A VARIATIONAL REPRESENTATION AND LARGE DEVIATIONS FOR FUNCTIONALS OF G-BROWNIAN MOTION

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Abstract. A variational representation for functionals of G-Brownian motion is established by a finite-dimensional approximate technique. As an application of the variational representation, we obtain a large deviation principle for stochastic flows driven by G-Brownian motion.

1. Introduction

Peng ([20]) proposed G-Brownian motion and G-expectation. The stochastic analysis under the G-expectation (G-stochastic calculus) has had many important progresses in recent years (cf. [22] and references therein). For a connection between Denis and Martini ([9]) and the G-stochastic integration theory of Peng ([20]), we refer to Denis, Hu and Peng ([10]), and Soner, Touzi and Zhang ([25]). The G-stochastic calculus also provides a framework for financial problems with uncertainty about the volatility and a stochastic method for fully nonlinear PDEs (cf. [9], [20], [26]).

The purpose of this paper is to establish a variational representation for functionals of G-Brownian motion and a large deviation principle for stochastic flows driven by G-Brownian motion. We obtain the following variational representation for functionals of G-Brownian motion:

\[ \mathbb{E}^G(e^{\Phi(B)}) = \exp \left\{ \sup_{\eta \in (M^2(0, T))^d} \mathbb{E}^G(\Phi(B^n) - H^G_T(\eta)) \right\}, \]

where \( \Phi \in L^1_G(\Omega_T) \) bounded, \( \{B_t, t \in [0, T]\} \) is a G-Brownian motion and \( \{(B)_t, t \in [0, T]\} \) is its quadratic variation process, \( B^n_t = B_t + \int_0^t \eta_s d\langle B\rangle_s \) and \( H^G_T(\eta) = \frac{1}{2} \sum_{i,j=1}^d \int_0^T \eta^i_s \eta^j_s d\langle B\rangle^{ij}_s \). The definitions of \( \mathbb{E}^G, L^1_G(\Omega_T), M^2(0, T), \) the G-Brownian motion and the quadratic variation process will be given in Section 2. As an application of the variational representation, we obtain a large deviation principle for stochastic flows driven by G-Brownian motion.

In the classical case, a variational representation of functionals of finite dimensional Brownian motion was first obtained by Boué and Dupuis ([4]). Chen and
Xiong ([7]) considered the variational representations under a $g$-expectation which is defined by a backward stochastic differential equation. The variational representations have been shown to be useful in deriving various asymptotic results in large deviations (cf. [4], [5], [6], [11] and [23]) and functional inequalities (cf. [3]).

Under $G$-expectation, the complicated measurable selection technique in [4] cannot be used and the Clark-Ocone formula is not available. In this paper, we will develop finite-dimensional approximate technique under $G$-expectation. We prove that a finite-dimensional functional for $G$-Brownian motion can be approximated by a sequence of $G$-stochastic differential equations, which plays an important role in the proof of the upper bound. The lower bound will be proved by the $G$-Girsanov transformation (cf. [29]) and bounded approximation. In particular, this also provides a new proof for the variational representations of Boué and Dupuis.

The remainder of the paper is organized as follows. In Section 2 we recall some basic conceptions and results under $G$-framework. The variational representation is proved in Section 3. An abstract large deviation principle for functionals of $G$-Brownian motion is presented in Section 4. A large deviation principle of stochastic flows driven by $G$-Brownian motion is established in Section 5.

2. $G$-EXPECTATION AND $G$-BROWNIAN MOTION

In this section, we briefly recall some basic conceptions and results about $G$-expectation and $G$-Brownian motion (see [10], [20], [21] and [22] for details).

2.1. Sublinear expectation. Let $\Omega$ be a given set and let $\mathcal{H}$ be a linear space of real valued functions defined on $\Omega$ with $c \in \mathcal{H}$ for all constants $c$, and satisfying: if $X_i \in \mathcal{H}$, $i = 1, \cdots, d$, then

$$\varphi(X_1, \cdots, X_d) \in \mathcal{H}, \text{ for all } \varphi \in \text{lip}(\mathbb{R}^d),$$

where $\text{lip}(\mathbb{R}^d)$ is the space of all bounded and Lipschitz continuous functions on $\mathbb{R}^d$.

A sublinear expectation $\hat{E}$ on $\mathcal{H}$ is a functional $\hat{E}: \mathcal{H} \mapsto \mathbb{R}$ satisfying the following properties:

- Monotonicity: If $X \geq Y$ then $\hat{E}(X) \geq \hat{E}(Y)$;
- Constant preserving: $\hat{E}(c) = c$;
- Sub-additivity: $\hat{E}(X) - \hat{E}(Y) \leq \hat{E}(X - Y)$;
- Positive homogeneity: $\hat{E}(\lambda X) = \lambda \hat{E}(X), \ \forall \lambda \geq 0$.

The triple $(\Omega, \mathcal{H}, \hat{E})$ is called a sublinear expectation space. $X \in \mathcal{H}$ is called a random variable in $(\Omega, \mathcal{H})$.

A $m$-dimensional random vector $X = (X_1, \cdots, X_m)$ is said to be independent of another $n$-dimensional random vector $Y = (Y_1, \cdots, Y_n)$ if

$$\hat{E}(\varphi(X, Y)) = \hat{E}(\hat{E}(\varphi(X, y))_{y=Y}), \text{ for } \varphi \in \text{lip}(\mathbb{R}^m \times \mathbb{R}^n).$$

Let $X_1$ and $X_2$ be two $d$-dimensional random vectors defined respectively in sublinear expectation spaces $(\Omega_1, \mathcal{H}_1, \hat{E}_1)$ and $(\Omega_2, \mathcal{H}_2, \hat{E}_2)$. They are called identically
distributed, denoted by $X_1 \sim X_2$, if
\[
\hat{\mathbb{E}}_1(\varphi(X_1)) = \hat{\mathbb{E}}_2(\varphi(X_2)), \quad \forall \varphi \in C_{b,\text{Lip}}(\mathbb{R}^n).
\]

A $d$-dimensional random vector $X = (X_1, \ldots, X_d)$ in a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called $G$-normal distributed if for each $a, b \geq 0$ we have $aX + b\tilde{X} \sim \sqrt{a^2 + b^2}X$, where $\tilde{X}$ is an independent copy of $X$. The letter $G$ denotes the function $G : S_d \mapsto \mathbb{R}$, $G(A) := \frac{1}{2}\hat{\mathbb{E}}(\langle AX, X \rangle)$, where $S_d$ is the collection of $d \times d$ symmetric matrices and $(\cdot, \cdot)$ denotes the inner product in $\mathbb{R}^d$, i.e., $(x, y) := \sum_{i=1}^d x^i y^i$ for any $x = (x^1, \ldots, x^d)$, $y = (y^1, \ldots, y^d) \in \mathbb{R}^d$.

Let $d$-dimensional random vector $X = (X_1, \ldots, X_d)$ in a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ be $G$-normal distributed. For each $\varphi \in \text{lip}(\mathbb{R}^d)$, set $u(t, x) = \hat{\mathbb{E}}(\varphi(x + \sqrt{t}X))$, $t \geq 0$, $x \in \mathbb{R}^d$. Then $u(t, x)$ is the unique viscosity solution of the following equation,
\[
\frac{\partial u}{\partial t} - G(D_x^2 u) = 0, \quad t \geq 0, \quad x \in \mathbb{R}^d, \quad u(0, x) = \varphi(x),
\]
where $D_x^2 u = (\partial^2_{x_i x_j} u)^{i,j=1}$ is the Hessian matrix of $u$.

The inner product in $S_d$ is defined by $(A_1, A_2) = \sum_{i,j=1}^d a_1(i, j)a_2(i, j)$ for $A_1 = (a_1(i, j))_{d \times d}, A_2 = (a_2(i, j))_{d \times d}$. Then the map $G : S_d \mapsto \mathbb{R}$ is a monotonic and sublinear function, i.e., for $A, \tilde{A} \in S_d$,
\[
\begin{cases}
G(A + \tilde{A}) \leq G(A) + G(\tilde{A}), \\
G(\lambda A) = \lambda G(A), \quad \text{for all } \lambda \geq 0, \\
G(A) \geq G(\tilde{A}), \quad \text{if } A \geq \tilde{A}.
\end{cases}
\]

For a monotonic and sublinear function $G : S_d \mapsto \mathbb{R}$ given, there exists a bounded, convex and closed subset $\Sigma \subset S_d^+$ (the non-negative elements of $S_d$) such that $G(A) = \frac{1}{2}\sup_{\sigma \in \Sigma}(A, \sigma)$. Throughout this paper, we assume that there exist constants $0 < \underline{\sigma} \leq \overline{\sigma} < \infty$ such that
\[
\Sigma \subset \{\sigma \in S_d ; \quad \underline{\sigma}I_{d \times d} \leq \sigma \leq \overline{\sigma}I_{d \times d}\}.
\]

2.2. $G$-Brownian motion and $G$-expectation. Let $\Omega$ denote the space of all $\mathbb{R}^d$-valued continuous paths $\omega : (0, +\infty) \ni t \mapsto \omega_t \in \mathbb{R}^d$, with $\omega_0 = 0$. Let $\mathcal{B}(\Omega), \mathcal{M}, L^0(\Omega), B_b(\Omega)$ and $C_b(\Omega)$ denote respectively the Borel $\sigma$-algebra of $\Omega$, the collection of all probability measure on $\Omega$, the space of all $\mathcal{B}(\Omega)$-measurable real functions, all bounded elements in $L^0(\Omega)$ and all continuous elements in $B_b(\Omega)$. For each $t \in [0, \infty)$, we also denote $\Omega_t := \{\omega_{\cdot t} : \omega \in \Omega\};$ $\mathcal{F}_t := \mathcal{B}(\Omega_t); L^0(\Omega_t):$ the space of all $\mathcal{B}(\Omega_t)$-measurable real functions; $B_b(\Omega_t) := B_b(\Omega) \cap L^0(\Omega_t); C_b(\Omega_t) := C_b(\Omega) \cap L^0(\Omega_t)$.

For each $t > 0$, set
\[
L_{\text{lip}}(\Omega_t) := \{\varphi(\omega_{t_1}, \omega_{t_2}, \ldots, \omega_{t_n}) : n \geq 1, t_1, \ldots, t_n \in [0, t], \varphi \in \text{lip}(([\mathbb{R}^d]^n))\}.
\]
Define $L_{\text{lip}}(\Omega) := \bigcup_{n=1}^\infty L_{\text{lip}}(\Omega_n) \subset C_b(\Omega)$. 

Let $G : S_d \mapsto \mathbb{R}$ be a given monotonic and sublinear function. A continuous process $\{B_t(\omega)\}_{t \geq 0}$ in a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E}^G)$ is called a $G$-Brownian motion if it has stationary and independent increments, $B_0 = 0$, $B_1$ is $G$-normal distributed and $\mathbb{E}^G(B_t) = -\mathbb{E}^G(-B_t) = 0$ for $t \geq 0$.

The topological completion of $L_{ip}(\Omega_t)$ (resp. $L_{ip}(\Omega)$) under the Banach norm $\| \cdot \|_{p,G} := (\mathbb{E}^G(|\cdot|^p))^1/p$ is denoted by $L^p_G(\Omega_t)$ (resp. $L^p_G(\Omega)$), where $p \geq 1$. $\mathbb{E}^G(\cdot)$ can be extended uniquely to a sublinear expectation on $L^1_G(\Omega)$. We denote also by $\mathbb{E}^G$ the extension. It is proved in [10] that $L^0(\Omega) \supset L^p_G(\Omega) \supset C_b(\Omega)$, and there exists a weakly compact family $\mathcal{P}$ of probability measures defined on $(\Omega, \mathcal{B}(\Omega))$ such that $\mathbb{E}^G(X) = \sup_{P \in \mathcal{P}} E_P(X)$ for $X \in C_b(\Omega)$. $\mathbb{E}^G(\cdot)$ has the following regularity ([10]): For each $\{X_n\}_{n=1}^\infty$ in $C_b(\Omega)$ with $X_n \downarrow 0$ on $\Omega$, $\mathbb{E}^G(X_n) \downarrow 0$. We also denote

$$\overline{\mathbb{E}}^G(X) = \sup_{P \in \mathcal{P}} E_P(X), \ X \in L^0(\Omega).$$

The natural Choquet capacity associated with $\mathbb{E}^G$ is defined by

$$c^G(A) := \sup_{P \in \mathcal{P}} P(A), \ A \in \mathcal{B}(\Omega).$$

A set $A \subset \Omega$ is polar if $c^G(A) = 0$. A property holds “quasi-surely” (q.s.) if it holds outside a polar set. A mapping $X$ on $\Omega$ with values in a topological space is said to be quasi-continuous (q.c.) if for any $\varepsilon > 0$, there exists an open set $O$ with $c^G(O) < \varepsilon$ such that $X|_O$ is continuous. $L^p_G(\Omega)$ also has the following characterization ([10]):

$$L^p_G(\Omega) = \{X \in L^0(\Omega); \ \lim_{n \to \infty} \mathbb{E}^G(|X|^p I_{\{|X| \geq n\}}) = 0, \ \text{and} \ X \text{ is } c^G\text{-quasi surely continuous}\}.$$

Let us recall the representation theorem of $G$-expectation. For a monotonic and sublinear function $G : S_d \mapsto \mathbb{R}$ given, there exists a bounded, convex and closed subset $\Sigma \subset S^+_d$ such that $G(A) = \frac{1}{2} \sup_{\sigma \in \Sigma} \langle A, \sigma \rangle$. Set $\Gamma := \{\gamma = \sigma^{1/2}, \sigma \in \Sigma\}$. Let $P$ be the Wiener measure on $\Omega$. Let $\mathcal{A}_0^\Gamma$ be the collection of all $\Gamma$-valued $\{\mathcal{F}_t, t \geq 0\}$-adapted processes on the interval $[0, +\infty)$, i.e., $\{\theta_t, t \geq 0\} \in \mathcal{A}_0^\Gamma$ if and only if $\theta_t$ is $\mathcal{F}_t$ measurable and $\theta_t \in \Gamma$ for each $t \geq 0$, and let $P_\theta$ be the law of the process $\{\int_0^t \theta_s d\omega_s, t \geq 0\}$ under the Wiener measure $P$. The representation theorem of $G$-expectation ([10]) is stated as follows: for all $X \in L^1_G(\Omega)$

$$\mathbb{E}^G(X) = \sup_{\theta \in \mathcal{A}_0^\Gamma} E_{P_\theta}(X). \quad (2.4)$$

2.3. $G$-Stochastic integral and quadratic variation process. Given $T > 0$. For $p \in [1, \infty)$, let $M^0_G(0, T)$ denote the space of $\mathbb{R}$-valued piecewise constant processes

$$\eta_t = \sum_{i=0}^{n-1} \eta_{ti} 1_{[t_i, t_{i+1})}(t)$$
where \( \eta \in L_G^p(\Omega_\eta) \), \( 0 = t_0 < t_1 < \cdots < t_n = T \). For \( \eta \in M_G^{p,0}(0,T) \), \( j = 1, \cdots, d \), the G-stochastic integral is defined by

\[
P^j(\eta) := \int_0^T \eta_s dB^j_s := \sum_{i=0}^{n-1} \eta_{t_i}(B^j_{t_{i+1}} - B^j_{t_i}).
\]

Let \( M_G^{2,0}(0,T) \) be the closure of \( M_G^{p,0}(0,T) \) under the norm: \( \|H\|_{M_G^p(0,T)} := \mathbb{E}\left(\int_0^T |\eta|^p dt\right) \). Then the mapping \( P^j : M_G^{2,0}(0,T) \to L_G^2(\Omega_T) \) is continuous, and so it can be continuously extended to \( M_G^2(0,T) \).

