THE ALON–MILMAN THEOREM FOR NON-SYMMETRIC BODIES

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Abstract. A classical theorem of Alon and Milman states that any $d$ dimensional centrally symmetric convex body has a projection of dimension $m \geq e^{c \sqrt{\ln d}}$ which is either close to the $m$-dimensional Euclidean ball or to the $m$-dimensional cross-polytope. We extended this result to non-symmetric convex bodies.

1. Introduction

Some fundamental results from the theory of normed spaces have been shown to hold in the more general setting of non-symmetric convex bodies. Dvoretzky’s theorem [Dvo61, Mil71] was extended in [LM75] and [Gor88]; Milman’s Quotient of Subspace theorem [Mil85] and duality of entropy results were extended in [MP00]. In this note, we extend the Alon–Milman Theorem.

A convex body is a compact convex set in $\mathbb{R}^d$ with non-empty interior. We denote the orthogonal projection onto a linear subspace $H$ or $\mathbb{R}^d$ by $P_H$. For $p = 1, 2, \infty$, the closed unit ball of $\ell_p^d$ centered at the origin is denoted by $B_p^d$. Let $K$ and $L$ be convex bodies in $\mathbb{R}^d$ with $L = -L$. We define their distance as

$$d(K, L) = \inf \{\lambda > 0 : L \subset T(K - a) \subset \lambda L \text{ for some } a \in \mathbb{R}^d \text{ and } T \in GL(\mathbb{R}^d)\}.$$ 

By compactness, this infimum is attained, and when $K = -K$, it is attained with $a = 0$.

Alon and Milman [AM83] proved the following theorem in the case when $K$ is centrally symmetric.

Theorem 1. For every $\varepsilon > 0$ there is a constant $C(\varepsilon) > 0$ with the property that in any dimension $d \in \mathbb{Z}^+$, and for any convex body $K$ in $\mathbb{R}^d$, at least one of the following two statements hold:

(i) there is an $m$-dimensional linear subspace $H$ of $\mathbb{R}^d$ such that $d(P_H(K), B_2^m) < 1 + \varepsilon$, for some $m$ satisfying $\ln \ln m \geq \frac{1}{2} \ln \ln d$, or

(ii) there is an $m$-dimensional linear subspace $H$ such that $d(P_H(K), B_1^m) < 1 + \varepsilon$, for some $m$ satisfying $\ln \ln m \geq \frac{1}{2} \ln \ln d - C(\varepsilon)$.

The main contribution of the present note is a way to deduce Theorem 1 from the original result of Alon and Milman, that is, the centrally symmetric case. By polarity, one immediately obtains

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Corollary 1. For every $\varepsilon > 0$ there is a constant $C(\varepsilon) > 0$ with the property that in any dimension $d \in \mathbb{Z}^+$, and for any convex body $K$ in $\mathbb{R}^d$ containing the origin in its interior, at least one of the following two statements hold:

(i) there is an $m$-dimensional linear subspace $H$ of $\mathbb{R}^d$ such that $d(H \cap K, B_2^m) < 1 + \varepsilon$, for some $m$ satisfying $\ln \ln m \geq \frac{1}{2} \ln \ln d$, or

(ii) there is an $m$-dimensional linear subspace $H$ such that $d(H \cap K, B_\infty^m) < 1 + \varepsilon$, for some $m$ satisfying $\ln \ln m \geq \frac{1}{2} \ln \ln d - C(\varepsilon)$.

2. Proof of Theorem \[1\]

For a convex body $K$ in $\mathbb{R}^d$, we denote its polar by $K^* = \{ x \in \mathbb{R}^d : \langle x, y \rangle \leq 1 \text{ for all } y \in K \}$. The support function of $K$ is $h_K(x) = \sup\{ \langle x, y \rangle : y \in K \}$. For basic properties, see \[\text{Sch14, BGVV14}].

First in Lemma \[2\] by a standard argument, we show that if the difference body $L - L$ of a convex body $L$ is close to the Euclidean ball, then so is some linear dimensional section of $L$. For this, we need Milman’s theorem whose proof (cf. \[\text{Mil71, FLM77, MS86}\]) does not use the symmetry of $K$ even if it is stated with that assumption.

Lemma 1 (Milman’s Theorem). For every $\varepsilon > 0$ there is a constant $C(\varepsilon) > 0$ with the property that in any dimension $d \in \mathbb{Z}^+$, and for any convex body $K$ in $\mathbb{R}^d$ with $B_2^d \subseteq K$, there is an $m$-dimensional linear subspace $H$ of $\mathbb{R}^d$ such that $(1 - \varepsilon)r(B_2^d \cap H) \subseteq K \subseteq (1 + \varepsilon)r(B_2^d \cap H)$, for some $m$ satisfying $m \geq C(\varepsilon)m^2d$, where

$$M = M(K) = \int_{S_2^{d-1}} ||x||_K d\sigma(x),$$

and $r = \frac{1}{M}$.

Lemma 2. Let $\alpha, \varepsilon > 0$ be given. Then there is a constant $c = c(\alpha, \varepsilon)$ with the property that in any dimension $m \in \mathbb{Z}^+$, and for any convex body $L$ in $\mathbb{R}^m$ with $d(L - L, B_2^m) < 1 + \alpha$, there is a $k$ dimensional linear subspace $F$ of $\mathbb{R}^m$ such that $d(P_F(L), B_2^k) < 1 + \varepsilon$ for some $k \geq cm$.

Proof. Let $\delta = d(L - L, B_2^m)$. We may assume that $\frac{1}{\delta}B_2^m \subseteq L - L \subseteq B_2^m$. Thus, $h_{L-L} \geq \frac{1}{\delta}$.

With the notations of Lemma \[1\] we have

$$M(L^*) = \int_{S_2^{d-1}} ||x||_L d\sigma(x) = \frac{1}{2} \int_{S_2^{d-1}} h_L(x) + h_L(-x) d\sigma(x)$$

$$= \frac{1}{2} \int_{S_2^{d-1}} h_{L-L}(x) d\sigma(x) \geq \frac{1}{2\delta} \geq \frac{1}{2(1 + \alpha)}.$$

Note that $L^* \supset (L - L)^* \supset B_2^d$, thus, by Lemma \[1\] and polarity, we obtain that $L$ has a $k$ dimensional projection $P_F$ with $d(P_F L, B_2^d \cap F) \leq 1 + \varepsilon$ and $k \geq C(\varepsilon)\frac{1}{4(1 + \alpha)^2}m$. Here, $C(\varepsilon)$ is the same as in Lemma \[1\]. \[\square\]
The novel geometric idea of our proof is the following. We call a convex body \( T = \text{conv} \left( T_1 \cup \{ \pm e \} \right) \) in \( \mathbb{R}^m \) a double cone if \( T_1 = -T_1 \) is convex set, \( \text{span} \ T_1 \) is an \( (m-1) \)-dimensional linear subspace, and \( e \in \mathbb{R}^m \setminus \text{span} \ T_1 \). Double cones are irreducible convex bodies, that is, for any double cone \( T \), if \( T = L-L \) then \( L = T/2 \), see [NV03, Yos91]. We prove a stability version of this fact.

**Lemma 3** (Stability of irreducibility of double cones). Let \( L \) be a convex body in \( \mathbb{R}^m \) with \( m \geq 2 \), and \( T \) be a double cone of the form \( T = \text{conv} \left( T_1 \cup \{ \pm e \} \right) \). Assume that \( T \subseteq L-L \subseteq \delta T \) for some \( 1 \leq \delta < \frac{3}{2} \). Then

\[
\left( \frac{3}{2} - \delta \right) T \subseteq L - a \subseteq \left( \delta - \frac{1}{2} \right) T.
\]

for some \( a \in \mathbb{R}^m \).

**Proof.** By the assumptions, \( e \in T \subseteq L-L \), thus, by translating \( L \), we may assume that \( o, e \in L \). Thus,

\[
L \subseteq (L-L) \cap (L-L+e) \subseteq \delta T \cap (\delta T + e).
\]

We claim that

\[
\delta T \cap (\delta T + e) = \frac{e}{2} + \left( \delta - \frac{1}{2} \right) T.
\]

Indeed, let \( H_\lambda \) denote the hyperplane \( H_\lambda = \lambda e + e^\perp \). To prove (3), we describe the sections of the right hand side and the left hand side by the hyperplanes \( H_\lambda \) for all relevant values of \( \lambda \). For any \( \lambda \in [-\delta, \delta] \), we have

\[
\delta T \cap H_\lambda = \delta (T \cap H_{\lambda/\delta}) = \lambda e + \delta \left( 1 - \frac{|\lambda|}{\delta} \right) T_1.
\]

For any \( \lambda \in [-\delta + 1, \delta + 1] \), we have

\[
(\delta T + e) \cap H_\lambda = e + (\delta T \cap H_{\lambda-1}) = \lambda e + \delta \left( 1 - \frac{|\lambda - 1|}{\delta} \right) T_1.
\]

Thus, for any \( \lambda \in [-\delta + 1, \delta] \), we have

\[
\delta T \cap (\delta T + e) \cap H_\lambda = \lambda e + \delta \left( 1 - \frac{1}{\delta} \max \{|\lambda|,|\lambda-1|\} \right) T_1.
\]

On the other hand, for any \( \lambda \in [-\delta + 1, \delta] \), we have

\[
(\frac{e}{2} + (\delta - 1/2) T) \cap H_\lambda = \lambda e + (\delta - 1/2) \left( 1 - \frac{|\lambda - 1/2|}{\delta - 1/2} \right) T_1.
\]

Combining these two equations yields (3).

Thus,

\[
T \subseteq L-L = \left( L - \frac{e}{2} \right) - \left( L - \frac{e}{2} \right) \subseteq \left( L - \frac{e}{2} \right) - \left( \delta - \frac{1}{2} \right) T.
\]
Using the fact that \( T = -T \), and \( 1 \leq \delta < 3/2 \), we obtain
\[
\left( \frac{3}{2} - \delta \right) T \subseteq L - \frac{\varepsilon}{2},
\]
finishing the proof of Lemma 3.

Now, we are ready to prove Theorem 1. With the notations of the theorem, let \( D = K - K \), and apply the symmetric version of the theorem for \( D \) in place of \( K \). We may assume that \( \varepsilon < 1/2 \). In case (i), we use Lemma 2 and loose a linear factor in the dimension of the almost-euclidean projection. In case (ii), we use Lemma 3 with \( T = B_m^1 \) and obtain the same dimension for the almost-\( \ell^m_1 \) projection.

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