Motivated by recent experiments in ultracold gases, we focus on the properties of the center of mass coordinate of an interacting one dimensional Fermi gas, displaying several distinct phases. While the variance of the center of mass vanishes in insulating phases such as phase separated and charge density wave phases, it remains finite in the metallic phase, which realizes a Luttinger liquid. By combining numerics with bosonization, we demonstrate that the autocorrelation function of the center of mass coordinate is universal throughout the metallic phase. It exhibits persistent oscillations and its short time dynamics reveal important features of the quantum liquid, such as the Luttinger liquid parameter and the renormalized velocity. The full counting statistics of the center of mass follows a normal distribution whose width is set solely by the Luttinger liquid parameter. Our results apply to non-integrable systems as well and are within experimental reach for e.g. carbon nanotubes and cold atomic gases.

Introduction. Strong correlations in combination with quantum mechanics in reduced dimensions have already provided a plethora of fascinating phenomena\cite{1, 2}, including spin-charge separation, charge fractionalization, Wigner crystals and non-Fermi liquid behaviour. Many of these pop up in a variety of fermionic and bosonic systems, including condensed matter, cold atomic systems\cite{3}, quantum optics\cite{4} and even in black holes\cite{5}. Not only compelling, but these systems promise to be relevant for possible application in topological quantum computation, spintronics and quantum information theory.

In classical mechanics, the concept of the center of mass coordinate plays a prominent role. Due to Newton’s third law, the action and reaction forces between the particles compensate each other, and the center of mass is influenced only by external forces. The very same program can be also carried out in quantum mechanics and the center of mass coordinate gets separated from the relative ones\cite{6}. However, this works only when the interaction depends on the relative position of the particles and not on their absolute position. In any realistic setting in condensed matter or cold atomic systems, an atomic or trapping potential is inevitably present, involving the absolute position of particles. Therefore, the center of mass contribution cannot be separated from the rest and its properties are influenced by strong correlations. Understanding how this happens is the main goal of this work, and low dimensional quantum systems featuring enhanced correlation effects represent an ideal playground for that.

The proper definition of the many-body position or center of mass coordinate has a long history\cite{2, 3, 6}, especially with periodic boundary conditions. With open boundary conditions (OBC), however, one can legitimately define the position operator in the conventional way\cite{4} as $\sum_{i} x_{i}$ by summing over the position operator of each particle. Moreover, experimentally realizations often imply OBC. In this context, a recent experiment on weakly interacting bosons in one dimension has already investigated the dynamics of the center of mass\cite{10}. Our aim is to shed light on the complementary, strongly correlated side of the problem, thus we focus on a strongly interacting one dimensional quantum liquid in one dimension with OBC\cite{11}. We find that the center of mass coordinate reveals universal behaviour and its variance vanishes in insulating phases. In the metallic phase, its temporal dynamics follow a universal scaling function, and give a direct measure of the two relevant parameters of the low energy theory. The full counting statistics of center of mass obey a normal distribution. The observation of all these features is within experimental reach.

Interacting fermions in 1D: lattice and continuum. We study one dimensional spinless fermions in a tight-binding chain with nearest neighbour interaction at half filling and open boundary condition (OBC)\cite{12} using several numerical techniques. This problem is equivalent to the 1D Heisenberg XXZ chain after a Jordan-Wigner transformation\cite{1, 2}. The Hamiltonian is

$$H = \sum_{m=1}^{N-1} \left[ \frac{J}{2} (c_{m+1}^{\dagger} c_{m} + \text{h.c.}) + J_{z} n_{m+1} n_{m} \right], \quad (1)$$

where $c$’s are fermionic operators, $n_{m} = c_{m}^{\dagger} c_{m}$ and $J_{z}$ denotes the nearest neighbour repulsion. This model realizes a Luttinger liquid (LL) for $|J_{z}| < J$ with LL parameter $K = \pi / 2 |1 - \arccos(J_{z}/J)|$ and renormalized velocity $v = a J \sqrt{1 - (J_{z}/J)^2 / 2 \arccos(J_{z}/J)}$ with $a$ the lattice constant. For $J_{z} > J$, the ground state becomes a charge density wave through a Kosterlitz-Thouless transition with broken $Z_{2}$ (corresponding to even/odd lattice sites) symmetry, while for $J_{z} < -J$, the ground state is phase separated through a first order phase transition,
i.e. all \(N/2\) fermions are "bound" together. This model is solved using exact diagonalization (ED) with Lanczos algorithm up to \(N = 26\) and by the density matrix renormalization group (DMRG) up to \(N = 80\).

By going to the continuum limit of Eq. (1) to capture the low energy dynamics, after bosonization the LL phase of this model with OBC is mapped onto\(^{11, 13}\)

\[
H = \sum_{q > 0} \omega(q) b_q^\dagger b_q, \tag{2}
\]

where \(b_q\) creates a bosonic density wave and the long wavelength part of the local charge density is \(\rho(x) = \partial_x \Theta(x)/\pi\) with

\[
\Theta(x) = i \sum_{q > 0} \sqrt{\frac{\pi K}{q L}} \sin(qx) \left[ b_q - b_q^\dagger \right] \tag{3}
\]

for OBC and \(K\) the LL parameter and \(\omega(q) = v q\) with \(v\) the Fermi velocity in the interacting systems and \(q = l\pi/L\) with \(l = 1, 2, 3, \ldots\).

Center of mass. We define the dimensionless center of mass operator for Eq. (1) as\(^{14}\)

\[
\hat{x} = \frac{1}{N} \sum_{m=1}^{N} \left( m - \sum_{m'=1}^{N} \frac{m'}{N} \right) n_m, \tag{4}
\]

where for simplicity, we have subtracted the equilibrium position of the center of mass coordinate such that \(\langle \hat{x} \rangle = 0\), irrespective of how the lattice sites are numbered. For identical particles, what we consider here, it is independent from their mass. This operator is also the normalized polarization operator\(^{15}\). Using bosonization, the very same quantity reads as

\[
\hat{x} = \int_0^L dx \frac{1}{\pi L} \partial_x \Theta(x), \tag{5}
\]

and we have neglected fast oscillating terms in the integrand\(^{12, 13}\) from short wavelength density fluctuations, which are expected to average out after the integral.

Since the expectation value of the center of mass operator is zero, let us start evaluating its standard deviation, \(\sigma_x\) from

\[
\sigma_x^2 = \langle \hat{x}^2 \rangle = \int_0^L dx \int_0^L dy \frac{xy(\partial_x \Theta(x) \partial_y \Theta(y))}{\pi^2 L^2} = \sum_{q > 0} \frac{K q}{\pi L} \left[ \int_0^L dx \cos(qx) \frac{x}{L} \right]^2 = \frac{7\zeta(3)}{2\pi^2} K, \tag{6}
\]

where \(\zeta(z)\) is the Riemann zeta function\(^{16}\) and \(\zeta(3) \approx 1.202\). The \(\sigma_x\) is a universal number and depends only on the LL parameter, \(K\), but is independent of the high energy degrees of freedom. Since \(K\) decreases with increasing \(J_z\), this implies that counterintuitively, the variance gets suppressed when moving from the attractive to the repulsive side. The numerical results from DMRG agree very nicely with Eq. (6), as seen in Fig. 1. The variance diverges as \(\sigma_x^2 \sim \sqrt{J/(J+J_z)}\) at the first order critical point. Slight deviations are visible close to \(J_z \sim J\), arising from the terms in the Hamiltonian, driving the Kosterlitz-Thouless transition, which are missing from Eq. (2). Nevertheless, the variance seems to remain finite at this critical point.

The above calculation can be extended to the gapped charge density wave phase, when the effective field theory of Eq. (1) is the sine-Gordon model\(^{11, 12}\). In this case, a Mott gap \(\Delta\) opens up in the spectrum. Within the realm of the semiclassical limit of this model, following Ref.\(^{17}\) and\(^{18}\), the variance of the center of mass is calculated with \(K\) replaced by \(\omega(q)/\sqrt{\omega^2(q) + \Delta^2}\) in Eq. (6) under the sum. This gives \((\sigma_x^2 \sim (L\Delta/v)^{-1}\) and vanishes in the thermodynamic limit, which is also corroborated by ED. For finite systems, the variance vanishes when the system size, \(L\) is much longer than the correlation length, \(v/\Delta\). Alternatively, the variance is negligible when the level spacing, \(v/L\) is much smaller than the actual gap. This would allow for estimating the gap size from measuring the variance of the center of mass in finite size systems.

In the phase separated regime, bosonization is not applicable, but the variance of the center of mass can be calculated. Since all \(N/2\) particles are bound together by the strong attractive interaction in the lattice of \(N\) sites, the ground state is in principle highly degenerate. As a result, \(\sigma_x \sim N/2\), which agrees with ED results on clean systems. However, any disorder or imperfection in the lattice, which is inevitably present in any real system, breaks this degeneracy and produces a unique ground state. Therein, the \(N/2\) particles occupy neighbouring lattice sites, their position is well defined and the variance is zero, as we also find from ED in the presence of weak impurities or disorder.

Dynamics of the center of mass. To gain further insight into the behaviour of the center of mass operator,
we evaluate its autocorrelation function as
\[ \chi_x(t) = \langle \hat{x}(t) \hat{x}(0) \rangle. \] (7)

Using \( b_q(t) = b_q \exp(-i\omega(q)t) \) in Eq. (3), we obtain
\[ \chi_x(t) = \frac{2K}{\pi^4} \sum_{l=1}^{\infty} \frac{1 - (-1)^l}{l^3} \exp(-iv\pi l/L) = \frac{2K}{\pi^4} \sum_{b=\pm} b \text{Li}_3(b \exp(-iv\pi/L)), \] (8)

with \( \text{Li}_s(z) \) the polylogarithm function, and gives \( \chi_x(0) = \sigma_x^2 \). Although Eq. (8) looks complicated at first, it is rather well approximated by \( \chi_x(t) \approx \sigma_x^2 \exp(-iv\pi/L) \). Similarly to the variance of the center of mass, \( \chi_x(t) \) is also independent of any cutoff and depends only on the universal combination \( vt/L \). Its initial temporal slope is
\[ c_x = i\partial_t \chi_x(t \to 0) = \langle [\hat{x}, H] \hat{x} \rangle = \frac{1}{2\pi L} vK, \] (9)

which depends only the the LL parameter \( K \) and the renormalized velocity of the interacting theory. Therefore, by measuring the variance of the center of mass and its initial dynamics, one can easily extract the two and only two essential ingredients of the LL theory, the velocity from \( v = 7\zeta(3)LC_x/\sigma_x^2\pi^3 \) and the LL parameter from \( \sigma_x \), as shown in Fig. 1. In addition, Eq. (8) predicts a universal data collapse of the center of mass oscillation, namely upon rescaling its magnitude by \( 1/K \) and its temporal evolution by \( v/L \), all curves should fall on top of each other, irrespective of the strength or even the sign of the interaction, as shown in Fig. 2. The time dependence spans several \( N/J \) periods (with \( N = 26 \)) and the agreement between numerics and Eq. (8) remains excellent, even though \( K \) and \( v \) decrease/increases by more than a factor of 2 from \( J_z/J = -0.6 \) to 0.6, respectively.

The center of mass autocorrelator is found to be universal at all timescales. This is somewhat surprising since the LL theory is designed to capture the low energy physics, thus it is expected to be universal in the long time limit. For \( \chi_x(t) \), on the other hand, already the short time dynamics turns out to be universal. The lattice model in Eq. (1) in integrable\([4, 5]\) therefore one may wonder whether these persistent oscillation arise due to the large number of constants of motion. Integrability is destroyed by adding a second nearest-neighbour density-density (i.e. \( J_z \sum_{n=2}^N n_{m+2}n_{m} \)) interaction\([10]\), what we have also studied numerically for several \( J_z \) and \( J'_{zz} \), yielding identical results to the integrable case: the persistent oscillations from Eq. (3) remain intact also for non-integrable LLs.

**Full counting statistics.** Already simple expectation values of physical quantities often display rather complex behavior. Higher moments of the observables contain, however, infinitely more information and encode unique information about e.g. non-local, multi-point correlators and entanglement, though they are typically difficult to access. Their information content is equivalent to determining the full distribution function of the quantity of interest.

Having studied simple correlation functions of the center of mass coordinate, we now address its full counting statistics\([20, 22]\). Its probability distribution function is
\[ P(X) = \langle \delta(X - \hat{x}) \rangle, \] (10)

whose characteristic function can easily be evaluated to yield \( G(p) = \langle \exp(ip\hat{x}) \rangle \). Note that \( G(p) \) is reminiscent to how the polarization operator is defined\([7, 8, 24]\) for periodic boundary condition, using only integer multiples of \( 2\pi \) for \( p \). Here, on the contrary, \( p \) takes any real values in the characteristic function and the normalized position and polarization operator, \( \hat{x} \) is defined by Eq. (4) without any ambiguity\([9]\).

Since \( \hat{x} \) in the exponent is a linear function of bosonic operators, and the low energy Hamiltonian is quadratic in Eq. (2), the expectation value is evaluated as\([23]\)
\[ G(p) = \exp(-p^2\langle \hat{x}^2 \rangle/2) = \exp(-p^2\sigma_x^2/2). \] (11)

This is evaluated also for Eq. (11) numerically using ED after finite size scaling, and plotted in Fig. 3 revealing excellent agreement between Eq. (11) and the numerical data. For smaller systems and especially for repulsive \( J_z \sim J \), slight deviations show up from the Gaussian behaviour for large \( p \), which stem from the fact that \( ||\hat{x}|| < N/2 \) is bounded for finite systems, therefore deviations appear in the tail, which diminish upon increasing the system size. Its Fourier transform gives the probability distribution function as a normal distribution with variance \( \sigma_x^2 \) as
\[ P(X) = \frac{1}{\sqrt{2\pi\sigma_x^2}} \exp\left(-\frac{X^2}{2\sigma_x^2}\right). \]

**Experimental ramifications.** There exists well-developed experimental technology to observe these
effects. LLs are routinely realized in both cold atomic settings, using either bosons or fermions, and condensed matter systems, including e.g. carbon nanotubes. The center of mass coordinate can be monitored using time of flight imaging, in-situ absorption imaging or scanning tunneling microscopy, allowing for the observation of its variance as well as its full distribution function, or at least some of its lower moments. These are all universal quantities, depending only on the LL parameter $K$.

The dynamics of the center of mass coordinate is measurable by e.g. tilting the lattice or applying a weak electric field at time $t = 0$, represented by the scalar potential of the force $F$, which creates a perturbation $H' = L\hat{x}F(t)$. Then, within linear response theory, the motion of the center of mass follows as $\langle \hat{x}(t) \rangle = -2L \int_0^t dt' \text{Im} \chi_x(t-t')F(t')$. For short times, $\text{Im} \chi_x(t) = -\nu Kt/2\pi L$, revealing the two LL characteristics in a universal manner. Therefore initially $\langle \hat{x}(t) \rangle = \nu Kt^2 F/2\pi$ for a constant, time independent force, corresponding to the classical motion of a particle in an external force $F$ with "mass" $\sim \pi/\nu K$. Based on Eq. (5) and Fig. 2 the $\langle \hat{x}(t) \rangle$ will exhibit persistent oscillations for longer times with frequency $\nu F/L$.

Both the short time and oscillatory behaviour is reminiscent to Bloch oscillations, albeit these persistent oscillations arise in a strongly correlated quantum liquid as opposed to the standard single particle picture behind Bloch oscillations. The analogy with Bloch oscillation is extended further by noting that the reflection on the boundary of the lattice in our study plays the role of Bragg reflection in the Bloch’s case. Note that the linear response results is exact within the realm of bosonization, there are no higher order corrections in $F$. This follows from the linear dispersion in $\omega(q) \sim q$, extending up to infinitely large energies, without any band bending, which could not accommodate Bloch oscillations. This is completely analogous to how the Born scattering limit of Dirac-delta potential is exact for the same linear dispersion.

As already noted, in the presence of electric field Bloch oscillations are also expected to show up for any realistic band structure at higher energies. Their typical timescale, $t_B = 1/\alpha F$ with $\alpha$ the lattice constant, represents the time during which the full Brillouin zone is swept through by the force. The timescale for the center of mass oscillation due to finite size effects from Eq. (8) is $L/\nu$, i.e. the timescale for sweeping through the real space lattice. Our results are observable with respect to Bloch oscillations as long as $L/\nu \ll t_B$, requiring weak forces and more importantly, short systems, which suits ideally the experimental conditions. In this context, this can be regarded as the poor man’s version of the Newton’s cradle experiment.

Conclusions. To sum up, we have shown that the variance of the center of mass coordinate vanishes in insulating phases and is universal at all timescales in the Luttinger liquid phase. Already its short time dynamics is able to reveal the basic characteristics of the underlying quantum liquid, such as theb Luttinger liquid parameter and velocity. Its correlation function as well as its full counting statistics follow a universal function, which are corroborated by analytical and numerical methods. These are within experimental reach both in condensed matter and cold atomic realizations, using setups similar to Bloch oscillations.

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项目名称：

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1. T. Giamarchi, Quantum Physics in One Dimension (Oxford University Press, Oxford, 2004).
2. A. O. Gogolin, A. A. Nersesyan, and A. M. Tsvelik, Bosonization and Strongly Correlated Systems (Cambridge University Press, Cambridge, 1998).
3. M. A. Cazalilla, R. Citro, T. Giamarchi, E. Orignac, and M. Rigol, One dimensional bosons: From condensed matter systems to ultracold gases, Rev. Mod. Phys. 83, 1405 (2011).
4. D. E. Chang, V. Gritsev, G. Morigi, V. Vuletic, M. D. Lukin, and E. A. Demler, Crystalization of strongly interacting photons in a nonlinear optical fibre, Nat. Phys. 4, 884 (2008).
5. V. Balasubramanian, I. n. García-Etxebarria, F. Larsen, and J. Simón, Helical luttinger liquids and three-dimensional black holes, Phys. Rev. D 84, 126012 (2011).
6. A. Messiah, Quantum Mechanics, Dover books on physics (Dover Publications, 1961).
7. R. Resta, Quantum-mechanical position operator in extended systems, Phys. Rev. Lett. 80, 1800 (1998).
REFERENCES

[8] R. Resta and S. Sorella, Electron localization in the insulating state, Phys. Rev. Lett. 82, 370 (1999).

[9] M. Rigol and B. S. Shastry, Drude weight in systems with open boundary conditions, Phys. Rev. B 77, 161101 (2008).

[10] Z. A. Geiger, K. M. Fujiiwara, K. Singh, R. Senaratne, S. V. Rajagopal, M. Lipatov, T. Shimasaki, R. Driben, V. V. Konotop, T. Meier, and D. M. Weld, Observation and uses of position-space Bloch oscillations in an ultracold gas, arXiv:1803.02456.

[11] M. A. Cazalilla, Bosonizing one-dimensional cold atomic gases, J. Phys. B: At. Mol. Opt. Phys. 37, S1 (2004).

[12] OBC arises by using holographic masks or by confining cold-atoms in a ring lattice and cutting it open or by using gates to confine electrons for carbon nanotubes.

[13] M. Fabrizio and A. O. Gogolin, Interacting one-dimensional electron gas with open boundaries, Phys. Rev. B 51, 17827 (1995).

[14] T. Vaughan, P. Drummond, and G. Leuchs, Quantum limits to center-of-mass measurements, Phys. Rev. A 75, 033617 (2007).

[15] G. D. Mahan, Many particle physics (Plenum Publishers, New York, 1990).

[16] I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series, and Products (Academic Press, New York, 2007).

[17] K. Maki and H. Takayama, Quantum-statistical mechanics of extended objects. i. kinks in the one-dimensional sine-gordon system, Phys. Rev. B 20, 3223 (1979).

[18] A. Iucci and M. A. Cazalilla, Quantum quench dynamics of the sine-gordon model in some solvable limits, New J. Phys. 12, 055019 (2010).

[19] K. Hallberg, E. Gagliano, and C. Balseiro, Finite-size study of a spin-1/2 heisenberg chain with competing interactions: Phase diagram and critical behavior, Phys. Rev. B 41, 9474 (1990).

[20] L. S. Levitov, H. Lee, and G. B. Lesovik, Electron counting statistics and coherent states of electric current, J. Math. Phys. 37, 4845 (1996).

[21] V. Gritsev, E. Altman, E. Demler, and A. Polkovnikov, Full quantum distribution of contrast in interference experiments between interacting one-dimensional Bose liquids, Nat. Phys. 2, 705 (2006).

[22] A. Silva, Statistics of the work done on a quantum critical system by quenching a control parameter, Phys. Rev. Lett. 101, 120603 (2008).

[23] M. Gring, M. Kuhnert, T. Langen, T. Kitagawa, B. Rauer, M. Schreitl, I. Mazets, D. A. Smith, E. Demler, and J. Schmiedmayer, Relaxation and prethermalization in an isolated quantum system, Science 337, 1318 (2012).

[24] R. Kobayashi, Y. O. Nakagawa, Y. Fukusumi, and M. Oshikawa, Scaling of polarization amplitude in quantum many-body systems in one dimension, arXiv:1802.01606.

[25] J. von Delft and H. Schoeller, Bosonization for beginners refermionization for experts, Ann. Phys. (Leipzig) 7, 225 (1998).

[26] I. Bloch, J. Dalibard, and W. Zwerger, Many-body physics with ultracold gases, Rev. Mod. Phys. 80, 885 (2008).

[27] J. H. Davies, The Physics of Low-dimensional Semiconductors: An Introduction (Cambridge University Press, 1997).

[28] T. Kinoshita, T. Wenger, and D. S. Weiss, A quantum Newton’s cradle, Nature 440, 900 (2006).