D0-brane realizations of the resolution of a reduced singular curve

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Abstract

Based on examples from superstring/D-brane theory since the work of Douglas and Moore on resolution of singularities of a superstring target-space $Y$ via a D-brane probe, the richness and the complexity of the stack of punctual D0-branes on a variety, and as a guiding question, we lay down a conjecture that any resolution $Y' \to Y$ of a variety $Y$ over $\mathbb{C}$ can be factored through an embedding of $Y'$ into the stack $\mathfrak{M}_r(Y)$ of punctual D0-branes of rank $r$ on $Y$ for $r \geq r_0$ in $\mathbb{N}$, where $r_0$ depends on the germ of singularities of $Y$. We prove that this conjecture holds for the resolution $\rho : C' \to C$ of a reduced singular curve $C$ over $\mathbb{C}$. In string-theoretical language, this says that the resolution $C'$ of a singular curve $C$ always arises from an appropriate D0-brane aggregation on $C$ and that the rank of the Chan-Paton module of the D0-branes involved can be chosen to be arbitrarily large.

Key words: D-brane, resolution, singularity; punctual D0-brane, stack; singular curve, normalization; embedding, separation of points, separation of tangents.

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Chien-Hao Liu dedicates this note to
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0. Introduction and outline.

The work [D-M] of Michael Douglas and Gregory Moore on resolution of singularities of a superstring target-space $Y$ via a D-brane probe (i.e., the realization of a resolution $Y'$ of $Y$ as a space of vacua – namely, a moduli space in quantum-field-theoretical sense – of the world-volume quantum field theory of the D-brane probe) has influenced many studies both on the mathematics and the string-theory side. (See also a related work [J-M] of Clifford Johnson and Robert Myers.) The attempt to understand the underlying geometry behind the setup of [D-M] is indeed part of the driving force that leads us to the current setting of D-branes in the project (cf. [L-Y1] and [L-Y2]). Based on examples[3] from superstring/D-brane theory since [D-M], the richness and the complexity of the stack $M_{\text{stack}}^{0,\infty}(Y)$ of punctual D0-branes on a variety $Y$, and as a guiding question, we lay down in this note[1] a conjecture that any resolution $Y' \rightarrow Y$ of a variety $Y$ over $\mathbb{C}$ can be factored through an embedding of $Y'$ into the stack $M_{\text{stack}}^{0,\infty}(Y)$ of punctual D0-branes of rank $r$ on $Y$ for $r \geq r_0$ in $\mathbb{N}$, where $r_0$ depends on the germ of singularities of $Y$; cf. Sec. 1. For the one-dimensional case, we prove that this conjecture holds for the resolution $\rho : C' \rightarrow C$ of a reduced singular curve $C$ over $\mathbb{C}$; cf. Sec. 2. In string-theoretical language, this says that the resolution $C'$ of a singular curve $C$ always arises from an appropriate D0-brane aggregation on $C$ and that the rank of the Chan-Paton module of the D0-branes involved can be chosen to be arbitrarily large.

Remark 0.1. [another aspect]. It should be noted that there is another direction of D-brane resolutions of singularities (e.g. [As], [Br], [Ch]), from the point of view of (hard/massive/solitonic) D-branes (or more precisely B-branes) as objects in the bounded derived category of coherent sheaves. Conceptually that aspect and ours (for which D-branes are soft in terms of string tension) are in different regimes of a refined Wilson’s theory-space of $d = 2$ supersymmetric field theory-with-boundary on the open-string world-sheet. Being so, there should be an interpolation between these two aspects. It would be very interesting to understand such details.

Convention. Standard notations, terminology, operations, facts in (1) algebraic geometry; (2) coherent sheaves; (3) resolution of singularities; (4) stacks can be found respectively in (1) [Ha]; (2) [H-L]; (3) [Hi], [Ko]; (4) [L-MB].

- All varieties, schemes and their products are over $\mathbb{C}$; a ‘curve’ means a 1-dimensional proper scheme over $\mathbb{C}$; a ‘stack’ means an Artin stack.
- The ‘support’ $\text{Supp}(\mathcal{F})$ of a coherent sheaf $\mathcal{F}$ on a scheme $Y$ means the scheme-theoretical support of $\mathcal{F}$; $\mathcal{I}_Z$ denotes the ideal sheaf of a subscheme of $Z$ of a scheme $Y$.
- The current note continues the study in [L-Y1] [arXiv:0709.1515 [math.AG], D(1)], [L-Y2] [arXiv:0901.0342 [math.AG], D(3)], and [L-Y3] [arXiv:0907.0263 [math.AG], D(4)] with some background from [L-L-S-Y] [arXiv:0809.2121 [math.AG], (2)]. A partial review of D-branes and Azumaya noncommutative geometry is given in [L-Y4] [arXiv:1003.1178 [math.SG], D(6)]. Notations and conventions follow these early works when applicable.

