Abstract

The effect of the electrodynamic forces on a charged particle in a propagating plane electromagnetic wave is investigated. First it is pointed out that for constant fields fulfilling the radiation condition there will be an acceleration in the direction of the Poynting vector. When oscillating fields are considered the Lorentz force on the particle only causes a drift, with constant average velocity, in the direction of propagation of the wave, i.e. the direction of the Poynting vector. Finally, when the radiative reaction (radiation damping) force is added the result is again an acceleration in the direction of wave propagation.

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1 Introduction

Recently Professor Thomas Gold [1] has published a manuscript on the Web stating that radiation pressure from the Sun does not exist, or at least that it cannot be used for propulsion with solar sails, as has been suggested [2]. I do not understand Gold’s, mainly thermodynamic, arguments but the phenomenon of radiation pressure is well established theoretically and experimentally so the manuscript is somewhat surprising. Usually the force of radiation pressure is explained by discussing how photons carry momentum which is either absorbed or reflected when impinging on a body. Here we will point out and elucidate the less known fact that the electrodynamic forces (Lorentz force and radiation damping force) on a charged particle from a plane electromagnetic wave accelerates the particle in the direction of propagation of the wave. This, at least, demonstrates that radiation pressure is an immediate consequence of the relativistic equation of motion of a charged particle in an electromagnetic field.

For simplicity of notation we will use Gaussian units. The relativistic equation of motion for a charged particle of charge $q$ and mass $m$, in an external electromagnetic field $F_{ab} = (E, B)$ (with the notation of Landau and Lifshitz [3]) is given by,

$$m \frac{du^a}{d\tau} = \frac{q}{c} F^a_{\ b} u^b,$$

(1)

if we, for the time being, neglect radiative reaction. Here $u^a = (1/c)dx^a/d\tau = \gamma(1, v/c) = (u^0, \ u)$, where $x^a = (ct, \ r)$, and $\tau$ is proper time, $d\tau = \gamma dt$, and, finally, $\gamma = u^0 = 1/\sqrt{1 - v^2/c^2}$. Later we will add $g^a$, the radiation damping force, on the right hand side. This is a correction needed since an accelerated charge radiates and this results in a reaction force. If one introduces the new independent variable

$$\zeta = \frac{q\tau}{mc},$$

(2)

one can write Eq. (1) in the form

$$\left( \frac{d\gamma}{d\zeta}, \frac{du}{d\zeta} \right) = (u \cdot E, \ \gamma E + u \times B)$$

(3)

for the time and space components respectively.
2 Motion due to the Lorentz force of constant fields

The general solution of Eq. (3) for constant electromagnetic fields \( F_{ab} = (E, B) \) is known and has been discussed extensively in the literature. In particular one can recommend the studies by Salingaros [4, 5]. Other illuminating contributions are by Hyman [6] and by Muños [7].

First we note that the case of radiation is very special and differs from the general case. For constant \( E \) and \( B \) one can in general make a Lorentz transformation to a reference frame in which these two vectors are parallel. Then the Poynting vector,

\[
S = \frac{c}{4\pi} E \times B,
\]

which represents the flux density of momentum in the electromagnetic field, becomes zero. In that frame therefore there can be no radiation pressure force. In such general fields charged particles will have a drift velocity equal to the velocity of the frame in which the fields are parallel. Radiation fields on the other hand are characterized by

\[
E \cdot B = 0, \quad |E| = |B|,
\]

and for such fields there is no such reference frame of zero Poynting vector. The general solution of Eq. (3) for constant \( E \) and \( B \) fulfilling the radiation conditions, for a particle starting at rest, is given by Salingaros [5] and is

\[
u = \zeta E + \frac{1}{2} \zeta^2 E \times B, \quad \gamma = 1 + \frac{1}{2} \zeta^2 |E|^2.
\]

The meaning of this result is that, for small \( \zeta \) the velocity will be essentially parallel (or anti-parallel) to the electric field as intuition demands, but for large \( \zeta \) the velocity will become more and more parallel (never anti-parallel) to the Poynting vector \( S = (c/4\pi) E \times B \).

