DISTANCE ONE SURGERIES ON THE LENS SPACE $L(n, 1)$

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Abstract. In this paper, we show that the lens space $L(s, 1)$ for $s \neq 0$ is obtained by a distance one surgery along a knot in the lens space $L(n, 1)$ with $n \geq 5$ odd only if $n$ and $s$ satisfy one of the following cases: (1) $n \geq 5$ is any odd integer and $s = \pm 1, n, n \pm 1$ or $n \pm 4$; (2) $n = 5$ and $s = -5$; (3) $n = 5$ and $s = -9$; (4) $n = 9$ and $s = -5$. As a corollary, we prove that the torus link $T(2, s)$ for $s \neq 0$ is obtained by a band surgery from $T(n, n)$ with $n \geq 5$ odd only if $n$ and $s$ are as listed above. Combined with the result of Lidman, Moore and Vazquez [12], it immediately follows that the only nontrivial torus knot $T(2, n)$ admitting chirally cosmetic banding is $T(2, 5)$.

The key ingredient of our proof is the Heegaard Floer mapping cone formula.

1. Introduction

The celebrated lens space realization problem solved by Greene [9] lists all lens spaces that can be obtained by a Dehn surgery along a knot in $S^3$. The lens space realization problem is part of Berge conjecture [1, 11], which remains open and has received considerable attention in recent years [2, 3, 8, 17, 19, 22, 23]. Instead of $S^3$, a natural generalization is to classify which lens spaces can be obtained by a Dehn surgery along a knot in another lens space.

The remarkable cyclic surgery theorem tells us that distance greater than one surgeries between lens spaces are solvable. In this case, the knot exterior is Seifert fibered, while Seifert fibered structures in lens spaces are well understood. So we focus on distance one surgeries between lens spaces. Here, distance one (respectively greater than one) surgeries refer to surgeries with the minimal geometric intersection number of the meridian of the surgered knot and the surgery slope equal to one (respectively greater than one).

Lidman, Moore and Vazquez [12] classified distance one surgeries between $L(3, 1)$ and $L(s, 1)$ for $s \in \mathbb{Z}$. In this paper, we are specifically concerned with distance one surgeries on the lens space $L(n, 1)$ with $n \geq 5$ odd yielding lens spaces of type $L(s, 1)$ for $s \neq 0 \in \mathbb{Z}$.

This problem is also related to band surgery problem, which is of interest in knot theory, in the study of surfaces in four-manifolds and in DNA topology (discussed below). Band surgery on a knot or link $L$ is defined as follows: embed an unit square $I \times I$ into $S^3$ by $b: I \times I \to S^3$ such that $L \cap b(I \times I) = b(\partial I \times I)$, then replace $L$ by $L' = (L - b(\partial I \times I)) \cup b(I \times \partial I)$. A fruitful technique of studying band surgery between knots or links is by lifting to their double branched covers. The lens space $L(n, 1)$ is the double branched cover of the torus link $T(2, n)$. As a consequence of the Montesinos trick, band surgeries on knots and links lift to distance one Dehn surgeries in their double branched covers. Therefore, studying our distance one surgery problem above can help to solve the band surgery problem on the torus link $T(2, n)$ yielding $T(2, s)$.

Why is $L(n, 1)$ (or $T(2, n)$)? Apart from $L(n, 1)$ (respectively $T(2, n)$) is the simplest lens space (respectively 2-bridge link) type, it is also motivated from DNA topology. In biology, circular DNA

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can be modeled as a knot or link, and torus knots or links $T(2, n)$ are a family of DNA knots or links occurring frequently in biological experiments. Additionally, there exist enzymatic complexes that mediate DNA recombination, during which strands of DNA are exchanged and the topology of the DNA molecule may be altered in the process. The mechanism of DNA recombination can be modeled by band surgery. This explains the biological motivation to study distance one surgeries between lens spaces of type $L(n, 1)$ and band surgery between torus link of type $T(2, n)$. It is due to technical reason that we only study the case $L(n, 1)$ with $n \geq 5$ odd: In such case, we can find a nice symmetry property in the Haagaard Floer mapping cone in Lemma 4.8.

Our main result is

**Theorem 1.1.** The lens space $L(s, 1)$ with $s \neq 0$ is obtained from a distance one surgery along a knot in $L(n, 1)$ with $n \geq 5$ odd only if $n$ and $s$ satisfy one of the following cases:

1. $n \geq 5$ is any odd integer and $s = \pm 1, n, n \pm 1$ or $n \pm 4$;
2. $n = 5$ and $s = -5$;
3. $n = 5$ and $s = -9$;
(4) \( n = 9 \) and \( s = -5 \).

Remark 1.2. Except for the cases (3) and (4), distance one surgeries for all other pairs of \( n \) and \( s \) in above list can be realized as the double branched covers of the band surgeries in Figure 1. The existence of distance one surgery between \( L(5, 1) \) and \( L(-9, 1) \) (or \( L(9, 1) \) and \( L(-5, 1) \)) is still unknown.

The above theorem directly implies the following corollary about band surgery once we lift to the double branched covers.

Corollary 1.3. The torus link \( T(2, s) \) with \( s \neq 0 \) is obtained by a band surgery from the torus knot \( T(2, n) \) with \( n \geq 5 \) odd only if \( n \) and \( s \) satisfy one of the following cases:

1. \( n \geq 5 \) is any odd integer and \( s = \pm 1, n, n \pm 1 \) or \( n \pm 4 \);
2. \( n = 5 \) and \( s = -5 \);
3. \( n = 5 \) and \( s = -9 \);
4. \( n = 9 \) and \( s = -5 \).

Chirally cosmetic banding. If a band surgery relates a knot with its mirror image, then it is called chirally cosmetic banding. Zeković [27] found a chirally cosmetic banding for the torus knot \( T(2, 5) \) as shown in Figure 1(f). Moore and Vazquez [15] Corollary 4.5 show that for all other torus knots \( T(2, m) \) with \( m > 1 \) odd and square free, there does not exist any chirally cosmetic banding. Livingston [14] Theorem 14, using Casson-Gordon theory, further remove the constraint that \( m \) is square free in Moore-Vazquez theorem, with the exception of \( m = 9 \). Combined with Lidman-Moore-Vazquez result [12] Theorem 1.1 about distance one surgery on \( L(3, 1) \), Theorem 1.1 immediately implies the following corollaries, which further improve Moore, Vazquez and Livingston’s result and give a complete classification about chirally cosmetic banding on the torus knot \( T(2, n) \). Here we do not use Casson-Gordon invariant to deduce the result.

Corollary 1.4. There exists a distance one surgery along a knot \( K \) in the lens space \( L(n, 1) \) with \( n > 0 \) odd yielding \( -L(n, 1) \) if and only if \( n = 1 \) or \( 5 \).

Corollary 1.5. The only nontrivial torus knot \( T(2, n) \) with \( n > 0 \) odd admitting a chirally cosmetic banding is \( T(2, 5) \).

The method we use to obstruct those pairs of lens spaces that are not listed in Theorem 1.1 is the Heegaard Floer mapping cone formula. A knot \( K \) in a rational homology sphere \( Y \) is called null-homologous if its homology class is trivial in \( H_1(Y) \), otherwise \( K \) is called homologically essential. For surgeries on a null-homologous knot in \( L(n, 1) \), we simply apply the \( d \)-invariant surgery formula essentially due to [10] Proposition 1.6. For homologically essential knots, we have to work harder to deduce a \( d \)-invariant surgery formula for \( L(n, 1) \) using the mapping cone formula and then apply it to obstruct unexpected distance one surgeries. In our previous paper [25], we give a \( d \)-invariant surgery formula for \( L(n, 1) \) with \( n \geq 5 \) prime, but we only give some partial results on distance one surgeries on \( L(n, 1) \) with \( n \geq 5 \) prime [25] Theorem 1.2-1.3. The following explains how we improve our previous result and give an almost complete classification on distance one surgeries on \( L(n, 1) \) with \( n \geq 5 \) odd, with the exception of distance one surgery between \( L(5, 1) \) and \( L(-9, 1) \) (or \( L(9, 1) \) and \( L(-5, 1) \)).

The challenges we meet in [25] to give a complete classification on distance one surgeries on \( L(n, 1) \) are: (1) We cannot deduce a \( d \)-invariant surgery formula for all surgery slopes. In fact, our \( d \)-invariant formula in [25] Proposition 4.4-4.5 only applies to “large” surgeries, precisely the surgery slope \( m \geq \frac{(n+k)k}{2n} + 1 \) or \( m \leq \frac{(3k-n)k}{2n} - 1 \), where \( k \) denotes the winding number which
will be defined in Section 2. (2) The computation of the \(d\)-invariants of the Seifert fibered spaces occurring in our \(d\)-invariant surgery formula is intractable. Thus we cannot make full use of our \(d\)-invariant formula to study our surgery problem. In fact, we only compute the \(d\)-invariants of the Seifert fibered spaces obtained by “positive large” surgeries in [25], precisely \(m \geq k + 3\).

For (1), in Proposition 4.4-4.5 we give an improved \(d\)-invariant surgery formula for \(L(n,1)\) with \(n \geq 5\) odd instead of prime and for all surgery slopes, that is, we get rid of the constraint on surgery slopes and generalize our surgery formula to \(n \geq 5\) odd. Recall that in [25] we use the so-called simple knots to fix the grading shift in the mapping cone formula. The key difficulty to deduce a \(d\)-invariant formula for all surgery slopes is to find where the element with grading equal to the \(d\)-invariant is supported in the mapping cone of a simple knot, since unlike surgered knots we do not assume that surgeries along simple knots yield \(L\)-spaces. The problem is addressed by using a key property of knot Floer complexes of knots in a same homology class shown in Lemma 3.4 (also Lemma 3.9), which along with Rasmussen’s notation for the hat version mapping cone formula helps us to easily trace where the element with grading equal to the \(d\)-invariant is supported. Besides, unlike \(L(n,1)\) with \(n \geq 5\) prime, knots in \(L(n,1)\) with \(n \geq 5\) odd are not all primitive, which means the set of relative Spin\(^c\) structures may contain some torsion part and it is more complicated. But fortunately, analysis on primitive knots applies to this case without much change.

In fact, we give a general \(d\)-invariant surgery formula for an arbitrary \(L\)-space admitting another distance one \(L\)-space surgery in the following theorem using the key property in Lemma 3.4 (also Lemma 3.9) and Rasmussen’s notation, where the terms in the formula will be explained in Section 3. The \(d\)-invariant surgery formula in Proposition 4.4-4.5 is derived from the general \(d\)-invariant formula. This formula is of independent interest since \(d\)-invariant has been very useful in many applications of Heegaard Floer homology.

**Theorem 1.6.** Let \(K\) be a knot in an \(L\)-space \(Y\) equipped with a positive framing \(\gamma\), and let \(K'\) be a Floer simple knot in \(Y\) with \([K] = [K'] \in H_1(Y)\). If \(\gamma\)-surgery along \(K\) produces an \(L\)-space \(Y_\gamma(K)\), then for any relative Spin\(^c\) structure \(\xi \in \text{Spin}^c(Y,K) \cong \text{Spin}^c(Y,K')\),

\[
d(Y_\gamma(K),G_{Y_\gamma(K),\gamma}(\xi)) = d(Y_\gamma(K'),G_{Y_\gamma(K'),\gamma}(\xi)) - 2 \max_{n \in \mathbb{Z}} \{ \min \{ V_{\xi + n,PD[\gamma]}(K), H_{\xi + n,PD[\gamma]}(K) \} \}.
\]

**Remark 1.7.** For a knot \(K\) in a rational homology sphere \(Y\) with a meridian \(\mu\) and a framing \(\gamma\), there is a pair of integers \(d_1\) and \(d_2\) with minimal absolute value such that

\[
d_1[\gamma] = d_2[\mu] \in H_1(Y - K),
\]

since \(K \subset Y\) has finite order. We call \(\gamma\) a positive framing if \(d_1\) and \(d_2\) have the same sign, and we call it a negative framing if \(d_1\) and \(d_2\) have opposite signs.

As a corollary, we give an estimate of \(d\)-invariants of \(L\)-spaces obtained by a distance one surgery from another \(L\)-space in Corollary 4.3, which is analogous to the result in \(S^3\) [10] Theorem 2.5], but our corollary is only for \(L\)-space surgery.

For (2), we show that these Seifert fibered spaces must be \(L\)-spaces in Proposition 4.1. It helps us to exclude Seifert fibered spaces which are not \(L\)-spaces, and these Seifert fibered spaces are those whose \(d\)-invariants are hard to compute. Almost all the rest are Seifert fibered spaces obtained by “negative large” surgeries, \(d\)-invariants of which are computed in Section 5.1.1 (see Lemma 5.8). In fact, the computation of \(d\)-invariants of an infinite family of Seifert fibered spaces is a technical and intractable problem, and our computation may have independent interests.

This paper is organized as follows. Section 2 gives some homological preliminaries and the four-dimensional perspective to our problem. Section 3 briefly introduces some preliminaries in Heegaard...
Floer homology, including $d$-invariant, the mapping cone formula, simple knots and Rasmussen’s notation for the hat version mapping cone formula. In Section 4, we deduce the general $d$-invariant surgery formula for $L$-spaces in Theorem 1.6 using the mapping cone formula, and then derive our improved $d$-invariant surgery formula for $L(n,1)$ with $n \geq 5$ odd. In Section 5, we compute the $d$-invariants of an infinite family of Seifert fibered spaces occurring in our $d$-invariant formula for $L(n,1)$ and apply our $d$-invariant formula to show Theorem 1.1.

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2. Homological preliminaries and the four-dimensional perspective

In our convention, $L(p,q)$ represents the lens space obtained from $p/q$-surgery of the unknot in $S^3$. Let $K$ be a knot in $Y = L(n,1)$ with $n \geq 5$ odd, then $K$ is either null-homologous or homologically essential.

When $K$ is null-homologous, the surgered manifold $Y_n(K)$ is well-defined. If $H_1(Y_n(K))$ is cyclic, then we have $H_1(Y_n(K)) = \mathbb{Z}_{mn}$. When $K$ is homologically essential, there are $2\frac{n-1}{d}$ different homology classes of $K$ in $H_1(Y) = \mathbb{Z}_n$ up to symmetry. To describe the homology class of $K$ properly, we need to define the winding number of $K$ and the key element $[c] \in H_1(Y)$ as follows. We first represent $Y$ by a Kirby diagram with an unknot $U$ with framing $n$ in $S^3$. Then $Y$ is obtained by gluing the solid torus $V_0$ to the unknot complement $V_1$. Let $c$ be the core of the solid torus $V_1$. We fix the orientations of $c$ and $U$ such that their linking number is one. Then we can choose the orientation of $K$ such that $[K] = k[c] \in H_1(Y)$ for some integer $1 \leq k \leq \frac{n-1}{2}$, in which case we say the mod $n$-winding number (or simply winding number) of $K$ is $k$. By possibly handlesliding $K$ over $U$ in the Kirby diagram, which is equivalent to isotopying $K$ in $Y$ over the meridian of $V_0$, we may further assume that the linking number of $K$ and $U$ is exactly $k$. We choose the meridian $\mu$ and longitude $\lambda$ for $K$ by regarding $K$ as a component of the link consisting of $K$ and $U$ in $S^3$. Similarly, we fix the meridian $\mu_0$ and longitude $\lambda_0$ for $U$. Then the first homology of the link complement

$$H_1(S^3 - U - K) = \mathbb{Z} \langle \mu_0 \rangle \oplus \mathbb{Z} \langle \mu \rangle$$

and

$$\left\{ \begin{array}{l} [\lambda_0] = k \cdot [\mu] \\ [\lambda] = k \cdot [\mu_0] \end{array} \right.$$  

Therefore, the first homology of the knot complement $Y - K$ is

$$H_1(Y - K) = H_1(S^3 - U - K)/(n\mu_0 + \lambda_0) = H_1(S^3 - U - K)/(n\mu_0 + k\mu).$$

Let $\theta = n'\mu_0 + k'\mu$ and $\vartheta = \frac{n}{d}\mu_0 + \frac{k}{d}\mu$, where $d = \gcd(n, k)$ and $\frac{n}{d}k' - \frac{k}{d}n' = 1$, then

$$H_1(Y - K) = \mathbb{Z} \langle \theta \rangle \oplus \mathbb{Z}_d \langle \vartheta \rangle.$$

One may check that

$$[\mu] = \frac{n}{d}[\theta] - n'[\vartheta] \in H_1(Y - K),$$

$$[\lambda] = -\frac{k^2}{d}[\theta] + kk'[\vartheta] \in H_1(Y - K).$$

Since we are interested in distance one surgery, we only consider $(m\mu + \lambda)$-surgery. We have

$$[m\mu + \lambda] = \frac{nnm - k^2}{d}[\theta] + (kk' - n'm)[\vartheta] \in H_1(Y - K).$$
Denote by $Y_{m\mu + \lambda}(K)$ the surgered manifold obtained by $(m\mu + \lambda)$-surgery along $K$. Then
\begin{equation}
H_1(Y_{m\mu + \lambda}(K)) = \mathbb{Z}_{\frac{nm - k^2}{2}} \oplus \mathbb{Z}_d.
\end{equation}
If $H_1(Y_{m\mu + \lambda}(K))$ is cyclic, then
\begin{equation}
H_1(Y_{m\mu + \lambda}(K)) = \mathbb{Z}_{|nm - k^2|}.
\end{equation}
For simplicity, we call the above surgery by $m$-surgery along $K$ with the understanding that $\mu$ and $\lambda$ are chosen as just described.

