Lecture Notes in Bioinformatics)*, volume 6199 LNCS, pages 297–308. Springer Verlag, 2010.

[42] Bob Coecke, Bill Edwards, and Robert W. Spekkens. Phase groups and the origin of non-locality for qubits. In *Electronic Notes in Theoretical Computer Science*, volume 270, pages 15–36. Elsevier, 2 2011.

[43] Bob Coecke and Raymond Lal. Time asymmetry of probabilities versus relativistic causal structure: An arrow of time. *Physical Review Letters*, 108(20), 5 2012.

[44] Bob Coecke and Raymond Lal. Causal Categories: Relativistically Interacting Processes. *Foundations of Physics*, 43(4):458–501, 4 2013.

[45] Chris Heunen and Jamie Vicary. Introduction to Categorical Quantum Mechanics. Technical report, 2013.

[46] Aleks Kissinger, Sean Tull, and Bas Westerbaan. Picture-perfect Quantum Key Distribution. *arXiv*, 4 2017.

[47] John H. Selby and Bob Coecke. A Diagrammatic Derivation of the Hermitian Adjoint. *Foundations of Physics*, 47(9):1191–1207, 9 2017.

[48] Peter Selinger. Dagger Compact Closed Categories and Completely Positive Maps. (Extended Abstract). *Electronic Notes in Theoretical Computer Science*, 170:139–163, 3 2007.

[49] Sean Tull. A CATEGORICAL RECONSTRUCTION OF QUANTUM THEORY. *Logical Methods in Computer Science*, 16(1):39, 2020.

[50] Nicola Pinzani, Stefano Gogioso, and Bob Coecke. Categorical Semantics for Time Travel. Technical report.

[51] Thomas D Galley, Flaminia Giacomini, and John H Selby. A no-go theorem on the nature of the gravitational field beyond quantum theory. Technical report.
[52] Aleks Kissinger and Sander Uijlen. A categorical semantics for causal structure. *Logical Methods in Computer Science*, 15(3), 2019.

[53] Bob Coecke, Tobias Fritz, and Robert W. Spekkens. A mathematical theory of resources. 9 2014.

[54] Ryuji Talaghi, Kun Wang, and Masahito Hayashi. Application of the resource theory of channels to communication scenarios. *Physical Review Letters*, 124(12), Mar 2020.

[55] Hlér Kristjánsson, Giulio Chiribella, Sina Salek, Daniel Ebler, and Matthew Wilson. Resource theories of communication. *New Journal of Physics*, 22(7):073014, 7 2020.

[56] Gilad Gour and Carlo Maria Scandolo. Dynamical entanglement. *Physical Review Letters*, 125(18):180505, 2020.

[57] Matt Wilson and Giulio Chiribella. Causality in higher order process theories. *Electronic Proceedings in Theoretical Computer Science*, 343:265–300, Sep 2021.

[58] Mathieu Huot and Sam Staton. Universal properties in quantum theory. In Peter Selinger and Giulio Chiribella, editors, *Proceedings 15th International Conference on Quantum Physics and Logic, QPL 2018, Halifax, Canada, 3-7th June 2018*, volume 287 of *EPTCS*, pages 213–223, 2018.

[59] Saunders Mac Lane. *Categories for the Working Mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer New York, New York, NY, 1971.

[60] Mathys Rennela and Sam Staton. Classical Control, Quantum Circuits and Linear Logic in Enriched Category Theory. *Logical Methods in Computer Science*, Volume 16, Issue 1, March 2020.

[61] Man Duen Choi. Completely positive linear maps on complex matrices. *Linear Algebra and Its Applications*, 10(3):285–290, 1975.

[62] G. M. Kelly. Basic concepts of enriched category theory. *Repr. Theory Appl. Categ.*, (10):vi+137, 2005. Reprint of the 1982 original [Cambridge Univ. Press, Cambridge; MR0651714].

[63] Jonathan Barrett, Robin Lorenz, and Ognyan Oreshkov. Cyclic Quantum Causal Models. Technical report.

[64] G S H Cruttwell. Normed Spaces and the Change of Base for Enriched Categories. 2008.

