GORENSTEIN COMPLEXES AND RECOLLEMENTS FROM COTORSION PAIRS

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Dedicated with gratitude to Mark Hovey

Abstract. We describe a general correspondence between injective (resp. projective) recollements of triangulated categories and injective (resp. projective) cotorsion pairs. This provides a model category description of these recollement situations. Our applications focus on displaying several recollements that glue together various full subcategories of $K(R)$, the homotopy category of chain complexes of modules over a general ring $R$. When $R$ is a Noetherian ring, these recollements involve complexes built from the Gorenstein injective modules. When $R$ is a (left) coherent ring for which all flat modules have finite projective dimension we obtain the duals. These results extend to a general ring $R$ by replacing the Gorenstein modules with the Gorenstein AC-modules introduced in [BGH13]. We also see that in any abelian category with enough injectives, the Gorenstein injective objects enjoy a maximality property in that they contain every other class making up the right half of an injective cotorsion pair.

1. Introduction

It has become clear that adjoint functors between homotopy categories of chain complexes are strongly connected with complete cotorsion pairs. For example, suppose $(\mathcal{F}, \mathcal{C})$ is a cotorsion pair in the category of chain complexes of modules over a ring $R$ and assume $\mathcal{F}$ is closed under suspensions. It is shown in [DEI12] that the full subcategory $K(\mathcal{F})$ of the usual homotopy category $K(R)$ of chain complexes is right admissible, meaning the inclusion has a right adjoint. On the other hand, $K(\mathcal{C})$ is left admissible. Knowing that the abstract categorical language for homotopy theory is that of model categories, and knowing that in algebraic settings model structures translate to cotorsion pairs, this paper aims to show there is a much deeper connection. In particular, we show how a recollement situation, which is a sort of elaborate gluing of triangulated categories with several adjoint pairs, may be obtained model categorically, using just the cotorsion pairs.

The idea relies on Becker’s work in [Bec12], where he finds and describes a beautiful way to localize two “injective” abelian model structures to obtain a third abelian model structure. The homotopy categories of the three involved model structures are then linked by a colocalization sequence and the localized model structure is in fact the right Bousfield localization of the first, killing the fibrant objects of the second. Becker goes on to recover Krause’s recollement $K_{cx}(\text{Inj}) \to K(\text{Inj}) \to D(R)$ from [Kra05] using the theory of abelian model categories. Here, $K(\text{Inj})$ is the homotopy category of all complexes of injective modules, $K_{cx}(\text{Inj})$ 

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is the full subcategory of exact complexes of injectives, and \( D(R) \) is the derived category of \( R \). After reading [Bec12] the author recalled seeing in his work instances of injective cotorsion pairs that were “linked” in the same way that allowed Becker to obtain Krause’s recollement. One of those situations were the three cotorsion pairs used by Becker to reconstruct the Krause recollement. But another two were “Gorenstein injective” versions of this. This inspired the author to look for general conditions that allowed for three injective cotorsion pairs to give a recollement. This appears as Theorem 4.6 but we describe it in more detail below. The theorem appears quite useful as it allowed the author to not only put together the two Gorenstein injective versions of the recollement alluded to above, but to find an unexpected third variation of the recollement and then without much effort to spot two other interesting recollements involving the categorical Gorenstein injective complexes. A converse to Theorem 4.6 appears as Corollary 4.16 and indicates that from the model category perspective, the conditions relating the three cotorsion pairs in Theorem 4.6 characterize (injective) recollement situations. So clearly many more recollement situations, perhaps even all of those that arise in practice, ought to be describable via cotorsion pairs. In fact, several more applications involving exact categories will appear in a sequel to the current paper. We now describe in more detail the main results in this paper.

1.1. Injective model structures and recollements from cotorsion pairs.

The theory of abelian model categories began in [Hov02] where it was shown that an abelian category with a compatible model structure is equivalent to two complete cotorsion pairs. The definition of cotorsion pair appears as Definition 2.2 and Hovey’s correspondence appears in Proposition 2.6. Our main goal in Sections 3 and 4 is to understand localization theory, in particular recollement situations, in terms of Hovey’s correspondence.

So let \( \mathcal{A} \) be an abelian category with enough injectives. We will call a complete cotorsion pair \((W, F)\) an injective cotorsion pair whenever \( W \) is thick (Definition 2.5) and \( W \cap F \) is the class of injective objects in \( \mathcal{A} \). As we will make clear, such a cotorsion pair is equivalent to an injective model structure on \( \mathcal{A} \), which as defined in [Gil11] is an abelian model structure on \( \mathcal{A} \) in which all objects are cofibrant. The canonical example of an injective cotorsion pair would have to be \((\mathcal{E}, dg\mathcal{L})\) in the category of chain complexes \( \text{Ch}(R) \), where here \( \mathcal{E} \) is the class of all exact complexes and \( dg\mathcal{L} \) is the class of DG-injective complexes. This cotorsion pair completely determines the standard injective model structure on \( \text{Ch}(R) \) for the derived category \( D(R) \). Focussing on the injective cotorsion pairs themselves is convenient because it puts all of the essential information for an injective model structure in one small package. In particular, \( F \) is the class of fibrant objects, \( W \) are the trivial objects, \( W \cap F \) are the injectives (trivially fibrant objects), and the entire model structure is determined from this information. This decluttering allows one to then focus on the relationships between the classes appearing in different cotorsion pairs, which turns out to be useful for spotting recollement situations.

Becker too considers injective cotorsion pairs in [Bec12], although he did not call them this. Becker shows that if \( \mathcal{M}_1 = (W_1, F_1) \) and \( \mathcal{M}_2 = (W_2, F_2) \) are injective cotorsion pairs (so model structures) with \( F_2 \subseteq F_1 \) then there is a colocalization sequence associated to the homotopy categories \( \text{Ho}(M_2) \xrightarrow{\text{R\text{id}}} \text{Ho}(M_1) \xrightarrow{\text{R\text{id}}} \text{Ho}(M) \) where \( M = M_1/M_2 \) is the right Bousfield localization of \( M_1 \) with respect to
killing the class $\mathcal{F}_2$ of fibrant objects in $\mathcal{M}_2$. This is central to this paper and the exact statement reappears in Proposition 3.11. We note two additions in how we state Becker’s result. First we define in Section 3 the notion of a weak injective model structure. We point out that Becker’s theorem has a converse in that every hereditary weak injective model structure is the right localization of two injective ones. We also add a uniqueness property that the class of trivial objects satisfy. These two additions are not hard but help give a bigger picture to the theory. More importantly, the uniqueness condition is quite useful for actually spotting a localization, or even potential recollement situation.

In light of the above, the very rough idea then behind spotting a recollement is to find three injective cotorsion pairs $\mathcal{M}_1 = (\mathcal{W}_1, \mathcal{F}_1)$, $\mathcal{M}_2 = (\mathcal{W}_2, \mathcal{F}_2)$, and $\mathcal{M}_3 = (\mathcal{W}_3, \mathcal{F}_3)$ for which $\mathcal{M}_1/\mathcal{M}_2$ is Quillen equivalent to $\mathcal{M}_3$ and vice versa $\mathcal{M}_1/\mathcal{M}_3$ is Quillen equivalent to $\mathcal{M}_2$. But to get a recollement one needs to “glue” together a localization sequence and a colocalization sequence. The exact condition allowing such a “gluing” is this: $\mathcal{F}_3 \subseteq \mathcal{F}_1$ and $\mathcal{W}_3 \cap \mathcal{F}_1 = \mathcal{F}_2$. A precise statement of the main theorem follows.

Theorem A. Let $\mathcal{A}$ be an abelian category with enough injectives and suppose we have three injective cotorsion pairs

$$\mathcal{M}_1 = (\mathcal{W}_1, \mathcal{F}_1), \quad \mathcal{M}_2 = (\mathcal{W}_2, \mathcal{F}_2), \quad \mathcal{M}_3 = (\mathcal{W}_3, \mathcal{F}_3)$$

such that $\mathcal{F}_3 \subseteq \mathcal{F}_1$ and $\mathcal{W}_3 \cap \mathcal{F}_1 = \mathcal{F}_2$. Then

1. The localization $\mathcal{M}_1/\mathcal{M}_2$ is Quillen equivalent to $\mathcal{M}_3$ while $\mathcal{M}_1/\mathcal{M}_3$ is Quillen equivalent to $\mathcal{M}_2$.
2. There is a recollement

$$\mathcal{F}_2/ \sim \quad \mathcal{F}_1/ \sim \quad \mathcal{F}_3/ \sim .$$

where here $f \sim g$ if and only if $g - f$ factors through an injective object.
3. There is a converse to (2) above.

Parts (1) and (2) appear in Theorem 4.6 while the converse appears in Corollary 4.10.

A few comments on Theorem A are in order. First, we point out that all the functors in the recollement are easily described. They are all just left or right derived identity functors between the three model structures, and in practice they correspond to taking precovers or preenvelopes using completeness of the cotorsion pairs.

Second, the author wishes to point out that the converse (3) and its proof were pointed out to him by the anonymous referee of an earlier version of this paper! In short, the converse says that starting with an ambient injective cotorsion pair $(\mathcal{W}, \mathcal{F})$ for which $\mathcal{F}/ \sim$ sits in the center of a recollement, then there must exist two injective cotorsion pairs $(\mathcal{W}', \mathcal{F}')$, and $(\mathcal{W}'', \mathcal{F}'')$ with $\mathcal{F}'' \subseteq \mathcal{F}$ and $\mathcal{W}'' \cap \mathcal{F} = \mathcal{F}'$, which recover the recollement. Though this converse is not used in the rest of the paper it clearly makes the theory more interesting as it indicates how (injective) recollements appear model categorically.

Third, while the applications of the above theorem in the current paper are to triangulated categories associated to $\text{Ch}(R)$, the author wishes to stress that Theorem 4.6 holds in the full generality of abelian categories with enough injectives and should prove useful in more general settings.
Last, the dual notion of a projective cotorsion pair is equally useful. There are versions of all our results for projective cotorsion pairs in abelian categories with enough projectives.

1.2. Gorenstein complexes and Gorenstein derived categories. The remainder of this paper, beginning in Section 5 is concerned with applications of Theorem A to Gorenstein homological algebra and its extension to arbitrary rings in [BCH13]. In Gorenstein homological algebra we essentially replace injective, projective, and flat modules with the Gorenstein projective, Gorenstein injective, and Gorenstein flat modules. We call an object $M$ in an abelian category with enough injectives Gorenstein injective if $M = Z_0 J$ for some exact complex of injectives $J$ which remains exact after applying $\text{Hom}(I, -)$ for any other injective object $I$. The Gorenstein projectives are the dual which we define in an abelian category with enough projectives. Ideally we would like classical results concerning the projectives, injectives and flats to have analogs in Gorenstein homological algebra. In particular, given a ring $R$, the most fundamental question in Gorenstein homological algebra is whether or not Gorenstein injective preenvelopes (resp. Gorenstein projective precovers) exist in $R$-$\text{Mod}$. Said another way, we wish to know when the Gorenstein injectives makes up the right half of a complete cotorsion pair. This first of all allows for the construction of Gorenstein derived functors which is the starting point of the theory; see [EJ00]. Second it leads to a generalized Tate cohomology theory which has been pursued in different ways by many authors. See for example [Ben97], [Jør07], [Iac05] and [AM02]. In light of this and Theorem A, it is natural to consider Gorenstein homological algebra in this paper because of the following result appearing as Theorem 5.4.

**Theorem B.** Let $\mathcal{A}$ be any abelian category with enough injectives. Let $\mathcal{I}$ denote the class of all injective objects in $\mathcal{A}$ and $\mathcal{GI}$ denote the class of Gorenstein injectives.

1. For any injective cotorsion pair $(\mathcal{W}, \mathcal{F})$, we have $\mathcal{I} \subseteq \mathcal{F} \subseteq \mathcal{GI}$.

2. Whenever $(\mathcal{GI}, \mathcal{GI})$ is a complete cotorsion pair it is automatically an injective cotorsion pair too.

In other words, the Gorenstein injectives are the largest possible class that can appear as the right half of an injective cotorsion pair. Therefore we obtain a recollement from Theorem A whenever $\mathcal{W}_3 \cap \mathcal{GI} = \mathcal{F}_2$, and this is the key to obtaining up to 10 recollements involving complexes of Gorenstein modules as we describe more below in Theorems C and D. This also leads us to investigate in Sections 6 and 7 a possible lattice structure on the class of all injective cotorsion pairs. For the Gorenstein injectives are the maximal element in the lattice while the usual injectives are the minimal element.

Categorically speaking, the abstract home for Tate cohomology is the *stable module category* which was defined in [Hov02] for Gorenstein rings by putting a model structure on the category of $R$-modules whose fibrant objects are the Gorenstein injectives. The existence of a nice stable module category boils down, again, to the existence of Gorenstein injective preenvelopes. In [BCH13] it is shown that the Gorenstein injective model structure on $R$-$\text{Mod}$ from [Hov02] extends to any (left) Noetherian ring $R$, and that the Gorenstein projective model structure extends to any (left) coherent ring $R$ for which all flat modules have finite projective dimension. But the most interesting aspect of the work in [BCH13] is that it provides a way to extend Gorenstein homological algebra to arbitrary rings. It introduces
two new classes of modules we call absolutely clean modules and level modules and proves that a perfect duality exists between these classes with respect to taking character modules. When $R$ is Noetherian, the absolutely clean modules are nothing more than the usual injective modules while the level modules are exactly the flat modules. When $R$ is coherent, the absolutely clean modules are the absolutely pure (or FP-injective) modules while still the level modules are the flat modules. This duality between the absolutely clean and level modules is key to extending Gorenstein homological algebra. For a general ring $R$, we then call a module $M$ Gorenstein AC-injective if $M = Z_0J$ for some exact complex of injectives $J$ for which $\text{Hom}_R(I, J)$ is also exact for any absolutely clean module $I$. To emphasize, when $R$ is Noetherian, the Gorenstein AC-injectives are just the usual Gorenstein injectives. There is the dual notion of Gorenstein AC-projective modules and these coincide with the usual Gorenstein projectives for sufficiently nice rings. It is shown in [BGH13] that for any ring $R$, $(W, \mathcal{GI})$ is an injective cotorsion pair, where $\mathcal{GI}$ is the class of Gorenstein AC-injectives. This provides a generalization of the stable module category to any ring $R$. On the other hand, we show $(\mathcal{GP}, V)$ is a projective cotorsion pair where $\mathcal{GP}$ is the class of Gorenstein AC-projectives. This provides another generalization of the stable module category, but for many interesting rings they will coincide.

In the current paper we first of all lift these results to the level of chain complexes by showing that each chain complex $X$ of $R$-modules has a Gorenstein injective preenvelope and an exact Gorenstein injective preenvelope when $R$ is Noetherian. It follows that $K(\text{GInj})$ and $K_{\text{ex}}(\text{GInj})$ each appear as the homotopy category of an injective model structure on $\text{Ch}(R)$. Here $K(\text{GInj})$ denotes the homotopy category of all (categorical) Gorenstein injective chain complexes and $K_{\text{ex}}(\text{GInj})$ denotes homotopy category of all exact Gorenstein injective complexes. But the more interesting thing is that we obtain several new recollements by using the correspondence between cotorsion pairs and recollements in Theorem A. For a general ring $R$, there are potentially 10 recollements we see (five injective ones and five projective ones) involving complexes of Gorenstein modules. We summarize the most interesting ones below in Theorems C and D.

**Theorem C.** Let $D(R)$ denote the derived category of $R$. Then the following hold.

1. If $R$ is (left) Noetherian ring, then there is a recollement

   \[
   K_{\text{ex}}(\text{GInj}) \quad \bigglarrow \quad K(\text{GInj}) \quad \bigglarrow \quad D(R)
   \]

   This says that the homotopy category of Gorenstein injective complexes is obtained by gluing the derived category together with the homotopy category of exact Gorenstein injective complexes.

2. If $R$ is (left) coherent ring and all flat modules have finite projective dimension, then the dual Gorenstein projective version of the above recollement in (1) exists. More generally, for any ring $R$ in which all level modules have finite projective dimension we have the same result.

3. Both the injective recollement of (1) and the projective recollement of (2) extend to arbitrary rings by replacing the Gorenstein injective complexes (resp. Gorenstein projective complexes) with the complexes of Gorenstein AC-injectives (resp. Gorenstein AC-projectives).

These results appear in Theorem 8.2, Theorem 8.3, and Theorem 10.2 where some other variations of these recollements also appear. As we explore in Sections 6

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and these other variations may or may not be distinct from the above recollements, nor need they be nontrivial. The farther away from Gorenstein the ring is, the more recollements we see.

Finally, we now describe one of two beautiful recollements that appear near the end of the paper. For any ring $R$, let $(\mathcal{W}, \mathcal{G})$ denote the injective cotorsion pair where $\mathcal{G}$ are the Gorenstein AC-injectives and $\mathcal{W}$ are the trivial objects in the corresponding stable module category. Now let $\mathcal{W}$ denote the class of all exact complexes $W$ for which each $W_n \in \mathcal{W}$. We will see that there is an injective model structure on $\text{Ch}(R)$ with $\mathcal{W}$ as its class of trivial objects. Let $\mathcal{D}(\mathcal{W})$ denote the associated triangulated homotopy category.

**Theorem D.** In the above set-up we have the following.

1. If $R$ is Noetherian then there is a recollement

$$K_{\mathcal{W}}(\text{Inj}) \rightleftarrows K(\mathcal{G}) \rightleftarrows \mathcal{D}(\mathcal{W}).$$

This says that the homotopy category of Gorenstein injective complexes is obtained by gluing $\mathcal{D}(\mathcal{W})$ together with the stable derived category $S(R) = K_{\mathcal{W}}(\text{Inj})$.

2. The injective recollement of (1) still holds for an arbitrary ring $R$ by replacing the Gorenstein injective complexes by the complexes of Gorenstein AC-injectives.

3. The projective dual to both (1) and (2) above hold. In particular, the projective dual of (1) holds if $R$ is coherent and all flat modules have finite projective dimension.

The outline of the paper is as follows. Preliminary notions and notation are laid out in Section 2. In Section 3 we look at the necessary notions of Hovey triples, injective cotorsion pairs, weak injective cotorsion pairs, and Becker’s localization theorem. In Section 4 we discuss recollements and the main Theorem A relating them to injective cotorsion pairs. In Section 5 we prove the maximality property of the Gorenstein injective cotorsion pair in Theorem B. In Section 6 we look at the semilattice of injective cotorsion pairs. Then in Section 7 we show how to lift an injective cotorsion pair from $R$-Mod to six in $\text{Ch}(R)$. Finally in Section 8 we look exclusively at the Gorenstein complexes and the Krause-like recollements of Theorem C parts (1) and (2). In Section 9 we conclude Theorem D parts (1) and (2), as well as an equally interesting variant of this recollement. In Section 10 we explain a little more detail on the work of [BGH13] which results in the extensions of our recollements to arbitrary rings $R$. That is, part (3) of Theorems C and D.

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2. PRELIMINARIES

Here we either recall or define fundamental ideas and set some notation which we use throughout the paper. By a ring $R$, we always mean a ring with 1. Central to this paper is Quillen’s notion of a model category \cite{Qui67}. Our basic reference is \cite{Hov99} and we are concerned with abelian model structures. The basic theory of abelian model structures is in \cite{Hov02}, but we will recall the correspondence with cotorsion pairs in this section below.

2.1. Chain complexes. We denote by $R$-$\text{Mod}$ the category of (left) $R$-modules and by $\text{Ch}(R)$ the category of chain complexes of (left) $R$-modules. We write chain complexes as $\cdots \to X_{n+1} \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} X_{n-1} \to \cdots$ so that the differentials lower the degree.

We let $S^n(M)$ denote the chain complex with all entries 0 except $M$ in degree $n$. We let $D^n(M)$ denote the chain complex $X$ with $X_n = X_{n-1} = M$ and all other entries 0. All maps are 0 except $d_n = 1_M$. Given $X$, the suspension of $X$, denoted $\Sigma X$, is the complex given by $(\Sigma X)_n = X_{n-1}$ and $(d_{\Sigma X})_n = -d_n$. The complex $\Sigma(\Sigma X)$ is denoted $\Sigma^2 X$ and inductively we define $\Sigma^n X$ for all $n \in \mathbb{Z}$.

Given two chain complexes $X$ and $Y$ we define $\text{Hom}(X, Y)$ to be the complex of abelian groups $\cdots \to \prod_{k \in \mathbb{Z}} \text{Hom}(X_k, Y_{k+n}) \xrightarrow{\delta_n} \prod_{k \in \mathbb{Z}} \text{Hom}(X_k, Y_{k+n-1}) \to \cdots$, where $(\delta_n f)_k = d_{k+n} f_k - (-1)^n f_{k-1} d_k$. This gives a functor $\text{Hom}(X, -) : \text{Ch}(R) \to \text{Ch}(\mathbb{Z})$ which is left exact, and exact if $X_n$ is projective for all $n$. Similarly the contravariant functor $\text{Hom}(-, Y)$ sends right exact sequences to left exact sequences and is exact if $Y_n$ is injective for all $n$.

Recall that $\text{Ext}^1_{\text{Ch}(R)}(X, Y)$ is the group of (equivalence classes) of short exact sequences $0 \to Y \to Z \to X \to 0$ under the Baer sum. We let $\text{Ext}^1_{d_{\text{Ch}(R)}}(X, Y)$ be the subgroup of $\text{Ext}^1_{\text{Ch}(R)}(X, Y)$ consisting of those short exact sequences which are split in each degree. We often make use of the following standard fact.

