LIMIT CYCLE BIFURCATIONS OF A PIECEWISE SMOOTH
HAMILTONIAN SYSTEM WITH A GENERALIZED
HETEROCLINIC LOOP THROUGH A CUSP

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Abstract. In this paper we study the limit cycle bifurcation of a piecewise
smooth Hamiltonian system. By using the Melnikov function of piecewise
smooth near-Hamiltonian systems, we obtain that at most 12n + 7 limit cycles
can bifurcate from the period annulus up to the first order in $\varepsilon$.

1. Introduction and the main results. The perturbed piecewise smooth diffe-
rential system has been attracted many researchers to study its limit cycles [1, 2,
8, 13, 16, 19]. The reason is that it can be applied in many natural fields such as
in nonlinear oscillations [17], automatic control [18], economics [7], biology [10] and
so on. Another interesting and important reason is that this problem can be seen
as an extension of the Hilbert’s 16th problem to the piecewise differential system.
We recall that the Hilbert’s 16th problem asks for the maximum number of limit
cycles that bifurcate from a perturbation of a period annulus. For the moment
this problem remains open, for more details on the 16th Hilbert’s problem see for
instance [9,12,20] and the references therein.

Consider a piecewise smooth differential system with discontinuity on the $y$ axis
\[
\dot{x} = f(x, y), \quad \dot{y} = g(x, y), \quad x \neq 0,
\]

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where

\[
(f(x, y), g(x, y)) = \begin{cases} 
(f^+(x, y), g^+(x, y)), & x > 0, \\
(f^-(x, y), g^-(x, y)), & x < 0,
\end{cases}
\]

and \(f^\pm, g^\pm \in C^\infty(\mathbb{R}^2)\). Then the functions define two systems below

\[
\dot{x} = f^+(x, y), \quad \dot{y} = g^+(x, y)
\]

and

\[
\dot{x} = f^-(x, y), \quad \dot{y} = g^-(x, y).
\]

We call (1.2) and (1.3) the right subsystem and left subsystem of (1.1) respectively. Denote \(\Sigma = \{(x, y)|x = 0\}\), following Filippov [5] we distinguish three open regions in the discontinuity straight line \(\Sigma\).

1) The sliding region \(\Sigma^s\) where the vectors \(f(p)\) and \(g(p)\) with \(p \in \Sigma\) point inward \(\Sigma\).

2) The escaping region \(\Sigma^e\) where the vectors \(f(p)\) and \(g(p)\) with \(p \in \Sigma\) point outward \(\Sigma\).

3) The sewing region \(\Sigma^s\) where the vectors \(f(p)\) and \(g(p)\) with \(p \in \Sigma\) point to the same direction and are transverse to \(\Sigma\).

Any segment contained in \(\Sigma^s \cup \Sigma^d\) is called a sliding segment. Any limit cycle \(\Gamma\) of (1.1) such that \(\Gamma \cap (\Sigma^s \cup \Sigma^d) = \emptyset\) is called a non-sliding limit cycle.

From [3, 5, 11], we know that the flow of (1.1), denoted by \(\phi(t, A)\), can be defined by using the flows \(\phi^\pm(t, A)\) of (1.2) and (1.3). For any point \(A \in \mathbb{R}_+^2 \cup \mathbb{R}_-^2\), where \(\mathbb{R}_\pm^2 = \{(x, y)|\pm x > 0\}\), we have

\[
\phi(t, A) = \phi^+(t, A) \quad \phi^-(t, A) \in \mathbb{R}_\pm^2.
\]

For a point \(A \notin \mathbb{R}_+^2 \cup \mathbb{R}_-^2\) satisfying \(f^+(A)f^-(A) > 0\), we define \(\phi(t, A)\) as follows

\[
\phi(t, A) = \begin{cases} 
\phi^-(t, A), & t < 0, \quad \phi^-(t, A) \in \mathbb{R}_-^2, \\
A, & t = 0, \\
\phi^+(t, A), & t > 0, \quad \phi^+(t, A) \in \mathbb{R}_+^2,
\end{cases}
\]

if \(f^+(t, A) > 0, f^-(t, A) > 0\); and

\[
\phi(t, A) = \begin{cases} 
\phi^+(t, A), & t < 0, \quad \phi^+(t, A) \in \mathbb{R}_+^2, \\
A, & t = 0, \\
\phi^-(t, A), & t > 0, \quad \phi^-(t, A) \in \mathbb{R}_-^2,
\end{cases}
\]

if \(f^+(t, A) < 0, f^-(t, A) < 0\).

Obviously, \(\phi(t, A) = A\) for all \(t\) if \(A \in \mathbb{R}_+^2\) or \(\mathbb{R}_-^2\) is a singular point of the corresponding system (1.2) or (1.3). In this case, \(A\) is also called a singular point of (1.1). For a point \(A \notin \mathbb{R}_+^2 \cup \mathbb{R}_-^2\) satisfying \(f^+(A)f^-(A) \leq 0, [6]\) gave the following definition.

**Definition 1.1.** [6] We call \(A \notin \mathbb{R}_+^2 \cup \mathbb{R}_-^2\) a generalized singular point of (1.1) and define \(\phi(t, A) = A\) for all \(t \in \mathbb{R}\) if \(f^+(A)f^-(A) \leq 0\).

In [6], Han and Zhang gave the following definitions.
Definition 1.2. [6] Let \( A_0 = (0, y_0) \) be a generalized singular point of (1.1). Suppose for some \( \varepsilon_0 > 0 \) the solution \( \phi(t, A) = (x(t, a), y(t, a)) \) of (1.1) starting at \( A = (0, a) \) for \( 0 < |a - y_0| < \varepsilon_0 \)

(i) \( T(a) = \min\{t > 0 | x(t, a) = 0, (a - y_0)(y(t, a) - y_0)\} \in (0, +\infty) \),
(ii) \( \lim_{a \to y_0} \phi(t, A) = A_0, \ 0 \leq a \leq T(a). \)

Define \( P(a) = \begin{cases} y(T(a), a), & 0 < |a - y_0| < \varepsilon_0, \\ y_0, & a = y_0. \end{cases} \)

We call \( P : (y_0 - \varepsilon_0, y_0 + \varepsilon_0) \to \mathbb{R} \) a Poincaré return map of (1.1) near \( A_0 \). Further, we call \( A_0 \) a center of (1.1) if \( P(a) = a \) for \( 0 < a - y_0 < \varepsilon_0 \). We call \( A_0 \) a focus of (1.1) if \( P(a) \neq a \) for \( 0 < a - y_0 < \varepsilon_0 \). We call \( A_0 \) is stable (resp., unstable) focus of (1.1) if \( P(a) < a \) (resp., \( P(a) > a \)) for \( 0 < a - y_0 < \varepsilon_0 \). We call \( A_0 \) a center-focus of (1.1) if it is neither a center nor a focus.

