The effective theory of low frequency fluctuations of selfinteracting scalar fields is constructed in the broken symmetry phase. The theory resulting from integrating fluctuations with frequencies much above the spontaneously generated mass scale \( p_0 >> M \) is found to be local. Non-local dynamics, especially Landau damping emerges under the effect of fluctuations in the \( p_0 \sim M \) region. A kinetic theory of relativistic scalar gas particles interacting via their locally variable mass with the low frequency scalar field is shown to be equivalent to this effective field theory for scales below the characteristic mass, that is beyond the accuracy of the Hard Thermal Loop (HTL) approximation.

1 Effective equation for the slow modes

The model investigated in this paper has the Lagrangian density

\[
L = \frac{1}{2}(\partial_\mu \varphi(x))^2 - \frac{1}{2}m^2 \varphi^2(x) - \frac{\lambda}{24} \varphi^4(x).
\]  

(1)

The field \( \varphi(x) \) is split into the sum of terms with low \( (p_0 < \Lambda) \) and high \( (p_0 > \Lambda) \) frequency Fourier-components, that is \( \varphi(x) = \tilde{\Phi}(x) + \phi(x), \langle \phi(x) \rangle = 0. \)

The classical equation of motion for the low frequency component \( \tilde{\Phi}(x) \) is the following:

\[
(\partial^2 + m^2)\tilde{\Phi}(x) + \frac{\lambda}{6} [\tilde{\Phi}^3(x) + 3\tilde{\Phi}(x)\phi^2(x) + 3\tilde{\Phi}^2(x)\phi(x) + \phi^3(x)] = 0.
\]  

(2)

The effective equation of motion arises upon averaging over the (quantum) fluctuations of the high frequency field \( \phi(x) \):

\[
(\partial^2 + m^2)\tilde{\Phi}(x) + \frac{\lambda}{6} [\tilde{\Phi}^3(x) + 3\tilde{\Phi}(x)\langle \phi^2(x) \rangle] = 0.
\]  

(3)

The last term on the left hand side represents the source induced by the action of the high frequency modes. It is a functional of \( \tilde{\Phi}(x) \). The action of the effective field theory may be reconstructed from this equation. This approach is
closely related to the Thermal Renormalisation Group equation of D’Attanasio and Pietroni.

In the broken phase the non-zero average value spontaneously generated below $T_c$ is separated from the low-frequency part, $\Phi(x) = \bar{\Phi} + \Phi(x)$. The expectation value $\bar{\Phi}$ is determined by the effective equation

$$m^2 + \frac{\lambda}{2} \langle \phi^2(x) \rangle^{(0)} + \frac{\lambda}{6} \bar{\Phi}^2 = 0,$$

where we have introduced the indexed expectation value

$$\langle \phi^2(x) \rangle^{(j)} \sim \Phi^j$$

to denote that part of the full expectation value which is "proportional" functionally to the $j$-th power of $\Phi(x)$.

Our present goal is to determine the effective linear dynamics of the $\Phi$-field, therefore it is sufficient to study the linearised effective equation for $\Phi(x)$:

$$(\partial^2 + \frac{\lambda}{3} \bar{\Phi}^2)\Phi(x) = -\frac{\lambda}{2} \bar{\Phi} \langle \phi^2(x) \rangle^{(1)}.$$  

(In this equation $\bar{\Phi}$ is the solution of (4)). Clearly, the linear response theory of (1) is contained in the induced current, determined by $\langle \phi^2(x) \rangle^{(1)}$.

2 Statistics of the high frequency modes

For the computation of the leading effect of the high-frequency modes in the low frequency projection of the equation of motion it is sufficient to study the two-point function $\langle \phi(x)\phi(y) \rangle$. For its determination we follow the procedure carefully described by Mrówecki and Danielewicz.

Introducing the Wigner transform by the relation

$$\langle \phi(x)\phi(y) \rangle = \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} \Delta(X, p), \quad X = \frac{x+y}{2}.$$  

one arrives at the following equations for $\Delta(X, p)$

$$(\frac{1}{4} \partial^2_X - p^2 + M^2(X))\Delta(X, p) = 0,$$

$$\langle \phi^2(x) \rangle^{(1)} = \int \frac{d^4p}{(2\pi)^4} \Delta^{(1)}(x = X, p).$$  

2
The important limitation on the range of validity of the effective dynamics is expressed by the assumption that the second derivative with respect to $X$ is negligible relative to $p^2$ and $M^2$ on the left hand side of the first equation of (8). Then this equation is transformed simply into a local mass-shell condition, while the second equation of (8) can be interpreted as a Boltzmann-equation for the phase-space “distribution function” $\Delta(X, p)$.

