Evolution of a Rotating Black Hole with a Magnetized Accretion Disk

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The effect of an accretion disk on the Blandford-Znajek process and the evolution of a black hole are discussed using a simplified system for the black hole - accretion disk in which the accretion rate is supposed to be dominated by the strong magnetic field on the disk. The evolutions of the mass and the angular momentum of the black hole are formulated and discussed with numerical calculations.

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1 Introduction

The system of a rotating black hole - magnetized accretion disk is an interesting object where the rotational energy of the black hole can be extracted along the field lines supported by the accretion disk. Recently, there has been increasing interest in the possibility that such a system with a stellar black hole can be the central engine for powering gamma - ray bursts via the Blandford-Znajek process originally proposed for AGN. The Blandford-Znajek power for a black hole with mass $M$ and angular momentum $J(=\tilde{a}M^2)$ is given by

$$P_{BZ} = \frac{1}{4} \tilde{a}^2 M^2 B_H^2 f(h),$$

which is estimated to be consistent with the observed cosmological gamma - ray bursts provided a strong magnetic field $B_H \sim 10^{15} \text{G}$ is on the black hole.
Since the power depends on the angular momentum parameter $\tilde{a}$ and the black hole mass $M$, the evolution of the black hole during the powering process may also be important for the time evolution of the power, which, in principle, is related to the complicated temporal structure of the gamma-ray spectrum. In the simple model proposed to explain the temporal structure, one of the important parameters is the evolution of the power in time.

The evolution of a rotating black hole in the Blandford-Znajek process, without the effects of an accretion disk but with a given magnetic field on the horizon, has been discussed in detail by Okamoto. While earlier discussions on the effects of the accretion disk can be found in Park and Vishniac, the effect of the accreting materials on the rotation of the black hole has continued to be an interesting subject.

In this paper, we will discuss the effects of an accretion disk, in which the magnetic torque is supposed to dominate the accretion rate, on black hole evolution. The magnetic field on the disk determines not only the accretion rates of energy and angular momentum into the black hole but also the rates of energy and angular momentum losses from the black hole since it is related to the magnetic field on the horizon. We suppose a very simple-minded model for a black hole-accretion disk, in which the transition region is simplified to a region in which nothing happens (no magnetic coupling between the horizon and the disk, for example) except the flow of accreting matter. The accretion disk is assumed to satisfy the condition for the axisymmetric solution proposed by Blandford in 1976 for an accretion disk in a force-free magnetosphere. Then, we can find an analytic relation between the magnetic fields on the horizon and the inner edge of the accretion disk. However, since there is insufficient information on the origin or the strength of the disk magnetic field, we take it as a parameter for the numerical calculations. Although the transition region between the horizon and the inner edge of the accretion disk is quite complicated, we try to infer some of the qualitative features of black-hole evolution from this simplified analysis.

2 Blandford-Znajek power and accretion rate

We consider the simplified system of a black hole-accretion disk. The environment around the system is assumed to be a force-free magnetosphere with axial symmetry. The accretion disk is known to be a very complicated
object compared to the black hole which is mathematically well defined. Provided with the mass \((M)\), the angular momentum \((J)\), and the charge \((Q)\) of a black hole, we can calculate its various physical quantities: for example, the horizon radius \((r_H)\) and the stable orbit of a test particle. However, the accretion disk is a rather complicated object with a number of physical parameters which define the accretion disk\(^1\). Among them are the shape parameters (thickness of the disk, for example), the density, the pressure, the temperature, and the viscosity parameter \((\alpha)\). While the magnetic field on the horizon is assumed to be well ordered \(^2\), the magnetic field on the disk can be decomposed into ordered and disordered parts. For an accretion disk where the ordered magnetic field dominates the disordered magnetic fields, we adopt axial symmetric and the steady - state solution obtained by Blandford for a force-free magnetosphere \(^1\).

