String Winding in a Black Hole Geometry.

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$U(1)$ zero modes in the $SL(2, \mathbb{R})_k/U(1)$ and $SU(2)_k/U(1)$ conformal coset theories, are investigated in conjunction with the string black hole solution. The angular variable in the Euclidean version, is found to have a double set of winding. Region III is shown to be $SU(2)_k/U(1)$ where the doubling accounts for the cut structure of the parafermionic amplitudes and fits nicely across the horizon and singularity. The implications for string thermodynamics and identical particles correlations are discussed.
Recently, a particularly interesting string solution was found [1], [2], [3]. In [3] the solution for the graviton-dilaton field equations, which shares many features with a 3+1 dimensional black hole, was also found as the conformal coset model $SL(2, \mathbb{R})/U(1)$. Duality, relating region I ending at the horizon to region V starting at the singularity, became promptly a particularly intriguing feature of this model [1], [3]. The later, also sets the formalism and notation used here.

We are going to discuss the $U(1)$ winding in this string solution. In the Euclidean version, the line element for the cigar shaped region I is given by

$$ds^2 = dr^2 + \tanh^2 r d\theta^2.$$  \hspace{1cm} (1)

We will find that $\theta$ has actually a double set of winding. This is in contrast with the usual case in conformal field theory, where right-moving modes and left-moving modes share the winding and the primary operators those imply [1]. Heuristically, the source for doubling is the topological $U(1)/U(1)$ theory [7], [8] hiding in the $SL(2, \mathbb{R})/U(1)$ coset construction as argued at [9], [10] (where the current-algebra $H \subset G$, was guaged to form $G/H$ by complexifying $H$ and then employing complex BRST to cancel the $H$ propagating degrees of freedom). The $U(1)/U(1)$ is classically the theory of flat $U(1)$ gauge connections, or extra winding for an angular variable $\theta$.

Before elaborating on winding doubling, we would like to present a somewhat broader context which makes flat gauge connections particularly interesting around black hole [3], a geometry which may have access to some non-perturbative aspects of string theory. Independently of the coupling constant it has a horizon and a singularity, with gravitational interactions becoming strong at the later. This non-perturbative flavor of the black hole solution is a motivation to the search for string features on which it differs from flat space-time. We will find that Euclidean winding is different around the black hole than in the flat space-time case.

Winding, has important roles in string theory. It is responsible for extra gauge symmetries of string origin [11]. It is also important in string thermodynamics. Strings in temperature $1/\beta$, are formulated by curling one (Euclidean space) dimension, to radius $\beta$ [12], [13]. Increasing the temperature by shrinking $\beta$, a string winding state, could have its $M^2 = -C + t \beta^2$ becoming tachyonic (as the second winding term cannot cancel the first

1 Doubling is thus related to ideas presented in [3] about orbifolds in complexified space-time and the separation of left and right movers.
tachyonic zero point term anymore). This was argued to indicate a transition temperature for the string, namely Hagedorn temperature, where the number of excited states wins over their Boltzman supression and the free energy is dominated by highly excited string states \([14], [15]\). The implications of the winding doubling for string thermodynamics will be discussed subsequently.

An additional string aspect which seems different between the black hole and flat space-time, is the way the Hilbert space is constrained to a ghost-free positive normed physical spectrum \([16]\). The well known mechanism of BRST cohomology does not seem to help, at least in its straightforward version, which requires a flat time-like direction \([17], [18]\). There are suggestions, based on the \(SL(2, \mathbb{R})_k\) current algebra, giving negative-norm state free spectrum for the black-hole string solution \([19]\). However, their compatibility with modular invariance has yet to be settled, along with finding the symmetry principle underlying them.