**Lemma 2.1.** For chain complexes $X$ and $Y$, we have

$$\text{Ext}^1_{d_{\text{Ch}(R)}}(X, \Sigma^{(n-1)} Y) \cong H_n \text{Hom}(X, Y) = \text{Ch}(R)(X, \Sigma^{-n} Y)/\sim,$$

where $\sim$ is chain homotopy.

In particular, for chain complexes $X$ and $Y$, $\text{Hom}(X, Y)$ is exact iff for any $n \in \mathbb{Z}$, any $f : \Sigma^n X \to Y$ is homotopic to 0 (or iff any $f : X \to \Sigma^n Y$ is homotopic to 0).

2.2. Cotorsion pairs. The most important concept in this paper is that of a cotorsion pair in an abelian category $\mathcal{A}$. A standard reference is \cite{EJ00}.

**Definition 2.2.** A pair of classes $(\mathcal{F}, \mathcal{C})$ in an abelian category $\mathcal{A}$ is a cotorsion pair if the following conditions hold:

1. $\text{Ext}^1_{\mathcal{A}}(F, C) = 0$ for all $F \in \mathcal{F}$ and $C \in \mathcal{C}$.
2. If $\text{Ext}^1_{\mathcal{A}}(F, X) = 0$ for all $F \in \mathcal{F}$, then $X \in \mathcal{C}$.
3. If $\text{Ext}^1_{\mathcal{A}}(X, C) = 0$ for all $C \in \mathcal{C}$, then $X \in \mathcal{F}$.

A cotorsion pair is said to have enough projectives if for any $X \in \mathcal{A}$ there is a short exact sequence $0 \to C \to F \to X \to 0$ where $C \in \mathcal{C}$ and $F \in \mathcal{F}$. We say it has enough injectives if it satisfies the dual statement. If both of these hold we say the cotorsion pair is complete. Whenever the category $\mathcal{A}$ has enough injectives and
projectives then a cotorsion pair $(\mathcal{F}, \mathcal{C})$ is complete iff $(\mathcal{F}, \mathcal{C})$ has enough injectives iff $(\mathcal{F}, \mathcal{C})$ has enough projectives. [EJ00, Proposition 7.1.7].

In $R$-Mod, the class of projectives is the left half of an obvious complete cotorsion pair while the class of injectives is the right half of an obvious complete cotorsion pair. The most well-known nontrivial example of a complete cotorsion pair is $(\mathcal{F}, \mathcal{C})$ where $\mathcal{F}$ is the class of flat modules and $\mathcal{C}$ are the cotorsion modules. A proof that this is a complete cotorsion theory can be found in [EJ00]. We will be concerned with cotorsion pairs of chain complexes in this paper and will use results from [Gil04] and [Gil08]. These results describe several cotorsion pairs in $\text{Ch}(R)$ that can be associated to a single given cotorsion pair in $R$-Mod. This will be encountered in Section 7 where we will recall the basic definitions.

2.3. **Hereditary cotorsion pairs.** Let $\mathcal{A}$ be an abelian category. We say a cotorsion pair $(\mathcal{F}, \mathcal{C})$ in $\mathcal{A}$ is **resolving** if $\mathcal{F}$ is closed under taking kernels of epimorphisms between objects of $\mathcal{F}$. That is, if for any short exact sequence $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$, we have $X \in \mathcal{F}$ whenever $Y$ and $Z$ are in $\mathcal{F}$. We say $(\mathcal{F}, \mathcal{C})$ is **coresolving** if the right hand class $\mathcal{C}$ satisfies the dual. Finally, we say that $(\mathcal{F}, \mathcal{C})$ is **hereditary** if it is both resolving and coresolving. The following is a standard test for checking to see if a given cotorsion pair is hereditary.

**Lemma 2.3** (Hereditary Test). Assume $(\mathcal{F}, \mathcal{C})$ is a cotorsion pair in an abelian category $\mathcal{A}$. Consider the statements below.

1. $\mathcal{F}$ is closed under taking kernels of epimorphisms between objects of $\mathcal{F}$.
2. $\mathcal{F}$ is syzygy closed, meaning $X \in \mathcal{F}$ whenever we have an exact $0 \rightarrow X \rightarrow P \rightarrow Z \rightarrow 0$ with $P$ projective and $Z \in \mathcal{F}$.
3. $\mathcal{C}$ is closed under taking cokernels of monomorphisms between objects of $\mathcal{C}$.
4. $\mathcal{C}$ is cosyzygy closed, meaning $Z \in \mathcal{C}$ whenever we have an exact $0 \rightarrow X \rightarrow I \rightarrow Z \rightarrow 0$ with $I$ injective and $X \in \mathcal{C}$.
5. $\text{Ext}^2_{\mathcal{A}}(\mathcal{F}, \mathcal{C}) = 0$ whenever $F \in \mathcal{F}$ and $C \in \mathcal{C}$.

Then we have the following implications depending on whether or not $\mathcal{A}$ has enough injectives or projectives.

- If $\mathcal{A}$ has enough injectives then we have (3) implies (4) implies (5) implies (1) implies (2).
- If $\mathcal{A}$ has enough projective then we have (1) implies (2) implies (5) implies (3) implies (4).
- (5) implies (1) – (4) without any assumption on enough injectives or projectives. (Just using the long exact sequence in Ext.)

In particular, when $\mathcal{A}$ is a category with enough projectives and injectives then all of the conditions (1) – (5) are equivalent.

**Proof.** See [GR99].

So suppose $\mathcal{A}$ has enough injectives and you have a cotorsion pair $(\mathcal{F}, \mathcal{C})$. Then from Lemma 2.3 the cotorsion pair is hereditary whenever it is known to just be coresolving, or even if $\mathcal{C}$ is just cosyzygy closed. However it can’t be used to conclude hereditary in the case that you only know it is resolving. But for a complete cotorsion pair, the following is a wonderfully convenient result that appears as Corollary 1.1.13 of [Bec12].

**Lemma 2.4** (Becker’s Lemma). Let $\mathcal{A}$ be an abelian category and let $(\mathcal{F}, \mathcal{C})$ be a complete cotorsion pair. Then the following are equivalent.
(1) \((F, C)\) is hereditary.
(2) \((F, C)\) is resolving.
(3) \((F, C)\) is coresolving.

2.4. Hovey’s correspondence. Hovey defines abelian model categories in [Hov02]. He then characterizes them in terms of cotorsion pairs as we now describe. So in fact one could even take the cotorsion pairs given in the correspondence below as the definition of an abelian model category. First, we need the definition of a thick subcategory.

**Definition 2.5.** By a thick subcategory of an abelian \(\mathcal{A}\) we mean a class of objects \(\mathcal{W}\) which is closed under direct summands and such that if two out of three of the terms in a short exact sequence are in \(\mathcal{W}\), then so is the third.

**Proposition 2.6.** Let \(\mathcal{A}\) be an abelian category with an abelian model structure. Let \(Q\) be the class of cofibrant objects, \(R\) the class of fibrant objects and \(W\) the class of trivial objects. Then \(\mathcal{W}\) is a thick subcategory of \(\mathcal{A}\) and both \((Q, W \cap R)\) and \((Q \cap W, R)\) are complete cotorsion pairs in \(\mathcal{A}\). Conversely, given a thick subcategory \(\mathcal{W}\) and classes \(Q\) and \(R\) making \((Q, W \cap R)\) and \((Q \cap W, R)\) each complete cotorsion pairs, then there is an abelian model structure on \(\mathcal{A}\) where \(Q\) are the cofibrant objects, \(R\) are the fibrant objects and \(W\) are the trivial objects.

We point out that the abelian model structure on \(\mathcal{A}\) is then completely determined by the classes \(Q\), \(W\) and \(R\). Indeed the cofibrations (resp. trivial cofibrations) are the monomorphisms with cokernel in \(Q\) (resp. \(Q \cap W\)) and the fibrations are the epimorphisms with kernel in \(R\) (resp. \(W \cap R\)). The weak equivalences are the maps which factor as a trivial cofibration followed by a trivial fibration. However, the description of the weak equivalences given by the Lemma below is sometimes more convenient. It follows from [Hov02, Lemma 5.8] and an application of the two out of three axiom.

**Lemma 2.7.** Say \(\mathcal{A}\) is an abelian category with an abelian model structure. Let \(\mathcal{W}\) denote the class of trivial objects. Then a map \(f\) is a weak equivalence if and only if it factors as a monomorphism with cokernel in \(\mathcal{W}\) followed by an epimorphism with kernel in \(\mathcal{W}\).

**Remark 1.** We note that one normally assumes the category \(\mathcal{A}\) is bicomplete for it to qualify as a model category. However, as explained in [Gil11] an abelian category with an abelian model structure already has enough colimits and limits to get the basic first results of homotopy theory. So we can always discuss model structures on abelian categories \(\mathcal{A}\), but we will deliberately reserve the term model category when \(\mathcal{A}\) is known to be bicomplete.

3. INJECTIVE AND WEAK INJECTIVE HOVEY TRIPLES

It was reading [Bec12] that led the author to denote abelian model structures as triples and this eventually led to the definition of a weak injective model structure as below. Throughout this entire section assume \(\mathcal{A}\) is an abelian category.

3.1. Hovey triples. We start with a convenient definition.

**Definition 3.1.** Let \(Q, W, R\) be three classes in \(\mathcal{A}\) as in Hovey’s correspondence. Then we call \((Q, W, R)\) a Hovey triple. By a hereditary Hovey triple we
mean that the two corresponding cotorsion pairs \((Q, W \cap R)\) and \((Q \cap W, R)\) are each hereditary.

**Notation.** Due to Hovey’s one-to-one correspondence between Hovey triples in \(A\) and abelian model structures on \(A\) we often will not distinguish between the Hovey triple and the actual model structure. For example, we may say \(M = (Q, W, R)\) is an abelian model structure and understand this to mean the model structure associated to the Hovey triple \((Q, W, R)\). On the other hand, we may say that an abelian model structure is hereditary and by this we mean that the Hovey triple is hereditary.

The following is an easy but useful fact about Hovey triples.

**Proposition 3.2** (Characterization of trivial objects). Suppose \((Q, W, R)\) is a Hovey triple in \(A\). Then the thick class \(W\) is characterized two ways as follows:

\[
W = \{ A \in A | \exists \text{s.e.s. } 0 \rightarrow A \rightarrow W_2 \rightarrow W_1 \rightarrow 0 \text{ with } W_2 \in W \cap R, W_1 \in Q \cap W \}
\]

\[
= \{ A \in A | \exists \text{s.e.s. } 0 \rightarrow W_2 \rightarrow W_1 \rightarrow A \rightarrow 0 \text{ with } W_2 \in W \cap R, W_1 \in Q \cap W \}
\]

Consequently, whenever \((Q, V, R)\) is a Hovey triple, then necessarily \(V = W\).

**Proof.** \((\supseteq)\) It is clear that each class described is contained in \(W\) since \(W\) is handed to us thick. \((\subseteq)\) Let \(W \in W\). Then apply enough injectives of \((Q, W \cap R)\) to get a short exact sequence \(0 \rightarrow W \rightarrow W_2 \rightarrow W_1 \rightarrow 0\) where \(W_2 \in W \cap R\) and \(W_1 \in Q\). But indeed \(W_1 \in Q \cap W\) since \(W\) is thick. So \(W\) is in the top class described. On the other hand, \(W\) is also in the bottom class described by a similar argument using enough projectives of the cotorsion pair \((Q \cap W, R)\).

Now if \((Q, V, R)\) is any other Hovey triple we must have \(V \cap R = W \cap R\) (since each equal \(Q^\perp\)) and \(Q \cap V = Q \cap W\) (since each equal \(R^\perp\)). So by what we just proved in the last paragraph it immediately follows that \(V = W\).

\[\square\]

### 3.2. Injective and weak injective Hovey triples.

**Definition 3.3.** We call a Hovey triple \((Q, W, R)\) **injective** if \(Q = A\). That is, if every object of \(A\) is cofibrant. By a **weak injective** Hovey triple we mean a Hovey triple \((Q, W, R)\) for which \(Q \cap W \cap R\) is exactly the class of injective objects in \(A\). We have the obvious dual notions of **projective** Hovey triples and **weak projective** Hovey triples.

We apply these same phrases to the actual model structures induced as well. So we speak of **injective**, **weak injective**, **projective**, and **weak projective** model structures on \(A\).

**Remark 2.** It is immediate that an injective Hovey triple is a weak injective Hovey triple. Also an injective Hovey triple is automatically hereditary by Becker’s Lemma [2.4] because \(W\) is thick.

Note that whenever \(M = (Q, W, R)\) is an injective Hovey triple then the class \(Q\) is redundant and need not be mentioned. Indeed as long as \(A\) has enough injectives, then an injective model structure on \(A\) is equivalent to a single complete cotorsion pair \(M = (W, R)\) where \(W\) is thick and \(W \cap R\) is the class of injective objects. We make this precise in the next definition.
Remark 3. We emphasize that we don’t use the term injective cotorsion pair unless $\mathcal{A}$ has enough injectives and we don’t use the phrase projective cotorsion pair unless $\mathcal{A}$ has enough projectives. The point is that a projective cotorsion pair IS a model structure on $\mathcal{A}$ but you don’t have a model structure if $\mathcal{A}$ lacks projectives. The definitions are useful though because now all of the essential information needed for such a model structure is packed into one simple idea - a single cotorsion pair.

The definition of an injective cotorsion pair is stronger that what we need. This isn’t just interesting but it is relevant for stating a converse to Becker’s localization theorem (Proposition 3.11). The author learned the Lemma below from Henrik Holm.

Lemma 3.5. Assume $(\mathcal{W}, \mathcal{F})$ is an hereditary cotorsion pair in $\mathcal{A}$. Suppose that for any $F \in \mathcal{F}$ we can find a short exact sequence $0 \to F' \to I \to F \to 0$ where $I$ is injective and $F' \in \mathcal{F}$. Then $\mathcal{W}$ is thick.

Proof. The assumptions imply that $\mathcal{W}$ is already closed under retracts and extensions and is resolving. So it remains to show that if

\[ (*) \quad 0 \to V \to W \to X \to 0 \]

is an exact sequence with $V, W \in \mathcal{W}$ then $X \in \mathcal{W}$ too. Applying $\text{Hom}_\mathcal{A}(-, F)$ to $(*)$, for any $F \in \mathcal{F}$, it follows that $\text{Ext}^{2}_\mathcal{A}(X, F) = 0$. To see that $\text{Ext}^{1}_\mathcal{A}(X, F) = 0$ for every $F \in \mathcal{F}$, pick a short exact sequence $0 \to F' \to I \to F \to 0$, where $I$ is injective and $F' \in \mathcal{F}$. Applying $\text{Hom}_\mathcal{A}(X, -)$ to this sequence gives $\text{Ext}^{1}_\mathcal{A}(X, F) \cong \text{Ext}^{1}_\mathcal{A}(F', X)$, which is zero by what we just proved. \qed

Proposition 3.6 (Characterizations of injective cotorsion pairs). Suppose $(\mathcal{W}, \mathcal{F})$ is a complete cotorsion pair in an abelian category $\mathcal{A}$ with enough injectives. Then each of the following statements are equivalent:

1. $(\mathcal{W}, \mathcal{F})$ is an injective cotorsion pair.
2. $(\mathcal{W}, \mathcal{F})$ is hereditary and $\mathcal{W} \cap \mathcal{F}$ equals the class of injective objects.
3. $\mathcal{W}$ is thick and contains the injective objects.

Proof. First, (1) implies (2) by definition and Becker’s Lemma 2.4. For (2) implies (3) note that for any $F \in \mathcal{F}$ using that the cotorsion pair $(\mathcal{W}, \mathcal{F})$ has enough projectives we can write $0 \to F' \to W \to F \to 0$ where $W \in \mathcal{W}$ and $F' \in \mathcal{F}$. Then since $\mathcal{F}$ is closed under extensions we have $W \in \mathcal{F}$. So by hypothesis we have $W$ is injective. Therefore (2) implies (3) follows from Lemma 3.5.

For (3) implies (1) first note that the hypothesis makes clear that the injective objects are in $\mathcal{W} \cap \mathcal{F}$. So we just wish to show that everything in $\mathcal{W} \cap \mathcal{F}$ is injective. So suppose $X \in \mathcal{W} \cap \mathcal{F}$. Then using that $\mathcal{A}$ has enough injectives find a short exact sequence $0 \to X \to I \to I/X \to 0$ where $I$ is injective. By hypothesis $I \in \mathcal{W}$ and also $\mathcal{W}$ is assumed to be thick, which means $I/X \in \mathcal{W}$. But now since $(\mathcal{W}, \mathcal{F})$ is a cotorsion pair the exact sequence splits. Therefore $X$ is a direct summand of $I$, proving $X \in \mathcal{I}$. \qed
By duality we have the following as well.

**Proposition 3.7** (Characterizations of projective cotorsion pairs). *Suppose* \((C, W)\) *is a complete cotorsion pair in an abelian category* \(A\) *with enough projectives. Then each of the following statements are equivalent:

1. \((C, W)\) is a projective cotorsion pair.
2. \((C, W)\) is hereditary and \(C \cap W\) equals the class of projective objects.
3. \(W\) is thick and contains the projective objects.

For a general Grothendieck category \(A\) it becomes important to know whether or not the left side of a cotorsion pair contains a set of generators for \(A\). We point out the following property of injective cotorsion pairs which could be of use in this setting. It won’t be used in this paper however.

**Proposition 3.8.** Let \(A\) be an abelian category with enough injectives. If \((W, F)\) is any injective cotorsion pair, then \(W\) contains all objects of finite injective dimension. If \(A\) also has enough projectives then \(W\) also contains all objects of finite projective dimension.

We have the dual for projective cotorsion pairs \((C, W)\).

**Proof.** The point is that \(W\) is *thick* and contains all injective and projective objects. So let \(M\) be of finite injective dimension. Then there exists an exact sequence

\[0 \to M \to I^0 \to I^1 \to \cdots \to I^n \to 0\]

with each \(I^n\) injective. Since each \(I^i \in W\) and \(W\) thick we conclude \(M \in \mathcal{I}\). If \(A\) has enough projectives, then the same argument works for projective dimension since projectives are always in the left side of a cotorsion pair.

\[\square\]

Note that Proposition 3.8 is saying that any object of finite injective dimension is trivial in the model structure associated to any injective cotorsion pair.

### 3.3. The homotopy category of a weak injective model structure.

It is convenient for us to explicitly define weak injective model structures for two reasons. First, the hereditary ones are precisely the right Bousfield localizations of two injective ones as we point out in Proposition 3.11. Second the homotopy relation on the subcategory of cofibrant-fibrant objects is characterized in a very nice way which we explain below. This will prove important for this paper. For example, it is automatic that for any weak injective (or weak projective) model structure on \(\text{Ch}(R)\), two maps from a cofibrant complex to a fibrant complex are formally homotopic if and only if they are chain homotopic in the usual sense.

First we point out that if \((Q, W, R)\) is any Hovey triple in \(A\) then the fully exact subcategory \(Q \cap R\) is a Frobenius category with \(Q \cap W \cap R\) being precisely the class of projective-injective objects. Therefore the stable category \(Q \cap R/\sim\) is naturally a triangulated category. Since \(A\) is abelian, it is a general fact that \(\text{Ho}(A)\) is also a triangulated category and is triangle equivalent to \(Q \cap R/\sim\). See [Gil11, Section 5] and [Hov99, Chapters 6 and 7] for more on all this. We now go on to make this a little more precise while focusing only on injective and weak injective model structures.

Recall that a *thick* subcategory of a triangulated category is a triangulated subcategory that is closed under retracts.
Proposition 3.9. Suppose $\mathcal{M} = (\mathcal{W}, \mathcal{F})$ is an injective cotorsion pair in $\mathcal{A}$.

1. If $Y$ is fibrant then two maps $f, g \colon X \to Y$ in $\mathcal{A}$ are homotopic, written $f \sim g$, if and only if $g - f$ factors through an injective object.

2. The fully exact subcategory $\mathcal{F}$ of fibrant objects is a Frobenius category with its projective-injective objects being precisely the injective objects of $\mathcal{A}$.

3. The inclusion functor $i : \mathcal{F} \to \mathcal{A}$ induces an inclusion of triangulated categories $\text{Ho} i : \mathcal{F}/ \sim \to \text{Ho}(\mathcal{A})$. It displays $\mathcal{F}/ \sim$ as an equivalent thick subcategory of $\text{Ho}(\mathcal{A})$.

4. The inverse of $\text{Ho} i$ is the functor $\text{Ho} Q \circ \text{Ho} R : \text{Ho}(\mathcal{A}) \to Q \cap R/ \sim$ and this is the fibrant replacement functor.

Proposition 3.10. Suppose $\mathcal{M} = (Q, \mathcal{W}, \mathcal{R})$ is an hereditary weak injective model structure in $\mathcal{A}$.

1. If $X$ is cofibrant and $Y$ is fibrant then two maps $f, g \colon X \to Y$ in $\mathcal{A}$ are homotopic, written $f \sim g$, if and only if $g - f$ factors through an injective object.

2. The fully exact subcategory $Q \cap R$ of cofibrant-fibrant objects is a Frobenius category with its projective-injective objects being precisely the injective objects of $\mathcal{A}$.

3. The inclusion functor $i : Q \cap R \to \mathcal{A}$ induces an inclusion of triangulated categories $\text{Ho} i : Q \cap R/ \sim \to \text{Ho}(\mathcal{A})$. It displays $Q \cap R/ \sim$ as an equivalent thick subcategory of $\text{Ho}(\mathcal{A})$.

4. The inverse of $\text{Ho} i$ is the functor $\text{Ho} Q \circ \text{Ho} R : \text{Ho}(\mathcal{A}) \to Q \cap R/ \sim$ and this is fibrant replacement followed by cofibrant replacement.