Definition 1.3. [6] Let \( A_0 = (0, y_0) \) be either a center, a focus or a center-focus of (1.1). We say it is elementary if one of the conditions is satisfied:

(i) \( A_0 \) is elementary as a singular point of both (1.2) and (1.3);
(ii) \( A_0 \) is elementary as a singular point of (1.2), and
\[
\begin{align*}
\text{f}^- (A_0) &= 0, \quad \text{f}^-_y (A_0) g^- (A_0) \neq 0; \\
\text{f}^+ (A_0) &= 0, \quad \text{f}^+_y (A_0) g^+ (A_0) \neq 0;
\end{align*}
\]
(iii) \( A_0 \) is elementary as a singular point of (1.3), and
\[
\begin{align*}
\text{f}^+ (A_0) &= 0, \quad \text{f}^+_y (A_0) g^+ (A_0) \neq 0; \\
\text{f}^- (A_0) &= 0, \quad \text{f}^-_y (A_0) g^- (A_0) \neq 0.
\end{align*}
\]
(iv) \( f^\pm (A_0) = 0, \ f^\pm_y (A_0) g^\pm (A_0) \neq 0. \)

By [4,5], system (1.1) has four possible types of foci as follows:

(i) Points of focus-focus type at \( A \in \Sigma \): both system (1.2) and (1.3) have a critical point at \( A \) with complex eigenvalues and their solutions turn around \( A \) counterclockwise.
(ii), (iii) Points of focus-parabolic (resp., parabolic-focus) type at \( A \in \Sigma \): the system defined in the right (resp., left) half plane has a critical point of focus type at \( A \) while the solutions of the system defined in neighbourhood of the left (resp., right) half plane have a parabolic contact (i.e., a second order contact point) with \( \Sigma \) at \( A \), the solution of which at this contact is locally contained in the right (resp., left) plane.
(iv) Points of parabolic-focus type at \( A \in \Sigma \): the solutions of both system have a parabolic contact at \( A \) with \( \Sigma \) in such a way that the flow induced by (1.1) turns around \( A \).

Observe that the parabolic-focus type can be reduced to the focus-parabolic case by applying the change of coordinates \( (x, y, t) \to (-x, -y, t) \) to (1.1).

In [15], Liu and Han considered the general form of a piecewise near-Hamiltonian system on the plane
\[
\begin{align*}
\dot{x} &= H_y + \varepsilon p(x, y), \\
\dot{y} &= -H_x + \varepsilon q(x, y),
\end{align*}
\]
(1.4)
where

\[ H(x, y) = \begin{cases} H^+(x, y), & x \geq 0, \\ H^-(x, y), & x < 0, \end{cases} \]

\[ p(x, y) = \begin{cases} p^+(x, y) = \sum_{i+j=0} a_{ij}^+ x^i y^j, & x \geq 0, \\ p^-(x, y) = \sum_{i+j=0} a_{ij}^- x^i y^j, & x < 0, \end{cases} \]

\[ q(x, y) = \begin{cases} q^+(x, y) = \sum_{i+j=0} b_{ij}^+ x^i y^j, & x \geq 0, \\ q^-(x, y) = \sum_{i+j=0} b_{ij}^- x^i y^j, & x < 0. \end{cases} \]

Assumption (I). There exist an interval \( J = (\alpha, \beta) \), and two points \( A(h) = (0, a(h)) \) and \( B(h) = (0, b(h)) \) such that for \( h \in J \)

\[ H^+(A(h)) = H^+(B(h)) = h, \quad H^-(A(h)) = H^-(B(h)) = \tilde{h}, \quad a(h) \neq b(h). \]

Assumption (II). The subsystem \( (1.6)|_{\varepsilon=0} \) has an orbital arc \( L_h^+ \) starting from \( A(h) \) and ending at \( B(h) \) defined by \( H^+(x, y) = h, \ x \geq 0 \); the subsystem \( (1.7)|_{\varepsilon=0} \) has an orbital arc \( L_h^- \) starting from \( B(h) \) and ending at \( A(h) \) defined by \( H^-(x, y) = H^-(B(h)), \ x < 0 \).

Under the Assumptions (I) and (II), \( (1.4)|_{\varepsilon=0} \) has a family of non-smooth periodic orbits \( L_h = L_h^+ \cup L_h^- \), \( h \in J \). For definiteness, we assume that the orbits \( L_h \) for \( h \in J \) orientate clockwise; see Fig.1. By Theorem 1.1 in [15], the first order Melnikov function of system (1.4) has the form

\[ M(h, \delta) = \frac{H_y^+(A)}{H_y^-(A)} \left[ \frac{H_y^-(B)}{H_y^+(B)} \int_{L_h^+} q^+ \, dx - p^+ \, dy + \int_{L_h^-} q^- \, dx - p^- \, dy \right], \quad h \in J. \]

Let

\[ M^+(h) = \int_{L_h^+} q^+ \, dx - p^+ \, dy, \quad M^-(h) = \int_{L_h^-} q^- \, dx - p^- \, dy, \]

\[ \tilde{M}^-(\tilde{h}) = \int_{H^-(x, y) = \tilde{h}, \ x \leq 0} q^- \, dx - p^- \, dy, \]

where \( \tilde{h} \) is given in Assumption (I). Then we know

\[ M^-(h) = \tilde{M}^-(H^-(B(h)), \delta), \quad h \in J. \]
As in the smooth case, a very important issue associated with (1.4) is to find the number of limit cycles and their distribution. Liang, Han and Romanovski [14] studied the following piecewise near Hamiltonian system
\[
\begin{align*}
\dot{x} &= -y + \varepsilon p^+(x, y), \quad x \geq 0, \\
\dot{y} &= 1 - x + \varepsilon q^+(x, y), \\
\dot{x} &= -y + \varepsilon p^-(x, y), \quad x < 0,
\end{align*}
\]
which has a generalize homoclinic loop with \( \varepsilon = 0 \). They studied the Hopf bifurcation, homoclinic bifurcation and Poincaré bifurcation of (1.11).

