Ermakov-Pinney and Emden-Fowler equations: 
new solutions from novel Bäcklund transformations

Sandra Carillo\textsuperscript{1} and Federico Zullo\textsuperscript{2}

Abstract

The class of nonlinear ordinary differential equations $y''y = F(z, y^2)$, where $F$ is a smooth function, is studied. Various nonlinear ordinary differential equations, whose applicative importance is well known, belong to such a class of nonlinear ordinary differential equations. Indeed, the Emden-Fowler equation, the Ermakov-Pinney equation and the generalized Ermakov equations are among them. Bäcklund transformations and auto Bäcklund transformations are constructed: these last transformations induce the construction of a ladder of new solutions admitted by the given differential equations starting from a trivial solution. Notably, the highly nonlinear structure of this class of nonlinear ordinary differential equations implies that numerical methods are very difficulty to apply.

Keywords: Non-linear ordinary differential equations, Bäcklund transformations, Schwartzian derivative, Ermakov-Pinney equation, Emden-Fowler equation.

1 Introduction

Since the 1990s, a number of results on discretization of ordinary differential equations describing integrable physical systems were achieved. Among them there are the Ruijsenaars-Schneider model \cite{21}, the Henon-Heiles, Garnier and Neumann systems \cite{14, 26}, the Ermakov-Pinney equation \cite{13}, the Euler top \cite{11}, the Lagrange \cite{16} and Kirchoff tops \cite{23}, the Chaplygin ball \cite{27}, the Gaudin systems \cite{22, 31} (see also \cite{30} and references therein). All these discretizations represent maps among solutions of the differential equations describing the corresponding integrable systems. Usually, these transformations are obtained from the Lax representation of the model, by constructing an appropriate dressing matrix intertwining the Lax representation of the system (see e.g. \cite{30} for details). These transformations turn out to be canonical in the phase space of the system. Furthermore, they enjoy some crucial properties inherited by the Lax representation and by the integrable structure of the system. In particular the maps are algebraic, and, if the Lax matrix is of order $N$, this algebraic structure enters through an auxiliary variable $\gamma$ which satisfies an irreducible polynomial equation of...
degree $N$ whose coefficients are rational functions of the dynamical variables (see e.g. [15]).

In this work an algebraic map among solutions of a class of nonlinear second order differential equations is obtained. Integrability of the equations is not requested, neither the transformations are obtained via the Lax representation of the system. Rather, both the differential equations considered and the transformations obtained are an extension of the results given in [2]: in that work the properties of the Schwarzian derivative were fundamental to achieve the results. In the present case, the Schwarzian derivative continues to play a key role in the construction of the transformations, allowing us to derive new results.

The present study concerns non-linear ordinary differential equations of second order of the form

$$yy'' = F(z,y^2), \quad y: \mathbb{R}^+ \rightarrow \mathbb{R}^+, \tag{1}$$

where, as usual, the prime sign denotes derivative with respect to the independent variable and $F$ is a suitably regular given function of its arguments. A large number of differential equation which model physical phenomena can be written in the form (1): we can mention the Ermakov-Pinney equation [5],[6] and the Emden-Fowler equation (see e.g. [4] and [9]). These two equations and their generalisations are further discussed in the subsequent sections. Finally, note that, when the function $F$ in (1) depends on the independent $z$ variable only, equation (1) reduces to the ordinary differential equation (1.1) in [2], which takes its physical origin in extended kinetic theory (see [2] and references therein).

It is well known, see [12] for instance, that the Schwarzian derivative plays a fundamental role in the theory of linear second order differential equations. Given any smooth enough function $f(z)$ defined on an open set $I \subset \mathbb{R}$, its Schwarzian derivative $\{f,z\}$ is defined via

$$\{f,z\} := \frac{f'''}{f'} - \frac{3}{2} \left(\frac{f''}{f'}\right)^2, \quad f'(z) \neq 0 \quad \forall z \in I. \tag{2}$$

The link between the Schwarzian derivative and the theory of linear differential equations is given by the following result (see e.g. [12], Theorem 10.1.1): if $g_0$ and $g_1$ are two independent solutions of the equation

$$g'' = 2P'g' + Qg, \tag{3}$$

where $P$ and $Q$ denote smooth functions, say $C^2$, of the independent variable, then the Schwarzian derivative of the ratio $g_0/g_1$, assuming $g_1 \neq 0$, depends only on the functions $P$ and $Q$. Specifically, it turns out

$$\{\frac{g_0}{g_1}, z\} = 2(P'' - P'^2 - Q). \tag{4}$$

The latter connects equations (3) and (1) via the Schwarzian derivative. In the following, Bäcklund transformations, admitted by (1) itself, are obtained. Then, new explicit solutions of (1) are constructed via Bäcklund transformations admitted by (1) itself.
The material is organised as follows. The opening Section 2 is concerned about the construction of a Bäcklund transformation needed to establish the subsequent results. In particular, conditions which imply it is an auto-Bäcklund transformation are established. The functional equation which guarantees the Bäcklund transformation is an auto-Bäcklund transformation are studied in Section 3. In the following Section 4, the relation between the linear equation (3), the equation (4) and the auto-Bäcklund of the ordinary differential equations under investigation are considered. Notably, they exhibit terms in which the Schwarzian derivative appears.