3 Motion due to the Lorentz force of of propagating wave

The result [7] by Salingaros and its interpretation does not survive when the field is allowed to have an oscillatory time dependence, as we will now show.
We start by rewriting Eq. (1),

\[ \frac{du^a}{d\zeta} = F^a_b u^b, \]  

(8)
on matrix form, as follows

\[ \frac{du}{d\zeta} = F(\zeta)u, \]  

(9)
and we will allow \( F \) to depend on time via \( \zeta \). Here the components of the matrices \( F \) and \( u \) are given by

\[
F = \begin{pmatrix}
0 & E_x & E_y & E_z \\
E_x & 0 & B_z & -B_y \\
E_y & -B_z & 0 & B_x \\
E_z & B_y & -B_x & 0
\end{pmatrix}, \quad \text{and} \quad u = \begin{pmatrix}
u^0 \\
u_x \\
u_y \\
u_z
\end{pmatrix},
\]

(10)
respectively. We now specialize to the radiation case and chose the x-axis in the direction of \( E \) so that \( E = E(\zeta)e_x \). If also chose the z-axis as the direction of the Poynting vector we must have \( B \) along the y-axis and of the same length as \( E \), so that \( B = E(\zeta)e_y \). The Poynting vector is then \( S = \frac{c}{4\pi}E^2(\zeta)e_z \), and the \( F \)-matrix is given by

\[
F = \begin{pmatrix}
0 & E(\zeta) & 0 & 0 \\
E(\zeta) & 0 & -E(\zeta) & 0 \\
0 & 0 & 0 & 0 \\
0 & E(\zeta) & 0 & 0
\end{pmatrix} \equiv E(\zeta)H,
\]

(11)
where we have defined the matrix

\[
H = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}.
\]

(12)
We have now found that, in the radiation case, the equation of motion, Eq. (9), becomes

\[ \frac{du}{d\zeta} = [E(\zeta)H]u. \]

(13)
Thus, differentiating \( u \) with respect to \( \zeta \) multiplies \( u \) with a matrix. The general solution of this equation is given by

\[
u(\zeta) = \exp \left[ \int_0^\zeta E(\eta) d\eta H \right] u_0.
\]

(14)
To get an explicit solution we put
\[ f(\zeta) = \int_0^\zeta E(\eta) d\eta, \] (15)
and use the series expansion of the exponential. For this we need the powers
of the matrix \( H \) and these are: \( H^0 = 1 \), the four by four unit matrix, \( H^1 = H \),
\[
H^2 = \begin{pmatrix}
1 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & -1
\end{pmatrix}.
\] (16)
All higher powers of \( H \) are zero matrices. We thus find that
\[
u(\zeta) = \left[ 1 + f(\zeta) H + \frac{1}{2} f^2(\zeta) H^2 \right] u_0.
\] (17)
We now assume that the initial condition is \( \tilde{u}_0 = (1 0 0 0) \), i.e. that the
particle is at rest. Explicit calculation then gives
\[
u(\zeta) = \begin{pmatrix}
1 + \frac{1}{2} f^2(\zeta) \\
f(\zeta) \\
0 \\
\frac{1}{2} f^2(\zeta)
\end{pmatrix}.
\] (18)
If we take \( E(\zeta) = \text{const.} \) we recover Salingaros’ solution (7).
If we instead assume that we have a simple harmonic wave, so that \( E(\zeta) = E_1 \cos(w\zeta) \) and \( f(\zeta) = E_1 \sin(w\zeta)/w \). We then find that the time average of
the four velocity (18) becomes
\[
<\nu(\zeta),\zeta> \equiv \frac{w}{2\pi} \int_0^{2\pi/w} \nu(\zeta) d\zeta = \begin{pmatrix}
1 + \frac{E_1^2}{4w^2} \\
0 \\
0 \\
\frac{E_1^2}{4w^2}
\end{pmatrix}.
\] (19)
The acceleration in the direction of the Poynting vector (the z-axis) that we
found in the case of constant fields has now become a drift with constant
average speed (originally calculated by McMillan [8]). The speed of this drift
is larger the smaller the frequency \( w \) is, but there is no acceleration and
thus no average force in the direction of the Poynting vector. For a recent
discussion of this problem, see McDonald and Shmakov [9].
4 The radiative reaction force from propagating plane wave

So far we have neglected the radiative reaction force and used the equation of motion, Eq. (1). To make it more accurate we must add the four force $g^a$ on the right hand side. Now we investigate the form of $g^a$ and how it will modify the solutions found in the previous sections.