Proof. Both follow from general results that can be found in [Gil11, Proposition 4.4 and Section 5] and Chapters 6 and 7 of [Hov99]. To show the triangulated subcategory $Q \cap R/ \sim$ is closed under retracts use the fact that $Q \cap R$ contains the injective objects and is itself is closed under direct sums and retracts.

3.4. Becker’s Theorem and a converse. We now look at a beautiful result from [Bec12, Proposition 1.4.2] giving a simple description, in terms of Hovey triples, of the right Bousfield localization of an injective model structure with respect to another. In our statement here we add a uniqueness property and a converse. These last two things are not hard at all but they give a complete picture and are in fact useful for spotting localizations. In fact the author had already witnessed on more than one occasion model structures on Ch$(R)$ arise in the following way. First, let $(\mathcal{W}_1, \mathcal{F}_1)$ and $\mathcal{M}_2 = (\mathcal{W}_2, \mathcal{F}_2)$ be two injective model structures on $\mathcal{A}$. Suppose you also have a thick subcategory $\mathcal{W}$ for which $\mathcal{M} = (\mathcal{W}, \mathcal{W}, \mathcal{F}_1)$ is a Hovey triple. Then these three model structures are obviously linked in some way. It was not until the result of Becker that the author learned the formal connection. It turns out that $\mathcal{M} = (\mathcal{W}_2, \mathcal{W}, \mathcal{F}_1)$ is the right Bousfield localization of $\mathcal{M}_1$ with respect to the fibrant objects in $\mathcal{M}_2$.

Proposition 3.11 (Characterization of Becker Localizations). Let $\mathcal{A}$ be an abelian category with enough injectives. Let $\mathcal{M}_1 = (\mathcal{W}_1, \mathcal{F}_1)$ and $\mathcal{M}_2 = (\mathcal{W}_2, \mathcal{F}_2)$ be two injective cotorsion pairs on $\mathcal{A}$ with $\mathcal{F}_2 \subseteq \mathcal{F}_1$. Then there exists a weak injective
hereditary Hovey triple $\mathcal{M}_1/\mathcal{M}_2 = (\mathcal{W}_2, \mathcal{W}, \mathcal{F}_1)$ on $\mathcal{A}$ where the thick class $\mathcal{W}$ is
\[ \mathcal{W} = \{ X \in \mathcal{A} \mid \exists \text{s.e.s. } 0 \to X \to F_2 \to W_1 \to 0 \text{ with } F_2 \in \mathcal{F}_2, W_1 \in \mathcal{W}_1 \} \]
\[ = \{ X \in \mathcal{A} \mid \exists \text{s.e.s. } 0 \to F_2 \to W_1 \to X \to 0 \text{ with } F_2 \in \mathcal{F}_2, W_1 \in \mathcal{W}_1 \} \]

We call $\mathcal{M}_1/\mathcal{M}_2$ the right localization of $\mathcal{M}_1$ with respect to $\mathcal{M}_2$ and we note the following uniqueness and converse:

1. (Uniqueness of Trivial Objects) Suppose $\mathcal{V}$ is a thick subcategory for which $\mathcal{M} = (\mathcal{W}_2, \mathcal{V}, \mathcal{F}_1)$ is a Hovey triple. Then $\mathcal{M} = \mathcal{M}_1/\mathcal{M}_2$.
2. (Converse) Let $\mathcal{M} = (\mathcal{Q}, \mathcal{W}, \mathcal{R})$ be a weak injective hereditary Hovey triple in $\mathcal{A}$. Then setting $\mathcal{M}_1 = (\mathcal{Q} \cap \mathcal{W}, \mathcal{R})$ and $\mathcal{M}_2 = (\mathcal{Q}, \mathcal{W} \cap \mathcal{R})$, these are each injective cotorsion pairs and $\mathcal{M} = \mathcal{M}_1/\mathcal{M}_2$.

Remark 4. Becker shows that his right localization $\mathcal{M}_1/\mathcal{M}_2$ is in fact the right Bousfield localization with respect to the maps $0 \to F$ where $F \in \mathcal{F}_2$.

Proof. This is Becker’s result. See [Bec12 Proposition 1.4.2]. We are simply noting the uniqueness and converse statements.

The uniqueness of the class of trivial objects is a corollary to Proposition 3.2.

To prove the converse statement, let $\mathcal{M} = (\mathcal{Q}, \mathcal{W}, \mathcal{R})$ be any weak injective and hereditary Hovey triple in $\mathcal{A}$. By definition of $\mathcal{M} = (\mathcal{Q}, \mathcal{W}, \mathcal{R})$ being hereditary we have that each of the cotorsion pairs $(\mathcal{Q} \cap \mathcal{W}, \mathcal{R})$ and $(\mathcal{Q}, \mathcal{W} \cap \mathcal{R})$ are also hereditary. Setting $\mathcal{M}_1 = (\mathcal{W}_1, \mathcal{F}_1) = (\mathcal{Q} \cap \mathcal{W}, \mathcal{R})$ and $\mathcal{M}_2 = (\mathcal{W}_2, \mathcal{F}_2) = (\mathcal{Q}, \mathcal{W} \cap \mathcal{R})$ we get that $\mathcal{M}_1$ and $\mathcal{M}_2$ are each injective cotorsion pairs by condition (2) of Proposition 3.6.

It now follows that $\mathcal{M} = \mathcal{M}_1/\mathcal{M}_2$ by the uniqueness property just mentioned above.

Example 3.12. It is easy to see that $\mathcal{M} = (\mathcal{W}, \mathcal{F})$ is an injective model structure if and only if it is the right localization of itself by the trivial model structure induced by the categorical injective cotorsion pair $(\mathcal{A}, \mathcal{I})$.

Example 3.13. Let $\mathcal{A} = \text{Ch}(R)$. Let $\mathcal{M}_1 = (\mathcal{W}_1, \mathcal{F}_1)$ be the Inj model structure from [BCHL13] which has as the fibrant complexes $\mathcal{F}_1 = du\mathcal{I}$, the class of all complexes of injective $R$-modules. Also let $\mathcal{M}_2 = (\mathcal{W}_2, \mathcal{F}_2)$ be the exact injective model structure from [BCHL13] which has as the fibrant complexes $\mathcal{F}_2 = ex\mathcal{I}$, the class of all exact complexes of injective $R$-modules. Denote the class of exact complexes by $\mathcal{E}$. Then it follows from Theorem 4.7 of [Gil08] that there is a hereditary weak injective Hovey triple $\mathcal{M} = (\mathcal{W}_2, \mathcal{E}, \mathcal{F}_1)$. It follows that $\mathcal{M} = \mathcal{M}_1/\mathcal{M}_2$. Its homotopy category is $\mathcal{D}(R)$ because the trivial complexes are precisely the exact complexes (and so it follows from Lemma 2.7 that the homology isomorphisms are the weak equivalences). This example is relevant to Becker’s approach to recovering Krause’s recollement. We see Gorenstein injective analogs in Section 8.

Remark 5. We have worked with injective cotorsion pairs and their right localizations in this section. We note that the dual statements concerning left localizations of projective cotorsion pairs also hold. We have omitted the projective statements in this section.

4. LOCALIZATION SEQUENCES AND RECOLLEMENTS FROM COTORSION PAIRS

The goal here is to describe how recollements are related to injective cotorsion pairs. This is Theorem 4.6 and its converse in Corollary 4.16. Since injective
cotorsion pairs are particular types of model structures the functors involved in the recollements are actually derived functors between simple Quillen adjunctions as we will describe. The method is a generalization of Becker’s approach from [Bec12] where he obtained Krause’s recollement from [Kra05], for a general ring $R$, using the theory of model categories.

Recall that the homotopy category of an abelian model category is always a triangulated category and has a set of weak generators whenever the model structure is cofibrantly generated. See Hovey [Hov02, Section 7] for more details. We start with the definition of a recollement. Loosely, a recollement is an “attachment” of two triangulated categories. The standard reference is [BBD82]. We give the definition that appeared in [Kra05] based on localization and colocalization sequences.

**Definition 4.1.** Let $\mathcal{T}' \xrightarrow{F} \mathcal{T} \xrightarrow{G} \mathcal{T}''$ be a sequence of exact functors between triangulated categories. We say it is a localization sequence when there exists right adjoints $F_{\rho}$ and $G_{\rho}$ giving a diagram of functors as below with the listed properties.

\[ \mathcal{T}' \xrightarrow{F} \mathcal{T} \xrightarrow{G} \mathcal{T}'' \]

1. The right adjoint $F_{\rho}$ of $F$ satisfies $F_{\rho} \circ F \cong \text{id}_{\mathcal{T}'}$.
2. The right adjoint $G_{\rho}$ of $G$ satisfies $G \circ G_{\rho} \cong \text{id}_{\mathcal{T}''}$.
3. For any object $X \in \mathcal{T}$, we have $GX = 0$ iff $X \cong FX'$ for some $X' \in \mathcal{T}'$.

A colocalization sequence is the dual. That is, there must exist left adjoints $F_{\lambda}$ and $G_{\lambda}$ with the analogous properties.

It is fair to say that a localization sequence is a sequence of right adjoints which in some sense “splits” at the level of triangulated categories. See [Kra05, Section 3] for the first properties of localization sequences which reflect this statement. Similarly, a colocalization sequence is a sequence of right adjoints with this property. It is true that if $\mathcal{T}' \xrightarrow{F} \mathcal{T} \xrightarrow{G} \mathcal{T}''$ is a localization sequence then $\mathcal{T}'' \xrightarrow{G_{\lambda}} \mathcal{T} \xrightarrow{F_{\rho}} \mathcal{T}'$ is a colocalization sequence and if $\mathcal{T}' \xrightarrow{F} \mathcal{T} \xrightarrow{G} \mathcal{T}''$ is a colocalization sequence then $\mathcal{T}'' \xrightarrow{G_{\lambda}} \mathcal{T} \xrightarrow{F_{\rho}} \mathcal{T}'$ is a localization sequence. This brings us to the definition of a recollement where the sequence of functors $\mathcal{T}' \xrightarrow{F} \mathcal{T} \xrightarrow{G} \mathcal{T}''$ is both a localization sequence and a colocalization sequence.

**Definition 4.2.** Let $\mathcal{T}' \xrightarrow{F} \mathcal{T} \xrightarrow{G} \mathcal{T}''$ be a sequence of exact functors between triangulated categories. We say $\mathcal{T}' \xrightarrow{F} \mathcal{T} \xrightarrow{G} \mathcal{T}''$ induces a recollement if it is both a localization sequence and a colocalization sequence as shown in the picture.

\[ \mathcal{T}' \xrightarrow{F} \mathcal{T} \xrightarrow{G} \mathcal{T}'' \]

So the idea will be to “glue” a colocalization sequence to a localization sequence to get the diagram of functors in the recollement diagram.
4.1. Colocalization sequences from weak injective model structures. Let’s look again at the setup to Proposition 3.11. We are working in an abelian category \( \mathcal{A} \) with enough injectives. We have two injective cotorsion pairs \( \mathcal{M}_1 = (\mathcal{W}_1, \mathcal{F}_1) \) and \( \mathcal{M}_2 = (\mathcal{W}_2, \mathcal{F}_2) \) with \( \mathcal{F}_2 \subseteq \mathcal{F}_1 \). Since the identity functor is exact it follows immediately that \( \mathcal{M}_2 \xrightarrow{id} \mathcal{M}_1 \xrightarrow{id} \mathcal{M}_1/\mathcal{M}_2 \) are right Quillen functors. That is, we have Quillen adjunctions \( \mathcal{M}_1/\mathcal{M}_2 \rightleftarrows \mathcal{M}_1 \rightleftarrows \mathcal{M}_2 \) consisting entirely of identity functors. The following comes from [Bec12, Corollary 1.4.5].

**Proposition 4.3.** Let \( \mathcal{A} \) be an abelian category with enough injectives and let \( \mathcal{M}_1 = (\mathcal{W}_1, \mathcal{F}_1) \) and \( \mathcal{M}_2 = (\mathcal{W}_2, \mathcal{F}_2) \) be two injective cotorsion pairs on \( \mathcal{A} \) with \( \mathcal{F}_2 \subseteq \mathcal{F}_1 \). Then the identity Quillen adjunctions \( \mathcal{M}_1/\mathcal{M}_2 \rightleftarrows \mathcal{M}_1 \rightleftarrows \mathcal{M}_2 \) descend to a colocalization sequence \( \text{Ho}(\mathcal{M}_2) \xrightarrow{\text{Rid}} \text{Ho}(\mathcal{M}_1) \xrightarrow{\text{Rid}} \text{Ho}(\mathcal{M}_1/\mathcal{M}_2) \) with left adjoints \( \text{Lid} \). In particular, on the level of the full subcategory of cofibrant-fibrant subobjects we have the colocalization sequence:

\[
\frac{\mathcal{F}_2/ \sim \xrightarrow{E(M_2)} I \xrightarrow{F_1/ \sim \xrightarrow{C(M_2)}} (\mathcal{F}_1 \cap \mathcal{W}_2)/ \sim}
\]

Here the functors \( I \) are each inclusion, the functor \( E(M_2) \) represents using enough injectives of the cotorsion pair \( M_2 = (\mathcal{W}_2, \mathcal{F}_2) \) (to get what would often be called a special \( \mathcal{F}_2 \)-reenvelope), and the functor \( C(M_2) \) represents using enough projectives of the cotorsion pair \( M_2 = (\mathcal{W}_2, \mathcal{F}_2) \) (to get what would often be called a special-\( \mathcal{W}_2 \) precover).

**Proof.** See [Bec12, Corollary 1.4.5] to see that we have the colocalization sequence \( \text{Ho}(\mathcal{M}_2) \xrightarrow{\text{Rid}} \text{Ho}(\mathcal{M}_1) \xrightarrow{\text{Rid}} \text{Ho}(\mathcal{M}_1/\mathcal{M}_2) \). In general, the right derived functor is defined on objects by first taking a fibrant replacement and then applying the functor, here the identity. Similarly, the left derived functor is defined by first taking a cofibrant replacement and then applying the functor. In any abelian model category \( \mathcal{M} = (\mathcal{Q}, \mathcal{W}, \mathcal{R}) \), taking cofibrant replacements corresponds to using enough projectives of the cotorsion pair \( (\mathcal{Q}, \mathcal{W} \cap \mathcal{R}) \) and taking fibrant replacements corresponds to using enough injectives of the cotorsion pair \( (\mathcal{Q} \cap \mathcal{W}, \mathcal{R}) \). Also as in Propositions 3.9 and 3.10 one uses cofibrant-fibrant replacement and inclusion when translating between the homotopy categories of \( \mathcal{M} \) and the full subcategory of cofibrant-fibrant objects. We conclude that the functors work as stated. \( \Box \)

4.2. Finding a recollement from three cotorsion pairs. With the same setup as in Subsection 4.1 now suppose we have three injective model structures \( \mathcal{M}_1 = (\mathcal{W}_1, \mathcal{F}_1), \mathcal{M}_2 = (\mathcal{W}_2, \mathcal{F}_2), \) and \( \mathcal{M}_3 = (\mathcal{W}_3, \mathcal{F}_3) \) having \( \mathcal{F}_2, \mathcal{F}_3 \subseteq \mathcal{F}_1 \). Then for \( i = 1, 2 \) we have from Proposition 4.3 that the identity functors give Quillen adjunctions \( \mathcal{M}_1/\mathcal{M}_i \rightleftarrows \mathcal{M}_i \rightleftarrows \mathcal{M}_i \) descending to give two colocalization sequences

\[
(*) \quad \text{Ho}(\mathcal{M}_2) \xrightarrow{\text{Rid}} \text{Ho}(\mathcal{M}_1) \xrightarrow{\text{Rid}} \text{Ho}(\mathcal{M}_1/\mathcal{M}_2)
\]

and

\[
(**) \quad \text{Ho}(\mathcal{M}_3) \xrightarrow{\text{Rid}} \text{Ho}(\mathcal{M}_1) \xrightarrow{\text{Rid}} \text{Ho}(\mathcal{M}_1/\mathcal{M}_3).
\]

However, we also have the following lemma.

**Lemma 4.4.** The identity adjunctions shown below are in fact Quillen adjunction:

\[
\mathcal{M}_3 \rightleftarrows \mathcal{M}_3, \quad \mathcal{M}_1/\mathcal{M}_2 \rightleftarrows \mathcal{M}_3, \quad \mathcal{M}_1/\mathcal{M}_3 \rightleftarrows \mathcal{M}_2
\]
Proof. The first is obvious and the second two are symmetric, so we show that \( \mathcal{M}_1/\mathcal{M}_2 \rightleftarrows \mathcal{M}_3 \) is a Quillen adjunction. That is, the identity from \( \mathcal{M}_3 \) to \( \mathcal{M}_1/\mathcal{M}_2 \) is right Quillen. For this just recall \( \mathcal{M}_1/\mathcal{M}_2 = (\mathcal{W}_2, \mathcal{W}, \mathcal{F}_1) \). Since the identity functor is exact it just boils down to noting that we are given \( \mathcal{F}_3 \subseteq \mathcal{F}_1 \) and that the injective objects are contained in \( \mathcal{W} \cap \mathcal{F}_1 = \mathcal{F}_2 \).

So reversing the direction of (**) above, and using the above lemma we get the following diagram of functors on the level of homotopy categories.

\[
\begin{array}{cccc}
\text{Ho}(\mathcal{M}_2) & \xrightarrow{\text{L id}} & \text{Ho}(\mathcal{M}_1) & \xleftarrow{\text{L id}} & \text{Ho}(\mathcal{M}_1/\mathcal{M}_2) \\
\text{R id} & & \text{R id} & & \text{R id} \\
\text{L id} & & \text{L id} & & \text{L id} \\
\text{R id} & & \text{R id} & & \text{R id} \\
\text{L id} & & \text{L id} & & \text{L id} \\
\text{R id} & & \text{R id} & & \text{R id} \\
\text{Ho}(\mathcal{M}_1/\mathcal{M}_3) & \xrightarrow{\text{L id}} & \text{Ho}(\mathcal{M}_1) & \xleftarrow{\text{L id}} & \text{Ho}(\mathcal{M}_3) \\
\end{array}
\]

(***)

This is a general form of Becker’s butterfly diagram \cite{Bec12, page 28}. We now make a few notes about the diagram before proceeding.

- We write all left adjoints on the top or on the left and we write all right adjoints on the bottom or right.
- The top row is a colocalization sequence.
- The bottom row is a localization sequence.
- The butterfly is pictured by identifying the two occurrences of \( \text{Ho}(\mathcal{M}_1) \) along the the middle vertical arrows. The middle vertical arrows are not literally the identity but these functors are canonical equivalences of \( \text{Ho}(\mathcal{M}_1) \).
- It is the identity however if you restrict to the full subcategory of cofibrant-fibrant objects.

Note that the condition \( \mathcal{F}_2, \mathcal{F}_3 \subseteq \mathcal{F}_1 \) is all that is required for the butterfly diagram setup to exist. We now give our main construction theorem, Theorem 4.6, giving simple criteria for an induced recollement situation. The proof will make use of the following proposition which relies on the uniqueness condition of Proposition 3.11. It can be useful on its own for spotting localizations of two injective model structures.

**Proposition 4.5.** Let \( \mathcal{W} \) be a thick class in \( \mathcal{A} \). Suppose \( \mathcal{M}_1 = (\mathcal{W}_1, \mathcal{F}_1) \) and \( \mathcal{M}_2 = (\mathcal{W}_2, \mathcal{F}_2) \) are injective model structures.

1. If \( \mathcal{W} \cap \mathcal{F}_1 = \mathcal{F}_2 \) and \( \mathcal{W}_1 \subseteq \mathcal{W} \). Then \( \mathcal{M}_1/\mathcal{M}_2 = (\mathcal{W}_2, \mathcal{W}, \mathcal{F}_1) \).
2. If \( \mathcal{W}_2 \cap \mathcal{W} = \mathcal{W}_1 \) and \( \mathcal{F}_2 \subseteq \mathcal{W} \). Then \( \mathcal{M}_1/\mathcal{M}_2 = (\mathcal{W}_2, \mathcal{W}, \mathcal{F}_1) \).

**Proof.** First, note that \( \mathcal{F}_2 \subseteq \mathcal{F}_1 \) by the given. So taking left perps we get \( \mathcal{W}_1 \subseteq \mathcal{W}_2 \). Also we are given that \( \mathcal{W}_1 \subseteq \mathcal{W} \), and so \( \mathcal{W}_1 \subseteq \mathcal{W}_2 \cap \mathcal{W} \) is automatic.

