A NOTE ON SOLVABLE MAXIMAL SUBGROUPS IN SUBNORMAL SUBGROUPS OF $GL_n(D)$

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Abstract. Let $D$ be a non-commutative division ring, $G$ a subnormal subgroup of $GL_n(D)$. In this note we show that if $G$ contains a non-abelian solvable maximal subgroup, then $n = 1$ and $D$ is a cyclic algebra of prime degree over $F$.

1. Introduction

In the theory of skew linear groups, one of unsolved difficult problems is that whether the general skew linear group over a division ring contains maximal subgroups. In [1], the authors conjectured that for $n \geq 2$ and a division ring $D$, the group $GL_n(D)$ contains no solvable maximal subgroups. In [2], this conjecture was shown to be true for non-abelian solvable maximal subgroups. In this paper, we consider the following more general conjecture.

Conjecture 1. Let $D$ be a division ring, $G$ a non-central subnormal subgroup of $GL_n(D)$. If $n \geq 2$, then $G$ contains no solvable maximal subgroups.

We note that this conjecture is not true if $n = 1$. Indeed, in [1], it was proved that the subgroup $C^* \cup C^*j$ is solvable maximal in the multiplicative group $H^*$ of the division ring of real quaternions $H$. In this note, we show that Conjecture 1 is true for non-abelian solvable maximal subgroups of $G$, that is, $G$ contains no non-abelian solvable maximal subgroups. This fact generalizes the main result in [2] and it is a consequence of Theorem 3.7 in the text.

2. Throughout this note, $D$ is a division ring with center $F$ and $D^*$ denotes the multiplicative group of $D$. For a positive integer $n$, $M_n(D)$ is the matrix ring of degree $n$ over $D$. We identify $F$ with $F I_n$ via the ring isomorphism $a \mapsto a I_n$, where $I_n$ is the identity matrix of degree $n$. If $S$ is a subset of $M_n(D)$, then $F[S]$ denotes the subring of $M_n(D)$ generated by the set $S \cup F$. Also, if $n = 1$, i.e., if $S \subseteq D$, then $F(S)$ is the division subring of $D$ generated by $S \cup F$. Recall that a division ring $D$ is locally finite if for every finite subset $S$ of $D$, the division subring $F(S)$ is a finite dimensional vector space over $F$. If $H$ and $K$ are two subgroups in a group $G$, then $N_K(H)$ denotes the set of all elements $k \in K$ such that $k^{-1} H k \leq H$, i.e., $N_K(H) = K \cap N_G(H)$. If $A$ is a ring or a group, then $Z(A)$ denotes the center of $A$.

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Let $V = D^n = \{ (d_1, d_2, \ldots, d_n) | d_i \in D \}$. If $G$ is a subgroup of $\text{GL}_n(D)$, then $V$ may be viewed as $D$-$G$ bimodule. Recall that a subgroup $G$ of $\text{GL}_n(D)$ is irreducible (resp. reducible, completely reducible) if $V$ is irreducible (resp. reducible, completely reducible) as $D$-$G$ bimodule. If $F'[G] = M_n(D)$, then $G$ is absolutely irreducible over $D$. An irreducible subgroup $G$ is imprimitive if there exists an integer $m \geq 2$ such that $V = \bigoplus_{i=1}^m V_i$ as left $D$-modules and for any $g \in G$ the mapping $V_i \to V_ig$ is a permutation of the set $\{ V_1, \ldots, V_m \}$. If $G$ is irreducible and not imprimitive, then $G$ is primitive.

2. Auxiliary lemmas

**Lemma 2.1.** Let $D$ be a division ring with center $F$, and $M$ a subgroup of $\text{GL}_n(D)$. If $M/M \cap F^*$ is a locally finite group, then $F[M]$ is a locally finite dimensional vector space over $F$.

