Random Activations in Primal-Dual Splittings for Monotone Inclusions with a priori Information

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Abstract In this paper, we propose a numerical approach for solving composite primal-dual monotone inclusions with a priori information. The underlying a priori information set is represented by the intersection of fixed point sets of a finite number of operators, and we propose and algorithm that activates the corresponding set by following a finite-valued random variable at each iteration. Our formulation is flexible and includes, for instance, deterministic and Bernoulli activations over cyclic schemes, and Kaczmarz-type random activations. The almost sure convergence of the algorithm is obtained by means of properties of stochastic Quasi-Fejér sequences. We also recover several primal-dual algorithms for monotone inclusions in the context without a priori information and classical algorithms for solving convex feasibility problems and linear systems. In the context of convex optimization with inequality constraints, any selection of the constraints defines the a priori information set, in which case the operators involved are simply projections onto half spaces. By incorporating random projections onto a selection of the constraints to classical primal-dual schemes, we obtain faster algorithms as we illustrate by means of a numerical application to a stochastic arc capacity expansion problem in a transport network.

Keywords Arc capacity expansion in traffic networks · Monotone operator theory · Primal-dual splitting algorithms · Randomized Kaczmarz algorithm · Stochastic Quasi-Fejér sequences.

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1 Introduction

We devote this paper to develop an efficient numerical algorithm for solving primal–dual monotone inclusions involving a priori information on the primal solutions. Primal-dual inclusions have found many applications, such as evolution inclusions [1, 2], variational inequalities [3–5], partial differential equations (PDEs) [4, 6], and Nash equilibria [7]. In particular, when the monotone operators are subdifferentials of convex functions, the inclusion we study reduces to an optimization problem with a priori information. This problem arises in many applications, such as arc capacity expansion in traffic networks [8], image recovery [9–11], and signal processing [12–14].

The a priori information is modeled by the intersection of fixed point sets of a finite number of averaged nonexpansive operators. In the particular case when these operators are projections onto closed convex sets, the a priori information is represented by their intersection. More generally, if the operators are resolvents, the a priori information set models common solutions to several convex optimization problems and monotone inclusions.

In the absence of a priori information, the studied problem can be solved by [15], and by [16] in the convex optimization case. These methods generalize several classical algorithms for monotone inclusions and convex optimization as, e.g., [17–19]. In the case when the a priori information is the fixed point set of a single operator, an extension of the method in [15] including the activation of the operator at each iteration is proposed in [20]. This method is applied to linearly constrained convex optimization problems, in which the a priori information set is a selection of the constraints. This formulation leads to an extension of the method in [16], which includes a projection onto the set defined by the selected constraints. This enforces feasibility on primal iterates on the chosen set, resulting in a more efficient algorithm as verified numerically.

The previous approach opens the question on finding an appropriate manner to select and project onto the constraints, in order to induce more efficient methods. In the particular context of convex feasibility, several projecting schemes are proposed in the literature, e.g., [21–24]. In the case of solving overdetermined consistent linear systems, a cyclic deterministic projection scheme over the hyperplanes generated by each of the equalities is proposed in [21]. A randomized version of the method in [21] is derived in [22], where the probability of activation of each hyperplane is proportional to the size of its normal vector. As a consequence, the method exhibits an exponential convergence rate in expectation. Beyond consistent linear systems, several projecting schemes are proposed in [23, 24] for the convex feasibility problem, including static, cyclic, and quasi-cyclic projections. A random block coordinate method using parallel Bernoulli activation is proposed in [25] for the resolution of monotone inclusion problems.

In the context of convex optimization with a priori information defined by the intersection of convex sets, this paper aims at combining previous projection schemes
with the primal-dual method in [15]. In the more general context of monotone inclusions with a priori information, our goal is to extend this idea to combine several activation schemes on the operators defining the a priori information set with the primal-dual splitting in [15] for monotone inclusions. As a result, we obtain a generalization of the methods in [15, 21, 22, 25] and a unified manner to activate the operators, including [21, 22, 25] and some schemes in [24, 25].

We illustrate the numerical efficiency of our method in the arc capacity expansion problem in transport networks, corresponding to a convex optimization with linear inequality constraints. We provide 13 algorithms by varying the projecting schemes available in the literature. In Section 4 we implement different activation schemes in the context of the arc capacity expansion problem in transport networks and we compare their efficiency with respect to the method without projections, justifying the advantage of our approach.

This paper is organized as follows. In Section 2 we introduce our notation and some preliminaries. In Section 3 we provide the main algorithm, we prove its almost sure weak convergence, and we exploit the flexibility of our approach obtaining several schemes in the literature. In Section 4 we implement different activation schemes in the context of the arc capacity expansion problem in transport networks and we compare their efficiency with respect to the method without any activation. Finally, we provide some conclusions and perspectives.

2 Notation and Preliminaries

Throughout this paper, $\mathcal{H}$ stands for a real separable Hilbert space, the identity operator on $\mathcal{H}$ is denoted by $\text{Id}$, and $\rightharpoonup$ and $\rightarrow$ denote weak and strong convergence in $\mathcal{H}$, respectively. The set of weak sequential cluster points of a sequence $(x_n)_{n \in \mathbb{N}}$ in $\mathcal{H}$ is denoted by $\mathfrak{W}(x_n)_{n \in \mathbb{N}}$. The projector operator onto a nonempty closed convex set $S \subset \mathcal{H}$ is denoted by $P_S$, its normal cone is denoted by $N_S$, and its strong relative interior is denoted by $\text{sri}(S)$. Given $\alpha \in [0, 1]$, an operator $T : \mathcal{H} \to \mathcal{H}$ is $\alpha$-averaged nonexpansive if, for every $x$ and $y$ in $\mathcal{H}$, we have $\|Tx - Ty\|^2 \leq \|x - y\|^2 - \frac{1-\alpha}{2}(\|\text{Id} - T\|)\|x - (\text{Id} - T)y\|^2$.

Let $M : \mathcal{H} \rightrightarrows \mathcal{H}$ be a set-valued operator. We denote by $\text{dom} M$ the domain of $M$, by $\text{ran}(M)$ its range of $M$, and by $\text{gra} M$ its graph. The inverse $M^{-1}$ of $M$ is the operator defined by $M^{-1} : u \mapsto \{x \in \mathcal{H} : u \in Mx\}$. Given $\rho \geq 0$, $M$ is $\rho$-strongly monotone iff, for every $(x, u)$ and $(y, v)$ in $\text{gra}(M)$, $\langle x - y, u - v \rangle \geq \rho \|x - y\|^2$, it is $\rho$-coercive iff $M^{-1}$ is $\rho$-strongly monotone, $M$ is monotone iff it is $0$-strongly monotone, and it is maximally monotone iff its graph is maximal, in the sense of inclusions in $\mathcal{H} \times \mathcal{H}$, among the graphs of monotone operators. The resolvent of $M$ is denoted by $J_M = (\text{Id} + M)^{-1}$. If $M$ is maximally monotone, then $J_M$ is single-valued and $1/2$—averaged nonexpansive operator, and $\text{dom} J_M = \mathcal{H}$. The parallel sum of $A : \mathcal{H} \rightrightarrows \mathcal{H}$ and $B : \mathcal{H} \rightrightarrows \mathcal{H}$ is defined by $A \square B = (A^{-1} + B^{-1})^{-1}$. We denote by $\mathcal{P} \mathcal{O} \mathcal{O}(\mathcal{H})$ the set of proper, lower semicontinuous and convex functions from $\mathcal{H}$ to $]-\infty, +\infty]$. The subdifferential of $f \in \mathcal{P} \mathcal{O} \mathcal{O}(\mathcal{H})$, denoted by $\partial f$, is maximally monotone; if $f$ is Gateaux differentiable in $x$ then $\partial f(x) = \{\nabla f(x)\}$, and we have $(\partial f)^{-1} = \partial f^*$, where $f^* \in \mathcal{P} \mathcal{O} \mathcal{O}(\mathcal{H})$ is the Fenchel conjugate of $f$. The infimal convolution of the two functions $f$ and $g$ from $\mathcal{H}$ to $]-\infty, +\infty]$ is defined by $f \boxplus g : x \mapsto \inf_{y \in \mathcal{H}} (f(y) + g(x - y))$. The proximal
operator of $f \in \Gamma_0(\mathcal{H})$ is defined by $\text{prox}_f : x \mapsto \arg\min_{y \in \mathcal{H}} f(y) + \frac{1}{2}\|x - y\|^2$ and we have $J_{\partial f} = \text{prox}_f$. Moreover, if $S \subset \mathcal{H}$ is a nonempty convex closed subset, then $\iota_S \in \Gamma_0(\mathcal{H})$. $N_S = \partial \iota_S$, and $J_{N_S} = P_S$, where $\iota_S$ assigns to $x \in \mathcal{H}$ the value $0$ if $x$ belongs to $S$ and $+\infty$, otherwise. For further results on monotone operator theory and convex optimization, the reader is referred to \cite{20}.