The origin of this force is the electromagnetic radiation that an accelerated charged particle sends out. This radiation carries energy and momentum and there for a reaction force on the particle itself. One can show that the force, due to dipole radiation, should be $\mathbf{f} = \frac{2q^2}{3\epsilon_0 c^3} \mathbf{\dot{v}}$. The four vector form of this should give the four force $g^a = \frac{2q^2}{3\epsilon_0 c^3} \mathbf{u}^a$. A four force must for kinematic reasons fulfill $g^a \mathbf{u}^a = 0$, the four scalar product with the four velocity must be zero. This is achieved by modifying $g^a$ by subtracting the component along the four velocity: $g^a_0 = g^a - (g^a_0 \mathbf{u}^b) \mathbf{u}^b$. Since $\mathbf{u}^a \mathbf{u}^a = 1$ this gives the desired property. Finally one can insert the expression for $\frac{d^2 \mathbf{u}^a}{dt^2}$ obtained by differentiating Eq. (1), and also the expression for $\frac{d \mathbf{u}^a}{dt}$, from the same equation. The force $g^a$ should, after all, be a perturbation compared to the Lorentz force. Thus one obtains

$$g^a = \frac{2q^2}{3\epsilon_0 c^3} \left\{ \frac{q}{mc} \left[ \frac{dF^a}{dt} \mathbf{u}^b + F^a_b \left( \frac{q}{mc} F^b_c \mathbf{u}^c \right) \right] \right\}$$

(20)

$$- \frac{2q^2}{3\epsilon_0 c^3} \left\{ \frac{q}{mc} \left[ \frac{dF^b_c \mathbf{u}^c + F^c_b \left( \frac{q}{mc} F^c_d \mathbf{u}^d \right) }{dt} \right] \right\} \mathbf{u}_b \mathbf{u}^a.$$  

(21)

Some algebra, and use of the fact that $F^{ab} \mathbf{u}_a \mathbf{u}_b = 0$ (due to the anti-symmetry of $F^{ab}$), turns this into

$$g^a = \frac{2q^2}{3mc^3} \frac{dF^a}{dt} \mathbf{u}^b + \frac{2q^2}{3m^2 c^5} \left( F^a_c F^c_b - F^d_c F^c_d \mathbf{u}^a \right) \mathbf{u}^b.$$  

(22)

The corresponding three dimensional force $\mathbf{f}$ can be obtained by noting that, with our conventions, $g^a = (\gamma/c)(\mathbf{f} \cdot \mathbf{v}/c, \mathbf{f})$.

Landau and Lifshitz [3] (§76, Problem 2) give the three dimensional form for this force as follows

$$\mathbf{f} = f_o + f_s = \frac{2}{3mc^3} \gamma \left( \frac{dE}{dt} + \frac{\mathbf{v}}{c} \times \frac{d\mathbf{B}}{dt} \right)$$

(23)

$$+ \frac{2}{3m^2 c^4} \left\{ \left[ \mathbf{E} \times \mathbf{B} + \left( \frac{\mathbf{v}}{c} \times \mathbf{B} \right) \times \mathbf{B} + \left( \frac{\mathbf{v}}{c} \cdot \mathbf{E} \right) \mathbf{E} \right] \right\}$$

(24)

$$- \frac{\mathbf{v}}{c} \gamma^2 \left[ \left( \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right)^2 - \left( \frac{\mathbf{v}}{c} \cdot \mathbf{E} \right)^2 \right].$$
Here \( \frac{d}{dt} = \frac{\partial}{\partial t} + \bm{v} \cdot \nabla \) and \( \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \). We will now analyze the effect of this force for a charge in an oscillatory plane wave.

