Slicing Swiss-Cheese with Constant Mean Curvature

Chul-Moon Yoo\textsuperscript{1,*} and Ken-ichi Nakao\textsuperscript{2,†}

\textsuperscript{1}Division of Particle and Astrophysical Science, Graduate School of Science, Nagoya University, Nagoya 464-8602, Japan

\textsuperscript{2}Department of Mathematics and Physics, Graduate School of Science, Osaka City University, 3-3-138 Sugimoto, Sumiyoshi, Osaka 558-8585, Japan

A sequence of Constant-Mean-Curvature (CMC) slices in the Swiss-Cheese (SC) Universe is investigated. We focus on the CMC slices which smoothly connect to the homogeneous time slices in the Einstein-de Sitter region in the SC universe. It is shown that the slices do not pass through the black hole region but white hole region.

\textsuperscript{*} yoo@gravity.phys.nagoya-u.ac.jp
\textsuperscript{†} knakao@sci.osaka-cu.ac.jp
I. INTRODUCTION

Numerical simulations of spacetime dynamics in cosmological settings came to be actively performed in recent years. One main motivation to simulate the cosmological nonlinear dynamics is to quantify the effect of the non-linear small scale inhomogeneity on the global expansion law of the universe[1–16]. Another significant motivation comes from primordial black holes[17, 18]. Spherically symmetric simulations of primordial black hole formation have ever been repeated in different settings[19–28]. Non-spherical simulation of gravitational collapse in an expanding background has been recently performed in Ref. [29].

When we analyze a spacetime dynamics with a numerical procedure, the dynamics is described as a sequence of time slices, that is, a foliation by a one-parameter family of spacelike hypersurfaces. Therefore, in order to understand the spacetime structure, the domain covered by the sequence of time slices should be correctly figured out. For this purpose, a sequence of time slices in a well-known analytic spacetime is often helpful. One of the useful time slice conditions is the so-called Constant-Mean-Curvature (CMC), which requires a uniform value of the trace of the extrinsic curvature of each time slice. A CMC slice is often taken as the initial hypersurface for numerical simulation because it simplifies the Hamiltonian and momentum constraint equations under certain assumptions. CMC slices in the Schwarzschild(Sch) spacetime may give a helpful insight to understand the intrinsic geometry and the embedding of the initial hypersurface for a dynamical simulation associated with black hole formation.

There are several works on CMC and other slices for well-known spacetimes(see e.g., [30–35]). In this paper, we investigate CMC slices in the Swiss-Cheese(SC) universe[36, 37]. The SC universe model is constructed by arbitrarily removing spherical regions from the Einstein-de Sitter (EdS) universe model in non-overlapping manner, and filling each removed region with a region of the Sch spacetime whose center is occupied by a black or white hole. Thus, the SC universe is composed of the interior Sch region and exterior EdS region which are matched each other with the Israel’s junction condition[38]. In the EdS region, we consider the trivial CMC slice, that is, the homogeneous slice on which the value of the trace of the extrinsic curvature is given by $3H$ with $H$ being the Hubble constant. Therefore, what we investigate in this paper is just a sequence of CMC slices in the Sch spacetime. The difference from previous studies on CMC slices in the Sch spacetime is in the boundary condition on the boundary between the Sch and EdS regions. The previous studies on CMC slices in the black hole spacetimes are applicable to only totally spherically symmetric spacetimes. By contrast, the CMC slices in the present study will be applicable to the situations in which black holes are randomly distributed in the expanding universe. The knowledge about the CMC slices in the SC universe may be helpful to get better insight into the geometry of the initial hypersurface.

This paper is organized as follows. In Sec. II, we review the SC universe deriving the equation describing the boundary between the Sch and EdS regions. The ordinary differential equations for CMC slices with the Kruskal coordinates are derived in Sec. III, and results are shown in Sec. IV.
We use the geometrized units in which both the speed of light and Newton’s gravitational constant are one. The Greek indices run from 0 to 3 and the Latin indices run from 1 to 3.