Motivated by [14, 15], in this paper, we consider a piecewise smooth near-Hamiltonian system of the form
\[
\begin{align*}
\dot{x} &= y + \varepsilon p^+(x, y), \quad x \geq 0, \\
\dot{y} &= 1 - x + \varepsilon q^+(x, y), \\
\dot{x} &= y + \varepsilon p^-(x, y), \quad x < 0,
\end{align*}
\]
where \( p^\pm(x, y) \) and \( q^\pm(x, y) \) are defined as (1.5). Then we have
\[
\begin{align*}
H^+(x, y) &= \frac{1}{2}y^2 + \frac{1}{2}x^3 - \frac{3}{2}x^2 + \frac{3}{2}x, \quad x \geq 0, \\
H^-(x, y) &= \frac{1}{2}y^2 - \frac{1}{2}x^2 - x - \frac{1}{2}, \quad x < 0.
\end{align*}
\]
Obviously, the limit cycles of system (1.12)|\( \varepsilon = 0 \) are all non-sliding limit cycles.

By Definitions 1.1-1.3, we know that (1.12)|\( \varepsilon = 0 \) has a hyperbolic saddle point (-1,0), a cusp (1,0) and a elementary center (0,0) which is a parabolic-parabolic type generalized singular point. For system (1.12)|\( \varepsilon = 0 \), there exist a family of periodic orbits as follows
\[
L_h = \{(x, y)|H^+(x, y) = \frac{h}{2}, x \geq 0\} \cup \{(x, y)|H^-(x, y) = \frac{\tilde{h}}{2}, x < 0\},
\]
with \( \tilde{h} = h - 1, \quad 0 < h < 1 \). For the sake of convenience, here we use \( h/2 \) instead of \( h \). If \( h \to 0 \), \( L_h \) approaches the origin. And if \( h \to 1 \), \( L_h \to L_0 \), where \( L_0 \) is a generalized heteroclinic loop with a cusp; see Fig. 2.
Figure 2. Phase portrait of system (1.12)|_{\varepsilon=0}

Note that $H^+_y(0, y) \equiv H^-_y(0, y)$ for $-1 < y < 1$. Then by (1.8), we have the first order Melnikov function of system (1.12) satisfying

$$M(\frac{h}{2}) = \int_{\hat{AB}} q^+ dx - p^+ dy + \int_{\hat{BA}} q^- dx - p^- dy = M^+(\frac{h}{2}) + M^-(\frac{h}{2}) := \mathcal{M}(h),$$

(1.15)

where $0 < h < 1$, and

$A = (0, \sqrt{h})$, $B = (0, -\sqrt{h})$,

$\hat{AB} = \{(x, y)|H^+(x, y) = \frac{h}{2}, x \geq 0\}$,

$\hat{BA} = \{(x, y)|H^-(x, y) = \frac{h}{2}, x < 0\}$.

Our main results are the following two theorems.

**Theorem 1.4.** For system (1.12), the first order Melnikov function has the following form

$$\mathcal{M}(h) = \sqrt{h}f_n(\sqrt{h}) + g_1(h)I_{10}(h) + \bar{g}_1(h)I_{20}(h),$$

(1.16)

where $0 < h < 1$, $f_i(u)$, $g_i(u)$, $\bar{g}_i(u)$ and $\bar{g}_i(u)$ are polynomial of degree $i$ in $u$, and

$$I_{10}(h) = \int_0^{\sqrt{h}} (1 + y^2 - h)^{\frac{3}{2}} dy \quad (i = 1, 2), \quad J_{10}(h) = \int_0^{\sqrt{h}} (1 + y^2 - h)^{\frac{5}{2}} dy.$$ (1.17)

**Theorem 1.5.** Using the first order Melnikov function, the upper bound for the number of limit cycles of the system (1.12) of degree $n$ bifurcating from the periodic orbits of system (1.12) with $\varepsilon = 0$ is $12n + 7$.

2. The first order Melnikov function.

**Lemma 2.1.** For system (1.12), the expansion of $M^+(\frac{h}{2})$ has the following form

$$M^+(\frac{h}{2}) = \sqrt{h} \sum_{k=0}^{[\frac{h}{2}]} \frac{2^{n-k} h^k}{2k + 1} - \sqrt{h} \mu_{1(\frac{3n-5}{2})}(h) - \mu_{1(\frac{n-1}{2})}(h)I_{10}(h) - \mu_{1(\frac{3n-5}{2})}(h)I_{20}(h),$$

where $0 < h < 1$, $\mu_i(u)$ is polynomial of degree $i$ in $u$, and $I_{10}(h)$ is defined as (1.17) for $i=1, 2$. 

**Proof.** Applying the Green’s formula we obtain

\[
M^+\left(\frac{h}{2}\right) = \int_{AB} q^+ dx - p^+ dy
\]

\[
= \int \int_{\text{int}(AB∪BA)} (p^+_x + q^+_y) dx dy + \int_{BA} p^+(0, y) dy
\]

\[
= \int_{AB} \bar{p}(x, y) dy + \int_{BA} p^+(0, y) dy
\]

where

\[
\bar{p}(x, y) = p^+(x, y) - p^+(0, y) + \int_0^x q^+_y(u, y) dy.
\]

(2.1)

Since \(AB\) can be represented as \(x = 1 - \sqrt{1 + y^2 - h} \equiv \Phi(y, h)\), we have

\[
M^+\left(\frac{h}{2}\right) = -\int_{-\sqrt{h}}^{\sqrt{h}} \bar{p}(\Phi(y, h), y) dy + \int_{-\sqrt{h}}^{\sqrt{h}} p^+(0, y) dy := -I_0(h) + I_1(h).
\]

(2.2)

Note that \(p^+(0, y) = \sum_{j=0}^n a^+_{0j} y^j\). It follows that

\[
I_1(h) = \int_{-\sqrt{h}}^{\sqrt{h}} p^+(0, y) dy = \sum_{j=0}^n a^+_{0j} (1 - (-1)^{j+1}) h^{j+1} = \sqrt{h} \sum_{k=0}^{[\frac{j}{2}]} \frac{2a^+_{0,2k}}{2k+1} h^k.
\]

(2.3)

By (2.1) we have

\[
\bar{p}(x, y) = \sum_{i+j=0}^n a^+_{ij} x^i y^j - \sum_{j=0}^n a^+_{0j} y^j + \sum_{i+j=0}^n \frac{b^+_{ij}}{i+1} x^{i+1} y^{j-1} = x \sum_{i+j=0}^n p^+_{ij} x^i y^j.
\]

(2.4)

where

\[
p^+_{ij} = a^+_{i+1,j} + \frac{j+1}{i+1} h^+_{i,j+1}.
\]

The definition of \(I_0(h)\) and (2.4) yield

\[
I_0(h) = \int_{-\sqrt{h}}^{\sqrt{h}} \bar{p}(\Phi(y, h), y) dy
\]

\[
= \sum_{i+j=0}^{n-1} p^+_{ij} \int_{-\sqrt{h}}^{\sqrt{h}} (1 - \sqrt{1 + y^2 - h})^{i+1} y^j dy
\]

\[
= 2 \sum_{i+2k=0}^{n-1} p^+_{i,2k} \int_0^{\sqrt{h}} (1 - \sqrt{1 + y^2 - h})^{i+1} y^{2k} dy.
\]

