Unitary unfoldings of a Bose–Hubbard exceptional point with and without particle number conservation

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The conventional non-Hermitian but $\mathcal{PT}$-symmetric three-parametric Bose–Hubbard Hamiltonian $H(\gamma, v, c)$ represents a quantum system of $N$ bosons, unitary only for parameters $\gamma$, $v$ and $c$ in a domain $D$. Its boundary $\partial D$ contains an exceptional point of order $K$ (EPK; $K = N + 1$) at $c = 0$ and $\gamma = v$, but even at the smallest non-vanishing parameter $c \neq 0$ the spectrum of $H(v, v, c)$ ceases to be real, i.e. the system ceases to be observable. In this paper, the question is inverted: all of the stable, unitary and observable Bose–Hubbard quantum systems are sought which would lie close to the phenomenologically most interesting EPK-related dynamical regime. Two different families of such systems are found. Both of them are characterized by the perturbed Hamiltonians $H(\lambda) = H(v, v, 0) + \lambda V$ for which the unitarity and stability of the system is guaranteed. In the first family the number $N$ of bosons is assumed conserved while in the second family such an assumption is relaxed. Attention is paid mainly to an anisotropy of the physical Hilbert space near the EPK extreme. We show that it is reflected by a specific, operationally realizable structure of perturbations $\lambda V$ which can be considered small.

1. Introduction

The recent growth of popularity of the study of Schrödinger-type evolution equations,

$$i \frac{d}{dt} \psi(t) = H \psi(t),$$

(1.1)
containing non-Hermitian generators (alias Hamiltonians $H \neq H^\dagger$) is reflected by the publication of dedicated books [1,2]. The choice appeared relevant not only in quantum physics but also far beyond this area [3,4]. One of the ‘hidden’ roots of the appeal of the innovative non-Hermitian models may look like a paradox: in these models, one can weaken the not always desirable robust stability of the systems controlled by self-adjoint Hamiltonians. *Pars pro toto*, let us recall the most impressive illustration in classical optics where, owing to the non-Hermiticity of $H$, one is even able to stop light, in principle at least [5].

In the language of mathematics, the main formal key to similar innovations of phenomenology may be seen in the non-Hermiticity-mediated accessibility of the eigenvalue degeneracies called exceptional points (EPs) [6]. This feature opened many new areas of research in physics (see [3,4] or the sample of references listed in our preceding paper [7]).

In 2008, Graefe et al. [8] followed the trend. Turning their attention to a non-Hermitian version of the popular Bose–Hubbard (BH) Hamiltonian (see also [9–11]) enabled them to enrich, in particular, the toy-model-mediated understanding of the Bose–Einstein condensation phenomenon [12]. In the context of mathematics they complemented the traditional numerical diagonalization approaches to the model [13,14] by pointing out the advantages of using dedicated versions of perturbation theory [15,16].

Today, 12 years later, we intend to revitalize the initiative of [8]. We will propose two extensions of the currently studied non-Hermitian BH-type Hamiltonians. In the first one, we will keep the number of bosons conserved. Indeed, such an assumption is still popular, mainly in light of the traditional experimental as well as theoretical role played by the BH model in the description of phase transitions in the systems of ultracold spin-zero atoms confined by an optical lattice [17–19]. Still, at present, the relevance of such a conventional requirement seems weakened by the recent shift of study of similar models (and, in particular, of their manifestly non-Hermitian versions) to the dynamical regime controlled by EPs, especially in optics and photonics [4,20]. In the second half of this paper, therefore, we shall omit the requirement for the conservation of the number of bosons as, after such an innovation of applications ranging up to field theory [18], it is over-restrictive.

For all of these purposes, in a formal parallel to paper [8], we intend to use and, occasionally, amend the techniques of perturbation theory. As a result, the physical scope of our present study will be narrower. In place of the open-quantum-system set-up of paper [8] characterized, basically, by the Feshbach-inspired effective Hamiltonians [1,21], we shall restrict our attention to the mere unitary, closed-quantum-system scenarios characterized, first of all, by the full, unreduced information about the dynamics.

Even when staying inside the unitary-evolution framework in which the energies are real we will keep in mind the warnings coming from rigorous mathematics [22]. Thus, we will accept, as often as possible, the bounded-operator constraints as recommended in [23,24]. We will also mostly employ the terminology used in these references, although we will also occasionally use some slightly misleading but still sufficiently well-understood popular abbreviations like ‘non-Hermitian operators’. After all, many of the similar terminological conventions and ambiguities have already been discussed and sufficiently thoroughly clarified elsewhere [25–27].

2. Conventional non-Hermitian Bose–Hubbard model

As explained in [8–11], one of the fairly realistic descriptions of the so-called Bose–Einstein condensation phenomenon is provided by the specific three-parametric bosonic Hamiltonian

$$\begin{align*}
H(\gamma, v, c) = & -i\gamma \left( a_1^\dagger a_1 - a_2^\dagger a_2 \right) + v \left( a_1^\dagger a_2 + a_2^\dagger a_1 \right) + c H_{\text{int}}, \\
H_{\text{int}} &= \frac{1}{2} \left( a_1^\dagger a_1 - a_2^\dagger a_2 \right)^2,
\end{align*}$$

(2.1)
where, for two modes taken into account, the symbols \( a_1, a_2 \) and \( a_1^\dagger, a_2^\dagger \) represent the respective annihilation and creation operators. The value of the coupling constant \( c \) controls the strength of the boson–boson interaction inside a double-well potential (we shall often consider just the interaction-free limit \( c \to 0 \) in what follows). Parameter \( v \) measures the intensity of the tunnelling through the barrier (for convenience we shall scale it to 1), while the tunable real quantity \( 2\gamma \) stands for an imaginary part of the on-site bosonic-energy difference [8,11].

(a) Matrix representation of Hamiltonian

The BH model of equation (2.1) is conservative in the sense that its Hamiltonian commutes with the number operator

\[
\hat{N} = a_1^\dagger a_1 + a_2^\dagger a_2.
\]  

(2.2)

For the sake of simplicity, we will set \( c = 0 \) (meaning that the mutual interaction between bosons is neglected) and \( v = 1 \) (reflecting just the choice of units) almost everywhere in what follows. We will also make use of the bases defined in [8]: in place of operators (2.1), we will work with their suitable matrix representations. In particular, for the systems in which the number of bosons \( N \) is conserved, Hamiltonian (2.1) may be given the block-diagonal infinite-dimensional matrix form

\[
H_{(BH)}(\gamma) = H_{(2)}^{(2+3+\ldots)}(\gamma) = \begin{pmatrix}
H_{(BH)}^{(2)}(\gamma) & 0 & 0 & \ldots \\
0 & H_{(BH)}^{(3)}(\gamma) & 0 & \ldots \\
0 & 0 & H_{(BH)}^{(4)}(\gamma) & \ldots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix},
\]  

(2.3)

which contains the separate fixed-\( N \) sub-Hamiltonians

\[
H_{(BH)}^{(2)}(\gamma) = \begin{bmatrix}
-i\gamma & 1 \\
1 & i\gamma
\end{bmatrix}, \quad H_{(BH)}^{(3)}(\gamma) = \begin{bmatrix}
-2i\gamma & \sqrt{2} & 0 \\
\sqrt{2} & 0 & \sqrt{2} \\
0 & \sqrt{2} & 2i\gamma
\end{bmatrix} \ldots.
\]  

(2.4)

In a historical perspective, the oldest versions of the BH models were based on the purely imaginary choices of \( \gamma \) (making all of the matrices (2.4) Hermitian, i.e. mathematically more user-friendly). This helped to simulate, first of all, the superfluid–insulator transitions [17]. By contrast, the present preference for the real-valued parameters \( \gamma \) may be perceived as mathematically less elementary but phenomenologically more promising, especially because, in contrast to the Hermitian case, the related EPs may now be reached (possibly even in an experiment). In the language of experimental physics, this means, therefore, that such a complementary choice opens the possibility of reaching a quantum-phase transition of the type simulating the Bose–Einstein condensation phenomenon [8]. Remarkably enough, even the instant \( \gamma = 0 \) of transition between the real and imaginary \( \gamma \)s can be given a specific physical interpretation of a broken-Hermiticity quantum-phase transition (see [28] for details).

