Invariant and coinvariant spaces for the algebra of symmetric polynomials in non-commuting variables

François Bergeron and Aaron Lauve

LaCIM, Université du Québec à Montréal, CP 8888, Succ. Centre-ville, Montréal (Québec) H3C 3P8, CANADA

Abstract. We analyze the structure of the algebra $K\langle x \rangle S_n$ of symmetric polynomials in non-commuting variables in so far as it relates to $K[x] S_n$, its commutative counterpart. Using the “place-action” of the symmetric group, we are able to realize the latter as the invariant polynomials inside the former. We discover a tensor product decomposition of $K\langle x \rangle S_n$ analogous to the classical theorems of Chevalley, Shephard-Todd on finite reflection groups. In the case $|x| = \infty$, our techniques simplify to a form readily generalized to many other familiar pairs of combinatorial Hopf algebras.

1 Introduction

One of the more striking results of the invariant theory of reflection groups is certainly the following: if $W$ is a finite group of $n \times n$ matrices, then there is a graded $W$-module decomposition of the polynomial ring $S = K[x]$, in variables $x = \{x_1, x_2, \ldots, x_n\}$, as a tensor product

$$S \simeq S_W \otimes S^W,$$ (1)

if and only if $W$ is a group generated by (pseudo) reflections. As usual, $S$ affords the natural $W$-module structure obtained by considering it as the symmetric space on the defining vector space $X^*$ for $W$, e.g.,

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(i) We assume throughout that $K$ is a field containing $\mathbb{Q}$.

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It is customary to denote by $S^W$ the ring of $W$-invariant polynomials for this action. To finish parsing (1), recall that $S_W$ stands for the **coinvariant space**, i.e., the $W$-module defined as

$$S_W := S/\langle S_W^+ \rangle,$$

the quotient of $S$ by the ideal generated by constant-term free $W$-invariant polynomials. We give $S$, $S^W$, and $S_W$ a grading by polynomial degree in $x$ (the latter being well-defined because $\langle S_W^+ \rangle$ is a homogeneous ideal). The motivation behind the quotient in (2) is to eliminate redundant copies of irreducible $W$-modules inside $S$. Indeed, if $V$ is such a module and $f(x)$ is any $W$-invariant polynomial with no constant term, then $Vf(x)$ is an isomorphic copy of $V$ living within $\langle S_W^+ \rangle$. As a result, the coinvariant space $S_W$ is the interesting part of the story.

The context for the present paper is the algebra $T = K\langle x \rangle$ of noncommutative polynomials, with $W$-module structure on $T$ obtained by considering it as the tensor space on the defining space $X^*$ for $W$. In the special case when $W$ is the symmetric group $S_n$, we elucidate a relationship between the space $S_W$ and the subalgebra $T_W$ of $W$-invariants in $T$. The subalgebra $T_W$ was first studied in [14, 5] with the aim of obtaining noncommutative analogs of classical results concerning symmetric function theory. Recent work in [12, 3] has extended a large part of the story surrounding (1) to this noncommutative context. In particular, there is an explicit $S_n$-module decomposition of the form $T \simeq T_{S_n} \otimes T_{S_n}$, cf. [3, Theorem 8.7].

By contrast, our work proceeds in a somewhat complementary direction. We consider $N = T_{S_n}$ as a tower of $S_d$-modules under the “place-action” and realize $S_{S_n}$ inside $N$ as a subspace $\Lambda$ of invariants for this action. This leads to a decomposition of $N$ analogous to (1). More explicitly, our main result is as follows.

**Theorem 1** There is an explicitly constructed subspace $\mathcal{C}$ of $N$ so that $\mathcal{C}$ and the place-action invariants $\Lambda$ exhibit a graded vector space isomorphism

$$N \simeq \mathcal{C} \otimes \Lambda.$$  

As an immediate corollary we derive the Hilbert series formula

$$\text{Hilb}_t(C) = \text{Hilb}_t(N) \prod_{i=1}^n (1 - t^i).$$

Here, as usual, the Hilbert series of a graded space $V = \bigoplus_{d \geq 0} V_d$ is the formal power series defined as

$$\text{Hilb}_t(V) = \sum_{d \geq 0} \dim V_d t^d,$$

where $V_d$ is the homogeneous degree $d$ component of $V$. The fact that (4) expands as a series in $\mathbb{N}[t]$ is not at all obvious, as one may check that the Hilbert series of $N$ is

$$\text{Hilb}_t(N) = 1 + \sum_{k=1}^n \frac{t^k}{(1 - t)(1 - 2t) \cdots (1 - kt)}$$

(taking $n = |x|$). We underline that the harder part of our work lies in working out the case $n < \infty$. This is accomplished in Section 6. If we restrict ourselves to the case $n = \infty$, both $N$ and $\Lambda$ become Hopf
algebras and things are much simpler. Our results are then consequences of a general theorem of Blattner, Cohen and Montgomery. As we will see in Section 5, stronger results hold in this simpler context. For example, (4) may be refined to a statement about “shape” enumeration.

