Abstract—We consider a relay channel where a relay helps the transmission of messages from one sender to one receiver. The relay is considered not only as a sender that helps the message transmission but as a wire-tapper who can obtain some knowledge about the transmitted messages. In this paper we study the coding problem of the relay channel under the situation that some of transmitted messages are confidential to the relay. A security of such confidential messages is measured by the conditional entropy. The rate region is defined by the set of transmission rates for which messages are reliably transmitted and the security of confidential messages is larger than a prescribed level. In this paper we give two definition of the rate region. We first define the rate region in the case of deterministic encoder and call it the deterministic rate region. Next, we define the rate region in the case of stochastic encoder and call it the stochastic rate region. We derive explicit inner and outer bounds for the above two rate regions and present a class of relay channels where two bounds match. Furthermore, we show that stochastic encoder can enlarge the rate region. We also evaluate the deterministic rate region of the Gaussian relay channel with confidential messages.

Index Terms—Relay channel, confidential messages, information security

I. INTRODUCTION

The security of communication systems can be studied from a information theoretical viewpoint by regarding them as a kind of cryptosystem in which some messages transmitted through communication channel should be confidential to anyone except for authorized receivers. The security of a communication system was first studied by Shannon [1] from a standpoint of information theory. He discussed a theoretical model of cryptosystems using the framework of classical one way noiseless channels and derived some conditions for secure communication. Subsequently, the security of communication systems based on the framework of broadcast channels were studied by Wyner [2] and Csiszár and Körner [3]. Maurer [4], Ahlswede and Csiszár[5], [6], Csiszár and Narayan [7], and Venkatesan and Anantharam [8] studied the problem of public key agreements under the framework of multi-terminal channel coding systems.

Various types of multiterminal channel networks have been investigated so far in the field of multi-user information theory. In those networks some kind of confidentiality of information transmitted through channels is sometimes required from the standpoint of information security. In this case it is of importance to analyze the security of communication from a viewpoint of multi-user information theory. The author [9] discussed the security of communication using relay channels. The author posed and investigate the relay channel with confidential messages, where the relay acts as both a helper and a wire-tapper. Recently, Liang and Poor [10] studied the security of communication using multiple access channel by formulating and investigating the multiple access channel with confidential messages.

In this paper we discuss the security of communication for relay channel under the framework introduced by the author in [9]. In the relay channel the relay is considered not only as a sender who helps the transmission of messages but as a wire-tapper who can learn something about the transmitted messages. The coding theorem for the relay channel was first established by Cover and El Gamal [11]. By carefully checking their coding scheme used for the proof of the direct coding theorem, we can see that in their coding scheme the relay helps the transmission of messages by learning all of them. Hence, this coding scheme is not adequate when some messages should be confidential to the relay.

The author [9] studied the security of communication for the relay channel under the situation that some of transmitted messages are confidential to the relay. For analysis of this situation the author posed the communication system called the relay channel with confidential messages or briefly said the RCC. In the RCC, a sender wishes to transmit two different types of message. One is a message called the common message which is sent to the receiver and the relay. The other is a message called the private message which is sent only to the receiver and is confidential to the relay as much as possible. The knowledge that the relay gets about private messages is measured by the conditional entropy of private messages conditioned by channel outputs that the relay observes. The author [9] defined the rate region by the set of transmission rates for which common and private messages are transmitted with arbitrary small error probabilities and the security of private message measured by the conditional entropy per transmission is larger than a prescribed level. The author [9] derived a inner bound of the capacity region of the RCC.

In this paper we study the coding problem of the RCC. In general two cases of encoding can be considered in the problem of channel coding. One is a case where deterministic encoders are used for transmission of messages and the other is a case where stochastic encoders are used. In the definition of the rate region by the author [9], deterministic encoders are implicitly assumed. In this paper we also consider the case of stochastic encoders. We define the rate region in the case where deterministic encoders are used for transmission and call it the deterministic rate region. We further define the rate region in the case of stochastic encoders and call it the stochastic rate region. We derive explicit inner and outer bounds for the above two rate regions and present a class of relay channels where inner and outer bounds match.
Furthermore, we give another class of relay channels, where the outer bound is very close to the inner bound. We also compare the results on stochastic and deterministic rate region, demonstrating that stochastic encoder can enlarge the rate region. We also study the Gaussian RCC, where transmissions are corrupted by additive Gaussian noise. We evaluate the deterministic rate region of the Gaussian RCC and derive explicit inner and outer bounds. We show that for some class of relay channels those two bounds match.

Recently, Liang and Veeravalli [12] and Liang and Kramer [13] posed and investigated a new theoretical model of cooperative relay communication network called the partially/fully cooperative relay broadcast channel (RBC). A special case of cooperative communication network called the partially/fully cooperative relay broadcast channel (RBC). A special case of the partially cooperative RBC coincides with the RCC in a framework of communication. However, in the problem setup, there seems to be an essential difference between them. The formulation of problem in the RBC is focused on an aspect of cooperation in relay channels. On the other hand, the formulation of problem by the author [9] is focused on an aspect of security in relay channels. Cooperation and security are two important features in communication networks. It is interesting to note that both cooperation and security simultaneously occur in relay communication networks.

II. RELAY CHANNELS WITH CONFIDENTIAL MESSAGES

Let $\mathcal{X}, \mathcal{S}, \mathcal{Y}, \mathcal{Z}$ be finite sets. The relay channel dealt with in this paper is defined by a discrete memoryless channel specified with the following stochastic matrix:

$$\Gamma = \{ \Gamma(y, z | x, s) \}_{(x, s, y, z) \in \mathcal{X} \times \mathcal{S} \times \mathcal{Y} \times \mathcal{Z}}.$$  \hspace{1cm} (1)

Let $X$ be a random variable taking values in $\mathcal{X}$ and $X^n = X_1 \times X_2 \cdots \times X_n$ be a random vector taking values in $\mathcal{X}^n$. We write an element of $\mathcal{X}^n$ as $x = x_1 x_2 \cdots x_n$. Similar notations are adopted for $S, Y,$ and $Z$.

In the RCC, we consider the following scenario of communication. Let $K_n$ and $\mathcal{M}_n$ be uniformly distributed random variables taking values in message sets $\mathcal{K}_n$ and $\mathcal{M}_n$, respectively. The random variable $\mathcal{M}_n$ is a common message sent to a relay and a receiver. The random variable $K_n$ is a private message sent only to the receiver and contains an information confidential to the relay. A sender transforms $K_n$ and $\mathcal{M}_n$ into a transmitted sequence $X^n$ using an encoder function $f_n$ and sends it to the relay and the receiver. For the encoder function $f_n$, we consider two cases; one is the case where $f_n$ is deterministic and the other is the case where $f_n$ is stochastic.

In the former case $f_n$ is a one to one mapping from $\mathcal{K}_n \times \mathcal{M}_n$ to $\mathcal{X}^n$. In the latter case $f_n : \mathcal{K}_n \times \mathcal{M}_n \rightarrow \mathcal{X}^n$ is a stochastic matrix defined by

$$f_n(k, m) = \{ f_n(x | k, m) \}_{x \in X^n}, (k, m) \in K_n \times \mathcal{M}_n.$$  

Here, $f_n(x | k, m)$ is the probability that the message $(k, m)$ is encoded as a channel input $x$. Channel inputs and outputs at the $i$th transmission is shown in Fig. 1. At the $i$th transmission, the relay observes the random sequence $Z^{i-1} = (Z_1, Z_2, \cdots, Z_{i-1})$ transmitted by the sender through noisy channel, encodes them into random variable $S_i$ and sends it to the receiver. The relay also wishes to decode the common message from observed channel outputs. The encoder function at the relay is defined by the sequence of functions $\{ g_i \}_{i=1}^n$. Each $g_i$ is defined by $g_i : Z^{i-1} \rightarrow \mathcal{S}$. Note that the channel input $S_i$ that the relay sends at the $i$th transmission depends solely on the output random sequence $Z^{i-1}$ that the relay previously obtained as channel outputs. The decoding functions at the receiver and the relay are denoted by $\psi_n$ and $\varphi_n$, respectively. Those functions are formally defined by $\psi_n : \mathcal{Y}^n \rightarrow \mathcal{K}_n \times \mathcal{M}_n$, $\varphi_n : \mathcal{Z}^n \rightarrow \mathcal{M}_n$. Transmission of messages via relay channel using $(f_n, \{ g_i \}_{i=1}^n, \psi_n, \varphi_n)$ is shown in Fig. 2. When $f_n$ is a deterministic encoder, error probabilities of decoding for transmitted pair $(k, m) \in \mathcal{K}_n \times \mathcal{M}_n$ are defined by

$$\lambda_1^{(n)}(k, m) \triangleq \sum_{y \in \mathcal{Y}, z \in \mathcal{Z}} \prod_{i=1}^n \Gamma(y_i, z_i | x_i(k, m), g(z^{i-1})),$$

$$\lambda_2^{(n)}(m) = \sum_{y \in \mathcal{Y}, z \in \mathcal{Z}} \prod_{i=1}^n \Gamma(y_i, z_i | x_i(k, m), g(z^{i-1})),$$

where $x_i(k, m)$ is the $i$th component of $x = f_n(k, m)$. The average error probabilities $\lambda_1^{(n)}$ and $\lambda_2^{(n)}$ of decoding are defined by

$$\lambda_1^{(n)} \triangleq \frac{1}{|\mathcal{K}_n| |\mathcal{M}_n|} \sum_{(k, m) \in \mathcal{K}_n \times \mathcal{M}_n} \lambda_1^{(n)}(k, m),$$

$$\lambda_2^{(n)} \triangleq \frac{1}{|\mathcal{M}_n|} \sum_{m \in \mathcal{M}_n} \lambda_2^{(n)}(m),$$

where $|\mathcal{K}_n|$ is a cardinality of the set $\mathcal{K}_n$. When $f_n$ is a stochastic encoder, error probabilities of decoding for transmitted pair $(k, m) \in \mathcal{K}_n \times \mathcal{M}_n$ are defined by

$$\mu_1^{(n)}(k, m) \triangleq \sum_{(x, y, z) : \psi_n(y) \neq (k, m)} \prod_{i=1}^n \Gamma(y_i, z_i | x_i(k, m), g(z^{i-1})) f_n(x | k, m),$$

$$\mu_2^{(n)}(m) \triangleq \sum_{x : \varphi_n(x) \neq m} \prod_{i=1}^n \Gamma(y_i, z_i | x_i(k, m), g(z^{i-1})) f_n(x | k, m),$$

where $\psi_n(y) \neq (k, m)$ and $\varphi_n(x) \neq m$.
\[ \triangleq \sum_{(x,y,z) : \varphi_n(z) \neq m} ^{\varphi_n(z)} \prod_{i=1}^{n} \Gamma(y_i, z_i | x_i(k, m), g(z^{-1})) f_n(x|k, m) . \]

The average error probabilities \( \mu_1^{(n)} \) and \( \mu_2^{(n)} \) of decoding are defined by
\[
\mu_1^{(n)} \triangleq \frac{1}{|K_n||M_n|} \sum_{(k,m) \in K_n \times M_n} \mu_1^{(n)}(k, m) ,
\mu_2^{(n)} \triangleq \frac{1}{|M_n|} \sum_{m \in M_n} \mu_2^{(n)}(m) .
\]

A triple \( (R_0, R_1, R_e) \) is achievable if there exists a sequence of quadruples \( \{(f_n, g_i^{(n)}), \psi_n, \varphi_n\} \) such that
\[
\lim_{n \to \infty} \lambda_1^{(n)} = \lim_{n \to \infty} \lambda_2^{(n)} = 0 , \quad \lim_{n \to \infty} \frac{1}{n} \log |M_n| = R_0 , \quad \lim_{n \to \infty} \frac{1}{n} \log |K_n| = R_1 , \quad \lim_{n \to \infty} \frac{1}{n} H(K_n|Z^n) \geq R_e .
\]

A Markov chain \( U \to X \to Y \to Z \to S \) means that random variables \( U, X, Y, Z \) form a Markov chain in this order. Set
\[
\tilde{\mathcal{R}}_d^{(in)}(\Gamma) \triangleq \{(R_0, R_1, R_e) : R_0, R_1, R_e \geq 0 , \quad R_0 \leq \min\{I(Y;US), I(Z;US)\} , \quad R_1 \leq I(X;Y|US) , \quad R_e \leq R_1 , \quad R_e \leq I(X;Y|ZUS) , \quad \text{for some } (U, X, S) \in \mathcal{P}_1 \} .
\]

Then, we have the following result.

**Theorem 2:** For any relay channel \( \Gamma \),
\[
\mathcal{R}_d(\Gamma) \subseteq \tilde{\mathcal{R}}_d^{(out)}(\Gamma) .
\]

An essential difference between inner and outer bounds of \( \mathcal{R}_d(\Gamma) \) is a gap \( \Delta \) given by
\[
\Delta \triangleq I(X;Y|ZUS) - [I(X;Y|US) - I(X;Z|US)] = I(X;ZYUS) = I(X;Z|YUS) .
\]

Observe that
\[
\Delta = H(Z|YUS) - H(Z|YXUS) \leq H(Z|YUS) - H(Z|YXUS) = I(X;Z|YUS) ,
\]

where \( \Delta \) follows from the Markov condition \( U \to XS \to YZ \). Hence, \( \Delta \) vanishes if the relay channel \( W = \{\Gamma(z, y|x, s) \}_{(x,y,z,s) \in X \times Y \times Z} \) satisfies the following:
\[
\Gamma(z, y|x, s, \Gamma(y|x, s) = \Gamma(z, y|x, s) \Gamma(y|x, s) .
\]

The above condition is equivalent to the condition that \( X, S, Y, Z \) form a Markov chain \( X \to SY \to Z \) in this order. Cover and El. Gamal [11] called this relay channel the reversed degraded relay channel. On the other hand, we have
\[
I(X;Y|ZUS) = H(Y|ZUS) - H(Y|ZXUS) \leq H(Y|ZS) - H(Y|ZXUS) = I(X;Y|ZS) ,
\]

where \( \Delta \) follows from the Markov condition \( U \to XSZ \to Y \). The quantity \( I(X;Y|ZUS) \) vanishes if the relay channel \( \Gamma \) satisfies the following:
\[
\Gamma(z, y|x, s) = \Gamma(y|x, s) \Gamma(z|x, s) .
\]

Hence, if the relay channel \( \Gamma \) satisfies \( \Delta \), then \( R_e \) should be zero. This implies that no security on the private messages is guaranteed. The condition \( \Delta \) is equivalent to the condition that \( X, S, Y, Z \) form a Markov chain \( X \to SZ \to Y \) in this order. Cover and El. Gamal [11] called this relay channel the degraded relay channel. Summarizing the above arguments, we obtain the following two corollaries.