By almost the same argument as in [23, Lemma 2.1], we conclude the following lemma which describes the surgery from the four-dimensional perspective.

**Lemma 2.1.** Let $Y'$ be the manifold obtained by a distance one surgery from $Y = \mathbb{L}(n, 1)$ with $n \geq 5$ odd, and let $W : Y \to Y'$ be the associated cobordism. Then $|H_1(Y')|$ is even if and only if $W$ is Spin.

3. Preliminaries in Heegaard Floer homology

In this section, we briefly introduce some preliminaries in Heegaard Floer homology including $d$-invariant, which is the main tool we use, and the Heegaard Floer mapping cone formula. We use $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$ coefficients throughout unless otherwise stated.

3.1. $d$-invariant. For a rational homology sphere $Y$ equipped with a Spin$^c$ structure $t$, the Heegaard Floer homology $HF^+(Y, t)$ can be decomposed as two summands as $\mathbb{F}[U]$-module. More precisely,
\begin{equation}
HF^+(Y, t) = \mathbb{F}[U, U^{-1}]/U \cdot \mathbb{F}[U] \oplus HF_{red}(Y, t).
\end{equation}
The first summand is abbreviated $T^+$, called the tower; the second summand is a torsion $\mathbb{F}[U]$-module. The $d$-invariant, or correction term, denoted by $d(Y, t)$, is the minimal $\mathbb{Q}$-grading of the tower. A rational homology sphere $Y$ is called an $L$-space if $HF^+(Y, t) = T^+$ for any Spin$^c$ structure $t \in \text{Spin}^c(Y)$. Note that any lens space is an $L$-space. We refer the readers to Ozsváth-Szabó [17] for details. The $d$-invariant of a lens space is computable according to the following recursive formula.

**Theorem 3.1** (Ozsváth-Szabó, Proposition 4.8 in [17]). Let $p > q > 0$ be relatively prime integers. Then there exists an identification $\text{Spin}^c(L(p, q)) \cong \mathbb{Z}_p$ such that
\begin{equation}
d(L(p, q), i) = -\frac{1}{4} + \frac{(2i + 1 - p - q)^2}{4pq} - d(L(q, r), j)
\end{equation}
where $r$ and $j$ are the reductions of $p$ and $i \pmod{q}$ respectively.

Under the identification $\text{Spin}^c(L(p, q)) \cong \mathbb{Z}_p$ in Theorem 3.1, the self-conjugate Spin$^c$ structures on $L(p, q)$ correspond to the set
\[ \mathbb{Z} \cap \left\{ \frac{p + q - 1}{2}, \frac{q - 1}{2} \right\}. \]
Note that when $p$ is odd, there is only one self-conjugate Spin$^c$ structure. In our case, the unique self-conjugate Spin$^c$ structure on the lens space $L(n, 1)$ with $n > 0$ odd corresponds to 0.

By the recursive formula above, we can compute the values of $d(L(n, 1), i)$ for $n > 0$ as follows,
\begin{equation}
d(L(n, 1), i) = -\frac{1}{4} + \frac{(2i - n)^2}{4n}.
\end{equation}
Note that $d$-invariants switch sign under orientation reversal.
There is a strong constraint on the \(d\)-invariant, when the cobordism associated to a distance one surgery between two \(L\)-spaces is Spin,

**Lemma 3.2** (Lidman-Moore-Vazquez, Lemma 2.7 in [12]). Let \((W, s) : (Y, t) \to (Y', t')\) be a Spin cobordism between \(L\)-spaces satisfying \(b^+_2(W) = 1\) and \(b^-_2(W) = 0\). Then

\[
d(Y', t') - d(Y, t) = -\frac{1}{4},
\]

Applying the above lemma for the Spin case which has been discussed in Lemma [7.1] almost the same argument as in [25] Theorem 1.1 shows the following proposition, which classifies distance one surgeries from \(L(n, 1)\) with \(n \geq 5\) odd to \(L(s, 1)\) with \(s\) even. Here we do not consider surgeries from \(L(n, 1)\) to \(S^1 \times S^2\).

**Proposition 3.3.** The lens space \(L(s, 1)\) with \(s \neq 0\) even is obtained from a distance one surgery along a knot in \(L(n, 1)\) with \(n \geq 5\) odd if and only if \(s\) is \(n + 1\) or \(n - 1\).

### 3.2. The mapping cone formula for rationally null-homologous knots.

In this section, we recall the mapping cone formula of Ozsváth and Szabó [21] for rationally null-homologous knots, which we use to deduce our \(d\)-invariant surgery formula.

Let \(Y\) be a rational homology sphere and \(K \subset Y\) an oriented knot. Let \(\mu\) be the meridian of \(K\) and \(\gamma\) a framing which is an embedded curve on the boundary of the tubular neighborhood of \(K\) intersecting \(\mu\) transversely once. Here, \(\gamma\) naturally inherits an orientation from \(K\).

We denote the set of relative Spin\(c\) structures over \(Y - K\) by \(\text{Spin}^c(Y, K)\), which is affinely isomorphic to \(H^2(Y, K)\). The natural map defined in [21] Section 2.2,

\[
G_{Y, \pm K} : \text{Spin}^c(Y, K) \to \text{Spin}^c(Y) \cong H^2(Y),
\]

sends a relative Spin\(c\) structure to a Spin\(c\) structure in the target manifold. It is equivariant with respect to the natural map \(i^* : H^2(Y, K) \to H^2(Y)\) induced from inclusion. Let \(-K\) denote \(K\) with the opposite orientation. We have

\[
G_{Y, -K}(\xi) = G_{Y, K}(\xi) + PD[\gamma].
\]

For each \(\xi \in \text{Spin}^c(Y, K)\), we can associate to it a \(Z \oplus Z\)-filtered knot Floer complex \(C^+_{\xi} = \text{CF}^\infty(Y, K, \xi)\), whose bifiltration is given by \((i, j) = (\text{algebraic}, \text{Alexander})\). Let

\[
A^+_\xi = C^+_\xi(\max\{i, j\} \geq 0) \quad \text{and} \quad B^+_\xi = C^+_\xi(\{i \geq 0\}).
\]

Basically, the complexes

\[
C^+_\xi(i \geq 0) = CF^+(Y, G_{Y, K}(\xi)) = B^+_\xi,
\]

\[
C^+_\xi(j \geq 0) = CF^+(Y, G_{Y, -K}(\xi)) = CF^+(Y, G_{Y, K}(\xi + PD[\gamma])) = B^+_{\xi + PD[\gamma]},
\]

while \(A^+_\xi\) is the Heegaard Floer homology of a large surgery \(Y_{r + \ell}(K)\) with \(l \gg 0\) in a certain Spin\(c\) structure. More details will be described below.

There are two natural projection maps

\[
v^+_\xi : A^+_\xi \to B^+_\xi, \quad h^+_\xi : A^+_\xi \to B^+_{\xi + PD[\gamma]},
\]

which can be identified with certain cobordism maps as follows. Fix \(l \gg 0\) and consider the negative definite two-handle cobordism \(W_1^+\) obtained by turning around the two-handle cobordism from \(-Y\) to \(-Y_{r + \ell}(K)\). Fix a generator \([F] \in H_2(W_1^+, \xi)\) such that \(PD[F]|_Y = PD[K]\). Given a Spin\(c\) structure \(\xi\) on \(Y_{r + \ell}(K)\), Ozsváth and Szabó [21] Theorem 4.1] show that there exist two
particular $\text{Spin}^c$ structures $v$ and $h = v + PD[F]$ on $W'_i$ which extend $t$ over $W'_i$ and an associated map $\Xi: \text{Spin}^c(Y_{\gamma+t\mu}(K)) \to \text{Spin}^c(Y, K)$ satisfying the following commutative diagrams:

\[
\begin{array}{ccc}
CF^+(Y_{\gamma+t\mu}(K), t) & \xrightarrow{\sim} & A^+_\xi \\
\downarrow f_{W'_i,s} & & \downarrow v^+_\xi \\
CF^+(Y, G_{Y,K}(\xi)) & \xrightarrow{\sim} & B^+_\xi \\
\end{array}
\quad \quad \begin{array}{ccc}
CF^+(Y_{\gamma+t\mu}(K), t) & \xrightarrow{\sim} & A^+_\xi \\
\downarrow f_{W'_i,h} & & \downarrow h^+_\xi \\
CF^+(Y, G_{Y,-K}(\xi)) & \xrightarrow{\sim} & B^+_{\xi+PD[\gamma]} \\
\end{array}
\]

where $\xi = \Xi(t)$, and $f_{W'_i,s}$ and $f_{W'_i,h}$ are $\text{Spin}^c$ cobordism maps in Heegaard Floer homology [20].

As we mentioned previously, the Heegaard Floer homology of any $\text{Spin}^c$ rational homology sphere contains a tower $T^+$. On homology, both $v^+_\xi$ and $h^+_\xi$ induce grading homogeneous maps between towers, which are multiplications by $U^N$ for some integer $N \geq 0$. We denote the non-negative integers corresponding to $v^+_\xi$ and $h^+_\xi$ by $V_\xi(K)$ and $H_\xi(K)$ respectively. These integers satisfy that for each $\xi \in \text{Spin}^c(Y, K)$,

\[
(3.5) \quad V_\xi(K) \geq V_{\xi+PD[\gamma]}(K) \geq V_\xi(K) - 1.
\]

There is another crucial property of $V_\xi(K)$ and $H_\xi(K)$, which will be used repeatedly in the later sections. Although it is well-known for experts, for clarity we will give a proof here.

**Lemma 3.4.** Let $K$ and $K'$ be two knots in a rational homology sphere $Y$ with $[K] = [K'] \in H_1(Y)$. Then for any relative $\text{Spin}^c$ structure $\xi \in \text{Spin}^c(Y, K) \equiv \text{Spin}^c(Y, K')$,

\[
(3.5) \quad V_\xi(K) - H_\xi(K) = V_\xi(K') - H_\xi(K').
\]

**Proof.** As we stated above, the two maps $v^+_\xi$ and $h^+_\xi$ are identified with certain $\text{Spin}^c$ cobordism maps. See Commutative Diagrams (3.3) and (3.4). By [20] Theorem 7.1, we have

\[
(3.6) \quad d(Y, G_{Y,K}(\xi)) - d(Y_{\gamma+t\mu}(K), t) = \frac{C_1(h)^2 - 3\sigma(W'_i) - 2\chi(W'_i)}{4} - 2V_\xi(K)
\]

\[
(3.7) \quad d(Y, G_{Y,-K}(\xi)) - d(Y_{\gamma+t\mu}(K), t) = \frac{C_1(h)^2 - 3\sigma(W'_i) - 2\chi(W'_i)}{4} - 2H_\xi(K)
\]

Comparing (3.6) and (3.7), we see that $V_\xi(K) - H_\xi(K)$ is completely determined by homological information, so we prove $V_\xi(K) - H_\xi(K) = V_\xi(K') - H_\xi(K')$.

Given any $s \in \text{Spin}^c(Y_\gamma(K))$, let

\[
\mathbb{A}^+_s = \bigoplus_{\{\xi \in \text{Spin}^c(Y_\gamma(K)), K_\gamma(\xi) = s\}} A^+_\xi,
\]

\[
\mathbb{B}^+_s = \bigoplus_{\{\xi \in \text{Spin}^c(Y_\gamma(K)), K_\gamma(\xi) = s\}} B^+_\xi,
\]

where $K_\gamma$ represents the oriented dual knot of the knot $K$ in the surgered manifold $Y_\gamma(K)$, and $G_{Y_\gamma(K), K_\gamma}: \text{Spin}^c(Y_\gamma(K), K_\gamma) \to \text{Spin}^c(Y_\gamma(K))$. Let

\[
D^+_s: \mathbb{A}^+_s \to \mathbb{B}^+_s, \quad (\xi, a) \mapsto (\xi, v^+_\xi(a)) + (\xi + PD[\gamma], h^+_\xi(a)).
\]

With this, we are ready to state the connection between the knot Floer complex of the knot $K$ and the Heegaard Floer homology of the manifold obtained from distance one surgery along $K$.

**Theorem 3.5** (Ozsváth-Szabó, Theorem 6.1 in [21]). For any $s \in \text{Spin}^c(Y_\gamma(K))$, the Heegaard Floer homology $\text{HF}^+(Y_\gamma(K), s)$ is the homology of the mapping cone $K^+_s$ of the chain map $D^+_s: \mathbb{A}^+_s \to \mathbb{B}^+_s$. 
Remark 3.6. There exist grading shifts on $A_0^+$ and $B_0^+$, which gives a consistent relative $\mathbb{Z}$-grading on $X_s^+$. Actually, the shift can be fixed such that the grading is the same as the absolute $\mathbb{Q}$-grading of $HF^+(Y, (K), s)$. It is important to point out that these shifts only depend on the homology class of the knot.

Denote
\[
A_0^+ = H_*(A_0^+) \quad \text{(respectively } B_0^+ = H_*(B_0^+)), \quad A_\xi^+ = H_*(A_\xi^+) \quad \text{(respectively } B_\xi^+ = H_*(B_\xi^+)).
\]
Let $v_\xi^+$, $h_\xi^+$ and $D_\xi^+$ denote the maps induced on homology by $v_\xi^+$, $h_\xi^+$ and $D_\xi^+$ respectively. Basically, $HF^+(Y, (K), s)$ is isomorphic to the direct sum of the kernel and cokernel of the map $D_\xi^+$.

There is an analogous mapping cone formula for the hat version of Heegaard Floer homology. One can define $\hat{A}_\xi$, $\hat{B}_\xi$, $\hat{D}_\xi$ and the mapping cone $\hat{\mathcal{K}}_s$ of $\hat{D}_\xi$, and the Heegaard Floer homology $\hat{HF}(Y, (K), s)$ can be calculated by the homology of $\hat{\mathcal{K}}_s$.

Remark 3.7. When $V_\xi(K) > 0$, the corresponding $\delta_\xi$ is trivial, and when $V_\xi(K) = 0$, $\delta_\xi$ is nontrivial. Similarly, when $H_\xi(K) > 0$, $\delta_\xi$ is trivial, and when $H_\xi(K) = 0$, $\delta_\xi$ is nontrivial.

3.3. Simple knots in lens spaces. To compute the $d$-invariant of the surgered manifold $Y, (K)$ via the mapping cone formula, we need to fix the grading shift. As pointed out in Remark 3.6, the grading shift only depends on the homology class of the knot. We will use a simple knot in the same homology class to fix it, since simple knots have the simplest knot Floer complex.

There is a standard genus one Heegaard diagram for a lens space $L(p, q)$ (e.g. $L(5, 1)$ in Figure 2), where we identify opposite sides of a rectangle to give a torus. The horizontal red curve represents the $\alpha$ curve which gives a solid torus $U_\alpha$, and the blue curve of slope $p/q$ represents the $\beta$ curve which gives a solid torus $U_\beta$. They intersect at $p$ points, $x_0, x_1, \ldots, x_{p−1}$, where we label them in the order they appear on the $\alpha$ curve. The simple knot $K(p, q, k) \subset L(p, q)$ is an oriented knot defined as the union of an arc joining $x_0$ to $x_k$ in the meridian disk of $U_\alpha$ and an arc joining $x_k$ to $x_0$ in the meridian disk of $U_\beta$.

![Figure 2](image)

Figure 2. An example of a simple knot $K(5, 1, 2)$ in $L(5, 1)$. (To draw $K(5, 1, 2)$ in the Heegaard diagram, we place two points $x_0'$ and $x_2'$ next to $x_0$ and $x_2$ respectively, and connect them in $U_\alpha$ and $U_\beta$.)

A knot $K$ in an $L$-space $Y$ is called Floer simple if $\text{rank } HF^+(Y, K) = \text{rank } HF(Y)$. We see that any simple knot $K(p, q, k)$ is Floer simple since the $p$ intersection points $x_0, \ldots, x_{p−1}$ represent $p$ different Spin$^c$ structures. In our case, we consider simple knots $K(n, 1, k)$ in $L(n, 1)$.
3.4. **Rasmussen’s notation for the hat version mapping cone formula.** If a knot $K$ satisfies that both its $\hat{A}_\xi$ and $\hat{B}_\xi$ are of rank one for any $\xi \in \text{Spin}^c(Y,K)$, then we can use Rasmussen’s notation [23], which can effectively simplify the computation of the hat version mapping cone formula. In fact, we will see in the next section that our case satisfies this condition. Besides, Floer simple knots are also the case.

