FEAST FOR DIFFERENTIAL EIGENVALUE PROBLEMS

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Abstract. An operator analogue of the FEAST matrix eigensolver is developed to compute the discrete part of the spectrum of a differential operator in a region of interest in the complex plane. Unbounded search regions are handled with a novel rational filter for the right half-plane. If the differential operator is normal or self-adjoint, then the operator analogue preserves that structure and robustly computes eigenvalues to near machine precision accuracy. The algorithm is particularly adept at computing high-frequency modes of differential operators that possess self-adjoint structure with respect to weighted Hilbert spaces.

Key words. FEAST, filtered subspace, differential eigenvalue problems, spectral methods

AMS subject classifications. 34L16, 65F15

1. Introduction. In this paper, we consider differential eigenvalue problems posed on the interval \([-1, 1]\], i.e.,

\[
Lu = \lambda u, \quad u(\pm 1) = \cdots = u^{(N/2)}(\pm 1) = 0.
\]

Here, \(L\) is a linear, ordinary differential operator of even order \(N\). A complex number \(\lambda\) and a function \(u\) satisfying (1.1) are called an eigenvalue and eigenfunction of \(L\), respectively. We focus on computing the eigenvalues of \(L\) contained in a simply connected region \(\Omega \subset \mathbb{C}\). Throughout, we assume that the boundary \(\partial \Omega\) is a rectifiable, simple closed curve and that the spectrum of \(L\), denoted by \(\lambda(L)\), does not intersect \(\partial \Omega\). We also assume that \(\Omega\) contains only a discrete portion of the spectrum of \(L\), i.e., finitely many eigenvalues counting multiplicities. To simplify discussion about the eigenfunctions of (1.1), we assume that there are eigenfunctions of \(L\) that form a basis for the invariant subspace of \(L\) associated with \(\Omega\).

Since the development of the QR algorithm in the 1960s, the standard methods for solving (1.1) have adopted a “discretize-then-solve” paradigm. These algorithms first discretize \(L\) to obtain a finite matrix eigenvalue problem and then solve the matrix eigenvalue problem with algorithms from numerical linear algebra [14,17,21,35]. Motivated by mathematical software for highly adaptive computations with functions [15], we propose an alternative strategy: an algorithm that solves (1.1) by directly manipulating \(L\) at the continuous level and only discretizes functions, not operators. By designing an eigensolver for \(L\) rather than intermediate discretizations, we are able to leverage spectrally accurate approximation schemes for functions while avoiding several pitfalls that plague spectral discretizations of (1.1) (for detailed accounts of such difficulties, see [52], [18, Ch. 2], and [51, Ch. 30]). For this reason, we view our proposed algorithms as adopting a “solve-then-discretize” paradigm. The solve-then-discretize paradigm has recently been applied to Krylov methods [19], iterative eigensolvers [23], and contour integral projection eigensolvers [7] for differential operators. Related techniques for computing with operators on infinite dimensional spaces have been proposed and studied in [12,32].

As an example of the advantages of our methodology, consider the simplest pos-
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Figure 1.1. Left: The eigenvalue condition numbers [5] for 4000 × 4000 discretizations of (1.2) obtained by collocation (blue dots), tau (red dots), Chebyshev–Galerkin (black dots), and ultraspherical (yellow dots) spectral methods are compared with the eigenvalue condition numbers (magenta dots) of (1.2), which are preserved by the operator analogue of FEAST. Right: The relative errors in the first 2000 eigenvalues of each spectral discretization of (1.2), computed with a backward stable eigensolver [20, p.385]. We observe fluctuations in the relative errors due to the ill-conditioning introduced by using nonsymmetric spectral discretizations of \( L \). In contrast, the relative errors (magenta dots) in the eigenvalues computed by \texttt{contFEAST}, a practical implementation of the operator analogue of FEAST (see section 4), are on the order of machine precision.

sible differential eigenvalue problem given by

\[
(1.2) \quad -\frac{d^2u}{dx^2} = \lambda u, \quad u(\pm 1) = 0.
\]

The eigenvalues of (1.2) are \( \lambda_k = (k\pi/2)^2 \), for \( k \geq 1 \), and are well-conditioned due to the fact that the eigenfunctions form a complete orthonormal set in the Hilbert space \( L^2([-1, 1]) \) [26, p. 382]. However, spectral discretizations of (1.2) lead to highly non-normal matrices with eigenvalues that are far more ill-conditioned than expected. Due to this ill-conditioning, the accuracy in the computed eigenvalues can be extremely variable and difficult to predict, ranging from a few digits to nearly full precision (see Figure 1.1). It is possible to use structure-preserving spectral discretizations to solve (1.2) accurately [11, 44]. However, there is a lack of literature on designing spectral discretizations of (1.1) when \( L \) is self-adjoint or normal with respect to an inner product other than \( L^2([-1, 1]) \). On the other hand, the solve-then-discretize methodology that we propose automatically preserves the normality or self-adjointness of \( L \) with respect to a relevant Hilbert space \( \mathcal{H} \), provided that the inner product \((\cdot, \cdot)_\mathcal{H}\) can be evaluated.

At the heart of our approach is an operator analogue of the FEAST matrix eigensolver, which we briefly outline:

1. We construct a basis for the eigenspace \( \mathcal{V} \) corresponding to \( \Omega \) by sampling the range of the associated spectral projector \( P_\mathcal{V} \).
2. We extract an \( \mathcal{H} \)-orthonormal basis for \( \mathcal{V} \) with a continuous analogue of the QR factorization [49].
3. We perform a Rayleigh–Ritz projection [41, p. 98] of \( L \) onto \( \mathcal{V} \) with the orthonormal basis in (2). We solve the resulting matrix eigenvalue problem to obtain approximations to the eigenvalues of \( L \) in \( \Omega \).

As with the FEAST matrix eigensolver, the spectral projector \( P_\mathcal{V} \) is applied approximately via a quadrature rule approximation. For matrices, this involves solving shifted linear systems, while for differential operators one needs to solve shifted linear differential equations. We solve these differential equations with the ultraspherical
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2. The FEAST matrix eigensolver. The FEAST matrix eigensolver uses approximate spectral projection to compute the eigenvalues of a matrix $A \in \mathbb{C}^{n \times n}$ in a region of interest $\Omega \subset \mathbb{C}$ [27] (see Figure 2.1). It is usually more computationally
Algorithm 2.1 The FEAST algorithm for matrix eigenvalue problems [39]. This is often viewed as a single iteration that is repeated to improve the accuracy of the computed eigenvalues and eigenvectors [47].

**Input:** \( A \in \mathbb{C}^{n \times n}, \Omega \subset \mathbb{C} \) containing \( m \) eigenvalues of \( A \), \( Y : \mathbb{C}^{n \times m} \).

1. Compute \( V = P_Y Y \).
2. Compute the QR factorization \( V = QR \).
3. Compute \( A_Q = Q^* A Q \) and solve the eigenvalue problem \( A_Q X = \Lambda X \) for \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_m) \) and \( X \in \mathbb{C}^{m \times m} \).

**Output:** Eigenvalues \( \lambda_1, \ldots, \lambda_m \) in \( \Omega \) and eigenfunctions \( U = Q X \).

efficient than standard eigensolvers when the number of eigenvalues in \( \Omega \) is much smaller than \( n \). The dominating computational cost of FEAST is solving several independent shifted linear systems, but these can be performed in parallel [27].

There are three essential ingredients to FEAST:

(i) **Spectral projector.** Let \( \lambda_1, \ldots, \lambda_m \) be the eigenvalues of \( A \) in \( \Omega \) and let \( V \) be the associated invariant subspace of \( A \), i.e., \( A V = V \). The spectral projector onto \( V \) is defined as

\[
P_V = \frac{1}{2\pi i} \int_{\partial \Omega} (zI - A)^{-1} \, dz.
\]

The important fact here is that \( \text{range}(P_V) = V \) and so \( P_V \) is a projection onto the invariant subspace of \( A \) [26].

(ii) **Basis for \( V \).** FEAST uses the spectral projector to construct a basis for \( V \). It begins with a matrix \( Y \in \mathbb{C}^{n \times m} \) with linearly independent columns that are not in \( \ker(P_V) \), then it computes \( Z = P_V Y \). The columns of \( Z \) span \( V \) and a QR factorization of \( Z \) provides an orthonormal basis, \( Q \), for \( V \).

(iii) **Rayleigh–Ritz projection.** Having obtained an orthonormal basis for \( V \), FEAST solves \( A_Q x = \lambda x \) using a dense eigensolver [39], where \( A_Q = Q^* A Q \). Since \( \text{range}(Q) = V \), the eigenvalues of \( A_Q \) are precisely the eigenvalues of \( A \) that lie inside \( \Omega \). When \( A_Q \) is diagonalizable, the eigenvectors \( x_1, \ldots, x_m \) of \( A_Q \) provide the coordinates of the eigenvectors \( u_1, \ldots, u_m \) of \( A \) in the basis \( Q \), i.e., \( u_i = Q x_i \) for \( i = 1, \ldots, m \).

For practical computation, FEAST approximates the contour integral in (2.1) with a quadrature rule. Given a quadrature rule with nodes \( z_1, \ldots, z_\ell \) and weights \( w_1, \ldots, w_\ell \), one can approximate \( P_V Y \) by

\[
P_V Y \approx \frac{1}{2\pi i} \sum_{k=1}^{\ell} w_k (z_k I - A)^{-1} Y.
\]

In this case, the range of \( Q \) only approximates \( V \) and the eigenpairs of \( A_Q \) provide approximations to the eigenpairs of \( A \), known as Ritz values and vectors [47]. To refine the accuracy of the Ritz values and vectors, a more accurate quadrature rule can be used to compute \( P_V Y \) [47]. FEAST also refines the approximate eigenvalues and eigenvectors by applying \( P_V \) to the \( n \times m \) block of approximate eigenvectors using a quadrature rule and iterating (ii) and (iii) until convergence. To fully understand this refinement process, one must examine FEAST through the lens of rational subspace iteration [47].