**Corollary 1:** For the reversed degraded relay channel \( \Gamma \), we have
\[
\tilde{\mathcal{R}}_d^{(in)}(\Gamma) = \mathcal{R}_d(\Gamma) = \tilde{\mathcal{R}}_d^{(out)}(\Gamma) .
\]
Corollary 2:  In the deterministic case, if the relay channel \( \Gamma \) is degraded, then no security on the private messages is guaranteed.

Next, we derive another inner bound and two other outer bounds of \( \mathcal{R}_d(\Gamma) \). Define a set of random triples \((U,X,S) \in \mathcal{U} \times \mathcal{X} \times \mathcal{S}\) by

\[
\mathcal{P}_2 = \{(U,X,S) : |U| \leq |Z||X||S| + 3, \quad U \rightarrow XSZ \rightarrow Y\}.
\]

It is obvious that \( \mathcal{P}_1 \subseteq \mathcal{P}_2 \). For given \((U,X,S) \in \mathcal{U} \times \mathcal{X} \times \mathcal{S}\), set

\[
\mathcal{R}(U,X,S|\Gamma)
\]

\[
\mathcal{R}_d^{(\text{in})}(\Gamma) \triangleq \bigcup_{(U,X,S) \in \mathcal{P}_1} \mathcal{R}(U,X,S|\Gamma),
\]

\[
\mathcal{R}_d^{(\text{out})}(\Gamma) \triangleq \bigcup_{(U,X,S) \in \mathcal{P}_2} \mathcal{R}(U,X,S|\Gamma).
\]

Then, we have the following.

Theorem 3:  For any relay channel \( \Gamma \),

\[
\mathcal{R}_d^{(\text{in})}(\Gamma) \subseteq \mathcal{R}_d(\Gamma) \subseteq \mathcal{R}_d^{(\text{out})}(\Gamma).
\]

Now we consider the case where the relay channel \( \Gamma \) satisfies

\[
\Gamma(y,z|x,s) = \Gamma(y|x,s)\Gamma(z|x).
\]

The above condition on \( \Gamma \) is equivalent to the condition that \( X, S, Y, Z \) satisfy the following two Markov chains:

\[
Y \rightarrow XS \rightarrow Z, S \rightarrow X \rightarrow Z.
\]

The first condition is equivalent to that \( Y \) and \( Z \) are conditionally independent given \( SX \) and the second is equivalent to that \( Z \) and \( S \) are conditionally independent given \( X \).

We say that the relay channel \( \Gamma \) belongs to the independent class if it satisfies \( (10) \). For the independent class of relay channels, we derive an outer bound of \( \mathcal{R}_d(\Gamma) \). To state our result, set

\[
\mathcal{R}_d^{(\text{out})}(\Gamma)
\]

\[
\mathcal{R}_d^{(\text{out})}(\Gamma) \triangleq \{(R_0, R_1, R_e) : R_0, R_1, R_e \geq 0,
\]

\[
R_0 \leq \min\{I(Y;US), I(Z;U|S)\},
\]

\[
R_0 + R_1 \leq I(X;Y|US) + \zeta(U,S,Y,Z)^+,
\]

\[
+ \min\{I(Z;U|S), I(Y;US)\},
\]

\[
R_e \leq R_1,
\]

\[
R_e \leq I(X;Y|US) - I(X;Z|US) + \zeta(U,S,Y,Z)^+,
\]

where we set

\[
\zeta(U,S,Y,Z) \triangleq I(XS;Y|U) - I(XS;Z|U) - [I(X;Y|US) - I(X;Z|US)]
\]

\[
= I(S;Y|U) - I(S;Z|U) = H(S|ZU) - H(S|UY).
\]

Then, we have the following.

Property 1: For any \((U,X,S) \in \mathcal{P}_2\),

\[
\zeta(U,S,Y,Z) \leq I(XS;Y|Z).
\]

Proof: We have the following chain of inequalities:

\[
\zeta(U,S,Y,Z)
\]

\[
= H(S|ZU) - H(S|UY)
\]

\[
\leq H(S|ZU) - H(S|ZYS)
\]

\[
= I(S;Y|ZU)
\]

\[
= H(Y|ZU) - H(Y|ZUS)
\]

\[
\leq H(Y|Z) - H(Y|ZXSU)
\]

\[
= H(Y|Z) - H(Y|ZXS) = I(XS;Y|Z),
\]

where the last equality follows from the Markov condition \( U \rightarrow ZXS \rightarrow Y \).

Our result is the following.

Theorem 4:  If \( \Gamma \) belongs to the independent class, we have

\[
\mathcal{R}_d(\Gamma) \subseteq \mathcal{R}_d^{(\text{out})}(\Gamma).
\]

B. Stochastic Case

In this subsection we state our results on inner and outer bounds of \( \mathcal{R}_e(\Gamma) \). Define two sets of random quadruples \((U,V,X,S) \in \mathcal{U} \times \mathcal{V} \times \mathcal{X} \times \mathcal{S}\) by

\[
\mathcal{Q}_1 \triangleq \{(U,V,X,S) : |U| \leq |X||S| + 3,
\]

\[
|V| \leq (|X||S|)^2 + 4|X||S| + 3,
\]

\[
U \rightarrow V \rightarrow XS \rightarrow YZ,
\]

\[
US \rightarrow V \rightarrow X,
\]

\[
\mathcal{Q}_2 \triangleq \{(U,V,X,S) : |U| \leq |Z||X||S| + 3,
\]

\[
|V| \leq (|Z||X||S|)^2 + 4|Z||X||S| + 3,
\]

\[
U \rightarrow V \rightarrow XSZ \rightarrow Y,
\]

\[
US \rightarrow VX \rightarrow Z,
\]

\[
US \rightarrow V \rightarrow X.
\]

It is obvious that \( \mathcal{Q}_1 \subseteq \mathcal{Q}_2 \). For given \((U,V,X,S) \in \mathcal{U} \times \mathcal{V} \times \mathcal{X} \times \mathcal{S}\), set

\[
\mathcal{R}(U,V,X,S|\Gamma)
\]

\[
\mathcal{R}_d^{(\text{in})}(\Gamma) \triangleq \bigcup_{(U,V,X,S) \in \mathcal{Q}_1} \mathcal{R}(U,V,X,S|\Gamma),
\]

\[
\mathcal{R}_d^{(\text{out})}(\Gamma) \triangleq \bigcup_{(U,V,X,S) \in \mathcal{Q}_2} \mathcal{R}(U,V,X,S|\Gamma).
\]

\[
\mathcal{R}(U,V,X,S|\Gamma)
\]

\[
\mathcal{R}_d^{(\text{in})}(\Gamma) \triangleq \bigcup_{(U,V,X,S) \in \mathcal{Q}_1} \mathcal{R}(U,V,X,S|\Gamma),
\]

\[
\mathcal{R}_d^{(\text{out})}(\Gamma) \triangleq \bigcup_{(U,V,X,S) \in \mathcal{Q}_2} \mathcal{R}(U,V,X,S|\Gamma).
\]
Then, if the following corollary.

**Theorem 5**: For any relay channel \( \Gamma \),

\[
\mathcal{R}_s^{\text{in}}(\Gamma) \subseteq \mathcal{R}_s(\Gamma) \subseteq \mathcal{R}_s^{\text{out}}(\Gamma).
\]

Similarly to the deterministic case, we estimate the quantity \( I(X;Y|US) - I(X;Z|US) \). We have the following chain of inequalities:

\[
\begin{align*}
I(V;Y|US) - I(V;Z|US) &\leq I(V;YZ|US) - I(V;Z|US) \\
&= I(V;Y|ZUS) \\
&= H(Y|ZUS) - H(Y|ZVUS) \\
&\leq H(Y|ZS) - H(Y|ZXVUS) \\
&= H(Y|ZS) - H(Y|ZXS) = I(X;Y|ZS),
\end{align*}
\]

where (13) follows from the Markov condition

\[ U \rightarrow V \rightarrow XSZ \rightarrow Y . \]

Then, if \( \Gamma \) is degraded, for any \( (U,V,S,X) \in \mathcal{Q}_2 \), we have

\[ I(V;Y|US) - I(V;Z|US) \leq 0 . \]

Hence, if the relay channel \( \Gamma \) is degraded, then \( R_s \) should be zero. This implies that no security on the private messages is guaranteed for the degraded relay channel. Thus, we obtain the following corollary.

**Corollary 3**: When the relay channel \( \Gamma \) is degraded, no security on the private messages is guaranteed even if \( f_n \) is a stochastic encoder.

**IV. SECRECY CAPACITIES OF THE RCC**

In this section we derive an explicit inner and outer bounds of the secrecy capacity region by using the results in the previous section.

**A. Deterministic Case**

We first consider the case where \( f_n \) is a deterministic encoder. The secrecy capacity region \( C_{ds}(\Gamma) \) for the RCC is defined by

\[
C_{ds}(\Gamma) = \{(R_0, R_1) : (R_0, R_1, R_1) \in \mathcal{R}_d(\Gamma)\}.
\]

From Theorems [1] and [2], we have the following corollary.

**Corollary 4**: For any relay channel \( \Gamma \),

\[
C_{ds}^{\text{in}}(\Gamma) \subseteq C_{ds}(\Gamma) \subseteq C_{ds}^{\text{out}}(\Gamma),
\]

where

\[
C_{ds}^{\text{in}}(\Gamma) = \{(R_0, R_1) : R_0, R_1 \geq 0, R_0 \leq \min\{I(Y;US), I(Z;US)\}, R_1 \leq [I(X;Y|US) - I(X;Z|US)]^+, \text{for some } (U,X,S) \in \mathcal{P}_1 \},
\]

\[
C_{ds}^{\text{out}}(\Gamma) = \{(R_0, R_1) : R_0, R_1 \geq 0, R_0 \leq \min\{I(Y;US), I(Z;US)\}, R_1 \leq I(X;Y|ZUS), \text{for some } (U,X,S) \in \mathcal{P}_1 \}.
\]

In particular, if \( \Gamma \) is reversely degraded, we have

\[
C_{ds}^{\text{in}}(\Gamma) = C_{ds}(\Gamma) = C_{ds}^{\text{out}}(\Gamma).
\]

From Theorem [3], we obtain the following corollary.

**Corollary 5**: For any relay channel \( \Gamma \),

\[
C_{ds}^{\text{in}}(\Gamma) \subseteq C_{ds}(\Gamma) \subseteq C_{ds}^{\text{out}}(\Gamma),
\]

where

\[
C_{ds}^{\text{out}}(\Gamma) = \{(R_0, R_1) : R_0, R_1 \geq 0, R_0 \leq \min\{I(Y;US), I(Z;US)\}, R_1 \leq [I(X;Y|US) - I(X;Z|US)]^+, \text{for some } (U,X,S) \in \mathcal{P}_1 \}.
\]

From Theorem [4], we have the following corollary.

**Corollary 6**: If \( \Gamma \) belongs to the independent class, we have

\[
C_{ds}(\Gamma) \subseteq C_{ds}^{\text{out}}(\Gamma),
\]

where

\[
C_{ds}^{\text{out}}(\Gamma) = \{(R_0, R_1) : R_0, R_1 \geq 0, R_0 \leq \min\{I(Y;US), I(Z;US)\}, R_1 \leq [I(X;Y|US) - I(X;Z|US)]^+, \text{for some } (U,X,S) \in \mathcal{P}_1 \}.
\]

Now, we consider the special case of no common message. Set

\[
\mathcal{R}_{d1e}(\Gamma) = \{(R_1, R_e) : (0, R_1, R_e) \in \mathcal{R}_d(\Gamma)\}
\]

and define the secrecy capacity by

\[
C_{ds}(\Gamma) = \max_{(R_1, R_e) \in \mathcal{R}_{d1e}(\Gamma)} R_1 = \max_{(0, R_1) \in \mathcal{R}_{ds}(\Gamma)} R_1.
\]

Typical shape of the region \( \mathcal{R}_{d1e}(\Gamma) \) and the secrecy capacity \( C_{ds}(\Gamma) \) is shown in Fig. [3].

From Theorems [1] and [2], we have the following corollary.

**Corollary 7**: For any relay channel \( \Gamma \),

\[
\mathcal{R}_{ds}^{\text{in}}(\Gamma) \subseteq \mathcal{R}_{ds}(\Gamma) \subseteq \mathcal{R}_{ds}^{\text{out}}(\Gamma),
\]
Theorem 3. Set

\[
\mathcal{R}_{d1e}^{(in)}(\Gamma) \triangleq \{(R_1, R_e): R_1, R_e \geq 0, \\
R_1 \leq I(X; Y|US), \\
R_e \leq R_1, \\
R_e \leq [I(X; Y|US) - I(X; Z|US)]^+ + \min\{I(Y; US), I(Z; U|S)\}, \\
\text{for some } (U, X, S) \in \mathcal{P}_1\right\},
\]

\[
\mathcal{R}_{d1e}^{(out)}(\Gamma) \triangleq \{(R_1, R_e): R_1, R_e \geq 0, \\
R_1 \leq I(X; Y|US), \\
R_e \leq R_1, \\
R_e \leq I(X; Y|ZUS), \\
\text{for some } (U, X, S) \in \mathcal{P}_1\right\}.
\]

Furthermore,

\[
\max_{(U, X, S) \in \mathcal{P}_1} [I(X; Y|US) - I(X; Z|US)]^+ \leq C_{ds}(\Gamma) \leq \max_{(U, X, S) \in \mathcal{P}_1} I(X; Y|ZUS).
\]

In particular, if \( \Gamma \) is reversely degraded, we have

\[
\mathcal{R}_{d1e}^{(in)}(\Gamma) = \mathcal{R}_{d1e}(\Gamma) = \mathcal{R}_{d1e}^{(out)}(\Gamma)
\]

and

\[
C_{ds}(\Gamma) = \max_{(U, X, S) \in \mathcal{P}_1} [I(X; Y|US) - I(X; Z|US)].
\]

Next, we state a result which is obtained as a corollary of Theorem 3.

