Contribution of heavy neutrinos to decay of standard-model-like Higgs boson $h \rightarrow \mu \tau$ in a 3-3-1 model with additional gauge singlets

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Abstract

In the framework of the improved version of the 3-3-1 models with right-handed neutrinos, which is added to the Majorana neutrinos as new gauge singlets, the recent experimental neutrino oscillation data is completely explained through the inverse seesaw mechanism. We show that the major contributions to $\text{Br}(\mu \rightarrow e\gamma)$ are derived from corrections at 1-loop order of heavy neutrinos and bosons. But, these contributions are sometime mutual destructive, creating regions of parametric spaces where the experimental limits of $\text{Br}(\mu \rightarrow e\gamma)$ are satisfied. In these regions, we find that $\text{Br}(\tau \rightarrow \mu\gamma)$ can achieve values of $10^{-10}$ and $\text{Br}(\tau \rightarrow e\gamma)$ may even reach values of $10^{-9}$ very close to the upper bound of the current experimental limits. Those are ideal areas to study lepton-flavor-violating decays of the standard model-like Higgs boson ($h_1^0$). We also pointed out that the contributions of heavy neutrinos play an important role to change $\text{Br}(h_1^0 \rightarrow \mu\tau)$, this is presented through different forms of mass mixing matrices ($M_R$) of heavy neutrinos. When $M_R \sim \text{diag}(1,1,1)$, $\text{Br}(h_1^0 \rightarrow \mu\tau)$ can get a greater value than the cases $M_R \sim \text{diag}(1,2,3)$ and $M_R \sim \text{diag}(3,2,1)$ and the largest that $\text{Br}(h_1^0 \rightarrow \mu\tau)$ can reach is very close $10^{-3}$.

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I. INTRODUCTION

The current experimental data has demonstrated that the neutrinos are masses and flavor oscillations [1]. This lead to a consequence that lepton-flavor-violating decays of charged leptons (cLFV) must be existence and it is strongly dependent on the flavor neutrinos oscillation. The cLFV is concerned in both theory and experiment. On the theoretical side, the processes $l_a \rightarrow l_b \gamma$ are loop induced, we therefore pay attention both neutrino and boson contributions, with special attention to the latter because the former is included active and exotic neutrinos, which can be very good solved through mixing matrix [2]. From the experimental side, branching ratio ($Br$) of cLFV have upper bound as given in Ref.[3].

\[
Br(\mu \rightarrow e\gamma) < 4.2 \times 10^{-13}, \\
Br(\tau \rightarrow e\gamma) < 3.3 \times 10^{-8}, \\
Br(\tau \rightarrow \mu\gamma) < 4.4 \times 10^{-8}.
\] (1)

After the discovery of the Higgs boson in 2012 [4, 5], lepton-flavor-violating decays of the standard model- like Higgs boson (LFVHDs) are getting more attention in the models beyond the standard model (BSM). The parameter space regions predicted from BSM for the large signal of LFVHDs is limited directly from both the experimental data and theory of cLFV [5–7]. Branching ratio of LFVHDs, such as ($h_1^0 \rightarrow \mu\tau, \tau e$), are stringent limited by the CMS Collaboration using data collected at a center-of-mass energy of 13TeV, as given $Br(h_1^0 \rightarrow \mu\tau, \tau e) \leq \mathcal{O}(10^{-3})$. Some published results show that $Br(h_1^0 \rightarrow \mu\tau)$ can reach values of $\mathcal{O}(10^{-4})$ in supersymmetric and non-supersymmetric models [8, 9]. In fact, the main contributions to $Br(h_1^0 \rightarrow \mu\tau)$ come from heavy neutrinos. If those contributions are minor or destructive, the $Br(h_1^0 \rightarrow \mu\tau)$ in a model is only about $\mathcal{O}(10^{-9})$ [10]. With the addition of heavy neutrinos, the models have many interesting features, for example, besides creating large lepton flavor violating sources, it can also explain the masses and mixing of neutrinos through the inverse seesaw (ISS) mechanism [11–14].

We recall that the 3-3-1 models with multiple sources of lepton flavor violating couplings have been introduced long ago [15, 16]. With the $\beta$ parameter, the general 3-3-1 model is separated into different layers, highlighting the properties of the neutral Higgs through contributions to rare decays that can be confirmed by experimental data [17–25]. However, LFVHDs have been mentioned only in the version with heavy neutral leptons assigned as
the third components of lepton (anti)triplets, where active neutrino masses are generated by effective operators \[26, 27\]. An other way of investigating LFVHDs is derived from the main contributions of gauge and Higgs bosons. Unfortunately, these works can only show that the largest values of \(Br(h_1^0 \rightarrow \mu\tau)\) is \(\mathcal{O}(10^{-5})\) \[28, 29\].

Recently, the 3-3-1 model with right-handed neutrinos (331RHN) is added heavy neutrinos which are gauge singlets (331ISS) has shown very well results when investigating LFVHDs \[30–32\]. As a result, the above model has predicted the parameter space regions, where satisfying the experimental upper limit of the \(Br(\mu \rightarrow e\gamma)\) and \(Br(h_1^0 \rightarrow \mu\tau)\) can reach value of \(\mathcal{O}(10^{-5})\) \[32\]. In contrast, 331ISS model still has some questions to be solved, such as: in the parameter space regions satisfying the experimental limits of \(Br(\mu \rightarrow e\gamma)\), are \(Br(\tau \rightarrow e\gamma)\) and \(Br(\tau \rightarrow \mu\gamma)\) excluded? What are the contributions of neutrinos, gauge and Higgs bosons to the LFVHDs? How does the parameterization of neutrinos mixing matrices (both active and exotic neutrinos) affect LFVHDs? In this work, we will solve those problems.

The paper is organized as follows. In the next section, we review the model and give masses spectrum of gauge and Higgs bosons. We then show the masses spectrum of the neutrinos through the inverse seesaw mechanism in Section \[III\]. We calculate the Feynman rules and analytic formulas for cLFV and LFVHDs in Section \[IV\]. Numerical results are discussed in Section \[V\]. Conclusions are in Section \[VI\]. Finally, we provide Appendix \[A\] \[B\] to calculate and exclude divergence in the amplitude of LFVHDs.

II. THE REVIEW MODEL

A. Particle content

We now consider 331ISS model, which is structured from the original 331RHN model as given in Ref. \[16\] and additional heavy Majorana neutrinos. The electric charge operator corresponding to the electroweak group \(SU(3)_L \otimes U(1)_X\) is \(Q = T_3 + \beta T_8 + X\), where \(\beta = -\frac{1}{\sqrt{3}}\) and \(T_{3,8}\) are diagonal \(SU(3)_L\) generators.

To avoid chiral anomalies, the left-handed component of lepton and third generations of quark are arranged into triplets, the two remaining generations of quark are in anti-triplets of \(SU(3)_L\). The right-handed component of all fermions are singlets of \(SU(3)_L\). Therefore,
the \((SU(3)C; SU(3)_L; U(1)_X)\) group structure of fermion fields are:

\[
L'_{aL} = \begin{pmatrix}
\nu'_{aL} \\
\nu'^c_{aL} \\
(N'_{aL})^c_L
\end{pmatrix}
\Rightarrow (1, 3, -1/3), \quad \left\{ \begin{array}{c}
l'_{aR} : (1, 1, -1) \\N'_{aR} : (1, 1, 0)
\end{array} \right.
\]

\[
Q'_{aL} = \begin{pmatrix}
d'_{aL} \\
-u'_{aL} \\
(D'_{aL})^c_L
\end{pmatrix}
\Rightarrow (3, 3^*, 0), \quad \left\{ \begin{array}{c}
d'_{aR} : (3, 1, -1/3) \\u'_{aR} : (3, 1, 2/3) \\D'_{aR} : (3, 1, -1/3)
\end{array} \right.
\]

\[
Q^3_L = \begin{pmatrix}
u'_{3L} \\
d'_{3L} \\
(U'_{3L})^c_L
\end{pmatrix}
\Rightarrow (3, 3, 1/3), \quad \left\{ \begin{array}{c}
u'_{3R} : (3, 1, 2/3) \\d'_{3R} : (3, 1, -1/3) \\U'_{3R} : (3, 1, 2/3)
\end{array} \right.
\]

where \(U'_{aL}\) and \(D'_{aL}\) for \(\alpha = 1, 2\) are three up- and down-type quark components in the flavor basis, while \(N'_{aL} \cong N'_{aR}\) are right-handed neutrinos added in bottom of lepton triplets.

The scalar sector consists of a triplet field \(\chi\), which provides the masses to the new heavy fermions, and two triplets \(\rho\) and \(\eta\), which give masses to the SM fermions at the electroweak scale. These scalar fields are assigned to the following \((SU(3)C; SU(3)_L; U(1)_X)\) representations.

\[
\eta = \begin{pmatrix}
\eta^0_1 \\
\eta^0_2 \\
\eta^0_3
\end{pmatrix}
\Rightarrow (1, 3, -1/3); \quad \rho = \begin{pmatrix}
\rho^+_1 \\
\rho^+_2 \\
\rho^+_3
\end{pmatrix}
\Rightarrow (1, 3, 2/3); \quad \chi = \begin{pmatrix}
\chi^0_1 \\
\chi^0_2 \\
\chi^0_3
\end{pmatrix}
\Rightarrow (1, 3, -1/3). \quad (3)
\]

There are two triplets \((\eta, \chi)\) have the same quantum numbers and different from a remain.

Neutral components of scalar triplets are shown relevantly with real and pseudo scalars as:

\[
\eta^0_1 = \frac{1}{\sqrt{2}}(u + R_1 + iI_1); \quad \eta^0_3 = \frac{1}{\sqrt{2}}(R'_1 + iI'_1)
\]

\[
\rho^0_2 = \frac{1}{\sqrt{2}}(v + R_2 + iI_2); \quad \chi^0_1 = \frac{1}{\sqrt{2}}(R'_3 + iI'_3); \quad \chi^0_3 = \frac{1}{\sqrt{2}}(\omega + R_3 + iI_3) \quad (4)
\]

The electroweak symmetry breaking (EWSB) mechanism follows

\[
SU(3)_L \otimes U(1)_X \xrightarrow{(\chi)} SU(2)_L \otimes U(1)_Y \xrightarrow{(\eta, \rho)} U(1)_Q,
\]

where the vacuum expectation values (VEVs) satisfy the hierarchy \(\omega \gg u, v\) as done in Refs. [19, 33]. In order to generate heavy neutrino masses at tree-level and arise mixing angles,
we use the ISS mechanism. Thus, the 331RHN model is extended, where three right-handed neutrinos which are singlets of $SU(3)_L$, $F'_{aR} \sim (1, 1, 0)$, $a = 1; 2; 3$ are added \[32\]. This model exits two global symmetries, one noted $L$ and $L'$ are the normal and new lepton numbers, respectively. They are related to each other by $L = \frac{4}{\sqrt{3}} T_8 + L'$ \[16\] \[34\]. Requiring $L$ must be softly broken, one add a Lagrangian terms, which is relevant to $F_a$ fields. The general Lagrangian Yukawa relates to leptons and heavy neutrinos is given follows:

$$
-L_Y^{LF} = h^{e}_{ab} \overline{L'}_{aL} \rho_{bR} - h^{\nu}_{ab} \epsilon^{ijk}(\overline{L}'_{aL})_i \rho^*_k + Y_{ab} \overline{L'}_{aL} \chi F'_{bR} + \frac{1}{2} (\mu_{F})_{ab}(\overline{F'}_{aR})^c F'_{bR} + \text{H.c.}, \tag{5}
$$

Two of first term in Eq.(5) generate masses for original charged leptons and neutrinos. The next term describes mixing between $N_a'$ and $F'_a$, and the fouth term generates masses for Majorana neutrinos $F'_a$.