Section 5 is devoted to show how some well known nonlinear ordinary differential equations are amenable to be treated via the method devised in the previous Sections. Specifically, the physically relevant cases of the Ermakov-Pinney and of the Emden-Fowler equation are studied in two different subsections where corresponding Bäcklund and auto Bäcklund transformations are constructed.

In the closing Section 6, further to some conclusions, also perspective investigations are mentioned.

2 Bäcklund transformations

In this Section an invariance shown in [2] is revisited and suitably generalised to adapt to the wider class of nonlinear ordinary differential equations under investigation. To start with, let us introduce the following Lemma.

Lemma 2.1 Consider the ordinary differential equation (1), where \( y : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \), then admitted solutions are related, via Bäcklund transformation, to solutions of the ordinary differential equation

\[
 v''v - \frac{1}{2}(v')^2 = 2vF(z,v). \tag{5}
\]

Proof Apply the transformation

\[
 T : \quad v = y^2 , \quad y, v : \mathbb{R}^+ \rightarrow \mathbb{R}^+ , \tag{6}
\]

and note that the kernel of \( T \) is empty since \( y \) and \( v \) are assumed to be real positive valued. Then, direct substitution of (6) in (1) gives (5).

The link can be graphically depicted as follows

\[
 \begin{array}{c|c}
 y''y = F(z, y^2) & v''v - \frac{1}{2}(v')^2 = 2vF(z,v) \\
 \end{array}
\]

Lemma 2.2 Consider the reciprocal transformation

\[
 R : \quad \begin{cases} 
 x = \Phi \\
 v = D_t \Phi 
 \end{cases} \quad D_t = vD_x , \quad D_x = [\Phi_t]^{-1}D_t \tag{7}
\]
wherein:

\[ D_t := \frac{d}{dt}, \quad D_x := \frac{d}{dx}, \quad \Phi_t := \frac{d}{dt} \Phi; \]  

(8)

it transforms (5), where \( v: \mathbb{R}^+ \rightarrow \mathbb{R}^+ \), in

\[ \{\phi, t\} = 2\phi_t F(\phi, \phi_t). \]  

(9)

**Proof** Direct application of the reciprocal transformation \( R \) produces the result.

□

The two Lemmata can be summarised in the following Bäcklund chart, according to the terminology in [3] where by Bäcklund chart is meant the net of links, represented by Bäcklund Transformations, among different equations.

\[ \begin{array}{c}
 y''y = F(z, y^2) \\
 \downarrow T \\
 v''v - \frac{1}{2} (v')^2 = 2vF(z, v) \\
 \downarrow R \\
 \{\phi, t\} = 2\phi_t F(\phi, \phi_t)
\end{array} \]

Hence, a correspondence between equations (11) and (9), we term Schwarzian equation, is established. This result, combined with known properties of the Schwarzian derivative, allows to prove further invariances enjoyed by (1).

**Proposition 2.3** Let \( g \) and \( \psi \) denote two suitably smooth maps, then assume \( \phi \) can be expressed via the composition of such two maps, namely \( \phi = g(\psi) \), then \( \psi \) satisfies a Schwarzian equation of the same form of (9), that is

\[ \{\psi, t\} = 2\psi_t \tilde{F}(\psi, \psi_t), \quad \text{where} \ \tilde{F} \ \text{is a smooth function}. \]  

(10)

**Proof** Computation of the Schwarzian derivative \( \{\phi, t\} \), where \( \phi = g(\psi) \), composition of the two differentiable maps \( g \) and \( \psi \), gives

\[ \{\phi, t\} = \psi_t^2 \{g, \psi\} + \{\psi, t\}. \]  

(11)

The latter, substituted in equation (9) can be re-written as

\[ \{\psi, t\} = 2\phi_t F(\phi, \phi_t) - \psi_t^2 \{g, \psi\}. \]  

(12)

Note that this equation is of the same form of (9). Indeed, the derivative \( \phi_t \) is \( g'(\psi_t) \), and hence

\[ \{\psi, t\} = 2g'\psi_t F(g, g'\psi_t) - \psi_t^2 \{g, \psi\}. \]  

(13)

Thus, if we define the new function \( \tilde{F} \) via

\[ \tilde{F}(\psi, \psi_t) := g'F(g, g'\psi_t) - \psi_t^2 \{g, \psi\}, \]  

and substitute it into equation (13), the thesis readily follows since equation (10), of the form of (9), is obtained.

□

The interested reader is referred to [3] and Ref.s therein to track early occurences of the term Bäcklund chart.
Now, the Bäcklund chart can be extended setting $B_1 : \phi - g(\psi) = 0$, so that the links are summarised as follows.

\[
y''y = F(z,y^2) \quad \text{T} \quad v''v - \frac{1}{2}(v')^2 = 2vF(z,v) \quad \text{R} \quad \{\phi,t\} = 2\phi_1 F(\phi,\phi_t) \quad B_1 \quad \{\psi,t\} = 2\psi_1 F(\psi,\psi_t)
\]

Accordingly, solutions admitted by the differential equation (1) can be constructed as stated in the following proposition.