Set

\[
\mathcal{R}_{1e}(U, X, S|\Gamma) \triangleq \mathcal{R}(U, X, S|\Gamma) \cap \{(R_0, R_1, R_e): R_0 = 0\}
\]

\[
= \{(R_1, R_e): R_1, R_e \geq 0, \\
R_1 \leq I(X; Y|US) + \min\{I(Y; US), I(Z; U|S)\}, \\
R_e \leq R_1, \\
R_e \leq [I(X; Y|US) - I(X; Z|US)]^+ \}.
\]

and

\[
\mathcal{R}_{d1e}^{(in)}(\Gamma) \triangleq \bigcup_{(U, X, S) \in \mathcal{P}_1} \mathcal{R}_{1e}(U, X, S|\Gamma),
\]

\[
\mathcal{R}_{d1e}^{(out)}(\Gamma) \triangleq \bigcup_{(U, X, S) \in \mathcal{P}_2} \mathcal{R}_{1e}(U, X, S|\Gamma).
\]

Then, we have the following.

Corollary 8: For any relay channel \( \Gamma \),

\[
\mathcal{R}_{d1e}^{(in)}(\Gamma) \subseteq \mathcal{R}_{d1e}(\Gamma) \subseteq \mathcal{R}_{d1e}^{(out)}(\Gamma).
\]

Furthermore,

\[
C_{ds}(\Gamma) \leq \max_{(U, X, S) \in \mathcal{P}_1} [I(X; Y|US) - I(X; Z|US)]^+ + \min\{I(Y; US), I(Z; U|S)\},
\]

\[
R_e \leq R_1,
\]

\[
R_e \leq [I(X; Y|US) - I(X; Z|US)]^+ + \zeta(U, Y, S, Z)^+,
\]

for some \((U, X, S) \in \mathcal{P}_1\).

Then we have the following.

Corollary 9: If \( \Gamma \) belongs to the independent class, we have

\[
\mathcal{R}_{d1e}(\Gamma) \subseteq \mathcal{R}_{d1e}^{(out)}(\Gamma).
\]

Furthermore,

\[
C_{ds}(\Gamma) \leq \max_{(U, X, S) \in \mathcal{P}_1} [I(X; Y|US) - I(X; Z|US)]^+ + \zeta(U, S, Y, Z)^+,
\]

\[
= \max_{(U, X, S) \in \mathcal{P}_1} [I(XS; Y|U) - I(XS; Z|U)]^+.
\]

B. Stochastic Case

The stochastic secrecy capacity region \( C_{ss}(\Gamma) \) for the RCC is defined by

\[
C_{ss}(\Gamma) = \{(R_0, R_1) : (R_0, R_1, R_e) \in \mathcal{R}_{ss}(\Gamma)\}. \tag{15}
\]

To describe our result set

\[
C_{ss}(U, V, X, S|\Gamma) \triangleq \mathcal{R}(U, V, X, S|\Gamma) \cap \{(R_0, R_1, R_e): R_1 = R_e\}
\]

\[
= \{(R_0, R_1): R_0, R_1 \geq 0, \\
R_0 \leq \min\{I(Y; US), I(Z; U|S)\}, \\
R_1 \leq [I(V; Y|US) - I(V; Z|US)]^+ \}.
\]

and

\[
C_{ss}^{(in)}(\Gamma) \triangleq \bigcup_{(U, V, X, S) \in \mathcal{Q}_1} C_{ss}(U, V, X, S|\Gamma),
\]

\[
C_{ss}^{(out)}(\Gamma) \triangleq \bigcup_{(U, V, X, S) \in \mathcal{Q}_2} C_{ss}(U, V, X, S|\Gamma).
\]

From Theorem 3 we obtain the following corollary.

Corollary 10: For any relay channel \( \Gamma \),

\[
C_{ss}^{(in)}(\Gamma) \subseteq C_{ss}(\Gamma) \subseteq C_{ss}^{(out)}(\Gamma).
\]

In particular, if \( \Gamma \) is degraded, we have

\[
C_{ss}^{(in)}(\Gamma) = C_{ss}(\Gamma) = C_{ss}^{(out)}(\Gamma).
\]
Next, set
\[ R_{s1e}(\Gamma) \triangleq \{(0, R_1, R_e) \in R_s(\Gamma)\} \]
and define the secrecy capacity by
\[ C_{ss}(\Gamma) \triangleq \max_{(R_1, R_e) \in R_{s1e}(\Gamma)} \{ R_1 \} = \max_{(0, R_1) \in C_{ss}(\Gamma)} R_1. \]
To describe our result, set
\[ R_{s1e}(U, V, X, S|\Gamma) \]
\[ \triangleq R(U, V, X, S|\Gamma) \cap \{(R_0, R_1, R_e) : R_0 = 0\} \]
\[ = \{(R_1, R_e) : R_1, R_e \geq 0, \]
\[ R_1 \leq I(V; Y|US) \]
\[ + \min\{I(Y; US), I(Z; U|S)\}, \]
\[ R_e \leq R_1, \]
\[ R_e \leq [I(V; Y|US) - I(V; Z|US)]^+ \} \]
and
\[ R_{s1e}^{(in)}(\Gamma) \triangleq \bigcup_{(U,V,X,S) \in Q_1} R_{s1e}(U, V, X, S|\Gamma), \]
\[ R_{s1e}^{(out)}(\Gamma) \triangleq \bigcup_{(U,V,X,S) \in Q_2} R_{s1e}(U, V, X, S|\Gamma). \]
From Theorem 5 we have the following corollary.

**Corollary 11:** For any relay channel \( \Gamma \),
\[ R_{s1e}^{(in)}(\Gamma) \subseteq R_{s1e}(\Gamma) \subseteq R_{s1e}^{(out)}(\Gamma). \]

Furthermore,
\[ \max_{(U,V,X,S) \in Q_1} [I(V; Y|US) - I(V; Z|US)]^+ \]
\[ \leq C_{ss}(\Gamma), \]
\[ \leq \max_{(U,V,X,S) \in Q_2} [I(V; Y|US) - I(V; Z|US)]^+. \]

V. Gaussian Relay Channels with Confidential Messages

In this section we study Gaussian relay channels with confidential messages, where two channel outputs are corrupted by additive white Gaussian noises. Let \((\xi_1, \xi_2)\) be correlated zero mean Gaussian random vector with covariance matrix
\[ \Sigma = \left( \begin{array}{cc} N_1 & \rho \sqrt{N_1 N_2} \\ \rho \sqrt{N_1 N_2} & N_2 \end{array} \right), |\rho| < 1. \]

Let \(\{(\xi_{1,i}, \xi_{2,i})\}_{i=1}^{\infty}\) be a sequence of independent identically distributed (i.i.d) zero mean Gaussian random vectors. Each \((\xi_{1,i}, \xi_{2,i})\) has the covariance matrix \(\Sigma\). The Gaussian relay channel is specified by the above covariance matrix \(\Sigma\). Two channel outputs \(Y_i\) and \(Z_i\) of the relay channel at the \(i\)th transmission are given by
\[ Y_i = X_i + S_i + \xi_{1,i}, \]
\[ Z_i = X_i + \xi_{2,i}. \]

Since \((\xi_{1,i}, \xi_{2,i}), i = 1, 2, \cdots, n\) have the covariance matrix \(\Sigma\), we have
\[ \xi_{2,i} = \rho \sqrt{\frac{N_2}{N_1}} \xi_{1,i} + \xi_{2|1,i}, \]
where \(\xi_{2|1,i}, i = 1, 2, \cdots, n\) are zero mean Gaussian random variable with variance \((1 - \rho^2)N_2\) and independent of \(\xi_{1,i}\). In particular if \(\Sigma\) satisfies \(N_1 \leq N_2\) and \(\rho = \sqrt{\frac{N_1}{N_2}}\), we have for \(i = 1, 2, \cdots, n,
\[ Y_i = X_i + S_i + \xi_{1,i}, \]
\[ Z_i = X_i + \xi_{1,i} + \xi_{2|1,i}. \]
(16)
which implies that for \(i = 1, 2, \cdots, n\), \(Z_i \rightarrow (Y_i, S_i) \rightarrow X_i\). Hence, the Gaussian relay channel becomes reversely degraded relay channel. Two channel input sequences \(\{X_i\}_{i=1}^{n}\) and \(\{S_i\}_{i=1}^{n}\) are subject to the following average power constraints:
\[ \frac{1}{n} \sum_{i=1}^{n} E[X_i^2] \leq P_1, \quad \frac{1}{n} \sum_{i=1}^{n} E[S_i^2] \leq P_2. \]
Let \(R_d(P_1, P_2|\Sigma)\) be a rate region for the above Gaussian relay channel when we use a deterministic encoder \(f_n\). To state our result set
\[ R_d^{(in)}(P_1, P_2|\Sigma) \]
\[ \triangleq \{(R_0, R_1, R_e) : R_0, R_1, R_e \geq 0, \]
\[ R_0 \leq \min_{0 \leq \eta \leq 1} \left\{ C\left(\frac{\eta P_1 + P_2 + 2\sqrt{\eta P_1 P_2}}{\eta P_1 + N_1}\right), \right. \]
\[ \left. C\left(\frac{\eta P_1}{\eta P_1 + N_2}\right)\right\} \]
\[ R_1 \leq C\left(\frac{\eta P_1}{\eta P_1 + N_2}\right), \]
\[ R_e \leq R_1, \]
\[ R_e \leq [C\left(\frac{\eta P_1}{\eta P_1 + N_2}\right) - C\left(\frac{2P_2}{N_2}\right)]^+, \]
for some \(0 \leq \theta \leq 1\}. \]
\[ R_d^{(out)}(P_1, P_2|\Sigma) \]
\[ \triangleq \{(R_0, R_1, R_e) : R_0, R_1, R_e \geq 0, \]
\[ R_0 \leq \min \left\{ C\left(\frac{\eta P_1 + P_2 + 2\sqrt{\eta P_1 P_2}}{\eta P_1 + N_1}\right), \right. \]
\[ \left. C\left(\frac{\eta P_1}{\eta P_1 + N_2}\right)\right\} \]
\[ R_1 \leq C\left(\frac{\eta P_1}{\eta P_1 + N_2}\right), \]
\[ R_0 + R_1 \leq C\left(\frac{P_1 + P_2 + 2\sqrt{\eta P_1 P_2}}{N_1}\right), \]
\[ R_e \leq R_1, \]
\[ R_e \leq \left[ C\left(\frac{\eta P_1}{\eta P_1 + N_2}\right) - C\left(\frac{2P_2}{N_2}\right)\right]^+, \]
for some \(0 \leq \theta \leq 1, 0 \leq \eta \leq 1\}. \]

where \(C(x) \triangleq \frac{1}{2} \log(1 + x)\). Our result is the following.

**Theorem 6:** For any Gaussian relay channel,
\[ R_d^{(in)}(P_1, P_2|\Sigma) \subseteq R_d(P_1, P_2|\Sigma) \subseteq R_d^{(out)}(P_1, P_2|\Sigma). \]
(17)
In particular, if the relay channel is reversely degraded, i.e., \(N_1 \leq N_2\) and \(\rho = \sqrt{\frac{N_1}{N_2}}\), then
\[ R_d^{(in)}(P_1, P_2|\Sigma) = R_d(P_1, P_2|\Sigma) = R_d^{(out)}(P_1, P_2|\Sigma). \]
Proof of the first inclusion in (17) in the above theorem is standard. The second inclusion can be proved by a converse coding argument similar to the one developed by Liang and Veeravalli [12]. Proof of Theorem 4 is stated in the next section.

Next, we study the secrecy capacity of the Gaussian RCCs. Define two regions by

\[ C_{ds}(P_1, P_2 | \Sigma) \triangleq \{(R_0, R_1) : (R_0, R_1, R_1) \in R_d(P_1, P_2 | \Sigma)\} , \]

\[ R_{dle}(P_1, P_2 | \Sigma) \triangleq \{(R_1, R_1, R_1) \in R_d(P_1, P_2 | \Sigma) \} . \]

Furthermore, define the secrecy capacity \( C_{ds}(P_1, P_2 | \Sigma) \) by

\[ C_{ds}(P_1, P_2 | \Sigma) \triangleq \max_{(R_0, R_1) \in C_{ds}(P_1, P_2 | \Sigma)} R_1 \]

\[ = \max_{(0, R_1) \in C_{ds}(P_1, P_2 | \Sigma)} R_1 \]

We obtain the following two results as corollaries of Theorem 6.

**Corollary 12:** For any Gaussian relay channel, we have

\[ C^{(in)}_{ds}(P_1, P_2 | \Sigma) \subseteq C_{ds}(P_1, P_2 | \Sigma) \subseteq C^{(out)}_{ds}(P_1, P_2 | \Sigma) , \]

where

\[ C^{(in)}_{ds}(P_1, P_2 | \Sigma) \triangleq \{(R_0, R_1) : R_0, R_1 \geq 0 , \]

\[ R_0 \leq \max_{0 \leq \eta \leq 1} \min \left\{ C \left( \frac{\theta P_1 + P_2 + 2 \sqrt{\theta P_1 P_2}}{\theta P_1 + N_1} \right) , \right. \]

\[ C \left( \frac{\theta P_1 + P_2 + 2 \sqrt{\theta P_1 P_2}}{\theta P_1 + N_1} \right) , \]

\[ R_1 \leq \left[ C \left( \frac{\theta P_1}{\eta N_1} \right) - C \left( \frac{\theta P_1}{\eta N_2} \right) \right] ^+ , \]

for some \( 0 \leq \theta \leq 1 \} . \]

\[ C^{(out)}_{ds}(P_1, P_2 | \Sigma) \triangleq \{(R_0, R_1) : R_0, R_1 \geq 0 , \]

\[ R_0 \leq \max_{0 \leq \eta \leq 1} \min \left\{ C \left( \frac{\theta P_1 + P_2 + 2 \sqrt{\theta P_1 P_2}}{\theta P_1 + N_1} \right) , \right. \]

\[ C \left( \frac{\theta P_1 + P_2 + 2 \sqrt{\theta P_1 P_2}}{\theta P_1 + N_1} \right) , \]

\[ R_1 \leq \left[ C \left( \frac{\theta P_1}{\eta N_1} \right) - C \left( \frac{\theta P_1}{\eta N_2} \right) \right] ^+ , \]

for some \( 0 \leq \theta \leq 1 \} . \]

In particular, if \( N_1 \leq N_2 \) and \( \rho = \sqrt{\frac{N_1}{N_2}} \), we have

\[ C^{(in)}_{ds}(P_1, P_2 | \Sigma) = C_{ds}(P_1, P_2 | \Sigma) = C^{(out)}_{ds}(P_1, P_2 | \Sigma) . \]

**Corollary 13:** For any Gaussian relay channel, we have

\[ R^{(in)}_{dle}(P_1, P_2 | \Sigma) \subseteq R_{dle}(P_1, P_2 | \Sigma) \subseteq R^{(out)}_{dle}(P_1, P_2 | \Sigma) , \]

where

\[ R^{(in)}_{dle}(P_1, P_2 | \Sigma) \triangleq \{(R_1, R_1) : R_1, R_1 \geq 0 , \]

\[ R_1 \leq C \left( \frac{P_1}{N_1} \right) , \]

\[ R_1 \leq \left[ C \left( \frac{P_1}{\eta N_1} \right) - C \left( \frac{P_1}{\eta N_2} \right) \right] ^+ . \]

**VI. PROOFS OF THE THEOREMS**

In this section we state proofs of Theorems 1-6 stated in the sections III and V.

