SPECTRA OF TUKEY TYPES OF ULTRAFILTERS ON BOOLEAN ALGEBRAS

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Abstract. Extending recent investigations on the structure of Tukey types of ultrafilters on \( \mathcal{P}(\omega) \) to Boolean algebras in general, we classify the spectra of Tukey types of ultrafilters for several classes of Boolean algebras. These include the classes of free Boolean algebras, superatomic Boolean algebras generated by almost disjoint families, interval algebras, tree algebras, and pseudo-tree algebras. We also give conditions guaranteeing the existence of an ultrafilter of maximum Tukey type, and show that each free algebra has exactly one Tukey type, the maximum.

1. Introduction

The investigation of the structure of Tukey types of ultrafilters on \( \omega \) is an area of much recent and ongoing research. This line of research was rekindled, after Isbell’s study in [10], by Milovich in [13]. One of the interests in this research stems from the connection via Stone duality between Tukey types of ultrafilters on \( \omega \) and cofinal types of neighborhood bases on the \( \check{\text{C}} \)ech-Stone compactification \( \beta\omega \) of the natural numbers and its \( \check{\text{C}} \)ech-Stone remainder \( \beta\omega \setminus \omega \). Since then, the structure of the Tukey types of ultrafilters on \( \omega \) has undergone an explosion of activity, starting with [8] of Dobrinen and Todorcevic, and continuing in work of Milovich in [13], Raghavan and Todorcevic in [15], further work of Dobrinen and Todorcevic in [9], [7], Blass, Dobrinen, and Raghavan in [1], Dobrinen, Mijares, and Trujillo in [6], and most recently Dobrinen in [4]. These lines of study have all been focused on clarifying the structure of Tukey types of ultrafilters on \( \omega \). In particular, the aforementioned works provide good understanding of the Tukey structure for p-points and other ultrafilters possessing some means of diagonalization. However, the exact structure of the Tukey types of ultrafilters on \( \omega \) still needs much more investigation.

Recalling that by an ultrafilter on \( \omega \), we actually mean an ultrafilter on the Boolean algebra \( \mathcal{P}(\omega) \) (or the Boolean algebra \( \mathcal{P}(\omega)/\text{fin} \) if it is nonprincipal), we expand the investigation of the Tukey structure of ultrafilters to the general class of all Boolean algebras. Recall that an ultrafilter on a Boolean algebra \( \mathcal{B} \) is simply a maximal filter on \( \mathcal{B} \). Letting \( \leq \) denote the natural partial ordering on a Boolean algebra \( \mathcal{B} \) (defined by \( a \leq b \) if and only if \( a \wedge b = a \)), we note that for any ultrafilter \( \mathcal{U} \) on a Boolean algebra \( \mathcal{B} \), \( (\mathcal{U}, \geq) \) is a partial ordering.

We now define Tukey reducibility for partial orderings in general. For a partial order \( (P, \leq_P) \), a subset \( X \subseteq P \) is cofinal in \( P \) if for each \( p \in P \), there is an \( x \in X \) such that \( p \leq_P x \). A map \( f : (P, \leq_P) \to (Q, \leq_Q) \) is cofinal if for each cofinal \( X \subseteq P \), the image \( f[X] \) is cofinal in \( Q \). A set \( X \subseteq P \) is unbounded if there is no upper bound for \( X \) in \( P \); that is, there is no \( p \in P \) such that for each \( x \in X \), \( x \leq_P p \). A map \( g : (P, \leq_P) \to (Q, \leq_Q) \) is unbounded or
Tukey if for each unbounded $X \subseteq P$, the image $g[X]$ is unbounded in $Q$. Schmidt showed in [17] that the existence of a cofinal map is equivalent to the existence of a Tukey map. We say that $(P, \leq_P)$ is Tukey reducible to $(Q, \leq_Q)$, and write $(P, \leq_P) \leq_T (Q, \leq_Q)$, if there is a cofinal map from $(Q, \leq_Q)$ to $(P, \leq_P)$ ([19]). We say that $(P, \leq_P)$ is Tukey equivalent to $(Q, \leq_Q)$, and write $(P, \leq_P) \equiv_T (Q, \leq_Q)$, if and only if $(P, \leq_P) \leq_T (Q, \leq_Q)$ and $(Q, \leq_Q) \leq_T (P, \leq_P)$.

Given a Boolean algebra $B$, it follows from Stone duality that the Tukey type of an ultrafilter $U$ on $B$ is the same as the cofinal type of the neighborhood basis of $U$ in the Stone space of $B$. Thus, the study of Tukey types of ultrafilters on Boolean algebras is equivalent to the study of cofinal types of neighborhood bases of Boolean spaces, that is, zero-dimensional compact Hausdorff spaces.

A partial order $(P, \leq)$ is directed if for any two $p, q \in P$, there is an $r \in P$ such that $p \leq r$ and $q \leq r$. Every ultrafilter $U$ on a Boolean algebra, when partially ordered by $\geq$, is a directed partial ordering. It follows from work of Schmidt in [17] and is implicit in work of Tukey [19] that, for each infinite cardinal $\kappa$, the directed partial ordering $([\kappa]^{<\omega}, \subseteq)$ is the maximum among all Tukey types of directed partial orderings of cardinality $\kappa$. That is, every directed partial ordering of cardinality $\kappa$ is Tukey reducible to $([\kappa]^{<\omega}, \subseteq)$. The minimal Tukey type is 1 (the one-element partially ordered set). An ultrafilter partially ordered by $\geq$ has this minimal Tukey type if and only if it is principal. We mention that for directed partial orderings, Tukey types and cofinal types coincide. (See [18] for more background.)

The two main guiding questions in the study of Tukey types of ultrafilters on $\omega$ are Isbell’s Problem and the question of how closely related Tukey reducibility is to Rudin-Keisler reducibility. (The interested reader is referred to the recent survey paper [5] of Dobrinen for more on this area of research.) In this paper, we focus on extensions of Isbell’s Problem to all Boolean algebras, as well as related questions which naturally arise. In [10], Isbell showed (in ZFC) that there is an ultrafilter on $\omega$ with maximum Tukey type $([c]^{<\omega}, \subseteq)$. He then asked the following.

**Question 1** (Isbell, [10]). Is there an ultrafilter on $\omega$ which has cofinal type strictly below the maximum?

This was answered in the positive by Milovich in [13] using the extra axiom $\diamondsuit$, and later by Dobrinen and Todorcevic in [8] using the weaker assumption of the existence of p-points. It is, however, still unknown whether there is a model of ZFC in which all ultrafilters on $\omega$ have the maximum Tukey type.

An ultrafilter $U$ on a Boolean algebra $B$ has maximum Tukey type if $(U, \geq) \equiv_T ([|B|]^{<\omega}, \subseteq)$, which is the maximum possible cofinal type for a directed partial ordering of cardinality $|B|$. Isbell’s result in [10] that there is an ultrafilter on $\omega$ which has maximum Tukey type initially led us to ask whether this is true for Boolean algebras in general: Given a Boolean algebra, is there an ultrafilter which has the maximum Tukey type? We will show that this is not the case: there are several classes of Boolean algebras which do not have any ultrafilters with the maximum Tukey type, the uncountable interval algebras being the simplest of these (see Theorem 21). This, along with the extension of Question 1 to Boolean algebras in general, led to the following collection of questions which we investigate. Our intuition was that Boolean algebras with strong structure would yield more precise descriptions of the structure of the Tukey types of their ultrafilters.
**Question 2.** For which Boolean algebras is there an ultrafilter which has the maximum Tukey type?

**Question 3.** Are there Boolean algebras on which all ultrafilters have the maximum Tukey type?

**Question 4.** Which Boolean algebras have ultrafilters with non-maximum Tukey type?

**Question 5.** Can we characterize those Boolean algebras all of whose ultrafilters have Tukey type strictly below the maximum?

We answer these questions for certain classes of Boolean algebras by obtaining the finer results of classifying their Tukey spectra. By the Tukey spectrum of a Boolean algebra \( B \), denoted \( Ts(B) \), we mean the set of all Tukey types of all ultrafilters on \( B \); this is exactly the structure of the Tukey types of ultrafilters on \( B \). We use the terminology Tukey spectra of a class \( C \) of Boolean algebras to mean the collection \( \{Ts(B) : B \in C\} \). The following is the main focus of this paper.

**Question 6.** For which classes of Boolean algebras can we classify their Tukey spectra?

We answer question 6 for the classes of free Boolean algebras, superatomic Boolean algebras generated by almost disjoint families, interval algebras, tree algebras, and pseudo-tree algebras. Section 2 is concerned with finding conditions which guarantee the existence of an ultrafilter with the maximum Tukey type, thus answering Question 2 for certain families of Boolean algebras. In particular, we show that countable completeness along with an independent family of maximal size guarantee an ultrafilter with maximum Tukey type. In Theorem 12, we show that the class of free Boolean algebras answers Question 3 positively; hence the Tukey spectrum of each free Boolean algebra consists exactly of the maximum Tukey type. In Section 19, we find the Tukey spectra of superatomic Boolean algebras generated by almost disjoint families. In particular, we show that the minimum and maximum Tukey types are always attained. Section 4 is concerned with the families of interval algebras, tree algebras, and pseudo-tree algebras. There, we classify the Tukey spectra for Boolean algebras in each of these classes.

We conclude the introduction with some notation and basic facts which will be useful throughout the paper. For \( \kappa, \mu \) cardinals, the partial order \( \leq \) on \( \kappa \times \mu \) is defined coordinate-wise: \((\alpha, \beta) \leq (\alpha', \beta') \) if \( \alpha \leq \alpha' \) and \( \beta \leq \beta' \). More generally, we fix the following notation.

**Definition 7.** Given a collection of cardinals \( \{\kappa_i : i \in I\} \) for some index set \( I \), \( \prod_{i \in I} \kappa_i \) denotes the collection of all functions \( f : I \to \bigcup \{\kappa_i : i \in I\} \) such that for each \( i \in I \), \( f(i) \in \kappa_i \). The partial ordering \( \leq \) on \( \prod_{i \in I} \kappa_i \) is coordinate-wise: For \( f, g \in \prod_{i \in I} \kappa_i \), \( f \leq g \) if and only if for all \( i \in I \), \( f(i) \leq g(i) \).

