THE LIMITING BEHAVIOUR OF HERMITIAN-YANG-MILLS FLOW OVER
COMPACT NON-KÄHLER MANIFOLDS

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Abstract. In this paper, we analyze the asymptotic behaviour of the Hermitian-Yang-Mills
flow over a compact non-Kähler manifold \((X, g)\) with the Hermitian metric \(g\) satisfying the
Gauduchon and Astheno-Kähler condition.

1. Introduction

Let \(X\) be an \(n\)-dimensional compact complex manifold and \(g\) a Hermitian metric with associ-
ated \((1, 1)\)-form \(\omega\). \(g\) is called to be Gauduchon if \(\omega\) satisfies \(\partial \bar{\partial} \omega^{n-1} = 0\). It has been proved by
Gauduchon \([12]\) that if \(X\) is compact, there exists a Gauduchon metric in the conformal class
of every Hermitian metric \(g\). If \(\partial \bar{\partial} \omega^{n-2} = 0\), the Hermitian metric \(g\) is said to be Astheno-Kähler
which was introduced by Jost and Yau in \([13]\).

Let \((X, \omega)\) be an \(n\)-dimensional compact complex manifold with \(\partial \bar{\partial} \omega^{n-1} = 0\) and \((L, h)\) a
Hermitian line bundle over \(X\). The \(\omega\)-degree of \(L\) is defined by

\[
\deg_\omega(L) := \int_X c_1(L, A_h) \wedge \omega^{n-1} / (n-1)!,
\]

where \(c_1(L, A_h)\) is the first Chern form of \(L\) associated with the induced Chern connection \(A_h\).
Since \(\partial \bar{\partial} \omega^{n-1} = 0\), \(\deg_\omega(L)\) is well defined and independent of the choice of metric \(h\) \([18\, p.\, 34-35]\). Now given a rank \(s\) coherent analytic sheaf \(\mathcal{F}\), we consider the determinant line bundle
\(\det \mathcal{F} = (\wedge^s \mathcal{F})^{**}\) associated with \(\mathcal{F}\). Define the \(\omega\)-degree of \(\mathcal{F}\) by

\[
\deg_\omega(\mathcal{F}) := \deg_\omega(\det \mathcal{F}).
\]

If \(\mathcal{F}\) is non-trivial and torsion free, the \(\omega\)-slope of \(\mathcal{F}\) is defined by

\[
\mu_\omega(\mathcal{F}) = \frac{\deg_\omega(\mathcal{F})}{\text{rank}(\mathcal{F})}.
\]

Let \((E, \bar{\partial}_E)\) be a rank \(r\) holomorphic vector bundle over \((X, \omega)\). A Hermitian metric \(H\) on \(E\)
is said to be \(\omega\)-Hermitian-Einstein if the Chern curvature \(F_H\) satisfies the Einstein condition

\[
\sqrt{-1} \Lambda_\omega F_H = \lambda \cdot \text{Id}_E,
\]

where \(\lambda = \frac{2\pi \mu_\omega(E)}{Vol(X)}\). When the \((1, 1)\)-form \(\omega\) is understood, we omit the subscript \(\omega\) in the
above definitions.

In this paper, we consider the following Hermitian-Yang-Mills flow on the holomorphic bundle
\((E, \bar{\partial}_E)\) with initial data \(H(0) = H_0\),

\[
(1.1) \quad H^{-1} \frac{\partial H}{\partial t} = -2 \left(\sqrt{-1} \Lambda_\omega F_H - \lambda \text{Id}_E\right)
\]

1991 Mathematics Subject Classification. 53C07, 58E15.

Key words and phrases. Gauduchon, Astheno-Kähler, Hermitian-Yang-Mills flow.

The authors were supported in part by NSF in China, No.11625106, 11571332, 11526212.
where $\lambda = \frac{2\pi \mu_\omega(E)}{Vol(X)}$ and $F_H$ is the curvature of the Chern connection with respect to $H$. The Hermitian-Yang-Mills flow (1.1) was introduced and studied by Donaldson in [10]. When $(X, \omega)$ is Kähler, Donaldson proved the long time existence and uniqueness of the solution for (1.1). Using this flow, Donaldson ([10]) obtained the existence of the irreducible Hermitian-Einstein metrics on stable bundles over algebraic manifolds which was extended by Uhlenbeck and Yau ([23]) to the Kähler case. On general Hermitian manifolds, the second author ([24]) got the long-time existence and uniqueness of the solution of (1.1).

Let’s consider the Hermitian vector bundle $(E, H_0)$. Denote the space of connections of $E$ compatible with $H_0$ by $A_{H_0}$ and the space of unitary integrable connections of $E$ by $A_{H_0}^{1,1}$. We denote by $G^C$ (resp. $G$, where $G = \{ \sigma \in G^C | \sigma^* H_0 \sigma = \text{Id} \}$) the complex gauge group (resp. unitary gauge group) of the Hermitian vector bundle $(E, H_0)$. $G^C$ acts on the space $A_{H_0}$ as follows: for $\sigma \in G^C$ and $A \in A_{H_0}$,

$$\bar{\partial}_{\sigma(A)} = \sigma \circ \bar{\partial}_A \circ \sigma^{-1}, \quad \partial_{\sigma(A)} = (\sigma^* H_0)^{-1} \circ \partial_A \circ \sigma^* H_0.$$  

Following Donaldson’s argument ([10]), we can show that the Hermitian-Yang-Mills flow (1.1) is gauge equivalent to the following heat flow

$$(1.3) \quad \begin{cases}
\frac{\partial A(t)}{\partial t} = \sqrt{-1}(\bar{\partial}_A - \partial_A)A_\omega F_A, \\
A(0) = A_0,
\end{cases}$$

where $A_0 = (\bar{\partial}_E, H_0)$. In fact, there is a family of complex gauge transformations $\sigma(t) \in G^C$ satisfying $\sigma(t)^* H_0 \sigma(t) = h(t) = H_0^{-1} H(t)$, where $H(t)$ is the long time solution of the Hermitian-Yang-Mills flow (1.1) with the initial metric $H_0$, such that $A(t) = \sigma(t)(A_0)$ is the long time solution of the heat flow (1.3) with the initial connection $A_0$.

When the underground manifold $(X, \omega)$ is Kähler, it is easy to see that the heat flow (1.3) is just the Yang-Mills flow by the Kähler identity. There are many interesting results on the convergence of the Yang-Mills flow, see references [1, 3, 4, 6, 7, 13, 17, 20]. In this article, we study the limiting behaviour of the Hermitian-Yang-Mills flow (or the heat flow (1.3)) under the assumption that $\omega$ is Gauduchon and Astheno-Kähler. We first give some basic properties of the heat flow (1.3) including energy inequality, monotonicity formula of certain quantities and small energy regularity. Then, following the argument of Hong and Tian in (1.3) and using Bando and Siu’s extension technique, we obtain the following convergence result of the heat flow (1.3)

**Theorem 1.1.** Let $(X, \omega)$ be an $n$-dimensional compact Hermitian manifold with $\omega$ satisfying $\partial \bar{\partial} \omega^{n-1} = \partial \bar{\partial} \omega^{n-2} = 0$. Suppose $A(t)$ is the global smooth solution of the heat flow (1.3) on the Hermitian vector bundle $(E, H_0)$ with the initial data $A_0$ over $(X, \omega)$. Then

(1) For every sequence $t_k \to \infty$, there is a subsequence $t_{k_j}$ such that as $t_{k_j} \to \infty$, $A(t_{k_j})$ converges modulo gauge transformations to a solution $A_\infty$ of equation

$$(1.4) \quad D_A A_\omega F_A = 0$$

on Hermitian vector bundle $(E_\infty, H_\infty)$ in $C^\infty_{loct}$ topology outside a subset $\Sigma \subset X$, where $\Sigma$ is a closed set of Hausdorff codimension at least 4.

(2) The limiting $(E_\infty, H_\infty, \bar{\partial}_{A_\infty})$ can be extended to the whole $X$ as a reflexive sheaf with a holomorphic orthogonal splitting

$$(E_\infty, H_\infty, A_\infty) = \bigoplus_{i=1}^l (E_{i\infty}^i, H_{i\infty}^i, A_{i\infty}^i),$$

where $H_{i\infty}^i$ is an admissible Hermitian-Einstein metric on reflexive sheaf $E_{i\infty}^i$. 

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The paper is organised as below. In Section 2, we present some basic properties of the heat flow \[1.3\]. In Section 3, we give the detailed proof of Theorem \[1.1\].

2. Existence of the heat flow and some basic estimates

Let \((X, \omega)\) be an \(n\)-dimensional compact Hermitian manifold and \((E, H_0)\) a rank \(r\) complex vector bundle over \((X, \omega)\). On the space \(A_{H_0}\), we define the Yang-Mills functional by

\[\text{YM}(A) = \int_X |F_A|^2 \, dV_\omega.\]

And the negative gradient flow of Yang-Mills functional is

\[\frac{\partial A}{\partial t} = -D_A^* F_A,\]

which is called the Yang-Mills flow.

