Boundedness of Linear Operators via Atoms on Hardy Spaces with Non-doubling Measures

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Abstract. Let \( \mu \) be a non-negative Radon measure on \( \mathbb{R}^d \) which only satisfies the polynomial growth condition. Let \( Y \) be a Banach space and \( H^1(\mu) \) the Hardy space of Tolsa. In this paper, the authors prove that a linear operator \( T \) is bounded from \( H^1(\mu) \) to \( Y \) if and only if \( T \) maps all \((p, \gamma)\)-atomic blocks into uniformly bounded elements of \( Y \); moreover, the authors prove that for a sublinear operator \( T \) bounded from \( L^1(\mu) \) to \( L^{1,\infty}(\mu) \), if \( T \) maps all \((p, \gamma)\)-atomic blocks with \( p \in (1, \infty) \) and \( \gamma \in \mathbb{N} \) into uniformly bounded elements of \( L^1(\mu) \), then \( T \) extends to a bounded sublinear operator from \( H^1(\mu) \) to \( L^1(\mu) \). For the localized atomic Hardy space \( h^1(\mu) \), corresponding results are also presented. Finally, these results are applied to Calderón-Zygmund operators, Riesz potentials and multilinear commutators generated by Calderón-Zygmund operators or fractional integral operators with Lipschitz functions, to simplify the existing proofs in the corresponding papers.

1 Introduction

The real-variable theory of Hardy spaces on \( \mathbb{R}^d \), which began with the remarkable work of Stein and Weiss [19], has been transformed into a rich theory. The well-known atomic and molecular characterizations of Hardy spaces enable one to deduce the boundedness on Hardy spaces of (sub)linear operators from their behaviors on atoms or molecules in principle. However, Meyer, Taibleson and Weiss [13] constructed an example of \( f \in H^1(\mathbb{R}^d) \) such that its norm can not be achieved by its finite atomic decompositions via \((1, \infty)\)-atoms. Inspired by this, Bownik [2] showed that there exists a linear functional, which maps all \((1, \infty)\)-atoms of \( H^1(\mathbb{R}^d) \) into bounded scalars but does not admit a bounded extension to \( H^1(\mathbb{R}^d) \). It turns out that the condition that a linear operator \( T \) maps all \((1, \infty)\)-atoms into a uniformly bounded subset of certain quasi-Banach space \( B \) fails to guarantee the extension of \( T \) to a bounded linear operator from the whole \( H^1(\mathbb{R}^d) \) to \( B \). Recently, Meda, Sjögren and Vallarino [11] proved that any linear operator mapping all \((1, q)\)-atoms with \( q \in (1, \infty) \) or all continuous \((1, \infty)\)-atoms into a uniformly bounded elements in a given Banach space \( B \) extends to a bounded linear operator from \( H^1(\mathbb{R}^d) \) to \( B \). Independently, in [28], a boundedness criterion was established as follows: a non-negative sublinear operator \( T \) extends to a bounded sublinear operator from Hardy spaces...
$H^p(\mathbb{R}^d)$ with $p \in (0, 1]$ to certain quasi-Banach space $\mathcal{B}$ if and only if $T$ maps all $(p, 2)$-atoms into uniformly bounded elements of $\mathcal{B}$. On the other hand, via making clear the dual and the completion of the space of finite linear combinations of $(p, \infty)$-atoms with $p \in (0, 1]$, Ricci and Verdera [18] further proved that if $T$ is a linear operator mapping all $(p, \infty)$-atoms with $p \in (0, 1)$ uniformly bounded to a Banach space $\mathcal{B}$, then $T$ extends to a bounded linear operator from $H^p(\mathbb{R}^d)$ to $\mathcal{B}$.

Let $\mu$ be a non-negative Radon measure on $\mathbb{R}^d$ which only satisfies the polynomial growth condition. Let $\mathcal{Y}$ be a Banach space and $H^1(\mu)$ the Hardy space of Tolsa (see [20, 23]). In this paper, we prove that a linear operator $T$ is bounded from $H^1(\mu)$ to $\mathcal{Y}$ if and only if $T$ maps all $(p, \gamma)$-atomic blocks ([20, 8]) into uniformly bounded elements of $\mathcal{Y}$; moreover, we show that for a sublinear operator $T$ bounded from $L^1(\mu)$ to $L^{1, \infty}(\mu)$, if $T$ maps all $(p, \gamma)$-atomic blocks with $p \in (1, \infty)$ and $\gamma \in \mathbb{N}$ into uniformly bounded elements of $L^1(\mu)$, then $T$ extends to a bounded sublinear operator from $H^1(\mu)$ to $L^1(\mu)$. For the localized atomic Hardy space $h^1(\mu)$ in [10], corresponding results are also presented.

Finally, these results are applied to Calderón-Zygmund operators, Riesz potentials and multilinear commutators generated by Calderón-Zygmund operators or fractional integral operators with Lipschitz functions, to simplify the existing proofs in the corresponding papers [3, 9, 12]. Moreover, these results seal a gap existing in the proof of [9, Theorem 1.1].

Recall that a non-negative Radon measure $\mu$ on $\mathbb{R}^d$ is called a non-doubling measure, if there exist positive constants $C$ and $n \in (0, d]$ such that for all $x \in \mathbb{R}^d$ and $r > 0$,

$$\mu(B(x, r)) \leq Cr^n,$$

where $B(x, r) \equiv \{y \in \mathbb{R}^d : |x - y| < r\}$. Such a measure $\mu$ is not necessary to be doubling, which is a crucial assumption in the classical theory of harmonic analysis. In recent years, it was shown that many classical results concerning the theory of Calderón-Zygmund operators and function spaces remain valid for non-doubling measures; see, for example, [14, 15, 20, 21, 22, 16, 17]. Moreover, the harmonic analysis for non-doubling measures plays an important role in the solution of several long-standing open questions related to analytic capacity, like Painlevé’s problem and Vitushkin’s conjecture; see [24, 25, 26, 27] for more details.

To state the main results of this paper, we first recall some notation and notions.

Throughout this paper, by a cube $Q \subset \mathbb{R}^d$, we mean a closed cube whose sides are parallel to the axes and centered at certain point of $\text{supp}(\mu)$, and we denote its side length by $l(Q)$ and its center by $x_Q$. For any given $\lambda \in (0, \infty)$ and cube $Q$, $\lambda Q$ denotes the cube concentric with $Q$ and having side length $\lambda l(Q)$. Given two cubes $Q$, $R \subset \mathbb{R}^d$, let $Q_R$ be the smallest cube concentric with $Q$ containing $Q$ and $R$. We also set $\mathbb{N} \equiv \{1, 2, \cdots\}$ and $\mathbb{Z}_+ \equiv \mathbb{N} \cup \{0\}$.

The following coefficient was first introduced by Tolsa in [20] and the Hardy space $H^1(\mu)$ by Tolsa in [23].

**Definition 1.1** Given two cubes $Q$, $R \subset \mathbb{R}^d$, define

$$\delta(Q, R) \equiv \max \left\{ \int_{Q \setminus R} \frac{1}{|x - x_Q|^n} d\mu(x), \int_{R \setminus R} \frac{1}{|x - x_R|^n} d\mu(x) \right\}.$$
Definition 1.2 Given \( f \in L^1_{\text{loc}}(\mu) \), set
\[
M_\Phi(f)(x) \equiv \sup_{\varphi \sim x} \left| \int_{\mathbb{R}^d} f \varphi \, d\mu \right|,
\]
where the notation \( \varphi \sim x \) means that \( \varphi \in L^1(\mu) \cap C^1(\mathbb{R}^d) \) and satisfies
(i) \( \| \varphi \|_{L^1(\mu)} \leq 1 \),
(ii) \( 0 \leq \varphi(y) \leq \frac{1}{|y-x|^\alpha} \) for all \( y \in \mathbb{R}^d \), and
(iii) \( |\nabla \varphi(y)| \leq \frac{1}{|y-x|^\alpha} \) for all \( y \in \mathbb{R}^d \), where \( \nabla = (\partial_{x_1}, \ldots, \partial_{x_d}) \).

