Research Article

Stationary Distribution and Extinction of a Stochastic Brucellosis Model with Standard Incidence

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In this paper, we proposed a stochastic SVEI brucellosis model with stage structure by introducing the effect of environmental white noise on transmission dynamics of brucellosis. By Has’minskii theory and constructing suitable Lyapunov functions, we established sufficient conditions on the existence of ergodic stationary distribution for the considered model. Moreover, we also established sufficient condition for extinction of the disease. Finally, two examples with numerical simulations are given to illustrate the main results of this paper.

1. Introduction

Brucellosis, which is recognized as a major public health problem, is a serious and economically devastating zoonosis which can infect animals, such as sheep, cattle, pig, and dogs. The disease is caused by bacteria of the genus Brucella, of which there are six species: B. abortus, B. melitensis, B. suis, B. ovis, B. canis, and B. neotomae [1]. Brucella can survive for long periods in dust, dung, water, slurry, aborted fetuses, soil, meat, and dairy products. In animals, brucellosis can be infected by contact with the infected animals (direct way of infection) and by contact of polluted environment (indirect way of transmission), and the disease mainly affects reproduction and fertility and reduces survival of newborns [2]. Brucellosis also can infect human being; the main transmission sources of human brucellosis include exposure to a contaminated environment by infected animals, direct contact with infected animals, and the ingestion of fresh milk or dairy products prepared from unpasteurized milk and unheated meat and animal liver [3]; there is no recorded cases of the infection between humans. Most of the human brucellosis cases are infected by Brucella melitensis (which is infected in sheep and goats), accounting for 84.5% of the total cases [4]. In humans, disease-related mortality is negligible, but the illness can last for several years [5]. Therefore, the key to solve the problem of this public health problem is the elimination of animal brucellosis.

It is worth noting that, mathematical models are widely used not only to study the transmission dynamics of brucellosis, but also to study the epidemiological characteristics of brucellosis [1–8]. Recently, the authors in [6] presented a sheep brucellosis model with immigration and proportional birth, considering both direct and indirect transmission. In [7], the authors proposed a multigroup SEIRV dynamical model with bidirectional mixed cross infection between cattle and sheep and investigated the influence of cross infection of mixed feeding on the brucellosis transmission. In [8], the authors proposed the following deterministic brucellosis transmission model:

\[
\begin{align*}
\frac{dS_1}{dt} &= A + \frac{b(S_1 + V)}{1 + \tau(S_2 + V)} - (\mu_1 + d + \eta_1)S_1, \\
\frac{dS_2}{dt} &= \eta_1S_1 - \beta S_2 I - \theta S_2 - \mu_2 S_2 + \delta V, \\
\frac{dV}{dt} &= \theta S_2 - \epsilon \beta VI - (\mu_2 + \delta) V, \\
\frac{dE}{dt} &= \beta \epsilon VI + \beta S_2 I - (\mu_2 + \eta_2) E, \\
\frac{dI}{dt} &= \eta_2 E - (\mu_2 + c) I, \\
\end{align*}
\]  

and studied the dynamical behavior of the model, where
sheep population is classified into five compartments: the susceptible young sheep $S_1(t)$, the susceptible adult (or sexually mature) sheep $S_2(t)$, the vaccinated sheep $V(t)$, the exposed sheep $E(t)$, and the infectious sheep $I(t)$. $A$ and $b$ are the input number of young sheep and the natural birth rate of sheep, and $\tau$ is the extent of the birth being delayed.
Figure 2: Continued.
μ₁ and μ₂ are the young sheep natural mortality rate and the elimination rate of adult sheep. η₁ and η₂ are the transfer rate from young sheep to adult sheep and exposed sheep to infected sheep. d is the output number of young sheep, δ is the vaccination rate, β is sheep-to-sheep transmission rate, ε is the ineffective vaccination rate, and c is the elimination rate caused by brucellosis. All the parameters are assumed to be positive.

However, the epidemics in the real world are often disturbed by some uncertain factors, such as environmental white noises. Therefore, it is difficult to describe these epidemic dynamics by using determined differential equation [9]. Thus, the deterministic models has some limitations in mathematical modeling of epidemics, and it is quite difficult to fitting data perfectly and to predicting the future dynamics of the epidemic system. In the past years, there has been a lot of researchers who are interested in the stochastic dynamical models [3, 9–28]. In particular, the stochastic epidemic models have been extensively studied [3, 10–25]. For example, in [3], the authors proposed and studied a periodic stochastic brucellosis model and obtained some conditions on the existence of nontrivial positive periodic solution of the model. In [11], the authors studied a stochastic SIRS epidemic model with standard incidence rate and partial immunity and obtained sufficient conditions on the extinction and existence of a stationary probability measure for the disease of the system. In [12], the authors studied a kind of stochastic SEIR epidemic model with standard incidence and obtained sufficient conditions for the existence of stationary distribution and the extinction of the disease in the system. In [24], the authors discussed a stochastic SIRS epidemic model with logistic growth and nonlinear incidence and obtained sufficient conditions on the ergodic stationary distribution and extinction of the considered model.

On the other hand, there are different approaches used in the literature to introduce random perturbations into population models, both from a mathematical and biological perspective [3, 9–28]. In this paper, in light of the above analysis and reasons, we consider the stochastic perturbations for deterministic system (1) and we employ the approach used in Mao et al. [26] and assume that the parameters involved in the model always fluctuate around some average value due to continuous fluctuations in the environment. This approach is reasonable and well justified biologically [27, 28]. By this approach, we study a stochastic S₁S₂VEI brucellosis model with standard incidence, and we assume that the environmental white noise affects the natural mortality rate, the elimination rate, transfer rate, and transmission rate. In order to obtain the stochastic S₁S₂VEI brucellosis model, we let $X(t) = (S_1(t), S_2(t), V(t), E(t), I(t))^T$, and then it is appropriate to model $X = (S_1, S_2, V, E, I)^T$ as a Markov process; thus, from [15] and model (1), we can get the following properties when $0 ≤ Δt ≪ 1$, the conditional mean

$$
E[S_1(t + Δt) – S_1(t) \mid X = x] = \left[ A + \frac{b(S_2 + V)}{1 + r(S_2 + V)} - (μ_1 + d + η_1)S_1 \right]Δt,
$$

$$
E[S_2(t + Δt) – S_2(t) \mid X = x] = [η_2S_1 - βS_2I - δS_2 - μ_2S_2 + δV]Δt,
$$

$$
E[V(t + Δt) – V(t) \mid X = x] = [δS_2 - εβVI - (μ_2 + δ)V]Δt,
$$

$$
E[E(t + Δt) – E(t) \mid X = x] = [βcVI + βS_2I - (μ_2 + η_2)E]Δt,
$$

$$
E[I(t + Δt) – I(t) \mid X = x] = [η_2E - (μ_2 + c)I]Δt,
$$

and the conditional covariance

$$
\text{Var}[S_1(t + Δt) – S_1(t) \mid X = x] = σ_1^2S_1^2Δt,
$$

$$
\text{Var}[S_2(t + Δt) – S_2(t) \mid X = x] = σ_2^2S_2^2Δt,
$$

$$
\text{Var}[V(t + Δt) – V(t) \mid X = x] = σ_3^2V^2Δt,
$$

$$
\text{Var}[E(t + Δt) – E(t) \mid X = x] = σ_4^2E^2Δt,
$$

$$
\text{Var}[I(t + Δt) – I(t) \mid X = x] = σ_5^2I^2Δt.
$$

**Figure 2:** Dynamic behaviors of the system.
Then, we derive the following stochastic form of system (1)

\[ \begin{align*}
  ds_I(t) &= \left[ \lambda + \frac{b(S_2 + V)}{1 + (S_2 + V)} - (\mu_1 + \delta)S_1 \right] dt + \sigma_1 S_1 dB(t),
  