**Fact 8.**
1. For any infinite cardinal \( \kappa \), every partial ordering of cardinality less than or equal to \( \kappa \) is Tukey reducible to \( ([\kappa]^<\omega, \subseteq) \).
2. Let \( \mathcal{U} \) be an ultralfilter on a Boolean algebra \( B \), and let \( \kappa = |B| \). Then \( (\mathcal{U}, \geq) \equiv_T ([\kappa]^<\omega, \subseteq) \) if and only if there is a subset \( \mathcal{X} \subseteq \mathcal{U} \) of cardinality \( \kappa \) such that for each infinite \( \mathcal{Y} \subseteq \mathcal{X} \), \( \mathcal{Y} \) is unbounded in \( \mathcal{U} \).
3. \( (\omega, \leq) \equiv_T ([\omega]^<\omega, \subseteq) \).
4. For any uncountable cardinal \( \kappa \), \( ([\kappa]^<\omega, \subseteq) >_T (\kappa, \leq) \).
5. For any infinite cardinal \( \kappa \) and any \( n < \omega \), \( (\kappa, \leq) \equiv_T (\prod_{i<n} \kappa_i, \leq) \).
(6) Let \( \mathcal{U} \) be an ultrafilter on a Boolean algebra \( \mathbb{B} \), and let \( G \subseteq \mathbb{B} \) be a filter base for \( \mathcal{U} \) that is closed under finite intersection. Then \( (\mathcal{U}, \supseteq) \equiv_T (G, \supseteq) \).

**Proof.** (1) follows from work of Schmidt in [17]. The proof of (2) is very similar to the proof of Fact 12 in [8]. (3) is due to Day in [3]. (4) follows from (1) along with the fact that every countably infinite subset of an uncountable cardinal \( \kappa \) is bounded in \( \kappa \). To show (5), let \( \kappa \geq \omega \) and \( n < \omega \). Define \( f : (\kappa, \leq) \to (\prod_{i<n} \kappa, \leq) \) by \( f(\alpha) = \langle \alpha, \ldots, \alpha \rangle \) (the element of \( (\prod_{i<n} \kappa, \leq) \) constant at \( \alpha \)). Then \( f \) is an unbounded cofinal map. For (6), note that a filter base \( G \) for \( \mathcal{U} \) that is closed under finite intersection is a cofinal subset of \( \mathcal{U} \). Then (6) follows by Fact 3 in [8]. \( \square \)

2. Conditions ensuring maximum Tukey type

It is well-known that for each infinite cardinal \( \kappa \), there is an ultrafilter on \( \mathcal{P}(\kappa) \) with maximum Tukey type. (This follows from combining work of Isbell in [10] and Schmidt in [17].) Such an ultrafilter may be constructed using an independent family on \( \kappa \) of cardinality \( 2^\kappa \). We begin by showing that this construction generalizes to any countably complete Boolean algebra with an independent family of cardinality \( |\mathbb{B}| \).

**Theorem 9.** If \( \mathbb{B} \) is an infinite countably complete Boolean algebra with an independent family of cardinality \( |\mathbb{B}| \), then there is an ultrafilter \( \mathcal{U} \) on \( \mathbb{B} \) such that \( (\mathcal{U}, \supseteq) \equiv_T ([|\mathbb{B}|]^{<\omega}, \subseteq) \).

**Proof.** Let \( \mathbb{B} \) be any countably complete Boolean algebra with an independent family \( \mathcal{I} = \{a_\alpha : \alpha < |\mathbb{B}|\} \), and let \( \kappa \) denote \( |\mathbb{B}| \). Let \( \mathcal{B} = \{\bigwedge_{\alpha \in F} a_\alpha : F \in [\kappa]^{<\omega}\} \cup \{\bigvee_{\alpha \in G} -a_\alpha : G \in [\kappa]^{\omega}\} \). Let \( \mathcal{F} \) be the filter generated by \( \mathcal{B} \).

First, note that \( \mathcal{F} \) is a proper filter. To see this, take any \( F_0, \ldots, F_m \in [\kappa]^{<\omega} \) and \( G_0, \ldots, G_n \in [\kappa]^{\omega} \), for some \( m, n \in \omega \), and let \( F = \bigcup_{i \leq m} F_i \). Then

\[
\bigwedge_{i \leq m} \bigwedge_{\alpha \in F_i} a_\alpha \land \bigwedge_{j \leq n} \bigvee_{\beta \in G_j} -a_\beta = \bigwedge_{\alpha \in F} a_\alpha \land \bigwedge_{j \leq n} \bigvee_{\beta \in G_j} -a_\beta
\]

\[
= \bigwedge_{\alpha \in F} a_\alpha \land \left( \bigvee_{\beta \in G_0} -a_\beta \right) \land \cdots \land \left( \bigvee_{\beta \in G_n} -a_\beta \right)
\]

\[
\geq \bigwedge_{\alpha \in F} a_\alpha \land \left( \bigvee_{\beta \in G_0 \setminus F} -a_\beta \right) \land \cdots \land \left( \bigvee_{\beta \in G_n \setminus F} -a_\beta \right).
\]

(1)
In order to apply the distributive law, for each \( j \leq n \), let the members of \( G_j \) be indexed as \( \beta_{j,k}, k \in \omega \). Then

\[
\bigwedge_{\alpha \in F} a_\alpha \land \left( \bigvee_{\beta \in G_0 \setminus F} -a_\beta \right) \land \cdots \land \left( \bigvee_{\beta \in G_n \setminus F} -a_\beta \right)
\]

\[
= \bigwedge_{\alpha \in F} a_\alpha \land \bigvee_{j \leq n} \bigwedge_{k < \omega} -a_{\beta_{j,k}}
\]

\[
= \bigwedge_{\alpha \in F} \left( \bigvee_{f : \omega \rightarrow n + 1} \bigwedge_{j \leq n} -a_{\beta_{j,f(j)}} \right)
\]

\[(2) \quad = \bigvee_{f : \omega \rightarrow n + 1} \left( \bigwedge_{\alpha \in F} \bigwedge_{j \leq n} -a_{\beta_{j,f(j)}} \right).\]

The right hand side is always non-zero, since \( \mathcal{I} \) is independent, and all \( \beta_{j,f(j)} \) are not in \( F \). Therefore, \( F \) is a proper filter.

**Claim.** \( (F, \geq) \equiv_T ([\kappa]^{<\omega}, \subseteq) \).

**Proof.** It suffices to show that \( (F, \geq) \equiv_T ([\kappa]^{<\omega}, \subseteq) \), since \( F \) being a directed partial order of size \( \kappa \) implies that \( (F, \geq) \leq_T ([\kappa]^{<\omega}, \subseteq) \). Define \( f : [\kappa]^{<\omega} \rightarrow F \) by \( f(F) = \bigwedge_{\alpha \in F} a_\alpha \), for each \( F \in [\kappa]^{<\omega} \). We claim that \( f \) is a Tukey map. To see this, let \( \mathcal{X} \subseteq [\kappa]^{<\omega} \) be unbounded in \(([\kappa]^{<\omega}, \subseteq)\). Then \( \mathcal{X} \) must be infinite. The \( f \)-image of \( \mathcal{X} \) is \( \{ \bigwedge_{\alpha \in F} a_\alpha : F \in \mathcal{X} \} \). Any lower bound \( b \) of \( \{ \bigwedge_{\alpha \in F} a_\alpha : F \in \mathcal{X} \} \) would have to have the property that \( b \leq \bigwedge_{\alpha \in \bigcup \mathcal{X}} a_\alpha \). But \( \bigvee_{\alpha \in \bigcup \mathcal{X}} -a_\alpha \) is in \( B \), since \( \bigcup \mathcal{X} \) is countably infinite; so the complement of \( \bigwedge_{\alpha \in \bigcup \mathcal{X}} a_\alpha \) is in \( F \). Since \( F \) is a proper filter, \( F \) contains no members below \( \bigwedge_{\alpha \in \bigcup \mathcal{X}} a_\alpha \). It follows that the \( f \)-image of \( \mathcal{X} \) is unbounded in \( F \). Therefore \( f \) is a Tukey map and the claim holds. \( \square \)

Let \( U \) be any ultrafilter on \( B \) extending \( F \). Then \( (F, \geq) \leq_T (U, \geq) \), since the identity map on \( (F, \geq) \) is a Tukey map. Hence, \( (U, \geq) \equiv_T ([\kappa]^{<\omega}, \subseteq) \). \( \square \)

The next theorem follows immediately by an application of the Balcar-Franek Theorem, which states that every infinite complete Boolean algebra has an independent subset of cardinality that of the algebra (see Theorem 13.6 in [12]).

**Theorem 10.** Every infinite complete Boolean algebra has an ultrafilter with maximum Tukey type.

**Proof.** By the Balcar-Franek Theorem, every infinite complete Boolean algebra \( B \) has an independent family of cardinality \(|B|\). By Theorem 9 there is an ultrafilter \( U \) on \( B \) such that \( (U, \geq) \equiv_T (|[B]|^{<\omega}, \subseteq) \). \( \square \)

We mention the following theorem of Shelah giving sufficient conditions for a Boolean algebra to have an independent family of maximal size.

**Theorem 11** (Shelah, (Theorem 10.1 in [12])). Assume \( \kappa, \lambda \) are regular infinite cardinals such that \( \mu^{<\kappa} < \lambda \) for every cardinal \( \mu < \lambda \), and that \( B \) is a Boolean algebra satisfying the \( \kappa \)-chain condition. Then every \( X \subseteq B \) of size \( \lambda \) has an independent subset of \( Y \) of size \( \lambda \).
Thus, if \( \mathcal{B} \) has the \( \kappa \)-chain condition, \( |\mathcal{B}| = \lambda \), and for all \( \mu < \lambda \), \( \mu^{<\kappa} < \lambda \), then \( \mathcal{B} \) contains an independent subset of size \( |\mathcal{B}| \). If \( \mathcal{B} \) is also countably complete, then Theorem \( \ref{thm:independence} \) implies that \( \mathcal{B} \) has an ultrafilter with maximum Tukey type.

The next theorem shows that every ultrafilter on a free Boolean algebra has maximum Tukey type. Thus, for each infinite cardinal \( \kappa \), the spectrum of the Tukey types of ultrafilters on \( \text{Clop}(2^\kappa) \) is precisely \( \{([\kappa]^{<\omega}, \subseteq)\} \).

**Theorem 12.** For each cardinal \( \kappa \geq \omega \) and each ultrafilter \( \mathcal{U} \) on \( \text{Clop}(2^\kappa) \), \( (\mathcal{U}, \geq) \equiv_T ([\kappa]^{<\omega}, \subseteq) \).