On the other hand, say \( X \in \mathcal{W}_2 \cap \mathcal{W} \). Since \( (\mathcal{W}_1, \mathcal{F}_1) \) is complete we can find a short exact sequence \( 0 \rightarrow \mathcal{B} \rightarrow \mathcal{A} \rightarrow \mathcal{X} \rightarrow 0 \) where \( \mathcal{B} \in \mathcal{F}_1 \) and \( \mathcal{A} \in \mathcal{W}_1 \). Since \( \mathcal{A} \)
and $X$ are in the thick class $W$ we get that $B$ is too. So $B \in \mathcal{F}_1 \cap W = \mathcal{F}_2$. Since $(W_2, \mathcal{F}_2)$ is a cotorsion pair the sequence $0 \to B \to A \to X \to 0$ must split, making $X$ a direct summand of $A$. But then $X$ must belong to $\mathcal{W}_1$ since the left side of a cotorsion pair is always closed under retracts. This shows $W_1 = W_2 \cap W$ and it follows that $(W_2, W, \mathcal{F}_1)$ is an injective Hovey triple and proves (1) because of the uniqueness part of Proposition 3.11. The proof of (2) is similar. The assumptions imply $\mathcal{F}_2 \subseteq W \cap \mathcal{F}_1$. On the other hand, for $X \in \mathcal{F}_1 \cap W$ use completeness of $(W_2, \mathcal{F}_2)$ to find a s.e.s. $0 \to X \to \mathcal{F}_2 \to W_2 \to 0$. But given that $\mathcal{F}_2 \in W$, thickness of $W$ implies $W_2 \in W \cap W_2 = W_1$. So the sequence splits.

\[ \square \]

**Theorem 4.6.** Let $\mathcal{A}$ be an abelian category with enough injectives and suppose we have three injective cotorsion pairs

\[ \mathcal{M}_1 = (\mathcal{W}_1, \mathcal{F}_1), \quad \mathcal{M}_2 = (\mathcal{W}_2, \mathcal{F}_2), \quad \mathcal{M}_3 = (\mathcal{W}_3, \mathcal{F}_3) \]

such that $\mathcal{F}_2, \mathcal{F}_3 \subseteq \mathcal{F}_1$. Then if $\mathcal{W}_3 \cap \mathcal{F}_1 = \mathcal{F}_2$ (or equivalently, $\mathcal{W}_2 \cap \mathcal{W}_3 = \mathcal{W}_1$ and $\mathcal{F}_2 \subseteq \mathcal{W}_3$), then $\mathcal{M}_1/\mathcal{M}_2$ is Quillen equivalent to $\mathcal{M}_3$ and $\mathcal{M}_1/\mathcal{M}_3$ is Quillen equivalent to $\mathcal{M}_2$. In fact, there is a recollement

\[ \begin{array}{ccc}
E(\mathcal{W}_2, \mathcal{F}_2) & \hookrightarrow & \mathcal{F}_2/ \sim \\
\longrightarrow & I & \longrightarrow \\
C(\mathcal{W}_3, \mathcal{F}_3) & \longrightarrow & \mathcal{F}_1/ \sim \\
\longrightarrow & & E(\mathcal{W}_3, \mathcal{F}_1) \\
& I & \longrightarrow \\
& \longrightarrow & \mathcal{F}_3/ \sim 
\end{array} \]

Here, the notation such as $E(\mathcal{W}_3, \mathcal{F}_3)$ means to take a special $\mathcal{F}_3$-preenvelope by using enough injectives of the cotorsion pair $(\mathcal{W}_3, \mathcal{F}_3)$. Similarly the notation $C(\mathcal{W}_3, \mathcal{F}_3)$ means to take a special $\mathcal{W}_3$-precover. Moreover the left adjoint $\lambda$ has essential image $(\mathcal{W}_2 \cap \mathcal{F}_1)/ \sim$ yielding an equivalence $(\mathcal{W}_2 \cap \mathcal{F}_1)/ \sim \cong (\mathcal{F}_3)/ \sim$.

**Proof.** First we show that the two conditions are equivalent. That is, $\mathcal{W}_3 \cap \mathcal{F}_1 = \mathcal{F}_2$ if and only if $\mathcal{W}_2 \cap \mathcal{W}_3 = \mathcal{W}_1$ and $\mathcal{F}_2 \subseteq \mathcal{W}_3$. For the “only if” part, the only part that is not clear is $\mathcal{W}_2 \cap \mathcal{W}_3 \subseteq \mathcal{W}_1$. So assume $W \in \mathcal{W}_2 \cap \mathcal{W}_3$. Use enough projectives of $(\mathcal{W}_1, \mathcal{F}_1)$ to find a short exact sequence $0 \to F_1 \to W_1 \to W \to 0$ with $F_1 \in \mathcal{F}_1$, and $W_1 \in \mathcal{W}_1 \subseteq \mathcal{W}_3$. Since $\mathcal{W}_3$ is thick we see that $F_1 \in \mathcal{W}_3 \cap \mathcal{F}_1 = \mathcal{F}_2$. So with $F_1 \in \mathcal{F}_2$ and $W \in \mathcal{W}_2$, the short exact sequence must split, making $W$ a retract of $W_1$. Hence $W$ must be in $\mathcal{W}_1$. For the “if” part, the analogous part to show is $\mathcal{W}_3 \cap \mathcal{F}_1 \subseteq \mathcal{F}_2$. This follows by a similar argument which starts by finding a short exact sequence $0 \to W \to \mathcal{F}_2 \to W_2 \to 0$ with $\mathcal{F}_2 \in \mathcal{F}_2$, and $W_2 \in \mathcal{W}_2$.

Having shown the two conditions are equivalent we point out that if either holds, Proposition 4.5 applies (with $\mathcal{W}_3$ in place of $W$) to conclude $\mathcal{M}_1/\mathcal{M}_2 = (\mathcal{W}_2, \mathcal{W}_3, \mathcal{F}_1)$. We now consider again the butterfly diagram $(\star \star \star)$ above consisting of all adjoint pairs.

First, note that $\mathcal{M}_1/\mathcal{M}_2$ and $\mathcal{M}_3$ have the same trivial objects. It follows then from Lemma 2.7 that the weak equivalences in $\mathcal{M}_1/\mathcal{M}_2$ coincide with those in $\mathcal{M}_3$. So by definition, the identity adjunction $\mathcal{M}_1/\mathcal{M}_2 \cong \mathcal{M}_3$ is a Quillen equivalence. This means the vertical maps on the far right of $(\star \star \star)$ are equivalences. As we noted above the middle vertical maps in $(\star \star \star)$ are already canonical equivalences.

Second, note that the full subcategory of $\mathcal{M}_1/\mathcal{M}_3 = (\mathcal{W}_3, \mathcal{V}, \mathcal{F}_1)$ consisting of the cofibrant-fibrant subobjects is $\mathcal{W}_3 \cap \mathcal{F}_1 = \mathcal{F}_2$. So we have $\text{Ho}(\mathcal{M}_1/\mathcal{M}_3) \cong \mathcal{M}_1/\mathcal{M}_2$. Therefore, $\mathcal{M}_2$ is Quillen equivalent to $\mathcal{M}_3$.
(\mathcal{W}_3 \cap \mathcal{F}_1)/ \sim = \mathcal{F}_2/ \sim$ through the canonical equivalence. Thus on the level of the homotopy category associated to the full subcategory of cofibrant-fibrant subobjects, the diagram $(\ast \ast \ast)$ becomes the following:

\[
\begin{array}{ccc}
\mathcal{F}_2/ \sim & \xrightarrow{E(\mathcal{W}_2, \mathcal{F}_2)} & \mathcal{F}_3/ \sim \\
& \xrightarrow{\text{Inclusion}} & \text{C}(\mathcal{W}_2, \mathcal{F}_2) \\
\downarrow \text{id} & \downarrow \text{id} & \downarrow \text{id} \\
\mathcal{F}_2/ \sim & \xrightarrow{\text{Inclusion}} & \mathcal{F}_3/ \sim \\
& \xrightarrow{E(\mathcal{W}_3, \mathcal{F}_3)} & \text{C}(\mathcal{W}_3, \mathcal{F}_3) \\
\end{array}
\]

In particular, the left hand square perfectly “glues” together along the inclusion $I: \mathcal{F}_2/ \sim \to \mathcal{F}_1/ \sim$ showing $I$ also has a left adjoint. It follows formally from [Kra06, Proposition 4.13.1] that the localization sequence which makes up the bottom row induces the recollement. However, we claim $\lambda = \text{C}(\mathcal{W}_2, \mathcal{F}_2)$. To see this, we point out that while the right square doesn’t commute on the nose, it is actually “glued” together by an isomorphism of functors. Indeed one can argue that there is a natural isomorphism from the composite functor $\mathcal{F}_1/ \sim \xrightarrow{E(\mathcal{W}_3, \mathcal{F}_3)/ \sim \text{C}(\mathcal{W}_3, \mathcal{F}_3)} \mathcal{F}_3/ \sim$ to the functor $\mathcal{F}_1/ \sim \xrightarrow{\text{C}(\mathcal{W}_2, \mathcal{F}_2)/ \sim \text{C}(\mathcal{W}_2, \mathcal{F}_2)} (\mathcal{W}_2 \cap \mathcal{F}_1)/ \sim$. From this it follows rather easily that the composite $\mathcal{F}_3/ \sim \xrightarrow{E(\mathcal{W}_3, \mathcal{F}_3)/ \sim \text{C}(\mathcal{W}_3, \mathcal{F}_3)} \mathcal{F}_3/ \sim$ is indeed the left adjoint to $\mathcal{F}_1/ \sim \xrightarrow{E(\mathcal{W}_3, \mathcal{F}_3)/ \sim \text{C}(\mathcal{W}_3, \mathcal{F}_3)} \mathcal{F}_3/ \sim$. That is, $\lambda = \text{C}(\mathcal{W}_2, \mathcal{F}_2)$. \hfill \Box

\textbf{Remark 6.} In the proof of Theorem 4.6 above, we described how to “glue” together the right square to get the recollement. But there is an alternate yet equally valid way to glue together the right square. Indeed there is also a natural isomorphism from the composite functor $\mathcal{F}_1/ \sim \xrightarrow{\text{C}(\mathcal{W}_2, \mathcal{F}_2)/ \sim \text{C}(\mathcal{W}_2, \mathcal{F}_2)} \mathcal{F}_3/ \sim$ to the functor $\mathcal{F}_1/ \sim \xrightarrow{E(\mathcal{W}_3, \mathcal{F}_3)/ \sim \text{C}(\mathcal{W}_3, \mathcal{F}_3)} \mathcal{F}_3/ \sim$. From this we deduce that the composite $(\mathcal{W}_2 \cap \mathcal{F}_1)/ \sim \xrightarrow{E(\mathcal{W}_3, \mathcal{F}_3)/ \sim \text{C}(\mathcal{W}_3, \mathcal{F}_3)} \mathcal{F}_3/ \sim \xrightarrow{\text{C}(\mathcal{W}_2, \mathcal{F}_2)/ \sim \text{C}(\mathcal{W}_2, \mathcal{F}_2)} (\mathcal{W}_2 \cap \mathcal{F}_1)/ \sim$. That is, we can restate Theorem 4.6 with a new recollement diagram emphasizing the quotient functor $\mathcal{F}_1/ \sim \xrightarrow{\text{C}(\mathcal{W}_2, \mathcal{F}_2)/ \sim \text{C}(\mathcal{W}_2, \mathcal{F}_2)} (\mathcal{W}_2 \cap \mathcal{F}_1)/ \sim$ with the inclusion as its left adjoint and with $\rho = \text{E}(\mathcal{W}_3, \mathcal{F}_3)$ as its right adjoint having essential image $\mathcal{F}_3/ \sim$. We will display all recollement diagrams in this paper however in the style appearing in the statement of Theorem 4.6.

\textbf{Theorem 4.7.} Let $\mathcal{A}$ be an abelian category with enough projectives and suppose we have three projective cotorsion pairs

$$\mathcal{M}_1 = (\mathcal{C}_1, \mathcal{W}_1), \quad \mathcal{M}_2 = (\mathcal{C}_2, \mathcal{W}_2), \quad \mathcal{M}_3 = (\mathcal{C}_3, \mathcal{W}_3)$$

such that $\mathcal{C}_2, \mathcal{C}_3 \subseteq \mathcal{C}_1$. Then if $\mathcal{W}_3 \cap \mathcal{C}_1 = \mathcal{C}_2$ (or equivalently, $\mathcal{W}_2 \cap \mathcal{W}_1 = \mathcal{W}_1$ and $\mathcal{C}_2 \subseteq \mathcal{W}_3$), then the left localization $\mathcal{M}_2 \backslash \mathcal{M}_1$ is Quillen equivalent to $\mathcal{M}_3$ and
$\mathcal{M}_3/\mathcal{M}_1$ is Quillen equivalent to $\mathcal{M}_2$. In fact, there is a recollement

$$
\begin{array}{ccc}
\mathcal{C}_2/\sim & \xrightarrow{E(C_3,W_3)} & \mathcal{C}_1/\sim \\
\xleftarrow{I} & & \xleftarrow{I} \\
\mathcal{C}(C_2,W_2) & & \mathcal{C}(C_3,W_3)
\end{array}
$$

Here, the notation such as $E(C_3,W_3)$ means to take a special $W_3$-preenvelope by using enough injectives of the cotorsion pair $(C_3,W_3)$. Similarly the notation $C(C_3,W_3)$ means to take a special $C_3$-precover. Moreover the right adjoint $\rho$ has essential image $(C_1 \cap W_2)/\sim$ yielding an equivalence $(C_1 \cap W_2)/\sim \cong C_3/\sim$.

**Example 4.8.** In terms of Theorem 4.6, Becker’s method of recovering Krause’s recollement boils down to the following observations. As in Example 3.13, let $dw\tilde{I}$ be the class of all complexes of injective $R$-modules, $ex\tilde{I}$ be the class of all exact complexes of injective $R$-modules and $dg\tilde{I}$ be the class of all DG-injective complexes. Then we have the following injective cotorsion pairs:

$\mathcal{M}_1 = (W_1, dw\tilde{I}), \quad \mathcal{M}_2 = (W_2, ex\tilde{I}), \quad \mathcal{M}_3 = (E, dg\tilde{I})$.

It is easy to see that $\mathcal{F}_2, \mathcal{F}_3 \subseteq \mathcal{F}_1$ and $\mathcal{E} \cap dw\tilde{I} = ex\tilde{I}$. So Theorem 4.6 gives a recollement which is Krause’s recollement from [Kra05]. See also the introduction to Section 8.

In an earlier version of this paper the author claimed without proof that this example extended to recover Krause’s recollement for quasi-coherent sheaves over any scheme $X$. This is false and a counterexample has been constructed and communicated to the author by Amnon Neeman.

4.3. **A converse: Lifting a recollement to the level of model categories.** Let $\mathcal{M} = (W,F)$ be a fixed injective cotorsion pair in an abelian category $A$ with enough injectives. We have just shown above that if $(W',F')$ and $(W'',F'')$ are also injective cotorsion pairs and if $F'' \subseteq F$ and $W'' \cap F = F'$, then there is a recollement $F'/\sim \to F/\sim \to F''/\sim$. The author was very pleased to learn from an anonymous referee that there is a converse of sorts. That is, all recollements with $F'/\sim$ in the middle come from injective cotorsion pairs in the way described in Theorem 4.6. In particular, this shows that all such recollements arise from Quillen functors between model structures on $A$. The same is true for just localization or colocalization sequences. Since we are now going in a new direction - building model structures on $A$ starting with thick subcategories of $\text{Ho}(A) = F/\sim$ we necessarily need different tools and theory in this section. In particular, we will apply the theory of torsion pairs and torsion triples in triangulated categories and we will relate these notions to cotorsion pairs (model structures). We refer the reader to [BR07] as a reference for torsion pairs and torsion triples in triangulated categories. We will only formulate the injective versions of our results here but of course there are projective versions as well.

For the remainder of this section, we let $\mathcal{M} = (W,F)$ be a fixed injective cotorsion pair in an abelian category $A$ with enough injectives. We point out again (see Proposition 3.9) that the full subcategory $F$ of fibrant objects naturally inherits the structure of a Frobenius category with its projective-injective objects being...
Lemma 4.10. All told we get the following well-known Lemma which we won’t prove in more.

Remark 7. Let \((\mathcal{W}, \mathcal{F})\) be another injective cotorsion pair with \(\mathcal{F}' \subseteq \mathcal{F}\). Then \(\mathcal{F}'\) is \(\mathcal{F}\)-thick and contains the injectives. The key point to check is that for a given short exact sequence \(0 \to F \to F' \to F'' \to 0\) with \(F \in \mathcal{F}\) and \(F', F'' \in \mathcal{F}'\) we indeed have \(F \in \mathcal{F}'\) too. To see this, use that \((\mathcal{W}', \mathcal{F}')\) is an injective cotorsion pair to find another short exact sequence \(0 \to F'' \to I \to F''' \to 0\) with \(F''' \in \mathcal{F}'\) and \(I \in \mathcal{W}' \cap \mathcal{F}'\) necessarily injective. Taking the pullback \(P\) of \(F''\) by \(F'''\) yields another short exact sequence \(0 \to F \to P \to I \to 0\) with \(P \in \mathcal{F}'\) since it is an extension of \(F'\) by \(F'''\). But this last s.e.s must split since \((\mathcal{W}, \mathcal{F})\) is an injective cotorsion pair. Since \(\mathcal{F}'\) is closed under retracts we get \(F \in \mathcal{F}'\).
Set-Up 4.11. We now fix a standard set-up along with notation that will be used throughout this section to prove Proposition 4.15. First, we have a fixed injective cotorsion pair \((W, \mathcal{F})\) for which \(\mathcal{F}/\sim\) serves as our ambient triangulated category. By Lemma 4.10 we speak interchangeably of \(\mathcal{F}\)-thick subcategories \(\mathcal{T} \subseteq \mathcal{F}\) which contain the injectives and thick subcategories \(\mathcal{T}/\sim\) of \(\mathcal{F}/\sim\). We assume we are given such a class \(\mathcal{T}\) and we define the following class in \(\mathcal{A}\):

\[ W_{\mathcal{T}} = \{ A \in \mathcal{A} \mid \text{there exists a s.e.s. } 0 \to A \to T \to W \to 0 \text{ with } T \in \mathcal{T}, W \in W \} \]

**Lemma 4.12.** The following hold for any \(W_{\mathcal{T}}\) class.

1. \(W_{\mathcal{T}} = R^{-1}(\mathcal{T}/\sim)\). That is, \(W_{\mathcal{T}}\) consists precisely of the objects of \(\mathcal{A}\) whose fibrant replacement in \(\mathcal{M} = (W, \mathcal{F})\) lies in \(\mathcal{T}\).
2. \(W_{\mathcal{T}}\) is a thick subcategory of \(\mathcal{A}\) and contains both \(\mathcal{T}\) and \(W\).
3. \(W_{\mathcal{T}} \cap \mathcal{F} = \mathcal{T}\)

**Proof.** For (1), recall that a fibrant replacement of \(A\) is obtained by using enough injectives in the cotorsion pair \((W, \mathcal{F})\) to get a s.e.s. \(0 \to A \to F \to W \to 0\) with \(F \in \mathcal{F}\) and \(W \in W\). If \(A \in W_{\mathcal{T}}\), then there is another s.e.s. \(0 \to A \to T \to W' \to 0\) with \(T \in \mathcal{T}\) and \(W' \in W\). One can argue that \(F\) is isomorphic to \(T\) in the stable category \(\mathcal{F}/\sim\), and from this it follows that there are injectives \(I\) and \(J\) such that \(F \oplus I \cong \mathcal{T} \oplus J\) in \(\mathcal{F}\). Since \(\mathcal{T}\) is thick and contains the injectives we get that \(F \in \mathcal{T}\) too.

(2) The proof of the thickness of \(W_{\mathcal{T}}\) follows just like the proof of the thickness of the class \(W\) given in [Bec12 Proposition 1.4.2]. Clearly \(\mathcal{T} \subseteq W_{\mathcal{T}}\) because for any given \(T \in \mathcal{T}\), there is the s.e.s. \(0 \to T \to 0 \to T \to 0\). Also, for any \(W \in W\), taking a fibrant replacement \(0 \to W \to F \to W' \to 0\) will have \(F \in \mathcal{F} \cap W\). So \(F\) is injective and hence in \(\mathcal{T}\) since we are assuming \(\mathcal{T}\) contains the injectives. This shows \(W \subseteq W_{\mathcal{T}}\).

(3) We already know \(\mathcal{T}\) is contained in both \(\mathcal{F}\) and \(W_{\mathcal{T}}\). On the other hand, if \(A \in W_{\mathcal{T}} \cap \mathcal{F}\) then there is a s.e.s. \(0 \to A \to T \to W \to 0\) with \(T \in \mathcal{T}\) and \(W \in W\). But with \(A\) being in \(\mathcal{F}\) we know that this sequence must split, making \(A\) a retract of \(T\). Hence \(A \in \mathcal{T}\) since \(\mathcal{T}\) is thick.

□

**Lemma 4.13.** Continuing with the notation in Set-Up 4.11 there is a bijection between the following two classes:

\[ \{ \text{\(\mathcal{F}\)-thick subcategories of } \mathcal{F} \text{ that contain the injectives} \} \leftrightarrow \{ \text{thick subcategories of } \mathcal{A} \text{ that contain } W \} \]

which acts by \(\mathcal{T} \to W_{\mathcal{T}}\) and has inverse \(W' \cap \mathcal{F} \leftrightarrow W'\).

**Proof.** The mapping \(\mathcal{T} \to W_{\mathcal{T}}\) is well-defined by Lemma 4.12. The alleged inverse mapping \(W' \cap \mathcal{F} \leftrightarrow W'\) is also well-defined, for if \(W'\) is a thick subcategory of \(\mathcal{A}\) containing \(W\), then \(W' \cap \mathcal{F}\) is clearly an \(\mathcal{F}\)-thick subcategory of \(\mathcal{F}\), and it contains the injectives since both \(\mathcal{F}\) and \(W\) do. To prove our claim that these are inverse mappings we must show \(W_{\mathcal{T}} \cap \mathcal{F} = \mathcal{T}\) and \(W_{W' \cap \mathcal{F}} = W'\). We already have \(W_{\mathcal{T}} \cap \mathcal{F} = \mathcal{T}\) from Lemma 4.12 so we focus on \(W_{W' \cap \mathcal{F}} = W'\).

\(W_{W' \cap \mathcal{F}} \supseteq W'\). Suppose \(W' \in W'\). We need to find a s.e.s. \(0 \to W' \to F \to W \to 0\) with \(F \in W' \cap \mathcal{F}\) and \(W \in W\). We simply apply enough injectives of the cotorsion pair \((W, \mathcal{F})\) to get such a s.e.s. \(0 \to W' \to F \to W \to 0\). Since \(W'\) is
thick and contains \( \mathcal{W} \) we conclude that \( F \in \mathcal{W}' \). So \( F \in \mathcal{W}' \cap \mathcal{F} \) and we have \( \mathcal{W}' = \mathcal{W}_{\mathcal{W}' \cap \mathcal{F}} \).