II. MATCHING EDS AND THE SCH SPACETIME

As mentioned, the SC universe model is constructed by removing spherical regions from the EdS universe model and filling each removed domain by a spherical domain of the Sch spacetime. The motion of a boundary between the EdS and the Sch regions is derived below.

A. Boundary on the EdS side

The line element of the EdS spacetime can be written as
\[ ds^2 = -d\tau^2 + a(\tau)^2 (d\chi^2 + \chi^2 d\Omega^2), \]
where \( a(\tau) \) is the scale factor. If we set \( a = a_h \) at \( \tau = \tau_h \), we have
\[ a(\tau) = a_h \left( \frac{\tau}{\tau_h} \right)^{2/3}. \]

The one-parameter family of the timelike hypersurfaces with constant \( \chi \) foliate the EdS spacetime. On each hypersurface of constant \( \chi \), we use the intrinsic coordinates \( \xi^i \) defined by
\[ \xi^i = (\tau, \theta, \phi). \]
Then, the induced metric \( h_{ij} \) on a hypersurface of constant \( \chi \) is given by
\[ h_{ij}d\xi^id\xi^j = -d\tau^2 + a(\tau)^2 \chi^2 d\Omega^2. \]

We remove a spherical region \( \chi < \chi_b \) in the EdS universe model and fill it with a region of the Sch spacetime. The boundary between the Sch and the EdS regions is a timelike hypersurface \( \Sigma_0 \) of
\[ \chi = \chi_b. \]

Defining \( A(\tau) \) as
\[ A(\tau) := a(\tau)\chi_b = A_h \left( \frac{\tau}{\tau_h} \right)^{2/3}, \]
we obtain the following form of the induced metric \( h_{ij}^b \) on \( \Sigma_0 \):
\[ h_{ij}^b d\xi^i d\xi^j = -d\tau^2 + A(\tau)^2 d\Omega^2, \]
where we have defined \( A_h \) by \( A_h = a_h\chi_b \).

As is well known, for the Gaussian normal coordinate, the extrinsic curvature tensor \( k_{ij} \) of \( \Sigma_0 \) is given by
\[ k_{ij} = \frac{1}{2} \ell^\mu \partial_\mu h_{ij} \big|_{\chi=\chi_b}, \]
where $\ell^\mu$ is the normalized vector which is normal to $\Sigma_0$. Since we have

$$\ell^\mu \partial_\mu = \frac{1}{a} \partial_\lambda,$$

(9)

nonzero components of the extrinsic curvature are give by

$$k_{\theta\theta} = \frac{k_{\phi\phi}}{\sin^2 \theta} = A(\tau).$$

(10)

B. Boundary on the Sch side

The metric of the Sch spacetime is given by

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2d\Omega^2,$$

(11)

where

$$f(r) = 1 - \frac{2M}{r}.$$  

(12)

The boundary between the EdS and the Sch regions, $\Sigma_0$, is described in the Sch side in the following manner:

$$t = t(\tau), \ r = r(\tau), \ \theta = \theta, \ \phi = \phi.$$  

(13)

The induced metric is given by

$$h^b_{ij}d\xi^i d\xi^j = \left(-f(r)t'^2 + \frac{r'^2}{f(r)}\right)d\tau^2 + r^2d\Omega^2,$$

(14)

where $t' = dt/d\tau$ and $r' = dr/d\tau$. From the 1st Israel Junction condition (equivalence of $h^b_{ij}$), we obtain

$$-ft'^2 + \frac{r'^2}{f(r)} = -1,$$

(15)

$$r = A_h \left(\frac{\tau}{\tau_h}\right)^{2/3}.$$  

(16)

Setting $x^\mu = (t, r, \theta, \phi)$ and $e^\mu_i := \partial x^\mu / \partial \xi^i$, we have the expression for the extrinsic curvature $k_{ij}$ as

$$k_{ij} = -\ell_\mu e^\mu_i \nabla_\nu e^\mu_j.$$

(17)

From this expression, we obtain

$$k_{\theta\theta} = \frac{1}{2} \ell_\nu g^{\nu\rho} \partial_\rho g_{\theta\theta}.$$  

(18)

Since

$$\ell_\mu = (-r', t', 0, 0),$$

(19)

we find

$$k_{\theta\theta} = t' r f(r).$$

(20)
In this paper, we do not allow any singular shell source on the boundary. Therefore, from the 2nd Junction condition, \( k_{ij} \) must have an identical value on each other side. Then, comparing the values of \( k_{\theta \theta} \), we obtain

\[
t' = \frac{1}{f(r)} = \left( 1 - \frac{2M}{A_h \left( \frac{\tau_h}{\tau} \right)^{2/3}} \right)^{-1}.
\]  

(21)

The other non-trivial component is \( k_{\tau \tau} \):

\[
k_{\tau \tau} = -\ell e_\nu e_\mu \nabla_\nu e_\mu.
\]  

(22)

Equation (15) is equivalent to \( g_{\mu \nu} e_\mu e_\nu = -1 \) and hence we have

\[
g_{\alpha \beta} e_\alpha e_\beta \nabla_\nu e_\tau = 0.
\]

Furthermore, we can easily find \( e_\nu \nabla_\nu e_\theta = 0 = e_\theta \nabla_\theta e_\phi \). Thus, Eq. (22) and the 2nd Junction condition on \( \tau-\tau \) component, \( k_{\tau \tau} = 0 \), lead to

\[
e_\nu \nabla_\nu e_\mu = 0.
\]  

(23)

This equation is just a geodesic equation.

The \( r \) component of Eq. (23) leads to

\[
r'' + \frac{1}{f} \partial_r f = 0,
\]  

(24)

where we have used Eq. (15). Using Eq. (16), we obtain

\[
\frac{2}{9} A_h^3 \tau_h^{-2} = M.
\]  

(25)

This condition implies that the mass inside the sphere specified by Eq. (5) in EdS is equivalent to the mass of the Sch spacetime. Actually, we can confirm it as follows:

\[
M = \rho \times \frac{4}{3} \pi A^3 = \frac{3}{8\pi} H^2 \times \frac{4}{3} \pi A^3 = \frac{2}{9} A_h^3 \tau_h^{-2},
\]  

(26)

where \( \rho \) and \( H \equiv a'/a = A'/A \) are the energy density and the Hubble constant, respectively, and we have used the Friedmann equation

\[
H^2 = \frac{8}{3\pi} \rho.
\]

From the \( t \)-component of the geodesic equation, we obtain

\[
t'' + \frac{1}{f^2} \partial_t f r' = 0.
\]  

(27)

We can check that this condition is automatically satisfied by (21).

Let us define \( \tau_h \) such that \( r = 2M \) at \( \tau = \tau_h \). Then, we obtain

\[
A_h = 2M = \frac{3}{2} \tau_h.
\]  

(28)
where we have used Eq. (26). Finally, we obtain
\[
\begin{align*}
    r &= \frac{3}{2} \tau_h \left( \frac{\tau}{\tau_h} \right)^{2/3}, \\
    t' &= \left[ 1 - \left( \frac{\tau_h}{\tau} \right)^{2/3} \right]^{-1}.
\end{align*}
\] (29)

The second equation can be integrated as
\[
\begin{align*}
    t &= \tau + 3 \tau_h \left( \frac{\tau}{\tau_h} \right)^{1/3} + \frac{3}{2} \tau_h \ln \left[ 1 + \left( \frac{\tau}{\tau_h} \right)^{1/3} \right] \\
    &= \tau + 3 \tau_h \left( \frac{\tau}{\tau_h} \right)^{1/3} - 3 \tau_h \text{Arccoth} \left[ \left( \frac{\tau}{\tau_h} \right)^{1/3} \right],
\end{align*}
\] (31)

where we have omitted the integration constant which can be chosen freely without loss of generality because of the time translational invariance. Eventually, we have only one parameter \( \tau_h \).

