Picard-Lefschetz Monodromy Groups of Quadratic Hypersurfaces

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Abstract

We study the topology of the space of affine hyperplanes $L \subset \mathbb{C}^n$ which are in general position with respect to a given generic quadratic hypersurface $A$, and calculate the monodromy action of the fundamental group of this space on the relative homology groups $H_\ast(\mathbb{C}^n, A \cup L)$ associated with such hyperplanes.

1 The statement of the problem and the relative homology group

$A$ is an non-degenerate quadratic hypersurface in $\mathbb{C}^n$.

For instance, $A$ could be the set \{(z_1, z_2, ... z_n) \in \mathbb{C}^n \mid z_1^2 + z_2^2 + ... + z_n^2 = 1\}.

$L$ is a complex hyperplane in $\mathbb{C}^n$.

By $\mathbb{CP}_\infty^n$ we denote the “infinitely distant” part $\mathbb{CP}^n \setminus \mathbb{C}^n$ of the projective closure of $\mathbb{C}^n$.

$\overline{A}$ is the closure of $A$ in $\mathbb{CP}^n$. Non-degeneracy of $A$ implies that $\overline{A}$ is smooth in $\mathbb{CP}^n$ and intersects $\mathbb{CP}_\infty^n$ transversally, and so $\overline{A} \cap \mathbb{CP}_\infty^n$ is a non-degenerate quadric hypersurface in $\mathbb{CP}_\infty^n$.

Let $\mathbb{CP}^n$ be the space of all hyperplanes in $\mathbb{CP}^n$.

Definition 1. $L$ is asymptotic for $A \subset \mathbb{C}^n$ if $\overline{L} \cap \mathbb{CP}_\infty^n$ is tangent to $\overline{A} \cap \mathbb{CP}_\infty^n$.

$L$ is not in general position with respect to $A$ if either it is tangent to $A$ at some point in $\mathbb{C}^n$, or it is asymptotic for $A$. 

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In other words, \( L \) is in general position with respect to \( A \) if and only if its closure \( \overline{L} \subset \mathbb{CP}^n \) is transversal to the (stratified) algebraic set \( A \cup \mathbb{CP}^{n-1}_\infty \).

**Notation.** Denote by \( \overset{\vee}{A} \) and \( \overset{*}{A} \) the subsets in \( \overset{\vee}{\mathbb{CP}^n} \) consisting of all tangent and asymptotic hyperplanes of \( A \), respectively; in addition, the point in \( \overset{\vee}{\mathbb{CP}^n} \) corresponding to the “infinitely distant” hyperplane also is by definition included into \( \overset{*}{A} \).

By the Thom’s isotopy lemma (see [2], [5]) the pairs of spaces \((\mathbb{CP}^n, A \cup L \cup \mathbb{CP}^{n-1}_\infty)\) form a locally trivial fiber bundle over the space \( \overset{\vee}{\mathbb{CP}^n} \setminus (\overset{\vee}{A} \cup \overset{*}{A}) \) of planes \( L \) which are in general position with respect to \( A \). Therefore the fundamental group of the latter space acts on all homology groups related with spaces \((\mathbb{CP}^n, A \cup L)\). The explicit calculation of this action is the main goal of this work; this is a sample result for a large family of similar problems concerning the hypersurfaces of higher degrees and/or non-generic ones.

This action is important in the problems of integral geometry, when the integration contour is represented by a relative chain in \( \mathbb{C}^n \) with boundary at \( A \cup L \), and integration \( n \)-form is holomorphic and has singularity at the infinity; see e.g. [5], Chapter III.

We always assume that \( n \geq 2 \), because otherwise the problem is trivial.

### 1.1 The representation space

**Proposition 1.** If \( L \) is in general position with respect to \( A \), then

\[
H_n(\mathbb{C}^n, A \cup L) \cong H_{n-1}(A \cup L) \cong \mathbb{Z}^2,
\]

and \( H_i(\mathbb{C}^n, A \cup L) \cong \tilde{H}_{i-1}(A \cup L) \cong 0 \) for all \( i \neq n \) (here \( \tilde{H} \) means homology group reduced modulo a point).

**Proof.** First, we have the long exact sequence for the pair \((\mathbb{C}^n, A \cup L)\):

\[
\ldots \rightarrow H_i(\mathbb{C}^n) \rightarrow H_i(\mathbb{C}^n, A \cup L) \rightarrow H_{i-1}(A \cup L) \rightarrow H_{i-1}(\mathbb{C}^n) \rightarrow \ldots
\]

(1)
The homology groups of $\mathbb{C}^n$ coincide with those of a point. So $H_i(\mathbb{C}^n, A \cup L) \cong \tilde{H}_{i-1}(A \cup L)$ for any $i$.