\( \mathcal{W}_{\mathcal{W}' \cap \mathcal{F}} \subseteq \mathcal{W}' \). Let \( A \in \mathcal{W}_{\mathcal{W}' \cap \mathcal{F}} \). Then there exists a s.e.s. \( 0 \to A \to \mathcal{W}' \to W \to 0 \) with \( \mathcal{W}' \in \mathcal{W}' \cap \mathcal{F} \) and \( W \in \mathcal{W} \). Since \( \mathcal{W}' \) is thick and contains \( \mathcal{W} \) we conclude that \( A \in \mathcal{W}' \).

**Notation.** Given a thick subcategory \( \mathcal{T}/\sim \) of \( \mathcal{F}/\sim \), then \( (\mathcal{T}/\sim)^\perp \) will denote the Hom-orthogonal in the triangulated category \( \mathcal{F}/\sim \). That is, \( (\mathcal{T}/\sim)^\perp = \{ F \in \mathcal{F}/\sim \mid \text{Hom}_{\mathcal{F}/\sim}(T, F) = 0 \text{ for all } T \in \mathcal{T}/\sim \} \). On the other hand, for a class \( \mathcal{W}' \) of objects in \( \mathcal{A} \), recall \( \mathcal{W}'^{\perp} \) denotes the Ext-orthogonal in \( \mathcal{A} \).

**Lemma 4.14.** Continuing with the notation in Set-Up 4.11 we have \( \mathcal{W}_T^+ \subseteq \mathcal{F} \) and \( (\mathcal{T}/\sim)^\perp = (\mathcal{W}_T^+)/\sim \).

**Proof.** Since \( \mathcal{W} \subseteq \mathcal{W}_T \) (Lemma 4.12) we get \( \mathcal{F} = \mathcal{W}_T^+ \supseteq \mathcal{W}_T^+ \).

We show \( (\mathcal{W}_T^+)/\sim \subseteq (\mathcal{T}/\sim)^\perp \). Let \( T \in \mathcal{T} \) and \( F \in \mathcal{W}_T^+ \). We wish to show \( \text{Hom}_{\mathcal{F}/\sim}(T, F) = 0 \). Since \( T, F \in \mathcal{F} \) and \( \mathcal{F} \) is Frobenius we have \( \text{Hom}_{\mathcal{F}/\sim}(T, F) = \text{Ext}_1^\mathcal{F}(\Sigma T, F) \). But \( \Sigma T \in \mathcal{T} \subseteq \mathcal{W}_T \), so \( \text{Ext}_1^\mathcal{F}(\Sigma T, F) = 0 \), proving \( \text{Hom}_{\mathcal{F}/\sim}(T, F) = 0 \).

Next we show \( (\mathcal{T}/\sim)^\perp \subseteq (\mathcal{W}_T^+)/\sim \). Let \( F \in (\mathcal{T}/\sim)^\perp \), so that \( \text{Hom}_{\mathcal{F}/\sim}(T, F) = 0 \) for all \( T \in \mathcal{T} \). Let \( A \in \mathcal{W}_T \) be given, so that it comes with a short exact sequence \( 0 \to A \to T \to W \to 0 \) with \( T \in \mathcal{T} \) and \( W \in \mathcal{W} \). We must show \( \text{Ext}_1^\mathcal{A}(A, F) = 0 \). We apply \( \text{Hom}_\mathcal{A}(\sim, F) \) to the s.e.s and get exactness of \( \text{Ext}_1^\mathcal{A}(T, F) \to \text{Ext}_1^\mathcal{A}(A, F) \to \text{Ext}_2^\mathcal{A}(W, F) \). But we have \( 0 = \text{Hom}_{\mathcal{F}/\sim}(\Sigma T, F) = \text{Ext}_1^\mathcal{A}(T, F) \), and also \( \text{Ext}_2^\mathcal{A}(W, F) = 0 \) since \( (\mathcal{W}, \mathcal{F}) \) is an hereditary cotorsion pair. It follows that \( \text{Ext}_1^\mathcal{A}(A, F) = 0 \).

We refer the reader to [BR07] for the definition of a torsion pair in a triangulated category.

**Proposition 4.15.** Continuing with the notation in Set-Up 4.11 the bijection \( \mathcal{T}/\sim \mapsto \mathcal{W}_T \) of Lemmas 4.10 and 4.13 restricts to a bijection between the following classes:

\[
\{ \mathcal{T} \mid (\mathcal{T}/\sim)^\perp \text{is a torsion pair in } \mathcal{F}/\sim \} \leftrightarrow \{ \mathcal{W}' \mid (\mathcal{W}', \mathcal{W}'^{\perp}) \text{is an injective cotorsion pair in } \mathcal{A} \text{ with } \mathcal{W}' \supseteq \mathcal{W} \}
\]

In particular, given \( \mathcal{T} \) with \( (\mathcal{T}/\sim)^\perp \) a torsion pair in \( \mathcal{F}/\sim \) we have that \( (\mathcal{W}_T, \mathcal{W}_T^+) \) is an injective cotorsion pair in \( \mathcal{A} \) with \( \mathcal{W}_T^+ \subseteq \mathcal{F} \) and \( (\mathcal{T}/\sim)^\perp = (\mathcal{W}_T^+)/\sim \). Conversely, given an injective cotorsion pair \( (\mathcal{W}', \mathcal{F}') \) in \( \mathcal{A} \) with \( \mathcal{F}' \subseteq \mathcal{F} \) we have that \( (\mathcal{W}' \cap \mathcal{F})/\sim, \mathcal{F}'/\sim \) is a torsion pair in \( \mathcal{F}/\sim \).

**Proof.** First suppose \( (\mathcal{T}/\sim)^\perp \) is a torsion pair in \( \mathcal{F}/\sim \). We wish to show \( (\mathcal{W}_T, \mathcal{W}_T^+) \) is a complete cotorsion pair in \( \mathcal{A} \). Let set \( \mathcal{F}' = \mathcal{W}_T^+ \). By Lemma 4.14 we have \( \mathcal{F}' \subseteq \mathcal{F} \) and \( (\mathcal{T}/\sim, \mathcal{F}'/\sim) \) is a torsion pair. We first show that this implies the following: For any \( F \in \mathcal{F}' \), there is a short exact sequence \( 0 \to F \to F'_1 \to T_1 \to 0 \) with \( F'_1 \in \mathcal{F}' \) and \( T_1 \in \mathcal{T} \) and another short exact sequence \( 0 \to F'_2 \to T_2 \to F \to 0 \) with \( F'_2 \in \mathcal{F}' \) and \( T_2 \in \mathcal{T} \). To find the first short exact sequence for a given \( F \), we use the definition of torsion pair to get an exact triangle \( T \to F \to \Sigma T \in \mathcal{F}/\sim \) with \( T \in \mathcal{T}/\sim \) and \( F \in \mathcal{F}/\sim \). Rotate the triangle to get an exact triangle \( F \xrightarrow{f} F' \to \Sigma T \to \Sigma F \), and note that \( \Sigma T \in \mathcal{T}/\sim \) since the class is thick.
Let $0 \to F \overset{i}{\to} I(F) \to \Sigma F \to 0$ be a short exact sequence with $I(F)$ injective and form the short exact sequence $0 \to F \overset{\alpha \equiv (i,-f)}{\to} I(F) \oplus F' \overset{\beta \equiv f'+i'}{\to} F'' \to 0$.

Here, $f'$ and $i'$ are the maps in the pushout square below and $F'' \in \mathcal{F}$ because $\mathcal{F}$ is closed under extensions. (See Proposition 2.12 of [B¨uh10] for more details on this construction. In the current case the cokernel $F''$ is automatically in $\mathcal{F}$ because the class is coresolving. But for constructing the dual short exact sequence $0 \to F'_2 \to T_2 \to F \to 0$ we need to use the dual of this construction.)

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & F & \overset{i}{\longrightarrow} & I(F) & \longrightarrow & \Sigma F & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F' & \overset{i'}{\longrightarrow} & F'' & \longrightarrow & \Sigma F & \longrightarrow & 0
\end{array}
\]

Now having in hand the short exact sequence $0 \to F \overset{\alpha \equiv (i,-f)}{\to} I(F) \oplus F' \overset{\beta \equiv f'+i'}{\to} F'' \to 0$ in $\mathcal{F}$, recall that by [Hap88, Lemma 2.7] there must be a map $\gamma : F'' \to \Sigma F$ in $\mathcal{F}$ giving an exact triangle $F \overset{\alpha}{\longrightarrow} I(F) \oplus F' \overset{\beta}{\longrightarrow} F'' \overset{\gamma}{\longrightarrow} \Sigma F$ in $\mathcal{F}/\sim$. It is easy to check that the left square below commutes in $\mathcal{F}/\sim$ and is an isomorphism (from $f$ to $\alpha$) in $\mathcal{F}/\sim$, and so extends to an isomorphism of triangles as shown.

\[
\begin{array}{cccccc}
F & \overset{f}{\longrightarrow} & F' & \longrightarrow & \Sigma T & \longrightarrow & \Sigma F \\
\downarrow & & \downarrow & & \downarrow & & \\
F & \overset{\alpha}{\longrightarrow} & I(F) \oplus F' & \overset{\beta}{\longrightarrow} & F'' & \overset{\gamma}{\longrightarrow} & \Sigma F
\end{array}
\]

Since $\Sigma T \in \mathcal{T}/\sim$ it follows that $F'' \in \mathcal{T}$. Therefore we have shown that the short exact sequence $0 \to F \overset{\alpha}{\longrightarrow} I(F) \oplus F' \overset{\beta}{\longrightarrow} F'' \to 0$ is the desired $0 \to F \to F'_1 \to T_1 \to 0$ with $F'_1 \in \mathcal{F}'$ and $T_1 \in \mathcal{T}$. A similar argument works to find the other short exact sequence $0 \to F'_2 \to T_2 \to F \to 0$ with $F'_2 \in \mathcal{F}'$ and $T_2 \in \mathcal{T}$. For this, first use the given torsion pair to get an exact triangle $T \to \Omega F \to F' \to \Sigma T$ in $\mathcal{F}/\sim$ with $T \in \mathcal{T}/\sim$ and $F' \in \mathcal{F}'/\sim$. Then rotate it twice to get $F' \to \Sigma T \to \Sigma \Omega F \to \Sigma F'$ and use the $(\mathcal{F}/\sim)$-isomorphism $\Sigma \Omega F \cong F$ to replace this with an exact triangle $F' \to \Sigma T \overset{\beta}{\longrightarrow} F \to \Sigma F'$. Now follow an argument dual to the above, replacing $\gamma$ with an epimorphism, to find the s.e.s $0 \to F'_2 \to T_2 \to F \to 0$.

Now we can see why $(\mathcal{W}_T, \mathcal{F})$ is a cotorsion pair. We have $\mathcal{F}' = \mathcal{W}_T^\perp$ by definition, so $\mathcal{W}_T \subseteq \mathcal{W}_T^\perp$ is automatic, which means it is left to show $\mathcal{W}_T \supseteq \mathcal{W}_T^\perp$.

So let $A \in \mathcal{A}$ be in $\mathcal{W}_T^\perp$. Using that $(\mathcal{W}, \mathcal{F})$ has enough injectives find a short exact sequence $0 \to A \to F \to W \to 0$ with $F \in \mathcal{F}$ and $W \in \mathcal{W}$. From the above paragraph we can also find a short exact sequence $0 \to F' \to T \to F \to 0$ with $F' \in \mathcal{F}'$ and $T \in \mathcal{T}$. Taking the pullback of $A \to F \to T$ we get a commutative
diagram with a bicartesian square (i.e. a pullback and pushout square) as shown.

\[
\begin{array}{cccccc}
0 & 0 & \downarrow & \downarrow & \downarrow & \downarrow \\
\downarrow & F' & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & P & \rightarrow & T & \rightarrow & W & \rightarrow & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & A & \rightarrow & F & \rightarrow & W & \rightarrow & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & 0 & \downarrow & \downarrow & \downarrow & \downarrow \\
\end{array}
\]

Note that by definition we have \( P \in W_T \). Since \( A \in \perp F' \), the left vertical column splits. We conclude \( A \in W_T \) since \( W_T \) is closed under retracts. This finishes the proof that \((W_T, F')\) is a cotorsion pair. If one follows the exact argument we just gave for an arbitrary \( A \in \mathcal{A} \) (rather than assuming \( A \in \perp F' \)), we see immediately that this cotorsion pair has enough projectives. To see that it has enough injectives we do a dual “pushout” argument. In detail, for an arbitrary \( A \in \mathcal{A} \) again use that \((W, F)\) has enough injectives to find a short exact sequence \( 0 \rightarrow A \rightarrow F \rightarrow W \rightarrow 0 \) with \( F \in \mathcal{F} \) and \( W \in \mathcal{W} \). From the last paragraph we also have a short exact sequence \( 0 \rightarrow F \rightarrow F' \rightarrow T \rightarrow 0 \) with \( F' \in \mathcal{F}' \) and \( T \in \mathcal{T} \). Taking the pushout of \( F' \leftarrow F \rightarrow W \) leads to a commutative diagram with bicartesian square as shown.

\[
\begin{array}{cccccc}
0 & 0 & \downarrow & \downarrow & \downarrow & \downarrow \\
\downarrow & 0 & \rightarrow & A & \rightarrow & F & \rightarrow & W & \rightarrow & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & A & \rightarrow & F' & \rightarrow & P & \rightarrow & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
T & \rightarrow & T & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & 0 & \downarrow & \downarrow & \downarrow & \downarrow \\
\end{array}
\]

Since \( W, T \in \mathcal{W}_T \) we get the extension \( P \in \mathcal{W}_T \). So the second row in the diagram shows that the cotorsion pair \((W_T, F')\) has enough injectives, and so it is complete.

On the other hand, say \((W_T, F')\) is an injective cotorsion pair. To see that \((\mathcal{T}/\sim, \mathcal{F}'/\sim)\) is a torsion pair in \( \mathcal{F}/\sim \) the only non-automatic thing required is to construct, for a given \( F \in \mathcal{F}/\sim \), an exact triangle \( T \rightarrow F \rightarrow F' \rightarrow \Sigma T \) with \( T \in \mathcal{T}/\sim \) and \( F' \in \mathcal{F}'/\sim \). But for a given \( F \in \mathcal{F} \) we use enough projectives of \((W_T, F')\) to get a short exact sequence \( 0 \rightarrow F' \rightarrow T \rightarrow F \rightarrow 0 \) with \( F' \in \mathcal{F}' \) and \( T \in \mathcal{W}_T \cap \mathcal{F} = \mathcal{T} \). This gives rise to an exact triangle \( F' \rightarrow T \rightarrow F \rightarrow \Sigma F' \) in
follows that \((\mathcal{T}/\sim, \mathcal{F}/\sim)\) is a torsion pair in \(\mathcal{F}/\sim\).

The remaining statements in the Proposition follow from the previous Lemmas.

We now are ready to present the converse to both Becker’s result in Proposition 3.11 and to the main Theorem 4.6. First, say \(\mathcal{T}' \xrightarrow{F'} \mathcal{T} \xrightarrow{G'} \mathcal{T}''\) is a localization sequence with right adjoints \(F'\) and \(G'\) as in Definition 4.11. Then \(F'\) and \(G'\) are each fully faithful and we will identify \(\mathcal{T}'\) and \(\mathcal{T}''\) with the thick subcategories \(\mathcal{T}' = \text{Im} F\) and \(\mathcal{T}'' = \text{Im} G'\). In this way \(\mathcal{T}'\) is identified as what we call a right admissible subcategory of \(\mathcal{T}\) (meaning the inclusion has a right adjoint) while \(\mathcal{T}''\) is identified as a left admissible subcategory of \(\mathcal{T}\) (its inclusion has a left adjoint). With these identifications we have \((\mathcal{T}', \mathcal{T}'')\) is a torsion pair in \(\mathcal{T}\). See [Kra04, discussion before Lemma 3.2]. Of course, if \(\mathcal{T}' \xrightarrow{F'} \mathcal{T} \xrightarrow{G'} \mathcal{T}''\) were a colocalization sequence from the start, we get in a similar way a torsion pair \((\mathcal{T}'', \mathcal{T}')\). Indeed the left adjoints \(F_\lambda\) and \(G_\lambda\) form a localization sequence \(\mathcal{T}' \xrightarrow{G_\lambda} \mathcal{T} \xrightarrow{F_\lambda} \mathcal{T}'\). So identifying \(\mathcal{T}'' = \text{Im} G_\lambda\) and \(\mathcal{T}' = \text{Im} F\) we get the torsion pair \((\mathcal{T}'', \mathcal{T}')\). Conversely any torsion pair \((\mathcal{X}, \mathcal{Y})\) in a triangulated category \(\mathcal{T}\) yields a (co)localization sequence. \(\mathcal{X}\) is right admissible while \(\mathcal{Y}\) is left admissible. See [BR07, Proposition 2.6] and also [Kra04, Lemma 3.2]. Finally, consider the recollement diagram of Definition 4.2. Then \(F, G_\rho,\) and \(G_\lambda\) are all fully faithful. Identifying \(\mathcal{T}' = \text{Im} F\) and setting \(\mathcal{T}_\lambda = \text{Im} G_\lambda\) and \(\mathcal{T}_\rho = \text{Im} G_\rho\), we see that the colocalization sequence gives a torsion pair \((\mathcal{T}_\lambda, \mathcal{T}')\) while the localization sequence gives a torsion pair \((\mathcal{T}', \mathcal{T}_\rho)\). We say \(\mathcal{T}' = \text{Im} F\) is an admissible subcategory of \(\mathcal{T}\) (for it has both a left and right adjoint) and also call \((\mathcal{T}_\lambda, \mathcal{T}', \mathcal{T}_\rho)\) a torsion triple. Conversely, an admissible subcategory gives rise to a torsion triple and a recollement. Finally we are able to state the promised converses as a corollary to Proposition 4.15.

**Corollary 4.16.** Proposition 4.15 gives the following.

1. Left admissible subcategories \(\mathcal{F}'/\sim\) of \(\mathcal{F}/\sim\) are in bijective correspondence with injective cotorsion pairs \((\mathcal{W}', \mathcal{F}')\) in \(\mathcal{A}\) with \(\mathcal{F}' \subseteq \mathcal{F}\).

2. Admissible subcategories \(\mathcal{F}'/\sim\) of \(\mathcal{F}/\sim\) are in bijective correspondence with two injective cotorsion pairs \((\mathcal{W}', \mathcal{F}')\) and \((\mathcal{W}'', \mathcal{F}'')\) in \(\mathcal{A}\) satisfying \(\mathcal{F}'' \subseteq \mathcal{F}\) and \(\mathcal{W}' \cap \mathcal{F} = \mathcal{F}'\).

We describe how the bijections work in the proof below.

**Proof.** As described above, a left admissible subcategory \(\mathcal{F}'/\sim\) of \(\mathcal{F}/\sim\) is equivalent to a colocalization sequence \(\mathcal{F}'/\sim \xhookrightarrow{\sim} \mathcal{F}/\sim \xrightarrow{\sim} \mathcal{T}/\sim\) where \(\mathcal{T}/\sim\) is another thick subcategory of \(\mathcal{F}/\sim\) and this in turn is equivalent to a torsion pair \((\mathcal{T}/\sim, \mathcal{F}'/\sim)\). In particular, this is equivalent to the injective cotorsion pair \((\mathcal{W}_T, \mathcal{F}')\) characterized by \(\mathcal{W}_T \cap \mathcal{F} = \mathcal{T}\).

In a similar way, an admissible subcategory \(\mathcal{F}'/\sim\) of \(\mathcal{F}/\sim\) is equivalent to a colocalization sequence \(\mathcal{F}'/\sim \xhookrightarrow{\sim} \mathcal{F}/\sim \xrightarrow{\sim} \mathcal{T}_\lambda/\sim\) along with a localization sequence \(\mathcal{F}'/\sim \xhookrightarrow{\sim} \mathcal{F}/\sim \xrightarrow{\sim} \mathcal{T}_\rho/\sim\) where \(\mathcal{T}_\lambda/\sim\) and \(\mathcal{T}_\rho/\sim\) are each thick subcategories of \(\mathcal{F}/\sim\). That is, it is equivalent to a torsion triple \((\mathcal{T}_\lambda/\sim, \mathcal{F}'/\sim, \mathcal{T}_\rho/\sim)\). Here, the torsion pair \((\mathcal{T}_\lambda/\sim, \mathcal{F}'/\sim)\) gives the injective cotorsion pair \((\mathcal{W}', \mathcal{F}')\) with \(\mathcal{W}' \cap \mathcal{F} = \mathcal{T}_\lambda\) while the torsion pair \((\mathcal{F}'/\sim, \mathcal{T}_\rho/\sim)\) gives another injective cotorsion pair \((\mathcal{W}'', \mathcal{F}'')\) with \(\mathcal{W}'' \cap \mathcal{F} = \mathcal{F}'\).
Note. The recollement diagram of Theorem 4.8 gives rise to the torsion triple \((W_2 \cap F_2)/\sim, F_2/\sim, F_3/\sim\) in the triangulated category \(F_1/\sim\). As a concrete example, using the notation of Example 4.8, Krause’s recollement corresponds to the torsion triple \((W_2 \cap dwI)/\sim, exI/\sim, dgI/\sim\) in \(dwI/\sim\).