2 The algebra $S^\mathfrak{S}$ of symmetric polynomials

2.1 Vector space structure of $S^\mathfrak{S}$

We specialize our introductory discussion to the group $W = \mathfrak{S}_n$ of permutation matrices. The action on $S = \mathbb{K}[x]$ is simply the permutation action $\sigma \cdot x_i = x_{\sigma(i)}$ and $S^\mathfrak{S}_n$ comprises the usual symmetric polynomials. We suppress $n$ in the notation and denote the subring of symmetric polynomials by $S^\mathfrak{S}$. (Note that upon sending $n$ to $\infty$, the elements of $S^\mathfrak{S}$ become formal series in $\mathbb{K}[x]$ of bounded degree; we still call them polynomials to affect a uniform discussion.) A monomial in polynomials. We suppress

\[
\prod_{i=1}^{n} x^a_i := \prod_{i=1}^{n} x_{\lambda(i)}^{a_i}, \quad \lambda = (\lambda(1), \ldots, \lambda(n)) \in \mathbb{N}^n
\]

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\]

is simply the monomial symmetric polynomial

\[
m_{\mu} = m_{\mu}(x) := \sum_{\lambda(\alpha) = \mu, \lambda \leq \alpha} y^{\alpha}.
\]

Letting $\mu = (\mu_1, \ldots, \mu_r)$ run over all partitions of $d = |\mu| = \mu_1 + \cdots + \mu_r$ gives a basis for $S^\mathfrak{S}_d$. As usual, we set $m_0 := 1$ and agree that $m_{\mu} = 0$ if $\mu$ has too many parts (i.e., $n < r$).

2.2 Dimension enumeration

A fundamental result in the invariant theory of $\mathfrak{S}_n$ is that $S^\mathfrak{S}$ is generated by a family $\{f_k\}_{1 \leq k \leq n}$ of algebraically independent symmetric polynomials, having respective degrees $\deg f_k = k$. (One may choose $\{m_k\}_{1 \leq k \leq n}$ for such a family.) It follows immediately that the Hilbert series of $S^\mathfrak{S}$ is

\[
\text{Hilb}_t(S^\mathfrak{S}) = \prod_{i=1}^{n} \frac{1}{1 - t^i}.
\]

Recalling that the Hilbert series of $S$ is $(1 - t)^{-n}$, we see from (1) and (6) that the Hilbert series for the coinvariant space $S^\mathfrak{S}$ is the well-known $t$-analog of $n!$:

\[
\prod_{i=1}^{n} \frac{1}{1 - t^i} = \prod_{i=1}^{n} (1 + t + \cdots + t^{i-1}).
\]

In particular, contrary to the situation in (4), the series $\text{Hilb}_t(S)/\text{Hilb}_t(S^\mathfrak{S})$ in $\mathbb{Z}[t]$ is obviously positive.

2.3 Algebra and coalgebra structures of $S^\mathfrak{S}$

Given partitions $\mu$ and $\nu$, there is an explicit formula for computing the product $m_{\mu} \cdot m_{\nu}$. In lieu of giving the formula, we refer the reader to [3, §4.1] and simply give an example:

\[
m_{21} \cdot m_{11} = 3m_{2111} + 2m_{221} + 2m_{311} + m_{32}.
\]
The extremal terms above are relevant to our coming discussion. Note that if \( n < 4 \), then the first term disappears. However, if \( n \) is sufficiently large then analogs of these terms always appear with positive integer coefficients for a given pair \((\mu, \nu)\). If \( \mu = (\mu_1, \ldots, \mu_r) \) and \( \nu = (\nu_1, \ldots, \nu_s) \) with \( r \leq s \), then the partition indexing the left-most term is denoted by \( \mu \cup \nu \) and is given by sorting the list \((\mu_1, \ldots, \mu_r, \nu_1, \ldots, \nu_s)\) in increasing order; the right-most term is indexed by \( \mu + \nu := (\mu_1 + \nu_1, \ldots, \mu_r + \nu_r, \nu_{r+1}, \ldots, \nu_s)\). Taking \( \mu = 31 \) and \( \nu = 221 \), we would have \( \mu \cup \nu = 32211 \) and \( \mu + \nu = 531 \).

The ring \( S^\Theta \) is also afforded a coalgebra structure with coproduct \( \Delta : S^\Theta \to \bigoplus_{k=0}^d S_k^\Theta \otimes S_{d-k}^\Theta \) and counit \( \varepsilon : S^\Theta \to \mathbb{K} \) given, respectively, by

\[
\Delta(m_{\mu}) = \sum_{\theta, \nu = \mu} m_{\theta} \otimes m_{\nu} \quad \text{and} \quad \varepsilon(m_{\mu}) = \delta_{\mu,0}.
\]

In the case \( n = \infty \), \( \Delta \) and \( \varepsilon \) are algebra maps, making \( S^\Theta \) a connected graded (by degree) Hopf algebra.

### 3 The algebra \( \mathcal{N} \) of noncommutative symmetric polynomials

#### 3.1 Vector space structure of \( \mathcal{N} \)

Suppose now that \( x \) denotes a set of non-commuting variables. The algebra \( T = \mathbb{K}(x) \) of noncommutative polynomials is graded by degree. A degree \( d \) noncommutative monomial \( z \in T_d \) is simply a length-\( d \) “word”:

\[
z = z_1 z_2 \cdots z_d, \quad \text{with each} \quad z_i \in x.
\]

In other terms, \( z \) is a function \( z : [d] \to x \), with \([d]\) denoting the set \( \{1, \ldots, d\} \). The permutation-action on \( x \) clearly extends to \( T \), giving rise to the subspace \( \mathcal{N} = T^\Theta \) of noncommutative \( \Theta \)-invariants. With the aim of describing a linear basis for the homogeneous component \( \mathcal{N}_d \), we next introduce set partitions of \([d]\) and the type of a monomial \( z : [d] \to x \). We write \( \mathbf{A} \vdash [d] \) when \( \mathbf{A} = \{A_1, \ldots, A_r\} \) is a set partition of \([d] \), i.e., \( A_1 \cup \ldots \cup A_r = [d] \), with \( A_i \neq \emptyset \) and \( A_i \cap A_j = \emptyset \) whenever \( i \neq j \). The type \( \tau(z) \) of a degree \( d \) monomial \( z : [d] \to x \) is the set partition

\[
\tau(z) := \{ z^{-1}(x) | x \in x \} \setminus \{\emptyset\} \quad \text{of} \quad [d],
\]

whose parts are the non-empty fibers of the function \( z \). For instance,

\[
\tau(x_1 x_8 x_1 x_5 x_8) = \{\{1, 3\}, \{2, 5\}, \{4\}\}.
\]

