2π-GRAFTINGS AND COMPLEX PROJECTIVE STRUCTURES I

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Abstract. Let $S$ be a closed orientable surface of genus at least two, and let $C$ and $C'$ be complex projective structures on $S$ with the same holonomy and orientation. We show that, if, via Thurston’s coordinates, the projection of $C'$ to $\mathcal{PML}(S)$ is sufficiently close to that of $C$, then $C$ and $C'$ are related by a $2\pi$-grafting along a multiloop $M$. Moreover $M$ is well-approximated by the difference of the measured laminations corresponding to $C$ and $C'$, calculated on a traintrack.

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1. Introduction

A (complex) projective structure on a connected and orientable surface $F$ is a $(\hat{\mathbb{C}}, \text{PSL}(2, \mathbb{C}))$-structure. That is, an atlas modeled on the Riemann sphere $\hat{\mathbb{C}}$ with transition maps lying in $\text{PSL}(2, \mathbb{C})$. Equivalently (see for example [19]), a projective structure on $F$ is a pair $(f, \rho)$ of

- an immersion $f : \tilde{F} \to \hat{\mathbb{C}}$ (developing map), where $\tilde{F}$ is the universal cover of $F$, and
- a representation $\rho : \pi_1(S) \to \text{PSL}(2, \mathbb{C})$ (holonomy)

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such that \( f \) is \( \rho \)-equivariant, i.e. \( f \cdot \gamma = \rho(\gamma) \cdot f \) for all \( \gamma \in \pi_1(F) \).

Throughout this paper, we consider only marked complex projective structures of fixed orientation. Our main focus is on projective structures on a closed orientable surface \( S \) of genus at least 2.

Then Gallo-Kapovich-Marden \( [8] \) showed that a representation \( \rho: \pi_1(S) \to \text{PSL}(2, \mathbb{C}) \) is the holonomy representation of some projective structure (on \( S \)) if and only if \( \rho \) satisfies:

- \( \text{Im}(\rho) \) is nonelementary, and
- \( \rho \) lifts to \( \tilde{\rho}: \pi_1(S) \to \text{SL}(2, \mathbb{C}) \).

In particular, holonomy representations are not necessarily discrete or faithful, in contrast with, for example, complete hyperbolic structures. Moreover, a representation \( \rho: \pi_1(S) \to \text{PSL}(2, \mathbb{C}) \) with the above conditions corresponds to infinitely many different projective structures (which is essentially in \( [8] \); see also \( [3] \)).

Thus a basic question is the characterization of projective structures with fixed holonomy, which has indeed been raised in \( [10, 14, 8, 6] \). However, answers to this question have been given only for some special discrete (and, in most cases, faithful) representations (\( [9, 11, 2] \)), using a certain surgery operation, called \( (2\pi)\)-grafting (see \( \S 3.1 \)).

Let \( \mathcal{P}_\rho \) denote the set of projective structures on \( S \) with fixed holonomy \( \rho: \pi_1(S) \to \text{PSL}(2, \mathbb{C}) \) and fixed orientation, and let \( \mathcal{PML}(S) \) denote the space of projective measured laminations on \( S \), which is homeomorphic to \( S^{6g-7} \). Our goal of this paper is to give a local characterization of \( \mathcal{P}_\rho \) using grafting, where the locality is measured in \( \mathcal{PML}(S) \) by a projection of \( \mathcal{P}_\rho \) into \( \mathcal{PML}(S) \) (see Theorem A). This characterization is given for arbitrary holonomy \( \rho \).

In the sequel paper \( [1] \), using this local characterization, we prove moreover that grafting indeed generates the entire set \( \mathcal{P}_\rho \) for every generic representation \( \rho \) in the \( \text{PSL}(2, \mathbb{C}) \)-character variety \( \chi(S) \) (see \( [8] \) Problem 12.1.2)).

1.1. Fuchsian projective structures. We review the characterization of \( \mathcal{P}_\rho \) when \( \rho: \pi_1(S) \to \text{PSL}(2, \mathbb{C}) \) is a discrete and faithful representation onto \( \text{PSL}(2, \mathbb{R}) \) (fuchsian representation). Accordingly \( \text{Im}(\rho) =: \Gamma \) is called a fuchsian group. Then quotienting out the domain of discontinuity of \( \Gamma \) by \( \Gamma \), we obtain two projective structures on \( S \) with the fuchsian holonomy \( \rho \), which have opposite orientations. Let \( C_0 \) denote the one in \( \mathcal{P}_\rho \).

**Theorem 1.1** (Goldman \( [9] \); also \( [13] \)). For every \( C \in \mathcal{P}_\rho \), \( C \) can be obtained by grafting \( C_0 \) along a multiloop \( M \) on \( C \).
Then we can naturally regard the multiloop $M$ as a measure laminination by assigning weight $2\pi$ to each loop of $M$.

Theorem 1.1 was proved, moreover, for quasifuchsian representations. Nevertheless, the proof easily boils down to the fuchsian case by a quasiconformal map.

1.2. **Thurston’s coordinates.** (see §3.2) Let $\mathcal{P}(S)$ be the space of projective structures on $S$. W. Thurston gave a parametrization of $\mathcal{P}(S)$ in a geometric manner:

$$\mathcal{P}(S) \cong \text{Teich}(S) \times \mathcal{ML}(S),$$

where $\text{Teich}(S)$ is the Teichmüller space of $S$ and $\mathcal{ML}(S)$ is the space of measured laminations on $S$. Namely, a pair $(\tau, L) \in \text{Teich}(S) \times \mathcal{ML}(S)$ represents a pleated surface in $\mathbb{H}^3$ and a projective structure $C \in \mathcal{P}(S)$ is modeled on $\hat{C}$; recalling $\partial \mathbb{H}^3 \cong \hat{C}$, a certain nearest-point projection from $\hat{C}$ to the pleated surface yields the correspondence between $C$ and $(\tau, L)$.

In this paper, we mostly use Thurston’s coordinates and write $C = (\tau, L)$, where $C \in \mathcal{P}(S)$, $\tau \in \text{Teich}(S)$ and $L \in \mathcal{ML}(S)$. We also write $C = (f, \rho)$ as a pair of the developing map $f$ and the holonomy representation $\rho$ of $C$. (We distinguish them by notations.)

As an example of Thurston’s coordinates, consider a projective structure $C = (\tau, L)$ with fuchsian holonomy $\rho: \pi_1(S) \to \text{PSL}(2, \mathbb{C})$. Then $\tau$ is $\mathbb{H}^2/\text{Im}(\rho)$, and $L$ is the multiloop $M$ in Theorem 1.1 for $C$ (see [9]).

1.3. **Local characterization of $\mathcal{P}_\rho$ in $\mathcal{PML}(S)$.** The goal of this paper is to prove:

**Theorem A** (see Theorem 6.1). Let $C = (\tau, L)$ be a projective structure on $S$ with arbitrary holonomy $\rho: \pi_1(S) \to \text{PSL}(2, \mathbb{C})$. Then there exists a neighborhood $U$ of $[L]$ in $\mathcal{PML}(S)$ such that, if $C' = (\tau', L') \in \mathcal{P}_\rho$ satisfies $[L'] \in U$, then $C$ and $C'$ are related by a grafting along a multiloop. More precisely, we have either

- (i) $C' = \text{Gr}_M(C)$ for some multiloop $M$ that is a good approximation of the difference $L' - L$, which is calculated on some traintrack carrying both $L$ and $L'$, or
- (ii) $[L] = [L'] \in \mathcal{PML}(S)$ and $C = \text{Gr}_M(C')$, where the multiloop $M = L - L'$.

We see that Theorem 1.1 is a special case of Theorem A (see Theorem 6.1 Case(III)): Let $\rho: \pi_1(S) \to \text{PSL}(2, \mathbb{C})$ be a fuchsian representation and let $C = C_0 \in \mathcal{P}_\rho$ as in §1.1. Then $C_0 = (\tau, \emptyset)$ in Thurston’s
coordinates, where \( \tau = \mathbb{H}^2 / \text{Im}(\rho) \in \text{Teich}(S) \). Then, in this special case, Theorem A holds for \( C = C_0 \) with \( U = \mathcal{PM}\mathcal{L}(S) \) by Theorem 1.1. More precisely, for every \( C' \in \mathcal{P}_\rho \), we have \( C' = \text{Gr}_{L'}(C_0) \), where \( C' = (\tau, L') \), and thus \( M = L' - \emptyset = L' \). However, we do not use Theorem 1.1 to prove Theorem A.

A main step of the proof of Theorem A is to prove the following theorem, which well explains Theorem A:

**Theorem B** (see Proposition 5.15 and also Theorem 5.4). Let \( C = (\tau, L) \) be a projective structure on \( S \) with holonomy \( \rho \). For every neighborhood \( V \) of \( \tau \) in \( \text{Teich}(S) \), there exists a neighborhood \( U \) of \( [L] \) in \( \mathcal{P}\mathcal{M}\mathcal{L}(S) \) such that, if a projective structure \( C' = (\tau', L') \) in \( \mathcal{P}_\rho \) satisfies \( [L'] \in U \), then \( \tau' \in V \).

By Theorem B, if \( U \) is sufficiently small, then \( \tau \) and \( \tau' \) are very close in \( \text{Teich}(S) \). Therefore, in Theorem A, the difference of the projective structures \( C = (\tau, L) \) and \( C' = (\tau', L') \) is captured, mostly, by the second coordinate \( \mathcal{M}\mathcal{L}(S) \) of Thurston’s coordinates.

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1.4. **Outline of the proofs.** Roughly speaking, we use the following ideas.

1.4.1. **Theorem B.** It suffices to show that, for every \( \epsilon > 0 \), if the neighborhood \( U \) of \( [L] \in \mathcal{P}\mathcal{M}\mathcal{L}(S) \) is sufficiently small, then \( \text{length}_{\tau'}(l') / \text{length}_{\tau}(l) < (1 + \epsilon) \) for all corresponding loops \( l \) and \( l' \) on \( \tau \) and \( \tau' \), respectively (Proposition 5.16). Let \( \beta : \mathbb{H}^2 \to \mathbb{H}^3 \) and \( \beta' : \mathbb{H}^2 \to \mathbb{H}^3 \) be the bending maps associated with the projective structures \( C \) and \( C' \), respectively, where their domains \( \mathbb{H}^2 \) correspond to the universal covers of \( \tau \) and \( \tau' \). Since \( \beta \) and \( \beta' \) are (almost everywhere) a locally isometry onto its image with intrinsic metric, we compare the lengths of \( l \) and \( l' \) by analyzing \( \beta(\tilde{l}) \) and \( \beta'(\tilde{l}') \), where \( \tilde{l} \) and \( \tilde{l}' \) are the lifts of \( l \) and \( l' \) to \( \mathbb{H}^2 \), respectively. Since \( \beta \) and \( \beta' \) are \( \rho \)-equivariant, they are \( \rho \)-equivariantly homotopic (Lemma 3.3). Moreover, since their bending laminations \( L \) and \( L' \) are sufficiently close, projected in \( \mathcal{P}\mathcal{M}\mathcal{L}(S) \), it turns out that \( \beta' \) is contained in a “small regular neighborhood” of \( \beta \) (§5.3), where \( \beta \) and \( \beta' \) are regarded as pleated surfaces in \( \mathbb{H}^3 \). Then we have a \( \rho \)-equivariant “near point projection” from \( \beta' \) to \( \beta \) in \( \mathbb{H}^3 \). Then this projection descends to a marking preserving map from \( \tau_i \) to \( \tau \). Since \( \beta' \) is contained in a sufficiently small neighborhood of \( \beta \), we can assume that this projection is \( (1 + \epsilon) \)-lipschitz (Lemma 5.18). Therefore \( \text{length}_{\tau'}(l') / \text{length}_{\tau}(l) < 1 + \epsilon \).
1.4.2. **Theorem A.** By Theorem B, we can identify \( \tau \) and \( \tau' \) by a smooth 
\((1 + \epsilon)\)-bilipschitz map, with sufficiently small \( \epsilon > 0 \), preserving the marking of the surface. Then, using this identification, we see that the bending maps \( \beta \) and \( \beta' \) are sufficiently close in \( C^0 \)-topology and, moreover, in \( C^1 \)-topology in the compliment of a small neighborhood of the bending laminations corresponding to \( L \) and \( L' \) (Theorem 5.4). Then, recalling that \( \tau, \tau' \in \text{Teich}(S) \), take a (fat) traintrack \( T \) on \( S \) that carries both \( L \) and \( L' \). Let \( \kappa: C \to \tau \) and \( \kappa': C' \to \tau' \) be the **collapsing maps**, which realize the correspondence of Thurston’s coordinates (§3.2.3). Then, since \( \beta \) and \( \beta' \) are \( C^1 \)-close in the compliment of the traintrack \( T \), we see that the subsurfaces of \( C \) and \( C' \) (approximately) corresponding to \( S \setminus T \) via \( \kappa \) and \( \kappa' \) are isomorphic (Proposition 6.11 (iii)). Thus the difference of the projective structures \( C' \) and \( C \) is in their subsurfaces corresponding to \( T \). The traintrack \( T \) is a union of branches (quadrangles), and each branch \( B \) of \( T \) corresponds to a quadrangular subsurface of \( C \) and of \( C' \). Then, by making \( T \) sufficiently **straight** and **slim** (Definition 6.8), the product structure on the branch \( B \) yields a product-like structures on the corresponding quadrangles on \( C \) and \( C' \) that are **supported** on a round cylinder in \( \hat{C} \) (Definition 6.4).

Then, as in Lemma 6.5, those corresponding quadrangular subsurfaces of \( C \) and \( C' \) differ by a grafting along a multiarc \( M_j \). Then the multiloop \( M \) in Theorem A is realized as the union of such multiarcs \( M_j \) over all branches \( B_j \) of \( T \). In addition the number of the multiarc \( M_j \) (times \( 2\pi \)) is well-approximated by the difference of the weights of \( L \) and \( L' \) on \( B_j \) (Proposition 6.10 (II- iii), §6.8). Thus \( M \) is a good approximation of the difference of \( L \) and \( L' \).

2. **Conventions and Notations**

We follow the followings, unless otherwise stated:

- By a component, we mean a **connected** component.
- For a geodesic metric space \( X \) and points \( x, y \in X \), we denote the geodesic segment connecting \( x \) to \( y \) by \([x, y]\).
- Let \( X \) be a manifold, and let \( Y \) be a subset of \( X \). Then the **total lift** of \( Y \) is \( \phi^{-1}(Y) \subset \tilde{X} \), where \( \phi: \tilde{X} \to X \) is the universal covering map.
- Let \( X \) be a subset of a hyperbolic space of dimension \( n \) with its ideal boundary, \( \mathbb{H}^n \cup \partial_{\infty}\mathbb{H}^n \) (\( \cong \mathbb{D}^n \)). Then, by Conv(\( X \)), we denote the convex hull of \( X \) in \( \mathbb{H}^n \).
- Let \( X \) be a metric space and \( A \) be a subset of \( X \). Then, for \( \epsilon > 0 \), we denote the \( \epsilon \)-neighborhood of \( A \) in \( X \) by \( N_\epsilon(A) \) or \( N_\epsilon(A, X) \).
• For $\epsilon > 0$, we say that $A$ and $B$ intersect $\epsilon$-orthogonally, if the intersection angle between $A$ and $B$ is $\epsilon$-close to $\pi/2$.

• By a [loop], we mean a simple closed curve.

3. Preliminaries

3.1. Grafting. (see also [9, 15].) Let $C = (f, \rho)$ be a projective structure on $S$. A loop $l$ on $C$ is called admissible if

(i) $\rho(l) \in \text{PSL}(2, \mathbb{C})$ is loxodromic, and

(ii) $f$ embeds $\tilde{l}$ into $\hat{C}$, where $\tilde{l}$ is a lift of $l$ to the universal cover of $S$.

Then, if $l$ is admissible, then the loxodromic element $\rho(l)$ generates a finite cyclic group $G$ in $\text{PSL}(2, \mathbb{C})$. Then the limit set $\Lambda(G)$ is the fixed point set of $\rho(l)$, and $G$ acts on the cylinder $\hat{C} \setminus \Lambda(G)$ freely and property discontinuously. Thus $\hat{C} \setminus \Lambda(G)/G$ is a projective structure $T_l$ on a torus (Hopf torus). Then, by (ii), $l$ is naturally embedded in $T_l$. Since $l$ is also embedded in $C$, there is a natural way to combine the projective structures $C$ and $T_l$ by cutting and pasting along $l$ as follows. We see that $T_l \setminus l$ is a cylinder and $C \setminus l$ is a surface with two boundary components. Thus we can obtain a new projective structure on $S$ by pairing up the boundary components of $T_l \setminus l$ and $C \setminus l$ and canonically identify them using the identification of $l \subset T$ and $l \subset C$. This surgery operation is called $(2\pi)$-grafting (of $C$ along $l$), and we denote the resulting projective structure by $Gr_l(C)$. It turns out that $C$ and $Gr_l(C)$ have the same holonomy representation $\rho$.

3.2. Thurston’s coordinates. (see [12, 16] and also [6, 17, 3].) We here explain more about the parametrization

(1) $\mathcal{P}(S) \cong \text{Teich}(S) \times \mathcal{ML}(S)$.

3.2.1. Bending maps. Let $(\tau, L) \in \text{Teich}(S) \times \mathcal{ML}(S)$. Then we regard $L$ as a geodesic lamination on $\tau$. Set $L = (\lambda, \mu)$, where $\lambda \in \mathcal{GL}(S)$ and $\mu$ is a transversal measure supported on $\lambda$. Let $\tilde{L} = (\tilde{\lambda}, \tilde{\mu}) \in \mathcal{ML}(\mathbb{H}^2)$ be the total lift of $L$ to $\mathbb{H}^2$. Then we can bend a copy of $\mathbb{H}^2$ inside $\mathbb{H}^3$ along $\tilde{\lambda}$ by the angle given by $\tilde{\mu}$. This map is called the bending map $\beta: \mathbb{H}^2 \to \mathbb{H}^3$ induced by $(\tau, L)$. Then $\beta$ is well-defined up to a postcomposition with an element of $\text{PSL}(2, \mathbb{C})$ (as the developing map $f$ is). If $C = (f, \rho) \in \mathcal{P}(S)$ corresponds to $(\tau, L) \in \text{Teich}(S) \times \mathcal{ML}(S)$ by (1), then we say that $\beta$ is the bending map associated with $C$. Since the $\pi_1(S)$-action preserves $\tilde{L}$, there is a representation of $\pi_1(S)$ into $\text{PSL}(2, \mathbb{C})$ under which $\beta$ is equivariant. Then this representation
is unique up to a conjugation by an element of $\text{PSL}(2, \mathbb{C})$, and, indeed, this representation coincides with the holonomy representation $\rho$ of $C$.

3.2.2. Maximal balls and collapsing maps. Let $C \in \mathcal{P}(S)$. Let $\bar{C}$ be the universal cover of $C = (f, \rho)$. An open topological ball $B$ in $\bar{C}$ is called a maximal ball if the developing map $f : \bar{C} \to \hat{\mathbb{C}}$ embeds $B$ onto a round open ball in $\hat{\mathbb{C}}$ and there is no such a ball in $\bar{C}$ properly containing $B$. Let $B$ be a maximal ball and let $H$ be the hyperplane in $\mathbb{H}^3$ bounded by the round circle $\partial f(B)$. Recalling $\hat{\mathbb{C}} = \partial \mathbb{H}^3$, let $\Phi : f(B) \to H$ be the canonical conformal map, obtained by continuously extending the nearest point projection of $\mathbb{H}^3$ onto $H$. Let $\partial_\infty B$ be $\text{cl}(f(B))$, where “$\text{cl}$” denotes closure; then $\partial_\infty B$ is a subset of the round circle $\partial f(B)$.

Suppose that $(\tau, L) \in \text{Teich}(S) \times \mathcal{ML}(S)$ corresponds to $C = (f, \rho)$ in $[\square]$. Let $\beta : \mathbb{H}^2 \to \mathbb{H}^3$ be the bending map induced by $(\tau, L)$. Then there is a leaf or the closure of a component of $\mathbb{H}^2 \setminus \hat{L}$, denoted by $X$, such that the bending map $\beta : \mathbb{H}^2 \to \mathbb{H}^3$ isometrically embeds $X$ onto $\text{Conv}(\partial_\infty B) \subset H$.

Clearly $\Phi \circ f$ embeds $B$ onto $H$. Then the core of the maximal ball $B$ is the subset of $B$ that $\Phi \circ f$ embeds onto $\text{Conv}(\partial B)$, which is denoted by $\text{Core}(B)$. Thus we have a unique embedding $\kappa_B$ of $\text{Core}(B)$ onto $X \subset \mathbb{H}^2$ such that $\beta \circ \kappa_B = \Phi \circ f$. Then, for each $x \in \text{Core}(B)$, the hyperplane $H$ is called a hyperbolic tangent plane of $\beta$ at $x$.

It turns out that $\text{Core}(B)$ are disjoint for different maximal balls $B$. Moreover $\bar{C}$ is the union of the cores $\text{Core}(B)$ of all maximal balls $B$ in $\bar{C}$. Thus we can define $\kappa : \bar{C} \to \mathbb{H}^2$ by $\kappa = \kappa_B$ on $\text{Core}(B)$. Then $\kappa$ commutes with the action of $\pi_1(S)$ and descends to the collapsing map $\nu : C \to \tau$. In addition, $\partial(\text{Core}(B))$ is the union of disjoint copies of $\mathbb{R}$ properly embedded in $\bar{C}$. Then, by taking the union of $\partial(\text{Core}(B))$ over all maximal balls $B$ in $\bar{C}$, we obtain a lamination $\tilde{\nu}$ on $\bar{C}$, such that $\kappa$ embeds each leaf of $\tilde{\nu}$ onto a leaf of $\bar{\lambda}$. Then the lamination $\tilde{\nu}$ descends to a lamination $\nu$ on $C$, and $\kappa$ embeds each leaf of $\mu$ onto a leaf of $\lambda$. Then $\nu$ is called the canonical lamination corresponding to $\lambda$ (via $\kappa$).

In particular the collapsing map $\kappa$ has the following properties. If $l$ is a periodic leaf of $L$, then $\kappa^{-1}(l)$ is a closed cylinder embedded in $C$. Besides this cylinder is foliated by closed leaves $m$ of $\nu$ such that $\kappa$ embeds $m$ onto $l$. Then, letting $\tilde{m}$ and $\tilde{l}$ be corresponding lifts of $m$ and $l$ to $\bar{C}$ and $\mathbb{H}^2$, respectively, $f$ embeds $\tilde{m}$ onto a circular arc on $\hat{C}$ whose endpoints are the endpoints of the geodesic $\beta(l)$. Let $h$ be the weight of $l$ with respect to $\mu$. Then $\kappa^{-1}(l)$ has a natural product structure.
$S^1 \times [0, h]$ so that $S^1 \times \{t\}$ is a closed leaf of $\nu$ for each $t \in [0, h]$ and $\kappa$ collapses $s \times [0, h]$ onto a single point on $l$ for each $s \in S^1$. Let $M$ be the union of all periodic leaves $l_1, l_2, \ldots, l_n$ of $L$. Then $\kappa^{-1}(M)$ is the union of the disjoint closed cylinders $\kappa^{-1}(l_1), \ldots, \kappa^{-1}(l_n)$. Then, the restriction of $\kappa : C \to \tau$ to $C \setminus \kappa^{-1}(M)$ is a diffeomorphism onto $\tau \setminus M$.  

3.2.3. Thurston’s metric on projective structures. We assign a canonical Euclidean metric to the cylinder $\kappa^{-1}(l_i) \cong S^1 \times [0, h_i]$ for each $i = 1, 2, \ldots, n$, such that its height $h_i$ is the weight of $l_i$ with respect to $\mu$ and that $\kappa$ isometrically embeds $S^1 \times \{t\}$ onto $l \subset \tau$ for each $t \in [0, h_i]$. (To be precise, for closed leaves $m = S^1 \times t$ and $m' = S^1 \times t'$ with $t, t' \in [0, h]$, let $\tilde{m}$ and $\tilde{m}'$ be corresponding lifts of $m$ and $m'$, respectively, to $\hat{C}$ that are invariant under the same element of $\pi_1(S)$; then the circular arcs $f(\tilde{m})$ and $f(\tilde{m}')$ share their corresponding endpoints and, thus, bound a crescent region in $\hat{C}$, and the angle at the veracities of the crescent region is $|t - t'| \mod 2\pi$.) On the other hand, we assign $C \setminus \kappa^{-1}(M)$ the pullback hyperbolic metric of $\tau$ via $\kappa$. Thus we have defined a piecewise Euclidean/hyperbolic metric on $C$, which is called Thurston’s metric.

We last see that the transversal measure $\mu$ supported on $\lambda$ naturally induces a transversal measure on $\nu$. On each Euclidean cylinder $\kappa^{-1}(l_i) = S^1 \times [0, h_i]$ in $C$, we assign a unique transversal measure to $\nu$ by the difference in the second coordinate $[0, h_i]$, so that the total measure of the cylinder is the weight of $l_i$ with respect to $\mu$. Since $\kappa$ isometrically embeds the hyperbolic subsurface of $C$ onto $\tau \setminus M$, we can isomorphically pull back $L \setminus M$ to a measured lamination on $C \setminus \kappa^{-1}(M)$ via $\kappa$. The union of those measured laminations on the Euclidean and hyperbolic subsurfaces of $C$ is called the canonical measured lamination on $C$ corresponding to $L$.

3.3. Intersection for measured laminations. (see [3])

Let $L$ be a measured geodesic lamination on $\mathbb{H}^2$. Let $X$ be a subset of $\mathbb{H}^2$.

Definition 3.1. The intersection of $L$ and $X$ is the minimal sublamination of $L$ that contains all leaves of $L$ intersecting $X$. We denote this intersection by $I(L, X)$.

3.4. Isomorphism via developing maps.

Definition 3.2. Let $C = (f, \rho)$ and $C' = (f', \rho)$ be projective structures on a surface $F$ with the same holonomy $\rho$, where $f : \hat{C} \to \hat{C}$ and $f' : \hat{C}' \to \hat{C}$ are their developing maps. Then $C$ and $C'$ are
isomorphic via \( f \) and \( f' \); if there is a marking preserving homeomorphism \( \tilde{\phi}: \tilde{C} \rightarrow \tilde{C}' \), such that, letting \( \tilde{\phi}: \tilde{C} \rightarrow \tilde{C} \) be the lift of \( \phi \), we have \( f = f' \circ \tilde{\phi}: \tilde{C} \rightarrow \tilde{C} \). Then, we also say that the isomorphism \( \tilde{\phi} \) is compatible with \( f \) and \( f' \).

3.5. Equivariant homotopies between Bending maps.

**Lemma 3.3.** Let \( C_1 = (\tau_1, \lambda_1) \) and \( C_2 = (\tau_2, \lambda_2) \) be projective structures on \( S \) with the same holonomy \( \rho: \pi_1(S) \rightarrow \text{PSL}(2, \mathbb{C}) \). Let \( \beta: \mathbb{H}^2 \rightarrow \mathbb{H}^3 \) and \( \beta': \mathbb{H}^2 \rightarrow \mathbb{H}^3 \) be the bending maps associated with \( C \) and \( C' \), respectively. Then \( \beta \) and \( \beta' \) are \( \rho \)-equivariantly homotopic (note that the domains of \( \beta \) and \( \beta' \) are identified via the markings \( \psi_1: S \rightarrow \tau_1 \) and \( \psi_2: S \rightarrow \tau_2 \) since \( \tau_1, \tau_2 \in \text{Teich}(S) \)).

**Proof.** Via the markings \( \psi_1 \) and \( \psi_2 \), we regard a subset of \( S \) also as a subset of \( \tau_1 \) and \( \tau_2 \).

*Step 1.* We first construct an equivariant homotopy for corresponding loops on \( \tau_1 \) and \( \tau_2 \). Let \( l \) be a loop on \( S \). Let \( l (\cong \mathbb{R}) \) be a lift of \( l \) to the universal cover \( \tilde{S} \) of \( S \). Then \( \beta_1|\tilde{l} \) and \( \beta_2|\tilde{l} \) are equivariant under the restriction of \( \rho \) to \( \langle l \rangle \subset \pi_1(S) \), the infinite cyclic group generated by \( l \in \pi_1(S) \). Then \( \rho(l) \) may be of any type of hyperbolic isometry, i.e. parabolic, elliptic, or loxodromic. In either case, we can easily construct a homotopy between \( \beta_1|\tilde{l} \) and \( \beta_2|\tilde{l} \) that is equivariant under \( \rho|\langle l \rangle \).

*Step 2.* Next let \( P \) be a pair of pants embedded in \( S \). Let \( l_1, l_2, l_3 \) be the boundary loops of \( P \). Pick disjoint arcs \( a_1, a_2, a_3 \) property embedded in \( P \) that decompose \( P \) into two hexagons. Let \( \tilde{P} \) be a lift of \( P \) to \( \tilde{S} \). Then we show that there is a homotopy between \( \beta_1|\tilde{P} \) and \( \beta_2|\tilde{P} \) equivariant under \( \rho|\pi_1(P) \). For each \( j = 1, 2, 3 \), pick a lift \( \tilde{l}_j \) of \( l_j \) to \( \tilde{P} \). Then, by Step 1, we have a homotopy connecting \( \beta_1|\tilde{l}_j \) an \( \beta_2|\tilde{l}_j \) that is equivariant under \( \rho|\pi_1(l_j) \). By equivariantly extending those homotopies, we have a homotopy \( \Phi_{\beta P}: \partial \tilde{P} \times [0, 1] \rightarrow \mathbb{H}^3 \) between \( \beta_1|\partial \tilde{P} \) and \( \beta_2|\partial \tilde{P} \) that is \( \rho|\pi_1(P) \)-equivariant. Then we can easily extend this homotopy \( \Phi_{\beta P} \) to the homotopy between the lifts of arcs \( a_i \) \((i = 1, 2, 3)\) to \( \tilde{P} \) and then to the entire lift \( \tilde{P} \), so that the extension is equivariant under \( \rho|\pi_1(P) \).