[65] M Wilson and G Chiribella. Manuscript in Preparation. Technical report.

[66] Mathieu Huot and Sam Staton. Universal properties in quantum theory. *Electronic Proceedings in Theoretical Computer Science*, 287:213–223, jan 2019.

[67] Matt Wilson, Giulio Chiribella, and Aleks Kissinger. Quantum Supermaps are Characterized by Locality. 5 2022.

[68] Matt Wilson and Giulio Chiribella. Free Polycategories for Unitary Supermaps of Arbitrary Dimension. 7 2022.

[69] Augustin Vanrietvelde and Giulio Chiribella. Universal control of quantum processes using sector-preserving channels. *Quantum Information and Computation*, 21(15&16):1320–1352, nov 2021.

24
P. T. Johnstone. BASIC CONCEPTS OF ENRICHED CATEGORY THEORY (London Mathematical Society Lecture Note Series, 64). Bulletin of the London Mathematical Society, 15(1):96–96, 1983.

Matt Wilson and Augustin Vanrietvelde. Composable constraints, 2021.

Augustin Vanrietvelde, Hlér Kristjánsson, and Jonathan Barrett. Routed quantum circuits. Advances in Mathematics, 129(1):99–157, 1997.

Paddy McCrudden. Balanced coalgebroids. Theory and Application of Categories, 7(6):71–147, 2000.

Rory B. B. Lucyshyn-Wright. Relative symmetric monoidal closed categories i: Autoenrichment and change of base. 2015.

Appendix

A Enriched Monoidal Categories

A key notion for the description of supermaps will be that of an enriched category, we begin with a brief overview of the basics of enriched category theory relevant to this paper, and then expand in detail.

Summary of Enriched Categories In the data of a category is a family of ‘composition’ functions $\circ_{A,B,C} : C(A, B) \times C(B, C) \to C(A, C)$, in other words a family of morphisms in the category Set of functions between sets. A key notion for the description of supermaps will be that of an enriched category [72], which generalises the notion of a Set-category $C$ to a $V$-category $C$ in which composition is specified as a morphism $\circ_{A,B,C} : [A, B] \otimes [B, C] \to [A, C]$ in a category $V$ rather than some function in SET. We relax the standard definition of $V$-enriched category $C$ to a $V_{\equiv}$ category $C$. In a $V_{\equiv}$-category $C$ from each pair of objects $A, B$ of $C$ an object $[A, B]$ is given in $V$ intended to represent the space of processes $C(A, B)$ as a set of states $V(I, [A, B])$ in $V$. The meaning of $\equiv$ is to express that the representation of $f \in C(A, B)$ as a state of $V$ is not exact but up to a bijection $\kappa : C(A, B) \to V(I, [A, B])$. A category $V_{\equiv}\text{Cat}$ can be constructed with each object a $V_{\equiv}$-category $C$ and each morphism termed a $V_{\equiv}$-functor, given by a functor $F^C : C \to C'$ and a family of morphisms $F_{A,B} : [A, B] \to [FA, FB]$ which represent on the states of $V$ the action of the functions $F^C : C(A, B) \to C(FA, FB)$ induced by the functor $F^C$. A functor $M : \text{MonCat} \to 2\text{Cat}$ can be constructed which assigns to each monoidal functor $F : V \to V'$ a new ‘change-of-base’ 2-functor $M(F) : V_{\equiv}\text{Cat} \to V'_{\equiv}\text{Cat}$.

A generalisation of the standard notion of a (Set) monoidal category $C$ with its parallel composition functions $\otimes_{A,A',B,B'} : C(A, A') \times C(B, B') \to C(A \otimes B, A' \otimes B')$ is given by a $V_{\equiv}$-monoidal category $C$ with morphisms $\otimes_{A,A',B,B'} : [A, A'] \times [B, B'] \to [A \otimes B, A' \otimes B']$. Furthermore one can construct a category $V_{\equiv}\text{MonCat}$ of $V_{\equiv}$-monoidal categories and $V_{\equiv}$-monoidal functors, along with a functor $M_{\text{mon}} : \text{MonCat} \to 2\text{Cat}$ which assigns to each monoidal functor $F : V \to V'$ a change-of-base 2-functor $M(F) : V_{\equiv}\text{MonCat} \to V'_{\equiv}\text{MonCat}$.
**Enriched Categories - Formally**  
Note that from any object $B$ of a monoidal category $\mathbf{V}$ and functor $[-, -] : \mathbf{C}^{op} \times \mathbf{C} \to \mathbf{V}$ a new functor $[-, B] \otimes [B, -] : \mathbf{C}^{op} \times \mathbf{C} \to \mathbf{V}$ can be constructed.