In the sequel, we lighten the heavy notation for set partitions, writing, e.g., \( \{1, 3\}, \{2, 5\}, \{4\}\) as \(13.25.4\). Clearly the type of a monomial is a finite set partition with at most \( n \) parts. Note that we may always order the parts in increasing order of their minimum elements. The shape \( \lambda(\mathbf{A}) \) of a set partition \( \mathbf{A} = \{A_1, \ldots, A_r\} \) is the (integer) partition \( \lambda(\{A_1, \ldots, A_r\}) \) obtained by sorting the part sizes of \( \mathbf{A} \) in increasing order. Observing that the permutation-action is type preserving, we are led to consider the monomial linear basis for the space \( \mathcal{N}_d \):

\[
m_{\mathbf{A}} = m_{\mathbf{A}}(x) := \sum_{\tau(z) = \mathbf{A}} z
\]

For example, with \( n = 2 \), we have \( m_{\emptyset} = 1 \), \( m_1 = x_1 + x_2 \), \( m_{12} = x_1^2 + x_2^2 \), \( m_{123} = x_1 x_2 + x_2 x_1 \), \( m_{123} = x_1 x_2 x_3 + x_3 x_1 x_2 \), \( m_{123} = x_1 x_2 x_3 + x_2 x_3 x_1 \), \( m_{123} = x_1 x_2 x_3 + x_2 x_3 x_1 \), \( m_{123} = 0 \). (Note that we set \( m_{\emptyset} = 1 \), taking \( \emptyset \) as the unique set partition of the empty set, and we agree that \( m_{\mathbf{A}} = 0 \) if \( \mathbf{A} \) is a set partition with more than \( n \) parts.)
3.2 Dimension enumeration and shape grading

Above, we determined that $\dim N_d$ is the number of set partitions of $d$ into at most $n$ parts. These are counted by the (length restricted) Bell numbers $B_d^{(n)}$. Then (5) follows from the fact that its right-hand side is the ordinary generating function for length restricted Bell numbers. See [9, §2]. We next highlight a finer enumeration, where we grade $N$ by shape rather than degree.

For each partition $\mu$, we may consider the submodule $N_\mu$ spanned by those $m_A$ for which $\lambda(A) = \mu$. This results in a direct sum decomposition $N_d = \bigoplus_{\mu \vdash d} N_\mu$. A simple dimension description for $N_d$ takes the form of a shape Hilbert series in the following manner. View commuting variables $q_i$ as marking parts of size $i$ and set $q_\mu := q_{\mu_1} q_{\mu_2} \cdots q_{\mu_r}$. Then

$$\text{Hilb}_q(N_d) = \sum_{\mu \vdash d} \dim N_\mu q_\mu = \sum_{\lambda \vdash [d]} q_{\lambda(A)}.$$  \hfill (9)

Here, $q_\mu$ is a marker for set partitions of shape $\lambda(A) = \mu$ and the sum is over all partitions into at most $n$ parts. Such a shape grading also makes sense for the form of a finer enumeration, where we grade

$$\text{Hilb}_q(S^\oplus_i) = \sum \mu \mu = \sum_{i \geq 1} \frac{1}{1-q_i}. \quad \hfill (10)$$

Using classical combinatorial arguments (cf. Chapter 2.3 of [2], Example 13), we see that the enumerator polynomials $\text{Hilb}_q(N_d)$ are naturally collected in the exponential generating function

$$\sum_{d=0}^{\infty} \frac{\text{Hilb}_q(N_d) t^d}{d!} = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \sum_{k=1}^{\infty} q_k \frac{t^k}{k!} \right)^n. \quad \hfill (11)$$

For example, with $n = 3$, we have

$$\text{Hilb}_q(N_d) = q_6 + 6 q_3 q_1 + 15 q_4 q_2 + 15 q_2^2 + 10 q_2^3 + 60 q_3 q_2 q_1 + 15 q_3^3,$$

thus $\dim N_222 = 15$ when $n \geq 3$. Evidently, the $q$-polynomials $\text{Hilb}_q(N_d)$ specialize to the length restricted Bell numbers $B_d^{(n)}$ when we set all $q_k$ equal to 1.

In view of (10), (11), and Theorem 1, we are led to claim the following refinement of (4).

**Corollary 2** For $n = \infty$, the shape Hilbert series of the space $\mathcal{C}$ is given by the expression

$$\text{Hilb}_q(\mathcal{C}) = \sum_{d \geq 0} d! \left| \exp \left( \sum_{k=1}^{\infty} q_k \frac{t^k}{k!} \right) \right|_{t^d} \prod_{i \geq 1} (1-q_i), \quad \hfill (12)$$

with $(-)|_{t^d}$ standing for the operation of taking the coefficient of $t^d$.