The authors of [8–11,29] offered a number of arguments showing that the choice of real \( \gamma \) may be given a phenomenologically consistent, experiment-oriented meaning. They found, in particular, that every element \( H_{(BH)}^{(K)}(\gamma) \) of the series of sub-Hamiltonians (2.4) with \( K = N + 1 \) can be assigned the \( K \)-plet of closed-form energy eigenvalues

\[
E_{n}^{(K)}(\gamma) = (1 - \gamma^2)^{1/2} (1 - K + 2n), \quad n = 0, 1, \ldots, K - 1
\]  

(2.5)

(see formula no. 28 and picture no. 1 in [8]). In [8] we also find that, in (2.4), only the diagonal matrix elements would be changed after the introduction of interaction \( H_{\text{int}} \) of equation (2.1).

(b) Exceptional point

From the point of view of traditional phenomenological applications of the Hermitian versions of the BH model [17–19], the number \( N \) of bosons in the system was always fixed and given in
advance. After a shift of attention to the non-Hermitian alternative of the model, the conservation of the number of particles was still considered useful, mainly for formal reasons. Indeed, at a fixed $N = K - 1$ (and in the present limit $c \to 0$, of course) one can easily determine the values of energies (2.5) and conclude that they remain all real if and only if $\gamma^2 \leq 1$, and non-degenerate unless $\gamma^2 = 1$. As long as $N < \infty$, the necessary mathematics remains elementary, showing only that, in the two end-of-unitarity limits $\gamma \to \pm 1$, the limiting sub-Hamiltonians $H_{(BH)}^{(K)}(\pm 1)$ cease to be diagonalizable. At all of the submatrix dimensions the energy (sub)spectra become $K$-times degenerate, $\lim_{\gamma \to \pm 1} E^{(K)}_n(\gamma) = 0$. The degeneracy applies also to the related $K$-plet of eigenvectors (see the detailed proof in [8]). Thus, the two special values of $\gamma = \pm 1$ acquire the status of Kato’s [6] exceptional point of order $K$ (EPK).

The picture of the physics becomes different when one recalls the full, infinite-dimensional-matrix Hamiltonian (2.3) and when one tentatively admits the existence of perturbations violating the commutativity of the Hamiltonian with operator $\hat{N}$ of equation (2.2). Formally speaking, a part of the spectrum of operator $H(\gamma, 1, 0)$ might then suddenly become infinitely degenerate after perturbation.

Naturally, this would be an exciting, entirely new mathematical phenomenon. Moreover, in both of the EPK limits $\gamma \to \pm 1$, both of the corresponding EPs might then also become infinitely degenerate. This could certainly open a number of new questions ranging from theoretical and experimental physics up to technology and applications.

The latter observation was a key motivation of our present study. For the sake of simplicity, let us now start the analysis by considering just one of the two EPKs, say, the positive one with $\gamma^{(EPK)} = 1$. At such a parameter, the diagonalization of submatrix $H_{(BH)}^{(K)}(1)$ is to be replaced by making it similar to a Jordan matrix,

$$H_{(BH)}^{(K)}(\gamma^{(EPK)}) = Q^{(K)} f^{(K)}(\eta) \left[ Q^{(K)} \right]^{-1}$$

and

$$f^{(K)}(\eta) = \begin{pmatrix} \eta & 1 & 0 & \ldots & 0 \\ 0 & \eta & 1 & \ldots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & \ldots & 0 & \eta & 1 \end{pmatrix},$$

where, in our case, we have $\eta = 0$. Symbol $Q^{(K)}$ denotes the transition matrix with the known closed form given in [7] and forming the sequence

$$Q^{(2)} = \begin{bmatrix} -i & 1 \\ 1 & 0 \end{bmatrix}, \quad Q^{(3)} = \begin{bmatrix} -2 & -2i & 1 \\ -2i \sqrt{2} & \sqrt{2} & 0 \\ 2 & 0 & 0 \end{bmatrix}, \ldots.$$  

Although the left-hand-side EP limit of the Hamiltonian in equation (2.6) is formally defined by prescription (2.3), it ceased to represent an acceptable generator of unitary evolution in quantum mechanics because such a matrix is not diagonalizable anymore.

(c) Unitarity-breaking perturbations at a fixed $N = K - 1$

In the vicinity of the manifestly unphysical EP-associated operator $H_{(BH)}(1)$, there may exist its mathematically well-defined and phenomenologically useful perturbed descendants

$$\delta \lambda(\lambda) = H_{(BH)}(1) + \lambda \mathbf{V}.$$  

We intend to show that, after the specification of an appropriate, quantum-theoretically consistent class of perturbations, operators (2.9) may really re-acquire the necessary diagonalizability (i.e.
a formal compatibility with quantum theory) as well as many new and attractive descriptive features.

At any fixed number of bosons $N$ and/or superscripted matrix dimension $K = N + 1$ the perturbed Hamiltonian of equation (2.9) degenerates to the mere finite-dimensional $K$ by $K$ matrix

$$
\mathcal{H}(K)(\lambda) = \mathcal{Q}(K)(\eta)[Q^{(K)}]^{-1} + \lambda \mathcal{V}^{(K)}. \tag{2.10}
$$

In 2008, Graefe et al. [8] attracted attention to the related spectral problem. In their paper, they decided to study some of the perturbation-theoretical aspects of the realistic as well as mathematically friendly BH Hamiltonian (2.1). In one of the dynamical scenarios of their interest they identified the perturbation term $\lambda \mathcal{V}^{(K)}$ of equation (2.10) with the difference of matrix operators $H(\gamma, 1, c) - H(1, 1, 0)$ at a fixed $N = K - 1$. This enabled them to reveal that the growth of the strength $c > 0$ diminishes, as a rule, the interval $\mathcal{D}$ of admissible $\gamma$s inside which the spectrum remains real and observable. Subsequently, they restricted attention to a $\gamma = v$ subset of the special perturbations

$$
\lambda \mathcal{V}^{(K)}_{(BH)} = H(1, 1, c) - H(1, 1, 0), \tag{2.11}
$$

which enabled them to identify the measure of the size of perturbation $\lambda$ directly with the boson–boson interaction strength $c \neq 0$. The conclusion was that at any non-vanishing perturbation strength $c \neq 0$ an abrupt breakdown of the reality of the perturbed spectrum is inevitable. In other words, one can say that in the vicinity of the EPK extreme the choice of perturbation $H_{\text{int}}$ as made in equation (2.1) has been found incompatible with the unitarity of the system. In this specific dynamical regime, indeed, the conventional BH model is not suitable for our purposes as it only admits the open-quantum-system probabilistic interpretation.

3. Modified non-Hermitian Bose–Hubbard models

In the present closed-quantum-system setting, a modification of the Hamiltonian is needed. Thus, our perturbed Hamiltonians (2.9) will represent, strictly speaking, a modified, non-BH family of certain new, amended, non-Hermitian (i.e. more precisely, quasi-Hermitian [23]) but still stable and strictly unitary BH-type quantum systems possessing the real energy spectra.

(a) Perturbations conserving the number of bosons

The block-diagonal matrix structure of Hamiltonian (2.3) can be interpreted as an infinite degeneracy of energy levels (2.5) with respect to the number of bosons $N = K - 1$. Although such an approach looks rather formal, it becomes relevant immediately after the conservation of the number of bosons happens to be broken, say, with the intention of making the model more realistic. A deeper understanding of such an option (i.e. of the consequences of the possible emergence of non-vanishing off-diagonal submatrices in equation (2.3)) has in fact been one of the key questions which motivated our present study.

