DENSITY OF ELLIPTIC CURVES OVER NUMBER FIELDS WITH
PRESCRIBED TORSION SUBGROUPS

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Abstract. Let $K$ be a number field. For positive integers $m$ and $n$ such that $m \mid n$, we let $\mathcal{K}_{m,n}$ be the set of elliptic curves $E/K$ defined over $K$ such that $E(K)_{\text{tors}} \supseteq \mathcal{T} \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$. We prove that if the genus of the modular curve $X_1(m,n)$ is 0, then 'almost all' $E \in \mathcal{K}_{m,n}$ satisfy that $E(K)_{\text{tors}} = \mathcal{T}$, i.e., no larger than $\mathcal{T}$. In particular, if $m = n = 1$, this result generalizes Duke's theorem ([Du97]) over $\mathbb{Q}$ to arbitrary number fields $K$ for the trivial torsion subgroup.

1. Introduction

The group structure of an elliptic curve over various fields and its application have been actively studied as one of the main topics in the area of number theory. Especially, for an elliptic curve $E$ over a number field $K$, it is known by the Mordell-Weil theorem ([Si09, VIII.6.7]) that the set of $K$-rational points $E(K)$ of $E$ called the Mordell-Weil group is a finitely generated abelian group, i.e., $E(K) \cong E(K)_{\text{tors}} \times \mathbb{Z}^r$, where $E(K)_{\text{tors}}$ is its finite torsion subgroup and an integer $r \geq 0$ is called the rank of $E(K)$ which is the number of independent non-torsion points in $E(K)$. But there hasn’t been yet found an efficient algorithm to compute the rank $r$ of $E(K)$, and there have been only some partial results on the finite abelian group structure or the order of $E(K)_{\text{tors}}$.

If we introduce several known results on the abelian group structure of $E(K)_{\text{tors}}$, Mazur ([Ma78]) has classified all realizable torsion subgroups of elliptic curves over $\mathbb{Q}$ as follows:

Theorem 1.1 ([Ma78]). For an elliptic curve $E/\mathbb{Q}$ defined over $\mathbb{Q}$, the torsion subgroup $E(\mathbb{Q})_{\text{tors}}$ of $E(\mathbb{Q})$ is isomorphic to one of the following groups:

- $\mathbb{Z}/m\mathbb{Z}$ where $m \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12\}$; or
- $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2n\mathbb{Z}$ where $n \in \{1, 2, 3, 4\}$.

Conversely, each group listed above can be realizable as a torsion subgroup $E(\mathbb{Q})_{\text{tors}}$ for infinitely many (non-isomorphic) elliptic curves $E$ defined over $\mathbb{Q}$.

Mazur’s result, Theorem 1.1 implies the following, which is obvious but worth mentioning for our interest in this paper.

Corollary 1.2. Each of the maximal classes of the torsion subgroups $E(\mathbb{Q})_{\text{tors}}$ of $E(\mathbb{Q})$ for elliptic curves $E$ over $\mathbb{Q}$ is isomorphic to one of the following groups:

- $\mathbb{Z}/m\mathbb{Z}$ where $m \in \{7, 9, 10, 12\}$; or
- $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2n\mathbb{Z}$ where $n \in \{3, 4\}$.
Conversely, each group $\mathcal{T}$ listed above is realizable as a torsion subgroup of an elliptic curve defined over $\mathbb{Q}$, and if there is an elliptic curve $E$ over $\mathbb{Q}$ with $E(\mathbb{Q})_{\text{tors}} \cong \mathcal{T}$, then for any finite subgroup $\mathcal{T}' \subseteq \mathcal{T}$, there are infinitely many (non-isomorphic) elliptic curves $E'$ over $\mathbb{Q}$ such that $E'(\mathbb{Q})_{\text{tors}} \cong \mathcal{T}'$.

It is natural to ask how many elliptic curves have each of possibly realizable group structures as their torsion subgroups over a given number field. In order to introduce the related questions as well as several related known results and our main result of this paper more precisely, we give some notations and definitions which are used throughout this paper:

- $K$: a number field.
- $\mathcal{U} := \{(s,t) \in \mathbb{A}^2 : 4s^3 + 27t^2 \neq 0\}$: a rational variety in $\mathbb{A}^2$.
- $E_{s,t}$: an elliptic curve whose affine equation is given by $y^2 = x^3 + sx + t$ with coefficients $s,t$ in the function field $K(\mathcal{U}) = K(s,t)$ of $\mathcal{U}$.
- $E_{A,B}$: the specialization of $E_{s,t}$ at $(A,B) \in \mathbb{A}^2(K)$.
- $E_{A,B}(K)_{\text{tors}}$: the torsion subgroup of the Mordell-Weil group $E_{A,B}(K)$ of $E_{A,B}$ over $K$.
- $H$: the absolute height on $\mathbb{P}^n(K)$ (see [Si09 VIII.5]).
- $\mathcal{H}(A,B) := H(A^3, B^2)$, the naive height of the elliptic curve $E_{A,B}$.
- $N_S(X)$: the number of points $P \in S \subseteq \mathbb{P}^n(K)$ such that $H(P) \leq X$ for a positive real number $X$.

Note that the specialization $E_{A,B}$ is an elliptic curve if and only if $(A,B) \in \mathcal{U}(K)$.

**Definition 1.3.** For two subsets $\mathcal{T}$ and $S$ of $\mathbb{P}^n(K)$ such that $\mathcal{T} \subseteq S$, we say that almost all points $P \in S$ belong to $\mathcal{T}$, if

$$\lim_{X \to \infty} \frac{N_{\mathcal{T}}(X)}{N_S(X)} = 1.$$  

In this paper, for any number field $K$, we are interested in the density of elliptic curves over $K$ whose torsion subgroups over $K$ are isomorphic to each of realizable torsion subgroups over $K$, by counting such elliptic curves up to height of the coefficients of their affine equations.

First, Harron and Snowden [HS17] have observed how often each torsion subgroup over $\mathbb{Q}$ listed in Theorem 1.1 appears:

**Theorem 1.4 ([HS17]).** For each finite abelian group $\mathcal{T}$ listed in Theorem 1.1, there is a constant $d_{\mathcal{T}} > 0$ such that

$$\frac{1}{d_{\mathcal{T}}} = \lim_{X \to \infty} \frac{\log \#\{(A,B) \in \mathcal{U}(\mathbb{Q}) : \mathcal{H}(A,B) \leq X, E_{A,B}(\mathbb{Q})_{\text{tors}} \cong \mathcal{T}\}}{\log \#\{(A,B) \in \mathcal{U}(\mathbb{Q}) : \mathcal{H}(A,B) \leq X\}}.$$  

Moreover, we have that $d_{\mathcal{T}} < d_{\mathcal{T}'}$, for any finite abelian group $\mathcal{T}'$ listed in Theorem 1.1 such that $\mathcal{T} \subsetneq \mathcal{T}'$.

Harron and Snowden [HS17] have computed the exact value of $d_{\mathcal{T}}$ for each possible torsion subgroup $\mathcal{T}$ over $\mathbb{Q}$ of an elliptic curve defined over $\mathbb{Q}$. On the other hand, if we take the asymptotic behaviour of the density as $\mathcal{T}$ gets bigger and so as $d_{\mathcal{T}}$ increases strictly, then Theorem 1.4 implies the following:

**Corollary 1.5.** For any finite abelian group $\mathcal{T}$ listed in Theorem 1.1, almost all elliptic curves over $\mathbb{Q}$ containing $\mathcal{T}$ have torsion subgroup $\mathcal{T}$ over $\mathbb{Q}$ exactly, i.e.,

$$\lim_{X \to \infty} \frac{\#\{(A,B) \in \mathcal{U}(\mathbb{Q}) : \mathcal{H}(A,B) \leq X, E_{A,B}(\mathbb{Q})_{\text{tors}} \cong \mathcal{T}\}}{\#\{(A,B) \in \mathcal{U}(\mathbb{Q}) : \mathcal{H}(A,B) \leq X, E_{A,B}(\mathbb{Q})_{\text{tors}} \supseteq \mathcal{T}\}} = 1.$$  

Motivated by Harron-Snowden’s result over $\mathbb{Q}$ [HS17] and extending it over arbitrary number fields, we raise the following questions.
**Question.** Let $K$ be a number field.

Q1. Assume that there is an elliptic curve over $K$ whose torsion subgroup over $K$ is isomorphic to a finite abelian group $T$. Then for any finite abelian subgroup $T' \subseteq T$, is there an elliptic curve $E$ over $K$ such that $E(K)_{\text{tors}} \cong T'$?

Q2. What are the maximal isomorphic classes of $E(K)_{\text{tors}}$ of an elliptic curve $E$ over $K$ (as an analogue of Corollary 1.2)?

Q3. Assume that there is an elliptic curve over $K$ with a finite abelian group $T$ as its torsion subgroup over $K$. Are there infinitely many (non-isomorphic) elliptic curves $E$ defined over $K$ such that $E(K)_{\text{tors}} \cong T$?

Q4. Assume that there is an elliptic curve over $K$ with a finite abelian group $T$ as its torsion subgroup over $K$. Then, do almost all elliptic curves $E$ over $K$ such that $E(K)_{\text{tors}} \supseteq T$ have its torsion subgroup $T$ over $K$ exactly, i.e., $E(K)_{\text{tors}} \cong T$?

The questions Q1–Q4 are still open for almost all number fields, even for quadratic extensions of $Q$. Note that if the answer to Q4 is positive, then so is the answer to Q1. Moreover, if Q1 has an affirmative answer, then we may consider only the maximal isomorphic classes among the torsion subgroups of elliptic curves defined over $K$ for Q2 and Q3.

If we introduce some partial answers to the questions Q1, Q3 over the quadratic extension $K = Q(\sqrt{-1})$, Kenku and Momose [KM88] have provided a list of isomorphic classes of groups to which every realizable torsion subgroup of an elliptic curve defined over $Q(\sqrt{-1})$ necessarily belongs, and Najman [Na11] has refined the list by finding some classes which are not realizable over $Q(\sqrt{-1})$. Then, Zhao [Z, Theorem 2.3] has computed $d_T$ in Theorem 1.4 over $Q(\sqrt{-1})$ for each torsion subgroup $T$ belonging to Najman’s refined list such that $T \neq 0, Z/2Z, Z/3Z$. We note that [Z, Theorem 2.3] shows that $d_{Z/13Z} = \infty$, and Najman [Na10] has proved that $Z/13Z$ would not be realizable as a torsion subgroup of any elliptic curve over $Q(\sqrt{-1})$. Moreover, Zhao [Z] has given a partial answer to Q4 over the quadratic extension $Q(\sqrt{-1})$ by proving that $d_T$ increases strictly as $T$ gets bigger.

Also, Trbović [Tr20] has provided a list of possible isomorphic classes of the torsion groups of elliptic curves over quadratic fields $Q(\sqrt{D})$ for square-free integers $2 \leq D < 100$, but without specifying which groups are actually realizable.

On the other hand, there are several partial answers to the questions Q1, Q3 in terms of small degrees of a given number field $K$ over $Q$. In [Gu21], Gužvić has given a good summary of the related results under various conditions. In particular, if we let $d = [K : Q]$, then for $d = 2$, some weakened answers to Q1, Q3 are given. In particular, Kemki-Momose [KM88] and Kamienny [K91] have given possible isomorphic classes of the torsion subgroup of an elliptic curve defined over a quadratic number field. For $3 \leq d \leq 6$, the lists of all possible and realizable isomorphic classes $T$ of groups such that there are infinitely many elliptic curves $E$ over some number fields $K$ of degree $d$ with $E(K)_{\text{tors}} \cong T$ are given [JKS04, Theorem 3.4] for $d = 3$, [JKP06, Theorem 3.6] for $d = 4$, and [DS04, Theorem 1.1] for $d = 5, 6$. Moreover, they have proved that for each $d = 3, 4, 5, 6$, if an abelian group $T$ is in their list, then so are the subgroups of $T$.

As a generalization of Theorem 1.4 to an arbitrary number field $K$, Bruin and Najman in [MN] considered elliptic curves with all level structures of the torsion subgroup $G$ such that the corresponding modular curves over $K$ satisfy certain conditions, and counted how many such elliptic curves have the $G$-level structure for each subgroup $G \subseteq \text{GL}_2(Z/NZ)$. More precisely, under the certain assumptions on the weights $\omega$ of the moduli stalks of such elliptic curves and their ‘reduced degrees’ $\epsilon(G)$, they proved that there exists a constant $\delta_G$ depending on $\omega$...
and \( e(G) \) such that
\[
\lim_{X \to \infty} \frac{\log \# \{(A,B) \in \mathcal{V}(K) : \mathcal{V}(A,B) \leq X, E_{A,B} \text{ admits } G\text{-level structure}\}}{\log X} = \frac{1}{\delta_G}.
\]
Note that since this constant \( \delta_G \) depends on the weights and the reduced degrees, it is not clear whether \( \delta_G \) increases or not as \( G \) gets bigger, so the answer to \( Q_4 \) over all number fields cannot be deduced directly by this result.

On the other hand, regarding \( Q_4 \), Duke gave an answer when \( K = \mathbb{Q} \) and \( \mathcal{T} \) is trivial as follows:

**Theorem 1.6** ([Du97]). For almost all elliptic curves \( E \) defined over \( \mathbb{Q} \), \( E(\mathbb{Q})_{\text{tors}} \) is trivial.

In this paper, we generalize Theorem 1.6 for certain torsion subgroups over all number fields and give an answer to \( Q_4 \) in these cases. Before stating our main theorem, we give a brief introduction to a modular curve \( \mathcal{X}(n) \) and give an answer to \( Q_4 \) in these cases. Before stating our main theorem, we give a brief introduction to a modular curve \( X_1(m,n) \). Let \( \mathbb{H}_n^* = \{ z \in \mathbb{C} : \text{Im} z > 0 \} \cup \mathbb{Q} \cup \{ \infty \} \). For positive integers \( m \) and \( n \) such that \( m \mid n \), the modular curve \( X_1(m,n) \) is defined by the quotient \( X_1(m,n) = \Gamma_1(m,n) \backslash \mathbb{H}_n^* \) of \( \mathbb{H}_n^* \) by the action of the congruence subgroup,
\[
\Gamma_1(m,n)
:= \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}_2(\mathbb{Z}) : a \equiv 1, b \equiv 0 \pmod{m \mathbb{Z}}, \text{ and } c \equiv 0, d \equiv 1 \pmod{n \mathbb{Z}} \right\}.
\]
We note that \( X_1(m,n) \) is an algebraic curve defined over \( \mathbb{Q}(\zeta_m) \) where \( \zeta_m \) is a primitive \( m \)th root of unity. We denote by \( g_{m,n} \) the genus of \( X_1(m,n) \).

We can refine the questions \( Q_3 \) and \( Q_4 \) a bit more depending on the genus \( g_{m,n} \). If \( g_{m,n} \geq 2 \), Faltings' theorem ([F83, Satz 7]) implies that \( X_1(m,n)(K) \) is finite for a number field \( K \) containing \( \mathbb{Q}(\zeta_m) \). Hence, \( Q_3 \) and \( Q_4 \) make sense only for a finite abelian group \( \mathcal{T} \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \) such that \( g_{m,n} \leq 1 \). If \( g_{m,n} = 1 \), then \( X_1(m,n)(K) \) can be finite or infinite depending on the given number field \( K \) containing \( \mathbb{Q}(\zeta_m) \). If \( g_{m,n} = 0 \), then \( X_1(m,n)(K) \) is infinite if and only if it is not empty. Moreover, the previous known results also reflect this phenomenon. For example, \( \mathcal{T} \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \) can be realized as a torsion subgroup of an elliptic curve over \( \mathbb{Q} \) and \( \mathbb{Q}(\sqrt{-1}) \) (see [Ma78], [Na11] and [Na10]) only if \( g_{m,n} = 0 \), considering the list of all \( (m,n) \) such that \( g_{m,n} = 0 \) given in [J08, Theorem 0.1], and for such a \( \mathcal{T} \), the answer for \( Q_4 \) is positive for the number fields \( \mathbb{Q} \) and \( \mathbb{Q}(\sqrt{-1}) \) (Corollary 1.5, [Z]).

Therefore, our interest of this paper is focused on the cases of the torsion subgroups isomorphic to \( \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \) when the genus \( g_{m,n} \) is 0. By [J08, Theorem 0.1], we note that for positive integers \( m \) and \( n \) such that \( m \mid n \),
\[
g_{m,n} = 0 \quad \text{if and only if} \quad (m,n) \in T_{g=0},
\]
where
\[
T_{g=0} = \{(m,n) \in \mathbb{Z}^+ \times \mathbb{Z}^+ : m \mid n, g_{m,n} = 0 \}.
\]
(1)

Finally, we are ready to state our main theorem:

**Theorem 1.7.** Let \( K \) be a number field. For each \( (m,n) \in T_{g=0} \), if there is an elliptic curve over \( K \) whose torsion subgroup over \( K \) contains a subgroup \( \mathcal{T} \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \), then almost all elliptic curves \( E \) defined over \( K \) such that \( E(K)_{\text{tors}} \supseteq \mathcal{T} \) satisfy that \( E(K)_{\text{tors}} \cong \mathcal{T} \) exactly, i.e.,
\[
\lim_{X \to \infty} \frac{\# \{(A,B) \in \mathcal{V}(K) : \mathcal{H}(A,B) \leq X, E_{A,B}(K)_{\text{tors}} \cong \mathcal{T} \}}{\# \{(A,B) \in \mathcal{V}(K) : \mathcal{H}(A,B) \leq X, E_{A,B}(K)_{\text{tors}} \supseteq \mathcal{T} \}} = 1.
\]
In particular, if there is an elliptic curve $E/K$ such that $E(K)_{\text{tors}} \supseteq \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$, then $\zeta_m \in K$.

The following is a direct application of Theorem 1.7.

**Corollary 1.8.** Let $K$ be a number field. For each $(m,n) \in T_{g=0}$, there is an elliptic curve $E_{A,B}$ over $K$ such that $E_{A,B}(K)_{\text{tors}} \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ if and only if $\zeta_m \in K$ where $\zeta_m$ is a primitive $m^{th}$ root of unity.