The first term, \( f_o \), in line (23) is then of an oscillatory character. It can, in fact, be combined with the Lorentz force, \( f_l(t) + f_o(t) = qE(t) + q\frac{v}{c} \times B(t) \) (25)

\[
\approx qE(t + \epsilon_t) + q\frac{v}{c} \times B(t + \epsilon_t) = f_l(t + \epsilon_t),
\]

(26)

and we see that the effect of this term is to make a tiny time translation of the Lorentz force with \( \epsilon_t = \frac{2}{3} \frac{q^2 mc^3}{\gamma^2} \). This essentially is the time needed for light to pass across the classical electron radius. This is unlikely to have any noticeable physical effects.

The essential radiation damping force is thus \( f_s \). All its terms (24) are quadratic in the fields so there is hope that this force, \( f_s \), produces some net work on the particle. In the rest frame \( (v = 0) \) we simply get,

\[
f_s(v = 0) = \frac{2}{3} \frac{q^4}{m^2 c^4} E \times B,
\]

(27)

and therefore the particle starts to accelerate in the direction of the Poynting vector. What happens when it picks up speed?

Assume that the field is of the type in Eq. (11) with oscillatory \( E = E(\zeta) \). Since \( E = Ee_x, B = Ee_y, \) and \( E \times B = E^2 e_z \), some algebra then shows that our radiation reaction force, quadratic in the fields, is

\[
f_s = \frac{2}{3} \frac{q^4}{m^2 c^4} E^2 \left( 1 - \frac{v_z}{c} \right) \left[ e_z - \gamma^2 \left( 1 - \frac{v_z}{c} \right) \frac{v}{c} \right].
\]

(28)

We see that the effect of this force is always to accelerate the particle in the direction, \( e_z \), of the Ponting vector. The dissipative, \( -v \), term is one order higher in \( v/c \) and will thus only become important near relativistic speeds. The force goes to zero when \( v \to ce_z \), but it does not change direction.

Surprisingly we find that this force, which usually is called the radiation damping force, does not damp anything in this case. Instead its effect is the acceleration of a charged particle in the direction of propagation of a travelling electromagnetic wave.

5 Conclusions

Above we have presented an elegant matrix method that yields an exact solution for the motion of a particle in an electromagnetic radiation field with
a given time dependence. This generalizes some older results for constant fields of Salingaros [4, 5]. The concise explicit formula of Eq. (28) for the radiation damping force in a plane wave might be also be a new contribution. The main aim of this article has been to illuminate the way in which classical electrodynamics leads to a pushing of charges along the direction of of wave propagation.

Just as in solar sailing, this is a very small force, but in space where there normally are no dissipative forces, even a small force can have very large effect in the long run. Note that there is no need for a force to be stronger than gravity to accelerate things outwards from the sun (a common misunderstanding). A typical object orbits the sun in a stationary ellipse. When a small force is added this ellipse will gradually change, and with time, these changes can become very large.

References

[1] Gold T 2003 The solar sail and the mirror, at URL: http://xxx.soton.ac.uk/html/physics/0306050

[2] Friedman L 1988 Starsailing: solar sails and interstellar travel (New York: Wiley), see also URL (2003): http://www.planetary.org/solarsail/ss_and_physics.html

[3] Landau L D and Lifshitz E M 1975 The classical theory of fields, 4th ed (Oxford: Pergamon)

[4] Salingaros N 1985 Phys. Rev. D 31 3150

[5] Salingaros N 1987 Am. J. Phys. 55 352

[6] Hyman A T 1997 Am. J. Phys. 65 195

[7] Muños G 1997 Am. J. Phys. 65 429

[8] McMillan E M 1950 Phys. Rev. 79 498

[9] McDonald K T and Shmakov K 1998 Classical ”dressing” of a free electron in a plane electromagnetic wave in http://www.arxiv.org/abs/physics/0003059