**Proposition 2.4** Let $f$ denote a smooth enough function such that $f'(z) \neq 0$, $\forall z \in \mathbb{R}_+^+$ introduce

\[
Y^2(z) := \frac{y^2(f(z))}{f'(z)}, \quad (15)
\]

where $y(z)$ is any solution of the equation (1), then $Y(z)$ is a solution of the equation

\[
YY'' = f'F(f,f'Y^2) - \frac{1}{2}\{f,z\}Y^2. \quad (16)
\]

**Proof** Note that (15) can be regarded as the Bäcklund transformation:

\[
B_2 : f'(z)Y^2(z) - y^2(f(z)) = 0 \quad (17)
\]

which, given the smooth function $f$, connects (10) to (11). Thus, explicit computations prove the result.

Let us discuss this result. The transformation (15) represents a map between corresponding solutions of equations (11) and (16). Two different cases can occur. That is, when the two equations (16) and (11) exhibit different forms, the Bäcklund transformation $B_2$, given by (17), connects solutions of equation (11) with solutions of equation (16).

A special case arises when equations (11) and (16) share the same form: in this case, $B_2$, given by (15), represents an auto-Bäcklund transformation between solutions of equation (11). The necessary and sufficient condition $B_2$ must satisfy to be an auto-Bäcklund transformation reads

\[
F(z,v) = f'F(f,f'v) - \frac{1}{2}\{f,z\}v. \quad (18)
\]

The latter is obtained via direct comparison between the r.h.s.s of the two equations (16) and (11). This result is stated in the following corollary.

**Corollary 2.5** Let $y(z)$ and $Y(z)$ be, in turn, solutions of the differential equations (11) and (16), then $B_2$, in (15), represents an auto-Bäcklund transformation, whenever $f$ denotes a solution of the functional differential equation (18).

Equation (18) is a functional differential equation: a characterisation of the solutions it admits is provided in the next section. Then, explicit examples are considered to show how it is possible to construct new solutions of equation (11) on application of the transformation (17).
3 The functional differential equation

This Section is concerned about the study of the functional equation (18) wherein the unknown is the function $f$, while $F$ is given. Thus, it is a 3rd order nonlinear ordinary differential equation subject to the condition $f' \neq 0$. Here, for the sake of simplicity, we do not look for $f$, which is a solution of (18), but we prefer to consider (18) as a linear non homogeneous ordinary differential equation in the unknown $F$, where, in addition, $F$ is assumed to depended on $v$ via a suitable power expansion. This approach allows us to say which forms the given function $F$ may assume in order to be amenable to the presented method. Specifically, given a suitable $F$, a method to construct solutions of equation (1), via Bäcklund transformations, is provided.

Accordingly, let us assume that the function $F(z, v)$ admits a formal power expansion in $v$, that is
\[ F(z, v) = \sum_n F_n(z) v^n, \] (19)
where the sum is intended on a suitable set of integers. For the sake of convenience, when $n \neq 1$, the coefficients $F_n(z)$ are looked for under the form $F_n(z) = (G'_n(z))^{n+1}$, where $G'_n$ denotes the derivative of a smooth (at least $C^1$) function of $z$. In the case $n = 1$, we set
\[ F_1(z) := (G'_1(z))^2 - \frac{1}{2}Q(z) \] (20)
where $Q(z)$ is a suitable function. Hence, substitution of the latter and of (19) in (18), gives
\[ 2 \sum_n ((f')G'_n(f) - (G'_n(f))^n + (f')^2Q(f)v + Q(z)v - \{f,z\}v = 0. \] (21)
which depends explicitly on $G_n$, so that, for example, $(G'_n(f))$ indicates that $G'_n$ is composed with $f(z)$.

The power series expansion in (21) represents a polynomial in $v$ whose first term multiplies $v^{-1}$, hence equality (21) is satisfied when all the coefficients of $v^k$ are set equal to zero. This means that the action of the map $f(z)$ on the functions $G_n(z)$ is a translation, that is, the functions $G_n(z)$ satisfy the linear functional equations
\[ G_n(f(z)) = G_n(z) + K_n \iff f'G'_n(f) = G'_n(z) \forall n, \] (22)
where $K_n$ are arbitrary constants. Substitution of (22) in (21) gives
\[ Q(z) = (f')^2Q(z) + \{f,z\}. \] (23)
Notably, the right hand side of the latter reminds the right hand side of equation (11) when, in turn, the variable $z$ is identified with the variable $t$ and the function $f$ with the function $\psi$. This correspondence suggests to identify $Q$ with a suitable Schwarzian derivative, that is, to set
\[ Q(z) := \{w,z\}. \] (24)
Substitution of this position in (22), and the subsequent comparison of (23) with (11) show that solutions of (23) are obtained whenever \( w(z) \) and \( f(z) \) are related via the functional equation

\[
\{ w(z), z \} = (f')^2 \{ w(f), f \} + \{ f, z \},
\]

wherein the Schwarzian derivative of the composition of two functions appears since \( (f')^2 \{ w(f), f \} + \{ f, z \} \equiv \{ w(f), z \} \). Accordingly, the function \( w(f(z)) \) and \( w(z) \) follow to be related via a fractional linear transformation, i.e.

\[
w(f(z)) = \frac{aw(z) + b}{cw(z) + d}, \quad ad - bc \neq 0.
\]

Hence, combination of the shown results, allows to prove the following proposition.