In a preparatory step towards such a generalization of the model, it is obvious that also the separate transition matrices (2.8) may be inserted in the definition of a global, infinite-dimensional block-diagonal transition matrix

$$
Q = Q^{(2+3+\ldots)} = 
\begin{pmatrix}
Q^{(2)} & 0 & 0 & \cdots \\
0 & Q^{(3)} & 0 & \cdots \\
0 & 0 & Q^{(4)} & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}, \tag{3.1}
$$
Introducing the infinite-dimensional direct-sum generalization of the single Jordan matrix, we only have to keep all of the limiting energy arguments equal

\[ J_\eta = J_\eta(2+3+\ldots)(\eta) = \begin{pmatrix} f_2^{(\eta)} & 0 & 0 & \ldots \\ 0 & f_3^{(\eta)} & 0 & \ldots \\ 0 & 0 & f_4^{(\eta)} & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \]  

(3.2)

This immediately leads to the full-space generalization

\[ H_{\text{BH}}(1) = Q J_\eta Q^{-1} \]  

(3.3)

of the Kth-subspace similarity relation (2.6).

It should be added that the confirmation of the EP-related non-diagonalizability status of matrix \( H_{\text{BH}}(1) \) was only rendered feasible by the exact, non-numerical tractability of its analysis. Indeed, it is well known [29] that such an EP-singularity status is fragile and highly sensitive to small random perturbations. In the numerically represented models, in particular, the random perturbations are always present because of the round-off errors. This makes the exact, non-numerical tractability of the conventional BH Hamiltonian (2.1) in its EP limit \( \gamma \to 1 \) particularly rare and important.

(b) Perturbations which do not violate the unitarity

Hamiltonian (2.10) is isospectral to its partner matrix

\[ H^{(K)}(\lambda) = \left[ Q^{(K)} \right]^{-1} S^{(K)}(\lambda) Q^{(K)}. \]

Parameter \( \lambda \) enters its perturbation-theory decomposition

\[ H^{(K)}(\lambda) = f^{(K)}(0) + \lambda W^{(K)}, \quad \lambda W^{(K)} = \left[ Q^{(K)} \right]^{-1} \gamma^{(K)} Q^{(K)}. \]  

(3.4)

Under two different philosophies, the EPK-related (i.e. perturbed BH) eigenvalue problem

\[ H^{(K)}(\lambda) |\Psi_n(\lambda)\rangle = E_n(\lambda) |\Psi_n(\lambda)\rangle, \quad n = 0, 1, \ldots, K - 1, \]  

(3.5)

has been studied in our two older papers [30,31]. Let us now briefly recall these results.

In [30], we followed the older methodical recommendation of review [23] so that we assumed the matrix of perturbation \( W^{(K)} \) acting in the most conventional Hilbert space \( \mathcal{K} = \mathbb{C}^K \) is \( \lambda \)-independent and bounded,

\[ W^{(K)} \in \mathcal{B}(\mathcal{K}). \]  

(3.6)

Under this assumption, we demonstrated that, in general, the spectrum ceases to be real even at the smallest non-vanishing couplings \( \lambda \neq 0 \). This just generalized the observations made, in the conventional open-system BH context, by the authors of [8]. Still, from our present, different, closed-quantum-system point of view, constraint (3.6) must be declared insufficient, not guaranteeing that the perturbations would be theoretically consistent and experimentally realizable.

These conclusions inspired our subsequent study [31]. In that study, we inverted the question, searching for a strengthening of the admissibility constraint (3.6) beyond the BH framework. The answer has been found and based on the introduction of \( \lambda \)-dependent matrices of perturbations

\[ W^{(K)} = W^{(K)}(\lambda) \in \mathcal{B}(\mathcal{K}). \]  

(3.7)

After such an enhancement of flexibility of the constraint and after its appropriate further amendment, perturbed Hamiltonians were made observable, keeping the states inside a preselected Hilbert space \( \mathcal{K} \).
As long as the required ultimate amendment of the theory is a rather technical matter, interested readers may find its detailed outline in appendix A. Here, let us only formulate the final result, recalling a suitable reparametrization of \( \lambda = 1/\Lambda^2 \) and of the class of the admissible, ‘sufficiently small’ perturbations

\[
W^{(K)}(\lambda) = V^{(K)}[\Lambda(\lambda)] + \text{higher-order corrections}, \quad \Lambda \gg 1.
\]  

(3.8)

The eligible and simplified leading-order matrices \( V^{(K)}(\Lambda) \) may be found displayed in appendix A. Under a useful though slightly artificial matrix-triangularity constraint

\[
V_{m,n} = \mathcal{O}\left(\frac{1}{\Lambda^0}\right), \quad m = 0, 1, \ldots, K - 2,
\]

(3.9)

the main part of the necessary condition of the reality of the perturbed spectrum has the form

\[
\begin{align*}
V_{m+1,m} &= \mathcal{O}\left(\frac{1}{\Lambda^0}\right), \quad m = 0, 1, \ldots, K - 2, \\
V_{n+2,n} &= \mathcal{O}(1/\Lambda^1), \quad n = 0, 1, \ldots, K - 3,
\end{align*}
\]

(3.10)

and so on, up to

\[
V_{K-1,0} = \mathcal{O}(1/\Lambda^{K-2}).
\]

More explicitly, we may write

\[
V_{m+1,m} = \frac{a^{(1)}_m}{\Lambda^0}, \quad m = 0, 1, \ldots, K - 2,
\]

(3.11)

and

\[
V_{n+2,n} = \frac{a^{(2)}_n}{\Lambda^1}, \quad n = 0, 1, \ldots, K - 3,
\]

and so on, up to

\[
V_{K-1,0} = \frac{a^{(K-1)}_0}{\Lambda^{K-2}},
\]

where all of the constants are bounded, \( a^{(j)}_k = \mathcal{O}(1) \). These coefficients may be arranged in a real array, which will be called a fundamental matrix

\[
C^{(K)} = \begin{bmatrix}
0 & 1 & 0 & \ldots & 0 \\
\frac{a^{(1)}_0}{\Lambda^0} & 0 & 1 & \ddots & \vdots \\
\frac{a^{(1)}_1}{\Lambda^0} & \frac{a^{(2)}_0}{\Lambda^1} & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 & 1 \\
\frac{a^{(K-1)}_0}{\Lambda^{K-2}} & \ldots & \frac{a^{(2)}_{K-3}}{\Lambda^{K-4}} & \frac{a^{(1)}_{K-2}}{\Lambda^{K-3}} & 0
\end{bmatrix}.
\]

(3.12)

Indeed, one immediately imagines that for the matrix elements lying inside a ‘physical’ domain \( D \) the spectrum of a given fundamental matrix can be real and non-degenerate.

**Lemma 3.1.** After the most elementary Kronecker-delta choice of \( a^{(j)}_k = \delta_{j,k} \), \( k = 0, 1, \ldots, K - 1 \), the set of eigenvalues \( \varepsilon_n \) of matrix (3.12) becomes defined in terms of the roots of classical orthogonal Chebyshev polynomials which are all real and non-degenerate [32].

Once the domain \( D \) is found non-empty, we may also recall and slightly reformulate the main result of [31].

**Theorem 3.2.** If the spectrum of fundamental matrix (3.12) is real and non-degenerate, then the quantum evolution controlled by Hamiltonian (3.4) is, in the leading-order approximation, unitary.

**Proof.** As explained in appendix A, the set of eigenvalues \( \{ \varepsilon_n \} \) of the fundamental matrix of coefficients \( C^{(K)} \) determines the leading-order bound state energies \( E_n(\lambda) \) in Schrödinger
The analysis of the role of the higher-order corrections remains non-trivial, especially when the spectrum of fundamental matrix $C^{(K)}$ remains real but degenerate. Still, the flexibility of the acceptable fundamental matrices as required by theorem 3.2 is large.