Thus we have the expansion

$$\text{Hilb}_q(\mathcal{C}) = 1 + 2 q_2 q_1 + (3 q_3 q_1 + 2 q_2^2 + 3 q_2 q_1^2)
+ (4 q_4 q_1 + 9 q_3 q_2 + 6 q_2 q_1^2 + 10 q_2^2 q_1 + 4 q_2 q_1^3) + \ldots$$

Corollary 2 will follow immediately from the explicit description of $\mathcal{C}$ and the isomorphism $\mathcal{C} \otimes \Lambda \to N$ in Section 5, which is not only degree preserving, but shape preserving as well.
3.3 Algebra and coalgebra structures of \( \mathcal{N} \)

Since the action of \( \mathfrak{S} \) on \( T \) is multiplicative, it is straightforward to see that \( \mathcal{N} \) is a subalgebra of \( T \). The multiplication rule in \( \mathcal{N} \), expressing a product \( m_{A} \cdot m_{B} \) as a sum of basis vectors \( \sum_{C} m_{C} \), is easy to describe. Since we make heavy use of the rule later, we develop it carefully here. We begin with an example (the digits corresponding to \( B = 1.2 \) appear in bold):

\[
m_{13.2} \cdot m_{1.2} = m_{13.24.5} + m_{13.25.4} + m_{13.24.5} + m_{13.24.5} + m_{13.25.4} + m_{13.24.5} + m_{13.24.5}
\]

(13)

Compare this to (8). Notice that the shapes indexing the first and last terms in (13) are the partitions \( \lambda(13.2) \cup \lambda(1.2) \) and \( \lambda(13.2) + \lambda(1.2) \). As was the case in \( S^\mathfrak{S} \), one of these shapes, namely \( \lambda(A) + \lambda(B) \), will always appear in the product, while appearance of the shape \( \lambda(A) \cup \lambda(B) \) depends on the cardinality of \( x \).

Let us now describe the multiplication rule. Given any \( D \subseteq \mathbb{N} \) and \( k \in \mathbb{N} \), we write \( D^{+k} \) for the set

\[
D^{+k} := \{ a + k \mid a \in D \}.
\]

By extension, for any set partition \( A = \{ A_{1}, \ldots, A_{r} \} \) we set \( A^{+k} := \{ A_{1}^{+k}, A_{2}^{+k}, \ldots, A_{r}^{+k} \} \). These definitions allow for the introduction of a bilinear (non-commutative) operation denoted by \( \cdot \) on formal linear combinations of set partitions. Given partitions \( A = \{ A_{1}, A_{2}, \ldots, A_{r} \} \) of \( [c] \) and a partition \( B = \{ B_{1}, B_{2}, \ldots, B_{s} \} \) of \( [d] \), the summands of \( A \omega B \) are set partitions of \( [c + d] \). The operation \( \omega \) is recursively defined by the rules:

(a) \( A \omega \emptyset = \emptyset \omega A = A \), with \( \emptyset \) denoting the unique set partition of the empty set;

(b) \( A \omega B = \{ A_{1} \} \cup (A' \omega B^{+c}) + \sum_{i=1}^{s} \{ A_{1} \cup B_{i}^{+c} \} \cup (A' \omega (B \setminus \{ B_{i} \})^{+c}) \),

with union \( \cup \) extended bilinearly and \( A' \) denoting \( \{ A_{2}, \ldots, A_{r} \} \).

As shown in [3, Prop. 3.2], the multiplication rule for \( m_{A} \) and \( m_{B} \) in \( \mathcal{N} \), is

\[
m_{A} \cdot m_{B} = \sum_{C \in A \cup B} m_{C}.
\]

(14)

The subalgebra \( \mathcal{N} \), like its commutative analog, is freely generated by certain monomial symmetric polynomials \( \{ m_{A} \}_{A \in \mathcal{A}} \), where \( \mathcal{A} \) is some carefully chosen collection of set partitions. This is the main theorem of Wolf [14]. See also [3, §7]. We use two such collections later, our choice depending on whether or not \( n < \infty \).

The operation \( (-)^{+k} \) has a left inverse called the standardization operator and denoted by \( "(-)^{k}\)”. It maps set partitions \( A \) of any cardinality-\( d \) subset \( D \subseteq \mathbb{N} \) to set partitions of \( [d] \), with \( A^{k} \) defined as the pullback of \( A \) along the unique increasing bijection from \( [d] \) to \( D \). For example, \( (18.4)^{1} = 13.2 \) and \( (18.4.67)^{1} = 15.2.34 \). The coproduct \( \Delta \) and counit \( \varepsilon \) on \( \mathcal{N} \) are given, respectively, by

\[
\Delta(m_{A}) = \sum_{B \subseteq C = A} m_{B^{k} \otimes m_{C^{k}}} \quad \text{and} \quad \varepsilon(m_{A}) = \delta_{A, \emptyset},
\]

where \( B \cup C = A \) means that \( B \) and \( C \) form complementary subsets of \( A \). In the case \( n = \infty \), the maps \( \Delta \) and \( \varepsilon \) are algebra maps, making \( \mathcal{N} \) a graded connected Hopf algebra.
4 The place-action of $\mathfrak{S}$ on $\mathcal{N}$

4.1 Swapping places in $T_d$ and $N_d$

On top of the permutation-action of the symmetric group $\mathfrak{S}_x$ on $T$, we also consider the “place-action” of $\mathfrak{S}_d$ on the degree $d$ homogeneous component $T_d$. Observe that the permutation-action of $\sigma \in \mathfrak{S}_x$ on a monomial $z$ corresponds to the functional composition $\sigma \circ z : [d] \xrightarrow{\sigma} x \xrightarrow{z} x$. By contrast, the place-action of $\rho \in \mathfrak{S}_d$ on $z$ gives the monomial $z \circ \rho : [d] \xrightarrow{\rho} [d] \xrightarrow{z} x$ composing $\rho$ with $z$ on the right. In the linear extension of this action to all of $T_d$, it is easily seen that $N_d$ (even each $N_{\mu}$) is an invariant subspace of $T_d$. Indeed, for any set partition $\Lambda = \{A_1, \ldots, A_r\} \vdash [d]$ and $\rho \in \mathfrak{S}_d$, one has (see [12, §2])

$$m_{\Lambda} \circ \rho = m_{\rho^{-1} \cdot \Lambda},$$

(15)

where as usual $\rho^{-1} \cdot \Lambda := \{\rho^{-1}(A_1), \rho^{-1}(A_2), \ldots, \rho^{-1}(A_r)\}$.

