GLOBAL ATTRACTOR FOR WEAKLY DAMPED GKDV EQUATIONS IN HIGHER SOBOLEV SPACES

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Abstract. Long time behavior of solutions for weakly damped gKdV equations on the real line is studied. With some weak regularity assumptions on the force $f$, we prove the existence of global attractor in $H^s$ for any $s \geq 1$. The asymptotic compactness of solution semigroup is shown by Ball's energy method and Goubet's high-low frequency decomposition if $s$ is an integer and not an integer, respectively.

1. Introduction. The $k$-generalized Korteweg-de Vries ($k$-gKdV) equation reads

$$
\begin{align*}
\frac{\partial}{\partial t} u + \partial_x^3 u + u^k \partial_x u &= 0, \quad x, t \in \mathbb{R}, \quad k \in \mathbb{Z}^+\\
u(x, 0) &= u_0(x).
\end{align*}
$$

The case $k = 1$ is the classical KdV equation, which models unidirectional propagation of nonlinear dispersive long waves\cite{12}. It is a completely integrable system, i.e., it can be solved by inverse scattering. The case $k = 2$ is called modified Korteweg-de Vries (mKdV) equation. The solutions of KdV and mKdV are connected by Miura transformation:

$$u_{mKdV} = M(u_{KdV}) = i \sqrt{6} \partial_x u_{KdV} + u_{KdV}^2.$$

Thus, mKdV is also a complete integrable system. But this does not hold for $k$-gKdV equations with $k \geq 3$, see e.g. \cite{13, 17}. It is proved in \cite{16, 18} that the solutions of $4$-gKdV associated with large initial data may blow up in finite time. Similar conclusions for $k$-gKdV with $k > 4$ are conjectured in \cite{17}. Therefore, $3$-gKdV is particular interesting as it is not a complete integrable and has global solutions. In fact, the well posedness, scattering and soliton stability of $3$-gKdV are considered in \cite{6, 11, 19, 30}. Motivated by these works, in this paper we consider the long time dynamics of solutions for the following forced $3$-gKdV equation with damping

$$
\begin{align*}
\frac{\partial}{\partial t} u + \partial_x^3 u + u^3 \partial_x u + \lambda u &= f(x), \quad t > 0, x \in \mathbb{R}\\u(x, 0) &= u_0(x).
\end{align*}
$$

Here, $\lambda > 0$ is a constant, $f$ is a given function independent of time, $u_0 \in H^s$. We shall first prove the following well posedness of (1).

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Theorem 1.1. Let $s \geq 1$, $f \in L^2 \cap H^{s-3}$. Then (1) has a global solution $u \in C([0, \infty); H^s)$, for any $T > 0$

$$\sup_{0 \leq t \leq T} \|u(t)\|_{H^s} \leq C(T, \lambda, f, u_0).$$

Moreover, for any $t > 0$, the map $S(t) : u_0 \mapsto u(t)$ is continuous from $H^s$ to $H^s$.

According to Theorem 1.1, $(H^s, S(t))$ defines a dynamical system in the terminology used by Chueshov and Lasiecka [2]. Now we state the main result on the asymptotic behavior of $\{S(t)\}_{t \geq 0}$.

Theorem 1.2. Let $s \geq 1$ and $f \in L^2 \cap H^\sigma$, $\sigma$ is given by

$$\sigma = \begin{cases} s - 3, & s \text{ is an integer,} \\ s - 3+, & \text{otherwise} \end{cases}$$

Then $\{S(t)\}_{t \geq 0}$ has a global attractor $\mathcal{A}$ in $H^s$.

Theorem 1.2 asserts the existence of the global attractor for 3–gKdV equation in higher sobolev spaces. This topic has been studied for other equations in references. Let $k \geq 1$ be an integer. The existence of the global attractor in $H^k_{\text{per}}$ for periodic BBM equation was obtained by Wang [32], that in $H^k \times H^k$ for Klein-Gordon-Schrödinger equation was proved in [14]. Moise and Rosa showed [20] that periodic KdV equation with force $f \in H^k_{\text{per}}$ has an global attractor in $H^k_{\text{per}}$, see also [31]. Similar results are announced in [5] for KdV equation on the real line. The proofs in [5, 20, 31] rely highly on the fact that KdV equation possesses an infinite number of polynomial invariants. However, this is not the case for 3–gKdV equation. Thus, their methods could not apply to our case directly. Moreover, unlike results in previous references, Theorem 1.2 also holds in fractional Sobolev spaces $H^s$. In this case, even the existence of an absorbing set in $H^s$ is not a trivial. Inspired by Bourgain[1], Staffilani [25] and Sohinger [24] on the growth norms for NLS and KdV, we overcome the difficulty by some dispersive estimates of Airy group and an iteration argument.

The existence of global attractor for KdV equations in $H^k$ is usually obtained with the same regular force $f \in H^k$, see Goubet and Rosa [5] and Rosa [22] for $k = 0, 1$, respectively. It’s should be noted that Theorem 1.2 allows the force belong to rough spaces below $H^s$. Results in similar manner are obtained for reaction diffusion equations[28, 36], strongly damped wave equations [35] and other equations [26, 27]. The theorem is sharp in sense that the equilibrium of (1) is at most $H^s$ if $f$ belongs to $H^{s-3}$. Thus, if $s$ is an integer, there is no asymptotic smoothing effect as [5]. Fortunately, in this case we can prove the asymptotic compactness of $\{S(t)\}_{t \geq 0}$ by Ball’s energy argument, and then obtain Theorem 1.2. However, this strategy seems difficult to be used if $s$ is not an integer. In this setting, the nonlinear term $u^3 u_x$ is linked to

$$\int_0^T e^{-\lambda (T-s)} (D^s (u^3 u_x), D^s u) d\tau$$

in the energy equation. Here $D^s$ denotes the $s$ order derivative (see next section for definition). Note that it is a nonlocal operator, nothing can be deduced from integration by parts. This difficulty shall be overcome by a fractional Leibniz rule in mixed type Lebesgue spaces when we derive the existence of global solutions and absorbing sets. The technique is still not sufficient to prove the corresponding
convergence of this term in the energy equation. This is the reason why we suppose that \( f \in H^{s-3} \), i.e., slightly more regular than integral case. We then adopt the high-low frequency decomposition of Goubet to show that \( \{S(t)\}_{t \geq 0} \) possesses a slightly asymptotic smoothing effect. Combining this fact and a tail estimates in \( L^2 \), we get the \( \omega \)-limit compactness of solution semigroup.

The paper is organized as follows. In section 2, we first recall the definition of some function spaces as well as some inequalities, and then show the local well posedness of \( 3-gKdV \) in \( H^s \). In section 3, we prove a uniform bound for solutions in \( H^1 \). Based on this estimate, we establish the existence of global solutions and absorbing sets in \( H^s \). In section 4 and section 5, we verify the existence of a global attractor in \( H^s \) if \( s \) is an integer and not an integer, respectively.

2. Preliminaries.

2.1. Function spaces. We first say a few words about the notations. We use \( H^s \), \( s \in \mathbb{R} \), to denote the Bessel potential space \( H^s(\mathbb{R}) \) defined by

\[
\| \varphi \|_{H^s} := \| \langle \xi \rangle^s \hat{\varphi} \|_{L^2(\mathbb{R})} < \infty
\]

where \( \hat{\varphi} = \mathcal{F} \varphi \) is the Fourier transform of \( \varphi \), \( \langle x \rangle = (1 + |x|^2)^{1/2} \). We also denote by \( s + (s-) \) that a constant equals \( s \) plus (minus) a small enough number, \( A \lesssim B \) means \( A \leq CB \) for some absolute constant \( C \), \( A \sim B \) means \( A \lesssim B \) and \( B \lesssim A \), and \( A \gg B \) means \( A/B \) is very big. We use \( (\cdot, \cdot) \) to denote the inner product of \( L^2 \).

Now we recall some function spaces that will be used in this paper. For \( s, b \in \mathbb{R} \), Bourgain spaces \( X^{s,b} \) are defined as the completion of Schwartz space \( \mathcal{S}(\mathbb{R}^2) \) with respect to the norm

\[
\| u \|_{X^{s,b}} := \| \langle \xi \rangle^s \hat{u}(\tau, \xi) \|_{L^2(\mathbb{R}^2)}
\]

where the space-time Fourier transform \( \hat{u} \) is given by

\[
\hat{u}(\tau, \xi) := \int_{\mathbb{R}^2} e^{-i(x \cdot \xi + \tau t)} u(t, x) dt dx.
\]

For an open interval \( I \subset \mathbb{R} \), it is convenient to introduce the restriction in time spaces \( X^{s,b}_I \) endowed with the norm

\[
\| u \|_{X^{s,b}_I} := \inf_{v \in X^{s,b}} \{ \| v \|_{X^{s,b}}, v(\cdot) = u(\cdot) \text{ on } I \}.
\]

If \( I = (0, \delta) \), we write \( X^{s,b}_\delta \) instead \( \| u \|_{X^{s,b}_I} \) for brevity. Grünrock proved the following multi-linear estimates, see Theorem 1 of [6].

Lemma 2.1. If \( -\frac{1}{6} < s \leq 0, -\frac{1}{2} < b' < s - \frac{1}{3}, b > \frac{1}{2}, \) then

\[
\left\| \partial_x \prod_{i=1}^{4} u_i \right\|_{X^{s,b'}} \lesssim \prod_{i=1}^{4} \| u_i \|_{X^{s,b}}.
\]

Corollary 1. For any \( s \geq 0, I \subset \mathbb{R} \), it holds that

\[
\left\| \partial_x \prod_{i=1}^{4} u_i \right\|_{X^{s,-\frac{1}{2}_I}} \lesssim \sum_{i=1}^{4} \| u_i \|_{X^{s,\frac{1}{2}}_I} \prod_{j \neq i} \| u_j \|_{X^{s,\frac{1}{2}}_I}.
\]
Proof. The case $s = 0$ follows from Corollary Lemma 2.1 directly, so suppose $s > 0$. Let $	ilde{u}_i \in X^{s, \frac{1}{2}+}$ and $\tilde{u}_i = u_i$ on $I$, $i = 1, 2, 3, 4$, then $\partial_x \prod_{i=1}^4 \tilde{u}_i = \partial_x \prod_{i=1}^4 u_i$ on $I$. It follows from Parseval identity and Lemma 2.1 that
\[
\left\| \int_\mathcal{D} \langle \tau - \xi^3 \rangle^{\frac{1}{2}+} \prod_{i=1}^4 \langle \tau_i - \xi_i^3 \rangle^{\frac{1}{2}+} \right\|_{L_2(\mathbb{R})} \lesssim \prod_{i=1}^4 \left\| \hat{\tilde{u}}_i \right\|_{L^2(\mathbb{R}^3)}
\]
where
\[
\int_\mathcal{D} = \int_{\xi=\sum_{i=1}^4 \xi_i, \tau=\sum_{i=1}^4 \tau_i} d\xi_1 d\xi_2 d\xi_3 d\tau_1 d\tau_2 d\tau_3.
\]
Combining this and the inequality
\[
\langle \xi \rangle^s \lesssim \sum_{i=1}^4 \langle \xi_i \rangle^s
\]
on $\xi = \sum_{i=1}^4 \xi_i$, we obtain
\[
\left\| \partial_x \prod_{i=1}^4 \tilde{u}_i \right\|_{X^{s, -\frac{1}{2}+}} \lesssim \sum_{i=1}^4 \left\| \tilde{u}_i \right\|_{X^{s, \frac{1}{2}+}} \prod_{j \neq i} \left\| \tilde{u}_j \right\|_{X^{b, \frac{1}{2}+}}.
\]
Since the inequality holds for any $\tilde{u}_i$, the Corollary follows.