The background-independent solution $\Delta^{(0)}$ is given by the well-known free correlator, slightly modified to account for the lower frequency cut appearing in the Fourier series expansion of $\phi(x)$:

$$\Delta^{(0)}(X, p) = (\Theta(p_0) + \tilde{n}(0))2\pi\delta(p^2 - M^2), \quad \tilde{n}(0) = \frac{1}{e^{\beta|p_0|} - 1} \Theta(|p_0| - \Lambda). \quad (10)$$

An iterated solution of the second equation of Eq.(8) starting from (10) yields

$$\Delta^{(1)}(X, p) = -\lambda^2 \Phi(p\partial_X)^{-1} \partial_{X\mu} \Phi \partial_{p\mu} \Delta^{(0)}(X, p). \quad (11)$$

3 The induced source

For the analysis of the induced source

$$j_{ind}(x) = \frac{\lambda}{2} \Phi(\phi^2(x))^{(1)} = \frac{\lambda^2 \bar{\Phi}^2}{4} \int \frac{d^4p}{(2\pi)^4} (p\partial_x)^{-1} \partial_x \Phi \partial_p \Delta^{(0)}(x, p) \quad (12)$$

one takes its Fourier-transform with respect to $x$. Using the explicit expression (10) one easily recognizes the only non-trivial (non-local) contribution arises from the integral

$$\int \frac{d^4p}{(2\pi)^4} \frac{1}{pk_0} \delta(p^2 - M^2) \frac{d\tilde{n}}{dp_0}. \quad (13)$$

Its imaginary part determines the rate of Landau-damping. Simple integration steps lead to

$$\text{Im} j(k_0, k) = -\frac{\lambda^2 \bar{\Phi}^2}{4e^{\beta|k_0|}} \Phi(k) \Theta(k^2 - k_0^2) \int_{t_0}^{\infty} dt \frac{d\tilde{n}}{dt},$$

$$\tilde{n}(t) = \frac{1}{e^{\beta Mt_0 - 1}} \Theta(Mt - \Lambda), \quad t_0 = 1/\sqrt{1 - (k_0/k)^2}. \quad (14)$$

This integral is zero if $\Lambda > Mt_0$, but for $\Lambda < Mt_0$ it gives

$$\text{Im} j(k_0, k) = \frac{\lambda^2 \bar{\Phi}^2}{16\pi} \frac{k_0}{k} \Phi(k) \Theta(k^2 - k_0^2) \frac{1}{e^{\beta M/\sqrt{1 - (k_0/k)^2} - 1}}. \quad (15)$$

independent of the value of $\Lambda$. 

3
The result has very transparent interpretation. In the HTL-limit, when only the modes with much higher frequencies than any mass scale in the theory are taken into account, no Landau-damping arises. The effective theory is local!

Going beyond HTL, ($k_0 << \Lambda << M$) the correct non-local dynamics (reflected also by the Landau damping) originating from the 1-loop self-energy contribution is recovered, when comparison with Boyanovsky et al. is made.

4 Classical mechanical representation of the non-local dynamics

Recently we proposed to superimpose on the scalar field theory (1) a gas of relativistic scalar particles with the action

$$S_{\text{mech}} = -\sum_i \int d\tau M_{\text{loc}}[\Phi, \Phi(\xi_i(\tau))], \quad M_{\text{loc}}^2[\Phi(\xi_i)] = m^2 + \frac{\lambda}{2} \Phi^2(\xi_i(\tau)), \quad (16)$$

where $\xi_i(\tau)$ denotes the world-line of the $i$-th particle of the gas. The mass of these particles varies with the field along the trajectory of the particles. The equation of motion of one of the particles is given by

$$M_{\text{loc}}(\xi) \frac{dp_\mu}{d\tau} = \frac{1}{2} \frac{\partial M_{\text{loc}}^2(\xi)}{\partial \xi_\mu}. \quad (17)$$

The kinetic equation for the collisionless evolution of the one-particle phase-space density $f(x, p)$ of this gas is

$$p_\mu \frac{\partial f(x, p)}{\partial x_\mu} + M_{\text{loc}} \frac{dp_\mu}{d\tau} \frac{\partial}{\partial p_\mu} f(x, p) = 0, \quad (18)$$

which clearly agrees with the second equation of (8), while the solution of the first one has dictated the choice of the effective mass expression in (16).

Variation of (16) with respect to $\Phi(x)$ leads to the term corresponding to the induced source density in the wave equation:

$$j_{\text{ind}} = \frac{\lambda}{2} \int \frac{d^3p}{(2\pi)^3p_0}(\Phi + \Phi(x))f(x, p) \quad (19)$$

($p_0^2 = p^2 + M^2$). This expression is obtained by averaging the contributions of the different particle-trajectories in the gas, passing nearby the point $x$ with any momentum $p$:

$$\langle \int d\tau \sum_i \delta^{(4)}(x - \xi_i(\tau)) \rangle = M_{\text{loc}} \int \frac{d^3p}{(2\pi)^3p_0} f(x, p). \quad (20)$$
Using the solution of the Boltzmann equation obtained upon iteration starting from the equilibrium Bose-Einstein factor, one finds the same expression for the non-local part of the source, as was found above implying the same result also for the rate of Landau damping.

5 Conclusion

We have presented two equivalent methods of treating the effective dynamics of the low-frequency fluctuations of self-interacting scalar fields in the broken phase of the theory. The dynamics is proved to be non-local if the effect of fluctuation modes below the mass scale $M$ is also taken into account. A fully local representation was proposed by superimposing a relativistic gas with specially chosen field dependent mass on the original field theory.

The range of validity of the fully local version of the effective model can be ascertained only from its comparison with the result of lowering the separation scale $\Lambda$ in the detailed integration over the fluctuations with different frequencies. From the comparison one learns that the combined kinetic plus field theory is equivalent to the effective theory for the modes with $k_0 << \Lambda < M$, that is its validity goes beyond the Hard Thermal Loop approximation.

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