**Proof.** Take any finite subset $\{ x_1, x_2, \ldots, x_k \} \subseteq F[M]$ and write

$$x_i = f_{i_1} m_{i_1} + f_{i_2} m_{i_2} + \cdots + f_{i_s} m_{i_s}.$$ 

Let $G = \langle m_{i_j} : 1 \leq i \leq k, 1 \leq j \leq s \rangle$ be the subgroup of $M$ generated by all $m_{i_j}$. Since $M/M \cap F^* \cong MF^*/F^*$ is locally finite, the group $GF^*/F^*$ is finite. Let $\{ y_1, y_2, \ldots, y_t \}$ be a transversal of $F^*$ in $GF^*$ and set

$$R = Fy_1 + Fy_2 + \cdots + Fy_t.$$ 

Then, $R$ is a finite dimensional vector space over $F$ containing $\{ x_1, x_2, \ldots, x_k \}$. □

**Lemma 2.2.** Every locally solvable periodic group is locally finite.

**Proof.** Let $G$ be a locally solvable periodic group, and $H$ be a finitely generated subgroup of $G$. Then, $H$ is solvable with derived series of length $n \geq 1$, say,

$$1 = H^{(n)} \lhd H^{(n-1)} \lhd \cdots \lhd H' \lhd H.$$ 

We shall prove that $H$ is finite by induction on $n$. For if $n = 1$, then $H$ is a finitely generated periodic abelian group, so it is finite. Suppose $n > 1$. It is clear that $H/H'$ is a finitely generated periodic abelian group, so it is finite. Hence, $H'$ is finitely generated. By induction hypothesis, $H'$ is finite, and as a consequence, $H$ is finite. □

**Lemma 2.3.** Let $D$ be a division ring with center $F$, and $G$ a subnormal subgroup of $D^*$. If $G$ is soluble-by-finite, then $G \subseteq F$.

**Proof.** Let $A$ be a soluble normal subgroup of finite index in $G$. Since $G$ is subnormal in $G$, so is $A$. By [L9] 14.4.4, we have $A \subseteq F$. This implies that $G/Z(G)$ is finite, so $G'$ is finite too. Therefore, $G'$ is a finite subnormal subgroup of $D^*$. In view of [K] Theorem 8), it follows that $G' \subseteq F$, hence $G$ is solvable. Again by [L9] 14.4.4, we conclude that $G \subseteq F$.

For our further use, we also need one result of Wehrfritz which will be restated in the following lemma for readers’ convenience.

**Lemma 2.4.** [L6] Proposition 4.1] Let $D = E(A)$ be a division ring generated as such by its metabelian subgroup $A$ and its division subring $E$ such that $E \leq C_D(A)$. Set $H = N_D(A), B = C_A(A'), K = E(Z(B)), H_1 = N_K(A) = H \cap K^*$, and let $T$ be the maximal periodic normal subgroup of $B$.
Proposition 3.1. Let $D$ be a division ring with center $F$, and $G$ a subnormal subgroup of $D^*$. If $M$ is a non-abelian solvable-by-finite maximal subgroup of $G$, then $M$ is abelian-by-finite and $[D : F] < \infty$.

Proof. Since $M$ is maximal in $G$ and $M \subseteq F(M)^* \cap G \subseteq G$, either $M = F(M)^* \cap G$ or $G \subseteq F(M)^*$. The first case implies that $M$ is a solvable-by-finite subnormal subgroup of $F(M)^*$, which yields $M$ is abelian by Lemma 2.3 a contradiction. Therefore, the second case must occur, i.e., $G \subseteq F(M)^*$. By Stuth’s theorem (see e.g. [13], 14.3.8), we conclude that $F(M) = D$. Let $N$ be a solvable normal subgroup of finite index in $M$. First, we assume that $N$ is abelian, so $M$ is abelian-by-finite. In view of [17], Corollary 24], the ring $F[N]$ is a Goldie ring, and hence it is an Ore domain whose skew field of fractions coincides with $F(N)$. Consequently, any $\alpha \in F(N)$ may be written in the form $\alpha = pq^{-1}$, where $q, p \in F[N]$ and $q \neq 0$. The normality of $N$ in $M$ implies that $F[N]$ is normalized by $M$. Thus, for any $m \in M$, we have