Throughout this paper $(\Omega, \mathcal{X}, \mathbb{P})$ is a fixed probability space. The space of all random variables $z$ with values in $\mathcal{H}$ such that $\|z\|$ is integrable is denoted by $L^1(\Omega, \mathcal{X}, \mathbb{P}; \mathcal{H})$. Given a $\sigma$-algebra $\mathcal{E}$ of $\Omega$, $x \in L^1(\Omega, \mathcal{X}, \mathbb{P}; \mathcal{H})$, and $y \in L^1(\Omega, \mathcal{X}, \mathbb{P}; \mathcal{H})$, $y$ is the conditional expectation of $x$ with respect to $\mathcal{E}$ iff, for every $E \in \mathcal{E}$, $\int_E x d\mathbb{P} = \int_E y d\mathbb{P}$, in which case we write $y = E(x | \mathcal{E})$. The characteristic function on $D \subset \Omega$ is denote by $\mathbbm{1}_D$, which is $1$ in $D$ and $0$ otherwise. An $\mathcal{H}$-valued random variable is a measurable map $x : (\Omega, \mathcal{X}) \to (\mathcal{H}, \mathcal{B})$, where $\mathcal{B}$ is the Borel $\sigma$-algebra. The $\sigma$-algebra generated by a family $\Phi$ of random variables is denoted by $(\sigma(\Phi))$. Let $\mathcal{X} = (\mathcal{X}_n)_{n \in \mathbb{N}}$ be a sequence of sub-$\sigma$ algebras of $\mathcal{X}$ such that $\big(\forall n \in \mathbb{N}\big)$ $\mathcal{X}_n \subset \mathcal{X}_{n+1}$. We denote by $\ell^\infty(\mathcal{X})$ the set of sequences of $[0, +\infty]$-valued random variables $(\xi_n)_{n \in \mathbb{N}}$ such that, for every $n \in \mathbb{N}$, $\xi_n$ is $\mathcal{X}_n$-measurable. We set

$$
(\forall \mu \in [0, +\infty]) \quad \ell^\infty_n(\mathcal{X}) := \left\{ (\xi_n)_{n \in \mathbb{N}} \in \ell^\infty(\mathcal{X}) \ : \ \sum_{n \in \mathbb{N}} \xi_n^\mu < +\infty \ \mathbb{P}\text{-a.s.} \right\}.
$$

(2.1)

Equalities and inequalities involving random variables will always be understood to hold $\mathbb{P}$--almost surely, even if the expression “$\mathbb{P}$--a.s.” is not explicitly written.

The following result is an especial case of \cite{27} Theorem 1 and is the main tool to prove the convergence of Stochastic Quasi-Fejér sequences.

\textbf{Lemma 2.1} \cite{25} Proposition 2.3] Let $\mathcal{Z}$ be a nonempty closed subset of a real Hilbert space $\mathcal{H}$, let $(\mathcal{X}_n)_{n \in \mathbb{N}}$ be a sequence of $\mathcal{H}$-valued random variables, and let $\mathcal{X} = (\mathcal{X}_n)_{n \in \mathbb{N}}$ be a sequence of sub-$\sigma$ algebras of $\mathcal{X}$ such that, for every $n \in \mathbb{N}$, $\mathcal{X}_n \subset \mathcal{X}_{n+1}$. Let $(b_n)_{n \in \mathbb{N}} \in \ell^\infty(\mathcal{X})$ be such that

$$
(\forall n \in \mathbb{N}) \quad \mathbb{E}(\|x^{n+1} - z\|^2 | \mathcal{X}_n) + b_n \leq \|x^n - z\|^2 \quad \mathbb{P} - a.s.
$$

(2.2)

Then, $(b_n)_{n \in \mathbb{N}} \in \ell^\infty(\mathcal{X})$. Moreover, if $\mathcal{M}(\mathcal{X}_n)_{n \in \mathbb{N}} \subset \mathcal{Z}$ $\mathbb{P}$--a.s., then $(\mathcal{X}_n)_{n \in \mathbb{N}}$ converges weakly $\mathbb{P}$--a.s. to a $\mathcal{Z}$--valued random variable.

\section{Main Problem and Algorithm}

We consider the following problem.

\textbf{Problem 3.1} Let $A : \mathcal{H} \rightrightarrows \mathcal{H}$, $B : \mathcal{G} \rightrightarrows \mathcal{G}$, and $D : \mathcal{G} \rightrightarrows \mathcal{G}$ be maximally monotone operators such that $D$ is $\delta$--strongly monotone, for some $\delta > 0$, let $C : \mathcal{H} \rightarrow \mathcal{H}$ be a $\mu$-- cocoercive operator, for some $\mu > 0$, and let $L : \mathcal{H} \rightarrow \mathcal{G}$ be a nonzero bounded linear operator. For every $i \in \{1, \ldots, m\}$, let $T_i : \mathcal{H} \rightarrow \mathcal{H}$ be a $\alpha_i$--averaged nonexpansive operator, for some $\alpha_i \in [0, 1]$, and set $S := \bigcap_{i=1}^m \text{Fix}(T_i)$. The problem is
to find \((x, u) \in (S \times G) \cap Z_0\), where \(Z_0\) is the set of primal-dual solutions to

\[
\begin{align*}
\text{find } (x, u) \in \mathcal{H} \times \mathcal{G} & \quad \text{s.t.} \quad \begin{cases} 
0 \in Ax + L^* u + Cx, \\
0 \in B^{-1} u - Lx + D^{-1} u,
\end{cases} \\
\end{align*}
\tag{3.1}
\]

and we assume that \(Z := (S \times G) \cap Z_0 \neq \emptyset\).

The set \(S\) represents a priori information on the primal solution. In the case when, for every \(i \in \{1, \ldots, m\}\), \(T_i = P_{S_i}\) for nonempty closed convex sets \((S_i)_{1 \leq i \leq m}\), the a priori information set is simply \(S = \cap_{i=1}^m S_i\). Alternatively if, for every \(i \in \{1, \ldots, m\}\), \(T_i = J_{M_i}\) for some maximally monotone operator \(M_i\) defined in \(\mathcal{H}\), Problem 3.1 reduces to find a common solution to a finite number of monotone inclusions, and this approach works in more general settings by choosing the operators \((T_i)_{1 \leq i \leq m}\) appropriately.

In [20], Problem 3.1 is solved when \(m = 1\), by including a deterministic activation of \(T_1\). This method reduces to the method in [15] in the case when \(T_1 = \text{Id}\) and, hence, \(S = \mathcal{H}\).

In the particular case when \(A = \partial F, B = \partial G, C = \nabla H,\) and \(D = \partial \ell\), where \(F \in \Gamma_0(\mathcal{H}), G \in \Gamma_0(\mathcal{G}), H : \mathcal{H} \rightarrow \mathbb{R}\) is a differentiable convex function with \(\mu^{-1}\)-Lipschitz gradient, and \(\ell \in \Gamma_0(\mathcal{G})\) is a \(\delta\)-strongly convex, every solution \((x, u) \in Z\) is a solution to the following primal optimization problem with a priori information

\[
\begin{align*}
\text{find } x \in S \cap \arg\min_{x \in \mathcal{H}} (F(x) + (G \square \ell)(Lx) + H(x)) & \quad (P_1) \\
\end{align*}
\]

and its dual problem

\[
\begin{align*}
\text{find } u \in \arg\min_{u \in \mathcal{G}} (G^*(u) + (F + H)^*(-L^* u) + \ell^*(u)) & \quad (D_1)
\end{align*}
\]

Moreover, if the following qualification condition is satisfied

\[
0 \in sri \left( L(\text{dom } F) - (\text{dom } G + \text{dom } \ell) \right).
\tag{3.2}
\]

then, by [23, Proposition 4.3] the sets of solutions coincide. In [19] a primal-dual algorithm for solving \((P_1)\) is proposed, in the case when \(H = 0\), \(\ell = \delta_{\{0\}}\), and \(S = \mathcal{H}\). In [16] the previous algorithm is extended to the case \(H \neq 0\). From [20] we obtain a method to solve the case when \(m = 1\) and \(S = \text{Fix } T_1\), which incorporates a deterministic activation of \(T_1\) at each iteration.

A particular instance of \((P_1)\) is the resolution of overdetermined consistent linear systems, in which \((P_1)\) reduces to find \(x \in S = \bigcap_{i=1}^m S_i\), where, for every \(i \in \{1, \ldots, m\}\), \(S_i\) is the hyperplane defined by a linear equation \(r_i^\top x = b_i\) in finite dimensions. In this context, the Kaczmarz method implements cyclic projections onto the hyperplanes are converging to a feasible solution to the problem in [21].

A randomized version of the Kaczmarz method is proposed in [22] for solving consistent and overdetermined linear systems. This Randomized Kaczmarz algorithm has an exponential convergence rate in expectation. In the convex feasibility setting, in which \((S_i)_{1 \leq i \leq m}\) are general close convex sets, alternative converging projecting schemes are proposed in [23][24].
Previous projecting schemes motivates the following result, which combines randomized/alternating activation of \( \{I, T_1, \ldots, T_m\} \) with the primal-dual method in [15]. Our method extends the fixed activation scheme proposed in [20]. We obtain a weakly convergent \( \mathbb{P} \)-a.s. algorithm, where \( \mathbb{P} \) is the probability measure associated with the the sequence of random variables modelling the operator activation. These sequences are defined in the probability space \((\Omega, \mathcal{X}, \mathbb{P})\).

**Theorem 3.1** Consider the setting of Problem 3.1. Let \( \tau \in [0, 2\mu] \), let \( \gamma \in [0, 2\delta] \) be such that

\[
\|L\|^2 < \left( \frac{1}{\gamma} - \frac{1}{2\delta} \right) \left( \frac{1}{\tau} - \frac{1}{2\mu} \right). \tag{3.3}
\]

Let \((x^0, \mathcal{Z}^0, u^0) \in \mathcal{H} \times \mathcal{H} \times \mathcal{G} \) be such that \( x^0 = \mathcal{Z}^0 \) and set \( I = \{0,1,\ldots,m\} \). Let \((\epsilon_k)_{k \in \mathbb{N}}\) be a sequence of independent random variables such that, for every \( k \in \mathbb{N} \), \( \epsilon_k(\Omega) = I_k \subset I \) and consider the following routine

\[
(\forall k \in \mathbb{N}) \begin{cases}
  u^{k+1} & = J_{\gamma B^{-1}} \left( u^k + \gamma \left( L x^k - D^{-1} u^k \right) \right) \\
p^{k+1} & = J_{\tau A} \left( x^k - \tau \left( L^* u^{k+1} + C x^k \right) \right) \\
x^{k+1} & = T_{\epsilon_k+1} p^{k+1} \\
\hat{x}^{k+1} & = x^{k+1} + p^{k+1} - x^k,
\end{cases} \tag{3.4}
\]

where \( T_0 = \text{Id} \). Suppose that one of the following hold:

(i) \( Z_0 \subset S \times \mathcal{G} \).

(ii) There exists \( N \in \mathbb{N} \setminus \{0\} \) such that \((\forall n \in \mathbb{N}) I = \bigcup_{k=n}^{n+N-1} I_k \) and

\[
0 < \zeta := \inf_{k \in \mathbb{N}} \min_{i \in I_k \setminus \{0\}} \pi^k_i, \tag{3.5}
\]

where, for every \( k \in \mathbb{N} \) and \( i \in I_k \), \( \pi^k_i = \mathbb{P}(\epsilon^k_1 \{i\}) \).

Then \(((x^k, u^k))_{k \in \mathbb{N}}\) converges weakly \( \mathbb{P} \)-a.s. to a \( Z \)-valued random variable.

**Proof.** Fix \( k \in \mathbb{N} \) and \((\hat{x}, \hat{u}) \in Z \). It follows from (3.4) that

\[
\frac{x^k - p^{k+1}}{\tau} - A p^{k+1} + L(x^k + p^{k} - x^{k-1}) - D^{-1} u^k \in B^{-1} u^{k+1}. \tag{3.6}
\]

Since \( A \) and \( B^{-1} \) are maximally monotone operators [26, Theorem 20.25], we deduce from (3.6) that

\[
\begin{aligned}
\left\langle x^k - p^{k+1} \frac{1}{\tau} - L^* (u^{k+1} - \hat{u}) | p^{k+1} - \hat{\tilde{x}} \right\rangle & - \left\langle C x^k - C \hat{x} | p^{k+1} - \hat{\tilde{x}} \right\rangle \\
+ \left\langle \frac{1}{\gamma} y^k - \hat{y}^{k+1} \right\rangle & + L(x^k + p^k - x^{k-1} - \hat{\tilde{x}}) | u^{k+1} - \hat{\tilde{u}} \right\rangle \\
- \left\langle D^{-1} u^k - D^{-1} \hat{\tilde{u}} | u^{k+1} - \hat{\tilde{u}} \right\rangle & \geq 0. \tag{3.7}
\end{aligned}
\]
Hence, it follows from \cite{20} Lemma 2.12(i) that
\[
\frac{\|x^k - \hat{x}\|^2}{\tau} + \frac{\|u^k - \hat{u}\|^2}{\gamma} \geq \frac{\|x^k - p^{k+1}\|^2}{\tau} + \frac{\|p^{k+1} - \hat{x}\|^2}{\tau} + \frac{\|p^{k+1} - u^{k+1}\|^2}{\gamma} + 2\langle L(p^{k+1} - \hat{x}) \mid u^{k+1} - \hat{u} \rangle
\]
\[
- 2\langle L(x^k + p^k - x^{k-1} - \hat{x}) \mid u^{k+1} - \hat{u} \rangle + 2 \langle D^{-1}u^k - D^{-1}\hat{u} \mid u^{k+1} - \hat{u} \rangle
\]
\[
+ 2 \langle Cx^k - C\hat{x} \mid p^{k+1} - \hat{x} \rangle.
\] (3.8)

Moreover, Cauchy-Schwartz inequality yields
\[
\langle L(p^{k+1} - \hat{x}) \mid u^{k+1} - \hat{u} \rangle - \langle L(x^k + p^k - x^{k-1} - \hat{x}) \mid u^{k+1} - \hat{u} \rangle
\]
\[
= \langle L(p^{k+1} - \hat{x}) \mid u^{k+1} - \hat{u} \rangle - \langle L(x^k - \hat{x}) \mid u^{k+1} - \hat{u} \rangle - \langle L(p^k - x^{k-1}) \mid u^{k+1} - \hat{u} \rangle
\]
\[
= \langle L(p^k - x^k) \mid u^{k+1} - \hat{u} \rangle - \langle L(p^k - x^{k-1}) \mid u^{k+1} - \hat{u} \rangle - \langle L(p^k - x^k) \mid u^{k+1} - \hat{u} \rangle - \langle L(p^k - x^{k-1}) \mid u^k - \hat{u} \rangle
\]
\[
\geq \langle L(p^{k+1} - x^k) \mid u^{k+1} - \hat{u} \rangle - \|L\|\|p^k - x^{k-1}\|\|u^{k+1} - u^k\|
\]
\[
- \langle L(p^k - x^{k-1}) \mid u^k - \hat{u} \rangle
\] (3.9)
\[
\geq \langle L(p^{k+1} - x^k) \mid u^{k+1} - \hat{u} \rangle - \nu\|L\|^2\|p^k - x^{k-1}\|^2
\]
\[
- \frac{1}{\nu}\|u^{k+1} - u^k\| - \langle L(p^k - x^{k-1}) \mid u^k - \hat{u} \rangle,
\] (3.10)

for every \(\nu > 0\). In addition, the cocoercivity of \(C\) and \(D^{-1}\) and the inequality \(ab \leq \frac{a^2}{4\mu} + \frac{b^2}{4\nu}\) yield
\[
\langle Cx^k - C\hat{x} \mid p^{k+1} - \hat{x} \rangle = \langle Cx^k - C\hat{x} \mid p^{k+1} - x^k \rangle + \langle Cx^k - C\hat{x} \mid x^k - \hat{x} \rangle
\]
\[
\geq -\|C\|\|x^k - \hat{x}\|\|p^{k+1} - x^k\| + \mu\|C\|\|x^k - C\hat{x}\|^2
\]
\[
\geq -\frac{\|p^{k+1} - x^k\|^2}{4\mu},
\] (3.11)

and, analogously, \(\langle D^{-1}u^k - D^{-1}\hat{u} \mid u^{k+1} - \hat{u} \rangle \geq -\frac{\|u^{k+1} - u^k\|^2}{4\nu}\).

Let \((\hat{x}, \hat{u}) \in Z\) and let \(\mathcal{F} = (X_k)_{k \in \mathbb{N}}\) be a sequence of sub-sigma-algebras of \(\mathcal{X}\) such that, for every \(k \in \mathbb{N}\), \(X_k = \sigma(x^0, \ldots, x^k)\). It follows from (3.10), the linearity of conditional expectation, and the mutual independence of \((\epsilon_k)_{k \in \mathbb{N}}\) that
\[
E(||x^{k+1} - \hat{x}||^2 \mid X_k) = E\left(\sum_{i \in I_{k+1}^0} 1_{\{\epsilon_{k+1} = i\}} \cdot ||T_i p^{k+1} - \hat{x}||^2 \mid X_k\right)
\]
\[
= \pi_{k+1}^i ||p^{k+1} - \hat{x}||^2 + \sum_{i \in I_{k+1}^0} \pi_{k+1}^i ||T_i p^{k+1} - \hat{x}||^2,
\] (3.12)
where, for every \( k \in \mathbb{N} \), \( \pi_k^i := I_k \setminus \{0\} \). Moreover, since, for every \( i \in \{1, \ldots, m\} \), \( T_i \) is \( \alpha_i \)-averaged nonexpansive, where \( \alpha_i \in ]0,1[ \), and \( \hat{x} \in \cap_{i \in I} \text{Fix} T_i \), we have

\[
\|T_i p^{k+1} - \hat{x}\|^2 \leq \|p^{k+1} - \hat{x}\|^2 - \left( \frac{1 - \alpha_i}{\alpha_i} \right) \|T_i p^{k+1} - p^{k+1}\|^2. \tag{3.13}
\]

Hence, since, for every \( k \in \mathbb{N} \), \( \sum_{i \in \mathcal{I}_k} \pi_k^i = 1 \), from (3.10) we deduce, \( \mathbb{P} \)-a.s.