When the dimension \( m \) of the invariant subspace \( V \) is unknown, there are several techniques for estimating \( m \) and selecting an appropriate value [27, 30, 47]. Most of
these can be incorporated into the operator analogue of FEAST in a straightforward manner. Consequently, we assume that \( m \) is known throughout the paper in order to focus on the algorithmic and theoretical aspects of FEAST that are particularly relevant in the operator setting.

Curiously, the originally proposed FEAST algorithm does not compute an orthonormal basis for \( \mathcal{V} \) before performing the Rayleigh–Ritz projection \([27,39]\).

However, when \( Q \) has orthonormal columns and \( \text{range}(Q) \) is an invariant subspace of \( A \), then the eigenvalues of the small matrix \( Q^*AQ \) are no more sensitive to perturbations than the original eigenvalues of \( A \). This highly desirable property follows from an examination of the structure of the left and right invariant subspaces of \( Q^*AQ \) or, alternatively, from the \( \varepsilon \)-pseudospectra of \( Q^*AQ \) \([51, \text{p. 382}]\).

3. An operator analogue of FEAST. The FEAST matrix algorithm provides a natural starting point for an operator analogue because it provides a recipe to construct a small matrix \( Q^*AQ \) whose eigenvalues coincide with those of \( A \) inside \( \Omega \) and have related invariant subspaces. The value of this is that the eigenstructure of \( Q^*AQ \) reflects the eigenstructure of \( A \) when the columns of \( Q \) are orthonormal. As the sensitivity of the eigenvalues of \( A \) depends intimately on the structure of the associated eigenvectors, \( Q^*AQ \) may be used to compute the desired eigenvalues of \( A \) efficiently without sacrificing accuracy. Here, we generalize FEAST so that it provides a recipe to construct a matrix whose eigenvalues coincide with those of a differential operator inside \( \Omega \).

3.1. FEAST for differential operators. In place of a matrix \( A \) acting on vectors from \( \mathbb{C}^n \), we now consider a differential operator \( \mathcal{L} \) acting on functions from a Hilbert space \( \mathcal{H} \). As described in section 2, the FEAST recipe prescribes a spectral projection to compute a basis for \( \mathcal{V} \), which is then used for the Rayleigh–Ritz projection to construct a matrix representation on \( \mathcal{V} \).

(i) Spectral projector. Unlike matrices, a differential operator \( \mathcal{L} \) is typically unbounded on \( \mathcal{H} \). To make sense of the spectral projector defined via contour integral, we require that \( \mathcal{L} \) be a closed operator\(^2\) and that its domain \( \mathcal{D}(\mathcal{L}) \) is dense in \( \mathcal{H} \). This is sufficient to ensure that the spectral projector onto \( \mathcal{V} \) is well-defined \([26, \text{p. 178}]\).

It is given by

\[
\mathcal{P}_\mathcal{V} = \frac{1}{2\pi i} \int_{\partial \Omega} (zI - \mathcal{L})^{-1} dz.
\]

(ii) Basis for \( \mathcal{V} \). With the spectral projector at our disposal, we apply \( \mathcal{P}_\mathcal{V} \) to functions \( f_1, \ldots, f_m \) in \( \mathcal{H} \setminus \ker(\mathcal{P}_\mathcal{V}) \) to obtain a basis of functions \( v_1, \ldots, v_m \) for \( \mathcal{V} \). Orthonormalizing \( v_1, \ldots, v_m \) with respect to the inner product \((\cdot, \cdot)_{\mathcal{H}}\) on \( \mathcal{H} \) gives us an \( \mathcal{H} \)-orthonormal basis \( q_1, \ldots, q_m \) for \( \mathcal{V} \).

(iii) Rayleigh–Ritz projection. To compute a matrix representation \( L \) of \( \mathcal{L} \) on \( \mathcal{V} \), the Rayleigh–Ritz projection is performed using the inner product on \( \mathcal{H} \). The elements of \( L \) are given by \( L_{ij} = (q_i, \mathcal{L}q_j)_{\mathcal{H}} \) for \( 1 \leq i, j \leq m \). The eigenvalues of \( L \) are precisely the eigenvalues \( \lambda_1, \ldots, \lambda_m \) of \( \mathcal{L} \) that lie inside \( \Omega \). The eigenvectors \( x_1, \ldots, x_m \) of \( L \) provide the coordinates of the eigenfunctions \( u_1, \ldots, u_m \) of \( \mathcal{L} \) with respect to the basis \( q_1, \ldots, q_m \), meaning that \( u_i = \sum_{k=1}^{m} x_i^{(k)} q_k \) where \( x_i^{(k)} \) is the \( k^{\text{th}} \) component of \( x_i \).

\(^1\)The FEAST algorithm for non-Hermitian matrices utilizes dual bases for the left and right eigenspaces to improve stability \([27]\).

\(^2\)An operator \( A : \mathcal{D}(A) \to \mathcal{H} \) is closed if the graph of \( A \) is a closed linear subspace of \( \mathcal{H} \times \mathcal{H} \) \([26, \text{p. 165}]\).
\textbf{Algorithm 3.1} An operator analogue of FEAST for differential operators.
\begin{itemize}
  \item[Input:] $\mathcal{L} : \mathcal{D}(\mathcal{L}) \to \mathcal{H}$, $\Omega \subset \mathbb{C}$ containing $m$ eigenvalues of $\mathcal{L}$, $F : \mathbb{C}^m \to \mathcal{H}$.
  \item[1:] Compute $V = \mathcal{P}_\lambda F$.
  \item[2:] Compute $V = QR$, where $Q : \mathbb{C}^m \to \mathcal{D}(\mathcal{L}) \subset \mathcal{H}$ has $\mathcal{H}$-orthonormal columns and $R \in \mathbb{C}^{m \times m}$ is upper triangular.
  \item[3:] Compute $L = Q^* \mathcal{L}Q$ and solve $LX = \Lambda X$ for $\Lambda = \text{diag}[\lambda_1, \ldots, \lambda_m]$ and $X \in \mathbb{C}^{m \times m}$.
\end{itemize}
\textbf{Output:} Eigenvalues $\lambda_1, \ldots, \lambda_m$ in $\Omega$ and eigenfunctions $U = QX$.

To avoid a clutter of indices, we employ the notation of quasimatrices.\textsuperscript{3} If $Q$ is the quasimatrix with columns $q_1, \ldots, q_m$, then the matrix $L$ whose elements are $L_{ij} = (q_i, \mathcal{L}q_j)_\mathcal{H}$ in (iii) is expressed compactly in quasimatrix notation as $L = Q^* \mathcal{L}Q$. Here, $Q^*$ is the conjugate transpose of the quasimatrix $Q$ so its rows are complex conjugates of the functions $q_1, \ldots, q_m$.

The analogue of FEAST for differential operators is succinctly summarized in Algorithm 3.1 using quasimatrix notation so that it closely resembles its matrix counterpart. Keep in mind that Algorithm 3.1 is a formal algorithm. In general, we cannot apply the spectral projector exactly, nor even represent the basis $V$ exactly with finite memory. A practical implementation is discussed in section 4.

\textbf{3.2. Condition number of the Ritz values.} As illustrated in Figure 1.1, the eigenvalues of matrix discretizations of $\mathcal{L}$ can be more sensitive to perturbations than the eigenvalues of $\mathcal{L}$. The advantage of our FEAST approach in subsection 3.1 is that the Ritz values, i.e., the eigenvalues of $Q^* \mathcal{L}Q$, are no more sensitive to perturbations than the original eigenvalues of $\mathcal{L}$ when range$(Q)$ is an invariant subspace of $\mathcal{L}$.

To see this, let $\lambda$ be a simple eigenvalue of a linear operator $\mathcal{L}$ that is closed and densely defined on a Hilbert space $\mathcal{H}$. Let $u, w \in \mathcal{H}$ satisfy $\mathcal{L}u = \lambda u$ and $\mathcal{L}^* w = \lambda w$. The \textit{condition number} of $\lambda$ is given by [26, p. 373]

$$\kappa_\mathcal{H}(\lambda) = \frac{\|u\|_\mathcal{H} \|w\|_\mathcal{H}}{(w, u)_\mathcal{H}}.$$  \hfill (3.2)

The condition number $\kappa_\mathcal{H}(\lambda)$ quantifies the worst-case first-order sensitivity of $\lambda$ to perturbations of $\mathcal{L}$. For instance, if we compute $\lambda$ using a backward stable algorithm in floating point arithmetic, we expect to achieve an accuracy of about $\kappa_\mathcal{H}(\lambda) \epsilon_{\text{mach}}$, where $\epsilon_{\text{mach}}$ is machine precision [50, Theorem 15.1]. It is worthwhile to note that the eigenvalue condition number defined in (3.2) is a generalization of the classical condition number for the eigenvalue of a matrix $A \in \mathbb{C}^{n \times n}$, which is recovered when one takes $\mathcal{H} = \mathbb{C}^n$ [46, p. 186].