(2.5)

It is easy to get that

\[
\int_0^{\sqrt{h}} y^{2k}(1 - \sqrt{1 + y^2 - h})^{i+1} dy = \sum_{r=0}^{i+1} C^r_{i+1} (-1)^r I_{rk}(h),
\]

where

\[
I_{rk}(h) = \int_0^{\sqrt{h}} (1 + y^2 - h)^{\frac{r}{2}} y^{2k} dy.
\]
If $r = 3l$, $l \in \mathbb{N}$, we see

$$I_{3l,k}(h) = \int_0^{\sqrt{h}} y^{2k}(1 + y^2 - h)^{l} dy = \sqrt{h} \nu_{l+k}(h), \quad (2.6)$$

where $\nu_{l+k}(h)$ is a polynomial of degree $l + k$ in $h$.

If $r = 3l + 1$, $l \in \mathbb{N}$, then

$$I_{3l+1,k}(h) = \int_0^{\sqrt{h}} y^{2k}(1 + y^2 - h)^{l + \frac{1}{2}} dy. \quad (2.7)$$

Using the formula

$$\int y^{2k}(1+y^2-h)^{l+\frac{1}{2}} dy = A(l)y^{2k+1}(1+y^2)(1-h)^{l+\frac{1}{2}} + B(l)(1-h) \int y^{2k}(1+y^2-h)^{l-\frac{1}{2}} dy,$$

where

$$A(l) = \frac{1}{2(l + \frac{1}{3}) + 2k + 1}, \quad B(l) = \frac{2(l + \frac{1}{3})}{2(l + \frac{1}{3}) + 2k + 1},$$

we have

$$I_{3l+1,k}(h) = A(l)h^{k+\frac{1}{2}} \varphi_{l-1}(h) + B(l)(1-h)I_{3l-2,k}(h), \quad l \geq 1, \quad k \geq 0.$$ 

It follows that

$$I_{3l+1,k}(h) = h^{k+\frac{1}{2}} \varphi_{l-1}(h) + \alpha(1-h)^{l}I_{1k}(h), \quad l \geq 1, \quad k \geq 0, \quad (2.8)$$

where

$$\varphi_{l-1}(h) = A(l) + \sum_{j=1}^{l-1} A(l-j)(1-h)^{j} \prod_{i=0}^{j-1} B(l-i), \quad \alpha = \prod_{i=0}^{l-1} B(l-i),$$

$\varphi_{l-1}(h)$ is a polynomial of degree $l - 1$ in $h$ and $\alpha$ is a constant. For $k \geq 0$, let

$$\varphi^*_l(h) = \begin{cases} \varphi_{l-1}(h), & l \geq 1, \\ 0, & l = 0, \end{cases} \quad \alpha^* = \begin{cases} \alpha, & l \geq 1, \\ 1, & l = 0. \end{cases}$$

By (2.8),

$$I_{3l+1,k}(h) = h^{k+\frac{1}{2}} \varphi^*_l(h) + \alpha^*(1-h)^{l}I_{1k}(h), \quad l \geq 0, \quad k \geq 0. \quad (2.9)$$

Further, using the following formula

$$\int y^{2k}(1+y^2-h)^{l+\frac{1}{2}} dy = A_1(k)y^{2k-1}(1+y^2-h)^{\frac{1}{2}} - B_1(k)(1-h) \int y^{2k-2}(1+y^2-h)^{\frac{1}{2}} dy, \quad (2.10)$$

where

$$A_1(k) = \frac{1}{2(k + \frac{1}{3})}, \quad B_1(k) = \frac{2k-1}{2(k + \frac{1}{3})},$$

we obtain

$$I_{1k}(h) = A_1(k)h^{k-\frac{1}{2}} - B_1(k)(1-h)I_{1,k-1}(h), \quad k \geq 1.$$ 

Thus

$$I_{1k}(h) = \sqrt{h} \xi_{k-1}(h) + \beta(1-h)^{k}I_{10}(h), \quad k \geq 1, \quad (2.11)$$
where

\[ \xi_{k-1}(h) = A_1(k)h^{k-1} + \sum_{j=1}^{k-1} (-1)^j A_1(k - j)h^{k-j-1}(1 - h)^j \prod_{i=0}^{j-1} B_1(k - i), \]

\[ \beta = (-1)^k \prod_{i=0}^{k-1} B_1(k - i), \]

\( \xi_{k-1}(h) \) is a polynomial of degree \( k - 1 \) in \( h \) and \( \beta \) is a constant. Let

\[ \xi^*_{k-1}(h) = \begin{cases} \xi_{k-1}(h), & k \geq 1, \\ 0, & k = 0, \end{cases} \]

\[ \beta^* = \begin{cases} \beta, & k \geq 1, \\ 1, & k = 0. \end{cases} \]

By (2.11),

\[ I_{1k}(h) = \sqrt{h}\xi^*_{k-1}(h) + \beta^*(1 - h)^k I_{10}(h), \quad k \geq 0. \quad (2.12) \]

From (2.9) and (2.12), we have

\[ I_{3l+1,k} = h^{k+\frac{1}{2}} \varphi^*_l(1 - h)^{\frac{1}{2}} + \sqrt{\alpha^*} \xi^*_{k-1}(h)(1 - h)^l + \alpha^* \beta^*(1 - h)^{l+k} I_{10}(h) \]

\[ \equiv \sqrt{h}\bar{\varphi}_{l+k-1}(h) + \bar{\alpha}(1 - h)^{l+k} I_{10}(h), \quad (2.13) \]

where \( \varphi_{l+k-1}(h) \) is a polynomial of degree \( l + k - 1 \) in \( h \), \( \bar{\alpha} \) is non-zero constant, \( 0 \leq k \leq \left[ \frac{n-1}{3} \right] \) and \( 0 \leq 3l + 1 \leq n \).