Using the Taylor expansion method, Demailly (\[8\]) showed that for any \(A \in A_{H_0}^{1,1}\), it holds

\[\partial^*_A = -\sqrt{-1}[\Lambda_\omega, \partial A] - \tau^*, \quad \partial_A = \sqrt{-1}[\Lambda_\omega, \partial^*_A] - \tau^*,\]

where \(\tau = [\Lambda_\omega, \partial \omega]\). From \[2.1\], we know that the heat flow \[1.3\] is equivalent to

\[\begin{cases}
\frac{\partial A(t)}{\partial t} = -D_A^* F_A - [\Lambda_\omega, dw]^* F_A, \\
A(0) = A_0.
\end{cases}\]

Using the result in \[24\] and following the argument of Donaldson (\[9\]), we can obtain the long time existence and uniqueness of solution of the heat flow \[1.3\]. Since the proof is similar as that in \[9\], we omit it.

**Theorem 2.1.** Let \((X, \omega)\) be an \(n\)-dimensional compact Hermitian manifold and \((E, H_0)\) a rank \(r\) Hermitian vector bundle over \(X\). Given any \(A_0 \in A_{H_0}^{1,1}\), the heat flow \[1.3\] has a unique long-time solution in the complex gauge orbit of \(A_0\) with the initial data \(A_0\).

2.1. Basic estimates. Suppose that \(A(t)\) is a smooth solution of the heat flow \[1.3\] and \(f\) a real smooth function over \(X\). It holds that

\[\begin{align*}
\frac{d}{dt} \int_X f^2 |F_A|^2 \, dV_g &= 2\text{Re} \int_X \left< f^2 F_A, D_A \frac{dA}{dt} \right> \, dV_g \\
&= 2\text{Re} \left\{ \int_X f^2 \left< D_A^* F_A, \frac{dA}{dt} \right> \, dV_g - \int_X \left< F_A, df^2 \wedge \frac{\partial A}{\partial t} \right> \, dV_g \right\} \\
&= -2 \int_X f^2 \left| \frac{\partial A}{\partial t} \right|^2 \, dV_g - 2\text{Re} \int_X f^2 \left< (\tau + \bar{\tau})^* F_A, \frac{\partial A}{\partial t} \right> \, dV_g \\
&\quad - 2\text{Re} \int_X \left< F_A, df^2 \wedge \frac{\partial A}{\partial t} \right> \, dV_g.
\end{align*}\]

Setting \(f \equiv 1\) on \(X\), we get

\[\frac{d}{dt} \int_X |F_A|^2 \, dV_g = -2 \int_X \left| \frac{\partial A}{\partial t} \right|^2 \, dV_g - 2\text{Re} \int_X \left< (\tau + \bar{\tau})^* F_A, \frac{\partial A}{\partial t} \right> \, dV_g.\]

**Proposition 2.2.** If the fundament form \(\omega\) satisfies \(\partial \bar{\partial} \omega^{n-1} = \partial \bar{\partial} \omega^{n-2} = 0\), there holds

\[\int_X \left< (\tau + \bar{\tau})^* F_A, \frac{\partial A}{\partial t} \right> \, dV_g = 0.\]
Proof. From Proposition 4.1 in [19], we have
\begin{equation}
(2.6) \quad \tau^* F_A = \frac{\ast (\bar{\partial}(\omega^{n-2}) \wedge F_A)}{(n-2)!} + \frac{\ast (\bar{\partial}(\omega^{n-1}) \Lambda_\omega F_A)}{(n-1)!}
\end{equation}
and
\begin{equation}
(2.7) \quad \tau^* F_A = -\frac{\ast (\bar{\partial}(\omega^{n-2}) \wedge F_A)}{(n-2)!} + \frac{\ast (\bar{\partial}(\omega^{n-1}) \Lambda_\omega F_A)}{(n-1)!}.
\end{equation}
At first, by (1.3), (2.6) and Stokes formula, we have
\[
\int_X \left\langle \tau^* F_A, \frac{\partial A}{\partial t} \right\rangle dV_g = \int_X \left\langle \tau^* F_A, \sqrt{-1} \bar{\partial} \Lambda_\omega F_A \right\rangle dV_g
\]
\[
= \int_X \left\langle \tau^* F_A, \sqrt{-1} \bar{\partial} (\omega^{n-2}) \wedge F_A \right\rangle dV_g
\]
\[
= \int_X \left\langle -\frac{\ast (\bar{\partial}(\omega^{n-2}) \wedge F_A)}{(n-2)!} + \frac{\ast (\bar{\partial}(\omega^{n-1}) \Lambda_\omega F_A)}{(n-1)!}, \sqrt{-1} \bar{\partial} \Lambda_\omega F_A \right\rangle dV_g
\]
\[
= \int_X \sqrt{-1} \left\langle \bar{\partial} \Lambda_\omega (\omega^{n-2}) \wedge F_A, \Lambda_\omega F_A \right\rangle dV_g - \int_X \sqrt{-1} \left\langle \frac{\bar{\partial} \Lambda_\omega (\omega^{n-2}) \wedge F_A}{(n-2)!}, \Lambda_\omega F_A \right\rangle dV_g
\]
\[
= \text{II}. \quad \text{I} + \text{II}.
\]
Then, by simple calculation, we have
\[
\text{I} = \int_X \sqrt{-1} \left\langle \frac{\bar{\partial} \Lambda_\omega (\omega^{n-2}) \wedge F_A}{(n-2)!}, \Lambda_\omega F_A \right\rangle dV_g
\]
\[
= \int_X \sqrt{-1} \left\langle \frac{\ast (\bar{\partial}(\omega^{n-2}) \wedge F_A)}{(n-2)!}, \Lambda_\omega F_A \right\rangle dV_g
\]
\[
= \int_X \sqrt{-1} \left\langle \frac{\ast (\bar{\partial}(\omega^{n-2}) \wedge F_A)}{(n-2)!}, \Lambda_\omega F_A \right\rangle dV_g
\]
\[
= 0
\]
and
\[
\text{II} = -\int_X \sqrt{-1} \left\langle \frac{\bar{\partial} \Lambda_\omega (\omega^{n-1}) \Lambda_\omega F_A}{(n-1)!}, \Lambda_\omega F_A \right\rangle dV_g
\]
\[
= -\int_X \sqrt{-1} \left\langle \frac{\ast (\bar{\partial}(\omega^{n-1}) \Lambda_\omega F_A)}{(n-1)!}, \Lambda_\omega F_A \right\rangle dV_g
\]
\[
= -\int_X \sqrt{-1} \left\langle \frac{\ast (\bar{\partial}(\omega^{n-1}) \Lambda_\omega F_A)}{(n-1)!}, \Lambda_\omega F_A \right\rangle dV_g + \int_X \sqrt{-1} \left\langle \frac{\ast (\bar{\partial}(\omega^{n-2}) \wedge F_A)}{(n-2)!}, \Lambda_\omega F_A \right\rangle dV_g
\]
\[
= -\int_X \sqrt{-1} \mathrm{tr} \left\{ \Lambda_\omega F_A \frac{\bar{\partial} (\omega^{n-1} \wedge \partial \Lambda_\omega F_A)}{(n-1)!} \right\}
\]
\[
= -\int_X \sqrt{-1} \partial | \Lambda_\omega F_A |^2 \wedge \frac{\bar{\partial} (\omega^{n-1})}{(n-1)!}
\]
\[
= 0
\]
In the same way, we have
\[
\int_X (\tau^* F_A, \frac{\partial A}{\partial t}) dV_g = 0.
\]
Therefore, there holds that

**Lemma 2.3.** Let $A(t)$ be a solution of the heat flow (1.3) with initial data $A_0$ over $X$. Then

\begin{equation}
YM(t) + 2 \int_0^t \int_X |\frac{\partial A}{\partial t}|^2 = YM(0).
\end{equation}

Let $f$ be a cut-off function with support inside $B_{2R}(x_0)$ and $f \equiv 1$ on $B_R(x_0)$ such that $0 \leq f \leq 1$ and $|df| \leq 2R^{-1}$. Set $e(A) = |F_A|^2$. From the identity (2.3), we have

\begin{equation}
\frac{d}{dt} \int_X f^2 e(A) + 2 \int_X f^2 \left| \frac{\partial A}{\partial t} \right|^2 \leq C \left( \int_X f|df| |F_A| \left| \frac{\partial A}{\partial t} \right| + \int_X f^2 |F_A| \left| \frac{\partial A}{\partial t} \right| \right)
\end{equation}

Then from (2.9), we can deduce the following local energy estimates:

**Lemma 2.4.** (Local energy estimates) Suppose $A(t)$ is a smooth solution of the heat flow (2.2). Fix $x_0 \in X$ and $R \in \mathbb{R}^+$ such that $B_{2R}(x_0) \subset X$. Then for any two finite numbers $s$ and $\tau$, we have

\[
\int_{B_R(x_0)} e(A)(\cdot, s) \, dV_g \leq \int_{B_{2R}(x_0)} e(A)(\cdot, \tau) \, dV_g + 2 \int_{\min\{s, \tau\}}^{\max\{s, \tau\}} \int_X \left| \frac{\partial A}{\partial t} \right|^2 \, dV_g \, dt
\]

\[
+ C \left( \frac{|s - \tau|}{R^2} YM(0) \int_{\min\{s, \tau\}}^{\max\{s, \tau\}} \int_X \left| \frac{\partial A}{\partial t} \right|^2 \, dV_g \, dt \right)^{\frac{1}{2}}
\]

\[
+ \left( C|s - \tau| YM(0) \int_{\min\{s, \tau\}}^{\max\{s, \tau\}} \int_X \left| \frac{\partial A}{\partial t} \right|^2 \right)^{\frac{1}{2}}.
\]