*Step 3.* Pick a maximal multiloop \( M \) on \( S \), which decompose \( S \) into \( 2(g - 1) \) pairs of pants \( P_k \) \((k = 1, 2, \ldots, 2(g - 1))\). Let \( \tilde{M} \) denote the total lift of \( M \) to \( \tilde{S} \). Then we can obtain a \( \rho \)-equivariant homotopy \( \Phi_M \) between \( \beta_1|\tilde{M} \) and \( \beta_2|\tilde{M} \) similarly as we obtained the homotopy \( \Phi_{\beta P} \) in Step 2. For arbitrary \( k \in \{1, 2, \ldots, 2(g - 1)\} \), let \( \tilde{P}_k \) be a lift of \( P_k \) to \( \tilde{S} \). Then \( \Phi_M \) induces a homotopy \( \Phi_{\beta P_k} \) between \( \beta_1|\partial \tilde{P}_k \) and
\( \beta_2 | \partial \tilde{P}_k \). Similarly to Step 2, we can extend this induced homotopy to a homotopy \( \Phi_{\tilde{P}_k} \) between \( \beta_1 | \tilde{P}_k \) and \( \beta_2 | \tilde{P}_k \) that is equivariant under \( \rho | \pi_1(\tilde{P}_k) \). Since \( \Phi_{\tilde{P}_k} \) and \( \Phi_{\tilde{M}} \) coincide on \( \partial \tilde{P}_k \) and \( \Phi_{\tilde{M}} \) is \( \rho \)-equivariant, We can \( \rho \)-equivalently extend all homotopies \( \Phi_{\tilde{P}_k} \) \( (k = 1, 2, \ldots, 2(g-1)) \) between \( \beta_1 | \tilde{P}_k \) and \( \beta_2 | \tilde{P}_k \) and obtain a desired homotopy between \( \beta_1 \) and \( \beta_2 \).

3.6. Quasi-isometries and bilipschitz maps.

**Definition 3.4.** Let \( X \) and \( Y \) are metric space. Let \( f, g: X \to Y \) be continuous maps. We say \( f \) is an \( \epsilon \)-perturbation of \( g \) for \( \epsilon > 0 \), if \( \text{dist}_Y(f(x), g(x)) < \epsilon \) for all \( x \in X \). If, in addition, \( f \) and \( g \) are homotopic, then we say that \( f \) and \( g \) are \( \epsilon \)-homotopic.

**Lemma 3.5.** Let \( \tau \) be a closed hyperbolic surface. For every \( \epsilon > 0 \), there exists \( D_{\tau} > 0 \), such that, if \( \eta: \tau \to \tau' \) is a \((1+\delta,\delta)\)-quasiisometry onto a hyperbolic surface \( \tau' \) with \( 0 < \delta < D_{\tau} \), then we can \( \epsilon \)-homotope \( \eta \) to a smooth \((1+\epsilon)\)-bilipschitz map.

**Proof.** (c.f. [15], the proof of Theorem 7.2.) **Step 1.** Pick \( \tau \in \text{Teich}(S) \). Fix a closed hyperbolic surface \( \tau \) homeomorphic to \( S \). First consider a simple geodesic segment \([A, B]\) on \( \tau \) that connects distinct points \( A \) and \( B \) on \( \tau \). For every \( \epsilon > 0 \), assuming that \( \delta > 0 \) is sufficiently small, if \( \eta: \tau \to \tau' \) is a \((1+\delta,\delta)\)-quasiisometry with \( \tau' \in \text{Teich}(S) \), then we can \( \epsilon \)-homotope \( \eta \) to \( \eta': \tau \to \tau' \) so that \( \eta(A) = \eta'(A), \eta(B) = \eta'(B), \) and \( \eta'|[A, B] \) is a \((1+\epsilon)\)-bilipschitz embedding onto a geodesic segment connecting \( \eta(A) \) to \( \eta(B) \).

**Step 2.** Next consider a finite triangulation \( \Delta = \{ \Delta_i \} \) \( (i = 1, 2, \ldots, n) \) of \( \tau \), where \( \Delta_i \) are hyperbolic triangles on \( \tau \) with disjoint interiors such that \( \bigcup_i \Delta_i = \tau \). Since there are only finitely many edges of \( \Delta \), for every \( \epsilon > 0 \), if \( \delta > 0 \) is sufficiently small, then given a \((1+\delta,\delta)\)-quasiisometry \( \eta: \tau \to \tau' \) onto a hyperbolic surface \( \tau' \), we can \( \epsilon \)-homotope \( \eta \) to \( \eta': \tau \to \tau' \) satisfying the property of \( \eta' \) in Step 1 for all edges of the triangulation \( \Delta \). Then, by taking sufficiently small \( \delta > 0 \), we can in addition assume that \( \eta' \) is a \((1+\epsilon)\)-bilipschitz embedding when restricted to each \( \Delta_i \). Therefore, since the triangulation is locally finite, if \( \delta > 0 \) is sufficiently small, then \( \eta \) is \( \epsilon \)-homotopic to a smooth \((1+\epsilon)\)-bilipschitz map. \( \Box \)

4. Bilipschitz curves and bending maps

**Definition 4.1.** Let \( \lambda \) and \( \lambda' \) be (possibly measured) geodesic laminations on a hyperbolic surface \( \tau \). Then the angle between \( \lambda \) and \( \lambda' \)
is
\[ \angle(\lambda, \lambda') := \sup_{p \in \lambda \cap \lambda'} \angle_p(l, l') \in [0, \pi/2], \]
over all \( p \in \lambda \cap \lambda' \), where \( l \) and \( l' \) are the (non-oriented) leaves of \( \lambda \) and \( \lambda' \), respectively, such that \( p \in l \cap l' \).

The following proposition is the main statement of this section:

**Proposition 4.2.** For every \( \delta > 0 \), there exists \( \epsilon > 0 \) such that, if \( L = (\lambda, \mu) \) is a measured geodesic lamination on \( \mathbb{H}^2 \) with \( \text{Area}_{\mathbb{H}^2}(\lambda) = 0 \) and \( l \) is a geodesic on \( \mathbb{H}^2 \) with \( \angle(l, L) < \epsilon \), then, letting \( \beta = \beta_L : \mathbb{H}^2 \to \mathbb{H}^3 \) be the bending map associated with \( L \),

(i) \( \beta_L|l : l \to \mathbb{H}^3 \) is a \((1 + \delta)\)-bilipschitz embedding,

and, letting \( m \) be the geodesic in \( \mathbb{H}^3 \) connecting the endpoints of \( \beta_L(L) \),

(ii) for each point \( x \in l \), \( \beta_L(x) \) is \( \delta \)-close to \( m \), and, if \( \beta_L \) is differentiable at \( x \), then the tangent vector of \( \beta_L|l \) at \( x \) is \( \delta \)-parallel to \( m \),

that is, the tangent vector of \( \beta_L|l \) in \( \mathbb{H}^3 \) at \( x \) is \( \delta \)-orthogonal to the totally geodesic hyperplane that is orthogonal to \( m \) and contains \( \beta(x) \).

**Remark 4.3.** The condition on \( \angle(l, L) \) is new. Similar statements are in \([5, 7, 3]\).

Letting \( \Phi_m : \mathbb{H}^3 \to m \) be the nearest point projection, we in addition have

**Corollary 4.4.** (iii) \( \Phi_m \circ \beta_L|l : l \to m \) is a \((1 + \delta)\)-bilipschitz map.

**Proof.** For each point \( y \in m \), \( \Phi_m^{-1}(y) \), is a totally geodesic hyperplan in \( \mathbb{H}^3 \) orthogonal to \( m \). Then \( \mathbb{H}^3 \) is foliated by such hyperplanes. Since \( \text{Area}_{\mathbb{H}^2}(\lambda) = 0 \) is \( \beta|l \) is differentiable almost everywhere. By (ii), the curve \( \beta_L|l \) stays in a small neighborhood of \( m \) and \( \delta \)-orthogonally intersects this foliation of \( \mathbb{H}^3 \) at almost every point at \( x \). By taking sufficiently small \( \epsilon > 0 \), we have the corollary.

We first prove a similar statements for geodesic segments of bounded lengths:

**Proposition 4.5.** For every (large) \( K > 0 \) and (small) \( \delta > 0 \), there exists \( \epsilon > 0 \) with the following properties:

(i) If \( L \) is a measured geodesic lamination and \( l (\cong \mathbb{R}) \) is a geodesic on \( \mathbb{H}^2 \) with \( \angle(l, L) < \epsilon \), then, if points \( x, y \) on \( l \) satisfies \( x < y \) and \( \text{dist}_{\mathbb{H}^2}(x, y) < K \), then we have \((1 - \delta) \cdot \text{dist}_{\mathbb{H}^2}(x, y) < \text{dist}_{\mathbb{H}^3}(\beta_L(x), \beta_L(y))\).
(ii) If $\beta|l$ is differentiable at $x$, for $y \in l$ with $y \neq x$, let $\theta_y(x) \in [0, \pi]$ denote the angle between the oriented geodesic segment $[\beta_L(x), \beta_L(y)]$ and the tangent vector $(\beta_L|l)'(x) \in T_{\beta(x)}(\mathbb{H}^3)$. Then $\theta_y(x) < \delta$ for all $x, y \in l$ with $\text{dist}(x, y) < K$ such that $\theta_y(x)$ is well-defined (see Figure 1).

**Proof.** First consider a right hyperbolic triangle $\triangle ABC$ in $\mathbb{H}^2$ (with geodesic edges) with $\angle C = \pi/2$, where $A, B, C$ are the vertices of the triangle. Then by elementary trigonometry, we can easily prove:

**Lemma 4.6.** For every (small) $\delta' \in (0, 1]$, there exists $\epsilon > 0$ depending only on $K$, such that, if $\angle B < \epsilon$ and $\text{length}(AB) < K$, then

$$(1 - \delta') \cdot \text{length}(AB) < \text{length}(CA) - \text{length}(BC),$$

and, for every point $A'$ in $\mathbb{H}^2$ with $\text{dist}(C, A') < \text{dist}(C, A)$, we have

$$\angle A'BC < \delta'.$$

Pick $K > 0$ and $\delta \in (0, 1]$. Then pick $\delta' > 0$ with $\delta' < \delta$. Let $\epsilon > 0$ be the number obtained by applying Lemma 4.6 for $K$ and $\delta'$. Then we can assume that $\delta' + \epsilon < \delta$, if necessarily by taking smaller $\epsilon > 0$.

Let $L$ be a measured (geodesics) lamination and $l(\cong \mathbb{R})$ be a geodesic on $\mathbb{H}^2$ with $\angle(l, L) < \epsilon$. Let $x$ and $y$ be distinct points on $l$ within distance less than $K$ such that $x < y$. Let $I \in \mathcal{ML}(\mathbb{H}^2)$ be the intersection $I([x, y], L)$ of $L$ and the geodesic segment $[x, y]$ connecting $x$ to $y$ (see §3.3). We can assume that $[x, y]$ intersects at least one leaf of $L$ transversally, since otherwise Proposition 4.5 clearly holds for such $x, y$. Let $m$ denote the leaf of $I$ closest to $x$. Then, there is a unique point $z \in \mathbb{H}^2$ such that $\triangle xyz$ is a hyperbolic right triangle with $\angle z = \pi/2$ and, letting $\eta: \mathbb{H}^2 \to \mathbb{H}^2$ be the orientation preserving isometry preserving $l$ and taking $l \cap m$ to $x$, we have $[x, z] \subset \eta(m)$. Then $[x, z]$ is disjoint from $I$, unless $m$ contains $x$. Thus, since $\angle(l, L) < \epsilon$, we have $\angle(yxz) < \epsilon$. Let $\beta_I: \mathbb{H}^2 \to \mathbb{H}^3$ be the bending map induced by $I$. Then $\beta_I$ isometrically embeds $[x, z]$ into $\mathbb{H}^3$. Therefore $\text{dist}_{\mathbb{H}^3}(\beta_I(x), \beta_I(z)) = \text{dist}_{\mathbb{H}^2}(x, z)$. Since bending maps are 1-lipschitz,
\[ \text{dist}(\beta_1(z), \beta_1(y)) \leq \text{dist}(z, y). \] By the triangle inequality, we have

\[
\begin{align*}
\text{dist}(\beta_1(x), \beta_1(y)) & \geq \text{dist}(\beta_1(x), \beta_1(z)) - \text{dist}(\beta_1(z), \beta_1(y)) \\
& \geq \text{dist}(x, z) - \text{dist}(z, y).
\end{align*}
\]

Then, by the first conclusion of Lemma 4.6, we have

\[
\text{dist}(\beta_1(x), \beta_1(y)) > (1 - \delta') \cdot \text{dist}(x, y).
\]

Since \( \beta_1 = \beta_L \) on \([x, y]\), \( \text{dist}(\beta_L(x), \beta_L(y)) > (1 - \delta') \cdot \text{dist}(x, y); \) thus we have shown i(i).

By applying the second conclusion of Lemma 4.6 to \( \triangle A'BC' = \triangle \beta_1(y)\beta_1(x)\beta_1(z), \) we have \( \angle \beta_1(y)\beta_1(x)\beta_1(z) < \delta'. \) By the triangle inequality, \( \theta_y(x) \leq \angle yxz + \angle \beta_1(y)\beta_1(x)\beta_1(z) < \epsilon + \delta' < \delta. \) Thus we have proved (ii).

**Proof of Proposition 4.2.** Set \( \beta = \beta_1 \) (i) Let \( L = (\lambda, \mu) \) be a geodesic lamination on \( \mathbb{H}^2 \) and let \( l \cong \mathbb{R} \) be a geodesic on \( \mathbb{H}^2. \) Let \( x, y \) be different points on \( l \) with \( x < y \) such that \( \beta|l \) is differentiable at \( y. \) Then, as before, let \( \theta_x(y) \in [0, \pi] \) denote the angle between the tangent vector of \( \beta|l \) at \( y \) and the oriented geodesic \([\beta(x), \beta(y)]\) in \( \mathbb{H}^3. \)

Fixing sufficiently small \( \delta' > 0, \) we show that there exists \( \epsilon > 0, \) depending only on \( \delta', \) such that if \( \angle(l, L) < \epsilon, \) then \( \theta_x(y) < \delta'. \) Pick \( K > 0 \) and \( \delta'' > 0 \) with \( \delta'' < \delta' / 2. \) Let \( \epsilon' = \epsilon(K, \delta'') > 0 \) be the number obtained by applying Lemma 4.5 to \( K \) and \( \delta''. \) Then divide the closed geodesics \([x, y]\) into subintervals \([p_0, p_1], [p_1, p_2], \ldots, [p_{n-1}, p_n]\) and \( x = p_0 < p_1 < \ldots < p_n = y \) so that

1. \( p_i \) are not on the leaves of \( L \) for \( i = 1, 2, \ldots, n - 1, \) and
2. \( K / 2 < \text{dist}_{\mathbb{H}^2}(p_i, p_{i+1}) < K \) for \( i = 0, 1, \ldots, n - 2 \)

(recall that \( \text{Area}_{\mathbb{H}^2}(\lambda) = 0. \) Consider the piecewise geodesic curve \( \bigcup_{i=0}^{n-1}[\beta(p_i), \beta(p_{i+1})] \) in \( \mathbb{H}^3, \) which connects \( \beta(x) \) to \( \beta(y). \) By Lemma 4.5 (ii), we have \( \theta_{p_{n-1}}(y) < \delta'' \) and also \( \pi - \angle(\beta(p_{n-1}), \beta(p_i), \beta(p_{i+1})) < 2\delta'' \) for all \( i = 1, 2, \ldots, n - 1. \) By Lemma 4.5 (i), \( \text{length}(\[\beta(p_i), \beta(p_{i+1})\]) > (1 - \delta'') \cdot (K / 2) \) for \( i = 0, 1, \ldots, n - 2. \) Thus, retaking sufficiently small \( \delta' > 0 \) if necessarily, the piecewise geodesic curve \( \bigcup_{i=0}^{n-1}[\beta(p_i), \beta(p_{i+1})] \) has sufficiently long segments (except \( [\beta(p_{n-1}), \beta(p_n)] \)) and the bending angle at its singularity points are sufficiently small. Therefore, assuming that \( \delta'' > 0 \) is sufficiently small, we have \( \angle \beta(x)\beta(y)\beta(p_{n-1}) < \delta' / 2 \) (see [5 §1.4.2]; also [7 3]). Then, by the triangle inequality,

\[ 0 < \theta_x(y) < \angle \beta(p_{n-1})\beta(y)\beta(x) + \theta_{p_{n-1}}(y) < \delta' / 2 + \delta''. \]
4.2

Thus we have that $0 < \theta_x(y) < \delta'$. We have

$$\frac{d\ \text{length}([\beta(x), \beta(y)])}{dy} = \cos(\theta_x(y))$$

(see [5] §1.4.2; also [7, 3]). Then, by taking sufficiently small $\delta' > 0$ if necessarily, we have $\frac{1}{1+\delta} < \cos(\theta_x(y)) \leq 1$ for all different $x, y$ on $l$ such that $\beta | l$ is differentiable at $y$. Since $\beta | l$ is differentiable at almost all points of $l$, $\beta | l$ is a $(1 + \delta)$-bilipschitz embedding.

(ii) Pick $\delta' > 0$ with $2\delta' < \delta$. Then, in the proof of (i), we have shown that there exists $\epsilon > 0$, such that, if $\angle(l, L) < \epsilon$, then $\theta_x(y) < \delta'$ for all different $x, y \in l$ such that $\beta | l$ is differentiable at $y$. Taking the limits as $x$ goes to the end points $\pm \infty$ of $l \cong \mathbb{R}$, we have $\theta_{-\infty}(y), \theta_{\infty}(y) \leq \delta'$. Thus we have $\beta(-\infty)\beta(y)\beta(\infty) > \pi - 2\delta'$, where $\beta(-\infty), \beta(\infty) \in \hat{\mathbb{C}}$ are the ideal endpoints of the geodesic $m$ bounded distance away from the bilipschitz curve $\beta | l$. It is well-known that the area of a triangle in $\mathbb{H}^2$ is equal to $\pi$ minus the sum of the angles of the vertices. Thus $\text{Area}_{\mathbb{H}^2}(\triangle \beta(-\infty)\beta(y)\beta(\infty)) < 2\delta'$. Thus, if necessary by taking smaller $\delta' > 0$, we can assume that $\text{dist}_{\mathbb{H}^2}(\beta(y), m) < \delta$. Recalling that $\Phi_m: \mathbb{H}^3 \to m$ is the nearest point projection, $\triangle \beta(-\infty)\beta(y)(\Phi_m \circ \beta)(y)$ has area less that $\delta'$. Using the same area formula, we have $\angle \beta(-\infty)\beta(y)(\Phi_m \circ \beta)(y) < \pi/2 - \delta'$. Since $\theta_{-\infty}(y) < \delta'$ and $2\delta' < \delta$, by the triangle inequality, we see that the tangent vector of $\beta | l$ at $y$ is $\delta$-parallel to $m$. (Alternatively, the $\delta$-closeness is immediate from (i) and Morse Lemma.)

5. Local stability of bending maps in $\mathcal{PML}$

5.1. Bending maps with a fixed bending lamination.

**Proposition 5.1.** Let $C = (\tau, L)$ and $C' = (\tau', L')$ be projective structures on $S$ with fixed holonomy $\rho$. Let $\beta$ and $\beta': \mathbb{H}^2 \to \mathbb{H}^3$ be the bending maps associated with $C$ and $C'$, respectively. Then, if $[L] = [L'] \in \mathcal{PML}(S)$, we have $\beta = \beta'$.

**Proof.** We first show that there exists a continuous shearing map $\eta: \mathbb{H}^2 \to \mathbb{H}^2$ such that $\beta \circ \eta = \beta'$. Let $\tilde{L}$ and $\tilde{L}'$ be the total lifts of $L$ and $L'$ to $\mathbb{H}^2$, respectively. Since $[L] = [L']$ in $\mathcal{PML}(S)$, we have $[L'] = [\tilde{L}] \in \mathcal{PML}(S)$. Then

**Lemma 5.2.** Let $l$ and $l'$ be corresponding leaves of $\tilde{L}$ and $\tilde{L}'$, respectively. In addition, assign the same orientation to $l$ and $l'$. Then $\beta(l) = \beta'(l')$ as oriented geodesics in $\mathbb{H}^3$; thus there exists an unique orientation-preserving isometry $\eta_l: l \to l'$ such that $\beta = \beta' \circ \eta_l$ on $l$. 

[42]
Proof. Let $l$ and $l'$ be as in the assumption. Then $\beta$ and $\beta'$ isometrically embed $l$ and $l'$, respectively, onto a geodesic in $\mathbb{H}^3$. By Lemma 3.3, $\beta, \beta': \mathbb{H}^2 \to \mathbb{H}^3$ are $\rho$-equivariantly homotopic. Thus the geodesic $\beta(l)$ and $\beta(l')$ has bounded Hausdorff distance, and therefore $\beta(l) = \beta'(l')$. Hence, there is an isometry $\eta_l: l \to l'$ such that $\beta = \beta' \circ \eta_l$. Since $\beta$ and $\beta'$ are $\rho$-equivariantly homotopic, that corresponding endpoints of $l$ and $l'$ (in $\partial S$) map to the same point on $\hat{\mathcal{C}}$. Thus $\eta_l$ preserves the orientation.

Corollary 5.3. Let $P$ and $P'$ are the closures of corresponding components of $\mathbb{H}^2 \setminus \lambda$ and $\mathbb{H}^2 \setminus \lambda'$, respectively. Then there exists a unique orientation preserving isometry $\eta_P: P \to P'$ such that $\beta' \circ \eta_P = \beta$ on $P$.

Proof. The equality $[L] = [L']$ induces an orientation-preserving homeomorphism between $P$ and $P'$ and, in particular, between $\partial P$ and $\partial P'$. By Lemma 5.2 for corresponding boundary geodesics $l$ and $l'$ of $P$ and $P'$, respectively, we have an orientation preserving isometry $\eta_l: l \to l'$ such that $\beta' \circ \eta_l = \beta$ on $l$. Then define $\eta_P: \partial P \to \partial P'$ by $\eta_P = \eta_l$ on each boundary geodesic $l$ of $P$. The bending maps $\beta$ and $\beta'$ isometrically embed $P$ and $P'$, respectively, into the same totally geodesic hyperplane in $\mathbb{H}^3$. Thus $\eta_P: \partial P \to \partial P'$ uniquely extends to an isometry $\eta: P \to P'$ such that $\beta' \circ \eta = \beta$ on $P$.

The hyperbolic plane $\mathbb{H}^2$ decomposes into components of $\mathbb{H}^2 \setminus \bar{L}$ and leaves of $\bar{L}$. Thus we can define $\eta: \mathbb{H}^2 \to \mathbb{H}^2$ by $\eta(x) = \eta_l(x)$ if $x$ is a leaf of $\bar{L}$ as in Lemma 5.2, and by $\eta(x) = \eta_P(x)$ if $x$ is in the closure of a component $P$ of $\mathbb{H}^2 \setminus \bar{L}$ as in Corollary 5.3. Then, on isolated leaves of $\bar{L}$, $\eta$ is well-defined and continuous by the uniqueness of $\eta_P$. Thus $\eta: \mathbb{H}^2 \to \mathbb{H}^2$ is well-defined, and $\beta = \beta' \circ \eta$. The bending maps $\beta$ and $\beta'$ are continuous, and moreover they are locally injective in the compliment of isolated leaves of weight $\pi$ modulo $2\pi$. Therefore, $\eta$ must be continuous since $\beta = \beta' \circ \eta$. Thus $\eta: \mathbb{H}^2 \to \mathbb{H}^2$ is a shearing map along $\bar{L}$ minus isolated leaves of $L$.

Last we show that the shearing map $\eta: \mathbb{H}^2 \to \mathbb{H}^2$ is an isometry. Then it suffice to show that for each $x \in \mathbb{H}^2$ contained in an irrational leaf of $\bar{L}$, there exists a small neighborhood $U$ of $x$ such that $\eta|U: U \to \mathbb{H}^2$ is an isometric embedding (i.e. there is no nontrivial shearing along leaves of $\bar{L}$ that intersects $U$). Let $U$ be a neighborhood of $x$ such that $\beta|U: U \to \mathbb{H}^3$ is injective. Then $\beta|U$ is an isometric embedding onto its image $\beta(U)$ with the intrinsic metric induced from $\mathbb{H}^3$. Then $\beta|U$ is an isometric embedding onto its image $\beta(U)$ with the intrinsic metric induced from $\mathbb{H}^3$. Then, since $\beta = \beta' \circ \eta$, $\beta'$ is injective on $\eta(U)$ and
5.2. Stability of bending maps with converging bending laminations. We fix a representation \( \rho : \pi_1(S) \to \text{PSL}(2, \mathbb{C}) \). Let \((C_i)_{i \in \mathbb{N}}\) be a sequence in \( \mathcal{P}_\rho \). For each \( i \in \mathbb{N} \), set \( C_i = (\tau_i, L_i) \in \text{Teich}(S) \times \mathcal{ML}(S) \) and \( L_i = (\lambda_i, \mu_i) \), where \( \lambda_i \) is in \( \mathcal{GL}(\tau_i) \) and \( \mu_i \) is a transversal measure supposed on \( \lambda_i \). Let \( \beta_i : \mathbb{H}^2 \to \mathbb{H}^3 \) be the bending map associates with \( C_i \). In Proposition 5.1 we have shown that, if the projective measured laminations of two projective structures coincide, then so do the bending maps. Our main goal of §5 is generalize Proposition 5.1 to the case that a sequence of projective measured laminations converges in \( \mathcal{PML}(S) \):

**Theorem 5.4.** Suppose that \( \lim_{i \to \infty} [L_i] = [L] \) in \( \mathcal{PML}(S) \). Then \( \beta_i \) converges to \( \beta \) as \( i \to \infty \). More precisely, we have

(i) For every \( \epsilon > 0 \), there exists \( I \in \mathbb{N} \), such that, if \( i > I \), then there exists a marking preserving \( (1+\epsilon)-\text{bilschitz} \) map \( \eta : \tau \to \tau_i \) such that \( \beta_i \circ \eta : \mathbb{H}^2 \to \mathbb{H}^3 \) is \( \epsilon \)-close to \( \beta \) in \( C^0 \)-topology, where \( \eta : \mathbb{H}^2 \to \mathbb{H}^3 \) is the lift of \( \eta_0 \).

(ii) Suppose, in addition, that \( \lambda_i \) converges to a lamination \( \lambda_\infty \) in \( \mathcal{GL}(S) \) as \( i \to \infty \) with the Chabauty topology. Then, we can in addition assume that \( \beta_0 \circ \eta_0 \) is \( \epsilon \)-close to \( \beta \) in \( C^1 \)-topology in the compliment of the \( \eta \)-neighborhood of \( \lambda_\infty \), the total lift of \( \lambda_\infty \) to \( \mathbb{H}^2 \).

We first generalize Lemma 5.2 to the setting of Theorem 5.4. From now on, we assume that \( \lambda_i \) converges to a lamination \( \lambda_\infty \) in \( \mathcal{GL}(S) \). Then, since \( [L_i] \to [L] \) in \( \mathcal{PML}(S) \), \( \lambda \) is a sublamination of \( \lambda_\infty \). Since \( \tau, \tau_i \in \text{Teich}(S) \), therefore \( \mathcal{GL}(S), \mathcal{GL}(\tau) \) and \( \mathcal{GL}(\tau_i) \) are canonically identified. Then, for each \( i \in \mathbb{N} \), let \( \nu_i \) be the geodesic lamination on \( \tau \) corresponding to \( \lambda_i \) on \( \tau_i \), and let \( \nu_\infty \) be the geodesic lamination on \( \tau \) corresponding to \( \lambda_\infty \) on \( S \). Then \( \nu_i \) converges to \( \nu_\infty \) as \( i \to \infty \) in the Chabauty topology. Letting \( \tilde{\nu}_i \in \mathcal{GL}(\mathbb{H}^2) \) denote the total lift \( \nu_i \in \mathcal{GL}(\tau_i) \) for each \( i \in \mathbb{N} \), we have

**Lemma 5.5.** For every \( \epsilon > 0 \), there exists \( I \in \mathbb{N} \) such that, if \( i > I \), then \( \beta_0 \) is a \( (1+\epsilon)\)-bilschitz embedding for all leaves \( l \) of \( \tilde{\nu}_i \).

**Proof.** For every \( \delta > 0 \), there exists \( I \in \mathbb{N} \), such that, if \( i > I \), then \( \angle \tau(\nu_i, \nu_\infty) < \delta \). Let \( \lambda \in \mathcal{GL}(\mathbb{H}^2) \) be the total lift of \( \lambda \in \mathcal{GL}(\tau) \). In addition, by Lemma 4.2, for every \( \epsilon > 0 \), there exists \( \delta > 0 \), such that if a geodesic \( l \) on \( \mathbb{H}^2 \), in particular a leaf \( \lambda \) of \( \tilde{\nu}_i \), satisfies \( \angle_{\mathbb{H}^2}(l, \lambda) < \delta \),
then \( \beta|\lambda \) is a \((1+\epsilon)\)-bilipschitz embedding. This completes the proof.

We next prove Theorem 5.4 (i) except the bilipschitz property, still assuming the convergence \( \lambda_i \to \lambda_\infty \):

**Proposition 5.6.** Every \( \epsilon > 0 \), there exists \( I \in \mathbb{N} \), if \( i > I \), then there exists a marking preserving homeomorphism \( \eta_i : \tau \to \tau_i \) such that \( \beta \) and \( \beta_i \circ \bar{\phi}_i \) are \( \epsilon \)-close in \( C^0 \)-topology.