$$mam^{-1} = mpq^{-1}m^{-1} = (mpm^{-1})(m^{-1}qm)^{-1} \in F(N).$$

In other words, $L := F(N)$ is a subfield of $D$ normalized by $M$. Let $\{x_1, x_2, \ldots, x_k\}$ be a transversal of $N$ in $M$ and set

$$\Delta = Lx_1 + Lx_2 + \cdots + Lx_k.$$ 

Then, $\Delta$ is a domain with $\dim_\mathbb{F} \Delta \leq k$, so $\Delta$ is a division ring that is finite dimensional over its center. It is clear that $\Delta$ contains $F$ and $M$, so $D = \Delta$ and $[D : F] < \infty$.

Next, we suppose that $N$ is a non-abelian solvable group with derived series of length $s \geq 1$. Then we have such a series

$$1 = N^{(s)} \leq N^{(s-1)} \leq \cdots \leq N' \leq N \leq M.$$ 

If we set $A = N^{(s-2)}$, then $A$ is a non-abelian metabelian normal subgroup of $M$. By the same arguments as above, we conclude that $F(A)$ is normalized by $M$ and we have $M \subseteq N_G(F(A)^*) \subseteq G$. By the maximality of $M$ in $G$, either $N_G(F(A)^*) = M$ or $N_G(F(A)^*) = G$. If the first case occurs, then $G \cap F(A)^*$ is a subnormal subgroup of $F(A)^*$ contained in $M$. Since $M$ is solvable-by-finite, so is $G \cap F(A)^*$. By Lemma 2.3 $A \subseteq G \cap F(A)^*$ is abelian, a contradiction. We may therefore assume that $N_G(F(A)) = G$, which says that $F(A)$ is normalized by $G$. In view of Stuth’s theorem, we have $F(A) = D$. From this we conclude that $Z(A) = F^* \cap A$ and $F = C_D(A)$. Set $H = N_{D^*}(A)$, $B = C_A(A')$, $K = F(Z(B))$, $H_1 = H \cap K^*$, and $T$ to be the maximal periodic normal subgroup of $B$. Then $H_1$ is an abelian group, $T$ is a characteristic subgroup of $B$ and hence of $A$. In view of Lemma 2.4 we have three possible cases:

Case 1: $T$ is not abelian.
Since $T$ is normal in $M$, we conclude that $M \subseteq N_G(F(T)^*) \subseteq G$. By the maximality of $M$ in $G$, either $M = N_G(F(T)^*)$ or $G = N_G(F(T)^*)$. The first case implies that $F(T)^* \cap G$ is subnormal in $F(T)^*$ contained in $M$. Again by Lemma 2.2, it follows that $T \subseteq F(T) \cap G$ is abelian, a contradiction. Thus, we may assume that $G = N_G(F(T)^*)$, which implies that $F(T) = D$ by Stuth’s theorem. By Lemma 2.2, $T$ is locally finite. In view of Lemma 2.1 we conclude that $D$ is a locally finite division ring. Since $M$ is solvable-by-finite, it contains no non-cyclic free subgroups. In view of [5, Theorem 3.1], it follows $[D:F] < \infty$ and $M$ is abelian-by-finite.