1 Unfamiliar readers are highly recommended to use keyword search to get a taste of the vast literature.
2 In part, for a subsection of a talk under the title ‘Azumaya noncommutative geometry and D-branes - an origin of the master nature of D-branes’ to be delivered in the workshop Noncommutative algebraic geometry and D-branes, December 12 – 16, 2011, organized by Charlie Beil, Michael Douglas, and Peng Gao, at Simons Center for Geometry and Physics, Stony Brook University, Stony Brook, NY.
3 For mathematicians: See [W-K] for the origin of the notion of Wilson’s theory-space and, for example, [H-I-V] and [H-H-P] for the case of $d = 2$ supersymmetric quantum field theories with boundary.
1 The stack of punctual D0-branes on a variety and an abundance conjecture.

We collect a few most essential definitions and setups for this sub-line of the project. Readers are referred to [L-Y4] for a more thorough review of the first part of the project and stringy-theoretical remarks on how inputs from [Po1], [Po2], and [Wi] lead to such a setting.

**D-branes as morphisms from Azumaya noncommutative spaces with a fundamental module.**

Our starting point is the following prototypical definition of D-branes that comes from a mathematical understanding of [Po1], [Po2] from Joseph Polchinski and [Wi] from Edward Witten based on how Alexandre Grothendieck developed the theory of schemes in modern (commutative) algebraic geometry:

**Definition 1.1. [D-brane].** Let \( Y \) be a variety (over \( \mathbb{C} \)). A D-brane on \( Y \) is a morphism \( \varphi \) from an Azumaya noncommutative space-with-a-fundamental-module \( (X^{A}, \mathcal{E}) := (X, \mathcal{O}^{A}_{X}, \mathcal{E}) \) to \( Y \). Here, \( X \) is a scheme over \( \mathbb{C} \), \( \mathcal{E} \) a locally free \( \mathcal{O}_{X} \)-module, and \( \mathcal{O}^{A}_{X} = \text{End}_{\mathcal{O}_{X}}(\mathcal{E}) \); and \( \varphi \) is defined through an equivalence class of gluing systems of ring homomorphisms given by \( \varphi^{\sharp} : \mathcal{O}_{Y} \to \mathcal{O}^{A}_{X} \). The rank of \( \mathcal{E} \) is called the rank of the D-brane.

Similar to the fact that the data of a morphism \( f : X \to Y \) between schemes can be encoded completely by its graph \( \Gamma_{f} \) as a subscheme in \( X \times Y \), the data of \( \varphi \) is also encoded completely by its graph \( \Gamma_{\varphi} \):

**Definition 1.2. [\( \varphi \) in terms of its graph \( \Gamma_{\varphi} \)].** The graph of a morphism in Definition 1.1 is given by an \( \mathcal{O}_{X \times Y} \)-module \( \hat{\mathcal{E}} \) that is flat over \( X \) and of relative dimension 0. In detail, let \( pr_{1} : X \times Y \to X \), \( pr_{2} : X \times Y \to Y \) be the projection map, and \( f_{\varphi} : \text{Supp}(\hat{\mathcal{E}}) \to Y \) be the restriction of \( pr_{2} \). Then \( \hat{\mathcal{E}} \) defines a morphism \( \varphi \) in Definition 1.1 as follows:

- \( \mathcal{E} = pr_{1 \ast} \hat{\mathcal{E}} \);
- note that \( \text{Supp}(\hat{\mathcal{E}}) \) is affine over \( X \); thus, the gluing system of ring homomorphisms
  - \( f^{\sharp}_{\varphi} : \mathcal{O}_{Y} \to \mathcal{O}_{\text{Supp}(\hat{\mathcal{E}})} \) defines a gluing system of ring-homomorphisms
  - \( \varphi^{\sharp} : \mathcal{O}_{Y} \to \text{End}_{\mathcal{O}_{X}}(\mathcal{E}) = \mathcal{O}^{A}_{X} \), which defines \( \varphi \).
It is worth emphasizing that, \textit{unlike} the standard setting for a morphism between ringed topological spaces in commutative geometry, in general $\varphi$ specifies only a correspondence from $X$ to $Y$ via the diagram

$$
\begin{array}{c}
X_{\varphi} \coloneqq \text{Supp}(\tilde{E}) \\
\pi_{\varphi} \downarrow \\
X
\end{array} \xymatrix{ \ar[r]^{f_{\varphi}} & Y }
$$

not a morphism from $X$ to $Y$.