After a suitable special choice of the parameters even some exactly solvable models can be obtained (see lemma 3.1). Multiple other, different choices of admissible perturbation-determining matrices $C^{(K)}$ (yielding the other admissible spectra $\{\epsilon_n\}$) determine different unfoldings of the initial $K$-tuple degeneracy of the conventional BH bound-state energy spectrum. In the opposite direction, every admissible fundamental matrix $C^{(K)}$ marks a path which connects the interior of the physical domain $D$ with its extreme EPK boundary. Thus, in the vicinity of that point the family of all of the perturbation-induced unfoldings forms a subdomain of stability.

Our initial input choice of perturbation (3.7) is restricted by the size-suppression rules of equation (3.10). This makes the standard $C^{(K)}$ norm of perturbation matrices irrelevant. As explained in [33] the admissible matrix elements of $W^{(K)}(\lambda)$ are in fact ordered in a hierarchy of size which merely reflects and copies the deformations of the geometry of the physical Hilbert space of states (in fact, its anisotropy increases with the decrease of $\lambda \to 0$ [34]).

4. Perturbations not conserving the number of bosons

Whenever the real experimental quantum dynamics admits the creation and annihilation of bosons the clear separation of the $K$-dimensional BH sub-Hamiltonians as sampled by equation (2.3) becomes unrealistic and artificial. At the same time the price to pay for the breakdown of such a separation would be high, suddenly making any constructive study of the spectrum almost prohibitively difficult in general.

One of the possible paths towards the necessary mathematical simplifications may parallel the older studies working with the ‘unperturbed’ infinite-dimensional matrix versions of BH Hamiltonians $H_{(BH)}(\gamma)$ having the block-diagonalized form. This assumption could simplify the study of influence of the non-conservative perturbations significantly. In the language of physics this assumption would mean that the ‘tractable’ generalized BH-type Hamiltonians would have the form of perturbations of the conventional $c = 0$ BH Hamiltonian $H_{(BH)}(\gamma)$,

$$H_{(GBH)}(\gamma, \lambda) = -i\gamma \left( a_1^\dagger a_1 - a_2^\dagger a_2 \right) + \left( a_1^\dagger a_2 + a_2^\dagger a_1 \right) + \lambda H_{\text{int}} \left( a_1^\dagger, a_1, a_2^\dagger, a_2 \right).$$

Next, using the same strategy as above we shall only study the models with $\gamma = 1$. We will see below that such a reduction of generality will further simplify the technicalities while keeping the potential physics behind the model still sufficiently interesting.

The finite-dimensional matrices of perturbations as studied above will now be replaced by the full partitioned matrices

$$V_{(GBH)}(\lambda) = \begin{pmatrix} V^{(2,2)}(\lambda) & V^{(2,3)}(\lambda) & V^{(2,4)}(\lambda) & \ldots \\ V^{(3,2)}(\lambda) & V^{(3,3)}(\lambda) & V^{(3,4)}(\lambda) & \ldots \\ V^{(4,2)}(\lambda) & V^{(4,3)}(\lambda) & V^{(4,4)}(\lambda) & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$  

Our main task will be to guarantee that these matrices really remain ‘sufficiently small’, i.e. that the related energy spectrum would not cease to be observable and real.
(a) Models with one off-diagonal pair of perturbation submatrices

In an elementary non-trivial realization of the EPK-unfolding Hamiltonian (2.9) with an off-diagonal perturbation (4.2) let us assume that, up to two exceptions (say, \( V^{(M,L)}(\lambda) \) and \( V^{(L,M)}(\lambda) \) with \( M < L \)), all of the block-off-diagonal perturbations vanish. Thus, one has to study the new class of partitioned Hamiltonians with a bounded but, otherwise, fully general matrix of perturbations

\[
S^{(M,L)}(\lambda) = H_{\text{BH}}(1) + \lambda \ W^{(M+L)}(\lambda),
\]

with a bounded but, otherwise, fully general matrix of ‘small’ boson–boson interactions

\[
W^{(M+L)}(\lambda) = V^{(M+L)}(\Lambda(\lambda)) + \text{higher-order corrections}, \quad \Lambda = \frac{1}{\sqrt{|\lambda|}} \gg 1.
\]

In such a scenario, one only has to use the special symbols for the direct sums of Jordan matrices

\[
\mathcal{J}^{(M+L)}(\eta) = \begin{bmatrix} f^{(M)}(\eta) & 0 \\ 0 & f^{(L)}(\eta) \end{bmatrix}
\]

(cf. equation (3.2)), which can then be used to replace equation (A 1) in appendix A by its partitioned form

\[
\begin{bmatrix} C^{(M)}(\Lambda) & 0 \\ 0 & C^{(L)}(\Lambda) \end{bmatrix} \begin{bmatrix} \mathcal{J}^{(M+L)}(-E) + \lambda \ W^{(M+L)}(\lambda) \\ |\psi^{(M)}(\lambda)\rangle \\ |\psi^{(L)}(\lambda)\rangle \end{bmatrix} = 0.
\]

(4.5)

This contains a partitioned upgrade of the eigenvector,

\[
\begin{bmatrix} |\phi^{(M)}(\lambda)\rangle \\ |\phi^{(L)}(\lambda)\rangle \end{bmatrix} = \begin{bmatrix} C^{(M)}(\Lambda) & 0 \\ 0 & C^{(L)}(\Lambda) \end{bmatrix} \begin{bmatrix} |\psi^{(M)}(\lambda)\rangle \\ |\psi^{(L)}(\lambda)\rangle \end{bmatrix}.
\]

(4.6)

Working again with the two equivalent versions of the ‘small’ parameters \( \lambda = 1/\Lambda^2 \) we re-scale the energy,

\[
E = E(\lambda) = \frac{\epsilon(\Lambda)}{\Lambda}.
\]

(4.7)

The Schrödinger equation then acquires the obvious partitioned structure, yielding the secular equation

\[
\det \begin{bmatrix} \mathcal{J}^{(M+L)}(-\epsilon(\Lambda)) + M^{(M+L)}(\Lambda) \end{bmatrix} = 0.
\]

(4.8)

In the full matrix of rescaled interactions

\[ M^{(M+L)}(\Lambda) = \begin{bmatrix} M^{(M)}(\Lambda) & A^{(M,L)}(\Lambda) \\ B^{(L,M)}(\Lambda) & M^{(L)}(\Lambda) \end{bmatrix}, \]

(4.9)

both of the diagonal blocks remain the same as in appendix A. As long as \( M < L \), the two off-diagonal non-square rescaled-interaction submatrices deserve an explicit display in terms of the respective abbreviations \( W_{mn}^{(P,Q)}(\lambda) = W_{mn} = \mathcal{O}(1) \) with superscripts \( (P, Q) = (M, L) \) or \( (P, Q) = (L, M) \), yielding

\[
A^{(M,L)}(\Lambda) = \begin{bmatrix}
A^{-1}W_{00} & A^{-2}W_{01} & A^{-3}W_{02} & \cdots & A^{-L}W_{0,L-1} \\
A^{0}W_{10} & A^{-1}W_{11} & A^{-2}W_{12} & \cdots & A^{-1}W_{1,L-1} \\
A^{1}W_{20} & A^{0}W_{21} & A^{-1}W_{22} & \cdots & A^{-2}W_{2,L-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
A^{M-2}W_{M-1,0} & A^{M-3}W_{M-1,1} & A^{M-4}W_{M-1,2} & \cdots & A^{M-1-L}W_{M-1,L-1}
\end{bmatrix},
\]

(4.10)

and, \textit{mutatis mutandis}, the analogous formula for \( B^{(L,M)}(\Lambda) \).