  ds_V(t) &= (\sigma_2 S_2 - \beta S_1 S_2 - \mu_2 S_2 + \delta V) dt + \sigma_2 S_2 dB(t),
  
  dV(t) &= \left[ \beta S_2 - \sigma_3 V + \beta S_1 I - (\mu_2 + \delta) V \right] dt + \sigma_3 V dB(t),
  
  dE(t) &= \left[ \beta E - (\mu_2 + \delta) E \right] dt + \sigma_4 EdB(t),
  
  dl(t) &= \left[ \eta_1 E - (\mu_2 + \epsilon) l \right] dt + \sigma_1 l dB(t),
\end{align*} \]

where \( B_1(t), B_2(t), B_3(t), B_4(t), \) and \( B_5(t) \) are the standard one-dimensional independent Brownian motions and \( \sigma_i^2 > 0 \) are the intensity of the white noises.

The main purpose of the present paper is to obtain the conditions for the existence of ergodic stationary distribution and extinction of the disease for model (4). In Section 2, we present some preliminaries which will be used in the following analysis. In Section 3, we show that there is a unique global ergodic stationary distribution for system (4) under certain conditions. In Section 5, we establish sufficient conditions for the disease extinction.

### 2. Preliminaries

Throughout this paper, let \((\Omega, F, \mathbb{P})\) be a complete probability space with a filtration \(\{F_t\}_{t \geq 0}\) satisfying the usual conditions (i.e., it is increasing and right continuous, while \(F_0\) contains all \(\mathbb{P}\) -null sets); \(B_i(t) (i = 1, 2, 3, 4)\) are defined on this complete probability space, and also let \(\mathbb{R}^d = \{x \in \mathbb{R}^d \mid x_i > 0, 1 \leq i \leq d\}\).

In general, consider the \(d\)-dimensional stochastic differential equation

\[ dx(t) = f(x(t), t) dt + g(x(t), t) dB(t) \quad \text{for } [t_0, \infty), \]

with initial value \(x(0) = x_0 \in \mathbb{R}^d\). \(B(t)\) denotes an \(n\)-dimensional standard Brownian motion defined on the complete probability space \((\Omega, F, \{F_t\}_{t \geq 0}, \mathbb{P})\). \(C^{\infty}(\mathbb{R}^d \times [t_0, \infty); \mathbb{R}_+)\) denotes the family of all nonnegative functions \(V(x, t)\) defined on \(\mathbb{R}^d \times [t_0, \infty)\) such that they are continuously twice differentiable in \(x\) and once in \(t\). The differential operator \(L\) of equation (5) is defined by [16].

\[ L = \frac{\partial}{\partial t} + \sum_{i=1}^{d} f_i(x, t) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{d} g_i(x, t) g_j(x, t) \frac{\partial^2}{\partial x_i \partial x_j}. \]

If \(L\) acts on a function \(V \in C^{\infty}(\mathbb{R}^d \times [t_0, \infty); \mathbb{R}_+)\), then

\[ LV(x, t) = V_t(x, t) + V_x(x, t) f(x, t) + \frac{1}{2} \text{trace} [g^T(x, t) V_{xx}(x, t) g(x, t)], \]

where

\[ V_j = \partial V/\partial t, V_x = ((\partial V/\partial x_1), \ldots, (\partial V/\partial x_d)), V_{xx} = (\partial^2 V/\partial x_i \partial x_j)_{i,j=1}^{d}. \]

By Itô’s formula, if \(x(t) \in \mathbb{R}^d\), then

\[ dV(x(t), t) = LV(x(t), t) dt + V_{xx}(x(t), t) g(x(t), t) dB(t). \]

Next, we present a result about the existence of stationary distribution (see Has’minskii [17]).

Let \(X(t)\) be a homogeneous Markov process in \(E_d\) (\(E_d\) denotes \(d\)-dimensional Euclidean space) and be described by the following stochastic differential equation:

\[ dX(t) = b(X) dt + \sum_{i=1}^{k} g_i(X) dB_i(t). \]

The diffusion matrix is defined as follows:

\[ A(x) = (a_{ij}(x)), \quad a_{ij}(x) = \sum_{i=1}^{k} g_i^T(x) g_j(x). \]

#### Lemma 1

The Markov process \(X(t)\) has a unique ergodic stationary distribution \(\Pi(\cdot)\) if there exists a bounded domain \(D \subset E_d\) with regular boundary \(\Gamma\) and

\[ A_j: \text{there is a positive number } M \text{ such that } \sum_{i=1}^{k} a_{ij}(x) \xi_i \geq M |\xi|^2, \quad x \in D, \xi \in \mathbb{R}^d; \]

\[ A_j: \text{there exists a nonnegative } C^2 \text{-function } V \text{ such that } L V \text{ is negative for any } E_d \setminus D. \text{ Then} \]

\[ \mathbb{P}_x \left\{ \lim_{T \to \infty} \frac{1}{T} \int_0^T f(x(t)) dt = \int_{E_d} f(x) \Pi(dx) \right\} = 1, \]

for all \(x \in E_d\), where \(f(\cdot)\) is a function integrable with respect to the measure \(\pi\).

### 3. Main Results

#### 3.1. Existence and Uniqueness of the Positive Solution

In studying the dynamical behavior of an epidemic model, the first importance is whether the solution is global and positive. Hence, in the following theorem, we will study the existence and uniqueness of the global positive solution, which is a prerequisite for researching the long-term behavior of model (4).

#### Theorem 2

For any initial value \(X_0 = (S_1(0), S_2(0), V(0), E(0), I(0)) \in \mathbb{R}^5_+\), there is a unique solution \(X(t) = (S_1(t), S_2(t), V(t), E(t), I(t))\) of system (4) on \(t \geq 0\), and the solution will remain in \(\mathbb{R}^5_+\) with probability one.

#### Proof

Since the coefficients of system (4) satisfy the local Lipschitz condition, then for any initial value \((S_1(0), S_2(0), V(0), E(0), I(0)) \in \mathbb{R}^5_+,\) there is a unique local solution \((S_1(t), S_2(t), V(t), E(t), I(t))\) on \([0, \tau_e]\), where \(\tau_e\) is the explosion time [16]. To show this solution is global, we only need to prove that \(\tau_e = \infty\) a.s. To this end, let \(n_0 > 0\) be sufficiently
large such that every component of $X_0$ lying within the interval $[1/n_0, n_0]$. For each integer $n > n_0$, define the stopping time as follows:

$$
\tau_n = \inf \left\{ t \in [0, \tau_\varepsilon) : \min \{ \sigma_1(t), \sigma_2(t), V(t), E(t), I(t) \} \leq \frac{1}{n} \text{ or } \max \{ \sigma_1(t), \sigma_2(t), V(t), E(t), I(t) \} \geq n \right\}. 
$$

(12)