4.2 The place-action structure of $\mathcal{N}$

Notice that the action in (15) is transitive on set partitions and is shape-preserving. It follows that a basis for the place-action invariants in $N_d$ is indexed by partitions. For such a basis we choose the polynomials

$$m_{\mu} := \frac{1}{(\dim N_{\mu}) \mu!} \sum_{\lambda(\Lambda) = \mu} m_{\Lambda},$$

(16)

with $\mu! = a_1! a_2! \cdots$ whenever $\mu = 1^{a_1} 2^{a_2} \cdots$. The normalizing coefficient will be explained in (19).

To simplify our discussion of the structure of $\mathcal{N}$ in this context, we will say that $\mathfrak{S}$ acts on $\mathcal{N}$ rather than being fastidious about underlying in each situation that individual $N_d$’s are being acted upon on the right by the corresponding group $\mathfrak{S}_d$. We also denote the set $N^{\otimes}$ of place-invariants by $\Lambda$. To summarize,

$$\Lambda = \text{span}\{m_{\mu} : \mu \text{ a partition of } d, d \in \mathbb{N}\}.$$  

(17)

The pair $(\mathcal{N}, \Lambda)$ begins to look like the pair $(\mathfrak{S}, \mathfrak{S}^{\otimes})$ from the introduction. This was the observation that originally motivated our search for Theorem 1.

We next decompose $\mathcal{N}$ into irreducible place-action representations. Although this can be worked out for any value of $n$, the results are more elegant when we send $n$ to infinity. Recall that the Frobenius characteristic of a $\mathfrak{S}_d$-module $V$ is the symmetric function

$$\text{Frob}(V) = \sum_{\mu \vdash d} v_{\mu} s_{\mu},$$

where $s_{\mu}$ is a Schur function—the character of “the” irreducible $\mathfrak{S}_d$ representation $\mathbb{V}_{\mu}$ indexed by $\mu$—and $v_{\mu}$ is the multiplicity of $\mathbb{V}_{\mu}$ in $V$. To reveal the $\mathfrak{S}_d$-module structure of $N_{\mu}$ we may use (15) and standard techniques from the theory of combinatorial species, cf. [2]. The Frobenius characteristic of $N_{\mu}$ is given by the following lemma.
Lemma 3  For a partition \( \mu = 1^{a_1}2^{a_2} \ldots k^{a_k} \), having \( a_i \) parts of size \( i \), we have

\[
\text{Frob}(\mu) = h_{d_1}[h_1][h_{d_2}[h_2] \cdots h_{d_k}[h_k]],
\]

(18)

with \( f[g] \) denoting plethysm of \( f \) and \( g \), and \( h_i \) denoting the \( i \)th homogeneous symmetric function.

Recall that the plethysm \( f[g] \) of two symmetric functions is obtained by linear and multiplicative extension of the rule \( p_k[p_\ell] := p_{k \ell} \), where the \( p_k \)'s denote the usual power sum symmetric functions (see [10, I.8] for notations and more details). For instance, one finds that \( h_3[h_2] = s_6 + s_{42} + s_{222} \). That is, \( N_{222} \) decomposes into 3 irreducible components, with the trivial representation \( s_6 \) coming from \( m_{222} \) inside \( \Lambda \).

4.3 \( \Lambda \) meets \( S^\infty \)

We begin by explaining the choice of coefficient in (16). From [12, Thm. 2.1], one learns that the restriction to \( N \) of the abelianization map \( ab : T \to S \) (the map making the variables commute) satisfies:

(a) \( ab(N) = S^\infty \), and

(b) \( ab(m_\mu) \) is a multiple of \( m_{\lambda(A)} \) depending only on \( \mu = \lambda(A) \), more precisely

\[
ab(m_\mu) = m_\mu.
\]

(19)

Formula (19) suggests that a natural right-inverse to \( ab(-) \) is given by

\[
\iota : S^\infty \hookrightarrow N, \quad \text{with } \iota(m_\mu) := m_\mu.
\]

(20)

The fact that the image of \( S^\infty \) in \( N \) is exactly the subspace \( \Lambda \) affords us a quick proof of Theorem 1 in the case \( n = \infty \). The isomorphism we construct for \( n < \infty \) still uses the map \( \iota \), but in a less essential way.

5 The coinvariant space of \( N \) (Case: \( n = \infty \))

5.1 Proof of main result

Suppose \( n = \infty \). Combining results of [3] and a theorem of Blattner, Cohen, and Montgomery [6], we may immediately deduce the existence of a subspace \( \mathcal{C} \) of \( N \) together with a vector space isomorphism \( N \cong \mathcal{C} \otimes \Lambda \). Indeed, from Propositions 4.3 and 4.5 of [3], we get that the map \( \iota \) is a coalgebra splitting of \( ab : N \to S^\infty \to 0 \), i.e.,

\[
ab \circ \iota = \text{id} \quad \text{and} \quad \Delta_N \circ \iota = (\iota \otimes \iota) \circ \Delta_S^\infty.
\]

Moreover \( ab \) is a morphism of Hopf algebras. In this context, Theorem 4.14 of [6] suggests that we let \( \mathcal{C} \) be the left Hopf kernel of the Hopf map \( ab \).

\[
\mathcal{C} = \{ h \in N : (\text{id} \otimes ab) \circ \Delta(h) = h \otimes 1 \}.
\]