**Proof.** From (2.9) and Hölder inequality, we have

\[
\left| \int_{\min\{s, \tau\}}^{\max\{s, \tau\}} \left( \frac{d}{dt} \int_X f^2 e(A) + 2 \int_X f^2 \left| \frac{\partial A}{\partial t} \right|^2 \right) \right|
\]

\[
\leq \int_{\min\{s, \tau\}}^{\max\{s, \tau\}} \left| \frac{d}{dt} \int_X f^2 e(A) + 2 \int_X f^2 \left| \frac{\partial A}{\partial t} \right|^2 \right|
\]

\[
\leq C/R \int_{\min\{s, \tau\}}^{\max\{s, \tau\}} \left( \int_X f^2 e(A) \, dV_g \right)^{\frac{1}{2}} \left( \int_X \left| \frac{\partial A}{\partial t} \right|^2 \right)^{\frac{1}{2}}
\]

\[
+ C \int_{\min\{s, \tau\}}^{\max\{s, \tau\}} \left( \int_X f^2 e(A) \, dV_g \right)^{\frac{1}{2}} \left( \int_X \left| \frac{\partial A}{\partial t} \right|^2 \right)^{\frac{1}{2}}
\]

\[
\leq \left( \frac{C|s - \tau|}{R^2} YM(0) \int_{\min\{s, \tau\}}^{\max\{s, \tau\}} \int_X \left| \frac{\partial A}{\partial t} \right|^2 \right)^{\frac{1}{2}}
\]

\[
+ \left( C|s - \tau| YM(0) \int_{\min\{s, \tau\}}^{\max\{s, \tau\}} \int_X \left| \frac{\partial A}{\partial t} \right|^2 \right)^{\frac{1}{2}}.
\]
This indicates that
\[
\int_{B_R(x_0)} e(A)(\cdot, s) \, dV_g \leq \int_{B_{2R}(x_0)} e(A)(\cdot, \tau) \, dV_g + 2 \int_{\min\{s, \tau\}}^{\max\{s, \tau\}} \int_X |\partial A/\partial t|^2 \, dV_g dt
\]
\[+ C \left( \frac{|s-\tau|}{R^2} YM(0) \int_{\min\{s, \tau\}}^{\max\{s, \tau\}} \int_X |\partial A/\partial t|^2 \, dV_g dt \right)^{1/2}
\]
\[+ \left( C|s-\tau|YM(0) \int_{\min\{s, \tau\}}^{\max\{s, \tau\}} \int_X |\partial A/\partial t|^2 \right)^{1/2}.
\]

\[\square\]

**Lemma 2.5.** Suppose \(A(t)\) is a smooth solution of the heat flow (1.3). Then it holds that
\[
\left( \frac{\partial}{\partial t} - 2\sqrt{-1} \Lambda_\omega \partial \Lambda_\omega \right) |\Lambda_\omega F_A|^2 = -2|D_\Lambda \Lambda_\omega F_A|^2 \leq 0.
\]
Furthermore, \(\| \Lambda_\omega F_A \|_{L^2_X} (t)\) is decreasing along the flow and the \(L^\infty\) norm of the \(\Lambda_\omega F_A\) is bounded.

**Proof.** By simple calculation, we have
\[
\frac{\partial}{\partial t} |\Lambda_\omega F_A|^2 = 2\text{Re} \left\langle \frac{\partial}{\partial t} \Lambda_\omega F_A, \Lambda_\omega F_A \right\rangle
\]
\[= 2\text{Re} \sqrt{-1} \Lambda_\omega \left\langle (\partial_\Lambda \overline{\partial} A - \overline{\partial} A \partial_\Lambda) \Lambda_\omega F_A, \Lambda_\omega F_A \right\rangle
\]
and
\[
\sqrt{-1} \Lambda_\omega \partial \overline{\partial} |\Lambda_\omega F_A|^2 = \sqrt{-1} \Lambda_\omega \partial \overline{\partial} \left( \Lambda_\omega F_A, \Lambda_\omega F_A \right) + \sqrt{-1} \Lambda_\omega \partial \left( \Lambda_\omega F_A, \partial_\Lambda \Lambda_\omega F_A \right)
\]
\[= \sqrt{-1} \Lambda_\omega \left( \partial_\Lambda \overline{\partial} A \Lambda_\omega F_A, \Lambda_\omega F_A \right) + |\overline{\partial} A \Lambda_\omega F_A|^2 + |\partial_\Lambda \Lambda_\omega F_A|^2
\]
\[+ \sqrt{-1} \Lambda_\omega \left( \Lambda_\omega F_A, \overline{\partial} A \partial_\Lambda \Lambda_\omega F_A \right)
\]
\[= \text{Re} \sqrt{-1} \Lambda_\omega \left( (\partial_\Lambda \overline{\partial} A - \overline{\partial} A \partial_\Lambda) \Lambda_\omega F_A, \Lambda_\omega F_A \right) + |D_\Lambda \Lambda_\omega F_A|^2.
\]
This implies that
\[
(2.10) \quad \left( \frac{\partial}{\partial t} - 2\sqrt{-1} \Lambda_\omega \partial \Lambda_\omega \right) |\Lambda_\omega F_A|^2 = -2|D_\Lambda \Lambda_\omega F_A|^2 \leq 0.
\]

Using the maximum principle, we have
\[
\sup_X |\Lambda_\omega F_A|^2(\cdot, t) \leq \sup_X |\Lambda_\omega F_A|^2(\cdot, 0).
\]

Integrating the two sides of (2.10) over \(X\), we have
\[
\frac{\partial}{\partial t} \int_X |\Lambda_\omega F_A|^2 \frac{\omega^n}{n!} - 2 \int_X \sqrt{-1} \Lambda_\omega \partial \overline{\partial} |\Lambda_\omega F_A|^2 \frac{\omega^n}{n!}
\]
\[= \frac{\partial}{\partial t} \int_X |\Lambda_\omega F_A|^2 \frac{\omega^n}{n!} - 2 \int_X \sqrt{-1} |\Lambda_\omega F_A|^2 \overline{\partial} \partial\omega^{n-1} \frac{(n-1)!}{n!}
\]
\[= \frac{\partial}{\partial t} \int_X |\Lambda_\omega F_A|^2 \frac{\omega^n}{n!} = - \int_X |D_\Lambda \Lambda_\omega F_A|^2 \frac{\omega^n}{n!} \leq 0.
\]
Therefore, \(\| \Lambda_\omega F_A \|^2_{L^2_X} (t)\) is decreasing along the flow. \(\square\)
Lemma 2.6. Suppose $A(t)$ is a smooth solution of (1.3) and set

$$I(t) = \int_X |D_A \Lambda \omega F_A|^2.$$

Then it holds that $I(t) \to 0$ as $t \to \infty$.

Proof. From equation (2.2), we have

$$\frac{d}{dt} D_A \Lambda \omega F_A = \left[ \frac{\partial A}{\partial t}, \Lambda \omega F_A \right] + D_A \Lambda \omega \frac{\partial F_A}{\partial t}$$

$$= \left[ \sqrt{-1}(\overline{\partial}_A - \partial_A) \Lambda \omega F_A, \Lambda \omega F_A \right] + D_A \Lambda \omega D_A \Lambda \omega \frac{\partial A}{\partial t}$$

$$= \left[ \sqrt{-1}(\overline{\partial}_A - \partial_A) \Lambda \omega F_A, \Lambda \omega F_A \right] + D_A \Lambda \omega D_A \sqrt{-1}(\overline{\partial}_A - \partial_A) \Lambda \omega F_A.$$

So

$$\frac{d}{dt} I(t) = 2 \text{Re} \int_X \left\langle \frac{d}{dt} (D_A \Lambda \omega F_A), D_A \Lambda \omega F_A \right\rangle$$

$$= 2 \text{Re} \int_X \left\langle [\sqrt{-1}(\overline{\partial}_A - \partial_A) \Lambda \omega F_A, \Lambda \omega F_A], D_A \Lambda \omega F_A \right\rangle$$

$$+ 2 \text{Re} \int_X \left\langle D_A \Lambda \omega D_A \sqrt{-1}(\overline{\partial}_A - \partial_A) \Lambda \omega F_A, D_A \Lambda \omega F_A \right\rangle$$

$$= 2 \text{Re} \int_X \left\langle [\sqrt{-1}(\overline{\partial}_A - \partial_A) \Lambda \omega F_A, \Lambda \omega F_A], D_A \Lambda \omega F_A \right\rangle$$

$$- \left[ |(\tau + \overline{\tau})^* D_A \Lambda \omega F_A| D_A \Lambda \omega F_A \right\rangle.$$ 