Definition 1.2 suggests another equivalent description of $\varphi$.

\textbf{Definition 1.3. [\(\varphi\) as morphism to stack of D0-branes].} Let $\mathcal{M}^{0_{A_{zf}}} (Y)$ be the stack of 0-dimensional $\mathcal{O}_Y$-modules. It follows from Definition 1.2 that this is precisely the stack of D0-branes on $Y$ in the sense of Definition 1.1 and, hence, the notation. Then, a morphism $\varphi$ in Definition 1.1 is specified by a morphism $\hat{\varphi} : X \rightarrow \mathcal{M}^{0_{A_{zf}}} (Y)$.

The stack of punctual D0-branes on a variety and an abundance conjecture.

\textbf{Definition 1.4. [stack of punctual D0-branes].} Let $Y$ be a variety. By a punctual 0-dimensional $\mathcal{O}_Y$-module, we mean a 0-dimensional $\mathcal{O}_Y$-module $\mathcal{F}$ whose $\text{Supp}(\mathcal{F})$ is a single point (with structure sheaf an Artin local ring). By Definition 1.2, $\mathcal{F}$ specifies a D0-brane on $Y$, which is called a punctual D0-brane. It is a morphism from an Azumaya point with a fundamental module to $Y$ that takes the fundamental module to a punctual 0-dimensional $\mathcal{O}_Y$-module. Let $\mathcal{M}_{r}^{0_{A_{zf}}}(Y)$ be the stack of punctual D0-branes of rank $r$ on a variety $Y$. It has an Artin stack with atlas constructed from Quot-schemes. There is a morphism $\pi_Y : \mathcal{M}_{r}^{0_{A_{zf}}}(Y) \rightarrow Y$ that takes $\mathcal{F}$ to $\text{Supp}(\mathcal{F})$ with the reduced scheme structure. $\pi_Y$ is essentially the Hilbert-Chow/Quot-Chow morphism.

The following two conjectures are motivated by the various examples in string theory concerning D-brane resolution of singularities of a superstring target-space and the richness and the complexity of the stack $\mathcal{M}^{0_{A_{zf}}} (Y)$:

\textbf{Conjecture 1.5. [D0-brane resolution of singularity].} Any resolution $Y' \rightarrow Y$ of a variety $Y$ can be factored through an embedding of $Y'$ into the stack $\mathcal{M}_{r}^{0_{A_{zf}}}(Y)$ of punctual D0-branes of rank $r$ on $Y$ for any $r \geq r_0$ in $\mathbb{N}$, where $r_0$ depends only on the germ of singularities of $Y$.

Conjecture 1.5 is a weaker form of the following stronger form of an abundance conjecture:

\textbf{Conjecture 1.6. [abundance].} Any birational morphism $Y' \rightarrow Y$ between varieties over $\mathbb{C}$ can be factored through an embedding of $Y'$ into the stack $\mathcal{M}_{r}^{0_{A_{zf}}}(Y)$ of punctual D0-branes of rank $r$ on $Y$ for any $r \geq r_0$ in $\mathbb{N}$, where $r_0$ depends only on the germ of singularities of $Y$ and the germ of singularities of $Y'$.

This says that all the birational models of and over $Y$ are already contained in the stack $\mathcal{M}_{r}^{0_{A_{zf}}}(Y)$ of punctual D0-branes on $Y$. All the birational transitions between birational models of and over $Y$ happens as correspondences inside $\mathcal{M}_{r}^{0_{A_{zf}}}(Y)$ (and hence the name of the conjecture) – an intrinsic stack over $Y$, locally of finite type, that is canonically associated to $Y$. 

3
Remark 1.7. [string-theoretical remark]. A standard setting (cf. [D-M]) in D-brane resolution of singularities of a (complex) variety $Y$ (which is a singular Calabi-Yau space in the context of string theory) is to consider a super-string target-space-time of the form $\mathbb{R}^{(9-2d)+1} \times Y$ and an (effective-space-time-filling) D$(9-2d)$-brane whose world-volume sits in the target space-time as a submanifold of the form $\mathbb{R}^{(9-2d)+1} \times \{p\}$. Here, $d$ is the complex dimension of the variety $Y$ and $p \in Y$ is an isolated singularity of $Y$. When considering only the geometry of the internal part of this setting, one sees only a D0-brane on $Y$. This explains the role of D0-branes in the statement of Conjecture 1.5 and Conjecture 1.6. In the physics side, the exact dimension of the D-brane (rather than just the internal part) matters since supersymmetries and their superfield representations in different dimensions are not the same and, hence, dimension does play a role in writing down a supersymmetric quantum-field-theory action for the world-volume of the D$(9-2d)$-brane probe. In the above mathematical abstraction, these data are now reflected into the richness, complexity, and a master nature of the stack $\mathcal{M}_{\rho}^{0\text{A}_f}$ $(Y)$ that is intrinsically associated to the internal geometry. The precise dimension of the D-brane as an object sitting in or mapped to the whole space-time becomes irrelevant.