This theorem gives an algebra isomorphism between \( N \) and the crossed product \( \mathcal{C} \#_\sigma S^\infty \). In fact, since \( \Delta_N \) is cocommutative, it is an isomorphism of Hopf algebras. We refer the interested reader to [6, §4] for the technical details. We mention only that: (i) the space \( \mathcal{C} \) is actually a Hopf subalgebra of \( N \) by construction; (ii) the crossed product \( \mathcal{C} \#_\sigma S^\infty \) is a certain algebra structure built on the tensor product \( \mathcal{C} \otimes S^\infty \) using a cocycle \( \sigma : S^\infty \times S^\infty \to \mathcal{C} \); and (iii) the isomorphism amounts to a cocycle twisting of simple multiplication: \( \mathcal{C} \otimes S^\infty \to \mathcal{C} \cdot \Lambda \). This completes the proof of Theorem 1. Moreover, since all spaces and morphisms are graded by degree, the Hilbert series for \( \mathcal{C} \) is the quotient of that for \( N \) by that for \( \Lambda \). This demonstrates (4).
5.2 Atomic set partitions.

Recall the result of Wolf that $N$ is a polynomial algebra, i.e., $N$ is freely generated by some collection of polynomials. We announce our first choice for this collection now, following the terminology of [4]. Let $\Pi$ denote the set of all set partitions (of $[d]$, $\forall d \geq 0$). We introduce the atomic set partitions $\Pi$. A set partition $A = \{A_1, \ldots, A_r\}$ of $[d]$ is atomic if there does not exist a pair $(s, c)$ ($1 \leq s < r, 1 \leq c < d$) such that $\{A_1, \ldots, A_s\}$ is a set partition of $[c]$. Conversely, $A$ is not atomic if there are set partitions $B$ of $[d']$ and $C$ of $[d'']$ splitting $A$ in two: $A = B \cup C^{+d'}$. We write $A = B\ C$ in this situation. A maximal splitting $A = A'\ A''\ \cdots\ A^{(r)}$ of $A$ is one where each $A^{(i)}$ is atomic. For example, the partition $12\ 7\ 23\ 5\ 68$ is atomic, while $12\ 34\ 57\ 8$ is not. The maximal splitting of the latter would be $12\ 12\ 45\ 678 \prec 13\ 2\ 45\ 678 \prec 13\ 24\ 578\ 6 \prec 14\ 23\ 578\ 6 \prec 17\ 235\ 4\ 68 \prec 12\ 34\ 57\ 8$.

It is proven in [4] that $N$ is freely generated by the atomic polynomials. To get a better sense of the structure, let us order $\Pi$ by giving $\Pi$ a total order “$<$” and then extending lexicographically. Given two atomic set partitions $A$ and $B$, we demand that $A \prec B$ if $A \upharpoonright [c]$ and $B \upharpoonright [d]$ with $c < d$. In case $A, B$ are partitions of the same set $[d]$, then any ordering will do for the current purpose... one interesting choice is to order $A$ and $B$ by ordering lexicographically their associated rhyme scheme words. Our convention for writing set partitions provides a bijection between set partitions and this special class of words, sending $A = \{A_1, A_2, \ldots, A_r\} \in \Pi_d$ to $w(A) = w_1 w_2 \cdots w_d$ defined by $w_i := k$ if and only if $i \in A_k$. For example, $w(13\ 2) = 121$ and $w(17\ 235\ 4\ 68) = 123\ 5\ 24\ 14$. Using this ordering on $\Pi$, we have the following chain within the set partitions of shape $3221$:

$$123^{45\ 678} \prec 13^{2\ 45\ 678} \prec 13^{24\ 578\ 6} \prec 14^{23\ 578\ 6} \prec 17\ 235\ 4\ 68 \prec 17\ 236\ 45\ 8.$$

In fact, $123^{45\ 678}$ is the unique minimal element of $\Pi(3221)$.

Define the leading term of a sum $\sum C \alpha C^{} m_C$ to be the monomial $m_C$ such that $C_0$ is lexicographically least among all $C$ with $\alpha C \neq 0$. Combined with (14), our choice for $<$ makes it clear that the leading term of $m_A \cdot m_B$ is $m_{A\ B}$. That is, multiplication in $N$ is shape-filtered. Since the left Hopf kernel $\mathfrak{c}$ is a subalgebra, it is shape-filtered as well. Finally, the isomorphism $\mathfrak{c} \otimes \Lambda \rightarrow N$ respects the shape structures on either side. This completes the proof of Corollary 2.

It is proven in [8] that $N$ is not only freely generated by the atomic polynomials $\{m_A|A \in \Pi\}$, but co-freely generated by them as well. By a classic theorem of Milnor and Moore [11], this means that $N$ is isomorphic to the universal enveloping algebra $\mathfrak{u}(\mathfrak{L}(\Pi))$ of the free Lie algebra $\mathfrak{L}(\Pi)$ on the set $\Pi$. This description will be useful in the next subsection. Let us finish this section with a few final remarks on atomic set partitions. First, note that set partitions with one part are trivially atomic. The set of these is denoted by $\Pi_1$. They are analogs of the generators $m_k$ for the algebra $S^0$. The remaining atomic set partitions

$$\Pi_r := \{\{A_1, \ldots, A_r\} \in \Pi: r > 1\}$$

are more interesting. They index a large portion of the generators for $\mathfrak{c}$. They are also the subject of an open question formulated at the end of Section 5.3.

