The Generalized Uncertainty Principle

Jun-Li Li and Cong-Feng Qiao*

School of Physical Sciences, University of Chinese Academy of Sciences
YuQuan Road 19A, Beijing 100049, China
Center of Materials Science and Optoelectronics Engineering & CMSOT,
University of Chinese Academy of Sciences, YuQuan Road 19A, Beijing 100049, China
Key Laboratory of Vacuum Physics, University of Chinese Academy of Sciences
YuQuan Road 19A, Beijing 100049, China

* To whom correspondence should be addressed; E-mail: qiaocf@ucas.ac.cn.

Abstract

The uncertainty principle lies at the heart of quantum physics, and is widely thought to be a fundamental limit on the measurement precisions of incompatible observables. Here we show that this mode of thought only pertains to the lowest order approximation of a generalized uncertainty principle, where the incompatibility between observables is interpreted as the statistical correlation. We derive out a generalized uncertainty relation which exhibits the full-order statistical correlations between observables. The new result extends the leading order linear correlation, the nature of Heisenberg type of uncertainty relations, to nonlinear correlation involved arbitrary high orders, and hence provides an alternative view to the foundation of quantum mechanics. The new finding will definitely enlarge the application of uncertainty relation in quantum information processing.
1 Introduction

The rise of the Heisenberg’s uncertainty principle is attributed to the early efforts to incorporate the wave and particle natures of each individual quantum. It states that the canonically conjugate quantities, $x$ and $p$, can be determined simultaneously only with a characteristic indeterminacy \[1\]. A well known formulation of the uncertainty relation is \[2\]

\[
\Delta X \Delta Y \geq \frac{1}{2} |\langle [X,Y] \rangle| ,
\]

where the variance $\Delta X \equiv \sqrt{\langle X^2 \rangle - \langle X \rangle^2}$ is a measure of the uncertainty for observable $X$ (similarly for $Y$) and the commutator $[X,Y] \equiv XY - YX$. In relation \[1\], $X$ and $Y$ no longer restrict to canonical variables but may stand for arbitrary observables. An improvement of the uncertainty relation was made by Schrödinger \[3\], gave

\[
(\Delta X)^2 (\Delta Y)^2 \geq \frac{1}{4} |\langle \{X,Y\} \rangle - 2\langle X \rangle \langle Y \rangle|^2 + \frac{1}{4} |\langle [X,Y] \rangle|^2 .
\]

Here the anti-commutator $\{X,Y\} \equiv XY + YX$. The uncertainty relations \[1\] and \[2\] normally explained as trade off relations of the uncertainties of incompatible observables, which is lower bounded by the expectation values of their (anti-)commutator. Or in other words, the variances of incompatible observables are not independent but correlated with each other.

The lower bounds of the above variance-based uncertainty relations depend on the quantum state and may reach zero in which case the uncertainty relation becomes trivial. Partly due to this reason, the entropic uncertainty relation was introduced \[4\], with a typical form of \[5\]

\[
H(X) + H(Y) \geq \log \frac{1}{c} ,
\]
where \( c = \max_{i,j} |\langle x_i | y_j \rangle|^2 \) is the maximum overlap of the eigenbases \( |x_i \rangle \) and \( |y_j \rangle \) of \( X \) and \( Y \) respectively and turns out to be measurement state independent. The Shannon entropy \( H(\cdot) \) is another measure of uncertainty for observable \( X \) or \( Y \) with regard to the probability distribution of the measurement outcomes. Studies indicate that these two superficially different forms of uncertainty relations are in fact interconvertible \([6]\). It was recently realized \([7]\) that the variance or entropy alone may be not sufficient to fully characterize the inner property of quantum uncertainty, some higher order moments of observables are nontrivial (note, variance is only the second-order central moment) \([6]\).

In this work, by interpreting the incompatibility as the statistical dependence of observables, which embraces the trade-off variance relation and more, we derive out a generalized uncertainty relation (GUR) capable of embodying the correlation of the measurement outcomes of observables to arbitrary order. The GUR can be expanded in terms of the statistical quantity Cumulant. According to the new formalism, the Heisenberg-like uncertainty relations represent the lowest order term in generalized uncertainty relation, and exhibit only the linear dependence of incompatible observables. To illustrate the non-linear correlation of incompatible observables in uncertainty relation, in practice one may calculate the “Skewness uncertainty relation” for third order correlation and “Kurtosis uncertainty relation” for forth order correlation and so forth in terms of the cumulant language. Here, the third order one will be presented explicitly as an example.

2 The generalized uncertainty principle

2.1 The complementary and uncertainty principle

To understand the wave-particle duality in the atomic phenomena, a fundamental concept “Complementarity Principle” was proposed by Bohr, which may be stated as “any given application of classical concepts precludes the simultaneous use of other classical
Figure 1. Two interpretations of complementarity. Regarding $X \cup Y$ as a whole, we have that: (I) the completeness of $X$ precludes the wholeness of $Y$; (II) $X$ and $Y$ cannot be whole simultaneously.

concepts which in a different connection are equally necessary for the elucidation of the phenomena” [8]. While Heisenberg put forward a more operational idea “that canonically conjugate quantities can be determined simultaneously only with a characteristic indeterminacy” [1]. These two different statements about the complementarity of two incompatible observables, $X$ and $Y$, are illustrated in Figure 1. While Bohr’s interpretation implies that the precise determination of one observable precludes the other, the Heisenberg’s interpretation reflects the fact that the two observables cannot be determined simultaneously. Notice that both Heisenberg’s uncertainty principle and Bohr’s interpretation concern about the impaired sections in the measurement, as shown in the Figure 1. In view of the overlap section, we are encouraged to think of a slightly different interpretation for the complementarity principle, i.e.

Observation 1 The generalized uncertainty principle: In quantum theory, the uncertainties of incompatible observables in a system are correlated to all orders in cumulant.
The generalized uncertainty principle indicates that measuring one observable may pro-
vide you some information as well on its incompatible partners, which in fact has no
conflict with the well-known interpretation of complementarity. To ascertain its physical
consequences, we need to quantify the generalized uncertainty principle. With this aim,
we need to obtain the statistical relation between physical observables first.

2.2 The statistical relation between physical observables

For a random variable $X$, we have the moments generating function

$$
\langle e^{sX} \rangle = \sum_{n=0}^{\infty} \langle X^n \rangle \frac{s^n}{n!}, \quad s \in \mathbb{C},
$$

(4)

where the bracket $\langle X \rangle$ means the expectation value of a variable $X$. The logarithm ex-
pansion of the equation (4) generates the cumulants [9]

$$
K(sX) \equiv \log(\langle e^{sX} \rangle) = \sum_{m=1}^{\infty} \kappa_m \frac{s^m}{m!},
$$

(5)

where $\kappa_m$ is called the cumulant of order $m$. The first few orders of the cumulants are

- Mean : $\kappa_1 = \langle X \rangle$, (6)
- Variance : $\kappa_2 = \langle X^2 \rangle - \langle X \rangle^2$, (7)
- Skewness : $\kappa_3 = \langle X^3 \rangle - 3\langle X^2 \rangle \langle X \rangle + 2\langle X \rangle^3$, (8)
- Kurtosis : $\kappa_4 = \langle X^4 \rangle - 4\langle X^3 \rangle \langle X \rangle - 3\langle X^2 \rangle^2 + 12\langle X^2 \rangle \langle X \rangle^2 - 6\langle X \rangle^4$, (9)

Here the mean $\kappa_1$ reflects expectation value of an observable with certain distribution;
the variance $\kappa_2$ measures the spread of the distribution; the skewness $\kappa_3$ measure the
distribution asymmetry; the kurtosis $\kappa_4$ measures the distribution tailedness; and so on.
For random variables $X$ and $Y$, the cumulants generating function turns to

$$K(sX, tY) \equiv \log[\langle e^{sX+tY} \rangle] = \sum_{m+n=1}^{\infty} \kappa_{mn} \frac{s^m t^n}{m! n!},$$

where $\kappa_{mn}$ are named cross cumulants \[10\]. While $\kappa_{m0}$ and $\kappa_{0n}$ have the similar expressions as equations (6)-(9), the first two terms of cross cumulants $\kappa_{mn}$ read

$$\kappa_{11} = \frac{1}{2} \langle XY + YX \rangle - \langle X \rangle \langle Y \rangle,$$

$$\kappa_{12} = \frac{1}{3} \langle XYY + YXY + YXY \rangle - (\langle X \rangle \langle Y^2 \rangle + \langle XY + YX \rangle \langle Y \rangle) + 2 \langle X \rangle \langle Y \rangle^2.$$

Here subscripts in $\kappa_{mn}$ indicate that there are $m$ $X$s and $n$ $Y$s in the expansion of moment. The cross cumulants for multiple variables can be similarly defined as

$$K(s_1 X_1, \ldots, s_N X_N) \equiv \log[\langle e^{s \cdot \vec{X}} \rangle].$$

According to the Corollary of Theorem I in Ref. \[11\] we further have:

**Observation 2** The cross cumulant $\kappa_{mn}$ is nonzero, if and only if its variables are statistically correlated.

Due to the Observation 2 the cumulants are capable of quantifying the statistical correlations of physical observables.

### 2.3 The generalized uncertainty relation

To exhibit the generalized uncertainty principle, we find the following generalized uncertainty relation exists:

**Theorem 1** For arbitrary observables $X$ and $Y$, there exists a generalized uncertainty relation

$$K[(s + s^*)X] + K[(t + t^*)Y] \geq K(Z_{st}) + c.c., \forall s, t \in \mathbb{C}.$$
Here $K$ signifies the generating function of cumulants to all orders; “c.c.” means the complex conjugation; $Z_{st} \equiv \{Z_{10}, Z_{01}, Z_{11}, \cdots \}$ is a set of operators defined as

$$\log(e^{sX}e^{tY}) = sX + tY + \frac{1}{2}[sX, tY] + \cdots = Z_{10} + Z_{01} + Z_{11} + \cdots ,$$

(15)

in light of the well-known Baker-Campbell-Hausdorff formula.

The uncertainty relation (14) is intriguing: the sum of the statistical properties of two observables, the left hand side, is lower bounded by their statistical dependence, the $K(Z_{st})$ on the right hand side in cross cumulants. Another primary merit of equation (14) lies in the fact that the quantum and classical correlations can be distinguished by high orders in $Z_{st}$, i.e., the noncommutative terms in the Baker-Campbell-Hausdorff formula. We shall show it by expanding the GUR (14) to several exemplifying orders.

2.3.1 First order: Mean value identity

The first cumulant is the mean value. Comparing the coefficients of $s$, $t$, and their complex conjugates on both sides of equation (14) gives

$$(s + s^*)\kappa_1(X) + (t + t^*)\kappa_1(Y) = \langle sX + tY \rangle + c.c. .$$

(16)

Here the cumulant is shown with arguments for corresponding observables, viz $\kappa_1(X)$. Note, (16) is an equality, which means that there is no contribution from the mean value to the generalized uncertainty relation (14).

2.3.2 Second order: Variance uncertainty relations

Expanding equation (14) to the second order we have
**Corollary 1** For two observables \(X\) and \(Y\), there exists the following uncertainty relation for cumulant \(\kappa_2\)

\[
|s|^2 \kappa_2(X) + |t|^2 \kappa_2(Y) \geq (\kappa_{11}(sX,tY) + \langle Z_{11} \rangle) + \text{c.c. \(\forall s, t \in \mathbb{C}\)} ,
\]

where \(Z_{11} = \frac{1}{2}[sX,tY]\) is defined in equation (15).

To compare with the traditional uncertainty relation, we write the cumulant in variance, and then equation (17) turns to

\[
|s|^2 \Delta X^2 + |t|^2 \Delta Y^2 \geq |st| \sqrt{\left( (X,Y) \right)^2 + \left( \{X,Y\} - 2 \langle X \rangle \langle Y \rangle \right)^2} \nonumber
= 2|st||\langle (X - \langle X \rangle)(Y - \langle Y \rangle) \rangle| .
\]

(18)

Notice, when \(|s| = \varepsilon \sqrt{\frac{\Delta Y}{\Delta X}}\) and \(|t| = \varepsilon \sqrt{\frac{\Delta X}{\Delta Y}}\), the uncertainty relation (18) implies (2), and it may further give out

\[
\rho_{X,Y} = \frac{|\langle (X - \langle X \rangle)(Y - \langle Y \rangle) \rangle|}{\Delta X \Delta Y} \leq 1 ,
\]

(19)

where \(\rho_{X,Y}\) is the Pearson correlation coefficient that characterizes the linear dependence between observables \(X\) and \(Y\).

The Corollary (1) spells out the contradiction between the quantum and classical theory as follows. First, the Heisenberg’s interpretation of the uncertainty relation does not fully distinguish the quantum theory from classical ones, i.e., the Pearson correlation coefficient are always less than one for both classical and quantum theories. Second, while interpreted as statistical correlations, the generalized uncertainty relation in terms of cumulant can distinguish the quantum theory from the classical ones, namely

**Classical UR** : \(|s|^2 \kappa_2(X) + |t|^2 \kappa_2(Y) \geq (\kappa_{11}(sX,tY) + \langle Z_{11} \rangle) + \text{c.c.} ,\) \hspace{2cm} (20)

**Quantum UR** : \(|s|^2 \kappa_2(X) + |t|^2 \kappa_2(Y) \geq (\kappa_{11}(sX,tY) + \langle Z_{11} \rangle) + \text{c.c.} .\) \hspace{2cm} (21)
Given a probability distribution function $p(x)$ of the random variable $X$, the skewness $\kappa_3$ describes the distribution asymmetry. $\kappa_3$ is zero when distribution is symmetric.

That is, when the cross cumulant $\kappa_{11} = 0$ the classical theory predicts no constraint on variances, while in quantum mechanics there remains a constraint induced by $Z_{11}$. Third, in case of $\kappa_{11} = 0$ and $\langle Z_{11} \rangle = 0$, the Heisenberg form of uncertainty relation becomes trivial, while the generalized uncertainty relation is still meaningful. This fact tells that the observables $X$ and $Y$ are linearly uncorrelated, rather independent, i.e., correlation may appear nonlinearly in high order expansion.

2.3.3 Third order: the skewness uncertainty relation

The generalized uncertainty relation enables us to explore the high order nonlinear correlations between incompatible observables, which stands as the most prominent characteristic of it. Expand equation (14) to the third order, i.e. $s^m t^n$ with $m + n = 3$, we have the following Corollary:
Corollary 2 For two observables $X$ and $Y$, following skewness uncertainty relation holds:

$$
|s|^2 \left[ \kappa_2(X) + \frac{s + s^*}{2} \kappa_3(X) \right] + |t|^2 \left[ \kappa_2(Y) + \frac{t + t^*}{2} \kappa_3(Y) \right] \\
\geq \left[ \kappa_{11}(sX, tY) + \frac{\kappa_{12}(sX, tY) + \kappa_{21}(sX, tY)}{2} + \langle Z_{11} + Z_{12} + Z_{21} \rangle + \frac{\{\{Z_1, Z_{11}\} - 2\langle Z_1 \rangle \langle Z_{11} \rangle\}}{2!} \right] + \text{c.c.}, \forall s, t \in \mathbb{C}.
$$

(22)

where $Z_1 = sX + tY$ and $Z_{ij}$ are

$$
Z_{11} = \frac{1}{2}[sX, tY], \quad Z_{21} = \frac{1}{12}[sX, [sX, tY]], \quad Z_{12} = \frac{1}{12}[tY, [tY, sX]].
$$

(23)

The third order cumulant $\kappa_3$ names the skewness, which characterizes the distribution asymmetry, as shown in Figure 2.

2.4 The quantum and classical uncertainty relations

Along the same line, there is no difficulty to derive the Kurtosis uncertainty relation and so on. In view of the second and third order uncertainty relations, it is clear that there are two types of terms on the right hand side of the generalized uncertainty relations. One is about the cross cumulants that signify the classical correlations between physical observables, the other is the higher order noncommutative one from the Baker-Campbell-Hausdorff formula signifying the quantum correlation.

We can further obtain the following generalized uncertainty relation distinguishing the classical from quantum in correlation for all orders:

Proposition 1 For two physical observables, there exist a classical uncertainty relation

$$
\langle e^{sX+s^*X} \rangle \langle e^{tY+t^*Y} \rangle \geq \left| \langle e^{sX+tY} \rangle \right|^2,
$$

(24)

and a quantum one

$$
\langle e^{sX+s^*X} \rangle \langle e^{tY+t^*Y} \rangle \geq \left| \langle e^{sX+tY+\frac{1}{2}[sX,tY]+\ldots} \rangle \right|^2.
$$

(25)
Here $e^{sX+ıtY+\frac{1}{2}[sX,ıtY]+\cdots} = e^{sX}e^{ıtY}$.

The quantum relation (25) may be regarded as a strengthened version of the classical relation (24). One may naturally conclude that traditionally the greater lower bound in relation (25) than (24) tells that the incompatible observables $X$ and $Y$ possess less uncertainty in classical theory. Whereas, from the GUR point of view, the greater lower bound means that statistically the incompatible observables $X$ and $Y$ is highly correlated in quantum theory.

To exhibit the quantum effect, through an explicit example we show the classical relation (24) may be violated in quantum theory. For two observables $X = \sigma_x$, $Y = \sigma_y$, and the qubit state $|\psi\rangle = \cos \frac{\theta}{2}|+\rangle + e^{ı\phi} \sin \frac{\theta}{2}|-\rangle$, the left hand side of relation (24) turns to

$$\left(\cosh 2 + \cos \phi \sin \theta \sinh 2\right)\left(\cosh 2 + \sin \phi \sin \theta \sinh 2\right),$$

(26)

while the right hand side is

$$\frac{1}{2} \left[\sqrt{2} \cosh \sqrt{2} + \sin \theta \left(\cos \phi + \sin \phi\right) \sinh \sqrt{2}\right]^2.$$

(27)

Here, we assume $s = t = 1$ for simplicity. Expressions (26) and (27) are numerically plotted in Figure 3 where violations of equation (24) evidently exist. Whereas, according to the generalized uncertainty principle, the two observables possessing non-classical statistical correlations fall in the violation region.

3 Discussions

We propose a generalized uncertainty principle based on an alternative interpretation of Bohr’s concept of complementarity, where the incompatibility between observables is interpreted as the statistical correlations between their measurement outcomes. To
Figure 3. The quantum violation of the classical uncertainty relation. The blue surface represents the quantum prediction for the left hand side of relation (24), while the dark yellow surface is the right hand side of it. The quantum state is taken to be $|\psi\rangle = \cos \frac{\theta}{2} |+\rangle + e^{i\phi} \sin \frac{\theta}{2} |-\rangle$ with two observables of $\sigma_x$ and $\sigma_y$. The violation happens in two circled region around $(\phi, \theta) = \{(\pi, \frac{\pi}{2}), (\frac{3\pi}{2}, \frac{\pi}{2})\}$, where the two observables have statistical correlations that cannot be explained by classical theory.
elucidate the principle, a generalized uncertainty relation is obtained, which exhibits a full-order statistical dependence between physical observables. The lowest order approximation yields linear dependence and gives out the Heisenberg type uncertainty relation. In scope of generalized uncertainty principle the lower bound of the Heisenberg uncertainty relation, null, turns out to be nontrivial. In this situation, though observables are uncorrelated, but they are not independent. The third order skewness uncertainty relation is given, which characterizes the correlation distribution asymmetry. One explicit example is given in order to show how classical uncertainty limit is violated by the quantum theory. It is highly expected that the emergence of high-order nonlinear correlations may lead some new applications in the quantum information processing.

Acknowledgements

This work was supported in part by the Strategic Priority Research Program of the Chinese Academy of Sciences, Grant No.XDB23030100; and by the National Natural Science Foundation of China(NSFC) under the Grants 11975236, 11635009, and 11375200.

References

[1] W. Heisenberg, Über den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik, Z. Phys. 43, 172 (1927); in Quantum theory and Measurement, edited by J. A. Wheeler and W. H. Zurek, (Princeton University press, Princeton, NJ, 1983), pp. 62-84.

[2] H. P. Robertson The Uncertainty Principle, Phys. Rev. 34, 163-164 (1929).

[3] E. Schrödinger, About Heisenberg uncertainty relation, Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.) 19, 296 (1930) (see also, arXiv: quant-ph/9903100).
[4] D. Deutsch, Uncertainty in quantum measurements, Phys. Rev. Lett. 50, 631-633 (1983).

[5] H. Maassen and J. B. M. Uffink, Generalized entropic uncertainty relations, Phys. Rev. Lett. 60, 1103-1106 (1988).

[6] Jun-Li Li and Cong-Feng Qiao, Equivalence theorem of uncertainty relations, J. Phys. A: Math. Theor. 50, 03LT01 (2017).

[7] Jun-Li Li and Cong-Feng Qiao, The optimal uncertainty relation, Ann. Phys. (Berlin) 531, 1900143 (2019).

[8] N. Bohr, Atomic theory and the description of nature, (Cambridge University Press, 1934), pp.10.

[9] A. Stuart and J. Keith Ord, Kendall’s Advanced Theory of Statistics, Vol 1: Distribution Theory, 6th edition (Wiley, 2010).

[10] J. Novak and M. LaCroix, Three lectures on free probability, arXiv:1205.2097

[11] R. Kubo, Generalized cumulant expansion method, J. Phys. Soc. Jpn. 17, 1100-1120 (1962).
Appendix

A Proof of Theorem 1

For two physical observables \( X \) and \( Y \), we considering two state vectors \( e^{sX}\psi \rangle \) and \( e^{tY}\psi \rangle \), where the Cauchy-Schwarz inequality tells

\[
\langle \psi | e^{sX} e^{s^*X} | \psi \rangle \langle \psi | e^{tY} e^{t^*Y} | \psi \rangle \geq \langle \psi | e^{sX} e^{tY} | \psi \rangle \langle \psi | e^{sX} e^{tY} | \psi \rangle^* .
\] (S1)

Applying the logarithm on both sides we have

\[
\log [\langle e^{(s+s^*)X} \rangle] + \log [\langle e^{(t+t^*)Y} \rangle] \geq \log (\langle e^{Z} \rangle) + c.c.
\] (S2)

Here \( Z = \log(e^{sX} e^{tY}) = Z_{01} + Z_{10} + Z_{11} + \cdots \). Using the cumulants generating function we arrive the Theorem 1.

B Proof of Corollary 1

From equation (17) we have \( \forall s, t \in \mathbb{C} \)

\[
|s|^2 \kappa_2(X) + |t|^2 \kappa_2(Y) \geq \frac{st \langle XY + YX \rangle - 2 \langle X \rangle \langle Y \rangle}{2} + \frac{st \langle [X,Y] \rangle}{2} + c.c.
\]

\[
= \text{Re}[st] (\langle XY + YX \rangle - 2 \langle X \rangle \langle Y \rangle) + \text{Im}[st] i \langle [X,Y] \rangle .
\] (S3)

The Cauchy-Schwarz inequality implies

\[
\text{Re}[st] (\langle XY + YX \rangle - 2 \langle X \rangle \langle Y \rangle) + \text{Im}[st] i \langle [X,Y] \rangle
\]

\[
\leq \sqrt{\text{Re}(st)^2 + \text{Im}(st)^2} \sqrt{|\langle XY + YX \rangle - 2 \langle X \rangle \langle Y \rangle|^2 + |\langle [X,Y] \rangle|^2}
\]

\[
= |st| \sqrt{|\langle XY + YX \rangle - 2 \langle X \rangle \langle Y \rangle|^2 + |\langle [X,Y] \rangle|^2} .
\] (S4)