In the first subsection we prove Theorem 1 the inclusion \( R^{(in)}_{d}(\Gamma) \subseteq R_{d}(\Gamma) \) in Theorem 3 and the inclusion \( R^{(in)}_{d}(\Gamma) \subseteq R_{s}(\Gamma) \) in Theorem 5. In the second subsection we prove Theorem 2 the inclusion \( R_{d}(\Gamma) \subseteq R^{(out)}_{d}(\Gamma) \) in Theorem 3 and the inclusion \( R_{s}(\Gamma) \subseteq R^{(out)}_{s}(\Gamma) \) in Theorem 5 Proof of Theorem 4 is given in the third subsection.

**A. Derivations of the Inner Bounds**

We first state an important lemma to derive inner bounds. To describe this lemma, we need some preparations. Let \( T_n, J_n, \) and \( L_n \) be three message sets to be transmitted by the sender. Let \( T_n, J_n, \) and \( L_n \) be uniformly distributed random...
variable over $\mathcal{T}_n$, $\mathcal{J}_n$ and $\mathcal{L}_n$ respectively. Elements of $\mathcal{T}_n$ and $\mathcal{J}_n$ are directed to the receiver and relay. Elements of $\mathcal{L}_n$ are only directed to the receiver. Encoder function $f_n$ is a one to one mapping from $T_n \times J_n \times K_n$ to $X^n$. Using the decoder function $\psi_n$, the receiver outputs an element of $T_n \times J_n \times K_n$ from a received message of $Y^n$. Using the decoder function $\phi_n$, the relay outputs an element of $T_n \times J_n \times K_n$ from a received message of $Z^n$. Formal definitions of $\psi_n$ and $\phi_n$ are $\psi_n : Y^n \to T_n \times J_n \times K_n$, $\phi_n : Z^n \to T_n \times J_n$. We define the average error probability of decoding at the receiver over $T_n \times J_n \times K_n$ in the same manner as the definition of $\lambda_1(n)$ and use the same notation for this error probability. We also define the average error probability of decoding at the relay over $T_n \times J_n$ in the same manner as the definition of $\lambda_2(n)$ and use the same notation for this probability. Then, we have the following lemma.

Lemma 1: Choose $(U, X, S) \in \mathcal{P}_1$ such that $I(X; Y | Y S) \geq I(X; Z | Y S)$. Then, there exists a sequence of quadruples $\{(f_n, (g_i), \psi_n, \phi_n)\}_{i=1}^\infty$ such that

$$\lim_{n \to \infty} \alpha_1(n) = \lim_{n \to \infty} \alpha_2(n) = 0,$$

$$\lim_{n \to \infty} \frac{1}{n} \log |\mathcal{T}_n| = \min\{I(Y; U S), I(Z; U S)\},$$

$$\lim_{n \to \infty} \frac{1}{n} \log |\mathcal{J}_n| = I(X; Z U S),$$

$$\lim_{n \to \infty} \frac{1}{n} \log |\mathcal{L}_n| = I(X; Y U S) - I(X; Z U S),$$

$$\lim_{n \to \infty} \frac{1}{n} H(L_n | Z^n) \geq I(X; Y U S) - I(X; Z U S).$$

The above lemma is proved by a combination of two coding techniques. One is the method that Csizsár and Körner [3] used for deriving an inner bound of the capacity regions of the broadcast channel with confidential messages and the other is the method that Cover and El Gamal [11] developed for deriving a lower bound of the capacity of the relay channel. Outline of proof of this lemma is given in Appendix A.

Proof of $R_d^{(\infty)}(\Gamma) \subseteq R_d(\Gamma)$: Set

$$I_0 \triangleq \min\{I(Y; U S), I(Z; U S)\},$$

$$I_1 \triangleq I(X; Y U S), I_2 \triangleq I(X; Z U S).$$

We consider the case that $I_1 \geq I_2$. The region $R(U, X, S|\Gamma)$ in this case is depicted in Fig. 4. From the shape of this region it suffices to show that for every

$$\alpha \in [0, \min\{I(Y; U S), I(Z; U S)\}],$$

the following $(R_0, R_1, R_e)$ is achievable:

$$R_0 = \min\{I(Y; U S), I(Z; U S)\} - \alpha,$$

$$R_1 = I(X; Y U S) + \alpha,$$

$$R_e = I(X; Y U S) - I(X; Z U S).$$

Choose $T'_n$ and $T''_n$ such that

$$T_n = T'_n \times T''_n,$$

$$\lim_{n \to \infty} \frac{1}{n} \log |T'_n| = \min\{I(Y; U S), I(Z; U S)\} - \alpha.$$

We take

$$M_n = T'_n, \ K_n = T''_n \times J_n \times L_n.$$ 

Then, by Lemma 1 we have

$$\lim_{n \to \infty} \alpha_1(n) = \lim_{n \to \infty} \alpha_2(n) = 0,$$

$$\lim_{n \to \infty} \frac{1}{n} \log |K_n| = I(X; Y U S) + \alpha,$$

$$\lim_{n \to \infty} \frac{1}{n} \log |M_n| = \min\{I(Y; U S), I(Z; U S)\} - \alpha,$$

$$\lim_{n \to \infty} \frac{1}{n} H(K_n | Z^n) \geq \lim_{n \to \infty} \frac{1}{n} H(L_n | Z^n) \geq I(X; Y U S) - I(X; Z U S).$$

To help understanding the above proof, information quantities contained in the transmitted messages are shown in Fig. 5.

Proof of Theorem 1: Since $R_d^{(\infty)}(\Gamma) \subseteq R_d(\Gamma)$, we have

Proof of $R_d^{(\infty)}(\Gamma) \subseteq R_d(\Gamma)$: Choose $(U, V, X, S) \in \mathcal{Q}_1$. The joint distribution of $(U, V, X, S)$ is given by

$$p_{U V X S}(u, v, x, s) = p_{U V}(u, s)p_{X|V}(x|v)\cdot (u, v, x, s) \in U \times V \times X \times S.$$

Consider the discrete memoryless channels with input alphabet $V \times S$ and output alphabet $Y \times Z$, and stochastic matrices defined by the conditional distribution of $(Y, Z)$ given $V, S$ having the form

$$\Gamma(y, z|v, s) = \sum_{x \in X} \Gamma(y, z|x, s)p_{X|V}(x|v).$$
Any deterministic encoder \( f_n' : K_n \times M_n \rightarrow \mathcal{Y}^n \) for this new RCC determines a stochastic encoder \( f_n \) for the original RCC by the matrix product of \( f_n' \) with the stochastic matrix given by \( p_{X|Y} = \{ p_{X|Y}(x|y) \}_{(x,y) \in \mathcal{X} \times \mathcal{Y}} \). Both encoders yield the same stochastic connection of messages and received sequences, so the assertion follows by applying the result of the first inclusion in Theorem 3 to the new RCC.

Cardinality bounds of auxiliary random variables in \( P_1 \) and \( Q_1 \) can be proved by the argument that Csiszár and Körner [3] developed in Appendix in their paper.

### B. Derivations of the Outer Bounds

In this subsection we derive the outer bounds stated in Theorems 2.15. We first remark here that cardinality bounds of auxiliary random variables in \( P_2 \) and \( Q_2 \) in the outer bounds can be proved by the argument that Csiszár and Körner [3] developed in Appendix in their paper.

The following lemma is a basis on derivations of the outer bounds.

**Lemma 2:** We assume \((R_0, R_1, R_e)\) is achievable. Then, we have

\[
\begin{align*}
nR_0 &\leq \min\{I(Y^n;M_n), I(Z^n;M_n)\} + n\delta_{1,n} \\
nR_1 &\leq I(K_n; Y^n|M_n) + n\delta_{2,n} \\
n(R_0 + R_1) &\leq I(Y^n; K_nM_n) + n\delta_{3,n} \\
nR_e &\leq nR_1 + n\delta_{4,n} \\
nR_e &\leq I(K_n; Y^n|M_n) - I(K_n; Z^n|M_n) + n\delta_{5,n}
\end{align*}
\]

where \(\{\delta_{i,n}\}_{i=1}^{\infty}, i = 1, 2, 3, 4, 5\) are sequences that tend to zero as \( n \to \infty \).

**Proof:** The above Lemma can be proved by a standard converse coding argument using Fano’s Lemma. We omit the detail. A similar argument is found in Csiszár and Körner [3] in Section V in their paper. We first prove \(\mathcal{R}_d(\Gamma) \subseteq \mathcal{R}_d^{(out)}(\Gamma)\). From Lemma 2 it suffices to derive upper bounds of

\[
I(Z^n;M_n), I(Y^n;M_n), I(K_n; Y^n|M_n), I(Y^n; K_nM_n), I(K_n; Y^n|M_n) - I(K_n; Z^n|M_n).
\]

For upper bound of the above five quantities, we have the following Lemma.

**Lemma 3:** Suppose that \( f_n \) is a deterministic encoder. Set

\[
U_i \triangleq M_nY^{-1}Z^{-1}, \quad i = 1, 2, \cdots, n.
\]

For \( i = 1, 2, \cdots, n, U_i, X_iS_i, \) and \( Y_iZ_i \) form a Markov chain \( U_i \rightarrow X_iS_i \rightarrow Y_iZ_i \) in this order. Furthermore, we have

\[
\begin{align*}
I(Y^n;M_n) &\leq \sum_{i=1}^{n} I(Y_i; U_iS_i) \quad (18) \\
I(Z^n;M_n) &\leq \sum_{i=1}^{n} I(Z_i; U_i|S_i) \quad (19) \\
I(Y^n;K_nM_n) &\leq \sum_{i=1}^{n} I(Y_i; X_iS_i) \quad (20) \\
I(K_n; Y^n|M_n) &\leq \sum_{i=1}^{n} I(X_i; Y_iZ_i|U_iS_i) \quad (21)
\end{align*}
\]

\[
I(K_n; Y^n|M_n) - I(K_n; Z^n|M_n) \leq \sum_{i=1}^{n} I(X_i; Y_iZ_i|U_iS_i). \quad (22)
\]

**Proof of Lemma 3** is given in Appendix B.

**Proof of Theorem 2.5** We assume that \((R_0, R_1, R_e)\) is achievable. Let \( Q \) be a random variable independent of \( K_nM_nX^nY^n \) and uniformly distributed over \( \{1, 2, \cdots, n\} \). Set

\[
X \triangleq X_Q, S \triangleq S_Q, Y \triangleq Y_Q, Z \triangleq Z_Q
\]

Furthermore, set

\[
U \triangleq U_QQ = Z^{Q-1}Y^{Q-1}M_nQ.
\]

Note that \( UXSYZ \) satisfies a Markov chain \( U \rightarrow X \rightarrow S \rightarrow Y \rightarrow Z \).

By Lemmas 2 and 3 we have

\[
\begin{align*}
R_0 &\leq \min\{I(Y;US|Q), I(Z;U|SQ)\} + \delta_{1,n} \\
R_1 &\leq \min\{I(Y;US), I(Z;U|S)\} + \delta_{1,n} \\
R_e &\leq (R_1 + \delta_{4,n}) + \delta_{5,n}.
\end{align*}
\]

Using memoryless character of the channel it is straightforward to verify that \( U \rightarrow X \rightarrow S \rightarrow Y \) and that the conditional distributions of and given coincide with the corresponding channel matrix. Hence by letting \( n \to \infty \) in (25), we obtain \((R_0, R_1, R_e) \in \mathcal{R}_d(\Gamma)\).

Next, we prove the inclusions \(\mathcal{R}_d(\Gamma) \subseteq \mathcal{R}_d^{(out)}(\Gamma)\) and \(\mathcal{R}_d(\Gamma) \subseteq \mathcal{R}_d^{(out)}(\Gamma)\). From Lemma 3 it suffices to derive upper bounds of the following five quantities:

\[
\begin{align*}
I(Z^n;M_n), I(Y^n;M_n), I(K_n; Y^n|M_n), I(K_n; Y^n|M_n) + I(Z^n;M_n), I(K_n; Y^n|M_n) - I(K_n; Z^n|M_n).
\end{align*}
\]

Since

\[
\begin{align*}
I(K_n; Y^n|M_n) + I(Z^n;M_n) \\
= I(K_n; Y^n|M_n) - I(K_n; Z^n|M_n) + I(K_nM_n; Z^n),
\end{align*}
\]

we derive an upper bound of (26) by estimating upper bounds of \( I(K_nM_n; Z^n) \) and (27).

The following two lemmas are key results to derive the outer bounds.