At first, one can easily check that

$$\text{Re} \int_X \langle [\sqrt{-1}(\overline{\partial}_A - \partial_A) \Lambda \omega F_A, \Lambda \omega F_A], D_A \Lambda \omega F_A \rangle \leq C(n, \text{rank} E, \| \Lambda \omega F_A \|_{L^\infty}) I(t).$$

Then, it holds

$$\int_X \langle D_A \Lambda \omega F_A, D_A \Lambda \omega F_A \rangle = \int_X \langle \Lambda \omega F_A, D_A^* D_A \Lambda \omega F_A \rangle$$

$$\leq \| \Lambda \omega F_A \|_{L^\infty} \int_X |D_A^* D_A \Lambda \omega F_A|$$

$$\leq \| \Lambda \omega F_A \|_{L^\infty} \text{Vol}(X)^{1/2} \left( \int_X |D_A^* D_A \Lambda \omega F_A|^2 \right)^{1/2}.$$

Inequality (2.11) implies

$$I(t)^2 \leq \| \Lambda \omega F_A \|^2_{L^\infty} \text{Vol}(X) \int_X |D_A^* D_A \Lambda \omega F_A|^2.$$

At last, it is easy to check that

$$\int_X \langle (\tau + \overline{\tau})^* D_A \Lambda \omega F_A, D_A^* D_A \Lambda \omega F_A \rangle = \int_X \langle D_A \Lambda \omega F_A, \Lambda \omega dw \wedge D_A^* D_A \Lambda \omega F_A \rangle$$

$$\leq \int_X |D_A^* D_A \Lambda \omega F_A|^2 + C^2 I(t).$$
From the above all, we have
\[
\frac{dI(t)}{dt} \leq CI(t) - \int_X |D_A D_A \Lambda \omega F_A|^2 + C^2 I(t) \leq CI(t) - CI(t)^2.
\]
From equality (2.9) and
\[
\int_X \left| \frac{\partial A}{\partial t} \right|^2 = \int_X \left| (\partial A - \partial A) \Lambda \omega F_A \right|^2 = \int_X |D_A \Lambda \omega F_A|^2,
\]
we have
\[
\int_0^\infty I(t) < YM(0).
\]
Using the technique in [11, Prop. 6.2.14], we have \( I(t) \to 0 \) as \( t \to \infty \).

Lemma 2.7. Suppose \( A(t) \) is a global smooth solution of heat flow (1.3). Then it holds that
\[
(\triangle - \frac{\partial}{\partial t})|F_A|^2 \geq 2|\nabla_A F_A|^2 - C(1 + |F_A| + |\text{Ric}| + |\text{Rm}|)|F_A|^2 - C|F_A||\nabla_A F_A|,
\]
where \( C \) is a positive constant depending on the geometry of \( X \).

Proof. First, using Bochner technique, we have
\[
\triangle |F_A|^2 = -2\langle \nabla_A \nabla_A F_A, F_A \rangle + 2|\nabla_A F_A|^2.
\]
Then by simple calculation, we have
\[
\frac{\partial}{\partial t} F_A = D_A \frac{\partial A}{\partial t} = -D_A^* D_A F_A - D_A \alpha,
\]
where \( \alpha = [\Lambda, d\omega]^* F_A \). Combining with the following Weitzenböck formula
\[
\triangle_A F_A = -D_A D_A^* F_A = \nabla_A^* \nabla_A F_A + \text{Ric}^\sharp F_A + F_A^\sharp F_A,
\]
we have
\[
\frac{\partial}{\partial t} F_A = -\nabla_A^* \nabla_A F_A - \text{Ric}^\sharp F_A - F_A^\sharp F_A - D_A \alpha.
\]
Therefore,
\[
(\triangle - \frac{\partial}{\partial t})|F_A|^2 \geq 2|\nabla_A F_A|^2 + 2\langle \text{Ric}^\sharp F_A + F_A^\sharp F_A + D_A \alpha, F_A \rangle
\]
\[
\geq 2|\nabla_A F_A|^2 - C(1 + |\text{Ric}| + |F_A|)|F_A|^2 - C|\nabla_A F_A||F_A|,
\]
where \( C \) is a constant depending on the geometry of \( X \). \( \square \)

2.2. Monotonicity formula. Let \((X, g)\) be an \( n \)-dimensional compact Hermitian manifold with fundamental \((1, 1)\)-form \( \omega \) satisfying \( \partial \bar{\partial} \omega^{n-1} = \partial \bar{\partial} \omega^{n-2} = 0 \). We regard \( X \) as a \( 2n \)-dimensional Riemannian manifold. For any \( x_0 \in X \), there exist normal geodesic coordinates \( \{x_i\}_{i=1}^{2n} \) in the geodesic ball \( B_r(x_0) \) centered at \( x_0 \) with radius \( r \leq i_X \) such that \( x_0 = (0, \ldots, 0) \) and
\[
|g_{ij}(x) - \delta_{ij}| \leq C(x_0) |x|^2, \quad \left| \frac{\partial g_{ij}}{\partial x_k} \right| \leq C(x_0) |x| \quad \forall x \in B_r,
\]
where \( i(X) \) is the infimum of the injectivity radius over \( X \) and \( C(x_0) \) a positive constant depending on \( x_0 \).
Let $u = (x, t)$ be a point in $X \times \mathbb{R}$. For a fixed point $u_0 = (x_0, t_0) \in X \times \mathbb{R}^+$, we write

$$S_r = X \times \{t = t_0 - r^2\},$$

$$T_r = \{u = (x, t) : t_0 - 4r^2 \leq t \leq t_0 - r^2, x \in X\},$$

$$P_r(u_0) = B_r(x_0) \times [t_0 - r^2, t_0 + r^2].$$

For simplicity, we denote $S_r, T_r$ and $P_r$ by $S_r, T_r$ and $P_r$ respectively.

For simplicity, we denote $G(0, 0, 0)$ by $G(x, t)$.

Assume that $A(t)$ is a smooth global solution of the heat flow (1.3) in $X \times \mathbb{R}^+$. Let $f$ be a smooth cut-off function such that $|f| \leq 1$, $f \equiv 1$ on $B_{R/2}$, $f = 0$ outside $B_R$ and $|\nabla f| \leq 2/R$, where $R \leq iX$. For any $(x, t) \in X \times [0, +\infty)$, we set

$$\Phi(r) = r^2 \int_{T_r(u_0)} e(A)f^2G_{u_0} dt$$

Then we have

**Theorem 2.8.** Assume that $A(t)$ is a solution of the heat flow (1.3) in $X \times \mathbb{R}^+$ with initial data $A_0$. Let $f$ be a smooth cut-off function such that $|f| \leq 1$, $f \equiv 1$ on $B_{R/2}$, $f = 0$ outside $B_R$ and $|\nabla f| \leq 2/R$, where $R \leq iX$. Then for any $r_1$ and $r_2$ with $0 < r_1 \leq r_2 \leq \min\{R/2, \sqrt{t_0}/2\}$, we have

$$\Phi(r_1) \leq C \exp(C(r_2 - r_1))\Phi(r_2) + C(r^2_2 - r^2_1)YM(0) + CR^{2-2n}\int_{P_r(u_0)} |F_A|^2 dV_g dt.$$

**Proof.** Choose normal geodesic coordinates $\{x_i\}_{i=1}^{2n}$ in the geodesic ball $B_R(x_0)$. Setting $x = r\bar{x}$, $t = t_0 + r^2\bar{t}$, we have

$$\Phi(r) = r^2 \int_{T_r(u_0)} e(A)f^2G_{u_0} dV_g dt$$

$$= r^2 \int_{t_0 - 4r^2}^{t_0 - r^2} \int_{\mathbb{R}^{2n}} e(A)(x, t)f^2(x)G_{u_0}(x, t) \det (g_{ij}) \ dx \ dt$$

$$= r^4 \int_{\tilde{T}_1} e(A)(r\bar{x}, t_0 + r^2\bar{t})f^2(\bar{x}) G(\bar{x}, \bar{t}) \det (g_{ij})(r\bar{x}) \ d\bar{x} \ d\bar{t}$$

where $\tilde{T}_1 = [-4, -1] \times \mathbb{R}^{2n}$. The $r$-direction derivative of $\Phi(r)$ is

$$d\Phi(r)/dr = 4r^3 \int_{\tilde{T}_1} e(A)(r\bar{x}, t_0 + r^2\bar{t})f^2(\bar{x}) G(\bar{x}, \bar{t}) \det (g_{ij})(r\bar{x}) \ d\bar{x} \ d\bar{t}$$

$$+ r^4 \int_{\tilde{T}_1} \frac{d}{dr}e(A)(r\bar{x}, t_0 + r^2\bar{t}) f^2(\bar{x}) G(\bar{x}, \bar{t}) \det (g_{ij})(r\bar{x}) \ d\bar{x} \ d\bar{t}$$

$$+ \int_{\tilde{T}_1} e(A)(r\bar{x}, t_0 + r^2\bar{t}) \left( \frac{d}{dr}f^2(\bar{x}) \det (g_{ij})(r\bar{x}) \right) G(\bar{x}, \bar{t}) d\bar{x} d\bar{t}$$

$$=: I_1 + I_2 + I_3.$$
where \( I_1 = \frac{4\Phi(r)}{r} \). At first, we calculate \( I_2 \). By simple calculation, we have

\[
\frac{\partial}{\partial r} e(A)(r\bar{x}^i, t_0 + r^2 \hat{t}) = \frac{\partial}{\partial x^i} e(A)(r\bar{x}^i, t_0 + r^2 \hat{t}) \frac{\partial x^i}{\partial r} + \frac{\partial}{\partial \hat{t}} e(A)(r\bar{x}^i, t_0 + r^2 \hat{t}) \frac{\partial \hat{t}}{\partial r}
\]

\[
= \bar{x}^i \frac{\partial}{\partial x^i} e(A) + 2r \hat{t} \frac{\partial}{\partial \hat{t}} e(A)
\]

\[
= \frac{x^i}{r} \frac{\partial}{\partial x^i} e(A) + \frac{2(t - t_0)}{r} \frac{\partial}{\partial \hat{t}} e(A).
\]