Remark 1. Since $H^{\frac{1}{2}+}$ is an algebra, similar to the proof of Corollary 1, we have for $2 \leq k \in \mathbb{N}$, $s > \frac{1}{2}$
\[
\| u \|_{H^s} \lesssim \| u \|_{H^s} \| u \|_{H^{k-1}}^{\frac{k}{k+1}}.
\]
Lemma 2.2. Let $s \in \mathbb{R}, -1/2 < b < b' \leq 0$ or $0 \leq b < b' < 1/2$, $\delta \in (0, 1)$. Then
\[
\| u \|_{X^{s, b}_\delta} \lesssim \delta^{b'-b} \| u \|_{X^{s, b'}}.
\]
Proof. Let $\psi : \mathbb{R} \rightarrow \mathbb{R}$ be a smooth function of time with $\psi = 1$ on $[0, 1]$ and $\text{supp} \psi \subset [-1, 2]$, and $\psi_{\delta}(\cdot) = \psi(\delta^{-1}(\cdot))$ for $\delta > 0$. The desired conclusion follows from the following estimates in [9]
\[
\| \psi_{\delta} u \|_{X^{s, b}_\delta} \lesssim \delta^{b'-b} \| u \|_{X^{s, b'}}.
\]
See Lemma 6.3 of [34] for more details.

Denote by $W(t)$ the group generated by Airy equation, namely
\[
W(t) \varphi = \frac{1}{2\pi} \int_\mathbb{R} e^{ix\xi} e^{it\xi^3} \hat{\varphi}(\xi) d\xi.
\]
From the linear estimates in [9], we can easily obtain their variant version in terms of $\| \cdot \|_{X^{s, b}_\delta}$.

Lemma 2.3. Let $s \in \mathbb{R}, b > 1/2$. Then the following inequalities hold
\[
\| W(t) u_0 \|_{X^{s, b}_\delta} \lesssim \| u_0 \|_{H^s},
\]
\[
\left\| \int_0^t W(t-t') u(t') dt' \right\|_{X^{s, b}_\delta} \lesssim \| u \|_{X^{s, b-1}_\delta}.
\]
We also need some space-time Lebesgue spaces. Let \( 1 \leq p, q \leq \infty \), \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) and \( I \subset \mathbb{R} \). Define
\[
\|f\|_{L^p_x L^q_t(I)} = \left( \int_{\mathbb{R}} \left( \int_I |f(x,t)|^{q} dt \right)^{p/q} dx \right)^{1/p}
\]
and
\[
\|f\|_{L^1_x L^\infty_t(I)} = \left( \int_{\mathbb{R}} \left( \int_I |f(x,t)|^p dx \right)^{q/p} dt \right)^{1/q}.
\]
With usual modification when \( p = \infty \) or \( q = \infty \). If \( I = \mathbb{R} \), we will omit \( I \) in the notation for brevity. Let \( Y \) be a Banach space, we also use \( \| \cdot \|_{L^\infty(0,T;Y)} \) to denote the norm \( \sup_{t \in (0,T)} \| \cdot \|_Y \).

For \( \alpha \geq 0 \), we define the fractional derivatives by
\[
D_x^\alpha f(x,t) = c_\alpha \int_{\mathbb{R}} e^{ix\xi} |\xi|^\alpha \hat{f}(\xi,t) d\xi
\]
where \( \hat{f}(x,t) \) denotes the partial Fourier transform of \( f \) in the \( x \) variable.

Now we recall some inequalities between these spaces and Bourgain spaces. It is easy to check by definition that
\[
X^{s,\frac{1}{2}+} \hookrightarrow L^\infty_{loc} (\mathbb{R}; H^s).
\]
Moreover, we have

**Lemma 2.4** (Lemma 2.4 [9]). If \( 0 \leq \theta \leq \frac{1}{8} \), then
\[
\|D_x^\theta u\|_{L^1_x L^4_t} \lesssim \|u\|_{X^{0,\frac{1}{2}+}}
\]
and
\[
\|u\|_{L^\infty_x L^\infty_t} \lesssim \|u\|_{X^{0,\frac{1}{2}+}}.
\]

**Lemma 2.5** (Lemma 2.6 [9]). If \( 0 \leq \theta \leq 1 \), then
\[
\|D_x^\theta u\|_{L^2_x L^\infty_t} \lesssim \|u\|_{X^{0,\frac{1}{2}+}}.
\]

**Remark 2.** It is easy to check that Lemma 2.4-2.5 still hold if we use the norms of \( L^p_x L^q_t(0,\delta) \) and \( X^{0,\frac{1}{2}+}_\delta \) instead.

**Lemma 2.6.** It holds that
\[
\|u\|_{L^2_x L^\infty_t(0,\delta)} \lesssim \|u\|_{X^{s,\frac{1}{2}+}}.
\]

**Proof.** It follows from Corollary 2.9 in [8] that
\[
\|W(t)\varphi\|_{L^2_x L^\infty_t(0,\delta)} \lesssim \|\varphi\|_{H^{s,\frac{1}{2}+}}.
\]
Combining the inequality and Lemma 2.9 in [29] implies the desired conclusion. \( \Box \)

**Lemma 2.7** (Fractional Leibniz rule). Let \( \alpha \in [0,1) \), \( \alpha_1, \alpha_2 \in [0,\alpha] \) with \( \alpha = \alpha_1 + \alpha_2 \). Let \( p, p_1, p_2, q, q_1, q_2 \in (1,\infty) \) be such that \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \) and \( \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} \). Then
\[
\|D_x^\alpha (fg) - fD_x^\alpha g - gD_x^\alpha f\|_{L^p_x L^q_t} \lesssim \|D_x^{\alpha_1} f\|_{L^p_x L^{q_1}_t} \|D_x^{\alpha_2} g\|_{L^{p_2}_x L^{q_2}_t}.
\]

**Remark 3.** The Lemma is concluded in Theorem A.8 of Kenig et al. [10]. It is obvious that the Lemma still holds if we use \( L^p_x L^q_t(0,\delta) \) norms instead.
2.2. Local well posedness. We consider a function

\[ \hat{Q} = \frac{\hat{f}(\xi)}{\lambda - i\xi^3} \]

such that \( Q \) be the solution of the stationary equation

\[ Q_{xxx} + \lambda Q = f. \]

**Lemma 2.8.** Let \( f \in H^{s-3, 0} \), \( 0 < \delta < 1 \), then for \( t \in \mathbb{R}^1 \)

\[ \| Q \|_{H^s} + \| Q \|_{X^{s, \frac{1}{2} +}} \lesssim \| f \|_{H^{s-3}}. \]

**Proof.** It's easy to see that \( |(\xi)^3/(\lambda - i\xi^3)| \leq \lambda^{-1} + 1 \), and then \( \| Q \|_{H^s} \lesssim \| f \|_{H^{s-3}} \), so it suffices to estimate the norm in Bourgain space. Let \( \chi(\cdot) \) be a smooth function, \( \chi = 1 \) on \([-1, 1]\) and \( \chi = 0 \) on \( \mathbb{R} \setminus [-2, 2] \). Denote \( \chi(t, \cdot) = \chi(t-\cdot) \), then \( \chi(t, \cdot)Q = Q \) on \((t, t+1)\). Thus

\[ \|Q\|_{X^{s, \frac{1}{2} +}} \lesssim \|\chi(t, \cdot)Q\|_{X^{s, \frac{1}{2} +}} \lesssim \|\chi\|_{H^\frac{1}{2} +} \|Q\|_{H^s} \lesssim \|f\|_{H^{s-3}} \]

as desired. \( \square \)

In order to establish the well posedness of (1), we split \( u = w + Q \), then \( w \) satisfies

\[ \begin{cases} w_t + w_{xxx} + \sum_{j=0}^4 c_j \partial_x (Q^j w^{4-j}) + \lambda w = 0 \\ w(0) = u_0 - Q. \end{cases} \tag{3} \]

Here, constants \( c_j \) are given by

\[ c_j = \binom{4}{j}/4. \]

The Duhamel formulation of \( w \) reads

\[ w(t) = W(t)w(0) - \int_0^t W(t-\tau) \left( \lambda w + \sum_{j=0}^4 c_j \partial_x (Q^j w^{4-j}) \right) d\tau. \tag{4} \]

Let \( 0 < \delta < 1 \). It follows from (4) and Lemma 2.3 that

\[ \|w\|_{X^{s, \frac{1}{2} +}} \lesssim \|w(0)\|_{H^s} + \|w\|_{X^{s, -\frac{1}{2} +}} + \left\| \sum_{j=0}^4 c_j \partial_x (Q^j w^{4-j}) \right\|_{X^{s, -\frac{1}{2} +}} \tag{5} \]

Thanks to Lemma 2.2, we have \( \|w\|_{X^{s, \frac{1}{2} +}} \lesssim \delta \|w\|_{X^{s, \frac{1}{2} +}} \) and the last term of (5) can be controlled by

\[ \lesssim \sum_{j=0}^4 \|\partial_x (Q^j w^{4-j})\|_{X^{s, -\frac{1}{2} +}} \lesssim \delta^{\frac{1}{2}} \sum_{j=0}^4 \|\partial_x (Q^j w^{4-j})\|_{X^{s, \frac{1}{2} -}}. \tag{6} \]

By virtue of Corollary 1, we have

\[ \|\partial_x (Q^j w^{4-j})\|_{X^{s, \frac{1}{2} -}} \lesssim \left( \|w\|_{X^{s, \frac{1}{2} +}} + \|Q\|_{X^{s, \frac{1}{2} +}} \right) \sum_{j=0}^3 \|Q\|_{X^{0, \frac{1}{2} +}} \|w\|_{X^{3-j, \frac{1}{2} +}}. \tag{7} \]

Gathering (5)-(7), using Lemma 2.8 we arrive at

\[ \|w\|_{X^{s, \frac{1}{2} +}} \lesssim \|w(0)\|_{H^s} + \delta^{\frac{1}{2}} \left( \|w\|_{X^{s, \frac{1}{2} +}} + \|f\|_{H^{s-3}} \right) \left( \|w\|_{X^{0, \frac{1}{2} +}} + \|f\|_{H^{s-3}} + 1 \right)^3. \]

\(^{1}\)Here and below, we use \( \| \cdot \|_{X^{s, b}_{(t, t+\delta)}} \) to denote the norm \( \| \cdot \|_{X^{s, b}_{L^I}} \) with \( I = (t, t+\delta) \).
Consider the set
\[ B := \{ u \in X^{s, \frac{1}{2}+}_s, \| u \|_{X^{s, \frac{1}{2}+}_s} \leq C(\| w(0) \|_{H^s} + \| f \|_{H^{s-3}}) \}. \]

Let \( w \in B \), we define \( \Gamma w \) as the right hand side of (4), then
\[ \| \Gamma w \|_{X^{s, \frac{1}{2}+}_s} \leq C_1 \| u_0 \|_{H^s} + C_2 \delta^{\frac{3}{2}} C(\| w(0) \|_{H^s} + \| f \|_{H^{s-3}})(\| w(0) \|_{H^s} + \| f \|_{H^{s-3}} + 1)^3 \]
if we set \( C = 2C_1 \) and
\[ \delta \sim (\| w(0) \|_{H^s} + \| f \|_{H^{s-3}} + 1)^{-18} \]
then we obtain
\[ \| \Gamma w \|_{X^{s, \frac{1}{2}+}_s} \leq C(\| u_0 \|_{H^s} + \| f \|_{H^{s-3}}). \]

Therefore, \( \Gamma B \subset B \). Moreover, one can check that
\[ \| \Gamma w_1 - \Gamma w_2 \|_{X^{s, \frac{1}{2}+}_s} \leq \alpha \| w_1 - w_2 \|_{X^{s, \frac{1}{2}+}_s} \]
on \( B \) with some \( \alpha \in (0, 1) \). Thus, \( \Gamma \) is a contraction on \( B \). Now we can state the main result in this subsection.

**Theorem 2.9.** Let \( s \geq 0, u_0 \in H^s, f \in H^{s-3} \), then (4) has a unique solution on \([0, \delta]\) such that
\[ \| w \|_{X^{s, \frac{1}{2}+}_s} \leq C(\| w(0) \|_{H^s} + \| f \|_{H^{s-3}}). \]

Moreover, the lifespan \( \delta \) satisfies
\[ \delta \sim (\| w(0) \|_{H^s} + \| f \|_{H^{s-3}} + 1)^{-18}, \]
and for \( t \in (0, \delta) \), the map \( w(0) \mapsto w(t) \) is continuous from \( H^s \) to \( H^s \).

3. Existence of absorbing sets.

3.1. Absorbing sets in \( H^1 \). In what follows, we derive some energy equalities for smooth solutions, which also hold for rough solutions after a limit process since the solution map is continuous from \( H^s \) to \( H^s \).