Definition 1.3 The Hardy space \( H^1(\mu) \) is defined to be the set of all functions \( f \in L^1(\mu) \) satisfying that \( \int_{\mathbb{R}^d} f \, d\mu = 0 \) and \( M_\Phi(f) \in L^1(\mu) \). Moreover, the norm of \( f \in H^1(\mu) \) is defined by
\[
\|f\|_{H^1(\mu)} \equiv \|f\|_{L^1(\mu)} + \|M_\Phi(f)\|_{L^1(\mu)}.
\]

We now recall atomic characterizations of the Hardy space \( H^1(\mu) \) and its localized variant in [20, 23, 8, 10].

Definition 1.4 Let \( \eta \in (1, \infty) \), \( \gamma \in \mathbb{N} \) and \( p \in (1, \infty] \). A function \( b \in L^1_{\text{loc}}(\mu) \) is called a \((p, \gamma)\)-atomic block if
(i) there exists certain cube \( R \) such that \( \text{supp}(b) \subset R \),
(ii) \( \int_{\mathbb{R}^d} b(x) \, d\mu(x) = 0 \),
(iii) for \( j = 1, 2 \), there exist functions \( a_j \) supported on cubes \( Q_j \subset R \) and numbers \( \lambda_j \in \mathbb{R} \) such that \( b = \lambda_1 a_1 + \lambda_2 a_2 \), and
\[
\|a_j\|_{L^p(\mu)} \leq [\mu(\eta Q_j)]^{1/p-1}[1 + \delta(Q_j, R)]^{-\gamma}.
\]

Then we define \( \|b\|_{H^1_{\text{atb}, \gamma}(\mu)} \equiv |\lambda_1| + |\lambda_2| \).

A function \( f \in L^1(\mu) \) is said to belong to the space \( H^1_{\text{atb}, \gamma}(\mu) \) if there exist \((p, \gamma)\)-atomic blocks \( \{b_i\}_{i \in \mathbb{N}} \) such that \( f = \sum_{i=1}^{\infty} b_i \) with \( \sum_{i=1}^{\infty} \|b_i\|_{H^1_{\text{atb}, \gamma}(\mu)} < \infty \). The \( H^1_{\text{atb}, \gamma}(\mu) \) norm of \( f \) is defined by
\[
\|f\|_{H^1_{\text{atb}, \gamma}(\mu)} \equiv \inf \left\{ \sum_{i=1}^{\infty} \|b_i\|_{H^1_{\text{atb}, \gamma}(\mu)} \right\},
\]
where the infimum is taken over all the possible decompositions of \( f \) as above.

Remark 1.1 If \( \gamma = 1 \), we denote \( H^1_{\text{atb}, \gamma}(\mu) \) simply by \( H^1_{\text{atb}}(\mu) \). The space \( H^1_{\text{atb}, \gamma}(\mu) \) when \( \gamma = 1 \) was introduced by Tolsa in [20], and when \( \gamma > 1 \) was introduced in [8]. It was proved in [20, 23, 8] that the definition of \( H^1_{\text{atb}, \gamma}(\mu) \) is independent of the chosen constant \( \eta \in (1, \infty) \) and that all the atomic Hardy spaces \( H^1_{\text{atb}, \gamma}(\mu) \) with \( \gamma \in \mathbb{N} \) and \( p \in (1, \infty] \) coincide with \( H^1(\mu) \) with equivalent norms. In the rest of this paper, unless explicitly stated, we always choose \( \eta = 2 \) and \( \gamma = 1 \) in the definition of \( H^1_{\text{atb}, \gamma}(\mu) \).
We now recall the notions of initial cubes and the localized atomic Hardy space, respectively, in [21] and [10].

**Definition 1.5** The Euclidean space \( \mathbb{R}^d \) is called an initial cube if \( \delta(Q, \mathbb{R}^d) < \infty \) for certain cube \( Q \) with \( l(Q) \in (0, \infty) \).

**Remark 1.2** In [21, p. 67], it was pointed out that if \( \delta(Q, \mathbb{R}^d) < \infty \) for certain cube \( Q \) with \( l(Q) \in (0, \infty) \), then \( \delta(Q', \mathbb{R}^d) < \infty \) for any cube \( Q' \) with \( l(Q') \in (0, \infty) \).

Let \( A \) be a big positive constant. In particular, as in [21, 23], we assume that \( A \) is much bigger than the constant \( \varepsilon_1 \) in Lemma 3.2 of [21]. In the case that \( \mathbb{R}^d \) is not an initial cube, let \( \{ R_j \}_{j \in \mathbb{Z}_+} \) be a sequence of increasing concentric ‘reference’ cubes as in [21] and

\[
\mathcal{D} = \{ Q \subset \mathbb{R}^d : \text{there exists a cube } P \subset Q \text{ and } j \in \mathbb{Z}_+ \text{ such that } P \subset R_j \text{ with } \delta(P, R_j) \leq (j + 1)A + \varepsilon_1 \}.
\]

If \( \mathbb{R}^d \) is an initial cube, we then define the set

\[
\mathcal{D} = \{ Q \subset \mathbb{R}^d : \text{there exists a cube } P \subset Q \text{ such that } \delta(P, \mathbb{R}^d) \leq A + \varepsilon_1 \}.
\]

It was pointed out in [10] that the definition of the set \( \mathcal{D} \) is independent of the chosen reference cubes \( \{ R_j \}_{j \in \mathbb{Z}_+} \) in the sense modulo certain small error; see also [21, p. 68].

**Definition 1.6** Let \( \eta \in (1, \infty), \gamma \in \mathbb{N} \) and \( p \in (1, \infty] \). A function \( b \in L^1_{\text{loc}}(\mu) \) is called a \((p, \gamma)\)-block if only (i) and (iii) of Definition 1.4 hold. Moreover, define \( |b|_{h^{1,p}_{\text{ath},\gamma}(\mu)} = \sum_{j=1}^2 |\lambda_j| \).

A function \( f \in L^1(\mu) \) is said to belong to the space \( h^{1,p}_{\text{ath},\gamma}(\mu) \) if there exist \((p, \gamma)\)-atomic blocks or \((p, \gamma)\)-blocks \( \{ b_i \} \) such that \( f = \sum_i b_i \) and \( \sum_i |b_i|_{h^{1,p}_{\text{ath},\gamma}(\mu)} < \infty \), where \( b_i \) is a \((p, \gamma)\)-atomic block as in Definition 1.4 if \( \text{supp}(b_i) \subset R_i \) and \( R_i \notin \mathcal{D} \), while \( b_i \) is a \((p, \gamma)\)-block if \( \text{supp}(b_i) \subset R_i \) and \( R_i \in \mathcal{D} \). Moreover, the \( h^{1,p}_{\text{ath},\gamma}(\mu) \) norm of \( f \) is defined by

\[
\|f\|_{h^{1,p}_{\text{ath},\gamma}(\mu)} = \inf \left\{ \sum_i |b_i|_{h^{1,p}_{\text{ath},\gamma}(\mu)} : f = \sum_i b_i \right\},
\]

where the infimum is taken over all decompositions of \( f \) as above.