(2.12)

Substituting (2.12) into \( I_2 \), we have

\[
I_2 = r \int_{T_r(u_0)} x^i \frac{d}{dx^i} e(A)(x, t) f^2(x) G u_0(x, t) \ dV_g dt
\]

\[
+ r \int_{T_r(u_0)} (2(t - t_0) \frac{d}{dt} e(A)(x, t)) f^2(x) G u_0(x, t) \ dV_g dt
\]

\[
=: I_{2.1} + I_{2.2}
\]

From the Bianchi identity

\[
D_A F_A = 0,
\]

we have

\[
0 = D_A F_A \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k} \right)
\]

\[
= \nabla_{A,\theta/\partial x_i} F_A \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k} \right) - \nabla_{A,\theta/\partial x_j} F_A \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k} \right) + \nabla_{A,\theta/\partial x_k} F_A \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k} \right).
\]

(2.13)

For simplicity, we set \( \nabla_{A,i} := \nabla_A \frac{\partial}{\partial x_i} \) and \( \nabla_i := \nabla \frac{\partial}{\partial x_i} \). Therefore, by (2.13), we have

\[
x^i \frac{\partial}{\partial x^i} |F_A|^2
\]

\[
= 2x^i \text{Re} \left\langle \nabla_{A,i} F_A, F_A \right\rangle
\]

\[
= x^i \text{Re} \left\langle \nabla_{A,i} F_A \left( \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k} \right) dx^j \wedge dx^k, F_A \right\rangle
\]

(2.14)

\[
= x^i \text{Re} \left\langle \left( \nabla_{A,j} F_A \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^k} \right) - \nabla_{A,k} F_A \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) \right) dx^j \wedge dx^k, F_A \right\rangle
\]

\[
= 2x^i \text{Re} \left\langle \nabla_{A,j} F_A \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^k} \right) dx^j \wedge dx^k, F_A \right\rangle
\]

\[
= 2 \text{Re} \left\langle \nabla_{A,j} x^i F_A \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^k} \right) dx^j \wedge dx^k, F_A \right\rangle - 2 \text{Re} \left\langle \delta^i \delta^k F_A \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^k} \right) dx^j \wedge dx^k, F_A \right\rangle
\]

\[
= 2 \text{Re} \left\langle dx^j \wedge (\nabla_{A,j} x^i F_A) \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^k} \right) dx^k, F_A \right\rangle - 4|F_A|^2
\]
and
\[ dx^i \wedge (\nabla_{A,j}(x^iF_A))^i(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})dx^k = dx^i \wedge \left( \nabla_{A,j}(x^iF_A) - x^iF_A(\nabla_{j} \frac{\partial}{\partial x^i}) - x^iF_A(\frac{\partial}{\partial x^i}, \nabla_{j} \frac{\partial}{\partial x^k}) \right) dx^k \]
\[ = \nabla_{A,j}(x^iF_A,ik)dx^j \wedge dx^k - x^iF_A(\nabla_{j} \frac{\partial}{\partial x^i}) dx^j \wedge dx^k \]
\[ = dx^i \wedge \nabla_{A,j}(x^iF_A,ik)dx^j - x^iF_A(\nabla_{j} \frac{\partial}{\partial x^i}) dx^j \wedge dx^k \]
\[ = D_A(x^iF_A,ik)dx^k - x^iF_A(\nabla_{j} \frac{\partial}{\partial x^i}) dx^j \wedge dx^k. \]

The reason for the second equality in (2.15) is that
\[ \sum_{j,k} F_A(\frac{\partial}{\partial x^j}, \nabla_{j} \frac{\partial}{\partial x^k})dx^j \wedge dx^k \]
\[ = \sum_{j<k} + \sum_{k<j} F_A(\frac{\partial}{\partial x^j}, \nabla_{j} \frac{\partial}{\partial x^k})dx^j \wedge dx^k \]
\[ = \sum_{j<k} (F_A(\frac{\partial}{\partial x^j}, \nabla_{j} \frac{\partial}{\partial x^k})dx^j \wedge dx^k + F_A(\frac{\partial}{\partial x^i}, \nabla_{k} \frac{\partial}{\partial x^j})dx^k \wedge dx^j) \]
\[ = 0. \]

And the reason for the forth equality in (2.15) is that \( dx^j \wedge \nabla_j dx^k = Ddx^k = 0. \) Substituting (2.15) into (2.14), we have
\[ x^i \frac{\partial}{\partial x^i}|F_A|^2 = 2\text{Re}(D_A(x^iF_A,ik)dx^k - x^iF_A(\nabla_{j} \frac{\partial}{\partial x^i}) dx^j \wedge dx^k, F_A) - 4|F_A|^2. \]
Noting that \( \frac{\partial G_{u\alpha}}{\partial x^x} = \frac{x^\alpha G_{u\alpha}}{2(t-t_0)}, \) we have
\[ \int_{T_r(u_0)} \langle d(f^2G_{u\alpha}) \wedge (x^iF_A,ik)dx^k, F_A \rangle dV_g dt = \int_{T_r(u_0)} \langle x^iF_A,ikdx^k, \nabla(f^2G_{u\alpha}).F_A \rangle dV_g dt \]
\[ = \int_{T_r(u_0)} \langle x^iF_A,ikdx^k, 2f g^{\alpha\beta} \frac{\partial f}{\partial x^\alpha} F_{A,\beta} dx^l \rangle G_{u\alpha} dV_g dt \]
\[ + \int_{T_r(u_0)} \langle x^iF_A,ikdx^k, \frac{g^{\alpha\beta}x^\alpha}{2(t-t_0)} F_{A,\beta} dx^l \rangle f^2 G_{u\alpha} dV_g dt \]
and
\[ \int_{T_r(u_0)} (t-t_0) \langle F_A, d(f^2G_{u\alpha}) \wedge \frac{\partial A}{\partial t} \rangle dV_g dt = \int_{T_r(u_0)} (t-t_0) \langle 2g^{\alpha\beta} \frac{\partial f}{\partial x^\alpha} F_{A,\beta} dx^l, \frac{\partial A}{\partial t} \rangle f G_{u\alpha} dV_g dt \]
\[ + \int_{T_r(u_0)} (t-t_0) \langle \frac{g^{\alpha\beta}x^\alpha}{2(t-t_0)} F_{A,\beta} dx^l, \frac{\partial A}{\partial t} \rangle f^2 G_{u\alpha} dV_g dt. \]
From the above all, we obtain

\[
I_{2.1} = 2 \text{Re} \int_{T_r(u_0)} \langle D_A(x^i F_{A,ik} dx^k), F_A \rangle f^2 G_{u_0} \, dV_g dt
- 2 \text{Re} \int_{T_r(u_0)} \left\langle x^i F_A \left( \nabla_j \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^k} \right) dx^j \wedge dx^k, F_A \right\rangle \langle f^2 G_{u_0}, dV_g \rangle dt
- 4r \int_{T_r(u_0)} |F_A|^2 f^2 G_{u_0} \, dV_g dt
= -2 \text{Re} \int_{T_r(u_0)} \left\langle (x^i F_{A,ik} dx^k), \frac{\partial A}{\partial t} \right\rangle f^2 G_{u_0} \, dV_g dt
- 2 \text{Re} \int_{T_r(u_0)} \langle x^i F_{A,ik} dx^k, [\Lambda_\omega, d\omega]^* F_A \rangle f^2 G_{u_0} \, dV_g dt
- 2 \text{Re} \int_{T_r(u_0)} \langle x^i F_{A,ik} dx^k, 2 f g^{\alpha\beta} \frac{\partial f}{\partial x^\alpha} F_{A,\beta l} dx^l \rangle f G_{u_0} \, dV_g dt
- 2 \text{Re} \int_{T_r(u_0)} \langle x^i F_A \left( \nabla_j \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^k} \right) dx^j \wedge dx^k, F_A \rangle f^2 G_{u_0} \, dV_g dt
- 4r \int_{T_r(u_0)} |F_A|^2 f^2 G_{u_0} \, dV_g dt,
\]