**Lemma 3.1.** Let \( u, w \) be a solution of (1) and (3), respectively. Then we have the following conserved quantities
\[ \frac{d}{dt} \| u \|_{L^2}^2 + 2\gamma \| u \|_{L^2}^2 = 2 \int f u dx, \]
(8)
\[ \frac{d}{dt} \varphi(w) + 2\gamma \varphi(w) = \psi(w), \]
(9)
where
\[ \varphi(w) = \int \left( \frac{1}{2} w^2 - \frac{1}{20} w^5 - \frac{1}{4} w^4 Q - \frac{1}{2} w^3 Q^2 - \frac{1}{2} w^2 Q^3 - \frac{1}{4} w Q^4 \right) dx, \]
\[ \psi(w) = \int \left( \frac{3\lambda}{20} w^5 + \frac{\lambda}{2} w^4 Q + \frac{\lambda}{2} w^3 Q^2 - \frac{1}{4} w Q^4 \right) dx. \]
Proof. Multiplying (1) with \( u \) and integrating on \( \mathbb{R} \), we obtain (8) immediately. To prove (9), we rewrite (3) as

\[ w_t + H(w, Q)_x + \lambda w = 0 \tag{10} \]

where \( H(w, Q) = w_{xx} + \frac{1}{4}(w^4 + 4w^3Q + 6w^2Q^2 + 4wQ^3 + Q^4) \). Taking the inner product of (10) with \(-H(w, Q)\) yields that

\[ (w_t + H(w, Q)_x + \lambda w, -H(w, Q)) = 0. \tag{11} \]

It’s easy to see that (\( H(w, Q)_x, -H(w, Q) \)) = (13), \( (w_t, -H(w, Q)) = \frac{d}{dt}\varphi(w) \) and

\[ (\lambda w, -H(w, Q)) = \lambda \int \left( w_x^2 - \frac{1}{4}w^4 - w^4Q - \frac{3}{2}w^2Q^2 - \frac{1}{4}w^4Q^3 + \frac{1}{2}wQ^4 \right) dx \]

\[ = 2\lambda \varphi(w) - \psi(w). \]

Inserting this into (11) implies (9). \( \square \)

Lemma 3.2. If \( f \in L^2, u_0 \in B, B \) is a bounded set in \( H^1 \), then there exists a positive number \( T = T(B) \) such that

\[ \|u\|_{H^1} \leq C, \quad t \geq T. \]

Proof. We first establish an \( L^2 \) bound of \( u \). By (8) and Hölder inequality, we get

\[ \frac{d}{dt}\|u\|_{L^2}^2 + \gamma \|u\|_{L^2}^2 \leq C\lambda^{-1}\|f\|_{L^2}^2. \]

Then using Gronwall lemma we have for \( t \geq 0 \)

\[ \|u(t)\|_{L^2}^2 \leq \|u_0\|_{L^2}^2 e^{-\lambda t} + C\lambda^{-1}\|f\|_{L^2}^2 \int_0^t e^{-2\lambda(T-\tau)} d\tau. \tag{12} \]

Let \( T_1 = \lambda^{-1} \ln(\|u_0\|_{L^2}^2 + 2) \). It follows that

\[ \|u(t)\|_{L^2} \leq C, \quad t \geq T_1. \]

Since \( w = u - Q \) and \( Q \) is bounded in \( L^2 \), we obtain

\[ \|w\|_{L^2} \leq C, \quad t \geq T_1. \]

Next, we obtain an \( L^2 \) bound of \( u_x \). It suffices to do this for \( w_x \), since \( Q \) is also bounded in \( H^1 \). The main tools are (9) and the interpolation inequalities

\[ \|w\|_{L^p} \leq \|w\|_{L^2}^p \|w\|_{L^\infty}^{p\gamma} \leq \|w\|_{H^1}^{p\gamma} \|w_x\|_{L^2}^p, \quad 2 < p < \infty. \tag{13} \]

Thanks to (13),

\[ \int w^5 dx \leq \|w\|_{L^5}^5 \leq C\|w\|_{L^2}^\gamma \|w_x\|_{L^2}^{\frac{3}{2}} \leq \varepsilon \int w_x^2 dx + c_\varepsilon \|w\|_{L^2}^{14} \]

holds for any \( \varepsilon > 0 \) and some constant \( c_\varepsilon \) depending only on \( \varepsilon \). Moreover, by Hölder and Young’s inequality, for \( j = 1, 2, 3, 4 \)

\[ \int w^j Q^{5-j} dx \leq \|w\|_{L^5}^j \|Q^{5-j}\|_{L^\infty} \leq \|w\|_{L^5}^j \|Q\|_{L^5}^{5-j} \leq \|w\|_{L^5}^5 + C\|Q\|_{L^5}^5 \]

\[ \leq \varepsilon \int w_x^2 dx + c_\varepsilon \|w\|_{L^2}^{14} + C\|Q\|_{L^5}^5. \]

Substituting these inequalities into (9), with proper \( \varepsilon = \varepsilon_0 \), we have for \( t \geq 0 \)

\[ \frac{d}{dt}\varphi(w) + \lambda \varphi(w) \leq C(\|w\|_{L^2}^{14} + \|Q\|_{L^5}^5). \tag{14} \]
By virtue of (12), we have \( \|u(t)\|_{L^2} \leq \|u_0\|_{L^2} + \lambda^{-1}\|f\|_{L^2}, \ t \geq 0 \). Note that \( \|\Omega\|_{L^2} \lesssim \|\Omega\|_{H^1} \leq (\lambda^{-1} + 1)\|f\|_{L^2} \) and \( w = u - \Omega \), we find (14) becomes
\[
\frac{d}{dt} \varphi(w) + \lambda \varphi(w) \leq C(\|w_0\|_{L^2} + \|f\|_{L^2})^4, \ t \geq 0.
\]
It follows that
\[
\varphi(w(t)) \leq e^{-\lambda t} \varphi(w(0)) + \int_0^t e^{-2\lambda(T-\tau)} C(\|w_0\|_{L^2} + \|f\|_{L^2})^4 d\tau.
\]
Thus, for any \( t \geq 0 \)
\[
\varphi(w(t)) \leq \varphi(w(0)) + C(\|w_0\|_{L^2} + \|f\|_{L^2})^4.
\]
This implies that for \( t \geq 0 \)
\[
\|w(t)\|_{H^1}^2 \lesssim \|w_0\|_{H^1}^2 + (\|w_0\|_{L^2} + \|f\|_{L^2})^4.
\]
(15)
If \( t \geq T_1 \), we use (14) on the interval \([T_1, t]\) and apply Gronwall lemma to obtain
\[
\varphi(w(t)) \leq e^{-\lambda T_1} \varphi(w(T_1)) + C \int_{T_1}^t e^{-2\lambda(T-\tau)} d\tau.
\]
(16)
Thanks to (15), we have \( |\varphi(w(T_1))| \leq C_0(\|w_0\|_{H^1}^2 + (\|w_0\|_{L^2} + \|f\|_{L^2})^4) \). Let \( T_1 \leq T_2 = \lambda^{-1}\ln(C_0(\|w_0\|_{H^1}^2 + (\|w_0\|_{L^2} + \|f\|_{L^2})^4) + 2) \), it follows from (16) that
\[
\varphi(w) \leq C, \ t \geq T_2.
\]
Using (13) again and the \( L^2 \) bound of \( w \), we obtain
\[
\|w_x\|_{L^2} \leq C, \ t \geq T_2.
\]
Note that \( u = w + Q \) and \( Q \) is bounded in \( H^1 \), we get the desired estimate. 

It follows from (15) that
\[
\|w\|_{L^\infty(0,2;H^1)} \leq \Phi(\|w(0)\|_{H^1})
\]
(17)
here and below \( \Phi(\cdot) \) is a polynomial, which may change from line to line.

**Corollary 2.** If \( f \in L^2 \cap H^{s-3}, s \geq 1 \), the lifespan in Theorem 2.9 can be improved to
\[
\delta \sim (\Phi(\|w(0)\|_{H^1}) + \|f\|_{H^{s-3}} + 1)^{-18}.
\]
**Proof.** From the proof of Theorem 2.9, it suffices to show that for \( \delta \in (0,1) \)
\[
\|w\|_{X^{\delta,\frac{1}{2}+}_{0,1}} \leq \Phi(\|w(0)\|_{H^1}).
\]
(18)
In fact, choose a smooth function \( \chi \) such that \( \chi = 1 \) on \([-1,1], \chi = 0 \) on \( \mathbb{R} \setminus (-2,2) \).
Then by definition,
\[
\|w\|_{X^{\delta,1}_{0,1}} \leq \|\chi(t)w\|_{X^{0,1}_{0,1}} \lesssim \|\chi(t)w\|_{L^2_x L^2_t} + \|\partial_t + \partial_x^2\chi(t)w\|_{L^2_x L^2_t}.
\]
Using (17) and (3) we obtain
\[
\|w\|_{X^{\delta,1}_{0,1}} \lesssim \|w\|_{L^2(0,2;L^2_x)} + \|\sum_{j=0}^4 c_j \partial_x(Q^j w^{4-j}) + \lambda w\|_{L^2(0,2;L^2_x)} \leq \Phi(\|w(0)\|_{H^1}),
\]
which implies (18) obviously. 

\( \square \)
3.2. Absorbing sets in $H^s$.

**Lemma 3.3.** Let $s \geq 1$, $f \in H^{s-3} \cap L^2$, $u_0 \in B$, $B$ is a bounded set in $H^s$, then there exist constants $C$ and $T_3 = T_3(B)$ such that

$$\|u(t)\|_{H^s} \leq C, \quad t \geq T_3.$$  

**Proof.** Since $u = w + Q$, it suffices to obtain an $H^s$ bound of $w$. To this end, acting $D^s$ on both sides of (3), multiplying with $D^s w$, and integrating over $[0, \delta] \times \mathbb{R}$ we have

$$\|D^s w(\delta)\|_{L^2}^2 = e^{-2\lambda \delta}\|D^s w(0)\|_{L^2}^2 - 2 \int_0^\delta e^{-2\lambda (\delta - \tau)} \left( \sum_{j=0}^4 c_j D^s \partial_x (Q^j w^{4-j}), D^s w \right) d\tau. \quad \text{(19)}$$

It amounts to deal with the integral on right hand side of (19). We first consider

$$I := \int_0^\delta e^{-2\lambda (\delta - \tau)} (D^s \partial_x (w^4), D^s w) d\tau.$$

**Estimates of $I$.** If $s$ is an integer, the proof is easier. So let $s = k + \alpha$, $k$ is an integer and $0 < \alpha < 1$. By Parseval identity, we rewrite $I$ as

$$I = \int_0^\delta e^{-2\lambda (\delta - \tau)} (D^\alpha \partial_x^{k+1} w^4, D^\alpha \partial_x^k w) d\tau = \sum_{\sum_i = k+1, 0 \leq j_i \leq k} c_{j_1, j_2, \ldots, j_s} \int_0^\delta e^{-2\lambda (\delta - \tau)} (D^\alpha \prod_{i=1}^4 \partial_x^{j_i} w, D^\alpha \partial_x^{j_i} w) d\tau + 4 \int_0^\delta e^{-2\lambda (\delta - \tau)} (D^\alpha (\partial_x^{k+1} w^3 w), D^\alpha \partial_x^{k} w) d\tau \quad \text{(20)}$$

where $c_{j_1, j_2, \ldots, j_s}$ are constants.