To use the simple Higgs spectra, we will choose the case of the Higgs potential discussed in Refs. \[28\] \[35\], namely,

$$
V = \mu_1^2 (\rho^\dag \rho + \eta^\dag \eta) + \mu_2^2 \chi^\dag \chi + \lambda_1 (\rho^\dag \rho + \eta^\dag \eta)^2 + \lambda_2 (\chi^\dag \chi)^2 + \lambda_3 (\rho^\dag \rho + \eta^\dag \eta) (\chi^\dag \chi) - \sqrt{2} f (\epsilon^{ijk} \rho^j \chi^k + \text{H.c.}), \tag{6}
$$

where $\lambda_1$, $\lambda_2$, $\lambda_3$ are the Higgs self-coupling constants, $f$ is a mass parameter and is imposed as real.

### B. Gauge and Higgs bosons

In the model under consider, we denote $g$ and $g'$ as the coupling constants of the electroweak symmetry ($SU(3)_L \otimes U(1)_{X}$). Then, the relation between coupling constants and sine of the Weinberg angle following:

$$
g = e s_W, \quad \frac{g'}{g} = \frac{3 \sqrt{2} s_W}{\sqrt{3 - 4 s_W^2}}, \tag{7}
$$

Gauge bosons in this model get masses through the covariant kinetic term of the Higgs bosons,

$$
\mathcal{L}^H = \sum_{H = \eta, \rho, \chi} (D_\mu H)^\dag (D_\mu H),
$$

The model comprises two pairs of singly charged gauge bosons, denoted as $W^{\pm}$ and $Y^{\pm}$, defined as

$$
W^\pm_{\mu} = \frac{W^1_{\mu} \mp i W^2_{\mu}}{\sqrt{2}}, \quad m^2_W = \frac{g^2}{4} (u^2 + v^2),
$$
The bosons $W^\pm$ as first line in Eq. (8) are identified with the SM ones, leading to $u^2 + v^2 \equiv v_0^2 = (246 \text{ GeV})^2$. In the remainder of the text, we will consider in detail the simple case $u = v = v_0/\sqrt{2} = \sqrt{2}m_W/g$ given in Refs. [28, 29, 35]. Under these imposing conditions, we get $h_0^1$ mixed with the three original states.

From the potential Higgs was given in Eq. (6), one can find the masses and the mass eigenstates of Higgs bosons. There are two pair of charged Higgs $H^{\pm}_{1,2}$ and Goldstone bosons of $W^\pm$ and $Y^\pm$, which are denoted as $G^{\pm}_W$ and $G^{\pm}_Y$, respectively. The relations between the original and physics states of the charged Higgs bosons are:

\[
\begin{pmatrix}
\rho^{\pm}_1 \\
\eta^{\pm}_2
\end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \end{pmatrix} \begin{pmatrix} G^{\pm}_W \\
H^{\pm}_1
\end{pmatrix},
\begin{pmatrix}
\rho^{\pm}_3 \\
\chi^{\pm}_2
\end{pmatrix} = \begin{pmatrix} -s_\alpha & c_\alpha \\
c_\alpha & s_\alpha
\end{pmatrix} \begin{pmatrix} G^{\pm}_Y \\
H^{\pm}_2
\end{pmatrix},
\]

with masses
\[
m^2_{H^+_1} = 2f_\omega, \quad m^2_{H^+_2} = f_\omega(t_\alpha^2 + 1), \quad m_{G^+_W} = m_{G^+_Y} = 0,
\]
and $c_\alpha \equiv \cos \alpha, \quad s_\alpha \equiv \sin \alpha, \quad t_\alpha \equiv \tan \alpha = \frac{v}{\omega}$.

With components of scalar fields are constructed as Eq. (4), the model contains four physical CP-even Higgs bosons $h^0_{1;2;3;4}$ and a Goldstone boson of the non-Hermitian gauge boson ($X^0$). The mixing of neutral Higgs $h^0_4$ and Goldstone of boson $X^0$ depends on $\gamma$ angle, ($t_\gamma \equiv \tan \gamma = \frac{u}{\omega}$). Three CP-even Higgs $h^0_{1;2;3}$ mutually mix and relate to their original components as:

\[
\begin{pmatrix}
R_1 \\
R_2 \\
R_3
\end{pmatrix} = \begin{pmatrix}
-c_\beta & s_\beta & \sqrt{\frac{2}{\lambda_3}} \\
-c_\beta & s_\beta & \sqrt{\frac{2}{\lambda_3}} \\
\sqrt{\frac{2}{\lambda_3}} & \sqrt{\frac{2}{\lambda_3}} & 0
\end{pmatrix} \begin{pmatrix}
h^0_1 \\
h^0_2 \\
h^0_3
\end{pmatrix},
\]

where $s_\beta = \sin \beta$ and $c_\beta = \cos \beta$, and they are defined by

\[
s_\beta = \frac{(4\lambda_1 - m^2_{h^0_4}/v^2)t_\alpha}{A}, \quad c_\beta = \frac{\sqrt{2}(\lambda_3 - f)}{A},
\]

\[
A = \sqrt{(4\lambda_1 - m^2_{h^0_4}/v^2)^2 t_\alpha^2 + 2\left(\lambda_3 - \frac{f}{w}\right)^2}.
\]

There is one neutral CP-even Higgs boson $h^0_4$ with a mass proportional to the electroweak scale and is identified with SM-like Higgs boson.
\[ m_{h_1^0}^2 = \frac{w^2}{2} \left[ 4\lambda_1 t_a^2 + 2\lambda_2 + \frac{f t_a^2}{w} - \sqrt{8t_a^2 \left( \frac{f}{w} - \lambda_3 \right)^2 + \left( 2\lambda_2 + \frac{f t_a^2}{w} - 4\lambda_1 t_a^2 \right)^2} \right]. \] (13)

The remaining two neutral Higgs \((h_2^0, h_3^0)\) in Eq.(11) have masses on the electroweak symmetry breaking scale \(m_{h_{2,3}} \sim \omega\), which are outside the range of LFVHDs so they are not given here.

### III. NEUTRINOS MASSES AND ISS MECHANISM

We now consider the Yukawa Lagrangian in Eq.(5). Charged leptons masses \(m_a\) are generated from first term and in order to avoid LFV processes at tree level, we can assume \(h_{ab}^e = \sqrt{2} \delta_{ab} m_a / v\). Thus, masses of original charged lepton are \(m_a = h_{a}^e v / \sqrt{2}\).

The second term in Eq. (5) is expanded by:

\[
h_{ab}^c e^{ijk} (L_{aL})(L_{bL})^c_i \rho_k = 2h_{ab}^c \left[ -\nu_{aL}(\nu_{bL})^c \rho_j^\pm + \nu_{aL}(N_{bL})^c \rho_j^\pm - \nu_{aL}(N_{bL})^c \rho_j^\mp \right] \] (14)

From the last term of Eq.(14), using antisymmetric properties of \(h_{ab}^c\) matrix and equality \(N_{aL}(\nu_{bL})^c = \nu_{bL}(N_{aL})^c\), we can contribute a Dirac neutrino mass term \(-L_{mass}^\nu = \nu_L m_D N_R^T + \text{H.c.}\), with basis, \(\nu_L \equiv (\nu_{1L}^e, \nu_{2L}^e, \nu_{3L}^e)^T\), \(N_R^T \equiv ((N_{1L})^e, (N_{2L})^e, (N_{3L})^e)^T\) and \(m_D\) has form \((m_D)_{ab} = \sqrt{2} h_{ab}^e v\).

The third term in Eq.(5) generates mass for heavy neutrinos, this consequence comes from the large value of Yukawa coupling \(Y_{ab}\). To describe mixing \(N_a\) and \(F_a\), \((M_R)_{ab} = Y_{ab} \frac{\omega}{\sqrt{2}}\) is introduced. The last term in Eq.(5) violates both \(L\) and \(L\), and hence \(\mu_F\) can be assumed to be small, in the scale of ISS models.

In the basis \(n_{1L}^c = (\nu_{1L}^e, N_{1L}^c, (F_1^c)^T)\) and \((n_{2L}^c)^T = ((\nu_{2L}^e), (N_{2L}^c), (F_2^c))^T\), Eq.(5) derives mass matrix following:

\[-L_{mass}^\nu = \frac{1}{2} \bar{n}_{L} M^\nu (n_{L}^c)^T + \text{H.c., where } M^\nu = \begin{pmatrix} 0 & m_D & 0 \\ m_D^T & 0 & M_R^T \\ 0 & M_R & \mu_F \end{pmatrix}. \] (15)

In the normal seesaw form, \(M^\nu\) can be written:

\[ M^\nu = \begin{pmatrix} 0 & M_D \\ M_D^T & M_N \end{pmatrix}, \] where \(M_D \equiv (m_D, 0)\), and \(M_N = \begin{pmatrix} 0 & M_R^T \\ M_R & \mu_F \end{pmatrix}. \] (16)
To obtain masses eigenvalues and physics states of neutrinos, one can diagonalize $M_\nu$ by $9 \times 9$ matrix $U_\nu$:

$$U_\nu^T M_\nu U_\nu = \hat{M}_\nu = \text{diag}(m_{n_1}, m_{n_2}, ..., m_{n_9}) = \text{diag}(\hat{m}_\nu, \hat{M}_N),$$ (17)

The relations between the flavor and mass eigenstates are

$$n'_L = U_\nu^* n_L, \quad \text{and} \quad (n'_L)^c = U_\nu^c (n_L)^c,$$ (18)

$$P_L n'_i = n'_L = U_{ij}^* n_{jL}, \quad \text{and} \quad P_R n'_i = n'_R = U_{ij}^* n_{jR}, \quad i,j = 1,2,...,9.$$ (19)

In general, $U_\nu$ is written in the form [36]

$$U_\nu = \Omega \begin{pmatrix} U & O \\ O & V \end{pmatrix},$$ (20)

$$\Omega = \exp \begin{pmatrix} O & R \\ -R^T & O \end{pmatrix} = \begin{pmatrix} 1 - \frac{1}{2}RR^T & R \\ -R^T & 1 - \frac{1}{2}R^TR \end{pmatrix} + O(R^3),$$ (21)

The matrix $U = U_{PMNS}$ is the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix [37, 38],

$$U_{PMNS} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix},$$ (22)

and $c_{ab} \equiv \cos \theta_{ab}$, $s_{ab} \equiv \sin \theta_{ab}$. The Dirac phase ($\delta$) and Majorana phases ($\sigma_1, \sigma_2$) are fixed as $\delta = \pi, \sigma_1 = \sigma_2 = 0$. In the normal hierarchy scheme, the best-fit values of neutrino oscillation parameters which satisfied the $3\sigma$ allowed values are given as [1]

$$s_{12}^2 = 0.32, \quad s_{23}^2 = 0.551, \quad s_{13}^2 = 0.0216,$$

$$\Delta m_{21}^2 = 7.55 \times 10^{-5} \text{ eV}^2, \quad \Delta m_{32}^2 = -2.50 \times 10^{-3} \text{ eV}^2, \quad (23)$$

where $\Delta m_{21}^2 = m_{n_2}^2 - m_{n_1}^2$ and $\Delta m_{32}^2 = m_{n_3}^2 - m_{n_2}^2$.