**Definition 7.** Let $\mathbf{C}$ be a category and $\mathbf{V}$ be a symmetric monoidal category, a $\mathbf{V}_\simeq$ enriched category $\mathbf{C}$ is a specification of

- A hom-functor $[A, B] : \mathbf{C} \times \mathbf{C} \to \mathbf{V}$

- A natural transformation $\circ_{ABC} : [A, B] \otimes [B, C] \to [A, C]$ (with a minor abuse of notation in the use of the symbol $\circ$)

- A natural isomorphism $\kappa : \mathbf{C}(A, B) \cong \mathbf{V}(I, [A, B])$

- Such that the following unitality and associativity conditions for the natural transformation $\circ_{ABC}$ hold:

\[
\begin{align*}
[A, B] & \circ_{B,C} \circ_{[A, B](C, D)} & = & \circ_{A,B} ([A, B](C, D)) \circ_{A,B} \\
[A, B] & \circ_{B,C} \circ_{[A, B](C, D)} & = & \circ_{A,B} ([A, B](C, D)) \circ_{A,B}
\end{align*}
\]  

From the definition of $\mathbf{V}$-categories one can define $\mathbf{V}_\simeq$-functors and $\mathbf{V}_\simeq$-natural transformations.

**Definition 8.** A $\mathbf{V}_\simeq$-functor is a functor $F^C : \mathbf{C} \to \mathbf{C}'$ and a natural transformation $F_{A,B} : [A, B] \to [F^C A, F^C B]$ satisfying:

\[
\begin{align*}
F_{A,B} & = F_{A,B} \circ F_{A,B} \\
F_{A,B} & = F_{A,B}
\end{align*}
\]

**Definition 9.** A $\mathbf{V}_\simeq$-natural transformation $\eta : F \Rightarrow G$ is a natural transformation $\eta^C : F^C \Rightarrow G^C$ satisfying:

\[
\begin{align*}
\eta_{A,B} & = \eta_{A,B} \circ \eta_{A,B} \\
\eta_{A,B} & = \eta_{A,B}
\end{align*}
\]

Indeed all together $\mathbf{V}_\simeq \mathbf{Cat}$ defines a monoidal 2-category, the monoidal product is given by taking $(\mathbf{V}, \mathbf{C}) \times (\mathbf{V}, \mathbf{D}) = (\mathbf{V}, \mathbf{E})$ where $\mathbf{E}$ is given by $\mathbf{E}((A, B), (A', B')) := \mathbf{V}(I, [A, A']\mathbf{C} \otimes [B, B']\mathbf{D})$ and all structure defined pairwise. The unit object is given by $\mathbf{V}(\mathbf{I}, \mathbf{I})$ with $\mathbf{I} := \{\bullet\}$ (the singleton category) with enrichment given by $[\bullet, \bullet] := I$. The notion of monoidal enrichment can then be immediately generalised to $\mathbf{V}\mathbf{Cat}_\simeq$.

**Definition 10.** A $\mathbf{V}_\simeq$-monoidal category $\mathbf{C}$ is a pseudomonoid in $\mathbf{V}_\simeq$. Explicitly this gives the category $\mathbf{C}$ monoidal structure and the category $\mathbf{V}$ an associative family of maps $\otimes_{A,A'B'} : [A, A'] \otimes [B, B'] \to [A \otimes_C B, A' \otimes_C B']$ in $\mathbf{V}$ which implement the monoidal structure of $\mathbf{C}$.