2 Realizations of resolution of singular curves via D0-branes.

Let $C$ be a reduced singular curve over $\mathbb{C}$ and

$$\rho : C' \rightarrow C$$

be the resolution of singularities of $C$. In the current 1-dimensional case, the singularities of $C$ are isolated and $\rho$ is realized by the normalization of $C$. In particular, $\rho$ is an affine morphism. The built-in $\mathcal{O}_C$-module homomorphism $\rho^* : \mathcal{O}_C \rightarrow \rho_* \mathcal{O}_{C'}$ determines a subsheaf $A_C \subset \mathcal{O}_{C'}$ of $\mathcal{O}$-subalgebras with the induced morphism $C' \rightarrow \text{Spec} A_C$ identical to $\rho$. Let $p' \in C'$ be a closed point, $p := \rho(p')$, and $m_{\rho'} = (t)$ (resp. $m_\rho$) be the maximal ideal of $\mathcal{O}_{C',p'}$ (resp. $\mathcal{O}_{C,p}$). Then $\rho^*(m_p) : \mathcal{O}_{C',p'} = (t^{n_{p'}})$ for some $n_{p'} \in \mathbb{N}$. $n_{p'} > 1$ if and only if $p \in C_{\text{sing}} := \text{the singular locus of } C$. We show in this section that:

**Proposition 2.1. [one-dimensional case].** Conjecture 1.5 holds for $\rho : C' \rightarrow C$. Namely, there exists an $r_0 \in \mathbb{N}$ depending only on the tuple $(n_{p'})_{p' \in C_{\text{sing}}}$ and a (possibly empty) set $\{b.i.i.(p) : p \in C_{\text{sing}}, C \text{ has multiple branches at } p\}$ (cf. Definition 2.6), both associated to the germ of $C_{\text{sing}}$ in $C$, such that, for any $r \geq r_0$, there exists an embedding $\tilde{\rho} : C' \hookrightarrow \mathcal{M}_{r}^{0\text{A}_f}(C)$ that makes the following diagram commute:

\[
\begin{array}{c}
\mathcal{M}_{r}^{0\text{A}_f}(C) \\
\downarrow \pi_C \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ quad
Lemma 2.2. [commutativity of push-forward and restriction]. Let \( p' \in C' \) be a closed point. Then \((\text{id}_{C'} \times \rho)_* (\mathcal{E}|_{\{p'\} \times C'}) = \mathcal{E}|_{\{p'\} \times C} \).

Proof. As \( \mathcal{E}' \) is flat over \( C' \) under \( p'_1 \), one has the exact sequence

\[
0 \rightarrow \mathcal{I}_{\{p'\} \times C'} \otimes_{\mathcal{O}_{C'}} \mathcal{E}' \rightarrow \mathcal{E}' \rightarrow \mathcal{E}|_{\{p'\} \times C} \rightarrow 0 .
\]

Since \( \text{id}_{C'} \times \rho \) is affine, \( (\text{id}_{C'} \times \rho)_* : \text{Coh}(C') \rightarrow \text{Coh}(C) \) is exact and one has

\[
0 \rightarrow (\text{id}_{C'} \times \rho)_* (\mathcal{I}_{\{p'\} \times C'} \otimes_{\mathcal{O}_{C'}} \mathcal{E}') \rightarrow \mathcal{E} \rightarrow (\text{id}_{C'} \times \rho)_* (\mathcal{E}|_{\{p'\} \times C'}) \rightarrow 0 .
\]

where the top horizontal line is an exact sequence. This proves the lemma.

Remark/Notation 2.3. [general restriction over a base]. Lemma 2.2 holds more generally with \( p' \) replaced by a subscheme of \( C' \), by the same proof with the replacement. We’ll denote the restriction of a coherent sheaf \( \mathcal{F}' \) (resp. \( \mathcal{F} \)) on \( C' \times C' \) (resp. \( C' \times C \)) over a subscheme \( Z' \) of the base \( C' \) by \( \mathcal{F}|_{Z'} \) (resp. \( \mathcal{F}|_{Z'} \)).