Throughout this paper, we set $\inf \emptyset = \infty$ (as usual $\emptyset$ denotes the empty set). It is easy to see that $\tau_n$ is increasing as $n \to \infty$. Let $\tau_\infty = \lim_{n \to \infty} \tau_n$, then $\tau_\infty \leq \tau_\varepsilon$ a.s. In what follows, we need to verify $\tau_\infty = \infty$ a.s. If this assertion is violated, there is a constant $T > 0$ and an $\varepsilon \in (0, 1)$ such that $\mathbb{P}\{\tau_\infty \leq T \} > \varepsilon$. As a result, there exists an integer $n_1 \geq n_0$ such that

$$
\mathbb{P}\{\tau_n \leq T \} \geq \varepsilon, n \geq n_1. 
$$

Define a $C^2$-function $V: \mathbb{R}^3_+ \to \mathbb{R}_+$ by

$$
V(S_1, S_2, V, E, I) = (S_1 - 1 - \ln S_1) + (S_2 - 1 - \ln S_2) + (V - 1 - \ln V) + (E - 1 - \ln E).
$$

(14)

Using Itô’s formula, we have

$$
dV(S_1, S_2, V, E, I) = LV(S_1, S_2, V, E, I) dt + \sigma_1(S_1 - 1) dB_1(t) + \sigma_2(S_2 - 1) dB_2(t) + \sigma_3(V - 1) dB_3(t) + \sigma_4(E - 1) dB_4(t) + \sigma_5(I - 1) dB_5(t),
$$

(15)

where

$$
LV = \left( 1 - \frac{1}{S_1} \right) \left( A + \frac{b(S_2 + V)}{1 + \tau(S_2 + V)} - (\mu_1 + d + \eta_1) S_1 \right) + \left( 1 - \frac{1}{S_2} \right) \left( \mu_2 S_2 \right) V - \mu_2 S_2 + \delta V + \left( 1 - \frac{1}{\delta} \right) (\mu_\delta + \varepsilon \beta V) - \left( \mu_\delta + \varepsilon \beta V \right) 
$$

$$
+ \left( 1 - \frac{1}{\delta} \right) (\beta \varepsilon V + \beta S_2 I - (\mu_2 + \eta_2) E) + \left( 1 - \frac{1}{T} \right) (\eta_2 E - (\mu_2 + c) I) + \frac{\sigma_1^2 + \sigma_3^2 + \sigma_4^2 + \sigma_5^2}{2}.
$$

(16)

By applying the following invariant set of model (1) which is obtained in [8]

$$
\Omega = \left\{ (S_1, S_2, V, E, I) \in \mathbb{R}^5_+ : S_1 + S_2 + V + E + I \leq \frac{A + b}{\mu \tau} \right\},
$$

(17)

and from the following inequalities

$$
\frac{b(S_2 + V)}{1 + \tau(S_2 + V)} \leq A + \frac{b(S_2 + V)}{\mu \tau} + \mu_1 + d + \eta_1 + \beta I + \theta + 4 \mu_2 + \varepsilon \beta I
$$

$$
+ \delta + \eta_2 + c - (\mu_1 + d) \frac{A S_1 - b(S_2 + V)}{S_1 [1 + \tau(S_2 + V)]} - \mu_2 S_2 + V + E + I - \mu_2 S_2 - \mu_1 S_1 - \frac{\beta \varepsilon V}{E}
$$

$$
- \frac{\beta S_2 I}{E} - cl - \frac{\eta_2 E}{I} + \frac{\sigma_1^2 + \sigma_3^2 + \sigma_4^2 + \sigma_5^2}{2} \leq A + \frac{b(\varepsilon + 1)(A + b)}{\mu \tau} + \mu_1 + d + \eta_1 + \theta + 4 \mu_2 + \delta + \eta_2 + c + \frac{\sigma_1^2 + \sigma_3^2 + \sigma_4^2 + \sigma_5^2}{2} = \zeta.
$$

(18)

Since $\zeta$ is positive constant which is independent of $S_1$, $S_2$, $V$, $E$, $I$, and $t$, we can get

$$
dV(S_1, S_2, V, E, I) \leq \zeta dt + \sigma_1(S_1 - 1) dB_1(t) + \sigma_2(S_2 - 1) dB_2(t) + \sigma_3(V - 1) dB_3(t) + \sigma_4(E - 1) dB_4(t) + \sigma_5(I - 1) dB_5(t).
$$

(19)

Integrating both sides (20) from 0 to $T \wedge \tau_\varepsilon$ and taking expectations, then we can obtain

$$
EV(S_1(\tau_\varepsilon \wedge T), S_2(\tau_\varepsilon \wedge T), V(\tau_\varepsilon \wedge T), E(\tau_\varepsilon \wedge T), I(\tau_\varepsilon \wedge T)) \leq V(S_1(0), S_2(0), V(0), E(0), I(0)) + \zeta T < \infty.
$$

(21)

Set $\Omega_\varepsilon = \{ \tau_\varepsilon \leq t \}$ for $n \geq n_1$ by (13), $P(\Omega_n) \geq \varepsilon$. Notice that for every $\omega \in \Omega_n$, there is at least one of $S_1(\tau_\varepsilon, \omega), S_2(\tau_\varepsilon, \omega), V(\tau_\varepsilon, \omega), E(\tau_\varepsilon, \omega), I(\tau_\varepsilon, \omega)$ that equals either $n$ or $1/n$. Hence, $S_1(\tau_\varepsilon, \omega), S_2(\tau_\varepsilon, \omega), V(\tau_\varepsilon, \omega), E(\tau_\varepsilon, \omega), I(\tau_\varepsilon, \omega)$ are no less than

$$
n - 1 - \log n \text{ or } \frac{1}{n} - 1 - \log n.
$$

(22)
Consequently,
\[
V(S_1(\tau', \omega), S_2(\tau', \omega), V(\tau', \omega), E(\tau', \omega), I(\tau', \omega)) \\
\geq (n - 1 - \log n) \wedge \left( \frac{1}{n} - 1 - \log n \right),
\]
where \( a \wedge b \) denotes the minimum of \( a \) and \( b \). In view of (21) and (23) we have
\[
V(S_1(0), S_2(0), V(0), E(0), I(0)) + \zeta T \\
\geq E[\mathbb{1}_\Omega(\omega)V(S_1(\tau, \Lambda T), S_2(\tau, \Lambda T), V(\tau, \Lambda T), E(\tau, \Lambda T), I(\tau, \Lambda T))] \\
\geq \delta \left( (n - 1 - \log n) \wedge \left( \frac{1}{n} - 1 + \log n \right) \right),
\]
where \( \mathbb{1}_\Omega(\omega) \) is the indicator function of \( \Omega_n \). Let \( n \to \infty \),
\[
\lim_{n \to \infty} n \geq \mathbb{E} \left[ \mathbb{1}_\Omega(\omega)V(S_1(\tau, \Lambda T), S_2(\tau, \Lambda T), V(\tau, \Lambda T), E(\tau, \Lambda T), I(\tau, \Lambda T)) \right] \\
\geq \delta \left( (n - 1 - \log n) \wedge \left( \frac{1}{n} - 1 + \log n \right) \right). 
\]

This leads to the contradiction
\[
\mathbb{E} > V(S_1(0), S_2(0), V(0), E(0), I(0)) + \zeta T = \infty. 
\]

Therefore, we must have \( \tau_{\infty} = \infty \) a.s.