A different way out is a non-linear physical state condition, possibly an interacting string field equation like \(Q\Psi + \Psi \star \Psi = 0\), for an appropriately defined string field theory (with product \(\star\)), instead of the condition \(Q\Psi = 0\) solved by BRST cohomology. A non-linear condition of this kind could fit with the non-perturbative nature of the black hole solution. Regretably, the relevant version of string field theory is not clear yet. Nonetheless, we could entertain the formal similarity between this physical state condition and a flat gauge connection. Although flat gauge configurations cannot be suggeted so far as a useful field theory approximation to the relevant string field theory\(^2\), we will proceed to study them.

Let us take stock of angular variables in the coset model \(SL(2, \mathbb{R})_k/U(1)\). It is convenient to specify the action of the \(U(1)\) gauge transformations in the WZW model \([20]\) in the Euler notation for \(g(z, \overline{z}) \in SL(2, \mathbb{R})\)

\[
g_{l, \text{euc.}}(z, \overline{z}) = e^{\frac{2}{3} \theta_L \sigma_2} e^{\frac{1}{2} \overline{\sigma}} e^{\frac{1}{2} \theta_R \sigma_2}.
\] (2)

The left handed (holomporhic) transformations act on \(\theta_L\) while the right handed (anti-holomporhic) ones act on \(\theta_R\). The choice of the relative sign of the right handed gauge

\(^2\) In string field theory \(Q\) is constructed out of Virasoro generators \(T(z)\) and ghosts \(b(z), c(z)\). There is, however, the Sugawara formula gives \(T(z)\) as a bilinear of \(SL(2, \mathbb{R})_k\) currents paralleled by \(b(z)\) expressed as a product of \(SL(2, \mathbb{R})_k\) ghost and a current \([7]\). With some luck, the similarity between the physical state condition, the highest weight condition with respect to the \(U(1)\) gauged in \(SL(2, \mathbb{R})_k\) and flat gauge condition may be more than formal.
action, gives vectorial or axial $U(1)$ gauge transformations \[ \text{and its results are discussed below. Anyway, gauging a } U(1) \text{ on the world-sheet will leave us with one angular field, } \theta(z, \bar{z}) \text{ and some doubts counting zero modes associated with it. The Euclidean region I can be found by gauging the vectorial } U(1) \text{ generated by the Pauli matrix } \sigma_2. \]

A convenient gauge choice \[ \text{is taking } g(z, \bar{z}) \text{ symmetric, leading to the line element } (1). \]

One could try other gauge choices, like taking $g(z, \bar{z})$ traceless, which yields a different angular variable $\tilde{\theta}$ (and the line element $ds^2 = dr^2 + \coth^2 r d\tilde{\theta}^2$). In both cases the coset has a single angular variable. $\theta$ is locally related to $\tilde{\theta}$ by a $U(1)$ gauge transformation. If, however, the world-sheet is topologically non-trivial, this would not mean that the two are equivalent.

It would rather mean that the ambiguity in the choice of an angular field, is given by an extra $U(1)$ flat connection as a global set of degrees of freedom.

In our discussion we will often use space-time (or Euclidean space) arguments for the black hole solution, which are usually justified semi-classically at infinite level $k$ which is weak WZW coupling constant. Since, to give $c = 26$, $k$ is small\[ \text{it is lucky that the two-dimensional nature of the solution opens a better justification to space-time arguments. They turn out to be exactly applicable for the “tachyon” states, which dominate this solution}. \]

We will check the effects of winding doubling in the parafermionic conformal field theory and substantiate them independently of space-time arguments (This check is encouraging for bolder applications of space-time arguments).

So far we were discussing the Euclidean region I formed gauging the vectorial $U(1)$ subalgebra generated by $\sigma_2$. Let us see how similar gauge conditions look through regions III and V. We will get the Euclidean picture of these regions starting from the Minkowski version. For region I the later can be found writing

\[ g_{1, \text{mink.}} = e^{\frac{1}{2} t L \sigma_3} e^{\frac{1}{2} r L \sigma_1} e^{\frac{1}{2} r R \sigma_3}. \] (3)

and gauging the $U(1)$ generated by $\sigma_3$. The gauge where $g$ is symmetric gives the Minkowski line element $ds^2 = dr^2 - \tanh^2 r dt^2$. The other regions follow in Kruskal coordinates $u, v$ \[ \text{where they can be put together. In region I, } u = \frac{1}{2} \sinh re^t, \ v = -\frac{1}{2} \sinh re^{-t}. \]