The outward current density along the disk, \(J_{\tilde{\omega}}\), is proportional to the toroidal component of the magnetic field on the disk, \(B_{\phi}\):

\[
2\pi J_{\tilde{\omega}} = -B_{\phi} \tag{2}
\]

where cylindrical coordinates \((\tilde{\omega}, \phi, z)\) are used. The \(z\)–direction is chosen to be perpendicular to the disk. \(\tilde{\omega}\) is the cylindrical radius from the origin. The torque exerted by an annular ring with width \(\Delta r\) due to the Lorentz force is given by

\[
\Delta T = -r 2\pi r J_r B_z \Delta r. \tag{3}
\]

For a steady - state accretion disk with a surface density \(\Sigma\), angular momentum conservation can be written as

\[
\Delta T = \Sigma v_r \frac{\partial (r^2 \Omega)}{\partial r} 2\pi r \Delta r = \dot{M}_+ \frac{\partial (r^2 \Omega)}{\partial r} \Delta r, \tag{4}
\]

where the torque due to the shear force of differential rotation is assumed to be negligible\(^3\). The accretion rate in the above equation is defined by

\[
\dot{M}_+ = \Sigma v_r 2\pi r. \tag{5}
\]

From Eqs.\(^4\) and \(^5\), using the Keplerian angular velocity, \(\partial (r^2 \Omega) / \partial r = r \Omega_{disk}/2\), we get

\[
\dot{M}_+ = 2r B_{\phi} B_z / \Omega_{disk}. \tag{6}
\]

\(^1\)For a nonrelativistic discussion of the disk, \(\tilde{\omega}\) is replaced by a radial distance \(r\) on the disk.
Adopting the axisymmetric solution \(^{10}\),
\[ B_\phi = 2r\Omega_{disk}B_z/c, \]  
we get the accretion rate given by
\[ \dot{M}_+ = 4r^2B_z^2/c. \]  

In this simple schematic model, the poloidal current flowing into the horizon is responsible for the toroidal magnetic field on the horizon and on the inner edge of the accretion disk. If there is no current source or sink in the transition region, current conservation implies that the total current flow, \( I \), onto the black hole should go into the inner edge \( (r_{in}) \) of the accretion disk. Then, Ampere’s law implies
\[ 2MB^H_\phi(\theta = \pi/2) = 4\pi I = \tilde{\omega}(r_{in})B^{disk}_\phi(r_{in}). \]  

On the other hand, from the boundary conditions on the horizon\(^3\) in the optimal case\(^3\),
\[ B^H_\phi = -\Omega_H MB_H, \]  
and on an accretion disk\(^{10}\) with angular velocity \( \Omega_D \),
\[ B^{disk}_\phi = -2\Omega_D rB_z, \]  
we get the relation between the poloidal magnetic fields on the horizon and the disk’s inner edge :
\[ \frac{B_z(r_{in})}{B^H_\phi} = \sqrt{\frac{GM}{r_{in}c^2}} \frac{\tilde{\omega}GMr_{in}}{2r_Hc^2\tilde{\omega}}, \]  
which implies that the poloidal magnetic field on the horizon is larger than that on the inner edge of the disk\(^4\). A similar observation has been discussed by Ghosh and Abramowicz\(^{14}\) by using the field configuration motivated by MacDonald\(^{15}\). However, it is not clear yet whether these field configurations can be realized by the conventional accretion dynamo\(^{16}\).

Using Eqs.\(^6\) and \(^{12}\), we can write the accretion rate in terms of the magnetic field on the horizon,
\[ \dot{M}_+ = \frac{Mr_{in}^2}{\tilde{\omega}r_H^2}M^2\tilde{a}^2B^2_H. \]  

\(^2\)The magnetic braking power from the disk in connection with the Blandford-Znajek power from the black hole has been discussed recently by Li\(^{12}\) and by Lee, Brown, and Wijers\(^{13}\).
The Blandford-Znajek power is given by

\[ P_{BZ} = \frac{1}{4} \tilde{a}^2 M^2 B_H^2 f(h), \]  

which can be written in terms of the accretion rate in Eq. (13) as

\[ P_{BZ} = \frac{1}{4} \omega^2 r_H^2 M \dot{M} f(h), \]  

where

\[ f(h) = \frac{1 + h^2}{h^2} [(h + \frac{1}{h}) \arctan h - 1], \]  

\[ h = \frac{\tilde{a}}{H}, \quad H = 1 + \sqrt{1 - \tilde{a}^2}. \]  

The horizon radius is \( r_H = H M \).