For any \( \eta = (\eta^1, \cdots, \eta^d) \in (M_G^2(0,T))^d \), define

\[
\int_0^T \eta_s dB_s = \sum_{i=1}^d \int_0^T \eta^i_s dB^i_s.
\]

The quadratic variation process of G-Brownian motion is defined by

\[
\langle B \rangle_t := ((\langle B \rangle^i_j)^1 \leq i,j \leq d = \left(B^i_t B^j_t - 2 \int_0^t B^i_s dB^j_s\right)_{1 \leq i,j \leq d}, \quad 0 \leq t \leq T.
\]

\( \langle B \rangle_t \) is a \( \mathcal{S}_d\)-valued process with stationary and independent increments.

For any \( 1 \leq i,j \leq d \), define a mapping \( M_G^{1,0}(0,T) \to L_G^2(\Omega_1) \) as follows:

\[
Q^{ij}_{0,T}(\eta) = \int_0^T \eta_s d\langle B \rangle^i_j_s := \sum_{k=0}^{n-1} \eta_{t_k}((\langle B \rangle^i_j)_{t_{k+1}} - \langle B \rangle^i_j_{t_k}).
\]

Then \( Q^{ij}_{0,T} \) can be uniquely extended to \( M_G^1(0,T) \). We still denote this mapping by

\[
\int_0^T \eta_s d\langle B \rangle^i_j_s = Q^{ij}_{0,T}(\eta), \quad \eta \in M_G^1(0,T).
\]

For \( \eta = (\eta^1, \cdots, \eta^d) \in (M_G^1(0,T))^d \), define

\[
\int_0^T \eta_s d\langle B \rangle_s = \left(\sum_{j=1}^d \int_0^T \eta^j_s d\langle B \rangle^j_s\right)_{d \times 1}.
\]

and for \( \eta = (\eta^{ij})_{d \times d} \in (M_G^1(0,T))^{d \times d} \), define

\[
\int_0^T \eta_s d\langle B \rangle_s = \sum_{i,j=1}^d \int_0^T \eta^{ij}_s d\langle B \rangle^{ij}_s.
\]

2.4. G-Girsanov formula. In one dimensional case, under a strong Novikov-type condition, Xu, Shang and Zhang ([29]) obtained a Girsanov formula under G-expectation based on the martingale characterization theorem ([28]) of G-Brownian motion. A multi-dimensional version of the G-Girsanov formula is presented in [19].
For $\eta = (\eta^1, \cdots, \eta^d) \in (M^2_G(0, T))^d$ satisfying the following strong Novikov condition:

$$
\mathbb{E}^G \left( \exp \left\{ \frac{1}{2}(1 + \epsilon) \sum_{i,j=1}^d \int_0^T \eta^i_s \eta^j_s d\langle B \rangle^{ij}_{s} \right\} \right) < \infty, \quad \text{for some } \epsilon > 0,
$$

we define

$$
\mathcal{E}^\eta_t = \exp \left\{ \int_0^t \eta_s dB_s - \frac{1}{2} \sum_{i,j=1}^d \int_0^t \eta^i_s \eta^j_s d\langle B \rangle^{ij}_{s} \right\}, \quad 0 \leq t \leq T.
$$

Then $\mathcal{E}^\eta_t$ is quasi-continuous. By the condition (2.5), $\mathcal{E}^\eta_t$, $t \in [0, T]$ is a martingale under each $P_\theta$, and $\mathcal{E}^\eta_t \in L^1_G(\Omega_t)$, $t \in [0, T]$ (cf. [29]). Set

$$
dP_{\theta, \eta} = \mathcal{E}^\eta_T dP_{\theta},
$$

and

$$
\mathbb{E}^{G,\eta}(X) = \sup_{\theta \in A_{\Omega_t}} P_{\theta, \eta}(X), \quad X \in L_{ip}(\Omega_T).
$$

Let $L^1_{G,\eta}(\Omega_t)$ be the completion of $(L_{ip}(\Omega_t), \mathbb{E}^{G,\eta}(| \cdot |))$. Then $\mathbb{E}^{G,\eta}$ can be extended to $L^1_{G,\eta}(\Omega_T)$, and the following G-Girsanov formula holds:

**G-Girsanov formula.** ([29], [19]) Under the condition (2.5),

$$
B_t^{-\eta} := B_t - \int_0^t \eta_s d\langle B \rangle_s, \quad t \in [0, T]
$$

is a $G$-Brownian motion under $\mathbb{E}^{G,\eta}$.

### 3. A variational representation for functionals of $G$-Brownian motion

The main result in this section is the following variation representation for functionals of $G$-Brownian motion.

**Theorem 3.1.** Let $\Phi \in L^1_G(\Omega_T)$ be bounded. Then

$$
\mathbb{E}^G(e^{\Phi(B)}) = \exp \left\{ \sup_{\eta \in (M^2_G(0, T))^d} \mathbb{E}^G \left( \Phi \left( B^\eta \right) - H^G_T(\eta) \right) \right\}
$$

$$
= \exp \left\{ \sup_{\eta \in (M^2_G(0, T))^d \cap B_{\theta}(\Omega_T)} \mathbb{E}^G \left( \Phi \left( B^\eta \right) - H^G_T(\eta) \right) \right\},
$$

where $B_t^\eta := B_t + \int_0^t \eta_s d\langle B \rangle_s$,

$$
H^G_t(\eta) = \frac{1}{2} \sum_{i,j=1}^d \int_0^t \eta^i_s \eta^j_s d\langle B \rangle^{ij}_{s}, \quad t \in [0, T].
$$
Remark 3.1. (1). The following proof of Theorem 3.1 is not depend on the representation for ordinary Brownian motion, the variational formula of the relative entropy, the measurable selection technique and the Clark-Ocone formula. In particular, this also gives a new proof of the representation for ordinary Brownian motion in [1].

(2). Notice that the $G$-expectation is the supremum of a collection of expectations so that the canonical map in Wiener space is a martingale under the expectations. If the representation for ordinary Brownian motion can be extended to continuous martingales, then Theorem 3.1 can be obtained from this extension. But the extension is not available.

Lemma 3.1. (cf. [22], [25]) Let $\eta_s^{ij} \in M^2_G(0, T)$, $\eta_s^{ij} = \eta_s^{ji}$, $i, j = 1, \ldots, d$ be given and set

$$M_t = 2 \int_0^t G(\eta_s)ds - \int_0^t \eta_s d\langle B \rangle_s, \ t \in [0, T].$$

Then $M_t \geq 0$, q.s. for all $t \in [0, T]$. In particular, $t \to M_t$ is increasing.

Proof. Take a sequence $\eta^{(N)} \in (M^2_{G^2}(0, T))^{d \times d}$, where

$$\eta_s^{(N)} = \sum_{k=1}^N \eta_s^{(N)} I_{[t_{k-1}^{(N)}, t_k^{(N)}]}(s), \ 0 = t_0^{(N)} < t_1^{(N)} < \cdots < t_N^{(N)} = T,$$

such that

$$\lim_{N \to \infty} \max_{1 \leq i, j \leq d} \mathbb{E}^G \left( \int_0^T |(\eta_s^{(N)})^{ij} - \eta_s^{ij}|^2 \right) ds = 0.$$

Then

$$\mathbb{E}^G \left( \int_0^T |G(\eta_s) - G(\eta_s^{(N)})| \right) ds \leq \mathbb{E}^G \left( \int_0^T \max\{G(\eta_s - \eta_s^{(N)}), G(\eta_s^{(N)} - \eta_s)\} \right) ds \leq \frac{1}{2} \mathbb{E}^G \left( \int_0^T \sup_{\sigma \in \Sigma} |(\eta_s - \eta_s^{(N)}, \sigma)| \right) ds \to 0 \text{ as } N \to \infty,$$

which yields that for any $t \in [0, T]$,

$$\lim_{N \to \infty} \mathbb{E}^G \left( \left| M_t - \left( 2 \int_0^t G(\eta_s^{(N)}) ds - \int_0^t \eta_s^{(N)} d\langle B \rangle_s \right) \right| \right) = 0.$$

Thus, it is sufficient to consider the case $\eta^{ij} \in M^2_{G^2}(0, T)$, $\eta_s^{ij} = \eta_s^{ji}$, $i, j = 1, \ldots, d$, i.e.,

$$\eta_s = \sum_{k=1}^N \eta_{t_{k-1}} I_{[t_{k-1}, t_k]}(s), \ 0 = t_0 < t_1 < \cdots < t_N = T.$$
By Jensen’s inequality, we have

\begin{align*}
M_t &= \sum_{k=1}^{N} (2G(\eta_{k-1}(t_k - t_{k-1})) - (\eta_{k-1}, \langle B \rangle_{t_k} - \langle B \rangle_{t_{k-1}})) \\
&= \sum_{k=1}^{N} \left( \sup_{\sigma \in \Sigma} (\eta_{k-1}(t_k - t_{k-1}), \sigma) - (\eta_{k-1}, \langle B \rangle_{t_k} - \langle B \rangle_{t_{k-1}}) \right) \\
&= \sum_{k=1}^{N} (t_k - t_{k-1}) \left( \sup_{\sigma \in \Sigma} (\eta_{k-1}, \sigma) - \left( \eta_{k-1}, \frac{\langle B \rangle_{t_k} - \langle B \rangle_{t_{k-1}}}{t_k - t_{k-1}} \right) \right) \geq 0.
\end{align*}

\[ \square \]

**Proof.** The proof of the lower bound of Theorem 3.1

**Step 1. Bounded case.** Let \( \eta \) be bounded. Then by the G-Girsanov-formula,

\[ \log \mathbb{E}^G(e^{\Phi(B)}) = \mathbb{E}^{G, \eta}(e^{\Phi(B')}) = \mathbb{E}^G \left( e^{\Phi(B')} \exp \left\{ - \int_0^T \eta_s dB_s - H^G_T(\eta) \right\} \right). \]

By Jensen’s inequality, we have

\[ \log \mathbb{E}^G(e^{\Phi(B)}) \geq \mathbb{E}^G \left( \Phi(B') - \int_0^T \eta_s dB_s - H^G_T(\eta) \right) = \mathbb{E}^G \left( \Phi(B') - H^G_T(\eta) \right). \]

**Step 2. General case.** Choose a sequence \( \{\Phi_n, n \geq 1\} \) of uniformly bounded and Lipschitz continuous functions such that

\[ \lim_{n \to \infty} \mathbb{E}^G(|\Phi_n - \Phi|^2) = 0. \]

For \( \eta \in (M^2_G(0, T))^d \) given, we can find a sequence \( \{\eta^{(m)}, m \geq 1\} \subset (M^2_G(0, T))^d \cap \mathcal{B}_b(\Omega_T) \) such that

\[ \lim_{m \to \infty} \mathbb{E}^G \left( \int_0^T |\eta^{(m)}_t - \eta_t|^2 dt \right) = 0. \]

Then

\[ \lim_{m \to \infty} \mathbb{E}^G \left( |H^G_T(\eta^{(m)}) - H^G_T(\eta)| \right) = 0, \]

and for each \( n \geq 1 \), by Lipschitz continuity of \( \Phi_n \), we also have

\[ \lim_{m \to \infty} \mathbb{E}^G \left( |\Phi_n(B') - \Phi_n(B'^{(m)})| \right) \leq l(n, \sigma) \lim_{m \to \infty} \mathbb{E}^G \left( \int_0^T |\eta^{(m)}_t - \eta_t| dt \right) = 0 \]

where \( l(n, \sigma) \) is a constant independent of \( m \). Therefore, in order to prove \( \mathbb{E}^G(e^{\Phi(B)}) \geq \mathbb{E}^G \left( \Phi(B') - H^G_T(\eta) \right) \), it only remains to verify

\[ \lim_{n \to \infty} \sup_{m \geq 1} \mathbb{E}^G \left( |\Phi_n(B'^{(m)}) - \Phi(B'^{(m)})| \right) = 0. \quad (3.3) \]
Fix $\epsilon > 0$. Let $M \in (0, \infty)$ be an uniform upper bound $|\Phi_n|, |\Phi|, n \geq 1$. Then

$$
\mathbb{E}^G \left( \left| \Phi_n \left( B_{\eta(n)}^m \right) - \Phi \left( B_{\eta(n)}^m \right) \right| \right)
= \sup_{\theta \in \mathcal{A}_{0,\infty}} E_{P_\theta} \left( \left| \Phi_n \left( B_{\eta(n)}^m \right) - \Phi \left( B_{\eta(n)}^m \right) \right| \right)
\leq 2M \sup_{\theta \in \mathcal{A}_{0,\infty}} P_\theta \left( \left| \Phi_n \left( B_{\eta(n)}^m \right) - \Phi \left( B_{\eta(n)}^m \right) \right| > \epsilon \right) + \epsilon.
$$

Therefore, we only need to show that for any $\epsilon > 0$,

$$
\lim_{n \to \infty} \sup_{m \geq 1} \sup_{\theta \in \mathcal{A}_{0,\infty}} P_\theta \left( \left| \Phi_n \left( B_{\eta(n)}^m \right) - \Phi \left( B_{\eta(n)}^m \right) \right| > \epsilon \right) = 0. \quad (3.4)
$$