**Case 2**: $T$ is abelian and contains an element $x$ of order 4 not in the center of $B = C_A(A')$.

It is clear that $x$ is not contained in $F$. Because $x$ is of finite order, the field $F(x)$ is algebraic over $F$. Since $(x)$ is a 2-primary component of $T$, it is a characteristic subgroup of $T$ (see the proof of [13, Theorem 1.1, p.132]). Consequently, $(x)$ is a normal subgroup of $M$. Thus, all elements of the set $x^M := \{m^{-1}xm | m \in M\} \subseteq F(x)$ have the same minimal polynomial over $F$. This implies $|x^M| < \infty$, so $x$ is an $FC$-element, and consequently, $[M : C_M(x)] < \infty$. Setting $C = \text{core}_M(C_M(x))$, then $C \trianglelefteq M$ and $[M : C]$ is finite. Since $M$ normalizes $F(C)$, we have $M \subseteq N_G(F(C)^*) \subseteq G$. By the maximality of $M$ in $G$, either $N_G(F(C)^*) = M$ or $N_G(F(C)^*) = G$. The last case implies that $F(C) = D$, and consequently, $x \in F$, a contradiction. Thus, we may assume that $N_G(F(C)^*) = M$. From this, we conclude that $G \cap F(C)^*$ is a subnormal subgroup of $F(C)^*$ which is contained in $M$. Thus, $C \subseteq G \cap F(C)^*$ is abelian by [13, 14.4.4]. Therefore, $C$ is an abelian normal subgroup of finite index in $M$. By the same arguments used in the first paragraph we conclude that $[D:F] < \infty$.

**Case 3**: $H = AH_1$.

Since $A' \subseteq H_1 \cap A$, we have $H/H_1 \cong A/A \cap H_1$ is abelian, and hence $H' \subseteq H_1$. Since $H_1$ is abelian, $H'$ is also abelian. Moreover, $M \subseteq H_1$, it follows that $M'$ is also abelian. In other words, $M$ is a metabelian group, and the conclusions follow from [4, Theorem 3.3]. \qed

Let $D$ be a division ring, and $G$ a subnormal subgroup of $D^*$. It was showed in [4, Theorem 3.3] that if $G$ contains a non-abelian metabelian maximal subgroup, then $D$ is cyclic of prime degree. The following theorem generalizes this phenomenon.

**Theorem 3.2.** Let $D$ be a division ring with center $F$, and $G$ a subnormal subgroup of $D^*$. If $M$ is a non-abelian solvable maximal subgroup of $G$, then the following conditions hold:

(i) There exists a maximal subfield $K$ of $D$ such that $K/F$ is a finite Galois extension with $\text{Gal}(K/F) \cong M/K^* \cap G \cong \mathbb{Z}_p$ for some prime $p$, and $[D:F] = p^2$.

(ii) The subgroup $K^* \cap G$ is the FC-center. Also, $K^* \cap G$ is the Fitting subgroup of $M$. Furthermore, for any $x \in M \setminus K$, we have $x^p \in F$ and $D = F[M] = \bigoplus_{i=1}^{p} Kx^i$. 
Theorem 3.4. Let $D$ be a division ring with center $F$, $G$ be a subnormal subgroup of $D^*$, and $M$ be a non-abelian maximal subgroup of $G$. Then $M$ cannot be finitely generated solvable-by-finite. In particular, $M$ cannot be polycyclic-by-finite.

Proof. Suppose that $M$ is solvable-by-finite. Then by Proposition 3.1, we conclude that $[D : F] < \infty$. In view of 10, Corollary 3, it follows that $M$ is not finitely generated. The rest of the corollary is clear.

Theorem 3.4. Let $D$ be a non-commutative locally finite division ring with center $F$, and $G$ a subnormal subgroup of $GL_n(D)$, $n \geq 1$. If $M$ is a non-abelian solvable maximal subgroup of $G$, then $n = 1$ and all conclusions of Theorem 3.2 hold.

Proof. By 5, Theorem 3.1, there exists a maximal subfield $K$ of $M_n(D)$ containing $F$ such that $K^* \cap G$ is a normal subgroup of $M$ and $M/K^* \cap G$ is a finite simple group of order $[K : F]$. Since $M/K^* \cap G$ is solvable and simple, we conclude $M/K^* \cap G \cong \mathbb{Z}/p$, for some prime number $p$. It follows that $[K : F] = p$ and $[M_n(D) : F] = p^2$, from which we have $n = 1$. Finally, all conclusions follow from Theorem 3.2.

Lemma 3.5. Let $R$ be a ring, and $G$ a subgroup of $R^*$. Assume that $F$ is a central subfield of $R$ and $A$ is a minimal abelian subgroup of $G$ such that $K = F[A]$ is normalized by $G$. Then $F[G] = \bigoplus_{g \in T} Kg$ for every transversal $T$ of $A$ in $G$.