3 Energy and angular momentum accretions into a black hole

The physics in the transition region may be quite complicated such that it is not certain how much energy and angular momentum can be accreted into the black hole horizon. It depends on the magnetic field and also on the particular types of accretions. In this work with a simple schematic model of a black hole and an accretion disk, it is assumed that the inner edge of the accretion disk is the last stable orbit defined by

\[ r_{in} = Z M, \]  

where

\[ Z = 3 + Z_2 - \sqrt{(3 - Z_1)(3 + Z_1 + 2Z_2)}, \]  

\[ Z_1 = 1 + (1 - \tilde{a}^2)^{\frac{1}{3}}[(1 + \tilde{a})^{\frac{1}{3}} + (1 - \tilde{a})^{\frac{1}{3}}], \]  

\[ Z_2 = \sqrt{3\tilde{a}^2 + Z_1^2}. \]  

It is assumed that the energy of a particle in the last stable orbit is accreted into the horizon without any radiation outward. The specific energy \( \tilde{E} \) (energy per unit rest mass, \( \tilde{E} = m\tilde{E} \)) at \( r_{in} \) is given by

\[ \tilde{E} = \frac{Z^2 - 2Z + \tilde{a} \sqrt{Z}}{Z \sqrt{Z^2 - 3Z + 2\tilde{a} \sqrt{Z}}}. \]
One can see that the accreted energy becomes smaller if the black hole is rotating: $\tilde{E}(\tilde{a} = 0) = 0.94 \rightarrow \tilde{E}(\tilde{a} = 1) = 0.58$. The rest of the specific energy is assumed to be dissipated during the accreting processes. It is interesting to note that for advection-dominated accretion\[17\], the specific energy accreted should be taken to be larger than $\tilde{E}$.

The specific angular momentum ($l = m\tilde{l}$) at the last stable orbit, which is carried into the horizon, is given by

$$\tilde{l} = M Z_l(\tilde{a}),$$

where

$$Z_l(\tilde{a}) = \frac{Z^2 - 2\tilde{a}\sqrt{Z} + \tilde{a}^2}{\sqrt{Z(Z^2 - 3Z + 2\tilde{a}\sqrt{Z})}}.$$ \hspace{1cm} (24)

In this simplified work, we assume that there is nothing particular between the horizon and the inner edge of the disk except the accretion flow, although recent works\[9\] show that energy and angular momentum extraction from a black hole onto a disk is possible, which makes $\tilde{E}$ and $\tilde{l}$ smaller.

### 4 Evolution of a black hole

The evolution rates of the black hole mass ($\dot{M}$) and the angular momentum ($\dot{J}$) are determined both by the accreted energy and the angular momentum, which increase the mass and the angular momentum, and by the Blandford-Znajek power in the opposite direction. From energy conservation, we get

$$\dot{M} = -P_{BZ} + \dot{M}_+ \tilde{E}$$

$$= (\tilde{E} - \tilde{P}_E) \dot{M}_+,$$ \hspace{1cm} (25)

$$= (\tilde{E} - \tilde{P}_E) \dot{M}_+,$$ \hspace{1cm} (26)

where

$$P_{BZ} = \dot{M}_+ \tilde{P}_E,$$ \hspace{1cm} (27)

$$\tilde{P}_E = \frac{f(h)H^2}{4Z}[1 + (\tilde{a}/Z)^2(1 + 2/Z)].$$ \hspace{1cm} (28)

Angular momentum conservation leads to

$$\dot{J} = -\frac{P_{BZ}}{\Omega_F} + \dot{M}_+ \tilde{l}.$$ \hspace{1cm} (29)
The first term on the right hand side is the angular momentum loss rate due to the Blandford-Znajek process, and the second term is the accreted angular momentum from the disk. Equation (29) can be written as

\[ \dot{J} = M \dot{M}_+ (Z_t - \tilde{P}_t), \]  

where

\[ \tilde{P}_t = \frac{f H^3}{\tilde{a}Z}[1 + (\tilde{a}/Z)^2(1 + 2/Z)]. \]  

The evolution of the angular momentum parameter \( \tilde{a} \) can be obtained from the angular momentum evolution as

\[ \frac{\dot{J}}{J} = \frac{\tilde{a}}{a} + \frac{\dot{M}}{M} \]  

or

\[ \dot{J} = (2 \frac{\dot{M}_+}{M} \tilde{a} + \dot{\tilde{a}}) M^2. \]  

Then, using Eq. (26), we get

\[ \dot{\tilde{a}} = \frac{\dot{M}_+}{M} [(Z_t - \tilde{P}_t) - 2\tilde{a}(\tilde{E} - \tilde{P}_E)], \]  

where the accretion rate is given by

\[ \dot{M}_+ = \frac{Z}{H^2[1 + (\tilde{a}/Z)^2(1 + 2/Z)]} M^2 \tilde{a}^2 B_H^2. \]  