\textbf{Theorem 3.1.} Let $\mathcal{L} : \mathcal{D}(\mathcal{L}) \to \mathcal{H}$ be a closed and densely defined operator on a Hilbert space $\mathcal{H}$, $Q : \mathbb{C}^m \to \mathcal{H}$ be an invariant subspace of $\mathcal{L}$ satisfying $Q^* Q = I$, and $L = Q^* \mathcal{L}Q$. Suppose that $u \in \text{range}(Q)$ satisfies $\mathcal{L}u = \lambda u$ and $w$ satisfies $\mathcal{L}^* w = \lambda w$, where $\mathcal{L}^*$ denotes the adjoint of $\mathcal{L}$ and $\lambda$ is a simple eigenvalue with condition number $\kappa_\mathcal{H}(\lambda)$. Then,

1) $LQ^* u = \lambda Q^* u$ and $L^* Q^* w = \lambda Q^* w$,
2) $(Q^* w, Q^* u)_{\mathbb{C}^m} = (w, u)_\mathcal{H}$, and
3) $\kappa_\mathcal{H}(\lambda) \leq \kappa_\mathcal{H}(\lambda)$.

\textsuperscript{3}A quasimatrix is a matrix whose columns (or rows) are functions defined on an interval $[a, b]$, in contrast to matrices whose columns (or rows) are vectors.
Proof. Denote \( x = Q^*u \) and \( y = Q^*w \). We prove the statements of the theorem in order. 1) Since \( u \in \text{range}(Q) \), we can write \( u = Qx \). Then, \( L(Qx) = \lambda(Qx) \) implies that \( Q^*LQx = \lambda x \) using the fact that \( Q^*Q = I \). For the left eigenvector, we write \( w = Qy + v \) for some \( v \in \text{range}(Q)^\perp \). Rewriting the adjoint equation for \( w \), we find that \( L^*(Qy + v) = \lambda(Qy + v) \) and multiplying by \( Q^* \) on both sides yields \( Q^*L^*Qy = \lambda y \). Here, we have used that \( \text{range}(Q)^\perp \) is an invariant subspace of \( L^* \) so that \( Q^*L^*y = 0 \). 2) By calculating \( (w, u)_H = (Qy + v, Qx)_H \) because \( v \in \text{range}(Q)^\perp \). Moreover, since \( Q^*Q = I \) we conclude that \( (Q, Qx)_H = (Qy + v, Qy + v)_H = ||y||_C^m + ||v||_H \). Therefore,

\[
\|u\|_H ||v||_H = ||x||_C^m (||y||_C^m + ||v||_H) \geq ||x||_C^m ||y||_C^m.
\]

Referring to 2) for equality of the inner products in the denominator, we have

\[
\kappa_C^m(\lambda) = \frac{||x||_C^m ||y||_C^m}{(w, u)_H} = \kappa_H(\lambda),
\]

which concludes the proof. \(\square\)

Theorem 3.1 shows that if \( L \) is a normal operator, then \( u = w \) and we have \( \kappa_{C^m}(\lambda) = \kappa_H(\lambda) = 1 \). For non-normal operators, item 3) of Theorem 3.1 may seem to erroneously indicate that ill-conditioning in the eigenvalues of \( L \) can be overcome by a Rayleigh–Ritz projection. However, when \( L \) is non-normal the spectral projector \( P_V \) is an oblique projection and computing the basis \( Q \) may be itself an ill-conditioned problem. In fact, the norm \( ||P_V||_H \) is closely related to the condition numbers of the eigenvalues \( \lambda_1, \ldots, \lambda_m \), defined in (3.2) [26, p. 79]. Since the smallest non-zero singular value of \( P_V \) is always 1, the condition number for the problem of computing \( Q \) and the condition numbers of the eigenvalues \( \lambda_1, \ldots, \lambda_m \) are intrinsically linked.

Theorem 3.1 illustrates why the operator analogue of FEAST leads to a well-conditioned matrix eigenvalue problem when the differential eigenvalue problem is well-conditioned. By computing an \( H \)-orthonormal basis for the Rayleigh–Ritz projection, the relevant structure of the left and right eigenspaces \( V \) and \( W \) is preserved. However, the first-order analysis above is limited to simple eigenvalues.

### 3.3. Pseudospectra of \( Q^*LQ \)

To go beyond first-order sensitivity analysis, we compare the \( \epsilon \)-pseudospectra of \( L \) and \( Q^*LQ \). Fix any \( \epsilon > 0 \) and let \( L : D(L) \rightarrow H \) be a closed operator with a domain \( D(L) \) that is dense in \( H \). The \( \epsilon \)-pseudospectrum of \( L \) is defined as the set [51, p. 31]

\[
\lambda_\epsilon(L) = \{ z \in \mathbb{C} : \|(zI - L)^{-1}\|_H > 1/\epsilon \}.
\]

The \( \epsilon \)-pseudospectrum set of \( L \) bounds the region in which the eigenvalues of the perturbed operator \( L + \mathcal{E} \) with \( \|\mathcal{E}\|_H < \epsilon \) can be found [51, p. 31]. This means that \( \lambda(L + \mathcal{E}) \subset \lambda_\epsilon(L) \). In fact, there is an equivalence so that [51, p. 31]

\[
\bigcup_{\|\mathcal{E}\|_H < \epsilon} \lambda(L + \mathcal{E}) = \lambda_\epsilon(L).
\]

This allows us to relate the sensitivity of the eigenvalues of \( L \) and \( Q^*LQ \) by comparing the resolvent norms \( \|(zI - L)^{-1}\|_H \) and \( \|(zI - Q^*LQ)^{-1}\|_C^m \), respectively.

A useful generalization of Theorem 3.1 is that the \( \epsilon \)-pseudospectrum of \( Q^*LQ \) is contained in the \( \epsilon \)-pseudospectrum of \( L \). Since this holds for any \( \epsilon > 0 \), it demonstrates that the eigenvalues (even those with multiplicity) of \( Q^*LQ \) are no more
sensitive to perturbations than those of $L$. This inclusion result is well-known in the matrix case where projection methods are a popular method for approximating the $\epsilon$-pseudospectra of large data-sparse matrices [51, p. 381].

**Theorem 3.2.** Let $L : \mathcal{D}(L) \to \mathcal{H}$ be a closed and densely defined operator on a Hilbert space $\mathcal{H}$. For a fixed $\epsilon > 0$, suppose that $Q : \mathbb{C}^m \to \mathcal{H}$ satisfies $Q^*Q = I$ and that range($Q$) is an invariant subspace of $L$. Then, $\lambda_{\epsilon}(Q^*LQ) \subset \lambda_{\epsilon}(L)$.

**Proof.** We follow the proof of Proposition 40.1 in [51, p. 382] for matrices, but with a closed operator. By definition of the resolvent and the fact that $Qx \in \mathcal{H}$ for any $x \in \mathbb{C}^m$, we have

$$\|(zI - L)^{-1}\|_\mathcal{H} = \max_{f \in \mathcal{H}, \|f\|_\mathcal{H} = 1} \|(zI - L)^{-1}f\|_\mathcal{H} \geq \max_{x \in \mathbb{C}^m, \|x\|_{\mathbb{C}^m} = 1} \|(zI - L)^{-1}Qx\|_\mathcal{H} = \max_{x \in \mathbb{C}^m, \|x\|_{\mathbb{C}^m} = 1} \|(zI - Q^*LQ)^{-1}x\|_{\mathbb{C}^m}.$$  

Here, the final equality holds because range($Q$) is an invariant subspace of $L$ and does not hold for general quasimatrices $Q$ with $\mathcal{H}$-orthonormal columns.

The inclusion in Theorem 3.2 may be strict, indicating that the eigenvalues of $Q^*LQ$ are less sensitive than those of $L$. For example, this may occur when the projection onto range($Q$) targets a subset of well-conditioned eigenvalues of $L$. However, we emphasize that ill-conditioning in the eigenvalues of $L$ cannot be overcome by a Rayleigh–Ritz projection: in general, the situation can be quite complicated [51, Ch. 40].

**Theorem 3.2** is useful for studying the stability of Algorithm 3.1. If an approximate eigenvalue $\hat{\lambda}$ of $Q^*LQ$ is computed with an error tolerance of $\epsilon > 0$, then

$$\hat{\lambda} \in \lambda_{\epsilon}(Q^*LQ) \subset \lambda_{\epsilon}(L).$$

From this, we know by (3.4) that $\hat{\lambda}$ is an eigenvalue of a perturbed operator $L + \mathcal{E}$ with $\|\mathcal{E}\|_\mathcal{H} < \epsilon$. In other words, the operator analogue of FEAST, Algorithm 3.1, is backward stable. As we see in section 5, Theorem 3.2 is also the starting point for a stability analysis when the spectral projection is no longer exact and the Rayleigh–Ritz projection is performed with a matrix $\hat{Q}$ that only approximates a basis for an invariant subspace of $L$.

**4. A practical differential eigensolver based on an operator analogue of FEAST.** The operator analogue of FEAST requires the manipulation of continuous objects such as differential operators, functions, and contour integrals (see Algorithm 3.1). For a practical implementation, these objects must be discretized; however, we avoid discretizing $L$ directly. Instead, we construct polynomial approximants to the basis for $\mathcal{V}$ by approximately solving shifted linear ODEs. These polynomial approximants are used in the Rayleigh–Ritz projection to compute the eigenvalues of $L$ in $\Omega$.

Let $z_1, \ldots, z_\ell$ and $w_1, \ldots, w_\ell$ be a set of quadrature nodes and weights to approximate the integral in (3.1). As FEAST does in the matrix case, we approximate $\mathcal{P}_\mathcal{V}$ in (3.1) with a quadrature rule as follows:

$$\hat{\mathcal{P}}_{\mathcal{V}} = \frac{1}{2\pi i} \sum_{k=1}^\ell w_k (z_k I - L)^{-1}.$$
Algorithm 4.1 A practical algorithm for computing the eigenvalues of a differential operator $L$, which we refer to as \texttt{contFEAST}.

\textbf{Input:} $L : D(L) \rightarrow \mathcal{H}$, $z_1, \ldots, z_\ell \in \partial \Omega$, $w_1, \ldots, w_\ell \in \mathbb{C}$, $F : \mathbb{C}^m \rightarrow \mathcal{H}$, $\epsilon > 0$.

1: \textbf{repeat}
2: \quad Solve $(z_k I - L)G_k = F$, $G_k(\pm 1) = 0$, $\ldots$, $G_k^{(N/2)}(\pm 1) = 0$, for $k = 1, \ldots, \ell$.
3: \quad Set $\hat{V} = \sum_{k=1}^{\ell} w_k G_k$.
4: \quad Compute $\hat{V} = \hat{Q} \hat{R}$, where $\hat{Q} : \mathbb{C}^m \rightarrow D(L) \subset \mathcal{H}$ has $\mathcal{H}$-orthonormal columns and $\hat{R} \in \mathbb{C}^{m \times m}$ is upper triangular.
5: \quad Compute $\hat{L} = \hat{Q}^* L \hat{Q}$ and solve $\hat{L} \hat{X} = \hat{X} \hat{\Lambda}$ for $\hat{\Lambda} = \text{diag}[\hat{\lambda}_1, \ldots, \hat{\lambda}_m]$ and $\hat{X} \in \mathbb{C}^{m \times m}$. Set $F = \hat{Q} \hat{X}$.
6: \textbf{until} $\|LF - F \hat{\Lambda}\|_\mathcal{H} \leq \epsilon \|\hat{\Lambda}\|_{\mathbb{C}^m}$.

\textbf{Output:} $\hat{\Lambda}$, $\hat{U} = \hat{Q} \hat{X}$.

We use the notation $\hat{P}_V$ to emphasize that (4.1) is an approximation to the spectral projector, i.e., that $\text{range}(\hat{P}_V) \approx V$.

If $F$ is a quasimatrix with columns $f_1, \ldots, f_m \in \mathcal{H}$, then $\hat{P}_V F$ is replaced by the approximation $\hat{P}_V F = \frac{1}{2\pi i} \sum_{k=1}^{N} w_k (z_k I - L)^{-1} F$. Therefore, to compute $\hat{P}_V F$ we need to solve $\ell$ shifted linear ODEs, each with $m$ righthand sides. That is, we need to solve

\begin{equation}
(z_k I - L)g_{i,k} = f_i, \quad g_{i,k}(\pm 1) = \cdots = g_{i,k}^{(N/2)}(\pm 1) = 0, \quad 1 \leq i \leq m.
\end{equation}

If the quasimatrix with columns $g_{1,k}, \ldots, g_{m,k}$ is denoted by $G_k$ for $k = 1, \ldots, \ell$, then we have $\hat{P}_V F = \sum_{k=1}^{\ell} w_k G_k$.

To construct a basis for $V$, it is important to choose $F$ so that the columns of $\hat{V} = \hat{P}_V F$ are linearly independent and, if possible, well-conditioned. In analogy with the implementation of matrix FEAST \cite{27, 39}, we obtain the columns of $F$ by selecting $m$ band-limited random functions\footnote{A periodic band-limited random function on $[-L, L]$ is a periodic function defined by a truncated Fourier series with random (e.g., standard Gaussian distributed) coefficients. In the non-periodic setting, the Fourier series is defined on a larger interval $[-L', L']$ and the domain is truncated to remove the traces of periodicity near the endpoints \cite{16}.} on $[-1, 1]$ \cite{16}. When $L$ is a normal operator, this typically yields a well-conditioned basis $V$.

We now outline the key implementation details of the practical differential eigensolver:

(i) \textbf{Approximate spectral projection.} To compute $\hat{V} = \hat{P}_V F$, we solve the shifted linear ODEs in (4.2) using the ultraspherical spectral method \cite{31}. The ultraspherical spectral method leads to well-conditioned linear systems and is capable of accurately resolving the functions $g_{i,k}$ even when they are highly oscillatory or have boundary layers. Moreover, an adaptive QR factorization automatically determines the degree of the polynomial interpolants needed to approximate the functions $g_{i,k}$ to near machine precision \cite{32, 33}. After accurately resolving the functions $g_{i,k}$, we can accurately compute a basis for $V$ provided that both the spectral projector is well-conditioned (i.e., $L$ is not highly non-normal) and the quadrature rule is sufficiently accurate.

(ii) \textbf{Orthonormal basis.} To compute an orthonormal basis $\hat{Q}$ for the columns of $\hat{V}$, we compute a QR factorization of the quasimatrix $\hat{V}$ by Householder triangularization \cite{49}. The Householder reflectors are constructed with respect to the inner product $(\cdot, \cdot)_\mathcal{H}$ so that the columns of $\hat{Q}$ are $\mathcal{H}$-orthonormal.
(iii) **Computing \( \hat{Q}^* L \hat{Q} \).** To construct the matrix \( \hat{L} = \hat{Q}^* L \hat{Q} \), we apply \( L \) to the columns of \( \hat{Q} \) and then evaluate the action of \( \hat{Q}^* \) on \( \hat{L} \hat{Q} \). Multiplying \( \hat{Q}^* \) with \( \hat{L} \hat{Q} \) involves taking the inner products

\[
(4.3) \quad \hat{L}_{ij} = (\hat{q}_i, L \hat{q}_j)_H, \quad 1 \leq i, j \leq m,
\]

where \( \hat{q}_i \) denotes the \( i \)th column of \( \hat{Q} \). The eigenvalues \( \hat{\lambda}_1, \ldots, \hat{\lambda}_m \) and eigenvectors \( \hat{x}_1, \ldots, \hat{x}_m \) of the matrix \( \hat{L} \) are computed using the QR algorithm [20, p. 385].

Critically, the inner product \((\cdot, \cdot)_H\) used in the QR factorization of \( \hat{V} \) and the construction of \( \hat{Q}^* L \hat{Q} \) depends on the choice of the Hilbert space \( H \). As long as we are able to evaluate \((\cdot, \cdot)_H\), we can exploit the fact that \( L \) is self-adjoint or a normal operator with respect to \((\cdot, \cdot)_H\) so that we can accurately compute the eigenvalues of \( L \) in \( \Omega \) (see Theorem 5.2). For this reason, our algorithm is able to accurately compute the eigenvalues and eigenfunctions of differential operators that are self-adjoint with respect to non-standard Hilbert spaces (see subsection 4.1).

Evaluating the inner product \((\cdot, \cdot)_H\) usually means computing an integral, which we approximate with a quadrature rule. For example, if \( H = L^2([-1, 1]) \),

\[
(f, g)_{L^2([-1, 1])} = \int_{-1}^{1} f^*(x) g(x) \, dx,
\]

where \( f^*(x) \) denotes the complex conjugate of \( f(x) \). Given the Gauss–Legendre quadrature nodes \( x_1, \ldots, x_p \) and weights \( w_1, \ldots, w_p \) on \([-1, 1]\), then one uses the approximation [10]

\[
(f, g)_{L^2([-1, 1])} \approx \sum_{k=1}^{p} w_k f^*(x_k) g(x_k).
\]

A practical implementation of the operator analogue of FEAST is presented in Algorithm 4.1. As with matrix FEAST, there are two approaches for improving the accuracy of the Ritz values \( \hat{\lambda}_1, \ldots, \hat{\lambda}_m \) and vectors \( \hat{Q} \hat{x}_1, \ldots, \hat{Q} \hat{x}_m \). The first is to improve the accuracy of the quadrature rule in (4.1). The second is to iterate the algorithm by replacing \( F \) by the quasimatrices \( \hat{U} \) with columns \( \hat{u}_i = \hat{Q} \hat{x}_i \) for \( 1 \leq i \leq m \), repeating the process if necessary. For normal operators, this iteration generates a sequence of quasimatrices \( \hat{Q}_k \) with \( H \)-orthonormal columns that converge to an \( H \)-orthonormal basis for the invariant subspace \( V \) as \( k \to \infty \) (see section 5). In the case of a matrix [41, p. 119] or a self-adjoint, closed operator [22], this latter approach falls within the framework of rational subspace iteration and geometric convergence of the Ritz pairs is typical.

Typically, solving the ODEs in (4.2) dominates the computational cost of Algorithm 4.1. With the ultraspherical spectral method, the computational complexity of solving the linear ODEs with \( m \) distinct right hand sides is \( \mathcal{O}(mM^2 N \log(N)) \) floating point operations (flops) [31]. Here, \( N \) and \( M \) are, respectively, the degrees of the truncated Chebyshev series needed to resolve the columns of \( G_k \) and the variable coefficients in \( L \) to within the tolerance \( \epsilon \) specified in Algorithm 4.1. In addition to the ODE solve, the QR factorization in (ii) requires \( \mathcal{O}(m^2 N) \) flops [49], while the dense eigenvalue computation with a small \( m \times m \) matrix in (iii) takes \( \mathcal{O}(m^3) \) flops [20, p. 391].

The complexity of one iteration of Algorithm 4.1 is therefore \( \mathcal{O}(mM^2 N \log(N) + m^3) \) flops. In practice, convergence to machine precision usually occurs within two or three iterations.
4.1. Computing high-frequency eigenmodes. Algorithm 4.1 adaptively and accurately resolves basis functions for highly oscillatory eigenmodes and preserves the sensitivity of the eigenvalues of the differential operator $L$, so it is well-suited to computing high-frequency eigenmodes when $L$ is self-adjoint or normal with respect to $(\cdot, \cdot)_H$. We provide two examples from Sturm–Liouville theory to illustrate the effectiveness of the solve-then-discretize methodology in the high-frequency regime.

4.1.1. A regular Sturm–Liouville eigenvalue problem. First consider a regular Sturm–Liouville eigenvalue problem (SLEP) given by

$$-\frac{d^2u}{dx^2} + x^2 u = \lambda \cosh(x) u, \quad u(\pm 1) = 0.$$  \hfill (4.4)

Classically, (4.4) defines a self-adjoint differential operator with respect to the inner product

$$(u, v)_w = \int_{-1}^{1} uv \cosh(x) \, dx.$$  \hfill (4.5)

Consequently, (4.4) possesses a complete $(\cdot, \cdot)_w$-orthonormal basis of eigenfunctions $u_1, u_2, u_3, \ldots$ and an unbounded set of real eigenvalues $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots$.

Asymptotics for the large eigenvalues of (4.4) are given by [2]

$$\sqrt{\lambda_n} \sim \frac{n\pi}{\int_{-1}^{1} \sqrt{\cosh(x)} \, dx}, \quad n \to \infty.$$  \hfill (4.6)

To accurately compute the large eigenvalues of (4.4) with Algorithm 4.1, we prescribe circular search regions with unit radius centered at the values given by the asymptotic formula in (4.6) (see Figure 4.1).

4.1.2. A singular Sturm–Liouville eigenvalue problem. Next, we consider the following singular SLEP:

$$-\frac{d^2u}{dx^2} = \lambda x^3 u, \quad u(\pm 1) = 0.$$  \hfill (4.7)
which is closely related to models of light propagation in a nonhomogeneous material [2,53]. Since the weight function $x^3$ changes sign at $x = 0$, (4.7) has a bi-infinite sequence of eigenvalues [6]. We index them in order as $\cdots \leq \lambda_{-2} \leq \lambda_{-1} < 0 < \lambda_1 \leq \lambda_2 \leq \cdots$. The asymptotics for the positive eigenvalues are given by [3]

$$\sqrt{\lambda_n} \sim \left(\frac{n - 1/4}{\pi}\right)^{1/2} \int_0^1 x^{3/2} \, dx, \quad \lambda_n > 0, \quad n \to \infty. \quad (4.8)$$

A similar expansion holds for the negative eigenvalues [3].

We use the leading order asymptotics in (4.8) to identify search regions that are likely to contain an eigenvalue of (4.7). However, because $x^3$ has a turning point at the origin, we must regard (4.7) as a generalized eigenvalue problem: $L_1 u = \lambda L_2 u$, where $L_1 u = -\frac{d^2 u}{dx^2}$ and $L_2 u = x^3 u$. The eigenvalues of the pencil $z L_2 - L_1$ are then computed with a straightforward generalization of Algorithm 4.1 that is based on the spectral projector for the generalized eigenvalue problem, i.e.,

$$P_N = \frac{1}{2\pi i} \int_{\partial \Omega} (z L_2 - L_1)^{-1} L_2 \, dz. \quad (4.9)$$

The eigenvalues and eigenfunctions are automatically resolved to essentially machine precision because of the use of the ultraspherical spectral method and the adaptive QR solver (see Figure 4.2).

5. Convergence and stability. The primary consequence of the approximations introduced in Algorithm 4.1 is that the spectral projector is no longer applied exactly. Therefore, the basis $Q$ computed for the Rayleigh–Ritz projection is not a basis for the invariant subspace $V$ of $L$. When $L$ is self-adjoint, the refinement of the basis $Q$ in Algorithm 4.1 generates a sequence of orthonormal quasimatrices $Q_k$ that converges geometrically to an orthonormal basis for $V$. More precisely, as $k \to \infty$, $\text{range}(Q_k) \to V$ geometrically fast with respect to a distance metric between subspaces [22]. In this section, we apply rational subspace iteration to normal operators (see subsection 5.1) and analyze the sensitivity of the Ritz values when the spectral projector is no longer applied exactly (see subsection 5.2). We conclude that Algorithm 4.1 computes elements in the $2\epsilon$-pseudospectrum of a normal differential
operator $\mathcal{L}$ when $\hat{Q}$ approximates an orthonormal basis for $\mathcal{V}$ with accuracy $\epsilon/(2\Lambda)$ (see Theorem 5.2). Here, $\Lambda$ is a positive constant depending on $\mathcal{L}$, $\mathcal{V}$, and the initial quasimatrix $F$ (see Lemma 5.3).

5.1. Rational subspace iteration for differential operators. Consider a rational function with poles $z_1, \ldots, z_t \in \mathbb{C}$ and weights $w_1, \ldots, w_r$, given by

$$
(5.1) \quad r(z) = \sum_{k=1}^{r} \frac{w_k}{z - z_k}, \quad z \in \mathbb{C} \setminus \{z_1, \ldots, z_t\}.
$$

Let $\mathcal{L} : \mathcal{D}(\mathcal{L}) \to \mathcal{H}$ be a normal, closed, and densely defined operator on a Hilbert space $\mathcal{H}$. Provided that $\lambda(\mathcal{L}) \cap \{z_1, \ldots, z_t\} = \emptyset$, then $r(\mathcal{L}) = \sum_{k=1}^{r} w_k(z_k I - \mathcal{L})^{-1}$ defines a bounded linear operator on $\mathcal{H}$. The spectral mapping theorem for unbounded normal operators ensures that $\lambda(r(\mathcal{L})) = r(\lambda(\mathcal{L}))$, where $r(\lambda(\mathcal{L}))$ denotes the closure of the set $r(\lambda(\mathcal{L}))$ [40, p.366]. In particular, if $\lambda_i$ is an eigenvalue of $\mathcal{L}$ with eigenfunction $u_i$, then $r(\lambda_i)$ is an eigenvalue of $r(\mathcal{L})$ with eigenfunction $u_i$.

The idea of rational subspace iteration is to compute a subset of eigenvalues $\lambda_1, \ldots, \lambda_m$ of $\mathcal{L}$ by choosing a rational function $r(\cdot)$ that is large on the discrete set $\{\lambda_1, \ldots, \lambda_m\}$ and small on $\lambda(\mathcal{L}) \setminus \{\lambda_1, \ldots, \lambda_m\}$. Subsequently, subspace iteration (also known as the block power method) [41, Ch. 5] can be used to compute the dominant eigenspace of $r(\mathcal{L})$. For example, the quadrature rule (4.1) approximating the spectral projector in (3.1) can be associated with a rational function approximating the characteristic function on $\Omega$. The rational function maps the eigenvalues inside of $\Omega$ to values near one and the eigenvalues outside of $\Omega$ to values near zero. Although subspace iteration is generally applied in the context of matrices, it may be extended to certain bounded linear operators on a Hilbert space (see Theorem 5.1).

To proceed, it is helpful to introduce the notions of the spectral radius and a dominant eigenspace of a bounded linear operator $\mathcal{B}$. The spectral radius of a bounded linear operator $\mathcal{B}$ on a Hilbert space $\mathcal{H}$ is the point of maximum modulus in the spectrum of $\mathcal{B}$, defined as [13, p. 99]

$$
(5.2) \quad \rho(\mathcal{B}) = \max \{ |z| : z \in \sigma(\mathcal{B}) \}.
$$

The spectral radius is useful in the analysis of subspace iteration because it characterizes the asymptotic behavior of $\|\mathcal{B}^k\|_{\mathcal{H}}$, in the sense that $\rho(\mathcal{B}) = \lim_{k \to \infty} \|\mathcal{B}^k\|_{\mathcal{H}}^{1/k}$ [13, p. 99]. Let $\mathcal{V}$ be an invariant subspace of $\mathcal{B}$ associated with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_m$ and a spectral projector $\mathcal{P}_\mathcal{V}$. We say that $\mathcal{V}$ is a dominant eigenspace of $\mathcal{B}$ if

$$
(5.3) \quad \rho((I - \mathcal{P}_\mathcal{V})\mathcal{B}) < |\lambda_m|.
$$

The following theorem is an extension of a standard convergence analysis [41, p. 119] for matrix subspace iteration to the setting of bounded linear operators with a dominant eigenspace.

**Theorem 5.1.** Let $\mathcal{B}$ be a bounded linear operator on a Hilbert space $\mathcal{H}$ with dominant eigenspace $\mathcal{V}$, defined in (5.3), having $\dim(\mathcal{V}) = m$. Select a quasimatrix $F : \mathbb{C}^m \to \mathcal{H}$ such that the columns of $\mathcal{P}_\mathcal{V} F$ are linearly independent and suppose the columns of the quasimatrix $Q_k : \mathbb{C}^m \to \mathcal{H}$ form an orthonormal basis for $\text{range}(\mathcal{B}^k F)$, for $k = 1, 2, 3, \ldots$. If $u \in \mathcal{V}$ is an eigenvector of $\mathcal{B}$ with eigenvalue $\lambda$, then there is a function $\hat{u}_k \in \text{range}(Q_k)$ such that

$$
\|\hat{u}_k - u\|_{\mathcal{H}} \leq ((\rho/\lambda) + \epsilon_k)^k \| (I - \mathcal{P}_\mathcal{V}) F x \|_{\mathcal{H}}, \quad k = 1, 2, 3, \ldots,
$$

where $\rho = \rho((I - \mathcal{P}_\mathcal{V})\mathcal{B})$, $\epsilon_k \to 0$ as $k \to \infty$, and $u = \mathcal{P}_\mathcal{V} F x$. 