That is, by appropriately chosen \( s \) and \( t \) the left hand side of equation [S3] could become that of equation [S4].
C  The expansions of the generalized uncertainty relation

Here we give a detailed expansion for the following cumulant uncertainty relation

\[ K[(s + s^*)X] + K[(t + t^*)Y] \geq K(Z_{st}) + \text{c.c.}, \ \forall s, t \in \mathbb{C}. \quad (S5) \]

Let \( Z = \log(e^{sX}e^{tY}) \), the Baker-Campbell-Hausdorff formula gives

\[
Z = sX + tY + \frac{1}{2}[sX, tY] + \frac{1}{12} \left( [sX, [sX, tY]] + [tY, [tY, sX]] \right)
- \frac{1}{24} [tY, [sX, [sX, tY]]] - \frac{1}{720} \left( [[[sX, tY], tY], tY] + [[[tY, sX], sX], sX] \right) + \frac{1}{360} \left( [[[sX, tY], tY], tY] + [[[tY, sX], sX], sX] \right) + \frac{1}{120} \left( [[[tY, sX], tY], sX], tY] + [[[sX, tY], sX], tY], sX] \right) + \cdots
\]

\[ = Z_{10} + Z_{01} + Z_{11} + Z_{21} + Z_{12} + Z_{22} + Z_{14} + Z_{41} + Z_{23} + Z_{32} \]

\[ Z_{23}^{(1)} + Z_{32}^{(2)} + \cdots \quad (S6) \]

which gives the elements of operators set \( Z_{st} \) as

\[
Z_{01} = sX, \ Z_{10} = tY, \ Z_{11} = \frac{1}{2}[sX, tY], \quad (S7)
\]
\[
Z_{21} = \frac{1}{12} [sX, [sX, tY]], \ Z_{12} = \frac{1}{12} [tY, [tY, sX]], \quad (S8)
\]
\[
Z_{22} = -\frac{1}{24} [tY, [sX, [sX, tY]]], \quad (S9)
\]
\[
Z_{14} = -\frac{1}{720} [[[sX, tY], tY], tY], \ Z_{41} = \frac{1}{720} [[[tY, sX], sX], sX] \quad (S10)
\]
\[
Z_{23}^{(1)} = \frac{1}{360} [[[sX, tY], tY], sX], \ Z_{32}^{(1)} = \frac{1}{360} [[[tY, sX], sX], sX], \quad (S11)
\]
\[
Z_{23}^{(2)} = \frac{1}{120} [[[tY, sX], tY], sX] \quad (S12)
\]
Different orders of uncertainty relations may be obtained via the expansion of both sides.

The left hand side of equation (S5) goes as

\[ K[(s + s^*)X] = (s + s^*)\kappa_1(X) + \frac{(s + s^*)^2}{2!}\kappa_2(X) + \frac{(s + s^*)^3}{3!}\kappa_3(X) + \cdots, \quad (S13) \]

\[ K[(t + t^*)Y] = (t + t^*)\kappa_1(Y) + \frac{(t + t^*)^2}{2!}\kappa_2(Y) + \frac{(t + t^*)^3}{3!}\kappa_3(Y) + \cdots. \quad (S14) \]

On the right hand side, we have

\[ K(Z_{st}) = \log \left( \sum_{n=0}^{\infty} \frac{Z^n}{n!} \right) = \log \left( 1 + \langle Z \rangle + \frac{1}{2!}\langle Z^2 \rangle + \frac{1}{3!}\langle Z^3 \rangle + \cdots \right) \]

\[ = \langle Z \rangle + \frac{1}{2!}\langle Z^2 \rangle + \frac{1}{3!}\langle Z^3 \rangle + \frac{1}{4!}\langle Z^4 \rangle + \frac{1}{5!}\langle Z^5 \rangle + \cdots - \frac{1}{2} \left( \langle Z \rangle + \frac{1}{2!}\langle Z^2 \rangle + \frac{1}{3!}\langle Z^3 \rangle + \frac{1}{4!}\langle Z^4 \rangle + \cdots \right)^2 \]