Proof. For the proof of this lemma, we use the similar techniques as in the proof of 2, Lemma 3.1. Since $K$ is normalized by $G$, it follows that $F[G] = \sum_{g \in T} Kg$ for every transversal $T$ of $A$ in $G$. Therefore, it suffices to prove that every finite subset of $\{g_1, g_2, \ldots, g_n\} \subseteq T$ is linearly independent over $K$. Assume by contradiction that there exists such a non-trivial relation $k_1g_1 + k_2g_2 + \cdots + k_ng_n = 0$, with $n$ chosen minimal. The minimality of $n$ implies $g_2g_1^{-1} \notin K$, and hence $g_2g_1^{-1} \notin A = C_G(A)$. Thus, there exists an element $x \in A$ such that $g_2g_1^{-1}x \neq xg_1^{-1}g_2$. For each $1 \leq i \leq n$, if we set $x_i = g_i^{-1}x$, then $x_i \neq x_j$. Since $G$ normalizes $K$, it follows $x_i \in K^*$ for all $1 \leq i \leq n$. Now, we have

$$(k_1g_1 + \cdots + k_ng_n)x - x_1(k_1g_1 + \cdots + k_ng_n) = 0.$$
Consequently,

\[(x_2 - x_1)k_2^2g_2 + \cdots + (x_n - x_1)k_ng_n = 0,\]

which is a non-trivial relation. This contradicts the minimal choice of \(n\). Therefore, \(T\) is linearly independent over \(K\). \(\square\)

**Remark 1.** In view of [8, Theorem 11], if \(D\) is a division ring with at least five elements and \(n \geq 2\), then any non-central subnormal subgroup of \(\text{GL}_n(D)\) contains SL\(_n(D)\) and hence is normal.

**Theorem 3.6.** Let \(D\) be non-commutative division ring with center \(F\), and \(G\) a subnormal subgroup of \(\text{GL}_n(D)\), \(n \geq 2\). Assume additionally that \(F\) contains at least five elements. If \(M\) is a solvable maximal subgroup of \(G\), then \(M\) is abelian.

**Proof.** Setting \(R = F[M]\), then \(M \subseteq R^* \cap G \subseteq G\). By the maximality of \(M\) in \(G\), either \(R^* \cap G = M\) or \(G \subseteq R^*\). We need to consider two possible cases:

**Case 1:** \(R^* \cap G = M\).

By Remark 1, \(M\) is a normal subgroup of \(R^*\). If \(M\) is reducible, then by [7, Lemma 1], it contains a copy of \(D^*\). Consequently, \(D^*\) is solvable, and hence it is commutative, a contradiction. We may therefore assume that \(M\) is irreducible. Consequently, \(R\) is a prime ring by [14, 1.1.14]. So, in view of [8, Theorem 2], either \(M \subseteq Z(R)\) or \(R\) is a domain. If the first case occurs, then we are done. Now, suppose that \(R\) is a domain. By [17, Corollary 24], we conclude that \(R\) is a Goldie ring, and hence \(R\) is an Ore domain. Let \(\Delta_1\) be the skew field of fractions of \(R\), which is contained in \(M_n(D)\) by [14, 5.7.8]. Since \(M \subseteq \Delta_1 \cap G \subseteq G\), either \(G \subseteq \Delta_1\) or \(M = \Delta_1 \cap G\). The first case implies \(\Delta_1 = M_n(D)\), which is impossible since \(n \geq 2\). Thus \(M = \Delta_1 \cap G\), and hence \(M\) is normal in \(\Delta_1^*\). Since \(M\) is solvable, it is contained in \(Z(\Delta_1)\) by [13, 14.4.4], so \(M\) is abelian.