For any $N \in (1, \infty)$,

$$
\sup_{m \geq 1} \sup_{\theta \in \mathcal{A}_{0,\infty}} P_\theta \left( \left| \Phi_n \left( B_{\eta(n)}^m \right) - \Phi \left( B_{\eta(n)}^m \right) \right| > \epsilon \right)
= \sup_{\theta \in \mathcal{A}_{0,\infty}} P_\theta \left( I_{\left\{ \left| \Phi_n \left( B_{\eta(n)}^m \right) - \Phi \left( B_{\eta(n)}^m \right) \right| \geq \epsilon \right\}} \mathcal{E}_{T}^{-\eta(n)} \left( \mathcal{E}_{T}^{-\eta(n)} \right)^{-1} \right)
\leq N \sup_{\theta \in \mathcal{A}_{0,\infty}} E_{P_\theta} \left( I_{\left\{ \left| \Phi_n \left( B_{\eta(n)}^m \right) - \Phi \left( B_{\eta(n)}^m \right) \right| \geq \epsilon \right\}} \mathcal{E}_{T}^{-\eta(n)} \right)
+ \sup_{\theta \in \mathcal{A}_{0,\infty}} E_{P_\theta} \left( I_{\left\{ \left| \Phi_n \left( B_{\eta(n)}^m \right) - \Phi \left( B_{\eta(n)}^m \right) \right| \geq \epsilon \right\}} I_{\{ \mathcal{E}_{T}^{-\eta(n)} \leq 1/N \}} \right).
$$

By Chebyshev’s inequality and the G-Girsanov-formula, for all $m \geq 1$,

$$
\sup_{\theta \in \mathcal{A}_{0,\infty}} E_{P_\theta} \left( I_{\left\{ \left| \Phi_n \left( B_{\eta(n)}^m \right) - \Phi \left( B_{\eta(n)}^m \right) \right| \geq \epsilon \right\}} \mathcal{E}_{T}^{-\eta(n)} \right)
\leq \frac{1}{\epsilon^2} \sup_{\theta \in \mathcal{A}_{0,\infty}} E_{P_\theta} \left( \left| \Phi_n \left( B_{\eta(n)}^m \right) - \Phi \left( B_{\eta(n)}^m \right) \right|^2 \mathcal{E}_{T}^{-\eta(n)} \right)
\leq \frac{1}{\epsilon^2} \mathbb{E}^G(|\Phi_n - \Phi|^2) \to 0 \text{ as } n \to \infty.
$$

We also have that

$$
\sup_{m \geq 1} \sup_{\theta \in \mathcal{A}_{0,\infty}} E_{P_\theta} \left( I_{\left\{ \left| \Phi_n \left( B_{\eta(n)}^m \right) - \Phi \left( B_{\eta(n)}^m \right) \right| \geq \epsilon \right\}} I_{\{ \mathcal{E}_{T}^{-\eta(n)} \leq 1/N \}} \right)
\leq \frac{1}{\log N} \sup_{m \geq 1} \sup_{\theta \in \mathcal{A}_{0,\infty}} E_{P_\theta} \left( - \log \mathcal{E}_{T}^{-\eta(n)} \right) = \frac{1}{\log N} \sup_{m \geq 1} \mathbb{E}^G \left( H_T^G(\eta(n)) \right) \to 0 \text{ as } N \to \infty.
$$
(3.4) is valid. Hence,
\[
E^G(e^{\Phi(B)}) \geq \exp \left\{ \sup_{\eta \in (M_2^d(0,T))^d} E^G(\Phi(B^n) - H_T^G(\eta)) \right\}.
\] 

The proof of the upper bound Theorem 3.1.
First, let \( \Phi = f(B_{t_1}, \ldots, B_{t_n}) \), where \( f \in \text{lip}(\mathbb{R}^d) \) and \( 0 \leq t_1 < \cdots < t_n = T \). Set
\[
\|f\| := \sup_{y \in \mathbb{R}^d} |f(y)|, \quad \|f\|_{\text{lip}} := \sup_{y, z \in \mathbb{R}^d, y \neq z} \frac{|f(y) - f(z)|}{|y - z|}.
\]
We want to show that there exists a constant \( C(\|f\|, \|f\|_{\text{lip}}) \in (0, \infty) \) that is only dependent on \( \|f\| \) and \( \|f\|_{\text{lip}} \), such that
\[
E^G(e^{\Phi(B)}) \leq \exp \left\{ \sup_{\eta \in (M_2^d(0,T))^d, |\eta| \leq \|f\|_{\text{lip}}} E^G(\Phi(B^n) - H_T^G(\eta)) \right\}.
\] 

Set \( \phi(x_1, \ldots, x_n) = e^{f(x_1, \ldots, x_n)} \). Then \( e^{-\|f\|} \leq \|\phi\| \leq e^{\|f\|} \), and there exists a constant \( C_1(\|f\|, \|f\|_{\text{lip}}) \in (0, \infty) \) that is only dependent on \( \|f\| \) and \( \|f\|_{\text{lip}} \), such that
\[
\|\phi\|_{\text{lip}} \leq e^{\|f\|} \sup_{y, z \in \mathbb{R}^d, y \neq z} \frac{|f(y) - f(z)| - 1}{|y - z|} \leq e^{\|f\|} \sup_{\lambda > 0} \frac{e^{\lambda \|f\|_{\text{lip}}} - 1}{\lambda} \leq C_1(\|f\|, \|f\|_{\text{lip}})
\]
For \( t_{n-1} \leq t \leq t_n \), set
\[
v_n(t, x_1, \ldots, x_{n-1}, x) = E^G(\phi(x_1, \ldots, x_{n-1}, x + B_{t_n} - B_t)) = E^G(\phi(x_1, \ldots, x_{n-1}, x + \sqrt{t_n - t}B_1)),
\]
and for any \( l = n-1, \ldots, 1, t \in [t_{l-1}, t_l] \), define
\[
v_l(t, x_1, \ldots, x_l, x) = E^G(v_{l+1}(t, x_1, \ldots, x_{l-1}, x + B_{t_l-n} - B_t, x + B_{t_l} - B_t))
= E^G(v_{l+1}(t, x_1, \ldots, x_{l-1}, x + \sqrt{t_l-n}B_1, x + \sqrt{t_l-n}B_1))
\]
Then by the definition of G-Brownian motion, \( v_l : [t_{l-1}, t_l] \times \mathbb{R}^d \rightarrow \mathbb{R}, l = 1, \ldots, n \) are the solutions of equations:
\[
\begin{cases}
\frac{\partial}{\partial t} v_l(t, x_1, \ldots, x_{l-1}, x) + G(D_x^2 v_l(t, x_1, \ldots, x_{l-1}, x)) = 0, & t \in [t_{l-1}, t_l),
\v_l(t, x_1, \ldots, x_{l-1}, x) = v_{l+1}(t, x_1, \ldots, x_{l-1}, x, x),
\v_n(t, x_1, \ldots, x_{n-1}, x) = \phi(x_1, \ldots, x_{n-1}, x).
\end{cases}
\] 

Since \( \phi \) is bounded, by the regularity result of Krylov (Theorem 6.4.3 in [17]), and Section 4 in Appendix C of [22], there exists a constant \( \alpha \in (0, 1) \) only depending on \( G, \varphi, d, \text{lip} \) and \( \|\phi\| \) such that for each \( l = 1, \ldots, n \), and for any \( \kappa \in (0, t_{l+1} - t_l) \),
\[
\sup_{(x_1, \ldots, x_l) \in \mathbb{R}^d} \|v_l(\cdot, x_1, \ldots, x_{l-1}, \cdot)\|_{C^{1+\alpha/2,2+\alpha}([t_l, t_{l+1} - \kappa] \times \mathbb{R}^d)} < \infty,
\]
where for given real function $u$ defined on $Q = [T_1, T_2] \times \mathbb{R}^d$, and given constants $\alpha, \beta \in (0, 1)$,

$$
\|u\|_{C^{\alpha, \beta}(Q)} = \sup_{x,y \in \mathbb{R}^d, x \neq y, s,t \in [T_1, T_2], s \neq t} \frac{|u(s, x) - u(t, y)|}{|r - s|^\alpha + |x - y|^\beta},
$$

$$
\|u\|_{C^{1+\alpha, 2+\beta}(Q)} = \|u\|_{C^{\alpha, \beta}(Q)} + \|\partial_t u\|_{C^{\alpha, \beta}(Q)} + \sum_{i=1}^d \|\partial_{x_i} u\|_{C^{\alpha, \beta}(Q)}
$$

$$
+ \sum_{i,j=1}^d \|\partial^2_{x_i x_j} u\|_{C^{\alpha, \beta}(Q)}.
$$

By the subadditivity of $\mathbb{E}^G$, for all $1 \leq l \leq n$, $(t, x_1, \ldots, x_{l-1}) \in [t_{l-1}, t_l] \times (\mathbb{R}^d)^{l-1},$

$$
\sup_{x,y \in \mathbb{R}^d, x \neq y} \frac{|v_l(t, x_1, \ldots, x_{l-1}, x) - v_l(t, x_1, \ldots, x_{l-1}, y)|}{|x - y|} \leq C_1(\|f\|, \|f\|_{\text{lip}}),
$$

and for all $1 \leq l \leq n$, $(x_1, \ldots, x_{l-1}, x) \in (\mathbb{R}^d)^l$,

$$
|v_l(t, x_1, \ldots, x_{l-1}, x) - v_l(s, x_1, \ldots, x_{l-1}, x)| \leq \bar{\sigma} C_1(\|f\|, \|f\|_{\text{lip}})|t - s|^{1/2}.
$$

Therefore, $x \to \nabla_x v_l(t, x_1, \ldots, x_{l-1}, x)$, $(t, x_1, \ldots, x_{l-1}) \in [t_{l-1}, t_l] \times (\mathbb{R}^d)^{l-1}$ are uniformly bounded.

Set $V_l(t, x_1, \ldots, x_{l-1}, x) = \log v_l(t, x_1, \ldots, x_{l-1}, x)$. Then for $(t, x) \in [t_{l-1}, t_l] \times \mathbb{R}^d$,

$$
\frac{\partial V_l}{\partial t} = -G \left( \frac{D_x^2 v_l}{v_l} \right), \ \nabla_x V_l = \frac{\nabla_x v_l}{v_l},
$$

and

$$
D_x^2 V_l = -(\partial_{x_i} V_l \partial_{x_j} V_l)_{i,j=1}^d + \frac{D_x^2 v_l}{v_l}.
$$

Therefore, $\mathbb{R}^d \ni x \to \nabla_x V_l(t, x_1, \ldots, x_{l-1}, x)$, $(t, x_1, \ldots, x_{l-1}, x) \in [t_{l-1}, t_l] \times (\mathbb{R}^d)^{l-1}$ are uniformly bounded, i.e., there exists a constant $C_2(\|f\|, \|f\|_{\text{lip}}) \in (0, \infty)$ that is only dependent on $\|f\|$ and $\|f\|_{\text{lip}}$, such that for all $1 \leq l \leq n$, $(t, x_1, \ldots, x_{l-1}, x) \in [t_{l-1}, t_l] \times (\mathbb{R}^d)^l$,

$$
|\nabla_x V_l(t, x_1, \ldots, x_{l-1}, x)| \leq C_2(\|f\|, \|f\|_{\text{lip}})
$$

and for any $\kappa \in (0, t_l - t_{l-1})$, $\mathbb{R}^d \ni x \to \nabla_x V_l(t, x_1, \ldots, x_{l-1}, x)$, $(t, x_1, \ldots, x_{l-1}) \in [t_{l-1}, t_l - \kappa] \times (\mathbb{R}^d)^{l-1}$ are uniformly Lipschitz continuous. Define

$$
U_l(t, x_1, \ldots, x_{l-1}, x) = \nabla_x V_l(t, x_1, \ldots, x_{l-1}, x) I_{[t_{l-1}, t_l)}(t).
$$

By the Picard iterative approach (cf. [14]), for any $\kappa \in (0, t_1)$, the stochastic differential equation

$$
\begin{cases}
  dX^{(1)}_t = U_1(t, X^{(1)}_t) dB_t + dB_t, & t \in [0, t_1 - \kappa], \\
  X^{(1)}_0 = 0,
\end{cases}
$$
has a unique continuous solution \( \{X_t^{(1)}, t \in [0, t_1 - \kappa] \} \). From the arbitrariness of \( \kappa \), and noting that for any \( p \geq 1 \),
\[
\mathbb{E}^G \left( |X_t^{(1)} - X_s^{(1)}|^p \right) \leq 2^p \left( \|U_1\|^p |t - s|^p + \tilde{\sigma}^{p/2}|t - s|^{p/2} \right)
\]
there exists a unique continuous process \( \{X_t^{(1)}, t \in [0, t_1] \} \) such that
\[
dX_t^{(1)} = U_1(t, X_t^{(1)})d\langle B \rangle_t + dB_t, \quad t \in [0, t_1],
\]
\[
X_0^{(1)} = 0.
\]

Recursively, for any \( 1 \leq l \leq n \), there exists a unique continuous process \( \{X_t^{(l)}, t \in [t_{l-1}, t_l] \} \), such that
\[
dX_t^{(l)} = U_l(t, X_t^{(l)}, \ldots, X_{t_{l-1}}^{(l-1)}, X_t^{(l-1)})d\langle B \rangle_t + dB_t, \quad t \in [t_{l-1}, t_l],
\]
\[
X_{t_{l-1}}^{(l)} = X_{t_{l-1}}^{(l-1)}.
\]

Define
\[
\tilde{\eta}_t = U_l(t, X_t^{(1)}, \ldots, X_{t_{l-1}}^{(l-1)}, X_t^{(l)}), \quad t \in [t_{l-1}, t_l], \quad l = 1, \ldots, n;
\]
and for any \( l = 1, \ldots, n \), and \( t \in [t_{l-1}, t_l] \),
\[
K_t^{(l)} = \int_{t_{l-1}}^t \left( G \left( \frac{D^2_x v_l}{v_l} \right)(t, X_{t_{l-1}}^{(1)}, \ldots, X_{t_{l-1}}^{(l-1)}, X_s^{(l)})ds + \frac{1}{2} D^2_x v_l(t, X_{t_{l-1}}^{(1)}, \ldots, X_{t_{l-1}}^{(l-1)}, X_s^{(l)})d\langle B \rangle_s \right).
\]

Then, by Proposition 1.4 in [22], \( \mathbb{E}^G(-K_t^{(l)}) = 0 \) for any \( t \in [t_{l-1}, t_l] \), and by Lemma 3.1 for any \( t \in [t_{l-1}, t_l] \), \( K_t^{(l)}(\omega) \geq 0 \) and \( t \to K_t^{(l)} \) is increasing for q.s. \( \omega \). Set
\[
K_t^{(l)} = \lim_{tt_{l-1}} K_t^{(l)}.
\]