In the following, we will extensively use the fact that

$$\left| \int_0^\delta e^{-\gamma (\delta - \tau)} \int_R g(x, \tau) h(x, \tau) dx d\tau \right| \lesssim \|g\|_{X^{s, \frac{1}{2}}_x} \|h\|_{X^{-s, -\frac{1}{2}}_x} \quad \text{(21)}$$

which can be proved following [5], see [21, p.26] for more details. For $I_1$, we have

$$|I_1| \lesssim \sup_{\sum_i = k+1, 0 \leq j_i \leq k} \left| \int_0^\delta e^{-2\lambda (\delta - \tau)} (D^\alpha \prod_{i=1}^4 \partial_x^{j_i} w, D^{2\alpha-1} \partial_x^{k} w) d\tau \right| \lesssim \sup_{\sum_i = k+1, 0 \leq j_i \leq k} \left\| \prod_{i=1}^4 \partial_x^{j_i} w \right\|_{X^{-s, -\frac{1}{2}}_x} \left\| D^{2\alpha-1} \partial_x^{k} w \right\|_{X^{s, \frac{1}{2}}_x} \lesssim \sup_{\sum_i = k+1, 0 \leq j_i \leq k} \|w\|_{X^{s+2\alpha-1, \frac{1}{2}}_x} \prod_{i=1}^4 \|w\|_{X^{s, \frac{1}{2}}_x} \quad \text{(22)}$$

where we have used Corollary 1. Then, by local well posedness and interpolation inequalities $\|w\|_{H^s} \lesssim \|w\|_{H^r} \|w\|_{H^t}$ for $r \in [1, s]$, (22) is bounded by...
By Hölder inequality,
\[ \|w(0)\|_{H^{k+2\alpha-1}} \leq \sup_{\sum_{i=1}^{4} j_i = k+1, 0 \leq j_i \leq k} \prod_{i=1}^{4} \|w(0)\|_{H_{j_i}}. \]
\[ \lesssim \|w(0)\|_{H^\alpha}^\max\left(\frac{4-\alpha}{2}, 0\right) \Phi(\|w(0)\|_{H^1}). \]
(23)

To estimate \(I_2\), we write the integral in \(I_2\) as
\[ I_{21} + I_{22} = \int_0^\delta e^{-2\lambda(\delta-r)} (w^3D^\alpha\partial_x^{k+1}w, D^\alpha\partial_x^kw) dt \]
\[ + \int_0^\delta e^{-2\lambda(\delta-r)} (D^\alpha(\partial_x^{k+1}w^3) - w^3D^\alpha\partial_x^{k+1}w, D^\alpha\partial_x^kw) dt. \]

Since \(D^\alpha\) commutes with \(\partial_x\), \((w^3D^\alpha\partial_x^{k+1}w, D^\alpha\partial_x^kw) = (w^3, \frac{1}{2}\partial_x(D^\alpha\partial_x^k)^2)\), using integration by parts we have
\[ I_{21} = \int_0^\delta e^{-2\lambda(\delta-r)} (-\frac{3}{2}w^2\partial_xw, (D^\alpha\partial_x^k)^2) dt \]
By Hölder inequality,
\[ |I_{21}| \lesssim \|w\|^3_{L^\infty_x L^{\infty}_t(0,\delta)} \|\partial_xw\|_{L^2_x L^2_t(0,\delta)} \|D^\alpha\partial_x^kw\|^2_{L^2_x L^2_t(0,\delta)}. \]
(24)

Thanks to Lemma (2.4), and (2),
\[ (24) \lesssim \|w\|^3_{X^{\frac{3}{2}, \frac{1}{2}+}} \|w\|^2_{X^{\frac{3}{2}, \frac{1}{2}+}}. \]
(25)

By local well posedness and interpolation inequality again, we find
\[ (25) \lesssim \|w(0)\|^3_{H^{\alpha}} \|w(0)\|^2_{H^{\frac{1}{2}-\frac{1}{4}}} \]
\[ \lesssim \|w(0)\|^\max\left(2, \frac{1}{\alpha-1}, 0\right) \Phi(\|w(0)\|_{H^1}). \]
(26)

For \(I_{22}\), by Hölder inequality and fractional Leibniz rule (see Lemma 2.7), we have
\[ |I_{22}| \lesssim \|D^\alpha w^3\|_{L^2_x L^2_t(0,\delta)} \|D^{k+1}w\|_{L^2_x L^2_t(0,\delta)} \|D^\alpha w\|_{L^2_x L^2_t(0,\delta)}. \]
(27)

By fractional Leibniz rule again,
\[ \|D^\alpha w^3\|_{L^2_x L^2_t(0,\delta)} \lesssim \|D^\alpha w\|_{L^2_x L^2_t(0,\delta)} \|w\|^2_{L^\infty_x L^\infty_t(0,\delta)}. \]
(28)

Note that
\[ \|w\|_{L^\infty_x L^\infty_t(0,\delta)} \lesssim \|w\|_{L^2_x L^2_t(0,\delta)} \|w\|^3_{L^\infty_x L^\infty_t(0,\delta)} \]
\[ \lesssim \|w\|_{L^2_x L^2_t(0,\delta)} \|w\|^3_{L^\infty_x L^\infty_t(0,\delta)} \]
\[ \lesssim \|w\|_{X^{\frac{3}{2}, \frac{1}{2}+}} \|w\|_{L^\infty_x L^\infty_t(0,\delta; H^\infty)} \]
\[ \lesssim \|w(0)\|_{H^1}. \]
(29)

By Lemma 2.4 and Lemma 2.5, we have
\[ \|D^\alpha w\|_{L^2_x L^2_t(0,\delta)} \lesssim \|w\|_{X^{\frac{3}{2}, \frac{1}{2}+}} \]
\[ \|D^\alpha w\|_{L^2_x L^2_t(0,\delta)} \lesssim \|w\|_{X^{\frac{3}{2}, \frac{1}{2}+}}. \]
(30)
(31)
Now let $\varepsilon, \sigma < 1$. One can show that for $0 < \varepsilon < 1$, we have

$$\|D^{k+1} w\|_{L^p_T L^q_x(0,\varepsilon)} \lesssim \|w\|_{X^s_{\delta} \frac{1}{2} +}.$$  \hspace{1cm} (32)

It follows from (27)-(32) that

$$|I_{22}| \lesssim \|w(0)\|_{H^1}^2 \|w\|_{X^s_{\delta} \frac{1}{2} +} \|w\|_{X^s_{\delta} \frac{1}{2} +} \|w\|_{X^s_{\delta} \frac{1}{2} +}$$

$$\lesssim \|w(0)\|_{H^1}^2 \|w(0)\|_{H^1} \|w(0)\|_{H^1}$$

$$\lesssim \|w(0)\|_{H^s}^{\max(2-\frac{1}{s},0)} \Phi(\|w(0)\|_{H^1}).$$  \hspace{1cm} (33)

Therefore, by (23)-(26) and (33) we have

$$|I| \lesssim \|w(0)\|_{H^s}^{\max(2-\frac{1}{s},0)} \Phi(\|w(0)\|_{H^1}).$$  \hspace{1cm} (34)

**Estimates of other terms.** Since $f \in H^{s-3}$, by Lemma 2.8 we know $Q \in X^s_{\delta} \frac{1}{2} +$. One can proceed as above to obtain

$$\left| \int_0^\varepsilon e^{-2\lambda(\delta - \tau)} \left( \sum_{j=1}^4 c_j D^j \partial_x Q^j D^j w \right) d\tau \right|$$

$$\lesssim \|w(0)\|_{H^s}^{\max(2-\frac{1}{s},0)} \Phi(\|w(0)\|_{H^1}).$$  \hspace{1cm} (35)

It follows from (34), (35) that

$$\|D^s w(\delta)\|_{L^2}^2 \leq e^{-2\lambda \delta} \|D^s w(0)\|_{L^2}^2 + C\|w(0)\|_{H^s}^{\max(2-\frac{1}{s},0)} \Phi(\|w(0)\|_{H^1}).$$

Thus,

$$\|w(\delta)\|_{H^s}^2 \leq e^{-2\lambda \delta} \|w(0)\|_{H^s}^2 + C\|w(0)\|_{H^s}^{\max(2-\frac{1}{s},0)} \Phi(\|w(0)\|_{H^1}).$$  \hspace{1cm} (36)

Since the case $2 - \frac{1}{s} < 0$ is easier, we assume that $2 - \frac{1}{s} > 0$ in the following. One can show that for $0 < \varepsilon, \sigma < 1$

$$\|w(0)\|_{H^s}^{2,\sigma} \leq \varepsilon \|w(0)\|_{H^s}^2 + C\varepsilon^{-\frac{2}{\sigma}}.$$

Choose $\varepsilon \sim \frac{1}{\varepsilon^2}, \sigma = \frac{1}{3(s-1)}$, we deduce from (36) that

$$\|w(\delta)\|_{H^s}^2 \leq \frac{1 + e^{-2\lambda \delta}}{2} \|w(0)\|_{H^s}^2 + \left( \frac{1 - e^{-2\lambda \delta}}{2} \right)^{16(s-1)} \Phi(\|w(0)\|_{H^1}).$$  \hspace{1cm} (37)

\footnote{Let $x > 0, 0 < \alpha < 2$. Using Young's inequality

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \quad a, b > 0, 1 < p < \infty, \frac{1}{p} + \frac{1}{q} = 1$$

with $a = (p\varepsilon^{\frac{1}{2}}x^\alpha), b = (p\varepsilon)^{-\frac{1}{2}}, p = \frac{2}{\alpha}, q = \frac{\alpha}{2}$. This yields that

$$x^\alpha \leq \varepsilon x^2 + C\varepsilon^{-\frac{1}{2}}.$$

Now let $\alpha = 2 - \sigma, x = \|w(0)\|_{H^s}$, and note $0 < \varepsilon < 1$, we find

$$\|w(0)\|_{H^s}^2 \leq \varepsilon \|w(0)\|_{H^s}^2 + C\varepsilon^{1-\frac{2}{\sigma}} \leq \varepsilon \|w(0)\|_{H^s}^2 + C\varepsilon^{-\frac{2}{\sigma}}.$$}
Thanks to Lemma 3.3, $w$ is bounded in $L^\infty(0, \infty; H^s)$. It follows from Corollary 2 that we can take $w(\delta)$ as a new data to derive an estimate of $w(2\delta)$. Repeat these process, after $n$ steps, we find
\[
\|w(n\delta)\|_{H^s}^2 \leq \frac{1 + e^{-2\lambda\delta}}{2} \|w((n-1)\delta)\|_{H^s}^2 + \left(\frac{1 - e^{-2\lambda\delta}}{2}\right)^{16(s-1)} \Phi(\|w(0)\|_{H^s}).
\]

Hence
\[
\|w(n\delta)\|_{H^s}^2 \leq \left(\frac{1 + e^{-2\lambda\delta}}{2}\right)^n \|w(0)\|_{H^s}^2 + \left(\frac{1 - e^{-2\lambda\delta}}{2}\right)^{16(s-1)} \Phi(\|w(0)\|_{H^s}) \sum_{j=0}^{n-1} \left(\frac{1 + e^{-2\lambda\delta}}{2}\right)^j.
\]

Note that for $\delta \in (0, 1)$
\[
\varepsilon = \frac{1 - e^{2\lambda\delta}}{2} \approx \delta.
\]

Thus,
\[
\|w(t)\|_{H^s}^2 \leq (1 - \varepsilon)^\frac{1}{2} \|w(0)\|_{H^s}^2 + \varepsilon^{-16(s-1)} \Phi(\|w(0)\|_{H^s}) \lesssim e^{-\omega t} \|w(0)\|_{H^s}^2 + (\Phi(\|w(0)\|_{H^s}) + \|f\|_{H^{s-3}} + 1)^{16(s-1) + 18 + \Phi(\|w(0)\|_{H^s})}
\]
for some $\omega > 0$ and any $t \geq 0$. Especially,
\[
\|w(t)\|_{H^s}^2 \lesssim \Phi(\|w(0)\|_{H^s}), \quad t \geq 0.
\]

(38)

If $t \geq T_2$, it’s convenient to regard $w(t)$ as a solution of (3) starts from $w(T_2)$. Then for $t \geq T_2$
\[
\|w(t)\|_{H^s}^2 \lesssim e^{-\omega(t-T_2)} \|w(T_2)\|_{H^s}^2
\]
\[
+ (\Phi(\|w(0)\|_{H^s}) + \|f\|_{H^{s-3}} + 1)^{16(s-1) + 18 + \Phi(\|w(0)\|_{H^s})}.
\]

By virtue of (38), let $T_3 = T_2 + \omega^{-1} \ln(\Phi(\|w(0)\|_{H^s}))$, we find for $t \geq T_3$
\[
\|w(t)\|_{H^s}^2 \lesssim 1
\]
and the Lemma 3.3 follows.

It follows from Lemma 3.3 that problem (1) has a bounded absorbing set $B_s$ in $H^s$, namely for any bounded set $B \subset H^s$, there exists $T_3(B)$ such that
\[
S(t)B \subset B_s, \quad t \geq T_3(B).
\]

Thus, problem (1) has a global solution in $H^s$, the energy equation (19) holds on any interval $[t_0, t]$,
\[
\|D^s w(t)\|_{L^2}^2 = e^{-2\lambda t} \|D^s w(t_0)\|_{L^2}^2
\]
\[
- 2 \int_{t_0}^{t} e^{-2\lambda(t-\tau)} \left( \sum_{j=0}^{4} c_j D^{s-j} (Q^j w^{4-j}, D^s w) \right) d\tau.
\]

(39)

Corollary 3. Let $u_1, u_2$ be the solutions of (1) with initial data $u_{01}, u_{02} \in H^s$, respectively. Then for $0 \leq \sigma \leq s$
\[
\|u_1 - u_2\|_{X(0,T)}^\frac{s}{4} \lesssim e^{KT} \|u_{01} - u_{02}\|_{H^s}.
\]
Proof. Let $U = u_1 - u_2$, then $U$ solves the integral equation
\[ U(t) = W(t)U(0) - \int_0^t W(t - \tau)\left(\lambda U + \frac{1}{4}\partial_x (U \sum_{j=0}^3 u_j^2 u_{2-j})\right) d\tau. \]

Similar to Section 2, we have
\[ \|u_1 - u_2\|_{X^{s, \frac{1}{2}+}} \lesssim \|U(0)\|_{H^s} + \delta^\frac{1}{2} \|u_1 - u_2\|_{X^{s, \frac{1}{2}+}}^{\frac{1}{2}} \left(\|u_2\|_{X^{s, \frac{1}{2}+}} + \|u_2\|_{X^{s, \frac{1}{2}+}} \right)^2. \]

Thus, if we choose $\delta \sim (\|u_1\|_{X^{s, \frac{1}{2}+}} + \|u_2\|_{X^{s, \frac{1}{2}+}})^{18-\sigma}$,
\[ \|u_1 - u_2\|_{X^{s, \frac{1}{2}+}} \lesssim \|U(0)\|_{H^s}. \]

Now let $T = n\delta$, we decompose
\[ [0,T] = [0, \delta] \cup [\delta, 2\delta] \cup \cdots \cup [(n-1)\delta, n\delta]. \]

It follows from (18) and Lemma 3.2 that $\|u_1\|_{X_{(j,k),(j+1)\delta}^{s,\frac{1}{2}+}}$, $\|u_2\|_{X_{(j,k),(j+1)\delta}^{s,\frac{1}{2}+}}$ are uniformly bounded with respect to $j$. Hence
\[ \|U\|_{X^{s, \frac{1}{2}+}} \lesssim \|U(j\delta)\|_{H^s}, \]

which of course implies
\[ \|U((j+1)\delta)\|_{H^s} \lesssim \|U(j\delta)\|_{H^s}. \]

Here, the implicit constants are independent of $j$. Therefore,
\[ \|U\|_{X^{s, \frac{1}{2}+}} \lesssim \sum_{j=0}^{n-1} \|U\|_{X_{(j,k),(j+1)\delta}^{s,\frac{1}{2}+}} \lesssim \sum_{j=0}^{n-1} \|U(j\delta)\|_{H^s} \lesssim e^{KT} \|U(0)\|_{H^s} \]

as desired. \hfill \Box

Proof of Theorem 1.1. It follows from Lemma 3.3 and Corollary 3 directly. \hfill \Box

4. Global attractor in $H^s$: Integral case. In this section, we assume that $s = k$, $k$ is an integer. Since problem (1) has a bounded absorbing set $B$ in $H^k$, thanks to the classical existence result [31], $\{S(t)\}_{t \geq 0}$ has a global attractor in $H^k$ if we can show that $\{S(t)\}_{t \geq 0}$ is asymptotically compact. In other words, it amounts to check that for any sequence $\{u_{0n}\} \subset B$, and a positive real number sequence $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$, then the sequence $\{S(t_n)u_{0n}\}$ is pre-compact in $H^k$.

Now let $u_n$ be the solutions of (1) associated with initial data $u_{0n}$, $w_n = u_n - Q$. For each $T > 0$, we only consider the sequence $\{w_n(t_n + \cdot)\}$ start with $n$ large enough such that $t_n - T \geq 0$. It follows from section 3 that
\[ \{w_n(t_n + \cdot)\} \text{ is bounded in } C([-T,T]; H^k) \]
and
\[ \left\{ \frac{dw_n}{dt}(t_n + \cdot) \right\}_n \text{ is bounded in } C([-T,T]; H^{k-3}). \]

Thus, $\{w_n(t_n + \cdot)\}_n$ is precompact in $C([-T,T]; H^{k-3}).$ By interpolation, we find that $\{w_n(t_n + \cdot)\}_n$ is precompact in $C([-T,T]; H^{k-3}).$ Hence, there exists a subsequence, still denoted by $\{w_n(t_n + \cdot)\}_n$, such that
\[ \{w_n(t_n + \cdot)\} \rightarrow \overline{w}(\cdot) \text{ strongly in } C([-T,T]; H^{k-3}). \]

Moreover,
\[ \{w_n(t_n + t)\} \rightarrow \overline{w}(t) \text{ weakly in } H^k \text{ for every } t \in \mathbb{R}. \]
and \( \overline{w}(t) \) is a solution of (3).

**Lemma 4.1** ([33]). Assume the assumptions of Theorem 1.2 are satisfied, \( u_0 \in B_\varepsilon \). Then, for any \( \varepsilon > 0 \), there exist constants \( T' = T(\varepsilon) \) and \( k' = k(\varepsilon) \) such that any solution of satisfies

\[
\int_{|x| \geq k} |u|^2 \, dx \leq \varepsilon, \quad \forall t \geq T', \quad k \geq k'.
\]

Using Lemma 4.1 and (40), it’s easy to check that

\[
w_n(t_n) \to \overline{w}(0) \text{ strongly in } H^{k'}.\tag{41}
\]

Now we are ready to prove the asymptotic compactness of \( \{S(t)\}_{t \geq 0} \). The energy equation (39) for \( w_n \) with \( t = t_n \) and \( t_0 = t_n - T \) reads

\[
\|D^k w_n(t_n)\|_{L^2}^2 = e^{-2\lambda T} \|D^k w_n(t_n - T)\|_{L^2}^2
\]

\[
- 2 \int_0^T e^{-2\lambda(T-\tau)} \left( \sum_{j=0}^{4} c_j D^k \partial_x (Q^j w_n^{4-j}) , D^k w_n \right) (t_n - T + \tau) \, d\tau. \tag{42}
\]

We first claim that

\[
\int_0^T e^{-2\lambda(T-\tau)} (D^k \partial_x (w_n^4), D^k w_n) (t_n - T + \tau) \, d\tau
\]

\[
\to \int_0^T e^{-2\lambda(T-\tau)} (D^k \partial_x (\overline{w}^4), D^k \overline{w}) (-T + \tau) \, d\tau \tag{43}
\]

as \( n \to \infty \). In view of (20), it suffices to show that

\[
\int_0^T e^{-2\lambda(T-\tau)} (w_n^3, \partial_x (D^k w_n)^2) (t_n - T + \tau) \, d\tau
\]

\[
\to \int_0^T e^{-2\lambda(T-\tau)} (\overline{w}^3, \partial_x (D^k \overline{w})^2) (-T + \tau) \, d\tau \tag{44}
\]

and for \( 0 \leq j \leq j_i + 1 \leq k \), \( \sum_{i=1}^{4} j_i = k + 1 \)

\[
\int_0^T e^{-2\lambda(T-\tau)} \left( \prod_{i=1}^{4} \partial_x^{j_i} w_n, D^k w_n \right) (t_n - T + \tau) \, d\tau
\]

\[
\to \int_0^T e^{-2\lambda(T-\tau)} \left( \prod_{i=1}^{4} \partial_x^{j_i} \overline{w}, D^k \overline{w} \right) (-T + \tau) \, d\tau \tag{45}
\]

as \( n \to \infty \).

In fact, by Lemma 2.4, Corollary 3 and (41) we have

\[
\left| \int_0^T e^{-2\lambda(T-\tau)} (\partial_x (w_n^3 - \overline{w}^3), (D^k w_n)^2) (t_n - T + \tau) \, d\tau \right|
\]

\[
\leq \|w_n - \overline{w}\|_{L^2(M)_{\varepsilon}(0,T)} (\|\partial_t w_n\|_{L^2(M)_{\varepsilon}(0,T)} + \|\partial_x \overline{w}\|_{L^2(M)_{\varepsilon}(0,T)}) \|D^k w_n\|_{L^2(M)_{\varepsilon}(0,T)}^2
\]

\[
+ \|\partial_x w_n - \partial_x \overline{w}\|_{L^2(M)_{\varepsilon}(0,T)} (\|w_n\|_{L^2(M)_{\varepsilon}(0,T)} + \|\overline{w}\|_{L^2(M)_{\varepsilon}(0,T)}) \|D^k w_n\|_{L^2(M)_{\varepsilon}(0,T)}^2
\]

\[
\leq \|w_n - \overline{w}\|_{X^{\frac{1}{2}}(\varepsilon)} (\|w_n\|_{X^{\frac{1}{2}}(\varepsilon)} + \|\overline{w}\|_{X^{\frac{1}{2}}(\varepsilon)}) \|w_n\|_{X^{\frac{1}{2}}(\varepsilon)} \to 0
\]

\[
\leq e^{K_T} \|w_n(t_n) - \overline{w}(0)\|_{H^{\frac{1}{2}}} \to 0
\]
as $n \to \infty$. Moreover,

\[
\left| \int_0^T e^{-2\lambda(T-\tau)} \left( \partial_x w^3, (D^k w_n)^2 \right) (t_n - T + \tau) d\tau \right| \\
\lesssim \|w\|_{L^\infty_x(0,T)} \|\partial_x w\|_{L^2_x(0,T)} \|D^k (w_n - w)\|_{L^2_x(0,T)} \|D^k (w + w)\|_{L^2_x(0,T)} \\
\lesssim \|w_n - w\|_{X^{k-\frac{1}{2}+\frac{1}{p}+\frac{1}{q}, \frac{1}{2}+\frac{1}{p}+\frac{1}{q}}_{(0,T)}} \|w\|_{X^{k-\frac{1}{2}+\frac{1}{p}+\frac{1}{q}, \frac{1}{2}+\frac{1}{p}+\frac{1}{q}}_{(0,T)}} \|w\|_{L^p_T L^q_x(0,T; H^1)} \\
\lesssim e^{KT} \|w_n(t_n) - \overline{w}(0)\|_{H^{k-\frac{1}{q}}} \to 0
\]

as $n \to \infty$. Thus, (44) follows.

To prove (45), rewrite

\[
\left( \prod_{i=1}^4 \partial_{x}^{j_i} w_n, D^k w_n \right) - \left( \prod_{i=1}^4 \partial_{x}^{j_i} \overline{w}, D^k \overline{w} \right) \\
= \left( \partial_{x}^{j_1} (w_n - \overline{w}) \partial_{x}^{j_2} w_n \partial_{x}^{j_3} w_n \partial_{x}^{j_4} w_n, D^k w_n \right) \\
+ \left( \partial_{x}^{j_1} \overline{w} \partial_{x}^{j_2} (w_n - \overline{w}) \partial_{x}^{j_3} w_n \partial_{x}^{j_4} w_n, D^k w_n \right) \\
+ \left( \partial_{x}^{j_1} \overline{w} \partial_{x}^{j_2} \overline{w} \partial_{x}^{j_3} (w_n - \overline{w}) \partial_{x}^{j_4} w_n, D^k w_n \right) \\
+ \left( \partial_{x}^{j_1} \overline{w} \partial_{x}^{j_2} \overline{w} \partial_{x}^{j_3} \overline{w} \partial_{x}^{j_4} (w_n - \overline{w}), D^k w_n \right) \\
:= I_1 + I_2 + I_3 + I_4 + I_5.
\]

For $I_1$, by Hölder inequality, Lemma 2.4 we have

\[
\left| \int_0^T e^{-2\lambda(T-\tau)} I_1 d\tau \right| \\
\lesssim \|D^{j_1} (w_n - \overline{w})\|_{L^\infty_x(0,T)} \|D^{j_2} w_n\|_{L^4_x L^4_T(0,T)} \|D^{j_3} w_n\|_{L^4_x L^4_T(0,T)} \\
\times \|D^{j_4} w_n\|_{L^4_x L^4_T(0,T)} \|D^k w_n\|_{L^2_x L^2_T(0,T)} \\
\lesssim \|w_n - \overline{w}\|_{X^{j_1+\frac{1}{2}+\frac{1}{q}+\frac{1}{2}+\frac{1}{2}, \frac{1}{2}+\frac{1}{q}+\frac{1}{2}+\frac{1}{2}}_{(0,T)}} \|w_n\|_{X^{j_2+\frac{1}{2}+\frac{1}{q}+\frac{1}{2}+\frac{1}{2}, \frac{1}{2}+\frac{1}{q}+\frac{1}{2}+\frac{1}{2}}_{(0,T)}} \|w_n\|_{X^{j_3+\frac{1}{2}+\frac{1}{q}+\frac{1}{2}+\frac{1}{2}, \frac{1}{2}+\frac{1}{q}+\frac{1}{2}+\frac{1}{2}}_{(0,T)}} \\
\times \|w_n\|_{X^{j_4+\frac{1}{2}+\frac{1}{q}+\frac{1}{2}+\frac{1}{2}, \frac{1}{2}+\frac{1}{q}+\frac{1}{2}+\frac{1}{2}}_{(0,T)}} \|w_n\|_{X^{\frac{k}{2}+\frac{1}{2}, \frac{1}{2}+\frac{1}{2}+\frac{1}{2}}_{(0,T)}}.
\]

Observe that $j_1 \leq k - 1, j_2, j_3, j_4 \leq k$, by Corollary 3 we find

\[
(46) \lesssim \|w_n - \overline{w}\|_{X^{\frac{k}{2}+\frac{1}{2}, \frac{1}{2}+\frac{1}{2}+\frac{1}{2}}_{(0,T)}} \|w_n\|_{X^{\frac{k}{2}+\frac{1}{2}, \frac{1}{2}+\frac{1}{2}+\frac{1}{2}}_{(0,T)}} \|w_n\|_{X^{\frac{k}{2}+\frac{1}{2}, \frac{1}{2}+\frac{1}{2}+\frac{1}{2}}_{(0,T)}} \\
\times \|w_n\|_{X^{\frac{k}{2}+\frac{1}{2}, \frac{1}{2}+\frac{1}{2}+\frac{1}{2}}_{(0,T)}} \|w_n\|_{X^{\frac{k}{2}+\frac{1}{2}, \frac{1}{2}+\frac{1}{2}+\frac{1}{2}}_{(0,T)}} \\
\lesssim C(T) \|w_n(t_n) - \overline{w}(0)\|_{H^{k-\frac{1}{2}}} \to 0
\]

as $n \to \infty$. Similarly,

\[
\lim_{n \to \infty} \left| \int_0^T e^{-2\lambda(T-\tau)} I_2 d\tau \right| \lesssim \lim_{n \to \infty} C(T) \|w_n(t_n) - \overline{w}(0)\|_{H^{k-\frac{1}{2}}} = 0,
\]

\[
\lim_{n \to \infty} \left| \int_0^T e^{-2\lambda(T-\tau)} I_3 d\tau \right| \lesssim \lim_{n \to \infty} C(T) \|w_n(t_n) - \overline{w}(0)\|_{H^{k-\frac{1}{2}}} = 0,
\]
Thus, (45) holds, and then the claim (43) follows.

Combining the weak convergence, we have

\[ \lim_{n \to \infty} \int_0^T e^{-\lambda(T-\tau)} I_4 d\tau \lesssim \lim_{n \to \infty} C(T) \| w_n(t_n) - \overline{w}(0) \|_{H^{k+\frac{1}{8}}} = 0, \]

\[ \lim_{n \to \infty} \int_0^T e^{-\lambda(T-\tau)} I_5 d\tau \lesssim \lim_{n \to \infty} C(T) \| w_n(t_n) - \overline{w}(0) \|_{H^{k-\frac{1}{8}}} = 0. \]

Thus, (45) holds, and then the claim (43) follows.

Similar to the proof of the claim, one can show that

\[ \lim_{n \to \infty} \int_0^T e^{-\lambda(T-\tau)} \left( \sum_{j=1}^{4} c_j D^k \partial_x (Q^j w_n^{1-j}), D^k w_n \right) (t_n - T + \tau) d\tau \]

\[ = \lim_{n \to \infty} \int_0^T e^{-\lambda(T-\tau)} \left( \sum_{j=1}^{4} c_j D^k \partial_x (Q^j \overline{w}^{1-j}), D^k \overline{w} \right) (t_n - T + \tau) d\tau. \]

Now, we pass to the limit as \( n \to \infty \) in (42) and obtain

\[ \lim_{n \to \infty} \| D^k w_n(t_n) \|_{L^2}^2 \leq C e^{-\lambda T} \]

\[ - 2 \int_0^T e^{-\lambda(T-\tau)} \left( \sum_{j=0}^{4} c_j D^k \partial_x (Q^j \overline{w}^{1-j}), D^k \overline{w} \right) (t_n - T + \tau) d\tau. \]

By virtue of energy equation for \( \overline{w} \), we find

\[ \lim_{n \to \infty} \| D^k w_n(t_n) \|_{L^2}^2 \leq 2 C e^{-\lambda T} + \| D^k \overline{w}(0) \|_{L^2}^2. \]

Since \( T > 0 \) can be arbitrary, we get

\[ \lim_{n \to \infty} \| D^k w_n(t_n) \|_{L^2}^2 \leq \| D^k \overline{w}(0) \|_{L^2}^2. \]

Combining the weak convergence, we have

\[ w_n(t_n) \to \overline{w}(0) \] strongly in \( H^k \).

This proves the asymptotic compactness of \( \{ S(t) \}_{t \geq 0} \), and, hence, the existence of global attractor in \( H^k \). Thus the proof of Theorem 1.2 in integral case is complete.

5. Global attractor in \( H^s \): Fractional case.

5.1. Splitting. Let \( P^N \) and \( P_N \) be Fourier projectors on higher frequency and lower frequency, respectively. Precisely,

\[ P_N \varphi = \mathcal{F}^{-1}(1_{|\xi| \leq N} \mathcal{F} \varphi(\xi)), \quad \varphi \in \mathcal{S} \]

and \( P^N = 1 - P_N \), where \( 1_{|\xi| \leq N} \) is the characteristic function on \( \{ \xi : |\xi| \leq N \} \). We split \( u = v + q \) such that

\[ u^3 u_x = v^3 v_x + \sum_{j=0}^{3} c_j \partial_x (v^j q^{4-j}) \]

where \( q, v \) satisfy the following equations:

\[ \begin{cases} q_t + q_{xxx} + \lambda q = -P^N \left( \sum_{j=0}^{3} c_j \partial_x (v^j q^{4-j}) \right), \\ q(0) = P^N u_0 \end{cases} \]  (47)
and
\[
\begin{align*}
    v_t + v_{xxx} + v^3 v_x + \lambda v &= f - P_N(\sum_{j=0}^{3} c_j \partial_x(v^j q^{4-j})), \\
    v(0) &= P_N u_0.
\end{align*}
\]  
(48)

Here and below, \( u_0 \) belongs to the absorbing set \( B_s \) obtained in Section 3.

5.2. Estimates of \( q \). Using \( v = u - q \), we rewrite (47) as
\[
\begin{align*}
    q_t + q_{xxx} + \lambda q &= -P_N(\sum_{j=0}^{3} c_j \partial_x(u^j q^{4-j})), \\
    q(0) &= P_N u_0
\end{align*}
\]  
(49)

for some different constants \( c_j \). Similar to Section 2, one can show that (49) has a unique solution \( q \in X^{0,\frac{1}{2}} \) with the bound
\[
\|q\|_{X^{0,\frac{1}{2}}} \lesssim \|P_N u_0\|_{L^2},
\]
the lifespan \( \delta \) satisfies that
\[
\delta \sim (\|w(0)\|_{H^s} + 1)^{-18-}.
\]

From the equation (49), we find that
\[
supp \hat{q} \subset \{\xi : |\xi| \geq N\}, \quad \forall t \geq 0.
\]

Thus,
\[
P_N q = q.
\]

Multiplying (47) with \( q \) and integrating over \([0, \delta] \times \mathbb{R} \) yields that
\[
\|q(\delta)\|_{L^2}^2 = e^{-2\Lambda \delta}\|q(0)\|_{L^2}^2 + \int_0^\delta e^{-2\Lambda(\delta - \tau)}(-2 \sum_{j=1}^{3} c_j \partial_x(u^j q^{4-j}), q) d\tau.
\]
(50)

Using integration by parts, Hölder inequality and Lemma 2.4, we obtain
\[
\left| \int_0^\delta e^{-2\Lambda(\delta - \tau)}(-2c_3 \partial_x(u^3 q), q) d\tau \right| \lesssim \int_0^\delta |u|^2 |u_x|^2 |q|^2 dx d\tau \lesssim \|u\|_{L_{x,t}^{\infty}(0,\delta)}^2 \|u_x\|_{L_{x,t}^{\infty}(0,\delta)}^2 \|q\|_{L_{x,t}^4(0,\delta)}^2 \
\lesssim \|u\|_{L_{x,t}^{\infty}(0,\delta; H^1)}^2 \|u_x\|_{L_{x,t}^{\infty}(0,\delta; L^2)}^2 \|q\|_{X^{0,\frac{1}{2}}}^2 \lesssim N^{-\frac{3}{2}} \|q(0)\|_{L^2}^2.
\]

Similarly, we have
\[
\left| \int_0^\delta e^{-2\Lambda(\delta - \tau)}(-2c_2 \partial_x(u^2 q^2), q) d\tau \right| \lesssim \int_0^\delta |u|^3 |u_x|^3 |q|^3 dx d\tau \lesssim \|u\|_{L_{x,t}^{\infty}(0,\delta)} \|u_x\|_{L_{x,t}^{\infty}(0,\delta)} \|q\|_{L_{x,t}^4(0,\delta)}^3 \
\lesssim \|u\|_{L_{x,t}^{\infty}(0,\delta; H^1)} \|u_x\|_{L_{x,t}^{\infty}(0,\delta; L^2)} \|q\|_{X^{0,\frac{1}{2}}}^3 \lesssim N^{-\frac{4}{3}} \|q(0)\|_{L^2}^3.
\]
Moreover,
\[
\left| \int_0^\delta e^{-2\lambda(\delta-\tau)}(-2c_1\partial_x(uq^3),q)\,d\tau \right| \lesssim \|u\|_{X^s_δ} \|\partial_x(q^4)\|_{X^s_δ} + \|u\|_{X^s_δ} \|q\|_{X^s_δ}^4
\lesssim \|u\|_{X^s_δ} \|q\|_{X^s_δ}^4
\lesssim N^{-\frac{5}{4}}\|q(0)\|_{L^2}^4.
\]

Thus,
\[
\|q(\delta)\|_{L^2}^2 \leq e^{-2\lambda\delta}\|q(0)\|_{L^2}^2 + CN^{-\frac{5}{4}}\|q(0)\|_{L^2}^2(1 + \|q(0)\|_{L^2} + \|q(0)\|_{L^2}^2).
\]

Now acting (49) with $D^s$, multiplying $D^s q$ and integrating on $[0, \delta] \times \mathbb{R}$ implies that
\[
\|D^s q(\delta)\|_{L^2}^2 = e^{-2\lambda\delta}\|D^s q(0)\|_{L^2}^2 + \int_0^\delta e^{-2\lambda(\delta-\tau)}(-2\sum_{j=0}^3 c_j\partial_x D^s(u^3q^{4-j}), D^s q)\,d\tau.
\]

Similar to Section 3, one can show that
\[
\left| \int_0^\delta e^{-2\lambda(\delta-\tau)}(-\sum_{j=0}^3 c_j\partial_x D^s(u^3q^{4-j}), D^s q)\,d\tau \right| \lesssim N^{-\min\left\{\frac{1}{2^{(s+1)}}, 2(s+1)\right\}}\|q(0)\|_{H^s}^2(1 + \|q(0)\|_{H^s} + \|q(0)\|_{H^s}^2).
\]