Since the system of a black hole - accretion disk we are considering for the central engine of the gamma-ray bursts is supposed to emerge in the final stage of the binary - merging processes like neutron star - neutron star and black hole - neutron star, the mass of the accretion disk should not be much greater than the solar mass and the lifetime (\( \tau \)) of an accretion disk with appreciable pressure for a strong magnetic field cannot be of an astronomical scale. From the large accretion rate due to the strong magnetic field, it is assumed to be less than a few thousand seconds. Since the presence of an accretion disk with an appreciable pressure or magnetic field is essential for the magnetic field on the horizon, the evolution of the accretion disk (lifetime with appreciable pressure in it) might be also responsible for the evolutions of the magnetic field on the disk and \( B_H \).
For the numerical calculations in this work, we tried to incorporate the effects of mass loss from the disk into the the time dependence of the magnetic field in the following form:

\[ B_H^2 = B_H^2(0)D(t), \]  

(36)

where we take

\[ D(t) = 1 - \left( \int_0^t \dot{M}_+/M_\odot \right). \]  

(37)

We take \( D(t) \) to be vanishing as the total accreting mass becomes a solar mass, which is supposed be a typical mass for an accretion disk emerging out of binary-merging processes.

5 Results and discussion

The evolution of the black hole is calculated with \( M(0) = 3M_\odot \) and \( B_H(0) = 10^{15}\text{G} \) and is shown in Fig. 1 for mass (\( M \)) and in Fig. 2 for angular momentum (\( \tilde{a} \)). When we take the initial angular momentum parameter to be \( \tilde{a}(0) = 1 \), the mass of the black hole increases up to \( \sim 3.6M_\odot \). Compared to the initial mass sum of \( 4M_\odot \), \( \sim 10\% \) of the rest-mass energy of the system is extracted by the Blandford-Znajek process, which is about the same fraction of energy that can be taken from the black hole only[3]. The interesting observation is that the rotation of the black hole does not stop, \( \tilde{a} \sim 0.82 \), even after the disk disappears (equivalently, no magnetic field on the black hole or no Blandford-Znajek process). For a system of only a black hole with a constant magnetic field on it, the angular momentum decreases rapidly within a few thousand seconds, and in the optimal process, the fraction of rotational energy extracted is about 9%[3]. This implies that the energy extracted from the black hole in the black hole-accretion disk system is not only from the black hole’s initial rotational energy but also from the energy accreted from the disk. The Blandford-Znajek power shown in Fig. 3 drops very rapidly within \( \sim 1000\text{s} \), as expected, and the pattern of the evolution with time is similar to the case with the black hole alone[3].

In this paper, we formulated the evolutions of the black hole mass and angular momentum, taking into account the flow of accreting matter from a strongly magnetized disk. Adopting the axisymmetric solution suggested by Blandford, we got analytic formulae for the evolutions and could see that the angular momentum accreted from the disk is comparable to the amount of
angular momentum extracted by the Blandford-Znajek process. A numerical calculation demonstrated that the accreted angular momentum was large enough to keep the black hole rotating rapidly while the Blandford-Znajek process was working.

Since we considered a very simplified situation, there are several issues to be discussed in the future. For the numerical calculation, we assumed a specific time dependence of the magnetic field on the disk, which was designed to be vanishing as the disk mass decreased. Although we think the general feature of this assumption may not be totally wrong, at least, for the isolated black hole - accretion disk system, we need more detailed analysis to find the relation between the magnetic field and the disk properties. Also, for a realistic accretion disk, we should consider the effects of the disordered magnetic field, the turbulence, and/or advection - dominated flow\cite{11}. It should be also noted that there are several works\cite{12, 21} demonstrating some difficulties in building up a strong poloidal magnetic field on the disk by means of conventional accretion and dynamo processes. In this work, we have not taken into account the recent observations that negative energy flow out to the disk from the black hole along the accreting flow is possible when there is a strong enough magnetic field in the transition region\cite{2}. That, together with a relativistic generalization of the Blandford solution and the accretion formulae used in this work, would be an interesting problem for future work.

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Figure 1. The evolutions of a black hole with $M(0) = 3M_\odot$ and $\dot{a}(0) = 1$
for a black hole only (solid line) and for a black hole - accretion
disk system (dotted line).

Figure 2. The evolution of the angular momentum in units of $M^2$.

Figure 3. The time dependences of the Blandford-Znajek power.
Figure 1:
Figure 3: