Fractal universe and cosmic acceleration in a Lemaître–Tolman–Bondi scenario

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Abstract

In this paper we attempt to answer the question: can cosmic acceleration of the universe have a fractal solution? We give an exact solution of a Lemaître–Tolman–Bondi (LTB) universe based on the assumption that such a smooth metric is able to describe, on average, a fractal distribution of matter. While the LTB model has a center, we speculate that, when the fractal dimension is not very different from the space dimension, this metric applies to any point of the fractal structure when chosen as center so that, on average, there is not any special point or direction. We examine the observed magnitude-redshift relation of type Ia supernovae (SNe Ia), showing that the apparent acceleration of the cosmic expansion can be explained as a consequence of the fractal distribution of matter when the corresponding space-time metric is modeled as a smooth LTB one and if the fractal dimension on scales of a few hundreds Mpc is $D = 2.9 \pm 0.02$.

Keywords: fractal universe, acceleration of the universe, Lemaître–Tolman–Bondi scenario

(Some figures may appear in colour only in the online journal)
The so-called concordance model of the universe is based on three fundamental assumptions. The first is that the dynamics of space-time is determined by Einstein’s field equations. The second is the generalization of the Copernican principle [1, 2]: all observers are equivalent and there are no special points and directions. The third is that matter distribution is spatially homogeneous, i.e. characterized by a spatially constant density \( \rho(r, t) = \rho(t) \). The Friedmann–Lemaître–Robertson–Walker (FLRW) model is derived under these three assumptions and it describes the geometry of the universe in terms of a single function, the scale factor, which obeys to the Friedmann equation [3]. In this framework one assumes exact translational and rotational invariance. When density fluctuations are introduced, their possible compatibility with the above framework depends on fluctuations statistical properties.

If matter distribution is a realization of a stationary stochastic point process [4], it is statistically homogeneous and isotropic thus still satisfying the Copernican requirement of the absence of special points or directions. However it can, or cannot be a spatially homogeneous stationary stochastic process: only in the latter case matter distribution satisfies the special and stronger case of the Copernican principle described by the Cosmological principle. Indeed, isotropy around each point together with the hypothesis that the matter distribution is a smooth function of position i.e. that this is analytical, implies spatial homogeneity [5, 6]. This is no longer the case for a non-analytic structure (i.e. not smooth), for which the obstacle to applying the FLRW solutions has in fact solely to do with the lack of spatial homogeneity [7].

Recently the attempts to construct cosmological models including spatial inhomogeneities have experienced a renewed interest in connection with both the detection of a complex network of galaxy clusters, filaments and void [8] and the evidences for a speeding up expansion of the universe as shown by the supernovae (SN) observations [9, 10]. Indeed, the deduction of the existence of dark energy is based on the assumption that the universe has a FLRW geometry. The underlying idea of inhomogeneous models is to interpret the acceleration derived from the SN observations as an apparent effect that arises in a too simplified solution of Einstein’s field equations, i.e. those derived with a density and pressure that are constant in space and depend only on time. It is possible to distinguish [11] between two main approaches to an inhomogeneous universe: (i) models that consider the spatial averaging of inhomogeneities and (ii) models placing the observer in a special point of the local universe.

It has been demonstrated that the spatial averaging of inhomogeneities gives rise to effective terms (named backreaction), in addition to the standard FLRW energy sources, that can play the role of dark energy on large scales [13]. Dark energy can be thus considered as an artifact due to the impact of inhomogeneities. There is an ongoing debate about whether an inhomogeneous model can thus evolve, on average, like the homogeneous FLRW solution in agreement with observations [12, 14–18].

Models of type (ii) instead use a very ad-hoc behavior of the spatial density implying a breakdown of the Copernican assumption on the Hubble scale: in particular it was assumed that we are near the center of spherically symmetric low density hole [11, 19]. In this context the simplest GR models are the spherically symmetric Lemaître–Tolman–Bondi (LTB) solutions with a central observer which clearly represent drastic simplification of the problem. LTB models without dark energy can fit supernovae, explaining the apparent acceleration of the universe by a Gpc-scale void around us [20], thus requiring fine-tuned and ad-hoc assumptions on the properties of matter distribution.

Another way to consider the effect of the inhomogeneities on the observations consists of studying the effect of randomly distributed inhomogeneous patches in a given background on the propagation of photon [21]. These attempts are known as Swiss cheese models. In this approach, light rays travel along a series of inhomogeneous patches which are usually modeled with radially inhomogeneous LTB regions [22, 23] or with Szekeres metric [24–26].
An interesting attempt, concerning a relativistic model for the observed large scale homogeneities was proposed by [27]: in this framework one makes the hypothesis that the large scale structures can be described as being a self similar fractal system.

In this paper we adopt a different approach that consists in modeling the spatial inhomogeneities in a way that is more close to what we can learn from observations of galaxy structures. Indeed, the statistical analysis of galaxy three dimensional surveys have shown that galaxy distribution is characterized by power-law correlations in the local universe [28–30]. More specifically it was found that the average conditional density decays as \( \langle n(r) \rangle_p \sim r^{-\gamma} \) where \( \gamma = 0.9 \pm 0.1 \) for \( r \in [0.1, 20] \) Mpc/h and \( \gamma = 0.2 \pm 0.1 \) for \( r \in [20, 100] \) Mpc h\(^{-1}\) [31]. Whether or not on scales \( r > 100 \) Mpc h\(^{-1}\) correlations decay and the distribution crossovers to uniformity, is still matter of considerable debate [30].

The power-law behavior of the conditional density can be interpreted as galaxy distribution having fractal properties at small scales. A fractal is a non-analytical point distribution: \( \langle n(r) \rangle \) decays (only) on average as a power law. From the \( i^{th} \) the (conditional) density decays as \( n_i(r) \sim f_i(r) \cdot r^{-\gamma} \) where the correction to scaling \( f_i(r) \) is such that \( \sigma_p = \langle n_i(r)^2 \rangle - \langle n_i(r) \rangle_p^2 = (\langle f_i(r) - 1 \rangle)^2 \approx \text{const.} \) [4]. If \( \sigma_p(r) < 1 \), as for the real galaxy structures [31], we can approximate the discrete matter source field as

\[
\rho_i(r) = \sum_i m_i \delta^0(\bar{r} - \bar{r}_i) \approx \langle n_i(r) \rangle. \tag{1}
\]

This situation allows us to use a smooth LTB metric for describing the spatial decay of the density without assuming the existence of a special position in the universe. Indeed, in LTB models isotropy is valid only for the privileged observer that makes measurements from the centre of coordinate system. Any other observer in a LTB universe far from the centre will experience a dipolar anisotropy [32–34] and thus the Cosmological principle is not valid in such a framework. On the other hand, a fractal distribution has the fundamental property that the density is seen to decay with the same power law for all the observers: as mentioned above it can be seen a stationary point process, i.e. a statistically homogeneous and isotropic distribution. To reconcile these two different properties of the metric of and of matter distribution, we assume to live in a local over-density and that any other observer, placed in any other galaxy, sees the same decaying radial density as us: this condition is approximately satisfied if \( \sigma_p(r) < 1 \) and thus equation (1) holds. This situation has a clear advantage with respect of assuming a single large-scale Gpc under-density. Indeed, while the LTB model has a center, we speculate that this metric applies to any point of the fractal structure when chosen as center so that, on average, there is not any special point or direction.

We are not able to quantify the perturbations neglected by making this assumption but we can assume that, as long as spatial fluctuations around the average behavior remain limited, i.e. \( \sigma_p(r) < 1 \), this model provide a reasonable description of the local metric of a fractal object. In this situation the local expansion rate around us would be smaller than the average expansion rate in the background: light-rays propagating from distance sources to us (or to any observer located in a local over-density) would therefore feel a decelerated expansion rate along their path: this is what we are going to show.

