Numerical solution of a coefficient inverse problem of the strength of a heat-loaded thin-walled structure

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Abstract. We consider the inverse problem of the strength of a thin-walled structure, which is affected by a complex of mechanical and thermal loads. The problem is approximated by the finite element method, using a superelement model that is naturally related to the construction in question. The finite-dimensional problem is a system of nonlinear algebraic equations with coefficients that depend on the vector of the required parameters of the model. To solve the inverse problem, the objective functional is used in the form of a squared discrepancy between the experimental and theoretical deformation values. A gradient method is used to find the minimum of the functional. Gradient information is output using the Lagrange function. The results of numerical experiments for a single caisson of a structure are presented, which confirm the efficiency of the proposed method.

1. Introduction
It is known that in conditions of high temperatures acting on a mechanically loaded structure, its plastic deformation and the drop in the magnitude of the elastic moduli are possible. In this case, the physical and mechanical characteristics of the construction material obtained by standard testing of the samples may differ from their respective values when operating in the structure. It seems relevant to restore the deformation diagrams based on the results of full-scale tests of the design of a construction as a whole. Mathematical models of such problems are inverse coefficient problems for systems of partial differential equations.

In the field of structural strength, identification methods have not been developed adequately, although the first works appeared in the mid-1970s. Among Russian authors it is necessary to note here, first of all, the work of Ya.M. Parkhomovskii [1], V.D. Ilyichev and V.V. Nazarova [2], N.M. Grevtsova [3] and others. Coefficient inverse problems, especially in the field of heat exchange, are associated with the work of the school of O.M. Alifanov and his colleagues [4]. The formulation, existing approaches and solution methods for the inverse problems of thin-walled structures are presented in the monograph by V.A. Kostin et al. [5]. In this book, the identification of physical and mechanical characteristics of materials and products made of them, boundary conditions, and loads acting on the aircraft is carried out from the standpoint of a unified methodology. One of the known approaches to solving the inverse boundary value problems is their formulation in the form of extremal problems with a PDE as a state problem. When solving such problems, we can use the methods of Lagrange multipliers.
Lagrangian functions and adjoint equations are widely used in solving optimization and inverse problems in a wide range of applications.

In the present paper, the inverse coefficient problem for a ten-section caisson of rectangular cross-section is solved. It is formulated in the form of a problem of minimizing a quadratic functional. This problem is approximated by the finite element method. When approximating the state equation, superelement technology is used, which are naturally related to the construction in question. To solve the finite-dimensional problem, a gradient method of minimization is used. Gradient information is obtained using the Lagrange function. The results of numerical calculations are presented and analyzed for an example with real input data.

2. Problem formulation and approximation

To simulate aeronautical thin-walled structures with longitudinal and transverse dialing, we use the mathematical model of Yu.G. Odinokov, modified by V.G. Shataev. It allows calculating the framed shells under the influence of an arbitrary external load and a temperature field. The system of differential equations has the form [6]:

\[ -(E_i S_i f'_i)' + \sum_{k=1}^{n} a_{ik} f'_k + \sum_{k=1}^{n} A_{ik} f_k = d_i, \quad i = 1, 2, \ldots n. \] (1)

The boundary conditions for (1) are written for each edge. In the case of fixed one end and loaded the other with an axial force, they have the form

\[ E_i S_i f'_i = P_i + P_i^T \quad \text{for} \quad z = l; \quad f_i = 0 \quad \text{for} \quad z = 0, \quad i = 1, 2, \ldots n. \] (2)

Above we use the following notations:
- \( f_i \) is the axial displacement of the edge;
- \( l \) and \( n \) are the length and number of longitudinal edges of the structure;
- \( E_i \) and \( \alpha_i \) are the modulus of elasticity and coefficient of linear expansion of the rib;
- \( S_i \) and \( T_i \) are the cross-sectional area and temperature of the \( i \)-th rib with attached skin;
- \( P_i \) is the axial force in the \( i \)-th edge of the structure.

By \( v' \) we denote the first order derivative of a function \( v \) with respect to the coordinate \( z \).

The deformation of the \( i \)-th edge is represented as a sum: \( f'_i = \frac{P_i}{E_i S_i} + \alpha_i T_i \), and the heating effect can be taken into account with the help of an additional term \( P_i^T = E_i S_i \alpha_i T_i \) in the boundary conditions (2). We consider it to be given, because for its knowledge it is sufficient to know the temperature of the structure. \( a_{ik} \) and \( A_{ik} \) are the coefficients that depend on the shape of the sections and characterize the shift work of the structure; \( d_i \) corresponds to load and takes into account the magnitude and nature of the load application, including temperature.