Let \( v_{p'} \simeq \text{Spec}(\mathbb{C}[\varepsilon]) \), where \( \varepsilon^2 = 0 \), be the subscheme of the base \( C' \) that corresponds to the \( \mathbb{C} \)-algebra quotient \( \mathcal{O}_{C',p'} \rightarrow \mathbb{C}[\varepsilon] \) with \( t \mapsto \varepsilon \). Then the restriction of \( \mathcal{E}' \) over \( v_{p'} \) determines an element \( \alpha_{p'} \in \text{Ext}_C^1(\mathcal{E}'|_{v_{p'}}, \mathcal{E}'|_{v_{p'}}) \). Similarly, the restriction of \( \mathcal{E} \) over \( v_{p'} \) determines an element \( \alpha_{p'} =: \rho_* \alpha_{p'} \in \text{Ext}_C^1(\mathcal{E}|_{v_{p'}}, \mathcal{E}|_{v_{p'}}) \). Let \( p := \rho(p') \) and recall \( t \in \mathcal{O}_{C',p'} \) and \( n_{p'} \in \mathbb{N} \) from the beginning of this section. Let us first state an elementary criterion for non-splitability of a short exact sequence, whose proof is immediate:

Lemma 2.4. [criterion of non-splitability]. Let \( W \) be a scheme and

\[
0 \rightarrow \mathcal{F}_2 \rightarrow \mathcal{G} \rightarrow \mathcal{F}_1 \rightarrow 0
\]

be an exact sequence of \( \mathcal{O}_W \)-modules that represents a class \( \beta \in \text{Ext}_W^1(\mathcal{F}_1, \mathcal{F}_2) \). Suppose that there exist a point \( w \in W \) and a local function \( f \in \mathcal{O}_{W,w} \) such that, for the associated \( \mathcal{O}_{W,w} \)-modules (still denoted the same), \( f \cdot \mathcal{F}_1 = f \cdot \mathcal{F}_2 = 0 \) while \( f \cdot \mathcal{G} \neq 0 \). Then, \( \beta \neq 0 \); namely, the above sequence doesn’t split.

Corollary 2.5. [push-forward of jet]. Continuing the main-line discussions and notations. Let \( \alpha' \) be given by the exact sequence

\[
0 \rightarrow \mathcal{E}|_{v_{p'}} \rightarrow \mathcal{F}' \rightarrow \mathcal{E}|_{v_{p'}} \rightarrow 0
\]

of \( \mathcal{O}_{C'} \)-modules. Denote the same for the associated exact sequence of \( \mathcal{O}_{C',p'} \)-modules. As such, suppose that there is an \( l \in \mathbb{N} \) such that \( (t^{n_{p'}})^l \cdot \mathcal{E}' = 0 \) while \( (t^{n_{p'}})^{l+1} \cdot \mathcal{F}' \neq 0 \). Then \( \alpha \neq 0 \) in \( \text{Ext}_C^1(\mathcal{E}|_{v_{p'}}, \mathcal{E}|_{v_{p'}}) \).

Proof. Note that the multiplication of \( t \) by an invertible element in \( \mathcal{O}_{C',p'} \) (i.e. by an element in \( \mathcal{O}_{C',p'} - \mathfrak{m}_{p'} \)) won’t alter its nilpotency behavior on the modules in question. The corollary follows immediately from Lemma 2.4 and the observation that, up to a multiplication by an invertible element in \( \mathcal{O}_{C',p'} \), one may assume that \( t^{n_{p'}} \in \rho^*(\mathcal{O}_{C,p}) \).

\[\Box\]
Separation of points in \( \rho^{-1}(p) \) via punctual D0-branes at \( p \).

Let \( p \in C_{\text{sing}} \) and \( \hat{C} \) be the formal neighborhood (as an ind-scheme) of \( p \) in \( C \). Then each irreducible component \( \hat{C}_i, i = 1, \ldots, k \), of \( \hat{C} \) corresponds to a branch of the germ of \( p \) in \( C \). Assume that \( k \geq 2 \). Then the intersection of two distinct components \( \hat{C}_i \) and \( \hat{C}_j \) of \( \hat{C} \) is represented by a punctual 0-dimensional subscheme \( Z_{ij} = Z_{ji} \) of \( C \) at \( p \) of finite length \( l_{ij} = l_{ji} \).