**Lemma 4:** Suppose that \( f_n \) is a deterministic encoder. Set

\[
U_i \triangleq Y^n_{i+1}Z^{-1}M_n, \quad i = 1, 2, \cdots, n.
\]

For \( i = 1, 2, \cdots, n, U_i, X_iS_i, \) and \( Y_i \) form a Markov chain
U_i \rightarrow X_i Z_i S_i \rightarrow Y_i in this order. Furthermore, we have
\[ I(Y^n; M_n) \leq \sum_{i=1}^{n} I(Y_i; U_i S_i) \]  
\[ I(Z^n; M_n) \leq \sum_{i=1}^{n} I(Z_i; U_i S_i) \]  
\[ I(Y^n; K_n M_n) \leq \sum_{i=1}^{n} I(Y_i; X_i U_i S_i) \]  
\[ I(Z^n; K_n M_n) \leq \sum_{i=1}^{n} I(Z_i; X_i U_i S_i) \]  
\[ I(K_n; Y^n|M_n) - I(K_n; Z^n|M_n) \]
\[ \leq \sum_{i=1}^{n} \{ I(X_i S_i; Y_i U_i) - I(X_i S_i; Z_i U_i) \} \]
\[ + I(U_i; Z_i|X_i S_i) \]
\[ = \sum_{i=1}^{n} \{ I(X_i; Y_i|U_i S_i) - I(X_i; Z_i|U_i S_i) \} \]
\[ + \zeta(U_i S_i Y_i Z_i) + I(U_i; Z_i|X_i S_i) \]  
where \( \hat{\delta}_{i,n} \triangleq \max\{\delta_{1,n} + \delta_{2,n}, \delta_{3,n}\} \).

Proof of \( R_d(\Gamma) \subseteq R_d^{(\text{out})}(\Gamma) \): We assume that \((R_0, R_1, R_e)\) is achievable. Let \( Q, X, Y, Z, S \) be the same random variables as those in the proof of Theorem 2. We set
\[ U \triangleq U_Q Q = Y^{Q-1} Z_{Q+1} M_n Q. \]

Note that \( U X S Y Z \) satisfies a Markov chain \( U \rightarrow X S Y Z \rightarrow Y \). Furthermore, if \( \Gamma \) belongs to the independent class, we have
\[ Z \rightarrow X S \rightarrow Y, \ U \rightarrow X S \rightarrow Z, \]
which together with \( U \rightarrow X S Y Z \rightarrow Y \) yields
\[ U \rightarrow X S Y Z \rightarrow Y. \]

By Lemmas 2 and 5 we have
\[ R_0 \leq \min\{I(Y; U S), I(Z; U S)\} + \delta_{1,n} \]
\[ R_0 + R_1 \leq I(X; Y U S) + \min\{I(Y; U S), I(Z; U S)\} + \delta_{3,n} \]
\[ R_e \leq R_1 + \delta_{4,n} \]
\[ R_e \leq I(X; Y U S) - I(X; Z U S) + \delta_{5,n} \]  
By Lemma 6, we set
\[ U_i \rightleftharpoons Y^{i-1} Z^n_{i+1} M_n, \ i = 1, 2, \cdots, n. \]

For \( i = 1, 2, \cdots, n, U_i, X_i S_i Z_i, \) and \( Y_i \) form a Markov chain \( U_i \rightarrow X_i Z_i S_i \rightarrow Y_i \) in this order. Furthermore, we have
\[ I(Y^n; M_n) \leq \sum_{i=1}^{n} I(Y_i; U_i S_i), \]  
\[ I(Z^n; M_n) \leq \sum_{i=1}^{n} I(Z_i; U_i S_i), \]  
\[ I(Y^n; K_n M_n) \leq \sum_{i=1}^{n} I(Y_i; X_i U_i S_i), \]  
\[ I(Z^n; K_n M_n) \leq \sum_{i=1}^{n} I(Z_i; X_i U_i S_i), \]
\[ I(K_n; Y^n|M_n) - I(K_n; Z^n|M_n) \]
\[ \leq \sum_{i=1}^{n} \{ I(X_i S_i; Y_i U_i) - I(X_i S_i; Z_i U_i) \} \]
\[ + I(U_i; Z_i|X_i S_i) \]
\[ = \sum_{i=1}^{n} \{ I(X_i; Y_i|U_i S_i) - I(X_i; Z_i|U_i S_i) \} \]
\[ + \zeta(U_i S_i Y_i Z_i) + I(U_i; Z_i|X_i S_i) \]  
Proofs of Lemmas 4 and 5 are given in Appendixes D and E, respectively.

Proof of \( R_d(\Gamma) \subseteq R_d^{(\text{out})}(\Gamma) \): We assume that \((R_0, R_1, R_e)\) is achievable. Let \( Q, X, Y, Z, S \) be the same random variables as those in the proof of Theorem 2. We set
\[ U \triangleq U_Q Q = Y^{Q-1} Z_{Q+1} M_n Q. \]

Note that \( U X S Y Z \) satisfies a Markov chain \( U \rightarrow X S Y Z \rightarrow Y \). By Lemmas 2 and 5 we have
\[ R_0 \leq \min\{I(Y; U S), I(Z; U S)\} + \delta_{1,n} \]
\[ R_0 + R_1 \leq I(X; Y U S) + \min\{I(Y; U S), I(Z; U S)\} + \delta_{3,n} \]
\[ R_e \leq R_1 + \delta_{4,n} \]
\[ R_e \leq I(X; Y U S) - I(X; Z U S) + \delta_{5,n} \]  

Proof of Lemma 6 is given in Appendix C.

Proof of \( R_e(\Gamma) \subseteq R_e^{(\text{out})}(\Gamma) \): Let \( Q, X, Y, Z, S, U \) be the same random variables as those in the proof of \( R_d(\Gamma) \subseteq R_d^{(\text{out})}(\Gamma) \). We further set \( V \triangleq U S K_n \). Note that \( U V X S Y Z \) satisfies the following Markov chains
\[ U \rightarrow V \rightarrow X S Y Z \rightarrow Y, \ U S \rightarrow V X \rightarrow Z, \]
\[ U S \rightarrow V \rightarrow X. \]
By Lemmas 2 and 6 we have
\[ R_0 \leq \min\{I(Y;US),I(Z;U|S)\} + \delta_{1,n} \]
\[ R_0 + R_1 \leq I(V;Y|US) + \min\{I(Y;US),I(Z;U|S)\} + \delta_{3,n} \]
\[ R_e \leq R_1 + \delta_{3,n} \]
\[ R_e \leq I(V;Y|US) - I(X;Z|US) + \delta_{5,n} \]
By letting \( n \to \infty \), we conclude that \((R_0, R_1, R_e) \in \mathcal{R}_n^{(out)}(\Gamma)\).

C. Computation of Inner and Outer Bounds for the Gaussian Relay Channel

In this subsection we prove Theorem 6. Let \((\xi_1, \xi_2)\) be a zero mean Gaussian random vector with covariance \(\Sigma\) defined in Section V. By definition, we have
\[ \xi_2 = \rho \sqrt{\frac{N_2}{N_1}} \xi_1 + \xi_{2|1} \]
where \(\xi_{2|1}\) is a zero mean Gaussian random variable with variance \((1 - \rho^2)N_2\) and independent of \(\xi_1\). We consider the Gaussian relay channel specified by \(\Sigma\). For two input random variables \(X\) and \(S\) of this Gaussian relay channel, output random variables \(Y\) and \(Z\) are given by
\[ Y = X + S + \xi_1 \]
\[ Z = X + \xi_2 = X + \rho \sqrt{\frac{N_2}{N_1}} \xi_1 + \xi_{2|1} \]
Define two sets of random variables by
\[ \mathcal{P}(P_1, P_2) \triangleq \{(U,X,S) : E[X^2] \leq P_1, E[S^2] \leq P_2, U \to XS \to YZ\} \]
\[ \mathcal{P}_G(P_1, P_2) \triangleq \{(U,X,S) : U,X,S\text{ are zero mean Gaussian random variables, } E[X^2] \leq P_1, E[S^2] \leq P_2, U \to XS \to YZ\} \]
Set
\[ \mathcal{R}_d^{(out)}(P_1, P_2|\Sigma) \]
\[ \triangleq \{(R_0, R_1, R_e) : R_0, R_1, R_e \geq 0, R_0 \leq \min\{I(Y;US),I(Z;U|S)\}, R_1 \leq I(X;Y|US), R_0 + R_1 \leq I(XS;Y), R_e \leq R_1, R_e \leq I(X;Y|ZUS), \}
\[ \text{for some } (U,X,S) \in \mathcal{P}(P_1, P_2).\} \]
\[ \mathcal{R}_d^{(in)}(P_1, P_2|\Sigma) \]
\[ \triangleq \{(R_0, R_1, R_e) : R_0, R_1, R_e \geq 0, R_0 \leq \min\{I(Y;US),I(Z;U|S)\}, R_1 \leq I(X;Y|US), R_e \leq R_1, R_e \leq I(X;Y|US) - I(X;Z|US), \}
\[ \text{for some } (U,X,S) \in \mathcal{P}_G(P_1, P_2).\} \]
Then, we have the following.

**Theorem 7:** For any Gaussian relay channel we have
\[ \hat{\mathcal{R}}_d^{(in)}(P_1, P_2|\Sigma) \subseteq \mathcal{R}_d(P_1, P_2|\Sigma) \subseteq \mathcal{R}_d^{(out)}(P_1, P_2|\Sigma) \]

**Proof:** The first inclusion can be proved by a method quite similar to that in the case of discrete memoryless channels. The second inclusion can be proved by a method quite similar to that in the proof of Theorem 2. We omit the detail of the proof of those two inclusions.

It can be seen from Theorem 2 that to prove Theorem 6 it suffices to prove
\[ \mathcal{R}_d^{(in)}(P_1, P_2|\Sigma) \subseteq \mathcal{R}_d^{(in)}(P_1, P_2|\Sigma), \]
\[ \mathcal{R}_d^{(out)}(P_1, P_2|\Sigma) \subseteq \mathcal{R}_d^{(out)}(P_1, P_2|\Sigma). \]
Proof of (49) is straightforward. To prove (50), we need some preparation. Set
\[ a \triangleq \frac{N_2 - 2\rho\sqrt{N_1 N_2}}{N_1 + N_2 - 2\sqrt{N_1 N_2}}. \]
Define random variables \(\tilde{Y}, \tilde{\xi}_1,\) and \(\tilde{\xi}_2\) by
\[ \tilde{Y} \triangleq aY + \tilde{a}Z, \]
\[ \tilde{\xi}_1 \triangleq a\xi_1 + \tilde{a}\xi_2 = \frac{(1-\rho^2)N_2\xi_1 + (N_1 - \rho\sqrt{N_1 N_2})\xi_{2|1}}{N_1 + N_2 - 2\rho\sqrt{N_1 N_2}}, \]
\[ \tilde{\xi}_2 \triangleq \xi_1 - \xi_2 = (1 - \rho) \sqrt{\frac{N_2}{N_1}} \xi_1 - \xi_{2|1}. \]
Let \(\tilde{N}_1 = E[\tilde{\xi}_1^2], i = 1, 2\). Then, by simple computation we can show that \(\tilde{\xi}_1\) and \(\tilde{\xi}_2\) are independent Gaussian random variables and
\[ \tilde{N}_1 = \frac{(1-\rho^2)N_1 N_2}{N_1 + N_2 - 2\rho\sqrt{N_1 N_2}}, \]
\[ \tilde{N}_2 = N_1 + N_2 - 2\rho\sqrt{N_1 N_2}. \]
We have the following relations between \(\tilde{Y}, Y,\) and \(Z:\)
\[ \tilde{Y} = X + aS + \tilde{\xi}_1, \]
\[ Y = \tilde{Y} + \tilde{a}S + \tilde{\xi}_2, \]
\[ Z = \tilde{Y} - aS - \tilde{\xi}_2. \]

The following is a useful lemma to prove (50).

**Lemma 7:** Suppose that \((U,X,S) \in \mathcal{P}(P_1, P_2).\) Let \(X(s)\) be a random variable with a conditional distribution of \(X\) for given \(S = s\). \(E_{X(s)}[\cdot]\) stands for the expectation with respect to the conditional distribution of \(X(s)\). Then, there exists a pair \((\alpha, \beta) \in [0, 1]^2\) such that
\[ E_S \left( E_{X(s)}[X(S)]^2 \right) = \alpha P_1, \]
\[ h(Y|S) \leq \frac{1}{2} \log \left( (2\pi e)(\alpha P_1 + \tilde{N}_1) \right), \]
\[ h(Z|S) \leq \frac{1}{2} \log \left( (2\pi e)(\alpha P_1 + \tilde{N}_2) \right), \]
\[ h(Y) \leq \frac{1}{2} \log \left( (2\pi e)(P_1 + P_2 + 2\sqrt{\alpha P_1 P_2 + N_1}) \right), \]
\[ h(\tilde{Y}|US) = \frac{1}{2} \log \left( (2\pi e)(\beta \alpha P_1 + \tilde{N}_1) \right), \]
\[ h(Y|US) \geq \frac{1}{2} \log \left( (2\pi e)(\beta \alpha P_1 + N_1) \right), \]
\[ h(Z|US) \geq \frac{1}{2} \log \left( (2\pi e)(\beta \alpha P_1 + N_2) \right). \]