For a satisfactory knot $K$, we represent the chain complex $\hat{D}_s : \hat{A}_s \to \hat{B}_s$ by a type of diagram shown in Figure 3. Here, the upper row of the diagram represents $\hat{A}_\xi$, while the lower row of the diagram represents $\hat{B}_\xi$. We denote $\hat{A}_\xi$ by $a +$ if $\hat{v}_\xi$ is nontrivial but $\hat{h}_\xi$ is trivial, and we denote $\hat{A}_\xi$ by $a −$ if $\hat{h}_\xi$ is nontrivial but $\hat{v}_\xi$ is trivial. Denote $\hat{A}_\xi$ by $a \circ$ if both $\hat{v}_\xi$ and $\hat{h}_\xi$ are nontrivial, and denote $\hat{A}_\xi$ by $a ∗$ if both $\hat{v}_\xi$ and $\hat{h}_\xi$ are trivial. Each $\hat{B}_\xi$ are represented by a filled circle. Nontrivial maps are indicated by arrows, and trivial maps are omitted.

**Remark 3.8.** For a Floer simple knot, its $\hat{A}_\xi$ can only be $+, −$ or $\circ$ and never be $∗$, and the same for a simple knot.

![Figure 3. An example of Rasmussen’s notation.](image)

The complex $\hat{D}_s : \hat{A}_s \to \hat{B}_s$ can be decomposed into summands corresponding to the connected components of the diagram. A $∗$ itself is always a summand, denoted by $[∗]$. For other summands, we denote them by an interval $[a, b]$, where $a$ and $b$ are labeled with $+, −$ or $∗$ and all the elements in between are $\circ$. See Figure 4. There may be more or no $\circ$ in the between, but for convenience we put one $\circ$ in every diagram in Figure 4. For the example in Figure 3, it can be decomposed into summands $[−, −]$, $[−, ∗]$, $[∗, +]$ and $[+, +]$.

![Figure 4. Acyclic summands.](image)

![Figure 5. Summands which have homology $F$ supported by an element in the bottom row.](image)
We see that summands of types $[+,-]$, $[-,-]$, $[-,*]$ and $[*,+]$ are acyclic and summands of types $[+,-]$, $[+,*]$, $[*,-]$, $[*,*]$, $[-,+]$ and $[*]$ have homology of rank one. Moreover, when the summand is type $[-,+]$ or $[*]$, the homology group $F$ is supported by an element in the top row (i.e. in the kernel of $\mathcal{D}_s$), and when the summand is type $[+,-]$, $[+,*]$, $[*,-]$ or $[*,*]$, the homology group $F$ is supported by an element in the bottom row (i.e. in the cokernel of $\mathcal{D}_s$). We see that the homology of our example in Figure 3 is of rank 3.

By Lemma 3.4 and the property of Floer simple knots in Remark 3.8 we immediately give the following lemma which is one of the key tricks we will use to deduce our $d$-invariant formula. Here we denote $\mathcal{A}_\xi$ of $K$ by $\mathcal{A}_\xi(K)$.

**Lemma 3.9.** Let $K$ be a knot in an $L$-space $Y$, and let $K'$ be a Floer simple knot in $Y$ with $[K] = [K'] \in H_1(Y)$. Suppose $\mathcal{A}_\xi(K)$ is of rank one for any relative Spin$^c$ structure $\xi \in \text{Spin}^c(Y,K) = \text{Spin}^c(Y,K')$. Then in Rasmussen's notation:

1. If $\mathcal{A}_\xi(K) = +,-, or \circ$, then $\mathcal{A}_\xi(K')$ is the same (i.e. $\mathcal{A}_\xi(K) = \mathcal{A}_\xi(K')$).
2. If $\mathcal{A}_\xi(K) = *$, then $\mathcal{A}_\xi(K')$ could be $+, -$ or $\circ$ (when $V_\xi(K) < H_\xi(K)$, $V_\xi(K) > H_\xi(K)$ or $V_\xi(K) = H_\xi(K)$ respectively).

4. $d$-INvariant SURGERY FORMULA

In this section, we first show Theorem 1.6. The $d$-invariant surgery formulas in Proposition 4.4 which we apply to analyze our distance one surgery problem, can be derived from Theorem 1.6.

Theorem 1.6 is deduced from the mapping cone formula. Fix a $\xi \in \text{Spin}^c(Y,K)$. Then the mapping cone $X^+_s$ consists of $A^+_{\xi+j,PD[\gamma]}$ and $B^+_{\xi+j,PD[\gamma]}$ for all $j \in \mathbb{Z}$, where $s = G_{Y_\gamma(K),K_\gamma}(\xi)$.

![Diagram](image)

**Figure 7.** The mapping cone $X^+_s$ with $s = G_{Y_\gamma(K),K_\gamma}(\xi)$.

As $Y_\gamma(K)$ is an $L$-space obtained by a distance one surgery from the $L$-space $Y$, it follows from Lemma 6.7 that

\[ \mathcal{A}_\xi(K) \cong \mathcal{B}_\xi \cong \mathcal{F}, \quad \mathcal{A}_\xi^+(K) \cong \mathcal{B}_\xi^+ \cong \mathcal{T}^+, \]
for all $\xi \in \text{Spin}^c(Y,K)$. Note that our setting satisfies the orientation condition in Lemma 6.7, since we assume $\gamma$ is a positive framing. This implies that $HF^+(Y,\gamma(K),s)$ for any $s \in \text{Spin}^c(Y,K))$ is completely determined by the integers $V_\xi(K)$ and $\hat{H}_\xi(K)$ for $\xi \in \text{Spin}^c(Y,K)$ with $G_{Y,\gamma(K)}(\xi) = s$. It also follows that Rasmussen’s notation is applicable.

In fact, for any $\xi$ and any positive integer $M$, there exists some positive integer $N$ such that

\[ V_{\xi+j,PD[\mu]}(K) = H_{\xi-j,PD[\mu]}(K) = 0, \]

\[ H_{\xi+j,PD[\mu]}(K) > M \text{ and } V_{\xi-j,PD[\mu]}(K) > M \text{ when } j > N. \]

Since we assume $\gamma$ is a positive framing, for all sufficiently large $j$, we have that $V_{\xi+j,PD[\gamma]}(K)$ and $H_{\xi+j,PD[\gamma]}(K)$ are greater than 0. It follows that $\tilde{h}_{\xi+j,PD[\gamma]}$ and $\tilde{H}_{\xi+j,PD[\gamma]}$ are nontrivial, and $\tilde{h}_{\xi-j,PD[\gamma]}$ and $\tilde{H}_{\xi-j,PD[\gamma]}$ are trivial, for all sufficiently large $j$. Therefore,

\[ \tilde{A}_{\xi+j,PD[\gamma]}(K) = + \text{ and } \tilde{A}_{\xi-j,PD[\gamma]}(K) = -, \]

for all sufficiently large $j$.

**Proof of Theorem 1.1** Given any relative $\text{Spin}^c$ structure $\xi$, we use the mapping cone $X_\xi^+(K)$ and $X_\xi^-(K')$ to show our $d$-invariant formula, where $s = G_{Y,\gamma(K),\xi}(\xi) = G_{Y,\gamma(K'),\xi}(\xi)$.

To find where the nonzero element of minimal grading in $HF^+(Y,\gamma(K),s)$ is supported in $X_\xi^+(K)$, we consider the hat version of the mapping cone $\tilde{X}_\xi(K)$ first. In Rasmussen’s notation, $\tilde{X}_\xi(K)$ contains at most one $*$, since otherwise there are two nontrivial summands of type $[\ast]$, which contradicts to

\[ \tilde{H}F(Y,\gamma(K),s) = \mathbb{F}. \]

We consider the following two cases.

Case i: there is no $*$ in the mapping cone $\tilde{X}_\xi(K)$. We claim that there is a unique summand with nontrivial homology if and only if all $+$’s appear to the right of all $-$’s. If there is a $-$ lying on the right of some $+$, then we have at least three nontrivial summands, two summands of type $[-,+]$ and one summand of type $[+,-]$, since $\tilde{A}_{\xi+j,PD[\gamma]}(K)$ and $\tilde{A}_{\xi-j,PD[\gamma]}(K)$ are labeled with a $+$ and $-$ respectively for all sufficiently large $j$. This gives a contradiction. Therefore, we have that $\tilde{X}_\xi(K)$ is like a sequence shown in the first row of the figure below, where for convenience, we delete all filled circles which stand for $\tilde{X}_{\xi+j,PD[\gamma]}$ or $\tilde{A}_{\xi-j,PD[\gamma]}$ in Rasmussen’s notation. We see the homology group $\mathbb{F}$ is supported in the summand $[-,+]$ in the middle. Choose an element $\tilde{A}_{\xi_1}(K)$ contained in this summand $[-,+]$. We see $\tilde{A}_{\xi_1}(K)$ could be $-, +$ or $\circ$. In any case, we have the natural quotient map

\[ \tilde{\Pi}_{\xi_1} : H_{\ast}(\tilde{X}_\xi(K)) \to \tilde{A}_{\xi_1}(K) \]

is an isomorphism. Then so is the natural quotient map

\[ \Pi_{\xi_1}^+ : H_{\ast}(X^+_{\xi}(K)) \to \mathcal{A}_{\xi_1}^+(K). \]

Hence, the nonzero element of minimal grading is supported in $\mathcal{A}_{\xi_1}^+(K)$, whose grading is the $d$-invariant of $Y_{\gamma}(K)$ in the $\text{Spin}^c$ structure $s$ after an appropriate grading shift. Let $\sigma(\xi_1)$ denote this grading shift. Then we have

\[ d(Y_{\gamma}(K),s) = d(Y,G_{Y,K}(\xi')) + \sigma(\xi_1), \]

for some $\xi'$ with $G_{Y,\gamma(K),\xi'}(\xi') = s$. In fact, when $\mathcal{A}_{\xi_1}^+(K) = +$, $\xi' = \xi_1 + PD[\gamma]$; when $\mathcal{A}_{\xi_1}^+(K) = -$ or $\circ$, $\xi = \xi_1$. 


\[ \hat{\mathcal{X}}_s(K) \begin{array}{c c c c c c c c c c} \ldots & - & 0 & 0 & - & 0 & 0 & + & + & 0 & + & \ldots \\ \| \end{array} \]
\[ \hat{\mathcal{X}}_s(K') \begin{array}{c c c c c c c c c c} \ldots & - & 0 & 0 & - & 0 & 0 & + & + & 0 & + & \ldots \\ \end{array} \]

Recall that grading shifts depend only on the homology class of the knot. So we use \( K' \) to compute the grading shift.

By Lemma 3.9, we have the Rasmussen’s notation for the mapping cone \( \hat{\mathcal{X}}_s(K') \) is the same as \( \hat{\mathcal{X}}_s(K) \). Thus

\[ \tilde{HF}(Y_\gamma(K'), s) = \mathbb{F}, \]

which is supported in the same place as in \( \hat{\mathcal{X}}_s(K) \). Therefore the quotient maps

\[ \tilde{\Pi}_{\xi_1}: H_*(\hat{\mathcal{X}}_s(K')) \to \tilde{\mathcal{X}}_{\xi_1}(K') \]

and

\[ \Pi_{\xi_1}^+ : H_* (\hat{\mathcal{X}}_s^+(K')) \to \mathcal{A}_{\xi_1}^+(K') \]

are isomorphisms. Once we understand where the nonzero element of minimal grading is supported, we can compute the \( d \)-invariant by the formula

\[ d(Y_\gamma(K'), s) = d(Y_\gamma, G_{Y,K'}(\xi')) + \sigma(\xi_1), \]

where \( G_{Y,K'}(\xi') = G_{Y,K}(\xi') \). Comparing (4.1) and (4.2), we get

\[ d(Y_\gamma(K), s) = d(Y_\gamma(K'), s). \]

In this case, we see that

\[ \max_{n \in \mathbb{Z}} \{ \min \{ V_{\xi + n \cdot PD[\gamma]}(K), H_{\xi + n \cdot PD[\gamma]}(K) \} \} = 0, \]

thus we obtain the equality

\[ d(Y_\gamma(K), s) = d(Y_\gamma(K'), s) - 2 \max_{n \in \mathbb{Z}} \{ \min \{ V_{\xi + n \cdot PD[\gamma]}(K), H_{\xi + n \cdot PD[\gamma]}(K) \} \}. \]

Case ii: there is one \( * \) in the mapping cone \( \hat{\mathcal{X}}_s(K) \). Assume that \( \tilde{\mathcal{X}}_{\xi_1}(K) = * \). We see that there is a unique summand with nontrivial homology if and only if all \( + \)'s appear to the right of the \( * \) and all \( - \)'s appear to the left of the \( * \). If there is a \( - \) lying on the right of \( * \), then there are at least two nontrivial summands of types \( [+] \) and \( [-,+] \) since \( \tilde{\mathcal{X}}_{\xi + PD[\gamma]}(K) \) is labeled with a \( + \) for all sufficiently large \( j \). It contradicts to

\[ \tilde{HF}(Y_\gamma(K), s) = \mathbb{F}. \]

Similarly, there is no \( + \) appearing to the left of the \( * \). Therefore \( \hat{\mathcal{X}}_s(K) \) is like a sequence in the first row of the figure below. We see that the homology group \( \mathbb{F} \) is supported in the summand \( [+] \), which implies that the nonzero element of minimal grading is supported in \( \mathcal{A}_{\xi_2}^+(K) \). Let \( \sigma(\xi_2) \) denote the grading shift. Then we have

\[ d(Y_\gamma(K), s) = d(Y, G_{Y,K}(\xi')) - 2 \min \{ V_{\xi_2}(K), H_{\xi_2}(K) \} + \sigma(\xi_2), \]

for some \( \xi' \) with \( G_{Y,K,K'}(\xi') = s \). In fact, if \( V_{\xi_2}(K) \leq H_{\xi_2}(K) \), then \( \xi' = \xi_2 \); if \( V_{\xi_2}(K) > H_{\xi_2}(K) \), then \( \xi' = \xi_2 + PD[\gamma] \).

Now we consider the mapping cone \( \hat{\mathcal{X}}_s(K') \). By Lemma 3.9, all elements except for \( \tilde{\mathcal{X}}_{\xi_2}(K') \) are the same as their corresponding elements in \( \hat{\mathcal{X}}_s(K) \), and \( \tilde{\mathcal{X}}_{\xi_2}(K') \) could be \( -, \circ \) or \( + \). In any case, we see that

\[ \tilde{HF}(Y_\gamma(K'), s) = \mathbb{F}, \]
and the homology group $\mathcal{F}$ is supported in the summand $[-,+]$ in the middle which contains the $+,−$ or $\circ$ corresponding to the $* \in \hat{\mathcal{F}}_d(K)$. It follows that the nonzero element of minimal grading is also supported in $\hat{\mathcal{F}}^+_d(K')$. We then have
\[
d(Y, (K'), s) = d(Y, G_{Y,K'}(\xi')) - 2 \min \{V_{\xi_2}(K'), H_{\xi_2}(K')\} + \sigma(\xi_2) = d(Y, G_{Y,K'}(\xi')) + \sigma(\xi_2),
\]
since $\min \{V_{\xi_2}(K'), H_{\xi_2}(K')\} = 0$ in any case.

Therefore, we show
\[
d(Y, (K), s) = d(Y, (K'), s) - 2 \min \{V_{\xi_2}(K), H_{\xi_2}(K)\}.
\]
We see that
\[
\min \{V_{\xi_2}(K), H_{\xi_2}(K)\} > 0,
\]
since $\hat{\mathcal{F}}^+_d(K) = *$, and
\[
\min \{V_\xi(K), H_\xi(K)\} = 0 \text{ for all other } \xi \neq \xi_2 \text{ with } G_{Y,(K),K_\xi}(\xi) = s.
\]
Therefore, we obtain the equality
\[
d(Y, (K), s) = d(Y, (K'), s) - 2 \max_{n \in \mathbb{Z}} \{\min \{V_{\xi+n-PD[1]}(K), H_{\xi+n-PD[1]}(K)\}\}.
\]

In fact, in the proof of Theorem 1.6, we show that for any relative Spin$^c$ structure $\xi \in \text{Spin}^c(Y, K')$,
\[
\hat{HFK}(Y, (K'), G_{Y,(K'),K_\xi'}(\xi)) = \mathcal{F}.
\]
Thus we have the following proposition, which generalizes Rasmussen’s result [23, First part of Theorem 2].

**Proposition 4.1.** Given $Y$, $K$ and $K'$ as in Theorem 1.6, let $\gamma$ be a framing. If $\gamma$-surgery along $K$ produces an $L$-space, then so does $K'$.

**Remark 4.2.** When $\gamma$ is a negative framing, we consider $−\gamma$-surgery along the mirror knot of $K$ in $−Y$.

We can also immediately conclude the following corollary which is analogous to the result in $S^3$ [16, Theorem 2.5], but this corollary is only for $L$-space surgery.