and

\[
I_{2.2} = 2 \int_{T_r(u_0)} (t - t_0) \frac{\partial}{\partial t} |F_A|^2 f^2 G_{u_0} \, dV_g dt
= 4 \text{Re} \int_{T_r(u_0)} (t - t_0) \left\langle F_A, D_A \frac{\partial A}{\partial t} \right\rangle f^2 G_{u_0} \, dV_g dt
= 4 \text{Re} \int_{T_r(u_0)} (t - t_0) \left\langle D_A F_A, \frac{\partial A}{\partial t} \right\rangle f^2 G_{u_0} \, dV_g dt
= 4 \text{Re} \int_{T_r(u_0)} (t - t_0) \left\langle F_A, d(f^2 G_{u_0}) \wedge \frac{\partial A}{\partial t} \right\rangle dV_g dt
= 4 \int_{T_r(u_0)} (t - t_0) \left( \frac{\partial A}{\partial t} \right)^2 f^2 G_{u_0} \, dV_g dt
- 4 \text{Re} \int_{T_r(u_0)} (t - t_0) \left\langle [\Lambda_\omega, d\omega]^* F_A, \frac{\partial A}{\partial t} \right\rangle f^2 G_{u_0} \, dV_g dt
- 4 \text{Re} \int_{T_r(u_0)} (t - t_0) \left\langle 2 g^{\alpha\beta} \frac{\partial f}{\partial x^\alpha} F_{A,\beta l} dx^l, \frac{\partial A}{\partial t} \right\rangle f G_{u_0} \, dV_g dt
- 4 \text{Re} \int_{T_r(u_0)} (t - t_0) \left\langle \frac{g^{\alpha\beta} x^\alpha}{2(t - t_0)} F_{A,\beta l} dx^l, \frac{\partial A}{\partial t} \right\rangle f^2 G_{u_0} \, dV_g dt.
\]

For simplicity, we set \( x \odot F_A = \frac{1}{2} x^i F_{A,ik} dx^k \), \( x \cdot F_A = \frac{1}{2} x^\alpha g^{\alpha\beta} F_{A,\beta l} dx^l \) and \( \nabla f \cdot F_A = 2g^{\alpha\beta} f^{-1} f_\alpha F_{A,\beta l} dx^l \), where \( f_\alpha = \frac{\partial f}{\partial x^\alpha} \). Substituting \( I_{2.1} \) and \( I_{2.2} \) into \( I_2 \), we have
By Cauchy inequality, it holds that

\[ I_2 = 4r \int_{T_r(u_0)} \frac{1}{|t-t_0|} \left| \frac{\partial A}{\partial t} - x \odot F_A \right|^2 f^2 G_{u_0} \, dV_g \, dt \]
\[ + 4r \int_{T_r(u_0)} \frac{1}{|t-t_0|} \left( x \cdot F_A - x \odot F_A, x \odot F_A - |t-t_0| \frac{\partial A}{\partial t} \right) f^2 G_{u_0} \, dV_g \, dt \]
\[ + 4r \int_{T_r(u_0)} \left( \nabla f \cdot F_A, |t-t_0| \frac{\partial A}{\partial t} - x \odot F_A \right) f^2 G_{u_0} \, dV_g \, dt \]
\[ + 4r \int_{T_r(u_0)} \left( [\Lambda, d\omega]^* F_A, |t-t_0| \frac{\partial A}{\partial t} - x \cdot F_A \right) f^2 G_{u_0} \, dV_g \, dt \]
\[ - r^2 \int_{T_r(u_0)} x^i F_A \left( \nabla_j \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^k} \right) d\omega^{jk}, F_A \right) f^2 G_{u_0} \, dV_g \, dt \]
\[ - 4r \int_{T_r(u_0)} |F_A|^2 f^2 G_{u_0} \, dV_g \, dt. \]

(2.17)

By simple calculation, we have

\[ I_3 = r \int_{T_r(u_0)} |F_A|^2 x^i \frac{\partial}{\partial x^i} \sqrt{\det(g_{ij})} G_{u_0} \, dx dt \]
\[ = r \int_{T_r(u_0)} |F_A|^2 x^i 2 f f_i G_{u_0} \, dV_g \, dt \]
\[ + r \int_{T_r(u_0)} |F_A|^2 x^i \frac{\sqrt{\det(g_{ij})}}{f} f^2 G_{u_0} \, dx dt \]
\[ = r \int_{T_r(u_0)} |F_A|^2 x^i 2 f f_i G_{u_0} \, dV_g dt \]
\[ + \frac{r}{2} \int_{T_r(u_0)} |F_A|^2 x^i \text{tr} \left( \frac{\partial g}{\partial x^i} g^{-1} \right) f^2 G_{u_0} \, dV_g \, dt. \]
Since $|g_{ij} - \delta_{ij}| \leq C|x|^2$, $\left| \frac{\partial g_{ij}}{\partial x^k} \right| \leq C|x|$ and $|\Gamma^i_{jk}| \leq C|x|$, there exists a constant $C_1$ such that

$$|x \cdot F_A - x \odot F_A|^2 = \frac{1}{2} x^i (\delta_j^i - g^{ij}) F_{A,ji} dx^j \leq C_1 |x|^6 |F_A|^2,$$

(2.19)

$$\langle x^i F_A (\nabla_j \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^k}) dx^j \wedge dx^k, F_A \rangle \leq C_1 |x|^2 |F_A|^2,$$

$$\text{tr} \left( \frac{\partial g}{\partial x^i} g^{-1} \right) \leq C_1 |x|.$$  

In addition, it is easy to check that there exists a constant $C_2 > 0$ such that

$$|\langle A_\omega, \omega \rangle | F_A|^2 \leq C_2 |F_A|^2.$$  

(2.20)

Substituting (2.19) and (2.20) into (2.18) and (2.17), we have there exists a constant $C_3 > 0$

$$\frac{d\Phi(r)}{dr} \geq r \int_{T_r(u_0)} \frac{1}{|t - t_0|} \left| \frac{\partial A}{\partial t} - x \odot F_A \right|^2 f^2 G_{u_0} \, dV_g dt - C_3 r \int_{T_r(u_0)} \frac{|x|^6}{|t - t_0|} |F_A|^2 f^2 G_{u_0} \, dV_g dt - 4 r \int_{T_r(u_0)} |t - t_0| |\nabla f \cdot F_A|^2 f^2 G_{u_0} \, dV_g dt - C_3 r \int_{T_r(u_0)} |t - t_0| |F_A|^2 f^2 G_{u_0} \, dV_g dt - C_3 r \int_{T_r(u_0)} |x|^2 |F_A|^2 f^2 G_{u_0} \, dV_g dt - 2 r \int_{T_r(u_0)} |x| |\nabla f| |f| |F_A|^2 G_{u_0} \, dV_g dt$$

From [5, p. 99], we have there exists a constant $C_4 > 0$ such that

$$r^{-1} |t - t_0| |x|^6 G_{u_0} \leq C_4 (1 + G_{u_0}),$$

$$r^{-1} |x|^2 G_{u_0} \leq C_4 (1 + G_{u_0})$$

holds on $T_r(u_0)$. Therefore, we get

$$- C_3 r \int_{T_r(u_0)} \left( \frac{|x|^6}{|t - t_0|} + |t - t_0| + |x|^2 \right) |F_A|^2 f^2 G_{u_0} \, dV_g dt \geq - C_5 \Phi(r) - C_5 r \text{YM}(0),$$

where $C_5$ is a positive constant depending on $C_3$ and $C_4$. At last, we estimate the remaining two terms

$$- 4 r \int_{T_r(u_0)} |t - t_0| |\nabla f \cdot F_A|^2 f^2 G_{u_0} \, dV_g dt$$

and

$$- 2 r \int_{T_r(u_0)} |x| |\nabla f| |f| |F_A|^2 G_{u_0} \, dV_g dt.$$

Since supp$(\nabla f) \subseteq B_R(x_0) \setminus B_{R/2}(x_0)$, $|\nabla f| \leq 2/R$ and $|t - t_0| \leq 4r^2$, we have

$$|t - t_0| |\nabla f \cdot F_A|^2 f^2 G_{u_0} \leq \frac{16}{(4\pi)^n R^{2-2n}} \exp \left( - \frac{R^2}{64 r^2} \right) |F_A|^2.$$
Theorem 2.9. Suppose that $A(t)$ is a smooth solution of the heat flow \( \text{(1.3)} \), then there exist positive constants $\varepsilon_0$ and $\delta_0$ such that if for some $0 < R < \min\{i_X/2, \sqrt{t_0}/2\}$, the inequality

$$R^{2-2n} \int_{P_R(u_0)} e(A) < \varepsilon_0$$

holds, then for any $\delta \in (0, \min\{\delta_0, 1/4\})$, we have

$$\sup_{P_R(x_0,t_0)} |F_A|^2 < 16(\delta R)^{-4}.$$
Proof. For any $\delta \in (0, 1/4]$, we define the function
\[
f(r) = (2\delta R - r)^4 \sup_{P_r(x_0,t_0)} |F_A|^2.
\]
Since $f(r)$ is continuous and $f(2\delta R) = 0$, we have $f(r)$ attains its maximum at a certain point $r_0 \in [0, 2\delta R)$. We claim that $f(r_0) < 16$, this means that for any $r \in [0, 2\delta R)$, we have
\[
(2\delta R - r)^4 \sup_{P_r(x_0,t_0)} |F_A|^2 < 16.
\]
In particular, when $r \in [0, \delta R]$, it holds that $\sup_{P_r(x_0,t_0)} |F_A|^2 < 16/(2\delta R - r)^4 < 16(\delta R)^{-4}$.