The line element is

\[ ds^2 = \frac{du \: dv}{1 - uv}. \] (4)

\[ ^3 \text{Rather than the geometrically suggested } c \sim 2, \text{ see } [22] \text{ for another way to get } c = 26. \]
Region I is \( uv < 0 \), region III \( 0 < uv < 1 \), bounded between the horizon \( uv = 0 \) and the singularity \( uv = 1 \) and region V is past the later. Kruskal description follows from \( SL(2, \mathbb{R}) \) when parametrizing

\[
g = \begin{pmatrix} a & u \\ -v & b \end{pmatrix}, \quad ab + uv = 1.
\]

We continue this description to region III, where \( u = \frac{1}{2} \sin re^t, \quad v = \frac{1}{2} \sin re^{-t} \) and \( ds^2 = dr^2 - \tan^2 r dt^2 \). \( SL(2, \mathbb{R}) \) is now parametrized

\[
g_{III, \text{ mink.}} = e^{\frac{i}{2} t_L \sigma_3} e^{\frac{i}{2} r \sigma_2} e^{\frac{i}{2} R \sigma_3}. \tag{6}
\]

Wick rotation in region I turns (3) into (2), changing the gauged \( U(1) \) subgroup \( e^{ia \sigma_3} \) into \( e^{i \sigma_2} \) (acting vectorially as a similarity transformation) and the line element into \( ds^2 = dr^2 + \tan^2 r d\theta^2 \). In region III (3) changes to

\[
g_{III, \text{ euc.}} = e^{\frac{i}{2} \theta_L \sigma_3} e^{\frac{i}{2} \sigma_2} e^{\frac{i}{2} R \sigma_3}. \tag{7}
\]

(\( \sigma_3 \) cannot turn into \( \sigma_2 \) already used by \( r \)) giving \( g_{III, \text{ euc.}} \in SU(2) \). The line element is \( ds^2 = dr^2 + \tan^2 r d\theta^2 \); as a check, near the horizon \( r = 0 \), it agrees with \( ds^2 \) for the sphere \( S^2 = SU(2)/U(1) \). \( SU(2) / U(1) \) in the Euclidean region III, will lead us to parafermions \([23]\) and their duality, after further examination of the Euclidean solution and its winding.

As a further check for the Euclidean regions we have found and in order to establish the way they are attached to each other, we will find a path in \( SL(2, \mathbb{R}) \) through all of them. This is done by imposing the condition \( \theta_R = 0, (a^2 + v^2 = b^2 + u^2 \) in the parametrization of (5) \) in addition to the gauge conditions. These read \( u + v = 0 \) (\( g_I \) symmetric) in region I, \( (u + v)^2 + (a + b)^2 = 4 \) in III (\( g \in SL(2, \mathbb{R}) \) always satisfies \( (u + v)^2 + (a + b)^2 \geq 4 \), saturated at the horizon \( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) and singularity \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \)) and \( a + b = 0, \) in V (dual to \( u + v = 0 \)). This path is parametrized by

\[
g_\rho = \begin{pmatrix} \cosh \rho & \sinh \rho \\ \sinh \rho & \cosh \rho \end{pmatrix} \quad \text{in I,} \quad g_\omega = \begin{pmatrix} \cos \omega & \sin \omega \\ -\sin \omega & \cos \omega \end{pmatrix} \quad \text{in III}
\]

and

\[
g_\sigma = \begin{pmatrix} \sinh \sigma & \cosh \sigma \\ -\cosh \sigma & \sinh \sigma \end{pmatrix} \quad \text{in V}.
\]

\footnote{The difference further on, shows that cosetting in current algebra does not give the coset manifold, but is rather done by integrating out the gauge field. The bilinear form in the gauge field \( A \) is thus responsible for the singularity, demonstrating the importance of string effects in this context.}

\footnote{providing a space-time description for left movers on the world-sheet, which is sometimes interesting by itself \[1\].}
$g_\omega$ belongs to $SU(2)$ as well (covering the intersection of the two groups).