Proof. Since \( u \in \mathcal{V} \) and the columns of \( \mathcal{P}_\mathcal{V} F \) are linearly independent, there is a vector \( x \in \mathbb{C}^n \) such that \( u = \mathcal{P}_\mathcal{V} F x \). Set \( E' = (I - \mathcal{P}_\mathcal{V}) F \) and write \( F x = u + Ex \). Apply \( B^k \) to \( F x \) to obtain that \( B^k F x = \lambda^k u + B^k Ex \). Since \( \hat{Q}_k \) is an orthonormal basis for \( \text{range}(B^k F) \), there is a matrix \( R_k \in \mathbb{C}^{n \times n} \) such that \( \hat{Q}_k R_k x = \lambda^k u + B^k Ex \). Now, let \( \mathcal{U} = \text{range}(I - \mathcal{P}_\mathcal{V}) \) and observe that \( \|B^k Ex\|_\mathcal{H} \leq \|(B|_{\mathcal{U}})^k\|_\mathcal{H}\|Ex\|_\mathcal{H} \), where \( B|_{\mathcal{U}} \) is the restriction of \( B \) to \( \mathcal{U} \). Therefore, \( \|\hat{Q}_k R_k x/\lambda^k - u\|_\mathcal{H} \leq \|(B|_{\mathcal{U}}/\lambda)^k\|_\mathcal{H}\|Ex\|_\mathcal{H} \). Since the spectral radius of \( B|_{\mathcal{U}} \) is \( \rho = \rho((I - \mathcal{P}_\mathcal{V})B) \), we have that \( \lim_{k \to \infty} \|(B|_{\mathcal{U}}/\lambda)^k\|_\mathcal{H}^{1/k} = |\rho/\lambda| \). We conclude that \( \|\hat{Q}_k R_k x/\lambda^k - u\|_\mathcal{H} \leq (|\rho/\lambda| + \epsilon_k)^k \|Ex\|_\mathcal{H} \), where \( \epsilon_k \to 0 \) as \( k \to \infty \). Since \( \hat{Q}_k R_k x/\lambda^k \in \text{range} \hat{Q}_k \), this concludes the proof. \( \square \)

Although we have neglected the effects of approximately solving the ODEs in (4.2) and the impact of round-off errors in our brief analysis of rational subspace iteration for normal differential operators, we mention two recent results for rational subspace iteration with matrices [42] and self-adjoint differential operators [22].

- For matrices, small errors made during application of the spectral projector do not alter the convergence behavior of subspace iteration for practical purposes [42]. In this case, the sequence \( \hat{Q}_k \) no longer converges to an exact basis for \( \mathcal{V} \). However, the matrices \( \hat{Q}_k \) approximate a basis for \( \mathcal{V} \) and the approximation error converges geometrically to a constant determined by the sizes of the errors introduced at each iteration [42].
- For self-adjoint differential operators (closed and densely defined on \( \mathcal{H} \)), rational subspace iteration converges to a subspace even when the resolvent operator is discretized to solve the ODEs in (4.2) [22]. The distance between the computed subspace and the target eigenspace (in a distance metric between subspaces) is proportional to the approximation error in the discretized resolvent [22].

We expect that similar statements hold for normal operators on \( \mathcal{H} \), but a rigorous and detailed convergence analysis is more subtle and beyond the scope of this paper.

5.2. A pseudospectral inclusion theorem. As \( \text{range}(\hat{Q}) \) is not an invariant subspace of \( \mathcal{L} \), the \( \epsilon \)-pseudospectrum of \( \hat{Q}^* \mathcal{L} \hat{Q} \) is not, in general, contained in the \( \epsilon \)-pseudospectrum of \( \mathcal{L} \). However, if \( \|\hat{Q} - Q\|_{\mathbb{C}^m \to \mathcal{H}} \) is sufficiently small, then the \( \epsilon \)-pseudospectrum of \( \hat{Q}^* \mathcal{L} \hat{Q} \) is contained in the \( \epsilon \)-pseudospectrum of \( \mathcal{L} \) for some \( \epsilon' > \epsilon \). How much larger \( \epsilon' \) is than \( \epsilon \) depends on the non-normality of \( \mathcal{L} \) as well as the size of \( \|\mathcal{L}(\hat{Q} - Q)\|_{\mathbb{C}^m \to \mathcal{H}} \) and the approximation error \( \|\hat{Q} - Q\|_{\mathbb{C}^m \to \mathcal{H}} \).

**Theorem 5.2.** Consider a closed operator \( \mathcal{L} \) with domain \( \mathcal{D}(\mathcal{L}) \) that is densely defined on a Hilbert space \( \mathcal{H} \). Fix \( \epsilon, \delta, \gamma > 0 \), let \( Q : \mathbb{C}^m \to \mathcal{H} \) satisfy \( Q^* Q = I \), and let \( \text{range}(Q) \) be an \( m \)-dimensional invariant subspace of \( \mathcal{L} \). Let \( E : \mathbb{C}^m \to \mathcal{D}(\mathcal{L}) \subset \mathcal{H} \) so that there is an \( \omega > 0 \) for which \( \|EE\|_{\mathbb{C}^m \to \mathcal{H}} \leq \omega \). Suppose that \( E \) satisfies \( \|E\|_{\mathbb{C}^m \to \mathcal{H}} \leq \delta \), \( Q^* E = 0 \), and \( \|Q^* E\|_{\mathcal{D}(\mathcal{L})} \leq \gamma \). If \( \delta < (\frac{\gamma}{\omega} - \gamma) / \omega \), then

\[
\lambda_{\epsilon}(\hat{Q}^* \mathcal{L} \hat{Q}) \subset \lambda_{\epsilon + (\omega - \delta)}(\mathcal{L}) \subset \lambda_{2\epsilon}(\mathcal{L}),
\]

where \( \hat{Q} = Q + E \).

**Proof.** Consider \( z \in \lambda_{\epsilon}(\hat{Q}^* \hat{Q}) \). If \( z \in \lambda_{\epsilon}(\hat{Q}^* \hat{Q}) \cap \lambda_{\epsilon}(\mathcal{L}) \), there is nothing to prove, so assume without loss of generality that \( z \notin \lambda_{\epsilon}(\mathcal{L}) \). Denote \( R_Q(z) = (zI - Q^* \mathcal{L} Q)^{-1} \) and \( R_{\hat{Q}}(z) = (zI - \hat{Q}^* \hat{Q})^{-1} \). We have that \( R_Q(z) = [R_Q(z)^{-1} - B]^{-1} \), where \( B = Q^* E E + E^* \mathcal{L} \mathcal{Q} + E^* \mathcal{L} E \). Employing a formula for the inverse of the sum of two matrices, we obtain \( R_{\hat{Q}}(z) = R_Q(z) - R_Q(z)(I - B R_Q(z))^{-1} B R_Q(z) \) [25].
We claim that \( \|BR_Q(z)\|_{C^m} \leq (\gamma + \omega \delta)/\epsilon \). To see this, consider the three terms whose sum comprises \( B \) and note that \( \|B\|_{C^m} \leq \gamma + \|E^*LQ\|_{C^m} + \|E^*LE\|_{C^m} \). The second term \( \|E^*\|_{H \rightarrow C^m} = \|E\|_{C^m \rightarrow H} \leq \delta \) [26, p. 256] and \( \|LE\|_{C^m \rightarrow H} \leq \omega \), so the final term satisfies

\[
\|E^*LE\|_{C^m} \leq \|E^*\|_{H \rightarrow C^m}\|LE\|_{C^m \rightarrow H} \leq \omega \delta.
\]

Therefore, \( \|B\|_{C^m} \leq \gamma + \omega \delta \). Finally, since \( z \notin \lambda_\epsilon(\mathcal{L}) \), we have that \( \|R_Q(z)\|_{C^m} \leq 1/\epsilon \) by Theorem 3.2.

For convenience, denote \( C = (\gamma + \omega \delta)/\epsilon \). Since \( \delta < (\epsilon - \gamma)/\omega \), we may use the Neumann series to compute \( (I - BR_Q(z))^{-1} = \sum_{k=0}^{\infty}(BR_Q(z))^k \). We see that \( R_Q(z) = R_Q(z)(I - \sum_{k=1}^{\infty}(BR_Q(z))^k) \) and therefore,

\[
\|R_Q(z)\|_{C^m} \leq \left(1 + \sum_{k=1}^{\infty}C^k\right)\|R_Q(z)\|_{C^m} = \frac{1}{1-C}\|R_Q(z)\|_{C^m}.
\]

Now, if \( z \in \lambda_\epsilon(\mathcal{Q}^*LQ) \), then \( \|R_Q(z)\|_{C^m} \geq \|R_Q(z)(1 - C) > (1-C)/\epsilon \). By Theorem 3.2, we have that \( \|(zI - L)^{-1}\|_{H} \geq \|R_Q(z)\|_{C^m} \). Collecting inequalities yields \( \|(zI - L)^{-1}\|_{H} > (1-C)/\epsilon \). Finally, \( \|(zI - L)^{-1}\|_{H} > 1/(2\epsilon) \), since \( \delta < (\gamma - \gamma)/\omega \) implies that \( C < 1/2 \).}

A consequence of Theorem 5.2 is that Algorithm 4.1 possesses a type of stability when \( \mathcal{L} \) is normal, provided that \( \mathcal{L} \) is uniformly bounded on the sequence \( E_1, E_2, E_3, \ldots \), where \( E_k = \hat{Q}_k - Q \) for \( k \geq 1 \). In Theorem 5.2, the normality of \( \mathcal{L} \) is measured by the norm of the matrix \( Q^*LE \) and controlled by \( \gamma \). For a normal operator, \( Q \) is a left invariant subspace of \( \mathcal{L} \) so that \( LQ \). Moreover, if \( \mathcal{L} \) is uniformly bounded on \( \{E_k\}_{k=1}^{\infty} \), then there is a \( \Lambda \geq 0 \) such that \( \sup_{k \geq 1}\|LQ_k\|_{C^m \rightarrow H} \leq \Lambda \). Applying Theorem 5.2 with \( \gamma = 0 \), we see that Algorithm 4.1 computes elements in the \( 2\epsilon \)-pseudospectra of \( \mathcal{L} \) provided that a basis for \( V \) is resolved to within \( \epsilon/(2\lambda) \).