The main strategy to prove Theorem 1.7, in particular, to prove that the desired torsion subgroup property in Theorem 1.7 holds for ‘almost all’ elliptic curves over a given number field $K$, is to apply the Hilbert irreducibility theorem (HIT) (see Theorem 2.5) to some parameterized polynomials defining $N$-torsion points of an elliptic curve and its specializations. If we describe it in more detail, let $V$ be an absolutely irreducible variety defined over $K$ and $f(X) = X^n + \sum_{1 \leq i \leq n} a_i X^{n-i} \in K(V)[X]$ be a monic polynomial over the function field $K(V)$ of $V$. For each $t \in V(K)$, let $f_t$ be the specialization of $f$ at $t$. Then, the Galois groups $G_t$ of $f_t$ over $K$ is a subgroup of the Galois group $G$ of $f$ over $K(V)$ in general, and HIT (Theorem 2.5) states that the Galois groups $G_t$ is exactly the Galois group $G$ for each $t \in V(K)$ outside a “thin set” (see Definition 2.1 for its definition). On the other hand, $Q_4$ asks whether torsion subgroups of almost all elliptic curves whose torsion subgroups contain a prescribed torsion subgroup $T$ are exactly $T$ (i.e., no larger than $T$), and HIT (Theorem 2.5) can be applied to give a partial answer to $Q_4$ by regarding ‘a thin set’ in terms of the Galois groups corresponding to the complement of a set of elliptic curves with the desired torsion subgroup property given in Theorem 1.7.

**Remark 1.9.** Most known results mentioned in the introduction (for example, [Du97], [Na11], and $\mathbb{Z}$) hold only over $\mathbb{Q}$ or certain number fields of bounded degrees, and also they need to know all classes of realizable torsion subgroups $T$ and compute both quantities $d_T$ and $d_T'$ (reciprocals of densities as in Theorem 1.4) explicitly for two realizable torsion subgroups $T \subsetneq T'$, to see if $d_T$ increases as $T$ gets bigger and deduce an answer to $Q_4$. However, the list of realizable torsion subgroups $T$ over general number fields $K$ is not known yet. In our paper, in order to give an answer to $Q_4$ over arbitrary number fields $K$, we only need to know the realization of a finite abelian group $T$ as a subgroup of the torsion subgroup of an elliptic curve over $K$, but there is no need to know either the realization of each $T'$ and each $T$ such that $T \subsetneq T'$, or compute the density of each of them, by applying more Galois group theoretic approach.

The structure of our paper is as follow: In Section 2, we define a thin set and give the proof of the Hilbert irreducibility theorem that is described above.

In Section 3, we interpret and transform the $N$-torsion subgroup problem into the Galois group problem of parameterized polynomials associated with the division polynomials (Definition 3.4) of elliptic curves. We give several equivalent statements for the torsion subgroup of an elliptic curve to contain a prescribed subgroup, and sufficient conditions for it to be exactly the given subgroup in terms of the Galois group conditions, which is the conclusion in Theorem 4.5.

In Section 4, we parameterize all elliptic curves over a given number field $K$ satisfying a sufficient condition (Condition $P(m,n)$ in (5) of Section 3) to have their torsion subgroups contain $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ for each $(m,n) \in T_{g=0}$ in (1) and prove that these parameterizations satisfy the Galois group conditions of Theorem 4.5. The genus 0 condition $(g_{m,n} = 0)$ of $X_1(m,n)$ is crucial to parameterize such elliptic curves, so we complete the proofs of Theorem 1.7 and Corollary 1.8.
2. Thin sets and Hilbert’s Irreducibility Theorem

First, we define a thin set. We refer to [Se08 §3.3] for more details.

**Definition 2.1.** For an irreducible $K$-variety $V$, we define the family $\mathcal{F}$ of thin sets in $V(K)$ by the minimal family satisfying the following conditions:

1. For a $K$-morphism $\phi : W \to V$ where $W$ is an irreducible $K$-variety such that $\dim W \leq \dim V$, $\phi(W(K)) \in \mathcal{F}$ is called thin, if one of the following statements is satisfied:
   (i) $\dim W < \dim V$.
   (ii) $\phi$ is geometrically surjective of $\deg \phi > 1$.
2. If $Z_1, Z_2 \in \mathcal{F}$, then $Z_1 \cup Z_2 \in \mathcal{F}$.
3. If $Z_1 \in \mathcal{F}$ and a subset $Z_2 \subseteq Z_1$, then $Z_2 \in \mathcal{F}$.

In order to prove Theorem 2.5 which is our main result of this section, called the Hilbert irreducibility theorem, we need the following fact.

**Proposition 2.2.** Let $S \subseteq \mathbb{P}^n(K)$ be a thin set. Then, for almost all points $P \in \mathbb{P}^n(K)$, $P \notin S$.

**Proof.** Referring to Definition 1.3 we show that

$$\lim_{X \to \infty} \frac{N_S(X)}{N_{\mathbb{P}^n(K)}(X)} = 0.$$  

This follows from [Se97 §2.5 Theorem(Sechannel)] and [Se97 §13.1 Theorem 3] which prove that for any $0 < \varepsilon$, $N_S(X) = O \left( X^{(n+1)/2 + \varepsilon} \right)$ and $N_{\mathbb{P}^n(K)}(X) \approx c_{n,K}X^{(n+1)d}$ where $d = [K : \mathbb{Q}]$ and $c_{n,K}$ is a positive constant depending on $n$ and $K$. □

Let $V$ be an absolutely irreducible variety defined over $K$ and $f(X) = X^n + \sum_{1 \leq i \leq n} a_i X^{n-i} \in K(V)[X]$ be a monic polynomial over the function field $K(V)$ of $V$. We define the specialization $f_t$ of $f$ at $t \in V(K)$ by $f_t(X) = X^n + \sum_{1 \leq i \leq n} a_i(t) X^{n-i} \in K(V)[X]$. We denote by $\mathcal{G}$ and $\mathcal{G}_t$ the Galois groups of $f$ over $K(V)$ and of $f_t$ over $K$, respectively. Then, in general, $\mathcal{G}_t \subseteq \mathcal{G}$. Let $V_f := \{(t,x) \in V \times \mathbb{A}^1 : f_t(x) = 0\}$ and let $W$ be the Galois closure of the covering $V_f \to V$ defined by $(t,x) \mapsto t$. Then, the Galois group of the cover $W \to V$ is $\mathcal{G}$.

**Proposition 2.3** ([Se08 Proposition 3.3.5]). Let $V/K$ be an absolutely irreducible smooth variety defined over a number field $K$ with $\dim(V) \geq 1$ and $f$ be a monic polynomial in $(K[V])[x]$ where $K(V)$ is the function field of $V$. Then, there is a thin set $S \subseteq V(K)$ such that any point $t \in V(K) - S$ satisfies that $\mathcal{G}_t = \mathcal{G}$.

**Remark 2.4.** In fact, [Se08 Proposition 3.3.5] assumes the irreducibility of $f$ which leads the irreducibility of the specializations $f_t$, but to prove our main theorem, we need neither the irreducibility of $f$ nor that of $f_t$ in Proposition 2.3. For more details, refer to [Se08 §3.3] and [Se08 Proposition 3.3.1] whose corollary is [Se08 Proposition 3.3.5].

Now, we are ready to prove the Hilbert irreducibility theorem (HIT) that we need.

**Theorem 2.5.** Let $V$ be a $K$-rational variety of dim $\geq 1$ and $f \in (K[V])[X]$ a monic polynomial over the function field $K(V)$ of $V$. Then, for almost all $t \in V(K)$, we have that $\mathcal{G}_t = \mathcal{G}$.

**Proof.** This follows from Proposition 2.2 and Proposition 2.3 □
In order to apply Theorem 2.5 for our main theorem, we need to transform and interpret the torsion subgroup problem into the Galois group problem of polynomials associated with torsion points, which we present in the following section.

3. N-Torsion subgroups and their associated Galois groups

For an elliptic curve $E/F$ defined over a field $F$ with characteristic 0 and an integer $N \geq 1$, we denote by $E(F)[N]$ and $E[N]$ the groups of $N$-torsion points defined over $F$ and over an algebraic closure $\overline{F}$ of $F$, respectively. Recalling that $E(F)[N] \subseteq E[N] \cong \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$, we consider them as subgroups of $\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$.

**Lemma 3.1.** For any positive integer $N$, if $\mathcal{A}$ is an abelian group such that $\mathcal{A} \cong \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$ and $\mathcal{B}$ is a subgroup of $\mathcal{A}$, then there exist positive divisors $m$ and $n$ of $N$ such that $m \mid n$ and $\mathcal{B} \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$, and moreover there exist $x, y \in \mathcal{A}$ such that $\mathcal{A} = \langle x, y \rangle$ and $\mathcal{B} = \langle \frac{N}{m}x, \frac{N}{n}y \rangle$.

**Proof.** We may assume that $\mathcal{B} \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \subseteq \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$ for some positive divisors $m$ and $n$ of $N$ such that $m \mid n$. Then there exist $h, k \in \mathcal{B}$ with $|h| = m$ and $|k| = n$ such that $\mathcal{B} = \langle h, k \rangle$ and $\langle h \rangle \cap \langle k \rangle = \{0\}$. We can find $y \in \mathcal{A}$ such that $|y| = N$ and $k = \frac{N}{n}y$ as follows: We let $\mathcal{C} := \{\alpha \in \mathcal{A}: n\alpha = 0\} \cong \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$,

$$X := \{\alpha \in \mathcal{A}: |\alpha| = N\}, \text{ and } Y := \{\alpha \in \mathcal{A}: |\alpha| = n\} = \{\alpha \in \mathcal{A}: |\alpha| = n\}.$$  

If we consider the function $f : X \to Y$ given by $\alpha \mapsto \frac{N}{n}\alpha$, then $f$ commutes with any $\sigma \in \text{Aut}(\mathcal{A})$, i.e., $f(\sigma(\alpha)) = \sigma(f(\alpha))$ for all $\sigma \in \text{Aut}(\mathcal{A})$ and all $\alpha \in X$. Hence, $\sigma(f^{-1}(\gamma)) = f^{-1}(\sigma(\gamma))$ for any $\sigma \in \text{Aut}(\mathcal{A})$ and $\gamma \in Y$. Since $\text{Aut}(\mathcal{C}) \cong \text{GL}_2(\mathbb{Z}/n\mathbb{Z})$ acts on $Y$ transitively and the natural homomorphism $\text{Aut}(\mathcal{A}) \cong \text{GL}_2(\mathbb{Z}/N\mathbb{Z}) \to \text{GL}_2(\mathbb{Z}/n\mathbb{Z}) \cong \text{Aut}(\mathcal{C})$ is surjective, $\text{Aut}(\mathcal{A})$ also acts on $Y$ transitively. Therefore, $f^{-1}(\gamma)$ is not empty for any $\gamma \in \mathcal{A}$ such that $|\gamma| = n$. Hence there is $y \in \mathcal{A}$ of order $N$ such that $\frac{N}{n}y = k$. Next, we choose $x \in \mathcal{A}$ such that $\mathcal{A} = \langle x, y \rangle$. Then $h = \frac{N}{m}(ax + by)$ for some integers $a$ and $b$. Then, we can show $\gcd(a, m) = 1$. In fact, if we let $d := \gcd(a, m)$, then since $d \mid m$ and $m \mid n$,

$$\frac{m}{d}h = \frac{a}{d}(Nx) + \frac{m}{d}\left(\frac{N}{m}by\right) = \frac{n}{d}b\left(\frac{N}{n}y\right) \in \langle h \rangle \cap \langle k \rangle = \{0\},$$

so $m \mid \frac{m}{d}$, which implies that $d = 1$. Therefore, there are two integers $s$ and $t$ such that $at + ms = 1$. Then, $\frac{N}{m}x = \frac{N}{m}(at + ms)x = \frac{N}{m}atx \in \langle \frac{N}{m}ax \rangle \subseteq \langle \frac{N}{m}x \rangle$ and so

$$\mathcal{B} = \left\langle \frac{N}{m}(ax + by), \frac{N}{n}y \right\rangle = \left\langle \frac{N}{m}ax, \frac{N}{n}y \right\rangle = \left\langle \frac{N}{n}x, \frac{N}{n}y \right\rangle.$$

$\square$

For any integer $N \geq 1$ and $(A, B) \in \mathcal{Y}(K)$, since the field $K(E_{A,B}[N])$ of definition of $E_{A,B}[N]$ over $K$ is Galois over $K$, we let its Galois group

$$G_{N,A,B} := \text{Gal}(K(E_{A,B}[N])/K).$$

Then for an ordered basis $\mathcal{B} = \{P, Q\}$ of $E_{A,B}[N] \cong \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$, we have ‘the right action’ of $G_{N,A,B}$ on $E_{A,B}[N]$, i.e., if $\sigma \in G_{N,A,B}$, $P^\sigma = \sigma(P) = a_xP + b_yQ$ and $Q^\sigma = \sigma(Q) = c_xP + d_yQ$, for some $a_x, b_y, c_x, d_y \in \mathbb{Z}/N\mathbb{Z}$, and $(R^\sigma)^T = R^T$, for $R \in E_{A,B}[N]$. Hence, by identifying $\text{GL}_2(\mathbb{Z}/N\mathbb{Z})$ with $\text{Aut}(E_{A,B}[N])$, and identifying each $\sigma \in G_{N,A,B}$
with \(\begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix} \in \text{GL}_2(\mathbb{Z}/N\mathbb{Z})\), we define an injective group homomorphism \(r_B : G_{N,A,B} \to \text{GL}_2(\mathbb{Z}/N\mathbb{Z})\) under the right action by
\[
r_B(\sigma) = \begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix}, \quad \text{where } \left( \sigma(P) \right) = \begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix} \left( \begin{pmatrix} P \\ Q \end{pmatrix} \right).
\]

**Lemma 3.2.** For any positive divisors \(m\) and \(n\) of \(N\) such that \(m \mid n\), and for \((A,B) \in \mathcal{U}(K)\), the following are equivalent:

(a) \(E_{A,B}(K)[N]\) contains \(\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}\).

(b) There is an \((\text{ordered})\) basis \(\mathcal{B}\) of \(E_{A,B}[N]\) such that \(r_B(G_{N,A,B}) \subseteq H_N(m,n)\), where \(H_N(m,n) = \begin{cases} a \equiv 1, b \equiv 0 \pmod{m\mathbb{Z}}, \text{ and } c \equiv 0, d \equiv 1 \pmod{n\mathbb{Z}} \end{cases}\).

**Proof.** If \(\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \subseteq E_{A,B}(K)[N]\), then by Lemma 3.1 there is a basis \(\mathcal{B} = \{P,Q\}\) of \(E_{A,B}[N]\) such that \(\begin{pmatrix} \frac{N}{m} \end{pmatrix} P, \begin{pmatrix} \frac{N}{n} \end{pmatrix} Q \in E_{A,B}(K)[N]\). The condition that \(\begin{pmatrix} \frac{N}{m} \end{pmatrix} P\) and \(\begin{pmatrix} \frac{N}{n} \end{pmatrix} Q\) are fixed under \(G_{N,A,B}\) implies the congruence conditions for matrix entries in \(H_N(m,n)\). The converse can be proved similarly.

Next, we recall the definition of the division polynomials of an elliptic curve.

**Definition 3.3** (the division polynomials, [Si09] III.Exercises 3.7]). For a non-negative integer \(n\), the \(n\)th division polynomial \(\psi_n\) of an elliptic curve \(E_{s,t} : y^2 = x^3 + sx + t\) over \(K(s,t) = K(\mathcal{U})\) is defined inductively by:

\[
\psi_0 = 0, \quad \psi_1 = 1, \quad \psi_2 = 2y,
\]

\[
\psi_3 = 3x^4 + 6sx^2 + 12tx - s^2, \quad \psi_4 = 4y(x^6 + 5sx^4 + 20tx^3 - 5s^2x^2 - 4stx - (8t^2 + s^2)),
\]

\[
\psi_{2k-1} = \psi_{k-1} \psi_k^3 - \psi_{k-2} \psi_k^3, \quad \text{and} \quad \psi_{2k} = \frac{\psi_k}{2y} (\psi_{k+1} \psi_{k-1} - \psi_{k-2} \psi_{k+1}), \quad \text{for } k \geq 3.
\]

For each integer \(n \geq 1\), let

\[
\phi_n = x \psi_n^2 - \psi_{n+1} \psi_{n-1}, \quad \text{and} \quad \omega_n = \frac{\psi_{n+1} \psi_{n-1} - \psi_{n-2} \psi_{n+1}}{4y}.
\]

Then, for any point \(P = (x,y) \in E_{s,t}, \begin{pmatrix} n \end{pmatrix} P = \left( \frac{\phi_n(P)}{(\psi_n(P))^2}, \frac{\omega_n(P)}{(\psi_n(P))^3} \right)\). In particular,

\[
x([n]P) = \frac{\phi_n(P)}{(\psi_n(P))^2} = \frac{\phi_n(x)}{(\psi_n(x))^2}.
\]

For any integer \(N \geq 1\), we recall the following (for example, see [Si09] III.Exercises 3.7):

- If \(N\) is odd, then \(\psi_N^x \in (\mathbb{Q}[\mathcal{U}])[x] \subseteq (K(\mathcal{U}))[x]\) is monic of \(\deg_x \psi_N^x = \frac{N^2-1}{2}\), and the zeros of \(\psi_N^x\) are the \(x\)-coordinates of points in \(E_{s,t}[N] - \{O\}\).

- If \(N\) is even, then \(\psi_N^y \in (\mathbb{Z}[\mathcal{U}])[x] \subseteq (K(\mathcal{U}))[x]\) is monic of \(\deg_x \psi_N = \frac{N^2-4}{2}\), and the zeros of \(\psi_N^y\) are the \(x\)-coordinates of points in \(E_{s,t}[N] - E[2]\).

Then, we construct the following polynomial whose solutions are exactly the \(x\)-coordinates of the points of order \(N\) by using the Mobius function \(\mu\).

