\( \mathcal{N} = 1 \) Super Yang-Mills from Supergravity: The UV-IR Connection

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We consider the Maldacena-Nuñez supergravity solution corresponding to \( \mathcal{N} = 1 \) super Yang-Mills in the IR. The MN gravity solution corresponds to a large number of NS or D fivebranes wrapped on a two sphere. This supergravity solution is regular and breaks the expected manner. Recently the authors in [2, 3] discussed the regular and breaks the supergravity solution. They considered in details the supergravity solutions corresponding to \( N \) D5-branes wrapped on a two-sphere that were first found in [1] and they then considered the massless open-string modes propagating on the flat part of the D5 world-volume. An effective action at energies where the higher string modes and the gravitational dual of the gaugino condensate. These authors considered in details the supergravity solutions corresponding to \( N \) D5-branes wrapped on a two-sphere that were first found in [1] and [2] and they then considered the massless open-string modes propagating on the flat part of the D5 world-volume. An effective action at energies where the higher string modes and the Kaluza-Klein excitations around the two-cycles decouple was found. The resulting theory is four-dimensional \( SU(N) \) super Yang-Mills theory. In the following we consider only the most interesting case \( \mathcal{N} = 1 \). The MN supergravity solution is dual to a gauge theory with four supercharges in four dimensions. In the \( SO(4) \) gauged supergravity Lagrangian one of the two \( SU(2) \) factors inside \( SO(4) \) is gauged by imposing an antiself duality constraint. This field has a Yang-Mills coupling \( \lambda \) which is dimensionfull. The inverse square of the four-dimensional coupling is proportional to the volume of the two-sphere on which the D5 branes are wrapped. It then turns out that the coupling constant \( g_{\text{YM}} \) and the vacuum angle \( \theta_{\text{YM}} \) can be expressed in terms of the ten-dimensional supergravity solution representing the \( N \) wrapped D5 branes in an explicit form,

\[ \frac{1}{g_{\text{YM}}^2} = G(\rho), \]  

found by Di Vecchia, Lerda and Merlatti [4]. Here \( \rho \) is a dimensionless radial variable the scale of which is given in terms of the dimensionfull coupling \( \lambda \), \( \rho = \lambda \rho \), and \( G \) is a known function. To obtain this result a Born-Infeld analysis was used.

In general the radial variable \( \rho \) should be considered as a function of the four-dimensional field theory scale \( \mu/\Lambda \), where \( \mu \) is the variable scale and \( \Lambda \) is a renormalization group invariant scale. The aim of this work is to investigate in some detail such a relation. In Reference [2] this connection was deduced by relating the MN solution to the non-vanishing of the gaugino condensate \( \langle \bar{\psi} \psi \rangle \) in \( \mathcal{N} = 1 \) super Yang-Mills without matter. As we shall see there is a considerable amount of freedom in the choice of the relation, and some guiding principle is needed. The relation [5] is, however, free of any such ambiguity, and hence we first compute the following beta-function

\[ \beta_{\rho}(g_{\text{YM}}) = \frac{\partial g_{\text{YM}}}{\partial \rho}. \]  

(2)

Since \( \rho = \rho(\mu/\Lambda) \), this is a well defined beta-function, which can be computed directly. As our guiding principle we then assume that any “physical” result which can be derived from \( \beta_{\rho} \), should hold also for

\[ \beta(g_{\text{YM}}) = \frac{\partial g_{\text{YM}}}{\partial \ln \mu/\Lambda}. \]  

(3)

It is well known that in general this \( \beta \) is rather arbitrary since it is scheme dependent. However, asymptotic freedom and a possible IR fixed point in \( \beta \) are expected to be scheme independent and as such to be relevant “physical” information.

Now it turns out that \( \beta_{\rho} \) possesses an IR fixed point. In going from \( \beta_{\rho} \) to \( \beta \), this fixed point can be “removed” only by an infrared singular transformation. According to the assumption that an IR fixed point is physical such a singular transformation is not allowed. This is a welcome feature. Indeed for the correspondence between supergravity and gauge theories to be meaningful we should not lose the precious information carried by supergravity computations.

Although the persistence of the IR fixed point in going from eq. (2) to eq. (3) imposes some constraint on the

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relation \( \rho = \rho(\mu/\Lambda) \) there is still a large amount of freedom. The order of the fixed point in \( \beta \) is, for example, not fixed yet. We can resolve this ambiguity by investigating the situation in the UV region. Here the asymptotically free first order perturbative beta-function was shown to follow from the duality relation between the gaugino condensate and the MN solution introduced in [3]. However, from this relation the second order term in the beta function does not occur with the “right” coefficient. It was shown by ’t Hooft [8] that if one restricts to analytic changes in the coupling, the second term in \( \beta \) is universal. Our modification of the relation \( \rho = \rho(\mu/\Lambda) \) used in [3] can be adjusted in such a way that the second term in \( \beta \) gets the right coefficient. At the same time this can be used to predict the IR fixed point to be of first order, and the slope of \( \beta \) can also be computed near this fixed point.