We now verify, with two mild constraints placed on the choice of the initial quasimatrix \( F \), that \( \mathcal{L} \) is uniformly bounded on the sequence \( \{\hat{Q}_k\}_{k=1}^{\infty} \) generated by Algorithm 4.1. Note that this implies that \( \mathcal{L} \) is uniformly bounded on \( \{E_k\}_{k=1}^{\infty} \), because \( E_k = \hat{Q}_k - Q \) and range(\( Q \)) \( \subset D(\mathcal{L}) \). The constraints on \( F \) are generically satisfied when \( F \) is selected as in section 4. In the statement of the bound on \( \|LQ_k\|_{C^m \rightarrow H} \), we use the notation \( \sigma_{min}(P_V F) \) and \( \sigma_{min}((I - P_V) F) \) to denote the smallest singular values of the quasimatrices \( P_V F \) and \( (I - P_V) F \), respectively.\(^5\)

**Lemma 5.3.** Consider a closed, normal operator \( \mathcal{L} \) with domain \( D(\mathcal{L}) \) that is densely defined on a Hilbert space \( H \). Let \( P_V \) be the bounded operator on \( \mathcal{H} \) defined in (4.1) and suppose that \( P_V \) has a dominant eigenspace of \( V \) (see (5.3)) with \( \dim(V) = m \). Let \( F \), and \( \{\hat{Q}_k\}_{k=1}^{\infty} \) be as in Theorem 5.1 with \( B = P_V \). Suppose that \( P_V^k F \) (for each \( k \geq 1 \)) and \( (I - P_V) F \) each have linearly independent columns and that range(\( F \)) \( \subset D(\mathcal{L}) \). Then, we have that

\[
\|LQ_k\|_{C^m \rightarrow H} \leq 2M\|LF\|_{C^m \rightarrow H}, \quad k = 1, 2, 3, \ldots,
\]

where \( M = \max\{1/\sigma_{min}(P_V F), 1/\sigma_{min}((I - P_V) F)\} \).

\(^5\)In analogy with the matrix setting, the singular value decomposition of a quasimatrix \( A : C^m \rightarrow H \) is the decomposition \( A = U\Sigma V^* \), where \( U : C^m \rightarrow H \) is a quasimatrix with \( H \)-orthonormal columns, \( \Sigma \in C^{m \times m} \) is a diagonal matrix with non-negative entries \( \sigma_1 \geq \cdots \geq \sigma_m \), and \( V \in C^{m \times m} \) is a unitary matrix [48].
Proof. Since $\hat{Q}_k$ is an orthonormal basis for $\mathcal{P}_V^k F$, there is a matrix $R_k \in \mathbb{C}^{n \times m}$ such that $\mathcal{P}_V^k F = Q_k R_k$. By the assumption that $\mathcal{P}_V^k F$ has linearly independent columns, we know that $R_k$ is invertible. We obtain that

$$Q_k = \hat{P}_V^k F R_k^{-1}. \tag{5.4}$$

We use the spectral projector $\mathcal{P}_V$ to rewrite (5.4) as

$$\hat{Q}_k = \hat{P}_V^k (P_V F + (I - P_V) F) R_k^{-1}. \tag{5.5}$$

Now, range($\mathcal{P}_V F$) and range((I − $\mathcal{P}_V$)F) are invariant under $\hat{P}_V$ [26, p. 178]. Consequently, there are matrices $D_1, D_2 \in \mathbb{C}^{m \times m}$ such that

$$\hat{P}_V^k \mathcal{P}_V F = P_V F D_1^k, \quad \hat{P}_V^k (I - \mathcal{P}_V) F = (I - \mathcal{P}_V) F D_2^k. \tag{5.6}$$

Substituting (5.6) into (5.5) yields the following useful equation for $\hat{Q}_k$:

$$\hat{Q}_k = (P_V F D_1^k + (I - \mathcal{P}_V) F D_2^k) R_k^{-1}. \tag{5.7}$$

Applying $\mathcal{L}$ to both sides of (5.7) and commuting with the spectral projectors $\mathcal{P}_V$ and $(I - \mathcal{P}_V)$ [26, p. 179], we obtain

$$\mathcal{L} \hat{Q}_k = (P_V \mathcal{L} F D_1^k + (I - \mathcal{P}_V) \mathcal{L} F D_2^k) R_k^{-1}. \tag{5.8}$$

Since range($F$) $\subset \mathcal{D}(\mathcal{L})$, we have that $\|\mathcal{L} F\|_{\mathbb{C}^m \to \mathcal{H}} < \infty$. Additionally, since $\mathcal{L}$ is normal, the spectral projectors have norms equal to 1 [26, p. 277]. Therefore, it remains to find a uniform bound for $\|D_1^k R_k^{-1}\|_{\mathbb{C}^m}$ and $\|D_2^k R_k^{-1}\|_{\mathbb{C}^m}$ as $k \to \infty$.

For brevity, we prove uniform boundedness of $\|D_1^k R_k^{-1}\|_{\mathbb{C}^m}$ and note that the proof for $\|D_2^k R_k^{-1}\|_{\mathbb{C}^m}$ is essentially identical. We begin by commuting $\hat{P}_V$ with the spectral projectors in (5.6) and substituting the QR factorization of $\hat{P}_V^k F$ to see that

$$\mathcal{P}_V \hat{Q}_k = (\mathcal{P}_V F) D_1^k R_k^{-1}. \tag{5.9}$$

Using the pseudoinverse ($\mathcal{P}_V^+$) of the quasimatrix $\mathcal{P}_V F$ and noting that $\mathcal{P}_V F$ has linearly independent columns, (5.9) implies that

$$D_1^k R_k^{-1} = (\mathcal{P}_V F)^+ \mathcal{P}_V \hat{Q}_k. \tag{5.10}$$

Now, we know that $\|\mathcal{P}_V \hat{Q}_k\|_{\mathcal{H}} \leq 1$, because $\|\mathcal{P}_V\|_{\mathcal{H}} = 1$ and $\hat{Q}_k$ has orthonormal columns. We conclude that

$$\|D_1^k R_k^{-1}\|_{\mathbb{C}^m} \leq \frac{1}{\sigma_{\min}(\mathcal{P}_V F)}. \tag{5.11}$$

A similar argument shows that

$$\|D_2^k R_k^{-1}\|_{\mathbb{C}^m} \leq \frac{1}{\sigma_{\min}((I - \mathcal{P}_V) F)}. \tag{5.12}$$

---

The pseudoinverse of a quasimatrix $A : \mathbb{C}^m \to \mathcal{H}$ may be defined via the SVD as $A^+ = V \Sigma^+ U^*$, where $\Sigma^+$ is the diagonal matrix with entries $\Sigma^+ = 1/\sigma$, if $\sigma \neq 0$ and 0 otherwise. It is easy to verify familiar properties from the matrix case [20, p. 290], i.e., if $A$ has linearly independent columns, then $A^+ A = I$ and $\|A^+\|_{\mathcal{H} \to \mathbb{C}^m} = 1/\sigma_{\min}(A)$. 

Taking norms in (5.8) and substituting the bounds from (5.11) and (5.12), we have that
\[ \|\tilde{L}\hat{Q}_k\|_{C^m \rightarrow \mathcal{H}} \leq \|LF\|_{C^m \rightarrow \mathcal{H}} \left( \frac{1}{\sigma_{\min}(P_V F)} + \frac{1}{\sigma_{\min}((I-P_V) F)} \right). \] The lemma follows immediately from (5.13).

Theorem 5.1, Theorem 5.2, and Lemma 5.3 provide a preliminary analysis to explain why Algorithm 4.1 accurately computes the eigenvalues of normal operators with a dominant eigenspace \( V \). Theorem 5.1 allows us to accurately resolve an orthonormal basis \( Q \) for \( V \) by refining the quasimatrix \( \hat{Q}_k \) with subspace iteration. Lemma 5.3 confirms that \( L\hat{Q}_k \) does not grow without bound as \( \hat{Q}_k \) is refined. Finally, Theorem 5.2 demonstrates that the eigenvalues are computed to the expected accuracy, provided that the basis for \( V \) has been resolved.

6. An operator analogue of the Rayleigh Quotient Iteration. It is useful to have operator analogues for other eigensolvers too; particularly, when the eigenvalues of interest are difficult to target with a pre-selected search region \( \Omega \subset \mathbb{C} \). The Rayleigh Quotient Iteration (RQI) is a generalization of the inverse iteration that incorporates dynamic shifting to obtain cubic (for Hermitian problems) or quadratic (non-Hermitian problems) convergence [38]. Given a matrix \( A \in \mathbb{C}^{n \times n} \) and an initial vector \( \tilde{y}_0 \in \mathbb{C}^n \), RQI computes the iterates
\[ \tilde{y}_{k+1} = (A - \beta_k I)^{-1} y_k, \quad \beta_k = y_k^* A y_k, \quad y_k = \frac{\tilde{y}_k}{\|\tilde{y}_k\|_2}, \quad k = 0, 1, 2, \ldots. \]
The vectors \( y_k \) typically converge to a nearby eigenvector of \( A \), while the sequence \( \beta_k \) converges to the associated eigenvalue of \( A \) [36]. In the matrix setting, (6.1) is often used to compute interior eigenvalues or refine an estimate of an invariant subspace [37,38].

Replacing a matrix \( A \) by a differential operator \( L: \mathcal{D}(L) \rightarrow \mathcal{H} \), as in (1.1), and the vectors \( \tilde{y}_k \) by functions \( f_k \in \mathcal{D}(L) \), we obtain an operator analogue of RQI. One needs to select an initial function \( f_0 \in \mathcal{D}(L) \) and solve a sequence of ODEs, i.e.,
\[ (L - \beta_k I) f_{k+1} = f_k, \quad f_{k+1}(\pm 1) = \cdots = f_{k+1}^{(N/2)}(\pm 1) = 0. \]
At each iteration, the shift \( \beta_k \) is computed from the Rayleigh Quotient \( (f_k, Lf_k)_{\mathcal{H}} \) (in strong form) and the solution \( f_{k+1} \) is normalized after each iteration. Just like in the matrix setting, we observe that the operator analogue of the Rayleigh Quotient Iteration converges cubically for self-adjoint operators and quadratically for normal operators [23].