**Definition 3.4** (the primitive division polynomials and their Galois groups). For each integer \(N \geq 1\), we define the \(N\)th primitive division polynomial \(\Psi_N\) as follows:

\[
\Psi_1 = 1, \quad \text{and} \quad \Psi_2 = x^3 + sx + t \in \mathbb{Q}[\mathcal{U}][x] \subseteq K(\mathcal{U})[x],
\]
For each integer \( N \geq 1 \), we denote by \( \overline{G}_N \) the Galois group of \( \Psi_N \) for \( E_{s,t} \) over \( K(s,t) \).
Let the polynomial \( \Psi_{N,A,B} \) over \( K \) be the specialization of \( \Psi_N \) at \( (A,B) \in \mathcal{U}(K) \).

First, we give some properties of the primitive division polynomials in Lemma 3.5 and Remark 3.6.

**Lemma 3.5.** If \( N \geq 3 \), \( \deg_x \Psi_N = \frac{1}{2} N^2 \prod_{\text{primes } p|N} (1 - \frac{1}{p^2}) \), which is the half of the number of torsion points in \( E_{s,t}(K(\mathcal{U})) \) of order \( N \).

**Proof.** The definition of \( \Psi_N \) proves that \( \delta = \deg_p \Psi_N \) is the half of the number of torsion points in \( E_{s,t}(K(\mathcal{U})) \) of exact order \( N \). Since \( E_{s,t}[N] \cong (\mathbb{Z}/N\mathbb{Z})^2 \) as abelian groups, we have that \( 2\delta = \#X \) where \( X = \{(a,b) \in (\mathbb{Z}/N\mathbb{Z})^2 : |(a,b)| = N\} \). The Chinese remainder theorem proves that \( \#X = \prod_{\text{primes } p|N} \#X_p \) where \( X_p = \{(a,b) \in (\mathbb{Z}/p^np\mathbb{Z})^2 : |(a,b)| = p^n \} \). The principle of inclusion–exclusion computes

\[
\#X_p = 2 \cdot p^{np} \cdot (p^{np} - p^{n-1}) - (p^{np} - p^{n-1})^2 = p^{2np} - p^{2n-2} = p^{2np}(1 - p^{-2})
\]

where the first term is the number of \( (a,b) \in (\mathbb{Z}/p^np\mathbb{Z})^2 \) such that at least one of \( a \) and \( b \) is relatively prime to \( p \) and the last term is the number of \( (a,b) \in (\mathbb{Z}/p^np\mathbb{Z})^2 \) such that both of \( a \) and \( b \) are relatively prime to \( p \).

**Remark 3.6.** For each integer \( N \geq 1 \), we note that:

1. \( \Psi_N \in (\mathbb{Q}[\mathcal{U}])[x] \subseteq (K(\mathcal{U}))[x] \) and \( \Psi_N \) is monic.
2. If \( N \geq 2 \), for \( (A,B) \in \mathcal{U}(K) \), the zeros of \( \Psi_{N,A,B} \) are the \( x \)-coordinates of torsion points in \( E_{A,B}(K) \) of order \( N \).
3. \( \Psi_N \) is separable over \( K(\mathcal{U}) \).
4. If we assign weights to the variables such that \( \deg(x) = 1 \), \( \deg(s) = 2 \), and \( \deg(t) = 3 \), then \( \Psi_N \in K(\mathcal{U})[x] \) is a homogeneous polynomial with \( \deg(\Psi_N) = \deg_x \Psi_N \), in other words, \( \Psi_{N,D^2s,D^3t}(Dx) = D^{\deg_x \Psi_{N,s,t}} \Psi_{N,s,t}(x) \) for any \( D \in K^{\times} \).

Next, we note that if there is \( \nu \in G_{N,A,B} \) such that \( \nu(P) = -P \) for all \( P \in E_{A,B}[N] \), then \( \langle \nu \rangle \leq G_{N,A,B} \), so we define the group,

\[
\overline{G}_{N,A,B} = \begin{cases} 
G_{N,A,B}/\langle \nu \rangle, & \text{if there is } \nu \in G_{N,A,B} \text{ such that } \nu(P) = -P \text{ for all } P \in E_{A,B}[N], \\
G_{N,A,B}, & \text{otherwise}.
\end{cases}
\]

Then, it turns out that \( \overline{G}_{N,A,B} \) is the Galois group of \( \Psi_{N,A,B} \) over \( K \) as follows.

**Lemma 3.7.** For any \( (A,B) \in \mathcal{U}(K) \) and each integer \( N \geq 3 \), the Galois group of \( \Psi_{N,A,B} \) over \( K \) is \( \overline{G}_{N,A,B} \). For a given basis \( B \) of \( E_{A,B}[N] \), we define the group homomorphism

\[
\tau_B : \overline{G}_{N,A,B} \to GL_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\} \text{ by } \sigma \mapsto r_B(\sigma)\{\pm I_2\},
\]

for each \( \sigma \in G_{N,A,B} \). Then, \( \tau_B \) is injective and

\[
\tau_B(\overline{G}_{N,A,B}) = \langle -I_2, r_B(G_{N,A,B}) \rangle / \{\pm I_2\}.
\]

In particular, if there is no \( \nu \in G_{N,A,B} \) such that \( \nu(P) = -P \), then

\[
\tau_B(\overline{G}_{N,A,B}) \cong r_B(G_{N,A,B}).
\]
Proof. Let \( P_1 := (\alpha_1, \beta_1) \) and \( P_2 := (\alpha_2, \beta_2) \in E_{A,B}^N \) form a basis of \( E_{A,B}^N \cong \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z} \). If we let \( P_3 = P_1 + P_2 := (\alpha_3, \beta_3) \), then \( P_3 \in E_{A,B}^N \). First, we show that the splitting field \( L \) of \( \Psi_{N,A,B} \) over \( K \) is \( K(\alpha_1, \alpha_2, \beta_1, \beta_2) \). It is enough to show that

\[
\{ \sigma \in G_{A,B,N} : \sigma(\gamma) = \gamma \text{ for all } \gamma \in L \} = \{ \sigma \in G_{A,B,N} : \sigma(\gamma) = \gamma \text{ for all } \gamma \in K(\alpha_1, \alpha_2, \beta_1, \beta_2) \}.
\]

If \( \sigma \in G_{A,B,N} \) fixes \( L \), then for each \( i = 1, 2, 3 \), \( \sigma(\alpha_i) = \alpha_i \) and \( \sigma(\beta_i) = \beta_i \) or \( -\beta_i \), i.e., \( \alpha_i \) are fixed under \( \sigma \) and \( P_i^3 = \pm P \) for any \( P \in E_{A,B}^N \). If we let \( \sigma(P_i) = \epsilon_i P_i \) for \( \epsilon_i \in \{ \pm 1 \} \), then we conclude that \( \epsilon_1 = \epsilon_2 = \epsilon_3 \). Hence, there is a sign \( \epsilon \in \{ \pm 1 \} \) such that

\[
P_i^3 = \epsilon_i P_i \text{ for all } i = 1, 2, 3, \quad \text{i.e., } \beta_1, \beta_2 \text{ is fixed by } \sigma.
\]

Conversely, if \( \sigma \in G_{A,B,N} \) fixes \( K(\alpha_1, \alpha_2, \beta_1, \beta_2) \), then for any integers \( m_1 \) and \( m_2 \), the \( x \)-coordinate of \( [m_1]P_1 + [m_2]P_2 \) is fixed under \( \sigma \). Hence \( L = K(\alpha_1, \alpha_2, \beta_1, \beta_2) \).

If \( \beta_1 \in L \) (so \( \beta_2 \in L \)), then there is no \( \nu \in G_{N,A,B} \) such that \( \nu(P) = -P \) for all \( P \in E_{A,B}^N \) and so \( \text{Gal}(L/K) = G_{N,A,B} = \overline{G}_{N,A,B} \). If \( \beta_1 \notin L \) (so \( \beta_2 \notin L \)), then \( \nu \in G_{N,A,B} \) and \( \text{Gal}(L/K) = G_{N,A,B} / \{ \nu \} = \overline{G}_{N,A,B} \) since \( \text{Gal}(K(P_1, P_2)/L) = \{ \nu \} \).

If there is \( \nu \in G_{N,A,B} \) such that \( \nu(P) = -P \) for all \( P \in E_{A,B}^N \), then \( r_B(\nu) = -I_2 \), so \( \overline{r_B} \) is well-defined. Also, if such a \( \nu \) exists, then

\[
\overline{r_B}(\overline{G}_{N,A,B}) = r_B(G_{N,A,B}) / \{ \pm I_2 \} = \langle -I_2, r_B(G_{N,A,B}) \rangle / \{ \pm I_2 \}.
\]

Otherwise, we have that

\[
\overline{r_B}(\overline{G}_{N,A,B}) = r_B(G_{N,A,B}) = \langle -I_2, r_B(G_{N,A,B}) \rangle / \{ \pm I_2 \}.
\]

We note that \( \overline{r_B} \) is injective since \( r_B \) is injective, and \( r_B(\nu) = -I_2 \) if there is a \( \nu \in G_{N,A,B} \) such that \( \nu(P) = -P \).

We define the following groups which are used throughout this paper.

**Definition 3.8.** For positive divisors \( m \) and \( n \) of \( N \) such that \( m \mid n \), we define three groups related with the groups \( H_N(m,n) \) defined in Lemma 3.2:

\[
\begin{align*}
(i) & \quad \overline{H}_N(m,n) := \langle H_N(m,n), -I_2 \rangle / \{ \pm I_2 \} \\
(ii) & \quad H_N^1(m,n) := H_N(m,n) \cap \text{SL}_2(\mathbb{Z}/N\mathbb{Z}) \\
(iii) & \quad \overline{H}_N^1(m,n) := \langle H_N^1(m,n), -I_2 \rangle / \{ \pm I_2 \}
\end{align*}
\]

We rewrite Lemma 3.2 in terms of \( \overline{G}_{N,A,B} \) instead of \( G_{N,A,B} \) for \( N \geq 3 \) as follows:

**Proposition 3.9.** For any integer \( N \geq 3 \), if \( m \) and \( n \) are positive divisors of \( N \) such that \( m \mid n \), then, for \( (A,B) \in \mathbb{Z}(K) \), the following are equivalent:

\[
\begin{align*}
(a) & \quad \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \subseteq E_{A,B}(K)[N]. \\
(b) & \quad \text{The image } \overline{r_B}(\overline{G}_{N,A,B}) \text{ of the Galois group } G_{N,A,B} \text{ of } \Psi_{N,A,B} \text{ over } K \text{ under } \overline{r_B} \text{ given in } \overline{3} \text{ is a subgroup of } \overline{H}_N(m,n) \text{ for some basis } B \text{ of } E_{A,B}(K)[N] \text{ and there is no } \nu \in G_{N,A,B} \text{ such that } r_B(\nu) = -I_2.
\end{align*}
\]

To sum up, the \( N \)-torsion subgroup problem is interpreted into the problem related with the Galois group of \( \Psi_{N,A,B} \) together with the non-existence of \( \nu \in G_{N,A,B} \) such that \( r_B(\nu) = -I_2 \). In order to observe the condition on the non-existence of such a \( \nu \in G_{N,A,B} \), we consider “quadratic twists” of an elliptic curve in Section 3.1.
3.1. Quadratic Twists. The multiplicative group $\mathbb{G}_m = \mathbb{A}^1 - \{0\}$ acts on $\mathbb{A}^2$ and $\mathbb{Z}$ by $D \cdot (A, B) = (D^2A, D^3B)$ for $D \in \mathbb{G}_m$ and $(A, B)$ in $\mathbb{A}^2$ or $\mathbb{Z}$. We call $D \cdot (A, B)$ the quadratic twist of $(A, B)$ by $D$. Equivalently, the quadratic twists of a given elliptic curve are defined as follows:

**Definition 3.10.** For $(A, B) \in \mathbb{A}^2$ and $D \in \mathbb{G}_m$, the curve $E_{D^2A,D^3B} : y^2 = x^3 + D^2Ax + D^3B$ is called the quadratic twist of $E_{A,B}$ by $D$ and denoted by $E_{A,B}^D$.

**Remark 3.11.**

(a) The morphism $\phi : E_{A,B}^D \to E_{A,B}$ defined by $(x, y) \mapsto \left(\frac{x}{D}, \frac{y}{D^3}\right)$ is an isomorphism defined over $K(\sqrt{D})$. Any curve $E_{A,B}/K$ and its quadratic twist $E_{A,B}^D$ is isomorphic over $K$ if and only if $D$ is a square in $K^\times$.

(b) Any quadratic twist preserves the Galois group of $\Psi_{N,A,B}$, i.e., $\overline{G}_{N,A,B} = \overline{G}_{N,D^2A,D^3B}$, for any $D \in K^\times$ and $(A, B) \in \mathbb{Z}(K)$.

(c) The action of every $D^2 \in (K^\times)^2$ on $\mathbb{Z}(K)$ preserves the torsion subgroups, i.e., $E_{A,B}^{D^2}(K)_{\text{tors}} \cong E_{A,B}(K)_{\text{tors}}$ for all $D \in K^\times$, since $E_{A,B}^{D^2}$ is isomorphic to $E_{A,B}$ over $K$.

Now, we give a certain condition which determines whether there exist torsion points of order $m$ and $n$ whose $x$-coordinates and $y$-coordinates are in $K$, by taking quadratic twists.

For $(A, B) \in \mathbb{Z}(K)$ and two positive integers $m$ and $n$ such that $m \mid n$, we define Condition $\mathcal{P}(m, n)$:

**Condition** $\mathcal{P}(m, n) : \Psi_{m,A,B}$ splits completely over $K$ and $\Psi_{n,A,B}$ has a zero in $K$. \hspace{1cm} (5)

**Proposition 3.12.** For any integer $N \geq 3$, if $m$ and $n$ are positive divisors of $N$ such that $m \mid n$, then for $(A, B) \in \mathbb{Z}(K)$, the following are equivalent:

(a) The image $\overline{\mathcal{P}_B}(\overline{G}_{N,A,B})$ of the Galois group $\overline{G}_{N,A,B}$ of $\Psi_{N,A,B}$ over $K$ is a subgroup of $\overline{\mathcal{P}_B}(\overline{H}_{N}(m, n))$ for some basis $\mathcal{B}$ for $E_{A,B}[N]$.

(b) Condition $\mathcal{P}(m, n)$ is satisfied.

(c) There exists $D \in K^\times$ such that $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \subseteq E_{A,B}^D(K)[N]$.

**Proof.** To show that Statement (a) implies Statement (b), let $\mathcal{B} = \{R_1, R_2\}$ be such a basis of $E_{A,B}[N]$. Then $\left[\frac{N}{m}\right] R_2 \in E_{A,B}(K)$ and $\left\{\left[\frac{N}{m}\right] a R_1 + \left[\frac{N}{m}\right] b R_2 : a, b \in \mathbb{Z}/m\mathbb{Z}\right\} = E_{A,B}[m] \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$. Hence, the $x$-coordinates $x\left(\left[\frac{N}{m}\right] R_2\right)$ and $x(P)$ are in $K$ if $P \in E_{A,B}[m]$ is of order $m$.

To show that Statement (b) implies Statement (c), let $\alpha \in K$ be a zero of $\Psi_{n,A,B}$ and let $D := \alpha^3 + A\alpha + B$. The point $Q := (D\alpha, D^2) \in E_{A,B}^D(K)$ is of order $n$. There is a basis $\mathcal{B} := \{R_1, R_2\}$ for $E_{A,B}[N]$ such that $\left[\frac{N}{m}\right] R_2 = Q$. Since $x\left(\left[\frac{N}{m}\right] R_1\right) \in K$ is a zero of $\Psi_{m,D^2A,D^3B}$ in $K$, \cite{3.13} in the proof of Lemma \cite{3.14} shows that $y\left(\left[\frac{N}{m}\right] R_1\right) \in K$. We conclude that $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \cong \left\{\left[\frac{N}{m}\right] R_1 \mid \left[\frac{N}{m}\right] R_2 \right\} \subseteq E_{A,B}^D(K)[N]$.

Statement (c) implies Statement (a) directly by Proposition \cite{3.14} and Remark \cite{3.11}(b). \hspace{1cm} $\square$

Next, we observe how many $D \in K^\times/(K^\times)^2$ satisfy $E_{A,B}^{D}(K)_{\text{tors}} \supseteq \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ for each $E_{A,B}/K$. 

Lemma 3.13. Let \((A, B) \in \mathcal{W}(K)\) and \(m\) and \(n\) be two positive integers such that \(m \mid n\). Assume that \((A, B)\) satisfies Condition \(P(m, n)\). Then, there exists \(D \in K^\times/(K^\times)^2\) such that \(\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \subseteq E_{A,B}^D(K)_{\text{tors}}\), and the \(y\)-coordinate of each point of \(E_{A,B}[m]\) is of the form \(a\sqrt{D}\) for some \(a \in K\). In particular, we have the following:

(a) If \(n \leq 2\), then for every \(D \in K^\times/(K^\times)^2\), we have that \(\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \subseteq E_{A,B}^D(K)_{\text{tors}}\).

(b) If \(n \geq 3\), then the number of such a \(D\) modulo \((K^\times)^2\) is at most 3.

(c) If \(m \geq 3\), then such a \(D\) is unique modulo \((K^\times)^2\).

Proof. First, for the existence of such a \(D \in K^\times/(K^\times)^2\), if \(m = 1\), then there is nothing to prove. Let \(m \geq 2\). Let \(\alpha \in K\) be a zero of \(\Psi_{m,A,B}\). Let \(D := \alpha^3 + A\alpha + B\). Then \(Q := (D\alpha, D^2) \in E_{A,B}^D(K)\) has order \(m\). Choose \(P \in E_{A,B}^D[m]\) such that \(\{\alpha \frac{n}{m}\} Q\) is a basis of \(E_{A,B}^D[m]\). If \(n = 2\), then \(y(P) = 0 \in K\). If \(n \geq 3\), then by Remark 3.6(4), \(\Psi_{m,D^3A,D^3B}\) splits completely over \(K\) and \(x(P), x\left(\alpha \frac{n}{m}\right)Q, x\left(P + \alpha \frac{n}{m}\right)Q \in K\). As in \([1]\) of the proof of Lemma 3.7, we have that \(y(P) y\left(\alpha \frac{n}{m}\right) \in K\) and \(y(P) \in K\). Hence, \(\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \cong \langle P, Q \rangle \subseteq E_{A,B}^D(K)_{\text{tors}}\). Considering the isomorphism \(\phi : E_{A,B}^D \rightarrow E_{A,B}\) defined by \((x, y) \mapsto \left(\frac{\sqrt{D}}{\alpha}, \frac{y}{\sqrt{D^3}}\right)\), we conclude that the \(y\)-coordinates of all \(m\)-torsion points of \(E_{A,B}(K)\) are of the form \(a\sqrt{D}\) for some \(a \in K\).