We have so shown that if the supergravity–gauge theory relation is valid the \( N = 1 \) Yang-Mills theory possesses in the IR an infrared stable fixed point. In the UV this theory displays asymptotic freedom.

In Section II we briefly summarize some known aspects of the \( N = 1 \) Yang-Mills theory relevant to this paper. In Section III we first study in some detail the beta function defined with respect to the supergravity dimensionless radial coordinate. After having shown that in the supergravity variable the theory possesses a fixed point we suggest how to introduce a general relation between the supergravity variable and the four dimensional renormalization scale \( \mu \) still leading to a fixed point. In Section IV we constrain the newly introduced relation by requiring the UV result to match the known two loop universal coefficients of the super Yang-Mills beta function. This fixes for us also the behavior in the IR. We then show that super Yang-Mills possesses a non perturbative IR stable fixed point. We comment our results when concluding in Section V.

II. ASPECTS OF THE SUPER YANG-MILLS THEORY

The \( SU(N) \) super Yang-Mills theory can be compactly written as:

\[
\mathcal{L} = \frac{1}{4\pi} \text{Im} \left[ \int d^2 \theta \, \tau_{YM} \, \text{Tr} \left[ W^a W_a \right] \right],
\]

with

\[
\tau_{YM} = \frac{\theta}{2\pi} + i \frac{4\pi}{g_{YM}^2}.
\]

and \( W_a = W^a T^a \) with \( a = 1, \ldots, N^2 - 1 \). We normalize our generators according to

\[
\text{Tr} \left[ T^a T^b \right] = \frac{1}{2} \delta^{ab}, \quad \left[ T^a, T^b \right] = i f^{abc} T^c.
\]

At the classical level the theory possesses a \( U(1)_R \) symmetry which does not commute with the supersymmetric algebra. The ABJ anomaly breaks the \( U(1)_R \) symmetry to \( Z_{2N} \). Non perturbative effects trigger the gluino condensation leading to further breaking of \( Z_{2N} \) symmetry to a left over \( Z_2 \) symmetry and we have \( N \) equivalent vacua. The gluino condensate is a relevant ingredient to constrain the theory, also at large \( N \), see [8], [9], and [10]. In the literature it has been computed in different ways and we recall here its expression in a much studied scheme [8]:

- **Pauli Villars Scheme**

\[
\langle \lambda^2 \rangle = \text{Const.} \mu^3 \text{Im} \left[ \frac{\tau_{YM}}{4\pi} \right] e^{i \frac{\pi}{2} \tau_{YM}} = \text{Const.} \mu^3 \frac{1}{g_{YM}^2} e^{-\frac{\kappa^2}{N g_{YM}^2} e^{\frac{i\pi}{2}}} \qquad (7)
\]

The Pauli-Villars scheme leads to an expression for the gluino condensate which is not holomorphic in \( \tau_{YM} \). The gluino condensate is a universal physical constant independent on the scheme. This fact allows us to compute the perturbative beta function in different schemes since the respective coupling constant must depend on the scale in such a way to compensate the dependence on \( \mu \). It also enables us to establish a relation between the coupling constants among different schemes.

It is possible to define a scheme in which the gluino possesses an holomorphic dependence in \( \tau_{YM} \). The holomorphic scheme is very constrained leading to a pure one-loop type of running. This is a welcome feature from the point of view of the supercurrent chiral multiplet [8]. The two coupling constants are related in the following way:

\[
\tau_{YM}^H = \tau_{YM} - \frac{N}{2\pi} \text{Im} \left[ \frac{\tau_{YM}}{4\pi} \right].
\]

Note that the two couplings are connected via a non analytical transformation. In the following we will use the Pauli-Villars scheme or any other scheme analytically related to it. Some of the aspects of the schemes non analytically related to the Pauli-Villars one from the point of view of supergravity will be investigated in [10].

The independence on the scale of the gluino condensate leads to the following beta function [8]:

\[
\beta(g_{YM}) = -\frac{3N}{16\pi^2} g_{YM}^3 \left[ 1 - \frac{N g_{YM}^2}{8\pi^2} \right]^{-1}.
\]

Where \( \beta(g_{YM}) = \frac{\partial g_{YM}}{\partial L} \) and \( L = \ln(\mu/\Lambda) \) while \( \mu \) is the renormalization scale and \( \Lambda \) a reference scale.

Although this coupling may capture the all order perturbative terms the non perturbative contributions (if any) to the complete beta functions are still missing.

