

\textbf{p-HARMONIC FUNCTIONS BY WAY OF INTRINSIC MEAN VALUE PROPERTIES}

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Abstract. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain satisfying the uniform exterior cone condition. We show that $p$-harmonic functions in $\Omega$ with continuous boundary values can be approximated by continuous solutions of an intrinsic nonlinear mean value property.

1. Introduction

1.1. Background. The aim of this paper is twofold. First we discuss existence and uniqueness of functions satisfying certain nonlinear mean value properties in a domain with prescribed continuous boundary values. Then we study in which instances such functions converge to the solution of the Dirichlet problem for the $p$-laplacian in the domain. The primary motivation is, of course, the usual mean value property for harmonic functions. By well known results of Gauss and Koebe, a continuous function $u$ in a domain $\Omega \subset \mathbb{R}^n$ is harmonic if and only if it satisfies the mean value property

\begin{equation}
(1.1) \quad u(x) = \int_{B(x, \rho)} u(y) \, dy
\end{equation}

for each $x \in \Omega$ and each $\rho > 0$ such that $0 < \rho < \text{dist}(x, \partial \Omega)$. The mean value property plays a relevant role in Geometric Function Theory and is indeed the fundamental key of the interplay between Classical Potential Theory, Probability and Brownian motion.

The problem of how many radii $\rho$ in (1.1) are necessary to guarantee harmonicity has also a very classical flavour. One of the most representative results in this direction is a theorem due to Volterra (for regular domains) and Kellogg (in the general case): if $\Omega$ is bounded, $u \in C(\Omega)$ and if for each $x \in \Omega$ there is a radius $\rho = \rho(x)$, with $0 < \rho \leq \text{dist}(x, \partial \Omega)$, such that (1.1) holds, then $u$ is harmonic in $\Omega$ ([Vol, Kel]). Therefore, under appropriate hypothesis, the mean value property for a single radius (depending on the point) implies harmonicity. See [NV] for a survey containing a variety of results related to the mean value property.

In the last decades, substantial efforts have been devoted to determine the stochastic structure of certain nonlinear PDE’s, a crucial step being the identification

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of the corresponding (nonlinear) mean value properties. In this paper we will focus on the \( p \)-laplacian which, for \( 1 < p < \infty \), is the divergence-form differential operator given by

\[
\Delta_p u := \text{div}(|\nabla u|^{p-2} \nabla u).
\]

Weak solutions \( u \in W^{1,p}_{\text{loc}}(\Omega) \) of \( \Delta_p u = 0 \) are said \( p \)-harmonic functions. Observe that the theory is nonlinear unless \( p = 2 \), in which case we recover harmonic functions. We refer to [Lin] for background and basic properties of \( p \)-harmonic functions.

Unfortunately, if \( p \neq 2 \) neither Gauss-Koebe nor Volterra-Kellog theorems have a \( p \)-harmonic counterpart and the nature of the connections between \( p \)-harmonic functions and mean value properties is more subtle than in the classical context. We start by some basic facts in the smooth case. If \( u \in C^2 \) and \( \nabla u \neq 0 \) then a direct computation gives

\[
(1.2) \quad \Delta_p u = |\nabla u|^{p-2} \left( \Delta u + (p-2) \frac{\Delta_\infty u}{|\nabla u|^2} \right),
\]

where

\[
\Delta_\infty u = \sum_{i,j=1}^n u_{x_i} u_{x_j} u_{x_i x_j}
\]

is the so called \( \infty \)-laplacian in \( \mathbb{R}^n \). Then (1.2) shows that, in the smooth case and away from the critical points, the \( p \)-laplacian can be understood as a linear combination of the usual laplacian and the normalized \( \infty \)-laplacian.

By using the viscosity characterization of \( p \)-harmonic functions ([JLM]), Manfredi, Parviainen and Rossi characterized \( p \)-harmonicity in terms of nonlinear mean value properties in [MPR10]. Namely, a function \( u \in C(\Omega) \) is \( p \)-harmonic in \( \Omega \subset \mathbb{R}^n \) if and only if \( u \) satisfies the asymptotic \( p \)-mean value property

\[
(1.3) \quad u(x) = \frac{p-2}{n+p} \left( \frac{1}{2} \sup_{B(x,\varepsilon)} u + \frac{1}{2} \inf_{B(x,\varepsilon)} u \right) + \frac{n+2}{n+p} \int_{B(x,\varepsilon)} u(y) \, dy + o(\varepsilon^2)
\]

in a viscosity sense for each \( x \in \Omega \). Moreover, if \( n = 2 \), this characterization holds also in the classical sense ([LM, AL16-2]), while for \( n \geq 3 \) it is not known whether \( p \)-harmonic functions satisfy (1.3) pointwise or not.

From a probabilistic point of view, the influential work of Peres, Schramm, Sheffield and Wilson ([PSSW]) established a game-theoretic interpretation of the \( \infty \)-laplacian, where the functional equation

\[
u_\varepsilon(x) = \frac{1}{2} \sup_{B(x,\varepsilon)} u_\varepsilon + \frac{1}{2} \inf_{B(x,\varepsilon)} u_\varepsilon
\]

appears as a dynamic programming principle of a two-player zero-sum tug-of-war game. A similar interpretation for the \( p \)-laplacian, \( p \in [2, \infty] \), was considered in [PS]. Manfredi, Parviainen and Rossi gave a systematic twist to the theory, from both an analytic and probabilistic point of view ([MPR10, MPR12]). In particular, in [MPR12] the term \( p \)-harmonious was introduced to denote (not necessarily
Continuous) solutions of the functional equation

\[(1.4) \quad u_\varepsilon(x) = \frac{p - 2}{n + p} \left( \frac{1}{2} \sup_{B(x, \varepsilon)} u_\varepsilon + \frac{1}{2} \inf_{B(x, \varepsilon)} u_\varepsilon \right) + \frac{n + 2}{n + p} \int_{B(x, \varepsilon)} u_\varepsilon(y) \, dy \]

for each \(x \in \Omega\). Note, however, that this dynamic programming principle raises some technical problems, coming from the fact that the balls \(B(x, \varepsilon)\) eventually escape the domain. In order to properly pose the Dirichlet problem for such \(p\)-harmonious functions, the authors in [MPR12] needed to extend a given \(f \in C(\partial \Omega)\) to the strip \(\{x \in \mathbb{R}^n \setminus \Omega : \text{dist}(x, \partial \Omega) \leq \varepsilon\}\) and proved that, if \(\Omega \subset \mathbb{R}^n\) is bounded and satisfies a so called boundary regularity condition, then there is a unique \(p\)-harmonious function \(u_\varepsilon\) having \(f\) as boundary values (in the extended sense). Furthermore, \(u_\varepsilon \to u\) uniformly in \(\Omega\) as \(\varepsilon \to 0\), where \(u\) is the unique \(p\)-harmonic function solving the Dirichlet problem in \(\Omega\) with boundary data \(f\). It should be noted that domains satisfying a uniform exterior cone condition (see Definition 1.4 below) satisfy the boundary regularity condition, in the sense of [MPR12]. See also [LPS, AHP, DMP] for further approaches.

In this paper we deal with a modified version of (1.4) in which the balls \(B(x, \varepsilon)\) are replaced by balls of variable radius \(B(x, \rho(x))\), where \(0 < \rho(x) \leq \text{dist}(x, \partial \Omega)\). We want to emphasize that, while the constant radius hypothesis is convenient for game-theoretic applications, the variable radius setting is natural for at least two reasons: it is closely related to the classical theory (\(p = 2\)) and it is intrinsic, in the sense that no extension of the domain is needed (this explains the term intrinsic in the title of the paper). In [AL16-1] and [AL18] we proved a number of results about existence, uniqueness and regularity of solutions of (1.4) with variable radius under different hypothesis on \(\rho\) and \(\Omega\). In this paper we substantially improve the existence results of [AL16-1] and provide also an approximation result. Before stating the main results we need some definitions.

1.2. Preliminary definitions and main results. From all the above considerations, it makes sense to introduce variable radius versions of (1.4). This is the aim of the following definitions.

**Definition 1.1.** Let \(\Omega \subset \mathbb{R}^n\) be a bounded domain. We say that a function \(\rho : \Omega \to (0, \infty)\) is an admissible radius function in \(\Omega\) if

1. \(0 < \rho(x) \leq \text{dist}(x, \partial \Omega)\) for every \(x \in \Omega\), and
2. \(\rho(x) = 0\) if and only if \(x \in \partial \Omega\).

Hereafter we will write \(B_\rho(x) := B(x, \rho(x))\).

**Definition 1.2.** Let \(\Omega \subset \mathbb{R}^n\) be a bounded domain and \(\rho\) an admissible radius function in \(\Omega\). Let \(S_\rho\) and \(M_\rho\) be the operators in \(L^\infty(\Omega)\) defined by

\[
S_\rho u(x) := \begin{cases} 
\frac{1}{2} \sup_{B_\rho(x)} u + \frac{1}{2} \inf_{B_\rho(x)} u & \text{if } x \in \Omega, \\
u(x) & \text{if } x \in \partial \Omega, 
\end{cases}
\]

and

\[
M_\rho u(x) := \begin{cases} 
\int_{B_\rho(x)} u(y) \, dy & \text{if } x \in \Omega, \\
u(x) & \text{if } x \in \partial \Omega, 
\end{cases}
\]
for every \( u \in L^\infty(\Omega) \). In addition, for a fixed \( p \in [2, \infty) \), we define the operator \( \mathcal{T}_{\rho,p} \) in \( L^\infty(\Omega) \) as the following linear combination of \( \mathcal{S}_\rho \) and \( \mathcal{M}_\rho \)

\[
\mathcal{T}_{\rho,p}(u) = \frac{p-2}{n+p} \mathcal{S}_\rho + \frac{n+2}{n+p} \mathcal{M}_\rho.
\]  

**Definition 1.3.** Let \( \Omega \subset \mathbb{R}^n \) be a domain and \( \rho \) be an admissible radius function in \( \Omega \). Let \( p \in [2, \infty) \). We say that a function \( u \) in \( \Omega \) is \( p \)-harmonious (w.r.t. \( \rho \)) if \( \mathcal{T}_{\rho,p}(u) = u \).