**Case 2:** \(G \subseteq R^*\).

In this case, remark 1 yields \(\text{SL}_n(D) \subseteq R^*\). Thus, by the Cartan-Brauer-Hua Theorem for the matrix ring, one has \(R = F[M] = M_n(D)\). According to [13, Theorem A], \(M\) is abelian-by-locally finite. Let \(A\) be a maximal abelian normal subgroup of \(M\) such that \(M/A\) is locally finite. By [14, 1.2.12], \(F[A]\) is a semisimple artinian ring. The Wedderburn-Artin Theorem implies that

\[F[A] \cong M_{n_1}(D_1) \times M_{n_2}(D_2) \cdots \times M_{n_s}(D_s),\]

where \(D_i\) are division \(F\)-algebras, \(1 \leq i \leq s\). Since \(F[A]\) is abelian, \(n_i = 1\) and \(K_i := D_i = Z(D_i)\) are fields that contain \(F\) for all \(i\). Therefore,

\[F[A] \cong K_1 \times K_2 \cdots \times K_s.\]

If \(M\) is imprimitive, then by [5, Lemma 2.6], we conclude that \(M\) contains \(\text{SL}_r(D)\) for some \(r > 1\). This is impossible since \(\text{SL}_r(D)\) is unsolvable if \(r > 1\). It follows that \(M\) is primitive. Therefore, in view of [2, Proposition 3.3], \(F[A]\) is an integral domain, so \(s = 1\). Hence, \(K := F[A]\) is a subfield of \(M_n(D)\) containing \(F\). Again by [2, Proposition 3.3], we conclude that \(L := C_{M_n(D)}(K) \cong M_n(\Delta_2)\) for some division \(F\)-algebra \(\Delta_2\). Since \(M\) normalizes \(K\), it also normalizes \(L\). Therefore, we have \(M \subseteq N_G(L^*) \subseteq G\). By the maximality of \(M\) in \(G\), either \(M = N_G(L^*)\) or \(G = N_G(L^*)\). The last case implies that \(L^*\) is normal in \(\text{GL}_n(D)\). By the Cartan-Brauer-Hua Theorem for the matrix ring, either \(L \subseteq F\) or \(L = M_n(D)\). If
\( L \subseteq F \) then \( M_n(D) = K = F[A] \) is a field, a contradiction. If \( L = M_n(D) \), then \( K = F[A] \subseteq F \). Consequently, \( M/M \cap F^* \) is locally finite, and hence \( D \) is a locally finite division ring by Lemma 24. If \( M \) is non-abelian, then by Theorem 23 we conclude that \( n = 1 \), a contradiction. Therefore \( M \) is abelian in this case. Now, we consider the case \( M = N_G(L^*) \), from which we have \( L^* \cap G \subseteq M \). In other words, \( L^* \cap G \) is a solvable normal subgroup of \( GL_m(\Delta_2) \). If \( m > 1 \), then in view of Remark 1 one has \( L^* \cap G \subseteq Z(\Delta_2) \) or \( SL_m(D) \subseteq L^* \cap G \). The last case implies that \( SL_2(\Delta_2) \) is solvable, a contradiction. Thus, we have \( L^* \cap G \subseteq Z(\Delta_2) \). If \( m = 1 \) then \( L = \Delta_2 \). According to \( [13, 14.4.4] \), we conclude that \( L^* \cap G \subseteq Z(\Delta_2) \). In any case, \( L^* \cap G \) is an abelian normal subgroup of \( M \) and \( M/L^* \cap G \) is locally finite. By the maximality of \( A \) in \( M \), it follows \( A = L^* \cap G = L^* \cap M = C_M(A) \). In other words, \( A \) is a maximal abelian subgroup of \( M \).

