REMARK ON ORDERED BRAID GROUPS

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Abstract. We recover the Dehornoy order on the braid group $B_{2g+n}$ from the tracial state on a cluster $C^*$-algebra $A(S_{g,n})$ associated to the surface $S_{g,n}$ of genus $g$ with $n$ boundary components. It is proved that the space of left-ordering of the fundamental group $\pi_1(S_{g,n})$ is a totally disconnected dense subspace of the projective Teichmüller space $\mathbb{P}T_{g,n} \cong S^{6g-7+2n}$. In particular, each left-ordering of $\pi_1(S_{g,n})$ defines the orbit of a Riemann surface $S_{g,n}$ under the geodesic flow on the space $T_{g,n}$.

1. Introduction

Let $S_{g,n}$ be an orientable surface of genus $g \geq 0$ with $n \geq 1$ boundary components and let $2g - 2 + n > 0$. The fundamental group $\pi_1(S_{g,n})$ is a free group of rank $2g + n - 1$. Such a group encodes the homotopy type of $S_{g,n}$. The surface $S_{g,n}$ can be endowed with a complex structure so that it becomes a Riemann surface with $n$ cusps. Such an assignment encodes geometry of $S_{g,n}$ and there exists infinitely many distinct Riemann surfaces assigned to the same $S_{g,n}$. The latter make a continuum $T_{g,n} \cong \mathbb{R}^{6g-6+2n}$ called the Teichmüller space of $S_{g,n}$. One can ask what an extra structure of the group $\pi_1(S_{g,n})$ is responsible for geometry of the surface $S_{g,n}$? The aim of our note is to answer this question, see Corollary 1.3.

Recall that the left (right, resp.) strict total order $\prec$ on the group $G$ is a relation between the elements of $G$, such that $x \prec y$ implies $zx \prec zy$ ($xz \prec yz$, resp.) for all $x, y, z \in G$. If both the left and right order exists, the group is said to be orderable. The orderable groups exist and the same group $\pi_1(S_{g,n})$ is orderable [Rolfsen 2014] [7, Proposition 2.9]. The left-orderings of $G$ have a natural topology and the corresponding space will be denoted by $LO(G)$ [Sikora 2004] [8].

To formalize our results, we need the following definitions. The cluster algebra $A(x, B)$ is a subring of the field of rational functions in variables $x = (x_1, \ldots, x_m)$ defined by a skew-symmetric matrix $B = (b_{ij}) \in M_n(\mathbb{Z})$. A new cluster $x' = (x_1, \ldots, x'_k, \ldots, x_m)$ and a new skew-symmetric matrix $B' = (b'_{ij})$ is obtained from $(x, B)$ by the exchange relations (2.2), see [Williams 2014] [9] for the motivation and examples. The seed $(x', B')$ is said to be a mutation of $(x, B)$ in the direction $k$, where $1 \leq k \leq m$. The algebra $A(x, B)$ is generated by the cluster variables $\{x_i\}_{i=1}^\infty$ obtained from the initial seed $(x, B)$ by the iteration of mutations in all possible directions $k$. The Laurent phenomenon says that $A(x, B) \subset \mathbb{Z}[x^\pm 1]$, where $\mathbb{Z}[x^\pm 1]$ is the ring of the Laurent polynomials in variables $x = (x_1, \ldots, x_m)$. Therefore each generator $x_i$ of the algebra $A(x, B)$ can be written as a Laurent polynomial in $n$ variables with the integer coefficients.

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The algebra $\mathcal{A}(x,B)$ has the structure of an additive abelian semigroup consisting of the Laurent polynomials with positive coefficients. In other words, the $\mathcal{A}(x,B)$ is a dimension group [6, Definition 3.5.2]. The cluster $C^*$-algebra $\mathbb{A}(x,B)$ is an AF-algebra, such that $K_0(\mathbb{A}(x,B)) \cong \mathcal{A}(x,B)$ [6, Section 4.4]. We denote by $\mathcal{A}(x,S_{g,n})$ the cluster algebra coming from a triangulation of the surface $S_{g,n}$ [Williams 2014] [9, Section 3.3]. The corresponding cluster $C^*$-algebra will be denoted by $\mathbb{A}(S_{g,n})$.

Let $B_m$ be the braid group given by the generators $\sigma_1, \ldots, \sigma_{m-1}$ subject to the relations $\sigma_i \sigma_j = \sigma_j \sigma_i$ when $|i - j| > 1$ and $\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j$ when $|i - j| = 1$. The group $B_m$ is left-orderable [2, Theorem I.1]. Recall that for $n \in \{1, 2\}$ there exists a faithful representation [6, Theorem 4.4.1]:

$$\{ \rho : B_{2g+n} \rightarrow \mathbb{A}(S_{g,n}) \mid \sigma_i \mapsto e_i + 1, \text{ where } e_i \text{ is projection} \}.$$ (1.1)

The representation (1.1) defines an ordering on the group $B_{2g+n}$ as follows. (We refer the reader to Sections 2.2 and 2.3 for the details.) The self-adjoint element $x \in \mathbb{A}(S_{g,n})$ is called positive if all its eigenvalues are non-negative real numbers. The set of all positive elements is a cone $\mathbb{A}^+(S_{g,n})$ in the $C^*$-algebra $\mathbb{A}(S_{g,n})$. Recall that $K_0(\mathbb{A}(S_{g,n})) \cong \mathcal{A}(x,S_{g,n})$ and let $\mathcal{A}^+(x,S_{g,n})$ be the additive semigroup consisting of the Laurent polynomials $\mathbb{Z}[x^{\pm 1}]$ with positive coefficients. The $K_0$-group preserves positivity of the respective semigroups, i.e. $K_0(\mathbb{A}^+(S_{g,n})) \cong \mathcal{A}^+(x,S_{g,n})$.