**Proof.** Step 1. We first define an appropriate homeomorphism \( \eta_i : \nu_i \to \lambda_i \) for sufficiently large \( i \in \mathbb{N} \). By Lemma 5.5, for every \( \epsilon > 0 \), there exists \( I \in \mathbb{N} \) such that, if \( i > I \), then, for every leaf \( l \) of \( \nu_i \), \( \beta|l \) is a \((1+\epsilon)\)-bilipschitz embedding. Let \( \lambda_i \in \mathcal{GL}(\mathbb{H}^2) \) be the total lift of \( \lambda_i \in \mathcal{GL}(\tau) \) for each \( i \in \mathbb{N} \), and let \( m (= m_i) \) be the leave of \( \lambda_i \) corresponding to \( l \). Then, by Lemma 4.2 (ii), we can in addition assume that the bilipschitz curve \( \beta|l \) is contained in the \( \epsilon \)-neighborhood of the geodesic \( \beta_i(m) \) in \( \mathbb{H}^3 \). Let \( \Phi_{\beta_i(m)} : \mathbb{H}^3 \to \beta_i(m) \) be the nearest point projection. Thus, by Corollary 4.4, for every \( \epsilon > 0 \), if \( i \) is sufficiently large, then, \( \Phi_{\beta_i(m)} \circ \beta|l : l \to \beta_i(m) \) is a \((1 + \epsilon)\)-bilipschitz map for all leaves \( l \) of \( \nu_i \). Thus there is a unique bilipschitz map \( \tilde{\eta}_i : l \to m \) such that \( \Phi_{\beta_i(m)} \circ \beta|l = \beta_i \circ \tilde{\eta}_i \). Then, for each \( x \in l \), \( \text{dist}_{\mathbb{H}^3}(\beta(x), \beta_i \circ \tilde{\eta}_i(x)) < \epsilon \).

Thus we can define a homeomorphism \( \tilde{\eta}_i : \nu_i \to \lambda_i \) by \( \tilde{\eta}_i(x) = \tilde{\eta}_i(x) \), where \( l \) is a leaf \( l \) of \( \nu_i \) containing \( x \). Thus \( \beta_i \circ \tilde{\eta}_i \) and \( \beta|\nu_i \) are \( \epsilon \)-close pointwise. Since \( \beta \) and \( \beta_i \) are \( \rho \)-equivariant, the action of \( \pi_1(S) \) commutes with \( \tilde{\eta}_i \). Thus \( \tilde{\eta} : \tilde{\nu}_i \to \lambda_i \) descends to a homeomorphism \( \eta_i : \nu_i \to \lambda_i \).

Step 2. We next extend \( \eta_i : \nu_i \to \lambda_i \) to a desired homeomorphism \( \eta_i : \tau \to \tau_i \). Let \( P \) be a component of \( \mathbb{H}^2 \setminus \tilde{\nu}_i \).

**Lemma 5.7.** For every \( \epsilon > 0 \), there exists \( I \in \mathbb{N} \), such that, for all components \( P \) of \( \mathbb{H}^2 \setminus \tilde{\nu}_i \), \( \beta|P \) is a \((1 + \epsilon, \epsilon)\)-quasiisometric embedding.

**Proof.** For a geodesic segment \( s \) on \( \mathbb{H}^2 \), let \( s_\epsilon \) denote \( s \) minus the \((\epsilon/2)\)-neighborhood of the endpoints of \( s \), so that \( s_\epsilon \) is a subsegment of \( s \) whose length is \( \text{length}(s) - \epsilon \). Recall that \( \nu_i \) converges to \( \nu_\infty \) as \( i \to \infty \) in \( \mathcal{GL}(\tau) \) and \( \lambda \) is a sublamination of \( \nu_\infty \). Thus, for every \( \epsilon > 0 \), for sufficiently large \( i \in \mathbb{N} \), \( \lambda \) is contained in the \( \epsilon \)-neighborhood \( \nu_i \). Therefore, for every \( \epsilon > 0 \), if \( i \) is sufficiently large, then, for every geodesic segment \( s \) in \( \mathbb{H}^2 \setminus \tilde{\nu}_i \), we have \( \angle(s_\epsilon, \lambda) \) \( < \epsilon \). Thus, by Proposition 4.2 (i), for arbitrary \( \epsilon > 0 \), if \( i \) is sufficiently large, then for every geodesic \( s \) in \( \mathbb{H}^2 \setminus \tilde{\nu}_i \), \( \beta|s_\epsilon \) is a \((1 + \epsilon)\)-bilipschitz map. Therefore \( \beta|s \) is a \((1 + \epsilon, \epsilon)\)-quasiisometric embedding. \( \square \)
Let $P$ be a component of $\mathbb{H}^2 \setminus \tilde{\nu}_i$, and let $Q$ be the component of $\mathbb{H}^2 \setminus \tilde{\lambda}_i$ corresponding to $P$. Noting that $\beta_i$ isometrically embeds $Q$ into a totally geodesic hyperplane in $\mathbb{H}^3$, let $\Phi_Q: \mathbb{H}^3 \to \beta_i(Q)$ denote the nearest point projection. By Lemma 3.3 and Lemma 5.7, we see that, for every $\epsilon > 0$, if $i$ is sufficiently large, then $\Phi_Q \circ \beta_i|P: P \to \beta_i(Q) \simeq Q$ is a $(1 + \epsilon, \epsilon)$-quasiisometry for all components $P$ of $\mathbb{H}^2 \setminus \tilde{\nu}_i$.

Let $\tilde{\eta}_P: P \to Q$ denote this quasiisometry. Then, we can in addition assume that $\beta_i \circ \tilde{\eta}_P$ and $\beta|P$ are $\epsilon$-close pointwise for all components $P$. Since $\beta$ and $\beta_i$ are $\rho$-equivariant, letting $\text{Stab}(P)$ be the maximal subgroup of $\pi_1(S)$ that preserves $P \subset \mathbb{H}^2$, $\text{Stab}(P)$ commutes with $\tilde{\eta}_P$. Since $\beta_i$ isometrically embeds $Q$, for every $\epsilon > 0$, if $i$ is sufficiently large, then, for every component $P$ of $\mathbb{H}^2 \setminus \tilde{\nu}_i$ and every boundary geodesic $l$ of $P$, $\tilde{\eta}_P[l]: l \to Q$ above and $\tilde{\eta}_i: l \to m$ in Step 1 are $\epsilon$-close pointwise. Therefore, if $i$ is sufficiently large, we can perturb $\tilde{\eta}_P$ so that $\tilde{\eta}_P|P \to Q$ is a homeomorphism and $\tilde{\eta}_P[l] = \tilde{\eta}_i$ for each boundary component $l$ of $P$, but preserving the properties that $\beta_i \circ \tilde{\eta}_P$ and $\beta|P$ are $\epsilon$-close pointwise and that $\tilde{\eta}_P$ commutes with $\text{Stab}(P)$. Therefore we can continuously extend a homeomorphism $\tilde{\phi}_i: \tilde{\nu}_i \to \tilde{\lambda}_i \subset \mathbb{H}^2$ to a homeomorphism $\tilde{\eta}_i: \mathbb{H}^2 \to \mathbb{H}^2$ by $\tilde{\eta}_i|P = \tilde{\eta}_P$ for all components $P$ of $\mathbb{H}^2 \setminus \tilde{\nu}_i$ so that $\tilde{\eta}_i$ commutes with the action of $\pi_1(S)$. Thus $\beta_i \circ \tilde{\eta}_i$ and $\beta$ are $\epsilon$-close, and $\tilde{\eta}_i$ descends to the desired homeomorphism $\eta: \tau \to \tau'$.

5.3. Regular neighborhoods of bending maps. Bending maps are in general (highly) non-injective. However, as a substitute, we construct a “regular neighborhood” of a bending map $\beta: \mathbb{H}^2 \to \mathbb{H}^3$.

Case 1. Suppose that $L$ contains no leaf of weight $\pi$ modulo $2\pi$. Then $\beta$ is locally injective. Let $M$ be the sublamination of $L$ consisting of the periodic leaves of $L$, and let $\tilde{M} \in \mathcal{ML}(\mathbb{H}^2)$ is the total lift of $M$. Then $\beta$ is $C^1$-smooth on each component of $\mathbb{H}^2 \setminus \tilde{M}$. Therefore, there is a regular neighborhood of $\beta$ stated as in the following. Since $\pi_1(S)$ acts on the universal cover of $\tau$, $\mathbb{H}^2$, as deck transformations, $\pi_1(S)$ naturally acts on $\mathbb{H}^2 \times (-1, 1)$ by $\gamma \cdot (x, t) = (\gamma \cdot x, t)$ for all $\gamma \in \pi_1(S)$, $x \in \mathbb{H}^2$ and $t \in (-1, 1)$. Then, letting $\zeta: \mathbb{H}^2 \to \mathbb{H}^2 \times (-1, 1)$ be the canonical embedding onto $\mathbb{H}^2 \times \{0\}$ defined by $x \mapsto (x, 0)$, we have

**Lemma 5.8** (regular neighborhood). There is an immersion $\iota: \mathbb{H}^2 \times (-1, 1) \to \mathbb{H}^3$ such that:

(i) $\beta = \iota \circ \zeta$,

(ii) $\iota$ is $\rho$-equivariant,
(iii) for every \( x \in \mathbb{H}^2 \), there exists a neighborhood \( U \) of \( x \) in \( \mathbb{H}^2 \) such that \( \iota|U \times (-1, 1) \) is an embedding, and

(iv) for the closure \( P \) of each component of \( \mathbb{H}^2 \setminus \tilde{M} \), \( \iota \) is \( C^1 \)-smooth on \( P \times (-1, 1) \).

Then, since \( S \) is closed, (iii) immediately implies that

(v) If \( \epsilon > 0 \) is sufficiently small, then, for every \( x \in \mathbb{H}^2 \), there is a unique topological 3-ball \( B_x \subset \mathbb{H}^2 \times (-1, 1) \) containing \( x \times \{0\} \) such that \( \iota \) embeds \( B_x \) onto the round ball of radius \( \epsilon \) centered at \( \iota((x, 0)) = \beta(x) \) in \( \mathbb{H}^3 \) and \( B_x \) changes continuously in \( x \in \mathbb{H}^2 \).

Consider the open 1-neighborhood of a totally geodesic hyperplane in \( \mathbb{H}^3 \). Then this 1-neighborhood has a canonical product structure, \( \mathbb{H}^2 \times (-1, 1) \), whose coordinates are given by the nearest point projection onto the hyperplane and the distance from the hyperplane. Then we canonically identify this 1-neighborhood with the domain of \( \iota \) in Lemma 5.8. Let \( N = \mathbb{H}^2 \times (-1, 1) \). Then we equip \( N \) with the pullback metric of \( \mathbb{H}^3 \) via \( \iota \), unless otherwise stated. However, we may also regard \( N \) as the 1-neighborhood contained in \( \mathbb{H}^3 \) and equip \( N \) with the metric induced from the ambient space, so that \( N \) is convex.

Recall that we have a sequence of bending maps \( \beta_i : \mathbb{H}^2 \to \mathbb{H}^3 \) \( (i \in \mathbb{N}) \). By Proposition 5.6, there is a homeomorphism \( \eta_i : \tau \to \tau_i \) for each \( i \in \mathbb{N} \), such that \( \beta_i \circ \eta_i \) converges to \( \beta \) uniformly in \( C^0 \) topology as \( i \to \infty \). Then \( \beta_i : \mathbb{H}^2 \to \mathbb{H}^3 \) naturally factors through a continuous map into \( N \):

**Proposition 5.9.** For every \( \delta > 0 \), there exists \( I \in \mathbb{N} \), such that, if \( i > I \), then there exists a unique continuous map \( \zeta_i : \mathbb{H}^2 \to N \) such that

(i) \( \beta_i = \iota \circ \zeta_i \),

(ii) the action of \( \pi_1(S) \) commutes with \( \zeta_i \),

(iii) \( \zeta_i \circ \tilde{\eta}_i(x) \in B_x \) for all \( x \in \mathbb{H}^2 \), and

(iv) \( \zeta_i \circ \tilde{\eta}_i(x) \) and \( \zeta(x) \) are \( \delta \)-close for all \( x \in \mathbb{H}^2 \).

**Proof.** By Proposition 5.6, for every \( \delta > 0 \), there exists \( I \in \mathbb{N} \), such that, if \( i > I \), such that \( \beta \) and \( \beta_i \circ \tilde{\eta}_i \) are \( \delta \)-close pointwise. Thus there exists a unique homotopy \( \vartheta : [0, 1] \times \mathbb{H}^2 \to \mathbb{H}^3 \) such that, letting \( \vartheta(t, x) = \vartheta_t(x) \),

- \( \vartheta_0 = \beta \),
- \( \vartheta_1 = \beta_i \circ \tilde{\eta}_i \), and
- for each \( x \in \mathbb{H}^2 \), \( \vartheta|[0,1] \times \{x\} \) is the geodesic segment in \( \mathbb{H}^3 \), parametrized by arc length, that connects \( \beta(x) \) to \( \beta_i \circ \tilde{\eta}_i(x) \).

Then \( \vartheta \) is \( \rho \)-equivariant and the geodesic segment \( \vartheta|[0,1] \times \{x\} \) has length less than \( \delta \) for all \( x \in \mathbb{H}^2 \).
Recall that, by Lemma 5.8 (v), for each point $x \in \mathbb{H}^2$, there exists a ball $B_x$ in $N$ such that: $B_x$ contains $\zeta(x)$; $B_x \subset N$ changes continuously in $x$; the immersion $\iota$ embeds $B_x$ isometrically onto a round ball of radius (fixed) $\epsilon$ centered at $\beta(x)$. Therefore, since we can assume $\delta < \epsilon$, the homotopy $\vartheta$ uniquely factors through to a homotopy $\xi: [0,1] \times \mathbb{H}^2 \to N$ so that, $\xi_0 = \zeta$ and $\iota \circ \xi = \vartheta$. Then, for all $x \in \mathbb{H}^2$, $\xi([0,1] \times \{x\})$ is a geodesic segment of length less than $\delta$ starting at $\zeta(x)$ and, thus, contained in $B_x$.

Since $\iota \circ \xi = \vartheta$, we have $\iota \circ \xi_1 = \beta \circ \tilde{\eta}_i$. Then let $\zeta_i = \xi_1 \circ \tilde{\eta}_i^{-1}: \mathbb{H}^2 \to N$. Then $\iota \circ \zeta_i = \beta_i$; thus (i) holds. Besides, for all $x \in \mathbb{H}^2$, $\zeta_i \circ \tilde{\eta}_i(x) = \xi(1,x) \in B_x$; thus (iii) holds. Since $\vartheta$ and $\iota$ are $\rho$-equivariant and $B_{\gamma,x} = \gamma B_x$ for all $x \in \mathbb{H}^2$ and $\gamma \in \pi_1(S)$, therefore $\zeta_i$ commutes with the action of $\pi_1(S)$; thus (ii) holds. In addition, we have $\zeta_i(x) = \xi_0(x)$ and $\zeta_i \circ \tilde{\eta}_i(x) = \zeta_i(x)$. Then, since the geodesic segment $\xi([0,1] \times \{x\})$ has length less than $\delta$, its endpoints $\zeta_i \circ \tilde{\eta}_i(x)$ and $\zeta_i(x)$ have length less than $\delta$ for all $x \in \mathbb{H}^2$; thus (iv) holds. \hfill \Box

Case 2. Suppose that $L$ contains a closed leaf with weight $\pi$ modulo $2\pi$. Let $M$ be the multiloop on $\tau$ consisting of all such closed leaves of $L$. In this case, for the closure $P$ of each components of $\mathbb{H}^2 \setminus M$, we will similarly have an open regular neighborhood $\iota_P: N_P \to \mathbb{H}^3$ of $\beta|P$.

Regard $P$ as a convex subset contained in a totally geodesic hyperplane in $\mathbb{H}^3$, and take an open $\epsilon$-neighborhood $N_P$ of $P$ in $\mathbb{H}^3$. Then let $\zeta_P: P \to \tilde{N}_P \subset \mathbb{H}^3$ denote the canonical embedding. We let $\text{Stab}(P)$ denote the maximal subgroup of $\pi_1(S)$ that preserves $P$. Then $\text{Stab}(P)$ isometrically acts on $N_P$. The nearest point projection of $\mathbb{H}^3$ to $P$ induces a projection $\Phi_P: N_P \to P$. Then $N_P \setminus P$ has a natural product structure $\partial N_P \times (0,1)$: Namely, we associate each point $x \in N_P \setminus P$ with the point $y \in \partial N_P$ such that the geodesic segment from $y$ to $\Phi_P(y)$ contains $x$ and with the distance $\text{dist}_{\mathbb{H}^3}(x,P) \in (0,\epsilon)$ times $1/\epsilon$. Thus we naturally set $N_P \cong \partial N_P \times [0,1)/ \sim$, where $(x,t_x) \sim (y,t_y)$ if $t_x = t_y = 0$ and $\Phi_P(x) = \Phi_P(y)$. This product structure is preserved under action of $\text{Stab}(P)$. Then, since $\beta|P$ is locally injective, similarly to Case 1, we have

**Lemma 5.10 (regular neighborhood).** There exists an immersion $\iota_P: N_P \to \mathbb{H}^3$ such that

(i) $\iota_P \circ \zeta_P = \beta|P$,
(ii) $\iota_P$ is $\rho|\text{Stab}(P)$-equivariant,
(iii) for every $x \in P$, there exists a neighborhood $U$ of $x$ in $P$ such that $\Phi_P^{-1}(U)$ embeds into $\mathbb{H}^3$ via $\iota_P$, and
Then, since $P/\text{Stab}(P)$ is compact, (ii) and (iii) implies

(v) there exists $\epsilon > 0$ such that, for each point $x \in P$, there exists a (unique) ball $B_x \subset N_P$ containing $x$ such that $\iota_P$ embeds $B_x$ onto a round ball of radius $\epsilon$ centered at $\iota_P(x) = \beta(x)$ and that $B_x$ changes continuously in $x \in P$.

Recalling that $P$ is a convex subset of $\mathbb{H}^2$, for a boundary component $l$ of $P$, let $N_\epsilon(l, P)$ denote the $\epsilon$-neighborhood of $l$ in $P$. Then let $R_\epsilon(l) = \Phi_P^{-1}(N_\epsilon(l, P)) \subset N_P$.

We equip $N_P$ with the pull back metric of $\mathbb{H}^3$ via $\iota_P$, unless otherwise stated. However we have first defined $N_P$ as a convex subset of $\mathbb{H}^3$ and, as in Case 1, we may use the metric induced by this inclusion when stated so. Indeed,

**Lemma 5.11.** If $\epsilon > 0$ is sufficiently small, then, in addition to (i)–(v) in Lemma 5.10, we have

(vi) if $l$ and $l'$ are different boundary components of $P$, then $R_\epsilon(l)$ and $R_\epsilon(l')$ are disjoint subsets of $N_P$, and

(vii) for each boundary component $l$ of $P$, the two metrics on $R_\epsilon(l)$ coincide and $\Phi_P|R_\epsilon(l)$ is the nearest point projection onto $N_\epsilon(l, P)$ with respect to the pullback metric via $\iota_P$ as well.

**Proof.** Since the action of $\text{Stab}(P)$ on $P$ is cocompact, there is a positive lower bound for the distance between any distinct boundary geodesics of $P \subset \mathbb{H}^2$. Thus, we have (vi).

Each periodic leaf of $L$ is isolated. Thus, for each boundary component $l$ of $P$, if $\epsilon > 0$ is sufficiently small, then $\iota_P$ isometrically embeds $N_\epsilon(l, P)$ into a totally geodesic hyperplane in $\mathbb{H}^3$. Thus we have (vii).

Let $l$ be a leaf of $\tilde{M}$; then $l$ bounds exactly two components of $\mathbb{H}^2 \setminus \tilde{M}$. Let $P_1$ and $P_2$ be the closures of those components. By Lemma 5.11 (vii), for each $j = 1, 2$, $N_\epsilon(l, P_j) \subset P_j$ isometrically embeds into $N_{P_j}$ via $\zeta_{P_j}$; then the image $\zeta_{P_j}(N_\epsilon(l, P_j))$ isometrically embeds, via $\iota_{P_j}$, into a totally geodesic hyperplane in $\mathbb{H}^3$. Moreover, since the weight of $l$ is $\pi$ modulo $2\pi$, we have $\iota_{P_1} \circ \zeta_{P_1}(N_\epsilon(l, P_1)) = \iota_{P_2} \circ \zeta_{P_2}(N_\epsilon(l, P_2))$. Thus, by Lemma 5.11 (vii), there is a unique isometry $\tilde{\psi}_l$ from $R_\epsilon(\zeta_{P_1}(l)) =: R_1 \subset N_{P_1}$ onto $R_\epsilon(\zeta_{P_2}(l)) =: R_2 \subset N_{P_2}$ taking $N_\epsilon(l_1, P_1)$ to $N_\epsilon(l_2, P_2)$, and therefore, $\iota_{P_2} \circ \tilde{\psi}_l = \iota_{P_1}$ on $R_1$. Thus we can quotient $P_1 \cup P_2 \subset \mathbb{H}^2$ by identifying $N_\epsilon(l, P_1)$ and $N_\epsilon(l, P_2)$ via $\tilde{\psi}_l$. Let $(P_1 \cup P_2)/\tilde{\psi}_l$ denote the
quotient, and let \( \tilde{\psi}_l : P_1 \cup P_2 \to (P_1 \cup P_2)/\tilde{\psi}_l \) also denote this quotient map. Note this quotient \((P_1 \cup P_2)/\tilde{\psi}_l \) is exactly the branched surface obtained from the convex subset \( P_1 \cup P_2 \subset \mathbb{H}^2 \) by quotienting the \( \epsilon \)-neighborhood of \( l, N_\epsilon(l, P_1 \cup P_2) \), by the isometric reflection fixing \( l \). Let \( m \subset \tau \) be the loop of \( M \) that lifts to \( l \subset \mathbb{H}^2 \). Then its \( \epsilon \)-neighborhood \( N_\epsilon(m) \) in \( \tau \) is a cylinder, and the isometric reflection of \( N_\epsilon(l, P_1 \cup P_2) \) about \( l \) descends to the isometric reflection of \( N_\epsilon(m) \) about \( m \).

Let \( \mathcal{R}_l \) denote \( \iota_{P_1}(R_1) = \iota_{P_2}(R_2) \subset \mathbb{H}^3 \). Then, clearly,

**Lemma 5.12.** Both projections \( \Phi_{P_1} : R_1 \to N_\epsilon(l, P_1) \) and \( \Phi_{P_2} : R_2 \to N_\epsilon(l, P_2) \), via \( \iota_{P_1} \) and \( \iota_{P_2} \), descend to the same nearest point projection

\[
\Phi_l : \mathcal{R}_l \to \iota_{P_1}(N_\epsilon(l, P_1)) = \iota_{P_2}(N_\epsilon(l, P_2))
\]

in \( \mathbb{H}^3 \), so that, by \( \iota_{P_1}(N_\epsilon(l, P_j)) \equiv N_\epsilon(l, P_j)/\tilde{\psi}_l \subset (P_1 \cup P_2)/\tilde{\psi}_l \), we have

\[
\Phi_l \circ \iota_{P_j} = \psi_\epsilon \circ \Phi_{P_j}
\]

on \( R_j \) for both \( j = 1, 2 \).

Consider the \( \epsilon \)-neighborhoods \( N_\epsilon(m) \) of all loops \( m \) of \( M \); then they are disjoint by Lemma 5.11 (vi). Let \( \psi_\epsilon : \tau \to \tau_\epsilon \) denote the quotient map that simultaneously quotients \( N_\epsilon(m) \) by the reflection about \( m \) for all loops \( m \) of \( M \). Then \( \tau_\epsilon \) is a branched hyperbolic surface with the boundary \( \psi_\epsilon(M) \) whose branched locus is the multiloop consisting of the \( \psi_\epsilon \)-image of \( \partial N_\epsilon(m) \) for all loops \( m \) of \( M \).

Regard \( \mathbb{H}^2 \) as the union of the closures \( P \) of all components of \( \mathbb{H}^2 \setminus M \). We can quotient \( \mathbb{H}^2 = \cup P \) by quotienting the \( \epsilon \)-neighborhoods of \( l \) by \( \tilde{\psi}_l \) for all leaves \( l \) of \( M \). Then this quotient is exactly the universal cover of \( \tau_\epsilon \) and the quotienting map descends \( \pi_1(S) \)-action on \( \mathbb{H}^2 \) to the deck transformation of \( \tau_\epsilon \). Observe that \( \psi_\epsilon : \tau \to \tau_\epsilon \) is a quasiisometry. Then \( \psi_\epsilon \) converges to an isometry as \( \epsilon \searrow 0 \). Thus we have

**Lemma 5.13.** For every \( \epsilon' > 0 \), there exists \( E > 0 \), such that, if \( E > \epsilon > 0 \), then

\[
(1 - \epsilon') \cdot \text{length}_\tau(l) < \text{length}_{\tau_\epsilon}(l')
\]

for all geodesic loops \( l \) on \( \tau \) and \( l' \) on \( \tau_\epsilon \) such that \( \psi_\epsilon(l) \) is homotopic to \( l' \).

Recall that we have the sequence of the bending maps \( \beta_i : \mathbb{H}^2 \to \mathbb{H}^3 \) \((i \in \mathbb{N}) \) associated with \( C_i \subset \mathcal{P}_\rho \) and the marking-preserving homeomorphism \( \eta_i : \tau \to \tau_i \) given by Proposition 5.6, such that \( \beta_i \circ \eta_i \) converging to \( \beta \) uniformly in \( C^0 \)-topology as \( i \to \infty \). Let \( M_i = \eta_i(M) \), a multiloop on \( \tau_i \), and let \( \tilde{M}_i \subset \mathbb{H}^2 \) be the total lift of \( M_i \). Then we have

**Proposition 5.14.** For every \( \delta > 0 \), if \( i \in \mathbb{N} \) is sufficiently large, then, for the closures \( Q \) and \( P \) of corresponding components of \( \mathbb{H}^2 \setminus \tilde{M}_i \) and
\[ \mathbb{H}^2 \setminus \tilde{M}, \text{ respectively, there is a unique continuous map } \zeta_{i,P}: Q \to N_P \text{ such that:} \]

(i) \( \beta_i = \iota_P \circ \zeta_{i,P} \) on \( Q \) and,
(ii) the action of \( \text{Stab}(Q) \cong \text{Stab}(P) \) commutes with \( \zeta_{i,P} \),
(iii) \( \zeta_{i,P} \circ \tilde{\eta}_i(x) \in B_x \) for all \( x \in P \),
(iv) \( \zeta_{i,P} \circ \tilde{\eta}_i(x) \) and \( \zeta(x) \) are \( \delta \)-close for all \( x \in P \).

The proof of this proposition is similar to the proof of Proposition 5.9.

5.4. Convergence of hyperbolic structures. The following proposition is the main step to prove Theorem 5.4.

**Proposition 5.15.** \( \tau_i \to \tau \) as \( i \to \infty \).

Proposition 5.15 follows immediately follows from Proposition 5.16 and Theorem 5.17 below.

**Proposition 5.16.** For every \( \epsilon > 0 \), there exists \( I \in \mathbb{N} \), such that, if \( i > I \), then \( \text{length}_{\tau}(l) < (1 + \epsilon) \cdot \text{length}_{\tau_i}(l_i) \) for all pairs of geodesic loops \( l \) and \( l_i \) on \( \tau \) and \( \tau_i \), respectively, that are homotopic (regarded as loops on \( S \)).

**Theorem 5.17.** (18) Consider the function \( K: \text{Teich}(S) \times \text{Teich}(S) \to \mathbb{R} \) given by
\[
K(\tau, \tau') = \sup \left( \ln \left( \frac{\text{length}_{\tau}(l)}{\text{length}_{\tau'}(l')} \right) \right),
\]
where the supremum runs over all pairs of geodesic loops \( l \) and \( l' \) on \( \tau \) and \( \tau' \), that are homotopic (regarded loops on \( S \)). Then \( K \) is well-defined and it defines an antisymmetric distance on \( \text{Teich}(S) \).

**Proof of Proposition 5.15 assuming Proposition 5.16.** By Proposition 5.16, \( K(\tau_i, \tau) \to 0 \) as \( i \to \infty \). Therefore, by Theorem 5.17, \( \tau_i \to \tau \) as \( i \to \infty \). \( \square \)

**Proof of Proposition 5.16.** Case 1. First suppose that \( L \) contains no closed leaf of weight \( \pi \) modulo 2\( \pi \). Then we have, by Lemma 5.8, a piecewise \( C^1 \)-smooth regular neighborhood \( \iota: N \to \mathbb{H}^3 \) of \( \beta \) where \( N \cong \mathbb{H}^2 \times (-1,1) \), and \( \beta \) factors through \( \zeta: \mathbb{H}^2 \to N \), namely \( \beta = \iota \circ \zeta \). Recall that \( N \) is equipped with the pullback metric of \( \mathbb{H}^3 \) under \( \iota \).

Let
\[
\Phi: N \cong \mathbb{H}^2 \times (-1,1) \to \mathbb{H}^2 \times \{0\}
\]
denote the canonical projection, i.e. \( (x,t) \mapsto (x,0) \). Then, since \( N \) is piecewise \( C^1 \)-smooth (Lemma 5.8(vi)), \( \Phi \) induces the map between the tangent space \( T(N) \) of \( N \) to the tangent space of \( \mathbb{H}^2 \) in \( N \):
\[
\Phi^*: T(N) \to T_{\mathbb{H}^2 \times \{0\}}(N).
\]
Then, since \( \Phi \) fixes the image of \( \zeta : \mathbb{H}^2 \to N \), by identifying \( T_{\mathbb{H}^2 \times \{0\}}(N) \) with \( T(\mathbb{H}^2) \), we have

**Lemma 5.18.** For every \( \epsilon > 0 \), if \( \delta \in (0, 1] \) is sufficiently small, then

\[
\Phi^* | T_{(x,t)}N : T_{(x,t)}N \to T_x \mathbb{H}^2
\]

is a \((1 + \epsilon)\)-lipschitz map for every \((x,t) \in \mathbb{H}^2 \times (-\delta, \delta) \subset N\).

Fix (small) \( \epsilon > 0 \) and then fix \( \delta > 0 \) obtained by Lemma 5.18. Let \( \zeta_i : \mathbb{H}^2 \to N \) be the continuous map obtained by Proposition 5.9 for sufficiently large \( i \in \mathbb{N} \). Then, by Proposition 5.9 (iv), \( \zeta_i \) maps into \( \mathbb{H}^2 \times (-\delta, \delta) \), for sufficiently large \( i \). Let \( l_i \) be a geodesic loop on \( \tau_i \). Regarding \( l_i \) as a simple closed curve on \( \tau \), let \( p_i : [0, 1] \to \mathbb{H}^2 \) be a path obtained by lifting \( l_i \) to the universal cover of \( \tau_i \), so that \( \text{length}_{\tau}(l_i) = \text{length}_{\mathbb{H}^2}(p_i) \). Let \( \alpha \) denote the element of \( \pi_1(S) \) that identifies the endpoints of \( p_i \). Let \( q_i = \zeta_i \circ p_i : [0, 1] \to N \). Then \( \text{length}_{\mathbb{H}^2}(p_i) = \text{length}_{\mathbb{N}}(q_i) \), since \( \beta_i \) is a local isometry onto its image, in \( \mathbb{H}^2 \), with intrinsic metric. Besides, if \( i \in \mathbb{N} \) is sufficiently large, the path \( q_i \) is contained in \( \mathbb{H}^2 \times (-\delta, \delta) \), for all geodesic loops \( l_i \) on \( \tau_i \). Let

\[
q'_i = \Phi \circ q_i : [0, 1] \to \text{Im} \zeta \cong \mathbb{H}^2.
\]

Then, by Lemma 5.18, we have

\[
\text{length}(q'_i) \leq (1 + \epsilon) \cdot \text{length}(q_i) = (1 + \epsilon) \cdot \text{length}(l_i).
\]

Since \( \alpha \) identifies the endpoints of \( p_i \) and the action of \( \pi_1(S) \) commutes with both \( \zeta_i \) (Proposition 5.9 (iii)) and \( \Phi \), therefore \( \alpha \) identifies the endpoints of \( q_i \) and, thus, the endpoints of \( q'_i \). Then \( q'_i \) descends to a closed curve on \( \tau \) that is homotopic to \( l_i \). Let \( l \) be the geodesic representative of this close curve on \( \tau \). Then \( \text{length}_{\tau}(l) \leq \text{length}_{\mathbb{H}^2}(q'_i) \). Therefore, we have

\[
\text{length}(l) \leq (1 + \epsilon) \cdot \text{length}(l_i)
\]

for sufficiently large \( i \in \mathbb{N} \).

**Case 2.** Suppose that \( L \) contains a close leaf of weight \( \pi \) modulo \( 2\pi \). Let \( M \) be the sublamination of \( L \) consisting of all such periodic leaves. For each component \( \mathbb{H}^2 \setminus \bar{M} \), letting \( \partial P \) be its closure, we have constructed a regular \( \epsilon \)-neighborhood of \( \partial P \) (Lemma 5.10 and Lemma 5.11). Then, recalling \( N_{\partial P} = \partial N_{\mathbb{H}^2} \times [0, 1] / \sim \), for \( \delta \in (0, 1] \), let \( N_{\partial P, \delta} = \partial N_{\mathbb{H}^2} \times [0, \delta] / \sim \). Then we can easily see an analogue of Lemma 5.18.

**Lemma 5.19.** For every \( \epsilon > 0 \), there exists \( \delta > 0 \), such that, if \( P \) is the closure of a component of \( \mathbb{H}^2 \setminus \bar{M} \), then, for all \( z \in N_{\partial P, \delta} \), the piecewise \( C^1 \)-smooth projection \( \Phi_P : N_P \to N_P \) induces a \((1 + \epsilon)\)-lipschitz map...
from the tangent space of $N_P$ at $z$ to the tangent space of $P$ at the point $x \in P$ such that $\zeta_P(x) = \Phi_P(z)$:

$$\Phi^*_P|T_zN_P : T_zN_P \to T_xP.$$ 

Given a regular neighborhood of $\beta|P$, since $\beta$ is $\rho$-equivariant, we can easily make a regular neighborhood of $\beta|\gamma P$ for all $\gamma \in \pi_1(S)$. Namely we set

$$N_{\gamma P} := N_P,$$

$$\Phi_{\gamma P} := \Phi,$$

$$\iota_{\gamma P} := \rho(\gamma) \cdot \iota_P : N_{\gamma P} \to \mathbb{H}^3$$

$$\zeta_{\gamma P} := \zeta_P \cdot \gamma^{-1} : \gamma P \to N_P,$$

which satisfy (i) - (vii) in Lemma 5.10 and Lemma 5.11. Therefore we can assume

Claim 5.20. The regular $\epsilon$-neighborhoods of $\beta|P$ and $\beta|\gamma P$ satisfy the above relations for every $\gamma \in \pi_1(S)$ and the closure $P$ of every component of $\mathbb{H}^2 \setminus \tilde{M}$.

Let $m$ be a geodesic loop on $\tau_i$. Then pick a partition of $m$ into finitely many disjoint arcs $m_j (j \in \mathbb{Z}_n)$ with some $n \in \mathbb{N}$ indexed in the cyclic order, such that, for each $j \in \mathbb{Z}_n$, $\eta_i^{-1}(m_j)$ is contained in the closure of a component of $\tau_i \setminus M$. Let $\tilde{m_j}$ be a lift of $m_j$ to the universal cover, $\mathbb{H}^2$, of $\tau_i$. Then $\tilde{\eta_i}^{-1}(\tilde{m_j})$ is contained in the closure of $P_j$ of a component of $\mathbb{H}^2 \setminus \tilde{M}$. By claim 5.20 we may change the choice of the lift $\tilde{m_j}$ by an element of $\pi_1(S)$ in the following argument, if necessarily.

Let $Q_j = \tilde{\eta_i}(P_j)$ and let $\zeta_{i,P_j} : Q_j \to N_{P_j}$ be the continuous map obtained by Proposition 5.14. Then we have $\iota_{P_j} \circ \zeta_{i,P_j}|\tilde{m_j} = \beta_i|\tilde{m_j}$. Then, since $\beta_i$ preserves length of a curve in its domain, so does $\zeta_{i,P_j}$. In particular, $\zeta_{i,P_j}$ preserves the length of $\tilde{m_j}$. Let $q_j$ denote the curve $\Phi_{P_j} \circ \zeta_{i,P_j}|\tilde{m_j}$ in $\text{Im} \zeta_{i,P_j} \simeq P_j$.

Proposition 5.21. For every $\epsilon > 0$, there exists $I > 0$, such that, if $i > I$ then, for every geodesic loop $m$ on $\tau_i$ and every finite partition of $m$ into segments $m_1, m_2, \ldots, m_n$ as above, we have

$$(1 + \epsilon) \cdot \text{length}_{\tau_i}(m_j) \geq \text{length}_{P_j}(q_j)$$

for each $j \in \{1, 2, \ldots, n\}$.

Proof. By Proposition 5.14, for every $\delta > 0$, if $I \in \mathbb{N}$ is sufficiently large, then $\zeta_{i,P_j}|\tilde{m_j}$ is contained in $N_{P_j,\delta}$ for each $j \in \{1, \ldots, n\}$. Then, similarly to Case 1, Lemma 5.19 implies the proposition. $\square$
Since \( m_j \) and \( m_{j+1} \) share an endpoint, accordingly, an endpoint of \( q_j \) corresponds to an endpoint \( q_{j+1} \). Let \( \Psi: \mathbb{H}^2 \to \tau \) be the universal covering map. Then, \( \Psi \) take the corresponding endpoints of \( q_j \subset P_j \) and \( q_{j+1} \subset P_{j+1} \) typically to different points on \( \tau \). However, recalling the quasiisometry \( \psi_\epsilon: \tau \to \tau_\epsilon \) from Lemma 5.22, we show

**Lemma 5.22.** The corresponding endpoints of \( q_j \) and \( q_{j+1} \) map to the same points under \( \psi_\epsilon \circ \Psi \) for each \( j \in \mathbb{Z}_n \); therefore \( \bigcup_j q_j \subset \mathbb{H}^2 \) descends to a closed curve on \( \tau_\epsilon \). Furthermore this closed curve is homotopic to \( \psi_\epsilon \circ \eta_\epsilon^{-1}(m) \).

**Proof.** Consider consecutive segments \( m_j \) and \( m_{j+1} \), sharing an endpoint \( v \). Then we can assume that \( \tilde{m}_j \) and \( \tilde{m}_{j+1} \) share an end point \( \tilde{v} \) that is a lift of \( v \) to \( \mathbb{H}^2 \) (by an element of \( \pi_1(S) \)). Thus, to show that \( \psi_\epsilon \circ \Psi(q_j) \) and \( \psi_\epsilon \circ \Psi(q_{j+1}) \) share an endpoint corresponding to \( v \), it suffices to show that \( \psi_\epsilon \circ \Phi_{P_j} \circ \zeta_{P_j}(\tilde{v}) = \psi_\epsilon \circ \Phi_{P_{j+1}} \circ \zeta_{P_{j+1}}(\tilde{v}) \in \tilde{\tau}_\epsilon \), where \( \tilde{\psi}_\epsilon: \tilde{\tau} \to \tilde{\tau}_\epsilon \) is the lift of \( \psi_\epsilon: \tau \to \tau_\epsilon \) to the map between universal covers, \( \mathbb{H}^2 \).

Since \( \tilde{m}_j \) and \( \tilde{m}_{j+1} \) share an endpoint, either \( P_j = P_{j+1} \) or \( P_j \) and \( P_{j+1} \) are the closures of adjacent components of \( \mathbb{H}^2 \setminus \tilde{M} \). Suppose that \( P_j = P_{j+1} \). Then, letting \( Q = \eta_l(P) \), the segments \( \tilde{m}_j \) and \( \tilde{m}_{j+1} \) are contained in \( Q \), sharing an endpoint \( \tilde{v} \). Thus the claim is clear.

Next suppose that \( P_j \) and \( P_{j+1} \) are adjacent. Then let \( l \) be the common boundary geodesic of \( P_j \) and \( P_{j+1} \), which is a leaf of \( \tilde{M} \). Then \( \eta_l^{-1}(\tilde{v}) \in l \). As in Case 2 of Proposition 5.14, for each \( k = j, j+1 \), let \( R_{k,\epsilon}(l) \subset N_{P_k} \) be \( \Phi_{P_k}^{-1}(N_\epsilon(l, P_k)) \), where \( N_\epsilon(l, P_k) \) is the \( \epsilon \)-neighborhood of \( l \) in \( P_k \) with \( \epsilon > 0 \) given by Lemma 5.11. Then we have a canonical isometry \( \tilde{\psi}_l: R_{k,\epsilon}(l_j) \to R_{k,\epsilon}(l_{j+1}) \), which coincides with the reflection \( \tilde{\psi}_l|_{P_j \cup P_{j+1}} \). Since \( \eta_l^{-1}(\tilde{v}) \in l \), by Proposition 5.14, assuming that \( \eta \in \mathbb{N} \) is sufficiently large, then \( \zeta_{P_k}(\tilde{v}) \) is contained in \( R_{k,\epsilon}(l_k) \) for each \( k = j, j+1 \). Then \( \tilde{\psi}_l \) identifies \( \zeta_{P_j}(\tilde{v}) \) and \( \zeta_{P_{j+1}}(\tilde{v}) \) since \( \iota_{P_j} = \iota_{P_{j+1}} \circ \tilde{\psi}_l \) on \( R_{k,\epsilon}(l_j) \). Therefore, by Lemma 5.12, we have \( \tilde{\psi}_l \circ \Phi_{P_{j+1}} \circ \zeta_{P_{j+1}}(\tilde{v}) = \tilde{\psi}_l \circ \Phi_{P_j} \circ \zeta_{P_j}(\tilde{v}) \in \tilde{\tau}_\epsilon \) as desired. Thus \( \bigcup_j q_j \) maps to a loop \( q \) via \( \psi_\epsilon \circ \Psi \).

We next show that \( \psi_\epsilon \circ \eta_\epsilon^{-1}(m) \) is homotopic to the loop \( q \) on \( \tau_\epsilon \). Consider a lift \( \tilde{m} \) of \( m \) to the universal cover of \( \tau_\epsilon \), and let \([m]\) be the element of \( \pi_1(S) \) that preserves \( \tilde{m} \). Then the partition of \( m \) into \( m_1, \ldots, m_n \) lifts an infinite partition of \( \tilde{m} \) that is invariant under the action of \([m]\) in \( \pi_1(S) \). Each segment of this partition of \( \tilde{m} \) is a lift \( \tilde{m}_j \) of the segment \( m_j \) with some \( j \in \mathbb{Z}_n \), and \( \Psi_{P_j} \circ \zeta_{P_j} \) takes \( \tilde{m}_j \) into \( P_j \subset \tilde{\tau} \). Then, since \( \eta_l: \tau \to \tau_\epsilon \) is a marking-preserving homeomorphism, the union of \( \Psi_{P_j} \circ \iota_{P_j}(\tilde{m}_j) \) over all such segments \( \tilde{m}_j \) of \( \tilde{m} \) is a quasigeodesic in \( \tilde{\tau} \) invariant under \([m]\), which corresponds to
the loop \( \tilde{\eta}_i^{-1}(m) \) on \( \tau \). Then \( \tilde{\psi}_\epsilon : \tilde{\tau} \to \tilde{\tau}_\epsilon \) takes this quasigeodesic to a (continuous) quasigeodesic that projects to \( q \) via the universal covering map. Therefore \( \psi_\epsilon \circ \eta_i^{-1}(m) \) is homotopic to \( q \).

\section*{Claim 5.23.} For every \( \epsilon > 0 \), there exists \( I > 0 \), such that if \( i > I \), then

\[(1 + \epsilon) \cdot \text{length}_{\tau_i}(m) \geq \text{length}_{\tau_\epsilon}(l)\]

for all geodesic loops \( m \) and \( l \) on \( \tau_i \) and \( \tau_\epsilon \), respectively, such that \( \psi_\epsilon \circ \eta_i^{-1}(m) \) is homotopic to \( l \).

\textbf{Proof.} By Proposition 5.21, for every \( \epsilon > 0 \), there exits \( I \in \mathbb{N} \) such that, if \( i > I \), given any geodesic loop \( m \) on \( \tau_i \) and a finite partition \( m = m_1 \cup m_2 \cup \ldots \cup m_n \) as above, such that \( (1 + \epsilon) \cdot \text{length}_{\tau_i}(m_j) \geq \text{length}_\tau(q_j) \) for each \( j \in \mathbb{Z}_n \), where \( q_j \) are the geodesic segments on \( \tilde{\tau} \) defined as above. Then

\[(1 + \epsilon) \cdot \text{length}_{\tau_i}(m) = (1 + \epsilon) \cdot \bigcup_j \text{length}_{\tau_i}(m_j) \geq \Sigma_j \text{length}_{\tau}(q_j) = (\Sigma_j \text{length}_{\tau}(q_j)).\]

By Lemma 5.22, \( \bigcup_j q_j \subset \tilde{\tau} \) descends to a loop on \( \tau_\epsilon \) under \( \psi_\epsilon \circ \psi_\tau \). Hence we have

\[(1 + \epsilon) \cdot \text{length}_{\tau_i}(m) \geq \text{length}_{\tau_\epsilon}(q).\]

Since \( l \) is the shortest loop homotopic to \( q \) on \( \tau_\epsilon \). Hence

\[(1 + \epsilon) \cdot \text{length}_{\tau_i}(m) \geq \text{length}_{\tau_\epsilon}(l).\]

\qed

Fix \( \epsilon' > 0 \). Then pick sufficiently small \( \epsilon > 0 \) and sufficiently large \( i \in \mathbb{N} \) so that they satisfy Lemma 5.13 and Proposition 5.23. Then for all geodesic loops \( l \) on \( \tau \), \( l_i \) on \( \tau_i \) and \( l_\epsilon \) on \( \tau_\epsilon \) such that \( \psi_\epsilon(l) \) is homotopic to \( l_\epsilon \) and \( \eta_i(l) \) is homotopic to \( l_i \), we have

\[(1 - \epsilon') \cdot \text{length}_{\tau}(l) < \text{length}_{\tau_\epsilon}(l_\epsilon) < (1 + \epsilon') \cdot \text{length}_{\tau_\epsilon}(l_i)\]

This completes the proof of Case 2.

\section*{5.5. Convergence of bending maps: the proof of Theorem 5.4.}

Recall that we have been assuming the \( \lambda_i \) converges to the geodesic lamination \( \lambda_\infty \) in \( \mathcal{GL}(S) \) as \( i \to \infty \). Then, in Proposition 5.15, we have shown that \( \tau_i \to \tau \in \text{Teich}(S) \). Thus we let \( \eta_i : \tau \to \tau_i \) be a marking-preserving diffeomorphism for each \( i \in \mathbb{N} \), so that \( \eta_i \) converges to the trivial isometry from \( \tau \) to itself. Then, for every \( \epsilon > 0 \), if \( i \in \mathbb{N} \) is sufficiently large, then \( \eta_i \) is \((1 + \epsilon)\)-bilipschitz. Clearly \((\tau_i, \lambda_i) \) converges
to \((\tau, \nu_\infty)\) in \(\text{Teich}(S) \times \mathcal{GL}(S)\). Let \(\varpi_i\) denote the geodesic lamination on \(\tau_i\) that corresponds to \(\nu_\infty \in \mathcal{GL}(\tau)\). Then \(\angle_{\tau_i}(\varpi_i, \lambda_i) \to 0\) as \(i \to \infty\).

Let \(P\) be the closure of a component of \(\mathbb{H}^2 \setminus \tilde{\nu}_\infty\), where \(\tilde{\nu}_\infty \in \mathcal{GL}(\mathbb{H}^2)\) is the total lift of \(\nu_\infty\). Then let \(P_i\) be the closure of a component of \(\mathbb{H}^2 \setminus \tilde{\varpi}_i\) corresponding to \(P\), where \(\tilde{\varpi}_i\) is the total lift of \(\varpi_i\). Since \(\eta_i: \tau \to \tau_i\) converges to the trivial isometry as \(i \to \infty\), letting \(\tilde{\eta}_i: \tilde{\tau} \to \tilde{\tau}_i\) denote the lift of \(\eta_i\) to an isometry between the universal covers of \(\tau\) and \(\tau_i\), we clearly have

**Lemma 5.24.** \(\tilde{\eta}_i(P_i)\) converges to \(P\) uniformly.

Moreover,

**Proposition 5.25.** for every \(\epsilon > 0\), if \(i \in \mathbb{N}\) is sufficiently large, then \(\beta_i \circ \tilde{\eta}_i\) and \(\beta\) are \(\epsilon\)-close pointwise in \(P\). Furthermore \(\beta_i \circ \tilde{\eta}_i\) converges to \(\beta\) in \(C^1\)-topology uniformly in \(P \setminus N_\epsilon(\partial P)\) as \(i \to \infty\); then since \(\eta_i\) are smooth and orientation preserving, the normal vectors also converges in the interior of \(P\).

**Proof.** We first show

**Lemma 5.26** (c.f. Lemma 5.7). For every \(\epsilon > 0\), if \(i \in \mathbb{N}\) is sufficiently large, then \(\beta_i | P_i: P_i \to \mathbb{H}^3\) is a \((1 + \epsilon, \epsilon)\)-quasiisometric embedding for the closure \(P_i\) of each component \(\mathbb{H}^2 \setminus \varpi_i\).

**Proof.** For a geodesic segment \(s\) on \(\mathbb{H}^2\), let \(s_\epsilon\) denote \(s\) minus the \((\epsilon/2)\)-neighborhood of the endpoints of \(s\), so that \(s_\epsilon\) is a subsegment of \(s\) whose length is \(\text{length}(s) - \epsilon\) (thus \(s_\epsilon = \emptyset\), if \(\text{length}(s) \leq \epsilon\)). For every \(\epsilon > 0\), if \(i\) is sufficiently large, then \(\lambda_i\) is contained in the \(\epsilon\)-neighborhood of \(\varpi_i\) in \(\tau_i\). Thus, if \(i\) is sufficiently, for all geodesic segments \(s\) in \(\mathbb{H}^2 \setminus \varpi_i\), we have \(\angle(s_\epsilon, \lambda_i) < \epsilon\), where \(\lambda_i\) is the total lift of \(\lambda_i\). Therefore, the lemma follows from Proposition 4.2(i). \(\square\)

**Corollary 5.27.** For every \(\epsilon > 0\), if \(i \in \mathbb{N}\) is sufficiently large, then \(\beta_i \circ \tilde{\eta}_i | P\) is a \((1 + \epsilon, \epsilon)\)-quasiisometric embedding for the closure \(P\) of each component of \(\mathbb{H}^2 \setminus \tilde{\nu}_\infty\).

**Proof.** By Lemma 5.24, for every \(\epsilon > 0\), if \(i\) is sufficiently large, then \(\tilde{\eta}_i(P)\) is contained in the \(\epsilon\)-neighborhood of \(P_i\) and, in addition, \(\tilde{\eta}_i\) is \((1 + \epsilon)\)-bilipschitz. Thus Lemma 5.26 immediately implies the corollary. \(\square\)

Since \(\beta | P\) is an isometric embedding into a totally geodesic hyper-plane in \(\mathbb{H}^3\), \(\beta | P: P \to \mathbb{H}^3\) continuously extends to a map from the ideal boundary \(\partial_\infty P\) of \(P\) into \(\hat{\mathbb{C}}\), where \(\partial_\infty P\) is a subset of \(\partial \mathbb{H}^2 \cong S^1\). In addition, for sufficiently small \(\epsilon > 0\), if \(i\) is sufficiently large, then, by Corollary 5.27, \(\beta_i \circ \tilde{\eta}_i | P\) is a \((1 + \epsilon, \epsilon)\)-quasiisometric embedding. Then
\[ \beta_i \circ \tilde{\eta}_i \mid P \] also extends to the ideal boundary \( \partial_\infty P \) of \( P \). Since \( \beta_i \circ \tilde{\eta}_i \) and \( \beta \) are \( \rho \)-equivalently homotopic (Lemma 3.3), \( \beta_i \circ \tilde{\eta}_i \mid \partial_\infty P = \beta \mid \partial_\infty P \). Since \( P \) is the closure of a component of \( \mathbb{H}^2 \setminus \nu_\infty \), \( \partial_\infty P \) contains at least 3 points. Therefore, Corollary 5.27 implies that \( \beta_i \circ \eta \) and \( \beta \) are \( \epsilon \)-close at each point in \( P \).

Last we show the \( C^1 \)-convergence in the interior of \( P \). For every \( \epsilon > 0 \), if \( i \) is sufficiently large, then \( P_i \setminus N_\epsilon (\partial P_i) \) and \( \tilde{\nu}_i \) are disjoint. Thus \( \beta_i \) embeds \( P_i \setminus N_\epsilon (\partial P_i) \) isometrically on to a totally geodesic hyperplane \( H_i \) in \( \mathbb{H}^3 \). This hyperplane \( H_i \) converges to the hyperplane containing \( \beta(P) \) as \( i \to \infty \), since we have already shown the \( C^0 \)-convergence. Thus, since \( \eta_i \) is smooth and it converges to the trivial isometry, \( \beta_i \circ \tilde{\eta}_i \) converges to \( \beta \) in \( C^1 \)-topology uniformly on \( P \setminus N_\epsilon (\partial P) \).

**Proof of Theorem 5.4.** First assume that \( \lambda_i \) converges to \( \lambda_\infty \in GL(S) \) as \( i \to \infty \) (as we have been assumed). Since there are only finitely many components of \( \tau \setminus \nu_\infty \), therefore, by Proposition 5.25, \( \beta_i \circ \tilde{\eta}_i \) converges to \( \beta \) uniformly in \( C^0 \)-topology on the closures of components of \( \mathbb{H}^2 \setminus \tilde{\nu}_\infty \) and moreover, for every \( \epsilon > 0 \), in \( C^1 \)-topology in \( \mathbb{H}^2 \setminus N_\epsilon (\tilde{\nu}_\infty) \). Then, since \( \beta_i \) are continuous and \( \tilde{\nu}_\infty \) has empty interior, the \( C^0 \)-convergence extends to the entire domain \( \mathbb{H}^2 \).

In general \( \lambda_i \) may not converge in \( GL(S) \). Thus suppose that \( \beta_i \circ \tilde{\eta}_i \) does not converge to \( \beta \) in \( C^0 \)-topology with a sequence of any marking-preserving homeomorphisms \( \eta_i : \tau \to \tau_i \). Then, taking a subsequence if necessarily, \( \beta_i \circ \tilde{\eta}_i \) does not converge to \( \beta \) in \( C^0 \)-topology for any subsequence of \( (\beta_i) \) and any sequence of marking-preserving homeomorphisms \( \eta_i : \tau \to \tau_i \). Thus, since \( GL(S) \) is compact, we can in addition assume that \( \lambda_i \) converges in \( GL(S) \) as \( i \to \infty \). This is a contradiction.

### 6. LOCAL CHARACTERIZATION OF PROJECTIVE STRUCTURES IN \( \mathcal{PML} \)

Recall that \( S \) is a closed orientable surface of genus at least 2 and projective structures have fixed orientation. The main theorem of this paper is:

**Theorem 6.1.** Let \( C = (\tau, L) \) be a projective structure on \( S \) with holonomy \( \rho \). Then there is a neighborhood \( U \) of \( [L] \) in \( \mathcal{PML}(S) \) such that, for every projective structure \( C' = (\tau', L') \in \mathcal{P}_\rho \) with \( [L'] \in U \), then
(I) if $[L] \neq [L'] \in \mathcal{PML}(S)$, then $\text{Gr}_M(C) = C'$ for some admissible multiloop $M$ on $C$ and, moreover, $M$ is a good approximation of $L' - L$, calculated on some traintrack $T$ carrying both $L$ and $L'$.

(II) if $[L] = [L'] \in \mathcal{PML}(S)$, then either

(i) $\text{Gr}_M(C) = C'$, where $M$ is $L' - L$, or
(ii) $\text{Gr}_M(C') = C$, where $M$ is $L - L'$, and

(III) if $L = \emptyset$ or $L' = \emptyset$, then either $\rho$ is fuchsian and $C' = \text{Gr}_L(C)$ or $C = \text{Gr}_{L'}(C')$, respectively, and $U = \mathcal{PML}(S)$.

In (I), by “good approximation”, we mean that, for every $\epsilon > 0$, if $U$ is sufficiently small, then, for each branch $B$ of $T$, the weight of $M$ on $B$ is $\epsilon$-close to the weight of $L' - L$ on $B$. In Case (II) if $L' - L > 0$ then (i) holds, if $L > L'$ then (ii) holds, and if $L = L'$ then $C = C'$.

Remark 6.2. We can regard (II) as the special case of (I), up to an exchange of $C$ and $C'$. In (III), we may regard $[L] = [L']$ since $L = 0 \cdot L'$ or $L' = 0 \cdot L'$; thus (III) is a special case of (II). Therefore (I) is essentially the main case. Note that (III) is Theorem 1.1.

The rest of this section is the proof of this theorem. We assume that projective surfaces are equipped with Thurston’s metric ($\mathfrak{3.2.3}$), unless otherwise stated.

### 6.1. Projective structures on a quadrangle supported on a cylinder

Let $\mathcal{A}$ be a round cylinder in $\hat{\mathbb{C}}$, that is, $\mathcal{A}$ is bounded by disjoint round circles $c_{-1}$ and $c_1$. Let $g$ denote the geodesic in $\mathbb{H}^3$ orthogonal to $c_{-1}$ and $c_1$, that is, $g$ is orthogonal to the totally geodesic hyperplanes, in $\mathbb{H}^3$, bounded by $c_{-1}$ and $c_1$. Then, there is a unique foliation $\mathcal{F}_\mathcal{A}$ of $\mathcal{A}$ whose leaves are round circles $\{c_t\}_{t \in [-1, 1]}$ orthogonal to $g$, which we call the canonical foliation on $\mathcal{F}_\mathcal{A}$.

**Definition 6.3.** Let $C = (f, \rho)$ be a projective structure on a simply connected surface $F$, so that $\rho$ is trivial. Let $e$ be a simple curve on $C$. Then we say that $e$ is supported on the round cylinder $\mathcal{A}$ if $f$ embeds $e$ properly into $\mathcal{A}$ so that $e$ transversally intersects all leaves $c_t$ of $\mathcal{F}_\mathcal{A}$.

Let $R$ be a quadrangle, and let $e_1, e_2, e_3, e_4$ denote the edges of $R$, cyclically indexed along $\partial R (\simeq S^1)$. Then

**Definition 6.4.** A projective structure $C = (f, \rho)$ on $R$ is supported on the round cylinder $\mathcal{A}$ if

(i) $e_1$ and $e_3$ immerse into $c_{-1}$ and $c_1$ via $f$, respectively, and
(ii) $e_2$ and $e_4$ are supported on $\mathcal{A}$. 


Then the support of $C$ is the round cylinder $A$ and the simple arcs $f(e_2)$ and $f(e_4)$ supported on $A$. The canonical foliation $F_A$ on $A$ induces a foliation on the quadrangle $C$ supported on $A$.

6.2. Grafting a quadrangle supported on a cylinder. (c.f. [2, §3.5].) Let $C$ be a projective structure on the quadrangle $R$ supported on the round cylinder $A$ as above. Let $m$ be a simple arc supported on $A$. Then we can regard $m$ also as an arc property embedded in $A$. Thus, similarly to a grafting along a loop (§3.1), we can combine two projective structures $C$ and $A$, by cutting and pasting along $m$, and obtain a new projective structure on $R$ supported on $A$. Namely, there is a unique way to pair up and identify the boundary arcs of $C \setminus m$ and the boundary arcs of $A \setminus m$ corresponding to $m$ to make a connected projective surface. We call this operation the grafting of $C$ along $m$ and denote this resulting projective structure by $\text{Gr}_m(C)$. We call $m$ an admissible arc on $C$ (as for grafting along a loop). Then $\text{Gr}_m(C)$ is also a quadrangle, and the support of $\text{Gr}_m(C)$ is the same as that of $C$. If there is a multiarc $M$ on $C$ consisting of arcs supported on $A$ (admissible multiarc), then we can graft $C$ along all arcs of $M$ simultaneously and obtain a new projective structure on $R$ with the same support. Then, we accordingly denote this resulting projective structure by $\text{Gr}_M(C)$.

Lemma 6.5. Let $C_1$ and $C_2$ be projective structures on the quadrangle $R$ that have the same support. Then, either $C_1 = \text{Gr}_M(C_2)$ or $C_2 = \text{Gr}_M(C_1)$ for some admissible multiarc $M$. Furthermore, the multiarc $M$ is unique up to an isotopy of $M$ on $R$ through admissible multiarcs.
Proof. Let $\mathcal{A}$ be the round cylinder supporting $C_1$ and $C_2$ (as above). Let $f_1: R \to \mathcal{A}$ and $f_2: R \to \mathcal{A}$ be the developing maps of $C_1$ and $C_2$, respectively. Let $\tilde{\mathcal{A}}$ be the universal cover of $\mathcal{A}$ and $\Psi: \tilde{\mathcal{A}} \to \mathcal{A}$ be the universal covering map. In addition, let $m_2$ and $m_4$ be the simple arcs property embedded in $\mathcal{A}$ that support $C_1$ and $C_2$, i.e, $m_2 = f_1(e_2) = f_2(e_2)$ and $m_4 = f_1(e_4) = f_2(e_4)$. Pick a lift $\tilde{m}_4$ of $m_4$ to $\tilde{\mathcal{A}}$. Then, for each $k = 1, 2$, $f_k: R \to \mathcal{A}$ uniquely lifts to $\tilde{f}_k: R \to \tilde{\mathcal{A}}$ so that $f_k = \Psi \circ \tilde{f}_k$ and $\tilde{f}_k$ embeds $e_4$ onto $\tilde{m}_4$. Clearly $\tilde{f}_k$ is an embedding (although $f_k$ may be not). Then $\tilde{f}_k(e_2)$ is a lift of $m_2$ to $\tilde{\mathcal{A}}$. Since projective structures have fixed orientation, $\tilde{f}_1(e_2)$ and $\tilde{f}_2(e_2)$ are in the same component of $\tilde{\mathcal{A}} \setminus \tilde{m}_4$. If $\tilde{f}_1(e_2) = \tilde{f}_2(e_2)$, then clearly $C_1 = C_2$. If $\tilde{f}_1(e_2) \neq \tilde{f}_2(e_2)$, without loss of generality, we can assume that $Im(\tilde{f}_2)$ is strictly contained in $Im(\tilde{f}_1)$, if necessarily, by exchanging $C_1$ and $C_2$. Thus we can naturally regard $Im(\tilde{f}_1) \setminus Im(\tilde{f}_2)$ as a projective structure on a quadrangle supported on $\mathcal{A}$, where its developing map is the restriction of $\Psi$ to $Im(\tilde{f}_1) \setminus Im(\tilde{f}_2)$. Then its supporting arc are both $m_4$. Let $d$ be the degree of the developing map of $Im(\tilde{f}_1) \setminus Im(\tilde{f}_2)$ (at a point in $\mathcal{A} \setminus m_4$). We see that, by the elliptic isometries $\phi_t$ ($t \in S^1$) of $\mathbb{H}^3$ fixing the geodesic $g$ orthogonal to $\mathcal{A}$, we can foliate $\mathcal{A}$ with the arcs $\phi_t(m_4)$.

Pick an admissible multiarc $M$ on $C_2$ consisting of $d$ disjoint arcs that embed, via $f_2$, onto a leaf $\phi_t(m_4)$ of the foliation. Then we see that $C_1 = Gr_M(C_2)$, since the union of the projective structures inserted to $C_2$ by $Gr_M$ is exactly $Im(\tilde{f}_1) \setminus Im(\tilde{f}_2)$.

Let $M'$ be an another admissible multiarc on $C_2$ consisting of $d$ arcs. Then it is easy to find a one-parameter family $M_s$ ($s \in [0, 1]$) of admissible multiarcs on $C_2$ that connects $M$ to $M'$. Then $Gr_{M_s}(C_2) = C_1$ for all $s \in [0, 1]$. Therefore the choice of $M$ is unique up to such an isotopy. \qed

6.3. Fat traintracks.

Definition 6.6 (c.f. [15]). Let $F$ be a topological surface. A topological (fat) traintrack on $F$ is a union $T = \cup_{j \in J} B_j$ of topological quadrangles $B_j \subset F$, such that, with some homeomorphisms $\phi_j: [-1, 1] \times [-1, 1] \to B_j \subset F$ ($j \in J$),

- there are locally finitely many quadrangles $B_j$ on $F$,
- the set of the outermost vertical edges $\phi_j(\{\pm 1\} \times [-1, 1])$ of $B_j$ for all $j \in J$ are (disjointly) divided into triples $\{e_{k,1}, e_{k,2}, e_{k,3}\}_{k \in K}$ and pairs $\{e_{h,1}, e_{h,2}\}_{h \in H}$ so that
for each $k \in K$, there is a point $p_k$ on $e_{k,1}$ that divides $e_{k,1}$ into $e_{k,2}$ and $e_{k,3}$, so that $e_{k,1} = e_{k,2} \cup e_{k,3}$,
- for each $h \in H$, we have $e_{h,1} = e_{h,2}$, and
- different quadrangles $B_{j_1}$ and $B_{j_2}$ may intersect only in their outermost vertical edges only in the way described above.

Each quadrangle $B_j$ is called a branch of the traintrack $T$. The (vertical) arc $\phi_j([t] \times [-1,1])$ is called a tie of the branch $B_j$ for each $t \in [-1,1]$. The (horizontal) arc $\phi_j([0,1] \times \{s\})$ is called a rail of the branch $B_j$ for each $s \in [-1,1]$. In particular, $\phi_j(\{\pm 1\} \times [-1,1])$ are called outermost ties of $B_j$. For each $k \in K$, the point $p_k$ is called a switch point of $T$. Then, each branch $B_j$ have two distinct foliations: the foliation by the rails and by the ties of $B_j$. Then the entire traintrack $T = \cup B_j$ is foliated by the ties.

**Definition 6.7.** Let $\lambda$ be a lamination on the surface $F$. Then the topological traintrack $T = \cup_{j \in J} B_j$ carries $\lambda$, if the interior of $T$ contains $\lambda$ and each component of $B_j \cap \lambda$ is an arc connecting the outermost ties of $B_j$ for each $j \in J$. If, in addition, $B_j \cap \lambda \neq \emptyset$ for all $j \in J$, then we say $T$ fully carries $\lambda$.

Suppose that $L = (\lambda, \mu)$ is a measured lamination carried by $T = \cup_{j \in J} B_j$. By $\mu(B_j) \in \mathbb{R}_{\geq 0}$, we mean the weight of $L$ on the branch $B_j$, i.e. $\mu(B_j) := \mu(\alpha)$, where $\alpha$ is an arc in $B_j$ intersecting each leaf of $L \cap B_j$ transversally in a single point.

Let $r$ be a rail of a branch $B_j$ and $v$ be an endpoint of $r$. If $v$ is a switch point, then it is identified with exactly two endpoints of other rails in adjacent branches. Otherwise, $v$ is identified with exactly one endpoint of a different rail. Thus, identifying the corresponding endpoints of rails, we have a singular foliation of $T$ where the singular points are exactly the switch points $p_k$.

Then a rail is an immersion $r$ of $\mathbb{R}$ into a leaf of this singular foliation of $T$, up to a homeomorphism of the domain, such that

- $r(t)$ does not converge as $t \to \infty, -\infty$ and
- if $r(t)$ is a switch point $p_k$ for some $t \in \mathbb{R}$, letting $B_{k,1}, B_{k,2}, B_{k,3}$ denote the branches of $T$ corresponding to the outermost ties $e_{k,1}, e_{k,2}, e_{k,3}$, respectively, for $p_k$ as in Definition 6.6, then $r$ embeds a small neighborhood of $t$ in $\mathbb{R}$ into either $B_{k,1} \cup B_{k,2}$ or $B_{k,1} \cup B_{k,3}$.

We assign a hyperbolic structure $\tau$ to the surface $F$. Then $T$ is called a smooth (fat) traintrack if all $\phi_i : [-1,1] \times [-1,1] \to F$ are smooth and
all rails of $T$ are smooth. The length of the branch $B_j$ is the maximal length of the rails of $B_j$.

**Definition 6.8** (c.f. [4]). For $\epsilon > 0$, a sooth traintrack $T$ on $\tau$ is called $\epsilon$-straight if every tie and rail of $T$ has curvature less than $\epsilon$ and, if a tie and a rail of $T$ intersects at a point, the angle of intersection is $\epsilon$-close to $\pi/2$. The traintrack $T$ is called $\epsilon$-slim if all ties have length less than $\epsilon$.

**Definition 6.9.** We say that a geodesic lamination $\lambda$ on $\tau$ is embedded in a smooth traintrack $T$, if $\lambda$ is carried by $T$ and each leaf of $\lambda$ is a rail of $T$. Furthermore, $\lambda$ is fully embedded in $T$, if, in addition, $\lambda$ is fully carried by $T$.

### 6.4. Decomposition of projective structures by traintracks

We state the main proposition for the proof of Theorem 6.1 (I).

**Proposition 6.10.** Let $C = (\tau, L) = (f, \rho)$ be a projective structure on $S$. Then there exists a neighborhood $U$ of $[L]$ in $\mathcal{PML}(S)$ such that, if a projective structure $C' = (\tau', L') = (f', \rho)$ on $S$ with the same holonomy $\rho$ satisfies $[L'] \in U \setminus \{L\}$, then there are a topological (fat) traintrack $T = \bigcup_{j=1}^n B_j$ on $S$ and marking homeomorphisms $\phi: S \to C$, $\phi': S \to C'$ with the following properties:

- (I) $\phi' \circ \phi^{-1}: C \to C'$ induces an isomorphism from $C \setminus \phi(T)$ to $C \setminus \phi'(T)$ compatible with the developing maps $f$ and $f'$.
- (II) (i) For all $j = 1, 2, \ldots, n$, $C'|_{\phi(B_j)}$ and $C'|_{\phi'(B_j)}$ are projective structures on a quadrangle supported on the same round cylinder with arcs and the outermost ties of $B_j$ correspond the boundary components of the round cylinder,
  
  (ii) $C'|_{\phi'(B_j)} = Gr_{M_j}(C|_{\phi(B_j)})$ for some admissible multiarc $M_j$, and

(iii) letting $\kappa: C \to \tau$ and $\kappa': C' \to \tau'$ be collapsing maps, the $\kappa \circ \phi$-image of $T = \bigcup_{j=1}^n B_j$ is a topological traintrack carrying $L = (\lambda, \mu)$ and the $\kappa' \circ \phi'$-image of $T = \bigcup_{j=1}^n B_j$ is a topological traintrack on $\tau'$ fully carrying $L' = (\lambda', \mu')$, and moreover $\mu'(\kappa' \circ \phi'(B_j)) - \mu(\kappa \circ \phi(B_j))$ is a good approximation of $M_j$.

We let $C = (\tau, L)$ be a projective structure on $S$ with holonomy $\rho: \pi_1(S) \to \text{PSL}(2, \mathbb{C})$. Let $(C_i)$ be a sequence in $\mathcal{P}_\rho(S)$ such that, setting $C_i = (\tau_i, L_i)$, the projective measure lamination $[L_i]$ converges to $[L]$ in $\mathcal{PML}(S)$. Then, it suffices to show Proposition 6.10 for $C' = C_i$ with sufficiently large $i \in \mathbb{N}$. Set $L_i = (\lambda_i, \mu_i)$, where $\lambda_i \in \mathcal{GL}(\tau_i)$ and $\mu_i$ is a transversal measure supported on $\lambda$. Since $\mathcal{GL}(S)$
is compact, we can in addition assume that $\lambda_i$ converges to a geodesic lamination $\lambda_\infty$ (in $\mathcal{G}\mathcal{L}(S)$). Then since $[L_i] \rightarrow [L]$ in $\mathcal{PML}(S)$, $\lambda$ is a sublamination of $\lambda_\infty$.

6.5. Construction of traintracks on $\tau$ and $\tau'$ for Proposition 6.10 (I). Let $\kappa: C \rightarrow \tau$ and $\kappa_i: C_i \rightarrow \tau_i$ be the collapsing maps for all $i \in \mathbb{N}$.

**Proposition 6.11.** There exists $K > 0$ with the following property: For every $\epsilon > 0$, there exists $I \in \mathbb{N}$, such that, if $i > I$, then there exist a fat train track $T = \cup_{j=1}^{n_j} B_j$ on $S$ and marking homeomorphisms $\psi: S \rightarrow \tau$ and $\psi_i: S \rightarrow \tau_i$ such that

(i) there is a marking-preserving smooth $(1 + \epsilon)$-bilipschitz map $\eta_i: \tau \rightarrow \tau_i$ such that $\eta_i \circ \psi = \psi_i$ and $\eta_i$ satisfies the conclusions of Proposition 5.4,

(ii) $\psi(T) = \cup_{j=1}^{n_j} \psi(B_j)$ is an $\epsilon$-straight and $\epsilon$-slim smooth train track on $\tau$ such that the branches $\psi(B_j)$ have length more than $K$ and the geodesic lamination $\lambda_\infty$ is fully embedded into $\psi(T)$; similarly $\psi_i(T) = \cup_{j} \psi_i(B_j)$ is an $\epsilon$-straight and $\epsilon$-slim smooth train track on $\tau_i$ such that the branches $\psi_i(B_j)$ have length more than $K$ and the geodesic lamination $\lambda_i$ is fully embedded into $\psi(T)$ (here, for the embedding property of $\lambda_\infty$ and $\lambda_i$, we may use different sets of homeomorphisms $\phi_j: [-1,1] \times [-1,1] \rightarrow B_j \subset S$), and

(iii) $\kappa^{-1} \circ \psi(S \setminus T) \subset C$ is isomorphic to $\kappa_i^{-1} \circ \psi_i(S \setminus T) \subset C_i$ via developing maps.

Moreover, if $\lambda_\infty$ contains no closed leaf, we can take an arbitrary number $K > 0$ and otherwise we can take $K > 0$ to be one third of the length of the shortest closed leaf of $\lambda_\infty$.

**Proof.** Step 1: Construction of $T$ and $\psi: S \rightarrow \tau$.

**Lemma 6.12.** Let $\nu$ be a geodesic lamination on a hyperbolic surface $\sigma$ homeomorphic to $S$. There exists $K > 0$, such that, for every $\epsilon > 0$, there exists an $\epsilon$-straight and $\epsilon$-slim train track $T = \cup_j R_j$ on $\sigma$ fully carrying $\nu$ such that

(i) each branch of $T$ has length more than $K$,

(ii) the ties of $T$ are geodesic segments,

(iii) each leaf of $\nu$ is a rail of $T$, and

If $\nu$ contains a closed leaf, then we can take the constant $K$ to be one third of the length of the shortest closed leaf of $\nu$. Otherwise, we can take $K$ to be any positive number.
Proof. Given \( \epsilon > 0 \), if \( \delta > 0 \) is sufficiently small, we can easily make \( N_\delta(\nu) \) into an \( \epsilon \)-straight and \( \epsilon \)-slim traintrack satisfying (ii) and (iii); let \( T_\delta = N_\delta(\nu) = \bigcup_{j=1}^n B_j \) denote this traintrack (with some \( n = n(\delta) \in \mathbb{N} \)).

If \( \nu \) contains a close leaf \( l \), then any traintrack \( T \) carrying \( \nu \) must have a branch of length equal to or less than \( \text{length}_\sigma(l) \). If, in addition, if some leaves of \( \nu \) spiral towards \( l \) from the both sides of \( l \), then \( l \) must be covered by at least 2 branches of \( T \) (Figure 3). Thus \( T \) must have a branch of length equal to or less than \( \frac{1}{2} \text{length}_\sigma(l) \). The number of the switch points of \( T_\delta \) is bounded for all \( \delta > 0 \), and thus we can assume that the number \( n = n(\delta) \) of the branches of \( T_\delta \) is also bounded. On the other hand, if \( \nu \) contains a non-periodic leaf, the total length of the branches \( B_j \) of \( T_\delta \) diverges to infinity, as \( \delta \to 0 \). Therefore, for every \( K > 0 \), we can split the traintrack \( T_\delta \) appropriately so that the length of every branch disjoint from periodic leaves of \( \nu \) is more than \( K \), keeping (ii) and (iii).

Hence we can take \( K \) to be one third of the length of the shortest closed leaf of \( \nu \).

Let \( T_\infty = \bigcup_{j=1}^n R_j \) denote the \( \epsilon \)-slim traintrack on \( \tau \) carrying \( \lambda_\infty \), obtained by Lemma \textbf{6.12}, where \( R_j \) are branches of \( T_\infty \). Then pick a topological traintrack \( T = \cup B_j \) on \( S \) so that \( (S,T) \) is topologically isomorphic to \( (\tau, T_\infty) \) via a marking homeomorphism \( \psi : S \to \tau \) of \( \tau \) and that \( \psi(B_j) = R_j \) for each \( j = 1,2,\ldots,n \).

Let \( \mathcal{L} \) be the canonical measured lamination on \( C \) corresponding to \( L \) via \( \kappa : C \to \tau \) (\textbf{3.2.2}). Then, since \( \tau \setminus T_\infty \) is disjoint from \( \lambda_\infty, \kappa^{-1} \) embeds \( \tau \setminus T_\infty \) onto its image. Let \( T_\infty = \kappa^{-1}(T_\infty) \), which is, at the moment, just a subset of \( C \). Fix a marking \( \phi : S \to C \) of \( C \) that takes
Thus we can pick a marking homeomorphism \( \phi \) on \( \eta \) and \( \eta \). Set \( \eta \colon \tau \to \tau_i \). In addition, by perturbing \( \eta_\lambda \colon \tau \to \tau \) (but preserving the properties from Proposition 5.4), we can assume that \( \eta_\lambda |_\psi(S \setminus T) \) corresponds to the isomorphism \( \psi_i : C \setminus \phi(T) \to C_i \setminus \phi_i(T) \) via \( \kappa \) and \( \kappa_i \), i.e. \( \kappa_i \circ \psi_i = \eta_\lambda \circ \kappa \) on \( \phi(S \setminus T) \). Set \( \psi_i = \eta_\lambda \circ \psi : S \to \tau_i \) for each \( i \in \mathbb{N} \). Thus we have constructed \( \psi \) and \( \psi_i \) satisfying (iii) and \( \eta_\lambda \) satisfying (i).

Step 3: (ii). We have already constructed a desired \( \psi : S \to \tau \) in Step 1. In Step 2, we set \( \psi_i = \eta_\lambda \circ \psi : S \to \tau_i \). Since \( \eta_\lambda : \tau \to \tau_i \) converges to the trivial isometry and \( (\tau_i, \lambda_i) \) converges to \( (\tau, \lambda_\infty) \) as \( i \to \infty \), the properties of \( \psi \) induce the desired properties for \( \psi_i \).

Pick sufficiently small \( \epsilon > 0 \). Let \( T_\infty = \bigcup_j R_j \) denote the \( \epsilon \)-straight and slim traintrack on \( \tau \) obtained by Proposition 6.11, where \( R_j \) (\( j = 1, 2, \ldots, n \)) are branches of \( T_\infty \). Then \( T_\infty \) fully carries \( \lambda_\infty \) and each branch \( R_j \) has length more than \( K \), where \( K > 0 \) is a constant which does not depend on \( \epsilon \). Let \( \tilde{\lambda}_\infty \) and \( \tilde{T}_\infty \) denote the total lifts of \( \lambda_\infty \) and \( T \) to \( \mathbb{H}^2 \), respectively. Then \( \tilde{T}_\infty \) is an \( \epsilon \)-straight and slim traintrack carrying \( \tilde{\lambda}_\infty \).

**Lemma 6.13.** For every \( \delta > 0 \), there exists \( \epsilon > 0 \) such that, if \( \epsilon < \epsilon \), then, for all leaves \( l \), \( m \) of \( \tilde{\lambda}_\infty \in \mathcal{GL}(\mathbb{H}^2) \) passing through the same branch \( R \) of \( \tilde{T}_\infty \), \( \beta(l \cap R) \) and \( \beta(m \cap R) \) are geodesics segments of length more than \( K \) in \( \mathbb{H}^3 \) that are \( \delta \)-close in Hausdorff topology.
Figure 4. $C^1$-close subsurfaces of the bending maps $\beta$ and $\beta_i$ correspond to isomorphic subsurfaces in $C$ an $C_1$.

**Proof.** Fix $\delta > 0$. Since all branches of $T_\infty$ have length more than $K$, for all leaves $l, m$ of $\tilde{\lambda}_\infty$ passing through a fixed branch $R$ of $\tilde{T}_\infty$, $l \cap R$ and $m \cap R$ are geodesic segments in $\mathbb{H}^2$ of length more than $K$. Since $\tilde{T}_\infty$ is $\epsilon$-slim, corresponding endpoints of $l \cap R$ and $m \cap R$ have distance less than $\epsilon$. Thus, by taking sufficiently small $\epsilon > 0$, we can assume that $l \cap R$ and $m \cap R$ are $\delta$-close in Hausdorff topology for all $R, l, m$ as above. Recall that $\beta$ is 1-lipschitz and $\beta$ isometrically embeds leaves of $\tilde{\lambda}_\infty$ onto geodesics in $\mathbb{H}^3$. Therefore, since $l \cap R$ and $m \cap R$ are $\delta$-close, $\beta(l \cap R)$ and $\beta(m \cap R)$ are also $\delta$-close in Hausdorff topology and they are geodesic segments of length more than $K$. □

For sufficiently large $i \in \mathbb{N}$, let $T_i = \bigcup_{j=1}^n R_{i,j}$ is the $\epsilon$-straight and slim traintrack on $\tau_i$ obtained by Proposition 6.11. Then each branch $R_j$ has length more than $K$. Similarly let $\tilde{\lambda}_i$ and $\tilde{T}_i$ denote the total lifts of $\lambda_i$ and $T_i$, respectively, to $\mathbb{H}^2$. Then $\tilde{T}_i$ is an $\epsilon$-straight and slim, $\tilde{\lambda}_i$ fully embeds into $\tilde{T}_i$ and each branch of $\tilde{T}_i$ has length at least $K$. Then, similarly

**Lemma 6.14.** For every $\delta > 0$, there exists $\epsilon > 0$ with the following property. If $\epsilon < \epsilon$, then for arbitrary leaves $l, m$ of $\tilde{\lambda}_i$ that intersect the same branch $R$ of $\tilde{T}_i$, which is $\epsilon$-straight and slim, $\beta(l \cap R)$ and $\beta(m \cap R)$ are geodesics segments of length more than $K$ (in $\mathbb{H}^3$) that are $\delta$-close in Hausdorff topology.

**Proof.** The proof is similar to the proof of Lemma 6.13. □
Recall that traintracks $T_\infty = \bigcup_{j=1}^{n} R_j$ on $\tau$ and $T_i = \bigcup_{j=1}^{n} R_{i,j}$ on $\tau_i$ are the images of $T = \bigcup_{j=1}^{n} B_j$ on $S$ under $\psi : S \to \tau$ and $\psi_i : S \to \tau$ (Proposition 6.11 (ii)). Let $\bar{T}$ be the total lift of $T$ to $\mathbb{H}^2$, and let $\psi : S \to \mathbb{H}^2$ and $\psi_i : \bar{S} \to \mathbb{H}^2$ be the lifts of $\psi$ and $\psi_i$, respectively. Then we have

**Lemma 6.15.** For every $\delta > 0$, there exist $e > 0$ and $I \in \mathbb{N}$ such that, if $T_\infty$ and $T_i$ are $\epsilon$-straight and slim with $\epsilon < e$ and $i > I$, then, for every branch $B$ of $\bar{T}$ and every leaf $l$ of $\lambda_\infty$ interesting the branch $\bar{\psi}(B)$ of $\bar{T}_\infty$ and leaf $m$ of $\bar{\lambda}_i$ intersecting the branch $\bar{\psi}_i(B)$ of $\bar{T}_i$, the geodesic segments $\beta(l \cap \bar{\psi}(B))$ and $\beta_i(m \cap \bar{\psi}_i(B))$ are of length more than $K$ and they are $\delta$-close with the Hausdorff metric.

**Proof.** Assuming $i \in \mathbb{N}$ is sufficiently large, since every branch of $T_\infty$ and $T_i$ has length at least $K$, then $\beta(l \cap \bar{\psi}(B))$ and $\beta_i(m \cap \bar{\psi}_i(B))$ are geodesic segments in $\mathbb{H}^2$ of length more than $K$ (for all $l, m, B$ as in the statements).

By Proposition 6.11 (and Proposition 5.4), for every $\delta > 0$, if $i \in \mathbb{N}$ is sufficiently large, then $\beta_i \circ \bar{\eta}_i$ and $\beta$ are $\delta$-close in $C^0$-topology. In addition, since $\bar{T}_\infty$ and $\bar{T}_i$ are $\epsilon$-slim, for every $\delta > 0$, if $i \in \mathbb{N}$ is sufficiently large and $\epsilon > 0$ is sufficiently small, then corresponding end points of $m \cap \bar{\psi}_i(B)$ and $\bar{\eta}_i(l \cap \bar{\psi}(B))$ are $\delta$-close. Thus, under the same assumption, corresponding endpoints of the geodesic segments $\beta(m \cap \bar{\psi}_i(B))$ and $\beta_i(l \cap \bar{\psi}(B))$ are $\delta$-close. \hfill $\Box$

**Lemma 6.16.** For every $N > 0$, there exists $I \in \mathbb{N}$ such that, if $[L] \neq [L_i]$ and $i > I$, then $\mu_i \circ \psi_i(B) > N \cdot \mu \circ \psi(B)$ for every branch $B$ of $T$.

**Proof.** We have set $T = \bigcup_{j=1}^{n} B_j$, where $n$ is the number of branches of $T$. Recall that $T$ corresponds to $T_i$ and $T_\infty$ via $\psi_i$ and $\psi$, respectively. Thus, since $L_i$ is fully carried by $T_i$, $L_i$ is identified with the $n$-tuple $(\mu_i \circ \psi_i(B_j))_{j=1}^{n} \in (\mathbb{R}_{>0})^n$. Similarly, since $L$ is carried by $T_\infty$, and it is identified with the $n$-tuple $(\mu \circ \psi(B_j))_{j=1}^{n} \in (\mathbb{R}_{>0})^n$. Since $[L_i]$ converges to $[L]$ in $\mathcal{PML}(S)$ as $i \to \infty$, then, for every $\delta > 0$, if $i$ is sufficiently large, we have

$$1 - \delta \frac{\mu \circ \psi(B_j)}{\mu \circ \psi(B_k)} \leq \frac{\mu_i \circ \psi_i(B_j)}{\mu_i \circ \psi_i(B_k)} \leq 1 + \delta \frac{\mu \circ \psi(B_j)}{\mu \circ \psi(B_k)} \tag{2}$$

for all branches $B_j$ and $B_k$ of $T$.

Suppose that the lemma fails. Then, taking a subsequence of $(C_i)_i$ if necessarily, we can assume that $\mu_i \circ \psi_i(B_j)$ indexed by $i \in \mathbb{N}$ is a bounded sequence in $\mathbb{R}_{>0}$ for some $j \in \{1, 2, \ldots, n\}$. Thus, up to a subsequence, by Inequalities (2), the sequence $\mu_i \circ \psi_i(B_j)$ converges as
Then, since $\delta > 0$ is arbitrarily fixed, by Inequalities (2), $L_i$ converges to some $c \cdot L$ with $c > 0$ as $i \to \infty$. In addition, we can assume that all $L_i$ are distinct. Since $\tau_i$ converges to $\tau$, thus $C_i = (\tau_i, L_i) \in \mathcal{P}_\rho$ converges to $(\tau, cL)$ in $\mathcal{P}(S)$. This is a contradiction to the fact that $\mathcal{P}_\rho$ is a discrete subset of $\mathcal{P}(S)$.

6.6. **Branches supported on a cylinder.** Given any $\epsilon > 0$, we have constructed an $\epsilon$-straight and slim traintrack $T_\infty(= T_\infty(\epsilon)) = \bigcup_{j=1}^n R_j$ on $\tau$ into which $\lambda_\infty$ is embedded (Proposition 6.11). In addition, for every $\delta > 0$, if $\epsilon > 0$ is sufficiently small, we can assume the conclusions in Lemma 6.13, Lemma 6.14, and Lemma 6.15. Since $\lambda_\infty$ is embedded in $T_\infty$, all ties of $T_\infty$ are transversal to $\lambda_\infty$ and thus, in particular, to $\lambda$. Therefore, recalling the collapsing map $\kappa: C \to \tau$, for all ties $t$ of $T_\infty$, $\kappa^{-1}(t)$ is a simple arc on $C$. Thus we can pull $T_\infty$ back to a topological (fat) traintrack on $C$ via $\kappa$. Namely, let $T_\infty = \kappa^{-1}(T_\infty)$ and $R_j = \kappa^{-1}(R_j)$ for each $j = 1, 2, \ldots, n$. Then $T_\infty = \bigcup_{j=1}^n R_j$ is a topological traintrack on $C$ that homeomorphic to $T_\infty = \bigcup_{j=1}^n R_j$ on $\tau$. Note that we have also pulled back the rails and the ties of $T_\infty$ to those of $T$ so that $L$ embeds into $T$ via $\kappa$, where $L$ is the canonical measured lamination on $C$.

**Proposition 6.17.** We can perturb the traintrack structure $T_\infty = \bigcup_j R_j$ (more precisely, branches and ties) so that the projective structure on each branch is supported on a round cylinder but $T_\infty$ is preserved a subset of $C$.

We prove this proposition in the rest of §6.6.

6.6.1. **Round circles on $\hat{C}$ corresponding to switches of $T_\infty$.** If two adjacent branches of the traintrack $T$ have a common outermost tie, then there is a no switch point on this outermost tie (this situation can be avoided if $\lambda_\infty$ contains no isolated periodic leaf). For each such an outermost tie $t$ of (a branch of) $T$, pick an endpoint of $t$ and call it marked point of the traintrack $T$. Accordingly, we call the corresponding point on $T_i$ and $T_\infty$ a marked point as well. Thus, every outermost tie of $T, T_\infty$ and $T_i$ contains a unique switch point or marked point.

Recall that $\tilde{T}_\infty = \tilde{T}_\infty(\epsilon)$ is the total lift of the traintrack $T_\infty$ on $\tau$ to $\mathbb{H}^2$. Letting $\tilde{\kappa}: \tilde{C} \to \mathbb{H}^2$ be the lift of $\kappa: C \to \tau$, we have

**Lemma 6.18.** For every $\delta > 0$, if $\epsilon > 0$ is sufficiently small, then then, for each switch point and marked point $p$ of $\tilde{T}_\infty(\epsilon)$, there exists a round circle $c_p$ on $\tilde{C}$ satisfying the followings:
(i) This assignment of round circles $c_p$ to switch points and marked points $p$ of $\partial \T_\infty$ is $\rho$-equivariant.

(ii) $c_p$ contains the point $f(\kappa^{-1}(p))$.

(iii) If a leaf $l$ of $\lambda_\infty$ passes through the tie of $\T_\infty$ containing $p$, then the geodesic $\beta(l)$ intersects the totally geodesic hyperplane $\Conv(c_p)$ transversely at an angle $\delta/2$ close to $\pi/2$.

(iv) Letting $q$ be the point on $l$ that maps into $\Conv(c_p)$, then we have $\dist_{\H^3}(p,q) < \delta$.

Proof. Let $p$ be a switch point of $\T_\infty$. Then, since $\text{int}(\T_\infty)$ contains $\lambda_\infty$, then $p \cap \lambda_\infty = \emptyset$. Thus $\kappa^{-1}(p)$ is a single point on $\hat{\C}$. Then $f(\kappa^{-1}(p)) \in \hat{\C}$ orthogonally projects $\beta(p)$ in the hyperbolic tangent plane of $\beta$ at $p$. Let $a_p$ be the tie of $\T_\infty$ whose interior contains $p$, which is a union of two outermost ties of different branches of $\T_\infty$. Since $a_p$ is a geodesic segment on $\H^2$ (Lemma 6.12), $\beta$ embeds a small neighborhood of $p$ in $a_p$ onto a geodesic segment in $\H^3$. Thus we let $c_p$ be the round circle on $\hat{\C}$ containing the point $f(\kappa^{-1}(p))$ such that the hyperplane $\Conv_{\H^3}(c_p)$ contains this geodesic segment (Condition (ii)). The $\pi_1(S)$-action on the universal cover of $\tau$ preserves the set of switch points on $\T_\infty$. Then, since $\beta$ is $\rho$-equivariant, this assignment of $c_p$ for switch points $p$ of $\T_\infty$ is accordingly $\rho$-equivariant (Condition (i)).

Let $m_p$ be the leaf of $\lambda_\infty$ closest to $p$. Since $\T_\infty$ is $\epsilon$-straight, $m_p$ intersects $a_p$ transversely, and $\angle(m_p,a_p)$ is $\epsilon$-close to $\pi/2$. Thus, if $0 < \epsilon < \delta/2$, the geodesic $\beta(m_p)$ intersects $\Conv_{\H^3}(c_p)$ at an angle $\delta/2$-close to $\pi/2$ for all switch points $p$ of $\T_\infty$. Thus, by lemma 6.13 if $\epsilon > 0$ is sufficiently small, then, for each leaf $l$ of $\lambda_\infty$ intersecting $a_p$, $\beta(l)$ transversely intersects $\Conv(c_p)$ at an angle $\delta$-close to $\pi/2$ (Condition (iii)). Since $\text{length}(a_p) < \epsilon$, the distance between the points $p$ and $l \cap a_p$ is also less than $\epsilon$. Then, since $\beta$ is $1$-lipschitz, the distance between $\beta(p)$ and $\beta(l \cap a_p)$ is also less than $\epsilon$. Let $q_l$ be the point on $l$ such that $\beta(q_l) \in \Conv(c_p)$. Then, since $\angle(\beta(l),\Conv(c_p))$ is $\delta$-close to $\pi/2$ for all $l$ and $p$ as above, if $\epsilon > 0$ is sufficiently small, then, in addition $\dist_{\H^2}(\beta(q_l),\beta(l \cap a_p)) = \dist_{\H^3}(q_l,l \cap a_p)$ is less than $\delta/2$ for all $l$ as above. Therefore, by the triangle inequality, $\dist_{\H^2}(p,q_l) < \delta/2 + \epsilon < \delta$ (Condition (vi)).

Let $p$ be a marked point on $\T_\infty$; then we can similarly define $c_p$ and show (i) - (iv). Then $\kappa^{-1}(p)$ is a single point on $\hat{\C}$. Let $a_p$ be the tie of $\T_\infty$ whose endpoint is $p$, which is a geodesic segment in $\H^2$. Since the interior of $\T_\infty$ contains $L$, then $\beta$ takes a sufficiently short subsegment of $a_p$ with an endpoint at $p$ onto a geodesic segment in $\H^3$. Then, let
Let $l$ be a leaf of $\tilde{\lambda}_\infty$, and let $a_1$ and $a_2$ be different outermost ties of branches of $T_\infty$ that intersect $l$. Accordingly, let $p_1$ and $p_2$ be the (different) switch or marked points on $\partial T_\infty$ contained in $a_1$ and $a_2$, respectively; let $c_1$ and $c_2$ be the round circles on $\hat{\mathbb{C}}$ corresponding to $p_1$ and $p_2$ given by Lemma 6.18; let $q_1$ and $q_2$ be the distinct points on $l$ such that $l$ intersects $\text{Conv}(c_1)$ and $\text{Conv}(c_2)$ at $\beta(q_1)$ and $\beta(q_2)$. In addition, let $P_1, P_2 \in \hat{\mathbb{C}}$ denote the ideal endpoints of the geodesic $\beta(l) \subset \mathbb{H}^3$ so that the geodesic rays from $\beta(q_1)$ to $P_1$ and from $\beta(q_2)$ to $P_2$ are disjoint. Recalling that $c_1, c_2$ depend on the choice of $\delta > 0$ in Lemma 6.18 we have

Lemma 6.19. There exists (sufficiently small) $\delta > 0$ and $\epsilon > 0$ such that, for all leaves $l$ of $\tilde{\lambda}_\infty$ and all outermost ties $a_1$ and $a_2$ of branches of $T_\infty = T_\infty(\epsilon)$ such that $a_1$ and $a_2$ intersect $l$,

- the corresponding round circles $c_1$ and $c_2$ on $\hat{\mathbb{C}}$ are disjoint, and
- $P_k$ is in the disk component of $\mathbb{C} \setminus (c_1 \cup c_2)$ bounded by $c_k$ for each $k = 1, 2$.

Proof. Recall that there is a lower bound $K > 0$ of the lengths of branches of $T_\infty(\epsilon)$, which does not depend on $\epsilon > 0$. Then the distance between the points $a_1 \cap l$ and $a_2 \cap l$ is more than $K$. By Lemma 6.18 we have $\text{dist}(q_k, p_k) < \delta$ for each $k = 1, 2$. Thus $\text{dist}(q_1, q_2) > K - 2\delta$. In addition, again by Lemma 6.18, $\beta(l)$ intersects $\text{Conv}(c_{p_k})$ at an angle $\delta$-close to $\pi/2$. Therefore, for sufficiently small $\delta > 0$, $c_{p_1}$ and $c_{p_2}$ are disjoint round circles on $\hat{\mathbb{C}}$ for all $l, a_1$ and $a_2$ as in the lemma. Observing the configuration of $\beta(l)$, $c_{p_1}$ and $c_{p_2}$ in $\mathbb{H}^3$, clearly the ideal point $P_k$ is in the disk component of $\mathbb{C} \setminus (c_1 \cup c_2)$ bounded by $c_k$ for each $k = 1, 2$. \hfill $\square$

More generally, let $l$ be a rail of $T_\infty$. Then consider all outermost ties $a_k$ ($k \in \mathbb{Z}$) intersecting $l$. We can naturally assume that the intersection points $a_k \cap l$ lie in $l$ in the order of the index $k \in \mathbb{Z}$. For each $k \in \mathbb{Z}$, let $c_k$ be the round circle, given by Lemma 6.18 corresponding to the switch point or the marked point of $T_\infty$ contained in $a_k$. Since $T_\infty$ fully carries $\tilde{\lambda}_\infty$, for every $k$, there is a leaf of $\tilde{\lambda}_\infty$ that intersects $a_{k-1}, a_k, a_{k+1}$. Then, by Lemma 6.19, $c_{k-1}, c_k, c_{k+1}$ are disjoint round circles on $\hat{\mathbb{C}}$ nested in the listed order. Therefore
Figure 5.

Corollary 6.20. The round circles $c_k$ are disjoint and nested in the order of the index $k \in \mathbb{Z}$. That is, $\mathcal{C} \setminus (\cup_k c_k \cup \{P_1, P_2\})$ is the union of disjoint round cylinders bounded by $c_k$ and $c_{k+1}$ for all $k \in \mathbb{Z}$.

Recalling that $\lambda$ is a sublamination of $\lambda_{\infty}$, let $\nu$ be the geodesic lamination on $C$ with respect to Thurston’s metric that is the union of the supporting lamination of the canonical lamination $\mathcal{L}$ and $\kappa^{-1}(\lambda_{\infty} \setminus \lambda)$. Then $\nu$ fully embeds into $\mathcal{T}_{\infty}$.

Let $\mathcal{R}$ and $\mathcal{R}$ be corresponding branches of $\tilde{T}_{\infty}$ and $\tilde{T}_{\infty}$, respectively. Recall that every outermost tie of a branch contains a switch point or marked point. Let $c_1$ and $c_2$ denote the round circles on $\hat{C}$, given by Lemma 6.18, corresponding to the switches on the outermost ties of $R$. Let $A$ denote the round annulus bounded by $c_1$ and $c_2$. Then we have

Proposition 6.21. For every $\delta > 0$, if $\epsilon > 0$ is sufficiently small, then for every rail $m$ of $\tilde{T}_{\infty}$ intersecting the branch $\mathcal{R}$ of $\tilde{T}_{\infty}$, letting $l$ be the corresponding rail of $\tilde{T}_{\infty}$, we have

- $f|m$ is a simple curve on $\hat{C}$ that connects the ideal ends points of $\beta(l)$ and intersects both $c_1$ and $c_2$ in a single point at an angle $\delta$-close to $\pi/2$; thus $f^{-1}(A) \cap m$ is a single arc, and
- the arc $f^{-1}(A) \cap m$ is $\delta$-close to the arc $\mathcal{R} \cap m$ with the Hausdorff metric.

Proof. Let $\tilde{\nu} \in \mathcal{GL}(\hat{C})$ be the total lift of the geodesic lamination $\nu$ on $C$ to $\hat{C}$. (Case 1.) We first prove the proposition for leaves of $\tilde{\nu}$. For each leaf $m$ of $\tilde{\nu}$, $f|m$ is a circular arc on $\hat{C}$ connecting the endpoints
of the geodesic $\beta(l)$ in $\mathbb{H}^3$, where $l$ is the leaf of $\tilde{\lambda}_\infty$ corresponding to $m$. Then the nearest point projection from $f(m)$ to $\beta(l)$ corresponds to $\tilde{\kappa}|m$: $m \to l$.

By Lemma 6.18 (iii) and (vi), for every $\delta > 0$, if $\epsilon > 0$ is sufficiently small, then, for every leaf $l$ of $\lambda_\infty$ passing through $R$, the geodesic $\beta(l)$ intersects the hyperplanes $\text{Conv}_{\mathbb{H}^3}(c_1)$ and $\text{Conv}_{\mathbb{H}^3}(c_2)$ at an angle $\delta$-close to $\pi/2$ and, in addition, $R \cap l$ is $\delta$-close to the geodesic segment $\beta^{-1}(\text{Conv}(A)) \cap l$.

Those properties for $l$ induce the desired properties for $m$.

(Case 2.) Suppose $m$ is a rail of $\tilde{T}_\infty$, intersecting $R$ that is not a leaf of $\tilde{\nu}$. Then $m$ is contained in a component $\mathcal{X}$ of $\tilde{C} \setminus \tilde{\nu}$. Let $m'$ be a leaf of $\tilde{\nu}$ bounding $\mathcal{X}$ and passing though thought $R$. We have seen that $m'$ has the desired property. Roughly, since $T_\infty$ is $\epsilon$-straight and slim, then $l$ and $l'$ are sufficiently close and thus $m$ also has the desired property, where $l'$ is the leaf of $\lambda_\infty$ corresponding to $m'$.

Letting $X = \kappa(\mathcal{X})$, a component of $\mathbb{H}^2 \setminus \lambda_\infty$.

Let $\Psi_X: D_X \to H_X$ be the orthogonal projection corresponding to $X$, where $D_X$ is the $f$-image of the maximal in $\tilde{C}$ containing $\mathcal{X}$ and $H_X$ is the totally geodesic hyperplane in $\mathbb{H}^3$ containing $\beta(X)$. Then we have

$$\Psi \circ f = \beta \circ \tilde{\kappa}$$

on $\mathcal{X}$ (see §3.2).

Since $T_\infty$ is $\epsilon$-straight, then $l$ has curvature less than $\epsilon$. For every $\delta > 0$, if $\epsilon > 0$ is sufficiently small, then the rails $l \cap R$ and $l' \cap R$ of $R$ are $\delta$-close for all $l, l'$ as above. Since $\beta$ isometrically embeds $X$ into $D_X$, then, if $\epsilon > 0$ is sufficiently small, then $\beta(l)$ intersects $\text{Conv}(c_1)$ and $\text{Conv}(c_2)$ in a single point $\delta$-orthogonally for all rails $l$ of $\tilde{T}_\infty$ intersecting $R$. Thus we can in addition assume that $R \cap l$ is $\delta$-close to $\beta^{-1}(\text{Conv}(A)) \cap l$. Those properties of $l$ induce the desired properties of $m$, by (3). □

For each rail $m$ of the traintrack $\tilde{T}_\infty$ passing through $R$, let $\alpha_m = m \cap R$, a rail of $R$. Let $\alpha'_m = m \cap f^{-1}(A)$. Then, by Proposition 6.21 $\alpha'_m$ is a simple arc in $m$ that is $\delta$-close to $m \cap R$, and $\alpha'_m$ is supported on $A$. Suppose that $m$ contains a switch point $p_k$ on $\partial R$ that corresponds to a switch point $p_k$ of $\partial R$ via $\tilde{\kappa}$. Then $p_k$ is an endpoint of $\alpha_m$. Since the round circle $c_k$ corresponding the switch point $p_k$ passes through $p_k$ by Lemma 6.18 (ii), the $\delta$-perturbation of $\alpha_m$ to $\alpha'_m$ preserves the endpoint $p_k$. In addition, since the developing map $f$ is a local homeomorphism, $\alpha'_m$ changes continuously when we vary the leaf $m$ passing through $R$ continuously. Set $R = \cup \alpha_m$ and
\[ \mathcal{R}' = \cup \alpha'_m, \] where the union runs over all rails \( m \) of \( \mathcal{T}_\infty \) passing through \( \mathcal{R} \). Therefore

**Corollary 6.22.** \( \mathcal{R}' \) is a quadrangle supported on \( \mathcal{A} \), that is \( \delta \)-close to \( \mathcal{R} \) with the Hausdorff metric, and this deformation preserves the switch point of \( \tilde{T}_\infty \) on \( \partial \mathcal{R} \).

We last show that this deformation of \( \mathcal{R} \) to \( \mathcal{R}' \) induces a desired deformation of the traintrack \( \mathcal{T}_\infty \). By Lemma 6.18 (i), the correspondence between the branches \( \mathcal{R} \) of \( \tilde{T}_\infty \) and the round cylinders \( \mathcal{A} \) in \( \hat{\mathcal{C}} \) is \( \rho \)-equivariant. Thus this deformation of \( \mathcal{R} \) to \( \mathcal{R}' \) commutes with the action of \( \pi_1(S) \) on \( \hat{\mathcal{C}} \). Pick an arbitrary rail \( m \) of \( \tilde{T}_\infty \). Let \( \{ \mathcal{R}_k \}_{k \in \mathbb{Z}} \) denote the branches of \( \tilde{T}_\infty = \cup_j \mathcal{R}_j \) intersecting \( m \). We can assume that this sequence is indexed along \( m \) so that \( \mathcal{R}_k \) and \( \mathcal{R}_{k+1} \) are adjacent for every \( k \in \mathbb{Z} \). Then we have a decomposition of \( m \) into rails \( m \cap \mathcal{R}_k \) of \( \mathcal{R}_k \). For each \( k \in \mathbb{Z} \), let \( \mathcal{A}_k \) be the round cylinder on \( \hat{\mathcal{C}} \) corresponding to \( \mathcal{R}_k \) as for Proposition 6.21. Thus, by Proposition 6.21 \( m \cap f^{-1}(\mathcal{A}_k) \) is a segment of \( m \) that is \( \delta \)-close to \( m \cap \mathcal{R}_k \) for each \( k \in \mathbb{Z} \). Then, since \( \mathcal{A}_k \) have disjoint interior, \( m \) is decomposed into the rails \( m \cap f^{-1}(\mathcal{A}_k) \) of \( \mathcal{R}'_k \) \((k \in \mathbb{Z}) \). Then, by Corollary 6.22 the decomposition of \( m \) into the rails \( m \cap \mathcal{R}_k \) of \( \mathcal{R}_k \) is \( \delta \)-close to that into the rails \( m \cap f^{-1}(\mathcal{A}_k) \) of \( \mathcal{R}'_k \). Since this decomposition holds for all rails \( m \) of \( \tilde{T}_\infty \) and the deformation preserves \( m \), if \( \mathcal{R} \) and \( \mathcal{Q} \) are different branches of \( \tilde{T}_\infty \), their deformations \( \mathcal{R}' \) and \( \mathcal{Q}' \) given by Corollary 6.22 have disjoint interiors. Thus we have a deformation of \( \tilde{T}_\infty = \cup \mathcal{R}_j \), preserving \( \tilde{T}_\infty \) as a subset of \( \hat{\mathcal{C}} \), such that each branch is a quadrangle supported on a round cylinder in \( \hat{\mathcal{C}} \). Since the deformation of each branch commutes with the action of \( \pi_1(S) \), this deformation of \( \tilde{T}_\infty \) descends to a desired deformation of \( \mathcal{T}_\infty \). We have thus completed the proof of Proposition 6.17.

**6.7. Traintrack on \( C_i \) for Proposition 6.11 (ii).** Similarly, we pull back the traintrack \( T_i \) on \( \tau_i \) to a traintrack on \( C_i \) via \( \kappa_i : C_i \to \tau_i \). Let \( \tilde{T}_i = \kappa_i^{-1}(T_i) \) and, for each \( j = 1, 2, \ldots, n \), let \( \mathcal{R}_{i,j} = \kappa_i^{-1}(\mathcal{R}_{i,j}) \). Then, similarly to the case of \( \mathcal{T}_\infty \), we see that \( \tilde{T}_i = \cup_{j=1}^n \mathcal{R}_{i,j} \) is a topological traintrack on \( C_i \) and the canonical geodesic lamination \( \nu_i \) on \( C_i \), which corresponds to \( \lambda_i \) via \( \kappa_i : C_i \to \tau_i \), fully embeds into \( \tilde{T}_i \). Let \( \tilde{T}_i \) be the total lift of \( T_i \) to \( C_i \). In Proposition 6.17 we have given an appropriate deformation of \( \mathcal{T}_\infty \) that yields the traintrack \( \phi(T) \) on \( C \) in Proposition 6.10 (II). We next construct an analogous deformation of \( \tilde{T}_i \) to complete the proof of Proposition 6.10 (II):

**Proposition 6.23.** For sufficiently large \( i \in \mathbb{N} \), there is a small deformation of topological traintrack \( \tilde{T}_i = \cup_{j=1}^n \mathcal{R}_{i,j} \) such that
(i) this perturbation preserves $\mathcal{T}_i$ as a subset of $C_i$, and
(ii) letting $\mathcal{R}_i$ and $\mathcal{R}$ be the corresponding branches of $\mathcal{T}_i$ and $\mathcal{T}_\infty$, respectively, then $\mathcal{R}_i$ is a rectangle supported on a round cylinder on $\hat{C}$ whose support is the support of $\mathcal{R}$.

**Proof.** (Recall that the proof of Proposition 6.17 is done by observing the bending map $\beta: \mathbb{H}^2 \to \mathbb{H}^3$. We have proved that $\beta_i: \mathbb{H}^2 \to \mathbb{H}^3$ converges to $\beta$ as $i \to \infty$ (Theorem 5.4). Thus we will prove Proposition 6.23 by imitating the proof of Proposition 6.17 for sufficiently large $i$.)

Let $\hat{C}_i$ be the universal cover of $C_i$. Then let $\mathcal{R}_i$ be a branch of $\hat{T}_i$, and let $\mathcal{R}_i$ be the branch of $\hat{T}_i$ corresponding to $\mathcal{R}_i$ via $\hat{\kappa}_i: \hat{C}_i \to \mathbb{H}^3$.

Accordingly let $\mathcal{R}$ be the branch of $\mathcal{T}_\infty$ corresponding to $\mathcal{R}_i$, and let $R$ be the branch of $\hat{T}_i$ corresponding to $\mathcal{R}$. Thus let $\mathcal{A}$ be the round cylinder supporting the quadrangle $\mathcal{R} \subset \hat{C}$. Then, recall that $\mathcal{A}$ is bounded by the round circles $c_1$ and $c_2$ corresponding to the switch points or marked points on $\partial R$, obtained by Lemma 6.18.

Let $\check{\nu}_i$ be the total lift of $\nu_i$ to $\hat{C}_i$. Then the leaves of $\nu_i$ are rails of $T_i$. Since $T_i$ is $\epsilon$-straight with sufficiently small $\epsilon > 0$, every rail $l_i$ of $T_i$ has curvature at most $\epsilon$. Thus, for every $\delta > 0$, by taking sufficiently small $\epsilon > 0$, the smooth curve $\beta_i|l_i$ is $(1 + \delta)$-bilipschitz for all rails $l_i$ of $\hat{T}_i(= \hat{T}_1(\epsilon))$. Then we have an analogue of Proposition 6.21 for $\hat{T}_i$.

**Proposition 6.24.** For every $\delta > 0$, if $i \in \mathbb{N}$ is sufficiently large and $\epsilon > 0$ is sufficiently small, then, for every rail $m_i$ of $\mathcal{T}_i$ passing through a branch $\mathcal{R}_i$, letting $l_i$ be the corresponding rail of $\mathcal{T}_i$ (passing through $R_i$), we have

(i) $f_i|m_i$ is a simple arc on $\hat{C}$ connecting the end points of the bilipschitz curve $\beta_i(l_i) \subset \mathbb{H}^3$ and it intersects $c_1$ and $c_2$ transversally in a single point; thus $f_i^{-1}(\mathcal{A}) \cap m_i$ is a single arc in $m_i$.

(ii) $f_i^{-1}(\mathcal{A}) \cap m_i$ is $\delta$-close to the rail $\mathcal{R}_i \cap m_i$ of $\mathcal{R}_i$ with the Hausdorff metric, and

(iii) if an endpoint of $\mathcal{R}_i \cap m_i$ is a switch point of $\hat{T}_i$, then the $\delta$-deformation of $\mathcal{R}_i \cap m_i$ to $f_i^{-1}(\mathcal{A}) \cap m_i$, given by (ii), preserves this switch point.

**Proof.** (Case 1.) Let $\mathcal{R}_i$ be a branch of $\hat{T}_i$. For every leaf $m_i$ of $\check{\nu}_i$ passing through $\mathcal{R}_i$, let $l_i$ be the leaf of $\check{\lambda}_i$ corresponding to $m_i$. Then $l_i$ intersects the branch $R_i$ of $\hat{T}_i$ corresponding to $\mathcal{R}_i$. Then $f|m_i$ is a circular arc on $\hat{C}$ connecting the endpoints of $\beta_i(l_i)$. By Lemma 6.15, for every $\delta > 0$, if $i \in \mathbb{N}$ is sufficiently large and $\epsilon > 0$ is sufficiently small, then, if $l_i$ and $l$ are rails of $\hat{T}$ an $\hat{T}$ passing through corresponding branches $R_i$ and $R$ of $\hat{T}_i$ and $\hat{T}$, respectively, then $\beta_i(l_i \cap R_i)$ and $\beta(l \cap R)$
are geodesic segments, in $\mathbb{H}^3$, of length at least $K$ that are $\delta$-close with the Hausdorff metric. We have seen that, for every $\delta > 0$, if $\epsilon > 0$ is sufficiently small, then for every leaf $m$ of $\tilde{\lambda}_i$, $\beta(l)$ intersects $\text{Conv}(c_1)$ and $\text{Conv}(c_2)$ at an angle $\delta$-close to $\pi/2$, and $\beta^{-1}(\text{Conv}(A)) \cap m$ is $\delta$-close to $m \cap R$ (Lemma 6.18 and Proposition 6.21). Since $\beta(l \cap R)$ and $\beta_i(l_i \cap R_i)$ are sufficiently close, we have

Claim 6.25. For every $\delta > 0$, if $\epsilon > 0$ is sufficiently small then, for every leaf $m_i$ of $\tilde{\lambda}_i$ passing through $R_i$, the geodesic $\beta_i(l_i)$ intersects $\text{Conv}(c_1)$ and $\text{Conv}(c_2)$ $\delta$-orthogonally, and $\beta^{-1}(\text{Conv}(A)) \cap m_i$ is $\delta$-close to $m_i \cap R_i$ in the Hausdorff metric.

Therefore, similarly to the proof of Proposition 6.21 (Case 1), we can prove that, if $i$ is sufficiently large, then, every leaf $m_i$ of $\tilde{\lambda}_i$ passing through a branch $R_i$ satisfies (i) and (ii). Since $m_i$ is a leaf of $\tilde{\nu}_i$, $\tilde{\nu}_i$ is contained in the interior of $\tilde{T}_i$, endpoints of the rail $m_i \cap R_i$ can not be a switch point of $\tilde{T}_i$.

(Case 2.) Suppose that $m_i$ is a rail of $\tilde{T}_i$ intersecting a branch $R_i$ of $\tilde{T}_i$. Then let $l_i$ be the corresponding rail of $\tilde{T}_i$ intersecting $R_i$. Pick a leaf $l'_i$ of $\tilde{\lambda}_i$ passing through $R_i$ that is closest to $l_i$ in $R_i$. Then there is a component $X_i$ of $\mathbb{H}^2 \setminus \tilde{\lambda}_i$ bounded by $l'_i$ and containing $l_i$. Then $\beta_i$ isometrically embeds $X_i$ into a totally geodesic hyperplane in $\mathbb{H}^3$. For every $\delta > 0$, if $\tilde{T}_i$ is $\epsilon$-slim with sufficiently small $\epsilon > 0$, then $l_i \cap R_i$ and $l'_i \cap R_i$ are $\delta$-close for all $R_i$ and all corresponding $l_i$ and $l'_i$ as above. Therefore, using Claim 6.25 and imitating the proof of Proposition 6.21 (Case 2), we can show that, every rail $m_i$ of $\tilde{T}_i$ passing through a branch $R_i$ of $\tilde{T}_i$ satisfies (i) and (ii).

Last we show (iii). Suppose that $m_i$ passes through a switch point $p_i$ on $\partial R_i$. Then let $p$ be the corresponding switch point on $\partial R$. Then $f(p)$ is on the boundary component of $A$ corresponding to $\tilde{\kappa}(p)$ (see Lemma 6.18). In addition, by Proposition 6.11 (iii), $f_i(p_i) = f(p)$. Therefore the switch point $p_i$ is preserved under the $\delta$-deformation of $m_i \cap R_i$ to $m_i \cap f_i^{-1}(A)$.

Fix small $\delta > 0$. Then assume that $i \in \mathbb{N}$ is sufficiently large so that the conclusions of Proposition 6.24 are satisfied for this $\delta$. Let $R_i$ be a branch of $\tilde{T}_i$, and let $\mathcal{R}$ is the corresponding branch of $\mathcal{T}_\infty$. Then $\mathcal{R} \subset C$ is a quadrangle supported on the round cylinder $A$. We set $\mathcal{R}_i = \cup(m_i \cap \mathcal{R}_i)$, where the union runs over all rail $m_i$ of $\tilde{T}_i$ passing through $\mathcal{R}_i$. Then let $\alpha_m = m_i \cap \mathcal{R}_i$ and let $\alpha'_m = m_i \cap f^{-1}(A)$. Then, by Proposition 6.24, $\alpha'_m$ and $\alpha_m$ are segments in $m_i$ that are $\delta$-close, and $\alpha'_m$ is supported on $A$. Therefore, similarly to Corollary 6.22 $\mathcal{R}_i'$ is

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a quadrangle supported on the round cylinder $A$. By Proposition 6.11 (iii), $\partial T_\infty$ and $\partial T_i$ are isomorphic via the developing maps. Therefore, we see that $R'_i$ and $R$ have the same pair of supporting arcs as well. By applying this $\delta$-deformation to all branches of $\tilde{T}_i$ simultaneously and descending to $T_i$, we obtain a desired deformation.

6.8. **Estimate of multiarcs for grafting by transversal measure.**

We have proved Proposition 6.10 except (II - ii) and (II - iii), which we prove in this section. Recalling Proposition 6.11, let $B$ be a branch of the traintrack $T$ on $S$. Set $R_i = \phi_i(B)$ and $R = \phi(B)$, which are the corresponding branches of $\tilde{T}_i$ and $\tilde{T}_\infty$, respectively. Let $R_i$ and $R$ denote the branches of $\tilde{T}_i$ and $\tilde{T}_\infty$ corresponding to $R_i$ and $R$, respectively, obtained by Proposition 6.17 and Proposition 6.23. Then, $R_i$ and $R$ are projective structures on a quadrangle with the same support. Thus $R_i$ and $R$ are equipped with the canonical foliations induced by the canonical foliation $F_A$ on $A$, and we assume that the ties of $R_i$ and $R$ are the leaves of those foliations by perturbing ties.

By Lemma 6.5, we have either $R = G_{\mathcal{M}}(R_i)$ or $R_i = G_{\mathcal{M}}(R)$ for some admissible multiarc $\mathcal{M}$ (II - ii). We are left to show

**Proposition 6.26.** For every $\delta > 0$, if $i$ is sufficiently large and $\epsilon > 0$ is sufficiently small, then, for all corresponding branches $R_i$ and $R$ of $\tilde{T}_i$ and $\tilde{T}_\infty$, we have $R_i = G_{\mathcal{M}}(R)$ for some admissible multiarc $\mathcal{M}$ on $\mathcal{R}$, and $(\bar{\mu}(R_i) - \bar{\mu}(R))/2\pi$ is $\delta$-close to the number of arcs of $\mathcal{M}$.

The rest of this section is the proof of this proposition. Pick a leaf $h$ of $F_A$, which is a round circle on $\hat{C}$. Since we have assumed that the ties of $R$ are induced by $F_A$, there is a unique tie $a$ of $R$ such that $f$ immerses $a$ into $h$. Set $\alpha = f|_a$. Similarly let $a_i$ be the tie of $R_i$ such that $f_i$ immerses $a_i$ into $h$. Set $\alpha_i = f_i|_a$.

As in §6.1, let $g$ be the geodesic in $\mathbb{H}^3$ orthogonal to $A$. Then, in particular, $g$ is orthogonal to $h$. We identify $\hat{C}$ with $S^2$, by an element of $\text{PSL}(2, \mathbb{C})$, so that $h \subset \hat{C}$ is the equator of $S^2$ and $g$ connects the north pole and the south pole of $S^2$. We assume that $S^2$ is the unit sphere, and then the spherical metric on $S^2$ induces a metric on $h$, so that $\text{length}_h(h) = 2\pi$. Then $\text{length}_h(\alpha)$ and $\text{length}_h(\alpha_i)$ denote the lengths of $\alpha$ and $\alpha_i$, respectively, with respect to this metric on $h$.

Recall that, given any $\epsilon > 0$, we can assume that $T$ and $T_i$, with sufficiently large $i$, are $\epsilon$-straight and slim (Proposition 6.11). Then

**Proposition 6.27.** For every $\delta > 0$, if $\epsilon > 0$ is sufficiently small, then we have
(i) \(|\text{length}_h(\alpha) - \tilde{\mu}(R)| < \delta \) for every branch \( R \) of \( \tilde{T}_\infty \) and every
tie \( a \) of the branch \( R \) of \( \tilde{T}_\infty \) corresponding to \( R \), and
(ii) if \( i \in \mathbb{N} \) is sufficiently large, we have, \(|\text{length}_h(\alpha_i) - \tilde{\mu}_i(R_i)| < \delta \)
for the branch \( R_i \) of \( \tilde{T}_i \) corresponding to \( R \).

Proof of Proposition 6.26 with Proposition 6.27 assumed. By Lemma
6.16, if \( i \) is sufficiently large, then we can assume that \( \tilde{\mu}_i(R_i) - \tilde{\mu}(R) > 0 \)
for all corresponding branches \( R_i \) and \( R \) of \( \tilde{T}_i \) and \( \tilde{T}_\infty \), respectively.
Pick corresponding ties \( a \) and \( a_i \) of \( R \) and \( R_i \), respectively, that map
to the same round circle \( \hat{h} \) on \( \hat{C} \) via the developing maps \( f \) and \( f_i \).
Then, by Proposition 6.27, for every \( \delta > 0 \), if \( \epsilon > 0 \) and sufficiently
small and \( i \in \mathbb{N} \) is sufficiently large, then \( \text{length}_h(\alpha_i) \) is \( \delta \)-close to
\( \mu_i(R_i) \) and \( \text{length}_h(\alpha) \) is \( \delta \)-close to \( \tilde{\mu}(R) \) for all corresponding \( R \) and
\( R_i \). Then, since \( R \) and \( R_i \) have the same support (Proposition 6.23
(ii)), \( \text{length}_h(\alpha_i) - \text{length}_h(\alpha) \) is a 2\( \pi \)-multiple. By Lemma 6.5 \( i \) we have \( R_i = Gr_M(R) \) for some admissible multiarc \( M \) on \( R \). By
the definition of grafting, we see that the number of arcs of \( M \) is exactly
\(|\text{length}_h(\alpha_i) - \text{length}_h(\alpha)|/2\pi \). Therefore \(|\tilde{\mu}_i(R_i) - \tilde{\mu}(R)|/2\pi \) is \( 2\delta \)-
close to the number of arcs of \( M \).

Proof of Proposition 6.27. We prove (i). The prove of (ii) is similar,
since we have a convergence \( \beta_i \to \beta \) (Theorem 5.4). Throughout this
proof, \( \delta > 0 \) is a sufficiently small number, which may be appropriately replaced by a smaller number by taking smaller \( \epsilon > 0 \), where \( T_\infty \) is \( \epsilon \)-
straight and slim; however \( \delta \) does not depend on the choice of the
branch \( R \) of \( T_\infty \).

Recall that we have a measured geodesic lamination \( L = (\lambda, \mu) \) on
\( \tau \) and that the geodesic lamination \( \lambda \) is a sublamination of \( \lambda_\infty \). Set
\( L_\infty = (\lambda_\infty, \mu) \), which fully embeds into \( \tilde{T}_\infty \) (Proposition 6.11). For
an arbitrary branch \( R \) of \( T_\infty \), let \( I \in \mathcal{ML}(\mathbb{H}^2) \) be the intersection
\( I(\tilde{L}, R) \) of \( \tilde{L} \) and \( R \) (see \$3.3). Set \( I = (\lambda_I, \mu_I) \), where \( \lambda_I \in \mathcal{G}\mathcal{L}(\mathbb{H}^2) \)
and \( \mu_I = \tilde{\mu}|\lambda_I \). Accordingly, let \( \beta_I : \mathbb{H}^2 \to \mathbb{H}^3 \) denote the bending map
induced by \( I \). The total measure of \( I \) is \( \tilde{\mu}(R) < \infty \), and thus, for every
geodesic segment \( s \) on \( \mathbb{H}^2 \) transversal to \( I \), we have \( \mu_I(s) \leq \tilde{\mu}(R) \in \mathbb{R}_{\geq 0} \).
Thus \( \beta_I : \mathbb{H}^2 \to \mathbb{H}^3 \) continuously extends to \( \partial \mathbb{H}^2 \cong \mathbb{S}^1 \).

Since each component of \( \mathbb{H}^2 \setminus \lambda_I \) is bounded by at most 2 leaves of
\( \lambda_I \), we can extend the geodesic lamination \( \lambda_I \) to a geodesic foliation
\( F_I \) on \( \mathbb{H}^2 \) (so that the dual tree of \( F_I \) is a copy of \( \mathbb{R} \)). Furthermore,
since \( T_\infty \) is \( \epsilon \)-slim, we can assume that, for every \( \delta > 0 \), if \( 0 < \epsilon < \delta \)
is sufficiently small, then if a leaf of \( F_I \) intersects a tie of \( R \) at some
point, then the intersection angle is \( \delta \)-close to \( \pi/2 \).
Let \( C_I \) denote the projective structure on the open disk \( \mathbb{D}^2 \) associated with the measured lamination \( I \) on \( \mathbb{H}^2 \). Since \( R \) is connected and \( I \) is a sublamination of \( \check{\partial} \), then \( C_I \) canonically embeds into \( \check{\mathbb{C}} \) (see [2]). In particular, since \( I = \check{\partial} \) on \( R \), then \( R \) canonically embeds into \( C_I \). Let \( f_I: \mathbb{D}^2 \rightarrow \check{\mathbb{C}} \) denote the developing map of \( C_I \), and let \( \mathcal{F}_I \) denote the canonical foliation on \( C_I \) corresponding to \( F_I \) via the collapsing map \( \kappa_I: C_I \rightarrow \mathbb{H}^2 \). Then the dual tree of \( \mathcal{F}_I \) is also a copy of \( \mathbb{R} \). Since \( C_I \) is Gromov-hyperbolic with Thurston’s metric, \( \partial C_I \cong \mathbb{S}^1 \). We then pick a continuous projection \( \Phi: C_I \rightarrow \partial C_I \) along leaves of \( \mathcal{F}_I \), i.e. for each point \( x \in C_I \), \( \Phi(x) \) is an endpoint of the leaf of \( \mathcal{F}_I \) containing \( x \). Since \( T_\infty \) is \( \epsilon \)-straight, each tie \( a \) of \( R \) is transversal to the foliation \( \mathcal{F}_I \). Thus \( \Phi \) projects \( a \) into a simple arc in \( \partial C_I \) (note that, if \( a \) intersects a Euclidean part of \( C_I \), then \( \Phi|a \) is not injective). Let \( b \) denote the simple arc \( \Phi(a) \).

Since \( \beta_I: \mathbb{H}^2 \rightarrow \mathbb{H}^3 \) extends to \( \partial \mathbb{H}^2 \), the collapsing map \( \kappa_I: C_I \rightarrow \mathbb{H}^2 \) continuously extends to a homeomorphism from \( \partial C_I \) to \( \partial \mathbb{H}^2 \), so that \( f_I = \beta_I \circ \kappa_I \) on \( \partial C_I \). In particular, \( \kappa_I \) homeomorphically takes \( b \) onto its image \( \kappa_I(b) \subset \partial \mathbb{H}^2 \). Then let \( \beta_b: b \rightarrow \check{\mathbb{C}} \) denote the restriction of \( \beta_I \) to \( b \).

The transversal measure \( \mu_I \) of \( I \) is defined for arcs in \( \mathbb{H}^2 \) transversal to \( I \). Since the total measure of \( \mu_I \) is finite, \( \mu_I \) continuously extends to arcs contained in \( \partial \mathbb{H}^2 \). Thus \( I \) continuously extends to a measured lamination on \( \partial \mathbb{H}^2 \) (whose leaves have dimension 0), and thus, in particular to \( b \).

Recall that \( h \) is a leaf of the foliation \( \mathcal{F}_A \) on the round cylinder \( A \subset \check{\mathbb{C}} \) supporting \( R \subset \check{\mathbb{C}} \) and that the geodesic \( g \subset \mathbb{H}^3 \) is orthogonal to \( A \). By Lemma 6.18 for every \( \delta > 0 \), if \( \epsilon > 0 \) is sufficiently small, then for all leaves \( l \) of \( F_I \) intersecting \( R \), the geodesic segments \( \beta_I(l \cap R) \) and \( g \cap \text{Conv}(A) \) are \( \delta \)-close with the Hausdorff metric. Thus, assuming \( \epsilon > 0 \) is sufficiently small, \( \text{Im}(\beta_h) \) is contained in a small neighborhood of an ideal endpoint \( O \) of \( g \). In particular, \( \text{Im}(\beta_h) \) is contained in a component \( D \) of \( \check{\mathbb{C}} \setminus h \), which is a round disk. In the following, \( D \) is equipped either with the spherical metric, regarded as a upper hemisphere (so that \( O \) is the north pole and \( h \) is the equator), or with the hyperbolic metric, regarded as the Poincare disk. Then, since \( \beta_h: b \rightarrow \check{\mathbb{C}} \) is the boundary of the bending map \( \beta_I: \mathbb{H}^2 \rightarrow \mathbb{H}^3 \), we see that

**Lemma 6.28.** The curve \( \beta_h: b \rightarrow D \) is smooth except at the end points of leaves of \( I \), and \( b \) is bent at those endpoints by angles corresponding to \( \mu_I \). In addition, for every \( \delta > 0 \), if \( \epsilon > 0 \) is sufficiently small, then
the curvature of $\beta_3$ is less than $\delta > 0$ at every smooth point (with either metric on $D$).

By the definition of $\Phi$: $C_I \rightarrow \partial C_I$, for each $x \in a \subset C_I$, there is a unique ray contained in a leaf $l$ of $\mathcal{F}_I$ that connects $x$ to $\Phi(x)$. Then $f_I$ homeomorphically takes this ray to a circular arc embedded in $\hat{C}$ that connects the point $f_I(x)$ to the point $f_I(\Phi(x)) = \beta_3(\Phi(x))$. Let $r_x: [0,1] \rightarrow D \subset \mathbb{S}^2$ denote this circular arc with $r_x(0) = f_I(\Phi(x))$ and $r_x(1) = f_I(x)$. Then, for every $\delta > 0$, if $\epsilon > 0$ is sufficiently small, then $r_x$ intersects the round circle $h$, $\delta$-orthogonally, at the end point $f_I(x)$, and $\gamma_x(0)$ is $\delta$-close to the center $O$.

There is a unique maximal ball in $C_I$ whose core contains $x$. By the definition of a maximal ball, its $f_I$-image is a round open ball in $\hat{C}$, which we denote by $D_x$. Then, $\partial D_x$ bounds a totally geodesic hyperplane in $\mathbb{H}^3$, and the circular arc $f_I(l) \subset \hat{C}$ orthogonally projects to the geodesic $\beta_3 \circ \kappa_I(l)$ contained in this hyperplane. Thus $r_x \subset f_I(l)$ orthogonally intersects $\partial D_x$ at the endpoint $r_x(0)$. For every $\delta > 0$, if $\epsilon > 0$ is sufficiently small, then, since $\kappa_I(l)$ is a leaf of $F_I$ intersecting $R$, the geodesic $\beta_3(\kappa_I(l))$ is $\delta$-orthogonal to the totally geodesic hyperplane $\operatorname{Conv}(h)$ in $\mathbb{H}^3$ (Lemma 6.18 (iii)). Therefore we can in addition assume that the curvature of $r_x: [0,1] \rightarrow D \subset \mathbb{S}^2$ is less than $\delta$ with respect to the spherical metric on $D$.

Let $r_x: [0,1] \rightarrow D$ be the hyperbolic geodesic ray in $D \cong \mathbb{H}^2$ connecting the endpoints of $r_x$. Then, since $r_x(0)$ is $\delta$-close to the center $O$ of $D$, for every $\delta > 0$, if $\epsilon > 0$ is sufficiently small, then $r_x$ and $r_x$ are $\delta$-close (with either metric on $D$) for all $x \in a$.

**Lemma 6.29.** For every $x \in a$, there exists a small neighborhood $U_x$ of $x$ in $a$, such that if $y \in U_x$, then $r_x|(0,1]$ and $r_y|(0,1]$ are disjoint.

**Proof.** Let $x \in a$. Assume that $y \in a$ is sufficiently close to $x$. If $r_x(0) = r_y(0)$, then we have $r_x|(0,1] \cap r_y|(0,1] = \emptyset$ since $f_I(x) \neq f_I(y) \in \partial D$. Thus we can assume that $r_x(0) \neq r_y(0)$. Then, since $\beta_3$ is a further restriction of $\beta_3|\partial \mathbb{H}^2$ and $x$ and $y$ are sufficiently close, the round circles $\partial D_x$ and $\partial D_y$ on $\hat{C}$ intersects at an angle sufficiently close to $0$. Let $l$ be the geodesic in $D \cong \mathbb{H}^2$ connecting $r_x(0)$ and $r_y(0)$. Then, since $r_x$ and $r_y$ are orthogonal to $\partial D_x$ and $\partial D_y$, respectively, they are disjoint circular arcs contained in the closure of a component of $D \setminus l$. Since $r_x$ and $r_y$ share endpoints with $r_x$ and $r_y$, respectively, $r_x$ and $r_y$ are disjoint geodesics rays in the closure of the component of $D \setminus l$. □

For each $x \in a$, let $\gamma_x: [0,1] \rightarrow D$ denote the circular arc on $D \subset \mathbb{S}^2$ connecting the center $O$ to $r_x(1) = f_I(x)$. Then $\gamma_x$ intersects $h = \partial D$ orthogonally at $\gamma_x(1)$. Identify $a$ with the closed interval $[0, \operatorname{length}_h(a)]$
so that \( f_1|a : a \rightarrow h \cong S^1 \) is an isometric immersion. Then, as \( x \in a \cong [0, \text{length}_1(a)] \) increases, the circular arc \( \gamma_x : [0, 1] \rightarrow D \) \((x \in [0, 1])\) continuously rotates in one direction by the rotation of \( D \) fixing the center \( O = \gamma_x(0) \). Define \( \gamma : a \times [0, 1] \rightarrow D \) by \( \gamma(x, t) = \gamma_x(t) \). Then \( \gamma(x, 1) = f_1(x) \) for all \( x \in a \). Thus \( \gamma|a \times [0, 1] \) is an immersion, and, more precisely, \( \gamma \) is a fan immersed into \( D \), where the vertex of the fan is \( O \) and the angle of the fan (at the vertex) is \( \text{length}_h(\alpha) \). Let \( \hat{F} \cong a \times [0, 1] \) denote the domain of \( \gamma \) and equip \( \hat{F} \) with the pull back metric of the spherical metric on \( D \) via \( \gamma \). Then we have

\[
\text{Area}(\hat{F}) = \text{Area}_{S^2}(D) \cdot \left( \text{length}_h(\alpha) / 2\pi \right) = \text{length}_h(\alpha).
\]

Similarly define \( r : a \times [0, 1] \rightarrow D \) by \( r(x, t) = r_x(t) \). Then, by Lemma 6.29, \( r|a \times (0, 1] \) is an immersion. Clearly \( r|\{x \in \{0\} \rightarrow D = \beta_b \circ \Phi: a \rightarrow D \) and \( r|a \times \{1\} = \alpha: a \rightarrow \partial D \). Let \( F \) be \( \text{Domain}(r) = a \times [0, 1] \) with the pull back metric of the spherical metric \( D \) via \( r \). Then the four boundary edges of \( F \) correspond to \( \alpha \), \( \beta_b \), \( r_0 \), and \( r_1 \). Applying the Gauss-Bonnet theorem to \( F \), we have

\[
\text{Area}(F) + \int_{\partial F} k ds + \Sigma \theta_p = 2\pi \cdot \chi(F),
\]

where \( k \) is the curvature along \( \partial F \) and \( \theta_p \) are the exterior angles at non-smooth points \( p \) of \( \partial F \), including infinitedesimal bending angles of \( \beta_b \) corresponding to \( \mu_\ell \) (Lemma 6.28). Then, for every \( \delta > 0 \), if \( \epsilon > 0 \) is sufficiently small, then the third term \( \Sigma \theta_p \) is \( \delta \)-close to \(-\mu(R) + 2\pi \): It is “\( \delta \)-close” since \( r_0 \) and \( r_1 \) are almost orthogonal to \( \partial D_0 \) and \( \partial D_{\text{length}(a)} \), respectively. Next consider the second term \( \int_{\partial F} k ds = \int_{\beta_b \cup r_0 \cup r_1 \cup a} k ds \).

Since \( \partial D \) is a geodesic loop on \( S^2 \), \( \int_{\partial F} k ds = 0 \). Since \( r_x \) is sufficiently close to \( r_x \) for each \( x \in a \), we can assume that, with respect the spherical metric on \( D \subset S^2 \), \( r_0 \) and \( r_1 \) have total curvature less than \( \delta \). Since \( T_\infty \) is \( \epsilon \)-slim and \( \text{Im}(\beta_b) \) is contained in a \( \delta \)-neighborhood of \( O \), \( \beta_b : b \rightarrow a \) is sufficiently short and it has sufficiently small curvature at smooth points. Therefore, for every \( \delta > 0 \), if \( \epsilon > 0 \) is sufficiently small, then \( \left| \int_{\partial F} k ds \right| < \delta \). Since \( F \) is topologically a disk, \( \chi(F) = 1 \). Thus we have

\textbf{Lemma 6.30.} For every \( \delta > 0 \), if \( \epsilon > 0 \) is sufficiently small, then,

\[
-\delta < \text{Area}(F) - \mu(R) < \delta.
\]

Next we have

\textbf{Lemma 6.31.} (i) For every subinterval \( d \) of \([0, \text{length}_h(\alpha)] \cong a \) of length less than \( 2\pi \), \( \gamma \) embeds \( d \times (0, 1] \) into \( D \). (ii) For every \( \delta > 0 \), if \( \epsilon > 0 \) sufficiently small, then \( r \) embeds \( d \times (0, 1] \) into \( D \) for every subinterval \( d \) of \([0, \text{length}_{S^2}(\alpha)] \cong a \) of length less than \( \pi - \delta \).
Proof. (i) If $d$ is a subinterval of $[0, \text{length}_{h_1}(a)]$, the restriction of $\gamma$ to $d \times [0, 1]$ is a fan centered at $O$ such that $\text{length}(d)$ is equal to the length of the circular arc of the fan. Thus, if $d < 2\pi$, then $\gamma: d \times (0, 1]$ is an embedding. (ii) Let $I = (\lambda_I, \mu_I)$ be the canonical measured lamination on $C_I$ corresponding to $I$ via the collapsing map $\kappa_I: C_I \to \mathbb{H}^2$. Since $\beta_b$ is the restriction on the (extended) bending map $\beta$ and it is a short curve contained in a small neighborhood of the center $O$, for every $\delta > 0$, if $\epsilon > 0$ is sufficiently small, then $\mu_I(d)$ is $\delta$-close to $\text{length}_h(f|d)$ for every subinterval $d$ of $a$ with bounded length (this “bounded” assumption is more important for Proposition 6.27 (ii)). In particular, assuming $\epsilon > 0$ is sufficiently small, if $\text{length}(d) < \pi - \delta$, then $\mu_I(d) < \pi - \delta/2 < \pi$. Then, for such $d$, $\beta_b|\Psi(d)$ is injective by the Gauss-Bonnet theorem, since $\beta_b$ has sufficiently small curvature at smooth points and $\beta_b|\Psi(d)$ is sufficiently short. Thus, since $r_a$ are $\delta$-orthogonal to $\beta_b$ at each $x \in a$, we see that $r_a|[(0, 1) \times (x \in a)]$ are pairwise disjoint. 

By [4] and Lemma 6.30, to prove Proposition 6.27 (i), it suffices to show

**Proposition 6.32.** For every $\delta > 0$, if $\epsilon > 0$ is sufficiently small, then

$$-\delta < \text{Area}(F) - \text{Area}(\hat{F}) < \delta$$

for every branch $R$ of $\bar{T}_\infty$ and every tie $a$ of the branch $\mathcal{R}$ of $\bar{T}_\infty$ corresponding to $R$.

Proof. Fix sufficiently small $\delta > 0$. For each $y \in b$, let $q_y: [0, 1] \to D$ be the geodesic segment connecting $O$ to $\beta_b(y)$. Since $\text{Im}(\beta_b)$ is contained in a sufficiently small neighborhood of $O$, we can assume that $\text{length}_{\mathbb{S}^2}(q_y)$ is less than $\delta$ for all $y \in b$. Define $q: b \times [0, 1] \to D$ by $q(y, t) = q_y(t)$. Let $Q$ be $b \times [0, 1]$ equipped with the 2-form obtained by pulling back the spherical Riemannian metric of $D$ via $q$. Since $\beta_b: b \to D \subset \mathbb{S}^2$ is locally injective, we assign a metric on $b$ by the arc length of $\beta_b$. Then $0 \leq \text{Area}(Q) < \int_{y \in b} \text{length}(q_y) dy$. For every $\delta > 0$, if $\epsilon > 0$ is sufficiently small, then $\text{Area}(Q) < \delta$, since $\text{length}(q_y)$ and $\text{length}(\beta_b)$ are sufficiently small.

Let $v_0$ and $v_1$ denote the endpoints of $a \cong [0, \text{length}_h(\alpha)]$ corresponding to 0 and $\text{length}(\alpha)$, respectively. For each $k = 0, 1$, let $E_k$ be the region in $D$ bounded by $r_{v_k}, \gamma_{v_k}$ and $q_{v_k}$. Since $q_{v_k}$ has length less than $\delta$, by taking smaller $\epsilon > 0$ if necessary, we can assume that $\text{Area}_{\mathbb{S}^2}(E_{v_k})$ ($k = 0, 1$) are less than $\delta$. Thus, to prove Proposition 6.32 (and therefore Proposition 6.27), it suffices to show:

**Claim 6.33.** $|\text{Area}(F) - \text{Area}(\hat{F})| < \text{Area}(Q) + \text{Area}(E_0) + \text{Area}(E_1)$. 

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Proof. We first show
\[ \text{Area}(\hat{F}) < \text{Area}(Q) + \text{Area}(E_0) + \text{Area}(E_1) + \text{Area}(F). \]

For this, it suffices to construct an almost-everywhere-defined embedding \( \iota: \hat{F} \to F \sqcup Q \sqcup E_0 \sqcup E_1 \) that is a piecewise isometry.

Observe that there is a unique continuous function \( \xi: a \to a \) such that, for \( x \in a \),
- if \( r_x \subset \gamma_x \) or \( r_x \supset \gamma_x \), then \( \xi(x) = x \),
- otherwise, letting \( d_x \) be the closed subinterval of \( a \) with the endpoints \( x \) and \( \xi(x) \), \( d_x \) is the largest subinterval containing \( x \) and satisfying \( r_x \cap \gamma_y \neq \emptyset \) for all \( y \in d_x \).

Note that, if \( \xi(x) = x \), we have \( d_x = \{ x \} \). Then, more precisely, each \( x \in a \) is of one of the following three types:

(i) \( r_y \) with \( y \in d_x \) contains \( O \) if and only if \( y = \xi(x) \) (Figure 6),
(ii) the point \( r_y(0) = \beta_h \circ \Phi(y) \) with \( y \in d_x \) intersects \( \gamma_x \) if and only if \( y = \xi(x) \) (Figure 7), or
(iii) \( \xi(x) \) is an endpoint of \( a \) (Figure 8).

In particular, if \( r_x \subset \gamma_x \), then \( x \) is of Type (ii) and if \( \gamma_x \subset r_x \), then \( x \) is of Type (i). Then since \( \xi: a \to a \) is continuous, we have

**Lemma 6.34.** the function \( d \) from \( a \) to the set of closed subintervals of \( a \) is continuous.

We also have
Lemma 6.35. For all $x \in a$,

$$\text{length}(d_x) < \pi.$$  

Moreover, for every $\delta > 0$, if $\epsilon > 0$ is sufficiently small, then if $x \in a$ is of Type (ii), then

$$\text{length}(d_x) < \pi/2 + \delta.$$
Proof. For each \( x \in a \), let \( l_x \) be the geodesic \( l_x \) in \( D \cong \mathbb{H}^2 \) containing \( \gamma_x \). Then \( l_x \) divides \( D \) into two semicircles. For each Type of (i), (ii), (iii), we see that \( f_1 \) embeds \( d_x \) into one of those semicircles. Thus \( \text{length}(d_x) < \pi \).

Suppose that \( x \) is of Type (ii). Then, since \( \gamma_x \cap r_y \neq \emptyset \) for all \( y \in d_x \) and \( r_{\xi(x)} \) is \( \delta \)-orthogonal to \( \beta_b \) at the point \( r_{\xi(x)}(0) \in \gamma_x \), the angle between \( r_{\xi(x)} \) and \( \gamma_x \) is less than \( \pi/2 + \delta \) at \( r_{\xi(x)}(0) \). Since \( r_{\xi(x)}(0) \) is sufficiently close to \( x \), then

\[
\text{length}(d_x) < \pi/2 + \delta.
\]

Observe that the correspondence between points \( x \) in \( a \) and Types (i), (ii) (iii) changes only at finitely many points on \( a \) : The type may change between (i) and (ii) when \( \beta_b \) is tangent to \( \gamma_x \); Type (iii) happens exactly in a (possibly empty) subinterval of \( a \) starting at an end point of \( a \) and, at the other endpoint of this subinterval, the type changes from (iii) to (ii).

Suppose that \( x \in a \) is of Type (i). Then \( \gamma_x : [0, 1] \to D \) is embedded in \( r(d_x \times [0, 1]) \). Moreover, since \( \beta_b \circ \Phi|d_x \) is disjoint from \( \gamma_x \), we see that \( \gamma_x \) factors through \( r \) via an unique isometric embedding \( \iota_x \) of \( [0, 1] \) into \( d_x \times [0, 1] = \text{Domain}(r) = F \).

Suppose that \( x \in a \) is of Type (ii). Then the points \( \beta_b \circ \Phi \circ \xi(x) \) divides \( \gamma_x \) into two segment: One segment \( \gamma_x^- \) is \( q(\Phi \circ \xi(x), *) \) and the other segment \( \gamma_x^+ \) is contained in \( r(d_x \times [0, 1]) \). By Lemma 6.35, we can assume that \( d_x < 2\pi/3 \). Thus, by Lemma 6.31, \( r|d_x \times (0, 1) \) is an embedding. Thus, \( \gamma_x^+ \) factors through \( r \) via an isometric embedding of the domain of \( \gamma_x^+ \) into \( d_x \times [0, 1] \subset F \). Clearly \( \gamma_x^- \) factors through \( q \) via the identification of \( \gamma_x^- \) and \( q(\Phi \circ \xi(x), *) \).

Suppose that \( x \in a \) is of Type (iii). Then \( \gamma_x \) is a curve embedded in \( r(d_x \times [0, 1]) \cup E_k \) with \( k \in \{0, 1\} \) satisfying \( v_k = \xi(x) \). Thus, similarly, \( \gamma_x : [0, 1] \to D \) factors through an isometrically embedding of the domain of \( \gamma_x \) into \( d_x \times [0, 1] \cup E_k \) via \( r \), if we remove, from the domain of \( \gamma_x \), the single point corresponding to the intersection of \( \gamma_x \) and \( r_{\xi(x)} \). Thus, for all \( x \in a \), let \( \iota_x \) be the above isometric embedding of (the domain of) \( \gamma_x \) possibly minus a single point into \( F \cup E_0 \cup E_1 \cup Q \). Then we see that \( \iota_x \) changes continuously in \( x \in a \). Let \( X \) be the set of the points \( (x, t) \in \hat{F} \cong a \times [0, 1] \) that are removed from the domain of \( \gamma_x \) to define \( \iota_x \). Then, since \( \iota_x \) is defined \([0, 1]\) minus at most a single point for each \( x \in a \), \( X \) has measure zero in \( \hat{F} \). In addition, since the configuration type (i.e. (i), (ii), (iii)) changes at only finitely many \( x \in a \), \( X \) decompose \( \hat{F} \) into finitely many (connected) components.

We then define \( \iota : \hat{F} \setminus X \to F \cup E_0 \cup E_1 \cup Q \) by \( \iota|_{\gamma_x} = \iota_x \) for all \( x \in a \). Since \( \iota_x \) changes continuously in \( x \), then \( \iota \) is a continuous. Moreover \( \iota \) is an isometric immersion (up to measure zero), since the Riemannian
metrics on \( \hat{F}, F, E_0, E_1, Q \) are all induced by the spherical metric on \( D \subset S^2 \). Thus the only thing left to show is that \( \iota \) is almost everywhere injective:

**Claim 6.36.** For almost all different \( x, y \in a \), \( \iota_x \) and \( \iota_y \) have disjoint images, up to measure zero.

**Proof.** We can assume that \( f(x) = f(y) \): Otherwise \( \gamma_x \) and \( \gamma_y \) are disjoint (except at the point \( O \)) and thus so are \( \text{Im}(\iota_x) \) and \( \text{Im}(\iota_y) \). Then \( \gamma_x = \gamma_y \) and \( |x - y| \geq 2\pi \).