As discussed below we do not need to assume that the fractal behavior extends up to an arbitrarily large scale: rather we show that it is sufficient that for a moderate value of the homogeneity scale \( l_0 \), beyond which \( \langle n(r) \rangle_p \approx \text{const.} \), the modification of the magnitude-redshift relation due to the inhomogeneous and power-law behavior of the (conditional) density provides a best fit to the SN data without the need of introducing dark energy.

Let us describe an inhomogeneous universe within the framework of the isotropic and inhomogeneous LTB metric in polar coordinates \( x^\mu = (t, r, \theta, \phi) \)
\[ ds^2 = -dt^2 + \frac{A'(t, r)^2}{1 - k(r)} dr^2 + A^2(t, r) \left[ d\theta^2 + \sin^2 \theta d\phi^2 \right] \] (2)

where \( A(t, r) \) is the radial inhomogeneous scale factor and \( k(r) \) is the inhomogeneous spatial curvature. Therefore the two independent Einstein’s equations which describe the model are:

\[
\frac{\dot{A}^2 + k}{A^2} + \frac{2\dot{A}' + k'(r)}{AA'} = 8\pi G \langle n(r) \rangle \tag{3}
\]

\[
\frac{\dot{A}^2 + 2\dot{A}' + k}{A^2} = 0 \tag{4}
\]

where \( \dot{} \equiv \partial_t \) and \( ' \equiv \partial_r \). By integrating the equation (4), we get:

\[
\left( \frac{\dot{A}}{A} \right)^2 = -k(r) + \frac{\alpha(r)}{A^3} \tag{5}
\]

which, for \( k = 0 \) can be exactly integrated with solution:

\[
A(t, r) = A_0(r) \left[ 1 + \frac{3}{2} \sqrt{\frac{\alpha(r)}{A_0^3(r)}} t \right]^{2/3}, \tag{6}
\]

where \( A_0(r) \) and \( \alpha(r) \) are two free functions due to the double integration in \( t \). Here we adopt the simplified choice \( k = 0 \) because it is supported by the value of spatial curvature on cosmological scales is highly constrained by the CMB best-fit parameters to be very small in the past and negligible today [35]. This assumptions implies that the inhomogeneous behavior on large scales is not due to curvature perturbations. Let us now define the Hubble function \( H(t, r) = \dot{A}/A \) and the comoving mass

\[
M(t, r) = \int_{S^3_P(r)} \langle n(r) \rangle A' A^2 4\pi r^2 dr \tag{7}
\]

where \( S^3_P(r) \) is the 3D sphere with radius \( r \) and centered in the observer position \( P \). By inserting the solution equation (6) in equation (3) and by integrating over the sphere; we get:

\[
\alpha(r) = 2GM(r) = 2G \int_{S^3_P(r)} \langle n(r) \rangle A'(t, r) A^2(t, r) 4\pi r^2 dr \tag{8}
\]

so that \( M(r) \) only depends by \( r \). From equation (5) we find

\[
2GM(r) = A_0^3(r) H_0^2(r), \tag{9}
\]

where \( H_0(r) \equiv H(0, r) \). Therefore we get that the solution is \( A(t, r) = A_0(r) \left[ 1 + \frac{3}{2} H_0(r) t \right]^{2/3} \) and the Hubble function at the present time is intimately related to the mass distribution by:

\[
H_0(r) = \sqrt{\frac{2GM(r)}{A_0^3(r)}}. \tag{10}
\]

