Scattering phases in the broken phase of the 4-d $O(4)$ non-linear $\sigma$-model

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Using Lüscher’s method we determine the elastic scattering phases in the broken phase of the 4-dimensional $O(4)$ non-linear $\sigma$-model from the two-particle energy spectrum in a Monte-Carlo study on finite lattices. In the isospin-0-channel we observe the $\sigma$-resonance and extract its mass and its width. In all scattering channels investigated the results are consistent with perturbative calculations.

1. INTRODUCTION

Lüscher established a relation between the energy spectrum of two-particle states in a finite box with periodic boundary conditions and elastic scattering phase shifts defined in infinite volume [1]. Since two-particle energy levels are calculable by Monte Carlo techniques, this relation opens the possibility to extract phase shifts from numerical simulations on finite lattices. We give a résumé of our results on the elastic scattering phases in the broken phase of the 4-dimensional $O(4)$ non-linear $\sigma$-model including the corrections to our preliminary results of the Lattice ’92 contribution [2]. For a more detailed discussion see ref. [3].

2. FROM THE TWO-PARTICLE SPECTRUM TO THE SCATTERING PHASES

We give the “pions” of our model a nonzero mass $m_\pi$ by means of an external source $J$ in the action:

$$ S = -2\kappa \sum_{x \mu=1}^{4} \Phi^\alpha_{x \mu} \Phi^\alpha_{x+\mu} + J \sum_{x} \Phi^4_{x} . \quad (1) $$

The scalar field is represented as a four-component vector $\Phi^\alpha_{x}$ of unit length: $\Phi^\alpha_{x} \Phi^\alpha_{x} = 1$.

Two-pion states are classified according to cubic symmetry and have “isospin” $I = 0, 1, 2$. For their investigation we define the operator:

$$ O^a_{i} (t) = \sum_{\vec{n} \in \mathbb{Z}^4_L} \tilde{f}_i (\vec{n}) \, \tilde{\Phi}^a_{\vec{n}, t} \, \tilde{\Phi}^b_{\vec{n}, t} \quad (2) $$

where $\tilde{f}_i (\vec{n})$, $i = 1, 2, \ldots$ is some basis of wave functions with correct cubic symmetry and $\tilde{\Phi}^a_{\vec{n}, t}$ is the spatial Fourier transform of the field $\Phi^a_{x, t}$ on a lattice of spatial extent $L$:

$$ \tilde{\Phi}^a_{\vec{n}, t} = L^{-3} \sum_{x \in \mathbb{Z}^4_L} \Phi^a_{x, t} \, e^{2\pi i \vec{n} \cdot \vec{x} / L} . \quad (3) $$

The simplest choice of cubically invariant wave functions is a sum over plane waves:

$$ \tilde{f}_i (\vec{n}) = \delta_{i, n^2} , \quad i = 0, 1, 2, \ldots . \quad (4) $$

We also used another set of (Lüscher-) wave functions [1], which gave the same final results (see [3] for further details).

In the isospin-0 channel, where the $\sigma$-resonance is expected, we use operators $O_i (t)$ given by

$$ O_i (t) = \frac{1}{\sqrt{3}} \sum_{a=1}^{3} O^a_{i} (t) . \quad (5) $$

Additionally we have to take into account the $\sigma$ field at zero momentum

$$ O_\sigma (t) = \tilde{\Phi}^4_{0, t} = \frac{1}{L^3} \sum_{\vec{x} \in \mathbb{Z}^4_L} \Phi^4_{\vec{x}, t} , \quad (6) $$

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since it has the correct quantum numbers and is expected to create a state with energy below the inelastic threshold.