For (a), if \(n = 1\), then it holds trivially for every \(D\) and if \(n = 2\), then it is also true since the \(y\)-coordinates of points in \(E_{A,B}^D\) of order 2 are 0.

Let’s prove (c) first. For the uniqueness of such a \(D\) modulo \((K^\times)^2\), if there exists such another \(D'\), as in the above, the \(y\)-coordinate of each \(m\)-torsion points of \(E_{A,B}(K)\) is of the form \(b\sqrt{D'}\) and \(b\sqrt{D}\) simultaneously, for some \(b, b' \in K^\times\) because \(m \geq 3\). In other words, \(D'/\mathbb{Z} \in K^\times/(K^\times)^2\), so this proves the uniqueness of \(D \in K^\times/(K^\times)^2\).

For (b), assume that there are four distinct \(D_1, D_2, D_3, D_4 \in K^\times/(K^\times)^2\) such that \(\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \subseteq E_{A,B}^{D_i}(K)_{\text{tors}}\) for each \(i = 1, 2, 3, 4\). We define the isomorphism \(\phi_i : E_{A,B}^{D_i} \rightarrow E_{A,B}\) by \((x, y) \mapsto \left(\frac{\sqrt{D_i}}{\sqrt{D_i}}, \frac{y}{\sqrt{i}}\right)\) over \(K(\sqrt{D_i})\) for each \(i = 1, 2, 3, 4\). Let \(R_i \in E_{A,B}^{D_i}(K)[n]\) be a point of order \(n\) for each \(i = 1, 2, 3, 4\). Then,

\[
\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \subseteq E_{A,B}(K(\sqrt{D_i}))_{\text{tors}} \quad \text{and} \quad 0 \neq y(\phi_i(R_i)) = a_i\sqrt{D_i} \quad \text{for some} \quad a_i \in K^\times.
\]

For each choice of two distinct \(i, j \in \{1, 2, 3, 4\}\), we have a positive divisor \(d_{ij}\) of \(n\) such that \(\mathbb{Z}/d_{ij}\mathbb{Z} \times \mathbb{Z}/d_{ij}\mathbb{Z} \cong \langle \phi_i(R_i), \phi_j(R_j) \rangle \subseteq E_{A,B}(K(\sqrt{D_i}, \sqrt{D_j}))\).

If \(d_{ij} = 1\) for some distinct \(i\) and \(j\), then \(\phi_i(R_i) \in \langle \phi_j(R_j) \rangle\) and since \(n \geq 3\), \(0 \neq y(\phi_i(R_i))\) is of the form both \(a\sqrt{D_i}\) and \(b\sqrt{D_j}\) for some \(a, b \in K^\times\), which is a contradiction since \(D_i\) and \(D_j\) are distinct modulo \((K^\times)^2\).

If \(d_{ij} \geq 3\) for some distinct \(i\) and \(j\), say \(i = 1\) and \(j = 2\), then letting \(d := d_{12}\). Since \(\mathbb{Z}/d\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z} \subseteq \langle \phi_1(R_1), \phi_2(R_2) \rangle\), we have that for \(k = 3, 4\), \([\frac{n}{d}] \phi_k(R_k) \in \langle \phi_1(R_1), \phi_2(R_2) \rangle\) and \(0 \neq y\left(\frac{n}{d} \phi_k(R_k)\right) \in K(\sqrt{D_1}, \sqrt{D_2})\). Hence, \(D_k\) is a square in \(K(\sqrt{D_1}, \sqrt{D_2})\). In other words, one of \(D_k\) and \(D_k/\alpha\) is a square in \(K\), which is a contradiction since \(D_1\), \(D_2\), \(D_3\), and \(D_4\) are distinct in \(K^\times/(K^\times)^2\).

If \(d_{ij} = 2\) for all distinct \(i\) and \(j\), then \(n\) is even and \([2] \phi_i(R_1) \in \langle \phi_j(R_2) \rangle\). In other words, \(y([2] \phi_i(R_1))\) is of form \(a\sqrt{D_i}\) and \(b\sqrt{D_j}\) for some \(a, b \in K\) simultaneously. Hence, \(y([2] \phi_i(R_1)) = 0\). But if \(n \geq 6\), then \(y([2] \phi_i(R_1)) \neq 0\) and this is a contradiction. If \(n = 4\), let \(\{P, \phi_i(R_1)\}\) be a basis of \(E_{A,B}[4]\). Then \(\phi_2(R_2) = [2]P \pm \phi_i(R_1) = \pm \phi_3(R_3)\) since each of the two groups \(\langle \phi_1(R_1), \phi_2(R_2) \rangle\) and \(\langle \phi_1(R_1), \phi_3(R_3) \rangle\) is isomorphic to \(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \subseteq E_{A,B}[4]\) and both groups contain \(\phi_1(R_1)\). But this is a contradiction since \(0 \neq y(\phi_2(R_2)) = \pm y(\phi_3(R_3))\) is
of the form both $a\sqrt{D_2}$ and $b\sqrt{D_3}$ for some $a, b \in K$, so $y(\phi_2(R_2)) = \pm y(\phi_3(R_3)) = 0$ again. This is a contradiction.

Next, we compare two densities of elliptic curves with prescribed subgroups in terms of $m$ and $n$ and those satisfying Condition $P(m, n)$, and we get the following result, which implies that the latter dominates the asymptotic behaviour of the former and this is the advantage of using Condition $P(m, n)$ with considering only $x$-coordinates but not necessarily considering the $y$-coordinates of torsion points.

**Proposition 3.14.** For a positive integer $N$ and its positive divisors $m, m', n,$ and $n'$ such that $m \mid n, m' \mid n', m \mid m', n \mid n'$, and $n \geq 3$, we assume that there exists an elliptic curve $E/K$ such that $E(K) \forall D \supseteq \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$. For a positive real number $X$, let

$$f(X) = \frac{\# \{(A, B) \in \mathcal{U}(K) : H(A, B) \leq X, \mathbb{Z}/m'\mathbb{Z} \times \mathbb{Z}/n'\mathbb{Z} \subseteq E_{A,B}(K)[N]\}}{\# \{(A, B) \in \mathcal{U}(K) : H(A, B) \leq X, \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \subseteq E_{A,B}(K)[N]\}}$$

and

$$g(X) = \frac{\# \{(A, B) \in \mathcal{U}(K) : H(A, B) \leq X, (A, B) \text{ satisfies } P(m', n')\}}{\# \{(A, B) \in \mathcal{U}(K) : H(A, B) \leq X, (A, B) \text{ satisfies } P(m, n)\}}.$$

Then, for any real number $X > 0$,

$$\frac{1}{3} g(X) \leq f(X) \leq g(X).$$

**Proof.** For $(A, B) \in \mathcal{U}(K)$, we consider two functions for $X \in \mathbb{R}^+$,

$$f_{A,B}(X) = \frac{\# \{D \in K^\times : H(D^2A, D^3B) \leq X, \mathbb{Z}/m'\mathbb{Z} \times \mathbb{Z}/n'\mathbb{Z} \subseteq E_{A,B}(K)[N]\}}{\# \{D \in K^\times : H(D^2A, D^3B) \leq X, \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \subseteq E_{A,B}(K)[N]\}},$$

and

$$g_{A,B}(X) = \frac{\# \{D \in K^\times : H(D^2A, D^3B) \leq X, D \cdot (A, B) \text{ satisfies } P(m', n')\}}{\# \{D \in K^\times : H(D^2A, D^3B) \leq X, D \cdot (A, B) \text{ satisfies } P(m, n)\}}.$$

Since the set $\{(A, B) \in \mathcal{U}(K) : H(A, B) \leq X\}$ for each positive number $X$ is finite, we can find a finite subset $\mathcal{R} \subseteq \mathcal{U}(K)$ such that any $(A, B) \in \mathcal{U}(K)$ with $H(A, B) \leq X$ is written as $(A, B) = (k^2A_0, k^3B_0)$ where $k \in K^\times$, for a unique $(A_0, B_0) \in \mathcal{R}$. Since the denominator of $f(X)$ is the sum of the denominators of $f_{A,B}(X)$ over all $(A, B) \in \mathcal{R}$. The denominators and numerators of $f(X)$ and $g(X)$ are written similarly. Hence, it is enough to show that $\frac{1}{3} g_{A,B}(X) \leq f_{A,B}(X) \leq g_{A,B}(X)$ for each $(A, B) \in \mathcal{U}(K)$.

If $(A, B)$ does not satisfy Condition $P(m', n')$, then neither does $D \cdot (A, B)$ for any $D \in K^\times$, and by Remark 3.11 (5) and Proposition 3.12 $\mathbb{Z}/m'\mathbb{Z} \times \mathbb{Z}/n'\mathbb{Z} \subseteq E_{A,B}(K)_{\text{tors}}$ for all $D \in K^\times$. Hence,

$$\{D \in K^\times : H(D^2A, D^3B) \leq X, D \cdot (A, B) \text{ satisfies } P(m', n')\} = \emptyset,$$

and

$$\{D \in K^\times : H(D^2A, D^3B) \leq X, \mathbb{Z}/m'\mathbb{Z} \times \mathbb{Z}/n'\mathbb{Z} \subseteq E_{A,B}(K)[N]\} = \emptyset,$$

so $f_{A,B}(X) = g_{A,B}(X) = 0$, in which case the statement holds trivially.

If $(A, B)$ satisfies Condition $P(m', n')$, then so do $D \cdot (A, B)$ for all $D \in K^\times$ by Proposition 3.12. Hence, in this case,

$$\{D \in K^\times : H(D^2A, D^3B) \leq X, D \cdot (A, B) \text{ satisfies } P(m', n')\} = \{D \in K^\times : H(D^2A, D^3B) \leq X\} = \{D \in K^\times : H(D^3A, D^3B) \leq X, D \cdot (A, B) \text{ satisfies } P(m, n)\},$$

so $g_{A,B}(X) = 1$. By Lemma 3.13 we see that

$$\emptyset \neq \{D \in K^\times / (K^\times)^2 : \mathbb{Z}/m'\mathbb{Z} \times \mathbb{Z}/n'\mathbb{Z} \subseteq E_{A,B}(K)[N]\}$$
Remark 3.15. In the proof of Proposition 3.14, the finiteness of points in the group action and get the set $\mathcal{F}$ is important. If there are infinitely many points in $\mathcal{F}$ with bounded height, we need the Axiom of Countable Choice to choose representatives of $\mathcal{F}$ up to the group action and get the set $\mathcal{F}$.

Finally, we prove that it is enough to consider only Condition $\mathcal{P}(m,n)$ instead of a torsion subgroup for our main goal.

Proposition 3.16. Let $N \geq 3$. For positive divisors $m, n$ of $N$ such that $m \mid n$, assume that there exists an elliptic curve $E/K$ over $K$ such that $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \subseteq E(K)_{\text{tors}}$. Then Statement (a) below implies Statement (b) and moreover, if $n \geq 3$, then (a) and (b) are equivalent:

(a) Let $m'$ and $n'$ be positive divisors of $N$ such that $m' \mid n'$, $m \mid m'$, and $n \mid n'$. If almost all elements of the set $\{(A,B) \in \mathcal{F} : (A,B) \text{ satisfies Condition } \mathcal{P}(m,n)\}$ satisfies Condition $\mathcal{P}(m',n')$, then $(m',n') = (m,n)$.

(b) Almost all $(A,B) \in \mathcal{F}$ such that $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \subseteq E_{A,B}(N)$ satisfy $E_{A,B}(N) \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$.

Proof. First, let $n \geq 3$ be an integer. If $(m,n) \neq (m',n')$ with $m' \mid n'$, $m \mid m'$ and $n \mid n'$, then Proposition 3.14 shows $f(X)$ approaches 0 as $g(X)$ goes to 0 and vice versa, so the following two statements are equivalent:

- Condition $\mathcal{P}(m',n')$ is not satisfied, for almost all $(A,B) \in \mathcal{F}$ satisfying Condition $\mathcal{P}(m,n)$.
- $E_{A,B}(N) \nsubseteq \mathbb{Z}/m'\mathbb{Z} \times \mathbb{Z}/n'\mathbb{Z}$, for almost all $(A,B) \in \mathcal{F}$ such that $E_{A,B}(N) \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$.

Since the number of pairs $(m',n')$ of positive divisors $m',n'$ of $N$ which satisfy the above divisibility conditions of $m, m', n, n'$ is finite, this completes the proof for $n \geq 3$. For $n = 1, 2$ and any $(A,B) \in \mathcal{F}$, we know that $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \subseteq E_{A,B}(N)$ if and only if $(A,B)$ satisfies Condition $\mathcal{P}(m,n)$ since the $y$-coordinate of each non-zero 2-torsion point of $E_{A,B}$ is 0.

For $(A,B) \in \mathcal{F}$ and each integer $N \geq 3$, we have considered the Galois group $G_{N,A,B}$ to determine whether $E_{A,B}(N) \subseteq \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ or not. The condition that $r_B(G_{N,A,B}) \subseteq H_N(m,n)$ for some basis $B$ of $E_{A,B}(N)$ implies $E_{A,B}(N) \subseteq \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ but not $E_{A,B}(N) \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$. Therefore, to prove that $E_{A,B}(N) \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$, we also need to find the condition to have that $r_B'(G_{N,A,B}) \nsubseteq H_N(m',n')$ for any basis $B'$ of $E_{A,B}(N)$ and divisors $m'$ and $n'$ of $N$ such that $m' \mid n'$, $m \mid m'$, and $n \mid n'$, and we give such a condition as follows.

Lemma 3.17. Let $m, m', n$, and $n'$ be positive divisors of $N \geq 3$ such that $m \mid n$ and $m' \mid n'$. Then, $\mathcal{P}_N(m,n) \subseteq \mathcal{P}_N(m',n')^{-1}$ for some $\gamma \in H_N(1,1)$ if and only if $m' \mid m$ and $n' \mid n$. 

Proof. The converse is obvious. It is obvious that
\[ \langle H_N^1(m, n), -I_2 \rangle \subseteq \langle \gamma H_N^1(m', n')\gamma^{-1}, -I_2 \rangle \]
if \( \overline{H}_N(m, n) \subseteq \overline{\tau H}_N(m', n')^{-1} \). First, we observe that for any positive divisor \( k \) of \( N \), \( H_N^1(k, k) \leq H_N^1(1, 1) \), since \( H_N^1(k, k) \) is the kernel of the the natural projection \( H_N^1(1, 1) \to H_k^1(1, 1) \). Hence,
\[
\begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix} \in H_N^1(n, n) = \gamma^{-1}H_N^1(n, n)\gamma \subseteq \langle \gamma^{-1}H_N^1(m, n)\gamma, -I_2 \rangle \subseteq \langle H_N^1(m', n'), -I_2 \rangle ,
\]
and
\[
\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \in H_N^1(m, n) \subseteq \langle \gamma H_N^1(m', n')\gamma^{-1}, -I_2 \rangle \subseteq \langle \gamma H_N^1(m', m')\gamma^{-1}, -I_2 \rangle \subseteq \langle H_N^1(m', m'), -I_2 \rangle .
\]
They happen only if \( m' \mid m \) and \( n' \mid n \).
\qed

4. The proofs of the main results

In Section 3, we have considered only \( N \)-torsion subgroups \( E(K)[N] \) of elliptic curves \( E/K \) for a given integer \( N \geq 3 \). To consider the full torsion subgroups \( E(K)_{\text{tors}} \) over \( K \), we apply Merel’s theorem([Me96]) which implies that \( E(K)_{\text{tors}} = E(K)[N_K] \) for some constant integer \( N_K \geq 3 \) depending only on \( K \).

For each \( (m, n) \), we find a parameterization \( \mathcal{E}_{r,u}^{m,n} \) of almost all elliptic curves satisfying Condition \( \mathcal{P}(m, n) \) and show that the Galois group of the \( N_K \)th primitive polynomial of \( \mathcal{E}_{r,u}^{m,n} \) is between \( \overline{\tau}^1_{N_K}(m, n) \) and \( \overline{\tau}^1_{N_K}(m, n) \). This Galois group condition is the conclusion of Theorem 4.5.

The parameterizations are given in Section 4.1 through Section 4.4 and the proofs of Theorem 4.7 and Corollary 4.8 are completed in Section 4.5.

4.1. The trivial torsion subgroup. We consider the trivial torsion subgroup first.

Extending the definition of the group homomorphism \( \tau_\mathcal{B} \) in ([3] of Lemma 3.7 and using the same notation, we define the group homomorphism
\[
\tau_\mathcal{B} : \text{Gal} \left( K(V)(\mathcal{A}_p[N])/K(V) \right) \to \text{GL}_2(\mathbb{Z}/N\mathbb{Z})
\]
for a given elliptic curve \( \mathcal{A}_p \) parameterized by a point \( p \in V(K) \) of a variety \( V/K \) with respect to a basis \( \mathcal{B} \) of \( \mathcal{A}_p[N] \).

Lemma 4.1. For each integer \( N \geq 3 \), \( \Psi_N \) is irreducible over the function field \( K(\mathcal{Y}) = K(s, t) \) and \( \overline{\tau}^1_N(1, 1) \subseteq \tau_\mathcal{B} \left( \overline{G}_N \right) \subseteq \overline{\tau}^1_N(1, 1) \) for any basis \( \mathcal{B} \) of \( E_{s,t}[N] \).