**Proof.** Let \( \kappa \) be an infinite cardinal. The basic clopen sets of \( \text{Clop}(2^\kappa) \) are the sets \( c_s = \{ f \in 2^\kappa : f \supseteq s \} \), where \( s \) is any function from a finite subset of \( \kappa \) into 2. Recall that the members of \( \text{Clop}(2^\kappa) \) can be represented as follows: To each \( b \in \text{Clop}(2^\kappa) \) there corresponds a unique finite set of \( s_i, i \leq k \), for some \( k \), such that

(i) For each \( i \leq k \), \( s_i \) is a function from some finite subset of \( \kappa \) into 2;
(ii) For all \( i < j \leq k \), \( c_{s_i} \cap c_{s_j} = \emptyset \);
(iii) \( b = \bigcup_{i \leq k} s_i \); and
(iv) Whenever \( t \) is a function from a finite subset of \( \kappa \) into 2, then \( c_t \leq b \) implies \( t \) is actually an extension of \( s_i \), for some \( i \leq k \).

We fix some notation. For each \( b \in \text{Clop}(2^\kappa) \), let \( \{s^b_i : i \leq k_b\} \) denote the representation of \( b \) satisfying (i) - (iv). Define \( h : \mathcal{U} \rightarrow [\kappa]^{<\omega} \) by \( h(b) = \bigcup_{i \leq k_b} \text{dom}(s^b_i) \).

**Claim.** \( h \) is a cofinal map.

**Proof.** Suppose not. Then there is a cofinal subset \( \mathcal{X} \) of \( \mathcal{U} \) such that \( h''\mathcal{X} \) is not cofinal in \( [\kappa]^{<\omega} \). Thus, there is some \( u \in [\kappa]^{<\omega} \) such that for each \( b \in \mathcal{X} \), \( h(b) \nsubseteq u \). It follows that for each \( b \in \mathcal{X} \), there is an \( i_b \in u \) such that \( i_b \notin \bigcup_{i \leq k_b} \text{dom}(s^b_i) \). Choose one such \( i_b \) for each \( b \in \mathcal{X} \). This partitions \( \mathcal{X} \) into \( u \) disjoint pieces, \( \mathcal{X}_i = \{ b \in \mathcal{X} : i_b \notin \bigcup_{j \leq k_b} \text{dom}(s^b_j) \} \), \( i \in u \).

We claim that at least one \( \mathcal{X}_i \) is cofinal. If not, then for each \( i \in u \), there is an \( a_i \in \mathcal{U} \) such that for each \( x \in \mathcal{X}_i \), \( x \nleq a_i \). Since \( \mathcal{U} \) is an ultrafilter, \( \bigwedge_{i \in u} a_i \in \mathcal{U} \). However, for each \( x \in \mathcal{X}_i \), \( x \nleq \bigwedge_{i \in u} a_i \), since there is an \( i \in u \) such that \( x \nleq a_i \). But this contradicts that \( \mathcal{X} \) is cofinal in \( \mathcal{U} \). Therefore, there is an \( i \in u \) such that \( \mathcal{X}_i \) is cofinal.

In particular, for each function \( s \) from a finite subset of \( \kappa \) into 2 such that \( i \in \text{dom}(s) \), there is a member \( x \in \mathcal{X}_i \) such that \( x \leq c_s \). But \( i \in \text{dom}(s) \setminus \bigcup_{j \leq k_s} \text{dom}(s^j_i) \) implies that \( x \) cannot be below \( c_s \), a contradiction. Therefore, \( h \) is a cofinal map.

Since \( (\mathcal{U}, \geq) \) is a directed partial ordering of cardinality \( \kappa \), \( (\mathcal{U}, \geq) \leq_T ([\kappa]^{<\omega}, \subseteq) \) holds. Therefore, \( \mathcal{U} \) has maximum Tukey type.

**Remark.** The Stone space of each free Boolean algebra is homogeneous; that is, given any infinite cardinal \( \kappa \), for any two ultrafilters \( \mathcal{U}, \mathcal{V} \) on \( \text{Clop}(2^\kappa) \), there is a homeomorphism from \( \text{Ult}(\text{Clop}(2^\kappa)) \) onto itself mapping \( \mathcal{U} \) to \( \mathcal{V} \) (see Exercise 4, page 139 in\textsuperscript{[12]}). Since a homeomorphism maps any neighborhood base of \( \mathcal{U} \) cofinally to any neighborhood base of \( \mathcal{U} \), and vice versa, \( \mathcal{U} \) is Tukey equivalent to \( \mathcal{V} \). In fact, homogeneity of the Stone space of any Boolean algebra implies all its ultrafilters have the same Tukey type. However, this says nothing about what that Tukey type is. We shall see in Section \( \ref{} \) that it is possible to have Boolean algebras in which all the ultrafilters have the same Tukey type, which is not the maximum type.
Next, we investigate the Tukey spectra of completions of free algebras. By Theorem 14, the completion of $\text{Clop}(2^\kappa)$, denoted $r.o.(\text{Clop}(2^\kappa))$, always has an ultrafilter of the maximum Tukey type $(2^\kappa)^{<\omega}, \subseteq)$. In particular, the Cohen algebra $r.o.(\text{Clop}(2^\omega))$ has an ultrafilter of type $(2^\omega)^{<\omega}, \subseteq)$. This leads us to the following question.

**Question 13.** Do all the ultrafilters in the completion of a free Boolean algebra have maximum Tukey type?

In Theorem 16, we will rule out some possible Tukey types below the top for all completions of free algebras. We begin with two propositions in which certain completeness or chain condition hypotheses rule out certain Tukey types of ultrafilters.

**Proposition 14.** If $\kappa \geq \omega$ and $B$ is a $\kappa$-complete atomless Boolean algebra, then $B$ has no ultrafilters of Tukey type $(\kappa, \leq)$.

*Proof.* Let $U$ be an ultrafilter on $B$ and suppose toward a contradiction that there is a strictly decreasing sequence $\langle b_\alpha : \alpha < \kappa \rangle$ cofinal in, and thus generating, $U$. Without loss of generality, we may assume that for each limit ordinal $\gamma < \kappa$, $b_\gamma = \bigwedge_{\alpha < \gamma} b_\alpha$ and that $b_0 = 1$. Since $\langle b_\alpha : \alpha < \kappa \rangle$ generates an ultrafilter, it follows that $\bigwedge_{\alpha < \kappa} b_\alpha = 0$. For each $\alpha < \kappa$, define $a_\alpha = b_\alpha \land -b_{\alpha+1}$. Since $B$ is atomless, there are non-zero $a_{\alpha,0}, a_{\alpha,1}$ partitioning $a_\alpha$. Let $c_i = \bigvee_{\alpha > \gamma} a_{\alpha,i}$, for $i < 2$. Then $c_0 \lor c_1 = 1$ and $c_0 \land c_1 = 0$; so exactly one of $c_0, c_1$ must be in $U$. But for each $i < 2$, we have that $c_i \not\geq b_\alpha$ for all $\alpha < \kappa$. Since $\langle b_\alpha : \alpha < \kappa \rangle$ generates $U$, this implies that neither of $c_0, c_1$ is in $U$, contradiction. \hfill $\Box$

**Proposition 15.** Let $\kappa$ be a regular uncountable cardinal. If $B$ is $\kappa$-c.c., then for all $\lambda \geq \kappa$, $B$ has no ultrafilters of Tukey type $(\kappa, \leq)$.

*Proof.* $B$ is $\kappa$-c.c. implies there are no strictly decreasing chains of order type $\lambda$ in $B$ for any $\lambda \geq \kappa$. In particular, no ultrafilter in $B$ can be generated by a strictly decreasing chain of order type $\lambda$. \hfill $\Box$

**Theorem 16.** Let $\kappa$ be an infinite cardinal. Then each ultrafilter on $r.o.(\text{Clop}(2^\kappa))$ is not Tukey equivalent to $\prod_{i \in I} \kappa_i$ for any collection $\{\kappa_i : i \in I\}$ such that $\prod_{i \in I} \kappa_i \not\equiv_T \prod_{i \in \omega} \omega$.

*Proof.* Let $\{\kappa_i : i \in I\}$ be any collection of cardinals, where $I$ is a nonempty index set. Without loss of generality, assume that each $\kappa_i \geq 2$. Suppose that $U$ is an ultrafilter on $r.o.(\text{Clop}(2^\kappa))$ which is Tukey equivalent to $\prod_{i \in I} \kappa_i$. We rule out every possibility except for $|I| = \omega$ with each $\kappa_i = \omega$.

Suppose at least one $\kappa_i \geq \aleph_1$. Then $U$ has a decreasing chain of order type $\aleph_1$. However, since $r.o.(\text{Clop}(2^\kappa))$ satisfies the countable chain condition, it has no decreasing chains of order type $\aleph_1$. Thus, we have a contradiction. Therefore, all $\kappa_i$ must be countable.

Now suppose that $I$ is uncountable. Then since each $\kappa_i \geq 2$, $\prod_{i \in I} \kappa_i$ has a strictly decreasing chain of order-type $\omega_1$, which is impossible in $r.o.(\text{Clop}(2^\kappa))$, again by the countable chain condition. Therefore, $I$ must be countable.

If $I$ is finite, then $\prod_{i \in I} \kappa_i$ is Tukey equivalent to $\omega$, which contradicts Proposition 14, since $r.o.(\text{Clop}(2^\kappa))$ is complete and atomless. If $|I| = \omega$ and all but finitely many $\kappa_i$ are finite, then $\prod_{i \in I} \kappa_i$ is Tukey equivalent to $\omega$, which, again by Proposition 14, is not possible. Thus, the only possibility we have not ruled out is that $|I| = \omega$, all $\kappa_i$ are countable, and infinitely many $\kappa_i = \omega$. In this case $\prod_{i \in I} \kappa_i$ is Tukey equivalent to $\prod_{i \in \omega} \omega$. \hfill $\Box$
In particular, the Cohen algebra has no ultrafilters of Tukey type $1$, $\omega$, $\omega_1$, $\mathfrak{c}$, $\omega \times \omega_1$, or $\omega_1^\omega$.

Question 17. Can r.o.$(\text{Clop}(2^\kappa))$ have an ultrafilter Tukey equivalent to $(\omega^\omega, \leq)$? In particular, can the Cohen algebra have an ultrafilter Tukey equivalent to $(\omega^\omega, \leq)$?