C. Kruskal extension

For later convenience, let us consider the Kruskal extension of the trajectory of \( \Sigma_0 \) in the Sch spacetime. In the outside the horizon, the Kruskal coordinates and the coordinates \((t, r)\) are related as
\[
\begin{align*}
    T &= \frac{1}{2} \left( \exp \left[ \frac{t + r + r_g \ln [(r - r_g)/r_g]}{2r_g} \right] - \exp \left[ \frac{-t + r + r_g \ln [(r - r_g)/r_g]}{2r_g} \right] \right), \\
    R &= \frac{1}{2} \left( \exp \left[ \frac{t + r + r_g \ln [(r - r_g)/r_g]}{2r_g} \right] + \exp \left[ \frac{-t + r + r_g \ln [(r - r_g)/r_g]}{2r_g} \right] \right).
\end{align*}
\] (32)

Substituting Eqs. (29) and (31) into these expressions, we get
\[
\begin{align*}
    T_b(\tau) &= \exp \left[ \frac{1}{2} \left( \frac{\tau}{\tau_h} \right)^{2/3} \right] \left\{ \left( \frac{\tau}{\tau_h} \right)^{1/3} \sinh \left[ \left( \frac{\tau}{\tau_h} \right)^{1/3} + \frac{\tau}{3\tau_h} \right] - \cosh \left[ \left( \frac{\tau}{\tau_h} \right)^{1/3} + \frac{\tau}{3\tau_h} \right] \right\}, \\
    R_b(\tau) &= \exp \left[ \frac{1}{2} \left( \frac{\tau}{\tau_h} \right)^{2/3} \right] \left\{ \left( \frac{\tau}{\tau_h} \right)^{1/3} \cosh \left[ \left( \frac{\tau}{\tau_h} \right)^{1/3} + \frac{\tau}{3\tau_h} \right] - \sinh \left[ \left( \frac{\tau}{\tau_h} \right)^{1/3} + \frac{\tau}{3\tau_h} \right] \right\}.
\end{align*}
\] (34)

This expression can be analytically extended to the region \( \tau < \tau_h \).

III. DIFFERENTIAL EQUATIONS FOR CMC SLICES

We are interested in CMC slices, that is, spacelike hypersurfaces with constant \( K \) in the SC universe, where \( K \) is the trace of the extrinsic curvature of the spacelike hypersurface.
In the EdS region, we choose homogeneous time slices. Then, we extend these time slices to the Sch region keeping $K = \text{const}$. For this purpose, we derive the differential equations for CMC slices in the Sch spacetime.

We use the Kruskal coordinate system, in which the line element is written as

$$ds^2 = \frac{4 r_g^3}{r(T, R)} \exp \left( -\frac{r(T, R)}{r_g} \right) (-dT^2 + dR^2) + r(T, R)^2 d\Omega^2,$$

where $r(T, R)$ is defined by

$$T^2 - R^2 = -\left( \frac{r - r_g}{r_g} \right) \exp \left( \frac{r}{r_g} \right).$$

We consider a spacelike hypersurface $\Sigma_1$ specified by the following parametric equation:

$$T = f_T(v),$$
$$R = f_R(v).$$

Covariant components of the vector normal to this surface is given by

$$N_{\mu} = \pm \left( -\dot{f}_R , \dot{f}_T, 0, 0 \right),$$

where we have chosen $\pm$ such that $N^\mu$ is future directed, i.e., + for $\dot{f}_R > 0$ and $-$ for $\dot{f}_R < 0$. The dot “$\cdot$” denotes $d/dv$. The contravariant components are given by

$$N^{\mu} = \pm \frac{r}{4 r_g^3} \exp \left( \frac{r}{r_g} \right) \left( \dot{f}_R , \dot{f}_T, 0, 0 \right).$$

The norm is calculated as

$$N_\mu N^\mu = \frac{r}{4 r_g^3} \exp \left( \frac{r}{r_g} \right) (-\dot{f}_R^2 + \dot{f}_T^2).$$

Using the freedom to choose the norm of $v$, we can set

$$N_\mu N^\mu = - \left[ \frac{r}{4 r_g^3} \exp \left( \frac{r}{r_g} \right) \right]^2. \quad \text{(43)}$$