**Remark 1.3** When \( \gamma = 1 \), we denote the space \( h^{1,p}_{\text{ath},\gamma}(\mu) \) simply by \( h^{1,p}_{\text{ath}}(\mu) \), which was introduced in [10]; moreover, it was proved there that the definition of \( h^{1,p}_{\text{ath}}(\mu) \) is independent of the chosen constant \( \eta \in (1, \infty) \), and that all the localized atomic Hardy spaces \( h^{1,p}_{\text{ath}}(\mu) \) with \( p \in (1, \infty) \) coincide with \( h^{1,\infty}_{\text{ath}}(\mu) \) with equivalent norms.

By the same argument as that used in the proof of Theorem 2.1 in [8], we have the following equivalent atomic characterization of \( h^{1,p}_{\text{ath},\gamma}(\mu) \). We omit the details here.
Proposition 1.1 Let $\eta \in (1,\infty)$, $\gamma \in \mathbb{N}$ with $\gamma > 1$ and $p \in (1,\infty]$. Then $h_{atb,\gamma}^{1,p}(\mu) = h_{atb}^{1,p}(\mu)$ with equivalent norms.

As a consequence of Remark 1.3 and Proposition 1.1, throughout this paper, we denote $h_{atb,\gamma}^{1,p}(\mu)$ simply by $h^1(\mu)$. Moreover, unless explicitly stated, in what follows, we always choose $\eta = 2$ and $\gamma = 1$ in the definition of $h_{atb,\gamma}^{1,p}(\mu)$.

The main results of this paper are as follows.

Theorem 1.1 Let $\eta \in (1,\infty)$, $\gamma \in \mathbb{N}$, $p \in (1,\infty)$, $T$ be a linear operator and $\mathcal{Y}$ a Banach space.

(i) If there exists a non-negative constant $C$ such that for all $(p,\gamma)$-atomic blocks $b$,

\begin{equation}
||Tb||_{\mathcal{Y}} \leq C|b|_{H_{atb,\gamma}^{1,p}(\mu)},
\end{equation}

then $T$ extends to a bounded linear operator from $H^1(\mu)$ to $\mathcal{Y}$.

(ii) If there exists a non-negative constant $\tilde{C}$ such that for all $(p,\gamma)$-atomic blocks $b$ with $\text{supp}(b) \subset R$ and $R \notin \mathcal{D}$, and all $(p,\gamma)$-blocks $b$ with $\text{supp}(b) \subset R$ and $R \in \mathcal{D}$,

\begin{equation}
||Tb||_{\mathcal{Y}} \leq \tilde{C}|b|_{h_{atb,\gamma}^{1,p}(\mu)},
\end{equation}

then $T$ extends to a bounded linear operator from $h^1(\mu)$ to $\mathcal{Y}$.

Remark 1.4 Observe that (1.1) (or (1.2)) is also necessary for an operator $T$ to be bounded from $H^1(\mu)$ (or $h^1(\mu)$) to $\mathcal{Y}$. From this fact and Theorem 1.1, we further deduce that if $T$ is linear, then $T$ extends to a bounded linear operator from $H^1(\mu)$ (or $h^1(\mu)$) to $\mathcal{Y}$ if and only if $T$ satisfies (1.1) (or (1.2)).

For sublinear operators bounded from $L^1(\mu)$ to $L^{1,\infty}(\mu)$, we also have the following conclusion.

Theorem 1.2 Let $\eta \in (1,\infty)$, $\gamma \in \mathbb{N}$, $p \in (1,\infty)$ and $T$ be a sublinear operator bounded from $L^1(\mu)$ to $L^{1,\infty}(\mu)$.

(i) If $T$ satisfies (1.1) with $\mathcal{Y} = L^1(\mu)$, then $T$ extends to a bounded sublinear operator from $H^1(\mu)$ to $L^1(\mu)$.

(ii) If $T$ satisfies (1.2) with $\mathcal{Y} = L^1(\mu)$, then $T$ extends to a bounded sublinear operator from $h^1(\mu)$ to $L^1(\mu)$.

Proofs of Theorems 1.1 and 1.2 are given in Section 2. We remark that the proof of Theorem 1.2 would be trivial if $T$ were linear. In fact, it is easy to see that if the linear operator $T$ is continuous from $L^1(\mu)$ to $L^{1,\infty}(\mu)$, and the image of atomic blocks (or blocks) is uniformly bounded in $L^1(\mu)$, then $T$ is automatically bounded from $H^1(\mu)$ (or $h^1(\mu)$) to $L^1(\mu)$. For sublinear operators, the proof of Theorem 1.2 requires only an easy additional measure theoretic argument.

In Section 3, we apply Theorem 1.1 to Calderón-Zygmund operators, Riesz potentials and multilinear commutators generated by Calderón-Zygmund operators or fractional integral operators with Lipschitz functions, to simplify the existing proofs in the corresponding
papers; see [3, Theorem 1], [9, Theorem 1.1] and [12, Theorems 3.1, 4.2]. In particular, we seal a gap existing in the proof that (III) implies (IV) of [9, Theorem 1.1] (see [9, pp. 379-381]). We also prove that if $\mathbb{R}^d$ is an initial cube, then the Calderón-Zygmund operator is bounded from $h^1(\mu)$ to $L^1(\mu)$.

We now make some conventions. Throughout this paper, we always use $C$ to denote a positive constant that is independent of the main parameters involved but whose value may differ from line to line. Constants with subscripts, such as $C_1$, do not change in different occurrences. If $f \leq Cg$, we then write $f \lesssim g$; and if $f \lesssim g \lesssim f$, we write $f \sim g$.

## 2 Proofs of Theorems 1.1 and 1.2

In this section, we show Theorems 1.1 and 1.2. To start with, we recall some useful notions and notation.

Let $p \in (1, \infty]$, $L^p_0(\mu)$ be the space of functions in $L^p(\mu)$ with compact support and $L^p_c(\mu)$ the space of functions in $L^p(\mu)$ having integral 0. Moreover, for each cube $Q$, we denote by $L^p(\mu)$ the subspace of functions in $L^p(\mu)$ supported in $Q$ and $L^p_0(\mu) \equiv L^p_c(\mu) \cap L^p(\mu)$. Then the unions of $L^p_0(\mu)$ and $L^p(\mu)$ as $Q$ varies over all cubes coincide with $L^p_c(\mu)$ and $L^p(\mu)$, respectively. Now let $\{Q_j\}_{j \in \mathbb{N}}$ be a sequence of increasing concentric cubes with $\mathbb{R}^d = \bigcup_{j \in \mathbb{N}} Q_j$. We topologize $L^p_c(\mu)$ (resp. $L^p(\mu)$) as the strict inductive limit of the spaces $L^p_0(Q_j)$ (resp. $L^p(Q_j)$) (see [1, II, p. 33] for the definition of the strict inductive limit topology). It is known that the definition of the topology of $L^p_c(\mu)$ (resp. $L^p(\mu)$) is independent of the choice of $\{Q_j\}_{j \in \mathbb{N}}$.

We now recall the definitions of RBMO (µ) of Tolsa in [20] and rbmo (µ) in [10].

**Definition 2.1** (i) Let $p \in [1, \infty)$. A function $f \in L^1_{\text{loc}}(\mu)$ is said to be in the space RBMO (µ) if there exists a nonnegative constant $C$ such that for any doubling cube $Q$,

$$
\left( \frac{1}{\mu(Q)} \int_Q |f(y) - m_Q(f)|^p \, d\mu(y) \right)^{1/p} \leq C, \tag{2.1}
$$

and for any two doubling cubes $Q \subset R$,

$$
|m_Q(f) - m_R(f)| \leq C[1 + \delta(Q, R)], \tag{2.2}
$$

where $m_Q(f)$ denotes the mean of $f$ over cube $Q$, namely, $m_Q(f) \equiv \frac{1}{\mu(Q)} \int_Q f(y) \, d\mu(y)$. Moreover, we define the RBMO (µ) norm of $f$ to be the minimal constant $C$ as above and denote it by $\|f\|_{\text{RBMO}(\mu)}$.

(ii) Let $p \in [1, \infty)$. A function $f \in L^1_{\text{loc}}(\mu)$ is said to be in the space rbmo (µ) if there exists a nonnegative constant $C$ such that (2.1) holds for any doubling cube $Q \notin D$, (2.2) holds for any two doubling cubes $Q \subset R$ with $Q \notin D$, and for any doubling cube $Q \in D$,

$$
\left( \frac{1}{\mu(Q)} \int_Q |f(y)|^p \, d\mu(y) \right)^{1/p} \leq C.
$$

Moreover, we define the rbmo (µ) norm of $f$ to be the minimal constant $C$ as above and denote it by $\|f\|_{\text{rbmo}(\mu)}$. 
Remark 2.1 In [20], Tolsa showed that RBMO $(\mu)$ is the dual space of $H^1(\mu)$. On the other hand, it was proved in [10] that rbmo $(\mu)$ is the dual space of $h^1(\mu)$.

Proof of Theorem 1.1. We first show (i) of Theorem 1.1. To this end, without loss of generality, we may assume $p = 2$. Moreover, by Remark 1.1, we choose $\eta = 2$ and $\gamma = 1$ in the definition of $H_{\text{atb},\gamma}^1(\mu)$. Let $Q$ be a fixed cube. If $f \in L^2_0(Q)$, then $f$ is a $(2,1)$-atomic block and

$$
|f|_{H_{\text{atb},2}^1(\mu)} \leq \|f\|_{L^2(\mu)}|\mu(2Q)|^{1/2}. \tag{2.3}
$$

Moreover, from this and (1.1), it follows that for any sequence of increasing concentric cubes $\{Q_j\}_{j \in \mathbb{N}}$ with $\mathbb{R}^d = \bigcup_{j \in \mathbb{N}} Q_j$, $T$ is bounded from $L^2_0(Q_j)$ to $\mathcal{Y}$ for each $j \in \mathbb{N}$. Then $T$ is bounded from $L^2_{\text{c},0}(\mu)$ to $\mathcal{Y}$, which implies that the adjoint operator $T^*$ of $T$ is bounded from the dual space $\mathcal{Y}^*$ of $\mathcal{Y}$ to $[L^2_{\text{c},0}(\mu)]^*$. Moreover, for all functions $f \in \mathcal{Y}^*$ and $(2,1)$-atomic blocks $b$, we have

$$
\left| \int_{\mathbb{R}^d} b(x) T^*(f)(x) \, d\mu(x) \right| = |\langle Tb, f \rangle| \lesssim \|f\|_{\mathcal{Y}^*} |b|_{H_{\text{atb},2}^1(\mu)}. \tag{2.4}
$$

We claim that for all $f \in \mathcal{Y}^*$, $T^* f \in \text{RBMO}(\mu)$ and $\|T^* f\|_{\text{RBMO}(\mu)} \lesssim \|f\|_{\mathcal{Y}^*}$. In fact, observe that for any doubling cube $Q$ and $\phi \in L^2(Q)$ with $\|\phi\|_{L^2(Q)} = 1$, $[\phi - m_Q(\phi)] \chi_Q$ is a $(2,1)$-atomic block, where and in what follows, $\chi_Q$ denotes the characteristic function of the set $Q$. From this, (2.3) and (2.4), we deduce that

\[
\left[ \int_Q |T^* f(x) - m_Q(T^* f)|^2 \, d\mu(x) \right]^{1/2} = \sup_{\|\phi\|_{L^2(Q)} = 1} \left| \int_Q \phi(x) [T^* f(x) - m_Q(T^* f)] \, d\mu(x) \right| \\
= \sup_{\|\phi\|_{L^2(Q)} = 1} \left| \int_Q \phi(x) - m_Q(\phi) |T^* f(x)| \, d\mu(x) \right| \\
\lesssim \|f\|_{\mathcal{Y}^*} \|\mu(Q)\|^{1/2},
\]

which implies that

$$
\left[ \frac{1}{\mu(Q)} \int_Q |T^* f(x) - m_Q(T^* f)|^2 \, d\mu(x) \right]^{1/2} \lesssim \|f\|_{\mathcal{Y}^*}. \tag{2.5}
$$

By (2.5) and Definition 2.1 (i), the claim is reduced to showing that for all doubling cubes $Q \subset R$,

$$
|m_Q(T^* f) - m_R(T^* f)| \lesssim [1 + \delta(Q, R)] \|f\|_{\mathcal{Y}^*}. \tag{2.6}
$$

Let

\[
a_1 \equiv \frac{|T^* f - m_R(T^* f)|^2}{T^* f - m_R(T^* f)} \chi_{Q \cap (T^* f \neq m_R(T^* f))},
\]

$a_2 \equiv C_R \chi_R$ and $b \equiv a_1 + a_2$, where $C_R$ is a constant such that $b$ has integral $0$. Then $b$ is a $(2,1)$-atomic block and

$$
|b|_{H_{\text{atb},2}^1(\mu)} \lesssim \|a_1\|_{L^2(\mu)} \|\mu(Q)\|^{1/2} [1 + \delta(Q, R)] + |C_R| \mu(R)
$$
\[
\int_Q |T^*f(x) - m_R(T^*f)|^2 \, d\mu(x) \\
= \int_{\mathbb{R}^d} a_1(x)|T^*f(x) - m_R(T^*f)| \, d\mu(x) \\
\leq \left[ \int_{\mathbb{R}^d} b(x)|T^*f(x)| \, d\mu(x) \right] + |C_R| \int_R |T^*f(x) - m_R(T^*f)| \, d\mu(x) \\
\lesssim \left[ \int_Q |T^*f(x) - m_R(T^*f)|^2 \, d\mu(x) \right]^{1/2} [\mu(Q)]^{1/2} [1 + \delta(Q, R)] \|f\|_{Y^*},
\]
which implies that
\[
\left[ \frac{1}{\mu(Q)} \int_Q |T^*f(x) - m_R(T^*f)|^2 \, d\mu(x) \right]^{1/2} \lesssim [1 + \delta(Q, R)] \|f\|_{Y^*}.
\]
From this, the Hölder inequality and (2.5), it then follows that
\[
|m_Q(T^*f) - m_R(T^*f)| \lesssim \frac{1}{\mu(Q)} \int_Q \|m_Q(T^*f) - T^*f(x)\| + |T^*f(x) - m_R(T^*f)| \, d\mu(x) \\
\lesssim [1 + \delta(Q, R)] \|f\|_{Y^*},
\]
which implies (2.6). By this together with (2.4), we obtain that \(T^*f \in \text{RBMO}(\mu)\) and \(\|T^*f\|_{\text{RBMO}(\mu)} \lesssim \|f\|_{Y^*}\). Thus, the claim is true.