The two-particle energies \( W_\nu \) are extracted from the matrices of connected correlation functions \([4]\). For the different isospin channels these matrices are given by

\[
\begin{align*}
C_0^0(t) &= \langle O_i(t) O_j(0) \rangle_c, \\
C_{ij}^1(t) &= \langle \text{Im} \, O_i^{ab}(t) \, \text{Im} \, O_j^{ab}(0) \rangle_c, \\
C_{ij}^2(t) &= \langle \text{Re} \, O_i^{ab}(t) \, \text{Re} \, O_j^{ab}(0) \rangle_c - C_{ij}^0(t).
\end{align*}
\]

As described in [1], for each lattice extent \( L \) and each two-particle energy level \( W_\nu < 4m_\sigma \) we get one value of the scattering phase shifts \( \delta_0^0, \delta_1^0, \) and \( \delta_2^0 \), respectively, in the elastic region \( 0 < k_\nu / m_\sigma < \sqrt{3} \). It is computed from the key relation

\[
\delta_k^l(k_\nu) = -\phi\left(\frac{k_\nu L}{2\pi}\right) \text{ mod } \pi, \tag{8}
\]

where \( \phi \) is a continuous function defined in [1], see also [2,3] \( (l=\text{angular momentum}). \) The momentum \( k_\nu \) corresponding to \( W_\nu \) has to be calculated with the help of the (lattice) energy momentum relation

\[
(2 \sinh^{1/2}(\frac{W_\nu}{2}))^2 = m_\sigma^2 + k_\nu^2. \tag{9}
\]

This relation does not determine \( k_\nu \) uniquely, since \( k_\nu \) depends not only on \( k_\nu \) but also on the direction of \( k_\nu \). We include this small ambiguity in the error estimate for \( k_\nu \).

3. NUMERICAL RESULTS

The data points in figs. 1-3 show the momenta extracted from the energy levels and the corresponding phase shifts \( \delta_k^0, \delta_k^1, \delta_k^2 \) at \( (\kappa = 0.315, J = 0.01) \).

How do our results compare with one-loop perturbation theory if we insert the renormalized coupling constant and masses as determined by our simulations? The dashed curves are the perturbative predictions based on an estimate \( \bar{m}_\sigma \) of the resonance mass \( m_\sigma \), which was obtained from a fit to the non-resonance \( \sigma \)-propagator in momentum space. We consider \( \bar{m}_\sigma \) to be an estimate only, since the \( \sigma \) particle is unstable. Fig. 1 shows that indeed \( m_\sigma \) lies below \( \bar{m}_\sigma \).

In figs. 2, 3 we observe no significant deviation from perturbation theory: For isospin 1 and 2 the perturbative predictions depend on \( m_\sigma \) only weakly.

In order to determine the resonance mass \( m_\sigma \) and the decay width \( \Gamma_\sigma \) from the measured scattering phases we have employed different methods:

A. Perturbative Fit

The one-loop perturbative formula for \( \delta_k^0 \) [5] (see [3] for details) as function of \( k_\nu, m_\sigma, \Gamma_\sigma \) can be used as a fit ansatz for all points in the elastic region. This leads to \( m_\sigma = 0.691(3) \) and \( \Gamma_\sigma = 0.112(7) \) in good agreement with the results from the Breit-Wigner fit below.

B. Breit-Wigner Fit

Fitting to the (relativistic) Breit-Wigner formula

\[
\tan \left( \delta_k^0 - \frac{\pi}{2} \right) = \frac{W^2 - m_\sigma^2}{m_\sigma \Gamma_\sigma} \tag{10}
\]

has the major advantage of being free of additional assumptions, but can only be applied near
the resonance at $\delta_0^0 = \pi/2$, so only few data points are used. Nevertheless we get for the resonance mass $m_\sigma$ and width $\Gamma_\sigma$: $m_\sigma = 0.706(2)$ and $\Gamma_\sigma = 0.130(9)$. These numbers have to be compared with the estimate $\tilde{m}_\sigma = 0.720(1)$ and the perturbative prediction $\bar{\Gamma}_\sigma = \Gamma_\sigma(\bar{m}_\sigma, \ldots) = 0.121(1)$.

4. DISCUSSION AND CONCLUSIONS

The results for the decay width agree reasonably well with the perturbative predictions, while the resonance masses $m_\sigma$ lie systematically (about 5\%) below the estimates $\tilde{m}_\sigma$ from the fit to the propagator in momentum space. However, this discrepancy should not be too surprising, because the width of the $\sigma$-resonance for our choice of parameters is rather large: $\Gamma_\sigma \approx 0.15 m_\sigma$.

Once again, renormalized perturbation theory has turned out to be very reliable in the four-dimensional $\Phi^4$-theory. Furthermore, we have demonstrated the applicability of Lüscher's method for studying particle scattering processes in massive quantum field theories on finite lattices – at least for this model.

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