Next, proceeding along the same lines as in appendix A we accept the boundedness assumption (3.6) and we replace the respective exact matrix-element functions \( W_{mn}^{(P,Q)} \) of cut-off \( \Lambda \) by their asymptotically dominant components \( V_{mn}^{(P,Q)} \). The real and non-degenerate set
of the leading-order energy eigenvalues $\epsilon_0 = \lim_{\Lambda \to \infty} \epsilon(\Lambda)$ should then be extracted from the leading-order version of the secular equation,

$$\det \left[ J^{(M+L)}(-\epsilon_0) + M^{(M+L)}_0(\Lambda) \right] = 0. \quad (4.11)$$

This is defined in terms of the leading-order version $M^{(M+L)}_0(\Lambda)$ of the interaction term. Its structure

$$M^{(M+L)}_0(\Lambda) = \begin{bmatrix} M^0(M)(\Lambda) & A^{(M,L)}_0(\Lambda) \\ B^{(L,M)}_0(\Lambda) & M^0(L)(\Lambda) \end{bmatrix} \quad (4.12)$$

exhibits important simplifications, with the diagonal blocks defined in appendix A (cf. equation (A 5)) and complemented by

$$A^{(M,L)}_0 = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ V_{10} & 0 & \cdots & 0 & \cdots & \cdots & \cdots \\ \Lambda V_{20} & \cdots & 0 & 0 & \cdots & \cdots & \cdots \\ \vdots & \ddots & \cdots & V_{M-2,M-3} & 0 & \cdots & \cdots \\ \Lambda^{M-2} V_{M-1,0} & \cdots & \Lambda V_{M-1,M-3} & V_{M-1,M-2} & 0 & \cdots & \cdots \end{bmatrix}, \quad (4.13)$$

and by the formula for $B^{(L,M)}(\Lambda)$ (with the display left to the reader). The latter two non-square matrices as well as their two diagonal-block square-matrix partners are all lower triangular, dominated by their respective lowest-left-corner elements. By analogy, we expect that the scope of theorem 3.2 might be extended to cover also the models with block-off-diagonal perturbations. Without proof, these expectations may be given, for Hamiltonians (4.3), the following explicit formulation.

**Conjecture 4.1.** To leading-order approximation, quantum evolution controlled by the $(M,L)$-partitioned Hamiltonians (4.3) will be unitary provided only that the spectrum of the corresponding $(M,L)$-partitioned analogue of the fundamental matrix (3.12) proves real and non-degenerate.

For any pair of integers $M$ and $L$ there exist only too many fundamental matrices $C^{(M+L)}$ with a real and non-degenerate spectrum. One can conclude that even in the non-conservative, partitioned-matrix BH-type quantum systems sampled by equation (4.3) the constructive guarantees of the reality of the spectra of energies unfolded along certain parametric paths remain mathematically feasible.

Equation (2.3) of §2 represents, in the EPK limit $\gamma \to 1$, our unperturbed block-diagonal Hamiltonian $H_{(BH)}(1)$. In this section such a Hamiltonian was considered to be endowed with a most elementary perturbation assumed to couple just the two basis-state sets representing the two arbitrary non-equal amounts of bosons, $N_1 (= M) < N_2 (= L)$. The idea can easily be generalized to the triply partitioned models with coupling of three preselected non-equal amounts of bosons $N_1 < N_2 < N_3$, etc. One immediately imagines that such steps would be too formal. From the point of view of the dynamics of the bosons, it makes much better sense to study just the systems with more partitions but with not too large a total number of mutually interacting bosons.

(b) Leading-order block-non-diagonal Hamiltonians

Although the most straightforward modification of the BH Hamiltonian (2.1) has been shown to require a change of its boson–boson interaction component $\hat{H}_{\text{int}}(\hat{a}_1, \hat{a}_2, \hat{a}_1^\dagger, \hat{a}_2^\dagger)$, such an obvious model-building strategy may have several weak points. In the basis where the unperturbed Hamiltonian remains block-diagonal, for example, technical obstacles might emerge in connection with the evaluation of matrix elements of the perturbation. In addition, one must often keep in mind that the underlying algebras of the creation and annihilation operators can only be realized in a strictly infinite-dimensional Hilbert space. For all of these reasons, an alternative strategy will be advocated and used in what follows. In a formally less ambitious
approach we will make ample use of the finite-dimensional, truncated-matrix versions of our Hamiltonians. In parallel, we will insist on staying inside a closed-quantum-system set-up and phenomenology. Thus, even our most complicated versions of perturbed BH Hamiltonians $H_{(\text{GBH})}(1, \lambda)$ will be required to possess the unitarity-compatible real spectra.

We shall keep in mind that even the most ambitious generalizations of the interactions should remain user-friendly. We felt inspired by the results outlined in appendix A. In light of these, one of the key features of our models will lie in their sparse-matrix form in the dominant-order approximation. Although the matrices will remain manifestly $\lambda$-dependent, their generic structure will be as follows:

$$H^{(2+3+\ldots)}_{\lambda} = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{bmatrix}. \quad (4.14)$$

Such a structure will result from a combination of the requirement for the reality of spectra with the physics-dictated step-by-step choice of the dimensions $N_1 = 2$, $N_2 = 3$, $\ldots$, $N_{\max} = L$ responsible for the boson-number partitioning. After truncation, the original infinite-dimensional matrix $H^{(2+3+\ldots)}_{\lambda}$ of equation (4.14) would acquire the finite-dimensional $K_L$ by $K_L$ matrix form $H^{(2+3+\ldots+L)}_{\lambda}$, where $K_L = (L^2 + L - 2)/2$.

With the formal proof postponed to a forthcoming study, our expectations concerning the reality of spectra have, at present, the following form.

**Conjecture 4.2.** To leading-order approximation, quantum evolution controlled by the truncated forms of Hamiltonian (4.14) will be unitary provided only that the spectrum of the fundamental matrix (i.e. of the corresponding $(N_1, N_2, \ldots, N_{\max})$-partitioned analogue of matrix (3.12)) proves real and non-degenerate.

In conjectures 4.1 and 4.2, we spoke about the leading-order approximations, which were only rather vaguely specified. Another formal weakness of these conjectures is that, at present, we do not know how one could include, sufficiently efficiently, the higher-order corrections, or how one could guarantee, in a systematic manner, the required spectral properties of the respective fundamental matrices. This is the reason why we do not provide the proofs here (converting our conjectures, in a more or less straightforward manner, into theorems), and why we prefer to add a few illustrative examples in the next section. We will shift our emphasis to physics (of the most elementary systems of ‘not too many’ bosons), and we will outline below a few more tricks that help to keep the spectra real and non-degenerate in practical applications.

**5. Examples**

The main technical challenge connected with the study of the models in the preceding section may be found formulated in conjecture 4.2: one needs to guarantee that the spectrum of a fundamental matrix is real and non-degenerate. Unfortunately, even at the smallest possible partition dimensions $N_1 (= M) = 2$ and $N_2 (= L) = 3$ the corresponding secular polynomial will be a polynomial of the fifth degree in the energy. For this reason, the necessary specification of its admissible, dynamics-determining coefficients (i.e. of the unitarity-compatible matrix elements of perturbations $\lambda W^{(M+L)}(\lambda)$ in equation (4.3)) seems to be a purely numerical task. Now, let us show that such a task can be reduced to a sequence of easier steps.
(a) Models with the bosonic pairs coupled to triples

(i) The most elementary case

In the above-mentioned most elementary generalized BH example with \( N_1 (= M) = 2 \) and \( N_2 (= L) = 3 \) one has to consider Hamiltonian

\[
H^{(2+3)}(\lambda) = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
\star & 0 & \star & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
\star & 0 & \star & 0 & 1 \\
\star & \star & \star & \star & 0
\end{bmatrix},
\]

(5.1)

with the potentially non-vanishing matrix elements marked by stars \( \star \). Up to scalar factor \( 1/\sqrt{\lambda} \) this Hamiltonian should be isospectral with the fundamental \((2, 3)\)-partitioned matrix

\[
C^{(2+3)} = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & a & 0 & b & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & e & f & g & h \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]

(5.2)

This is an ansatz which varies with eight free parameters representing the dominant components of the interaction. Now, in spite of the quintic-polynomial (i.e. exactly unsolvable) nature of the corresponding secular equation, we seek the criteria of admissibility of these parameters in the unitary evolution regime.