Then, the normalized vector which is normal to $\Sigma_1$ is given by

$$n^\mu = \pm \left( \dot{f}_R , \dot{f}_T, 0, 0 \right).$$

We note that the following equation is satisfied:

$$-\dot{f}_R^2 + \dot{f}_T^2 = - \frac{r}{4 r_g^3} \exp \left( \frac{r}{r_g} \right). \quad \text{(45)}$$
For later convenience, we differentiate Eq. (45) with respect to $R$ or $T$ to obtain

$$-\dot{f}_R \partial_R \dot{f}_R + \dot{f}_T \partial_T \dot{f}_T = -\frac{\partial_R r}{8r_g^2} \exp\left(\frac{r}{r_g}\right) \left(1 + \frac{r}{r_g}\right), \quad (46)$$

$$-\dot{f}_R \partial_T \dot{f}_R + \dot{f}_T \partial_T \dot{f}_T = -\frac{\partial_T r}{8r_g^3} \exp\left(\frac{r}{r_g}\right) \left(1 + \frac{r}{r_g}\right), \quad (47)$$

where $\dot{f}_T$ and $\dot{f}_R$ are treated as fields on the $T-R$ plane.

The basic equation is given by

$$K = \nabla_{\mu} v^\mu = 3H = \text{spatially constant.} \quad (48)$$

This equation leads to

$$\pm \frac{1}{r} \exp\left(\frac{r}{r_g}\right) \left\{ \partial_T \left[ r \exp\left(-\frac{r}{r_g}\right) \dot{f}_R \right] + \partial_R \left[ r \exp\left(-\frac{r}{r_g}\right) \dot{f}_T \right] \right\} = 3H. \quad (49)$$

Using

$$\ddot{f}_R = \dot{f}_R \partial_R \dot{f}_R + \dot{f}_T \partial_T \dot{f}_R,$$

we obtain

$$\partial_T \dot{f}_R = \frac{1}{f_T} \left( \dot{f}_R - \dot{f}_R \partial_R \dot{f}_R \right),$$

and hence, we can rewrite $\partial_T \dot{f}_R + \partial_R \dot{f}_T$ as

$$\partial_T \dot{f}_R + \partial_R \dot{f}_T = \frac{f_R}{f_T} + \frac{1}{f_T} \left( -\dot{f}_R \partial_R \dot{f}_R + \dot{f}_T \partial_T \dot{f}_T \right)$$

$$= \frac{f_R}{f_T} - \frac{\partial_R r}{8f_T r_g^3} \exp\left(\frac{r}{r_g}\right) \left(1 + \frac{r}{r_g}\right). \quad (50)$$

By a similar procedure, we obtain

$$\partial_R \dot{f}_T = \frac{1}{f_R} \left( \dot{f}_T - \dot{f}_T \partial_T \dot{f}_T \right)$$

and hence, we have

$$\partial_T \dot{f}_R + \partial_R \dot{f}_T = \frac{\dot{f}_T}{f_R} - \frac{1}{f_R} \left( \dot{f}_T \partial_T \dot{f}_T - \dot{f}_R \partial_T \dot{f}_R \right)$$

$$= \frac{\dot{f}_T}{f_R} + \frac{\partial_T r}{8f_T r_g^3} \exp\left(\frac{r}{r_g}\right) \left(1 + \frac{r}{r_g}\right). \quad (51)$$

From Eqs. (49) and (50), we obtain

$$\ddot{f}_R = f_T \left[ \pm K - \left(\frac{1}{r} - \frac{1}{r_g}\right) \left( \partial_T r \dot{f}_R + \partial_R r \dot{f}_T \right) \right] + \frac{\partial_R r}{8r_g^3} \exp\left(\frac{r}{r_g}\right) \left(1 + \frac{r}{r_g}\right). \quad (52)$$

Similarly, from Eqs. (49) and (51), we obtain

$$\ddot{f}_T = f_R \left[ \pm K - \left(\frac{1}{r} - \frac{1}{r_g}\right) \left( \partial_T r \dot{f}_R + \partial_R r \dot{f}_T \right) \right] - \frac{\partial_T r}{8r_g^3} \exp\left(\frac{r}{r_g}\right) \left(1 + \frac{r}{r_g}\right). \quad (53)$$
We solve these two equations with a constraint equation (45).