It is worth mentioning that the universality argument for the two loop beta function coefficient is valid for all of the infinite renormalization schemes whose coupling constants are related by analytical transformations [8]. It is also expected that since the presence of a fixed point
carries physical information as such should be renormalization scheme independent [3].

If supergravity-gauge theory correspondence is valid we should be able to adopt different renormalization schemes when connecting the supergravity solutions to gauge theories. Besides for this correspondence to be meaningful we should not lose the precious information carried in supergravity calculations. We will address these issues in the forthcoming sections.

III. WRAPPED BRANES: THE NONPERTURBATIVE FIXED POINT

In [3] the coupling $g_{YM}^2 N$ was studied as a function of the (dimensionless) radial variable $\rho$. In particular, it was shown that

$$\frac{1}{g_{YM}^2 N} = \frac{Y(\rho)}{16\pi^2} E\left(\sqrt{\frac{Y(\rho) - 1}{Y(\rho)}}\right),$$  \hspace{1cm} (10)

where the function $Y$ is given by

$$Y(\rho) = 4\rho \coth 2\rho - 1,$$  \hspace{1cm} (11)

and $E$ is a complete elliptic integral

$$E(k) = \int_0^{\pi/2} d\phi \sqrt{1 - k^2 \sin^2 \phi}.$$  \hspace{1cm} (12)

In deriving this result supergravity was used. In principle, in the infrared limit $\rho \to 0$ the superstring (to which supergravity is an approximation) should be used. However, the supergravity metric is non-singular for the $N' = 1$ case, in contrast to the $N' = 2$ case. Therefore it still makes sense to ask for the infrared behavior even in the supergravity approximation. We therefore expand eq. (10) around $\rho = 0$,

$$g_{YM} \sqrt{N} \approx g_c \sqrt{N}(1 - \rho^2 + \frac{29}{15} \rho^4 + ...),$$  \hspace{1cm} (13)

where the critical coupling $g_c \sqrt{N} = 4\sqrt{2\pi}$ is reached for $\rho = 0$, which is a non-singular point in the supergravity metric, as already mentioned.

From eq. (13) it thus seems as if the coupling “stops” at some finite value for $\rho \to 0$. Ultimately we would like to have a connection between $\rho$ and the field theory scale $\mu$. Without specifying any explicit relation, let us therefore consider the function $\rho = \rho(\mu/\Lambda)$, where $\Lambda$ is a fixed renormalization group invariant scale. Usually one considers the variation of the coupling with $\mu$ in terms of the $\beta$ function. However, since $\rho$ is a function of $\mu$, we could equally well define the beta-function

$$\beta_\rho(g_{YM}) \equiv \frac{\partial g_{YM}}{\partial \rho} \approx -2g_c \rho,$$  \hspace{1cm} (14)

where we used eq. (13) in the last step. From eq. (14) we see that the reason that the coupling “stops” for $\rho \to 0$ is that the $\beta_\rho$ function has a zero for $g_{YM} \to g_c$. The square root behavior of the zero in $\beta_\rho$ is not expected to occur in field theory with a conventional scale $\mu$. However, the scale $\rho$ is not conventional in supersymmetric Yang-Mills theory, and in general $\rho$ is expected to be a complicated function of $\mu/\Lambda$.

To obtain a conventional beta-function we should use the relation

$$\beta(g_{YM}) = \beta_\rho(g_{YM}) \frac{\partial \rho}{\partial L}, \quad L \equiv \ln \frac{\mu}{\Lambda}.$$  \hspace{1cm} (15)

From this relation it follows that the conventional beta-functions also have a zero at $g_{YM} = g_c$ provided the Jacobian

$$\frac{\partial \rho}{\partial L}$$

is non-singular when $g_{YM} = g_c$.

It is, of course, possible to choose a singular Jacobian [4] such that the zero in $\beta_\rho$ is removed. However, from the point of view of the supergravity dual this appears unnatural, since there is clearly a zero in the $\beta_\rho$ function. From the point of view of the field theory (forgetting about the supergravity) this also appears quite unnatural, since then the coupling $g_{YM}$ anyhow “stops”, but this time for no (good) reason, as we shall see in the following. Thus, based on the supergravity/super-Yang-Mills correspondence, it appears natural to assign the zero in the beta-function a physical value.

In [2, 3] the gravitational dual of the gaugino condensate was shown to be given by the function

$$a(\rho) = \frac{2\rho}{\sinh 2\rho},$$  \hspace{1cm} (17)

which is bounded between 0 ($\rho \to \infty$) and 1 ($\rho \to 0$). On the basis of the behavior of the gaugino condensate,

$$< \Lambda^2 > = \Lambda^3 = \text{renormalization group inv.},$$  \hspace{1cm} (18)

it was then argued in [3] that the supergravity dual $a(\rho)$ behaves like

$$a(\rho) = \frac{\Lambda^3}{\mu^3}.$$  \hspace{1cm} (19)

This fixes the connection between $\rho$ and $\mu$. However, since $a$ is bounded by 1, it is clear that one can never reach the infrared limit $\mu \to 0$, where $\Lambda/\mu$ becomes infinite. Instead, the lowest possible scale is $\mu = \Lambda$.