Second, Milnor theorem shows that $A$ is homotopy equivalent to $S^{n-1}$, and $A \cap L$ is homotopy equivalent to $S^{n-2}$. Thus $H_k(A) = \begin{cases} 0 & \text{for others;} \\
\mathbb{Z} & \text{for } k = 0 \text{ or } n - 1, \end{cases}$

$L$ is homeomorphic to $\mathbb{C}^{n-1}$, so $H_k(L) = 0$ for $k \geq 1$.

Third, we have the Mayer-Vietoris sequence for $A$ and $L$:

$$\cdots \to H_{n-1}(A \cap L) \to H_{n-1}(A) \oplus H_{n-1}(L) \to H_{n-1}(A \cup L) \to H_{n-2}(A \cap L) \to H_{n-2}(A) \oplus H_{n-2}(L) \to \cdots$$

which in the case $n > 2$ is as follows:

$$\cdots \to 0 \to \mathbb{Z} \oplus 0 \to H_{n-1}(A \cup L) \to \mathbb{Z} \to 0 \oplus 0 \to \cdots$$

Therefore $H_{n-1}(A \cup L) \cong \mathbb{Z}^2$.

The case $n = 2$ is obvious.

The same arguments with $n$ replaced by any other dimension show that all groups $\tilde{H}_i(A \cup L)$ with $i \neq n - 1$ are trivial.

\[ \square \]

2 The fundamental group of the space of generic hyperplanes

In this section we calculate the fundamental group $\pi_1(\mathbb{CP}^n \setminus (\mathbb{V} \cup \mathbb{A}))$, and in the next one we describe its action on $H_{n-1}(A \cup L)$.

**Theorem 1.** If $n \geq 3$ then the group $\pi_1(\mathbb{CP}^n \setminus (\mathbb{V} \cup \mathbb{A}))$ is generated by three elements $\alpha, \beta, \kappa$ with relations $\kappa \alpha = \beta \kappa$, $\kappa^2 = 1$.

**Remark 1.** Obviously, this presentation of the group can be reduced to one with only two generators $\alpha, \kappa$ with the single relation $\kappa^2 = 1$. However, the previous more symmetric presentation is more convenient for us.

Denote by $P^* \mathbb{A}$ the set of all hyperplanes in $\mathbb{CP}^{n-1}$, which are tangent to the hypersurface $\partial \mathbb{A} \equiv \mathbb{A} \setminus A$ of “infinitely distant” points of $\mathbb{A}$.
Thus $P^*A = (\overline{A \setminus A})$.

Associating with any affine hyperplane in $\mathbb{C}^n$ its infinitely distant part, we obtain the down-left arrow in the commutative diagram of maps:

$$
\begin{array}{ccc}
\mathbb{C}P^n \setminus (\mathbb{C} \cup ^* A) & \xrightarrow{\text{inclusion}} & \mathbb{C}P^n \setminus ^* A \\
\downarrow & & \downarrow \\
\mathbb{C}P^{n-1} \setminus ^* P^\infty A & \xrightarrow{} & \mathbb{C}P^{n-1} \setminus P^\infty
\end{array}
$$

Indeed, an affine hyperplane belongs to $^* A$ if and only if its image under this map belongs to $P^* A$. On the other hand, the fiber of this map over any point of $\mathbb{C}P^{n-1} \setminus P^\infty$ consists of a pencil of affine hyperplanes parallel to one another, so it is a line bundle. Any such fiber $\mathbb{C}^1$ intersects the set $^\infty A$ at exactly two points: indeed, for any non-asymptotic hyperplane there are exactly two hyperplanes parallel to it and tangent to $A$.

Considering the fiber bundle represented by the left-hand part of the diagram (2),

$$
E \\
\downarrow F \\
B
$$

let $F = \mathbb{C}^1 \setminus \{2 \text{ points}\}, E = \mathbb{C}P^n \setminus (\mathbb{C} \cup ^* A), B = \mathbb{C}P^{n-1} \setminus P^* A$.

We have the exact sequence for the fiber bundle.

$$
... \rightarrow \pi_2(E) \rightarrow \pi_2(B) \rightarrow \pi_1(F) \rightarrow \pi_1(E) \rightarrow \pi_1(B) \rightarrow \pi_0(F)...
$$

(3)

$F$ is connected, so the rightmost arrow is trivial.

**Lemma 1.** If $n > 2$ then $\pi_1(B) = \mathbb{Z}_2$; if $n = 2$ then $\pi_1(B) = \mathbb{Z}$.

Proof. The statement for $n = 2$ is obvious: in this case $B$ is the complex projective line less two points. For $n = 3$ this statement follows by the Zariski
theorem (using the case \( n = 2 \) as the base), see e.g. \([4]\), Chapter 6, §3. Finally, for \( n > 3 \) it follows from the case \( n = 3 \) by the strong Lefschetz theorem, see \([2]\).

\[\begin{align*}
\textbf{Lemma 2.} \quad \text{Let } C \text{ be a smooth quadratic hypersurface in } \mathbb{CP}^{n-1}. \text{ If } n \neq 3 \text{ then } \pi_2(\mathbb{CP}^{n-1} \setminus C) \text{ is trivial. } \\
\pi_2(\mathbb{CP}^2 \setminus C) \cong \mathbb{Z}.
\end{align*}\]

In particular, this is true for the base of our fiber bundle \((3)\).

\begin{proof}
Let \([C] \subset \mathbb{C}^n\) be the union of lines corresponding to the points of \(C\). We have a fiber bundle

\[
\begin{array}{c}
\mathbb{C}^n \setminus [C] \\
\downarrow \\
\mathbb{C}^* \\
\mathbb{CP}^{n-1} \setminus C
\end{array}
\]

This fiber bundle is trivial because it is a restriction of the tautological bundle of \(\mathbb{CP}^{n-1}\) on the complement of a non-trivial divisor, so its first Chern class is equal to 0.

Therefore \(\pi_2(\mathbb{C}^n \setminus [C]) = \pi_2(\mathbb{CP}^{n-1} \setminus C) \oplus \pi_2(\mathbb{C}^*) = \pi_2(\mathbb{CP}^{n-1} \setminus C)\).

Let \(\varphi : \mathbb{C}^n \to \mathbb{C}\) be the quadratic polynomial defining the sets \([C]\) and \(C\). It defines the Milnor fibration \(\varphi : \mathbb{C}^n \setminus [C] \to \mathbb{C}^*\).

Let \(E' = \mathbb{C}^n \setminus [C], B' = \mathbb{C}^*, F' = V_\lambda\). In this notation, \(\pi_2(B) = \pi_2(\mathbb{CP}^{n-1} \setminus C) = \pi_2(\mathbb{C}^n \setminus [C]) = \pi_2(E')\).

We have the exact sequence for the fiber bundle.

\[
\ldots \pi_3(B') \to \pi_2(F') \to \pi_2(E') \to \pi_2(B') \to \pi_1(F') \to \pi_1(E') \to \pi_1(B') \ldots
\]

The base \(B'\) is homotopy equivalent to \(S^1\), in particular the groups \(\pi_3(B')\) and \(\pi_2(B')\) are trivial.

Also, according to the Milnor theorem, \(F'\) is homotopy equivalent to \(S^{n-1}\).
Thus $\pi_2(B) = \pi_2(E') = \pi_2(F') = \pi_2(S^{n-1}) = \begin{cases} 0 & \text{for } n \neq 3; \\ \mathbb{Z} & \text{for } n = 3. \end{cases}$

So for $n \neq 3$ the interesting fragment of the exact sequence (3) reduces to

$$1 \to \pi_1(F) \to \pi_1(E) \to \pi_1(B) \to 1.$$ (4)

**Lemma 3.** In the case $n = 3$ the map $\pi_2(E) \to \pi_2(B)$ in (3) is epimorphic.

**Proof.** By the construction of the generator of the group $\pi_2(B) \sim \mathbb{Z}$ in this case, this generator can be realised by the sphere consisting of complexifications of all oriented planes through the origin in $\mathbb{R}^3$. All these planes do not meet the set $\hat{A} \cup \hat{A}$, and hence define a 2-spheroid in $E$. □

So, the map $\pi_2(B) \to \pi_1(F)$ in (3) is trivial, and we can use the exact sequence (4) also in the case $n = 3$.

$$\pi_1(F) = \mathbb{Z} \ast \mathbb{Z}, \pi_1(B) = \mathbb{Z}_2$$

Thus $\pi_1(E)$ has three generators $\alpha, \beta, \kappa$, where $\alpha$ and $\beta$ are two free generators of $\pi_1(F)$, and $\kappa$ is an element of the coset $\pi_1(E) \setminus \pi_1(F)$.

We can realize these elements as follows. Choose the linear coordinates in $\mathbb{C}^n$ in which $A$ is given by the equation $z_1^2 + \cdots + z_n^2 = 1$. Take for the base point in $\mathbb{CP}^n \setminus (\hat{A} \cup \hat{A})$ the hyperplane $\{z_1 = 0\}$. The fiber $F$ containing this point consists of all complex hyperplanes $\{z_1 = \text{const}\}$ parallel to this one, they are characterized by the corresponding value of $z_1$. The exceptional points of intersection with $\hat{A}$ in this fiber correspond to the values 1 and $-1$.

Then for $\alpha$ and $\beta$ we take the classes of two simplest loops in $\mathbb{C}^1$ going along line intervals from 0 to the points $1 - \varepsilon$ (respectively, $-1 + \varepsilon$), $\varepsilon > 0$ very small, then turning counterclockwise around the point 1 (respectively, $-1$) along a circle of radius $\varepsilon$, and coming back to 0.

For $\kappa$ we take the 1-parameter family of planes given by the equation $(\cos \tau)z_1 + (\sin \tau)z_2 = 0$, $\tau \in [0, \pi]$. 

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Lemma 4. The element $\kappa$ thus defined does not belong to the image of $\pi_1(F)$ in $\pi_1(E)$ under the second map in (4), i.e. its further map to $\pi_1(B)$ defines a generator of the latter group.

Indeed, it is easy to check this in the case $n = 2$, which provides (via the Zariski theorem) the generator of the latter group.

The loop $\kappa$ defines also a loop in the base of our fiber bundle. Moving the fibers over it and watching the corresponding movement of two exceptional points, we get that $\kappa$ acts on $\pi_1(F)$ by permuting $\alpha$ and $\beta$.

Theorem 1 is proved.

3 Monodromy representation

We know that

$$H_n(\mathbb{C}^n, A \cup L) = \mathbb{Z}^2,$$

see Proposition 1.

Proposition 2. For any $n$, the monodromy action of the group $\pi_1(\mathbb{C}P^n \setminus (\hat{A} \cup \hat{A}))$ on $H_n(\mathbb{C}^n, A \cup L)$ has a 1-dimensional invariant subspace.

Proof. This subspace is the image of the group $H_n(\mathbb{C}^n, A) \cong \mathbb{Z}$ under the obvious map $H_n(\mathbb{C}^n, A) \to H_n(\mathbb{C}^n, A \cup L)$; it corresponds via the boundary isomorphism $H_n(\mathbb{C}^n, A \cup L) \to H_{n-1}(A \cup L)$ in (1) to the image of the map $H_{n-1}(A) \to H_{n-1}(A \cup L)$. Indeed, this image does not depend on $L$.

It is convenient to fix the generators of this group (5) as follows. Suppose again that $A$ is given by the equation

$$z_1^2 + \cdots + z_n^2 = 1,$$

and the basepoint $L_0$ in the space of planes is given by $z_1 = 0$. Then we have two relative cycles in $\mathbb{C}^n$ (and even in $\mathbb{R}^n$) modulo $A \cup L_0$: they are given by the two half-balls bounded by the the surface (6) and (the real part of) the hyperplane $L_0$; we supply these half-balls with the orientations induced from a fixed orientation of $\mathbb{R}^n$. It follows immediately from the proof of Proposition 1 that these two chains indeed generate the group $H_n(\mathbb{C}^n, A \cup L) = \mathbb{Z}^2$. 

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Denote these two generators by $a$ and $b$. Namely, $a$ (respectively, $b$) is the part placed in the half-space where $z_1 > 0$ (respectively, $z_1 < 0$). The invariant subspace of the monodromy action is then generated by the sum of these two elements: indeed, it is a relative cycle mod $A$ only.

Let we study the action of loops $\alpha$, $\beta$, and $\kappa$ on $a$ and $b$.

**Proposition 3.** For any $n$, $\kappa(a) = b$, $\kappa(b) = a$.

Proof. This follows immediately from the construction of both cycles $a$ and $b$ and of the loop $\kappa$: when the hyperplane $L_\tau$ moves along this loop, the parts of the space $\mathbb{R}^n$ bounded by the sphere $\{6\}$ and real parts of these hyperplanes move correspondingly and permute at the end of this movement. \(\square\)

**Proposition 4.** If $n$ is odd, then the action of both loops $\alpha$ and $\beta$ is trivial. If $n$ is even, then $\alpha(a) = -a$, $\alpha(b) = 2a + b$, $\beta(b) = -b$, $\beta(a) = 2b + a$.

Proof. Both these statements follow immediately from the Picard–Lefschetz formula, see Chapter III in [5]. \(\square\)

So, in the case of odd $n$ the monodromy action reduces to that of the group $\mathbb{Z}_2$. In the case of even $n$ the monodromy group is infinite: for instance the orbit of any generating element $a$ or $b$ consists of all points of the integer lattice $\mathbb{Z}^2$ satisfying the conditions $u - v = 1$ or $u - v = -1$.  

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