Note that \( \mathcal{T}_{\rho,2} = \mathcal{M}_\rho \) and that, formally, \( \mathcal{T}_{\rho,\infty} = \mathcal{S}_\rho \). As the following proposition says, \( \mathcal{T}_{\rho,p} \) preserves the class \( C(\Omega) \), provided the admissible radius function \( \rho \) is continuous.

**Proposition 1.1 ([AL18, Proposition 4.1]).** If \( \rho \in C(\Omega) \), then \( \mathcal{T}_{\rho,p} : C(\Omega) \to C(\Omega) \).

From now on we will take \( C(\Omega) \) as the natural function space where the operators \( \mathcal{T}_{\rho,p} \) are defined. In addition, \( \mathcal{T}_{\rho,p} \) satisfies the following properties:

1. **Affine invariance:** if \( a, b \in \mathbb{R} \) and \( u \in C(\Omega) \), then \( \mathcal{T}_{\rho,p}(au+b) = a\mathcal{T}_{\rho,p}u + b \).
2. **Monotonicity:** if \( u,v \in C(\Omega) \) such that \( u \leq v \), then \( \mathcal{T}_{\rho,p}u \leq \mathcal{T}_{\rho,p}v \).
3. **Non-expansiveness:** if \( u,v \in C(\Omega) \), then \( \|\mathcal{T}_{\rho,p}u - \mathcal{T}_{\rho,p}v\|_\infty \leq \|u - v\|_\infty \).
4. \( \inf_{B_\rho(x)} u \leq \mathcal{T}_{\rho,p}u(x) \leq \sup_{B_\rho(x)} u \) for every \( x \in \Omega \).

It is easy to check that the \( k \)-th iteration of \( \mathcal{T}_{\rho,p} \), denoted by \( \mathcal{T}_{\rho,p}^k \), also satisfies the above four properties.

Let \( f \in C(\partial\Omega) \). In this paper we are interested in existence and uniqueness of solutions of the Dirichlet problem

\[
\begin{cases}
\mathcal{T}_{\rho,p}u = u & \text{in } \Omega, \\
u &= f & \text{on } \partial\Omega.
\end{cases}
\]

We will also discuss assumptions under which normalized solutions of (1.6) converge to the corresponding solution of the Dirichlet problem for the \( p \)-laplacian.

Notice that (1.6) is equivalent to the problem of finding a fixed point of \( \mathcal{T}_{\rho,p} \) among all continuous functions with prescribed continuous boundary data \( f \). Given \( f \in C(\partial\Omega) \), we define \( \mathcal{K}_f \) as the set of all norm-preserving continuous extensions of \( f \) to \( \Omega \),

\[
\mathcal{K}_f := \{ u \in C(\overline{\Omega}) : u|_{\partial \Omega} = f \text{ and } \|u\|_{\infty,\Omega} = \|f\|_{\infty,\partial\Omega} \},
\]

and we observe that, by Proposition 1.1 and the non-expansiveness of the operator, \( \mathcal{T}_{\rho,p}(\mathcal{K}_f) \subset \mathcal{K}_f \). Then (1.6) has a solution in \( C(\Omega) \) if and only if \( \mathcal{T}_{\rho,p} \) has a fixed point in \( \mathcal{K}_f \).

It is worth to recall at this point that the Dirichlet problem (1.6) for \( p = \infty \) was first studied by Le Gruyer and Archer in [LA] in the context of metric spaces. There, the functions satisfying \( \mathcal{S}_\rho u = u \) were originally called harmonious functions and studied in connection to extension problems of continuous functions in metric spaces. It follows, as a particular case of results in [LA], that if \( \Omega \subset \mathbb{R}^n \) is a bounded
and convex domain and $\rho$ is 1-Lipschitz then the Dirichlet problem
\begin{equation}
\begin{aligned}
S_\rho u &= u \text{ in } \Omega, \\
u &= f \text{ on } \partial \Omega,
\end{aligned}
\end{equation}
has a unique solution for each $f \in C(\overline{\Omega})$. One of the most important features of the operator $S_\rho$ (with 1-Lipschitz $\rho$) is that it preserves the concave modulus of continuity: if $\tilde{\omega}_u$ is the lowest concave modulus of continuity of $u$ in $\Omega$, then $\tilde{\omega}_{S_\rho u} \leq \tilde{\omega}_u$. This invariance property allows the use of Schauder’s fixed point theorem to prove existence in (1.8). Unfortunately, the operators $T_{\rho,p}$ do not preserve in general the modulus of continuity, thus Schauder’s Theorem is no longer available and different strategies are required to obtain fixed points.

In order to state the main theorems of this work, we previously need to impose certain geometrical condition on the boundary of the domain.

**Definition 1.4** (Uniform Exterior Cone Condition). Let $\alpha \in (0, \frac{\pi}{2})$ and $r > 0$. We denote by $K_{\alpha,r}$ the truncated circular cone
\[ K_{\alpha,r} = \{ x \in \mathbb{R}^n : x_1 \leq -|x| \cos \alpha \text{ and } |x| \leq r \}. \]
We say that a domain $\Omega \subset \mathbb{R}^n$ satisfies the uniform exterior cone condition if there exist constants $\alpha \in (0, \frac{\pi}{2})$ and $r > 0$ such that for every $\xi \in \partial \Omega$ there is a rotation $R \in SO(n)$ in $\mathbb{R}^n$ such that
\[ \xi + R(K_{\alpha,r}) \subset \mathbb{R}^n \setminus \Omega. \]

**Remark.** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Then $\Omega$ is Lipschitz if and only if both $\Omega$ and $\mathbb{R}^n \setminus \Omega$ satisfy the uniform exterior cone condition (see [Gri]).

We now state the first main result of this paper (compare with [AL16-1] where the same result was proven under the assumption that $\Omega$ is strictly convex and the admissible radius function is 1-Lipschitz).

**Theorem 1.** Let $\Omega \subset \mathbb{R}^n$ be a domain satisfying the uniform exterior cone condition and $p \in [2, \infty)$. Suppose that $\rho \in C(\overline{\Omega})$ is a continuous admissible radius function in $\Omega$ satisfying
\[ \lambda \text{dist}(x, \partial \Omega)^{\beta} \leq \rho(x) \leq \Lambda \text{dist}(x, \partial \Omega) \]
for all $x \in \Omega$, where
\begin{equation}
\beta \geq 1, \quad 0 < \Lambda < 1 - \left( \frac{p - 2}{n + p} \right)^{1/\beta} \quad \text{and} \quad 0 < \lambda \leq \Lambda \left( \frac{\text{diam } \Omega}{2} \right)^{1-\beta}.
\end{equation}
Then for any $f \in C(\partial \Omega)$ there exists a unique solution $u_\rho \in C(\overline{\Omega})$ to the Dirichlet problem
\begin{equation}
\begin{aligned}
T_{\rho,p} u_\rho &= u_\rho \quad \text{in } \Omega, \\
u_\rho &= f \quad \text{on } \partial \Omega,
\end{aligned}
\end{equation}
where $T_{\rho,p}$ is the averaging operator defined in (1.5). Furthermore, for any norm-preserving continuous extension $u \in C(\overline{\Omega})$ of $f$, the sequence of iterates $\{ T_{\rho,p}^k u \}_{k}$ converges uniformly to $u_\rho$ in $\overline{\Omega}$.

By letting $\beta = 1$, we obtain the following corollary as an immediate consequence.
Corollary 1.1. Let $\Omega \subset \mathbb{R}^n$ be a domain satisfying the uniform exterior cone condition and $p \in [2, \infty)$. Suppose that $\rho \in C(\overline{\Omega})$ is a continuous admissible radius function in $\Omega$ satisfying

$$\lambda \operatorname{dist}(x, \partial \Omega) \leq \rho(x) \leq \Lambda \operatorname{dist}(x, \partial \Omega)$$

for all $x \in \Omega$, where

$$0 < \lambda \leq \Lambda < \frac{n + 2}{n + p}. $$

Then for any $f \in C(\partial \Omega)$ there exists a unique solution $u_\rho \in C(\overline{\Omega})$ to the Dirichlet problem (1.10). Furthermore, for any norm-preserving continuous extension $u \in C(\overline{\Omega})$ of $f$, the sequence of iterates $\{T_{\rho, p}^k u\}_k$ converges uniformly to $u_\rho$ in $\overline{\Omega}$.

The existence part in Theorem 1 is obtained from the equicontinuity and subsequent uniform convergence in $\overline{\Omega}$ of the iterates $\{T_{\rho, p}^k u\}_k$, for $u \in C(\overline{\Omega})$. When $p = 2$, the fact that the sequence $\{M_{\rho}^k u\}_k$ converges uniformly in $\overline{\Omega}$ to the solution of the (harmonic) Dirichlet Problem in $\Omega$ with boundary data $f = u|_{\partial \Omega}$ was already observed by Lebesgue, in the case that $\Omega$ is regular and $\rho(x) = \operatorname{dist}(x, \partial \Omega)$ ([Leb], see also [Car] for a more general approach in this direction). A significative difference between Lebesgue’s setting and ours is that we actually use the convergence of iterates to prove existence of solution to the Dirichlet problem while in Lebesgue’s note, existence is taken for granted and convergence is obtained as a consequence. Interior equicontinuity is more straightforward and was obtained, in different contexts, in [AL16-1] and [AL18]. However, boundary equicontinuity turns out to be a much more delicate matter. In [AL16-1] boundary equicontinuity was established under the assumption that $\Omega$ is strictly convex and $\rho$ is 1-Lipschitz.

Our approach here is based on the construction of explicit barriers for $T_{\rho, p}$, having the additional advantage that they work for domains satisfying the uniform exterior cone condition. We would like to point out that, since the operators $T_{\rho, p}$ are not local, some steps in Perron’s method (like Poisson’s modification) do not work in our setting and we need ad hoc arguments to prove existence. Our approach gives, in particular, a more constructive proof of the existence of solution to the Dirichlet problem for the $p$-laplacian in domains satisfying a uniform exterior cone condition.

The next theorem is our second main result of the paper. It says that, when considering a family of admissible radius functions going to zero in an appropriate way, the corresponding solutions given by Theorem 1 converge uniformly to the $p$-harmonic solution of the Dirichlet problem.

Theorem 2. Let $\Omega \subset \mathbb{R}^n$ be a domain satisfying the uniform exterior cone condition and $p \in [2, \infty)$. Suppose that $\{\rho_\varepsilon\}_{0 < \varepsilon \leq 1}$ is a collection of continuous admissible radius functions in $\Omega$ satisfying

$$\lambda \operatorname{dist}(x, \partial \Omega)^\beta \leq \frac{\rho_\varepsilon(x)}{\varepsilon} \leq \Lambda \operatorname{dist}(x, \partial \Omega)$$

for all $x \in \Omega$ and every $0 < \varepsilon \leq 1$, where $\beta$, $\lambda$ and $\Lambda$ are as in (1.9). Given any continuous boundary data $f \in C(\partial \Omega)$, let $u_\varepsilon$ be the solution of

$$\begin{cases}
T_{\rho_\varepsilon, p} u_\varepsilon = u_\varepsilon & \text{in } \Omega, \\
u_\varepsilon = f & \text{on } \partial \Omega.
\end{cases}$$
Then \( u_\varepsilon \rightarrow u_0 \) uniformly in \( \overline{\Omega} \), where \( u_0 \) is the unique \( p \)-harmonic function in \( \Omega \) solving
\[
\begin{aligned}
\Delta_p u_0 &= 0 \quad \text{in } \Omega, \\
u_0 &= f \quad \text{on } \partial \Omega.
\end{aligned}
\]

A fundamental step in the proofs of Theorems 1 and 2 is the construction of explicit barriers in the complement of a cone. As an additional feature, the barriers that we construct do not depend on \( p \) and turn out to be simultaneously barriers for the operators \( T_{p,p} \) and for the \( p \)-laplacian. We believe that the construction has an independent interest which might be useful in other situations.

**Definition 1.5.** Let \( \Omega \subset \mathbb{R}^n \) be a domain and \( \xi \in \partial \Omega \). We say that a function \( w_\xi \in C(\overline{\Omega}) \) is a \( T_{p,p} \)-barrier at \( \xi \) if \( w_\xi > 0 \) in \( \overline{\Omega} \setminus \{\xi\} \), \( w_\xi(\xi) = 0 \), and \( w_\xi \geq T_{p,p} w_\xi \) in \( \Omega \). In this case, we say that \( \xi \) is a \( T_{p,p} \)-regular point. Moreover, a bounded domain \( \Omega \subset \mathbb{R}^n \) is \( T_{p,p} \)-regular if every point on \( \partial \Omega \) is \( T_{p,p} \)-regular.

**Theorem 3.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain satisfying the uniform exterior cone condition with constants \( \alpha \in (0, \frac{n}{2}) \) and \( r > 0 \) as in Definition 1.4. Choose \( \gamma \) such that
\[
0 < \gamma < \frac{8(\sin \alpha)^{n-2}}{n(13\pi^2 + 4\pi)}.
\]

Then for each \( \xi \in \partial \Omega \) there exists a function \( w_\xi \in C(\overline{\Omega}) \) such that \( w_\xi(\xi) = 0 \), \( w_\xi > 0 \) in \( \overline{\Omega} \setminus \{\xi\} \),
\[
\begin{aligned}
w_\xi(x) &\geq \int_{B(x,\varrho)} w_\xi(y) \, dy, \\
w_\xi(x) &\geq \frac{1}{2} \sup_{B(x,\varrho)} w_\xi + \frac{1}{2} \inf_{B(x,\varrho)} w_\xi
\end{aligned}
\]
for every ball \( B(x,\varrho) \subset \Omega \) and
\[
\mathcal{L}(|x - \xi|) \leq w_\xi(x) \leq \gamma^{-2}|x - \xi|^{\gamma}
\]
for every \( x \in \overline{\Omega} \), where \( \mathcal{L}(t) := \alpha^2 - n \min\{t, r\}^\gamma \). In particular, for each \( p \in [2, \infty] \) and any admissible radius function \( \rho \) in \( \Omega \), \( w_\xi \) is, simultaneously, a \( T_{p,p} \)-barrier and a barrier for the \( p \)-laplacian at \( \xi \) in \( \Omega \).

The rest of the paper is organized as follows: in section 2 we show the existence of \( T_{p,p} \)-barriers for domains satisfying the uniform exterior cone condition (Theorem 3). Then in sections 3 and 4 we use these barriers to prove existence of fixed points of \( T_{p,p} \) (Theorem 1) and their convergence to \( p \)-harmonic functions (Theorem 2), respectively. For the sake of convenience and, whenever the role of \( p \in [2, \infty] \) causes no confusion, we will write \( T_\rho \) instead of \( T_{p,p} \) in what follows.

2. Barriers for \( T_\rho \)

Our goal is to construct a \( T_\rho \)-barrier at each boundary point and consequently, to show that each point on the boundary is \( T_\rho \)-regular. Fix \( \xi \in \partial \Omega \). Recalling the definition of the uniform exterior cone condition, there exist constants \( \alpha \in (0, \frac{n}{2}) \) and \( r > 0 \) and a rotation \( R_\xi \in SO(n) \) such that
\[
\xi + R_\xi(K_{\alpha,r}) \subset \mathbb{R}^n \setminus \Omega,
\]
where
\[
K_{\alpha,r} = \{ x \in \mathbb{R}^n : x_1 \leq -|x| \cos \alpha \text{ and } |x| \leq r \}.
\]
We observe that after a translation and a rotation, we can assume that $\xi = 0$ and $R_\xi = \text{Id}$, in which case we define a bigger domain $\Omega_{\alpha,r} = \mathbb{R}^n \setminus K_{\alpha,r}$ so that $\Omega \subset \Omega_{\alpha,r}$.

Then our aim is to construct a function $w$ in $\Omega_{\alpha,r}$ such that its restriction to $\Omega$, $w|_\Omega$, verifies $w \geq T_\rho w$ for every admissible radius function $\rho$ in $\Omega$.

We split the construction of such function in two steps. First, we construct the barrier at 0 for the complement of an unbounded cone along the negative $x_1$-axis. Second, we adapt the argument to work for the complement of a truncated cone.

2.1. Barrier for the complement of a whole cone. Let $\alpha \in (0, \frac{\pi}{2})$ and define

$$\Omega_\alpha := \{ x \in \mathbb{R}^n : x_1 > -|x| \cos \alpha \}.$$ 

We will use polar coordinates with respect to the $x_1$-axis, that is, we assign a pair $(R, \theta)$ to each $x \in \mathbb{R}^n$, where $R = |x|$ and $\theta = \arccos \left( \frac{x_1}{|x|} \right) \in [0, \pi)$ is the angle between $x$ and the positive $x_1$-axis. Then,

$$\Omega_\alpha = \{ x \in \mathbb{R}^n : 0 \leq \theta < \pi - \alpha \}.$$ 

Before stating the main result of this section, we define an auxiliary function $\phi : (-\pi, \pi) \to [0, \infty)$ as the solution of the differential equation

$$\begin{cases} \phi''(\theta) + (n-2)\phi'(\theta) \cot \theta = 1, \\ \phi(0) = \phi'(0) = 0, \end{cases}$$

which has the integral form

$$\phi(\theta) = \int_0^{[\theta]} \int_0^t \left( \frac{\sin s}{\sin t} \right)^{n-2} ds \, dt$$

for every $\theta \in (-\pi, \pi)$ (see [HK, Lemma 2.4]). We review some of the properties of the auxiliary function $\phi$ in the following lemma.

**Lemma 2.1.** The function $\phi : (-\pi, \pi) \to [0, \infty)$ defined in (2.3) satisfies:

i) $\phi \in C^2(-\pi, \pi)$.

ii) $\phi$ is increasing in $(0, \pi)$ and convex in $(-\pi, \pi)$.

iii) For every $|\theta| \leq \pi - \alpha$,

$$0 \leq \phi(\theta) \leq \frac{\pi^2}{8} + \frac{\pi}{2(\sin \alpha)^{n-2}} \quad \text{and} \quad \phi'(\theta) \leq \frac{\pi}{(\sin \alpha)^{n-2}}.$$ 

**Proof.** It is easy to check that $\phi \in C^2(-\pi, \pi)$. Hereafter, we restrict the analysis to the interval $[0, \pi)$. By differentiation of (2.3),

$$\phi'(\theta) = \frac{1}{(\sin \theta)^{n-2}} \int_0^\theta (\sin t)^{n-2} \, dt \geq 0,$$

so $\phi$ is increasing in $(0, \pi)$. Next, since $\phi$ satisfies (2.2), then

$$\phi''(\theta) = \frac{(\sin \theta)^{n-1} - (n-2) \cos \theta \int_0^\theta (\sin t)^{n-2} \, dt}{(\sin \theta)^{n-1}}$$

for $0 < \theta < \pi$. Observe that if $\pi/2 \leq \theta < \pi$, then $\cos \theta \leq 0$, so $\phi'' \geq 0$ in $[\pi/2, \pi)$. For $0 \leq \theta \leq \pi/2$ define

$$\psi(\theta) = (\sin \theta)^{n-1} - (n-2) \cos \theta \int_0^\theta (\sin t)^{n-2} \, dt.$$
and observe that $\psi(0) = 0$ and
\[
\psi'(\theta) = \cos\theta (\sin\theta)^{n-2} + (n-2) \sin\theta \int_0^\theta (\sin t)^{n-2} \, dt \geq 0
\]
for $0 \leq \theta \leq \frac{\pi}{2}$. Therefore $\phi$ is convex in $(0, \pi)$.

To show (2.4) note first that
\[
\hat{\theta}(\sin t) t^{n-2} \leq \theta \max_{\frac{\pi}{2} \leq t \leq \theta} \{ (\sin t)^{n-2} \},
\]
for each $0 < \theta < \pi$, so
\[
\phi'(\theta) \leq \begin{cases} 
\theta & \text{if } 0 \leq \theta \leq \frac{\pi}{2}, \\
\frac{\theta}{(\sin\theta)^{n-2}} & \text{if } \frac{\pi}{2} \leq \theta < \pi.
\end{cases}
\]

Since $\phi$ is convex in $(0, \pi)$, then $\phi'$ is increasing in $(0, \pi)$, and recalling that $\alpha \in (0, \frac{\pi}{2})$ we obtain that
\[
\phi'(\theta) \leq \phi'(\pi - \alpha) \leq \frac{\pi}{(\sin\alpha)^{n-2}},
\]
for every $0 \leq \theta \leq \pi - \alpha$, which is the second inequality in (2.4).

On the other hand, for $\frac{\pi}{2} \leq \theta < \pi$ we get
\[
\int_\frac{\pi}{2}^\theta \frac{t}{(\sin t)^{n-2}} \, dt \leq \left( \theta - \frac{\pi}{2} \right) \max_{\frac{\pi}{2} \leq t \leq \theta} \left\{ \frac{1}{(\sin t)^{n-2}} \right\} = \frac{\theta - \frac{\pi}{2}}{\sin\theta^{n-2}},
\]
Integrating (2.5) we obtain
\[
0 \leq \phi(\theta) \leq \frac{\pi^2}{8} + \frac{\theta - \frac{\pi}{2}}{(\sin\theta)^{n-2}}
\]
for every $\frac{\pi}{2} \leq \theta < \pi$. In particular, since $\phi$ is increasing,
\[
\phi(\theta) \leq \phi(\pi - \alpha) \leq \frac{\pi^2}{8} + \frac{\frac{\pi}{2} - \alpha}{(\sin(\pi - \alpha))^{n-2}} \leq \frac{\pi^2}{8} + \frac{\pi}{2(\sin\alpha)^{n-2}},
\]
and the first in equality (2.4) follows. $\square$

**Lemma 2.2.** For $\alpha \in (0, \frac{\pi}{2})$ let
\[
\Omega_\alpha := \{ x \in \mathbb{R}^n : x_1 > -|x| \cos\alpha \}
\]
and $U : \overline{\Omega}_\alpha \to \mathbb{R}$ be the function defined as
\[
U(x) = \begin{cases} 
|x| \gamma(A - \phi(\theta)), \\
\theta = \arccos \left( \frac{\alpha}{|x|} \right),
\end{cases}
\]
where $\phi : (-\pi, \pi) \to [0, \infty)$ is the auxiliary function defined in (2.3) and $A > 0$, $\gamma \in (0, \frac{1}{2}]$ are constants satisfying
\[
\frac{\pi^2}{8} + \frac{3\pi^2 + \pi}{2(\sin\alpha)^{n-2}} \leq A \leq \frac{1}{\gamma(\gamma + n - 2)}.
\]
Then $U \in C^2(\Omega_\alpha) \cap C(\overline{\Omega}_\alpha)$, $U(0) = 0$, $U > 0$ in $\overline{\Omega}_\alpha \setminus \{0\}$,
\[
U(x) \geq \int_{B(x, \varepsilon)} U(y) \, dy, \quad U(x) \geq \frac{1}{2} \sup_{B(x, \varepsilon)} U + \frac{1}{2} \inf_{B(x, \varepsilon)} U
\]
for every ball $B(x, ρ) ⊂ Ω_α$, and

$$α^{2-n} |x|^γ ≤ U(x) ≤ γ^{-2} |x|^γ$$

(2.9)

for every $x ∈ Ω_α$. In particular, for each $p ∈ [2, ∞]$ and any admissible radius function $ρ$ in $Ω_α$, $U$ is, simultaneously, a $T_{p,p}$-barrier and a barrier for the $p$-laplacian at 0 in $Ω_α$.

2.2. Proof of Lemma 2.2. The regularity of $U$ is a direct consequence of its construction. To see (2.9) we recall (2.4) together with (2.7) to get that, for every $0 ≤ θ ≤ π − α$,

$$0 < \frac{3π^2}{2(\sin α)^{n-2}} ≤ A − φ(θ) ≤ \frac{1}{γ(γ + n - 2)}.$$

Then (2.9) follows.

In order to show (2.8), let us recall from [HK, Lemma 2.4] the expression of the laplacian of $U$ in the polar coordinates $x ⇔ (R, θ)$,

$$ΔU = R^{γ-2} [-φ''(θ) − (n-2)φ'(θ) cot θ + γ(γ + n - 2)(A − φ(θ))],$$

which, together with (2.2), gives

$$ΔU = -R^{γ-2} [1 - γ(γ + n - 2)(A − φ(θ))].$$

Since $φ ≥ 0$ and $A$ and $γ$ satisfy (2.7), it turns out that $ΔU ≤ 0$. That is, $U$ is superharmonic and the first inequality in (2.8) follows by the mean value property for superharmonic functions.

Before proving the second inequality in (2.8), we first note that, since $U$ is rotationally invariant with respect to the $x_1$-axis, the problem is actually bidimensional. Therefore we replace $x ∈ R^n$ by the complex number $z = Re^{iθ}$, where $R = |x|$, $cos θ = \frac{Re}{|x|}$ and assume that $Ω_α$ lies in the complex plane, so

$$Ω_α = \{z = Re^{iθ} : R > 0, |θ| < π − α\}.$$

Then the second inequality in (2.8) is equivalent to

$$U(z_0) ≥ \frac{1}{2} \sup_{B(z_0, r)} U + \frac{1}{2} \inf_{B(z_0, r)} U$$

(2.11)

for each $z_0 = R_0e^{iθ_0}$ and $0 < r < R_0$ such that $B(z_0, r) ⊂ Ω_α$. Here we assume, by symmetry, that $0 ≤ θ_0 < π − α$.

Observe that $\overline{B}(z_0, r)$ lies in the cone $\{Re^{iθ} : |θ − θ_0| ≤ t_m\}$, where

$$t_m = \arcsin \left(\frac{r}{R_0}\right).$$

Given $|t| ≤ t_m$, elementary computations show that the ray $\{Re^{i(θ_0+t)} : R > 0\}$ intersects $\partial B(z_0, r)$ at two points $R_+(t)e^{i(θ_0+t)}$ and $R_-(t)e^{i(θ_0+t)}$, where

$$R_±(t) = R_0 \left(\cos t ± \sqrt{\left(\frac{r}{R_0}\right)^2 - \sin^2 t}\right).$$

(2.12)
By Lemma 2.1, \( \phi \) is increasing and even, \( \phi \geq 0 \) and \( \phi(0) = 0 \). It follows that \( \sup_{B(z_0,r)} U \) must be of the form \( R_+^\gamma(t)(A - \phi(\vartheta_0 - t)) \) for some \( 0 \leq t \leq t_m \). Then

\[
\sup_{B(z_0,r)} U + \inf_{B(z_0,r)} U \leq R_+^\gamma(t)(A - \phi(\vartheta_0 - t)) + R_-^\gamma(t)(A - \phi(\vartheta_0 + t)),
\]

and, since \( U(z_0) = R_0^\gamma(A - \phi(\vartheta_0)) \) by definition, the desired inequality (2.11) will follow from the next lemma.

**Lemma 2.3.** Let \( A > 0 \) and \( \gamma \in (0, \frac{1}{2}] \) satisfy (2.7). For \( z_0 = R_0 e^{i \vartheta_0} \) and \( 0 < r < R_0 \) such that \( B(z_0,r) \subset \Omega_\alpha \), the inequality

\[
R_+^\gamma(t)(A - \phi(\vartheta_0 - t)) + R_-^\gamma(t)(A - \phi(\vartheta_0 + t)) \leq 2R_0^\gamma(A - \phi(\vartheta_0))
\]

holds for every \( |t| \leq \arcsin \left( \frac{r}{R_0} \right) \), where \( R_\pm(t) \) were defined in (2.12).

**Proof.** Let us denote

\[
\lambda_\pm = \lambda_\pm(t) = \frac{1}{2} \left( \frac{R_\pm(t)}{R_0} \right)^\gamma
\]

for simplicity. Then (2.13) is equivalent to

\[
F(t) := \frac{\phi(\vartheta_0) - (\lambda_+ \phi(\vartheta_0 - t) + \lambda_- \phi(\vartheta_0 + t))}{1 - (\lambda_+ + \lambda_-)} \leq A
\]

for every \( 0 \leq t \leq \arcsin \left( \frac{r}{R_0} \right) \). We show that the previous inequality holds true. Observe that after a rearrangement of the terms we can write

\[
F(t) = \phi(\vartheta_0) + \frac{\lambda_+ \lambda_-}{1 - (\lambda_+ + \lambda_-)} \left[ \phi(\vartheta_0) - \frac{\lambda_+}{\lambda_+ + \lambda_-} \phi(\vartheta_0 - t) - \frac{\lambda_-}{\lambda_+ + \lambda_-} \phi(\vartheta_0 + t) \right].
\]

Let us focus on the term in brackets. From the convexity of \( \phi \) we can estimate the term in brackets as follows

\[
\phi(\vartheta_0) - \frac{\lambda_+}{\lambda_+ + \lambda_-} \phi(\vartheta_0 - t) - \frac{\lambda_-}{\lambda_+ + \lambda_-} \phi(\vartheta_0 + t) \leq \phi(\vartheta_0) - \phi \left( \vartheta_0 - \frac{\lambda_+ - \lambda_-}{\lambda_+ + \lambda_-} t \right) \leq \frac{\lambda_+ - \lambda_-}{\lambda_+ + \lambda_-} t \phi'(\vartheta_0).
\]

Thus

\[
F(t) \leq \phi(\vartheta_0) + \frac{\lambda_+ - \lambda_-}{1 - (\lambda_+ + \lambda_-)} t \phi'(\vartheta_0).
\]

Notice that, since the function \( \phi \) is increasing in \((0, \pi)\) and \( \vartheta_0 \geq 0 \) by assumption, then \( \phi'(\vartheta_0) \geq 0 \). Next, using Lemma A.1 (see Appendix A) we get

\[
\lambda_+ + \lambda_- = \frac{R_+^\gamma(t) + R_-^\gamma(t)}{2R_0^\gamma} \leq \frac{1}{2} \left( 1 + \frac{r}{R_0} \right)^\gamma + \frac{1}{2} \left( 1 - \frac{r}{R_0} \right)^\gamma \leq \frac{1}{2} \left( 1 + \frac{r}{R_0} \right) \gamma + \frac{1}{2} \left( 1 - \frac{r}{R_0} \right) \gamma,
\]

which together with \( t \leq \arcsin \left( \frac{r}{R_0} \right) \leq \frac{\pi}{2R_0} \) yields

\[
F(t) \leq \phi(\vartheta_0) + \frac{\pi}{2} \left[ \frac{r}{2R_0^\gamma} \left[ \left( 1 + \frac{r}{R_0} \right) \gamma - \left( 1 - \frac{r}{R_0} \right) \gamma \right] + \frac{1}{2} \left( \frac{r}{R_0} \gamma + \left( 1 - \frac{r}{R_0} \right) \gamma \right) \right] \phi'(\vartheta_0).
\]

By Lemma A.2 together with the fact that \( \gamma \in (0, \frac{1}{2}] \) we get

\[
F(t) \leq \phi(\vartheta_0) + 2\pi \varphi'(\vartheta_0) \leq \frac{\pi^2}{8} + \frac{3\pi^2 + \pi}{2(\sin \alpha)^{n-2}}.
\]
where in the second inequality we have recalled the estimates (2.4). Then the result follows from the choice of $A$ in (2.7).