To solve these equations, we need explicit expressions for $\partial_T r$, $\partial_R r$ and $r$. Differentiating Eq.(37), we obtain

$$
\partial_T r = \frac{T}{2r_g (-\dot{f}_R^2 + \dot{f}_T^2)}, \tag{54}
$$

$$
\partial_R r = \frac{-R}{2r_g (-\dot{f}_R^2 + \dot{f}_T^2)}. \tag{55}
$$

Combining Eqs. (37) and (45), we find

$$
r = r_g \ln \left[ T^2 - R^2 - 4r_g^2 \left(-\dot{f}_R^2 + \dot{f}_T^2\right) \right]. \tag{56}
$$

This final expression for $r$ is much more convenient than Eq.(37) in our numerical integration.

IV. CMC SLICES IN THE SC UNIVERSE

A. Boundary conditions

The cosmic time $\tau$ and the areal radius $r$ on the sphere $\Sigma_0 \cap \Sigma_1$ are denoted by $\tau_b$ and $r_b$, respectively. Then, the Hubble constant on $\Sigma_0 \cap \Sigma_1$ is given by

$$
H = \frac{a'}{a} = \frac{2}{3 \tau_b}, \tag{57}
$$

and, from Eq. (29), we have

$$
r_b = \frac{3}{2} \tau_b \left( \frac{\tau_b}{\tau_h} \right)^{2/3} = \frac{3}{2} \tau_h \left( \frac{2}{3 \tau_h H} \right)^{2/3}. \tag{58}
$$

As mentioned in Sec. I, we assume that the CMC hypersurface $\Sigma_2$ in the EdS region agrees with $\tau = \tau_b$, and hence $\ell^\mu$ is tangent to $\Sigma_2$ in the EdS side on $\Sigma_1 \cap \Sigma_2$. Since the tangent space is continuous at $\Sigma_1 \cap \Sigma_2$, $\ell^\mu$ is also tangent to $\Sigma_2$ on the Sch side. Since we have $e^\mu_\nu \propto (T'_b, R'_b, 0, 0)$ in the Sch side, the relation $e^\mu_\nu \ell^\mu = 0$ leads to

$$
\ell^\mu = C \left( R'_b, T'_b, 0, 0 \right),
$$

where

$$
C = \sqrt{\frac{r \exp[r/r_g]}{4r_g^3 (T'_b^2 - R'_b^2)}}. \tag{59}
$$

Thus, we have

$$
(\dot{f}_T, \dot{f}_R, 0, 0) = -C(R'_b, T'_b, 0, 0) \tag{60}
$$
on $\Sigma_1 \cap \Sigma_2$, where the negative sign has been assigned, so that the value of $v$ increases inward.
From Eqs. (34) and (35), we obtain

\[ T'_b(\tau) = \frac{1}{3\tau_h} \left( \frac{\tau}{\tau_h} \right)^{1/3} \exp \left[ \frac{1}{2} \left( \frac{\tau}{\tau_h} \right)^{2/3} \right] \cosh \left[ \left( \frac{\tau}{\tau_h} \right)^{1/3} + \frac{\tau}{3\tau_h} \right], \quad (61) \]

\[ R'_b(\tau) = \frac{1}{3\tau_h} \left( \frac{\tau}{\tau_h} \right)^{1/3} \exp \left[ \frac{1}{2} \left( \frac{\tau}{\tau_h} \right)^{2/3} \right] \sinh \left[ \left( \frac{\tau}{\tau_h} \right)^{1/3} + \frac{\tau}{3\tau_h} \right]. \quad (62) \]

By using these equations, Eq. (60) gives \( \dot{f}_T \) and \( \dot{f}_R \) on \( \Sigma_1 \cap \Sigma_2 \) as \( \dot{f}_T = -CR'_b(\tau_b) \) and \( \dot{f}_R = -CT'_b(\tau_b) \), while the values of \( f_T \) and \( f_R \) on \( \Sigma_1 \cap \Sigma_2 \) are given as \( f_T = T_b(\tau_b) \) and \( f_R = R_b(\tau_b) \) with Eqs. (34) and (35). Then, we can integrate Eqs. (52) and (53) to find a CMC slice.