If \( r = 3l + 2 \), \( l \in \mathbb{N} \), then

\[ I_{3l+2,k}(h) = \int_0^{\sqrt{h}} y^{2k}(1 + y^2 - h)^{l+\frac{3}{2}} dy. \quad (2.14) \]

Similar to formula (2.8), we have

\[ I_{3l+2,k}(h) = h^{k+\frac{1}{2}} \psi^*_{l-1}(h) + \gamma(1 - h)^l I_{2k}(h), \quad l \geq 1, \quad k \geq 0, \quad (2.15) \]

where

\[ \psi^*_{l-1}(h) = \frac{1}{2l + 2k + 7/3} + \sum_{j=1}^{l-1} \frac{(1 - h)^j}{2l + 2k - 2j + 7/3} \prod_{i=0}^{j-1} \frac{2l - 2i + 4/3}{2l + 2k - 2i + 5/3}, \]

\[ \gamma = \prod_{i=0}^{l-1} \frac{2l - 2i + 4/3}{2l + 2k - 2i + 5/3}, \]

\( \psi_{l-1}(h) \) is a polynomial of degree \( l - 1 \) in \( h \) and \( \gamma \) is a constant. For \( k \geq 0 \), let

\[ \psi^*_{l-1}(h) = \begin{cases} \psi_{l-1}(h), & l \geq 1, \\ 0, & l = 0, \end{cases} \]

\[ \gamma^* = \begin{cases} \gamma, & l \geq 1, \\ 1, & l = 0. \end{cases} \]

we get

\[ I_{3l+2,k}(h) = h^{k+\frac{1}{2}} \psi^*_{l-1}(h) + \gamma^*(1 - h)^l I_{2k}(h), \quad l \geq 0, \quad k \geq 0. \quad (2.16) \]

Similar to formula (2.11), we have

\[ I_{2k}(h) = \sqrt{h}\eta_{k-1}(h) + \delta(1 - h)^k I_{20}(h), \quad k \geq 1, \quad (2.17) \]
where
\[ \eta_{k-1}(h) = \frac{h^{k-1}}{2k + 14/3} + \sum_{j=1}^{k-1} \frac{h^{k-j-1}(h-1)^j}{2k - 2j + 14/3} \prod_{i=0}^{j-1} \frac{2k - 2i - 1}{2k - 2i + 14/3}, \]
\[ \delta = (-1)^k \prod_{i=0}^{k-1} \frac{2k - 2i - 1}{2k - 2i + 14/3}. \]

\( \eta_{k-1}(h) \) is a polynomial of degree \( k - 1 \) in \( h \) and \( \delta \) is a constant. Let
\[ \eta_{k-1}^*(h) = \begin{cases} \eta_{k-1}(h), & k \geq 1, \\ 0, & k = 0, \end{cases} \]
\[ \delta^* = \begin{cases} \delta, & k \geq 1, \\ 1, & k = 0. \end{cases} \]

By (2.17),
\[ I_{2k}(h) = \sqrt{h} \eta_{k-1}^* (1-h)^k I_{20}(h), \quad k \geq 0. \tag{2.18} \]

From (2.16) and (2.18), we obtain
\[ I_{3l+2,k} = h^{k+\frac{1}{2}} \psi_{*+1}^*(h) + \sqrt{h} \gamma^* \eta_{k-1}^* (1-h)^{l+k} + \gamma^* \delta^* (1-h)^{l+k} I_{10}(h) \]
\[ := \sqrt{h} \psi_{*+k-1}(h) + \gamma (1-h)^{l+k} I_{20}(h), \tag{2.19} \]

where \( \psi_{*+k-1}(h) \) is a polynomial of degree \( l + k - 1 \) in \( h \), \( \gamma \) is non-zero constant, \( 0 \leq k \leq \lfloor \frac{n-1}{2} \rfloor \) and \( 0 \leq 3l + 2 \leq n \).

Noting that \( \lfloor \frac{i+1}{3} \rfloor + k \leq \lfloor \frac{3n-1}{6} \rfloor \), \( \lfloor \frac{i}{3} \rfloor + k \leq \lfloor \frac{a-1}{2} \rfloor \) and \( \lfloor \frac{i-1}{3} \rfloor + k \leq \lfloor \frac{3n-5}{6} \rfloor \), and substituting (2.6), (2.13) and (2.19) into (2.5), we obtain
\[ I_0(h) = 2 \sum_{i+2k=0}^{n-1} \psi_{i+2k}^* \left( \sum_{r=0}^{i+1} (-1)^r C_{i+1}^r I_{rk} \right) + \sum_{r=0}^{i+1} (-1)^r C_{i+1}^r I_{rk} + \sum_{r=0}^{i+1} (-1)^r C_{i+1}^r I_{rk} \]
\[ = 2 \sum_{i+2k=0}^{n-1} \psi_{i+2k}^* \left[ \sqrt{h} \mu_{[\frac{i+1}{3}]+k}(h) + \mu_{[\frac{i}{3}]}(h)(1-h)^k I_{10}(h) \right] \]
\[ + \mu_{[\frac{i-1}{3}]}(h)(1-h)^k I_{20}(h), \]

where \( \mu_j(h) \) denotes a polynomial of \( h \) of degree \( j \). From (2.2), (2.3) and (2.20), we have
\[ M^+ \left( \frac{h}{2} \right) = -I_0(h) + I_1(h) \tag{2.21} \]
\[ = \sqrt{h} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{2^{2k+1} h^{k}}{2k+1} - \sqrt{h} \mu_{[\frac{i+1}{3}]}(h) - \mu_{[\frac{i-1}{3}]}(h)(1-h)^k I_{10}(h) - \mu_{[\frac{n-1}{6}]}(h) I_{20}(h). \]

**Lemma 2.2.** For system (1.12), the expansion of \( M^- \left( \frac{h}{2} \right) \) has the following form
\[ M^- \left( \frac{h}{2} \right) = -\sqrt{h} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{2^{2k+1} h^{k}}{2k+1} + \sqrt{h} \tilde{\mu}_{[\frac{1}{3}]}(h) + \tilde{\mu}_{[\frac{n-1}{6}]}(h) I_{10}(h). \]
where \( 0 < h < 1 \), \( \mu_i(u) \) is a polynomial of degree \( i \) in \( u \), and \( J_{10}(h) \) is defined as (1.17).