To analyze in more details what happens, let us consider the behavior of $a$ near $\rho = 0$, where it follows from eqs. (17) and (19) that

$$e^{-3L} \approx 1 - \frac{2}{3} \rho^2 + \frac{14}{45} \rho^4 + ...,$$  \hspace{1cm} (20)
leading to the Jacobian
\[ \frac{\partial \rho}{\partial L} \approx \frac{9}{4 \rho} + \frac{3}{5} \rho + \ldots . \] (21)

Thus we see that the connection \[\rho\rightarrow\rho\] leads to a singularity at \[\rho = 0\], and since from eq. (14) \[\beta_\rho \propto \rho\], it follows that the zero has been transformed away by a singular Jacobian. For \[g_{\text{YM}} \approx g_c\] we can easily compute the beta function based on the assumption (14),
\[\beta(g_{\text{YM}}) \approx g_c \left( -\frac{9}{2} + \frac{81}{5} \rho^2 + \ldots \right), \] (22)

which can also be expressed in terms of the coupling through eq. (13),
\[\beta(g_{\text{YM}}) \approx g_c \left( -\frac{9}{2} + \frac{81}{5} \frac{g_c - g_{\text{YM}}}{g_c} + \ldots \right). \] (23)

Thus we see that from the field theory point of view it appears as if one can go to higher coupling than \[g_{\text{YM}}\], since \[\beta\] has no singularities (poles or cuts) for \[g_{\text{YM}} > g_c\]. For somebody who does not know the supergravity connection, there is no way to understand why the coupling should come to a stop for \[\mu = \Lambda\]. Thus the field theory appears to be intrinsically mysterious.

The peculiar behavior discussed above is due to the use of eq. (13), which implies that \[\mu\] can never be smaller than \[\Lambda\] due to \[a \leq 1\]. Of course, one can argue that these peculiarities are due to the use of the supergravity approximation to the superstring, and hence one cannot trust the infrared behavior. However, as mentioned before, the metric is regular for \[\rho \to 0\], so there is no obvious reason to ignore the supergravity approximation. In any case, it is of interest to see if this approximation can be brought into a satisfactory context in super-Yang-Mills field theory. To this end, let us notice that eq. (13) a priori can be replaced by
\[a(\rho) = f(g_{\text{YM}}) \frac{\Lambda^3}{\mu^3}. \] (24)

where \[f\] is some function of the coupling \[g_{\text{YM}}\]. It should be emphasized that due to the considerable freedom in the choice of renormalization schemes, corresponding to different couplings, the function \[f\] is by no means unique.

Let us consider the IR limit \[\mu \to 0\], corresponding to \[\rho \to 0\]. From eq. (24) we clearly need \[f(g_{\text{YM}}) \to 0\] in order that \[a \to 1\]. Furthermore, since
\[\frac{\mu}{\Lambda} = \exp \left( \int_{g_c}^{g_{\text{YM}}} \frac{dx}{\beta(x)} \right) \] (25)

with \(g_* = g_{\text{YM}}(\mu = \Lambda)\), and since \(a \to 1\), we need
\[f(g_{\text{YM}}) \approx \exp \left( +3 \int_{g_c}^{g_{\text{YM}}} \frac{dx}{\beta(x)} \right) \to 0. \] (26)

This implies that
\[\int_{g_c}^{g_{\text{YM}}} \frac{dx}{\beta(x)} \to -\infty. \] (27)

This is the condition for having no Landau singularity. In order for this condition to be satisfied when \[g_{\text{YM}} \to g_c\], we clearly need a zero in the beta function.

The rest of the analysis is standard. Assume the existence of a zero,
\[\beta(g_{\text{YM}}) = -\beta_0(g_c - g_{\text{YM}})^\alpha, \] (28)

where \(\alpha \geq 1\) in order to satisfy the condition (27), we obtain for \(\alpha = 1\),
\[f(g_{\text{YM}}) \approx \text{const.} \left( g_c - g_{\text{YM}} \right)^{\frac{3}{2}}, \] (29)

and for \(\alpha > 1\) we obtain
\[f(g_{\text{YM}}) \approx \text{const.} \exp \left( \frac{3}{\beta_0(1 - \alpha)} (g_c - g_{\text{YM}})^{1-\alpha} \right). \] (30)