By Itô formula for \( G \)-Brownian motion (cf. [14]), for any \( t \in [t_{l-1}, t_l] \),
\[
V_t(t, X_{t_{l-1}}^{(1)}, \ldots, X_{t_{l-1}}^{(l-1)}, X_t^{(l)}) - V_t(t_{l-1}, X_{t_{l-1}}^{(1)}, \ldots, X_{t_{l-1}}^{(l-1)}, X_t^{(l)})
\]
\[
= \int_{t_{l-1}}^t \frac{\partial V_t}{\partial t}(t, X_{t_{l-1}}^{(1)}, \ldots, X_{t_{l-1}}^{(l-1)}, X_s^{(l)})ds + \int_{t_{l-1}}^t \nabla_x V_t(t, X_{t_{l-1}}^{(1)}, \ldots, X_{t_{l-1}}^{(l-1)}, X_s^{(l)})dX_s^{(l)}
\]
\[
+ \frac{1}{2} \int_{t_{l-1}}^t D^2_x V_t(t, X_{t_{l-1}}^{(1)}, \ldots, X_{t_{l-1}}^{(l-1)}, X_s^{(l)})d\langle B \rangle_s
\]
\[
= -K_t^{(l)} + H_t^{G}(\tilde{\eta}) - H^{G}_{t_{l-1}}(\tilde{\eta}) + \int_{t_{l-1}}^t \tilde{\eta}_s dB_s.
\]

Since \( \mathbb{R}^d \ni x \to V_t(t, x_1, \ldots, x_{l-1}, x) \), \( (t, x_1, \ldots, x_{l-1}) \in [t_{l-1}, t_l] \times (\mathbb{R}^d)^{l-1} \) are uniformly Lipschitz continuous, and \( \{t_{l-1}, t_l \} \ni t \to V_t(t, x_1, \ldots, x_{l-1}, x) \), \( (x_1, \ldots, x_{l-1}, x) \in (\mathbb{R}^d)^l \) are 1/2-uniformly Hölder continuous, we have that
\[
\lim_{t \downarrow t_{l-1}} \mathbb{E}^G \left( |V_t(t, X_{t_{l-1}}^{(1)}, \ldots, X_{t_{l-1}}^{(l-1)}, X_t^{(l)}) - V_t(t, X_{t_{l-1}}^{(1)}, \ldots, X_{t_{l-1}}^{(l-1)}, X_t^{(l)})| \right) = 0.
\]
Hence, $K_{t_l} \in L^1_c(\Omega_T)$, and $\lim_{t\to t_l} \mathbb{E}^G \left( |K_{t_l}^{(t)} - K_{t_l}^{(l)}| \right) = 0$, $\mathbb{E}^G \left( -K_{t_l}^{(t)} \right) = 0$, and

$$V_t(t, X_{t_1}^{(1)}, \ldots, X_{t_{l-1}}^{(l-1)}, X_{t_l}^{(l)}) - H_{t_l}^G(\tilde{\eta}) = V_{t-1}(t, X_{t_1}^{(1)}, \ldots, X_{t_{l-1}}^{(l-1)}) - H_{t_{l-1}}^G(\tilde{\eta}) - K_{t_l}^{(t)} + \int_{t_{l-1}}^{t} \tilde{\eta}_s dB_s,$$

which yields that

$$\Phi(B^\tilde{\eta}) - H_T^G(\tilde{\eta}) = V_n(t, X_{t_1}^{(n)}, \ldots, X_{t_n}^{(n)}) - H_T^G(\tilde{\eta}) = V_1(0, 0) - \sum_{l=1}^{n} K_{t_l}^{(l)} + \int_{0}^{T} \tilde{\eta}_s dB_s.$$

Therefore,

$$\Phi(B^\tilde{\eta}) - H_T^G(\tilde{\eta}) - \int_{0}^{T} \tilde{\eta}_s dB_s = V_1(0, 0) - \sum_{l=1}^{n} K_{t_l}^{(l)}$$

and

$$\mathbb{E}^G \left( \Phi(B^\tilde{\eta}) - H_T^G(\tilde{\eta}) \right) = V_1(0, 0).$$

Since $\sum_{l=1}^{n} K_{t_l}^{(l)} \geq 0$, q.s., we obtain that

$$\mathbb{E}^G \left( e^{\Phi(B)} \right) = \mathbb{E}^G \left( \exp \left\{ \Phi(B^\tilde{\eta}) - \int_{0}^{T} \tilde{\eta}_s dB_s - H_T^G(\tilde{\eta}) \right\} \right)$$

$$= e^{V_1(0, 0)} \mathbb{E}^G \left( e^{-\sum_{l=1}^{n} K_{t_l}^{(l)}} \right)$$

$$\leq \exp \left\{ \mathbb{E}^G \left( \Phi(B^\tilde{\eta}) - H_T^G(\tilde{\eta}) \right) \right\}.$$

Now, for $m, N \geq 2/\min_{1 \leq i \leq n}(t_l - t_{l-1})$, define

$$\eta_{s_{\min}}^{(m, N)} = \sum_{l=1}^{n} \sum_{k=1}^{N} \tilde{\eta}_{t_{l-1} + (k-1)(t_l - t_{l-1} - 1/m)/N} I_{\left[ t_{l-1} + (k-1)(t_l - t_{l-1} - 1/m)/N, t_{l-1} + k(t_l - t_{l-1} - 1/m)/N \right]}(s).$$

Then $\eta_s^{(m, N)} \in M^2_G(0, T)$, $|\eta_s^{(m, N)}| \leq C_2(\|f\|, \|f\|_{lip})$, and

$$\lim_{m \to \infty} \limsup_{N \to \infty} \mathbb{E}^G \left( \int_{0}^{T} \left| \eta_s^{(m, N)} - \tilde{\eta}_s \right|^2 \right) = 0.$$

Therefore,

$$\mathbb{E}^G \left( \Phi(B^\tilde{\eta}) - H_T^G(\tilde{\eta}) \right) \leq \sup_{\eta \in (M^2_G(0, T))^{d}, |\eta| \leq C_2(\|f\|, \|f\|_{lip})} \mathbb{E}^G \left( \Phi(B^\eta) - H_T^G(\eta) \right),$$

and so (3.6) holds.

For general bounded function $\Phi \in L^1_c(\Omega_T)$, choose a sequence $\{\Phi_n, n \geq 1\} \subset L_{lip}(\Omega_T)$ of uniformly bounded and Lipschitz continuous functions such that

$$\lim_{n \to \infty} \mathbb{E}^G(|\Phi_n - \Phi|^2) = 0.$$
Then
\[ \lim_{n \to \infty} \mathbb{E}^G(|e^{\Phi_n} - e^\Phi|) = 0. \]

By the above proof, there exists a sequence of positive constants \( C_n \) such that
\[ \log \mathbb{E}^G(\exp\{\Phi_n\}) \leq \sup_{\eta \in \mathbb{D}} \mathbb{E}^G(\Phi_n(B^n) - H_T^G(\eta)). \]

Set \( \mathbb{D} = \bigcup_{n \geq 1} \{ \eta \in (M^{2,0}_G(0,T))^d; |\eta| \leq C_n \} \). Then
\[ \log \mathbb{E}^G(\exp\{\Phi_n\}) \leq \sup_{\eta \in \mathbb{D}} \mathbb{E}^G(\Phi_n(B^n) - H_T^G(\eta)). \]

Since
\[ \left| \sup_{\eta \in \mathbb{D}} \mathbb{E}^G(\Phi_n(B^n) - H_T^G(\eta)) - \sup_{\eta \in \mathbb{D}} \mathbb{E}^G(\Phi(B^n) - H_T^G(\eta)) \right| \leq \sup_{\eta \in \mathbb{D}} \mathbb{E}^G(|\Phi_n(B^n) - \Phi(B^n)|), \]

and the same proof as (3.3) yields that
\[ \lim_{n \to \infty} \sup_{\eta \in \mathbb{D}} \mathbb{E}^G(|\Phi_n(B^n) - \Phi(B^n)|) = 0, \]
we obtain
\[ \mathbb{E}^G(e^\Phi) \leq \exp\left\{ \sup_{\eta \in \mathbb{D}} \mathbb{E}^G(\Phi(B^n) - H_T^G(\eta)) \right\} \quad (3.10) \]
which with together (3.3) yields the conclusion of Theorem 3.1.

4. AN ABSTRACT LARGE DEVIATION PRINCIPLE FOR FUNCTIONALS OF 
   G-BROWNIAN MOTION

In this section, we apply the variation representation to study the large deviations for functionals of G-Brownian motion. The inverse of the Varadhan Lemma under a G-expectation is presented. An abstract large deviation principle for functionals of G-Brownian motion is obtained.

Let \((\mathcal{Y}, \rho)\) be a Polish space and let \(\Psi^\epsilon: \Omega_T \times \mathcal{A} \to \mathcal{Y}\) be a map. Set
\[ Z^\epsilon := \Psi^\epsilon(\sqrt{\epsilon}B, \langle B \rangle). \]

Define
\[ \mathbb{H} = \left\{ f(\cdot) = \int_0^T f'(s)ds; f' \in L^2([0, T], \mathbb{R}^d) \right\}, \quad ||f||_H = ||f'||_{L^2}; \quad (4.1) \]
\[ \mathbb{G} = \left\{ g(\cdot) = \int_0^T g'(s)ds; g' \in L^2([0, T], \mathbb{R}^{d	imes d}) \right\}, \quad ||g||_G = \int_0^T ||g'(t)||_{HS}dt; \quad (4.2) \]
and
\[ \mathcal{A} = \left\{ g \in \mathbb{G}; g'(s) \in \Sigma \text{ for any } s \in [0, T] \right\}. \quad (4.3) \]
Then \((A, \| \cdot \|_G)\) is a closed convex subset of \(G\). We also denote
\[
\mathbb{H}_s = \{ f \in \mathbb{H}; f'(t) = \theta_1 I_{[0,t_1]}(t) + \sum_{i=2}^m \theta_i I_{(t_{i-1},t_i]}(t), 0 < t_1 < \cdots < t_n = T, \theta_i \in \mathbb{R}^d \}
\]
and
\[
\|g\| := \sup_{t \in [0,T]} \|g(t)\|_{HS}, \quad g \in A; \quad \|f\| := \sup_{t \in [0,T]} |f(t)|, \quad f \in \mathbb{H},
\]
where \(\|A\|_{HS} := \sqrt{\sum_{i,j} a_{ij}^2}\) is the Hilbert-Schmidt norm of a matrix \(A = (a_{ij})\).

Define
\[
\rho_{HG}((f_1, g_1), (f_2, g_2)) = \|f_1 - f_2\| + \|g_1 - g_2\|, \quad (f_1, g_1), (f_2, g_2) \in \mathbb{H} \times A.
\]

We introduce the following Assumption \((A)\):

\((A0)\). For any \(\Phi \in C_b(Y), \Phi(Z^\tau)\) is quasi-continuous;

There exists a map \(\Psi : \mathbb{H} \times A \to Y\) such that the following conditions \((A1), (A2)\) and \((A3)\) hold:

\((A1)\). For each \(N > 1\), if \(f_n, g_n \in \mathbb{H}\) and \(g \in A\) satisfy that \(\|f_n\|_H \leq N, \|f\|_H \leq N, \|f_n - f\|_H \to 0\) and \(\|g_n - g\|_G \to 0\), then
\[
\Psi(f_n, g_n) \to \Psi(f, g);
\]

\((A2)\). For \(\Phi \in C_b(Y)\), for each \(r > 0\),
\[
\lim_{\varepsilon \to 0} \sup_{\eta \in (A^2_{0,(0,T)} \cap \mathbb{H} \times A)} \mathbb{E}^G \left( \left\| \Phi \circ \Psi^\varepsilon \left( \sqrt{\varepsilon} B + \int_0^T \eta_s d\langle B \rangle_s, \langle B \rangle \right) - \Phi \circ \Psi \left( \int_0^T \eta_s d\langle B \rangle_s, \langle B \rangle \right) \right\| \right) = 0;
\]

\((A3)\). There exists a sequence of continuous maps \(\Psi^{(N)} : (\mathbb{H} \times A, \rho_{HG}) \to (Y, \rho)\) such that for each \(l \in (0, \infty),\)
\[
\lim_{N \to \infty} \sup_{\|f\|_H \leq l, g \in A} \rho(\Psi(f, g), \Psi^{(N)}(f, g)) = 0.
\]

**Remark 4.1.** Assumption \((A)\) is slightly different from the classical case(cf. [3]).

We have an additional condition \((A0)\). In the classical case, the condition \((A0)\) is always true.

Let \(I : Y \to [0, \infty)\) be defined by
\[
I(y) = \inf_{(f,g) \in \mathbb{H} \times A} \left\{ \frac{1}{2} \int_0^T (f'(s), g'(s)f'(s)) ds; \quad y = \Psi \left( \int_0^T g'(s)f'(s) ds, g \right) \right\}. \quad (4.4)
\]

Since \(\varphi(t) = \int_0^t g'(s)f'(s) ds, t \in [0,T]\) yields that \(f'(s) = g'(s)^{-1} \varphi'(s)\), it is easy to get the following representation of \(I(y)\):
\[
I(y) = \inf_{(f,g) \in \mathbb{H} \times A} \left\{ J(f, g), \quad y = \Psi(f, g) \right\}, \quad y \in Y, \quad (4.5)
\]
where
\[
J(f, g) = \begin{cases} 
\frac{1}{2} \int_0^T (f'(s), (g'(s))^{-1}f'(s))ds, & (f, g) \in H \times \mathbb{A}, \\
+\infty, & \text{otherwise}.
\end{cases}
\] (4.6)

Lemma 4.1. (1). Let \( \Upsilon : (\mathbb{A}, \| \cdot \|) \to \mathbb{R} \) be a bounded continuous function. Then
\[
E^G(\Upsilon(B)) = \sup_{g \in \mathbb{A}} \Upsilon(g).
\] (4.7)

(2). Let \( \Phi \in C_0(\mathcal{Y}) \), and let \( \Psi : H \times \mathbb{A} \to \mathcal{Y} \) satisfy (A1) and (A3). Then for each function \( f \in H \),
\[
E^G(\Phi \circ \Psi \left( \int_0^T f'(s)d\langle B \rangle_s, \langle B \rangle \right) - H_T^G(f)) = \sup_{g \in \mathbb{A}} \left( \Phi \circ \Psi \left( \int_0^T g'(s)f'(s)ds, g \right) - \frac{1}{2} \int_0^T (f'(s), g'(s)f'(s))ds \right).\] (4.8)

Proof. (1). Firstly, let us show (4.7) for
\[
\Upsilon(g) = \psi(g_{t_1}, g_{t_2} - g_{t_1}, \ldots, g_{t_m} - g_{t_{m-1}})
\]
where \( \psi \) is bounded continuous in \((\mathbb{R}^d \times \mathbb{R})^m\), and \( 0 < t_1 < t_2 < \cdots < t_m \leq T \). Since \( \langle B \rangle_t - \langle B \rangle_s \) is independent of \( \Omega_s \) for any \( s < t \), by Chapter III, Theorem 5.3 in [22], we have that
\[
E^G(\psi(\langle B \rangle_{t_1}, \langle B \rangle_{t_2} - \langle B \rangle_{t_1}, \cdots, \langle B \rangle_{t_m} - \langle B \rangle_{t_{m-1}})) = \sup_{\theta \in \Sigma} \psi(\theta_{t_1}, \theta_{t_2} - \theta_{t_1}, \cdots, \theta_{t_m} - \theta_{t_{m-1}}).
\]

For any \( \theta_1, \theta_2, \cdots, \theta_m, g \in \Sigma, \) set \( g'(t) = \theta_1 I_{[0,t]} + \sum_{i=2}^m \theta_i I_{[t_{i-1}, t_i]} \). Then \( \psi(\theta_{t_1}, \theta_{t_2} - \theta_{t_1}, \cdots, \theta_{t_m} - \theta_{t_{m-1}}) = \psi(g(t_1), g(t_2) - g(t_1), \cdots, g(t_m) - g(t_{m-1})). \) Therefore,
\[
\sup_{\theta_1, \theta_2, \cdots, \theta_m, g \in \Sigma} \psi(\theta_{t_1}, \theta_{t_2} - \theta_{t_1}, \cdots, \theta_{t_m} - \theta_{t_{m-1}}) \leq \sup_{g \in \mathbb{A}} \psi(g(t_1), g(t_2) - g(t_1), \cdots, g(t_m) - g(t_{m-1})).
\]

On the other hand, for any \( g \in \mathbb{A}, \) set \( \theta_1 = g(t_1)/t_1, \) and \( \theta_i = (g(t_i) - g(t_{i-1}))/t_i - t_{i-1} \) for \( i = 2, \cdots, m. \) Then \( \theta_i \in \Sigma \) and \( g(t_i) - g(t_{i-1}) = \theta_i(t_i - t_{i-1}) \) for any \( i = 1, \cdots, m, \) where \( t_0 = 0. \) Therefore,
\[
\sup_{\theta_1, \theta_2, \cdots, \theta_m, g \in \Sigma} \psi(\theta_{t_1}, \theta_{t_2} - \theta_{t_1}, \cdots, \theta_{t_m} - \theta_{t_{m-1}}) \geq \sup_{g \in \mathbb{A}} \psi(g(t_1), g(t_2) - g(t_1), \cdots, g(t_m) - g(t_{m-1})).
\]

Therefore, (4.7) holds in this case.
Next, we assume that $\Upsilon$ is Lipschitz continuous with respect to the uniform topology, i.e., there exists a constant $l > 0$ such that

$$|\Upsilon(g) - \Upsilon(f)| \leq l \sup_{t \in [0,T]} \|g(t) - f(t)\|_{HS} \quad \text{for all} \quad g, f \in \mathbb{A}.$$ 

For each $N \geq 1$, set $t_i^N = \frac{T}{N}$, $1 \leq i \leq N$. For any $(x_1, x_2, \ldots, x_N) \in \Sigma^N$, set

$$x^{(N)}(t) := x_1(t \wedge t_1) + \sum_{i=2}^{N} x_i(t \wedge t_i - t \wedge t_{i-1}), \quad t \in [0,T],$$

and $\tilde{\psi}(x_1, x_2, \ldots, x_N) = \Upsilon\left(x^{(N)}\right)$. We can extend continuously $\tilde{\psi}(x_1, x_2, \ldots, x_N)$ to $(\mathbb{R}^{d \times d})^N$. Define

$$\psi(x_1, x_2, \ldots, x_N) := \tilde{\psi}\left(\frac{x_1}{t_1}, \frac{x_2}{t_2 - t_1}, \ldots, \frac{x_N}{t_N - t_{N-1}}\right), \quad (x_1, x_2, \ldots, x_N) \in (\mathbb{R}^{d \times d})^N.$$ 

For $g \in \mathbb{A}$, set

$$g^{(N)}(t) := \frac{g(t_1)}{t_1}(t \wedge t_1) + \sum_{i=2}^{N} \frac{(g(t_i) - g(t_{i-1}))}{t_i - t_{i-1}}(t \wedge t_i - t \wedge t_{i-1}), \quad t \in [0,T],$$

and define

$$\Upsilon^{(N)}(g) := \Upsilon(g^{(N)}) = \psi(g(t_1), g(t_2) - g(t_1), \ldots, g(t_N) - g(t_{N-1})).$$

Then

$$\mathbb{E}^G\left(\Upsilon^{(N)}\left(\langle B \rangle\right)\right) = \sup_{g \in \mathbb{A}} \Upsilon^{(N)}(g).$$

and

$$|\Upsilon(g) - \Upsilon^{(N)}(g)| \leq 2l \max_{1 \leq i \leq N} \sup_{t \in [t_{i-1}, t_i]} \|g(t) - g(t_{i-1})\|_{HS} \leq 2l\sigma T/N.$$ 

Therefore,

$$\mathbb{E}^G\left(\Upsilon\left(\langle B \rangle\right)\right) = \lim_{N \to \infty} \mathbb{E}^G\left(\Upsilon^{(N)}\left(\langle B \rangle\right)\right) = \lim_{N \to \infty} \sup_{g \in \mathbb{A}} \Upsilon^{(N)}(g) = \sup_{g \in \mathbb{A}} \Upsilon(g).$$

Now, by the proof of Lemma 3.1, Chapter VI in [22], for general bounded continuous $\Upsilon$, we can choose a sequence of Lipschitz functions $\Upsilon_N$ such that $\Upsilon_N \uparrow \Upsilon$. Therefore

$$\mathbb{E}^G\left(\Upsilon\left(\langle B \rangle\right)\right) = \sup_{\theta \in \mathbb{A}, N \geq 1} \sup_{\mathbb{P}_\theta} \mathbb{E}_{\mathbb{P}_\theta}\left(\Upsilon_N\left(\langle B \rangle\right)\right)$$

$$= \sup_{N \geq 1} \mathbb{E}^G\left(\Upsilon_N\left(\langle B \rangle\right)\right) = \sup_{N \geq 1} \sup_{g \in \mathbb{A}} \Upsilon_N(g) = \sup_{g \in \mathbb{A}} \Upsilon(g).$$

(2) Choose a sequence of simple functions $f_N^T = \theta_1^N I_{[0,t_1^N]} + \sum_{i=2}^{N} \theta_i^N I_{[t_{i-1}^N,t_i^N]}$ such that $\sup_{N \geq 1} \|f_N\|_H < \infty$ and $\int_0^T |f'(s) - f_N'(s)|^2 \, ds \to 0$ as $N \to \infty$. Then

$$\sup_{g \in \mathbb{A}} \left| \int_0^T (f'(s), g'(s)f'(s)) \, ds - \int_0^T (f_N'(s), g'(s)f_N'(s)) \, ds \right| \to 0,$$
Then and by (1),

\[ \lim_{N \to \infty} \sup_{g \in A} \left| \int_0^t g'(s)f'(s)ds - \int_0^t g'(s)f_N'(s)ds \right| = 0. \]

Let \( \Psi^{(N)} : (\mathcal{H} \times \mathcal{A}, \rho_{HG}) \to (\mathcal{Y}, \rho) \) such that for any \( l \in (0, \infty), \)

\[ \lim_{N \to \infty} \sup_{\|\varphi\|_H \leq L, g \in A} \rho(\Psi(\varphi, g), \Psi^{(N)}(\varphi, g)) = 0. \]

Define

\[ \Phi^{(N)}(g) = \Phi \circ \Psi_N \left( \int_0^t g'(s)f'(s)ds, g \right) - \frac{1}{2} \int_0^T (f_N'(s), g'(s)f_N'(s))ds \]

Then

\[ \lim_{N \to \infty} \sup_{g \in A} \left| \Phi^{(N)}(g) \right| = 0. \]

and by (1),

\[ \mathbb{E}^{\mathcal{G}}(\Phi^{(N)}([B])) = \sup_{g \in A} \Phi^{(N)}(g), \]

Therefore, (4.8) holds. \( \square \)

**Lemma 4.2.** Let (A2) hold. Then for any \( \Phi \in C_b(\mathcal{Y}) \) and each \( N \geq 1, \)

\[ \lim_{\epsilon \to 0} \sup_{\eta \in (M_{2,0}^G(0,T))^d \cap B_0(\Omega_T)} \mathbb{E}^{\mathcal{G}}\left( \left| \Phi \circ \Psi^\epsilon \left( \sqrt{\epsilon}B + \int_0^\epsilon \eta_s d\langle B \rangle_s, \langle B \rangle \right) \right. \right. \]

\[ \left. - \Phi \circ \Psi \left( \int_0^\epsilon \eta_s d\langle B \rangle_s, \langle B \rangle \right) \right| = 0; \]

**Proof.** For \( \eta \in (M_{2,0}^G(0,T))^d \cap B_0(\Omega_T), \) we can write \( \eta_s = \sum_{k=1}^n \eta_{k-1} I_{[t_{k-1},t_k)}(s). \) For \( r \in (0, \infty) \) fixed, for any \( \delta > 0, \) let \( \phi(x) \in \text{lip}(\mathbb{R}) \) satisfy \( 0 \leq \phi \leq 1, \) \( \phi(x) = 1 \) for all \( |x| \leq r \) and \( \phi(x) = 0 \) for all \( |x| \geq r + \delta. \) Define

\[ \hat{\eta}_t = \eta_t \phi \left( \int_0^t |\eta_s|^2 ds \right). \]

Then

\[ \left| \int_0^T |\hat{\eta}_s|^2 ds \right| \leq r + \delta, \quad \left\{ \int_0^T |\eta_s|^2 ds \leq r \right\} \subset \{ \hat{\eta}_s = \eta_s \text{ for any } s \in [0,T] \}. \]

Set

\[ \hat{\eta}_s^N := \sum_{k=1}^n \sum_{j=1}^N \hat{\eta}_{k-1} + (j-1)(t_k-t_{k-1})/N I_{[t_{k-1} + (j-1)(t_k-t_{k-1})/N, t_{k-1} + j(t_k-t_{k-1))/N)}(s). \]

Then

\[ \mathbb{E}^{\mathcal{G}}\left( \int_0^T |\hat{\eta}_s^N - \hat{\eta}_s| ds \right) \leq \frac{\|\phi\|_{\text{lip}}}{2} \mathbb{E}^{\mathcal{G}} \left( \sum_{k=1}^n \sum_{j=1}^N |\eta_{k-1}|^3 (t_k-t_{k-1})^2 \right) \to 0 \text{ as } N \to \infty. \]
Therefore, for any \( t \in [0, T] \),
\[
\mathbb{E}^G \left( \left| \int_0^t \hat{\eta}_s^N d\langle B \rangle_s - \int_0^t \hat{\eta}_s d\langle B \rangle_s \right| \right) \to 0 \text{ as } N \to \infty.
\]

In particular, on \( \left\{ \int_0^T |\eta_s|^2 ds \leq r \right\} \),
\[
\int_0^t \eta_s d\langle B \rangle_s = \int_0^t \hat{\eta}_s d\langle B \rangle_s, \text{ q.s.,}
\]
and
\[
\Psi^G \left( \sqrt{\epsilon} \mathcal{B} + \int_0^t \hat{\eta}_s d\langle B \rangle_s, \langle B \rangle \right) = \Psi^G \left( \sqrt{\epsilon} \mathcal{B} + \int_0^t \eta_s d\langle B \rangle_s, \langle B \rangle \right), \text{ q.s.}
\]
Therefore, for any \( \Phi \in C_b(\mathcal{Y}) \) and each \( N \geq 1 \), for all \( \eta \in (M^{2,0}_G(0, T))^d \cap \mathcal{B}_b(\Omega_T) \) with \( \int_0^T \mathbb{E}^G(|\eta_s|^2) ds \leq N \),
\[
\mathbb{E}^G \left( \left| \Phi \circ \Psi^G \left( \sqrt{\epsilon} \mathcal{B} + \int_0^t \eta_s d\langle B \rangle_s, \langle B \rangle \right) - \Phi \circ \Psi \left( \int_0^t \eta_s d\langle B \rangle_s, \langle B \rangle \right) \right| \right)
\leq \frac{2\|\Phi\|_N}{r} + \mathbb{E}^G \left( \left| \Phi \circ \Psi^G \left( \sqrt{\epsilon} \mathcal{B} + \int_0^t \eta_s d\langle B \rangle_s, \langle B \rangle \right) - \Phi \circ \Psi \left( \int_0^t \eta_s d\langle B \rangle_s, \langle B \rangle \right) \right| \left| I_{\left\{ \int_0^T |\eta_s|^2 ds \leq r \right\}} \right)
\leq \frac{2\|\Phi\|_N}{r} + \sup_{\eta \in (M^{2,0}_G(0, T))^d \cap \mathcal{B}_b(\Omega_T)} \mathbb{E}^G \left( \left| \Phi \circ \Psi^G \left( \sqrt{\epsilon} \mathcal{B} + \int_0^t \eta_s d\langle B \rangle_s, \langle B \rangle \right) - \Phi \circ \Psi \left( \int_0^t \eta_s d\langle B \rangle_s, \langle B \rangle \right) \right| \right).
\]

First, letting \( \epsilon \to 0 \), then \( r \to \infty \), by (A2), we obtain the conclusion of the lemma. 

\[ \square \]

**Theorem 4.1.** Suppose that the assumption (A) holds. Then
(1). For any \( L \in [0, \infty) \), \( C_L := \{ y ; I(y) \leq L \} \) is compact in \( \mathcal{Y} \);
(2). For any \( \Phi \in C_b(\mathcal{Y}) \),
\[
\lim_{\epsilon \to 0} \epsilon \log \mathbb{E}^G \left( \exp \left\{ \frac{\Phi(Z^\epsilon)}{\epsilon} \right\} \right) - \sup_{y \in \mathcal{Y}} \{ \Phi(y) - I(y) \} = 0. \tag{4.9}
\]

**Proof.** (1). First, we prove that \( C_L = \cap_{n \geq 1} \Gamma_{L+1/n} \), where
\[
\Gamma_{L+1/n} = \left\{ (f, g) : J(f, g) \leq L + \frac{1}{n}, (f, g) \in \mathcal{H} \times \mathcal{A} \right\}.
\]
In fact, for \( y \in C_L \) given, for each \( n \geq 1 \), choose \( f_n \in \mathcal{H}, g_n \in \mathcal{A} \) such that \( y = \Psi(f_n, g_n) \) and \( J(f_n, g_n) \leq L + \frac{1}{n} \). Since \( n \geq 1 \) is arbitrary, we have \( C_L \subseteq \cap_{n \geq 1} \Gamma_{L+1/n} \). Conversely, suppose \( y \in \cap_{n \geq 1} \Gamma_{L+1/n} \). Then, for some \( f_n \in \mathcal{H}, g_n \in \mathcal{A} \)
with \( y = \Psi(f_n, g_n) \), we have that \( J(f_n, g_n) \leq L + 1/n \). Therefore, \( I(y) \leq L + 1/n \).