**Corollary 4.3.** Given $Y$, $K$, $K'$ and $\gamma$ as in Theorem 1.6, suppose $\gamma$-surgery along $K$ produces an $L$-space $Y_\gamma(K)$, then for any relative Spin$^c$ structure $\xi \in \text{Spin}^c(Y, K) \cong \text{Spin}^c(Y, K')$,
\[
d(Y_\gamma(K), G_{Y_\gamma(K'),K_\xi'}(\xi)) \leq d(Y_\gamma(K'), G_{Y_\gamma(K'),K_\xi'}(\xi)).
\]

The equality holds for all Spin$^c$ structures in $Y_\gamma(K)$ and $Y_\gamma(K')$ if and only if $\hat{HFK}(Y, K) \cong \hat{HFK}(Y, K')$ with respect to the Alexander grading.
Proof. First by [24] Proof of Lemma 3.2 and [19] Theorem 1.2, \( \hat{HFK}(Y, K) \) consists of positive chains (see [24] Definition 3.1). Note that our assumption about the surgery slope \( \gamma \) coincides with the positive surgery definition in [24] Proof of Lemma 3.2. We see in the proof of Theorem 1.6 that the equality holds for all Spin\(^c\) structures only when there is no \( \ast \) in the mapping cone \( \widehat{\Delta}_{\ast}(K) \), and in addition \( \widehat{\Delta}_{\ast}(K) = \widehat{\Delta}_{\ast}(K') \), for any Spin\(^c\) structure \( \ast \). It implies that \( \hat{HFK}(Y, K, \ast) \) contains only one generator for any \( \ast \in \text{Spin}\(^c\)(Y) \) since otherwise some generator will contribute a \( \ast \) in the mapping cone. This means \( K \) is Floer simple. Since \( \widehat{\Delta}_{\ast}(K) = \widehat{\Delta}_{\ast}(K') \) for any Spin\(^c\) structure \( \ast \), it also follows that \( \hat{HFK}(Y, K) \cong \hat{HFK}(Y, K') \) with respect to the Alexander grading, which can also be obtained by [20] Theorem 1.5.

Now we are ready to deduce the \( d \)-invariant surgery formula we use to study our surgery problem. For the convenience of calculation, we choose two Spin\(^c\) structures, which are easy to compute, to give the \( d \)-invariant formula. We see that the even case has been solved in Proposition 3.3. Here, by even case we mean the case that the surgery result is \( L(s, 1) \) with \( s \) even. For the odd case, Ni and Wu’s \( d \)-invariant surgery formula can help to analyze null-homologous knot surgeries (see [16] Proposition 1.6 and [25] Proposition 3.2). As for homologically essential knots of the odd case, in our previous paper, we give a \( d \)-invariant surgery formula only for “large” surgeries on \( L(n, 1) \) with \( n \geq 5 \) prime [25] Proposition 4.4 and 4.5. Now we can remove the constraint on surgery slopes and give an improved \( d \)-invariant surgery formula for all surgery slopes as follows. In addition, our new \( d \)-invariant surgery formula applies to \( L(n, 1) \) with \( n \geq 5 \) odd instead of prime. We split it into two cases \( m > k^2/n \) and \( m < k^2/n \), which correspond to a positive framing and negative framing respectively.

**Proposition 4.4.** Let \( Y = L(n, 1) \) with \( n \geq 5 \) odd and \( K \subset Y \) be a homologically essential knot with winding number \( 1 \leq k \leq (n-1)/2 \), and let \( K' \) be a simple knot in \( Y \) with \( [K] = [K'] \in H_1(Y) \). Suppose \( \gamma = (m\mu + \lambda) \)-surgery with \( m > k^2/n \) along \( K \) yields an \( L \)-space \( Y_\gamma(K) \) with \( |H_1(Y_\gamma(K))| \) odd. Then \( \gamma \)-surgery along \( K' \) produces the Seifert fibered \( L \)-space \( M(0, 0; (m-k, 1), (n-k, 1), (k, 1)) \), abbreviated by \( M \), and

\[
(4.3) \quad d(Y_\gamma(K), \ast) = d(M, t_M) - 2V_{\xi_0}(K),
\]

where \( t \) and \( t_M \) are the unique self-conjugate Spin\(^c\) structures on \( Y_\gamma(K) \) and \( M \) respectively, and \( V_{\xi_0}(K) \) is a non-negative integer. Furthermore, if \( V_{\xi_0}(K) \geq 2 \), then

\[
(4.4) \quad d(Y_\gamma(K), \ast + 1_i PD[\mu]) = d(M, t_M + 1_i PD[\mu]) - 2V_{\xi_0+PD[\mu]}(K),
\]

where \( i : Y - K \to Y_\gamma(K) \) or \( Y - K' \to M \) is inclusion, and \( V_{\xi_0+PD[\mu]}(K) \) is a non-negative integer satisfying \( V_{\xi_0}(K) - 1 \leq V_{\xi_0+PD[\mu]}(K) \leq V_{\xi_0}(K) \).

**Proposition 4.5.** Given \( Y, K, K' \) and \( k \) as in Proposition 4.4, suppose \( \gamma = (m\mu + \lambda) \)-surgery with \( m < k^2/n \) along \( K \) yields an \( L \)-space \( Y_\gamma(K) \) with \( |H_1(Y_\gamma(K))| \) odd. Then \( \gamma \)-surgery along \( K' \) produces the Seifert fibered \( L \)-space \( M(0, 0; (m-k, 1), (n-k, 1), (k, 1)) \), abbreviated by \( M \), and

\[
(4.5) \quad d(Y_\gamma(K), \ast) = d(M, t_M) + 2V_{\xi_0}(K),
\]

where \( t \) and \( t_M \) are the unique self-conjugate Spin\(^c\) structures on \( Y_\gamma(K) \) and \( M \) respectively, and \( V_{\xi_0}(K) \) is a non-negative integer. Furthermore, if \( V_{\xi_0}(K) \geq 2 \), then

\[
(4.6) \quad d(Y_\gamma(K), \ast + 1_i PD[\mu]) = d(M, t_M + 1_i PD[\mu]) + 2V_{\xi_0+PD[\mu]}(K),
\]

where \( i : Y - K \to Y_\gamma(K) \) or \( Y - K' \to M \) is inclusion, and \( V_{\xi_0+PD[\mu]}(K) \) is a non-negative integer satisfying \( V_{\xi_0}(K) - 1 \leq V_{\xi_0+PD[\mu]}(K) \leq V_{\xi_0}(K) \).
Remark 4.6. In our notation for the Seifert fibered space $M(0, 0; (m − k, 1), (n − k, 1), (k, 1))$, the two 0’s means the base space for $M$ is of genus 0 and without boundary, and $(m − k, 1), (n − k, 1)$, and $(k, 1)$ specify the type of its exceptional fibers.

Remark 4.7. Our surgery formulas in Proposition 4.4 and 4.5 still hold when $n = 3$ (See Proposition 4.1 and 4.2). Besides, our surgery formulas are also true for $k = 0$ (i.e. $K$ is null-homologous), but Ni-Wu’s formula is more general and easy to apply, thus we do not include $k = 0$ in our results.

By (2.1) and (2.3), we see that when $nm − k^2 > 0$, $d_1$ and $d_2$ have the same sign, that is, $\gamma$ is a positive framing. We will discuss the case $nm − k^2 > 0$ in detail, and the other case $nm − k^2 < 0$ can be obtained by reversing the orientation.

Fix $l > 0$, and choose the parity of $l$ such that $|H_1(Y_{\mu, \lambda})| = nl − k^2$ is odd. Then there exists only one self-conjugate Spin$^c$ structure on $Y_{\mu, \lambda}$, denoted by $t_0$. Consider the relative Spin$^c$ structure $\xi_0 = \Xi(t_0)$, which is induced from (3.3). There are some nice properties of $\xi_0$ given in the following lemmas, which we can prove by the same arguments as in [12].

Lemma 4.8 (Proposition 4.5 in [12]). Let $[\alpha] \in H_1(Y − K)$. Then $V_{\xi_0 + P \alpha}(K) = H_{\xi_0 − P \alpha}(K)$.

Lemma 4.9 (Lemma 4.7 in [12]). The Spin$^c$ structure $s_0 = G_{\gamma(K), K, \alpha}(\xi_0) \in \text{Spin}^c(Y_{\gamma(K)})$ is self-conjugate.

Proof of Proposition 4.4. First, we apply Theorem 1.6 for the relative Spin$^c$ structure $\xi_0$ to show

$$d(Y_{\gamma(K)}, s_0) = d(M, s_0) − 2V_{\xi_0}(K).$$

We can see that $\gamma$-surgery along $K' = K(n, 1, k)$ gives the Seifert fibered space $M(0, 0; (m − k, 1), (n − k, 1), (k, 1))$ which must be an $L$-space by Proposition 4.1. This computation is standard (cf. [6, Lemma 9]). By Lemma 4.9, $s_0$ is self-conjugate, and it is the unique self-conjugate Spin$^c$ structure on $Y_{\gamma(K)}$ and $M$ since $|H_1(Y_{\gamma(K)})| = |H_1(M)|$ is assumed to be odd.

To find

$$\max_{n \in \mathbb{Z}} \{\min\{V_{\xi_0 + n \cdot P \gamma}(K), H_{\xi_0 + n \cdot P \gamma}(K)\}\},$$

we consider the hat version mapping cone $\hat{\Xi}_{s_0}(K)$ first. According to Lemma 4.8, $\xi_0$ has a nice symmetric property, then $\hat{\Xi}_{s_0}(K)$ must be labeled with a $\circ$ or $\ast$ since neither $\ast$ nor $\circ$ satisfies $\hat{\xi}_0 = \xi_0$.

We claim that there is no $\ast$ in $\hat{\Xi}_{s_0}(K)$ except possibly $\hat{\Xi}_{s_0}(K)$. If there is a $\ast$, say on the left of $\hat{\Xi}_{s_0}(K)$, then by Lemma 4.8, there is a $\ast$ on the right of it, which gives two nontrivial summands of type $\ast$. It contradicts to

$$\hat{HF}(Y_{\gamma(K)}, s_0) = \emptyset.$$

In fact, all elements on the right of $\hat{\Xi}_{s_0}(K)$ must be $\ast$ or $\circ$, and all elements on the left of it must be $\ast$ or $\circ$.

Hence, if $\hat{\Xi}_{s_0}(K) = \circ$, we have

$$\max_{n \in \mathbb{Z}} \{\min\{V_{\xi_0 + n \cdot P \gamma}(K), H_{\xi_0 + n \cdot P \gamma}(K)\}\} = 0 = V_{\xi_0}(K).$$

Therefore

$$d(Y_{\gamma(K)}, s_0) = d(M, s_0) − 2 \max_{n \in \mathbb{Z}} \{\min\{V_{\xi_0 + n \cdot P \gamma}(K), H_{\xi_0 + n \cdot P \gamma}(K)\}\}$$

$$= d(M, s_0) − 2V_{\xi_0}(K).$$
If \( \widehat{\mathcal{H}}_{\xi_0}(K) = \ast \), then by Lemma 4.8 and our claim,

\[
\max_{n \in \mathbb{Z}} \{ \min \{ V_{\xi_0+n \cdot PD[\gamma]}(K), H_{\xi_0+n \cdot PD[\gamma]}(K) \} \} = V_{\xi_0}(K) = H_{\xi_0}(K).
\]

Therefore

\[
d(Y_{\gamma}(K), s_0) = d(M, s_0) - 2 \max_{n \in \mathbb{Z}} \{ \min \{ V_{\xi_0+n \cdot PD[\gamma]}(K), H_{\xi_0+n \cdot PD[\gamma]}(K) \} \}
\]

\[
= d(M, s_0) - 2 V_{\xi_0}(K).
\]

To prove the second equality (4.4), we apply Theorem 1.6 for the relative \( \text{Spin}^c \) structure \( \xi_0 + PD[\mu] \). If \( V_{\xi_0}(K) \geq 2 \), then by (3.5),

\[
1 \leq V_{\xi_0+n \cdot PD[\mu]}(K) \leq V_{\xi_0}(K) \quad \text{and} \quad V_{\xi_0}(K) \leq V_{\xi_0+PD[\mu]}(K).
\]

Since by Lemma 4.8,

\[
H_{\xi_0+PD[\mu]}(K) = V_{\xi_0-PD[\mu]}(K),
\]

we have

\[
1 \leq V_{\xi_0+PD[\mu]}(K) \leq H_{\xi_0+PD[\mu]}(K).
\]

Thus the hat version \( \widehat{\mathcal{H}}_{\xi_0+PD[\mu]}(K) \) is \( \ast \). There is no other \( \ast \) in \( \widehat{\mathcal{X}}_{\xi_0+i \cdot PD[\mu]}(K) \), since otherwise there are two nontrivial summands of type \( \ast \), which gives a contradiction. Therefore

\[
\max_{n \in \mathbb{Z}} \{ \min \{ V_{\xi_0+PD[\mu]+n \cdot PD[\gamma]}(K), H_{\xi_0+PD[\mu]+n \cdot PD[\gamma]}(K) \} \} = \min \{ V_{\xi_0+PD[\mu]}(K), H_{\xi_0+PD[\mu]}(K) \}
\]

\[
= V_{\xi_0+PD[\mu]}(K),
\]

which implies

\[
d(Y_{\gamma}(K), s_0 + i \cdot PD[\mu]) = d(M, s_0 + i \cdot PD[\mu]) = 2 V_{\xi_0+PD[\mu]}(K).
\]

This completes the proof. \( \square \)

**Proof of Proposition 4.2.** The proof is similar to the case \( nm - k^2 > 0 \). We consider \(-Y_{\gamma}(K)\) as an \( L \)-space obtained from \((-m\mu + \lambda)\)-surgery along a knot in \( L(-n,1) \) and then apply the same argument. \( \square \)

5. Distance one surgeries on the lens space \( L(n,1) \)

![Diagram](image_url)

(a) When \( m \geq k + 3 \).

(b) When \( m \leq k - 3 \).

**Figure 8.** The plumping diagram of \( M(0,0; (m-k,1), (n-k,1), (k,1)) \) when \( k > 1 \).
In this section, we will apply our \(d\)-invariant surgery formulas in Proposition 4.4-4.5 to analyze our surgery problem. For different homology classes of the surgered knot \(K\), we will apply different \(d\)-invariant formulas. Besides, to apply our \(d\)-invariant formulas, we need to compute \(d\)-invariant of \(M\), and for different types of \(M\) (see below), we have to compute them separately. Therefore we divide distance one surgeries on \(L(n, 1)\) into the following cases:

Case 1: \(K\) is null-homologous;
Case 2: \(K\) is homologically essential:
   i \(k = 1\) (\(M\) is a lens space);
   ii \(k > 1\):
      (a) \(m \geq k + 3\) (\(M\) is a Seifert fibered space with plumbing diagram shown in Figure 8(a));
      (b) \(m = k - 1\) or \(k + 1\) (\(M\) is a lens space);
      (c) \(m \leq k - 3\) (\(M\) is a Seifert fibered space with plumbing diagram shown in Figure 8(b)).

Note that to make \(|H_1(M)|\) odd, \(k\) and \(m\) must have different parities. For the convenience of readers, we list our results and strategies we adopt for corresponding cases in Table 1.

| Cases          | Strategy                                           | Proved in         |
|----------------|----------------------------------------------------|-------------------|
| Even case      | Applying Lemma 3.2                                 | Proposition 3.3   |
| Odd case       |                                                     |                   |
| Null-homologous| Applying Ni-Wu’s formula and [15, Corollary 3.7]   | Proposition 5.1   |
| Homologically  | Applying Proposition 4.4.4.5                        | Proposition 5.3   |
| essential      |                                                   |                   |
| \(k = 1\)     |                                                   |                   |
| \(m \geq k + 3\)| Applying Proposition 4.4.4.5                        | Proposition 5.5   |
| \(m = k - 1\) | Applying Proposition 4.4.4.5                        | Proposition 5.11  |
| or \(k + 1\)  |                                                   | (in Section 5.2)  |
| \(m \leq k - 3\)| Applying Proposition 4.4.4.9 and Lemma 5.8          | Proposition 5.10  |
| and Lemma 5.8 |                                                   | (in Section 5.1)  |
| The rest       | Considering symmetry of surgeries                   | Section 5.3       |

Table 1. Strategies and results for different cases.

We first deal with Case 1, Case 2.i and Case 2.ii.(a), in which \(d\)-invariants of \(M\) are easier to compute. In fact, we analyze the similar cases in [25].

**Proposition 5.1.** Let \(K\) be a null-homologous knot in the lens space \(L(n, 1)\) with \(n \geq 5\) odd. The lens space \(L(s, 1)\) with \(s\) odd is obtained by a distance one surgery along \(K\) if and only if \(s = n\), or \(n = 5\) and \(s = -5\).

**Proof.** We first use Ni-Wu’s \(d\)-invariant surgery formula for null-homologous knots (see [25, Proposition 3.2] and [16, Proposition 1.6]) to obstruct all surgery results except the case \(s = \pm n\). The computation is almost the same as in [25, Proof of Theorem 1.2(i)], where we show a similar result as this proposition under the assumption that \(n \geq 5\) is prime, but we do not use this assumption in the computation.

Moore and Vazquez [15, Corollary 3.7] show that a distance one surgery along any knot in \(L(n, 1)\) yielding \(L(-n, 1)\) with \(n > 1\) square-free and odd exists if and only if \(n = 5\). In fact, in the proof of [15, Corollary 3.7], the assumption that \(n\) is square-free is just used to show that any knot admitting
this type of surgery must be null-homologous. Here we already assume that \( K \) is null-homologous, thus the argument of [15, Corollary 3.7] can also be applied to rule out the case \( s = -n \) except when \( n = 5 \) and \( s = -5 \) in our setting.