\[
\mathbb{E}(\|x^{k+1} - \hat{x}\|^2 | \mathcal{X}_k) + \sum_{i \in \mathcal{P}_{k+1}} \frac{\pi_{k+1}^i (1 - \alpha_i)}{\alpha_i} \|T_i p^{k+1} - p^{k+1}\|^2 \leq \|p^{k+1} - \hat{x}\|^2. \tag{3.14}
\]

Replacing (3.10), (3.11), and (3.13) in (3.8) we have, \( \mathbb{P} \)-a.s.

\[
\frac{\|x^k - \hat{x}\|^2}{\tau} + \frac{\|u^k - \hat{u}\|^2}{\gamma} \geq \frac{1}{\tau} \mathbb{E}(\|x^{k+1} - \hat{x}\|^2 | \mathcal{X}_k) + \left( \frac{1}{\tau} - \frac{1}{2 \mu} \right) \|x^k - p^{k+1}\|^2 \\
+ \frac{\|u^{k+1} - \hat{u}\|^2}{\gamma} + \left( \frac{1}{\gamma} - \frac{1}{2 \beta} - \frac{1}{\nu} \right) \|u^{k+1} - u^k\|^2 \\
+ 2(L(p^{k+1} - x^k) | u^{k+1} - \hat{u}) \\
- 2(L(p^k - x^{k-1}) | u^k - \hat{u}) - \nu \|L\|^2 \|p^k - x^{k-1}\|^2 \\
+ \sum_{i \in \mathcal{I}_{k+1}} \frac{\pi_{k+1}^i (1 - \alpha_i)}{\tau \alpha_i} \|T_i p^{k+1} - p^{k+1}\|^2. \tag{3.15}
\]

In particular, if we consider \( \nu := 2 \left( \frac{1}{\gamma} - \frac{1}{2 \beta} + \frac{2 \mu \|L\|^2}{2 \mu + \gamma} \right)^{-1} \) and \( \rho := \frac{1}{2} \left( \frac{1}{\gamma} - \frac{1}{2 \beta} - \frac{2 \mu \|L\|^2}{2 \mu + \gamma} \right) > 0 \), we have \( \nu \|L\|^2 = \left( \frac{1}{\gamma} - \frac{1}{2 \beta} \right) (1 - \nu \rho) \) and by (3.11) we obtain, \( \mathbb{P} \)-a.s.

\[
\frac{\|x^k - \hat{x}\|^2}{\tau} + \frac{\|u^k - \hat{u}\|^2}{\gamma} \geq \frac{1}{\tau} \mathbb{E}(\|x^{k+1} - \hat{x}\|^2 | \mathcal{X}_k) + \left( \frac{1}{\tau} - \frac{1}{2 \mu} \right) \|x^k - p^{k+1}\|^2 \\
+ \frac{\|u^{k+1} - \hat{u}\|^2}{\gamma} + 2(L(p^{k+1} - x^k) | u^{k+1} - \hat{u}) \\
- 2(L(p^k - x^{k-1}) | u^k - \hat{u}) \\
- \left( \frac{1}{\tau} - \frac{1}{2 \mu} \right) \|p^k - x^{k-1}\|^2 \\
+ \rho \|u^{k+1} - u^k\|^2 + \nu \rho \left( \frac{1}{\tau} - \frac{1}{2 \mu} \right) \|p^k - x^{k-1}\|^2 \\
+ \sum_{i \in \mathcal{I}_{k+1}} \frac{\pi_{k+1}^i (1 - \alpha_i)}{\tau \alpha_i} \|T_i p^{k+1} - p^{k+1}\|^2. \tag{3.16}
\]

Now, let us consider the self-adjoint linear operator \( V : \mathcal{H} + \mathcal{G} + \mathcal{H} \rightarrow \mathcal{H} + \mathcal{G} + \mathcal{H} \) defined by

\[
V : (x, u, p) \mapsto \left( \frac{x}{\tau} + \frac{u}{\gamma} + Lp, \left( \frac{1}{\tau} - \frac{1}{2 \mu} \right) p + L^* u \right). \tag{3.17}
\]
Note that, for every $x := (x, u, p) \in \mathcal{H} \oplus \mathcal{G} \oplus \mathcal{H}$, from (3.3) we deduce
\[
\langle x \mid V x \rangle = \frac{\|x\|^2}{\tau} + \frac{\|u\|^2}{\gamma} + 2(Lp \mid u) + \left(\frac{1}{\tau} - \frac{1}{2\mu}\right) \|p\|^2 \quad (3.18)
\]
\[
\geq \frac{\|x\|^2}{\tau} + \frac{\|u\|^2}{\gamma} - 2\|L\||p||u|| + \left(\frac{1}{\tau} - \frac{1}{2\mu}\right) \|p\|^2 \geq \frac{\|x\|^2}{\tau} + \frac{\|u\|^2}{\gamma} - 2\|L\||p||u|| + (\gamma\|L\|^2 + \epsilon)\|p\|^2 \geq \frac{\|x\|^2}{\tau} + \frac{\|u\|^2}{\gamma} - \frac{1}{\epsilon} \|p\|^2 + (\gamma - \epsilon)\|L\|^2 + \epsilon\|p\|^2,
\]
where $\epsilon = \left(\frac{1}{\tau} - \frac{1}{2\mu}\right) - \gamma\|L\|^2 > 0$ and $\epsilon > 0$ is arbitrary. Hence, by taking $\gamma < e < \gamma + \epsilon/\|L\|^2$ we deduce that $V$ is strongly monotone in $\mathcal{H} \oplus \mathcal{G} \oplus \mathcal{H}$. Define the scalar product $\langle \cdot \mid \cdot \rangle_{\mathcal{V}} = \langle \cdot \mid V \cdot \rangle$ and set $\mathcal{H}$ be the real Hilbert space $\mathcal{H} \times \mathcal{G} \times \mathcal{H}$ endowed with this scalar product. We denote by $\|\cdot\|_{\mathcal{V}} = \sqrt{\langle \cdot \mid \cdot \rangle_{\mathcal{V}}}$ the associated norm. Note that, for every $k \in \mathbb{N},$
\[
\left(\frac{1}{\tau} - \frac{1}{2\mu}\right) \|p^{k+1} - x^k\|^2 + 2(L(p^{k+1} - x^k) \mid u^{k+1} - \hat{u}) + \frac{\|u^{k+1} - \hat{u}\|^2}{\gamma} \quad (3.19)
\]
is $X_k$-measurable. Therefore, by defining, for every $k \in \mathbb{N},$ $x^k = (x^k, u^k, p^k - x^{k-1}) \in \mathcal{H}$ and set $\tilde{x} = (\tilde{x}, \tilde{u}, 0) \in Z \times \{0\}$, we deduce from (3.10) and (3.18) that
\[
E\left(\|x^{k+1} - \tilde{x}\|^2_{\mathcal{V}} \mid X_k\right) + b_k \leq \|x^k - \tilde{x}\|^2_{\mathcal{V}}, \quad (3.20)
\]
where
\[
b_k := \rho\|u^{k+1} - u^k\|^2 + \left(\frac{1}{\tau} - \frac{1}{2\mu}\right) \|p^k - x^{k-1}\|^2 + \sum_{i \in I_{k+1}^p} \frac{\pi_{k+1}^i (1 - \alpha_i)}{\tau \alpha_i} \|T_i p^{k+1} - p^{k+1}\|^2 \quad (3.21)
\]
defines a sequence in $\ell_+(\mathcal{X})$. By denoting $Z = Z \times \{0\}$, we deduce from Lemma 2.1 that, $P$-a.s.,
\[
\sum_{k \in \mathbb{N}} \|u^{k+1} - u^k\|^2 < +\infty, \quad \sum_{k \in \mathbb{N}} \|p^k - x^{k-1}\|^2 < +\infty,
\]
and
\[
\sum_{k \in \mathbb{N}} \sum_{i \in I_{k+1}^p} \frac{\pi_{k+1}^i (1 - \alpha_i)}{\tau \alpha_i} \|T_i p^{k+1} - p^{k+1}\|^2 < +\infty. \quad (3.22)
\]
Let $\tilde{\mathcal{O}}$ be the set such that $P(\tilde{\mathcal{O}}) = 1$ and (3.22) holds, and fix $w \in \tilde{\mathcal{O}}$. Let $x(w) := (x(w), u(w), z(w)) \in \mathcal{M}(x^k(w))_{k \in \mathbb{N}},$ say
\[
x^{k_n}(w) = (x^{k_n}(w), u^{k_n}(w), p^{k_n}(w) - x^{k_n-1}(w)) \rightarrow (x(w), u(w), z(w)). \quad (3.23)
\]
Note that, (3.22) yields \( z(w) = 0 \) and, therefore, in order to prove that \( x(w) \in Z = Z \times \{0\} \) it is enough to prove that \( (x(w), u(w)) \in Z \). By defining
\[
M : (x,u) \mapsto (Ax + L^*u) \times (B^{-1}u - Lx)
\]
and
\[
\begin{align*}
y^k & := \frac{x^k - p^{k+1} + C\mu^{k+1} - Cx^k}{\gamma} \\
v^k & := \frac{u^k - u^{k+1} + L(x^k - p^{k+1} + p^k - x^{k-1}) + D^{-1}u^{k+1} - D^{-1}u^k}{\gamma}
\end{align*}
\]
we obtain from (3.10) that
\[
(p^k, v^k) \in (M + Q)(p^{k+1}, u^{k+1}).
\]
Moreover, from [3] Proposition 2.7(iii) we have that \( M \) is maximally monotone. Since \( D \) is \( \delta \)-strongly monotone, [26] Proposition 22.11(ii) yields its surjectivity. Hence, since \( C \) is \( \mu \)-cocoercive and, from [20] Theorem 21.1, that \( \text{dom} Q = \mathcal{H} \times \mathcal{G} \). Therefore, by [20] Corollary 25.5(i) we conclude that \( M + Q \) is maximally monotone. Moreover, it follows from (3.22), (3.4), (3.5), and (3.22) that, for every \( k \in \mathbb{N} \),
\[
\begin{align*}
\|y^k - y^{k-1}\| & = \|T_{\alpha_k}(w)p^k(w) - x^{k-1}(w)\| \\
& \leq \left\| T_{\alpha_k}(w)p^k(w) - p^k(w) \right\| + \left\| p^k(w) - x^{k-1}(w) \right\| \\
& \leq \sum_{i \in I^c_0} \|T_ip^k(w) - p^k(w)\| + \|p^k(w) - x^{k-1}(w)\| \\
& \leq \frac{1}{\zeta(1 - \pi)} \sum_{i \in I^c_0} \tau_{k+1} \|T_ip^k(w) - p^k(w)\| \\
& \quad + \|p^k(w) - x^{k-1}(w)\|
\end{align*}
\]
(3.27)
\[
\rightarrow 0,
\]
(3.28)
where \( \pi = \max_{i=1,\ldots,m} \alpha_i \). Now, let \( N \in \mathbb{N} \) be the integer provided by assumption (ii) fix \( i \in I \), and fix \( n \in \mathbb{N} \). Therefore, (ii) ensures the existence of \( j_n \) such that \( i \in I_{j_n} \) and \( k_n + 1 \leq j_n \leq k_n + N \) and, from (3.28), we deduce
\[
\begin{align*}
\|x^{j_n}(w) - x^{k_n}(w)\| & \leq \sum_{l=k_n}^{j_n-1} \|x^{l+1}(w) - x^l(w)\| \\
& \leq \sum_{l=k_n}^{k_n+N-1} \|x^{l+1}(w) - x^l(w)\| \\
& \rightarrow 0.
\end{align*}
\]
(3.29)
Hence, from \( x^{k+1} \rightarrow x(w), \) \( \text{(3.21)} \), \( \text{(3.22)} \), \( \text{(3.23)} \), and \( \text{(3.24)} \) we deduce that
\[
p^\infty(w) \rightarrow x(w) \quad \text{and} \quad (\text{Id} - T_i)p^\infty(w) \rightarrow 0. \tag{3.30}
\]
Therefore, since \( \text{Id} - T_i \) is maximally monotone \text{[26] Example 20.29} and its graph is closed in the weak-strong topology \text{[25] Proposition 20.38}, we deduce \( x(w) \in \text{Fix}(T_i) \). Since \( i \in I \) is arbitrary, we conclude \( x(w) \in \bigcap_{i=0}^m \text{Fix}(T_i) = S \) and the result follows from Lemma \( \text{[21]} \). \( \square \)