**Proposition 3.1** If the function \( F(z, v) \) can be represented as a power series expansion in \( v \), then the solution of the functional differential equation

\[
F(z, v) = f'^2 F(f, f'v) - \frac{1}{2} \{ f, z \} v.
\]

is given by

\[
F(z, v) = \sum_n a_n (G'_n(z))^{n+1} v^n - \frac{1}{2} \{ w(z), z \} v
\]

where the functions \( G_n(z) \) and \( w(z) \) satisfy the functional equations \( a, b, c, d \) and \( K_n \) are arbitrary constants

\[
G_n(f(z)) = G_n(z) + K_n, \quad w(f(z)) = \frac{aw(z) + b}{cw(z) + d}, \quad ad - bc \neq 0.
\]

The previous proposition allows to find differential equations possessing the auto-Bäcklund transformations [17]. A corollary of this proposition reads as follows.

**Corollary 3.2** Let \( y_0(z) \) be a solution of the differential equation

\[
y'' = \sum_n a_n (G'_n(z))^{n+1} y^{n-1} - \frac{1}{2} \{ w(z), z \} y,
\]

where the functions \( G_n(z) \) and \( w(z) \) are specified in Proposition 3.1, then

\[
y_1(z)^2 = \frac{y_0(f(z))^2}{f'(z)}
\]

represents an auto-Bäcklund transformation admitted by equation (29).

These results can be enriched and complemented thanks to the following two propositions.
Proposition 3.3  The general solution of the functional equation

\[ G(f(z)) = G(z) + K \]  \hspace{1cm} (30)

is given by

\[ G(z) = G_0(z) + \Phi(G_0(z)) \]  \hspace{1cm} (31)

where \( G_0(z) \) is a particular solution of equation (30) and \( \Phi(z) \) is an arbitrary periodic function of period \( K \).

Proposition 3.4  Let \( w(z) \) be defined as

\[ w(z) := \frac{DG(z) - B}{A - CG(z)}, \quad A, B, C, D \in \mathbb{R} \]  \hspace{1cm} (32)

where \( G(z) \) is any function satisfying equation (30), then \( w(z) \) is a solution of

\[ w(f(z)) = \frac{(\Delta + DCK)w(z) + D^2K}{(\Delta - DCK) - w(z)C^2K} \]  \hspace{1cm} (33)

where \( \Delta = AD - BC \neq 0 \).

These propositions can be easily proved via direct computations.

4  Auto-Bäcklund transformations

In this Section auto-Bäcklund transformations are related to linear second order ordinary differential equations. Specifically, the attention is focussed on how solutions admitted by second order linear ordinary differential equations lead to the construction of the transformation (17).

First of all, the general ideas are presented. Consider the link between two independent solutions admitted by (3) and let \( w(z) \) denote their ratio, namely

\[ w(z) := \frac{g_0(z)}{g_1(z)}, \quad g_1(z) \neq 0 \]  \hspace{1cm} (34)

Then, according to (4), the Schwarzian derivative of \( w \) is given by

\[ -\frac{1}{2} \{w, z\} = Q + (P')^2 - P'', \]

where \( P \) and \( Q \) are the smooth coefficients of the ordinary differential equation (1). Assume, now, a function \( f(z) \) exists such that \( w(f(z)) \) can be expressed as a fractional linear transformation of \( w(z) \), i.e. it is such that it satisfies the hypotheses of proposition 3.1. Consider, then, the fractional linear transformation:

\[ w(f(z)) = \frac{(\Delta + DCK)w(z) + D^2K}{(\Delta - DCK) - w(z)C^2K}, \quad A, B, C, D, K \in \mathbb{R} \]  \hspace{1cm} (35)
where $\Delta = AD - BC \neq 0$. Let the function $G(z)$ be defined via

$$G(z) := \frac{A w(z) + B}{C w(z) + D},$$

then, the functional relation $G(f(z)) = G(z) + K$ holds, $\forall z$. Note that the function $w(z)$ is given by \([34]\); hence, the derivative of the function $G(z)$ can be written, on use of the Wronskian of solutions of equation \([3]\), as

$$G'(z) = \Delta C_1 \left( \frac{e^{P(z)}}{C g_0(z) + D g_1(z)} \right)^2,$$

(36)

where $C_1$ is a suitably chosen constant. When the arbitrary constants, denoted as $a_n$, are inserted as coefficients, the following, consequence of Corollary \([3.2]\) is proved.

**Corollary 4.1** Let $y_0(z)$ be a solution of the differential equation

$$y'' = \sum_n a_n \left( \frac{e^{P(z)}}{C g_0(z) + D g_1(z)} \right)^{2n+2} y^{2n-1} + \left( Q + (P')^2 - P'' \right) y,$$

(37)

where the functions $P(z)$ and $Q(z)$ are arbitrary and $g_0$ and $g_1$ are two independent solutions of the linear differential equation \([3]\). Then, if it is possible to find a function $f(z)$ such that equation \([35]\) holds, the map

$$y_1(z)^2 = \frac{y_0(f(z))^2}{f'(z)},$$

(38)

defines an auto-Bäcklund transformation admitted by equation \([34]\).

The relevance of this result is stressed in the next section where the results \([37]-[38]\) are applied to different examples.

## 5 Applications

The class of differential equations \([29]\) can be shown to embrace, as particular cases, a large number of nonlinear differential equations. The aim of this Section is to show implications on the well known Ernako-Pinney and Emden-Fowler nonlinear equations whose applicative relevance is undoubtable. In addition, some generalisations of these equations are also considered.