We also remark that the standard behavior near the fixed point, i.e.,
\[g_c - g_{\text{YM}} \approx \text{const.} \exp \left( \frac{\beta_0}{1 - \alpha} L \right), \] (31)

for \(\alpha = 1\), and
\[g_c - g_{\text{YM}} \approx (\beta_0(1 - \alpha)L)^{\frac{1}{1-\alpha}}, \] (32)

for \(\alpha > 1\), taken together with eq. (13), lead to
\[\rho \approx \text{const.} \left( \frac{\mu}{\Lambda} \right)^{\frac{\beta_0}{2}} \] for \(\alpha = 1\) and
\[\rho \approx \left( \frac{1}{\sqrt{g_c}} \right) (\beta_0(1 - \alpha)L)^{1/2(1-\alpha)} \] for \(\alpha > 1\). (33)

These equations turn out to be quite important later.

**IV. LINKING THE UV TO THE IR**

In this section we shall analyze the UV limit carefully. We start from eq. (10), noticing first that
\[Y(\rho) = 4\rho - 1 + 8\rho \sum_{n=1}^{\infty} e^{-4n\rho}, \] (34)

and hence the argument \(k\) of the complete elliptic integral becomes for large values of \(\rho\) (i.e. in the UV)
\[k^2 = 1 - \frac{1}{Y(\rho)} \approx 1 - \frac{1}{4\rho} + \ldots . \] (35)

The quantity \(k' = \sqrt{1 - k^2}\) becomes \(k'^2 \approx 1/4\rho\). We hence need to expand around \(k' = 0\), where \(E(k)\) is not
analytic in $k'$. Using the well known expansion of $E$ near $k = 1$

$$
E(k) \approx 1 + \frac{1}{2} \left(\ln \frac{4}{k'} - \frac{1}{2}\right) k'^2 + O(k'^4 \ln k').
$$

we obtain from these equations and eq. (10)

$$
\frac{16\pi^2}{g_{YM}^2 N} \approx 4\rho + \frac{1}{2} \ln(8\sqrt{\rho}) - \frac{5}{4} + O\left(\frac{\ln \rho}{\rho}\right) + O(\rho e^{-4\rho}).
$$

(37)

This shows that $1/g_{YM}^2$ is not analytic in $\rho$ for large $\rho$. From this result we can compute $\beta_\rho$ by differentiation,

$$
\beta_\rho(g_{YM}) \approx -\frac{g_{YM}^2 N}{8\pi^2} \left(1 + \frac{1}{16\rho} + \ldots\right).
$$

(38)

The $\beta$–function referring to the scale $L = \ln(\mu/\Lambda)$ is:

$$
\beta(g_{YM}) \approx -\frac{g_{YM}^2 N}{8\pi^2} \left(1 + \frac{1}{16\rho} + \ldots\right) \frac{\partial \rho}{\partial L}.
$$

(39)

In passing we mention that eq. (37) can be inverted (iteratively) to give

$$
\rho \approx \frac{4\pi^2}{g_{YM}^2 N} + \frac{1}{16} \ln \frac{g_{YM}^2 N}{256\pi^2} + \frac{5}{16} + \ldots,
$$

(40)

showing that $\rho$ is not analytic in $1/g_{YM}^2$ for a small YM coupling.

We now return to eq. (24). Since $g_{YM}$ is a function of $\rho$ from eq. (10), we can equally well write eq. (24) as

$$
\frac{\Lambda^2}{\mu^2} = g(\rho) \ a(\rho).
$$

(41)

It should again be emphasized that since the function $f$ in eq. (24) is not unique, but scheme dependent, the same applies to the related function $g(\rho)$.

We now want to determine the unknown function $g(\rho)$ from the following three requirements:

(i) The first two coefficients in the beta-function (39) should be correctly reproduced, since they are universal for a wide range of renormalization schemes.

(ii) There should be an infrared limit, corresponding to $\Lambda/\mu \rightarrow \infty$.

(iii) Point (i) and (ii) are achieved (hopefully) assuming the function $g(\rho)$ to be a simple global factor.

We now claim that the following version of eq. (11), which satisfies (iii),

$$
\frac{\Lambda^3}{\mu^3} = \rho^{-p} \ a(\rho),
$$

(42)

with some power $p$ to be determined from the requirement (i), also satisfies the two requirements (i) and (ii) simultaneously. We notice that the case $p = 0$ corresponds to the choice in [3], so the calculations of these authors are included in the following.