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Quoting Bill Blewett from [13, A000110], “a rhyme scheme is a string of letters (e.g. abba) such that the leftmost letter is always $a$ and no letter may be greater than one more than the greatest letter to its left. Thus abc is not valid since $c$ is more than one greater than $a$. For example, $[\# \Pi_2 = 5]$ because there are 5 rhyme schemes on 3 letters: aab, aba, aba, abc.”
5.3 Explicit description of the Hopf algebra structure of \( \mathcal{C} \)

It is not too hard to find elements in the left Hopf kernel of the abelianization map \( ab \). Consider the following simple calculation. The sum of monomials \( m_{13,2} := m_{13,2} - m_{12,3} \) is primitive. Indeed,

\[
\Delta(\tilde{m}_{13,2}) = 1 \otimes m_{13,2} + m_{12} \otimes m_1 + m_1 \otimes m_{12} + m_{13,2} \otimes 1 \\
= 1 \otimes m_{12} - m_{12} \otimes m_1 - m_1 \otimes m_{12} - m_{13,2} \otimes 1
\]

We conclude that \(( id \otimes ab ) \circ \Delta(\tilde{m}_{13,2}) = \tilde{m}_{13,2} \otimes 1 \). In other terms, \( \tilde{m}_{13,2} \in \mathcal{C} \). The linear map \( \Delta \) may be split as \( \Delta = \Delta^r + \Delta' \), the sum of its primitive and imprimitive parts respectively. What we have just done in the example is to find a modification \( \tilde{m}_{13,2} \) of \( m_{13,2} \) satisfying \( \Delta'(\tilde{m}_{13,2}) = 0 \). This suggests the following proposition.

**Proposition 4** There is a primitive element

\[
\tilde{m}_A = m_A + \sum_{B : \lambda(B) = \lambda(A)} \alpha_B m_B
\]

associated to each \( A \in \Pi_\mu \) such that \( \sum_B \alpha_B = -1 \) and \( B \in \Pi \Rightarrow \alpha_B = 0 \).

The existence of primitives comes from the Milnor-Moore isomorphism of \( N \) with \( U(\mathfrak{L}(\Pi)) \). Showing that they can be chosen with the above properties is a simple calculation, inducting on the number of parts \( r \) of an atomic set partition \( A = \{ A_1, \ldots, A_r \} \) and applying \( (\Delta)^r \).

The ideas behind the proposition and the preceding example yield several immediate corollaries: (i) each \( \tilde{m}_A \) from Proposition 4 belongs to \( \mathcal{C} \); (ii) \( \mathcal{C} \) is shape-graded, i.e., if \( h \in \mathcal{C} \) is written as \( \sum h_\mu \), then each \( h_\mu \) belongs to \( \mathcal{C} \) as well; (iii) for any \( g \in N \) and \( h \in \mathcal{C} \), we have that \( [g,h] = gh - hg \) also belongs to \( \mathcal{C} \); (iv) if \( A \) and \( B \) belong to \( \Pi_\mu \), then \( [m_A,m_B] \) belongs to \( \mathcal{C} \). These points essentially account for all of \( \mathcal{C} \), as the next result suggests. First, recall that \( S^\mathfrak{L} \) is also a universal enveloping algebra of a Lie algebra. Namely, the abelian Lie algebra \( \mathfrak{A}(\{m_1,m_2,\ldots\}) \), where all Lie brackets \( [m_j,m_k] \) are zero. Since the integers \( k = 1,2,\ldots \) are in 1-1 correspondence with \( \Pi_\mu \), we have a natural map from \( \mathfrak{L}(\Pi) \) to \( \mathfrak{A}(\{m_1,m_2,\ldots\}) \). Our final characterization of \( \mathcal{C} \) is as follows.

**Corollary 5** Let \( \mathcal{C} \) be the kernel of the map \( \pi \) from the free Lie algebra on \( \Pi \) to the free abelian Lie algebra on \( \Pi_\mu \). Then the coinvariant space \( \mathcal{C} \) is the universal enveloping algebra of the Lie algebra \( \mathcal{C} \).

Before turning to the case \( n < \infty \), we remark that we have left unanswered the question of finding a systematic procedure (e.g., a closed formula in the spirit of Möbius inversion) that constructs a primitive element \( \tilde{m}_A \) for each \( A \in \Pi_\mu \).

6 The coinvariant space of \( N \) (Case: \( n < \infty \))

We repeat our example of Section 3.3 in the case \( n = 3 \). The leading term with respect to our previous order would be \( m_{112,4,5} \), except that this term does not appear because \( 13,2,4,5 \) has more than \( n = 3 \) parts. Fortunately, the rhyme scheme bijection \( w \) reveals a more useful leading term:

\[
m_{121} \cdot m_{12} = 0 + m_{12113} + m_{12131} + m_{12123} + m_{12132} + m_{12121} + m_{12112}.
\]
The concatenation 121|12 is the lexicographically smallest word appearing above. This is generally true: if \( w(A) = u \) and \( w(B) = v \), then \( uv \) is the smallest element of \( w(A \cup B) \). Let us call a rhyme scheme word a verse if it cannot be written as the concatenation of two shorter rhyme schemes. The splitting of a rhyme scheme \( w \) is the maximal deconcatenation \( w = w'\mid w'' \cdots \mid w^{(r)} \) of \( w \) into verses \( w^{(i)} \). For example, 12314 is a verse while 11232411 is a string of four verses versus 1|12|32|4|11. It is easy to see that if \( a, b, c, \) and \( d \) are verses, then \( a|c = b|d \) if and only if \( a = b \) and \( c = d \). The preceding observations make it clear that \( N \) is verse-filtered and that \( N \) is freely generated by the monomials \( \{ m_{w(A)} \mid w(A) \text{ is a verse} \} \). This is the collection of monomials originally chosen by Wolf, cf. [3, §7] for details.