Remark 3.1 In this remark, we explore the flexibility of our formulation.

(i) \textbf{Random projections in convex optimization:} Consider the context of primal-dual convex optimization problems \text{[24], [25]}. For every \( i \in \{1, \ldots, m\} \), suppose that \( T_i = P_{S_i} \), where \( S_i \) is a nonempty closed convex subset of \( \mathcal{H} \). Then, \( S := \bigcap_{i=1}^m S_i \neq \emptyset \) and \( \text{(3.3)} \) reduces to
\[
(\forall k \in \mathbb{N}) \quad \begin{cases} u^{k+1} = \text{prox}_{p^\gamma}(u^k + \gamma(\tilde{L}u^k - \nabla f^*(u^k))) \\ v^{k+1} = \text{prox}_{g^\gamma}(v^k - \tau(\nabla u^{k+1} + \nabla H(u^k))) \\ x^{k+1} = P_{S_i}p^{k+1} \\ x_k = x^{k+1} + p^{k+1} - x^k, \end{cases} \tag{3.31}
\]
which solves \text{(24)–(25)} if \((\epsilon_k)_{k \geq 1} \) satisfies \text{(31)}. This algorithm will be revisited in the application studied in Section 4.

(ii) \textbf{Primal-dual with cyclic Bernoulli random activations:} Suppose that, for every \( k \in \mathbb{N}, I_k = \{0, i(k)\} \), where \( i : \mathbb{N} \rightarrow I \) is a function such that, for every \( n \in \mathbb{N} \), \( i(n, \ldots, n+N-1) = \{1, \ldots, m\} \) and \( N \) is defined in \text{[22]} (ii). Moreover, define \((\epsilon_k)_{k \in \mathbb{N}} \) as the sequence of independent \((0,1)\)–valued random variables such that, for every \( k \in \mathbb{N}, \epsilon_k^{-1}(\{0\}) = \epsilon_k^{-1}(\{0\}) \). Hence, we have that, for every \( n \in \mathbb{N} \),
\[
\bigcup_{k=n}^{n+N-1} I_k = \{0\} \cup i(n, \ldots, n+N-1) = I, \tag{3.32}
\]
holds, and the activation step in \text{[26]} is equivalent to
\[
(\forall k \in \mathbb{N}) \quad x^{k+1} = p^{k+1} + \epsilon_{k+1}(T_i(k+1)p^{k+1} - p^{k+1}) = T_i p^{k+1}, \tag{3.33}
\]
where \( T_0 = \text{Id} \). Therefore, we deduce that a particular instance of \text{[3.3]} is the primal-dual method with random Bernoulli activations as it is used, e.g., in \text{[26]}.

In particular, if \( i: k \mapsto (k \bmod m) + 1 \), the random Bernoulli activation is applied over a cyclic order of the operators \( (T_i)_{1 \leq i \leq m} \). This is a generalization of the primal-dual method with a priori information developed in \text{[26]}. In addition, if for every \( k \in \mathbb{N}, \epsilon_k^{-1}(\{0\}) = \emptyset \), the activations become deterministic and \text{[22]} holds. In this context, we recover \text{[26]} by setting \( m = 1 \) and \( T_1 = T \).

(iii) \textbf{Projections onto convex sets:} In the context of Remark \text{[26]} (i), suppose that \( f = g = h = L = 0, \ell = \epsilon_{\{0\}} \), for every \( k \in \mathbb{N}, I_k = \{0, (k \mod m) + 1\} \), and that \( \epsilon_k^{-1}(\{0\}) = \emptyset \). Then \( \text{prox}_f(x) = x \), \( \text{prox}_g(x) = x - \text{prox}_y(x) = 0 \) \text{[26] Proposition 24.8(ix)]}. (ii) holds, the activation scheme in \text{[3.31]} reduces to
\[
(\forall k \in \mathbb{N}) \quad x^{k+1} = P_{S_k \cap \mathcal{H}_m} x^k, \tag{3.34}
\]
and we recover the convergence of the method in \text{[26] Corollary 5.26]. We can also recover more general projection schemes as, for instance, cyclic activations \text{[21]}, which inspire condition (iii).
(iv) **Kaczmarz algorithm:** In the context of Remark 3.4(iv), assume that \( \mathcal{H} = \mathbb{R}^n \), let \( R \) be a full rank \( m \times n \) matrix such that \( m \leq n \), denote the rows of \( R \) by \( r_1, ..., r_m \in \mathbb{R}^n \), and set \( b = (b_1, ..., b_m)^\top \in \mathbb{R}^m \). Moreover, suppose that \( S = \{ x \in \mathcal{H} : Rx \leq b \} = \bigcap_{i=1}^m S_i \neq \emptyset \), where, for every \( i \in \{1, ..., m\}, S_i = \{ x \in \mathbb{R}^n : r_i^\top x = b_i \} \) is nonempty, closed, and convex. Then, (3.34) reduces to

\[
(\forall k \in \mathbb{N}) \quad x^{k+1} = x^k + \frac{b_i(k+1) - r_i^\top x^k}{\|r_i(k+1)\|^2} r_i(k+1),
\]

which is the Kaczmarz method proposed in [21] and its convergence follows from Theorem 3.1.

(v) **Randomized Kaczmarz:** Consider the setting described in Remark 3.4(iv) and let \((\epsilon_k)_{k \in \mathbb{N}}\) be a sequence of independent \( I \)-valued random variables, with \( \epsilon_k^{-1}\{0\} \equiv \emptyset \), and for every \( i \in I \setminus \{0\} \) and \( k \in \mathbb{N} \), \( \pi_k^i \) is proportional to \( \|r_i\|^2 \). Therefore, as in Remark 3.4(iv) (3.3) reduces to

\[
(\forall k \in \mathbb{N}) \quad x^{k+1} = P_{S_{\epsilon_{k+1}}} x^k + \frac{b_{\epsilon_{k+1}} - r_{\epsilon_{k+1}}^\top x^k}{\|r_{\epsilon_{k+1}}\|^2} r_{\epsilon_{k+1}},
\]

which is the method proposed in [22] and its convergence is deduced from Theorem 3.1. Note that in [22] the authors obtain exponential convergence rate in expectation, while our method has \( F \)-a.s. convergence.