Equation (10) naturally appears from the solution of the Einstein equations and is completely general (modulo the absence of spatial curvature) and it basically relates the inhomogeneous Hubble flow the mass content. For a pure fractal we have \( M(r) \sim r^D \) where \( D = 3 - \gamma < 3 \) is
the fractal dimension [4]. The last free function \( A_0(r) \) can be chosen by exploiting the residual freedom that we have in redefining the radial coordinate \( r \) in the equation (2). In fact, thanks to this we can always specify the function \( A(t, r) \) at a given time \( t_* \). Thanks to this we can choice \( A_0(r) = r \). Being this just due to a redefinition of coordinates that leaves us within the LTB metric, this choice cannot affect any observables. We can write the mass \( M(r) \) as

\[
M(r) = \Phi \rho^D
\]

where \( \Phi \) is the amplitude of the fractal distribution, related to the average distance between nearest galaxies [4]. Therefore, we can rewrite the Hubble function in a more intuitive way as:

\[
H_0(r) = \frac{B}{r^{D+3}}
\]

where \( B \equiv \sqrt{2G \Phi} \). From the mathematical point of view, let us notice that our expression for \( H_0(r) \) diverges when \( r \to 0 \). Nevertheless, we will argue later that this bad behavior can be easily fixed by requiring an appropriate lower cutoff.

In order to compare this model with the observations of the supernova Ia data, let us consider the well-known formula of the luminosity distance/redshift relation for on-center LTB models [36] (see also [37] for similar interesting approaches). Recently [38] within a purely inhomogeneous and anisotropic framework, an exact geometrical expression for the angular distance has been evaluated by solving the Sachs equation. As shown in [34], this solution reduces to the well-known formula for the angular distance of on-center LTB models by performing a coordinate transformation. Hence, thanks to the Etherington relation, we relate the angular and the luminosity distance as both function of redshift \( z \) and we use the usual expression, given by:

\[
d_L = (1 + z)^2 A(t(z), r(z))
\]

where

\[
\frac{dt}{dz} = \frac{A'(t(z), r(z))}{(1 + z)A'(t(z), r(z))},
\]

\[
\frac{dr}{dz} = \frac{1}{(1 + z)A'(t(z), r(z))}.
\]

Let us notice that for \( z = 0 \) we get \( d_L(0) = A(t(0), r(0)) = A_0(r) = r \), thanks to the partial redefinition of coordinate discussed before. Equation (14) come from the geodesic null condition \( ds^2 = 0 \) for a photon traveling towards the center of the coordinates system, where the affine parameter is labeled by the redshift (see [39] for the generalization of this procedure when \( k(r) \neq 0 \)). Given the high non-linearity of the solution (6), equation (14) can be solved only numerically up to larger redshift. However this is already enough for the purposes of this work. This is important to stress that in a pure fractal distribution, the center is established only after the integration over the sphere. Hence, following our approach, the expression for off-center observer [34, 36] has not to be considered. Let us then consider the distance modulus:

\[
\mu(z) = 5 \log_{10} \left[ \frac{d_L(z)}{1 \text{ Mpc}} \right] + 25
\]

that is immediately comparable with the experimental data, UNION2 data set, that consist of redshift-magnitude of 557 supernovae Ia in which we have for each supernova the observed distance modulus. We do likelihood analysis with \( B \) and \( D \) free parameters in order to find the best fit values. We do the comparison between the observed \( \mu_{\text{obs}}(z_i) \pm \Delta \mu(z_i) \) and theoretical distance modulus \( \mu_{\text{th}} \) by performing a standard \( \chi^2 \) analysis with
we find the best-fit values by requiring the minimization of the $\chi^2$. The minimization has been obtained using MINUIT package from CERLIB. The results are shown in figure 1. The red curve corresponds to the best fit values for our model i.e. $D = 2.87 \pm 0.02$ and $B = 96 \pm 4 \text{ km s}^{-1} \text{ Mpc}^{-1}$ with a $\chi^2 / \text{dof} = 1.19$. Therefore, by introducing a fractal exponent significantly different from 3, we are able to reproduce the supernova data without referring to any dark energy. Furthermore, our results are compatible with no transition to homogeneity.