Proof of Lemma 7 is given in Appendix F. Using this lemma, we can prove Theorem 6.
Proof of Theorem \[\text{(51)}\] We first prove \[\text{(49)}\]. Choose \((U, X, S) \in \mathcal{P}_G\) such that
\[
\mathbf{E}[X^2] = P_1, \quad \mathbf{E}[S^2] = P_2,
\]
\[
U = \sqrt{\frac{\theta P_1 + P_2 + 2\sqrt{\theta P_2 P_2}}{\theta P_1 + P_2 + 2}}, \quad X = U + \tilde{X},
\]
where \(\tilde{U}\) and \(\tilde{X}\) are zero mean Gaussian random variables with variance \(\theta P_1\) and \(\theta P_2\), respectively. The random variables \(X, S, \tilde{U}, \) and \(\tilde{X}\) are independent. For the above choice of \((U, X, S)\), we have
\[
I(Y; US) = C \left( \frac{\theta P_1 + P_2 + 2\sqrt{\theta P_2 P_2}}{\theta P_1 + P_2 + 2} \right),
\]
\[
I(Z; U|S) = C \left( \frac{\theta P_1}{N_1} \right),
\]
\[
I(X; Y|US) = C \left( \frac{\theta P_1}{N_2} \right), \quad I(X; Z|US) = C \left( \frac{\theta P_2}{N_2} \right).
\]
Thus, \[\text{(49)}\] is proved. Next, we prove \[\text{(50)}\]. By Lemma \[\text{(7)}\] we have
\[
I(Y; US) = h(Y) - h(Y|US) \leq C \left( \frac{(1-\beta)P_1 + P_2 + 2\sqrt{\alpha P_1 P_2}}{\alpha P_1 + P_2 + 2} \right), \quad (52)
\]
\[
I(Z; U|S) = h(Z|S) - h(Z|US) \leq C \left( \frac{\beta P_1}{N_1} \right),
\]
\[
I(X; Y|US) = h(Y) - h(Y|US) \leq C \left( \frac{(1-\beta)P_1 + P_2 + 2\sqrt{\alpha P_1 P_2}}{\alpha P_1 + P_2 + 2} \right), \quad (52)
\]
\[
I(X; Z|US) = h(Z) - h(Z|US) \leq C \left( \frac{\beta P_2}{N_2} \right),
\]
\[
I(X; YZ|US) = h(Y) - h(Y|US) \leq C \left( \frac{(1-\beta)P_1 + P_2 + 2\sqrt{\alpha P_1 P_2}}{\alpha P_1 + P_2 + 2} \right), \quad (52)
\]
\[
I(X; Y|US) = h(Y) - h(Y|US) \leq C \left( \frac{(1-\beta)P_1 + P_2 + 2\sqrt{\alpha P_1 P_2}}{\alpha P_1 + P_2 + 2} \right), \quad (52)
\]
\[
\text{where } \text{(52)} \text{ follows from}
\]
\[
h(Z|\tilde{Y}US) = h(Z|\tilde{Y}XS) = h(Z|\tilde{Y}S) = \frac{1}{2} \log \left( (2\pi e)\alpha^2 \hat{N}_2 \right).
\]
From \[\text{(55)}\] and \[\text{(57)}\], we have
\[
I(X; Y|US) \leq C \left( \frac{(1-\beta)P_1 + P_2 + 2\sqrt{\alpha P_1 P_2}}{\alpha P_1 + P_2 + 2} \right) - C \left( \frac{\beta P_2}{N_2} \right). \quad (58)
\]
Here we transform the variable pair \((\alpha, \beta) \in [0,1]^2\) into \((\eta, \theta) \in [0,1]^2\) in the following manner:
\[
\theta = \beta \alpha, \quad \eta = 1 - \frac{\alpha}{\theta} = \frac{\alpha - \theta}{1 - \theta}. \quad (59)
\]
This map is a bijection because from \[\text{(59)}\], we have
\[
\alpha = 1 - \eta \theta \geq \theta, \quad \beta = \frac{\theta}{\alpha}, \quad (60)
\]
Combining \[\text{(52)}\], \[\text{(54)}\], \[\text{(57)}\], \[\text{(58)}\], and \[\text{(60)}\], we have \[\text{(50)}\].

APPENDIX

A. Outline of Proof of Lemma \[\text{(7)}\]

Let
\[
\mathcal{T}_n = \{1, 2, \ldots, 2^{nR_0(n)}\}, \quad \mathbb{L}_n = \{1, 2, \ldots, 2^{nR_1(n)}\}, \quad \mathbb{J}_n = \{1, 2, \ldots, 2^{nR_2(n)}\},
\]
where \(|x|\) stands for the integer part of \(x\) for \(x > 0\). Furthermore, set
\[
\mathcal{W}_n \triangleq \{1, 2, \ldots, 2^{nR(n)}\}.
\]

We consider a transmission over \(B\) blocks, each with length \(n\). For each \(i = 0, 1, \ldots, B - 1\), let \((w_i, t_i, j_i, l_i) \in \mathcal{W}_n \times \mathbb{T}_n \times \mathbb{J}_n \times \mathbb{L}_n\) be a quadruple of messages to be transmitted at the \(i\)th block. For \(i = 0\), the constant message vector \((w_0, t_0, j_0, l_0) = (1, 1, 1, 1)\) is transmitted. For fixed \(n\), the rate triple \((R_0(n), R_1(n), R_2(n))\) approaches \((R_0, R_1, R_2)\) as \(B \to \infty\).

We use random codes for the proof. Fix a joint probability distribution of \((U, S, X, Y, Z)\):
\[
p_{USXYZ}(u, s, x, y, z) = p_{SU}(u|s)p_{X|US}(x|u, s)\Gamma(y, z|x, s),
\]
where \(U\) is an auxiliary random variable that stands for the information being carried by the message that is to be sent to the receiver and the relay. In the following, we use \(A_e\) to denote the jointly \((e, t)\)-typical set based on this distribution. A formal definition of \(A_e\) is in \([15, \text{Chapter 14.2}]\).

Random Codebook Generation: We generate a random code book by the following steps.

1. Generate \(2^{nR(n)}\) i.i.d. \(s \in \mathcal{S}^n\) each with distribution \(\Pi_{s=1} p_{S}(s_i)\). Index \(s(w_i), w_i \in \mathcal{W}_n\).
2. For each \(s(w_i)\), generate \(2^{nR_0(n)}\) i.i.d. \(u \in \mathcal{U}^n\) each with distribution \(\Pi_{s=1} p_{U}(u_i|s_i)\). Index \(u(w_i, t_i), t_i \in \mathcal{T}_n\).
3. For each \(u(t_i, w_i)\) and \(s(w_i)\), generate \(2^{nR_1(n)}\) i.i.d. \(x \in \mathcal{X}^n\) each with distribution \(\Pi_{s=1} p_{X}(x_i|s_i, u_i)\). Index \(x(w_i, t_i, j_i, l_i), (w_i, t_i, j_i, l_i) \in \mathcal{W}_n \times \mathbb{T}_n \times \mathbb{J}_n \times \mathbb{L}_n\).

Random Partition of Codebook \(\mathcal{T}_n\): We define the mapping \(\phi : \mathcal{T}_n \to \mathcal{W}_n\) in the following manner. For each \(t_i \in \mathcal{T}_n\), choose \(w_i \in \mathcal{W}_n\) at random according to the uniform distribution over \(\mathcal{W}_n\) and map \(t_i\) to \(w_i\). The random choice is independent for each \(t_i \in \mathcal{T}_n\). For each \(w_i \in \mathcal{W}_n\), define \(\mathcal{T}_n(w_i) \triangleq \{t_i \in \mathcal{T}_n : \phi(t_i) = w_i\}\).

Encoding: At the beginning of block \(i\), let \((t_{i-1}, j_i, l_i)\) be the new message triple to be sent from the sender in block \(i\) and \((t_{i-1}, j_{i-1}, l_{i-1})\) be the message triple to be sent from the sender in previous block \(i - 1\).

At the beginning of block \(i\), the relay has decoded the message \(t_{i-1}\). It then compute \(w_i = \phi(t_{i-1})\) and send the codeword \(s(w_i)\).

Decoding: Let \(y_i \in \mathcal{Y}^n\) and \(z_i \in \mathcal{Z}^n\) be the sequences that the receiver and the relay obtain at the end of block \(i\),
respectively. The decoding procedures at the end of block $i$ are as follows.

1. Decoder 2a at the Relay: The relay declares that the message $t_i$ is sent if there is a unique $\hat{t}_i$ such that 
\[
(s(w_i), u(w_i, \hat{t}_i), z_i) \in A_{SUZ, \epsilon},
\]
where $A_{SUZ, \epsilon}$ is a projection of $A_{\epsilon}$ along with $(U, S, Z)$, that is
\[
A_{SUZ, \epsilon} = \{(s, u, z) \in S^n \times U^n \times Z^n : (s, u, x, y, z) \in A_{\epsilon}, \text{ for some } x, y \in X^n \times Y^n\}.
\]
For projections of $A_{\epsilon}$, similar definition and notations are used for other random variables. It can be shown that the decoding error $e^{(n)}_{2a}$ in this step is small for sufficiently large $n$ if
\[
R_0^{(n)} < I(U; Z|S). \tag{61}
\]

2. Decoder 2b at the Relay: For $(w, t, j) \in W_n \times T_n \times J_n$, set
\[
\mathcal{D}(w, t, j) \triangleq \{x : x = (w, t, j, l) \text{ for some } l \in \mathcal{L}_n\}.
\]
The relay, having known $w_i$ and $\hat{t}_i$, declares that the message $\hat{t}_i$ is sent if there is a unique $\hat{t}_i$ such that
\[
\mathcal{D}(w_i, \hat{t}_i, j_l) \cap A_{SUZ, \epsilon}(s(w_i), u(w_i, \hat{t}_i), z_{i-1}) \neq \emptyset,
\]
where
\[
A_{SUZ, \epsilon}(s(w_i), u(w_i, \hat{t}_i), z_{i-1}) \triangleq \{x \in X^n : (s(w_i), u(w_i, \hat{t}_i), x, z_{i-1}) \in A_{SUZ, \epsilon}\}.
\]
It can be shown that the decoding error $e^{(n)}_{2b}$ in this step is small for sufficiently large $n$ if
\[
r_2^{(n)} < I(X; Z|US). \tag{62}
\]

3. Decoders 1a and 1b at the Receiver: The receiver declares that the message $\hat{w}_i$ is sent if there is a unique $\hat{w}_i$ such that
\[
(s(\hat{w}_i), y_i) \in A_{SY, \epsilon}.
\]
It can be shown that the decoding error $e^{(n)}_{1a}$ in this step is small for sufficiently large $n$ if
\[
r^{(n)} < I(Y; S). \tag{63}
\]
The receiver, having known $w_{i-1}$ and $\hat{w}_i$, declares that the message $\hat{t}_{i-1}$ is sent if there is a unique $\hat{t}_{i-1}$ such that
\[
(s(w_{i-1}), u(w_{i-1}, \hat{t}_{i-1}), y_{i-1}) \in A_{SUZ, \epsilon}
\]
and $\hat{t}_{i-1} \in T_n(\hat{w}_i)$.
It can be shown that the decoding error $e^{(n)}_{1b}$ in this step is small for sufficiently large $n$ if
\[
R_0^{(n)} < I(Y; U|S) + r^{(n)} < I(Y; U|S) + I(Y; S) = I(Y; US). \tag{63}
\]

4. Decoder 1c at the Receiver: The receiver, having known $w_{i-1}, \hat{t}_{i-1}$, declares that the message pair $(\hat{z}_{i-1}, \hat{t}_{i-1})$ is sent if there is a unique $(\hat{z}_{i-1}, \hat{t}_{i-1})$ such that
\[
(s(w_{i-1}), u(w_{i-1}, \hat{t}_{i-1}), x(w_{i-1}, \hat{t}_{i-1}, \hat{z}_{i-1}, \hat{t}_{i-1}, y_{i-1}) \in A_{SUZ, \epsilon}.
\]
It can be shown that the decoding error $e^{(n)}_{1c}$ in this step is small for sufficiently large $n$ if
\[
r_1^{(n)} < I(X; Y|US). \tag{64}
\]
For convenience we show the encoding and decoding processes at the blocks $i - 1, i, i + 1$ in Fig. 6. Summarizing the above argument, it can be shown that for each block $i = 1, 2, \cdots, B - 1$, there exists a sequence of code books
\[
\{(s(w), u(w, t), x(w, t, j, l)) \in W_n \times T_n \times J_n \times \mathcal{L}_n
\}
\]
for $n = 1, 2, \cdots$, such that
\[
\lim_{n \to \infty} \frac{1}{n} \log |\mathcal{T}_n| = \lim_{n \to \infty} R_0^{(n)} = \min\{I(Y; US), I(Z; U|S)\} \tag{65}
\]
\[
\lim_{n \to \infty} \frac{1}{n} \log |\mathcal{J}_n| = \lim_{n \to \infty} r_2^{(n)} = I(X; Z|US) \tag{66}
\]
\[
\lim_{n \to \infty} \frac{1}{n} \log |\mathcal{L}_n| = \lim_{n \to \infty} r_1^{(n)} = I(X; Y|US) - I(X; Z|US) - I(Y; S) = I(Y; US). \tag{67}
\]

**Computation of Security Level:** Suppose that $T_n, L_n, J_n$ are random variables corresponding the messages to be transmitted at the block $i$. For simplicity of notations we omit the
suffix $i$ indicating the block number in those random variables. For each block $i = 1, 2, \cdots, B - 1$, we estimate a lower bound of $H(L_n|Z^n)$. Let $W_n$ be a random variable over $\mathcal{W}_n$ induced by $\phi$ and the uniform random variable $T_n$ over $T_n$ corresponding to the messages to be transmitted at the block $i - 1$. Formally, $W_n = \phi(T_n)$. On lower bound of $H(L_n|Z^n)$, we have the following:

$$H(L_n|Z^n) = H(L_n, J_n|Z^n) - H(J_n|Z^n)$$

By Fano’s inequality, we have

$$\frac{1}{n} H(J_n|Z^n) \leq r_2^{(n)} e_{2b}^{(n)} + \frac{1}{n}. \quad (69)$$

The right member of (69) tends to zero as $n \to \infty$. Hence, it suffices to evaluate a lower bound of $H(L_n, J_n|Z^n, T_n, W_n)$. On this lower bound we have the following chain of inequalities:

$$H(L_n, J_n|Z^n, W_n, T_n, J_n, L_n) = H(J_n, L_n|Z^n, W_n, T_n) - H(Z^n|W_n, T_n)$$

$$\geq \log \{|J_n||L_n\} + H(Z^n|W_n, T_n, J_n, L_n) - H(Z^n|W_n, T_n)$$

$$\geq n[r_2^{(n)} + r_1^{(n)}] - 2$$

$$+ H(Z^n|W_n, T_n, J_n, L_n) - H(Z^n|W_n, T_n). \quad (70)$$

We first estimate $H(Z^n|W_n, T_n, J_n, L_n)$. To this end we set

$$\mathcal{A}^* = \{(w, t, j, l, z) : (s(w), (x(w, t, j, l), z) \in A_{SUZ, \epsilon}\}$$

By definition of $\mathcal{A}^*$, if $(w, t, j, l, z) \in \mathcal{A}^*$, we have

$$- \frac{1}{n} \log p_{Z^n|X^n, \mathcal{S}^n}(z|x(w, t, j, l), s(w)) - H(Z|X, \mathcal{S}) \leq 2\epsilon.$$ 

