LATTICE COPIES AND APPLICATIONS FOR WEAK L- AND M-WEAKLY COMPACT OPERATORS ON BANACH LATTICES

ZHANGJUN WANG¹ AND ZILI CHEN²

Abstract. Several recent papers investigated lattice copies and unbounded convergences in Banach lattices. In this paper, we first solve the problem of Rincón-Villamizar and Leal-Archila which is an extension of the well-known James distortion theorem. Using lattice copies of $\ell_1$, $\ell_\infty$ and $ba(2^N)$ and unbounded convergence, then we introduce weak L- and M-weakly compact operators on Banach lattices and research the relationship between these operators and L- and M-weakly compact operators. Finally, we study the compactness of weak L-weakly compact and weak M-weakly compact operators.

1. Introduction

Let us begin with some preliminary knowledge to drawing off our research background.

For a set $\Gamma$, $\ell_p(\Gamma)(1 \leq p < \infty)$ stands the Banach space of the all bounded families $(a_\gamma)_{\gamma \in \Gamma}$ endowed with the $\ell_p$ norm. When $\Gamma$ is countable, these spaces are denoted as $\ell_p$. If $X$ and $Y$ are Banach spaces, we recall that $Y$ contains a copy of $X$ whenever $Y$ contains a subspace isomorphic to $X$; we also say that $Y$ contains almost isometric copies of $X$ if for any $\epsilon > 0$, there is an isomorphism $T_\epsilon$ from $X$ into a subspace of $Y$ satisfying $\|T_\epsilon\|\|T_\epsilon^{-1}\| \leq 1 + \epsilon$. Moreover, if $T_\epsilon$ is also lattice isomorphism, $Y$ is said to contains lattice-almost isometric copies of $X$.

For two Banach lattices $E$ and $F$ such that $E$ contains copy of $F$, a problem is determining if $E$ contains lattice-almost isometric copies of $F$. In [3], James proved that if a Banach space $E$ contains a copy of $c_0$ (resp. $\ell_1$), then $E$ contains a almost isometric copy of $c_0$ (resp. $\ell_1$). The corresponding statement for $\ell_\infty$ was proved by Partington in [4]. In [5, 6], Chen showed that if a Banach lattice contains a lattice copy of $c_0$ (resp. $\ell_1$, $\ell_\infty$), then it contains lattice-almost isometric copies of $c_0$ (resp. $\ell_1$, $\ell_\infty$). Recently, Rincón-Villamizar and Leal-Archila solved the problem (in [4]) for $c_0(\Gamma)$, $\ell_1(\Gamma)$ and $\ell_\infty(\Gamma)$ for dual Banach lattice and raise the problem for $\ell_p(\Gamma)$. The aim of Section 1 of this paper

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is the lattice-almost isometric copies of $\ell_p(\Gamma)$ and solve the problems of Rincón-Villamizar and Leal-Archila in [7].

A net $(x_\alpha)$ in a Banach lattice $E$ is unbounded order (resp. norm, absolute weak) convergent to some $x$, denoted by $x_\alpha \xrightarrow{uo} x$ (resp. $x_\alpha \xrightarrow{un} x$, $x_\alpha \xrightarrow{uaw} x$), if the net $(|x_\alpha - x| \wedge u)$ converges to zero in order (resp. norm, weak) for all $u \in E_+$. A net $(x'_\alpha)$ in a dual Banach lattice $E'$ is unbounded absolute weak* convergent to some $x'$, denoted by $x'_\alpha \xrightarrow{uaw^*} x'$, if $|x'_\alpha - x'| \wedge u' \xrightarrow{w^*} 0$ for all $u' \in E'_+$. For the basic theory of $uo$, $un$, $uaw$ and $uaw^*$-convergence, we refer to [8–11].

In [12, 13], we studied the continuity functionals and operators for different types of unbounded convergences in Banach lattices, and showed the characterizations of continuous functionals, L-weakly compact sets, L- and M-weakly compact operators on Banach lattices by $uo$, $uaw$ and $uaw^*$-convergence. Based on the above results, we can find that the $un$-convergence is special. Therefore, we will try to study these sets and operators by $un$-convergence. In Section 3, we use unbounded convergence and and lattice copies of $\ell_1$, $\ell_\infty$ and $ba(2^\mathbb{N})$ to introduce and research new classes of sets and operators so called weak (weak*) L-weakly compact sets, weak (weak*) L-weakly compact operators and weak M-weakly compact operators, which contrast with L-weakly compact sets and L-(M-)weakly compact operators. At the end of the paper, we investigate compactness of weak L-weakly compact and weak M-weakly compact operators by these lattice copies. For undefined terminology, notation and basic theory of Riesz space, Banach lattice and linear operator, we refer to [1, 2].

2. LATTICE-ALMOST ISOMETRIC COPIES OF $\ell_p(\Gamma)$

Let us determine the lattice-almost isometric copies of $\ell_p(\Gamma)$ in Banach lattices. If $\tau$ is an ordinal, $|\Gamma|$ (resp. $[\Gamma]^{<\tau}$) denotes the cardinality of $\Gamma$ (resp. the family of all subsets of $\Gamma$ with cardinality less than $\tau$). We denote by $2^\Gamma$ the set of all subsets of $\Gamma$.

**Theorem 2.1.** A Banach lattice $E$ contains a lattice copy of $\ell_p(\Gamma)$ ($1 \leq p < \infty$) iff it contains lattice-almost isometric copies of $\ell_p(\Gamma)$.

