Large Mixing Angle MSW and Atmospheric Neutrinos from Single Right-Handed Neutrino Dominance and $U(1)$ Family Symmetry

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Abstract

Single right-handed neutrino dominance (SRHND) in the 23 sector of the light effective neutrino mass matrix has been proposed as a natural explanation for the concurrent large 23 mixing angle and large 23 mass hierarchy. In this paper we show how large 12 mixing angles, suitable for the large mixing angle (LMA) MSW solution to the solar neutrino problem, may arise from SRHND. In order to understand the conditions for SRHND and LMA MSW we first consider the case of one and two right-handed neutrinos, and obtain simple analytic conditions which are then extended to the case of three right-handed neutrinos. We then introduce a single $U(1)$ family symmetry and show how these analytic conditions may be translated into $U(1)$ charge assignments and perform a systematic search for the simplest examples.
1 Introduction

The latest atmospheric neutrino preliminary results from Super Kamiokande [1] are best fit by $\nu_\mu \rightarrow \nu_\tau$ oscillations $\sin^2 2\theta_{23} = 0.99$ and $\Delta m^2_{23} = 3.1 \times 10^{-3} \text{eV}^2$. The 90% confidence level ranges are (approximately) $\sin^2 2\theta_{23} > 0.85$ and $1.5 \times 10^{-3} \text{eV}^2 < \Delta m^2_{23} < 5.5 \times 10^{-3} \text{eV}^2$.

Super Kamiokande is also beginning to provide important clues concerning the correct solution to the solar neutrino problem. The latest preliminary evidence from Super Kamiokande [2] provides some support for a day-night asymmetry, as expected from the large mixing angle MSW solution [3], while the energy spectrum distortion is consistent with being flat (up to a turn up at the end which may be accounted for by hep neutrinos) which disfavours the small mixing angle MSW solution (SMA MSW) [3], and data on the seasonal variation expected by the vacuum oscillation (VO) approach [4] is too statistics dominated to draw any real conclusion. Although it is too early to draw any firm conclusions, the above indications provide some impetus for considering the large mixing angle MSW solution (LMA MSW) [5] to the solar neutrino problem. The best fit for the LMA MSW solution including the day-night effect is $\sin^2 2\theta_{12} \approx 0.76$ and $\Delta m^2_{12} \approx 2.7 \times 10^{-5} \text{eV}^2$.

From a theoretical point of view any indication of neutrino mass is very exciting since it represents new physics beyond the standard model. The see-saw mechanism [6] implies that the three light neutrino masses arise from some large mass scales corresponding to the Majorana masses of some heavy “right-handed neutrinos” $N^p_R$, $M^{pq}_{RR}$ ($p, q = 1, \cdots, Z$) whose entries take values around or below the unification scale $M_U \sim 10^{16} \text{GeV}$. The presence of electroweak scale Dirac mass terms $m^{ip}_{LR}$ (a $3 \times Z$ matrix) connecting the left-handed neutrinos $\nu^i_L$ ($i = 1, \ldots, 3$) to the right-handed neutrinos $N^p_R$ then results in a very light see-saw suppressed effective $3 \times 3$ Majorana
mass matrix

\[ m_{LL} = m_{LR} M_{RR}^{-1} m_{LR}^T \]  

for the left-handed neutrinos \( \nu^i_L \), which are the light physical degrees of freedom observed by experiment.

If the neutrino masses arise from the see-saw mechanism then it is natural to imagine that the Dirac neutrino mass matrices are related somehow to those of the charged quarks and leptons, perhaps by some relations present in unified theories, and are thus hierarchical in nature. The existence of a physical neutrino mass hierarchy \( m_{\nu_1} \ll m_{\nu_2} \ll m_{\nu_3} \) has the attractive feature that \( \Delta m_{23}^2 \approx m_{\nu_3}^2 \) and \( \Delta m_{12}^2 \approx m_{\nu_2}^2 \), which fixes \( m_{\nu_3} \approx 5.6 \times 10^{-2} \text{eV} \), and (assuming the LMA MSW solution) \( m_{\nu_2} \approx 5.2 \times 10^{-3} \text{eV} \), with rather large errors. Thus \( m_{\nu_2}/m_{\nu_3} \approx 0.1 \). In view of such a 23 mass hierarchy the presence of a large 23 mixing angle looks a bit surprising at first sight, especially given our experience with small quark mixing angles. Of course many explanations have been put forward subsequently, including the possibility that the 23 subdeterminant happens to be accidentally small \[7\]; that there is large mixing in the charged lepton sector \[8\]; that the mixing angles are magnified by renormalisation group effects \[9\]; that the contributions to the 23 block of the light effective Majorana matrix come predominantly from a single right-handed neutrino \[10\] which causes the 23 subdeterminant to approximately vanish.

The last mechanism above, called single right-handed neutrino dominance (SRHND), was further developed in the framework of theories of flavour based on a single \( U(1) \) family symmetry \[11\]. Within this framework general conditions for achieving SRHND were established in terms of \( U(1) \) charges of the various fields. The analysis was directed towards achieving the SMA MSW solution rather than the LMA MSW solution, although the conditions for achieving either solution were not discussed \[11\]. The purpose of the present paper is to discuss analytically the conditions under which
SRHND and LMA MSW may both be achieved in these models, and to provide a systematic study of the charge assignments under a single $U(1)$ family symmetry which can satisfy both these conditions in a natural way. In general there has been relatively little model building dedicated to the LMA MSW solution (compared to the SMA MSW solution) and most of the models that are capable of accounting for both atmospheric and LMA MSW neutrinos involve rather complicated symmetries which stretch one’s credulity. We therefore find the results in this paper to be quite significant, namely that the simplest possible models containing a single $U(1)$ family symmetry possess a wealth of possible charge assignments which are capable both of describing the LMA MSW solution and naturally able to account for the atmospheric neutrino data, via SRHND.

We begin in the next section by introducing $Z$ right-handed neutrinos into the standard model, or its supersymmetric extension. In section 3 we consider the case of a single right-handed neutrino, developing a simple geometrical argument which provides an intuitive understanding of the see-saw mechanism in this case, and showing how it can account for the atmospheric neutrino data. In section 4 we add a further subdominant right-handed neutrino as a perturbation, and present a geometrical argument in this case, which provides useful insight into the see-saw mechanism in this case, and then explore the conditions for SRHND and the LMA MSW solution. In section 5 we discuss the case of three right-handed neutrinos, for three different heavy Majorana textures and, using our experience gained in the two-right-handed neutrino case, we present conditions for LMA MSW solution. In section 6 we introduce a single $U(1)$ family symmetry where the $U(1)$ charges of the three right-handed neutrinos, and three lepton doublets leads to an expansion of the Yukawa and heavy Majorana matrix in powers of the Wolfenstein parameter $\lambda$ [12]. In each case we obtain general conditions on the $U(1)$ charges for achieving the LMA MSW solution, consistent with the requirements of SRHND and the CHOOZ constraint [13] and tabulate the
simplest examples of successful charge assignments for each heavy texture.

2 The (Supersymmetric) Standard Model with Z Right-handed neutrinos

The ideas in this paper apply equally well to the standard model or one of its supersymmetric extensions involving two Higgs doublets. To fix the notation we consider the Yukawa terms with two Higgs doublets augmented by Z right-handed neutrinos, which, ignoring the quarks, are given by

\[ \mathcal{L}_{yuk} = \epsilon_{ab} \left[ Y^e_{ij} H_d^a L_i^b E_j^c - Y^{\nu}_{ip} H_u^a L_i^b N^c_p + \frac{1}{2} Y_{R}^{pq} \Sigma N^c_p N^c_q \right] + H.c. \] (2)

where \( \epsilon_{ab} = -\epsilon_{ba} \), \( \epsilon_{12} = 1 \), and the remaining notation is standard except that the Z right-handed neutrinos \( N^p_R \) have been replaced by their CP conjugates \( N^c_p \) with \( p, q = 1, \ldots, Z \) and we have introduced a singlet field \( \Sigma \) whose vacuum expectation value (VEV) induces a heavy Majorana matrix \( M_{RR} = < \Sigma > Y_{RR} \). When the two Higgs doublets get their VEVs \( < H_u^2 > = v_2, < H_d^1 > = v_1 \) with \( \tan \beta \equiv v_2/v_1 \) we find the terms

\[ \mathcal{L}_{yuk} = v_1 Y^e_{ij} E_i^c E_j^c + v_2 Y^{\nu}_{ip} N_i^c N^c_p + \frac{1}{2} M^{pq}_{RR} N^c_p N^c_q + H.c. \] (3)

Replacing CP conjugate fields we can write in a matrix notation

\[ \mathcal{L}_{yuk} = \bar{E}_L v_1 Y^e E_R + \bar{N}_L v_2 Y^{\nu} N_R + \frac{1}{2} M^{pq}_{RR} N^c_p N^c_q + H.c. \] (4)

where we have assumed that all the masses and Yukawa couplings are real and written \( Y^* = Y \). The diagonal mass matrices are given by the following unitary transformations

\[ v_1 Y^e_{diag} = V_{eL} v_1 Y^e V_{eR}^\dagger = \text{diag}(m_e, m_\mu, m_\tau), \]

\[ M^{diag}_{RR} = \Omega_{RR} M_{RR} \Omega_{RR}^\dagger = \text{diag}(M_{R1}, \ldots, M_{RZ}), \] (5)

\(^1\)In the case of the standard model we replace one of the two Higgs doublets by the charge conjugate of the other, \( H_d = H_u^\ast \), and none of the results in this paper will change.
where the unitary transformations are also orthogonal. From Eq.\[4\] the light effective left-handed Majorana neutrino mass matrix is
\[
m_{LL} = v_2^2 Y_\nu M_{RR}^{-1} Y_\nu^T \tag{6}\]
Having constructed the light Majorana mass matrix it must then be diagonalised by unitary transformations,
\[
m_{LL}^{diag} = V_{\nu L} m_{LL} V_{\nu L}^\dagger = \text{diag}(m_{\nu 1}, m_{\nu 2}, m_{\nu 3}). \tag{7}\]

The leptonic analogue of the CKM matrix is the MNS matrix defined as [14]
\[
V_{MNS} = V_{e L} V_{\nu L}^\dagger. \tag{8}\]

which may be parametrised by a sequence of three rotations about the 1,2 and 3 axes, as in the standard CKM parametrisation,
\[
V_{MNS} = R_{23} R_{13} R_{12} \tag{9}\]
where
\[
R_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix}, \quad R_{13} = \begin{pmatrix} c_{13} & 0 & s_{13} \\ 0 & 1 & 0 \\ -s_{13} & 0 & c_{13} \end{pmatrix}, \quad R_{12} = \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{10}\]
where $s_{ij} = \sin \theta_{ij}$, $c_{ij} = \cos \theta_{ij}$, and $\theta_{ij}$ refer to lepton mixing angles. Note that we completely ignore CP violating phases in this paper.

3 One right-handed neutrino

3.1 Geometrical argument

We begin by recalling [10] our simple atmospheric neutrino model consisting of a single right-handed neutrino $N_{R3}$ with heavy Majorana mass $M_{R3 R3} = Y$ added to the (supersymmetric) standard model. This allows Dirac couplings of the form
\[
v_2 \bar{N}_{R3} (d N_{L1} + e N_{L2} + f N_{L3}) \tag{11}\]
where we have written $Y_{13}^\nu = d$, $Y_{23}^\nu = e$, $Y_{33}^\nu = f$. Only one linear combination of left-handed neutrinos couples to $\bar{N}_{R3}$ so this combination defines the (unnormalised) mass eigenstate

$$\nu_{L3} = dN_{L1} + eN_{L2} + fN_{L3}$$

(12)

The two linear combinations of left-handed neutrinos orthogonal to $\nu_{L3}$ are two massless eigenstates $\nu_{L1}, \nu_{L2}$. There is clearly a massless degenerate subspace, so the choice of basis $\nu_{L1}, \nu_{L2}$ is arbitrary. Assuming that we are in a basis where the charged leptons are diagonal (and hence $V_{MNS} = V^\dagger_{\nu_L}$) the MNS angles are given by the sequence of rotations required to rotate the vector $\nu_{L3}$ to lie along the $N_{L3}$ axis. This is achieved by first rotating $\nu_{L3}$ about the $N_{L1}$ axis with $\tan \theta_{23} = e/f$, which puts the vector into the $N_{L1}, N_{L3}$ plane,

$$R_{23}^\dagger \begin{pmatrix} d \\ e \\ f \end{pmatrix} = \begin{pmatrix} d \\ 0 \\ \sqrt{e^2 + f^2} \end{pmatrix}$$

(13)

Then rotating the resultant vector about the $N_{L2}$ axis with $\tan \theta_{13} = d/\sqrt{e^2 + f^2}$, which puts it along the $N_{L3}$ axis, as desired,

$$R_{13}^\dagger R_{23}^\dagger \begin{pmatrix} d \\ e \\ f \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \sqrt{d^2 + e^2 + f^2} \end{pmatrix}$$

(14)

The final rotation $R_{12}^\dagger$ about the $N_{L3}$ axis is clearly not uniquely defined due to the massless degeneracy.

### 3.2 See-saw with $Z = 1$

The above example with a single right-handed neutrino $N_{R3}$ may be dealt with more formally as a special case of the general see-saw model with $Z = 1$, so that $M_{RR}$ is a trivial $1 \times 1$ matrix $Y$ and $m_{LR}$ is a $3 \times 1$ column matrix $(d, e, f)^T v_2$. Since $M_{RR}$ is trivially invertible the light effective mass matrix is simply given by

$$m_{LL} = \frac{m_{LR}m_{LR}^T}{Y} = \begin{pmatrix} d^2 Y & d e Y & d f Y \\ e d Y & e^2 Y & e f Y \\ f d Y & f e Y & f^2 Y \end{pmatrix} v_2^2.$$

(15)
The matrix in Eq.15 has vanishing determinant and rank one which implies two zero eigenvalues. Note how the single right-handed neutrino coupling to the 23 sector implies vanishing determinant of the 23 submatrix. This provides a natural explanation of both large 23 mixing angles and a hierarchy of neutrino masses in the 23 sector at the same time. The mass matrix may be diagonalised by the same sequence of rotations as required to put the mass eigenvector along the \( N_{L3} \) axis. Namely, the 23 rotation with \( t_{23} = e/f \) clearly gives simultaneous zeroes in the 12,22,23 (and 21,32) positions since it corresponds geometrically to rotating \( \nu_{L3} \) into the 13 plane,

\[
R_{23}^\dagger m_{LL} R_{23} = \begin{pmatrix}
\frac{e^2}{Y} & 0 & \frac{d\sqrt{e^2 + f^2}}{Y} \\
0 & 0 & 0 \\
\frac{d\sqrt{e^2 + f^2}}{Y} & 0 & \frac{e^2 + f^2}{Y}
\end{pmatrix} v_2^2.
\]

(16)

And the subsequent 13 rotation with \( t_{13} = d/\sqrt{e^2 + f^2} \) completes the diagonalisation since it corresponds to aligning rotating \( \nu_{L3} \) to lie along the 3 axis,

\[
R_{13}^\dagger R_{23}^\dagger m_{LL} R_{23} R_{13} = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & \frac{d^2 + e^2 + f^2}{Y}
\end{pmatrix} v_2^2.
\]

(17)

where the mass eigenvalue is

\[
m_{\nu_3} = \frac{(d^2 + e^2 + f^2)}{Y} v_2^2.
\]

(18)

### 3.3 Physical Interpretation

According to the above arguments the mixing angles are therefore given as,

\[
\tan \theta_{23} = \frac{e}{f}, \quad \tan \theta_{13} = \frac{d}{\sqrt{e^2 + f^2}},
\]

(19)

with \( \tan \theta_{12} \) undefined. In order to explain the atmospheric neutrino data and CHOOZ constraint we need \( \tan \theta_{23} \approx 1 \) and \( \tan \theta_{13} \leq \lambda \), which implies that the three Yukawa couplings satisfy the relations:

\[
d \ll e \approx f
\]

(20)
The interpretation of the atmospheric neutrino mixing is now clear. There is a single massive neutrino whose approximately normalised form is

$$\nu_{L3} = t_{13} \nu_e + s_{23} \nu_\mu + c_{23} \nu_\tau$$

(21)

with a Majorana mass $m_{\nu_3} \approx 5.6 \times 10^{-2} \text{ eV}$, which mixes strongly with a massless state approximately given by

$$\nu_{L0} \approx c_{23} \nu_\mu - s_{23} \nu_\tau$$

(22)

(where we have written $N_{L1} = \nu_e$, $N_{L2} = \nu_\mu$, $N_{L3} = \nu_\tau$, assuming we are in the diagonal charged lepton basis) and gives rise to the approximate two-state near maximal mixing observed by Super Kamiokande.

In order to account for the solar neutrino data a small mass perturbation is required to lift the massless degeneracy. In our original approach \cite{10,11} we introduced additional right-handed neutrinos in order to provide a subdominant contribution to the effective mass matrix in Eq.15. We shall first consider the effect of a second right-handed neutrino $N_{R2}$ which gives subdominant contributions to the 23 submatrix of $m_{LL}$, leads to a non-zero $m_{\nu_2}$, and fixes $\tan \theta_{12}$. The conditions for achieving large mixing angles $\theta_{12}$ corresponding to the LMA MSW solution will then become apparent. Subsequently we shall extend the analysis to the case of three right-handed neutrinos, where our experience gained with the two right-handed neutrinos will guide us towards the LMA MSW solutions in that case.

4 Two right-handed neutrinos

\footnote{Another approach \cite{15} which does not rely on additional right-handed neutrinos is to use SUSY radiative corrections so that the one-loop corrected neutrino masses are not zero but of order $10^{-5}$ eV suitable for the vacuum oscillation solution.}
4.1 Geometrical argument

With two right-handed neutrinos the Dirac couplings are extended to

\[ v_2 \bar{N}_{R2}(aN_{L1} + bN_{L2} + cN_{L3}) + v_2 \bar{N}_{R3}(dN_{L1} + eN_{L2} + fN_{L3}) \]  
(23)

We shall require that the two right-handed neutrinos have an approximately diagonal Majorana mass matrix, since \( N_{R3} \) must be isolated to implement the SRHND mechanism,

\[ M_{RR} = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \]  
(24)

There are now two linear combinations of left-handed neutrinos which couple to right-handed neutrinos, defined by the unnormalised states

\[ \nu_{L2} = aN_{L1} + bN_{L2} + cN_{L3} \]
\[ \nu_{L3} = dN_{L1} + eN_{L2} + fN_{L3} \]  
(25)

Neither of the states \( \nu_{L2}, \nu_{L3} \) are eigenvectors since they are not mutually orthogonal. However the vector \( \nu_{L1} \) which is orthogonal to \( \nu_{L2}, \nu_{L3} \) corresponds to a massless eigenstate since it has no coupling to right-handed neutrinos. By repeating the same rotations as in Eq.14 we can put \( \nu_{L3} \) along the 3 axis. The effect of these rotations on \( \nu_{L2} \) is

\[ R_{13}^\dagger R_{23}^\dagger \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} c_{13}a - s_{13}(s_{23}b + c_{23}c) \\ c_{23}b - s_{23}c \\ s_{13}a + c_{13}(s_{23}b + c_{23}c) \end{pmatrix} \]  
(26)

Since \( \nu_{L1} \) is orthogonal to both of \( \nu_{L2}, \nu_{L3} \), we can easily construct it after the above rotations have been performed. Up to a normalisation \( \nu_{L1} \) is given in this basis by

\[ \nu_{L1} = \begin{pmatrix} \frac{c_{23}b - s_{23}c}{c_{13}a - s_{13}(s_{23}b + c_{23}c)} \\ -\frac{c_{23}b - s_{23}c}{c_{13}a - s_{13}(s_{23}b + c_{23}c)} \\ 0 \end{pmatrix} \]  
(27)

Since \( \nu_{L3} \) lies along the 3 axis, it is clear that \( \nu_{L1} \) lies in the 12 plane. We may now perform a 12 rotation to put \( \nu_{L1} \) along the 1 axis. Such a rotation is given by

\[ \tan \theta_{12} = \frac{c_{13}a - s_{13}(s_{23}b + c_{23}c)}{c_{23}b - s_{23}c} \]  
(28)
Such a 12 rotation leaves $\nu_{L3} \sim (0, 0, 1)^T$ unchanged of course, and by design gives $\nu_{L1} \sim (1, 0, 0)^T$. It also rotates $\nu_{L2}$ into the 23 plane, since $\nu_{L2}$ is orthogonal to $\nu_{L1}$,

$$R_{12}^I R_{13}^I R_{23}^I \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ s_{12}(c_{13}a - s_{13}(s_{23}b + c_{23}c)) + c_{12}(c_{23}b - s_{23}c) \\ s_{13}a + c_{13}(s_{23}b + c_{23}c) \end{pmatrix}$$ (29)

After having identified the massless eigenvector $\nu_{L1}$, and rotated it to lie along the 1 axis, we are left with the problem of finding the two massive eigenvectors (recall that $\nu_{L2}$ and $\nu_{L3}$ are not mutually orthogonal). However since $\nu_{L3}$ lies along the 3 axis and $\nu_{L2}$ is in the 23 plane we shall shortly show that in terms of $m_{LL}$ the problem has been reduced to diagonalising a $2 \times 2$ matrix involving further 23 rotations. Furthermore we shall show that the additional 23 rotation that is required is actually rather small and gives only a small correction to $\theta_{23}$ (with $\theta_{13}$ and $\theta_{12}$ unchanged).

### 4.2 See-saw with $Z = 2$

In constructing $m_{LL}$ the example with two right-handed neutrinos $N_{R2}, N_{R3}$ is again a special case of the general see-saw model but now with $Z = 2$. Since $M_{RR}$ is easily invertible the light effective mass matrix is simply obtained

$$m_{LL} = \begin{pmatrix} \frac{e^2}{Y} + \frac{a^2}{X} & \frac{de}{Y} + \frac{ab}{X} & \frac{df}{Y} + \frac{ac}{X} \\ \frac{de}{Y} + \frac{ab}{X} & \frac{e^2}{Y} + \frac{b^2}{X} & \frac{ef}{Y} + \frac{bc}{X} \\ \frac{df}{Y} + \frac{ac}{X} & \frac{ef}{Y} + \frac{bc}{X} & \frac{f^2}{Y} + \frac{c^2}{X} \end{pmatrix} v_2^2$$ (30)

We may now state the conditions for SRHND precisely as the requirement that the $1/Y$ terms in the 23 submatrix dominate over the $1/X$ terms in the full matrix,

$$\frac{e^2}{Y} \sim \frac{ef}{Y} \sim \frac{f^2}{Y} \gg \frac{xy}{X}$$ (31)

where $x, y \in a, b, c$. The matrix in Eq.(30) has vanishing determinant and rank two which implies one zero eigenvalue. We now discuss the effect of the above sequence of rotations which were followed to put $\nu_{L3}$ along the 3-axis and $\nu_{L1}$ along the 1-axis, and show that after these rotations have been performed $m_{LL}$ is approximately diagonal.
with only small additional 23 rotations required to diagonalise it. To begin with, the
23 rotation in Eq.19, corresponding geometrically to rotating \( \nu_{L3} \) into the 13 plane, gives
\[
R_{23}^\dagger m_{LL} R_{23} = \begin{pmatrix}
\frac{d^2 + \frac{a^2}{X}}{Y} & \frac{e_{23ab} - \frac{e_{23ac}}{X}}{Y} & \frac{d\sqrt{e^2 + f^2}}{Y} + \frac{e_{23ac} + \frac{e_{23ab}}{X}}{Y} \\
\frac{e_{23ab} - \frac{e_{23ac}}{X}}{Y} & \frac{d\sqrt{e^2 + f^2}}{Y} + \frac{e_{23ac} + \frac{e_{23ab}}{X}}{Y} & \frac{e^{2} + f^{2}}{Y} + \frac{(e_{23c}+e_{23b})^{2}}{X} \\
\frac{d\sqrt{e^2 + f^2}}{Y} + \frac{e_{23ac} + \frac{e_{23ab}}{X}}{Y} & \frac{e^{2} + f^{2}}{Y} + \frac{(e_{23c}+e_{23b})^{2}}{X} & v_{2}^{2}
\end{pmatrix}
\]
(32)
The reason for the cancellation of leading 1/Y terms in the 12,22,23 (and 21,32) elements of Eq.32 is the same as in Eq.16. The subsequent 13 rotation in Eq.19, corresponding to rotating \( \nu_{L3} \) to lie along the 3-axis, will clearly remove the 1/Y terms in the 11,13 (and 31) positions, as in Eq.17, and will lead to a rather complicated matrix of the form
\[
R_{13}^\dagger R_{23}^\dagger m_{LL} R_{23} R_{13} = \begin{pmatrix}
O(\frac{1}{X}) & O(\frac{1}{X}) & O(\frac{1}{X}) \\
O(\frac{1}{X}) & O(\frac{1}{X}) & O(\frac{1}{X}) \\
O(\frac{1}{X}) & O(\frac{1}{X}) & O(\frac{1}{X})
\end{pmatrix} v_{2}^{2}
\]
(33)
whose essential feature is that the leading 1/Y terms only appear in the 33 position.
The 12 rotation in Eq.28, previously required to rotate the eigenvector \( \nu_{L1} \) to lie along the 1-axis, then gives zeroes in the 12,13 (and 21,31) positions. This is because the ratio of the 12/22 elements and 13/23 elements in Eq.33 are both equal to \( t_{12} \) above, as is easy to verify. Since the eigenvalue corresponding to \( \nu_{L1} \) is massless it also leads to a zero in the 11 position. Therefore the 12 rotation leads to a matrix of the form
\[
R_{12}^\dagger R_{13}^\dagger R_{23}^\dagger m_{LL} R_{23} R_{13} R_{12} = \begin{pmatrix}
0 & 0 & 0 \\
0 & O(\frac{1}{X}) & O(\frac{1}{X}) \\
0 & O(\frac{1}{X}) & O(\frac{1}{X})
\end{pmatrix} v_{2}^{2}
\]
(34)
and from Eq.31 we conclude that the remaining 23 rotations required to diagonalise the matrix are small.
4.3 Physical Interpretation

According to the above arguments the mixing angles with a dominant right-handed neutrino $N_{R3}$ and a subdominant right-handed neutrino $N_{R2}$ are therefore given as,

$$\tan \theta_{23} = \frac{e}{f}, \tan \theta_{13} = \frac{d}{\sqrt{e^2 + f^2}}, \tan \theta_{12} = \frac{c_{13}a - s_{13}(s_{23}b + c_{23}c)}{c_{23}b - s_{23}c}$$  \hspace{1cm} (35)

The effect of the second subdominant right-handed neutrino is to give a non-zero mass to one of the two previously massless neutrinos (the other one remaining massless) with an eigenvalue of order the 22 element of matrix in Eq.32, leaving the heaviest neutrino mass unchanged at leading order,

$$m_{\nu_1} = 0, \ m_{\nu_2} \sim \frac{(c_{23}b - s_{23}c)^2}{X} v^2_2, \ m_{\nu_3} = \frac{(d^2 + e^2 + f^2)}{Y} v^2_2.$$  \hspace{1cm} (36)

Since the 12 mixing angle is the angle relevant for the MSW effect, the conditions under which the LMA solution may be achieved corresponds to

$$\tan \theta_{12} \sim \frac{a}{b - c} \sim 1$$  \hspace{1cm} (37)

where we have used Eq.35 with the approximation that $\theta_{13}$ is small and $\theta_{23}$ is large.

This condition may be traced back to the fact that the 12 angle originates from the ratio of the 12 and 22 elements in Eq.32. after the cancellation of the 1/Y terms.

We can even state conditions for LMA MSW in terms of the elements of the original matrix in Eq.30. LMA MSW requires at least one of the 1/X terms in the 12 or 13 elements in Eq.30 to be of order the largest of the 1/X terms in the 22, 23 and 33 elements of Eq.30,

$$\max\left(\frac{ab}{X}, \frac{ac}{X}\right) \sim \max\left(\frac{b^2}{X}, \frac{bc}{X}, \frac{c^2}{X}\right)$$  \hspace{1cm} (38)

Being able to spot the LMA MSW case by inspection of the original light Majorana matrix proves very useful when dealing with more complicated situations involving three right-handed neutrinos, as we now discuss.
5 Three right-handed neutrinos

With three right-handed neutrinos the Dirac couplings are extended to

\[
Y^T_{\nu} = \begin{pmatrix}
a' & b' & c' \\
a & b & c \\
d & e & f
\end{pmatrix}
\] (39)

There are now three distinct textures for the heavy Majorana neutrino matrix which maintain the isolation of the dominant right-handed neutrino \( N_{R3} \), namely the diagonal, democratic and off-diagonal textures introduced previously[11]. We consider each of them in turn.

5.1 Diagonal Texture

\[
M_{RR}^{\text{diag}} = \begin{pmatrix}
X' & 0 & 0 \\
0 & X & 0 \\
0 & 0 & Y
\end{pmatrix}
\] (40)

We may invert the heavy Majorana matrix and construct the light Majorana matrix using the see-saw mechanism,

\[
m_{LL}^{\text{diag}} = \begin{pmatrix}
\frac{d^2}{Y} + \frac{a^2}{X} + \frac{a'^2}{X'} \\
\frac{e}{Y} + \frac{ab}{X} + \frac{a'b'}{X'} \\
\frac{f}{Y} + \frac{ac}{X} + \frac{a'c'}{X'}
\end{pmatrix} v_2^2
\] (41)

which compared to Eq.30 contains extra terms proportional to \( 1/X' \) from the right-handed neutrino \( N_{R1} \). If these extra terms are subdominant compared to the \( 1/X \) terms so that the SRHND condition in Eq.31 is extended to

\[
e^2 \left( \frac{Y}{Y} \right) \sim \frac{e f}{Y} \sim \frac{f^2}{Y} \gg \frac{xy}{X} \gg \frac{x'y'}{X'}
\] (42)

where \( x, y \in a, b, c \) and \( x', y' \in a', b', c' \), then the resulting neutrino spectrum will be very similar to the case of two right-handed neutrinos with the mixing angles as in Eq.37 up to some small perturbations, and a hierarchy of neutrino masses similar to that in Eq.36,

\[
m_{\nu_1} \ll m_{\nu_2} \ll m_{\nu_3}.
\] (43)
Thus the condition for LMA MSW is as in Eq. 37 in this case. On the other hand if both $1/X$ and $1/X'$ terms are important so that the SRHND condition in Eq. 31 is extended to

\[ \frac{e^2}{Y} \sim \frac{ef}{Y} \sim \frac{f^2}{Y} \gg \frac{xy}{X} \sim \frac{x'y'}{X'} \]  

(44)

then the condition for LMA MSW is a simple generalisation of Eq. 38,

\[ \text{max} \left( \frac{ab}{X'}, \frac{ac}{X'}, \frac{a'b'}{X'}, \frac{a'c'}{X'} \right) \sim \text{max} \left( \frac{b^2}{X'}, \frac{bc}{X'}, \frac{c^2}{X'}, \frac{b'c'}{X'}, \frac{c'^2}{X'} \right) \]  

(45)

and the lightest two neutrinos will have similar masses rather than being hierarchical,

\[ m_{\nu_1} \sim m_{\nu_2} \ll m_{\nu_3}. \]  

(46)

5.2 Democratic Texture

\[ M_{RR}^{\text{dem}} = \begin{pmatrix} X & X & 0 \\ X & X & 0 \\ 0 & 0 & Y \end{pmatrix} \]  

(47)

The democratic case (assuming the Majorana masses in the upper block are of the same order but are not exactly equal) will also give results qualitatively similar to the two right-handed neutrino case in Eq. 30 but the analytic expression will be more complicated, depending on the inverse of the 2 by 2 heavy Majorana matrix. The order of magnitude of the terms is as follows

\[ m_{LL}^{\text{dem}} = \begin{pmatrix} \frac{e^2}{Y} + O\left(\frac{a^2}{X}\right) + O\left(\frac{a'^2}{X'}\right) & \frac{de}{Y} + O\left(\frac{ab}{X}\right) + O\left(\frac{a'b'}{X'}\right) & \frac{df}{Y} + O\left(\frac{ac}{X}\right) + O\left(\frac{a'^c}{X'}\right) \\ \frac{e'^2}{Y} + O\left(\frac{b^2}{X}\right) + O\left(\frac{b'^2}{X'}\right) & \frac{e'f}{Y} + O\left(\frac{bc}{X}\right) + O\left(\frac{b'c'}{X'}\right) & \frac{e'^c}{Y} + O\left(\frac{bc'}{X}\right) + O\left(\frac{c'^2}{X'}\right) \end{pmatrix} v_2^2 \]  

(48)

In this case we might expect democracy to lead to the primed couplings being of the same order as the unprimed couplings \[3\] in which case the SRHND conditions are

\[ \frac{e^2}{Y} \sim \frac{ef}{Y} \sim \frac{f^2}{Y} \gg \frac{xy}{X} \sim \frac{x'y'}{X} \]  

(49)

where $x, y \in a, b, c$ and $x', y' \in a', b', c'$. The mixing angles in this case have the same pattern as in Eq. 35 but the expression for $\tan \theta_{12}$ is now only correct qualitatively.

\[3\] This is the case in the examples in Table 2 for instance.
and will contain additional primed terms. The LMA MSW condition is a simple
generalisation of that in Eq.38, similar to Eq.45,
\[
\max \left( \frac{ab}{X}, \frac{ac}{X}, \frac{a'b'}{X}, \frac{ac'}{X} \right) \sim \max \left( \frac{b}{X}, \frac{bc}{X}, \frac{c}{X}, \frac{b'}{X}, \frac{c'}{X} \right) \quad (50)
\]
The main difference compared to the two right-handed neutrino case is that in the
democratic case if the primed terms are of order the corresponding unprimed terms
we would expect a spectrum as in Eq.46.

5.3 Off-Diagonal Texture

\[
M_{RR}^{\text{off-diag}} = \begin{pmatrix} 0 & X & 0 \\ X & 0 & 0 \\ 0 & 0 & Y \end{pmatrix} \quad (51)
\]
The off-diagonal case is qualitatively different from the other two cases and gives
\[
m_{LL}^{\text{off-diag}} = \begin{pmatrix} \frac{d^2}{Y} + \frac{2a'X}{X} & \frac{de}{Y} + \frac{a'b + ab'}{X} & \frac{df}{Y} + \frac{a'c + ac'}{X} \\ \frac{e^2}{Y} + \frac{2bb'}{X} & \frac{ef}{Y} + \frac{b'c + bc'}{X} \end{pmatrix} v^2 \quad (52)
\]
SRHND is now defined by the conditions
\[
\frac{e^2}{Y} \sim \frac{ef}{Y} \sim \frac{f^2}{Y} \gg \frac{xx'}{X} \quad (53)
\]
where \( x \in a, b, c \) and \( x' \in a', b', c' \). In this case the geometrical picture we developed for
the single right-handed neutrino is still valid, and the mixing angles in Eq.19 remain
unchanged. However the geometrical arguments in section 4.1 are no longer applicable
to this case (where all three right-handed neutrinos play an important role) and the
12 rotation required in this case is obtained by following the procedure in section
4.2 for the case of two right-handed neutrinos. The result is a more complicated
expression,
\[
t_{12} = \frac{c_{13} [c_{23}(a'b + ab') - s_{23}(a'c + ac')] - s_{13}[(b'c + bc')(c_{23}^2 - s_{23}^2) + 2s_{23}c_{23}(bb' - cc')]}{2c_{23}^2bb' + 2s_{23}^2cc' - 2c_{23}s_{23}(b'c + bc')} \quad (54)
\]
which reduces to Eq.28 in the limit $a' = a$, $b' = b$, $c' = c$. The LMA MSW solution condition is again a straightforward generalisation of Eq.38,

$$\max\left(\frac{a'b'}{X}, \frac{ab'}{X}, \frac{a'c}{X}, \frac{ac}{X}\right) \sim \max\left(\frac{bb'}{X}, \frac{b'c}{X}, \frac{bc}{X}, \frac{cc'}{X}\right)$$

(55)

As in the democratic case the large 12 mixing of the right-handed neutrinos implies a spectrum as in Eq.16.

6 U(1) Family Symmetry

6.1 General Considerations

Since the heavy Majorana masses and Dirac Yukawa couplings are free parameters it is always possible to choose them to satisfy the previous conditions. One way to constrain the theory and give some insight into the fermion mass spectrum is via a broken family symmetry [16], [17], [18]. Such family symmetries have recently been applied to neutrino masses [7], [8], [11], [19], [20], [21]. For definiteness we shall focus on a particular class of model based on a single pseudo-anomalous $U(1)$ gauged family symmetry [18]. We assume that the $U(1)$ is broken by the equal VEVs of two singlets $\theta, \bar{\theta}$ which have vector-like charges $\pm 1$ [18]. The $U(1)$ breaking scale is set by $\langle \theta \rangle = \langle \bar{\theta} \rangle$ where the VEVs arise from a Green-Schwartz mechanism [22] with computable Fayet-Iliopoulos $D$-term which determines these VEVs to be one or two orders of magnitude below $M_U$. Additional exotic vector matter with mass $M_V$ allows the Wolfenstein parameter [12] to be generated by the ratio [18]

$$\frac{\langle \theta \rangle}{M_V} = \frac{\langle \bar{\theta} \rangle}{M_V} = \lambda \approx 0.22$$

(56)

The idea is that at tree-level the $U(1)$ family symmetry only permits third family Yukawa couplings (e.g. the top quark Yukawa coupling). Smaller Yukawa couplings are generated effectively from higher dimension non-renormalisable operators corresponding to insertions of $\theta$ and $\bar{\theta}$ fields and hence to powers of the expansion parameter.
in Eq.56, which we have identified with the Wolfenstein parameter. The number of powers of the expansion parameter is controlled by the $U(1)$ charge of the particular operator. As discussed in ref.11, this simple picture may be more complicated, but it is sufficient for our purposes here. The fields relevant to neutrino masses $L_i$, $N_p$, $H_u$, $\Sigma$ are assigned $U(1)$ charges $l_i$, $n_p$, $h_u$, $\sigma$. From Eqs.56, the neutrino Yukawa couplings and Majorana mass terms may then be expanded in powers of the Wolfenstein parameter,

$$Y_{\nu ip} \sim \lambda |l_i + n_p + h_u|, \quad M_{RR}^{pq} \sim \lambda |n_p + n_q + \sigma| < \Sigma > . \quad (57)$$

In dealing with the neutrino sector it is convenient to absorb the Higgs charge $h_u$ into the definition of the lepton charges $l_i$ so that Eq.57 becomes

$$Y_{\nu ip} \sim \lambda |l_i + n_p|, \quad M_{RR}^{pq} \sim \lambda |n_p + n_q + \sigma| < \Sigma > \quad (58)$$

The light Majorana matrix may then be constructed from Eq.58. If we were to assume positive definite values for $l_i + n_p$ and $n_p + n_q + \sigma$ then the modulus signs could be dropped and the right-handed neutrino charges $n_p$ would cancel when $m_{LL}$ is constructed using the see-saw mechanism with Eq.58 [23]. From the point of view of SRHND it is therefore important that such a cancellation does not take place, and so we shall require that at least some of the combinations $l_i + n_p$ and $n_p + n_q + \sigma$ take negative values. In such a case the choice of right-handed neutrino charges will play an important role in determining $m_{LL}$, and each particular choice of $n_p$ must be analysed separately.

The conditions on the Yukawa couplings and heavy Majorana masses in the SRHND approach developed earlier may now be translated into conditions on the $U(1)$ charges, via Eq.58. The heavy Majorana matrix in Eq.58 is explicitly

$$M_{RR} \sim \begin{pmatrix}
\lambda |2n_1 + \sigma| & \lambda |n_1 + n_2 + \sigma| & \lambda |n_1 + n_3 + \sigma| \\
\lambda |2n_2 + \sigma| & \lambda |n_2 + n_3 + \sigma| \\
\lambda |2n_3 + \sigma| & \lambda |n_2 + n_3 + \sigma| & \lambda |2n_3 + \sigma|
\end{pmatrix} < \Sigma > \quad (59)$$
The conditions which ensure that the third dominant neutrino is isolated require that the elements $\lambda^{|n_1+n_3+\sigma|}, \lambda^{|n_2+n_3+\sigma|}$ be sufficiently small. The diagonal texture condition is

$$|n_1 + n_2 + \sigma| > \min(|2n_1 + \sigma|, |2n_2 + \sigma|), \quad 2|n_1 + n_2 + \sigma| \geq |2n_1 + \sigma| + |2n_2 + \sigma| \quad (60)$$

leading to an approximate texture as in Eq.40

$$M_{\text{diag}}^{RR} \sim \begin{pmatrix} \lambda^{2n_1+\sigma} & 0 & 0 \\ 0 & \lambda^{2n_2+\sigma} & 0 \\ 0 & 0 & \lambda^{2n_3+\sigma} \end{pmatrix} < \Sigma > \quad (61)$$

The democratic texture condition is

$$|n_1 + n_2 + \sigma| = |2n_1 + \sigma| = |2n_2 + \sigma| \quad (62)$$

leading to an approximate texture as in Eq.47

$$M_{\text{dem}}^{RR} \sim \begin{pmatrix} \lambda^{2n_1+\sigma} & \lambda^{n_1+n_2+\sigma} & 0 \\ \lambda^{n_1+n_2+\sigma} & \lambda^{2n_2+\sigma} & 0 \\ 0 & 0 & \lambda^{2n_3+\sigma} \end{pmatrix} < \Sigma > \quad (63)$$

The off-diagonal texture condition is

$$|n_1 + n_2 + \sigma| < |2n_1 + \sigma|, |2n_2 + \sigma|, \quad (64)$$

leading to an approximate texture as in Eq.51

$$M_{\text{off}^{-}\text{diag}}^{RR} \sim \begin{pmatrix} 0 & \lambda^{n_1+n_2+\sigma} & 0 \\ \lambda^{n_1+n_2+\sigma} & 0 & 0 \\ 0 & 0 & \lambda^{2n_3+\sigma} \end{pmatrix} < \Sigma > \quad (65)$$

The transpose of the Dirac Yukawa matrix in Eq.58 is explicitly

$$Y_{\nu}^T \sim \begin{pmatrix} \lambda^{n_1+l_1} & \lambda^{n_1+l_2} & \lambda^{n_1+l_3} \\ \lambda^{n_2+l_1} & \lambda^{n_2+l_2} & \lambda^{n_2+l_3} \\ \lambda^{n_3+l_1} & \lambda^{n_3+l_2} & \lambda^{n_3+l_3} \end{pmatrix} \quad (66)$$

which may be compared to the notation in Eq.39. The requirement of large 23 mixing and small 13 mixing expressed in Eq.20 becomes

$$|n_3 + l_2| = |n_3 + l_3|, \quad |n_3 + l_1| - |n_3 + l_3| = 1 \text{ or } 2 \quad (67)$$
The remaining conditions for the $U(1)$ charges depend on the specific heavy Majorana texture under consideration. For instance the basic SRHN D requirement will be different for the three heavy textures, as may be seen by comparing Eqs. 42, 44, 49, 73. These conditions may be translated into conditions on the $U(1)$ charges using Eqs. 51, 53, 65, 66, for the various textures in Eqs. 40, 47, 51, and by requiring dominance at the order of $\lambda$ or $\lambda^2$. Similarly the LMA MSW charge conditions in Eqs. 45, 50, 55 are readily translated into charge conditions.

6.2 Examples

We have performed a computer search over charges which satisfy all the conditions given above, and so provide a natural account of the atmospheric and solar neutrinos via the LMA MSW effect. Our first observation is that the diagonal and democratic textures may only satisfy all the conditions if the charges are allowed to be half-integer. This is similar to the SMA MSW results for the two right-handed neutrino case in ref. 11. Moreover all the LMA MSW solutions found have the feature that the subdominant contributions to $m_{LL}$ are suppressed by $\lambda^2$ and also $\tan\theta_{13} \sim \lambda$, corresponding physically to $m_{\nu_2}/m_{\nu_3} \sim \lambda^2$ and a CHOOZ angle near the current limit. Examples of the charges in the diagonal and democratic cases which lead to LMA MSW are given in Tables 1, 2.

Turning to the off-diagonal texture, we find that in this case both integer and half-integer charge solutions are allowed for the LMA MSW solution with the SRHND interpretation of the atmospheric neutrino results. The subdominant contributions to $m_{LL}$ are suppressed by $\lambda$ or $\lambda^2$, and the CHOOZ angle may be $\tan\theta_{13} \sim \lambda$ or $\tan\theta_{13} \sim \lambda^2$. Examples corresponding to integer charges with $m_{\nu_2}/m_{\nu_3} \sim \lambda^2$ and a CHOOZ angle $\tan\theta_{13} \sim \lambda^2$ well below the current limit are given in Table 3.

In general the dominant right-handed neutrino may be lighter, heavier or equal...
| $l_1$ | $l_2$ | $l_3$ | $n_1$ | $n_2$ | $n_3$ | $\sigma$ |
|-------|-------|-------|-------|-------|-------|--------|
| -1    | 1     | 1     | 0     | 1/2   | -1/2  | -1     |
| -1    | 1     | 1     | 1/2   | 0     | -1/2  | -1     |
| -1/2  | 1/2   | 1/2   | 0     | 1/2   | -1    | -1     |
| -1/2  | 1/2   | 1/2   | 0     | 1/2   | -1/2  | -1     |
| -1/2  | 1/2   | 1/2   | 1/2   | 0     | -1    | -1     |
| -1/2  | 1/2   | 1/2   | 1/2   | 0     | -1/2  | -1     |
| 0     | -1    | -1    | -1/2  | 1/2   | 1     | 1      |
| 0     | -1    | -1    | 0     | 1/2   | 1     | 0      |
| 0     | -1    | -1    | 0     | 1/2   | 1     | 1/2    |
| 0     | -1    | -1    | 0     | 1/2   | 1     | 1      |
| 0     | -1    | 1/2   | -1/2  | 1     | 1      |
| 0     | -1    | 1/2   | 0     | 1     | 1/2    |
| 0     | -1    | 1/2   | 0     | 1     | 1      |
| 0     | 1     | 1     | -1/2  | 0     | -1    | -1     |
| 0     | 1     | 1     | -1/2  | 0     | -1    | -1     |
| 0     | 1     | 1     | -1/2  | 1     | -1    | -1     |
| 0     | 1     | 1     | 0     | -1/2  | -1    | -1     |
| 0     | 1     | 1     | 0     | -1/2  | -1    | -1     |
| 0     | 1     | 1     | 1/2   | -1/2  | -1    | -1     |
| 0     | 1     | 1     | 1/2   | -1/2  | -1    | -1     |
| 1/2   | -1/2  | -1/2  | -1/2  | 0     | 1/2   | 1      |
| 1/2   | -1/2  | -1/2  | -1/2  | 0     | 1     | 1      |
| 1/2   | -1/2  | -1/2  | 0     | -1/2  | 1/2   | 1      |
| 1/2   | -1/2  | -1/2  | 0     | -1/2  | 1     | 1      |
| 1     | -1    | -1    | -1/2  | 0     | 1/2   | 1      |

Table 1: Examples of charges which satisfy the conditions of SRHND, and lead to a diagonal heavy Majorana texture with hierarchical neutrino masses $m_{\nu_2}/m_{\nu_3} \sim \lambda^2$, large $\theta_{23} \sim 1$ (Super-Kamiokande), small $\theta_{13} \sim \lambda$ (CHOOZ), and large $\theta_{12} \sim 1$ (LMA MSW). From these charges the neutrino mass matrices of interest may be explicitly constructed using Eqs.61,66 together with Eqs.39,40,41.
| $l_1$ | $l_2$ | $l_3$ | $n_1$ | $n_2$ | $n_3$ | $\sigma$ |
|-------|-------|-------|-------|-------|-------|-------|
| -1    | 0     | 0     | 1/2   | 1/2   | -1    | -1    |
| -1    | 0     | 0     | 1/2   | 1/2   | -1/2  | -1    |
| -1    | 0     | 0     | 1/2   | 1/2   | 0     | -1    |
| -1    | 0     | 1     | 1/2   | 1/2   | -1/2  | -1    |
| -1    | 1     | 0     | 1/2   | 1/2   | -1/2  | -1    |
| -1    | 1     | 1     | 0     | 0     | -1/2  | -1/2  |
| -1    | 1     | 1     | 0     | 0     | -1/2  | 0     |
| -1/2  | 1/2   | 1/2   | 0     | 0     | -1    | -1    |
| -1/2  | 1/2   | 1/2   | 0     | 0     | -1    | 0     |
| -1/2  | 1/2   | 1/2   | 0     | 0     | -1/2  | -1    |
| -1/2  | 1/2   | 1/2   | 0     | 0     | -1/2  | -1/2  |
| -1/2  | 1/2   | 1/2   | 0     | 0     | -1/2  | 0     |
| 0     | -1    | -1    | 1/2   | 1/2   | 1     | -1    |
| 0     | -1    | -1    | 1/2   | 1/2   | 1     | 1/2   |
| 0     | -1    | -1    | 1/2   | 1/2   | 1     | 1     |
| 0     | 1     | 1     | -1/2  | -1/2  | -1    | -1    |
| 0     | 1     | 1     | -1/2  | -1/2  | -1    | -1/2  |
| 0     | 1     | 1     | -1/2  | -1/2  | -1    | 0     |
| 0     | 1     | 1     | -1/2  | -1/2  | -1    | 1/2   |
| 0     | 1     | 1     | -1/2  | -1/2  | -1    | 1     |
| 1/2   | -1/2  | -1/2  | 0     | 0     | 1/2   | 0     |
| 1/2   | -1/2  | -1/2  | 0     | 0     | 1/2   | 1/2   |
| 1/2   | -1/2  | -1/2  | 0     | 0     | 1/2   | 1     |
| 1/2   | -1/2  | -1/2  | 0     | 0     | 1     | 0     |
| 1/2   | -1/2  | -1/2  | 0     | 0     | 1     | 1/2   |
| 1/2   | -1/2  | -1/2  | 0     | 0     | 1     | 1     |
| 1     | -1    | -1    | 0     | 0     | 1/2   | 0     |
| 1     | -1    | -1    | 0     | 0     | 1/2   | 1/2   |
| 1     | -1    | -1    | 0     | 0     | 1/2   | 1     |
| 1     | -1    | 0     | -1/2  | -1/2  | 1/2   | 1     |
| 1     | 0     | -1    | -1/2  | -1/2  | 1/2   | 1     |
| 1     | 0     | 0     | -1/2  | -1/2  | 0     | 1     |

Table 2: Examples of charges which satisfy the conditions of SRHND, and lead to a democratic heavy Majorana texture with hierarchical neutrino masses $m_{\nu_2}/m_{\nu_3} \sim \lambda^2$, large $\theta_{23} \sim 1$ (Super-Kamiokande), small $\theta_{13} \sim \lambda$ (CHOOZ), and large $\theta_{12} \sim 1$ (LMA MSW). From these charges the neutrino mass matrices of interest may be explicitly constructed using Eqs.63,66 together with Eqs.47,48.
Table 3: Examples of integer charges which satisfy the conditions of SRHND, and lead to an off-diagonal heavy Majorana texture with hierarchical neutrino masses $m_{\nu_2}/m_{\nu_3} \sim \lambda^2$, large $\theta_{23}$, small $\theta_{13}$ (Super-Kamiokande), small $\theta_{13} \sim \lambda^2$ (CHOOZ), and large $\theta_{12} \sim 1$ (LMA MSW). From these charges the neutrino mass matrices of interest may be explicitly constructed using Eqs.65,66 together with Eqs.39,51,52. In this off-diagonal case there are many more integer solutions with a larger $\theta_{13} \sim \lambda$ (CHOOZ), and many more half-integer solutions not displayed.

In this case $N_{R3}$ gives a dominant contribution to the 23 block from the fact that the product of the Dirac coupling of $N_{L2}, N_{L3}$ to $N_{R1}$ times the coupling of $N_{L2}, N_{L3}$ to $N_{R2}$ is $\lambda$ times smaller than the coupling of $N_{L2}, N_{L3}$ to $N_{R3}$ squared, with $N_{R3}$ being...
\[ \lambda \text{ times lighter.} \]

The second set of charges in Table 3 has the dominant right-handed neutrino being heavier by a factor of \( \lambda \) than the other two,

\[
M_{RR} \sim \begin{pmatrix}
0 & \lambda & 0 \\
\lambda & 0 & 0 \\
0 & 0 & 1
\end{pmatrix} < \Sigma >, \quad Y_{\nu}^T \sim \begin{pmatrix}
\lambda^4 & \lambda^2 & \lambda^2 \\
\lambda & \lambda & \lambda \\
\lambda^2 & 1 & 1
\end{pmatrix}
\]

leading to dominant and sub-dominant contributions to \( m_{LL} \) (from \( N_{R3} \) and \( N_{R1}, N_{R2} \) respectively)

\[
m_{\text{off-diag}} \sim \begin{pmatrix}
\lambda^4 & \lambda^2 & \lambda^2 \\
\lambda^2 & 1 & 1 \\
\lambda^2 & 1 & 1
\end{pmatrix} < \Sigma > + \begin{pmatrix}
\lambda^4 & \lambda^2 & \lambda^2 \\
\lambda^2 & \lambda^2 & \lambda^2 \\
\lambda^2 & \lambda^2 & \lambda^2
\end{pmatrix} < \Sigma >
\]

In this case \( N_{R3} \) gives a dominant contribution to the 23 block from the fact that the product of the Dirac coupling of \( N_{L2}, N_{L3} \) to \( N_{R1} \) times the coupling of \( N_{L2}, N_{L3} \) to \( N_{R2} \) is \( \lambda^3 \) times smaller than the coupling of \( N_{L2}, N_{L3} \) to \( N_{R3} \) squared, which overcomes the fact that \( N_{R1} \) and \( N_{R2} \) are \( \lambda \) times lighter than \( N_{R3} \).

The third set of charges in Table 3 implies that the dominant right-handed neutrino is equal in mass to the other two,

\[
M_{RR} \sim \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix} < \Sigma >, \quad Y_{\nu}^T \sim \begin{pmatrix}
\lambda^4 & \lambda & \lambda \\
\lambda^3 & \lambda & \lambda \\
\lambda^2 & 1 & 1
\end{pmatrix}
\]

leading to dominant and sub-dominant contributions to \( m_{LL} \) (from \( N_{R3} \) and \( N_{R1}, N_{R2} \) respectively)

\[
m_{\text{off-diag}} \sim \begin{pmatrix}
\lambda^4 & \lambda^2 & \lambda^2 \\
\lambda^2 & 1 & 1 \\
\lambda^2 & 1 & 1
\end{pmatrix} < \Sigma > + \begin{pmatrix}
\lambda^4 & \lambda^2 & \lambda^2 \\
\lambda^2 & \lambda^2 & \lambda^2 \\
\lambda^2 & \lambda^2 & \lambda^2
\end{pmatrix} < \Sigma >
\]

In this case \( N_{R3} \) gives a dominant contribution to the 23 block from the fact that the product of the Dirac coupling of \( N_{L2}, N_{L3} \) to \( N_{R1} \) times the coupling of \( N_{L2}, N_{L3} \) to \( N_{R2} \) is \( \lambda^2 \) times smaller than the coupling of \( N_{L2}, N_{L3} \) to \( N_{R3} \) squared, with all right-handed neutrinos being approximately degenerate.

\footnote{This is similar to one of the examples in ref. [20].}
In each of the three examples above the hierarchy of neutrino masses $m_{\nu_2}/m_{\nu_3} \sim \lambda^2$ follows from the fact that $N_{R3}$ dominates the contribution to the 23 block by a factor of $\lambda^2$. The large angle $\theta_{23} \sim 1$ (Super-Kamiokande), and the small angle $\theta_{13} \sim \lambda^2$ (CHOOZ) are both determined by the dominant contribution, while $\theta_{12} \sim 1$ (LMA MSW) arises from the sub-dominant contribution due to the cancellation effect of the dominant contribution discussed earlier. Thus the order of the MNS matrix in these examples is given from Eqs.9, 10 as

$$V_{MNS} \sim \begin{pmatrix} 1 & 1 & \lambda^2 \\ 1 & 1 & 1 \\ \lambda^2 & 1 & 1 \end{pmatrix}$$ (74)

The form of $V_{MNS}$ in Eq.74 therefore resembles the so-called bimaximal mixing form. However it should be remembered that the $U(1)$ charge assignments only give the order of all the entries in powers of the expansion parameter $\lambda$, and so we would not expect exact bimaximal mixing, only large 12 and 23 angles. Furthermore in obtaining our estimates of $V_{MNS}$ we have neglected the contribution from the charged lepton sector. Since the lepton doublets of the second and third families tend to have equal $U(1)$ charges $l_2 = l_3$ we would in fact expect order unity contributions to the 23 mixing angle from the lepton sector also. Including the charged lepton contributions we would still expect the 23 mixing angle to remain of order unity, but it may be enhanced or reduced depending on the phase of the lepton contribution. All we can say is that the 23 mixing angle receives a contribution of order unity from both the neutrino sector and the charged lepton sector. Alternatively it is possible that the contribution to 23 mixing coming from the charged lepton sector is suppressed either because $l_2 \neq l_3$ (for examples see near the bottom of Table 2) or for some other dynamical reason.

Finally we note that throughout this paper we have labelled the dominant right-handed neutrino as $N_{R3}$, and the subdominant right-handed neutrinos as $N_{R1}, N_{R2}$. This is completely without loss of generality since any relabelling of the right-handed
neutrinos simply corresponds to exchanging rows of $Y_T^T$. However it is natural to associate rows with the largest Yukawa couplings to the third family which justifies the labelling of $N_{R3}$ as the dominant right-handed neutrino in Eqs.70,72 and is consistent with Eq.68. In the case of a single $U(1)$ family symmetry all the examples of SRHND involve the natural assignment of $N_{R3}$ as the dominant right-handed neutrino. However in theories with two $U(1)'s$ it is possible to make the dominant right-handed neutrino so light that its Yukawa couplings are smaller than those of one of the subdominant right-handed neutrinos, and in this case it is natural to relabel the right-handed neutrinos $N_{R3} \leftrightarrow N_{R2}$ so that $N_{R2}$ becomes the dominant right-handed neutrino. In such theories the Dirac mixing angles may all be small [24].

7 Conclusion and Discussion

The SRHND mechanism gives a very nice understanding of the hierarchy $m_{\nu_2} \ll m_{\nu_3}$ due to an approximately vanishing subdeterminant of the light mass matrix. The large Super Kamiokande mixing angle and small CHOOZ angle arise from the conditions in Eq.20. The solar neutrino spectrum is accounted for by including the effect of additional right-handed neutrinos which give a small perturbation to the effect of the dominant right-handed neutrino. As originally formulated the SRHND mechanism was applied to the SMA MSW solution [10, 11], and it was not clear to what extent it could be applied to the LMA MSW solution. Although there are isolated examples of LMA MSW solutions in $U(1)$ family models in the literature [20] there has been no systematic study of the conditions for achieving the LMA MSW solution in these models to our knowledge, and certainly not in the framework of SRHND. In this paper we have developed simple analytical arguments, beginning with the one and two right-handed neutrino cases, then extending the arguments to the three right-handed neutrino cases, which determine the conditions for SRHND and the LMA.
MSW solutions. Having established the criteria for the LMA MSW solution, we then examined whether these conditions could be met in the framework of theories containing an abelian family symmetry. Our main conclusion is that the conditions for SRHND and LMA MSW are very easy to satisfy, and may be achieved in the simplest type of theory based on a single $U(1)$ family symmetry, where the abundance of examples is very encouraging. From the results in Tables 1-3 all the neutrino matrices of interest may be readily constructed, and this was done explicitly for three examples to demonstrate that the dominant right-handed neutrino may be lighter, heavier or equal in mass to the subdominant right-handed neutrinos.

One of the basic consequences of SRHND is a hierarchical mass spectrum with either Eq.43 or Eq.46. The fully hierarchical spectrum in Eq.43 requires the specific condition in Eq.42, which amounts to a sort of double right-handed neutrino dominance (DRHND) which imposes an extra restriction on the theory, so the more generic expectation is Eq.46. We emphasise that there is no possibility of an inverted mass spectrum with

$$m_{\nu_1} \ll m_{\nu_2} \sim m_{\nu_3}. \quad (75)$$

It is clear that SRHND has no preference for the LMA MSW solution over the SMA MSW solution, and without going beyond the basic framework here it is not possible to make any further predictions. The question then naturally arises of how to include the ideas of SRHND in a unified theory which accounts for all the quark and lepton masses and mixing angles? Although this question is really beyond the scope of the present paper, we would like to make a few closing remarks on this subject.

There are two main effects which come from embedding SRHND into a unified theory. The first is the question of RG running of the parameters from the unification scale down to the weak scale, and the second is that of the constraints imposed by the unified theory on the quark and lepton Yukawa couplings. Regarding the first
point, several authors have recently analysed the effect of RG running, and have all
concluded that in the case of hierarchical neutrino masses considered here the effects
are very small [25]. Unless the τ Yukawa coupling is quite large at the unification
scale (as may be the case for large tan β ) the corrections are always quite small in
general, and the fact that our scheme here involves no fine-tuning of any kind (as
evidenced by the fact that all our arguments have involved an order of magnitude
expansion in terms of the Wolfenstein parameter λ) means that it is quite robust to the
expected small radiative corrections to Yukawa couplings. As far as the second point
is concerned, unification may in general give powerful restrictions on the choice of
U(1) charges which may be assumed for the leptons. For example the combination of
SU(2)R and a quark-lepton symmetry will provide a powerful restriction on the lepton
charges which must essentially be chosen to be equal to the quark charges. However
the effect of group theoretical Clebsch coefficients, which must necessarily be present
in order to account for other features of the quark and lepton mass spectrum, will
be expected to have an important effect on neutrino physics. These questions will be
addressed in detail in a future publication [26].

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