By Lemma 3.5 \( F[M] = \oplus_{m \in T} K_m \) for some transversal \( T \) of \( A \) in \( M \). Thus, for any \( x \in L \), there exist \( k_1, k_2, \ldots, k_t \in K \) and \( m_1, m_2, \ldots, m_t \in T \) such that \( x = k_1m_1 + k_2m_2 + \cdots + k_tm_t \). Take an arbitrary element \( a \in A \), by the normality of \( A \) in \( M \), there exist \( a_i \in A \) such that \( ma_i = a_im_i \) for all \( 1 \leq i \leq t \). Since \( xa = ax \), it follows

\[
(k_1a_1 - k_1a)m_1 + (k_2a_2 - k_2a)m_2 + \cdots + (k_ta_t - k_ta)m_t = 0.
\]

Because \( \{m_1, m_2, \ldots, m_t\} \) is linearly independent over \( K \), we have \( a = a_1 = \cdots = a_t \). Consequently, \( ma_i = a_im_i \) for all \( a \in A \), and thus \( m_1 \in C_M(A) = A \) for all \( 1 \leq i \leq t \). This means \( x \in K \), and hence \( L = K \).

Next, we prove that \( M/A \) is simple. Suppose that \( N \) is an arbitrary normal subgroup of \( M \) properly containing \( A \). Note that by the maximality of \( A \) in \( M \), we conclude that \( N \) is non-abelian. We claim that \( Q := F[N] = M_n(D) \). Indeed, since \( N \) is normal in \( M \), we have \( M \subseteq N_G(Q^*) \subseteq G \), and hence either \( N_G(Q^*) = M \) or \( N_G(Q^*) = G \). First, we suppose the former case occurs. Then \( Q^* \cap G \subseteq M \), hence \( Q^* \cap G \) is a solvable normal subgroup of \( R^* \). In view of \( [2, Proposition 3.3] \), \( Q \) is a prime ring. It follows by \( [8, Theorem 2] \) that either \( Q^* \cap G \subseteq Z(Q) \) or \( Q \) is a domain. If the first case occurs, then \( N \subseteq Q^* \cap G \) is abelian, which contradicts to the choice of \( N \). If \( Q \) is a domain, then by Goldie’s theorem, it is an Ore domain. Let \( \Delta_2 \) be the skew field of fractions of \( Q \), which is contained in \( M_n(D) \) by \( [14, 5.7.8] \). Because \( M \) normalizes \( Q \), it also normalizes \( \Delta_2 \), from which we have \( M \subseteq N_G(\Delta_2^*) \subseteq G \). Again by the maximality of \( M \) in \( G \), either \( N_G(\Delta_2^*) = M \) or \( N_G(\Delta_2^*) = G \). The first case implies that \( \Delta_2^* \cap G \) is a solvable normal subgroup of \( \Delta_2^* \). Consequently, \( N \subseteq \Delta_2^* \cap G \) is abelian by \( [13, 14.4.4] \), a contradiction. If \( N_G(\Delta_2^*) = G \), then \( \Delta_2 = M_n(D) \) by Stuth’s theorem, which is impossible since \( n \geq 2 \). Therefore, the case \( N_G(Q^*) = M \) cannot occur. Next, we consider the case \( N_G(Q^*) = G \). In this case we have \( Q^* \cap G \subseteq G \), hence \( SL_n(D) \subseteq Q^* \). By the Cartan-Brauer-Hua theorem for the matrix ring, we conclude \( Q = M_n(D) \) as claimed. In other words, we have \( F[N] = F[M] = M_n(D) \).

For any \( m \in M \subseteq F[N] \), there exist \( f_1, f_2, \ldots, f_s \in F \) and \( n_1, n_2, \ldots, n_s \in N \) such that

\[
m = f_1n_1 + f_2n_2 + \cdots + f_sn_s.
\]

Let \( H = \langle n_1, n_2, \ldots, n_s \rangle \) be the subgroup of \( N \) generated by \( n_1, n_2, \ldots, n_s \). Set \( B = AH \) and \( S = F[B] \). Since \( M/A \) is locally finite, the group \( B/A \) is finite. Let \( \{x_1, \ldots, x_k\} \) be a transversal of \( A \) in \( B \). The maximality of \( A \) in \( M \) implies that \( A \)
is a maximal abelian subgroup of $B$ that is also normal in $B$. By Lemma 3.3