**Remark 5.2.** The results listed in Table 3 show that only null-homologous knot in \( L(n, 1) \) with \( n \geq 3 \) odd may admit a distance one surgery yielding \( L(n, 1) \) itself. Combined with Gainullin’s result [7, Theorem 8.2], we have that if a knot \( K \subset L(n, 1) \) with \( n \geq 3 \) odd admits a distance one surgery yielding itself, then \( K \) is the unknot. Therefore, the band surgery in Figure 1(b) is the only one which transforms \( T(2, n) \) with \( n \geq 3 \) odd into itself.

**Proposition 5.3.** Let \( K \) be a homologically essential knot with winding number \( k = 1 \) in the lens space \( L(n, 1) \) with \( n \geq 5 \) odd. The lens space \( L(s, 1) \) with \( s \) odd is obtained by a distance one surgery along \( K \) only if \( s = \pm 1 \) or \( n = 5 \) and \( s = -9 \).

**Proof.** The Seifert fibered space \( M(0, 0; (m - k, 1), (n - k, 1), (k, 1)) \) in this case reduces to the lens space \( L(mn - 1, n) \), whose \( d \)-invariant is easy to compute by the recursive formula (3.1). Applying Proposition 4.4 and 4.5 directly, the result follows by almost the same computation as in the proof of [25, Theorem 1.2 (ii)], where we show a similar result as this proposition under the assumption that \( n = 5 \) is prime but we do not utilize this assumption in the computation. Still we can not obstruct distance one surgery between \( L(5, 1) \) and \( L(-9, 1) \) when \( k = 1 \).

**Remark 5.4.** In fact, distance one surgery between \( L(5, 1) \) and \( L(-9, 1) \) (or \( L(9, 1) \) and \( L(-5, 1) \)) is the only one unexpected surgery in Theorem 1.1 as pointed out in Remark 1.2. We try to use the \( d \)-invariant formula in Theorem 1.6 for all other \( \text{Spin}^c \) structures to obstruct this surgery. But unfortunately, it fails.

**Proposition 5.5.** Let \( K \) be a homologically essential knot with winding number \( 1 < k \leq (n - 1)/2 \) in the lens space \( L(n, 1) \) with \( n \geq 5 \) odd. There does not exist a distance one surgery along \( K \) yielding a lens space \( L(s, 1) \) with \( s \) odd when the surgery slope \( m \geq k + 3 \).

**Proof.** This is also obtained by applying Proposition 4.4 and 4.5 directly. In [25, Theorem 1.2 (iii)], we conclude a similar result as this proposition, where we assume \( n \geq 5 \) is prime, but we do not use the assumption in the computation. Thus to get the same computation as in [25, Proof of Theorem 1.2 (iii)] shows this proposition.

5.1. **Distance one surgery on \( L(n, 1) \) with \( n \geq 9 \) odd, \( k > 1 \) and \( m \leq k - 3 \).** The goal of this section is analyzing Case 2.ii.(c). The case \( n = 5, 7 \) has been discussed in [25, Theorem 1.3], thus we only consider \( n \geq 9 \).

By Proposition 4.4 and 4.5 the Seifert fibered space \( M = M(0, 0; (m - k, 1), (n - k, 1), (k, 1)) \) must be an \( L \)-space. According to [4, 10, 13], the Seifert fibered space \( M(0, 0; (-p_3, 1), (p_1, 1), (p_2, 1)) \) with \( p_1, p_2, p_3 \geq 2 \) is an \( L \)-space if and only if

1. \( p_3 \geq \min\{p_1, p_2\} \) or \( p_3 = \min\{p_1, p_2\} - 1 \) and \( \max\{p_1, p_2\} \leq 2p_3 + 1 \).

In our case, \( k - m \geq 3 \) and \( n - k > k \geq 2 \) since \( m \leq k - 3 \), \( n \geq 9 \) and \( 1 < k \leq \frac{n - 1}{2} \). Therefore, \( M \) is an \( L \)-space if and only if

- \( m \leq 0 \) or
\[ m = 1 \text{ and } k \geq \frac{n+1}{3}. \]

Hence we only need to consider the above two cases.

5.1.1. \emph{d-invariant} of \( M(0,0;(m-k,1),(n-k,1),(k,1)) \) with \( n \geq 9 \text{ odd, } 1 < k \leq (n-1)/2 \text{ and } m \leq 0 \). To utilize our \( d \)-invariant surgery formula, we must compute the \( d \)-invariant of the Seifert fibered space \( M = M(0,0;(m-k,1),(n-k,1),(k,1)) \) with \( n \geq 9 \text{ odd, } 1 < k \leq (n-1)/2 \text{ and } m \leq 1 \). When \( m = 1 \), the \( d \)-invariant of \( M \) is hard to compute, thus we only deal with the case \( m \leq 0 \) in this section. In fact, the computation of \( d \)-invariants for an infinite family of Seifert fibered spaces is a technical and intractable problem. Our computation may have independent interests since \( d \)-invariant is very useful in answering a number of questions in 3-dimensional topology and knot theory. The computation is due to Ozsváth and Szabó’s algorithm which computes the \( d \)-invariant of a larger class of 3-manifolds, namely, the plumbed 3-manifolds [18]. The Seifert fibered space \( M \) with \( n \geq 9 \text{ odd, } m \leq 0 \text{ and } 1 < k \leq (n-1)/2 \) is the boundary of the plumbed 4-manifold, denoted by \( X \), which is constructed by plumbing disc bundles over \( S^2 \) according to the plumbing diagram, denoted by \( G \), shown in Figure [8(b)]. We have the following exact sequence.

\[
\begin{array}{ccccccc}
\text{Spin}^c(X) & \longrightarrow & \text{Spin}^c(M) \\
\downarrow c_2 & & \downarrow c_1 \\
0 & \longrightarrow & H_2(X) & \xrightarrow{Q} & H^2(X) & \longrightarrow & H^2(M) & \longrightarrow & 0
\end{array}
\]

Let \( \Sigma_1, \ldots, \Sigma_n \) be a basis of \( H_2(X) \) determined by the \( n \) spheres corresponding to the \( n \) vertices in \( G \). Since \( X \) is simply-connected, the cohomology \( H^2(X) = \text{Hom}(H_2(X), \mathbb{Z}) \), which is a free \( \mathbb{Z} \)-module over the Hom-dual basis \( \Sigma_1^*, \ldots, \Sigma_n^* \). With these choices of bases, the map \( H_2(X) \rightarrow H^2(X) \) is represented by the matrix of the intersection form \( Q : H_2(X) \times H_2(X) \rightarrow \mathbb{Z} \).

The set of characteristic vectors for \( G \) is defined as

\[ \text{Char}(G) = \{ w \in H^2(X) \mid \langle w, \Sigma_i \rangle \equiv Q(\Sigma_i, \Sigma_i) \pmod{2} \text{ for all } 1 \leq i \leq n \}. \]

The set of \( \text{Spin}^c \) structures on \( X \) is in one-to-one correspondence with \( \text{Char}(G) \) by the first Chern class \( c_1 \). The \( \text{Spin}^c \) structures on \( M \) are in bijection with \( 2H^2(M) \) via \( c_1 \). Since \( |H^2(M)| \) is odd, the \( \text{Spin}^c \) structures are in one-to-one correspondence with \( H^2(M) \) and thus also with \( \text{coker}Q \). We use \( t(w) \) to represent the \( \text{Spin}^c \) structure on \( M \) that is determined by the equivalence class of \( w \in \text{Char}(G) \) in coker \( Q \). If \( w, w' \in \text{Spin}^c(X) \) restrict to the same \( \text{Spin}^c \) structure on \( M \), then their corresponding characteristic vectors \( w, w' \in \text{Char}(G) \) are congruent modulo the image of \( 2H_2(X) \) in \( H^2(X) \); equivalently, \( (w - w')Q^{-1} \in H_2(X) \).

A vertex in \( G \) is called \emph{bad} if its weight is strictly greater than the minus of its valence. Ozsváth and Szabó [18] show that if \( Q \) is negative definite and \( G \) contains at most one bad vertex, then

\[
(5.1) \quad d(M, t) = \max_{\{w \in \text{Char}(G) : t(w) = t\}} \frac{w^TQ^{-1}w + |G|}{4}.
\]

Moreover, they give an algorithm to find the characteristic vector \( w \) that maximises (5.1), which we review below.

We start with a \( w \in \text{Char}(G) \) satisfying

\[
(5.2) \quad e_i + 2 \leq \langle w, \Sigma_i \rangle \leq -e_i \text{ for all } 1 \leq i \leq n,
\]
where $e_i = Q(\Sigma_i, \Sigma_i)$. Let $w_0 = w$. We then construct $w_k$ inductively as follows: choose any $1 \leq j \leq n$ such that $\langle w_j, \Sigma_j \rangle = -e_j$, then we let $w_{k+1} = w_k + 2PD(\Sigma_j)$ and call this action a pushing down the value of $w_k$ on $\Sigma_j$. The path $\{w_0, w_1, \ldots\}$ will terminate at some $w_m$ when one of the followings happens:

- $e_i \leq \langle w_m, \Sigma_i \rangle \leq -e_i - 2$ for all $1 \leq i \leq n$. In this case, the path is called maximising, and we say $w_0$ supports a maximising path.
- $\langle w_m, \Sigma_i \rangle > -e_i$ for some $i$. In this case, the path is called non-maximising.

Ozsváth and Szabó proved that the maximiser of (5.1) is contained in the set of characteristic vectors which support a maximising path.

In our case, the plumbing diagram $G$ contains exactly one bad vertex, and the intersection form $Q$ is negative definite, shown as follows. Thus Ozsváth and Szabó’s algorithm is applicable.

The next lemma lists all the characteristic vectors which support a maximising path.

**Lemma 5.6.** There are exactly $-nm+k^2$ number of characteristic vectors supporting a maximising path and corresponding to $-nm+k^2$ number of Spin$^c$ structures on $M(0,0; (m-k, 1), (n-k, 1), (k, 1))$ with $n \geq 9$, $m \leq 0$ and $1 < k \leq (n-1)/2$. They are listed as follows:

1. $w_1(i, j_i) = (0, \ldots, 0, 2, 0, \ldots, 0|0, \ldots, 0|j_i|0)$. Here, the integer $i \in [1, k-1]$ denotes the place where 2 appears, and $j_i$ is an integer satisfying $m - k + 2 \leq j_i \leq -m + k - 2i$ (e.g. $w_1(1, -m + k - 2) = (2, \ldots, 0, 0, \ldots, 0|0, \ldots, 0| - m + k - 2|0)$).
2. $w_2(i, j_i) = (0, \ldots, 0|0, \ldots, 0, 2, 0, \ldots, 0|j_i|0)$. Here the integer $i$ denotes the place where 2 appears.
   - (i) If $m < 0$, then $i$ takes value in the interval $[1, n - k - 1]$, and $j_i$ is an integer satisfying $m - k + 2 \leq j_i \leq -m + k - 2i$ when $1 \leq i \leq k - 1$; when $i \geq k$, $j_i$ satisfies $m - k + 2 \leq j_i \leq -m - k$.
   - (ii) If $m = 0$, then $i$ takes value in the interval $[1, k - 1]$, and $j_i$ is an integer satisfying $m - k + 2 \leq j_i \leq -m + k - 2i$.
3. $w_3(j) = (0, \ldots, 0|0, \ldots, 0|j|2)$, where the integer $j \in [m - k + 2, -m - k]$. This type of vectors exists only when $m < 0$.
4. $w_4(j) = (0, \ldots, 0|0, \ldots, 0|j|0)$, where the integer $j \in [m - k + 2, -m + k]$. 

\[
Q = \begin{pmatrix}
\begin{array}{cccc}
-1 & 2 & & \\
2 & -1 & 2 & \\
 & & \ddots & 1 \\
 & & 1 & -2 \\
 1 & & & \\
\end{array}
\end{pmatrix}
\]
In the above notation, we divide vectors by vertical bars to 4 blocks, which contain $k - 1$, $n - k - 1$, 1 and 1 elements respectively. The subscripts in $w$ are used to distinguish different types of those vectors.

Proof. We start with a vector $w$ such that

$$0 \leq \langle w, \Sigma_i \rangle \leq 2 \text{ for } i \neq n - 1,$$

$$m - k + 2 \leq \langle w, \Sigma_{n-1} \rangle \leq k - m.$$

We first claim that if $w$ supports a maximising path, then $w$ contains no substring of the form $(2, 0, \ldots, 0, 2)$ in one of the block in the vector notation above. Because otherwise we push down the 2's from left to right in the substring, which will eventually produce a 4 at the last spot of the substring. Therefore, there are at most three 2's in any vector which supports a maximising path, since otherwise the pigeonhole principle implies that there must be two of them in the same block, which gives a substring of the form $(2, 0, \ldots, 0, 2)$.

Now we consider the following 4 cases.

Case 1: there are three 2’s in $w$. Then the vector $w$ much be like $(0, \ldots, 0, 2, 0, \ldots, 2, 0, \ldots, 0, 2)$ for some $j \in [m-k+2, -m+k]$. Pushing down the 2’s in the first block, we will eventually obtain a 4 in the last block. So this $w$ initiates a non-maximising path.

Case 2: there are two 2’s in $w$. If $w$ is like $(0, \ldots, 0, 2, 0, \ldots, 0, 2, 0, \ldots, 0, 2)$ for some $j \in [m-k+2, -m+k]$, then like Case (1), pushing down the 2’s in the first or second block, we will eventually obtain a 4 in the last block. So this $w$ initiates a non-maximising path.

Case 3: there is only one 2 in $w$. Then we consider the following three subcases.

Case 3.i: $w$ is of the form $(0, \ldots, 0, 2, 0, \ldots, 0, 0, \ldots, 0, j|0)$ for some $j \in [m-k+2, -m+k]$, after pushing down the 2’s in the first and second block, we get a 4 in the last block. So this $w$ also initiates a non-maximising path.

Case 3: there is only one 2 in $w$. Then we consider the following three subcases.

Case 3.ii: $w$ is of the form $(0, \ldots, 0, 2, 0, \ldots, 0, 0, \ldots, 0, j|0)$ for some $j \in [m-k+2, -m+k]$. In fact, to make $w$ support a maximising path, the different places that the 2 lies in determine the different ranges of value that $j$ could be. Precisely, we claim that if the 2 lies in the $i$th entry in the first block for $1 \leq i \leq k - 1$, then $j$ can take a value in the interval $[m-k+2, -m+k-2i]$.

Before proving our claim, we define several moves on characteristic vectors due to Ozsváth and Szabó’s algorithm.

$$M1: (0, \ldots, 0, 2, 0, \ldots, 0) \Rightarrow (-2, \ldots, 0, 2, 0, \ldots, 0)$$

$$M2: (0, \ldots, 0, -2, 0, \ldots, 0) \Rightarrow (0, \ldots, 0, -2, 0, \ldots, 0)$$

$$M3: (0, \ldots, 0, 0, \ldots, 0, 2) \Rightarrow (0, \ldots, 0, 0, \ldots, 0, 2)$$

$$M4: (0, \ldots, 0, 2) \Rightarrow (0, \ldots, 0, 2)$$

$$M5: (0, \ldots, 0, 0, \ldots, 0, 2) \Rightarrow (0, \ldots, 0, 0, \ldots, 0, 2)$$

$$M6: (0, \ldots, 0, 2) \Rightarrow (0, \ldots, 0, 2)$$

Suppose that

$$w = (0, \ldots, 0, 2, 0, \ldots, 0|0),$$

for some $j \in [m-k+2, -m+k]$ and $1 \leq i \leq k - 1$, supports a maximising path. According to Ozsváth and Szabó’s algorithm, we transform $w$ as follows.
Following the similar but strenuous steps as in Case 3.i, one can show that only when \( j + 2i > -m + k \), we see obviously \( w \) initiates a non-maximising path. Note that for \( m = 0 \), we need carefully deal with the case that the 2 lies at the last spot in the first block, in which only when \( w = w_1(k - 1, 2 - k) \), \( w \) initiates a maximising path.

Case 3.ii: \( w \) is of the form \((0, \ldots, 0, 2, 0, \ldots, 0 | j | 0)\). Suppose the 2 lies in the \( i \)th entry in the second block. We divide it into two cases to analyze: a. \( 1 \leq i \leq k - 1 \); b. \( k \leq i \leq n - k - 1 \). Following the similar but strenuous steps as in Case 3.i, one can show that only when \( w = w_2(i, j) \), \( w \) supports a maximising path.

Case 3.iii: \( w \) is of the form \((0, \ldots, 0 | j | 0)\). Also the similar argument as in Case 3.i implies that only when \( w = w_3(j) \) supports a maximising path when \( m < 0 \); no vector of this form supports a maximising path when \( m = 0 \).

Case 4: there is no 2 in \( w \). Then \( w \) is of the form \((0, \ldots, 0 | j | 0)\). When \( m - k + 2 \leq j \leq -m + k - 2 \), \( w \) obviously supports a maximising path. Following the similar argument as in Case 3.i, we show that when \( j = -m + k \), \( w \) also initiates a maximising path.