Letting \( n \to \infty \), we obtain \( I(y) \leq L \). Thus \( y \in C_L \), and in turn, \( \cap_{n \geq 1} \Gamma_{L+1/n} \subseteq \Gamma_L \) follows.

(2). From Theorem 3.1 we have

\[
\epsilon \log \mathbb{E}^G \left( \exp \left\{ \frac{1}{\epsilon} \Phi(Z^\epsilon) \right\} \right) = \sup_{\eta \in (M_G^2, (0, T))^d \cap \mathcal{B}_T(\Omega_T)} \mathbb{E}^G \left( \Phi \circ \Psi^\epsilon \left( \sqrt{\epsilon} \mathcal{C}_n, \langle B \rangle \right) - H_T^G(\sqrt{\epsilon} \eta) \right) = \sup_{\eta \in (M_G^2, (0, T))^d \cap \mathcal{B}_T(\Omega_T)} \mathbb{E}^G \left( \Phi \circ \Psi^\epsilon \left( \sqrt{\epsilon} B + \int_0^T \eta_s d\langle B \rangle_s, \langle B \rangle \right) - H_T^G(\eta) \right)
\]

where \( \| \Phi \| = \sup_{y \in \mathcal{Y}} |\Phi(y)| \), and the last equality is due to that if \( \int_0^T \mathbb{E}^G(|\eta_s|^2)ds > \frac{4\| \Phi \|}{2} \), then

\[
\mathbb{E}^G \left( \Phi \circ \Psi^\epsilon \left( \sqrt{\epsilon} B + \int_0^T \eta_s d\langle B \rangle_s, \langle B \rangle \right) - H_T^G(\eta) \right) \leq -\| \Phi \|.
\]

Therefore, by (A2) and Lemma 4.2 as \( \epsilon \to 0 \),

\[
\lim_{\epsilon \to 0} \left| \epsilon \log \mathbb{E}^G \left( \exp \left\{ \Phi(Z^\epsilon) / \epsilon \right\} \right) - \sup_{\eta \in (M_G^2, (0, T))^d \cap \mathcal{B}_T(\Omega_T)} \mathbb{E}^G \left( \Phi \circ \Psi \left( \int_0^T \eta_s d\langle B \rangle_s, \langle B \rangle \right) - H_T^G(\eta) \right) \right| = 0.
\]

Since for each \( \eta \in (M_G^2, (0, T))^d \cap \mathcal{B}_T(\Omega_T) \),

\[
\Phi \circ \Psi \left( \int_0^T \eta_s d\langle B \rangle_s, \langle B \rangle \right) - H_T^G(\eta) \leq \sup_{(f, g) \in \mathcal{H} \times \mathcal{A}} \left( \Phi \circ \Psi \left( \int_0^T g'(s)f'(s)ds, g \right) - \frac{1}{2} \int_0^T (f'(s), g'(s) f'(s))ds \right)
\]

\[
= \sup_{y \in \mathcal{Y}} \left( \Phi(y) - \frac{1}{2} \int_0^T (f'(s), g'(s) f'(s))ds \right)
\]

\[
= \sup_{y \in \mathcal{Y}} \left\{ \Phi(y) - I(y) \right\}, \text{ q.s.,}
\]

we obtain the upper bound:

\[
\lim_{\epsilon \to 0} \sup \epsilon \log \mathbb{E}^G \left( \exp \left\{ \Phi(Z^\epsilon) / \epsilon \right\} \right) \leq \sup_{y \in \mathcal{Y}} \left\{ \Phi(y) - I(y) \right\}.
\]
From Theorem 3.1, we also have that
\[
\epsilon \log \mathbb{E}^G \left( \exp \left\{ \frac{1}{\epsilon} \Phi(Z^\epsilon) \right\} \right)
\geq \sup_{f \in H, \|f\| \leq \frac{4\|\Phi\|}{\epsilon}} \mathbb{E}^G \left( \Phi \circ \Psi \left( \sqrt{\epsilon} B + \int_0^T f'_s d\langle B \rangle_s, \langle B \rangle \right) - H^G_T(f) \right).
\]

Thus, by (A2), (A3) and Lemma 4.1,
\[
\liminf_{\epsilon \to 0} \epsilon \log \mathbb{E}^G \left( \exp \left\{ \frac{\Phi(Z^\epsilon)}{\epsilon} \right\} \right)
\geq \sup_{f \in H, \|f\| \leq \frac{4\|\Phi\|}{\epsilon}} \mathbb{E}^G \left( \Phi \circ \Psi \left( \int_0^T g'(s)f'(s)ds, g \right) - \frac{1}{2} \int_0^T (f'(s), g'(s)f'(s))ds \right)
= \sup_{\|f\| \leq \frac{4\|\Phi\|}{\epsilon}, g \in A} \sup_{\Phi \circ \Psi \left( \int_0^T g'(s)f'(s)ds, g \right) - \frac{1}{2} \int_0^T (f'(s), g'(s)f'(s))ds}.
\]

Then, letting \( N \to \infty \), we obtain the lower bound:
\[
\liminf_{\epsilon \to 0} \epsilon \log \mathbb{E}^G \left( \exp \left\{ \frac{\Phi(Z^\epsilon)}{\epsilon} \right\} \right) \geq \sup_{y \in \mathcal{Y}} \{ \Phi(y) - I(y) \}.
\]

Therefore, (4.9) is valid.

\[\square\]

**Theorem 4.2.** Suppose that the assumption (A) holds. Then \( \{Z^\epsilon, \epsilon > 0\} \) satisfies the large deviation principle in \( \mathcal{Y} \) with the rate function \( I(y) \), i.e., for any closed subset \( F \subset \mathcal{Y} \),
\[
\limsup_{\epsilon \to 0} \epsilon \log c^G(Z^\epsilon \in F) \leq - \inf_{y \in F} I(y),
\]
and for any open subset \( O \subset \mathcal{Y} \),
\[
\liminf_{\epsilon \to 0} \epsilon \log c^G(Z^\epsilon \in O) \geq - \inf_{y \in O} I(y).
\]

**Proof.** This is a consequence of Theorem 4.1. Its proof is the same as probability measure case. For given open set \( O \), for any \( y \in O \), choose continuous map \( \Phi : \mathcal{Y} \to [0,1] \) such that \( \Phi(y) = 1 \) and for any \( z \in O^c, \Phi(z) = 0 \). For any \( m \geq 1 \), set \( \Phi_m(z) := m(\Phi(z) - 1) \), \( z \in \mathcal{Y} \). Then
\[
\mathbb{E}^G \left( \exp \left\{ \frac{1}{\epsilon} \Phi_m(Z^\epsilon) \right\} \right) \leq e^{-m} c^G(Z^\epsilon \in O^c) + c^G(Z^\epsilon \in O).
\]
Therefore,

\[
\max \left\{ \liminf_{\epsilon \to 0} \epsilon \log e^{G(Z^\epsilon \in O)}, -m \right\} \\
\geq \liminf_{\epsilon \to 0} \epsilon \log E^G \left( \exp \left\{ \frac{1}{\epsilon} \Phi_m(Z^\epsilon) \right\} \right) = \sup_{z \in Y} \{\Phi(z) - I(z)\} \geq -I(y).
\]

Letting \( m \to +\infty \), we obtain the lower bound.

Next, let us show the upper bound. For closed set \( F \) given, for any \( y \notin F \), choose continuous map \( \Phi_y : E \to [0, 1] \) such that \( \Phi_y(y) = 1 \) and for any \( z \in F \), \( \Phi_y(z) = 0 \). For any finite set \( A \subset F^c \), set \( \Phi_A(z) = \max_{y \in A} \Phi_y(z) \). Then

\[
\limsup_{\epsilon \to 0} \epsilon \log e^{G(Z^\epsilon \in F)} \leq \inf_{A \subset F^c} \liminf_{\epsilon \to 0} \epsilon \log E^G \left( \exp \left\{ \frac{-m\Phi_A(Z^\epsilon)}{\epsilon} \right\} \right) \\
= - \sup_{A \subset F^c} \inf_{z \in Y} \{m\Phi_A(z) + I(z)\}.
\]

Without loss of generality, we assume that \( l := \sup_{A \subset F^c} \inf_{z \in Y} J_A(z) < \infty \), where \( J_A(z) = m\Phi_A(z) + I(z) \). Then \( \{z : J_A(z) \leq l\} \) is nonempty compact set for any finite \( A \). Therefore, \( \cap_{A \subset F^c} \inf_{z \in Y} \{z : J_A(z) \leq l\} \) is nonempty, and so

\[
l \geq \inf_{z \in Y} \sup_{A \subset F^c} J_A(z) = \min \left\{ m + \inf_{z \in F^c} I(z), \inf_{z \in F} I(z) \right\} \xrightarrow{m \to \infty} \inf_{z \in F} I(z),
\]

which yields (4.11).

\[\square\]

5. LARGE DEVIATIONS FOR STOCHASTIC FLOWS DRIVEN BY \( G \)-BROWNIAN MOVEMENT

The homeomorphic property with respect to initial values of the solution for stochastic differential equations driven by \( G \)-Brownian motion was obtained in [14] and the large deviations for solutions \( \{X^\epsilon(t), t \in [0, T]\} \subset C([0, T], \mathbb{R}^p) \) of small perturbation stochastic differential equations starting from \( x \) (fixed) by \( G \)-Brownian motion were studied in [15] by exponential estimates and discretization/approximation techniques. In this section, we consider large deviations for the flows \( \{X^\epsilon(t), (x, t) \in \mathbb{R}^p \times [0, T]\} \subset C(\mathbb{R}^p \times [0, T], \mathbb{R}^p) \). The quasi continuity of the flows is proved. A Kolmogorov criterion on weak convergence under \( G \)-expectations is given. A large deviation principle for the flows is established under the Lipschitz condition. In the classical framework, Large deviations for stochastic flows have been studied extensively (see [1], [2], [6], [13], [18], [23] and references therein). For general theory of large deviations and random perturbations, we refer to [8], [11], and [12].

For positive number \( p \geq 1 \) given, for each \( N \geq 1 \), \( \psi \in C(\mathbb{R}^p \times [0, T], \mathbb{R}^p) \), set

\[\|\psi\|_N = \sup_{x \in [-N, N]^p, t \in [0, T]} |\psi(x, t)|,\]
and define
\[
\rho(\psi_1, \psi_2) = \sum_{N=1}^{\infty} \frac{1}{2^N} \min\{\|\psi_1 - \psi_2\|_N, 1\}, \ \psi_1, \psi_2 \in C(\mathbb{R}^p \times [0, T], \mathbb{R}^p).
\]

Then \((C(\mathbb{R}^p \times [0, T], \mathbb{R}^p), \rho)\) is a separable metric space.

Consider the following small perturbation stochastic differential equation driven by a \(d\)-dimensional \(G\)-Brownian motion \(B\):
\[
X^\varepsilon(x, t) = x + \int_0^t b^\varepsilon(X^\varepsilon(x, s))ds + \sqrt{\varepsilon} \int_0^t \sigma^\varepsilon(X^\varepsilon(x, s))dB_s + \int_0^t h^\varepsilon(X^\varepsilon(x, s))d\langle B \rangle_s,
\]

where
\[
b^\varepsilon : \mathbb{R}^p \to \mathbb{R}^p; \quad \sigma^\varepsilon = (\sigma^\varepsilon_{i,j})_{1 \leq i \leq p, 1 \leq j \leq d} : \mathbb{R}^p \to \mathbb{R}^p \otimes \mathbb{R}^d,
\]

and
\[
h^\varepsilon = (h^\varepsilon)_{1 \leq k \leq p} = ((h^\varepsilon_{i,j})_{1 \leq i \leq p, 1 \leq j \leq d})_{1 \leq k \leq p} : \mathbb{R}^p \to (\mathbb{R}^{d \times d})^p, \ \varepsilon \geq 0
\]
satisfy the following conditions:

\((H1)\). \(b^\varepsilon, \sigma^\varepsilon\) and \(h^\varepsilon, \varepsilon \geq 0\) are uniformly Lipschitz continuous, i.e., there exists a constant \(L > 0\) such that for any \(x, y \in \mathbb{R}^p\),
\[
\max\left\{|b^\varepsilon(x) - b^\varepsilon(y)|, \|\sigma^\varepsilon(x) - \sigma^\varepsilon(y)\|_{HS}, \max_{1 \leq k \leq p} \|h^\varepsilon(x) - h^\varepsilon(y)\|_{HS}\right\} \leq L|x - y|.
\]

\((H2)\). \(b^\varepsilon, \sigma^\varepsilon\) and \(h^\varepsilon\) converge uniformly to \(b := b^0, \sigma := \sigma^0\) and \(h := h^0\) respectively, i.e.,
\[
\lim_{\varepsilon \to 0} \sup_{x \in \mathbb{R}^p} \max\left\{|b^\varepsilon(x) - b(x)|, \|\sigma^\varepsilon(x) - \sigma(x)\|_{HS}, \max_{1 \leq k \leq p} \|h^\varepsilon(x) - h(x)\|_{HS}\right\} = 0.
\]

Then by Theorem 4.1 in [14] and the Kolmogorov criterion under \(G\)-expectation (cf. Theorem 1.36, Chapter VI in [22]), the SDE (5.1) has a unique solution \(X^\varepsilon = \{X^\varepsilon(x, t), x \in \mathbb{R}^p, t \in [0, T]\} \subset C(\mathbb{R}^p \times [0, T], \mathbb{R}^p)\) and \(X^\varepsilon(x, t) \in L^2_G(\Omega_T)\) for all \((x, t) \in \mathbb{R}^p \times [0, T]\). Furthermore, there exists a map \(\Psi^\varepsilon : \Omega_T \times \mathbb{A} \to C(\mathbb{R}^p \times [0, T], \mathbb{R}^p)\) such that
\[
\Psi^\varepsilon(\sqrt{\varepsilon}B, \langle B \rangle) = X^\varepsilon.
\]