Hence, the following seesaw relations are valid [36]:

$$R^* \simeq (-m_D M^{-1}, \quad m_D (M_R)^{-1}),$$ (24)

$$m_D^{-1} m^T_D \simeq m_\nu \equiv U_{PMNS}^\dagger \hat{m}_\nu U_{PMNS}^\dagger,$$ (25)

$$V^* \hat{M}_N V^\dagger \simeq M_N + \frac{1}{2} R^T R^* M_N + \frac{1}{2} M_N R^\dagger R,$$ (26)
where
\[ M \equiv M_R^T \mu_F^{-1} M_R. \] (27)

In framework of 331RHN model adding $F_a$ as flavor singlets, the Dirac neutrino mass matrix $m_D$ must be antisymmetric. From results in Ref. [30], with the aim of giving regions of parameter space with large LFVHDs, $m_D$ can be chosen in form to suit for both the inverse and normal hierarchy cases of active neutrino masses as:

\[ m_D \equiv \varrho \begin{pmatrix} 0 & 1 & x_{13} \\ -1 & 0 & x_{23} \\ -x_{13} & -x_{23} & 0 \end{pmatrix}, \] (28)

Therefore, $m_D$ has only three independent parameters $x_{13}, x_{23}$, and $\varrho = \sqrt{2} v h_\nu^{\nu}$. In general, the matrix $m_{\nu}$ in Eq. (25) is symmetric, $(m_{\nu})_{ij} = (m_{\nu})_{ji}$, their components are given by:

\[ (m_{\nu})_{ij} = (m_D)_{ik}(M^{-1})_{kl}(m_D^T)_{lj}, \quad (k \neq i, l \neq j) \] (29)

We can calculate in detail,

\[
\begin{align*}
(m_{\nu})_{ij} - (m_{\nu})_{ji} & \sim x_{13} \left[ M_{13}^{-1} - M_{31}^{-1} \right] + x_{23} \left[ M_{23}^{-1} - M_{32}^{-1} \right] + (M_{12}^{-1} - M_{21}^{-1}), \quad \text{for} \quad i \neq j \\
(m_{\nu})_{11} & = M_{22}^{-1} + x_{13}(M_{23}^{-1} + M_{32}^{-1}) + x_{13}^2 M_{33}^{-1}, \\
(m_{\nu})_{22} & = -M_{11}^{-1} - x_{23}(M_{13}^{-1} + M_{31}^{-1}) + x_{23}^2 M_{33}^{-1}, \\
(m_{\nu})_{33} & = x_{13}^2 M_{11}^{-1} + x_{13} x_{23}(M_{12}^{-1} + M_{21}^{-1}) + x_{23}^2 M_{22}^{-1},
\end{align*}
\] (30)

From Eq. (30), we have two solutions $x_{13}, x_{23}$ and one equation, which express the relation of components of matrix $m_{\nu}$:

\[
\begin{align*}
x_{13} & = \frac{(m_{\nu})_{23} \left[ (m_{\nu})_{12}^2 - (m_{\nu})_{11} (m_{\nu})_{33} \right] + (m_{\nu})_{13} \sqrt{\left[ (m_{\nu})_{12}^2 - (m_{\nu})_{11} (m_{\nu})_{33} \right] \left[ (m_{\nu})_{33}^2 - (m_{\nu})_{22} (m_{\nu})_{33} \right]}}{(m_{\nu})_{13} (m_{\nu})_{22} - (m_{\nu})_{11} (m_{\nu})_{23}}, \\
x_{23} & = \frac{(m_{\nu})_{13} \left[ (m_{\nu})_{23}^2 - (m_{\nu})_{22} (m_{\nu})_{33} \right] + (m_{\nu})_{23} \sqrt{\left[ (m_{\nu})_{23}^2 - (m_{\nu})_{22} (m_{\nu})_{33} \right] \left[ (m_{\nu})_{33}^2 - (m_{\nu})_{22} (m_{\nu})_{33} \right]}}{(m_{\nu})_{13} (m_{\nu})_{22} - (m_{\nu})_{11} (m_{\nu})_{23}}, \\
(m_{\nu})_{11} & = (m_{\nu})_{23}^2 + (m_{\nu})_{22} (m_{\nu})_{13}^2 + (m_{\nu})_{33} (m_{\nu})_{12}^2 = (m_{\nu})_{11} (m_{\nu})_{22} (m_{\nu})_{33} + 2 (m_{\nu})_{12} (m_{\nu})_{13} (m_{\nu})_{23}.
\end{align*}
\] (31)
Based on experimental data of neutrinos oscillation in Eq.(23), the matrix $m_D$ is parameterized and only depends on $\varrho$.

$$m_D \simeq \varrho \times \begin{pmatrix}
0 & 1 & 0.7248 \\
-1 & 0 & 1.8338 \\
-0.7248 & -1.8338 & 0
\end{pmatrix}. \tag{32}$$

It should be emphasized that $m_D$ has a form like Eq.(32) and differ from Ref.[32]. This is caused Eq.(30) can have many different solutions, but the use of Eq.(32) is suited very well to investigate LFVHDs.

IV. COUPLINGS AND ANALYTIC FORMULAS

In this section, we will calculate amplitudes and branching ratios of the LFVHDs in term of $M^\nu$ and physical neutrino masses. With this aim, all vertices are presented in term of physical masses and mixing parameters. From relation in Eq.(17), one can derive equation as follow:

$$M_{ab}^\nu = \left(U^{\nu s} M^\nu U^{\nu^t}\right)_{ab} = 0 \rightarrow U^{\nu s}_{ak} U^{\nu^s}_{bk} m_{nk} = 0, \tag{33}$$

Here $a,b = 1,2,3$ and $m_{nk}$ is mass of neutrino $n_k$, with $k$ run taken over $1,2,...,9$. It is interesting that, the relation in Eq.(33) lead to represent Yukawa couplings in term of $M^\nu$ and physical neutrino masses.

$$\sqrt{2} v h_{ab}^\nu = (m_D)_{ab} = (M^\nu)_{a(b+3)} = (U^{\nu s} \hat{M}^\nu U^{\nu^t})_{a(b+3)} = U^{\nu s}_{ak} U^{\nu^s}_{(b+3)k} m_{nk};$$

$$\frac{w}{\sqrt{2}} Y_{ab} = (M_R)_{ab} = (M^\nu)_{(a+3)(b+6)} = U^{\nu s}_{(a+3)k} U^{\nu^s}_{(b+6)k} m_{nk}. \tag{34}$$

We then pay attention the relevant couplings of LFVHDs. These couplings are derived by Lagrangian Yukawa, Lagrangian kinetics of lepton (or scalar) fields and Higgs potential. From the first term in Eq.(5), we can give couplings between leptons and Higgs boson as follow:

$$-h_{ab}^\nu \bar{L}_a \rho_{lR}^\nu_l + \text{h.c.} = -\frac{g m_a}{m_W} \left[ \bar{\nu}_{lL}^\nu L a_R \rho_1^+ + \bar{\nu}_{lL}^\nu a_L \rho_2^+ + N_{lL}^\nu a_R \rho_3^+ + \text{h.c.} \right]$$

$$\supset \frac{g m_a c_\beta}{2m_W} h_{lR}^0 a_L a_L + \frac{g m_a}{\sqrt{2} m_W} \left[ \left( U_{a L}^\nu P_{RL} H_1^+ + U_{a L}^\nu P_{LR} H_2^+ \right) - \frac{g m_a}{m_W} \left[ c_\alpha \left( U_{(a+3)}^\nu P_{RL} H_1^+ + U_{(a+3)}^\nu P_{LR} H_2^+ \right) \right] \right]. \tag{35}$$
Neutrinos interact with gauge bosons based on the kinetic terms of the leptons. When we can define symmetry coefficient \( \lambda \)

The relevant couplings in the second term of the Lagrangian in Eq. (5) are

\[
h_{ab}^\nu e^{ijk} [L_{aL}^i](L_{bL}^j) \rho_k^* + \text{h.c.}
\]

\[
= 2h_{ab}^\nu \left[ -\bar{L}_{aL}^i(\nu_{bL}^\nu)^c \rho_3^c - \bar{\nu}_{aL}^c(N_{bL}^\nu)^c \rho_2 + \bar{L}_{aL}^i(N_{bL}^\nu)^c \rho_1 \right]
\]

\[
= \frac{g \lambda}{2 m_W} h_1^0 \left[ \sum_{i=1}^3 U_{aL}^i U_{\nu}^{\nu^*} \overline{m}_i \left( m_{n_i} P_L + m_{n_j} P_R \right) n_j \right]
\]

\[
= \frac{g \lambda}{m_W} \left[ (m_D)_{ab} U_{\nu}^{\nu^*} H^2 \overline{t}_a P_R n_i + \text{h.c.} \right] + \frac{g}{\sqrt{2} m_W} \left[ (m_D)_{ab} U_{\nu}^{\nu^*} H^{-1} \overline{t}_a P_R n_i + \text{h.c.} \right].
\]

The couplings get contributions of \( M_R \) matrix given by:

\[
- Y_{ab} L_{aL}^i X F_{bR}^i + \text{h.c.}
\]

\[
= - \frac{g \lambda}{\sqrt{2} m_W} (M_R)_{ab} \left[ \bar{L}_{aL}^i \chi_1 \chi_0 - \bar{\nu}_{aL}^0 + \bar{N}_{aL}^0 \right] F_{bR}^i + \text{h.c.}
\]

\[
\equiv - \frac{g \lambda}{\sqrt{2} m_W} (M_R)_{ab} \left[ s_\beta U_{(a+3)i} U_{(b+6)j} \overline{n}_i P_R n_j h_1^0 + \sqrt{2} s_\alpha U_{(a+6)i} \overline{t}_a P_R n_i H_2^+ + \text{h.c.} \right],
\]

Neutrinos interact with gauge bosons based on the kinetic terms of the leptons. When we only concern with the couplings of the charged gauge bosons, the results are.

\[
L^{\ell\nu} = L_{aL}^i \gamma^\mu P_\mu L_{aL}^i \ni \text{h.c.}
\]

\[
= \frac{g}{\sqrt{2}} \left( \bar{L}_{aL}^i \gamma^\mu \nu_{aL}^\nu W^-_\mu + \bar{L}_{aL}^i \gamma^\mu N_{aL}^\nu Y^- \right) + \text{h.c.}
\]

\[
= \frac{g}{\sqrt{2}} \left[ U_{aL}^i \overline{t}_a \gamma^\mu P_L n_i W^-_\mu + U_{aL}^i \overline{n}_i \gamma^\mu P_L t_a W^+_\mu + U_{(a+3)i} \overline{t}_a \gamma^\mu P_L n_i Y^-_\mu + U_{(a+3)i} \overline{n}_i \gamma^\mu P_L t_a Y^+_\mu \right],
\]

To calculate \( h_{ij}^0 \overline{n}_i n_j \) coupling, we use results in Eqs. [36] [38] as given above. Furthermore, we can define symmetry coefficient \( \lambda_{ij}^0 = \lambda_{ji}^0 \) as Ref. [32], the result therefore obtained.