Now we can look closely for winding in the Euclidean solution. It is interestingly contrasted with a cylinder, which is the Euclidean thermal version of flat Minkowski space \cite{12}, whose set of winding is topologically stable. For special values of the radius namely $k$ times the self-dual radius, they lead to $k + 1$ conformal blocks. The Euclidean black hole is a bit different. The cigar shaped region I by itself, would drive us into the conclusion that winding is not topologically stable here. This is not the case, since we have already established that region III is attached to region I at $r = 0$, by threading a path between them. Argued differently, the horizon, a Minkowski light-cone in Kruskal coordinate is Wick rotated, giving $r = 0$, which is the point common to the regions. In fact, we are going to argue that the winding is actually doubled, by closely studying the parafermionic theory in region III and latter continuing it to the other regions.

The study of parafermionic conformal field theories \cite{23} was motivated by critical two-dimensional $Z_k$ clock models in Statistical Mechanics \cite{24}. They provide one of the best studied coset models, $SU(2)_k/U(1)_k$. In addition to the role of a useful laboratory for $U(1)$ cosetting they also happened to serve as the centerpiece for the Euclidean string black hole solution. Most notable in these models, is the wealth of observables ($\sim k^3$). We will concentrate in this note on the set of order operators $\sigma_l$ and the set of disorder operators $\mu_l$. The $\mu_l$'s are known to be non-local with respect to the $\sigma_l$'s and correlation functions containing both have cuts \cite{25}. $\sigma$’s and $\mu$’s follow from gauging different $U(1)$ subalgebras of $SU(2)_k$, the vectorial (electric) $U(1)$ gives the $\sigma$’s and the axial (magnetic) $U(1)$ gives the $\mu$’s. In other words, they result from quantizing different sets of zero modes (windings) in the theory thereby demonstrating doubling.

To see that this is indeed the case we note that $\sigma$’s and $\mu$’s are related by Kramers-Wannier duality \cite{26}. This duality, which relates magnetic cosetting to electric cosetting, works the same way as the duality found in the string black hole solution \cite{4} \cite{5}. For more careful examination, we observe that the cut structure in the parafermionic amplitudes is the inverse of the cut structure between electric and magnetic $U(1)$ vertices. In chapter 5 of \cite{23} the $SU(2)_k$ holomorphic current $J^3(z) = \partial_z \phi(z)$, is taken to be the holomorphic

\footnote{In addition to giving the primary operators, they also have effects, like holomorphic factorization in the $\tau$ dependence of the torus partition function, whose subtlety in our case we will see soon.}
$U(1)$ current, $\phi(z)$ being the holomorphic part of the scalar field $\Phi(z, \bar{z})$. By formula (5.10) in \cite{23}

$$
\sigma_l(z, \bar{z}) : \exp \left( \frac{i l}{\sqrt{k}} (\phi(z) + \overline{\phi}(\bar{z})) \right) = \sigma_l(z, \bar{z}) : \exp \left( \frac{i l}{\sqrt{k}} \Phi(z, \bar{z}) \right):
$$

(9)
gives a cutless $SU(2)_k$ correlator with

$$
\mu_n(z, \bar{z}) : \exp \left( \frac{i n}{\sqrt{k}} (\phi(z) - \overline{\phi}(\bar{z})) \right) = \mu_n(z, \bar{z}) : \exp \left( \frac{i n}{\sqrt{k}} \tilde{\Phi}(z, \bar{z}) \right):
$$

(10)