\[ + \frac{1}{3} \left( \langle Z \rangle + \frac{1}{2!}\langle Z^2 \rangle + \frac{1}{3!}\langle Z^3 \rangle + \frac{1}{4!}\langle Z^4 \rangle + \cdots \right)^3 \]

\[ - \frac{1}{4} \left( \langle Z \rangle + \frac{1}{2!}\langle Z^2 \rangle + \frac{1}{3!}\langle Z^3 \rangle + \frac{1}{4!}\langle Z^4 \rangle + \cdots \right)^4 + \cdots. \quad (S15) \]

Here \( Z = Z_{10} + Z_{01} + Z_{11} + Z_{21} + Z_{12} + Z_{22} + Z_{14} + Z_{41} + \cdots \) as defined in equation (S6).

To the second order, the coefficients of the terms \( s^2 \), \( t^2 \), and \( st \) on right hand side are

\[ K^{(2)}(Z_{st}) = \langle Z_{11} \rangle + \frac{1}{2!}\langle (Z_{10} + Z_{01})^2 \rangle - \frac{1}{2}\langle Z_{10} + Z_{01} \rangle^2 \]

\[ = \frac{\langle [sX, tY] \rangle}{2} + \frac{\langle [sX + tY] \rangle^2 - \langle sX + tY \rangle^2}{2} \]

\[ = \frac{st\langle [X, Y] \rangle}{2} + \frac{s^2\kappa_2(X) + t^2\kappa_2(Y) + st\kappa_{11}(X, Y) + st\kappa_{11}(Y, X)}{2}. \quad (S16) \]

Now considering the third order, the right hand side for \( s^3 \), \( t^3 \), \( s^2t \), and \( st^2 \) gives

\[ K^{(3)}(Z_{st}) = \langle Z_{12} + Z_{21} \rangle + \frac{\langle (Z_{10} + Z_{01})Z_{11} + Z_{11}(Z_{10} + Z_{01}) \rangle}{2!} + \frac{\langle (Z_{10} + Z_{01})^3 \rangle}{3!} \]

\[ - \frac{\langle Z_{10} + Z_{01} \rangle (\langle (Z_{10} + Z_{01})^2 \rangle + 2\langle Z_{11} \rangle)}{2} + \frac{\langle Z_{10} + Z_{01} \rangle^3}{3!} \]

\[ = \frac{\langle [X, [X, Y]] \rangle}{12} + \frac{\langle [Y, [Y, X]] \rangle}{4} + \frac{\langle [X + Y, [X, Y]] \rangle}{3} + \frac{\langle (X + Y)^3 \rangle}{3!} \]

\[ - \frac{\langle X + Y \rangle (\langle (X + Y)^2 \rangle + \langle [X, Y] \rangle)}{2} + \frac{\langle X + Y \rangle^3}{3}. \quad (S17) \]
where we have write \( sX \mapsto X \) and \( tY \mapsto Y \) for simplicity. Further simplification leads to

\[
\begin{align*}
K^{(3)}(Z_{st}) &= \frac{\langle [X, [X, Y]] \rangle + \langle [Y, [Y, X]] \rangle}{12} + \frac{\langle \{X + Y, [X, Y]\} \rangle}{4} \\
&\quad - \frac{\langle X + Y \rangle \langle [X, Y] \rangle}{2} + \frac{\kappa_3(X + Y)}{3!} \\
&= \frac{\langle [X, [X, Y]] \rangle + \langle [Y, [Y, X]] \rangle}{12} + \frac{\langle \{X + Y, [X, Y]\} \rangle - 2\langle X + Y \rangle \langle [X, Y] \rangle}{4} \\
&\quad + \frac{\kappa_3(X + Y)}{3!}.
\end{align*}
\]

\[(S18)\]

Here \( \kappa_3(X + Y) = \kappa_3(X) + 3\kappa_{12}(X, Y) + 3\kappa_{21}(X, Y) + \kappa_3(Y). \)