**Remark.** We want to emphasize that, in the proof of Lemma 2.2, the definition of $\phi$ as solution of the differential equation (2.2) is used exclusively to show the first inequality in (2.8), while for the second inequality we only need to require the convexity of $\phi$ in $(-\pi, \pi)$ and the fact that $\phi$ is increasing in $[0, \pi)$.

The following proposition says that the function $U$ is also a $p$-superharmonic for each $p \in [2, \infty]$.

**Proposition 2.1.** Let $U$ be the function defined in (2.6) with $A > 0$ and $\gamma \in (0, \frac{1}{2}]$ as in (2.7). Then $\Delta_p U \leq 0$ in $\Omega_\alpha$ for each $p \in [2, \infty]$.

**Proof.** From the representation (1.2) and the fact that $p \geq 2$ it is enough to check that $\Delta U \leq 0$ and $\Delta_\infty U \leq 0$.

The choices of $A$ and $\gamma$ in the expression of $\Delta U$ in (2.10) easily give that $\Delta U \leq 0$. We also need the expression of $\Delta_\infty U$ in polar coordinates (see [DS]):

$$\Delta_\infty U = -R^{3\gamma-4} \left[ \gamma^3 (1 - \gamma)(A - \phi) + \gamma (1 - 2\gamma)(A - \phi)(\phi')^2 + (\phi'\phi'') \right].$$

Observe that, since $\gamma \in (0, \frac{1}{2}]$, $A - \phi > 0$ and $\phi'' \geq 0$, the term in brackets is positive, so $\Delta_\infty U \leq 0$. $\square$

2.3. **Barrier for the complement of a truncated cone.** Let $\alpha \in (0, \frac{\pi}{2})$, $r > 0$ and define

$$\Omega_{\alpha,r} = \mathbb{R}^n \setminus K_{\alpha,r}$$

where $K_{\alpha,r}$ is as in (2.1). Note that $\Omega_{\alpha,r}$ is the complement of a truncated cone and that $\Omega_{\alpha,r} \supset \Omega_{\alpha}$. Let $U : \Omega_{\alpha} \rightarrow \mathbb{R}$ be the function defined in (2.6) for $A > 0$ and $\gamma \in (0, \frac{1}{2}]$ as in (2.7). Then, from the first inequality in (2.9) it follows that

$$m = \inf \{ U(x) : x \in \Omega_{\alpha} \setminus B(0, r) \} \geq \alpha^{2-n} r^\gamma > 0.$$

**Lemma 2.4.** Let $U$ be as in (2.6) and define $w : \Omega_{\alpha,r} \rightarrow \mathbb{R}$ by

$$w(x) = \begin{cases} \min \{ U(x), m \} & \text{if } x \in \Omega_{\alpha,r} \cap B(0, r), \\ m & \text{if } x \in \Omega_{\alpha,r} \setminus B(0, r). \end{cases}$$

Then $w \in C(\overline{\Omega_{\alpha,r}})$, $w(0) = 0$, $w > 0$ in $\Omega_{\alpha,r} \setminus \{0\}$,

$$w(x) \geq \int_{B(x, \tilde{r})} w(y) \, dy, \quad w(x) \geq \frac{1}{2} \sup_{B(x, \tilde{r})} w + \frac{1}{2} \inf_{B(x, \tilde{r})} w$$

for every ball $B(x, \tilde{r}) \subset \Omega_{\alpha,r}$ and

$$\mathcal{L}(|x|) \leq w(x) \leq \gamma^2 |x|^\gamma$$

for every $x \in \overline{\Omega_{\alpha,r}}$, where

$$\mathcal{L}(t) := \alpha^{2-n} \min \{ t, r \}^\gamma.$$

In particular, for each $p \in [2, \infty]$ and any admissible radius function in $\Omega_{\alpha,r}$, $w$ is, simultaneously, a $T_{p,p}$-barrier and a barrier for the $p$-laplacian at 0 in $\Omega_{\alpha,r}$.  


Proof: Since $U \in C(\overline{\Omega}_\alpha)$, the continuity of $w$ only needs to be checked at $\Omega_\alpha \cap \partial B(0,r)$. Fix $x_0 \in \Omega_\alpha \cap \partial B(0,r)$. Then $U(x_0) \geq m$. From the continuity of $U$ it follows that $\lim_{x \to x_0} w(x) = \min\{U(x_0), m\} = m = w(x_0)$. The inequalities in (2.16) follow from the definition of $w$ and (2.9). To prove (2.15), choose any ball $B(x, \rho) \subset \Omega_\alpha$.

We distinguish two cases:

1) If $w(x) < m$ then observe that $B(x, \rho) \subset \Omega_\alpha$ and that $w \leq U$ in $B(x, \rho)$. It follows from Lemma 2.2 that

$$\int_{B(x,\rho)} w(y) \, dy \leq \int_{B(x,\rho)} U(y) \, dy \leq U(x) = w(x)$$

and

$$\frac{1}{2} \sup_{B(x,\rho)} w + \frac{1}{2} \inf_{B(x,\rho)} w \leq \frac{1}{2} \sup_{B(x,\rho)} U + \frac{1}{2} \inf_{B(x,\rho)} U \leq U(x) = w(x).$$

2) If $w(x) = m$ then (2.15) follows immediately since $w \leq m$.

From (2.15) it is immediate that $T_{\rho,\rho} w \leq w$ for every admissible radius function $\rho$ in $\Omega_\alpha$, and each $\rho \in [2, \infty]$. Finally, that $w$ is $p$-superharmonic is consequence of the $p$-superharmonicity of $U$, the invariance of $p$-superharmonic functions by rotations and the Pasting Lemma ([HKM, Lemma 7.9]).

\[\square\]

2.4. Proof of Theorem 3. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain satisfying the uniform exterior cone condition with constants $\alpha \in (0, \frac{\pi}{2})$ and $r > 0$. From (1.11) it follows in particular that $\gamma \in (0, \frac{1}{2}]$ and

$$\frac{1}{\gamma (\gamma + n - 2)} > \frac{1}{\gamma n} > \frac{13\pi^2 + 4\pi}{8(\sin \alpha)^{n-2}} \geq \frac{\pi^2}{8} + \frac{3\pi^2 + \pi}{2(\sin \alpha)^{n-2}},$$

which allows to choose $A > 0$ so that (2.7) holds and, subsequently, to construct $U$ and $w$ as in (2.6) and (2.14), respectively.

Recalling Definition 1.4, there exists, for every $\xi \in \partial \Omega$, a rotation $R_\xi \in SO(n)$ in $\mathbb{R}^n$ such that $\xi + R_\xi(K_\alpha, r) \subset \mathbb{R}^n \setminus \Omega$ or, equivalently,

$$R_\xi^{-1}(\Omega - \xi) \subset \Omega_\alpha.$$ 

Then we define $w_\xi : \overline{\Omega} \to \mathbb{R}$ by

$$(2.18) \quad w_\xi(x) = w(R_\xi^{-1}(x - \xi)), $$

where $w$ is the barrier function given by (2.14). Recalling Lemma 2.4, we observe that $w_\xi$ is non-negative in $\overline{\Omega}$ and $w_\xi(x) = 0$ if and only if $x = \xi$. On the other hand, if $\rho$ is an admissible radius function in $\Omega$, since $\xi + R_\xi(B_\rho(x)) = B(\xi + R_\xi(x), \rho(x)) \subset \Omega_\alpha$, then

$$T_{\rho} w_\xi(x) = T_{\rho} w \left( R_\xi^{-1}(x - \xi) \right) \leq w \left( R_\xi^{-1}(x - \xi) \right) = w_\xi(x), $$

Thus (1.12) follows from (2.15) and, in particular, $w_\xi$ is a $T_{\rho}$-barrier at $\xi \in \partial \Omega$. The fact that $w_\xi$ is also a barrier for the $p$-laplacian follows in a similar way. Finally, from (2.16) we get (1.13) for every $x \in \overline{\Omega}$, where $\mathcal{L}$ is given by (2.17). This finishes the proof of the theorem.
3. Existence of solutions

We split the section in two parts. In the first part we show the equicontinuity of the sequence \( \{ T^k u \} \) in \( \overline{\Omega} \). In the second one we establish existence and uniqueness of the Dirichlet Problem for \( T_\rho \).

3.1. Equicontinuity results. At this point we refer to [AL18, Theorem 4.5] for the equicontinuity of the sequence \( \{ T^k \} \) at interior points of \( \Omega \), where \( u \in \mathcal{K}_f \) (see also [AL16-1, Proposition 2.6]).

**Theorem 4 ([AL18, Theorem 4.5]).** Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain and \( p \in [2, \infty) \). Suppose that \( \rho \in C(\overline{\Omega}) \) is a continuous admissible radius function in \( \Omega \) satisfying

\[
\lambda \operatorname{dist}(x, \partial \Omega)^{\beta} \leq \rho(x) \leq \Lambda \operatorname{dist}(x, \partial \Omega)
\]

for all \( x \in \Omega \), where

\[
\beta \geq 1, \quad 0 < \Lambda < 1 - \left( \frac{p - 2}{n + p} \right)^{1/\beta} \quad \text{and} \quad 0 < \lambda \leq (\operatorname{diam} \Omega)^{1-\beta} \Lambda.
\]

Then, for any \( u \in C(\overline{\Omega}) \), the sequence of iterates \( \{ T^k u \} \) is locally uniformly equicontinuous in \( \Omega \).

Therefore it only remains to show that, given a function \( u \in \mathcal{K}_f \), the sequence of iterates \( \{ T^k u \} \) is equicontinuous at each \( T_\rho \)-regular point of the boundary.

**Proposition 3.1.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain and \( f \in C(\partial \Omega) \). For any \( u \in \mathcal{K}_f \), the sequence of iterates \( \{ T^k u \} \) is equicontinuous at each \( T_\rho \)-regular point of \( \partial \Omega \).

**Proof.** Since \( u \in \mathcal{K}_f \), then \( u \) is uniformly continuous in \( \overline{\Omega} \), that is, for each \( \eta > 0 \) there exists small enough \( \delta > 0 \) such that \( |u(x) - u(y)| < \eta \) for every \( x, y \in \overline{\Omega} \) satisfying \( |x - y| < \delta \). Fix \( C = C_{u, \eta} = 2\|u\|_\infty / \mathcal{L}(\delta) \), where \( \mathcal{L} \) is the non-decreasing continuous function defined in (2.17). Then

\[
|u(x) - u(y)| \leq C \mathcal{L}(|x - y|) + \eta
\]

for every \( x, y \in \overline{\Omega} \). Therefore, if \( \xi \in \partial \Omega \) is \( T_\rho \)-regular we obtain, recalling (1.13), that

\[
|u(x) - f(\xi)| \leq C w_\xi(x) + \eta
\]

for every \( x \in \overline{\Omega} \), where \( w_\xi \) is a \( T_\rho \)-barrier at \( \xi \). Let \( k \in \mathbb{N} \). By the affine invariance and the monotonicity of \( T^k_\rho \),

\[
|T^k_\rho u(x) - f(\xi)| = |T^k_\rho (u - f(\xi))(x)| \leq T^k_\rho (C w_\xi + \eta)(x) \leq C w_\xi(x) + \eta
\]

for every \( x \in \overline{\Omega} \), where in the second inequality we have used that \( w_\xi \geq T^k_\rho w_\xi \). Finally, taking limits it turns out that

\[
0 \leq \limsup_{x \to \xi} |T^k_\rho u(x) - f(\xi)| \leq C \limsup_{x \to \xi} w_\xi(x) + \eta = \eta
\]

for each \( k \in \mathbb{N} \) and every \( \eta > 0 \). Thus the sequence of iterates \( \{ T^k u \} \) is equicontinuous at \( \xi \) and the proof is finished. \( \square \)
Remark. The proof of the above result only requires the affine invariance and the monotonicity of $T_p$. On the other hand, the proof does not require any assumption on the admissible radius function. Thus, the equicontinuity estimates obtained in the previous result are independent of the particular choice of $\rho$ in the definition of the operator $T_p$.

In view of Theorem 4 and Proposition 3.1, we have proved the following.

**Theorem 5.** Under the assumptions in Theorem 4, assume in addition that $\Omega$ is $T_p$-regular. If $u \in K_f$, then the sequence of iterates $\{T_p^k u\}_k$ is equicontinuous in $\overline{\Omega}$.

3.2. Existence and uniqueness. We start with the following comparison principle, that uses a standard argument (see also [AL16-1, Proposition 4.1]).

**Proposition 3.2.** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $\rho$ an admissible radius function in $\Omega$. Assume that $u, v \in C(\overline{\Omega})$ satisfy $u \leq T_p u, v \geq T_p v$ in $\Omega$ and $u \leq v$ on $\partial \Omega$. Then $u \leq v$ in $\Omega$.

**Proof.** Let $m = \max_{\overline{\Omega}} (u - v)$. We show that $m \leq 0$ by contradiction: suppose that $m > 0$ and let $A := \{x \in \Omega : u(x) - v(x) = m\}$. Since $u - v$ is upper semicontinuous in $\overline{\Omega}$ and $u - v \leq 0$ on the boundary, then $A$ is a nonempty closed subset of $\Omega$. The contradiction will then follow by proving that $A$ is also open, so $A = \Omega$ and $u(x) - v(x) = m > 0$ for every $x \in \Omega$.

To see that $A$ is open, we choose any $a \in A$ and we show that $B_\rho(a) \subset A$. Recalling that $u$ and $v$ are sub and super-solutions of $T_p$ we obtain that

$$T_p u(a) \geq u(a) = m + v(a) \geq m + T_p v(a),$$

and by the definition of $T_p$,

$$\frac{p-2}{n+p} S_p u(a) + \frac{n+2}{n+p} M_p u(a) \geq \frac{p-2}{n+p} (m + S_p v(a)) + \frac{n+2}{n+p} (m + M_p v(a)).$$

Hence, by the monotonicity of $S_p$ and $M_p$, and since $p \in [2, \infty)$, it turns out that

$$M_p u(a) = m + M_p v(a),$$

and recalling the definition of $M_p$,

$$m = \int_{B_\rho(a)} (u - v).$$

Since $m$ is defined as the maximum in $\overline{\Omega}$ of $u - v$, then $u(x) - v(x) = m$ for every $x \in B_\rho(a)$. Then $B_\rho(a) \subset A$, and so $A$ is an open set. Therefore, since $\Omega$ is connected, $A = \Omega$ and $u - v \equiv m > 0$ in $\Omega$, which contradicts the assumption $u \leq v$ on $\partial \Omega$.

Uniqueness of fixed points follows immediately as a corollary.

**Corollary 3.1.** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $f \in C(\partial \Omega)$. Suppose that $u, v \in K_f$ are fixed points of $T_p$. Then $u = v$ in $\overline{\Omega}$.

In order to show the existence of fixed points for $T_p$ in $K_f$, we will make use of the following technical result, which can be stated in the more general context of Banach spaces.
Lemma 3.1. Let \((X, \|\cdot\|)\) be a Banach space, \(\emptyset \neq K \subset X\) any closed subset and \(T : K \to K\) a non-expansive operator. Fix \(x \in K\). If \(y \in K\) is any limit point of the sequence \(\{T^k x\}_k\) then
\[
\lim_{k \to \infty} ||T^{k+1}x - T^k x|| = ||Ty - y||.
\]

Proof. Observe first that, since \(T\) is non-expansive, the sequence of non-negative real numbers \(\{||T^{k+1}x - T^k x||\}_k\) is non-increasing, and thus every subsequence converges to the same limit. Next, take any convergent subsequence \(\{T^k x\}_j\) and denote the limit by \(y \in K\). The triangle inequality and the non-expansiveness of \(T\) yield
\[
\left|\left|T^{k+1}x - T^k x\right\| - |Ty - y|\right| \leq ||(T^{k+1}x - Ty) - (T^k x - y)|| \\
\leq 2||T^k x - y||,
\]
for each \(j \in \mathbb{N}\). Then (3.1) follows after taking limits as \(j \to \infty\). \(\square\)

Now we are ready to prove Theorem 1.

Proof of Theorem 1. Since \(\Omega\) is \(T_p\)-regular by assumption, the sequence of iterates \(\{T_p^\ell u\}_k\) is equicontinuous at each point in \(\overline{\Omega}\) for any \(u \in \mathcal{K}_f\), by Theorem 5. Then the Arzelà-Ascoli’s theorem yields the existence of at least one subsequence converging uniformly to a function \(v \in \mathcal{K}_f\). Furthermore, \(T_p^\ell v\) is also a limit point of \(\{T_p^\ell u\}_k\) for each \(\ell = 0, 1, 2, \ldots\). Therefore, since \(\mathcal{K}_f\) is a closed subset of \(C(\overline{\Omega})\) and \(\tilde{T} = T_p : \mathcal{K}_f \to \mathcal{K}_f\) is non-expansive, Lemma 3.1 implies that
\[
||T^{\ell+1}v - T^\ell v||_\infty = \lim_{k \to \infty} ||T^{k+1}u - T^k u||_\infty = : d \geq 0
\]
for every \(\ell = 0, 1, 2, \ldots\) In consequence, if \(d = 0\) we have in particular that \(T_p v = v\), so \(v\) is a fixed point of \(T_p\).

In order to show that actually \(d = 0\), let us assume on the contrary that \(d > 0\) and argue by contradiction. Let \(\ell \in \mathbb{N}\) to be fixed later. Since \(T^{\ell+1}v - T^\ell v\) is a continuous function vanishing on \(\partial \Omega\), we can choose an interior point \(x_0 \in \Omega\) such that
\[
|T^{\ell+1}v(x_0) - T^\ell v(x_0)| = d.
\]
We assume that \(T^{\ell+1}v(x_0) - T^\ell v(x_0) = d\) since otherwise the proof goes in an analogous way. Recalling the definition of \(T = T_p\) and \(M = M_p\), it turns out that
\[
d = \frac{p - 2}{n + p} [\mathcal{S}(T^\ell v)(x_0) - \mathcal{S}(T^{\ell-1} v)(x_0)] + \frac{n + 2}{n + p} \mathcal{M}(T^\ell v - T^{\ell-1} v)(x_0). \tag{3.3}
\]

From (3.2) and the non-expansiveness of \(\mathcal{S}\) and \(\mathcal{M}\), it follows that
\[
\mathcal{S}(T^\ell v)(x_0) - \mathcal{S}(T^{\ell-1} v)(x_0) \leq d \quad \text{and} \quad \mathcal{M}(T^\ell v - T^{\ell-1} v)(x_0) \leq d,
\]
which, together with (3.3), implies that \(\mathcal{M}(T^\ell v - T^{\ell-1} v)(x_0) = d\). Equivalently,
\[
\int_{B_p(x_0)} (T^\ell v(y) - T^{\ell-1} v(y)) \, dy = d,
\]
and by (3.2), the integrand must be equal to \(d\) in \(B_p(x_0)\). In particular, \(T^\ell v(x_0) - T^{\ell-1} v(x_0) = d\) so we can repeat this argument iteratively until we finally get that
\[
T^\ell v(x_0) = v(x_0) + \ell d.
\]
Recalling (1.7) and that \( \mathcal{T} = \mathcal{T}_\rho : \mathcal{K}_f \to \mathcal{K}_f \) we get
\[
\|f\|_\infty \geq \mathcal{T}^\ell v(x_0) = v(x_0) + \ell d \geq -\|f\|_\infty + \ell d.
\]
Hence, choosing \( \ell \) such that
\[
\ell > \frac{2\|f\|_\infty}{d}
\]
we obtain the desired contradiction.