**Proof.** Similar to formula (2.2), we obtain

\[
M^{-\left(\frac{h}{2}\right)} = \int_{\mathbb{B}^n} q^+ dx - p^- dy = \int_{-\sqrt{h}}^{\sqrt{h}} \tilde{p}(\Phi(y, h), y) dy - \int_{-\sqrt{h}}^{\sqrt{h}} p^-(0, y) dy \tag{2.22}
\]

where

\[
\Phi(y, h) = \sqrt{1 + y^2 - h} - 1, \quad \tilde{p}(x, y) = p^-(x, y) - p^-(0, y) + \int_0^x q^+_0(u, y) dy. \tag{2.23}
\]

It is easy to get that

\[
J_1(h) = \int_{-\sqrt{h}}^{\sqrt{h}} p^-(0, y) dy = \sqrt{h} \sum_{k=0}^{[\frac{r}{2}]} \frac{2a_{0,2k}}{2k + 1} h^k. \tag{2.24}
\]

(2.23) gives

\[
\tilde{p}(x, y) = \sum_{i+j=0}^{n} a_{ij} x^i y^j - \sum_{j=0}^{n} a_{0j} y^j + \sum_{i+j=0}^{n} \frac{j b_{ij}}{i+1} x^{i+1} y^{j-1} := x \sum_{i+j=0}^{n-1} p_{ij} x^i y^j, \tag{2.25}
\]

where

\[
p_{ij} = a_{i+1,j} + \frac{j+1}{i+1} b_{i,j+1}.
\]

By the definition of \( J_0(h) \) and (2.23), we have

\[
J_0(h) = \int_{-\sqrt{h}}^{\sqrt{h}} \tilde{p}(\Phi(y, h), y) dy
\]

\[
= \sum_{i+j=0}^{n-1} p_{ij} \int_{-\sqrt{h}}^{\sqrt{h}} (\sqrt{1 + y^2 - h} - 1)^{i+1} y^j dy
\]

\[
= 2 \sum_{i+2k=0}^{n-1} p_{i,2k} (-1)^{i+1} \int_{0}^{\sqrt{h}} (\sqrt{1 + y^2 - h} - 1)^{i+1} y^{2k} dy
\]

\[
= 2 \sum_{i+2k=0}^{n-1} p_{i,2k} (-1)^{i+1} \sum_{r=0}^{i+1} C_{i+1}^r (-1)^r J_{rk}(h),
\]

where

\[
J_{rk}(h) = \int_{0}^{\sqrt{h}} (1 + y^2 - h)^\frac{r}{2} y^{2k} dy.
\]

If \( r = 2l, l \in \mathbb{N} \), we see

\[
I_{2l,k}(h) = \int_{0}^{\sqrt{h}} (1 + y^2 - h)^l y^{2k} dy = \sqrt{h} \tilde{v}_{l+k}(h), \tag{2.27}
\]

where \( \tilde{v}_{l+k}(h) \) is a polynomial of degree \( l + k \) in \( h \).

If \( r = 2l + 1, l \in \mathbb{N} \), then

\[
I_{2l+1,k}(h) = \int_{0}^{\sqrt{h}} (1 + y^2 - h)^{l+\frac{1}{2}} y^{2k} dy. \tag{2.28}
\]
Using the formula
\[
\int y^{2k}(1+y^2-h)^{l+\frac{1}{2}}dy = A_3(l)y^{2k+1}(1+y^2-h)^{l+\frac{1}{2}} + B_3(l)(1-h)\int y^{2k}(1+y^2-h)^{l-\frac{1}{2}}dy,
\]
where
\[
A_3(l) = \frac{1}{2(l+k+1)}, \quad B_3(l) = \frac{2l+1}{2(l+k+1)},
\]
which means that
\[
J_{2l+1,k}(h) = A_3(l)h^{k+\frac{1}{2}} + B_3(l)(1-h)J_{2l-1,k}(h), \quad l \geq 1, \quad k \geq 0.
\]
It follows that
\[
J_{2l+1,k}(h) = h^{k+\frac{1}{2}}\tilde{\varphi}_{l-1}(h) + \tilde{\alpha}(1-h)J_1(k)(h), \quad l \geq 1, \quad k \geq 0, \quad (2.29)
\]
where
\[
\tilde{\varphi}_{l-1}(h) = A_3(l) + \sum_{j=1}^{l-1} A_3(l-j)(1-h)^j \prod_{i=0}^{j-1} B_3(l-i), \quad \tilde{\alpha} = \prod_{i=0}^{l-1} B_3(l-i),
\]
\[
\tilde{\varphi}_{l-1}(h) \text{ is a polynomial of degree } l-1 \text{ in } h \text{ and } \tilde{\alpha} \text{ is a constant. For } k \geq 0, \text{ let}
\]
\[
\tilde{\varphi}_{l-1}(h) = \begin{cases} \tilde{\varphi}_{l-1}(h), & l \geq 1, \\ 0, & l = 0, \end{cases} \quad \tilde{\alpha} = \begin{cases} \tilde{\alpha}, & l \geq 1, \\ 1, & l = 0. \end{cases}
\]
By (2.29),
\[
J_{2l+1,k}(h) = h^{k+\frac{1}{2}}\tilde{\varphi}_{l-1}(h) + \tilde{\alpha}(1-h)J_1(k)(h), \quad l \geq 0, \quad k \geq 0. \quad (2.30)
\]
Using the formula
\[
\int y^{2k}(1+y^2-h)^{l+\frac{1}{2}}dy = A_4(k)y^{2k-1}(1+y^2-h)^{l+\frac{1}{2}} - B_4(k)(1-h)\int y^{2k-2}(1+y^2-h)^{l+\frac{1}{2}}dy,
\]
where
\[
A_4(k) = \frac{1}{2(k+1)}, \quad B_4(k) = \frac{2k-1}{2(k+1)}.
\]
It follows that
\[
J_{1k}(h) = A_4(k)h^{k-\frac{1}{2}} - B_4(k)(1-h)J_{1,k-1}(h), \quad k \geq 1.
\]
Thus
\[
J_{1k}(h) = \sqrt{h}\tilde{\xi}_{k-1}(h) + \tilde{\beta}(1-h)^k J_{10}(h), \quad k \geq 1, \quad (2.31)
\]
where
\[
\tilde{\xi}_{k-1}(h) = A_4(k)h^{k-1} + \sum_{j=1}^{k-1} (-1)^j A_4(k-j)h^{k-j-1}(1-h)^j \prod_{i=0}^{j-1} B_4(k-i), \quad \tilde{\beta} = (-1)^k \prod_{i=0}^{k-1} B_4(k-i),
\]
\[
\tilde{\xi}_{k-1}(h) \text{ is a polynomial of degree } k-1 \text{ in } h \text{ and } \tilde{\beta} \text{ is a constant. Let}
\]
\[
\tilde{\xi}_{k-1}(h) = \begin{cases} \tilde{\xi}_{k-1}(h), & k \geq 1, \\ 0, & k = 0, \end{cases} \quad \tilde{\beta} = \begin{cases} \tilde{\beta}, & k \geq 1, \\ 1, & k = 0. \end{cases}
\]
By (2.31),
\[
J_{1k}(h) = \sqrt{h}\tilde{\xi}_{k-1}(h) + \tilde{\beta}(1-h)^k J_{10}(h), \quad k \geq 0. \quad (2.32)
\]
From (2.30) and (2.32), we obtain
\[
J_{2l+1,k} = h^{k+\frac{1}{2}}\tilde{\varphi}_{l-1}(h) + \sqrt{h}\tilde{\alpha}\tilde{\xi}_{k-1}(h)(1-h)^l + \tilde{\alpha}(1-h)^l J_{2l+1,k}(h) := \sqrt{h}\tilde{\alpha}(1-h)^l J_{10}(h), \quad (2.33)
\]
where $\tilde{\phi}_{i+k-1}(h)$ is a polynomial of degree $l+k-1$ in $h$, $\alpha$ is non-zero constant, $0 \leq k \leq \lfloor \frac{n-3}{2} \rfloor$ and $0 \leq 2l + 1 \leq n$.