\[
\lambda_{ij}^0 = \sum_{k=1}^3 \left( U_{kL}^l U_{kj}^{\nu^*} m_{n_i} + U_{kL}^{\nu^*} U_{kj}^l m_{n_j} \right) - \sum_{k,q=1}^3 \sqrt{2} t_\alpha t_\beta (M_R)^{cd} \left[ U_{(k+3)i} U_{(q+6)j}^{\nu^*} + U_{(k+3)j} U_{(q+6)i}^{\nu^*} \right].
\]

This result is coincidence with the Feynman rules given in Ref. [39]. In such way, the \( h_{ij}^0 \overline{n}_k n_j \) coupling can be written in the symmetric form \( h_{ij}^0 \overline{n}_k n_j \sim h_{ij}^0 \overline{n}_k (\lambda_{kj}^0 P_L + \lambda_{jk}^0 P_R) n_j \). For brevity, we also define the coefficients related to the interaction of charged Higgs and fermions as follows:

\[
\lambda_{ak}^{L,1} = - \sum_{i=1}^3 (m_D)_{ai} U_{(i+3)k}^{\nu^*}, \quad \lambda_{ak}^{R,1} = m_a U_{ak}^{\nu^*},
\]

\[
\lambda_{ak}^{L,2} = \sum_{i=1}^3 \left[ (m_D)_{ai} U_{ik}^{\nu^*} + t_\alpha (M_R^*)_{ai} U_{(i+6)k}^{\nu^*} \right], \quad \lambda_{ak}^{R,2} = m_a U_{(a+3)k}^{\nu^*}.
\]
The couplings related to LFVHDs are given in Tab. (I). Especially, based on the characteristics of this model, some couplings of $h_1^0$ such as $h_1^0 H_1^+H_2^-, h_1^0 Y^+W^-, h_1^0 Y^+H_1^-, h_1^0 W^±H_{1,2}^±$ are zero.

| Vertex               | Coupling                                                                 |
|----------------------|--------------------------------------------------------------------------|
| $h_1^0 l_al_a$       | $\frac{igma}{2mWc_\beta}$                                              |
| $h_0^0 l_\mu n_\mu$ | $\frac{igma}{2mW}\left(\lambda_{k2}^0 P_L + \lambda_{k2}^0 P_R\right)$ |
| $h_0^0 H_2^+H_2^-$  | $i\lambda_{H_2}^\pm = -iw\left[2s_\beta s_a^2 \lambda_2 + s_\beta c_a^2 \lambda_3 - \sqrt{2}\left(2c_\beta c_a^2 \lambda_1 + c_\beta s_a^2 \lambda_3\right) t_\alpha - \frac{\sqrt{2}}{\omega} f c_\beta c_a s_\alpha\right]$ |
| $h_0^0 H_1^+H_1^-$  | $i\lambda_{H_1}^\pm = -iw\left(-2\sqrt{2}c_\beta \lambda_1 + \frac{s_\beta v_3 \lambda_3 + s_\beta f}{v}\right)$ |
| $H_2^a \bar{n}_b$, $H_2^a \bar{n}_b$ | $-i\frac{iga}{\sqrt{mW}}\left(\lambda_{kk}^{L,2} P_L + \lambda_{kk}^{R,2} P_R\right)$ |
| $H_1^a \bar{n}_b$, $H_1^a \bar{n}_b$ | $-i\frac{iga}{\sqrt{mW}}\left(\lambda_{kk}^{L,1} P_L + \lambda_{kk}^{R,1} P_R\right)$ |
| $W_\mu \bar{n}_b$, $W_\mu \bar{n}_b$ | $i\frac{iga}{\sqrt{2}} U_{kk}^\gamma P_L$, $i\frac{iga}{\sqrt{2}} U_{kk}^\gamma P_L$ |
| $Y_\mu \bar{n}_b$, $Y_\mu \bar{n}_b$ | $i\frac{iga}{\sqrt{2}} U_{(b+3)k}^\gamma P_L$, $i\frac{iga}{\sqrt{2}} U_{(a+3)k}^\gamma P_L$ |
| $Y_\mu \bar{n}_b h_1^0$, $Y_\mu \bar{n}_b h_1^0$ | $\frac{iga}{\sqrt{2}}\left(c_\alpha c_\beta + \sqrt{2}s_\alpha s_\beta\right)\left(p_{h_1} - p_{H_2}\right)^\mu$, $\frac{iga}{\sqrt{2}}\left(c_\alpha c_\beta + \sqrt{2}s_\alpha s_\beta\right)\left(p_{H_2} - p_{h_1}\right)^\mu$ |
| $h_0^0 W_\mu^+W_-^\nu$ | $-igmwc_\beta g^{\mu\nu}$ |
| $h_0^0 Y_\mu^+Y_-^\nu$ | $igm\frac{iga}{\sqrt{2}}\left(\sqrt{2}s_\beta c_\alpha - c_\beta s_\alpha\right) g^{\mu\nu}$ |

TABLE I: Couplings related to the SM-like Higgs decay ($h_1^0 \rightarrow l_al_b$) in the 331ISS model. All momenta in the Feynman rules corresponding to these vertices are incoming.

From Tab. (I), we can show all Feynman diagrams at one-loop order of the $l_a \rightarrow l_b\gamma$ decays in the unitary gauge as Fig. (I).

The regions of parameter space predicting large branching ratios ($Br$) for LFVHDs are affected strongly by the current experimental bound of $Br(l_a \rightarrow l_b\gamma)$, with $(a > b)$. Therefore, we will simultaneously investigate the LFV decay of charged leptons and LFVHDs. In the limit $m_a, b \rightarrow 0$, where $m_a, b$ are denoted for the masses of charged leptons $l_a, b$, respectively, we can derive the result, which is a very good approximate formula for branching ratio of cLFV as given in Ref. (2)

$$Br(l_a \rightarrow l_b\gamma) \simeq \frac{12\pi^2}{G_F^2} |D_R|^2 Br(l_a \rightarrow l_b\nu_a),$$

(40)
FIG. 1: Feynman diagrams at one-loop order of $l_a \rightarrow l_b \gamma$ decays in the unitary gauge. In diagram (2), $H^\pm_s$ ($s = 1, 2$) is charged Higgs bosons in this model where $D_R = D^W_R + D^Y_R + D^H_R$ and $G_F = \frac{g^2}{4\sqrt{2}m_W^2}$. The analytic forms are represented as:

\[
D^W_R = -\frac{eg^2}{32\pi^2 m_W^2} \sum_{k=1}^9 U^\nu_{ak} U^\nu_{bk} F(t_{kW}),
\]

\[
D^Y_R = -\frac{eg^2}{32\pi^2 m_Y^2} \sum_{k=1}^9 U^\nu_{(a+3)k} U^\nu_{(b+3)k} F(t_{KY}),
\]

\[
D^H_R = -\frac{eg^2 f_s}{16\pi^2 m_W^2} \sum_{k=1}^9 \left[ \frac{\lambda^{L,s}_{ak} \lambda^{L,s}_{bk}}{m^2_{H^+_s}} \times \frac{1 - 6t_{ks} + 3t_{ks}^2 + 2t_{ks}^3 - 6t_{ks}^2 \ln(t_{ks})}{12(t_{ks} - 1)^4} \\
+ \frac{m_{n_k} \lambda^{L,s}_{ak} \lambda'^{R,s}_{bk}}{m^2_{H^+_s}} \times \frac{-1 + t_{ks}^2 - 2t_{ks} \ln(t_{ks})}{2(t_{ks} - 1)^3} \right],
\]

(41)

Some quantities are defined as below. Especially, the $F(x)$ function is derived from the characteristics of PV functions in this model.

\[
t_{kW} \equiv \frac{m_{n_k}^2}{m_W^2}, \quad t_{kY} \equiv \frac{m_{n_k}^2}{m_Y^2}, \quad t_{ks} \equiv \frac{m_{n_k}^2}{m^2_{H^+_s}},
\]

\[
f_1 \equiv \frac{e^2}{a}, \quad f_2 = \frac{1}{2}, \quad \lambda'^{R,1}_{bk} \equiv U^\nu_{bk}, \quad \lambda'^{R,2}_{bk} \equiv U^\nu_{(b+3)k},
\]

\[
F(x) \equiv -\frac{10 - 43x + 78x^2 - 49x^3 + 4x^4 + 18x^3 \ln(x)}{12(x - 1)^4}.
\]

(42)

For different charge lepton decays, we use experimental data $\text{Br}(\mu \rightarrow e\nu_e\nu_\mu) = 100\%$, $\text{Br}(\tau \rightarrow e\nu_e\nu_\tau) = 17.82\%$, $\text{Br}(\tau \rightarrow \mu\nu_\mu\nu_\tau) = 17.39\%$ as given in Ref. 3.

To investigate the LFVHDs of the SM-like Higgs boson $h^0 \rightarrow l^+_a l^-_b$, we use scalar factors $\Delta_{(ab)L}$ and $\Delta_{(ab)R}$, which was first obtained in Ref. 32. Therefore, the effective Lagrangian of these decays is $\mathcal{L}_{\text{LFVH}}^{\text{eff}} = h^0 \left( \Delta_{(ab)L} \tilde{l}_a P_L l_b + \Delta_{(ab)R} \tilde{l}_a P_R l_b \right) + \text{h.c.}$ Based on the couplings in
Tab. we obtain the one-loop Feynman diagrams contributing to these LFVHDs amplitude in the unitary gauge are shown in Fig.2. The scalar factors $\Delta_{(ab) L,R}$ arise from the loop contributions. Here, we only pay attention to corrections at one-loop order. The partial width of $h_1^0 \rightarrow l_a^+ l_b^-$ is

$$\Gamma(h_1^0 \rightarrow l_a^+ l_b^-) = \frac{m_{h_1^0}}{8\pi} \left( |\Delta_{(ab) L}|^2 + |\Delta_{(ab) R}|^2 \right), \quad (43)$$

With conditions $p_{1,2}^2 = m_{a,b}^2$, $(p_1 + p_2)^2 = m_{h_1^0}^2$ and $m_{h_1^0}^2 \gg m_{a,b}^2$, we obtain branching ratio is $Br(h_1^0 \rightarrow l_a l_b) = \Gamma(h_1^0 \rightarrow l_a l_b)/\Gamma_{h_1^0}^{\text{total}}$, where $\Gamma_{h_1^0}^{\text{total}} \simeq 4.1 \times 10^{-3}\text{GeV}[3, 41]$. All Feynman diagrams at one-loop order in unitary gauge contributing to $h_1^0 \rightarrow l_a l_b$ decay are given follow.

FIG. 2: Feynman diagrams at one-loop order of $h_1^0 \rightarrow l_a l_b$ decays in unitary gauge.

Factors contribute to the partial width of $h_1^0 \rightarrow l_a^+ l_b^-$ are

$$\Delta_{(ab) L,R} = \sum_{k=1,4,5,8} \Delta_{(ab) L,R}^{W(k)} + \sum_{k=1,2,3,4,5,8} \Delta_{(ab) L,R}^{Y(k)} + \sum_{k=6,7,9,10} \Delta_{(ab) L,R}^{H_s(k)} \quad (44)$$

where the analytic forms of $\Delta_{(ab) L,R}^{W,Y,H_s}$ are calculated using the unitary gauge and shown in the App.[A]. The amplitudes of each diagram (denoted by $k$) in Fig.2 are represented analytically by PV (Passarino-Veltman) functions. In which, only $C_i$ functions, $i = 0, 1, 2$, are finite functions, the rest are diverging. However, the divergence cancellation of the total amplitude in Eq.(44) is proved analytically by techniques similar to Ref.[32, 42] and presented as App.[B]. Here, we show the term groups for which the divergence has been
eliminated.

\[
\Delta_{1,L,R} = \Delta_{(ab),L,R}^{(1)W} + \Delta_{(ab),L,R}^{(8)W} + \Delta_{(ab),L,R}^{(6)H_1} + \Delta_{(ab),L,R}^{(9+10)H_1},
\]

\[
\Delta_{2,L,R} = \Delta_{(ab),L,R}^{(1)Y} + \Delta_{(ab),L,R}^{(2)Y} + \Delta_{(ab),L,R}^{(3)Y} + \Delta_{(ab),L,R}^{(8)Y} + \Delta_{(ab),L,R}^{(6)H_2} + \Delta_{(ab),L,R}^{(9+10)H_2},
\]

\[
\Delta_{3,L,R} = \Delta_{(ab),L,R}^{(7)H_1} + \Delta_{(ab),L,R}^{(7)H_2} + \Delta_{(ab),L,R}^{(4+5)W} + \Delta_{(ab),L,R}^{(4+5)Y}.
\]

Based on the finite terms \(\Delta_{1,L,R}, \Delta_{2,L,R}, \Delta_{3,L,R}\) (\(\Delta_i\), \(i = 1,3\) - for short), we can investigate the change in total amplitude of \(h_1^0 \rightarrow l_al_b\) with the mass of the heavy neutrinos and other parameters of the model.

V. NUMERICAL RESULTS OF CLFV AND LFVHD

A. Setup parameters

We use the well-known experimental parameters \[1, 3\]: the charged lepton masses \(m_e = 5 \times 10^{-4}\) GeV, \(m_\mu = 0.105\) GeV, \(m_\tau = 1.776\) GeV, the SM-like Higgs mass \(m_{h_1^0} = 125.1\) GeV, the mass of the W boson \(m_W = 80.385\) GeV and the gauge coupling of the \(SU(2)_L\) symmetry \(g \simeq 0.651\).

To numerically investigate the \(l_a \rightarrow l_b\gamma\) and the LFVHDs, we choose the free parameters are: mass of charged gauge boson \(m_Y\), Higgs self-coupling constants \(\lambda_1, \lambda_3\), mass of charged Higgs \(m_{H_1^\pm}\). Therefore, the dependent parameters are given follows.

\[
v = \frac{\sqrt{2}m_W}{g}, \quad s_\alpha = \frac{m_W}{\sqrt{2}m_Y}, \quad \omega = \frac{2m_Y}{gc_\alpha},
\]

\[
f = \frac{gc_\alpha m_{H_1^\pm}^2}{4m_Y}, \quad m_{H_1^\pm}^2 = \frac{m_{H_1^\pm}^2}{2} \left( t_\alpha^2 + 1 \right),
\]

\[
\lambda_2 = \frac{t_\alpha^2}{2} \left( \frac{m_{h_1^0}^2}{v} - \frac{m_{H_1^\pm}^2}{2\omega^2} \right) + \frac{\left( \lambda_3 - \frac{m_{H_1^\pm}^2}{2\omega^2} \right)^2}{4\lambda_1 - \frac{m_{h_1^0}^2}{2\omega^2}}. \quad (46)
\]

The heavy charged gauge boson mass \(m_Y\) is related to the lower constraint of neutral gauge boson \(Z'\) in 3-3-1 models, which have also been mentioned in Refs.\[43, 44\]. To satisfy those constraints, we choose the default value \(m_Y = 4.5\) TeV. The values of the Higgs self-couplings must satisfy theoretical conditions of unitarity and the Higgs potential must be bounded from below, which also guarantee that all couplings of the SM-like Higgs boson approach the SM limit when \(v \ll \omega\). For the above reasons, the Higgs self-couplings are fixed as
\( \lambda_1 = 1, \lambda_3 = -1 \). Based on recent data of neutral meson mixing \( B_0 - \overline{B}_0 \) \cite{17}, we can choose the lower bound of \( m_{H_1^\pm} \geq 500\text{GeV} \). This is also consistent with Ref.\cite{32}. Characteristic for the scale of the matrix \( m_D \) is the parameter \( \varrho \) as shown in Eq.(32), considered in the range of the perturbative limit, \( \varrho = \sqrt{2}v h_{23}^\nu \leq 617\text{GeV} \). In the calculations below, we fix the values for \( \varrho \) to be: 100, 200, 400, 500 and 600 GeV. To represent masses of heavy neutrinos \( (F_a) \), we parameterize the matrix \( M_R \) in the form of a diagonal. In particular, the hierarchy of a diagonal matrix \( M_R \) can yield large results for the LFVHDs.

### B. Numerical results of cLFV

In this section, we numerically investigate of \( l_a \to l_b \gamma \) decays with \( a > b \) use a diagonal and non-hierarchical \( M_R \) matrix. That means \( M_R \sim \text{diag}(1,1,1) \). We overhaul regions mentioned in Ref.\cite{32}, where \( \varrho \) was chosen to be from a hundred of GeV to 600GeV, and \( M_R \) was in form \( M_R = k \varrho \text{diag}(1,1,1) \), with \( k \) is small. As a result, it is shown the existence of the narrow regions of parameter space where can satisfy the experimental bound on \( Br(\mu \to e\gamma) < 4.2 \times 10^{-13} \) and change fastly with the change of \( m_{H_1^\pm} \).

To indicate the origin, we numerically investigate the contributions to \( Br(\mu \to e\gamma) \) in Eq.(41). We choose \( M_R = 9 \varrho \text{diag}(1,1,1) \), \( \varrho \) is fixed 200, 400, 600GeV and \( m_{H_1^\pm} \) is in range \((0.1, 5)\text{TeV}\). The contributions of gauge and Higgs boson defend on \( m_{H_1^\pm} \) as shown in Fig.3. In the parameter space under consideration, \( D_{R}^{W\pm} \) and \( D_{R}^{H_1^\pm} \) have the same order of size \( 10^{-9} \) (left), while \( D_{R}^{Y\pm} \) and \( D_{R}^{H_2^\pm} \) are of size \( 10^{-36} \) (right). Therefore, the contributions of \( D_{R}^{W\pm} \) and \( D_{R}^{H_1^\pm} \) to \( Br(\mu \to e\gamma) \) are dominant parts. An interesting result is that, while \( D_{R}^{W\pm} \) is always positive and almost unchanged for fixed values of \( \varrho \), \( D_{R}^{H_1^\pm} \) is negative and decreases as \( \varrho \) increases. It is the reason that the contributions of gauge and Higgs boson are destructive, creating the narrow regions of parameter space where satisfy \( Br(\mu \to e\gamma) < 4.2 \times 10^{-13} \).

In a similar way, we can investigate the contributions of gauge and Higgs boson to \( Br(\tau \to e\gamma) \) and \( Br(\tau \to \mu\gamma) \) according to change of \( m_{H_1^\pm} \) and find out the regions of parameter space which comply with current experimental limits \( Br(\tau \to e\gamma) < 3.3 \times 10^{-8} \) and \( Br(\tau \to \mu\gamma) < 4.4 \times 10^{-8} \) \cite{3}. However, the parameter space is only really meaningful when all the experimental limits are satisfied. For the above reasons, we will examine \( Br(\tau \to e\gamma) \) and \( Br(\tau \to \mu\gamma) \) in narrow space regions allowed to satisfy the experimental limit of \( Br(\mu \to e\gamma) \).

We choose \( k = 9 \) and fix \( \varrho = 200, 400, 600\text{GeV} \), the range of \( m_{H_1^\pm} \) is from \( 500\text{GeV} \) to \( 10\text{TeV} \),
FIG. 3: Contributions of $D_R^{W\pm}, D_R^{H^\pm_2}$ (left) and $D_R^{Y\pm}, D_R^{H^\pm_1}$ (right) to $Br(\mu \to e\gamma)$ as function of $m_{H_1^\pm}$ with fixed $\rho = 200, 400, 600$ GeV.

$Br(l_a \to l_b\gamma)$ depend on $m_{H_1^\pm}$ are shown in Fig. 4.

FIG. 4: Plots of $Br(l_a \to l_b\gamma)$ depend on $m_{H_1^\pm}$ (first row) and contour plots of $Br(l_a \to l_b\gamma)$ as function of $\rho$ and $m_{H_1^\pm}$ (second row).

We illustrate how $Br(l_a \to l_b\gamma)$ change with $m_{H_1^\pm}$, in the case $k = 9$ and the fixed values of $\rho = 200, 400, 600$ GeV, corresponding to the plots in the first row of Fig. 4. Here, we
obtain narrow parameter spaces that satisfy the experimental limits of the $Br(\mu \to e\gamma)$, respectively, shown in the second row. The expected space (colorless) is between the two curves 4.2, the remain rules out of the experimental limit (green). It should be emphasized that, for other fixed values of $\varrho$ within the limits of the perturbation theory, we can also investigate in the same way.

In each allowed narrow space, where the $Br(\mu \to e\gamma)$ is within the experimental limits, $Br(\tau \to e\gamma)$ and $Br(\tau \to \mu\gamma)$ also satisfy the upper bound limits of the experiment. In particular, thee values of $Br(\tau \to e\gamma)$ can reach as high as $10^{-9}$ and $Br(\tau \to \mu\gamma)$ is about $10^{-10}$, close to the accuracy found in today’s large accelerators. These results are shown in Tab.II.

| $\varrho$[GeV] | Values of $m_H^2$[TeV] satisfy $Br(\mu \to e\gamma) < 4.2 \times 10^{-13}$ | Values of $Br(\tau \to e\gamma) \times 10^9$ | Values of $Br(\tau \to \mu\gamma) \times 10^{10}$ |
|----------------|-------------------------------------------------|---------------------------------|-----------------
| 200            | 1.112 → 1.114                                   | 2.269 → 2.304                   | 8.569 → 8.872  |
| 400            | 2.256 → 2.261                                   | 2.158 → 2.198                   | 7.813 → 8.161  |
| 600            | 3.396 → 3.405                                   | 2.124 → 2.172                   | 7.579 → 7.986  |

TABLE II: The ranges of $Br(\tau \to e\gamma)$ and $Br(\tau \to \mu\gamma)$ in narrow space regions where the experimental limits of $Br(\mu \to e\gamma)$ are satisfied.

The $l_a \to l_b\gamma$ processes have also been studied previously in the context of the 331 models such as Ref.[45]. According to the result, the parameter space areas satisfy the experimental limit (comply with Refs.[46, 47]) of the $l_a \to l_b\gamma$ are given. However, it has two restrictions: i) the limit of $\mu \to e\gamma$ is not tight ($2.4 \times 10^{-12}$), ii) has not shown the region of the parameter space suitable for all $l_a \to l_b\gamma$ decay. These restrictions have been overcome as shown in Tab.II. This is very interesting result given in the framework of this model and a suggestion for the verification of physical effects in the model from current experimental data.

C. Numerical results of LFVHD

We consider the narrow spatial regions where the $Br(l_a \to l_b\gamma)$ approach the experimental upper limit, this may be predicting large LFVHD. Therefore, we will investigate
the contributions of $\Delta_{i,R,L}, i = 1, 3$ in Eq. (45) to $Br(h_1^0 \to l_a l_b)$ and then, we continue to examine $Br(h_1^0 \to l_a l_b)$ in the narrow spaces mentioned above. This is done both in the case of hierarchical and non-hierarchical $M_R$.