Thus, all the parafermionic $Z_k$ cut structure is between the electric $\Phi(z, \bar{z})$ and magnetic $\tilde{\Phi}(z, \bar{z}) = \phi(z) - \overline{\phi}(\bar{z})$ vertices. A bit more algebraic version of the description suggested here for $\sigma$ and $\mu$ is found introducing the rational torus $U(1)_k \subset SU(2)_k$ by which we coset each time. Then $\sigma$ are characterized as the operators which are local with respect to $J^3(z)$ and $J^+(z)^k \tilde{J}^-(\bar{z})^k$. $\mu$ are rather local with respect to $J^+(z)^k \tilde{J}^-(\bar{z})^k$. Doubling of winding is noted realizing that different sets of winding give the $\sigma$'s and the $\mu$'s upon quantization, since they came about, cosetting different $U(1)$'s. All the operators in the parafermionic theory will be discussed in this language elsewhere.

The cut structure in parafermions provides an important property shared with non-critical strings, or Liouville theory and hence an additional relationship \cite{27} between the black hole solution and non-critical strings. If we consider the $\sigma$'s as operators in the bulk of the system, then the $\mu$'s seem to introduce boundaries in the form of cut lines between the various $\mu$'s. It is also natural to specify modified boundary conditions on the cylinder (or torus) as though a cut is running between two $\mu$'s on the boundary\cite{28} \cite{29}. This results in more modular invariant combinations, under a subgroup of the mapping class group\cite{30}. This distinction between operators, naturally specified on points and operators associated with boundaries was made by Seiberg and Moore\cite{30} \cite{31} (the later were associated with normalizable wave functions) in the context of two dimensional gravity. In our case operators and states are dual.

Further understanding of the parafermionic theories as coset model is required in order to get semiclassical intuition. Large $k$ is currently under investigation and seems hopeful, since self dual $Z_k$ clock models fall within a Kosterlitz-Thouless phase for $k > 4$ and

\footnote{Along with the ambiguity in specifying the chiral algebra and the cuts in the amplitudes, this makes the definition of conformal blocks tricky. For example, the Ising model has 3 blocks under the Virasoro algebra but only 2 blocks when the chiral algebra consists of a free Majorana fermion. In this case, the modular invariance is under the group generated by $S$ and $T^2$. These subtleties in defining conformal blocks are typical to doubling of winding.}
should not be too sensitive to $1/k$ corrections (as well as to moving off criticality, possibly by Ginzburg-Landau formulation. A further speculation will be that lattice models could do as well as their continuum limits, in accord with discrete or topological approaches to 2-d gravity.). It should also be mentioned that parafermionic models continues smoothly accross the horizon to region I with $\Phi$ as $\theta$. Instead of worrying about continuation across the singularity, we will simply use the parafermionic duality to see that $\bar{\Phi}$ continues to $\bar{\theta}$, further empasizing that Kramers-Wannier duality is equivalent to the duality found in the black hole solution[4] [5].

The importance of winding for string thermodynamics was already mentioned above. $\ell$ fold Euclidean winding states becoming massless, was also interpreted in string thermodynamics [32]. Much the same way that by becoming massless, a minimal winding state signifies that higher string excitations start to dominate the string free energy; these $\ell$ fold windings become massless when highly excited $\ell$ identical strings states get to dominate the ensemble of $\ell$ identical string excitations. This interpretation was shown to be consistent with Bose-Einstein as well as Fermi-Dirac quantum statistics, in the bosonic and fermionic string theories. It is tempting to abstract from this theromodynamical argument that the Euclidean winding states are related to quantum correlations between identical particles. By that we would learn that identical particles correlations could behave differently in the black hole case. One could speculate that the doubling of winding states, would naively present itself as a $Z_2$ additional quantum number (“color”). Along with implications to black-hole thermodynamics, this should be examined more carefully.

A calculation of the partition function on the torus or other amplitudes will be very helpful to clarify this issues as well as the physical spectrum. It will also be interesting to relate the arguments found here to the extra twisted states found at [33] in the $SL(2,\mathbb{R})$ theory.

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