Combining (51)-(53) together yields
\[
\|q(\delta)\|_{H^s}^2 \leq \left(1 + \frac{e^{-2\lambda\delta}}{2}\right)\|q(0)\|_{H^s}^2 + \|q(0)\|_{H^s}^2 \left( CN^{-\min\left\{\frac{1}{2^{(s+1)}}, 2(s+1)\right\}} \cdot (1 + \|q(0)\|_{H^s} + \|q(0)\|_{H^s}^2) - \frac{1 - e^{-2\lambda\delta}}{2} \right).
\]

Since $q(0)$ is bounded in $H^s$, we choose $N$ big enough, say $N = N_0$, $N_0$ depends only on the absorbing set, such that the second term on right hand side is negative, thus
\[
\|q(\delta)\|_{H^s} \leq \left(1 + \frac{e^{-2\lambda\delta}}{2}\right)\|q(0)\|_{H^s}.
\]

Then an iteration argument yields that for some $\omega = C\lambda$
\[
\|q(t)\|_{H^s} \lesssim e^{-\omega t}\|q(0)\|_{H^s}, \quad t \geq 0.
\]

5.3. Estimates of $v$. Since $v = u - q$, from previous discussion, we get the global existence of $v$ in $H^s$. Moreover, $P_N v = P_N u$ is smooth and satisfies
\[
\|P_N v\|_{H^{s+\varepsilon}} + \|P_N v\|_{X^{s+\varepsilon, \frac{1}{2}}_{(t,x)}} \lesssim N^\sigma
\]
for all $\sigma \geq 0, t \geq 0, \delta \in (0, 1)$. Thus, it suffices to focus on an estimate of $P_N v$. To this end, we introduce $Z = P_N v - P_N Q$, which is a solution of
\[
\begin{cases}
Z_t + Z_{xxx} + P_N(v^3 v_x) + \lambda Z = 0, \\
Z(0) = -P_N Q.
\end{cases}
\]

Obviously, $Z$ has the following bound
\[
\|Z(t)\|_{H^s} \lesssim 1, \quad t \geq 0.
\]
Lemma 5.1. Let $0 < \delta < 1$. It holds that
\[ \| P^N (v^3 v_x) \|_{X^{s+, \frac{1}{2} +}} \lesssim C(N) + \| Z \|_{X^{s+, \frac{1}{2} +}}. \]

Proof. Since $v = P_N v + Z + P^N Q$, it follows that
\[ v^3 v_x = \sum_{j,k,l \geq 0, j+k+l=4} c_{j,k,l} \partial_x (Z^j (P_N v)^k (P^N Q)^l) \]
for some constants $c_{j,k,l}$. By Lemma 2.8 and Corollary 2, we have
\[ \| P_N v \|_{X^{s+, \frac{1}{2} +}} + \| Q \|_{X^{s+, \frac{1}{2} +}} \lesssim 1 \]
and thus
\[ \| P_N v \|_{X^{s+, \frac{1}{2} +}} \lesssim N^{0+}. \]

It is easy to see that
\[ \| P^N \partial_x (Z^j (P_N v)^k (P^N Q)^l) \|_{X^{s+, -\frac{1}{2} -}} \lesssim \| \partial_x (Z^j (P_N v)^k (P^N Q)^l) \|_{X^{s+, -\frac{1}{2} -}}. \]

In the case $j = 1, k = 1, l = 2$, it follows from Corollary 1, (55)-(56) that
\[ (57) \lesssim \| Z \|_{X^{s+, \frac{1}{2} +}} \| P_N v \|_{X^{0, \frac{1}{2} +}} \| Q \|_{X^{0, \frac{1}{2} +}} + \| P_N v \|_{X^{s+, \frac{1}{2} +}} \| Z \|_{X^{s, \frac{1}{2} +}} \| Q \|_{X^{s, \frac{1}{2} +}} \]
\[ \lesssim C(N) + \| Z \|_{X^{s+, \frac{1}{2} +}}. \]

The other cases are similar. Thus the proof is complete. \qed

Remark 4. As a byproduct of the proof of Lemma 5.1, we have for $j = 0$
\[ \| \partial_x (Z^j (P_N v)^k (P^N Q)^l) \|_{X^{s+, -\frac{1}{2} +}} \lesssim C(N) \]
and for $j \geq 2$
\[ \| \partial_x (Z^j (P_N v)^k (P^N Q)^l) \|_{X^{s+, -\frac{1}{2} +}} \lesssim N^{-s} \| Z \|_{X^{s+, \frac{1}{2} +}} \]
and for $j = 1, l \geq 1$
\[ \| \partial_x (Z^j (P_N v)^k (P^N Q)^l) \|_{X^{s+, -\frac{1}{2} +}} \lesssim N^{-s} \| Z \|_{X^{s+, \frac{1}{2} +}}. \]

Now we write $Z(t + t_0)$ as a solution of (54) starts from $Z(t)$, namely
\[ Z(t + t_0) = W(t_0) Z(t) - \int_0^{t_0} W(t_0 - \tau) \left( \lambda Z(t + \tau) + P^N (v^3 v_x) \right) d\tau. \]

Using Corollary 1 and Lemma 5.1, we have
\[ \| Z \|_{X^{s+, \frac{1}{2} +}} \lesssim \| Z(t) \|_{H^{s+}} + \delta^{0+} (C(N) + \| Z \|_{X^{s+, \frac{1}{2} +}}). \]

Hence
\[ \| Z \|_{X^{s+, \frac{1}{2} +}} \lesssim \| Z(t) \|_{H^{s+}} + C(N) \]
for all $t \geq 0, \delta \sim \delta_0, \delta_0$ is small enough depends only on the absorbing set $B_\delta$. 
Lemma 5.2. Let

Now we need the following lemma.

On one hand, using integration by parts, we find

\[ g \leq \|D^{s+}Z(t)\|_{L^2}^2 \]

\[ -2 \int_t^{t+\delta} e^{-2\lambda(s+t-\tau)} \left( \sum_{j,k,l \geq 0, j+k+l=4} c_{j,k,l} D^{s+} \partial_x (Z^j(P_N v)^k(P_N Q)^l), D^{s+}Z \right) d\tau. \quad (62) \]

Now we need the following lemma.

Lemma 5.2. Let \( j, k, l \geq 0, j + k + l = 4 \), then

\[ \left| \int_t^{t+\delta} e^{-2\lambda(s+t-\tau)} (D^{s+} \partial_x (Z^j(P_N v)^k(P_N Q)^l), D^{s+}Z) d\tau \right| \]

\[ \lesssim N^{-\frac{1}{2}} \|Z\|^2_{X^{s+\frac{1}{2}+}} + C(N). \quad (63) \]

Proof. We divide the proof in several cases. If \( j = 0 \), by (21) and (58) we find

\[ LHS(63) \lesssim \|D^{s+} \partial_x (Z^j(P_N v)^k(P_N Q)^l)\|_{X^{s+\frac{1}{2}+}} \|D^{s+}Z\|_{X^{s+\frac{1}{2}+}} \]

\[ \lesssim \|\partial_x (Z^j(P_N v)^k(P_N Q)^l)\|_{X^{s+\frac{1}{2}+}} \|Z\|^2_{X^{s+\frac{1}{2}+}} \]

\[ \lesssim C(N) \|Z\|^2_{X^{s+\frac{1}{2}+}} \]

\[ \lesssim N^{-s} \|Z\|^2_{X^{s+\frac{1}{2}+}} + C'(N). \]

Since \( s \geq 1 \), this proves (63) in this case. If \( j \geq 2 \) or \( j = 1, l \geq 1 \), using (59) and (60) respectively we have

\[ LHS(63) \lesssim N^{-s} \|Z\|^2_{X^{s+\frac{1}{2}+}}. \]

Finally, we consider the most difficult case \( j = 1, l = 0 \), then \( k = 3 \). Let \( s+ = m+\alpha \), \( 0 \leq \alpha < 1 \). Then by Parseval identity

\[(D^{s+} \partial_x (Z(P_N v)^3), D^{s+}Z) = (D^\alpha \partial_x^{m+1} (Z(P_N v)^3), D^\alpha \partial_x^m Z) \]

\[ = \sum_{n=0}^{m+1} c_{mn} (D^\alpha \partial_x^n Z \partial_x^{m+1-n} (P_N v)^3), D^\alpha \partial_x^m Z). \quad (64) \]

Let \( g = \partial_x^{m+1-n} (P_N v)^3 \). Rewrite

\[ D^\alpha (\partial_x^n Z g) = D^\alpha \partial_x^n Z g + D^\alpha (\partial_x^n Z g) - D^\alpha \partial_x^n Z g. \quad (65) \]

On one hand, using integration by parts, we find

\[ (D^\alpha \partial_x^n Z g, D^\alpha \partial_x^m Z) = \begin{cases} \sum_{i=\min\{m,n\}}^{m+n+1} c_i ((D^\alpha \partial_x^i Z)^2, \partial_x^{m+n-2i}g), \text{ if } m + n \text{ is odd;} \\ \sum_{i=\min\{m,n\}}^{m+n} c_i ((D^\alpha \partial_x^i Z)^2, \partial_x^{m+n-2i}g), \text{ if } m + n \text{ is even.} \end{cases} \quad (66) \]
It is easy to see that $i \leq m$ in both cases. Now by (66), using Hölder inequality and Lemma 2.4 we have
\[
\left| \int_t^{t+\delta} e^{-2\lambda(\delta+t-\tau)} (D^\alpha \partial_x^m Z g, D^\alpha \partial_x^m Z) d\tau \right| \lesssim \sum_{\min\{m,n\} \leq i \leq m} \left| \int_t^{t+\delta} e^{-2\lambda(\delta+t-\tau)} c_i ((D^\alpha \partial_x^m Z)^2, \partial_x^{m+n-2i} g) d\tau \right| \quad (67)
\]
\[
\lesssim \sum_{\min\{m,n\} \leq i \leq m} \|D^\alpha \partial_x^m Z\|_{L^2_xL^1_t(t,t+\delta)}^2 \|\partial_x^{m+n-2i} g\|_{L^2_xL^2_t(t,t+\delta)}^2 \lesssim \sum_{\min\{m,n\} \leq i \leq m} \|Z\|^2_{X^{s+\alpha-\frac{1}{2}+rac{1}{4},+}_x(t,t+\delta)} \lesssim \sum_{\min\{m,n\} \leq i \leq m} N^{-2(m+\frac{1}{2}-i)} \|Z\|^2_{X^{s+\alpha-\frac{1}{2}+rac{1}{4},+}_x(t,t+\delta)} \lesssim N^{-s+\frac{1}{4}+\frac{1}{4}} \|Z\|^2_{X^{s+\alpha-\frac{1}{2}+rac{1}{4},+}_x(t,t+\delta)} \quad (68)
\]

Thanks to Remark 1, note that $\text{supp} \tilde{Z} \subset \{\xi : |\xi| \geq N\}$ and $\|v\|_{H^s} \lesssim 1$ we obtain
\[
(67) \lesssim \sum_{\min\{m,n\} \leq i \leq m} \|Z\|^2_{X^{s+\alpha-\frac{1}{2}+rac{1}{4},+}_x(t,t+\delta)} \text{ with } \max\{0,m+1-2i-\alpha\} = 0 \quad (69)
\]

It follows from (67)-(69) that
\[
\left| \int_t^{t+\delta} e^{-2\lambda(\delta+t-\tau)} (D^\alpha \partial_x^m Z g, D^\alpha \partial_x^m Z) d\tau \right| \lesssim N^{-s+\frac{1}{4}+\frac{1}{4}} \|Z\|^2_{X^{s+\alpha-\frac{1}{2}+rac{1}{4},+}_x(t,t+\delta)} \quad (70)
\]

On the other hand, similar to section 3, by Hölder inequality, Lemma 2.4-2.5 and Lemma 2.7 we have
\[
\left| \int_t^{t+\delta} e^{-2\lambda(\delta+t-\tau)} (D^\alpha \partial_x^m Z g, D^\alpha \partial_x^n Z - D^\alpha \partial_x^n Z) d\tau \right| \lesssim \|D^\alpha \partial_x^m Z g\|_{L^2_x(t,t+\delta)} \|D^\alpha \partial_x^n Z - D^\alpha \partial_x^n Z\|_{L^2_xL^1_t(t,t+\delta)} \lesssim N^{-\frac{1}{2}-(m+1-\alpha)} \|Z\|^2_{X^{s+\alpha-\frac{1}{2}+rac{1}{4},+}_x(t,t+\delta)} \|g\|_{L^2_x(t,t+\delta)} \|D^\alpha \partial_x^n Z\|_{L^2_xL^1_t(t,t+\delta)} \quad (71)
\]