We note that block generalizations of RQI (RSQR and GRQI [1]) are also easily extended to the differential operator setting. In this case, a sequence of quasimatrices \( \hat{Q}_k \) with \( \mathcal{H} \)-orthonormal columns are generated to approximate an invariant subspace of \( L \) and a Rayleigh–Ritz projection is performed to compute approximate eigenvalues and eigenvectors. As with the operator analogue of FEAST, Theorem 5.2 implies that the iteration (6.2) accurately computes eigenvalues of normal differential operators when the basis for the target eigenspace is sufficiently resolved.

6.1. Free vibrations of an airplane wing. The improved convergence rate of RQI can offer much faster computation time than subspace iteration, often requiring only 3 or 4 ODE solves to reach an accuracy of essentially machine precision [23]. We now employ (6.2) for the rapid computation of vibrational modes of an airplane wing.
An airplane wing may be crudely modeled as a thin, cantilevered beam of length $L$ with a linear taper. The governing equation for free vibrations is [24]

$$
\frac{d^2}{dx^2} \left( (1 + x) \frac{d^2 u}{dx^2} \right) = \lambda u, \quad u(0) = u'(0) = 0, \quad u''(L) = u'''(L) = 0.
$$

The variable coefficient $1 + x$ accounts for the linear taper of the wing, while the boundary conditions on $u''$ and $u'''$ at $x = 1$ express the natural requirement that the bending moment and shear force vanish at the endpoint.

To compute a few of the smoothest modes of (6.3) we use the eigenfunctions $w_n$ of the cantilevered beam equation with constant coefficients, given in closed form by [24]

$$
w_n(x) = \cosh \beta_n x - \cos \beta_n x + \frac{\cos \beta_n L + \cosh \beta_n L}{\sin \beta_n L + \sinh \beta_n L} (\sin \beta_n x + \sinh \beta_n x).
$$

Here $\beta_n$ is the $n$th root of the function $g(\beta) = \cosh(\beta L) \cos(\beta L) + 1$ [24]. We target a mode of (6.3) by setting $f_0(x) = w_n(x)$. Figure 6.1 shows the modes that are computed using initial guesses $w_1, \ldots, w_4$, corresponding to the smallest four positive roots of $g(\beta)$.

7. Computing eigenvalues in unbounded regions. The stability analysis of solutions to time-dependent partial differential equations (PDEs) provides an abundant source of differential eigenvalue problems. Consider the initial boundary value problem (IBVP) with periodic boundary conditions

$$
u_t = Lu + N(u), \quad u_t(x, 0) = g(t), \quad u(-1, t) = u(1, t).
$$

Here, $L$ and $N$ are linear and nonlinear ordinary differential operators, respectively. In many instances, (7.1) supports steady-states, traveling wave states, or other phenomena whose stability is of critical importance in the physical problem under study [4,29,
43]. When \( L \) is self-adjoint or normal, the stability analysis often reduces to determining whether or not the eigenvalues of \( L \) are contained in one half-plane \([4, 28, 43, 51]\).

We now show how to modify the spectral projector in (3.1) to derive a practical rational filter to compute (finitely many) eigenvalues of \( L \) in the right half-plane.

7.1. A rational filter for the half-plane. Let \( L \) be a closed linear operator that is densely defined on a Hilbert space \( \mathcal{H} \). Suppose that \( L \) is a normal operator with a spectrum in the left half-plane \( \text{Re}(z) < 0 \) except for finitely many eigenvalues \( \lambda_1, \ldots, \lambda_m \) (including multiplicities) such that \( \text{Re}(\lambda_i) > 0 \) for \( 1 \leq i \leq m \). Denote the eigenspace associated with \( \lambda_1, \ldots, \lambda_m \) by \( \mathcal{V} \) and consider search regions that are semi-circles of radius \( R \), i.e.,

\[
\Omega_R = \{ z \in \mathbb{C} : |z| < R, \text{Re}(z) > 0 \}, \quad R > \max_{1 \leq i \leq m} |\lambda_i|.
\]

To construct a computable spectral projector onto the right half-plane we consider taking \( R \to \infty \). We adopt the following strategy:

(i) Introduce a \( 1/R \) decay into the integrand of the spectral projector (3.1) as \( R \to \infty \), while preserving the projection onto \( \mathcal{V} \).

(ii) Split the projector into an integral over the vertical part of \( \partial \Omega_R \) and an integral over the circular arc of \( \partial \Omega_R \). By taking \( R \to \infty \), we observe that the contribution from the circular arc goes to 0 due to the additional \( 1/R \) decay introduced into the integrand.

(iii) Map the imaginary axis to the interval \([-1, 1]\) and approximate the spectral projector by a quadrature rule.

Select \( a \in \mathbb{R}^+ \) and consider the family of functions that are analytic in the right half-plane defined by

\[
P_R(\lambda) = \frac{1}{2\pi i} \int_{\partial \Omega_R} \frac{(z + a)^{-1}(z - \lambda)^{-1}}{\partial z} \, dz.
\]

By Cauchy’s Integral Formula, we know that \( P_R(\lambda) = (\lambda + a)^{-1} \) if \( \lambda \in \Omega_R \) and is zero otherwise [45]. Taking the limit \( R \to \infty \), we obtain

\[
P(\lambda) = \lim_{R \to \infty} P_R(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{(iy + a)^{-1}(iy - \lambda)^{-1}}{\partial y} \, dy.
\]

Using functional calculus for unbounded normal operators we can extend \( P(\lambda) \) to an operator-valued function \( P(\mathcal{L}) \) [34]. Moreover, one can show that the spectrum of
\(P(\mathcal{L})\) is discrete with non-zero eigenvalues \(P(\lambda_i)\) for \(1 \leq i \leq m\), and that we have \(\text{range}(P(\mathcal{L})) = V\) [40, pp. 362–366].

Now, take the change-of-variables \(x = \frac{2}{\pi} \tan^{-1} y\) in (7.4) to obtain

\[
(7.5) \quad P(\mathcal{L}) = \frac{1}{4} \int_{-1}^{1} \left( i \tan \left( \frac{\pi x}{2} \right) + a \right)^{-1} \left( i \tan \left( \frac{\pi x}{2} \right) I - \mathcal{L} \right)^{-1} \sec^{2} \left( \frac{\pi x}{2} \right) dx.
\]

Using Gauss–Legendre quadrature nodes \(x_1, \ldots, x_L\) and weights \(w_1, \ldots, w_L\) on \([-1, 1]\), we can approximate \(P(\mathcal{L})\) by

\[
(7.6) \quad \tilde{P}(\mathcal{L}) = \frac{1}{4} \sum_{k=1}^{L} w_k \frac{1 - z_k^2}{z_k + a} (z_k I - \mathcal{L})^{-1}, \quad z_k = i \tan \left( \frac{\pi x_k}{2} \right).
\]

Figure 7.1 (right) displays a colormap of the derived rational filter \(\tilde{P}(\lambda)\) in the complex plane.

### 7.2. Stability of thin fluid films.
To demonstrate the utility of the filter in (7.6), we assess the stability of the steady-state solutions to a PDE governing the motion of a thin film of fluid supported below by a flat substrate. The PDE can be written as

\[
(7.7) \quad u_t = \partial_x^4 u + \partial_x (u \partial_x u),
\]

where \(u(x, t)\) is a positive, periodic function representing the thickness of the fluid [29]. The nonlinear term in (7.7) models gravitational effects and substrate-fluid interactions [29].

A so-called droplet steady-state \(u_{\text{ss}}(x)\) of (7.7), rescaled to have a contact angle of \(\pi/4\) and so that it is supported on \([0, l]\), is stable if all the eigenvalues of a fourth-order differential operator are in the left half-plane. The associated differential eigenvalue problem is [28]

\[
(7.8) \quad \frac{d^4 u}{dx^4} + \frac{d}{dx} \left( u_{\text{ss}} \frac{du}{dx} \right) = \lambda u, \quad u(0) = u(l) = 0, \quad u''(0) = u''(l) = 0.
\]

We compute the steady-state \(u_{\text{ss}}(x)\) by solving the second order nonlinear ODE [29]

\[
(7.9) \quad \frac{du_{\text{ss}}}{dx} + \frac{1}{2} u_{\text{ss}}^2 - \delta = 0, \quad u_{\text{ss}}(0) = 0, \quad u_{\text{ss}}'(0) = 1.
\]
Here, $\delta$ is a dimensionless quantity relating the rescaled problem to the original contact angle [29]. The parameter $\delta$ and the length $l$ of the droplet's base are interdependent and may be calculated analytically [29].

In Figure 7.2, a numerical approximation to the rescaled steady-state $u_{ss}$ is displayed along with the right-most eigenvalues of the corresponding differential eigenvalue problem (7.8). Using the rational filter in (7.6) to perform the approximate spectral projection in Algorithm 4.1, we are able to identify an eigenvalue of (7.8) in the right half-plane, which indicates that the droplet (see Figure 7.2 (left)) is unstable.

**Conclusions.** An operator analogue of the FEAST matrix eigensolver is derived to solve differential eigenvalue problems without discretizing the operator. This approach leads to an algorithm which can exploit fast, well-conditioned spectral methods for computing with functions while preserving the key structures from $L$. The result is an efficient, automated, and highly accurate eigensolver for normal and self-adjoint differential operators. This eigensolver is particularly adept in the high-frequency regime and may provide a new direction towards robust high-frequency eigenvalue computations in one, two, and three dimensions.

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