Let \(H^{1,2}_{\text{fin}}(\mu)\) be the set of all finite linear combinations of \((2,1)\)-atomic blocks. Then \(H^{1,2}_{\text{fin}}(\mu)\) is dense in \(H^{1}(\mu)\). On the other hand, \(H^{1,2}_{\text{fin}}(\mu)\) coincides with \(L^{2}_{1,0}(\mu)\) as vector spaces. Then by Remark 2.1 and the above claim, we have that for all \(g \in H^{1,2}_{\text{fin}}(\mu)\) and \(f \in Y^*\) with \(\|f\|_{Y^*} = 1\), \(\|Tg, f\| = \|g, T^*f\| \lesssim \|g\|_{H^{1}(\mu)} \|T^*f\|_{\text{RBMO}(\mu)} \lesssim \|g\|_{H^{1}(\mu)}\). From this and (1.1), it follows that \(Tg \in \mathcal{Y}\) and \(\|Tg\|_{\mathcal{Y}} \lesssim \|g\|_{H^{1}(\mu)}\), which via a density argument then completes the proof of Theorem 1.1 (i).

We now prove (ii). Similarly to (i), without loss of generality, we may assume that \(p = 2\) and we choose \(\eta = 2\) and \(\gamma = 1\) in the definition of \(h^{1,p}_{\text{atb}, \gamma}(\mu)\). Using an argument similar to (i), we see that if \(T\) satisfies (1.2), then \(T\) is bounded from \(L^{2}_{1,0}(\mu)\) to \(\mathcal{Y}\), which implies that \(T^*\) is bounded from \(Y^*\) to \(L^{2}_{1}(\mu)^*\). Moreover, we have that for all \(f \in Y^*\) and \((2,1)\)-atomic blocks or \((2,1)\)-blocks \(b\) as in (1.2),

\[
\left[ \int_{\mathbb{R}^d} b(x)|T^*(f)(x)| \, d\mu(x) \right] = |\langle Tb, f \rangle| \lesssim \|f\|_{Y^*} \|b\|_{h^{1,2}_{\text{atb}}(\mu)},
\]

We claim that for all \(f \in Y^*\), \(T^*f \in \text{rbmo}(\mu)\) and \(\|T^*f\|_{\text{rbmo}(\mu)} \lesssim \|f\|_{Y^*}\). In fact, we first prove that for any doubling cube \(Q \in \mathcal{D}\),

\[
|m_Q(T^*f)| \lesssim \|f\|_{Y^*}.
\]
Let \( Q \in \mathcal{D} \) be doubling. Observe that for any doubling cube \( Q \) and \( \phi \in L^2(Q) \) with \( \|\phi\|_{L^2(Q)} = 1 \), \( \phi \) is a \((2,1)\)-block. From this and (2.7), it follows that

\[
\left[ \int_Q |T^* f(x)|^2 \, d\mu(x) \right]^{1/2} = \sup_{\|\phi\|_{L^2(Q)} = 1} \left| \int_Q \phi(x) T^* f(x) \, d\mu(x) \right| \lesssim \|f\|_{Y^*} \|\phi\|_{h^{1,2}(\mu)} \lesssim \|f\|_{Y^*} [\mu(Q)]^{1/2},
\]

which via the Hölder inequality yields (2.8).

By the proof of (2.5), we also have that for any doubling cube \( Q \notin \mathcal{D} \),

\[
\tag{2.9}
\frac{1}{\mu(Q)} \int_Q |T^* f(x) - m_Q(T^* f)| \, d\mu(x) \lesssim \|f\|_{Y^*}.
\]

By this and (2.8) together with Definition 2.1 (ii), to show the claim, it suffices to prove that for any two doubling cubes \( Q \subset R \) with \( Q \notin \mathcal{D} \),

\[
\tag{2.10}
|m_Q(T^* f) - m_R(T^* f)| \lesssim [1 + \delta(Q, R)] \|f\|_{Y^*}.
\]

In fact, if \( R \notin \mathcal{D} \), then by the proof of (2.6), we obtain (2.10). Now suppose that \( R \in \mathcal{D} \). We set \( a = a_1 \), where \( a_1 \) is as in the proof of (2.6). Then \( a \) is a \((2,1)\)-block with \( \text{supp}(a) \subset R \) and

\[
|a|_{h^{1,2}_a(\mu)} \lesssim \left[ \int_Q |T^* f(x) - m_R(T^* f)|^2 \, d\mu(x) \right]^{1/2} \left[ \mu(Q) \right]^{1/2} \left[ 1 + \delta(Q, R) \right].
\]

By this, (2.7), (2.8) with \( Q \) replaced by \( R \) and the Hölder inequality, we see that

\[
\int_Q |T^* f(x) - m_R(T^* f)|^2 \, d\mu(x)
\]

\[
= \int_Q [T^* f(x) - m_R(T^* f)] a(x) \, d\mu(x)
\]

\[
\leq \left| \int_Q T^* f(x) a(x) \, d\mu(x) \right| + |m_R(T^* f)| \int_Q |a(x)| \, d\mu(x)
\]

\[
\lesssim [1 + \delta(Q, R)] \|f\|_{Y^*} \left[ \int_Q |T^* f(x) - m_R(T^* f)|^2 \, d\mu(x) \right]^{1/2} \left[ \mu(Q) \right]^{1/2}.
\]

This in turn implies that

\[
\left[ \frac{1}{\mu(Q)} \int_Q |T^* f(x) - m_R(T^* f)|^2 \, d\mu(x) \right]^{1/2} \lesssim [1 + \delta(Q, R)] \|f\|_{Y^*},
\]

which together with (2.9) and the Hölder inequality yields (2.10). Combining (2.8), (2.9) and (2.10) implies the claim.

Let \( h^{1,2}_{\text{fin}}(\mu) \) be the set of all finite linear combinations of all \((2,1)\)-atomic blocks or \((2,1)\)-blocks \( b \) as in (1.2). Then \( h^{1,2}_{\text{fin}}(\mu) \) is dense in \( h^1(\mu) \). On the other hand, \( h^{1,2}_{\text{fin}}(\mu) \)
Since $\eta > 1.1$, we choose $\eta$.

Proof of Theorem 1.2. By similarity we only prove (i). As in the proof of Theorem 1.1, the argument then completes the proof of Theorem 1.1 (ii). This finishes the proof of Theorem 1.1.

**Proof of Theorem 1.2.** By similarity we only prove (i). As in the proof of Theorem 1.1, we choose $\eta = 2$ and $\gamma = 1$ in the definitions of $H^1_{\text{atb}, \gamma}(\mu)$. Let $f \in L^1(\mu)$ and $f = \sum_{i=1}^{\infty} b_i$, where for each $i \in \mathbb{N}$, $b_i$ is a $(p, 1)$-atomic block with $p$ as in the theorem. Since $H^1(\mu) \subset L^1(\mu)$ and $T$ is bounded from $L^1(\mu)$ to $L^1, \infty(\mu)$, we see that $Tf$ is well defined. Furthermore, by the boundedness of $T$ from $L^1(\mu)$ to $L^1, \infty(\mu)$, we have that for any $\epsilon > 0$,

$$\lim_{N \to \infty} \mu \left( \left\{ x \in \mathbb{R}^d : T \left( \sum_{i=N+1}^{\infty} b_i \right)(x) > \epsilon \right\} \right) \leq \lim_{N \to \infty} \frac{1}{\epsilon} \sum_{i=N+1}^{\infty} \| b_i \|_{L^1(\mu)} = 0.$$ 