First of all, a simple case is treated. Note that if, in \([29]\), here re-written for convenience

$$y'' = \sum_n a_n (G_n(z))^{n+1} y^{2n-1} - \frac{1}{2} \{w(z), z\} y,$$

(29)

we set $G_n(z) := G(z)$, $\forall n$, it is compatible with the further positions $K_n = K$ and $\Phi = 0$. Indeed, in \([28]\) the arbitrariness in the choice of the functions $G_n(z)$ is reflected...
in the arbitrariness of the parameters $K_n$ and of the periodic function $\Phi$, respectively, in equation (22) and (31). Furthermore, the (tacitly) assumed convergence, in a given open set $\Omega \subset \mathbb{R}$, of the series in (29), implies that the sequences $\{a_n\} \in \mathbb{R}$ is such that, in $\Omega \subset \mathbb{R}$, the following convergent series can be introduced

$$H(z) := \sum_n a_n z^n. \quad (39)$$

The Corollary 3.2 takes, in this case, the following simplified form.

**Corollary 5.1** Let $y_0(z)$ be a solution of the differential equation

$$yy'' = G' H(G'y^2) - \frac{1}{2}\{w, z\} y^2, \quad (40)$$

where the functions $G(z)$ and $w(z)$, as specified in Proposition 3.1, satisfy the functional equations (28), that is

$$G(f(z)) = G(z) + K, \quad w(f(z)) = aw(z) + b, \quad cw(z) + d, \quad a, b, c, d, K \in \mathbb{R}. \quad (28)$$

Then, the map

$$y_1(z)^2 = \frac{y_0(f(z))^2}{f'(z)}$$

defines an auto-Bäcklund transformation of the equation (40).

In the next subsections, differential equations with relevant physical meaning are obtained corresponding to *ad hoc* choices, in (40), of the function $H(z)$.

### 5.1 The Ermakov-Pinney equation

A first remarkable case is represented by the Ermakov-Pinney equation studied in this subsection. Specifically, if, in the equation (37), the function $P(z)$ as well as all the coefficients $a_n$ are all set equal to zero except $a_{-1}$, denoted via $a_1 = -\alpha$, then, the Ermakov-Pinney equation $^4$

$$y'' = Q(z)y - \frac{\alpha}{y^3} \quad (41)$$

is obtained. This is equivalent to choose $H(z) = -\frac{\alpha}{z}$ in corollary 5.1.

The Ermakov-Pinney equation was introduced in 1880 by Ermakov $^6$, who investigated solvability conditions of second order ordinary differential equations. The Ermakov-Pinney equation is closely connected to the harmonic oscillator with a time-dependent frequency. It is used to describe various problems, in quantum mechanics, such as the motion of charged particles in the Paul trap $^{18}$, in atomic transport theory $^{25}$ and Bose-Einstein condensate theory $^{20}$. Furthermore, it plays a fundamental

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$^4$On introduction of the Kronecker symbol $\delta_{k,l} := \begin{cases} 0 & k \neq l \\ 1 & k = l \end{cases}$, the condition on the coefficients $a_n$ can be written as $a_n = -\alpha \delta_{-1,n}$. 

$^6$
role in the description of the unitary evolution of quantum non-autonomous systems [28]. It appears also in cosmology [10]. A set of Bäcklund transformation and an exact discretization admitted by equation (41) is due to A. Hone [13], who, on application of two different transformations, provides its exact discretization.

Now, on applications of results in the previous sections, a set of Bäcklund transformations admitted by the Ermakov-Pinney equation are obtained. In particular, since equation (41) corresponds to $P = 0$ in equation (37), the admitted Bäcklund transformations are specified by the following proposition.

**Proposition 5.2** Suppose that $y_0$ is a solution of the Ermakov-Pinney equation. Define the function $w(z) = \frac{g_0}{g_1}$, where $g_0$ and $g_1$ are two independent solutions of the linear differential equation $y'' = Qy$. Then, if it is possible to find a function $f(z)$ such that equation (26) holds, the map

$$y_1(z)^2 = \frac{y_0(f(z))^2}{f'(z)}$$

is an auto-Bäcklund transformation of the equation (41).

**Proof** The proposition follows on specialisation, in Corollary 4.1, of all the parameters via $a_k = -\alpha \delta_{k,-1}$ and $P \equiv 0$.

Notable examples are represented by the case when the function $Q$ exhibits a dependence on $z^2$: in particular, it is a linear combination of a term proportional to $z^{-2}$ with a term proportional to $z^{4k}$ for some integer $k \in \mathbb{N}$. Both these functions, taken separately, have application in scalar field cosmologies (see [10], eqs. (10)-(11)-(31)-(38)). Specifically, consider $Q$ to be

$$Q(z) = p^2(2k+1)^2\frac{z^{4k}}{4} + \frac{k(k+1)}{z^2}, \quad p \in \mathbb{R}^+.$$  

In this case, the function $w$ is obtained

$$w(z) = e^{pz^{2k+1}}$$  

and the function $f(z)$ can be defined by

$$f^{2k+1} = \frac{1}{p} \ln \left( \frac{aw(z) + b}{cw(z) + d} \right), \quad ad - bc \neq 0.$$  

Hence, given a solution $y_0$ of equation (41), further solutions are represented by

$$y_1(z)^2 = \frac{y_0(f)^2}{f'}.$$  

Indeed, chosen $Q$ in (42), a particular solution of equation (41) is given by

$$y_0(z) = \frac{\beta}{z^k}, \quad \beta^4 := \frac{4\alpha}{p^2(2k+1)^2}.$$
Thus, on application of the transformation (45), after few manipulations, we find a new set of solutions, say $y_1(z)$, as follows

$$y_1(z)^2 = \frac{\beta^2}{2k} \frac{(aw(z) + b)(cw(z) + d)}{w(z)(ad - bc)}, \quad ad - bc \neq 0,$$

(47)

where $w(z)$ is defined by equation (43). Note that the previous equation defines the general solution of equation (41) which corresponds to $Q$ in (42).