We define the semigroup $B^+_{2g+n}$ as the composition of mappings $\rho$ and $K_0$ shown on the commutative diagram in Figure 1. Our main results can be formulated as follows.

**Theorem 1.1.** The ordering of the braid group $B_{2g+n}$ defined by $B^+_{2g+n}$ is:

(i) left-invariant;

(ii) isomorphic to the Dehornoy order.

**Remark 1.2.** If $g = 0$, $n = 2$ ($g = n = 1$, resp.) then the cluster algebra $\mathcal{A}^+(x,S_{0,2})$ ($\mathcal{A}^+(x,S_{1,1})$, resp.) consists of the Jones (HOMFLY, resp.) polynomials [6, Section 4.4.6]. In this case the map $B_{2g+n}^+ \rightarrow \mathcal{A}^+(x,S_{g,n})$ can be described explicitly. Namely, such a map acts by the formula $b \mapsto V_\mu(t)$ ($b \mapsto W_\mu(s,t)$, resp.), where $V_\mu(t)$ ($W_\mu(s,t)$, resp.) are the Jones (HOMFLY, resp.) polynomials of the closure $\hat{b}$ of the braid $b$. This phenomenon was reported by [Franks & Williams 1987] [3, Theorem 2.2 (1)] and [Ito 2020] [4], respectively. A generalization to the multivariable Laurent polynomials is given by [6, Theorem 4.4.1].
Let \( T^t : T_{g,n} \to T_{g,n} \mid t \in \mathbb{R} \) be the Teichmüller geodesic flow on \( T_{g,n} \) [5, Section 4]. By a projective Teichmüller space \( \mathbb{P}T_{g,n} \cong S^{6g-7+2n} \) we understand the space of orbits of the flow \( T^t \), where \( S^{6g-7+2n} \) is the sphere of dimension \( 6g-7+2n \). Denote by \( \mathcal{X} \) a totally disconnected dense subset of \( S^{6g-7+2n} \) consisting of the vectors with the rationally independent components.

**Corollary 1.3.** \( \text{LO}(\pi_1(S_{g,n})) \cong \mathcal{X} \subset S^{6g-7+2n} \cong \mathbb{P}T_{g,n} \). In particular, each left-ordering of \( \pi_1(S_{g,n}) \) defines the orbit \( \{T^t(S_{g,n}) \mid t \in \mathbb{R} \} \) of a Riemann surface \( S_{g,n} \) under the geodesic flow \( T^t \).

The article is organized as follows. The preliminary facts can be found in Section 2. Theorem 1.1 and corollary 1.3 are proved in Section 3. In Section 4, we apply corollary 1.3 to evaluate the space \( \text{LO}(F_m) \) of the free group of rank \( m \).

2. Preliminaries

We briefly review the ordered braid groups, positivity for the C*-algebras and cluster C*-algebras. We refer the reader to [Dehornoy, Dynnikov, Rolfsen and Wiest 2002] [2], [Williams 2014] [9] and [5, Section 4.4] for the details.

2.1. Positive braids.

2.1.1. **Ordered groups.** A strict ordering of a set \( X \) is a binary relation \( \prec \) which is transitive, i.e. \( x \prec y \) and \( y \prec z \) implies \( x \prec z \), and anti-reflexive, i.e. \( x \prec x \) cannot hold for all \( x, y, z \in X \). A strict ordering of \( X \) is called linear (or total) if for all \( x, y \in X \) one and only one of the three relations is possible: \( x \prec y \), \( y \prec x \) or \( x = y \).

**Definition 2.1.** The group \( G \) is left-orderable (right-orderable, resp.) if \( G \) admits a total order which is invariant by the left (right, resp.) multiplication, i.e. for all \( x, y, z \in G \) if \( x \prec y \) then \( zx \prec zy \) (\( xz \prec yz \), resp.).

**Remark 2.2.** Every subgroup \( H \) of the left-orderable group \( G \) is left-orderable.

If \( h : G \to H \) is a group homomorphism, then \( H \) is left-orderable if \( G \) is left-orderable. Indeed, the order \( h(x) \prec h(y) \) of \( H \) induced by \( x \prec y \) is left-invariant, since \( h(z)h(x) \prec h(z)h(y) \) for all \( h(z) \in H \). The converse is false in general.

**Remark 2.3.** The group \( G \) is left-orderable if and only if \( G \) is right-orderable. Indeed, given a left-order \( \prec \) on \( G \) one can define a new order \( \prec^* \) by letting \( x \prec^* y \) whenever \( y^{-1} \prec x^{-1} \). It is verified directly that \( \prec^* \) is a right-invariant order. The converse is proved similarly.

An element \( x \) of the left-orderable group \( G \) is called positive (negative, resp.) if \( Id \prec x \) \( (x \prec Id, \text{resp.}) \). The set of all positive (negative, resp.) elements is a semigroup \( G^+ (G^-, \text{resp.}) \) and \( G = G^+ \cup G^- \cup \{Id\} \) is a disjoint union, provided the order is total. The semigroup \( G^+ (G^-, \text{resp.}) \) will be called a positive (negative, resp.) cone in \( G \).