This via the Riesz theorem implies that there exists a subsequence $\{T(\sum_{i=1}^{j_k} b_i)\}_{j_k}$ such that for $\mu$-a.e. $x \in \mathbb{R}^d$,

$$|Tf(x)| \leq \left| T \left( \sum_{i=1}^{j_k-1} b_i \right)(x) \right| + T \left( \sum_{i=j_k}^{\infty} b_i \right)(x) \leq \sum_{i=1}^{j_k-1} |Tb_i(x)| + T \left( \sum_{i=j_k}^{\infty} b_i \right)(x) \to \sum_{i=1}^{\infty} |Tb_i(x)|, j_k \to \infty.$$ 

Since $T$ is sublinear, then from this fact, we deduce that for $\mu$-a.e. $x \in \mathbb{R}^d$, $|Tf(x)| \lesssim \sum_{i=1}^{\infty} |Tb_i(x)|$, which together with (1.1) in turn implies that

$$\| Tf \|_{L^1(\mu)} \lesssim \sum_{i=1}^{\infty} \| T b_i \|_{L^1(\mu)} \lesssim \sum_{i=1}^{\infty} \| b_i \|_{H^1_{\text{atb}}(\mu)}.$$ 

By this, we have that $Tf \in L^1(\mu)$ and $\| Tf \|_{L^1(\mu)} \lesssim \| f \|_{H^1(\mu)}$. This finishes the proof of Theorem 1.2 (i), and hence, the proof of Theorem 1.2.

### 3 Applications

In this section, we apply Theorems 1.1 to Calderón-Zygmund operators, Riesz potentials and multilinear commutators generated by Calderón-Zygmund operators or fractional integral operators with Lipschitz functions, to simplify the existing proofs in the corresponding papers. We also prove that if $\mathbb{R}^d$ is an initial cube, then the Calderón-Zygmund operator is bounded from $h^1(\mu)$ to $L^1(\mu)$. 
3.1 Calderón-Zygmund operators and Riesz potentials

Recall that a $\mu$-locally integrable function $K$ on $\mathbb{R}^d \times \mathbb{R}^d \setminus \{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d : x = y\}$ is called a Calderón-Zygmund kernel if there exists a positive constant $C$ such that for all $x, y \in \mathbb{R}^d$ with $x \neq y$,

$$|K(x, y)| \leq C \frac{1}{|x - y|^n},$$

and for all $x, x' \in \mathbb{R}^d$ with $|x - x'| \leq |x - y|/2$,

$$|K(x, y) - K(x', y)| + |K(y, x) - K(y, x')| \leq C \frac{|x - x'|}{|x - y|^{n+1}}.$$  

(3.1) \hspace{1cm} (3.2)

For all $\epsilon \in (0, \infty)$, $x \in \mathbb{R}^d$ and $f \in L^2(\mu)$, define the truncated operator $T_\epsilon$ by

$$T_\epsilon f(x) \equiv \int_{|x-y| \geq \epsilon} K(x, y) f(y) d\mu(y).$$

It is known that if the operators $\{T_\epsilon\}_{\epsilon > 0}$ are bounded on $L^2(\mu)$ uniformly, then there exists an operator $T$ which is the weak limit as $\epsilon \to 0$ of certain subsequence of $\{T_\epsilon\}_{\epsilon > 0}$; see [20]. It was proved in [3] that the operator $T$ is also bounded on $L^2(\mu)$ and satisfies that for all $f \in L^2(\mu)$ with bounded support and $x \notin \text{supp}(f)$,

$$T f(x) \equiv \int_{\mathbb{R}^d} K(x, y) f(y) d\mu(y).$$

(3.3)

The following Proposition 3.1 was claimed in [3] without a proof. Using Theorem 1.1, we can give a simpler proof of Proposition 3.1 as below.

**Proposition 3.1** Let $T$ be a bounded linear operator on $L^2(\mu)$ as in (3.3) with the kernel $K$ satisfying (3.1) and (3.2). Then $T$ extends to a bounded linear operator from $H^1(\mu)$ to $L^1(\mu)$.

**Proof.** Let $b \equiv \lambda_1 a_1 + \lambda_2 a_2$ be any $(2,1)$-atomic block as in Definition 1.4. Since $T$ is linear, we write

$$\|Tb\|_{L^1(\mu)} \leq \sum_{j=1}^{2} |\lambda_j| \int_{2Q_j} |Ta_j(x)| \, d\mu(x) + \sum_{j=1}^{2} |\lambda_j| \int_{(2\sqrt{d}R) \setminus (2Q_j)} \cdots + \int_{\mathbb{R}^d \setminus (2\sqrt{d}R)} |Tb(x)| \, d\mu(x) \equiv I_1 + I_2 + I_3,$$

where for $j = 1, 2$, $Q_j$ and $R$ are as in Definition 1.4. Using the Hölder inequality, the boundedness of $T$ on $L^2(\mu)$ and Definition 1.4, we have that $I_1 \lesssim \sum_{j=1}^{2} |\lambda_j|$. By an argument similar to that used in [20, p.113-114], we obtain $I_2 + I_3 \lesssim \sum_{j=1}^{2} |\lambda_j|$, which combined the estimate of $I_1$ implies (1.1) with $\mathcal{Y} = L^1(\mu)$. This together with Theorem 1.1 and $H^1(\mu) = H^{1,2}_{\text{at}}(\mu)$ with equivalent norms yields the boundedness of $T$ from $H^1(\mu)$ to $L^1(\mu)$, which completes the proof of Proposition 3.1.
Let $T$ be as in Proposition 3.1. Recall that $T^* 1 = 0$ means that for any bounded function $b$ with compact support and $\int_{\mathbb{R}^d} b(x) \, d\mu(x) = 0$,
\[
\int_{\mathbb{R}^d} Tb(x) \, d\mu(x) = 0;
\]
see [3]. By Proposition 3.1, this makes sense.

If $T^* 1 = 0$, using Proposition 3.1 again, we can also complete the proof of Theorem 1 in [3] as follows.

**Proposition 3.2** Let $T$ be the same as in Proposition 3.1 and $T^* 1 = 0$. Then $T$ extends to a bounded linear operator on $H^1(\mu)$.

**Proof.** Let $\mathcal{M}_\Phi$ be as in Definition 1.2 and take $\eta = 4$ in the definition of $H^{1, 2}_{atb, 2}(\mu)$. For any $f \in H^{1, 2}_{atb, 2}(\mu)$, by Definition 1.4, there exist $(2, 2)$-atomic blocks $\{b_i\}_{i=1}^\infty$ such that $f = \sum_{i=1}^\infty b_i$ and $\sum_{i=1}^\infty |b_i|_{H^{1, 2}_{atb, 2}(\mu)} < \infty$. By an argument similar to that used in the proof of Theorem 1 in [8], we have that for all $b_i$,
\[
\|\mathcal{M}_\Phi(Tb_i)\|_{L^1(\mu)} \lesssim |b_i|_{H^{1, 2}_{atb, 2}(\mu)}.
\]

On the other hand, by an argument similar to the proof of Theorem 4.2 in [20], we obtain
\[
\left\| \sum_{i=1}^\infty |Tb_i| \right\|_{L^1(\mu)} \leq \sum_{i=1}^\infty \|Tb_i\|_{L^1(\mu)} \lesssim \sum_{i=1}^\infty |b_i|_{H^{1, 2}_{atb, 2}(\mu)} < \infty.
\]