A 2-category can immediately be defined, the category $\mathbf{V}\mathbf{MonCat}_\simeq$ given by the free construction of a 2-category of pseudomonoids from any monoidal 2-category. For every lax monoidal functor $F : \mathbf{V} \to \mathbf{V}'$ one can construct the change-of-base 2-functor $M(F) : \mathbf{V}\mathbf{Cat}_\simeq \to \mathbf{V}'\mathbf{Cat}_\simeq$. This functor is given by sending $(\mathbf{V}, \mathbf{C})$ to $(\mathbf{V}', \mathbf{U})$ where $\mathbf{U}$ is the underlying category of the standard change of base of enriched categories.
Lemma 4. For every monoidal functor $\mathcal{F}$ the functor $\mathcal{M}(\mathcal{F})$ is a monoidal 2-functor.

Proof. In [64] it is shown that the change of base functor for standard enriched categories $\mathcal{M}_e(\mathcal{F}) : \mathcal{VCat} \to \mathcal{VCat}'$ is a monoidal 2-functor. $\mathcal{M}$ is given by $\gamma_V \circ \mathcal{M}_e \circ \chi_V$, where $\chi_V : \mathcal{VCat}_\sim \to \mathcal{VCat}$ is the monoidal 2-functor which replaces each category $C$ with the underlying category $U$ from enrichment and $\gamma_V : \mathcal{VCat} \to \mathcal{VCat}_\sim$ is the monoidal 2-functor which views each $V$ monoidal category $U$ as a $V_\sim$ monoidal category $U$ using the identity function as bijection.

We can extend the definition of a $V_\sim$-monoidal category to a $V_\sim$ symmetric monoidal category using symmetric pseudomonoids [73, 74] rather than pseudomonoids.

Definition 11. A $V_\sim$-smc $C$ is a symmetric pseudomonoid [74] on $C$ in $\mathcal{VCat}_\sim$.

One can construct freely construct an entire 2-category of symmetric pseudomonoids [74, 75] $\text{SymPsMon}[\mathcal{VCat}_\sim]$, whilst keeping change-of-base theorems. This provides a higher level way to see that pm-functors as written explicitly in the main text can be defined and composed without explicitly checking all coherences.

Lemma 5. There is a functor $\mathcal{M}_{smc} : \text{SymMonCat} \to \mathcal{2Cat}$ which assigns to each $\mathcal{F} : V \to V'$ a 2-functor $\mathcal{M}_{smc}(\mathcal{F}) : \text{SymPsMon}[\mathcal{VCat}_\sim] \to \text{SymPsMon}[\mathcal{VCat}_\sim]$.

Proof. When $V$ is a symmetric monoidal category and $\mathcal{F} : V \to V'$ is a symmetric monoidal functor the change of base functor $M : \text{SymMonCat} \to \text{SM2Cat}$ assigns to each $\mathcal{F} : V \to V'$ a symmetric monoidal 2-functor $\mathcal{M}(\mathcal{F}) : \mathcal{VCat}_\sim \to \mathcal{VCat'}_\sim$ indeed the braiding coherence required for $\mathcal{M}(\mathcal{F})$ to be a symmetric monoidal 2-functor is directly inherited from the coherence for $\mathcal{F}$. The 2-functor $\text{SymPsMon} : \text{SM2Cat} \to \mathcal{2Cat}$ constructed in [75] can be applied to return $\mathcal{M}_{smc}(\mathcal{F}) := \text{SymPsMon}[\mathcal{M}(\mathcal{F})] : \text{SymPsMon}[\mathcal{VCat}_\sim] \to \text{SymPsMon}[\mathcal{VCat}_\sim]$.

Given a functor $\mathcal{M} : D \to \mathcal{Cat}$ its Grothendieck construction $\mathcal{G}(\mathcal{M})$ has as objects pairs $(d,c)$ where $x$ is an object of $D$ and $c$ is an object of $\mathcal{M}(d)$. Morphisms $(f,d,c) \to (f',d',c')$ are given by pairs $(f,g)$ of morphisms $f : d_1 \to d_2$ and $g : \mathcal{M}(f)(c_1) \to c_2$. The Grothendieck construction $\mathcal{PM} := \mathcal{G}(\mathcal{M}_{smc})$ of the functor $\mathcal{M}_{smc}$ consequently has for each object a pair of an smc $V$ and a $V_\sim$-smc $C$ and for each morphism a pair of a symmetric monoidal functor $\mathcal{F} : V \to V'$ and an enriched symmetric monoidal functor from $\mathcal{M}(\mathcal{F}')(V,C)$ to $(V',C')$ explicitly meaning a pair of a symmetric monoidal functor $\mathcal{F}^C : C \to C'$ and a family of natural transformations $\mathcal{F}_{A,B} : \mathcal{F}[A,A'] \to [\mathcal{F}^C A, \mathcal{F}^C A']$.