Intuitively, the existence of the octuplet of free parameters might prove sufficient for an adjustment to any input quintuplet of eigenvalues \( \varepsilon_n \). One can rapidly find that such a strategy is not viable. Safer conclusions can only result from a construction based on the explicit form,

\[
\det(C^{(2+3)} - \varepsilon I) = 0,
\]

of the underlying quintic-polynomial secular equation.

One way to an efficient simplification of the problem lies in the fact that in the secular polynomial the coefficient at the second power of \( \varepsilon \) is equal to \( g \). Thus, the term drops out after one selects \( g = 0 \). This immediately implies that, when we further set \( e = 0 \), the five roots of the secular polynomial acquire the following elementary form:

\[
0, \pm \sqrt{a + d + h \pm \sqrt{(4b(c + f) + (d + h - a)^2)}}/\sqrt{2}.
\]

(5.3)

Owing to the elementary nature of the model with \( M = 2 \) and \( L = 3 \) (where the selection of \( g = e = 0 \) was ‘obvious’) the problem is solved. Unfortunately, the similar helpful selection ceases to be available whenever \( M > 2 \) and/or \( L > 3 \).

Let us now describe an alternative, more robust procedure by which the existence of the necessary fundamental matrices with real spectra could be proved, in a systematic iterative manner, even at the larger \( M > 2 \) and/or \( L > 3 \). In the first step let us return to the benchmark model (5.3) and let us make a ‘wrong’ choice, having set all of the remaining parameters equal to 1. Naturally, we get an unsatisfactory answer because the spectrum appears real but partially degenerate. Still, although the trial-and-error guesswork did not work, a subsequent return to the \( g \neq 0 \) tentative amendment using \( f = 1/100 \) and \( g = 1/2 \) already appears to serve the purpose. What is obtained is the numerical (i.e. approximate) quintuplet of well-separated real eigenvalues

\[-1.5118, -0.4630, 0.0000, 0.4630, 1.5118].\]
This offers an alternative proof of the existence of at least one fundamental matrix with the required properties.

Incidentally, as long as the model with $K_L = 5$ is not yet too large, the alternative proof based on the ‘wrong’ choice of $g \neq 0$ can still be made non-numerical. Indeed, besides the obvious exact root $\varepsilon_0 = 0$, the other four non-vanishing ones can also be given the closed form

$$\varepsilon = \varepsilon_{\pm, \pm} = \left( \pm \sqrt{390} \pm \sqrt{110} \right)/20,$$

which parallels formula (5.3).

(ii) Boundary $\partial D$ of the corridor of unitarity

In the preceding subsection, we demonstrated that the corridor $D = D^{(2+3)}$ of the admissible parameters of unitary unfoldings of the twice degenerate BH $EP = EP_2 + EP_3$ extreme is a non-empty domain. This opens a number of new questions concerning the shape and properties of the boundary $\partial D^{(2+3)}$ of stability.

In a preparatory step, we notice that in the original ‘eight-star’ Hamiltonian (5.1) only six stars represent the dominant components of the interaction (i.e. for the $O(\Lambda^0)$ matrix elements of $V$, see the first line of equation (3.10)). For this reason, we will omit the study of the role of the two next-order $O(\Lambda^{-1})$ contributions (see the second line in equation (3.10)) and we will set again, for methodical reasons, $e = g = 0$ in our five by five fundamental matrix (5.2). Then, the related secular polynomial

$$P = z^5 - (a + d + h) z^3 + (a(d + h) - b(f + c)) z$$

(5.4)
defines, via the condition of reality and non-degeneracy of all of its five roots (5.3), the whole six-dimensional physical domain $D^{(2+3)}$ of parameters $a, b, c, d, f$ and $h$ of dynamical relevance.

The ‘most common’ parts of boundary $\partial D^{(2+3)}$ could now be interpreted as the separate submanifolds of pairwise EP2 mergers of the roots. The boundary may also contain the lower-dimensional parts supporting the higher-order mergers, up to the most interesting EP5 extreme if any.

As long as one of the roots of the secular polynomial is constant, $z_0 = 0$, the localization of the EP5 boundary manifold $\partial D^{(2+3)}$ is facilitated. Indeed, the secular polynomial must degenerate to monomial, $P^{(EP5)} = z^5$. In light of equations (5.3) and (5.4), this leads to two necessary-condition equations yielding the $b \neq 0$ ‘solution A’,

$$h = -a - d, \quad f = -a^2/b - c,$$

(5.5)
and its $b = 0$ complement, ‘solution B’,

$$a = b = 0, \quad h = -d.$$

(5.6)

In the former case ‘A’, condition (5.5) is also sufficient. This means that condition (5.6) of case ‘B’ merely leads to the less singular types of degeneracy. Its analysis may be found in appendix B.

Once we return to the spectral-degeneracy limit of type ‘A’ our fundamental matrix $C^{(2+3)}_{(A)}$ is really easily shown to satisfy the EP5 analogue of equation (2.6)

$$C^{(2+3)}_{(A)} = Q_{(A)} f^{(5)}(0) \left[ Q_{(A)} \right]^{-1}.$$

(5.7)
As long as such a limiting matrix still contains four freely variable real parameters $a, b, c$ and $d$, the corresponding part of the boundary $\partial D^{(2+3)}$ is also a four-dimensional manifold. For its compact description, it makes sense to abbreviate $-a^2/b - c = F(a, b, c) = F$. Then it is easy to display the
transition matrix

\[ Q(A) = \begin{bmatrix}
  -Fb & 0 & a & 0 & 1 \\
  0 & -Fb & 0 & a & 0 \\
  aF & 0 & c & 0 & 0 \\
  0 & aF & 0 & c & 0 \\
  -(a^2 + Fb + ad)F & 0 & -(2ca + cd + a^3/b) & 0 & 0
\end{bmatrix} \]

and to prove that it is invertible since \( \det Q(A) = -F^5b^2 \).

**(b) Bosonic pairs and triples coupled to quadruples \( L = 4, K_L = 9 \)**

Let the dominant-order Hamiltonian have the following sparse, nine by nine matrix form:

\[ \delta_0^{(2+3+4)}(\lambda) = \begin{bmatrix}
  0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  * & 0 & * & 0 & 0 & * & 0 & 0 & 0 \\
  0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
  * & 0 & * & 0 & 1 & * & 0 & 0 & 0 \\
  * & * & * & * & 0 & * & * & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
  0 & 0 & * & 0 & 0 & * & 0 & 1 & 0 \\
  * & 0 & * & 0 & * & * & 0 & 1 & 0 \\
  * & * & * & * & * & * & * & * & 0
\end{bmatrix}. \] (5.8)

The stars mark the \( \lambda \)-dependent matrix elements which should again characterize the leading-order components of the admissible, real-spectrum-supporting boson–boson interaction \( \lambda \mathcal{V}_0^{(2+3+4)} \).

After an appropriate rescaling made in the spirit of equation (3.11), these components become proportional to the 29 relevant parameters entering the fundamental matrix with the same sparse-matrix structure. For pedagogical reasons, we will still omit all of the representatives of the subdominant perturbations (for example, in equation (3.10) we would only keep the elements of the first row). In this way, our candidate for the fundamental matrix will have the following simplified, 17-parametric form:

\[ \mathcal{C}^{(2+3+4)} = \begin{bmatrix}
  0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  a & 0 & b & 0 & 0 & c & 0 & 0 & 0 \\
  0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
  d & 0 & e & 0 & 1 & f & 0 & 0 & 0 \\
  0 & h & 0 & j & 0 & l & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
  o & 0 & m & 0 & 0 & n & 0 & 1 & 0 \\
  0 & u & 0 & q & 0 & s & 0 & 1 & 0 \\
  0 & 0 & 0 & 0 & x & 0 & 0 & \omega & 0
\end{bmatrix}. \] (5.9)

This matrix is a \((2+3+4)\) analogue of its \((2+3)\) partitioned predecessor (5.2) as well as of the even simpler, unpartitioned matrix (3.12). Thus, to leading-order approximation, quantum evolution controlled by Hamiltonian (5.8) will be unitary provided only that the spectrum of fundamental matrix (5.9) proves real and non-degenerate.

Our final task is to prove the existence of the unitarity-supporting boson–boson interactions.

**Lemma 5.1.** There exists a non-empty domain of matrix elements in (5.9) yielding the real and non-degenerate spectrum of the fundamental matrix.

**Proof.** The process of proof will be similar to the preceding case. In the first step, we set all parameters equal to 1 and obtain the elementary secular polynomial \( z^9 - 6z^7 + 3z^5 \) with the nine
real roots
\[ \{0, 0, 0, 0, 0, -(3 + 6^{1/2})^{1/2}, (3 + 6^{1/2})^{1/2}, -(3 - 6^{1/2})^{1/2}, (3 - 6^{1/2})^{1/2}\}, \]
i.e. numerically,
\[ \{\pm 2.334414218, \pm 0.7419637843, 0., 0., 0., 0.\}. \]

The quintuple degeneracy was weakened by a modification of \( b = 1 + v \). This yielded a modified secular polynomial which was linear in \( v \). The behaviour of the easily obtained curve \( v = v(z) \) near the origin indicated a weakening of the degeneracy for \( v \in (-1/4, 0) \), so we chose \( v = -1/10 \) and obtained the amended spectrum
\[ \{\pm 2.327224413, \pm 0.7154619694, \pm 0.2685902108, 0., 0., 0.\}. \]

With the degeneracy reduced to 3, we iterated the process and added a new auxiliary variable \( w \) to elements \( a, l \) and \( u \), having evaluated the (this time, double-branched) function \( w(z) \). As long as this function has two branches,
\[ w_1(z) = \left( -\frac{1}{10} + \frac{319}{200} z^2 + O(z^4) \right) \]
and
\[ w_2(z) = \left( -z^2 + O(z^4) \right), \]
forming a small circle below the real line at small \( z \), we concluded that the small negative value will work. Indeed, with \( w = -1/100 \) we obtained
\[ \{\pm 2.325373957, \pm 0.7112721146, \pm 0.2586335768, \pm 0.09917969302, 0.\}, \]
i.e. the sample spectrum we needed for the proof.

We may conclude that the introduction of the new non-vanishing matrix elements safely removed the degeneracy while still keeping the other eight roots almost unchanged. In addition, the latter spectrum yields, after premultiplication by factor \( \sqrt{\lambda} \), the ultimate bound-state energies.

### 6. Conclusion

The main mathematical message delivered by our present paper is that there are not too many really deep conceptual differences between the models with the conventional, simple EPs (sampled by equations (2.6) + (2.7)) and the generalized models with the much less usual, degenerate EPs (sampled, for example, by equations (3.3) + (3.2)). Most importantly, what is shared is the correlation between the natural physical requirement of the unitarity of the model (i.e. of the reality of the spectrum) and the highly artificial-looking mathematical requirements by which the matrix elements of the underlying ‘admissible’ perturbation matrices must exhibit an ordering in size as sampled by equation (3.10).

Our present results may be read as a climax of recent developments in the field. First of all, the formal, purely algebraic-geometry origin of the necessity of the apparently strongly counterintuitive hierarchy of the matrix elements of admissible perturbation matrices may be found described, in pedagogical detail, in Section Nr. IV of [31]. Second, in the context of the physics of closed quantum systems, a deeper clarification of the apparent paradox of irrelevance of the conventional norms of matrices of perturbations should be sought in [30]. The conclusion is that, as long as our quantum system of interest lives, by assumption, in the vicinity of its loss-of-the-unitarity boundary \( \partial D \), the enormous differences in the influence of separate matrix elements of the interaction just reflect the deformations of the geometry of the physical Hilbert space. Indeed, this geometry becomes, near \( \partial D \), increasingly anisotropic (for more details, see also an extensive commentary on this topic in [34]).

Besides these mathematical results, our attention was attracted by the descriptive features of the conventional BH Hamiltonian of equation (2.1). From this point of view, the main message
delivered by our present paper was threefold. First, in reference to the extensive study [8] of the role of the specific boson–boson interaction perturbations \( c H_{\text{int}} \) we emphasized that, near the EPK extreme of boundary \( \partial D \), such a perturbation would be out of our present closed-system-oriented interest because, owing to its complex energies, it only describes the physical reality in the traditional open, unstable quantum-system setting.

Second, guided by the results presented in appendix A we found a new domain of applicability of the closed-quantum-system philosophy under the assumption that the number of bosons is constant. For this purpose we recommended the replacement of the conventional, ‘too large’ boson–boson interaction term \( c H_{\text{int}} \) by its ‘sufficiently small’ boson–boson interaction amendment as specified by theorem 3.2.

Third, we observed that the mathematics behind the latter amendment need not necessarily remain restricted to the perturbations which commute with the boson number operator \( \hat{N} \) of equation (2.2). Subsequently we imagined that such an innocent-looking extension of mathematics leads to an enormous extension of the descriptive phenomenological capacity of the model. Indeed, in the dominant-order approximation we managed to reduce the problem of the guarantee of the unitarity of evolution of the system in question (i.e. of the reality of the \( \lambda \)-dependent bound-state energies) to the purely mathematical analysis of spectra of certain auxiliary, sparse and \( \lambda \)-independent ‘fundamental’ matrices \( C \).

The feasibility of application of the latter criterion was finally demonstrated via the first two non-trivial illustrative examples in which the number of bosons was not conserved (see their respective partitioned-matrix Hamiltonians in equations (5.1) and (5.8)) and in which the EP singularities acquired the extremely interesting degenerate-EP structures of the form of the direct sums \( E_2+E_3 \) and \( E_2+E_3+E_4 \), respectively.

It seems worth adding that the numbers of the relevant variable parameters which were ‘multi-indexing’ the corresponding five- and nine-dimensional matrices of perturbations were 8 and 27, respectively. However, after a number of trial and error ‘experiments’ we eliminated some of the less relevant variables and, ultimately, we managed to prove the non-emptiness of the respective ‘physical’, unitarity-supporting domains \( D \) of the parameters. Moreover, last but not least, we accompanied the latter proofs of existence by an explicit localization and EP classification of a few parts of the end-of-unitarity quantum-phase transition boundary \( \partial D^{(2+3)} \).

Data accessibility. This article has no additional data.

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Appendix A. Conservative BH perturbations (3.7) and the necessary and sufficient condition of the reality of the spectrum

The demonstration of the reality of the spectra as described in [31] is strongly model-dependent and hardly applicable in the present general BH-related set-up. Let us, therefore, propose and describe a different, shorter version of the method.