Assuming that the claim is not true. This means $f(r_0) \geq 16$. Set
\[
\rho_0 = (2\delta R - r_0)f(r_0)^{-1/4} < \frac{1}{2}(2\delta R - r_0) = \delta R - r_0/2.
\]
Rescale the Riemannian metric $\tilde{g} = \rho_0^{-2}g$ and $t = t_1 + \rho^2 \tilde{t}$, where $(x_1, t_1)$ satisfies
\[
|F_A|^2(x_1, t_1) = \sup_{P_{\rho_0}(x_0,t_0)} |F_A|^2.
\]
Setting
\[
e_{\rho_0}(x, \tilde{t}) = |F_A|_{\tilde{g}}^2 = \rho_0^4 |F_A|_g^2,
\]
we have
\[
e_{\rho_0}(x_1, 0) = \rho_0^4 |F_A|_{\tilde{g}}^2(x_1, t_1) = \rho_0^4 \times \frac{f(r_0)}{(2\delta R - r_0)^4} = 1
\]
and
\[
\sup_{(x,\tilde{t}) \in \tilde{P}_1(x_1,0)} e_{\rho_0}(A)(x,\tilde{t}) \sup_{P_{\rho_0}(x_1,t_1)} e(A) = \rho_0^4 \sup_{P_{2\delta R + \rho_0}(x_1,t_1)} e(A) \leq \rho_0^4 \left( \frac{2\delta R + \rho_0}{2} \right)^{-4} f(r_0) = 16, \tag{2.21}
\]
where $\tilde{P}_1(x_1,0) = B_{\rho_0}(x_0) \times [-1,1]$. From the Bochner type inequality and (2.21), on $\tilde{P}_1(x_1,0)$, it holds that
\[
(\frac{\partial}{\partial \tilde{t}} - \Delta_{\tilde{g}})e_{\rho_0}(x,\tilde{t}) = \rho_0^6 (\frac{\partial}{\partial \tilde{t}} - \Delta g)e(A)(x,t)
\]
\[
\leq C \rho_0^6 (|\text{Ric}|_g + |\text{Rm}|_g + |F_A|_g) e(A)
\]
\[
\leq C' e_{\rho_0}.
\]
By the parabolic mean value inequality, we have
\[
1 = e_{\rho_0}(x_1,0) \leq C'' \int_{\tilde{P}_1(x_1,0)} |F_A|_{\tilde{g}}^2 \, dV_{\tilde{g}} d\tilde{t},
\]
where $C > 0$ is a constant depending on the geometry of $X$ and the initial connection $A_0$.

Choose normal geodesic coordinates centred at $x_1$ and construct cut-off function $\varphi \in C^\infty_0(B_{R/2}(x_1))$ such that $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ on $B_{R/4}(x_1)$ and $|\nabla \varphi| \leq 8/R$. Taking $r_1 = \rho$ and $r_2 = \min\{1/4, \delta_0\}R$.
and applying the monotonicity formula, we have
\[ \int_{P_t(x_1,0)} |F_A|^2 dV_g dt = \rho_0^{-2n} \int_{P_{t_0}(x_1,t_1)} |F_A|^2 dV_g dt \]
\[ \leq C \rho_0^2 \int_{P_{t_0}(x_1,t_1)} |F_A|^2 G(x_1,t_1+2\rho_0^2) \nu^2 dV_g dt \]
\[ \leq C \rho_0^2 \int_{T_{t_0}(x_1,t_1)} |F_A|^2 G(x_1,t_1+2\rho_0^2) \nu^2 dV_g dt \]
\[ \leq C e^{C(r_2-\rho_0)r_2^2} \int_{T_{t_2}(x_1,t_1+2\rho_0^2)} |F_A|^2 G(x_1,t_1+2\rho_0^2) \nu^2 dV_g dt \]
\[ + C(r_2^2-\rho_0^2)YM(0) + C(R/2)^2-2n \int_{P_{R/2}(x_1,t_1)} |F_A|^2 dV_g dt \]
\[ \leq C \delta_0^{2-2n} R^{2-2n} \int_{P_{R}(x_0,t_0)} |F_A|^2 dV_g dt + C\delta_0^2 R^2 YM(0) \]
\[ + C \delta_0^{2-2n} R^{2-2n} \int_{P_{R}(x_0,t_0)} |F_A|^2 dV_g dt \]
\[ \leq \tilde{C} (\delta_0^{2-2n} \varepsilon_0 + \delta_0^2 R^2 YM(0)) \]
where the constants depend on the geometry of \( X \) and the initial data \( A_0 \). From (2.22) and (2.23), we have \( 1 \leq C'' \tilde{C} (\delta_0^{2-2n} \varepsilon_0 + \delta_0^2 R^2 YM(0)) \). Thus, choosing \( \delta_0 \) and \( \varepsilon_0 \) properly, we can obtain a contradiction.

3. PROOF OF THEOREM 1.1

Using the same argument as that in the proof of Theorem 2 in Bando and Siu’s paper (2), we have

Theorem 3.1. Let \( X \) be an \( n \)-dimensional complex manifold, \( g \) a Hermitian metric on \( X \) with associated \((1,1)\)-form \( \omega \). Let \((E,h)\) be a holomorphic vector bundle with a Hermitian metric \( h \) over \( X \setminus S \), where \( S \) is a closed subset with locally finite Hausdorff measure of real co-dimension 4. If the curvature \( F_h \) is locally integrable, then

(1) \( E \) can be extended to the whole \( X \) as a reflexive sheaf \( \mathcal{E} \), and for any local section \( s \in \Gamma(U, \mathcal{E}) \), \( \log^+ h(s,s) \) belongs to \( H^1_{loc} \);

(2) If \( A_\omega F \) is locally bounded, then \( h \) is locally bounded and \( h \in W^{2,p}_{loc} \) for any finite \( p \) where \( \mathcal{E} \) is locally free;

(3) If \((E,h)\) is Hermitian-Einstein, then \( h \) smoothly extends as a Hermitian-Einstein metric over the place where \( \mathcal{E} \) is locally free.

Proof of Theorem 1.1

(1) Step 1. Construct the closed set \( \Sigma \) of Hausdorff codimension at least 4

From Lemma 2.3, we have for arbitrary sequence \( t_k \to \infty \) and \( a > 0 \), it holds that

\[ \int_{t_k-a}^{t_k+a} \int_X \left| \frac{\partial A}{\partial t} \right|^2 dV_g dt \to 0, \quad t_k \to \infty. \]

Then for arbitrary \( \epsilon > 0 \), there exists \( K \in \mathbb{Z}^+ \), such that when \( k \geq K \), it holds that

\[ \int_{t_k-a}^{t_k+a} \int_X \left| \frac{\partial A}{\partial t} \right|^2 dV_g dt < \epsilon. \]
Construct the set
\[ \Sigma = \bigcap_{0 < r < \bar{r} X} \{ x \in X, \liminf_{k \to \infty} r^{4-2n} \int_{B_r(x)} e(A) dV_g \geq \varepsilon_1 \} \]
where \( \varepsilon_1 \) is determined below.
For \( x_1 \in X \setminus \Sigma \), there exists \( r_1 > 0 \), such that when \( t_k \) is sufficiently large, we have
\[ r_1^{4-2n} \int_{B_{r_1}(x_1)} e(A)(\cdot, t_k) dV_g < \varepsilon_1. \]
Set \( s = t_k - r_1^2 \), \( \tau = t_k + r_1^2 \). Applying Lemma 2.4 for any \( t \in [s, \tau] \), we have
\[ \int_{B_{r_1/2}(x_1)} e(A)(\cdot, t) dV_g \leq \int_{B_{r_1}(x_1)} e(A)(\cdot, t_k) + 2 \int_{t_k - r_1^2}^{t_k + r_1^2} \int_{X} \left| \frac{\partial A}{\partial t} \right|^2 e(A)(\cdot, t) dV_g dt \]
\[ + C \left( YM(0) \int_{t_k - r_1^2}^{t_k + r_1^2} \int_{X} \left| \frac{\partial A}{\partial t} \right|^2 dV_g dt \right)^{1/2} \]
\[ + CiX \left( YM(0) \int_{t_k - r_1^2}^{t_k + r_1^2} \int_{X} \left| \frac{\partial A}{\partial t} \right|^2 dV_g dt \right)^{1/2} \]
\[ \leq \int_{B_{r_1}(x_1)} e(A)(\cdot, t_k) + 2\varepsilon + C(1 + iX)(YM(0)\varepsilon)^{1/2}. \]
Consider
\[ r_1^{2-2n} \int_{P_{r_1/2}(x_1,t_k)} e(A)(\cdot, t) dV_g dt = r_1^{2-2n} \int_{t_k -(r_1/2)^2}^{t_k +(r_1/2)^2} \int_{B_{r_1/2}(x_1)} e(A)(\cdot, t) dV_g dt \]
Substituting (3.1) into (3.2), we have
\[ r_1^{2-2n} \int_{t_k -(r_1/2)^2}^{t_k +(r_1/2)^2} \int_{B_{r_1/2}(x_1)} e(A)(\cdot, t) dV_g dt \]
\[ \leq r_1^{2-2n} \int_{t_k -(r_1/2)^2}^{t_k +(r_1/2)^2} \int_{B_{r_1}(x_1)} e(A)(\cdot, t_k) dV_g dt \]
\[ + Cr_1^{4-2n} \left( \varepsilon + (1 + iX)(YM(0)\varepsilon)^{1/2} \right) \]
\[ = \frac{1}{2} r_1^{4-2n} \int_{B_{r_1}(x_1)} e(A)(\cdot, t_k) dV_g + Cr_1^{4-2n} \left( \varepsilon + (1 + iX)(YM(0)\varepsilon)^{1/2} \right) \]
Choosing \( \varepsilon_1 = \varepsilon_0/4^{n-1} \) and \( \varepsilon \) such that \( C2^{2n-2}r_1^{4-2n}(\varepsilon + (1 + iX)(YM(0)\varepsilon)^{1/2}) \leq \varepsilon_0/2 \), we have
\[ \left( \frac{r_1}{2} \right)^{2-2n} \int_{P_{r_1/2}(x_1,t_k)} e(A)(\cdot, t) dV_g dt \leq \varepsilon_0. \]
where \( \varepsilon_0 \) is the constant in Theorem 2.9. Applying the small energy regularity theorem, we have
\[ \sup_{P_{\varepsilon_0r_1}(x_1,t_k)} e(A)(\cdot, \cdot) \leq C(\delta_0 r_1)^{-4}, \]
where \( \delta_0 \) is the constant in Theorem 2.9.
It is easy to check that for any \( x \in B_{\delta_0 r_1}(x_1) \), we can choose small enough \( r_1 \) such that \( B_{r_1}(x) \subseteq B_{\delta_0 r_1}(x_1) \) and