Remark. Milovich in [14] defines a preorder to be cofinally scalene if it is cofinally equivalent to the product of some collection of regular cardinals. Theorem 16 shows that all ultrafilters on the completions of free Boolean algebras are not cofinally scalene for any type other than possibly $\prod_{\omega \in \omega} \omega$. We will show in Section 4 Theorem 21 that there are Boolean algebras (namely interval algebras) which have only cofinally scalene ultrafilters.

The following simple fact shows that each finite-c cofinite Boolean algebra has Tukey spectrum of size two, consisting exactly of the minimum and the maximum Tukey types.

Fact 18. Let $X$ be any infinite set, and let $B$ denote the finite-c cofinite algebra on $X$. Then the ultrafilters on $B$ consist exactly of the the principal ultrafilters and the cofinite ultrafilter. Thus, the Tukey spectrum of $B$ is $\{ (1, \leq), (\|[X]\|^\leq, \subseteq) \}$.

Proof. The collection of cofinite subsets of $X$ is certainly an ultrafilter on $B$, and its Tukey type is exactly $\|[X]\|^\leq$, since the intersection of any infinite subset of cofinite subsets of $X$ is not cofinite. Moreover, it is the only nonprincipal ultrafilter on $B$. For if $U$ contains a finite subset of $X$, then it contains a singleton. For each $x \in X$, $\{x\}$ generates a principal ultrafilter, hence with Tukey type 1.

Lastly, we state a theorem that will be proved in Section 4: a tree algebra $\text{Treealg}T$ of size $\kappa$ has an ultrafilter with maximum Tukey type if and only if the underlying tree $T$ has an initial chain with $\kappa$-many immediate successors. (This is Theorem 24; we will also prove in Corollary 30 a more general version for pseudo-tree algebras.)

3. Tukey spectra of Superatomic Boolean algebras constructed from almost disjoint families

In this section, we find the Tukey spectra of Boolean algebras which are generated by an almost disjoint family on some infinite set. Let $\lambda$ be an infinite cardinal. A family $A \subseteq \mathcal{P}(\lambda)$ is almost disjoint if for all pairs $a, b \in A$, $|a \cap b| < \omega$. Given an almost disjoint family $A \subseteq \mathcal{P}(\lambda)$, the almost disjoint Boolean algebra generated by $A$ is the subalgebra of $\mathcal{P}(\lambda)$ generated by $A$. Recall that any Boolean algebra generated from an almost disjoint family is superatomic. (See Example 0.1 on page 721 in [16].)

Theorem 19. Let $A$ be an almost disjoint family on an infinite cardinal $\lambda$ with $|A| = \kappa$, and let $B$ denote the subalgebra of $\mathcal{P}(\lambda)$ generated by $A$. Then every ultrafilter on $B$ has Tukey type either $1$ or else $[\mu]^\leq$ for some $\omega \leq \mu \leq \kappa$. The minimum and maximum types $(1$ and $[\kappa]^\leq)$ are always realized. Moreover, for $\omega \leq \mu \leq \kappa$, $[\mu]^\leq$ is realized as the Tukey type of some ultrafilter on $B$ if and only if there is an $a \in A$ such that the set $\{a \cap -b : b \in A \setminus \{a\}\}$ has cardinality $\mu$.

Proof. Let $A$ be an almost disjoint family on an infinite cardinal $\lambda$. Let $B$ denote the subalgebra of $\mathcal{P}(\lambda)$ generated by $A$, and let $\kappa = |B|$. Since $B$ is superatomic, there are many ultrafilters on $B$ generated by an atom, and these all are Tukey equivalent to $1$. 8
Suppose that $\mathcal{U}$ is a nonprincipal ultrafilter on $\mathbb{B}$. Then there is at most one $a \in \mathcal{A}$ such that $a \in \mathcal{U}$. For suppose $\mathcal{U}$ contains two members $a$ and $b$ in $\mathcal{A}$. Then $a \cap b$ is finite. If $a \cap b$ is empty, then $\mathcal{U}$ is the whole algebra $\mathbb{B}$, and we are not considering improper ultrafilters. If $a \cap b$ is nonempty, then $\mathcal{U}$ would be principal.

Fix any $a \in \mathcal{A}$ and enumerate $\mathcal{A} \setminus \{a\}$ as $\{a_\alpha : \alpha < \kappa\}$. Note that $\{a\} \cup \{-a_\alpha : \alpha < \kappa\}$ has the finite intersection property: For any finite $F \subseteq \kappa$, for each $\alpha \in F$, $a \cap a_\alpha$ is finite, so $a \cap -a_\alpha$ is cofinite in $a$. It follows that $a \cap (\bigcap_{\alpha \in F} -a_\alpha)$ is also cofinite in $a$. Thus, $\{a\} \cup \{-a_\alpha : \alpha < \kappa\}$ generates a filter base, each member of which is cofinite in $a$. Let $\mathcal{U}$ be the ultrafilter generated by $\{a\} \cup \{-a_\alpha : \alpha < \kappa\}$.

Let $K \subseteq \kappa$ be a maximal subset of $\kappa$ such that for all $\alpha \neq \beta$ in $K$, $a \cap -a_\alpha \neq a \cap -a_\beta$. Let $\mu = |K|$. Note that $\mu \geq \omega$ if and only if $\mathcal{U}$ is nonprincipal. Principal ultrafilters have been discussed above; they are Tukey equivalent to $1$. Suppose $\mathcal{U}$ is nonprincipal. Define a map $g : [K]^{<\omega} \to \mathcal{U}$ by $g(F) = a \cap (\bigcap_{\alpha \in F} -a_\alpha)$, for each $F \in [K]^{<\omega}$. Then $g$ is a Tukey map. To see this, let $\mathcal{X}$ be an unbounded subset of $[K]^{<\omega}$. Then $G := \bigcup \mathcal{X}$ must be infinite. Suppose by way of contradiction that there is a bound $b \in \mathcal{U}$ for $\{g(F) : F \in \mathcal{X}\}$. Then it must be the case that $b \subseteq a \cap (\bigcap_{\alpha \in F} -a_\alpha)$, for each $F \in \mathcal{X}$. It follows that $b \subseteq a \cap (\bigcap_{\alpha \in G} -a_\alpha)$. By definition of $K$, for each $F \in \mathcal{X}$, there is an $\alpha \in K$ such that $a \cap (\bigcap_{\beta \in F} -a_\beta \cap -a_\alpha) \subseteq a \cap (\bigcap_{\beta \in G} -a_\beta)$. Thus, $a \cap (\bigcap_{\alpha \in G} -a_\alpha)$ is not cofinite in $a$. Hence, there is no such bound $b$. Therefore, $g$ is a Tukey map, and hence $([\mu]^{<\omega}, \subseteq) \leq_T (\mathcal{U}, \geq)$.

To show that $([\mu]^{<\omega}, \subseteq) \equiv_T (\mathcal{U}, \geq)$, recall that since $\mathcal{U}$ is the upwards closure in $\mathbb{B}$ of $\mathcal{B} := \{a \cap (\bigcap_{\alpha \in F} -a_\alpha) : F \in [K]^{<\omega}\}$, it follows that $\mathcal{B}$ is Tukey equivalent to $\mathcal{U}$. Define a map $h : \mathcal{B} \to [\mu]^{<\omega}$ as follows: Given $b \in \mathcal{B}$, let $h(b) = F$ for some $F \in [K]^{<\omega}$ such that $b = a \cap (\bigcap_{\alpha \in F} -a_\alpha)$. If $\mathcal{Y}$ is an unbounded subset of $\mathcal{B}$, then in particular, $\mathcal{Y}$ must be infinite. Thus, the set $\{h(b) : b \in \mathcal{Y}\}$ is infinite, and hence unbounded in $[K]^{<\omega}$. Therefore, $h$ is a Tukey map, and thus $([\mu]^{<\omega}, \subseteq) \equiv_T (\mathcal{U}, \geq)$.

The last possible case is that $\mathcal{U}$ is the ultrafilter generated by the set $\{-a : a \in \mathcal{A}\}$. In this case, index the members of $\mathcal{A}$ so that $\mathcal{A} = \{a_\alpha : \alpha < \kappa\}$. This ultrafilter is nonprincipal. To see this, let $F$ be any finite subset of $\kappa$, and let $\beta \in \kappa \setminus F$. Then $a_\beta \cap (\bigcup_{\alpha \in F} a_\alpha)$ is finite, since $a_\beta \cap a_\alpha$ is finite for each $\alpha \in F$. Thus, $a_\beta \not\subseteq \bigcup_{\alpha \in F} a_\alpha$, which implies that $-a_\beta \not\subseteq \bigcap_{\alpha \in F} -a_\alpha$. Therefore, $-a_\beta \cap \bigcap_{\alpha \in F} -a_\alpha$ is strictly contained in $\bigcap_{\alpha \in F} -a_\alpha$. Hence, $\mathcal{U}$ has no minimal member, so $\mathcal{U}$ is nonprincipal. Similarly as before, it is easy to check that the map $g : [\kappa]^{<\omega} \to \mathcal{U}$, given by $g(F) = \bigcap_{\alpha \in F} -a_\alpha$, is a Tukey map. Therefore, $([\mu]^{<\omega}, \subseteq) \equiv_T (\mathcal{U}, \geq)$, since $|\mathbb{B}| = \kappa$.

Thus, every ultrafilter on $\mathbb{B}$ has Tukey type minimum, maximum, or else has Tukey type exactly $[\mu_\alpha]^{<\omega}$, where $a \in \mathcal{A}$ and $\mu_\alpha$ is the cardinality of the set $\{a \cap -b : b \in \mathcal{A} \setminus \{a\}\}$. We have shown there are ultrafilters realizing all of these possibilities. □

**Remark.** If $\mathcal{A}$ is a maximal almost disjoint family of subsets of $\lambda$ all of whose members have size $\lambda$, with $|\mathcal{A}| = \kappa$, then the proof above implies that the almost disjoint Boolean algebra generated by $\mathcal{A}$ has ultrafilters only of the minimal type 1 and the maximal type $([\kappa]^{<\omega}, \subseteq)$.