Let us underline that our solution is decelerated so the acceleration of the universe is only an apparent homogeneous effect. In fact, as shown in figure 2, our solution (red curve) is the superposition of different homogeneous FLRW—CDM models with different $H_0$ (black curves); in particular, at low redshifts, the curve can be view as a FLRW model with $H_0 = H_0(1 \text{ Mpc})$ but, at high redshifts, the LTB curve is well described by a FLRW model with $H_0 = H_0(10,000 \text{ Mpc})$.

The validity of our study has to be understood in this sense: the actual observations about a fractal distribution of matter involve structures up to a $\sim 100 \text{ Mpc}$. In this regime, all the cosmological distances are degenerate so a distribution of matter as considered in equation (11) can safely mimic the observed distribution. Then the application of our result in this range for SNe IA leads to figure 3. For higher redshift, the relation between distances becomes non-linear and the implementation of a possible fractal behavior on larger scales may be done more rigourosly on the light-cone rather than on constant-time hypersurfaces. However the latter extrapolation, as presented in figure 1, just assumes that the strongly inhomogeneous behavior extends on scales which are larger than the observed ones today. We then use the luminosity distance-redshift relation as an indirect probe of the possible fractal behavior of matter on largest scales.

\[\chi^2 = \sum_{i=1}^{557} \left( \frac{\mu_{\text{obs}}(z_i) - \mu_{\text{th}}(z_i, \Delta H, r_0, \Delta)}{\Delta \mu(z_i)} \right)^2,\] (16)

In order to make independent the dimension of $B$ by $D$, we considered the radial coordinate as measured with respect 1 Mpc, namely $r = \frac{r}{1 \text{ Mpc}}$.  

\[B = (96.1 \pm 3.8) \text{ km sec}^{-1} \text{ Mpc}^{-1}\]

\[D = 2.869 \pm 0.014\]

\[\chi^2 / \text{dof} = 1.19\]
Figure 2. Log–Log plot for the distance modulus in term of the redshift. Thick red curve indicates the fractal inhomogeneous model. Thin black curves refer to several pure matter FLRW models with different $H_0$ taken at particular values of $H_0(r)$, from $r = 1$ Mpc (bottom) to $r = 10000$ Mpc (top).

Figure 3. Distance modulus and Hubble function in term of redshift at low $z$. 
According to what we have said up to here, one concern has still to be addressed i.e. the un-physical divergence of $H_0$. We note that, in our model, $H_0$ is finite at low redshift ($0.01 < z < 0.05$). Indeed, let us refer to figure 3, where we considered the lowest redshift data from the UNION2 catalog. As we can see, $H_0$ changes its value from 72 to 67 Km s$^{-1}$ Mpc$^{-1}$ within the low redshift range. Hence we assume that up to a few Mpc the Hubble law is quiet and that at smaller scale our simple description breaks down. Nevertheless, we find a very good agreement with the supernovae data even at the smallest scale of the UNION2 catalog, as shown in figure 3.

In summary we have provided an analytical solution for a fractal distribution of matter in the framework of a LTB model. This task is not trivial at all and has been achieved by properly take care about statistical equivalence of the observer’s position. Indeed, this is really different from the homogeneity of distribution of matter, which consists of a stronger hypothesis, even if people usually don’t realize it. Moreover, we used this simplified, but observationally supported, scenario to show that it is possible to explain the apparent acceleration of the universe by means of radial inhomogeneities without introducing dark energy. Nevertheless we found that a simple and observationally motivated description of large scale galaxy inhomogeneities can describe the Supernovae data as well as other exactly inhomogeneous models. Furthermore by using LTB model to compute the magnitude-redshift relation we have found that the best fit to the supernovae data corresponds to a fractal dimension $D = 2.87$ at large scales $r > 100$ Mpc h$^{-1}$, which is in good agreement with galaxy data [30]. A recent work [41] with a different oversimple inhomogeneous fractal cosmological model the authors obtain an exponent $D = 3.36$ by means a best fit for UNION 2 supernovae data.