\[
\int_{B_{r_1}(x)} e(A)(\cdot, t_k) \leq \int_{B_{r_1}(x)} r_1^{4-2n} e(A)(\cdot, t_k) \leq r_1^{4-2n} \frac{r_1}{\delta_0 r_1} C(\delta_0 r_1)^{-4} = C \left( \frac{r_1}{\delta_0 r_1} \right)^4 < \varepsilon_1,
\]

that is \( B_{\delta_0 r_1}(x_1) \subseteq X \setminus \Sigma \) and \( \Sigma \) is closed.

In fact \( \mathcal{H}^{2n-4}(\Sigma) < \infty \). Since \( \Sigma \) is closed, we have for any \( \delta > 0 \), there exist finite geodesic balls \( \{B_{r_i}(x_i)\}_{i \in \Gamma} \), where \( x_i \in \Sigma \), \( r_i < \delta \), such that

- \( \Sigma \subseteq \cup_{i \in \Gamma} B_{r_i} \),
- when \( i \neq j \), \( B_{r_i/2}(x_i) \cap B_{r_j/2}(x_j) = \emptyset \).

Since \( x_i \in \Sigma \), we have

\[
\int_{B_{r_i/2}} e(A)(\cdot, t_k) \, dV_g > 2^{2n-4} \varepsilon_1,
\]

for \( t_k \) is sufficiently large. This implies

\[
\int_{B_{r_i/2}} e(A)(\cdot, t_k) \, dV_g < \varepsilon_1^{-1} 2^{2n-4} \int_{B_{r_i/2}} e(A)(\cdot, t_k) \, dV_g.
\]

Thus we have

\[
\sum_{i \in \Gamma} r_i^{4-2n} \leq \varepsilon_1^{-1} 2^{4-2n} \int_{\cup B_{r_i/2}} e(A)(\cdot, t_k) < \varepsilon_1^{-1} 2^{4-2n} \text{YM}(0) < +\infty.
\]

This implies that \( \mathcal{H}^{2n-4}(\Sigma) \leq \varepsilon_1^{-1} 2^{4-2n} \text{YM}(0) < \infty \).

Step 2 The convergence in \( X \setminus \Sigma \)

From the above, we have that for any \( x_0 \in X \setminus \Sigma \), there exists \( r_0 \), when \( t_k \) is sufficiently large, it holds that

\[
\sup_{P_{r_0}(x_0, t_k)} e(A)(\cdot, \cdot) \leq C.
\]

Applying the Uhlenbeck weak compactness theorem ([21]), there exists subsequence \( \{t_{k'}\} \subseteq \{t_k\} \) and gauge transformations \( \sigma(k') \) such that \( \sigma(k')(A(t_{k'})) \) converges to the connection \( A_\infty \) of limiting bundle \( (E_\infty, H_\infty) \) in weak \( W^{1,2}_{\text{loc}}(X \setminus \Sigma) \) sense and \( A_\infty \) satisfies

\[
D_{A_\infty} A_\omega F_{A_\infty} = 0.
\]

In fact, over \( X \setminus \Sigma \), it holds that \( \sigma(k')(A(t_k)) \) converges to \( A_\infty \) in \( C^\infty_{\text{loc}} \) sense. For any \( x_0 \in X \setminus \Sigma \), there exists small enough \( r_0 \), such that when \( t_k \) is sufficiently large, we have

\[
\sup_{P_{r_0}(x_0, t_k)} |F_A|^2 \leq C.
\]

Therefore, over \( P_{r_0}(x_0, t_k) \), we have

\[
(\triangle - \frac{\partial}{\partial t}) |F_A|^2 \geq 2|\nabla_A F_A|^2 - C(1 + |\text{Ric}| + |Rm| + |F_A| |F_A|^2 - C |F_A| |\nabla_A F_A|
\]

\[
\geq |\nabla_A F_A|^2 - C |F_A|^2.
\]

Assume that there exist \( r_j, j = 0, \cdots, l - 1 \) such that

\[
\sup_{P_{r_j}(x_0, t_k)} |\nabla_A F_A|^2 \leq C,
\]

Therefore, over \( P_{r_0}(x_0, t_k) \), we have
By a similar proof as that in Lemma 2.7, we have there exists \( r_1 \) such that
\[
(\Delta - \frac{\partial}{\partial t})|\nabla_A^l F_A|^2 \geq 2|\nabla_A^{l+1} F_A|^2 - C|\nabla_A^l F_A||\nabla_A^{l+1} F_A| - C|\nabla_A^l F_A|^2
\geq |\nabla_A^{l+1} F_A|^2 - C|\nabla_A^l F_A|^2 - C.
\]
in \( P_{r_1}(x_0, t_k) \). This implies for any \( j = 1, \cdots, l \), we have
\[
\int_{P_{r_1}(x_0, t_k)} |\nabla_A^j F_A|^2 \leq C.
\]
From the parabolic mean value inequality, there exists \( \delta > 0 \) such that
\[
\sup_{P_{\delta r_1}(x_0, t_k)} |\nabla_A^l F_A|^2 \leq C.
\]
Using Donaldson’s diagonal technique in [11, Theorem 4.4.8], there exists subsequence \( \{t_{k_i}\} \) and smooth gauge transformation \( \{\sigma_{k_i}\} \) such that
\[
\sigma_{k_i}(A(t_{k_i})) \text{ converges to } A_\infty \text{ in } C^\infty_{\text{loc}}(X \setminus \Sigma) \text{ sense}
\]
and \( A_\infty \) satisfies equation (1.4).

(2) Since \( \sqrt{-1} \Lambda_\omega F_{A_\infty} \) is parallel and \( (\sqrt{-1} \Lambda_\omega F_{A_\infty})_{H_\infty} = \sqrt{-1} \Lambda_\omega F_{A_\infty} \), we can decompose \( (E_\infty, H_\infty) \) according to the eigenvalues of \( \sqrt{-1} \Lambda_\omega F_{A_\infty} \) over \( X \setminus \Sigma \)
\[
E_\infty = \bigoplus_{i=1}^k E_i^\infty.
\]
Setting \( H_i^\infty = H_\infty|_{E_i^\infty}, A_i^\infty = A_\infty|_{E_i^\infty}, \) we have \( A_i^\infty \) is a Hermitian-Einstein connection on \( (E_i^\infty, H_i^\infty) \), i.e.
\[
\sqrt{-1} \Lambda_\omega F_{A_i^\infty} = \lambda_i \text{Id}_{E_i^\infty}.
\]
From Lemma 2.3 we have YM(t) is decreasing along the flow. So it holds that
\[
\int_{X \setminus \Sigma} |F_{A_\infty}|^2_{H_\infty} < \infty.
\]
In addition that \( H_{2n-4}(\Sigma) < \infty \) and \( H_i^\infty \) satisfies the Hermitian-Einstein equation, we have that every \( (E_i^\infty, D_{A_i^\infty}) \) can be extended to the whole \( X \) as a reflexive sheaf (also denoted by \( (E_i^\infty, \overline{D}_{A_i^\infty}) \)) and \( H_i^\infty \) can be smoothly extended over the place where the sheaf \( (E_i^\infty, \overline{D}_{A_i^\infty}) \) is locally free by Theorem 3.4.

\[\square\]

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