Observe that for each $x \in \mathbb{R}^d$ and $\varphi \sim x$, there exists a positive constant $M$, depending on $x$, such that for all $y \in \mathbb{R}^d$, $0 \leq \varphi(y) \leq M$. Moreover, by Proposition 3.1, we obtain that $Tf = \sum_{i=1}^\infty Tb_i$ in $L^1(\mu)$. These two facts together with (3.5) and the Lebesgue dominated convergence theorem yield that
\[
\int_{\mathbb{R}^d} \varphi(y) Tf(y) \, d\mu(y) = \int_{\mathbb{R}^d} \sum_{i=1}^\infty \varphi(y) Tb_i(y) \, d\mu(y) = \sum_{i=1}^\infty \int_{\mathbb{R}^d} \varphi(y) Tb_i(y) \, d\mu(y).
\]

From this, it further follows that for all $x \in \mathbb{R}^d$,
\[
\mathcal{M}_\Phi(Tf)(x) \leq \sum_{i=1}^\infty \mathcal{M}_\Phi(Tb_i)(x),
\]
which together with the Levi lemma and (3.4) yields that
\[
\|\mathcal{M}_\Phi(Tf)\|_{L^1(\mu)} \leq \left\| \sum_{i=1}^\infty \mathcal{M}_\Phi(Tb_i) \right\|_{L^1(\mu)} \leq \sum_{i=1}^\infty \|\mathcal{M}_\Phi(Tb_i)\|_{L^1(\mu)} \lesssim \sum_{i=1}^\infty |b_i|_{H^{1, 2}_{atb, 2}(\mu)}.
\]

This together with Definition 1.2 and $H^1(\mu) = H^{1, 2}_{atb, 2}(\mu)$ with equivalent norms in turn implies that $Tf \in H^1(\mu)$ and $\|Tf\|_{H^1(\mu)} \lesssim \|f\|_{H^1(\mu)}$, which completes the proof of Proposition 3.2.
Proposition 3.3 Let \( \mathbb{R}^d \) be an initial cube and \( T \) as in Proposition 3.1. Then \( T \) extends to a bounded linear operator from \( h^1(\mu) \) to \( L^1(\mu) \).

**Proof.** By Theorem 1.1 (ii), we only need to prove that \( T \) satisfies (1.2) with \( \mathcal{Y} = L^1(\mu) \). Following the proof of Proposition 3.1, we see that for all \((2,1)\)-atomic blocks \( b \) with \( \text{supp} (b) \subset R \) and \( R \notin \mathcal{D} \), \( \| Tb \|_{L^1(\mu)} \lesssim |b|_{h^1_{atb}(\mu)} \). Now assume that \( b \) is a \((2,1)\)-block with \( \text{supp} (b) \subset R \) and \( R \in \mathcal{D} \). Since \( T \) is linear, we write \( \| Tb \|_{L^1(\mu)} \leq I_1 + I_2 + I_3 \), where \( I_j \), \( j = 1, 2, 3 \), are as in Proposition 3.1. By the same argument as in the proof of Proposition 3.1, we have that \( I_1 + I_2 \lesssim |b|_{h^1_{atb}(\mu)} \). It remains to estimate \( I_3 \). Since \( R \in \mathcal{D} \) and \( \mathbb{R}^d \) is an initial cube, we see that \( \delta (R, \mathbb{R}^d) \lesssim 1 \). From this together with (3.1), (3.3), Definition 1.6 and the fact that for all \( x \in \mathbb{R}^d \setminus (2\sqrt{d}R) \) and \( y \in R \), \( |x - x_R| \lesssim |x - y| \), it follows that

\[
I_3 \lesssim \int_{\mathbb{R}^d \setminus (2\sqrt{d}R)} \frac{|b|}{|x - y|^n} d\mu(y) \, d\mu(x)
\]

\[
\lesssim \int_{\mathbb{R}^d \setminus (2\sqrt{d}R)} \frac{\|b\|_{L^1(\mu)}}{|x - x_R|^n} d\mu(x) \lesssim \delta (R, \mathbb{R}^d) \|b\|_{L^1(\mu)} \lesssim |b|_{h^1_{atb}(\mu)}.
\]

This combined with the estimates of \( I_1 \) and \( I_2 \) implies for all \((2,1)\)-blocks \( b \) with \( \text{supp} (b) \subset R \) and \( R \in \mathcal{D} \), \( \| Tb \|_{L^1(\mu)} \lesssim |b|_{h^1_{atb}(\mu)} \), which together with the estimate for \((2,1)\)-atomic blocks \( b \) with \( \text{supp} (b) \subset R \) and \( R \notin \mathcal{D} \) further completes the proof of Proposition 3.3.

We now consider Riesz potentials in [4]. Let \( \alpha \in (0, n) \) and \( K_\alpha \) be a locally integrable function on \( \mathbb{R}^d \times \mathbb{R}^d \setminus \{x = y\} \) satisfying that there exists a positive constant \( C \) such that for all \( x, y \in \mathbb{R}^d \) with \( x \neq y \),

\[
(3.6) \quad |K_\alpha(x, y)| \leq C \frac{1}{|x - y|^{n - \alpha}}.
\]

and for all \( x, x' \) and \( y \in \mathbb{R}^d \) with \( |x - x'| \leq |x - y|/2 \),

\[
(3.7) \quad |K_\alpha(x, y) - K_\alpha(x', y)| + |K_\alpha(y, x) - K_\alpha(y, x')| \leq C \frac{|x - x'|^\delta}{|x - y|^{n - \alpha + \delta}},
\]

where \( \delta \in (0, 1] \). The operator \( T^\alpha \) associated to the above kernel \( K_\alpha \) and the measure \( \mu \) is defined by setting, for all \( f \in L^2(\mu) \) with bounded support and \( x \notin \text{supp} (f) \),

\[
(3.8) \quad T^\alpha f(x) \equiv \int_{\mathbb{R}^d} K_\alpha(x, y) f(y) \, d\mu(y).
\]

The operator \( T^\alpha \) was introduced by García-Cuerva and Gatto in [4]. By sealing the gap existing in the proof that (III) implies (IV) of Theorem 1.1 in [9], we have the boundedness of \( T^\alpha \) as follows.

**Proposition 3.4** Let \( \alpha \in (0, n) \) and \( T^\alpha \) be a linear operator as in (3.8) with the kernel \( K_\alpha \) satisfying (3.6) and (3.7). Then \( T^\alpha \) extends to a bounded linear operator from \( H^1(\mu) \) to \( L^{2/(n - \alpha)}(\mu) \).
Proof. Take $\eta = 4$ in the definition of $H_{\text{atb}}^{1, n/\alpha} (\mu)$ and let $b \equiv \lambda_1 a_1 + \lambda_2 a_2$ be any $(n/\alpha, 1)$-atomic block. Since $T^\alpha$ is linear, we write

$$
\|T^\alpha b\|_{L^{n/(n-\alpha)} (\mu)} \lesssim \sum_{j=1}^{2} |\lambda_j|^{n/(n-\alpha)} \int_{2Q_j} |T^\alpha a_j(x)|^{n/(n-\alpha)} \, d\mu(x) + \sum_{j=1}^{2} |\lambda_j|^{n/(n-\alpha)} \int_{(2\sqrt{R}) \setminus (2Q_j)} \cdots \int_{(2\sqrt{R}) \setminus (2Q_j)} |T^\alpha b(x)|^{n/(n-\alpha)} \, d\mu(x) \equiv L_1 + L_2 + L_3,
$$

where for $j = 1, 2, Q_j$ and $R$ are as in Definition 1.4. Recall that $T^\alpha$ is bounded from $L^p(\mu)$ to $L^q(\mu)$ for all $p \in (1, n/\alpha)$ and $q$ with $1/q = 1/p - n/\alpha$ (see [4]). By an argument similar to the proof in [9, pp. 376-380], we have that for all cubes $Q$ and functions $a \in L^{n/\alpha} (\mu)$ supported in $Q$,

$$
\int_Q |T^\alpha a(x)|^{n/(n-\alpha)} \, d\mu(x) \lesssim \|a\|_{L^{n/\alpha} (\mu)} (2Q).
$$