**Question 20.** What are the Tukey spectra of superatomic Boolean algebras in general?

4. Spectra of Tukey Types of Interval Algebras, Tree Algebras, and Pseudo-Tree Algebras

Basic facts about interval algebras and tree algebras can be found in Volume 1 of the Handbook of Boolean Algebras [12]; basic facts about pseudo-tree algebras which generalize
from corresponding tree algebra facts can be found in [11]. Since pseudo-trees are probably the least well-known of these classes, we provide some background on them here. We follow the notation in [11].

A pseudo-tree is a partially ordered set \((T, \leq)\) such that for each \(t \in T\), the set \(T \downarrow t = \{s \in T : s \leq t\}\) is linearly ordered. The pseudo-tree algebra on a pseudo-tree \(T\) is generated in the same way as a tree algebra: Treealg\((T)\) is the algebra of sets generated by the “cones” \(T \uparrow t = \{s \in T : s \geq t\}\), for \(t \in T\). A pseudo-tree algebra is thus a generalization of both an interval algebra and a tree algebra, and the following discussion of the correspondence between ultrafilters and initial chains applies to all three classes of Boolean algebras.

Let \(T\) be an infinite pseudo-tree with a single root. (It does no harm to assume that all of our pseudo-trees have single roots; any pseudo-tree algebra is isomorphic to a pseudo-tree algebra on a pseudo-tree with a single root (see [11]).) An initial chain in \(T\) is a non-empty chain \(C \subseteq T\) such that if \(c \in C\) and \(t < c\) then \(t \in C\). There is a one-to-one correspondence between ultrafilters \(U\) on Treealg\((T)\) and initial chains in \(T\), given by

\[\phi(U) = \{t \in T : T \uparrow t \in U\} \]

This set of generators is closed under finite intersection, so that \((U, \supseteq) \equiv_T (H_C, \supseteq)\).

Let \(C \subseteq T\) be an initial chain. Call a subset \(R \subseteq T\) a set of approximate immediate successors of \(C\) if

(i) \(r > C\) for all \(r \in R\), and

(ii) for all \(s > C\), there is an \(r \in R\) such that \(C < r \leq s\).

Then define

\[\varepsilon_C = \min\{|R| : R \text{ is a set of approximate immediate successors of } C\}\]

The character \(\chi\) of an ultrafilter \(U\) on Treealg\((T)\) is the minimum size of a generating set for \(U\). If \(C\) is the initial chain corresponding to \(U\), then \(\chi(U) = \max\{\varepsilon_C, \text{cf } C\}\) (see [2]). If \(T\) is a tree, then the set \(\text{imm}(C)\) of immediate successors of an initial chain \(C\) in \(T\) is well-defined, even if \(C\) does not have a top element, and \(\varepsilon_C = |\text{imm}(C)|\).

We describe all possible Tukey types of ultrafilters on interval algebras in Theorem [21]. From this, we classify the spectra of Tukey types of all interval algebras. In the terminology of Milovich [14], ultrafilters on interval algebras have cofinally scalene Tukey types. We assume that all linear orders mentioned have least elements. (Where a linear order does not naturally have a least element, we add an element \(-\infty\) to \(L\) and proceed to build Intalg\(L\) as in the Handbook.)

For \(L\) a linear order and \(X \subseteq L\), the coinitiality of \(X\), denoted \(\text{ci}(X)\), is the least cardinal \(\mu\) such that \(\mu^*\) is coinitial in \(X\).

**Theorem 21.** Let \(L\) be a linear ordering. Let \(P\) denote the set of pairs of regular cardinals \((\kappa, \mu)\) for which there is an initial chain \(C\) in \(L\) such that the cofinality of \(C\) is \(\kappa\) and the coinitiality of \(L \setminus C\) is \(\mu^*\). Then the Tukey spectrum of Intalg\(L\) is exactly \(\{(\kappa \times \mu, \leq) : (\kappa, \mu) \in P\}\).
Proof. Let $L$ be a linear order with a first element and set $A = \text{Intalg} \ L$. Since $A$ is also a pseudo-tree algebra, its ultrafilters are associated with initial chains as are those of pseudo-tree algebras (see below). Let $\mathcal{U} \in \text{Ult} \ A$ and let $C$ be the initial chain associated with $\mathcal{U}$. Let $\kappa = \text{cf} \ C$ and $\{c_\alpha : \alpha < \kappa\}$ be an increasing cofinal sequence in $C$. Let $\mu$ be such that $\mu^*$ is the coinitiality of $L \setminus C$, and let $\{\beta : \beta < \mu\}$ be a decreasing coinitial sequence in $L \setminus C$.

Case 1: $C$ has a greatest element $z$ and $L \setminus C$ has a least element $z'$; that is, $\kappa = \mu = 1$. Then $\mathcal{U}$ is generated by the singleton $\{z\} = [z, z']$, so $(\mathcal{U}, \supseteq)$ has minimal Tukey type, $(1, \subseteq)$.

Case 2: $C$ has a greatest element $z$ but $L \setminus C$ has no least element; that is, $\kappa = 1$ and $\mu \geq \omega$. Then $\mathcal{U}$ is generated by

$$G_2 = \{[z, l_\beta) : \beta < \mu\}.$$ 

As $G_2$ is closed under finite intersection, it is cofinal in $\mathcal{U}$; hence $(G_2, \supseteq) \equiv_T (\mathcal{U}, \supseteq)$. Define $f : (G_2, \supseteq) \to (\mu, \leq)$ by $f([z, l_\beta)) = \beta$ for $\beta < \kappa$. One can check that $f$ is an unbounded cofinal map, so $(G_2, \supseteq) \equiv_T (\mu, \leq)$.

Case 3: $C$ has no greatest element but $L \setminus C$ has a least element $z'$; that is, $\kappa \geq \omega$ and $\mu = 1$. Then $\mathcal{U}$ is generated by

$$G_3 = \{[c_\alpha, z') : \alpha < \kappa\}.$$ 

As $G_3$ is closed under finite intersection, $(G_3, \supseteq) \equiv_T (\mathcal{U}, \supseteq)$. Define $f : (G_3, \supseteq) \to (\kappa, \leq)$ by $f([c_\alpha, z')) = \alpha$. One can check that $f$ is an unbounded cofinal map, so $(G_3, \supseteq) \equiv_T (\kappa, \leq)$.

Case 4: $C$ has no greatest element and $L \setminus C$ has no least element; that is, $\kappa \geq \omega$ and $\mu \geq \omega$. Then $\mathcal{U} = \langle G_4 \rangle$ where

$$G_4 = \{[c_\alpha, l_\beta) : \alpha < \kappa, \beta < \mu\}.$$ 

As $G_4$ is closed under finite intersection, $(G_4, \supseteq) \equiv_T (\mathcal{U}, \supseteq)$. Define $f : (G_4, \supseteq) \to (\kappa \times \mu)$ by $f([c_\alpha, l_\beta)) = (\alpha, \beta)$. One can check that $f$ is an unbounded cofinal map, so $(G_4, \supseteq) \equiv_T (\kappa \times \mu, \leq)$.

Thus if $L$ is a linear order and $\mathcal{U}$ is an ultrafilter on $\text{Intalg} \ L$, then there are only three possibilities for the Tukey type of $(\mathcal{U}, \supseteq)$. Letting $C$ be the initial chain corresponding to $\mathcal{U}$, either

1. $(\mathcal{U}, \supseteq) \equiv_T (\kappa, \leq)$ where $\text{cf} \ C = \kappa$, or
2. $(\mathcal{U}, \supseteq) \equiv_T (\mu, \leq)$ where $\text{ci} \ (L \setminus C) = \mu$, or
3. $(\mathcal{U}, \supseteq) \equiv_T (\kappa \times \mu, \leq)$ where $\text{cf} \ C = \kappa$ and $\text{ci} \ (L \setminus C) = \mu$.

(Recall from Fact 8 (5) that if $\text{cf} \ C = \kappa = \text{ci} \ (L \setminus C)$ then $(\mathcal{U}, \supseteq) \equiv_T (\kappa \times \kappa, \leq) \equiv_T (\kappa, \leq)$.)

Observe that Fact 8 (4) then implies that no uncountable interval algebra $\text{Intalg} \ L$ has an ultrafilter of maximal Tukey type $([|L|]^{<\omega}, \subseteq)$. Countably infinite linear orders always have ultrafilters of top Tukey type: let $L$ be a countably infinite linear order. Then $L$ contains an initial chain $C$ with $\text{cf} \ C = \omega$ or $\text{ci} \ (L \setminus C) = \omega$; and in any case, if $\mathcal{U}$ is the ultrafilter corresponding to $C$, $(\mathcal{U}, \supseteq) \equiv_T (\omega, \leq) \equiv_T ([\omega]^{<\omega}, \subseteq)$.

Fact 22. Given any collection $P$ of pairs of regular cardinals, there is a linear order $L$ whose Tukey spectrum includes $\{(\kappa \times \mu, \leq) : (\kappa, \mu) \in P\}$.

Proof. Say $P = \{(\kappa_\alpha, \mu_\alpha) : \alpha < \lambda\}$ for some $\lambda$. For $\alpha < \lambda$, let $X_\alpha$ be a sequence of order type $\kappa_\alpha$, and let $Y_\alpha$ be a sequence of order type $\mu^*_\alpha$. Define a linear order $L$ by $L = \sum_{\alpha < \lambda}(X_\alpha + Y_\alpha)$, ordered as follows: for $l, m \in L$, $l < m$ if (i) there is an $\alpha < \lambda$ such that $l, m \in X_\alpha$ and $l < m$ in $X_\alpha$, or (ii) there is an $\alpha < \lambda$ such that $l, m \in Y_\alpha$ and $l < m$ in $Y_\alpha$, or (iii) there is
an $\alpha < \lambda$ such that $l \in X_\alpha$ and $y \in Y_\alpha$, or (iv) there are $\alpha < \beta < \lambda$ such that $l \in X_\alpha \cup Y_\alpha$ and $m \in X_\beta \cup Y_\beta$.

Fix $\alpha < \lambda$. Let $C_\alpha = \{ l \in L : l < Y_\alpha \}$ (that is, those element of $L$ that are below every element of $Y_\alpha$). Then $\cf C_\alpha = \cf X_\alpha = \kappa_\alpha$ and $\cf (L \setminus C_\alpha) = \cf Y_\alpha = \mu_\alpha$. It follows – letting $U_\alpha$ be the ultrafilter corresponding to $C_\alpha$ – that $(U_\alpha, \supseteq) \equiv_T (\kappa_\alpha \times \mu_\alpha, \leq)$. Thus $T \subseteq (\text{Intalg } L)$ includes each $(\kappa_\alpha \times \mu_\alpha, \leq)$ for $\alpha < \lambda$.