Proof. Let \( \mathcal{B} \) be a basis of \( E_{s,t}[N] \). It is obvious that \( \tau_\mathcal{B} \left( \overline{G}_N \right) \subseteq \text{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{ \pm I_2 \} = \overline{\tau}^1_N(1, 1) \). First, we show that \( \overline{\tau}^1_N(1, 1) = \text{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{ \pm I_2 \} \subseteq \tau_\mathcal{B} \left( \overline{G}_N \right) \). By [DS05 Corollary 7.5.3], we have that \( \text{SL}_2(\mathbb{Z}/N\mathbb{Z}) \subseteq \text{Gal} \left( \mathbb{C}(s, t) \left( E_{s,t}[N] \right)/\mathbb{C}(s, t) \right) \). For a variable \( j \) and an elliptic curve
\[
\mathcal{E}_j : y^2 = x^3 - \frac{27j}{4(j - 1728)} x - \frac{27j}{4(j - 1728)}
\]
defined over the function field \( \mathbb{C}(j) \), we deduce by [DS05 Corollary 7.5.3] that
\[
\text{SL}_2(\mathbb{Z}/N\mathbb{Z}) \cong \text{Gal} \left( \mathbb{C}(j) \left( \mathcal{E}_j[N] \right)/\mathbb{C}(j) \right) .
\]
Since $\mathcal{E}_j = E_{-g/4,-g/4}$, a specialization of $E_{s,t}$ where $g = \frac{27j}{j-1728}$, we get
\[
\text{SL}_2(\mathbb{Z}/N\mathbb{Z}) \cong \text{Gal} \left( \mathbb{C} (j) (\mathcal{E}_j[N]) / \mathbb{C} (j) \right) \subseteq \text{Gal} \left( \mathbb{C} (s,t) (E_{s,t}[N]) / \mathbb{C} (s,t) \right)
\]
(for example, see [SE]). Moreover,
\[
\text{Gal} \left( \mathbb{C} (s,t) (E_{s,t}[N]) / \mathbb{C} (s,t) \right) \cong \text{Gal} \left( K (s,t) (E_{s,t}[N]) / K (s,t) \cap K (s,t) (E_{s,t}[N]) \right)
\subseteq \text{Gal} \left( K (s,t) (E_{s,t}[N]) / K (s,t) \right),
\]
so we conclude that
\[
\mathcal{H}^1_N(1,1) \subseteq \tau_B (\overline{G}_N).
\]
Since the group $\text{Gal} (K (s,t) (E_{s,t}[N]) / K (s,t))$ permutes transitively the torsion points of $E_{s,t}$ of order $N$, the group $\overline{G}_N$ permutes transitively the zeros of $\Psi_N$. This implies that $\Psi_N$ is irreducible.

Finally, we prove our main result for the trivial torsion subgroup as a generalization of Theorem 1.6 to an arbitrary number field $K$.

**Theorem 4.2.** For almost all $(A,B) \in \mathcal{U} (K)$, $E_{A,B}(K)_{\text{tors}}$ is trivial.

**Proof.** By Merel’s theorem ([Mc02]), there exists an integer $N_K \geq 3$ such that $E_{A,B}(K)_{\text{tors}} = E_{A,B}(K)[N_K]$ for all $(A,B) \in \mathcal{U} (K)$. Theorem 2.5 and Lemma 3.1 show that $\mathcal{H}^1_N(1,1) \subseteq \tau_B (\overline{G}_{N,A,B}) \subseteq \mathcal{H}_N(1,1)$ for some basis $B$ of $E_{A,B}[N_K]$ for almost all $(A,B) \in \mathcal{U} (K)$. Then, Proposition 3.12 and Lemma 3.17 prove that Proposition 3.16(a) holds, hence, Proposition 3.16(b) holds.

4.2. **Torsion subgroups of exponent 2.** In this subsection, we parameterize all elliptic curves $E/K$ satisfying Condition $\mathcal{P}(m,2)$. Recall that $\Psi_{2,s,t}(x) = x^3 + sx + t$.

Condition $\mathcal{P}(1,2)$ implies that $x^3 + sx + t$ has a zero for $x$. If $x^3 + sx + t$ has a zero $u \neq 0$, then we can parametrize as
\[
x^3 + sx + t = (x - u)(x^2 + ux + u^2 r) = x^3 + u^2(r - 1)x - u^3 r
\]
\[
= : x^3 + u^2 s_{1,2}(r) x + u^3 t_{1,2}(r),
\]
where
\[
s_{1,2}(r) = r - 1 \quad \text{and} \quad t_{1,2}(r) = -r \in K[r].
\]
If $x = 0$ is a zero of $x^3 + sx + t$, then $t = 0$ so we have that
\[
E_{s,0}: y^2 = x^3 + sx.
\]
Condition $\mathcal{P}(2,2)$ implies that $x^3 + sx + t$ splits completely over $K$. We have a non-zero solution $u$ of $x^3 + sx + t = 0$. Then we can let $ur$ be another zero for some $r$, and we have that
\[
x^3 + sx + t = (x - u)(x - ru)(x + (r + 1) u) = x^3 - u^2(r^2 + r + 1)x + u^3 r(r + 1)
\]
\[
= : x^3 + u^2 s_{2,2}(r) x + u^3 t_{2,2}(r),
\]
where
\[
s_{2,2}(r) = -(r^2 + r + 1), \quad \text{and} \quad t_{2,2}(r) = r(r + 1) \in K[r].
\]
To sum up, for \((m, n) = (1, 2)\) or \((2, 2)\), if we define the elliptic curve \(E_{r,u}^{m,n}\) by
\[
E_{r,u}^{m,n}: y^2 = x^3 + u^2 s_{m,n}(r)x + u^3 t_{m,n}(r)
\]
over the function field \(K(\mathcal{V}_{m,n})\), where \(\mathcal{V}_{m,n} = \{(r, u) \in A^2 : u \neq 0, (s_{m,n}(r), t_{m,n}(r)) \in \mathcal{W}\}\), then by the definitions of the rational varieties \(\mathcal{V}_{m,n}\), the specialization of \(E_{r,u}^{m,n}\) is non-singular if and only if \((r, u) \in \mathcal{V}_{m,n}(K)\), and we conclude that:

- \(E_{r,u}^{1,2}\) and \(E_{s,0}\) parameterize all elliptic curves \(E_{A,B}/K\) satisfying Condition \(P(1, 2)\).
- \(E_{r,u}^{2,2}\) parameterizes all elliptic curves \(E_{A,B}/K\) satisfying Condition \(P(2, 2)\).

### 4.3. Torsion subgroups of exponent 3

In this subsection, we parameterize all elliptic curves \(E/K\) satisfying Condition \(P(3, 3)\).

**Lemma 4.3.** Let \((A, B) \in \mathcal{W}(K)\).

If \(A \neq 0\), then we have the following:

- (a) \(\Psi_{3,A,B}\) has a zero \(u\) in \(K\) if and only if \(u \neq 0\) and \(A = ru^2\) and \(B = \frac{r^2 - 6ru - 3u^3}{12}\) for some \(r \in K\) such that \(\left(\frac{r, \frac{r^2 - 6ru - 3u^3}{12}}{K}\right) \notin \mathcal{W}(K)\).
- (b) \(\Psi_{3,A,B}\) splits completely over \(K\) with a zero \(u \in K\) if and only if \(u \neq 0\) and \(A, B\) have zeros \(v = 3\rho^2 - 12\sqrt{-3}(3\rho^2 + 4\rho + 1)\).

For such \(A, B\), \(\Psi_{3,0,B}\) has four zeros, \(0, u, \frac{1 \pm \sqrt{-3}}{2}u\).

**Proof.** First, suppose \(A \neq 0\). We note that \(\Psi_{3,A,B}\) has a zero \(u \in K\) if and only if \(A^2 - 6Au^2 - 3u^4 - 12Bu = 0\) by recalling Definition 3.4. Since \(A \neq 0\), clearly \(u \neq 0\), and we have that \(B = \frac{A^2 - 6Au^2 - 3u^4}{12u}\). Replacing \(A\) with \(u^2r\), we can write \(A\) and \(B\) in terms of \(u\) and \(r\) and get the desired expressions for \(A\) and \(B\). Suppose that \(\Psi_{3,A,B}\) splits completely over \(K\). Let \(v \in K - \{0, u\}\) be a zero of \(\Psi_{3,A,B}\). Since \(\Psi_{3,A,B}(u) = \Psi_{3,A,B}(v) = 0\),
\[
\frac{A^2 - 6Au^2 - 3u^4}{12u} = B = \frac{A^2 - 6Av^2 - 3v^4}{12v}.
\]
In other words, \(A\) is a solution of the quadratic equation,
\[
s^2 + 6uv + 3(u^2 + uv + v^2)uv = 0
\]
for \(s\). Hence, \(A = -3uv \pm (u - v)\sqrt{-3uv} \in K\) and \(-3uv\) is a square in \(K^\times\). Then \(v = -3u\rho^2\) for some \(\rho \in K^\times\). As in (3) of the proof of Lemma 3.7, since the product of the \(y\)-coordinates is in \(K\), we have
\[
K = K\left(\sqrt{u^3 + Au + B\sqrt{v^3 + Av + B}}\right) = K\left(\sqrt{\frac{A^2 + 6Au^2 + 9u^4}{12u}} \sqrt{\frac{A^2 + 6Av^2 + 9v^4}{12v}}\right)
\]
\[ = K \left( \frac{(A + 3u^2)(A + 3v^2)}{12\sqrt{uv}} \right) = K (\sqrt{-3}). \]

Hence, \( \sqrt{-3} \in K \). Now we find the parametrizations of \( A \) and \( B \). Recall that \( x - u \) and \( x - v \) are factors of \( \Psi_{3,A,B} \). Let \( \psi_{3,A,B} = 3(x - u)(x - v)(x^2 + ax + b) \). To find the factorization of \( \Psi_{3,A,B} \), we compare the coefficients of both sides and we conclude that \( -A^2 = 3buv, 4B = auv - b(u + v), 2A = b + uv - (u + v)a, \) and \( 0 = a - u - v \). Hence,
\[
\begin{align*}
a &= u + v \quad \text{and} \quad A = -3uv \pm (u - v)\sqrt{-3uv}.
\end{align*}
\]

We note that switching \( u \) and \( v \) gives the same result, so substituting \(-3u\rho^2 \) for \( v \), and switching \( u \) and \( v \) to avoid the double signs, we obtain
\[
\begin{align*}
A/u^2 &= 3\rho(3\rho^2 + 3\rho + 1), \\
b/u^3 &= -(A/u^2)^2(3\rho^2 + 3\rho + 1)^2, \\
B/u^3 &= \left(\frac{u + v}{4}\right)(uv - b) = \left(\frac{1 + v/u}{4}\right)(v/u - b/u^2) = \left(\frac{3\rho^2 - 1}{4}\right)(9\rho^4 + 18\rho^3 + 18\rho^2 + 6\rho + 1).
\end{align*}
\]

To prove the converse, let \( u = -3u\rho^2 \) and we show that \( \Psi_{3,A,B}(u) = 0 \) and \( \Psi_{3,A,B}(v) = 0 \). Again, the conclusion \( \Box \) in the proof of Lemma 3.7 implies that the splitting field of \( \Psi_{3,A,B} \) is \( K \) itself and the zeros of \( \Psi_{3,A,B}(x) \) can be obtained by solving \( \Psi_{3,A,B}(x) = 3(x - u)(x - v)(x^2 + ax + b) = 0 \).

We can prove for \( A \) similarly.

Finally, Lemma 4.3 gives the parametrizations of elliptic curves over \( K \) satisfying Condition \( \mathcal{P}(1,3) \) and Condition \( \mathcal{P}(3,3) \). To describe it precisely, let polynomials \( s_{1,3}, t_{1,3}, s_{3,3}, t_{3,3} \) in \( K[r] \) as:
\[
s_{1,3}(r) = r, \quad t_{1,3}(r) = \frac{r^2 - 6r - 3}{12},
\]
\[
s_{3,3}(r) = 3r^2 - 3r + 1, \quad \text{and} \quad t_{3,3}(r) = \frac{(3r^2 - 1)(9r^4 + 18r^3 + 18r^2 + 6r + 1)}{4}.
\]

For \((m, n) = (1, 3) \) or \((3, 3) \), we define the elliptic curve \( E_{m,n}^{3} \) by
\[
E_{m,n}^{3} : y^2 = x^3 + u^2 s_{m,n}(r)x + u^3 t_{m,n}(r)
\]
defined over the function field \( K(\mathcal{V}_{m,n}) \), where \( \mathcal{V}_{m,n} = \{ (r, u) \in K^2 : u \neq 0, (s_{m,n}(r), t_{m,n}(r)) \in \mathcal{W} \} \). By the definitions of the rational varieties \( \mathcal{V}_{m,n} \), the specialization of \( E_{r,u}^{m,n} \) is non-singular if and only if \( (r, u) \in \mathcal{V}_{m,n}(K) \). Note that \( E_{0,-u^3/4}^{3} = E_{0,u}^{3,3} \). Lemma 4.3 implies:

- \( E_{r,u}^{3,3} \) and \( E_{0,t}^{m,n} \) parameterize all elliptic curves \( E_{A,B}/K \) satisfying Condition \( \mathcal{P}(1,3) \).
- If \( \sqrt{-3} \in K \), then \( E_{r,u}^{3,3} \) parameterizes all elliptic curves satisfying Condition \( \mathcal{P}(3,3) \).

Conversely, if there is an elliptic curve \( E/K \) satisfying Condition \( \mathcal{P}(3,3) \), then \( \sqrt{-3} \in K \).

4.4. Torsion subgroups \( \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \) when \( g_{m,n} = 0 \). In Section 4.2 and 4.3 we have considered the parameterized elliptic curves of the form
\[
E_{r,u}^{m,n} : y^2 = x^3 + u^2 s_{m,n}(r)x + u^3 t_{m,n}(r)
\]
defined over the function field \( K(\mathcal{V}_{m,n}) \), where \( \mathcal{V}_{m,n} = \{ (r, u) \in K^2 : u \neq 0, (s_{m,n}(r), t_{m,n}(r)) \in \mathcal{W} \} \) is a \( K \)-rational variety and \( s_{m,n}, t_{m,n} \in K[r] \). The elliptic curve \( E_{r,u}^{m,n} \) parameterize all elliptic curves which satisfy Condition \( \mathcal{P}(m,n) \) for each \((m, n) \in \{(1, 2), (2, 2), (1, 3), (3, 3)\} \).

In this subsection, we consider the pairs \((m, n) \) of positive integers such that \( n \geq 4, m \mid n \), and \( g_{m,n} = 0 \), and for each of such pairs \((m, n) \), we parameterize almost all elliptic curves
$E_{r,u}^{m,n}$ satisfying Condition $\mathcal{P}(m,n)$. Referring to Proposition 3.16, Lemma 3.17 and Proposition 3.12, we need to show that the primitive polynomials of $E_{r,u}^{m,n}$ satisfy certain Galois group conditions and we give some sufficient conditions on it in Theorem 4.5 below.

To prove Theorem 4.5, we will need the following well-known fact.

**Fact 4.4** ([DF §13.2, Exercises #18]). Let $F$ and $u$ be a field and an indeterminate $u$. Two non-zero relatively prime polynomials $f, g \in F[X]$, let $r_0 \in F(u)$ be a solution of the equation $u = \frac{f(X)}{g(X)}$ for $X$. Then $[F(r_0) : F(u)] = \max\{\deg f, \deg g\}$.

**Theorem 4.5.** For indeterminates $r$ and $u$, and for non-zero relatively prime polynomials $f, g \in K[r]$, we let an elliptic curve

$$E_{r,u}': y^2 = x^3 + u^2 f(r)x + u^3 g(r)$$

over the function field $K(\mathcal{W})$ where $\mathcal{W} := \{(r, u) \in \mathbb{A}^2 : u \neq 0, (f(r), g(r)) \in \mathcal{W}\}$. For each integer $k \geq 3$, we let $\Psi_k' = \Psi_{k,u^2,uf,g}$ and denote its Galois group over $K(\mathcal{W})$ by $\overline{\mathcal{G}}_k'$. For a positive integer $N$ and positive divisors $m$ and $n$ of $N$ such that $m \mid n$, if

(i) $\Psi_m'$ splits completely over $K(\mathcal{W})$,
(ii) $\Psi_n'$ has a zero in $K(\mathcal{W})$, and
(iii) $\max\{3\deg f, 2\deg g\} = \left[\overline{\mathcal{P}}_N(1,1) : \overline{\mathcal{P}}_N(m,n)\right]$,

then we have an (ordered) basis $\mathcal{B}$ of $E_{r,u}'[N]$ such that

$$\overline{\mathcal{P}}_N(m,n) \subseteq \mathcal{B} \subseteq \overline{\mathcal{P}}_N(m,n).$$

**Proof.** First, let $r_0 \in \overline{\mathbb{C}(u)}$ be a solution of $u = \frac{f(r)}{g(r)}$ for $r$. Observe that $r_0$ is transcendental over $\mathbb{C}$. The specialization $E_{r_0,f(r_0)/g(r_0)}'$ of $E_{r,u}'$ at $(r, u) = (r_0, f(r_0)/g(r_0))$ is

$$\mathcal{E}_{j} : y^2 = x^3 - \frac{27j}{4(j - 1728)}x - \frac{27j}{4(j - 1728)},$$

where $j = 1728\left(\frac{4(f(r_0))^3}{4(f(r_0))^3 + 27(g(r_0))^2}\right) \in \mathbb{C}(r_0)$. Since $j \in \mathbb{C}(r_0)$ is transcendental over $\mathbb{C}$, [DS05, Corollary 7.5.3] implies that for any basis $\mathcal{B} = \{P, Q\}$ of $\mathcal{E}_j[N]$, we have

$$\mathcal{B} \left(\text{Gal}(\mathbb{C}(j) (x(\mathcal{E}_j[N]) / \mathbb{C}(j)) \right) = \overline{\mathcal{P}}_N(1,1)$$

and

$$\mathcal{B} \left(\text{Gal} (\mathbb{C}(j) (x(\mathcal{E}_j[N]) / \mathbb{C}(j)) (x(\frac{\mathcal{X}}{m}) P), x(\frac{\mathcal{X}}{n}) Q)))) = \overline{\mathcal{P}}_N(m,n).$$

(6)

Let $\beta \in \mathbb{C}(r_0)$ be a zero of $\Psi_{n,r_0,f(r_0)/g(r_0)}'$ by the condition (ii), and

$$Q' := \left(\beta, \sqrt{\beta^3 + \beta((f(r_0))^3/(g(r_0))^2) + ((f(r_0))^3/(g(r_0))^2)}\right) \in \mathcal{E}_j(\mathbb{C}(r_0))[n].$$

Then, $Q'$ has order $n$ by Remark 3.6. Since $n \mid N$, there is a point $Q' \in \mathcal{E}_j[N]$ such that $x(\frac{\mathcal{X}}{m}) Q = Q'$. Then, we can choose a point $P \in \mathcal{E}_j[N]$ such that $\mathcal{B} = \{P, Q\}$ is a basis of $\mathcal{E}_j[N]$. Since $\Psi_{m,r_0,f/g(r_0)}'$ splits completely over $\mathbb{C}(r_0)$ by the condition (i), we have that $x(\frac{\mathcal{X}}{m}) P \in \mathbb{C}(r_0)$. 