### 4 Application to the arc capacity expansion problem of a directed graph

In this section we aim at solving the traffic assignment problem with arc-capacity expansion on a network with minimal cost under uncertainty. Let \( \mathcal{A} \) be the set of arcs and let \( \mathcal{O} \) and \( \mathcal{D} \) be the sets of origin and destination nodes of the network, respectively. The set of routes from \( o \in \mathcal{O} \) to \( d \in \mathcal{D} \) is denoted by \( R_{od} \) and \( \mathcal{R} = \bigcup_{(o,d) \in \mathcal{O} \times \mathcal{D}} R_{od} \) is the set of all routes. The arc-route incidence matrix \( N \in \mathbb{R}^{|A| \times |\mathcal{R}|} \) is defined by \( N_{a,r} = 1 \), if arc \( a \) belongs to the route \( r \), and \( N_{a,r} = 0 \), otherwise.

The uncertainty is modeled by a finite set \( \Xi \) of possible scenarios. For every scenario \( \xi \in \Xi \), \( \pi_{\xi} \in [0,1] \) is its probability of occurrence, \( b_{od,\xi} \in \mathbb{R}_+ \) is the forecasted demand from \( o \in \mathcal{O} \) to \( d \in \mathcal{D} \), \( c_{a,\xi} \in \mathbb{R}_+ \) is the corresponding capacity of the arc \( a \in \mathcal{A} \), \( f_{r,\xi} : \mathbb{R}_+ \to \mathbb{R}_+ \) is an increasing and \( \beta_{a,\xi} \)-Lipschitz continuous travel time function on arc \( a \in \mathcal{A} \), for some \( \beta_{a,\xi} > 0 \), and the variable \( f_{r,\xi} \in \mathbb{R}_+ \) stands for the flow in route \( r \in \mathcal{R} \).

In the problem of this section, we consider the expansion of flow capacity at each arc in order to improve the efficiency of the network operation. We model this decision making process in a two-stage stochastic problem. The first stage reflects the investment in capacity and the second corresponds to the operation of the network in an uncertain environment.

In order to solve this problem, we take a non-anticipativity approach [8], letting our first stage decision variable depend on the scenario and imposing a non-anticipativity constraint. We denote by \( x_{a,\xi} \in \mathbb{R}_+ \) the variable of capacity expansion on arc \( a \in \mathcal{A} \) in scenario \( \xi \in \Xi \) and the non-anticipativity condition is defined.
by the constraint

\[ W = \{ x \in \mathbb{R}^{|A| \times |\Xi|} : (\forall (\xi, \xi') \in \Xi^2) \ x_\xi = x_{\xi'} \}, \]

where \( x_\xi \in \mathbb{R}^{|A|} \) is the vector of capacity expansion for scenario \( \xi \in \Xi \) and we denote \( f_\xi \in \mathbb{R}^{|R|} \) analogously. We restrict the capacity expansion variables by imposing, for every \( a \in A \) and \( \xi \in \Xi \), \( x_\xi \in M := \times_{a \in A} [0, M_a] \subset \mathbb{R}^{|A|} \), where \( M_a > 0 \) represents the upper bound of capacity expansion, for every \( a \in A \). Additionally, we model the investment cost of expansion via a quadratic function defined by a symmetric positive definite matrix \( Q \in \mathbb{R}^{|A| \times |A|} \).

**Problem 4.1** The problem is to

\[
\min_{(x, f) \in (W \cap M^{|\Xi|}) \times \mathbb{R}^{|R||\Xi|}} \sum_{\xi \in \Xi} p_\xi \left[ \sum_{a \in A} (Nf_a)_a \int_{0}^{\infty} l_{a, \xi}(z) dz + \frac{1}{2} x_\xi^\top \Sigma_x \xi x_\xi \right]
\]

s.t.

\[
(\forall \xi \in \Xi)(\forall a \in A) \quad (Nf_a)_a - x_{a, \xi} \leq c_{a, \xi}, \quad (4.1)
\]

\[
(\forall \xi \in \Xi)(\forall (o, d) \in O \times D) \quad \sum_{r \in R_{od}} f_{r, \xi} = h_{od, \xi}, \quad (4.2)
\]

under the assumption that the solution set \( Z_1 \) is nonempty.

The first term of the objective function in Problem 4.1 represents the expected operational cost of the network. The optimality conditions of the optimization problem with this objective cost related to the pure traffic assignment problem, defines a Wardrop equilibrium [29]. The second term in the objective function is the expansion investment cost. Constraints in (4.1) represent that, for every arc \( a \in A \), the flow cannot exceed the expanded capacity \( c_{a, \xi} + x_{a, \xi} \) at each scenario \( \xi \in \Xi \), while (4.2) are the demand constraints.

### 4.1 Formulation and Algorithms

Note that Problem 4.1 can be equivalently written as

\[
\text{find } (x, f) \in S \cap \left( \arg\min_{(x, f) \in \mathcal{H}} F(x, f) + G(L(x, f)) + H(x, f) \right), \quad (P)
\]
where

\[
\begin{align*}
H &:= \mathbb{R}^{|A|} \times \mathbb{R}^{|R|}, \\
(\forall \xi \in \Xi) \quad V^+_{\xi} &:= \left\{ f \in \mathbb{R}^{|R|} : (\forall (o, d) \in \mathcal{O} \times \mathcal{D}) \sum_{r} f_r = h_{o,d,\xi} \right\}, \\
A &:= (\mathcal{M} \Xi | \cap W) \times \left( \times_{\xi} V^+_{\xi} \right), \\
F &:= \{ \iota \}, \\
(\forall \xi \in \Xi) \quad H_{\xi} &:= \left\{ (x, u) \in \mathbb{R}^{|A|} \times \mathbb{R}^{|A|} : (\forall a \in \mathcal{A}) u_a - x_a \leq c_{a,\xi} \right\} \quad (4.3), \\
G &:= (x, u) \mapsto h_{\xi}, \\
L : (x, f) &\mapsto (x, a, \xi) \in \Xi \\
S &:= \cap_{i=1}^{m} \mathcal{S}_i \supset \text{dom} (G \circ L), \\
H : (x, f) &\mapsto \sum_{\xi \in \Xi} p_{\xi} \left[ \sum_{a \in \mathcal{A}} \int_{0}^{(Nf_{\xi})} t_{a,\xi}(z)dz + \frac{1}{2} x_{\xi}^\top Qx_{\xi} \right],
\end{align*}
\]

and \((S_i)_{1 \leq i \leq m}\) are nonempty closed convex sets such that \(S \neq \emptyset\). Note that \(F\) and \(G\) are lower semicontinuous convex proper functions, and \(L\) is linear and bounded with \(\|L\| \leq \max \{1, \|N\|\}\). Moreover, note that, since \((t_{a,\xi})_{a \in \mathcal{A}, \xi} \in \Xi\) are increasing, \(N\) is linear, and \(Q\) is definite positive, \(H\) is a separable convex function. In addition, by defining

\[
\psi_{\xi} : \mathbb{R}^{|R|} \rightarrow \mathbb{R}^{|R|} : f \mapsto N^\top (t_{a,\xi}(Na))_{a \in \mathcal{A}},
\]

simple computations yield

\[
\nabla H : (x, f) \mapsto \left( (p_{\xi} Qx_{\xi})_{\xi \in \Xi}, (\psi_{\xi}(f))_{\xi \in \Xi} \right),
\]

which is Lipschitz continuous with constant

\[
\mu^{-1} = \max_{\xi \in \Xi} \left( p_{\xi} \max_{a \in \mathcal{A}, \xi} \left\{ \|Q\|, \|N\|^2 \max_{a \in \mathcal{A}, \xi} \beta_{a,\xi} \right\} \right).
\]

Altogether, \([20]\) is a particular case of \([21]\). Assume that the following Slater condition

\[
\exists (\hat{x}, \hat{f}) \in \mathcal{A} \quad \text{such that} \quad (\forall \xi \in \Xi) (\forall a \in \mathcal{A}) \sum_{r \in R} N_{a,r} \hat{f}_{a,\xi} - \hat{x}_{a,\xi} < c_{a,\xi} \quad (4.4)
\]

holds. Then by \([26]\) Proposition 27.21 the qualification condition \([32]\) is satisfied and, therefore, \([24]\) is a particular case of Problem 3.1.