**B. CMC slices in the Kruskal diagram**

Performing numerical integrations, we finally obtain the results shown in Fig. 1. It is

![Diagram](image)

**FIG. 1.** CMC slices in the Sch region with the Kruskal coordinate. Each CMC slice is described by a union of solid segment and dotted segment separated by the \( \alpha = 0 \) point on it: \( \alpha \) is positive on the solid segment, whereas \( \alpha \) is negative on the dotted segment.

found from this figure that CMC slices do not pass through the black hole region but the white hole region. A similar slice is observed in an analysis of massless scalar field collapse
in an expanding background[29]. This is an example showing that the knowledge about the sequence of CMC slices obtained in this paper actually helps in understanding the spacetime structure of the numerical spacetime solution.

Fig. 1 shows that the CMC hypersurfaces intersect with each other. This fact implies that the lapse function $\alpha$ associated with the foliation by CMC hypersurfaces has zero points. In Appendix B, we derive the following necessary and sufficient condition for the appearance of a zero point of $\alpha$:

$$C := \dot{f}_R \partial_\tau T - \dot{f}_T \partial_\tau R = 0,$$

where the coordinate system has been set as $(\tau, v, \theta, \phi)$ with $\tau = 2K$. The $\tau$ derivatives $\partial_\tau T$ and $\partial_\tau R$ can be numerically calculated by getting a nearby CMC hypersurface specified by a slightly different value of $\tau$. From the results, we plot the curve $T = C(R)$ on which $\alpha$ vanishes in Fig. 1. We also plot the curves on which $v$ is constant. In Fig. 1, each CMC slice is described by a union of solid segment and dotted segment separated by the $\alpha = 0$ point on it. It can be found that there is no intersection between the solid segments and also no intersection between the dotted segments.

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Appendix A: Zero points of the lapse function

A CMC hypersurface is specified by the trace of the extrinsic curvature $K = 2/\tau$, or equivalently, $\tau$, and a simple radial coordinate on it is $v$. We refer to the coordinate system $(\tau, v, \theta, \phi)$ as the CMC coordinate system in this paper. Then, we have

$$g_{\tau\tau} = (\partial_\tau T)^2 \Omega,$$

$$g_{vv} = -\dot{T} \partial_\tau T + \dot{R} \partial_\tau R \Omega,$$

where

$$\Omega = \frac{4r^3}{\sigma} \exp \left( -\frac{r}{\sigma} \right),$$

and $\sigma = \frac{2M}{\rho}$.
a dot represents the derivative with respect to \( v \) with \( \tau \) fixed, and we have used Eq. (45) in the last equality of the third equation. The spatial metric \( \gamma_{ij} \), its inverse \( \gamma^{ij} \) and the shift vector \( \beta_i \) in the CMC coordinate system are

\[
\gamma_{ij} = \text{diag} \left[ 1, r^2, r^2 \sin^2 \theta \right],
\gamma^{ij} = \text{diag} \left[ 1, \frac{1}{r^2}, \frac{1}{r^2 \sin^2 \theta} \right],
\beta_i = (g_{\tau v}, 0, 0),
\]

respectively, and hence we have \( \beta^i \beta_i = (g_{\tau v})^2 \). From \( g_{\tau \tau} = -\alpha^2 + \beta^i \beta_i \), we obtain

\[
\alpha^2 = -g_{\tau \tau} + (g_{\tau v})^2 = \Omega^2 \left( \dot{R} \partial_{\tau} T - \dot{T} \partial_{\tau} R \right)^2,
\]

where we have used Eq. (45). The zero point of \( \alpha \) appears if and only if

\[
\dot{R} \partial_{\tau} T - \dot{T} \partial_{\tau} R = 0 \quad (A1)
\]

holds. This equation implies that the zero point of \( \alpha \) appears if and only if the normal vector to the CMC hypersurface is orthogonal to the time coordinate basis \( \partial / \partial \tau \).

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