Then, we have the following diagram of subfields of $\mathbb{C}(r_0) (x (\sigma_j[N]))$ over $\mathbb{C}(j)$:

\[
\begin{array}{ccc}
\mathbb{C}(j) (x (\sigma_j[N])) & \mathbb{C}(r_0) (x (\sigma_j[N])) & \mathbb{C}(r_0) \\
\mathbb{C}(j) (x (\sigma_j[N])) \cap \mathbb{C}(r_0) & \mathbb{C}(j) (x ([N_m] P), x ([N_n] Q)) & \mathbb{C}(j) \\
\end{array}
\]

First, we show that

$$\mathbb{C}(r_0) = \mathbb{C}(j) (x ([N_m] P), x ([N_n] Q)).$$

The inclusion ‘⊇’ is obvious. To prove the other inclusion ‘⊆’, we note that $\mathbb{C}(j) (x ([N_m] P), x ([N_n] Q))$ is the fixed subfield of $\mathbb{C}(j) (x (\sigma_j[N]))$ under the group $T_N^j(m, n)$ by (6), so its degree over $\mathbb{C}(j)$ is equal to the index $[T_N^{i}(1, 1) : T_N^{i}(m, n)]$, and by the condition (iii) and Fact 4.3 we have that

$$[T_N^{i}(1, 1) : T_N^{i}(m, n)] = \max\{3\deg f, 2\deg g\} = [\mathbb{C}(r_0) : \mathbb{C}(j)].$$

Hence since $\mathbb{C}(r_0)$ and $\mathbb{C}(j) (x ([N_m] P), x ([N_n] Q))$ have the same degrees over $\mathbb{C}(j)$, so they are equal to each other.

Next, we show that

$$\mathbb{C}(r_0) (x (\sigma_j[N])) = \mathbb{C}(j) (x (\sigma_j[N])).$$

The inclusion ‘⊇’ is obvious. To prove the other inclusion ‘⊆’, it is enough to show that their extension degrees over $\mathbb{C}(j) (x ([N_m] P), x ([N_n] Q))$ are the same. In fact, by the above, we have that

$$[\mathbb{C}(r_0) (x (\sigma_j[N])) : \mathbb{C}(j) (x ([N_m] P), x ([N_n] Q))] = [\mathbb{C}(r_0) (x (\sigma_j[N])) : \mathbb{C}(r_0)]$$

$$= [\mathbb{C}(j) (x (\sigma_j[N])) : \mathbb{C}(j) (x (\sigma_j[N])) \cap \mathbb{C}(r_0)].$$

The last equality follows by restricting to $\mathbb{C}(j)$, since the field extension $\mathbb{C}(j) (x (\sigma_j[N])) / \mathbb{C}(j)$ is Galois over $\mathbb{C}(j)$. Since

$$\mathbb{C}(j) (x ([N_m] P), x ([N_n] Q)) \subseteq \mathbb{C}(j) (x (\sigma_j[N])) \cap \mathbb{C}(r_0)$$

$$\subseteq \mathbb{C}(r_0) = \mathbb{C}(j) (x ([N_m] P), x ([N_n] Q)),$$

we conclude that

$$[\mathbb{C}(j) (x (\sigma_j[N])) : \mathbb{C}(j) (x (\sigma_j[N])) \cap \mathbb{C}(r_0)] = [\mathbb{C}(j) (x (\sigma_j[N]) : \mathbb{C}(j) (x ([N_m] P), x ([N_n] Q))].$$

Finally, recalling that $\pi_3$ is injective by Lemma 3.7,

$$\pi_3^{-1} \left( T_N^j(m, n) \right) = \text{Gal} \left( \mathbb{C}(j) (x (\sigma_j[N])) / \mathbb{C}(j) (x ([N_m] P), x ([N_n] Q)) \right)$$

$$= \text{Gal} \left( \mathbb{C}(r_0) (x (\sigma_j[N])) / \mathbb{C}(r_0) \right) \quad \text{by the above}$$

$$\subseteq \text{Gal} \left( \mathbb{C}(r, u) (x (E_{r,u}[N])) / \mathbb{C}(r, u) \right)$$

$$\cong \text{Gal} \left( K(r, u) (x (E'_{r,u}[N])) / \mathbb{C}(r, u) \cap K(r, u) (x (E'_{r,u}[N])) \right).$$
\[ \subseteq \text{Gal} \left( K(r, u)(x(E_{r,u}[N]))/K(r, u) \right) = \overline{G}_N \]

and so \( \overline{H}_N(m, n) \subseteq \tau_B \left( \overline{G}_N \right) \). Obviously, \( \tau_B \left( \overline{G}_N \right) \subseteq \overline{H}_N(m, n) \) since \( \left[ \frac{N}{m} \right] P, \left[ \frac{N}{n} \right] Q \in E_{r,u}(K(r, u))[N] \). \( \square \)

Now, for each pair \((m, n)\) of positive integers \(m\) and \(n\) such that \(m \mid n\) and \(g_m n = 0\), we parameterize all elliptic curves \(E_{r,u}^{m,n}\) satisfying Condition \(\mathcal{P}(m, n)\) and show the parametrizations satisfy the group condition of Theorem 3.5, that is, the Galois group of the primitive polynomial of \(\theta\) parameterize all elliptic curves \(G\).

Conversely, we show that any \(s(\text{RF, Theorem 4.21})\) and so \(\overline{H}_N(m, n) \subseteq \tau_B \left( \overline{G}_N \right) \subseteq \overline{H}_N(m, n)\) since \(\left[ \frac{N}{m} \right] P, \left[ \frac{N}{n} \right] Q \in E_{r,u}(K(r, u))[N] \). \( \square \)

We will need the following well-known fact.

**Fact 4.6** (RF Theorem 4.21). Let \(C/K\) be a rational plane curve defined by an irreducible polynomial \(\theta(x, y) \in K[x, y]\). Then, there is a birational morphism \((A_\lambda B_\lambda : B_\lambda) : \mathbb{P}^1 \to C\) for some \(A_\lambda, B_\lambda \in K[v]\) such that two fractions \(\frac{A_\lambda}{A_\lambda'}\) and \(\frac{B_\lambda}{B_\lambda'}\) are reduced. Moreover, \(\max\{\deg A_\lambda, \deg A_\lambda'\} = \deg_\theta\) and \(\max\{\deg B_\lambda, \deg B_\lambda'\} = \deg_\theta\).

To parameterize all elliptic curves satisfying Condition \(\mathcal{P}(m, n)\), we consider the modular curve \(X_1(m, n)\) which is introduced in Section 11. For positive integral weights \(d_0, d_1, \ldots, d_k\), we define the weighted projective space

\[ \mathbb{P}^{d_0, d_1, \ldots, d_k} := (\mathbb{A}^{k+1} - \{0\}) / \sim \]

where the equivalent relation \(\sim\) is given by \((a_0, a_1, \ldots, a_k) \sim (\lambda^{d_0} a_0, \lambda^{d_1} a_1, \ldots, \lambda^{d_k} a_k)\) for \((a_0, a_1, \ldots, a_k) \in \mathbb{A}^{k+1} - \{0\}\) and all \(\lambda \in \mathbb{A}^1 - \{0\}\). We denote the equivalent class of \((a_0, a_1, \ldots, a_k)\) by \([a_0 : a_1 : \ldots : a_k]\).

We consider when \(m = 1\). We show that the quasi-variety

\[ U_{1,n} := \left\{ [s : t : x_2] \in \mathbb{P}^{2,3,1} : \text{there is } Q \in E_{s,t}^D \text{ of order } n \text{ such that } x(Q) = D x_2 \text{ for some } D \in \mathbb{G}_m \right\} \]

is birational to \(X_1(1, n)\). First, we show that

\[ U_{1,n} = \left\{ [s : t : x_2] \in \mathbb{P}^{2,3,1} : \Psi_{n,s,t}(x_2) = 0, 4s^3 + 27t^2 \neq 0 \right\} \]

and \(U_{1,n}\) is an open subset of the curve defined \(\Psi_{n,s,t}(x_2) = 0\). For any \([s : t : x_2] \in U_{1,n}\), then, obviously \(4s^3 + 27t^2 \neq 0\) and there exist non-zero \(D \in K(s, t, x_2)^x\) and a point \(Q \in E_{s,t}^D\) of order \(n\) such that \(x(Q) = D x_2\). The orders of \(Q\) implies the equality \(\Psi_{n,s,t}(x_2) = 0\). Conversely, we show that any \([s : t : x_2] \in \mathbb{P}^{2,3,1}\) such that \(\Psi_{n,s,t}(x_2) = 0\) and \(4s^3 + 27t^2 \neq 0\) is in \(U_{1,n}\). For \(D = x_2^3 + sx_2 + t\), we have that \(Q = (D x_2, D^2) \in E_{s,t}^D(K(s, t, x_1, x_2))\). Note that \(Q\) has order \(n\). The polynomial \(\Psi_{n,s,t}(x)\) in variables \(s, t,\) and \(x\) is irreducible over \(K(s, t)\) and \(K[s, t]\) by Lemma 3.1 and Gauss’s lemma. Next, we show that the open subset \(Y_1(1, n) = \Gamma_1(1, n) \setminus \{z \in \mathbb{C} : \text{Im} z > 0\}\) of \(X_1(1, n)\) is isomorphic to the open subset \(U_{1,n}\). Following [DS05] §1.5, we recall the structure of the open set \(Y_1(1, n)\) as the moduli space

\[ \{(E, Q) : \text{an elliptic curve } E, \text{ and } Q \in E \text{ such that } |Q| = n.\} / \sim_K, \]

where the equivalence relation \(\sim_K\) is defined by \((E, Q) \sim_K (E', Q')\) if and only if there is an isomorphism \(\phi/K : E \to E'\) as an isogeny such that \(\phi'(Q) = Q\) (In the notations of [DS05] §1.5, \(\Gamma_1(1, n) = \Gamma_1(n)\) and \(Y_1(1, n) = Y_1(n)\)). Obviously, \((E, Q) = (E, -Q) \in Y_1(1, n)\) via the isomorphism \(E \to E\) defined by \(X \mapsto -X\). We construct an isomorphism \(Y_1(1, n) \to U_{1,n}\)
by following [Si09] III.10. Recall that two elliptic curves $E_{A,B}$ and $E_{A',B'}$ defined over $K$ are isomorphic over $K$ if and only if there is $u \in K^\times$ such that $A' = Au^3$ and $B' = Bu^6$. Moreover, each isomorphism $E_{A,B} \to E_{A',B'}$ is of the form $(x,y) \mapsto ((\zeta u)^2x, (\zeta u)^3y)$ where $\zeta$ is a $k$th root of unity, where

$$k = \begin{cases} 
2, & \text{if } A \neq 0 \text{ and } B \neq 0, \\
4, & \text{if } B = 0, \\
6, & \text{if } A = 0.
\end{cases}$$

Hence, we correspond $(E, Q) \in Y_1(1, n)$ to $[s : t : x(Q)] \in \mathbb{P}^{4,6,2}$, uniquely. Composing with the natural morphism $\mathbb{P}^{4,6,2} \to \mathbb{P}^{2,3,1}$, we have constructed the morphism $\iota : Y_1(1, n) \to U_{1,n} \subseteq \mathbb{P}^{2,3,1}$ so far. The morphism $\iota$ is an isomorphism, since each $[s : t : x_2] \in U_{1,n}$ is the image of $(E_{D_s,t}, (Dx_2, D^2)) \in X_1(1, n)$ where $D = x_2^3 + sx_2 + t \neq 0$, recalling that $n \geq 3$. To sum up, $Y_1(1, n)$ and $U_{1,n}$ are isomorphic and $U_{1,n}$ and $X_1(1, n)$ are birational.

**Proposition 4.7.** Let $n \geq 2$ be an integer such that $g_{1,n} = 0$. If there is an elliptic curve $E/K$ such that $E(K)_{\text{tors}} \supseteq \mathbb{Z}/n\mathbb{Z}$, then there are non-zero pairwise relatively prime polynomials $f, g, h \in K[r]$ satisfying the following three conditions:

(i) For $n \neq 2, 3$, the elliptic curves of the form $E_{r,u}^{1,n}: y^2 = x^3 + u^2 f(r)x + u^3 g(r)$ of parameters $r$ and $u$ parameterize all but finitely many elliptic curves $E_{A,B}$ (up to quadratic twists) satisfying Condition $\mathcal{P}(1,n)$.

(ii) For $n = 2$, $E_{r,2}^{1,2}$ and $E_{s,0}$ parameterize all elliptic curves satisfying Condition $\mathcal{P}(1,2)$.

(iii) For $n = 3$, $E_{r,3}^{1,3}$ and $E_{0,1}$ parameterize all elliptic curves satisfying Condition $\mathcal{P}(1,3)$.

(ii) $uh(r)$ is a zero of the $n$th division polynomial of the elliptic curve $E_{r,u}^{1,n}$.

(iii) $\deg f \geq 2 \deg h$, $\deg g \geq 3 \deg h$, and

$$\max\{3 \deg f, 2 \deg g\} = \left[\mathcal{H}_N(1, 1) : \mathcal{H}_N^{1}(1, n)\right] = \delta,$$

where $\delta = \deg_x \Psi_{n,s,t}$. Moreover, if $n \geq 4$, then $3 \deg f = 2 \deg g = \delta$.

**Proof.** The proofs for $n = 2$ and $3$ are given in Section 4.2 and Section 4.3 respectively.

Let $n \geq 4$. We write the $n$th primitive division polynomial of $E_{s,t}$ as

$$\Psi_{n,s,t}(x) = \sum_{0 \leq k \leq \delta} \left(\sum_{2i + 3j = \delta - k} c_{i,j} s^i t^j\right) x^k \in \mathbb{Q}[s,t][x],$$

and let $\theta(s, t, x) := \Psi_{n,s,t}(x)$ for convenience. By Lemma 4.1 and Gauss’s lemma, $\theta(s, t, x)$ is irreducible over $K(\mathbb{Z})$ and $K[s,t]$. Let $Z_{1,n} \subseteq \mathbb{P}^{2,3,1}$ be the curve defined by the equation $\theta(s, t, x) = 0$. Then, $U_{1,n} = \{ s : t : x \in Z_{1,n} : 4s^2 + 27t^2 \neq 0 \}$ is a dense open subset of $Z_{1,n}$. The assumption that $g_{1,n} = 0$ implies that all but not finitely many points of $\theta(s, t, x) = 0$ can be parameterized by $[f(r_0) : g(r_0) : h(r_0)]$ where $f, g, h \in K(r)$ are rational functions of a variable $r$, which shows that the condition (i) holds.

Since $\theta(f, g, h) = 0$ and $uh(r)$ is a zero of the $n$th division polynomial $\Psi_{n,u^2 f(r), u^3 g(r)}(x) = u^\delta \theta(f(r), g(r), u^{-1} x)$ of the elliptic curve $y^2 = x^3 + u^2 f(r)x + u^3 g(r)$, the condition (ii) holds.

Note that $6 \mid \delta$ by Lemma 3.3 since $n \geq 4$. First, we show that $c_{i,j} |_{i,j=0,0} = c_{i,j} |_{i,j=3,0} = 0$, i.e., $s, t \mid \theta(s, t, 0) \in \mathbb{Q}[s,t]$. If $s \mid \theta(s, t, 0)$, then $0, \sqrt{t} \in E_{0,t}(K(t))$ is a torsion point of order $n$. A direct computation shows that $x(2(0, \sqrt{t})) = 0$ and the point $(0, \sqrt{t})$ is of order 3, which is a contradiction since $n \geq 4$. If $t \mid \theta(s, t, 0)$, then $(0, 0) \in E_{s,0}(K(s))$ is a torsion point of order $n$. Obviously, the order of the point $(0, 0)$ is 2, which is again a contradiction.
We may assume that they are polynomials since \( f d^2, g d^3, h d \in K[r] \) and \([f d^2 : g d^3 : h d] = [f : g : h] : \mathbb{P}^1 \to \mathbb{P}^{2,3,1}\) for some \( d \in K[r] \). Moreover, we may assume that

\[
\text{at least one of } p^2 \nmid f, p^3 \nmid g, \text{ and } p \nmid h \text{ is satisfied for each prime } p \in K[r]. \tag{7}
\]

In fact, if there exists a prime \( p \in K[r] \) such that \( p^2 \nmid f, p^3 \nmid g, \) and \( p \nmid h, \) then we replace \( f, g, \) and \( h \) by \( f/p^2, g/p^3, \) and \( h/p \) respectively, since \([f/p^2 : g/p^3 : h/p] = [f : g : h] \) and we repeat this argument so that at least one of \( p^2 \nmid f, p^3 \nmid g, \) and \( p \nmid h \) holds.