From (42) and the asymptotic UV behavior

$$
a(\rho) \approx 4\rho e^{-2\rho} (1 + e^{-4\rho} + \ldots),
$$

(43)

we easily obtain

$$
\frac{\partial \rho}{\partial L} \approx \frac{3}{2} \left(1 - \frac{1}{2p}\right) \approx \frac{3}{2} \left(1 + \frac{1 - p}{2\rho}\right) .
$$

(44)

It should be noticed that since we consider the large $\rho$ limit, the expression $1/(1 - (1 - p)/2\rho)$ is not more accurate than $1 + (1 - p)/2\rho$, since terms of order $1/\rho^2$ arises also from other sources, for example the further expansion of the complete elliptic integral.

The beta-function (39) then becomes

$$
\beta(g_{YM}) \approx -\frac{g_{YM}^2 N}{16\pi^2} \left[1 + \left(\frac{9}{8}\right) \rho \ g_{YM}^2 N \ + \ldots\right].
$$

(45)

In the fifth order term there are three contributions, namely a term $1/16\rho$ coming from the expansion of the complete elliptic integral, a term $-p/2\rho$ coming from the factor $\rho^{-p}$ in eq. (42), and a term $+1/2\rho$ coming from the function $a(\rho)$.

It is also instructive to display the beta function containing the next order in the expansion in $1/\rho$.

$$
\beta(g_{YM}) \approx -\frac{3g_{YM}^3 N}{16\pi^2} \left[1 + \left(\frac{9}{8} - p\right) + \frac{1}{4\rho^2} \times \left(1 - p\right) \left(\frac{9}{8} - p\right) + \right.

\left.\frac{3}{4^3} \left(\frac{35}{12} - \ln(8\sqrt{\rho})\right)\right] + \ldots ,
$$

(46)

which as function of the coupling constant reads:

$$
\beta(g_{YM}) \approx -\frac{3g_{YM}^3 N}{16\pi^2} \left[1 + \left(\frac{9}{8} - p\right) \ g_{YM}^2 N \ + \ldots\right] + 2g_{YM}^4 N^2 \left[8 \left(p^2 - \frac{3}{2} p + \frac{143}{256}\right)\right] + \ldots
$$

(47)

In showing this result, for completeness, we kept not only the leading logarithmic corrections but we also display the constant terms appearing in the coefficient of the order $g_{YM}^2$ term. The $g_{YM}^2$ term in eq. (47) contains a logarithmic term which does not occur in a straightforward perturbative calculation of the beta function. However, by a coupling constant transformation it is possible to remove the logarithmic term. Likewise eq. (47) can be put in agreement with the correspondent term in eq. (3) via a coupling constant transformation.
Thus we write the beta function we derived as:

$$\beta(g) = c_1 g^3 + c_2 g^5 + g^7 (c_3 + c_4 \ln g) + \ldots,$$

with the obvious identification of the $c_i$ coefficients and coupling constant $g$ with our previous expression. Let’s perform the following coupling constant change

$$g = \tilde{g} + h_1 \tilde{g}^3 + \tilde{g}^5 (h_2 + h_3 \ln \tilde{g}) + \ldots.$$  

(49)

In the new coupling constant the first two terms in the beta function are unchanged (as expected by the universality argument) while only the coefficient of the third term (i.e. the $g^7$ term) changes:

$$\tilde{\beta}(\tilde{g}) = c_1 \tilde{g}^3 + c_2 \tilde{g}^5 + \tilde{g}^7 [c_3 + 2c_2 h_1 + c_1 (3h_2^2 - 2h_2 - h_3) + (c_4 - 2c_1 h_3) \ln \tilde{g}]$$

(50)

Taking $h_3 = c_4/2c_1$ we arrive at a beta function without the logarithmic term. Clearly higher orders can always be fixed by a suitable non universal redefinition of the coupling constant $\tilde{g}$.

At this point we can focus directly on the universal terms. From (51) we see that irrespective of the value of $\rho$ we always obtain the leading asymptotically free term in the beta function. However, the next term fixes the value of $\rho$, since the first two universal terms should correspond to the beta function

$$\beta(g_{YM}) \approx -\frac{3g_{YM}^3 N}{16\pi^2} \left(1 + \frac{Ng_{YM}^2}{8\pi^2}\right).$$

(51)

Thus $9/8 - p = 1$, or

$$p = \frac{1}{8}.$$  

(52)

This value of $p$ thus ensures the validity of the condition (i).

At this stage it should be pointed out that if we had started with an arbitrary function $g(\rho)$ in eq. (41), we would then have found that in order to satisfy the condition (i) we would need that $g(\rho) \approx \rho^{-1/8}$ to leading order when $\rho \to \infty$. In this sense eq. (41) is a consequence of (i) in the UV. Promoting (41) to a global equation is, of course, a separate assumption.

To see what happens in the infrared, we notice that $a(\rho) \to 1$ for $\rho \to 0$. Thus, from eq. (42) we see

$$\rho \approx \left(\frac{\mu}{\Lambda}\right)^{\frac{1}{p}}.$$  

(53)

This asymptotic behavior can be directly compared to the behavior in eq. (33), giving the result

$$\beta_0 = \frac{6}{p} = 48,$$  

(54)

valid for a first order zero in the beta function. We remark that higher order zeros are not possible due to the requirement (iii), since they require $\rho$ to behave like a power of $-L = -\ln \mu/\Lambda$. Such a power could only arise if $\rho^{-p}$ in eq. (42) is replaced by

$$\rho^{-p} \to \exp\left[\frac{3(g_{YM}(\rho^{1/3})^{(1-\alpha)})}{\beta_0(\alpha - 1)}\right],$$

(55)

where $\alpha > 1$ is the order of the fixed point. In the IR the expression on the right hand side behaves like $e^{-3L}$ due to the second eq. (45). In the UV this would lead to

$$\frac{\partial \rho}{\partial L} \approx \frac{3}{2} \left(1 + \frac{1}{2\rho} - \frac{3}{\rho_0} \left(\frac{g_{YM}^2}{2^3}\right)^{1-\alpha} \right).$$

(56)

To satisfy the requirement (i) we need that

$$\frac{3}{\rho_0} g_{YM}^{1-\alpha} \rho^{1-2\alpha} \approx \frac{p}{2\rho} \ldots,$$

(57)

which is impossible due to the condition $\alpha > 1$ for a higher than first order fixed point. Therefore we need $\alpha = 1$.

Consequently we see that the beta-function behaves like

$$\beta(g_{YM}) \approx -48 \left(g_{YM} - g_{YM}\right) \approx -\frac{6\sqrt{N}}{2\pi} \left(g_{c}^2 - g_{YM}^2\right).$$

(58)

This result shows that the behavior of beta near the IR zero has been fixed by requiring that the second term in the UV beta function has the correct universal value.

Our computations and results can be straightforwardly applied to Ref. [12] which make use of a different non-singular supergravity solution based on the warped deformed conifold found in [13].

V. CONCLUSIONS

We investigated the $N = 1$ four-dimensional super Yang-Mills theory with gauge group $SU(N)$ using the approach introduced in [14]. Here the authors exploited the connection between the gaugino condensate and the function $a(\rho)$. A remarkable feature of this connection is that it leads to asymptotic freedom with the correct coefficient. On the other hand this relation allows more freedom which we used to investigate the IR limit $\mu \to 0$ of the theory.

The supergravity theory provides a dimensionless scale $\rho$ respect to which we defined a beta function $\beta_{\rho}$. The latter is of direct relevance for the super Yang-Mills theory when making the mild assumption that $\rho$ is some function of the field theory scale $\mu$. $\beta_{\rho}$ is completely determined and displays an infrared fixed point. Assuming that this has a physical meaning, the beta functions computed with respect to the scale $\mu$ should then have this behavior as well. This still leaves much freedom. However, if the relation between the gaugino condensate and
the function $a(\rho)$ contains a “global” (i.e., the same factor in the IR and the UV) factor, then the order of the fixed point and the slope of the beta function in the IR can be uniquely fixed using the UV information. The UV input is due to the “universality” of the first two terms of the perturbative UV beta function. Clearly this input is not intrinsic to the approach presented in [3], but must be taken as a kind of boundary condition from the known results regarding the super Yang-Mills theory in the UV.

The existence of an IR fixed point in our approach leads to the natural question concerning the physical meaning of this. For example, one may ask if the four dimensional theory is still confining. On this account it worth recalling that the renormalization invariant scale $\Lambda$ is always there, and survives also in the infrared, where the vanishing of $\mu$ is exactly compensated by the zero in the beta function in the construction of $\Lambda$. Hence one needs independent arguments to settle the confinement question. For a recent discussion of this question we refer to [3].

The most natural interpretation of a Landau singularity is that the original, weak coupling (semi-classical) vacuum is unstable (tachyonic), or meaningless (ghost-like), and is replaced by another ground state. Similarly, the existence of an infrared fixed point can be interpreted as the persistence of the original vacuum state. In the case of $\mathcal{N} = 1$ super Yang-Mills theory this can be related to the persistence of the gaugino condensate, i.e., the expectation value

$$<0|\lambda^2|0> = \Lambda^3. \quad (59)$$

There is no sign that the vacuum state $|0>$ needs to decay to another state $|\overline{0}>$. The constant $\Lambda$ is well defined in the whole range of the scale $\mu$ from $\infty$ to 0. Therefore the physical meaning of the IR fixed point may be that it leaves the gaugino condensate as a genuine physical feature of the theory. This has to be contrasted with the case where there is a Landau singularity. The renormalization group invariant $\Lambda$, originally defined in the weak coupling regime, would then only be defined up to a scale $\mu = \Lambda$, beyond which the theory with vacuum $|0>$ would be ill defined. In a new vacuum, which hopefully exists beyond the Landau singularity, other condensations than $\overline{0}$ may then be preferred. We therefore think that the IR fixed point advocated by us is a physically reasonable feature of $\mathcal{N} = 1$ super Yang-Mills theory, since it keeps the gaugino condensation unchanged at large couplings.