We note that the Tukey spectrum of the interval algebra in the proof of Fact [22] may also contain types not among $\{ (\kappa \times \lambda, \leq) : (\kappa, \lambda) \in P \}$. For example, if $\lambda > \omega$, let $C = \{ l \in L : L < X_\omega \}$ and let $U$ be the ultrafilter corresponding with $C$. Then $(U, \supseteq) \equiv_T (\omega, \leq)$, which might not be among the types $(\kappa \times \lambda, \leq)$ for and $(\kappa, \lambda) \in P$.

Now we attend to the class of tree algebras. All products we mention in what follows will be of the following sort: for an index set $I$ and a collection of cardinals $\{ \kappa_i : i \in I \}$, let $\prod_{i \in I} \kappa_i$ denote the collection of all functions $f \in \prod_{i \in I} \kappa_i$ such that for all but finitely many $i \in I$, $f(i) = 0$; again the partial ordering is coordinate-wise. The next two theorems characterize those tree $T$ for which $\text{Treealg } T$ has an ultrafilter of maximal Tukey type $([|T|]^{<\omega}, \subseteq)$.

**Proposition 23.** If $T$ is a tree of size $\omega$ with a single root, then $\text{Treealg } T$ has an ultrafilter of type $([\omega]^{<\omega}, \subseteq)$.

**Proof.** Let $T$ be a tree of size $\omega$ with a single root. Then also $|\text{Treealg } T| = \omega$.

First suppose some $z \in T$ has $\omega$-many immediate successors $\{ s_i : i < \omega \}$. Set $C = T \downarrow z$ and let $U = \varphi^{-1}(C)$ be the corresponding ultrafilter. Define $a_n \in \text{Treealg } T$, for $n < \omega$, by $a_n = (T \uparrow z) \setminus \bigcup_{i < n} (T \uparrow s_i)$. Note that for all $n < \omega$, $a_n \in U$. Then the $a_n$ form an infinite descending chain in $\text{Treealg } T$ – that is, a chain of type $\omega$ in $(U, \supseteq)$. Since $(U, \supseteq)$ contains a chain of type $\omega$, $(U, \supseteq) \equiv_T (\omega, \leq) \equiv_T ([\omega]^{<\omega}, \subseteq)$.

Next suppose that no $z \in T$ has $\omega$-many immediate successors. Since $T$ has a single root, this means that for all $n < \omega$, $|\text{Lev}_n(T)| < \omega$; that is, $T$ is an $\omega$-tree. By König’s Lemma, $T$ has an infinite chain $\{ c_n : n < \omega \}$. Let $C$ be minimal among initial chains containing $\{ c_n : n < \omega \}$ (so $C$ is formed by taking the downwards closure in $T$ of $\{ c_n : n < \omega \}$). Let $U = \varphi^{-1}(C)$ be the ultrafilter corresponding to $C$. For each $n < \omega$, set $a_n = T \uparrow c_n$. Then the $a_n$ form a chain of type $\omega$ in $(U, \supseteq)$: $a_0 \supseteq a_1 \supseteq \cdots$, so that $(U, \supseteq) \equiv_T (\omega, \leq)$. Since $(\omega, \leq)$ has the maximum Tukey type for $\omega$, $(U, \supseteq) \equiv_T ([\omega]^{<\omega}, \subseteq)$. \hfill \Box

**Theorem 24.** Let $\kappa > \omega$ and let $T$ be a tree of size $\kappa$. Then $\text{Treealg } T$ has an ultrafilter with type $([\kappa]^{<\omega}, \subseteq)$ if and only if $T$ has an initial chain with $\kappa$-many immediate successors.

**Proof.** Let $\kappa > \omega$ and let $T$ be a tree of size $\kappa$. Suppose $T$ has an initial chain $C$ such that $|\text{imm}(C)| = \kappa$. List the elements of $\text{imm}(C)$ as $\{ s_\alpha : \alpha < \kappa \}$. Let $U$ be the ultrafilter associated with $C$. Note that $U$ is generated by $G = \{ (T \uparrow t) \setminus \bigcup_{a \in F} (T \uparrow s_a) : t \in C, F \in [\kappa]^{<\omega} \}$, and that $(U, \supseteq) \equiv_T (G, \supseteq)$. Fix any element $c \in C$. Define $g : [\kappa]^{<\omega} \to U$ by $g(F) = (T \uparrow c) \setminus \bigcup_{a \in F} (T \uparrow s_a)$, for $F \in [\kappa]^{<\omega}$. We claim that $g$ is an unbounded map. Let $X \subseteq [\kappa]^{<\omega}$ be unbounded, and suppose by way of contradiction that $g[X]$ is bounded in $U$. Then there is an $F \in [\kappa]^{<\omega}$ and a $t \in C$ such that $g(X) \supseteq (T \uparrow t) \setminus \bigcup_{a \in F} (T \uparrow s_a)$ for all $X \subseteq X$. – But then $X \subseteq F$ for all $X \subseteq X$, which is a contradiction since $F$ is finite and $|\bigcup X|$ is infinite. Thus $g$ is an unbounded map, and so $([\kappa]^{<\omega}, \subseteq) \equiv_T (\omega, \leq)$. By the maximality of $(([\kappa]^{<\omega}, \subseteq), ([\kappa]^{<\omega}, \subseteq)) \equiv_T (U, \supseteq)$.

Now suppose that $C$ is any initial chain in $T$ such that $(U, \supseteq) \equiv_T ([\kappa]^{<\omega}, \subseteq)$, where $U$ is the ultrafilter corresponding to $C$. Let $\{ c_\beta : \beta < \cf C \}$ be an increasing cofinal sequence in $C$,
and list the immediate successors of $C$ as $\text{imm}(C) = \{s_\alpha : \alpha < \lambda\}$. (Note that $\text{imm}(C) \neq \emptyset$, since if $C$ had no successors, $U$ would be generated by $\{T \uparrow c_\beta : \beta < \text{cf } C\}$, and so would be Tukey equivalent to $(\text{cf } C, \leq)$.) Then

$$H = \{(T \uparrow c_\beta) \setminus \bigcup_{\alpha \in F} (T \uparrow s_\alpha) : \beta < \text{cf } C \text{ and } F \in [\lambda]^{<\omega}\}$$

is a generating set for $U$ of minimum size. Suppose by way of contradiction that $|\text{imm}(C)| < \kappa$. Since $(H, \supseteq) = U$ and $H$ is closed under finite intersection, $(H, \supseteq) \equiv_T (U, \supseteq) \equiv ([\kappa]^{<\omega}, \subseteq)$; so by Fact 8(2), there is an $X \subseteq H$ of size $\kappa$ such that every infinite subset of $X$ is unbounded in $(H, \supseteq)$. There are some $I \subseteq \text{cf } C$ and $F \subseteq [\lambda]^{<\omega}$ such that $X \subseteq \{(T \uparrow c_\beta) \setminus \bigcup_{\alpha \in F} (T \uparrow s_\alpha) : \beta \in I \text{ and } F \in F\}$. Since we have supposed that $\lambda < \kappa$, it must be that $|I| = \kappa = \text{cf } C$.

By the pigeonhole principle, choose an $F \in F$ such that $\kappa$-many elements of $X$ are of the form $(T \uparrow c_\beta) \setminus \bigcup_{\alpha \in F} (T \uparrow s_\alpha)$ for some $\beta < \kappa$. Let $X'$ be the set of all such elements, and write $X' = \{(T \uparrow c_\beta) \setminus \bigcup_{\alpha \in F} (T \uparrow s_\alpha) : \gamma < \kappa\}$, where $\gamma < \gamma' < \kappa$ implies that $c_{\beta, \gamma} < c_{\beta', \gamma'}$ in $T$. Consider $Y = \{(T \uparrow c_\beta) \setminus \bigcup_{\alpha \in F} (T \uparrow s_\alpha) : \gamma < \omega\}$. $Y$ is an infinite subset of $X$, but $(T \uparrow c_{\beta_0}) \setminus \bigcup_{\alpha \in F} (T \uparrow s_\alpha) \subseteq X' \subseteq H$ is an upper bound for $Y$ in $H$, and this is a contradiction. Thus $\text{imm}(C) = \kappa$.

**Theorem 25.** Let $T$ be a tree, let $U$ be an ultrafilter on $T$, and let $C$ be the initial chain corresponding to $U$. Let $\{c_\alpha : \alpha < \text{cf } C\}$ be an increasing cofinal sequence in $C$, and let $\{s_\beta : \beta < \mu\}$ be the set of immediate successors of $C$ in $T$. Then $(U, \supseteq) \equiv_T (\text{cf } C \times \prod_{\beta < \mu} \{0, 1\}, \leq)$.

**Proof.** $U$ is generated by $G = \{(T \uparrow c_\alpha) \setminus \bigcup_{\beta \in F} (T \uparrow s_\beta) : \alpha < \text{cf } C, F \in [\mu]^{<\omega}\}$. Define $f : (G, \supseteq) \rightarrow (\text{cf } C \times \prod_{\beta < \mu} \{0, 1\}, \leq)$ by $f((T \uparrow c_\alpha) \setminus \bigcup_{\beta \in F} (T \uparrow s_\beta)) = \langle \alpha \rangle \bowtie \langle e_\beta : \beta < \mu \rangle$ where

$$e_\beta = \begin{cases} 1 & \text{if } \beta \in F \\ 0 & \text{if } \beta \not\in F \end{cases}$$

We claim that $f$ is an unbounded cofinal map. Let $X \subseteq G$ be unbounded. Then either (i) the set

$$\{\alpha < \text{cf } C : (T \uparrow c_\alpha) \setminus \bigcup_{\beta \in F} (T \uparrow s_\beta) \in X \text{ for some } F \in [\mu]^{<\omega}\}$$

is unbounded in $\text{cf } C$, or (ii) the set

$$\{\gamma < \mu : (T \uparrow c_\alpha) \setminus \bigcup_{\beta \in F} (T \uparrow s_\beta) \in X \text{ for some } \alpha < \text{cf } C \text{ and } F \in [\mu]^{<\omega} \text{ with } \gamma \in F\}$$

is infinite. In case (i), the set of first coordinates of elements of $f[X]$ is unbounded in $\text{cf } C$. In case (ii), there are infinitely many $\beta < \mu$ at which some element of $f[X]$ has the value $1$. In either case, $f[X]$ is unbounded in $(\text{cf } C \times \prod_{\beta < \mu} \{0, 1\}, \leq)$.

