Viviani’s Theorem and its Extensions Revisited; Canghareeb, Sanghareeb and a New Definition of the Ellipse

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A two dimensional outsider called ”Canghareeb”1 consisting of a head and three legs, lives inside a triangle under the following conditions;

• For better balance and movement he puts its legs on the sides of the triangle perpendicularly.

• He can extend one leg or two legs at the expense of the others so that the sum of the lengths of the three legs stays constant.

• He can not bend its legs.

The following questions arise; Where does Canghareeb live inside the triangle? Are there necessary conditions for Canghareeb to live inside a given triangle? What happens if the triangle is equilateral, isosceles or scalene?

In Fig. 1 Canghareeb is illustrated by a point (head) and three perpendiculars (legs). The questions are intended to describe the set of points formed by the traces of the head when Canghareeb moves inside the triangle.

These questions are related to Viviani’s theorem and its extensions (see [1]), as will be illustrated.

Canghareeb

We shall rely on the following basic theorem (see [1] Theorems 1 and 2) to conclude the results in this section. At first recall the following terminology: Let \( \mathcal{P} \) be a polygon consisting of both boundary and interior points. Define the distance sum function \( \mathcal{V} : \mathcal{P} \to \mathbb{R} \), where for each point \( P \in \mathcal{P} \) the value \( \mathcal{V}(P) \) is defined as the sum of the distances from \( P \) to the sides of \( \mathcal{P} \).

**Theorem 1** Any triangle can be divided into parallel line segments on which \( \mathcal{V} \) is constant. Furthermore, the following conditions are equivalent:

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1The word "Canghareeb" comes from Arabic, and is an abbreviation of two words "caen ghareeb" which means outsider.
• $V$ is constant on the triangle $\triangle ABC$.

• There are three non-collinear points, inside the triangle, at which $V$ takes the same value.

• $\triangle ABC$ is equilateral.

**Theorem 2**  
(a) Any convex polygon $P$ can be divided into parallel line segments, on which $V$ is constant.

(b) If $V$ takes equal values at three non-collinear points, inside a convex polygon, then $V$ is constant on $P$.

**Equilateral triangle**

By Viviani’s theorem; the sum of the distances from the sides of any point inside an equilateral triangle is constant. The constant sum is the height of the equilateral triangle. Thus we have the following conclusion:

**Conclusion 1**  If the total length of the legs of Canghareeb equals the height of an equilateral triangle then Canghareeb can live inside the triangle and move freely from one point inside the triangle to another.

**Isosceles triangle**

Since an isosceles triangle has a reflection symmetry across the height we conclude that if Canghareeb can reach a point $P$ inside the triangle then he can reach the reflection point $P'$ across the height. Thus we have:

**Conclusion 2**  Canghareeb can live on a line segment parallel to the base of an isosceles triangle provided the total length of its legs ranges between the lengths of the smallest and the largest altitudes of the triangle.

**Proof.** If Canghareeb can reach point $P$ inside the triangle then he can reach the reflection point $P'$ across the height. Indeed, In Fig. 2 if the line segment $DF$ is parallel to the base $BC$ then when $P$ moves along $DF$, the length $a$ remains constant and $b + c = DP\sin\alpha + PE\sin\alpha = DE\sin\alpha$ which is also constant. By Theorem 1 Canghareeb can not live at any other point inside the triangle unless it is equilateral.

**Scalene triangle**

Similarly, by Theorem 1 Canghareeb can live inside a scalene triangle, provided the following necessary condition is satisfied: the total length of its legs ranges between the lengths of the smallest and the largest altitudes of the triangle. This is true due to the fact that the distance sum function $V$ is a linear continuous function in two variables hence; it takes on every value between its minimum and its maximum.

Thus we have:
Conclusion 3. Canghareeb can live on a line segment inside a scalene triangle provided the total length of its legs ranges between the lengths of the smallest and the largest altitudes of the triangle.

The question is how can we determine this segment for a general triangle? We shall illustrate the method explained in [1] by the following example:

Example 1. Let $ABC$ be the right angled triangle with vertices $(0, 0), (0, 3), (4, 0)$ respectively (see Fig. 3). If the total length of Canghareeb’s legs is $l$, $3 \leq l \leq 4$, then Canghareeb can live on a line segment inside the triangle parallel to the line $2x + y = 0$.

In Fig. 3 $l = 3.16743$ and the living area is the line segment $HI$.

Proof. The smallest altitude of the triangle is 3 and the largest altitude is 4. Hence by the previous conclusion, Canghareeb can live inside the triangle across a line segment. To find this line segment, we compute the distance sum function $V$; The equation of the hypotenuse is $3x + 4y = 12$. Hence,

$$V = x + y - \frac{3x + 4y - 12}{5} = \frac{2}{5}x + \frac{1}{5}y - \frac{12}{5}.$$

Therefore, the lines $V = c$ are parallel to the line $2x + y = 0$ and the result follows.

The following example will be left to the reader but at the end of the paper a clue will be given.

Example 2. One Canghareeb proclaimed that he can reach the three points $(-2, 0), (\frac{1}{2}, \frac{1}{2}), (2, 0)$ inside a triangle. Show that the triangle is equilateral. If the coordinates of two of the vertices are $(-2, 0), (2, 0)$ find the coordinates of the third vertex of the triangle.

Sanghareeb

Motivated by Canghareeb’s story, another mysterious creature called Sanghareeb lives in the plane relative to a triangle with the following conditions:

- He puts its legs on the sides of the triangle or their extensions perpendicularly.
- He can extend one leg or two legs at the expense of the others so that the sum of the squares of the lengths of the three legs stays constant.
- He can not bend its legs.

One day Sanghareeb met Canghareeb inside the triangle $ABC$ shown in Fig. 3 with vertices $(0, 0), (0, 3), (4, 0)$. The total length of Canghareeb’s legs was 3.16743 and the sum of the squares of the lengths of Sanghareeb’s legs was 5. Sanghareeb boasted to Canghareeb that he can reach points outside the
triangle. When the latter asked about the former’s secret, Sanghareeb showed Canghareeb the journey he could make. In fact Sanghareeb walked on a closed smooth curve, called the ellipse. He made the following mathematics;

Referring to Fig. 3, $QR^2 + QS^2 + QT^2 = 5$ implies the following equation of a quadratic curve;

$$x^2 + y^2 + \left( \frac{3x + 4y - 12}{5} \right)^2 = 5.$$  

Simplifying, one gets the equivalent equation;

$$34x^2 + 41y^2 + 24xy - 72x - 96y + 19 = 0.$$  

This is exactly the equation of the ellipse shown in Fig. 3 and Sanghareeb could meet Canghareeb at exactly two points $L$ and $J$ inside the triangle.

### A new definition of the ellipse

In view of the previous discussion we may state the following theorem, which gives a new definition of the ellipse;

**Theorem 3** An ellipse is the locus of points which have a constant sum of squares of distances from the sides of a given triangle.

**Proof.** We shall prove the following two claims;

(a) Given a triangle, the locus of points which have a constant sum of squares of distances from the sides is an ellipse.

(b) Given an ellipse, there is a triangle for which the sum of the squares of the distances from the sides for all points on the ellipse is constant.

Using analytic geometry, we choose for part (a) the coordinate system such that the vertices of the triangle lie on the axes.

Suppose the coordinates of the vertices of the triangle are $A(0, a), B(-b, 0), C(c, 0)$, where $a, b, c > 0$ (see Fig. 4). Let $(x, y)$ be any point in the plane and let $d_1, d_2, d_3$ be the distances of $(x, y)$ from the sides of the triangle $ABC$. Clearly we have

$$\sum_{i=1}^{3} d_i^2 = \left( \frac{ax + cy - ac}{a^2 + c^2} \right)^2 + \left( \frac{ax - by + ab}{a^2 + b^2} \right)^2 + y^2.$$  

Hence, $\sum_{i=1}^{3} d_i^2 = k$ (constant) if and only if the point $(x, y)$ lies on the quadratic curve

$$\left( \frac{ax + cy - ac}{a^2 + c^2} \right)^2 + \left( \frac{ax - by + ab}{a^2 + b^2} \right)^2 + y^2 = k.$$  \hspace{1cm} (1)  

In general, a quadratic equation in two variables;
\[ Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, \]

represents an ellipse provided that the discriminant
\[ \Delta = B^2 - 4AC < 0. \]

It is easily seen that this is the case for equation (1).

For part (b), after rotation or translation of the axes we may assume that the ellipse has a canonical form:
\[ \frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1, \quad \alpha \geq \beta. \]

Because of the symmetry of the ellipse, we shall take an isosceles triangle with vertices \( A(0, a), B(-b, 0), C(b, 0) \) and compute the sum of the squares of the distances from its sides. Substituting \( b=c \) in equation (1), we have
\[
\left( ax + by - ab \right)^2 + \left( ax - by + ab \right)^2 + y^2 = k.
\]

Equivalently, we get a translation of a canonical ellipse;
\[
\frac{x^2}{a^2 + 3b^2} + \left( y - \frac{2ab^2}{a^2 + 3b^2} \right)^2 = \frac{(a^2 + b^2)k - 2a^2b^2}{2a^2(a^2 + 3b^2)} + \frac{2b^4}{(a^2 + 3b^2)^2}.
\]

Thus, it is enough to take
\[
\frac{(a^2 + b^2)k - 2a^2b^2}{2a^2(a^2 + 3b^2)} + \frac{2b^4}{(a^2 + 3b^2)^2} = 1,
\]

and therefore,
\[
k = \frac{2a^2(a^4 + 7a^2b^2 + 10b^4)}{(a^2 + b^2)(a^2 + 3b^2)}. \tag{2}
\]

Translating downward by \( \frac{-2ab^2}{a^2 + 3b^2} \), we get that the ellipse \( \frac{x^2}{a^2 + 3b^2} + \frac{y^2}{b^2} = 1 \) is the locus of points which have a constant sum of squares of distances from the sides of the triangle with vertices \( A(0, a - \frac{2ab^2}{a^2 + 3b^2}), B(-b, -\frac{2ab^2}{a^2 + 3b^2}), C(b, -\frac{2ab^2}{a^2 + 3b^2}) \).

Moreover, this constant is given by equation (2) \( \blacksquare \).

The following table exhibits some examples;

| a   | b   | \( \alpha^2 = a^2 + 3b^2 \) | \( \beta^2 = 2a^2 \) | Ellipse | \( k \) | Figure |
|-----|-----|-----------------------------|-----------------------|--------|--------|--------|
| 1   | 1   | 4                           | 2                     | \( \frac{x^2}{4} + \frac{y^2}{2} = 1 \) | \( \frac{9}{2} \) | 5      |
| 1   | 2   | 13                          | 2                     | \( \frac{x^2}{13} + \frac{y^2}{2} = 1 \) | \( \frac{378}{65} \) | 6      |
| \( \sqrt{3} \) | 1   | 6                           | 6                     | \( x^2 + y^2 = 6 \) | 10     | 7      |

Notice that the locus of points in the third example is a circle. In general, the locus of points is a circle whenever we have \( \alpha^2 = \beta^2 \) or \( a^2 + 3b^2 = 2a^2 \) which is equivalent to \( a = \sqrt{3}b \). In this case the vertices of the triangle are \( A(0, \frac{2a}{\sqrt{3}}), B(-b, -\frac{2a}{\sqrt{3}}) \) and \( C(b, -\frac{2a}{\sqrt{3}}) \). This implies that the triangle is equilateral. Therefore, we have the following:
**Conclusion 4**  The locus of points which have a constant sum of squares of distances from the sides of a given triangle is a circle if and only if the triangle is equilateral.

**Concluding remarks**

Canghareeb and Sanghareeb are described to be creatures living in the plane relative to a triangle. We may extend the idea to any \( n \)-gon. In this case the creature will be two dimensional with \( n \) legs. It may be extended to polyhedra in which case it will be three dimensional with \( n \) legs where \( n \) is the number of faces.

Viviani (1622-1703), who was a student and assistant of Galileo, discovered that equilateral triangles satisfy the following property; the sum of the distances from the sides of any point inside an equilateral triangle is constant.

Viviani’s theorem can be easily proved by using areas. Joining a point \( P \) inside the triangle to its vertices divides it into three parts. The sum of their areas will be equal to the area of the original one. Therefore, the sum of the distances from the sides will be equal to the height of the triangle and the theorem follows.

Kawasaki [3, p. 213], with a proof without words, used only rotations to establish Viviani’s theorem.

Samelson [4, p. 225] gave a proof of Viviani’s theorem that uses vectors and Chen & Liang [2, p. 390-391] used this vector method to prove a converse: if inside a triangle there is a circular region in which the sum of the distances from the sides is constant then the triangle is equilateral. In [1] this theorem is generalized for convex polygons (and for polyhedra as well); If inside a convex polygon there are three non-collinear points for which the sum of the distances from the sides is constant, then the sum of the distances from the sides for each point inside the polygon is constant.

This means that if Canghareeb can reach three non-collinear points inside some triangle then the triangle is equilateral and he can reach every point inside the triangle. This fact is the clue for solving Example 2.

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**References**

[1] E. Abboud, Viviani’s Theorem and its Extension, *College Math. J.* **41** (2010), 207-215.

[2] Z. Chen and T. Liang, The converse of Viviani’s theorem, *College Math. J.* **37** (2006), 390-391.
[3] K. Kawasaki, Proof Without Words: Viviani’s theorem, *Math. Mag.* **78** (2005), p. 213.

[4] H. Samelson, Proof without words: Viviani’s theorem with vectors, *Math. Mag.* **76** (2003), p. 225.

Figure 1: Canghareeb with head P and legs a, b and c
Figure 2: Canghareeb lives on a segment parallel to the base of an isosceles triangle
Figure 3: Canghareeb the total length of whose legs 3.16743, lives on the segment HI parallel to AD inside the triangle ABC
Figure 4: Sanghareeb moves along an ellipse
Figure 5: Sanghareeb $G$ with legs $GH, GI, GJ$ moves along an ellipse with $k = 4.5$
Figure 6: Sanghareeb $G$ moves along an ellipse with $k = \frac{278}{65}$.
Figure 7: Sanghareeb $G$ moves along a circle with $k = 10$
GH^2 + GI^2 + GJ^2 = 4.50000

\( ax - by = -ab \)

\( ax + cy = ac \)
GH^2 + GI^2 + GJ^2 = 10.00000
PE + PF + PG = 3.16743
QR^2 + QS^2 + QT^2 = 5.00000