Substituting (2.27) and (2.33) into (2.26), and noting that $\lfloor \frac{l+1}{2} \rfloor \leq \lfloor \frac{n-2}{2} \rfloor$ and $\lfloor \frac{j}{2} \rfloor \leq \lfloor \frac{n-2}{2} \rfloor$, we obtain

$$\begin{align*}
J_0(h) &= 2 \sum_{i+2k=0}^{n-1} p^{-}_{i,2k} \left( \sum_{r=0, r=2l}^{i+1} C_{r+1}^{r} I_{rk} - \sum_{r=0, r=2l+1}^{i+1} C_{r+1}^{r} I_{rk} \right) \\
&= 2 \sum_{i+2k=0}^{n-1} p^{-}_{i,2k} \left[ \sqrt{h} \tilde{\mu}_{\lfloor \frac{n-1}{2} \rfloor + k}(h) + \tilde{\mu}_{\lfloor \frac{j}{2} \rfloor}(h)(1-h)^{k} J_{10}(h) \right]
\end{align*} \tag{2.34}$$

where $\tilde{\mu}_{j}(h)$ denotes a polynomial of $h$ of degree $j$. From (2.22), (2.24) and (2.34), we have

$$
M^{-}\left(\frac{h}{2}\right) = J_{0}(h) - J_{1}(h)
= -\sqrt{h} \sum_{k=0}^{\lfloor \frac{j}{2} \rfloor} \frac{2a_{0,2k}}{2k+1} h^{k} + \sqrt{h} \tilde{\mu}_{\lfloor \frac{j}{2} \rfloor}(h) + \tilde{\mu}_{\lfloor \frac{n-1}{2} \rfloor}(h) J_{10}(h) \tag{2.35}
$$

Now we give the proof of Theorem 1.4. From (1.15), Lemmas 2.1 and 2.2, we have

$$
\begin{align*}
\overline{M}(h) &= M^{+}\left(\frac{h}{2}\right) + M^{-}\left(\frac{h}{2}\right) \\
&= \sqrt{h} \sum_{k=0}^{\lfloor \frac{j}{2} \rfloor} \frac{2a_{0,2k}}{2k+1} h^{k} - \sqrt{h} \tilde{\mu}_{\lfloor \frac{n-1}{2} \rfloor}(h) - \tilde{\mu}_{\lfloor \frac{n-1}{2} \rfloor}(h) I_{10}(h) - \tilde{\mu}_{\lfloor \frac{n-1}{2} \rfloor}(h) I_{20}(h)
\end{align*} \tag{2.36}$$

$$
\begin{align*}
&-\sqrt{h} \sum_{k=0}^{\lfloor \frac{j}{2} \rfloor} \frac{2a_{0,2k}}{2k+1} h^{k} + \sqrt{h} \tilde{\mu}_{\lfloor \frac{j}{2} \rfloor}(h) + \tilde{\mu}_{\lfloor \frac{n-1}{2} \rfloor}(h) J_{10}(h) \\
&:= \sqrt{h} f_{n}(\sqrt{h}) + g_{\lfloor \frac{n-1}{2} \rfloor}(h) I_{10}(h) + g_{\lfloor \frac{n-1}{2} \rfloor}(h) I_{20}(h) + g_{\lfloor \frac{n-1}{2} \rfloor}(h) J_{10}(h),
\end{align*}
$$

where $0 < h < 1$, $f_i(u)$, $g_i(u)$, $\tilde{g}_i(u)$ and $\tilde{g}_i(u)$ are polynomial of degree $i$ in $u$, and $I_{10}(h)$ ($i = 1, 2$) and $J_{10}(h)$ are defined as (1.17). Hence, the conclusions of Theorem 1.4 hold.

3. Proof of Theorem 1.5. From Theorem 1.4 the first order Melnikov function is

$$
\overline{M}(h) = \sqrt{h} f_{n}(\sqrt{h}) + g_{\lfloor \frac{n-1}{2} \rfloor}(h) I_{10}(h) + g_{\lfloor \frac{n-1}{2} \rfloor}(h) I_{20}(h) + g_{\lfloor \frac{n-1}{2} \rfloor}(h) J_{10}(h), \tag{3.1}
$$

where $0 < h < 1$,

$$
I_{10}(h) = \int_{0}^{\sqrt{h}} (1 + y^2 - h)^{\frac{i}{2}} dy, \tag{3.2}
$$

$$
I_{20}(h) = \int_{0}^{\sqrt{h}} (1 + y^2 - h)^{\frac{i}{2}} dy, \tag{3.2}
$$

$$
J_{10}(h) = \int_{0}^{\sqrt{h}} (1 + y^2 - h)^{\frac{i}{2}} dy.
$$
Let $y = \sqrt{1 - h^2} x$, we obtain

\[
I_{10}(h) = (1 - h) \int_0^1 \left( 1 + x^2 \right)^{\frac{h}{2}} dx,
\]
\[
I_{20}(h) = (1 - h) \int_0^1 \left( 1 + x^2 \right)^{\frac{h}{2}} dx,
\]
\[
J_{10}(h) = (1 - h) \int_0^1 \left( 1 + x^2 \right)^{\frac{h}{2}} dx.
\]