Without loss of generality when studying $Br(h_1^0 \to l_a l_b)$, we will choose $Br(h_1^0 \to \mu \tau)$. $M_R$ matrix is chosen non-hierarchically in the form $M_R = 9\varrho \times \text{diag}(1, 1, 1)$ and hierarchically in the form $M_R = 9\varrho \times \text{diag}(1, 2, 3)$ and $M_R = 9\varrho \times \text{diag}(3, 2, 1)$.

In case $M_R = 9\varrho \times \text{diag}(1, 1, 1)$, the contributions of $\Delta_{i,R,L}, i = 1, 3$ to $Br(h_1^0 \to \mu \tau)$ are given in Fig. 5. With the parameter domain of this model selected in $\text{VA}$, for each fixed value of $m_{H_1^\pm}$, $\Delta_{i,R,L}, i = 1, 3$ increase with $\varrho$ and contribution of $\Delta_3$ was very small compared to ones of $\Delta_{1,2}$. Thus, we can ignore the contribution of $\Delta_3$ to $Br(h_1^0 \to \mu \tau)$.

![Figure 5](image.png)

**FIG. 5:** Plots of $\Delta_{i,R,L}, i = 1, 3$ as function of $\varrho$ (left) and contour plots of $\Delta_{1,2,R,L}$ as functions of $m_{H_1^\pm}$ and $\varrho$ (right). In right panel, the black, blue, dashed-black and dashed-blue curves present constant values of $\Delta_{1,R}, \Delta_{2,R}, \Delta_{1,L}, \Delta_{2,L}$, respectively. All plots are investigated in case $M_R = 9\varrho \times \text{diag}(1, 1, 1)$

In right panel of Fig. 5, the green, pink, red present the value ranges of $1 < \Delta_{1,R} \times 10^4 < 3, 3 < \Delta_{1,R} \times 10^4 < 4, \Delta_{1,R} \times 10^4 > 4$, respectively. The yellow, magenta illustrate areas of $\Delta_{2,R} > 1 \times 10^{-4}$. We can find that the magenta region may gives the largest $Br(h_1^0 \to \mu \tau)$, with $\Delta_{(\mu \tau)R} \sim 6 \times 10^{-4}$ and $\Delta_{(\mu \tau)L} \sim 0.4 \times 10^{-4}$.

All contributions to $Br(h_1^0 \to \mu \tau)$ in case $M_R = 9\varrho \text{diag}(1, 1, 1)$ are presented in Fig. 6. Here, we chose the fixed values of $\varrho = 100, 200, 400, 500, 600\text{GeV}$ and $m_{H_1^\pm}$ in the range $500\text{GeV}$ to $10\text{TeV}$. As the result, $Br(h_1^0 \to \mu \tau)$ increases with value of $\varrho$ and changes very
slowly with the change of large $m_{H^\pm}$. The maximum value that $Br(h_1^0 \rightarrow \mu\tau)$ can reach is about $0.71 \times 10^{-3}$. This result is very close upper limit of current experimental data Ref.[3].

\[ m_H \pm 1. \]

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\[ \varrho = 100 \text{ GeV} \]
\[ \varrho = 200 \text{ GeV} \]
\[ \varrho = 400 \text{ GeV} \]
\[ \varrho = 500 \text{ GeV} \]
\[ \varrho = 600 \text{ GeV} \]

\[ 10^{-3} \]
\[ 5 \times 10^{-5} \]
\[ 0.001 \]
\[ 0.100 \]

\[ m_{H^\pm} \pm 2 \]

\[ MR = 9 \varrho \times \text{diag}(1,1,1) \]

\[ \varrho = 100 \text{ GeV} \]
\[ \varrho = 200 \text{ GeV} \]
\[ \varrho = 400 \text{ GeV} \]
\[ \varrho = 500 \text{ GeV} \]
\[ \varrho = 600 \text{ GeV} \]

\[ 10^{-3} \]
\[ 5 \times 10^{-5} \]
\[ 0.001 \]
\[ 0.100 \]

\[ m_{H^\pm} \pm 1 \]

\[ MR = 9 \varrho \times \text{diag}(1,1,1) \]

\[ MR = 9 \varrho \times \text{diag}(1,1,1) \]

\[ MR = 9 \varrho \times \text{diag}(3,2,1) \]

\[ MR = 9 \varrho \times \text{diag}(1,2,3) \]

\[ \Delta \mathbf{R} > 1 \times 10^{-4} \]

\[ \Delta (\mu\tau)_R \sim 3.5 \times 10^{-4} \]

\[ \Delta (\mu\tau)_L \sim 0.23 \times 10^{-4} \]

\[ 0.164 \times 10^{-3} \]

\[ 0.164 \times 10^{-3} \]

\[ 0.164 \times 10^{-3} \]

FIG. 6: Plots of $Br(h_1^0 \rightarrow \mu\tau)$ as function of $m_{H^\pm}$ with fixed value $\varrho = 100, 200, 400, 500, 600$ GeV and $MR = 9 \varrho \text{diag}(1,1,1)$ (left) and density plots of $Br(h_1^0 \rightarrow \mu\tau)$ as function of $m_{H^\pm}$ and $\varrho$ with $MR = 9 \varrho \times \text{diag}(1,1,1)$ (right).

We use free parameters derived from $M_R$ and $m_D$ matrices to give the results above. It is necessary to emphasize the difference with Ref.[32] in parameterizing the matrix $m_D$, we parameterize the matrix $m_D$ in the same form as Eq.(32). Using that consequence and including all contributions, especially heavy neutrinos, we have shown that $Br(h_1^0 \rightarrow \mu\tau)$ is close to $10^{-3}$.

A very interesting result in this model is $Br(h_1^0 \rightarrow \mu\tau)$ can strongly changes when choosing matrix $M_R$ with hierarchical form. To prove this statement, we investigate $Br(h_1^0 \rightarrow \mu\tau)$ in cases $MR = 9 \varrho \times \text{diag}(3,2,1)$ and $MR = 9 \varrho \times \text{diag}(1,2,3)$.

Ignoring contributions of $\Delta_3$ to $Br(h_1^0 \rightarrow \mu\tau)$, the main contribution in case $MR = 9 \varrho \times \text{diag}(3,2,1)$ is shown on the right panel of Fig.7. The yellow, magenta present areas of $\Delta_{2,R} > 1 \times 10^{-4}$. It is easy to point out that the magenta region may gives the largest $Br(h_1^0 \rightarrow \mu\tau)$, with $\Delta(\mu\tau)_R \sim 3.5 \times 10^{-4}$ and $\Delta(\mu\tau)_L \sim 0.23 \times 10^{-4}$. These results lead to contributions to $Br(h_1^0 \rightarrow \mu\tau)$ as shown in left panel of Fig.8.

In the parameter space satisfying the experimental limits of $l_a \rightarrow l_b\gamma$, $Br(h_1^0 \rightarrow \mu\tau)$ could reach $0.164 \times 10^{-3}$ (right panel in Fig.8), smaller than the corresponding value in the case $MR = 9 \varrho \times \text{diag}(1,1,1)$.
FIG. 7: Plots of $\Delta_{i,R,L}, i = 1, 3$ as function of $\varrho$ (left) and contour plots of $\Delta_{1,2,R,L}$ as functions of $m_{H^\pm_1}$ and $\varrho$ (right). In right panel, the black, blue, dashed-black and dashed-blue curves present constant values of $\Delta_{1,R}, \Delta_{2,R}, \Delta_{1,L}, \Delta_{2,L}$, respectively. All plots are investigated in case $M_R = 9\varrho \times diag(3,2,1)$.

FIG. 8: Plots of $Br(h_1^0 \to \mu\tau)$ as function of $m_{H^\pm_1}$ with fixed value $\varrho = 100, 200, 400, 500, 600$ GeV and $M_R = 9\varrho diag(3,2,1)$ (left) and density plots of $Br(h_1^0 \to \mu\tau)$ as function of $m_{H^\pm_1}$ and $\varrho$ with $M_R = 9\varrho \times diag(3,2,1)$ (right).

Similarly, we can obtain the pink area in right panel of Fig. 9 that is likely to give the largest value of $Br(h_1^0 \to \mu\tau)$ when $\Delta_{(\mu\tau)R} \sim 3.2 \times 10^{-4}$ and $\Delta_{(\mu\tau)L} \sim 0.22 \times 10^{-4}$ in case $M_R = 9\varrho \times diag(1,2,3)$.

The change rule of $\Delta_{i,R,L}, i = 1, 3$ in Fig. 9 produces the survey results of $Br(h_1^0 \to \mu\tau)$.
FIG. 9: Plots of $\Delta_{i,R,L}, i = 1,3$ as function of $\varrho$ (left) and contour plots of $\Delta_{1,2,R,L}$ as functions of $m_{H_1^\pm}$ and $\varrho$ (right). In right panel, the black, blue, dashed-black and dashed-blue curves present constant values of $\Delta_{1,R}, \Delta_{2,R}, \Delta_{1,L}, \Delta_{2,L}$, respectively. All plots are investigated in case $M_R = 9 \varrho \times \text{diag}(1,2,3)$ as shown in Fig. 10. The largest value of $Br(h_1^0 \rightarrow \mu\tau)$ as presented in the right panel of Fig. 10 is about $0.160 \times 10^{-3}$. This value is approximately to corresponding ones in case $M_R = 9 \varrho \times \text{diag}(3,2,1)$, but also smaller when $M_R = 9 \varrho \times \text{diag}(1,1,1)$.

FIG. 10: Plots of $Br(h_1^0 \rightarrow \mu\tau)$ as function of $m_{H_1^\pm}$ with fixed value $\varrho = 100, 200, 400, 500, 600$ GeV and $M_R = 9 \varrho \text{diag}(1,2,3)$ (left) and density plots of $Br(h_1^0 \rightarrow \mu\tau)$ as function of $m_{H_1^\pm}$ and $\varrho$ with $M_R = 9 \varrho \times \text{diag}(1,2,3)$ (right)
In fact, when $M_R$ is chosen in different diagonal form, the heavy neutrinos ($N_a, F_a$) have different masses. This is caused that the contributions of charged Higgs and gauge boson will be destructive interference at the different $m_{H^±}$. These are consequences that the narrow regions of parameter space where satisfy the experimental limits of $Br(l_a \rightarrow l_b \gamma)$ have different ranges of $m_{H^±}$ as shown in right panels of Fig. 6, Fig. 8, Fig. 10.

However, the values of $Br(h_1^0 \rightarrow \mu \tau)$ are only really meaningful when considered in narrow spaces that satisfy the experimental limits of $Br(l_a \rightarrow l_b \gamma)$. These allowed spaces are confined to the two curves 4.2 (black) on the right part of Fig. 6, Fig. 8 and Fig. 10. In these regions, $Br(h_1^0 \rightarrow \mu \tau)$ can reach $0.71 \times 10^{-3}$ in case $M_R = 9 \varrho \times diag(1, 1, 1)$. This value is close to the upper bound of the experimental limit and can be detected by large accelerators to confirm the validity of this model.