B PM morphisms between layers of the Grothendieck Construction

Lemma 6. Given any $C^2_\sim$ monoidal category $C^2$ and $C^2_\sim$ monoidal category $C^1$ there exists a morphism $\Gamma$ between them in $\mathcal{PM}$ given by given by $\Gamma := (R^2_1, R^2_2, \gamma^1)$ where $\gamma^1_{A,B} : R^2_1([A,B]^2) \to [R^2_1(A), R^2_1(B)]^3$ is given by

\begin{equation}
\begin{array}{c}
\text{Val} \\
\Rightarrow \\
\Rightarrow \\
\Rightarrow \\
\Rightarrow \\
\Rightarrow \\
\text{Val}
\end{array}
\end{equation}

(22)
Proof. Including the isomorphisms associated to the monoidal functors in the definition of a pm-morphism (which were omitted in the main text for readability) gives the following three conditions:

\[ \phi \circ (\circ) = \phi \circ (\circ) \]

\[ \phi \circ (\otimes) = \phi \circ (\otimes) \]

\[ \phi \circ (\otimes) = \phi \circ (\otimes) \]

For readability of the proofs we take all categories \( \mathbf{C}^i \) to be strict. The generic non-strict case is proven identically up to additional book-keeping of associators and unitors. The first state preservation condition is immediate so we begin by checking the second condition for pm-functors: preservation of sequential composition.

\[ \phi \circ (\circ) = \phi \circ (\circ) \]

\[ \phi \circ (\otimes) = \phi \circ (\otimes) \]

\[ \phi \circ (\otimes) = \phi \circ (\otimes) \]

Then the third condition, for preservation of parallel composition:

\[ \phi \circ (\otimes) = \phi \circ (\otimes) \]

\[ \phi \circ (\otimes) = \phi \circ (\otimes) \]

\[ \phi \circ (\otimes) = \phi \circ (\otimes) \]
Lemma 7. A category $\mathcal{C}$ is closed symmetric monoidal if and only if $\mathcal{C}$ is a symmetric monoidal and furthermore

- $\mathcal{C}$ is linked
- $\mathcal{C}$ has faithful usage

Proof. We begin by showing that the above two bullet points give closed monoidal structure. Let the three bullet points be true for $\mathcal{C}$, then to each pair $A, B \in \mathcal{C}$ assign the candidate for evaluation

$$\text{eval} := \begin{array}{c}
\otimes_{\mathcal{C}(A,B)} \\
\eta^{-1}
\end{array}$$

(25)

Since every $\otimes$ is completely injective by assumption, so is every $\text{eval}$. Since $\eta : A \rightarrow [I, A]$ is a natural isomorphism for any $f \in \mathcal{C}(A, B)$ there exists a morphism $\kappa f$ such that

$$\text{eval} := \begin{array}{c}
\otimes_{\mathcal{C}(A,B)} \\
\eta^{-1}
\end{array} = \begin{array}{c}
\eta^{-1} \\
\eta
\end{array}$$

(26)

One can apply the isomorphism $\eta$ to the partial insertion operation to generate a partial insertion using a lower level type $Y$ as opposed to the higher level type $[I, Y]$.

$$\begin{array}{c}
\otimes_{\mathcal{C}(A \otimes Y, Z)} \\
\eta
\end{array}$$

(27)

This partial insertion operation can be used to construct the curried version of any process $f$ from its static version $\kappa(f)$, since

$$\begin{array}{c}
\otimes_{\mathcal{C}(A,B)} \\
\eta
\end{array}$$

(28)

It follows that for every process $f$ its curried version exists, that is, the co-universal arrow definition of a closed symmetric monoidal category is satisfied.