3.2. Stationary Distribution and Ergodicity. The difference between model (1) and the stochastic model is that the stochastic model does not have the endemic equilibrium. Hence, we cannot study the persistence of the disease by studying the stability of the endemic equilibrium and turn to check out the existence and uniqueness of the stationary distribution for the system (4) which implies the persistence of the disease in some sense. In this section, based on the theory of Has’minskii [17], we verify that there is an ergodic stationary distribution, which reveals the persistence of the disease.

Define a parameter
\[
R_0' = \frac{\eta_1 \eta_2 \beta e^\theta}{(\mu_1 + d + \eta_1 + (\sigma_1^2/2)) (\theta + \mu_2 + (\sigma_2^2/2)) (\mu_2 + \sigma_2^2 + c + (\sigma_3^2/2))}.
\]

**Theorem 3.** Assume that \( R_0' > 1 \), then system (4) has a unique stationary distribution \( \Pi(\cdot) \) and it has the ergodic property.

**Proof.** In view of Theorem 2, we have obtained that for any initial value \( (S_1(0), S_2(0), V(0), E(0), I(0)) \in \mathbb{R}_+^5 \), there is a unique global solution \( (S_1, S_2, V, E, I) \in \mathbb{R}_+^5 \).

The diffusion matrix of system (4) is given by
\[
A = \begin{pmatrix}
\sigma_1^2 S_1^2 & 0 & 0 & 0 & 0 \\
0 & \sigma_2^2 S_2^2 & 0 & 0 & 0 \\
0 & 0 & \sigma_3^2 V^2 & 0 & 0 \\
0 & 0 & 0 & \sigma_4^2 E^2 & 0 \\
0 & 0 & 0 & 0 & \sigma_5^2 I^2
\end{pmatrix}. 
\]

Choose \( M = \min_{(S_1, S_2, V, E, I) \in \mathcal{D}_0} \{ \sigma_1^2 S_1^2, \sigma_2^2 S_2^2, \sigma_3^2 V^2, \sigma_4^2 E^2, \sigma_5^2 I^2 \} \); one can get that
\[
\sum_{i,j=1}^{5} a_{ij}(S_1, S_2, V, E, I) \xi_i \tilde{\xi}_j = \sigma_1^2 S_1^2 \xi_1^2 + \sigma_2^2 S_2^2 \xi_2^2 + \sigma_3^2 V^2 \xi_3^2 \\
+ \sigma_4^2 E^2 \xi_4^2 + \sigma_5^2 I^2 \xi_5^2 \geq M |\xi|^2, 
\]
where \( \chi \) is a constant satisfying \( 0 < \chi < 2 \mu_1^2 + \sigma_3^2 \nu_\alpha \nu_\beta \nu_\gamma \nu_\delta \nu_\eta \nu_\zeta \).

Then the condition \( A_1 \) in Lemma 1 is satisfied.

Construct a \( C^2 \)-function \( Q : \mathbb{R}_+^5 \to \mathbb{R} \) in the following from
\[
Q(S_1, S_2, V, E, I) = M(S_1 + S_2 + V + E + I - c_1 \ln S_1 \\
- c_2 \ln S_2 - c_3 \ln E - c_4 \ln I - \ln V) \\
+ \frac{1}{\chi + 1} (S_1 + S_2 + V + E + I)^{x+1} \\
- \ln S_1 - \ln S_2 - \ln E - \ln V \\
+ (S_1 + S_2 + V + E + I) \\
= MV_1 + V_2 + V_3 + V_4 + V_5 + V_6, 
\]

where \( \chi \) is a constant satisfying \( 0 < \chi < 2 \mu_1^2 + \sigma_3^2 \nu_\alpha \nu_\beta \nu_\gamma \nu_\delta \nu_\eta \nu_\zeta \).

Choose \( c_1 = \frac{A}{\mu_1 + d + \eta_1 + (\sigma_1^2/2)}, c_2 = \frac{A}{\theta + \mu_2 + (\sigma_2^2/2)}, c_3 = \frac{A}{\mu_2 + \sigma_2^2 + c + (\sigma_3^2/2)}, c_4 = \frac{A}{\mu_2 + \sigma_2^2 + c + (\sigma_3^2/2)}, \)

and \( M > 0 \) satisfies the following condition
\[
-M \lambda + C \leq -2, 
\]
\[
\lambda = 5A \left\{ \left[ \frac{\eta_1 \eta_2 \beta c \theta}{(\mu_1 + d + \eta_1 + (\sigma_1^2/2)) (\theta + \mu_2 + (\sigma_2^2/2)) (\mu_2 + \eta_2 + (\sigma_3^2/2)) (\mu_2 + c + (\sigma_3^2/2))} \right]^{1/5} - 1 \right\} = 5A \left( \frac{(R_0)^{1/5} - 1}{R_0} \right) > 0, \quad (32)
\]

\[
C = \sup_{(S_0, S_1, V, E, I) \in B^2} \left\{ -\frac{1}{2} \left[ \mu - \frac{1}{2} \sigma (\sigma_1^2 \nu_2 \sigma_3^2 \sigma_4^2) \right] + \delta + A + \frac{b}{r} + M \left( \frac{\sigma_2^2}{2} \mu_2 + \frac{\sigma_2^2 + \sigma_3^2 + \sigma_4^2}{2} \right) \right\}.
\]

It is easy to check that

\[
\lim_{k \to \infty} \inf_{Q(S_1, S_2, V, E, I) \in B^2} Q(S_1, S_2, V, E, I) = \infty, \quad (34)
\]

where \( U_k = \{(1/k, k) \times (1/k, k) \times (1/k, k) \times (1/k, k) \times (1/k, k) \} \). Furthermore, \( Q(S_1, S_2, V, E, I) \) is a continuous function. Hence, \( Q(S_1, S_2, V, E, I) \) must have a minimum point \((S_1(0), S_2(0), V(0), E(0), I(0))\) in the interior of \( R^5_+ \). Then we define a nonnegative \( C^1 \)-function \( V : R^5_+ \rightarrow R_+ \) as follows:

\[
V(S_1, S_2, V, E, I) = Q(S_1, S_2, V, E, I) - Q(S_1(0), S_2(0), V(0), E(0), I(0)).
\]

Making use of Itô’s formula, we have

\[
LV_1 = -\left( \frac{\mu_2 (S_2 + V + E + I) + (\mu_1 + d) S_1 + \frac{c_i A}{S_1} + \frac{c_B S_1 (b(S_1 + V) + \frac{c_B S_1 \beta c \theta}{S_1})}{S_1} + \frac{c_B S_1 (b(S_1 + V) + \frac{c_B S_1 \beta c \theta}{S_1})}{S_1} - \frac{c_B S_1 \beta c \theta}{S_1} + \frac{c_B \theta c \mu_2 + \sigma_2^2}{2} + c_i \left( \frac{\mu_2 + d + \eta_1 + (\sigma_2^2/2)}{2} \right) + c_i \left( \frac{\mu_2 + c + \sigma_2^2}{2} \right) \right) \left( \frac{b(S_1 + V)}{1 + r(S_1 + V)} \right) + \frac{b(S_1 + V)}{1 + r(S_1 + V)} \left( \frac{\sigma_2^2}{2} \right)
\]