In a summary, there are \(-nm + k^2\) number of vectors supporting a maximising path, which is exactly the number of Spin\(^c\) structures on \( M \). \( \square \)

Next goal is to find the maximisers of Formula [5.1] in the equivalent classes corresponding to the Spin\(^c\) structures \( \tau_M \) and \( t_M + i^*PD[\mu] \). Since there are the same number of vectors supporting a maximising path as the number of Spin\(^c\) structures by Lemma [5.6], they are all maximisers of Formula [5.1] in each equivalent class for corresponding Spin\(^c\) structure. However, it is inconvenient to consider the characteristic vector supporting a maximising path which corresponds to \( t_M \). In fact, \( w_i^TQ^{-1}w_i \) is a constant for any \( w_i \) in a maximising path \( \{w_0, w_1, \ldots\} \). Therefore, we can find some characteristic vector which is contained in the maximising path corresponding to \( t_M \), and such characteristic vector is also a maximiser of Formula [5.1] in the equivalent class for \( t_M \). The following lemma determines the corresponding maximisers we use to compute \( d(M, t_M) \) and \( d(M, t_M + i^*PD[\mu]) \).
Lemma 5.7. Let \( M = M(0,0; (m-k,1),(n-k,1),(k,1)) \) with \( n \geq 9 \) odd, \( m \leq 0 \) and \( 1 < k \leq (n-1)/2 \).

1. When \( k \) is even,
   - the characteristic vector \( v_1 = (0,\ldots,0,-2,0,\ldots,0|0,\ldots,0| -m + k|0) \) is a maximiser of Formula (5.1) in the equivalent class corresponding to the Spin\( ^c \) structure \( t_M \), where \(-2\) is in the \( \frac{k}{2} \)th entry in the first block;
   - the characteristic vector \( w_1(\frac{k}{2},m-k+2) = (0,\ldots,0,2,0,\ldots,0|0,\ldots,0|m-k+2|0) \) is a maximiser of Formula (5.1) in the equivalent class corresponding to the Spin\( ^c \) structure \( t_M + i^*PD[\mu] \).

2. When \( k \) is odd,
   - the characteristic vector \( v_2 = (0,\ldots,0|0,\ldots,0,-2,0,\ldots,0| -m + k|0) \) is a maximiser of Formula (5.1) in the equivalent class corresponding to the Spin\( ^c \) structure \( t_M \), where \(-2\) is in the \( \frac{k-1}{2} \)th entry in the second block;
   - when \( m \neq 0 \), the characteristic vector \( w_2(\frac{m-k}{2},m-k+2) = (0,\ldots,0,0,2,0,\ldots,0|m-k+2|0) \) is a maximiser of Formula (5.1) in the equivalent class corresponding to the Spin\( ^c \) structure \( t_M + i^*PD[\mu] \);
   - when \( m = 0 \), let \( i \equiv \frac{n-k}{2} \) (mod \( k \)) with \( 0 \leq i \leq k-1 \), then when \( i = 0 \) (respectively \( i > 0 \)), \( w_2(k-2) \) (respectively \( w_2(i,-k+2) \)) is a maximiser of Formula (5.1) in the equivalent class corresponding to the Spin\( ^c \) structure \( t_M + i^*PD[\mu] \) up to Spin\( ^c \) conjugation.

Proof. Since the first Chern class \( c_1 \) of a self-conjugate Spin\( ^c \) structure is 0, the unique self-conjugate Spin\( ^c \) structure \( t_M \) must correspond to the equivalence class of characteristic vectors which are in the image of \( Q \).

When \( k \) is even, we can see that \( v_1 \) satisfies this; more precisely, \( v_1 = Qu^T \), where
\[
v = (1,2,\ldots,\frac{k-2}{2},\frac{k}{2},\frac{k-2}{2},\ldots,2,1|0,\ldots,0|-1|0).
\]

When \( k \) is odd, \( v_2 \) is in the image of \( Q \); more precisely, \( v_2 = Qu^T \), where
\[
v = (0,\ldots,0|1,2,\ldots,\frac{n-k-2}{2},\frac{n-k}{2},\frac{n-k-2}{2},\ldots,2,1|-1|0).
\]
Thus \( v_1 \) (respectively \( v_2 \)) is in the equivalence class corresponding to \( t_M \), when \( k \) is even (respectively odd).

When \( m \leq -1 \), we claim that when \( k \) is even, the characteristic vector \( w_2(\frac{k}{2},-m) \) initiates a maximising path which contains \( v_1 \); when \( k \) is odd, if \( k < \frac{3}{2} \) (respectively \( k = \frac{3}{2} \) and \( k > \frac{3}{2} \)), then \( w_2(\frac{n+k}{2},-m-k) \) (respectively \( w_3(-m-k) \) and \( w_1(\frac{n-k}{2},-m-n+2k) \)) initiates a maximising path which contains \( v_2 \). When \( m = 0 \), we claim that if \( \frac{n-k}{2} \equiv i \) (mod \( k \)) with \( 0 \leq i \leq k-1 \), then when \( i = 0 \), \( w_2(k-2) \) initiates a maximising path which contains \( v_2 \); when \( i > 0 \), \( w_1(i,k-2i) \) initiates a maximising path which contains \( v_2 \). One may check our claims by following Ozsváth and Szabó’s algorithm. Thus \( v_1 \) (respectively \( v_2 \)) is a maximiser of Formula (5.1) in the equivalent class corresponding to \( t_M \), when \( k \) is even (respectively odd).

To find the maximiser corresponding to \( t_M + i^*PD[\mu] \), we need to find the preimage of \( i^*PD[\mu] \) in \( H^2(X) \) first. Consider the framed link
\[
\mathcal{L} = ((K_1,-2),\ldots,(K_{n-2},-2),(K_{n-1},m-k),(K_n,-2))
\]
with linking matrix given by the matrix $Q$, where each $K_i$ for $1 \leq i \leq n$ is an unknot and they are linked in the simplest manner, and $-2$'s and $m - k$ represent the corresponding framing coefficients. Then $M$ can be regarded as the boundary of the plumbed 4-manifold $X$ given by the framed link $L$. Denote by $D_i$ a small normal disk to $K_i$, and $\partial D_i = \mu_i$ the meridian of $K_i$. A systematic yet strenuous computation of homology shows that $\mu = \mu_{n-1}$. Thus, $i^*PD[\mu]$ corresponds to $PD[D_{n-1}] \in H^2(X)$ which is represented by the vector $(0, 0, \ldots, 0|0, \ldots, 0|0)$.

When $m \leq -1$, we claim $w_1(\frac{k}{2}, m - k + 2)$ (respectively $w_2(\frac{n-k}{2}, m - k + 2)$) is a maximiser of Formula (5.1) in the equivalence class corresponding to $t_M + i^*PD[\mu]$, when $k$ is even (respectively odd). We see that $-\nu_1$ (respectively $-\nu_2$) is also in the equivalence class corresponding to $t_M$, when $k$ is even (respectively odd), although it may be not in a maximising path. Thus $w_1(\frac{k}{2}, m - k + 2)$ (respectively $w_2(\frac{n-k}{2}, m - k + 2)$) is in the equivalence class corresponding to $t_M + i^*PD[\mu]$, when $k$ is even (respectively odd). Since $w_1(\frac{k}{2}, m - k + 2)$ (respectively $w_2(\frac{n-k}{2}, m - k + 2)$) is in the list of Lemma 5.6 when $m \leq -1$ and $k$ is even (respectively odd), the claim follows.

When $m = 0$, let $i = \frac{n-k}{2}$ (mod $k$) with $0 \leq i \leq k - 1$. As we mentioned above, when $i = 0$, $w_4(k)$ initiates a maximising path corresponding to $t_M$. Thus $w_4(k - 2)$ is in the equivalence class corresponding to $t_M + i^*PD[\mu]$. Since $w_4(k - 2)$ is in the list of Lemma 5.6 when $m = 0$, it is a maximiser of Formula (5.1) in the equivalent class corresponding to $t_M + i^*PD[\mu]$. When $i > 0$, as we mentioned above, $w_1(i, k - 2i)$ initiates a maximising path corresponding to $t_M$, and one may check that the maximising path always contains the characteristic vector $u_i = (0, \ldots, 0|0, \ldots, 0, -2, 0, \ldots, 0|0)$, where the $-2$ lies in the $i$th entry in the second block. Thus $u_i$ is in the equivalence class corresponding to $t_M$, and so is $-u_i$. Then we see that the characteristic vector $w_2(i, -k + 2)$ is in the equivalence class corresponding to $t_M + i^*PD[\mu]$, which is also in the list of Lemma 5.6 when $m = 0$. Hence, $w_2(i, -k + 2)$ is a maximiser of Formula (5.1) in the equivalent class corresponding to $t_M + i^*PD[\mu]$. $\square$

Now we are ready to compute $d(M, t_M)$ and $d(M, t_M + i^*PD[\mu])$ using Formula (5.1).

**Lemma 5.8.** Let $M = M(0, 0; (m-k, 1), (n-k, 1), (k, 1))$ with $n \geq 9$ odd, $m \leq 0$ and $1 < k \leq \frac{n-1}{2}$, where $m$ and $k$ have different parities. Let $t_M$ be the unique self-conjugate Spin$^c$ structure on $M$.

(i) When $k$ is odd ($m$ is even),

$$d(M, t_M) = \frac{m}{4} \quad \text{when } m \leq 0,$$

$$d(M, t_M + i^*PD[\mu]) = \frac{nm^2 + (4n - 2^k)m + 4n - 4k^2}{4(nm - k^2)} \quad \text{when } m \leq -2,$$

$$d(M, t_M + i^*PD[\mu]) = \frac{2ki + 2n - kn}{-k^2} \quad \text{when } m = 0 \text{ and } i \equiv \frac{n - k}{2} \pmod{k} \quad \text{with } 0 \leq i \leq k - 1.$$
(ii) When \( k \) is even (\( m \) is odd),

\[
d(M, t_M) = \frac{m - 2k + n}{4},
\]

(5.6) \[ d(M, t_M + i^*PD[\mu]) = \frac{nm^2 + (4n + n^2 - 2k^n - k^2)m + 2k^3 - 4k^2 + 4n - nk^2}{4(nm - k^2)}. \]

(5.7)

Remark 5.9. We can compute \( d \)-invariants of \( M(0, 0; (m - k, 1), (n - k, 1), (k, 1)) \) with \( n \geq 9, m \leq 0 \) and \( 1 < k \leq \frac{n - 1}{2} \) for all Spin\(^c\) structures using Lemma 5.6.

5.1.2. Distance one surgery on \( L(n, 1) \) with \( n \geq 9 \) odd, \( k > 1 \) and \( m \leq k - 3 \). In this section, we use our \( d \)-invariant surgery formula to study Case 2.ii.(c). We show the following proposition.

**Proposition 5.10.** Let \( K \) be a homologically essential knot with winding number \( 1 < k \leq (n - 1)/2 \) in the lens space \( L(n, 1) \) with \( n \geq 9 \) odd. Suppose the surgery slope \( m \leq k - 3 \). Then the lens space \( L(s, 1) \) with \( s \) odd is obtained by a distance one surgery along \( K \) only if \( s = 9, \pm(n - k^2) \) or \( n + 4 \). In addition, a distance one surgery from \( L(n, 1) \) to \( L(\pm(n - k^2), 1) \) exists only when \( k \geq (n + 1)/3 \) is even.

**Proof.** Suppose \( L(s, 1) \) with \( s \) odd is obtained by \((m\mu + \lambda)\)-surgery along \( K \) for some \( m \leq k - 3 \). As we state at the beginning of Section 5.1, we only need to deal with the case when \( m \leq 1 \). Since we can not compute \( d(M, t_M) \) for \( m = 1 \), we only analyze the case when \( m \leq 0 \) here. That is why we do not rule out the case \( s = \pm(n - k^2) \) in the proposition. When \( m \leq 0 \), we have \( mn - k^2 < 0 \). So we can apply the \( d \)-invariant surgery formulas in Proposition 4.5. We divide it into 2 cases due to Lemma 5.8.

Case 1: \( k \) is even (\( m \) is odd). By (2.5), \( |s| = -nm + k^2 \).

If \( s = nm - k^2 \), Formula 4.5 gives

\[-d(L(-nm + k^2, 1), 0) = d(M, t_M) + 2V_{\xi_0}(K).\]

Using (3.2) and (5.6), we have

\[ V_{\xi_0}(K) = \frac{(n - 1)m - k^2 - n + 1 + 2k}{8} < 0 \]

since \( (n - 1)m < 0, 2k - k^2 \leq 0 \) and \( 1 - n < 0 \). It contradicts to \( V_{\xi_0}(K) \geq 0 \).

If \( s = -nm + k^2 \), Formula 4.5 implies

\[ d(L(-nm + k^2, 1), 0) = d(M, t_M) + 2V_{\xi_0}(K).\]

Plugging in 5.6 and 3.2, we have

\[ V_{\xi_0}(K) = \frac{-(n + 1)m + k^2 + 2k - n - 1}{8}. \]

When \( m = -1 \) and \( k = 2 \), we have \( L(-nm + k^2, 1) = L(n + 4, 1) \). There is a distance one surgery from \( L(n, 1) \) to \( L(n + 4, 1) \) given by the double branched cover of the band surgery in Figure 1(c). When \( m = -1 \) and \( k \geq 4 \) or \( m \leq -3 \) and \( k \geq 2 \), we have \( V_{\xi_0}(K) \geq 2 \). We can thus apply Formula 4.6 and get

\[ d(L(-nm + k^2, 1), j) = d(M, t_M + i^*PD[\mu]) + 2V_{\xi_0 + PD[\mu]}(K). \]
The axis of symmetry of the function $j$ is $j = \frac{k^2 - mn}{2}$, and $f(0) = nm - k^2 + n$ is $j = \frac{k^2 - mn}{2}$, and $f(0) = nm + k^2 + n > 0$, $f(1) = n + 1 > 0$ and $f(2) = nm - k^2 + n < 0$ when $m = -1$ and $k \geq 4$ or $m \leq -3$ and $k \geq 2$. Therefore, the roots of $f(j)$ lie in $(1, 2)$ and $(-nm + k^2 - 2, -nm + k^2 - 1)$, which are not integers. This gives a contradiction.

Case 2: $k$ is odd ($m$ is even). If $s = nm - k^2$, Formula (4.5) implies

$$-d(L(-nm + k^2, 1), 0) = d(M, t_M) + 2V_{\xi_0}(K).$$

Using (3.2) and (5.3), we compute when $m \leq 0$

$$V_{\xi_0}(K) = \frac{(n - 1)m - k^2 + 1}{8} < 0,$$

which is a contradiction.

If $s = -nm + k^2$, Formula (4.5) gives

$$d(L(-nm + k^2, 1), 0) = d(M, t_M) + 2V_{\xi_0}(K),$$

which implies

$$V_{\xi_0}(K) = \frac{-(n + 1)m + k^2 - 1}{8}.$$

When $m = 0$ and $k = 3$, we have $V_{\xi_0}(K) = 1 < 2$. So Formula (4.6) is not applicable here. In this case, we cannot obstruct distance one surgery from $L(n, 1)$ to $L(9, 1)$ by our $d$-invariant surgery formula. When $k \geq 5$ or $m \leq -2$, we have $V_{\xi_0}(K) \geq 2$. We can thus apply Formula (4.6) and get

$$d(L(-nm + k^2, 1), j) = d(M, t_M + i*PD[\mu]) + 2V_{\xi_0 + PD[\mu]}(K),$$

for some $j \in [0, -nm + k^2 - 1]$. Since $d(M, t_M + i*PD[\mu])$ takes different values when $m = 0$ and $m \leq -2$, we further divide it into 2 subcases.

Case 2.a: $m = 0$. Applying (5.3) and (3.2), we get

$$V_{\xi_0 + PD[\mu]}(K) = \frac{1}{8} + \frac{(2j - k^2)^2}{8k^2} + \frac{2ki + n - kn}{2k^2},$$

for some $j \in [0, -nm + k^2 - 1]$. By (3.2) and (5.7), it follows

$$(5.8)$$

$$V_{\xi_0 + PD[\mu]}(K) = -\frac{1}{8} + \frac{(2j - (-nm + k^2))^2 - nm^2 - (4n + n^2 - 2kn - k^2)m - 2k^3 + 4k^2 - 4n + nk^2}{8(-nm + k^2)},$$

where

$$V_{\xi_0 + PD[\mu]}(K) = \frac{-(n + 1)m + k^2 + 2k - n - 1}{8} \text{ or } \frac{-(n + 1)m + k^2 + 2k - n - 1}{8} - 1.$$
Thus, \( V_{\xi_0 + pD[\mu]}(K) = \frac{k^2 - 1}{8} \) or \( \frac{k^2 - 1}{8} - 1 \). In the case of \( \frac{k^2 - 1}{8} \), Equation (5.11) can be simplified as

\[
j^2 - k^2j + 2ik + n - kn = 0.
\]

The axis of symmetry of the function \( f(j) = j^2 - k^2j + 2ik + n - kn \) is \( j = \frac{k^2}{2} \) and \( f(0) = 2ik + n - kn \leq 2(k - 1)k + (1 - k)(2k + 1) = 1 - k < 0 \) since \( n \geq 2k + 1, i \leq k - 1 \) and \( k \geq 5 \). Thus \( f(j) \) has no integral root in \([0, k^2 - 1]\). So this case is impossible. For the case of \( \frac{k^2 - 1}{8} - 1 \), Equation (5.11) can be simplified as

\[
j^2 - k^2j + 2ik + n - kn + 2k^2 = 0.
\]

Let \( f(j) = j^2 - k^2j + 2ik + n - kn + 2k^2 \). Since \( i \equiv \frac{n+k}{2} \) (mod \( k \)), we have \( n = (2l + 1)k + 2i \) for some \( l \in \mathbb{N} \) and \( 1 \leq i \leq k - 1 \). Then

\[
f(j) = j^2 - k^2j + 2ik + (1 - k)(2l + 1)k + 2i + 2k^2 = j^2 - k^2j - (2l + 1)k^2 + (2l + 1)k + 2i + 2k^2.
\]

Its axis of symmetry is \( j = \frac{k^2}{2} \), and \( f(0) = -(2l + 1)k^2 + (2l + 1)k + 2i + 2k^2 \). We see that if \( l \geq 1 \), then \( f(0) < 0 \) since \( k \geq 5 \) and \( i \leq k - 1 \), thus \( f(j) \) has no root in \([0, k^2 - 1]\), which gives a contradiction. If \( l = 0 \), then we have \( n = k + 2i \), and \( f(j) = j^2 - k^2j + n + k^2 \). We see that \( f(0) = n + k^2 > 0 \), \( f(1) = 1 + n > 0 \) and \( f(2) = 4 + n - k^2 = -k^2 + k + 2i + 4 < 0 \) since \( k \geq 5 \) and \( i \leq k - 1 \). Therefore, the roots of \( f(j) \) lie in \((1, 2)\) and \((k^2 - 2, k^2 - 1)\), which are not integers. This gives a contradiction.