Now we demonstrate the converse. Let $\mathcal{C}$ be a closed SMC, then there exist sequential and parallel composition morphisms defined as adjuncts to circuits of evaluation morphisms. Concretely the definition of closed monoidal category enforces that there must exist processes $\otimes$ and $\circ$ satisfying.

$$\begin{array}{c}
\otimes_{\mathcal{C}(A,B)} \\
\eta^{-1}
\end{array} = \begin{array}{c}
\otimes_{\mathcal{C}(A,B)} \\
\eta^{-1}
\end{array}$$

(29)

29
which satisfy the coherence conditions for a symmetric monoidal category. The uniqueness property for co-universal arrows lifts to faithful usage for each sequential composition maps. Finally a monoidal natural isomorphism $A \cong [I, A]$ for the induced functor $[I, -]$ must be constructed. Indeed, up to unitor the inverse $\eta$ of $\text{eval}_{f \Rightarrow A}$, being an isomorphism by assumption, is such a candidate. $\text{eval}_{f \Rightarrow A}$ is natural for any closed monoidal category, so $\eta$ being its inverse is immediately also natural. Furthermore $\eta$ is easily checked to be monoidal. 

This completes the proof. 

\[ \eta = \text{eval}_{I \Rightarrow A} \]

\[ (30) \]

D The Apex of a Merger is Closed Monoidal

Here we prove our main technical result. As a recap, from a series of enriched monoidal categories $C_{i-1}, C_{i}, C_{i+1}$ a chain of raising functors $R_{i}^{i+1} : C_{i-1} \rightarrow C_{i}$ can be written down with $R_{i}^{i+1}(-) = [I_{i-1}, -]$.

**Lemma 8.** In any merger of infinite order, the following condition holds for the isomorphism $\mu_{i}^{i+1} := \phi^{i+1} \circ \eta_{i}^{i+1}$

\[ (31) \]

\[ (32) \]

Proof. Indeed the above property is the key ingredient in the construction of our main result. We work with the following definition of a closed symmetric monoidal category
Definition 12. An SMC $C$ is closed if for every $A, B \in \text{Ob}(C)$ there exists an object $A \Rightarrow B$ and a morphism $\text{eval}_{A \Rightarrow B} : A \otimes A \Rightarrow B \to B$, called the evaluation morphism, such that for all $f : A \otimes C \to B$ there exists a unique $\bar{f} : C \to (A \Rightarrow B)$ such that $\text{eval}_{A \Rightarrow B} \circ (\text{id} \otimes \bar{f}) = f$.

Theorem 2. The apex $C$ of any Merger of infinite order is a closed symmetric monoidal category

Proof. Since the coproduct $\coprod_i F_i$ is essentially surjective, each object $A$ can be assigned an object $X_A$ an “index” $l_A$ and an isomorphism $L_A$ such that $L_A : A \to F_{l_A}(X_A)$. A compact notation can be introduced for combinations of functors of the form $[I_i, -]$.

- $R^{i+1}_i := [I_i, -]$
- $R^j_i := R^j_{i-1} \circ R^{i-1}_i$

Furthermore the function $l : \text{ob}(C) \to \mathbb{N}$ can be extended to lists by

$$l_{AB} := \max(l_A, l_B)$$

After which one can define the object representing the space of morphisms from $A$ to $B$ by

$$A \Rightarrow B := F_{l_{AB}+1}(R^l_{l_A}(X_A), R^l_{l_B}(X_B))$$

This is the object representing the lifting of both $A$ and $B$ in to the $C^{l_{AB}}$ which contains them both, and then using the process object in the next category $C^{l_{AB}+1}$ to represent the processes between them. For each $A, B$ an evaluation $e_{A \Rightarrow B} : A \otimes (A \Rightarrow B) \to B$ can be defined by,

$$e_{A \Rightarrow B} := R^{l_{AB}+1}_A$$

For $C$ to be closed monoidal one must show that for every $A, B, C$ and for every $f : A \otimes C \to B$ there exists a unique $\bar{f} : C \to (A \Rightarrow B)$ such that,

$$\text{eval}_{A \Rightarrow B} \circ (\text{id} \otimes \bar{f}) = f$$
Indeed such a map \( \bar{f} \) can be constructed. Firstly defining \( g \) such that

\[
\eta_{\mathcal{ABC}} = \mu_{\mathcal{ABC}} + 1 \Delta \eta_{\mathcal{ABC}} + 1 \nu_{\mathcal{ABC}} + 1 \kappa g \quad (35)
\]