### 5.2 The Emden-Fowler equation

This subsection is devoted to a second example of an ordinary differential equation which admits Bäcklund transformations whose construction is possible according to the general results comprised in Sections 2-4. The generalised Emden-Fowler equation of the first kind in the unknown $q(x)$ reads (see e.g. [9])

$$xq_{xx} + \alpha q_x + \beta x^{\nu} q^n = 0.$$

(48)

This equation appears in a wide variety of mathematical physics problems: specifically, when $\nu = 1$ and $\alpha = 3$ it describes the hydrostatic and thermodynamic equilibrium of stars. In addition, it appears in investigations on the Einstein’s field equations [11] or in the mean-field description of critical adsorption [8] (see [9] and references therein for further examples). When the parameter $\alpha$ is integer, equation (48) is used to model spherically symmetric steady state solutions of evolution problems involving the Laplace operator in a $\alpha$-dimensional space. More precisely, when $r$ denotes the radial coordinate, looking for spherically symmetric solutions, the reduction of the equation

$$\nabla^2 q + \beta r^{\nu-1} q^n = 0$$

directly gives equation (48). A particularly interesting generalisation is represented by the modified Emden-Fowler equation, which is obtained on substitution of the factor $x^{\nu}$, in (48), with a function of $x$:

$$xq_{xx} + \alpha q_x + \beta r(x) q^n = 0.$$

(49)

In this section two examples are provided: one for equation (48) and the other for equation (49). Equation (48) is the particular case of equation (49), which corresponds to the choice $r(x) = x^{\nu}$. Likewise, our first example is a particular case of the second one.

**Example 1.** In this example, let $\nu = 1 - 2\alpha$ in (48) and, for later convenience, set $n = 2m - 1$. The differential equation (48) then reads

$$xq_{xx} + \alpha q_x + \beta x^{1-2\alpha} q^{2m-1} = 0.$$

(50)

The following change of variables

$$q(x) = \frac{y(z)}{z}, \quad z = x^{\alpha-1}$$

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in (50) reduces to
\[ y'' + \frac{\beta}{(\alpha - 1)^2} \frac{y^{2m-1}}{z^{2m+2}} = 0. \] (51)

The latter can be compared with equation (37): that is, according to Corollary 4.1 when we set \( Q = P'' - (P')^2 \) and \( a_m = -\frac{\beta}{(\alpha - 1)^2} \delta_{m,n} \), equation (51) is obtained. Indeed, the choice \( Q = P'' - (P')^2 \) entails that the functions \( g_0 \) and \( g_1 \) are linear combinations of \( e^P \) and \( z e^P \) (independent solutions of equation (3)). Let, now choose
\[ g_0 = e^P(D - bz), \quad g_1 = e^P(az - C), \quad aD - bC = 1 \]
then, equation (37) reads exactly as (51). The next step is to show, solving the functional equation (33), that the function \( f(z) \) is given by
\[ f(z) = \frac{\Delta z}{Kz + \Delta}, \]
which allows to find the Bäcklund transformations
\[ y_1^2(z) = y_0^2 \left( \frac{\Delta z}{Kz + \Delta} \right) \frac{(Kz + \Delta)^2}{\Delta^2}. \] (52)

A particular solution of equation (48) is given in [9]: it corresponds to the case when
\[ y_0(z) = p z^{m-1} \] (53)
and represents a solution of equation (51), where the constant \( p \) is required to satisfy
\[ \beta(m - 1)^2 p^{2m-2} + m(\alpha - 1)^2 = 0. \]

Note that the latter, in general, given real \( \beta \) and \( m \), admits complex solutions \( p \in \mathbb{C} \); hence, to obtain \( p \in \mathbb{R} \), \( \beta \) and \( m \) are required to be, in turn, positive and negative, or vice versa. The new solution (52) reads
\[ y_1^2(z) = p^2 \left( \frac{\Delta z}{Kz + \Delta} \right)^{2m-1} \frac{(Kz + \Delta)^2}{\Delta^2} \] (54)
and represents two different one-parameter solutions of equation (51), the parameter we can choose is \( \frac{K}{\Delta} \). Note also that further iterations of the same transformation, where each iteration is identified by the pair of parameters \( \Delta_n \) and \( K_n \), do not add further constants to the solution. Indeed, the \( n^{th} \) iteration can be written as
\[ y_n^2(z) = p^2 \left( \frac{R_n z}{S_n z + R_n} \right)^{2m-1} \frac{(S_n z + R_n)^2}{R_n^2} \]
where the coefficients \( R_n \) and \( S_n \) solve the recurrence relations
\[ R_{n+1} = \Delta_{n+1} R_n, \quad S_{n+1} = \Delta_{n+1} S_n + K_{n+1} R_n. \]
with the initial values $R_0 = 1$, $S_0 = 0$ (and obviously $\Delta_1 = \Delta$ and $K_1 = K$).