Then, we have

$$H(Z^n|W_n, T_n, J_n, L_n) \geq n[H(Z|X, \mathcal{S}) - 2\epsilon] + n\Pr\{(W_n, T_n, J_n, L_n, Z^n) \in \mathcal{A}^*\}$$

$$\geq n[H(Z|X, \mathcal{S}) - 2\epsilon] - n(e_{2b}^{(n)} - e_{2b}^{(n)}). \quad (71)$$

Next, we derive an upper bound of $H(Z^n|W_n, T_n)$. To this end we set

$$\mathcal{B}^* = \{(w, t, z) : (s(w), u(w, t), z) \in A_{SUZ, \epsilon}\}$$

By definition of $\mathcal{B}^*$, if $(w, t, z) \in \mathcal{B}^*$, we have

$$- \frac{1}{n} \log p_{Z^n|U^n, S^n}(z|u(w, t), s(w)) - nH(Z|U, S) \leq 2\epsilon.$$ 

Then, we have

$$H(Z^n|W_n, T_n) \leq n[H(Z|U, S) + 2\epsilon] + n\kappa \Pr\{(W_n, T_n, Z^n) \notin \mathcal{B}^*\}$$

$$\leq n[H(Z|U, S) + 2\epsilon] + n\kappa e_{2b}^{(n)}, \quad (72)$$

where $\kappa = \max\{s, u, z\} \log[p_{Z|US}(z|u, s)^{-1}]$. Combining (68) and (72), we have

$$\frac{1}{n} H(L_n|Z^n) \geq r_2^{(n)} + r_1^{(n)} - I(X; Z|US) - 4\epsilon - \frac{3}{n} - \kappa e_{2a}^{(n)} - [r_2^{(n)} + H(Z|XS)]e_{2b}^{(n)}. \quad (73)$$

From (65) - (67), and (73), we have

$$\lim_{n \to \infty} \frac{1}{n} H(L_n|Z^n) \geq I(X; Y|US) - I(X; Z|US) - 4\epsilon.$$ 

Since $\epsilon$ can be made arbitrary small, we have

$$\lim_{n \to \infty} \frac{1}{n} H(L_n|Z^n) \geq I(X; Y|US) - I(X; Z|US).$$

For $n = 1, 2, \cdots$, we choose block $B = B_n$ so that

$$B_n = \left[\left(\max\{e_{1b}^{(n)}, e_1^{(n)}, e_{2a}^{(n)}, e_{2b}^{(n)}\}\right)^{-1/2}\right]. \quad (74)$$

Define $\{g_i\}_{i=1}^{n_B}$ by

$$g_i \triangleq \begin{cases} \phi, & \text{if } i \mod n = 0, \\ \text{constant}, & \text{otherwise}. \end{cases}$$

Then, we obtain the desired result for a sequence of block codes $\left\{\{f_nB_n, \{g_i\}_{i=1}^{n_B}, \psi_nB_n, \varphi_nB_n\}\right\}_{n=1}^{\infty}$. Thus, the proof of Lemma \cite{1} is completed.

\section*{B. Proof of Lemma \cite{3}}

In the following bounding argument we frequently use equalities or data processing inequalities based on the fact that for $i = 1, 2, \cdots, n$, $S_i = g_i(Z_i|Z_i^{-1})$ is a function of $Z_i^{-1}$. The notation $[i]$ stands for $\{1, 2, \cdots, n\} - \{i\}$.

**Proof of Lemma \cite{4}** We first prove (18). We have the following chain of inequalities:

$$I(Y^n; M_n) = H(Y^n) - H(Y^n|M_n)$$

$$= \sum_{i=1}^{n} \{H(Y_i) - H(Y_i|Y_i^{-1}, M_n)\}$$

$$\leq \sum_{i=1}^{n} \{H(Y_i) - H(Y_i|Y_i^{-1}, Z_i^{-1}M_n)\}$$

$$= \sum_{i=1}^{n} \{H(Y_i) - H(Y_i|Y_i^{-1}, Z_i^{-1}S_iM_n)\}$$

$$= \sum_{i=1}^{n} I(Y_i; U_iS_i).$$
Next, we prove (19). We have the following chain of inequalities:

\[
\begin{align*}
I(Z^n; M_n) &= H(Z^n) - H(Z^n | M_n) \\
&= \sum_{i=1}^{n} \{ H(Z_i | Z^{i-1} ) - H(Z_i | Z^{i-1} M_n) \} \\
&\leq \sum_{i=1}^{n} \{ H(Z_i | S_i ) - H(Z_i | Y^{i-1} Z^{i-1} S_i M_n) \} \\
&= \sum_{i=1}^{n} I(Z_i; U_i | S_i).
\end{align*}
\]

Thirdly, we prove (20). We have the following chain of inequalities:

\[
\begin{align*}
I(Y^n; K_n M_n) &= \sum_{i=1}^{n} \{ H(Y_i | Y^{i-1} ) - H(Y_i | Y^{i-1} K_n M_n) \} \\
&= \sum_{i=1}^{n} \{ H(Y_i | Y^{i-1} ) - H(Y_i | Y^{i-1} X^n) \} \quad (75) \\
&\leq \sum_{i=1}^{n} \{ H(Y_i ) - H(Y_i | X^n) \} \\
&= \sum_{i=1}^{n} I(Y_i; X^n) \\
&= \sum_{i=1}^{n} I(Y_i; X_i S_i).
\end{align*}
\]

where

\[ (75): X^n = f_n(K_n, M_n) \text{ and } f_n \text{ is a one-to-one mapping.} \]

\[ (76): Y_i \rightarrow X_i S_i \rightarrow Y^{i-1} X_i. \]

Next, we prove (21). We have the following chain of inequalities:

\[
\begin{align*}
I(K_n; Y^n | M_n) &\leq I(K_n; Y^n Z^n | M_n) \\
&= H(Y^n Z^n | M_n) - H(Y^n Z^n | K_n M_n) \\
&= H(Y^n Z^n | M_n) - H(Y^n Z^n | X^n M_n) \quad (77) \\
&= \sum_{i=1}^{n} \{ H(Y_i Z_i | Y^{i-1} Z^{i-1} M_n) \\
&\quad - H(Y_i Z_i | Y^{i-1} Z^{i-1} S_i X^n M_n) \} \\
&= \sum_{i=1}^{n} \{ H(Y_i Z_i U_i S_i) - H(Y_i Z_i U_i S_i X^n) \} \\
&= \sum_{i=1}^{n} \{ H(Y_i Z_i U_i S_i) - H(Y_i Z_i | X_i S_i) \} \\
&\leq \sum_{i=1}^{n} \{ H(Y_i Z_i U_i S_i) - H(Y_i Z_i | U_i S_i) \} \\
&= \sum_{i=1}^{n} I(X_i; Y_i Z_i U_i S_i),
\end{align*}
\]

where

\[ (77): X^n = f_n(K_n, M_n) \text{ and } f_n \text{ is a one-to-one mapping.} \]

Finally, we prove (22). We have the following chain of inequalities:

\[
\begin{align*}
I(K_n; Y^n | M_n) &= I(K_n; Y^n Z^n | M_n) - I(K_n; Z^n | M_n) \\
&\leq I(K_n; Y^n Z^n | M_n) - I(K_n; Z^n | M_n) \\
&= I(K_n; Y^n Z^n | M_n) \\
&= H(Y^n | Z^n M_n) - H(Y^n | Z^n K_n M_n) \\
&= H(Y^n | Z^n M_n) - H(Y^n | Z^n X^n) \quad (79) \\
&\leq \sum_{i=1}^{n} \{ H(Y_i | Y^{i-1} Z^n M_n) - H(Y_i | Y^{i-1} Z^n X^n) \} \\
&= \sum_{i=1}^{n} \{ H(Y_i | U_i S_i Z_i) - H(Y_i | U_i S_i Z_i X_i) \} \\
&\leq \sum_{i=1}^{n} \{ H(Y_i | U_i S_i Z_i) - H(Y_i | U_i S_i Z_i X_i) \} \\
&= \sum_{i=1}^{n} I(X_i; Y_i | U_i S_i),
\end{align*}
\]

where

\[ (79): X^n = f_n(K_n, M_n) \text{ and } f_n \text{ is a one-to-one mapping.} \]

\[ (80): Y_i \rightarrow Z_i X_i S_i \rightarrow Y^{i-1} Z^n S_i X[i]. \]

Thus, the proof of Lemma 5 is completed.

\[ \blacksquare \]

\[ C. \text{ Proof of Lemma 6} \]

The following is a key lemma to prove Lemma 6.

\[ \text{Lemma 8:} \]

\[ \begin{align*}
I(Y^n; M_n) &\leq \sum_{i=1}^{n} I(Y_i; Y_{i+1}^{n+1} Z^{i-1} M_n S_i), \\
I(Z^n; M_n) &\leq \sum_{i=1}^{n} I(Z_i; Y_{i+1}^{n+1} Z^{i-1} M_n | S_i),
\end{align*} \]

\[ \begin{align*}
I(K_n; Y^n | M_n) + I(Y^n; M_n) &\leq \sum_{i=1}^{n} \{ I(K_n; Y_i | Y_{i+1}^{n+1} Z^{i-1} M_n S_i) \\
&\quad + I(Y_i; Y_{i+1}^{n+1} Z^{i-1} M_n S_i) \}, \\
I(K_n; Y^n | M_n) + I(Z^n; M_n) &\leq \sum_{i=1}^{n} \{ I(K_n; Y_i | Y_{i+1}^{n+1} Z^{i-1} M_n S_i) \\
&\quad + I(Z_i; Y_{i+1}^{n+1} Z^{i-1} M_n S_i) \}, \\
I(Y^n; K_n | M_n) - I(Z^n; K_n | M_n) &\leq \sum_{i=1}^{n} \{ I(K_n; Y_i | Y_{i+1}^{n+1} Z^{i-1} M_n S_i) \\
&\quad - I(K_n; Z_i | Y_{i+1}^{n+1} Z^{i-1} M_n S_i) \}. \quad (85)
\end{align*} \]
Lemma 6 immediately follows from the above lemma. We omit the detail. In the remaining part of this appendix we prove Lemma 8.

Proof of Lemma 8 We first prove (81) and (82). We have the following chains of inequalities:

\[
I(Y^n; M_n) = \sum_{i=1}^{n} \{ H(Y_i | Y_{i+1}^n, M_n) - H(Y_i | Y_{i+1}^n) \} \\
\leq \sum_{i=1}^{n} \{ H(Y_i) - H(Y_i | Y_{i+1}^n Z^{i-1} S_i M_n) \} \\
= \sum_{i=1}^{n} I(Y_i; Y_{i+1}^n Z^{i-1} S_i M_n), \\
I(Z^n; M_n) = \sum_{i=1}^{n} \{ H(Z_i | Z_{i-1}) - H(Z_i | Z_{i-1}, S_i M_n) \} \\
\leq \sum_{i=1}^{n} \{ H(Z_i | S_i) - H(Z_i | Y_{i+1}^n Z^{i-1} S_i M_n) \} \\
= \sum_{i=1}^{n} I(Z_i; Y_{i+1}^n Z^{i-1} M_n | S_i).
\]

Next, we prove (83). We have the following chain of inequalities:

\[
I(K_n; Y^n | M_n) + I(Y^n; M_n) = H(Y^n) - H(Y^n | K_n M_n) \\
= \sum_{i=1}^{n} \{ H(Y_i | Y_{i+1}^n) - H(Y_i | Y_{i+1}^n K_n M_n) \} \\
\leq \sum_{i=1}^{n} \{ H(Y_i) - H(Y_i | Y_{i+1}^n Z^{i-1} S_i K_n M_n) \} \\
= \sum_{i=1}^{n} I(Y_i; Y_{i+1}^n Z^{i-1} S_i K_n M_n) \\
= \sum_{i=1}^{n} \{ I(Y_i; K_n Y_{i+1}^n Z^{i-1} S_i M_n) + I(Y_i; Y_{i+1}^n Z^{i-1} S_i M_n) \}.
\]

Finally, we prove (84) and (85). We first observe the following two identities.

\[
H(Y^n | M_n) - H(Z^n | M_n) = \sum_{i=1}^{n} \{ H(Y_i) - H(Y_{i+1}^n Z^{i-1} M_n) \} - H(Z_i | Y_{i+1}^n Z^{i-1} K_n M_n) \\
= \sum_{i=1}^{n} \{ H(Y_i) - H(Y_i | X_i U_i) \} - H(Z_i | Y_{i+1}^n Z^{i-1} X_i U_i) \\
= I(Y_i; X_i U_i | S_i),
\]

where

(90): \( X^n = f_n(K_n, M_n) \) and \( f_n \) is a one-to-one mapping.

(91): \( Y_i \rightarrow X_i S_i \rightarrow Y_{i+1} X_{[i]} \).

Thus, the proof of Lemma 8 is completed.

D. Proof of Lemma 9

In this appendix we prove Lemma 4. We first present a lemma necessary to prove this lemma.

Lemma 9: Suppose that \( f_n \) is a deterministic encoder. Set \( X^n = f_n(K_n, M_n) \). For any sequence \( \{ U_i \}_{i=1}^{n} \) of random variables, we have

\[
I(Y^n; K_n M_n) \leq \sum_{i=1}^{n} I(Y_i; X_i U_i S_i) \tag{88}
\]

\[
I(Z^n; K_n M_n) \leq \sum_{i=1}^{n} I(Z_i; X_i U_i S_i) \tag{89}
\]

Proof: We first prove (88). We have the following chain of inequalities:

\[
I(Y^n; K_n M_n) = H(Y^n) - H(Y^n | K_n M_n) \\
= H(Y^n) - H(Y^n | X^n) \\
= \sum_{i=1}^{n} \{ H(Y_i) - H(Y_i | Y_{i+1}^n X^n) \} \\
\leq \sum_{i=1}^{n} \{ H(Y_i) - H(Y_i | Y_{i+1}^n X^n) \} \\
= \sum_{i=1}^{n} I(Y_i; X_i U_i S_i) \tag{90}
\]

Finally, we prove (89). We first observe the following two identities.

\[
H(Y^n | M_n) - H(Z^n | M_n) = \sum_{i=1}^{n} \{ H(Y_i | X_i U_i) - H(Z_i | Y_{i+1}^n Z^{i-1} X_i U_i) \} - H(Z_i | Y_{i+1}^n Z^{i-1} X_i U_i) \\
= \sum_{i=1}^{n} \{ H(Y_i | X_i U_i) - H(Y_i | X_i U_i S_i) \} - H(Z_i | Y_{i+1}^n Z^{i-1} X_i U_i) \\
= \sum_{i=1}^{n} I(Y_i; X_i U_i | S_i),
\]

where

(90): \( X^n = f_n(K_n, M_n) \) and \( f_n \) is a one-to-one mapping.