For any \((f, g) \in \mathbb{H} \times \mathbb{A}\), let \(\Psi(f, g)(x, t) \in C(\mathbb{R}^p \times [0, T], \mathbb{R}^p)\) be a unique solution of the following ordinary differential equation:
\[
\Psi(f, g)(x, t) = x + \int_0^t b(\Psi(f, g)(x, s))ds + \int_0^t \sigma(\Psi(f, g)(x, s))f(s)ds + \int_0^t h(\Psi(f, g)(x, s))dg(s),
\]

\[(5.3)\]
Theorem 5.1. Let (H1) and (H2) hold. Let \( X^\varepsilon = \{X^\varepsilon(x,t), x \in \mathbb{R}^p, t \in [0,T]\} \) be a unique solution of the SDE (5.1). Then

(1) For any \( \Phi \in C_b(C(\mathbb{R}^p \times [0,T], \mathbb{R}^p)) \),

\[
\lim_{\varepsilon \to 0} \left| \varepsilon \log \mathbb{E} \left( \exp \left( \frac{\Phi(X^\varepsilon)}{\varepsilon} \right) \right) - \sup_{\psi \in C(\mathbb{R}^p \times [0,T], \mathbb{R}^p)} \{ \Phi(\psi) - I(\psi) \} \right| = 0, \tag{5.4}
\]

where

\[
I(\psi) = \inf_{(f,g) \in \mathcal{H} \times \mathcal{A}} \{ J(f,g), \psi = \Psi(f,g) \}, \quad \psi \in C(\mathbb{R}^p \times [0,T], \mathbb{R}^p). \tag{5.5}
\]

(2) For any closed subset \( F \) in \( (C(\mathbb{R}^p \times [0,T], \mathbb{R}^p), \rho) \),

\[
\lim_{\varepsilon \to 0} \varepsilon \log e^F(\varepsilon) \leq -\inf_{\psi \in F} I(\psi), \tag{5.6}
\]

and for any open subset \( O \) in \( (C(\mathbb{R}^p \times [0,T], \mathbb{R}^p), \rho) \),

\[
\lim_{\varepsilon \to 0} \inf_{\psi \in O} \varepsilon \log e^O(\varepsilon) \geq -\inf_{\psi \in O} I(\psi). \tag{5.7}
\]

Proof. By Theorem 4.2 we only need to verify the conditions (A0), (A1), (A2) and (A3) for \( \mathcal{Y} = C(\mathbb{R}^p \times [0,T], \mathbb{R}^p) \), \( Z^\varepsilon = X^\varepsilon \) and \( \Psi \) defined by (5.3). These will be given in Lemma 5.1, Lemma 5.2, and Lemma 5.3. \( \square \)

Remark 5.1. In particular, Theorem 5.1 yields that \( \{\sqrt{\varepsilon}B_t, t \in [0,T]\}, \varepsilon > 0 \) satisfies a large deviation principle, which was first obtained in [15] by the subadditive method.

Lemma 5.1. Assume that (H1) and (H2) hold. Let \( X = \{X(x,t), x \in \mathbb{R}^p, t \in [0,T]\} \) be a unique solution of the SDE:

\[
X(x,t) = x + \int_0^t b(X(x,s))ds + \int_0^t \sigma(X(x,s))dB_s + \int_0^t h(X(x,s))d\langle B \rangle_s. \tag{5.8}
\]

Then for any \( \Phi \in C_b(C(\mathbb{R}^p \times [0,T], \mathbb{R}^p)) \), \( \Phi(X) \) is quasi-continuous.

Proof. First, we assume that \( \Phi \in C_b(C(\mathbb{R}^p \times [0,T], \mathbb{R}^p)) \) is Lipschitz continuous, i.e., there exists a constant \( l > 0 \) such that

\[ |\Phi(\psi) - \Phi(\varphi)| \leq l\rho(\psi, \varphi) \quad \text{for all} \quad \psi, \varphi \in C(\mathbb{R}^p \times [0,T], \mathbb{R}^p). \]

For any \( N \geq 1 \), for each \( \psi \in C(\mathbb{R}^p \times [0,T], \mathbb{R}^p) \), set \( \psi^{(N)}(x,t) = \psi(((-N) \vee x) \wedge N, t) \), where \( (-N) \vee x \wedge N = ((-N) \vee x_1 \wedge N, \ldots, (-N) \vee x_p \wedge N) \). For given \( N \geq 1 \), for each \( \psi = (\psi_1, \ldots, \psi_p) \in C(\mathbb{R}^p \times [0,T], \mathbb{R}^p) \), for any \( (x_1, \ldots, x_p, x_{p+1}) \in [0,1]^{p+1} \), set

\[ \tilde{\psi}_j(x_1, \ldots, x_p, x_{p+1}) = \psi^{(N)}(N(2x_1 - 1), \ldots, N(2x_p - 1), Tx_{p+1}). \]

For any \( m \geq 1 \), the Bernstein polynomial of \( \tilde{\psi}_j \) is defined by

\[ B_m(\tilde{\psi}_j)(x_1, \ldots, x_p, x_{p+1}) = \sum_{1 \leq i_1, \ldots, i_{p+1} \leq m} \tilde{\psi}_j \left( \frac{i_1}{m}, \ldots, \frac{i_p}{m}, \frac{i_{p+1}}{m} \right) \prod_{k=1}^{p+1} \binom{m}{i_k} x_k^{i_k}(1-x_k)^{m-i_k}. \]
Then, by Bernstein’s theorem (cf. Theorem 3.1 and its proof in [16]), we have that
\[
\left| \tilde{\psi}_j(x_1, \ldots, x_p, x_{p+1}) - B_m(\tilde{\psi}_j)(x_1, \ldots, x_p, x_{p+1}) \right| \\
\leq \sup_{\sum_{k=1}^{p+1} |x_k-y_k|^2 \leq 1/m} \left| \tilde{\psi}_j(x_1, \ldots, x_p, x_{p+1}) - \tilde{\psi}_j(y_1, \ldots, y_p, y_{p+1}) \right| \\
+ \frac{p + 1}{2m} \sup_{(x_1, \ldots, x_p, x_{p+1}) \in [0,1]^{p+1}} \left| \tilde{\psi}_j(x_1, \ldots, x_p, x_{p+1}) \right|.
\]
Since \(X(t) \in L^2_G(\Omega_T)\) for all \((x,t) \in \mathbb{R}^p \times [0,T]\), we have that
\[
\tilde{X}_j \left( \frac{i_1}{m}, \ldots, \frac{i_p}{m}, \frac{i_{p+1}}{m} \right) \in L^2_G(\Omega_T), j = 1, \ldots, p, 1 \leq i_1, \ldots, i_{p+1} \leq m;
\]
and
\[
B_m(\tilde{X}_j)(x_1, \ldots, x_p, x_{p+1}) \in L^2_G(\Omega_T) \text{ for all } (x_1, \ldots, x_p, x_{p+1}) \in [0,1]^{p+1}.
\]
For \(x \in \mathbb{R}^p, t \in [0,T]\), Set
\[
X^{N,m}(x,t) = \left( B_m(\tilde{X}_1) \left( -1 \lor \frac{x + N}{2N} \land 1, \frac{t}{T} \right), \ldots, B_m(\tilde{X}_p) \left( -1 \lor \frac{x + N}{2N} \land 1, \frac{t}{T} \right) \right).
\]
Noting that \(\Phi(X^{N,m})\) is a continuous function of \(\tilde{X}_j \left( \frac{i_1}{m}, \ldots, \frac{i_p}{m}, \frac{i_{p+1}}{m} \right), j = 1, \ldots, p, \)
\(1 \leq i_1, \ldots, i_{p+1} \leq m\), we obtain \(\Phi(X^{N,m}) \in L^2_G(\Omega_T)\).
By Theorem 4.1 in [14], for any \(q \geq 2\),
\[
\mathbb{E}^G \left( |X(x,t) - X(y,s)|^q \right) \leq C_{q,T}(|x-y|^q + |s-t|^{q/2}).
\]
This yields by the Kolmogorov criterion under \(G\)-expectation (cf. Theorem 1.36, Chapter VI in [22]) that for each \(1 \leq j \leq p\),
\[
\lim_{m \to \infty} \mathbb{E}^G \left( \sup_{\sum_{k=1}^{p+1} |x_k-y_k|^2 \leq 1/m} \left| \tilde{X}_j(x_1, \ldots, x_p, x_{p+1}) - \tilde{X}_j(y_1, \ldots, y_p, y_{p+1}) \right|^2 \right) = 0,
\]
and
\[
\mathbb{E}^G \left( \sup_{(x_1, \ldots, x_p, x_{p+1}) \in [0,1]^{p+1}} \left| \tilde{X}_j(x_1, \ldots, x_p, x_{p+1}) \right|^2 \right) < \infty.
\]
Therefore,
\[
\lim_{m \to \infty} \mathbb{E}^G \left( \sup_{x \in [-N,N]^p, t \in [0,T]} \left| X(x,t) - X^{N,m}(x,t) \right|^2 \right) = 0,
\]
and by
\[
|\Phi(X) - \Phi(X^{N,m})| \leq l \sup_{x \in [-N,N]^p, t \in [0,T]} |X(x,t) - X^{N,m}(x,t)| + \frac{l}{2N-1},
\]
we obtain
\[
\lim_{N \to \infty} \lim_{m \to \infty} \mathbb{E}^G \left( |\Phi(X) - \Phi(X^{N,m})|^2 \right) = 0,
\]
which implies that \( \Phi(X) \in L^1_G(\Omega_T) \), and so \( \Phi(X) \) is quasi-continuous.

For general \( \Phi \in C_b(C(\mathbb{R}^p \times [0,T], \mathbb{R}^p)) \), set \( M = \sup_{\psi \in C(\mathbb{R}^p \times [0,T], \mathbb{R}^p)} |\Phi(\psi)| \). For any \( N \geq 1 \), set
\[
\Phi^{(N)}(\psi) = \inf_{\varphi \in C(\mathbb{R}^p \times [0,T], \mathbb{R}^p)} \{ \Phi(\varphi) + N\|\psi - \varphi\|_N, \psi \in C(\mathbb{R}^p \times [0,T], \mathbb{R}^p) \}.
\]
Then (cf. Lemma 3.1, Chapter VI in [22]), \( |\Phi^{(N)}| \leq M \),
\[
|\Phi^{(N)}(\psi) - \Phi^{(N)}(\varphi)| \leq N\|\psi - \varphi\|_N, \psi, \varphi \in C(\mathbb{R}^p \times [0,T], \mathbb{R}^p),
\]
and for any \( \psi \in C(\mathbb{R}^p \times [0,T], \mathbb{R}^p) \), \( \Phi^{(N)}(\psi) \uparrow \Phi(\psi) \) as \( N \to \infty \). Therefore, \( \Phi^{(N)}(X) \) is quasi-continuous for all \( N \geq 1 \). For any \( \delta > 0 \), choose a compact subset \( K \subset \Omega_T \) such that \( c^G(K^c) < \delta \) and for all \( N \geq 1 \), \( \Phi^{(N)}(X) \) is continuous on \( K \). By Dini’s Theorem, \( \Phi^{(N)}(X) \) converges uniformly to \( \Phi(X) \) on \( K \), and so \( \Phi(X) \) is continuous on \( K \). Thus, \( \Phi(X) \) is quasi-continuous.

\[\square\]

**Lemma 5.2.** Assume that \((H1)\) and \((H2)\) hold.

(1). For each \( N \geq 1 \), if \( f_n, n \geq 1, f \in H \), \( g_n \in A \) and \( g \in A \) satisfy that \( \|f_n\|_H \leq N, \|f\|_H \leq N, \|f_n - f\|_H \to 0 \) and \( \|g_n - g\|_G \to 0 \), then
\[
\Psi(f_n, g_n) \to \Psi(f, g).
\]

(2). For \( \Phi \in C_b(C(\mathbb{R}^p \times [0,T], \mathbb{R}^p)) \), for each \( N \geq 1 \),
\[
\lim_{\epsilon \to 0} \sup_{\eta \in (\mathbb{M}_2(\mathbb{R}^2))^d \cap \mathbb{B}_h(0_T)} \mathbb{E}^G \left( \left| \Phi \circ \Psi^\epsilon \left( \sqrt{\epsilon}B + \int_0^t \eta_s d\langle B \rangle_s, \langle B \rangle \right) \right| \right) = 0. \tag{5.10}
\]

**Proof.** (1). For any \( m \geq 1 \), set
\[
M_m = \sup_{|x| \leq m, t \in [0,T]} (\|\sigma(\Psi(f, g)(x, s))\|_{HS} + \max_{1 \leq k \leq p} \|h^k(\Psi(f, g)(x, s))\|_{HS}).
\]
Then, there exists a constant \( M \in (0, \infty) \) such that, on \( \{ |x| \leq m, t \in [0,T] \} \),
\[
|\Psi(f, g)(x, t) - \Psi(f_n, g_n)(x, t)| \leq M \int_0^t |\Psi(f, g)(x, s) - \Psi(f_n, g)(x, s)| (1 + |f_n'(s)| + \|g_n'(s)\|_{HS}) ds + M_m \int_0^t (|f_n'(s) - f'(s)| + \|g_n'(s) - g'(s)\|_{HS}) ds.
\]
By Gronwall's inequality,
\[
\sup_{|x| \leq m, t \in [0,T]} |\Psi(f, g)(x, t) - \Psi(f_n, g)(x, t)| \\
\leq M_m \int_0^T (|f'_n(s) - f'(s)| + \|g'_n(s) - g'(s)\|_{HS}) ds e^{M \int_0^T (1 + |f'_n(t)| + \|g'_n(0)\|_{HS}) dt} \\
\leq M_m \left( \sqrt{T} \|f_n - f\|_H + \|g_n - g\|_G \right) e^{M(T + \sqrt{NT + p\theta_T})} \to 0 \text{ as } n \to \infty.
\]

(2) For any \( \eta \in (M^2_G(0, T))^d \) with \( \int_0^T |\eta_s|^2 ds \leq r \), set \( X^{\eta, \epsilon} = \Psi^\epsilon(\sqrt{\epsilon}B + \int_0^\cdot \eta_s d\langle B \rangle_s, \langle B \rangle) \). Then
\[
X^{\eta, \epsilon}(x, t) = x + \int_0^t b^\epsilon(X^{\eta, \epsilon}(x, s)) ds + \sqrt{\epsilon} \int_0^t \sigma^\epsilon(X^{\eta, \epsilon}(x, s)) dB_s + \int_0^t \sigma^\epsilon(X^{\eta, \epsilon}(x, s)) \eta_s d\langle B \rangle_s + \int_0^t h^\epsilon(X^{\eta, \epsilon}(x, s)) d\langle B \rangle_s.
\]

and there exists a constant \( M = M(\bar{\sigma}) \) such that
\[
\left| \int_0^t \sigma^\epsilon(X^{\eta, \epsilon}(x, s)) \eta_s d\langle B \rangle_s \right| \leq \left( \int_0^t \|\sigma^\epsilon(X^{\eta, \epsilon}(x, s))\|^2 ds \right)^{1/2} M r^{1/2}.
\]