Let $\lambda = \sqrt{h}$, $\lambda \in (0, 1)$, (3.1) and (3.3) yield

\[
\overline{M}(h) = \lambda f_n(\lambda) + \rho_1(\lambda) I_1(\lambda) + \rho_2(\lambda) I_2(\lambda) + \rho_3(\lambda) I_3(\lambda), \quad 0 < \lambda < 1,
\]

where

\[
I_1(\lambda) = \int_0^1 \sqrt{1 - \lambda^2} \left( 1 + x^2 \right)^{\frac{h}{2}} dx,
\]
\[
I_2(\lambda) = \int_0^1 \sqrt{1 - \lambda^2} \left( 1 + x^2 \right)^{\frac{h}{2}} dx,
\]
\[
I_3(\lambda) = \int_0^1 \sqrt{1 - \lambda^2} \left( 1 + x^2 \right)^{\frac{h}{2}} dx,
\]

\[
\rho_1(\lambda) = g_{\frac{n-1}{2}}(\lambda^2)(1 - \lambda^2)^{\frac{h}{2}} = f_{2[n-1]}(\lambda)(1 - \lambda^2)^{\frac{h}{2}},
\]
\[
\rho_2(\lambda) = g_{\frac{n-3}{2}}(\lambda^2)(1 - \lambda^2)^{\frac{h}{2}} = f_{2[n-3]}(\lambda)(1 - \lambda^2)^{\frac{h}{2}},
\]
\[
\rho_3(\lambda) = g_{\frac{n-5}{2}}(\lambda^2)(1 - \lambda^2)^{\frac{h}{2}} = f_{2[n-5]}(\lambda)(1 - \lambda^2)^{\frac{h}{2}},
\]

where $f_i(\lambda)$ is a polynomial of degree $i$ in $\lambda$. For the sake of simplicity, in the sequel, we always omit the $\lambda$ in the following functions in case of no ambiguity. Then we have

\[
M_1(\lambda) = \frac{d}{d\lambda} \left( \overline{M}(\rho_1) \right) = \left( \frac{\lambda f_n}{\rho_1} \right)' (1 - \lambda^2)^{-\frac{h}{2}} + \left( \frac{\rho_2}{\rho_1} \right)' I_2 + \frac{\rho_2}{\rho_1} (1 - \lambda^2)^{-2}
\]
\[
= \left( \frac{\lambda f_n}{\rho_1} \right)' \rho_1 - \lambda f_n \rho_1' + \rho_2^2 (1 - \lambda^2)^{-\frac{h}{2}} + \rho_1 \rho_2 (1 - \lambda^2)^{-\frac{h}{2}} + \rho_1 \rho_3 (1 - \lambda^2)^{-\frac{h}{2}}
\]
\[
= \left( \frac{\rho_2}{\rho_1} \right)' I_2 + \left( \frac{\rho_3}{\rho_1} \right)' I_3.
\]

\[
M_2(\lambda) = \frac{d}{d\lambda} \left( \left( \frac{\rho_2}{\rho_1} \right)' \right)^{-1} M_1(\lambda) = \frac{\left( f_{n+2[n-1]+2}[\lambda^2](1 - \lambda^2)^{-\frac{h}{2}} \right)}{(\rho_2^2 \rho_1 - \rho_2 \rho_1')},
\]
\[
+ (1 - \lambda^2)^{-\frac{h}{2}} + \frac{\rho_3 \rho_1 - \rho_3 \rho_1'}{\rho_2^2 \rho_1 - \rho_2 \rho_1'} (1 - \lambda^2)^{-2} + \left( \frac{\rho_3 \rho_1 - \rho_3 \rho_1'}{\rho_2^2 \rho_1 - \rho_2 \rho_1'} \right)' I_3
\]
\[
= \frac{\left( f_{n+4[n-1]+4}[\lambda^2](1 - \lambda^2)^{-\frac{h}{2}} \right)}{(\rho_2^2 \rho_1 - \rho_2 \rho_1')^2} + \left( \frac{\rho_3 \rho_1 - \rho_3 \rho_1'}{\rho_2^2 \rho_1 - \rho_2 \rho_1'} \right)' I_3.
\]
Similarly again, we have
\[
M_3(\lambda) = \frac{d}{d\lambda} \left( \left[ \frac{\rho_3^2 \rho_1 - \rho_3 \rho_1^2}{\rho_2^2 \rho_1 - \rho_2 \rho_1^2} \right]^{-1} M_2(\lambda) \right)
\]
\[
= \left( \frac{f_{10}[\frac{n-1}{2}+4\frac{3n-5}{6}]+n+16(\lambda)(1-\lambda^2)^{-\frac{13}{6}}}{\Delta(\lambda)} \right) + (1-\lambda^2)^{-2}
\]
(3.8)

where
\[
\Delta(\lambda) = (\rho_3^2 \rho_1 - \rho_3 \rho_1^2)(\rho_2 \rho_1 - \rho_2 \rho_1^2) - (\rho_3^2 \rho_1 - \rho_3 \rho_1^2)(\rho_2^2 \rho_1 - \rho_2 \rho_1^2)
\]

Obviously,
\[
\rho_2 \rho_1 - \rho_2 \rho_1^2 = g_{[\frac{n-1}{2}]+[3n-5]}(\lambda^2)(1-\lambda^2),
\]
\[
\Delta(\lambda) = (\rho_3^2 \rho_1 - \rho_3 \rho_1^2)(\rho_2 \rho_1 - \rho_2 \rho_1^2) - (\rho_3^2 \rho_1 - \rho_3 \rho_1^2)(\rho_2^2 \rho_1 - \rho_2 \rho_1^2)
\]
(3.9)

Hence, we have
\[
M_3(\lambda) = \frac{f_{10}[\frac{n-1}{2}+4\frac{3n-5}{6}]+n+16(\lambda)(1-\lambda^2)^{-2}}{f_{12}[\frac{n-1}{2}+4\frac{3n-5}{6}]+14(\lambda)}
\]
(3.10)

We use the notation \( \# \{ \lambda \in (0,1) | f(\lambda) = 0 \} \) to indicate the number of zeros of the function \( f(\lambda) \) in the interval \( (0,1) \) taking into account their multiplicities. Obviously, the number of zeros of \( \overline{M}(\lambda) \) and \( \overline{M}(h) \) in \( (0,1) \) are same. Therefore, we get from (3.5)-(3.10)
\[
\# \{ h \in (0,1) | \overline{M}(h) = 0 \} = \# \{ \lambda \in (0,1) | \overline{M}(\lambda) = 0 \}
\]
\[
\leq \# \{ \lambda \in (0,1) | \rho_1(\lambda) = 0 \} + \# \{ \lambda \in (0,1) | \rho_2 \rho_1 - \rho_2 \rho_1^2 = 0 \}
\]
\[
+ \# \{ \lambda \in (0,1) | \Delta(\lambda) = 0 \} + \# \{ \lambda \in (0,1) | M_3(\lambda) = 0 \} + 3
\]
\[
\leq \left[ \frac{n-1}{2} \right] + \left[ \frac{n-1}{2} \right] + \left[ \frac{3n-5}{6} \right] + 1 + 3 \left[ \frac{n-1}{2} \right] + \left[ \frac{3n-5}{6} \right]
\]
\[
+ 3 + 10 \left[ \frac{n-1}{2} \right] + 4 \left[ \frac{3n-5}{6} \right] + n + 16
\]
\[
= n + 15 \left[ \frac{n-1}{2} \right] + 6 \left[ \frac{3n-5}{6} \right] + 20
\]
\[
\leq 12n + 7,
\]

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