2.1.2. **Dehornoy order.** The \( m \)-strand braid group \( B_m \) is given by the presentation

\[
\langle \sigma_1, \ldots, \sigma_{m-1} \mid \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j| \geq 2, \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j \text{ for } |i-j| = 1 \rangle.
\]

**Definition 2.4.** An element \( x \in B_m \) is called \( \sigma_i \)-positive if it contains \( \sigma_i \) but does not contain \( \sigma_i^{-1} \) or \( \sigma_j^{\pm 1} \) for \( j < i \).

**Remark 2.5.** It is verified directly that the set \( B_m^+ \) of all \( \sigma_i \)-positive elements of \( B_m \) is a semigroup closed under the multiplication operation.
Theorem 2.6. (Dehornoy [2]) The semigroup $B_m^+$ defines a left-invariant linear order of the braid group $B_m$.

2.2. Positivity for $C^*$-algebras.

2.2.1. $C^*$-algebras. The $C^*$-algebra is an algebra $A$ over $C$ with a norm $a \mapsto ||a||$ and an involution $a \mapsto a^*$ such that it is complete with respect to the norm and $||ab|| \leq ||a|| ||b||$ and $||a^*a|| = ||a||^2$ for all $a, b \in A$.

The positive cone $K$ of $A$ is the set of complex numbers $\lambda$ such that $\lambda 1$ is not invertible. An element $a \in A$ is self-adjoint if $a^* = a$. The spectrum of a self-adjoint element is a subset of the real line. If element $a$ is self-adjoint and $Spec a \subseteq [0, \infty)$, then $a$ is called positive. The set of all positive elements of the $C^*$-algebra $A$ is denoted by $A^+$. The set $A^+$ is a closed cone in $A$, i.e. if $x, y \in A^+$ then $x + y \in A^+$ and $\lambda x \in A^+$ for any real $\lambda > 0$.

A linear functional $\phi$ on $A$ is positive if $\phi(x) \geq 0$ whenever $x \in A^+$. A positive linear functional of norm 1 is called a state on $A$. The set of all states $S(A)$ is the state space of $A$. A tracial state $\tau \in S(A)$ is a state satisfying $\tau(xy) = \tau(xy)$ for all $x, y \in A$.

2.2.2. $K$-theory. The algebraic direct limit of the $C^*$-algebras $M_n(A)$ under the embeddings $a \mapsto \text{diag}(a, 0)$ will be denoted by $M_\infty(A)$. Two projections $p, q \in M_\infty(A)$ are equivalent, if there exists an element $v \in M_\infty(A)$, such that $p = v^*v$ and $q = vv^*$. Let $[p]$ be the equivalence class of projection $p$. We write $V(A) := \{[p] : p = p^* = p^2 \in M_\infty(A)\}$ for all equivalence classes of projections in the $C^*$-algebra $M_\infty(A)$. The set $V(A)$ has the natural structure of an abelian semi-group with the addition operation defined by the formula $[p] + [q] := \text{diag}(p, q) = [p' \oplus q']$, where $p' \sim p, \ q' \sim q$ and $p' \perp q'$. The identity of the semi-group $V(A)$ is given by $[0]$, where $0$ is the zero projection. By the $K_0$-group $K_0(A)$ of the unital $C^*$-algebra $A$ one understands the Grothendieck group of the abelian semi-group $V(A)$, i.e. a completion of $V(A)$ by the formal elements $[p] - [q]$. The image of $V(A)$ in $K_0(A)$ is a positive cone $K_0^+(A)$ defining the order structure $\leq$ on the abelian group $K_0(A)$. The pair $(K_0(A), K_0^+(A))$ is known as a dimension group of the $C^*$-algebra $A$.

Remark 2.7. The $K_0$-group preserves the positive cone $A^+$. Namely, a tracial state $\tau$ on $A$ extends to such on $M_\infty(A)$ so that the abelian semigroup $V(A)$ corresponds to the positive cone $A^+$.

2.3. Cluster $C^*$-algebras.

2.3.1. Cluster algebras. The cluster algebra of rank $n$ is a subring $A(x, B)$ of the field of rational functions in $n$ variables depending on variables $x = (x_1, \ldots, x_n)$ and a skew-symmetric matrix $B = (b_{ij}) \in M_n(Z)$. The pair $(x, B)$ is called a seed. A new cluster $x' = (x_1', x_2', \ldots, x_n')$ and a new skew-symmetric matrix $B' = (b'_{ij})$ is obtained from $(x, B)$ by the exchange relations [Williams 2014] [9, Definition 2.22]:

$$x_{i,k}x_k' = \prod_{i=1}^n x_i^{\max(b_{ik}, 0)} + \prod_{i=1}^n x_i^{\max(-b_{ik}, 0)},$$

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k \\ b_{ij} + \frac{|b_{ik} + b_{kj}|}{2} & \text{otherwise.} \end{cases} \quad (2.2)$$
The seed \((x', B')\) is said to be a mutation of \((x, B)\) in direction \(k\). where \(1 \leq k \leq n\). The algebra \(A(x, B)\) is generated by the cluster variables \(\{x_i\}_{i=1}^{\infty}\) obtained from the initial seed \((x, B)\) by the iteration of mutations in all possible directions \(k\).