We prove that \( h \neq 0 \) by showing that there are only finitely many points of the form \([s : t : 0]\) in the curve \( Z_{1,n} \). Note that there is no point of the form \([s : 0 : 0]\) in \( Z_{1,n} \) since \( \theta(s,0,0) = c_{\delta/2,0,0} s^{\delta/2} \) but we have shown that \( c_{0,\delta/3,0} \neq 0 \). So, since \( t \neq 0, \)

\[
t^{-\delta/3} \theta(s,t,0) = \sum_{0 \leq k \leq \delta/6} c_{3k,\delta/3-2k,0} \left( \frac{s^3}{t^2} \right)^k =: \phi \left( \frac{s^3}{t^2} \right)
\]

for some polynomial \( \phi \in K[X] \) of degree \( \delta/6 \) noting that \( \theta(s,t,0) = \sum_{2i+3j=\delta} c_{ij0} s^i t^j \). Hence, \( \phi \) has only finitely many zeros. Let \( a \in \overline{K} \) be a zero of \( \phi \). Then, \( a \neq 0 \), since \( c_{0,\delta/3,0} \neq 0 \). The condition that \( s^3/t^2 = a \) is equivalent to that \( (s,t) = (a \lambda^2, a^3 \lambda^3) \) for some \( \lambda \in \overline{K}^\times \). Hence \([s : t : 0] = [a : a : 0] \in \mathbb{P}^{2,3,1}\). In other words, each zero of \( \phi(X) \) for \( X \) gives only one point of the curve \( \theta(s,t,0) = 0 \). Since \([f : g : h] : \mathbb{P}^1 \to Z_{1,n} \) is birational, the image of \([f : g : h] \) is an infinite set and so \( h \neq 0 \).

Next, to show that \( f, g, \) and \( h \) are pairwise relatively prime, we denote the valuation function on \( K[r] \) by \( v_p \) at each prime \( p \in K[r] \). Let \( s' = f/h^2 \) and \( t' = g/h^3 \). First, if there is a prime \( p \in K[r] \) such that \( v_p(s') < 0, \) then we want to show that \( 2v_p(t') \leq 3v_p(s') \). If not, i.e., if \( 3v_p(s') < 2v_p(t') \), then \( v_p((s')^{\delta/2}) \) is the unique minimal value among \( \{v_p(s^i t^j) : 0 \leq 2i + 3j \leq \delta \} \). Since \( c_{\delta/2,0,0} \neq 0, \) \( \infty = v_p(0) = v_p(\theta(s',t')) = \frac{\delta}{2} v_p(s') < 0, \) which is a contradiction. Hence \( 2v_p(t') \leq 3v_p(s'), \) and moreover \( v_p(t') < 0 \). Similarly, if there is a prime \( p \in K[r] \) such that \( v_p(t') < 0, \) then we can show that \( 3v_p(s') \leq 2v_p(t'), \) and we conclude that \( v_p(s') < 0 \) as well. Hence, \( v_p(s') < 0 \) if and only if \( v_p(t') < 0, \) and in this case, we have that

\[
3v_p(f) = 3(v_p(s') + 2v_p(h)) = 2(v_p(t') + 3v_p(h)) = 2v_p(g).
\]

Suppose that there is a common prime factor \( p \) of \( fg \) and \( h \). If \( v_p(s') \geq 0 \) and \( v_p(t') \geq 0, \) then \( v_p(f) = v_p(s') + 2v_p(h) \geq 2 \) and \( v_p(g) = v_p(t') + 3v_p(h) \geq 3 \) which is a contradiction to (7). Thus, \( v_p(s') < 0, \) so \( v_p(t') < 0 \) and \( 3v_p(f) = 2v_p(g) \) by the above argument. Hence \( v_p(s'), v_p(t') < 0 \) and \( 3v_p(f) = 2v_p(g), \) and since \( p \mid f \) \( g, \) we have that \( p^2 \mid f \) \( p^3 \mid g, \) which is again a contradiction to (7). Hence we conclude that \( \gcd(f,h) = 1 \) and \( \gcd(g,h) = 1 \). If \( f \) and \( g \) have a common prime factor \( p \), then \( p \mid 0 = \sum_{2i+3j+k=\delta} c_{ijk} f^i g^j h^k \) and so \( fg \) and \( h \) have a common prime factor, which is impossible by the above argument. So we conclude that \( \gcd(f,g) = 1 \). Since \( h \neq 0, \) and \( f, g \) and \( h \) are pairwise relatively prime, they are non-zero too.

Next, we show the condition (iii). If \( \deg f < 2 \deg h \) and \( \deg g \geq 3 \deg h, \) then \( \deg s' < 0 \leq \deg t' \) and

\[
-\infty = \deg(0) = \left( \sum_{2i+3j+k=\delta} c_{ijk} (s')^i (t')^j \right) = \frac{\delta}{3} \deg t' \geq 0
\]
Weil pairing (see [Si09, III.8]) and we fix a primitive 

\[ U_{n,s,t} \]

\[ \zeta \]

is isomorphic to \( U_{m,n} \) defined before, we define the morphism

\[ A \]

A similar argument works even if \( m \) equals the cardinality of the orbit of \( (0, 1) \). Obviously, \( (E, P, Q) \) is an isomorphism

\[ \varphi \]

\[ H \]

\[ \{ \zeta_m \} \]

\[ \zeta_m \in K. \]

We show that the quasi-variety

\[ U_{m,n} := \left\{ \begin{array}{l} [s : t : x_1 : x_2] \in \mathbb{P}^{2,3,1,1} : \\
4s^3 + 27t^2 \neq 0, \\
\text{there are } P, Q \in E_{s,t} \text{ of order } m \text{ and } n \text{ such that} \\
x(P) = Dx_1, x(Q) = Dx_2 \text{ for some } D \in \mathbb{G}_m, \text{ and} \\
\epsilon_m \left( P, \left[ \frac{n}{m} \right] Q \right) = \zeta_m \end{array} \right\} \]

over \( K \) is birational to the modular curve \( X_1(m, n) \). Postponing the proof that \( U_{m,n} \subsetneq \mathbb{P}^{2,3,1,1} \) is a quasi-variety later, we assume that \( U_{m,n} \) is a quasi-variety and first, we construct an isomorphism

\[ \iota : Y_1(m, n) := \Gamma_1(m, n) \setminus \{ z \in \mathbb{C} : \text{Im} z > 0 \} \rightarrow U_{m,n}. \]

The open set \( Y_1(m, n) \subseteq X_1(m, n) \) is the moduli space

\[ \left\{ (E, P, Q) : \right. \\
\text{an elliptic curve } E, \text{ and } P, Q \in E \text{ such that} \\
|P| = m, |Q| = n, \text{ and } \epsilon_m \left( P, \left[ \frac{n}{m} \right] Q \right) = \zeta_m \left\} \right/ \sim_K, \\
\]

where the equivalence relation \( \sim_K \) is defined by \( (E, P, Q) \sim_K (E', P', Q') \) if and only if there is an isomorphism \( \phi : E \rightarrow E' \) as an isogeny defined over \( K \) such that \( \phi(P) = P' \) and \( \phi(Q) = Q' \). Obviously, \( (E, P, Q) = (E, -P, -Q) \in Y_1(m, n) \) via the isomorphism \( E \rightarrow E \) defined by \( X \mapsto -X \) (see [DS05, §1.5] to consider the case when \( m = n \), or equivalently, \( \Gamma_1(n, n) = \Gamma(n) \)). A similar argument works even if \( m \neq n \). As for the morphism \( Y_1(1, n) \rightarrow U_{1,n} \subsetneq \mathbb{P}^{2,3,1} \) defined before, we define the morphism \( \iota : Y_1(m, n) \rightarrow U_{m,n} \subsetneq \mathbb{P}^{2,3,1,1} \). To show that \( Y_1(m, n) \) is isomorphic to \( U_{m,n} \) over \( K \), we show that \( \iota \) is surjective and injective. The definition of \( U_{m,n} \) proves that \( \iota \) is surjective. Assume that two points \( (E_{s,t}, P, Q) \) and \( (E_{s',t'}, P', Q') \) of
$Y_1(m, n)$ have the same image in $U_{m,n}$ via $\iota$. In other words, we have $D \neq 0$ such that $[s' : t' : x(P') : x(Q')] = [D^2 s : D^3 t : D x(P) : D x(Q)]$. Let $x_1 = x(P)$ and $x_2 = x(Q)$. Since $n \geq 4$, we have that $y(Q') \neq 0, y(Q) \neq 0$ and $y(Q') = y(Q)\sqrt{D^3}$. In other words, $D = u^2$ for some $u \in \mathbb{G}_m$. The morphism $\phi : E_{s,t} \rightarrow E_{s',t'}$ defined by $(x,y) \mapsto (u^2x, u^3y)$ over $K$ is an isomorphism. Moreover, considering the $x$-coordinates of the points $P, P', Q, \text{ and } Q'$, we have that $\phi(P) = \pm P'$ and $\phi(Q) = \pm Q'$. The signs are same because $\phi$ preserves the Weil pairing. Hence, $(E, P, Q) = (E', P', Q') \in X_1(m, n)$ and $\iota$ is injective.

Next, we prove that $U_{m,n} \subseteq \mathbb{P}^{2,3,1,1}$ is a quasi-variety. First, let

$$W_{m,n} := \{[s : t : x_1 : x_2] \in \mathbb{P}^{2,3,1,1} : \Psi_{m,s,t}(x_1) = 0, \Psi_{n,s,t}(x_2) = 0\}.$$ 

From now on, we describe how to define a rational function

$$\eta : W_{m,n} \rightarrow M := \{\zeta_m^k + \zeta_m^{-k} : k \in \mathbb{Z}/m\mathbb{Z}\}.$$ 

We will show that

$$U_{m,n} = \{[s : t : x_1 : x_2] \in \eta^{-1} (\zeta_m + \zeta_m^{-1}) : 4s^3 + 27t^2 \neq 0\}. $$

For a given $Q = (x_2, y_2) \in E_{s,t}$ is of order $n$, there is a rational function $h \in K(s, t, x_2)(E_{s,t})$ such that $\text{div}(h) = \sum_{R \in E_{s,t}[m]} (Q' + R) - (R)$ where $Q' \in E_{s,t}$ is a point such that $[m]Q' = \overline{[m]}Q$. For a given $P \in E_{s,t}[m]$, the rational function $\frac{h(X+P)}{h(X)} : E \rightarrow \mathbb{P}^1$ is constant as a value in $\{\zeta_m^k : k \in \mathbb{Z}/m\mathbb{Z}\}$, and this is the Weil pairing $e_m(P, \overline{[m]}Q)$ (see [Si09, III.8] for more detail of the rational function $h$). Any Galois conjugate $\sigma$ in the absolute Galois group of $K(s, t, x_2)$ fixes the elliptic curve $E_{s,t}$ and sends $Q$ to $Q - Q$. Hence, $h = b_0 + b_1y_2$ where $b_0, b_1 \in K(s, t, x_2)(x)$. Letting $h' := b_0 - b_1y_2$, we have that $\text{div}(h') = \sum_{R \in E_{s,t}[m]} (-Q' + R) - (R)$ since $h' = h \circ [-1]$. Hence, for each $P \in E_{s,t}[m], e_m(P, \overline{[m]}Q) = \frac{h'(X+P)}{h(X)}$. We consider the rational function $\eta \in (K(s, t, x_1, x_2)(y_1, y_2))(E_{s,t})$ in the variable $X = (x, y) \in E_{s,t}$ defined by:

$$\eta : E_{s,t} \rightarrow \mathbb{P}^1, \quad \eta(X) = \frac{h(X + P)h'(X) + h(X)h'(X + P)}{h(X)} = \frac{h(X + P)}{h(X)} + \frac{h'(X + P)}{h'(X)},$$

where $P = (x_1, y_1) \in E_{s,t}$. Obviously, $\eta$ is a constant function as a value in $M$. We show that $\eta \in K(s, t, x_1, x_2)$. Let $X = (x, y) \in E_{s,t}$. The denominator $(hh')(X) \in K(s, t, x_2, y_2) \subset K(s, t, x_2, x, y)$ of $\eta$ in the above form is in $K(s, t, x_2, x)$ since $y_2^2, y^2 \in K(s, t, x_2, x)$. To show that the numerator of $\eta$ in the above form is in $K(s, t, x_2, x_1, x)$, we let $h(X + P) = \sum a_{ijk}y_1^iy_2^jk$ for some $a_{ijk} \in K(s, t, x_1, x_2)(x)$ where $i, j, k$ run over $\{0, 1\}$. Since

$$\sum_{i,j,k}(-1)^i a_{ijk}y_1^iy_2^jk = h'(X + P) = h(-X - P) = \sum_{i,j,k}(-1)^{i+k} a_{ijk}y_1^iy_2^jk,$$

we have that $i + k$ and $j$ have the same parity, i.e., $i + j + k$ is even. Then

$$h(X + P)h'(X) + h(X)h'(X + P) = \sum_{i,j,k} \sum_q ((-1)^q + (-1)^q) a_{ijk}b_qy_1^iy_2^jy^k(y_2y)^q$$

$$= 2 \sum_{i,j,k} (-1)^i a_{ijk}b_1y_1^iy_2^jy^j + \sum_{i,j,k} (-1)^i a_{ijk}b_1y_1^kjy_2^iy^k + \sum_{i,j,k} (-1)^i a_{ijk}b_1y_1^iy_2^jy^{-i+j+k}(y_1y)^1,$$
where \( q \) runs over \( \{0, 1\} \). Since \(-i + j + k \) is also even, and \( y^{-i+j+k} \in K(s, t, x) \), the coefficient \( 2 \sum_{j,k} (-1)^i a_{ijk} b y^{j+k} \) of \( (y_1 y)^i \) is in \( K(s, t, x, x_1, x_2, x) \). We denote it by \( c_i \in K(s, t, x_1, x_2, x) \). In other words, \( h(X + P) h'(X) + h(X) h'(X + P) = c_0 + c_1 y_1 y \in K(s, t, x_1, x_2)(y_1 y) \). Since \(-X = (x, -y)\), \( \eta(X) = \eta(-X) \) and \( (hh')(X) = (hh')(-X) \), we have that

\[
c_0 + c_1 y_1 y = h(X + P) h'(X) + h(X) h'(X + P)
\]

\[
= h(-X + P) h'(-X) + h(-X) h'(-X + P) = c_0 - c_1 y_1 y.
\]

Therefore, \( h(X + P) h'(X) + h(X) h'(X + P) = c_0 \in K(s, t, x_1, x_2)(x) \). Moreover, \( \eta \in K(s, t, x_1, x_2) \) since \( \eta \) does not depend on \( X \). Note that \( \eta(s, t, x_1, x_2) = \eta(D^2 s, D^3 t, D x_1, D x_2) \), since the Weil pairing is invariant under any isomorphism, and in particular, under the isomorphism \( E_{s,t} \to E_{s,t}^D \) defined by \( (x, y) \mapsto (D x, \sqrt{D} y) \).

Now we show that \( U_{m,n} = \{ [s : t : x_1 : x_2] \in \eta^{-1} (\zeta_m + \zeta_m^{-1}) : 4 s_3^3 + 27 t^2 \neq 0 \} \). Let \( [s : t : x_1 : x_2] \in U_{m,n} \). Then, obviously \( 4 s_3^3 + 27 t^2 \neq 0 \), and there exist non-zero \( D \in K(s, t, x_1, x_2)^{\times} \) and points \( P, Q \in E_{s,t}^D \) of order \( m \) and \( n \) such that \( x(P) = D x_1 \) and \( x(Q) = D x_2 \). The orders of \( P \) and \( Q \) imply the equalities \( \Psi_{m,s,t}(x_1) = 0 \) and \( \Psi_{n,s,t}(x_2) = 0 \), and by the definition of \( \eta \),

\[
\eta(s, t, x_1, x_2) = \eta(D^2 s, D^3 t, D x_1, D x_2) = \frac{h(X + P)}{h(X)} + \frac{h'(X + P)}{h'(X)}.
\]

Conversely, we show that any \( [s : t : x_1 : x_2] \in W_{m,n} \) such that \( \eta(s, t, x_1, x_2) = \zeta_m + \zeta_m^{-1} \) and \( 4 s_3^3 + 27 t^2 \neq 0 \) is in \( U_{m,n} \). For \( D = x_3^3 + s_2 x_2 + t \), we have that \( Q = (D x_2, D^2) \in E_{s,t}^D(K(s, t, x_1, x_2)) \). Let \( P \) be a point of \( E_{s,t}^D \) such that \( x(P) = D x_1 \). Note that two points \( P \) and \( Q \) have orders \( m \) and \( n \), respectively. Let \( r = r_m \left( \left[ \frac{a}{m} \right] Q \right) \). Then,

\[
r + r^{-1} = \frac{h(X + P)}{h(X)} + \frac{h'(X + P)}{h'(X)} = \eta(D^2 s, D^3 t, D x_1, D x_2) = \eta(s, t, x_1, x_2) = \zeta_m + \zeta_m^{-1}
\]

and \( r = \zeta_m^\epsilon \) for some \( \epsilon \in \{ \pm 1 \} \). Hence, two points \( \epsilon P, Q \in E_{s,t}^D \) are of order \( m \) and \( n \) and \( e_m \left( \left[ \frac{a}{m} \right] Q \right) = \zeta_m \). Lastly, we show that \( P \in E_{s,t}^D(K(s, t, x_1, x_2)) \). For any \( \sigma \) in the absolute Galois group \( G_{K(s,t,x_1,x_2)} \) of \( K(s, t, x_1, x_2) \),

\[
\zeta_m^{\pm 1} = e_m \left( \pm \epsilon P, \left[ \frac{a}{m} \right] Q \right) = e_m \left( \epsilon \sigma(P), \sigma \left( \left[ \frac{a}{m} \right] Q \right) \right) = \sigma \left( \zeta_m \right) = \zeta_m
\]

since \( \zeta_m \) and \( x(P) \in K(s, t, x_1, x_2) \). Hence, we conclude that if \( m = 2 \), then \( y(P) = 0 \in K(s, t, x_1, x_2) \), and if \( m \geq 3 \), then \( \sigma(e_P) = \epsilon P \) for every \( \sigma \in G_{K(s,t,x_1,x_2)} \), and \( y(P) \in K(s, t, x_1, x_2) \).