**Acknowledgments**

It is a pleasure to thank Poul Henrik Damgaard, Paolo di Vecchia, Alberto Lerda and Raffaele Marotta for helpful discussions and comments. The work of F.S. is supported by the Marie-Curie fellowship under contract MCFI-2001-00181.

### APPENDIX A: A DIFFERENT CHOICE

As repeatedly emphasized the choice of the interpolating function $\rho^{-\rho}$ exhibited in (12) is not unique. To illustrate this point we now consider the Bessel function $I_\nu(z)$ with a suitable relation between $z$ and $\rho$. The asymptotic behaviors are:

$$\lim_{z \to \infty} I_\nu(z) \approx \frac{e^z}{\sqrt{2\pi z}} + \ldots, \quad (A1)$$

$$\lim_{z \to 0} I_\nu(z) \approx \frac{1}{\Gamma(\nu + 1)} \left(\frac{z}{2}\right)^\nu \left[1 + \frac{1}{\nu + 1} \left(\frac{z}{2}\right)^2 + \ldots \right]. \quad (A2)$$

Now in the IR we need an exponential behavior (which corresponds to $z \to \infty$), so we need to set

$$z = 3 \left(\frac{g_c \rho^2}{\beta_0(\alpha - 1)}\right)^{\frac{1}{\alpha - 1}}. \quad (A3)$$

Here we have $\alpha > 1$.

On the other hand, in the UV (i.e. $z \to 0$) we need

$$\rho^{-1/8} \propto \rho^{2\nu(1-\alpha)}, \quad (A4)$$

leading to

$$\alpha = 1 + \frac{1}{16\nu}. \quad (A5)$$

which is indeed larger than 1. Hence we can have the following behavior,

$$\frac{\Lambda^3}{\mu^3} = I_\nu \left(\frac{3 \left(\frac{g_c \rho^2}{\beta_0(\alpha - 1)}\right)^{1-\alpha} \cdot a(\rho)}{\beta_0(\alpha - 1)}\right) \quad (A6)$$

and

$$\beta(g_{YM}) = -\beta_0(g_c - g_{YM})^{1+\frac{1}{4\nu}}. \quad (A7)$$

So according to this result it seems that by a different choice of the function $g(\rho)$ (still allowing a universal behavior in the UV) one can have a higher order IR fixed point.

However, if we consider the asymptotic expansion (A2) we see that the correction terms are of order $z^2$ relative to 1, i.e. of order $\rho^{4(1-\alpha)}$. Since this is a perturbative correction in the UV, and since $\rho \propto 1/g_{YM}^2$, we observe that this correction corresponds to $(g_{YM}^2)^{4(\alpha - 1)}$. In order for this to be really perturbative, we require

$$4(\alpha - 1) = 1, \quad \text{i.e.} \quad \alpha = 1.25. \quad (A8)$$

Although this power is larger than one, it is fractional. This would lead to a non analytical behavior of the beta function with undesirable branch cuts not expected in field theory.

We mention that the result (A8) also results if one uses eq. (A6) in the UV,

$$e^{-3L} \approx \rho^{2\nu(1-\alpha)}e^{-2\rho \left(1 + \text{const.} \cdot \rho^{4(1-\alpha)}\right)} \quad (A9)$$
leading to
\[
\frac{\partial \rho}{\partial L} \approx \frac{3}{2} \left( 1 + \frac{1 + 2\nu(1 - \alpha)}{2\rho} + \text{const.} \rho^{3-4\alpha} \right). \tag{A10}
\]
Demanding that the third term on the right hand side of this equation is of order $1/\rho^2$ again gives eq. (A8). We also see that in order to have the right coefficient in the second term (1-1/8) we again obtain eq. (A5).

We cannot completely exclude the possibility that by a suitable choice of the interpolating function the IR fixed point turns into a higher order one. However the previous computations show that such a possibility is unlikely.

If we insist that there should be a first order zero in the beta function we see from the work in the main text that this requires a power behavior of the function $g(\rho)$ in the infrared, $\rho \to 0$. Also, to satisfy the requirement (i) we always need the power -1/8. Consequently, in this case there is very little freedom in the choice of $g(\rho)$. Taking into account that the functional dependence can anyhow be changed by coupling constant transformations (even if $\mathcal{N} = 1$ super Yang-Mills theory were completely solved), it appears that the simple choice $g(\rho) = \rho^{-1/8}$ is reasonable.

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