By the BDG inequality under \( G \)-expectation and Gronwall’s equality, we can get that (cf. [14]) for \( q \geq 2 \), for any \( m \geq 1 \), there exists a constant \( \beta = \beta(m, q, r, \bar{\sigma}) \) such that
\[
\sup_{f_0^T |\eta_s| ds \leq \epsilon, \epsilon \in [0, 1]} \sup_{|x| \leq m} \mathbb{E}^G \left( \sup_{t \in [0,T]} |X^{\eta, \epsilon}(x, t)|^q \right) \leq \beta
\]
and for any \( x, y \in \mathbb{R}^q \), for any \( s, t \in [0, T] \),
\[
\sup_{f_0^T |\eta_s| ds \leq \epsilon, \epsilon \in [0, 1]} \mathbb{E}^G \left( |X^{\eta, \epsilon}(x, t) - X^{\eta, \epsilon}(y, s)|^q \right) \leq \beta(|x - y|^q + |s - t|^{q/2}). \tag{5.12}
\]

Set
\[
\theta(\epsilon) = \sup_{x \in \mathbb{R}^p} \max \left\{ |b^\epsilon(x) - b(x)|, \|\sigma^\epsilon(x) - \sigma(x)\|_{HS}, \max_{1 \leq k \leq p} \|h^{\epsilon, k}(x) - h^k(x)\|_{HS} \right\}
\]
\[ Z^{\eta,\epsilon}(x,t) = X^{\eta,\epsilon}(x,t) - X^{\eta,0}(x,t). \] Then
\[
Z^{\eta,\epsilon}(x,t) = \sqrt{\epsilon} \int_0^t \sigma^\epsilon(X^{\eta,\epsilon}(x,s))dB_s + \int_0^t (b^\epsilon(X^{\eta,\epsilon}(x,s)) - b(X^{\eta,\epsilon}(x,s)))ds \\
+ \int_0^t (\sigma^\epsilon(X^{\eta,\epsilon}(x,s)) - \sigma(X^{\eta,\epsilon}(x,s)))\eta_s d\langle B \rangle_s \\
+ \int_0^t (h^\epsilon(X^{\eta,\epsilon}(x,s)) - h(X^{\eta,\epsilon}(x,s)))d\langle B \rangle_s, \\
\]
and so for any \( q \geq 2 \), by the BDG inequality under \( G \)-expectation and Gronwall’s equality, there exists a function \( \gamma(\epsilon,\theta(\epsilon),q,r,\bar{\sigma}) \) satisfying
\[
\gamma(\epsilon,\theta(\epsilon),q,r,\bar{\sigma}) \to 0 \quad \text{as} \quad \epsilon \to 0 \\
\text{such that (cf. \[14\])}
\]
\[
\sup_{f,\eta_0} \sup_{|x| \leq m} \mathbb{E}^G \left( \sup_{t \in [0,T]} |Z^{\eta,\epsilon}(x,t)|^q \right) \leq \gamma(\epsilon,\theta(\epsilon),q,r,\bar{\sigma}), \\
\]
which yields that
\[
\lim_{\epsilon \to 0} \sup_{f,\eta_0} \sup_{|x| \leq m} \mathbb{E}^G \left( \sup_{t \in [0,T]} |Z^{\eta,\epsilon}(x,t)|^q \right) = 0. \quad (5.13)
\]
Finally, by the below Lemma 5.4, (5.10) is a consequence of (5.12) and (5.13). \( \square \)

For given \( N \geq 1 \), for each \( f \in \mathcal{H}, g \in \mathcal{A}, \) let \( \Psi^{(N)}(f,g) \in C(\mathbb{R}^p \times [0,T],\mathbb{R}^p) \) be defined by
\[
\Psi^{(N)}(f,g)(x,t) \\
= x + \sum_{k=1}^N b \left( \Psi^{(N)}(f,g) \left( x, \left( \frac{k-1}{N} \right) T \right) \right) \left( \frac{kT}{N} \wedge t - \left( \frac{k-1}{N} \right) T \wedge t \right) \\
+ \sum_{k=1}^N \sigma \left( \Psi^{(N)}(f,g) \left( x, \left( \frac{k-1}{N} \right) T \right) \right) \left( f \left( \frac{kT}{N} \wedge t \right) - f \left( \left( \frac{k-1}{N} \right) T \wedge t \right) \right) \\
+ \sum_{k=1}^N h \left( \Psi^{(N)}(f,g) \left( x, \left( \frac{k-1}{N} \right) T \right) \right) \left( g \left( \frac{kT}{N} \wedge t \right) - g \left( \left( \frac{k-1}{N} \right) T \wedge t \right) \right),
\]
Then it is obvious that for any \( N \geq 1, (\mathbb{H} \times \mathbb{A}, \rho_{HG}) \ni (f, g) \rightarrow \Psi^{(N)}(g) \) is continuous and

\[
\Psi^{(N)}(f, g)(x, t) = x + \int_0^t b(\Psi^{(N)}(f, g)(x, \pi_N(s))) \, ds \\
+ \int_0^t \sigma(\Psi^{(N)}(f, g)(x, \pi_N(s))) \, df(s) \\
+ \int_0^t h(\Psi^{(N)}(f, g)(x, \pi_N(s))) \, dg(s)
\]

where \( \pi_N(s) = \frac{(k-1)T}{N} \), for \( s \in [(k-1)T/N, kT/N), \) \( k = 1, \ldots, N \).

**Lemma 5.3.** Assume that \((H1)\) and \((H2)\) hold. Then for any \( l \in (0, \infty) \),

\[
\lim_{N \to \infty} \sup_{\|f\|_H \leq l, g \in \mathbb{A}} \rho(\Psi(f, g), \Psi^{(N)}(f, g)) = 0.
\]

**Proof.** Firstly, by the Lipschitz condition, there exists a constant \( L_1 \in (0, \infty) \) such that for any \( x \in \mathbb{R}^p, t \in [0, T], f \in \mathbb{H}, g \in \mathbb{A}, \)

\[
|\Psi(f, g)(x, t)| \leq |x| + L_1 \int_0^t (1 + |\Psi(f, g)(x, s)|)(1 + |f'(s)| + \|g'(s)\|_{HS}) \, ds.
\]

Therefore, by Gronwall’s inequality, for any \( m \geq 1, \)

\[
\bar{M}_m := \sup_{\|f\|_H \leq l, g \in \mathbb{A}} \sup_{|x| \leq m, t \in [0, T]} |\Psi(f, g)(x, t)| < \infty.
\]

Furthermore, there exist positive constants \( L_2, L_3 \) such that for any \( |x| \leq m, t \in [0, T], \|f\|_H \leq l, g \in \mathbb{A}, \)

\[
|\Psi(f, g)(x, \pi_N(t)) - \Psi(f, g)(x, t)| \\
\leq L_2 \max_{1 \leq k \leq N} \max_{t \in [(k-1)T/N, kT/N]} \left( \int_{(k-1)T/N}^t |f'(s)| + \|g'(s)\|_{HS} \right) ds \leq \frac{L_3}{\sqrt{N}}.
\]
Therefore, by Gronwall lemma, we obtain that for any $\lambda, \epsilon$ and $t \in [0,T]$, there exist positive constants $L_4, L_5$ such that for any $|x| \leq m$, $t \in [0,T]$, $\|f\|_H \leq l$, $g \in \mathbb{A}$,

$$\max_{s \in [0,t]} |\Psi^{(N)}(f,g)(x,s) - \Psi(f,g)(x,s)| \leq L_4 \frac{1}{\sqrt{N}} + L_5 \int_0^t \max_{u \in [0,s]} |\Psi^{(N)}(f,g)(x,u) - \Psi(f,g)(x,u)| (1 + |f'(s)| + \|g'(s)\|_{H^S}) \, ds.$$

Therefore, by Gronwall lemma, we obtain that for any $m \geq 1$,

$$\lim_{N \to \infty} \sup_{|x| \leq m, t \in [0,T], \|f\|_H \leq l, g \in \mathbb{A}} |\Psi^{(N)}(f,g)(x,t) - \Psi(f,g)(x,t)| = 0.$$

\[\square\]

**Lemma 5.4.** Let $T > 0$ and let $\{Y_{\lambda, \epsilon} = \{Y_{\lambda, \epsilon}(t), t \in [0,T]^m\}; \epsilon \in [0,1], \lambda \in \Lambda\}$ be a family of $\mathbb{R}^p$-valued continuous processes such that $Y_{\lambda, \epsilon}(t)$ is quasi-continuous for all $\lambda, \epsilon$ and $t$. Assume that there exists constants $L \in (0, +\infty)$, $q > 0$ and $\kappa > 0$ such that

$$\sup_{\lambda \in \Lambda, \epsilon \in [0,1]} \mathbb{E}^G (|Y_{\lambda, \epsilon}(t) - Y_{\lambda, \epsilon}(s)|^q) \leq C |t - s|^{m+\kappa}, \quad s, t \in [0,T]^m. \tag{5.14}$$

Then

$$\sup_{\lambda \in \Lambda, \epsilon \in [0,1]} \mathbb{E}^G \left( \left( \sup_{s \neq t} \frac{|Y_{\lambda, \epsilon}(t) - Y_{\lambda, \epsilon}(s)|}{|t - s|^{\alpha}} \right)^q \right) < \infty, \tag{5.15}$$

for every $\alpha \in [0, \kappa/q)$. As a consequence, $\{Y_{\lambda, \epsilon}(t), t \in [0,T]^m\}; \epsilon \in [0,1], \lambda \in \Lambda\}$ is tight under $\mathbb{E}^G$, i.e., for any $\delta > 0$, there exists a compact $K_\delta \subset C([0,T]^m, \mathbb{R}^p)$ such that

$$\sup_{\lambda \in \Lambda, \epsilon \in [0,1]} \mathbb{E}^G (Y_{\lambda, \epsilon} \in K_\delta) < \delta. \tag{5.16}$$
Furthermore, if for $t \in [0, T]^m$ and any $\delta > 0$,
\[
\limsup_{\epsilon \to 0} c^G (|Y_{\lambda, \epsilon}(t) - Y_\lambda(t)| \geq \delta) = 0,
\]
where $Y_\lambda(t) := Y_{\lambda,0}(t)$, then $Y_{\lambda, \epsilon}$ converges uniformly to $Y_\lambda$ in distribution under $\mathbb{E}^G$, i.e., for any $\Phi \in C_b(C([0, T]^m, \mathbb{R}^p))$,
\[
\limsup_{\epsilon \to 0} \mathbb{E}^G (|\Phi(Y_{\lambda, \epsilon}) - \Phi(Y_\lambda)|) = 0.
\]  

**Proof.** First, from the proof of the Kolmogorov criterion under $G$-expectation (cf. Theorem 1.36, Chapter VI in [22]), we can obtain (5.15). Since for each $\Phi$, we have that for all $y \in C([0, T]^m, \mathbb{R}^p)$,\[\begin{align*}
\sup_{s \neq t} |y(t) - y(s)| &\leq (t - s)^{\alpha}, \\
&\leq (t - s)^{\alpha} \leq r
\end{align*}\]
is compact subset for any $r \in (0, \infty)$, by Chebyshev’s inequality and (5.15), for any $\delta > 0$, there exists a compact $K_\delta \subset C([0, T]^m, \mathbb{R}^p)$ such that (5.16) holds.

If for each $t \in [0, T]^m$ and $\delta > 0$, (5.17) holds. Take $\alpha \in (0, \kappa/q)$. For any $\delta > 0$, choose $r = r(\delta) \in (0, \infty)$ such that
\[
\sup_{\lambda \in \Lambda, \epsilon \in [0, 1]} c^G (Y_{\lambda, \epsilon}(t) \in K_\delta^c) < \delta,
\]
where $K_\delta = \left\{ y \in C([0, T]^m, \mathbb{R}^p); \sup_{s \neq t} |y(t) - y(s)| \leq r \right\}$.

By continuity of $\Phi$ and compactness of $K_r$, there exists $\zeta > 0$ such that for any $\psi, \varphi \in C([0, T]^m, \mathbb{R}^p) \cap K_r$ with $|\psi - \varphi| \leq \zeta$, $|\Phi(\psi) - \Phi(\varphi)| < \delta$.

By the definition of $K_r$, there exist $l \geq 1$, $\tau \in (0, (\zeta/3)^{1/\alpha}/r)$ and $t_1, \cdots, t_l$ such that $[0, T]^m = \bigcup_{i=1}^l U(t_i, \tau)$, and
\[
\sup_{\lambda \in \Lambda} c^G \left( \max_{1 \leq i \leq l} \sup_{t \in U(t_i, \tau)} |Y_\lambda(t) - Y_\lambda(t_i)| \geq \zeta/3, Y_\lambda \in K_r \right) = 0,
\]
where $U(t_i, \tau) = \{ t \in [0, T]^m; |t - t_i| < \tau \}$.

By (5.17), there exists $\epsilon_0$ such that for all $\epsilon \in (0, \epsilon_0)$,
\[
\max_{1 \leq i \leq l} \sup_{\lambda \in \Lambda} c^G (|Y_{\lambda, \epsilon}(t_i) - Y_\lambda(t_i)|) < \delta/3,
\]
By triangle inequality
\[
|Y_{\lambda, \epsilon}(t) - Y_\lambda(t)| \leq |Y_{\lambda, \epsilon}(t) - Y_{\lambda, \epsilon}(t_i)| + |Y_{\lambda, \epsilon}(t_i) - Y_\lambda(t_i)| + |Y_\lambda(t) - Y_\lambda(t_i)|,
\]
we have that for all $\epsilon \in (0, \epsilon_0)$,
\[
\sup_{\lambda \in \Lambda} c^G \left( Y_{\lambda, \epsilon} \in K_r, Y_\lambda \in K_r, \sup_{t \in [0, T]^m} |Y_{\lambda, \epsilon}(t) - Y_\lambda(t)| \geq \zeta \right) < \delta.
\]
Therefore, for all $\epsilon \in (0, \epsilon_0)$, $\sup_{\lambda \in \Lambda} \mathbb{E}^G (|\Phi(Y_{\lambda, \epsilon}) - \Phi(Y_\lambda)|) \leq (1 + 3M)\delta$, where $M := \sup_{y \in C([0, T]^m, \mathbb{R}^p)} |\Phi(y)|$. This yields (5.18).

$\square$
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