Observe that the a priori information \(S\) is redundant with the objective function, because \(S = \cap_{i=1}^{m} \mathcal{S}_i \supset \text{dom} (G \circ L)\). We will show in Section 4.3 that this redundant formulation has important numerical advantages. In what follows, we exploit the splittable structure of \(S\) and provide an application of the primal-dual splitting with random projection detailed in \([43]\) in order to solve Problem 4.1.
next result is a consequence of Theorem 3.1 applied to the context of (2) as in Remark 4.1. We denote by $P_{\xi}:\mathbb{R}^{\lfloor A\rfloor}\times\mathbb{R}^{\lfloor R\rfloor}\to\mathbb{R}^{\lfloor A\rfloor}\times\mathbb{R}^{\lfloor R\rfloor}$: $(x,f)\mapsto(x_{\xi},f_{\xi})$ the orthogonal projection onto the scenario $\xi\in\Xi$.

**Corollary 4.1** Consider the setting of Problem (2). Let $\gamma > 0$ and let $\tau \in [0,2\mu]$ be such that

$$\max\{1,\|N\|^2\} < \frac{1}{\gamma} \left( \frac{1}{\tau} - \frac{1}{2\mu} \right).$$

(4.5)

Let $(u^{0},v^{0}) \in G$, let $(x^{0},f^{0})$, $(\bar{x}^{0},\bar{f}^{0}) \in \mathcal{H}_{2}$ be such that $(x^{0},f^{0}) = (\bar{x}^{0},\bar{f}^{0})$ and, set $I = \{0,1,\ldots,m\}$. Let $(\epsilon_{k})_{k\in\mathbb{N}}$ be a sequence of independent random variables such that, for every $k \in \mathbb{N}$, $\epsilon_{k}(\Omega) = I_{k} \subset I$ and consider the following routine

$$\begin{aligned}
(\forall k \in \mathbb{N}) & \quad \begin{cases}
\left( u^{k+1}_{\xi}, v^{k+1}_{\xi} \right) = \left( u^{k}_{\xi} + \gamma x^{k}_{\xi}, v^{k}_{\xi} + \gamma N_{\xi}^{k} \right) \\
\left( u^{k+1}, v^{k+1} \right) = \left( \bar{u}^{k+1}_{\xi}, \bar{v}^{k+1}_{\xi} \right) - \gamma P_{H_{\xi}} \left( \gamma^{-1} \left( \bar{u}^{k+1}_{\xi}, \bar{v}^{k+1}_{\xi} \right) \right) \\
\left( p^{k+1}_{\xi}, g^{k+1}_{\xi} \right) = x^{k}_{\xi} - \tau \left( u^{k+1}_{\xi} + p^{k}_{\xi} \bar{Q} x^{k}_{\xi} \right), \\
\left( p^{k+1}, g^{k+1} \right) = \left( P_{\xi} \left( P_{M\xi\cap W} \left( p^{k+1} \right) \right), P_{\xi} \left( g^{k+1}_{\xi} \right) \right) - \tau (N^{\top} \bar{u}^{k+1}_{\xi} + p^{k}_{\xi} \bar{Q} x^{k}_{\xi}) \\
\left( x^{k+1}, f^{k+1} \right) = \left( x^{k+1}_{\xi}, f^{k+1}_{\xi} \right) + \left( p^{k+1}, g^{k+1} \right) - \left( x^{k}, f^{k} \right),
\end{cases}
\end{aligned}$$

(4.6)

where $S_{0} = \mathcal{H}$. Then $(x^{k}, f^{k})_{k\in\mathbb{N}}$ converges $\mathbb{P}$-a.s. to a $Z_{1}$-valued random variable.

**Remark 4.1** In order to implement the algorithm in (4.6) we need to compute the following projections:

1. It follows from [26, Proposition 24.11] that

$$\begin{aligned}
(\forall \xi \in \Xi) & \quad P_{H_{\xi}}: (x,u) \mapsto \left( P_{H_{\xi}}(x_{\alpha,\xi},u_{\alpha,\xi}) \right)_{\alpha \in A}.
\end{aligned}$$

(4.7)

and, by applying [26, Example 29.20], we obtain, for every $\alpha \in A$ and $\xi \in \Xi$,

$$P_{H_{\alpha,\xi}}: (\eta,\nu) \mapsto \begin{cases}
\left( \frac{\eta + \nu - c_{\alpha,\xi}}{2}, \frac{\eta + \nu + c_{\alpha,\xi}}{2} \right), & \text{if } \eta - \nu + c_{\alpha,\xi} < 0; \\
(\eta,\nu), & \text{if } \eta - \nu + c_{\alpha,\xi} \geq 0.
\end{cases}$$

(4.8)

2. We deduce from [26, Theorem 3.16] that

$$\begin{aligned}
P_{M:|W|}: & \mathbb{R}^{\lfloor A\rfloor}\times|W| \to \mathbb{R}^{\lfloor A\rfloor}\times|W|: x \mapsto \left( \text{mid} \left( 0,\bar{x},M_{a} \right) \right)_{\alpha \in A, \xi \in \Xi},
\end{aligned}$$

(4.9)

where $\bar{x} = \frac{1}{|\Xi|} \sum_{\xi \in \Xi} x_{\xi}$ and mid: $(a,b,c) \mapsto P_{[a,c]}$ is the median function.
3. Note that

\[(\forall \xi \in \Xi) \quad V_\xi^+ = \bigcap_{(o,d) \in O \times D} \left\{ f_{od} \in \mathbb{R}^{|R_{o,d}|} \bigg| \sum_{r \in R_{o,d}} f_{od,r} = h_{od,\xi} \right\}, \quad (4.10)\]

and it follows from [26, Proposition 24.11] that

\[P_{V_\xi^+} : \mathbb{R}^{|R|} \to \mathbb{R}^{|R|} : f \mapsto \left( P_{V_{od,\xi}} f_{od} \right)_{(o,d) \in O \times D}. \quad (4.11)\]

In order to compute the projection onto \(V_{od,\xi}^+\), we use the algorithm proposed in [30], which follows from quasi-Newton method onto the dual of the knapsack problem.

### 4.2 Selection criteria for random projections

Now we establish several criteria for selecting half-spaces from the constraints as a priori information sets \((S_i)_{1 \leq i \leq m}\). The resulting choice of \((S_i)_{1 \leq i \leq m}\) lead to different families of algorithms. In Section 4.3, we compare their numerical performance with respect to the base scheme, in which no projection is applied.

Note that the feasible set defined by constraints (4.1) in Problem 4.1 can be written as

\[(\forall \xi \in \Xi) \quad \bigcap_{a \in A} P_{\xi}(\Sigma_{a,\xi}) , \quad (4.12)\]

where

\[(\forall \xi \in \Xi) (\forall a \in A) \quad \Sigma_{a,\xi} := \left\{ (x,f) \in \mathbb{R}^{|A||\Xi|} \times \mathbb{R}^{|R||\Xi|} : N_a f_{x} - x_{a,\xi} \leq c_{a,\xi} \right\}. \quad (4.13)\]

The a priori information sets \((S_i)_{1 \leq i \leq m}\) are chosen as particular intersections of \((\Sigma_{a,\xi})_{(a,\xi) \in A \times \Xi}\) for guaranteeing feasibility of primal iterates. In order to obtain simple projection implementations, note that (4.13) and [26, Proposition 29.23] yields

\[(\forall (a,a') \in A^2)(\forall (\xi,\xi') \in \Xi^2) \quad \xi \neq \xi' \Rightarrow P_{\Sigma_{a,\xi} \cap \Sigma_{a',\xi'}} = P_{\Sigma_{a,\xi}} \circ P_{\Sigma_{a',\xi'}} , \quad (4.14)\]

because the normal vectors to \(\Sigma_{a,\xi}\) and \(\Sigma_{a',\xi'}\) are orthogonal. This property also holds for a vector of finite different scenarios \((\xi_1, \ldots, \xi_m) \in \Xi^m\) and arbitrary vector of arcs \((a_1, \ldots, a_m) \in A^m\). Based on this property, we propose four classes of algorithms in which, for every \(i \in \{1, \ldots, m\}\), \(S_i\) corresponds to the intersection of a selection of \((\Sigma_{a,\xi})_{(a,\xi) \in A \times \Xi}\) with different scenarios. This allows us to obtain explicit projections in our methods.
For defining the sets onto which we project, for every \( l \in \{1, \ldots, |\Xi|\} \), let \( \iota: \mathcal{A}^l \times D_l \to \{1, \ldots, m\} \) be a bijection, where \( m = |\mathcal{A}^l \times D_l| \) and
\[
D_l = \{ (\xi_1, \ldots, \xi_l) \in \Xi^l \mid (\forall i, j \in \{1, \ldots, l\}) \ i \neq j \Rightarrow \xi_i \neq \xi_j \}.
\]
(4.15)

Observe that \( D_1 = \Xi \). In order to obtain simple projection implementations we define, for every \( a = (a_i)_{i=1}^l \in \mathcal{A}^l \) and \( \xi = (\xi_i)_{i=1}^l \in D_l \),
\[
K_{l(a,\xi)}^l = \bigcap_{i=1}^l \Sigma_{a_i, \xi_i}.
\]
(4.16)

Note that, for every \( i \in \{1, \ldots, m\} \), \( K_i^l \) is an intersection of a selection of \( l \) sets from \( (\Sigma_{a,\xi})_{a \in \mathcal{A}, \xi \in \Xi} \), where the considered scenarios are all different and the arcs are arbitrary. Since the selected scenarios are different, we obtain from (4.14) the explicit formula
\[
(\forall a \in \mathcal{A}^l)(\forall \xi \in D_l) \quad P_{K_{l(a,\xi)}}^l = \prod_{i=1}^l P_{\Sigma_{a_i, \xi_i}}.
\]
(4.17)

For every \( l \in \{1, \ldots, |\Xi|\} \), we propose four classes of algorithms based on Corollary 4.1 depending on the selection of the convex sets \( (K_i^l)_{1 \leq i \leq m} \) in which they project, for different values of \( m \in \{1, \ldots, |\mathcal{A}^l||D_l|\} \).