In addition, we stress that our description is not in contradiction with the Copernican principle, as the center point of the LTB model can be chosen to be in any galaxy, i.e. in any local peak of the conditional density. A more refined analysis, which will take the CMB Planck data into account will be presented in a forthcoming work.

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References

[1] Bondi H 1952 Cosmology (Cambridge: Cambridge University Press)
[2] Clifton T and Ferreira P G 2008 Phys. Rev. Lett. 101 131302
[3] Weinberg S 2008 Cosmology (Oxford: Oxford University Press)
[4] Gabrielli A, Sylos Labini F, Joyce M and Pietronero L 2004 Statistical Physics for Cosmic Structures (Berlin: Springer)
[5] Trumper M 1967 Z. Phys. Astrophys. 66 215
[6] Straumann N 1974 Helv. Phys. Acta 47 379
[7] Joyce M et al 2000 Europhys. Lett. 50 416
[8] York D et al 2000 Astron. J. 120 1579
[9] Riess A G et al 1998 Astron. J. 116 1009
[10] Perlmutter S et al 1999 Astrophys. J. 517 565
[11] Anderson L and Coley A 2011 Class. Quantum Grav. 28 160301
[12] Bonnor W B 1974 Mon. Not. R. Astron. Soc. 167 55
[13] Buchert T 2000 Gen. Relativ. Gravit. 32 105
[14] Ellis G 2008 Nature 452 158
[15] Buchert T 2008 Gen. Relativ. Gravit. 40 467
[16] Buchert T et al 2015 Class. Quantum Grav. 32 215021
[17] Kolb E W, Marra V and Matarrese S 2010 Gen. Relativ. Gravit. 42 1399
[18] Larena J, Alimi J M, Buchert T, Kunz M and Corasaniti P S 2009 Phys. Rev. D 79 083011
[19] Celerier M N 2012 Astron. Astrophys. 543 A71
[20] Celerier M N 2000 Astron. Astrophys. 353 63
[21] Kantowski R 1969 Astrophys. J. 155 89
[22] Marra V, Kolb W E, Matarrese S and Riotto A 2007 Phys. Rev. D 76 123004
[23] Marra V and Kolb S W E 2008 Phys. Rev. D 77 023503
[24] Bolejko K and Celerier M N 2010 Phys. Rev. D 82 103510
[25] Peel A, Troxel M A and Ishak M 2014 Phys. Rev. D 90 123536
[26] Koksbjerg S M 2017 Phys. Rev. D 95 063532
[27] Ribeiro M B 1992 Astrophys. J. 388 41
[28] Syllos Labini F et al 1998 Phys. Rep. 293 66
[29] Hogg D W et al 2005 Astrophys. J. 624 54
[30] Labini F S 2011 Class. Quantum Grav. 28 164003
[31] Antal T, Labini F S, Vasilyev N L and Baryshev Y V 2009 Europhys. Lett. 88 59001
[32] Alnes H and Amarzguioui M 2006 Phys. Rev. D 74 103520
[33] Alnes H and Amarzguioui M 2007 Phys. Rev. D 75 023506
[34] Fanizza G, Gasperini M, Marozzi G and Veneziano G 2015 J. Cosmol. Astropart. Phys. JCAP02(2015)002
[35] Planck Collaboration 2018 arXiv:1807.06209
[36] Cosmai L, Fanizza G, Gasperini M and Tedesco L 2013 Class. Quantum Grav. 30 095011
[37] Nogueira F A M G 2013 arXiv:1312.5005 [gr-qc]
[38] Fanizza G, Gasperini M, Marozzi G and Veneziano G 2013 J. Cosmol. Astropart. Phys. JCAP11(2013)019
[39] Romano A E and Sasaki M 2012 Gen. Relativ. Gravit. 44 353
[40] Amanullah A et al (The Supernova Cosmolgy Project) 2010 Astrophys. J. 716 712
[41] Ruffini R and Stahi C (https://doi.org/10.1142/97898132266090595)