From this and Definition 1.4, it follows that $L_1 \lesssim \sum_{j=1}^{2} |\lambda_j|^{n/(n-\alpha)}$. Moreover, arguing as the proof in [9, p. 381], we obtain that $L_2 + L_3 \lesssim \sum_{j=1}^{2} |\lambda_j|^{n/(n-\alpha)}$. This together with the estimate of $L_1$ implies (1.1) with $Y = L^{n/(n-\alpha)} (\mu)$, from which, Theorem 1.1 and $H^1 (\mu) = H_{\text{atb}}^{1, n/\alpha} (\mu)$ with equivalent norms, it follows that $T^\alpha$ extends to a bounded linear operator from $H^1 (\mu)$ to $L^{n/(n-\alpha)} (\mu)$. This finishes the proof of Proposition 3.4.

3.2 Multilinear commutators

This subsection is devoted to the boundedness of multilinear commutators of Calderón-Zygmund operators and fractional integral operators with Lipschitz functions. We begin with the definition of Lipschitz functions in [5].

Definition 3.1 Let $\beta \in (0, \infty)$. A function $f \in L^1_{\text{loc}} (\mu)$ is said to belong to the space $\text{Lip} (\beta, \mu)$ if there exists a positive constant $C$ such that for $\mu$-almost every $x$ and $y \in \text{supp} (\mu)$,

$$
|f(x) - f(y)| \leq C|x - y|^\beta.
$$

Moreover, we define the $\text{Lip} (\beta, \mu)$ norm of $f$ to be the minimal constant $C$ in (3.9) and denote it by $\|f\|_{\text{Lip} (\beta, \mu)}$.

Let $T$ be a bounded linear operator on $L^2 (\mu)$ as in (3.3) with the kernel $K$ satisfying (3.1) and (3.2), $m \in \mathbb{N}$, $\beta_i \in (0, 1]$ and $h_i \in \text{Lip} (\beta_i, \mu)$, $i = 1, \cdots, m$. The multilinear commutator $T_{\tilde{h}}$ is formally defined by

$$
T_{\tilde{h}}(f) \equiv [h_m, \cdots, [h_2, [h_1, T]] \cdots](f),
$$

where $\tilde{h} \equiv (h_1, h_2, \cdots, h_m)$ and

$$
[h_1, T](f) \equiv h_1 T(f) - T(h_1 f).
$$

The operator $T_{\tilde{h}}$ was introduced in [12] and the following Proposition 3.5 was also obtained there (see [12, Theorem 3.1]). Using Theorem 1.1, we can also give a simpler proof of this proposition as below.
Proposition 3.5 Let \( m \in \mathbb{N}, \beta_i \in (0,1], h_i \in \text{Lip} (\beta_i, \mu) \) for \( i = 1, \ldots, m \), and \( T_h \) be as in (3.10). If \( \beta \equiv \sum_{i=1}^{m} \beta_i < n \) and \( 1/q = 1 - \beta/n \), then \( T_h \) extends to a bounded linear operator from \( H^1(\mu) \) to \( L^q(\mu) \).

Proof. Take \( \eta = 4 \) in the definition of \( H_{\text{atb}}^{1,n/\beta}(\mu) \). Repeating the proof of Theorem 3.1 in [12], we have that for all \( (n/\beta,1) \)-atomic blocks \( b \), \( \| T_h(b) \|_{L^q(\mu)} \lesssim \| b \|_{H_{\text{atb}}^{1,n/\beta}(\mu)} \). This implies (1.1) with \( Y = L^q(\mu) \). Since \( T_h \) is linear, then an application of Theorem 1.1 together with \( H^1(\mu) = H_{\text{atb}}^{1,n/\beta}(\mu) \) with equivalent norms yields that \( T_h \) extends to a bounded linear operator from \( H^1(\mu) \) to \( L^q(\mu) \), which completes the proof of Proposition 3.5.

We now consider multilinear commutators generated by fractional integral operators and Lipschitz functions in [12]. To be precise, let \( \alpha \in (0,n), x \in \text{supp} (\mu) \) and \( f \in L^\infty(\mu) \) with bounded support. The fractional integral operator \( I_\alpha \) is defined by

\[
I_\alpha(f)(x) = \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{n-\alpha}} \, d\mu(y).
\]

In [6], García-Cuerva and Martell introduced the operator \( I_\alpha \) and proved that \( I_\alpha \) is bounded from \( L^p(\mu) \) to \( L^{q,\infty}(\mu) \) with \( p \in [1, n/\alpha) \) and \( 1/q = 1/p - \alpha/n \). For \( m \in \mathbb{N} \) and \( h_i \in \text{Lip} (\beta_i, \mu) \), where \( \beta_i \in (0,1], i = 1, \ldots, m \), and \( \alpha + \sum_{i=1}^{m} \beta_i < n \), define the multilinear commutator \( I_{\alpha,\vec{h}} \) by setting, for all \( f \in L^2(\mu) \) with bounded support and \( x \notin \text{supp} (f) \),

\[
I_{\alpha,\vec{h}}(f)(x) = \int_{\mathbb{R}^d} \prod_{i=1}^{m} [h_i(x) - h_i(y)] \frac{f(y)}{|x-y|^{n-\alpha}} \, d\mu(y).
\]

(3.11)

The commutator \( I_{\alpha,\vec{h}} \) was also introduced in [12] and the following Proposition 3.6 is Theorem 4.2 in [12]. Applying Theorem 1.1, we can also give a simpler proof of Proposition 3.6 as below.

Proposition 3.6 Let \( \alpha \in (0,n), m \in \mathbb{N}, \beta_i \in (0,1], h_i \in \text{Lip} (\beta_i, \mu) \) for \( i = 1, \ldots, m \), and \( I_{\alpha,\vec{h}} \) be as in (3.11). If \( \beta \equiv \alpha + \sum_{i=1}^{m} \beta_i < n \) and \( 1/q = 1 - \beta/n \), then \( I_{\alpha,\vec{h}} \) extends to a bounded linear operator from \( H^1(\mu) \) to \( L^q(\mu) \).

Proof. Take \( \eta = 4 \) in the definition of \( H_{\text{atb}}^{1,n/\beta}(\mu) \). Arguing as in the proof of Theorem 3.1 in [12], we see that for all \( (n/\beta,1) \)-atomic blocks \( b \), \( \| I_{\alpha,\vec{h}}(b) \|_{L^q(\mu)} \lesssim \| b \|_{H_{\text{atb}}^{1,n/\beta}(\mu)} \), which implies (1.1) with \( Y = L^q(\mu) \). From this together with Theorem 1.1 and \( H^1(\mu) = H_{\text{atb}}^{1,n/\beta}(\mu) \) with equivalent norms, it follows that \( I_{\alpha,\vec{h}} \) extends to a bounded linear operator from \( H^1(\mu) \) to \( L^q(\mu) \). This finishes the proof of Proposition 3.6.

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