Remark 4.8. The open subset \( Y_1(m, n) \) of the modular curve \( X_1(m, n) \) and the quasi-variety \( U_{m,n} \) do not depend on the choice of a primitive \( m \)th root \( \zeta_m \) of unity. Any primitive \( m \)th root of unity is of the form \( \zeta_a^\alpha \) for some \( a \in (\mathbb{Z}/m\mathbb{Z})^\times \). Let \( b \in (\mathbb{Z}/m\mathbb{Z})^\times \) be the inverse of \( a \). If \( Y_1(m, n) \) is defined by \( \zeta_a^\alpha \) instead of \( \zeta_m \) in the definition of \( Y_1(m, n) \), we define the morphism \( Y_1(m, n) \to Y_1(m, n) \) by \( (E, P, Q) \mapsto (E, [b] P, Q) \). Obviously, its inverse is the morphism defined by \( (E, P, Q) \mapsto (E, [b] P, Q) \). Similarly, if \( U_{m,n} \) is defined by \( \zeta_a^\alpha \) instead of \( \zeta_m \) in the definition of \( U_{m,n} \), then the morphism \( U_{m,n} \to U_{m,n} \) defined by \( [s : t : x_1 : x_2] \mapsto [s : t : \phi_{b,a}^\alpha(x_1), x_2] \) is an isomorphism, since \( \phi_{b,a}^\alpha(x_1) \) is the \( x \)-coordinate of the multiplication by \( a \) of a point whose \( x \)-coordinate is \( x_1 \). (See the explanation below Definition 3.3.)
Proposition 4.9. Let \((m, n) \in T_{g=0}\) with \((m, n) \neq (1, 1)\). If there is an elliptic curve \(E/K\) such that \(E(K)_{\text{tors}} \supseteq \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}\), then \(\zeta_m \in K\) and there are non-zero pairwise relatively prime polynomials \(f, g, h \in K[\rho]\) satisfying the following three conditions:

(i) For \((m, n) \neq (1, 2), (1, 3)\), the elliptic curves of the form \(E_{\rho, u}^{m,n} : y^2 = x^3 + u^2 f(\rho)x + u^3 g(\rho)\) of parameters \(\rho\) and \(u\) parameterize all but finitely many elliptic curves \(E_{A,B}\) (up to quadratic twists) satisfying Condition \(\mathcal{P}(m, n)\).

(ii) \(u h(\rho)\) is a zero of the \(n\)th division polynomial of \(E_{\rho, u}^{m,n}\).

(iii) \(\deg f \geq 2 \deg h, \deg g \geq 3 \deg h,\) and \(\max\{3 \deg f, 2 \deg g\} = \left[\mathcal{T}_n^1(1, 1) : \mathcal{T}_n^1(m, n)\right] = m\delta\), where \(\delta\) is the degree of the primitive \(n\)th division polynomial \(\Psi_{n, \rho, u}(x)\) of \(E_{\rho, u}^{m,n}\).

Moreover, if \(n \geq 4\), then \(3 \deg f = 2 \deg g = m\delta\).

Proof. By the Weil pairing, we know that \(\zeta_m \in K[\mathbb{S}[10] III. Corollary 8.1.1]\). Proposition 4.7 proves for \(m = 1\). Note that \(n \geq 2\) since \((m, n) \neq (1, 1)\). The proofs for \(n = 2\) and 3 are given in Section 4.2 and Section 4.3, respectively. Let \(m \geq 2\) and \(n \geq 4\).

Proposition 4.7 provides non-zero pairwise relatively prime polynomials \(f_0, g_0, h_0\) satisfying the following two conditions:

(i‘) The elliptic curves of the form \(E_{r, u}^{1,n} : y^2 = x^3 + f_0(r)x + u^3 g_0(r)\) of parameters \(r\) and \(u\) parameterize all but finitely many elliptic curves \(E_{A,B}\) (up to quadratic twists) satisfying Condition \(\mathcal{P}(1, n)\).

(ii‘) \(u h_0(r)\) is a zero of the \(n\)th division polynomial of \(E_{r, u}^{1,n}\).

(iii‘) \(\deg f_0 \geq 2 \deg h_0\) and \(\deg g_0 \geq 3 \deg h_0\) and \(3 \deg f_0 = 2 \deg g_0 = \left[\mathcal{T}_n^1(1, 1) : \mathcal{T}_n^1(1, n)\right] = \delta\).

\(\xi\) is the inverse of the birational map \((f_0, g_0, h_0)\):

\[
\begin{array}{ccc}
\mathbb{P}^1 & \xrightarrow{\varphi} & \mathbb{P}^1 \\
\downarrow \phi & & \downarrow \psi \\
U_{m,n} & \xrightarrow{\gamma} & U_{1,n}
\end{array}
\]

Since \(f = f_0 \circ \varphi, g = g_0 \circ \varphi,\) and \(\left[\mathcal{T}_n^1(1, 1) : \mathcal{T}_n^1(m, n)\right] = n\), it is enough to show that \(\varphi\) is a polynomial of degree \(m\) to verify the condition (iii) considering (iii‘).

We consider the natural right group action \(H_n^1(1, 1)\) on the row vector space \((\mathbb{Z}/N\mathbb{Z})^2\) in the proof of Proposition 4.7. Let \(X = \{v \in (\mathbb{Z}/N\mathbb{Z})^2 : |v| = m\} / \sim \) be the quotient of the subset of elements in the group \((\mathbb{Z}/N\mathbb{Z})^2\) of order \(m\) where the equivalence relation \(\sim\) is defined by \(v \sim -v\) for each \(v\) of order \(m\). We denote the equivalence class \(\{v, -v\}\) by \(\pm v\).

Naturally, we define the right group action of \(\mathcal{T}_n^1(1, 1)\) on \(X\) by \((\pm v \cdot (A \{\pm I_2\}) = \pm (vA)\) for \(v \in X\) and \(A \in H_n(1, 1)\). We denote this element in \(X\) by \(\pm vA\). Let \(\Psi_{n,r,u}^l\) be the \(N\)th division polynomial of \(E_{r, u}^{1,n} : y^2 = x^3 + u^2 f_0(r)x + u^3 g_0(r)\) and \(\mathcal{T}_n^I = \text{Gal}\left(K(r, u)(x(E_{r, u}^{1,n}[N]))/K(r, u)\right)\) be the
Galois group of $\Psi_{K,\mathbb{P},u}$ over $K(r, u)$. Since $\zeta_\mathbb{P} \in K$, and the Galois conjugations and the Weil pairing commute with each other, we have that
\[
\mathcal{H}^1_{N}(1, n) \subseteq \tau_B(G_N) \subseteq \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid \{\pm I_2\} \in \mathcal{H}(1, n) : a \equiv 1 \pmod{m} \right\} / \{\pm I_2\} =: \mathcal{H}
\]
by Theorem 1.7. Since the quotient sets $X / \mathcal{H}^1_{N}(1, n)$ and $X / \mathcal{H}$ are the same, we also have that $X / \tau_B(G_N) = X / \mathcal{H}(1, n)$. It is easy to show that for $a, b \in \mathbb{Z}$, $(\frac{N}{m}a, \frac{N}{b}b) \in (\mathbb{Z/NZ})^2$ and $(0, \frac{N}{m}) \in (\mathbb{Z/NZ})^2$ generate $\frac{N}{m}(\mathbb{Z/NZ})^2$ if and only if $\gcd(a, m) = 1$. Moreover, the orbit of $\pm (\frac{N}{m}, 0)\tau_B(G_N)$ is $\{ \pm (\frac{N}{m}, \frac{N}{k}) : k \in \mathbb{Z} \}$. In other words, $\Psi'_{n,r,1}(x) \in K[r][x]$ has a prime factor $k_r(x) \in K[r][x]$ of degree $m$ by Gauss’s lemma, and the set of all zeros of $k_r$ is $\{ x \left( [\frac{N}{m}] P + [\frac{N}{k}] Q \right) : k \in \mathbb{Z} \} \subseteq K(r)$. 

By letting $k(r, x) := k_r(x) \in K[r, x]$, we show that the morphism $(\varphi, \phi) : \mathbb{P}^1 \to C : k(r, x) = 0$ is birational, in other words, $k(\varphi(\rho), \phi(\rho)) = 0$ and all but finitely many rational points of $C$ are of the form $(\varphi(\rho), \phi(\rho))$.

First, we show that $k(\varphi(\rho), \phi(\rho)) = 0$. Since $(f, g, \phi, h)(\rho) \in U_{m,n}$, there are $D \in K^\times$ and two points $R_1, R_2 \in E^D_{f(g),\phi(\rho)} = E^{1,n}_{\phi(\rho),k}$ of order $m$ and $n$ with $x(R_1) = D\phi(\rho)$ and $x(R_2) = D\psi(\rho)$ such that $\epsilon_m(R_1, \frac{N}{m}) R_2 = \zeta_n$. We can take a basis $\{P, Q\}$ of $E^{1,n}_{\varphi(\rho),k}$ such that $R_1 = \frac{N}{m} P$ and $R_2 = \frac{N}{n} Q$. Therefore, $k(\varphi(\rho), \phi(\rho)) = 0$.

If $(r, x) \in C$, then $(f_0(r), g_0(r), x, h_0(r)) \in U_{m,n}, \text{i.e.,} (f_0(r), g_0(r), x, h_0(r)) = (f, g, \phi, h)(\rho)$ and

\[
\varphi(\rho) = (\tau \circ \gamma \circ (f, g, \phi, h))(\rho) = (\tau \circ (f_0, g_0, h_0))(r) = r.
\]

If there exists an irreducible polynomial $p \in K[r][x]$ in the variable $x$ such that $v_p(\varphi) < 0$, then $\infty = v_p(0) = m v_p(\varphi) < 0$, since $k_r \in K[r][x]$ and $\deg k_r = m$, which is a contradiction, so $\varphi \in K[r]$. Hence $\deg \varphi \neq \deg x k = m$ by Fact 1.6. Since $(f_0, g_0) = 1$, we have the two polynomials $v, w \in K[r]$ such that $v f_0 + w g_0 = 1$, and so we have that $(v \circ \varphi)f + (w \circ \varphi)g = 1$. Similarly, $f, g$, and $h$ are pairwise relatively prime.

4.5. Proof of the main theorem. Finally, we complete the proofs of our main theorem, Theorem 1.4 and Corollary 1.8.

**Proof of Theorem 1.7.** If $(m, n) = (1, 1)$, then Theorem 1.2 completes the proof.

Suppose $(m, n) \in T_{g=0}$ given in (11) with $(m, n) \neq (1, 1)$. Merel’s theorem ([Me96]) gives a positive integer $N_K \geq 4$ such that $E_{A,B}(K)_{\text{tors}} = E_{A,B}(K)/N_K$ for all $(A, B) \in \Psi'(K)$. Then, $n$ divides $N_K$ since there is an elliptic curve $E/K$ such that $E(K)_{\text{tors}} \geq \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ by assumption.

We consider polynomials $f, g \in K[\rho]$ and elliptic curve
\[
E^m_{\rho, u} : y^2 = x^3 + u^2 f(\rho)x + u^3 g(\rho)
\]
obtained in Proposition 4.9 defined over the function field $\mathcal{F}(K)$ where $\mathcal{F} = \{(\rho, u) \in K^2 : u \neq 0, (f(\rho), g(\rho)) \in \Psi'\}$ is a $K$-rational variety. For $(\rho, u) \in \mathcal{F}(K)$, we let $\overline{G}_{m,n,u}$ be the Galois groups of $\Psi_{K,\mathbb{P},u}$ over $K$.

Theorem 4.3, Proposition 4.9, and Theorem 2.5 show that $\overline{H}_1_{N}(m, n) \subseteq \tau_B(G'_{N,K,m,n}) \subseteq \overline{H}(m, n)$ for some basis $B$ of $E^m_{\rho, u}[N_K]$ for almost all $(\rho_1, u_1) \in \mathcal{F}(K)$. [Si09, VIII. Theorem.8.5.6] shows that the values of $\frac{H(a_2 f(\rho), w g(\rho))}{H(f(\rho), u g(\rho))}$ is bounded below and above by positive constants where $\delta$ is the degree of the $n$th primitive division polynomial of the elliptic curve $E_{s,t}$. 
Hence, Proposition 3.12 and Lemma 3.17 prove that, for any divisors \( m' \) and \( n' \) of \( N_K \) such that \( m' \mid n' \) and \( n \mid n' \), if almost all \((u^2f(\rho), u^3g(\rho)) \in \mathcal{U}(K)\) satisfy Condition \( \mathcal{P}(m', n') \), then \((m', n') = (m, n)\).

If \((m, n) \neq (1, 2), (1, 3)\), then all but finitely many (up to quadratic twists) elliptic curves \( E/K \) such that \( E(K)_{\text{tors}} \supseteq \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \) can be given as specializations of \( E_{\rho,u}^{m,n} \). Hence, Proposition 3.16 (a) is satisfied, hence, Proposition 3.16 (b) holds.

If \((m, n) = (1, 2), (1, 3)\), then almost all elliptic curves \( E_{A,B}/K \) satisfying Condition \( \mathcal{P}(m, n) \) are parameterized by \( E_{\rho,u}^{m,n}, E_s,0, \) and \( E_{0,t} \). Recalling Proposition 3.16 it is enough to prove the following statements:

(a) Let \( m' \) and \( n' \) be positive divisors of \( N_K \) such that \( m' \mid n' \) and \( n \mid n' \). If almost all \((u^2f(\rho), u^3g(\rho)) \in \mathcal{U}(K)\) satisfy Condition \( \mathcal{P}(m', n') \), then \((m', n') = (m, n)\).

(b) Let \( m' \) and \( n' \) be positive divisors of \( N_K \) such that \( m' \mid n' \) and \( 3 \mid n' \). If almost all \((0, B) \in \mathcal{U}(K)\) satisfy Condition \( \mathcal{P}(m', n') \), then \((m', n') = (1, 3)\).

(c) Let \( m' \) and \( n' \) be positive divisors of \( N_K \) such that \( m' \mid n' \) and \( 2 \mid n' \). If almost all \((A, 0) \in \mathcal{U}(K)\) satisfy Condition \( \mathcal{P}(m', n') \), then \((m', n') = (1, 2)\).

The statement (a) is proved just before. To prove (b), we show that

\[
U := \left\{ b \in K^\times : \begin{array}{l}
(0, b) \in \mathcal{U}(K) \text{ satisfies } \mathcal{P}(m', n') \\
\text{for all } (m', n') \neq (1, 3) \text{ such that } 3 \mid n'
\end{array}\right\} \subseteq \bigcup_{\text{finite}} b_i(K^\times)^3
\]

for some finite number of \( b_i \in K^\times \), which implies that \( U \) is a thin set in \( K^\times \). Merel’s theorem ([Me96]) and Proposition 3.12 admit only finite number of pairs \((m', n')\) such that \( 3 \mid n' \) and Condition \( \mathcal{P}(m', n') \) is satisfied by \((0, b)\) for some \( b \in K^\times \). Hence it is enough to show that

\[
\{ b \in K^\times : (0, b) \text{ satisfies } \mathcal{P}(m', n') \} \subseteq \bigcup_{\text{finite}} b_i(K^\times)^3
\]

for all \((m', n') \neq (1, 3)\) such that \( 3 \mid n' \). Such \((m', n')\) is \((3, 3)\) or satisfies \( n' \geq 4 \). For \((m', n') = (3, 3)\), referring to Section 4.3 if \((0, b)\) satisfies Condition \( \mathcal{P}(3, 3) \), then \((0, b) = (s_{3,3}(\rho)u^2, s_{3,3}(\rho)v^3)\) for some \( \rho \in K \) and \( u \in K^\times \) where \( F_{\rho,u}^{3,3} : y^2 = x^3 + s_{3,3}(\rho)u^2x + t_{3,3}(\rho)v^3 \). \( s_{3,3} \) has three zeros. Hence \( b \in \bigcup_{i=1}^{3} \rho_i(K^\times)^3 \) for zeros \( \rho_i \) of \( s_{3,3} \).

If \( n' \geq 4 \), we observe that the \( n' \)th primitive division polynomial \( \Psi_{n',0,0}(x) \in \mathbb{Q}[t][x] \) of \( E_{0,t} \) is \( F(x^3, t)\) for some homogeneous polynomial \( F(X, t) \in \mathbb{Q}[X, t] \) which is not divisible by \( X \) and \( t \). This observation follows by the following facts:

- \( x \) and \( t \) do not divide \( \Psi_{n',0,0}(x) \).
- The polynomial \( \Psi_{n',0,0}(x) \in \mathbb{Q}[t][x] = \mathbb{Q}[x, t] \) is homogeneous if we assign weights to the variables \( x \) and \( t \) such that \( \text{wt}(x) = 1 \) and \( \text{wt}(t) = 3 \).

Because \( X, t \nmid F(X, t) \), we have that \( F(X, t) = \prod_{i=1}^{3/3} (X-c_i t) \) for some \( c_i \in \mathbb{Q}^\times \). In other words, \( \Psi_{n',0,0}(x) = \prod_{i=3/3}^{3/3} (x^3 - c_i t) \) and any \((0, b)\) satisfies \( \mathcal{P}(m', n') \) only if \( b \in \bigcup_{i=1}^{3/3} c_i^{-1}(K^\times)^3 \cap K \).

To prove (c), we apply a similar argument with showing that

\[
U' := \left\{ a \in K^\times : \begin{array}{l}
(a, 0) \in \mathcal{U}(K) \text{ satisfies } \mathcal{P}(m', n') \\
\text{for all } (m', n') \neq (1, 2) \text{ such that } 2 \mid n'
\end{array}\right\} \subseteq \bigcup_{\text{finite}} a_i(K^\times)^2
\]

Hence, \( U' \) is thin. Note that \( \Psi_{n',s,0}(x) \) is \( G(x^2, s) \) for some \( G(X, s) \in \mathbb{Q}[X, s] \) such that \( X, s \nmid G \) for any \( n' \geq 4 \). So this completes the proof. \( \square \)
Proof of Corollary 1.8. First, suppose that $\zeta_m \in K$. For each $(m, n) \neq (5, 5)$, [LMFDB] provides an elliptic curve $E/\mathbb{Q}(\zeta_m)$ such that

$$E(\mathbb{Q}(\zeta_m))_{\text{tors}} \supseteq \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}.$$ 

For $(m, n) = (5, 5)$, by [JKP06] Theorem 3.3, there exists an elliptic curve $E/\mathbb{Q}(\zeta_5)$ such that $E(\mathbb{Q}(\zeta_5))_{\text{tors}} \supseteq \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$ since the modular curve $X_1(5, 5)$ is the moduli space of the elliptic curves whose Mordell-Weil groups of rational points contain $\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$. (For more details, refer to the explanation after Proposition 4.7.)

The converse is nothing but [Si09 III.Corollary 8.1.1].

\[ \square \]

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