Now suppose $X \subseteq G$ is a cofinal subset. Let $p = \langle \alpha, e_0, e_1, \ldots, e_\beta, \ldots \rangle \in (\text{cf } C \times \prod_{\beta < \mu} \{0, 1\}, \leq)$. Set $F = \{\beta < \mu : e_\beta = 1\}$. As $X$ is cofinal in $G$, there is some $x \in X$ such that $x \subseteq (T \uparrow c_\alpha) \setminus \bigcup_{\beta \in F} (T \uparrow s_\beta)$. Then $f(x) \geq p$. Thus $f[X]$ is cofinal in $(\text{cf } C \times \prod_{\beta < \mu} \{0, 1\}, \leq)$.

Then since $G$ is closed under finite intersection and generates $U$,

$$(U, \supseteq) \equiv_T (G, \supseteq) \equiv_T (\text{cf } C \times \prod_{\beta < \mu} \{0, 1\}, \leq).$$
From Theorem 25 we can describe the Tukey spectrum of a tree algebra.

Corollary 26. Let $T$ be a tree. The Tukey spectrum of $\text{Treealg} T$ consists of those Tukey types $(\langle C, \subseteq \rangle)$ where each $C \subseteq T$ is an initial chain with $\mu$-many immediate successors.

Finally, we turn our attention to the broader class of pseudo-tree algebras.

Theorem 24 showed that the tree algebra on a tree $T$ has an ultrafilter of top Tukey type if there is an initial chain $C$ in $T$ with $\vert \text{imm}(C) \vert = \vert T \vert$. One possible pseudo-tree analog of “having an initial chain with $\vert T \vert$-many immediate successors” would be “having an initial chain $C$ with $\varepsilon C = \vert T \vert$”, and so one could ask whether the ultrafilter corresponding to such an initial chain has top Tukey type. The answer is “no”. For example, let $T$ be a linear order of order type $\omega_1 + 1 + \omega_1^*$ (this is also a pseudo-tree), and let $C$ consist of the first $\omega_1$-many elements of $T$. Then by Theorem 21, the ultrafilter corresponding to $C$ has type $(\omega_1, \subseteq)$, which is strictly less than the top type for $\text{Treealg} T$ by Fact 9 (4).

The next Fact shows that uncountable pseudo-tree algebras always have ultrafilters whose Tukey type is strictly below the maximum type.

Fact 27. Let $T$ be an uncountable pseudo-tree. Then there is an ultrafilter $U$ on $\text{Treealg} T$ whose Tukey type is strictly less than that of $\langle \vert T \vert \rangle^{<\omega}, \subseteq \rangle$.

Proof. Let $T$ be an uncountable pseudo-tree. Let $C$ be any initial chain such that for no $t \in T$ is $t > C$, and let $U$ be the ultrafilter corresponding to $C$. Then $C$ is generated by $G = \{ T \uparrow c : c \in C \}$, so that $(U, \supseteq) \equiv_T (G, \supseteq) \equiv_T (\kappa, \subseteq)$ for some cardinal $\kappa$. Since $T$ is uncountable, $(\kappa, \subseteq) \not\equiv_T \langle \vert T \vert \rangle^{<\omega}, \subseteq \rangle$.

Proposition 28. Let $\lambda$ be any cardinal (finite or infinite) and let $\{ \kappa_\delta : \delta < \lambda \}$ be a set of regular cardinals. Then there is a pseudo-tree $T$ and an ultrafilter $U$ on $\text{Treealg} T$ such that $(U, \supseteq) \equiv_T \prod_{\delta < \lambda} \kappa_\delta$.

Proof. Let regular cardinals $\kappa_\delta$, for $\delta < \lambda$, be given. Let $T$ be the pseudo-tree constructed by putting above a single root $r$ $\lambda$-many pairwise-incomparable linear orders $T_\delta = \{ t_\alpha^\delta : \alpha < \kappa_\delta \}$ where each $T_\delta$ is an inverted copy of $\kappa_\delta$ – that is, $T_\delta$ is isomorphic to $\kappa_\delta^*$. Let $A = \text{Treealg} T$ be the pseudo-tree algebra on $T$. Let $C = \{ r \}$ and let $U$ be the ultrafilter associated with $C$. Then $U = \langle G \rangle$ where

$$ G = \{ (T \uparrow r) \setminus \bigcup_{s \in S} (T \uparrow s) : S \text{ is a finite antichain of elements above } r \}. $$

Since any such antichain $S$ consists of at most one element $t_\alpha^\delta$ from each $T_\delta$, for $\delta < \lambda$, a typical element of $G$ is of the form $(T \uparrow r) \setminus \bigcup_{j < k} (T \uparrow t_{\alpha_j}^\delta)$ for some $k < \omega$. Also note that $G$ is closed under finite intersection, so that $(G, \supseteq) \equiv_T (U, \supseteq)$.

Define $f : (G, \supseteq) \rightarrow (\prod_{\delta < \lambda} \kappa_\delta, \subseteq)$ by

$$ f \left( (T \uparrow r) \setminus \bigcup_{j < k} (T \uparrow t_{\alpha_j}^\delta) \right) = \langle \beta_\delta : \delta < \lambda \rangle $$
where, for $\delta < \lambda$,

$$\beta_\delta = \begin{cases} 
0 & \text{if } \delta \not\in \{\delta_0, \ldots, \delta_{k-1}\} \\
\alpha_i & \text{if } \delta \in \{\delta_0, \ldots, \delta_{k-1}\}
\end{cases}$$

(That is: in those coordinates where no part of the chain $T_\delta$ has been excluded, put a 0; and in those coordinates where $T_\delta$ has been cut-off at $t^\delta_{\alpha_{\delta_j}}$, put $\alpha_{\delta_j}$.)

Suppose $X \subseteq G$ is unbounded. Then either there is at least one $\delta < \lambda$ for which the set of $t^\delta_{\alpha_i}$ excluded by some $x \in X$ is coinitial with $T_\delta$, so that the set of $\delta$th coordinates of $f(x)$, for $x \in X$, is unbounded in $\kappa_\delta$; or the set of $\delta < \lambda$ such that some $x \in X$ has a non-zero entry in coordinate $\delta$ is infinite. In either case, $f[X]$ is unbounded in $\prod_{\delta<\lambda}^w \kappa_\delta$, and so $f$ is an unbounded map.

Now suppose $X$ is a cofinal subset of $G$. Let $\langle \alpha_\delta : \delta < \lambda \rangle \in \prod_{\delta<\lambda}^w \kappa_\delta$. Only finitely many of the entries $\alpha_\delta$ are non-zero; say $\{\alpha_\delta : j < k\}$ are non-zero for some $k < \omega$. As $X$ is cofinal in $G$, there is an $x \in X$ such that $(T \uparrow r) \setminus \bigcup_{j<k} (T \uparrow t_{\delta_j}) \supseteq x$. Then $\langle \alpha_\delta : \delta < \lambda \rangle \leq f(x)$ in $\prod_{\delta<\lambda}^w \kappa_\delta$. Thus $f[X]$ is cofinal in $\prod_{\delta<\lambda}^w \kappa_\delta$, and so $f$ is a cofinal map.

Thus $\langle U, \supseteq \rangle \equiv_T (G, \supseteq) \equiv_T (\prod_{\delta<\lambda}^w \kappa_\delta, \leq)$.

**Theorem 29.** Let $T$ be a pseudo-tree and let $U$ be an ultrafilter on $\text{Treealg} T$. Then $U$ is Tukey equivalent to the product of a regular cardinal and a weak product (finite or infinite) of regular cardinals under coordinate-wise $\leq$.

**Proof.** Let $T$ be a pseudotree and let $U$ be an ultrafilter on $\text{Treealg} T$. Let $C = \phi(U)$ be the initial chain corresponding to $U$ and let $S \subseteq T$ be a set of approximate immediate successors of $C$ of minimal cardinality. Let $\kappa = |S|$, and enumerate the members of $S$ as $s_\alpha$, $\alpha < \kappa$. For each $\alpha < \kappa$, let $\gamma(\alpha)$ be the least $\gamma < \kappa$ such that $((T \downarrow s_\alpha) \setminus C) \cap ((T \downarrow s_\gamma) \setminus C) \neq \emptyset$.

Let $\lambda$ be the cardinality of the set $\{\gamma(\alpha) : \alpha < \kappa\}$. $\lambda$ could be less than $\kappa$, since there could be many $\alpha \neq \alpha'$ for which $\gamma(\alpha) = \gamma(\alpha')$.

**Subclaim 1.** Each $(T \downarrow s_\alpha) \setminus C$ is “closed downwards above $C$” – that is, if $t \in (T \downarrow s_\alpha) \setminus C$ and $C < s \leq t$, then $s \in (T \downarrow s_\alpha) \setminus C$.

**Proof of Subclaim:** Suppose $t \in (T \downarrow s_\alpha) \setminus C$ and $C < s \leq t$. Then $s \notin C$; and since $s \leq t \leq s_\alpha$, $s \leq s_\alpha$; so $s_\alpha \in (T \downarrow s_\alpha) \setminus C$.

**Subclaim 2.** $((T \downarrow s_\alpha) \setminus C) \cap ((T \downarrow s_\beta) \setminus C) \neq \emptyset$ if and only if $\gamma(\alpha) = \gamma(\beta)$.

**Proof of Subclaim:** First suppose $((T \downarrow s_\alpha) \setminus C) \cap ((T \downarrow s_\beta) \setminus C) \neq \emptyset$. Since also $((T \downarrow s_\alpha) \setminus C) \cap ((T \downarrow s_{\gamma(\alpha)}) \setminus C) \neq \emptyset$ and all of these sets are downwards-closed above $C$, we have $((T \downarrow s_\beta) \setminus C) \cap ((T \downarrow s_{\gamma(\alpha)}) \setminus C) \neq \emptyset$. Then $\gamma(\beta) \leq \gamma(\alpha)$ by minimality of $\gamma(\beta)$. Similarly $\gamma(\alpha) \leq \gamma(\beta)$.