Case 2.b: \( m \leq -2 \). Applying (5.4) and (3.2), we get

\[
(5.12) \quad V_{\xi_0 + pD[\mu]}(K) = -\frac{1}{8} + \frac{(2j - (-nm + k^2))^2 + nm^2 + (4n - k^2)m + 4n - 4k^2}{8(-nm + k^2)},
\]

where

\[
V_{\xi_0 + pD[\mu]}(K) = \begin{cases} \frac{-(n + 1)m + k^2 - 1}{8} & \text{or} & \frac{-(n + 1)m + k^2 - 1}{8} - 1, \end{cases}
\]

In the case of \( \frac{-(n + 1)m + k^2 - 1}{8} \), Equation (5.12) can be simplified as

\[
j^2 - (k^2 - mn)j + nm - k^2 + n = 0,
\]

which is the same as Equation (5.9). A similar argument shows that the function \( f(j) = j^2 - (k^2 - mn)j + nm - k^2 + n \) has no integral root in \([0, -nm + k^2 - 1]\), thus this case can be ruled out. For the case of \( \frac{-(n + 1)m + k^2 - 1}{8} - 1 \), we rewrite Equation (5.12) as

\[
j^2 - (k^2 - mn)j - nm + k^2 + n = 0,
\]

which is the same as Equation (5.10). A similar argument gives a contradiction.

To sum up, we conclude that in this setting the lens space \( L(s, 1) \) is obtained by a distance one surgery from \( L(n, 1) \) only if \( s = 9, n + 4, \) or \( \pm(n - k^2) \). Here a distance one surgery from \( L(n, 1) \) to \( L(n + 4, 1) \) can be realized as the double branched cover of the band surgery in Figure 1(c). As we discussed at the beginning of Section 5.1, the case \( s = \pm(n - k^2) \) occurs only when \( k \geq (n + 1)/3 \) is even.  
\[\square\]
5.2. **Distance one surgery on** $L(n, 1)$ **with** $n \geq 9$ **odd**, $k > 1$ **and** $m = k - 1$ **or** $k + 1$. In this section, we focus on Case 2.ii.(b). Since we have discussed the case $n = 5$ or $7$ in [25, Theorem 1.3], we only need to deal with the case $n \geq 9$ odd. Applying Proposition 5.12, we conclude the following result.

**Proposition 5.11.** Let $Y = L(n, 1)$ with $n \geq 9$ odd, and $K \subset Y$ be a homologically essential knot with winding number $1 < k \leq (n - 1)/2$. Suppose the surgery slope $m = k - 1$ or $k + 1$. Then the lens space $L(s, 1)$ with $s$ odd is obtained by a distance one surgery along $K$ only if $n$ and $s$ satisfy one of the following cases:

1. $n \geq 9$ is any odd integer and $s = n - 4$;
2. $n = 9$ and $s = -5$;
3. $n = 13$ and $s = -17$.

We divide the surgeries into 4 cases shown in Table 2, since for different cases the corresponding values of $d$-invariants are different. More details will be given below. We also list our results for corresponding cases in Table 2 and Proposition 5.11 is obtained by combining these results.

| Cases | Proved in |
|-------|-----------|
| $m = k - 1$ | $k$ even: Proposition 5.12; $k$ odd: Proposition 5.13 |
| $m = k + 1$ | $k$ even: Proposition 5.14; $k$ odd: Proposition 5.15 |

We start with the case $m = k - 1$, in which the Seifert fibered space $M(0, 0; (m - k, 1), (n - k, 1), (k, 1))$ reduces to the lens space $L(nk - n - k^2, k - 1)$. Note that the unique self-conjugate Spin$^c$ structure $t_M$ corresponds to $\frac{k - 2}{2}$ when $k$ is even, and corresponds to $\frac{nk - n - k^2 + k - 2}{2}$ when $k$ is odd. By carefully tracing $i^*PD[\mu] \in H_1(L(nk - n - k^2, k - 1))$, we see that up to Spin$^c$ conjugation the Spin$^c$ structure $t_M + i^*PD[\mu]$ corresponds to $\frac{3k - 2}{2}$ when $k$ is even, and corresponds to $\frac{nk - n - k^2 - k - 2}{2}$ when $k$ is odd, because the difference of the two Spin$^c$ structures $\frac{k - 2}{2}$ and $\frac{3k - 2}{2}$ is $\pm i^*PD[\mu]$, and so is the difference of the Spin$^c$ structures $\frac{nk - n - k^2 + k - 2}{2}$ and $\frac{nk - n - k^2 - k - 2}{2}$. See [5, Section 6] for more details about differences of Spin$^c$ structures in a lens space. For the convenience of readers, we compute the $d$-invariants of the relevant lens spaces using the recursive formula (3.1) as follows.

For $m = k - 1$ and $k$ even,

$$d(L(nk - n - k^2, k - 1), \frac{k - 2}{2}) = \frac{n - k - 3}{4},$$

$$d(L(nk - n - k^2, k - 1), \frac{3k - 2}{2}) = \frac{(2k - nk + n + k^2)^2 + (11 - 2k)(nk - n - k^2)}{4(k - 1)(nk - n - k^2)}$$

when $k \geq 4$,

$$d(L(nk - n - k^2, k - 1), \frac{3k - 2}{2}) = d(L(n - 4, 1), 2) = \frac{1}{4} + \frac{(8 - n)^2}{4(n - 4)}$$

when $k = 2$. 

For $m = k - 1$ and $k$ odd,

\begin{align}
(5.16) & \quad d(L(nk - n - k^2, k - 1), \frac{nk - n - k^2 + k - 2}{2}) = \frac{k - 3}{4}, \\
(5.17) & \quad d(L(nk - n - k^2, k - 1), \frac{nk - n - k^2 - k - 2}{2}) = \frac{(nk - n - k^2)(k^2 - 8k + 11) + 4k^2}{4(k - 1)(nk - n - k^2)}.
\end{align}

Now we are ready to apply our surgery formulas.

**Proposition 5.12.** Given $Y$ and $K$ as in Proposition 5.11, suppose the surgery slope $m = k - 1$ and $k$ is even. Then the lens space $L(s, 1)$ with $s$ odd is obtained by a distance one surgery along $K$ only if $s = n - 4$ or $n = 9$ and $s = -5$.

**Proof.** Suppose the lens space $L(s, 1)$ with $s$ odd is obtained by $((k - 1)\mu + \lambda)$-surgery along $K$. By (2.5), we have $s = nk - n - k^2$. Here, $(k - 1)n - k^2 > 0$ since $k > 1$ and $n \geq 9$.

If $s = nk - n - k^2$, Formula (4.3) gives

\[ d(L(nk - n - k^2, 1), 0) = d(L(nk - n - k^2, k - 1), \frac{k - 2}{2}) - 2V_{\xi_0}(K), \]

where $0$ corresponds to the unique self-conjugate Spin$^c$ structure on $L(nk - n - k^2, 1)$. We compute from (3.2) and (5.13)

\[ V_{\xi_0}(K) = \frac{k^2 - nk - n + 2n - 2}{8} = \begin{cases} 0 & \text{when } k = 2 \\
< 0 & \text{when } k \geq 4 \text{ and } n \geq 9. \end{cases} \]

When $k \geq 4$, it gives a contradiction since $V_{\xi_0}(K)$ is non-negative. When $k = 2$, we have $L(s, 1) = L(n - 4, 1)$. In fact, $L(nk - n - k^2, k - 1) = L(n - 4, 1)$ which means $(\mu + \lambda)$-surgery along the simple knot $K(n, 1, 2) \subset L(n, 1)$ yields $L(n - 4, 1)$. This is also the surgery given by the double branched cover of the band surgery in Figure 1(c) (the reverse direction).

If $s = -nk + n + k^2$, Formula (4.3) gives

\[ -d(L(nk - n - k^2, 1), 0) = d(L(nk - n - k^2, k - 1), \frac{k - 2}{2}) - 2V_{\xi_0}(K), \]

which implies

\[ V_{\xi_0}(K) = \frac{-k^2 + nk - k - 4}{8}. \]

When $n = 9$ and $k = 2$, we have $V_{\xi_0}(K) = 1 < 2$, so Formula (4.4) is not applicable here. In this case, we cannot obstruct distance one surgery from $L(9, 1)$ to $L(-5, 1)$ by our $d$-invariant surgery formula. When $n = 9$ and $k = 4$ or $n = 11$ and $k = 2$, we have $V_{\xi_0}(K) = \frac{3}{2}$, which contradicts to the fact that $V_{\xi_0}(K)$ is an integer.

Otherwise, we have $n = 11$ and $k \geq 4$ or $n \geq 13$ and $k \geq 2$, which implies $V_{\xi_0}(K) \geq 2$, thus we can apply Formula (4.4). It yields

\[ -d(L(nk - n - k^2, 1), j) = d(L(nk - n - k^2, k - 1), \frac{3k - 2}{2}) - 2V_{\xi_0 + PD[\mu]}(K), \]

for some $j \in [0, nk - n - k^2 - 1]$. Since $k = 2$ and $k \geq 4$ give different $d$-invariant values for $L(nk - n - k^2, k - 1)$, we discuss them separately.

When $k = 2$, plugging in (3.2) and (5.15), it gives

\[ V_{\xi_0 + PD[\mu]}(K) = \frac{(8 - n)^2 + (2j - (n - 4))^2}{8(n - 4)} - \frac{1}{4}. \]
where \( V_{0+kPD[n]}(K) = \frac{n-5}{4} \) or \( \frac{n-9}{4} \). In the case of \( \frac{n-5}{4} \), Equation (5.18) can be simplified to
\[
j^2 - (n - 4)j + 12 - 2n = 0.
\]
There is no integral root of the function \( f(j) = j^2 - (n - 4)j + 12 - 2n \) between 0 and \( n - 5 \), since its axis of symmetry is \( j = \frac{n-4}{2} \) and \( f(0) = 12 - 2n < 0 \) when \( n \geq 11 \). This gives a contradiction. For the case of \( \frac{n-9}{4} \), Equation (5.18) can be simplified to
\[
j^2 - (n - 4)j + 4 = 0.
\]
There is no integral root of the function \( f(j) = j^2 - (n - 4)j + 4 \) between 0 and \( n - 5 \), since its axis of symmetry is \( j = \frac{n-4}{2} \), and \( f(0) = 4 > 0 \) and \( f(1) = 9 - n < 0 \) when \( n \geq 11 \), which means the roots of \( f(j) \) lie in \((0, 1)\) and \((n-5, n-4)\). Therefore this case is impossible.

When \( k \geq 4 \), combined with (3.2) and (5.14), it yields
\[
(5.19) \quad V_{0+kPD[n]}(K) = \frac{2k - nk + n + k^2}{8(k-1)(n - k^2)} + \frac{2j - (nk - n - k^2)^2}{8(nk - n - k^2)} - \frac{1}{8},
\]
where \( V_{0+kPD[n]}(K) = \frac{-k^2 + nk - k - 4}{8} \) or \( \frac{-k^2 + nk - k - 12}{8} \). In the case of \( \frac{-k^2 + nk - k - 4}{8} \), we simplify Equation (5.19) as
\[
(5.20) \quad j^2 - (nk - n - k^2)j + 2n + k^2 - nk = 0.
\]
There is no integral root of the function \( f(j) = j^2 - (nk - n - k^2)j + 2n + k^2 - nk \) between 0 and \( nk - n - k^2 - 1 \) since its axis of symmetry is \( j = \frac{nk-n-k^2}{2} \) and \( f(0) = 2n + k^2 - nk < 0 \) when \( n \geq 11 \) and \( k \geq 4 \), which gives a contradiction. For the case of \( \frac{-k^2 + nk - k - 12}{8} \), Equation (5.19) can be simplified as
\[
(5.21) \quad j^2 - (nk - n - k^2)j + nk - k^2 = 0.
\]
The axis of symmetry of the function \( f(j) = j^2 - (nk - n - k^2)j + nk - k^2 \) is \( j = \frac{nk-n-k^2}{2} \), and we see that \( f(0) = nk - k^2 > 0 \), \( f(1) = n + 1 > 0 \), and \( f(2) = k^2 - nk + 2n + 4 < 0 \) when \( n \geq 11 \) and \( k \geq 4 \). Therefore, the roots of \( f(j) \) lie in \((1, 2)\) and \((nk - n - k^2 - 2, nk - n - k^2 - 1)\), which are not integers. This complete the proof. \( \Box \)

**Proposition 5.13.** Given \( Y \) and \( K \) as in Proposition 5.11, suppose the surgery slope \( m = k - 1 \) and \( k \) is odd. Then the lens space \( L(s, 1) \) with \( s \) odd is obtained by a distance one surgery along \( K \) only if \( n = 13 \) and \( s = -17 \).

**Proof.** By (2.5), we have \(|s| = nk - n - k^2 \). Here, \((k-1)n - k^2 > 0 \) since \( k > 1 \) and \( n \geq 9 \).

If \( s = nk - n - k^2 \), Formula (4.3) gives
\[
d(L(nk - n - k^2, 1), 0) = d(L(nk - n - k^2, k - 1), \frac{nk - n - k^2 + k - 2}{2}) - 2V_{\xi_0}(K),
\]
where 0 corresponds to the unique self-conjugate Spin\(^c\) structure on \( L(nk - n - k^2, 1) \). Combined with (3.2) and (5.16), this yields
\[
V_{\xi_0}(K) = \frac{k^2 + k - nk + n - 2}{8} < 0, \text{ when } n \geq 9 \text{ and } k \geq 3.
\]
It contradicts to that \( V_{\xi_0}(K) \) is non-negative.

If \( s = -nk + n + k^2 \), applying Formula (4.3) we have
\[
-d(L(nk - n - k^2, 1), 0) = d(L(nk - n - k^2, k - 1), \frac{nk - n - k^2 + k - 2}{2}) - 2V_{\xi_0}(K),
\]
which implies the integer

$$V_{\xi_0}(K) = \frac{-k^2 + kn + k - n - 4}{8}.$$  

When \(n = 9\) and \(k = 3\), by (5.24) we have \(H_1(Y_{n+\lambda}) = \mathbb{Z}_3 \oplus \mathbb{Z}_3\) which is noncyclic, thus we rule out this case. When \(n = 11\) and \(k = 3\), we have \(V_{\xi_0}(K) = \frac{3}{2}\), which is not an integer.