5. The Gorenstein injective cotorsion pair

Let \(\mathcal{A}\) be any abelian category with enough injectives. It is clear that the canonical injective cotorsion pair \(\mathcal{A}, I\) satisfies \(I \subseteq F\) whenever \((W, F)\) is another injective cotorsion pair. We now show two things. First, for any injective cotorsion pair \((W, F)\) we have \(F \subseteq GI\), where \(GI\) is the class of Gorenstein injective objects. Second, whenever the class \(GI\) forms the right half of a complete cotorsion pair \((^\perp GI, GI)\), then this is automatically an injective cotorsion pair too. These facts have an immediate application (Corollary 8.1) to a simple characterization of the Gorenstein injective cotorsion pair, cogenerated by a set, and that it is in fact an injective cotorsion pair.

We start with the definition of a Gorenstein injective object in \(\mathcal{A}\).

Definition 5.1. Let \(\mathcal{A}\) be an abelian category with enough injectives and let \(M \in \mathcal{A}\). We call \(M\) Gorenstein injective if \(M = Z_0I\) for some exact complex \(I\) of injectives which remains exact after applying \(\text{Hom}_{\mathcal{A}}(I, -)\) for any injective object \(I\). We will also call such a complex \(I\) a complete injective resolution of \(M\).

See [EJ00] for a basic reference on Gorenstein injective \(R\)-modules. In [BGHR] it is shown that whenever \(R\) is a (left) Noetherian ring then \((^\perp GI, GI)\) is a complete cotorsion pair, cogenerated by a set, and that it is in fact an injective cotorsion pair.

Theorem 5.2. Let \(\mathcal{A}\) be an abelian category with enough injectives and let \(GI\) denote the class of Gorenstein injectives in \(\mathcal{A}\). Then we have \(F \subseteq GI\) whenever \((W, F)\) is an injective cotorsion pair. Moreover, whenever \((^\perp GI, GI)\) is a complete cotorsion pair it is automatically an injective cotorsion pair too.

Proof. First, suppose \((W, F)\) is an injective cotorsion pair and let \(F \in F\). We must show that \(F\) is Gorenstein injective. We will construct a complete injective resolution of \(F\). First, since \(\mathcal{A}\) has enough injectives we may take a usual injective coresolution of \(F\). But this coresolution will actually remain exact after applying \(\text{Hom}_{\mathcal{A}}(I, -)\) for any injective \(I\) since \((W, F)\) is hereditary and since \(W\) contains any injective \(I\). So this gives us the right half \(F \rightarrow J^*\) of a complete injective resolution of \(F\).

To build the left half of a complete injective resolution use that \((W, F)\) has enough injectives to get a short exact sequence

\[ (\ast) \quad 0 \rightarrow F' \rightarrow J \rightarrow F \rightarrow 0, \]

where \(J \in W\) and \(F' \in F\). As \(F\) is closed under extensions, and since \(F, F' \in F\), it follows that \(J \in W \cap F\). That is, \(J\) must be injective. Since \(F' \in F\) we again have \(\text{Ext}^1_{\mathcal{A}}(I, F') = 0\) for all injectives \(I\). Hence the exact sequence \((\ast)\) stays exact under application of the functor \(\text{Hom}_{\mathcal{A}}(I, -)\) whenever \(I\) is injective. Iterating this process allows us to construct the left half of a complete injective resolution \(J^* \rightarrow F\). We compose to set \(J = J_0 \rightarrow F \leftarrow J^*\) with \(M = Z_0J\) making \(J\) the desired complete injective resolution of \(M\).
Next, set $\mathcal{V} = \mathcal{GI}$ and suppose that $(\mathcal{V}, \mathcal{GI})$ is a complete cotorsion pair. If $M$ is Gorenstein injective then it follows from the definition that $\mathrm{Ext}^n_A(I, M) = 0$ for any injective object $I$ and $n \geq 1$. So $\mathcal{V}$ contains the injective objects. Next, we claim that the class $\mathcal{GI}$ is cosyzygy closed. Indeed it is clear that if $M$ is Gorenstein injective with complete injective resolution $J$, then $Z_n J$ are also all Gorenstein injective by definition. In particular we have the short exact sequence $0 \rightarrow M \rightarrow J_0 \rightarrow Z_{-1} J \rightarrow 0$ with $Z_{-1} J$ also Gorenstein injective. So if $0 \rightarrow M \rightarrow I \rightarrow Z \rightarrow 0$ is any other short exact sequence with $I$ injective we get $\mathrm{Ext}^2_A(V, M) \cong \mathrm{Ext}^2_A(V, Z) \cong \mathrm{Ext}^1_A(V, Z_{-1} J) = 0$. So the class $\mathcal{GI}$ is cosyzygy closed. Since $A$ has enough injectives we conclude $(\mathcal{V}, \mathcal{GI})$ is hereditary from Lemma 2.3. Now $(\mathcal{V}, \mathcal{GI})$ satisfies the hypotheses of Lemma 3.5 and so $\mathcal{V}$ is thick. Finally, since we have shown that $\mathcal{V}$ is thick and contains the injectives we get from part (2) of Proposition 3.6 that $(\mathcal{V}, \mathcal{GI})$ is an injective cotorsion pair.

5.1. The Gorenstein projective cotorsion pair. We now state the dual result concerning the Gorenstein projectives.

**Definition 5.3.** Let $A$ be an abelian category with enough projectives and let $M \in A$. We call $M$ Gorenstein projective if $M = Z_0 Q$ for some exact complex $Q$ of projectives which remains exact after applying $\mathrm{Hom}_A(-, P)$ for any projective object $P$. We will also call such a complex $Q$ a complete projective resolution of $M$.

Again, see [EJ00] for a basic reference on Gorenstein projective $R$-modules. It is shown in [BGH13] that $(\mathcal{GP}, \mathcal{GP}^\perp)$ is in fact cogenerated by a set, so complete, whenever $R$ is a ring for which all level modules have finite projective dimension. We define level modules in Section 10. But this includes all (left) coherent rings in which all flat modules have finite projective dimension. We pause now to comment on the extraordinarily large class of rings satisfying the condition that all flat modules have finite projective dimension.

In [Sim74], Simson gives a short proof of the following: If a ring $R$ has cardinality at most $\aleph_n$ then the maximal projective dimension of a flat module is at most $n + 1$. This amazing result doesn’t depend on whether the ring is commutative or Noetherian or anything and so there is an abundance of rings with the property that all flat modules have finite projective dimension. However, there is the continuum hypothesis. Putting cardinality aside, Enochs, Jenda and López-Ramos have considered rings in [EJLR] they call $n$-perfect. These are rings in which all flat modules have projective dimension at most $n$. In that paper and in other papers coauthored by Enochs they give numerous examples of $n$-perfect rings. In particular, a perfect ring is 0-perfect and any $n$-Gorenstein ring is $n$-perfect. In this language the above result of Simson says that any ring $R$ with cardinality at most $\aleph_n$ is $(n + 1)$-perfect. There is more. A main example of an $n$-perfect ring is a commutative Noetherian ring of finite Krull dimension $n$. This follows from [Jen70] and [GR71]. Finally, we point out a generalization of this due to Peter Jørgensen. In the article [Jør05], he shows that every flat module has finite projective dimension whenever $R$ is right-Noetherian and has a dualizing complex. Note that our condition of saying $R$ has finite projective dimension for each flat module is more general than saying $R$ is $n$-perfect because we are not assuming an upper bound on the projective dimensions.
Theorem 5.4. Let \( \mathcal{A} \) be an abelian category with enough projectives and let \( \mathcal{GP} \) denote the class of Gorenstein projectives in \( \mathcal{A} \). Then we have \( \mathcal{C} \subseteq \mathcal{GP} \) whenever \((\mathcal{C}, \mathcal{W})\) is a projective cotorsion pair. Moreover, whenever \((\mathcal{GP}, \mathcal{GP}^\perp)\) is a complete cotorsion pair it is automatically a projective cotorsion pair too.

6. The semilattice of injective cotorsion pairs in \( R\text{-Mod} \) and \( \text{Ch}(R) \)

We now let \( R \) be a ring and we assume for the remainder of the paper that \( \mathcal{A} \) is either the category of \( R \)-modules or the category of chain complexes of \( R \)-modules. That is, now \( \mathcal{A} \) denotes either \( R\text{-Mod} \) or \( \text{Ch}(R) \). In light of Theorem 5.2 and Corollary 4.10 it makes sense to consider whether or not the injective cotorsion pairs form a lattice. We now make a brief investigation, again focusing on injective or projective cotorsion pairs. But following either ordering, of model categories it is best to change the ordering depending on whether we are considering either the classes on the left side or the classes on the right side. For the theory of model categories it is best to change the ordering depending on whether we are focusing on injective or projective cotorsion pairs. But following either ordering, Lemma 6.1 guarantees suprema and infima so that a partial ordering gives a complete lattice on the class of all cotorsion pairs. It is nontrivial that restricting the partial ordering to just those cotorsion pairs which are cogenerated by a set forms a complete sublattice. But this follows from the fact that \( \cap_{i \in I} \mathcal{C}_i \) is deconstructible (as defined in [St'o13]) whenever each \((\mathcal{C}_i, \mathcal{D}_i)\) is cogenerated by a set [St'o13, Proposition 2.9].

6.1. The semilattice of injective cotorsion pairs. Suppose we have two injective cotorsion pairs \( \mathcal{M}_1 = (\mathcal{W}_1, \mathcal{F}_1) \) and \( \mathcal{M}_2 = (\mathcal{W}_2, \mathcal{F}_2) \) in \( \mathcal{A} \). Then we define \( \mathcal{M}_2 \preceq_r \mathcal{M}_1 \) if and only if \( \mathcal{F}_2 \subseteq \mathcal{F}_1 \). Note that this happens if and only if the inclusion functor \( \mathcal{M}_2 \to \mathcal{M}_1 \) is a right Quillen functor. With respect to \( \preceq_r \) we have that the canonical injective cotorsion pair is the least element with respect to this ordering and from Theorem 5.2 the Gorenstein injective cotorsion pair, whenever it exists, is the maximum element. We know it exists whenever \( R \) is Noetherian by [BCH13].

Proposition 6.2. Let \( \{ (\mathcal{W}_i, \mathcal{F}_i) \}_{i \in I} \) be a collection of injective cotorsion pairs each cogenerated by a set \( \mathcal{S}_i \). Then its supremum \( \bigvee_{i \in I} (\mathcal{W}_i, \mathcal{F}_i) = (\cap_{i \in I} \mathcal{W}_i, (\cap_{i \in I} \mathcal{W}_i)^\perp) \) is also an injective cotorsion pair cogenerated by a set.
Proof. Since each \((\mathcal{W}_i, \mathcal{F}_i)\) is cogenerated by a set, each class \(\mathcal{W}_i\) is deconstructible by \([\mathcal{S}^1]'\). Then by \([\mathcal{S}^1]'\) Proposition 2.9 (2) it follows that \(\cap_{i \in I} \mathcal{W}_i\) is also deconstructible. This implies the cotorsion pair \((\cap_{i \in I} \mathcal{W}_i, (\cap_{i \in I} \mathcal{W}_i)^{\perp})\) is cogenerated by a set, and so is complete. Since each \(\mathcal{W}_i\) is thick and contains the injective objects we get that \(\cap_{i \in I} \mathcal{W}_i\) is also thick and contains the injectives. So \((\cap_{i \in I} \mathcal{W}_i, (\cap_{i \in I} \mathcal{W}_i)^{\perp})\) is an injective cotorsion pair by Proposition 3.6.

\[\square\]

Remark 8. Proposition 6.2 says that the ordering on the class of all injective cotorsion pairs that are cogenerated by a set is a “complete join-semilattice” sitting inside the complete lattice of all cotorsion pairs that are cogenerated by a set. By a complete join-semilattice we mean that the supremum of any given set exists. The author doesn’t know whether or not infima exist. That is, let \(\{ (\mathcal{W}_i, \mathcal{F}_i) \}_{i \in I}\) be a collection of injective cotorsion pairs each cogenerated by a set \(\mathcal{S}_i\). Then we know from Lemma 6.1 that \((\cap_{i \in I} \mathcal{F}_i), \cap_{i \in I} \mathcal{F}_i)\) is the infimum in the lattice of all cotorsion pairs that are cogenerated by a set. We would like to know if this cotorsion pair is injective. (The only thing not clear is whether or not the left class is closed under taking cokernels of monomorphisms between its objects. Equivalently, does \([\cap_{i \in I} \mathcal{F}_i]) \cap [\cap_{i \in I} \mathcal{F}_i)\) consist only of injective modules?)

Finally, suppose \(R\) is a ring with the Gorenstein injective cotorsion pair \((^\perp \mathcal{GI}, \mathcal{GI})\) being complete. Then \(\mathcal{GI}\) becomes a Frobenius category and we point out that the semilattice of injective cotorsion pairs in \(R\)-Mod embeds inside the lattice of thick subcategories of the stable category \(\mathcal{GI}/^\sim\) by Corollary 4.16.

6.2. The semilattice of projective cotorsion pairs. On the other hand we have the semilattice of projective cotorsion pairs in \(\mathcal{A}\). We give the simple proof of the dual result below. It is necessarily different due to the fact that we must always cogenerate a cotorsion pair by a set to get completeness.

Suppose that \(\mathcal{M}_1 = (\mathcal{C}_1, \mathcal{W}_1)\) and \(\mathcal{M}_2 = (\mathcal{C}_2, \mathcal{W}_2)\) are projective cotorsion pairs in \(\mathcal{A}\). Then we define \(\mathcal{M}_2 \preceq_1 \mathcal{M}_1\) if and only if \(\mathcal{C}_2 \subseteq \mathcal{C}_1\). Note that this happens if and only if the inclusion functor \(\mathcal{M}_2 \rightarrow \mathcal{M}_1\) is a left Quillen functor. With respect to \(\preceq_1\) we have that the canonical projective cotorsion pair is the least element with respect to the ordering and from Theorem 5.4 the Gorenstein projective cotorsion pair, whenever it exists, is the maximum element. We know it exists whenever \(R\) is a coherent ring in which all flat modules have finite projective dimension by \([\mathcal{B}^1CH13]\).

Proposition 6.3. Let \(\{ (\mathcal{C}_i, \mathcal{W}_i) \}_{i \in I}\) be a collection of projective cotorsion pairs each cogenerated by a set \(\mathcal{S}_i\). Then its supremum, which is the cotorsion pair \(\vee_{i \in I}(\mathcal{C}_i, \mathcal{W}_i) = (\perp(\cap_{i \in I} \mathcal{W}_i), \cap_{i \in I} \mathcal{W}_i)\), is also a projective cotorsion pair cogenerated by a set.

Proof. Here it is clear that \(\cap_{i \in I} \mathcal{W}_i\) is thick and contains the projectives since each \(\mathcal{W}_i\) does. Also, if \(\{ \mathcal{S}_i \}_{i \in I}\) represents the cogenerating sets, then \(\cup_{i \in I} \mathcal{S}_i\) is a cogenerating set by Lemma 6.1. So \(\vee_{i \in I}(\mathcal{C}_i, \mathcal{W}_i)\) is complete and so a projective cotorsion pair by Proposition 6.7.

\[\square\]

6.3. Examples. In general, the semilattice of injective (or projective) cotorsion pairs on \(\text{Ch}(R)\) is always more interesting than the one on \(R\)-Mod and depends much on the global (Gorenstein) dimension of \(R\). We will look at examples concerning \(\text{Ch}(R)\) at the end of Section 7.
For a generic ring $R$, there seems to be just two injective cotorsion pairs on $R$-Mod that are typically of interest. The first is the canonical injective cotorsion pair $(\mathcal{A}, \mathcal{I})$ and the second the Gorenstein AC-injective cotorsion pair $(\mathcal{W}, \mathcal{GI})$ discussed more in Section 7. When $R$ is Noetherian this is exactly the Gorenstein injective cotorsion pair. But a ring can certainly have more than these two injective cotorsion pairs. For example, if $R$ is Noetherian then as we see in the next section there are several injective cotorsion pairs on $\text{Ch}(R)$. But $\text{Ch}(R)$ is really just a graded version of the category of $R[x]/(x^2)$-modules, and $R[x]/(x^2)$-Mod will have these analogous injective cotorsion pairs as well.

We note the following simplification for a ring of finite global dimension.

Example 6.4. Suppose $R$ is a ring with $\text{gl.dim}(R) < \infty$. Then the Gorenstein injective modules coincide with the injective modules. So in this case there is only one injective cotorsion pair in $R$-Mod, the categorical one. Similarly there is only one projective cotorsion pair in $R$-Mod, as the canonical projective cotorsion pair coincides with the Gorenstein projective cotorsion pair.

7. Lifting from modules to chain complexes

The author has considered before the problem of lifting any given cotorsion pair $(\mathcal{F}, \mathcal{C})$ in an abelian category $\mathcal{A}$ to various cotorsion pairs in $\text{Ch}(R)$. In particular, see [Gil04] and [Gil08]. We now show that when the ground pair $(\mathcal{F}, \mathcal{C})$ is an injective (resp. projective) cotorsion pair, then these lifted pairs are also injective (resp. projective). We start with the following notations which were introduced in [Gil04] and [Gil08].

Definition 7.1. Given a class of $R$-modules $\mathcal{C}$, we define the following classes of chain complexes in $\text{Ch}(R)$.

1. $\text{deg} \mathcal{C}$ is the class of all chain complexes with $C_n \in \mathcal{C}$.
2. $\text{ex} \mathcal{C}$ is the class of all exact chain complexes with $C_n \in \mathcal{C}$.
3. $\mathcal{C}$ is the class of all exact chain complexes with cycles $Z_n C \in \mathcal{C}$.

The “dw” is meant to stand for “degreewise” while the “ex” is meant to stand for “exact”. When $\mathcal{C}$ is the class of projective (resp. injective, resp. flat) modules, then $\mathcal{C}$ are the categorical projective (resp. injective, resp. flat) chain complexes.

Moreover, if we are given any cotorsion pair $(\mathcal{F}, \mathcal{C})$ in $R$-Mod, then following [Gil04] we will denote $\overset{\text{dg}}{\mathcal{F}}$ by $\mathcal{F}$ and $\overset{\text{dg}}{\mathcal{C}}$ by $\mathcal{C}$.

7.1. Injective cotorsion pairs in $\text{Ch}(R)$ from one in $R$-Mod. Let $R$ be any ring and $(\mathcal{W}, \mathcal{F})$ be an injective cotorsion pair in $R$-Mod which is cogenerated by a set. We see below that this information allows for the construction of six injective cotorsion pairs, also each cogenerated by a set, in $\text{Ch}(R)$. However, depending on the pair $(\mathcal{W}, \mathcal{F})$ that we start with and the particular ring $R$, some of these six will coincide. See the examples ahead in Section 7.3.

Proposition 7.2. Let $(\mathcal{W}, \mathcal{F})$ be an injective cotorsion pair of $R$-modules cogenerated by some set. Then the following are also each injective cotorsion pairs in $\text{Ch}(R)$, and cogenerated by sets.

1. $(\overset{\text{dg}}{\mathcal{W}}, \mathcal{F})$
2. $(\overset{\text{dg}}{\mathcal{F}}, \mathcal{W})$
Proof. This time we give the proofs for the projective case and this appears in Proposition 7.3 below. We note that the only difference between the projective and injective case is that we are always cogenerating by a set to get completeness of the cotorsion pair. For (1)–(3) one can cogenerate by a set due to [Gil08, Propositions 4.3, 4.4, and 4.6] and for (4)–(6) one can use the theory of deconstructible classes in [ˇSt’o13]. □

7.2. Projective cotorsion pairs in \(\text{Ch}(R)\) from one in \(R\)-Mod. Let \(R\) be any ring and \((C, W)\) be a projective cotorsion pair in \(R\)-Mod which is cogenerated by a set. We have the dual statement to Proposition 7.2 which we now prove.

Proposition 7.3. Let \((C, W)\) be a projective cotorsion pair of \(R\)-modules cogenerated by some set. Then the following are also each projective cotorsion pairs in \(\text{Ch}(R)\), and cogenerated by sets.

(1) \((\text{du}\bar{C}, (\text{du}\bar{C})^+)\)
(2) \((\text{ex}\bar{C}, (\text{ex}\bar{C})^+)\)
(3) \((\text{dg}\bar{W}, \bar{W})\)
(4) \((\text{ex}\bar{W}, \text{ex}\bar{W})\)
(5) \((\text{dg}\bar{C}, \bar{C})\)

Proof. The proofs for (1)–(3) are all similar as are the proofs for (4)–(6).

Let us prove (6). First, it follows from [Gil04, Section 3] that \((\text{ dg}\bar{W}, \bar{W})\) is a cotorsion pair. It is easy to show that \(\bar{W}\) is thick because \(W\) is thick. (In particular, note that any short exact sequence \(0 \to W' \to W \to W'' \to 0\) of complexes in \(\bar{W}\) gives rise to a short exact sequence \(0 \to Z_nW' \to Z_nW \to Z_nW'' \to 0\) on the level of cycles because the complex \(W'\) is exact.) Next, recall that a projective chain complex is one which is exact with projective cycles. Since \(W\) contains the projective modules, we get that \(\bar{W}\) contains the projective complexes. So since \(\bar{W}\) is thick and contains the projectives it will follow from Proposition 6.7 that \((\text{ dg}\bar{W}, \bar{W})\) is a projective cotorsion pair once we know it is complete. Note here that if we had started with \((\text{ du}\bar{W}, \text{ dw}\bar{W})\) or \((\text{ ex}\bar{W}, \text{ ex}\bar{W})\) then again we get that each of these is a cotorsion pair by [Gil08, Propositions 3.2 and 3.3] and similarly we argue that both \(\text{ du}\bar{W}\) and \(\text{ ex}\bar{W}\) are thick and contain the projectives. But finally, it was shown in [Gil08, Propositions 4.3–4.6] that each of the cotorsion pairs \((\text{ du}\bar{W}, \text{ dw}\bar{W})\), \((\text{ ex}\bar{W}, \text{ ex}\bar{W})\) and \((\text{ ex}\bar{W}, \bar{W})\) are cogenerated by a set. This proves (4)–(6).

Now let us prove (1)–(3). It follows again from [Gil04] and [Gil08] that these are all cotorsion pairs. Let us focus on (2) for example, since (1) and (3) will be similar. First, from [Gil08, Proposition 3.3] we note that for this cotorsion pair \((\text{ ex}\bar{C}, (\text{ ex}\bar{C})^+)\)
the right class $(ex\tilde{C})^\perp$ equals the class of all complexes $W$ for which $W_n \in W$ and such that $Hom(C, W)$ is exact whenever $C \in ex\tilde{C}$. ($Hom$ is defined in Section 2).

By Lemma 2.1 we get that $Hom(C, W)$ is exact if and only if any chain map $f : \Sigma^n C \to W$ is null homotopic if and only if any chain map $f : C \to \Sigma^n W$ is null homotopic.) In light of Proposition 3.7 we wish to show that this class $(\Sigma f)$ is thick and contains the projectives and that the cotorsion pair $(null \ homotopic.)$ In light of Proposition 3.7 we wish to show that this class $\Sigma f$ is thick.

By Lemma 2.1 we get that $\text{Hom}(C, W)$ is exact if and only if any chain map $f : \Sigma^n C \to W$ is null homotopic if and only if any chain map $f : C \to \Sigma^n W$ is null homotopic.) In light of Proposition 3.7 we wish to show that this class $\Sigma f$ is thick and contains the projectives and that the cotorsion pair $(ex\tilde{C}, (ex\tilde{O})^\perp)$ is complete. The projective complexes are easily seen to be in $(ex\tilde{O})^\perp$ by its above description (recall $W$ contains the projectives and any chain map into a projective is null). Next, $(ex\tilde{O})^\perp$ is clearly closed under retracts since it is the right side of a cotorsion pair. To complete the thickness claim suppose that $0 \to W'' \to W \to W''' \to 0$ is a short exact sequence of complexes. If any two of the three $W', W, W''$ are in $(ex\tilde{O})^\perp$ then note that since $W$ is thick we get that all of the $W'_n, W_n, W''_n$ are in $W$. It now follows that for any $C \in ex\tilde{O}$ we will always get a short exact sequence of $\text{Hom}$-complexes $0 \to \text{Hom}(C, W') \to \text{Hom}(C, W) \to \text{Hom}(C, W'') \to 0$ (because Ext vanishes degreewise). So now if any two of the three complexes $\text{Hom}(C, W'), \text{Hom}(C, W), \text{Hom}(C, W'')$ are exact then the third is automatically exact due to the long exact sequence in homology. This completes the proof that $(ex\tilde{O})^\perp$ is thick.

Finally it is left to show that $(ex\tilde{C}, (ex\tilde{O})^\perp)$ is complete. We use the results in [St'o13] pointing out that a class which is the left side of a cotorsion pair is deconstructible if and only if that cotorsion pair is cogenerated by set. It follows from [St'o13] Theorem 4.2 that $dw\tilde{C}$ and $\tilde{C}$ are deconstructible since $\tilde{C}$ is. So the only cotorsion pair left is $(ex\tilde{C}, (ex\tilde{O})^\perp)$. But here note that $ex\tilde{C} = dw\tilde{C} \cap E$ where $E$ is the class of exact complexes. Since $E$ is the left side of a cotorsion pair cogenerated by a set it is deconstructible and since $dw\tilde{C}$ is also deconstructible, it follows from [St'o13] Proposition 2.9 that $ex\tilde{C}$ is also deconstructible.

7.3. Examples and homological dimensions. We briefly discussed the semilattices of injective and projective cotorsion pairs on $R$-$\text{Mod}$ in Section 6.3. We continue that discussion now by looking in more detail at the basic examples of model structures on $\text{Ch}(R)$ induced from the categorical injective (resp. projective) cotorsion pair and the Gorenstein injective (resp. Gorenstein projective) cotorsion pair. The basic theme is that for a Noetherian ring $R$, the semilattice for $\text{Ch}(R)$ becomes more complicated as we move from $R$ having finite global dimension, to $R$ being Gorenstein, to a general Noetherian $R$. In the next section we look in more detail at the models induced by the Gorenstein injective and Gorenstein projective pairs.

We will give projective examples here and leave the obvious dual statements concerning injective cotorsion pairs to the reader. Let $\mathcal{P}$ denote the class of projective modules and $(\mathcal{P}, \mathcal{A})$ the canonical projective cotorsion pair. Consider the six projective cotorsion pairs on $\text{Ch}(R)$ induced by $(\mathcal{P}, \mathcal{A})$ using Proposition 1.3. We see that they are in fact only four distinct pairs. They are $(dw\bar{\mathcal{P}}, (dw\bar{\mathcal{P}})^\perp)$, $(ex\bar{\mathcal{P}}, (ex\bar{\mathcal{P}})^\perp)$, $(\mathcal{P}, \text{Ch}(R))$, and $(dg\bar{\mathcal{P}}, E)$. We have the following observation.

**Proposition 7.4.** Let $R$ be any ring with $\text{gl.dim}(R) < \infty$. Then we have $ex\bar{\mathcal{P}} = \bar{\mathcal{P}}$ and $dw\bar{\mathcal{P}} = dg\bar{\mathcal{P}}$.

**Proof.** $dg\bar{\mathcal{P}} \subseteq dw\bar{\mathcal{P}}$ is automatic. To show $dw\bar{\mathcal{P}} \subseteq dg\bar{\mathcal{P}}$ let $P \in dw\bar{\mathcal{P}}$ and let $E \in E$. We must show $\text{Ext}^1(P, E) = 0$. Since $\text{gl.dim}(R) = d < \infty$, it is easy to argue that $E$ has finite projective dimension in $\text{Ch}(R)$ because any $d$th syzygy must be exact.
and inherit projective cycles. Letting
\[ 0 \to P^d \to P^{d-1} \to \cdots \to P^1 \to P^0 \to E \to 0 \]
be a finite projective resolution we conclude by dimension shifting that \( \Ext^1(P, E) = \Ext^{d+1}(P, P^d) \). But \( \Ext^{d+1}(P, P^d) = 0 \) because \((dw\widehat{P}, (dw\widehat{P})^\perp)\) is a projective cotorsion pair. Thus \( dw\widehat{P} = dg\widehat{P} \). Also, \( ex\widehat{P} = \widehat{P} \) because given any \( P \in ex\widehat{P} \) and an \( R \)-module \( N \), we can dimension shift \( \Ext^1(Z_nP, N) = \Ext^{d+1}(Z_{n-d}P, N) = 0 \).

\[ \square \]

So the next two examples clarify further the models obtained on \( \text{Ch}(R) \) from \((\mathcal{P}, \mathcal{A})\) using Proposition 7.3.

**Example 7.5.** Say \( R \) has finite global dimension. Then we saw in Example 6.4 that \((\mathcal{P}, \mathcal{A}) = (GP, W)\) is the only projective cotorsion pair on \( R \)-Mod. Proposition 7.4 tells us that we only get two distinct projective cotorsion pairs coming from Proposition 7.3 in this case. This first is \((dw\widehat{P}, E)\) which is the projective model for the derived category of \( R \) and the other is the trivial projective model structure \((\widehat{P}, \mathcal{A})\). It follows from Corollary 8.3 that \((dw\widehat{P}, E)\) is actually the Gorenstein projective pair in \( \text{Ch}(R) \), sitting on top of the semilattice while \((\widehat{P}, \mathcal{A})\) is the canonical projective pair sitting at the bottom of the semilattice. The author doesn’t know whether or not there are others on \( \text{Ch}(R) \) besides these two when \( \text{gl.dim}(R) < \infty \).

**Example 7.6.** Now suppose that \( R \) has infinite global dimension. Then we have all four generally distinct pairs \((dw\widehat{P}, (dw\widehat{P})^\perp)\), \((ex\widehat{P}, (ex\widehat{P})^\perp)\), \((\widehat{P}, \text{Ch}(R))\), and \((dg\widehat{P}, E)\). We note \( P \) is the class of categorical projective complexes and so \((P, P^\perp)\) is trivial as a model structure, and not of interest. Next, \( dg\widehat{P} \) is the class of DG-projective complexes and \((dg\widehat{P}, E)\) is the usual projective model structure on \( \text{Ch}(R) \) having homotopy category the usual derived category \( D(R) \). The model structure associated to \((dw\widehat{P}, (dw\widehat{P})^\perp)\) appears in \[BGH13\] where it is called the Proj model structure on \( \text{Ch}(R) \). This model structure has also appeared in \[Pos11\] where its homotopy category was called the contraderived category of \( R \). The model structure \((ex\widehat{P}, (ex\widehat{P})^\perp)\) also appears in \[BGH13\] where it is called the exact Proj model structure on \( \text{Ch}(R) \). Its homotopy category is the (projective) stable derived category \( S(R) \) introduced in \[BGH13\]. For a general ring \( R \) we have the portion of the semilattice shown below.

![Semilattice Diagram](image)

We point out that there are two more model structures on \( \text{Ch}(R) \) distinct from the above that exist whenever \( R \) is a coherent ring in which all flat modules have finite projective dimension. These will appear in \[BGH13\].

Having considered projective models on \( \text{Ch}(R) \) induced from the canonical projective pair \((\mathcal{P}, \mathcal{A})\) via Proposition 7.3 we now turn to models induced from the
Gorenstein projective cotorsion pair \((\mathcal{GP}, W)\). More on this appears in the next Section. We now just look at the case when \(R\) is a Gorenstein ring.

**Example 7.7.** Let \(R\) be a Gorenstein ring and again assume \(\text{gl.dim}(R) = \infty\). Recall that \(R\) must have finite injective dimension when considered as a module over itself and that these dimensions must coincide. Here assume \(\text{id}(R) = d\). Also recall that a module \(M\) over \(R\) has finite injective dimension if and only if it has finite projective dimension if and only if it has finite flat dimension and that if this is the case then all these dimensions must be \(\leq d\). Call these the modules of finite \(R\)-dimension and let \(W\) denote the class of all these modules. Then it was shown in [Hov02] that \((W, GI)\) is a complete cotorsion pair where \(GI\) are the Gorenstein injective complexes and that \((\mathcal{GP}, W)\) is a complete cotorsion pair where \(GP\) are the Gorenstein projective complexes. Then we have the following Gorenstein version of Proposition 7.4.

**Proposition 7.8.** Let \(R\) be a Gorenstein ring with \(\text{id}(R) = d\) and let \((\mathcal{GP}, W)\) be the Gorenstein projective cotorsion pair. Then \(dw\tilde{\mathcal{GP}} = dg\tilde{\mathcal{GP}}\) and \(ex\tilde{\mathcal{GP}} = \tilde{\mathcal{GP}}\).

**Proof.** Let \(G \in ex\tilde{\mathcal{GP}}\). We wish to show \(Z_nG \in \mathcal{GP}\). So let \(W \in W\) and we must show \(\text{Ext}^1_R(Z_nG, W) = 0\). But since \(\text{Ext}^1_R(C, W) = 0\) for all \(C \in \mathcal{GP}\) we can dimension shift to get \(\text{Ext}^1_R(Z_nG, W) \cong \text{Ext}^{d+1}_R(Z_{n-d}G, W)\). But since \(W\) must have finite injective dimension less than or equal to \(d\) we get that this last group equals 0. Therefore \(G \in \mathcal{GP}\). So \(ex\tilde{\mathcal{GP}} = \mathcal{GP}\). The fact that \(dw\tilde{\mathcal{GP}} = dg\tilde{\mathcal{GP}}\) is true by combining [GH10, Theorem 3.11] with Corollary 8.1 below.

We conclude that when \(R\) is Gorenstein of infinite global dimension, applying Proposition 7.4 to both the categorical projective and the Gorenstein projective pairs generally leads to 8 model structures on \(\text{Ch}(R)\). This is illustrated concretely in the next example where all 8 model structures are distinct.

**Example 7.9.** Let \(R = \mathbb{Z}_4\), the ring of integers mod 4 and consider \(\text{Ch}(R)\). Then as described in Example 7.6 the projective cotorsion pair \((\mathcal{P}, \mathcal{A})\) on \(R\)-Mod gives rise to the four projective cotorsion pairs \((dw\mathcal{P}, (dw\mathcal{P})^\perp)\), \((ex\mathcal{P}, (ex\mathcal{P})^\perp)\), \((\mathcal{P}, \text{Ch}(R))\), and \((dg\mathcal{P}, \mathcal{E})\) in \(\text{Ch}(R)\). These classes are indeed distinct because the complex \(
\cdots \mathbb{Z}/4 \xrightarrow{x^2} \mathbb{Z}/4 \xrightarrow{x^2} \mathbb{Z}/4 \cdots \)

is in \(ex\mathcal{P}\) but not \(\mathcal{P}\), and so also \(dw\mathcal{P}\) but not \(dg\mathcal{P}\). Recall that \(R\) is quasi-Frobenius meaning that the class of injective modules coincides with the class of projective modules. It follows that the Gorenstein projective cotorsion pair on \(R\)-Mod is \((\mathcal{A}, \mathcal{I})\) where \(\mathcal{I}\) is the class of injective/projective \(R\)-modules. The four associated projective cotorsion pairs from Example 7.7 turn out to be \((\text{Ch}(R), \mathcal{I})\) and \((\mathcal{E}, dg\mathcal{I})\) and \((ex\mathcal{I}, ex\mathcal{I})\) and \((dw\mathcal{I}, dw\mathcal{I})\). These eight classes of cofibrant objects are distinct and are related to each other as shown in the cube shaped lattice below.
8. Gorenstein models for the derived category and recollements

Assume $R$ is any Noetherian ring and let $\mathcal{GI}$ denote the class of Gorenstein injective modules. From [BGH13] we know that the Gorenstein injectives are part of an injective cotorsion pair $(W, \mathcal{GI})$ giving rise to a model structure on $R$-Mod. The resulting model structure on $R$-Mod is called the Gorenstein injective model structure on $R$-Mod and coincides with the one in [Hov02] when $R$ is a Gorenstein ring. Now applying Proposition 7.2 we potentially get 6 injective model structures on $Ch(R)$. As illustrated by the examples in the previous section, the number of cotorsion pairs on $Ch(R)$ induced by $(W, \mathcal{GI})$ and the canonical $(A, I)$ increases as we consider more general rings. In particular, when $R$ is non-Gorenstein there are indeed many injective model structures on $Ch(R)$. Three particular model structures appearing in [BGH13] are the following:

1. $M_1 = (\perp_{dw\overline{I}}, dw\overline{I}) = \text{The injective model for the homotopy category of all complexes of injectives (or coderived category in the language of [Pos11]).}$
2. $M_2 = (\perp_{e\overline{xI}}, e\overline{xI}) = \text{The injective model for the stable derived category.}$
3. $M_3 = (E, dg\overline{I}) = \text{The injective model for the usual derived category.}$

These are linked through Krause’s recollement [Kra05] indicated below.

Recall that $K(DG\text{-}Inj) \cong D(R)$ and here the notation such as $E(M_3)$ and $C(M_3)$ respectively represent using enough injectives or enough projectives with respect to that cotorsion pair. This corresponds to taking special preenvelopes or precovers.

It is not our purpose at this point to make a detailed study of all of the analogous Gorenstein derived categories. We simply wish to illustrate the usefulness of Theorem 4.6 by presenting three Gorenstein analogs of Krause’s recollement. These appear in Theorem 8.2 and the Gorenstein projective analogs appear in Theorem 8.4. In the next section we point out two new and interesting recollement
situations involving these derived categories and in Section 10 we see an extension to arbitrary rings $R$. We now set some language following the language used in [BCH13].

- $dw\tilde{\mathcal{I}}$ is the class of (categorical) Gorenstein injective complexes by Corollary 8.1 below. We call the model structure $(dw\tilde{\mathcal{I}}, dw\tilde{\mathcal{I}})$ the Gorenstein injective model structure on $\text{Ch}(R)$.
- $ex\tilde{\mathcal{I}}$ is the class of exact Gorenstein injective complexes. We call the model structure $(ex\tilde{\mathcal{I}}, ex\tilde{\mathcal{I}})$ the exact Gorenstein injective model structure on $\text{Ch}(R)$.
- $dg\tilde{\mathcal{I}} = \tilde{\mathcal{W}}\perp$ is the class of DG-Gorenstein injective complexes. We call the model structure $(\tilde{\mathcal{W}}, dg\tilde{\mathcal{I}})$ the DG-Gorenstein injective model structure on $\text{Ch}(R)$.
- $\tilde{\mathcal{I}}$ is the class of exact DG-Gorenstein injective complexes. We call the model structure $(dg\tilde{\mathcal{W}}, \tilde{\mathcal{I}})$ the exact DG-Gorenstein injective model structure on $\text{Ch}(R)$.

By Proposition 3.9, two chain maps $f, g : X \to F$ in $\text{Ch}(R)$ (where $F$ is fibrant) are formally homotopic in any of these model structures if and only if their difference factors through an injective complex. But injective complexes are contractible and it follows that the two maps are homotopic if and only if they are chain homotopic in the usual sense. So, for example, the homotopy category of the Gorenstein injective model structure is equivalent to $K(GInj)$, the chain homotopy category of the Gorenstein injective complexes. Similarly, the homotopy category of the exact Gorenstein injective model structure will be denoted $K_{ex}(GInj)$. Then we have the DG-versions which we will denote by $K(DG-GInj)$ and $K_{ex}(DG-GInj)$.

**Remark 9.** We resist any urge to give names to the complexes in $dw\tilde{\mathcal{W}}$ and $ex\tilde{\mathcal{W}}$ and $dg\tilde{\mathcal{W}}$ at this point. However, note the following. Since $\mathcal{W}$ contains all the projective modules, $dw\tilde{\mathcal{W}}$ (resp. $ex\tilde{\mathcal{W}}$, resp. $dg\tilde{\mathcal{W}}$) contains all complexes of projectives (resp. exact complexes of projectives, resp. DG-projective complexes). Moreover, since $\mathcal{W}$ contains the injectives, $dw\tilde{\mathcal{W}}$ contains all complexes of injectives and $ex\tilde{\mathcal{W}}$ contains the exact complexes of injectives.

We have the following portion of the semilattice of injective cotorsion pairs in $\text{Ch}(R)$. But note that we have not even included the model structures corresponding to $dw\tilde{\mathcal{W}}\perp$ and $ex\tilde{\mathcal{W}}\perp$, which are still interesting especially in light of Theorem 9.1.
8.1. Gorenstein injective complexes. We now wish to characterize the categorical Gorenstein injective complexes, show that these are the fibrant objects in a model structure on $\text{Ch}(R)$ having $\mathcal{D}(R)$ as its homotopy category, and embed these homotopy categories in a Gorenstein version of Krause’s recollement.

It has been known for some time that over certain rings, especially Gorenstein rings, that the Gorenstein injective (resp. Gorenstein projective) complexes are precisely those complexes $X$ for which each $X_n$ is Gorenstein injective (resp. Gorenstein projective). For example, see [GR99]. Enochs, Estrada, and Iacob have shown this for Gorenstein projective complexes over a commutative Noetherian ring admitting a dualizing complex [EE05]. Moreover, we see from [YL11] that this is true for any ring $R$. But Theorem 5.2 says that the Gorenstein injective complexes sit on the top of the semilattice of injective cotorsion pairs. So the following is a quick and elegant proof of the above fact which works at this point for any Noetherian ring $R$.

**Corollary 8.1.** Let $R$ be any ring for which we know that the Gorenstein injective cotorsion pair $(\mathcal{W}, \mathcal{GI})$ is cogenerated by a set. Then $(\perp \text{dw}\mathcal{GI}, \text{dw}\mathcal{GI})$ is also cogenerated by a set and $\text{dw}\mathcal{GI}$ is exactly the class of Gorenstein injective complexes.

**Proof.** We choose to prove the projective version this time. See proof of Corollary 8.3.

**Theorem 8.2.** Let $R$ be any Noetherian ring. Then for the three choices of $\mathcal{M}_1$ and $\mathcal{M}_2$ as indicated below these are injective cotorsion pairs in $\text{Ch}(R)$ with $\mathcal{M}_2 \leq_r \mathcal{M}_1$ having right localization $\mathcal{M}_1/\mathcal{M}_2$ a model for the derived category $\mathcal{D}(R)$.

1. $\mathcal{M}_1 = (\perp \text{dw}\mathcal{GI}, \text{dw}\mathcal{GI})$ and $\mathcal{M}_2 = (\perp \text{ex}\mathcal{GI}, \text{ex}\mathcal{GI})$.
2. $\mathcal{M}_1 = (\mathcal{W}, \text{dg}\mathcal{GI})$ and $\mathcal{M}_2 = (\text{dg}\mathcal{W}, \mathcal{GI})$.
3. $\mathcal{M}_1 = (\text{ex}\mathcal{W}, (\text{ex}\mathcal{W})^\perp)$ and $\mathcal{M}_2 = (\text{dw}\mathcal{W}, (\text{dw}\mathcal{W})^\perp)$.

Furthermore, each case gives a recollement with the usual derived category as indicated by Theorem 4.6. For example, the first gives the recollement

$\begin{array}{ccc}
K_{\text{ex}}(\text{GI}n) & \xrightarrow{E(\mathcal{M}_2)} & K(\text{GI}n) \\
\downarrow & & \downarrow \\
C(\mathcal{M}_3) & \xrightarrow{I} & K(\text{DG-Inj}) \\
\end{array}$

Recall that $K(\text{DG-Inj}) \cong \mathcal{D}(R)$ and the notation $E(\mathcal{M}_3)$ and $C(\mathcal{M}_3)$ respectively represent using enough injectives or enough projectives with respect to the cotorsion pair $\mathcal{M}_3 = (\mathcal{E}, \text{dg}\mathcal{I})$. (So taking DG-injective preenvelopes or Exact precovers.)