Toward locating \( \mathcal{C} \) within \( N \), we first locate \( S^\oplus \). Consider the partition \( \mu = 32211 \). Note that the lexicographically least rhyme scheme word of shape \( \mu \) is \( w(123.45.67.8.9) = 111223345 \). We are led to introduce the words

\[
w(\mu) := 1^{\mu_1}2^{\mu_2} \cdots k^{\mu_k}
\]

associated to partitions \( \mu = (\mu_1, \mu_2, \ldots, \mu_k) \); we call these descending rhymes since \( \mu_1 \geq \cdots \geq \mu_k \).

Finally, we want to view \( \mathcal{C} \) as the rhymes that don’t involve a descending rhyme. Then, by the fact that \( N \) is verse-filtered, we will get an easy vector space isomorphism \( \mathcal{C} \otimes \Lambda \to N \) given by multiplication. Toward that end, we introduce the notion of vexillary rhymes.

A vexillary rhyme is a word that begins with a maximal (but possibly empty) descending rhyme, followed by one extra verse. The vexillary decomposition of a rhyme scheme \( w \) is the expression of \( w \) as a product \( w = w'\mid w'' \cdots \mid w^{(r)} \mid w^{(r+1)} \), where \( w', \ldots, w^{(r)} \) are vexillary rhymes and \( w^{(r+1)} \) is a possibly empty descending rhyme (which we call a tail). For a given word \( w \), this decomposition is accomplished by first splitting \( w \) into verses, then recombining, from left to right, consecutive verses to form vexillary rhymes. For instance, the splitting of 112212 is 1|12|22|12. The first two factors combine to make one vexillary rhyme; the last factor is a descending tail: 112212 \( \mapsto \) 1|12|22|12. Similarly,

\[
1231231411122311 \mapsto 123|12314|111223\mid11 \mapsto \overline{123} \overline{12314} \overline{111} \overline{1223} \overline{1} \overline{1}.
\]

Suppose now that \( u \) and \( v \) are rhyme schemes and that the vexillary decomposition of \( u \) is tail-free. Then by construction, the vexillary decomposition of \( uv \) is the concatenation of the respective vexillary decompositions of \( u \) and \( v \). We are ready to identify \( \mathcal{C} \) as a subalgebra of \( N \).

**Theorem 6** Let \( \mathcal{C} \) be the subalgebra of \( N \) generated by vexillary rhymes. Then \( \mathcal{C} \) has a basis indexed by rhyme scheme words \( w \) whose vexillary decompositions are tail-free. Moreover, the map \( \mathcal{C} \otimes \Lambda \to N \) given by \( m_{w(u)}m_{w(v)} \cdots m_{w(\mu)} \otimes m_{(\mu_1, \ldots, \mu_k)} \mapsto m_{w(u\mid w(v)\cdots w(\mu))} \) is a vector space isomorphism.

### 7 Other directions

We conclude with another advertisement for the Blattner-Cohen-Montgomery theorem. The authors’ present investigation into coinvariant spaces began by moving vertically within the commuting diagram (cube) of Hopf algebras depicted in Figure 1 (whereas in previous work, it was customary to move from left to right, cf. [1]). One may just as well move in other directions within the cube. To illustrate, we apply the Blattner-Cohen-Montgomery theorem to two other edges of interest (leaving aside any comments on group actions). The first of these concerns the downward arrow on the front-right side of the cube. Recall that, from a purely combinatorial perspective, bases in \( K[x]^{\sim, \oplus} \) are indexed by “set compositions” (ordered set partitions), and those in \( K[x]^{\sim, \oplus} \) by integer compositions (here “\( \sim \)” indicates the quasi-action.
of Hivert, cf. [7, §3]). One may find a coalgebra splitting from \( \mathbb{K}[x,y]^{S^\Theta} \) to \( \mathbb{K}(x)^{S^\Theta} \) and an associated coinvariant subalgebra in the spirit of our \((N,S)\) investigation.

Another direction is to consider the Hopf algebra morphism \( sp : \mathbb{K}[x,y]^{S^\Theta} \to \mathbb{K}(x)\!\!\!\langle x \rangle^{S^\Theta} \) (the bottom-right arrow going from NW to SE in Figure 1). These are the **diagonally quasi-symmetric functions** and **quasi-symmetric functions** respectively. For details omitted below, we refer the reader to [1]. The space \( \mathbb{K}[x,y]^{S^\Theta} \) is defined as the \( S \)-invariants, inside \( \mathbb{K}[x,y] \), under the diagonal embedding of \( S \) in \( S \times S \). (The quasi-action of Hivert passes easily through this diagonal embedding.) A basis for \( \mathbb{K}[x,y]^{S^\Theta} \) is given by the “monomial functions” \( m_{a,b} \), indexed by “bicompositions”, i.e., elements \((a,b)\) in \( \mathbb{N}^{2\times r} \) such that \( a_i + b_i > 0 \). These \( m_{a,b} \) conveniently map to the quasi-symmetric function \( m_{a+b} \) under the specialization map \( sp \) sending \( y_i \) to \( x_i \). It is straightforward to show that the map sending \( m_a \) to \( m_{a,0} \) is a coalgebra splitting. We may thus analyze this situation in a manner analogous to our main result. Perhaps more surprising than the fact that the quotient

\[
\frac{\text{Hilb}_t(\mathbb{K}[x,y]^{S^\Theta})}{\text{Hilb}_t(\mathbb{K}[x]^{S^\Theta})}
\]

belongs to \( \mathbb{N}[[t]] \) is the fact that the objects it counts have already been named. We discover a connection between compositions, set compositions, and “L-convex polyominoes.” See [13, A003480].

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**Fig. 1:** The Hopf algebras of symmetric and quasisymmetric functions in one and two sets of commuting and noncommuting variables.
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