VI. CONCLUSION

In the 331ISS model, when the Marajona neutrinos ($F_a$), which are $SU(3)_L$ singlets, were added, the neutrinos were mixed and massed according to an inverse seesaw mechanism. Therefore, lepton flavor violating couplings are generated. The gauge bosons and the charged Higgs in this model make a major contribution to the $l_a \rightarrow l_b \gamma$ decay. Investigating the participation of heavy neutrinos to these major contributions, we show that these components sometimes mutual destructive. Due to the interference of major contributions, narrow regions of the parameter space satisfying the experimental limits of the $Br(\mu \rightarrow e \gamma)$ are created and in those regions, $k$ is small and $\varrho$ is large ($k$ is the ratio factor when parameterizing the matrix $M_R$ and the matrix $m_D$). In particular, in these allowed narrow spaces, $Br(\tau \rightarrow e \gamma)$ can reach about $10^{-9}$ and $Br(\tau \rightarrow \mu \gamma)$ may achieves $10^{-10}$, these results are very close to the upper bound of the experimental limits.

Performing numerical investigation, we point out that $\Delta_{1, 2}$ are main contributions to $Br(h_1^0 \rightarrow \mu \tau)$ while $\Delta_3$ is ignored because it is very small compared to $\Delta_{1, 2}$. All of these contributions are less than $10^{-3}$ in the selected parameter space of this model.

We also found that the contributions of heavy neutrinos through $\Delta_i$, $i = 1, 3$ lead to the change of $Br(h_1^0 \rightarrow \mu \tau)$. This is presented through the hierarchy of the mixing matrix of heavy neutrinos ($M_R$). In case $M_R \sim diag(1, 1, 1)$, $Br(h_1^0 \rightarrow \mu \tau)$ has a greater value than the cases $M_R \sim diag(3, 2, 1)$ and $M_R \sim diag(1, 2, 3)$. The largest value that $Br(h_1^0 \rightarrow \mu \tau)$
can reach is about $O(10^{-3})$ in the context of this model.

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Appendix A: Form factors of LFVHDs in the unitary gauge

In this appendix we use Passarino-Veltman (PV) functions \cite{28, 48} for representing all analytic formulas of one-loop contributions to LFVHDs defined in Eq. (43). We also use notations for one-loop integral of PV functions, such as $D_0 = (k^2 - M_0^2 + i\delta)$, $D_1 = (k - p_1)^2 - M_1^2 + i\delta$, $D_2 = (k + p_2)^2 - M_2^2 + i\delta$ where $\delta$ is an infinitesimal positive real quantity.

\[
B_{0,\mu}^{(i)} = \frac{(2\pi\mu)^{4-D}}{i\pi^2} \int \frac{d^4k \{1, k\}}{D_0 D_i}, \quad B_{0}^{(12)} = \frac{(2\pi\mu)^{4-D}}{i\pi^2} \int \frac{d^4k}{D_1 D_2},
\]

\[
C_{0,\mu} = C_{0,\mu}(M_0, M_1, M_2) = \frac{1}{i\pi^2} \int \frac{d^4k \{1, k\}}{D_0 D_1 D_2},
\]

\[
B_{\mu}^{(i)} = B_{1,\mu}^{(i)}, \quad C_{\mu} = C_{1,\mu} + C_{2,\mu}.
\]

The analytic expressions for $\Delta_{L,R}^{(k)W} \equiv \Delta_{(ab)_{L,R}}^{(k)W}$, $\Delta_{L,R}^{(k)H_+} \equiv \Delta_{(ab)_{L,R}}^{(k)H_+}$ and $\Delta_{L,R}^{(k)Y} \equiv \Delta_{(ab)_{L,R}}^{(k)Y}$, $\Delta_{L,R}^{(k)YH_{\pm}^0} \equiv \Delta_{(ab)_{L,R}}^{(k)YH_{\pm}^0}$, where $k$ implies the diagram (k) in Fig. 2 are divided into the following sections.

Donations with the participation of $W^\pm$ -boson

\[
\Delta_{L}^{(1)W} = \frac{g^3 c_\beta m_a}{64\pi^2 m_W^2} \sum_{i=1}^{9} U_{\alpha i}^{\nu} U_{\mu i}^{\nu} \left\{ m_{n_i}^2 \left[ B_{1}^{(1)} - B_{0}^{(1)} - B_{0}^{(2)} \right] - m_a^2 B_{1}^{(2)} + \left( 2m_W^2 + m_{h_1^0}^2 \right) m_{n_i}^2 C_0 \right\},
\]

\[
\Delta_{R}^{(1)W} = \frac{g^3 c_\beta m_b}{64\pi^2 m_W^2} \sum_{i=1}^{9} U_{\alpha i}^{\nu} U_{\mu i}^{\nu} \left\{ -m_{n_i}^2 \left[ B_{1}^{(2)} + B_{0}^{(1)} + B_{0}^{(2)} \right] + m_a^2 B_{1}^{(1)} + \left( 2m_W^2 + m_{h_1^0}^2 \right) m_{n_i}^2 C_0 \right\},
\]

\[
\Delta_{L}^{(4+5)W} = \frac{g^3 m_a m_b c_\beta}{64\pi^2 m_{h_1^0}^2 (m_a^2 - m_b^2)} \sum_{i=1}^{9} U_{\alpha i}^{\nu} U_{\mu i}^{\nu} \left\{ 2m_{n_i}^2 \left[ B_{0}^{(1)} - B_{0}^{(2)} \right] - \left( 2m_W^2 + m_{n_i}^2 \right) \left[ B_{1}^{(1)} + B_{1}^{(2)} \right] - m_a^2 B_{1}^{(1)} - m_b^2 B_{2}^{(1)} \right\},
\]
\[ \Delta_{(4+5)W} = \frac{m_a}{m_b} \Delta_{(4+5)W}^{(a,b)} , \]
\[ \Delta_{L}^{(8)W} = \frac{g^3 c_\beta m_a}{64 \pi^2 m_W^3} \sum_{i,j=1}^9 U_{\alpha i}^{\nu *} U_{\beta j}^{\nu} \left\{ \lambda_{ij}^{0*} m_{n_i} \left[ B_0^{1(12)} - m_W^2 C_0 + \left( 2m_W^2 + m_{n_j}^2 - m_a^2 \right) C_1 \right] \right. \]
\[ + \left. \lambda_{ij}^{0} m_{n_i} \left[ B_1^{1(1)} + \left( 2m_W^2 + m_{n_j}^2 - m_b^2 \right) C_1 \right] \right\} , \]
\[ \Delta_{R}^{(8)W} = \frac{g^3 c_\beta m_b}{64 \pi^2 m_W^3} \sum_{i=1}^9 U_{\alpha i}^{\nu *} U_{\beta 0}^{\nu} \left\{ \lambda_{ij}^{0*} m_{n_i} \left[ B_0^{1(12)} - m_W^2 C_0 - \left( 2m_W^2 + m_{n_j}^2 - m_b^2 \right) C_2 \right] \right. \]
\[ - \left. \lambda_{ij}^{0*} m_{n_i} \left[ B_1^{(2)} + \left( 2m_W^2 + m_{n_j}^2 - m_a^2 \right) C_2 \right] \right\} . \quad (A1) \]

Donations with the participation of \( Y^\pm \)-boson

\[ \Delta_{L}^{(1)Y} = -\frac{g^3 m_a (\sqrt{2}s_\beta c_\alpha - c_\beta s_\alpha)}{64 \sqrt{2} \pi^2 m_Y^3} \sum_{i=1}^9 U_{\alpha (a+3)i}^{\nu *} U_{(b+3)i}^{\nu} \left\{ m_{n_i} \left[ B_1^{(1)} - B_0^{(1)} - B_0^{(2)} \right] - m_b^2 B_1^{(2)} \right\} , \]
\[ + \left( 2m_Y^2 + m_{h_0}^2 \right) m_{n_i} C_0 - \left[ 2m_Y^2 \left( 2m_Y^2 + m_{n_i}^2 + m_a^2 - m_b^2 \right) + m_{n_i}^2 m_{h_0}^2 \right] C_1 \]
\[ + \left[ 2m_Y^2 \left( m_a^2 - m_{h_0}^2 \right) + m_b^2 m_{h_0}^2 \right] C_2 \right\} , \]
\[ \Delta_{R}^{(1)Y} = -\frac{g^3 m_b (\sqrt{2}s_\beta c_\alpha - c_\beta s_\alpha)}{64 \sqrt{2} \pi^2 m_Y^3} \sum_{i=1}^9 U_{(a+3)i}^{\nu *} U_{(b+3)i}^{\nu} \left\{ -m_{n_i} \left[ B_1^{(2)} + B_0^{(1)} + B_0^{(2)} \right] + m_a^2 B_1^{(1)} \right\} , \]
\[ + \left( 2m_Y^2 + m_{h_0}^2 \right) m_{n_i} C_0 - \left[ 2m_Y^2 \left( m_b^2 - m_{h_0}^2 \right) + m_a^2 m_{h_0}^2 \right] C_1 \]
\[ + \left[ 2m_Y^2 \left( 2m_Y^2 + m_{n_i}^2 - m_a^2 + m_b^2 \right) + m_{n_i}^2 m_{h_0}^2 \right] C_2 \right\} , \]
\[ \Delta_{L}^{(2)Y} = \frac{g^3 c_a (c_\beta c_\alpha + \sqrt{2}s_\beta s_\alpha)}{64 \pi^2 m_W m_Y^2} \sum_{i=1}^9 U_{(a+3)i}^{\nu} \left\{ \lambda_{bi}^{L,2} m_{n_i} \left[ B_0^{(1)} - B_1^{(1)} + \left( m_Y^2 + m_{H_1}^2 - m_{h_0}^2 \right) C_0 + \left( m_Y^2 - m_{H_1}^2 + m_{h_0}^2 \right) C_1 \right] \right. \]
\[ + \left. \lambda_{bi}^{R,2} m_b \left[ 2m_Y^2 C_1 - \left( m_Y^2 + m_{H_1}^2 - m_{h_0}^2 \right) C_2 \right] \right\} , \]
\[ \Delta_{R}^{(2)Y} = \frac{g^3 c_a (c_\beta c_\alpha + \sqrt{2}s_\beta s_\alpha)}{64 \pi^2 m_W m_Y^2} \sum_{i=1}^9 U_{(a+3)i}^{\nu} \left\{ \lambda_{ai}^{L,2} m_{n_i} \left[ -2m_Y^2 C_0 - \left( m_Y^2 - m_{H_1}^2 + m_{h_0}^2 \right) C_2 \right] \right. \]
\[ + \left. \lambda_{ai}^{R,2} \left[ -m_{n_i}^2 B_0^{(1)} + m_a^2 B_1^{(1)} + m_{n_i} \left( m_Y^2 - m_{H_1}^2 + m_{h_0}^2 \right) C_0 \right] \right. \]
\[ + \left. \left[ 2m_Y^2 \left( m_{h_0}^2 - m_b^2 \right) - m_{n_i}^2 \left( m_Y^2 - m_{H_1}^2 + m_{h_0}^2 \right) \right] C_1 + 2m_b^2 m_Y^2 C_2 \right\} , \]
\[ \Delta_{L}^{(3)Y} = \frac{g^3 c_a (c_\beta c_\alpha + \sqrt{2}s_\beta s_\alpha)}{64 \pi^2 m_W m_Y^2} \sum_{i=1}^9 U_{(b+3)i}^{\nu} \left\{ \lambda_{ai}^{L,2} m_{n_i} \left[ -2m_Y^2 C_0 + \left( m_Y^2 - m_{H_1}^2 + m_{h_0}^2 \right) C_1 \right] \right. \]
\[ + \left. \lambda_{ai}^{R,2} \left[ -m_{n_i}^2 B_0^{(1)} + m_a^2 B_1^{(1)} + m_{n_i} \left( m_Y^2 - m_{H_1}^2 + m_{h_0}^2 \right) C_0 \right] \right. \]
\[ + \left. \left[ 2m_Y^2 \left( m_{h_0}^2 - m_b^2 \right) - m_{n_i}^2 \left( m_Y^2 - m_{H_1}^2 + m_{h_0}^2 \right) \right] C_1 + 2m_b^2 m_Y^2 C_2 \right\} , \]
\[ \Delta_{R}^{(3)Y} = \frac{g^3 c_a (c_\beta c_\alpha + \sqrt{2}s_\beta s_\alpha)}{64 \pi^2 m_W m_Y^2} \sum_{i=1}^9 U_{(b+3)i}^{\nu} \left\{ \lambda_{ai}^{L,2} m_{n_i} \left[ -2m_Y^2 C_0 + \left( m_Y^2 - m_{H_1}^2 + m_{h_0}^2 \right) C_1 \right] \right. \]
\[ + \left. \lambda_{ai}^{R,2} \left[ -m_{n_i}^2 B_0^{(1)} + m_a^2 B_1^{(1)} + m_{n_i} \left( m_Y^2 - m_{H_1}^2 + m_{h_0}^2 \right) C_0 \right] \right. \]
\[ + \left. \left[ 2m_Y^2 \left( m_{h_0}^2 - m_b^2 \right) - m_{n_i}^2 \left( m_Y^2 - m_{H_1}^2 + m_{h_0}^2 \right) \right] C_1 + 2m_b^2 m_Y^2 C_2 \right\} . \]
\[\Delta^{(3)Y}_R = \frac{g^3 m_a c_\alpha (c_\beta c_\alpha + \sqrt{2}s_\beta s_\alpha)}{64\pi^2 m_W m_Y^2} \sum_{i=1}^{9} U_{(b+3)i}^\nu \left\{ \lambda^{L,2*}_{R,i} m_n \left[ B_0^{(2)} + B_1^{(2)} \right] + \left( m_Y^2 + m_{H_1^\pm}^2 - m_h^2 \right) C_0 - \left( m_Y^2 - m_{H_1^\pm}^2 + m_h^2 \right) C_2 \right\}, \]