**Example 2.** In this example we look at the modified Emden-Fowler equation. Again, for later convenience, we set $n = 2m - 1$. We specify the function $r(x)$ as

\[ r(x) = \frac{1}{x^{2\alpha}} \left( \frac{x^{\alpha-1}}{\eta + \gamma x^{\alpha-1}} \right)^{2m+2}, \]  

which corresponds to the following differential equation in the unkown $q(x)$

\[ xq_{xx} + \alpha q_x + \beta x^{1-2\alpha} \left( \frac{x^{\alpha-1}}{\eta + \gamma x^{\alpha-1}} \right)^{2m+2} q^{2m-1} = 0. \]  

When the special values $(\eta, \gamma) = (0, 1)$ are chosen, (56) reduces to (50) of the previous example. The subsequent change of variables in (56)

\[ q(x) = \frac{y(z)}{z}, \quad z = x^{\alpha-1}, \]

gives

\[ y'' + \frac{\beta}{(\alpha - 1)^2 (\eta + \gamma z)^{2m+2}} y^{2m-1} = 0. \]  

This equation can be compared with equation (37): according to Corollary 4.1, we must set $Q = P'' - (P')^2$ and $a_m = -\frac{\beta}{(\alpha - 1)^2} \delta_{m,n}$. Then, the functions $g_0$ and $g_1$ are linear combinations of $e^P$ and $ze^P$ (that is, they are two independent solutions of equation (3)). Let

\[
\begin{align*}
g_0 &= e^P(a_0 z + b_0), \\
g_1 &= e^P(a_1 z + b_1).
\end{align*}
\]

These choices allow to check that equation (37) reduces exactly to (57). Then it can be shown that the function $f(z)$, solution of the functional equation (33), is given by

\[ f(z) = \frac{(K \eta \gamma + \Delta \delta)z + K \eta^2}{(\Delta \delta - K \eta \gamma) - K \gamma^2 z}, \quad \delta := a_0 b_1 - a_1 b_0. \]

According to Corollary 4.1 the Bäcklund transformations admitted by equation (57) are

\[ y_1(z)^2 = y_0 \left( \frac{(K \eta \gamma + \Delta \delta)z + K \eta^2}{(\Delta \delta - K \eta \gamma) - K \gamma^2 z} \right)^2 \left( \frac{(\Delta \delta - K \eta \gamma) - K \gamma^2 z}{\Delta \delta} \right)^2. \]  

A particular solution of (56) is given by

\[ y_0(z) = p(\eta + \gamma z)^{\frac{m}{m-1}}, \]  

where the constant $p$ represents a solution of

\[ \beta(m - 1)^2 p^{2m-2} + m \gamma^2 (\alpha - 1)^2 = 0. \]

Note that the solution (59) reduces to (53) when $(\eta, \gamma) = (0, 1)$ and, then, the transformation (58) coincides with the transformation (54).

The new solution is given by

\[ y_1(z)^2 = p^2 \left( \frac{\Delta \delta \eta + \gamma z}{\Delta \delta - K \eta \gamma - K \gamma^2 z} \right)^{\frac{2m}{m-1}} \left( \frac{(\Delta \delta - K \eta \gamma) - K \gamma^2 z}{\Delta \delta} \right)^2. \]
6 Conclusions

A class of nonlinear differential equations of second order of the form $yy'' = F(z, y^2)$, which represents a genuine generalisation of $y y'' = S(x)$, in [2], is considered in this article. The studied equations are proved to admit Bäcklund transformations (15) which, in the general case, represent maps from solutions of a nonlinear differential equations of the form (1) to solutions of another equation of the same class. Notably, these Bäcklund transformations comprise also, as special cases, auto-Bäcklund transformations. These special cases are investigated; thus, whenever $F(z, v) and f(z)$ satisfy the differential functional equation (18), it is shown that the map (17) represents an auto Bäcklund transformations admitted by equation (1). These auto Bäcklund transformations are algebraic and, indeed, in the complex domain, they establish multi-valued correspondences among solutions of the same differential equations; however, when restrictions to positive real valued functions are considered, one-to-one correspondences are obtained. In particular, in the very special case when $F$ depends only on $z$, i.e. $F(z, v) = S(z)$, and, furthermore, the function $f$ is a linear fractional transformation, i.e. $f(z) = (az + b)/(cz + d)$, the transformation in [2] is obtained as a particular case of (17). In addition, a variety of applications of equation (1) are provided: they represents only some of the examples in which the constructed Bäcklund transformations can be applied to obtain new results. Further systems of physical interest, which, for instance arise as reductions of partial differential equations, belong to the class of equations we investigate: in the present article we restrict our attention to some physically relevant examples. Further results, in particular as far as the structure of equation (1) as well as its solutions in the complex domain are in progress. Also, currently under investigation are relations between the obtained Bäcklund transformations, integrability properties and, possibly, the Hamiltonian structure of the linked equations, aiming to specialise the results in [7].

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References

[1] A.I. Bobenko, B. Lorbeer, Yu B. Suris Integrable discretizations of the Euler top, J. Math. Phys., 39, (1998), 6668-6683.