**Definition 2.6. [branch intersection index].** For \( k \geq 2 \), define the branch intersection index \( b.i.i.(p) \) at \( p \in C_{\text{sing}} \) to be

\[
b.i.i.(p) := \max\{l_{ij} : 1 \leq i, j \leq k; i \neq j\}.
\]

Let \( p \in C_{\text{sing}}, \rho^{-1}(p) = \{p'_1, \ldots, p'_k\} \), and \( \hat{C}'_i \) be the formal neighborhood of \( p'_i \) in \( C' \). Then \( \rho : C' \to C \) induces a morphism \( \hat{\rho}_i : \hat{C}'_i \to \hat{C}_i \) of ind-schemes, for \( i = 1, \ldots, k \). The image \( \hat{\rho}_i(\hat{C}'_i) \) is a branch of \( \hat{C} \), which we may assume to be \( \hat{C}_i \), after relabeling, since different \( \hat{C}'_i \)'s are mapped to different branches of \( \hat{C} \) under \( \hat{\rho}_i \). Let \( m_{p'_i} = (u_i) \) be the maximal ideal of \( \mathcal{O}_{C', p'_i} \).

- \( \mathcal{F}'_{i,l} \) be the 0-dimensional \( \mathcal{O}_{C', \mathcal{F}'} \)-module \( \mathcal{O}_{C', \mathcal{F}'} / (u_i^{n_{p'_i}^{(l)})} \);
- \( \hat{\mathcal{F}}_{i,l} \) be the \( \mathcal{O}_{\mathcal{C}', \mathcal{F}'} \)-module associated to \( \mathcal{F}'_{i,l} \);
- \( \mathcal{F}_{i,l} \) be the \( \mathcal{O}_{C} \)-module \( \rho_* \mathcal{F}'_{i,l} \);
- \( \hat{\mathcal{F}}_{i,l} \) be the \( \mathcal{O}_{C} \)-module \( \hat{\rho}_i \ast \hat{\mathcal{F}}_{i,l} = \rho_* \mathcal{F}_{i,l} \).

Then, one has the following lemma:

**Lemma 2.7. [separation by punctual modules].** \( \text{length}(\text{Supp}(\mathcal{F}_{i,l})) \geq l \) and \( \text{Supp}(\hat{\mathcal{F}}_{i,l}) \subset \hat{C}_i \). In particular, if \( l > b.i.i.(p) \), then \( \mathcal{F}_{1,l}, \ldots, \mathcal{F}_{k,l} \) are punctual 0-dimensional \( \mathcal{O}_C \)-modules at \( p \) that are non-isomorphic to each other.

**Proof.** As in the previous theme, we may assume that \( u_i^{n_{p'_i}^{(l)}} = \rho^2(f_i) \) for some \( f_i \in m_{p} \subset \mathcal{O}_{C,p} \).

Let \( h \in \mathbb{C}[x] \) be a polynomial in one variable. Then, by construction, \( h(u_i^{n_{p'_i}^{(l)}}) \cdot \mathcal{F}_{i,l} = 0 \) if and only if \( h \in (x^l) \). In other words, \( h(f_i) \cdot \mathcal{F}_{i,l} = 0 \) if and only of \( h \in (x^l) \). It follows that there exists a local section \( m_{i,l} \) of \( \mathcal{F}_{i,l} \) such that \( f_i^{l-1} \cdot m_{i,l} \neq 0 \). Consider the sub-\( \mathcal{O}_C \)-module \( \mathcal{O}_C \cdot m_{i,l} \simeq \mathcal{O}_C / \text{Ann}(m_{i,l}) \) of \( \mathcal{F}_{i,l} \), where \( \text{Ann}(m_{i,l}) \) is the annihilator of \( m_{i,l} \) in \( \mathcal{O}_{C,p} \). Then,

\[
m_{i,l}, f_i \cdot m_{i,l}, \ldots, f_i^{l-1} \cdot m_{i,l}
\]

are \( \mathbb{C} \)-linearly independent in \( \mathcal{F}_{i,l} \), which implies that

\[
1, f_i, \ldots, f_i^{l-1}
\]

are \( \mathbb{C} \)-linearly independent in \( \mathcal{O}_{C,p} \). Since

\[
\text{Span}_\mathbb{C}\{1, f_i, \ldots, f_i^{l-1}\} \cap \text{Ann}(m_{i,l}) = 0
\]

as \( \mathbb{C} \)-vector subspaces in \( \mathcal{O}_{C,p} \), one has that \( \text{length}(\text{Supp}(\mathcal{O}_{C,p} / \text{Ann}(m_i))) \geq l \) and, hence, that \( \text{length}(\text{Supp}(\mathcal{F}_{i,l})) \geq l \). The rest of the lemma are immediate.

We say that \( p'_1, \ldots, p'_k \in \rho^{-1}(p) \subset C' \) are separated by the punctual \( \mathcal{O}_C \)-modules \( \mathcal{F}_{1,l}, \ldots, \mathcal{F}_{k,l} \) at \( p \in C \) when \( \mathcal{F}_{1,l}, \ldots, \mathcal{F}_{k,l} \) as constructed above are non-isomorphic to each other.
Construction of embeddings $C' \to \mathcal{M}^{0\mathcal{A}_f}_{r_0}(C)$ that descend to $\rho$.

We now proceed to prove Proposition 2.1 in three steps.

Step (a): Examination of a local model.

Consider the local ring $\mathcal{O}_{C' \times C', (p', p')} = \mathcal{O}_{C', p'} \otimes_{\mathbb{C}} \mathcal{O}_{C', p'}$. (For simplicity of phrasing, here we use ‘$\simeq$’ to mean ‘standard canonical isomorphism’.) Let $m_{p'} = (t_1) \subset \mathcal{O}_{C', p'}$ be the maximal ideal of the first factor and $m_{p'} = (t_2) \subset \mathcal{O}_{C', p'}$ be the maximal ideal of the second factor. Given $r \in \mathbb{N}$, compare the following two quotient $\mathcal{O}_{C' \times C', (p', p')}$-modules:

$$M_1 := \frac{\mathcal{O}_{C', p'} \otimes_{\mathbb{C}} \mathcal{O}_{C', p'}}{(t_1 \otimes 1 - 1 \otimes t_2)^r, t_1 \otimes 1} \quad \text{and} \quad M_2 := \frac{\mathcal{O}_{C', p'} \otimes_{\mathbb{C}} \mathcal{O}_{C', p'}}{(t_1 \otimes 1 - 1 \otimes t_2)^r, t_1^r \otimes 1}.$$  

$M_1$ corresponds to the restriction of the $\mathcal{O}_{C', p'} \otimes_{\mathbb{C}} \mathcal{O}_{C', p'}$-module $\mathcal{O}_{C', p'} \otimes_{\mathbb{C}} \mathcal{O}_{C', p'}/(t_1 \otimes 1 - 1 \otimes t_2)^r$, which is flat over $C'$ (the first factor), to over $p' \in C'$ (the first factor) while $M_2$ corresponds to the restriction of the same $\mathcal{O}_{C', p'} \otimes_{\mathbb{C}} \mathcal{O}_{C', p'}$-module to over $v_{p'} \simeq \text{Spec}(\mathbb{C}[t_1]/(t_1^r)) \simeq \text{Spec}(\mathbb{C}[\varepsilon]) \subset C'$ (the first factor). They fit into an exact sequence, representing a class in $\text{Ext}^1_{\mathcal{O}_C}(M_1, M_1)$ (here $C' = \text{the second factor}$),

$$0 \to M_1 \xrightarrow{a} M_2 \xrightarrow{b} M_1 \to 0$$

of $\mathbb{C}[\varepsilon]$-modules with

$$M_1 = \text{Span}_{\mathbb{C}} \left\{ 1 \otimes 1, 1 \otimes t_2^2, \ldots, 1 \otimes t_2^{r-1} \right\};$$  

$$M_2 = \text{Span}_{\mathbb{C}[\varepsilon]} \left\{ 1 \otimes 1, 1 \otimes t_2^2, \ldots, 1 \otimes t_2^{r-1} \right\}$$

$$\quad = \text{Span}_{\mathbb{C}} \left\{ 1 \otimes 1, 1 \otimes t_2^2, \ldots, 1 \otimes t_2^{r-1}, \varepsilon \otimes 1, \varepsilon \otimes t_2^2, \ldots, \varepsilon \otimes t_2^{r-1} \right\},$$

where $a = \text{multiplication by } \varepsilon$, and $b = \text{quotient by } \varepsilon M_1$. As $\mathbb{C}[\varepsilon]$-modules and with respect to the above bases (and with a vector identified as a column vector),

$$t_2 \text{ on } M_1 = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & 1 & 0 \\ \cdots & \cdots & \vdots & 1 \end{bmatrix}_{r \times r}$$

$$\text{and} \quad t_2 \text{ on } M_2 = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & 1 & \varepsilon \\ \cdots & \cdots & \cdots & 1 \end{bmatrix}_{r \times r}.$$  

Here all the missing entries in the $r \times r$-matrices are 0. It follows that, as $\mathcal{O}_{C', p'}$ (the second factor) -modules,  

$$t_2^l \cdot M_1 = 0 \quad \text{if and only if } \ l \geq r \quad \text{while} \quad t_2^l \cdot M_2 = 0 \quad \text{if and only if } \ l \geq r + 1.$$  

In particular, the above short exact sequence (of $\mathcal{O}_{C', p'}$-modules) doesn’t split.

Step (b): Construction of a local embedding $C' \to \mathcal{M}^{0\mathcal{A}_f}_{r_0}(C)$ that descend to $\rho$, for some $r_0 \in \mathbb{N}$.

Let

$$l_0 := 1 + \max\left\{ \text{b.i.i.}(p) : p \in C_{\text{sing}}, C \text{ has multiple branches at } p \right\}$$

(by convention, $l_0 = 1$ if $C$ has only single branch at each $p \in C_{\text{sing}}$) and

$$r_0 := l_0 \cdot \text{l.c.m.}\{n_{p'} : p' \in C'\} \in \mathbb{N}.$$  

(Here, l.c.m. = the ‘least common multiple’ in $\mathbb{N}$.) Since $n_{p'} = 1$ except for $\rho(p')$ in the finite set $C_{\text{sing}}$, $r_0$ is well-defined. Furthermore, since $\{n_{p'}\}_{\rho(p') \in C_{\text{sing}}}$ and $\{\text{b.i.i.}(p) : p \in C_{\text{sing}}\}$ (possibly
empty) depend only on the germ of $C_{\text{sing}}$ in $C$, $r_0$ depends only on the germ of $C_{\text{sing}}$ in $C$. Let $\mathcal{E}'$ be the $\mathcal{O}_{C' \times C'}$-module
\[
\mathcal{E}' = \mathcal{O}_{C' \times C'}/I_{\Delta_c'}
\]
and $\tilde{\mathcal{E}} := (id_{C'} \times \rho)_*(\mathcal{E}')$ on $C' \times C$. Then, it follows from the construction and Lemma 2.7 that the induced morphism
\[
\tilde{\rho}_0 : C' \to \mathcal{M}^{0,0,l}_{\rho}(C)
\]
descends to $\rho$ and sends distinct closed points of $C'$ to distinct geometric points on $\mathcal{M}^{0,0,l}_{\rho}(C)$ (i.e. $\tilde{\rho}$ separates points of $C'$). Furthermore, it follow from the local study in Step (a) and Corollary 2.5 that all the extension classes $\alpha_{p'} \in \text{Ext}^1_C(\tilde{\mathcal{E}}_{p'}, \tilde{\mathcal{E}}_{p'})$, $p' \in C'$, $\tilde{\mathcal{E}}$ specifies are non-zero. This shows that $\tilde{\rho}_0$ separates also tangents of $C'$ and hence is an embedding.

**Step (c): Embeddings** $C' \hookrightarrow \mathcal{M}^{0,0,l}_{\rho}(C)$ that descend to $\rho$, for all $r > r_0$.

Finally, to obtain an embedding $\tilde{\rho} : C' \to \mathcal{M}^{0,0,l}_{\rho}(C)$ for $r > r_0$ that descends to $\rho$, observe that the $\mathcal{O}_{C' \times C'}$-module $\mathcal{O}_{\Gamma\rho}$ has the following properties:

- The corresponding extension class $\tilde{\alpha}_{p'}$ in $\text{Ext}^1_C(\mathcal{O}_p, \mathcal{O}_p)$, where $p := \rho(p')$, vanishes if and only if $p \in C_{\text{sing}}$.

This implies that all the extension classes $\hat{\alpha}_{p'} \in \text{Ext}^1_C(\hat{\mathcal{E}}_{p'}, \hat{\mathcal{E}}_{p'})$, $p' \in C'$, as specified by the direct sum
\[
\hat{\mathcal{E}} := \hat{\mathcal{E}} \oplus \mathcal{O}_{\Gamma\rho}^{\oplus(r-r_0)}
\]
of $\mathcal{O}_{C' \times C'}$-modules, remain non-zero. Furthermore,
\[
\text{Supp}((\hat{\mathcal{E}} \oplus \mathcal{O}_{\Gamma\rho}^{\oplus(r-r_0)})|_{p' \times C}) = \text{Supp}(\hat{\mathcal{E}}_{p'}) \quad \text{for all } p' \in C'.
\]

It follows that the morphism $\tilde{\rho} : C' \to \mathcal{M}^{0,0,l}_{\rho}(C)$ specified by $\hat{\mathcal{E}}$ on $C' \times C$ separates both points and tangents of $C'$ and, hence, is an embedding that descend to $\rho$.

This concludes the proof of Proposition 2.1.

**Remark 2.8.** [non-uniqueness]. In general there can be other embeddings of $C'$ into $\mathcal{M}^{0,0,l}_{\rho}(C)$ that descend also to $\rho$. Hence, the one constructed in the proof above is by no means unique.
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