\[\Delta^{(4+5)Y}_L = \frac{m_a}{m_b} \Delta^{(4+5)Y}_L, \]

\[\Delta^{(8)Y}_L = \frac{g^3 c_\beta m_a}{64\pi^2 m_W m_Y^2} \sum_{i,j=1}^{9} U_{(a+3)i}^\nu U_{(b+3)j}^\nu \left\{ \lambda^{0*}_{ij} m_m \left[ B_0^{(12)} - m_Y^2 C_0 + \left( 2m_Y^2 + m_{n_j}^2 - m_b^2 \right) C_1 \right] + \lambda^{0*}_{ij} m_m \left[ B_1^{(1)} + \left( 2m_Y^2 + m_{n_j}^2 - m_b^2 \right) C_1 \right] \right\}, \]

\[\Delta^{(8)Y}_R = \frac{g^3 c_\beta m_b}{64\pi^2 m_W m_Y^2} \sum_{i,j=1}^{9} U_{(a+3)i}^\nu U_{(b+3)j}^\nu \left\{ \lambda^{0*}_{ij} m_m \left[ B_0^{(12)} - m_Y^2 C_0 - \left( 2m_Y^2 + m_{n_j}^2 - m_b^2 \right) C_2 \right] - \lambda^{0*}_{ij} m_m \left[ B_1^{(2)} + \left( 2m_Y^2 + m_{n_j}^2 - m_b^2 \right) C_2 \right] \right\}. \]
\[ + \lambda_{ij}^{0s} \left[ \lambda_{ai}^{L,s} \lambda_{bj}^{R,s} m_n C_0 + \lambda_{ai}^{L,s} \lambda_{bj}^{L,s} m_n m_b (C_0 + C_2) \right] \]

\[ \Delta_{(7)H_L} = \frac{g^2 \lambda_{H_s}^2 f_s}{16 \pi^2 m_W^2} \sum_{i=1}^{9} \left[ -\lambda_{ai}^{R,s} \lambda_{bi}^{L,s} m_n C_0 - \lambda_{ai}^{L,s} \lambda_{bi}^{L,s} m_a C_1 + \lambda_{ai}^{R,s} \lambda_{bi}^{R,s} m_a m_b C_2 \right], \]

\[ \Delta_{(7)H_R} = \frac{g^2 \lambda_{H_s}^2 f_s}{16 \pi^2 m_W^2} \sum_{i=1}^{9} \left[ -\lambda_{ai}^{L,s} \lambda_{bi}^{R,s} m_n C_0 - \lambda_{ai}^{R,s} \lambda_{bi}^{R,s} m_a C_1 + \lambda_{ai}^{L,s} \lambda_{bi}^{L,s} m_a m_b C_2 \right], \]

\[ \Delta_{(9+10)H_L} = -\frac{g^3 c_\beta f_s}{32 \pi^2 m_W^3} \frac{1}{(m_a^2 - m_b^2)} \left[ m_a m_b m_n \lambda_{ai}^{L,s} \lambda_{bi}^{R,s} (B_0^{(1)} - B_0^{(2)}) + m_n \lambda_{ai}^{R,s} \lambda_{bi}^{L,s} (m_b^2 B_0^{(1)} - m_a^2 B_0^{(2)}) \right] \]

\[ \Delta_{(9+10)H_R} = -\frac{g^3 c_\beta f_s}{32 \pi^2 m_W^3} \frac{1}{(m_a^2 - m_b^2)} \left[ m_a m_b m_n \lambda_{ai}^{R,s} \lambda_{bi}^{L,s} (B_0^{(1)} - B_0^{(2)}) + m_n \lambda_{ai}^{L,s} \lambda_{bi}^{R,s} (m_b^2 B_0^{(1)} - m_a^2 B_0^{(2)}) \right] \]

**Appendix B: The divergent cancellation in amplitudes**

The divergent parts in terms as shown in App[A] only contain B functions, we note: \( \text{div} B_0^{(1)} = \text{div} B_0^{(2)} = \text{div} B_0^{(12)} = 2 \text{div} B_1^{(1)} = -2 \text{ div} B_1^{(2)} = \Delta_e \). Ignoring the common factor of \( g^3/(64\pi^2 m_W^3) \) and using \( 1/m_Y = \sqrt{2}s_\alpha/m_W \), the divergent parts of \( \Delta_L \) derived from Eq. \( \text{[A1][A2][A3]} \) are

\[ \text{div} \left[ \Delta_L^{(1)W} \right] = m_a \Delta_e \times \left( -\frac{3c_\beta}{2} \right) \sum_{i=1}^{9} U_{ai}^{\nu_s} U_{bi}^{\nu_s} m_n^2, \]

\[ \text{div} \left[ \Delta_L^{(8)W} \right] = m_a \Delta_e \times c_\beta \sum_{i,j=1}^{9} U_{ai}^{\nu_s} U_{bj}^{\nu_s} \left( \lambda_{ij}^{0s} m_n + \frac{1}{2} \lambda_{ij}^{0s} m_n \right), \]

\[ \text{div} \left[ \Delta_L^{(4+5)W} \right] = \text{div} \left[ \Delta_L^{(4)Y} \right] = \text{div} \left[ \Delta_L^{(4+5)Y} \right] = 0, \]

\[ \text{div} \left[ \Delta_L^{(1)Y} \right] = m_a \Delta_e \times 3 s_\alpha^3 \left( \sqrt{2}s_\beta c_\alpha - c_\beta s_\alpha \right) \sum_{i=1}^{9} U_{ai}^{\nu_s} U_{bj}^{\nu_s} \lambda_{(a+3)i}^{L,s} \lambda_{(b+3)i}^{L,s} m_n, \]

\[ \text{div} \left[ \Delta_L^{(2)Y} \right] = m_a \Delta_e \times s_\alpha^2 c_\alpha \left( c_\beta c_\alpha + \sqrt{2}s_\beta s_\alpha \right) \sum_{i=1}^{9} U_{ai}^{\nu_s} U_{bj}^{\nu_s} \lambda_{bi}^{L,1} m_n, \]

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\[
\text{div} \left[ \Delta^{(3)Y}_L \right] = m_\Delta \Delta_\epsilon \times \left[ -2s_\alpha^2 c_\alpha \left( c_\beta c_\alpha + \sqrt{2} s_\beta s_\alpha \right) \right] \sum_{i=1}^{9} U^\nu_{(a+3)i} U^\nu_{(b+3)i} m^2_{n_i},
\]
\[
\text{div} \left[ \Delta^{(8)Y}_L \right] = m_\Delta \Delta_\epsilon \times 2s_\beta^2 c_\beta \sum_{i,j=1}^{9} U^\nu_{(a+3)i} U^\nu_{(b+3)j} \left( \lambda^0_{ij} m_{n_j} + \frac{1}{2} \lambda^0_{ij} m_{n_i} \right),
\]
\[
\text{div} \left[ \Delta^{(6)H^1}_L \right] = m_\Delta \Delta_\epsilon \times (-2c_\beta^2 c_\alpha^2) \sum_{i,j=1}^{9} U^\nu_{ai} \lambda^0_{ij} \lambda^L_{bj},
\]
\[
\text{div} \left[ \Delta^{(6)H^2}_L \right] = m_\Delta \Delta_\epsilon \times (-c_\beta^2) \sum_{i,j=1}^{9} U^\nu_{ai} \lambda^L_{ij} \lambda^L_{bj},
\]
\[
\text{div} \left[ \Delta^{(9+10)H^1}_L \right] = m_\Delta \Delta_\epsilon \times (2c_\beta^2 c_\alpha^2) \sum_{i=1}^{9} U^\nu_{ai} \lambda^L_{bi} \lambda^L_{bj},
\]
\[
\text{div} \left[ \Delta^{(9+10)H^2}_L \right] = m_\Delta \Delta_\epsilon \times c_\beta \sum_{i=1}^{9} U^\nu_{ai} \lambda^L_{bi} m_{n_i},
\]

Similarly, the divergences of the $\Delta^{(k)W,Y,H^\pm}_L$ are shown. Using the equalities $M^\nu = U^\nu M U^{\nu\dagger}$, we can prove that

\[
\text{div} [\Delta_{1,L,R}] = \text{div} \left[ \Delta^{(1)W}_{L,R} + \Delta^{(8)W}_{L,R} + \Delta^{(6)H^1}_{L,R} + \Delta^{(9+10)H^1}_{L,R} \right] = 0,
\]
\[
\text{div} [\Delta_{2,L,R}] = \text{div} \left[ \Delta^{(1)Y}_{L,R} + \Delta^{(2)Y}_{L,R} + \Delta^{(3)Y}_{L,R} + \Delta^{(8)Y}_{L,R} + \Delta^{(6)H^2}_{L,R} + \Delta^{(9+10)H^2}_{L,R} \right] = 0,
\]
\[
\text{div} [\Delta_{3,L,R}] = \text{div} \left[ \Delta^{(7)H_1}_{L,R} + \Delta^{(7)H_2}_{L,R} + \Delta^{(4+5)W}_{L,R} + \Delta^{(4+5)Y}_{L,R} \right] = 0,
\]

\[\text{(B1)}\]

\[\text{(B2)}\]

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