The Laurent phenomenon says that \(A(x, B) \subset \mathbb{Z}[x^{\pm 1}]\), where \(\mathbb{Z}[x^{\pm 1}]\) is the ring of the Laurent polynomials in variables \(x = (x_1, \ldots, x_n)\) [Williams 2014] [9, Theorem 2.27]. In particular, each generator \(x_i\) of the algebra \(A(x, B)\) can be written as a Laurent polynomial in \(n\) variables with the integer coefficients.

### 2.3.2. Cluster \(C^*\)-algebras

The cluster algebra \(A(x, B)\) has the structure of an additive abelian semigroup consisting of the Laurent polynomials with positive coefficients. In other words, the \(A(x, B)\) is a dimension group [5, Definition 3.5.2]. We define the cluster \(C^*\)-algebra \(\mathbb{A}(x, B)\) as an AF-algebra, such that \(K_0(\mathbb{A}(x, B)) \cong A(x, B)\) [5, Section 4.4].

### 2.3.3. Surface cluster algebras and braid groups

Denote by \(S_{g,n}\) the Riemann surface of genus \(g \geq 0\) with \(n \geq 0\) cusps. Let \(A(x, S_{g,n})\) be the cluster algebra coming from a triangulation of the surface \(S_{g,n}\) [Williams 2014] [9, Section 3.3]. We shall denote by \(\mathbb{A}(S_{g,n})\) the corresponding cluster \(C^*\)-algebra. Since \(\mathbb{A}(S_{g,n})\) is an AF-algebra, it is characterized by its K-theory and thus can be generated by a series of projections \(\{e_i\}_{i=1}^{\infty}\). We let \(e_i\) be such a projection in the cluster \(C^*\)-algebra \(\mathbb{A}(S_{g,n})\).

**Theorem 2.8.** ([6, Theorem 4.4.1]) The formula \(\sigma_i \mapsto e_i + 1\) defines a faithful representation

\[
\rho : \begin{cases} 
B_{2g+1} \to \mathbb{A}(S_{g,1}) \\
B_{2g+2} \to \mathbb{A}(S_{g,2}). 
\end{cases}
\]

If \(b \in B_{2g+1}\) (\(b \in B_{2g+2}\), resp.) is a braid, there exists a Laurent polynomial \([\rho(b)] \in K_0(\mathbb{A}(S_{g,1}))\) (\([\rho(b)] \in K_0(\mathbb{A}(S_{g,2}))\), resp.) with the integer coefficients depending on \(2g\) (\(2g+1\), resp.) variables, such that \([\rho(b)]\) is a topological invariant of the closure of \(b\).

### 3. Proof

#### 3.1. Proof of theorem 1.1.

**3.1.1. Part I.** We split the proof of item (i) of theorem 1.1 in a series of lemmas.

**Lemma 3.1.** Let \(A_{2g+n}\) be the norm-closure in the strong operator topology of the algebra \(C(\rho(\sigma_1), \ldots, \rho(\sigma_{2g+n-1}))\), where \(\rho\) is given by the formula (1.1). Then:

(i) \(A_{2g+n}\) is a \(C^*\)-subalgebra of the AF-algebra \(\mathbb{A}(S_{g,n})\);

(ii) \((K_0(A_{2g+n}), K_0^+(A_{2g+n})) \cong (K_0(\mathbb{A}(S_{g,n})), K_0^+(\mathbb{A}(S_{g,n})))\) is an isomorphism of the dimension groups.

**Proof.** (i) Since \(\rho(\sigma_i) = e_i + 1\), we conclude that \(\rho^*(\sigma_i) = (e_i + 1)^* = e_i^* + 1^* = e_i + 1 = \rho(\sigma_i)\), i.e. \(\rho(\sigma_i)\) is a self-adjoint element. It is verified directly, that the braid relations are invariant of the \(*\)-conjugacy. Therefore, the polynomial algebra \(C(\rho(\sigma_1), \ldots, \rho(\sigma_{2g+n-1}))\) is an \(*\)-algebra. Clearly, the norm-closure in the strong operator topology of the \(*\)-algebra is a \(C^*\)-algebra. Thus \(A_{2g+n}\) is a \(C^*\)-subalgebra of the AF-algebra \(\mathbb{A}(S_{g,n})\). Item (i) is proved.

(ii) In view of item (i), one gets an injective \(*\)-homomorphism \(A_{2g+n} \to \mathbb{A}(S_{g,n})\). The latter gives rise to an injective homomorphism between the abelian groups
$K_0(A_{2g+n}) \to K_0(\mathbb{A}(S_{g,n}))$ by the main property of the functor $K_0$. In other words, we obtain an inclusion of the dimension groups:

$$(K_0(A_{2g+n}), K_0^+(A_{2g+n})) \subseteq (K_0(\mathbb{A}(S_{g,n})), K_0^+(\mathbb{A}(S_{g,n}))). \quad (3.1)$$

Let us show that (3.1) is an isomorphism of the dimension groups. Indeed, assume to the contrary that $(K_0(A_{2g+n}), K_0^+(A_{2g+n})) \subset (K_0(\mathbb{A}(S_{g,n})), K_0^+(\mathbb{A}(S_{g,n})))$ is a strict inclusion. Recall [5, Theorem 2] that there exists an order-ideal:

$I_\theta \subset (K_0(\mathbb{A}(S_{g,n})), K_0^+(\mathbb{A}(S_{g,n}))) := G$, \quad (3.2)

such that $G/I_\theta$ is a simple dimension group. Then $I_\theta \cap (K_0(A_{2g+n}), K_0^+(A_{2g+n})) := I'_\theta$ is an order-ideal of the dimension group $(K_0(A_{2g+n}), K_0^+(A_{2g+n})) := G'$ such that $G'/I'_\theta$ is a non-trivial ideal of $G/I_\theta$. The latter contradicts simplicity of the dimension group $G/I_\theta$. Thus (3.1) must be an isomorphism between the corresponding dimension groups. Item (ii) follows.

Lemma 3.1 is proved. \hfill \Box

**Corollary 3.2.** $K_0(A_{2g+n}) \cong A(x, S_{g,n})$.

**Proof.** By definition $K_0(\mathbb{A}(S_{g,n})) \cong A(x, S_{g,n})$, where $A(x, S_{g,n})$ is the additive group of the surface cluster algebra. Corollary 3.2 follows from item (ii) of lemma 3.1. \hfill \Box

**Remark 3.3.** By the construction, the $C^*$-algebra $A_{2g+n}$ is isomorphic to a reduced group $C^*$-algebra $C^*_r(B_{2g+n})$ of the braid group $B_{2g+n}$. Indeed, since $\rho(\sigma_i) = e_i + 1$ one finds its inverse $\rho(\sigma_i^{-1}) = \frac{1}{2}(2 - e_i)$ and therefore $\rho$ is a unitary representation. Moreover, it can be shown that $\rho$ is weakly contained in the regular representation of $B_{2g+n}$. Thus corollary 3.2 yields an explicit formula:

$$K_0(C^*_r(B_{2g+n})) \cong A(x, S_{g,n}) \subset \mathbb{Z}[x^\pm 1]. \quad (3.3)$$

In other words, the $K_0$-group of the reduced group $C^*$-algebra of the braid group consists of the Laurent polynomials with the integer coefficients. To the best of our knowledge, the expression (3.3) is new. An independent study of the K-theory for the pure braids was undertaken in the recent paper [Azzali, Browne, Gomez Aparicio, Ruth & Wang 2022] [1] and is based on the Baum-Connes isomorphism.

**Lemma 3.4.** The positive cone $B_{2g+n}^+$ defines a left-invariant ordering of the braid group $B_{2g+n}$.

**Proof.** (i) In view of remark 2.7, the positive cone $B_{2g+n}^+$ consists of the words which correspond to the elements of $A^+(x, S_{g,n})$, i.e. the Laurent polynomials with positive coefficients. Let us show that the ordering of the group $B_{2g+n}$ induced by $B_{2g+n}^+$ is left-invariant.

(ii) Roughly speaking, the property of invariance of the ordering under left-multiplication follows from the property $\tau(vu) = \tau(uv)$ of the tracial state $\tau$ holding for all $u, v \in C^*_r(B_{2g+n})$. Namely, it is immediate that the left-multiplication invariance is equivalent to:

$$\tau(zx) = \tau(x), \quad \forall z \in B_{2g+n}. \quad (3.4)$$
(iii) Therefore it is necessary and sufficient to show that the following system of equations in variables $u$ and $v$ is solvable for any $x, z \in B_{2g+n}$:

\[
\begin{aligned}
    vu &= zx \\
    uv &= x.
\end{aligned}
\]  

(3.5)

(iv) It is easy to see that (3.5) is equivalent to:

\[
\begin{aligned}
    v &= u^{-1}x \\
    zx &= u^{-1}xu.
\end{aligned}
\]  

(3.6)

(v) To solve (3.6), recall [2, Proposition 1.1.5] that for every braid $\beta \in B_{2g+n}$ there exist positive braids $\beta_1, \beta_2 \in B_{2g+n}^+$, such that:

\[
\beta = \beta_1^{-1}\beta_2.
\]  

(3.7)

(vi) We let $\beta = x$ and $\beta_1 = z$. Then (3.7) takes the form:

\[
zx = \beta_2.
\]  

(3.8)

(vii) Recall that each $x \in B_{2g+n}$ is equivalent to a positive element $\beta_2 \in B_{2g+n}^+$, see e.g. [2, p. 9]. In other words, $\beta_2 = u^{-1}xu$ for some $u \in B_{2g+n}$.

(viii) Therefore there exists $u \in B_{2g+n}$ such that equation (3.8) takes the form:

\[
zx = u^{-1}xu
\]  

(3.9)

(ix) It remains to notice that $u$ in (3.9) is a root of the second equation (3.6). The second root $v = u^{-1}x$ is defined by the values of $u$ and $x$. Thus equations (3.5) are solvable for the variables $u$ and $v$. As explained, the latter is equivalent to the order-invariance of the left multiplication in the braid group $B_{2g+n}$.

Lemma 3.4 is proved. \hfill \Box

3.1.2. Part II. Our proof of item (ii) of theorem 1.1 is based on an observation, that the closure of positive braids in the Dehornoy order gives rise to positive Laurent polynomials, see remark 1.2. This fact was proved by [Franks & Williams 1987] [3, Theorem 2.2 (1)] and [Ito 2020] [4] for the Jones and HOMFLY polynomials, respectively. We split the proof in two steps.

**Lemma 3.5.** The order on the braid group is Dehornoy if and only if the positive braids correspond to the positive Laurent polynomials.

**Proof.** (i) It follows from [Franks & Williams 1987] [3, Theorem 2.2 (1)] and [Ito 2020] [4] that the ordering of $B_{2g+n}$ is Dehornoy if and only if the skein relations applied to the closure of braids $x, y, z \in B_{2g+n}$ preserve the semigroup $B_{2g+n}^+$. (In other words, if $x, y \in B_{2g+n}^+$, then $z \in B_{2g+n}^+$, where the Laurent polynomials of $\hat{x}, \hat{y}, \hat{z}$ are connected by the skein relation.)

(ii) On the other hand, it is known [6, Section 4.4.6] that the skein relations transform to the exchange relations in the algebra $A(x, S_{g,n})$. Therefore, the ordering on $B_{2g+n}$ is Dehornoy if and only if the exchange relations preserve the semigroup of the Laurent polynomials with positive coefficients.
(iii) But the exchange relations always preserve the positive semigroup of the cluster algebra $\mathcal{A}(x, S_{g,n})$. The latter is known as the Positivity Conjecture proved for all surface cluster algebras [Williams 2014] [9, Conjecture 2.28].

Lemma 3.5 follows from the items (i)-(iii).  

**Corollary 3.6.** The braid ordering (i) in theorem 1.1 is Dehornoy’s ordering.

**Proof.** In view of remark 2.7 and commutative diagram in Figure 1, we conclude that the positive braids correspond to the positive Laurent polynomials. It remains to apply the conclusion of lemma 3.5. Therefore the braid ordering (i) in theorem 1.1 is isomorphic to the Dehornoy ordering. Corollary 3.6 is proved.  

3.2. **Proof of corollary 1.3.** For the sake of clarity, let us outline the main ideas. Consider a $C^*$-algebra $\mathcal{A}_{\Theta} := \mathcal{A}(S_{g,n})/I_{\Theta}$, where $I_{\Theta} \subset \mathcal{A}(S_{g,n})$ is the primitive ideal and $\Theta = (1, \theta_1, \ldots, \theta_{6g-7+2n}) \in \mathbb{FT}_{g,n}$ [5, Theorem 2]. (The ideal $I_{\Theta}$ is given a subgraph of the Bratteli diagram of the AF-algebra $\mathcal{A}(S_{g,n})$ obtained by an excision of the diagram along an infinite path convergent to $\Theta$, see [5] for the details and examples.) The $\mathcal{A}_{\Theta}$ is a simple AF $C^*$-algebra and $K_0(\mathcal{A}_{\Theta}) \cong \mathbb{Z} + \mathbb{Z}\theta_1 + \cdots + \mathbb{Z}\theta_{6g-7+2n}$. We prove that $C^*_{\alpha} \left( \pi_1(S_{g,n}) \right) \rtimes_{\alpha} \mathbb{Z} \subset \mathcal{A}_{\Theta}$, where $\alpha \in \text{Aut } C^*_{\alpha} \left( \pi_1(S_{g,n}) \right)$ and $\pi_1(S_{g,n})$ the profinite completion of the fundamental group $\pi_1(S_{g,n})$. The latter inclusion gives rise to a positive semigroup $\pi_1^+ (S_{g,n})$ depending on $\Theta \in \mathbb{FT}_{g,n}$. Thus one gets a space of different left-invariant ordering of $\pi_1(S_{g,n})$ dual to the projective Teichmüller space of the surface $S_{g,n}$. We pass to a detailed argument by splitting the proof in a series of lemmas.

**Lemma 3.7.** For each $\Theta \in \mathbb{FT}_{g,n}$ there exists an automorphism $\alpha_{\Theta}$ of $C^*_{\alpha} \left( \pi_1(S_{g,n}) \right)$ such that

$$C^*_{\alpha} \left( \pi_1(S_{g,n}) \right) \rtimes_{\alpha} \mathbb{Z} \subset \mathcal{A}_{\Theta}. \quad (3.10)$$

**Proof.** (i) Let $b \in B_{2g+n}$ be a braid. Denote by $\mathcal{L}_b$ a link obtained by the closure of $b$ and let $\pi_1(\mathcal{L}_b)$ be the fundamental group of $\mathcal{L}_b$. Recall that

$$\pi_1(\mathcal{L}_b) \cong \langle x_1, \ldots, x_{2g+n} \mid x_i = r(b)x_i, \ 1 \leq i \leq 2g+n \rangle, \quad (3.11)$$

where $x_i$ are generators of the free group $F_{2g+n}$ and $r : B_{2g+n} \to \text{Aut } (F_{2g+n})$ is the Artin representation of $B_{2g+n}$. Let $I_b$ be a two-sided ideal in the algebra $\mathcal{A}(S_{g,n})$ generated by relations (3.11). It is easy to see that the ideal $I_b$ is self-adjoint and thus (1.1) induces a representation

$$R : \pi_1(\mathcal{L}_b) \to \mathcal{A}(S_{g,n}) / I_b := \mathcal{A}_b. \quad (3.12)$$

Notice that the $\mathcal{A}_b$ is an AF-algebra of stationary type [6, Section 3.5.2] given by a positive matrix $A \in GL_{6g-6+2n}(\mathbb{Z})$. Therefore it is stably isomorphic to an AF-algebra $\mathcal{A}_{\Theta}$, where $\Theta = (1, \theta_1, \ldots, \theta_{6g-7+2n})$ is an eigenvector corresponding to the Perron-Frobenius eigenvalue of the matrix $A$.

(ii) Since $\pi_1(S_{g,n}) \cong F_{2g+n-1}$, one can write (3.11) in the form:

$$\pi_1(\mathcal{L}_b) \cong \pi_1(S_{g,n}) \rtimes_{\alpha} \mathbb{Z}, \quad (3.13)$$

$$\pi_1(S_{g,n}) \cong \pi_1(S_{g,n}) \rtimes_{\alpha} \mathbb{Z}, \quad (3.13)$$
where the crossed product is taken by an automorphism $\alpha \in Aut(\pi_1(S_{g,n}))$. It follows from (3.12) that for an $\alpha_\Theta \in Aut(C^*_r(\pi_1(S_{g,n})))$ one gets an inclusion of the $C^*$-algebras:

$$C^*_r(\pi_1(S_{g,n})) \rtimes_{\alpha_\Theta} \mathbb{Z} \subset \mathbb{A}_\Theta.$$  \hspace{1cm} (3.14)

(iii) It remains to extend the inclusion (3.14) to the profinite completion of the fundamental group $\pi_1(S_{g,n})$ as follows. Since $\pi_1(S_{g,n})$ is a residually finite group, we obtain a dense embedding of the groups:

$$\pi_1(S_{g,n}) \hookrightarrow \widehat{\pi_1(S_{g,n})}. \hspace{1cm} (3.15)$$

Passing to the reduced group $C^*$-algebras, we get a dense embedding of the $C^*$-algebras:

$$C^*_r(\pi_1(S_{g,n})) \hookrightarrow C^*_r(\widehat{\pi_1(S_{g,n})}). \hspace{1cm} (3.16)$$

The crossed product

$$C^*_r(\pi_1(S_{g,n})) \rtimes_{\alpha_\Theta} \mathbb{Z} \subset \mathbb{A}_\Theta \hspace{1cm} (3.17)$$

can be defined as the norm-closure of (3.14), where $\alpha_\Theta$ converge to an automorphism $\alpha_\Theta \in Aut(C^*_r(\pi_1(S_{g,n})))$. Clearly, the inclusion (3.17) is strict because $\mathbb{A}_\Theta$ is an AF $C^*$-algebra with the trivial $K_1$-group unlike the embedded crossed product $C^*$-algebra.

This argument finishes the proof of lemma 3.7. \hfill \Box

**Lemma 3.8.** $K_0\left(C^*_r\left(\pi_1(S_{g,n})\right) \rtimes_{\alpha_\Theta} \mathbb{Z}\right) \cong K_0(\mathbb{A}_\Theta).$

**Proof.** Roughly speaking, the proof follows from lemma 3.1 (ii). Namely, an isomorphism of the dimension groups

$$(K_0(A_{2g+n}), K_0^+(A_{2g+n})) \cong (K_0(\mathbb{A}(S_{g,n})), K_0^+(\mathbb{A}(S_{g,n}))) \hspace{1cm} (3.18)$$

implies that for an order-ideal $K_0(\mathbb{I}_s)$ of $K_0^+(\mathbb{A}(S_{g,n}))$, one gets from (3.12) an isomorphism:

$$K_0\left(C^*_r(\pi_1(S_{g,n})) \rtimes_{\alpha_\Theta} \mathbb{Z}\right) \cong K_0(\mathbb{A}_\Theta). \hspace{1cm} (3.19)$$

Passing to the profinite completion at the LHS of (3.19), we obtain the conclusion of lemma 3.8. \hfill \Box
Lemma 3.9. Let $\pi_1^+(S_{g,n})$ be a semigroup obtained as a pull back of the positive cone $\mathbb{Z}^+\cdot\mathbb{Z}\theta_1 + \cdots + \mathbb{Z}\theta_{6g-7+2n} > 0$ on the commutative diagram in Figure 2, where $i$ is an embedding induced by (3.15). Then each $(1, \theta_1, \ldots, \theta_{6g-7+2n}) \in \mathbb{P}T_{g,n}$ defines a left-invariant order on the fundamental group $\pi_1(S_{g,n})$ if and only if $\theta_i \in \mathbb{R}$ are rationally independent.

Proof. (i) Let us show that the semigroup $\pi_1^+(S_{g,n})$ depends on $6g-7+2n$ variables $\theta_i$. Indeed, lemma 3.7 says that automorphisms $\alpha_\Theta$ of $C^*_r(\hat{\pi}_1(S_{g,n}))$ are parameterized by $\Theta = (1, \theta_1, \ldots, \theta_{6g-7+2n}) \in \mathbb{P}T_{g,n}$. From lemma 3.8 and remark 2.7 we conclude that the semigroup $\pi_1^+(S_{g,n})$ obtained from the diagram in Figure 2, will vary as $(1, \theta_1, \ldots, \theta_{6g-7+2n})$ runs all rationally independent values of $\theta_i$.

Remark 3.10. It is easy to see that if $\theta_i$ are rationally dependent, then the rank of the algebra $A_\Theta$ in (3.12) is less than $6g-6+2n$ and thus $\mathcal{R}$ is no longer a faithful representation. Therefore the condition of rational independence for the $\theta_i$ cannot be omitted.

(ii) Let us show that the order of $\pi_1(S_{g,n})$ defined by the semigroup $\pi_1^+(S_{g,n})$ is invariant of the left-multiplication. Roughly speaking, the proof repeats all the steps in the proof of lemma 3.4. Namely, one takes the canonical tracial state $\tau$ on the $C^*$-algebra $C^*_r(\hat{\pi}_1(S_{g,n})) \rtimes_{\alpha_\Theta} \mathbb{Z}$ and writes equations (3.4)–(3.9) for $\tau$. The details are left to the reader.

Lemma 3.9 is proved. \hfill $\square$

Corollary 3.11. $\text{LO}(\pi_1(S_{g,n})) \cong \mathcal{K} \subset \mathbb{P}T_{g,n}$.

Proof. Using lemma 3.9, we identify the space $\text{LO}(\pi_1(S_{g,n}))$ of all left-invariant orderings of the fundamental group $\pi_1(S_{g,n})$ with the a totally disconnected subset $\mathcal{K} \subset \mathbb{P}T_{g,n}$ consisting of the vectors $(1, \theta_1, \ldots, \theta_{6g-7+2n})$ with the rationally independent components $\theta_i$. \hfill $\square$

Corollary 3.12. Each left-ordering of $\pi_1(S_{g,n})$ defines a Riemann surface $S_{g,n}$ up to the action of the Teichmüller geodesic flow on $T_{g,n}$.

Proof. Recall that the Teichmüller space $T_{g,n}$ is a fiber bundle over $\mathbb{P}T_{g,n}$ with the fiber $\mathbb{R}$ [5, Theorem 2]. Moreover, each fiber $\mathbb{R}$ is an orbit of the Teichmüller geodesic flow $\{T^t : T_{g,n} \to T_{g,n} \mid t \in \mathbb{R} \}$ [5, Section 4].

On the other hand, corollary 3.11 says that each left-ordering of $\pi_1(S_{g,n})$ defines a fiber $\mathbb{R}$ in $T_{g,n}$. The fiber consists of all Riemann surfaces $S_{g,n}$ which lie at the same orbit of the geodesic flow $T^t$. Corollary 3.12 follows. \hfill $\square$

Corollaries 3.11 and 3.12 finish the proof of corollary 1.3.
4. The ordering space of free groups

The free groups $F_m$ are orderable [Rolfsen 2014] [7, Proposition 2.9]. The space $LO(F_m)$ can be evaluated using corollary 1.3 combined with the well-known formula $\pi_1(S_{g,n}) \cong F_{2g+n-1}$.

Corollary 4.1.

$$LO(F_m) \cong \begin{cases} \mathcal{K} \subset S^{6k-5}, & \text{if } m = 2k \\ \mathcal{K} \subset S^{6k-3}, & \text{if } m = 2k + 1. \end{cases}$$

(4.1)

Proof. (i) Since the number of cusps $n \geq 1$, we conclude that $\pi_1(S_{g,n}) \cong F_{2g+n-1}$ is a free group. Therefore corollary 1.3 says that:

$$LO(F_{2g+n-1}) \cong \mathcal{K} \subset PT_{g,n} \cong S^{6g-7+2n}.$$  

(4.2)

(ii) On the other hand, from the representation (2.3) we have a restriction $n \in \{1, 2\}$. Letting $g = k$ in (4.2), one gets the formula (4.1) for $n = 1$ and $n = 2$. □

Example 4.2. We conclude by an example of the free group $F_2$ of rank 2. Such a group is the fundamental group of the once-punctured torus $S_{1,1}$ [5, Example 1]. In other words, we have $g = n = k = 1$. From corollary 4.1, one gets:

$$LO(F_2) \cong \mathcal{K} \subset S^1,$$

(4.3)

where $\mathcal{K}$ is the set of all irrational numbers of the circle $S^1 = \mathbb{R}/\mathbb{Z}$. On the other hand, by corollary 1.3 each irrational $\theta \in \mathcal{K}$ defines a Riemann surface $S^1$ modulo the action of the Teichmüller flow $T^\theta$. This fact can be proved independently using the notion of a non-commutative torus $\mathcal{A}_\theta$ [6, Section 1.3].

Conflict of interest

The authors declare that they have no conflict of interest.

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References

1. S. Azzali, S. L. Browne, M. P. Gomez Aparicio, L. C. Ruth, H. Wang, K-homology and K-theory of pure braid groups, New York J. Math. 28 (2022), 1256-1294.
2. P. Dehornoy, I. Dynnikov, D. Rolfsen, B Wiest, Why are braids orderable?, Panoramas et Synthèses 14, Société Mathématique de France, 2002.
3. J. Franks and R. Williams, Braids and the Jones polynomial, Trans. Amer. Math. Soc. 303 (1987), 97-108.
4. T. Ito, A note on HOMFLY polynomials of positive braid links, arXiv:2005.13188
5. I. V Nikolaev, On cluster $C^*$-algebras, J. Funct. Spaces 2016, Article ID 9639875, 8 p. (2016)
6. I. V Nikolaev, Noncommutative Geometry, Second Edition, De Gruyter Studies in Math. 66, Berlin, 2022.
7. D. Rolfsen, Low-dimensional topology and ordering groups, Mathematica Slovaca 64 (2014), 579-600.
8. A. S. Sikora, Topology on the spaces of orderings of groups, Bull. London Math. Soc. 36 (2004), 519-526.
9. L. K. Williams, Cluster algebras: an introduction, Bull. Amer. Math. Soc. 51 (2014), 1-26.