Otherwise, we have \(n = 11\) and \(k \geq 5\) or \(n \geq 13\) and \(k \geq 3\), which implies \(V_{\xi_0}(K) \geq 2\), thus we can apply Formula (4.4). It gives

$$-d(L(nk - n - k^2), j) = d(L(nk - n - k^2, k - 1), \frac{nk - n - k^2 - k - 2}{2}) - 2V_{\xi_0 + PD[\mu]}(K),$$

for some \(j \in [0, nk - n - k^2 - 1]\). Combined with (3.2) and (5.17), this yields

$$V_{\xi_0 + PD[\mu]}(K) = \frac{(nk - n - k^2)(k^2 - 8k + 11) + 4k^2}{8(k-1)(nk - n - k^2)} + \frac{(2j - (nk - n - k^2))^2 - 1}{8(nk - n - k^2)} - \frac{1}{8},$$

where \(V_{\xi_0 + PD[\mu]}(K) = \frac{-k^2 + kn + k - n - 4}{8}\) or \(\frac{-k^2 + kn + k - n - 12}{8}\). In the case of \(\frac{-k^2 + kn + k - n - 4}{8}\), Equation (5.22) can be simplified as

$$j^2 - (nk - n - k^2)j + 2n + k^2 - nk = 0,$$

which is the same as (5.20). A similar argument shows that the function \(f(j) = j^2 - (nk - n - k^2)j + 2n + k^2 - nk\) has no integral root in \([0, nk - n - k^2 - 1]\) when \(n \geq 11\) and \(k \geq 3\). Thus this case is impossible. For the case of \(\frac{-k^2 + kn + k - n - 12}{8}\), we rewrite Equation (5.22) as

$$j^2 - (nk - n - k^2)j + nk - k^2 = 0.$$  

The axis of symmetry of the function \(f(j) = j^2 - (nk - n - k^2)j + nk - k^2\) is \(j = \frac{2nk - n - k^2}{2}\), and we have that \(f(0) = nk - k^2 \geq 0\), \(f(1) = n + 1 \geq 0\) and \(f(2) = k^2 - nk + 2n + 4\). We see that \(f(2) = 0\) when \(n = 13\) and \(k = 3\), which means we cannot obstruct distance one surgery from \(L(13, 1)\) to \(L(-17, 1)\) by our surgery formula in this case. For the remaining cases, namely when \(n = 11\) or 13 and \(k \geq 5\) and when \(n \geq 15\) and \(k \geq 3\), we have \(f(2) < 0\). It means the roots of \(f(j)\) lie in \((1, 2)\) and \((nk - n - k^2 - 2, nk - n - k^2 - 1)\), which are not integers. This completes the proof. \(\square\)

Now we focus on the case \(m = k + 1\), in which the Seifert fibered space \(M(0, 0; (m - k, 1), (n - k, 1), (k, 1))\) reduces to the lens space \(L(nk + n - k^2, k + 1)\). Depending on the different parities of \(k\), the unique self-conjugate Spin\(^c\) structure \(t_M\) is \(\frac{k}{2}\) or \(\frac{nk + n - k^2 + k}{2}\), and the Spin\(^c\) structure \(t_M + i^* PD[\mu]\) corresponds to \(\frac{k}{2}\) or \(\frac{nk + n - k^2 - k}{2}\) up to Spin\(^c\) conjugation. For the convenience of readers, we compute the \(d\)-invariants of the relevant lens spaces as follows.

For \(m = k + 1\) and \(k\) even,

$$d(L(nk + n - k^2, k + 1), \frac{k}{2}) = \frac{n - k - 1}{4},$$

$$d(L(nk + n - k^2, k + 1), \frac{3k}{2}) = \frac{(2k - nk - n + k^2)^2 - (2k + 1)(nk + n - k^2)}{4(k + 1)(nk + n - k^2)}.$$  

For \(m = k + 1\) and \(k\) odd,

$$d(L(nk + n - k^2, k + 1), \frac{nk + n - k^2 + k}{2}) = \frac{k - 1}{4},$$

$$d(L(nk + n - k^2, k + 1), \frac{nk + n - k^2 - k}{2}) = \frac{4k^2 + (k^2 - 4k - 1)(nk + n - k^2)}{4(k + 1)(nk + n - k^2)}.$$
We now are ready to apply our surgery formula.

**Proposition 5.14.** Given $Y$ and $K$ as in Proposition 5.11 suppose the surgery slope $m = k + 1$ and $k$ is even. Then there does not exist a distance one surgery along $K$ yielding a lens space $L(s, 1)$ with $s$ odd.

**Proof.** Suppose a lens space $L(s, 1)$ with $s$ odd is obtained by $((k + 1)\mu + \lambda)$-surgery along $K$, then $\vert s \vert = nk + n - k^2$ by (2.5). Here, obviously $(k + 1)n - k^2 > 0$.

If $s = nk + n - k^2$, Formula (4.3) gives

$$d(L(nk + n - k^2, 1, 0) = d(L(nk + n - k^2, k + 1), \frac{k}{2}) - 2V_{\xi_0}(K).$$

Combined with (3.2) and (5.23), it follows

$$V_{\xi_0}(K) = \frac{k^2 - nk - k}{8} < 0,$$

which contradicts to the fact that $V_{\xi_0}(K)$ is non-negative.

If $s = -nk - n + k^2$, Formula (4.3) shows

$$-d(L(nk + n - k^2, 1, 0) = d(L(nk + n - k^2, k + 1), \frac{k}{2}) - 2V_{\xi_0}(K),$$

which implies

$$V_{\xi_0}(K) = \frac{nk + 2n - k^2 - 2}{8} \geq 2,$$

since $n \geq 9$ and $k \geq 2$. Thus we can apply Formula (4.4) and get

$$-d(L(nk + n - k^2, 1, j) = d(L(nk + n - k^2, k + 1), \frac{3k}{2}) - 2V_{\xi_0+PD[\mu]}(K),$$

for some $j \in [0, nk + n - k^2 - 1]$. Plugging (3.2) and (5.24) into it, we have

$$(5.27) \quad V_{\xi_0+PD[\mu]}(K) = \frac{(2k - nk - n + k^2)^2 - (2k + 1)(nk + n - k^2)}{8(k + 1)(nk + n - k^2)} + \frac{(2j - (nk + n - k^2))^2}{8(nk + n - k^2)} - \frac{1}{8},$$

where $V_{\xi_0+PD[\mu]}(K) = \frac{nk + 2n - k^2 - k - 2}{8}$ or $\frac{nk + 2n - k^2 - k - 10}{8}$. In the case of $\frac{nk + 2n - k^2 - k - 2}{8}$, Equation (5.27) can be simplified as

$$(5.28) \quad j^2 - (nk + n - k^2)j + k^2 - kn = 0.$$

The function $f(j) = j^2 - (nk + n - k^2)j + k^2 - kn$ has no integral root between 0 and $nk + n - k^2 - 1$, since its axis of symmetry is $j = \frac{nk + n - k^2}{2}$, and $f(0) = k^2 - nk < 0$. For the case of $\frac{nk + 2n - k^2 - k - 10}{8}$, we can rewrite Equation (5.27) as

$$(5.29) \quad j^2 - (nk + n - k^2)j + nk + 2n - k^2 = 0.$$

Considering the function $f(j) = j^2 - (nk + n - k^2)j + nk + 2n - k^2$, its axis of symmetry is $j = \frac{nk + n - k^2}{2}$, and we see $f(0) = nk + 2n - k^2 > 0$, $f(1) = n + 1 > 0$ and $f(2) = k^2 - nk + 4 < 0$ since $n \geq 2k + 1$ and $k \geq 2$. Therefore the roots of $f(j)$ lie in $(1, 2)$ and $(nk + n - k^2 - 2, nk + n - k^2 - 1)$, which are not integers. This gives a contradiction.

Therefore, $L(s, 1)$ can not be obtained by a distance one surgery from $L(n, 1)$ in this setting. □

**Proposition 5.15.** Given $Y$ and $K$ as in Proposition 5.11 suppose the surgery slope $m = k + 1$ and $k$ is odd. Then there does not exist a distance one surgery along $K$ yielding a lens space $L(s, 1)$ with $s$ odd.
Proof. Suppose a lens space $L(s, 1)$ with $s$ odd is obtained by $((k + 1)\mu + \lambda)$-surgery along $K$, then $|s| = nk + n - k^2$ by [25]. Here, obviously $(k + 1)n - k^2 > 0.$

If $s = nk + n - k^2$, Formula (4.3) gives

$$d(L(nk + n - k^2, 1), 0) = d(L(nk + n - k^2, k + 1), \frac{nk + n - k^2 + k}{2}) - 2V_\xi(K).$$

Combined with (3.2) and (5.25), we get

$$V_\xi(K) = \frac{k^2 + k - nk - n}{8} < 0,$$

which contradicts to the fact that $V_\xi(K)$ is non-negative.

If $s = -nk + n + k^2$, Formula (4.3) shows

$$-d(L(nk + n - k^2, 1), 0) = d(L(nk + n - k^2, k + 1), \frac{nk + n - k^2 + k}{2}) - 2V_\xi(K),$$

which implies

$$V_\xi(K) = \frac{nk + k + n - k^2 - 2}{8} \geq 2,$$

since $n \geq 9$ and $k \geq 3$. Hence we apply Formula (4.4) and get

$$-d(L(nk + n - k^2, 1), j) = d(L(nk + n - k^2, k + 1), \frac{nk + n - k^2 - k}{2}) - 2V_\xi + PD(j)(K),$$

for some $j \in [0, nk + n - k^2 - 1]$. Plugging (3.2) and (5.26) into it, we have

$$V_\xi(K) = \frac{4k^2 + (k^2 - 4k - 1)(nk + n - k^2)}{8(k + 1)(nk + n - k^2)} + \frac{(2j - (nk + n - k^2))^2}{8(nk + n - k^2)} - \frac{1}{8},$$

where $V_\xi + PD(j)(K) = \frac{nk + k + n - k^2 - 2}{8}$ or $\frac{nk + k + n - k^2 - 10}{8}$. In the case of $\frac{nk + k + n - k^2 - 2}{8}$, we can simplify Equation (5.30) as

$$j^2 - (nk + n - k^2)j + k^2 - kn = 0,$$

which is the same as (5.28). A same argument shows that the function $f(j) = j^2 - (nk + n - k^2)j + k^2 - kn$ has no integral root in $[0, nk + n - k^2 - 1]$. For the case of $\frac{nk + k + n - k^2 - 10}{8}$, we can rewrite Equation (5.30) as

$$j^2 - (nk + n - k^2)j + nk + 2n - k^2 = 0,$$

which is the same as (5.29). Also, a similar argument implies that the function $f(j) = j^2 - (nk + n - k^2)j + nk + 2n - k^2$ has no integral root in $[0, nk + n - k^2 - 1]$. This completes the proof.

5.3. Proof of Theorem 1.1 In this section, we obstruct all remaining unexpected surgery results to prove Theorem 1.1 by considering symmetry of surgeries.

Proof of Theorem 1.1 We combine all results in this paper, our previous paper [25] and result of Lidman, Moore and Vazquez [12] as listed in Table 3. Then, we see $L(s, 1)$ with $s \neq 0$ is obtained by a distance one surgery along a knot in $L(n, 1)$ with $n \geq 5$ odd only if $s = \pm 1$, $n$, $n \pm 1$, $n \pm 4$, $\pm(n - k^2)$, $9$, $n = 5$ and $s = -5$, $n = 5$ and $s = -9$, $n = 9$ and $s = -5$ or $n = 13$ and $s = -17$. In addition, the case $s = \pm(n - k^2)$ may occur only when $k \geq \frac{n + 1}{2}$ is even and $n \geq 9$, and note that $n - k^2 < 0$ in this case. In fact, distance one surgeries for the cases $s = \pm 1$, $n$, $n \pm 1$, $n \pm 4$ and $n = 5$ and $s = -5$ can be realized as the double branched covers of the band surgeries in Figure 1. We want to rule out the cases $s = 9$, $\pm(n - k^2)$ and $n = 13$ and $s = -17$. 
We know that if there exists a distance one surgery from $L(n_1, 1)$ to $L(n_2, 1)$, then there is a distance one surgery from $L(n_2, 1)$ to $L(n_1, 1)$ and a distance one surgery from $L(-n_1, 1)$ to $L(-n_2, 1)$. We will use this symmetric property to obstruct unexpected surgeries. Obviously, we can not obstruct the cases when $n = 5$ and $s = -9$ and when $n = 9$ and $s = -5$ by this trick.

If there exists a distance surgery from $L(n, 1)$ to $L(9, 1)$, then there exists a distance one surgery from $L(9, 1)$ to $L(n, 1)$. Since $n \geq 5$, $n$ can only be $9, 9 \pm 1, 9 \pm 4$ or $k^2 - 9$ with $\frac{n+1}{3} \leq k < \frac{n-1}{2}$. If $n = 9, 9 \pm 1$ or $9 \pm 4$, then we leave it aside. If $n = k^2 - 9$ for some $\frac{10}{3} \leq k \leq 4$, then we have $k = 4$. But there is no distance one surgery from $L(7, 1)$ to $L(9, 1)$ by [25, Theorem 1.3 (ii)]. This gives a contradiction.

If there exists a distance one surgery from $L(13, 1)$ to $L(-17, 1)$, then there is a distance one surgery from $L(17, 1)$ to $L(-13, 1)$. We must have $-13 = 17 - k^2$ for some $\frac{17+1}{3} \leq k < \frac{17-1}{2}$, which implies $k^2 = 30$. It is impossible.

If there exists a distance one surgery from $L(n, 1)$ to $L(n-k^2, 1)$ with $\frac{n+1}{3} \leq k < \frac{n-1}{2}$, then there is a distance one surgery from $L(k^2 - n, 1)$ to $L(-n, 1)$. Here, $k^2 - n > 0$. If $k^2 - n = 3$, then $n$ must be $1, 2$ or $6$, which contradicts to that $n \geq 9$ is odd. Otherwise we must have $-n = (k^2 - n) - k^2$ for some $\frac{k^2 - n + 1}{3} \leq k \leq \frac{k^2 - n - 1}{2}$, which implies $k = k_1$. Now we have $k$ must satisfy

\[\frac{k^2 - n + 1}{3} \leq k \text{ and } \frac{n+1}{3} \leq k \leq \frac{n-1}{2},\]

which implies

\[(5.31) \quad k^2 - 3k - n + 1 \leq 0 \text{ and } \frac{n+1}{3} \leq k \leq \frac{n-1}{2}.

The axis of symmetry of the function $f(x) = x^2 - 3x - n + 1$ is $\frac{3}{2}$, thus

\[\min_{\frac{n+1}{3} \leq x \leq \frac{n-1}{2}} f(x) = f\left(\frac{n+1}{3}\right) = \frac{n^2 - 16n + 1}{9},\]
which is strictly greater than 0 when \( n \geq 17 \). It gives a contradiction when \( n \geq 17 \).

If there exists a distance one surgery from \( L(n,1) \) to \( L(k^2-n,1) \) with \( \frac{n+1}{3} \leq k \leq \frac{n+3}{3} \), then there is a distance one surgery from \( L(k^2-n,1) \) to \( L(n,1) \). Since \( n \geq 9 \) and \( k^2-n > 0 \), \( n \) can only be \( k^2-n, k^2-n+1, k^2-n+4 \), or \( k^2-(k^2-n) \) with \( \frac{k^2-n+1}{3} \leq k \leq \frac{k^2-n-1}{3} \). If \( n = k^2-n, k^2-n+1 \) or \( k^2-n+4 \), then we leave it aside. If \( n = k^2-(k^2-n) \) for some \( \frac{k^2-n+1}{3} \leq k \leq \frac{k^2-n-1}{3} \), then we have

\[
k_1 = k, \quad \frac{k^2-n+1}{3} \leq k_1 \quad \text{and} \quad \frac{n+1}{3} \leq k \leq \frac{n-1}{2},
\]

which implies

\[
k^2-3k-n+1 \leq 0 \quad \text{and} \quad \frac{n+1}{3} \leq k \leq \frac{n-1}{2}.
\]

It is the same as \( \text{(5.31)} \). Hence we conclude that there is no distance one surgery from \( L(n,1) \) to \( L(k^2-n,1) \) when \( n \geq 17 \).

Now we deal with the cases when \( s = \pm(n-k^2) \) and \( n = 9, 11, 13 \) or 15.

When \( n = 9 \), if there is a distance one surgery from \( L(9,1) \) to \( L(\pm(9-k^2),1) \), then we have \( \frac{9+1}{3} \leq k \leq \frac{9-1}{3} \). Since in this case \( k \) is even, \( k \) must be 4, which implies \( L(\pm(9-k^2),1) = L(\mp7,1) \). But there is no distance one surgery from \( L(7,1) \) to \( L(\pm9,1) \) by \( \text{[23 Theorem 1.3 (ii)]} \), which gives a contradiction.

When \( n = 11 \), if there is a distance one surgery from \( L(11,1) \) to \( L(\pm(11-k^2),1) \), then we have \( \frac{11+1}{3} \leq k \leq \frac{11-1}{3} \). Since in this case \( k \) is even, \( k \) must be 4, which implies \( L(\pm(11-k^2),1) = L(\mp5,1) \). By \( \text{[23 Theorem 1.3 (i)]} \), there does not exist a distance one surgery from \( L(5,1) \) to \( L(\pm11,1) \). It gives a contradiction.

When \( n = 13 \), if there is a distance one surgery from \( L(13,1) \) to \( L(\pm(13-k^2),1) \), then we have \( \frac{13+1}{3} \leq k \leq \frac{13-1}{3} \). Since in this case \( k \) is even, \( k \) must be 6. Putting \( k = 6 \) in \( k^2-3k-n+1 \) in \( \text{(5.31)} \), we have \( k^2-3k-n+1 = 6 > 0 \), which gives a contradiction.

When \( n = 15 \), if there is a distance one surgery from \( L(15,1) \) to \( L(\pm(15-k^2),1) \), then we have \( \frac{15+1}{3} \leq k \leq \frac{15-1}{3} \). Since in this case \( k \) is even, \( k \) must be 6. Putting \( k = 6 \) in \( k^2-3k-n+1 \) in \( \text{(5.31)} \), we have \( k^2-3k-n+1 = 4 > 0 \), which gives a contradiction.

In summary, we conclude that \( L(s,1) \) with \( s \neq 0 \) is obtained by a distance one surgery from \( L(n,1) \) with \( n \geq 5 \) odd only if \( s = \pm1, n, n \pm 1, n \pm 4, n = 5 \) and \( s = -5, n = 5 \) and \( s = -9 \) or \( n = 9 \) and \( s = -5 \).

\[ \square \]

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