Now suppose that $\gamma(\alpha) = \gamma(\beta)$. Then $((T \downarrow s_\alpha) \setminus C) \cap ((T \downarrow s_{\gamma(\alpha)}) \setminus C) \neq \emptyset$ and $((T \downarrow s_\beta) \setminus C) \cap ((T \downarrow s_{\gamma(\alpha)}) \setminus C) \neq \emptyset$. Then by Subclaim 1, $((T \downarrow s_\alpha) \setminus C) \cap ((T \downarrow s_\beta) \setminus C) \neq \emptyset$.

**Subclaim 3.** If $s_{\gamma(\alpha)} \neq s_{\gamma(\beta)}$ for some $\alpha, \beta < \kappa$, then $s_{\gamma(\alpha)} \downarrow s_{\gamma(\beta)}$.

**Proof of Subclaim:** Suppose $s_{\gamma(\alpha)} \neq s_{\gamma(\beta)}$ and suppose, by way of contradiction, that $s_{\gamma(\alpha)}$ and $s_{\gamma(\beta)}$ are comparable; say $s_{\gamma(\beta)} < s_{\gamma(\alpha)}$. $s_{\gamma(\beta)}$ was chosen because $\gamma(\beta)$ was the least ordinal such that $((T \downarrow s_{\gamma(\beta)}) \setminus C) \cap ((T \downarrow s_{\gamma(\alpha)}) \setminus C) \neq \emptyset$. As $s_{\gamma(\beta)} < s_{\gamma(\alpha)}$, $((T \downarrow s_{\gamma(\alpha)}) \setminus C) \supseteq ((T \downarrow s_{\gamma(\beta)}) \setminus C)$, and so also $((T \downarrow s_\beta) \setminus C) \cap ((T \downarrow s_{\gamma(\alpha)}) \setminus C) \neq \emptyset$. Then $\gamma(\alpha) \geq \gamma(\beta)$ by the minimality of $\gamma(\beta)$.
\(s_{\gamma(\alpha)}\) was chosen because \(\gamma(\alpha)\) was the least ordinal such that \(((T \downarrow s_{\alpha}) \setminus C) \cap ((T \downarrow s_{\gamma(\alpha)}) \setminus C) \neq \emptyset\). As \(((T \downarrow s_{\alpha}) \setminus C)\) is closed downwards in \(C\) and as \(s_{\gamma(\beta)} < s_{\gamma(\alpha)}\), also \(((T \downarrow s_{\alpha}) \setminus C) \cap ((T \downarrow s_{\gamma(\beta)}) \setminus C) \neq \emptyset\). Then by the minimality of \(\gamma(\alpha)\), \(\gamma(\beta) \geq \gamma(\alpha)\).

Then \(\gamma(\alpha) = \gamma(\beta)\), so that \(s_{\gamma(\alpha)} = s_{\gamma(\beta)}\); but this is a contradiction. Thus \(s_{\gamma(\alpha)} \not\subseteq s_{\gamma(\beta)}\).

By Subclaim 2, \((T \downarrow s_{\alpha}) \setminus C\) and \((T \downarrow s_{\beta}) \setminus C\) are disjoint if and only if \(\gamma(\alpha) \neq \gamma(\beta)\). Let \(\lambda\) be the cardinality of the set \(\{\gamma(\alpha) : \alpha < \kappa\}\), and enumerate the set \(\{s_{\gamma(\alpha)} : \alpha < \kappa\}\) as \(\{b_{\delta}^\alpha : \delta < \lambda\}\). For each \(\delta < \lambda\), let \(\theta_\delta\) be the coinitiality of \((T \downarrow b_\delta) \setminus C\), and let \(\{b_{\eta_\delta}^\delta : \eta_\delta < \theta_\delta\}\) be a decreasing coinitial sequence in \((T \downarrow b_0^\delta) \setminus C\).

**Subclaim 4.** Let \(S' = \{b_{\eta_\delta}^\delta : \delta < \lambda, \eta_\delta < \theta_\delta\}\). Then \(S'\) is a set of approximate immediate successors of \(C\).

Proof of Subclaim: First note that \(C < S'\). Suppose \(r > C\). As \(S\) is a set of approximate immediate successors of \(C\), there is an \(s_{\alpha} \in S\) with \(C < s_{\alpha} \leq r\). We have \(((T \downarrow s_{\alpha}) \setminus C) \cap ((T \downarrow s_{\gamma(\alpha)}) \setminus C) \neq \emptyset\). Say \(s_{\gamma(\alpha)} = b_\delta\) for some \(\delta < \lambda\). Pick \(\eta_\delta\) large enough so that \(b_{\eta_\delta}^\delta \in ((T \downarrow s_{\alpha}) \setminus C) \cap ((T \downarrow s_{\gamma(\alpha)}) \setminus C)\). Then \(b_{\eta_\delta}^\delta \in S'\) and \(C < b_{\eta_\delta}^\delta \leq s_{\alpha} \leq r\). Thus \(S'\) is a set of approximate immediate successors of \(C\).

Let \(\{c_{\alpha} : \alpha < \cf C\}\) be an increasing cofinal sequence in \(C\). By Subclaim 4, \(U = \langle G \rangle\) where

\[G = \{(T \uparrow c_{\alpha}) \setminus \bigcup_{s' \in F} (T \uparrow s') : \alpha < \cf C\text{ and }F \text{ is a finite antichain in }S'\}\.

We claim that \((G, \supseteq) \equiv_T (\cf C \times \prod_{\delta < \lambda} w_{\theta_\delta}, \subseteq)\). Define a map \(f : G \to \cf C \times \prod_{\delta < \lambda} w_{\theta_\delta}\) by \(f((T \uparrow c_{\alpha}) \setminus \bigcup_{s' \in F} (T \uparrow s')) = \langle \alpha \rangle \prec \langle \varepsilon_\delta : \delta < \lambda \rangle\), where

\[\varepsilon_\delta = \begin{cases} 0, & \text{if } b_{\eta_\delta}^\delta \not\in F \text{ for all } \eta_\delta < \theta_\delta \\ \eta_\delta, & \text{if } b_{\eta_\delta}^\delta \in F \end{cases}\]

We claim that \(f\) is an unbounded cofinal map. Suppose \(X \subseteq G\) is unbounded. Then \(X\) contains elements of the form \((T \uparrow c_{\alpha}) \setminus \bigcup_{s' \in F} (T \uparrow s')\) where either the \(c_{\alpha}\) are cofinal in \(C\), or the union of the sets \(F\) is infinite, or the union of the sets \(F\) is coinitial in one of the sets \(\{b_{\eta_\delta}^\delta : \eta_\delta < \theta_\delta\}\). In the first case, \(f[X]\) is unbounded in the first coordinate; in the second case, the set of coordinates in which some element of \(f[X]\) has a non-zero element is infinite; and in the third case, \(f[X]\) is unbounded in one of the coordinates \(\delta\) for some \(\delta < \lambda\). In any case, \(f[X]\) is unbounded in \(\cf C \times \prod_{\delta < \lambda} w_{\theta_\delta}\).

Now we define a map \(g\) from \(\cf C \times \prod_{\delta < \lambda} w_{\theta_\delta}\) to \(G\) as follows: let \(x = \langle \alpha, \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_\delta, \ldots \rangle \in \cf C \times \prod_{\delta < \lambda} w_{\theta_\delta}\). Set \(E = \{\delta < \lambda : \varepsilon_\delta \neq 0\}\). Note that \(E\) is a finite subset of \(\lambda\). Let \(F = \{b_{\eta_\delta}^\delta : \delta \in E\}\). Note that \(F\) is a finite antichain in \(S'\). Define \(g(x) = (T \uparrow c_{\alpha}) \setminus \bigcup_{s' \in F} (T \uparrow s')\). One may also check that \(g\) is an unbounded map.

Thus \((U, \subseteq) \equiv_T (G, \supseteq) \equiv_T \cf C \times \prod_{\delta < \lambda} w_{\theta_\delta}\). Noting that \(\cf C\) and all of the \(\theta_\delta\) are regular cardinals, we are done. \(\square\)

Note that this is consistent with what we already know about ultrafilters on tree algebras from Theorem 24, because for any \(\kappa\), \([\kappa]^{<\omega}, \subseteq \equiv_T \prod_{\alpha < \kappa} \{0, 1\}\).

Theorem 29 together with Fact 8(2), tells us exactly when a pseudo-tree has an ultrafilter with maximum Tukey type. First we introduce some terminology: let \(T\) be a pseudo-tree, let \(C\) be an initial chain in \(T\), and let \(\kappa\) be a cardinal. We say that there is a \(\kappa\)-fan above
Corollary 30. Let $T$ be a pseudo-tree with a single root. The Tukey spectrum of $Treealg T$ consists of exactly those types $(cf C \times \prod_{\alpha < \kappa} \theta_{\alpha}, \leq)$ where $C$ is an initial chain having a set of approximate immediate successors $S = \{ s^{\alpha}_{\beta} : \alpha < \kappa, \beta < \theta_{\alpha} \}$ that is a $\kappa$-fan.

In particular, if $|T| = \kappa$, then $T$ has an ultrafilter with maximal type $([\kappa]^{<\omega}, \subseteq)$ if and only if there is an initial chain $C \subseteq T$ having a set $S$ of approximate immediate successors such that $S = \{ s^{\alpha}_{\beta} : \alpha < \kappa, \beta < \theta_{\alpha} \}$ is a $\kappa$-fan above $C$ where $\kappa$-many of the cardinals $\theta_{\alpha}$ are 1.

5. Questions

We conclude by restating some problems and questions.

Question 31. Characterize those Boolean algebras that have an ultrafilter of the maximum Tukey type, and characterize those Boolean algebras that have an ultrafilter of Tukey type strictly below the maximum.

Question 32. If $B$ is an infinite Boolean algebra such that all ultrafilters on $B$ have maximum Tukey type, is $B$ necessarily a free algebra?

Or is the following possible?

Question 33. Does the completion of a free Boolean algebra have only ultrafilters which are of maximum Tukey type?

If not, can we at least rule out the remaining possible case of a cofinally scalene ultrafilter?

Question 34. Can the completion of a free Boolean algebra have an ultrafilter Tukey equivalent to $\langle \omega^\omega, \leq \rangle$? In particular, can the Cohen algebra have an ultrafilter Tukey equivalent to $\langle \omega^\omega, \leq \rangle$?

Note that a positive answer to Question 34 would imply a negative answer to Question 33 and a positive answer to Question 34 would imply a negative answer to Question 32.

Question 35. What are the Tukey spectra of superatomic Boolean algebras?

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