Finally, to see that the sequence of iterates \( \{T^k_\rho u\}_k \) actually converges uniformly to the unique fixed point \( v \in \mathcal{K}_f \), suppose on the contrary that there exist \( \eta > 0 \) and a subsequence \( \{T^k_\rho u\}_j \) such that \( \|T^k_\rho u - v\|_\infty \geq \eta \) for each \( j \in \mathbb{N} \). We can assume that this subsequence converges uniformly to a function \( w \in \mathcal{K}_f \) (otherwise, by equicontinuity and Arzelà-Ascoli’s theorem we could take a further subsequence), which would be a limit point of \( \{T^k_\rho u\}_k \) in \( \mathcal{K}_f \), and thus a fixed point of \( T_\rho \). Then the contradiction follows by uniqueness and the fact that \( \|w - v\|_\infty \geq \eta \).

\[\square\]

4. Convergence to \( p \)-harmonic functions

In this section we study the convergence of solutions \( u_\rho \) to (1.10) as the admissible radius function converges to zero in \( \Omega \). Before moving into details, it is worth to recall that one of the main connections between mean value properties and \( p \)-harmonic functions arises from the asymptotic expansion for the \( p \)-laplacian of a twice-differentiable function \( \phi \) at a non-critical point \( x \). This expansion can be expressed in terms of the average operator \( T_\rho \) as follows
\[
T_\rho \phi(x) = \phi(x) + \frac{\rho(x)^2}{2(n + p)} \Delta^N_\rho \phi(x) + o(\rho(x)^2) \quad (\rho(x) \to 0),
\]
where \( \Delta^N_\rho \phi \) stands for the normalized \( p \)-laplacian of \( \phi \) defined as
\[
\Delta^N_\rho \phi := \Delta \phi + (p - 2) \frac{\Delta \phi}{|\nabla \phi|^2}.
\]

Heuristically speaking, if the fixed points \( T^k_\rho u_\rho = u_\rho \) converge to a function \( u_0 \) as \( \rho \to 0 \), then it is reasonable to expect that this function is \( p \)-harmonic. Indeed, this is one of the key ideas required in the proof of Theorem 2.

To this end, we need first to impose appropriate conditions in order to ensure that \( \rho(x) \) converges to zero in a uniform way. Given a bounded domain \( \Omega \subset \mathbb{R}^n \), let us consider a collection of continuous admissible radius functions \( \{\rho_\varepsilon\}_{0 < \varepsilon \leq 1} \) satisfying
\[
\lambda \text{dist}(x, \partial \Omega)^\beta \leq \frac{\rho_\varepsilon(x)}{\varepsilon} \leq \Lambda \text{dist}(x, \partial \Omega)
\]
for all \( x \in \Omega \) and every \( 0 < \varepsilon \leq 1 \), where \( \beta, \lambda \) and \( \Lambda \) are as in (1.9). Since \( \Omega \) is bounded, \( \|\rho_\varepsilon\|_\infty \) decreases as fast as, at least, a constant multiple of \( \varepsilon \). In consequence \( \rho_\varepsilon(x) = O(\varepsilon) \) uniformly for every \( x \in \Omega \), and the asymptotic expansion for \( T_{\rho_\varepsilon} \) becomes
\[
T_{\rho_\varepsilon} \phi(x) = \phi(x) + \frac{\varepsilon^2}{2(n + p)} \left( \frac{\rho_\varepsilon(x)}{\varepsilon} \right)^2 \Delta^N_{\rho_\varepsilon} \phi(x) + o(\varepsilon^2) \quad (\varepsilon \to 0)
\]
for every \( x \in \Omega \).

On the other hand, \( \rho_\varepsilon \) is an admissible radius function satisfying the hypothesis of Theorem 1 for each \( 0 < \varepsilon \leq 1 \). Therefore, assuming that \( \Omega \) satisfies the uniform
exterior cone condition, Theorem 1 yields, for any fixed \( f \in C(\partial \Omega) \), a function \( u_\varepsilon \in C(\overline{\Omega}) \) satisfying
\[
\begin{cases}
\mathcal{T}_\varepsilon u_\varepsilon = u_\varepsilon & \text{in } \Omega, \\
u_\varepsilon = f & \text{on } \partial \Omega,
\end{cases}
\]
for each \( 0 < \varepsilon \leq 1 \), where \( \mathcal{T}_\varepsilon := \mathcal{T}_{\partial \Omega} \).

The strategy to prove Theorem 2 is inspired by the method for convergence of numerical schemes established by Barles and Souganidis in the 90’s ([BS]) and in a more recent result by del Teso, Manfredi and Parviainen on the convergence of dynamic programming principles for the \( p \)-laplacian ([DMP]). The steps in the proof can be split into two parts. First, by taking pointwise limits as \( \varepsilon \to 0 \), we define semicontinuous functions
\[
u(x) := \liminf_{y \to x, \varepsilon \to 0} u_\varepsilon(y) \leq \limsup_{y \to x, \varepsilon \to 0} u_\varepsilon(y) =: \overline{u}(x),
\]
and, using the asymptotic expansion (4.2), we show that \( \underline{u} \) and \( \overline{u} \) are \( p \)-superharmonic and \( p \)-subharmonic, respectively. In a second part we prove that \( \overline{u} \leq \underline{u} \) in \( \overline{\Omega} \) with the aid of the comparison principle for \( p \)-subharmonic and \( p \)-superharmonic functions [JLM, Theorem 2.7], so both functions coincide with \( u_0 \), the unique \( p \)-harmonic function satisfying
\[
\begin{cases}
\Delta_p u_0 = 0 & \text{in } \Omega, \\
u_0 = f & \text{on } \partial \Omega.
\end{cases}
\]

We remind that, by [JLM, Theorem 2.5], the concepts of \( p \)-subharmonic function and viscosity \( p \)-subsolution coincide. Hence, for the sake of simplicity, we will use hereafter the viscosity characterization.

**Definition 4.1.** Let \( p \in [2, \infty] \).

1. We say that an upper semicontinuous function \( u \) in \( \Omega \) is \( p \)-subharmonic if \( u \not\equiv -\infty \) and for every \( x \in \overline{\Omega} \) and any \( \phi \in C^2(\Omega) \) such that \( \nabla \phi(x) \neq 0 \) and \( u - \phi < u(x) - \phi(x) = 0 \) in \( \Omega \setminus \{x\} \) we have that \( \Delta_p \phi \geq 0 \).
2. We say that a lower semicontinuous function \( u \) in \( \Omega \) is \( p \)-superharmonic if \( u \not\equiv \infty \) and for every \( x \in \Omega \) and any \( \phi \in C^2(\Omega) \) such that \( \nabla \phi(x) \neq 0 \) and \( u - \phi > u(x) - \phi(x) = 0 \) in \( \Omega \setminus \{x\} \) we have that \( \Delta_p \phi \leq 0 \).
3. We say that \( u \in C(\Omega) \) is \( p \)-harmonic if it is both \( p \)-subharmonic and \( p \)-superharmonic.

**Proposition 4.1.** Let \( p \in [2, \infty) \), \( \Omega \subset \mathbb{R}^n \) a bounded domain satisfying the uniform exterior cone condition, \( f \in C(\partial \Omega) \) and \( u_\varepsilon \in \mathcal{K}_f \) the unique solution of (4.3) provided by Theorem 1, for \( 0 < \varepsilon \leq 1 \). Let \( \underline{u} \) and \( \overline{u} \) be the functions defined in (4.4). Then \( \underline{u} \) is \( p \)-superharmonic and \( \overline{u} \) is \( p \)-subharmonic in \( \overline{\Omega} \).

*Proof.* We show that \( \overline{u} \) is \( p \)-subharmonic. Fix any \( x \in \Omega \) and \( \phi \in C^2(\Omega) \) such that \( \nabla \phi(x) \neq 0 \) and \( \overline{u} - \phi < \overline{u}(x) - \phi(x) = 0 \) in \( \Omega \setminus \{x\} \). That is, \( x \) is a strict global maximum of \( \overline{u} - \phi \) in \( \overline{\Omega} \). We need to check that \( \Delta_p \phi(x) \geq 0 \).