First we show that \( \text{Im}(\iota_x) \cap F \) and \( \text{Im}(\iota_y) \cap F \) are disjoint. Then \( \text{Im}(\iota_x) \cap F \) and \( \text{Im}(\iota_y) \cap F \) are a single arc contained in \( F \cong a \times [0, 1] \) with their endpoints at \((x, 1)\) and \((y, 1)\), respectively. We have seen that \( r \) immerses \( a \times (0, 1] \) into \( D \). Therefore, since \( r = \gamma \circ \iota \) in \( \iota^{-1}(F) \) and \( \gamma_x = \gamma_y \), if \( \iota_x \) and \( \iota_y \) intersect at a point in \( a \times (0, 1] \subset F \), then \( \text{Im}(\iota_x) \cap \text{Im}(\iota_y) \cap F \) contains an arc whose endpoint is on \( a \times \{1\} \). This implies that \( x = y \), which is a contradiction.

We next show that \( \text{Im}(\iota_x) \cap Q \) and \( \text{Im}(\iota_y) \cap Q \) are disjoint up to measure zero. Then, without loss of generality, we can assume that \( x \) and \( y \) are of Type (ii) and that \( x < y \). In addition, by lemma 6.35 we can assume that \( \text{length}(d_x) < \frac{\pi}{2} + \delta \) and \( \text{length}(d_y) < \frac{\pi}{2} + \delta \) with sufficiently small \( \delta \). Therefore, since \( |x - y| \geq 2\pi \), we must have \( d_x \cap d_y = \emptyset \). In particular, \( \xi(x) < \xi(y) \), and then, the length of the interval \( [\xi(x), \xi(y)] \) is more than \( \pi - 2\delta > 0 \). Suppose that \( \text{Im}(\iota_x) \cap \text{Im}(\iota_y) \cap Q \neq \emptyset \); then we must have \( \Phi \circ \xi(x) = \Phi \circ \xi(y) \). Since the non-injectivity of \( \Phi \mid a : a \to b \) corresponds to the non-injectivity of the collapsing maps \( \kappa_l : C_l \to \mathbb{H}^3 \), for all \( z_1, z_2 \in a = [0, \text{length}_h(a)] \) with \( z_1 < z_2 \), we have \( \Phi(z_1) \leq \Phi(z_2) \). Then \( \Phi \) must collapse the closed interval \( [\xi(x), \xi(y)] \) to a single point, so that \( r_z(0) \) is constant for all \( z \in [\xi(x), \xi(y)] \). Then we can observe that, for all but finitely many distinct points in \( (x, y) \) are of Type (i) and those finitely many exceptional points \( z \) are of Type (ii) with \( r_z \subset \gamma_z \). Thus \( \text{Im}(\iota_x) \cap \text{Im}(\iota_y) \cap Q \neq \emptyset \) for only finitely many distinct pairs of points, \( (x, y) \), of \( a \).

We last show that, for each \( k = 0, 1 \), if \( x, y \in a \) are distinct, then \( \text{Im}(\iota_x) \cap E_k \) and \( \text{Im}(\iota_y) \cap E_k \) are disjoint. It suffices to show for \( k = 0 \) by the symmetry. Recall that \( \xi(x) = 0 \) if and only if \( x \) in a (possibly empty) interval \( [0, w] \subset [0, \text{length}(a)] \) \( \cong a \) with some \( w \). Then, since \( E_0 \) is an (ideal) geodesic triangle with a vertex at \( O \) and \( (\gamma_z \setminus O) \cap E_0 \neq \emptyset \) for all \( z \in [0, w] \), we see that \( w < \pi \). Therefore \( \gamma_z \setminus O \) are disjoint for different \( x \in [0, w] \). Therefore, \( \text{Im}(\iota_x) \cap \text{Im}(\iota_y) \cap E_0 \) is at most a single point, which corresponds to \( O \), for all different \( x, y \in a \). \( \square \)
Next we show

\[ \text{Area}(F) < \text{Area}(Q) + \text{Area}(E_0) + \text{Area}(E_1) + \text{Area}(\hat{F}) \]

to complete the proof of Claim 6.33. For this, we similarly construct an almost-everywhere-defined map \( \iota : F \to \hat{F} \sqcup Q \sqcup E_0 \sqcup E_1 \) that is a piecewise isometric-embedding but, in contrast with the previous case, not necessarily an embedding. Nevertheless, this map will help us to compare the areas.

We see that there is a unique continuous function \( \xi : a \to a \) such that

- if \( r_x \subset \gamma_x \) or \( r_x \supset \gamma_x \) then \( \xi(x) = x \), and
- otherwise, letting \( d_x \) be the subinterval of \( a \) bounded by \( x \) and \( \xi(x) \), \( d_x \) is the largest subinterval of \( a \) containing \( x \) and satisfying \( r_x \cap \gamma_y \neq \emptyset \) for all \( y \in d_x \).

In addition, if \( r_x \subset \gamma_x \) or \( r_x \supset \gamma_x \), we set \( d_x = \{x\} \). Moreover, for each \( x \in a \), we have either,

1. \( \gamma_y \) with \( y \in d_x \) contains the point \( r_x(0) = \beta_b \circ \Phi(x) \) only at \( y = \xi(x) \) (Figure 9), or
2. \( \xi(x) \) is an endpoint of \( a \) (Figure 10).

Then, since \( r_y \) and \( \gamma_y \) are geodesic rays in \( D \cong \mathbb{H}^2 \) and \( \gamma_y(0) = O \) for all \( y \in a \), we have

**Lemma 6.37.**

(i) \( \text{length}(d_x) < \pi \).
(ii) \( \gamma \) embeds \( d_x \times (0,1] \) into \( D \) for each \( x \in a \).

Suppose that \( x \) is of Type (I). Then \( r_x \) is contained in \( \gamma(d_x \times [0,1]) \). Thus \( r_x \) factors through \( a \) via a unique embedding of the domain of \( r_x \) to \( d_x \times [0,1] \subset \hat{F} \).

Suppose that \( x \) is of Type (II). We assume that \( \xi(x) = 0 \), since the case of \( \xi(x) = \text{length}(\alpha) \) is similar. Let \( w \) be the maximal number in \( a \cong [0,\text{length}(\alpha)] \) such that \( \gamma(0) \cap r_x \neq \emptyset \) for all \( x \in [0,w] \). Then we have either

1. \( \text{II-i) } r_w(0) \) is on \( r_0 \) (Figure 11) or
2. \( \text{II-ii) } r_w \) contains \( O \) (Figure 12).

Suppose (II-i) holds. Then \( x \in a \) satisfies \( \xi(x) = 0 \) if and only if \( x \in [0,w] \). In addition, for no point \( x \) in \( [0,w] \), we have \( \gamma_x \subset r_x \) or \( \gamma_x \supset r_x \). For every \( \delta > 0 \), if \( \epsilon > 0 \) is sufficiently small, then \( \angle(\gamma_y, r_y) < \delta \) for all \( y \in a \) (thus in particular at \( y = 0 \)) and the angle between \( r_w \) and \( \gamma_0 \) at \( \beta_b \circ \Phi(w) \) is at most \( \pi/2 + \delta \). Therefore, since \( r_y \) does not intersect \( O \) for all \( y \in [0,w] \), we see that \( w < \pi/2 + \delta \) with sufficiently small \( \delta > 0 \) and in particular \( w < \pi \). Let \( P \) the triangle bounded by \( \beta_b(\Phi([0,w])) \), \( r_0 \), \( \gamma_0 \); then \( P \) is contained in \( E_0 \). Since
$w < \pi$, $\gamma$ embeds $[0, w] \times (0, 1]$ into $D$. Then $\gamma([0, w] \times [0, 1])$ and $P$ have disjoint interiors and their union is topologically a disk. Thus, for each $x \in [0, w]$, the geodesic segment $r_x$ is naturally embedded in this disk, and we can naturally decompose the domain of $r_x$ into two segments and embed them into $[0, w] \times [0, 1] \subset \hat{F}$ and $E_0$, so that $r_x$ factors through this embedding via $\gamma$.

Next suppose (II-ii) holds. Then $\xi(x) = 0$ for $x \in a$ if and only if $x \in [0, w]$. The geodesic in $D \cong \mathbb{H}^2$ containing $\gamma_0$ divides $\partial D \cong S^1$ into
two semicircles. Then \( f_I([0, w]) \) is strictly contained in one of those semicircles, and therefore we have \( w < \pi \). Then \( \gamma_0 \) and \( \beta_b \circ \Phi([0, w]) \) are disjoint, and hence there is a disk in \( D \) bounded by \( \gamma_0, r_0, q_w, \) and \( \beta_b \circ \Phi([0, w]) \), which we denote by \( P \). This disk \( P \) is immersed into \( D \) but it may not be embedded in \( D \). Nonetheless this immersion is at most two-to-one, and the image, in \( D \), of such non-injective points is contained in a small neighborhood of the curve \( \beta_b \circ \Phi([0, w]) \). Let \( Q_0 \) be \([0, w] \times [0, 1] \subset Q\). Then we can decompose \( P \) into two simply
connected regions, by cutting along \( q \), with some appropriate \( z \in [0, 1] \), and then isometrically embeds those regions into the disjoint union of \( E_0 \) and \( Q_0 \). Therefore \( \text{Area}(P) < \text{Area}(E_0) + \text{Area}(Q_0) \). Let \( \hat{F}_0 \) be the \( \gamma \)-image of \( [0, w] \times [0, 1] \). Then \( \hat{F}_0 \) and \( P \) are topologically disks and their boundary circles share \( \gamma_0 \). Thus the union of \( \hat{F}_0 \) and \( P \) along \( \gamma_0 \) is again a disk immersed into \( D \). Then \( r \) naturally embeds \([0, w] \times (0, 1]\) (essentially) onto this disk \( \hat{F}_0 \cup_{\gamma_0} P \).

Therefore, letting \( F_0 \) denote \([0, w] \times [0, 1] \subset \text{Domain}(\gamma) \), we have

\[
(5) \quad \text{Area}(F_0) < \text{Area}(\hat{F}_0) + \text{Area}(E_0) + \text{Area}(Q_0).
\]

Then, for each \( x \in [0, w] \), we can naturally decompose the domain of \( r \), minus at most two points into \( \hat{F} \subset E_0 \sqcup E_1 \sqcup Q \) defined as above for all types (I), (II-i), and (II-ii). Those points deleted from \([0, 1] \) change continuously in \( x \in a \). Let \( X \) be the set of the deleted points \( \{(x, t) \in a \times [0, 1] : 0 < t < 1, r_x \text{ noncontinuous at } t \} \). Then \( X \) has measured zero in \( \hat{F} \) and it decomposes \( F \) into finitely many regions. Recalling that \( F = a \times [0, 1] \), define \( \iota : F \setminus X \to \hat{F} \sqcup E_0 \sqcup E_1 \sqcup Q \) by \( \iota | \{x\} \times [0, 1] = r_x \), which is continuous on \( \{x\} \times [0, 1] \) minus at most two points, for each \( x \in a \). Then \( \iota \) is an isometric immersion (on each component of \( F \setminus X \)). Although \( \iota \) is not necessarily an embedding (even up to measure zero as we see below), we have

**Lemma 6.38.** For every \( \delta > 0 \), if \( \epsilon > 0 \) is sufficiently small, then, for different \( x, y \in a \), if \( \text{Im}(\iota_x) \cap \text{Im}(\iota_y) \cap \hat{F} \neq \emptyset \), then \( \pi - \delta < |x - y| < 2\pi \) and the \( r \)-image of \( \text{Im}(\iota_x) \cap \text{Im}(\iota_y) \) is contained in the \( \delta \)-neighborhood of \( O \).

**Proof.** For \( x, y \in a \), we have seen that \( \text{Im}(\iota_x) \cap \hat{F} \subset d_x \times [0, 1] \) and \( \text{Im}(\iota_y) \cap \hat{F} \subset d_y \times [0, 1] \). Assume \( \text{Im}(\iota_x) \cap \text{Im}(\iota_y) \neq \emptyset \). Then \( d_x \cap d_y \neq \emptyset \). By Lemma 6.37 (i), we must have \( |x - y| < 2\pi \). By Lemma 6.31, for every \( \delta > 0 \), if \( \epsilon > 0 \) is sufficiently small, then \( r \) embeds \( d \times (0, 1) \) into \( D \) for all subintervals \( d \) of \( a \) with length less than \( \pi - \delta \). Thus we have \( \pi - \delta < |x - y| < 2\pi \) with sufficiently small \( \delta > 0 \).

Since \( \gamma \) is a fan, there is a natural quotient map from \( \hat{F} \cong a \times [0, 1] \) to the quotient \( a \times [0, 1]/a \times \{0\} \), which collapses \( a \times \{0\} \) to a single point. Thus, for different \( x \) and \( y \) in \( a \), \( \{x\} \times [0, 1] \) and \( \{y\} \times [0, 1] \) intersect only at \( \{x\} \times \{0\} = \{y\} \times \{0\} \) in this quotient space \( a \times [0, 1]/a \times \{0\} \). If \( \epsilon > 0 \) is sufficiently small, then \( \iota_x \) and \( \iota_y \) are \( \delta \)-close to \( \gamma(x) \) and \( \gamma(y) \), respectively.
respectively, which correspond to \( \{x\} \times [0, 1] \) and \( \{y\} \times [0, 1] \). Thus, for every \( \delta > 0 \), if \( \epsilon > 0 \) is sufficiently small, then \( \iota_x \cap \iota_y \) is contained in \( a \times [0, \delta] \subset \hat{F} \) for all different \( x, y \in a \).

Consider all points \( x \) on \( a \) such that the \( r_x \)-image of \( (0, 1) \) intersects the center \( O \). There are only finitely many such points \( x \), since the restriction of \( r \) to \( a \times (0, 1) \) is an immersion. Set \( 0 = x_0 < x_1 < \ldots < x_{n+1} = \text{length}(\alpha) \) to be those points \( x \) together with the endpoints of \( a \). For each \( k = 0, 1, \ldots, n \), set \( F_k = [x_k, x_{k+1}] \times [0, 1] \subset F \). The center \( O \) divides \( r_{x_k} \) into \( \gamma_{x_k} \) and \( q_{x_k} \). Then, accordingly, let \( \hat{F}_k = [x_k, x_{k+1}] \times [0, 1] \subset \hat{F} \) and \( Q_k = [\Phi(x_k), \Phi(x_{k+1})] \times [0, 1] \subset Q \) for each \( k = 0, 1, \ldots, n \).

**Lemma 6.39.** Assume that \( \iota_x \cap \iota_y \cap \hat{F} \neq \emptyset \) for some \( x, y \in a \). Then

- Both \( x, y \) are contained in the same interval \([x_k, x_{k+1}]\) for some \( k \in \{0, 1, \ldots, n\} \).
- All points in \([x_k, x_{k+1}]\) are of Type (I).
- There exist \( p, p' \) with \( x_k < p < p' < x_{k+1} \) such that \( \iota_x \cap \iota_y \cap \hat{F} \neq \emptyset \) for different \( x, y \in [x_k, x_{k+1}] \) with \( x < y \) if and only if \( x \in [x_k, p] \) and \( y \in [p', x_{k+1}] \).

**Proof.** Suppose that \( \iota_x \cap \iota_y \neq \emptyset \) for some \( x, y \in a \) with \( x < y \). By Lemma 6.38, \( \pi - \delta < |x - y| < 2\pi \) with sufficiently small \( \delta > 0 \). Since \( \mu = \Phi([x, y]) \) and \( \text{length}_b([x, y]) \) are \( \delta \)-close, we can assume that \( \pi - \delta < \mu = \Phi([x, y]) < 2\pi + \delta \) by taking sufficiently small \( \epsilon > 0 \). Since \( \mu \) corresponds to bending angles of \( \beta_b \), the restriction of \( \beta_b : b \rightarrow D \) to \( \Phi([x_k, x_{k+1}]) \) has at most a single self intersection. Therefore, \( \iota_x \) and \( \iota_y \) must intersect as in Figure 13 or 14. Then, in particular, for every \( z \in (x, y) \), \( \iota_z \) must be disjoint from \( O \). Thus \( x \) and \( y \) are contained in the same interval \([x_k, x_{k+1}]\) for some \( k \in \{0, 1, \ldots, n\} \). Let \( Z \) be the set of pairs \( (x, y) \) of points in \([x_k, x_{k+1}]\) such that \( x < y \) and \( \iota_x \cap \iota_y \neq \emptyset \). From Figure 13 or 14, we see that \( Z = [x_k, p] \times [p', x_{k+1}] \) for some \( p, p' \) with \( x_k < p < p' < x_{k+1} \). Then \( x_k \) and \( x_{k+1} \) must be of Type (I), and therefore all points of \([x_k, x_{k+1}]\) are of Type (I).

We have seen that Type (II) occurs exactly when \( x \) is in a (possibly empty) interval containing an endpoint of \( x \). Since \( x_1, x_2, \ldots, x_n \) are of Type (I), this interval is contains in \([x_0, x_1]\) or \([x_n, x_{n+1}]\). Thus every \( x \in [x_1, x_n] \) is of Type (I).

**Claim 6.40.** Assume that all \( x \) in \([x_k, x_{k+1}]\) are of Type (I) for some \( k \in \{0, 1, \ldots, n\} \). Then \( \text{Area}(F_k) \leq \text{Area}(\hat{F}_k) + \text{Area}(Q_k) \).
Proof. Since $\gamma_{x_k} \subseteq r_{x_k}$ and $\gamma_{x_{k+1}} \subseteq r_{x_{k+1}}$, we see that $\iota(F_k)$ is contained in $\hat{F}_k$. Then, if $\iota|\hat{F}_k$ is injective, the claim clearly holds. Thus, by Lemma 6.39 we can assume that there exists $p, p'$ with $x_k < p < p' < x_{k+1}$ such that $t_x \cap t_y \cap \hat{F} \neq \emptyset$ for different $x, y \in [x_k, x_{k+1}]$ with $x < y$ if and only if $x \in [x_k, p]$ and $y \in [p', x_{k+1}]$ as in Figure 13 (if $k \in \{1, 2, \ldots, n-1\}$) or Figure 14 (if $k = 0, n$). Thus $\iota(F_k) = \hat{F}_k$. Suppose, in addition, $k \in \{1, 2, \ldots, n-1\}$. Then consider the disk bounded by $\beta_b \circ \Phi([x_k, x_{k+1}])$, $r_{x_k}$, and $r_{x_{k+1}}$, which we denote
by $P_k$ (see Figure 15). Then, since $q_{x_k} \subset r_{x_k}$ and $q_{x_k} \subset r_{x_k}$, we see that $P_k$ is contained in $q(Q_k)$. Thus $\text{Area}(P_k) \leq \text{Area}(Q_k)$. We see that $r$ is injective on $F_k \setminus r^{-1}(P_k)$. In addition $r^{-1}(P_k) \cap F_k$ has exactly two connected components, and $r$ isometrically embeds each component onto $P_k$. Therefore we have

$$\text{Area}(F_k) = \text{Area}(\iota(F_k)) + \text{Area}(P_k) \leq \text{Area}(\hat{F}_k) + \text{Area}(Q_k)$$

as desired.

The proof is similar when $k = 0$ or $k = n$. □

**Claim 6.41.** Suppose that $[x_0, x_1]$ contains a subinterval of Type (II-i). Then $\iota$ embeds $F_0 \setminus X$ into $\hat{F}_0 \sqcup E_0$; therefore $\text{Area}(F_0) \leq \text{Area}(\hat{F}_0) + \text{Area}(E_0)$. Similarly, suppose that $[x_n, x_{n+1}]$ contains a subinterval of Type (II-i). Then $\iota$ embeds $F_n \setminus X$ into $\hat{F}_n \sqcup E_n$; therefore $\text{Area}(F_n) \leq \text{Area}(\hat{F}_n) + \text{Area}(E_n)$.

**Proof.** Let $[0, w] \subset [0, \text{length}(\alpha)]$ be the (maximal) subinterval of Type (II-i), if there exists. Then $[0, w]$ is contained in $[0, x_1]$. By Lemma 6.39 $\iota$ embeds $F_0 \cap \iota^{-1}(\hat{F})$ into $\hat{F}_0$. In addition we have seen that $\iota$ embeds $[0, w] \times [0, 1]$ into the disjoint union of $[0, w] \times [0, 1] \subset \text{Domain}(\gamma)$ and $E_0$. Thus $\iota$ embeds $F_0 \setminus X$ into $\hat{F}_0 \sqcup E_0$.

We can similarly prove the analogous claim about $F_n$. □

In addition, by (5), we have an analogous statement for Type (II-ii):
Claim 6.42. Suppose that \([x_0, x_1]\) contains a subinterval of Type (II-ii). Then \(\iota\) embeds \(F_0 \setminus X\) into \(\hat{F}_0 \sqcup E_0 \sqcup Q_0\); therefore \(\text{Area}(F_0) \leq \text{Area}(\hat{F}_0) + \text{Area}(E_0) + \text{Area}(Q_0)\). Similarly suppose that \([x_n, x_{n+1}]\) contains a subinterval of Type (II-ii). Then \(\iota\) embeds \(F_n \setminus X\) into \(\hat{F}_n \sqcup E_1 \sqcup Q_n\); therefore \(\text{Area}(F_n) \leq \text{Area}(\hat{F}_n) + \text{Area}(E_1) + \text{Area}(Q_n)\).

By Claim 6.40, Claim 6.41 and Claim 6.42, we have

\[
\text{Area}(F) = \sum_{k=0}^{n} \text{Area}(F_k) \\
\leq \sum_{k=0}^{n} \text{Area}(\hat{F}_k) + \sum_{k=0}^{n} \text{Area}(Q_k) + \text{Area}(E_0) + \text{Area}(E_1) \\
= \text{Area}(\hat{F}) + \text{Area}(Q) + \text{Area}(E_0) + \text{Area}(E_1).
\]

6.9. The proof of Theorem 6.1. Recall that there are three cases in the theorem.

6.9.1. Case (II). This case is equivalent to the following claim:

Proposition 6.43. Let \(C = (\tau, L)\) and \(C' = (\tau', L')\) be different projective structures with the same holonomy \(\rho: \pi_1(S) \to \text{PSL}(2, \mathbb{C})\). Assume \([L] = [L'] \in \mathcal{PML}\) and \(L \leq L'\). Then \(Gr_{L-L}(C) = C'\).

Proof of Claim 6.43. Let \(\beta: \mathbb{H}^2 \to \mathbb{H}^3\) and \(\beta': \mathbb{H}^2 \to \mathbb{H}^3\) denote the bending maps associated with \(C = (\tau, L)\) and \(C' = (\tau', L')\), respectively. Since \([L] = [L']\), by Proposition 5.1, we have \(\beta = \beta'\) and, in particular, \(\tau = \tau'\). Then

Lemma 6.44. \(L\) and \(L'\) have only periodic leaves.

Proof. Assume that the projective measured lamination \([L] = [L']\) contains a minimal irrational lamination. Then, this irrational lamination supports sublaminations \(N\) and \(N'\) of \(L\) and \(L'\), respectively. Set \(L = (\lambda, \mu)\) and \(L' = (\lambda', \mu')\), where \(\lambda\) and \(\lambda'\) are geodesic laminations on \(\tau\) and \(\mu\) and \(\mu'\) are transversal measures. Recall that the transversal measures \(\mu\) and \(\mu'\) correspond to (infinitesimal) exterior angles of the bending maps \(\beta\) and \(\beta'\), respectively (\([12, 16]\)). Then, since \(\beta = \beta'\) and the lamination \(N\) and \(N'\) are irrational, we must have \(N = N'\). Therefore, since \([L] = [L'] \in \mathcal{PML}\) and the laminations \(N\) and \(N'\) are sublaminations of \(L\) and \(L'\), we have \(L = L'\). Since \(\tau = \tau'\), this is a contradiction to the assumption that \(C \neq C'\). \(\square\)
that C is the support of L. Then, since \( \beta = \beta' \), the weight of each loop of \( L' - L \) must be a multiple of \( 2\pi \). This completes the proof. \( \square \)

6.9.2. Case (III). Suppose that \( L = \emptyset \) in \( \mathcal{ML}(S) \). Then clearly \( \rho \) is fuchsian. Given a projective structures \( C' = (\tau', L') \) with the same holonomy \( \rho \), letting \( L' = (\lambda', \mu') \), we can regard \( L = \emptyset \) as \( 0 \cdot L' = (\lambda', 0) \). In this way, we can regard \( L \) and \( L' \) as having the same supporting lamination. Therefore we can apply the proof of Case (II), and see that \( C' = Gr_{L'}(C) \).

6.9.3. Case (I). Let \( C = (f, \rho) \) be a projective structure on \( S \) with holonomy \( \rho \). Set \( C = (\tau, L) \) to be Thurston’s coordinates of \( C \). Let \( U \) be the neighborhood of \([L] \) in \( \mathcal{PL}(S) \), obtained by applying Proposition 6.10 to \( C \). Then, for every \( C'' = (f', \rho) \in \mathcal{P}\mathcal{L} \) with \([L'] \in U \), there are a topological (fat) traintrack \( T = \cup_j B_j \) \((j = 1, 2, \ldots, n) \) on \( S \), where \( B_j \) are branches of \( T \), and marking homeomorphisms \( \phi: S \to C \) and \( \phi': S \to C' \) that satisfy (I) and (II) in Proposition 6.10. Set \( C' = (\tau', L') \) to be Thurston’s coordinates of \( C'' \). Then we assume that \([L] \neq [L'] \) since it is an assumption of Case (I).

Regard \( S \) as the union of \( S \setminus T \) and \( T = \cup_j B_j \). Then, this decomposition of \( S \) induces decompositions of \( C \) and of \( C'' \) via \( \phi \) and \( \phi' \), respectively:

\[
C' = (C' \setminus \phi'(T)) \cup \phi'(T) = (C' \setminus \phi'(T)) \cup (\cup_j \phi'(B_j)).
\]

Then Proposition 6.10 (I), \( \phi' \circ \phi^{-1} \) yields an isomorphism from \( (C \setminus \phi(T)) \) to \( (C' \setminus \phi'(T)) \) compatible with the developing maps \( f \) and \( f' \). In addition, by Proposition 6.10 (II - ii), for each \( j = 1, 2, \ldots, n \), we have \( \phi'(B_j) = Gr_{M_j}(\phi(B_j)) \) for some admissible multiarc \( M_j \) on \( \phi(B_j) \). Then, by the definition of an admissible multiarc, each arc of \( M_j \) connects the outermost ties of \( \phi(B_j) \).

Let \( \kappa: C \to \tau \) and \( \kappa': C' \to \tau' \) be the collapsing maps. Then by Proposition 6.10 (iii), \( \kappa \) and \( \kappa' \) descend the traintracks \( \phi(T) \) on \( C \) and \( \phi'(T) \) on \( C'' \) to traintracks on \( \tau \) and \( \tau' \) respectively; setting \( L = (\lambda, \mu) \) and \( L' = (\lambda', \mu') \), indeed \( \frac{1}{2\pi} [\mu'(\kappa' \circ \phi'(B_j)) - \mu(\kappa \circ \phi(B_j))] \) is a good approximation of the number of arcs of \( M_j \), for each \( j = 1, 2, \ldots, n \).

Since \( \kappa' \circ \phi'(T) \) carries \( L' \), the \( n \)-tuple \( \{\mu'(\kappa' \circ \phi'(B_j))\}_{j=1}^{n} \in \mathbb{R}_{\geq 0}^{n} \) satisfies the switch conditions of the traintrack \( \kappa' \circ \phi'(T) \) on \( \tau' \) and similarly, since \( \kappa \circ \phi(T) \) carries \( L \), the \( n \)-tuple \( \{\mu(\kappa \circ \phi(B_j))\}_{j=1}^{n} \in \mathbb{R}_{\geq 0}^{n} \) satisfies the switch conditions of the traintrack \( \kappa \circ \phi(T) \) on \( \tau \). Since \( \kappa' \circ \phi': S \to \tau \) and \( \kappa' \circ \phi': S \to \tau \) identify the traintrack \( T \) on \( S \) with
those image traintracks on $\tau$ and $\tau'$, respectively, the $n$-tuple of their differences \( \{\mu'((\kappa' \circ \phi'(B_j)) - \mu((\kappa \circ \phi(B_j)))\}_{j=1}^{n} \in \mathbb{R}_{\geq 0}^{n} \) satisfies the switch conditions as well. Therefore, by the approximation, the $n$-tuple of the numbers of the arcs of $M_j$ ($j = 1, 2, \ldots, n$) also satisfies the switch conditions. Thus the union $\bigcup_j M_j =: M$ is a multiloop carried by the traintrack $\phi(T) \subset C$, up to proper isotopies of $M_j$ on $\phi(B_j)$ through admissible multiarcs.

For each $j = 1, 2, \ldots, n$, $\phi'(B_j) = Gr_{M_j}(\phi(B_j))$ still holds after the isotopy of $M_j$ (Lemma 6.5). Since $\phi(T)$ carries $L$ and $\phi'(T)$ carries $L'$, we can regard $L' - L$ as a measured lamination on $S$ carried by $T$ whose weight on the branch $B_j$ is $\mu'((\kappa' \circ \phi'(B_j)) - \mu((\kappa \circ \phi(B_j)))$ for each $j$. Thus $L' - L$ is a good approximation of $M$.

We last compare $\phi(T) \subset C$ and $\phi'(T) \subset C'$ as projective structures on $T$. Let $B_i$ and $B_j$ be adjacent branches of $T$ and let $m_i$ and $m_j$ be arcs of $M_i$ and $M_j$, respectively, that share an endpoint, so that $m_i \cup m_j$ is a simple arc on $B_i \cup B_j$. Since $B_j$ and $B_i$ are supported on a round cylinder, the projective structure inserted by the grafting $Gr_{m_i \cup m_j}$ of $B_i \cup B_j$ is exactly the union of projective structures inserted by the graftings $Gr_{m_i}$ of $B_i$ and $Gr_{m_j}$ of $B_j$. Since this happens for all adjacent arcs, we have

$$\phi'(T) = \bigcup_j \phi'(B_j) = \bigcup_j Gr_{M_j}(\phi(B_j)) = Gr_M(\phi(T))$$

(a similar argument is used in \[2\]). Hence

$$C' = (C' \setminus \phi'(T)) \cup \phi'(T) = (C \setminus \phi(T)) \cup [Gr_M(\phi(T))] = Gr_M(C).$$

\[6.1\]

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