Such a \( g \) must exist since each functor \( \mathcal{F}_i \) is full. In terms of this \( g \) define \( \bar{f} \) by

\[
\eta_{\mathcal{ABC}} = \mu_{\mathcal{ABC}} + 1 \Delta \eta_{\mathcal{ABC}} + 1 \nu_{\mathcal{ABC}} + 1 \kappa g = \eta_{\mathcal{ABC}} + 1 \nu_{\mathcal{ABC}} + 1 \Delta \eta_{\mathcal{ABC}} + 1 \kappa g \quad (36)
\]

Then to prove the required identity first requires repeated application of lemma (16),

\[
\eta_{\mathcal{ABC}} = \mu_{\mathcal{ABC}} + 1 \Delta \eta_{\mathcal{ABC}} + 1 \nu_{\mathcal{ABC}} + 1 \kappa g = \eta_{\mathcal{ABC}} + 1 \nu_{\mathcal{ABC}} + 1 \Delta \eta_{\mathcal{ABC}} + 1 \kappa g \quad (37)
\]
and then using the defining identity for the partial insertion operation $\Delta$.

\[
\begin{align*}
&= \eta_{ABC} + 1 \circ \langle \eta_{ABC} + 1 \rangle_{L} B \\
&= \eta_{ABC} + 1 \\
&= \eta_{ABC} + 1 \\
&= \eta_{ABC} + 1 \\
&= \eta_{ABC} + 1
\end{align*}
\]

and finally using monoidal naturality of the transformation $\eta_{ABC} + 1$.

\[
\begin{align*}
&= \eta_{ABC} + 1 \\
&= \eta_{ABC} + 1 \\
&= \eta_{ABC} + 1 \\
&= \eta_{ABC} + 1
\end{align*}
\]

The morphism $\bar{f}$ satisfying $e \circ (A \otimes \bar{f}) = f$ must be demonstrated to be unique. Every $\mu^i_j$ is an isomorphism by fully faithful-ness of the sequence of enriched monoidal categories, as a result every morphism $h : C \to A \Rightarrow B$ can be written in the form

\[
\begin{align*}
&= \eta_{ABC} + 1 \\
&= \eta_{ABC} + 1 \\
&= \eta_{ABC} + 1
\end{align*}
\]
Where in the last line fullness of each $\mathcal{F}_i$ is used. Assuming $h$ and $h'$ have decomposition in terms of $m$ and $m'$ respectively both evaluate to the same morphism $e \circ (A \otimes h) = e \circ (A \otimes h')$:

\[
\begin{align*}
\eta_{ABC} &\otimes L_A \otimes L_B \otimes m_{ABC+1} \otimes m'_{ABC+1} = \\
\mathcal{F}_{ABC+1} &\otimes \mathcal{F}_{ABC+1} \otimes \mathcal{F}_{ABC+1} \otimes \mathcal{F}_{ABC+1},
\end{align*}
\]

which in turn implies

\[
\begin{align*}
\eta_{ABC} &\otimes L_A \otimes L_B \otimes m_{ABC+1} \otimes m'_{ABC+1} = \\
\mathcal{F}_{ABC+1} &\otimes \mathcal{F}_{ABC+1} \otimes \mathcal{F}_{ABC+1} \otimes \mathcal{F}_{ABC+1}.
\end{align*}
\]

Since each $\eta$ and $L$ is an isomorphism, and each composition morphism $\circ$ is part of the structure of a faithful monoidal enrichment, and each $\mathcal{F}_i$ is faithful this entails that $m = m'$ and as a result that $\bar{f} = \bar{f}'$. It follows that $\bar{f}$ is the unique morphism satisfying the evaluation condition for $f$. 

\[\square\]