Now we need to bound the norms involved with $g$. Firstly,
\[
\|g\|_{L^\infty_x(t,t+\delta)} \lesssim \|P_N v\|^3_{L^\infty_x(t,t+\delta; H^{m+1-\alpha+\frac{1}{2}+})} \lesssim N^{\max\{0,\frac{3}{2}-n-\alpha\}} \quad (72)
\]

Moreover, observe that
\[
g = \partial_x^{m+1-n} (P_N v)^3 = \sum_{j_1+j_2+j_3=m+1-1} c(j_1,j_2,j_3) \partial_x^{j_1} (P_N v) \partial_x^{j_2} (P_N v) \partial_x^{j_3} (P_N v). \quad (73)
\]
Thus, using Lemma 2.7 and Hölder inequality we have

\[
\|D^\alpha g\|_{L^\infty_t L^1_x(t, t+\delta)} \lesssim \sum_{j_1+j_2+j_3 = m+1-n} \|D^\alpha \partial^j_x (P_N v)\|_{L^\infty_t L^6_x(t, t+\delta)} \cdot \|\partial^j_x (P_N v)\|_{L^\infty_t L^{16}_x(t, t+\delta)}.
\]  

(74)

Proceed as the proof of (29), we get

\[
(74) \lesssim \sum_{j_1+j_2+j_3 = m+1-n} \|P_N v\|_{X^{j_1+\alpha, \frac{1}{4}, \frac{1}{4}} (t, t+\delta)} \cdot \|P_N v\|_{X^{j_2+\alpha, \frac{1}{4}, \frac{1}{4}} (t, t+\delta)} \cdot \|P_N v\|_{X^{j_3+\alpha, \frac{1}{4}, \frac{1}{4}} (t, t+\delta)}
\]  

\[
\lesssim \sum_{j_1+j_2+j_3 = m+1-n} N^{\max\{0, j_1-m+\}} N^{\frac{1}{4} \alpha} \max\{0, j_2-m-\alpha+\frac{1}{4}+\}
\]  

\[
\lesssim \sum_{j_1+j_2+j_3 = m+1-n : j_1 \leq j_2 \leq j_3} N^{\max\{0, j_1-m+\}} N^{\max\{0, j_2-m-\alpha+1+\}} N^{\max\{0, j_3-m-\alpha+1\}}
\]  

(75)

Since \(j_1 > m\) is impossible in the sum, use \(j_2 + j_3 \leq m + 1 - n\), we find

\[
(75) \lesssim \sum_{j_1+j_2+j_3 = m+1-n : j_1 \leq j_2 \leq j_3} N^{\max\{0, j_2-m-\alpha+1, j_3-m-\alpha+1, j_2-j_3-2m-2\alpha+2\}} \lesssim N^{\max\{0, 2-n-\alpha, n+1-m-2\alpha+\}}.
\]  

(76)

Inserting (72)-(76) into (71) yields that

\[
(71) \lesssim N^{-\frac{1}{4} - (m+1-n)} \|Z\|_{X^{j_1+\alpha, \frac{1}{4}, \frac{1}{4}} (t, t+\delta)}^2
\]  

\[
\lesssim N^{-\frac{1}{4}} \|Z\|_{X^{j_1+\alpha, \frac{1}{4}, \frac{1}{4}} (t, t+\delta)}^2
\]  

for \(n \geq 2\). Combining this and (70) proves that

\[
\left| \sum_{n=2}^{m+1} \int_t^{t+\delta} e^{-2\lambda (\delta+t-\tau)} (D^\alpha \partial_x^n g, D^\alpha \partial_x^m Z) d\tau \right| \lesssim \int_t^{t+\delta} e^{-2\lambda (\delta+t-\tau)} (D^\alpha \partial_x^n g, D^\alpha \partial_x^m Z) d\tau \lesssim N^{-\frac{1}{4}} \|Z\|_{X^{j_1+\alpha, \frac{1}{4}, \frac{1}{4}} (t, t+\delta)}^2.
\]  

(77)

Now if \(n = 0\) or \(1\), using integration by parts, it follows from (73) that

\[
\left| \int_t^{t+\delta} e^{-2\lambda (\delta+t-\tau)} (D^\alpha \partial_x^n g, D^\alpha \partial_x^m Z) d\tau \right|
\]  

\[
= \left| \int_t^{t+\delta} e^{-2\lambda (\delta+t-\tau)} (\partial_x^n (\partial_x Z), Z d\tau \right|
\]  

\[
\lesssim \sum_{j_1+j_2+j_3 = m+1-n} \left| \int_t^{t+\delta} (\partial_x^n Z \partial_x^j (P_N v) \partial_x^j (P_N v) \partial_x^j (P_N v), D^\alpha \partial_x^m Z) d\tau \right|
\]  

(78)
The integral in the sum of (78), using (21) and Corollary 1, is bounded by
\[ \lesssim \| \partial_x \partial^\alpha_x Z \partial^\alpha_x (P_N v) \partial^\alpha_x (P_N v) \|_{X_{(t, t+\delta)}^{\alpha+1}} \| D^{2\alpha} \partial^\alpha_x \|_{X_{(t, t+\delta)}^{\alpha+1}} \]
\[ \lesssim \| Z \|_{X_{(t, t+\delta)}^{\alpha+1}} \| P_N v \|_{X_{(t, t+\delta)}^{\alpha+1}} \| P_N v \|_{X_{(t, t+\delta)}^{\alpha+1}} \| Z \|_{X_{(t, t+\delta)}^{\alpha+1}} \]  
\[ \lesssim N^{-(m+1-n)} \| Z \|_{X_{(t, t+\delta)}^{\alpha+1}}^2 \]
\[ \lesssim \begin{cases} N^{-(m+\alpha)+} \| Z \|_{X_{(t, t+\delta)}^{\alpha+1}}^2, & \text{if } n = 0; \\ N^{-m} \| Z \|_{X_{(t, t+\delta)}^{\alpha+1}}^2, & \text{if } n = 1. \end{cases} \]

Combining (64), (77) and (79) completes the proof of the lemma in case \( j = 1, l = 0, k = 3 \).

It follows from (62) and Lemma 5.2 we infer that
\[ \| D^+ Z(t + \delta) \|_{L_x^2}^2 \leq e^{-2\lambda t} \| D^{s+} Z(t) \|_{L_x^2}^2 + C N^{-\frac{1}{2}+} \| D^{s+} Z(t) \|_{L_x^2}^2 + C(N). \]  

Choose \( N \) large enough, say \( N = N_0 \), such that
\[ N^{-\frac{1}{2}+} \leq \frac{1 - e^{-2\lambda \delta}}{2}. \]

Then (80) now becomes
\[ \| D^{s+} Z(t + \delta) \|_{L_x^2}^2 \leq \frac{1 + e^{-2\lambda \delta}}{2} \| D^{s+} Z(t) \|_{L_x^2}^2 + C(N_0). \]  

Note that (81) holds for any \( t \geq 0 \), thus
\[ \| D^{s+} Z(t) \|_{L_x^2}^2 \lesssim e^{-\omega t} \| D^{s+} Z(0) \|_{L_x^2}^2 + C(N_0). \]

Note that \( Z(0) = -P^N Q \) is bounded in \( H^{s+} \), so
\[ \| D^{s+} Z(t) \|_{L_x^2}^2 \lesssim 1, \quad t \geq 0. \]

Thus, we obtain that
\[ \| v(t) \|_{H^{s+}} \lesssim 1, \quad t \geq 0. \]

5.4. **Proof of main results.** In this subsection we prove Theorem 1.2 when \( s \) is not an integer. To this end, we review the definition of the Kuratowski measure of non-compactness. Let \( X \) be a Banach space and \( A \) be a bounded subset of \( X \). The Kuratowski measure of non-compactness \( \kappa(A) \) of \( A \) is defined by
\[ \kappa(A) = \inf \{ \delta > 0 \mid A \text{ has a finite open cover of sets of diameter } \delta \}. \]

It’s obvious that Kuratowski measure \( \kappa(A) \) depends on the metric of \( X \). Sometimes we shall write \( \kappa_X(A) \) instead, to emphasis the metric used, in the following. Some important properties of \( \kappa(A) \) are summarized as follows, see e.g. [7].

**Lemma 5.3.** \( \kappa(A) \) satisfies the following properties:

(1) \( \kappa(A) = 0 \) if and only if \( \overline{A} \) is compact, where \( \overline{A} \) is the closure of \( A \);

(2) \( \kappa(A) \leq d(A), d(A) \) denotes the diameter of \( A \);

(3) \( \kappa(A + B) \leq \kappa(A) + \kappa(B) \) for any \( A, B \subset X \).
For the convenience of the reader, we recall the following criterion of the existence of global attractor, see [7, 15, 23].

**Proposition 1.** Let $X$ be a Banach space and $\{S(t)\}_{t \geq 0}$ be a continuous semigroup on $X$. Then $\{S(t)\}_{t \geq 0}$ has a global attractor in $X$ provided that the following conditions hold:

(1) $\{S(t)\}_{t \geq 0}$ has a bounded absorbing set in $X$;

(2) for any bounded subset $B$ of $X$, we have $\kappa(S(t)B) \to 0$, as $t \to \infty$.

Let $B$ be a bounded set in $H^s$, it follows from the results in section 3 that $S(t)B \subset \mathcal{B}_s$ for $t \geq T_3(B)$, where $\mathcal{B}_s$ is bounded in $H^s$. Choose a smooth cut off function $\phi_k$ such that $0 \leq \phi_k \leq 1$, $\phi_k = 0$ if $|x| \leq k$ and $\phi_k = 1$ if $|x| \geq 2k$. Thanks Lemma 4.1, we find for $t \geq T_4 = T_3 + T'(\varepsilon)$, $k \geq K(\varepsilon)$

$$\|\phi_k S(t)B\|_{L^2} \leq \varepsilon.$$ 

Now let $u_0 \in B$, split

$$S(t)u_0 = u(t) = q(t) + v(t) := S_1(t)u_0 + S_2(t)u_0.$$ 

For any $\varepsilon > 0$, it follows from subsection 5.2 that for $t \geq T_5 = T_3 + C \lambda^{-1} \ln(2 + \|u_0\|_{H^s})$

$$\|S_1(t)B\|_{H^s} \leq \varepsilon$$

which implies that

$$\kappa_{H^s}(S_1(t)B) \leq \varepsilon.$$ 

Thus, for $t \geq \max\{T_4, T_5\}$

$$\|\phi_k S_2(t)B\|_{L^2} \leq \|\phi_k S_1(t)B\|_{L^2} + \|\phi_k S(t)B\|_{L^2} \leq 2\varepsilon.$$ 

Since $\{S_2(t)B\}$ is bounded in $H^{s+}$ for $t \geq T_3$, by interpolation

$$\|\phi_k S_2(t)B\|_{H^s} \leq \|\phi_k S_2(t)B\|_{L^2}^{\frac{1}{s+}} \leq \|\phi_k S_2(t)B\|_{H^{s+}}  \leq C\varepsilon^{\frac{s}{s+}}.$$ 

Hence

$$\kappa_{H^s}(\phi_k S_2(t)B) \lesssim \varepsilon^{\frac{s}{s+}}.$$ 

Also note that the support of element of $\{(1 - \phi_k)S_2(t)B\}$ is contained in $\{x : |x| \leq 2k\}$, thus by the compact embedding $H^{s+}(|x| \leq 2k) \hookrightarrow H^s(|x| \leq 2k)$

$$\kappa_{H^s}((1 - \phi_k)S_2(t)B) = 0.$$ 

Therefore for $t \geq \max\{T_4, T_5\}$

$$\kappa_{H^s}(S(t)B) \lesssim \varepsilon^{\frac{s}{s+}}.$$ 

Then Theorem 1.2 in fractional case follows from Proposition 1.

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