Let us consider a ten-section caisson of rectangular section, which is a typical representative of the power structure of the inter-spar wing part. One end of the caisson is rigidly fixed, while the other end has axial and moment loads. The upper and lower panels of the caisson are subjected to temperature loads that are constant along the length of the structure (Fig. 1).

We assume that there are experimental values of the values of the deformations of the ribs \( f'_{exp}(z, T) = f'_d \) obtained in full-scale bench tests of the aircraft as a whole or its wing, when the loads exceed the operational loads and plastic deformations (physical nonlinearity) are possible. It is required to determine the true stiffness \( E_i S_i \) of the structural elements, which is a variable quantity in the coefficients of the system of differential equations due to a change of \( E_i \) with increasing temperature and load. Guided by the variable elasticity parameters method of I.A. Birger, we consider \( E_i \) as a set of secant models corresponding to the nonlinear part of the deformation diagram. We also assume that the engineering hypothesis about the only shifting
work of a shell under conditions of plastic deformation at elevated temperatures is confirmed very well by aligning the shifting stresses.

In the numerical solution of the formulated problem we apply a variant of finite element method. To establish the correspondence between the number of strain gauges and the theoretical value of deformations, we use not a classical finite element method, but a variant in the form of superelements. A sufficiently "smooth change" of the stress-strain state makes it possible to use this method correctly from the point of view of the mechanics of the process. At the same time, the complexity of the solution is substantially reduced by decreasing the order of the resolving system of equations. Below we briefly describe the technology of superelement construction (for more details see [6]).

According to the method of superelements, the components of the displacement of an arbitrary point of the construction with respect to the \( z \) and \( s \) coordinates are given in the following form:

\[
\begin{align*}
    u(x, y, z) &= \sum_{i=1}^{M} \theta_i(x, y) \varphi_i(z), \quad i = 1, 2, \ldots, M, \\
    f(x, y, z) &= \sum_{j=1}^{N} r_j(x, y) \xi_j(z), \quad j = 1, 2, \ldots, N.
\end{align*}
\]

Above \( z \) is the axis perpendicular to the plane of the section and \( s \) is the axis along the tangent to the contour of the cross section. The functions \( \varphi_i(z) \) and \( \xi_j(z) \) are the required generalized displacements, which depend only on the coordinate \( z \). The location of the transverse and longitudinal displacements \( \theta_i(x, y) \) and \( r_j(x, y) \) can be considered as generalized coordinates. Thus, the design model in section \( z \) has \( M \) degrees of freedom in the transverse and \( N \) in the longitudinal directions. The point of intersection of a section with the \( z \)-axis is taken as the generalized node. Each compartment actually becomes a superelement with generalized displacements at the nodes. It is assumed that \( n \) ribs are one-dimensional tensile-compressive elements, and the \( m \) panels are shifted two-dimensional elements of a real power construction.

On the basis of the Lagrange principle, expressing the variation of the potential energy of deformation and the work of external forces on possible displacements through the previous forms of generalized displacements, we obtain the equilibrium equation for superelements:

\[
KF = P. \tag{3}
\]
Above

\[ K = K \] is the global stiffness matrix, which has a block-diagonal form: 

\[ K = \operatorname{diag}(K^1, K^2, \ldots, K^c) \] 

\( c \) is the number of superelements, and \( K_e(\beta) \) is \( 2(M + N) \times 2(M + N) \) stiffness matrix of a superelement \( e \), which entries depends on the refined rigidity (control parameters) \( \beta_{ij} = (E S)_{ij} \).

\[ F = (F^1, F^c, \ldots, F^c)^T \] is the global vector of generalized displacements with 

\[ F_e = \left\{ \varphi^1, \ldots, \varphi^M, \xi^1, \ldots, \xi^N \right\}^T \]

the right-hand side \( P = P_M + P_T \) is the load, composed of mechanical \( P_M \) and thermal \( P_T \) parts. Corresponding system of the equations for a superelement is

\[ K_e F_e = P_e. \]  

with

\[ P_e = P_M^e + P_T^e. \]

In what follows we neglect the deformations of the ribs in the plane. When solving the discrete problem, according to (3) and (4) and the known relations between \( F \) and \( F_e \) we find the numerical vector of displacement \( f \) and numerical values of axial deformations \( Bf \), where \( B \) is the matrix corresponding to numerical differentiation.