First, in our perturbed Schrödinger equation (3.5) let us preselect any ground- or excited-state subscript \( n = n_0 \). Then, in a shorthand notation, let us just remember this information and simplify \( |\Psi_{n_0}(\lambda)\rangle \rightarrow |\Psi(\lambda)\rangle \) and \( E_{n_0}(\lambda) \rightarrow E(\lambda) \). Second, let us introduce a large, \( \lambda \)-dependent cut-off parameter \( \Lambda = \Lambda(\lambda) = 1/\sqrt{\lambda} \) entering a one-parametric auxiliary diagonal matrix \( C^{(K)}(\Lambda) \) with ‘increasingly large’ elements \( C_{kk}^{(K)}(\Lambda) = \Lambda^k, k = 0, 1, \ldots, K - 1 \). Next, let us re-write the perturbed Schrödinger equation (3.5) in a preconditioned \( K \) by \( K \) matrix form

\[
C^{(K)}(\Lambda) \left[ I^{(K)}(0) + \lambda W^{(K)}(\lambda) - E(\lambda) I \right] \left[ C^{(K)}(\Lambda(\lambda)) \right]^{-1} |\Phi(\lambda)\rangle = 0,
\]

\[
|\Phi(\lambda)\rangle = C^{(K)}(\Lambda) |\Psi(\lambda)\rangle. \tag{A1}
\]
Finally, let us re-scale the energy $E(\lambda) = \epsilon(A)/\Lambda$ and transform our initial, conventional secular equation

$$\det \left\{ G^{(K)}[A(\lambda)] \left[ f^{(K)}(0) + \lambda W^{(K)}(\lambda) - E(\lambda)f^{(K)} \right] \left[ G^{(K)}(A(\lambda)) \right]^{-1} \right\} = 0 \quad (A 2)$$

into the following equivalent equation:

$$\det \left[ f^{(K)}(0) + M^{(K)}(A) - \epsilon(A)f^{(K)} \right] = 0. \quad (A 3)$$

The re-scaled matrix of perturbations is defined in terms of elements $W_{mn} = W_{mn}^{(K)}(\lambda)$ multiplied by powers of our cut-off-resembling large parameter $\Lambda = 1/\sqrt{\lambda}$,

$$M^{(K)}(A) = \begin{bmatrix}
\Lambda^{-1}W_{00} & \Lambda^{-2}W_{01} & \Lambda^{-3}W_{02} & \cdots & \Lambda^{-K}W_{0,K-1} \\
\Lambda^0W_{10} & \Lambda^{-1}W_{11} & \Lambda^{-2}W_{12} & \cdots & \Lambda^{-1}W_{1,K-1} \\
\Lambda^1W_{20} & \Lambda^0W_{21} & \Lambda^{-1}W_{22} & \cdots & \Lambda^{-2}W_{2,K-1} \\
& \ddots & \ddots & \ddots & \ddots \\
\Lambda^{-K}W_{K-1,0} & \Lambda^{-K}W_{K-1,1} & \Lambda^{-K-1}W_{K-1,2} & \cdots & \Lambda^{-1}W_{K-1,K-1}
\end{bmatrix}. \quad (A 4)$$

Under the boundedness assumption (3.6) and using a leading-order-coefficient simplification

$$W_{mn}^{(K)}(\lambda) = V_{mn}A^{\text{const.}} + \text{corrections}, \quad m, n = 0, 1, \ldots, K - 1,$$

we get the lower triangular leading-order matrix of perturbations

$$M_0^{(K)}(A) = \begin{bmatrix}
0 & 0 & \cdots & 0 & 0 \\
V_{10} & 0 & \cdots & \cdots & \cdots \\
\Lambda V_{20} & \cdots & \cdots & 0 & 0 \\
& \ddots & \ddots & \ddots & \ddots \\
\Lambda^{-K}V_{K-1,0} & \cdots & \Lambda V_{K-1,K-3} & V_{K-1,K-2} & 0
\end{bmatrix}. \quad (A 5)$$

The insertion of this matrix in equation (A 3) with $\epsilon(A) = \epsilon_0 + \text{corrections}$ yields the leading-order secular equation

$$\det \begin{bmatrix}
-\epsilon_0 & 1 & 0 & \cdots & 0 \\
V_{10} & -\epsilon_0 & 1 & \cdots & \cdots \\
\Lambda V_{20} & \cdots & \cdots & 0 & 0 \\
& \ddots & \ddots & \ddots & \cdots \\
\Lambda^{-K}V_{K-1,0} & \cdots & \Lambda V_{K-1,K-3} & V_{K-1,K-2} & -\epsilon_0
\end{bmatrix} = 0. \quad (A 6)$$

After its systematic analysis as sampled in [30,31] one comes to the conclusion that the spectrum cannot be real unless one accepts the assumption that in the above-mentioned conventional Hilbert space $\mathcal{H} = C^K$ matrix (A 5) is also kept bounded

$$M_0^{(K)} \in B(\mathcal{H}). \quad (A 7)$$

This type of constraint was rendered possible by the above-mentioned requirement (3.7) of an explicit variability of the separate matrix elements with the cut-off or strength of perturbation $\lambda = 1/\Lambda^2$. Thus, besides the matrix-triangularity rule (3.9) as also used in equation (A 5), the reality of the perturbed spectrum is, in general, guaranteed by equations (3.10) and (3.11). These equations represent the two alternative versions of the necessary condition of the unitarity of the evolution. In the present notation, this means that the underlying fundamental matrix (3.12) must have a real and discrete spectrum. Then, with all of its $K(K - 1)/2$ variable parameters such a matrix specifies an admissible perturbation. In this sense, any such matrix could be interpreted as a definition of one of the eligible paths through a corridor of unitary unfolding of the EPK spectral degeneracy.
Appendix B. A few non-EP5 components of the boundary of stability \( \partial D^{(2+3)} \)

Let us recall condition (5.6) in case ‘B’ of §5a(ii). It is easily shown to lead to factorization

\[
C^{(2+3)}_{(B)} = Q_{(B)} I_{(B)} \left[ Q_{(B)} \right]^{-1},
\]

where \( I_{(B)} = J^{(4+1)}(0) \). This means that we have to deal with the mere EP4 boundary of the physical parametric domain \( D \). Indeed, using abbreviation \( f + c = \alpha = \alpha(f, c) \) and assumption \( c \neq 0 \) the related three-parametric transition matrix with \( \det Q_{(B)} = \alpha^3 / c \) looks particularly elementary

\[
Q_{(B)} = \begin{bmatrix}
0 & 0 & 1 & -c^{-1} & -c^{-1} \\
0 & 0 & 0 & 1 & 0 \\
\alpha & 0 & 0 & 0 & 0 \\
0 & \alpha & 0 & 0 & 0 \\
-\alpha d & 0 & f & 1 & 1
\end{bmatrix}.
\]

In the singular limit \( c \to 0 \) with \( f \neq 0 \), remarkably enough, the transition matrix remains regular

\[
Q_{(B,0)} = \begin{bmatrix}
0 & 0 & 1 & -f^{-1} & -f^{-1} \\
0 & 0 & 0 & 1 & 0 \\
f & 0 & 0 & 0 & 0 \\
0 & f & 0 & 0 & 0 \\
-fd & 0 & f & 0 & 0
\end{bmatrix},
\]

with \( \det Q_{(B,0)} = \alpha^3 / c \).

Obviously, the subsequent singular limit of \( f \to 0 \) still deserves a separate treatment. In equation (A 1), this leads, by direct computations, to

\[
J_{(B)} = J_{(B,0,0)} = J^{(3+2)}(0) \text{ and }
\]

\[
Q_{(B)} = Q_{(B,0,0)} = \begin{bmatrix}
0 & 1 & 0 & -1 & 0 \\
0 & 0 & 1 & 0 & -1 \\
d & 0 & 1 & 0 & 0 \\
0 & d & 0 & 0 & 0 \\
-d^2 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]

This is an invertible matrix since \( \det Q_{(B,0,0)} = d^3 \).

In the singular limit of \( d \to 0 \), the fundamental matrix itself acquires an EP2+EP3 form. It is probably worth adding that its conversion into a reordered canonical form EP3+EP2 still requires a non-trivial, non-permutation parameter-free transition matrix

\[
Q_{(B)} = Q_{(B,0,0,0)} = \begin{bmatrix}
0 & 1 & 0 & -1 & 0 \\
0 & 0 & 1 & 0 & -1 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0
\end{bmatrix},
\]

with unit determinant.

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