[2] S. Carillo, A novel Bäcklund invariance of a nonlinear differential equation, Journal of Mathematical Analysis and Applications, 252: 2, (2000), 828-839.

[3] S. Carillo, M. Lo Schiavo, C. Schiebold, Bäcklund Transformations and Non Abelian Nonlinear Evolution Equations: a novel Bäcklund Chart, Symmetry, Integrability and Geometry: Methods and Applications (SIGMA), 12, (2016), 087, 17 pages.
[4] S. Chandrasekhar: *An introduction to the study of stellar structures*, Dover, New York, 1958.

[5] A.K. Common, M. Musette, *Two discretisations of the Ermakov-Pinney equation* Physics Letters, Section A: General, Atomic and Solid State Physics, 235:6, (1997), 574-580.

[6] V. Ermakov: *Second order differential equations. Conditions of complete integrability*, Universita Izvestia Kiev Series III 9, (1880), 1-25. English translation: A.O. Harin, Redactor: P.G.L. Leach, Applied Analysis and Discrete Mathematics, 2 (2008), 123-145.

[7] B. Fuchssteiner, S. Carillo: *The Action-Angle Transformation for Soliton Equations*, Physica A, 166, (1990), 651-676.

[8] S. Gbutzmann, U. Ritschel: *Analytic solution of Emden-Fowler equation and critical adsorption in spherical geometry*, Z. Phys. B, 96 (1995), 391-393.

[9] H. Goenner, P. Havas: *Exact solutions of the generalized Lane–Emden equation*, Journal of Mathematical Physics, 41, (2000), 7029.

[10] R.L. Hawkins, J.E. Lidsey: *Ermakov-Pinney equation in scalar field cosmologies*, Physical Review D, 66 (2002), 023523.

[11] E. Hertl: *Spherically symmetric nonstatic perfect fluid solutions with shear*, General Relativity and Gravitation, 28: 8 (1996), 919–934.

[12] E. Hille, *Ordinary differential equations in the complex domain*, Wiley, 1976, Toronto.

[13] A.N.W. Hone: *Exact discretization of the Ermakov–Pinney equation*, Physics Letters A, 263 (1999), 347–354.

[14] A.N.W. Hone, V.B. Kuznetsov, O. Ragnisco, *Bäcklund transformations for many-body systems related to KdV*, J. Phys. A: Math. Gen., 32 (1999), 299-306.

[15] A. N. W. Hone, O. Ragnisco, F. Zullo, *Algebraic entropy for algebraic maps*, J. Phys. A: Math. Theor. 49: 2 (2016), 02LT01.

[16] V.B. Kuznetsov, M. Petrera, O. Ragnisco, *Separation of variables and Bäcklund transformations for the symmetric Lagrange top*, J. Phys. A: Math. Gen., 37 (2004), 8495-8512.

[17] K. Kefalas: *On smooth solutions of non linear dynamical systems* $f_{n+1} = u(f_n)$, *PART I*, Physics International 5: 1 (2014), 112-127.

[18] F. Major, V.N. Gheorghe, G. Werth, *Charged Particle Traps - Physics and Techniques of Charged Particle Field Confinement*, Springer, Berlin, 2005.
[19] J. Moser, A.P. Veselov: *Discrete versions of some classical integrable systems and factorization of matrix polynomials*, Commun. Math. Phys., 139 (1991), 217-243.

[20] A.I. Nicolin: *Resonant wave formation in Bose-Einstein condensates*. Phys. Rev. E, 84 (2011), 056202.

[21] F.W. Nijhoff, O. Ragnisco, V.B. Kuznetsov: *Integrable time-discretisation of the Ruijsenaars-Schneider model*, Commun. Math. Phys., 176 (1996), 681-700.

[22] O. Ragnisco, F. Zullo, *Bäcklund transformations as exact integrable time discretizations for the trigonometric Gaudin model*, J. Phys. A, 43 (2010), 434029.

[23] O. Ragnisco, F. Zullo: *Bäcklund transformation for the Kirchhoff top* Symmetry, Integrability and Geometry: Methods and Applications (SIGMA), 7 (2011), 001, 13 pages.

[24] C. Small: *Functional Equations and How to Solve Them*, Springer, 2007, New York.

[25] E. Torrontegui, S. Ibáñez, X. Chen, A. Ruschhaupt, D. Guéry-Odelin, and J. G. Muga: *Fast atomic transport without vibrational heating* Phys. Rev. A, 83 (2011), 013415.

[26] A.V. Tsiganov: *On Auto and Hetero Bäcklund Transformations for the Hénon–Heiles Systems*, Phys. Lett. A, 379 (2015), 2903–2907.

[27] A.V. Tsiganov: *Integrable Euler Top and Nonholonomic Chaplygin Ball*, J. Geom. Mech., 3: 3 (2011), 337–362.

[28] D.G. Vergel, E.J.S. Villasenor: *The time-dependent quantum harmonic oscillator revisited: Applications to quantum field theory*, Annals of Physics, 324 (2009), 1360–1385.

[29] E.T. Whittaker, G.N. Watson: *A course of modern analysis*, Cambridge University Press, 1927.

[30] F. Zullo: *Bäcklund transformations and Hamiltonian flows*, J. Phys. A, 46: 14, (2013), 145203.

[31] F. Zullo: *Bäcklund transformations for the elliptic Gaudin model and a Clebsch system*, J. Math. Phys., 52 (2011), 073507.