$$S = Kx_1 \oplus Kx_2 \oplus \cdots \oplus Kx_k,$$

which says that $S$ is an artinian ring. Since $C_{M_n(D)}(A) = L$ is a field, in view of \cite[Proposition 3.3]{2}, $A$ is irreducible. Because $B$ contains $A$, it is irreducible too. By \cite[1.1.14]{14}, it follows that $S$ is a prime ring. Now, $S$ is both prime and artinian, so it is simple and $S \cong M_{n_0}(\Delta_0)$ for some division $F$-algebra $\Delta_0$. If we set $F_0 = Z(\Delta_0)$, then $Z(S) = F_0$. Since $B$ is abelian-by-finite, the group ring $FB$ is a PI-ring by \cite[Lemma 11, p.176]{12}. Thus, as a homomorphic image of $FB$, the ring $S = F[B]$ is also a PI-ring. By Kaplansky’s theorem, we conclude that $[S : F_0] < \infty$.

If we set $K_0 = F_0[A]$, then $K \subseteq K_0$ and $F_0[B] = S$. By Lemma 3.3 we conclude that $S = K_0x_1 \oplus \cdots \oplus K_0x_k$. For dimensional reason, one has $K = K_0$ and $F_0 \subseteq K$. Hence $K$ is a finite extension field over $F_0$. Recall that $A$ is normal in $B$, so for any $b \in B$, the mapping $\theta_b : K \rightarrow K$ given by $\theta_b(x) = bxb^{-1}$ is well defined. It is clear that $\theta_b$ is an $F_0$-automorphism of $K$. Thus, the mapping

$$\psi : B \rightarrow \text{Gal}(K/F_0)$$

defined by $\psi(b) = \theta_b$ is a group homomorphism with

$$\ker\psi = C_B(K^*) = C_B(A) = A.$$

Since $F_0[B] = S$, it follows that $C_S(B) = F_0$. Therefore, the fixed field of $\psi(B)$ is $F_0$, and hence $K/F_0$ is a Galois extension. By the fundamental theorem of Galois theory, one has $\psi$ is a surjective homomorphism. Hence, $B/A \cong \text{Gal}(K/F_0)$.

Setting $M_0 = M \cap S^*$, then $B \subseteq M_0$ and $F_0[M_0] = F_0[B] = S$. It is clear that $A$ is a maximal abelian subgroup of $M_0$ that is also normal in $M_0$. By replacing $B$ by $M_0$ and by the same arguments used in the preceding paragraph, we also have $M_0/A \cong \text{Gal}(K/F_0)$. Consequently, $B/A \cong \text{Gal}(K/F_0) \cong M_0/A$. The conditions $B \subseteq M_0$ and $B/A \cong M_0/A$ imply $B = M_0$. Hence, $m \in M_0 = B \subseteq N$. Since $m$ was chosen arbitrarily, it follows that $M = N$, which implies the simplicity of $M/A$. Since $M/A$ is simple and solvable, one has $M/A \cong Z_p$, for some prime number $p$. By Lemma 3.3 it follows $\dim_K M_n(D) = |M/A| = p$, which forces $n = 1$, a contradiction.

Now, we are ready to get the main result of this note which gives in particular, the positive answer to Conjecture 4 for non-abelian case.

**Theorem 3.7.** Let $D$ be a non-commutative division ring with center $F$, $G$ a subnormal subgroup of $\text{GL}_n(D)$. Assume additionally that $F$ contains at least five elements if $n > 1$. If $M$ is a non-abelian solvable maximal subgroup of $G$, then $n = 1$ and the following conditions hold:

(i) There exists a maximal subfield $K$ of $D$ such that $K/F$ is a finite Galois extension with $\text{Gal}(K/F) \cong M/K^* \cap G \cong Z_p$ for some prime $p$, and $[D : F] = p^2$.

(ii) The subgroup $K^* \cap G$ is the FC-center. Also, $K^* \cap G$ is the Fitting subgroup of $M$. Furthermore, for any $x \in M \setminus K$, we have $x^p \in F$ and $D = F[M] = \bigoplus_{i=1}^p Kx^i$.

**Proof.** Combining Theorem 3.2 and Theorem 3.6 we get the results. \qed
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