**Proof.** Suppose that $E$ contains a lattice copy of $\ell_p(\Gamma)$, then there exists a disjoint family $(x_\gamma)_{\gamma \in \Gamma}$ in $E_+$ and two positive constants $C_1, C_2$ such that

$$
C_1 \sum_{\gamma \in A} |a_\gamma|^p \leq \left\| \sum_{\gamma \in A} a_\gamma x_\gamma \right\|^p \leq C_2 \sum_{\gamma \in A} |a_\gamma|^p,
$$
for all $A \in 2^\Gamma$ and every family of scalars $\{a_\gamma : \gamma \in \Gamma\}$. Let $\tau$ be an ordinal such that $|\Gamma| = \tau$. For $L \in [\Gamma]^{< \tau}$, put

$$K_L = \inf\{\| \sum_{\gamma \in M} a_\gamma x_\gamma \|^p : \sum_{\gamma \in M} a_\gamma^p = 1, a_\gamma \geq 0, M \in 2^\Gamma, L \cap M = \emptyset\}.$$ 

Clearly, $\{K_L : L \in [\Gamma]^{< \tau}\}$ is bounded. Also, if $L, L' \in [\Gamma]^{< \tau}$ and $L \subset L'$, then $K_L \leq K_{L'}$. Since $C_1 \leq K_L \leq C_2$ for all $L$, so we let

$$K = \sup\{K_L : L \in [\Gamma]^{< \tau}\}.$$ 

Take $0 < \epsilon < 1$ be fixed, and $0 < \theta < 1 < \lambda$ with $\theta/\lambda \geq (1 - \epsilon)^p$. Set $L_0 \in [\Gamma]^{< \tau}$ such that $\theta K < K_{L_0} \leq K < \lambda K$. Thus there is $M_0 \in 2^\Gamma$ with $M_0 \cap L_0 = \emptyset$ and $(b_\gamma^{M_0})_{\gamma \in M_0}$ such that

$$\theta K < K_{L_0} \leq \| \sum_{\gamma \in M_0} b_\gamma^{M_0} x_\gamma \|^p < \lambda K, \sum_{\gamma \in M_0} (b_\gamma^{M_0})^p = 1.$$ 

Now let $\alpha - 1 = \beta$ be an ordinal with $\alpha < \tau$. Assume that $\{M_\eta \in 2^\Gamma : \eta \leq \beta\}$ has been defined in a way such that:

1. for each $\eta \leq \beta$, we have

$$\theta K < K_{L_0} \leq \| \sum_{\gamma \in M_\eta} b_\gamma^{M_\eta} x_\gamma \|^p < \lambda K, \sum_{\gamma \in M_\eta} (b_\gamma^{M_\eta})^p = 1.$$ 

2. The family $\{M_\eta \in 2^\Gamma : \eta \leq \beta\}$ is a family of disjoint finite sets and $M_\eta \cap L_0 = \emptyset$ for each $\eta \leq \beta$.

Since $|\bigcup_{\eta \leq \beta} M_\eta| < |\Gamma|$, so $\Gamma/\bigcup_{\eta \leq \beta} M_\eta \neq \emptyset$. If $N_\beta = \bigcup_{\eta \leq \beta} M_\eta$, then $\theta K < K_{L_0} \leq K_{N_\beta \cup L_0} < \lambda K$. Hence, there exists $M \in 2^\Gamma$ such that $M \cap (N_\beta \cup L_0) = \emptyset$ and

$$\theta K < K_{L_0} \leq K_{N_\beta \cup L_0} \leq \| \sum_{\gamma \in M} b_\gamma^M x_\gamma \|^p < \lambda K, \sum_{\gamma \in M} (b_\gamma^M)^p = 1.$$ 

Therefore, we take $M_\alpha := M$. Now, we suppose that $\alpha$ is a limit ordinal and $\{M_\beta : \beta < \alpha\}$ has been defined. It follows from

$$|\bigcup_{\beta < \alpha} M_\beta| \leq \alpha < \tau = |\Gamma|$$

that $\Gamma/\bigcup_{\beta < \alpha} M_\beta \neq \emptyset$. If $N_\alpha = \bigcup_{\beta < \alpha} M_\beta$, then $\theta K < K_{L_0} \leq K_{N_\alpha \cup L_0} < \lambda K$. So, there is $M' \in 2^\Gamma$ such that $M' \cap (N_\alpha \cup L_0) = \emptyset$ and

$$\theta K < K_{L_0} \leq K_{N_\alpha \cup L_0} \leq \| \sum_{\gamma \in M'} b_\gamma^{M'} x_\gamma \|^p < \lambda K, \sum_{\gamma \in M'} (b_\gamma^{M'})^p = 1.$$ 

Let $M_\alpha := M'$. By the way, we have constructed a family $\mathcal{M} = \{M_\alpha : \alpha < \tau\} = \{M_\gamma : \gamma \in \Gamma\}$ in $2^\Gamma$ and a family $\mathcal{Y} = \{y_{M_\gamma} : \gamma \in \Gamma\}$ in $E_+$ with $|\mathcal{M}| = |\mathcal{Y}| = |\Gamma|$ satisfying
(1) for any $\gamma \in \Gamma$, there exists a set of positive scalars \{${b}_{\gamma}' : \gamma' \in M_{\gamma}$\} for all $\gamma' \in M_{\gamma}$ such that

$$y_{M_{\gamma}} = \sum_{\gamma' \in M_{\gamma}} {b}_{\gamma}' x_{\gamma'}, \sum_{\gamma' \in M_{\gamma}} (b_{\gamma}')^p = 1,$$

and $\theta K < K_{L_0} \leq ||y_{M_{\gamma}}||^p < \lambda K$.

(2) The family $M$ is a family of disjoint finite sets with $M_{\gamma} \cap L_0 = \emptyset$ for each $\gamma \in \Gamma$.

Let $z_\gamma = (\lambda K)^{-1/p}y_{M_{\gamma}}$, clearly, $(z_\gamma : \gamma \in \Gamma)$ is a disjoint family in $B_E^+$. For each $\mu := (\mu_\gamma)_{\gamma \in \Gamma} \in l_p(\Gamma)$, we have

$$\left\| \sum_{\gamma \in \Gamma} \mu_\gamma z_\gamma \right\|^p \leq \left\| \sum_{\gamma \in \Gamma} \mu_\gamma \right\|^p (\sup_{\gamma \in \Gamma} ||z_\gamma||^p) \leq \left\| \sum_{\gamma \in \Gamma} \mu_\gamma \right\|^p.$$

On the other hand,

$$\left\| \sum_{\gamma \in \Gamma} \mu_\gamma z_\gamma \right\|^p = (\lambda K)^{-1} \left\| \sum_{\gamma \in \Gamma} \mu_\gamma y_\gamma \right\|^p$$

$$= (\lambda K)^{-1} \left\| \mu \right\|^p \left\| \sum_{\gamma \in \Gamma} |\mu_\gamma| y_\gamma / ||\mu|| \right\|^p$$

$$\geq (\lambda K)^{-1} K_{L_0} ||\mu||^p \geq (1 - \epsilon)^p ||\mu||^p.$$

Now we define $T_{\epsilon} \mu = \sum_{\gamma \in \Gamma} \mu_\gamma z_\gamma$, it can be easily verified that, $T_\epsilon$ is a lattice embedding from $\ell_p(\Gamma)$ into $E$ such that $(1 - \epsilon)||\mu|| \leq ||T_\epsilon \mu|| \leq ||\mu||$. The proof is completed. \hfill \Box

3.2. WEAK L-WEAKLY COMPACT AND WEAK M-WEAKLY COMPACT OPERATORS

A vector $e$ in a Riesz space $E$ is said to be strong order unit if the ideal $I_e$ generated by $e$ equal to $E$. According to [4], Theorem 4.21 and Theorem 4.29, a Banach lattice $E$ is lattice and norm isomorphic to $C(K)$ for some compact Hausdorff space $K$ whenever $E$ has a strong order unit. It follows from [4], Theorem 2.3 that the un-convergence implies norm convergence in $E$ iff $E$ has strong order unit.

Since every bounded un-null sequence in $l_\infty$ is norm convergent to zero. It is clearly that every un-null sequence in $B_{l_\infty}$ converges uniformly to zero on $B_{l_1}$ and $B_{ba(2^\Gamma)}$. So for the identical operator $I : l_1 \to l_1$, every un-null sequence in $B_{l_\infty}$ converges uniformly to zero on $I(B_{l_1})$ and $I''(B_{ba(2^\Gamma)})$. And $I'x_n' \to 0$ for every un-null sequence $(x_n') \subset B_{l_\infty}$.

Clearly, $B_{l_1}$ and $B_{ba(2^\Gamma)}$ are not L-weakly compact sets, $I$ and $I''$ are not L-weakly compact operators and $I'$ is not M-weakly compact. Therefore, we introduce these sets and operators.

**Definition 3.1.** Let $E$ be a Banach lattice, a bounded subset $A \subset E(B \subset E')$ is called weak(weak*) L-weakly compact set whenever $\sup_{x \in A} |x_n'(x)| \to 0 (\sup_{x' \in B} |x'(x_n)| \to 0)$ for
every bounded \textit{un}-null sequence \( \{x'_n\}\{(x_n)\} \) in \( E'(E) \). Clearly, \( A(B) \) is weak (weak*) L-weakly compact set iff for each sequence \( \{x_n\}\{(x'_n)\} \) in \( A(B) \), \( x'_n(x_n) \rightarrow 0 \) for every bounded \textit{un}-null sequence \( \{x'_n\}\{(x_n)\} \) in \( E'(E) \).

Respectively, let \( X \) be a Banach space. A continuous operator \( T : X \rightarrow E(T : X \rightarrow E') \) is said to be \textit{weak} (weak*) \textit{L-weakly compact} operator if \( T(B_X) \) is weak (weak*) L-weakly compact set in \( E(E') \). Clearly, \( T \) is weak (weak*) L-weakly compact iff \( f_n(Tx_n) \rightarrow 0 \) for every sequence \( (x_n) \subset B_X \) and every bounded \textit{un}-null sequence \( (f_n) \) in \( E'(E) \).

A continuous operator \( T : E \rightarrow Y \) is said to be \textit{weak M-weakly compact operator} whenever \( Tx_n \rightarrow 0 \) for every \textit{un}-null sequence \( (x_n) \subset B_E \).

For an operator \( T : E \rightarrow F \) between two Riesz spaces we shall say that its modulus \( |T| \) exists (or that \( T \) possesses a modulus) whenever \( |T| := T \vee (-T) \) exists. The carrier of \( T \) is denoted by \( CT \) with \( CT := \{x \in E : |T|(|x|) = 0\}^d \). According to [12, Theorem 2.2], the carriers of the \textit{uo-continuous}, \textit{un-continuous}, \textit{uaw-continuous}, \textit{uaw*-continuous} and disjoint continuous operators on atomic Banach lattice are finite-dimensional. Hence, we assume that these unbounded convergence sequences for these operators are norm bounded. The following basic properties of weak and weak* L-weakly compact sets in Banach lattice can be obtained.

\textbf{Proposition 3.2.} For a Banach lattice \( E \), the following statements hold.

(1) Every subset and finite union of weak L-weakly compact set in \( E \) is weak L-weakly compact set in \( E \).

(2) Every subset and finite union of weak* L-weakly compact set in \( E' \) is weak* L-weakly compact set in \( E' \).

(3) The solid convex hull of weak L-weakly compact set in \( E \) is weak L-weakly compact set in \( E \).

(4) The solid convex hull of weak* L-weakly compact set in \( E' \) is weak* L-weakly compact set in \( E' \).

\textbf{Proof.} (1) and (2). Obvious.

(3) For a weak L-weakly compact subset \( A \) of Banach lattice \( E \), let \( Sol(A) \) denote the solid hull of \( A \). \( \sup_{x \in Sol(A)} |x'_n(x)| = \sup_{x \in A} \{|y'_n(x)| : |y'_n| \leq |x'_n|\} \). Clearly, \( y'_n \rightarrow 0 \) since \( x'_n \rightarrow 0 \). So we have \( \sup_{x \in Sol(A)} |x'_n(x)| \rightarrow 0 \), therefore \( Sol(A) \) is also weak L-weakly compact set.

let \( B \) denote the solid convex hull of \( A \) as

\[
B := \left\{ \sum_{i=1}^{n} \lambda_i x_i : x_i \in Sol(A), \lambda_i \geq 0, \sum_{i=1}^{n} \lambda_i = 1 \right\}.
\]
Then \( |x'_n(x)| = |x'_n(\sum_{i=1}^n \lambda_i x_i)| \leq \sum_{i=1}^n |\lambda_i x'_n(x_i)| \to 0 \) for every bounded \( un \)-null sequence \( (x'_n) \) in \( E' \). Hence \( \sup_{x \in B} |x'_n(x)| \to 0 \), therefore \( B \) is weak L-weakly compact set.

(4) is similar to (3). □

According to the above results, it is clear that every L-weakly compact set and operator is weak and weak* L-weakly compact and every M-weakly compact operator is weakly M-weakly compact, but the converse does not hold in general. Then, we consider that when are weak (weak*) L- and M-weakly compact L- and M-weakly compact.

The following results are some characterizations of order continuous Banach lattice by weak L-weakly compact, weak* L-weakly compact and weak M-weakly compact operators.

**Lemma 3.3.** Let \( E \) and \( F \) be Banach lattices, the following holds.

1. A continuous operator \( T : E \to F \) is weak L-weakly compact iff \( T' : F' \to E' \) is weak M-weakly compact.
2. A continuous operator \( T : E \to F \) is weak M-weakly compact iff \( T' : F' \to E' \) is weak* L-weakly compact.

**Proof.** (1) Since \( y'(y) = y'(Tx) = (T'y)(x) \), hence \( \|T'y_n\| = \sup_{y \in T(B_E)} |y_n(y)| \). Assume that \( T \) is weak L-weakly compact, then for a \( un \)-null sequence \( (y'_n) \subset B_{F'} \), we have \( \|T'y'_n\| = \sup_{y \in T(B_E)} |y'_n(y)| \to 0 \), hence \( T' \) is weak M-weakly compact. The converse is similar.

(2) is similar to (1). □

**Theorem 3.4.** Let \( E \) be a Dedekind \( \sigma \)-complete Banach lattice, the following conditions are equivalent.

1. \( E \) has order continuous norm.
2. For each Banach space \( F \), every weak M-weakly compact operator \( T : E \to F \) is \( M \)-weakly compact.
3. Every positive weak M-weakly compact operator \( T : E \to l_\infty \) is \( M \)-weakly compact.
4. For each Banach space \( X \), every adjoint weak* L-weakly compact operator \( T' : X' \to E' \) for continuous operator \( T : E \to X \) is \( L \)-weakly compact.
5. Every positive adjoint weak* L-weakly compact operator \( T' : \text{ba}(2^\mathbb{N}) \to E' \) for continuous operator \( T : E \to l_\infty \) is \( L \)-weakly compact.

**Proof.** (1) ⇒ (2) For a weak M-weakly compact operator \( T : E \to F \) and a bounded disjoint sequence \( (x_n) \subset E \). It is easy to see that \( x_n \overset{un}{\to} 0 \) by \[1\, Proposition 3.5\]. So we have \( Tx_n \to 0 \), therefore \( T \) is \( M \)-weakly compact.

(2) ⇒ (3) Obvious.

(3) ⇒ (1) Assume that \( E \) is not order continuous, it follows from \[1\, Theorem 4.51\] that \( E \) contains lattice copy of \( l_\infty \). According to \[2\, Proposition 1.5.10(1)\], the identical
operator $I : l_\infty \to l_\infty$ can extension to all of $E$. Moreover, $I$ has a positive extension to all of the $E$ by Exercise 1.5.E1. Therefore, there exists a positive projection $P : E \to l_\infty$.

Let $T = P : E \to l_\infty$. For a bounded un-null sequence $(x_n) \subset E$, it follows from Theorem 4.3 that $Tx_n \xrightarrow{un} 0$. Since $l_\infty$ has strong order unit, hence $Tx_n \to 0$ by Theorem 2.3. Hence, $T$ is positive weak M-weakly compact. But, $T$ is not M-weakly compact. Indeed, for the disjoint unit vectors $(e_n)$ of $l_\infty$, $\|Te_n\| = 1 \not\rightarrow 0$, this leads to contradiction. Therefore, $E$ has order continuous norm.

\( (1) \Rightarrow (4) \). Every disjoint sequence $(x_n) \subset E$ is un-null.

\( (4) \Rightarrow (5) \). Obvious.

\( (5) \Rightarrow (4) \) According to Lemma 3.3.

Dually, we have a similar result of dual Banach lattice.

**Theorem 3.5.** Let $E$ be a Banach lattice, then the following statements are equivalent.

1. $E'$ has order continuous norm.
2. For each Banach space $X$, every weak L-weakly compact operator $T : X \to E$ is L-weakly compact.
3. Every positive weak L-weakly compact operator $T : l_1 \to E$ is L-weakly compact.

**Proof.**

\( (1) \Rightarrow (2) \). Every disjoint sequence $(x'_n) \subset E'$ is un-null.

\( (2) \Rightarrow (3) \). Obvious.

\( (3) \Rightarrow (1) \). For a positive weak L-weakly compact operator $T$, according to Lemma 3.3, $T' : E' \to l_\infty$ is positive weakly M-weakly compact. It follows from Theorem 3.4 that $T'$ is M-weakly compact. Therefore, $T$ is L-weakly compact by Theorem 5.64.

The following results are some characterizations of weak (weak*) L-weakly compact sets and weak (weak*) L-weakly compact operators about disjoint sequence.

**Theorem 3.6.** Let $E$ be a Banach lattice, $A$ a bounded solid subset of $E$ and $B$ a bounded solid subset of $E'$. The following statements hold.

1. If $E$ has order continuous norm, then $A$ is weak L-weakly compact iff $x'_n(x_n) \to 0$ for every positive disjoint sequence $(x_n)$ in $A$ and each bounded un-null sequence $(x'_n)$ in $E$.
2. If $E'$ has order continuous norm, then $B$ is weak* L-weakly compact iff $x'_n(x_n) \to 0$ for every positive disjoint sequence $(x'_n)$ in $B$ and each bounded un-null sequence $(x_n)$ in $E$.

**Proof.**

\( (1) \Rightarrow \). Clearly.
(1) ⇐ Let \((x'_n)\) be a bounded \(un\)-null sequence in \(E'\). To finish the proof, we have to show that \(\sup_{x \in A} |x'_n(x)| \to 0\). Assume by way of contradiction that \(\sup_{x \in A} |x'_n(x)| \to \epsilon\) for all \(n\). Note that the equality \(\sup_{x \in A} |x'_n(x)| = sup_{0 \leq x \in A} |x'_n(x)|\) holds, since \(A\) is solid. \(|x'_n| \xrightarrow{w^*} 0\) since \(E'\) is order continuous. Let \(n_1 = 1\). Because \(|x'_n|(4x_{n_1}) \to 0\), there exists some \(1 < n_2 \in \mathbb{N}\) such that \(|x'_n_2|(4x_{n_1}) < \frac{1}{2}\). It is easy to see that we can find a strictly increasing subsequence \((n_k)_{k=1}^{\infty} \subset \mathbb{N}\) such that 
\[ |x'_{n_{m+1}}|\left(4^m \sum_{k=1}^{m} x_{n_k}\right) < \frac{1}{m} \text{ for all } m. \]
Let 
\[ x = \sum_{k=1}^{\infty} 2^{-k} x_{n_k}, y_m = (x_{n_{m+1}} - 4^m \sum_{k=1}^{m} x_{n_k} - 2^{-m} x)^+. \]
According to [1], Lemma 4.35], \((y_m)\) is a disjoint sequence in \(A \cap E_+\). Now, we have 
\[ |x'_{n_{m+1}}|(y_m) = |x'_{n_{m+1}}|(x_{n_{m+1}} - 4^m \sum_{k=1}^{m} x_{n_k} - 2^{-m} x)^+ \]
\[ \geq |x'_{n_{m+1}}|(x_{n_{m+1}} - 4^m \sum_{k=1}^{m} x_{n_k} - 2^{-m} x) \]
\[ = |x'_{n_{m+1}}|(x_{n_{m+1}}) - |x'_{n_{m+1}}|(4^m \sum_{k=1}^{m} x_{n_k}) - 2^{-m}|x'_{n_{m+1}}| x \]
\[ > \epsilon - \frac{1}{m} - 2^{-m}|x'_{n_{m+1}}| x. \]
Let \(m \to \infty\), it is clear that \(2^{-m}|x'_{n_{m+1}}| x \to 0\). Hence, \(|x'_{n_{m+1}}|(y_m) \to 0\). This leads to a contradiction.

The proof of (2) is similar. \(\square\)

Using similar proof methods, we also have the following result.

**Theorem 3.7.** Let \(E\) and \(F\) be Banach lattices, for a positive operator \(T : E \to F\), the following statements hold.

1. If \(F\) has order continuous norm, then \(T\) is weak \(L\)-weakly compact iff \(y_n(Tx_n) \to 0\) for each positive disjoint sequence \((x_n) \subset B_E\) and every bounded \(un\)-null sequence \((y_n)\) in \(F'\).

2. For the positive adjoint operator \(T' : F' \to E'\) of \(T\), if \(E'\) has order continuous norm, then \(T\) is weak* \(L\)-weakly compact iff \(T'y_n(x_n) \to 0\) for each positive disjoint sequence \((y_n) \subset B_{F'}\) and every bounded \(un\)-null sequence \((x_n)\) in \(E\).

**Proof.** (1) \(\Rightarrow\) Obvious.
(1) $\Leftarrow$ Let $(y'_n)$ be an arbitrary bounded un-null sequence in $F'$. $|y'_n| \xrightarrow{w^*} 0$ since $F$ is order continuous. Hence, $|T'(y'_n)(z)| = \sup_{y \in T([-z, z])} |y'_n(y)| \to 0$ for each $z \in E_+$. Without loss of generality, $y'_n \geq 0$ for all $n$. To finish the proof, we have to show that $\sup_{x \in B_E} |y'_n(Tx)| \to 0$. Assume by way of contradiction that $\sup_{x \in B_E} |y'_n(Tx)| \not\to 0$. Then, by passing to a subsequence if necessary, we can suppose that there would exist some $\epsilon > 0$ such that $\sup_{x \in B_E} |y'_n(Tx)| > \epsilon$ for all $n$. Note that the equality $\sup_{x \in B_E} |y'_n(Tx)| = \sup_{0 \leq x \in B_E} \{|y'_n \circ T|(x)| \}$ since $A$ is solid. For every $n$, there exists $z_n$ in $B_E \cap E_+$ such that $|T'(y'_n)(z_n)| > \epsilon$. It is similar to the proof of Theorem 3.3 that there exists a subsequence $(y_n)$ of $(z_n)$ and a subsequence $(g_n)$ of $(y'_n)$ such that

$$|g_n \circ T|(y_n) > \epsilon, |g_{n+1} \circ T|(\sum_{i=1}^n y_i) < \frac{1}{n}.$$ 

Let $x = \sum_{i=1}^\infty 2^{-i}y_i$ and $x_n = (y_{n+1} - 4^n(\sum_{i=1}^n y_i) - 2^{-n}x)^+$, according to [1, Lemma 4.35], $(x_n)$ is positive and disjoint. Hence,

$$|g_{n+1} \circ T|(x_n) \geq |g_{n+1} \circ T|(y_{n+1} - 4^n(\sum_{i=1}^n y_i) - 2^{-n}x) > \epsilon - \frac{1}{n} - 2^{-n}|g_{n+1} \circ T|x.$$ 

Therefore, $|g_{n+1} \circ T|(x_n) \not\to 0$. Clearly, there exists a sequence $(u_n)$ in $E$ satisfying $|u_n| \leq x_n$ such that $|g_{n+1}(T(u_n))| = |g_{n+1} \circ T|(x_n)$. As applications of

$$|g_{n+1}(T(u_n^+))| + |g_{n+1}(T(u_n^-))| \geq |g_{n+1}(T(u_n))| = |g_{n+1} \circ T|(x_n) \not\to 0,$$

we have $g_{n+1}(T(u_n^+)) \not\to 0$. This leads to a contradiction.

The rest of the proof is similar. 

\[ \square \]

4. Compactness of weak L-weakly compact and weak M-weakly compact operators

Compact operator is not weak L-weakly compact and weak M-weakly compact in general. Considering the compact operator (rank is 1) $T : l_1 \to l_\infty$ define as $T(x_n) = (\sum_{n=1}^\infty x_n, \sum_{n=1}^\infty x_n, ...)$ $\in l_\infty$ for each $(x_n) \in l_1$. It is clear that $T$ is not weak (weak*) L-weakly compact and weak M-weakly compact.

Weak (weak*) L-weakly compact and weak M-weakly compact operators are also not compact in general. Considering the identical operator $I : l_1 \to l_1$, $I$ is weak L-weakly compact, $I'$ is weak M-weakly compact and $I''$ is weak* L-weakly compact. But $I$, $I'$ and $I''$ are not compact.

In this section, we research when weak (weak*) L-weakly compact and weak M-weakly compact operator is compact and the converse.
It is easy to see that every semi-compact operator is L-weakly compact whenever the range space is order continuous. Therefore, the following result can be obtained immediately.

**Proposition 4.1.** Let $X$ be a Banach space and $E$ be a Banach lattice, then the following hold.

1. If $E$ has order continuous norm, then every compact operator from $X$ into $E$ is weak $L$-weakly compact.
2. If $E'$ has order continuous norm, then every compact operator from $X$ into $E'$ is weak* $L$-weakly compact.
3. If $E'$ has order continuous norm, then every compact operator from $E$ into $X$ is weak $M$-weakly compact.

Recall that a vector $e > 0$ in a Banach lattice $E$ is an atom if for any $u, v \in [0, e]$ with $u \wedge v = 0$, either $u = 0$ or $v = 0$. In this case, the band generated by $e$ is $\text{span}\{e\}$. Moreover, the band projection $P_e : E \to \text{span}\{e\}$ defined by

$$P_e x = \sup_n (x^+ \wedge ne) - \sup_n (x^- \wedge ne)$$

exists, and there is a unique positive linear functional $f_e$ on $E$ such that $P_e(x) = f_e(x)e$ for all $x \in E$. We call $f_e$ the coordinate functional with the atom $e$. Clearly, the span of any finite set of atoms is also a projection band. A Banach lattice $E$ is called atomic if $E$ has a complete disjoint system consisting of atoms.

The order continuous part $E^a$ of a Banach lattice $E$ is given by

$$E^a = \{x \in E : \text{every monotone increasing sequence in } [0, |x|] \text{ is norm convergent}\}.$$

According to [2, Corollary 2.3.6], it is equivalent to

$$E^a = \{x \in E : \text{every disjoint sequence in } [0, |x|] \text{ is norm convergent}\}.$$

A Banach lattice $E$ is said to be order continuous whenever $\|x_\alpha\| \to 0$ for every net $x_\alpha \downarrow 0$ in $E$. By [2, Proposition 2.4.10], $E^a$ is the largest closed ideal with order continuous norm of $E$.

It is natural to consider that when are weak(weak*) L-weakly and M-weakly compact operator compact. The following results answer the question.

**Theorem 4.2.** For a Banach lattice $F$, the following statements are equivalent.

1. $F^a$ is atomic and $F'$ has order continuous norm.
2. For each Banach space $X$, every weak $L$-weakly compact operator $T : X \to F$ is compact.
For each Banach lattice $E$ without order continuous dual, every positive weak $L$-weakly compact operator $T : E \to F$ is compact.

Proof. (1) $\Rightarrow$ (2) Since $F'$ has order continuous norm, so every weak $L$-weakly compact operator $T : X \to F$ is $L$-weakly compact by Theorem 3.5. According to [2, Proposition 3.6.2], for any $\epsilon > 0$, there exists some $y \in F'_+$ such that

$$T(B_E) \subset [-y, y] + \epsilon \cdot B_E.$$ 

Since $F'_a$ is atomic, it follows from [14, Theorem 6.1(5)] that the order interval $[-y, y]$ is norm compact. Using [1, Theorem 3.1], we have $T(B_E)$ is relatively compact set in $F$, so $T$ is also compact.

(2) $\Rightarrow$ (3). Obvious.

(3) $\Rightarrow$ (1) We claim that $F'$ is order continuous. Assume that $F'$ is not order continuous, then $E'$ and $F'$ contain the lattice copy of $l_1$, moreover there exists a positive projection from $E'$ to $l_1$. Let $R : E \to l_1$ be the positive projection, $S : l_1 \to F$ be the canonical injection from $l_1$ into $F$ and $T = S \circ R : E \to l_1 \to F$. It is clear that $T$ is weak $L$-weakly compact since $T'$ is weak $M$-weakly compact, but not compact. Therefore, $F'$ has order continuous norm.

Then we prove that $F'_a$ is atomic. Since $E'$ is not order continuous then, by [2, Theorem 2.4.14], there is a norm bounded disjoint sequence $(u_n)$ of positive elements in $E$ which does not converge weakly to zero. Without loss of generality, we may assume that $\|u_n\| \leq 1$ for all $n$ and that there are $\phi \in E'_+$ and $\epsilon > 0$ such that $\phi(u_n) > \epsilon$ for all $n$. It follows from [15, Theorem 116.3] that the components $\phi_n$ of $\phi$, in the carriers $C_{u_n}$, form an order bounded disjoint sequence in $E'_+$ such that $\phi_n(u_n) = \phi(u_n)$ for all $n$ and $\phi_n(u_m) = 0$ if $n \neq m$. Note that $0 \leq \phi_n \leq \phi$ for all $n$.

Assume that $F'_a$ is not atomic, it follows from [14, Theorem 6.1] that there exists some $0 \leq y \in F'_a$ such that $[0, y]$ is not norm compact. Now, fix a sequence $(y_n)$ in $[0, y]$ which has no norm convergent subsequence in $F'_a$ and none in $F$.

Define an operator $T : E \to F$ by

$$T(x) = \sum_{n=1}^{\infty} \left( \frac{\phi_n(x)}{\phi(u_n)} \right) y_n$$

for $x \in E$. Note that in view of the inequality

$$\sum_{n=1}^{\infty} \| \left( \frac{\phi_n(x)}{\phi(u_n)} \right) y_n \| \leq \frac{1}{\epsilon} \| y \| \sum_{n=1}^{\infty} \phi_n(|x|) \leq \frac{1}{\epsilon} \| y \| \phi(|x|)$$
for each $x \in E$, the series defining $T$ converges in norm for each $x \in E$ and $T(u_n) = y_n$ for all $n$. Hence the operator $T$ is well defined and it is also easy to see that $T$ is a positive operator.

Since $(y_n)$ has no norm convergent subsequence in $F$, so $T$ is not compact by Grothendieck theorem ([1, Theorem 5.3]). However, $T$ maps norm bounded subset in $E$ to an order bounded subset in $F$. To see this, note that for all $x \in B_E$, we have

$$|T(x)| \leq T(|x|) = \sum_{n=1}^{\infty} \frac{\phi_n(|x|)}{\phi(u_n)} y_n \leq \frac{1}{\epsilon} \left( \sum_{n=1}^{\infty} \phi_n(|x|) \right) y \leq \frac{1}{\epsilon} \phi(|x|) y \leq \frac{1}{\epsilon} \|\phi\| y.$$ 

So $T$ is L-weakly compact operator, hence $T$ is weak L-weakly compact operator. This leads contradiction, so $F^a$ is atomic. □

The following result shows that when do weakly L-weakly compact operators and compact operators coincide.

**Theorem 4.3.** Let $F$ be a Banach lattice, then the following statements are equivalent.

1. $F$ is atomic and both $F$ and $F'$ are order continuous.
2. For each Banach space $X$, every continuous operator $T : X \to F$ is compact operator iff $T$ is weak L-weakly compact.
3. For each Banach lattice $E$ without order continuous dual, every positive operator $T : E \to F$ is compact operator iff $T$ is weak L-weakly compact.

**Proof.** (1) $\Rightarrow$ (2). Since $F$ is order continuous, so every compact operator $T : E \to F$ is weak L-weakly compact. Since $F$ is an atomic Banach lattice with order continuous dual, it follows form Theorem 4.2 that every weak L-weakly compact operator is compact.

(2) $\Rightarrow$ (3). Obvious.

(3) $\Rightarrow$ (1). According to Theorem 4.2, we have $F^a$ is atomic and $F'$ is order continuous. We claim that $F$ has order continuous norm.

Suppose that $F$ is not order continuous, by [2, Corollary 2.4.3], there exists a disjoint sequence $(x'_n) \subset B_{F'}$ such that $x'_n \to 0$ in $\sigma(F', F)$. That is, there is some $y \in F_+$ with $x'_n(y) \to 0$. As $E \neq \{0\}$, we may fix $u \in E$ and pick a $\phi \in E'$ such that $\phi(u) = \|u\| = 1$ holds.

Now, we consider operator $T : E \to F$ defined by

$$T(x) = \phi(x) \cdot y$$

for each $x \in E$. Clearly, $T$ is a positive compact operator (its rank is 1). But it is not an weak L-weakly compact operator. If not, as the singleton $\{u\}$ and $T(u) = \phi(u)y = y$. For a norm bounded disjoint sequence $(y'_n) \subset F'$, clearly, $y'_n(y) \to 0$, but $y'_n \not\to 0$. Therefore,
Then we study that when is weak M-weakly compact operator compact.

**Theorem 4.4.** Let $E$ be a Dedekind $\sigma$-complete Banach lattice, then the following is equivalent.

1. $(E')^a$ is atomic and $E$ has order continuous norm.
2. For each Banach space $Y$, every weak M-weakly compact operator $T : E \to Y$ is compact.
3. For each Banach lattice $F$ without order continuous norm, every positive weak M-weakly compact operator $T : E \to F$ is compact.

**Proof.** (1) $\Rightarrow$ (2). For a weakly M-weakly compact operator $T : E \to Y$. Since $E$ has order continuous norm, according to Theorem 3.4, $T$ is M-weakly compact. It follows from [1, Theorem 5.64] that $T' : Y' \to E'$ is L-weakly compact.

According to [2, Proposition 3.6.2], for any $\epsilon > 0$, there exists some $x' \in (E')^a$ such that $T'(B_{Y'}) \subset [-x', x'] + \epsilon \cdot B_{E'}$.

Since $(E')^a$ is atomic, it follows from [14, Theorem 6.1(5)] that the order interval $[-x', x']$ is norm compact. Using [1, Theorem 3.1], we have $T(B_{Y'})$ is relatively compact set in $E'$, so $T'$ is also compact. It follows Schauder theorem ([1, Theorem 5.2]) that $T : E \to Y$ is compact.

(2) $\Rightarrow$ (3). Obvious.

(3) $\Rightarrow$ (1). First, we prove that $E$ has order continuous norm. Assume that $E$ is not order continuous, then $E$ and $F$ contain the lattice copy of $l_\infty$, moreover there exists a positive projection from $E$ to $l_\infty$. Let $R : E \to l_\infty$ be the positive projection, $S : l_\infty \to F$ be the canonical injection from $l_1$ into $F$ and $T = S \circ R : E \to l_\infty \to F$. Clearly, $T$ is a positive weak M-weakly compact operator, but not compact. Therefore, $E$ has order continuous norm.

Then, we claim that $(E')^a$ is atomic. We assume that $(E')^a$ is not atomic and construct a positive weak M-weakly compact operator from $E$ into $F$ which is not compact.

Since the norm of $F$ is not order continuous, According to [1, Theorem 4.14], there exists a disjoint order bounded sequence $(y_n)$ in $F_+$ which does not converge to zero in norm. We may assume that $0 \leq y_n \leq y$ and $\|y_n\| = 1$ for all $n$ and some $y \in F_+$.

Since $(E')^a$ is not atomic, there exists a $0 \leq \psi \in (E')^a$ such that the order interval $[0, \psi]$ is not norm compact by [14, Theorem 6.1]. Choose a sequence $(\phi_n)$ in $[0, \psi]$ which has no norm convergent subsequence in $E'$. Also, since $\psi \in (E')^a$, by [2, Theorem 2.4.2], the order
interval $[0, \psi]$ is weakly compact. By the Eberlein-Smulian Theorem ([1, Theorem 3.40]), we may assume, by extracting a subsequence if necessary, that $(\phi_n)$ converges weakly to some $\phi \in [0, \psi]$. So $(\phi_n)$ converges weakly* to $\phi$.

Now, define two operators $S, T : E \to F$ by

$$S(x) = \phi(x)y + \sum_{n=1}^{\infty} (\phi_n - \phi)(x)y_n, \quad T(x) = \psi(x)y$$

for each $x \in E$. It follows from the proof of Wickstead in [16, Theorem 1] that $0 \leq S \leq T$ and that $S$ is not compact.

We claim that $S$ is weak M-weakly compact. For this, note that $0 \leq S' \leq T'$ and $T'(h) = h(y)\psi$ for all $h \in F'$. Then for every $h \in B_{F'}$, we have

$$|S'(h)| \leq S'(|h|) \leq T'(|h|) \leq |h|y\psi \leq \|y\|\psi,$$

Since $[-\psi, \psi]$ is L-weakly compact set, so $S'$ is L-weakly compact operator, moreover $S'$ is weak* L-weakly compact operator. According to Proposition 3.3, $S$ is weak M-weakly compact. Therefore, $(E')^a$ is atomic.

Dually, we have:

**Corollary 4.5.** Let $E$ be a Dedekind $\sigma$-complete Banach lattice, then the following statements are equivalent.

1. $(E')^a$ is atomic and $E$ has order continuous norm.
2. For each Banach space $Y$, every weak* L-weakly compact adjoint operator $T' : Y' \to E'$ for continuous operator $T : E \to Y$ is compact.
3. For each Banach lattice $F$ without order continuous norm, every positive weak* L-weakly compact adjoint operator $T' : F' \to E'$ for continuous operator $T : E \to F$ is compact.

The following result shows that when an operator is both compact and weak M-weakly compact.

**Theorem 4.6.** Let $E$ be a Dedekind $\sigma$-complete Banach lattice, then the following conditions are equivalent.

1. $E'$ is atomic and both $E$ and $E'$ are order continuous.
2. For each Banach space $Y$, every continuous operator $T : E \to Y$ is weak M-weakly compact iff $T$ is compact.
3. For each Banach lattice $F$ without order continuous norm, every positive operator $T : E \to F$ is weak M-weakly compact iff $T$ is compact.
Proof. (1) ⇒ (2). For a weakly M-weakly compact operator $T : E \to Y$. Since $E'$ is atomic and $E$ is order continuous, it follows from Theorem 4.4 that $T$ is compact. According to $E'$ is order continuous and Schauder theorem ([1, Theorem 5.2]), the converse is obtained immediately.

(2) ⇒ (3). Obvious.

(3) ⇒ (1) According to Theorem 4.4, we have $(E')^a$ is atomic and $E$ is order continuous. We claim that $E'$ is order continuous.

Assume that $E'$ is not order continuous. There exists a positive projection $P : E \to l_1$. Fix a vector $0 < y \in F_+$. Define the operator $S : l_1 \to F$ as follows:

$$S(\lambda_n) = \left(\sum_{n=1}^{\infty} \lambda_n \right) y$$

for each $(\lambda_n) \in l_1$. Obviously, the operator $S$ is well defined. Let $T = S \circ P : E \to l_1 \to F$ then $T$ is a positive compact operator since $S$ is a finite rank operator (rank is 1). Let $(e_n)$ be the standard basis of $l_1$. Clearly, $e_n \overset{un}{\to} 0$, but $T(e_n) = y \not\to 0$. Hence, $T$ is not a weak M-weakly compact operator. This leads contradiction. Therefore $E'$ is order continuous, moreover $E'$ is atomic. \qed

The dual result is obtained immediately.

Corollary 4.7. Let $E$ be a Dedekind σ-complete Banach lattice, then the following statements are equivalent.

1. $(E')^a$ is atomic and both $E$ and $E'$ are order continuous.
2. For each Banach space $Y$, every adjoint operator $T' : Y' \to E'$ for continuous operator $T : E \to Y$ is weak* L-weakly compact iff $T$ is compact.
3. For each Banach lattice $F$ without order continuous norm, every positive adjoint operator $T' : F' \to E'$ for continuous operator $T : E \to F$ is weak* L-weakly compact iff $T$ is compact.

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1 The first author: School of Mathematics, Southwest Jiaotong University, Chengdu, Sichuan, China, 610000.
   Email address: zhangjunwang@my.swjtu.edu.cn

2 The second author: School of Mathematics, Southwest Jiaotong University, Chengdu, Sichuan, China, 611756.
   Email address: zlchen@home.swjtu.edu.cn