(F) **Fixed selection:** We fix \( j \in \{1, \ldots, |\mathcal{A}^l||D_l|\} \), we set \( S = S_j = K_j^l \) and, for every iteration \( k \in \mathbb{N} \), we set \( \epsilon_k^{-1}(\{j\}) = \emptyset \). In this class, we project deterministically in a fixed block of size \( l \), in which arcs and scenarios are \( \iota^{-1}(j) \in \mathcal{A}^l \times D_l \).

(BA) **Bernoulli alternating selection:** We fix the bijection \( \iota: \mathcal{A}^l \times D_l \to \{1, \ldots, |\mathcal{A}^l||D_l|\} \), fix \( m \leq |\mathcal{A}^l||D_l| \), and, for every \( i \in \{1, \ldots, m\} \), we set \( S_i = K_i^l \). For every iteration \( k \in \mathbb{N} \), we set \( I_k = \{0, (k \text{ mod } m) + 1\} \). In this class, we follow a \( \{0,1\} \)–Bernoulli random process, as illustrated in Remark 3.2 to project onto blocks of size \( l \) assigned cyclically along iterations.

(DA) **Deterministic alternating selection:** We fix the bijection \( \iota: \mathcal{A}^l \times D_l \to \{1, \ldots, |\mathcal{A}^l||D_l|\} \), fix \( m \leq |\mathcal{A}^l||D_l| \), and, for every \( i \in \{1, \ldots, m\} \), we set \( S_i = K_i^l \). For every iteration \( k \in \mathbb{N} \), we set \( \epsilon_k^{-1}(\{0\}) = \emptyset \). In this class, we project deterministically onto blocks of size \( l \) assigned cyclically along iterations.

(RK) **Random Kaczmarz selection:** We fix \( m = |\mathcal{A} \times D_l| \), for every \( i \in \{1, \ldots, m\} \), we set \( S_i = K_i^l \), for every \( k \in \mathbb{N} \), we set \( \epsilon_k \) as a \( \{1, \ldots, m\} \)-valued random variable such that, for every \( i \in \{1, \ldots, m\} \), \( \pi_i^k = \frac{j}{m} \). In this class of algorithms, we randomly project onto a block of \( l \) constraints from different scenarios at each iteration.

4.3 Numerical experiences

In this section, we apply the four classes of algorithms defined in Section 4.2 to a specific instance of Problem 4.1. We consider the network presented in [31] (see also [8]), represented in Figure 1. We set \( |\Xi| = 18 \), \( p_k = \frac{k}{18} \), and \( (\epsilon_\xi)_{\xi \in \Xi} \) as a
Figure 1 Network with $|A| = 19$, $O = \{1, 4\}$, $D = \{2, 3\}$, $|R_{1,2}| = 8$, $|R_{4,3}| = |R_{1,3}| = 6$, $|R_{4,2}| = 5$, and $|R| = 25$. 

Table 1 Numerical values of $c$, $\kappa$, and $\eta$ on every arc.

| Arcs | $c$  | $\kappa$ | $\eta$ |
|------|------|----------|--------|
| 1    | 1100 | 15       | 7      |
| 2    | 484  | 6.6      | 9      |
| 3    | 154  | 2.1      | 9      |
| 4    | 1100 | 15       | 12     |
| 5    | 330  | 4.5      | 3      |
| 6    | 484  | 6.6      | 9      |
| 7    | 1100 | 15       | 5      |
| 8    | 220  | 3        | 13     |
| 9    | 220  | 3        | 5      |
| 10   | 220  | 6        | 9      |

The demand $(h_{\xi})_{\xi \in \Xi}$ is obtained as a sample of the random variable $d + s \cdot \text{Beta}(50, 10)$, where $d = (d_{1,2}, d_{1,3}, d_{4,2}, d_{4,3}) = (300, 700, 500, 350)$ and $s = (120, 120, 120, 120)$ and we consider the capacity expansion limits, for every $a \in A$, $M_a = 200 \cdot \kappa_a$. The matrix of the quadratic cost of expansion is given by $Q = \text{Id}_{|A|}$ and we consider the travel time function

$$(\forall \xi \in \Xi)(\forall a \in A) \quad t_{a,\xi}(u) := \eta_a + \tau_a \frac{u}{c_{a,\xi}},$$

where $\eta$ is in Table 1 and $\tau := 0.15 \eta$. Hence, for every $\xi \in \Xi$ and $a \in A$, $\beta_{a,\xi} := \frac{\tau a_{a,\xi}}{c_{a,\xi}}$.

In this context, we apply the four classes of algorithms defined in Section 4.2 with $l \in \{1, 9, 18\}$, which give raise to 12 algorithms. We compare their numerical...
performance with respect to the method without any projection proposed in [15, 16], called Algorithm 1. This comparison is performed by creating 20 random instances of the problem, obtained via the random function of MATLAB and using the same seed. In Table 2 we detail the algorithm labelling according to the class and the number of constraints onto which we project ($l \in \{1, 9, 18\}$). For the class of fixed selections, we choose to project onto the polyhedron related to capacity constraints of arc $a = 16$, i.e., $\bigcap_{j=1}^{l} \sum_{16, \xi_j}^{l}$, where $(\xi_1, \ldots, \xi_l) \in D_l$. This is naturally justified from the topology of the network in Figure 1, since the total demand arriving to node 3 exceeds largely the capacity in arcs 16 and 19. Thus, it is mandatory to expand the capacity of those arcs. Moreover, the arc 16 has the lowest capacity.

All algorithms stop at the first iteration when the relative error is less than a tolerance of $10^{-10}$, where the relative error of the iteration $k \in \mathbb{N}$ is

$$
\epsilon_k = \sqrt{\frac{\|x^{k+1} - x^k\|^2 + \|f^{k+1} - f^k\|^2 + \|u^{k+1} - u^k\|^2 + \|v^{k+1} - v^k\|^2}{\|x^k\|^2 + \|f^k\|^2 + \|u^k\|^2 + \|v^k\|^2}}. \quad (4.19)
$$

In Table 3 we provide the average execution time and the average number of iterations of Algorithms 1 – 13, obtained from 20 random realizations of vectors $(c_\xi)_{\xi \in \Xi}$ and $(d_\xi)_{\xi \in \Xi}$. We see a considerable decrease on the average time and number of iterations of the algorithms in classes (BA), (DA), and (RK), as we increase the number of constraints $l$ considered in the projections. The algorithms in class (F) remains comparable with respect to Algorithm 1.

In Figure 2 we provide the boxplot of % of improvement in terms of number of iterations of Algorithms 2 – 13 with respect to Algorithm 1 (without projections). A similar boxplot in terms of computational time is provided in Figure 3. We verify that Algorithms 2, 3 & 4 belonging to the (F) class, are comparable in efficiency to Algorithm 1. We see that all other algorithms have a superior performance at the
exception of few outliers. In particular, Algorithms 10 (DA) and 13 (RK) exhibit larger gains in performance, reaching up to 35% of improvement in iterations and up to 38% in computational time. We also observe that the algorithms in which we project onto a larger number of constraints (larger \( l \)) have better performance.

**Figure 2** Boxplot of % of improvement in number of iterations for Algorithms 2-13 with respect to Algorithm 1.

**Figure 3** Boxplot of % of improvement in time for Algorithms 2-13 with respect to Algorithm 1.
Table 4  Worst scenario flow excess on arcs (max_{\xi} (u_{a,\xi} - c_{a,\xi})) \in A and arc capacity expansion vector \((x_a)_{a \in A}\) at the optimum.

Figure 4  Graphical representation of the expanded arcs (in red, \(x_a > 0\)) at the optimum.

In terms of the obtained solution, 7 arcs are expanded and the expansion capacity coincides with the extra flow needed in the equilibrium for the worst scenario. Finally, in Table 4 we show, at the optimum, the flow excess at each arc in the worst scenario and the corresponding arc capacity expansion for one of the 20 random realizations. We verify that the arc capacity expansion coincides with the worst scenario flow excess on arcs where the excess is strictly positive. In the arcs in which there is a slack on the capacity, the expansion is zero. In Figure 4 we represent the expanded arcs.
5 Conclusions

In this work, we provide a new primal-dual algorithm for solving monotone inclusions with a priori information. The a priori information is represented via fixed point sets of a finite number of nonexpansive operators, which are activated randomly/deterministically in our proposed method. We apply four classes of algorithms with different activation schemes for solving convex optimization with a priori information and, in particular, to the arc capacity expansion problem on traffic networks. We observe an improvement up to 35% in computational time for the algorithms including randomized and alternating projections with respect to the method without projections, justifying the advantage of our approach.

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