(91): \( Y_i \rightarrow X_i S_i \rightarrow Y_{i+1} X_{[i]} \).
Next, we prove \[89\]. We have the following chain of inequalities:

\[
I(Z^n; K_n M_n) = H(Z^n) - H(Z^n | K_n M_n) = H(Z^n) - H(Z^n | X^n) = n \sum_{i=1}^{n} \{H(Z_i | Z^{i-1}) - H(Z_i | Z^{i-1} X^n)\} = n \sum_{i=1}^{n} \{H(Z_i | S_i) - H(Z_i | S_i)\} = n I(Z_i; X_i U_i | S_i)
\]

where

- (92): \( X^n = f_n(K_n, M_n) \) and \( f_n \) is a one-to-one mapping.
- (93): \( Z_i \rightarrow X_i S_i \rightarrow Z_i^{i-1} X_i^{i-1} \).

Thus, the proof of Lemma \[89\] is completed.

**Proof of Lemma \[89\]** Set \( U_i = Y_{i+1}^{n} Z_{i+1}^{-1} M_n \). It can easily be verified that \( U_i, X_i S_i Z_i, Y_i \) form a Markov chain \( U_i \rightarrow X_i S_i Z_i \rightarrow Y_i \) in this order. From (81), (82), and (85) in Lemma \[87\] we obtain

\[
I(Y^n; M_n) \leq \sum_{i=1}^{n} I(Y_i; U_i S_i),
I(Z^n; M_n) \leq \sum_{i=1}^{n} I(Z_i; U_i | S_i),
\]

and

\[
I(Y^n; K_n | M_n) - I(Z^n; K_n | M_n) \leq \sum_{i=1}^{n} \{I(K_n; Y_i U_i S_i) - I(K_n; Z_i U_i S_i)\},
\]

respectively. From (88), (89) in Lemma \[87\] we obtain

\[
I(Y^n; K_n M_n) \leq \sum_{i=1}^{n} I(Y_i; X_i U_i S_i),
I(Z^n; K_n M_n) \leq \sum_{i=1}^{n} I(Z_i; X_i U_i | S_i),
\]

respectively. It remains to evaluate an upper bound of

\[
I(K_n; Y_i U_i S_i) - I(K_n; Z_i U_i S_i).
\]

We have the following chain of inequalities:

\[
I(K_n; Y_i U_i S_i) - I(K_n; Z_i U_i S_i) = H(Y_i | U_i S_i) - H(Z_i | U_i S_i) + H(Z_i | K_n M_n U_i S_i) - H(Y_i | U_i S_i) = H(Y_i | U_i S_i) - H(Y_i | Z_i X^n U_i S_i) + H(Z_i | Y_i X^n U_i S_i) - H(Z_i | U_i S_i) \leq H(Y_i | U_i S_i) - H(Y_i | Z_i X^n U_i S_i) + H(Z_i | Y_i X^n U_i S_i) - H(Z_i | U_i S_i) \leq H(Y_i | U_i S_i) - H(Y_i | Z_i X^n U_i S_i) + H(Z_i | Y_i X^n U_i S_i) - H(Z_i | U_i S_i) = I(Y_i; Z_i X_i U_i S_i) - I(Z_i; Y_i X_i U_i S_i) = I(X_i; Y_i U_i S_i) - I(X_i; Z_i U_i S_i),
\]

where

- (95): \( X^n = f_n(K_n, M_n) \) and \( f_n \) is a one-to-one mapping.
- (96): \( Y_i \rightarrow Z_i X_i S_i \rightarrow U_i X_i | S_i \).

Thus, the proof of Lemma \[87\] is completed.

**E. Proof of Lemma \[87\]**

In this appendix we prove Lemma \[87\].

**Proof of Lemma \[87\]** Set \( U_i = Y_{i+1}^{n} Z_{i+1}^{-1} M_n \). It can easily be verified that \( U_i, X_i S_i Z_i, Y_i \) form a Markov chain \( U_i \rightarrow X_i S_i Z_i \rightarrow Y_i \) in this order. In a manner similar to the proof of Lemma \[87\] we can derive the following two bounds

\[
I(Y^n; M_n) \leq \sum_{i=1}^{n} I(Y_i; Y_i^{i-1} Z_{i+1}^{-1} M_n S_i),
I(Z^n; M_n) \leq \sum_{i=1}^{n} I(Z_i; Y_i^{i-1} Z_{i+1}^{-1} M_n | S_i).
\]

Hence, we have

\[
I(Y^n; M_n) \leq \sum_{i=1}^{n} I(Y_i; U_i S_i),
I(Z^n; M_n) \leq \sum_{i=1}^{n} I(Z_i; U_i | S_i).
\]

Furthermore, from (88), (89) in Lemma \[87\] we obtain

\[
I(Y^n; K_n M_n) \leq \sum_{i=1}^{n} I(Y_i; X_i U_i S_i),
I(Z^n; K_n M_n) \leq \sum_{i=1}^{n} I(Z_i; X_i U_i | S_i),
\]

respectively. It remains to evaluate an upper bound of

\[
I(K_n; Y^n | M_n) - I(K_n; Z^n | M_n).
\]
Since $f_n$ is a deterministic, we have
\[
I(K_n; Y^n | M_n) - I(K_n; Z^n | M_n) = H(Y^n | M_n) - H(Z^n | M_n) - H(Y^n | X^n) + H(Z^n | X^n) .
\] (99)
We separately evaluate the following two quantities:
\[
H(Y^n | M_n) - H(Z^n | M_n), \quad H(Y^n | X^n) - H(Z^n | X^n).
\]

We observe the following two identities:
\[
H(Y^n | M_n) - H(Z^n | M_n)
= \sum_{i=1}^{n} \{ H(Y_i | Y_{i+1}^n Z_{i+1}^n M_n) - H(Z_i | Y_i Z_i Z_{i+1}^n M_n) \} , \quad (100)
\]
\[
- H(Y^n | X^n) + H(Z^n | X^n)
= \sum_{i=1}^{n} \{ - H(Y_i | Y_{i+1}^n Z_{i+1}^n X^n) + H(Z_i | Y_i Z_i Z_{i+1}^n X^n) \} . \quad (101)
\]
Those identities follow from an elementary computation based on the chain rule of entropy. From (100), we have
\[
H(Y^n | M_n) - H(Z^n | M_n)
= \sum_{i=1}^{n} \{ H(Y_i | U_i) - H(Z_i | U_i) \} . \quad (102)
\]
Next, we evaluate an upper bound of
\[
- H(Y_i | Y_{i+1}^n Z_{i+1}^n X^n) + H(Z_i | Y_i Z_i Z_{i+1}^n X^n).
\]
Set $\tilde{U}_i \triangleq Y_{i+1}^n Z_{i+1}^n X_i$. We have the following chain of inequalities:
\[
- H(Y_i | Y_{i+1}^n Z_{i+1}^n X^n) + H(Z_i | Y_i Z_i Z_{i+1}^n X^n)
= - H(Y_i | Z_i X_i \tilde{U}_i) + H(Z_i | Z_i X_i \tilde{U}_i)
= - H(Y_i | Z_i X_i \tilde{U}_i) + I(Y_i; Z_i | X_i \tilde{U}_i)
+ H(Z_i | X_i \tilde{U}_i) - I(Y_i; Z_i | X_i \tilde{U}_i)
= - H(Y_i | Z_i X_i \tilde{U}_i) + H(Z_i | X_i \tilde{U}_i)
\leq - H(Y_i | Z_i X_i S_i) + H(Z_i | X_i S_i), \quad (103)
\]
where (103) follows from $Y_i \rightarrow Z_i X_i \tilde{U}_i \rightarrow \tilde{U}_i$. Combining (99), (101), and (104), we obtain
\[
I(K_n; Y^n | M_n) - I(K_n; Z^n | M_n)
\leq \sum_{i=1}^{n} \{ H(Y_i | U_i) - H(Z_i | U_i)
- H(Y_i | X_i S_i) + H(Z_i | X_i S_i) \}
\leq \sum_{i=1}^{n} \{ H(Y_i | U_i) - H(Z_i | U_i)
- H(Y_i | X_i S_i U_i) + H(Z_i | X_i S_i) \}
= \sum_{i=1}^{n} \{ I(X_i S_i; Y_i | U_i) - I(X_i S_i; Z_i | U_i)
+ I(U_i; Z_i | X_i S_i) \}
= \sum_{i=1}^{n} \{ I(X_i; Y_i | U_i S_i) - I(X_i; Z_i | U_i S_i)
+ \zeta(S_i, U_i, Y_i, Z_i) + I(U_i; Z_i | X_i S_i) \}.
\]
Thus, the proof of Lemma 5 is completed. 

F. Proof of Lemma 2
We first observe that by the Cauchy-Schwarz inequality we have
\[
E_S \left( E_{X(S)} X(S) \right)^2 \leq E_S \left( \sqrt{E_{X(S)} X^2(S)} \sqrt{E_{X(S)} 1} \right)^2
= E_S E_{X(S)} X^2(S) \leq P_1 .
\]
Then, there exists $\alpha \in [0, 1]$ such that
\[
E_S \left( E_{X(S)} X(S) \right)^2 = \alpha P_1 .
\]
We derive an upper bound of $h(Y)$. We have the following chain of inequalities:
\[
h(Y)
= h(X + S + \xi_1)
\leq \frac{1}{2} \log \left\{ (2\pi e) \left( E_{X_S} |X + S|^2 + N_1 \right) \right\}
\leq \frac{1}{2} \log \left\{ (2\pi e) \left( E_X |X|^2 + 2E_{X_S} X S + E_S S^2 + N_1 \right) \right\}
\leq \frac{1}{2} \log \left\{ (2\pi e) \left( P_1 + P_2 + 2E_{X_S} X S + N_1 \right) \right\} . \quad (105)
\]
By the Cauchy-Schwarz inequality we have
\[
E_{X_S} X S
\leq \sqrt{E_S S^2} \sqrt{E_S \left( E_{X(S)} X(S) \right)^2} = \sqrt{P_2} \sqrt{\alpha P_1} . \quad (106)
\]
From (105) and (106), we have
\[
h(Y) \leq \frac{1}{2} \log \left\{ (2\pi e) \left( P_1 + P_2 + \sqrt{\alpha P_1} \right) \right\} .
\]
Next, we estimate an upper bound of $h(Y | S)$. We have the following chain of inequalities:
\[
h(Y | S) = E_S \left[ h(X(S) + \xi_1) \right]
\leq E_S \left[ \frac{1}{2} \log \left\{ (2\pi e) \left( V_{X(S)} |X(S)| + N_1 \right) \right\} \right]
\leq E_S \left[ \frac{1}{2} \log \left\{ (2\pi e) \left( E_{X(S)} |X|^2 |S| \right) \right\} \right]
\leq \frac{1}{2} \log \left\{ (2\pi e) \left( E_{X(S)} |X|^2 + N_1 \right) \right\}
\leq \frac{1}{2} \log \left\{ (2\pi e) \left( \alpha P_1 + N_1 \right) \right\} .
\]
Similarly, we obtain
\[
h(Z | S) \leq \frac{1}{2} \log \left\{ (2\pi e) \left( \alpha P_1 + N_2 \right) \right\} ,
\]
\[
h(\tilde{Y} | S) \leq \frac{1}{2} \log \left\{ (2\pi e) \left( \alpha P_1 + \tilde{N}_1 \right) \right\} . \quad (107)
\]
Since
\[ h(Y|S) \geq h(Y|XS) = \frac{1}{2} \log \left\{ (2\pi e)\tilde{N}_1 \right\} \]
and (109), there exists \( \beta \in [0, 1] \) such that
\[ h(Y|US) = \frac{1}{2} \log \left\{ (2\pi e) \left( \beta \alpha \rho N_1 + \tilde{N}_1 \right) \right\} . \]
Finally, we derive lower bounds of \( h(Y|US) \) and \( h(Z|US) \). We recall the following relations between \( Y, Z \), and \( Y \):
\[ Y = \tilde{Y} + \tilde{a}S + \tilde{a} \xi_2 . \tag{108} \]
\[ Z = \tilde{Y} - aS - a \xi_2 . \tag{109} \]
Applying entropy power inequality to (108), we have
\[ \frac{1}{2\pi e} 2^{2h(Y|US)} \geq \frac{1}{2\pi e} 2^{2h(\tilde{Y}|US)} + \frac{1}{2\pi e} 2^{2h(\tilde{a} \xi_2)} \]
\[ = \beta \alpha \rho N_1 + \tilde{N}_1 + a^2 \tilde{N}_2 \]
\[ = \beta \alpha \rho N_1 + \tilde{N}_1 + \frac{(1+\rho^2)N_1N_2}{N_1+N_2-2\rho \sqrt{N_1N_2}} \]
\[ = \beta \alpha \rho N_1 + N_1. \]
Hence, we have
\[ h(Y|US) \geq \frac{1}{2} \log \left\{ (2\pi e) (\beta \alpha \rho N_1 + N_1) \right\} . \]
Applying entropy power inequality to (109), we have
\[ \frac{1}{2\pi e} 2^{2h(Z|US)} \geq \frac{1}{2\pi e} 2^{2h(\tilde{Y}|US)} + \frac{1}{2\pi e} 2^{2h(\tilde{a} \xi_2)} \]
\[ = \beta \alpha \rho N_1 + \tilde{N}_1 + a^2 \tilde{N}_2 \]
\[ = \beta \alpha \rho N_1 + \tilde{N}_1 + \frac{(1+\rho^2)N_1N_2}{N_1+N_2-2\rho \sqrt{N_1N_2}} \]
\[ = \beta \alpha \rho N_1 + N_2. \]
Hence, we have
\[ h(Z|US) \geq \frac{1}{2} \log \left\{ (2\pi e) (\beta \alpha \rho N_1 + N_2) \right\} . \]
Thus the proof of Lemma 4 is completed.

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