**Proof.** We set $\mathcal{M}_3 = (\mathcal{E}, \text{dg}\mathcal{I})$. According to Theorem 4.6 in each case we just need to show $F_2, F_3 \subseteq F_1$ and one of the other two conditions. For statements (1) and (2) we will show $\mathcal{E} \cap F_1 = F_2$. Then we use the other condition in Theorem 4.6 to prove (3).
Take the first pair, \( \mathcal{M}_1 = (\perp \text{dg}\mathcal{G}I, \text{dg}\mathcal{G}I) \) and \( \mathcal{M}_2 = (\perp \text{ex}\mathcal{G}I, \text{ex}\mathcal{G}I) \). They are injective cotorsion pairs in \( \text{Ch}(R) \) by Proposition 7.2. Since \( \mathcal{F}_1 = \text{dg}\mathcal{G}I \) lies at the top of the semilattice it clearly contains both \( \mathcal{F}_2 = \text{ex}\mathcal{G}I \) and \( \mathcal{F}_3 = \text{dg}\mathcal{G}I \). It is also clear that \( \mathcal{E} \cap \text{dg}\mathcal{G}I = \text{ex}\mathcal{G}I \). So we get the recollement for (1).

Now take the second pair, \( \mathcal{M}_1 = (\tilde{W}, \text{dg}\mathcal{G}I) \) and \( \mathcal{M}_2 = (\text{dg}\tilde{W}, \text{ex}\mathcal{G}I) \). They are injective cotorsion pairs in \( \text{Ch}(R) \) by Proposition 7.2. We have \( \mathcal{E} \cap \text{dg}\mathcal{G}I = \text{ex}\mathcal{G}I \) since this in general holds from \cite{Gil04}. So all that is left is to show \( \text{dg}\mathcal{G}I \) in each degree. Furthermore any map \( f \) is DG-injective then it is injective in each degree, so Gorenstein injective in each degree. Furthermore any map \( f : E \to I \) where \( E \) is exact must be null homotopic. In particular, any map \( f : W \to I \) where \( W \in \tilde{W} \) must be null homotopic. This completes the proof of statement (2).

Next take the third pair, \( \mathcal{M}_1 = (\text{ex}\tilde{W}, (\text{ex}\tilde{W})^\perp) \) and \( \mathcal{M}_2 = (\text{dw}\tilde{W}, (\text{dw}\tilde{W})^\perp) \). In this setup we first need to see that the DG-injective complexes are in \( (\text{ex}\tilde{W})^\perp \). But \( (\text{ex}\tilde{W})^\perp \) is the class of all complexes \( Y \) of Gorenstein injective modules with the property that any map \( f : W \to Y \) is null homotopic whenever \( W \in \text{ex}\tilde{W} \). So we see that the DG-injectives are in this class. So following the setup to Theorem 4.11 we do have \( \mathcal{F}_2, \mathcal{F}_3 \subseteq \mathcal{F}_1 \) and we will now use the second condition of Theorem 4.10. That is, we will finish by showing \( \mathcal{W}_2 \cap \mathcal{W}_3 = \mathcal{W}_1 \) and \( \mathcal{F}_2 \subseteq \mathcal{W}_3 \). But \( \mathcal{W}_2 \cap \mathcal{W}_3 = \mathcal{W}_1 \) is clear in this case so lets look at \( \mathcal{F}_2 \subseteq \mathcal{W}_3 \). It is required to show that every complex in \( \mathcal{F}_2 = (\text{dw}\tilde{W})^\perp \) is exact. But as mentioned in the above Remark, the DG-projective complexes must be in \( \text{dw}\tilde{W} \), and so \( (\text{dw}\tilde{W})^\perp \) must be contained in the class of exact complexes. This completes the proof and we get the three recollements.

**Remark 10.** Theorem 8.2 holds for any ring \( R \) for which we know that the Gorenstein injective cotorsion pair \( (\mathcal{W}, \mathcal{G}I) \) is cogenerated by a set. In fact, whenever \( (\mathcal{W}, \mathcal{F}) \) is an injective cotorsion pair in \( R\text{-Mod} \) which is cogenerated by a set, then the analogous six lifted cotorsion pairs will give rise to three (not necessarily distinct or nontrivial) recollements. For example, starting with the canonical injective cotorsion pair \( (A, I) \), the analog of recollement (1) recovers Krause’s recollement. However, the analogs to recollements (2) and (3) are trivial!

**8.2. The Gorenstein projective complexes.** We now look at the dual situation. Here we assume that \( R \) is any coherent ring in which each flat module has finite projective dimension. (See Subsection 5.1 for examples of such coherent rings. But as we have already alluded to, one can replace this with the more general hypothesis of any ring \( R \) in which the so-called level modules have finite projective dimension. This is explained in Section 10.) With this hypothesis on \( R \) we have from [BGH13] that if \( \mathcal{GP} \) denotes the class of all Gorenstein projective modules then there is a projective cotorsion pair \( (\mathcal{GP}, \mathcal{W}) \) which is cogenerated by a set. (The class \( \mathcal{W} \) is in general different than the class \( \mathcal{W} = \perp \mathcal{GP} \), unless we know \( R \) is Gorenstein or some other nice conditions on the ring.) Applying Proposition 7.3 to this projective cotorsion pair results in 6 lifted projective model structures on \( \text{Ch}(R) \), all analogous to the Gorenstein injective versions considered above. We record the duals of the injective results, starting with the promised fact that the Gorenstein projective
complexes are precisely the complexes having a Gorenstein projective module in each degree.

**Corollary 8.3.** Let $R$ be any ring for which we know that the Gorenstein projective cotorsion pair $(\mathcal{GP}, \mathcal{W})$ is cogenerated by a set. Then $(\mathcal{dGP}, (\mathcal{dGP})^\perp)$ is also cogenerated by a set and $\mathcal{dGP}$ is exactly the class of Gorenstein projective complexes.

**Proof.** $(\mathcal{dGP}, (\mathcal{dGP})^\perp)$ is a projective cotorsion pair by Proposition 7.3. So by the “lattice theorem” Theorem 5.4 we have that $\mathcal{dGP}$ is contained in the class of all Gorenstein projective complexes. To finish we just need to show that if $G$ is Gorenstein projective then each $G_n$ must be Gorenstein projective. For this suppose $G$ is a Gorenstein projective chain complex so that there is an exact sequence of projective chain complexes

$$P = \cdots \to P^2 \to P^1 \to P^0 \to P^{-1} \to \cdots$$

for which $G = \ker (P^0 \to P^{-1})$ and which will remain exact after application of $\text{Hom}_{\text{Ch}(R)}(-, Q)$ for any projective chain complex $Q$. Then for any projective module $M$ taking $Q$ to be the projective complex $D^{n+1}(M)$ we get exactness of the complex $\text{Hom}_{\text{Ch}(R)}(P, D^{n+1}(M)) \cong \text{Hom}_R(P_n^*, M)$, where $P_n^*$ denotes the complex

$$P_n^* = \cdots \to P_n^2 \to P_n^1 \to P_n^0 \to P_n^{-1} \to \cdots$$

Of course $P_n^*$ is an exact complex of projective modules and since we also have $G_n = \ker (P_n^0 \to P_n^{-1})$ we conclude that $G_n$ is a Gorenstein projective module.

**Remark 11.** We use the notation $\mathcal{M}_2 \backslash \mathcal{M}_1$ from [Bec12] to denote the left localization of a projective cotorsion pair $\mathcal{M}_1$ by another $\mathcal{M}_2$ having $\mathcal{C}_2 \subseteq \mathcal{C}_1$. This notation emphasizes a left localization.

**Theorem 8.4.** Let $R$ be a coherent ring in which all flat modules have finite projective dimension. Then for the three choices of $\mathcal{M}_1$ and $\mathcal{M}_2$ as indicated below these are projective cotorsion pairs in $\text{Ch}(R)$ with $\mathcal{M}_2 \subseteq \mathcal{M}_1$ having left localization $\mathcal{M}_2 \backslash \mathcal{M}_1$ a model for the derived category $\mathcal{D}(R)$.

1. $\mathcal{M}_1 = (\mathcal{dGP}, (\mathcal{dGP})^\perp)$ and $\mathcal{M}_2 = (\mathcal{exGP}, (\mathcal{exGP})^\perp)$.
2. $\mathcal{M}_1 = (\mathcal{dGP}, \mathcal{W})$ and $\mathcal{M}_2 = (\mathcal{GP}, \mathcal{dW})$.
3. $\mathcal{M}_1 = (\mathcal{exW}, \mathcal{exW})$ and $\mathcal{M}_2 = (\mathcal{dW}, \mathcal{dW})$.

Furthermore, each case gives a recollement with the usual derived category as indicated by Theorem 4.7. For example, the first gives the recollement

$$
\begin{array}{ccc}
E(\mathcal{M}_3) & \overset{I}{\longrightarrow} & K(\text{GP} \text{proj}) \\
\downarrow & & \downarrow \\
C(\mathcal{M}_2) & & K(\text{DG} \text{-proj})
\end{array}
$$

Recall that $K(\text{DG} \text{-proj}) \cong \mathcal{D}(R)$ and the notation $E(\mathcal{M}_3)$ and $C(\mathcal{M}_3)$ respectively represent using enough injectives or enough projectives with respect to the cotorsion pair $\mathcal{M}_3 = (\mathcal{dGP}, \mathcal{E})$. (So taking Exact preenvelopes or DG-projective precovers.)
**Remark 12.** Theorem 8.4 holds for any ring \( R \) for which we know that the Gorenstein projective cotorsion pair \((GP, W)\) is cogenerated by a set. So as described in Section 10, it holds for any ring \( R \) for which all level modules have finite projective dimension.

### 9. More recollement situations

In this section we point out two more interesting recollements, both injective and projective versions, arising from the classes of Gorenstein complexes introduced in Section 8. Note that since the class \( dw\tilde{\mathcal{I}} \) of Gorenstein injective complexes sits on the top of the semilattice of injective cotorsion pairs, Theorem 4.6 says we get a recollement situation whenever we notice \( W \cap dw\tilde{\mathcal{I}} = \mathcal{F}_2 \).

**Theorem 9.1.** Let \( R \) be a Noetherian ring. Then we have the two recollement situations below.

\[
\begin{array}{c}
\text{K(Inj)} & \xrightarrow{E(M_2)} & \text{K(GInj)} \\
\downarrow I & & \downarrow I \\
\text{C(M_3)} & & \text{K((dw\tilde{\mathcal{W}})\perp)} \\
\end{array}
\]

AND

\[
\begin{array}{c}
\text{K(ex(Inj))} & \xrightarrow{E(M_2)} & \text{K(GInj)} \\
\downarrow I & & \downarrow I \\
\text{C(M_3)} & & \text{K((ex\tilde{\mathcal{W}})\perp))} \\
\end{array}
\]

**Proof.** Apply Theorem 4.6. For the first, we take \( M_1 = (\downarrow dw\tilde{\mathcal{I}}, dw\tilde{\mathcal{I}}) \), \( M_2 = (\downarrow dw\tilde{\mathcal{I}}, dw\tilde{\mathcal{I}}) \) and \( M_3 = (dw\tilde{\mathcal{W}}, (dw\tilde{\mathcal{W}})^\perp) \). It is clear that \( dw\tilde{\mathcal{W}} \cap dw\tilde{\mathcal{I}} = dw\tilde{\mathcal{I}} \) since \( \mathcal{W} \cap \mathcal{I} \) is the class of injective modules.

For the second, we take \( M_1 = (\downarrow dw\tilde{\mathcal{I}}, dw\tilde{\mathcal{I}}) \), \( M_2 = (\downarrow ex\tilde{\mathcal{I}}, ex\tilde{\mathcal{I}}) \) and \( M_3 = (ex\tilde{\mathcal{W}}, (ex\tilde{\mathcal{W}})^\perp) \). Again, it is clear that \( ex\tilde{\mathcal{W}} \cap dw\tilde{\mathcal{I}} = ex\tilde{\mathcal{I}} \) since \( \mathcal{W} \cap \mathcal{I} \) is the class of injective modules. \( \square \)

**Remark 13.** The dual to Theorem 9.1 holds whenever \( R \) is a ring for which we know that the Gorenstein projective cotorsion pair is cogenerated by a set. In particular, we have the dual whenever \( R \) is coherent and all flat modules have finite projective dimension.

**Example 9.2.** Let \( R \) be the quasi-Frobenius ring \( \mathbb{Z}/4 \) as in Example 7.9. Then the second recollement in Theorem 9.1 above gives a recollement \( S(R) \xrightarrow{K} K(R) \xrightarrow{K} K(W_{\text{proj}}) \) where \( S(R) = K_{\text{ex}}(\text{Inj}) \) is the stable derived category of \( R \), \( K(R) \) is the usual homotopy category and \( W_{\text{proj}} = ex\tilde{\mathcal{P}}\perp = ex\tilde{\mathcal{I}}\perp \) are the trivial objects in the model for the projective stable derived category, \( S_{\text{proj}}(R) \), of \( [BGH13] \). As shown in \( [BGH13] \), we have \( W_{\text{proj}} \neq W_{\text{inj}} \) but \( S_{\text{proj}}(R) \cong S(R) \) through a natural Quillen equivalence.
10. RECOLLEMENTS INVOLVING COMPLEXES OF GORENSTEIN AC-INJECTIVES

In this section we extend the results of Sections 8 and 9 to general rings. This depends on a generalization of Gorenstein homological algebra to arbitrary rings, provided by \[\text{BGH13}\]. We now briefly explain this, but we refer the reader to \[\text{BGH13}\] for the full details.

A main idea of \[\text{BGH13}\] is that Gorenstein homological algebra only seems to work well over Noetherian rings. We see from \[\text{Gil10}\] that the theory extends to coherent rings if we alter the definition of Gorenstein injective (resp. Gorenstein projective) to get the so-called Ding injective (resp. Ding projective) modules. In \[\text{BGH13}\] we see that the generalization goes beyond coherent rings to general rings \(R\). Here we get what we call the Gorenstein AC-injective and Gorenstein AC-projective modules which we define shortly. When \(R\) is coherent, they agree with the Ding modules of \[\text{Gil10}\]. For Noetherian rings, the Gorenstein AC-injective modules coincide with the usual Gorenstein injectives. The Gorenstein AC-projective modules are a bit more subtle. They do coincide with the usual Gorenstein projectives whenever \(R\) is Noetherian and has a dualizing complex. But as we indicate below, it is not because the ring is Noetherian. It is because certain modules we call level, which for Noetherian rings are the flat modules, have finite projective dimension. It is the existence of a dualizing complex which forces this \[\text{Jør05}\].

In more detail, a module \(N\) is said to be of type \(FP_{\infty}\) if it has a projective resolution by finitely generated projectives. Over coherent rings these are precisely the finitely presented modules, and over Noetherian rings these are precisely the finitely generated modules. Loosely speaking, finitely generated modules are well-behaved over Noetherian rings, while finitely presented modules are well-behaved over coherent rings. So as we relax the ring from Noetherian to coherent we need to sharpen the notion of a “finite” module. So the idea is that for a general ring \(R\), it is the type \(FP_{\infty}\) modules that are the well-behaved “finite” modules.

Now let \(I\) be an \(R\)-module. Note that \(I\) is injective if and only if \(\text{Ext}^1_R(N, I) = 0\) for all finitely generated modules \(N\). More generally, \(I\) is called absolutely pure (or \(FP\)-injective) if \(\text{Ext}^1_R(N, I) = 0\) for all finitely presented modules \(N\). Still more generally, for reasons described in \[\text{BGH13}\] we call \(I\) absolutely clean (or \(FP_{\infty}\)-injective) if \(\text{Ext}^1_R(N, I) = 0\) for all modules \(N\) of type \(FP_{\infty}\). For coherent rings, we have absolutely clean = absolutely pure. For Noetherian rings, absolutely clean = absolutely pure = injective. We point out that \(FP_{\infty}\)-modules have been studied by Livia Hummel in \[\text{Mil08}\].

Recall that for a left \(R\)-module \(N\), its character module is the right \(R\)-module \(N^+ = \text{Hom}_R(N, Q/Z)\). For Noetherian rings \(R\), it is known that \(N\) is flat if and only if \(N^+\) is injective and also \(N\) is injective if and only if \(N^+\) is flat. (See \[\text{LJ00}\].) For a coherent ring \(R\) it is known that \(N\) is flat if and only if \(N^+\) is absolutely pure and \(N\) is absolutely pure if and only if \(N^+\) is flat. (See \[\text{FH72}\].) In order for this duality to extend to non-coherent rings we need the following definition, which is an extensions of the notion of flatness: A left \(R\)-module \(N\) is called level if \(\text{Tor}^1_R(M, N) = 0\) for all right \(R\)-modules \(M\) of type \(FP_{\infty}\). Now for a general ring \(R\), we get \(N\) is level if and only if \(N^+\) is absolutely clean and \(N\) is absolutely clean if and only if \(N^+\) is level. This duality captures the above dualities for the special cases of coherent and Noetherian rings. Moreover there are interesting characterizations of coherent rings in terms of level modules and Noetherian rings.
rem 8.2 while the fourth and fifth correspond to Theorem 9.1. When

Note that the first three recollements correspond to those in The o-

Remark 14.

Proof. We now define a module \( M \) to be Gorenstein AC-

injective if \( M = Z_0 X \) for some exact complex of injectives \( X \) for which \( \text{Hom}_R(I, X) \) remains exact for any absolutely clean module \( I \). We see that when \( R \) is coherent, the Gorenstein AC-injective modules coincide with the Ding injective modules of [GHI10] and when \( R \) is Noetherian, the Gorenstein AC-injective modules coincide with the usual Gorenstein injective modules. On the other hand we define a module \( M \) to be Gorenstein AC-projective if \( M = Z_0 X \) for some exact complex of projectives \( X \) for which \( \text{Hom}_R(X, L) \) remains exact for all level modules \( L \). Then for coherent rings, these coincide with the Ding projective modules of [GHI10]. If every level module has finite projective dimension then they coincide with the usual Gorenstein projectives. In particular, when \( R \) is coherent and satisfies that all flat modules have finite projective dimension, then the Gorenstein AC-projectives are exactly the Gorenstein projectives. A main result of [BGH13] is the following, which

appear there as Theorem 5.5/Proposition 5.10 and Theorem 8.5/Proposition 8.10.

**Fact 10.1.** Let \( R \) be any ring. Denote the class of Gorenstein AC-injective modules by \( \mathcal{GI} \) and the class of Gorenstein AC-projective modules by \( \mathcal{GP} \). Then \( (\mathcal{GI}, \mathcal{GI}) \) is an injective cotorsion pair and \( (\mathcal{GP}, \mathcal{GP}) \) is a projective cotorsion pair. Each are cogenerated by a set.

This result allows for the following extension of the recollements in Sections 8

and 9 to arbitrary rings.

**Theorem 10.2.** Let \( R \) be any ring. Denote the class of all Gorenstein AC-injective modules by \( \mathcal{GI} \), and set \( \mathcal{W} = \perp \mathcal{GI} \). Then all of the pairs below are injective cotorsion pairs in \( \text{Ch}(R) \). For each choice of \( \mathcal{M}_1, \mathcal{M}_2, \) and \( \mathcal{M}_3 \) shown below, Theorem 4.9 yields a recollement.

1. \( \mathcal{M}_1 = (dw\mathcal{GI}, dw\mathcal{GI}), \quad \mathcal{M}_2 = (ex\mathcal{GI}, ex\mathcal{GI}), \quad \mathcal{M}_3 = (\mathcal{E}, dg\mathcal{I}) \)

2. \( \mathcal{M}_1 = (\mathcal{W}, dg\mathcal{GI}), \quad \mathcal{M}_2 = (dg\mathcal{W}, \mathcal{GI}), \quad \mathcal{M}_3 = (\mathcal{E}, dg\mathcal{I}) \)

3. \( \mathcal{M}_1 = (ex\mathcal{W}, (ex\mathcal{W})'), \quad \mathcal{M}_2 = (dw\mathcal{W}, (dw\mathcal{W})'), \quad \mathcal{M}_3 = (\mathcal{E}, dg\mathcal{I}) \)

4. \( \mathcal{M}_1 = (dw\mathcal{GI}, dw\mathcal{GI}), \quad \mathcal{M}_2 = (dw\mathcal{I}, dw\mathcal{I}), \quad \mathcal{M}_3 = (dw\mathcal{W}, (dw\mathcal{W})') \)

5. \( \mathcal{M}_1 = (ex\mathcal{GI}, ex\mathcal{GI}), \quad \mathcal{M}_2 = (ex\mathcal{I}, ex\mathcal{I}), \quad \mathcal{M}_3 = (ex\mathcal{W}, (ex\mathcal{W})') \)

**Remark 14.** Note that the first three recollements correspond to those in Theorem 8.2 while the fourth and fifth correspond to Theorem 9.1. When \( R \) is Noetherian they coincide with those recollements. Moreover, we have the dual statements corresponding to the three recollements of Theorem 8.4 and the two in Remark 13. They coincide with those recollements whenever all level modules have finite projective dimension. In particular, whenever \( R \) is coherent and all flat modules have finite projective dimension.

**Proof.** Since \( (\mathcal{W}, \mathcal{GI}) \) is an injective cotorsion pair in \( \text{R-Mod} \) which is cogenerated by a set, it follows then from Proposition 7.2 that all the above are injective cotorsion pairs in \( \text{R-Mod} \). Now the proof of the first three recollements follows exactly
as in Theorem 8.2 while the fourth and fifth follows just as in Theorem 9.1. It all works the same way since $W \cap GI$ equals the class of injective modules, making it easy to verify the hypotheses of Theorem 4.6.

\[ \square \]

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