Below we use the following form of writing of state equation (3):

\[ K(\beta)f = P, \]  

where \( K(\beta) \) is the symmetric and positive definite stiffness matrix, \( f \) is a vector of displacements in a rib in the \( z \)-direction.

3. Optimization problem and solution method

Turning to the mathematical formulation of the problem for a thin-walled construction of general form, we obtain the optimization problem to minimize the function

\[ J(f, \beta) = \frac{1}{2} \| Bf - f'_d \|^2, \quad f'_d \in \mathbb{R}^n \]  

under the constraint (5). Hereafter \( \|v\| \) and \((v, v)\) are Euclidean norm and inner product in the space \( \mathbb{R}^n \). For the optimization problem (5), (6) we construct the Lagrange function

\[ L(f, \beta, \lambda) = J(f, \beta) + (\lambda, K(\beta)f - P). \]  

A critical point of Lagrange function satisfies the following system of equations - first order optimal conditions:

\[ \nabla_\lambda L(f, \beta, \lambda) = 0 \iff K(\beta)f - P = 0, \]

\[ \nabla_f L(f, \beta, \lambda) = 0 \iff K^T(\beta)\lambda + \nabla_f J(f, \beta) = 0, \]

\[ \nabla_\beta L(f, \beta, \lambda) = 0 \iff \nabla_\beta J(f, \beta) + (\lambda, \nabla_\beta K(\beta)f) = 0, \]

where

\[ \nabla_f J(f, \beta) = \left( \frac{\partial J(f, \beta)}{\partial f_1}, \frac{\partial J(f, \beta)}{\partial f_2}, \ldots, \frac{\partial J(f, \beta)}{\partial f_n} \right) = B^T(Bf - f'_d) \] is the gradient of the objective function with respect to \( f \),

\[ \nabla_\beta K(\beta) = (K_1, K_2, \ldots, K_n) \] with \( n \times n \) matrices \( K_i = \frac{\partial K(\beta)}{\partial \beta_i} \).

The information obtained is used to apply the gradient method of solving the problem.

We set the initial approximation \( \beta^0 \), for \( k = 0, 1, \ldots, \)
(i) we find displacement vector $f^{k+1}$ from the equation $K(\beta^k)f^{k+1} = P$;
(ii) we find Lagrange multipliers $\lambda^{k+1}$ from the equation $K^T(\beta^k)\lambda^{k+1} = -B^T(Bf^{k+1} - f'_d)$;
(iii) we update the parameters $\beta$:

$$\beta^{k+1} = \beta^k - \mu \left( \nabla_{\beta} J(f^{k+1}, \beta^k) + (\lambda^{k+1}, \nabla_{\beta} K(\beta^k)f^{k+1}) \right)$$

Iterative parameter $\mu > 0$ is determined on the basis of computational experiments.

4. Numerical example

The proposed method was implemented with the following initial data: caisson length $l = 1m$, panel width (distance between stringers) $s = 0.15m$, panel thickness $\delta = 0.001m$ (for all panels), $E = 5.8810^8 H/m^2$, area $F = 0.0003 m^2$, temperature $T = 250^C$. It was assumed that we know the diagrams of the deformations $\sigma - \varepsilon$, obtained during testing the standard samples and taking into account temperature. The axial load applied to the ribs 1, 3, 6 and 8 was the same and equal $P = 5250N$; moment $M_z = 1000 Hm$.

Because of the lack of data from the physical experiment, the deformation values were obtained numerically by solving the finite element approximation of the direct problem. This nonlinear problem is solved on the basis of secant moduli by the variable elasticity parameter method. In order to use these results as if they are values of the physical experiment, random errors of about 5 percents are introduced in them, which is typical for measurements when fixing the deformations of the structure and the method of loading it in a full-scale experiment. The iterative step in the gradient method was determined in the numerical experiments. We used the fact that the value of the sought elastic parameters was only refined relative to the tabulated values, and not determined anew.

The graphs in Fig. 2,3 demonstrate the decreasing of the difference between calculated deformations and moduli of elasticity with their exact values.
Figure 2. Deformations: experimental, first and last iterations; for the 1-st rib on the left and for the second rib on the right

Figure 3. Moduli of elasticity: experimental, first and last iterations; for the 1-st rib on the left and for the second rib on the right

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