From the definition of \( \overline{u} \), we pick sequences \( \varepsilon_j \to 0 \) and \( z_j \to x \) such that \( u_{\varepsilon_j}(z_j) \to \overline{u}(x) \). For each \( j \), let \( y_j \in \partial \Omega \) such that \( y_j \) is a maximum of \( u_{\varepsilon_j} - \phi \) with \( \nabla \phi(y_j) \neq 0 \). We claim that \( y_j \to x \) as \( j \to \infty \). Otherwise, there would be a further
subsequence (still denoted by \(\{y_j\}\)) converging to \(x' \neq x\). Then
\[
\overline{u}(x') - \phi(x') \geq \limsup_{j \to \infty} (u_{y_j}(y_j) - \phi(y_j)) \geq \limsup_{j \to \infty} (u_{x_j}(z_j) - \phi(z_j)) = \overline{u}(x) - \phi(x),
\]
so we obtain a contradiction with the fact that \(x\) is a strict global maximum of \(\overline{u} - \phi\). Then \(y_j \to x\) and \(u_{x_j} - u_{y_j}(y_j) \leq \phi - \phi(y_j)\) in \(\overline{\Omega}\). In consequence, by the monotonicity and the affine invariance of \(\mathcal{T}_{x_j}\) we obtain
\[
0 = \mathcal{T}_{x_j} u_{x_j} - u_{y_j}(y_j) \leq \mathcal{T}_{x_j} \phi(y_j) - \phi(y_j)
\]
for every \(j \in \mathbb{N}\). Notice that the left-hand side of the inequality is equal to zero due to the fact that \(\mathcal{T}_{x} u_{x} = u_{x}\) for every \(0 < \varepsilon \leq 1\). Recall (4.2), rearrange terms and divide by \(\varepsilon_j^2\) to obtain
\[
\left(\frac{\rho_j(y_j)}{\varepsilon_j}\right)^2 \Delta_N^p \phi(y_j) \geq o(1) \quad (j \to \infty).
\]
Assume for a moment that \(\Delta_N^p \phi(x) < 0\). Then \(\Delta_N^p \phi(y_j) < 0\) for every large enough \(j \in \mathbb{N}\), and recalling (4.1) we get
\[
(\lambda \text{dist}(y_j, \partial\Omega)^{\beta})^2 \Delta_N^p \phi(y_j) \geq o(1) \quad (j \to \infty).
\]
Hence, taking limits as \(j \to \infty\),
\[
(\lambda \text{dist}(x, \partial\Omega)^{\beta})^2 \Delta_N^p \phi(x) \geq 0,
\]
and thus \(\Delta_N^p \phi(x) \geq 0\). \(\square\)

In the next proposition we give a uniform boundary equicontinuity estimate for \(\{u_{\varepsilon}\}_{0 < \varepsilon \leq 1}\). This estimate is crucial to prove that the functions \(u\) and \(\overline{u}\) attach the right values near the boundary.

**Proposition 4.2.** Let \(p \in [2, \infty)\), \(\Omega \subset \mathbb{R}^n\) a bounded domain satisfying the uniform exterior cone condition (with constants \(\alpha, r\), \(\gamma \in (0, \frac{1}{2}]\) as in (1.11) and \(f \in C(\partial\Omega)\). Under the assumptions of Theorem 2, let \(u_{\varepsilon} \in \mathcal{K}_f\) be the unique solution of (4.3), for each \(0 < \varepsilon \leq 1\). Then for each \(\eta > 0\), there exists a constant \(C > 0\) depending only on \(\Omega\), \(f\) and \(\eta\) such that
\[
(4.5) \quad |u_{\varepsilon}(x) - f(\xi)| \leq C\gamma^{-1} |x - \xi|^{\gamma} + \eta,
\]
for every \(x \in \overline{\Omega}\) and \(\xi \in \partial\Omega\).

**Proof.** By uniform continuity, there exists small enough \(\delta > 0\) such that \(|f(\zeta) - f(\xi)| < \eta\) for every \(\xi, \zeta \in \partial\Omega\) satisfying \(|\zeta - \xi| < \delta\). Fix \(C = C_{f, \eta} = 2 \|f\|_{\infty} / \mathcal{L}(\delta)\), where \(\mathcal{L}\) is the function defined in (2.17). Then
\[
|f(\zeta) - f(\xi)| \leq C \mathcal{L}(|\zeta - \xi|) + \eta
\]
and from (1.13) it follows that
\[
(4.6) \quad |f(\zeta) - f(\xi)| \leq C w_{\varepsilon}(\zeta) + \eta
\]
for every \(\xi, \zeta \in \partial\Omega\), where \(w_{\varepsilon}\) is the \(\mathcal{T}_{\varepsilon}\)-barrier at \(\xi\) defined in (2.18). From the fact that \(w_{\varepsilon}\) is a \(\mathcal{T}_{\varepsilon}\)-barrier at \(\xi\) (Theorem 3) and the Comparison Principle (Proposition 3.2) it follows that
\[
(4.7) \quad |u_{\varepsilon}(x) - f(\xi)| \leq C w_{\varepsilon}(x) + \eta
\]
for each \(x \in \overline{\Omega}\) which, together with (1.13), implies (4.5). \(\square\)
Remark. If \( u \) is the \( p \)-harmonic function in \( \Omega \) with boundary data \( f \) then (4.5) holds with \( u_\varepsilon \) replaced by \( u \). This follows from (4.6), the fact that \( w_\xi \) is \( p \)-superharmonic and the Comparison Principle for the \( p \)-laplacian.

Proof of Theorem 2. We know that the semicontinuous functions \( \underline{u} \) and \( \overline{u} \) defined in (4.4) satisfy \( \underline{u} \leq \overline{u} \) by construction. We claim that the theorem follows from the reverse inequality. Indeed, suppose that \( \underline{u} \) and \( \overline{u} \) agree over the whole domain. This allows to define a continuous function \( u_0 = \overline{u} = \underline{u} \in K_f \) as the pointwise limit
\[
 u_0(x) = \lim_{\varepsilon \to 0} u_\varepsilon(x)
\]
for each \( x \in \overline{\Omega} \). Then, by Proposition 4.1, \( u_0 \) is both \( p \)-subharmonic and \( p \)-superharmonic, so \( u_0 \) is \( p \)-harmonic in \( \Omega \) and \( u_0 \mid _{\partial \Omega} = f \).

Our strategy to show that \( \underline{u} \leq \overline{u} \) relies on the uniform equicontinuity estimate from Proposition 4.2 to show that \( \underline{u} \) and \( \overline{u} \) take the right values near the boundary. The desired inequality will then follow as a consequence of the comparison principle for \( p \)-subharmonic and \( p \)-superharmonic functions.

Fix an arbitrary small \( \eta > 0 \) and choose \( C > 0 \) as in Proposition 4.2. By the definition of \( \overline{u} \) we have that
\[
 u(x) - f(\xi) = \limsup_{y \to x, \varepsilon \to 0} (u_\varepsilon(y) - f(\xi)) \leq C\gamma^{-2}|x - \xi|^\gamma + \eta,
\]
for \( x \in \overline{\Omega}, \xi \in \partial \Omega \) and \( \varepsilon > 0 \), where in the inequality we have used the estimate (4.5). Taking limits as \( x \to \xi \) we get
\[
 \limsup_{x \to \xi} (\overline{u}(x) - f(\xi)) \leq \eta
\]
for arbitrary small \( \eta > 0 \). Repeating an analogous argument for \( \underline{u} \) we obtain
\[
 \limsup_{x \to \xi} \underline{u}(x) \leq f(\xi) \leq \liminf_{x \to \xi} \underline{u}(x)
\]
for every \( \xi \in \partial \Omega \). Since \( \underline{u} \) and \( \overline{u} \) are respectively \( p \)-superharmonic and \( p \)-subharmonic, by the comparison principle [JLM, Theorem 2.7] we finally obtain that \( \overline{u} \geq \underline{u} \) in \( \Omega \).

\( \square \)

Appendix A. Auxiliary Lemmas

Lemma A.1. Let \( T \in [0, \frac{\pi}{2}] \) and \( \gamma > 0 \). Then
\[
 \left( \cos t + \sqrt{\sin^2 T - \sin^2 t} \right)^\gamma \pm \left( \cos t - \sqrt{\sin^2 T - \sin^2 t} \right)^\gamma \leq (1 + \sin T)^\gamma \pm (1 - \sin T)^\gamma
\]
whenever \( |t| \leq T \).

Proof. Let \( a \in [0, 1] \) and define \( \varphi_\pm : [a, 1] \to \mathbb{R} \) by
\[
 \varphi_\pm(x) = \left( x + \sqrt{x^2 - a^2} \right)^\gamma \pm \left( x - \sqrt{x^2 - a^2} \right)^\gamma.
\]
Direct computation shows that
\[
 \varphi'_\pm(x) = \frac{\gamma}{\sqrt{x^2 - a^2}} \varphi_\pm(x) \geq 0,
\]
therefore \( \varphi_\pm \) is positive and increasing in \([a, 1]\). In particular, \( \varphi_\pm(x) \leq \varphi_\pm(1) \) for every \( x \in [a, 1] \). Then the result follows by letting \( a = \cos T \) and performing the change of variables \( x = \cos t \). \( \square \)
Lemma A.2. Let $\gamma \in (0, 1)$. Then
\[
(\text{A.1}) \quad \frac{x^2}{2} [(1 + x)^\gamma - (1 - x)^\gamma] \leq \frac{2}{1 - \gamma} \quad \frac{1}{2} [(1 + x)^\gamma + (1 - x)^\gamma]
\]
for all $x \in (0, 1]$.

Proof. Let us recall the Taylor series of $f(x) = (1 + x)^\gamma$:
\[
(\text{A.2}) \quad (1 + x)^\gamma = 1 + \sum_{k=1}^{\infty} \frac{\gamma}{k!} x^k
\]
for $|x| \leq 1$, where
\[
\binom{\gamma}{k} = (-1)^{k-1} \gamma(1 - \gamma)(2 - \gamma) \cdots (k - 1 - \gamma)
\]
for each $k \in \mathbb{N}$. Observe that, since $\gamma \in (0, 1)$, we have that
\[
\binom{\gamma}{2k-1} > 0 \quad \text{and} \quad \binom{\gamma}{2k} < 0 \quad \text{for each} \quad k \in \mathbb{N}.
\]
We can rewrite the left-hand side in (A.1) by replacing (A.2),
\[
\frac{x^2}{2} [(1 + x)^\gamma - (1 - x)^\gamma] = \sum_{k=1}^{\infty} \binom{\gamma}{2k-1} x^{2k} - \sum_{k=1}^{\infty} \binom{\gamma}{2k} x^{2k}.
\]
Hence, (A.1) follows from the fact that
\[
\sum_{k=1}^{\infty} \left[ \binom{\gamma}{2k-1} + \frac{2}{1 - \gamma} \binom{\gamma}{2k} \right] x^{2k} \leq 0
\]
for every $x \in (0, 1)$. In fact, every coefficient in the above series is nonpositive, that is
\[
\binom{\gamma}{2k-1} + \frac{2}{1 - \gamma} \binom{\gamma}{2k} = \binom{\gamma}{2k-1} \left[ 1 + \frac{2}{1 - \gamma} \cdot \frac{\gamma - 2k + 1}{2k} \right] \leq 0
\]
for every $k \in \mathbb{N}$.

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