Using the inequality \( a + b + c + d + e \geq 5 \sqrt[5]{abcde} \), \( a, b, c, d, e > 0 \) leads to

\[
LV_1 = \left( A + \frac{b(S_2 + V)}{1 + r(S_1 + V)} - (\mu_1 + d) S_1 - \mu_2 (S_2 + V + E + I) - cl \right) + \frac{1}{2} \left( \frac{b(S_1 + S_2 + V + E + I)}{1 + r(S_1 + V)} \right)^{x-1} \times \left( \frac{c_i (S_1 + S_2 + V + E + I)^{x-1}}{2} \right) \leq \left( S_1 + S_2 + V + E + I \right)^{x-1} \left( A + \frac{b}{r} - \mu (S_1 + S_2 + V + E + I) \right) + \frac{1}{2} \left( \frac{b(S_1 + S_2 + V + E + I)}{1 + r(S_1 + V)} \right)^{x-1} \times \left( \frac{c_i (S_1 + S_2 + V + E + I)^{x-1}}{2} \right) \]

\[
= \left( A + \frac{b}{r} \right) (S_1 + S_2 + V + E + I)^{x-1} - \left( \frac{1}{2} \right) \left( \frac{b(S_1 + S_2 + V + E + I)}{1 + r(S_1 + V)} \right)^{x-1} (S_1 + S_2 + V + E + I)^{x-1}.
\]

(38)
Thus, we can construct a compact subset $D$ such that the condition $A_2$ in Lemma 1 holds. Define the bounded closed set

$$D = \left\{ \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5 \in [0, \infty) : \epsilon_1 \leq \epsilon_2 \leq \epsilon_3 \leq \epsilon_4 \leq \epsilon_5 \leq 1 \right\},$$

(47)

where $\epsilon_i > 0 (i = 1, 2, 3, 4, 5)$ are sufficiently small constants satisfying the following conditions:

$$-\frac{1}{\epsilon_1} + F \leq -1,$$

(48)

$$-M\lambda - \frac{\epsilon_5}{\epsilon_2} + C \leq -1,$$

(49)

$$-M\lambda + M\beta\epsilon_3 \left( \frac{1 + \epsilon}{M} + c_2 + \epsilon \right) + C \leq -1,$$

(50)

$$-\frac{\beta\epsilon_2\epsilon_4}{\epsilon_3} + F \leq -1,$$

(51)

$$-\frac{\theta S_2}{\epsilon_5} + F \leq -1,$$

(52)

$$-\frac{1}{\epsilon_1^2} \mu \left( -\frac{1}{\epsilon_2^2} \frac{1}{\epsilon_3^2} \frac{1}{\epsilon_4^2} \frac{1}{\epsilon_5^2} \right) + \frac{c_2 + \epsilon}{M} + G \leq -1,$$

(53)

$$-\frac{1}{\epsilon_1^2} \mu \left( -\frac{1}{\epsilon_2^2} \frac{1}{\epsilon_3^2} \frac{1}{\epsilon_4^2} \frac{1}{\epsilon_5^2} \right) \leq -1,$$

(54)

$$-\frac{1}{\epsilon_1^2} \mu \left( -\frac{1}{\epsilon_2^2} \frac{1}{\epsilon_3^2} \frac{1}{\epsilon_4^2} \frac{1}{\epsilon_5^2} \right) + J \leq -1,$$

(55)

$$-\frac{1}{\epsilon_1^2} \mu \left( -\frac{1}{\epsilon_2^2} \frac{1}{\epsilon_3^2} \frac{1}{\epsilon_4^2} \frac{1}{\epsilon_5^2} \right) + K \leq -1,$$

(56)

$$-\frac{1}{\epsilon_1^2} \mu \left( -\frac{1}{\epsilon_2^2} \frac{1}{\epsilon_3^2} \frac{1}{\epsilon_4^2} \frac{1}{\epsilon_5^2} \right) + L \leq -1,$$

(57)

where $F, G, H, J, K, \text{and } L$ are positive constants which can be seen from (60), (68), (70), (72), (74), and (76), respectively. Note that for sufficiently small $\epsilon_i, i = 1, 2, 3, 4, 5$. For
convenience, we divide \( \mathbb{R}_c^2 \setminus D \) into ten domains

\[
D_1 = \{(S_1, S_2, V, E, I) \in \mathbb{R}_c^2 : 0 < S_1 < \varepsilon_1\},
\]

\[
D_2 = \{(S_1, S_2, V, E, I) \in \mathbb{R}_c^2 : 0 < S_1 < \varepsilon_2, V \geq \varepsilon_3\},
\]

\[
D_3 = \{(S_1, S_2, V, E, I) \in \mathbb{R}_c^2 : 0 < I < \varepsilon_3\},
\]

\[
D_4 = \{(S_1, S_2, V, E, I) \in \mathbb{R}_c^2 : S_1 \geq \varepsilon_2, I \geq \varepsilon_3, 0 < E < \varepsilon_4\},
\]

\[
D_5 = \{(S_1, S_2, V, E, I) \in \mathbb{R}_c^2 : 0 < V < \varepsilon_5, S_2 \geq \varepsilon_4\},
\]

\[
D_6 = \{(S_1, S_2, V, E, I) \in \mathbb{R}_c^2 : S_1 > \frac{1}{\varepsilon_1}\},
\]

\[
D_7 = \{(S_1, S_2, V, E, I) \in \mathbb{R}_c^2 : S_2 > \frac{1}{\varepsilon_2}\},
\]

\[
D_8 = \{(S_1, S_2, V, E, I) \in \mathbb{R}_c^2 : I > \frac{1}{\varepsilon_3}\},
\]

\[
D_9 = \{(S_1, S_2, V, E, I) \in \mathbb{R}_c^2 : E > \frac{1}{\varepsilon_4}\},
\]

\[
D_{10} = \{(S_1, S_2, V, E, I) \in \mathbb{R}_c^2 : V > \frac{1}{\varepsilon_5}\}.
\]

(58)

Next, we will show that \( LV(S_1, S_2, V, E, I) \leq -1 \) on \( \mathbb{R}_c^2 \setminus D \), which is equivalent to proving it on the above ten domains.

**Case 1.** If \((S_1, S_2, V, E, I) \in D_1\), one can get that

\[
LV \leq \frac{A}{S_1} + M \beta I \left( c_2 + \varepsilon + \frac{1 + \varepsilon}{M} \right) - \frac{1}{2} \left[ \mu - \frac{1}{2} \chi \left( \sigma^1 \sigma_2^2 \sigma_3^3 \sigma_4^4 \sigma_5^5 \right) \right]
\]

\[
\times \left( S_1^{x+1} + S_2^{x+1} + V^{x+1} + E^{x+1} + I^{x+1} \right)
\]

\[
+ \mu_1 + d + \eta_1 + \theta + 3 \mu_2 + \eta_2 + \delta + A + \frac{b}{\tau}
\]

\[
+ M \left( \frac{\sigma^2_2}{2} + \mu_2 + \delta + \frac{b}{\tau} \right) + \frac{\sigma^1_2 + \sigma^2_3 + \sigma^3_3 + \sigma^4_4}{2}
\]

\[
\leq -\frac{A}{S_1} + F \leq -\frac{A}{\varepsilon_1} + F \leq -1,
\]

where

\[
F = \sup_{(S_1, S_2, V, E, I) \in \mathbb{R}_c^2} \left\{ M \beta I \left( c_2 + \varepsilon + \frac{1 + \varepsilon}{M} \right) - \frac{1}{2} \left[ \mu - \frac{1}{2} \chi \left( \sigma^1 \sigma_2^2 \sigma_3^3 \sigma_4^4 \sigma_5^5 \right) \right]
\]

\[
\times \left( S_1^{x+1} + S_2^{x+1} + V^{x+1} + E^{x+1} + I^{x+1} \right)
\]

\[
+ \mu_1 + d + \eta_1 + \theta + 3 \mu_2 + \eta_2 + \delta + A + \frac{b}{\tau}
\]

\[
+ M \left( \frac{\sigma^2_2}{2} + \mu_2 + \delta + \frac{b}{\tau} \right) + \frac{\sigma^1_2 + \sigma^2_3 + \sigma^3_3 + \sigma^4_4}{2}.
\]

According to (48), we have

\[
LV \leq -1, \text{ for any } (S_1, S_2, V, E, I) \in D_1.
\]

(61)

**Case 2.** If \((S_1, S_2, V, E, I) \in D_2\), we have

\[
LV \leq -M \lambda - \frac{\delta V}{S_2} - \frac{1}{2} \left[ \mu - \frac{1}{2} \chi \left( \sigma^1 \sigma_2^2 \sigma_3^3 \sigma_4^4 \sigma_5^5 \right) \right]
\]

\[
\times \left( S_1^{x+1} + S_2^{x+1} + V^{x+1} + E^{x+1} + I^{x+1} \right) + \mu_1 + d + \eta_1 + \theta + 3 \mu_2 + \eta_2 + \delta + A + \frac{b}{\tau}
\]

\[
+ M \left( \frac{\sigma^2_2}{2} + \mu_2 + \delta + \frac{b}{\tau} \right) + \frac{\sigma^1_2 + \sigma^2_3 + \sigma^3_3 + \sigma^4_4}{2}
\]

\[
\leq -M \lambda - \frac{\delta V}{S_2} + C \leq -M \lambda - \frac{\varepsilon_2}{\varepsilon_2} + C.
\]

(62)

where \( C \) is defined in (33).

In view of (49), we can obtain that for sufficiently small \( \varepsilon_i (i = 2, 5) \), \( LV \leq -1 \) for any \((S_1, S_2, V, E, I) \in D_2\).

**Case 3.** If \((S_1, S_2, V, E, I) \in D_3\), one can see that

\[
LV \leq -M \lambda + M \beta I \left( c_2 + \varepsilon + \frac{1 + \varepsilon}{M} \right)
\]

\[
- \frac{1}{2} \left[ \mu - \frac{1}{2} \chi \left( \sigma^1 \sigma_2^2 \sigma_3^3 \sigma_4^4 \sigma_5^5 \right) \right]
\]

\[
\times \left( S_1^{x+1} + S_2^{x+1} + V^{x+1} + E^{x+1} + I^{x+1} \right) + \mu_1 + d + \eta_1 + \theta + 3 \mu_2 + \eta_2 + \delta + A + \frac{b}{\tau}
\]

\[
+ M \left( \frac{\sigma^2_2}{2} + \mu_2 + \delta + \frac{b}{\tau} \right) + \frac{\sigma^1_2 + \sigma^2_3 + \sigma^3_3 + \sigma^4_4}{2}.
\]

(63)

We obtain that

\[
LV \leq -1 \text{ for any } (S_1, S_2, V, E, I) \in D_3.
\]

(64)

**Case 4.** If \((S_1, S_2, V, E, I) \in D_4\), one can see that

\[
LV \leq -\frac{\beta S_1}{E} + M \beta I \left( c_2 + \varepsilon + \frac{1 + \varepsilon}{M} \right)
\]

\[
- \frac{1}{2} \left[ \mu - \frac{1}{2} \chi \left( \sigma^1 \sigma_2^2 \sigma_3^3 \sigma_4^4 \sigma_5^5 \right) \right]
\]

\[
\times \left( S_1^{x+1} + S_2^{x+1} + V^{x+1} + E^{x+1} + I^{x+1} \right) + \mu_1 + d + \eta_1 + \theta + 3 \mu_2 + \eta_2 + \delta + A + \frac{b}{\tau}
\]

\[
+ M \left( \frac{\sigma^2_2}{2} + \mu_2 + \delta + \frac{b}{\tau} \right) + \frac{\sigma^1_2 + \sigma^2_3 + \sigma^3_3 + \sigma^4_4}{2}
\]

\[
\leq -\frac{\beta S_1}{E} + F \leq -\frac{\beta \varepsilon \varepsilon_4}{\varepsilon_4} + F.
\]

(65)

In view of (51), we can obtain that for sufficiently small \( \varepsilon_i (i = 3, 4) \), \( LV \leq -1 \) for any \((S_1, S_2, V, E, I) \in D_4\).
Case 5. If \((S_1, S_2, V, E, I) \in D_5\), one can see that
\[
LV \leq -\frac{\theta S_2}{V} + M\beta\left(c_2 + \varepsilon + \frac{1 + \varepsilon}{M}\right) - \frac{1}{2} \left[\mu - \frac{1}{2} \chi(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2)\right] \\
\left(\prod^{\text{1st}} + \prod^{\text{2nd}} + V^{\text{1st}} + E^{\text{1st}} + I^{\text{1st}}\right) + \mu_1 + d + \eta_1 + \theta + 3\mu_2 + \eta_2 + \delta + A + \frac{b}{\tau} + M\left(\frac{\sigma_2}{2} + \mu_2 + \delta + \frac{b}{\tau}\right) + \frac{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2}{2} \\
\leq -\frac{\theta S_2}{V} + F \leq \frac{\theta c_2}{\varepsilon} + F.
\]

We can obtain that for sufficiently small \(\varepsilon\) \((i = 2, 5)\), \(LV \leq -1\) for any \((S_1, S_2, V, E, I) \in D_5\).

Case 6. If \((S_1, S_2, V, E, I) \in D_6\), one can see that
\[
LV \leq -\frac{1}{4} \left[\mu - \frac{1}{2} \chi(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2)\right] S_{1 \text{st}}^{\text{11}} - \frac{1}{4} \left[\mu - \frac{1}{2} \chi(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2)\right] \times S_{2 \text{nd}}^{\text{11}} - \frac{1}{2} \left[\mu - \frac{1}{2} \chi(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2)\right] \left(\prod^{\text{1st}} + V^{\text{1st}} + E^{\text{1st}} + I^{\text{1st}}\right) + \mu_1 + d + \eta_1 + \theta + 3\mu_2 + \eta_2 + \delta + A + \frac{b}{\tau} + M\left(\frac{\sigma_2}{2} + \mu_2 + \delta + \frac{b}{\tau}\right) + \frac{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2}{2} \\
+ G \leq -\frac{1}{4} \left[\mu - \frac{1}{2} \chi(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2)\right] \frac{1}{\varepsilon_1^{\text{11}}} + G,
\]

where
\[
G = \sup_{(S_1, S_2, V, E, I) \in D_6} \left\{ -\frac{1}{4} \left[\mu - \frac{1}{2} \chi(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2)\right] S_{1 \text{st}}^{\text{11}} - \frac{1}{2} \left[\mu - \frac{1}{2} \chi(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2)\right] \times \left(\prod^{\text{1st}} + V^{\text{1st}} + E^{\text{1st}} + I^{\text{1st}}\right) + \mu_1 + d + \eta_1 + \theta + 3\mu_2 + \eta_2 + \delta + A + \frac{b}{\tau} + M\left(\frac{\sigma_2}{2} + \mu_2 + \delta + \frac{b}{\tau}\right) + \frac{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2}{2}\right\}.
\]

By (53), we conclude that \(LV \leq -1\) on \(D_6\).

Case 7. If \((S_1, S_2, V, E, I) \in D_7\), one can see that
\[
LV \leq -\frac{1}{4} \left[\mu - \frac{1}{2} \chi(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2)\right] S_{1 \text{st}}^{\text{11}} - \frac{1}{4} \left[\mu - \frac{1}{2} \chi(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2)\right] \times S_{2 \text{nd}}^{\text{11}} - \frac{1}{2} \left[\mu - \frac{1}{2} \chi(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2)\right] \left(\prod^{\text{1st}} + V^{\text{1st}} + E^{\text{1st}} + I^{\text{1st}}\right) + \mu_1 + d + \eta_1 + \theta + 3\mu_2 + \eta_2 + \delta + A + \frac{b}{\tau} + M\left(\frac{\sigma_2}{2} + \mu_2 + \delta + \frac{b}{\tau}\right) + \frac{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2}{2} \\
+ M\beta\left(c_2 + \varepsilon + \frac{1 + \varepsilon}{M}\right) + \frac{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2}{2} \\
\leq -\frac{1}{4} \left[\mu - \frac{1}{2} \chi(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2)\right] \frac{1}{\varepsilon_1^{\text{11}}} + H.
\]

where
\[
H = \sup_{(S_1, S_2, V, E, I) \in D_7} \left\{ -\frac{1}{4} \left[\mu - \frac{1}{2} \chi(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2)\right] S_{1 \text{st}}^{\text{11}} - \frac{1}{4} \left[\mu - \frac{1}{2} \chi(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2)\right] \times S_{2 \text{nd}}^{\text{11}} - \frac{1}{2} \left[\mu - \frac{1}{2} \chi(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2)\right] \left(\prod^{\text{1st}} + V^{\text{1st}} + E^{\text{1st}} + I^{\text{1st}}\right) + \mu_1 + d + \eta_1 + \theta + 3\mu_2 + \eta_2 + \delta + A + \frac{b}{\tau} + M\left(\frac{\sigma_2}{2} + \mu_2 + \delta + \frac{b}{\tau}\right) + \frac{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2}{2} \right\}.
\]

Together with (54), we can deduce that \(LV \leq -1\) on \(D_7\).

Case 8. If \((S_1, S_2, V, E, I) \in D_8\), one can see that
\[
LV \leq -\frac{1}{4} \left[\mu - \frac{1}{2} \chi(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2)\right] \left(\prod^{\text{1st}} + V^{\text{1st}} + E^{\text{1st}} + I^{\text{1st}}\right) + \mu_1 + d + \eta_1 + \theta + 3\mu_2 + \eta_2 + \delta + A + \frac{b}{\tau} + M\left(\frac{\sigma_2}{2} + \mu_2 + \delta + \frac{b}{\tau}\right) + \frac{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2}{2} \\
+ M\beta\left(c_2 + \varepsilon + \frac{1 + \varepsilon}{M}\right) + \frac{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2}{2} \\
\leq -\frac{1}{4} \left[\mu - \frac{1}{2} \chi(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2)\right] \frac{1}{\varepsilon_1^{\text{11}}} + f.
\]
where

\[
J = \sup_{(S_S, S_T, V, E, I) \in \mathbb{R}^+} \left\{ \frac{1}{4} \left[ -\mu - I \frac{1}{2} \chi \left( \sigma_1^2 \nu \sigma_3^2 \nu \sigma_5^2 \nu \sigma_7^2 \right) \right] \right\}^{V_t+1} \\
- \frac{1}{2} \left[ \mu - \frac{1}{2} \chi \left( \sigma_1^2 \nu \sigma_3^2 \nu \sigma_5^2 \nu \sigma_7^2 \right) \right] \times \left( S_t^{V_t+1} + S_{t+1}^V + V_{t+1} + E_{t+1} \right) + \mu_1 + d + \eta_1 + \theta + 3 \mu_2 \\
+ \eta_3 + \delta + A + \frac{b}{\tau} + M \left( \frac{\sigma_2^2}{2} + \mu_2 + \delta + \frac{b}{\tau} \right) \\
+ M \beta I \left( c_2 + \epsilon + \frac{1}{M} \right) + \frac{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2}{2} \right),
\]

(72)

which together with (55) implies that \( LV \leq -1 \) on \( D_y \).

Case 9. If \((S_S, S_T, V, E, I) \in D_y\), we obtain

\[
LV \leq \left\{ \frac{1}{4} \left[ -\mu - I \frac{1}{2} \chi \left( \sigma_1^2 \nu \sigma_3^2 \nu \sigma_5^2 \nu \sigma_7^2 \right) \right] \right\}^{V_t+1} \\
- \frac{1}{2} \left[ \mu - \frac{1}{2} \chi \left( \sigma_1^2 \nu \sigma_3^2 \nu \sigma_5^2 \nu \sigma_7^2 \right) \right] \times \left( S_t^{V_t+1} + S_{t+1}^V + V_{t+1} + E_{t+1} \right) \\
+ \mu_1 + d + \eta_1 + \theta + 3 \mu_2 + \eta_3 + \delta + A + \frac{b}{\tau} + M \left( \frac{\sigma_2^2}{2} + \mu_2 + \delta + \frac{b}{\tau} \right) \\
+ M \beta I \left( c_2 + \epsilon + \frac{1}{M} \right) + \frac{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2}{2} \right) \\
\leq \left\{ \frac{1}{4} \left[ -\mu - I \frac{1}{2} \chi \left( \sigma_1^2 \nu \sigma_3^2 \nu \sigma_5^2 \nu \sigma_7^2 \right) \right] \right\}^{V_t+1} + K \\
\leq - \frac{1}{4} \left[ -\mu - I \frac{1}{2} \chi \left( \sigma_1^2 \nu \sigma_3^2 \nu \sigma_5^2 \nu \sigma_7^2 \right) \right] \frac{1}{e^K} + K \\
\leq - \frac{1}{4} \left[ -\mu - I \frac{1}{2} \chi \left( \sigma_1^2 \nu \sigma_3^2 \nu \sigma_5^2 \nu \sigma_7^2 \right) \right] \frac{1}{e^K} + K,
\]

(73)

where

\[
K = \sup_{(S_S, S_T, V, E, I) \in \mathbb{R}^+} \left\{ \frac{1}{4} \left[ -\mu - I \frac{1}{2} \chi \left( \sigma_1^2 \nu \sigma_3^2 \nu \sigma_5^2 \nu \sigma_7^2 \right) \right] \right\}^{V_t+1} \\
- \frac{1}{2} \left[ \mu - \frac{1}{2} \chi \left( \sigma_1^2 \nu \sigma_3^2 \nu \sigma_5^2 \nu \sigma_7^2 \right) \right] \times \left( S_t^{V_t+1} + S_{t+1}^V + V_{t+1} + E_{t+1} \right) \\
+ \mu_1 + d + \eta_1 + \theta + 3 \mu_2 + \eta_3 + \delta + A + \frac{b}{\tau} + M \left( \frac{\sigma_2^2}{2} + \mu_2 + \delta + \frac{b}{\tau} \right) \\
+ M \beta I \left( c_2 + \epsilon + \frac{1}{M} \right) \right) + \frac{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2}{2} \right).
\]

(74)

By (56), we can conclude that \( LV \leq -1 \) on \( D_y \).

Case 10. If \((S_S, S_T, V, E, I) \in D_{10}\), it follows that

\[
LV \leq - \frac{1}{4} \left[ \mu - \frac{1}{2} \chi \left( \sigma_1^2 \nu \sigma_3^2 \nu \sigma_5^2 \nu \sigma_7^2 \right) \right] V_t^{V_t+1} \\
\leq - \frac{1}{4} \left[ \mu - \frac{1}{2} \chi \left( \sigma_1^2 \nu \sigma_3^2 \nu \sigma_5^2 \nu \sigma_7^2 \right) \right] \times \left( S_t^{V_t+1} + S_{t+1}^V + V_{t+1} + E_{t+1} \right) \\
+ \mu_1 + d + \eta_1 + \theta + 3 \mu_2 + \eta_3 + \delta + A + \frac{b}{\tau} + M \left( \frac{\sigma_2^2}{2} + \mu_2 + \delta + \frac{b}{\tau} \right) + M \beta I \left( c_2 + \epsilon + \frac{1}{M} \right) \right) + \frac{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2}{2} \right)
\]

(75)

where

\[
L = \sup_{(S_S, S_T, V, E, I) \in \mathbb{R}^+} \left\{ \frac{1}{4} \left[ -\mu - I \frac{1}{2} \chi \left( \sigma_1^2 \nu \sigma_3^2 \nu \sigma_5^2 \nu \sigma_7^2 \right) \right] \right\}^{V_t+1} \\
- \frac{1}{2} \left[ \mu - \frac{1}{2} \chi \left( \sigma_1^2 \nu \sigma_3^2 \nu \sigma_5^2 \nu \sigma_7^2 \right) \right] \times \left( S_t^{V_t+1} + S_{t+1}^V + V_{t+1} + E_{t+1} \right) \\
+ \mu_1 + d + \eta_1 + \theta + 3 \mu_2 + \eta_3 + \delta + A + \frac{b}{\tau} + M \left( \frac{\sigma_2^2}{2} + \mu_2 + \delta + \frac{b}{\tau} \right) + M \beta I \left( c_2 + \epsilon + \frac{1}{M} \right) \right) + \frac{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2}{2}
\]

(76)

Combining with (57) yields \( LV \leq -1 \) on \( D_{10} \).

Obviously, \( A_3 \) in Lemma 2.1 is satisfied. According to Lemma 2.1, we can obtain that system (4) is ergodic and has a unique stationary distribution.

**Remark 4.** Theorem 3 reveals that system (4) has a unique ergodic stationary distribution \( \pi(\cdot) \) if \( R_0 = \eta_1 \eta_2 \beta \theta / ((\mu_1 + d + \eta_1 + (\sigma_2^2/2))(\mu_2 + (\sigma_3^2/2))(\mu_2 + \epsilon + (\sigma_4^2/2))) > 1 \). Note that the expression of \( R_0 \) coincide with the threshold \( R_0 \) of the deterministic system (1) if there is no stochastic perturbation. This shows that we generalize the result of the deterministic system.

### 3.3. Extinction of the Disease

As it is well known, one of the main concern of epidemiology is how we regulate the disease dynamics in order to eradicate the disease in the long term. Moreover, in [10], Allen et al. proposed and studied several types of stochastic epidemic models and pointed out that the stochastic models should suit the question of disease.
extinction better. Hence, in this section, we shall establish some sufficient conditions for extinction of the disease in stochastic model (4).

**Theorem 5.** Let \( S_i(t), S_2(t), V(t), E(t), I(t) \) be the solution of system (4) with any initial value \( S_i(0), S_2(0), V(0), E(0), I(0) \) \( \in \mathbb{R}_+^5 \). If \( (\beta (e + 1)(Ar + b)(\mu_2 + \eta_2))/\mu T \) \(<\sigma_1^2/2) + \mu_2 + c)\wedge(\sigma_2^2/2) \), then the disease \( I(t) \) will extinct exponentially with probability one, i.e., moreover

\[
\lim_{t \to \infty} \sup_{t \in [0,T]} I(t) \leq \frac{\eta_2}{\mu_2} \exp \left( (x_0 + \bar{e})t \right), \forall t \geq T.
\] (80)

**Remark 6.** Theorem 5 suggests that the disease will become extinct if \( (\beta (e + 1)(Ar + b)(\mu_2 + \eta_2))/\mu T \) \(<\sigma_1^2/2) + \mu_2 + c)\wedge(\sigma_2^2/2) \).

### 4. Numerical Examples

In this section, we will give two numerical examples to illustrate the main theoretical results obtained in this paper. The numerical simulation method can be found in [9, 22, 23]. The following is a corresponding discrete equations of system (4):

**Example 1.** We take parameters as \( A = 45, \beta = 0.05, \mu_1 = 0.05, \mu_2 = 0.06, e = 3.2, r = 0.01, b = 0.3, \eta_1 = 0.25, \eta_2 = 0.5, \) and \( \theta = 1.1, \delta = 0.01, c = 0.02, \sigma_1 = 0.2, \sigma_2 = 0.05, \sigma_3 = 0.3, \sigma_4 = 0.03, \) and \( \sigma_5 = 0.02. \) It is clear that conditions of Theorem 3 are satisfied; by calculating, we have the basic reproduction number \( R_0^2 = \eta_1 \eta_2 \beta e \theta ((\mu_1 + d + \eta_1 + (\sigma_1^2/2))(\theta + \mu_2 + (\sigma_2^2/2))) = 1.17080128 \). The histogram and the smoothing curves of the probability density functions of \( S_1(t), S_2(t), V(t), E(t), I(t) \) are given in Figure 1.
Example 2. We take parameters as $A = 1000$, $\beta = 0.0001$, $\mu_1 = 0.1$, $\mu_2 = 0.25e = 0.18$, $\tau = 0.002$, $b = 1.5$, $\eta_1 = 1.06$, $\eta_2 = 3.4$, and $\theta = 0.1$, $\delta = 0.4$, $c = 0.05$, $\sigma_1 = 0.2$, $\sigma_2 = 0.5$, $\sigma_3 = 0.31$, $\sigma_4 = 1.65$, and $\sigma_5 = 1.5$. It is clear that conditions of Theorem 5 are satisfied.

The curves on the persistence of $S_1(t), S_2(t), V(t), E(t)$ and extinction of $I(t)$ for stochastic model (4) are given in Figure 2, where the initial value is $(S_1(0), S_2(0), V(0), E(0), I(0)) = (1, 1.5, 1, 1, 1)$.

Remark 7. In this paper, we consider the stochastic perturbations for deterministic model (1) and derived model (4). Thus, model (4) can be specialized as models (1). Hence, model (4) can be seen as a general model compared to model (1), and the theoretical results obtained in this article can be seen as the extensions and supplements of the model and the theoretical results obtained in [8].

5. Conclusion

In this paper, firstly, we have considered the stochastic perturbations for deterministic system (1) and established corresponding stochastic system (4). Secondly, under the condition $R_0 > 1$ and applying the theory of stochastic differential equations, Has'minskii theory, Ito's formula, and Lyapunov function method, we obtained some sufficient conditions on the existence of ergodic stationary distribution of model (4). We also established sufficient conditions on the extinction of the disease. Finally, two examples are presented to validate the main results of this paper. The results obtained in this paper suggest that stochastic perturbations have remarkable effects on the disease in model (4). Especially, from the numerical simulations, we can see that, under the stochastic perturbations, the disease in the stochastic system will become extinct more quickly than the corresponding deterministic one.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

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